Special homological dimensions and Intersection Theorem

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Abstract

Let \((R, \mathfrak{m})\) be commutative Noetherian local ring. It is shown that \(R\) is Cohen–Macaulay ring if there exists a Cohen–Macaulay finite (i.e. finitely generated) \(R\)-module with finite upper Gorenstein dimension. In addition, we show that, in the Intersection Theorem, projective dimension can be replaced by quasi-projective dimension.

1. Introduction

Let \(M\) and \(N\) be finite \(R\)-modules and \(\text{pd}_R M < \infty\). The New Intersection Theorem of Peskine and Szpiro [PS], Hochster [H], and P. Roberts [R1], [R2] yields an inequality

\[
\text{dim}_R N \leq \text{dim}_R (M \otimes_R N) + \text{pd}_R M. 
\]

By applying the inequality (1) with \(N = R\) one derives,

\[
\text{dim}_R \leq \text{dim}_R M + \text{pd}_R M. 
\]

By the Auslander–Buchsbaum Formula, the inequality (2) is equivalent to

\[
\text{cmd} R \leq \text{cmd}_R M 
\]

where \(\text{cmd}_R M = \text{dim}_R M - \text{depth}_R M\) is the Cohen–Macaulay defect of \(M\) that is a non-negative integer which determines the failure of \(M\) to be Cohen–Macaulay; we set \(\text{cmd} R = \text{cmd}_R R\).

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It is shown that the projective dimension can be replaced by the quasi–projective dimension (cf. [AGP]) in (1).

Also we show that the projective dimension can be replaced by upper Gorenstein dimension (cf. [V]) in (2) and (3). In addition, it is shown that the grade of a module of finite upper Gorenstein dimension is actually equal to its codimension.

(1.1) **Setup and Notions.** Throughout, the rings will denote a non–trivial, commutative, local, Noetherian ring and the modules are finite (that means finitely generated). A diagram of local homomorphisms $R \rightarrow R' \leftarrow Q$, with $R \rightarrow R'$ a flat extension and $R' = Q/(x)$ where $x = x_1, x_2, \cdots, x_c$ is a Q–regular sequence, is called a quasi–deformation of codimension $c$. The quasi–projective dimension of the $R$–module $M$ is defined by Avramov, Gasharov, and Peeva [AGP] as

$$qpd_R M = \inf \{ pd_Q(M \otimes_R R') | R \rightarrow R' \leftarrow Q \text{ is a quasi–deformation} \}.$$ 

A local surjection $\pi: Q \rightarrow R$ is a Gorenstein deformation if Ker $(\pi)$ is a Gorenstein ideal. A Gorenstein quasi–deformation of $R$ is a diagram of local homomorphisms $R \rightarrow R' \leftarrow Q$, with $R \rightarrow R'$ a flat extension and $R' \leftarrow Q$ a Gorenstein deformation. The upper Gorenstein dimension of the $R$–module $M$ is defined by Veliche [V] as

$$G^*\dim_R M = \inf \{ pd_Q(M \otimes_R R') - pd_Q R' | R \rightarrow R' \leftarrow Q \text{ is a Gorenstein quasi–deformation} \}.$$ 

The $n$th Betti number of $M$ over $R$ is defined by $\beta^n_R(M) = \text{rank}_k(\text{Ext}_R^n(M, k))$. The complexity of $M$ is defined by

$$\text{cx}_RM = \inf \{ d \in \mathbb{N}_0 | \beta^n_R(M) \leq an^{d-1} \text{ for some positive real } a \text{ and } n \gg 0 \}.$$ 

By [AGP; Thm. 5.11] for any $R$–module $M$ there is an equality $qpd_R M = \text{CI-dim}_R M + \text{cx}_R M$ where CI-dim$_R M$ is the complete intersection dimension of $M$.

