On the Multi Trace Superpotential and Corresponding Matrix Model

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Abstract

We study $\mathcal{N} = 1$ supersymmetric $U(N)$ gauge theory coupled to an adjoint scalar superfiled with a cubic superpotential containing a multi trace term. We show that the field theory results can be reproduced from a matrix model which its potential is given in terms of a linearized potential obtained from the gauge theory superpotential by adding some auxiliary nondynamical field. Once we get the effective action from this matrix model we could integrate out the auxiliary field getting the correct field theory results.

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1 Introduction

Recently it has been proposed [1] that the exact superpotential and gauge coupling for a wide class of $\mathcal{N} = 1$ supersymmetric gauge theories can be obtained using perturbative computations in a related matrix model. Given an $\mathcal{N} = 1$ SYM theory the potential of the corresponding matrix model is given in terms of the gauge theory superpotential. Even more interesting the nonperturbative results of gauge theory can be obtained from just planar diagrams of matrix model without taking any large $N$ limit in the gauge theory. This conjectured based on earlier works [2] -[6] and has recently been verified perturbatively using superspace formalism [7] or anomalies [8, 9]. Further developments can be found in [10]-[14].

To make the proposal precise and also to fix our notation consider a $U(N)$ gauge theory with $\mathcal{N} = 1$ supersymmetry coupled to a chiral superfield in the adjoint representation of $U(N)$. Moreover, in general, we take the following superpotential

\[ W(\phi) = \sum_{k=1}^{n+1} \frac{g_k}{k} \text{Tr}(\phi^k) \]  

for some $n$. To get a supersymmetric vacuum we need to impose D- and F-term conditions. Taking $\phi$ to be diagonal would satisfy the D-term and for F-term we need to set $W'(x) = 0$. This equation has $n$ roots $a_i$ and thus one can write $W'(x) = g_{n+1} \prod_{i=1}^{n} (x - a_i)$. Therefore by taking $\phi$ to have eigenvalue $a_i$ with multiplicity $N_i$, the gauge symmetry $U(N)$ is broken down to $\prod_{i=1}^{n} U(N_i)$ with $\sum_{i=1}^{n} N_i = N$.

If the roots $a_i$ are all distinct, the chiral superfields are all massive and can then be integrated out to get an effective action for low energy theory. The chiral part of the low energy effective Lagrangian can be written as [9]

\[ L_{\text{eff}} = \int d^2 \theta \, W_{\text{eff}} + c.c. , \quad W_{\text{eff}} = f(S_k, g_k) + \sum_{i,j} \tau_{ij} \omega_{\alpha i} \omega^\alpha_j , \]  

where $S_k = -\frac{1}{32\pi^2} \text{Tr} W_{\alpha i} W^{\alpha i}$ and $\omega_{\alpha i} = \frac{1}{4\pi} \text{Tr} W_{\alpha i}$ with $W^{\alpha i}$ being the gauge superfields of $U(N_i)$ gauge group.

The main point in the Dijkgraaf-Vafa’s proposal is that the chiral part of the effective action can be given by a holomorphic function $F_G(S_k)$, such that

\[ W_{\text{eff}} = \sum_{i=1}^{n} N_i \frac{\partial F_G}{\partial S_i} + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 F_G}{\partial S_i \partial S_j} \omega_{\alpha i} \omega^\alpha_j . \]  

Now what is left to be determined is the function $F_G$. In fact it is the goal of the Dijkgraaf-Vafa’s proposal to identify $F_G$ as the free energy of an auxiliary nonsupersymmetric matrix model which has for its ordinary potential the same function $W$ that is the superpotential of the four dimensional supersymmetric gauge theory. The matrix model free energy is given by

\[ e^{-\frac{1}{g_s} F_0} = \frac{1}{\text{Vol}(U(M))} \int D\phi \, e^{\left(-\frac{1}{g_s} W(\phi)\right)} , \]  

where $
where \( \phi \) is an \( M \times M \) matrix belongs to \( U(M) \). For the model we are considering one needs also to take \( \phi \) in such a way that the \( U(M) \) symmetry is broken down to \( \prod_{i=1}^{n} U(M_i) \) such that \( \sum_{i=1}^{n} M_i = M \). Moreover one should also identify \( S_i = g_s M_i \).

Taking large \( M \) limit one can compute \( F_0 \) order by order using only planar diagrams in the matrix perturbation theory. Now the prescription [7] is that, for example, the \( l \)th instanton contribution to the effective action can be reproduced from a perturbative contribution with \( l \) loops in the auxiliary matrix model. Actually having the matrix model free energy the effective superpotential is obtained by

\[
W_{\text{eff}} = \sum_{i=1}^{n} \left( N_i \frac{\partial F_0}{\partial S_i} - 2\pi i \tau_0 S_i \right) .
\]  

(5)

where \( \tau_0 \) is the bare coupling of the theory.

By now there are huge number of papers devoted to this proposal where only superpotentials with single trace operators have been studied. Recently superpotentials containing multi trace operators has also been considered in [15] where the authors showed that taking naively \( W \) with multi trace as the potential of matrix model would lead to incorrect matrix model. By “incorrect” they mean that one cannot reproduce the corresponding gauge theory results, though the obtained matrix model could be an auxiliary matrix model of some gauge theory which, of course, is not what we started with. More precisely it has been shown that although the diagrams surviving the large \( M \) limit of the matrix model with multi trace potential are exactly the graphs that contribute to the effective action of the field theory with multi trace tree level superpotential, one cannot compute the effective superpotential of the field theory by taking a derivative \( \frac{\partial F_0}{\partial S} \).

This problem can, of course, be solved [15] using the linearized superpotential in the matrix model. In fact starting with a multi trace operators in the superpotential one can linearized it using some nondynamical background fields \( A_i \). Then the potential would contain only single trace operators with \( A_i \)'s dependent coefficients. Once we find \( W_{\text{eff}} \) from matrix model, we can integrate out \( A_i \)'s fields getting the correct gauge theory result.

