Energy-based operator splitting approach for the time discretization of coupled systems of partial and ordinary differential equations for fluid flows: The Stokes case

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Abstract

The goal of this work is to develop a novel splitting approach for the numerical solution of multiscale problems involving the coupling between Stokes equations and ODE systems, as often encountered in blood flow modeling applications. The proposed algorithm is based on a semi-discretization in time based on operator splitting, whose design is guided by the rationale of ensuring that the physical energy balance is maintained at the discrete level. As a result, unconditional stability with respect to the time step choice is ensured by the implicit treatment of interface conditions within the Stokes substeps, whereas the coupling between Stokes and ODE substeps is enforced via appropriate initial conditions for each substep. Notably, unconditional stability is attained without the need of subiterating between Stokes and ODE substeps. Stability and convergence properties of the proposed algorithm are tested on three specific examples for which analytical solutions are derived.

Keywords: multiscale fluid flow, operator splitting, partial and ordinary differential equations, blood flow simulations

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1. Introduction

Multiscale coupling between systems of partial differential equations (PDEs) and ordinary differential equations (ODEs) is often necessary when modeling complex problems arising in science, engineering and medicine. In particular, the present work is motivated by applications to blood flow modeling through the cardiovascular system, even though the resulting conceptual framework may be meaningful and applicable to a more general context of hydraulic networks.

The coupling between PDEs and ODEs in blood flow modeling has been utilized in different ways depending on the specific modeling needs. Many studies have used Windkessel-like models [1] as boundary conditions for three-dimensional (3D) blood flow simulations in regions of particular interest, as in [2, 3, 4, 5]. In addition, ODEs have been used to provide systemic descriptions of the cardiovascular system where 3D regions are embedded, as in [6, 7, 8, 9]. In all these applications, the PDE/ODE coupling leads to interface conditions enforcing the continuity of mass and the balance of forces, which should also be preserved at the discrete level when solving the problem numerically.

Many strategies have been proposed for the numerical solution of coupled PDE/ODE systems in the context of blood flow modeling. In particular, monolithic and splitting (or partitioned) schemes have been proposed, where the PDE and ODE systems are solved simultaneously or in separate substeps, respectively. An extensive discussion about advantages and limitations of monolithic and splitting approaches can be found in [10, 11]. The present contribution focuses on splitting techniques and on the properties of their modular structure. Among the many interesting contributions in this area, we mention here those that are most closely related to our work. In [12], Quarteroni et al consider the multiscale coupling between the Navier-Stokes equations in a rigid domain and a lumped parameter model. A splitting strategy based on subiterations between PDE and ODE solvers at each time step is proposed and assessed in different meaningful configurations. The splitting formulation was then used as an effective tool to prove the well-posedness of the coupled problem [13], in com-
bination with appropriate fixed point results. In [14], Fouchet-Incaux et al compare the numerical stability of explicit and implicit coupling between the Stokes or Navier-Stokes equations and circuit-based models containing resistances and capacitances. Unconditional stability was proved in the implicit case, whereas conditional stability was proved in the explicit case. In [15], Moghadam et al propose a time implicit approach to couple general lumped parameter models with a finite-element based solution of a Navier-Stokes problem in a 3D domain. The algorithm combines the stability properties of monolithic approaches with the modularity of splitting algorithms. The method is based on a Newton type iterative scheme, where data are exchanged between the two domains at each Newton iteration of the nonlinear Navier-Stokes solver to ensure convergence of both domains simultaneously. To the best of our knowledge, the splitting schemes that have been proposed for coupled PDE/ODE systems in the context of fluid flow modeling so far, require subiterations between substeps, usually involving the values of pressure and flow rate at the multiscale interfaces, in order to achieve convergence of the overall algorithm. Depending on the mathematical properties of the models in each substep and of the coupling between them, the convergence of such subiterations might become an issue, especially in the case of nonlinear problems.

The present study aims at providing a novel splitting scheme for coupled PDE/ODE systems for fluid flow that does not require subiterations between substeps to achieve stability for the overall algorithm. The scheme stability follows from ensuring that the physical energy balance is maintained at the discrete level via a suitable application of operator splitting techniques [16, 17] to semi-discretize the problem in time, as in [18, 19]. As a result, the proposed algorithm allows us to: (i) solve in separate substeps potential nonlinearities within the systems of PDEs and/or ODEs; (ii) maintain some flexibility in choosing the numerical method for the solution of each subproblem; (iii) ensure unconditional stability without the need of subiterations between substeps. Indeed, the physical consistency of the coupling conditions at the discrete level is a major issue in multiscale numerical simulations, which also includes the
coupling between PDEs of different types arising, for example, in the context of 3d-1d modeling of blood flow [20].

We remark that, in the present work, we aim at developing the main skeleton of the splitting algorithm, focusing on the scheme stability and performance with respect to the time step size in the norms dictated by the energy balance of the system. Thus, even though the mathematical framework will be presented in a general way, here we will focus on the Stokes problem as PDE system and resistive connections between Stokes regions and lumped networks. In addition, we will present numerical results obtained in the case of two-dimensional (2d) Stokes problems coupled with zero-dimensional (0d) lumped circuit models. However, the basic skeleton presented in this article could serve as a starting point for further extensions and improvements, including other PDE models, e.g. Navier-Stokes or porous media, and numerical variants that provide increased accuracy in time by including suitable time-extrapolations of quantities of particular interest, in the same spirit as [21, 22], or by symmetrization of the splitting algorithm as discussed in [17].

The paper is organized as follows. The mathematical framework is described in Section 2 and the energy identity for the fully coupled system is derived in Section 3. The energy-based operator splitting approach is presented in Section 4 where we also study its stability properties with respect to the choice of the time step. Section 5 explores the properties of the proposed splitting algorithm by comparing analytical and numerical solutions in three particular examples. Conclusions and future perspectives are outlined in Section 6.

2. Mathematical model

The present work focuses on the coupling between systems of PDEs, representing the motion of an incompressible viscous fluid in a bounded domain \( \Omega \subset \mathbb{R}^d, \ d = 2 \) or 3, and systems of ODEs, representing lumped descriptions of the flow of a viscous fluid through a complex hydraulic network. In order to maintain the focus on the splitting strategy with respect to the coupling
interface conditions, in this article we will assume that $\Omega$ is a non-deformable domain and that the fluid is Newtonian. Extensions to deformable domains and non-Newtonian fluids are beyond the scope of this article, even though they are within reach, as outlined in Section.

**Geometrical architecture of the coupled system.** A schematic representation of the geometrical coupling considered in this paper is provided in Figure. It consists of: (i) $L$ regions of space denoted by $\Omega_l \subset \mathbb{R}^d$, with $l \in \mathcal{L} = \{1, \ldots, L\}$ and $d = 2$ or 3, where the fluid flow is described by the Stokes equations; (ii) $M$ lumped hydraulic circuits denoted by $\Upsilon_m$, with $m \in \mathcal{M} = \{1, \ldots, M\}$, where the fluid flow is described by the hydraulic analog of Kirchoff laws of currents and voltages. In order to clarify the geometrical setting of the coupling conditions, let us assume that the boundary of each domain $\Omega_l$, denoted by $\partial \Omega_l$, is the union of three portions, namely $\partial \Omega_l = \Gamma_l \cup \Sigma_l \cup S_l$, where different types of boundary and interface conditions are imposed. Specifically, Dirichlet conditions are imposed on $\Gamma_l$, Neumann conditions are imposed on $\Sigma_l$ and Stokes-circuit coupling conditions are imposed on $S_l$, as described below. In particular, each region $\Omega_l$ may have $j_{\Omega_l} \geq 1$ Stokes-circuit connections, implying that each boundary portion $S_l$ may be written as $S_l = \bigcup_{m \in \mathcal{M}_l} S_{lm}$, with $l \in \mathcal{L}$. We remark that, for each $l \in \mathcal{L}$, the set $\mathcal{M}_l \subseteq \mathcal{M}$ identifies the circuits $\Upsilon_m$ that are connected to $\Omega_l$. Similarly, each circuit $\Upsilon_m$ may have $j_{\Upsilon_m} \geq 1$ Stokes-circuit connections and the set $\mathcal{L}_m \subseteq \mathcal{L}$ identifies the Stokes regions $\Omega_l$ that are connected to $\Upsilon_m$. For example, in the specific architecture depicted in Figure, we have that $j_{\Omega_1} = 1$ and $\mathcal{M}_1 = \{1\}$, $j_{\Omega_2} = 4$ and $\mathcal{M}_2 = \{1, 2, 3\}$, $j_{\Upsilon_3} = 3$ and $\mathcal{L}_3 = \{2, 3\}$, $j_{\Upsilon_4} = 1$ and $\mathcal{L}_4 = \{3\}$. It may also happen that the same Stokes region $\Omega_l$ and the same circuit $\Upsilon_m$ enjoy multiple connections, as for $\Omega_2$ and $\Upsilon_3$ in Figure.

Thus, an additional subscript is introduced to distinguish between the various connections, so that we can write $S_{lm} = \bigcup_{k=1}^{j_{\Omega_l,\Upsilon_m}} S_{lm,k}$, where $j_{\Omega_l,\Upsilon_m}$ is the total number of connections between $\Omega_l$ and $\Upsilon_m$. For example, for the architecture depicted in Figure, we have $j_{\Omega_2,\Upsilon_3} = 2$. Note that with these notations, we also have $j_{\Omega_l} = \sum_{m \in \mathcal{M}_l} j_{\Omega_l,\Upsilon_m}$ and $j_{\Upsilon_m} = \sum_{l \in \mathcal{L}_m} j_{\Omega_l,\Upsilon_m}$.
We remark that, for the cases considered in this study: (i) Dirichlet conditions are imposed on a (at least) portion of the boundary of each domain $\Omega_l$, namely $\Gamma_l \neq \emptyset$; (ii) each domain $\Omega_l$ is connected to (at least) one circuit, namely $S_l \neq \emptyset$; and (iii) Neumann conditions may not be imposed on the boundary of $\Omega_l$, namely $\Sigma_l = \emptyset$, as it happens in Example 3 described hereafter.

**Stokes problems in $\Omega_l$.** Let $v_l = v_l(x,t)$ and $p_l = p_l(x,t)$, for $l \in \mathcal{L}$, denote the velocity vector field and the pressure field, respectively, pertaining to the fluid in each domain $\Omega_l \times (0,T)$, with $\Omega_l \subset \mathbb{R}^d$, $d = 2, 3$ and $T > 0$. Then, for
$l \in \mathcal{L}$, we can write the Stokes equations as
\begin{align}
\nabla \cdot \mathbf{v}_l &= 0 \quad \text{in } \Omega_l \times (0,T), \quad (1) \\
\rho \frac{\partial \mathbf{v}_l}{\partial t} &= -\nabla p_l + \mu \Delta \mathbf{v}_l + \rho \mathbf{f}_l \quad \text{in } \Omega_l \times (0,T), \quad (2)
\end{align}

where $\rho$ and $\mu$ are positive given constants representing the fluid density and dynamic viscosity, respectively, and $\mathbf{f}_l$ are given body forces per unit of mass.

