OKA PRINCIPLE ON THE MAXIMAL IDEAL SPACE OF $H^\infty$

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Abstract. The classical Grauert and Ramspott theorems constitute the foundation of the Oka principle on Stein spaces. In this paper we establish analogous results on the maximal ideal space $M(H^\infty)$ of the Banach algebra $H^\infty$ of bounded holomorphic functions on the open unit disk $\mathbb{D} \subset \mathbb{C}$. We illustrate our results by some examples and applications to the theory of operator-valued $H^\infty$ functions.

1. Introduction

Let $H^\infty$ be the Banach algebra of bounded holomorphic functions in the open unit disk $\mathbb{D} \subset \mathbb{C}$ equipped with pointwise multiplication and supremum norm. In this paper, following our earlier work [Br1], [Br2], [Br3], we investigate further the relationship between certain analytic and topological objects on the maximal ideal space of $H^\infty$. The subject is intertwined with the area of the so-called Oka Principle which in a broad sense means that on Stein spaces (closed complex subvarieties of complex coordinate spaces) cohomologically formulated analytic problems have only topological obstructions (for recent advances in the theory see, e.g., survey [FL]). This principle can be transferred, to some extent, to the theory of commutative Banach algebras to reveal (via the Gelfand transform) some connections between algebraic structure of a Banach algebra and topological properties of its maximal ideal space. (The most general results there are due to Novodvorski, Taylor and Raeburn, see, e.g., [R] and references therein.)

Recall that for a unital commutative complex Banach algebra $A$ the maximal ideal space $M(A)$ is the set of all nonzero homomorphisms $A \to \mathbb{C}$. Since norm of each $\varphi \in M(A)$ is at least one, $M(A)$ is a subset of the closed unit ball of the dual space $A^*$. It is a compact Hausdorff space in the weak* topology induced by $A^*$ (called the Gelfand topology). Let $C(M(A))$ be the Banach algebra of continuous complex-valued functions on $M(A)$ equipped with supremum norm. An element $a \in A$ can be thought of as a function in $C(M(A))$ via the Gelfand transform $\hat{a} : A \to C(M(A)), \hat{a}(\varphi) := \varphi(a)$. Map $\hat{a}$ is a nonincreasing-norm morphism of Banach algebras. Algebra $A$ is called uniform if the Gelfand transform is an isometry (as for $H^\infty$).

For $H^\infty$ evaluation at a point of $\mathbb{D}$ is an element of $M(H^\infty)$, so $\mathbb{D}$ is naturally embedded into $M(H^\infty)$ as an open subset. The famous Carleson corona theorem [C] asserts that $\mathbb{D}$ is dense in $M(H^\infty)$. In general, if a Stein space $X$ is embedded as an open dense subset into a normal topological space $\tilde{X}$, it is natural to introduce an analog of the complex analytic structure on $\tilde{X}$ regarding it as a ringed space with the structure sheaf $\mathcal{O}_{\tilde{X}}$ of germs of complex-valued continuous functions on open subsets $U$ of $\tilde{X}$ whose restrictions to $U \cap \tilde{X}$ are holomorphic. Then one can ask whether analogs of classical results of complex analysis (including the Oka Principle) are valid on $(\tilde{X}, \mathcal{O}_{\tilde{X}})$. Some results in this direction for $\tilde{X}$ being a fiberwise compactification of an unbranched covering $X$ of a Stein manifold have been obtained in [BK1], [BK2]. Another example, $X = \mathbb{D}$ and

2010 Mathematics Subject Classification. Primary 30H05. Secondary 32L05.

Key words and phrases. Oka principle, Maximal ideal space of $H^\infty$, Grauert theorem, Ramspott theorem.

Research supported in part by NSERC.
\[ X = M(H^\infty), \] has been the main subject of papers \[ Br1, Br2, Br3 \] where the Stein-like theory analogous to the classical complex function theory on Stein spaces (see, e.g., [GR]) has been developed and then applied to the celebrated Sz.-Nagy operator corona problem [SN]. In the present paper we continue this line of research and establish (in the framework has been developed and then applied to the celebrated Sz.-Nagy operator corona problem theory analogous to the classical complex function theory on Stein spaces (see, e.g., [GR])

Theorem 3.5 asserts that up to homotopy equivalence holomorphic maps

\[ 2.10) \text{asserting that inclusion of the category of holomorphic principal bundles on } M(H^\infty) \text{ and continuous maps of} \]

\[ M \text{ and } \] continuous maps of

\[ one correspondence between sets of path-connected components of spaces of holomorphic

\[ H^\infty \text{ and that of Novodvorski, Taylor and Raeburn.} \]

Section 2 is devoted to the Oka Principle for holomorphic principal bundles. We work in a more general setting of unital subalgebras \[ H^\infty_I \subseteq H^\infty \] generated by closed ideals \[ I \subset H^\infty \]. In Proposition 2.1 we describe the topological structure of maximal ideal spaces \[ M(H^\infty_I) \]. Then we define holomorphic principal bundles on \[ M(H^\infty_I) \] with fibres complex Banach Lie groups and prove Grauert-Bungart-type theorems for them (Theorems 2.9, 2.10) asserting that inclusion of the category of holomorphic principal bundles on \[ M(H^\infty_I) \] to the category of topological ones induces a bijective map between the corresponding sets of isomorphism classes of bundles. Further, in Theorem 2.14 we prove that holomorphic principal bundles on \[ M(H^\infty_I) \] are isomorphic to holomorphic bundles pulled back from certain Stein domains by maps of the inverse limit construction describing \[ M(H^\infty_I) \] as the inverse limit of the inverse system of maximal ideal spaces of finitely generated subalgebras of \[ H^\infty_I \] (see, e.g., [RG]). This establishes a connection between our intrinsic analytic theory of \[ M(H^\infty_I) \] and that of Novodvorski, Taylor and Raeburn.

Section 3 deals with the Oka Principle for holomorphic maps from \[ M(H^\infty_I) \] to complex Banach homogeneous spaces.

In Section 3.1 we prove a Ramspott-type theorem (Theorem 3.2) providing a one-to-one correspondence between sets of path-connected components of spaces of holomorphic and continuous maps of \[ M(H^\infty_I) \] into complex Banach homogeneous spaces \[ X \]. In turn, Theorem 3.5 asserts that up to homotopy equivalence holomorphic maps \[ M(H^\infty_I) \rightarrow X \] can be uniformly approximated by holomorphic maps into \[ X \] pulled back from certain Stein domains by the inverse limit projections of Theorem 2.11.

Section 3.2 is devoted to a Runge-type theorem (Theorem 3.7) for holomorphic maps from some subsets of \[ M(H^\infty_I) \] to complex Banach homogeneous spaces \[ X \] and a nonlinear interpolation problem for such maps (Theorem 3.8) going back to Carleson [C] (\[ X = \mathbb{C} \]) and Treil [T6] (\[ X = \mathbb{C}^* := \mathbb{C} \setminus \{0\} \]). For instance, we prove that if \[ X \] is simply connected, then a holomorphic map into \[ X \] defined on a neighbourhood of a holomorphically convex subset \[ K \subset M(H^\infty_I) \] can be uniformly approximated on \[ K \] by holomorphic maps \[ M(H^\infty_I) \rightarrow X \] and that if \[ X \] is a complex Banach Lie group and \[ K \] is the zero locus of the image under the Gelfand transform of \[ I \] in \[ C(M(H^\infty_I)) \], then for each holomorphic map \[ F : U \rightarrow X \] defined on a neighbourhood \[ U \] of \[ K \] there is a holomorphic map \[ \hat{F} : M(H^\infty_I) \rightarrow X \] such that \[ \hat{F}|_K = F|_K \].

