AN EMBEDDING, AN EXTENSION, AND AN INTERPOLATION OF ULTRAMETRICS

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Abstract. The notion of the ultrametrics can be considered as a zero-dimensional analogue of ordinary metrics, and it is expected to prove ultrametric versions of theorems on metric spaces. In this paper, we provide ultrametric versions of the Arens–Eells isometric embedding theorem of metric spaces, the Hausdorff extension theorem of metrics, the Niemytzki–Tychonoff characterization theorem of the compactness, and the author’s interpolation theorem of metrics and theorems on dense subsets of spaces of metrics.

1. Introduction

Let $X$ be a set. A metric $d$ on $X$ is said to be an ultrametric or a non-Archimedean metric if for all $x, y, z \in X$ we have
\begin{equation}
    d(x, y) \leq d(x, z) \lor d(z, y),
\end{equation}
where the symbol $\lor$ stands for the maximum operator on $\mathbb{R}$. The inequality (1.1) is called the strong triangle inequality. We say that a set $S$ is a range set if $S \subset [0, \infty)$ and $0 \in S$. For a range set $S$, we say that a metric $d : X^2 \to [0, \infty)$ on $X$ is $S$-valued if $d(X^2) \subset S$. Note that $[0, \infty)$-valued ultrametrics are nothing but ultrametrics. The $S$-valued ultrametrics are studied as a special case and a reasonable restriction of ultrametrics. Gao and Shao [13] studied $R$-valued Urysohn universal ultrametric spaces for a countable range set $R$. Brodskiy, Dydak, Higes and Mitra [3] utilized $\{\{0\}\cup\{3^n \mid n \in \mathbb{Z}\}$-valued ultrametrics for their study on the 0-dimensionality in categories of metric spaces.

The notion of the ultrametrics can be considered as a 0-dimensional analogue of ordinary metrics, and it is often expected to prove ultrametric versions of theorems on metric spaces. In this paper, for every range set $S$, we provide $S$-valued ultrametric versions of the Arens–Eells isometric embedding theorem [11] of metric spaces, the Hausdorff extension theorem [17] of metrics, the Niemytzki–Tychonoff characterization [27] of the compactness, and the author’s interpolation theorem of metrics and theorems on dense $G_\delta$ subsets of spaces of metrics [22].

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In 1956, Arens and Eells [1] established the result which today we call the Arens–Eells embedding theorem, stating that for every metric space \((X, d)\), there exist a real normed linear space \(V\) and an isometric embedding \(I : X \rightarrow V\) such that

(1) \(I(X)\) is closed in \(V\);
(2) \(I(X)\) is linearly independent in \(V\).

Before stating our first main result, we introduce some basic notions.

Let \(R\) be a commutative ring, and let \(V\) be an \(R\)-module. A subset \(S\) of \(V\) is said to be \(R\)-independent if for every finite subset \(\{f_1, \ldots, f_n\}\) of \(S\), and for all \(N_1, \ldots, N_n \in R\), the identity \(\sum_{i=1}^{n} N_i f_i = 0\) implies \(N_i = 0\) for all \(i\). A function \(\|\ast\| : V \rightarrow [0, \infty)\) is said to be an ultra-norm on \(V\) if the following are satisfied:

(1) \(\|x\| = 0\) if and only if \(x = 0\);
(2) for every \(x \in V\), we have \(\| - x\| = \|x\|\);
(3) for all \(x, y \in V\), we have \(\|x + y\| \leq \|x\| \vee \|y\|\).

The pair \((V, \|\ast\|)\) is called an ultra-normed \(R\)-module (see also [49]).

Megrelishvili and Shlossberg [28] have already proven an ultrametric version of the Arens–Eells embedding theorem, stating that every ultra-metric space is isometrically embeddable into an ultra-normed Boolean group (a \(\mathbb{Z}/2\mathbb{Z}\)-module) as a closed set of it, which is a consequence of their study on free non-Archimedean topological groups and Boolean groups with actions from topological groups. By introducing module structures into universal ultrametric spaces of Lemin–Lemin type [24], we obtain a more general \(S\)-valued ultrametric version of the Arens–Eells embedding theorem.

**Theorem 1.1.** Let \(S\) be a range set. Let \(R\) be an integral domain, and let \((X, d)\) be an \(S\)-valued ultrametric space. Then there exist an \(S\)-valued ultra-normed \(R\)-module \((V, \|\ast\|)\), and an isometric embedding \(I : X \rightarrow V\) such that

(1) \(I(X)\) is closed in \(V\);
(2) \(I(X)\) is \(R\)-independent in \(V\).

Moreover, if \((X, d)\) is complete, then we can choose \((V, \|\ast\|)\) as a complete metric space.

**Remark 1.1.** Let \(R\) be an integral domain. Let \(t_R\) be the trivial valuation on \(R\) defined by \(t_R(x) = 1\) if \(x \neq 0\); otherwise \(t_R(x) = 0\). Let \(V\) be a torsion-free \(R\)-module. Then every ultra-norm \(\|\ast\|\) on \(V\) is compatible with \(t_R\), i.e., for every \(r \in R\) and for every \(x \in V\), we have \(\|r \cdot x\| = t_R(r)\|x\|\). For every finite field, there exist no valuations on it except the trivial valuation. Thus we can consider that Theorem [11] includes the Arens–Eells embedding theorem into normed spaces over all finite fields. The author does not know whether such an embedding theorem into normed spaces over all non-Archimedean valued fields holds true or not.
**Remark 1.2.** There are various isometric embedding theorems from an ultrametric space into a metric space with algebraic structures. For instance, Schikhof [37] constructed an isometric embedding from an ultrametric space into a non-Archimedean valued field. The existence of an isometric embedding from an ultrametric space into a Hilbert space was first proven by Timan and Vestfrid [42] in a separable case, and was proven by A. J. Lemin [25] in a general setting. The papers [43], [47] and [12] also contain related results.

For a range set $S$, and for a topological space $X$, we denote by $\text{UM}(X,S)$ (resp. $\text{M}(X)$) the set of all $S$-valued ultrametrics (resp. metrics) on $X$ that generate the same topology as the original one of $X$. We denote by $\text{UM}(X)$ the set $\text{UM}(X,[0,\infty))$. We say that a topological space $X$ is $S$-valued ultrametrizable (resp. ultrametrizable) if $\text{UM}(X,S) \neq \emptyset$ (resp. $\text{UM}(X) \neq \emptyset$). We say that $X$ is completely $S$-valued ultrametrizable (resp. completely ultrametrizable) if there exists a complete metric $d \in \text{UM}(X,S)$ (resp. $d \in \text{UM}(X)$).

We say that a range set $S$ has countable coinitiality if there exists a non-zero strictly decreasing sequence $\{r_i\}_{i \in \mathbb{N}}$ in $S$ with $\lim_{i \to \infty} r_i = 0$.

**Remark 1.3.** For a range set $S$ with the countable coinitiality, and for a topological space $X$, it is worth clarifying a relation between the ultrametrizability and the $S$-valued ultrametrizability. In Proposition 2.12, we show that these two properties are equivalent to each other.

In 1930, Hausdorff [17] proved the extension theorem stating that for every metrizable space $X$, for every closed subset $A$ of $X$, and for every metric $e \in \text{M}(A)$, there exists a metric $D \in \text{M}(X)$ with $D|_A = e$.

By using the Arens–Eells embedding theorem, Toruńczyk [44] provided a simple proof of the Hausdorff extension theorem. Due to Toruńczyk’s method and Theorem 1.2, we can prove an $S$-valued ultrametric version of the Hausdorff extension theorem.

**Theorem 1.2.** Let $S$ be a range set. Let $X$ be an $S$-valued ultrametrizable space, and let $A$ be a closed subset of $X$. Then for every $e \in \text{UM}(A,S)$, there exists $D \in \text{UM}(X,S)$ with $D|_A = e$. Moreover, if $X$ is completely metrizable and $e \in \text{UM}(A,S)$ is complete, then we can choose $D$ as a complete $S$-valued ultrametric.

**Remark 1.4.** There are several studies on extending a partial or continuous ultrametrics (see [10], [37], [38], or [45]).

In 1928, Niemytzki and Tychonoff [27] proved that a metrizable space $X$ is compact if and only if all metrics in $\text{M}(X)$ are complete. Hausdorff [17] gave a simple proof of their characterization theorem by applying his extension theorem of metrics mentioned above. By using Hausdorff’s argument and Theorem 1.2, we obtain an ultrametric version of the Niemytzki–Tychonoff theorem.
Corollary 1.3. Let $S$ be a range set with the countable coinitiality. Let $X$ be an $S$-valued ultrametrizable space. Then the space $X$ is compact if and only if for every ultrametric $d \in \text{UM}(X, S)$ is complete.

To state our next results, for a topological space $X$, and for a range set $S$, we define a function $UD^S_X : \text{UM}(X, S)^2 \to [0, \infty]$ by assigning $UD^S_X(d, e)$ to the infimum of $\epsilon \in S \sqcup \{\infty\}$ such that for all $x, y \in X$ we have

$$d(x, y) \leq e(x, y) \lor \epsilon,$$

and

$$e(x, y) \leq d(x, y) \lor \epsilon.$$

The function $UD^S_X$ is an ultrametric on $\text{UM}(X, S)$ valued in $\text{CL}(S) \sqcup \{\infty\}$, where $\text{CL}(S)$ stands for the closure of $S$ in $[0, \infty)$. We also define a function $D_X : \text{M}(X) \times \text{M}(X) \to [0, \infty]$ by

$$D_X(d, e) = \sup_{(x, y) \in X^2} |d(x, y) - e(x, y)|.$$

The function $D_X$ is a metric on $\text{M}(X)$ valued in $[0, \infty]$.

Remark 1.5. Qiu [40] introduced the strong $\epsilon$-isometry in the study on the non-Archimedean Gromov–Hausdorff distance (see [51]). This concept is an analogue for ultrametric spaces of the $\epsilon$-isometry in the study on the ordinary Gromov–Hausdorff distance (see [4]). Roughly speaking, for a range set $S$, for an $S$-valued ultrametrizable space $X$, and for $S$-valued ultrametrics $d, e \in \text{UM}(X, S)$, the inequality $UD^S_X(d, e) \leq \epsilon$ is equivalent to the statement that the identity maps $1_X : (X, d) \to (X, e)$ and $1_X : (X, e) \to (X, d)$ are strong $\epsilon$-isometries.

The author [22] proved an interpolation theorem of metrics with the information of $D_X$ (see [22, Theorem 1.1]). As its application, the author proved that for every non-discrete metrizable space $X$ the set of all metrics in $\text{M}(X)$ with a transmissible property, which is a geometric property determined by finite subsets (see Definition 6.2), is dense $G_\delta$ in the metric space $(\text{M}(X), D_X)$ (see [22, Theorem 1.2]), and also proved a local version of it (see [22, Theorem 1.3]).

By using Theorems 1.1 and 1.2, we can prove an ultrametric version of the author’s interpolation theorem.

A family $\{H_i\}_{i \in I}$ of subsets of a topological space $X$ is said to be discrete if for every $x \in X$ there exists a neighborhood of $x$ intersecting at most single member of $\{H_i\}_{i \in I}$. For a range set $S$, and for a subset $E$ of $S$, we denote by $\text{sup } E$ the supremum of $E$ taken in $[0, \infty]$, not in $S$. For $C \in [1, \infty)$, we say that $S$ is $C$-quasi-complete if for every bounded subset $E$ of $S$, there exists $s \in S$ with $\text{sup } E \leq s \leq C \cdot \text{sup } E$. We say that $S$ is quasi-complete if $S$ is $C$-quasi-complete for some $C \in [1, \infty)$. Note that a range set is 1-quasi-complete if and only if it is closed under the supremum operator.
Theorem 1.4. Let $C \in [1, \infty)$, and let $S$ be a $C$-quasi-complete range set. Let $X$ be an ultrametrizable space, and let $\{A_i\}_{i \in I}$ be a discrete family of closed subsets of $X$. Then for every $S$-valued ultrametric $d \in \text{UM}(X, S)$, and for every family $\{e_i\}_{i \in I}$ of ultrametrics with $e_i \in \text{UM}(A_i, S)$ for all $i \in I$, there exists an $S$-valued ultrametric $m \in \text{UM}(X, S)$ satisfying the following:

1. For every $i \in I$ we have $m|_{A_i^2} = e_i$;
2. $\sup_{i \in I} \text{UD}_{A_i}^S(e_i, d|_{A_i^2}) \leq \text{UD}_{X}^S(m, d) \leq C \cdot \sup_{i \in I} \text{UD}_{A_i}^S(e_i, d|_{A_i^2})$.

Moreover, if $X$ is completely metrizable, and if each $e_i \in \text{UM}(A_i, S)$ is complete, then we can choose $m \in \text{UM}(X, S)$ as a complete metric.

Similarly to [22], Theorem 1.4 enables us to prove theorems on dense $G_\delta$ subsets of $\text{UM}(X, S)$. The definitions of the transmissible parameter, the transmissible property, and the $S$-ultra-singularity for a range set can be seen in Definitions 6.1, 6.2 and 6.3.

Theorem 1.5. Let $S$ be a quasi-complete range set with the countable coinitiality. Let $\mathcal{G}$ be an $S$-ultra-singular transmissible parameter. Then for every non-discrete ultrametrizable space $X$, the set of all $d \in \text{UM}(X, S)$ for which $(X, d)$ satisfies the anti-$\mathcal{G}$-transmissible property is dense $G_\delta$ in the space $(\text{UM}(X, S), \text{UD}_{X}^S)$.

For a property $P$ on metric spaces, we say that a metric space $(X, d)$ satisfies the local $P$ if every non-empty open metric subspace of $X$ satisfies the property $P$.

Theorem 1.6. Let $S$ be a quasi-complete range set with the countable coinitiality. Let $X$ be a second countable, locally compact locally non-discrete ultrametrizable space. Then for every $S$-ultra-singular transmissible parameter $\mathcal{G}$, the set of all $d \in \text{UM}(X, S)$ for which $(X, d)$ satisfies the local anti-$\mathcal{G}$-transmissible property is a dense $G_\delta$ set in the space $(\text{UM}(X, S), \text{UD}_{X}^S)$.

For example, the doubling property is equivalent to a $\mathcal{G}$-transmissible property for some ultra-singular transmissible parameter $\mathcal{G}$.