An $R$-complex $X$ is a sequence of $R$-modules $X_\ell$ and $R$-linear maps $\partial^X_\ell, \ell \in \mathbb{Z}$,

$$X = \cdots \rightarrow X_{\ell+1} \xrightarrow{\partial^X_{\ell+1}} X_\ell \xrightarrow{\partial^X_\ell} X_{\ell-1} \rightarrow \cdots$$

such that $\partial^X_\ell \partial^X_{\ell+1} = 0$ for all $\ell \in \mathbb{Z}$. The module $X_\ell$ is called the module in degree $\ell$, and the map $\partial^X_\ell : X_\ell \rightarrow X_{\ell-1}$ is called the $\ell$-th differential. An $R$-module $M$ is thought of as the
complex $M = 0 \rightarrow M \rightarrow 0$. The supremum and infimum of $X$ are defined by

$$\sup X = \sup \{ \ell \in \mathbb{Z} | H_\ell(X) \neq 0 \}$$

$$\inf X = \inf \{ \ell \in \mathbb{Z} | H_\ell(X) \neq 0 \}$$

A morphism $\alpha : X \rightarrow Y$ is said to be a quasi-isomorphism if the induced morphism $H(\alpha) : H(X) \rightarrow H(Y)$ is an isomorphism. The derived category $\mathcal{D}(R)$ of the category of $R$–complexes is the category of $R$–complexes localized at the class of all quasi–isomorphisms. The full subcategory $\mathcal{D}^b(R)$ consist of complexes $X$ with $H_\ell(X)$ a finite $R$–module for all $\ell$ and $H_\ell(X) = 0$ for $|\ell| \gg 0$. The left derived functor of the tensor product functor of $R$-complexes is denoted by $- \otimes^L_R -$. For a complex $X$, the dimension of $X$ is defined by

$$\dim_R X = \sup \{ \dim R/p - \inf X_p | p \in \text{Spec } R \}.$$ 

When $M$ is an $R$–module, this notion agrees with the usual definition of $\dim_R M$.

Let $R$ be a local and $X$ be a homologically finite complex of $R$–modules. the CI-dim$_R X$ is defined by Sather-Wagstaff in [S] as follows

$$\text{CI-dim}_R X = \inf \{ \text{pd}_Q(X \otimes_R R') - \text{pd}_Q R' | R \rightarrow R' \leftarrow Q \text{ is a quasi-deformation} \}.$$ 

2. Upper Gorenstein dimension.

In this section it is shown that the ring $R$ is Cohen–Macaulay if there exists a Cohen–Macaulay finite $R$–module $M$ of finite upper Gorenstein dimension. (The converse is easy.) In addition it is shown that the grade of a module of finite upper Gorenstein dimension is actually equal to its codimension.

(2.1) Theorem. Let $M$ be a finite $R$–module with finite upper Gorenstein dimension. Then the following hold

(a) cmd $R \leq$ cmd $R M$.

(b) $\dim R \leq \dim_R M + G^* - \dim_R M$. 

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Proof. (a) Since $G^*\dim M < \infty$, there exists a quasi-deformation $R \to R' = Q/J \leftarrow Q$ with $\text{pd}_Q M' < \infty$, where $M' = M \otimes_R R'$. By the Intersection Theorem, $\text{cmd} Q \leq \text{cmd} Q M'$. It is well-known that $\text{cmd} Q M' = \text{cmd} R M'$. Now since $R \to R'$ is a flat extension, by [AF; 1.2] the following hold

$$\text{cmd} R' = \text{cmd} R + \text{cmd} R R'/mR';$$

$$\text{cmd} R M' = \text{cmd} R M + \text{cmd} R R'/mR'.$$

On the other hand by [AFH; Cor. 3.12] we have $\text{cmd} Q = \text{cmd} R'$. Therefore

$$\text{cmd} R + \text{cmd} R R'/mR' = \text{cmd} Q \leq \text{cmd} Q M' = \text{cmd} R M + \text{cmd} R R'/mR'.$$

(b) This part is obtained by applying (a) and the equality $\text{depth} R = \text{depth} R M + G^*\dim R M$, cf. [V; Prop. 2.4]. \hfill $\square$

Let $M$ be a finite $R$–module. The grade of $M$, $\text{grade} R M$, defined by Rees to be the maximal length of $R$–regular sequence in the annihilator of $M$. Also, the codimension $\text{codim} R M$ of the support of $M$ in the spectrum of $R$ is defined as the height of the annihilator of $M$. In [AF] Avramov and Foxby have studied some properties of the codimension of a module with finite projective dimension. Now as an application of Theorem (1.2), we give a generalization of [AF; Prop. 2.5] for a module with finite upper Gorenstein dimension. Often it is convenient to compute $\text{grade} R M$ and $\text{codim} R M$ from the formulas

$$\text{grade} R M = \inf\{\text{depth} R_p | p \in \text{Supp} R M\};$$

$$\text{codim} R M = \inf\{\text{dim} R_p | p \in \text{Supp} R M\}.$$

(2.2) Proposition. (See [AF; Prop. 2.5]) If $M$ is a non-zero finite $R$–module of finite upper Gorenstein dimension, then $\text{grade} R M = \text{codim} R M$ and there exists a prime ideal $p$ minimal in $\text{Supp} R M$ such that $R_p$ is Cohen–Macaulay of dimension $\text{grade} R M$.

