How to gamble with non-stationary $\mathcal{X}$-armed bandits and have no regrets

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August 23, 2019

Abstract

In $\mathcal{X}$-armed bandit problem an agent sequentially interacts with environment which yields a reward based on the vector input the agent provides. The agent’s goal is to maximise the sum of these rewards across some number of time steps. The problem and its variations have been a subject of numerous studies, suggesting sub-linear and some times optimal strategies. The given paper introduces a novel variation of the problem. We consider an environment, which can abruptly change its behaviour an unknown number of times. To that end we propose a novel strategy and prove it attains sub-linear cumulative regret. Moreover, in case of highly smooth relation between an action and the corresponding reward, the method is nearly optimal. The theoretical result are supported by experimental study.

1 Introduction

Numerous studies consider variations of an $\mathcal{X}$-armed bandit problem, a problem where an agent at each time step $t$ sequentially interacts with environment, supplying a vector $X_t$ and receiving a reward $y_t$, depending (presumably) on $X_t$. The reward is immediately made known to the agent, so the subsequent actions can be based on it. Typically, the reward is obfuscated by some noise.

This model was found to be useful is such applications as clinical trials, pricing, finance, logistics, advertisement and recommendation [4, 18, 3, 2, 1, 11, 17] and has (along with stochastic optimization in general) unsurprisingly attracted immense attention in the recent decades. Initially, though, the research was focused on multi-armed bandits, where only the actions $X_t$ from a finite set are feasible [9]. Consideration of $\mathcal{X}$-armed bandits (with continuous feasible set $\mathcal{X}$) was not long in coming [13, 5].

Currently due to ever-changing nature of our world, the studies on the multi-armed bandits are increasingly more concerned with the environments, changing their behavior in the course of time [10, 6, 14]. For instance, an active line of
work are restless bandits, referring to a class of problems where the environment switches between the internal states according to a known stochastic law \[15\]. Also, there are settings where no such knowledge is available, like \[4\], presuming a bound on the total variation of the mean reward associated with each arm, or \[7\] implying no such bound, yet presuming these changes to be relatively rare. The latter is indeed the setting we extend to the case of \(X\)-armed bandits. Particularly, we let the underlying relationship between the action \(X_t\) and the reward \(y_t\) abruptly change. The agent has no knowledge of when to expect such a change, neither any information on the nature of the change is revealed. As usual, in a bandit setting the goal is to minimize the cumulative regret – a discrepancy between the received reward and the largest possible one.

Formally, consider a number of stationary periods \(K \in \mathbb{N}\) and a sequence of functions \(F := \{f_1, f_2, ..., f_K\}\) mapping from the normed vector space \(X\) to \(\mathbb{R}\). Further, we introduce a sequence of time points (being called change-points) \(0 = \tau_0 \leq \tau_1 < \tau_2 < ... < \tau_K = K\), when the environment switches between the functions \(f_i\). Also consider a piece-wise-constant map \(k : \mathbb{N} \to \{1, 2, ..., K\}\), such that \(k(1) = 1, k(T) = K, k(i+1) = k(i) + 1\) if \(\exists k : i = \tau_k\) and \(k(i+1) = k(i)\) otherwise.

At time points \(t = \{1, 2, ..., T\}\) the agent consequently interacts with the environment, providing a vector \(X_t \in X\) and receives a reward

\[y_t = f_{k(t)}(X_t) + \varepsilon_t,\]

where \(\varepsilon_t\) denotes i.i.d. centered noise.

Neither the functions \(f_i\) nor the locations \(\tau_i\) nor the number \(K\) is known to the agent. The goal is to minimize the cumulative regret \(\mathbb{E}[R_T]\), where

\[R_T := \sum_{t=1}^{T} \max_{x \in X} f_{k(t)}(x) - f_{k(t)}(X_t).\]

The algorithm is said to be no-regret, if \(\lim_{T \to \infty} \mathbb{E}[R_T] / T = 0\), which is exactly what we demonstrate. Moreover, the bound we establish approaches the lower bound for highly smooth functions, so the suggested approach is nearly optimal. Our contribution is outlined as follows

- We propose a novel approach for a non-stationary \(X\)-armed bandit problem.
- We establish a sub-linear bound on cumulative regret under mild assumptions.
- The bound meets the minimax lower bound in case of highly smooth functions \(f_i\) up to slowly growing factors, implying near-optimality of our approach.
- The theoretical findings are verified empirically. A comparative study has also been conducted.
• Our approach is adaptive to the number of change-points (unlike in [7], where $K$ is deemed known).

• We propose an algorithm, generally suited for detection of a change-point in regression, not only in a bandit setting.

The paper is organized as follows. Section 2 introduces the suggested approach along with the necessary background and followed by a rigorous theoretical study given in Section 3. The theoretical results are put to a test in Section 4 describing the empirical study. We conclude the paper with Section 5 outlining the directions of the future research.

2 The proposed strategy

This section presents a novel algorithm in sub-section 2.3. We also develop a novel change-point detection algorithm as its necessary building block and summarize it in sub-section 2.2. Both of these algorithms rely on Gaussian Process Regression, therefore we open the section with a brief description of this well-known approach.

2.1 Background: Gaussian Process Regression

In the given study we rely on a well known black-box non-parametric approach known as Gaussian Process Regression [12]. Formally, we model the noise with a normal distribution and impose the zero-mean Gaussian Process prior with covariance function $k(\cdot, \cdot)$ on the regression function

$$f \sim \mathcal{GP}(0, \rho k(\cdot, \cdot)),$$

$$y_j \sim \mathcal{N}(f(X_j), \sigma^2) \text{ for } j \in 1..n,$$

where $n$ is the number of covariate-response pairs under consideration and $\rho$ is a regularization parameter.