The system is equipped with the initial conditions
\[
\mathbf{v}_l(x,t) = \mathbf{v}_{l,0}(x) \quad \text{in } \Omega_l, \quad (3)
\]

and the boundary and interface conditions
\begin{align}
\mathbf{v}_l &= 0 \quad \text{on } \Gamma_l \times (0,T), \quad (4) \\
\left(-p_l \mathbf{I} + \mu \nabla \mathbf{v}_l\right) \mathbf{n}_l &= -\mathbf{p}_l \mathbf{n}_l \quad \text{on } \Sigma_l \times (0,T), \quad (5) \\
\left(-p_l \mathbf{I} + \mu \nabla \mathbf{v}_l\right) \mathbf{n}_{lm,k} &= \mathbf{g}_{lm,k} \quad \text{on } S_{lm,k} \times (0,T), \quad (6)
\end{align}

where $\mathbf{I}$ is the $d \times d$ identity tensor, $\mathbf{n}_l$ is the outward unit normal vector to $\Sigma_l$, and $\mathbf{p}_l = \mathbf{p}_l(t)$ are given functions of time. For $m \in \mathcal{M}_l$ and $k = 1, \ldots, j_{\Omega_l,\Upsilon_m}$, the vector $\mathbf{n}_{lm,k}$ denotes the outward unit normal vector to $S_{lm,k}$ and the functions $\mathbf{g}_{lm,k}$ are defined via the coupling conditions (9).

**Lumped hydraulic circuits in $\Upsilon_m$.** Let the dynamics in each lumped hydraulic circuit $\Upsilon_m$, for $m \in \mathcal{M} = \{1, \ldots, M\}$, be described by the vector
\[
\mathbf{y}_m = [y_{m1}, y_{m2}, \ldots, y_{md_m}]^T
\]
of state variables satisfying the following system of (possibly nonlinear) ODEs
\[
\frac{d\mathbf{y}_m}{dt} = \mathbf{A}_m(\mathbf{y}_m, t)\mathbf{y}_m + \mathbf{r}_m(\mathbf{y}_m, t) \quad (7)
\]
equipped with the initial conditions
\[
\mathbf{y}_m(t = 0) = \mathbf{y}_{m,0}, \quad (8)
\]

where $\mathbf{y}_m$ and $\mathbf{r}_m$ are $d_m$-dimensional vector-valued functions and $\mathbf{A}_m$ is a $d_m \times d_m$ tensor. The tensor $\mathbf{A}_m$ embodies topology and physics of the connections among the circuit nodes and the vector-valued function $\mathbf{r}_m$ comprises two
contributions: (i) sources and sinks within the circuit, including generators of current and voltage; and (ii) connections with Stokes regions. In this study, we will focus on lumped circuits involving resistive, capacitive and inductive elements, also known as RCL circuits. As a consequence, typical choices for the electrical state variables would be potential, voltage, charge, current or magnetic flux, which, in hydraulic terms, correspond to pressure, pressure difference, volume, volumetric flow rate or linear momentum flux. Since these state variables are characterized by different physical units, it follows that the ODE system in (7) is not homogeneous in terms of units.

**Remark 1.** The physical units of state variables and equations in system (7) differ also depending on whether the circuits are coupled with 3D or 2D Stokes regions, as summarized in Table 1. In the 3D case, namely when \( \Omega_l \subset \mathbb{R}^d \) with \( d = 3 \) for all \( l \in L \), the physical units of the state variables are kg m\(^{-1}\)s\(^{-2}\) for pressure and pressure difference, m\(^3\) for volume, m\(^3\)s\(^{-1}\) for flow rate and kg m\(^{-1}\)s\(^{-1}\) for linear momentum flux. Consequently, the physical units for the corresponding differential equations are kg m\(^{-1}\)s\(^{-3}\) if the state variable is a pressure or a pressure difference, m\(^3\)s\(^{-1}\) if the state variable is a volume, m\(^3\)s\(^{-2}\) if the state variable is a flow rate and kg m\(^{-1}\)s\(^{-2}\) if the state variable is a linear momentum flux. In the 2D case, namely when \( \Omega_l \subset \mathbb{R}^d \) with \( d = 2 \) for all \( l \in L \), pressures and pressure differences are physical variables whose units are still kg m\(^{-1}\)s\(^{-2}\) and whose corresponding differential equations have the units of kg m\(^{-1}\)s\(^{-3}\). However, volumes and volumetric flow rates should be interpreted as quantities per unit of length, and therefore their units in the 2D case are m\(^2\) and m\(^2\)s\(^{-1}\), respectively, and their corresponding differential equations have the units of m\(^2\)s\(^{-1}\) and m\(^2\)s\(^{-2}\), respectively. Note that the linear momentum flux is a quantity per unit of area, hence, in the 2D case its unit is still kg m\(^{-1}\)s\(^{-1}\) and the corresponding differential equations have the units of kg m\(^{-1}\)s\(^{-2}\). This remark is very important to fully understand the meaning of the coupling conditions detailed hereafter.
State variable | Units of state variable | Units of differential equation
---|---|---
(Coupling dimension) | $(\Omega_l \in \mathbb{R}^2)$ | $(\Omega_l \in \mathbb{R}^3)$
pressure | kg m$^{-1}$s$^{-2}$ | kg m$^{-1}$s$^{-3}$
pressure difference | kg m$^{-1}$s$^{-2}$ | kg m$^{-1}$s$^{-3}$
volume | m$^2$ | m$^3$
flow rate | m$^2$s$^{-1}$ | m$^3$s$^{-1}$
linear momentum flux | kg m$^{-1}$s$^{-1}$ | kg m$^{-1}$s$^{-2}$

Table 1: Physical units of state variables for the lumped hydraulic circuits and their corresponding differential equations, in the case where the circuits are coupled with two-dimensional or three-dimensional Stokes regions, namely $(\Omega_l \in \mathbb{R}^2)$ or $(\Omega_l \in \mathbb{R}^3)$, respectively.

**Coupling conditions.** A domain $\Omega_l$ is connected to a lumped circuit $\Upsilon_m$ via the interfaces $S_{lm,k}$, with $k = 1, \ldots, j_{\Omega_l, \Upsilon_m}$ as indicated in Figure 2, where we impose the condition

$$g_{lm,k}(x, t) = -P_{lm,k}(t)n_{lm,k}(x) \quad \text{for} \quad x \in S_{lm,k} \quad \text{and} \quad t \in (0, T), \quad (9)$$

where $P_{lm,k}$ is the pressure at the node of the circuit sitting on $S_{lm,k}$. Under some geometric assumptions on the domain, this condition corresponds to imposing that the average pressure on the interface $S_{lm,k}$ is equal to the nodal pressure, see [23, 24, 25]. In addition, the continuity of mass, and consequently flow rate, across $S_{lm,k}$ implies that

$$Q_{lm,k}(t) = \int_{S_{lm,k}} v_l(x, t) \cdot n_{lm,k}(x) dS_{lm,k} \quad \text{for} \quad t \in (0, T). \quad (10)$$

For $t \in \mathcal{L}_m$ and $k = 1, \ldots, j_{\Omega_l, \Upsilon_m}$, each term $Q_{lm,k}$ contributes as source/sink for the circuit $\Upsilon_m$; thus, it is convenient to rewrite $r_m$ in (7) as

$$r_m(y_m, Q_{lm,k}, P_{lm,k}, t) = s_m(y_m, t) + b_m(Q_{lm,k}, P_{lm,k}, t), \quad (11)$$

where $s_m(y_m, t)$ represents the contribution of sources and sinks within the circuit (generators of current and voltage) and $b_m(Q_{lm,k}, P_{lm,k}, t)$ gathers all contributions from the $j_{\Upsilon_m}$ Stokes-circuit connections.
Figure 2: Schematic representation of the coupling between the Stokes region $\Omega_l$ and the lumped circuit $\Upsilon_m$. Coupling conditions for the pressure $P_{lm,k}$ and the flow rate $Q_{lm,k}$ should be imposed on the interface $S_{lm,k}$.

**Fully coupled problem.** The fully coupled problem consists in finding $v_l$, $p_l$, $P_{lm,k}$, $Q_{lm,k}$ and $y_m$, for $l \in \mathcal{L}$, $m \in \mathcal{L}$ and $k = 1, \ldots, j_{\Omega_l}, \Upsilon_m$ satisfying equations (1), (2) and (7), subject to the coupling conditions (6), (9) and (10), the boundary conditions (4) and (5), and the initial conditions (3) and (8). Existence of solutions to the fully coupled problem has been proved in some particular cases. For example, in [13], Quarteroni et al. proved local in time existence of a solution when the connections are made through bridging regions. In [26], Baffico et al. proved the existence of a strong solution for small data when the Navier-Stokes equations are connected with a single resistor.

3. Energy identity of the fully coupled problem

The fully coupled problem satisfies an energy identity that embodies the main mechanisms governing the physics of the system. Let us begin by introducing the norms that will be used in the sequel. For $d = 2$ or $3$, $d_z \in \mathbb{N}^*$, $T > 0$ and $\Omega \subset \mathbb{R}^d$, given the vector valued function $z: (0, T) \to \mathbb{R}^{d_z}$, the vector field $\Phi: \Omega \times (0, T) \to \mathbb{R}^d$ and the tensor field $K: \Omega \times (0, T) \to \mathbb{R}^{d_z \times d_z}$, we introduce the following notations

$$\|z\| = \sqrt{\sum_{i=1}^{d_z} z_i^2}, \quad \|\Phi\|_{L^2(\Omega)} = \sqrt{\int_{\Omega} \Phi \cdot \Phi \, d\Omega} \quad \text{and} \quad \|K\|_{L^2(\Omega)} = \sqrt{\int_{\Omega} \underline{K} : \underline{K} \, d\Omega},$$

(12)

where the spaces $\mathbb{R}^d$ and $\mathbb{R}^{d_z \times d_z}$ are endowed with the usual Euclidean inner products and, for the sake of clarity, the time dependence is omitted. The
notation $L^2(\Omega)$ denotes the space of square integrable real functions and the corresponding space of vector valued functions $[L^2(\Omega)]^d$ is denoted by $L^2(\Omega)$. By a slight abuse of notations, we will use in the sequel the same symbol $\| \cdot \|_{L^2(\Omega)}$ for the associated norms, see (12). It is useful to recall that, given $K \in \mathbb{R}^{d \times d}$ symmetric and positive definite, then we can define the norm $\|K^{1/2}z\| = \sqrt{z^T K z}$, where the superscript $T$ means transpose.