This is the aim of this article to further study a superpotential containing multi trace operators. In fact we shall study \( \mathcal{N} = 1 \) \( U(N) \) SYM theory coupled to an adjoint scalar superfield with the cubic superpotential given by

\[
W_{\text{tree}} = \frac{1}{3} \text{Tr}(\phi^3) + \frac{1}{2} m \text{Tr}(\phi^2) + \lambda \text{Tr}(\phi) + \frac{1}{2} g \text{Tr}(\phi) \text{Tr}(\phi^2) .
\]  

(6)

We will see that, using the linearized form of the superpotential, one can reproduce the gauge theory results in the cases with and without gauge symmetry breaking. We note that in [15] the authors have only considered a model where the gauge symmetry is not broken. As we will see the procedure works in the case with broken gauge symmetry as well.

\[\mathcal{N} = 1 \] supersymmetric \( U(N) \) gauge theory with cubic single trace superpotential has been extensively studied, for example, in [16]-[20].
The organization of this paper is as follows. In section 2 we will review $\mathcal{N} = 1$ $U(N)$ SYM theory with cubic single trace superpotential. We shall consider two different cases in which the gauge group may or may not be broken. In section 3 we will study the same theory with a multi trace term added to the superpotential. Regarding the fact that this model can be thought of as a deformation of $\mathcal{N} = 2$ theory we will find the effective superpotential using the factorization of Seiberg-Witten curve. In section 4 we will reproduce the same field theory results using linearized superpotential. In section 5 we will see how the corresponding matrix model can be treated. The last section is devoted to conclusions. Some technical computation of factorization of Seiberg-Witten curve is presented in the appendix.

2 Single trace superpotential

In this section we shall review the $\mathcal{N} = 1$ supersymmetric $U(N)$ gauge theory coupled with an adjoint scalar hypermultiplet with cubic superpotential containing only single trace operators

$$W_{\text{tree}} = \frac{1}{3} \text{Tr}(\phi^3) + \frac{1}{2} m \text{Tr}(\phi^2) + \lambda \text{Tr}(\phi).$$  

(7)

Taking $\phi$ diagonal one just needs to set $W'(\phi) = 0$ to get the supersymmetric vacuum, and therefore the derivative of superpotential can be recast to

$$W'(x) = (x - a_1)(x - a_2), \quad a_{1,2} = -\frac{m}{2} \pm \frac{1}{2} \sqrt{m^2 - 4\lambda}.$$

(8)

In general we can take $\phi$ to have eigenvalues $a_1$ or $a_2$ with multiplicity $N_1$ and $N_2$, respectively. This will break gauge symmetry to $U(N_1) \times U(N_2)$ with $N_1 + N_2 = N$. Of course as an special case one can, for example, take $N_2 = 0$ which corresponds to the supersymmetric vacuum without gauge symmetry breaking. In the following we shall consider both cases.

2.1 Unbroken gauge symmetry

In this section for the case where the gauge symmetry is not broken, we will first review how the exact superpotential can be obtained using the factorization of the Seiberg-Witten curve. In fact the model we are interested in can be obtained from $\mathcal{N} = 2$ supersymmetric $U(N)$ gauge theory perturbed by a general tree level superpotential given by

$$W_{\text{tree}} = \sum_{k=1}^{n+1} \frac{1}{k} g_k \text{Tr}(\phi^k).$$

(9)

A generic point in the moduli space of the $U(N)$ $\mathcal{N} = 2$ theory will be lifted by adding such a superpotential. The points which are not lifted are precisely where at least $N - n$ mutually local monopoles become massless. This can be seen from
the following argument. The gauge group in the $\mathcal{N} = 1$ theory is broken down to $\prod_{i=1}^{\mathcal{N}} U(N_i)$, and each $SU(N_i)$ factors are confined. We expect condensation of $N_i - 1$ magnetic monopoles in each of these $SU(N_i)$ factors and a total of $N - n$ condensed magnetic monopoles. These monopoles condense at the points on the $\mathcal{N} = 2$ moduli space where $N - n$ mutually local monopoles become massless. These are precisely the points which are not lifted by addition of the superpotential.

These considerations are equivalent to the requirement that the corresponding Seiberg-Witten curve has the factorization

$$P_N^{2}(x, u) - 4\Lambda^{2N} = H_{N-n}^{2}(x) F_{2n}(x),$$

where $P_N(x, u)$ is an order $N$ polynomial in $x$ with coefficients determined by the (vevs of) the $u_k$, $\Lambda$ is an ultraviolet cut-off, $H$ and $F$ are order $N - n$ and $2n$ polynomials in $x$, respectively.

The $N - n$ double roots place $N - n$ conditions on the original variables $u_k$. We can parameterize all the $\langle u_k \rangle$ by $n$ independent variables $\alpha_j$. In other words, the $\alpha_j$’s then correspond to massless fields in the low-energy effective theory. If we know the exact effective action for these fields, to find the vacua, we simply minimize $S_{\text{eff}}$. Furthermore, substituting $\langle u_k \rangle$ back into the effective action gives the action for the vacua.

In general the factorization problem is hard to solve, but for the confining vacuum where all $N - 1$ monopoles have condensed, there is a general solution given by Chebyshev polynomials.\footnote{This was worked out first by Douglas and Shenker \cite{Douglas:1995nw}.} In our case, we have the solution

$$\langle u_p \rangle = \frac{N}{p!} \sum_{q=0}^{[p/2]} C_p^q C_2^q \Lambda^q z^{p-2q}, \quad C_n^p := \binom{n}{p} = \frac{n!}{p!(n-p)!},$$

where $z = \langle u_1 \rangle N$. We note, however, that the above procedure is not the best form to be compared with the matrix model result because there is no gluino condensate $S$. To make the comparison, we need to “integrate in” \cite{Douglas:1995kn} the glueball superfield as in \cite{Witten:1995im}.

In the model we are considering the exact superpotential which has to be minimized is

$$W_{\text{exact}} = \langle u_3 \rangle + m \langle u_2 \rangle + \lambda \langle u_1 \rangle,$$

where

$$\langle u_1 \rangle = Nz, \quad \langle u_2 \rangle = \frac{N}{2} (z^2 + 2\Lambda^2), \quad \langle u_3 \rangle = \frac{N}{3} (z^3 + 6\Lambda^2 z).$$

The “integrate in” procedure can be done by setting $B := \Lambda^2$, and use the equation

$$NS = B \frac{\partial W_{\text{exact}}}{\partial B} = NB(m + 2z),$$

where $B := \Lambda^2$. This was worked out first by Douglas and Shenker \cite{Douglas:1995nw}.\footnote{This was worked out first by Douglas and Shenker \cite{Douglas:1995nw}.}
to solve for $B$ in terms of $S$. Then we find $z$ by solving
\[ 0 = \frac{\partial W_{\text{exact}}}{\partial z} = N(z^2 + mz + \lambda + 2B). \] (15)

Now the effective superpotential for the glueball superfield $S$ can be written as
\[ W_{\text{eff}}(S, g, \Lambda) = -S \log \left( \frac{B}{\Lambda^2} \right)^N + W_{\text{exact}}(S, g). \] (16)