For a given covariate $X^*$ the predictive distribution is also Gaussian with mean

$$\mu_* = k^* \mathcal{K}^{-1} y$$

and variance

$$\sigma_*^2 = k(X^*, X^*) - \langle k^* \mathcal{K}^{-1}, k^* \rangle,$$

where $y = [y_i]_{i=1..n}$, $\mathcal{K} = [\rho k(X_i, X_j) + \sigma^2 \delta_{ij}]_{i,j=1..n}$ and $k^* = [k(X^*, X_i)]_{i=1..n}$.

2.2 Change-point detection procedure

Our approach requires a change-point detection procedure as its crucial building block. To that end we suggest a novel approach, namely, Algorithm 1. Given a sequence of covariate and response pairs $\{(X_t, y_t)\}_{t=1}^{2n}$, we train a Gaussian Process Regression twice – using only the first and only the second half of the
given data respectively. This way we obtain two predictive functions $\mu_1$ and $\mu_2$. Next we make predictions for all the provided covariates and calculate the discrepancy between these predictions

$$\hat{\Delta}^2 := \frac{1}{n} \sum_{i=1}^{2n} (\mu_1(X_t) - \mu_2(X_t))^2.$$ 

Finally, we compare the discrepancy against some predetermined threshold $\theta_n$. Intuitively, if the covariate-response pairs were generated with the same functional relationship, $\hat{\Delta}^2$ should be small, while violation of this assumption should lead to larger values.

**Algorithm 1: CPD**

**Data:** Covariate-response pairs $\{(X_t, y_t)\}_{i=1}^{2n}$, threshold $\theta_n$, regularization parameter $\rho_{CP}$

**Result:** True if a break is detected, False otherwise

1. $\mu_1(\cdot) \leftarrow$ train GPR on $\{(X_t, y_t)\}_{i=1}^{n}$ w. $\rho_{CP}$
2. $\mu_2(\cdot) \leftarrow$ train GPR on $\{(X_t, y_t)\}_{i=n+1}^{2n}$ w. $\rho_{CP}$
3. $\hat{\Delta}^2 \leftarrow \frac{1}{n} \sum_{i=1}^{2n} (\mu_1(X_t) - \mu_2(X_t))^2$
4. return $\hat{\Delta}^2 > \theta_n$

2.3 GP-UCB-CPD algorithm

A well known approach called CP-UCB was proven [13] to attain sub-linear regret in a non-stochastic setting. The main idea behind the algorithm is to train at each time point $t$ Gaussian Process Regression, using the history of rewards. Denote the obtained predictive mean $\mu_t(\cdot)$ and predictive variance $\sigma_t^2(\cdot)$. The next input vector is chosen using the optimistic rule

$$X_t = \arg \max_{X \in \mathcal{X}} \mu_t(X) + \sqrt{\beta_t \sigma_t(X)},$$

obtaining an exploration-exploitation trade-off.

In a non-stationary setting we cannot hope for good performance of GP-UCB anymore, as non-stationarity of the underlying distribution violates assumptions of GPR consistency results. To that end we suggest to use Algorithm 1 in order to detect a change and suggest to abandon the history acquired before it. Unfortunately, we cannot use the history acquired by (2.3), as the chosen vectors might concentrate in a vicinity of $\arg \max_x f_i(x)$, which will be the case after some number of break-free iterations. Therefore, we dedicate some portion of steps to uniform exploration. It is chosen adaptively with respect to the number of change-points $K$ (see line 11). The idea can be back-ported into an earlier method suggested by [7] for multi-armed bandits, which requires such knowledge. Namely, at each iteration we check whether we have accumulated enough $(2n)$ uniformly sampled points (uniformlySampled). If so, we apply Algorithm 1 to them (line 6). If a change-point is detected, we abandon all
the data we have accumulated so far (lines 7-8). Further, in the line 11 it is
 decided whether this step should be dedicated to the uniform exploration. If so,
an input $X_t$ is chosen uniformly from $\mathcal{X}$. Otherwise, the rule (2.3) is used. In
both cases the reward is received and stored along with the input in $\text{history}$
and if it was an exploration step, also in $\text{uniformlySampled}$.

Algorithm 2: GP-UCB-CPD

```
Data: Compact $\mathcal{X}$, threshold $\theta_n$, budget $T$, sequence of positive real
numbers $\{\beta_t\}$, real parameter $\xi > 0$, regularization parameters
$\rho_{\text{UCB}}$ and $\rho_{\text{CP}}$
1 history ← [ ] // empty list
2 uniformlySampled ← [ ] // empty list
3 for $t \in 1, 2, ..., T$ do
4     if $|\text{uniformlySampled}| \geq 2n$ then
5         tail ← uniformlySampled$[-2n:]$ // take the last 2n elements
6         if CDP(tail, $\theta_n$, $\rho_{\text{CP}}$) then
7             uniformlySampled ← [ ]
8             history ← [ ]
9         end
10    end
11    if $|\text{uniformlySampled}| \leq \xi \sqrt{|\text{history}|}$ then
12        Sample $X_t \sim U[\mathcal{X}]$
13        Sample $y_t \leftarrow f(X_t) + \varepsilon_t$
14        Append $(X_t, y_t)$ to uniformlySampled
15    else
16        $t' = |\text{history}|
17        (\mu_t(\cdot), \sigma^2_t(\cdot)) \leftarrow \text{train GPR on } \text{history } w. \rho_{\text{UCB}}$
18        $X_t \leftarrow \arg\max_{X \in \mathcal{X}} \mu_t(X) + \sqrt{\beta_t} \sigma_t(X)$
19        Sample $y_t \leftarrow f(X_t) + \varepsilon_t$
20    end
21    Append $(X_t, y_t)$ to history
22 end
```

3 Theoretical analysis of GP-UCB-CPD

First of all, we assume, the noise $\varepsilon_t$ has light tails. Formally, we presume them
to be sub-Gaussian.