Section 4 contains some applications and examples of the obtained results.

In Section 4.1 we describe the structure of spaces of holomorphic maps of \[ M(H^\infty_I) \] into complex flag manifolds and tori, the latter by means of BMOA functions (Theorem 4.2).

In Section 4.2 we prove some results about holomorphic maps from \[ M(H^\infty_I) \] into the space of idempotents of a complex unital Banach algebra \[ \mathbb{A} \] (Theorem 4.4). In particular, we show that in some cases (e.g., if \[ \mathbb{A} \] is the algebra of bounded linear operators on a Hilbert or \[ L^p \] space) such \[ \mathbb{A} \]-valued holomorphic idempotents can be transformed by appropriate holomorphic similarity transformations to constant idempotents of \[ \mathbb{A} \].
In Section 4.3 we study analogs of the Sz.-Nagy operator corona problem for holomorphic maps of \( M(H^\infty) \) into the space of left-invertible elements of a complex Banach algebra \( \mathfrak{A} \) with unit \( 1_\mathfrak{A} \). In particular, we solve a general problem considered by Vitse \cite{V} proving that a holomorphic map \( F \) on \( \mathbb{D} \) with a relatively compact image in \( \mathfrak{A} \) has a holomorphic left inverse with a relatively compact image \( G \) (i.e., such that \( G(z)H(z) = 1_\mathfrak{A} \) for all \( z \in \mathbb{D} \)) if and only if for every \( z \in \mathbb{D} \) there is a left inverse \( G_z \in \mathfrak{A} \) of \( F(z) \) such that the family \( \{G_z\}_{z \in \mathbb{D}} \subset \mathfrak{A} \) is uniformly bounded.

Finally, in Example 4.11 we discuss a more general than the Sz.-Nagy problem, the so-called Completion Problem, asking about extension of a bounded \( H^\infty \) operator-valued function to an invertible one (see, e.g., \cite{T5} and references therein).

Sections 5–12 contain proofs of results of the paper.

2. Oka Principle for Principal Bundles

2.1. Maximal Ideal Spaces of Algebras \( H^\infty_I \). We work in a more general setting of (uniform) Banach algebras \( H^\infty_I := \mathbb{C} + I \), where \( I \subset H^\infty \) is a closed ideal. Such algebras arise naturally in the theory of bounded holomorphic functions in balls and polydisks, see \cite{AM1}, \cite{AM2}. They were also studied in the framework of the theory of univariate \( H^\infty \) functions, see \cite{Gam}, \cite{MSW}.

The corona theorem for algebra \( H^\infty_I \) can be derived from the Carleson corona theorem, see \cite{MSW, Th. 1.6}. It states that for a \( n \)-tuple of functions \( f_1, \ldots, f_n \in H^\infty_I, \ n \in \mathbb{N} \), satisfying the corona condition

\[
\sum_{j=1}^n |f_j(z)| \geq \delta > 0 \quad \text{for all} \quad z \in \mathbb{D},
\]

there exist functions \( g_1, \ldots, g_n \in H^\infty_I \) such that

\[
\sum_{j=1}^n f_j g_j = 1 \quad \text{on} \quad \mathbb{D}.
\]

From here using some basic results due to Suárez \cite{S1} and Treil \cite{T6} on the structure of \( M(H^\infty) \) one obtains the following topological description of \( M(H^\infty_I) \).

For an ideal \( I \subset H^\infty \), we define

\[
hull(I) := \{ x \in M(H^\infty) : \hat{f}(x) = 0 \quad \forall f \in I \}.
\]

**Proposition 2.1.** (a) There is a continuous surjective map \( Q_Z : M(H^\infty) \to M(H^\infty_I) \), \( Z := \hull(I) \), sending \( Z \) to a point and one-to-one outside of \( Z \).

(b) Covering dimension \( \dim M(H^\infty_I) = 2 \).

(c) Čech cohomology group \( H^2(M(H^\infty_I), Z) = 0 \).

(Recall that for a normal space \( X \), \( \dim X \leq n \) if every finite open cover of \( X \) can be refined by an open cover whose order \( \leq n + 1 \). If \( \dim X \leq n \) and the statement \( \dim X \leq n - 1 \) is false, we say that \( \dim X = n \).)

**Remark 2.2.** (1) Part (a) of the proposition says that \( M(H^\infty_I) \) is homeomorphic to the (Alexandroff) one-point compactification of space \( M(H^\infty) \setminus Z \) and \( \mathbb{D} \setminus Z \) is an open dense subset of \( M(H^\infty_I) \).

(2) Proposition 2.1 implies that algebra \( H^\infty_I \) is projective free (i.e., every projective \( H^\infty_I \)-module is free), see, e.g., \cite{BS, Cor. 1.4].
A closed subset \( Z \subset M(H^\infty) \) such that \( Z = \text{hull}(I) \) for an ideal \( I \subset H^\infty \) is called a hull. For a hull \( Z \) by \( \mathcal{A}_Z \) we denote the partially ordered by inclusion set of all algebras \( H^\infty_I \) for which \( \text{hull}(I) = Z \). E.g., if \( Z = \emptyset \), then \( \mathcal{A}_Z = \{H^\infty\} \). But, in general, set \( \mathcal{A}_Z \) may be even uncountable.

**Example 2.3.** Let \( v \) be a (unbounded) holomorphic function on \( \mathbb{D} \) such that \( u := e^v \in H^\infty \) is an inner function. For each \( \alpha \in (0, 1) \) we define the inner function \( u_\alpha := e^{\alpha v} \in H^\infty \). Let \( I(u_\alpha) \subset H^\infty \) be the principal ideal generated by \( u_\alpha \). Then \( \text{hull}(I(u_\alpha)) = \text{hull}(I(u_1)) =: Z \) is a nonempty compact subset of \( M(H^\infty) \setminus \mathbb{D} \). Moreover, all ideals \( I(u_\alpha) \) are closed and \( I(u_\alpha) \subsetneq I(u_\beta) \) for \( \beta < \alpha \). Thus, the corresponding set \( \mathcal{A}_Z \) contains a subset of the cardinality of the continuum.

Each set \( \mathcal{A}_Z \) contains the unique maximal subalgebra \( H^\infty_{I(Z)} \), where \( I(Z) := \{f \in H^\infty : \hat{f}(x) = 0 \ \forall x \in Z\} \). For instance, if \( Z \) is a single point, then \( I(Z) \subset H^\infty \) is a maximal ideal and \( H^\infty_{I(Z)} = H^\infty \). In turn, if \( Z \) is the zero locus of \( \hat{b} \), where \( b \) is a Blaschke product with simple zeros, then \( H^\infty_{I(Z)} = H^\infty_{I(b)} \).

**Convention.** In what follows we assume that map \( Q_Z \) of Proposition 2.1 satisfies \( Q_Z|_{M(H^\infty)\setminus Z} = \text{id} \). Then maximal ideal spaces of algebras in \( \mathcal{A}_Z \) coincide with \( M(H^\infty_{I(Z)}) \). This particular space will be denoted by \( M(\mathcal{A}_Z) \).

### 2.2. Oka Principle for Holomorphic Principal Bundles on \( M(\mathcal{A}_Z) \)

Let \( U \subset M(\mathcal{A}_Z) \) be an open subset and \( X \) be a complex Banach manifold (i.e., a complex manifold modelled on a complex Banach space).