One can explicitly solve the equations (14) and (15) to find $z$ and $B$ in power series of $S$. The results up to $O(S^6)$ are
\[ B = \frac{S}{\Delta} + 4 \frac{S^2}{\Delta^4} + 40 \frac{S^3}{\Delta^7} + 512 \frac{S^4}{\Delta^{10}} + 7392 \frac{S^5}{\Delta^{13}}, \]
\[ z = -\frac{m + \Delta}{2} - 2 \frac{S}{\Delta^2} - 12 \frac{S^2}{\Delta^5} - 128 \frac{S^3}{\Delta^8} - 1680 \frac{S^4}{\Delta^{11}} - 24576 \frac{S^5}{\Delta^{14}}. \] (17)

Plugging these solutions to the above expression of effective superpotential, one gets
\[ W_{\text{eff}} = -NS(\log(\frac{S}{\Delta \Lambda^2}) - 1) - \frac{2N}{3} \frac{S^2}{\Delta^3} \left( 3 + 16 \frac{S}{\Delta^3} + 140 \frac{S^2}{\Delta^6} + 512 \frac{S^3}{\Delta^9} \right), \] (18)

which is the exact effective action up to 5 instanton.

Using the Dijkgraaf-Vafa’s proposal one will be able to reproduce this result using a nonsupersymmetric matrix model with the potential given by (7). Since we are interested in the case where the gauge symmetry is not broken one considers the expansion around a classical solution as the following
\[ \phi = a_1 1_{M \times M} + \varphi, \] (19)

and therefore the potential of matrix model reads
\[ W(\varphi) = W(a_1) + \frac{1}{3} \text{Tr}(\varphi^3) + \frac{1}{2} \Delta \text{Tr}(\varphi^2), \] (20)

where $\Delta = a_1 - a_2$. Here $\phi$ is $M \times M$ matrix belongs to $U(M)$ group. One can now write the Feynman rules and thereby evaluate the matrix model free energy order by order using (4). Here we shall take a limit in which $M$ is large and keeping the ’t Hooft coupling $S = g_s M$ fixed, and thus only planar diagrams would contribute. In this limit the free energy is found to be [16]
\[ \mathcal{F}_0(S) = \frac{1}{2} S^2 \log \left( \frac{S}{\Delta^3} \right) - S^2 \log \left( \frac{\Lambda}{\Delta} \right) + \frac{2}{3} \frac{S^3}{\Delta^3} \left( 1 + 4 \frac{S}{\Delta^3} + 28 \frac{S^2}{\Delta^6} + \cdots \right) \] (21)

up to 4-loop. Using this expression the exact superpotential is given by (see also [3])
\[ W_{\text{exact}} = -NS \left( \log \left( \frac{S}{\Delta \Lambda^2} \right) - 1 \right) - \frac{2N}{3} \frac{S^2}{\Delta^3} \left( 3 + 16 \frac{S}{\Delta^3} + 140 \frac{S^2}{\Delta^6} + \cdots \right), \] (22)

which is in exact agreement with the field theory computation (18). As we see the $l$th loop contribution to the matrix model free energy is the same as $l$ instantons contribution to the effective action.
2.2 Broken gauge symmetry

In this subsection we shall review the case where the gauge symmetry is broken to two parts. In other words we consider a matrix model where \( \text{U}(M) \) group is broken down to \( \text{U}(M_1) \times \text{U}(M_2) \). To get such a matrix model we take

\[
\phi = \begin{pmatrix}
  a_11_{M_1 \times M_1} & 0 \\
  0 & a_21_{M_2 \times M_2}
\end{pmatrix}
+ \begin{pmatrix}
  \varphi_{11} & \varphi_{12} \\
  \varphi_{21} & \varphi_{22}
\end{pmatrix}
\]  

(23)

here \( M_1 + M_2 = M \). Moreover we will consider the large \( M_1 \) and \( M_2 \) limit while keeping \( S_1 = g_sM_1 \) and \( S_2 = g_sM_2 \) fixed. These means that only planar diagrams would be important. To do the explicit computation one can use a gauge in which \( \varphi_{12} \) and \( \varphi_{21} \) are set to zero. This can be done using Faddeev-Popov ghost field, and therefore the matrix model action is found as [16]

\[
W = \frac{1}{2} \Delta \left( \text{Tr}(\varphi_{11}^2) - \text{Tr}(\varphi_{22}^2) \right) + \frac{1}{3} \left( \text{Tr}(\varphi_{11}^3) + \text{Tr}(\varphi_{22}^3) \right)
+ \Delta \left( \text{Tr}(B_{21}C_{12}) - \text{Tr}(B_{12}C_{21}) \right) + \text{Tr}(B_{21}\varphi_{11}C_{12} + C_{21}\varphi_{11}B_{12})
+ \text{Tr}(B_{12}\varphi_{22}C_{21} + C_{11}\varphi_{22}B_{21}),
\]  

(24)

where \( B \) and \( C \) are corresponding ghost fields.

It is now easy to write down the Feymann rules for double line Feymann diagrams and thereby to compute the matrix model free energy order by order. The result up to 4-loop is [16]

\[
\mathcal{F}_0(S_1, S_2) = -\frac{1}{2} \sum_i S_i^2 \log \left( \frac{S_i}{\Delta^3} \right) + (S_1 + S_2)^2 \log \left( \frac{\Lambda}{\Delta^3} \right)
+ \frac{1}{3\Delta^3} \left( 2S_1^3 - 15S_1^2S_2 + 15S_1S_2^2 - 2S_2^3 \right)
+ \frac{1}{3\Delta^6} \left( 8S_1^4 - 91S_1^3S_2 + 177S_1^2S_2^2 - 91S_1S_2^3 - 8S_2^4 \right)
+ \frac{1}{3\Delta^9} \left( 56S_1^5 - 871S_1^4S_2 + 2636S_1^3S_2^2 - 2636S_1^2S_2^3 + 871S_1S_2^4 - 56S_2^5 \right).
\]  

(25)

Having the matrix model free energy the effective superpotential for the case where the gauge symmetry is broken as \( \text{U}(N) \rightarrow \text{U}(N_1) \times \text{U}(N_2) \) can be found as following

\[
W_{\text{eff}} = \sum_{i=1}^{2} \left( N_i \frac{\partial F_0}{\partial S_i} - 2\pi i \tau_0 S_i \right).
\]  

(26)

where \( \tau_0 \) is the bare coupling of the theory.

