More on $\mathcal{N} = 8$ Attractors

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ABSTRACT

We examine few simple extremal black hole configurations of $\mathcal{N} = 8$, $d = 4$ supergravity. We first elucidate the relation between the BPS Reissner-Nördstrom black hole and the non-BPS Kaluza-Klein dyonic black hole. Their classical entropy, given by the Bekenstein-Hawking formula, can be reproduced via the attractor mechanism by suitable choices of symplectic frame. Then, we display the embedding of the axion-dilaton black hole into $\mathcal{N} = 8$ supergravity.
1 Introduction

It has been known for some time \[1\] that extremal BPS black hole (BH) states coming from string and M theory compactifications to four and five dimensions, preserving various fractions of the original \( \mathcal{N} = 8 \) supersymmetry, can be invariantly classified in terms of orbits of the fundamental representations of the exceptional groups \( E_{7(7)} \) and \( E_{6(6)} \). These are the duality groups of the low energy actions, whose discrete subgroups appear as symmetries of the non-perturbative spectrum of BPS states \[2\]. These orbits, which have been further studied in \((3, 4, 5)\), correspond to well defined categories of allowed entropies of extremal BHs in \( d = 5 \) and in \( d = 4 \), given in terms of the cubic \( E_{6(6)} \) invariant \( \mathcal{I}_3 \) \((1, 4, 6)\) and the quartic \( E_{7(7)} \) invariant \( \mathcal{I}_4 \) \((7, 8, 9)\). There are three types of orbits depending on whether the BH background preserves \( 1/2, 1/4 \) or \( 1/8 \) of the original supersymmetry. Only \( 1/8 \) BPS states have non-vanishing entropy and regular horizons, while \( 1/4 \) and \( 1/2 \) BPS configurations lead to vanishing classical entropy.

The \( \mathcal{N} = 8 \) attractors have been explored in \[9\] by solving the criticality condition for the suitable BH effective potential and extending the lore of \( \mathcal{N} = 2 \) special Kähler geometry \[10\].

In this note we focus on some specific simple configurations in \( \mathcal{N} = 8, d = 4 \) supergravity which capture some representatives of the regular (sometimes, called “large”), \textit{i.e.} with non-vanishing classical entropy, extremal BPS and non-BPS BH charge orbits of the theory. One is the Reissner-Nördstrom (RN) dyonic BH, with electric and magnetic charge \( e \) and \( m \) respectively, and Bekenstein-Hawking entropy (in unit of Planck mass) \[11\]

\[ S_{RN} = \pi \left( e^2 + m^2 \right). \tag{1.1} \]

Another one is the Kaluza-Klein (KK) dyonic BH, with a KK monopole charge \( p \) and a KK momentum \( q \), which is dual to a \( D0-D6 \) brane configuration in Type II A supergravity. Its Bekenstein-Hawking entropy reads

\[ S_{KK} = \pi |pq|. \tag{1.2} \]

One more interesting example is the extremal axion-dilaton BH, a subsector of pure \( \mathcal{N} = 4 \) supergravity in \( d = 4 \) which was considered in the past in \[12, 13\].

Our aim is to show how the entropies of these BHs can be obtained in the context of \( \mathcal{N} = 8, d = 4 \) supergravity by exploiting the attractor mechanism \[14, 15, 10, 16\] for extremal BPS and non-BPS BHs. Earlier studies for some specific cases where examined in \[17, 18\].

It is in fact known that while the BH charge configuration with entropy given by (1.1) is \( 1/8 \) BPS \[11\], the entropy (1.2) is related to a non BPS one. Indeed, the \( E_{7(7)} \) quartic invariant \( \mathcal{I}_4 \) on these configurations reduces to

\[ \sqrt{\mathcal{I}_4^{RN}} = e^2 + m^2; \tag{1.3} \]
\[ \sqrt{-\mathcal{I}_4^{KK}} = |pq|. \tag{1.4} \]

In particular we note that , if the magnetic (or electric) charge is switched off, the RN BH remains regular, whereas the KK BH reaches zero entropy (\( \mathcal{I}_4 = 0 \)) and becomes 1/2 BPS \[3\].
The simplest way to obtain these configurations is to observe that the BPS and non-BPS charge orbits with $\mathcal{I}_4 \neq 0$ in $\mathcal{N} = 8, d = 4$ supergravity are given by [1]:

\begin{align}
\mathcal{O}_{1/8-\text{BPS}} : & \quad \frac{E_{7(7)}}{E_{6(2)}}, \quad \mathcal{I}_4 > 0; \\
\mathcal{O}_{\text{non-}\text{-BPS}} : & \quad \frac{E_{7(7)}}{E_{6(6)}}, \quad \mathcal{I}_4 < 0.
\end{align}

The moduli spaces corresponding to the above disjoint orbits are [1 9]

\begin{align}
\mathcal{M}_{1/8-\text{BPS}} = & \quad \frac{E_{6(2)}}{SU(6) \times SU(2)} \\
\mathcal{M}_{\text{non-}\text{-BPS}} = & \quad \frac{E_{6(6)}}{USp(8)}.
\end{align}

Hence, a convenient representative of these orbits is given by the (unique) $E_6$-singlets in the decomposition of the fundamental representation 56 of $E_{7(7)}$ into the two relevant non-compact real forms of $E_6$:

\begin{align}
\text{RN} \quad \mathcal{O}_{1/8-\text{BPS}} : & \quad \left\{ \begin{array}{l}
E_{7(7)} \to E_{6(2)} \times U(1) ; \\
56 \to (27, 1) + (1, 3) + (27', -1) + (1, -3) ;
\end{array} \right. \\
\text{KK} \quad \mathcal{O}_{\text{non-}\text{-BPS}} : & \quad \left\{ \begin{array}{l}
E_{7(7)} \to E_{6(6)} \times SO(1, 1) ; \\
56 \to (27, 1) + (1, 3) + (27', -1) + (1', -3) ,
\end{array} \right.
\end{align}

where the $U(1)$ charges and $SO(1, 1)$ weights are indicated, and the prime denotes the contravariant representations. Notice that, consistently with the group factors $U(1)$ and $SO(1, 1)$, 27 is complex for $E_{6(2)}$, whereas it is real for $E_{6(6)}$. Both $E_{6(2)} \times U(1)$ and $E_{6(6)} \times SO(1, 1)$ are maximal non-compact subgroups of $E_{7(7)}$, with symmetric embedding.

Our result is simply stated as follows.

The two extremal BH charge configurations determining the embedding of RN and KK extremal BHs into $\mathcal{N} = 8, d = 4$ supergravity with entropies (1.1) and (1.2), are given by the two $E_6$-singlets in the decompositions (1.8) and (1.9).

The two situations can be efficiently associated to two different parametrizations of the real symmetric scalar manifold $\frac{E_{7(7)}}{SU(8)}$ ($dim_\mathbb{R} = 70, rank = 7$) of $\mathcal{N} = 8, d = 4$ supergravity.

For the branching (1.8), pertaining to the RN extremal BH, the relevant parametrization is the SU(8)-covariant one. This corresponds to the Cartan’s decomposition basis, where the coset coordinates $\phi_{ijkl}$ ($i = 1, \ldots, 8$) sit in the four-fold antisymmetric self-real irrep 70 of SU(8). The attractor mechanism implies that at the horizon

$$\phi_{ijkl,H} = 0,$$

i.e. the scalar configuration at the event horizon of the 1/8-BPS extremal BH is given by the origin of $\frac{E_{7(7)}}{SU(8)}$. Some care should be taken with regards to “flat” directions [8 19]. Due to the existence of the moduli space $\frac{E_{6(2)}}{SU(6) \times SU(2)}$ ($dim_\mathbb{R} = 40, rank = 4$) of the $\frac{1}{8}$-BPS
attractor solutions, strictly speaking 40 scalar degrees of freedom out of 70 are actually undetermined at the event horizon of the given $1/2$-BPS RN extremal BH. In other words, 40 real scalar degrees of freedom, spanning the quaternionic symmetric coset $\frac{E_{6(2)}}{SU(6) \times SU(2)}$, is strictly speaking 40 scalar degrees of freedom out of 70 are actually undetermined at the event horizon of the given $1/2$-BPS RN extremal BH. In other words, 40 real scalar degrees of freedom, spanning the quaternionic symmetric coset $\frac{E_{6(2)}}{SU(6) \times SU(2)}$, can be set to any real value, without affecting the RN BH entropy (1.1).

It should be noticed that, consistently with the Gaillard-Zumino formulation of electric-magnetic duality in presence of scalar fields [21], the solution (1.10) to the attractor equations is the only one allowed in presence of a compact underlying symmetry (in this case $U(1)$).

On the other hand, the best parametrization for the branching (1.9), pertaining to the KK extremal BH, is given by the KK radius

$$r_{KK} \equiv V^{1/3} \equiv e^{2\varphi},$$

by the 42 real scalars $\psi_{ijkl}$ ($i = 1, \ldots, 8$) sitting in the 42 of $USp(8)$, and by the 27 real axions $a^I$ ($I = 1, \ldots, 27$) sitting in the 27 of $USp(8)$ (or equivalently, in the 27 of $E_{6(6)}$).

In virtue of the attractor mechanism, the KK radius is stabilized as follows [22]:

$$r_{KK,H}^3 \equiv V_H \equiv e^{6\varphi_H} = 4 \left| \frac{q}{p} \right|,$$

while all axions vanish:

$$a^I_H = 0.$$

The 42 real scalars $\psi_{ijkl}$ are actually undetermined at the event horizon of the non-BPS KK BH, without affecting its entropy (1.2). Indeed, they span the moduli space $\frac{E_{6(6)}}{USp(8)}$ ($\dim_{\mathbb{R}} = 42$, rank = 6) of the non-BPS attractor solutions, which is the real symmetric scalar manifold of $\mathcal{N} = 8, d = 5$ supergravity [19].