The organization of this paper is as follows: In Section 2, we review the basic or classical statements on $S$-valued ultrametric spaces and 0-dimensional spaces. We give proofs of some of them. In Section 3, we observe that a construction of universal ultrametric spaces and isometric embeddings of Lemin–Lemin-type [24] can be applied to $S$-valued ultrametric spaces for all range set $S$. We also discuss algebraic structures on universal ultrametric spaces of Lemin–Lemin-type. After that, we prove Theorem 1.1. In Section 4, we prove Theorem 1.2 and Corollary 1.3 by following Toruńczyk and Hausdorff’s methods, and by using Theorem 1.1. In Section 5, we prove Theorem 1.4 by converting the author’s proof of [22, Theorem 1.1] into an $S$-valued ultrametric proof with Theorems 1.1 and 1.2. In Section 6, we introduce transmissible...
property, originally defined in [22], and we prove Theorem 1.5 as an application of Theorem 1.4. In Section 7, we first show that \( UM(X, S) \) is a Baire space for a range set \( S \), and for a second countable locally compact \( X \) (see Lemma 7.6). We next prove Theorem 1.6, which is a local version of Theorem 1.5.

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2. Preliminaries

In this paper, we denote by \( \mathbb{N} \) the set of all positive integers. For a set \( E \), we denote by \( \text{card}(E) \) the cardinality of \( E \). For a metric space \((X, d)\), and for a subset \( A \) of \( X \), we denote by \( \delta_d(A) \) the diameter of \( A \). We denote by \( B(x, r) \) (resp. \( U(x, r) \)) the closed (resp. open) ball centered at \( x \) with radius \( r \). We also denote by \( B(A, r) \) the set \( \bigcup_{a \in A} B(a, r) \).

To emphasize metrics under consideration, we sometimes denote by \( B(x, r; d) \) (resp. \( U(x, r; d) \)) the closed (resp. open) ball in \((X, d)\). For a range set \( S \), we define \( S^+ = S \setminus \{0\} \).

2.1. Modification of ultrametrics.

Lemma 2.1. Let \( S \) be a range set. Let \((X, d)\) be an \( S \)-valued ultrametric space. Let \( \varepsilon \in S^+ \). Then the function \( e : X^2 \to [0, \infty) \) defined by \( e = \min\{d, \varepsilon\} \) belongs to \( UM(X, S) \).

Proof. For all \( a, b, c \in \mathbb{R} \), we have \((a \lor b) \land c = (a \land c) \lor (b \land c)\), where \( \land \) stands for the minimum operator on \( \mathbb{R} \). This leads to the lemma. \( \square \)

Let \((X, d)\) and \((Y, e)\) be metric spaces. Define a function \( d \times_\infty e \) on \((X \times Y)^2\) by

\[
(d \times_\infty e)((x, y), (z, w)) = d(x, z) \lor e(y, w).
\]

It is well-known that \( d \times_\infty e \) is a metric on \( X \times Y \), and it generates the product topology of \( X \times Y \). In the case of ultrametrics, we have:

Lemma 2.2. Let \( S \) be a range set. Let \((X, d)\) and \((Y, e)\) be \( S \)-valued ultrametric spaces. Then the metric \( d \times_\infty e \) belongs to \( UM(X \times Y, S) \).

The following proposition is known as an amalgamation of ultrametrics. The proof can be seen in, for example, [2, Theorem 2.2], or, by replacing the symbol “+” with the symbol “\( \lor \)” in the proof of [22, Propositions 3.3], we can prove the following proposition.

For a mutually disjoint family \( \{T_i\}_{i \in I} \) of topological spaces, we consider that the set \( \bigsqcup_{i \in I} T_i \) is equipped with the direct sum topology.

Proposition 2.3. Let \( S \) be a range set. Let \((X, d_X)\) and \((Y, d_Y)\) be \( S \)-valued ultrametric spaces. If \( X \cap Y = \emptyset \), then for every \( r \in S^+ \), there exists an \( S \)-valued ultrametric \( h \in UM(X \sqcup Y, S) \) such that

1. \( h|_{X^2} = d_X \);
(2) \( h|_{Y^2} = d_Y \);
(3) for all \( x \in X \) and \( y \in Y \) we have \( r \leq h(x, y) \).

As a consequence of Proposition\(^{2.3} \) we can construct a one-point extension of an ultrametric space.

**Corollary 2.4.** Let \( S \) be a range set. Let \( (X, d) \) be an \( S \)-valued ultrametric space, and let \( o \notin X \). Then there exists an \( S \)-valued ultrametric \( D \in \text{UM}(X \cup \{o\}, S) \) with \( D|_{X^2} = d \).

### 2.2. Invariant metrics on modules.

Let \( R \) be a commutative ring, and let \( V \) be an \( R \)-module. We say that a metric on \( d \) on \( V \) is invariant, or invariant under the addition if for all \( a, x, y \in V \) we have
\[
d(x + a, y + a) = d(x, y).
\]

By the definitions of the ultra-norms and invariant metrics, we obtain:

**Lemma 2.5.** Let \( R \) be a commutative ring, and let \( (V, \| \cdot \|) \) be an ultra-normed \( R \)-module. Then the metric \( d \) on \( V \) defined by \( d(x, y) = \| x - y \| \) is an invariant ultrametric on \( V \). Conversely, if \((V, d)\) is a pair of an \( R \)-module \( V \) and an invariant ultrametric \( d \), then the function \( \| \cdot \| : V \to [0, \infty) \) defined by \( \|x\| = d(x, 0) \) is an ultra-norm on \( V \).

Based on Lemma\(^{2.5} \) in what follows, we will use a pair \((V, d)\) of an \( R \)-module and an invariant ultrametric \( d \) on \( V \), rather than a pair \((V, \| \cdot \|)\) of \( V \) and an ultra-norm \( \| \cdot \| \) on \( V \).

By the definition of the ultra-norms, we obtain:

**Lemma 2.6.** Let \( R \) be a commutative ring, and let \( (V, d) \) be an ultra-normed \( R \)-module. Then the addition \( + : V \times V \to V \) and the inversion \( m : V \to V \) defined by \( m(x) = -x \) are continuous with respect to the topology induced from \( d \).

The next lemma is utilized in the proof of Theorem\(^{1.1} \)

**Lemma 2.7.** Let \( R \) be a commutative ring, and let \( (V, d) \) be an ultra-normed \( R \)-module. If for every non-zero \( r \in R \) and for every \( v \in V \) we have \( d(r \cdot v, 0) = d(v, 0) \), then the completion of \((V, d)\) becomes an ultra-normed \( R \)-module which contains \( V \) as an \( R \)-submodule.

**Proof.** We introduce an \( R \)-module structure into the completion \((W, D)\) of \((V, d)\). For all \( x, y \in W \), take sequences \( \{x_n\}_{n \in \mathbb{N}} \) and \( \{y_n\}_{n \in \mathbb{N}} \) in \( V \) such that \( x_n \to x \) and \( y_n \to y \) as \( n \to \infty \). Then we define an addition on \( W \) by
\[
x + y = \lim_{n \to \infty} (x_n + y_n) \tag{1}
\]
Since \( d \) is an ultra-norm, this addition is well-defined.

For every \( r \in R \), we define a scalar multiplication on \( W \) by
\[
r \cdot x = \lim_{n \to \infty} r \cdot x_n \tag{2}
\]
By the assumption on the scalar multiplication on $V$ and the ultra-norm, the scalar multiplication on $W$ is well-defined. By these definitions, $(W, D)$ becomes an ultra-normed $R$-module which contains $V$ as an $R$-submodule. This finishes the proof. □

2.3. Basic properties of ultrametric spaces. The next lemma follows from the strong triangle inequality.

**Lemma 2.8.** Let $X$ be a set, and let $w : X^2 \to \mathbb{R}$ be a symmetric map. Then $w$ satisfies the strong triangle inequality if and only if for all $x, y, z \in X$ the inequality $w(x, z) < w(y, z)$ implies $w(y, z) = w(x, y)$.

By this lemma, we see that in an ultrametric space, every triangle is isosceles, and the side-length of the legs of the isosceles triangle is equal to or greater than the side-length of base.

The property in the next proposition follows from the strong triangle inequality (see (12) in [7, Theorem 1.6]).

**Proposition 2.9.** Let $S$ be a range set, and let $(X, d)$ be an $S$-valued ultrametric space. Then the completion of $(X, d)$ is an $S$-valued ultrametric space.

**Remark 2.1.** By Proposition 2.9 for every separable ultrametric space $(X, d)$, the set \{ $d(x, y) \mid x, y \in X$ \} is countable. This phenomenon is a reason why we consider $S$-valued ultrametrics for a range set $S$.

We now prove that for every range set $S$ with the countable coinitiality, the ultrametrizability and the $S$-valued ultrametrizability are equivalent to each other.

**Lemma 2.10.** Let $S$ be a range set with the countable coinitiality. Let $\{r(i)\}_{i \in \mathbb{N}}$ be a non-zero strictly decreasing sequence in $S$ such that $r(i) \to 0$ as $i \to \infty$. Put $T = \{0\} \cup \{r(i) \mid i \in \mathbb{N}\}$. If $\text{UM}(X, S) \neq \emptyset$, then $\text{UM}(X, T) \neq \emptyset$.

**Proof.** Take $h \in \text{UM}(X, S)$. Put $d = \min\{h, r(1)\}$. Then by Lemma 2.11 we have $d \in \text{UM}(X, S)$. Define a symmetric function $e : X^2 \to T$ by

$$
e(x, y) = \begin{cases} r(n) & \text{if } r(n + 1) < d(x, y) \leq r(n); \\ 0 & \text{if } d(x, y) = 0. \end{cases}$$

Then $e$ is a $T$-valued ultrametric. Take $x \in X$. For each $s \in S$, take $n \in \mathbb{N}$ with $r(n + 1) < s \leq r(n)$. By the definition of $e$, we have $U(x, r(n + 1); e) \subset U(x, s; d)$. On the other hand, for all $n \in \mathbb{N}$, we also have $U(x, r(n); d) \subset U(x, r(n); e)$. Thus the ultrametric $e$ generates the same topology as $X$. □

**Lemma 2.11.** Let $X$ be a topological space. Let $S$ and $T$ be two range sets, and let $\psi : S \to T$ be a bijective monotone map. Then for every $d \in \text{UM}(X, S)$ the function $\psi \circ d$ belongs to $\text{UM}(X, T)$. 

Proof. Since \( \psi \) is monotone, the function \( \psi \circ d \) satisfies the strong triangle inequality, and hence it is a \( T \)-valued ultrametric. For every \( x \in X \) and for every \( s \in S \), we have \( U(x, s; d) = U(x, \psi(s); \psi \circ d) \). This implies that \( \psi \circ d \in \text{UM}(X, T) \). □

Proposition 2.12. Let \( S \) be a range set with the countable coinitiality, and let \( X \) be a topological space. Then \( X \) is ultrametrizable if and only if \( X \) is \( S \)-valued ultrametrizable.

Proof. It suffices to show that if \( X \) is ultrametrizable, then \( \text{UM}(X, S) \neq \emptyset \). Let \( \{r(i)\}_{i \in \mathbb{N}} \) be a non-zero strictly decreasing sequence in \( S \) such that \( r(i) \to 0 \) as \( i \to \infty \). Put \( T = \{0\} \cup \{ r(i) \mid i \in \mathbb{N} \} \), and put \( A = \{0\} \cup \{ 2^{-i} \mid i \in \mathbb{N} \} \). Then there exists a bijective monotone map \( \psi : A \to T \). Since \( \text{UM}(X) \neq \emptyset \), Lemmas 2.10 and 2.11 imply that \( \text{UM}(X, T) \neq \emptyset \). From \( \text{UM}(X, T) \subset \text{UM}(X, S) \), the proposition follows. □

Remark 2.2. If \( S \) does not have countable coinitiality, Proposition 2.12 does not hold true. In this case, all \( S \)-valued ultrametrizable spaces are discrete (see Lemma 2.16).

A topological space \( X \) is said to be \( 0 \)-dimensional if for every pair of disjoint two closed subsets \( A \) and \( B \) of \( X \), there exists a clopen subset \( Q \) of \( X \) such that \( A \subset Q \) and \( Q \cap B = \emptyset \). Such a space is sometimes also said to be ultranormal. Note that a metric space \((X, d)\) is \( 0 \)-dimensional if and only if every finite open covering of \( X \) has a refinement covering of \( X \) consisting of mutually disjoint finite open sets. This equivalence follows from the fact that the large inductive dimensions and the covering dimensions coincide on metric spaces (see e.g., [33, Theorem 5.4]).

The following was proven by de Groot [14] (see also [7]).

Proposition 2.13. All ultrametrizable spaces are \( 0 \)-dimensional.

We now clarify a relation between the complete \( S \)-valued ultrametrizability for a range set \( S \) and the complete metrizability. The proofs of the following lemma and proposition are \( S \)-valued ultrametric analogues of [50, Theorem 24.12].

Lemma 2.14. Let \( S \) be a range set with the countable coinitiality. Let \( X \) be a completely \( S \)-valued ultrametrizable, and let \( G \) be an open subset of \( X \). Then \( G \) is completely \( S \)-valued ultrametrizable.

Proof. Since \( X \) is \( 0 \)-dimensional, and since all open sets of metric spaces are \( F_\sigma \), there exists a sequence \( \{O_n\}_{n \in \mathbb{N}} \) of clopen sets of \( X \) such that

1. for each \( n \in \mathbb{N} \), we have \( O_n \subset O_{n+1} \);
2. \( G = \bigcup_{n \in \mathbb{N}} O_n \).

