Generalizations of Pauli channels

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Abstract

The Pauli channel acting on $2 \times 2$ matrices is generalized to an $n$-level quantum system. When the full matrix algebra $M_n$ is decomposed into pairwise complementary subalgebras, then trace-preserving linear mappings $M_n \to M_n$ are constructed such that the restriction to the subalgebras are depolarizing channels. The result is the necessary and sufficient condition of complete positivity. The main examples appear on bipartite systems.

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In this paper we consider particular subalgebras of a full matrix algebra $M_n = M_n(\mathbb{C})$. (By a subalgebra we mean *-subalgebra with unit.) An F-subalgebra is a subalgebra isomorphic to a full matrix algebra $M_k$. (“F” is the abbreviation of “factor”, the center of such a subalgebra is minimal, $\mathbb{C}I$.) An M-subalgebra is a maximal Abelian subalgebra, equivalently, it is isomorphic to $\mathbb{C}^n$. (“M” is an abbreviation of “MASA”, the center is maximal, it is the whole subalgebra.) If $\mathcal{A}_1$ and $\mathcal{A}_2$ are subalgebras, then they are called quasi-orthogonal or complementary if the subspaces $\mathcal{A}_1 \oplus \mathbb{C}I$ and $\mathcal{A}_2 \oplus \mathbb{C}I$ are orthogonal (with respect to the Hilbert-Schmidt inner product $\langle A, B \rangle = \text{Tr} A^*B$). Concerning complementary subalgebras we refer to [8], see also [6, 7, 9].

Complementary M-subalgebras can be given by mutually unbiased bases. Assume that $\xi_1, \xi_2, \ldots, \xi_n$ and $\eta_1, \eta_2, \ldots, \eta_n$ are orthonormal bases such that

$$|\langle \xi_i, \eta_j \rangle| = \frac{1}{\sqrt{n}} \quad (1 \leq i, j \leq n).$$
If $\mathcal{A}_1$ is the algebra of all operators with diagonal matrix in the first basis and $\mathcal{A}_2$ is defined similarly with respect to the second basis, then $\mathcal{A}_1$ and $\mathcal{A}_2$ are complementary M-subalgebras.

There are examples such that $M_n$ is the linear span of pairwise complementary subalgebras in the case when $n$ is a power of a prime number. If $M_n$ is decomposed into complementary subalgebras, then we construct trace-preserving mappings $M_n \to M_n$ which are completely positive under some conditions.

1 Introduction

If the pairwise complementary subalgebras $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r$ of $M_n$ are given and they linearly span the whole algebra $M_n$, then any operator is the sum of the components in the subspaces $\mathcal{A}_i \ominus \mathbb{C}I$ ($1 \leq i \leq r$) and $\mathbb{C}I$:

$$A = -\frac{(r - 1)\text{Tr} A}{n} I + \sum_{i=1}^r E_i(A),$$

where $E_i : M_n \to \mathcal{A}_i$ is the trace-preserving conditional expectation (which is nothing else but the orthogonal projection with respect to the Hilbert-Schmidt inner product, see [10] about details). It is easier to formulate things for matrices of trace 0. If $\text{Tr} B = 0$, then it has orthogonal decomposition

$$B = \sum_{i=1}^r E_i(B).$$

As a generalization of the Pauli channel on a qubit, we define a linear mapping $\alpha : M_n \to M_n$ such that

$$\alpha(B) = \sum_{i=1}^r \lambda_i E_i(B)$$

or for an arbitrary $A$

$$\alpha(A) = \left(1 - \sum_{i=1}^r \lambda_i\right) \frac{\text{Tr} A}{n} I + \sum_{i=1}^r \lambda_i E_i(A), \quad (1)$$

where $\lambda_i \in \mathbb{R}$, $1 \leq i \leq r$. We want to find the condition for complete positivity. The motivation is the following well-known example in which the complementary subalgebras are generated by the Pauli matrices [10].