**Definition 3.1 (Sub-Gaussianity).** We say, a centered random variable $x$ is
sub-Gaussian with $g^2$ if

$$\mathbb{E}[\exp(sz)] \leq \exp(g^2 s^2/2), \ \forall s \in \mathbb{R}.$$ 

We say, a centered random vector $X$ is sub-Gaussian with $g^2$ if for all unit
vectors $u$ the product $\langle u, X \rangle$ is sub-Gaussian with $g^2$.

Further, we consider a broad class of smooth functions
Definition 3.2 (Sobolev class). Consider an orthonormal basis \( \{ \psi_j \} \) in \( L_2(\mathbb{R}^p) \) and a function \( f = \sum_j f_j \psi_j \in L_2(\mathbb{R}^p) \). We call it \( \alpha \)-smooth Sobolev for \( \alpha > 1/2 \) if

\[
\exists B : \sum_{j=1}^{\infty} j^{2\alpha} f_j^2 \leq B^2
\]

For the sake of simplicity we restrict ourselves to \( \mathcal{X} \subset \mathbb{R} \) and Matrn covariance function

\[
k(x, x') := 2^{1-\alpha} r^\alpha B_\alpha(r)/\Gamma(\alpha),
\]

where \( r = \sqrt{2\alpha} \|x - x'\|/l \), \( \alpha > 1/2 \) controls smoothness, \( l \) is the lengthscale and \( B_\alpha \) denotes modified Bessel function of the second kind. Similar results can be established for the multivariate case (yet, not for a high-dimensional one, which is left for the future research) as well as for other classes of kernels (e.g. squared exponential).

In the theoretical part of the paper we use \( \|\cdot\| \) to denote Euclidean norm, \( \|\cdot\|_\infty \) denotes sup-norm, while \( \|\cdot\|_k \) stands for the norm of the reproducing kernel Hilbert space.

Theorem 3.1. Let \( \varepsilon_i \) be sub-Gaussian with \( g^2 \), functions \( f_i \) be Sobolev with \( \alpha \) and \( B \), and uniformly bounded: \( \exists F : \sup_i \|f_i\|_\infty \leq F \), choose some positive \( \xi \),

\[
\sigma^2 = g^2,
\]

\[
n = (c\Delta)^{-\max\{\frac{2\alpha}{\pi - 1/4}, 1\}} T^{1/(2\alpha+4)},
\]

\[
\rho_{\text{UCB}} = ((12\alpha + 26) \log T)^{-1}, \quad \rho_{\text{CP}} = B^2 / \log n,
\]

\[
\beta_t = D t^{1/(\alpha+1)} \log^3 t,
\]

\[
\theta = 2C \left( \frac{\log n}{n} \right)^{(\alpha-1/2)/\alpha}
\]

for some \( D, C \) and \( c \) depending only on \( F, B, \text{Var} \{\varepsilon_1\} \) and \( \alpha \). Finally, assume, there is enough space between the change points

\[
n \left( \frac{2}{\xi^2} + \frac{1}{\xi^2} \right) \leq \sqrt{|\tau_i - \tau_{i+1}|},
\]

Then

\[
E[R_T] = O \left( \sqrt{KT^{\frac{\alpha+3}{2\alpha}}} \log^3 T \right).
\]

We defer the proof to Appendix A.

For highly smooth Sobolev functions (\( \alpha \gg 1 \)) the bound Theorem 3.1 boils down to \( \sqrt{KT^\kappa} \) with \( \kappa \approx 1 \) up to the logarithmic factor. The obtained bound matches the lower bound for \( \mathcal{X} \)-armed bandits with \( K \) stationary periods. Really, the lower bound for a non-stochastic \( \mathcal{X} \)-armed bandit is \( \sqrt{T} \). Let the
Figure 1: Here we present the dependence of cumulative regret $R_T$ on the horizon $T$ under fixed number of stationary periods $K = 4$. The measurements are marked with round dots. The dashed line depicts the fitted curve $3.525T^{0.619}$. Both axis are in log scale.

periods between the changes in the non-stationary setting be $T_i$, then the lower bound is $\sum_{i=1}^k \sqrt{T_i}$. But clearly,

$$\max_{\{T_i\} \text{ s.t. } \sum_i T_i = T} \sum_{i=1}^K \sqrt{T_i} = \sqrt{KT}$$

and hence the lower bound in case of $K$ stationary points is $\sqrt{KT}$.

Further, in [7] the length of stationary periods is presumed to be at least $\sim \sqrt{T}$. The assumption we make is notably weaker replacing the root with $T^{1/(\alpha+2)}$. In fact, the necessary minimal interval between the change points barely depends on $T$ for highly smooth functions.

