A GENERAL FRAMEWORK FOR PARALLELIZING DYKSTRA SPLITTING

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ABSTRACT. We show a general framework of parallelizing Dykstra splitting that includes the classical Dykstra’s algorithm and the product space formulation as special cases, and prove their convergence. The key idea is to split up the function whose conjugate takes in the sum of all dual variables in the dual formulation.

1. Introduction

Let $X$ be a finite dimensional Hilbert space. Consider the problem

$$\begin{align*}
(P1) \quad \min_{x \in X} \sum_{i=1}^{r} h_i(x) + g(x),
\end{align*}$$

where $h_i : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are proper closed convex functions. The (Fenchel) dual of $(P1)$ is

$$\begin{align*}
(D1) \quad \max_{z \in X} \sum_{i=1}^{r} \langle h_i^*(z_i) \rangle - g^* \left( - \sum_{i=1}^{r} z_i \right).
\end{align*}$$

A particular case of $(P1)$ that is well studied is when $g(x) := \frac{1}{2} \|x - x_0\|^2$ for some $x_0 \in X$, and

$$g^*(y) = \frac{1}{2} \|y\|^2 + x_0^T y = \frac{1}{2} \|y + x_0\|^2 - \frac{1}{2} \|x_0\|^2.$$

The resulting $(D1)$ would be the sum of a block separable concave function and a smooth concave function. For the problem $(D1)$, if the map $z \mapsto g^*(- \sum_{i=1}^{r} z_i)$ is smooth, then $(D1)$ can be solved by block coordinate minimization (BCM);
Specifically, one maximizes a particular $z_i$, say $z_{i^*}$, while keeping all other $z_j$, where $j \neq i^*$, fixed, and the index $i^*$ is cycled over all indices in $\{1, \ldots, r\}$. (It would be a minimization if $(D_1)$ were written in a minimization form.)

Dykstra’s algorithm was proposed in [Dyk83], and it was separately recognized to be the BCM on $(D_1)$ with $h_i(\cdot) \equiv \delta_{C_i}(\cdot)$ for all $i \in \{1, \ldots, r\}$ and $g(x) := \frac{1}{2} \|x - x_0\|^2$ in [Han88, GM89]. An advantage of Dykstra’s algorithm is that it decomposes the complicated problem $(P1)$ so that the proximal operation is applied to only one function of the form $h_i(\cdot)$ (or $h^*_i(\cdot)$) at a time so that one can solve the larger problem in hand. Dykstra’s algorithm was shown to converge to the primal minimizer in [BD85], even when a dual minimizer does not exist. For more information Dykstra’s algorithm, we refer to [Deu01a, Deu01b, BC11, ER11].

The extension of considering general $h_i(\cdot)$ was done in [Han89] and [Tse93]. We now refer to this as Dykstra’s splitting. It is quite easy to see that the convergence to the dual objective value implies the convergence to the primal minimizer. (See for example, the end of the proof in Theorem 4.5.) In [Han89] and [Tse93], they proved that Dykstra’s splitting converges, but under a constraint qualification. The paper [BC08] proved that Dykstra’s splitting converges without constraint qualifications, but only for the case where $r = 2$. In [Pan17], we proved the convergence of Dykstra’s splitting in finite dimensions for any $r \geq 2$.

The BCM algorithm is related to block coordinate gradient descent, but we shall only mention them in passing as we do not deal directly with these algorithms in this paper. Much research on the BCM and related algorithms is on nonasymptotic convergence rates when a minimizer to $(D1)$ exists and the level sets are bounded.

A shortcoming of Dykstra’s splitting is that it requires that the proximal operations on $h_i(\cdot)$ be taken one at a time in order. If data on the functions $h_i(\cdot)$ were distributed on different agents, then these agents would be idling as they wait for their turn. Another parallel method for solving $(P1)$ when $r > 2$ is to use the product space formulation largely due to [Pie84]. (See the first paragraph in Section 3 for more details.) A shortcoming of this product space formulation is that a central controller needs to compute the average of all intermediate primal variables before the next iteration can proceed. This can be a tedious task depending on the communication model, and if it were easy to do, then accelerated proximal algorithms might be preferred (see for example [BT09, Tse08], who built on the work of [Nes83]). Parallelizations of the BCM were suggested in [Cal16, RT16], but we note that their approach is different from what we will discuss in this paper, and the approach in [RT16] requires random sampling.

Dykstra’s splitting falls under the larger class of proximal methods. See a survey in [CP11].

1.1. Contributions of this paper. The contribution of this paper is to parallelize the BCM problem arising from Dykstra’s algorithm so that agents that otherwise would have been idle in Dykstra’s algorithm can be actively decreasing the dual objective value. This is achieved by breaking up the $g(\cdot)$ in $(P1)$ so that smaller versions of the problem of the form $(D1)$ can be solved in parallel with few communication requirements between the agents involved. We also show that our algorithm generalizes the product space formulation. We also prove its convergence to the primal minimizer in the spirit of [BD85, GM89], even when a dual minimizer may not exist.
2. Algorithm Description

Consider the problem \((P1)\). Let \(\{\lambda_i\}_{i=0}^m\) be constants such that \(\sum_{i=0}^m \lambda_i = 1\) and \(\lambda_i \geq 0\). Define \(h_j : X \rightarrow \mathbb{R}\) to be
\[h_j(x) = \lambda_j - rg(x)\]
for all \(j \in \{r+1, \ldots, r+m\}\). (2.1)

Then \((P2)\) can be rewritten as
\[
\min_x \sum_{i=1}^{r+m} h_i(x) + \lambda_0 g(x),
\]
which in turn has dual
\[
\max_{z \in X^{r+m}} F(z) := - \sum_{i=1}^{r+m} h_i^*(z_i) - \lambda_0 g^* \left( -1 \sum_{i=1}^{r+m} z_i \right),
\]
where
\[
h_i^*(z_i) = \lambda_i - rg^* \left( \frac{z_i}{\lambda_i} \right) \text{ for all } i \in \{r+1, \ldots, r+m\}. \quad (2.4)
\]

Algorithm 2.1 describes a method to solve \((D2)\) (and equivalently, \((P1)\) and \((P2)\)).

**Algorithm 2.1. (A general framework for Dykstra’s splitting)** This algorithm solves \((2.3)\), which can in turn find the primal minimizer of \((1.1)\). Let \(\bar{w}\) be a fixed number.

