Fix Your Features: Stationary and Maximally Discriminative Embeddings using Regular Polytope (Fixed Classifier) Networks

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Abstract

Neural networks are widely used as a model for classification in a large variety of tasks. Typically, a learnable transformation (i.e. the classifier) is placed at the end of such models returning a value for each class used for classification. This transformation plays an important role in determining how the generated features change during the learning process.

In this work we argue that this transformation not only can be fixed (i.e. set as non trainable) with no loss of accuracy, but it can also be used to learn stationary and maximally discriminative embeddings.

We show that the stationarity of the embedding and its maximal discriminative representation can be theoretically justified by setting the weights of the fixed classifier to values taken from the coordinate vertices of three regular polytopes available in $\mathbb{R}^d$, namely: the $d$-Simplex, the $d$-Cube and the $d$-Orthoplex. These regular polytopes have the maximal amount of symmetry that can be exploited to generate stationary features angularly centered around their corresponding fixed weights.

Our approach improves and broadens the concept of a fixed classifier, recently proposed in [1], to a larger class of fixed classifier models. Experimental results confirm both the theoretical analysis and the generalization capability of the proposed method.

1. Introduction

Deep Convolutional Neural Networks (DCNNs) have achieved state-of-the-art performance on several classification and representation tasks in Computer Vision [2, 3]. In DCNNs, both representation and classification are jointly learned in a single network. The representation for an input sample is the feature vector $f$ generated by the penultimate layer, while the last layer (i.e. the classifier) outputs score values according to the inner product as:

$$z_i = w_i^T \cdot f$$  \hspace{1cm} (1)

for each class $i$, where $w_i$ is the weight vector of the classifier for the class $i$. To evaluate the loss, the scores are further normalized into probabilities via the Softmax function [4].

Since the values of $z_i$ can be also expressed as $z_i = w_i^T \cdot f = \|w_i\| \|f\| \cos(\theta)$, where $\theta$ is the angle between $w_i$ and $f$, the score for the correct label with respect to the other labels is obtained by optimizing $\|w_i\|$, $\|f\|$ and $\theta$. This simple formulation of the final classifier provides the intuitive explanation of how feature vector directions and weight vector directions align simultaneously with each other at training time so that their average angle is made as small as possible.

In this paper we exploit the fact that fixed weights can induce some form of stationarity on the generated features. Indeed, if the parameters $w_i$ of the classifier in Eq. 1 are fixed (i.e. set as non trainable), only the feature vector directions can align toward the classifier weight vector directions and not the opposite. Therefore weights can be regarded as fixed angular references to which features align.

According to this, we provide a precise result on the spatio-temporal statistical properties of the generated fea-
network is basically forced to learn class feature representation into the predetermined symmetric set of subspaces.

We show that our approach can be theoretically justified and easily implemented by setting the classifier weights to values taken from the coordinate vertices of a regular polytope in the embedding space. Regular polytopes are the generalization in any number of dimensions of regular polygons and regular polyhedra (i.e. Platonic Solids). Although, there are infinite regular polygons in $\mathbb{R}^2$ and 5 regular polyhedra in $\mathbb{R}^3$, there are only three regular polytopes in $\mathbb{R}^d$ with $d \geq 5$, namely the $d$-Simplex, the $d$-Cube and the $d$-Orthoplex. Having different symmetry, geometry and topology, each regular polytope will reflect its properties into the classifier and the embedding space which defines. Fig. 1 illustrates the three basic architectures defined by the proposed approach termed Regular Polytope Networks (RePoNet). Fig. 2 provides a first glance at our main result in a 2D embedding space. Specifically, the main evidence from Fig. 2a and 2b is that the features learned by RePoNet remain aligned with their corresponding fixed weights and maximally exploit the available representation space directly from the beginning of the training phase.

We apply our method to multiple vision datasets showing that it is possible to generate stationary and maximally discriminative features without reducing the generalization performance of DCNN models.

