Holographic hydrodynamics: models and methods

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Abstract

We review recent developments in holographic hydrodynamics. We start from very basic discussion on hydrodynamic systems and motivate why string theory is an essential tool to deal with these systems when they are strongly coupled. The main purpose of this review article is to understand different holographic techniques to compute transport coefficients (first order and higher order) and their corrections in presence of higher derivative terms in the bulk Lagrangian. We also mention some open challenges in this subject.

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1 Introduction and motivation

In this article we shall study the characteristics of hydrodynamic system from string theory perspective.

Hydrodynamics is an effective theory, describing the dynamics of some field theory at large distances and time-scales. A fluid system is considered to be a continuous medium. When we talk about a small volume element (or “fluid particle”) in fluid it has to be remembered that the small volume element consists of a large number of molecules (specifically the size of the fluid particle is much much greater than the mean free path of the system). The equations of hydrodynamics assume that the fluid is in local thermodynamic equilibrium at each point in space and time, even though different thermodynamic quantities like fluid velocity ($\vec{v}(\vec{x},t)$), energy ($e(\vec{x},t)$), pressure ($p(\vec{x},t)$), fluid density ($\rho(\vec{x},t)$) etc may vary. Fluid mechanics applies only when the length scales of variation of thermodynamic variables are large compared to equilibration length scale of the fluid, namely mean free path $\lambda$.

Hydrodynamic description does not follow from the action principle rather it is normally formulated in the language of equations of motion. The reason for this is the presence of dissipation in thermal media. Due to internal friction called viscosity, a dissipative fluid loses its energy over time as it propagates. The fluid without any viscous drag is called ideal fluid. In the simplest case, the hydrodynamic equations are just the laws of conservation of energy
and momentum (we are considering the fluid without any global charge or current),
\[ \nabla_\mu T^{\mu \nu} = 0. \]  
(1)

At any space time point the fluid is characterized by \( d + 1 \) variables (in \( d \) dimensions): velocities \( u^\mu(x) \) (of fluid particles) and its temperature \( T(x) \). All these variables are functions of space and time. Since velocities are time like they follow \( u_\mu u^\mu = -1 \), therefore total number of variables describing the fluid is \( d \) which is equal to the number of equations of motion as in equation (1).

In hydrodynamics we express \( T_{\mu \nu} \) in terms of \( T(x) \) and \( u^\mu(x) \). Following the standard procedure of effective field theories, we expand energy momentum tensor in powers of spatial derivatives. As we have already explained local thermodynamic quantities (velocity and temperature) vary very slowly over space-time therefore their derivatives are very small. Thus it is legitimate to express the energy-momentum tensor in powers of derivatives of local quantities i.e., \( \partial s \ll (\partial s)^2, \partial^2 s \ll (\partial s)^3, \partial^3 s, \ldots \ll \ldots \) where, \( s \) stands for any local quantity.

At the zeroth order, \( T_{\mu \nu} \) is given by the familiar formula for ideal fluids,
\[ T_{\mu \nu} = (e(x) + p(x)) u_\mu u_\nu(x) + p(x) g_{\mu \nu}(x) \]  
(2)

where \( e(x) \) is energy density and \( p(x) \) is pressure\(^2\) \( g_{\mu \nu} \) is the metric tensor of background spacetime. If we also consider the fluid system has conformal invariance (a system which is scale invariant) then its stress tensor becomes traceless i.e. \( T^\mu_\mu = 0 \). This implies that \( p = \frac{1}{d-1} e \). This is the thermodynamic equation of state.

At the next order in derivative expansion fluid energy-momentum tensor is given by,
\[ T_{\mu \nu} = \left[ (e + p) u_\mu u_\nu + p g_{\mu \nu} - 2 \sigma_{\mu \nu} \right] + \frac{\eta}{2} P^\alpha_\mu P^\beta_\nu \left[ \nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{2}{d-1} g_{\alpha \beta} \nabla \cdot u \right] + \zeta g_{\mu \nu} \nabla \cdot u, \]  
\[ P^\mu_\nu = g^\mu_\nu + u^\mu u^\nu \]  
(3)

where \( \sigma^{\mu \nu} \) is proportional to derivatives of \( u^\mu(x) \) and is termed as dissipative part of \( T^{\mu \nu}(x) \). \( \sigma_{\mu \nu} \) is a symmetric tensor. It has been divided into a traceless part and a trace part. The coefficient of traceless part is denoted by \( \eta \) and that of trace part is denoted by \( \zeta \). The coefficient \( \eta \) is called shear viscosity coefficient and \( \zeta \) is called bulk viscosity coefficient. Shear viscosity coefficient describes fluid’s reaction against applied shear stress where as bulk viscosity coefficient measures reaction against volume stress. \( \eta \) and \( \zeta \) are called the first order transport coefficients as they appear at the first order in derivative expansion of energy-momentum tensor. For conformal fluid bulk viscosity coefficient vanishes. Since trace of stress tensor is proportional to \( \zeta \) (from equation (3)) i.e.,
\[ T^\mu_\mu \sim \zeta \nabla \cdot u. \]

In a similar fashion one can write the expression for stress tensor to the second order in derivative expansion. We shall come to the second order hydrodynamics later in section\(^7\).

Before we go ahead let us first motivate why we use string theory to study different properties of fluid dynamics.

\(^1\)Other thermodynamic variables can be expressed as a function of temperature.

\(^2\)Next time onwards we shall drop the functional dependence of local variables i.e. we shall write \( u_\nu(x) \) as \( u_\mu \) and similarly for other variables.
1.1 Why string theory?

The hydrodynamic behavior of a system is characterized by a set of transport coefficients, like shear viscosity, bulk viscosity etc as mentioned above. After Relativistic Heavy Ion Collider (RHIC) experiments[2], the study of shear viscosity to entropy density ratio of gauge theory plasma has developed lots of attention. The QGP (Quark-Gluon Plasma) produced at RHIC behaves like viscous fluid with very small shear viscosity coefficient (near-perfect fluid). Such a low ratio of shear viscosity to entropy density is very hard to describe with conventional methods. The temperature of the gas of quarks and gluons produced at RHIC is approximately 170 $MeV$ which is very close to the confinement temperature of QCD. Therefore, at this high temperature they are not in the weakly coupled regime of QCD. In fact near the transition temperature the gas of quarks and gluons belongs to the non-perturbative realm of QCD. Thus usual perturbative gauge theory computations are not applicable to explain RHIC results. On the other hand in Lattice gauge theory (a technique to study the properties of strongly coupled system) it is difficult to compute real time correlators of energy momentum tensor as the theory is formulated on Euclidean lattice[3]. The AdS/CFT correspondence, at this point, appears to be a technically powerful tool to deal with strongly coupled (conformal) field theory in terms of weakly coupled (super)-gravity theory in AdS space. Holographic techniques (motivated from the AdS/CFT correspondence) to compute hydrodynamic transport coefficients exhibit a remarkable quantitative agreement with those arising from numerical fits to RHIC data.

This motivates us to study properties of hydrodynamic systems from the point of view of string theory, in particular, the AdS/CFT correspondence.

Now, we pause our discussion on fluid system a little and discuss the basic idea and working principle of the AdS/CFT correspondence.

2 The AdS/CFT conjecture and fluid/gravity correspondence

In this section we briefly review the AdS/CFT conjecture[3]. The correspondence is itself a vast subject and there are lots of good review articles on this[4]. Therefore, here we shall briefly discuss the important points of this conjecture to introduce ourselves to the notations which will be used throughout this article.

The original conjecture states the equivalence between two seemingly unrelated theories: Type IIB string theory on $AdS_5 \times S^5$, where both $AdS_5$ and $S^5$ has radius $b$, with a five form field strength $F_5$, which has integer flux $N$ over $S^5$, and complex string coupling $\tau_S = a + ie^{-\phi}$ where $a$ is axion and $\phi$ is dilaton field and $\mathcal{N} = 4$ SYM theory in 4 dimension, with gauge group $SU(N)$, Yang-Mills coupling $g_{YM}$ and instanton angle $\theta_I$ (together define a complex coupling $\tau_{YM} = a + 4\pi i g_{YM}^2$) in its superconformal phase, with $g_S = \frac{g_{YM}^2}{4\pi}$, $a = \frac{\theta_I}{2\pi}$ and $b = 4\pi g_S N(\alpha')^2$.

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3 Please look at [4] where the author have found a lattice technique to compute the ratio. But their result is far from universality($\eta/s = 1/4\pi$).

4 See [5] for example

4
The best understood examples of the AdS/CFT correspondence relate the strongly coupled dynamics of certain conformal field theories (CFT) to the dynamics of gravitational systems in AdS spaces. The description is holographic, as the CFT is living on the boundary of the bulk AdS space and thus has one lower dimension than the full bulk spacetime. All the characteristics or information of a lower dimensional theory is captured in a theory living in one higher dimension and vice versa. The description is also called a duality because when the field theory is strongly coupled the gravity theory is weakly coupled and vice versa.

These boundary theories are quantum field theories of a particular kind. They pertain to strongly coupled systems which are difficult if not impossible to study perturbatively. The AdS/CFT correspondence seems to be a powerful tool to deal with these strongly coupled (conformal) field theory. Since our main target is to understand the properties of strongly coupled plasma produced at RHIC, lets us first understand how such field theory can be dealt with via weak gravity computations.

2.1 The Large ‘t Hooft coupling ($\lambda$) Limit

The large ‘t Hooft coupling limit corresponds to taking $\lambda = g_{YM}^2 N = g_S N \to \infty$, while $N \to \infty$ also. This is the weakest form of the conjecture. In this large $N$ limit, the super Yang-Mills theory becomes nonperturbative i.e. a strongly coupled super Yang-Mills theory, but the string theory side becomes very much tractable. This can be realized as follows: from the relation $b^4 = 4\pi g_S N(\alpha')^2$, one can easily check that large $\lambda$ limit corresponds to the radius of curvature $b$ being much much larger than the string length, $b^2 \gg \alpha'$. Now, the classical string theory effective action is given by,

$$\mathcal{L} \sim a_1 \alpha' \mathcal{R} + a_2 \alpha'^2 \mathcal{R}^{(2)} + a_3 \alpha'^3 \mathcal{R}^{(3)} + \cdots$$

(4)

where, $\mathcal{R}^{(n)}$ is a $2n$ derivative term constructed out of Ricci tensor or Riemann tensor or Ricci scalar. Since the near horizon geometry of $D3$ brane spacetime is asymptotically AdS, with both the $AdS_5$ and $S_5$ has radius of curvature $b$, the scale of Riemann tensor is set by,

$$\mathcal{R} \sim \frac{1}{b^2} \sim \frac{1}{\sqrt{g_S N \alpha'}} \sim \frac{1}{\sqrt{\lambda \alpha'}}.$$  

(5)

Therefore the expansion of effective Lagrangian in powers of $\alpha'$ effectively becomes an expansion in powers of $(\lambda)^{-\frac{1}{2}},$

$$\mathcal{L} \sim a_1 (\lambda)^{-\frac{1}{2}} + a_2 (\lambda)^{-1} + a_3 (\lambda)^{-\frac{3}{2}} + \cdots.$$  

(6)

Thus taking $\lambda \to \infty$ limit corresponds to considering the classical supergravity action instead of full classical string theory.

\footnote{In general, QCD does not have any conformal invariance ($\beta$ function is not zero). However, at high enough temperature, the QCD plasma is well described by some conformal field theory and thus AdS/CFT correspondence can approximate it by a gravity dual. With this assumption we apply this correspondence to explain the properties of strongly coupled QGP produced at RHIC.}
2.2 The Field Operator Mapping

Another important machinery in the AdS/CFT correspondence is the mapping between bulk fields and boundary operators. In [6] Witten proposed that the boundary values of supergravity fields acts as a source to the corresponding operator in the field theory side. To be more explicit, let us define the string theory partition function as,

$$Z_S = \int \prod_i [d\chi_i] \exp \left[ -S[\chi_i] \right]$$

where we collectively denote all string theory fields by $\chi_i$. Since AdS is a space with a boundary, we need to specify the boundary values for these fields to compute the partition function. Let us denote the boundary value of the field $\chi_i$ by $\chi_i^B$. Therefore the partition function is, in general a function of the boundary values $\chi_i^B$’s of the fields,

$$Z_S[\chi_i^B] = \int \left[ \prod_i [d\chi_i] \exp \left[ -S[\chi_i] \right] \right]_{\chi_i \to \chi_i^B}.$$  \hspace{1cm} (8)

On the other hand in the filed theory side the correlation functions of any operators $O_i$ are given by,

$$\langle O_{i_1}(x_1)O_{i_2}(x_2) \cdots O_{i_n}(x_n) \rangle = \frac{\delta^n Z_{CFT}[\{J_i\}]}{\delta J_{i_1}(x_1)\delta J_{i_2}(x_2) \cdots \delta J_{i_n}(x_n)}|_{J_i=0}$$  \hspace{1cm} (9)

where $Z_{CFT}[\{J_i\}]$ is given by,

$$Z_{CFT}[\{J_i\}] = \int \prod_i [d\phi_i] \exp \left[ -S[\phi_i] + \sum_i \int [d^4x] \sqrt{-g_{CFT}} J_i(x) O_i(x) \right].$$  \hspace{1cm} (10)

According to [6] there is an one to one correspondence between $\chi_i$’s and $O_i$’s such that

$$Z_{CFT}[\{J_i\}] = Z_S[\chi_i^B], \hspace{0.5cm} \text{with} \hspace{0.5cm} \chi_i^B = J_i.$$ \hspace{1cm} (11)

For example, if we want to compute correlation function of boundary stress tensor $T_{\mu\nu}$ then the corresponding bulk field is metric $g_{\mu\nu}$. We need to first calculate the bulk partition function (string theory/gravity partition function). Then following equation (9) we can find the n-point correlation function of boundary operators by taking functional derivative of the bulk partition function with respect to the boundary values of the corresponding bulk fields. We shall use these results later in this article.

2.3 Fluid/gravity correspondence

The power of AdS/CFT is not confined to characterizing only the thermodynamic properties of boundary field theories. If we consider a black object with translation invariant horizon, for example black $D3$ brane geometry, one can also discuss hydrodynamics - long wave length deviation (low frequency fluctuation) from thermal equilibrium. In addition to the thermodynamic
quantities the black brane is also characterized by the hydrodynamic parameters like viscosity, diffusion constant, etc.. The black $D3$ brane geometry with low energy fluctuations (i.e. with hydrodynamic behavior) is dual to some finite temperature gauge theory plasma living on boundary with hydrodynamic fluctuations. Therefore studying the hydrodynamic properties of strongly coupled gauge theory plasma using the AdS/CFT duality is an interesting subject of current research.

The first attempt to study hydrodynamics via AdS/CFT was in [7], where authors related the shear viscosity coefficient $\eta$ of strongly coupled $\mathcal{N} = 4$ gauge theory plasma in large $N$ limit with the absorption cross-section of low energy gravitons by black $D3$ brane. Other hydrodynamic quantities like speed of sound, diffusion coefficients, drag force on quarks etc can also be computed in the context of AdS/CFT.

3 Goal of this article

In this article we explain how to compute different transport coefficients of strongly coupled gauge theory plasma using gauge/gravity correspondence.

The most famous transport coefficient is shear viscosity coefficient. We plan to study a systematic procedure to compute this particular transport coefficient of dual plasma in presence of possible stringy correction to the bulk Lagrangian. We first discuss about the famous Kubo formula in section 4. In section 5 we show how the shear viscosity coefficient of strongly coupled boundary gauge theory plasma depends on horizon value of the effective coupling of transverse graviton moving in black brane background. However to completely specify the boundary plasma, it is necessary to understand its higher order transport coefficients (coefficients which appear in second order in derivative expansion). In section 7 we study the second order hydrodynamics. We define a response function in bulk and show that the response function flows non-trivially with the radial direction and depends on the full black hole geometry. We study the radial flow of this response function (of energy-momentum tensor of dual gauge theory) in presence of generic higher derivative terms in bulk Lagrangian and solve these flow equations analytically to obtain second order transport coefficients of boundary plasma. We consider explicit example to explain this formalism in section 8. Finally we end this article with some concluding remarks in section 11.

4 Holographic hydrodynamics

As we have already mentioned, at present the best available analytical tool one has to study the properties of strong coupling hydrodynamics is precisely the AdS/CFT correspondence. In this section we briefly discuss how one can apply the knowledge of the correspondence to capture the hydrodynamic properties of the plasma. The goal of this section is to introduce Kubo formula and use this formula in the context of gauge/gravity duality to compute different transport coefficients of strongly coupled plasma.
4.1 Kubo formula

The Kubo formula relates transport coefficients of plasma with their thermal correlators. Let us consider the response of the fluid to small metric fluctuation $g_{xy} = \eta_{xy} + h_{xy}$ and consider $h_{xy} = h_{xy}(t, x_3)$. The expression for energy momentum tensor in fluid’s rest frame is given by (from equation (3) and also considering conformal fluid),

$$T_{xy} = ph_{xy} + \eta \frac{\partial h_{xy}}{\partial t} + O(\partial^2). \quad (12)$$

At the level of linear response theory the one-point function of $T_{xy}$ is linear in $h_{xy}$ and when expressed in Fourier space the proportionality constant is simply the thermal retarded correlator $G_{xy,xy}^{(R)}$ (Green’s function),

$$\langle T_{xy} \rangle = G_{xy,xy}^{(R)} h_{xy}. \quad (13)$$

Therefore writing the expression of equation (12) in momentum space we get,

$$G_{xy,xy}^{(R)} = p - i\omega \eta + O(\omega^2). \quad (14)$$

Thus the shear viscosity coefficient is given by

$$\eta = \lim_{\omega \to 0} - \frac{\Im G_{xy,xy}^{(R)}}{\omega}. \quad (15)$$

Hence, to compute shear viscosity coefficient we need to find the real time retarded Green’s function of EM tensor. Now, the usual prescription of AdS/CFT to compute boundary correlator is Euclidean. In principle, some real time Green’s function can be obtained by analytic continuation of the corresponding Euclidean ones. However, in many cases it is actually very difficult to get. In particular the low frequency low momentum limit Green’s function (which is interesting for hydrodynamics) is difficult to obtain from analytic continuation of Euclidean one. The difficulty here is, we need to analytically continue from a discrete set of points in Euclidean frequencies (the Matsubra frequencies) $\omega = 2\pi in$ (n integer) to real values of $\omega$. The smallest value of the Matsubra frequency is quite large. Hence to get information in small $\omega$ limit that we are interested in, is quite difficult. The authors of [8] have done a detailed analysis of this difficulty and have given a prescription to compute the real time correlator. This is a well-defined holographic method to compute thermal correlators of boundary theory and capture the values of different transport coefficients (for example shear viscosity in first order hydrodynamics). However we will follow a different path to compute boundary thermal correlator. In the next two sections we try to elaborate on this new technique to compute transport coefficients holographically.

In the next section(s), we use this formula holographically to extract the values of different transport coefficients for strongly coupled boundary fluid system.

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6 Similarly other transport coefficients are also captured in $G^{(R)}$. We shall consider them in section 7.
5 First order hydrodynamics

We plan to discuss the properties of hydrodynamic system order by order in derivative expansion. As we have already mentioned at the first order we encounter shear viscosity, the only non-trivial transport coefficient, for conformal fluid without any other conserved current. This section is dedicated to discuss the holographic techniques to compute this transport coefficient in generic higher derivative gravity.

5.1 Hydrodynamic limit in AdS/CFT and Membrane paradigm

In this subsection we briefly review the proposal described in [9] relating the hydrodynamic limit of AdS/CFT to the membrane paradigm. This proposal relates a generic transport coefficient of the boundary theory to some geometric quantities evaluated at the black hole horizon in the bulk. We will concentrate on the shear viscosity coefficient of the boundary fluid and confine ourselves at the level of linear response and low frequency limit of the strongly coupled gauge theory.

Membrane side

Let us start with classical black hole membrane paradigm, which says that the black hole has a fictitious fluid on its horizon. In general, the black hole action can be expressed as,

\[ S_{\text{eff}} = S_{\text{out}} + S_{\text{surf}}, \]

where \( S_{\text{out}} \) contains integration over space time out side the horizon and \( S_{\text{surf}} \) is the boundary term on the horizon. Physically, \( S_{\text{surf}} \) represents the effect of the horizon fluid on the spacetime. Let us consider a general black hole background,

\[ dS^2 = g_{MN} dx^M dx^N = g_{rr} dr^2 + g_{\mu\nu} dx^\mu dx^\nu, \]

where \( M, N \) runs over \( d + 1 \) dimensional bulk spacetime and \( \mu, \nu \) runs over \( d \) dimensional boundary spacetime. This black hole has a horizon at \( r_h \) and asymptotic boundary (where the dual gauge theory sits) at \( r_b \). We assume \( SO(3) \) invariance in the boundary spatial directions, that is all the metric components and the couplings in the theory are only functions of \( r \). We consider a small perturbation \( h_{xy} \) in the \( SO(3) \) tensor sector of this metric. We use \( \phi(r, x^\mu) = h^r_y \)

as the off diagonal component of graviton and in the Fourier space, the perturbation looks like,

\[ \phi(r, k_\mu) = \int \frac{d^d x}{(2\pi)^d} \phi(r, x^\mu) e^{ik_\mu x^\mu}, \quad k_\mu = (-\omega, \vec{k}). \]

The action for this massless perturbation can be written as,

\[ S_{\text{out}} = -\int_{r > r_h} d^{d+1} x \sqrt{-g} \frac{1}{q(r)} (\nabla \phi)^2, \quad S_{\text{surf}} = \int_{\Sigma} d^d x \sqrt{-\gamma} \left( \frac{\Pi_{r_h, x}}{\sqrt{-\gamma}} \right) \phi(r_h, x) \]

9
here, $\Pi$ is the conjugate momentum to $\phi$ for the $r-$foliation and $\gamma$ is the induced metric on the horizon. We can interpret $S_{surf}$ as the effect of the membrane fluid on the spacetime and $\Pi_{mb} = (\frac{\Pi(r_h)}{\sqrt{\gamma}})$ as the “membrane $\phi-$charge”.