A continuous map \( f : U \to X \) is said to be holomorphic (written as \( f \in \mathcal{O}(U, X) \)) if restriction \( f|_{U \cap (\mathbb{D} \setminus Z)} \) is a holomorphic map of complex manifolds.

For \( X = \mathbb{C} \) we set \( \mathcal{O}(U) := \mathcal{O}(U, \mathbb{C}) \).

Using \([\text{Br}1, \text{Prop. 1.3}]\) one obtains the following description of \( X \)-valued holomorphic functions on \( U \cap (\mathbb{D} \setminus Z) \) having continuous extensions to \( U \).

**Proposition 2.4.** A map \( f \in \mathcal{O}(U \cap (\mathbb{D} \setminus Z), X) \) extends to a map in \( \mathcal{O}(U, X) \) if and only if there exist open covers \( (U_\alpha)_{\alpha \in A} \) of \( U \) and \( (V_\beta)_{\beta \in B} \) of \( X \) and a map \( \tau : A \to B \) such that

(a) for each \( \beta \in B \) holomorphic functions on \( V_\beta \) separate points and

\[
\bigcap_{W \subset Q_Z^{-1}(U) : W \cap Z \neq \emptyset} f(W \cap (\mathbb{D} \setminus Z)) \neq \emptyset.
\]

(b) \( f(U_\alpha \cap (\mathbb{D} \setminus Z)) \in V_{\tau(\alpha)} \) for all \( \alpha \in A \);

(Here for topological spaces \( S, Y \) the implication \( S \subset Y \) means that the closure \( \tilde{S} \) of \( S \) in \( Y \) is compact and \( \tilde{S} \) stands for the interior of \( S \).)

**Remark 2.5.** (1) Condition (a) implies that map \( Q_Z f := f \circ Q_Z \) extends continuously to \( M(H^\infty) \) and then (b) guarantees that this extension is constant on \( Z \). If \( Z = \mathbb{Z} \cap \mathbb{D} \), then instead of (b) one can assume that \( Q_Z f \) extends to a continuous map on \( \mathbb{D} \) taking the same value on \( Z \).

(2) If \( X \) is a complex submanifold of a complex Banach space, \( f \in \mathcal{O}(\mathbb{D} \setminus Z, X) \) extends to a map in \( \mathcal{O}(M(\mathcal{A}_Z), X) \) iff \( f|_{\mathbb{D} \setminus Z} \in X \) and for each \( g \in \mathcal{O}(X) \) the Gelfand transform \( g \circ f \) is constant on \( Z \). (Note that \( g \circ f \in \mathcal{O}(\mathbb{D} \setminus Z) \) extends to a function in \( H^\infty \) by the Riemann extension theorem.) Here the first condition implies that \( Q_Z f \) extends continuously to \( M(H^\infty) \) by \([\text{Br}1, \text{Prop. 1.3}]\), and the second one that this extension is constant on \( Z \).
Let $U$ be an open subset of $M(\mathcal{A}_Z)$. A topological principal $G$-bundle $\pi : P \to U$ with fibre a complex Banach Lie group is called holomorphic if it is defined on an open cover $\{U_i\}_{i \in I}$ of $U$ by a cocycle $\{g_{ij} \in \mathcal{O}(U_i \cap U_j, G)\}_{i,j \in I}$. In this case, $P|_{U \cap (\mathbb{D} \setminus Z)}$ is a holomorphic principal $G$-bundle on $U \cap (\mathbb{D} \setminus Z)$ in the usual sense.

Recall that $P$ is defined as the quotient space of disjoint union $\bigsqcup_{i \in I} U_i \times G$ by the equivalence relation:

\begin{equation}
U_j \times G \ni u \times g \sim u \times gg_{ij}(u) \in U_i \times G.
\end{equation}

The projection $\pi : P \to U$ is induced by natural projections $U_i \times G \to U_i, i \in I$.

A bundle isomorphism $\varphi : (P_1, G, \pi_1) \to (P_2, G, \pi_2)$ of holomorphic principal $G$-bundles on $U$ is called holomorphic if $\varphi|_{U \cap (\mathbb{D} \setminus Z)} : P_1|_{U \cap (\mathbb{D} \setminus Z)} \to P_2|_{U \cap (\mathbb{D} \setminus Z)}$ is a biholomorphic map of complex Banach manifolds.

We say that a holomorphic principal $G$-bundle $(P, G, \pi)$ on $U$ is trivial if it is holomorphically isomorphic to the trivial bundle $U \times G$. (For basic facts of the theory of bundles, see, e.g., [Hus].)

For a complex Banach Lie group $G$ by $G_0$ we denote the connected component containing unit $1_G \in G$. Then $G_0$ is a clopen normal subgroup of $G$. By $q : G \to G/G_0 =: C(G)$ we denote the continuous quotient homomorphism onto the discrete group of connected components of $G$. Let $\pi : P \to U (\subset M(\mathcal{A}_Z))$ be a holomorphic principal $G$-bundle defined on an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of $U$ by a cocycle $g = \{g_{ij} \in \mathcal{O}(U_i \cap U_j, G)\}_{i,j \in I}$. By $P_{C(G)}$ we denote the principal bundle on $U$ with discrete fibre $C(G)$ defined on cover $\mathcal{U}$ by locally constant cocycle $q(g) = \{q(g_{ij}) \in C(U_i \cap U_j, C(G))\}_{i,j \in I}$. Let $K \subset U$ be a compact subset. The next result used in the applications in particular characterizes trivial holomorphic bundles on $M(\mathcal{A}_Z)$.

**Theorem 2.6.** $P$ is trivial over a neighbourhood of $K$ if and only if the associated bundle $P_{C(G)}$ is topologically trivial over $K$.

**Corollary 2.7.** $P$ is trivial over a neighbourhood of $K$ in one of the following cases:

1. Group $G$ is connected;
2. $U = M(H^\infty)$ and images of all maps $q(g_{ij})$ belong to a finite subgroup of $C(G)$ (e.g., this is true if $G$ has finitely many connected components);
3. $P$ is trivial over $K$ in the category of topological principal $G$-bundles.

In turn, not all principal bundles on $M(\mathcal{A}_Z)$ are trivial:

**Proposition 2.8.** Let $G$ be a complex Banach Lie group such that $C(G)$ has a nontorsion element. Then there exists a nontrivial holomorphic principal $G$-bundle on $M(\mathcal{A}_Z)$.

Let $\mathcal{P}_G^O$ and $\mathcal{P}_G^C$ be the sets of isomorphism classes of holomorphic and topological principal $G$-bundles on $M(\mathcal{A}_Z)$, respectively.

The following two results (analogous to the classical results of Grauert [Gra] and Bungart [Bun]) show that the natural map $i : \mathcal{P}_G^O \to \mathcal{P}_G^C$ induced by inclusion of the category of holomorphic principal bundles on $M(\mathcal{A}_Z)$ to the category of topological ones is a bijection. These constitute the Oka Principle for holomorphic principal bundles on $M(\mathcal{A}_Z)$.

**Theorem 2.9 (Injectivity of $i$).** If two holomorphic principal $G$-bundles on $M(\mathcal{A}_Z)$ are isomorphic as topological bundles, they are holomorphically isomorphic.

**Theorem 2.10 (Surjectivity of $i$).** Each topological principal $G$-bundle on $M(\mathcal{A}_Z)$ is isomorphic to a holomorphic one.
Let $A \in \mathcal{A}_Z$, i.e., $A = H^\infty_I$ for some closed ideal $I \subset H^\infty$ such that $\text{hull}(I) = Z$. Recall that we assumed that $M(A)$ coincides with $M(\mathcal{A}_Z)$.