Proof. Choose $q \in \text{Supp} R M$ such that $\text{grade} R M = \text{depth} R_q$, and then choose $p$ contained in $q$ and minimal in $\text{Supp} R M$. By using [V; prop. 2.4] and [V; prop. 2.10] we conclude from the
choices of \( q \) and \( p \) that

\[
\text{grade}_R M = \text{depth}_R q \geq G^* - \dim_R M_q \geq G^* - \dim_R p M_p = \text{depth}_R p \geq \text{grade}_R M.
\]

Therefore \( \text{grade}_R M = \text{depth}_R p \). Since \( M_p \) is an \( R_p \)-module of finite length and of finite upper Gorenstein dimension, the ring \( R_p \) is Cohen–Macaulay by (1.2). Now by the following inequalities

\[
\text{grade}_R M = \dim_R p \geq \text{codim}_R M \geq \text{grade}_R M,
\]

the assertion holds. \( \square \)

3. Quasi–projective dimension.

The main result in this section is the Theorem 3.3 That is a generalization of the Intersection Theorem.

(3.1) **Definition.** For \( X \in \mathcal{D}_b^f (R) \) the quasi-projective dimension of \( X \) is defined as

\[
qpd_R X = \inf \{ \text{pd}_Q (X \otimes_R R') \mid R \to R' \leftarrow Q \text{ is a quasi–deformation} \}.
\]

(3.2) **Lemma.** If \( \pi : R \to S \) is a surjection local homomorphism and \( Y, Z \in \mathcal{D}_b^f (S) \), then

\[
\dim_R (Y \otimes^L_R Z) = \dim_S (Y \otimes^L_S Z).
\]

**Proof.** Let \( I = \ker(\pi) \). Then we have \( R/\pi^{-1}(q) \cong S/q \) for every prime ideal \( q \) of \( S \).

Claim 1. \( \text{Supp}_R (Y \otimes^L_R Z) \subseteq V(I) \). Fix \( p \in \text{Supp}_R (Y \otimes^L_R Z) = \text{Supp}_R (Y) \cap \text{Supp}_R (Z) \). Then \( p \in \text{Supp}_R (H_i(Y)) \) for some \( i \). Since \( H_i(Y) \) is an \( S \)-module, one has \( I \subseteq \text{Ann}_R (H_i(Y)) \) and therefore \( p \in \text{Supp}_R (H_i(Y)) = V(\text{Ann}_R (H_i(Y))) \subseteq V(I) \).

Claim 2. For a prime ideal \( q \) of \( S \), one has \( q \in \text{Supp}_S (Y \otimes^L_S Z) \) if and only if \( \pi^{-1}(q) \in \text{Supp}_R (Y \otimes^L_R Z) \). Let \( p = \pi^{-1}(q) \). First note that the complexes \( Y_q \) and \( Y_p \) are isomorphic over \( R_p \). In particular, \( \inf(Y_p) = \inf(Y_q) \) and \( p \in \text{Supp}_R (Y) \) if and only if \( q \in \text{Supp}_S (Y) \). As we just argued, \( p \in \text{Supp}_R (Y \otimes^L_R Z) = \text{Supp}_R (Y) \cap \text{Supp}_R (Z) \) if and only if \( q \in \text{Supp}_S (Y \otimes^L_S Z) = \text{Supp}_S (Y) \cap \text{Supp}_S (Z) \). This proves the claim.
Claim 3. For a prime ideal $q$ of $S$, let $p = \pi^{-1}(q)$; then
\[
\inf((Y \otimes_{R_k} Z)_p) = \inf((Y \otimes_{S} Z)_q).
\]

One has
\[
\begin{align*}
\inf((Y \otimes_{R_k} Z)_p) &= \inf(Y_p \otimes_{R_p} Z_p) \\
&= \inf(Y_p) + \inf(Z_p) \\
&= \inf(Y_q) + \inf(Z_q) \\
&= \inf(Y_q \otimes_{S_q} Z_q) \\
&= \inf((Y \otimes_{S} Z)_q).
\end{align*}
\]