4 Experimental study

In this section we experimentally support the theoretical results and present a comparative study. We consider Matrn kernel with smoothness index $\alpha = 5/2$ and lengthscale $l = 1$, denoting it $k_0(\cdot, \cdot)$. The functions $f_i$ are drawn independently from $\mathcal{GP}(0, k_0(\cdot, \cdot))$. The noise $\varepsilon_i$ is independent, centered and Gaussian, its standard deviation is 0.05. The chosen domain $\mathcal{X} = [0, 5]$ is discretized into 1000 evenly spaced points. The change-points are chosen to be evenly spaced, as this is obviously the most hostile setting. For all the experiments we choose the parameter controlling the portion of the steps, dedicated to the uniform sampling $\xi = \sqrt{3}$, the covariance function the Gaussian Process Regression uses is $k_0(\cdot, \cdot)$, the standard deviation of the noise $\sigma = 0.05$. $\rho_{\text{UCB}}$ and $\rho_{\text{CP}}$ are chosen to be equal to 1 as their variation prescribed by the theorem induces only marginal change to the overall performance. $D = 0.02$, threshold of change-point
Figure 2: Keeping the horizon $T$ fixed we let the number of stationary intervals run from 1 to 9 and depict the cumulative regret with round dots. The fitted curve $305.7 T^{0.5346}$ is shown with a dashed line. Both axis are in log scale.

Figure 3: The plot demonstrates averaged cumulative regret of several algorithms interacting with an environment changing its behaviour every 300 points. The compared algorithms are Algorithm 2 (denoted GP-UCB-CPD), Algorithm 2 using an oracle change-point detector (GP-UCB-Oracle), Algorithm 2 using a change-point detector, which never detects a change point (Algorithm 2 with $\theta = +\infty$, denoted GP-UCB-NO-Detector) and finally, GP-UCB, suggested in [13] (Algorithm 2 with $\xi = 0$).
detection algorithm $\theta = 0.2$, while its window size $n = 20$. The experiments are repeated 100 times and the results are averaged.

In the first experiment we examine dependence of cumulative regret $R_T$ on horizon $T$ under fixed number of stationary periods $K$. Namely, we run Algorithm 2 for $K = 4$ and $T \in \{1200, 1700, 2200, 2700, 3200\}$. The results are shown in the Figure 1. We also fit a parametric curve $CT^c$ and the optimal power coefficient is $c = 0.619$ with 95% confidence interval being $(0.56, 0.69)$. Moreover, Theorem 3.1 suggests $c = 0.6111\ldots$ based on the $\alpha = 5/2$. Overall, we conclude the dependence to clearly be sub-linear and to closely follow the discovered theoretical properties.

The next step is to keep the horizon fixed $T = 2700$ and access $K = 2, \ldots, 9$. The results are reported in Figure 2 along with the fitted curve $305.7T^{0.5346}$. The 95% confidence interval for the power coefficient is $(0.481, 0.588)$. Again, the dependence is evidently sub-linear and agrees with the theoretical result, dictating the power coefficient of 0.5.

We conclude the section with a comparative study. Here we choose $T = 1200$ and $K = 4$. The findings are presented in Figure 3. Algorithm 2 is denoted as GP-UCB-CPD. For the sake of comparison we also consider a version of Algorithm 2, equipped with an oracle change-point detector $d$ and is referred to as GP-UCB-Oracle. The other two algorithms we compare against the algorithm equipped with a change-point detector, never detecting a change-point and the algorithm, abandoning the change-point detection (and the uniform sampling) altogether (via the choice $\xi = 0$). We call these approaches GP-UCB-NO-Detector and GP-UCB.

As we can see on the Figure 3, before the first change-point GP-UCB performs best, which is not surprising as other algorithms have accumulate regret during uniform sampling. The fact that GP-UCB-Oracle and GP-UCB-CPD accumulate approximately equal regret during this period implies low probability of false positive decision by the change-point detection algorithm. After the first break GP-UCB-CPD performs notably worse, than GP-UCB-CPD for some period of time, which is due to an unavoidable delay of change point detection. This behaviour is repeated when the subsequent change points happen. In spite of the stellar score before the first change-point, performance of GP-UCB greatly deteriorates after the first break. Moreover, it turns out to perform only marginally better than GP-UCB-NO-Detector.

Overall, unsurprisingly the lowest average regret is achieved by GP-UCB-Oracle, the method aware of the location of the change-point. The second best is GP-UCB-CPD.

5 Future work

In the study we have considered a realistic setting of an $A'$-armed bandit problem and suggested a novel strategy achieving sub-linear cumulative regret and being nearly optimal for highly smooth functions. This conclusion follows from both theoretical and empirical studies. Yet, many questions remain unanswered. The
lines of our future work can be foreseen as follows

- As long as our approach relies on Gaussian Process Regression, whose performance deteriorates in high dimension, the suggested methodology is only effective in a low-dimensional setting. High-dimensional $\mathcal{X}$-armed bandits have already attracted researchers’ interest in the past [8], but non-stationary setting is yet to be analysed.

- Switching from GPR-based to tree-based approaches (see [5]) can yield an approach attaining nearly-optimal performance for wider classes of functions $f_t$.

- A different setting will also be considered (akin to the one suggested in [4]), where we allow the environment to change its behaviour at every step $t$, yet impose a bound for the total variation.

- Gaussian Process Regression is notorious for its cubic time complexity, which renders it ineffective on large samples of data which are common nowadays. Thankfully, numerous linear-time approximate approaches have been developed. [12] and can be used instead to alleviate the issue. Moreover, as we have to deal with ever-growing datasets, suggestion of a distributed approach is another valuable step.

Acknowledgements

The research of “Project Approximative Bayesian inference and model selection for stochastic differential equations (SDEs)” has been partially funded by Deutsche Forschungsgemeinschaft (DFG) through grant CRC 1294 “Data Assimilation”, “Project Approximative Bayesian inference and model selection for stochastic differential equations (SDEs)”.

Further, we would like to thank Vladimir Spokoiny and Alexandra Carpentier for the discussions which have greatly improved the manuscript.