This effective superpotential should be compared with that obtained from gauge theory computation. The gauge theory result may be found using factorization of Seiberg-Witten curve, though, in general the factorization procedure is difficult to
be done. Nevertheless for an special case this can easily be worked out. For example consider the SYM theory with gauge group $U(3N)$ broken down into $U(2N) \times U(N)$. Actually the analysis of this theory is equivalent to SYM theory with gauge group $U(3)$ broken down into $U(2) \times U(1)$ where the effective superpotential is turned out to be $^{5}[23]$

$$W_{\text{eff}} = u_3 + m u_2 + \lambda u_1 \pm 2 \Lambda^3.$$ (27)

Of course it is not a suitable form for comparison with the matrix model result. Actually to compare these two results one can, for example, integrate out the $S_1$ and $S_2$ fields from the effective superpotential obtained from the matrix model. Doing so, one can see that the matrix model reproduce the correct result (27) order by order $^{3}$.

### 3 Multi trace superpotential

In this section we will study $\mathcal{N} = 1$ $U(N)$ SYM theory coupled to an adjoint scalar superfield with a superpotential containing a multi trace operator

$$W_{\text{tree}} = \frac{1}{3} \text{Tr}(\phi^3) + \frac{1}{2} m \text{Tr}(\phi^2) + \lambda \text{Tr}(\phi) + \frac{1}{2} g \text{Tr}(\phi) \text{Tr}(\phi^2).$$ (28)

To get the supersymmetric vacuum one needs to impose F- and D-terms conditions. Taking diagonal $\phi$ would satisfy the D-term condition and for F-term we need to solve $W'_{\text{tree}}(\phi) = 0$. This equation has two solutions, $b_{1,2}$, and therefore in general $\phi$ can be taken such that to have eigenvalue $b_i$ with multiplicity $N_i$. This will break the gauge symmetry down to $U(N_1) \times U(N_2)$ with $N_1 + N_2 = N$.

To find the eigenvalues $b_i$’s we note that the adjoint scalar has been taken as $\phi = \text{diag}(b_1^{N_1 \times N_1}, b_2^{N_2 \times N_2})$, and thus the superpotential is given by

$$W_{\text{tree}} = \frac{1}{3} (N_1 b_1^3 + N_2 b_2^3) + \frac{m}{2} (N_1 b_1^2 + N_2 b_2^2) + \lambda (N_1 b_1 + N_2 b_2)$$

$$+ \frac{g}{2} (N_1 b_1 + N_2 b_2)(N_1 b_1^2 + N_2 b_2^2),$$ (29)

so, the F-term condition reads

$$\lambda + m b_1 + b_1^2 + \frac{g}{2} (N_1 b_1^2 + N_2 b_2^2) + g b_1 (N_1 b_1 + N_2 b_2) = 0$$

$$\lambda + m b_2 + b_2^2 + \frac{g}{2} (N_1 b_1^2 + N_2 b_2^2) + g b_2 (N_1 b_1 + N_2 b_2) = 0.$$ (30)

One can now solve these equations to find $b_1$ and $b_2$. The solution is

$$b_1 = - \frac{m}{1 + N_1 g} - \frac{1 + N_2 g}{1 + N_1 g} b_2,$$ (31)

$^{5}$In appendix A we have presented the factorization of Seiberg-Witten curve for the general case where the unbroken gauge symmetry has only one nonabelian factor of $U(2)$. 

and \( b_2 \) satisfies
\[
b_2^2 + \tilde{m} b_2 + \tilde{\lambda} = 0 ,
\]
where
\[
\tilde{m} = \frac{(1 + 2 N_1 + N_1 N_2 g^2) m}{(1 + (N_1 + N_2) g (3 + N_1 N_2 g^2) - 1) ,}
\]
\[
\tilde{\lambda} = \frac{2 \lambda (1 + N_1 g)^2 + m^2 N_1 g}{(1 + (N_1 + N_2) g (3 + N_1 N_2 g^2) - 1) .}
\]

Thus in general one can write \( W'(x) = (x - b_1)(x - b_2) \).

### 3.1 Unbroken gauge symmetry

By making use of the fact that this model can be obtained from deformation of \( \mathcal{N} = 2 \ U(N) \) SYM theory by adding the superpotential (28), the effective superpotential can be obtained from the factorization of Seiberg-Witten curve. In fact since the gauge symmetry is not broken the factorization can be obtained for the confining vacuum where all \( N - 1 \) monopoles have condensed in terms of Chebyshev polynomials. Indeed the solution is the same as (11). Therefore the effective superpotential reads
\[
W_{\text{exact}} = N (\lambda + 2 \Lambda^2 + g N \Lambda^2) z + \frac{mN}{2} (z^2 + 2 \Lambda^2) + \frac{N}{3} (1 + \frac{3 g N}{2}) z^3 ,
\]

Setting \( B := \Lambda^2 \) we can proceed to integrate in the glueball field \( S \) as follows. First we find \( B \) in terms of \( S \) from the equation
\[
N S = B \frac{\partial W_{\text{exact}}}{\partial B} = N B (m + 2 z + g N z) .
\]

Then we find \( z \) by solving
\[
0 = \frac{\partial W_{\text{exact}}}{\partial z} = N \left( (1 + \frac{3 g N}{2}) z^2 + m z + \lambda + 2 B + g N B \right) .
\]

The effective action for the glueball superfield \( S \) can be written as
\[
W_{\text{eff}}(S, g, \Lambda) = -S \log \left( \frac{B}{\Lambda^2} \right)^N + W_{\text{exact}}(S, g) .
\]

To write the effective superpotential explicitly let us, for simplicity, set \( \lambda = 0 \). In this case one finds the following solutions for \( z \) and \( B \) in power series of \( S \) up to order \( O(S^6) \)
\[
B = \frac{S}{m} + \frac{(2 + g N)^2 S^2}{m^4} + \frac{(10 + 7 g N)(2 + g N)^3 S^3}{2 m^7} .
\]
\[ z = -\frac{(2 + gN)^2S^2}{m^2} - \frac{(6 + 5gN)(2 + gN)^2S^2}{2m^5} - \frac{(16 + 16gN + 11g^2N^2)(2 + gN)^3S^3}{m^8} - \frac{5(6 + 5gN)(28 + 44gN + 19g^2N^2)(2 + gN)^4S^4}{8m^{11}} - \frac{(3072 + 9768gN + 11940g^2N^2 + 6654g^3N^3 + 1427g^4N^4)(2 + gN)^5S^5}{4m^{14}}. \]

Plugging these solutions into (37) one gets the effective superpotential as follows

\[
W_{\text{eff}} = -NS\log\left(\frac{S}{\Lambda^2}\right) - 1 - \frac{N(2 + gN)^2S^2}{2m^5} - \frac{N(4 + 3gN)(2 + gN)^3S^3}{3m^6} - \frac{N(140 + 212gN + 83g^2N^2)(2 + gN)^4S^4}{24m^9} - \frac{N(128 + 292gN + 228g^2N^2 + 61g^3N^3)(2 + gN)^5S^5}{4m^{12}}.
\]

As a check for this expression we note that setting \( g = 0 \) we will get the same result as that in the single trace case.