01 Start with dual variables \(\{z_i^0\}_{i=1}^{r+m}\) to the problem \((2.3)\).
02 For \(n = 1, \ldots\)
03 For \(w = 1, \ldots, \bar{w}\)
04 Choose a (possibly empty) subset \(S_{n,w} \subset \{1, \ldots, r+m\}\).
05 Let \(\{z_i^{n,w}\}_{i \in S_{n,w}}\) be a minimizer of
\[
\min_{z_i^i : i \in S_{n,w}} \sum_{i \in S_{n,w}} h_i^*(z_i) + \lambda_0 g^* \left( -1 \sum_{i \in S_{n,w}} z_i + \sum_{i \notin S_{n,w}} z_i^{n,w-1} \right) \quad (2.5)
\]
06 Let \(\{z_i^{n,w}\}_{i \notin S_{n,w}}\) be such that
\[
\sum_{i \notin S_{n,w}} h_i^*(z_i^{n,w}) \leq \sum_{i \notin S_{n,w}} h_i^*(z_i^{n,w-1}) \quad (2.6a)
\]
and
\[
\sum_{i \notin S_{n,w}} z_i^{n,w} = \sum_{i \notin S_{n,w}} z_i^{n,w-1}. \quad (2.6b)
\]
07 End for
08 Let \(z_i^{n+1,0} = z_i^{n,w}\).
09 End for

Dykstra’s algorithm is usually expressed in terms of the primal variables. We refer to Proposition 4.1.

We note that in Algorithm 2.1 both problems \((2.3)\) and \((2.6)\) are aimed to increase the dual objective value in \((2.3)\).

We show a direct connection between \((D1)\) and \((D2)\), whose proof is just elementary convexity.
Proposition 2.2. (Direct connection between (D1) and (D2)) Consider the problem

$$\min_{z_i : r + 1 \leq i \leq r + m} \lambda_0 g^* \left( \frac{1}{\lambda_0} \left( \sum_{i=1}^{r} \bar{z}_i + \sum_{i=r+1}^{r+m} z_i \right) \right) + \sum_{i=r+1}^{r+m} \lambda_{j-r} g^* \left( \frac{1}{\lambda_{j-r}} z_i \right). \tag{2.7}$$

Suppose further that $g^*(\cdot)$ is strictly convex. Then (2.7) has a minimum value of $g^*(- \sum_{i=1}^{r} \bar{z}_i)$ with minimizer

$$z_j = -\lambda_{j-r} \sum_{i=1}^{r} \bar{z}_i \text{ for all } j \in \{r+1, \ldots, r+m\}. \tag{2.8}$$

This establishes the equivalence of (D1) and (D2) directly without appealing to the primal problems (P1) and (P2).

2.1. Finding $\{z_{i,w}^{r,n}\}_{i \in S_n,w}$ in (2.6). We now suggest methods for finding $\{z_{i,w}^{r,n}\}_{i \in S_n,w}$ satisfying (2.6). Let $S'$ be a subset of $\{1, \ldots, r + m\} \setminus S_{n,w}$, and suppose $j \in \{r + 1, \ldots, r + m\} \setminus S'$. Consider the problem

$$\min_{z_i : i \in S' \cup \{j\}} \sum_{i \in S' \cup \{j\}} h_i^*(z_i) \tag{2.9}$$

$$\text{s.t.} \sum_{i \in S' \cup \{j\}} z_i = \sum_{i \in S' \cup \{j\}} z_i^{r,n,w-1}.$$

Such a problem can be solved by other methods like the ADMM. But intermediate iterates of the other methods may not satisfy the equality constraint of (2.9), so one has to check whether these intermediate iterates are indeed more useful than what we had started off with. We now show an option that is close to the spirit of BCM.

Proposition 2.3. (Reduced unconstrained problem) Let $j \in \{r + 1, \ldots, r + m\}$, and suppose $j \notin S'$. Recall that $h_j(x) = \lambda_{j-r} g(x).$ For the problem (2.9), $\{z_i : i \in S' \cup \{j\}\}$ is a minimizer to (2.9) if and only if $\{z_i : i \in S'\}$ is a minimizer to

$$\min_{z_i : i \in S'} \sum_{i \in S'} h_i^*(z_i) + \lambda_{j-r} g^* \left( \frac{1}{\lambda_{j-r}} \left( \sum_{i \in S' \cup \{j\}} z_i^{r,n,w-1} - \sum_{i \in S'} z_i \right) \right). \tag{2.10}$$

Proof. The linear constraint in (2.9) can be removed by expressing $z_j$ in terms of the other variables. \qed

Similar to the relationship between (P1) and (D1), (2.10) has dual

$$\min_x \sum_{i \in S'} h_i(x) + \lambda_{j-r} g(x) - \left( \sum_{i \in S' \cup \{j+1\}} z_i^{r,n,w-1} \right)^T x \tag{2.11}$$

This problem (2.10) can then be solved by BCM. Given that $|S'|$ is now smaller compared to $\{|1, \ldots, r + m|\}$, other methods might be better. For example, an accelerated proximal gradient method can be used if the problem size is small enough that communication problems are less significant. Interior point methods can be considered if the subproblem size is small enough. The set $\{1, \ldots, r + m\} \setminus S_{n,w}$ can be partitioned into at most $m$ such subsets, with each subset taking a term of the form $\lambda_{j-r} g(x)$ so that these problems can be solved in parallel.
3. Product space formulation is a subcase

Consider the problem of projecting $x_0$ onto $\cap_{i=1}^r C_i$. This problem can be equivalently formulated using the product space formulation largely attributed to [Pie84] and also studied in [IP91]; specifically, the projection of $(x_0, \ldots, x_0) \in X^r$ onto $D \cap C$, where $D \subset X^r$ is the set $\{(x, \ldots, x) : x \in X\}$ and $C$ is the set $C_1 \times \cdots \times C_r$, gives $P_{\cap_{i=1}^r C_i}(x_0)$ in each component. Dykstra’s algorithm can then be applied on this formulation, which can then be rewritten as Algorithm 3.1 below. In this section, we show that Algorithm 3.1 is a special case of Algorithm 2.1.