Take a sequence of \( \{a_n\}_{n \in \mathbb{N}} \) in the field \( \mathbb{Q}_2 \) of all 2-adic numbers such that for each \( m \in \mathbb{N} \) the sum \( \sum_{i=m}^{\infty} a_i \) is convergent to a non-zero 2-adic number (for example, we can take \( a_n = 2^n - 2^{n+1} \)). Define a function
\(F : X \to \mathbb{Q}_2\) by \(F(x) = \sum_{i=1}^{\infty} a_i \cdot \chi_{O_i}(x)\), where \(\chi_{O_i}\) is the characteristic function of \(O_i\). Then \(F\) is continuous. By the assumption on \(\{a_n\}_{n \in \mathbb{N}}\), for every \(x \in G\), we have \(F(x) \neq 0\) and \(F|_{X\setminus G} = 0\). Take a complete \(S\)-valued ultrametric \(d \in \text{UM}(X, S)\). We denote by \(v_2 : \mathbb{Q}_2 \to \mathbb{Z}[1/\infty]\) the 2-adic valuation on \(\mathbb{Q}_2\). Take a non-zero strictly decreasing sequence \(\{r(i)\}_{i \in \mathbb{N}}\) in \(S\) with \(\lim_{i \to \infty} r(i) = 0\). We put \(r(\infty) = 0\). Then a metric \(W : (\mathbb{Q}_2)^2 \to S\) defined by

\[
W(x, y) = v_2(x - y) \lor 1
\]

belongs to \(\text{UM}(\mathbb{Q}_2, S)\), and it is complete. Define a metric \(d\) on \(G\) by

\[
D(x, y) = W\left(\frac{1}{F(x)}, \frac{1}{F(y)}\right) \lor d(x, y).
\]

Since the function \(1/F\) is continuous on \(G\), we have \(D \in \text{UM}(G, S)\). We next show that \(D\) is complete. Assume that \(\{x_n\}_{n \in \mathbb{N}}\) is Cauchy in \((G, D)\). Then \(\{1/F(x_n)\}_{n \in \mathbb{N}}\) and \(\{x_n\}_{n \in \mathbb{N}}\) are Cauchy in \((\mathbb{Q}_2, W)\) and \((X, d)\), respectively. Thus, there exist \(A \in \mathbb{Q}_2\) and \(B \in X\) such that \(1/F(x_n) \to A\) in \((\mathbb{Q}_2, W)\), and \(x_n \to B\) in \((X, d)\). If \(B \notin G\), then we have \(F(x_n) \to 0\). This contradicts to \(1/F(x_n) \to A\). Thus \(B \in G\), and hence \(D\) is complete. Therefore we conclude that \(G\) is completely \(S\)-valued ultrametrizable. \(\square\)

**Proposition 2.15.** Let \(S\) be a range set with the countable coinitiality. A topological space \(X\) is completely \(S\)-valued ultrametrizable if and only if \(X\) is completely metrizable and \(S\)-valued ultrametrizable.

**Proof.** It suffices to show that if \(X\) is completely metrizable and \(S\)-valued ultrametrizable, then \(X\) is completely \(S\)-valued ultrametrizable. Take a non-zero strictly decreasing sequence \(\{r(i)\}_{i \in \mathbb{N}}\) in \(S\) with \(\lim_{i \to \infty} r(i) = 0\). We put \(T = \{0\} \cup \{r(i) \mid i \in \mathbb{N}\}\). Then \(T\) is a range set with \(T \subset S\). Note that \(T\) has countable coinitiality and it is a closed set of \([0, \infty)\). By Proposition 2.12, we can take an ultrametric \(d \in \text{UM}(X, T)\). Let \((Y, D)\) be a completion of \((X, d)\). By Proposition 2.13, the space \((Y, D)\) is a \(T\)-valued ultrametric space. Since \(X\) is completely metrizable, \(X\) is \(G_{\delta}\) in \(Y\) (see [11], Theorem 4.3.24). Thus there exists a sequence \(\{G_n\}_{n \in \mathbb{N}}\) of open sets in \(Y\) such that \(X = \bigcap_{n \in \mathbb{N}} G_n\). By Lemmas 2.4 and 2.4, we can take a sequence \(\{e_n\}_{n \in \mathbb{N}}\) of complete \(T\)-valued ultrametrics such that \(e_n \in \text{UM}(G_n, T)\) and \(e_n(x, y) \leq r(n)\) for all \(x, y \in G_n\) and for all \(n \in \mathbb{N}\). Define an \(S\)-valued ultrametric \(D \in \text{UM}(X, S)\) by

\[
D(x, y) = \sup_{n \in \mathbb{N}} e_n(x, y).
\]

Then \(D\) is complete. Since \(T\) is a closed set of \([0, \infty)\), we have \(D \in \text{UM}(X, T)\). By \(\text{UM}(X, T) \subset \text{UM}(X, S)\), we obtain a complete \(S\)-valued ultrametric \(D \in \text{UM}(X, S)\). \(\square\)

If a range set does not have countable coinitiality, we obtain:
Lemma 2.16. Let $S$ be a range set which does not have the countable coinitiality. Then every $S$-valued ultrametric space $(X,d)$ is discrete and complete.

Proof. Since $S$ does not have countable coinitiality, there exists $r \in [0,\infty)$ such that $[0,r) \cap S = \{0\}$. Thus for every $x \in X$ we have $U(x,r) = \{x\}$. This implies the lemma. □

Combining Lemma 2.16 and Proposition 2.15, we obtain:

Proposition 2.17. Let $S$ be a range set. A topological space $X$ is completely $S$-valued ultrametrizable if and only if $X$ is completely metrizable and $S$-valued ultrametrizable.

2.4. Continuous functions on 0-dimensional spaces. The following theorem was stated in [9, Theorem 1.1], and a Lipschitz version of it was proven in [3, Theorem 2.9].

Proposition 2.18. Let $X$ be an ultrametrizable space, and let $A$ be a closed subset of $X$. Then there exists a retraction from $X$ into $A$; namely, there exists a continuous map $r : X \rightarrow A$ with $r|_A = 1_A$.

Corollary 2.19. Let $X$ be an ultrametrizable space, and let $A$ be a closed subset of $X$. Let $Y$ be a topological space. Then every continuous map $f : A \rightarrow Y$ can be extended to a continuous map from $X$ to $Y$.

Proof. By Proposition 2.18, there exists a retraction $r : X \rightarrow A$. Put $F = f \circ r$, then $F$ is a continuous extension of $f$. □

Let $Z$ be a metrizable space. We denote by $C(Z)$ the set of all non-empty closed subsets of $Z$. For a topological space $X$ we say that a map $\phi : X \rightarrow C(Z)$ is lower semi-continuous if for every open subset $O$ of $Z$ the set $\{x \in X \mid \phi(x) \cap O \neq \emptyset\}$ is open in $X$.

The following theorem is known as the 0-dimensional Michael continuous selection theorem. This was stated in [31], essentially in [30] (see also [29, Proposition 1.4]).

Theorem 2.20. Let $X$ be a 0-dimensional paracompact space, and $A$ a closed subsets of $X$. Let $Z$ be a completely metrizable space. Let $\phi : X \rightarrow C(Z)$ be a lower semi-continuous map. If a continuous map $f : A \rightarrow Z$ satisfies $f(x) \in \phi(x)$ for all $x \in A$, then there exists a continuous map $F : X \rightarrow Z$ with $F|_A = f$ such that for every $x \in X$ we have $F(x) \in \phi(x)$.

By the invariance of the ultra-norms under the addition, we have:

Proposition 2.21. Let $R$ be a commutative ring, and let $(V,d)$ be an ultra-normed $R$-module. Let $x,y \in V$. Then for every $r \in (0,\infty)$ we have

$$\mathcal{H}(B(x,r),B(y,r)) \leq d(x,y),$$

where $\mathcal{H}$ is the Hausdorff distance induced from $d$. 

AN EMBEDDING, AN EXTENSION, AND AN INTERPOLATION 11
Proof. For every \( w \in B(y, r) \), the invariance of \( d \) under the addition implies that \( x + w - y \in B(x, r) \) and \( d(w, x + w - y) = d(x, y) \). Thus, \( B(y, r) \subset B(B(x, r), d(x, y)) \). Similarly, we obtain \( B(x, r) \subset B(B(y, r), d(x, y)) \). Therefore, \( \mathcal{H}(B(x, r), B(y, r)) \leq d(x, y) \). \( \square \)

**Corollary 2.22.** Let \( X \) be a topological space, Let \( R \) be a commutative ring, and let \( (V, d) \) be an ultra-normed \( R \)-module. Let \( H : X \to V \) be a continuous map and \( r \in (0, \infty) \). Then a map \( \phi : X \to C(V) \) defined by \( \phi(x) = B(H(x), r) \) is lower semi-continuous.

**Proof.** For every open subset \( O \) of \( V \), and for every point \( a \in X \) with \( \phi(a) \cap O \neq \emptyset \), choose \( u \in \phi(a) \cap O \) and \( l \in (0, \infty) \) with \( U(u, l) \subset O \). By Proposition 2.21 and the continuity of \( H \), we can take a neighborhood \( N \) of the point \( a \) in \( X \) such that for every \( x \in N \) we have
\[
\mathcal{H}(\phi(x), \phi(a)) \leq d(H(x), H(a)) < l.
\]
Then we have \( \phi(x) \cap U(u, l) \neq \emptyset \), and hence \( \phi(x) \cap O \neq \emptyset \). Therefore the set \( \{ x \in X \mid \phi(x) \cap O \neq \emptyset \} \) is open in \( X \). \( \square \)

The following theorem is known as the Stone theorem on the paracompactness, which was proven in [39].

**Theorem 2.23.** All metrizable spaces are paracompact.

By Theorem 2.23 and Proposition 2.13 we can apply Theorem 2.20 to all ultrametrizable space.

2.5. **Baire spaces.** A topological space \( X \) is said to be **Baire** if an intersection of every countable family of dense open subsets of \( X \) is dense in \( X \).

The following is known as the Baire category theorem.

**Theorem 2.24.** Every completely metrizable space is a Baire space.

Since \( G_\delta \) subset of completely metrizable space is completely metrizable (see, e.g. [50, Theorem 24.12]), we obtain the following:

**Lemma 2.25.** Every \( G_\delta \) subset of a completely metrizable space is a Baire space.

3. **AN EMBEDDING THEOREM OF ULTRAMETRIC SPACES**

In this section, we prove Theorem 1.1.

3.1. **Proof of Theorem 1.1.** We first discuss general algebraic facts.

**Lemma 3.1.** Let \( R \) be a commutative ring, and let \( V \) be an \( R \)-module. Let \( P \) be an \( R \)-independent set of \( V \), and let \( Q \) be a subset of \( P \). Let \( H \) be an \( R \)-submodule of \( V \) generated by \( Q \). Then \( H \cap P = Q \).

**Proof.** By the definition of \( H \), first we have \( Q \subset H \cap P \). Since \( P \) is \( R \)-independent, we have \( (P \setminus Q) \cap H = \emptyset \). Thus every \( x \in H \cap P \) must belong to \( Q \), and hence we conclude that \( H \cap P \subset Q \). \( \square \)
Let \( R \) be a commutative ring. Let \( X \) be a set, and let \( o \not\in X \). We denote by \( F(R, X, o) \) the free \( R \)-module \( M \) satisfying that

1. \( X \sqcup \{o\} \subset M \);
2. \( o \) is the zero element of \( M \);
3. \( X \) is an \( R \)-independent generator of \( M \).

Note that by the construction of free modules, \( F(R, X, o) \) uniquely exists up to isomorphism.

For two sets \( A, B \), we denote by \( \text{Map}(A, B) \) the set of all maps from \( A \) into \( B \). Let \( R \) be a commutative ring, and let \( V \) be an \( R \)-module. Let \( E \) be a set. Then the set \( \text{Map}(E, V) \) becomes an \( R \)-module with the coordinate-wise addition and scalar multiplication. Note that the zero element of \( \text{Map}(E, V) \) is the zero function of \( \text{Map}(E, V) \); namely the constant function valued at the zero element of \( V \). In what follows, the set \( \text{Map}(E, V) \) will be always equipped with this module structure.

We next discuss a construction of universal ultrametric spaces of Lemin–Lemin type [24]. Let \( S \) be a range set. Let \( M \) be a set, and let \( o \in M \) be a fixed base point. A map \( f : S_+ \to M \) is said to be eventually \( o \)-valued if there exists \( C \in S_+ \) such that for every \( q > C \) we have \( f(q) = o \). We denote by \( L(S, M, o) \) the set of all eventually \( o \)-valued maps from \( S_+ \) to \( M \). Define a metric \( \Delta \) on \( L(S, M, o) \) by

\[
\Delta(f, g) = \sup \{ q \in S_+ \mid f(q) \neq g(q) \}.
\]

Note that \( \Delta \) takes values in the closure \( \text{CL}(S) \) of \( S \) in \([0, \infty)\).

The next lemma follows from the definitions of \( L(S, M, o) \) and \( \Delta \).

**Lemma 3.2.** For every range set \( S \), for every set \( M \) and for every point \( o \in M \), the space \( (L(S, M, o), \Delta) \) is a complete \( \text{CL}(S) \)-valued ultrametric space.

In the next theorem, we review the Lemin–Lemin construction [24] of embeddings into their universal spaces in order to obtain more detailed information of their construction.

**Theorem 3.3.** Let \( S \) be a range set. Let \((X \sqcup \{o\}, d)\) be an \( S \)-valued ultrametric space with \( o \not\in X \). Let \( K \) be a set with \( X \sqcup \{o\} \subset K \). Then there exists an isometric embedding \( L : X \to L(S, K, o) \) such that

1. for every \( q \in S_+ \) we have \( L(o)(q) = o \);
2. for every \( x \in X \) the function \( L(x) \) is valued in \( X \sqcup \{o\} \);
3. for all \( x, y \in X \) we have

\[
(0, d(x, y)) \cap S_+ = \{ q \in S_+ \mid L(x)(q) \neq L(y)(q) \}.
\]

**Proof.** Let \( X \sqcup \{o\} = \{x(\alpha)\}_{\alpha<\kappa} \) be an injective index with \( x(0) = o \), where \( \kappa \) is a cardinal. By following the Lemin–Lemin’s way [24], we construct an isometric embedding \( L : X \to L(S, K, o) \) by transfinite induction. Put \( L(x(0)) = o \). Let \( \gamma < \kappa \). Assume that an isometric embedding \( L : \{x(\alpha) \mid \alpha < \gamma \} \to L(S, K, o) \) is already defined. Set \( D_\gamma = \inf \{ d(x(\alpha), x(\gamma)) \mid \alpha < \gamma \} \).
Case 1. (There exists an ordinal $\beta < \gamma$ with $D_\gamma = d(x(\beta), x(\gamma))$.) We define an eventually $o$-valued map $L(x(\gamma)) : S_+ \to K$ by

$$L(x(\gamma))(q) = \begin{cases} x(\gamma) & \text{if } q \in (0, D_\gamma]; \\ L(x(\beta))(q) & \text{if } q \in (D_\gamma, \infty). \end{cases}$$

Case 2. (No ordinal $\beta < \gamma$ satisfies $D_\gamma = d(x(\beta), x(\gamma))$.) Take a sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ with $\alpha_n < \gamma$ and $d(x(\alpha_n), x(\gamma)) < D_\gamma + 1/n$ for all $n \in \mathbb{N}$. We define an eventually $o$-valued map $L(x(\gamma)) : S_+ \to K$ by

$$L(x(\gamma))(q) = \begin{cases} x(\gamma) & \text{if } q \in (0, D_\gamma]; \\ L(x(\alpha_n))(q) & \text{if } D_\gamma + 1/n < q. \end{cases}$$

Similarly to [24], the map $L : X \cup \{o\} \to L(S, K, o)$ is well-defined and isometric, and we see that the conditions (1) and (2) are satisfied.