Now, following membrane paradigm, the horizon is a regular place for the in-falling observer, hence, any physical deformation of the system has to satisfy the in-falling boundary condition. The in-falling boundary condition implies that near the horizon $r_h$,

- the deformation should behave as $\phi \sim (r - r_h)^i\omega^\beta$, for some constant $\beta$ and
- the solution should be a function of the non singular “Eddington-Finklestein” co-ordinate $v$ defined as,

$$ dv = dt + \sqrt{\frac{g_{rr}}{g_{tt}}} dr. $$

(19)

This implies near the horizon $r_h$, the deformation satisfies,

$$ \partial_r \phi = \sqrt{\frac{g_{rr}}{g_{tt}}} \partial_t \phi. $$

(20)

The above equation (20) puts constraint on the constant $\beta$ as,

$$ \beta = \sqrt{\frac{g_{rr}(r - r_h)^2}{g_{tt}}} \bigg|_{r_h}. $$

(21)

With equations (17,20) and some redefinition of the time, we can also express the membrane charge as,

$$ \Pi_{mb} = -\frac{1}{q(r_h)} \partial_t \phi(r_h). $$

(22)

As per our interpretation, $\Pi_{mb}$ is the response of the membrane fluid induced by $\phi$ and using equation (17), in linear response, we define a shear viscosity coefficient for the membrane fluid as,

$$ \Pi_{mb} = i\omega \eta_{mb} \phi $$

(23)

and thus we get,

$$ \eta_{mb} = \frac{1}{q(r_h)}. $$

(24)

**Boundary Side**

With this much of analysis of the membrane fluid, we concentrate on the boundary side where the gauge theory lives. This is a interacting theory at finite temperature and behaves as a fluid at sufficiently long length scale or low energy. The real time (Lorentzian signature) finite temperature version of AdS/CFT correspondence allows to compute various hydrodynamic quantities of this gauge theory at strong coupling by doing some supergravity calculations in
In the AdS space. Now using the Kubo formula, the shear viscosity of the boundary fluid is given in terms of retarded Green’s function, the response of graviton to the boundary stress tensor. In [8], the authors have given a simple prescription to compute the boundary correlator using the bulk field \( \phi \), the off diagonal component of the graviton. Their prescription requires to find a solution for the graviton which is in-falling at the horizon and constant at the boundary. Then one computes the on-shell action with this solution and the retarded Green’s function is related to the surface term of the on-shell action at the boundary. Taking \( \phi(k_\mu, r) = f(k_\mu, r) \phi_0(k_\mu) \) with normalization \( f(k_\mu, r_b) \to 1 \),

\[
S = - \sum_{r=r_h, r_b} \int \frac{d^d k}{(2\pi)^d} \phi_0(k_\mu) G(k_\mu, r) \phi_0(-k_\mu)
\]

\[
G^R(k_\mu) = \lim_{r \to r_b} 2G(k_\mu, r) = \lim_{r \to r_b} 2\sqrt{-g} g^{rr} \frac{\partial_r f(k_\mu, r)}{q(r)}
\]

(25)

For this profile of the graviton, we can evaluate its conjugate momenta \( \Pi \), it readily gives us the relation,

\[
G^R(k_\mu) = - \lim_{r \to r_b} \frac{\Pi(k_\mu, r)}{\phi(k_\mu, r)}
\]

(26)

Hence, the shear viscosity coefficient \( \eta \) can be written as,

\[
\eta = \lim_{k_\mu \to 0} \lim_{r \to r_b} \frac{\Pi(k_\mu, r)}{i \omega \phi(k_\mu, r)}
\]

(27)

Important point to note is that, in the low frequency limit \( k_\mu \to 0, \text{ with } \Pi, \omega \phi \text{ fixed} \), the flow of \( \Pi \) and \( \omega \phi \) in the \( r \)-direction are trivial. Hence, we can actually compute the shear viscosity coefficient of the boundary fluid at any constant \( r \)-slice and its value would be same. We compute it at the horizon and get,

\[
\eta = \frac{1}{q(r_h) \sqrt{-g}} = \frac{1}{q(r_h)} A
\]

(28)

Comparing equations (28) and (23), we see that the viscosity coefficient of the boundary fluid is related to that of the membrane fluid and more importantly they are given as just the value of the inverse effective coupling of the transverse graviton evaluated at the horizon. To emphasize, equation (20) plays a crucial role in this equivalence. The AdS/CFT response of the graviton is almost same as that of the membrane except that the membrane now has to sit in the boundary. The in-falling boundary condition of the graviton field in AdS/CFT is precisely the regularity condition given by equation (20) of the membrane paradigm. In the low frequency limit, we can place a fictitious membrane at each constant \( r \) and define the transport coefficient as \( \eta(r) \). Since the flow is trivial in this limit, \( \eta(r) \) actually comes out to be a constant, \( \frac{1}{q(r_h)} \).
5.2 Example: Two derivative gravity

In this section explain how to calculate the shear viscosity coefficient of the boundary fluid from the effective coupling constant of transverse graviton in Einstein-Hilbert gravity.

We first fix the background spacetime. We start with the following Einstein-Hilbert action in AdS space.

\[ I = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left( R + 12 \right). \]  

Here we have taken the radius of the AdS space to be 1. The background spacetime is given by the following metric\(^7\)

\[ ds^2 = -h_t(r)dt^2 + \frac{dr^2}{h_r(r)} + \frac{1}{r}d\vec{x}^2, \quad h_t(r) = \frac{1-r^2}{r}, \quad h_r(r) = 4r^2(1-r^2). \]  

The black hole has horizon at \( r_0 = 1 \) and the temperature is given by, \( T = \frac{1}{\pi} \).

We consider the following metric perturbation,

\[ g_{xy} = g^{(0)}_{xy} + h_{xy}(r,x) = g^{(0)}_{xy}(1 + \epsilon \Phi(r,x)) \]  

where \( \epsilon \) is an order counting parameter. We consider terms up to order \( \epsilon^2 \) in the action of \( \Phi(r,x) \). The action (in momentum space) is given by, (up to some total derivative terms in the action)\(^8\)

\[ S = \frac{1}{16\pi G_5} \int \frac{d\omega d^3\vec{k}}{(2\pi)^4} dr \left( A^{(0)}_1(r)\phi'(r, -\vec{k})\phi'(r, \vec{k}) + A^{(0)}_0(r, \vec{k})\phi(r, \vec{k})\phi(r, -\vec{k}) \right) \]  

where,

\[ A^{(0)}_1(r) = \frac{r^2 - 1}{r}, \quad A^{(0)}_0(r, \vec{k}) = \frac{\omega^2}{4r^2(1-r^2)}. \]  

This can be viewed as an action for minimally coupled scalar field \( \phi(r, \vec{k}) \) with effective coupling given by,

\[ K_{\text{eff}}(r) = \frac{1}{16\pi G_5} \frac{A^{(0)}_1(r)}{\sqrt{-g^{(0)}g^{rr}}}. \]  

Therefore according to \(^9\) the effective coupling \( K_{\text{eff}} \) calculated at the horizon \( r_0 \) gives the shear viscosity coefficient of boundary fluid,

\[ \eta = r_0^{-\frac{3}{2}}(-2K_{\text{eff}}(r_0)) = \frac{1}{16\pi G_5}. \]  

However, when we add higher derivative terms (effect of string theory) in bulk action the situation becomes complicated. In the next two subsections we generalize this idea to higher derivative gravity theories.

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\(^7\)We are working in a coordinate frame where asymptotic boundary is at \( r \to 0 \).

\(^8\)Though throughout this article we have written the four vector \( k \), but in practice we have worked in \( \vec{k} \to 0 \) limit. In all the expressions we have dropped the terms proportional to \( \vec{k} \) or its power.
5.3 The Effective Action

Having understood the above procedure of determining the shear viscosity coefficient from the effective coupling of transverse graviton it is tempting to generalize this method for any higher derivative gravity. The first problem one faces is that the action for transverse graviton no more has the canonical form as in equation (32). For generic 'n' derivative gravity theory the action can have terms with (and up to) ‘n’ derivatives of \( \Phi(r, x) \). Therefore, from that action it is not very clear how to determine the effective coupling. In this section we try to address this issue.

We construct an effective action which is of form given by equation (32) with different coefficients capturing higher derivative effects. We determine these two coefficients by claiming that the equation of motion for \( \phi(r, k) \) coming from these two actions (general action and effective action) are same up to first order in perturbation expansion (in coefficient of higher derivative term). Once the effective action for transverse graviton is obtained in canonical form then one can extract the effective coupling from the coefficient of \( \phi'(r, k)\phi'(r, -k) \) term in the action. Needless to say, our method is perturbatively correct.

5.3.1 The General Action and Equation of Motion

Let us start with a generic ‘n’ derivative term in the action with coefficient \( \mu \). We study this system perturbatively and all our expressions are valid up to order \( \mu \). The action is given by,

\[
S = \frac{1}{16\pi G_5} \int d^5x \left( R + 12 + \mu \mathcal{R}^{(n)} \right)
\]  

(36)

where, \( \mathcal{R}^{(n)} \) is any \( n \) derivative Lagrangian. \( \mu \) is the perturbation parameter. The metric in general is given by (assuming planar symmetry),

\[
ds^2 = -(h_t(r) + \mu h_t^{(n)}(r))dt^2 + \frac{dr^2}{h_r(r) + \mu h_r^{(n)}(r)} + \frac{1}{r}(1 + \mu h_s^{(n)}(r))d\vec{x}^2
\]

(37)

where \( h_t^{(n)} \), \( h_r^{(n)} \) and \( h_s^{(n)} \) are higher derivative corrections to the metric.

Substituting the background metric with fluctuations in the action given in equation (36) (we call it general action or original action) for the scalar field \( \phi(r, k) \) we get,

\[
S = \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} dr \sum_{p,q=0}^{n} A_{p,q}(r, k) \phi^{(p)}(r, -k)\phi^{(q)}(r, k)
\]

(38)

where, \( \phi^{(p)}(r, k) \) denotes the \( p^{th} \) derivative of the field \( \phi(r, k) \) with respect to \( r \) and \( p + q \leq n \). The coefficients \( A_{p,q}(r, k) \) in general depends on the coupling constant \( \mu \). \( A_{p,q} \) with \( p + q \geq 3 \) are proportional to \( \mu \) and vanishes in \( \mu \to 0 \) limit , since the terms \( \phi^{(p)}\phi^{(q)} \) with \( p + q \geq 3 \) appears as an effect of higher derivative terms in equation (36). Up to some total derivative
terms, the general action in equation \([38]\) can also be written as,

\[ S = \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} dr \sum_{p=0}^{n/2} A_p(r, k) \phi^{(p)}(r, -k) \phi^{(p)}(r, k), \quad n \text{ even} \]

\[ = \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} dr \sum_{p=0}^{(n-1)/2} A_p(r, k) \phi^{(p)}(r, -k) \phi^{(p)}(r, k), \quad n \text{ odd}. \]

The equation of motion for the scalar field \(\phi(r, k)\) is given by,

\[ \sum_{p=0}^{n/2} \left( -\frac{d}{dr} \right)^p \frac{\partial \mathcal{L}(\{\phi^{(m)}\})}{\partial \phi^{(p)}(r, k)} = 0 \text{ (n even)}, \quad \sum_{p=0}^{(n-1)/2} \left( -\frac{d}{dr} \right)^p \frac{\partial \mathcal{L}(\{\phi^{(m)}\})}{\partial \phi^{(p)}(r, k)} = 0 \text{ (n odd)} \]

where \(\mathcal{L}(\{\phi^{(m)}\})\) is given as

\[ \mathcal{L}(\{\phi^{(m)}\}) = \sum_p A_p(r, k) \phi^{(p)}(r, -k) \phi^{(p)}(r, k). \]

We analyze the general action for the scalar field \(\phi(r, k)\) and their equation of motion perturbatively and write an effective action for the field \(\phi(r, k)\).

The generic form of the equation of motion (varying the general action) up to order \(\mu\) is given by,

\[ A_0(r, k) \phi(r, k) - A_1(r, k) \phi'(r, k) - A_1(r, k) \phi''(r, k) = \mu \tilde{F}(\{\phi^{(p)}\}) + \mathcal{O}(\mu^2) \]

where \(\tilde{F}(\{\phi^{(p)}\})\) is some linear function of double and higher derivatives of \(\phi(r, k)\), coming from two or higher derivative terms in equation \([38]\). The zeroth order \((\mu \to 0)\) equation of motion is given by,

\[ A_0^{(0)}(r, k) \phi(r, k) - A_1^{(0)}(r, k) \phi'(r, k) - A_1^{(0)}(r, k) \phi''(r, k) = 0 \]

where, \(A_p^{(0)}\) is the value of \(A_p\) at \(\mu \to 0\). From this equation we can write \(\phi''(r, k)\) in terms of \(\phi'(r, k)\) and \(\phi(r, k)\) in \(\mu \to 0\) limit.

\[ \phi''(r, k) = \frac{A_0^{(0)}(r, k)}{A_1^{(0)}(r, k)} \phi(r, k) - \frac{A_1^{(0)}(r, k)}{A_1^{(0)}(r, k)} \phi'(r, k). \]

Then the full equation of motion can be written in the following way,

\[ A_0^{(0)}(r, k) \phi(r, k) - A_1^{(0)}(r, k) \phi'(r, k) - A_1^{(0)}(r, k) \phi''(r, k) = \mu \tilde{F}(\phi(r, k), \phi'(r, k), \phi''(r, k), \ldots) + \mathcal{O}(\mu^2). \]

Since the right hand side of equation \([45]\) is proportional to \(\mu\), we can replace the \(\phi''(r, k)\) and other higher (greater than 2) derivatives of \(\phi(r, k)\) by its leading order value in equation \([44]\). Therefore up to order \(\mu\) the equation of motion for \(\phi\) is given by,

\[ A_0^{(0)}(r, k) \phi(r, k) - A_1^{(0)}(r, k) \phi'(r, k) - A_1^{(0)}(r, k) \phi''(r, k) = \mu \mathcal{F}(\phi(r, k), \phi'(r, k)) + \mathcal{O}(\mu^2) \]

\[ = \mu(\mathcal{F}_1 \phi'(r, k) + \mathcal{F}_0 \phi(r, k)) + \mathcal{O}(\mu^2) \]

where \(\mathcal{F}_0\) and \(\mathcal{F}_1\) are some function of \(r\). This is the perturbative equation of motion for the scalar field \(\phi(r, k)\) obtained from the general action in equation \([38]\).
5.3.2 Strategy to Find The Effective Action

In this subsection we describe the strategy to write an effective action for the field \(\phi(r, k)\) whose equation of motion has the form of (32) but with different functions. The prescription is following:

(a) We demand the equation of motion for \(\phi(r, k)\) obtained from the original action and the effective action are same up to order \(\mu\). This will fix the coefficients of \(\phi'^2\) and \(\phi^2\) terms in effective action.

Let us start with the following form of the effective action.

\[
S_{\text{eff}} = \frac{1}{16\pi G_5} \int \frac{d\omega d^3 \vec{k}}{(2\pi)^4} dr \left[ (A_1^{(0)}(r, k) + \mu B_1(r, k))\phi'(r, -k)\phi'(r, k) + (A_0^{(0)}(r, k) + \mu B_0(r, k))\phi(r, k)\phi(r, -k) \right].
\] (47)

The functions \(B_0\) and \(B_1\) are yet to be determined. We determine these functions by claiming that the equation of motion for the scalar field \(\phi(r, k)\) obtained from this effective action is same as equation (46) up to order \(\mu\). The equation of motion for \(\phi(r, k)\) from the effective action is given by,

\[
A_0^{(0)}(r, k)\phi(r, k) - A_1^{(0)}(r, k)\phi'(r, k) - A_1^{(0)}(r, k)\phi''(r, k) = \mu \left( B_1'(r, k) - \frac{A_1^{(0)}(r, k)}{A_1^{(0)}(r, k)} B_1(r, k) \right) \phi'(r, k)
+ \mu \left( B_1(r, k) - \frac{A_0^{(0)}(r, k)}{A_1^{(0)}(r, k)} - B_0(r, k) \right) \phi(r, k) + \mathcal{O}(\mu^2).
\] (48)

Therefore comparing with equation (46) we get,

\[
B_1'(r, k) - \frac{A_1^{(0)}(r, k)}{A_1^{(0)}(r, k)} B_1(r, k) - F_1(r, k) = 0
\] (49)

and

\[
B_0(r, k) = B_1(r, k) - \frac{A_0^{(0)}(r, k)}{A_1^{(0)}(r, k)} - F_0(r, k).
\] (50)

The solutions are given by,

\[
B_1(r, k) = A_1^{(0)}(r, k) \int dr \frac{F_1(r, k)}{A_1^{(0)}(r, k)} + \kappa A_1^{(0)}(r, k) = \bar{B}_1(r, k) + \kappa A_1^{(0)}(r, k)
\] (51)

and

\[
B_0 = \bar{B}_0(r, k) + \kappa A_0^{(0)}
\] (52)

for some constant \(\kappa\). We need to fix this constant.

(b) Condition (a) can not fix the overall normalization factor of the effective action. In particular we can multiply it by \((1 + \mu \Gamma)\) (for some constant \(\Gamma\)) and still get the same equation
of motion. Considering this normalization, the effective action is given by,

\[ S_{\text{eff}} = \frac{1 + \mu \Gamma}{16\pi G_5} \int \frac{d\omega d^3k}{(2\pi)^4} dr \left[ (A_1^{(0)}(r, k) + \mu B_1(r, k))\phi'(r, -k)\phi(r, k) 
+ (A_0^{(0)}(r, k) + \mu B_0(r, k))\phi(r, k)\phi(r, -k) \right]. \] (53)

Substituting the values of \( B \)'s in equations (51) and (52) we get,

\[ S_{\text{eff}} = (1 + \mu(\Gamma + \kappa))S^{(0)} + \mu \int dr \left[ \tilde{B}_1(r, k)\phi'(r, -k)\phi'(r, k) + \tilde{B}_0(r, k)\phi(r, -k)\phi(r, k) \right]. \] (54)

where \( S^{(0)} \) is the effective action at \( \mu \to 0 \) limit. This implies that the integration constant \( \kappa \) can be absorbed in the overall normalization constant \( \Gamma \). Henceforth we will denote this combination as \( \Gamma \).

Our prescription is to take \( \Gamma \) to be zero from the following observation.

• The shear viscosity coefficient of boundary fluid is related to the imaginary part of retarded Green function in low frequency limit. The retarded Green function \( G_{xy,xy}^R(k) \) is defined in the following way: the on-shell action for graviton can be written as a surface term,

\[ S \sim \int \frac{d^4k}{(2\pi)^4} \phi_0(k) G_{xy,xy}(k, r) \phi_0(-k) \] (55)

where \( \phi_0(k) \) is the boundary value of \( \phi(r, k) \) and \( G_{xy,xy}^R \) is given by,

\[ G_{xy,xy}^R(k) = \lim_{r \to 0} 2G_{xy,xy}(k, r) \] (56)

and shear viscosity coefficient is given by\(^9\)

\[ \eta = \lim_{\omega \to 0} \left[ \frac{1}{\omega} \text{Im} G_{xy,xy}^R(k) \right] \quad \text{(computed on-shell)} \] . (57)

• Now it turns out that the imaginary part of this retarded Green function obtained from the original action and effective action are same up to the normalization constant \( \Gamma \) in presence of generic higher derivative terms in the bulk action. Therefore it is quite natural to take \( \Gamma \) to be zero as it ensures that starting from the effective action also one can get same shear viscosity using Kubo machinery. To show that the above statement is true we do not need to know the full solution for \( \phi \), in other words to find the difference between the two Green functions one does not need to calculate the Green functions explicitly. Assuming the following general form of solution for \( \phi \)

\[ \phi \sim (1 - r^2)^{-i\omega/\beta} (1 + i\omega/\mu \xi(r)) \] (58)

it can be shown generically. For a detailed proof see \[10\].

\(^9\)To calculate this number one has to know the exact solution, i.e. the form of \( \xi \) and the value of \( \beta \) in equation (58).
Because of the canonical form of the effective action, it follows from the argument in [9] and the statement above, that the shear viscosity coefficient of boundary fluid is given by the horizon value of the effective coupling obtained from the effective action in presence of any higher derivative terms in the bulk action.

(c) After getting the effective action for \( \phi(r, k) \), the effective coupling is given by,

\[
K_{\text{eff}}(r) = \frac{1}{16\pi G_5} \frac{A_1^{(0)}(r, k) + \mu B_1(r, k)}{\sqrt{-g}g^{rr}} \tag{59}
\]

where \( g^{rr} \) is the 'rr' component of the inverse perturbed metric and \( \sqrt{-g} \) is the determinant of the perturbed metric. Hence the shear viscosity coefficient is given by,

\[
\eta = r_0^{-\frac{3}{2}}(-2K_{\text{eff}}(r = r_0)) \tag{60}
\]

where \( r_0 \) is the corrected horizon radius. To summaries, we have obtained a well defined procedure to find the correction (up to order \( \mu \)) to the coefficient of shear viscosity of the boundary fluid in presence of general higher derivative terms in the action.

5.4 Membrane fluid in higher derivative gravity

Let us also consider the effect of the higher derivative terms on the membrane fluid. In higher derivative gravity, since the canonical form of the action (32) breaks down, it is not very obvious how to define the membrane charge \( \Pi_{\text{mb}} \). Instead of the original action if we consider the effective action (47) for graviton then it is possible to write the membrane action perturbatively and define the membrane charge (\( \Pi_{\text{mb}} \)) in higher derivative gravity. As if the membrane fluid is sensitive to the effective action \( S_{\text{eff}} \) in higher derivative gravity.

Following [9] we can write the membrane action and charge in the following way (in momentum space)

\[
S_{\text{mb}} = \int_\Sigma \frac{d^4k}{(2\pi)^4} \sqrt{-\sigma} \left( \frac{\Pi(r_0, k)}{\sqrt{-\sigma}} \phi(r_0, -k) \right) \tag{61}
\]

where \( \sigma_{\mu\nu} \) is the induced metric on the membrane and \( \Pi(r, k) \) is conjugate momentum of \( \phi \) with respect to \( r \) foliation where \( \Pi(r, k) = (A_1^{(0)}(r, k) + \mu B_1(r, k))\phi'(r, -k) \). The membrane charge is then given by,

\[
\Pi_{\text{mb}} = \frac{\Pi(r_0, k)}{\sqrt{-\sigma}} = -\tilde{K}_{\text{eff}}(r_0) \sqrt{g^{(0)rr}} \partial_r \phi(r, k) \bigg|_{r_0}. \tag{62}
\]

Imposing the in-falling wave boundary condition on \( \phi \), it can be shown that the membrane charge \( \Pi_{\text{mb}} \) is the response of the horizon fluid to the bulk graviton excitation and the membrane fluid transport coefficient is given by,

\[
\eta_{\text{mb}} = \tilde{K}_{\text{eff}}(r_0). \tag{63}
\]

Hence, we see that even in higher derivative gravity the shear viscosity coefficient of boundary fluid is captured by the membrane fluid.
Motivated by the fact that like entropy of boundary theory, shear viscosity coefficient is also determined in terms of data specified on the black hole horizon, we can think that if we know the near-horizon geometry of an asymptotically AdS black hole space time, it would be enough to compute the ratio $\eta/s$. In the next section we elaborate this particular issue with an example.

6 Near-horizon analysis of $\eta/s$

As we know that it is possible to find the entropy of boundary field theory by only specifying the complete near horizon geometry of its dual black hole spacetime, one can also ask in the same spirit, whether different hydrodynamics coefficients of the boundary plasma can be found from the knowledge of bulk near-horizon geometry. There are enough hints from the above analysis which seems to point to an affirmative answer to this question.

In last section we have shown that in the low frequency limit ($\omega \to 0$) the shear viscosity coefficient $\eta$ is related to transport coefficient of membrane fluid, which in turn is given by the effective coupling constant of graviton ($h_{xy}$) evaluated on the horizon. We have also described how to generalize this prescription has been generalized for any higher derivative gravity. All these approaches indicate that there can be an ”IR” description of transport coefficients $\eta$. However, it is not quite clear how other transport coefficients like $\tau_{\pi}$ etc. which appear in next order in $\omega$ can be calculated only in terms of horizon data. The derivation of [9, 10] was strictly valid only in $\omega \to 0$ limit.