### 3.2 Broken gauge symmetry

In this case to get a closed form for the effective superpotential we will consider the case where the gauge symmetry \( U(3N) \) is broken down to \( U(2N) \times U(N) \). Essentially this is equivalent to study the case with \( U(3) \rightarrow U(2) \times U(1) \) symmetry breaking. To find the effective superpotential one can use the factorization of Seiberg-Witten curve as we presented in the appendix.

Regarding the fact that the Seiberg-Witten curve for \( U(N) \) theory is given by

\[ y^2 = (x^3 - s_1x^2 - s_2x - s_3)^2 - 4\Lambda^6 \]

basically we need to minimize the total superpotential

\[ W_T = u_3 + mu_2 + \lambda u_1 + gu_1u_2 + L \left( p^3 - s_1p^2 - s_2p - s_3 \pm 2\Lambda^3 \right) + Q(3p^2 - 2s_1p - s_2), \]

where \( p \) could be either \( b_1 \) or \( b_2 \). To be specific we set \( p = b_1 \). Classically one has

\[ s_1^{\text{class}} = 2b_1 + b_2, \quad s_2^{\text{class}} = -2b_1b_2 + b_1^2, \quad s_3^{\text{class}} = b_1^2b_2 \]

where the \( b_i \) are the solutions found previously.


while from the total superpotential, quantum mechanically we find
\[ s_i = s_i^{\text{class}} \pm 2\Lambda^3 \delta_{i,3}. \]  
(43)

Therefore the effective superpotential reads
\[ W_{\text{eff}} = u_3^{\text{class}} + mu_2^{\text{class}} + \lambda u_1^{\text{class}} + gu_1^{\text{class}} u_2^{\text{class}} \pm 2\Lambda^3, \]  
(44)

where
\[ u_1^{\text{class}} = 2b_1 + b_2, \quad u_2^{\text{class}} = b_1^2 + \frac{1}{2} b_2^2, \quad u_3^{\text{class}} = \frac{2}{3} b_1^2 + \frac{1}{3} b_2^2. \]  
(45)

4 Linearized superpotential

4.1 Field theory description

Following [15] one can recast the superpotential to the form with only single trace operators using auxiliary fields. In our case we need two fields \( A_1 \) and \( A_2 \) and the superpotential may be written as
\[ W_{\text{tree}} = \frac{1}{3} \text{Tr}(\phi^3) + \frac{1}{2} (m + gA_1) \text{Tr}(\phi^2) + (\lambda + gA_2) \text{Tr}(\phi) - gA_1A_2. \]  
(46)

Since \( A_1 \) and \( A_2 \) have no dynamics, one can integrate them out and refined the multi trace superpotential (28). These fields can be treated as constant background fields and therefore the theory can be solved using single trace superpotential. This will generate an effective superpotential \( W_{\text{eff}}^{\text{single}}(A_1, A_2, S) \) as a function of \( A_i \)'s and glueball superfield \( S \). This function is the same as that in the model without multi trace but with \( A_i \)'s dependent couplings.

For example in the case where the gauge group is not broken the effective superpotential can be read from the single trace result (12)
\[ W_{\text{exact}}^{\text{single}}(A_1, A_2) = \langle u_3 \rangle + m' \langle u_2 \rangle + \lambda' \langle u_1 \rangle, \]  
(47)

where \( m' = m + gA_1, \lambda' = \lambda + gA_2 \) and
\[ \langle u_1 \rangle = Nz, \quad \langle u_2 \rangle = \frac{N}{2} (z^2 + 2\Lambda^2), \quad \langle u_3 \rangle = \frac{N}{3} (z^3 + 6\Lambda^2 z). \]  
(48)

The same as in the previous section one can proceed to “integrate in” the glueball superfield. To do this one sets \( B := \Lambda^2 \), and uses the equation
\[ NS = B \frac{\partial W_{\text{exact}}^{\text{single}}}{\partial B} = NB(m' + 2z), \]  
(49)

to solve for \( B \) in terms of \( S \). One can also find \( z \) by solving
\[ 0 = \frac{\partial W_{\text{exact}}^{\text{single}}}{\partial z} = N(z^2 + m'z + \lambda' + 2B). \]  
(50)
Now the effective action for the glueball superfield $S$ can be written as

$$W_{\text{eff}}(A_1, A_2, S) = -S \log \left( \frac{B}{\Lambda^2} \right)^N + W_{\text{exact}}(A_1, A_2, S).$$ (51)

Using the result of single trace we get

$$W_{\text{eff}}(A_1, A_2, S) = -NS \log(\frac{S}{\Delta A^2}) - 1 - 2N \frac{S^2}{\Delta^3} \left( 3 + 16 \frac{S}{\Delta^3} + 140 \frac{S^2}{\Delta^6} + 512 \frac{S^3}{\Delta^9} \right),$$ (52)

where $\Delta^2 = (m + gA_1)^2 - 4(\lambda + gA_2)$. One should also add to this the $-gA_1A_2$ term and the final answer for the superpotential is

$$W_{\text{eff}}(A_1, A_2, S) = W_{\text{eff}}(A_1, A_2, S) - gA_1A_2.$$ (53)

To get the final result for the effective superpotential with multi trace operator we need to integrate out $A_i$’s using their equations of motion

$$\frac{\partial W_{\text{eff}}(A_1, A_2, S)}{\partial A_1} - gA_2 = 0, \quad \frac{\partial W_{\text{eff}}(A_1, A_2, S)}{\partial A_2} - gA_1 = 0.$$ (54)

These equation can be solved to find $A_i$’s in terms of glueball superfield and then plugging back the results into the (53) one can obtain the effective superpotential for the theory with tree level superpotential (28). This should reproduce the field theory result of multi trace superpotential (39). This can be seen as follows.