**Algorithm 3.1.** (Product space formulation for Dykstra’s algorithm) For the problem of projecting $x_0$ onto $\cap_{i=1}^r C_i$, the product space formulation gives the following algorithm. (Compare this to [IP91, page 267])

01 Start with dual variables $\{z_i^n\}_{i=1}^r$, and let $x_1 = x_0 - \frac{1}{r} \sum_{i=1}^r z_i^n$.
02 For $n = 2, \ldots$
03 For $i = 1, \ldots, r$
04 $u_i^n = x_i^{n-1} + z_i^{n-1}$
05 $x_i^n = P_{C_i}(u_i^n)$
06 $z_i^n = u_i^n - x_i^n$
07 End for
08 $x^n = \frac{1}{r} \sum_{i=1}^r x_i^n$.
09 End for

Algorithm 3.2 is a particular way to solve (2.6) in Algorithm 2.1 which will be used throughout this section and the next section.

**Algorithm 3.2.** (Calculating (2.6)) Suppose that (2.6) in Algorithm 2.1 is performed using the following step:

01 Find disjoint subsets $\{S_{n,w,j}'\}_{j=r+1}^{r+m}$ such that
$$S_{n,w} \cap \left( \bigcup_{j=r+1}^{r+m} S_{n,w,j}' \right) = \emptyset,$$
and for all $j \in \{r+1, \ldots, r+m\}$
$$\begin{cases} S_{n,w,j}' \subset \{1, \ldots, r\} \cup \{j\}, & \text{and} \\ S_{n,w,j}' \neq \emptyset \text{ implies } S_{n,w,j}' \supseteq \{j\}. \end{cases}$$
02 For $j = r+1, \ldots, r+m$
03 Define $\{z_i^{n,w}\}_{i \in S_{n,w,j}'}$ by
$$\{z_i^{n,w}\}_{i \in S_{n,w,j}'} \in \arg \min_{z_i \in S_{n,w,j}'} \sum_{i \in S_{n,w,j}'} h_i(z_i),$$
$s.t.$
$$\sum_{i \in S_{n,w,j}'} z_i = \sum_{i \in S_{n,w,j}'} z_i^{n,w-1}.$$ (3.1a)
04 End for
05 If $i \notin S_{n,w} \cup \bigcup_{j=r+1}^{r+m} S_{n,w,j}'$, then $z_i^{n,w} = z_i^{n,w-1}$.
We now present our result showing that Algorithm 3.1 is a particular case of Algorithm 2.1.

**Proposition 3.3.** (Product space formulation is a particular case) Suppose Algorithm 2.1 is run using Algorithm 3.3 for the subproblem (2.6). Suppose the parameters are set to be \( \bar{w} = 2, m = r - 1 \), \( g(\cdot) = \frac{1}{2} \| \cdot \|_{(2)}^{2} \) and 

(i) \( h_{i}(\cdot) = \delta_{C_{i}}(\cdot) \) and for all \( i \in \{1, \ldots, r\} \),

(ii) \( \lambda_{i} = \frac{1}{r} \) for all \( i \in \{0, \ldots, r - 1\} \),

(iii) \( S_{n, 1} = \{r + 1, \ldots, 2r - 1\} \) and \( S_{n, 2} = \{r\} \) for all \( n \geq 0 \), and

(iv) \( S_{r, 1, j}^{\prime} = \emptyset \) and \( S_{r, 2, j}^{\prime} = \{j - r, j\} \) for all \( n \geq 0 \) and \( j \in \{r + 1, \ldots, 2r - 1\} \).

If \( z_{i}^{1} \) of Algorithm 3.1 equals \( z_{i}^{1,0} \) of Algorithm 2.1 for all \( i \in \{1, \ldots, r\} \), then \( z_{i}^{n} = z_{i}^{n,0} \) for all \( n \geq 0 \) and \( i \in \{1, \ldots, r\} \).

**Proof.** We prove our result by induction. Consider the equalities

\[
{z}_{i}^{k} = {z}_{i}^{k,0} \quad \text{for all} \quad i \in \{1, \ldots, r\} \quad (3.2a)
\]

and

\[
x^{k} = x_{0} - \frac{1}{r} \sum_{i=1}^{r} z_{i}^{k}. \quad (3.2b)
\]

For \( k' = 1 \), (3.2a) follows from the induction hypothesis, and (3.2b) follows from line 1 of Algorithm 3.1. We shall show that if (3.2) holds for \( k' = k \), then (3.2) holds for \( k' = k + 1 \).

First, \( \{z_{i}^{k,1}\}_{i \in \{r + 1, \ldots, 2r - 1\}} \) are calculated through (2.5). Proposition 2.2 shows that

\[
z_{i}^{k,1} = \frac{x_{k+1} - \delta_{C_{i}}(z_{i})}{2} = \frac{x_{k+1} - \frac{1}{r} \sum_{i=1}^{r} z_{i}^{k,0}}{2} = -\frac{1}{r} \sum_{i=1}^{r} z_{i}^{k,0}. \quad (3.3)
\]

Next, for \( i \in \{1, \ldots, r - 1\} \), solving the problems (3.1) with \( S_{r, 2, i+r}^{\prime} = \{i, i + r\} \) gives

\[
(z_{i}^{k,2}, z_{i+r}^{k,2}) = \arg \min_{(z_{i}, z_{i+r})} \delta_{\overline{C}_{i}}(z_{i}) + \frac{1}{2} \| r(z_{i+r} + x_{0}) \|^{2} - \frac{1}{2} \| rx_{0} \|^{2} \quad (3.4a)
\]

s.t. \( z_{i} + z_{i+r} = z_{i}^{k,1} + z_{i+r}^{k,1} \). \quad (3.4b)

One can calculate that

\[
z_{i+r} + x_{0} \overset{(3.4b)}{=} z_{i}^{k,1} + z_{i+r}^{k,1} - z_{i} + x_{0} \overset{(3.3)}{=} z_{i}^{k,1} - \frac{1}{r} \sum_{i'=1}^{r} z_{i'}^{k,0} - z_{i} + x_{0} \overset{(3.5)}{=} \]

and

\[
z_{i}^{k,1} - \frac{1}{r} \sum_{i'=1}^{r} z_{i'}^{k,0} - z_{i} + x_{0} \overset{(3.2b)}{=} z_{i}^{k,1} + x^{k} - z_{i} \overset{(3.6)}{=} \]

Alg. 3.2 line 5 \( z_{i}^{k,0} + x^{k} - z_{i} \) Alg. 3.1 line 4 \( u_{i}^{k+1} - z_{i} \).