Let us now proceed to derive the energy identity for the coupled system. For each $l \in \mathcal{L}$, let us multiply (2) by $v_l$ in $L^2(\Omega_l)$ and integrate over $\Omega_l$. After utilizing the divergence free condition (1) and the boundary and interface conditions (4)-(6), we obtain:

$$\rho \frac{d}{dt} \|v_l\|_{L^2(\Omega_l)}^2 + \mu \|\nabla v_l\|_{L^2(\Omega_l)}^2 = \rho \int_{\Omega_l} f_l \cdot v_l \, d\Omega_l - \rho \int_{\Sigma_l} v_l \cdot n_l \, d\Sigma_l$$

$$+ \sum_{m \in \mathcal{M}_l} \sum_{j=1}^{J_{lm}} \int_{S_{lm,k}} v_l \cdot g_{lm,k} \, dS_{lm,k}.$$ 

The last term on the right hand side can be further manipulated using the coupling conditions (9) and (10) to write

$$\int_{S_{lm,k}} v_l \cdot g_{lm,k} \, dS_{lm,k} = -P_{lm,k} \int_{S_{lm,k}} v_l \cdot n_{lm,k} \, dS_{lm,k} = -P_{lm,k}Q_{lm,k}.$$ 

Combining the above relationships, for each $l \in \mathcal{L}$ we obtain

$$\rho \frac{d}{dt} \|v_l\|_{L^2(\Omega_l)}^2 + \mu \|\nabla v_l\|_{L^2(\Omega_l)}^2 = \rho \int_{\Omega_l} f_l \cdot v_l \, d\Omega_l - \rho \int_{\Sigma_l} v_l \cdot n_l \, d\Sigma_l$$

$$+ \sum_{m \in \mathcal{M}_l} \sum_{j=1}^{J_{lm}} P_{lm,k}Q_{lm,k}.$$ 

Let us now consider the system of differential equations in (7) describing the dynamics of $\Upsilon_m$, for $m \in \mathcal{M}$. Since the equations might not be homogeneous in terms of physical units, see Remark 1, for each $m \in \mathcal{M}$ we need to perform the inner product between (7) and the vector valued function $U_m \Upsilon_m$, in such a way that each of the resulting scalar equations has the physical dimensions of a rate of change of energy, namely $Kg \text{ m}^2 \text{ s}^{-3}$, in the case of 3D Stokes regions, or that of a rate of change of energy per unit length, namely $Kg \text{ m s}^{-3}$, in the case of 2D Stokes regions. Thus, the tensor $U_m$ is diagonal and its entries $U_{mj}$,
The appropriate choices for $U_{m_j}$ have been summarized in Table 2 along with the corresponding physical units in the cases where the circuit is connected to 2D or 3D Stokes regions. We remark that the specific choice for capacitances and inductances appearing in $U_{m_j}$ is determined by the corresponding circuit element pertaining to $y_{mj}$, as detailed in the examples hereafter. We also remark that, in general, $U_{m} = U_{m}(y_{m}, t)$. Thus, performing the scalar product between (7) and $U_{m}y_{m}$ we obtain:

$$\frac{1}{2} \frac{d}{dt} (y_{m}^T U_{m} y_{m}) + y_{m}^T B_{m} y_{m} = r_{m}^T U_{m} y_{m},$$  \hspace{1cm} (16)$$

where

$$B_{m} = -U_{m} A_{m} - \frac{1}{2} \frac{d}{dt} U_{m}.$$  \hspace{1cm} (17)$$
Using the coupling conditions (11) and the fact that $U_m$ is symmetric and positive definite, we finally obtain
\[
\frac{1}{2} \frac{d}{dt} \|U_m^{1/2} y_m\|^2 + y_m^T B_m y_m = b_m^T U_m y_m + s_m^T U_m y_m.
\] (18)

Now, summing (15) over $l \in \mathcal{L}$, (18) over $m \in \mathcal{M}$, and adding the resulting equations, we obtain the following energy identity for the fully coupled system
\[
\frac{d}{dt} (E_{\Omega} + E_{\Upsilon}) + D_{\Omega} + U_T = F_{\Omega} + F_{\Upsilon} + G,
\] (19)

where
\[
E_{\Omega} = \frac{1}{2} \sum_{l \in \mathcal{L}} \rho \|v_l\|^2_{L^2(\Omega_l)}, \quad E_{\Upsilon} = \frac{1}{2} \sum_{m \in \mathcal{M}} \|U_m^{1/2} y_m\|^2,
\] (20)
\[
D_{\Omega} = \sum_{l \in \mathcal{L}} \mu \|\nabla v_l\|^2_{L^2(\Omega_l)}, \quad U_T = \sum_{m \in \mathcal{M}} y_m^T B_m y_m,
\] (21)
\[
F_{\Omega} = \sum_{l \in \mathcal{L}} \left( \rho \int_{\Omega_l} f_l \cdot v_l \, d\Omega_l - p_l \int_{\Sigma_l} v_l \cdot n_l \, d\Sigma_l \right),
\] (22)
\[
F_{\Upsilon} = \sum_{m \in \mathcal{M}} s_m^T U_m y_m,
\] (23)
\[
G = - \sum_{m \in \mathcal{M}} \sum_{l \in \mathcal{L}} P_{lm,k} Q_{lm,k} + \sum_{m \in \mathcal{M}} b_m^T U_m y_m.
\] (24)

$E_{\Omega}$ represents the total kinetic energy in the Stokes regions, $E_{\Upsilon}$ represents the total energy (kinetic + potential) characterizing the lumped circuits, $D_{\Omega}$ represents the viscous dissipation in the Stokes regions, $U_T$ represents all the contributions from resistive, capacitive and inductive elements in the lumped circuits, $F_{\Omega}$ represents the forcing on the system due to body forces and external pressures acting on the Stokes regions, $F_{\Upsilon}$ represents the forcing on the system due to generators of current and voltage within the lumped circuits, and $G$ represents the contribution from the Stokes-circuit connections.

We emphasize that $E_{\Omega}(t) \geq 0$, $E_{\Upsilon}(t) \geq 0$ and $D_{\Omega}(t) \geq 0$ for all $t$, whereas the sign of $U_T(t)$ depends on the properties of the tensor $B_m$. The functionals $F_{\Omega}$ and $F_{\Upsilon}$ do not have a definite sign since they depend on the external forcing. Lastly, the functional form of $G$ depends on the type of lumped elements.
involved in the Stokes-circuit connections.

The case of resistive connections. In order to clearly elucidate the main rationale behind the proposed splitting algorithm, in this article we will focus on resistive connections between Stokes regions and lumped circuits. Resistive connections are among the most common in blood flow modeling \[11\]. In some applications, though, capacitive and inductive elements might be needed. Capacitive connections are used when the fluid pressure in \( \Omega_l \) influences the fluid flow in \( \Upsilon_m \) and, simultaneously, the fluid pressure in \( \Upsilon_m \) influence the fluid flow in \( \Omega_l \), without having actual fluid flow between \( \Omega_l \) and \( \Upsilon_m \). Thus, some portions of the boundary of \( \Omega_l \) must be deformable, leading to a fluid-structure interaction problem that goes beyond the scope of this article and might be considered as future research direction, as outlined in Section 6. Inductive connections are used when the regime of interest is such that inertial effects become important. Since in the present paper we are neglecting inertial effects by adopting the Stokes equations in each \( \Omega_l \), we consistently neglect inertial effects in the connections between the Stokes regions and the lumped circuits. We remark that the particular elements allowed in the connections might lead to different initial problems, as pointed out in \[12\]. We also remark that resistive, inductive and capacitive elements may all be present in the lumped circuit \( \Upsilon_m \).

Let us then consider the case where a resistor connects a Stokes region \( \Omega_l \) with the circuit \( \Upsilon_m \), therefore allowing us to choose pressures as state variables at both ends. For the sake of simplicity, let us also assume that the resistor node within the circuit is set to ground via a capacitor, in the same spirit as \[13\], as shown in Figure 3. We denote the resistance and capacitance in the connection by \( R_{lm,k} \) and \( C_{lm,k} \), respectively, and the pressures at the Stokes and circuit ends by \( P_{lm,k} \) and \( \pi_{lm,k} \), respectively. Thus, for each \( m \in M \), we can rewrite the vector of state variables \( y_m \) as

\[
y_m = [\pi_m, \omega_m]^T,
\]

where the \( j_{\Upsilon_m} \)-dimensional column vector \( [\pi_{lm,k}]^T \), with \( l \in L_m \) and \( k = 1, \ldots, j_{\Omega_l,\Upsilon_m} \), gathers all pressures at the circuit end of the connecting resistors, whereas the \( (d_m - j_{\Upsilon_m}) \)-dimensional column vector \( [\omega_{lm,k}]^T \), with \( l \in L_m \) and \( k = 1, \ldots, j_{\Omega_l,\Upsilon_m} \), gathers all pressures at the Stokes end of the connecting resistors.
Figure 3: Schematic representation of the particular type of coupling between a Stokes region \( \Omega_l \) and a circuit \( \Upsilon_m \) considered in this work. A resistive connections between \( \Omega_l \) and \( \Upsilon_m \), in addition to the capacitive connection to the ground at the circuit side of the resistor, allows us to adopt pressures as state variables at both ends.

1, \ldots, j_{\Omega_l, \Upsilon_m} \), gathers the remaining state variables. Then, for each \( m \in \mathcal{M} \), we can rewrite system (7) as

\[
\frac{d}{dt} \begin{bmatrix} \pi_m \\ \omega_m \end{bmatrix} = A_m \begin{bmatrix} \pi_m \\ \omega_m \end{bmatrix} + s_m + b_m. \tag{25}
\]

Recalling that \( b_m \) gathers the contributions due to the Stokes-circuit connections, we can write

\[
b_m = [b_m, 0]^T,
\]

where \( b_m = [\beta_{l_{m,k}}]^T \), with \( \beta_{l_{m,k}} = \frac{Q_{l_{m,k}}}{C_{l_{m,k}}} \) for \( l \in \mathcal{L}_{m} \) and \( k = 1, \ldots, j_{\Omega_l, \Upsilon_m} \). Thus, denoting by \( \mathcal{G}_{RC} \) the functional defined in equation (24) in the case of the Stokes-circuit connections depicted in Figure 3, and noticing that the entries of \( U_m \) corresponding to \( \pi_{l_{m,k}} \) are simply given by \( C_{l_{m,k}} \), we can write

\[
\mathcal{G}_{RC} = - \sum_{m \in \mathcal{M}_{l_{1}}} \sum_{k=1}^{j_{\Omega_l, \Upsilon_m}} P_{l_{m,k}} Q_{l_{m,k}} + \sum_{m \in \mathcal{M}_{l_{1}}} \sum_{k=1}^{j_{\Omega_l, \Upsilon_m}} \beta_{l_{m,k}} C_{l_{m,k}} \pi_{l_{m,k}}, \tag{26}
\]

Furthermore, since \( Q_{l_{m,k}} \) is the flow rate through the resistor of resistance \( R_{l_{m,k}} \), we can write

\[
Q_{l_{m,k}} = \frac{(P_{l_{m,k}} - \pi_{l_{m,k}})}{R_{l_{m,k}}},
\]

thus obtaining

\[
\mathcal{G}_{RC} = - \sum_{m \in \mathcal{M}_{l_{1}}} \sum_{k=1}^{j_{\Omega_l, \Upsilon_m}} R_{l_{m,k}} (Q_{l_{m,k}})^2 = -D_{RC} \leq 0 \quad \forall t \geq 0, \tag{27}
\]

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Figure 4: Example 1. The two-dimensional Stokes region $\Omega_1$ is connected to the lumped circuit $\Upsilon_1$ via a resistive element with resistance $R_{11,1}$.

showing that $\mathcal{G}_{RC}$ contributes to the energy dissipation of the fully coupled system, consistently with the physics of a resistive connection. As a consequence, in the case of Stokes-circuit connections illustrated in Figure 3 and considered hereafter, the fully coupled system satisfies the energy identity

$$\frac{d}{dt} \left( E_{\Omega} + E_{\Upsilon} \right) + D_{\Omega} + D_{RC} + U_{\Upsilon} = F_{\Omega} + F_{\Upsilon}.$$  \hspace{1cm} (28)

To better understand the mathematical formulation of the problem and the various contributions in the energy identity, particularly those coming from $\mathcal{G}_{RC}$, we examine three illustrative examples that differ by: (i) number of Stokes regions and lumped circuits; (ii) number of connections between Stokes regions and lumped circuits; and (iii) type of elements within the lumped circuits. In all these examples, each domain $\Omega_l \in \mathbb{R}^d$, with $d = 2$ and $l \in \mathcal{L}$, is defined as the rectangle $(0, L) \times (-H/2, H/2)$, with $H, L > 0$ given.