It should be clear from our discussion that the possibility of having a non-vanishing scalar stabilized at the horizon of the KK extremal BH is related to the presence of a singlet in the relevant decomposition of the 70 scalars. This in turn is related to the existence of an underlying non-compact symmetry ($SO(1,1)$ in the present case), admitting no compact sub-symmetry.

An alternative way to obtain eqs. (1.1) and (1.2) is to use appropriate truncations for the bare charges in the corresponding expression of the quartic invariant $I_4$, which is known to be related to the Bekenstein-Hawking entropy by the formula

$$S = \sqrt{|I_4|}.$$  

The manifestly $SU(8)$-invariant expression of $I_4$ reads as follows:

$$I_4 = Tr \left(ZZ^\dagger\right)^2 - \frac{1}{4} Tr^2 \left(ZZ^\dagger\right) + 8 Re Pf (Z),$$

where $Z \equiv Z_{AB}$ ($\phi$) is the central charge $8 \times 8$ skew-symmetric matrix. Since (1.15) is moduli-independent, it can be evaluated at $\phi = 0$ without loss of generality, and in such a case $Z_{AB}$ is replaced by $Q_{AB}$, the bare charge matrix in the $SU(8)$ basis.
Considering the RN black hole, we will see that a suitable truncation of the $\mathcal{N} = 8$ bare charge matrix $Q_{AB}$ ($A, B = 1, \ldots, 8$), reduces it to the form

$$Q_{AB}^{\text{RN}} \to (z\epsilon_{ab}, 0), \quad z \equiv e + im,$$

where $a, b = 1, 2$ and $\epsilon^T = -\epsilon$). Thus one obtains

$$\mathcal{I}_4 = |z|^4 = (e^2 + m^2)^2,$$

which is nothing but Eq. (1.13) and it is also the same result as in pure $\mathcal{N} = 2, d = 4$ supergravity, which has a $U(1)$ global $R$-symmetry [11].

On the other hand, the manifestly $E_6(6)$-invariant expression of $\mathcal{I}_4$ in terms of the cubic invariant $\mathcal{I}_3$, as function of the bare electric and magnetic charges is given by [1, 23, 5]:

$$\mathcal{I}_4 = -\left(p^0 q_0 + p^i q_i\right)^2 + 4\left[q_0 \mathcal{I}_3 (p) - p^0 \mathcal{I}_3 (q) + \{\mathcal{I}_3 (p), \mathcal{I}_3 (q)\}\right].$$

(1.18)

By truncating the fluxes in such a way that

$$p^i = 0 = q_i,$$

(1.19)

one obtains ($p^0 \equiv p$, $q_0 \equiv q$)

$$\mathcal{I}_4 = - (pq)^2,$$

(1.20)

which now coincides with Eq. (1.14).

We will show that there is yet another way to obtain the two entropies for RN and KK black holes (1.1) and (1.2). This consists in using the attractor equations for the effective black hole potential $\frac{\partial V_{BH}}{\partial \phi} = 0$ and the expression of the entropy as the value of such potential at the critical point,

$$S = \pi V_{BH}|_{\text{crit}}.$$

(1.21)

The plan of this paper is as follows.

In Sect. 2 we consider various bases of $\mathcal{N} = 8, d = 4$ supergravity, namely the $SL(8, \mathbb{R})$, $SU(8)$- and $USp(8)$-covariant ones, exploiting the relevant branchings of the $U$-duality group $E_7(7)$. Then, Sect. 3 is devoted to the computation of the fundamental quantities for the geometry of the scalar manifold $E_7(7)$ in the $SL(8, \mathbb{R})$-covariant basis. Then, Sect. 4 analyses the $E_6(6)$-covariant basis, with the goal of exhibiting the connection with $\mathcal{N} = 8, d = 5$ supergravity: the $d = 4$ effective BH potential is recast in a manifestly $d = 5$ covariant form. Moreover, the charge configurations of this potential leading to vanishing axion fields are studied along with the corresponding attractor solutions. In Sect. 5 the embedding of the axion-dilaton extremal BH in $\mathcal{N} = 8, d = 4$ supergravity, through an intermediate embedding into $\mathcal{N} = 4, d = 4$ theory with 6 vector multiplets, is analyzed. Finally, Sect. 6 contains an outlook, as well as some concluding comments and remarks. The paper also contains in an Appendix the embedding of the $d = 5$ uplift of the $stu$ model (the so-called $(SO(1,1))^2$ model) into $d = 5$ maximal supergravity.
2 Symplectic Frames

The de Wit-Nicolai [24] formulation of $N = 8, d = 4$ supergravity is based on a symplectic frame where the maximal non-compact symmetry of the Lagrangian is $SL(8, \mathbb{R})$ [25], according to the decomposition

$$E_7(7) \rightarrow SL(8, \mathbb{R}),$$

$$56 \rightarrow 28 + 28^\prime,$$

where $SL(8, \mathbb{R})$ is a maximal non-compact subgroup of $E_7(7)$, and $28$ is its two-fold antisymmetric irreducible representation. Since the theory is pure, the $R$-symmetry, namely $SU(8)$, is the stabilizer of the scalar manifold. It is not a symmetry of the Lagrangian, but only of the equations of motion. The maximal compact symmetry of the Lagrangian is the intersection of $SL(8, \mathbb{R})$ with $SU(8)$, which is $SO(8)$ (the maximal compact subgroup of $SL(8, \mathbb{R})$ itself).

Another symplectic frame corresponds to the decomposition (1.9). In this case, the maximal non-compact symmetry of the Lagrangian is $E_6(6) \times SO(1, 1) \otimes_{s} T_{27}$, with “$\otimes_{s}$” denoting the semi-direct group product and $T_{27}$ standing for the 27-dimensional Abelian subgroup of $E_7(7)$. The maximal compact symmetry is now $USp(8)$, which is also the maximal compact symmetry of the Lagrangian. Note that all terms in the Lagrangian are $SU(8)$ invariant, with the exception of the vector kinetic terms, which are $SU(8)$-invariant only on-shell.

Let us decompose $E_7(7)$ along two different maximal non-compact subgroups according to the following diagram:

$$
\begin{array}{ccc}
E_7(7) & \rightarrow & SL(8, \mathbb{R}) \\
\downarrow & & \downarrow \\
E_6(6) \times SO(1, 1) & \rightarrow & SL(6, \mathbb{R}) \times SL(2, \mathbb{R}) \times SO(1, 1).
\end{array}
$$

If one goes first horizontally, the 56 of $E_7(7)$ decomposes as

$$56 \rightarrow 28 + 28^\prime \rightarrow \left\{ \begin{array}{l}
(15, 1, 1) + (6, 2, -1) + (1, 1, -3) + \\
+ (15', 1, -1) + (6', 2, 1) + (1, 1, 3).
\end{array} \right.$$

Alternatively, one can first go downward, and use that

$$E_6(6) \rightarrow SL(6, \mathbb{R}) \times SL(2, \mathbb{R});$$

$$27 \rightarrow (15, 1) + (6', 2),$$

$$1 \rightarrow (1, 1),$$

thus obtaining:

$$56 \rightarrow (27, 1) + (1, 3) + (27', -1) + (1, -3) \rightarrow \left\{ \begin{array}{l}
(15, 1, 1) + (6', 2, 1) + (1, 1, 3) + \\
+ (15', 1, -1) + (6, 2, -1) + (1, 1, -3).
\end{array} \right.$$
Therefore, either way on the diagram and irrespectively of the intermediate decomposition, one obtains the same irreducible representations of $SL(6,\mathbb{R}) \times SL(2,\mathbb{R}) \times SO(1,1)$, which enjoys a unique embedding in the $U$-duality group $E_{7(7)}$. In particular, one sees that the singlets are indeed the same in the two cases, and the alternative decompositions are related by the interchange of $(15,1,1)$ with $(15',1,-1)$. Then one concludes that these two formulations, corresponding to two different symplectic frames, can be interchanged by dualizing 15 out of the 28 vector fields.

An analogous argument holds if one decomposes $E_{7(7)}$ according to two different maximal compact subgroups along the diagram

$$E_{7(7)} \quad \rightarrow \quad SU(8)$$

$$\downarrow \quad \downarrow$$

$$E_{6(2)} \times U(1) \quad \rightarrow \quad SU(6) \times SU(2) \times U(1).$$

This time, going first horizontally along the diagram, the result reads:

$$56 \rightarrow 28 + 2\overline{8} \rightarrow \begin{cases} (15,1,1) + (6,2,-1) + (1,1,-3) + \\ + (\overline{15},1,-1) + (\overline{6},2,1) + (1,1,3). \end{cases}$$

Equivalently, one can first go vertically on the diagram and use

$$E_{6(2)} \rightarrow SU(6) \times SU(2);$$

$$27 \rightarrow (15,1) + (\overline{6},2),$$

$$1 \rightarrow (1,1),$$

thus obtaining:

$$56 \rightarrow (27,1) + (\overline{27},-1) + (1,3) + (1,-3) \rightarrow \begin{cases} (15,1,1) + (\overline{6},2,1) + (1,1,3) + \\ + (\overline{15},1,-1) + (6,2,-1) + (1,1,-3). \end{cases}$$

Again, either of the two alternative branchings in (2.6), which are related by the interchange of $(15,1,1)$ with $(\overline{15},1,-1)$, yield the same decomposition into irreducible representations of $SU(6) \times SU(2) \times U(1)$. Moreover, the $U(1)$ singlet which commutes with $SU(6) \times SU(2)$ is the same as the one which commute with $E_{6(2)}$.

