ON THE ZETA-FUNCTION OF A POLYNOMIAL AT INFINITY

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Abstract. We use the notion of Milnor fibres of the germ of a meromorphic function and the method of partial resolutions for a study of topology of a polynomial map at infinity (mainly for calculation of the zeta-function of a monodromy). It gives effective methods of computation of the zeta-function for a number of cases and a criterium for a value to be atypical at infinity.

§1.- Introduction

The main idea of the paper is to bring together methods of [7] and [8] for computing the zeta-function of the monodromy at infinity of a polynomial. Let $P$ be a complex polynomial in $(n + 1)$ variables. It defines a map from $\mathbb{C}^{n+1}$ to $\mathbb{C}$ which also will be denoted by $P$. It is known ([13]) that there exists a finite set $B(P) \subset \mathbb{C}$ such that the map $P$ is a $C^\infty$ locally trivial fibration over its complement. The monodromy transformation $h$ of this fibration corresponding to the loop $z_0 \cdot \exp(2\pi i \tau)$ $(0 \leq \tau \leq 1)$ with $\|z_0\|$ big enough is called the geometric monodromy at infinity of the polynomial $P$. Let $h_*$ be its action in the homology groups of the fibre (the level set) \( \{P = z_0\} \).

Definition. The zeta-function of the monodromy at infinity of the polynomial $P$ is the rational function

$$
\zeta_P(t) = \prod_{q \geq 0} \{ \det [id - t h_*|_{H_q(\{P = z_0\}; \mathbb{C})}] \}^{(-1)^q}.
$$

Remark 1. We use the definition from [2], which means that the zeta-function defined this way is the inverse of that used in [1].

The degree of the zeta-function (the degree of the numerator minus the degree of the denominator) is equal to the Euler characteristic $\chi_P$ of the (generic) fibre \( \{P = z_0\} \). Formulae for the zeta-functions at infinity for certain polynomials were given in particular in [6], [9].

Key words and phrases. Complex polynomial function, monodromy, zeta-function, bifurcation set.

First author was partially supported by Iberdrola, INTAS–96–0713, RFBR 96–15–96043. Last two authors were partially supported by DGCYT PB94-0291.
§2.- ZETA-FUNCTION OF A POLYNOMIAL VIA
ZETA-FUNCTIONS OF MEROMORPHIC GERMS

A polynomial function $P : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ defines a meromorphic function $P$ on the projective space $\mathbb{C}P^{n+1}$. At each point $x$ of the infinite hyperplane $\mathbb{C}P^{n}_\infty$ the germ of the meromorphic function $P$ has the form $F(u, x_1, \ldots, x_n)$ where $u, x_1, \ldots, x_n$ are local coordinates such that $\mathbb{C}P^{n}_\infty = \{u = 0\}$, $F$ is the germ of a holomorphic function, and $d$ is the degree of the polynomial $P$.

In [8], for a meromorphic germ $f = \frac{F}{G}$, there were defined two Milnor fibres (the zero and the infinite ones), two monodromy transformations, and thus two zeta-functions $\zeta_0^f(t)$ and $\zeta_\infty^f(t)$. Let $\zeta^P_{x}(\cdot \in [0, \infty))$ be the corresponding zeta-function of the germ of the meromorphic function $P$ at the point $x \in \mathbb{C}P^{n}_\infty$.

For the aim of convenience, in [8] we considered only meromorphic germs $f = \frac{F}{G}$ with $F(0) = G(0) = 0$. At a generic point of the infinite hyperplane $\mathbb{C}P^{n}_\infty$ the meromorphic function $P$ has the form $\frac{1}{u^d}$. For a germ of the form $f = \frac{1}{u}$ with $G(0) = 0$, it is reasonable to give the following definition: its infinite Milnor fibre coincides with the (usual) Milnor fibre of the holomorphic germ $G$ and its zero Milnor fibre is empty. Thus $\zeta_0^f(t) = 1$ and $\zeta_\infty^f(t) = \zeta_G(t)$. According to this definition, for the germ $\frac{1}{u^d}$, its infinite zeta-function is equal to $(1 - t^d)$.