2. Related Work

A recent paper [1] explores the idea of excluding the parameters $w_i$ in Eq.1 from learning. The work shows that a fixed classifier causes little or no reduction in classification performance for common datasets while allowing a noticeable reduction in trainable parameters, especially when the number of classes is large. Setting the last layer as not trainable also reduces the computational complexity for training as well as the communication cost in distributed learning. The described approach sets the classifier with the coordinate vertices of orthogonal vectors taken from the columns.

Fig. 1: The three basic architectures of RePoNet. Fig. 1(a) illustrates the $d$-Simplex, Fig. 1(b) the $d$-Cube and Fig. 1(c) the $d$-Orthoplex.

Fig. 2: Feature learning on the MNIST dataset in a 2D embedding space. Fig. (a) and Fig. (c) show the 2D features learned by RePoNet and by a standard trainable classifier respectively. The two methods achieve the same classification accuracy. Fig. (b) and Fig. (d) show the training evolution of the classifier weights (dashed) and their corresponding class feature means (solid) respectively. Both are expressed according to their angles.
of the Hadamard\textsuperscript{1} matrix. Although the work uses a fixed classifier, no mention is made about the fact that the generated features may have stationary properties. A major limitation of this method is that, when the number of classes is greater than the dimension of the feature space, it is not possible to have mutually orthogonal columns and therefore some of the classes are constrained to lie in a common subspace causing a reduction in classification performance. We improve and generalize this work by finding a novel set of unique directions that overcomes the limitations of the Hadamard matrix.

Some papers exploit Eq. 1 to train DCNNs by direct angle optimization [5, 6, 7, 8]. For these papers, among others, the performance is not only dominated by the predicted labels as in [1], but features are also required to be compact and well separable in terms of their angular directions. From a semantic point of view the angle encodes the required discriminative information for class recognition. The wider the angles the better the classes are separated from each other and, accordingly, their representation is more discriminative. The common idea of these works is that of constraining the features and/or the classifier weights to be unit normalized. The works [9], [10] and [8] optimize both features and weights, while the work [5] normalizes the features only and [6] normalizes the weights only. Specifically, [5] also proposes adding a scale parameter after feature normalization based on the property that increasing the norm of samples can decrease the Softmax loss [11]. Despite not being involved in learning discriminative feature, the work [1] in addition to fixing the classifier, normalizes both the weights and the features and exploits the multiplicative scale parameter. In accordance with [12, 1] and [8] we found that feature normalization and the multiplicative scale parameter are hard to optimize for general datasets, having a significant dependence on image quality. We follow the work [6] that normalizes the classifier weights and sets its biases to zero. This allows to optimize angles, enabling the network to learn angularly distributed features.

From a statistical point of view normalizing weights is equivalent to consider features distributed on the unit hypersphere according to the von Mises-Fisher distribution [10]. Under this model each class weight represents the mean of its corresponding features and the scalar factor (i.e. the concentration parameter) is inversely proportional to their standard deviations. Several methods implicitly follow this statistical interpretation [8, 6, 12, 13, 14] in which the weights act as a summarizer or as parameterized prototype of the features of each class. Eventually, as conjectured in [8] if all classes are well-separated they will roughly correspond to the means of features in each class. We improve upon these works by showing that our proposed classifiers produce features exactly centered around to their fixed weights as the training process advances.

While all the above works impose large angular distances between the classes, they provide solutions to enforce such constraint in a local manner without considering global interclass separability and intraclass compactness. For this purpose, very recently the work [15] adds a regularization loss to specifically force the classifier weights to be far from each other in a global manner. The work draws inspiration from a well-known problem in physics – the Thomson problem [16], where given $K$ charges confined to the surface of a sphere, one seeks to find an arrangement of the charges which minimizes the total electrostatic energy. Electrostatic force repels charges each other inversely proportional to their mutual distance. In [15] global equiangular features are obtained by adding to the standard categorical Cross-Entropy loss a further loss inspired by the Thomson problem. We follow a similar principle for global separability and compactness by considering that minimal energies are often concomitant with special geometric configurations of charges that recall the geometry of Platonic Solids in high dimensional spaces [17].

