QCD Parameters Correlations from Heavy Quarkonia

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Correlations between the QCD coupling $\alpha_s$, the gluon condensate $\langle \alpha_s G^2 \rangle$, and the $c,b$-quark running masses $\overline{m}_{c,b}$ in the $\overline{MS}$-scheme are explicitly studied (for the first time) from a global analysis of the (axial-)vector and (pseudo)scalar charmonium and bottomium spectra using optimized ratios of Laplace sum rules (LSR) evaluated at the $\mu$-subtraction stability point where PT $@$N2LO, N3LO, $\langle \alpha_s G^2 \rangle$ $@$NLO and LO $D = 6 - 8$-dimensions non-perturbative condensates corrections are included. Our results clarify the (apparent) discrepancies between different estimates of $\langle \alpha_s G^2 \rangle$ from $J/\psi$ sum rule but also shows the sensitivity of the sum rules on the choice of the $\mu$-subtraction scale which does not permit a high-precision estimate of $\overline{m}_{c,b}$. We obtain from the (axial-)vector [resp. (pseudo)scalar] channels: $\langle \alpha_s G^2 \rangle = (8.5 \pm 3.0) [\text{resp. } (6.34 \pm 0.39)] \times 10^{-2} \text{ GeV}^4$, $\overline{m}_c(\overline{m}_c) = 1256(30) [\text{resp. } 1266(16)] \text{ MeV}$ and $\overline{m}_b(\overline{m}_b) = 4192(15) \text{ MeV}$. Combined with our recent determinations from vector channel, one can deduce the average: $\overline{m}_c(\overline{m}_c)$ $|_{\text{average}} = 1263(14) \text{ MeV}$ and $\overline{m}_b(\overline{m}_b)$ $|_{\text{average}} = 4184(11) \text{ MeV}$. Adding the two above values of the gluon condensate to previous estimates in Table 1, one obtains the new sum rule average: $\langle \alpha_s G^2 \rangle |_{\text{average}} = (6.35 \pm 0.35) \times 10^{-2} \text{ GeV}^4$. The mass-splittings $M_{\chi_0^{(c,b)}} - M_{\eta_c(b)}$ give $\langle \alpha_s G^2 \rangle$ $|_{\text{average}} = 0.1183(19)(3)$ in good agreement with the world average (see more detailed discussions in the section: addendum).

Keywords: QCD spectral sum rules, Perturbative and Non-Pertubative calculations, Hadron and Quark masses, Gluon condensates.

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1. Introduction

Gluon condensates introduced by SVZ$^{1-3}$ play important rôle in gluodynamics and in the QCD spectral sum rules analysis where they enter as high-dimension operators in the OPE of the hadronic correlators. In particular, this is the case for the heavy quark systems and the pure Yang-Mills gluonia/glueball channels$^{4-6}$ where the light quark loops and condensates are absent to leading order. The heavy quark condensate contribution can be absorbed into the gluon one through the relation: $^{1,2}$

$$ \langle \bar{Q}Q \rangle = -\langle \alpha_s G^2 \rangle / (12\pi M_Q) + ... $$

$^*$Some preliminary versions of this work have been presented @ QCD17, Montpellier - FR and @ HEPMAD16 and 17, Antananarivo-MG.
Table 1. Selected determinations of \(\langle \alpha_s G^2 \rangle\) from charmonium, bottomium and light quark systems. The numbers marked with * are not included in the average. This average does not take into account the new results of this paper. Estimates from variants of the SVZ sum rules using some weight functions are not considered here. The ones from high-moments of \(\tau\) decays and from the lattices are only mentioned for comparisons.

| Sources                     | \((\alpha_s G^2) \times 10^2\) [GeV\(^4\)] | References                           |
|-----------------------------|---------------------------------------------|---------------------------------------|
| **Charmonium**              |                                             |                                       |
| \(q^2 = 0\)-moments        | 4 ± 2                                       | SVZ 79\(^{1,2}\) (guessed error)     |
| \(q^2 \neq 0\)-moments     | 5.3 ± 1.2                                   | Miller-Obsson 82\(^{13}\)            |
| -                           | 9.2 ± 3.4                                   | Broadhurst et al. 94\(^{15}\)        |
| -                           | \(\approx 6.6^*\)                           | Ioffe-Zyablyuk 07\(^{16,17}\)       |
| -                           | 2.8 ± 2.2                                   | Narison 12a\(^{18}\)                |
| Exponential                 | 7.0 ± 1.3                                   | Narison 12b\(^{27}\)                |
| Exponential \(M_{\Psi} - M_{\eta_c}\) | 10 ± 4                                    | Narison 96\(^{28,29}\)             |
| **Bottomium**              |                                             |                                       |
| Exponential \(M_{\chi_b} - M_{\Upsilon}\) | 6.5 ± 2.5                                  | Narison 96\(^{28,29}\)             |
| Non-rel. moments            | 5.5 ± 3                                     | Yndurain 99\(^{30}\)                |
| \(e^+e^- \rightarrow I=1\) Hadrons | \(1.0 \sim 6.6^*\)                         | Eidelman et al. 79\(^{31}\)         |
| Exponential                 | 4 ± 1                                       | Launer et al. 84\(^{32}\)           |
| FESR                        | 13 ± 6                                      | Bertlmann et al. 88\(^{33,34}\)     |
| Infinite norm               | \(1.0 \sim 30^*\)                          | Causse-Mennessier \(^{35}\)         |
| \(\tau\)-like decay        | 7 ± 1                                       | Narison 95\(^{36,37}\)              |
| \(\tau\)-decay             |                                             |                                       |
| Axial spectral function     | 6.9 ± 2.6                                   | Dominguez-Sola 88\(^{38}\)          |
| **Sum Rule Average**        | \textbf{6.25 ± 0.45}                        | Prior \textbf{2017}                  |
| \(\tau\)-decay with high moments |                                         |                                       |
| ALEPH collaboration         | 6.3 ± 1.2                                   | Duflot 95\(^{39}\)                  |
| CLEO II collaboration       | 2.4 ± 1.0                                   | Duflot 95\(^{39}\)                  |
| OPAL collaboration          | \(-0.9 \sim +4\)                           | Ackerstaff et al. 99\(^{40}\)       |
| ALEPH collaboration         | \(-5 \sim +6\)                             | Schaef et al. 05\(^{41}\)           |
| ALEPH collaboration         | \(-12 \sim -0.6\)                          | Davier et al. 14\(^{42}\)           |
| **Lattice**                |                                             |                                       |
| \(O(\alpha_s^{12})\)       | \(\approx 13\)                              | Rakow 05\(^{43,45}\)               |
| \(O(\alpha_s^{35})\)       | \(\approx 27\)                              | Bali-Pineda 15\(^{46,47}\)         |
| Average plaquette           | \(\approx 44\)                              | Lee 14\(^{48}\)                     |