so (3.4) can be rewritten as

\[
z_{i}^{k,2} \overset{\text{arg min}}{\overset{(3.7)}{z_{i}}} \delta_{\overline{C}_{i}}(z_{i}) + \frac{1}{2} \| r(z_{i+r} + x_{0}) \|^{2} \]

and

\[
z_{i}^{k,2} \overset{\text{arg min}}{\overset{(3.7)}{z_{i}}} \delta_{\overline{C}_{i}}(z_{i}) + \frac{1}{2} \| z_{i} - u_{i}^{k+1} \|^{2}. \]

Also, we have

\[
-z_{r} - \sum_{i=1}^{r-1} z_{i}^{k,1} - \sum_{i=r+1}^{2r-1} z_{i}^{k,1} + x_{0} \overset{(3.8)}{=} z_{r}^{k,1} - \frac{1}{r} \sum_{i=1}^{r} z_{i}^{k,1} + x_{0} - z_{r}. \]

PARALLEL Dijkstra Splitting 6
Next, solving the problem \((2.5)\) with \(S_{k,2} \ni \{r\}\) gives

\[
z_r^{k,2} = \arg \min_{z_r} \delta^*_C(z_r) + \frac{1}{2r} \left\| r \left( -z_r - \sum_{i=1}^{r-1} z_{i}^{k,1} - \sum_{i=r+1}^{2r-1} z_{i}^{k,1} + x_0 \right) \right\|^2.
\]

From \((3.7)\) and \((3.9)\), we can see that the forms for \(z_r^{k,2}\) and \(z_r^{k,2}\) are identical. Similar to the relationships between \((1.1)\) and \((1.2)\), these problems are the (Fenchel) dual problems to

\[
\min_x \delta_C(x) + \frac{1}{2} \| x - u_i^{k+1} \|^2,
\]

which has primal solution \(P_{C_i}(u_i^{k+1})\) and dual solution \(u_i^{k+1} - P_{C_i}(u_i^{k+1})\). Specifically, one has \(0 \in \partial \delta_C(P_{C_i}(u_i^{k+1})) + P_{C_i}(u_i^{k+1}) - u_i^{k+1}\), which gives \(P_{C_i}(u_i^{k+1}) \in \partial \delta^*_C(u_i^{k+1}) - P_{C_i}(u_i^{k+1})\), and thus \(u_i^{k+1} - P_{C_i}(u_i^{k+1})\) minimizes \((3.7)\). Therefore,

\[
z_i^{k,2} = u_i^{k+1} - P_{C_i}(u_i^{k+1}) \text{ Alg 3.1, lines 5,6,} \]

Combining with the fact that \(z_i^{k+1,0} = z_i^{k,2}\), we have \(z_i^{k+1,0} = z_i^{k+1}\) for all \(i \in \{1, \ldots, r\}\) as needed. To complete our induction, note that from Algorithm 3.1 we have

\[
x^{k+1} \text{ line 8 } \frac{1}{r} \sum_{i=1}^{r} x_i^{k+1} \text{ line 6 } \frac{1}{r} \sum_{i=1}^{r} (u_i^{k+1} - z_i^{k+1}) \]

\[
\text{ line 4 } \frac{1}{r} \sum_{i=1}^{r} (x_k + z_i^{k} - z_i^{k+1}) \text{ (4.2a)} \equiv x_0 - \frac{1}{r} \sum_{i=1}^{r} z_i^{k+1}. \]

\[
□
\]

4. Convergence

In this section, we prove a convergence result for Algorithm \(2.1\) combined with Algorithm \(3.2\). Our result would cover the case of the product space decomposition as well as the original Dykstra’s algorithm.

Throughout this section, we make the following assumption on \(g(\cdot)\) and \(\{\lambda_i\}_{i=0}^{r-1}\).

**Assumption 4.1.** Assume \(m \geq 0, \lambda_i = \frac{1}{m+1}\) for all \(i \in \{0, \ldots, m\}\), and \(g(x) = \frac{m+1}{2} \| x - x_0 \|_2^2\). The term \(\lambda_0 g^*(\cdot, \cdot)\) in \((2.5)\) becomes

\[
\frac{1}{2} \left\| \sum_{i \in S_{n,w}} z_i - \sum_{i \notin S_{n,w}} z_i^{n,w-1} + x_0 \right\|^2 - \frac{1}{2} \| x_0 \|^2.
\]

We define \(v^{n,w} \in X\) and \(x^{n,w} \in X\) to be

\[
\begin{align*}
v^{n,w} := & \sum_{i=1}^{r+m} z_i^{n,w} \tag{4.2a} \\
x^{n,w} := & x_0 - v^{n,w} \tag{4.2b}
\end{align*}
\]

**Assumption 4.2.** For each \(n\) and \(i\), there is an index \(p(n,i)\) in \(\{1, \ldots, \bar{w}\}\) such that

(A) For all \(i \in \{1, \ldots, r + m\}\),
(i) \( i \notin S_{n,w} \) and \( i \notin S'_{n,w,j} \) for all \( w \in \{p(n,i) + 1, \ldots, \bar{w}\} \) and \( j \in \{r + 1, \ldots, r + m\} \), and

(ii) Either \( i \in S_{n,p(n,i)} \), or \( i \in S'_{n,p(n,i),j} \) for some \( j \in \{r + 1, \ldots, r + m\} \).

This can be easily checked using line 5 of Algorithm 3.2 to lead to
\[
z_i^{n,p(n,i)} = z_i^{n,p(n,i)+1} = \cdots = z_i^{n,w} \quad \text{for all } i \in \{1, \ldots, r + m\}.
\] (4.3)

(B) If \( i \in \{1, \ldots, r\} \) and \( i \in S'_{n,p(n,i),j} \) for some (unique) \( j \in \{r + 1, \ldots, r + m\} \) (including the case \( i = j \)).

- There is some \( q(n,i) \) in \( \{1, \ldots, p(n,i) - 1\} \), such that \( j \in S_{n,q(n,i)}, \) and

\[
S'_{n,p(n,i),j} \cap S_{n,w} = \emptyset \quad \text{and} \quad S'_{n,p(n,i),j} \cap S'_{n,w,j'} = \emptyset
\]

for all \( w \in \{q(n,i) + 1, \ldots, p(n,i) - 1\} \) and \( j' \in \{r + 1, \ldots, r + m\} \).