Let $\mathcal{S} = \{\Xi\}$ be a prestratification of the infinite hyperplane $\mathbb{C}P^{n}_\infty$ (that is a partitioning of $\mathbb{C}P^{n}_\infty$ into semi-analytic subspaces without any regularity conditions) such that, for each stratum $\Xi$ of $\mathcal{S}$, the infinite zeta-function $\zeta_\infty^P_{x}(t)$ does not depend on $x$, for $x \in \Xi$. Let us denote this zeta-function by $\zeta_\Xi^P(t)$ and by $\chi_\Xi^\infty$ its degree $\deg \zeta_\Xi^P(t)$. A straightforward repetition of the arguments from the proof of Theorem 1 in [7] gives

**Theorem 1.**

$$\zeta_P(t) = \prod_{\Xi \in \mathcal{S}} [\zeta_\Xi^P(t)]^{\chi(\Xi)},$$

$$\chi_P = \sum_{\Xi \in \mathcal{S}} \chi_\Xi^\infty \cdot \chi(\Xi).$$

**Remark 2.** One can write the formula for $\chi_P$ in the form of an integral with respect to the Euler characteristic

$$\chi_P = \int_{\mathbb{C}P^{n}_\infty} \chi_{P,x} \, d\chi$$

in the sense of Viro ([14]).

**Remark 3.** Let $P_d$ be the (highest) homogeneous part of degree $d$ of the polynomial $P$. Then at each point $x \in \mathbb{C}P^{n}_\infty \setminus \{P_d = 0\}$ the germ of the meromorphic function $P$ is of the form $\frac{1}{u^d}$. The set $\Xi^n = \mathbb{C}P^{n}_\infty \setminus \{P_d = 0\}$ can be considered as the $n$-dimensional stratum of a partitioning. It brings the factor $(1 - t^d)\chi(\Xi^n)$ into the zeta-function $\zeta_P(t)$.

§3.- EXAMPLES

3.1. Yomdin-at-infinity polynomials. This name was introduced in [4]. For a polynomial $P \in \mathbb{C}[z_0, z_1, \ldots, z_n]$ let $P_d$ be its homogeneous part of degree $d$. Let a
polynomial $P$ be of the form $P = P_d + P_{d-k} + \text{terms of lower degree, } k \geq 1$. Let us consider hypersurfaces in $\mathbb{C}P^n$ defined by $\{P_d = 0\}$ and $\{P_{d-k} = 0\}$. Let $\text{Sing}(P_d)$ be the singular locus of the hypersurface $\{P_d = 0\}$ (including all points where $\{P_d = 0\}$ is not reduced). One says that $P$ is a Yomdine-at-infinity polynomial if $\text{Sing}(P_d) \cap \{P_{d-k} = 0\} = \emptyset$ (in particular it implies that $\text{Sing}(P_d)$ is finite).

Y. Yomdin ([15]) has considered critical points of holomorphic functions which are local versions of such polynomials. He gave a formula for their Milnor numbers. The generic fibre (level set) of a Yomdin-at-infinity polynomial is homotopy equivalent to the bouquet of $n$-dimensional spheres ([5]). Its Euler characteristic $\chi_P$ (or rather the (global) Milnor number) has been determined in [4]. For $k = 1$, the zeta-function of such a polynomial has been obtained in [6].

Let $P(z_0, z_1, \ldots, z_n) = P_d + P_{d-k} + \ldots$ be a Yomdin-at-infinity polynomial. Let $\text{Sing}(P_d)$ consist of $s$ points $Q_1, \ldots, Q_s$. One has the following natural stratification of the infinite hyperplane $\mathbb{CP}^n_{\infty}$:

1. the $n$-dimensional stratum $\Xi^n = \mathbb{C}P^n_{\infty} \setminus \{P_d = 0\}$;
2. the $(n-1)$-dimensional stratum $\Xi^{n-1} = \{P_d = 0\} \setminus \{Q_1, \ldots, Q_s\}$;
3. the 0-dimensional strata $\Xi_i^n (i = 1, \ldots, s)$, each consisting of one point $Q_i$.