where a similar relation holds for the mixed heavy quark-gluon condensate \(\langle \bar{Q}GQ \rangle\). \(G\) is the short hand notation for the gluon field strength \(G_{\mu\nu}\) and \(M_Q\) is the pole mass. The SVZ original value\(^{1,2}\)

\[
\langle \alpha_s G^2 \rangle \simeq 0.04 \text{ GeV}^4 ,
\]

extracted (for the first time) from charmonium sum rules\(^{1,2}\) has been challenged by different authors (for reviews, see e.g.\(^{7,10}\) and Table 1). One can see in Table 1 that the results from standard SVZ and FESR sum rules for heavy and light quark systems vary in a large range but all of them are positive numbers, while the ones from analysis of the modified \(\tau\)-decays moments allow negative values. However, one should notice from the original QCD expression of the \(\tau\)-decay rate\(^{49,50}\) that
the $\langle \alpha_s G^2 \rangle$ gluon condensate contribution is absent to leading order indicating that it is a bad place for extracting a such quantity.\textsuperscript{51} The presence of $\langle \alpha_s G^2 \rangle$ in the analysis of\textsuperscript{39–42} is only an artifact of the high-moments where the systematic errors needs to be better controlled. Earlier lattice calculations indicate a non-zero positive value of $\langle \alpha_s G^2 \rangle$\textsuperscript{52–55} while recent estimates in Table 1 give positive values about 2-7 times higher than the phenomenological estimates. However, the subtraction of the perturbative contribution in the lattice analysis which is scheme dependent is not yet well-understood\textsuperscript{48} and does not permit a direct comparison of the lattice results obtained at large orders of PT series with the ones from the truncated PT series used in the phenomenological analysis. These previous results indicate that $\langle \alpha_s G^2 \rangle$ is not yet well determined and motivate a reconsideration of its estimate.

A first step for the improvement of the estimate of the gluon condensate was the recent direct determination of the ratio of the dimension-six gluon condensate $\langle g^3 f_{abc} G^3 \rangle$ over $\langle \alpha_s G^2 \rangle$ from the heavy quark systems with the value:\textsuperscript{18,27,56}

$$\rho \equiv \langle g^3 f_{abc} G^3 \rangle / \langle \alpha_s G^2 \rangle = (8.2 \pm 1.0) \text{ GeV}^2,$$ \hspace{1cm} (3)

which differs significantly from the instanton model estimate\textsuperscript{57–59} and may question the validity of this approximation. Earlier lattice results in pure Yang-Mills found: $\rho \approx 1.2 \text{ GeV}^2$\textsuperscript{52–55} such that it is important to have new lattice results for this quantity. Note however, that the value given in Eq.3 might also be an effective value of the unknown high-dimension condensates not taken into account in the analysis of\textsuperscript{18,27,56} when requiring the fit of the data by the truncated OPE at that order. We shall see that the effect of this term is a small correction at the stability region where the optimal results are extracted.

In this paper, we pursue a such program by reconsidering the extraction of the lowest dimension QCD parameters from the (axial-)vector and (pseudo)scalar charmonium and bottomonium spectra taking into account the correlations between $\alpha_s$, the gluon condensate $\langle \alpha_s G^2 \rangle$, and the $c,b$-quark running masses. We shall use these parameters for predicting the known masses of the (pseudo)scalar heavy quarkonia ground states and also re-extract $\alpha_s$ and $\langle \alpha_s G^2 \rangle$ from the mass-splittings $M_{\psi(0S)} - M_{\eta_c(1S)}$. In so doing, we shall work with the example of the QCD Laplace sum rules (LSR) where the corresponding Operator Product Expansion (OPE) in terms of condensates is more convergent than the moments evaluated at small momentum.

2. The QCD Laplace sum rules

- Form of the sum rule

We shall work with the Finite Energy version of the QCD Laplace sum rules (LSR) and their ratios:

$$L_n^c(\tau) = \int_{4m^2_0}^{t_c} dt \ t^n \ e^{-\tau t} \text{Im} \Pi_{V(A)}(t), \hspace{1cm} R_n^c(\tau) = \frac{L_n^c}{L_n^0},$$ \hspace{1cm} (4)
where \( \tau \) is the LSR variable, \( t_c \) is the threshold of the "QCD continuum" which parametrizes, from the discontinuity of the Feynman diagrams, the spectral function \( \text{Im}\Pi_{V(A)}(t, m_Q^2, \mu) \) associated to the transverse part \( \Pi_{V(A)}(q^2, m_Q^2, \mu) \) of the two-point correlator:

\[
\Pi_{V(A)}^{\mu\nu}(q^2) = i \int d^4x \ e^{-iqx} \langle 0 | J_{V(A)}^\mu(x) \big( J_{V(A)}^\nu(0) \big)^\dagger | 0 \rangle \\
= - \left( g^{\mu\nu} q^2 - q^{\mu} q^{\nu} \right) \Pi_{V(A)}(q^2) + \frac{q^{\mu} q^{\nu}}{Q^2} \Pi_{V(A)}^{(0)}(q^2),
\]

where \( J_{V(A)}^\mu(x) = \bar{Q} \gamma^{\mu} (\gamma_5) Q(x) \) is the heavy quark local vector (axial-vector) current. In the (pseudo)scalar channel associated to the local current \( J_{S(P)} = \bar{Q} i(\gamma_5) Q(x) \), we work with the correlator:

\[
\Psi_{S(P)}(q^2) = i \int d^4x \ e^{-iqx} \langle 0 | T J_{S(P)}(x) \big( J_{S(P)}(0) \big)^\dagger | 0 \rangle,
\]

which is related to the longitudinal part \( \Pi_{V(A)}^{(0)}(q^2) \) of the (axial-)vector one through the Ward identity:

\[
q^2 \Pi_{A(V)}^{(0)}(q^2) = \Psi_{P(S)}(q^2) - \Psi_{P(S)}(0).
\]