Let us now turn to the scalar sector. As mentioned above, the coordinate system for the scalar manifold $E_{7(7)}/SU(8)$ based on the Cartan decomposition, the real scalars $\phi_{ijkl}$ sit in the $70$ (four-fold antisymmetric and self-real irreducible representation) of $SU(8)$ with $i = 1,\ldots,8$. The embedding of the RN extremal BH is related to the further decomposition

$$SU(8) \rightarrow SU(6) \times SU(2) \times U(1),$$

$$70 \rightarrow (20,2,0) + (15,1,-2) + (\overline{15},1,2).$$
On the other hand, for describing the KK extremal BH one decomposes $SU(8)$ under its maximal subgroup $USp(8)$:

$$SU(8) \rightarrow USp(8),$$

$$70 \rightarrow 42 + 27 + 1,$$

where 42 and 27 are respectively the four-fold and two-fold antisymmetric irreducible representations (both skew-traceless and self-real) of $USp(8)$.

The crucial difference between (2.10) and (2.11) is that the latter decomposition contains a real singlet, whereas the first one does not. This is related to an underlying maximal compact ($U(1)$ symmetry which is present for (2.10) and not for (2.11). This feature explains the different behaviour of the two solutions at the attractor point: the RN solution has the behaviour (1.10) while the KK solution is given by (1.12)-(1.13).

3 \hspace{0.5cm} SL(8, \mathbb{R})-Basis

In this section we aim at making contact between the symplectic formalism for extended supergravities reviewed in [26] and the original formulation of $\mathcal{N} = 8$ supergravity of [24] for some of the key geometrical objects that are relevant for the present investigation (see also [27] for recent developments).

We start by considering the coset representative for $E_{7(7)}/SU(8)$, which is parametrized as [24]

$$V = \begin{pmatrix} u_{ij}^I & v_{ijkl} \\ v_{iJ}^K & u_{KL}^j \end{pmatrix}. \quad (3.1)$$

The sub-matrices $u$ and $v$ carry indices of both $E_{7(7)}$ and $SU(8)$ ($I = 1, \ldots, 8$, $I = 1, \ldots, 8$) but one can choose a suitable $SU(8)$ gauge for the fields, and then retain only manifest invariance with respect to the rigid diagonal subgroup of $E_{7(7)} \times SU(8)$, without distinction among the two types of indices. Comparing the notation of [24] (in particular the appendix B) with the symplectic formalism of [21, 26], we can identify

$$\begin{cases} \phi_0 \equiv u \\ \phi_1 \equiv v \end{cases} \quad \rightarrow \quad u_{ij}^{kl} = (P^{1/2})_{ij}^{kl}, \quad v_{ijkl} = -(P^{1/2})_{mn}^{ij} \gamma^{mnkl},$$

so that

$$\begin{cases} f = \frac{1}{\sqrt{2}}(\phi_0 + \phi_1) = \frac{1}{\sqrt{2}}(u + v) \\ i\hbar = \frac{1}{\sqrt{2}}(\phi_0 - \phi_1) = \frac{1}{\sqrt{2}}(u - v) \end{cases}. \quad (3.2)$$

Since sections are sub-matrices of the symplectic representation, relatively to electric and magnetic subgroups, their explicit indices components are given by

$$f_{ij}^{kl} = \frac{1}{\sqrt{2}} \left( (P^{1/2})_{ij}^{kl} - (\overline{P}^{1/2})_{mn}^{ij} \gamma_{mnkl} \right),$$

$$h_{ij,kl} = \frac{-i}{\sqrt{2}} \left( (P^{1/2})_{ij}^{kl} + (\overline{P}^{1/2})_{mn}^{ij} \gamma_{mnkl} \right), \quad (3.3)$$

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where, in matrix notation,
\[
P = 1 - YY\dagger, \quad Y = B\tanh\sqrt{B\dagger B}, \quad B_{ij,kl} = -\frac{1}{2\sqrt{2}}\delta_{ijkl},
\]
the last definition coming from the choice of the symmetric gauge for the coset representative in Eq. (B.1) of [24]. If one defines
\[
\hat{P} = 1 - Y\dagger Y,
\]
and uses the identity
\[
(\hat{P}^{-1/2})Y\dagger = Y\dagger(\hat{P}^{-1/2}),
\]
the following simple expressions for \(f\) and \(h\) are finally achieved:
\[
f = \frac{1}{\sqrt{2}} \left[ P^{-1/2} - (\hat{P}^{-1/2})Y\dagger \right] = \frac{1}{\sqrt{2}} [1 - Y\dagger] \frac{1}{\sqrt{1 - YY\dagger}},
\]
\[
h = -\frac{i}{\sqrt{2}} \left[ P^{-1/2} + (\hat{P}^{-1/2})Y\dagger \right] = -\frac{i}{\sqrt{2}} [1 + Y\dagger] \frac{1}{\sqrt{1 - YY\dagger}}.
\]
The above notations are such that
\[
P^{1/2} = \sqrt{1 - YY\dagger} \quad \Rightarrow \quad P^{kl}_{ij} = \delta_{ij}^{kl} - y_{imn}\bar{y}^{mnkl}
\]
\[
\hat{P}^{1/2} = \sqrt{1 - Y\dagger Y} \quad \Rightarrow \quad \hat{P}^{kl}_{ij} = \delta_{ij}^{kl} - \bar{y}^{klmn}y_{mnij}
\]
It is easily checked that the symplectic sections satisfy the usual relations
\[
i(f\dagger h - h\dagger f) = 1,
\]
\[
h^T f - f^T h = 0.
\]
These are obtained writing the symplectic sections as in (3.7) and (3.8), and using the identity
\[
Y\hat{P}^{-1} = P^{-1}Y.
\]
The kinetic matrix is given in terms of the symplectic sections by [26]
\[
\mathcal{N} = hf^{-1}.
\]
Therefore, Eqs. (3.7) and (3.8) yield
\[
\mathcal{N}_{ijkl} = -i(\delta_{mn}^{kl} + \bar{y}^{mnkl})(\delta_{mn}^{ij} - \bar{y}^{ijnm})^{-1}.
\]
We now turn to the central charge function, which is defined by

\[ Z_{ij} = f_{ijkl} q_{kl} - h_{ijkl} p_{kl} \tag{3.14} \]

where electric and magnetic charges are in the same \( SO(8) \) adjoint representation as vector fields. Using the definitions in (3.3), one obtains

\[ Z_{ij} = \frac{1}{\sqrt{2}} \left( (P^{-1/2})_{ij}^{kl} - (\bar{P}^{-1/2})_{ij}^{mn} \bar{y}^{mnkl} \right) q_{kl} + \frac{i}{\sqrt{2}} \left( (P^{-1/2})_{ij}^{kl} + (\bar{P}^{-1/2})_{ij}^{mn} \bar{y}^{mnkl} \right) p_{kl} = \]

\[ = \left( \frac{1}{\sqrt{1 - YY}} \right)^{ij}_{kl} Q_{kl} - \left( \frac{1}{\sqrt{1 - YY}} \right)^{ij}_{mn} \bar{y}^{mnkl} \bar{Q}_{kl} \right] \tag{3.15} \]

where the complex charges

\[ Q_{ij} \equiv \frac{1}{\sqrt{2}} (q_{ij} + ip_{ij}) \tag{3.16} \]

have been introduced.

Then one can also give an expression for the BH potential, which is given by

\[ V_{BH} = \frac{1}{2} Z_{ij} Z^{ij} = \]

\[ = \frac{1}{4} \left[ (1 - YY)^{-1} Q_{kl} \bar{Q}_{ij} + \right. \]

\[ - \left( \sqrt{1 - YY} \right)^{-1} Q_{ab} \left( \sqrt{1 - YY} \right)^{-1} \bar{Q}_{cd} Y_{ijkl} Q_{kl} + \]

\[ - \left( \sqrt{1 - YY} \right)^{-1} \bar{Q}_{ab} \bar{Y}^{ijkl} \bar{Q}_{kl} \left( \sqrt{1 - YY} \right)^{-1} \bar{Q}_{cd} + \]

\[ + (1 - YY)^{-1} Y_{ijkl} Y^{jkl} Q_{kl} \right] \tag{3.17} \]

Thus, in the expansion around the zero field configuration, the BH receives contribution from the term

\[ V_{BH}(\phi = 0) = \frac{1}{4} Q_{ij} \bar{Q}^{ij} \tag{3.18} \]

The linear term in the expansion of the BH potential near the point \( \phi = 0 \) receives contributions from the second and third row of Eq. (3.17), yielding the condition

\[ Q_{ij} \bar{Q}_{ijkl} Q_{kl} - Q_{ij} \bar{Q}_{ijkl} Q_{kl} = 0 \tag{3.19} \]

\[ \Downarrow \]

\[ Q_{ij} Q_{kl} \delta_{ij}^{mnpq} - \frac{1}{4!} Q_{ij} \bar{Q}_{kl} \epsilon^{ijklmnpq} = 0 \tag{3.20} \]

The configuration corresponding to charges \( Q_{AB} \) in the singlet of \( SU(2) \times SU(6) \) trivially satisfies condition (3.20). Furthermore, it sets to zero the linear term for all values of \( \phi \), implying the \( \phi = 0 \) point to be an attractor point for this configuration.