3. Main Contributions

Our technical contributions can be summarized as follows: (1) We generalize the concept of fixed classifiers. (2) We show that they can generate stationary and maximally discriminative features at training time. (3) We provide a theoretical model that justifies our result.

4. Regular Polytopes and Maximally Discriminative Stationary Embeddings

We are basically concerned with the following question: How should the non trainable weights be distributed in the embedding space such that they generate stationary and maximally discriminative features?

Let $X = \{(x_i, y_i)\}_{i=1}^N$ be the training set containing $N$ samples, where $x_i$ is the raw input to the DCNN and $y_i \in \{1, 2, \cdots, K\}$ is the label of the class that supervises the output of the DCNN. Then, the Cross Entropy loss can be written as:

$$\mathcal{L} = -\frac{1}{N} \sum_{i=1}^N \log \left( \frac{\exp(w_{yi}^\top f_i + b_{y_i})}{\sum_{j=1}^K \exp(w_j^\top f_i + b_j)} \right),$$

(2)

where $W = \{w_j\}_{j=1}^K$ are the classifier weight vectors for the $K$ classes. Following the discussion in [8] we normalize the weights and zero the biases ($\hat{w}_j = \frac{w_j}{||w_j||}, \ b_j = 0$) to directly optimize angles, enabling the network to learn angularly distributed features.

Angles therefore encode the required discriminative information for class recognition and the wider they are,

\textsuperscript{1}The Hadamard matrix is a square matrix whose entries are either $+1$ or $1$ and whose rows are mutually orthogonal.
If we further consider, the feature vector parametrized by its unit vector as \( \mathbf{f}_i = \kappa_i \hat{\mathbf{f}}_i \) where \( \kappa_i = \|\mathbf{f}_i\| \) and \( \hat{\mathbf{f}}_i = \frac{\mathbf{f}_i}{\|\mathbf{f}_i\|} \), then Eq. 2 can be rewritten as:

\[
\mathcal{L} = -\frac{1}{N} \sum_{i=1}^{N} \log \left( \frac{\exp(\kappa_i \mathbf{w}_y^\top \hat{f}_i)}{\sum_{j=1}^{K} \exp(\kappa_i \mathbf{w}_j^\top \hat{f}_i)} \right)
\]

The equation above can be interpreted as if \( N \) realizations from a set of \( K \) von Mises-Fisher distributions with different concentration parameters \( \kappa_i \) are passed through the Softmax function. The probability density function of the von Mises-Fisher distribution for the random \( d \)-dimensional unit vector \( \hat{f} \) is given by: \( P(\hat{f}; \mathbf{w}, \kappa) \propto \exp(\kappa \mathbf{w}^\top \hat{f}) \) where \( \kappa \geq 0 \). Under this parameterization \( \mathbf{w} \) is the mean direction on the hypersphere and \( \kappa \) is the concentration parameter. The greater the value of \( \kappa \) the higher the concentration of the distribution around the mean direction \( \hat{f} \). The distribution is unimodal for \( \kappa > 0 \) and is uniform on the sphere for \( \kappa = 0 \). The loss in Eq. 3 optimizes for large values of \( \kappa \) and small angles providing intraclass compactness.

As with this formulation each weight vector is the mean direction of its associated features on the hypersphere, equiangular features maximizing the available space can be obtained by arranging accordingly their corresponding weight vectors around the origin. This problem is equivalent to distributing points uniformly on the sphere and is a well-known geometric problem, called Tammes problem [18] which is a generalization of the physic problem firstly addressed by Thomson [16]. In 2D the problem is that of placing \( K \) points on a circle so that they are as far as possible from each other. In this case the optimal solution is that of placing the points at the vertices of a regular \( K \)-sided polygon.

The 3D analogues of regular polygons are Platonic Solids. However, the five Platonic solids are not the unique solutions of the Thomson problem. In fact, only the tetrahedron, octahedron and the icosahedron are the unique solutions for \( K = 4, 6 \) and 12 respectively. For \( K = 8 \): the cube is not optimal in the sense of the Thomson problem. This means that the energy stabilizes at a minimum in configurations that are not symmetric from a geometric point of view. The unique solution in this case is provided by the vertices of an irregular polytope [19].