**Example 1** Let $\sigma_0 = I$ and $\sigma_1, \sigma_2, \sigma_3$ be Pauli matrices, i.e.,

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
and let $\mathcal{E} : M_2 \to M_2$ be defined as
\[
\mathcal{E} (w_0\sigma_0 + (w_1, w_2, w_3) \cdot \sigma) = w_0\sigma_0 + (\lambda_1 w_1, \lambda_2 w_2, \lambda_3 w_3) \cdot \sigma
\] (2)
for $\omega_i \in \mathbb{C}$, where $\lambda_i \in \mathbb{R}$ and
\[
(w_1, w_2, w_3) \cdot \sigma = w_1\sigma_1 + w_2\sigma_2 + w_3\sigma_3.
\]

Density matrices are sent to density matrices if and only if
\[-1 \leq \lambda_i \leq 1.
\]

It is not difficult to compute the representing block matrix
\[
X := \sum_{i,j} E_{ij} \otimes E_{ij},
\]
we have
\[
X = \begin{bmatrix}
\frac{1 + \lambda_3}{2} & 0 & 0 & \frac{1 + \lambda_3}{2} \\
0 & \frac{1 - \lambda_1}{2} & \frac{1 - \lambda_2}{2} & 0 \\
0 & \frac{\lambda_1 - \lambda_2}{2} & \frac{1 - \lambda_3}{2} & 0 \\
\frac{\lambda_1 + \lambda_2}{2} & 0 & 0 & \frac{1 + \lambda_3}{2}
\end{bmatrix}.
\]

According to Choi’s theorem [2] the positivity of this matrix is equivalent to the complete positivity of $\mathcal{E}$. $X$ is unitarily equivalent to the matrix
\[
\begin{bmatrix}
\frac{1 + \lambda_3}{2} & \frac{1 + \lambda_3}{2} & 0 & 0 \\
\frac{\lambda_1 + \lambda_2}{2} & \frac{\lambda_1 + \lambda_2}{2} & 0 & 0 \\
0 & 0 & \frac{1 - \lambda_3}{2} & \frac{1 - \lambda_3}{2} \\
0 & 0 & \frac{\lambda_1 - \lambda_2}{2} & \frac{\lambda_1 - \lambda_2}{2}
\end{bmatrix}.
\]

This matrix is obviously positive if and only if
\[1 \pm \lambda_3 \geq |\lambda_1 \pm \lambda_2|.
\] (3)

This is necessary and sufficient condition of complete positivity.

It is not obvious that condition (3) is symmetric in the three variables $\lambda_1, \lambda_2, \lambda_3$. Condition (3) actually determines the tetrahedron which is the convex hull of the points $(1, 1, 1), (1, -1, -1), (-1, 1, -1)$ and $(-1, -1, 1)$.

Now we show the idea leading to the generalization. The mapping $\mathcal{E}$ in (2) has the form
\[
\mathcal{E}(\cdot) = \sum_{i=0}^{3} \mu_i \sigma_i(\cdot)\sigma_i.
\]

From the expansion of $\mathcal{E}(\sigma_j)$ we can get equations and the solution is the following:
\[
\begin{aligned}
\mu_0 &= \frac{1}{4} (1 + \lambda_1 + \lambda_2 + \lambda_3), & \mu_1 &= \frac{1}{4} (1 + \lambda_1 - \lambda_2 - \lambda_3), \\
\mu_2 &= \frac{1}{4} (1 - \lambda_1 + \lambda_2 - \lambda_3), & \mu_3 &= \frac{1}{4} (1 - \lambda_1 - \lambda_2 + \lambda_3).
\end{aligned}
\]
If $\mu_i \geq 0$ for every $i$, then $\mathcal{E}$ is a completely positive mapping. Therefore,

$$1 + \lambda_3 \geq \pm (\lambda_1 + \lambda_2), \quad 1 - \lambda_3 \geq \pm (\lambda_1 - \lambda_2)$$

or together this is $(3)$. (Actually, this argument gives that $(3)$ is a sufficient condition for the complete positivity.)