Working with \( \Psi_{P(S)}(0) \) is safe as \( \Psi_{P(S)}(0) \) should affect the \( Q^2 \)-moments and the exponential sum rules derived from \( \Pi_{A(V)}^{(0)}(q^2) \) which is not accounted for in e.g 13,22,26.

Originally named Borel sum rules by SVZ because of the appearance of a factorial suppression factor in the non-perturbative condensate contributions into the OPE, it has been shown by 61 that the PT radiative corrections satisfy instead the properties of an inverse Laplace sum rule though the present given name here.

- **Parametrisation of the spectral function**

\( \text{Im}\Pi_V(t) \) is related to the ratio \( R_{e^+e^-} \) of the total cross-section of \( \sigma(e^+e^- \rightarrow \text{hadrons}) \) over \( \sigma(e^+e^- \rightarrow \mu^+\mu^-) \) through the optical theorem. Expressed in terms of the leptonic widths and meson masses, it reads in a narrow width approximation (NWA):

\[
R_{e^+e^-} \equiv 12\pi \text{Im}\Pi_V(t) = \frac{9\pi}{Q^2_V} \alpha^2 \sum M_V \Gamma_{V \rightarrow e^+e^-} \delta(t - M_V^2),
\]

where \( M_V \) and \( \Gamma_{V \rightarrow e^+e^-} \) are the mass and leptonic width of the \( J/\psi \) or \( \Upsilon \) mesons; \( Q^2_V = 2/3(-1/3) \) is the charm (bottom) electric charge in units of e; \( \alpha = 1/133 \) is the running electromagnetic coupling evaluated at \( M_V^2 \). We shall use the experimental values of the \( J/\psi \) and \( \Upsilon \) parameters compiled by PDG. 62 We include the contributions of the \( \psi(3097) \) to \( \psi(4415) \) and \( \Upsilon(9460) \) to \( \Upsilon(11020) \) within NWA. The high-energy part of the spectral function is parametrized by the “QCD continuum” from a threshold \( t_c \) (we use \( \sqrt{t_c} = 4.6 \) GeV and \( \sqrt{t_c} = 11.098 \) GeV just above the last resonance).
QCD parameters correlations...

In the case of the axial-vector and (pseudo)scalar channels where there are no complete data, we use the duality ansatz:

\[ \text{Im}\{\Pi(t); \Psi(t)\} \simeq f_H^2 M_H^{(0;2)} \delta(t - M_H^2) + \Theta(t - t_c)^\text{"QCD continuum"}; \]  

where \( M_H \) and \( f_H \) are the lowest ground state mass and coupling analogue to \( f_\rho \) and \( f_\pi \). This implies:

\[ R_n^c \equiv R \simeq M_H^2, \]  

indicating that the ratio of moments appears to be a useful tool for extracting the masses of hadrons.\(^7\)–\(^10\) We shall work with the lowest ratio of moments \( R_0^c \). Exponential sum rules have been used successfully by SVZ for light quark systems\(^1\),\(^2\),\(^7\)–\(^10\) and extensively by Bell and Bertlmann for heavy quarkonia in their relativistic and non-relativistic versions.\(^19\)–\(^26\),\(^28\),\(^29\)

**QCD Perturbative expressions @N2LO**

The perturbative QCD expression of the vector channel is deduced from the well-known spectral function to order \( \alpha_s \) within the on-shell renormalization scheme.\(^63\),\(^64\) The one of the axial-vector current has been obtained in.\(^13\),\(^65\)–\(^67\) To order \( \alpha_s^2 \) (N2LO), the spectral functions are usually parametrized as:

\[ R^{(2)} = C_F R_A^{(2)} + C_A C_F R_{NA}^{(2)} + C_F T_Q n_l R_i^{(2)} + C_F T_Q (R_F^{(2)} + R_S^{(2)} + R_G^{(2)}) , \]  

which are respectively the abelian (A), non-abelian (NA), massless (I) and heavy (F) internal quark loops, singlet (S) and double bubble gluon (G) contributions. \( C_F = 4/3, C_A = 3, T_Q = 1/2 \) are usual SU(3) group factors and \( n_l \) is the number of light quarks. We use the (approximate) but complete result in the on-shell scheme given by\(^68\) for the abelian and non-abelian contributions. The one from light quarks comes from.\(^69\)–\(^71\) The one from heavy fermion internal loop comes from\(^72\) for the vector current while the one from the axial current is (to our knowledge) not available. The singlet one due to double triangle loop comes from.\(^73\) The one from the gluonic double-bubble reconstructed from massless fermions comes from.\(^69\),\(^70\),\(^72\) The previous on-shell expressions are transformed into the \( \overline{MS} \)-scheme through the relation between the on-shell \( M_Q \) and running \( \overline{m}_Q(\mu) \) quark masses\(^7\)–\(^10\),\(^74\)–\(^83\) @N2LO:

\[ M_Q = \overline{m}_Q(\mu) \left[ 1 + \frac{4}{3} a_s + (16.2163 - 1.0414 n_l) a_s^2 \right. \]

\[ + \ln \left( a_s + (8.8472 - 0.3611 n_l) a_s^2 \right) + \ln^2 \left( 1.7917 - 0.0833 n_l \right) a_s^2 + \cdots \right], \]

for \( n_l \) light flavours where \( \mu \) is the arbitrary subtraction point and \( a_s \equiv \alpha_s / \pi \), \( \ln \equiv \ln (\mu / M_Q)^2 \).
**QCD Non-Perturbative expressions @LO**