The non geometric symmetry between the locations causes the global charge to be different from zero. Therefore in general, when the number of charges is arbitrary, their position on the sphere cannot reach a configuration for which the global charge vanishes to zero. A similar argument holds in higher dimensions for the so called generalized Thomson problem [17]. According to this, we argue that, the geometric limit to obtain a zero global charge in the generalized Thomson problem is equivalent to the impossibility to obtain a maximally discriminative classifier for an arbitrary number of classes.

However, since the classification problem it is not confined in a three dimensional space, our approach addresses this irregularity by selecting the appropriate dimension of the embedding space so as to have access to symmetrical fixed classifiers directly from regular polytopes. In dimensions 5 and higher, there are only three ways to do that (See Tab. 1) and they involve the symmetry properties of the three well known regular polytopes available in high dimensional space [20]. These three special classes exist in every dimensionality and are: the \( d \)-Simplex, the \( d \)-Cube and the \( d \)-Orthoplex. In the next paragraphs the three fixed classifiers derived from them are presented.

### The \( d \)-Simplex Fixed Classifier.

In geometry, a simplex is a generalization of the notion of a triangle or tetrahedron to arbitrary dimensions. Specifically, a \( d \)-simplex is a \( d \)-dimensional polytope which is the convex hull of its \( d + 1 \) vertices. A regular \( d \)-simplex may be constructed from a regular \( (d - 1) \)-simplex by connecting a new vertex to all original vertices by the common edge length. According to this, the weights for this classifier can be computed as:

\[
\mathbf{W}_S = \left\{ e_1, e_2, \ldots, e_{d-1}, \alpha \sum_{i=1}^{d-1} e_i \right\}
\]

where \( \alpha = \frac{1 - \sqrt{d+1}}{d} \) and \( e_i \) with \( i \in \{1, 2, \ldots, d - 1\} \) denotes the standard basis in \( \mathbb{R}^{d-1} \). The final weights will be shifted about the centroid and normalized. The \( d \)-Simplex fixed classifier defined in an embedding space of dimension \( d \) can accomodate a number of classes equal to its number of vertices:

\[
K = d + 1.
\]

This classifier has the largest number of classes that can be embedded in \( \mathbb{R}^d \) such that their corresponding class features are equidistant from each other. It can be shown that the angle subtended between any pair of weights is equal to:

\[
\theta_{\mathbf{w}_i, \mathbf{w}_j} = \arccos\left(-\frac{1}{d}\right) \quad \forall i, j \in \{1, 2, \ldots, K\} : i \neq j.
\]

| Dimension \( d \) | 1 | 2 | 3 | 4 | \( \geq 5 \) |
|------------------|---|---|---|---|---------|
| Number of Regular Polytopes | 1 | \( \infty \) | 5 | 6 | 3 |

Table 1: Number of regular Polytopes as dimension \( d \) increases.
The $d$-Orthoplex Fixed Classifier. This classifier is derived from the $d$-Orthoplex (or Cross-Polytope) regular polytope that is defined by the convex hull of points, two on each Cartesian axis of an Euclidean space, that are equidistant from the origin. The weights for this classifier can therefore defined as:

$$W_d = \{\pm e_1, \pm e_2, \ldots, \pm e_d\}.$$  

Since it has $2d$ vertices the derived fixed classifier can accommodate in its embedding space of dimension $d$:

$$K = 2d$$  

(6)

different classes. Each vertex is connected to other $d - 1$ vertices and the angle between connected vertices is

$$\theta_{w_i, w_j} = \frac{\pi}{2} \forall i, j \in \{1, 2, \ldots, K\} : j \in C(i)$$  

(7)

Where each $j \in C(i)$ is a connected vertex and $C$ is the set of connected vertices defined as $C(i) = \{j : (i, j) \in E\}$. $E$ is the set of edges of the graph $G = (W_d, E)$. The $d$-Orthoplex is the dual polytope of the $d$-Cube and vice versa (i.e. the normals of the $d$-Orthoplex faces correspond to the the directions of the vertices of the $d$-Cube).