Pauli channels form an important and popular subject in quantum information theory [1, 3, 4]. The mappings $(1)$ were studied in the paper [5] in the case when the subalgebras are maximal Abelian and pairwise complementary. Our method is different and we allow non-commutative subalgebras as well.

The mapping $(1)$ restricted to $\mathcal{A}_i$ has the form

$$D \mapsto \lambda_i D + (1 - \lambda_i) \frac{I}{n}$$

on density matrices $D$. If $0 \leq \lambda_i \leq 1$, then we can say that $D$ does not change with probability $\lambda_i$ and with probability $1 - \lambda_i$ it is sent to the tracial state. Such mappings are usually called as depolarizing channels [10].

A simple example including non-commutative subalgebras is the following.

**Example 2** Consider $M_4 = M_2 \otimes M_2$ and the complementary F-subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_4$ generated by the following triplets of unitaries:

\[
\begin{array}{cccc}
\sigma_0 \otimes \sigma_1 & \sigma_0 \otimes \sigma_2 & \sigma_0 \otimes \sigma_3, \\
\sigma_1 \otimes \sigma_0 & \sigma_2 \otimes \sigma_1 & \sigma_3 \otimes \sigma_1, \\
\sigma_2 \otimes \sigma_0 & \sigma_3 \otimes \sigma_2 & \sigma_1 \otimes \sigma_2, \\
\sigma_3 \otimes \sigma_0 & \sigma_1 \otimes \sigma_3 & \sigma_2 \otimes \sigma_3.
\end{array}
\]

We take also the M-subalgebra $\mathcal{A}_5$ generated by $\sigma_1 \otimes \sigma_1, \sigma_2 \otimes \sigma_2, \sigma_3 \otimes \sigma_3$. The conditional expectations $E_j : M_4 \to \mathcal{A}_j$ are convex combinations of automorphisms

$$E_j(A) = \frac{1}{4} \sum_{i=1}^{4} U_{ji}^* AU_{ji}, \quad (4)$$

where $U_{j1} = I$ and $U_{ji}$'s are orthogonal unitaries from $\mathcal{A}_j'$. Since $\mathcal{A}_5$ is an M-subalgebra, $\mathcal{A}_5' = \mathcal{A}_5$. The subalgebras $\mathcal{A}_1', \ldots, \mathcal{A}_4'$ are F-subalgebras generated by the following unitaries:

\[
\begin{array}{cccc}
\sigma_1 \otimes \sigma_0 & \sigma_2 \otimes \sigma_0 & \sigma_3 \otimes \sigma_0, \\
\sigma_0 \otimes \sigma_1 & \sigma_1 \otimes \sigma_2 & \sigma_1 \otimes \sigma_3, \\
\sigma_2 \otimes \sigma_1 & \sigma_0 \otimes \sigma_2 & \sigma_2 \otimes \sigma_3, \\
\sigma_3 \otimes \sigma_1 & \sigma_3 \otimes \sigma_2 & \sigma_0 \otimes \sigma_3.
\end{array}
\]

(The above triplets generating $\mathcal{A}_j$ and $\mathcal{A}_j'$ ($1 \leq j \leq 4$) are Pauli triplets, see [7] for details.) Moreover,

$$\text{(Tr } A)I = \frac{1}{4} \left( A + \sum_{j=1}^{5} \sum_{k=2}^{4} U_{jk}^* AU_{jk} \right).$$
The linear mapping (1) has the concrete form

\[ \alpha(A) = \left(1 - \sum_{i=1}^{5} \lambda_i\right)\frac{\text{Tr} A}{4}I + \sum_{i=1}^{5} \lambda_i E_i(A), \]

where the conditional expectations \( E_j \) is expressed by the commutant, see (4). (The condition for complete positivity of \( \alpha \) is in Theorem 4.)