The Euler characteristic of the stratum $\Xi^n$ is equal to

$$\chi(\mathbb{C}P^n_{\infty}) - \chi(\{P_d = 0\}) = (n+1) - \chi(n, d) + (-1)^{n-1} \sum_{i=1}^s \mu_i,$$

where $\chi(n, d) = (n+1) + \frac{(1-d)^{n+1}-1}{d}$ is the Euler characteristic of a non-singular hypersurface of degree $d$ in the complex projective space $\mathbb{C}P^n_{\infty}$, $\mu_i$ is the Milnor number of the germ of the hypersurface $\{P_d = 0\} \subset \mathbb{C}P^n_{\infty}$ at the point $Q_i$. At each point of the stratum $\Xi^n$, the germ of the meromorphic function $P$ has (in some local coordinates $u, y_1, \ldots, y_n$) the form $\frac{1}{w^d} (\mathbb{C}P^n_{\infty} = \{u = 0\})$ and its infinite zeta-function $\zeta^\infty_{\Xi^n}(t)$ is equal to $(1 - t^d)$.

At each point of the stratum $\Xi^{n-1}$, the germ of the polynomial $P$ has (in some local coordinates $u, y_1, \ldots, y_n$) the form $\frac{1}{w^d}$. Its infinite zeta-function $\zeta^\infty_{\Xi^{n-1}}(t)$ is equal to 1 and thus it does not contribute a factor to the zeta-function of the polynomial $P$.

At a point $Q_i$ ($i = 1, \ldots, s$), the germ of the meromorphic function $P$ has the form $\varphi(u, y_1, \ldots, y_n) = \frac{g_i(y_1, \ldots, y_n) + u^k}{u^d}$, where $g_i$ is a local equation of the hypersurface $\{P_d = 0\} \subset \mathbb{C}P^n_{\infty}$ at the point $Q_i$. Thus $\mu_i$ is its Milnor number.

To compute the infinite zeta-function $\zeta^\infty_{\varphi}(t)$ of the meromorphic germ $\varphi$, let us consider a resolution $\pi : (\mathcal{X}, \mathcal{D}) \to (\mathbb{C}^n, 0)$ of the singularity $g_i$, i.e., a proper modification of $(\mathbb{C}^n, 0)$ which is an isomorphism outside the origin in $\mathbb{C}^n$ and such that, at each point of the exceptional divisor $\mathcal{D}$, the lifting $g_i \circ \pi$ of the function $g_i$ to the space $\mathcal{X}$ of the modification has (in some local coordinates) the form $y_1^{m_1} \ldots y_n^{m_n}$ ($m_i \geq 0$).

Let us consider the modification $\tilde{\pi} = id \times \pi : (\mathbb{C}_u \times \mathcal{X}, 0 \times \mathcal{D}) \to (\mathbb{C}^{n+1}, 0) = (\mathbb{C}_u \times \mathbb{C}^n, 0)$ of the space $(\mathbb{C}^{n+1}, 0)$ – the trivial extension: $(u, x) \mapsto (u, \pi(x))$. Let $\tilde{\varphi} = \varphi \circ \tilde{\pi}$ be the lifting of the meromorphic function $\varphi$ to the space $\mathbb{C}_u \times \mathcal{X}$ of the modification $\tilde{\pi}$. Let $\mathcal{M}^\infty_{\varphi} = \tilde{\pi}^{-1}(\mathcal{M}^\infty_{\varphi})$ $(\mathcal{M}^\infty_{\varphi}$ is the infinite Milnor fibre of the germ $\varphi$) be the local level set of the meromorphic function $\tilde{\varphi}$ (close to the infinite point). In the natural way one has the (infinite) monodromy $h^\infty_{\varphi}$ acting on $\mathcal{M}^\infty_{\varphi}$ and its zeta function $\zeta^\infty_{\tilde{\varphi}}(t)$. 
Theorem 2.
\[ \zeta_{\varphi}^\infty(t) = (1 - t^{d-k})^{\chi(D)} \zeta_{\varphi}^\infty(t). \]