The $d$-Cube Fixed Classifier. The $d$-Cube (or Hyper-cube) is the regular polytope formed by taking two congruent parallel hypercubes of dimension $(d - 1)$ and joining pairs of vertices, so that the distance between them is 1. A $d$-cube of dimension 0 is one point. The fixed classifier derived from the $d$-Cube is constructed by creating a vertex for each binary number in a string of $d$ bits. Each vertex is a $d$-dimensional boolean vector with binary coordinates $-1$ or 1. Weights are finally obtained from the normalized vertices:

$$W_c = \left\{ w \in \mathbb{R}^d : \left[-\frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d}}\right]^d \right\}.$$  

The $d$-Cube can accommodate:

$$K = 2^d$$  

(8)

classes. The vertices are connected by an edge whenever the Hamming distance of their binary numbers is one therefore forming a $d$-connected graph. The angle of a vertex with its connected vertices is:

$$\theta_{w_i, w_j} = \arccos\left(\frac{d - 2}{d}\right), \forall i, j \in \{1, \ldots, K\} : j \in C(i)$$  

(9)

where $C(i)$ is the set of vertices connected to the vertex $i$.

Given a classification problem with $K$ classes, the three RePoNet fixed classifiers can be simply instantiated by defining a non trainable fully connected layer of dimension $d$, where $d$ is computed from Eqs. 4, 6 and 8 as summarized in the following table:

| RePoNet     | $d$-Simplex | $d$-Cube | $d$-Orthoplex |
|-------------|-------------|----------|---------------|
| Layer dim.  | $d = K - 1$ | $d = \lceil \log_2(K) \rceil$ | $d = \lceil K/2 \rceil$ |

Fig. 3 shows the angle between a weight and its connected weights computed from Eqs. 5, 7 and 9 as the dimension of the embedding space increases. Having the largest angle between its weights, the $d$-Simplex fixed classifier achieves the best intra-class separability. However, as the embedding space dimension increases, its angle tends towards $\pi/2$. Therefore the largest the dimension of the space the more it becomes similar to the $d$-Orthoplex classifier. The main difference between the two classifiers is in their neighbors connectivity. The different connectivity of the three regular polytope classifiers has a direct influence on the evaluation of the loss. In the case of the $d$-Simplex classifier, all the summed terms in the loss of Eq. 3 have always comparable magnitudes in a mini batch.

The $d$-Cube classifier has the most compact feature embedding and the angle between each weight and its $d$ neighbors decreases as the dimension increases. Accordingly, it’s the hardest to optimize.

![Figure 3: The angular space defined by RePoNet classifiers.](image)

Figure 3: The angular space defined by RePoNet classifiers. Curves represent the angle between a weight and its connected weights as the dimension of the embedding space increases. The angle between class features follows the same trend.

5. Theoretical Analysis

The joint density $P(f_1, f_2, \ldots, f_K)$ specified by a learned DCNN model encapsulates the dependence among the features of the $K$ classes. In general, there is a large number of ways about the form in which such dependencies can be learned, but there are some simple forms which accurately describe the way in which features are learned by the proposed approach. Suppose that, in considering the joint
distribution above, the labels identifying the individual features are uninformative, in the sense that the information the features contain is independent of the order in which the labels \( y_i \) are presented to the output of the classifier. This order independence is captured by the property of Exchangeability (or symmetrical dependency) \([21, 22]\). With reference to Fig. 4, exchanging the weights \( w_i \) is equivalent to exchange the associated labels and consequently their associated learned features. However, due to the symmetrical configuration of the weights, the form that shapes the dependency between the features does not change. More formally:

**Definition 1.** Exchangeability. Let \( \pi \) denote an arbitrary permutation on \( n \) elements (that is, a function that maps \( \{1, \ldots, n\} \) onto itself). The finite sequence \( X_1, \ldots, X_n \) is said to be exchangeable if the joint distribution of the permuted random vector \( (X_{\pi(1)}, \ldots, X_{\pi(n)}) \) is the same no matter which is chosen. The infinite sequence \( X_1, X_2, \ldots \) is said to be exchangeable if any finite subsequence is exchangeable.