Our main result is the necessary and sufficient condition for the complete positivity of mappings like (1) which can be called generalized Pauli channel.

2 Generalized Pauli channels

Let \( \mathcal{A} \) be a (unital \( * \)-) subalgebra of \( M_n \). Our aim is to describe the conditional expectation onto \( \mathcal{A} \) by means of an orthogonal system in the commutant.

Up to unitary equivalence, a subalgebra \( \mathcal{A} \) of \( M_n \) can be written as

\[ \mathcal{A} = \bigoplus_{i=1}^{k} M_{n_i} \otimes I_{m_i}. \]

The commutant \( \mathcal{A}' \) in \( M_n \) is

\[ \mathcal{A}' = \bigoplus_{i=1}^{k} I_{n_i} \otimes M_{m_i}. \]

Let \( N = \sum_{i=1}^{k} n_i^2 \) and let \( P_i \) be a minimal central projection of \( \mathcal{A} \), that is, \( P_i = I_{n_i} \otimes I_{m_i} \).

**Proposition 1** Let \( \{U_i\}_{i=1}^{N} \) be an orthonormal basis of \( \mathcal{A} \). Then the completely positive map \( F \) from \( M_n \) onto \( \mathcal{A}' \) given by

\[ F(X) = \sum_{i=1}^{N} U_i^* X U_i \quad (X \in M_n) \]

is equal to

\[ F(X) = \bigoplus_{i=1}^{k} \frac{n_i}{m_i} \text{Tr}_{n_i}(P_i X P_i), \]

where \( \text{Tr}_{n_i} \) is a partial trace from \( M_{n_i} \otimes M_{m_i} \) onto \( M_{m_i} \).

In particular, if all \( n_i/m_i \) are equal, then \( \frac{n}{\dim \mathcal{A}} F \) is the trace-preserving conditional expectation from \( M_n \) onto \( \mathcal{A}' \).
Proof: If all $n_i/m_i$ are equal, then their ratio is equal to $\frac{\dim A}{n}$. Therefore it is sufficient to prove the first assertion.

Let $\{e_{ij}^{(l)}\}_{i,j=1}^{n_l}$ and $\{f_{ij}^{(l)}\}_{i,j=1}^{m_l}$ be matrix units of $M_{n_l}$ and $M_{m_l}$, respectively. Then $U_i$ is written by

$$U_i = \sum_{l=1}^{k} \sum_{s,t=1}^{n_l} U_{i,sts}^{(l)} e_{st}^{(l)}$$

for some $U_{i,sts}^{(l)} \in \mathbb{C}$. The operator $W \in M_N$ given in terms of its matrix entries by the formula

$$W_{i,sts} = \sqrt{m_l} U_{i,sts}^{(l)}$$

for $1 \leq i \leq N$, $1 \leq l \leq k$ and $1 \leq s,t \leq n_l$ is unitary. Indeed, $W$ can be considered as the matrix which takes the orthonormal basis $\{\sqrt{m_l} e_{st}^{(l)}\}$ of $\mathcal{A}$ into the orthonormal basis $\{U_i\}$. Hence we have

$$\sum_{i=1}^{N} W_{i,sts}^{(l)} W_{i,s't't'} = \delta_{ll'} \delta_{ss'} \delta_{tt'}.$$  

and therefore

$$\sum_{i=1}^{N} U_{i,sts}^{(l)} U_{i,s't't'} = \delta_{ll'} \delta_{ss'} \delta_{tt'} \frac{1}{m_l}. \quad (5)$$