Proof. The infinite monodromy transformation of the function \( \tilde{\varphi} \) can be described in the following way. Let \( h^\infty_{\varphi}: \mathcal{M}^\infty_{\varphi} \to \mathcal{M}^\infty_{\varphi} \) be the infinite monodromy transformation of the germ \( \varphi \). One can suppose that it preserves the intersection of the Milnor fibre \( \mathcal{M}^\infty_{\varphi} \) with the line \( \mathbb{C}_u \times \{0\} \). There it coincides with the infinite monodromy transformation of the restriction \( \varphi|_{\mathbb{C}_u \times \{0\}} = \frac{u^k}{u^d} \) of the germ \( \varphi \) to this line, i.e., with a cyclic permutation of \( (d-k) \) points. The zeta-function of a cyclic permutation of \( (d-k) \) points is equal to \( (1 - t^{d-k}) \). The projection \( \tilde{\pi} : \mathcal{M}^\infty_{\varphi} \to \mathcal{M}^\infty_{\varphi} \) is an isomorphism outside \( \mathcal{M}^\infty_{\varphi} \cap (\mathbb{C}_u \times \{0\}) \), the preimage of each point from \( \mathcal{M}^\infty_{\varphi} \cap (\mathbb{C}_u \times \{0\}) \) is isomorphic to the exceptional divisor \( D \). This means that the transformation (the diffeomorphism) \( h^\infty_{\varphi}: \mathcal{M}^\infty_{\varphi} \to \mathcal{M}^\infty_{\varphi} \) can be constructed in such a way that it preserves \( \tilde{\pi}^{-1}(\mathcal{M}^\infty_{\varphi} \cap (\mathbb{C}_u \times \{0\})) \) and acts on it by a cyclic permutation of \( (d-k) \) copies of \( D \). The zeta-function of this transformation of \( \{(d-k) \text{ points}\} \times D \) is equal to \( (1 - t^{d-k})^{\chi(D)} \). The result follows from the multiplication property of the zeta-function of a transformation (see [2] p. 94).

For \( \tilde{m} = (m_1, m_2, \ldots, m_n) \) with integer \( m_1 \geq m_2 \geq \ldots \geq m_n \geq 0 \), let \( S_{\tilde{m}} \) be the set of points of the exceptional divisor \( D \) of the resolution \( \pi \) at which the lifting of the germ \( g_i \) has the form \( y_1^{m_1} \cdot \ldots \cdot y_n^{m_n} + u^k \). Thus, for fixed \( \tilde{m} \), the infinite zeta-function \( \zeta_{\varphi,x}^\infty(t) \) of the germ of the meromorphic function \( \tilde{\varphi} \) at a point \( x \) from \( \{0\} \times S_{\tilde{m}} \) is one and the same. It can be determined by the Varchenko type formula from [8]. If there are more than one integers \( m_i \) different from zero, \( \zeta_{\varphi,x}^\infty(t) = (1 - t^{d-k})^{-1} \). For \( x \in \{0\} \times S_m \),
\[ \zeta_{\varphi,x}^\infty(t) = (1 - t^{d-k})(1 - t^{\frac{m(d-k)}{g.c.d.(m,k)}})^{-g.c.d.(m,k)}. \]

According to Theorem 1
\[ \zeta_{\varphi}^\infty(t) = (1 - t^{d-k})^{\chi(D)} \prod_{m \geq 1} \left( 1 - t^{\frac{m(d-k)}{g.c.d.(m,k)}} \right)^{-g.c.d.(m,k)} \chi(S_m) \]
and by Theorem 2
\[ \zeta_{\varphi}^\infty(t) = (1 - t^{d-k}) \prod_{m \geq 1} \left( 1 - t^{\frac{m(d-k)}{g.c.d.(m,k)}} \right)^{-g.c.d.(m,k)} \chi(S_m). \]

The zeta-function \( \zeta_h(t) \) of a transformation \( h : X \to X \) of a space \( X \) into itself determines the zeta-function \( \zeta^k_h(t) \) of the \( k \)-th power \( h^k \) of the transformation \( h \). In particular, if \( \zeta_h(t) = \prod_{m \geq 1} (1 - t^m)^{a_m} \), then
\[ \zeta^k_h(t) = \prod_{m \geq 1} \left( 1 - t^{\frac{m}{g.c.d.(k,m)}} \right)^{g.c.d.(k,m) a_m}. \]
The formulae (1) and (2) mean that
\[ \zeta_\varphi^\infty(t) = (1 - t^{d-k}) \left( \zeta_k(t^{d-k}) \right)^{-1} \] (3).