Let $D$ be the set of all finite subsets of $A$ directed by inclusion. If $\alpha = \{f_1, \ldots, f_n\} \in D$ we let $A_\alpha$ be the unital closed subalgebra of $A$ generated by $\alpha$. By $M(A_\alpha)$ we denote the maximal ideal space of $A_\alpha$. It is naturally identified with the polynomially convex hull of the image of $F_\alpha : M(\mathcal{A}_Z) \rightarrow \mathbb{C}^n$, $F_\alpha(x) := (\hat{f}_1(x), \ldots, \hat{f}_n(x))$ (here $\hat{\cdot}$ is the Gelfand transform for algebra $H^\infty_I(x)$). If $\alpha, \beta \in D$ with $\alpha \supseteq \beta$, then linear map $F^\#_\beta : \mathbb{C}^\#_\alpha \rightarrow \mathbb{C}^\#_\beta$, $\mathbb{C}^\#_\alpha \ni (z_1, \ldots, z_\#_\alpha) \mapsto (z_1, \ldots, z_\#_\beta) \in \mathbb{C}^\#_\beta$, sends $M(A_\alpha)$ to $M(A_\beta)$. Thus we obtain the inverse system of compacta $\{M(A_\alpha), F^\#_\beta\}$ whose limit is naturally identified with $M(\mathcal{A}_Z)$ and the limit projections coincide with maps $F_\alpha$ (see, e.g., [Ro] for details).

Let $P_i$ be holomorphic principal $G$-bundles defined on open neighbourhoods $O_i \subset \mathbb{C}^\#_\alpha$ of $M(A_\alpha)$, $i = 1, 2$. We say that $P_1$ and $P_2$ are isomorphic if they are holomorphically isomorphic on an open neighbourhood $O \subset O_1 \cap O_2$ of $M(A_\alpha)$. By $(\mathcal{P}_G^\alpha)_\alpha$ we denote the set of isomorphism classes of holomorphic principal $G$-bundles defined on neighbourhoods of $M(A_\alpha)$). Projections $F^\#_\beta$ induce maps $(\mathcal{P}_G^\alpha)_\beta \rightarrow (\mathcal{P}_G^\alpha)_\alpha$ assigning to the isomorphism class of a bundle $P$ the isomorphism class of its pullback $(F^\#_\beta)^*P$. Thus we obtain the direct system of sets $\{(\mathcal{P}_G^\beta)_\alpha, (\mathcal{P}_G^\alpha)_\alpha\}$.

Similarly, limit projection $F_\alpha$ induces a map $\mathcal{F}_\alpha : (\mathcal{P}_G^\alpha)_\beta \rightarrow (\mathcal{P}_G^\alpha)_\alpha$ assigning to the isomorphism class of a bundle $P$ the isomorphism class of its pullback $F^*\alpha P$. Since $F^\#_\beta = F^\alpha \circ (\mathcal{P}_G^\alpha)_\alpha$ for all $\alpha \supseteq \beta$ in $D$, the family of maps $\{\mathcal{F}_\alpha\}_{\alpha \in D}$ induces a map $\mathcal{F}_A : \lim_{\rightarrow} (\mathcal{P}_G^\alpha)_\alpha \rightarrow \mathcal{P}_G^\alpha$ is a bijection.

In particular,

$$\mathcal{P}_G^\alpha = \bigcup_{\alpha \in D} \mathcal{F}_\alpha^*((\mathcal{P}_G^\alpha)_\alpha).$$

The statement consists of two parts:

1. (Surjectivity of $\mathcal{F}_A$). For each holomorphic principal $G$-bundle $P$ on $M(\mathcal{A}_Z)$ there exist $\alpha \in D$ and a holomorphic principal $G$-bundle $\hat{P}$ defined on a neighbourhood of $M(A_\alpha)$ such that bundles $P$ and $F^\alpha_\alpha \hat{P}$ are holomorphically isomorphic.

2. (Injectivity of $\mathcal{F}_A$). If holomorphic principal $G$-bundles $P_1, P_2$ defined on a neighbourhood of $M(A_\beta)$ are such that $F^\beta_\beta P_1$ and $F^\beta_\beta P_2$ are holomorphically isomorphic bundles, then there exist $\alpha \supseteq \beta$ and a neighbourhood $U$ of $M(A_\alpha)$ such that bundles $(F^\alpha_\alpha)^*P_1$ and $(F^\alpha_\alpha)^*P_2$ are defined on $U$ and holomorphically isomorphic.

3. Oka Principle for Maps to Complex Homogeneous Spaces

3.1. Holomorphic Maps from $M(\mathcal{A}_Z)$ to Banach Homogeneous Spaces. Let $K \subset M(\mathcal{A}_Z)$ be a compact subset and $X$ a complex Banach manifold. By $\mathcal{O}(K, X) \subset \mathcal{C}(K, X)$ we denote the set of continuous maps holomorphic on neighbourhoods of $K$ and by $\mathcal{A}(K, X)$ the closure of $\mathcal{O}(K, X)$ in the topology of uniform convergence of $\mathcal{C}(K, X)$. If $G$ is a complex Banach Lie group with Lie algebra $\mathfrak{g}$ and exponential map $\exp_G : \mathfrak{g} \rightarrow G$, set $\mathcal{A}(K, G)$ is a complex Banach Lie group with respect to pointwise product of maps with Lie algebra $\mathcal{A}(K, \mathfrak{g})$ and the exponential map being the composition of $\exp_G$ with elements of $\mathcal{A}(K, \mathfrak{g})$.

The following result interesting in its own right will be used in the subsequent proofs.

**Theorem 3.1.** If $G$ is simply connected, then $\mathcal{A}(K, G)$ is path-connected.

To formulate further results we invoke the definition of a complex Banach homogeneous space (see, e.g., [R] Sec. 1)].
Suppose a complex Banach Lie group \( G \) with unit \( e \) acts holomorphically and transitively on a complex Banach manifold \( X \), i.e., there is a holomorphic map \((g, p) \mapsto g \cdot p\) of \( G \times X \) onto \( X \) satisfying \( g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p \) and \( e \cdot p = p \) for all \( g_1, g_2 \in G, p \in X \), and for each pair \( p, q \in X \) there is some \( g \in G \) with \( g \cdot p = q \). For \( p \in X \) consider holomorphic surjective map \( \pi^p : G \to X, \pi^p(g) := g \cdot p, g \in G \). Let \( G_e (\cong g) \) and \( X_p \) be tangent spaces of \( G \) at \( e \) and \( X \) at \( p \). By \( d\pi^p_e : G_e \to X_p \) we denote the differential of \( \pi^p \) at \( e \).

Theorem 3.2. Let \( X \) be a complex Banach homogeneous space. Then \( \Theta \) is a bijection.

The statement consists of two parts:

1. (Injectivity of \( \Theta \)). If two maps in \( \mathcal{O}(M(\mathcal{A}_Z), X) \) are homotopic in \( C(M(\mathcal{A}_Z), X) \), they are homotopic in \( \mathcal{O}(M(\mathcal{A}_Z), X) \).

2. (Surjectivity of \( \Theta \)). Every map in \( C(M(\mathcal{A}_Z), X) \) is homotopic to a map in \( \mathcal{O}(M(\mathcal{A}_Z), X) \).