\(^1\)The expression with explicit indices is given by

\[ \bar{P}_{ij}^{kl} = (\bar{P})_{ij}^{kl} \]
4 $E_{6(6)}$-Basis and Relation to $d = 5$

This section is aimed to establish the relation between the $\mathcal{N} = 8, d = 4$ theory and $\mathcal{N} = 8, d = 5$ supergravity ([28, 29]), especially for what concerns the effective BH potential.

In our normalisations the kinetic Lagrangian for vector fields in the $\mathcal{N} = 2$ theory reads (with $F_{\mu\nu} \equiv \frac{1}{2} (\partial_{\mu} A_\nu - \partial_{\nu} A_\mu) = \partial_{[\mu} A_{\nu]}$) [30, 31]
\[
\mathcal{L} = \ldots - \text{Im} \mathcal{N}_{\Lambda \Sigma} F^\Lambda F^{\Sigma} - \text{Re} \mathcal{N}_{\Lambda \Sigma} F^\Lambda \star F^{\Sigma},
\]
where $\mathcal{N}_{\Lambda \Sigma}$ is the $d = 4$ vector kinetic matrix, with $\Lambda, \Sigma = 0, 1, \ldots, 27$. The effective BH potential is given by [16]
\[
V_{BH} = -\frac{1}{2} Q^T \mathcal{M}(\mathcal{N}) Q,
\]
where $Q$ is the symplectic charge vector $Q = \left( p^\Lambda \right)$, and the matrix $\mathcal{M}$ reads [16]
\[
\mathcal{M}(\mathcal{N}) = \begin{pmatrix}
\text{Im} \mathcal{N} + \text{Re} \mathcal{N}(\text{Im} \mathcal{N})^{-1} \text{Re} \mathcal{N} & -\text{Re} \mathcal{N}(\text{Im} \mathcal{N})^{-1} \\
-(\text{Im} \mathcal{N})^{-1} \text{Re} \mathcal{N} & (\text{Im} \mathcal{N})^{-1}
\end{pmatrix}.
\]

The $d = 5$ $U$-duality group $E_{6(6)}$ acts linearly on the 27 vectors $\hat{A}_I^{\hat{\mu}}$, with $\hat{\mu} = 1, \ldots, 5$ and $I = 1, \ldots, 27$. The $d = 5$ vector kinetic matrix $\mathcal{N}_{IJ}$ is a function of the scalar fields spanning the $d = 5$ scalar manifold $\frac{E_{6(6)}}{USp(8)}$ ($\dim_R = 42, \text{rank} = 6$).

According to the splitting $\Lambda = \{0, I\}$, the $d = 4$ kinetic vector matrix assumes the block form
\[
\mathcal{N}_{\Lambda \Sigma} = \begin{pmatrix}
\mathcal{N}_{00} & \mathcal{N}_{0J} \\
\mathcal{N}_{I0} & \mathcal{N}_{IJ}
\end{pmatrix}.
\]

By using to the formulae obtained in [32], which determine $\mathcal{N}_{\Lambda \Sigma}$ in terms of five-dimensional quantities, in a normalization[^2] that is suitable for comparison to $\mathcal{N} = 2$, one obtains
\[
\mathcal{N}_{\Lambda \Sigma} = \begin{pmatrix}
\frac{1}{3} d_{IJK} a^I a^J a^K - i \left( e^{2\phi} a_{IJ} a^I a^J + e^{6\phi} \right) & -\frac{i}{2} d_{IJK} a^I a^K + i e^{2\phi} a_{KJ} a^K \\
-\frac{1}{2} d_{IKL} a^K a^L + i e^{2\phi} a_{IK} a^K & d_{IJK} a^K - i e^{2\phi} a_{IJ}
\end{pmatrix}.
\]

[^2]: Compared to the notation of [32], here we use $\mathcal{N}_{\Lambda \Sigma} \rightarrow 4\mathcal{N}_{\Lambda \Sigma}$, $2\mathcal{N}_{IJ} \rightarrow a_{IJ}$, $d_{IJK} \rightarrow -d_{IJK}/4$ and $a^I \rightarrow -a^I$. 

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Since the $d_{IKL}$ tensor, the $a^I$ fields, the $d = 5$ vector kinetic matrix $a_{IJ}$ and the field $\phi$ are real, the expressions for $\text{Im} \mathcal{N}$ and $\text{Re} \mathcal{N}$ are given by

$$\text{Im} \mathcal{N}_{\Lambda \Sigma} = -e^{6\phi} \begin{pmatrix} 1 + e^{-4\phi}a_{IJ}a^J & -e^{-4\phi}a_{KJ}a^K \\ -e^{-4\phi}a_{JK}a^K & e^{-4\phi}a_{IJ} \end{pmatrix}; \quad (4.6)$$

$$\text{Re} \mathcal{N}_{\Lambda \Sigma} = \begin{pmatrix} \frac{1}{3}d_{KLM}a^K a^L a^M & -\frac{1}{2}d_{JLM}a^L a^M \\ -\frac{1}{2}d_{JLM}a^L a^M & d_{IKJ}a^K \end{pmatrix} = \begin{pmatrix} \frac{1}{3}d & -\frac{1}{2}d_J \\ -\frac{1}{2}d_J & d_{IJ} \end{pmatrix}, \quad (4.7)$$

where the following shorthand notation has been introduced:

$$d \equiv d_{IKL}a^I a^J a^K, \quad d_I \equiv d_{IKJ}a^J a^K, \quad d_{IJ} \equiv d_{IKJ}a^K. \quad (4.8)$$

The inverse matrix $(\text{Im} \mathcal{N}_{\Lambda \Sigma})^{-1} \equiv \text{Im} \mathcal{N}^{\Lambda \Sigma}$ can be determined by noticing the block structure of $(4.6)$. Then, by performing computations analogous to those of [22], one finds

$$\left(\text{Im} \mathcal{N}^{-1}\right)^{\Lambda \Sigma} = -e^{-6\phi} \begin{pmatrix} 1 & a^J \\ a^I & a^I a^J + e^{4\phi}a^{IJ} \end{pmatrix}, \quad (4.9)$$

where $a^{IJ} \equiv (a_{IJ})^{-1}$. Inserting the above expressions into Eq. $(4.12)$, the $\mathcal{N} = 8$, $d = 4$ effective BH potential can finally be rewritten in a $d = 5$ language:

$$V_{BH} = (p^0)^2 \left[ \frac{1}{2}e^{2\phi}a_{IJ}a^Ia^J + \frac{1}{2}e^{6\phi} + \frac{1}{8}e^{-6\phi} \left( \frac{d^2}{9} + e^{4\phi}a^{IJ}d_Jd_I \right) \right] +
+p^0p^J \left[ -e^{2\phi}a_{IJ}a^J - \frac{1}{4}e^{-6\phi} \left( \frac{1}{3}d_Jd_I + 2e^{4\phi}a^{KJ}d_Kd_I \right) \right] +
+p^Ip^J \left[ \frac{1}{2}e^{2\phi}a_{IJ} + \frac{1}{8}e^{-6\phi} \left( d_Id_J + 4e^{4\phi}a^{KL}d_Kd_Ld_J \right) \right] +
+\frac{1}{6}g_0p^0 e^{-6\phi}d + \frac{1}{6}g_1p^0 e^{-6\phi} \left[ da^I + 3e^{4\phi}a^{K}d_K \right] +
+\frac{1}{2}g_0p^I e^{-6\phi}d_I - \frac{1}{2}g_1p^I e^{-6\phi} \left[ d_Ja^I + 2e^{4\phi}a^{KI}d_Jd_K \right] +
+\frac{1}{2}(q^0)^2 e^{-6\phi} + q_0q_I e^{-6\phi}a^I + \frac{1}{2}y_{IJ} e^{-6\phi} \left[ a^Ia^J + e^{4\phi}a^{IJ} \right]. \quad (4.10)$$
Notice that this formula becomes identical to the corresponding one of [22] concerning (purely cubic) \( N = 2 \) geometries [33, 34], where \( a_{IJ} = 4 e^{3\phi} g_{ij} \) and \( V \equiv e^{6\phi} \).