We now prove the condition (3). Note that for each $\alpha < \kappa$, the function $L(x(\alpha))$ is valued in $\{x(\beta) \mid \beta \leq \alpha\}$. Let $\gamma < \kappa$. Assume that for all $\alpha, \beta < \gamma$, the condition (3) is satisfied for $x = x(\alpha)$ and $y = x(\beta)$. We prove that for every $\alpha < \gamma$, the condition (3) is satisfied for $x = x(\alpha)$ and $y = x(\gamma)$.

In Case 1, by the definition of $D_\gamma$ and $\beta < \gamma$, we have $d(x(\beta), x(\gamma)) \leq d(x(\alpha), x(\gamma))$. From this inequality and the strong triangle inequality (or Lemma 2.8, it follows that $d(x(\alpha), x(\beta)) \leq d(x(\alpha), x(\gamma))$). Thus, by the hypothesis of transfinite induction and the definition of $L(x(\gamma))$, we conclude that the condition (3) is satisfied.

In Case 2, by the definition of $D_\gamma$, we have $D_\gamma < d(x(\alpha), x(\gamma))$, and for all sufficiently large $n \in \mathbb{N}$, we obtain $d(x(\alpha_n), x(\gamma)) < d(x(\alpha), x(\gamma))$. Lemma 2.8 implies that $d(x(\alpha_n), x(\alpha)) = d(x(\alpha), x(\gamma))$. Since on the set $(D_\gamma + 1/n, d(x(\alpha), x(\alpha_n)])$ the function $L(x(\gamma))$ coincides with $L(x(\alpha_n))$, by the hypothesis of transfinite induction we have

$$(D_\gamma + 1/n, d(x(\alpha), x(\alpha_n)]) \cap S_+ \subset \{ q \in S_+ \mid L(x(\alpha))(q) \neq L(x(\gamma))(q) \}.$$  

By $L(x(\alpha))(S_+) \subset \{ x(\beta) \mid \beta \leq \alpha \}$ and $L(x(\gamma))(0, D_\gamma] = x(\gamma)$, we also have

$$(0, D_\gamma] \cap S_+ \subset \{ q \in S_+ \mid L(x(\alpha))(q) \neq L(x(\gamma))(q) \}.$$  

These imply the condition (3) for $x = x(\alpha)$ and $y = x(\gamma)$. \qed

**Remark 3.1.** An ultrametric space $X$ is said to be *universal* for a class of ultrametric spaces if every ultrametric space in the class is isometrically embeddable into $X$. In [24], for each cardinal $\tau$, Lemin and Lemin constructed a universal ultrametric space for the class of all ultrametric spaces of topological weight $\tau$. There are several studies on universal ultrametric spaces. Vaughan [46] studied universal ultrametric spaces of Lemin–Lemin-type. Vestfrid [47], Gao and Shao [13], and Wan [48] studied universal ultrametric spaces of Urysohn-type.
The following lemma plays a central role in the proof of our embedding theorem from ultrametric spaces into ultra-normed modules.

**Lemma 3.4.** Let $S$ be a range set. Let $R$ be a commutative ring. If $M$ is an $R$-module, then the following are satisfied:

1. the space $L(S, M, 0)$ becomes an $R$-submodule of $\text{Map}(S_+, M)$;
2. the ultrametric $\Delta$ on $L(S, M, 0)$ is invariant under the addition; namely, $(L(S, M, 0), \Delta)$ is ultra-normed;
3. if $M$ is torsion-free, then for every $r \in R \setminus \{0\}$, and for every $x \in L(S, M, o)$, we have $\Delta(r \cdot x, 0) = \Delta(x, 0)$.

**Proof.** The condition (1) follows from $L(S, M, 0) \subset \text{Map}(S_+, M)$ and the definition of the eventually $0$-valued maps. We prove the condition (2). For all $f, g, h \in L(S, M, 0)$, and for every $q \in S_+$, we have $f(q) \neq g(q)$ if and only if $f(q) + h(q) \neq g(q) + h(q)$. Thus, by the definition, $\Delta$ is invariant under the addition. By the similar way, since $R$ is an integral domain, we see that the condition (3) is satisfied. □

**Lemma 3.5.** Let $R$ be a commutative ring. Let $S$ be a range set. Let $(X \cup \{o\}, D)$ be an $S$-valued ultrametric space with $o \notin X$. Put $M = F(R, X, o)$. Let $L : (X \cup \{o\}) \to L(S, M, o)$ be an isometric embedding constructed in Theorem 3.3. Then $L(X)$ is $R$-independent in the $R$-module $L(S, M, o)$.

**Proof.** In this proof, we denote by $0_M$ the zero element $o$ of $M$. Let $C = \{x_1, \ldots, x_n\}$ be an arbitrary finite subset of $X$. Assume that

$$\sum_{i=1}^n N_i \cdot L(x_i) = 0_L,$$

where $N_i \in R$ for all $i$ and $0_L$ stands for the zero function of $L(S, M, o)$. Put

$$b = \min\{\Delta(L(x), L(y)) \mid x, y \in C \cup \{o\} \text{ and } x \neq y\}.$$

Then $b > 0$. Take $c \in S_+$ with $c < b$. By the definition of $\Delta$ and the conditions (1) and (3) stated in Theorem 3.3, we see that for all $i, j = 1, \ldots, n$ we have $L(x_i)(c) \neq L(x_j)(c)$, and for each $i$ we have $L(x_i)(c) \neq 0_M$. By the definition of $F(R, X, o)$, we see that the set $\{L(x_1)(c), \ldots, L(x_n)(c)\}$ is $R$-independent in $M$. Since

$$\sum_{i=1}^{n+1} N_i \cdot L(x_i)(c) = 0_M,$$

we have $N_i = 0$ for all $i$. Thus $\{L(x_1), \ldots, L(x_n)\}$ is $R$-independent in $L(S, M, o)$. Since $S = \{x_1, \ldots, x_n\}$ is arbitrary, we see that $L(X)$ is $R$-independent in $L(S, M, o)$. □

**Lemma 3.6.** Let $R$ be a commutative ring. Let $S$ be a range set. Let $(X \cup \{o\}, D)$ be an $S$-valued ultrametric space with $o \notin X$. Put
$M = F(R, X, o)$. Let $L : (X \sqcup \{o\}) \to L(S, M, o)$ be an isometric embedding constructed in Theorem 3.3. Let $Q$ be an $R$-submodule of $L(S, M, o)$ generated by $L(X)$. Then the metric $\Delta|_Q$ takes values in the range set $S$.

**Proof.** In this proof, we denote by $0_M$ the zero element $o$ of $M$.

By the invariance of $\Delta$ under the addition, it suffices to show that for every $x \in Q$ we have $\Delta(x, 0_L) \in S$, where $0_L$ is the zero function of $L(S, N, o)$. Take $x \in Q$. Then there exist a finite subset $\{x_1, \ldots, x_n\}$ of $X$ and a finite subset $\{N_1, \ldots, N_n\}$ of $R \setminus \{0\}$ such that $x = \sum_{i=1}^n N_i L(x_i)$. Let $p_0, p_1, \ldots, p_k$ be a sequence in $S$ such that

1. $p_0 = 0$;
2. $p_j < p_{j+1}$ for all $j$;
3. $\{d(x_i, 0) \mid i = 1, \ldots, n\} \cup \{d(x_i, x_j) \mid i \neq j\} = \{p_1, \ldots, p_k\}$.

For $l \in \{0, \ldots, k-1\}$, we put $I_l(j) = (p_j, p_{j+1}] \cap S$, and we put $I_l(k) = (p_k, \infty) \cap S$. By the definition of $\{p_j\}_{j=0}^k$, and by the properties (2) and (3) of the map $L$ stated in Theorem 3.3, we obtain:

(A) for all $a \in \{1, \ldots, n\}$ we have $L(x_a) = 0_M$ on $I_l(k)$;
(B) for every $a \in \{1, \ldots, n\}$, and for every $j \in \{0, \ldots, k\}$, if there exists $c \in I_l(j)$ with $L(x_a)(c) = 0_M$, then we have $L(x_a) = 0_L$ on $I_l(j)$;
(C) for all $a, b \in \{1, \ldots, n\}$, and for every $j \in \{0, \ldots, k\}$, if there exists $c \in I_l(j)$ with $L(x_a)(c) = L(x_b)(c)$, then we have $L(x_a) = L(x_b)$ on $I_l(j)$.

Suppose that $\Delta(x, 0) \notin S$. By the property (A), we can take $j \in \{0, \ldots, k-1\}$ such that $\Delta(x, 0) \notin I_l(j)$. By the definition of $\Delta$, there exists $p \in I_l(j)$ with $x(p) \neq 0_M$, and we see that $x(p_{j+1}) = 0_M$. Put $q = p_{j+1}$. Take a subset $\{y_1, \ldots, y_m\}$ of $\{x_1, \ldots, x_n\}$ such that

(a) $L(y_1)(q), \ldots, L(y_m)(q)$ are not equal to the zero element $0_M$ of $M$, and they are different to each other;
(b) $m$ is maximal in cardinals of all subsets of the set $\{x_1, \ldots, x_n\}$ satisfying the property (a).

By the properties (B) and (C), the set $\{L(y_1)|_{I_l(j)}, \ldots, L(y_m)|_{I_l(j)}\}$ is a maximal $R$-independent subset of $\{L(x_1)|_{I_l(j)}, \ldots, L(x_n)|_{I_l(j)}\}$ in the $R$-module $\text{Map}(I_l(j), M)$. Then there exists a subset $\{C_1, \ldots, C_m\}$ of $R$ such that

$$x|_{I_l(j)} = \sum_{l=1}^m C_l L(y_l)|_{I_l(j)}.$$ 

Since $x(q) = 0_M$, we have

$$\sum_{l=1}^m C_l L(y_l)(q) = 0_M.$$ 

Since $\{L(y_1)(q), \ldots, L(y_m)(q)\}$ is a subset of $X$, it is $R$-independent in $M$. Thus we have $C_l = 0$ for all $l \in \{1, \ldots, m\}$. This implies
that $x = 0_M$ on $I(j)$. This contradicts the existence of $p \in I(j)$ with $x(p) \neq 0_M$. Therefore, $\Delta(x, 0_L) \in S$. This completes the proof. □

Before proving Theorem 1.1, we recall that every free module on every integral domain is torsion-free.

**Proof of Theorem 1.1.** Let $S$ be a range set. Let $R$ be a commutative ring, and let $(X, d)$ be an ultrametric space.

We first deal with the case where $(X, d)$ is complete. Take $o \not\in X$. Put $M = F(R, X, o)$. Let $(X \cup \{o\}, D)$ be a one-point extension of $(X, d)$ (see Corollary 2.4). Let $L : (X \cup \{o\}, D) \to (L(S, M, o), \Delta)$ be an isometric embedding stated in Theorem 3.3. Let $Q$ be an $R$-submodule of $L(S, M, o)$ generated by $L(X)$, and let $(V, \Xi)$ be the completion of $(Q, \Delta|Q)$.

By Lemmas 2.7, 3.6, and Proposition 2.9, the space $(V, \Xi)$ is an $S$-valued ultra-normed $R$-module. Since complete metric subspaces are closed in metric spaces, Lemma 3.5 implies that $(V, \Xi)$ and $L : (X, d) \to (V, \Xi)$ satisfy the conditions (1) and (2) stated in Theorem 1.1. Moreover, the latter part of the theorem is also proven.

In the case where $(X, d)$ is not complete, let $(Y, e)$ be the completion of $(X, d)$. As in the above, we can take an ultra-normed $R$-module $(W, D)$ and an isometric embedding $I : Y \to W$ satisfying the conditions (1) and (2) in Theorem 1.1. Let $H$ be an $R$-submodule of $W$ generated by $I(X)$. Since $I(Y)$ is $R$-independent, Lemma 3.1 yields $H \cap I(Y) = I(X)$. Thus $I(X)$ is closed in $H$, and hence $(H, D|H^2)$ and $I$ are desired ones. This completes the proof of Theorem 1.1. □

Remark 3.2. If a range set $S$ is closed under the supremum operator, then we can replace the assumption that $R$ is an integral domain in the statement of Theorem 1.1 with the condition that $R$ is a commutative ring. In this case, the space $(L(S, M, o), \Delta)$ is an $S$-valued ultrametric space, and in the proof of Theorem 1.1 we can use the space $(L(S, M, o), \Delta)$ instead of the space $(V, \Xi)$.

### 3.2. Ultrametrics taking values in general totally ordered sets.

We say that an ordered set is **bottomed** if it has a minimal element. Let $(T, \leq_T)$ be a bottomed totally ordered set. Let $X$ be a set. A function $d : X \times X \to T$ is said to be a $(T, \leq_T)$-valued ultrametric on $X$ if the following are satisfied:

1. for all $x, y \in X$ we have $d(x, y) = 0_T$ if and only if $x = y$, where $0_T$ stands for the minimal element of $(T, \leq_T)$;
2. for all $x, y \in X$ we have $d(x, y) = d(y, x)$;
3. for all $x, y, z \in X$ we have $d(x, y) \leq_T d(x, z) \lor_T d(z, y)$, where $\lor_T$ is the maximal operator of $(T, \leq_T)$.

Such general ultrametric spaces, or general metric spaces on which distances are valued in a totally ordered Abelian group are studied for a long time (see e.g., [35], [5], [31], [32] and [8]).
The construction of universal ultrametric space of Lemin–Lemin-type mentioned above and the proof of Theorem 1.1 are still valid for \((T, \leq_T)\)-valued ultrametric spaces for all bottomed totally ordered set \((T, \leq_T)\). For simplicity, and for necessity of our study, we omit the details of the proof of the following:

**Theorem 3.7.** Let \((T, \leq_T)\) be a bottomed totally ordered set. Let \(R\) be an integral domain, and let \((X, d)\) be a \((T, \leq_T)\)-valued ultrametric space. Then there exist a \((T, \leq_T)\)-valued ultra-normed \(R\)-module \((V, \| \ast \|)\), and an isometric embedding \(I : X \to V\) such that

1. \(I(X)\) is closed in \(V\);
2. \(I(X)\) is \(R\)-independent in \(V\).

Moreover, if \((X, d)\) is complete, then we can choose \((V, \| \ast \|)\) as a complete \((T, \leq_T)\)-valued ultrametric space.