Suppose we have been able to solve the equations (49) and (50) exactly. Therefore we have the exact form of $z$ and $B$ as a function of $S, A_1$ and $A_2$

$$B = B(A_1, A_2, S), \quad z = z(A_1, A_2, S).$$ (55)

plugging these into the effective superpotential (53) one gets

$$W_{\text{eff}}(A_1, A_2, S) = \frac{N}{3}(z^3 + 6\Lambda^2 z) + \frac{N}{2}(m + gA_1)(z^2 + 2\Lambda^2) + (\lambda + gA_2)NZ - S \log \left( \frac{B}{\Lambda^2} \right)^N - gA_1A_2.$$ (56)

Thus the equations of motion of $A_i$’s read

$$\frac{\partial W_{\text{eff}}}{\partial A_1} + \frac{\partial B}{\partial A_1} \frac{\partial W_{\text{eff}}}{\partial B} + \frac{\partial z}{\partial A_1} \frac{\partial W_{\text{eff}}}{\partial z} - gA_2 = 0,$$
\[
\frac{\partial W_{\text{eff}}}{\partial A_2} + \frac{\partial B}{\partial A_2} \frac{\partial W_{\text{eff}}}{\partial B} + \frac{\partial z}{\partial A_2} \frac{\partial W_{\text{eff}}}{\partial z} - g A_1 = 0, \tag{57}
\]

which are

\[
0 = \frac{g N}{2} (z^2 + 2B) - g A_2 + N \frac{\partial B}{\partial A_1} \left( - \frac{S}{B} + (m + z + g A_1) \right) \nonumber \\
+ N \frac{\partial z}{\partial A_1} \left( z^2 + (m + g A_1)z + \lambda + g A_2 + B \right), \nonumber \\
0 = g N z - g A_1 + N \frac{\partial B}{\partial A_2} \left( - \frac{S}{B} + (m + z + g A_1) \right) \nonumber \\
+ N \frac{\partial z}{\partial A_2} \left( z^2 + (m + g A_1)z + \lambda + g A_2 + B \right). \tag{58}
\]

By making use of the equations (49) and (50) we find

\[
A_2 = \frac{N}{2} (z^2 + 2B), \quad A_1 = Nz. \tag{59}
\]

Now one has to plug these solution into the effective superpotential to get the final result which is, of course, what we have found in the previous section (39).

In the case where the gauge group is broken to two parts we can follow the same procedure. To be specific we consider \(U(3) \rightarrow U(2) \times U(1)\) where we will be able to write a closed form for the exact superpotential. More precisely using the field theory result in the single trace case the effective superpotential reads

\[
W_{\text{eff}}(A_1, A_2) = (\lambda + g A_2) u_1^{\text{class}} + (m + g A_1) u_2^{\text{class}} + u_3^{\text{class}} \pm 2\Lambda^3 - g A_1 A_2, \tag{60}
\]

where

\[
u_1^{\text{class}} = 2a'_1 + a'_2, \quad \nu_2^{\text{class}} = a_2'^2 + \frac{a_2'^2}{2}, \quad \nu_3^{\text{class}} = \frac{2a_1'^3}{3} + \frac{a_2'^3}{3} \pm 2\Lambda^3, \tag{61}
\]

with \(a_{1,2}' = -\frac{m'}{2} \pm \frac{1}{2} \sqrt{m'^2 - 4\lambda'}\). We should now show that upon integrating out the auxiliary fields \(A_1\) and \(A_2\) the obtained effective action is the same as that in the field theory computation with multi trace operator (44). To see this we note that

\[
\frac{\partial W_{\text{eff}}}{\partial A_1} = g(\nu_2^{\text{class}} - A_2) + \left( \lambda + g A_2 \right) \frac{\partial \nu_1^{\text{class}}}{\partial A_1} + (m + g A_1) \frac{\partial \nu_2^{\text{class}}}{\partial A_1} + \frac{\partial \nu_3^{\text{class}}}{\partial A_1} = 0, \nonumber \\
\frac{\partial W_{\text{eff}}}{\partial A_2} = g(\nu_1^{\text{class}} - A_1) + \left( \lambda + g A_2 \right) \frac{\partial \nu_1^{\text{class}}}{\partial A_2} + (m + g A_1) \frac{\partial \nu_2^{\text{class}}}{\partial A_2} + \frac{\partial \nu_3^{\text{class}}}{\partial A_2} = 0, \tag{62}
\]

which leads to the following solution for \(A_i\)’s

\[
A_1 = \nu_1^{\text{class}}, \quad A_2 = \nu_2^{\text{class}}. \tag{63}
\]

From these expressions one can find \(A_1\) and \(A_2\) and plugging them into the effective superpotential (60). Doing so we will get the same result as (44).
4.2 Matrix model description

In this section we study the matrix model of the gauge theory with multi trace operators. As it was shown in [15] taking the naively \( W \) including a multi trace operator as the potential of the corresponding matrix model would lead to an incorrect result. And in fact we should work with the linearized form of the superpotential. Therefore we consider \( U(M) \) matrix model with the following cubic potential

\[
W_{\text{tree}} = \frac{1}{3} \text{Tr}(\phi^3) + \frac{1}{2} m' \text{Tr}(\phi^2) + \lambda' \text{Tr}(\phi) - g A_1 A_2 .
\] (64)

This can be thought of as a matrix model with the single trace potential while treating \( A_i \)'s as constant background fields plus a shift of the form \(- g A_1 A_2\). For the single trace part, the potential has two critical point \( a'_1 \) and \( a'_2 \) such that

\[
W'(x) = (x - a'_1)(x - a'_2), \quad a'_{1,2} = -\frac{m'}{2} \pm \frac{1}{2} \sqrt{m'^2 - 4\lambda'} .
\] (65)

In the case where the gauge symmetry is not broken one can take the following small fluctuations

\[
\phi = a'_1 1_{M \times M} + \varphi ,
\] (66)

and therefore the potential of matrix model reads

\[
W(\varphi) = W(a'_1) + \frac{1}{3} \text{Tr}(\varphi^3) + \frac{1}{2} \Delta' \text{Tr}(\varphi^2) ,
\] (67)

where \( \Delta = a'_1 - a'_2 \). We can now write down the Feynman rules and thereby evaluate the free energy order by order. Here we shall also consider the large \( M \) limit while keeping \( g_s M = S \) fixed. Thus only planar diagram would contribute. Basically using the single trace result as that in section 2 we find

\[
\mathcal{F}_0^{\text{single}}(A_1, A_2, S) = -\frac{1}{2} S^2 \log \left( \frac{S}{\Delta'^3} \right) + S^2 \log \left( \frac{\Lambda}{\Delta'} \right) + \frac{2}{3} \frac{S^3}{\Delta'^3} \left( 1 + 4 \frac{S}{\Delta'^3} + 28 \frac{S^2}{\Delta'^6} \right)
\] (68)

up to 4-loop. Using this expression, the exact superpotential is given by

\[
W_{\text{eff}}^{\text{single}}(A_1, A_2, S) = -NS \left( \log \left( \frac{S}{\Delta' A^2} \right) - 1 \right) - \frac{2N}{3} \frac{S^2}{\Delta'^3} \left( 3 + 16 \frac{S}{\Delta'^3} + 140 \frac{S^2}{\Delta'^6} \right)
\] (69)

Finally the effective superpotential for the multi trace model can be found by integrating out \( A_1 \) and \( A_2 \) from the total superpotential given by

\[
W_{\text{eff}}(A_1, A_2, S) = W_{\text{eff}}^{\text{single}}(A_1, A_2, S) - g A_1 A_2 ,
\] (70)

This, of course, is the same expression as (53) and thus would lead to correct answer. Therefore we might conclude that the linearized superpotential would give a
correct matrix model for an $\mathcal{N} = 1$ gauge theory with a multi trace operators in the superpotential.