Let $T$ be a partial isometry with $T^*T = e_{s_1s_1}^{(l_1)} \otimes f_{t_1t_1}^{(l_1)}$ and $TT^* = e_{s_2s_2}^{(l_2)} \otimes f_{t_2t_2}^{(l_2)}$. Then we obtain

$$F(T) = \sum_{i=1}^{N} U_i^{*} T U_i = \sum_{i=1}^{N} \sum_{p=1}^{n_2} \sum_{q=1}^{n_1} U_{i,ps}^{(l_2)} U_{i,sq}^{(l_1)} e_{ps}^{(l_2)} T e_{s}^{(l_1)}$$

$$= \delta_{l_1l_2} \delta_{s_1s_2} \delta_{pq} \sum_{p=1}^{n_1} \frac{1}{m_{l_1}} e_{ps}^{(l_1)} T e_{s}^{(l_1)}$$

by (5) so that $F$ maps the off-diagonal part to 0, that is, if $l_1 \neq l_2$ then $F(T) = 0$.

Now let $T = e_{s_2s_2}^{(l)} \otimes f_{t_2t_1}^{(l)}$. Then we obtain

$$F(T) = \sum_{p=1}^{n_l} \frac{1}{m_l} e_{ps}^{(l)} \otimes f_{t_2t_1}^{(l)} = \delta_{s_1s_2} \frac{1}{m_l} I_{n_l} \otimes f_{t_2t_1}^{(l)}$$

$$= \frac{n_l}{m_l} \text{Tr}_{n_l}(T)$$

which shows the first assertion. □

The commutant of M-subalgebras and F-subalgebras are again M-subalgebras and F-subalgebras, and in both types it is possible to choose an orthogonal basis consisting of unitaries, only. Thus by an application of the previous proposition, for such a
subalgebra \( \mathcal{A} \), the trace-preserving conditional expectation is the convex combination of automorphisms:

\[
X \mapsto \frac{1}{\dim \mathcal{A}'} \sum_{i=1}^{m} U_i^* X U_i' \quad (X \in M_n),
\]

where \( \{U_i'\} \) is an orthogonal basis of \( \mathcal{A}' \) consisting of unitaries. Bases consisting of unitaries are important also in quantum state teleportation [13].

**Theorem 1** Let \( \{U_i : 1 \leq i \leq m\} \) be an orthonormal system in \( M_n \). Then the linear mapping

\[
\alpha(A) = \sum_{i=1}^{m} \mu_i U_i^* A U_i
\]

is completely positive if and only if \( \mu_i \geq 0 \) for every \( 1 \leq i \leq m \).

**Proof:** If \( \mu_i \geq 0 \) for every \( 1 \leq i \leq m \), it is clear that \( \alpha \) is completely positive. To prove the converse, we first show that

\[
\sum_{i,j} W^* E_{ij} W \otimes E_{ij}
\]

is a projection if \( \text{Tr} \, W W^* = 1 \). This is obviously self-adjoint and we can compute that it is idempotent:

\[
\left( \sum_{i,j} W^* E_{ij} W \otimes E_{ij} \right) \left( \sum_{k,l} W^* E_{kl} W \otimes E_{kl} \right) = \sum_{i,j,l} W^* E_{ij} W W^* E_{jl} \otimes E_{il}
\]

\[
= \text{Tr} \, W W^* \left( \sum_{i,l} W^* E_{il} W \otimes E_{il} \right).
\]

It follows that

\[
P_k := \sum_{i,j} U_k^* E_{ij} U_k \otimes E_{ij}
\]

is a projection for every \( 1 \leq k \leq m \). To show that they are pairwise orthogonal, we compute the trace of \( P_k P_l \):

\[
\text{Tr} \, P_k P_l = \text{Tr} \sum_{i,j,u,v} U_k^* E_{ij} U_k U_l^* E_{uv} U_l \otimes E_{ij} E_{uv}
\]

\[
= \sum_{i,j} \text{Tr} \, U_k^* E_{ij} U_k U_l^* E_{ji} U_l = \sum_{i,j} \text{Tr} \, E_{ij} U_k^* U_l^* E_{ji} U_l U_k^*.
\]

Due to the Lemma 1 below this equals \( \text{Tr} \, U_k^* U_l^* \text{Tr} \, U_l U_k^* = 0 \) when \( k \neq l \).