For a bottomed totally ordered set \((T, \leq_T)\), we define the coinitiality \(\text{coi}(T, \leq_T)\) of \(T\) as the minimal cardinal \(\kappa > 0\) such that there exists a strictly decreasing map \(f : \kappa + 1 \to T\) with \(f(\kappa) = 0_T\) such that for every \(t \in T\), there exists \(\alpha < \kappa\) with \(f(\alpha) \leq t\). Note that a range set \(S\) has countable coinitiality if and only if \(\text{coi}(\text{CL}(S), \leq) = \omega_0\). Some readers may think our results such as Corollary 1.3 and Theorems 1.2–1.6 in this paper can be generalized for \((T, \leq_T)\)-valued ultrametrics for a bottomed totally ordered set \((T, \leq_T)\) satisfying \(\text{coi}(T, \leq_T) > \omega_0\). Unfortunately, it seems to be quite difficult. Our proofs of Theorems 1.2–1.6 require the extension theorem (Corollary 2.19) of continuous functions on ultrametric spaces. An analogue for \((T, \leq_T)\)-valued ultrametric spaces of Corollary 2.19 seems not to hold true.

### 4. An Extension Theorem of Ultrametrics

In this section, by following the methods of Toruńczyk [44] and Hausdorff [17], we prove Theorem 1.2 and Corollary 1.3. Since Toruńczyk’s proof of Lemma in [44] on real linear spaces does not depend on the coefficient ring \(\mathbb{R}\), we can apply that method to all ultra-normed modules over all commutative rings. Toruńczyk used the Dugundji extension theorem in the proof of Lemma in [44]. Instead of the Dugundji extension theorem, we use Corollary 2.19 which is an extension theorem for continuous functions on ultrametrizable spaces.

**Lemma 4.1.** Let \(R\) be a commutative ring. Let \((E, D_E)\) and \((F, D_F)\) be two ultra-normed \(R\)-modules. Let \(K\) and \(L\) be closed subsets of \(E\) and \(F\), respectively. Let \(f : K \to L\) be a homeomorphism. Let \(g : K \times \{0\} \to \{0\} \times L\) be a homeomorphism defined by \(g(x, 0) = (0, f(x))\), where we consider \(K \times \{0\} \subset E \times F\) and \(\{0\} \times L \subset E \times F\). Then there exists a homeomorphism \(h : E \times F \to E \times F\) with \(h|_{K \times \{0\}} = g\).

**Proof.** By Corollary 2.19 we obtain a continuous map \(\beta : F \to E\) which is an extension of \(f^{-1} : L \to K\). Define a map \(J : E \times F \to E \times F\)
by \( J(x, y) = (x + \beta(y), y) \). Lemma \(^{2,7}\) implies that the addition and the inversion on \( E \) is continuous, and hence \( J \) is continuous. The map 
\[ Q : E \times F \to E \times F \]
defined by \( Q(x, y) = (x - \beta(y), y) \) is also continuous, and it is the inverse map of \( J \), and hence \( J \) is a homeomorphism. Similarly, by Corollary \(^{2,19}\) we obtain a continuous map \( \alpha : E \to F \) which is an extension of \( f : K \to L \). Define a map \( I : E \times F \to E \times F \) by \( I(x, y) = (x, y + \alpha(x)) \). Then \( I \) is a homeomorphism. Define a homeomorphism \( h : E \times F \to E \times F \) by \( h = J^{-1} \circ I \). Since for every \( x \in K \) we have \( I(x, 0) = (x, \alpha(x)) = (x, f(x)) \), we obtain 
\[ h(x, 0) = J^{-1}(x, f(x)) = Q(x, f(x)) = (x - \beta(f(x)), f(x)) = (x - f^{-1}(f(x)), f(x)) = (0, f(x)) = g(x, 0), \]
and hence \( h \) is an extension of \( g \). 

**Proof of Theorem \(^{1,2}\)** Let \( S \) be a range set. Let \( X \) be an \( S \)-valued ultrametrizable space, and let \( A \) be a closed subset of \( X \). Let \( e \in UM(A, S) \). Take \( d \in UM(X, S) \). Theorem \(^{1,1}\) implies that there exist an \( S \)-valued ultra-normed \( Z \)-module \((E, D_E)\) and a closed isometric embedding \( i : (X, d) \to (E, D_E) \). Similarly, there exist an \( S \)-valued ultra-normed \( Z \)-module \((F, D_F)\) and a closed isometric embedding \( j : (A, e) \to (F, D_F) \).

Since \( A \) is closed in \( X \), the set \( i(A) \) is closed in \( E \). Since \( i \) and \( j \) are topological embeddings, \( i(A) \) and \( j(A) \) are homeomorphisms. Define a map \( f : i(A) \to j(A) \) by \( f = j \circ (i|_A)^{-1} \), and by applying Lemma \(^{1,1}\) to \( f \), we obtain a homeomorphism \( h : E \times F \to E \times F \) which is an extension of the map \( g : i(A) \times \{0\} \to \{0\} \times j(A) \) defined by \( g(i(a), 0) = (0, j(a)) \).

Let \( k : E \to E \times \{0\} \) be a natural embedding defined by \( k(x) = (x, 0) \).
The map \( H : X \to E \times F \) defined by \( H = h \circ k \circ i \) is a topological embedding. Define a metric \( D \) on \( X \) by 
\[ D(x, y) = (D_E \times D_F)(H(x), H(y)). \]
Then \( D \in UM(X, S) \). Since for every \( a \in A \) we have \( H(a) = (0, j(a)) \), and since \( j : (A, \rho) \to (F, D_F) \) is an isometric embedding, we have \( D|_{A^2} = e \). This completes the proof of the former part.

We next show the latter part. Assume that \( X \) is completely metrizable, and \( e \in UM(A, S) \) is complete. Then by Proposition \(^{2,17}\) we can choose \( d \in UM(X, S) \) as a complete \( S \)-valued ultrametric. Thus, we can choose \((E, D_E)\) and \((F, D_F)\) as complete ultrametric spaces, and hence the metric space \((X, D)\) can be regarded as a closed metric subspace of the complete metric space \((E \times F, D_F \times D_E)\). Therefore \( D \) is complete. This finishes the proof. \( \Box \)

**Remark 4.1.** In the proof of Theorem \(^{1,2}\) for simplicity, we use \( Z \)-modules. The proof described above is still valid even if we use any integral domain as a coefficient ring.
We next prove Corollary 1.3 which characterizes the compactness in terms of the completeness of ultrametrics.

**Lemma 4.2.** Let $S$ be a range set with the countable coinitiality. Let $M$ be a countable discrete space. Then there exists a non-complete $S$-valued ultrametric $d \in \text{UM}(M,S)$.

**Proof.** Take a non-zero strictly decreasing sequence $\{a(i)\}_{i \in \mathbb{N}}$ in $S$ with $\lim_{i \to \infty} a(i) = 0$. We may assume that $M = \mathbb{N}$. Define a metric $d$ on $M$ by

$$d(n,m) = \begin{cases} a(n) \vee a(m) & \text{if } n \neq m; \\ 0 & \text{if } n = m. \end{cases}$$

Then $d \in \text{UM}(M,S)$ and $d$ is non-complete. In particular, $\{n\}_{n \in \mathbb{N}}$ is Cauchy, and it does not have any limit point in $(M,d)$. □

**Proof of Corollary 1.3.** Assume that $X$ is not compact. Then there exists a closed countable discrete subset $M$ of $X$. By Theorem 1.2 and Lemma 4.2, we obtain a non-complete $S$-valued ultrametric $D$ on $X$ with $D \in \text{UM}(X,S)$. This implies Corollary 1.3. □

### 5. An Interpolation Theorem of Ultrametrics

In this section, we prove Theorem 1.4.

#### 5.1. Amalgamations.

The following lemma is a specialized version of [22, Proposition 3.2] for our study on $S$-valued ultrametrics.

**Lemma 5.1.** Let $S$ be a range set. Let $(X,d_X)$ and $(Y,d_Y)$ be $S$-valued ultrametric spaces, and let $Z = X \cap Y$. Assume that

(A) $Z \neq \emptyset$;
(B) $d_X|_Z = d_Y|_Z$;
(C) there exists $s \in S_+$ such that for every $x \in X \setminus Z$ we have $\inf_{z \in Z} d_X(x,z) = s$.

Then there exists an $S$-valued ultrametric $h$ on $X \cup Y$ such that

(1) $h|_{X^2} = d_X$;
(2) $h|_{Y^2} = d_Y$.

**Proof.** We define a symmetric function $h : (X \cup Y)^2 \to [0, \infty)$ by

$$h(x,y) = \begin{cases} d_X(x,y) & \text{if } x, y \in X; \\ d_Y(x,y) & \text{if } x, y \in Y; \\ \inf_{z \in Z} (d_X(x,z) \vee d_Y(z,y)) & \text{if } (x, y) \in X \times Y. \end{cases}$$

Since $d_X|_Z = d_Y|_Z$, the function $h$ is well-defined. By the definition, $h$ satisfies the conditions (1) and (2).
We next prove that $h$ satisfies the strong triangle inequality. In the case where $x, y \in X$ and $z \in Y$, for all $a, b \in Z$ we have
\[
h(x, y) = d_X(x, y) \leq d_X(x, a) \vee d_X(a, b) \vee d_X(b, y)
\]
\[
= d_X(x, a) \vee d_Y(a, b) \vee d_X(b, y)
\]
\[
\leq (d_X(x, a) \vee d_Y(a, z)) \vee (d_Y(z, b) \vee d_X(b, y)),
\]
and hence we obtain $h(x, y) \leq h(x, z) \vee h(z, y)$. In the case where $x, z \in X$ and $y \in Y$, for all $a \in Z$ we have
\[
h(x, y) \leq d_X(x, a) \vee d_Y(a, y)
\]
\[
\leq d_X(x, z) \vee (d_X(z, a) \vee d_Y(a, y)),
\]
and hence we have $h(x, y) \leq h(x, z) \vee h(z, y)$. By replacing the role of $X$ with that of $Y$, we see that $h$ satisfies the strong triangle inequality.

We now prove that $h$ takes values in $S$. It suffices to show that for all $x \in X \setminus Z$ and $y \in Y \setminus Z$, we have $h(x, y) \in S$. By the assumption (C) and the definition of $h$, we obtain $s \leq h(x, y)$. If $s = h(x, y)$, then $h(x, y)$ is in $S$. If $s < h(x, y)$, by the assumption (C), there exists $z \in Z$ with $h(x, z) < h(x, y)$. Lemma 2.8 implies that $h(x, y) = h(z, y)$. Since $h(z, y) = d_Y(z, y)$, we have $h(x, y) \in S$. This completes the proof. \(\square\)

Let $X$ and $Y$ be two sets, and let $\tau : X \to Y$ be a bijective map. For a metric $d$ on $Y$, we denote by $\tau^{*}d$ the metric on $X$ defined by $(\tau^{*}d)(x, y) = d(\tau(x), \tau(y))$. Remark that the map $\tau$ is an isometry from $(X, \tau^{*}d)$ into $(Y, d)$.

The following Proposition 5.2 and Lemmas 5.3 and 5.4 are ultrametric versions of Proposition 3.1, Lemma 3.4, Lemma 3.5.

**Proposition 5.2.** Let $S$ be a range set. Let $X$ be an ultrametrizable space. Let $r \in S_+$ and $d, e \in UM(X, S)$ satisfy $\mathcal{U}D_S^X(d, e) \leq r$. Put $X_0 = X$, and let $X_1$ be a set with $\text{card}(X_1) = \text{card}(X_0)$ and $X_0 \cap X_1 = \emptyset$. Let $\tau : X_0 \to X_1$ be a bijection. Then there exists an ultrametric $h \in UM(X_0 \cup X_1, S)$ such that

1. $h|_{X_0^2} = d$;
2. $h|_{X_1^2} = (\tau^{-1})^*e$;
3. for every $x \in X_0$ we have $h(x, \tau(x)) = r$.

**Proof.** We define a symmetric function $h : (X_0 \cup X_1)^2 \to [0, \infty)$ by
\[
h(x, y) = \begin{cases} 
    d(x, y) & \text{if } x, y \in X_0; \\
    e(x, y) & \text{if } x, y \in X_1;
    \inf_{a \in X_0} (d(x, a) \vee r \vee e(\tau(a), y)) & \text{if } (x, y) \in X_0 \times X_1.
\end{cases}
\]
By the definition, for every $x \in X$, we have $h(x, \tau(x)) \geq r$, and
\[
h(x, \tau(x)) \leq d(x, x) \vee r \vee e(\tau(x), \tau(x)) = r.
\]
Therefore for every $x \in X$ we have $h(x, \tau(x)) = r$. 

\[\]
We now prove that \( h \) satisfies the strong triangle inequality. In the case where \( x, y \in X_0 \) and \( z \in X_1 \), for all \( a, b \in X_0 \), by \( \mathcal{UD}_X^S(d, e) \leq r \) we have
\[
\begin{align*}
  h(x, y) &= d(x, y) \leq d(x, a) \vee d(a, b) \vee d(b, y) \\
  &\leq d(x, a) \vee r \vee e(\tau(a), \tau(b)) \vee d(b, y) \\
  &\leq d(x, a) \vee r \vee e(\tau(a), z) \vee e(\tau(b), z) \vee d(b, y) \\
  &\leq (d(x, a) \vee r \vee e(\tau(a), z)) \vee (d(y, b) \vee r \vee e(\tau(b), z)),
\end{align*}
\]
and hence we obtain \( h(x, y) \leq h(x, z) \vee h(y, z) \). In the case where \( x, z \in X_0 \) and \( y \in X_1 \), for all \( a \in X_0 \) we have
\[
\begin{align*}
  h(x, y) &\leq d(x, a) \vee r \vee e(\tau(a), y) \\
  &\leq d(x, z) \vee (d(z, a) \vee r \vee e(\tau(a), y)),
\end{align*}
\]
and hence \( h(x, y) \leq h(x, z) \vee h(y, z) \). By replacing the role of \( X_0 \) with that of \( X_1 \), we see that \( h \) satisfies the strong triangle inequality. By the property (3), we also see that \( h \in \text{UM}(X_0 \cup X_1) \).

We next prove that \( h \) takes values in \( S \). It suffices to show that for all \((x, y) \in X_0 \times X_1\), we have \( h(x, y) \in S \). By the definition of \( h \), we have \( r \leq h(x, y) \). If \( r = h(x, y) \), then \( h(x, y) \) is in \( S \). If \( r < h(x, y) \), by \( h(x, \tau(x)) = r \), we have \( h(x, \tau(x)) < h(x, y) \). From Lemma 2.8 it follows that \( h(x, y) = h(\tau(x), y) \). Since \( h(\tau(x), y) = e(\tau(x), y) \in S \), we conclude that \( h \) takes values in \( S \).