Using the OPE à la SVZ, the non-perturbative contributions to the two-point correlator can be parametrized by the sum of higher dimension condensates:

\[
\text{Im} \Pi(t) = \sum C_{2n}(t, m^2, \mu^2) \langle O_{2n} \rangle : n = 1, 2, \ldots
\]  

(13)

where \(C_{2n}\) are Wilson coefficients calculable perturbatively and \(\langle O_{2n} \rangle\) are non-perturbative condensates. In the exponential sum rules, the order parameter is the sum rule variable \(\tau\) while for the heavy quark systems the relevant condensate contributions at leading order in \(\alpha_s\) are the gluon condensate \(\langle \alpha_s G^2 \rangle\) of dimension-four \(^{1,2}\) the dimension-six gluon \(\langle g^3 f_{abc} G^3 \rangle\) and light four-quark \(\alpha_s \langle \bar{u}u \rangle^2\) condensates. \(^{11,12}\) The condensates of dimension-8 entering in the sum rules are of seven types. \(^{57}\) They can be expressed in different basis depending on how each condensate is estimated (vacuum saturation \(^{57}\) or modified vacuum saturation \(^{84}\)). Our estimate of these \(D=8\) condensates is the same as in. \(^{18}\) For the vector channel, we use the analytic expressions of the different condensate contributions given by Bertlmann. \(^{22}\) We shall not include the eventual \(D = 2\) operator induced by a tachyonic gluon mass \(^{85,86}\) as it is dual to the contribution of large order terms, \(^{87}\) which we estimate using a geometric growth of the PT series. In various examples, its contribution is numerically negligible. \(^{88}\)

**Initial QCD input parameters**

In the first iteration, we shall use the following QCD input parameters:

\[
\alpha_s(M_\tau) = 0.325^{+0.008}_{-0.016}, \quad \langle \alpha_s G^2 \rangle = (0.07 \pm 0.04) \text{ GeV}^4.
\]

\[
\overline{m}_c(\overline{m}_c) = (1261 \pm 17) \text{ MeV}, \quad \overline{m}_b(\overline{m}_b) = (4177 \pm 11) \text{ MeV},
\]

(14)

The central value of \(\alpha_s\) comes from \(\tau\)-decay. \(^{51,92-96}\) The range covers the one allowed by PDG \(^{62,94-96}\) (lowest value) and the one from our determination from \(\tau\)-decay (highest value). The values of \(\overline{m}_{c,b}(\overline{m}_{c,b})\) are the average from our recent determinations from charmonium and bottomium sum rules \(^{18,56}\). The value of \(\langle \alpha_s G^2 \rangle\) almost covers the range from different determinations mentioned in Table 1 and reviewed in \(^{7,8,28,29}\) We shall use the ratio of condensates given in Eq. 3. For the light four-quark condensate, we shall use the value:

\[
\alpha_s \langle \bar{u}u \rangle^2 = (5.8 \pm 1.8) \times 10^{-4} \text{ GeV}^6,
\]

(15)

obtained from the original \(\tau\)-decay rate \(^{51}\) where the gluon condensate does not contribute to LO \(^{49,50}\) and by some other authors from the light quark systems \(^{7,8,32,89-91}\) where a violation by a factor about 3–4 of the vacuum saturation assumption has been found.
QCD parameters correlations...

3. Charmonium Ratio of Moments $R_{J/\psi(\chi_{c1})}$

- **Convergence of the PT series**

In so doing, we shall work with the renormalized (but non-resummed renormalization group) perturbative (PT) expression where the subtraction point $\mu$ appears explicitly. We include the known N2LO terms. The $D = (6)8$ condensates contributions are included for the (axial-)vector current. The value of $\sqrt{\tau_c} = 4.6$ GeV is chosen just above the $\psi(4040)$ mass for the vector current where the sum of all

![Graph a)

![Graph b)](image-url)
lower mass $\psi$ state contributions are included in the spectral function. For the axial current, we use (as mentioned) the duality ansatz and leave $t_c$ as a free parameter which we shall fix after an optimisation of the sum rule. We evaluate the ratio of moments at $\mu = 2.8$ GeV and for a given value of $t_c = 20$ GeV$^2$ for the $\chi_{c1}$ around which they will stabilize (as we shall show later on). The analysis is illustrated in Fig. 1. On can notice the importance of the N2LO contribution which is dominated by the abelian and non-abelian contributions. The N2LO effects go towards the good direction of the values of the experimental masses.

- **LSR variable $\tau$-stability and Convergence of the OPE**

The OPE is done in terms of the exponential sum rule variable $\tau$. We show in Fig. 2 the effects of the condensates of different dimensions. One can notice that the presence of condensates are vital for having $\tau$-stabilities which are not there for the PT-terms alone. The $\tau$-stability is reached for $\tau \simeq 0.6$ GeV$^{-2}$. At a given order of the PT series, the contributions of the $D = 8$ condensates are negligible at the $\tau$-stability region while the $D = 6$ contribution goes again to the right track compared with the data.

- **Continuum threshold $t_c$-stability for $R_{\chi_{c1}}$**

We show the analysis in Fig. 3 where the curves correspond to different $t_c$-values. We find nice $t_c$-stabilities where we take the value:

$$t_c \simeq (17 \sim 22) \text{ GeV}^2,$$

where the lowest value corresponds to the phenomenological estimate $M_{\chi_{c1}}(2P) - M_{\chi_{c1}}(1P) \approx M_{\psi}(2S) - M_{\psi}(1S)$ while the higher one corresponds to the beginning of $t_c$-stability. This range of $t_c$-values induces an error of about 8 MeV in the meson mass determination.