The complete positivity implies that

\[
\sum_{i,j} \left( \sum_k \mu_k U_k^* E_{ij} U_k \otimes E_{ij} \right) = \sum_k \mu_k \left( \sum_{i,j} U_k^* E_{ij} U_k \otimes E_{ij} \right) = \sum_k \mu_k P_k
\]

is positive, therefore \( \mu_k \geq 0 \).
Lemma 1

\[ \sum_{i,j} \text{Tr} E_{ij} X E_{ji} Y = (\text{Tr} X)(\text{Tr} Y). \]

Proof: Since both sides are bilinear in the variables \( X \) and \( Y \), it is enough to check the case \( X = E_{ab} \) and \( Y = E_{cd} \). Simple computation gives that left-hand-side is \( \delta_{ab}\delta_{cd} \).

A physicist might make a different proof of the lemma:

\[ \sum_{i,j} \text{Tr} E_{ij} X E_{ji} Y = \sum_{i,j} \text{Tr} |e_i\rangle\langle e_j| X |e_j\rangle\langle e_i| Y = \sum_{i,j} \langle e_j| X |e_j\rangle\langle e_i| Y |e_i\rangle \]

and the right-hand-side is \( (\text{Tr} X)(\text{Tr} Y) \).

We also need the next lemma; the proof can be found in [13].

Lemma 2 Let \( V_1, V_2, \ldots, V_{n^2} \) be matrices in \( M_n \). Then the following conditions are equivalent:

1. \( \text{Tr} V_i^* V_j = \delta_{ij} \quad (1 \leq i, j \leq n^2) \),
2. \( \sum_{i=1}^{n^2} V_i^* A V_i = (\text{Tr} A)I \) for every \( A \in M_n \).

The next result includes important particular cases which are formulated afterwards.

Theorem 2 Let \( \mathcal{A}_1, \ldots, \mathcal{A}_r \) be pairwise complementary subalgebras of \( M_n \) such that their commutants \( \mathcal{A}_1', \ldots, \mathcal{A}_r' \) are pairwise complementary as well. Then the trace-preserving conditional expectations \( E_j : M_n \to \mathcal{A}_j \) can be expressed by the orthonormal bases \( U_{j1}, U_{j2}, \ldots, U_{jn(j)} \in \mathcal{A}_j' \), where \( U_{j1} = \frac{1}{\sqrt{n}} I \), via the formula

\[ E_j(A) = \frac{n}{\dim \mathcal{A}_j'} \sum_{i=1}^{n(j)} U_{ji}^* A U_{ji} \] (6)

and the generalized Pauli channel [1] is completely positive if and only if

\[ 1 + \frac{n^2 \lambda_i}{\dim \mathcal{A}_i'} \geq \sum_j \lambda_j \]

for every \( 1 \leq i \leq r \) and

\[ \sum_j \lambda_j \left( \frac{n^2}{\dim \mathcal{A}_j'} - 1 \right) \geq -1. \]

To prove this theorem we prepare the following proposition.

Proposition 2 Let \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) be complementary subalgebras of \( M_n \). Then \( \mathcal{A}_1' \) and \( \mathcal{A}_2' \) are complementary if and only if \( \mathcal{A}_1 \mathcal{A}_2 \) linearly spans \( M_n \). Moreover, in this case the trace-preserving conditional expectation \( E_1 : M_n \to \mathcal{A}_1' \) can be expressed as

\[ E_1(X) = \frac{n}{\dim \mathcal{A}_1} \sum_i U_i^* X U_i \quad (X \in M_n), \]
where \( \{U_i\} \) is an orthonormal basis of \( A_1 \).