A Proof of Theorem 3.1

Proof of Theorem 3.1. Denote $\omega = 6(\alpha+2)$. Choose $\beta_t$ as prescribed by Lemma A.1. Now it applies here, since (Lemma C.3) is implied by (Theorem 3.1) and the other assumptions are explicitly required to hold. Hence,

$$R_T \leq \sqrt{\frac{8KT\beta_T\eta_T}{\log(1+\sigma^{-2})}} + F \left(\left(\frac{2}{\xi} + \frac{1}{\xi^2}\right)T^{3/\omega} + 1\right) \sqrt{KT}.$$

on a set of probability at least $1 - 4/T^2$ (here we also used $\omega > 3$). Using the fact that $\mathbb{E}[R_T] \leq FT$ we have,

$$\mathbb{E}[R_T] \leq \sqrt{\frac{8KT\beta_T\eta_T}{\log(1+\sigma^{-2})}} + F \left(\left(\frac{2}{\xi} + \frac{1}{\xi^2}\right)T^{3/\omega} + 1\right) \sqrt{KT} + 4F/T.$$
Substitution of (Theorem 3.1) and the bound from Lemma B.8 completes the proof.

**Lemma A.1.** Let \( \varepsilon_t \) be sub-Gaussian with \( g^2 \), functions \( f_i \) be \( \alpha \)-Sobolev, denote
\[
F := \max_{i=1..K} \{ \max(\|f_i\|_k, \|f_i\|_\infty) \}
\]
and choose some positive \( \xi \),
\[
\sigma^2 = g^2,
\]
\[
\rho_{UCB} = (2(\omega + 1) \log T)^{-1} \text{ for any } \omega > 1,
\]
\[
\beta_t = 80000(\eta_t \log t + F^2) \log t,
\]
\[
\theta = 2C \left( \frac{\log n}{n} \right)^{(\alpha-1/2)/\alpha}
\]
and \( n \) such that
\[
(c\Delta)^{2 \max\{\alpha/(\alpha-1/2), 2\}} \geq \frac{\log n}{n},
\]
(A.1)
for some \( C \) and \( c \) depending only on \( F, B, \Var[\varepsilon_1] \) and \( \omega \). Then on a set of probability at least \( 1 - 3T/n^2 - T^{-(\omega-1)} \)
\[
R_T \leq \sqrt{8KT\beta_T \eta_T / \log(1 + \sigma^{-2}) + F \left( \frac{2}{\xi} + \frac{1}{\xi^2} \right) + 1} \sqrt{KT}.
\]

**Proof.** We apply Lemma C.3 to each instance of usage of Algorithm 1. The statement of the lemma holds for all all of them on a set of probability at least \( 1 - 3T/n^2 - T^{-(\omega-1)} \). The rest of the argument is conditioned on this set. Denote the regret accumulated between \( \tau_i \) and \( \tau_{i+1} \) at the line 19 (we exclude uniform sampling from consideration for now just like the iterations when the change has happened, but was not detected yet) as \( R_i \). Denote \( T_i = \tau_{i+1} - \tau_i \). Now we apply Lemma B.7 for each interval between the changes. Its statement holds on a set of probability at least \( 1 - K/T^\omega \), but as long as \( K \leq T \), the probability is at least \( 1 - T^{-(\omega-1)} \).

\[
\sum_{i=1}^K R_i \leq \sum_{i=1}^K \sqrt{8T_i \beta_T \eta_T / \log(1 + \sigma^{-2})}
\]
\[
\leq \sqrt{8\beta_T \eta_T / \log(1 + \sigma^{-2}) \sum_{i=1}^K T_i},
\]
where we have used monotonicity of the bound yielded by Lemma B.7, \( \beta_t \) and \( \eta_t \). Now we use the fact that \( \sum_{i=1}^K \sqrt{T_i} \leq \sqrt{K \sum_{i=1}^K T_i} \), which yields
\[
\sum_{i=1}^K R_i \leq \sqrt{8KT \beta_T \eta_T / \log(1 + \sigma^{-2})},
\]
(A.2)
The bound (A) does not handle the regret accumulated during the uniform sampling (at most \( \sum_{i} \sqrt{T_i} \) iterations) and the periods when the change has happened, yet was remaining undetected. In order to estimate the delay of detection of \( i+1 \)-st change point, consider an equation.

\[
\sqrt{T_i} + \gamma - \sqrt{T_i} = 1/\xi,
\]

characterizing the maximal number of iterations \( \gamma \) between two consecutive uniform sampling steps. Clearly,

\[
\gamma = \left(2\sqrt{T_i} + 1/\xi\right)/\xi \leq \left(2/\xi + 1/\xi^2\right)\sqrt{T_i}.
\]

Hence, the total regret accumulated during the delay of detection is at most \( \sum_{i} n \left(2/\xi + 1/\xi^2\right)\sqrt{T_i} \leq n \left(2/\xi + 1/\xi^2\right)\sqrt{KT} \). Incorporation of this observation with (A) constitutes the claim.

\[\square\]

B Analysis of UCB rule

This section adapts the regret bound for UCB obtained in [13]. In this section we assume the environment is stationary, i.e. for \( t = 1, 2, ..., T \)

\[
y_t = f(X_t) + \epsilon_t.
\]

Denote the set of time-steps when condition in the line 11 of Algorithm 2 computes to True as \( \mathcal{T} \), its complement as \( \bar{\mathcal{T}} \) and \( T := \{1, 2, ..., T\} \). Also let \( \eta_n \) denote the highest information gain possible from \( n \) observations. Further, for a sequence of real values \( \{a_i\} \) and a set of indexes \( B \) we write \( a_B := \{a_i\}_{i \in B} \). If not said otherwise, in this section we choose \( \rho_{UCB} = 1 \).