Exchangeable sequences are basically random sequences that are invariant to the class of transformations representing permutations. This property is directly related to strict stationarity of random sequences.

**Definition 2.** Stationarity. The sequence \( X_1, X_2, \ldots \) is said to be stationary if, for a fixed \( k \geq 0 \), the joint distribution of \( (X_i, \ldots, X_{i+k}) \) is the same no matter what positive value of \( i \) is chosen.

Under the assumption (empirically confirmed in Sec. 6.1) that a DCNN with a fixed regular polytope classifier has sufficient expressive power to learn any permutation of its labels, then the generated features are stationary.

**Theorem 1.** A fixed classifier DCNN with weights set to values taken from the coordinate vertices of a regular polytope generates at training time stationary features.

**Proof.** It directly follows from the Def. 1 that an exchangeable process is stationary.

6. Experimental Results

We used standard datasets to evaluate both the correctness and the performance of our approach. All the experiments are conducted with the well known MNIST, FashionMNIST \([23]\), EMNIST \([24]\), CIFAR-10 and CIFAR-100 \([25]\) datasets. We train all the networks using Adam \([26]\), and the network initialization follows \([27]\). All the networks use Batch Normalization \([28]\) and ReLU if not otherwise specified. MNIST and FashionMNIST contain 50,000 training images and 10,000 test images. The images are in grayscale and the size of each image is 28 \( \times \) 28 pixels. There are 10 possible classes of digits and clothes respectively. The EMNIST dataset (balanced split) holds 112,800 training images, 18,800 test images and has 47 classes including lower/upper case letters and digits. CIFAR-10 contains 50,000 training images and 10,000 test images. The images are in color and have a resolution of 32 \( \times \) 32 pixels. There are 10 classes of various animals and vehicles. CIFAR-100 holds the same number of images of same size, but contains 100 different classes.

6.1. Exchangeability Assumption

In order to provide empirical support for the theoretical assumption of exchangeability made in this paper we ran a set of experiments to specifically evaluate the expressive power of our approach to learn any permutation of its labels. According to this, we generate random permutations of the labels and a new model is learned for each generated permutation. Since fixed classifiers cannot rely on an adjustable set of subspaces for class feature representation we want to test if some permutations are harder than others for our proposed method. The presence of such hard permutations not only would preclude the underlying exchangeability property but it would also preclude the general applicability of our method. The standard trainable classifier does not suffer this problem since it is not forced to learn feature representations into predetermined set of subspaces. When features cannot be well separated a trainable classifier can rearrange its (randomly initialized) feature subspaces directions so that the previous convolutional layers can better disentangle the non-linear interaction between complex data patterns.

Fig. 5 shows the mean and the 95\% confidence interval computed from the accuracy curves of the learned models. To provide further insights about this analysis, 20 out of 500 accuracy curves computed for each datasets are also
shown. Specifically, the evaluation is performed on three different datasets with an increasing level of complexity (i.e. MNIST, CIFAR-10 and CIFAR-100). All the models are trained for 200 epochs to make sure that the models trained with CIFAR-100 achieve convergence.

In order to address the most severe possible outcomes that may happen, for this experiment we used the $d$-Cube fixed classifier. Being the hardest to optimize, this experiment can be regarded as a worst case analysis scenario for our method. As evidenced from the figure, the performance is substantially insensitive to both permutations and datasets. The average reduction in performance at the end of the training process is substantially negligible and the confidence intervals reflect the complexity of the datasets. Although the space of permutations cannot be exhaustively evaluated even for a small number of classes, we have achieved proper convergence for the whole set of 1500 learned models. The experiment took 5 days on a Nvidia DGX-1 (8 Tesla V100).

We further report qualitative results on the exchangeability property and on the capability of deciding where the features of each class will be projected before starting the training phase. Fig. 6a shows features learned in a $k$-sided polygon ($2d$ embedding space) on the MNIST dataset. In particular the model is learned with the permutation (manually selected) of the labels that places even and odd digits features respectively on the positive and negative half-space of the abscissa. Fig. 6b shows the features of CIFAR10 learned with a similar 10-sided-polygon. It can be noticed that features are distributed following the polygonal pattern shown in Fig. 6a.