Combining the computations for the stratification \( \{ \Xi^n, \Xi^{n-1}, \Xi^0 \} \) of the infinite hyperplane \( \mathbb{CP}^n_\infty \), one has

**Theorem 3.** For a Yomdin-at-infinity polynomial \( P = P_d + P_{d-k} + \ldots \), its zeta-function at infinity is equal to

\[ \zeta_P(t) = (1 - t^{d-k}) \chi(\Xi^n) (1 - t^{d-k})^s \left( \prod_{i=1}^s \zeta_{g_i}(t^{d-k}) \right)^{-1}, \]

where \( \chi(\Xi^n) = \frac{1-(1-d)^{n+1}}{d} + (-1)^{n-1} \sum_{i=1}^s \mu(g_i) \) and \( g_i \) is a local equation of the hypersurface \( \{ P_d = 0 \} \subset \mathbb{CP}^n_\infty \) at its singular point \( Q_i \).

**3.2.** Let \( n + 1 \) be equal to 3, \( P = P_d + P_{d-k} + \ldots \), \( P_d = 0 \) is a curve in \( \mathbb{CP}^2_\infty \). Let \( C_{q_1}^1 + \ldots + C_{q_r}^r \) be its decomposition into irreducible components. Let \( \{ P_d = 0 \}_{\text{red}} \) be the reduced curve \( C_1 + \ldots + C_r \) and let \( \text{Sing}(\{ P_d = 0 \}_{\text{red}}) \) consist of \( s \) points \( \{ Q_1, \ldots, Q_s \} \). Suppose that:

1. the curve \( \{ P_{d-k} = 0 \} \) is reduced;
2. \( Q_i \notin \{ P_{d-k} = 0 \} \), \( i = 1, \ldots, s \);
3. for each \( j \) with \( q_j > 1 \), the curves \( C_j \) and \( \{ P_{d-k} = 0 \} \) intersect transversally, i.e., the set \( C_j \cap \{ P_{d-k} = 0 \} \) consists of \( d_j(d-k) \) different points \( d_j = \deg C_j \).

The generic fibre of the polynomial \( P \) is homotopy equivalent to the bouquet of 2-dimensional spheres. In this case the number of these spheres is equal to \( \mu(P) = \dim \mathbb{C}[x, y, z]/\text{Jac}(P) \) and is equal to

\[ (d-1)^3 - k \cdot \left( \chi(\{ P_d = 0 \}) + d(2d-d-3) \right) + k^2 \cdot (d-d), \]

where \( \tilde{d} = d_1 + \ldots + d_r \) is the degree of the (reduced) curve \( \{ P_d = 0 \}_{\text{red}} \), [4]. Let us consider the following partitioning of the infinite hyperplane \( \mathbb{CP}^2_\infty \):

1. the 0-dimensional stratum \( \Xi^0 \) consisting of one point \( Q_i \) each \( i = 1, \ldots, s \);
2. the 0-dimensional stratum \( \Lambda^0_j = C_j \cap \{ P_{d-k} = 0 \} \), for each \( j = 1, \ldots, r \);
3. the 1-dimensional stratum \( \Xi_j^1 = C_j \setminus \{ Q_i \} \cup \Lambda^0_j \), for each \( j = 1, \ldots, r \);
4. the 2-dimensional stratum \( \Xi^2 = \mathbb{CP}^2_\infty \setminus \{ P_d = 0 \} \).

At each point of the stratum \( \Xi^2 \), the germ of the meromorphic function \( P \) has the form \( \frac{1}{u^r} (\mathbb{CP}^2_\infty = \{ u = 0 \}) \). Its infinite zeta-function is equal to \( (1 - t^d) \). The Euler characteristic \( \chi(\Xi^2) \) of the stratum \( \Xi^2 \) is equal to

\[ \chi(\mathbb{CP}^2_\infty) - \chi(\{ P_d = 0 \}) = 3 - 3\tilde{d} + \tilde{d}^2 - \sum_{i=1}^s \mu_i, \]

where \( \mu_i \) is the Milnor number of the (reduced) curve \( \{ P_d = 0 \}_{\text{red}} \) at the point \( Q_i \).
At each point of the stratum $\Xi^1_\lambda$, the germ of the meromorphic function $P$ has the form $g_{ij}u^k$. Its infinite zeta-function can be determined by the Varchenko type formula from [8] and is equal to
\[
(1 - t^{d-k})(1 - t^{g.c.d.(q_j,k)} - g.c.d.(q_j,k)).
\]
The Euler characteristic of the stratum $\Xi^1_\lambda$ is equal to
\[
\chi(C_j) - d_j(d - k) - \#\{C_j \cap \{Q_i : i = 1, \ldots, s\}\}.
\]

At each point of the stratum $\Lambda^0$, the germ of the meromorphic function $P$ has the form $\frac{g_{ij}u^k + y_2}{u^d}$. Its infinite zeta-function is equal to 1.