**Example 1.** In Example 1, see Figure 4, the 2D Stokes region $\Omega_1$ is connected to the lumped circuit $\Upsilon_1$; as a consequence, here we have $l = m = j_{\Omega_1} = j_{\Upsilon_1} = j_{\Omega_1} = 1$ and $\mathcal{L} = \mathcal{L}_1 = \mathcal{M} = \mathcal{M}_1 = \{1\}$. The circuit $\Upsilon_1$ is described by the vector of state variables $y_1 = [\pi_{11,1}, \omega_{11}]^T$ whose dimension is $d_1 = 2$. This circuit includes a voltage generator, resistances and capacitances, some of which nonlinear, as often needed in blood flow modeling applications [27, 28].

Due to the particular structure of the circuit, it is natural to choose $\pi_{11,1}$ as being the nodal pressure and $\omega_{11}$ the nodal volume. In this example, in the
ODE system (7), we have

\[
\mathbf{A}_1(y_1, t) = \begin{bmatrix}
-\frac{1}{R_a(y_1,t)C_{11,1}} & \frac{1}{R_a(y_1,t)C_{11,1}C_{11,1}} \\
\frac{1}{R_a(y_1,t)} & \frac{1}{R_a(y_1,t)C_a(y_1,t)} - \frac{1}{R_bC_a(y_1,t)}
\end{bmatrix},
\]  
(29)

\[
r_1(y_1, Q_{11,1}, P_{11,1}, t) = s_1(t) + b_1(Q_{11,1}, t) = \begin{bmatrix} 0, \frac{\bar{p}(t)}{R_b} \end{bmatrix}^T + \begin{bmatrix} Q_{11,1}(t) / C_{11,1}, 0 \end{bmatrix}^T,
\]
(30)

with

\[
Q_{11,1}(t) = \frac{P_{11,1}(t) - \pi_{11,1}(t)}{R_{11,1}},
\]  
(31)

Moreover, we have

\[
\mathbf{U}_1(y_1, t) = \begin{bmatrix} C_{11,1} & 0 \\
0 & \frac{1}{C_a(y_1,t)} \end{bmatrix},
\]  
(32)

and we observe that, we chose \(C_{11,1}\) for the entry in \(U_{11}\) as it refers to the variable \(\pi_{11,1}\) and, similarly, we chose \(1/C_a\) for the entry \(U_{12}\) as it refers to the variable \(\omega_{11}\).

Using the definition of \(\mathbf{U}_1\) given in (32), the functionals \(E_\Upsilon\) and \(F_\Upsilon\) can be written as

\[
E_\Upsilon = \frac{1}{2} \left( C_{11,1} \pi_{11,1}^2 + \frac{\omega_{11}^2}{C_a} \right), \quad F_\Upsilon = \frac{\bar{p}}{R_bC_a} \omega_{11},
\]  
(33)

respectively. The former functional represents the fluid potential energy stored in the capacitors of the lumped circuit, whereas the latter accounts for the forcing on the system due to the generator of voltage. Similarly, using the definition of \(\mathbf{b}_1\) given in (17), the functional \(U_\Upsilon\) can be written as

\[
U_\Upsilon = \frac{\pi_{11,1}^2}{R_a} - 2 \frac{\pi_{11,1} \omega_{11}}{R_aC_a} + \left( \frac{1}{R_a} + \frac{1}{R_b} + \frac{1}{2} \frac{dC_a}{dt} \right) \frac{\omega_{11}^2}{C_a^2}.
\]  
(34)

Clearly, if \(C_a\) is constant, then \(U_\Upsilon \geq 0\), thereby providing energy dissipation for the system. Using the definition of \(\mathbf{b}_1\) given in (30) and the definition of \(Q_{11,1}\) given in (31), we can write the functional \(G_{RC}\) as

\[
G_{RC} = -R_{11,1} (Q_{11,1})^2 := -D_{RC} \leq 0, \text{ for all } t \in (0, T),
\]  
(35)
which shows that $\mathcal{G}_{RC}$ contributes via $\mathcal{D}_{RC}$ to the mechanisms of energy dissipation in the coupled system.

**Example 2.** In Example 2, see Figure 5, the 2D Stokes regions $\Omega_1$ and $\Omega_2$ are connected to the lumped circuit $\Upsilon_1$; as a consequence, here we have $l = 2$, $m = 1$, $j_{\Omega_1} = j_{\Omega_2} = j_{\Omega_2}$, $r_1 = 1$, $j_{r_1} = 2$, and $\mathcal{M} = \mathcal{M}_1 = \{1\}$ for $l = 1, 2$, $\mathcal{L} = \mathcal{L}_1 = \{1, 2\}$. The circuit $\Upsilon_1$ is described by the vector of state variables $y_1 = [\pi_{11,1}, \pi_{21,1}, \omega_{11}]^T$ whose dimension is $d_1 = 3$, and it includes also an inductive element, in addition to a voltage generator, resistances and capacitances as occurring in systemic modeling of blood flow, see for instance [29]. Due to the particular structure of the circuit, it is natural to choose $\pi_{11,1}$ and $\pi_{21,1}$ as the nodal pressures and $\omega_{11}$ as the flow rate, respectively. In this example, in the ODE system (7), we have

$$\begin{bmatrix}
0 & 0 & -\frac{1}{C_{11,1}} \\
0 & -\frac{1}{C_{21,1} R_b} & \frac{1}{C_{21,1}} \\
\frac{1}{L_a} & -\frac{1}{L_a} & -\frac{R_a}{L_a}
\end{bmatrix}, \quad (36)$$

$$r_1(y_1, t) = s_1(t) + b_1(Q_{11,1}, Q_{21,1}, t) = \begin{bmatrix} 0, \frac{\ddot{p}(t)}{C_{21,1} R_b} \end{bmatrix}^T + \begin{bmatrix} \frac{Q_{11,1}(t)}{C_{11,1}}, \frac{Q_{21,1}(t)}{C_{21,1}}, 0 \end{bmatrix}^T, \quad (37)$$
where
\[ Q_{11,1}(t) = \frac{P_{11,1}(t) - \pi_{11,1}(t)}{R_{11,1}} \quad l = 1, 2. \] (38)

Moreover, in this case
\[ U_{1}(y_{1}, t) = \begin{bmatrix} C_{11,1} & 0 & 0 \\ 0 & C_{21,1} & 0 \\ 0 & 0 & L_{a} \end{bmatrix}. \] (39)

Proceeding as in the case of Example 1, we can write explicitly the functionals \( \mathcal{E}_{\Psi}, \mathcal{U}_{\Psi} \) and \( \mathcal{F}_{\Psi} \) as
\[ \mathcal{E}_{\Psi} = \frac{1}{2} (C_{11,1} \pi_{11,1}^{2} + L_{a} \omega_{11}^{2} + C_{21,1} \pi_{21,1}^{2}), \] (40)
\[ \mathcal{U}_{\Psi} = R_{a} \omega_{11}^{2} + \frac{1}{R_{b}} \pi_{21,1}^{2}, \quad \mathcal{F}_{\Psi} = \frac{\bar{p}}{R_{b}} \pi_{21,1}, \] (41)
where we can identify the contributions of both potential and kinetic energy in \( \mathcal{E}_{\Psi} \). In addition, the term \( \mathcal{U}_{\Psi}(t) \geq 0 \) for all \( t \), thereby contributing to the overall energy dissipation. Using the definition of \( b_{1} \) given in (37), \( Q_{11,1} \) and \( Q_{21,1} \) given in (38), we conclude that
\[ q_{RC} = -R_{11,1} (Q_{11,1})^{2} - R_{21,1} (Q_{21,1})^{2} := -\mathcal{D}_{RC} \leq 0, \text{ for all } t \in (0, T). \] (42)

**Example 3.** In Example 3, see Figure 6, the 2D Stokes region \( \Omega_{1} \) is connected to the closed lumped circuit \( \Psi_{1} \); as a consequence, here we have \( l = m = 1, \ j_{01} = j_{01}, \ \tau_{1} = 2 \) and \( L = L_{1} = M = M_{1} = \{1\} \). The circuit \( \Psi_{1} \) is described by the vector of state variables \( y_{1} = [\pi_{11,1}, \pi_{11,2}, \omega_{11}]^{T} \) whose dimension is \( d_{1} = 3 \).