This can be easily checked using line 5 of Algorithm 3.2 to lead to
\[
z_i^{n,q(n,i)} = z_i^{n,q(n,i)+1} = \cdots = z_i^{n,p(n,i)-1} \quad \text{for all } i' \in S'_{n,p(n,i),j'}.
\] (4.4)

Assumptions 4.1 and 4.2(B) can be further generalized, but we feel that they are enough to capture the main ideas needed for more general cases. Moreover, one can check that the classical Dykstra’s algorithm and the product space formulation satisfy these assumptions. Assumptions 4.2(A) implies that all the components of \( z \) are updated in an iteration.

4.1. **Proof of convergence.**

Claim 4.3. For all \( i \in S_{n,w} \), we have

(a) \(-x_{n,w} + \partial h_i^*(z_i^{n,w}) \ni 0,\)

(b) \(-z_i^{n,w} + \partial h_i(x_{n,w}) \ni 0,\)

(c) \(h_i(x_{n,w}) + h_i^*(z_i^{n,w}) = \langle x_{n,w}, z_i^{n,w} \rangle.\)

Proof. By taking the optimality conditions in (2.5) with respect to \( z_i \) for \( i \in S_{n,w} \), we have

\[
0 \mathrel{\overset{(2.5), (6.1)}{\in}} \partial h_i^*(z_i^{n,w}) + \sum_{i \in S_{n,w}} z_i^{n,w} + \sum_{i \notin S_{n,w}} z_i^{n,w-1} - x_0
\]

\[
\mathrel{\overset{(2.6), (6.2)}{=}} \partial h_i^*(z_i^{n,w}) + \sum_{i=1}^r z_i^{n,w} - x_0 \mathrel{\overset{(4.2)}{=}} \partial h_i^*(z_i^{n,w}) - x_{n,w},
\]

so (a) holds. The equivalences of (a), (b) and (c) is standard. \(\square\)

Another rather standard and elementary result is as follows. We refer to Pan17 for its (short) proof.

**Proposition 4.4.** (On solving (2.5)) If a minimizer \( z_{n,w} \) for (2.5) exists, then the \( x_{n,w} \) in (4.2) satisfies

\[
x_{n,w} = \arg \min_{x \in X} \sum_{i \in S_{n,w}} h_i(x) + \frac{1}{2} \left\| x - \left( x_0 - \sum_{i \notin S_{n,w}} z_i^{n,w} \right) \right\|^2.
\] (4.5)

Conversely, if \( x_{n,w} \) solves (4.5) with the dual variables \( \{z_i^{n,w}\}_{i \in S_{n,w}} \) satisfying

\[
\z_i^{n,w} \in \partial h_i(x_{n,w}) \quad \text{and} \quad x_{n,w} - x_0 + \sum_{i \notin S_{n,w}} z_i^{n,w} + \sum_{i \in S_{n,w}} z_i^{n,w} = 0,
\] (4.6)

then \( \{z_i^{n,w}\}_{i \in S_{n,w}} \) solves (2.5).
For any $x \in X$ and $z \in X^{r+m}$, the analogue of [GM89] (8) is
\begin{equation}
\frac{1}{2} \|x_0 - x\|^2 + \sum_{i=1}^{r+m} h_i(x) - F(z) \quad (4.7)
\end{equation}
\begin{equation}
\frac{1}{2} \|x_0 - x\|^2 + \sum_{i=1}^{r+m} [h_i(x) + h_i^*(z_i)] - \left< x_0, \sum_{i=1}^{r+m} z_i \right> + \frac{1}{2} \| \sum_{i=1}^{r+m} z_i \|^2
\end{equation}
Fenchel duality
\begin{equation}
\geq \frac{1}{2} \|x_0 - x\|^2 + \sum_{i=1}^{r+m} \langle x, z_i \rangle - \left< x_0, \sum_{i=1}^{r+m} z_i \right> + \frac{1}{2} \| \sum_{i=1}^{r+m} z_i \|^2
\end{equation}
\begin{equation}
= \frac{1}{2} \left\| x_0 - x - \sum_{i=1}^{r+m} z_i \right\|^2 \geq 0.
\end{equation}

We now prove our convergence result.

**Theorem 4.5. (Convergence result)** Suppose Assumptions (4.1) and (4.2) hold. Consider the sequence $\{z_{n,w}\}_{0 \leq n < \infty}$ generated by Algorithm 2.1 with Algorithm 3.2 used to calculate $F_{n,w}$. Suppose that
- The value of (4.1) (i.e., the primal objective value) is $\alpha$ and is finite, and the value of (4.2) (i.e., the dual objective value) is $\beta$.
- $\|z_{n,w}\| \in O(\sqrt{n})$.
- Minimizers can be obtained for the problems (4.5) and (4.6).

The sequences $\{v_{n,w}\}_{0 \leq n < \infty}$ and $\{x_{n,w}\}_{0 \leq n < \infty}$ are then deduced from (4.5), and we have:

(i) The sum
\begin{equation}
\sum_{n=1}^{\infty} \sum_{w=1}^{w_{n}} \left\| v_{n,w} - v_{n,w-1} \right\|^2 + \sum_{j, r+1 \leq j \leq r+m} \left\| z_{n,w,j} - z_{n,w-1,j} \right\|^2
\end{equation}

is finite, and $\{F(z_{n,w})\}_{n=1}^{\infty}$ is nondecreasing, where $F(\cdot)$ is as defined in (4.3).

(ii) There is a constant $C$ such that $\|v_{n,w}\|^2 \leq C$ for all $n \in \mathbb{N}$ and $w \in \{1, \ldots, w_{n}\}$.

(iii) Let
\begin{equation}
\gamma_n := \sum_{w=1}^{w_{n}} \left( \| v_{n,w} - v_{n,w-1} \| + \sum_{j, r+1 \leq j \leq r+m} \| z_{n,w,j} - z_{n,w-1,j} \| \right). \quad (4.8)
\end{equation}

There exists a subsequence $\{v_{n_k,w_k}\}_{k=1}^{\infty}$ of $\{v_{n,w}\}_{n=1}^{\infty}$ which converges to some $v^* \in X$ and that
\begin{equation}
\lim_{k \to \infty} \gamma_{n_k} \sqrt{n_k} = 0. \quad (4.9)
\end{equation}

(iv) For all $i \in \{1, \ldots, r+m\}$ and $n \in \mathbb{N}$, we can find $x^n_i \in \partial h_i^*(z_{n,w})$ such that $\| x^n_i - (x_0 - v_{n,w}) \| \leq \gamma_n$.