Next, recall that a path-connected topological space \( X \) is \( n \)-simple if for each \( x \in X \) the fundamental group \( \pi_1(X, x_0) \) acts trivially on the \( n \)-homotopy group \( \pi_n(X, x) \) (see, e.g., [Hu] Ch. IV.16) for the corresponding definitions and results). For instance, \( X \) is \( n \)-simple if group \( \pi_n(X) = 0 \) and \( 1 \)-simple if and only if group \( \pi_1(X) \) is abelian. Also, it is worth noting that every path-connected topological group is \( n \)-simple for all \( n \).

Corollary 3.3. Let \( X \) be a complex Banach homogeneous space \( n \)-simple for all \( n \leq 2 \). Then there is a natural one-to-one correspondence between elements of \( [M(\mathcal{A}_Z), X]_\mathcal{O} \) and the Čech cohomology group \( H^1(M(\mathcal{A}_Z), \pi_1(X)) \). In particular, if \( X \) is simply connected, space \( \mathcal{O}(M(\mathcal{A}_Z), X) \) is path-connected.

Remark 3.4. (1) The correspondence of the corollary is described in Remark [10.4]

(2) Group \( H^1(M(\mathcal{A}_Z), \mathbb{Z}) \) is always nontrivial, see Lemma [7.4].

Recall that for each \( A \in \mathcal{A}_Z \) space \( M(\mathcal{A}_Z) \) can be presented as the inverse limit of the inverse limit system \( \{M(A_\alpha), F^\alpha_\beta\}, \alpha \in D, \) see Section 2.2. Two maps \( M(A_\alpha) \to X \) holomorphic in a neighbourhood \( U \) of \( M(A_\alpha) \) are said to be homotopic in \( \mathcal{O}(M(A_\alpha), X) \) if their restrictions to a neighbourhood \( V \subset U \) of \( M(A_\alpha) \) can be joined by a path in \( \mathcal{O}(V, X) \). This homotopy relation is an equivalence relation with the set of equivalence classes denoted by \( [M(A_\alpha), X]_\mathcal{O} \).

Projections \( F^\alpha_\beta \) induce maps \( (\mathfrak{F}_\beta^\alpha)^* : [A_\beta, X]_\mathcal{O} \to [A_\alpha, X]_\mathcal{O} \) assigning to the homotopy class of a map \( f \in \mathcal{O}(M(A_\beta), X) \) the homotopy class of its pullback \( (F^\alpha_\beta)^* f \). Thus we obtain the direct system of sets \( \{[M(A_\alpha), X]_\mathcal{O}, (\mathfrak{F}_\beta^\alpha)^*\} \).

Similarly, limit projection \( F^\alpha_\alpha \) induces a map \( \mathfrak{F}^\alpha_\alpha : [M(A_\alpha), X]_\mathcal{O} \to [M(\mathcal{A}_Z), X] \) assigning to the homotopy class of \( f \in \mathcal{O}(M(A_\alpha), X) \) the homotopy class of its pullback \( F^\alpha_\alpha f \). Since \( \mathfrak{F}^\alpha_\beta = \mathfrak{F}^\alpha_\alpha \circ (\mathfrak{F}_\alpha^\beta)^* \) for all \( \alpha \geq \beta \) in \( D \), the family of maps \( \{\mathfrak{F}^\alpha_\alpha\}_{\alpha \in D} \) induces a map \( \mathfrak{F}_A \) of the direct limit \( \bigcup_{\alpha \in D} [M(A_\alpha), X]_\mathcal{O} \) into \( [M(\mathcal{A}_Z), X]_\mathcal{O} \).

Theorem 3.5. Let \( X \) be a complex Banach homogeneous space. Then \( \mathfrak{F}_A \) is a bijection.
In particular,

\[ [M(\mathcal{A}_Z), X]|_O = \bigcup_{\alpha \in D} \mathfrak{F}_\alpha^*([M(A_\alpha), X]|_O). \]

To prove the result we establish the following:

1. (Surjectivity of \( \mathfrak{F}_A \)). For each holomorphic map \( f : M(\mathcal{A}_Z) \to X \) there exist \( \alpha \in D \) and a holomorphic map into \( X \) defined on a neighbourhood of \( M(A_\alpha) \) such that maps \( f \) and \( F_\alpha^*f \) are homotopic in \( \mathcal{O}(M(\mathcal{A}_Z), X) \).

2. (Injectivity of \( \mathfrak{F}_A \)). If holomorphic maps \( f_1, f_2 \) into \( X \) defined on a neighbourhood of \( M(A_\beta) \) are such that \( F_\beta^*f_1 \) and \( F_\beta^*f_2 \) are homotopic in \( \mathcal{O}(M(\mathcal{A}_Z), X) \), then there exist \( \alpha \supseteq \beta \) and a neighbourhood \( U \) of \( M(A_\alpha) \) such that maps \( (F_\beta^*)^{-1}f_1 \) and \( (F_\beta^*)^{-1}f_2 \) are defined on \( U \) and homotopic in \( \mathcal{O}(U, X) \).

Let \( X \) be a complex Banach homogeneous manifold under the action of a complex Banach Lie group \( G \). It is known, see, e.g., \([R, \text{Prop. 1.4}]\), that the stabilizer of a point \( p \in X \), \( G(p) := \{ g \in G : \pi^p(g) = p \} \), is a closed complex Banach Lie subgroup of \( G \) and stabilizers of different points are conjugate in \( G \) by inner automorphisms.

The following result will be used in applications.

**Theorem 3.6.** Let \( X \) be a complex Banach homogeneous space under the action of a complex Banach Lie group \( G \). Assume that group \( G(p) \subset G, p \in X, \) is connected. Then for every \( f \in \mathcal{O}(M(\mathcal{A}_Z), X) \) and each \( p \in X \) there is a map \( \tilde{f}_p \in \mathcal{O}(M(\mathcal{A}_Z), G) \) such that \( f(x) = \tilde{f}_p(x) \cdot p \) for all \( x \in M(\mathcal{A}_Z) \).

3.2. Nonlinear Approximation and Interpolation Problems. A compact subset \( K \subset M(\mathcal{A}_Z) \) is called holomorphically convex if for any \( x \notin K \) there is \( f \in \mathcal{O}(M(\mathcal{A}_Z)) \) such that

\[ \max_{K} |f| < |f(x)|. \]

Note that for a natural number \( l \) any subset of \( M(\mathcal{A}_Z) \) of the form

\[ \{ x \in M(\mathcal{A}_Z) : \max_{1 \leq j \leq l} |f_j(x)| \leq 1, f_j \in \mathcal{O}(M(\mathcal{A}_Z)), 1 \leq j \leq l \} \]

is holomorphically convex and each holomorphically convex \( K \subset M(\mathcal{A}_Z) \) is intersection of such subsets.

Let \( U \) be an open neighbourhood of a holomorphically convex set \( K \subset M(\mathcal{A}_Z) \) and \( X \) be a complex Banach homogeneous space. The following result is a nonlinear analog of the Runge approximation theorem.

**Theorem 3.7.**

1. Suppose a map in \( \mathcal{O}(U, X) \) can be uniformly approximated on \( K \) by maps in \( C(M(\mathcal{A}_Z), X) \). Then it can be uniformly approximated on \( K \) by maps in \( \mathcal{O}(M(\mathcal{A}_Z), X) \).

2. If \( X \) is simply connected, then each map in \( \mathcal{O}(U, X) \) can be uniformly approximated on \( K \) by maps in \( \mathcal{O}(M(\mathcal{A}_Z), X) \).

Let \( Z \subset M(H^\infty) \) be a hull and \( U \) be an open neighbourhood of \( Z \).