In addition to including resistive, capacitive and inductive elements, the main feature here is that the circuit \( \Psi_{1} \) is closed, as the circulatory system. Due to the particular structure of the circuit, it is natural to choose \( \pi_{11,1} \) and \( \pi_{11,2} \) as being the nodal pressures and \( \omega_{11} \) as being the flow rate, respectively. In this example, in the ODE system (7), we have
\[
\mathbb{A}_{1}(y_{1}, t) = \begin{bmatrix} -\frac{1}{\rho_{c} C_{11,1}} & 0 & -\frac{1}{c_{11,1}} \\ 0 & -\frac{1}{\rho_{c} C_{11,2}} & \frac{1}{c_{11,2}} \\ \frac{1}{L_{c}} & -\frac{1}{L_{c}} & -\frac{\bar{p}}{L_{c}} \end{bmatrix}, \] (43)
\[ r_1(y_1, Q_{11,1}, P_{11,1}, Q_{11,2}, P_{11,2}, t) = s_1(t) + b_1(Q_{11,1}, Q_{11,2}, t) \] (44)

\[ = \left[ \frac{\tilde{p}_a(t)}{R_a C_{11,1}}, \frac{\tilde{p}_b(t)}{R_b C_{11,2}} \right]^T + \left[ \frac{Q_{11,1}(t)}{C_{11,1}}, \frac{Q_{11,2}(t)}{C_{11,2}} \right]^T, \]

where

\[ Q_{11,k}(t) = \frac{P_{11,k}(t) - \pi_{11,k}(t)}{R_{11,k}} \quad k = 1, 2. \] (45)

Moreover, the tensor \( U_1 \) is given by

\[ U_1(y_1, t) = \begin{bmatrix} C_{11,1} & 0 & 0 \\ 0 & C_{11,2} & 0 \\ 0 & 0 & L_c \end{bmatrix}. \] (46)

The functionals \( \mathcal{E}_T, \mathcal{U}_T \) and \( \mathcal{F}_T \) are given by

\[ \mathcal{E}_T = \frac{1}{2} \left( C_{11,1} \pi_{11,1}^2 + C_{11,2} \pi_{11,2}^2 + L_c \omega_{11}^2 \right), \] (47)

\[ \mathcal{U}_T = \frac{1}{R_a} \pi_{11,1}^2 + \frac{1}{R_b} \pi_{11,2}^2 + R_c \omega_{11}^2, \quad \mathcal{F}_T = \frac{\tilde{p}_a}{R_a} \pi_{11,1} + \frac{\tilde{p}_b}{R_b} \pi_{11,2}. \] (48)

We can identify the contributions of both potential and kinetic energy in \( \mathcal{E}_T \). Moreover, it follows that \( \mathcal{U}_T(t) \geq 0 \) for all \( t \), thereby contributing to the overall...
energy dissipation. Using the definition of $b_1$ given in (44), $Q_{11,1}$ and $Q_{11,2}$ given in (45), we conclude that

$$G_{RC} = -R_{11,1}(Q_{11,1})^2 - R_{11,2}(Q_{11,2})^2 := -D_{RC} \leq 0,$$

for all $t \in (0,T)$. (49)

4. Energy-based operator splitting approach

The above examples showed that resistive Stokes-circuit connections contribute to the energy dissipation of the fully coupled system, namely $G_{RC} = -D_{RC}$, with $D_{RC}(t) \geq 0$ for all $t$. Thus, the energy identity (28) holds.

In particular, if all forcing terms are zero, namely $F_\Omega(t) = F_\Upsilon(t) = 0$ for $t \geq 0$, and the circuit properties are such that $\mathcal{U}_\Upsilon(t) \geq 0$ for $t \geq 0$, then from (28) we obtain that $\mathcal{E} = \mathcal{E}_\Omega + \mathcal{E}_\Upsilon$ is a decreasing function of time, namely

$$\mathcal{E}(t) \leq \mathcal{E}(0) \quad \forall t \geq 0. \quad (50)$$

This essential mathematical and physical property must be preserved at the discrete level, and this provides the main rationale for our splitting scheme. We begin by adopting an operator splitting technique, see e.g. [17, Chap. II], to perform a semi-discretization in time to solve sequentially in separate substeps the PDE systems associated with the Stokes regions and the ODE systems associated with the lumped hydraulic circuits. The most important feature of our scheme is that the substeps are designed so that the energy at the semi-discrete level mirrors the behavior of the energy of the fully coupled system, thereby providing unconditional stability to the proposed splitting method via an upper bound in the norms of the solution similar to that provided by (50). The version of the method detailed below yields, at most, a first-order accuracy in time, since it includes only two substeps; however, the scheme can be generalized to attain second-order accuracy using symmetrization techniques [17, Chap. VI].

**First-order splitting algorithm.** For the sake of simplicity, let $\Delta t$ denote a fixed time step, let $t^n = n\Delta t$ and let $\varphi^n = \varphi(t^n)$ for any general expression $\varphi$. 

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Let $v_l^0 = v_{l,0}$ for all $l \in L$ and $y_m^0 = y_{m,0}$ for all $m \in M$. Then, for any $n \geq 0$ solve

**Step 1** For each $l \in L$, $m \in M$ and $k = 1, \ldots, j_{l}, \tau_{m}$, given $v_l^n$ and $y_m^n$, find $v_l$ and $y_m$ such that

\[
\nabla \cdot v_l = 0 \quad \text{in } \Omega_l \times (t^n, t^{n+1}), \quad (51)
\]
\[
\rho \frac{\partial v_l}{\partial t} = -\nabla p_l + \mu \Delta v_l + \rho f_l \quad \text{in } \Omega_l \times (t^n, t^{n+1}), \quad (52)
\]
\[
\frac{dy_m}{dt} = b_m(Q_{lm,k}, P_{lm,k}, t) \quad \text{in } (t^n, t^{n+1}), \quad (53)
\]

with the initial conditions

\[
v_l(x, t^n) = v_l^n(x) \quad \text{in } \Omega_l, \quad (54)
\]
\[
y_m(t^n) = y_m^n, \quad (55)
\]

and the boundary conditions

\[
v_l = 0 \quad \text{on } \Gamma_l \times (t^n, t^{n+1}), \quad (56)
\]
\[
\left( -p_l I + \mu \nabla v_l \right) n_l = -\bar{p}_l n_l \quad \text{on } \Sigma_l \times (t^n, t^{n+1}), \quad (57)
\]
\[
\left( -p_l I + \mu \nabla v_l \right) n_{lm,k} = -P_{lm,k} n_{lm,k} \quad \text{on } S_{lm,k} \times (t^n, t^{n+1}), \quad (58)
\]

with

\[
\int_{S_{lm,k}} v_l(x, t) \cdot n_{lm,k}(x, t) dS_{lm,k} = Q_{lm,k}(t) \quad \text{in } (t^n, t^{n+1}), \quad (59)
\]

and then set

\[
v_l^{n+\frac{1}{2}} = v_l(x, t^{n+1}), \quad p_l^{n+1} = p_l(x, t^{n+1}) \quad \text{and} \quad y_m^{n+\frac{1}{2}} = y_m(t^{n+1}). \quad (60)
\]

**Step 2** For each $l \in L$, $m \in M$ and $k = 1, \ldots, j_{l}, \tau_{m}$, given $v_l^{n+\frac{1}{2}}$ and $y_m^{n+\frac{1}{2}}$, find $v_l$ and $y_m$ such that

\[
\rho \frac{\partial v_l}{\partial t} = 0 \quad \text{in } \Omega_l \times (t^n, t^{n+1}), \quad (61)
\]
\[
\frac{dy_m}{dt} = A_m(y_m, t) y_m + s_m(y_m, t) \quad \text{in } (t^n, t^{n+1}), \quad (62)
\]

with
with the initial conditions

\[ v_l(x, t^n) = v_l^{n+\frac{1}{2}}(x) \quad \text{in } \Omega_l, \quad (63) \]

\[ y_m(t^n) = y_m^{n+\frac{1}{2}}, \quad (64) \]

and set

\[ v_l^{n+1} = v_l(x, t^{n+1}) \quad \text{and} \quad y_m^{n+1} = y_m(t^{n+1}). \quad (65) \]

**Remark 2.** In the particular case of resistive connections described in Section 3 we have that (53) reduces to

\[ \frac{d\pi_{lm,k}}{dt} = \frac{Q_{lm,k}}{C_{lm,k}} \quad \text{and} \quad \frac{d\omega_m}{dt} = 0, \quad \text{for } l \in \mathcal{L}, \ m \in \mathcal{M}_l, \ k = 1, \ldots, \mathcal{J}_l, \mathcal{J}_m. \]

Some of the main features of the proposed algorithm are the following:

1. The solution at time \( t^{n+1} \) is obtained from the solution at time \( t^n \) after solving sequentially Step 1 and Step 2, without the need of subiterations between the two steps. The scheme is stable even without subiterations because each substep satisfies an energy identity similar to that of the fully coupled problem, as discussed later in this Section.

2. Pressure and flow rate at the coupling interfaces, namely \( P_{lm,k} \) and \( Q_{lm,k} \), are solved for simultaneously, thereby implicitly, even though the algorithm is partitioned. This is not the case in other splitting schemes, see for example Quarteroni et al [12], where pressures computed from the lumped circuits are used as inputs for the Stokes problem and flow rates computed in the Stokes problem are used as inputs for the lumped circuits.

3. Steps 1 and 2 communicate via the initial conditions. In particular, the state variables \( y_m \), with \( m \in \mathcal{M} \), are updated in Step 1 and their value at \( t^{n+1} \) provides the initial conditions for Step 2.

4. Steps 1 and 2 are defined on the discrete time interval \( (t^n, t^{n+1}) \), but the differential operators have yet to be fully discretized in time and space. Even though specific choices will have to be made for the time and space discretization of Steps 1 and 2, the overall splitting scheme described above
is independent of these choices, keeping in mind that first-order convergence in time can be achieved only if all the substeps are solved with numerical methods that are at least first-order in time.

5. By treating a subset of the ODE systems jointly with the Stokes problems in Step 1, see Equations (58) and (59), the interface conditions can be dealt with implicitly and ensure proper energy balance at the multiscale interfaces. On the other hand, by treating the main part of the ODE systems in Step 2, the splitting algorithm provides the flexibility to: (i) modify only one step in the scheme should the modeling application require the integration of new ODE models; and (ii) adopt a numerical scheme that best addresses potential nonlinearities affecting the ODE system.

6. Equation (61) effectively means that the velocity vector fields $v_l$, for $l \in \mathcal{L}$, are not updated in Step 2, thereby implying that $v_l^{n+1} = v_l^{n+1/2}$.

Stability analysis for the first-order splitting algorithm. The stability analysis will be performed on a simplified problem that preserves the main difficulties associated with the PDE/ODE coupling considered in this work.

**Theorem 1.** Consider the Stokes-circuit coupled problem described in Section 2, in the case of resistive connections considered in Section 3. Under the assumptions that

(i) the circuit properties are such that the tensor $A_m$ is constant and $B_m$ is positive definite;

(ii) there are no forcing terms, namely $\mathcal{F}_\Omega(t) = \mathcal{F}_\Gamma(t) = 0$ for all $t$;

the first-order splitting algorithm (51)-(65) is unconditionally stable.