The potential (4.11), because of the definitions (4.8), can be seen to be a polynomial of degree up to sixth in the axion fields, whose general solutions are hard to determine. However, one can consider in particular attractor solutions with vanishing axion fields. These are given by specific charge configurations that solve the following attractor equations:

\[
\frac{\partial V_{BH}}{\partial a^I} \bigg|_{a^J = 0} = -e^{2\phi} p^0 p^K a_{KI} - e^{-2\phi} q_J p^K d_{ILK} a^{JL} + q_0 q_I e^{-6\phi} = 0 . \quad (4.11)
\]

Therefore, the BH charge configurations \( Q = (p^0, p^I, q_0, q_I) \) supporting axion-free solutions fall into three classes:

a) Electric BH:

\[
V_{BH}(\phi, p^0, q_I) \bigg|_{a^J = 0} = \frac{1}{2} e^{6\phi} (p^0)^2 + \frac{1}{2} e^{-2\phi} a^{IJ} q_I q_J . \quad (4.13)
\]

b) Magnetic BH:

\[
V_{BH}(\phi, q_0, p^J) \bigg|_{a^I = 0} = \frac{1}{2} e^{-6\phi} (q_0)^2 + \frac{1}{2} e^{2\phi} a^{IJ} p_I p^J . \quad (4.14)
\]

c) BH charged with respect to the KK vector:

\[
V_{BH}(\phi, q_0, p^0) \bigg|_{a^I = 0} = \frac{1}{2} e^{-6\phi} (q_0)^2 + \frac{1}{2} e^{6\phi} (p^0)^2 . \quad (4.15)
\]

In order to recover the complete attractor solution, one also has to stabilize \( e^{\phi} \). For the KK charged BH one gets,

\[
\frac{\partial V_{BH}^{KK}}{\partial \phi} \bigg|_{a^I = 0} = 0 \quad \iff \quad e^{6\phi} = \left| \frac{q_0}{p^0} \right| , \quad (4.16)
\]

thus yielding

\[
V_{BH}^{KK}(q_0, p^0) \bigg|_{a^I = 0} = \left| q_0 p^0 \right| . \quad (4.17)
\]

In the electric case it holds that

\[
\frac{\partial V_{BH}^e}{\partial \phi} \bigg|_{a^I = 0} = 0 \quad \iff \quad e^{2\phi} = \left( \frac{a^{IJ} q_I q_J}{3(p^0)^2} \right)^{\frac{1}{2}} , \quad (4.18)
\]

implying the critical value

\[
V_{BH}^e(q_I, p^0) \bigg|_{a^I = 0} = 2 |p^0|^{1/2} \left( \frac{a^{IJ} q_I q_J}{3} \right)^{3/4} . \quad (4.19)
\]
Analogously, for the magnetic BH one finds
\[ \frac{\partial V_{BH}^m}{\partial \phi} \bigg|_{a^I = 0} = 0 \quad \iff \quad e^{2\phi} = \left( \frac{a_{IJ} p^I p^J}{3 q_0^2} \right)^{-\frac{1}{4}}, \quad (4.20) \]
yielding
\[ V_{BH}^m(q_0, p^I) \bigg|_{a^I = 0} = 2 |q_0|^{1/2} \left( \frac{a_{IJ} p^I p^J}{3} \right)^{3/4}. \quad (4.21) \]

In virtue of the Bekenstein-Hawking entropy-area formula, the above expressions for the critical electric and magnetic BH potentials must be compared with appropriate powers of the \( E_6(6) \) cubic invariants \( I_3(p) \equiv \frac{1}{3!} d_{IJK} p^I p^J p^K \) and \( I_3(q) \equiv \frac{1}{3!} d_{IJK} q^I q^J q^K \). Indeed, in \( d = 5 \) it must hold that \( S \sim V^{3/4} \big|_{crit} \sim |I_3|^{1/2} \),
\[ (4.22) \]
Defining the electric and magnetic \( d = 5 \) effective potentials respectively as
\[ V_5^e = a^{IJ} q_I q_J, \quad V_5^m = a_{IJ} p^I p^J \quad (4.23) \]
one obtains
\[ V_{crit}^e = 2 |p^0|^{1/2} \left( \frac{V_5^e}{3} \right)^{3/4} \bigg|_{crit} \quad (4.24) \]
and
\[ V_{crit}^m = 2 |q_0|^{1/2} \left( \frac{V_5^m}{3} \right)^{3/4} \bigg|_{crit}. \quad (4.25) \]
By comparison with \( \mathcal{N} = 2 \) symmetric \( d- \)geometries having
\[ V_5^e \big|_{crit} = |I_3(q)|^{2/3} = |q_1 q_2 q_3|, \quad (4.26) \]
one obtains the expressions for the critical potential of the four dimensional electric and magnetic BHs:
\[ V_{BH \, crit}^e(q_I, p^0) = 2 \sqrt{\frac{|p^0| d_{IJK} q_I q_J q_K}{3!}}, \quad (4.27) \]
and
\[ V_{BH \, crit}^m(q_0, p^I) = 2 \sqrt{\frac{|q_0| d_{IJK} p^I p^J p^K}{3!}}. \quad (4.28) \]

More generally, these solutions can be compared with the embedding of the \( \mathcal{N} = 2 \) purely cubic supergravities into \( \mathcal{N} = 8 \) supergravity, and using the above critical values of the BH potential in (1.21), one finds for the three family of configurations under exam the correct result:
\[ \frac{S_{BH}}{\pi} = \sqrt{|I_4|}. \quad (4.29) \]

It is interesting to remark that the KK black hole can be connected to the RN solution by performing an analytic continuation of the charges, as one can see from the redefinition
\[ p^0 \to p + iq, \quad q_0 \to p - iq, \]
13
in, which allows one to recover the RN entropy

\[ S_{RN} = \pi \left( p^2 + q^2 \right). \]  (4.30)

We conclude this Section by pointing out that the 70 scalars of \( \mathcal{N} = 8, \ d = 4 \) supergravity have been decomposed according to representations of \( USp(8) \) (maximal compact subgroup of \( E_{6(6)} \times SO(1,1) \)) as follows:

\[ 70 \rightarrow 42 + 27 + 1. \]  (4.31)

The 42 unstabilized fields are the coordinates of the corresponding moduli space [19]. The non-compact form of the exceptional group, \( E_{6(6)} \), in fact, enters in the expression of the coset

\[ \frac{E_{6(6)}}{USp(8)}, \]  (4.32)

which is the moduli space of the \( d = 4 \) non-BPS, \( Z_{AB} \neq 0 \) extremal BHs, whose orbit is precisely

\[ \mathcal{O} = \frac{E_{7(7)}}{E_{6(6)}}. \]  (4.33)

Indeed, the KK BH is indeed a non supersymmetric solution (see also Sect. [1]).

5 Embedding of the Axion-Dilaton Extremal BH

The embedding of the axion-dilaton BH in \( \mathcal{N} = 8, \ d = 4 \) supergravity can be performed by a three step supersymmetry reduction, which can be schematically indicated as

\[ \mathcal{N} = 8 \rightarrow \mathcal{N} = 4, \ n_V = 6 \rightarrow \text{pure} \mathcal{N} = 4 \rightarrow \mathcal{N} = 2 \quad \text{quadratic}, \ n_V = 1, \]  (5.1)

where \( n_V \) denotes the number of vector multiplets coupled to the supergravity multiplet. More precisely, the first step consists in truncating \( \mathcal{N} = 8 \) supergravity to an \( \mathcal{N} = 4 \) theory interacting with six matter (vector) multiplets. In the second step, \( \mathcal{N} = 4 \) reduces to the pure theory, while in the last reduction one obtains \( \mathcal{N} = 2 \) supergravity quadratic theory with a single vector multiplet.

Let us examine more precisely each intermediate step.

1) In the first step, the \( \mathcal{N} = 8 \) central charge matrix \( Z_{AB} \) assumes the block form \( (a,b = 1,..,4, \ i, j = 1,...,4) \):

\[ Z_{AB} \rightarrow \begin{pmatrix} Z_{ab} & 0 \\ 0 & i Z_{ij} \end{pmatrix}. \]  (5.2)

where \( Z_{ab} \) is the \( \mathcal{N} = 4 \) central charge matrix and \( Z_{ij} \) are the matter charges of the 6 vector multiplets (sitting in the two-fold antisymmetric of \( SU(4) \), or equivalently in the vector representation of \( SO(6) \sim SU(4) \)).
Consequently, the $\mathcal{N} = 8$ scalar manifold $E_{7(7)}/SU(8)$, reduces to
\[
\frac{SL(2, \mathbb{R}) \times SO(6, 6)}{U(1) \times SO(6) \times SO(6)} = \frac{SL(2, \mathbb{R}) \times SO(6, 6)}{U(1) \times SU(4) \times SU(4)},
\]
which admits three orbits. This is the scalar manifold for $\mathcal{N} = 4$ supergravity coupled to 6 vector multiplets.

2) In the second step, the 6 vector multiplets are eliminated and $Z_{ij} = 0$; this corresponds to retaining only states which are singlets with respect to the second $SU(4)$ in the stabilizer of the coset (5.3), and the theory becomes pure $\mathcal{N} = 4$, with $U$-duality $SL(2, \mathbb{R}) \times SU(4)$:
\[
\begin{pmatrix}
Z_{ab} & 0 \\
0 & iZ_{ij} & \\
\end{pmatrix} \rightarrow \begin{pmatrix}
Z_{ab} & 0 \\
0 & 0 & \\
\end{pmatrix}, \tag{5.4}
\]
with $\epsilon = (0, 1)$. Accordingly, the scalar manifold reduces to $\frac{SL(2, \mathbb{R})}{U(1)}$. Notice that, the presence of the axion-dilaton $s$ spanning $\frac{SL(2, \mathbb{R})}{U(1)}$, in the $\mathcal{N} = 4$ supergravity multiplet, only an $SU(4)$ out of the whole (local) $\mathcal{N} = 4 \mathbb{R}$-symmetry $U(4)$ gets promoted to (global) $U$-duality symmetry.

3) In the last step, 4 out of 6 graviphotons drop out, reducing the overall gauge symmetry from $U(1)^6$ to $U(1)^2$, with resulting $U$-duality $SL(2, \mathbb{R}) \times SU(2)$.

Therefore, at the $\mathcal{N} = 2$ level one can have both BPS attractors ($D_s Z = 0$) and the non-BPS ($Z = 0$) ones.

On a group theoretical side, this step correspond to performing the decomposition
\[
SU(4) \rightarrow SU(2) \times SU(2) \times U(1),
\]
\[
4 \rightarrow (2, 1, \frac{1}{2}) + (1, 2, -\frac{1}{2}) \tag{5.5},
\]
\[
6 \rightarrow (2, 2, 0) + (1, 1, 1) + (1, 1, -1),
\]
and to retaining only the singlets of $SU(2) \times SU(2)$.