6.2. Performance Evaluation

We evaluate the classification performance of RePoNets on the following datasets: MNIST, EMNIST, FashionMNIST, CIFAR-10 and CIFAR-100. The proposed method is compared with the fixed classifier method reported in [1], here implemented for different architectures and different dimensions of the embedding space. Standard CNN baselines with learnable classifiers are also included. Except for the final fixed classifier all the compared methods have exactly the same architecture and training settings as the one that RePoNet uses.

We trained the so called LeNet++ architecture [29] on all the MNIST family datasets. The network is a modification of the LeNets [30] to a deeper and wider network including parametric rectifier linear units (pReLU) [31]. We further trained the VGG network of [32] on the CIFAR-10 and CIFAR-100 datasets using two networks of depth 13 and 19 and the same hyper-parameters used in the original work. Also in this case, we compared all the variants of our approach including the two architectures with different dimensions of the feature space.

All the results are reported in Tab. 2. Each entry in the table reports the test-set accuracy. The subscript indicates the specific feature space dimension $d$ used for that experi-

![Figure 5: Average accuracy curves and confidence interval computed from the MNIST, CIFAR10 and CIFAR100 datasets under different random permutations of the labels.](image)

![Figure 6: The distribution of features learned using a 10-sided regular polygon. (a): A special permutation of classes is shown in which the MNIST even and odd digits are placed in the positive and negative half-space of the abscissa respectively. (b): The features learned using the CIFAR10 dataset.](image)

![Figure 7: Learning speed. Comparing training error (top) and test accuracy (bottom) on the CIFAR100 dataset.](image)
The results reveal and confirm that our fixed classifier models achieve comparable classification accuracy of other trainable classifier models. This evidence is in agreement on all the combinations of datasets, architectures and feature space dimensions. All the RePoNet variants exhibit similar behavior even in complex combinations such as in the case of the CIFAR100 dataset in low dimensional feature space. For example, the RePoNet d-Cube fixed classifier implemented with the VGG19 architecture achieves an accuracy of 64.02% in a \( d = 7 \) dimensional feature space. A fully trainable classifier in a feature space of dimension \( d = 512 \) (i.e. two orders of magnitude larger) achieves 68.69%. With a reasonable shorter feature dimension of \( d = 50 \) RePoNet d-Orthoplex improves the accuracy to 69.07%. Results further show that the number of classes has not an about equal number of unique weight directions in the embedding space (i.e. \( d < K \)), as in the Hadamard fixed classifier [1], no proper learning can be obtained. This effect is also present for simple datasets as the MNIST digits dataset. When \( d \approx K \) or \( d > K \) classification performance are similar. However, as shown in Fig. 7, RePoNet has faster learning speed matching the trainable baseline. Our conjecture is that with our symmetrical fixed classifiers, each term in the loss function tends to have the same magnitude (i.e. von Mises-Fisher distribution is similar to the Normal distribution) and therefore the average is a good estimator. Contrarily, in Hadamard classifier the terms may have different magnitudes and “important” errors may be not taken correctly into account by averaging.

We finally evaluate our approach in the more sophisticated DenseNet-169 architecture [33]. Tab.3 shows that RePoNet classifiers are architecture-agnostic being able to improve accuracy following the expressive power of more competitive architectures.

### 7. Conclusion

We have shown that a special set of fixed classifiers based on regular polytopes generates stationary features by maximally exploiting the available representation space. The proposed method is simple to implement and theoretically correct. Experimental results confirm both the theoretical analysis and the generalization capability of the approach. RePoNet improves and generalizes the concept of a fixed classifier, recently proposed in [1], to a larger class of fixed classifier models exploiting the inherent symmetry of regular polytopes in the feature space.

Our finding may have implications in all of those Deep Neural Network learning contexts in which a classifier must be robust against changes of the feature representation while learning as in incremental and continual learning settings.
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