(v) For the $v^*$ in (iii), $x_0 - v^*$ is the minimizer of the primal problem (P1), and $\lim_{n \to \infty} F(z_{n,w}) = \frac{1}{2} \|v^*\|^2 + \sum_{i=1}^{r+m} h_i(x_0 - v^*)$.

The properties (i) to (v) in turn imply that $\lim_{n \to \infty} x_{n,w}$ exists, and $x_0 - v^*$ is the primal minimizer of (1.1).
Proof. We first remark on the proof of this result. The proof in \cite{Pan17} was adapted from \cite{GM89}. Part (iv) is new, and arises from considering (4.13). This also results in changes to the statements of the other parts of the corresponding result in \cite{Pan17}.

We first show that (i) to (v) implies the final assertion. For all \( n \in \mathbb{N} \) we have, from weak duality,

\[
F(z^{n,w}) \leq \beta \leq \alpha \leq \frac{1}{2} \|x_0 - (x_0 - v^*)\|^2 + \sum_{i=1}^{r+m} h_i(x_0 - v^*), \tag{4.10}
\]

hence \( \beta = \alpha = \frac{1}{2} \|x_0 - (x_0 - v^*)\|^2 + h(x_0 - v^*) \), and that \( x_0 - v^* = \arg \min_x \sum_{i=1}^{r+m} h_i(x) + \frac{1}{2} \|x - x_0\|^2 \). Since the values \( \{F(z^{n,w})\}_{n=1}^\infty \) are nondecreasing in \( n \), we have

\[
\lim_{n \to \infty} F(z^{n,w}) = \frac{1}{2} \|x_0 - (x_0 - v^*)\|^2 + \sum_{i=1}^{r+m} h_i(x_0 - v^*),
\]

and (substituting \( x = x_0 - v^* \) in (4.11))

\[
\frac{1}{2} \|x_0 - (x_0 - v^*)\|^2 + h(x_0 - v^*) - F(z^{n,w}) \geq \frac{1}{2} \|x_0 - (x_0 - v^*) - v^{n,w}\|^2 \tag{4.12}
\]

\[
\frac{1}{2} \|x_0 - (x_0 - v^*)\|^2 - F(z^{n,w}) = \frac{1}{2} \|x_0 - (x_0 - v^*) - v^{n,w}\|^2. \tag{4.13}
\]

Hence \( \lim_{n \to \infty} x^{n,w} \) is the minimizer of (P1).

It remains to prove assertions (i) to (v).

Proof of (i): For \( j \in \{r, \ldots, r + m\} \), let \( z^{n,w,j} \in X^{r+m} \) be the vector such that

\[
z_i^{n,w,j} = \begin{cases} z_i^{n,w} & \text{if } i \in S_{n,w} \text{ or } i \in S'_{n,w,j'} \text{ for some } j' \leq j \\ z_i^{n,w-1} & \text{otherwise.} \end{cases}
\]

From the fact that \( \{z_i^{n,w,r}\}_{i \in S_{n,w}} = \{z_i^{n,w}\}_{i \in S_{n,w}} \) is a minimizer of the outer problem \((2.16)\), we have

\[
F(z^{n,w,r}) + \frac{1}{2} \left\| \sum_{i \in S_{n,w}} z_i^{n,w} - \sum_{i \in S_{n,w}} z_i^{n,w-1} \right\|^2 \leq F(z^{n,w-1}). \tag{4.11}
\]

Next, we note that for \( j \in \{r + 1, \ldots, r + m\} \) such that \( S'_{n,w,j} \neq 0 \), solving the inner problems sequentially like in line 3 of Algorithm \text{3.2} where each optimization problem has the form \((2.11)\), gives

\[
F(z^{n,w,j}) + \frac{1}{2} \left\| \sum_{i \in S'_{n,w,j} \backslash \{j\}} z_i^{n,w} - \sum_{i \in S'_{n,w,j} \backslash \{j\}} z_i^{n,w-1} \right\|^2 \leq F(z^{n,w,j-1}). \tag{4.12}
\]

In view of \((3.14)\), we have

\[
\sum_{i \in S'_{n,w,j} \backslash \{j\}} z_i^{n,w} - \sum_{i \in S'_{n,w,j} \backslash \{j\}} z_i^{n,w-1} = z_j^{n,w-1} - z_j^{n,w}.
\]

Observe that \( z_j^{n,w+r+m} = z_j^{n,w} \). We can combine \((4.11)\) and \((4.12)\) to get

\[
F(z^{n,w}) + \frac{1}{2} \|v^{n,w} - v^{n,w-1}\|^2 + \sum_{j \in \{r + 1, \ldots, r + m\}} \frac{1}{2} \|z_j^{n,w-1} - z_j^{n,w}\|^2 \leq F(z^{n,w-1}). \tag{4.13}
\]

Next, \( F(z^{n,w}) \leq \alpha \) by weak duality. The proof of the claim follows from summing \((4.13)\) over all \( n \).
Proof of (ii): Substituting $x$ in (4.7) to be the primal minimizer $x^*$ and $z$ to be $z^{n,w}$, we have

\[ \frac{1}{2} \| x_0 - x^* \|^2 + \sum_{i=1}^{r+m} h_i(x^*) - F(z^{1,0}) \geq \frac{1}{2} \| x_0 - x^* \|^2 + \sum_{i=1}^{r+m} h_i(x^*) - F(z^{n,w}) \]

part (i)

\[ \geq \frac{1}{2} \| x_0 - x^* \|^2 - \sum_{i=1}^{r+m} z_i^{n,w} \]

Lemma (4.2a) \[ = \frac{1}{2} \| x_0 - x^* - v^{n,w} \|^2. \]

The conclusion is immediate.

Proof of (iii): We first show that

\[ \liminf_{n \to \infty} \gamma_n \sqrt{n} = 0. \]

(4.14)

Seeking a contradiction, suppose instead that there is an $\epsilon > 0$ and $\bar{n} > 0$ such that if $n > \bar{n}$, then $\gamma_n \sqrt{n} > \epsilon$. By the Cauchy Schwarz inequality, we have

\[ \frac{\epsilon^2}{n} \leq \gamma_n^2 \leq \frac{\epsilon^2}{\bar{n}} \left( \sum_{i=1}^{\bar{n}} \left\| z_i^{n,w} - v_i^{n,w-1} \right\|^2 + \sum_{r+1 \leq r+m} \left\| z_j^{n,w} - z_j^{n,w-1} \right\|^2 \right). \]

This contradicts the earlier claim in (i).