At a point $Q_i$, the germ of the meromorphic function $P$ has the form $\frac{g_{ij}(y_1,y_2) + u^k}{u^d}$, where $\{g_i = 0\}$ is the local equation of the (non-reduced) curve $\{P_d = 0\}$ at the point $Q_i$. Its infinite zeta-function is equal to
\[
(1 - t^{d-k})\left(\epsilon_q^k(t^{d-k})\right)^{-1}.
\]

**Remark 4.** We can not apply the formula (3) directly since the singularity of the germ $g_i$ is, in general, not isolated. However, it is not difficult to see that, actually, the proof of this formula uses only the fact that the singularity of the germ $g_i$ can be resolved by a modification which is an isomorphism outside the origin. This is so for a curve singularity.

Thus one obtains
\[
\zeta_P(t) = (1 - t^d)\chi(\Xi^2)(1 - t^{d-k})(3d - d^2 - d(d-k) + \sum \mu_i) \times
\prod_{j=1}^r \left(1 - t^{g_{ij}(d-k)}\right)^{-g.c.d.(q_j,k)\chi(\Xi^1_\lambda)} \cdot \prod_{i=1}^s \left(\epsilon_q^k(t^{d-k})\right)^{-1}.
\]

§4.- **ON THE BIFURCATION SET OF A POLYNOMIAL MAP**

As we have mentioned, a polynomial map $P : \mathbb{C}^{n+1} \to \mathbb{C}$ defines a locally trivial fibration over the complement to a finite set in $\mathbb{C}$. The minimal set $B(P)$ with this property is called the bifurcation set of $P$. The bifurcation set consists of critical values of the polynomial $P$ (in the affine part) and of atypical ("critical") values at infinity.

In order to consider a level set $\{P = c\}$, one can substitute the polynomial $P$ by the polynomial $(P - c)$ and consider the zero level set. Thus let us consider the zero level set $V_0 = \{P = 0\} \subset \mathbb{C}^{n+1}$ of the polynomial $P$. Let us suppose that the level set $V_0$ of the polynomial $P$ has only isolated singular points (in the affine part $\mathbb{C}^{n+1}$). For $\rho > 0$, let $B_\rho$ be the open ball of radius $\rho$ centred at the origin in $\mathbb{C}^{n+1}$ and $S_\rho = \partial B_\rho$ be the $(2n + 1)$-dimensional sphere of radius $\rho$ with the centre at the origin. There exists $R > 0$ such that, for all $\rho \geq R$, the sphere $S_\rho$ is transversal to the level set $V_0 = \{P = 0\}$ of the polynomial map $P$. The restriction $P|_{\mathbb{C}^{n+1} \setminus B_R} : \mathbb{C}^{n+1} \setminus B_R \to \mathbb{C}$ of the function $P$ to the complement of the ball $B_R$ defines a $C^\infty$ locally trivial fibration over a punctured neighbourhood of the origin in $\mathbb{C}$. The loop $\varepsilon_0 \cdot \exp(2\pi i \tau)$ (0 $\leq$ $\tau$ $\leq$ 1, $\|\varepsilon_0\|$ small enough) defines the monodromy transformation $h : V_{\varepsilon_0} \setminus B_R \to V_{\varepsilon_0} \setminus B_R$. Let us denote its zeta-function $\zeta_h(t)$ by $\zeta^0(t)$. We use the following definition.
Definition. The value 0 is atypical at infinity for the polynomial $P$ if the restriction $P|_{C^{n+1}\setminus B_R}$ of the function $P$ to the complement of the ball $B_R$ is not a $C^\infty$ locally trivial fibration over a neighbourhood of the origin in $\mathbb{C}$.