The above three step reduction can be viewed from the point of view of the classification of large charge orbits. One starts with the $\mathcal{N} = 8$ scalar manifold $E_{7(7)}/SU(8)$ admitting the two regular orbits (1.5) and (1.6). The large charge orbits of $\mathcal{N} = 4$, $d = 4$ supergravity coupled to 6 vector multiplets are:
\[
\begin{cases}
\mathcal{O}_{1/4\text{BPS}} : & SL(2, \mathbb{R}) \times \frac{SO(6, 6)}{SO(2) \times SO(6, 4)}; \\
\mathcal{O}_{\text{non-BPS}, Z_{ab}=0} : & SL(2, \mathbb{R}) \times \frac{SO(6, 6)}{SO(2) \times SO(6, 4)}; \\
\mathcal{O}_{\text{non-BPS}, Z_{ab}\neq 0} : & SL(2, \mathbb{R}) \times \frac{SO(6, 6)}{SO(1, 1) \times SO(5, 5)}.
\end{cases} \tag{5.7}
\]
where the coincidence of the first two orbits is due to the symmetry between the gravity and the matter sector.

The corresponding moduli spaces for the $\mathcal{N} = 4$, $n = 6$ attractor solutions, exploiting the hidden symmetries of the above charge orbits, are given by:

$$\begin{align*}
\mathcal{M}_{BPS} &= \frac{SO(6,4)}{SU(4) \times SU(2) \times SU(2)}; \\
\mathcal{M}_{non\,BPS,Z_{ab}=0} &= \frac{SO(6,4)}{SO(6) \times SO(4)}; \\
\mathcal{M}_{non\,BPS,Z_{ab} \neq 0} &= SO(1,1) \times \frac{SO(5,5)}{SO(5) \times SO(5)} = SO(1,1) \times \frac{SO(5,5)}{USp(4) \times USp(4)}.
\end{align*}$$

Notice that $\mathcal{M}_{1/4\,BPS}$ (and $\mathcal{M}_{non\,-BPS,Z_{ab}=0}$) are homogeneous symmetric quaternionic manifolds, as in the $\mathcal{N} = 4 \to \mathcal{N} = 2$ reduction they become the hypermultiplets’ scalar manifold [26].

The truncation of the $\mathcal{N} = 8$ theory into $\mathcal{N} = 4$ is based on the decomposition

$$E_{7(7)} \to SL(2, R) \times SO(6, 6)$$

and on the following group embeddings

$$SO(6, 4) \times SO(2) \subseteq E_{6(2)};$$

$$SO(5, 5) \times SO(1, 1) \subseteq E_{6(6)}.$$ 

Therefore, one can readily establish that the orbits 1/4 BPS and non BPS, $Z_{ab} = 0$ descend from the $\mathcal{N} = 8$, BPS orbit $E_{7(7)} / E_{6(2)}$, whereas the orbit $\mathcal{O}_{non\,BPS, Z_{ab} \neq 0}$ comes from the $\mathcal{N} = 8$, non-BPS orbit $E_{7(7)} / E_{6(6)}$.

There is also another way to interpret the three step reduction (5.1), that is in terms of $U$-duality invariant representations. At group level, the embedding of the axion-dilaton extremal BH into $\mathcal{N} = 8$, $d = 4$ supergravity is based on the decomposition of $E_{7(7)} \to SU(8)$ and

$$SU(8) \to SU(4) \times SU(4) \times U(1),$$

$$8 \to (4, 1, \frac{1}{2}) + (1, 4, -\frac{1}{2}),$$

$$28 \to (4, 4, 0) + (6, 1, 1) + (1, 6, -1),$$

$$\overline{28} \to (\overline{4}, \overline{4}, 0) + (6, 1, -1) + (1, 6, 1),$$

where $SU(4) \times SU(4) \times U(1)$ is a maximal subgroup of $SU(8)$.

Then, the first truncation ($\mathcal{N} = 8 \to \mathcal{N} = 4$, $n = 6$) consists in setting

$$(4, 4, 0) = 0 = (\overline{4}, \overline{4}, 0),$$

which gives rise to the decomposition (5.2).

We recall that the quartic invariant of the $U$-duality group $SL(2, \mathbb{R}) \times SO(6, n)$ of $\mathcal{N} = 4$, $d = 4$ supergravity coupled to $n$ vector multiplets is [8]

$$I_4 = S_1^2 - |S_2|^2,$$
where the three $SO(6,n)$ invariants $S_1$, $S_2$ and $\overline{S}_2$ are defined by $(a,b = 1, \ldots, 4, I = 1, \ldots, n)$:

$$
S_1 \equiv \frac{1}{2} Z_{ab} \overline{Z}^{ab} - Z_I \overline{Z}^I ;
$$

$$
S_2 \equiv \frac{1}{4} \epsilon^{abcd} Z_{ab} Z_{cd} - \overline{Z}_I \overline{Z}^I .
$$

(5.15)

(5.16)

The case $n = 6$ is remarkably symmetric, as the symmetry of the gravity and matter sector is the same and furthermore, due to the isomorphism $SU(4) \sim SO(6)$, the $SO(6)$-vector $Z_I$ of matter charges can be equivalently represented as the $SU(4)$-antisymmetric tensor $i \overline{Z}_{ij}$ ($i,j = 1, \ldots, 4$). Consequently, for $n = 6$ we have

$$
S_{1,n=6} \equiv \frac{1}{2} Z_{ab} \overline{Z}^{ab} - \frac{1}{2} Z_{ij} \overline{Z}^{ij} ;
$$

$$
S_{2,n=6} \equiv \frac{1}{4} \epsilon^{abcd} Z_{ab} Z_{cd} - \frac{1}{4} \epsilon_{ijkl} Z^{ij} Z^{kl} .
$$

(5.17)

(5.18)

Notice that $O_{1/4BPS}$ and $O_{nonBPS}$, $Z_{ab}=0$ in Eq. (5.7) correspond to the two disconnected branches of the same manifold, classified by the sign of the real $SO(6,6)$-invariant [26]. Indeed, $S_{1,n=6} > 0$ for $O_{1/4BPS}$ and $S_{1,n=6} < 0$ for $O_{nonBPS}$, $Z_{ab}=0$.

By a suitable $U(1) \times SU(4) \times SU(4)$ transformation, one can reach the normal frame for both gravity sector and matter sector, such that the two matrices $Z_{ab}$ and $Z_{ij}$ are simultaneously skew-diagonalized, obtaining

$$
Z_{ab} \rightarrow \left( \begin{array}{cc} Z_1 & \varepsilon \\ Z_2 & \varepsilon \end{array} \right) \otimes \epsilon ;
$$

$$
Z_{ij} \rightarrow e^{i\theta} \left( \begin{array}{cc} Z_3 & \varepsilon \\ Z_4 & \varepsilon \end{array} \right) \otimes \epsilon ,
$$

(5.19)

(5.20)

where $Z_1, Z_2 \in \mathbb{R}^+$, and $Z_3, Z_4 \in \mathbb{R}^+$, $\theta \in [0, 2\pi)$. Thus, in the normal frame one obtains

$$
S_{1,n=6} = |Z_1|^2 + |Z_2|^2 - |Z_3|^2 - |Z_4|^2 ;
$$

$$
S_{2,n=6} = 2 \left( Z_1 Z_2 - Z_3 \overline{Z}_4 \right) ;
$$

$$
I_{4,n=6} = S^2_{1,n=6} - |S_{2,n=6}|^2 = \sum_{i=1}^4 |Z_i|^4 - 2 \sum_{i<j=1}^4 |Z_i|^2 |Z_j|^2 + 4 \prod_{i=1}^4 Z_i + \prod_{i=1}^4 \overline{Z}_i .
$$

(5.21)

(5.22)

(5.23)

Eq. (5.23) coincides with the expression of the quartic invariant of $\mathcal{N} = 8$, $d = 4$ supergravity, as given by [7] (see also [3]). Considering now the second step of the reduction, where one reaches the pure $\mathcal{N} = 4$ theory, one sets $Z_{ij} = 0$, or equivalently $Z_3 = 0 = Z_4$ in the normal frame (that is, retaining only states which are singlets with respect to the second $SU(4)$ in the stabilizer of the coset (5.3)). Notice that, by doing so, $I_{4,n=0}$ becomes a perfect square:

$$
I_{4,n=0} = S^2_{1,n=0} - |S_{2,n=0}|^2 = (|Z_1|^2 - |Z_2|^2)^2 = (Z_1^2 - Z_2^2)^2 .
$$

(5.24)

Eq. (5.24) implies that $I_{4,n=0}$ is (weakly) positive, and as a consequence an unique class of large attractor exists, namely the 1/4-BPS one. The (weak) positivity of $I_{4,n=0}$ is
consistent with the known expression of \( I_{4,n=0} \) in terms of the magnetic and electric charges \((p^A, q)\) \((\Lambda = 1, \ldots, 6)\):

\[
I_{4,n=0} = 4 \left[ p^2 q^2 - (p \cdot q)^2 \right],
\]

(5.25)

where here \( p^2 \equiv p^A p^B \delta_{AB} \), \( q^2 \equiv q_A q_B \delta_{AB} \) and \( p \cdot q \equiv p^A q^B \delta_{AB} \). Notice that in the basis of bare charges \( I_{4,n=0} \), as given by Eq. (5.25), is (weakly) positive due to the Schwarz inequality, and not because it is a non-trivial perfect square of an expression of the bare magnetic and electric charges [37].