**Proof:** Assume \( A'_1 \) and \( A'_2 \) are complementary. Let \( \{U'_i\} \) and \( \{V'_j\} \) be orthonormal bases of \( A'_1 \) and \( A'_2 \), respectively, which consist of scalar multiple of their matrix units. Then the trace-preserving conditional expectations onto \( A_1 \) and \( A_2 \) are given by the linear combinations of \( U'_i^* \cdot U'_i \) and \( V'_j^* \cdot V'_j \), respectively, thanks to Proposition 1.

Since \( A'_1 \) and \( A'_2 \) are complementary subalgebras, \( \{V'_j U'_i\}_{i,j} \) is an orthogonal system. Moreover the trace is written by the linear combination of \( U'_i V'_j^* \cdot V'_j U'_i \), because \( A_1 \) and \( A_2 \) are complementary subalgebras if and only if the composition of two conditional expectations equals to \( \frac{1}{n} \text{Tr} \). But this shows that \( \{V'_j U'_i\}_{i,j} \) linearly spans the whole \( M_n \) thanks to Lemma 2.

Conversely assume \( A_1 A_2 \) linearly spans the whole space \( M_n \). Since \( A_1 \) is a subalgebra of \( M_n \), \( A_1 \) can be written as

\[
A_1 = \bigotimes_{l=1}^k M_{n_l} \otimes I_{m_l}.
\]

Let \( Q \) be a minimal central projection in \( A_2 \) and let \( \{U^{(s)}_i\} \) and \( \{V_j\} \) are orthonormal bases of \( A_1 \) and \( A_2 \), respectively, with the assumption \( U^{(s)}_i \in M_{n_s} \otimes I_{m_s} \). Since \( A_1 \) and \( A_2 \) are complementary and \( \text{span}\{A_1 A_2\} = M_n \), \( \{\sqrt{n} U^{(s)}_i V_j\} \) is an orthonormal basis of \( M_n \). Therefore by Lemma 2 and Proposition 1, we have

\[
\sum_{s,i,j} U^{(s)*}_i V_j^* Q V_j U^{(s)}_i = \frac{1}{n} \text{Tr} Q \cdot I
\]

and

\[
\sum_j V_j^* Q V_j = cQ
\]

for some \( c > 0 \). These equations imply, for \( 1 \leq s \leq k \),

\[
\sum_i U^{(s)*}_i Q U^{(s)}_i = \frac{\text{Tr} Q}{cn} P_s,
\]

where \( P_s \) is a central projection \( I_{n_s} \otimes I_{m_s} \). Now we take the trace to the above equation. Then we have

\[
\text{Tr} \left( \sum_i^{n^2} U^{(s)*}_i Q U^{(s)}_i \right) = \sum_i^{n^2} \frac{1}{n} \text{Tr} \left( U^{(s)*}_i U^{(s)}_i \right) \text{Tr} Q = \frac{n^2}{n} \text{Tr} Q
\]

and

\[
\frac{\text{Tr} Q}{cn} \text{Tr} P_s = \frac{\text{Tr} Q}{cn} n_s m_s
\]

so that

\[
\frac{n_s}{m_s} = \frac{1}{c}.
\]
Hence \( n_s/m_s \) is equal to \( 1/c = \frac{\dim A_s}{n} \) for all \( 1 \leq s \leq k \) and so

\[
E_1 = \frac{n}{\dim A_1} \sum_{s,i} U_i^{(s)^*} (\cdot) U_i^{(s)}
\]

is the trace-preserving conditional expectation onto \( A_1' \) by Proposition 1. Similarly,

\[
E_2 = \frac{n}{\dim A_2} \sum_j V_j^{*} (\cdot) V_j.
\]

is the trace-preserving conditional expectation onto \( A_2' \). Since

\[
\sum_{s,i,j} U_i^{(s)^*} V_j^{*} (\cdot) V_j U_i^{(s)} = \sum_{s,i,j} V_j^{*} U_i^{(s)^*} (\cdot) U_i^{(s)} V_j
\]

is the normalized trace on \( M_n \) by Lemma 2, we obtain

\[
\frac{n^2}{\dim A_1 \dim A_2} = 1
\]

and so the composition \( E_1 \circ E_2 \) equals to \( \frac{1}{n} \text{Tr} \).