**Theorem 3.8.** Let \( X \) be a complex Banach homogeneous space and \( f \in \mathcal{O}(U, X) \).

1. Restriction \( f|_Z \in C(Z, X) \) extends to a map in \( \mathcal{O}(M(H^\infty), X) \) if and only if it extends to a map in \( C(M(H^\infty), X) \).

2. \( f|_Z \) extends to a map in \( \mathcal{O}(M(H^\infty), X) \) if \( X \) is \( n \)-simple for all \( n \leq 2 \).

**Remark 3.9.** (1) The quantitative version of the second part of Theorem 3.8 for \( X = \mathbb{C} \) and \( Z \) the zero locus of the Gelfand transform of a Blaschke product was proved by Carleson \([\text{C}]\). For \( X = \mathbb{C}^n \) the result of part (2) follows from Treil’s theorem \([\text{T6}]\) via its cohomological interpretation due to Suárez \([\text{S1}, \text{Th. 1.3}]\) and the Arens-Royden theorem.
under the action of group $G$. It has the natural structure of a simply connected compact complex homogeneous space $(d \text{ of connected and consists of all maps of the form } \alpha \in GL(n) \text{ of } h \text{ representation, we also obtain (see Remark 2.5(2)) that } O \cap \mathbb{D} \text{ isomorphic to the group of nonsingular block upper triangular matrices with dimensions of blocks } d_i - d_{i-1} \text{ with } d_0 := 0. \text{ Thus space } O(M(\mathbb{A}_Z), F(d_1, \ldots, d_k)) \text{ is path-connected and consists of all maps of the form } f \cdot p_0, \text{ where } f \in O(M(\mathbb{A}_Z), G) \text{ and } p_0 \text{ is the orbit of the unit of } G. \text{ Since } G \text{ admits a faithful linear representation, we also obtain (see Remark 2.5(2)) that } g \in O(\mathbb{D} \setminus Z, X) \text{ extends to a map in } O(M(\mathbb{A}_Z), X) \text{ iff } g = f \cdot p_0 \text{ for some } f \in O(\mathbb{D} \setminus Z, G) \text{ such that } f(\mathbb{D} \setminus Z) \subseteq G \text{ and for every } h \in O(G) \text{ the Gelfand transform } h \circ f \text{ is constant on } Z. $

Example 4.1. The complex flag manifold $F(d_1, \ldots, d_k)$ is the space of all flags of type $(d_1, \ldots, d_k)$ in $\mathbb{C}^n$, $n := d_k$, i.e., of increasing sequences of subspaces $\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k = \mathbb{C}^n$ with $\dim \mathbb{C} V_i = d_i$. 

It has the natural structure of a simply connected compact complex homogeneous space under the action of group $GL_n(\mathbb{C})$. The stabilizer of a flag is the connected subgroup of $GL_n(\mathbb{C})$ isomorphic to the group of nonsingular block upper triangular matrices with dimensions of blocks $d_i - d_{i-1}$ with $d_0 := 0$. Thus space $O(M(\mathbb{A}_Z), F(d_1, \ldots, d_k))$ is path-connected and consists of all maps of the form $f \cdot p_0$, where $f \in O(M(\mathbb{A}_Z), GL_n(\mathbb{C}))$ and $p_0$ is the flag of subspaces $V_i^0 := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : z_1 = z_2 = \cdots = z_{n-d_i} = 0\}, 0 \leq i \leq k - 1$, $V_k := \mathbb{C}^n$. Here the image of $V_i^0$ under $f(x)$, $x \in M(\mathbb{A}_Z)$, is the subspace of $\mathbb{C}^n$ generated by the last $d_i$ column vectors of matrix $f(x)$.  

Example 3.10. Let $F \subset \{z \in \mathbb{C} : |z| = 1\}$ be a closed subset of Lebesgue measure zero. By $I_F \subset H^\infty$ we denote the ideal generated by all functions of the disk-algebra $A(\mathbb{D}) (= H^\infty \cap C(\mathbb{D}))$ equal zero on $F$. According to the Rudin-Carleson theorem for each $f \in C(F)$ there exists a function $\tilde{f} \in A(\mathbb{D})$ such that $\tilde{f}|_F = f$ and $\|\tilde{f}\|_{A(\mathbb{D})} = \|f\|_{C(F)}$. The map transposed to embedding $A(\mathbb{D}) \hookrightarrow H^\infty$ induces a continuous surjection of the maximal ideal spaces $p : M(H^\infty) \rightarrow \hat{\mathbb{D}} (= M(A(\mathbb{D})))$ such that $Z := \text{hull}(I_F) = p^{-1}(F)$. Let $O \subset \mathbb{D}$ be a relatively open subset containing $F$. Then $U := p^{-1}(O)$ is an open neighbourhood of $Z$ and $U \cap \mathbb{D} = O \cap \mathbb{D}$. Theorem 3.8 implies the following nonlinear version of the Rudin-Carleson theorem: Let $G$ be a connected complex Banach Lie group and $f \in O(U \cap \mathbb{D}, G)$ satisfy (c). Then there exists a map $\tilde{f} \in O(\mathbb{D}, G)$ satisfying (c) such that $\lim_{z \to x} \tilde{f}(z) f(z)^{-1} = 1_G$ for all $x \in F$; here $1_G \in G$ is the unit. 

4. Applications and Examples

4.1. Holomorphic Maps of $M(\mathbb{A}_Z)$ into Flag Manifolds and Complex Tori. For basic results of the Lie group theory, see, e.g., [OV].

Recall that the maximal connected solvable Lie subgroup $B$ of a connected (finite-dimensional) complex Lie group $G$ is called a Borel subgroup. A Lie subgroup $P \subset G$ containing some Borel subgroup is called parabolic. The set of orbits $X = G/P$ under the action of a parabolic subgroup $P \subset G$ on $G$ by right multiplications has the natural structure of a complex homogeneous space and is called the flag manifold. Since $P$ contains the radical of $G$, one may assume that in the definition of $X$ group $G$ is semisimple. Each flag manifold is simply connected and thus by Corollary 3.3 and Theorem 3.6 space $O(M(\mathbb{A}_Z), X)$ is path-connected and consists of all maps of the form $f \cdot p_0$, where $f \in O(M(\mathbb{A}_Z), G)$ and $p_0$ is the orbit of the unit of $G$. Since $G$ admits a faithful linear representation, we also obtain (see Remark 2.5(2)) that $g \in O(\mathbb{D} \setminus Z, X)$ extends to a map in $O(M(\mathbb{A}_Z), X)$ iff $g = f \cdot p_0$ for some $f \in O(\mathbb{D} \setminus Z, G)$ such that $f(\mathbb{D} \setminus Z) \subseteq G$ and for every $h \in O(G)$ the Gelfand transform $h \circ f$ is constant on $Z$. 

Example 4.1. The complex flag manifold $F(d_1, \ldots, d_k)$ is the space of all flags of type $(d_1, \ldots, d_k)$ in $\mathbb{C}^n$, $n := d_k$, i.e., of increasing sequences of subspaces $\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k = \mathbb{C}^n$ with $\dim \mathbb{C} V_i = d_i$. 