In this section, we observe that only with the knowledge of near horizon geometry of a black hole spacetime one can easily calculate the thermodynamic and hydrodynamic quantities, namely entropy and shear viscosity of boundary fluid and their ratio. One does not need to know the full analytic solutions of Einstein equations. This observation is helpful when bulk Lagrangian is very complicated. For example when gravity is coupled to various matter fields (gauge field, scalars etc.) in non-trivial way, it may not be always possible to find an analytic solution[13]. But if a black hole solution exists then it is possible to find the near horizon geometry, i.e. how the metric and other fields behave in near horizon limit. The method is very simple. We do not need to solve any differential equation to find the near horizon geometry. First we write the field equations. Then take a suitable near horizon ansatz for different fields. In general different components of the metric goes like $g_{tt} \sim a_1(r-r_h) + a_2(r-r_h)^2 + \cdots$, $g_{rr} \sim \frac{b_1}{r-r_h}(1+b_2(r-r_h)+\cdots)$ (for non-extremal case), scalar fields behaves as $\varphi \sim \varphi_h + \varphi_1(r-r_h) + \cdots$ and gauge fields goes as $A_t = q(r-r_h) + \cdots$. Substituting these in the field equations we find the coefficients consistently order by order in $(r-r_h)$. Once we find the near-horizon structure of the spacetime we find entropy of the system using Wald’s formula. which says we only need to calculate the horizon values of some quantities. To calculate shear viscosity we adopt the method proposed in last section[9, 10]. We write the effective action for transverse graviton in black hole near-horizon region and find the coupling constant. For non-extremal black hole usually (in presence of higher derivative terms) the effective coupling depends on radial coordinate and we need to take $r \to r_h$ limit at the end to find the horizon value of the coupling constant. See [11] for detailed discussion and examples.
However we would like to clarify the following important points:

Using the AdS/CFT conjecture we are studying the low frequency hydrodynamic properties of field theory plasma which has a gravity dual. Our observation is certainly valid for those fluids whose dual gravity satisfies the asymptotic AdS boundary condition. The presence of a cosmological constant term in the bulk action ensures that the black hole solution must be asymptotically AdS. We found near-horizon geometry by solving the background equations of motion in presence of the cosmological constant term. This ensures that we are solving the near horizon geometry of an asymptotically AdS black hole space time.

One can also consider Lagrangian with different higher derivative terms. Higher derivative corrections typically make energy conditions e.g. weak energy conditions in general relativity hard to interpret, as a result of which the underlying solutions could be singular. This means that it is not necessarily true that the solutions are asymptotically AdS. However in many cases, the higher derivative terms of \( R^{(n)} \) kind (contraction of Riemann, Ricci or scalar) can be treated perturbatively. Usually if the zeroth order black hole solution is non singular then these higher derivative terms (when treated perturbatively) do not create any potential hazard in the bulk theory (blowing up of some invariant quantity between horizon and infinity) and one gets solutions which are asymptotically AdS [10], [12], [13]. For more generic situations, if the higher derivative terms are treated exactly, one has to take special care to show that there exists a non-singular solution everywhere between the horizon and the boundary which is asymptotically AdS.

The asymptotic AdS boundary condition is true for all the gravity models we studied in this chapter. For generic models (where we think our result is most useful) of realistic boundary plasmas such as QGP or more generically QCD, one should encounter a dual gravity theory with non-trivial matter coupling. A complete analytic solution for such a background is almost impossible. For these cases, the existence of the asymptotic AdS solution can be checked numerically, and the corresponding near-horizon geometry can be obtained analytically. In these scenarios, our observation will be useful to extract information about the coefficient \( \eta \) for the plasma.

Now we will present a particular model where the above observation will be useful and we will compute the shear viscosity of the boundary fluid dual to this system just from the knowledge of the near-horizon geometry.

### 6.1 The model

We will focus on a 5-dimensional theory of gravity coupled to a massless scalar and an abelian electromagnetic field whose action is\(^{10}\)

\[
S_{\text{EM}} = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left( R + 12 - e^{\alpha \varphi(r)} F_{\mu\nu} F^{\mu\nu} - \partial_{\mu} \varphi \partial^{\mu} \varphi \right) + S_{\text{CS}}, \tag{64}
\]

\[
S_{\text{CS}} = \frac{\zeta}{16\pi G_5} \int d^5x \epsilon^{\mu\nu\rho\sigma\gamma} A_{\mu} F_{\nu\rho} F_{\sigma\gamma}, \tag{65}
\]

\(^{10}\)We also have to add counterterms to regularize the action — for [64], the counterterms are given in [14] (see, also, the nice review [16] and references therein for a more detailed discussion).
In this section, we consider a constant moduli potential, \( V(\phi) = 2\Lambda = -12/l^2 \), and also fix the radius of AdS to be \( l = 1 \).

Since the equations of motion for the gauge field simplify, we choose the constant in front of the Chern-Simons term to be \( \frac{\zeta}{3} \). The action (64) with various values for \( \alpha \) resembles (truncated) actions obtained in string compactifications [14]. The coupling \( \zeta \) captures the strength of the anomaly of the boundary current.

The equations of motion for the metric, scalar, and electromagnetic field \( (F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}) \) are
\[
R_{\mu\nu} + 4g_{\mu\nu} + e^{\alpha\varphi(r)} \left( \frac{1}{3} F^2 g_{\mu\nu} - 2 F_{\mu\rho} F_{\nu\sigma} g^{\rho\sigma} \right) - \partial_\mu \varphi \partial_\nu \varphi = 0 ,
\]
\[
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu \varphi) = - \frac{1}{2} \alpha e^{\alpha\varphi(r)} F_{\mu\nu} F^{\mu\nu} ,
\]
\[
e^{\alpha\varphi(r)} \partial_\nu \left( \sqrt{-g} F^{\nu\mu} \right) + \frac{\zeta}{4} \epsilon^{\mu\rho\sigma\gamma\delta} F_{\rho\sigma} F_{\gamma\delta} = 0 ,
\]
where we have varied the scalar and the electromagnetic field independently. The Bianchi identities for the gauge field are
\[
F_{\mu\nu;\lambda} = 0 .
\]

Since we are interested in a theory for which the Chern-Simons term has a non-trivial contribution, we consider the following ansatz for the gauge field
\[
A = E(r) dt - \frac{B}{2} y dx + \frac{B}{2} x dy - P(r) dz .
\]
The magnetic field, \( B \), is fixed to be constant by the Bianchi identities. Thus, the field strength is
\[
F = \begin{pmatrix}
0 & E'(r) & 0 & 0 & -P'(r) \\
-E'(r) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & B & 0 \\
0 & 0 & -B & 0 & 0 \\
P'(r) & 0 & 0 & 0 & 0
\end{pmatrix} ,
\]
where ‘ denotes derivatives with respect to \( r \).

Our analysis is on time-independent black hole solutions and so we consider the following ansatz for the metric
\[
ds^2 = \frac{dr^2}{U(r)} - U(r) dt^2 + e^{2V(r)} \left( dx^2 + dy^2 \right) + e^{2W(r)} \left( dz + Z(r) dt \right)^2 ,
\]
which is compatible with the symmetries of the problem.

In this case, the horizon is located at (the biggest root of) \( U(r_h) = 0 \) and the temperature can be easily computed on the Euclidean section — we obtain
\[
T = \frac{U'(r_h)}{4\pi r} .
\]

Please note that our \( P'(r) \) is the same as \( P(r) \) in [19] and our \( Z(r) \) in the ansatz of the metric (71) is their \( C(r) \).

Our convention for the coordinates is \((r, t, x, y, z)\).
By using the metric ansatz \([71]\), we can rewrite the Maxwell equations as

\[
[Q(r) e^{2V(r)+W(r)+\alpha\varphi(r)}]'' - 2\zeta BP'(r) = 0, 
\]

\[e^{2V(r)+W(r)+\alpha\varphi(r)} \left(U(r)e^{-2W(r)} P'(r) - Q(r) Z(r)\right)'' - 2\zeta BE'(r) = 0 \tag{74}
\]

where

\[Q(r) = E'(r) + Z(r) P'(r). \tag{75}\]

It is easier to work with combinations of Einstein equations rather than using directly \([66]\). First, we extract the expressions of second derivatives of the functions that characterize the metric in the following way: we obtain \(W''(r)\) from \((rr)\)-, \(V''(r)\) from \((xx)\)-, and \(U''(r)\) from \((zz)\)-component of Einstein equations.

Let us now consider the \((tz)\)-component of Einstein equations in which we replace \(W''(r)\), \(V''(r)\), and \(U''(r)\) — we obtain

\[e^{2W(r)} \left[2V''(r) Z'(r) + 3W''(r) Z'(r) + Z''(r)\right] - 4Q(r) P'(r) e^{\alpha\varphi(r)} = 0. \tag{76}\]

then considering the

An important observation, which can be drawn by studying the system of equations \([73]-[76]\), is that a non-zero magnetic field is not compatible with a constant function \(Z(r)\) (and, also, \(P(r)\)).

The other (independent) combinations of Einstein equations are obtained as follows: by replacing \(U''(r)\) in the \((rr)\)-component of Einstein equations we get

\[2B^2 e^{\alpha\varphi(r)-4V(r)} + 2Q(r)^2 e^{\alpha\varphi(r)} + U(r)[2V'(r) + W'(r)]^2 
+ [U(r) (2V'(r) + W'(r))]' + \frac{1}{2} e^{2W(r)} Z'(r)^2 - 12 = 0 \tag{77}\]

by replacing \(W''(r)\) in the \((zz)\)-component of Einstein equations

\[-4Q(r)^2 e^{\alpha\varphi(r)} + U''(r) + U'(r)[2V'(r) - W'(r)] - 2e^{2W(r)} Z'(r)^2 
+ 2U(r)[2V''(r) - 2V'(r)W'(r) + 2V'(r)^2 + \varphi'(r)^2] = 0 \tag{78}\]

and the last one is, in fact, the \((xx)\)-component of Einstein equations

\[e^{2W(r)} \left[-4B^2 e^{\alpha\varphi(r)} - 3e^{4V(r)} U'(r) V'(r) + 12e^{4V(r)}] - 2Q(r)^2 e^{4V(r)} + 2W(r) + \alpha\varphi(r) 
+ U(r) e^{4V(r)} [2P'(r)^2 e^{\alpha\varphi(r)} - 3e^{2W(r)} (V''(r) + V'(r) W'(r) + 2V'(r)^2)] \right] = 0. \tag{79}\]

We also use the ansatz of the metric in the equation of motion for the scalar \([67]\) and so this equation becomes

\[\left[U(r) e^{2V(r)+W(r)} \varphi'(r)\right]' + \alpha e^{2V(r)+W(r)+\alpha\varphi(r)[B^2 e^{-4V(r)} + U(r) e^{-2W(r)} P'(r)^2 - Q(r)^2] = 0. \tag{80}\]

Due to the non-trivial coupling between the scalar and gauge fields, the equation \([67]\) has a non-trivial right hand side. The non-vanishing electromagnetic field may also be understood as a source for the scalar field. Thus, the scalar charge is determined by the electric and magnetic
charges and so it is not an independent parameter that characterizes the system — this charge plays an important role when the asymptotic value of the scalar is not fixed (see [17]).

We conclude this section with an observation on the Hamiltonian constraint. A vanishing Hamiltonian is a characteristic feature of any theory that is invariant under arbitrary coordinate transformations — for our system, we can obtain a first order differential equation by replacing $W''(r), V''(r)$, and $U''(r)$ in the $(tt)$-component of Einstein equations.

The Hamiltonian constraint, which can be enforced as an initial condition, has the following expression:

$$
2B^2e^{\alpha(r)-4W(r)} + U(r)[-2P'(r)^2e^{\alpha(r)-2W(r)} + 4V'(r)W'(r) + 2V'(r)^2 - \varphi'(r)^2] + 2Q(r)^2e^{\alpha(r)} + 2U'(r)V'(r) + U'(r)W'(r) + \frac{1}{2}e^{2W(r)}Z'(r)^2 - 12 = 0. \quad (81)
$$

### 6.2 Non-extremal Near horizon geometry

In this section, we will first find the near-horizon geometry of the non-extremal black hole, which we will need to compute the shear viscosity to entropy density ratio. In the extremal limit, due to the attractor mechanism, the near horizon geometry is universal regardless of the asymptotic values of the scalars. We will not present a detailed analysis of the extremal branches of solutions here, those can be found in [18]. Here we only present the near-horizon geometry of the non-extremal case.

As in [19], we work with a coordinate system in which the solution takes the canonical form at the horizon. That is, the field strength $F_H$ and the metric $ds^2_H$ are

$$
F_H = q \, dr \wedge dt + B \, dx \wedge dy - p \, dr \wedge dz, \\
\text{and } ds^2_H = r_H^2(dx^2 + dy^2 + dz^2), \quad (82)
$$

where $q$ and $B$ are the charge density (of the black brane) and the magnetic field at the horizon, respectively. In this way, the gauge freedom is removed and the initial conditions are

$$
U(r_h) = Z(r_h) = P(r_h) = 0, \quad V(r_h) = W(r_h) = \ln(r_h). \quad (83)
$$

A similar analysis (and numerical solutions) in the presence of the Gauss-Bonnet term but without the Chern-Simons term was presented in [20].

The generic solutions have a non-degenerate horizon. Near the event horizon, they admit a power series expansion of the form (using the definition of the temperature [72] in the expression of $U$):

$$
U(r) = 4\pi T(r-r_h) + u_2(r-r_h)^2 + \cdots, \\
V(r) = \ln(r_h) + v_1(r-r_h) + v_2(r-r_h)^2 + \cdots, \\
W(r) = \ln(r_h) + w_1(r-r_h) + w_2(r-r_h)^2 + \cdots, \\
Z(r) = z_1(r-r_h) + z_2(r-r_h)^2 + \cdots, \\
E(r) = q(r-r_h) + q_1(r-r_h)^2 + q_2(r-r_h)^3 + \cdots, \\
P(r) = p(r-r_h) + p_1(r-r_h)^2 + \cdots, \\
\varphi(r) = \varphi_h + \varphi_1(r-r_h) + \varphi_2(r-r_h)^2 + \cdots \quad (84)
$$
It is important to emphasize that, what is generally called near horizon geometry for a non-extremal black hole is just a truncation of the above series expansion. To compute the shear viscosity, though, we need also some data at the order \((r - r_h)^2\).

Another observation is that, in principle, one can use a boost transformation in \(z\) direction to set \(p = 0\) (see [19]). However, the boost transformation is singular at some point outside the black hole horizon. In our analysis we keep the value of \(p\) non-zero and determine it in terms of other horizon data. We will see in section 6.3 that the expressions for the entropy, shear viscosity, and their ratio remain unchanged if we set the horizon value of \(P'(r)\) to be zero (in other words, they do not depend of \(p\)). This is expected due to the fact that the physical quantities should be invariant under the boost transformations.

By substituting the ansatz (84) in the field equations, we get the following expressions for the coefficients at the order \((r - r_h)^2\):

\[
\begin{align*}
v_1 &= -\frac{2B^2 e^{\alpha \varphi_h} + r_h^4 (q^2 e^{\alpha \varphi_h} - 6)}{6\pi Tr_h^4}, \\
w_1 &= \frac{4B^2 e^{\alpha \varphi_h} - r_h^4 (4q^2 e^{\alpha \varphi_h} + 3z_1^2 r_h^2 - 24)}{24\pi Tr_h^4}, \\
p &= \frac{q (2B\zeta e^{-\alpha \varphi_h} + z_1 r_h^3)}{4\pi Tr_h}, \\
\varphi_1 &= \frac{\alpha e^{\alpha \varphi_h} (q^2 r_h^2 - B^2)}{4\pi Tr_h^4}, \\
q_1 &= \frac{e^{-2\alpha \varphi_h}}{16\pi Tr_h^4} \left( 2B^2 \left[ (\alpha^2 + 2) e^{2\alpha \varphi_h} + 4\zeta^2 \right] - r_h^4 e^{2\alpha \varphi_h} [2q^2 (\alpha^2 - 2) e^{\alpha \varphi_h} + z_1^2 r_h^2 + 24] \right).
\end{align*}
\] (85)

However, in higher derivative gravity theories the only coefficient at the order \((r - r_h)^2\), which we need for viscosity bound computation, is \(u_2\). But, for completeness, we present the expressions of all the other coefficients that appear at order \((r - r_h)^2\), in Appendix D.

A non-extremal charged scalar black hole is characterized by four independent parameters: the mass, electric charge, magnetic field, and also the value of the scalar at the horizon, \(\varphi_h\). In this case, the horizon radius (and so the entropy) and the horizon value of the scalar depend of the asymptotic boundary data \(\varphi_\infty\). We will see in the next subsection that this is in contrast with the extremal case for which we obtain an attractor behavior at the horizon.

At first sight, it may seem surprising that the data (85, 246) we need to compute the entropy and shear viscosity depend also on \(z_1\), a coefficient that we do not compute explicitly. However, we will see in the next subsection that the final values of the physical quantities depend in fact just on four independent parameters, namely \((q, B, r_h, \varphi_h)\) — we ‘trade’ the mass for the horizon radius and so the independent parameters that completely characterize the black hole are the ones mentioned above.

\footnote{We obtain the results as functions of the coefficients \((T, q, B, \varphi_h, r_h, \text{and} \ z_1)\) — this will simplify the computations of the shear viscosity.}
6.3 Shear Viscosity to Entropy Density Ratio

Interestingly, as has already been observed that only with the knowledge of the near horizon geometry one can easily calculate the shear viscosity of boundary fluid. One does not need to know the full analytic solutions of Einstein equations — this method is especially useful in higher derivative AdS gravity theories.

At first sight, it seems that the dual gravitational mode (4.1) does not generally decouple. Interestingly enough, the decoupling occurs when the momentum vanishes and this is what we need for the computation of the viscosity in the hydrodynamics limit — a detailed derivation of this claim has been provided in Appendix E.

By plugging (31) in the action and keeping the terms at order $\epsilon^2$ (at the first order in $\epsilon$, we obtain the equations of motion for gravitons), we get the effective action for the perturbation of the form equation (38):

\[ S = \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} dr [A_1(r,k)\phi'(r,k)\phi'(r,-k) + A_0(r,k)\phi(r,k)\phi(r,-k)], \]  

(86)

where

\[ A_1(r,k) = -\frac{1}{2} e^{2V(r)+W(r)}U(r), \quad A_0(r,k) = \frac{e^{2V(r)+W(r)}\omega^2}{2U(r)}. \]  

(87)

At this point, it is important to emphasize that there are many total derivatives in this action that do not affect the equations of motion for the graviton. For the computation of the imaginary part of the two-point function, the coefficient of the term $\phi'\phi'$ in the bulk action is important. The other total derivatives in the bulk action and the Gibbons-Hawking surface term contribution exactly cancel on the boundary [21]. It was also shown [21] (straightforward to check for present case) that, in the case of Einstein gravity, the ratio of viscosity and entropy density is not affected when the matter fields are minimally coupled. The effective coupling [22, 21] is

\[ K_{eff} = \frac{1}{16\pi G_5} \frac{A_1(r,k)}{\sqrt{-g_{rr}}} = -\frac{1}{32\pi G_5} \]  

(88)

and so the viscosity coefficient of the boundary fluid stress tensor is

\[ \eta = e^{2V(r_h)+W(r_h)}(-2K_{eff}(r_h)). \]  

(89)

In this case, the shear viscosity to entropy density ratio turns out to be universal, namely

\[ \frac{\eta}{s} = \frac{1}{4\pi}. \]  

(90)

\[ ^{14}\text{The terms that contain the derivatives with respect to the spatial coordinates, } \vec{\xi}, \text{ combine in terms whose coefficient is proportional with } \vec{p}^2. \text{ Since we work in the hydrodynamic approximation } \vec{p} = 0, \text{ these terms do not play any role in our analysis.} \]
6.3.1 Four derivative action

Let us now consider the action (64) supplemented with the most general four-derivative interactions \[23\]:

\[
S_{HD} = S_{EM} + \frac{\alpha'}{16\pi G_5} \int d^5 x \sqrt{-g} \left[ c_1 R_{abcd} R^{abcd} + c_2 R_{abcd} F^{ab} F^{cd} + c_3 (F^2)^2 \right. \\
\left. + c_4 F^4 + c_5 \epsilon^{abdec} A_a R_{bcfg} R_{defg} \right].
\]  

Since in supergravity actions the gauge kinetic terms couple to various scalars, it will be interesting to understand the role of the moduli in computing the viscosity bound. Unlike [23], our action contains a scalar, \(\phi\), and the coefficients \(c_i\) depend on the value of \(\phi\). This resembles the four-derivative supergravity action [24].

We treat the higher derivative terms perturbatively and apply the effective action method of section 5.3 to compute the shear viscosity coefficient of the boundary fluid. However, to obtain the viscosity bound we also need the entropy density. We start by using the Noether charge formalism of Wald [25] (see, also, [26, 20] for a discussion in AdS) to compute the entropy density — we will need just the data in Section 3.1 and Appendix B.

When we add higher derivative corrections to the action, the entropy is no longer given by the area law — instead, we use a general formula proposed by Wald

\[
s = -2\pi \int_{\mathcal{H}} \frac{\partial L}{\partial R_{abcd}} \epsilon_{ab} \epsilon_{cd},
\]  

where \(\epsilon_{ab}\) is the binormal to the surface \(\mathcal{H}\).

By using (92), we obtain the following expression for the entropy density:

\[
s = \frac{r_h^3}{4G_5} \alpha' + \frac{r_h^3}{4G_5} \left[ c_1 (3z_1 r_h^2 - 4u_2) - 2c_2 q^2 \right] + O(\alpha'^2).
\]  

We use the expression of \(u_2\) given in Appendix B to rewrite this expression as

\[
s = \frac{r_h^3}{4G_5} - \frac{\alpha' r_h^3}{G_5} \left[ \frac{c_1}{3} \left( \frac{5B^2}{r_h^4} e^{\alpha \varphi_h} + 7q^2 e^{\alpha \varphi_h} - 6 \right) + \frac{c_2}{2} q^2 \right] + O(\alpha'^2).
\]  

As expected, the entropy density depends on four independent parameters, namely \((r_h, q, B, \varphi_h)\). Since in Wald formula only the four derivative interactions that involve the curvature tensor are important, the entropy only depends on \(c_1\) and \(c_2\) (\(c_5\) does not appear because the binormal has just \(r\)t components and the contribution from this term vanishes).

To compute the four derivative corrections to the shear viscosity coefficient, we have to find the quadratic action for the transverse graviton moving in the background spacetime. As in previous section, we consider again the following metric perturbation

\[
g_{xy} = g^{(0)}_{xy} + h_{xy}(r, x) = g^{(0)}_{xy} \left[ 1 + \epsilon \Phi(r, x) \right],
\]
where $\epsilon$ is an order counting parameter. We find the effective action for transverse graviton following the prescription described in previous section. From that effective action one can extract the form of effective coupling.

