Manipulability of aggregation procedures
in Impartial Anonymous Culture

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Abstract

Aleskerov et al. \cite{1-2} estimated the degree of manipulability for the case of multi-valued choice (without using any tie-breaking rule) and for Impartial Culture (IC). In our paper, we address the similar question for the multi-valued choice and for Impartial Anonymous Culture (IAC). We use Nitzan-Kelly’s (NK) index to estimate the degree of manipulability, which is calculated as the share of all manipulable voting situations, and calculate indices for 3 alternatives and up to 10000 voters. We have found that for the case of 3 alternatives Nanson’s procedure shows the best results. Hare’s procedure shows close, but a bit higher results. The worst aggregation procedure in terms of manipulability is Plurality rule. Additionally, it turned out that NK indices for IAC are smaller than NK indices for IC.

Keywords: manipulability; aggregation procedures; social choice

1. Introduction

Gibbard \cite{3} and Satterthwaite \cite{4} have shown that if some rather weak conditions hold, every aggregation procedure is manipulable on unrestricted domain. Duggan, Swartz \cite{5}, Ching, Zhou \cite{6} and Benoit \cite{7} demonstrated the similar result for the case of multi-valued choice. This leads to an interesting question: to which extent every aggregation procedure is manipulable? This question was studied for the first time by Chamberlin \cite{8} and Nitzan \cite{9}. As well as Kelly \cite{10} and Aleskerov, Kurbanov \cite{11} these papers studied the degree of manipulability with the assumption of Impartial Culture (all profiles are equally likely) and alphabetical tie-breaking rule. Lepelley, Valognes \cite{12}, Favardin, Lepelley \cite{13} and Pritchard, Wilson \cite{14}
studied the same problem for Impartial Anonymous Culture assumption (all voting situations are equally likely) and alphabetical tie-breaking rule.

Aleskerov et al. [1-2] estimated the degree of manipulability for the case of multi-valued choice (without using any tie-breaking rule) and for Impartial Culture (IC). In our paper, we address the similar question for the multi-valued choice and for Impartial Anonymous Culture (IAC). We use Nitzan-Kelly’s index to estimate the degree of manipulablity, which is calculated as the share of all manipulable voting situations, and calculate indices for 3 alternatives and up to 10000 voters. We compute those indices for 7 aggregation procedures: Plurality rule, Approval voting, Borda’s rule, Black’s procedure, Hare’s rule, Nanson’s rule and Threshold rule.

The structure of this paper is as follows. Section 2 introduces the basic notation and concepts. Section 3 defines NK index to measure the degree of manipulability of aggregation procedures and explains the computational scheme. Section 4 presents the aggregation procedures under study. Section 5 presents and discusses the results.

2. Basic Notations

Here we use the same notations and almost the same model as in Aleskerov et al. [1-2]. The main difference is in computation scheme, which is defined in the next section. We consider a finite set \( A \) consisting of \( m \) alternatives, \( m = 3 \). Let \( A = 2^d \setminus \emptyset \) denote the set of all non-empty subsets of \( A \). Each agent from a finite set \( N = \{1, \ldots, n\} \), \( n > 1 \), is assumed to have a preference \( P_i \in L \) over alternatives where \( L \) is the set of linear orders on \( A \).

An ordered \( n \)-tuple of preferences \( P_i \) is called a (preference) profile, \( \vec{P} \). A group decision is made by an aggregation procedure based on \( \vec{P} \) and is considered to be an element of \( A \). Thus we define an aggregation procedure as a mapping \( C : L^n \rightarrow A \).

Using an aggregation procedure we may get a situation when two alternatives are both chosen. In this paper we use the concept of multiple choice allowing the social choice to consist of multiple alternatives. Every agent \( i \) is assumed to have an extended preference \( EP_i \) over \( A \) which is induced by her preference \( P_i \) over \( A \). The main goal of extended preferences is to allow each agent to compare all possible multi-valued social choices.

There are many preference extension axioms. One can find them, for example, in Barbera [15] and Kelly [16]. The detailed survey can be found in Barbera et al. [17]. In this paper we use two lexicographical extensions: Leximin and Leximax, as described by Ozyurt and Sanver [18].