**Lemma 5.3.** Let \( S \) be a range set, and let \( s \in S_+ \). Let \( \{(A_i, e_i)\}_{i \in I} \) be a mutually disjoint family of \( S \)-valued ultrametric spaces. Then there exists an ultrametric \( h \in \text{UM}(\bigcup_{i \in I} A_i, S) \) such that
1. for every \( i \in I \) we have \( h|_{A_i^2} = e_i \);
2. for all distinct \( i, j \in I \), and for all \( x \in A_i \) and \( y \in A_j \), we have \( s \leq h(x, y) \).

**Proof.** We may assume that \( I \) is an ordinal. By transfinite induction, we define a desired ultrametric \( h \) as follows: Let \( a \in I + 1 \). Assume that for every \( b < a \) we already define ultrametrics \( \{h_b\}_{b < a} \) such that
1. if \( i < j < a \), then for all \( x, y \in A_i \) we have \( h_j(x, y) = h_i(x, y) \);
2. for every \( i < a \) we have \( h_b \in \text{UM}(\bigcup_{i < b} A_i, S) \);
3. if \( i \neq j \) and \( x \in A_i \) and \( y \in A_j \), then we have \( s \leq h_b(x, y) \).

If \( a = b+1 \), we can define an \( S \)-valued ultrametric \( h_a \in \text{UM}(\bigcup_{i < a} A_i, S) \) by using Proposition 2.3 for \( X = \bigcup_{i < b} A_i \), \( Y = A_a \) and \( r = s \). Assume next that \( a \) is a limit ordinal. We define a function \( h_a \) on \( (\bigcup_{i < a} A_i)^2 \) by
\[
h_a(x, y) = h_i(x, y),
\]
where \( i < a \) is the first ordinal with \( x, y \in \bigcup_{i < i} A_k \). By the inductive hypothesis (1), the function \( h_a \) is well-defined. From the inductive hypotheses (2) and (3), it follows that \( h_a \in \text{UM}(\bigcup_{i < a} A_i, S) \). Put \( h = h_j \), then the proof is completed. \( \Box \)
Lemma 5.4. Let $S$ be a range set. Let $X$ be an ultrametrizable space, and let $\{A_i\}_{i \in I}$ be a discrete family of closed subsets of $X$. Let $d \in \text{UM}(X, S)$, and let $\{e_i\}_{i \in I}$ be a family of ultrametrics such that $e_i \in \text{UM}(A_i, S)$. Assume that $\sup_{i \in I} \text{UD}^S_{A_i}(e_{A_i}, d|_{A_i^2}) < \infty$. Let $\eta$ be a member in $S_+$ such that
\[
\sup_{i \in I} \text{UD}^S_{A_i}(e_{A_i}, d|_{A_i^2}) \leq \eta.
\]
Let $\{B_i\}_{i \in I}$ be a mutually disjoint family of sets such that for all $i \in I$ we have $\text{card}(B_i) = \text{card}(A_i)$ and $X \cap B_i = \emptyset$. Let $\tau: \coprod_{i \in I} A_i \to \coprod_{i \in I} B_i$ be a bijection such that for each $i \in I$ the map $\tau_i = \tau|_{A_i}$ is a bijection between $A_i$ and $B_i$. Then there exists an $S$-valued ultrametric $h$ on $X \cup \coprod_{i \in I} B_i$ such that

1. for every $i \in I$ we have $h|_{B_i^2} = (\tau_i^{-1})^* e_i$;
2. $h|_{X^2} = d$;
3. for every $x \in \coprod_{i \in I} A_i$ we have $h(x, \tau(x)) = \eta$.

Proof. By Proposition 5.2, for every $i \in I$, we find an $S$-valued ultrametric $l_i \in \text{UM}(A_i \cup B_i, S)$ such that

1. $l_i|_{A_i^2} = d|_{A_i^2}$;
2. $l_i|_{B_i^2} = (\tau_i^{-1})^* e_i$;
3. for every $x \in A_i$ we have $l_i(x, \tau(x)) = \eta$.

By Lemma 5.3 we obtain an $S$-valued ultrametric $k$ which is a member of $\text{UM}(\coprod_{i \in I} (A_i \cup B_i), S)$ such that

1. for each $i \in I$ we have $k|_{(A_i \cup B_i)^2} = l_i$;
2. for all distinct $i, j \in I$, and for all $x \in A_i \cup B_i$ and $y \in A_j \cup B_j$, we have $\eta \leq h(x, y)$.

Since $X \cap \left( \coprod_{i \in I} (A_i \cup B_i) \right) = \coprod_{i \in I} A_i$, and since the ultrametric $k$ satisfies the assumptions stated in Lemma 5.1, we obtain an $S$-valued ultrametric $h$ on $X \cup \coprod_{i \in I} B_i$ such that

1. $h|_{X^2} = d$;
2. $h|_{(\coprod_{i \in I} B_i)^2} = k|_{(\coprod_{i \in I} B_i)^2}$.

By the definitions of ultrametrics $l_i$ and $k$, we conclude that $h$ is an $S$-valued ultrametric as required. \hfill \qed

5.2. Proof of Theorem 1.4. Before proving Theorem 1.4, we recall:

Proposition 5.5. Let $T$ be a topological space, and let $\{S_i\}_{i \in I}$ be a discrete family of closed subsets of $T$. Then $\bigcup_{i \in I} S_i$ is closed in $T$.

In the proof of [22, Theorem 1.1], the author used the Michael continuous selection theorem for paracompact spaces. Instead of that continuous selection theorem, to prove Theorem 1.4, we now use the 0-dimensional Michael continuous selection theorem (Theorem 2.20).
Proof of Theorem 1.4. Let \( C \in [1, \infty) \), and let \( S \) be a \( C \)-quasi-complete range set. Let \( X \) be an ultrametrizable space. Let \( \{A_i\}_{i \in I} \) be a discrete family of closed subsets of \( X \). Let \( d \in \text{UM}(X, S) \), and let \( \{e_i\}_{i \in I} \) be a family of \( S \)-valued ultrametrics with \( e_i \in \text{UM}(A_i, S) \).

If \( \sup_{i \in I} \text{UD}^S_{A_i}(e_i, d|_{A_i^2}) = \infty \), then Theorem 1.4 follows from Lemma 5.3 and Theorem 1.2. We may assume that \( \sup_{i \in I} \text{UD}^S_{A_i}(e_i, d|_{A_i^2}) < \infty \).

Let \( \eta \) be a member in \( S \) such that

\[
\sup_{i \in I} \text{UD}^S_{A_i}(e_i, d|_{A_i^2}) \leq \eta \leq C \cdot \sup_{i \in I} \text{UD}^S_{A_i}(e_i, d|_{A_i^2}).
\]

Let \( \{B_i\}_{i \in I} \), and let \( \tau : \prod_{i \in I} A_i \to \prod_{i \in I} B_i \) be the same family and the same map as in Lemma 5.4, respectively. Put \( Z = X \cup \prod_{i \in I} B_i \). By Lemma 5.4 we find an \( S \)-valued ultrametric \( h \) on \( Z \) such that

1. For every \( i \in I \) we have \( h|_{B_i^2} = (\tau_i^{-1})^*e_i \);
2. \( h(X^2) = d \);
3. For every \( x \in \prod_{i \in I} A_i \) we have \( h(x, \tau(x)) = \eta \).

By Theorem 1.1 we can take an isometric embedding \( H \) from \((Z, h)\) into a complete \( S \)-valued ultra-normed \( Z \)-module \((Y, D_Y)\). Define a map \( \phi : Z \to C(Y) \) by \( \phi(x) = B(H(x), \eta) \). By Corollary 2.22 the map \( \phi \) is lower semi-continuous. We define a map \( f : \bigcup_{i \in I} A_i \to Y \) by \( f_i(x) = H(\tau(x)) \). Then \( f \) is continuous. By the property (3) of \( h \), for every \( x \in \bigcup_{i \in I} A_i \) we have \( f(x) \in \phi(x) \).

Since \((Y, D_Y)\) is complete, we can apply the 0-dimensional Michael continuous selection theorem (Theorem 2.20) to the map \( f \), and hence we obtain a continuous map \( F : X \to Y \) such that \( F|_{\bigcup_{i \in I} A_i} = f \) and for every \( x \in X \) we have \( F(x) \in \phi(x) \). Note that \( F(x) \in \phi(x) \) means that \( D_Y(F(x), H(x)) \leq \eta \).

By Lemma 5.3 we obtain an ultrametric \( k \in \text{UM}(\bigcup_{i \in I} A_i, S) \) such that for every \( i \in I \) we have \( k|_{A_i^2} = e_i \). Since the \( S \)-valued ultrametric \( k \) generates the same topology as \( \bigcup_{i \in I} A_i \), and since \( \bigcup_{i \in I} A_i \) is closed in \( X \) (see Proposition 5.5), we can apply Theorem 1.2 to the \( S \)-valued ultrametric \( k \), and hence there exists an \( S \)-valued ultrametric \( r \in \text{UM}(X, S) \) such that for every \( i \in I \) we have \( r|_{A_i^2} = e_i \). Put \( l = \min\{r, \eta\} \). Note that by Lemma 2.1 we have \( l \in \text{UM}(X, S) \).

Put \( D = D_Y \times \infty l. \) Then \( D \) is an \( S \)-valued ultrametric on \( Y \times X \). Take a base point \( o \in X \). Define a map \( E : X \to Y \times X \) by

\[
E(x) = (F(x), x).
\]

Since the second component of \( E \) is a topological embedding, so is \( E \).

We also define a map \( K : X \to Y \times X \) by

\[
K(x) = (H(x), o).
\]

Then, by the definition of the ultrametric \( D \) on \( Y \times X \), the map \( K \) from \((X, d)\) to \((Y \times X, D)\) is an isometric embedding. Since for every
x ∈ X we have \(D_Y(F(x), H(x)) \leq \eta \) and \(l(x, o) \leq \eta \), we obtain

\[
D(E(x), K(x)) = D_Y(F(x), H(x)) \lor l(x, o) \leq \eta.
\]

Define a function \(m : X^2 \to [0, \infty) \) by \(m(x, y) = D(E(x), E(y))\), then \(m\) is an \(S\)-valued ultrametric on \(X\). Since \(E\) is a topological embedding, we see that \(m \in \text{UM}(X, S)\). For every \(i \in I\), and for all \(x, y \in A_i\), we have \(D_Y(F(x), F(y)) = e_i(x, y)\) and

\[
l(x, y) \leq r(x, y) = e_i(x, y);
\]

thus we obtain

\[
D(E(x), E(y)) = D_Y(F(x), F(y)) \lor l(x, y) = e_i(x, y),
\]

and hence \(m|_{A_i^2} = e_i\). Moreover, we have

\[
\sup_{i \in I} \mathcal{U}D_{A_i}^S(e_i, d|_{A_i^2}) \leq \mathcal{U}D_X^S(m, d).
\]

We also obtain the inequality \(\mathcal{U}D_X^S(m, d) \leq \eta\); indeed, for all \(x, y \in X\),

\[
m(x, y) = D(E(x), E(y))
\]

\leq \(D(E(x), K(x)) \lor D(K(x), K(y)) \lor D(K(y), E(y))\)

\leq \(D(K(x), K(y)) \lor \eta = d(x, y) \lor \eta\)

and

\[
d(x, y) = D(K(x), K(y))
\]

\leq \(D(K(x), E(x)) \lor D(E(x), E(y)) \lor D(E(y), K(y))\)

\leq \(D(E(x), E(y)) \lor \eta = m(x, y) \lor \eta\).

Therefore \(\mathcal{U}D_X^S(m, d) \leq \eta\), and hence we conclude that

\[
\sup_{i \in I} \mathcal{U}D_{A_i}^S(e_i, d|_{A_i^2}) \leq \mathcal{U}D_X^S(m, d) \leq C \cdot \sup_{i \in I} \mathcal{U}D_{A_i}^S(e_i, d|_{A_i^2}).
\]

This completes the proof of the former part of Theorem 1.4.

By the latter part of Theorem 1.2, we can choose \(l\) as a complete \(S\)-valued ultrametric. Then \(m\) becomes a complete \(S\)-valued ultrametric. This leads to the proof of the latter part of Theorem 1.4 \(\square\).

In Corollary 5.6, by letting \(I\) be a singleton, we obtain the following:

**Corollary 5.6.** Let \(C \in [1, \infty)\), and let \(S\) be a \(C\)-quasi-complete range set. Let \(X\) be an ultrametrizable space, and let \(A\) be a closed subset of \(X\). Then for every \(d \in \text{UM}(X, S)\), and for every \(e \in \text{UM}(A, S)\), there exists an ultrametric \(m \in \text{UM}(X, S)\) satisfying the following:

1. \(m|_{A^2} = e\);
2. \(\mathcal{U}D_A^S(e, d|_{A^2}) \leq \mathcal{U}D_X^S(m, d) \leq C \cdot \mathcal{U}D_A^S(e, d|_{A^2})\).

Moreover, if \(X\) is completely ultrametrizable, and if \(e \in \text{UM}(A, S)\) is a complete \(S\)-valued ultrametric, then we can choose \(m \in \text{UM}(X, S)\) as a complete metric.
6. Transmissible properties and ultrametrics

In this section we introduce the transmissible property, originally defined in [22], and we prove Theorem 1.5.

6.1. Transmissible properties on metric spaces. Let $\mathcal{P}^*(\mathbb{N})$ be the set of all non-empty subsets of $\mathbb{N}$. For a topological space $T$, we denote by $\mathcal{F}(T)$ the set of all closed subsets of $T$. For a subset $W \in \mathcal{P}^*(\mathbb{N})$, and for a set $E$, we denote by $\text{Seq}(W,E)$ the set of all finite injective sequences $\{a_i\}_{i=1}^n$ in $E$ with $n \in W$.