Evaluating the effective coupling in the near horizon we obtain the shear viscosity coefficient

$$\eta = \frac{1}{16\pi G_5} + \alpha' \left( c_1 r_h^4 \left( 8 q^2 e^{\alpha \phi_h} + 3 z_1^2 r_h^2 - 32 \pi T v_1 - 36 \pi T w_1 - 10 u_2 + 48 \right) - B^2 \left( c_2 - 8 c_1 e^{\alpha \phi_h} \right) \right)$$

$$+ \mathcal{O}(\alpha'^2)$$

which can be rewritten as (we use the results in Appendix [D])

$$\eta = \frac{r_h^3}{16\pi G_5} - \frac{\alpha' r_h^3}{2\pi G_5} \left[ c_1 \left( q^2 - \frac{B^2}{r_h^4} \right) e^{\alpha \phi_h} - 6 \right] + \mathcal{O}(\alpha'^2).$$

The ratio of the shear viscosity and entropy density turns out to be

$$\frac{\eta}{s} = \frac{1}{4\pi} + \frac{\alpha'}{\pi} \left[ \frac{c_1}{3} \left( q^2 - \frac{B^2}{r_h^4} \right) e^{\alpha \phi_h} - 6 \right] + \mathcal{O}(\alpha'^2).$$

In $B \to 0$ limit this result matches with [23].

Let us end up this section with a discussion of the extremal limit. In the absence of the moduli, the extremality condition is $2B^2 + r_h^4 (q^2 - 6) = 0$ and so the shear viscosity to entropy density ratio becomes

$$\frac{\eta}{s} = \frac{1}{4\pi} + \frac{\alpha'}{\pi} \left[ -c_1 \frac{B^2}{r_h^4} + \frac{3c_2}{2} \left( 2 - \frac{B^2}{r_h^4} \right) \right] + \mathcal{O}(\alpha'^2).$$

Therefore, there is a drastic change when the magnetic field is turned on. That is, unlike the electrically charged solution studied in [23], the leading correction of $\eta/s$ in the extremal limit depends on both, $c_1$ and $c_2$. As expected, in $B \to 0$ limit our result matches with the one of [23].

Thus we see, with the knowledge of near-horizon geometry we have been able to compute the ratio $\eta/s$ of a boundary system which is dual to some non-trivial bulk theory, whose analytic solution is not known completely.

In the next section we will discuss about the transport coefficients, which appear at the second order of derivative expansion in energy momentum tensor. Other examples of higher derivative correction to first order transport coefficients have been discussed in section 8.

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Note that our ‘$q$’ is different than ‘$q$’ of [23]. In [23], $q$ is the physical charge (up to some normalization). In our case, the physical charge is $\sim t_h^6 q$. 

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26
7 Second order Hydrodynamics

In the introduction section we have discussed that hydrodynamics is an effective description of field theory in terms of derivative expansions of its local variables. One can write the energy-momentum tensor of the system in terms of derivative expansion. We have written the form of energy-momentum tensor up to first order in derivative expansion in equation (3). In this section we discuss about the second order expansion of energy-momentum tensor. At the second order we encounter many more transport coefficients. Before discussing the holographic method to compute these transport coefficients we must mention that second order hydrodynamic description is necessary in a relativistic theory, as the first order description is not consistent with causality issues [28, 29, 30]. For example, consider a fluid system with some conserved current \( J^\mu \). The current satisfies the conservation equation and diffusion equation,

\[
\partial_\mu J^\mu = 0, \quad J_i = -D \partial_i \rho, \quad (100)
\]

where \( D \) is the diffusion constant and \( \rho \) is corresponding charge density. Using these two equations one can write,

\[
\partial_t \rho - D \nabla^2 \rho = 0. \quad (101)
\]

This is a parabolic equation (first order in \( t \) and second order in \( x_i \)'s), and does not satisfy causality (see [28, 29, 30] for details). In order to restore causality one needs to make the diffusion equation hyperbolic. The only possible way to do that is to modify the Fick’s law, i.e. to introduce a new term in the second equation of (100),

\[
J_i = -D \partial_i \rho - \tau \partial t J_i. \quad (102)
\]

The coefficient \( \tau \) is called the relaxation time. Now using equation (102) and conservation equation we can write,

\[
\tau \partial_t^2 \rho + \partial_t \rho - D \nabla^2 \rho = 0. \quad (103)
\]

This is hyperbolic equation. A propagating solution \( \rho = \exp[i\omega t + iqz] \) leads to dispersion relation

\[
-\tau \omega^2 - i\omega + Dq^2 = 0. \quad (104)
\]

Therefore in \( q \to \infty \) limit, the wavefront velocity turns out to be \( v \sim \sqrt{D/\tau} \), which is consistent as long as \( v < c \). Thus, the second order terms in diffusion equation or dispersion relation saves the causality problem which appear at the first order hydrodynamics (i.e. if we consider first order terms only).

Like the conserved current \( J^\mu \) the first order energy momentum tensor also faces the same problem. Therefore one has to add higher derivative terms to energy-momentum tensor of a relativistic fluid. From the conformal symmetry of the system the form of the second order energy momentum tensor can be determined (like first order). In this article we will not discuss about the derivation of the second order EM tensor rather, we start with the expression. The derivation can be found in [16, 27, 28, 29]. In [27], the author has developed a Weyl-covariant formalism which simplifies the study of conformal hydrodynamics.

\[\text{In appendix G.1 we have briefed the method of [28].}\]
The energy-momentum tensor up to second order in derivative expansion is given by,

\[
T^\mu_\nu = (\epsilon + P)u^\mu u_\nu + Pg^{\mu_\nu} - \sigma^{\mu_\nu} + \Theta^{\mu_\nu} \\
\Theta^{\mu_\nu} = \eta \tau \left[ (D\sigma^{\mu_\nu}) + \frac{1}{d-1} \sigma^{\mu_\nu}(\nabla \cdot u) \right] + \kappa \left[ R^{(\mu_\nu)} - (d-2)u_\alpha R^{(\mu_\nu)\beta_\alpha} u_\beta \right] + \lambda_1 \sigma^{(\mu} \sigma^{\nu)} + \lambda_2 \sigma^{(\mu_\lambda} \Omega^{\nu_\lambda)} + \lambda_3 \Omega^{(\mu_\lambda} \Omega^{\nu_\lambda)}.
\]

The notations are following,

\[
\Omega^{\mu_\nu} = \frac{1}{2} \Delta^{\mu_\alpha} \Delta^{\nu_\beta} (\nabla_\alpha u_\beta - \nabla_\beta u_\alpha),
\]

\[
\Delta^{\mu_\nu} = u^\mu u^\nu + g^{\mu_\nu},
\]

\[
D = u^\mu \nabla_\mu
\]

and for a second rank tensor tensor \(A^{\mu_\nu}\),

\[
\langle A^{\mu_\nu} \rangle \equiv \frac{1}{2} \Delta^{\mu_\alpha} \Delta^{\nu_\beta} (A_{\alpha_\beta} + A_{\beta_\alpha}) - \frac{1}{d-1} \Delta^{\mu_\nu} \Delta^{\alpha_\beta} A_{\alpha_\beta} \equiv A^{<\mu_\nu>}
\]

It is an easy exercise to check that \(\langle A^{\mu_\nu} \rangle\) is transverse and traceless i.e. \(u^{<\mu} A^{\mu_\nu}> = 0\) and \(g^{<\mu_\nu}> = 0\). Thus \(T^{\mu_\nu}\) is also transverse and traceless up to second order in derivative expansion.

We see that, we encounter five second order transport coefficients which are named \(\tau \Pi, \kappa, \lambda_{1,2,3}\) after \([29]\). For non-conformal fluid there are eight other transport coefficients appear in the second order.

Note that if we consider the hydrodynamic system in flat space, then \(\kappa\) term does not appear, as the Ricci and Riemann tensor vanishes for flat metric. However, \(\kappa\) contributes to the two-point Greens function of the stress-energy tensor (please look at equation \((108)\)). The \(\lambda_{1,2,3}\) terms are nonlinear in velocity. They also do not appear when we consider small perturbations and thus do not appear in Green’s function. The parameter \(\tau \Pi\) has dimension of time and can be thought of as the relaxation time. This \(\tau \Pi\) term is enough to get rid of the causality problem.

The main goal of this section is to discuss a holographic technique to compute the second order transport coefficients. If we follow the analysis given in section \([4]\), we find that the expression for Green’s function up to second order in frequencies is given by,

\[
G^{xy,xy}_R(\omega, k) = p - i\eta \omega + \eta \tau \omega^2 - \frac{\kappa}{2} [(d-3)\omega^2 + k^2] + \mathcal{O}(k^3).
\]

One important thing to note here (also mentioned earlier) that other three transport coefficients do not appear in the expression of Green’s function. Therefore we can not compute those three transport coefficients calculating retarded Green’s function. One needs to adopt different holographic methods \([28, 29]\).

In this section we study the radial evolution of response function defined by equation \((113)\). The retarded Green’s function of boundary fluid is given by the asymptotic value of this response function. We show that, at low frequencies, the evolution of this response function is
independent of the radial direction and hence it can be computed either at the horizon or at the boundary. Thus, computing the response function at horizon, the shear-viscosity coefficient, which is a first-order transport coefficient of the boundary plasma, can be obtained from the characteristics of the membrane fluid. In order to completely specify the boundary plasma, it is necessary to understand its higher-order transport coefficients and one needs to move away from the low frequency limit. In this case, the response function flows non-trivially with the radial direction and depends on the full black hole geometry. Hence, although the boundary plasma and the membrane fluid have the same shear-viscosity coefficients, other transport coefficients (higher order) can differ and it is not clear how the two are related. One possibility is that the higher-order coefficients of the two are related by renormalization-group flow equations.

We again look at leading Einstein-Hilbert (E-H) action with a negative cosmological constant in 4+1 dimensions and study the motion of a transverse graviton in this background\textsuperscript{17}. The action and the solution is given by, equations (29) and (30).

We study the graviton’s fluctuation in this background like equation (31). Here we use Fourier transform to work in the momentum space $k = \{-\omega, \vec{k}\}$. From this action, we can find the conjugate momentum $\Pi(r, k_\mu)$ of the transverse graviton (for r-foliation) and the equation of motion from action given in equation (32)\textsuperscript{18},

\[ \Pi(r, k_\mu) = 2 A^{(0)}_1(r, k) \phi'(r, k), \quad \Pi'(r, k_\mu) - 2 A^{(0)}_0(r, k) \phi(r, k) = 0. \]  

(109)

The on-shell action reduces to the following surface term\textsuperscript{19},

\[ S = \sum_{r=0,1} \int \frac{d^4k}{(2\pi)^4} (A^{(0)}_1(r, k) \phi'(r, k) \phi(r, -k)). \]  

(110)

Following the AdS/CFT prescription given in \textsuperscript{8} (also look at section 5), the boundary retarded Green’s function is given as,

\[ G_R(k_\mu) = \lim_{r \to 0} \frac{2 A^{(0)}_1(r, k) \phi'(r, k) \phi(r, -k)}{\phi_0(k) \phi_0(-k)}, \]  

(111)

where, $\phi_0(k_\mu)$ is the value of the graviton fluctuation at boundary. Full solution of the graviton can be written as $\phi(r, k_\mu) = \phi_0(k_\mu) F(r, k_\mu)$, where $F(r, k_\mu)$ goes to identity at the boundary. We can rewrite the boundary retarded Green’s function as,

\[ G_R(k_\mu) = \lim_{r \to 0} \frac{\Pi(r, k_\mu)}{\phi(r, k_\mu)}. \]  

(112)

Let us define a response function of the boundary theory as\textsuperscript{20},

\[ \tilde{\chi}(k_\mu, r) = \frac{\Pi(r, k_\mu)}{i\omega \phi(r, k_\mu)} \]  

(113)

\textsuperscript{17}Here we restrict ourselves to five space-time dimensions, but the discussions are quite generic and can be extended to arbitrary dimensions.

\textsuperscript{18}The factor $1/16\pi G_5$ has been absorbed in $A^{(0)}_1$.

\textsuperscript{19}Please look at \textsuperscript{31} for detailed discussion on other boundary terms.

\textsuperscript{20}We set the zero frequency part of $G$ to zero, as it gives contact terms.
where $\omega = k_0$. This function is defined for all $r$ and $k_\mu$. Therefor the boundary Green’s function is given by,

$$G_R(k_\mu) = \lim_{r \to 0} i\omega \tilde{\chi}(k_\mu, r).$$  \hfill (114)

We will study the radial evolution of the response function $\tilde{\chi}(k_\mu)$ from horizon to boundary. Differentiating equation (113) and using the equations of motion (109) we get,

$$\partial_r \tilde{\chi}(k_\mu, r) = i\omega \sqrt{-g} \frac{g_{rr}}{g_{tt}} \left[ \frac{\tilde{\chi}(k_\mu, r)^2}{\Sigma(r)} - \frac{\Upsilon(r)}{\omega^2} \right],$$  \hfill (115)

where we define

$$\Sigma(r) = -2A_1^{(0)}(r, k_\mu) \sqrt{-g} \frac{g_{rr}}{g_{tt}}, \quad \Upsilon(r) = 2A_0^{(0)}(r, k_\mu) \sqrt{-g}.$$  \hfill (116)

As mentioned earlier, the flow equation (115) is valid for any value of momentum. This is a first order differential equation and we need to specify one boundary condition to solve this equation. That naturally comes from the behavior of the equation at the horizon. Demanding the solution to be regular at the horizon, we get the following condition,

$$\tilde{\chi}(k_\mu, r)^2 \bigg|_{r=1} = \Sigma(r) \Upsilon(r) \bigg|_{r=1}. \quad (117)$$

For two derivative gravity this boundary condition implies that

$$\tilde{\chi}(k_\mu, 1) = -\sqrt{\frac{\Sigma(1) \Upsilon(1)}{\omega^2}} = -\frac{1}{16\pi G_5}$$  \hfill (118)

which is independent of $k_\mu$. Therefore the full momentum response at the horizon corresponds to only to the zero momentum limit of boundary response, $\tilde{\chi}(k_\mu, 1) = \tilde{\chi}(k_\mu \to 0, r \to 0)$. \hfill (22)

With this boundary condition, one can integrate out the differential equation (115) from horizon to asymptotic boundary and obtain the AdS/CFT response for all momentum $k_\mu$. In particular, it is trivial to see that at $(\omega, k_i) \to 0$ limit, the flow is trivial $\partial_r \tilde{\chi}(k_\mu, r) = 0$ and using the boundary condition in equation (117) we get the first order transport coefficient of boundary fluid, i.e. the shear viscosity coefficient turns out to be $\eta = \frac{1}{16\pi G_5}$.

In this section, we will go away from $(\omega, k_i \to 0)$ limit. As we have already mentioned, it is possible to integrate the flow equation for any momentum (perturbatively) and we can easily find the higher order transport coefficients. The usual Kubo approach to compute these coefficients requires the full profile of the transverse graviton in black hole background background (solving a second order differential equation), where as, using the flow equation, one can get these transport coefficients without explicit knowledge of the graviton’s profile.

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\(^{21}\)We choose the negative brunch. The sign of the boundary condition in equation (117) depends on the choice of coordinate. In our coordinate the boundary is at $r \to 0$ hence we need to choose the negative branch.

\(^{22}\)However, in higher derivative gravity we will see that the $\tilde{\chi}(k_\mu, 1)$ depends on spatial momentum.
7.1 A renormalized response function

When we solve the flow equation (115) to get the boundary response function in general it involves divergence at the boundary \((r \to 0)\). These are usual UV divergences and to remove them we need to re-normalize the response function properly.

We follow the holographic renormalization prescription of \([32, 16]\). As the graviton is massless, we only need to add the following counterterm to the graviton’s action,

\[
S_C = \frac{1}{16\pi G_5} \int_{r=\delta} d^4x \sqrt{-\gamma} \frac{1}{4} \Phi(\epsilon, x) \Box \Phi(\epsilon, x).
\]

(119)

In momentum space,

\[
S_C = \frac{1}{64\pi G_5} \int_{r=\delta} \frac{d^4k}{(2\pi)^4} \sqrt{-\gamma} \phi(\delta, k)(g^{tt}\omega^2 + k_i k^i)\phi(\delta, -k).
\]

(120)

Therefore the renormalized Green’s function is given by,

\[
G_R = \lim_{r \to 0} \left[ \Pi(r, k_\mu) + \sqrt{-\gamma} \frac{g^{tt}\omega^2 + k_i k^i}{32\pi G_5} \right].
\]

(121)

However we will study the flow of un-renormalized response function defined in equation (113) and we define our renormalized response function as,

\[
\bar{\chi}_{\text{Ren}}(r, k_\mu) = \bar{\chi}(r, k_\mu) + \frac{1}{i\omega} \sqrt{-\gamma} \frac{g^{tt}\omega^2 + k_i k^i}{32\pi G_5}.
\]

(122)

The counter term will cancel the UV divergences appearing in the expression of \(\bar{\chi}\) and we will get a finite result at the boundary, i.e. \(\lim_{r \to 0} \bar{\chi}_{\text{Ren}}(r, k_\mu)\) will be finite. From the above analysis, we understand that one can get rid of the UV divergences appearing in the response function by following the holographic renormalization technique. But, an important observation is, this counter term does not add any finite contribution to the result, it only cancels out the divergences. Thus, one can study the flow of the un-renormalized response function and ignore the divergences piece to get the finite contribution at the boundary.

7.2 Second order transport coefficients from flow equation

In this subsection, we compute the higher order transport coefficients by solving the flow equation (115) perturbatively up to order \(\omega^2\) and \(k_i^2\). This is a non-linear first order differential equation. Now, the right hand side of this equation is proportional to \(\omega\). Hence, to solve \(\bar{\chi}\) to order \(\omega^2\), we can replace the leading order solution for \(\bar{\chi}\) in the right hand side of equation (115). This simplifies the situation a lot as the non-linear equation becomes linear. Now, to leading order, \(\bar{\chi} = -\eta = -\frac{1}{16\pi G_5}\). Therefore up to order \(\omega^2\), we get,

\[
\partial_r \bar{\chi}(k_\mu, r) = i\omega \sqrt{-g_{rr}} \left[ \frac{\eta^2}{g_{tt}} \left( \frac{\Sigma(r)}{\omega^2} - \frac{\Upsilon(r)}{\omega} \right) \right] + \mathcal{O}(\omega^2, k_i^2).
\]

(123)
The integration constant for the equation can be fixed from the boundary condition in equation \(\square\). Putting the value of the constant, the solution takes the form,

\[
\imath \omega \bar{\chi}(k_\mu, 0) = \lim_{r \to 0} \frac{1}{96 \pi G_5 r} \left[ 3q^2 (r - 1) + \omega (3 \omega + r (\omega (\log(8) - 3) + 6i)) \right] + \mathcal{O}(q \omega^2, \omega q^2, q^3, \omega^3)
\]

\[
= -\imath \omega \left( \frac{1}{16 \pi G_5} \right) + \omega^2 \left[ \frac{1}{2} (1 - \ln 2) \left( \frac{1}{16 \pi G_5} \right) \right] - \frac{q^2}{2} \left( \frac{1}{16 \pi G_5} \right) + \mathcal{O}(\frac{1}{r}). \tag{124}
\]

Here we have chosen the four momentum to be \(k = \{\omega, 0, 0, q\}\).

This expression has divergence as \(r \to 0\) (UV divergence) and can be removed by adding suitable counter term (as explained in the last section).

Comparing the finite piece of equation (124) at \(r \to 0\) with the generic expansion of the retarded Green’s function in equation (108)\(\square\), we get,

\[
\eta = \frac{T^3 \pi^3}{16 \pi G_5}, \quad \tau = \frac{2 - \ln 2}{2 \pi T}, \quad \kappa = \frac{\eta}{\pi T}. \tag{125}
\]

Here \(T = \frac{1}{\pi}\). These results are in agreement with \cite{28, 29, 30}. Thus, we see that, studying the flow equation of the response function we can compute the higher order transport coefficients perturbatively. Here, we present the results for the second order transport coefficients, but, in general it is possible to go beyond second order.

At this point, it is not clear why only considering the boundary term from action in equation \(\square\) is enough to get the correct results. In usual Kubo approach, one needs to take into account the Gibbons-Hawking term also. But as the action in equation \(\square\) has well defined variational principle, one does not need to add any Gibbons-Hawking term with it. It has been explained in great details in \cite{31} that the boundary terms coming from the original action and the corresponding Gibbons-Hawking action are exactly same as the boundary terms coming from the action in equation \(\square\) up to terms proportional to \(\phi^2\) and pure divergence terms. The \(\phi^2\) terms do not contribute to any transport coefficients\(\square\). The divergent terms will get canceled by the proper counterterms and hence are not important for finding the transport coefficients. Thus it is clear that the effective action will give us the correct transport coefficients for the boundary plasma. This observation also holds for higher derivative gravity theory\(\square\).

### 7.3 Higher derivative correction to flow equation

So far we have discussed the flow equation of two point correlation function of energy-momentum tensor of boundary theory whose gravity dual is given by Einstein-Hilbert action (two derivative action). But it is not obvious how to generalize this for higher derivative case. The construction in the previous section was based on the canonical form of graviton’s action given in equation \(\square\). As discussed in section 5.3, in presence of arbitrary higher derivative terms in the bulk, the general action for the perturbation \(h_{xy}\) does not have the above form as equation \(\square\).

---

\(\square\): The overall sign depends on the choice of coordinate.

\(\square\): They only contribute to pressure of the boundary theory.

\(\square\): In \cite{31}, it has been proved explicitly for \(R^{(n)}\) gravity theory.
Rather it will have more than two derivative (with respect to $r$) terms like $\phi' \phi''$, $\phi''^2$ etc. In presence of these terms it is not possible to bring this action into a canonical form (up to some total derivative terms). In this section we consider generic higher derivatives terms in the bulk Lagrangian. We follow the prescription of section 5.3.2 to construct an effective action "$S_{\text{eff}}$" for transverse graviton in canonical form in presence of generic higher derivative terms in the bulk. The effective action and original action give same equation of motion perturbatively in the coupling of the higher derivative terms.

Let us consider a gravity set-up with $n$ derivative action in equation (36). We write an effective action for transverse graviton in canonical form,

$$S_{\text{eff}} = \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} dr \left[ A_1^{\text{HD}}(r,k) \phi'(r,k) \phi'(r,-k) + A_0^{\text{HD}}(r,k) \phi(r,k) \phi(r,-k) \right]$$

(126)

with some unknown function $A_1^{\text{HD}}$ and $A_0^{\text{HD}}$. We fix these functions by demanding that the equations of motion obtained from the effective action and the original action are same perturbatively in $\mu$ (see section 5.3.2).

The generalized canonical momentum and equation of motion are given by,

$$\Pi^{\text{HD}}(r,k) = 2A_1^{\text{HD}}(r,k) \phi'(r,k), \quad \left( \Pi^{\text{HD}}(r,k) \right)' = 2A_0^{\text{HD}}(r,k) \phi(r,k).$$

(127)

Once we find the effective action for the graviton, we follow the procedure in the previous section to obtain the flow equation for the boundary Green’s function in generic higher derivative gravity.