Here we employ the concept of information gain \([?]\), defined as mutual information between the function \( f \sim GP(0,k(\cdot,\cdot)) \) and the observations \( y_{\mathcal{T}} \)

\[
i(y_{\mathcal{T}}; f) = H(y_{\mathcal{T}}) - H(y_{\mathcal{T}} | f),
\]

where \( H(\cdot) \) denotes entropy and in our case

\[
i(y_{\mathcal{T}}; f) = \frac{1}{2} \log \det \left( I + \sigma^{-2}K_{\mathcal{T}} \right).
\]

In order to extend the results by [13] for the case allowing for uniform sampling (see line 12) we prove the following trivial lemma.

**Lemma B.1.**

\[
i(y_{\mathcal{T}}; f) \leq i(y_{\bar{\mathcal{T}}}; f)
\]

**Proof.** The claim follows from the fact that the eigenvalues of \( i_{\mathcal{T}} + \sigma^{-2}K_{\mathcal{T}} \) and \( i_{\bar{\mathcal{T}}} + \sigma^{-2}K_{\mathcal{T}} \) are larger or equal to 1, while \( |\mathcal{T}| \leq |\bar{\mathcal{T}}| \). \[\square\]
The next result connects information gain and predictive variance of GPR.

**Lemma B.2** (extension of Lemma 5.3 by [13]).

\[ I(y_{\bar{T}}; f) = \frac{1}{2} \sum_{t \in \bar{T}} (1 + \sigma^{-2} \sigma_t^2(X_t)) \]

**Lemma B.3** (extension Lemma 7.1 by [13]). For positive \( \zeta \)

\[ \frac{1}{2} \sum_{t \in \bar{T}} \max\{\sigma^{-2} \sigma_t^2(X_t), \zeta\} \leq 2 \frac{2 \zeta}{\log(1 + \zeta)} \eta |T| \]

*Proof.* The proof consists in combination of Lemma B.2, Lemma B.1 and the fact that \( \min\{r, \zeta\} \leq \zeta \log(1 + r) / \log(1 + \zeta) \) for positive \( r \). \( \Box \)

Next we extend GPR consistency result for to the case of sub-Gaussian noise.

**Lemma B.4** (Theorem 6 in [13]). Let \( \delta \in (0, 1) \) and \( \sup_{t \in T} |\varepsilon_t| \leq \sigma \) and choose

\[ \beta_t = 2 \|f\|^2_k + 300 \eta_t \ln^3(t/\delta) \]

Then on a set of probability at least \( 1 - \delta \) for all \( t \)

\[ |\mu_t(x) - f(x)| \leq \beta_t^{1/2} \sigma_t(x) \]

**Lemma B.5.** Let \( \varepsilon_t \) be sub-Gaussian with \( g^2 \), \( \delta \in (0, 1) \) and \( u > 0 \). Choose

\[ \sigma = g \sqrt{2(u + \log T)} \]

and

\[ \beta_t = 2 \|f\|^2_k + 300 \eta_t \log^3(t/\delta) \]

Then on a set of probability at least \( 1 - \delta - \exp(-u) \) for all \( t \)

\[ |\mu_t(x) - f(x)| \leq \beta_t^{1/2} \sigma_t(x) \]

*Proof.* Due to sub-Gaussianity for all \( t \) for any positive \( x \)

\[ \mathbb{P}\{|\varepsilon_t| > x\} \leq 2 \exp\left(-\frac{x^2}{2g^2}\right) \]

and uniformly

\[ \mathbb{P}\left\{\sup_{t \leq T} |\varepsilon_t| > x\right\} \leq 2T \exp\left(-\frac{x^2}{2g^2}\right) \]

Change of variables yields for positive \( u \)

\[ \mathbb{P}\left\{\sup_{t \leq T} |\varepsilon_t| > g \sqrt{2(u + \log T)}\right\} \leq \exp(-u) \]

Finally, choose \( \sigma = g \sqrt{2(u + \log T)} \) and apply Lemma B.4. \( \Box \)
Substitution of Lemma B.5 in place of Theorem 6 extends Theorem 3 in the desired way bounding the regret of GP-UCB-CP in the absence of change-points.

**Lemma B.6.** Let \( \varepsilon_t \) be sub-Gaussian with \( g^2 \), choose
\[
\sigma^2 = 2g^2(u + \log T)
\]
and
\[
\beta_t = 2\|f\|^2_k + 300\eta_t \log^3(t/\delta).
\]
Then on a set of probability at least \( 1 - \delta - \exp(-u) \)
\[
R_T \leq \sqrt{8T\beta_T\eta_T/\log(1 + \sigma^{-2})}.
\]

As one can see, Lemma B.6 suggests a choice of \( \sigma^2 \) depending on the horizon \( T \). For the sake of uniformity we note, this scheme is equivalent to the one with \( \sigma^2 \) independent of \( T \) at cost of non-trivial regularization parameter \( \rho \). Really, Lemma B.6 suggest to use \( k(\cdot, \cdot) + 2g^2(u + \log T)\delta(\cdot, \cdot) \) as a covariance function of responses. Clearly, one can use \( k(\cdot, \cdot)/(2(u + \log T)) + g^2\delta(\cdot, \cdot) \) instead. By the means of simple algebra one verifies, the posterior mean remains the same, while the posterior variance shall be \( 2(u + \log T) \) times smaller, and hence, \( \beta_t \) shall be adjusted accordingly. The next lemma summarizes this observation and changes the variables driving the probability of the set its statement is conditioned on.