**Definition 6.1** ([22]). Let $Q$ be an at most countable set, $P$ a topological space. Let $F : Q \to \mathcal{F}(P)$ and $G : Q \to \mathcal{P}^*(\mathbb{N})$ be maps. Let $Z$ be a set. Let $\phi$ be a correspondence assigning a pair $(q,X)$ of $q \in Q$ and a metrizable space $X$ to a map $\phi^{q,X} : \text{Seq}(G(q),X) \times Z \times \mathcal{M}(X) \to P$. We say that a sextuple $(Q,P,F,G,Z,\phi)$ is a transmissible paremeter if for every metrizable space $X$, for every $q \in Q$, and for every $z \in Z$ the following are satisfied:

(TP1) for every $a \in \text{Seq}(G(q),X)$ the map $\phi^{q,X}(a,z) : \mathcal{M}(X) \to P$ defined by $\phi^{q,X}(a,z)(d) = \phi^{q,X}(a,z,d)$ is continuous, where $\mathcal{M}(X)$ is equipped with the topology induced from $\mathcal{D}_X$;

(TP2) for every $d \in \mathcal{M}(X)$, if $S$ is a subset of $X$ and $a \in \text{Seq}(G(q),S)$, then we have $\phi^{q,X}(a,z,d) = \phi^{q,S}(a,z,d|_S)$.

We introduce a property determined by a transmissible parameter.

**Definition 6.2** ([22]). Let $\mathfrak{G} = (Q,P,F,G,Z,\phi)$ be a transmissible parameter. Let $(X,d)$ be a metric space. We say that $(X,d)$ satisfies the $\mathfrak{G}$-transmissible property if there exists $q \in Q$ such that for every $z \in Z$ and for every $a \in \text{Seq}(G(q),X)$ we have $\phi^{q,X}(a,z,d) \in F(q)$. We say that $(X,d)$ satisfies the anti-$\mathfrak{G}$-transmissible property if $(X,d)$ satisfies the negation of the $\mathfrak{G}$-transmissible property; namely, for every $q \in Q$ there exist $z \in Z$ and $a \in \text{Seq}(G(q),X)$ with $\phi^{q,X}(a,z,d) \in X \setminus F(q)$.

A property on metric spaces is a transmissible property (resp. anti-transmissible property) if it is equivalent to the $\mathfrak{G}$-transmissible property (resp. anti-$\mathfrak{G}$-transmissible property) for some transmissible parameter $\mathfrak{G}$.

By the condition (TP2) in Definition 6.1 we obtain the following:

**Lemma 6.1.** Let $\mathfrak{G}$ be a transmissible parameter. If a metric space $(X,d)$ satisfies the $\mathfrak{G}$-transmissible property, then so does every metric subspace of $(X,d)$.

6.2. Transmissible properties on ultrametric spaces. The following concept is an $S$-valued ultrametric version of the singularity of the transmissible parameters (see [22, Definition 1.3]).

**Definition 6.3.** Let $S$ be a range set. Let $\mathfrak{G} = (Q,P,F,G,Z,\phi)$ be a transmissible parameter. We say that $\mathfrak{G}$ is $S$-ultra-singular if for each
Lemma 6.2. Let \( R, d \) be a complete ultrametric space \((G, \text{ultrametric space})\), and let \( d, e \) and for all \( q \in Q \) there exist \( z \in Z \), a finite \( S \)-valued ultrametric space \((R, d_R)\), and an index \( R = \{ r_i \}_{i=1}^{\text{card}(R)} \) such that

1. \( \delta_{d_R}(R) \leq \epsilon \);
2. \( \text{card}(R) \in G(q) \);
3. \( \phi^{q,R}(\{r_i\}_{i=1}^{\text{card}(R)}, z, d_R) \in X \setminus F(q) \).

By the definitions of \( D_X \) and \( UD_X^S \), we obtain:

**Lemma 6.2.** Let \( S \) be a range set. For every ultrametrizable space \( X \), and for all \( d, e \in UM(X, S) \) we have

\[
D_X(d, e) \leq UD_X^S(d, e).
\]

In particular, the identity map

\[
1_{UM(X,S)} : (UM(X,S), UD_X^S) \rightarrow (UM(X,S), D_X|_{UM(X,S)^2})
\]

is continuous.

Let \( S \) be a range set. Let \( X \) be an ultrametrizable space, and let \( \mathfrak{G} = (Q, P, F, G, Z, \phi) \) be a transmissible parameter. For \( q \in Q \), for \( a \in \text{Seq}(G(q), X) \) and for \( z \in Z \), we denote by \( US(X, S, \mathfrak{G}, q, a, z) \) the set of all \( d \in UM(X, S) \) such that \( \phi^{q,X}(a, z, d) \in X \setminus F(q) \). We also denote by \( US(X, S, \mathfrak{G}) \) the set of all \( d \in UM(X, S) \) such that \( (X, d) \) satisfies the anti-\( \mathfrak{G} \)-transmissible property.

**Proposition 6.3.** Let \( S \) be a range set. Let \( X \) be an ultrametrizable space, and let \( \mathfrak{G} = (Q, P, F, G, Z, \phi) \) be a transmissible parameter. Then for all \( q \in Q \), \( a \in \text{Seq}(G(q), X) \) and \( z \in Z \), the set \( US(X, S, \mathfrak{G}, q, a, z) \) is open in \((UM(X,S), UD_X^S)\).

**Proof.** Fix \( q \in Q \), \( a \in \text{Seq}(G(q), X) \) and \( z \in Z \). Since the map \( \phi^{q,X}(a, z) : M(X) \rightarrow P \) is continuous, Lemma 6.2 implies that the map \( \phi^{q,X}(a, z)|_{UM(X,S)} : UM(X,S) \rightarrow P \) is also continuous, where \( UM(X,S) \) is equipped with the topology induced from \( UD_X^S \). Since

\[
US(X, S, \mathfrak{G}, q, a, z) = (\phi^{q,X}(a, z)|_{UM(X,S)})^{-1}(X \setminus F(q)),
\]

the set \( US(X, S, \mathfrak{G}, q, a, z) \) is open in \((UM(X,S), UD_X^S)\). \( \Box \)

**Corollary 6.4.** Let \( S \) be a range set. Let \( \mathfrak{G} = (Q, P, F, G, Z, \phi) \) be a transmissible parameter. Let \( X \) be an ultrametrizable space. Then the set \( US(X, S, \mathfrak{G}) \) is \( G_5 \) in \( UM(X,S) \). Moreover, if the set \( Q \) is finite, then \( US(X, S, \mathfrak{G}) \) is open in \((UM(X,S), UD_X^S)\).

**Proof.** By the definitions of \( US(X, S, \mathfrak{G}) \) and \( US(X, S, \mathfrak{G}, q, a, z) \), we have

\[
US(X, S, \mathfrak{G}) = \bigcap_{q \in Q} \bigcup_{a \in \text{Seq}(G(q), X)} \bigcup_{z \in Z} US(X, S, \mathfrak{G}, q, a, z).
\]

This equality together with Proposition 6.3 proves the lemma. \( \Box \)
We say that a topological space is an \((\omega_0 + 1)\)-space if it is homeomorphic to the one-point compactification of the countable discrete topological space.

**Lemma 6.5.** Let \(S\) be a range set with the countable coinitiality. A transmissible parameter \(\mathfrak{G}\) is \(S\)-ultra-singular if and only if there exists an \(S\)-valued ultrametric \((\omega_0 + 1)\)-space with arbitrary small diameter satisfying the anti-\(\mathfrak{G}\)-transmissible property.

**Proof.** Let \(\mathfrak{G} = (Q, P, F, G, Z, \phi)\). First assume that there exists an \((\omega_0 + 1)\)-ultrametric space with arbitrary small diameter satisfying the anti-\(\mathfrak{G}\)-transmissible property. By the definition of anti-\(\mathfrak{G}\)-transmissible property, we see that \(\mathfrak{G}\) is \(S\)-ultra-singular.

Next assume that \(\mathfrak{G}\) is \(S\)-ultra-singular. Take a non-zero strictly decreasing sequence \(\{r(i)\}_{i \in \mathbb{N}}\) with \(\lim_{i \to \infty} r(i) = 0\). Fix \(\epsilon \in (0, \infty)\) and take a surjective map \(\theta : \mathbb{N} \to Q\). Take \(N \in \mathbb{N}\) such that for every \(n > N\), we have \(r(n) < \epsilon\). Then there exists a sequence \(\{(R_i, d_i)\}_{i \in \mathbb{N}}\) of finite ultrametric spaces such that for each \(i \in \mathbb{N}\) there exist \(z_i \in Z\) and an index \(R_i = \{r_{i,j}\}_{j=1}^{\text{card}(R_i)}\) satisfying

\[
\begin{align*}
(1) & \quad \delta_{d_i}(R_i) \leq r(N + i); \\
(2) & \quad \text{card}(R_i) \in G(\theta(i)); \\
(3) & \quad \phi^{\theta(i),R_i} \left(\{r_{i,j}\}_{j=1}^{\text{card}(R_i)}, z_i, d_i\right) \in X \setminus F(\theta(i)).
\end{align*}
\]

Put

\[L = \{\infty\} \sqcup \bigsqcup_{i \in \mathbb{N}} R_i,\]

and define a metric \(d_L\) on \(L\) by

\[
d_L(x, y) = \begin{cases} 
\delta_{d_i}(x, y) & \text{if } x, y \in X_i \text{ for some } i; \\
\delta_{d_i}(x, y) & \text{if } x, y \in X_i \text{ for some } i \neq j; \\
\delta_{d_i}(x, y) & \text{if } x = \infty, y \in X_i \text{ for some } i; \\
\delta_{d_i}(x, y) & \text{if } x, y = \infty \text{ for some } i.
\end{cases}
\]

Note that this construction is a specific version of the telescope space defined in \([20]\). The \((L, d_L)\) is a metric \((\omega_0 + 1)\)-space with \(\delta_{d_L}(L) \leq \epsilon\). By the definition, the metric \(d_L\) is an \(S\)-valued ultrametric (see also \([20]\) Lemma 3.1). By the properties (2) and (3) of \(\{R_i, d_i\}_{i \in \mathbb{N}}\), the metric space \((L, d_L)\) satisfies the anti-\(\mathfrak{G}\)-transmissible property. \(\square\)

Let \(S\) be a range set. Let \(\mathfrak{G}\) be a transmissible parameter. For a non-discrete ultrametrizable space \(X\), and for an \((\omega_0 + 1)\)-subspace \(R\) of \(X\), we denote by \(UT(X, S, R, \mathfrak{G})\) the set of all \(d \in UM(X, S)\) for which \((R, d_{R_E})\) satisfies the anti-\(\mathfrak{G}\)-transmissible property.

**Corollary 5.6** and **Lemma 6.5** imply the following:

**Proposition 6.6.** Let \(C \in [1, \infty)\), and let \(S\) be a \(C\)-quasi-complete range set with the countable coinitiality. Let \(\mathfrak{G} = (Q, P, F, G, Z, \phi)\)
be an \( S \)-ultra-singular transmissible parameter. Then for every non-discrete ultrametrizable space \( X \), and for every \((\omega_0+1)\)-subspace \( R \) of \( X \), the set \( UT(X, S, R, \mathcal{G}) \) is dense in \((UM(X, S), \mathcal{UD}_X^S)\).

**Proof.** Let \( \epsilon \in (0, \infty) \) be an arbitrary number. Let \( d \in UM(X, S) \). Take an \((\omega_0+1)\)-subspace \( R \) of \( X \) with \( \delta_d(R) \leq \epsilon \). By Lemma 6.5 there exists an \( S \)-valued ultrametric \( e \in UM(R, S) \) with \( \delta_e(R) \leq \epsilon \) such that \((R, e)\) satisfies the anti-\( \mathcal{G} \)-transmissible property. Since \( \delta_d(R) \leq \epsilon \) and \( \delta_e(R) \leq \epsilon \), by the definition of \( \mathcal{UD}_R^S \) we have \( \mathcal{UD}_R^S(d|_{R^2}, e) \leq \epsilon \). By applying Corollary 6.6 to \( d \) and \( e \), there exists an \( S \)-valued ultrametric \( m \in UM(X, S) \) with

1. \( m|_{R^2} = e \);
2. \( \mathcal{UD}_X^S(d, m) \leq C \cdot \mathcal{UD}_R^S(d|_{R^2}, e) \leq C \cdot \epsilon \).

By Lemma 6.1 we see that \((X, m)\) satisfies the anti-\( \mathcal{G} \)-transmissible property. Since \( \epsilon \) is arbitrary, the proposition follows. \( \square \)

**Proof of Theorem 7.3** Let \( C \in [1, \infty) \), and let \( S \) be a \( C \)-quasi-complete range set with the countable coinitiality. Let \( X \) be a non-discrete metrizable space, and let \( \mathcal{G} \) be an \( S \)-ultra-singular transmissible parameter. Since \( X \) is non-discrete, there exists an \((\omega_0+1)\)-subspace \( R \) of \( X \). By the definitions, we have

\[ UT(X, S, R, \mathcal{G}) \subset US(X, S, \mathcal{G}). \]

From Proposition 6.6 and Corollary 6.4 it follows that \( US(X, S, \mathcal{G}) \) is dense \( G_\delta \) in \((UM(X, S), \mathcal{UD}_X^S)\). This finishes the proof. \( \square \)

For a range set \( S \), and for a complete metrizable space \( X \), we denote by \( CUM(X, S) \) the set of all complete metrics in \( UM(X, S) \). From the latter part of Corollary 6.6 we deduce the following:

**Theorem 6.7.** Let \( S \) be a quasi-complete range set with the countable coinitiality. Let \( \mathcal{G} \) be an \( S \)-ultra-singular transmissible parameter. For every non-discrete completely ultrametrizable space \( X \), the set of all \( d \in CUM(X, S) \) for which \((X, d)\) satisfies the anti-\( \mathcal{G} \)-transmissible property is dense \( G_\delta \) in the ultrametric space \((CUM(X, S), \mathcal{UD}_X^S|_{CUM(X,S^2)})\).

**Remark 6.1.** Let \( C \in [1, \infty) \), and let \( S \) be a \( C \)-quasi-complete range set with the countable coinitiality. We can prove an \( S \)-valued ultrametric analogue of [22, Proposition 4.15], which states that satisfying a metric inequality on metric spaces is a transmissible property.

6.3. **Examples.** We show some examples of transmissible properties.

6.3.1. **The doubling property.** For a metic space \((X, d)\) and for a subset \( A \) of \( X \), we set \( \alpha_d(A) = \inf \{ d(x, y) \mid x, y \in A \text{ and } x \neq y \} \). A metic space \((X, d)\) is said to be doubling if there exist \( C \in (0, \infty) \) and \( \alpha \in (0, \infty) \) such that for every finite subset \( A \) of \( X \) we have

\[ \text{card}(A) \leq C \left( \frac{\delta_d(A)}{\alpha_d(A)} \right)^\alpha. \]
Note that \((X,d)\) is doubling if and only if \((X,d)\) has finite Assouad dimension (see e.g., [19] Section 10).

Similarly to [22, Proposition 4.9], we obtain:

**Proposition 6.8.** Let \(S\) be a range subset with the countable coinitiality. The doubling property is a transmissible property with an \(S\)-ultra-singular parameter.