The boundary Green’s function is given by,

$$G^{\text{HD}}_R(k_\mu) = \lim_{r \to 0} 2A_1^{\text{HD}}(r,k_\mu) \phi'(r,k_\mu) \phi(r,-k_\mu) / \phi_0(k) \phi_0(-k),$$

(128)

which can be written using the definition of canonical momentum as,

$$G^{\text{HD}}_R(k_\mu) = \lim_{r \to 0} \frac{\Pi^{\text{HD}}(r,k_\mu)}{\phi(r,k_\mu)}.$$

(129)

Let us define a response function of the boundary theory in higher derivative theory as,

$$\bar{\chi}^{\text{HD}}(k_\mu, r) = \frac{\Pi^{\text{HD}}(r,k_\mu)}{i\omega \phi(r,k_\mu)}.$$ 

(130)

Therefore the flow equation is given by,

$$\partial_r \bar{\chi}^{\text{HD}}(k_\mu, r) = i \omega \sqrt{-g_{rr} g_{tt}} \left[ \frac{\bar{\chi}^{\text{HD}}(k_\mu, r)^2}{\Sigma^{\text{HD}}(r,k)} - \frac{\Upsilon^{\text{HD}}(r,k)}{\omega^2} \right],$$

(131)

where we define

$$\Sigma^{\text{HD}}(r,k) = -2A_1^{\text{HD}}(r,k_\mu) \sqrt{-g_{rr} / g_{tt}}, \quad \Upsilon^{\text{HD}}(r,k) = 2A_0^{\text{HD}}(r,k_\mu) \sqrt{-g_{tt} / g_{rr}}.$$ 

(132)
This is the flow equation for two point correlation function of energy-momentum tensor in presence of generic higher derivative term in the bulk action. Therefore integrating this equation from horizon to asymptotic boundary one can find the higher derivative correction to the transport coefficients at any order in frequency/momentum.

Like two derivative case here also we need to provide a boundary condition to solve this equation. The response function $\chi^{\text{HD}}(k_\mu, r)$ should be well-defined at horizon. This implies,

$$\chi^{\text{HD}}(k_\mu, r) \bigg|_{r=r_h} = \sqrt{\frac{\gamma^{\text{HD}}(r)}{\omega^2}} \bigg|_{r=r_h}$$

here the horizon is located at $r = r_h$.

One important point to mention here is that unlike two derivative gravity where $\tilde{\chi}(k_\mu, r_h)$ was independent of $k_\mu$, $\chi^{\text{HD}}(k_\mu, r_h)$ can in general depend on $k_\mu$. We will see this explicitly in the next section. Therefore the full momentum response at the horizon may not be able to correspond only to the zero momentum limit of boundary response in higher derivative theory.

Here we have solved the equation (131) numerically and plotted the function in fig. 1. We can see that for non-zero $k$ the horizon value of real part of the response function is different than that for zero momentum.

![Figure 1: Flow of response function for Higher derivative bulk action.](image)

Like two derivative case, the response function in higher-derivative gravity theory also contains UV divergences. We need to add proper counter term following the holographic renormalization procedure to cancel these divergences. A little more thinking also says that in presence of any higher-derivative term in the action the structure of the counterterm remains same as equation (119). Only the overall normalization constant depends on higher-derivative coupling. Thus, similar to the leading gravity, the counterterm in higher derivative gravity also cancels out the divergence and does not add any finite contribution to the boundary response function. One can study the flow equation of the un-renormalized response function and read off the transport coefficients from its finite piece.

After learning the generic techniques to compute the first and second order transport coefficients of strongly coupled boundary fluid in section 5 and 7, we would like to discuss some examples where we can apply these methods to compute them. In the next section we plan to discuss some examples of higher derivative gravity, motivated from string theory.
8 Examples of 2nd Order Transport Coefficients for Higher Derivative Theories

8.1 Eight derivative correction

In this section we apply the effective action approach for eight derivative terms in the Lagrangian. We consider the well known $Weyl^{(4)}$ term. This term appears in type II string theory. Adding this term in the bulk action corresponds to $\frac{1}{\lambda^{3/2}}$ correction in dual large $N$ theory. The five dimensional bulk action is given by,

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left( R + 12 + \mu W^{(4)} \right)$$

where,

$$W^{(4)} = C^{nmnk}C_{pmnq}C_{h}^{rsp}C_{rsk}^{q} + \frac{1}{2} C^{nhmn}C_{pqn}C_{h}^{rsp}C_{rsk}^{q}$$

and the weyl tensors $C_{abcd}$ are given by,

$$C_{abcd} = R_{abcd} + \frac{1}{3} (g_{ad}R_{cb} + g_{bc}R_{ad} - g_{ac}R_{db} - g_{bd}R_{ca}) + \frac{1}{12} (g_{ac}g_{bd} - g_{ad}g_{cb}) R .$$

The background metric is given by [33] (with horizon at $r_0 = 1$),

$$ds^2 = -\frac{(1 - r^2)}{r} \left( 1 + 45 \mu r^6 - 75 \mu r^4 - 75 \mu r^2 \right) dt^2$$

$$+ \frac{1}{4(1 - r^2)r^2} \left( 1 - 285 \mu r^6 + 75 \mu r^4 + 75 \mu r^2 \right) dr^2 + \frac{1}{r} dx^2 .$$

The temperature of this black hole is given by $T = \frac{1}{\pi} (1 + 15 \mu)$ .

8.2 The General Action

Putting the perturbed metric in equation (134) we get the general action for the scalar field $\phi(r,k)$. As mentioned in section 5.3, we see that in presence of higher derivative terms the general action does not have canonical form. The action for $\phi$ is given by,

$$S = \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} \int dr \left[ A_1^W (r,k) \phi(r,k) \phi(r,-k) + A_2^W (r,k) \phi'(r,k) \phi'(r,-k) \\
+ A_3^W (r,k) \phi''(r,k) \phi''(r,-k) + A_4^W (r,k) \phi(r,k) \phi'(r,-k) \\
+ A_5^W (r,k) \phi(r,k) \phi''(r,-k) + A_6^W (r,k) \phi'(r,k) \phi'(r,-k) \right] .$$

Expressions for $A_i^W$ can be found in [10]. Up to some total derivative terms this action can be written as,

$$S = \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} \int dr \left[ A_0^W \phi(r,k) \phi(r,-k) + A_1^W \phi'(r,k) \phi'(r,-k) + A_2^W \phi''(r,k) \phi''(r,-k) \right]$$
where,
\[ A^W_0 = A^W_1(r, k) - \frac{A^W_4(r, k)}{2} + \frac{A^W_5(r, k)}{2} \]
\[ A^W_1 = A^W_2(r, k) - A^W_5(r, k) - \frac{A^W_6(r, k)}{2}, \quad A^W_2 = A^W_3(r, k). \] (139)

With this form of action it is not possible to define the effective coupling constant of transverse graviton or a response function (following equations (34) and (113) respectively). Therefore we find the effective action for transverse graviton \( \phi \) in presence of \( W^4 \) term in the action.

### 8.3 The Effective Action and Shear Viscosity

Following the general discussion in section 5.3.2 we find the effective action for graviton in presence of \( W^4 \) term. We write the effective action for the scalar field in the following way,
\[ S^W_{\text{eff}} = \frac{(1 + \Gamma \mu)}{16 \pi G_5} \int \frac{d^4k}{(2\pi)^4} \left[ (A^{(0)}_1(r, k) + \mu B^W_1(r, k))\phi'(r, -k)\phi'(r, k) \right. \\
\left. + (A^{(0)}_0(r, k) + \mu B^W_0(r, k))\phi(r, k)\phi(r, -k) \right]. \] (140)

The functions \( B^W_0 \) and \( B^W_1 \) are given by,
\[ B^W_0(r, k) = -\frac{\omega^2 (663r^6 - 573r^4 + 75r^2)}{4r^2(r^2 - 1)}, \quad B^W_1(r, k) = \frac{(r^2 - 1)(129r^6 + 141r^4 - 75r^2)}{r}. \] (141)

We set the normalization constant \( \Gamma = 0 \) (see [10] for detailed discussion). Then the effective coupling constant is given by equation (59),
\[ K_{\text{eff}}(r) = \frac{1}{16 \pi G_5} \frac{A^{(0)}_1(r, k) + \mu B^W_1(r, k)}{\sqrt{-gg^{rr}}} = \frac{1}{16 \pi G_5} \left( -\frac{1}{2} \left( 1 + 36\mu r^4(6 - r^2) \right) \right). \] (142)

Therefore the shear viscosity is given by,
\[ \eta = r_0^{-\frac{3}{2}} (-2K_{\text{eff}}(r_0)) = \frac{1}{16 \pi G_5} (1 + 180 \mu), \quad (r_0 = 1) \] (143)

and shear viscosity to entropy density ratio
\[ \frac{\eta}{s} = \frac{1}{4\pi} (1 + 120 \mu) \] (144)

where entropy density \( s \) is given by \( s = \frac{1}{4\pi G_5} (1 + 60 \mu) \) [33]. These results agree with the one in the literature.
8.4 String theory correction to flow equation

Now let us compute string theory corrections to the second order transport coefficients following the general discussion given in section 7.3.

From the effective action for transverse graviton, computed in equation (140), the flow equation is given by,

\[ \partial_r \bar{\chi}^{W^4}(k, r) = i \omega \sqrt{-\frac{g_{rr}}{g_{tt}}} \left[ \frac{\bar{\chi}^{W^4}(k, r)^2}{\Sigma^{W^4}(r, k)} - \frac{\Upsilon^{W^4}(r, k)}{4 \omega^2} \right], \tag{145} \]

where,

\[ \Sigma^{W^4}(r, k) = -2 A_{W^4}^1(r, k) \sqrt{-\frac{g_{rr}}{g_{tt}}}, \quad \Upsilon^{W^4}(r, k) = 2 A_{W^4}^0(r, k) \sqrt{-\frac{g_{tt}}{g_{rr}}} \] \tag{146}

where,

\[ A_{W^4}^1 = A_{W^4}^0 + \mu B_{W^4}^1, \quad A_{W^4}^0 = A_{W^4}^0 + \mu B_{W^4}^0. \] \tag{147}

The explicit expressions for \( \Sigma \) and \( \Upsilon \) can be obtained by using \( A_{W^4}^0 \) and \( A_{W^4}^1 \). From the regularity of \( \bar{\chi}^{W^4}(k, r) \) at horizon we get,

\[ \bar{\chi}^{W^4}(k, 1) = \sqrt{\frac{\Sigma^{W^4}(1)\Upsilon^{W^4}(1)}{\omega^2}} = \frac{r_0^3}{16 \pi G_5} + \frac{\mu r_0}{4 \pi G_5} \left( 45 r_0^2 + 11 q^2 \right). \tag{148} \]

Here, we see that unlike the two-derivative gravity, the horizon value of the response function depends on spatial momenta \( q \). With this boundary condition we solve the flow equation up to order \( \omega^2 \) and \( q^2 \) (ignoring \( \mathcal{O}(\omega q^2) \) term). Here we write the final result \[26\]

\[ i \omega \bar{\chi}^{W^4}(k, 0) = -i(1 + 180 \mu) \frac{r_0^3}{16 \pi G_5} \omega + \left[ \frac{1}{2} \left( - \log(2) \right) + \frac{5}{4} \mu (199 - 66 \log(2)) \right] \frac{r_0^3}{16 \pi G_5} \omega^2 \]

\[ - \frac{1}{2} (1 + 20 \mu) \frac{r_0^3}{16 \pi G_5} q^2 + \mathcal{O}(q^3, q^2\omega^2, q^3, \omega^3). \] \tag{149}

Comparing this result with equation (108) we get

\[ \frac{\eta}{\pi^3 T^3} = 1 + 135 \mu + \mathcal{O}(\mu^2), \quad \kappa = \frac{\eta}{\pi T} (1 - 145 \mu) + \mathcal{O}(\mu^2) \]

\[ \tau \pi T = \frac{2 - \log(2)}{2 \pi} + \frac{375 \mu}{4 \pi} + \mathcal{O}(\mu^2). \] \tag{150}

These results are in agreement with [34] who applied usual Kubo formula to obtain these results. The agreement provides a non-trivial check to this approach of obtaining higher order transport coefficients from the flow equation [115].

\[26k = \{ \omega, 0, 0, q \} \text{ and we ignore the UV divergence piece.} \]
8.5 Four derivative correction

Next, we will concentrate on the generic four derivative corrections to Einstein-Hilbert action. These terms arise in the effective action for the heterotic string theory. In fact, the complete super-symmetrized $R^2$ correction to effective Heterotic string theory is known and one way to obtain it is the super-symmetrization of the Lorentz Chern-Simons terms [35, 36]. This terms also arises in the context of Type IIB string theory [13, 37], where the theory is on $AdS_5 \times X^5$, the compact space $X^5$ being $S^5/Z_2$. The dual theory is $\mathcal{N} = 2Sp(N)$ gauge theory with 4 fundamental and 1 antisymmetric traceless hyper-multiplets. This super-conformal theories arises in the context of $N$D3-branes sitting inside 8 D7-branes coincident on an orientifold 7-plane. In this case, generic four derivative $R^2$ correction comes form the DBI action of the branes.

Here we compute the generic four derivative correction to the second order transport coefficients, the relaxation time $\tau_\pi$ and $\kappa$. We can choose the coefficients of the higher derivative terms to be the four-dimensional Euler density and get pure Gauss-Bonnet correction to these coefficients.

The action

$$I = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left[ R + 12 + \alpha' \left( \beta_1 R^2 + \beta_2 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \beta_3 R_{\mu\nu} R^{\mu\nu} \right) \right].$$

(151)

In particular for Gauss-Bonnet correction, $\beta_1 = 1, \beta_2 = 1, \beta_3 = -4$. One can get rid of the $\text{Ricci}^2$ and $\text{Scalar}^2$ terms by a field redefinition and therefore all physical quantities should depend on the coefficient $\beta_2$ only. Here, we prefer to work with the generic case as it would be easier for us the get the results for pure Gauss-Bonnet combination at every step.

The background solution is given by [38],

$$ds^2 = f(r) dt^2 + \frac{g(r)}{4r^3} dr^2 + \frac{1}{r} dx^2$$

(152)

where $f(r)$ and $g(r)$ are given by,

$$f(r) = r - \frac{1}{r} - 2r(r^2 - 1) \beta_2 \alpha'$$

(153)

and

$$g(r) = \frac{r}{1 - r^2} + \frac{2r(10\beta_1 + (1 - 3r^2)\beta_2 + 2\beta_3)\alpha'}{3(r^2 - 1)}.$$ 

(154)

This is the background metric corrected up to order $\alpha'$. We have fixed the integration constant such that the boundary metric is Minkowskian and the horizon is located at $r = 1$. The temperature of the black brane is given by,

$$T = \frac{1}{\pi} + \frac{10\beta_1 - 5\beta_2 + 2\beta_3}{3\pi} \alpha'.$$

(155)
Similar to the Weyl case, we can write the following effective action for this model,

$$S_{\text{eff}} = \frac{1}{16\pi G_5} \int \frac{d^4 k}{(2\pi)^4} dr \left[ A_{1GB}(r,k)\phi(r,k)\phi'(r,-k) + A_{0GB}(r,k)\phi(r,k)\phi(r,-k) \right]$$

where, $A_{1GB}$ and $A_{0GB}$ are given in appendix C. Now, it is straightforward to write the corresponding flow equation (115) in this case,

$$\partial_r \bar{\chi}^{GB}(k, r) = i\omega \sqrt{-g_{rr} g_{tt}} \left[ \frac{\bar{\chi}^{GB}(k, r)}{\Sigma^{GB}(r, k)} - \frac{\Upsilon^{GB}(r, k)}{\omega^2} \right],$$

where we define

$$\Sigma^{GB}(r, k) = -2A_{1GB}(r, k) \sqrt{-g_{rr} g_{tt}} \quad (158)$$

$$\Upsilon^{GB}(r, k) = 2A_{0GB}(r, k) \sqrt{-g_{tt} g_{rr}} \quad (159)$$

Now, the boundary condition (117) takes the following form,

$$\bar{\chi}^{GB}(k, 1) = \frac{1}{16\pi G_5} \left[ 1 + \left( (q^2 - 8) \beta_3 - 40\beta_1 \right) \alpha' \right]. \quad (160)$$

As mentioned earlier, we see that even in this case, the boundary condition depends on spatial momenta $q$ through the coefficient $\beta_3$. With this boundary condition, one can solve the flow equation (157) and the solution is given by,

$$i\omega \bar{\chi}^{GB}(k, 0) = \frac{1}{16\pi G_5} \left[ -i(1 - (40\beta_1 + 8\beta_3) \alpha')\omega + \frac{\omega^2}{2} \left( 1 - \log 2 \right) + \frac{\alpha'}{6}(130\beta_1(\log 2 - 1) - \beta_2(5\log 2 - 2) + 26\beta_3(\log 2 - 1)) - \frac{q^2}{2} \left[ 1 - \frac{1}{3}(130\beta_1 + 25\beta_2 + 26\beta_3)\alpha' \right] \right]$$

$$+ \mathcal{O}(q\omega^2, \omega q^2, q^3, \omega^3). \quad (161)$$

From this expression we get the following transport coefficients,

$$\eta = \frac{1}{16\pi G_5} (1 - 8(5\beta_1 + \beta_3) \alpha') + \mathcal{O}(\alpha'^2). \quad (162)$$

This matches with results in [13, 39, 40]. Now, we find the higher order coefficients,

$$\kappa = \frac{\eta}{\pi T} (1 - 10\beta_2 \alpha') + \mathcal{O}(\alpha'^2)$$

$$\tau_{\pi T} = \frac{2 - \ln 2}{2\pi} - \frac{11\beta_2}{2\pi} \alpha' + \mathcal{O}(\alpha'^2). \quad (163)$$
As we can see, the physical quantities $\eta/s, \kappa, \tau T$ only depend on the coefficient $\beta_2$. In particular to Gauss-Bonnet combination, the corrections are,

$$\kappa = \frac{\eta}{\pi T} (1 - 10\alpha') + \mathcal{O}(\alpha'^2)$$

(164)

$$\tau T = \frac{2 - \ln 2}{2\pi} - \frac{11}{2\pi} \alpha' + \mathcal{O}(\alpha'^2).$$

(165)

Thus, we show that studying the flow of the response function (constructed from the effective action), we can find out all higher order transport coefficients systematically.

### 8.5.1 Exact result for Gauss-Bonnet black hole

As we have done the above computation perturbatively, the above expressions are valid only at order $\alpha'$. But, one can consider the Gauss-Bonnet term exactly in coupling. For pure Gauss-Bonnet combination the equations of motion remain second order differential equation and hence it is easy to solve exactly to find the background space-time. We solve the flow equation in this background exactly in coupling constant, and find the exact expressions for relaxation time $\tau_\pi$ and $\kappa$. In this section we briefly outline the result.

The action and the solution is given by,

$$\mathcal{I}_{GB} = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left[ R + 12 + \frac{\lambda_{gb}}{2} \left( R^2 + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} \right) \right]$$

$$ds^2 = r^2 \left( -\frac{f(r)}{f_\infty} dt^2 + d\vec{x}^2 \right) + \frac{dr^2}{r^2 f(r)}$$

(166)

where,

$$f_{\pm}(r) = \frac{1}{2\lambda_{gb}} \left[ 1 \pm \sqrt{1 - 4\lambda_{gb} \left( 1 - \frac{r_0^4}{r^4} \right)} \right]$$

(167)

and

$$f_\infty = \lim_{r \to \infty} f(r) = \frac{1 - \sqrt{1 - 4\lambda_{gb}}}{2\lambda_{gb}}.$$  

(168)

In this coordinate the boundary metric is $\eta$. We also consider only the $'-'$ branch of $f_{\pm}$ which corresponds to a non-singular black hole solution with non-degenerate horizon.

The black hole temperature is given by,

$$T = \frac{r_0}{\pi f_\infty}.$$  

(169)

With exact GB, the effective action for fluctuation has a canonical form. Therefore we derive the flow equation for the response function (as we did in section 7) and solving this
equation we get,
\[ \eta = \frac{1}{16\pi G_5} (1 - 4\lambda_{gb}), \quad \kappa = \frac{2\lambda_{gb} (8\lambda_{gb} - 1)}{(1 - \sqrt{1 - 4\lambda_{gb}}) (4\lambda_{gb} - 1)} \]
\[ \tau T = \frac{1}{4\pi (-1 + 4\lambda_{gb})} \left[ -8\lambda_{gb}^2 + 12\sqrt{1 - 4\lambda_{gb}}\lambda_{gb} + 10\lambda_{gb} - 2\sqrt{1 - 4\lambda_{gb}} - 4 \log(2)\lambda_{gb} ight. \\
+ (1 - 4\lambda_{gb}) \log \left( -4\lambda_{gb} + \sqrt{1 - 4\lambda_{gb}} + 1 \right) + (4\lambda_{gb} - 1) \log (1 - 4\lambda_{gb}) - 2 + \log(2) \right]. \]

One can easily check that up to first order in \( \lambda_{gb} \), the results in (170) reduces to the one in (164). In [41] the authors obtained the relation between second order transport coefficients and \( \lambda_{gb} \) numerically, however we are able to present the result exactly.

**Violation of bound on shear viscosity to entropy density ratio**

In our units the entropy density \( s \) turns out to be \( s = \frac{1}{4G_5} \) and hence shear viscosity to entropy density ratio turns out to be
\[ \frac{\eta}{s} = \frac{1}{4\pi} (1 - 4\lambda_{gb}), \]
which violates the famous KSS bound for \( \lambda_{gb} > 0 \) [40, 13]. In [42] the authors conjectured a bound on shear viscosity to entropy density ratio for gauge theory plasmas which have a holographic dual, i.e.,
\[ \frac{\eta}{s} \geq \frac{1}{4\pi}. \]

In [40] the authors argue that when \( \lambda_{gb} \geq \frac{9}{100} \) the theory violates microcausality and is inconsistent. Therefore, for (3+1)-dimensional CFT duals of (4+1)-dimensional Gauss-Bonnet gravity, consistency of the theory requires
\[ \frac{\eta}{s} \geq \frac{1}{4\pi} \cdot \frac{16}{25}. \]

As we have mentioned in introduction that the flow equation is a first order non-linear differential equation but one can reduce this equation to a second order linear differential equation. This second order differential equation is related to the equation of motion for transverse graviton (gauge invariant excitations). Therefore we can use this equation to study causality violation in Gauss-Bonnet gravity. In [41, 43] it was found that to preserve causality of a conformal fluid there exists a bound on second order transport coefficients,
\[ \tau T - 2\frac{\eta}{s} \geq 0. \]

In Fig 2 we plot \( \tau T - 2\frac{\eta}{s} \) for our result and find the following bound on \( \lambda_{gb} \) which is in agreement with [41],
\[ -0.711 \leq \lambda_{gb} \leq 0.113. \]
9 Flow equation for charged black holes

Electrically charged black holes in five dimensions have drawn a lot of interest in the context of AdS/CFT. The electric charge of these black holes are mapped to the global R-charge of the dual field theory. Because of the presence of the electric charges, the thermodynamics and the phase structure of these black holes are rather complicated and also interesting at the same time. There have been a lot of study of thermodynamics and phase transitions of these charged black hole with different horizon topologies (see [44] and references therein).

The goal of the present section is to apply the AdS/CFT correspondence to understand how non-vanishing chemical potentials effect the hydrodynamic behavior of strongly coupled gauge theories. We study the second order hydrodynamics in two cases: (a) Generic R-charge black holes and (b) Charge black holes in higher-derivative gravity.