Under the Leximax extension, two sets are compared according to their best elements. If they are the same, then the ordering is made according to the second best elements, etc. The elements according to which the sets are compared will disagree at some step – except possibly when one set is a subset of the other, in which case the smaller set is preferred. The case of 3 alternatives gives us the following order for Leximax:

\[
\{a\} \succ \{a, b\} \succ \{a, b, c\} \succ \{a, c\} \succ \{b\} \succ \{b, c\} \succ \{c\}
\]

The concept of the Leximin extension is defined similarly so that it is based on the ordering of two sets according to a lexicographical comparison of their worst elements. Again the elements according to which the sets are compared will disagree at some step – except possibly when one set is a subset of the other, in which case the larger set is preferred. The case of 3 alternatives gives us the following order for Leximin:

\[
\{a\} \succ \{a, b\} \succ \{b\} \succ \{a, c\} \succ \{a, b, c\} \succ \{b, c\} \succ \{c\}
\]

We define manipulation in the case of multiple choice as follows. Let \( \vec{P} = \{P_1, \ldots, P_n\} \) be a profile of sincere preferences, and \( \vec{P}_{-i} = \{P_1, \ldots, P_{i-1}, P_i', P_{i+1}, \ldots, P_n\} \)
be some profile in which all agents but $i$ declare their sincere preferences, and $P_i'$ is agent $i$'s deviation from her sincere preference $P_i$. We say that $C : L^m \rightarrow A$ is manipulable by $i$ at $P_i'$ if $C(P_i') \in EP(C(P))$ for some $P_i'$, where $EP_i$ is the extended preference of $P_i$. In other words, we suppose that outcome when the $i$-th agent deviates from her true preference is more preferable according to her extended preference over sets (with respect to her true preference over alternatives) than in the case when she reveals her sincere preference.

3. Manipulability indices and computation scheme

We use the same indices as [1] but modify them to the Impartial Anonymous Culture problem. Number of alternatives being $m$, the total number of possible linear orders is equal to $m!$, and the total number of profiles with $m$ agents in Impartial Anonymous Culture is equal to $\binom{m+n}{m}$. Nitzan [9] introduces the following index, which was also used by Kelly [10]. We modify it to the Impartial Anonymous culture case and call this index as Nitzan-Kelly's index and denote as $NK$, to measure the degree of manipulability of aggregation procedures

$$NK = \frac{d_0}{C^m_{m!+n-1}},$$

where $d_0$ is the number of voting situations in which manipulation takes place.

We use the following computer modelling scheme to calculate $NK$ index for given number of alternatives, given number of agents and given social choice rule:

1. Generate 1,000,000 profiles,
2. For each profile we determine whether it is manipulable or not. A profile is considered manipulable if there is at least one agent who may manipulate,
3. If the profile is manipulable, we increase $d_0$ by 1,
4. Calculate $NK$ index by dividing $d_0$ by the total number of generated profiles, i.e. 1,000,000.

For each profile under consideration, all manipulating orderings for each voter are generated and the respective choice sets of manipulating voting situations are compared with the choice of the original voting situation. We calculate $NK$ indices for 7 aggregation procedures described in the next section.

4. Aggregation procedures

In this paper we consider 7 aggregation procedures.

1. **Plurality Rule:** Choose alternatives that are ranked first by the maximum number of agents, i.e.

   $$a \in C(P) \iff \forall x \in A \quad n^+(a, P) \geq n^+(x, P),$$

   where $n^+(a, P) = \text{card}\{i \in N \mid \forall y \in A \quad a \in P_i y\}$.