**Proof.** Let $\Delta t = t^{n+1} - t^n$ and let us begin by considering Step 1. Under assumptions (i)-(ii), using an implicit Euler scheme for the time-discretization and following a similar procedure to that detailed in Section 3, we can obtain
the following energy identity at the time-discrete level for Step 1

\[
\frac{1}{\Delta t} \mathcal{E}_I^{n+\frac{1}{2}} + \mathcal{D}_{\Omega,I}^{n+\frac{1}{2}} + \mathcal{D}_{RC,I}^{n+\frac{1}{2}} = \\
\frac{1}{\Delta t} \left( \sum_{l \in \mathcal{L}} \rho \int_{\Omega_l} v^n_l \cdot v^{n+\frac{1}{2}}_l \, d\Omega_l + \sum_{m \in \mathcal{M}} (y^n_m)^T \frac{U_m}{\sqrt{\frac{1}{2}}} y^{n+\frac{1}{2}}_m \right), 
\]

(66)

where \(\mathcal{E}_I^{n+\frac{1}{2}} = \mathcal{E}_{\Omega,I}^{n+\frac{1}{2}} + \mathcal{E}_{\mathcal{T},I}^{n+\frac{1}{2}}\) and

\[
\mathcal{E}_{\Omega,I}^{n+\frac{1}{2}} = \sum_{l \in \mathcal{L}} \left( \rho \|v^n_l\|_{L^2(\Omega_l)} + \|v^{n+\frac{1}{2}}_l\|_{L^2(\Omega_l)} \right), \quad \mathcal{E}_{\mathcal{T},I}^{n+\frac{1}{2}} = \sum_{m \in \mathcal{M}} \|U_{1/2} \frac{y^n_m}{\sqrt{\frac{1}{2}}} |_{L^2(\Omega)} \|^2,
\]

\[
\mathcal{D}_{\Omega,I}^{n+\frac{1}{2}} = \sum_{l \in \mathcal{L}} \mu \|\nabla v^{n+\frac{1}{2}}_l\|_{L^2(\Omega_l)}, \quad \mathcal{D}_{RC,I}^{n+\frac{1}{2}} = \sum_{m \in \mathcal{M}} \sum_{l \in \mathcal{L}} R_{lm,k} \left(Q_k^{n+\frac{1}{2}} \right)^2.
\]

(67)

(68)

Using Young’s inequality on the first term on the right hand side of (66), yields

\[
\sum_{l \in \mathcal{L}} \rho \int_{\Omega_l} v^n_l \cdot v^{n+\frac{1}{2}}_l \, d\Omega_l \leq \sum_{l \in \mathcal{L}} \frac{\rho}{2} \left( \|v^n_l\|_{L^2(\Omega_l)} + \|v^{n+\frac{1}{2}}_l\|_{L^2(\Omega_l)} \right) = \frac{1}{2} \mathcal{E}_{\Omega,I}^{n+\frac{1}{2}} + \frac{1}{2} \mathcal{E}_{\mathcal{T},I}^{n+\frac{1}{2}}.
\]

(69)

To estimate the second term on the right hand side of (66), we first use Cauchy-Schwarz inequality and then Young’s inequality to obtain

\[
\sum_{m \in \mathcal{M}} (y^n_m)^T \frac{U_m}{\sqrt{\frac{1}{2}}} y^{n+\frac{1}{2}}_m \leq \sum_{m \in \mathcal{M}} \left( (y^n_m)^T \frac{U_m}{\sqrt{\frac{1}{2}}} y^n_m \right) \left( (y^{n+\frac{1}{2}}_m)^T \frac{U_m}{\sqrt{\frac{1}{2}}} y^{n+\frac{1}{2}}_m \right) = \frac{1}{2} \mathcal{E}_{\mathcal{T},I}^{n+\frac{1}{2}} + \frac{1}{2} \mathcal{E}_{\mathcal{T},I}^{n+\frac{1}{2}}.
\]

(70)

Note that, Eq. (70) holds since \(\frac{U_m}{\sqrt{\frac{1}{2}}}\) is symmetric and positive definite. Combining (66), (69) and (70), we obtain

\[
\frac{1}{2\Delta t} \mathcal{E}_I^{n+\frac{1}{2}} + \mathcal{D}_{\Omega,I}^{n+\frac{1}{2}} + \mathcal{D}_{RC,I}^{n+\frac{1}{2}} \leq \frac{1}{2\Delta t} \mathcal{E}_I^n,
\]

(71)

for which it follows that \(\mathcal{E}_I^{n+\frac{1}{2}} - \mathcal{E}_I^n \leq -2\Delta t \left( \mathcal{D}_{\Omega,I}^{n+\frac{1}{2}} + \mathcal{D}_{RC,I}^{n+\frac{1}{2}} \right) \leq 0\), and, finally, we can conclude that

\[
\mathcal{E}_I^{n+\frac{1}{2}} \leq \mathcal{E}_I^n.
\]

(72)
Let us now consider the energy identity for Step 2. Using an implicit Euler scheme for the time-discretization, we can obtain the following energy identity at the time-discrete level for Step 2

$$\frac{1}{\Delta t} \mathcal{E}_{II}^{n+1} + \mathcal{U}_{II}^{n+1} = \frac{1}{\Delta t} \left( \sum_{l \in L} \rho \int_{\Omega_l} v_l^{n+\frac{1}{2}} \cdot v_l^{n+1} \, d\Omega_l + \sum_{m \in M} \left( y_m^{n+\frac{1}{2}} \right)^T \mathbf{U}_m v_m^{n+1} \right),$$

(73)

where $\mathcal{E}_{II}^{n+1} = \mathcal{E}_{\Omega,II}^{n+1} + \mathcal{E}_{\Upsilon,II}^{n+1}$ and

$$\mathcal{E}_{\Omega,II}^{n+1} = \sum_{l \in L} \rho \| v_l^{n+1} \|_{L^2(\Omega_l)}^2, \quad \mathcal{E}_{\Upsilon,II}^{n+1} = \sum_{m \in M} \| U_m^{1/2} y_m^{n+1} \|_{L^2}^2,$$

(74)

$$\mathcal{U}_{II}^{n+1} = \sum_{m \in M} (y_m^{n+1})^T \mathbf{B}_m y_m^{n+1}.$$  

(75)

Now, to estimate the right hand side of (73), we use the same procedure utilized for Step 1 in (69)-(70), and we obtain the following inequality

$$\frac{1}{2\Delta t} \mathcal{E}_{II}^{n+1} + \mathcal{U}_{II}^{n+1} \leq \frac{1}{2\Delta t} \left( \sum_{l \in L} \rho \| v_l^{n+\frac{1}{2}} \|_{L^2(\Omega_l)}^2 + \sum_{m \in M} \left( y_m^{n+\frac{1}{2}} \right)^T \mathbf{U}_m v_m^{n+\frac{1}{2}} \right) = \frac{1}{2\Delta t} \mathcal{E}_{II}^{n+\frac{1}{2}}.$$  

(76)

We remark that assumption (i) guarantees that $\mathcal{U}_{II}^{n+1} \geq 0$. Thus (76) implies $\mathcal{E}_{II}^{n+1} - \mathcal{E}_{II}^{n+\frac{1}{2}} \leq -2\Delta t \mathcal{U}_{II}^{n+1} \leq 0$, which leads to

$$\mathcal{E}_{II}^{n+1} \leq \mathcal{E}_{II}^{n+\frac{1}{2}}.$$  

(77)

Thanks to the fact that the initial conditions for Step 2 coincide with the final solution of Step 1 as stated in (60), it follows that $\mathcal{E}_{II}^{n+\frac{1}{2}} = \mathcal{E}_{I}^{n+\frac{1}{2}}$. Thus, combining (72) and (77), we obtain the following inequality

$$\mathcal{E}_{II}^{n+1} \leq \mathcal{E}_{II}^{n+\frac{1}{2}} = \mathcal{E}_{I}^{n+\frac{1}{2}} \leq \mathcal{E}_{I}^{n} \quad \text{for } n \geq 1,$$

(78)

which provides an upper bound for the norm of the solution regardless of the time step size, thereby ensuring unconditional stability of the algorithm.

\[\square\]

Remark 3. Unconditional stability is a direct consequence of treating implicitly in Step 1 the contributions from the $j_{\kappa,m}$ Stokes-circuit connections, represented...
by \( b_m(Q_{l,m}, P_{l,m}, t) \), see (53). This splitting choice allows us to evaluate at the same time instant all the quantities in \( \mathcal{G}_{RC} \), ensuring that \( \mathcal{G}_{RC} \) can be expressed as a dissipation \( D_{RC} \) even at the discrete level. As a consequence, the proposed splitting algorithm does not introduce uncontrolled artificial terms in the energy \([10, 14]\), ensuring numerical stability with respect to the time step.

5. Numerical results

In this section we evaluate the performance of the proposed splitting method by comparing the numerical solutions of the three illustrative examples constructed in Section 3 with their exact solutions reported in the Appendix. In particular, we assess the convergence properties of the method for different choices of time step and we show that the expected first-order convergence in time is actually achieved. Table 3 lists the parameter values utilized in each example, whose order of magnitudes as similar to those arising in blood flow applications and are reported in the cgs system for ease of reference. A common feature of the exact solutions for the three examples considered here is their periodicity in time with given period \( \tau = 2\pi/\omega \). The global time step \( \Delta t \) is determined by the number of intervals in each time period, denoted \( N_\tau \), according to the formula \( \Delta t = \tau/N_\tau \). Next, each substep is discretized with different time steps, denoted \( \Delta t_1 \) and \( \Delta t_2 \), respectively, using an implicit Euler scheme. In order to check that the computed numerical solution is periodic of period \( \tau \), we introduce the superscript \( \text{per} \) to denote the computed solution over each time interval \( (n-1)\tau, n\tau \), for \( n \geq 1 \), so that, for a general expression \( \varphi \), we define its discretization \( \varphi^{[\text{per}]} \) over one period of time as

\[
\varphi^{[\text{per}]} = \left[ \varphi^{(\text{per} - 1)N_\tau}, \varphi^{(\text{per} - 1)N_\tau + 1}, \ldots, \varphi^{\text{per}N_\tau} \right]^T \quad \text{for } \text{per} \geq 1. \tag{79}
\]

Finally, we use the following criterion:

\[
\max_{m \in M} \left\{ \frac{\|\varphi_1^{[\text{per}]} - \varphi_1^{[\text{per} - 1]}\|_{L^2(\Omega)}}{\|\varphi_1^{[\text{per} - 1]}\|_{L^2(\Omega)}}, \frac{\|\varphi_2^{[\text{per}]} - \varphi_2^{[\text{per} - 1]}\|_{L^2(\Omega)}}{\|\varphi_2^{[\text{per} - 1]}\|_{L^2(\Omega)}}, \frac{\|\varphi_3^{[\text{per}]} - \varphi_3^{[\text{per} - 1]}\|_{L^2(\Omega)}}{\|\varphi_3^{[\text{per} - 1]}\|_{L^2(\Omega)}} \right\} < \varepsilon_{\text{per}}, \tag{80}
\]
to identify the numerical quantities to be compared with the exact solution over one time period. The results reported in this section have been obtained for \( \varepsilon_{\text{per}} = 10^{-6} \). Once numerical periodicity is reached, results are extracted and the following normalized errors \cite{ref30} Sec. 15.8 are computed

\[
\text{Err}_v = \left( \frac{1}{2} \left( \Delta t \sum_{n=(\text{per}-1)\times N_{\tau}}^{\text{per} \times N_{\tau}} \sum_{l \in L} \| v^n_l - v_{l,ex}(t^n) \|_{L^2(\Omega_l)}^2 \right)^{1/2}, \tag{81}
\]

\[
\text{Err}_p = \left( \frac{1}{2} \left( \Delta t \sum_{n=(\text{per}-1)\times N_{\tau}}^{\text{per} \times N_{\tau}} \sum_{l \in L} \| p^n_l - p_{l,ex}(t^n) \|_{L^2(\Omega_l)}^2 \right)^{1/2}, \tag{82}
\]

\[
\text{Err}_y = \left( \frac{1}{2} \left( \Delta t \sum_{n=(\text{per}-1)\times N_{\tau}}^{\text{per} \times N_{\tau}} \sum_{m \in M} \| U^{1/2}_m(t^n) y^n_m - U^{1/2}_{m,ex}(t^n) y_{m,ex}(t^n) \|_{L^2(\Omega)}^2 \right)^{1/2}, \tag{83}
\]

where the subscript \( \text{ex} \) denotes the exact solution. We remark that the norms appearing in the definitions for the errors are dictated by the norms in the energy functionals \cite{ref20} associated with the Stokes domains and the circuits, respectively, since the energy estimates provide the main rationale behind the design of the numerical algorithm.