On the other hand for the case where the gauge group is broken, we consider the large $M \, U(M)$ matrix model and take the small fluctuations as follows

\[
\phi = \begin{pmatrix} a_{11}^1 M_1 \times M_1 & 0 \\ 0 & a_{22}^2 M_2 \times M_2 \end{pmatrix} + \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix},
\]

with $M_1 + M_2 = M$. Therefore the gauge symmetry is broken down to $U(M_1) \times U(M_2)$. We shall also consider the large $M_1$ and $M_2$ limit while keeping $S_1 = g_s M_1$ and $S_2 = g_s M_2$ fixed. Using the single trace result the matrix model action is found to be

\[
W = \frac{1}{2} \Delta' \left( \text{Tr}(\varphi_{11}^2) - \text{Tr}(\varphi_{22}^2) \right) + \frac{1}{3} \left( \text{Tr}(\varphi_{11}^3) + \text{Tr}(\varphi_{22}^3) \right)
\]

\[
+ \Delta' \left( \text{Tr}(B_{21} C_{12}) - \text{Tr}(B_{12} C_{21}) \right) + \text{Tr}(B_{21} \varphi_{11} C_{12} + C_{21} \varphi_{11} B_{12})
\]

\[
+ \text{Tr}(B_{12} \varphi_{22} C_{21} + C_{11} \varphi_{22} B_{21}).
\]

(72)

Correspondingly the matrix model free energy up to 4-loop reads

\[
\mathcal{F}_0(A_1, A_2, S_1, S_2) = -\frac{1}{2} \sum S_i^2 \log \left( \frac{S_i}{\Delta^3} \right) + (S_1 + S_2)^2 \log \left( \frac{\Lambda}{\Delta^3} \right)
\]

\[
+ \frac{1}{3 \Delta^3} \left( 2 S_1^3 - 15 S_1^2 S_2 + 15 S_1 S_2^2 - 2 S_2^3 \right)
\]

\[
+ \frac{1}{3 \Delta^4} \left( 8 S_1^4 - 91 S_1^3 S_2 + 177 S_1^2 S_2^2 - 91 S_1 S_2^3 - 8 S_2^4 \right)
\]

\[
+ \frac{1}{3 \Delta^5} \left( 56 S_1^5 - 871 S_1^4 S_2 + 2636 S_1^3 S_2^2 - 2636 S_1^2 S_2^3 - 871 S_1 S_2^4 - 56 S_2^5 \right).
\]

(73)

Having the explicit expression for the matrix model free energy with symmetry breaking as $U(M) \to U(M_1) \times U(M_2)$ one can find the effective superpotential $\mathcal{W}_{\text{eff}}(A_i, S_i)$ for the gauge theory where the gauge group is broken as $U(N) \to U(N_1) \times U(N_2)$ by making use of (5). Then the effective superpotential for the multi trace theory can be obtained by integrating out the auxiliary fields $A_i$’s from

\[
\mathcal{W}_{\text{eff}}(A_i, S_i) = \mathcal{W}_{\text{eff}}^{\text{single}}(A_i, S_i) - g A_1 A_2.
\]

(74)

To check the result one might consider the model with $N_1 = 2$ and $N_2 = 1$ where the field theory result is known. Doing the same analysis as before one can see that this does give the correct answer.

5 Conclusions

In this paper we have studied $\mathcal{N} = 1$ supersymmetric $U(N)$ gauge theory coupled to an adjoint scalar superfield with a cubic superpotential containing a multi trace
operator. Then we have looked for the corresponding matrix model in the context of the Dijkgraaf-Vafa’s proposal.

Following [15] we have considered a matrix model in which its potential is given by linearized form of the superpotential of the corresponding gauge theory using some auxiliary fields. In this way the problem can be recast to the single trace case with, of course, coefficients which now depend on the auxiliary fields. Using this matrix model one can find the free energy and thereby the effective superpotential using the Dijkgraaf-Vafa’s proposal. At the end we should integrate out the auxiliary fields finding the final result of the exact superpotential for the theory with multi trace in the tree level superpotential. As it was noticed in [15] it is crucial when the auxiliary fields are integrated out.

In this paper we have only considered the multi trace operator with the form $\text{Tr}(\phi)\text{Tr}(\phi^2)$, while we could have also considered other multi trace operators like $(\text{Tr}(\phi))^3$. In this paper we have studied two different models: one with gauge symmetry breaking and the other without gauge symmetry breaking. In both cases we have seen that linearized matrix model does give the correct field theory result.

In fact one of our motivation for doing this project was whether the Dijkgraaf-Vafa’s proposal can be also applied for the exceptional group. We note, however, that the tree level superpotential of a gauge theory with an exceptional group has usually multi trace operators. For example $\mathcal{N} = 1$ supersymmetric gauge theory with gauge group $G_2$ can be obtained from $\mathcal{N} = 2$ $G_2$ SYM theory by a tree level superpotential given by

$$W_{\text{tree}} = \frac{m}{4} \text{Tr}(\phi^2) + \frac{g}{6} \left( \text{Tr}(\phi^6) - \frac{1}{16} \text{Tr}(\phi^3)^3 \right).$$

So the first step to study these theories is to increase our knowledge about the physics of multi trace operators. We hope to address this issue in the future.