2. **q-Approval Rule, q=2:** Let us define

   $$n^+(a, P, q) = \text{card}\{i \in N \mid \text{card}\{D_i(a)\} \leq q - 1\},$$

   where $D_i(a) = \{y \in A : yP_i a\}$ is the upper contour set of $a \in A$ in $P_i \in L$. Let $n^+(a, P, q)$ be the number of agents for which $a$ is ranked among the first $q$ alternatives in their preference ordering. The integer $q$ can be called as the degree of the procedure. We define $q$-Approval as follows

   $$a \in C(P) \iff \forall x \in A \quad n^+(a, P, q) \geq n^+(x, P, q),$$

   i.e., the alternatives which are admitted to be among the $q$ best by the highest number of agents are chosen. It can be easily seen that Plurality Rule is a special case of $q$-Approval where $q = 1$. Below we use $q=2$.

3. **Borda's Rule:** Let $r_i(x, P)$ be the cardinality of the lower contour set of $x \in A$ in $P_i \in \hat{P}$, i.e.

   $$r_i(x, P) = |L_i(x)| = |\{b \in A : xP_i b\}|.$$

   The sum of $r_i(x, P)$ over all $i \in N$ is called the Borda score of
alternative $a$.

$$r(a, \tilde{P}) = \sum_{i=1}^{n} r_i(a, P_i).$$

The alternatives with maximum Borda score are chosen, i.e.

$$a \in C(\tilde{P}) \iff \forall b \in A, \ r(a, \tilde{P}) \geq r(b, \tilde{P}).$$

4. **Black's Procedure**: Let us define the majority relation $\mu$ for a given profile $\tilde{P}$

$$x \mu y \iff \text{card} \{i \in N | xP_i y\} > \text{card} \{i \in N | yP_i x\}.$$  

Concordet winner $CW(\tilde{P})$ in the profile $\tilde{P}$ is an element undominated in the majority relation $\mu$ (constructed according to the profile), i.e.

$$CW(\tilde{P}) = \{a \mid \exists x \in A, x \mu a\}.$$ 

Black's rule picks the unique Concordet winner if it exists and the Borda winner(s) otherwise.

5. **Threshold rule** [1]: Let $v_i(x)$ be the number of agents for which the alternative $x$ is the worst in their ordering, $v_i(x)$ is the number of agents placing $x$ the second worst, and so on, $v_m(x)$ is the number of agents considering the alternative $x$ as their best one. Then we order the alternatives lexicographically. The alternative $x$ is said to $V$-dominate the alternative $y$ if $v_i(x) < v_i(y)$ or, if there exists $k$ not more than $m$, s.t. $v_i(x) = v_i(y)$, $i = 1,...,k-1$, and $v_k(x) < v_k(y)$. In other words, first, the number of worst places are compared, if these numbers are equal then the number of second worst places are compared and so on. The alternatives which are not dominated by other alternatives via $V$ are chosen.

6. **Hare's Procedure**. First, if an alternative is chosen by a simple majority of voters, then the procedure stops. Otherwise, the alternative $a$ with the minimum number of votes is omitted. Then the procedure is applied to the set $X = A \setminus \{a\}$ and to the profile $\tilde{P}/X$ until the alternative ranked first by a simple majority is found.

7. **Nanson's Procedure (modified)**. For each alternative Borda’s count is calculated. Then average count is calculated, and alternatives $c \in A$ are omitted for which Borda’s count is less than average count. Then the set $X = \{a \in A : r(a, \tilde{P}) \geq r\}$ is considered, and the procedure is applied to the profile $\tilde{P}/X$. Such procedure is repeated until choice set will not be empty.

5. **Results**

We will study the results for 3 alternatives and for 7 defined aggregation procedures for Impartial Anonymous Culture under Leximin and Leximax extended preferences.

A chart with Leximin extended preferences for 7 aggregation procedures in IAC for up to 100 agents is given on Figure 1.

We can point out to several observations:

1. All NK indices for IAC are decreasing with the growing number of agents. This is the same as in Impartial Culture. The explanation is pretty simple: as soon as we consider individual manipulability, the weight of one’s preferences and its influence are decreasing with the growing number of agents.

2. For most of aggregation procedures we can notice periods of either 2 or 3 agents in their values. The same was true for NK index in Impartial Culture.

3. Nanson’s procedure is the least manipulable aggregation procedure out of 7 aggregation procedures which we are considering. It shows the smallest values of NK index which are not higher than 0.15.