**Lemma B.7.** Let \( \varepsilon_t \) be sub-Gaussian with \( g^2 \), denote \( F := \|f\|^2_k \) and choose
\[
\rho_{\text{UCB}} = (2(\omega + 1) \log T)^{-1} \text{ for any } \omega > 0,
\]
and
\[
\beta_t = 80000(\eta_t \log t + F^2) \log t.
\]
Then on a set of probability at least \( 1 - T^{-\omega} \)
\[
R_T \leq \sqrt{8T\beta_T\eta_T/\log(1 + \sigma^{-2})}.
\]

**Proof.** As seen above, the desired bound can be established on a set of probability at least \( 1 - \delta - \exp(-u) \) under the choice
\[
\sigma^2 = 2g^2,
\]
\[
\rho_{\text{UCB}} = (2(u + \log T))^{-1},
\]
and
\[
\beta_t = 2(u + \log T) \left( 2\|f\|^2_k + 300\eta_t \log^3(t/\delta) \right).
\]
Now we parameterize \( \delta \) with a positive \( v \) as \( \delta = \exp(-v) \). Next we choose \( v = u = \omega \log T \). Substitution yields the claim. \( \square \)

In conclusion, we cite a result bounding the information gain.

**Lemma B.8** (Theorem 5 by [13]). Let \( k(\cdot, \cdot) \) be Matrn covariance function with smoothness index \( \alpha \).
\[
\eta_T = O \left( T^{1/(\alpha+1)} \log T \right).
\]
C Formal treatment of Algorithm 1

In this section we establish two theoretical results about our change-point detection procedure. Namely, Lemma C.1 provides an upper bound of $\Delta^2$ in the absence of a change-point, while Lemma C.2 gives its lower bound. These two results combined induce a proper choice of the threshold $\theta$.

First, assume $\{X_t\}_{t=1}^{2n} \overset{iid}{\sim} U(\mathcal{X})$ and let

$$y_t = f(X_t) + \varepsilon_t$$

where $\varepsilon_t$ denotes i.i.d. centered noise.

**Lemma C.1.** Let $\varepsilon$ be sub-Gaussian, $k(\cdot, \cdot)$ satisfy Assumption D.1 and Assumption D.2 and $f$ be Sobolev with $\alpha$ and $B$. Choose $\rho_{\text{CP}} = B^2/\log n$. Then for some positive $\omega$ and constant $C$, which depends only on $\text{Var}[\varepsilon_1]$, $\omega$ and $B$,

with probability at least $1 - 2\omega$,

$$\hat{\Delta}^2 \leq C \left( \frac{\log n}{n} \right)^{(\alpha-1/2)/\alpha} (\sigma^2/\log n) \left( \frac{\sigma^2}{n} \right)^{\alpha-1/2/\alpha}.$$

**Proof.** The proof consists in two applications of Lemma D.1, yielding concentration of $\mu_1$ and $\mu_2$ around $f$ and a piece of straightforward algebra.

$$\hat{\Delta}^2 = \frac{1}{n} \sum_{i=1}^{2n} (\mu_1(X_i) - \mu_2(X_i))^2$$

$$= \frac{1}{n} \sum_{i=1}^{2n} (\mu_1(X_i) - f(X_i) + f(X_i) - \mu_2(X_i))^2$$

$$\leq 4\Delta_f^2$$

$$\lesssim B^{1/\alpha} \left( \frac{\sigma^2 \log n}{n} \right)^{(\alpha-1/2)/\alpha},$$

where $\Delta_f$ comes from Lemma D.1.

On the other hand, let $\{X_t\}_{t=1}^{2n} \overset{iid}{\sim} U(\mathcal{X})$ as before and let there be two functions $f_1$ and $f_2$ such that

$$y_t = f_1(X_t) + \varepsilon_t \text{ for } t \leq n$$

and

$$y_t = f_2(X_t) + \varepsilon_t \text{ for } t > n.$$

Needless to say, an ability of the algorithm to detect a change-point depends on some measure of discrepancy between the two functions. We suggest to consider $L_2$-norm.

$$\Delta^2 := \int_{X \in \mathcal{X}} (f_1(X) - f_2(X))^2 dX$$

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**Lemma C.2.** Let the functions $f_1$ and $f_2$ be bounded: $F := \max\{\|f_1\|_{\infty}, \|f_2\|_{\infty}\}$. Choose $\rho_{CP} = B^2 / \log n$. Then for any positive $v$, $\omega$ and a positive constant $C$ depending only on $F$, $B$, $\Var[\varepsilon_1]$ and $\omega$

\[
\mathbb{P}\left\{ \hat{\Delta}^2 \geq \Delta^2 - F\sqrt{\frac{v}{n}} - C\left(\frac{\log n}{n}\right)^{(\alpha-1/2)/\alpha} \right\} \geq 1 - 2\exp(-v) - 2n^{-\omega}.
\]

**Proof.** First, consider

\[
\hat{\Delta}^2 := \frac{1}{n} \sum_{1}^{2n} (f_1(X_t) - f_2(X_t))^2.
\]

Clearly, $\mathbb{E}[\hat{\Delta}^2] = \Delta^2$, the summands are i.i.d. and bounded, hence Hoeffding’s inequality applies here and yields for any positive $v$

\[
\mathbb{P}\left\{ \left| \Delta^2 - \hat{\Delta}^2 \right| > F\sqrt{\frac{v}{n}} \right\} \leq 2\exp(-v). \tag{C.1}
\]

In the next piece of algebra for a function $\phi : X \to B$ we write $\phi^{1..2n}$ to refer an element-wise application $\phi^{1..2n} := (\phi(X_1), \phi(X_2), ..., \phi(X_{2n}))^T$.