9.1 R-charged black holes

We consider a conformal field theory with conserved charge (density) in addition to energy and momentum. This is especially an interesting extension of the hydrodynamics of the uncharged fluids.

The second order hydrodynamics of charged fluid has been studied in [45, 46]. They consider Reissner-Nordstrom black hole in five dimensions and found the effect of chemical potential on second order transport coefficients in some limits of chemical potential. One important outcome of their analysis was to find a new non-dissipative contribution to the charge current. However, we consider generic parity preserving $R$-charged black holes with three (unequal) charges (chemical potentials) and find the exact expressions for second order transport coefficients in presence of three chemical potentials. As we have mentioned in the introduction that solving the flow equation (of retarded Green’s function of energy momentum tensor) we can only find two second order transport coefficients whereas in [45, 46] all other second order transport coefficients have been reported.

We consider R-charged black holes in five dimensions. A consistent truncation of $\mathcal{N} = 8$, $D = 5$ gauged supergravity with $SO(6)$ Yang-Mills gauge group, which can be obtained by $S^5$ reduction of type $IIB$ supergravity, gives rise to $\mathcal{N} = 2$, $D = 5$ gauge supergravity with...
The same theory can also be obtained by compactifying eleven dimensional supergravity, low energy theory of M theory, on a Calabi-Yau three folds. The bosonic part of the action of $\mathcal{N} = 2$, $D = 5$ gauged supergravity is given by \cite{44}. We follow the notation of \cite{47}.

\[
\mathcal{I}_{\text{sugra}} = \frac{1}{16\pi G_5} \int d^5 x \sqrt{-g} \left[ R + V(X) - \frac{1}{2} G_{IJ}(X) F^I_{\mu\nu} F^{\mu\nu\gamma} - G_{IJ}(X) \partial_{\mu} X_I \partial^{\mu} X_J \right] + \frac{\xi}{16\pi G_5} \int d^5 x \epsilon^{\mu\nu\rho\sigma\gamma} A_{\mu} F_{\nu\rho} F_{\sigma\gamma}
\]

where, $X^I$'s are three real scalar fields, subject to the constraint $X^1 X^2 X^3 = 1$. $F^I$'s, which are field strengths of three Abelian gauge fields (I,J=1,2,3), and the scalar potential $V(X)$ is given by,

\[
F^I_{\mu\nu} = 2 \partial_{[\mu} A^I_{\nu]}, \quad G_{IJ} = \frac{1}{2} \text{diag} [(X^1)^{-2}, (X^2)^{-2}, (X^3)^{-2}], \quad V(X) = 2 \sum I \frac{1}{X_I}
\]

The three-charge non-extremal STU solution is specified by the following background values of the metric

\[
ds^2 = -\mathcal{H}^{-2/3} f_k dt^2 + \mathcal{H}^{1/3} \left( f_k^{-1} dr^2 + r^2 d\Omega^2_{3,k} \right),
\]

\[
f_k = k - \frac{m_k}{r^2} + r^2 \mathcal{H}, \quad H_i = 1 + \frac{q_i}{r^2}, \quad H = H_1 H_2 H_3,
\]

as well as the scalar and the gauge fields

\[
X^i = \frac{\mathcal{H}^{1/3}}{H_i}, \quad A_i^I = \sqrt{\frac{kq_i + m_k}{q_i}} \left( 1 - H_i^{-1} \right)^{-1}.
\]

The parameter $k$ determines the spatial curvature of $d\Omega^2_{3,k}$: $k = 1$ corresponds to the metric on the three-sphere of unit radius, $k = 0$ - to the metric on $\mathbb{R}^3$. As the Hydrodynamic approximation is valid only in the case of a translational-invariant horizon, in our case we set $k = 0$ and

\[
d\Omega^2_{3,0} \rightarrow (dx^2 + dy^2 + dz^2).
\]

Replacing the radial coordinate $r \rightarrow r_0/\sqrt{r}$, where $r_0$ is the largest root of the equation $f(r) = 0$, the background solution in this new coordinate is given by,

\[
ds^2_{5} = -\mathcal{H}^{-2/3} \left( \frac{\pi T_0}{r} \right)^2 f dt^2 + \mathcal{H}^{1/3} \left( \frac{1}{4f r^2} dr^2 + \mathcal{H}^{1/3} \frac{\pi T_0}{r} \right)^2 (dx^2 + dy^2 + dz^2)
\]

\[
f(r) = \mathcal{H}(r) - r^2 \prod_{i=1}^{3} (1 + \kappa_i), \quad H_i = 1 + \kappa_i r, \quad \kappa_i \equiv \frac{q_i}{r_0^2}.
\]
where $\kappa_i'$s are chemical potentials and

$$T_0 = r_0/\pi.$$  \hfill (183)

The scalar fields and the gauge fields are given by

$$X^i = \frac{\mathcal{H}^{1/3}}{H_i(u)}, \quad A_i^i = \frac{\tilde{\kappa}_i \sqrt{2} u}{LH_i(u)}$$  \hfill (184)

where,

$$\tilde{\kappa}_i = \sqrt{q_i \prod_{i=1}^{3} (1 + \kappa_i)^{1/2}}.$$  \hfill (185)

The Hawking temperature of the background (181) is given by

$$T_H = \frac{2 + \kappa_1 + \kappa_2 + \kappa_3 - \kappa_1 \kappa_2 \kappa_3}{2 \sqrt{(1 + \kappa_1)(1 + \kappa_2)(1 + \kappa_3)}} T_0.$$  \hfill (186)

We perturb the $xy$ component of background metric and the action for transverse graviton is given by,

$$S_{\text{eff}} = \frac{1}{16 \pi G_5} \int \frac{d^4k}{(2\pi)^4} dr \left[ A^Q_1(r,k) \phi'(r,k) \phi'(r,-k) + A^Q_0(r,k) \phi(r,k) \phi(r,-k) \right]$$  \hfill (187)

where,

$$A^Q_1 = -\frac{r_0^2 f(r)}{r}$$  \hfill (188)

and

$$A^Q_0 = \frac{r_0^2}{4r^2} \left( \frac{H_1 H_2 H_3}{f(r)} - q^2 \right).$$  \hfill (189)

Therefore the flow equation is given by,

$$\partial_r \tilde{\chi}^Q(k, r) = i\omega \frac{g_{rr}}{g_{tt}} \left[ \frac{\tilde{\chi}^Q(k, r)^2}{\Sigma^Q(r,k)} - \frac{\Upsilon^Q(r,k)}{\omega^2} \right].$$  \hfill (190)

Solving this equation perturbatively in $\omega$ and $q$ we get

$$\tilde{\chi}^Q(k, r) = -\frac{r_0^2 \prod_{i=1}^{3} (1 + \kappa_i)^{1/2}}{16 \pi G_5} + \frac{i r_0^2 (q^2 - \omega^2)}{2 \omega} \frac{1}{16 \pi G_5} \left( 1 - \frac{1}{r} \right) + \frac{i \omega r_0^2 \prod_{i=1}^{3} (1 + \kappa_i)}{16 \pi G_5 \sqrt{4 P_\kappa + (1 + S_\kappa)^2}} \left[ \frac{1 + S_\kappa - 2 P_\kappa - \sqrt{P_\kappa + (1 + S_\kappa)^2}}{1 + S_\kappa - 2 r S_\kappa - \sqrt{4 P_\kappa + (1 + S_\kappa)^2}} \right] + O(q^2, \omega q^2, q^3, \omega^3)$$  \hfill (191)
where,
\[ S_\kappa = \sum_i \kappa_i, \quad P_\kappa = \prod_i \kappa_i. \]  \hfill (192)

Computing the response function at the boundary (throwing away the divergent piece) we get the following transport coefficients,
\[ \eta = \frac{r_0^3}{16\pi G_5} \prod_i (1 + \kappa_i)^{1/2} \]  \hfill (193)
and
\[ \kappa = \frac{\eta}{\pi T} \frac{1 + S_\kappa/2 - P_\kappa/2}{\prod_i (1 + \kappa_i)} \]
\[ \tau_{\pi T} = \frac{2 + S_\kappa - P_\kappa}{4\pi \prod_i (1 + \kappa_i)} \left[ 2 - \frac{\prod_i (1 + \kappa_i)}{\sqrt{4P_\kappa + (1 + S_\kappa)^2}} \ln \left( \frac{3 + S_\kappa + \sqrt{4P_\kappa + (1 + S_\kappa)^2}}{3 + S_\kappa - \sqrt{4P_\kappa + (1 + S_\kappa)^2}} \right) \right]. \]  \hfill (194)

These are the new results in this paper. It is easy to check that for \( \kappa_i \to 0 \) limit we recover the results in section 7.

To complete the discussion on the second order transport coefficients for \( R \)-charged black holes one should find the flow of Green’s functions for two point correlation functions of \( R \)-currents. As we will mention in section 10 that in presence of finite charges (or chemical potentials) it is very hard to solve the Riccati equation even perturbatively in \( \omega \) and \( q \). We find it very difficult to get any analytic solution for \( R \)-current Green’s function. However we consider a simple model in section 10 and study the flow of \( R \)-current Green’s function numerically.

### 9.2 Charged black holes in higher derivative gravity

In this section, we will study five-dimensional gravity in presence of a negative cosmological constant and coupled to \( U(1) \) gauge field. The model has been studied in [23, 48, 49], the action is given as,
\[ S = \frac{1}{16\pi G_5} \int d^5 x \sqrt{-g} \left[ R + 12 - \frac{1}{4} F^2 + \frac{c_1}{3} \epsilon^{abcde} A_a F_{bc} F_{de} + \alpha' \left( c_1 R_{abcd} R^{abcd} ight. \right. \]
\[ + c_2 R_{abcd} F^{ab} F^{cd} + c_3 (F^2)^2 + c_4 F^4 + c_5 \epsilon^{abcde} A_a R_{bcfg} R^{fg} \left. \right) \].  \hfill (195)

Here, \( F^2 = F_{ab} F^{ab}, F^4 = F_{ab} F^{bc} F_{cd} F^{da} \), and the AdS radius is set to unity. The action includes the Chern-Simon term and also a generic set of four derivative terms. All the four derivative terms will be treated perturbatively in our computation and here \( \alpha' << 1 \) is the perturbation parameter. In [23], it was shown that, within perturbative approach, after using field-redefinition, this is the most generic four derivative action that one can write down. In
this section, we will closely follow their work. The background metric and the gauge field in presence of these higher derivative terms have the following form,

$$ds^2 = -r^2 f(r) dt^2 + \frac{1}{r^2 g(r)} dr^2 + r^2 (dx^2 + dy^2 + dz^2), \quad A = h(r) dt,$$

where,

$$f(r) = f_0(r) (1 + \alpha' F(r)), \quad g(r) = f_0(r) (1 + \alpha' (F(r) + G(r))),
$$

$$h(r) = h_0(r) + \alpha' H(r).$$

(197)

Here $f_0(r), g_0(r)$ and $h_0(r)$ are the solution of the background in absence of the higher-derivative terms in the action and they are given as,

$$f_0 = g_0 = \left(1 - \frac{r_0^2}{r^2}\right) \left(1 + \frac{r_0^2}{r^2} - \frac{Q^2}{r_0^2 r^4}\right), \quad h_0 = \sqrt{3} Q \left(\frac{1}{r_0^2} - \frac{1}{r^2}\right).$$

(198)

Here, $Q$ is related to the physical charge of the system and $r_0$ is the position of the horizon. From (197), it is clear that even in presence of the higher-derivative terms, the horizon remains at $r_0$. The higher-derivative corrections to this background are given by the functions $F(r), G(r)$ and $H(r)$. The form of these functions are given in [23]. We would not write those expressions and refer the reader to that paper.

Using the flow equation, we will study the higher order transport coefficient of the plasma theory dual to this gravity model. For this, we will write the effective action for the metric fluctuation in (31), as we have done in previous sections,

$$S_{\text{eff}} = \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} dr \left[ A_{1CB} (r, k) \phi'(r, k) \phi'(r, -k) + A_{0CB} (r, k) \phi(r, k) \phi(r, -k) \right]$$

(199)

where, $A_{1CB}$ and $A_{0CB}$ are given in appendix F. The corresponding the flow equation (115) for this case with the coefficients $A_{1CB}$ and $A_{0CB}$ is,

$$\partial_r \chi_{CB}(k, r) = i\omega \sqrt{-\frac{g_{rr}}{g_{tt}}} \left[ \frac{\chi_{CB}(k, r)}{\Sigma_{CB}(r, k)} - \frac{\Upsilon_{CB}(r, k)}{\omega^2} \right],$$

(200)

where we define

$$\Sigma_{CB}(r, k) = -2A_{1CB}(r, k) \sqrt{-\frac{g_{rr}}{g_{tt}}}, \quad \Upsilon_{CB}(r, k) = 2A_{0CB}(r, k) \sqrt{-\frac{g_{tt}}{g_{rr}}}. \quad (201)$$

We solve this flow equation to find the effect of higher derivative terms and chemical potential (or charge) on transport coefficient. Here, we present our result for small $Q$ only, though it is possible to find the results for any $Q$.

The boundary condition (117) will take the following form:

$$\chi(k, r_0) = \frac{r_0^3}{16\pi G_5} \left[ 1 - \alpha' \frac{24 c_1 Q^2}{r_0^6} \right].$$

(202)

\footnote{Note that we are working in a different coordinate where $r \to \infty$ is the boundary, therefore we choose the positive branch of the boundary condition (117).}
In this case, the horizon value of the response function is independent of momenta. With this boundary condition, we can solve the flow equation (200), and the solution is given by,

\[
\begin{align*}
\chi(k_\mu, \infty) &= i\omega \frac{r_0^3}{16\pi G_5} \left[ 1 - \alpha' \frac{24c_1 Q^2}{r_0^6} \right] - \omega^2 \frac{r_0^2}{32\pi G_5} \left[ (1 - \ln 2) - \frac{Q^2}{2r_0^6}(3 - \ln 2) \right] \\
&\quad + \frac{\alpha'}{6} \left( c_1(2 - 5\ln 2) - \frac{Q^2}{2r_0^6}(c_1(35 - 58\ln 2 - 48c_2(2 - \ln 2))) \right) \\
&\quad + \frac{q^2 r_0^2}{32\pi G_5} \left[ 1 - \frac{\alpha'}{3} \left( 25c_1 + \frac{Q^2}{r_0^6}(32c_1 + 24c_2) \right) \right] + \mathcal{O}(q^2, \omega q^2, q^3, \omega^3) + \mathcal{O}(Q^4).
\end{align*}
\]

(203)

It is easy to read off the transport coefficients from this expression.

\[
\begin{align*}
\eta &= \frac{r_0^3}{16\pi G_5} \left[ 1 - \alpha' \frac{24c_1 Q^2}{r_0^6} \right] + \mathcal{O}(Q^4) \\
\kappa &= \frac{\eta}{\pi T} \left[ \left( 1 - \frac{Q^2}{2r_0^6} \right) - \alpha' \left( 10c_1 - \frac{Q^2}{3r_0^6}(37c_1 - 48c_2) \right) \right] + \mathcal{O}(Q^4) \\
\tau_\pi T &= \frac{2 - \ln 2}{2\pi} - \frac{Q^2(5 - 3\ln 2)}{4\pi r_0^6} + \alpha' \left[ -\frac{11c_1}{2\pi} + \frac{Q^2}{4\pi r_0^6}(-16c_2 + 5c_1(11 - 4\ln 2)) \right] + \mathcal{O}(Q^4).
\end{align*}
\]

(204)

where, the temperature \( T \) of the system is given by,

\[
T = \frac{r_0}{\pi} \left[ \left( 1 - \frac{Q^2}{2r_0^6} \right) - \frac{\alpha'}{3} \left( 5c_1 + \frac{Q^2}{2r_0^6}(31c_1 + 48c_2) \right) \right] + \mathcal{O}(Q^4).
\]

(205)

We see that the first order as well as the second order transport coefficients coming from retarded Green’s function of energy momentum tensor only depends on two coefficients \( c_1, c_2 \). This feature was observed in [23] for entropy density \( s \) and first-order transport coefficients \( \eta \). The coefficients \( c_3, c_4 \), which parameterize couplings in the four point function of the dual \( U(1) \) current does not play any role in these hydrodynamic coefficients. They should be important for the computation of conductivity, which comes from the Green’s function of the boundary \( R \)-current. Two other coefficients \( \zeta, c_5 \) also do not appear in the expressions. One can find a magnetic brane solution of the action (195) like [19]. In that case it would be interesting to find the effect of magnetic field on transport coefficients.

So far we have discussed the flow equation for energy-momentum tensor. However, in general the fluid can have other conserved currents like \( R \) current, if the theory has a global \( R \) symmetry. Therefore, in the same spirit, one can also discuss the flow of \( R \) current in the context of fluid/gravity correspondence. This is the final topic of our discussion.

\[28\] It would be interesting to study the flow of retarded Green’s function for boundary \( R \)-current. We found it to be difficult to get any analytical solution for response function in presence of finite chemical potential and higher derivative terms. However, it would be nice to know the higher derivative corrections to other second order transport coefficients appear in \( R \) current [15, 40].
10 Flow of retarded Green’s function of boundary $R$ current

Finally, in this section, we study the flow of retarded Green’s function of boundary $R$-current,

$$G^{R}_{i,j}(k) = -i \int dt d^3 x e^{ik \cdot x} \langle [J_i(x), J_j(0)] \rangle$$  \hspace{1cm} (206)

where $J_\mu(x)$ is the CFT current dual to a bulk gauge field $A_\mu$.

In hydrodynamic approximation one can express the current in powers of boundary derivatives. Up to first order in derivative expansion it has the following form,

$$J_\nu = -\kappa P^\alpha_\nu \partial_\alpha \frac{\mu}{T} + \Omega l_\nu + \mathcal{O}(\partial^2)$$  \hspace{1cm} (207)

where, $\kappa$ and $\Omega$ are two first order transport coefficients, $\mu$ is chemical potential, $T$ is temperature and

$$P_{\mu\nu} = u_\mu u_\nu + \eta_{\mu\nu}$$  

$$l_\mu = \epsilon_{\mu}^{\alpha\beta\gamma} u_\alpha \partial_\beta u_\gamma.$$  \hspace{1cm} (208)

The expression of $J_\mu$ up to second order in derivative expansion can be found in \cite{45, 46}. From conformal invariance of the theory it is possible to write all possible second order transport coefficients appear in the expression of $J_\mu$. However, like energy momentum tensor, from the expression of retarded Green’s function it is not possible to compute all the transport coefficients that appear in different order of derivative expansion.

In this section we study the flow equation of retarded Green’s function of boundary $R$-current. Unfortunately we find it difficult to solve the flow equation analytically to extract any transport coefficient. We present our calculation how to write the flow equation for retarded Green’s function of $R$-currents in presence of generic higher derivative terms in bulk Lagrangian and some numerical results.

We start with Einstein-Maxwell action

$$S = \frac{1}{16\pi G_5} \int d^5 x \sqrt{-g} \left( R + 12 - \frac{1}{4} F^2 \right).$$  \hspace{1cm} (209)

Solution is given by equation \cite{196} with $\alpha' = 0$. The temperature of the black hole is given by,

$$T = \frac{r_0}{\pi} \left( 1 - \frac{Q^2}{2r_0^6} \right)$$  \hspace{1cm} (210)

and the chemical potential is given by,

$$\mu = \frac{\sqrt{3} Q}{r_0^2}.$$  \hspace{1cm} (211)
For technical advantage we write the metric and gauge field in a different coordinate. We change the radial coordinate \( r \rightarrow r_0 \sqrt{r} \). In this coordinate the metric and gauge field is given by,

\[
\begin{align*}
  ds^2 &= -\frac{r_0^2 U(r)}{r} \, dt^2 + \frac{dr^2}{4r^2 U(r)} + \frac{r_0^2}{r} (dx^2) \\
  A_t(r) &= E(r)
\end{align*}
\]  

(212)

where,

\[
U(r) = (1 - r)(1 + \frac{Q^2 r^2}{r_0^2}) , \quad E(r) = \frac{\sqrt{3} Q}{r_0^2} (1 - r) .
\]  

(213)

We turn on small fluctuations for the component of gauge fields. Since the \( A_t \) component of the bulk vector is non-vanishing in this background, the perturbations \( A_x \) can couple to the \( tx \) component of graviton. Therefore we also need to consider small metric fluctuations for components \( g_{tx} \). Writing them in momentum space

\[
\begin{align*}
  A_x(r, x) &= \int \frac{d^4 k}{(2\pi)^4} e^{ik.x} A_1(r, k) \\
  g_{tx}(r, x) &= \int \frac{d^4 k}{(2\pi)^4} e^{ik.x} \Phi(r, k)
\end{align*}
\]  

(214)

However, there exists a constraint relation between \( A_x \) and \( g_{tx} \). We use this relation to replace \( g_{tx} \) from equation of motion of \( A_x \).

The on-shell action for gauge field fluctuations are given by,

\[
S_A = \frac{1}{16\pi G_5} \int \frac{d^4 k}{(2\pi)^4} \left[ - r_0^2 U(r) A_x^2(r, k) + \left( \frac{\omega^2}{4r U(r)} - \frac{q^2}{4r} \right) A_x^2(r, k) \right].
\]  

(215)

The current corresponding to \( A_x \) fluctuation is given by,

\[
J_x(r, k) = \frac{\delta S_A}{\delta A_x'(r, k)} = -2r_0^2 U(r) A_x'(r, k).
\]  

(216)

The equation of motion for \( A_x(r, k) \) is given by,

\[
(U(r) A_x'(r, k))' = -\frac{1}{4r_0^2} \left( \frac{\omega^2}{r U(r)} - \frac{q^2}{r} \right) A_x(r, k) - E'(r) \phi'(r, k)
\]

\[
= -\frac{1}{4r_0^2} \left( \frac{\omega^2}{r U(r)} - \frac{q^2}{r} \right) A_x(r, k) + \frac{r E'(r)^2}{r_0^2} A_x(r, k).
\]  

(217)

Using the constraint relation (coming from \( rx \) component of Einstein equations),

\[
\phi'(r, k) = -\frac{r E'(r) A_x(r, k)}{r_0^2}.
\]  

(218)
and the equations of motion one gets,

\[ J'_x(r, k) = \frac{1}{2} \left( \frac{\omega^2}{rU(r)} - \frac{q^2}{r} \right) A_x(r, k) - 2rE'(r)A_x(r, k). \]  \hspace{1cm} (219)

Next we define a response function

\[ \sigma(r, k) = \frac{J_x(r, k)}{i\omega A_x(r, k)}. \]  \hspace{1cm} (220)

Taking the derivative with respect to \( r \) and using the equation of motion we find the flow is given by,

\[ \sigma'(r, k) = \frac{i\omega}{2r_0U(r)} \left[ \sigma(r, k)^2 - \left( \frac{r_0}{r} \left( 1 - \frac{q^2U(r)}{\omega^2} \right) - \frac{4r_0rE'(r)^2U(r)}{\omega^2} \right) \right]. \]  \hspace{1cm} (221)

From the regularity of the response function at the horizon we find the boundary condition is given by,

\[ \sigma(1, k)^2 = 1. \]  \hspace{1cm} (222)

With this boundary condition one can integrate this nonlinear equation to find finite frequency response of boundary Green’s function.