Numerical results are obtained in the case \( d = 2 \), since the aim of the present work is to set the basis of a methodological approach for the time-discretization of geometrical multiscale fluid flow models. However, the method applies also to the case \( d = 3 \) in a straightforward manner. The spatial discretization of the Stokes problem is handled via a triangular uniform mesh of 4000 elements for each domain \( \Omega_l \) and an inf-sup stable finite element pair, namely (Taylor-Hood) \( P_2/P_1 \) elements \cite{ref30} Chap. 15. The comparison between numerical approximations and exact solutions is performed over one time period, once periodicity has been reached, for three different global time steps, namely \( \Delta t = 0.01, 0.005, 0.001 \). Results are obtained with the time subsets \( \Delta t_1 = \Delta t \) and \( \Delta t_2 = \Delta t/s \), with \( s = 5 \) for Example 1 and \( s = 10 \) for Examples 2 and 3. The computational framework relies on the finite element library Freefem \cite{ref31}.

**Example 1.** The results presented here refer to the case of nonlinear vari-
able resistance $R_a$ and capacitance $C_a$, whose analytical expression is reported in the Appendix. We also tested the method in the case of constant $R_a$ and $C_a$ against the exact solution and obtained similar performances as in the nonlinear case. Figure 7 displays a comparison between the exact solution and different numerical approximations of physical quantities of interest, namely pressure $P_{11,1}$ and flow rate $Q_{11,1}$ at the Stokes-circuit interface $S_{11,1}$ (upper panel) and pressure $\pi_{11,1}$ and volume $\omega_{11}$ in the 0D circuit (lower panel). We remark that the nonlinearities in $R_a$ and $C_a$ have been treated explicitly, which means that $R_a$ and $C_a$ are evaluated at the previous time step. A good agreement is ob-

![Interface quantities](image1)

![0D unknowns](image2)

Figure 7: Example 1. Comparison between the exact solution and the corresponding numerical approximation for interface quantities and 0D unknowns, for three time steps $\Delta t = 0.01, 0.005, 0.001$, over one time period once the numerical periodicity has been reached.

tained for $P_{11,1}$, $\pi_{11,1}$ and $\omega_{11}$ for $\Delta t = 0.01$, and the error decreases as $\Delta t$ is reduced. The numerical approximation of $Q_{11,1}$ captures the periodicity of the solution even for $\Delta t = 0.01$, but the peaks are lower than those exhibited by
the exact solution. As $\Delta t$ is the reduced, the computed $Q_{11,1}$ approaches the exact peaks of the interface flow rate, thereby capturing the full dynamics of the problem. Figure 7 also shows that the numerical solution is not affected by spurious oscillations or instabilities, even for the largest time step. These findings confirm that the choice of the time step affects the accuracy of the computed solution but not the stability of the numerical scheme, thereby supporting the unconditional stability result proved in Theorem 1.

Example 2. Figure 8 displays a comparison between the exact solution and pressures and flow rates at the Stokes-circuit interfaces, namely $P_{11,1}$ and $Q_{11,1}$ at the interface $S_{11,1}$ and $P_{21,1}$ and $Q_{21,1}$ at the interface $S_{21,1}$ (upper panel) and the state variables $\pi_{11,1}$, $\pi_{21,1}$ and $\omega_{11}$ pertaining to the 0D circuit $\Upsilon_1$ (lower panel). Similarly to Example 1, the comparison shows very good agreement in the pressures $P_{11,1}$, $P_{21,1}$, $\pi_{11,1}$ and $\pi_{21,1}$, and a satisfactory approximation in the flow rates $Q_{11,1}$, $Q_{21,1}$ and $\omega_{11}$, especially when the time step is small.

Example 3. Figure 9 displays a comparison between the exact solution and pressures and flow rates at the Stokes-circuit interfaces, namely $P_{11,1}$ and $Q_{11,1}$ at the interface $S_{11,1}$ and $P_{11,2}$ and $Q_{11,2}$ at the interface $S_{11,2}$ (upper panel) and the state variables $\pi_{11,1}$, $\pi_{11,2}$ and $\omega_{11}$ pertaining to the 0D circuit $\Upsilon_1$ (lower panel). Similarly to Examples 1 and 2, the comparison shows very good agreement in the pressures, both for the interface pressures $P_{11,1}$ and $P_{11,2}$ and for the nodal pressures $\pi_{11,1}$ and $\pi_{11,2}$. The approximation of the interface flow rates $Q_{11,1}$ and $Q_{11,2}$ improves as $\Delta t$ decreases, capturing periodicity and peaks. Conversely, the numerical approximation of the flow rate $\omega_{11}$ shows a very good agreement even for $\Delta t = 0.01$. Note that, in contrast with Examples 1 and 2, the Stokes problem in $\Omega_1$ does not include any external forcing term. Thus, the fully coupled problem is driven only by the time evolution of the pressures $\tilde{p}_a$ and $\tilde{p}_b$ imposed by the voltage generators within the 0D circuit.

Order of convergence in time. The time-discretization scheme presented in this article is based on a first order operator splitting technique, which is known to be first-order accurate in time. We performed a standard time refinement
Figure 8: Example 2. Comparison between the exact solution and the corresponding numerical approximation for interface quantities and 0D unknowns, for three time steps $\Delta t = 0.01, 0.005, 0.001$, over one period once the periodicity is reached.
Figure 9: Example 5. Comparison between the exact solution and the corresponding numerical approximation for interface quantities and 0D unknowns, for three time steps \( \Delta t = 0.01, 0.005, 0.001 \), over one time period once the numerical periodicity has been reached.
study for Example 1, Example 2 and Example 3, using the errors defined in (81)-(83), that confirmed this property, as shown in Figure 10.

Figure 10: Plot of the energy norm errors defined in (81)-(83) in logarithmic scale as a function of the global time step $\Delta t = 0.01, 0.005, 0.001$ for the three examples considered.

6. Conclusions and Future Perspectives

This work presents a novel energy-based operator splitting approach for the numerical solution of multiscale problems involving the coupling between Stokes equations and ODE systems. Unconditional stability with respect to the time step choice is ensured by the implicit treatment of interface conditions within the Stokes substeps, whereas the coupling between Stokes and ODE substeps is enforced via appropriate initial conditions for each substep. Notably, the splitting design has been driven by the rationale of ensuring that the physical energy balance is maintained at the discrete level and, as a consequence, unconditional stability is attained without the need of subiterating between substeps. Results were presented in the case of 2d-0d coupling, for the sake of simplicity. However, the 3d-0d case does not present any additional conceptual issue from the splitting viewpoint. Upon comparison with exact solutions, numerical results show a better agreement for pressures than for flow rates at the Stokes-circuit interfaces. This might be a consequence of the simple finite element approach adopted here, where pressures are primal variables but flow rates are not.
In this work, the splitting algorithm is presented in its simplest version, which is at most first-order accurate in time. It has the advantages of easy implementation, low computational cost and good stability and robustness properties. In addition, the framework presented here could serve as a basis for extensions to (i) higher order time-discretization methods via, for example, symmetrization [17] or time-extrapolation [21] [22]; (ii) different PDE systems in each $\Omega_i$ region deriving, for example, from Navier-Stokes equations [18], fluid-structure interactions [19] [32], non-Newtonian fluid flows [33] or porous media flows [34] [35]; and (iii) different spatial discretization methods for each PDE problem including, for example, higher order finite element methods [36] or hybridizable discontinuous Galerkin methods [34] [37] [38]. The extensions mentioned above are relatively straight-forward, since the algorithms described in the corresponding references are directly implementable on that presented in this work. On the other hand, extensions to more general coupling architectures require further investigation, possibly involving defective interface conditions, as discussed in [39], or special numerical strategies depending on the physics of the multiscale connections, as discussed in [15].

The case involving the Navier-Stokes equations deserves particular mention for its relevance to the multiscale modeling of blood flow. In this case, the mathematical problem to be solved in Step 1 would also include nonlinearities due to advection which, if not treated properly, might disrupt the physical energy balance at the multiscale interface [20]. Following the approach described in [17] and implemented in [18] [19], we would again adopt the operator splitting viewpoint and modify the overall algorithm by introducing an additional step where the advection is treated separately using energy-preserving schemes, such as the wave-like method described in [17]. By doing so, the physical energy balance is maintained from the continuous to the discrete level also in the case of the Navier-Stokes equations, thereby providing stability for the overall algorithm.
Appendix: Exact solutions of numerical examples

Example 1. A direct computation shows that the coupled problem \([1]-[10]\) with \([29]-[31]\) admits the exact solution

\[
v_1(x_1, x_2, t) = [s(t) \gamma(x_2), 0]^T, \quad p_1(x_1, x_2, t) = s(t) \mathcal{P}(x_1),
\]

(84)

\[
\pi_{11,1}(t) = P_{11,1}(t) - R_{11,1}Q_{11,1}(t),
\]

(85)

\[
\omega_{11}(t) = \frac{1}{2\gamma_1} \left\{ -1 \right.
\]

\[
+ \sqrt{1 + 4\gamma_1 C_a \left[ \pi_{11,1}(t) - R_a(y_1, t) \left( Q_{11,1}(t) - C_{11,1} \frac{d\pi_{11,1}}{dt}(t) \right) \right]} \right\},
\]

(86)

where

\[
s(t) = s_0 + s_1 \sin(\omega t), \quad \gamma(x_2) = V_0 \cos^2 \left( \frac{\pi x_2}{H} \right),
\]

(87)

\[
\mathcal{P}(x_1) = a_0 + a_1 \exp(-k x_1), \quad P_{11,1}(t) = s(t) \mathcal{P}(L),
\]

(88)

\[
Q_{11,1}(t) = \frac{V_0 H}{2} s(t), \quad R_a(y_1, t) = \overline{R_a} + \frac{\alpha_0}{1 + \alpha_1 e^{-\alpha_2 \pi_{11,1}(t)}},
\]

(89)

\[
C_a(y_1, t) = \frac{\overline{C_a}}{1 + \gamma_1 \omega_{11}(t)},
\]

(90)

provided that the forcing terms are given by

\[
f_1(x_1, x_2, t) = \left[ \frac{ds}{dt}(t) \gamma(x_2) - \frac{\mu}{\rho} s(t) \frac{d^2 \gamma}{dx_2^2}(x_2) + \frac{s(t)}{\rho} \frac{d\mathcal{P}}{dx_1}(x_1), 0 \right]^T,
\]

(91)

\[
\overline{p}_1(t) = s(t) \mathcal{P}(0),
\]

(92)

\[
\overline{\rho}(t) = R_b \frac{d \omega_{11}}{dt}(t) - \frac{R_b}{R_a(y_1, t)} \pi_{11,1}(t) + \frac{R_b}{C_a(y_1, t)} \left( \frac{1}{R_a(y_1, t)} + \frac{1}{R_b} \right) \omega_{11}(t),
\]

(93)

and the initial conditions are equal to the exact solution evaluated at \(t = 0\).