□

Proof of Theorem 2. The first assertion is already proven in the above proposition. Due to the Lemma 2, we have

\[
(\text{Tr} A)I = \frac{A}{n} + \sum_{j=0}^{n} \sum_{k=2}^{n} U_{jk}^* A U_{jk} + \sum_{t=1}^{l} W_t^* A W_t,
\]

where orthonormal system \( W_t \) extend the orthonormal system \( U_{jk} \) to a complete system in the linear space \( M_n \). In formula (1) we use this expression for \( (\text{Tr} A)I \) and the assumed decomposition of the conditional expectations. So in the expansion of \( \alpha(A) \) the coefficient of \( U_{jk}^* A U_{jk} \) is

\[
\frac{1}{n} \left( 1 - \sum_i \lambda_i \right) + \frac{n \lambda_j}{\dim A_j'}
\]

and the coefficient of \( \frac{A}{n} = \left( \frac{1}{\sqrt{n}} I \right) A \left( \frac{1}{\sqrt{n}} I \right) \) is

\[
\frac{1}{n} \left( 1 - \sum_j \lambda_j \right) + \sum_j \frac{n \lambda_j}{\dim A_j'}.
\]

Theorem 1 tells us that completely positivity holds if and only if both are positive. □

Corollary 1 Assume that \( M_n \) contains pairwise complementary \( M \)-subalgebras \( A_1, \ldots, A_r \). Then the generalized Pauli channel is completely positive if and only if

\[
1 + n \lambda_i \geq \sum_j \lambda_j \geq -\frac{1}{n - 1}
\]

for every \( 1 \leq i \leq r \).

This result appeared also in [5].
3 Bipartite channels

In this section we consider subalgebras of $M_n \otimes M_n$. A subalgebra isomorphic to $M_n$ will be called F-subalgebra. An M-subalgebra is a maximal Abelian subalgebra. Both kinds of subalgebras are subspaces of dimension $n^2$.

**Theorem 3** Assume that $A_1$ and $A_2$ are F- or M-subalgebras of $M_n \otimes M_n$. If they are complementary, then the commutants $A_1'$ and $A_2'$ are complementary as well.

**Proof:** Since both kinds of subalgebras are subspaces of dimension $n^2$, the dimension of $A_1A_2$ is $n^4$ so that $A_1A_2 = M_n \otimes M_n$. Therefore the commutants $A_1'$ and $A_2'$ are complementary by Proposition 2.

**Theorem 4** Assume that $M_n \otimes M_n$ is decomposed to pairwise complementary F- and M-subalgebras $A_i$ ($1 \leq i \leq n^2 + 1$). The trace-preserving conditional expectation of $M_n \otimes M_n$ onto $A_i$ is denoted by $E_i$. The linear trace-preserving mapping acting as

$$\alpha(B) = \sum_{i=1}^{n^2+1} \lambda_i E_i(B) \quad (B \in M_n \otimes M_n, \text{Tr } B = 0)$$

is completely positive if and only if

$$1 + n^2 \lambda_i \geq \sum_j \lambda_j \geq -\frac{1}{n^2 - 1}$$

for every $1 \leq i \leq n^2 + 1$.

**Proof:** Theorem 3 allows to use Theorem 2 and the result follows.

The theorem can be applied in Example 2. Note that decompositions of $M_2 \otimes M_2$ into F- and M-subalgebras are discussed in [9], while decomposition of $M_n \otimes M_n$ into F-subalgebras is constructed in [6] if $n = p^k$ with a prime number $p > 2$.

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