- **Subtraction point $\mu$-stability**

The subtraction point $\mu$ is an arbitrary parameter. It is popularly taken between 1/2 and 2 times an “ad hoc” choice of scale. However, the physical observables should be not quite sensitive to $\mu$ even for a truncated PT series. In the following, like in the previous case of external (unphysical) variable, we shall fix its value by looking for a $\mu$-stability point if it exists at which the observable will be evaluated. This procedure has been used recently for improving the LSR predictions on molecules and four-quark charmonium and bottoming states.\textsuperscript{97–101} Taking here the example of the ratios of moments, we show in Fig. 4 their $\mu$-dependence. We notice that $R_\psi$ is a smooth decreasing function of $\mu$ while $R_{\chi_{c1}}$ presents a slight stability at:

$$\mu = (2.8 \sim 2.9) \text{ GeV},$$

at which we shall evaluate the two ratios of moments. On can notice that at a such higher scale, one has a better convergence of the $\alpha_s(\mu)$ PT series.
QCD parameters correlations...

• Correlations of the QCD parameters

Once fixed these preliminaries, we are now ready to study the correlation between \( \alpha_s \), the gluon condensate \( \langle \alpha_s G^2 \rangle \), and the \( c \)-quark running masses \( m_c(\mu_c) \). In so doing we request that the \( \sqrt{\mathcal{R}_{J/\psi}} \) sum rule reproduces within (2-3) MeV accuracy the experimental measurement, while the \( \chi_{c1} \) mass is reproduced within (8–10) MeV which is the error induced by the choice of \( t_c \) in Eq. 16. The results of the analysis are obtained at the \( \tau \)-stability points which are about 1.1 (resp. 0.6) GeV\(^{-2}\) for the

Fig. 2. The same as in Fig. 1 but for different truncation of the OPE: a) \( J/\psi \) and b) \( \chi_{c1} \).
Fig. 3. Behaviour of the ratio of moments $\mathcal{R}_{\chi c_1}$ versus $\tau$ in GeV$^{-2}$. The input and the meaning of each curve are given in the legend.

Fig. 4. Behaviour of the ratio of moments $\mathcal{R}_J/\psi$ and $\mathcal{R}_{\chi c_1}$ versus $\mu$ for $t_c = 20$ GeV$^2$. The inputs and the meaning of each curve are given in the legends.

$J/\psi$ (resp. $\chi c_1$) channels. They are shown in Fig. 5 for the two values of $\mu$ given in Eq. 34. One can notice that, $\langle \alpha_s G^2 \rangle$ decreases smoother from the $\chi c_1$ (grey region) than from the $J/\psi$ sum rule when $\overline{m}_c$ increases. In the $J/\psi$ sum rule, it moves from 0.15 to 0.02 GeV$^4$ for $\overline{m}_c(\overline{m}_c)$ varying from 1221 to 1301 MeV. This feature may explain the apparent discrepancy of the results reviewed in the introduction from
Fig. 5. Correlation between $\langle \alpha_s G^2 \rangle$ and $\overline{m}_c(\overline{m}_c)$ for the range of $\alpha_s$ values given in Eq. 14 and for $\mu$ given in Eq. 20.

One should notice that the results from the $J/\psi$ sum rules are quite sensitive to the choice of the subtraction point (no $\mu$-stability) which then does not permit accurate determinations of $\langle \alpha_s G^2 \rangle$ and $\overline{m}_c(\overline{m}_c)$. Some accurate results reported in the literature for an "ad hoc" choice of $\mu$ may be largely affected by the $\mu$ variation.

One can also see from Fig. 5 that within the alone $J/\psi$ sum rule the values of $\langle \alpha_s G^2 \rangle$ and $\overline{m}_c(\overline{m}_c)$ cannot be strongly constrained. Once the constraint from the $\chi_{c1}$ sum rule is introduced, one obtains a much better selection. Taking as a conservative result the range covered by the change of $\mu$ in Eq. 34, one deduces:

$$\langle \alpha_s G^2 \rangle = (8.5 \pm 3.0) \times 10^{-2} \text{ GeV}^4, \quad \overline{m}_c(\overline{m}_c) = (1256 \pm 30) \text{ MeV}. \quad (18)$$

We improve this determination by including the N3LO PT corrections and NLO $\langle \alpha_s G^2 \rangle$ gluon condensate (using the parametrization in $^{16,17}$ contributions $^{15}$). The effects of these quantities on $\sqrt{R_{J/\psi}}$ and $\sqrt{R_{\chi_{c1}}}$ is about (1 $\sim$ 2) MeV at the optimization scales which induces a negligible change such that the results quoted in Eq. 18 remain the same $^{16}$ at N3LO PT and $^{17}$ NLO gluon condensate approximations. This value of $\langle \alpha_s G^2 \rangle$ is in good agreement with the one $(7.5 \pm 2.0) \times 10^{-2}$ GeV$^4$ from our previous analysis of the charmonium Laplace sum rules using resummed PT series $^{27}$ indicating the self-consistency of the results. However, these results do not favor lower ones quoted in Table 1. Taking the weighted average of different sum rule determinations given in Table 1 with the new result in Eq. 18, we obtain the sum rule average:

$$\langle \alpha_s G^2 \rangle|_{\text{average}} = (6.30 \pm 0.45) \times 10^{-2} \text{ GeV}^4, \quad (19)$$

Similar relations from vector moments have been obtained $^{16,17}$ while the ones between $\alpha_s$ and $\overline{m}_c$ have been studied in $^{102,103}$.
where the error may be optimistic but comparable with the one of the most precise predictions given in Table 1. These results agree within the errors within our recent estimates of $\langle \alpha_s G^2 \rangle$ and $m_c(m_c)$\textsuperscript{18,27,56} obtained from the moments and their ratios subtracted at finite $Q^2 = n \times 4m_c^2$ with $n = 0, 1, 2$, and from the heavy quark mass-splittings.\textsuperscript{28,29} Hereafter, we shall use the value of $\langle \alpha_s G^2 \rangle$ in Eq. 19.

Fig. 6. Behaviour of the ratio of moments $R_\Upsilon$ versus $\tau$ in GeV$^{-2}$ for different truncation of the OPE. The input and the meaning of each curve are given in the legend.