Through (4.14), we find a sequence $\{v^{n,k}\}_{k=1}^\infty$ such that $\lim_{k \to \infty} \gamma_n k \sqrt{n} = 0$, and by part (ii), we can assume $\lim_{k \to \infty} v^{n,k}$ exists, say $v^*$. This completes the proof of (iii).

Proof of (iv):

If $i \in S_{n,p(n,i)}$, then $z_i^{n,w} \equiv z_i^{n,p(n,i)}$, and by Claim 4.2, we have $x_i^{n,p(n,i)} \in \partial h_i(z_i^{n,p(n,i)})$. We also have

\[ \| x_i^{n,p(n,i)} \| \leq \gamma_n. \]

(4.15)

So in this case, $x_i^n$ can be chosen to be $x_i^{n,p(n,i)}$.

Next, if $i \notin S_{n,p(n,i)}$, then $i \in S_{r+1}^{n,p(n,i),j}$ for some $j \in \{r+1, \ldots, r+m\}$. We first consider the case where $i \in \{1, \ldots, r\}$. We claim that $x_i^n$ can be chosen to be

\[ x_i^n := x_0 - \sum_{i' \in S_{r+1}^{n,p(n,i),j}} z_i^{n,p(n,i)} - \sum_{i' \notin S_{r+1}^{n,p(n,i),j}} z_i^{n,q(i)}. \]

(4.16)

We look at how $x_i^n$ is related to $x_i^{n,q(i)}$. Recall that $x_i^{n,q(i)}$ is derived from $z_i^{n,q(i)}$ with $j \in S_{n,q(i)}$, and $\{z_i^{n,q(i)}\}_{i \in S_{n,q(i)}}$ is a minimizer of

\[ \min_{i' \in S_{n,q(i)}} \sum_{i' \in S_{n,q(i)}} h_{i'}(z_{i'}) + \frac{1}{2} \left\| - \sum_{i' \in S_{n,q(i)}} z_i^{n,q(i)} + z_{i'}^{n,q(i)-1} + x_0 \right\|^2. \]

By looking at the variable $z_i^{n,q(i)}$ only in (2.7), we have

\[ z_i^{n,q(i)} = \arg \min_{z_i} h_i^*(z_i) + \frac{1}{2} \left\| - \sum_{i' \notin j} z_i^{n,q(i)} - z_j + x_0 \right\|^2 + \frac{1}{2} \| x_0 \|^2. \]
The formula above gives conditions there to get this. Now, we have \( \nabla h_j^*(\cdot) = \cdot + x_0 \), which gives 0, \( z_{j,q}^{n,q(n,i)} + x_0 \) + \( \sum_{i'=1}^{r+m} z_{j,q}^{n,q(n,i)} - x_0 \), or
\[
 z_{j,q}^{n,q(n,i)} = - \sum_{i'=1}^{r+m} z_{j,q}^{n,q(n,i)}.
\] (4.19)

Next, recall (4.18). The set of variables \( \{z_{i,q}^{n,p(n,i)}\}_{i \in S_{n,p(n,i),j}} \) is a minimizer of the problem (3.1), which can be written through Proposition (2.3) as
\[
\min_{z_{i,q}^{n,p(n,i)}} \sum_{i' \in S'_{n,p(n,i),j}} h_{i'}^*(z_{i'}) + \frac{1}{2} \left\| \sum_{i' \in S'_{n,p(n,i),j}} z_{i'}^{n,p(n,i)} - \sum_{i' \in S'_{n,p(n,i),j}} z_{i'} + x_0 \right\|^2.
\] (4.20)

Now,
\[
\sum_{i' \in S'_{n,p(n,i),j}} z_{i'}^{n,p(n,i)} + \sum_{i' \in S'_{n,p(n,i),j}} z_{i'}^{n,q(n,i)} = \sum_{i' \in S'_{n,p(n,i),j}} z_{i'}^{n,q(n,i)},
\] (4.21)

Since \( z_{i,q}^{n,p(n,i)} \) is a component of a minimizer of (4.20), we can use the optimality conditions there to get
\[
0 \in \partial h_i^*(z_{i,q}^{n,p(n,i)}) + \sum_{i' \in S'_{n,p(n,i),j}} z_{i'}^{n,p(n,i)} - \sum_{i' \in S'_{n,p(n,i),j}} z_{i'}^{n,p(n,i)} - x_0
\] (4.22)

The formula above gives \( x_i^n \in \partial h_i^*(z_{i,q}^{n,p(n,i)}) \). Now
\[
\|x_i^n - (x_0 - v^{n,q(n,i)})\| \leq \left\| \sum_{i' \in S'_{n,p(n,i),j}} (z_{i'}^{n,p(n,i)} - z_{i'}^{n,q(n,i)}) \right\| \leq \sum_{i' \in S'_{n,p(n,i),j}} \|z_{i'}^{n,p(n,i)} - z_{i'}^{n,q(n,i)}\| \leq \gamma_n.
\] (4.23)

Lastly, we consider the case where \( i \in \{r+1, \ldots, r+m\} \) and \( i \in S'_{n,p(n,i),i} \). For this \( i \), recall that \( h_i^*(\cdot) = \frac{1}{2} \| \cdot + x_0 \|^2 - \frac{1}{2} \| x_0 \|^2 \). Hence \( \nabla h_i^*(\cdot) = \cdot + x_0 \), which would
mean that $z_i^{n,p(n,i)} + x_0 \in \partial h_i^*(z_i^{n,p(n,i)})$. Let $x_i^n$ be $x_0 + z_i^{n,p(n,i)}$. Then

$$\|x_i^n - (x_0 - v^n, \tilde{w})\| = \|z_i^{n,p(n,i)} + v^n, \tilde{w}\| = \|z_i^{n,p(n,i)} - z_i^{n,q(n,i)} - v^n, \tilde{w}\| = \|z_i^{n,p(n,i)} - z_i^{n,q(n,i)} + v^n, \tilde{w}\| = \|z_i^{n,p(n,i)} - z_i^{n,p(n,i)} - 1\| + \|v^{n,q(n,i)} + v^n, \tilde{w}\| \leq \gamma_n. \tag{4.23}$$

This ends the proof of part (iv).