It may be useful to notice that the expression for the volume \(\omega_{11}\) in equation \(86\) results from enforcing the volume-pressure relationship

\[
\omega_{11}(t) = C_a(y_1, t) \left[ \pi_{11,1}(t) - R_a(y_1, t) \left( Q_{11,1}(t) - C_{11,1} \frac{d\pi_{11,1}}{dt}(t) \right) \right],
\]

(94)

where \(C_a\) is given by \(90\). All the parameter values involved in the exact solution are listed in Table 3. Since this example is set in a 2D context, flow rates should
be interpreted per unit of length and the units of resistance and capacitance should scale accordingly. We remark that the case of constant resistance and capacitance, namely $R_a(y_1, t) = \overline{R}_a$ and $C_a(y_1, t) = \overline{C}_a$, corresponds to the choice of $\alpha_0 = 0$ and $\gamma_1 = 0$. In this case though, the expressions for $\omega_{11}(t)$ in (86) simplifies to

$$\omega_{11}(t) = \overline{C}_a \left[ \pi_{11,1}(t) - \overline{R}_a \left( Q_{11,1}(t) - C_{11,1} \frac{d\pi_{11,1}}{dt}(t) \right) \right].$$  \hspace{1cm} (95)

**Example 2.** Using a similar approach to the one developed for Example 1, it can be shown that the coupled problem (1)-(10) with (36)-(38) admits the following exact solution

$$v_l(x_1, x_2, t) = [s_l(t) \mathcal{Y}_l(x_2), 0]^T, \quad p_l(x_1, x_2, t) = s_l(t) \mathcal{P}_l(x_1) \quad l = 1, 2, \hspace{1cm} (96)$$

$$\pi_{11,1}(t) = P_{11,1}(t) - R_{11,1} Q_{11,1}(t), \hspace{1cm} (97)$$

$$\pi_{21,1}(t) = \pi_{11,1}(t) - R_a \omega_{11}(t) - L_a \frac{d\omega_{11}}{dt}(t), \hspace{1cm} (98)$$

$$\omega_{11}(t) = Q_{11,1}(t) - C_{11,1} \frac{d\pi_{11,1}}{dt}(t), \hspace{1cm} (99)$$

where

$$\mathcal{Y}_l(x_2) = V_0 \cos \left( \frac{\pi x_2}{H} \right), \quad \mathcal{P}_l(x_1) = a_{0l} + a_{1l} \exp(-k x_1), \hspace{1cm} (100)$$

$$P_{11,1}(t) = s_1(t) \mathcal{P}_l(L) \quad l = 1, 2, \hspace{1cm} (101)$$

$$s_1(t) = s_0 + s_1 \sin(\omega t), \quad s_2(t) = \frac{1}{a_{02} + a_{12} + R_{21,1} V_0 \pi_{21,1}(t)}, \hspace{1cm} (102)$$

$$Q_{11,1}(t) = \frac{V_0 H}{2} s_1(t), \quad Q_{21,1}(t) = -\frac{V_0 H}{2} s_2(t), \hspace{1cm} (103)$$

provided that the forcing terms are given by

$$f_l(x_1, x_2, t) = \left[ \frac{ds_l}{dt}(t) \mathcal{Y}_l(x_2) - \frac{\mu}{\rho} s_l(t) \frac{d^2 \mathcal{Y}_l}{d x_2^2}(x_2) + \frac{s_l(t)}{\rho} \frac{d\mathcal{P}_l}{d x_1}(x_1), 0 \right]^T, \hspace{1cm} (104)$$

$$\overline{p}_l(t) = s_l(t) \mathcal{P}_l(0), \hspace{1cm} (105)$$

$$\overline{\rho}(t) = -R_b \omega_{11}(t) + \pi_{21,1}(t) - R_b Q_{21,1}(t) + R_b C_{21,1} \frac{d\pi_{21,1}}{dt}(t), \hspace{1cm} (106)$$

for $l = 1, 2$, and the initial conditions are equal to the exact solution evaluated at $t = 0$. 

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All the parameter values involved in the exact solution are listed in Table 3 with the same convention on the units used in the previous examples.

**Example 3.** Proceeding as in the previous examples, it can be shown that the coupled problem (1)-(10) with (43)-(45) admits the exact solution

\[
v_1(x_1, x_2, t) = [s(t) \mathcal{V}(x_2), 0]^T, \quad p_1(x_1, x_2, t) = s(t) \mathcal{P}(x_1),
\]

\[
\pi_{11,1}(t) = P_{11,1}(t) - R_{11,1}Q_{11,1}(t), \quad \pi_{11,2}(t) = P_{11,2}(t) - R_{11,2}Q_{11,2}(t),
\]

\[
\omega_{11}(t) = \Lambda_1 + \Lambda_2 \sin(\omega t) + \Lambda_3 \cos(\omega t),
\]

where

\[
s(t) = s_0 + s_1 \sin(\omega t), \quad \mathcal{V}(x_2) = V_0 \cos^2 \left( \frac{\pi x_2}{H} \right),
\]

\[
\mathcal{P}(x_1) = a_0 + a_1 \exp(-kx_1), \quad P_{11,1}(t) = s(t) \mathcal{P}(L),
\]

\[
P_{11,2}(t) = s(t) \mathcal{P}(0), \quad Q_{11,1}(t) = -Q_{11,2}(t) = \frac{V_0 H}{2} s(t),
\]

\[
\Lambda_1 = s_0 \mathcal{G}, \quad \Lambda_2 = s_1 \mathcal{G} \left[ \left( \frac{\omega L_c}{R_c} \right)^2 + 1 \right]^{-1},
\]

\[
\Lambda_3 = -\frac{\omega \Lambda_2 L_c}{R_c}, \quad \mathcal{G} = \frac{\mathcal{P}(L) - \mathcal{P}(0) - (R_{11,1} + R_{11,2}) \frac{V_0 H}{2}}{R_c},
\]

provided that the forcing terms are \(f_1(x_1, x_2, t)\) as in (91) and

\[
\tilde{p}_a(t) = R_a C_{11,1} \frac{d\pi_{11,1}}{dt}(t) + \pi_{11,1}(t) + (\omega_{11}(t) - Q_{11,1}(t)) R_a,
\]

\[
\tilde{p}_b(t) = R_b C_{11,2} \frac{d\pi_{11,2}}{dt}(t) + \pi_{11,2}(t) - (\omega_{11}(t) + Q_{11,2}(t)) R_b,
\]

and the initial conditions are equal to the exact solution evaluated at time \(t = 0\).

Note that in (109), for simplicity, we consider only the particular solution of the ODE corresponding to \(\omega_{11}\), setting to zero the coefficient corresponding to the homogeneous solution. All the parameter values involved in the exact solution are listed in Table 3 with the same convention on the units used in the previous examples.
| Param. | Value       | Param. | Value       | Param. | Value       |
|--------|-------------|--------|-------------|--------|-------------|
| $H$    | 2 cm        | $L$    | 10 cm       | $\rho$ | 1 g cm$^{-3}$ |
| $\mu$  | 1 g cm$^{-1}$ s$^{-1}$ | $V_0$ | 2 cm s$^{-1}$ | $\omega$ | $\pi$ s$^{-1}$ |
| $k$    | 0.1 cm$^{-1}$ | $s_0$ | 2           | $s_1$ | 1           |

**Example 1 parameters**

| Param. | Value       | Param. | Value       | Param. | Value       |
|--------|-------------|--------|-------------|--------|-------------|
| $R_{11,1}$ | 10 g cm$^{-3}$ s$^{-1}$ | $\bar{R}_a$ | 10 g cm$^{-3}$ s$^{-1}$ | $R_0$ | 10 g cm$^{-3}$ s$^{-1}$ |
| $\alpha_0$ | 10 g cm$^{-3}$ s$^{-1}$ | $\alpha_1$ | 1           | $\alpha_2$ | 0.001 g cm$^{-1}$ s$^{-2}$ |
| $C_{11,1}$ | 0.001 g$^{-1}$ cm$^3$ s$^2$ | $\bar{C}_a$ | 0.01 g$^{-1}$ cm$^3$ s$^2$ | $\gamma_1$ | 1 cm$^{-2}$ |
| $a_0$ | 150 g cm$^{-1}$ s$^{-2}$ | $a_1$ | 1000 g cm$^{-1}$ s$^{-2}$ |

**Example 2 parameters**

| Param. | Value       | Param. | Value       | Param. | Value       |
|--------|-------------|--------|-------------|--------|-------------|
| $R_{11,1}$ | 10 g cm$^{-3}$ s$^{-1}$ | $R_a$ | 10 g cm$^{-3}$ s$^{-1}$ | $R_{21,1}$ | 10 g cm$^{-3}$ s$^{-1}$ |
| $R_0$ | 10 g cm$^{-3}$ s$^{-1}$ | $C_{11,1}$ | 0.001 g$^{-1}$ cm$^3$ s$^2$ | $C_{21,1}$ | 0.001 g$^{-1}$ cm$^3$ s$^2$ |
| $L_a$ | 0.003 g cm$^{-3}$ | $a_{01}$ | 150 g cm$^{-1}$ s$^{-2}$ | $a_{11}$ | 1000 g cm$^{-1}$ s$^{-2}$ |
| $a_{02}$ | 75 g cm$^{-1}$ s$^{-2}$ | $a_{12}$ | 500 g cm$^{-1}$ s$^{-2}$ |

**Example 3 parameters**

| Param. | Value       | Param. | Value       | Param. | Value       |
|--------|-------------|--------|-------------|--------|-------------|
| $R_{11,1}$ | 10 g cm$^{-3}$ s$^{-1}$ | $R_{11,2}$ | 50 g cm$^{-3}$ s$^{-1}$ | $R_a$ | 10 g cm$^{-3}$ s$^{-1}$ |
| $R_0$ | 10 g cm$^{-3}$ s$^{-1}$ | $R_c$ | 70 g cm$^{-3}$ s$^{-1}$ | $L_c$ | 0.003 g cm$^{-3}$ |
| $C_{11,1}$ | 0.001 g$^{-1}$ cm$^3$ s$^2$ | $C_{11,2}$ | 0.001 g$^{-1}$ cm$^3$ s$^2$ | $a_0$ | 150 g cm$^{-1}$ s$^{-2}$ |
| $a_1$ | 1000 g cm$^{-1}$ s$^{-2}$ |

Table 3: Parameter values for Examples 1, 2, and 3. Examples are set in a two-dimensional context, hence flow rates are per unit of length and the units of resistance and capacitance scale accordingly.
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