Fig. 7. Behaviour of the ratio of moments $R_{\chi b1}$ versus $\tau$ for different values of $t_c$. The input and the meaning of each curve are given in the legend.
4. Bottomium Ratios of Moments $\mathcal{R}_{\Upsilon(\chi_{b1})}$

- $\tau$ and $t_c$-stabilities and test of convergences

The analysis is very similar to the previous $J/\psi$ sum rule. The relative perturbative and non-perturbative contributions are very similar to the curves in Figs. 1 to 2. We use the value; $\mu = 9.5$ GeV which we shall justify later on. However, it is informative to show in Fig.6 the $\tau$-behaviour of $\mathcal{R}_\Upsilon$ for different truncation of the OPE where $\tau$-stability is obtained at $\tau \simeq 0.22$ GeV$^{-2}$. In Fig. 7, we show the $\tau$-behaviour of $\mathcal{R}_{\chi_{b1}}$, for different values of $t_c$ from which we deduce a stability at $\tau \simeq 0.28$ GeV$^{-2}$ and $t_c$-stability which we shall take to be $\sqrt{t_c} \simeq 11$ GeV. A much better convergence of the $\alpha_s$ series is observed as the sum rule is evaluated at a higher scale $\mu$. The OPE converges also faster as $\tau$ is smaller here.

- $\mu$-stability

The two sum rules are smooth decreasing functions of $\mu$ but does not show $\mu$-stability. Instead, their difference presents $\mu$-stability at:

$$\mu \simeq (9 \sim 10) \text{ GeV},$$

(20)
as shown in Fig.8 at which we choose to evaluate the two sum rules.

**Mass of $\chi_{b1}(1^{++})$ from $\mathcal{R}_{\chi_{b1}}$**

Using the previous value of the QCD parameters, we predict from the ratio of $\chi_{b1}$ moments:

$$M_{\chi_{b1}} \simeq 9677(26)t_c(8)\alpha_s(11)G^2(9)m_b(99)\mu \text{ MeV},$$

(21)
Fig. 9. Behaviour of $\Delta \alpha_s(M_\tau)$ versus $m_b(m_b)$ from the ratio of moments $R_\Upsilon$. The horizontal band corresponds to the range of $\alpha_s$ value given in Eq. 14. The input and the meaning of each curve are given in the legend.

which is (within the error) about 100 MeV lower than the experimental mass $M_{\chi b_1}^{\text{exp}} = 9893$ MeV. The agreement between theory and experiment may be improved when more data for higher states are available or/and by including Coulombic corrections shown to be small for the vector current (see e.g.\textsuperscript{56}) and not considered here.

- **Correlation between $\alpha_s(\mu)$ and $m_b(m_b)$ from $R_\Upsilon$**

From the previous analysis, one can notice that the $\chi_{b1}$ channel cannot help from a precise study of the correlation between $\alpha_s$ and $m_b(m_b)$. We show in Fig. 9 the result of the analysis from the $\Upsilon$ channel by requiring that the experimental value of $\sqrt{R_\Upsilon}$ is reproduced within $(1 \sim 2)$ MeV accuracy. First, one can notice that the error due to the gluon condensate with the value given in Eq. 18 is negligible. Given the range of $\alpha_s$ quoted in Eq. 14, one can deduce the prediction:

$$m_b(m_b) = 4192(15)(8)_{\text{coul}} \text{ MeV},$$

(22)

where we have added in Eq. 22 an error of about 8 MeV from Coulombic corrections as estimated in.\textsuperscript{27} The previous result in Eq. 22 corresponds to:

$$\alpha_s(M_\tau) = 0.321(12) \implies \alpha_s(M_Z) = 0.1186(15)(3)$$

(23)

given by the range in Eq. 14. The running from $M_\tau$ to $M_Z$ due to the choice of the thresholds induces the last error (3). This result is consistent with the ones from moments sum rules quoted in Eq. 14 and with:\textsuperscript{27}

$$m_b(m_b) = 4212(32) \text{ MeV},$$

(24)
from LSR with RG resummed PT expressions. Taking the average of our three determinations, we obtain the final estimate:

\[ m_b \langle m_b \rangle_{\text{average}} = (4184 \pm 11) \text{ MeV}, \]

where the errors come from the most precise determination. Due to the large errors induced by the subtraction scale as shown in Fig 9, one cannot accurately extract the value of \( \alpha_s \) given the present value of \( m_b \).

![Fig. 10. Behaviour of \( M_{\eta_c} \) versus \( \tau \) for different values of \( t_c \).](image1)

![Fig. 11. Behaviour of \( M_{\chi c^0} \) versus \( \tau \) for different values of \( t_c \).](image2)
5. (Pseudo)scalar charmonium

In these channels, we shall work with the ratio of sum rules associated to the two-point correlator $\psi_{P(S)}(q^2)$ defined in Eq. 6 which is not affected by $\psi_{P(S)}(0)$. We shall use the PT expression known at N2LO,\textsuperscript{69–73} the contribution of the gluon condensates of dimension 4 and 6 to LO.\textsuperscript{11,12}

\begin{itemize}
  \item $\eta_c$ and $\chi_{c0}$ masses
\end{itemize}

The $\eta_c$ sum rule shows a smooth decreasing function of $\mu$ but does not present a $\mu$-stability. Then, we choose the value of $\mu$ given in Eq. 34 for evaluating it. We show in Fig. 10 the $\tau$-behaviour of the $\eta_c$-mass for different values of $t_c$ which we take from 10 GeV\textsuperscript{2} [around the mass squared of the $\eta_c(2P)$ and $\eta_c(3P)$] until 13 GeV\textsuperscript{2} ($t_c$-stability). Similar analysis is done for the $\chi_{c0}$ associated to the scalar current $\bar{Q}(i)Q$ which is shown in Fig. 11, where we take $t_c \simeq (16 \sim 24)$ GeV\textsuperscript{2}. Using the averaged values of $\langle \alpha_s G^2 \rangle$ and $m_c(m_c)$ in Eqs. 19 and 28, we deduce the optimal result in units of MeV:

$$
M_{\eta_c} = 2979(5)_{\alpha_s(30)}_{m_c(10)} \mu_{(11)} t_c(11) \alpha_s(30) m_c(10) G^2,
$$

$$
M_{\chi_{c0}} = 3411(1)_{\alpha_s(26)}_{m_c(20)} \mu_{(17)} t_c(26) \alpha_s(30) m_c(20) G^2,
$$

in good agreement within the errors with the experimental masses: $M_{\eta_c} = 2984$ MeV and $M_{\chi_{c0}} = 3415$ MeV but not enough accurate for extracting with precision the QCD parameters.