6.3.2. The rich \(S\)-ultra-pseudo-cones property. Let \((X,d)\) be a metric space. Let \(\{A_i\}_{i \in \mathbb{N}}\) be a sequence of subsets of \(X\), and let \(\{u_i\}_{i \in \mathbb{N}}\) be a sequence in \((0,\infty)\). We say that a metric space \((P,d_P)\) is a pseudo-cone of \(X\) approximated by \((\{A_i\}_{i \in \mathbb{N}},\{u_i\}_{i \in \mathbb{N}})\) if

\[
\lim_{i \to \infty} \mathcal{GH}(\langle A_i, u_i \cdot d|_{A_i^2} \rangle, (P, d_P)) = 0
\]

(see [21]), where \(\mathcal{GH}\) is the Gromov–Hausdorff distance (see [4]). For a metric space \((X,d)\), we denote by \(\mathcal{P}(X,d)\) the class of all pseudo-cones of \((X,d)\). Let \(S\) be a range set, and let \(T\) be a range subset of \(S\) which is countable dense subset of \(S\). Let \(\mathcal{U}_T\) be the class of all finite ultrametric spaces on which all distances are in \(T\). We say that a metric space \((X,d)\) has rich \(S\)-ultra-pseudo-cones if \(\mathcal{U}_T\) is contained in \(\mathcal{P}(X,d)\) for some countable dense range subset \(T\) of \(S\).

**Lemma 6.9.** Let \(S\) be a range set, and let \(T\) be a countable dense range subset of \(S\). Let \(X\) be a finite discrete space, and let \(d \in \text{UM}(X,S)\). For every \(\epsilon \in (0,\infty)\), there exists a \(T\)-valued ultrametric \(e \in \text{UM}(X,T)\) such that for all \(x,y \in X\) we have \(|d(x,y) - e(x,y)| < \epsilon\).

**Proof.** Let \(a_0, a_1, \ldots, a_m\) be a sequence in \(S\) with \(\{d(x,y) \mid x,y \in X\} = \{a_0, a_1, \ldots, a_m\}\). We may assume that \(a_0 = 0\) and \(a_i < a_{i+1}\) for all \(i\). Put \(q_0 = a_0(=0)\). Since \(T\) is dense in \(S\), we can take a sequence \(q_1, \ldots, q_m\) in \(T_+\) such that \(|a_i - q_i| < \epsilon\) and \(q_i < q_{i+1}\) for all for all \(i \in \{1, \ldots, m\}\). Define a function \(e : X \times X \to T\) by putting \(e(x,y) = q_i\) if \(d(x,y) = a_i\). Lemma 2.11 implies that \(e\) is an ultrametric. By the definition, the ultrametric \(e\) satisfies the conditions as required. \(\square\)

Since every compact ultrametric space has a finite \(\epsilon\)-net for all \(\epsilon \in (0,\infty)\), Lemma 6.9 implies that for every range set \(S\), every compact \(S\)-valued ultrametric space is arbitrarily approximated by members of \(\mathcal{U}_T\) in the sense of Gromov–Hausdorff for every countable dense range subset \(T\) of \(S\). Thus we have:

**Corollary 6.10.** Let \(S\) be a range set. Let \((X,d)\) be an \(S\)-valued ultrametric space. The the following are equivalent to each other

1. \((X,d)\) has rich \(S\)-ultra-pseudo-cones
2. \(\mathcal{P}(X,d)\) contains all compact \(S\)-valued ultrametric spaces.
3. \(\mathcal{P}(X,d)\) contains \(\mathcal{U}_T\) for all countable dense range subset \(T\) of the range set \(S\).
A metric space is said to be rich pseudo-cones if all compact metric spaces are pseudo-cones of it. In [22, Proposition 4.12], it was proven that the rich pseudo-cones property is an anti-transmissible property with a singular parameter. Similarly, we obtain:

**Proposition 6.11.** Let $S$ be a range set with the countable coinitiality. The rich $S$-ultra-pseudo-cones property is an anti-transmissible property with an $S$-ultra-singular transmissible parameter.

### 7. Local transmissible properties and ultrametrics

In this section, we first investigate the basic properties on a specific ultrametric $u_S$ on a range set $S$. These properties help us to prove Lemma 7.1. We also prove Theorem 1.6 which is a local version of Theorem 1.5.

We define an ultrametric $u_S$ on a range set $S$ in such a way that $u_S(x, y)$ is the infimum of $\epsilon \in (0, \infty)$ such that $x \leq y + \epsilon$ and $y \leq x + \epsilon$. We denote by $d_E$ the Euclidean metric on $S$ defined by $d_E(x, y) = |x - y|$.

By the definition of $u_S$, we obtain:

**Lemma 7.1.** Let $S$ be a range set. Then for all distinct $x, y \in [0, \infty)$, we have $u_S(x, y) = x \lor y$. Hence $u_S$ is an $S$-valued ultrametric on $S$.

By the definitions of $d_E$ and $u_S$, we have:

**Lemma 7.2.** Let $S$ be a range set. For all $a, b \in S$, we have

$$d_E(a, b) \leq u_S(a, b).$$

Moreover, the identity map $1_S : (S, u_S) \to (S, d_E)$ is continuous.

**Lemma 7.3.** Let $S$ be a range set. Then the ultrametric space $(S, u_S)$ is complete.

*Proof.* Let $\{a_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $(S, u_S)$. Assume that there exists $a \in S$ such that $\{ n \in \mathbb{N} \mid a_n = a \}$ is infinite. Since $\{a_n\}_{n \in \mathbb{N}}$ is Cauchy, it is convergent to $a$. Assume next that for every $a \in S$, the set $\{ n \in \mathbb{N} \mid a_n = a \}$ is finite. For every $\epsilon \in (0, \infty)$, we can take $N \in \mathbb{N}$ such that for all $n, m > N$, we have $u_S(a_n, a_m) \leq \epsilon$. By the assumption, for every $n \in \mathbb{N}$, there exists $m > N$ with $a_n \neq a_m$. Thus by Lemma 7.1 we have $a_n \leq \epsilon$. This implies that $\lim_{n \to \infty} a_n = 0$. Therefore the space $(S, u_S)$ is complete. \qed

Let $S$ be a range set. Let $H$ be a topological space, and let $C(H, S)$ be the set of all continuous function from $H$ into $S$, where $S$ is equipped with the Euclidean topology. We define an ultrametric $U^S_H$ on $C(H, S)$ by

$$U^S_H(f, g) = \min \left\{ 1, \sup_{x \in H} u_S(f(x), g(x)) \right\}.$$
We also define a metric $E_H$ on $C(T, [0, \infty))$ by

$$E_H(f, g) = \min \left\{ 1, \sup_{x \in H} |f(x) - g(x)| \right\}.$$

Note that $(C(H, [0, \infty)), E_H)$ is complete.

**Remark 7.1.** Let $S$ be a range set. The space $(\text{UM}(X, S), \mathcal{U}D_X^S)$ and $(M(X), \mathcal{D}_X)$ can be considered as a topological subspace of the spaces $(C(X^2, S), \mathcal{U}X^2)$ and $(C(X^2, [0, \infty)), E_{X^2})$, respectively. Namely,

1. for every ultrametrizable space $X$, we see that $\text{UM}(X, S) \subset C(X^2, S)$, and the metric $\mathcal{U}X^2$ on $\text{UM}(X, S)$ generates the same topology as that induced from $\mathcal{D}_X$.
2. for every metrizable space $X$, we have $M(X) \subset C(X^2, [0, \infty))$, and the metric $E_{X^2}$ on $M(X)$ generates the same topology as that induced from $\mathcal{D}_X$.

By the definitions of $E_H$ and $\mathcal{U}_H^S$, and by Lemma 7.2, we have:

**Lemma 7.4.** Let $S$ be a range set. Let $H$ be a topological space. For all $f, g \in C(H, S)$, we have

$$E_H(f, g) \leq \mathcal{U}_H^S(f, g).$$

Moreover, the inclusion map $(C(H, S), \mathcal{U}_H^S) \to (C(H, [0, \infty)), E_H)$ is continuous.

Similarly to Lemma 7.3, Lemma 7.4 and the completeness of the space $(C(H, [0, \infty)), E_H)$ lead to the following:

**Lemma 7.5.** Let $S$ be a range set. Let $H$ be a topological space. Then the ultrametric space $(C(H, S), \mathcal{U}_H^S)$ is complete.

**Proof.** Let $\{f_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $C(H, S)$. Then for every $x \in H$, we find that $\{f_n(x)\}_{n \in \mathbb{N}}$ is Cauchy in $(S, u_S)$. By Lemma 7.3, $\{f_n(x)\}_{n \in \mathbb{N}}$ have a limit. Let $F(x) \in S$ be a limit of $\{f_n(x)\}_{n \in \mathbb{N}}$. By Lemma 7.4, the sequence $\{f_n\}_{n \in \mathbb{N}}$ is also Cauchy in $(C(H, [0, \infty)), E_H)$, and it has a limit $G \in (C(H, [0, \infty)), E_H)$. Note that $G$ is continuous. Lemma 7.3 yields $F = G$. Therefore $\{f_n\}_{n \in \mathbb{N}}$ has a limit in $C(H, S)$. This finishes the proof.

In the proof of Theorem 1.6 to apply the intersection property of Baire spaces to dense $G_\delta$ subsets, we need the following:

**Lemma 7.6.** Let $S$ be a range set. For every second countable locally compact ultrametrizable space $X$, the space $\text{UM}(X, S)$ is a Baire space.

**Proof.** By Lemma 7.5, the space $(C(X^2, S), \mathcal{U}X^2)$ is completely metrizable. By Lemma 2.25, in order to prove the lemma, it suffices to show that $\text{UM}(X, S)$ is $G_\delta$ in $(C(X^2, S), \mathcal{U}X^2)$.

We denote by $Q$ the set of all $f \in C(X^2, [0, \infty))$ such that

1. for every $x \in X$ we have $f(x) \geq 0$ and $f(x, x) = 0$;
Namely, $Q$ is the set of all continuous pseudo-ultrametrics on $X$. The set $Q$ is a closed subset in the space $(C(X^2, [0, \infty)), \mathcal{E}_X^2)$. Since all closed subsets of a metric space are $G_\delta$ in the whole space, the set $Q$ is $G_\delta$ in the space $(C(X^2, [0, \infty)), \mathcal{E}_X^2)$.

Since $X$ is second countable and locally compact, we can take a sequence $\{D_n\}_{n \in \mathbb{N}}$ of compact subsets of $X^2$ with $\bigcup_{n \in \mathbb{N}} D_n = X^2 \setminus \Delta_X$, where $\Delta_X$ is the diagonal set of $X^2$, and we can take a sequence $\{K_n\}_{n \in \mathbb{N}}$ of compact subsets of $X$ with $K_n \subset \text{INT}(K_{n+1})$ and $\bigcup_{n \in \mathbb{N}} K_n = X$, where INT means the interior.

As in the proof of [22, Theorem 5.1], for every $n \in \mathbb{N}$, let $L_n$ be the set of all $f \in C(X^2, [0, \infty))$ for which there exist $c \in (0, \infty)$ and $N \in \mathbb{N}$ such that for each $k > N$ we have

$$\inf_{x \in K_n} \inf_{y \in X \setminus K_k} f(x, y) > c.$$  

For each $n \in \mathbb{N}$, let $E_n$ be the set of all $f \in C(X^2, [0, \infty))$ such that for each $(x, y) \in D_n$, we have $0 < f(x, y)$. In the proof of [22, Theorem 5.1], it was proven that each $L_n$ and each $E_n$ are open subsets of the space $(C(X^2, [0, \infty)), \mathcal{E}_X^2)$.

Similarly to the proof of [22, Theorem 5.1], we obtain

$$\text{UM}(X) = Q \cap \left( \bigcap_{n \in \mathbb{N}} L_n \right) \cap \left( \bigcap_{n \in \mathbb{N}} E_n \right)$$

as subsets of $(C(X^2, [0, \infty)), \mathcal{E}_X^2)$; namely, $\text{UM}(X)$ is a $G_\delta$ subset of the space $(C(X^2, [0, \infty)), \mathcal{E}_X^2)$. Since the inclusion map from the space $(C(X^2, S), \mathcal{U}_X^S)$ into the space $(C(X^2, [0, \infty)), \mathcal{E}_X^2)$ is continuous (Lemma 7.4), and since $\text{UM}(X, S) = \text{UM}(X) \cap C(X^2, S)$, we conclude that $\text{UM}(X, S)$ is $G_\delta$ in the space $(C(X^2, S), \mathcal{U}_X^S)$. This completes the proof of the lemma. 

\begin{proof}[Proof of Theorem 7.4] Let $S$ be a quasi-complete range set with the countable coinitiality. Let $X$ be a second countable, locally compact, non-discrete ultrametrizable space, and let $\mathfrak{G} = (Q, P, F, G, Z, \phi)$. Let $E$ be the set of all $S$-valued ultrametrics $d \in \text{UM}(X, S)$ for which $(X, d)$ satisfies the local anti-$\mathfrak{G}$-transmissible property. Let $\{U_i\}_{i \in \mathbb{N}}$ be a countable open base of $X$, and let $\{R_i\}_{i \in \mathbb{N}}$ be a family of $(\omega_0 + 1)$-subspaces of $X$ with $R_i \subset U_i$. Since $\{U_i\}_{i \in \mathbb{N}}$ is an open base of $X$, by Lemma 6.4 we have

$$E = \bigcap_{i \in \mathbb{N}} \bigcap_{q \in Q} \bigcup_{z \in Z} \bigcup_{a \in \text{Seq}(G(q), U_i)} \text{US}(X, S, \mathfrak{G}, q, a, z).$$

\end{proof}
Corollary 6.4 implies that $E$ is $G_\delta$ in $\text{UM}(X,S)$. By the definitions, for each $i \in \mathbb{N}$, the set
\[
\bigcap_{q \in Q} \bigcup_{a \in \text{Seq}(G(q),U_i)} US(X,S,G,q,a,z)
\]
contains $UT(X,S,R_i,G)$. From Proposition 6.6 it follows that each set $UT(X,S,R_i,G)$ is dense in $\text{UM}(X,S)$. By Lemma 7.6 the space $\text{UM}(X,S)$ is a Baire space. Since $E$ is an intersection of countable dense $G_\delta$ sets in a Baire space $\text{UM}(X,S)$, the set $E$ is dense $G_\delta$ in $\text{UM}(X,S)$. This completes the proof. □

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