Notice that \( \sqrt{I_{4,n=0}} \) (with \( I_{4,n=0} \) given by Eq. (5.25)) must coincide with the value of the effective BH potential of the pure \( \mathcal{N} = 4 \) theory at its critical points. This can be understood (see the recent discussion given in [26] and [38]) because this potential reads as follows \((\Lambda = 1, \ldots, 6)\):

\[
V_{BH,\text{pure},\mathcal{N}=4} (\phi, a, p, q) = e^{2\phi} (sp_A - q_A)(s p_A - q_A) = (e^{2\phi} a^2 + e^{-2\phi}) p^2 + e^{2\phi} q^2 - 2ae^{2\phi} p \cdot q,
\]

(5.26)

where the complex (axion-dilaton) field

\[
s \equiv a + ie^{-2\phi}
\]

(5.27)

parametrizes the coset \( SU(1,1)/U(1) \) of \( \mathcal{N} = 4, d = 4 \) pure supergravity [39].

By computing the criticality conditions of \( V_{BH,\text{pure},\mathcal{N}=4} \), one obtains the following stabilization equations for the axion \( a \) and the dilaton \( \phi \) at criticality, \((\phi, a) = (\phi_H (p, q), a_H (p, q))\):

\[
\frac{\partial V_{BH}}{\partial a} \bigg|_{\text{crit}} = 0 \iff a_H (p, q) = \frac{p \cdot q}{p^2},
\]

(5.28)

\[
\frac{\partial V_{BH}}{\partial \phi} \bigg|_{\text{crit}} = -e^{-4\phi} p^2 + q^2 - a_H (p, q) p \cdot q = -e^{-4\phi} p^2 + q^2 - \frac{(p \cdot q)^2}{p^2} = 0;
\]

\[
\updownarrow
\]

\[
e^{-2\phi_H (p, q)} = \sqrt{\frac{p^2 q^2 - (p \cdot q)^2}{p^2}}.
\]

(5.29)

Thus, the Bekenstein-Hawking BH entropy is computed to be

\[
S_{BH} (p, q) = \frac{A_H (p, q)}{4} = \pi V_{BH} (\phi_H (p, q), a_H (p, q), p, q) = 2\pi \sqrt{p^2 q^2 - (p \cdot q)^2} = \pi \sqrt{I_{4,n=0}}.
\]

(5.30)

The third and last step, when the pure \( \mathcal{N} = 4 \) theory reduces to the \( \mathcal{N} = 2 \) quadratic theory with \( n_V = 1 \), is performed through the truncation \((U (1))^6 \to (U (1))^2\) of the overall Abelian gauge invariance \((\Lambda = 1, \ldots, 6 \to \Lambda = 1, 2)\). In this case, \( I_{4,n=0, (U (1))^6 \to (U (1))^2} \) is a perfect square in both the basis of \( Z_{ab} \) and in the basis of charges \((p^A, q)\), and it actually is the square of the quadratic invariant \( I_{2(n=1)} \) of the axion-dilaton system:

\[
I_{4,n=0, (U (1))^6 \to (U (1))^2} = \left( |Z_1|^2 - |Z_2|^2 \right)^2 = 4 \left( p^1 q_2 - p^2 q_1 \right)^2 = I_{2(n=1)}^2;
\]

\[
\updownarrow
\]

\[
I_{2(n=1)} = \pm 2 \left| p^1 q_2 - p^2 q_1 \right|
\]

(5.31)

(5.32)
implying that the axion-dilaton system exhibits two types of attractors: the \( \frac{1}{2} \)-BPS one \( (I_2(n=1) > 0) \) and the non-BPS \( Z = 0 \) one \( (I_2(n=1) < 0) \).

By further putting
\[
p_1 = 0 = q_2, \quad p_2^2 \equiv p, \quad q_1 \equiv q
\]
\((\Rightarrow p \cdot q = 0)\), one obtains:
\[
I_4^* = I_2^*(n=0, U(1)^6 \rightarrow U(1)^2) = 4 (pq)^2;
\]
\[
I_2(n=1) = \pm 2 |pq|,
\]
where \( I^* \) means the evaluation along Eq. (5.33). For a recent treatment of the axion-dilaton-Maxwell-Einstein-(super)gravity system and of the extremal BH attractors therein, see e.g. Sects. 6 and 7 of [38].

The similarity between the r.h.s.’s of Eqs. (1.4) and (5.35) is only apparent. In fact, the KK extremal BH has \( \sqrt{-I_{4,KK}} \), which necessarily implies that it is non-BPS \((Z_{AB} \neq 0 \text{ in } N = 8 \text{ and } Z \neq 0 \text{ in } N = 2)\). On the other hand, the axion-dilaton extremal BH has \( I_2(n=1) \) and a “±” in the r.h.s., so that it can be both \( \frac{1}{2} \)-BPS and non-BPS \( Z = 0 \) in \( N = 2 \). Moreover, the choice (5.33) leads to vanishing axion \( a \) (see Eq. (5.28)), and this explains that Eqs. (5.35) has \( SO(1,1) \) symmetry, as Eq. (1.4).

### 5.1 Truncations of the scalar sector

As reported e.g. in Sects. 6 and 7 of [38], one can see that the attractor mechanism stabilizes the complex axion-dilaton s at the event-horizon of the axion-dilaton extremal BH itself, while, as given by Eqs. (1.12) and (1.13) within the branching (2.11), only one real scalar degree of freedom, namely the KK radius \( r_{KK} \) defined by Eq. (1.11), is stabilized at the event horizon of the extremal KK BH.

The relevant branching of the scalar sector for the embedding of the axion-dilaton extremal BH into \( N = 8 \), \( d = 4 \) supergravity is given by:

\[
SU(8) \rightarrow SU(4) \times SU(4) \times U(1),
\]
\[
70 \rightarrow (1, 1, 2) + (1, 1, -2) + (6, 6, 0) + (\overline{4}, 4, 1) + (4, \overline{4}, -1).
\]

Eq. (5.36) is the analogue of Eqs. (2.10) and (2.11), holding respectively for the \( (N = 8, d = 4 \) embedding of the) RN and KK \( d = 4 \) extremal (and asymptotically flat) BHs.

A remarkable feature characterizing the branchings (2.10), (2.11) and (5.36) is the possible presence of a singlet in their r.h.s.’s. The decomposition (5.36) contains two \( SU(4) \times SU(4) \) singlets, whereas the decomposition (2.11) contains a real singlet, and the decomposition (2.10) does not contain any singlet. The presence of the singlet may lead to an underlying maximal compact symmetry \( U(1) \) for (2.10), absent for (2.11), and \( SU(4) \) for (5.36).

1. The first truncation \((N = 8 \rightarrow N = 4, n_V = 6)\) corresponds to setting
\[
(\overline{4}, 4, 1) = 0 = (4, \overline{4}, -1).
\]

\(^3\)Notice the difference with respect to the analogue truncation condition (5.13) for the decomposition of the 28 and \( 2\overline{8} \) of \( SU(8) \).
Indeed, by applying the condition (5.37), one obtains the correct quantum numbers of the scalar manifold \( SL(2,\mathbb{R}) \times U(1) \times SO(6) \times SO(6) \) of the \( \mathcal{N} = 4, d = 4 \) supergravity coupled to 6 vector multiplets.

2. The second truncation \( (\mathcal{N} = 4, n_V = 6 \rightarrow \text{pure } \mathcal{N} = 4) \) simply consists in implementing the condition

\[
(6, 6, 0) = 0, \tag{5.38}
\]

which is consistently symmetric under the exchange of the gravity sector and the matter sector. Through condition (5.38), one achieves the correct quantum numbers of the scalar manifold \( SL(2,\mathbb{R}) \times U(1) \) of the pure \( \mathcal{N} = 4, d = 4 \) supergravity.

3. The third and last step \( (\text{pure } \mathcal{N} = 4 \rightarrow \mathcal{N} = 2 \text{ quadratic}, n_V = 1) \) does not change anything with respect to the previous one. Indeed, the scalar sector is unaffected by this third truncation, and the scalar manifold remains \( SL(2,\mathbb{R}) \times U(1) \).

6 Conclusions

In the present investigation, we have considered some examples of extremal BH configurations in the framework of BH attractors of \( \mathcal{N} = 8 \) supergravity.

The effective BH potential has been computed in different bases, namely in the manifestly \( SU(8) \)-covariant basis, as well as in the \( USp(8) \)-covariant one. The former is suitable to describe the (BPS) Reissner-Nordstrom extremal BH with its \( U(1) \) symmetry, as a consequence of the attractor point to be the origin of the \( d = 4 \) scalar manifold \( E_{7(7)} / SU(8) \). The latter has \( d = 5 \) origin, and it is appropriate in order to describe the non-BPS Kaluza-Klein extremal BH, with its \( SO(1,1) \) symmetry arising from the non-trivial attractor value of the KK radial mode.

We have also considered the axion-dilaton system, whose BPS or non-BPS nature depends on whether it is embedded in \( \mathcal{N} = 2 \) quadratic or in \( \mathcal{N} = 4, d = 4 \) supergravity. The axion-dilaton extremal BH is obtained as a particular case of the attractor equations of the maximal \( d = 4 \) theory. In that case, all 70 scalars other than the \( SU(4) \times SU(4) \)-singlets in the decomposition (5.36) are set to vanish, and correspondingly only 12 graviphoton electric and magnetic charges are taken to be nonzero (see Eq. (5.12)). At the level \( \mathcal{N} = 2 \), this attractor solution is obtained by retaining only 4 (2 electric and 2 magnetic) non-vanishing charges, according to the decomposition (5.6) of \( SU(4) \).