\begin{itemize}
  \item Correlation between $m_c(m_c)$ and $\langle \alpha_s G^2 \rangle$
\end{itemize}

We study the correlation between $m_c(m_c)$ and $\langle \alpha_s G^2 \rangle$ by requiring that the sum rules reproduce the masses of the $\eta_c$ and $\chi_{c0}$ within the error induced by the choice of $t_c$ respectively 11 and 17 MeV. We show the result of the analysis in Fig. 12 keeping only the strongest constraint from $M_{\eta_c}$. We deduce:

$$
m_c(m_c) = 1266(16) \text{ MeV},
$$

in good agreement with the one in \textsuperscript{105} from pseudoscalar moments. We combine our determinations in Eqs. 18 and 27 with the two determinations\textsuperscript{18,56} from vector moments sum rules quoted in Eq. 14. As a final result, we quote the average from exponential and moment sum rules from a global fit of the quarkonia spectra:

$$
m_c(m_c)|_{\text{average}} = (1263 \pm 14) \text{ MeV},
$$

where we have retained the error from the most precise prediction rather than from the weighted average. It is remarkable that this value agrees with the original SVZ estimate\textsuperscript{1,2} of the euclidian mass.
Fig. 12. Behaviour of $\Delta m_c(m_c)$ versus $\Delta \langle \alpha_s G^2 \rangle$ from $M_{\eta_b}$. The dashed region corresponds to $\Delta \alpha_s = 0$ and for different values of $t_c \simeq 10 \sim 13$ GeV$^2$. The two extremal lines correspond to $\Delta \alpha_s \pm 12$. We use the range of $\alpha_s$ in Eq. 14 and of $\langle \alpha_s G^2 \rangle$ in Eq. 19.

Fig. 13. Behaviour of $M_{\eta_b}$ versus $\tau$ for different values of $t_c$.

6. (Pseudo)scalar bottomium

- $\eta_b$ and $\chi_{b0}$ masses

The masses of the $\eta_b(0^{-+})$ and $\chi_{b0}(0^{++})$ are extracted in a similar way using the value of $\mu$ in Eq. 20 and the parameters in Eqs. 19 and 25. We take the range $\sqrt{t_c} = (9.5 \sim 12)$ [resp. $(10.5 \sim 13)$] GeV for the $\eta_b$ [resp. $\chi_{b0}$] channels, as shown in Figs 13 and 14 from which we deduce in units of MeV:

$$M_{\eta_b} = 9394(16)t_c(30)\alpha_s(16)m_b(8)G^2,$$
$$M_{\chi_{b0}} = 9844(7)t_c(35)\alpha_s(6)m_b(17)G^2,$$

(29)

in good agreement with the data $M_{\eta_b} = 9399$ MeV and $M_{\chi_{b0}} = 9859$ MeV.
• **Correlation between \( \overline{m}_b(\overline{m}_b) \) and \( \langle \alpha_s G^2 \rangle \)**

The analysis done for charmonium is repeated here where we request that the sum rule reproduces the \( \eta_b \) and \( \chi_{b0} \) masses with the error induced by the choice of \( t_c \).

Unfortunately, this constraint is too weak and leads to \( \overline{m}_b(\overline{m}_b) \) with an accuracy of about 40 MeV which is less interesting than the estimate from the vector channel in Eq. 22.

7. \( \alpha_s \) and \( \langle \alpha_s G^2 \rangle \) from \( M_{\chi_{c(b)}(0)} - M_{\eta_{c(b)}} \)

As the sum rules reproduce quite well the absolute masses of the (pseudo)scalar states, we can confidently use their mass-splittings for extracting \( \alpha_s \) and \( \langle \alpha_s G^2 \rangle \). We shall not work with the Double Ratio of LSR\(^7,8,106–116\) as each sum rule does not optimize at the same points. We check that, in the mass-difference, the effect of the choice of the continuum threshold is reduced and induces an error from 6 to 14 MeV instead of 11 to 35 MeV in the absolute value of the masses. The effect due to \( \overline{m}_{c,b} \) in Eqs. 28 and 25 and to \( \mu \) in Eqs. 34 and 20 induce respectively an error of about (1–2) MeV and 8 MeV.

The largest effects are due to the changes of \( \alpha_s \) and \( \langle \alpha_s G^2 \rangle \). We show their correlations in Fig 15 where we have runned the value of \( \alpha_s \) from \( \mu = 2.85 \text{ GeV} \) to \( M_c \) in the charm channel and from \( \mu = 9.5 \text{ GeV} \) to \( M_b \) in the bottom one where the values of \( \mu \) correspond to the scales at which the sum rules have been evaluated. We have requested that the method reproduces within the errors the experimental mass-splittings by about 2-3 MeV. With the central values given in Eqs. 19 and 23, the allowed region leads to our final predictions:

\[
\alpha_s(M_c) = 0.318(15) \implies \alpha_s(M_Z) = 0.1183(19)(3),
\]

\[
\langle \alpha_s G^2 \rangle = (6.34 \pm 0.39) \times 10^{-2} \text{ GeV}^4.
\]

Adding into the analysis the range of input \( \alpha_s \) values given in Eq. 14 (light grey horizontal band in Fig. 15), one can deduce stronger constraints on the value of
\[ \langle \alpha_s G \rangle = (6.39 \pm 0.35) \times 10^{-2} \text{ GeV}^4. \]  

Combining the previous values in Eqs. 18, 30 and 31 with the ones in Table 1, one obtains the new sum rule average:

\[ \langle \alpha_s G \rangle |_{\text{average}} = (6.35 \pm 0.35) \times 10^{-2} \text{ GeV}^4, \]

where we have retained the error from the most precise determination in Eq. 31 instead of the weighted error of 0.23. This result definitely rules out some eventual lower and negative values quoted in Table 1.