It has the natural structure of a simply connected compact complex homogeneous space under the action of group $GL_n(\mathbb{C})$. The stabilizer of a flag is the connected subgroup of $GL_n(\mathbb{C})$ isomorphic to the group of nonsingular block upper triangular matrices with dimensions of blocks $d_i - d_{i-1}$ with $d_0 := 0$. Thus space $O(M(\mathbb{A}_Z), F(d_1, \ldots, d_k))$ is path-connected and consists of all maps of the form $f \cdot p_0$, where $f \in O(M(\mathbb{A}_Z), GL_n(\mathbb{C}))$ and $p_0$ is the flag of subspaces $V_i^0 := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : z_1 = z_2 = \cdots = z_{n-d_i} = 0\}, 0 \leq i \leq k - 1$, $V_k := \mathbb{C}^n$. Here the image of $V_i^0$ under $f(x)$, $x \in M(\mathbb{A}_Z)$, is the subspace of $\mathbb{C}^n$ generated by the last $d_i$ column vectors of matrix $f(x)$.
For example, in the case of $\mathbf{F}(1, n)$, the $(n-1)$-dimensional complex projective space, we obtain that space $O(M(\mathcal{A}_Z), \mathbf{F}(1, n))$ consists of all maps $f$ of the form $f(x) = [f_1(x) : f_2(x) : \cdots : f_n(x)], x \in M(\mathcal{A}_Z)$, where $f_i \in O(M(\mathcal{A}_Z))$ and satisfy the corona condition on $M(\mathcal{A}_Z)$ (cf. (2.1)). (Note that the column vector composed of such $f_i$ extends automatically to an invertible matrix with entries in $O(M(\mathcal{A}_Z))$ because of projective freeness of this algebra, see Remark 2.2.)

Let $\mathbb{CT}^n$ be the complex torus obtained as the quotient of $\mathbb{C}^n$ by a lattice $\Gamma (\cong \mathbb{Z}^{2n})$ (acting on $\mathbb{C}^n$ by translations). By $\pi : \mathbb{C}^n \to \mathbb{CT}^n$ we denote the holomorphic projection map. In the next result we describe the structure of space $O(M(\mathcal{A}_Z), \mathbb{CT}^n)$.

Recall that a holomorphic function $f$ on $\mathbb{D}$ belongs to the BMOA space if it has boundary values a.e. on $\mathbb{T} := \{ z \in \mathbb{C} : | z | = 1 \}$ satisfying

$$\sup_I \frac{1}{| I |} \int_I | f(\theta) - f_I | d\theta < \infty,$$

where $I \in \mathbb{T}$ is an open arc of arclength $| I |$ and $f_I := \frac{1}{| I |} \int_I f(\theta) d\theta$.

**Theorem 4.2.** A map $f \in O(\mathbb{D}, \mathbb{CT}^n)$ extends continuously to $M(H^\infty)$ iff it is factorized as $f = \pi \circ \tilde{f}$ for some $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_n) \in O(\mathbb{D}, \mathbb{C}^n)$ with all $\tilde{f}_i \in$ BMOA.

**Remark 4.3.** (1) Let $Q_Z : M(H^\infty) \to M(\mathcal{A}_Z)$ be the continuous map transposed to embedding $H^\infty_I \to H^\infty$. Clearly, $f \in O(\mathbb{D} \setminus Z, \mathbb{CT}^n)$ extends continuously to $M(\mathcal{A}_Z)$ iff $Q_Z f$ extends to a map $C(M(H^\infty), \mathbb{CT}^n)$ constant on $Z$.

(2) It follows from the proof of Theorem 3.2 that if $f, g \in O(M(\mathcal{A}_Z), \mathbb{CT}^n)$ are homotopic, then there is a map $h \in O(M(\mathcal{A}_Z), \mathbb{C}^n)$ such that $f(x) = g(x) + (\pi \circ h)(x)$ for all $x \in M(\mathcal{A}_Z)$. (Here $+$ stands for addition on $\mathbb{CT}^n$ induced from that on $\mathbb{C}^n$.)

Space $O(M(\mathcal{A}_Z), \mathbb{CT}^n)$ is an abelian complex Banach Lie group under the pointwise addition of maps. Thus, according to Corollary 3.3 $H^1(M(\mathcal{A}_Z), \mathbb{Z}^{2n})$ is naturally isomorphic to the quotient of group $O(M(\mathcal{A}_Z), \mathbb{CT}^n)$ by the connected component of its unit $\pi(O(M(\mathcal{A}_Z), \mathbb{C}^n))$.

### 4.2. Structure of Spaces of $H^\infty$ Idempotents.

Let $\mathfrak{A}$ be a complex Banach algebra with unit $1_\mathfrak{A}$. By $\text{id} \mathfrak{A} = \{ a \in \mathfrak{A} : a^2 = a \}$ we denote the set of idempotents of $\mathfrak{A}$. It is a closed complex Banach submanifold of $\mathfrak{A}$ which is a discrete union of connected complex Banach homogeneous spaces, see [R, Cor.1.7]. Specifically, let $\mathfrak{A}_0^{-1}$ be the connected component containing the unit $1_\mathfrak{A}$ of the complex Banach Lie group $\mathfrak{A}_0^{-1}$ of invertible elements of $\mathfrak{A}$. Then each connected component of $\text{id} \mathfrak{A}$ is a complex Banach homogeneous space under the action $\mathfrak{A}_0^{-1} \times \text{id} \mathfrak{A} \to \text{id} \mathfrak{A}$ by similarity transformations $\alpha (a, p) \to \alpha a p^{-1}$.

Analogously for unital complex Banach algebras $C(M(\mathcal{A}_Z), \mathfrak{A})$ and $O(M(\mathcal{A}_Z), \mathfrak{A})$ their sets of idempotents $\text{id} C(M(\mathcal{A}_Z), \mathfrak{A})$ and $\text{id} O(M(\mathcal{A}_Z), \mathfrak{A})$ coincide with $C(M(\mathcal{A}_Z), \text{id} \mathfrak{A})$ and $O(M(\mathcal{A}_Z), \text{id} \mathfrak{A})$ (because $M(\mathcal{A}_Z)$ is connected) and are discrete unions of connected complex Banach homogeneous spaces. Each connected component of $C(M(\mathcal{A}_Z), \text{id} \mathfrak{A})$ or $O(M(\mathcal{A}_Z), \text{id} \mathfrak{A})$ is the complex homogeneous space under the action of the complex Banach Lie group $C(M(\mathcal{A}_Z), \mathfrak{A})_0^{-1}$ or $O(M(\mathcal{A}_Z), \mathfrak{A})_0^{-1}$ by similarity transformations.

**Theorem 4.4.** (a) Embedding $O(M(\mathcal{A}_Z), \text{id} \mathfrak{A}) \hookrightarrow C(M(\mathcal{A}_Z), \text{id} \mathfrak{A})$ induces bijection between connected components of these complex Banach manifolds.

(b) If the stabilizer $\mathfrak{A}_0^{-1}(p) \subset \mathfrak{A}_0^{-1}$ of a point $p \in \text{id} \mathfrak{A}$ is connected, then for each $f \in O(M(\mathcal{A}_Z), \text{id} \mathfrak{A})$ whose image belongs to the connected component containing $p$ there is $g \in O(M(\mathcal{A}_Z), \mathfrak{A}_0^{-1})$ such that $f = gpg^{-1}$.

(c) If $A \in \mathcal{A}_Z$, then for each $f \in O(M(\mathcal{A}_Z), \text{id} \mathfrak{A})$ there are $\alpha \in D$, a map $h \in O(M(\mathcal{A}_Z), \text{id} \mathfrak{A})$ uniformly approximated by maps of the algebraic tensor product $\tilde{A}_{\alpha} \otimes \mathfrak{A}$ and a map $g \in O(M(\mathcal{A}_Z), \mathfrak{A}_0^{-1})$ such that $gfg^{-1} = h$. 