In Appendix A, we have finally considered the embedding of the \( stu \) model in \( \mathcal{N} = 8, d = 4 \) and \( d = 5 \) supergravity, is considered. In the \( d = 4 \) case, all non-singlet charges in the decomposition of \( E_{7(7)} \) with respect to \( SO(4, 4) \times (SL(2,\mathbb{R}))^3 \) are set to vanish \( [H] \), whereas for \( d = 5 \) one obtains an axion-free framework, given by non-zero values for \((p^0, q_1, q_2, q_3) \) or \((q_0, p^1, p^2, p^3) \).

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\section{Appendix}

\subsection{Truncation of $\mathcal{N} = 8, d = 5$ supergravity to the $d = 5$ uplift of the $stu$ model}

The bosonic sector of the $\mathcal{N} = 8, d = 5$ supergravity theory consists in the metric $g_{\mu\nu}$ ($\mu, \nu = 1, \ldots, 5$), 27 vectors $A_\mu^A$ and 42 scalars $\phi_{abcd}$ parametrizing the coset $\frac{E_{6(6)}}{USp(8)}$. The index $A = 1, \ldots, 27$ is in the 27 of $E_{6(6)}$, and it can be traded for a couple of flat antisymmetric indices $(ab)$ of $USp(8)$. Thus, the vectors $A_\mu^{ab}$ transform in the 27 of $USp(8)$, that is

$$27 \text{ of } E_{6(6)} \longrightarrow 27 \text{ of } USp(8).$$

(A.1)

The 42 scalars $\phi_{abcd}$ are in the traceless self-real 4-fold antisymmetric representation 42 of $USp(8)$.

Upon performing the $d = 5 \rightarrow d = 4$ reduction, one gets 70 scalars, which split into the following irreps. of $USp(8)$:

$$70 = 42 + 27 + 1.$$  

(A.2)

Here 27 accounts for the axions coming from the $A_5^{ab}$ vectors of $E_{6(6)}$, 1 is the KK radius $r_{KK}$ (see the definition (1.11)), and 42 corresponds to the scalars in $\frac{E_{6(6)}}{USp(8)}$.

In order to extract the $stu$ model, we notice that its $d = 5$ uplift is the $(SO(1, 1))^2$ model with cubic hypersurface \cite{33, 34} (see e.g. the treatment given in \cite{22})

$$\tilde{\lambda}^1 \tilde{\lambda}^2 \tilde{\lambda}^3 = 1.$$  

(A.3)

The $\mathcal{N} = 8 \rightarrow \mathcal{N} = 2, d = 5$ supersymmetry reduction corresponds, at the level of $E_{6(6)}$, to taking the decomposition

$$E_{6(6)} \longrightarrow SO(1, 1) \times SO(5, 5) \longrightarrow (SO(1, 1))^2 \times SO(4, 4),$$

(A.4)

so that (weights with respect to $SO(1, 1)$’s are disregarded, irrelevant for our purposes)

$$27 \rightarrow 1 + 16 + 10 \rightarrow 1 + 8_a + 8_c + 1 + 1 + 8_u.$$  

(A.5)
Thus, three $SO(4,4)$-singlets are generated; they correspond to the three Abelian vector fields of the $d = 5$ uplift of the $stu$ model. By further reducing to $d = 4$, one gets a further vector from the KK vector (alias the $d = 4$ graviphoton). This can be easily seen by completing the decomposition \textcolor{red}{(A.4)} starting from the $U$-duality group $E_{7(7)}$ of $d = 4$ maximal supergravity:

$$E_{7(7)} \longrightarrow SO(1,1) \times E_{6(6)} \longrightarrow (SO(1,1))^2 \times SO(5,5) \longrightarrow (SO(1,1))^3 \times SO(4,4), \quad (A.6)$$

so that Eq. \textcolor{red}{(A.5)} gets completed as (as above, neglecting weights with respect to $SO(1,1)$, as they are irrelevant for our purposes)

$$28 \rightarrow 27 + 1 \rightarrow 1 + 16 + 10 + 1 \rightarrow 1 + 8_s + 8_c + 1 + 8_v + 1, \quad (A.7)$$

containing four $SO(4,4)$ singlets in the last term.

It is worth pointing out that at $d = 4$ the $(SO(1,1))^3$ commuting with $SO(4,4)$ gets enhanced to $(SL(2,\mathbb{R}))^3$. By further decomposing $SO(4,4) \rightarrow (SL(2,\mathbb{R}))^4$, (A.8)

this yields the $(SL(2,\mathbb{R}))^7$, used for the seven qubit entanglement in quantum information theory \textcolor{red}{[40, 41]}.

Notice that the presence of three different $8$’s of $SO(4,4)$ in the r.h.s. of the decomposition \textcolor{red}{(A.5)} (as well as of \textcolor{red}{(A.7)}) is the origin of the triality symmetry \textcolor{red}{[42, 43]} of the $stu$ model \textcolor{red}{[44]}.

The $(SO(1,1))^2$ factor in the r.h.s. of the branching \textcolor{red}{(A.4)} is nothing but the scalar manifold of the $d = 5$ counterpart of the $stu$ model (spanned by $\lambda$, $\hat{\lambda}^1$, $\hat{\lambda}^2$, and $\hat{\lambda}^3$ satisfying the cubic constraint \textcolor{red}{(A.8)}). On the other hand, the $(SO(1,1))^3$ factor in the r.h.s. of the branching \textcolor{red}{(A.7)} is spanned by the (unconstrained, strictly positive) $d = 4$ dilatons $\lambda^1 \equiv -Im(s)$, $\lambda^2 \equiv -Im(t)$ and $\lambda^3 \equiv -Im(u)$. They are related to their hatted counterparts by $\lambda^i \equiv r_{KK} \hat{\lambda}^i$, $i = 1, 2, 3$, implying (see Eqs. \textcolor{red}{(A.3)} and \textcolor{red}{(A.11)}; see also e.g. \textcolor{red}{[22]})

$$\lambda^1 \lambda^2 \lambda^3 = r_{KK}^3 \equiv V. \quad (A.9)$$

The decomposition of the $d = 5$ stabilizer (analogue to the decomposition \textcolor{red}{(A.4)} of the $U$-duality group of the $d = 5$ maximal supergravity) reads as follows:

$$USp(8) \rightarrow USp(4) \times USp(4) = Spin(5) \times Spin(5) \rightarrow Spin(4) \times Spin(4) = (SU(2))^2 \times (SU(2))^2, \quad (A.10)$$

yielding the following decomposition of the fundamental $8$ of $USp(8)$:

$$8 \rightarrow (4, 1) + (1, 4) \rightarrow (2, 1, 1, 1) + (1, 1, 2, 1) + (1, 1, 1, 2). \quad (A.11)$$

This allows one to compute the corresponding branchings of the $27 = (8 \times 8)_{A,0}$ and $42 = (8 \times 8 \times 8)_{A,0}$ (the subscript “$A,0$” standing for “antisymmetric traceless”) of $USp(8)$ (the intermediate decompositions with respect to $USp(4) \times USp(4)$ are omitted, because irrelevant for our purposes):

$$27 \rightarrow (2, 2, 1, 1) + (2, 1, 2, 1) + (2, 1, 1, 2) + (1, 2, 2, 1) + (1, 2, 1, 2) + (1, 1, 2, 2) + 3(1, 1, 1, 1); \quad (A.12)$$

$$42 \rightarrow (2, 2, 2, 2) + (2, 2, 1, 1) + (2, 1, 2, 1) + (2, 1, 1, 2) + (1, 2, 2, 1) + (1, 2, 1, 2) + (1, 1, 2, 2) + 2(1, 1, 1, 1) \quad (A.13)$$
Consistently with previous statements, the three \((SU(2))^4\)-singlets in the r.h.s. of the decomposition \((A.12)\) and the two \((SU(2))^4\)-singlets in in the r.h.s. of the decomposition \((A.13)\) respectively are the three Abelian vector fields (including the \(d = 5\) graviphoton) and the two independent real scalars (say, \(\hat{\lambda}^1\) and \(\hat{\lambda}^2\)) in the bosonic spectrum of the \((SO(1, 1))^2\) model, which is the \(d = 5\) uplift of the \(stu\) model.

Reducing to \(d = 4\), the six real scalar degrees of freedom of the \(stu\) model are the radius \(r_{KK}\) (see Eqs. \((1.11)\) and \((A.9)\)), the two scalars \(\hat{\lambda}^1\) and \(\hat{\lambda}^2\), and the three axions (coming from the fifth component \(A^I_5\) \((I = 1, 2, 3)\) of the three \(d = 5\) vectors). As previously mentioned, the four \(d = 4\) vectors come from the three \(d = 5\) vectors and from the KK vector \(g_{\mu\nu}\) \((\mu = 1, ..., 4)\).

Finally, it should be notice that \(\lambda^1\lambda^2\lambda^3\) (defining the volume of the \(d = 5\) cubic hypersurface through Eqs. \((1.11)\) and \((A.9)\)) can be obtained through a consistent truncation of the \(E_6(6)\)-invariant expression (\(\Lambda, \Sigma, \Delta = 1, ..., 27\))

\[
\frac{1}{3!} d_{\Lambda\Sigma\Delta} \lambda^\Lambda \lambda^\Sigma \lambda^\Delta \quad (A.14)
\]

to \((SO(1, 1))^2\), by retaining only the three singlets of \(SO(4, 4)\) (see the decompositions \((A.4)\) and \((A.5)\) above).

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