![Figure 3: Flow of R current correlation function.](image)

In presence of generic higher derivative terms in the Lagrangian the on-shell action for fluctuation \( A_x \) may not have canonical form like (215). In that case one has to write an effective action for \( A_x \) like transverse graviton. The effective will have the same form as (215) only the coefficients will depend on coupling constant of higher derivative terms.

We conclude this section by presenting some numerical solutions of the flow equation (221) for both non-extremal extremal black holes in fig. 3.

11 Conclusion and future directions

In this review we have discussed the hydrodynamic behavior of strongly coupled systems in the context of the gauge/gravity correspondence. We have learnt how we can apply the holographic methods to compute different transport coefficients of boundary fluid systems. We have
also elaborated the effect of higher derivative terms, which come from string theory, on these coefficients.

First, we have shown how the shear-viscosity coefficient can be viewed as the effective coupling of transverse graviton in a two derivative theory. Next, we generalized this statement to any arbitrary higher derivative theory. We have developed a procedure to construct an effective action for transverse graviton in canonical form in presence of any higher derivative terms in bulk and showed that the horizon value of the effective coupling obtained from the effective action gives the shear viscosity coefficient of boundary fluid. Our results are valid up to first order in $\mu$ (coefficient of higher derivative term). We discussed two non-trivial examples to check the method. We have considered four derivative and eight derivative ($Weyl^4$) Lagrangian and calculated the correction to the shear viscosity using our method. We found complete agreement between our result and the results obtained using other methods.

In section 6 we have discussed that the shear viscosity to entropy density ratio is controlled by the near horizon geometry of the dual black hole spacetime. Therefore the knowledge of the near-horizon behavior of different bulk fields are enough to compute the ratio. That is why the computations are not very complicated even in the presence of higher derivative terms. We would like to emphasize that there is no ambiguity in defining the overall coefficient of the effective action and the results are consistent. We apply this idea to compute $\eta/s$ in section 6.3 and see that the ratio is controlled by the horizon values of the scalars and so, in the non-extremal case, an operator deformation in the QFT will produce an interpolating non-trivial flow in which the moduli approach the (IR) black hole horizon. In the extremal case, though, the horizon moduli values are fixed and so the shear viscosity to entropy density ratio does not depend on the asymptotic values of the scalars. Therefore, QFTs with different UV fixed points can flow to the same IR fixed point.

One important point to mention here is that the radial independence of retarded Green’s function (response function) depends on the massless properties of transverse graviton $h_{xy}$ at $k \to 0$ limit. We have seen that the mass term is proportional to $k^2$ and it vanishes in low frequency limit. However, in general this observation is not true. If we have other matter fields in the Lagrangian (like magnetic field considered in section 6.1) then all the components of transverse graviton may not be massless. Please look at appendix E. The isotropy (SO(3) symmetry) in spatial direction is lost in presence of a magnetic field. Therefore though $xy$ component of transverse graviton ($h_{xy}$) is massless but other components $h_{xz}, h_{yz}$ are not decoupled from gauge field perturbations and hence not massless even in low frequency limit. Therefore the retarded Green’s function (response function) corresponding to these massive fluctuations are not independent of radial direction. Hence, horizon value and asymptotic values are different in these case. As a result the isotropy of viscosity tensor is also lost\(^{29}\).

We have also studied the flow equations for retarded Green’s function of boundary theory analytically and found higher order transport coefficients of the boundary plasma solving this equation. We have generalized the analysis for generic higher derivative gravity theory. The flow equation for Green’s function is a first order non-linear differential equation of Riccati type. Because of its non-linear nature it is hard to solve this equation exactly. After a change

\(^{29}\)Since, $[T_{xy}, T_{xy}]$ correlator is different than $[T_{xz}, T_{xz}]$ or $[T_{yz}, T_{yz}]$ correlators.
of variable one can reduce this non-linear equation to a second order linear homogeneous differential equation. But to solve this we need to specify two boundary conditions. In this article we have dealt with the non-linear equation and specify the boundary condition at the horizon. Therefore the hydrodynamic characteristic of the field theory at UV fix point is determined by IR boundary condition. In this way of computing the transport coefficients has an advantage over usual Kubo approach. In Kubo approach, one has to first find the transverse graviton by solving a second order differential equation and then compute regarded Green’s function. Instead, the flow equation is a first order differential equation (although non-linear). As we want a perturbative expansion of Green’s function in powers of \( \omega \) and \( q \) the equation turns out to be a linear first order differential equation. Thus, technically, it is simpler to get results for causal hydrodynamics, particularly when the dual bulk theory is complicated.

It would be interesting to use renormalization − group flow techniques to understand how the response function evolves as a function of the radial direction. One can also apply Blackfold techniques of [52] and the idea of Wilsonian RG flow explained in [53]. Blackfolds are higher dimensional black holes with two separate length scales (mass and angular-momentum) and their equations of motions are obtained from fluid correspondence (in leading Einstein gravity). Generalizing this technique in reverse, one should obtain equations of boundary plasma from the Blackfold equations in a generic gravity theory. It would be really nice to check this explicitly.

There are other methods to compute first and second order transport coefficients holographically. We have discussed two of them in little details in appendix C. In C.1 we have briefed the method developed in [28]. This is an elegant method to understand the properties of boundary fluid from boosted black brane geometry. This method shows that, all the constituent equations of fluid system are captured in the Einstein equations. In [39], this method has been extended to higher derivative gravity (Gauss-Bonnet gravity) and shown that the above mentioned statement is true even in presence of higher derivative terms. Although this method captures almost all transport coefficients (except \( \kappa \), the second order transport coefficient) up to second order in derivative expansion, but it is technically very hard to extend this idea to study higher derivative corrections. One problem is that the exact form of boundary stress tensor is not known for any generic higher derivative gravity, for example Weyl\(^4\) term.

In C.2 we have discussed another method to compute first order transport coefficients developed in [54]. In [22], the authors proposed a Wald like formula for shear viscosity coefficient in generic higher derivative gravity. Though the formula works for generic four derivative term in bulk action, but unfortunately it fails to produce the correct result for Weyl\(^4\) term [10]. Motivated by this formula the author in [54] proposed a new Wald like formula for the first order transport coefficients. The author has shown that the transport coefficients can simply be obtained by evaluating the residue of the pole of the quadratic action near the horizon. It only assumes the massless behavior of low energy perturbation and regularity of the field at horizon. The result is independent of the boundary terms and thus one can neglect them.

We conclude this review with some discussion on the currents trends on this subject.

In this article we have mainly considered the characteristic of conformal fluid which has zero bulk viscosity at first order. Bulk viscosity measures the reaction of fluid against any bulk stress. The holographic computation of bulk viscosity is also an interesting topic in this area of research. In [55] the author has proposed a holographic bound on bulk viscosity to entropy
density ratio of strongly coupled gauge theory plasma,

\[ \frac{\zeta}{s} \geq \frac{1}{2\pi} \left( \frac{1}{p} - c_s^2 \right), \]  

(223)

where \( c_s \) is the speed of sound and \( p \) is the spatial dimension of the system. This bound is dynamical, i.e. the right hand side depends on temperature unlike the bound on shear viscosity to entropy density ratio. They observed that the bound is saturated by the \( p + 1 \) space-time dimensional gauge theory plasma holographically dual to a stack of near-extremal flat \( Dp \)-branes at leading order (i.e. infinitely large \('t\) Hooft coupling). The bulk viscosity bound 223 is also saturated in toroidal compactifications of conformal theories 55, 56. It has been found that in various examples of string theory embedding of the holographic gauge-gravity correspondence the bound is saturated. For example, \( \mathcal{N} = 2^* \) gauge theory plasmas, gauge theories with adjoint \( R \) charges, cascading gauge theories etc. There are further evidences in favor of saturation of this bound in 57, 58 for phenomenological models of fluid-gravity correspondence. However 59, 60 considered some other phenomenological models where the bound is violated. Recently in 61 the author addressed the question if the violation of bulk viscosity bound is only limited to the phenomenological models of gauge/gravity correspondence. They considered strongly coupled \( \mathcal{N} = 4 \) supersymmetric Yang-Mills plasma compactified on a two-manifold of constant curvature and found that the resulting \((1 + 1)\)-dimensional hydrodynamic system can have bulk viscosity coefficients which violates the bound if the curvature of the compact manifold is negative. It would be also interesting to check this bound explicitly in presence of some higher derivative terms in bulk.

Non-relativistic generalization of the original AdS/CFT correspondence has opened up new directions in current research after a series of beautiful experiments on cold atoms at unitarity. The correspondence has been extended to explore the holographic duals of strongly coupled non-relativistic conformal field theories. Cold atoms are system of fermions interacting through a short-range potential which can be fine-tuned to obtain a massless bound state. Since the theory is scale invariant, it can be described as a non-relativistic conformal field theory with Schrödinger symmetry. This symmetry consists of the usual Galilean invariance, the scaling symmetry as well as the particle number symmetry. There has been some activity along this direction where solution generating techniques have been used to obtain bulk geometries with asymptotic Schrödinger symmetry.

At the first order in derivative expansion, we have one interesting transport coefficient, which is shear viscosity coefficient. It turns out that for non-relativistic fluid the shear viscosity to entropy density ratio saturates the KSS bound 62. However, 62 has also computed other transport coefficients like, thermal conductivity kinematic viscosity, thermal diffusivity, Prandtl number holographically, using the dual geometry proposed in 63, 64, 65.

As mentioned earlier, fluid dynamics in general can be thought of as an effective field theory with an infinite number of irrelevant terms obtained as usual in a derivative expansion. In relativistic case consideration of second order hydrodynamics is essential because the first order formalism is inconsistent with causality issues. However in non-relativistic setup one is not forced to consider the second order (beyond ideal and viscous terms) terms in stress tensor. But it is interesting to study those terms if we want to view hydrodynamics as a derivative
expansion of its local variables.

One interesting problem in this direction is to study the light-cone reduction of relativistic conformal/non-conformal hydrodynamic stress tensor [28, 29, 30, 46, 45]. In second order we encounter different transport coefficients in case of relativistic theory (see section 7). It is possible to reduce the theory along one of the light-cone coordinates and understand how the higher dimensional transport coefficients descend down to new transport coefficients for the lower dimensional non-relativistic theory. The reduction of first order stress tensor has been done in [63]. Following the same procedure one can write the second order corrections to the stress tensor of a charged fluid with Schrödinger symmetry. Once we have all possible second order transport coefficient of a non-relativistic fluid we can evaluate their holographic values using the non-relativistic gauge/gravity correspondence.

To compute the second order non-relativistic charged fluid’s transport coefficients holographically, one has to take the bulk spacetime to be five dimensional Reissner-Nordström black hole (embedded in 10 dimensional geometry) and perform TsT transformation to get asymptotically Schrödinger spacetime. Then one should turn on hydrodynamic fluctuations in this background and study the flow of corresponding response function (retarded Green’s function) like [31]. Evaluating the response function at asymptotic boundary it is possible to compute the non-relativistic second order transport coefficients of 2 (spatial)-dimensional dual fluid system. However, in this way of computing the transport coefficients (Kubo method) may not capture all possible coefficient (like relativistic case). Applying the method described in [28] ([63] has already applied this method to evaluate first order transport coefficient for non-relativistic fluid) would be a good idea to compute the second order transport coefficients holographically.

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Appendix
A Second order transport coefficients from Kubo formula

In this appendix we re-derive second order transport coefficients \( \kappa \) and \( \tau \) from usual Kubo approach. Let us now consider the action with solution given in section 7,

\[
S = \frac{1}{16\pi G_5} \int d^5 x \sqrt{-g} [R + 12].
\] (224)

For well defined variation of this action we need to add a Gibbons-Hawking boundary term. Also, requiring the on-shell action being finite at boundary, we have to add counter-terms following usual approach of holographic renormalization. They are as follows:

\[
S_{GH} = \frac{1}{8\pi G_5} \int d^4 x \sqrt{-\gamma} K,
\]

\[
S_{CT} = \frac{1}{16\pi G_5} \int d^4 x \sqrt{-\gamma} [6 + \frac{1}{2} \mathcal{R}]
\] (225)

where \( \gamma \) and \( \mathcal{R} \) are boundary metric and Ricci scalar (constructed out of \( \gamma \)) respectively.

We consider the following metric perturbation,

\[
g_{xy} = g_{xy}^{(0)} + h_{xy}(r, x) = g_{xy}^{(0)}(1 + \epsilon \Phi(r, x)),
\] (226)

where \( \epsilon \) is an order counting parameter. We are interested in quadratic on-shell action for this transverse graviton \( \Phi(r, x) \). Let us define the Fourier transform,

\[
\phi(r, k) = \int \frac{d^4 x}{(2\pi)^4} e^{-ik \cdot x} \Phi(r, x),
\]

and \( k = \{-\omega, \vec{k}\} \). Substituting this fluctuation in action (224), we get (227). Now, we rewrite this action (224) as equation of motion piece (which vanishes on-shell) and boundary term. Thus, on-shell \( S, S_{GH}, S_{CT} \) become,

\[
S = \frac{1}{16\pi G_5} \int_{r=\delta} \frac{d^4 k}{(2\pi)^4} L(\phi(r, k)),
\]

\[
S_{GH} = \frac{1}{8\pi G_5} \int_{r=\delta} \frac{d^4 k}{(2\pi)^4} L_{GH}(\phi(r, k)),
\]

\[
S_{CT} = \frac{1}{16\pi G_5} \int_{r=\delta} \frac{d^4 k}{(2\pi)^4} L_{CT}(\phi(r, k)),
\] (228)

where we define,

\[
L(\phi(r, k)) = a_2(\epsilon) \phi(r, k) \phi^*(r, k) + \frac{a_4(r) - a_5'(r)}{2} \phi(r, k) \phi^*(r, k),
\] (229)

\[
L_{GH}(\phi(r, k)) = 2(g_1) \phi(r, k) \phi^*(r, k) + 2(g_2) \phi' (r, k) \phi^*(r, k),
\] (230)

\[
L_{CT}(\phi(r, k)) = (c_0 + c_1 \omega^2 + c_2 q^2) \phi(r, k) \phi^*(r, k).
\] (231)

\[a_3, a_6\] are zero here

\[\text{We ignore contribution from horizon as}\] [50]
The coefficients $a_2, a_4, a_5$ are given in (237) and other coefficients are,

$$g_1 = \frac{2 - r^2}{r^2}, \quad g_2 = -2\frac{1 - r^2}{r}, \quad c_0 = -3\frac{\sqrt{1 - r^2}}{r^2}, \quad c_1 = \frac{1}{4r\sqrt{1 - r^2}}, \quad c_2 = -\frac{\sqrt{1 - r^2}}{4r} \quad (232)$$

The retarded Green’s function is defined at asymptotic infinity as,

$$G^R(k) = 2\lim_{r \to 0} \frac{(L + L_{GH} + L_{CT})|_{on-shell}}{\phi_0(k)\phi_0(-k)}, \quad (233)$$

Here, $\phi_0(k)$ is the boundary value of the fluctuation (227). The retarded Green’s function $G^R$ is a function of boundary momenta $k_\mu = (\omega, 0, 0, q)$. Now the leading action and the Gibbons-Hawking action get divergences from $\phi'(r, k)\phi^*(r, k)$ and $\phi(r)\phi^*(r, k)$ parts. Both these divergences get canceled by the counter-term action which is always proportional to only $\phi(r)\phi^*(r, k)$. In this case of leading Einstein’s gravity, it is even more simplified. Divergences coming from $\phi(r)\phi^*(r, k)$ piece of leading and Gibbons-Hawking action gets canceled by momentum independent piece of Counter-term action. It turns out that there is a cancellation among the corresponding coefficients as,

$$\lim_{r \to 0} \left( \frac{1}{2}(a_4(r) - a_5'(r)) + 2g_1 + c_0 \right) = \frac{1}{2}, \quad (234)$$

i.e. the final contribution from $\phi(r)\phi^*(r, k)$ piece is only a finite number $\frac{1}{2}$. As the graviton fluctuation $\phi(r, \omega, \vec{k}) = \phi_0(1 + F(r, \omega, \vec{k}))$ and moreover $\lim_{r \to 0} F(r, \omega, \vec{k}) = 0$, we see that the $\phi(r)\phi^*(r)$ term above only contribute to pressure (the $\omega$ independent piece of $G^R$). It would never contribute to any transport coefficient.

Also, the divergences coming from $\phi'(r, k)\phi^*(r, k)$ piece of original and Gibbons-Hawking action, get canceled with the piece of the counter-term proportional to $\omega^2, q^2$ ($c_1$ and $c_2$ are purely divergent at boundary). Here, the situation is more subtle, as there is no cancellation among the coefficients. One actually needs to put the solution of $\phi(r, \omega, \vec{k})$ to see the cancellation.

The overall lesson from this detailed analysis is that counter-term only cancel the UV divergences in usual holographic renormalization process and at most contribute to pressure of the boundary plasma. It has no effects on any transport coefficients. In [29], the author have computed second order transport coefficients for the plasmas dual to leading Einstein’s gravity following this usual approach. The results are as follows,

$$\tau_\pi = \frac{2 - \ln 2}{2\pi T}, \quad \kappa = \frac{\eta}{\pi T}. \quad (235)$$

These results match with the one we obtained in (125) by solving the flow equations.

### B Equivalence of Boundary Terms

In this appendix we will show explicitly why the transport coefficients computed from the original action and the effective action are same, even for any higher derivative theory. It was
already noticed \[10\], that the two would give same first-order transport coefficient $\eta$ with a suitable choice of the overall normalization constant. Here, we show that, not just the first order transport coefficients, rather any higher order transport coefficients computed from the original action and the effective action are same.

We consider a general class of action for $\phi$ which appears when the higher derivative terms are made of different contraction of Ricci tensor, Riemann tensor, Weyl tensor, Ricci scalar etc. or their different powers. Since, all these tensors involve two derivatives of metric they can only have terms like $\partial_a \partial_b \Phi(r, x)$ and its lower derivatives. Therefor the most generic quadratic (in $\Phi(r, x)$, in linear response theory) action for this kind of higher derivative gravity has the following form (in momentum space)\[32\]

$$
S = \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} dr \left[ a_1(r)\phi(r)^2 + a_2(r)\phi'(r)^2 + a_4(r)\phi(r)\phi'(r) + a_6(r)\phi''(r)\phi'(r) + a_3(r)\phi''(r)^2 + a_5(r)\phi(r)\phi''(r) \right]
$$

(236)

where,

$$
a_1(r) = \frac{-8r^2 + \omega^2 r + 8}{4r^3 - 4r^5} + \alpha' f_2(r), \quad a_2(r) = -3r + \frac{3}{r} + \alpha' h_2(r)
$$

$$
a_4(r) = -\frac{6}{r^2} - 2 + \alpha' g_2(r) \quad a_5(r) = -4r + \frac{4}{r} + \alpha' j_2(r)
$$

(237)

and $a_3(r), a_6(r), j_2(r), g_2(r), h_2(r)$ and $f_2(r)$ depends on higher derivative terms in the action and hence are computed purely from the background solution with $\alpha' \to 0$. Among these coefficients $a_3$ is special, as, it couples to $\phi'^2$. All four derivatives act on the graviton fluctuation and thus $a_3$ only depends on background metric functions and there r-derivatives. It is easy to convince ourselves that $a_3 \propto r(r^2 - 1)^2 f(r, \alpha')$, where $f(r, \alpha')$ is a function that depend on the higher derivative terms and finite (constant or 0) at the boundary $r \to 0$. Now let us write the effective Lagrangian as follows,

$$
S_{\text{eff}} = \frac{1 + \alpha'\kappa}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} dr \left[ \frac{4r(r^2 - 1)^2 \phi'(r)^2 - \omega^2 \phi(r)^2}{4r^2(r^2 - 1)} + \alpha' \left( b_2(r)\phi(r)^2 + b_1(r)\phi'(r)^2 \right) \right].
$$

(238)

Demanding that the equation of motion (up to order $\alpha'$) of $\phi$ derived from the original action and the above action are same we get,

$$
b_1(r) = \frac{1}{2r(r^2 - 1)^2} \left[ (-4r^3 - 12r + \omega^2)a_3(r) + (r^2 - 1)(2\kappa r^4 - a_6'(r)r^3 - 4\kappa r^2 + 2a_3'(r)r^2 + 2(r^2 - 1)h_2(r)r - 2(r^2 - 1)j_2(r)r + a_6'(r)r + 2\kappa + 2a_3'(r)) \right]
$$

(239)

\[32\]In all the expressions we have omitted $k$ dependence of $\phi$.  

57
\[
b_2(r) = -\frac{1}{16r^2 (r^2 - 1)^4} \left[ (\omega^4 + 144r^3 \omega^2) a_3(r) + 4(r^2 - 1) \left( -4r^2 f_2(r)(r^2 - 1)^3 
+ (\omega^2 \kappa - 2r^2(r^2 - 1)j2''(r))(r^2 - 1) + 2r^2 g_2'(r)(r^2 - 1)^2 + r\omega^2 a_3''(r))(r^2 - 1)
+ (1 - 11r^2)\omega^2 a_3'(r) \right) \right].
\]

The boundary terms coming from the original action (after adding Gibbons-Hawking boundary terms) are given by \[^{33}\]

\[
S^B = \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} \left[ -\frac{\phi(r)^2}{r^2} + \phi(r)^2 + r\phi'(r)\phi(r) - \frac{\phi(r)\phi'(r)}{r} + \alpha' \left( \frac{1}{2} g_2(r)\phi(r)^2 - \frac{1}{2} j2'(r)\phi(r)^2 + (h_2(r) - j2(r) - \frac{a6'(r)}{2})\phi'(r)\phi(r) + \frac{a3'(r)\phi(r)\omega^2 + 4(r^4 - 1)\phi'(r)}{4r(r^2 - 1)^2} \right) \phi(r) - \frac{a3(r)(6r\phi(r)\phi'(r)\omega^2)}{4r(r^2 - 1)^3} \right] \left( -\frac{\phi(r)\omega^2}{2r(r^2 - 1)^2} - \frac{(r^4 - 1)\phi'(r)}{r(r^2 - 1)^2} \right) \right].
\]

And the boundary terms coming from the effective action are given by,

\[
S^B_{\text{eff}} = \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} \left[ (r - \frac{1}{r}) \phi(r)\phi'(r) + \frac{\alpha'}{2r(r^2 - 1)^2} \left( \phi(r)(2\Gamma(r^2 - 1)^3 + (-a6'(r)r^3 + 2a3'(r)r^2 + 2(r^2 - 1) h_2(r)r - 2(r^2 - 1) j2(r)r + a6'(r)r + 2a3'(r))(r^2 - 1)
+ (-4r^3 - 12r + \omega^2) a3(r)\phi'(r) \right) \right].
\]

Now, it is interesting to compute the difference between these two boundary terms and the result is \[^{34}\]

\[
S^B - S^B_{\text{eff}} = \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} \left[ \phi(r)^2 \frac{r^2}{r^2} + \phi(r)^2 + \alpha' \left( \frac{1}{2} g_2(r)\phi(r)^2 - \frac{1}{2} j2'(r)\phi(r)^2 \right) + \frac{a3'(r)\omega^2\phi(r)^2}{4r(r^2 - 1)^2} - \frac{a3(r)(6r\omega^2)\phi(r)^2}{4r(r^2 - 1)^3} \right]
\]

The term proportional to \(a_3\) in the parenthesis of (243) vanishes at the boundary whereas the term proportional to \(a_3'\) gives a pure UV divergent piece and a vanishing piece, due to the property of \(a_3\) mentioned above. This is true irrespective of the choice of the higher derivative terms. Thus, we see that the two boundary terms differ only by terms which are either purely

\[^{33}\]There was a sign error in \[^{10}\]

\[^{34}\]It has been shown in \[^{10}\] that \(\Gamma = 0\).
divergent or of the form \( g(r)\phi^2 \), where \( g(r) \) is any function of \( r \). The divergent terms would get canceled once appropriate boundary terms are added (which has been discussed in sections 7 and 7.3). The terms proportional to \( \phi^2 \) can only contribute to pressure of the boundary theory and are not important for the computation of transport coefficients of the boundary plasma. Thus we see that, it is obvious that the transport coefficients coming from the original action and the boundary action are same.