4. 2-Approval rule is the most manipulable. It shows the biggest values of NK index for almost all numbers of agents (except for 3 and 6 agents). The highest value is about 0.45, which means that almost half of all profiles are manipulable.

Now we will have a look on Figure 2. This is the chart with NK index for IAC for Leximax preferences extension method.
Figure 1. NK index for Leximin preferences extension method in IAC

Figure 2. NK index for Leximax preferences extension method in IAC
Here even more interesting results are observed in comparison with Leximin:

1. Periods for aggregation procedures by the numbers of agents exist and seem to be the same as in the case of Leximin preferences extension method.

2. 2-Approval rule is again the worst in terms of manipulability. Now its values start even from 0.64, which stands for more than half manipulable profiles.

3. There is no one aggregation procedure which is the least manipulable. Hare’s procedure shows best results at the beginning (small number of agents), but then it is surpassed by Nanson’s rule with the growing number of agents.

Now we will compare the results of 3 aggregation procedures: Plurality rule (as the worst aggregation procedure in terms of manipulability), Nanson’s procedure and Hare’s procedure (as the best aggregation procedures in terms of manipulability) for Leximin preferences extensions method for both Impartial Culture and Impartial Anonymous Culture. The results are given on Figure 3.

Let us compare the results of IAC manipulability with the result of IC manipulability. They are given in the Table 1 for the case of up to 20 agents.

Table 1. Results for IAC and IC for up to 20 agents

| Method   | Number of voters | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  | 15  | 16  | 17  | 18  | 19  | 20  |
|----------|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| IC Leximin3 |                 | B | N | B | N | I | B | N | H | N | H | N | N | H | N | H | N | N | N | N | H |
| IC Leximax3 |                 | P | H | B | I | N | H | N | H | N | N | N | N | N | N | N | N | N | H | H | H |
| IAC Leximin3 |                 | N | N | N | H | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N |
| IAC Leximax3 |                 | P | H | B | I | N | H | N | H | H | H | N | N | N | N | N | N | N | N | N | H | H | H |

Here capital letters stand for aggregation procedures: B = Black Procedure, H = Hare Procedure, N = Nanson’s Rule
As we can see, though some differences take place in the least manipulable procedure, in most cases the least manipulable aggregation procedures turned out to be the same for both IAC and IC. At the same time, the absolute values of NK index for IAC are less than the values of NK index for IC.

Overall, we can notice that

1. The values of NK index for IAC are generally lower than the values of NK index in IC. Starting from approximately 30 agents it can be clearly seen that all values of NK index in IAC for all three aggregation procedures are lower than the values of those aggregation procedures in IAC. Even the worst aggregation procedure in terms of manipulability – Plurality rule in IAC is less manipulable than Nanson’s procedure – the best aggregation procedure in IC.

2. All values of NK index for all three aggregation procedures in both IC and IAC are decreasing with the growing number of agents. All aggregation procedures have a local maximum at the beginning of the chart, but then they are decreasing.

3. The best aggregation procedure in terms of manipulability – Nanson’s procedure in IAC for 100 agents has NK index equal to 0.012. It means that only a bit more than 1% of profiles in that case are manipulable.

4. All three aggregation procedures have periods of either 2 or 3 agents.

5. The least manipulable aggregation procedures for similar cases in IAC and IC are mostly the same, but in approximately 20% of situations there are differences in the least manipulable aggregation procedures.

All these points can be put into further investigation. For example, we showed that for three aggregation procedures IAC results are better than IC results. The question whether this is true for other aggregation procedures should be considered.

Finally, we will have a look on the chart with larger numbers of agents. We calculated values of NK index for 7 aggregation procedures for Leximin extended preferences and 10,000 agents.

![Figure 4. 7 aggregation procedures and y=1/n line for n=1000..10000 agents](image_url)
We can see that all our results are the same for 10,000 agents. Additionally, we put $y=1/n$ line on the chart, and it turned out that it lays near Nanson’s Procedure’s line. That maybe a good way to approximate the degree of manipulability of this aggregation procedure, but it requires further investigation.

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