Now from claims 1–3 we have
\[
\dim_R(Y \otimes_{R_k} Z) = \sup\{\dim(R/p) - \inf((Y \otimes_{R_k} Z)_p) | p \in \text{Supp}_R(Y \otimes_{R_k} Z)\} \\
= \sup\{\dim(S/q) - \inf((Y \otimes_{S} Z)_q) | q \in \text{Supp}_S(Y \otimes_{S} Z)\} \\
= \dim_S(Y \otimes_{S} Z).
\]

and this is the desired equality. \hfill \Box

(3.3) **Theorem.** Let $Y \in \mathcal{D}^f_b(R)$ with finite quasi–projective dimension. Then for $X \in \mathcal{D}^f_b(R)$,
\[
\dim_R X \leq \dim_R(Y \otimes_{R_k} X) + \text{qpd}_R Y.
\]

**Proof.** Since $\text{qpd}_R Y < \infty$, so there is a quasi–deformation $R \rightarrow R' \leftarrow Q$ with $\text{pd}_Q Y' < \infty$ where $Y' = Y \otimes_R R'$. By the Intersection Theorem (finite version) for complexes, cf. [F; 18.5],
\[
\dim_Q X' \leq \dim_Q(Y' \otimes_{Q} X') + \text{pd}_Q Y'.
\]

We have the following
\[
\dim_Q(Y' \otimes_{Q} X') = \dim_{R'}(Y' \otimes_{R'} X') \\
= \dim_{R'}(Y \otimes_{R} X) \\
= \dim_{R'}(Y \otimes_{R} X) + \dim R'/mR'
\]

where the the first equality comes from lemma 3.2. It is also easy to see that
\[
\dim_Q X' = \dim_{R'} X' = \dim_R Y + \dim R'/mR'.
\]

Thus the proof is completed. \hfill \Box
In the Theorem (3.3), if we put finite $R$–modules $M$ and $N$ instead of the complexes $Y$ and $X$ then we get the following result.

(3.4) **Corollary.** Let $M$ be a finite $R$–module with finite complete intersection dimension. Then for a finite $R$–module $N$;

$$\dim N \leq \dim (N \otimes_R M) + \text{qpd}_RM.$$  

**Proof.** By $[F; 16.22]$, $\dim (N \otimes^L_R M) = \dim (N \otimes_R M)$. Therefore the assertion is obtained by applying Theorem 3.3. \hfill $\square$

The following example shows that we can not replace quasi-projective dimension with complete intersection dimension in Theorem 3.3.

**Example** Let $Q = k[[x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n]]$ where $k$ is a field. Let $z_i = x_iy_i$ for $i = 1, 2, \cdots, n$. Consider the $Q$–ideals $I = (z_1, z_2, \cdots, z_n)$, $J = (x_1, x_2, \cdots, x_n)$, and $L = (y_1, y_2, \cdots, y_n)$. Let $R = Q/I$. Then $R$ is complete intersection but is not regular. Consider the $R$–modules $A$ and $B$ as $A = R/JR$ and $B = R/LR$. Then $A \otimes_R B = R/(J+L)R$ and hence $\dim A \otimes_R B = 0$. Since $R$ is complete intersection we have $\text{CI-dim}_RA < \infty$ and hence, $\text{CI-dim}_RA = \text{depth}R - \text{depth}_RA = 0$. On the other hand $\dim B = n$ so $\dim B > \text{CI-dim}_RA + \dim A \otimes_R B$.

Let $M$ and $N$ be $R$–modules. Then we define

$$\text{grade}_R(M, N) = \inf \{i | \text{Ext}_R^i(M, N) \neq 0\}.$$  

If $\text{Ext}_R^i(M, N) = 0$ for all $i$, then $\text{grade}_R(M, N) = \infty$. In $[AY; \text{Theorem 3.1}]$ Araya and Yoshino have used the Intersection Theorem to prove “Let $M$ and $N$ be finite $R$–modules with $\text{pd}_RN < \infty$ and $\text{Tor}_i^R(M, N) = 0$ for all $i > 0$ then for any finite $R$–module $L$, we have the following inequalities

$$(3) \quad \text{grade}_R(L, M) - \text{pd}_RN \leq \text{grade}_R(L, M \otimes_R N) \leq \text{grade}_R(L, M).$$  

In the following Theorem we show that, in (3), projective dimension can be replaced by quasi-projective dimension.