**Proof of (v):** Let $x_i^n$ be as chosen in (iv). Since $x_i^n \in \partial h_i^*(z_i^{n,\tilde{w}})$, we have $h_i(x_i^n) + h_i^*(z_i^{n,\tilde{w}}) = (x_i^n, z_i^{n,\tilde{w}})$. From earlier results, we obtain

$$- \sum_{i=1}^{r+m} h_i(x_0 - v^*) \leq \frac{1}{2}\|x_0 - (x_0 - v^*)\|^2 - F(z_i^{n,\tilde{w}}) \leq \frac{1}{2}\|v^*\|^2 + \sum_{i=1}^{r+m} h_i(z_i^{n,\tilde{w}}) - \langle x_0, v^n, \tilde{w} \rangle + \frac{1}{2}\|v^n, \tilde{w}\|^2 \leq \frac{1}{2}\|v^*\|^2 - \frac{1}{2}\|v^n, \tilde{w}\|^2 + \sum_{i=1}^{r+m} (-h_i(x_i^n) + \langle x_i^n, z_i^{n,\tilde{w}} \rangle) - \langle x_0, v^n, \tilde{w} \rangle + \frac{1}{2}\|v^n, \tilde{w}\|^2 \leq \frac{1}{2}\|v^*\|^2 - \frac{1}{2}\|v^n, \tilde{w}\|^2 + \sum_{i=1}^{r+m} \langle x_i^n - (x_0 - v^n, \tilde{w}), z_i^{n,\tilde{w}} \rangle. \tag{4.25}$$

In view of part (iv), we can choose $x_i^n$ to satisfy

$$\langle x_i^n - (x_0 - v^n, \tilde{w}), z_i^{n,\tilde{w}} \rangle \leq \|x_i^n - (x_0 - v^n, \tilde{w})\| \|z_i^{n,\tilde{w}}\| \leq \gamma_n \|z_i^{n,\tilde{w}}\|. \tag{4.26}$$

Recall the assumption that $\|z_i^{n,\tilde{w}}\| \in O(\sqrt{n})$, and from part (iii) that $\lim_{k \to \infty} \gamma_n \sqrt{n} = 0$ and $\lim_{k \to \infty} v^n, \tilde{w} = v^*$. We thus have

$$\lim_{k \to \infty} \gamma_n \|z_i^{n,\tilde{w}}\| = 0 \text{ for all } i \in \{1, \ldots, r + m\}. \tag{4.27}$$

We now look at (4.25). The first two terms in the final line have limit 0 as $k \to \infty$. In view of (4.26) and (4.27), the limit of the last term in the final line equals

$$\lim_{k \to \infty} \sum_{i=1}^{r+m} (-h_i(x_i^n) + \langle x_i^n, z_i^{n,\tilde{w}} \rangle) = \lim_{k \to \infty} \sum_{i=1}^{r+m} h_i(x_i^n) \leq - \sum_{i=1}^{r+m} h_i(x_0 - v^*).$$

Therefore this ends the proof of the result at hand.

**Remark 4.6.** (On $\|z_i^{n,\tilde{w}}\| \in O(\sqrt{n})$) The level sets of the dual problem may be unbounded. (See for example, [Han88, page 9] and [GM89, Section 4].) In such a case the condition $\|z_i^{n,\tilde{w}}\| \in O(\sqrt{n})$ controls the rate of growth of the dual variable $z_i^{n,\tilde{w}}$, so that the proof of convergence carries through. If Algorithm 2.1 were run with Algorithm 3.2, then a sufficient condition for $\|z_i^{n,\tilde{w}}\| \in O(\sqrt{n})$ is that $|S_n, w| = 1$ for all $n \in \mathbb{N}$ and $w \in \{1, \ldots, \tilde{w}\}$, and that if $S'_{n, w, j} \neq \emptyset$, then $|S'_{n, w, j} \setminus \{j\}| = 1$. Depending on the structure of the functions $h_i(\cdot)$, it is still possible to obtain
\[ \|z^{n,\bar{w}}\| \in O(\sqrt{n}) \text{ even if } |S_{n,w}| > 1 \text{ or } |S_{n,w,j}^r\{j\}| > 1. \] We refer to Pan17 for more details. Note that there are options other than Algorithm 4.2 to carry out (3.1). For example, one can use the strategies mentioned in Subsection 2.1. A generalization of Theorem 4.5 for such situations would either need \( \|z^{n,\bar{w}}\| \in O(\sqrt{n}) \) or a new method of proof.

### 4.2. Parallel computations satisfying Assumption 4.2

Consider a simple example where \( r = 2, m = 2, \bar{w} = 4 \), and that for all \( n \geq 1 \), we have

\[ S_{n,1} = \{3\}, \quad S_{n,2} = \{1\}, \quad S_{n,3} = \{2\}, \quad S_{n,4} = \{4\}. \tag{4.28a} \]

If \( S_{n,w,j}^r = \emptyset \) for all \( j \in \{3, 4\} \) and \( w \in \{1, 2, 3, 4\} \), then Assumption 4.2 is satisfied, and the convergence theory of the earlier part of this section holds. If we have

\[ S_{n,3,3}^r = \{1, 3\} \quad \text{and} \quad S_{n,4,3}^r = \{2, 3\}, \tag{4.28b} \]

with all other \( S_{n,j}^r = \emptyset \) instead, then we can check that Assumption 4.2(B) is not satisfied. (Specifically, note that \( p(n, 3) = 4 \). But \( 3 \notin S_{n,3} \) and \( 3 \notin S_{n,3,3}^r \).)

But since calculations involving \( S_{n,3,3}^r \) and \( S_{n,4,3}^r \) do not affect calculations involving \( S_{n,3} \) and \( S_{n,4} \), we can move them to the next iteration counter \( n + 1 \); Specifically, we have \( \bar{w} = 6 \), and for all \( n \geq 1 \),

\[ \tilde{S}_{n,1} = \tilde{S}_{n,2} = \emptyset, \quad \tilde{S}_{n,3} = \{3\}, \quad \tilde{S}_{n,4} = \{1\}, \quad \tilde{S}_{n,5} = \{2\}, \quad \tilde{S}_{n,6} = \{4\} \]

and

\[ \tilde{S}_{n+1,1,3} = \{1, 3\}, \quad \tilde{S}_{n+1,2,3} = \{2, 3\}. \tag{4.29} \]

The calculations for (4.28) and (4.29) are the same, but transferring the calculations from \( S_{n,3,3}^r \) and \( S_{n,4,3}^r \) to \( S_{n+1,1,3}^r \) and \( S_{n+1,2,3}^r \) allows the convergence theory in the earlier part of this section to go through.

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