Here we have considered only \( R^{(n)} \) gravity theory. A more rigorous proof is required for theories involving covariant derivatives of curvature tensors and scalars.

C Functions appeared in four derivative action

\[
A^0_{GB}(r, k) = -\frac{q^2(r^2 - 1) + \omega^2}{4r^2(r^2 - 1)} + \frac{\alpha'}{12r^2(r^2 - 1)} \left( q^2(2\beta_3(13r^2 - 3\omega^2 - 13) + 130(r^2 - 1)\beta_1 + (-36r^4 + 25r^2 + 11)\beta_2) + \omega^2((6r^2 - 11)\beta_2 + 130\beta_1 + 26\beta_3) \right)
\]

(244)

and

\[
A^1_{GB}(r, k) = r - \frac{1}{r} - \frac{(r^2 - 1)((18r^2 - 13)\beta_2 + 110\beta_1 + 22\beta_3)\alpha'}{3r}.
\]

(245)

D Coefficients in expansion (84) at \( \mathcal{O}(r - r_h)^2 \)

For completeness, in this appendix we present the other coefficients that appear in the near horizon expansion of different fields in eq. (84). They are

\[
u_2 = e^{\alpha\phi_h} \left( \frac{5B^2}{3r_h^4} + \frac{7q^2}{3} \right) + \frac{3}{4}z_1^2r_h^2 - 2
\]

\[
v_2 = \frac{1}{288\pi^2 T^2} \left( e^{-2\alpha\phi_h} \left( 24 B^4 \xi^2 e^{\alpha\phi_h} + 24 \left( 5B^2 + 2q^2 \right) e^{3\alpha\phi_h} - 72B^2\xi^2 \right) + \left( 6B^4(\alpha^2 - 4) - B^2 q^2 \left( 9\alpha^2 + 8 \right) + q^4 \left( 3\alpha^2 - 4 \right) \right) e^{4\alpha\phi_h} - 144 e^{2\alpha\phi_h} \right).
\]

\[
w_2 = \frac{e^{-4\alpha\phi_h}}{288\pi^2 T^2} \left( -36 B^4 \xi^4 - 24B^2 \xi^2 \left( B^2 + q^2 \right) e^{3\alpha\phi_h} + 48 \left( q^2 - 2B^2 \right) e^{5\alpha\phi_h} + 72B^2 e^{2\alpha\phi_h} \right) \left( 3\alpha^2 - 4 \right) + \left( -3B^4(\alpha^2 - 4) - 8B^2 q^2 + q^4 \left( 3\alpha^2 - 4 \right) \right) e^{6\alpha\phi_h} - 144 e^{4\alpha\phi_h} \right)
\]

\[
z_2 = -\frac{B\zeta \left( B^2 \left( 9\xi^2 e^{-3\alpha\phi_h} + 1 \right) + 5 \left( q^2 - 6e^{-\alpha\phi_h} \right) \right)}{6\pi T}
\]

\[
p_1 = \frac{Bq\zeta e^{-3\alpha\phi_h} \left( (3\alpha^2 + 4) \left( B^2 - q^2 \right) e^{3\alpha\phi_h} - 12B^2\xi^2 + 24 e^{2\alpha\phi_h} \right)}{96\pi^2 T^2}.
\]

(246)
E Decoupling of $h_{xy}$ mode

In what follows, we provide a detailed derivation of the decoupling of the dual gravitational mode (4.1). We have explicitly checked that the $h_{xy} = e^{i t \omega + 2 V(r)} e \Phi(r)$ mode does not couple with any other field when the momentum vanishes.

For two derivative gravity theory, this can be easily seen from the equations of motion (66, 67, 68). However, we are interested in the most general four-derivative action (91). In this case, instead of writing the equations of motion in the presence of higher derivative terms, we will explicitly compute the action up to order $\epsilon^2$. In this way, it can be explicitly checked that there is no coupling between $h_{xy}$ and the other fields.

Let us turn on the following perturbations of the metric

$$g_{\alpha\beta} = g_{\alpha\beta}^{(0)} + \epsilon h_{\alpha\beta}$$

$$= \begin{pmatrix}
\frac{1}{U} + e^{i t \omega} \epsilon \xi_1 & e^{i t \omega} \epsilon \xi_2 & e^{i t \omega} \epsilon \xi_3 & e^{i t \omega} \epsilon \xi_4 & e^{i t \omega} \epsilon \xi_5 \\
e^{i t \omega} \epsilon \xi_2 & e^{2 W} Z^2 - U & e^{i t \omega} \epsilon \Upsilon & 0 & e^{i t \omega + 2 V} \epsilon \Phi \\
e^{i t \omega} \epsilon \xi_3 & e^{i t \omega} \epsilon \Upsilon & e^{2 V} & e^{i t \omega + 2 V} \epsilon \Phi & e^{i t \omega} \epsilon \chi \\
e^{i t \omega} \epsilon \xi_4 & 0 & e^{i t \omega + 2 V} \epsilon \Phi & e^{2 V} & 0 \\
e^{i t \omega} \epsilon \xi_5 & e^{2 W} Z + e^{i t \omega} \epsilon \Upsilon & e^{i t \omega} \epsilon \chi & 0 & e^{2 W}
\end{pmatrix}$$

(247)

and gauge gauge fields

$$A_\alpha = A_\alpha^{(0)} + \epsilon f_\alpha$$

$$= \left( e^{i t \omega} e a_x(r), e^{i t \omega} e a_y(r) - \frac{B y}{2} + e^{i t \omega} e a_x(r), -\frac{B x}{2} + e^{i t \omega} e a_y(r), e^{i t \omega} e a_z(r) + P(r) \right)$$

(248)

Here $g_{\alpha\beta}^{(0)}$ and $A_\alpha^{(0)}$ are the background metric (71) and the background gauge field; $\epsilon$ dependent terms are the perturbations.

Using these field excitations, we can now compute the action.\footnote{We use the Mathematica notebook for this computation. We emphasize that our system is symmetric in $x-$ and $y-$directions.} The result is complicated, but for our purpose it is enough to pick up the $\Phi(r)$-dependent terms — for concreteness, let us write the $\Phi(r)$-dependent part of the action:
\[ S_{\Phi,\Phi} = \Phi(r)^2 \left[ \frac{1}{4U(r)} \left( (e^{-2V(r)} - W(r))(-U(r)e^{2W(r)}(4B^2e^{\alpha_\varphi(r)} + 4Q(r)^2e^{4V(r)} + \alpha_\varphi(r)} ight. \\
-2e^{4V(r)}U''(r) - 4e^{4V(r)}U'(r)(2V'(r) + W'(r)) + e^{4V(r)} + 2W'(r))Z'(r)^2 + 24e^{4V(r)} \\
+2U(r)^2e^{4V(r)}(2P'(r)^2e^{\alpha_\varphi(r)} + e^{2W(r)}(4V''(r) + 4V'(r)W'(r) + 6V'(r)^2 \\
+2W''(r) + 2W'(r)^2 - \varphi'(r)^2)) + 14\omega^2e^{4V(r)} + 2W'(r)) \right] \]

\[ + \Phi(r)\Phi'(r) \left[ (2e^{2V(r)} + W(r))(U''(r) + U(r)(3V'(r) + W'(r))) \\
-2(2\alpha'e^{W(r)} - 2V(r) - 2U(r)V'(r))(U(r)(B^2c_2 + c_1e^{4V(r)} + 2W(r))Z'(r)^2) - 2c_1U(r)^2e^{4V(r)} \\
(2V''(r) + 3V'(r)^2 + W'(r)^2) + 2c_1\omega^2e^{4V(r)} + 2c_1U(r)e^{4V(r)}U'(r)^2V'(r) \\
+c_1e^{4V(r)}U'(r)(2U(r)^2(V''(r) + 3V'(r)^2 + \omega^2))\right] / U(r) \]

\[ + \Phi(r)\Phi''(r) \left[ (2U(r)e^{2V(r)} + W(r) - 4c_1\alpha'U(r)e^{2V(r)} + W(r))(U'(r)V'(r) + 2U(r)(V''(r) + V'(r)^2)) \right] \]

\[ + \Phi'(r) \left[ (\alpha'e^{W(r)} - 2V(r))(U(r)(B^2c_2 + 2c_1e^{4V(r)}U'(r)V'(r) - c_1e^{4V(r)} + 2W(r))Z'(r)^2) \right] \\
+ c_1e^{4V(r)}(U'(r)^2 + 4\omega^2) + 2c_1U(r)^2e^{4V(r)}(-2V''(r) + V'(r)^2 + W'(r)^2) + \frac{3}{2}U(r)e^{2V(r)} + W(r) \]

\[ + \Phi'(r)\Phi''(r) \left[ (2c_1\alpha'U(r)e^{2V(r)} + W(r))(U'(r) + 4U(r)V'(r)) \right] \]

\[ + \Phi''(r)\left[ 2c_1\alpha'U(r)^2e^{2V(r)} + W(r) \right] \]

(249)

Since there is no mixing between \( \Phi \) and the other modes in the zero momentum limit, this mode remains massless and a computation of the related Green’s function from the near-horizon data is still possible.

However, the other metric fluctuations namely \( h_{xz} \) or \( h_{yz} \) are not decoupled from the gauge field perturbations in the zero momentum limit (even in the absence of higher derivative terms) — these coupled terms are proportional to \( B \) or \( \zeta \) (CS term). Therefore, these modes are not massless in the zero momentum limit and the near horizon geometry is not sufficient to compute the related two point correlation functions, i.e. \( < [T_{xz}, T_{xz}] > \) or \( < [T_{yz}, T_{yz}] > \) .
Functions appeared in Higher-derivative Charged black-hole action

\[ A^{CB}_1 = \frac{r^5(\omega^2 - q^2) + q^2 r_0^4 r}{2(r^4 - r_0^4)} + \frac{c_1 \alpha' (11 r^8(\omega^2 - q^2) - r_0^4 r^4(25 q^2 + 6 \omega^2) + 36 q^2 r_0^8)}{6 r^3(r^4 - r_0^4)} \]

\[ + \frac{Q^2}{2 r_0^2(r^2 - r_0^2)(r^2 + r_0^2)^2} \left[ r^3 \omega^2 + \frac{\alpha'}{3 r_0^5} \left( c_1 [28 q^2 r_0^6 r^2 - r^8(36 q^2 + 127 \omega^2) - 4 r_0^2 r_6(7 q^2 - 3 \omega^2) - 8 q^2 r_0^8 + 4 r_0^4 r^4(11 q^2 + 3 \omega^2)] - 24 c_2 [r_0^2 r_6(2 q^2 - 3 \omega^2) - 2 q^2 r_0^6 r^2 \\ + r_0^4 r^4(2 q^2 - 3 \omega^2) - 2 q^2 r_0^8 + 4 r^8 \omega^2] \right] \right] + O(Q^4) \quad (250) \]

\[ A^{CB}_1 = \frac{1}{2} (r^4 - r_0^4) - \frac{c_1 \alpha'}{6 r^3} (13 r^4 - 18 r_0^4) (r^4 - r_0^4) + \frac{Q^2 (r^2 - r_0^2)}{2 r_0^2} \left[ 1 - \frac{\alpha'}{3 r_0^4} \left( 24 c_2 (4 r^4 - 3 r_0^2 r^2 - 3 r_0^4) \\ + c_1 (101 r^4 - 156 r_0^2 r^2 - 120 r_0^4) \right) \right] + O(Q^4). \quad (251) \]

Other methods of holographic hydrodynamics

In this appendix we outline two other methods of computing various transport coefficients very briefly. First we discuss about the boosted black brane method by [28]. Next, we talk about the pole methods by [54].

G.1 Boosted black branes and fluid dynamics

In this section we will briefly sketch the working procedure of [28]. For detailed discussion readers are referred to the original paper. We will also skip the technical details in this section.

- Consider the Einstein-Hilbert action with negative cosmological constant

\[ I = -\frac{1}{16 \pi G_5} \int d^5 x \sqrt{-g} \left( R + \frac{12}{L^2} \right) \quad (252) \]

where \( L \) is the radius of AdS space.

- The equation of motions are given by [36]

\[ E_{MN} = R_{MN} - \frac{1}{2} R g_{MN} - \frac{6}{L^2} g_{MN} = 0. \quad (253) \]

\[ x^M = \{ v, r, \vec{x} \}. \]
• There exists a class of solutions to these equations of motion given by the “boosted black branes”\textsuperscript{37}

\[
ds^2 = -2u_\mu dx^\mu dr - \frac{r^2}{L^2} f(br)u_\mu u_\nu dx^\mu dx^\nu + \frac{r^2}{L^2} (u_\mu u_\nu + \eta_{\mu\nu}) dx^\mu dx^\nu
\]

(254)

with,

\[
f(r) = 1 - \frac{1}{r^4},
\]

\[u_v = -\gamma \quad \text{and} \quad u_i = \gamma \beta_i
\]

(255)

where, \(\gamma = 1/\sqrt{1 - \beta^2}\).

• Putting the values of \(u_\mu\)'s the metric can also be written as,

\[
ds^2 = 2\gamma dvdr - \frac{r^2}{L^2} \gamma^2 f(br)dv^2 + \frac{r^2}{L^2} dx^i dx^i
+ \frac{r^2}{L^2} (\gamma^2 - 1)dv^2 - 2\gamma \beta_i dx^i dr - 2 \frac{r^2}{L^2} \gamma^2 (1 - f(br)) \beta_i dx^i dv
+ \frac{r^2}{L^2} \gamma^2 (1 - f(br)) \beta_i \beta_j dx^i dx^j.
\]

(256)

The solution is parametrized by four constant parameters \(b\) and \(\beta_i\)'s.

• The black brane horizon is located at \(r_H = 1/b\) and the temperature of this black brane is given by,

\[
T = \frac{1}{\pi b L^2}.
\]

(257)

• Consider the metric (256) and replace the constant parameters \(b\) and \(\beta_i\)'s by slowly varying functions \(b(x^\mu)\) and \(\beta_i(x^\mu)\)'s of boundary coordinates \(x^\mu\)

\[
ds^2 = 2\gamma dvdr - \frac{r^2}{L^2} \gamma^2 f(b(x^\alpha)r)dv^2 + \frac{r^2}{L^2} dx^i dx^i
+ \frac{r^2}{L^2} (\gamma^2 - 1)dv^2 - 2\gamma \beta_i(x^\alpha) dx^i dr - 2 \frac{r^2}{L^2} \gamma^2 (1 - f(b(x^\alpha)r)) \beta_i(x^\alpha) dx^i dv
+ \frac{r^2}{L^2} \gamma^2 (1 - f(b(x^\alpha)r)) \beta_i(x^\alpha) \beta_j(x^\alpha) dx^i dx^j.
\]

(258)

We will call this metric \(g^{(0)}(b(x^\alpha), \beta_i(x^\alpha))\).

• In general the metric (258) is not a solution to Einstein equations unless one adds some corrections to the metric and also the parameters \(b(x^\alpha), \beta_i(x^\alpha)\) satisfy some set of equations, which turn out to be the equations of boundary fluid mechanics.

\textsuperscript{37}x^\mu = \{v, \vec{x}\}.
• Write the parameters $b(x^\alpha)$ and $\beta_i(x^\alpha)$ and the metric as a derivative expansion of the parameters. Up to first order in derivative expansion,

$$ g = g^{(0)}(b(x^\alpha), \beta_i(x^\alpha)) + \epsilon g^{(1)}(b(x^\alpha), \beta_i(x^\alpha)), \quad (259) $$

$$ b(x^\alpha) = b^{(0)}(x^\alpha) \quad (260) $$

and

$$ \beta_i(x^\alpha) = \beta_i^{(0)}(x^\alpha) \quad (261) $$

where $\epsilon$ is a dimension less parameter whose power counts the number of (boundary)spacetime derivatives acting on the parameters. Since $b^{(1)}(x^\alpha)$ and $\beta^{(1)}_i(x^\alpha)$ do not enter in to the first order equation of motions, we have kept the expansion for $b$ and $\beta_i$’s up to leading order.

• In general one can write the metric and parameters as power series of $\epsilon$. Then plug the metric in Einstein equations and solve the metric and the parameters order by order (in $\epsilon$). For example in our case since we are interested up to first order, we will plug the metric in Einstein equations and solve for $g^{(1)}$ and the constraint equations imply some relations between the zero $^{th}$ order parameters. We will work in a particular gauge,

$$ Tr((g^{(0)})^{-1}g^{(1)}) = 0. \quad (262) $$

• After finding the metric with first order fluctuations one can find the boundary stress tensor (using the definition given in [66]). The form of the boundary stress up to first order in derivative expansion is given by,

$$ 16\pi G_5 T_{\mu \nu} = T^4 \pi^4 L^3 (4 u_\mu u_\nu + \eta_{\mu \nu}) - 2 T^3 \pi^3 L^3 \sigma_{\mu \nu}, \quad (263) $$

where $\sigma_{\mu \nu}$ is given by,

$$ \sigma_{\mu \nu} = P_\mu^\alpha P_\nu^\beta \partial_{(\alpha} u_{\beta)} - \frac{1}{3} P_{\mu \nu} \partial_{\alpha} u^\alpha \quad (264) $$

and $P_{\mu \nu} = u_\mu u_\nu + \eta_{\mu \nu}$.

One can follow this procedure to second order in derivative expansion. We are not discussing that lengthy algebra here. Rather we are going to present the final answer for stress tensor up to second order in derivative expansion.

$$ T^{\mu \nu} = (\pi T)^4 (\eta^{\mu \nu} + 4 u^\mu u^\nu) - 2 (\pi T)^3 \sigma^{\mu \nu} $$

$$ + (\pi T)^2 \left( (\ln 2) T_{2a}^{\mu \nu} + 2 T_{2b}^{\mu \nu} + (2 - \ln 2) \left[ \frac{1}{3} T_{2e}^{\mu \nu} + T_{2d}^{\mu \nu} + T_{2e}^{\mu \nu} \right] \right) \quad (265) $$

64
where

\[
\begin{align*}
\sigma^{\mu\nu} &= P^{\mu\alpha} P^{\nu\beta} \partial_{(\alpha} u_{\beta)} - \frac{1}{3} P^{\mu\nu} \partial_{\alpha} u^\alpha \\
T_{2a}^{\mu\nu} &= \epsilon^{\alpha\beta\gamma}(\mu) \sigma_{\gamma}^{\nu} u^\alpha \mu \\
T_{2b}^{\mu\nu} &= \sigma^{\mu\alpha} \sigma_{\alpha}^{\nu} - \frac{1}{3} P^{\mu\nu} \sigma^{\alpha\beta} \sigma_{\alpha\beta} \\
T_{2c}^{\mu\nu} &= \partial_{\alpha} u^\alpha \sigma^{\mu\nu} \\
T_{2d}^{\mu\nu} &= D u^\mu D u^\nu - \frac{1}{3} P^{\mu\nu} D u^\alpha D u_{\alpha} \\
T_{2e}^{\mu\nu} &= P^{\mu\alpha} P^{\nu\beta} D (\partial_{(\alpha} u_{\beta)}) - \frac{1}{3} P^{\mu\nu} P^{\alpha\beta} D (\partial_{\alpha} u_{\beta}) \\
l_{\mu} &= \epsilon_{\alpha\beta\gamma\mu} u^\alpha \partial^\beta u^\gamma.
\end{align*}
\]

(266)

Our conventions are \(\epsilon_{0123} = -\epsilon^{0123} = 1\) and \(D \equiv u^\alpha \partial_\alpha\) and the brackets () around the indices to denote symmetrization, i.e., \(a^{(\alpha b}\beta) = (a^{\alpha b\beta} + a^{\beta a\beta})/2\).

From this expression of second order stress tensor, one can read-off the transport coefficients,

\[
\eta = \frac{\pi}{8} N^2 T^3, \quad \tau_\Pi = \frac{2 - \ln 2}{\pi T}, \quad \lambda_1 = \frac{2 \eta}{\pi T}, \quad \lambda_2 = \frac{2 \eta \ln 2}{\pi T}, \quad \lambda_3 = 0.
\]

(267)

As the coefficient \(\kappa\) does not enter the equations of fluid dynamics in flat space (see equation (105)), this analysis leaves this coefficient undetermined.

**G.2 Pole Method**

The method is very well explained in the original paper [54]. Here we outline the working formula in brief. Let us consider a Lagrangian containing arbitrary higher-derivative terms. As usual, we treat the higher-derivative terms perturbatively small and the equations of motions can still be treated as quadratic. The most generic form of the quadratic action in Fourier space (at zero spatial momenta) is given as follows:

\[
S = \int \Pi_{i=1}^{d-1} dx^i \int \frac{d\omega}{2\pi} \left( S_z + S_t + S_B[\phi(z, x^i, \omega)] \right),
\]

(268)

Where \(S_z\) is radial action, containing at least two radial \((z)\) derivative and no time derivative. \(S_t\) contains time derivatives and are proportional to \(\omega^2\). \(S_B\) is the boundary Lagrangian. The only constraint that has been put to write the above Lagrangian density is that the field \(\phi\) is massless. Studying the generic structure of the boundary terms, it can logically be shown that these boundary terms do not contribute to the computation of first order transport coefficients.

The next step is to compute the quadratic action near the horizon. Now, by demanding the regularity at the horizon, it can be shown that the metric perturbation \(\phi\) has following solution (independent of the details of the action),

\[
\phi_{\text{sol}} = \phi_0 \exp -i \frac{\omega}{4\pi T} \log z,
\]

(269)
where, $T$ is the temperature of the background. The Lagrangian density computed on this solution will always have a pole and other regular terms in the radial variable $z$. The first order transport coefficient is then given by a simple formula as,

$$
\Xi = 8\pi T \frac{{\text{Residuez} = 0\mathcal{L}_2}}{\omega^2},
$$

(270)

where, $\mathcal{L}_2$ is the quadratic Lagrangian density evaluated at $\phi_{sol}$. This method is really simple and easily usable for theories with arbitrary higher-derivative couplings (powers of curvature tensors as well as their covariant derivatives). But its extension to higher order transport coefficient is rather unclear.

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