(3.5) **Theorem.** Let $M$ and $N$ be finite $R$–modules with $\text{qpd}_RN < \infty$ and $\text{Tor}_i^R(M, N) = 0$ for any $i > 0$. Then
grade$_R(L, M) - qpd_R N \leq grade_R(L, M \otimes_R N) - cx_R N \leq grade_R(L, M).

proof. By [BH; Prop. 1.2.10] there exists $p \in \text{Supp}_R L \cap \text{Supp}_R (M \otimes_R N)$ such that

grade$_R(L, M \otimes_R N) = \text{depth}_{R_p} (M_p \otimes_{R_p} N_p).

By [AGP; Prop. 1.6] CI-dim$_{R_p} N_p < \infty$ and so by applying [I; Thm. 4.3] we have

grade$_R(L, M \otimes_R N) = \text{depth}_{R_p} N_p + \text{depth}_{R_p} M_p - \text{depth}_{R_p} R_p.$

On the other hand $\text{depth}_{R_p} N_p = \text{CI-dim}_{R_p} N_p$, see [AGP; Thm. 1.4]. Thus grade$_R(L, M \otimes_R N) = \text{depth}_{R_p} M_p + \text{CI-dim}_{R_p} N_p$. Therefore by applying [BH; Prop. 1.2.10]

grade$_R(L, M \otimes_R N) \geq \text{grade}_R(L, M) - \text{CI-dim}_R N.$

Now the left inequality is obtained by applying the equality CI-dim$_R N = qpd_R N - cx_R N$.

For the right inequality, there exists $p \in \text{Supp}_M \cap \text{Supp}_L$ such that grade$_R(L, M) = \text{depth}_{R_q} M_p$. Let $q$ be a minimal element of the set $\text{Supp}_R (R/p_R \otimes_R M_q)$. Then $p \subseteq q$ and so $q \in \text{Supp}_L$. By Corollary 3.2, $\dim_{R_q} (R_q/p_R_q \otimes_{R_q} M_q) \leq qpd_{R_q} N_q$ and hence by [AGP; Thm 5.11] $\dim_{R_q} (R_q/p_R_q \otimes_{R_q} M_q) \leq \text{CI-dim}_{R_q} N_q + cx_{R_q} N_q$. The following inequalities hold:

\[
\begin{align*}
\text{CI-dim}_{R_q} N_q & \geq \dim_{R_q} (R_q/p_R_q \otimes_{R_q} M_q) - cx_{R_q} N_q \\
& \geq \dim_{R_q} (R_q/p_R_q \otimes_{R_q} M_p) + \dim_{R_q} (R_q/p_R_q) - cx_{R_q} N_q \\
& \geq \text{depth}_{R_q} M_q - \text{depth}_{R_q} M_p + cx_{R_q} N_q \\
& \geq \text{depth}_{R_q} M_q - \text{depth}_{R_q} R_q.
\end{align*}
\]

Therefore

\[
\begin{align*}
\text{cx}_R N + \text{grade}_R(L, M) & \geq \text{cx}_R N + \text{depth}_{R_q} M_p \\
& \geq \text{depth}_{R_q} M_q - \text{CI-dim}_{R_q} N_q \\
& = \text{depth}_{R_q} (M_q \otimes_{R_q} N_q) \\
& \geq \text{grade}_R(L, M \otimes_R N).
\end{align*}
\]

Now the assertion holds. \qed

The following example shows that the term “cx$_R N$” is necessary in the Theorem (3.5).

Example. Let $R = k[[X,Y]]/(XY)$, $N = R/yR$ where $x$ (resp. $y$) is image of $X$ (resp. $Y$) in $R$. Since $R$ is complete intersection so CI-dim$_R N < \infty$. Since depth$_R N = 1$ we have
CI-dim$_{R}N = 0$, the quasi–deformation can be chosen as $R = R'$ and $Q = k[[X,Y]]$. Then it is easy to see that $c_{x_{R}}(N) = 1$. Set $M = R$ and $L = R/xR$. Then grade$_{R}(L,M) = 0$ and grade$_{R}(L,M \otimes_{R} N) = 1$.

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