\[
\hat{\Delta}^2 = \frac{1}{n} \|\mu_1^{1..2n} - \mu_2^{1..2n}\|^2
\]
\[
= \frac{1}{n} \|\mu_1^{1..2n} - f_1^{1..2n} + (f_1^{1..2n} - f_2^{1..2n}) + f_2^{1..2n} - \mu_2^{1..2n}\|^2
\]
\[
\geq \hat{\Delta}^2 - \frac{1}{n} \left( \|\mu_1^{1..2n} - f_1^{1..2n}\|^2 + \|\mu_2^{1..2n} - f_2^{1..2n}\|^2 \right)
\]

Now we make use of Lemma D.1 (defining $\Delta_f$) and (C) and obtain

\[
\hat{\Delta}^2 \geq \Delta^2 - F\sqrt{\frac{v}{n}} - C\Delta_f^2
\]

on a set of probability at least $1 - 2\exp(-v) - 2n^{-\omega}$. \hfill \Box

Finally, we are ready to describe the behavior of Algorithm 1.

**Lemma C.3.** Let $f$, $f_1$ and $f_2$ be Sobolev with $\alpha$ and $B$, further let these functions be bounded: $F := \{\|f\|_{\infty}, \|f_1\|_{\infty}, \|f_2\|_{\infty}\} \leq +\infty$. Choose $\rho_{CP} = B^2 / \log n$ and an arbitrary positive $\omega$. There exist positive constants $c$ and $C$ depending only on $F$, $B$, $\Var[\varepsilon_1]$ and $\omega$, such that the choice of the threshold

\[
\theta = 2C\left(\frac{\log n}{n}\right)^{(\alpha-1/2)/\alpha}
\]

and the length of the sample $n$ satisfying

\[
(c\Delta)^{\max\{\alpha/2, 1/2\}} \geq \frac{\log n}{n} \tag{C.2}
\]

imply no false alarm if the data is not subject to a change and guarantees detection of a change if such is present with probability at least $1 - 3n^{-\omega}$. 

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Proof. Lemma C.1 bound $\hat{\Delta}^2$ in the absence:

$$\hat{\Delta}^2 \leq C \left( \frac{\log n}{n} \right)^{(\alpha-1/2)/\alpha}.$$ 

Hence, the choice of the threshold $\theta = 2C \left( \frac{\log n}{n} \right)^{(\alpha-1/2)/\alpha}$ allows for at most $2n^{-\omega}$ first type error rate. Now, using Lemma C.2 we see, it is sufficient to show, that

$$\Delta^2 - F \sqrt{\frac{\nu}{n} - C \left( \frac{\log n}{n} \right)^{(\alpha-1/2)/\alpha}} \geq \theta.$$  \hspace{1cm} (C.3)

Below we use $C$ to denote a generalized constant, which depends only on $F$, $B$, $\text{Var}[\varepsilon_1]$, its value might change from line to line. (C) is equivalent to

$$\Delta^2 \geq C \left( \frac{\log n}{n} \right)^{(\alpha-1/2)/\alpha} + C \sqrt{\frac{\nu}{n}}.$$ 

Under the choice of $v = \omega \log n$ it is sufficient to obtain

$$\Delta^2 \geq C \left( \frac{\log n}{n} \right)^{\min\{\frac{(\alpha-1/2, 1/2}\}}$$

which follows from (Lemma C.3). \hfill \Box

D Consistency of Gaussian Process Regression

by [16]

In this section we quote a consistency result for predictions of Gaussian Process Regression. It imposes the following two assumptions on the covariance function $k(\cdot, \cdot)$.

Assumption D.1. Let there exist $C_\psi$ and $L_\psi$ s.t. for eigenfunctions $\{\psi_j(\cdot)\}_{j=1}^\infty$ of covariance function $k(\cdot, \cdot)$

$$\max_j \|\psi_j\|_\infty \leq C_\psi$$

and for all $t, s \in \mathbb{R}^p$

$$|\psi_j(t) - \psi_j(s)| \leq jL_\psi \|t - s\|.$$ 

Assumption D.2. Let for the eigenvalues $\{\mu_j\}_{j=1}^\infty$ of covariance function $k(\cdot, \cdot)$ exist positive $c$ and $C$ s.t. $c j^{-2\alpha} \leq \mu_j \leq C j^{-2\alpha}$ for $\alpha > 1/2$.

Matérn kernel with smoothness index $\alpha - 1/2$ satisfies these assumptions. In [16] the authors claim, their results also hold for kernels with non-polynomially decaying eigenvalues, like RBF and polynomial kernels. And as long as we do not use these assumptions in our proofs directly, so do ours.
Lemma D.1 (Corollary 2.1 in [16]). Assume $\varepsilon_i$ are sub-Gaussian and $f$ is $\alpha$-smooth Sobolev. Assume, $X_i \sim U[\mathcal{X}]$. Further, let $k(\cdot, \cdot)$ satisfy Assumption D.1 and Assumption D.2 and choose

$$\rho = \frac{B^2}{\log n},$$

where $n$ is the size of the training sample. Then, with probability at least $1 - n^{-\omega}$ for any positive $\omega$ we have

$$\|f - \mu\|_\infty \leq \Delta_f := A(\omega)B^{1/2\alpha} \left( \frac{\text{Var}[\varepsilon_1] \log n}{n} \right)^{(\alpha-1/2)/(2\alpha)},$$

where $\mu$ denotes the predictive function and $A(\omega)$ is a constant depending only on $\omega$.

This result is demonstrated in [16] for $\omega = 10$, however this choice is purely arbitrary.

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