Remark 5. This definition does not depend on a choice of coordinates, i.e., it is invariant with respect to polynomial diffeomorphisms of the space $\mathbb{C}^{n+1}$. One can see that an atypical at infinity value is atypical, i.e. it belongs to the bifurcation set $B(P)$ of the polynomial $P$. Moreover the bifurcation set $B(P)$ is the union of the set of critical values of the polynomial $P$ (in $\mathbb{C}^{n+1}$) and of the set of values atypical at infinity in the described sense. If the singular locus of the level set $V_0 = \{P = 0\}$ is not finite, the value 0 hardly can be considered as typical at infinity. Thus, one should consider this definition as a (possible) general definition of a value atypical at infinity. In fact the same definition was used in [10].

Let $S$ be a prestratification of the infinite hyperplane $\mathbb{CP}_\infty^n$ such that, for each stratum $\Xi$ of $S$, the zero zeta-function $\zeta^0_{P,x}(t)$ of the germ of the meromorphic function $P$ at a point $x \in \mathbb{CP}_\infty^n$ does not depend on the point $x$, for $x \in \Xi$ (let it be $\zeta^0_\Xi(t)$ and let its degree be $\chi_{\Xi}^0$).

Theorem 4.

$$\zeta^0_P(t) = \prod_{\Xi \in S} [\zeta^0_{\Xi}(t)]^{\chi(\Xi)},$$

$$\chi(V_{\varepsilon} \setminus B_R) = \sum_{\Xi \in S} \chi^0_{\Xi} \cdot \chi(\Xi).$$

The proof is essentially the same as that of Theorem 1. Since the Euler characteristic of the set $V_\varepsilon \setminus B_R$ is equal to 0, one has

Corollary 1. If $\zeta^0_P(t) \neq 1$, then the value 0 is atypical at infinity for the polynomial $P$.

In several papers (see, e.g., [3], [11], [12]) there was considered an integer $\lambda_P(c)$ ($c \in \mathbb{C}$) such that

$$\chi(\{P = c\}) = \chi(\{P = c + \varepsilon\}) + (-1)^{n+1} \left(\sum \mu_i + \lambda_P(c)\right),$$

where $\mu_i$ are the Milnor numbers of the (isolated) singular points of the level set $\{P = c\} \subset \mathbb{C}^{n+1}$. Theorem 4 gives the following formula for this invariant:

Corollary 2.

$$\lambda_P(0) = (-1)^n \deg \zeta^0_P(t) = (-1)^n \sum_{\Xi \in S} \chi^0_{\Xi} \cdot \chi(\Xi) \left(-1^n \int_{\mathbb{CP}_\infty^n} \chi^0_{P,x} \, d\chi\right).$$

Example. Let $P(x,y,z) = x^a y^b (x^c y^d - z^{c+d}) + z$, $(ad - bc) \neq 0$, and let $D = \deg(P) = a + b + c + d$. The curve $\{P_D = 0\} \subset \mathbb{CP}^2_\infty$ consists on three components: the line $C_1 = \{x = 0\}$ with multiplicity $a$, the line $C_2 = \{y = 0\}$ with multiplicity $b$, and the reduced curve $C_3 = \{x^c y^d - z^{c+d} = 0\}$. Let $Q_1 = C_1 \cap C_3 = (1 : 0 : 0)$, $Q_2 = C_1 \cap C_3 = (0 : 1 : 0)$, $Q_3 = C_1 \cap C_2 = (0 : 0 : 1)$. At each point $x$ of the infinite hyperplane $\mathbb{CP}^2_\infty$ except $Q_1$ and $Q_2$, one has $\zeta^0_{P,x}(t) = 1$. At the point $Q_1$, the germ of the meromorphic function $P$ has the form $y^b (y^d - z^{c+d}) + zu^{D-1}$. 


Its zero zeta-function can be obtained by the Varchenko type formula from [8]. If \((ad - bc) < 0\), then \(\zeta_{P,Q}^0(t) = 1\). If \((ad - bc) > 0\), then
\[
\zeta_{P,Q}^0(t) = (1 - t^{\frac{ad-bc}{G.C.D.}})^{G.C.D.},
\]
where \(G.C.D. = g.c.d(c, d) \cdot g.c.d(\frac{ad-bc}{g.c.d(c,d)}, D - 1)\). At the point \(Q_2\), we have just the symmetric situation. Finally
\[
\zeta_p^0(t) = (1 - t^{\frac{|ad-bc|}{G.C.D.}})^{G.C.D.}.
\]
It means that the value 0 is atypical at infinity. In the same way \(\zeta_{P-c}^0(t) = 1\), for \(c \neq 0\).

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