Fig. 15. Correlation between \( \alpha_s \) and \( \langle \alpha_s G \rangle \) by requiring that the sum rules reproduce the (pseudo)scalar mass-splittings.

8. Summary and Conclusions

- We have explicitly studied (for the first time) the correlations between \( \alpha_s \), \( \langle \alpha_s G \rangle \) and \( \overline{m}_{c,b} \) using ratios of Laplace sum rules @N3LO of PT QCD and including the gluon condensate \( \langle \alpha_s G \rangle \) of dimension 4 @NLO and the ones of dimension 6-8 @LO in the (axial-)vector charmonium and bottomium channels. We have used the criterion of \( \mu \)-stability in addition to the usual sum rules stability ones (sum rule variable \( \tau \) and continuum threshold \( t_c \)) for extracting our optimal results.

- Our final result from the \( J/\psi \) channel in Eq. 31 and the sum rule average in Eq. 32 including this new value confirm and improve our previous estimates of \( \langle \alpha_s G \rangle \) from moments within \( n \) (number of moments) stability criterion 18 and from Laplace sum rules within \( \tau \) stability criterion 27 in the vector channels quoted in Table 1 and from the heavy quark mass-spittings obtained in 28,29.
The corresponding values of $\bar m_{c,b}$ from vector moments and Laplace sum rules quoted in Eqs 14 and 24 are also confirmed by the present determinations given in Eqs. 18 to 28 and in Eqs. 22 to 25.

We have extended the analysis to the (pseudo)scalar channels where the experimental masses of the lowest ground states are reproduced quite well. The $\eta_c$ sum rule also leads to an alternative prediction of $\bar m_c$ in Eq. 27.

The $\chi_{c0}(b_0) - \eta_c(b)$ mass-splittings lead to improved values of the gluon condensate $\langle \alpha_s G^2 \rangle$ in Eqs. 30 and 31 which give the new sum rule average in Eq. 32.

The $\chi_{c0}(b_0) - \eta_c(b)$ mass-splittings also provide a new prediction of $\alpha_s$ in Eq. 30 in good agreement with the world average.\textsuperscript{62,94–96}

9. Addendum : $\alpha_s(\mu)$ from $M_{\chi_{c0}(ab)} - M_{\eta_c(b)} \oplus$N2LO

In this complementary note, we present a more detailed discussion of the $\alpha_s$-results obtained previously $\oplus$ N2LO in Section 7 for two different subtraction scales $\mu$ from the (pseudo)scalar heavy quarkonia mass-splittings $M_{\chi_{c0}(ab)} - M_{\eta_c(b)}$. This complementary discussion is useful for a much better understanding of these results.

- Optimized subtraction scales

Besides the usual sum rules optimization procedure (sum rule variables and QCD continuum threshold) studied in details in previous sections, we deduce from Figs. 4 and 8 that the ratios of charmonium and bottomium moments are optimized respectively at the values of the subtraction scales:

$$\mu_c = (2.8 \sim 2.9) \text{ GeV} \quad \text{and} \quad \mu_b = (9 \sim 10) \text{ GeV}. \quad (33)$$

- $\alpha_s$ and $\langle \alpha_s G^2 \rangle$ correlation

We study, in Fig.15, the correlation between $\alpha_s$ and $\langle \alpha_s G^2 \rangle$ where the charmonium (resp. bottomium) sum rules have been evaluated at $\mu_c$ (resp. $\mu_b$) but runned to the scale $M_\tau$ for a global comparison of the results. For the range of $\langle \alpha_s G^2 \rangle$ values allowed by different analysis ($x$-axis) and requiring that the sum rule reproduces the experimental mass-splittings $M_{\chi_{c0}} - M_{\eta_c}$ by about $2 \sim 3$ MeV, one obtains the grey band limited by the two green (continuous) curves in Fig.15 which lead to:

$$\alpha_s(2.85) = 0.262(9) \sim \alpha_s(M_\tau) = 0.318(15) \sim \alpha_s(M_Z) = 0.1183(19)(3). \quad (34)$$

In the same way, the $M_{\chi_{b0}} - M_{\eta_b}$ bottomium sum rule evaluated at the optimization scale $\mu_b = 9$ GeV gives (sand colour band limited by two dotted red curves):

$$\alpha_s(9.50) = 0.180(8) \sim \alpha_s(M_\tau) = 0.312(27) \sim \alpha_s(M_Z) = 0.1175(32)(3). \quad (35)$$
Fig. 16. Comparison with the running of the world average $\alpha_s(M_Z) = 0.1181(11)^{62,94}$ (grey band limited by the two green curves) of our predictions at three different scales: $\mu = M_\tau$ for the original $\tau$-decay width $^{51}$ (open circle), $\mu_c = 2.85$ GeV for $M_{\chi_{c0}} - M_{\eta_c}$ (full triangle) and $\mu_b = 9.5$ GeV for $M_{\chi_{b0}} - M_{\eta_b}$ (full square).^7

- **Comparison with the world average**

These values of $\alpha_s(\mu)$ estimated at different $\mu$-scales are shown in Fig. 9 where they are compared with the running of the world average $\alpha_s(M_Z) = 0.1181(11)^{62,94}$. We have added, in the figure, your previous estimate of $\alpha_s(M_\tau)^{51}$ obtained from the original $\tau$-decay rate (lowest moment): $^{49,50}$

$$\alpha_s(M_\tau) = 0.325(8), \quad (36)$$

where one should note that non-perturbative corrections beyond the standard OPE (tachyonic gluon mass and duality violations) do not affect sensibly the above value of $\alpha_s(M_\tau)$ as indicated by the coincidence of the central value with the recent one from high-moments.$^{93}$

Our most precise prediction for $\alpha_s$ from the heavy-quarkonia mass-splittings comes from the (pseudo)scalar charmonium one in Eq. 34 which corresponds to:

$$\alpha_s(M_Z) = 0.1183(19)(3), \quad (37)$$

which agrees with the world average: $\alpha_s(M_Z) = 0.1181(11)^{62,94}$

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