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**A Appendix**

In this appendix we show how the factorization of Seiberg-Witten curve can be worked out for the case where the gauge symmetry $U(N)$ is broken down to $U(2) \times U(1)^{N-1}$. To do this Consider $\mathcal{N} = 2$ SYM theory with $U(N)$ gauge group. The corresponding Seiberg-Witten curve is given by [24, 25]

$$y^2 = P(x, s_i)^2 - 4\Lambda^{2N}, \quad \text{with} \quad P(x, s_i) = x^N - \sum_{i=1}^{N} s_i x^{N-i},$$

(76)
where
\[ k s_k + \sum_{i=1}^{k} i s_{k-i} u_i = 0 \], \quad k = 1, 2, \ldots, N, \quad (77)\]

with \( s_0 = -1 \). Here \( u_k \)'s are the gauge invariant parameters of the theory defined in terms of \( N = 1 \) adjoint superfield \( \phi \) as following
\[ u_k = \frac{1}{k} \text{Tr}(\phi^k) \quad (78)\]

We would like to perturb the theory by adding the following a tree-level superpotential
\[ W = \sum_{n=1}^{N} \frac{g_n}{n} \text{Tr}(\phi^n) \] \quad (79)

The supersymmetric \( \mathcal{N} = 1 \) vacua of the theory are determined by F-term condition \( W' = \sum_n g_n \phi^{n-1} = 0 \). Then the roots \( a_i \) of
\[ W'(x) = \sum_{n=1}^{N} g_n x^{n-1} = g_N \prod_{i=1}^{N-1} (x - a_i) \quad (80)\]
give the eigenvalues of \( \phi \). In particular we are interested in the vacuum where the gauge group \( U(N) \) is broken down to \( U(2) \times U(1)^{N-1} \) in which in low energies we are left with an \( \mathcal{N} = 1 \) \( SU(2) \) SYM theory which is in confining phase and the photon multiplets for \( U(1)^{N-1} \) are decoupled. Therefore we take
\[ \phi = \text{diag}(a_1, a_1, a_2, a_3, \ldots, a_{N-1}) \quad (81)\]

This \( \mathcal{N} = 1 \) vacuum where one monopole becomes massless is parameterized by the set of moduli \( \tilde{s}_i \) where the Seiberg-Witten curve factorizes in such a way that (76) has one double root and \( 2(N-1) \) single roots:
\[ P(x, \tilde{s}_i)^2 - 4\Lambda^{2N} = H_1^2(x) F_{2(N-1)}(x) \] \quad (82)

where \( H_1 \) is given by \( H_1(x) = x - x_0 \) with some \( x_0 \) to be determined. Therefore the factorization (82) is equivalent to
\[ P(x_0, \tilde{s}_i) \pm 2\Lambda^N = 0 \], \quad \frac{\partial P(x, \tilde{s}_i)}{\partial x}|_{x=x_0} = 0 \quad (83)\]

Now the taste is to minimize the superpotential (79) subject to the above constraints. Thus the total superpotential can be written as
\[ W_T = \sum_{n=1}^{N} g_n u_n + L \left( P(x_0, \tilde{s}_i) \pm 2\Lambda^N \right) + Q \left( P'(x_0, \tilde{s}_i) \right) \] \quad (84)
Here $L, Q$ and $x_0$ should be treated as Lagrange multipliers. From the total superpotential the equations of motion for $L, Q$ and $x_0$ read

\[
\frac{\partial}{\partial L} : P(x_0, \tilde{s}_i) \pm 2\Lambda^N = 0 ,
\]

\[
\frac{\partial}{\partial Q} : P'(x_0, \tilde{s}_i) = 0 ,
\]

\[
\frac{\partial}{\partial x_0} : Q P''(x_0, \tilde{s}_i) = 0 .
\]  

(85)

Moreover the equation of motion for $u_k$ leads to

\[
g_k = L \frac{\partial}{\partial u_k} P(x_0, \tilde{s}_i) = -L \sum_{i=2}^{N} x_0^{N-i} \frac{\partial \tilde{s}_i}{\partial u_k} = -L \sum_{i=2}^{N} x_0^{N-i} \tilde{s}_{i-k} .
\]

(86)

Here in the last equality we have used the Newton’s relation to get $\frac{\partial \tilde{s}_i}{\partial u_k} = \tilde{s}_{i-k}$. The equations (85) and (86) can be solved for the parameters. Doing so one finds

\[
Q = 0, \quad L = g_N, \quad x_0 = a_1, \quad \tilde{s}_i = s_i \pm 2\Lambda^N \delta_{iN} .
\]  

(87)

Plugging these solutions into the total superpotential one gets

\[
W_{eff} = \sum_{k=1}^{N} g_k u_k \pm 2g_N \Lambda^N .
\]  

(88)

This is the same expression obtained in [26]. It can also be show that

\[
F_{2(N-1)} = \prod_{i=1}^{N-1} (x - a_i)^2 \mp 4\Lambda^N \prod_{i=2}^{N-1} (x - a_i)
\]

\[
= \frac{1}{g_N^2} \left( W'(x)^2 + f_{N-2} \right)
\]

(89)

with $f_{N-2} = \mp 4g_N^2 \Lambda^N \prod_{i=2}^{N-1} (x - a_i)$ being the quantum correction. This means that the quantum dynamics of the $\mathcal{N} = 1$ $U(1)^{N-2}$ at low energies is captured by the following curve

\[
y^2 = W'(x)^2 + f_{N-2} .
\]  

(90)

Having the reduced curve explicitly one can proceed to evaluate the periods of the curve. The periods are given in terms of integral of a one form over different one cycles of the curve

\[
S_i = \oint_{\alpha_i} y dx = \oint_{\alpha_i} dx \ g_N \sqrt{\prod_{i=1}^{N-1} (x - a_i)^2 \mp 4\Lambda^N \prod_{i=2}^{N-1} (x - a_i)} ,
\]  

(91)
where $\alpha_i$ is a one cycle loops around the $i$th cut of the reduced curve. In particular one has

$$S = \sum_i S_i = \oint_C dx \ g_N \sqrt{\prod_{i=1}^{N-1} \frac{(x-a_i)^2}{(x-a_i)} + 4\Lambda^N \prod_{i=2}^{N-1} (x-a_i)} ,$$  \hfill (92)

where $C$ is a loop at infinity. Therefore we get

$$S = \oint_C dx \ g_N \prod_{i=1}^{N-1} (x-a_i) \left( 1 \mp \frac{2\Lambda^N}{(x-a_1) \prod_{i=1}^{N-1} (x-a_i)} + \cdots \right)$$

$$= \pm 2g_N \Lambda^N ,$$  \hfill (93)

which means

$$\frac{\partial W_{\text{eff}}}{\partial (\log \Lambda^2)} = -\frac{b_{N-2}}{4g_N} ,$$  \hfill (94)

where $b_{N-2} = \mp 4g_N^2 \Lambda^N$ is the numerical coefficient of $x^{N-2}$ term in the $f_{N-2}$. In other words we get

$$\Lambda^2 \frac{\partial W_{\text{eff}}}{\partial \Lambda^2} = NS ,$$  \hfill (95)

which can be interpreted as the Konishi anomaly [27, 28, 9].

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