Here $\widehat{A}_\alpha$ is the image of $A_\alpha$ under the Gelfand transform $\hat{\cdot}: A \to C(M(\mathcal{A}_Z))$.

Remark 4.5. (1) If $\mathcal{A}_0^{-1}$ is simply connected and the hypothesis of part (b) holds for all $p \in \mathfrak{A}$, then, since in this case $\mathcal{O}(M(\mathcal{A}_Z), \mathcal{A}_0^{-1})$ coincides with $\mathcal{O}(M(\mathcal{A}_Z), \mathfrak{A})_0^{-1}$ (cf. Theorem 3.1), embedding $\text{id} \mathfrak{A} \hookrightarrow \mathcal{O}(M(\mathcal{A}_Z), \text{id} \mathfrak{A})$ assigning to each $p$ the map of the constant value $p$ induces bijection between sets of connected components of $\text{id} \mathfrak{A}$ and $\mathcal{O}(M(\mathcal{A}_Z), \text{id} \mathfrak{A})$.

(2) Part (c) follows from Theorem 3.5 and the fact that $F_0^*(\mathcal{O}(M(A_\alpha), \mathfrak{A}))$ coincides with $A_\alpha \otimes_\mathbb{C} \mathfrak{A}$, the injective tensor product of these algebras (cf. the arguments in Section 12).

(3) Let $H^\infty_Z(\mathfrak{A})$ be the Banach algebra of bounded maps $F \in \mathcal{O}(\mathfrak{D}, \mathfrak{A})$ equipped with pointwise product and addition with norm $\|F\| := \sup_{z \in \mathfrak{D}} \|F(z)\|_\mathfrak{A}$ such that for each $\varphi \in \mathfrak{A}^*$ the Gelfand transform of $\varphi \circ F \in H^\infty$ is constant on $Z$. Let $H^\infty_Z(\mathfrak{A}) \subset H^\infty_Z(\mathfrak{A})$ be the closed subalgebra of maps with relatively compact images. (If $Z$ is empty or a single point we omit $z$ in the indices.) Each map in $H^\infty_Z(\mathfrak{A})$ admits a continuous extension to $M(H^\infty)$ constant on $Z$, see, e.g., [Br1] Prop. 1.3. This implies that algebras $H^\infty_Z(\mathfrak{A})$ and $\mathcal{O}(M(\mathcal{A}_Z), \mathfrak{A})$ are isometrically isomorphic. Thus Theorem 4.4 describes the structure of set $\text{id} H^\infty_Z(\mathfrak{A})$.

Example 4.6. Let $L(X)$ be the Banach algebra of bounded linear operators on a complex Banach space $X$ equipped with the operator norm. By $I_X \in L(X)$ we denote the identity operator and by $GL(X) \subset L(X)$ the set of invertible bounded linear operators on $X$. Clearly, $L(X)^{-1} := GL(X)$. By $GL_0(X) \subset GL(X)$ we denote the connected component of $I_X$. Each $P \in L(X)$ determines a direct sum decomposition $X = X_0 \oplus X_1$, where $X_0 := \ker(P)$ and $X_1 := \ker(I_X - P)$. It is easily seen that the stabilizer $GL_0(X)(P) \subset GL_0(X)$ consists of all operators $B \in GL_0(X)$ such that $B(X_k) \subset X_k$, $k = 1, 2$. In particular, restrictions of operators in $GL_0(X)(P)$ to $X_k$ determine a monomorphism of complex Banach Lie groups $S_P : GL_0(X)(P) \to GL(X_1) \oplus GL(X_2)$. Moreover, $S_P$ is an isomorphism if $GL(X_i), i = 1, 2$ are connected. Now, Theorem 4.4 leads to the following statement:

(1) Suppose $P \in \text{id} L(X)$ is such that groups $GL(X_1)$ and $GL(X_2)$ are connected. Then for each $F \in \mathcal{O}(M(\mathcal{A}_Z), \text{id} L(X))$ whose image belongs to the connected component containing $P$ there is $G \in \mathcal{O}(M(\mathcal{A}_Z), GL_0(X))$ such that $G\text{F}G^{-1} = P$.

In particular, the result is valid for $X$ being one of the spaces: a finite-dimensional space, a Hilbert space, $c_0$ or $\ell^p$, $1 \leq p \leq \infty$. Indeed, $GL(X)$ is connected if $\dim X < \infty$ and contractible for other listed above spaces, see, e.g., [M] and references therein. Moreover, each subspace of a Hilbert space is Hilbert, and each infinite-dimensional complemented subspace of $X$ being either $c_0$ or $\ell^p$, $1 \leq p \leq \infty$, is isomorphic to $X$, see [Pd, Lin]. This gives the required conditions.

It is worth noting that there are complex Banach spaces $X$ for which groups $GL(X)$ are not connected, see, e.g., [D].

In turn, part (c) of the theorem implies in this case:

(2) Let $A \in \mathcal{A}_Z$. For each $F \in \mathcal{O}(M(\mathcal{A}_Z), \text{id} L(X))$ there exist a finitely generated unital subalgebra $B \subset A$ and maps $H \in \text{id} B \otimes_\mathbb{R} L(X), G \in \mathcal{O}(M(\mathcal{A}_Z), GL_0(X))$ such that $G\text{F}G^{-1} = H$.

4.3. Extension of Operator-valued $H^\infty$ Functions. Let $\mathfrak{A}$ be a complex Banach algebra with unit $1_\mathfrak{A}$. Let $\mathfrak{A}_l^{-1} = \{a \in \mathfrak{A} : \exists b \in \mathfrak{A}$ such that $ba = 1_\mathfrak{A}\}$ be the set of left-invertible elements of $\mathfrak{A}$. Clearly, $\mathfrak{A}_l^{-1}$ is an open subset of $\mathfrak{A}$ and complex Banach Lie group $\mathfrak{A}_l^{-1}$ acts holomorphically on each connected component of $\mathfrak{A}_l^{-1}$ by left multiplications: $(g, a) \mapsto ga$. In fact, we have
Proposition 4.7. Each connected component of $\mathfrak{A}^{-1}_t$ is a complex Banach homogeneous space under the action of $\mathfrak{A}^{-1}_0$.

Next, similarly to Theorem 4.4 the following result holds.

Theorem 4.8. (a) If the stabilizer $\mathfrak{A}^{-1}_0(p) \subset \mathfrak{A}^{-1}_0$ of a point $p \in \mathfrak{A}^{-1}_t$ is connected, then for each $f \in O(M(\mathfrak{A}_Z), \mathfrak{A}^{-1}_t)$ with image in the connected component containing $p$ there is $g \in O(M(\mathfrak{A}_Z), \mathfrak{A}^{-1}_t)$ such that $f = gp$.

(b) If $\mathfrak{A} \subset \mathfrak{A}_Z$, then for each $f \in O(M(\mathfrak{A}_Z), \mathfrak{A}^{-1}_t)$ there are $\alpha \in D$, a map $h \in O(M(\mathfrak{A}_Z), \mathfrak{A}^{-1}_t)$ uniformly approximated by maps of the algebraic tensor product $\hat{A}_\alpha \otimes \mathfrak{A}$ and a map $g \in O(M(\mathfrak{A}_Z), \mathfrak{A}^{-1}_t)$ such that $gf = h$.

The analog of Theorem 4.4(a) for Banach algebras $C(M(\mathfrak{A}_Z), \mathfrak{A})$ and $O(M(\mathfrak{A}_Z), \mathfrak{A})$ is also true.

Theorem 4.9. $C(M(\mathfrak{A}_Z), \mathfrak{A}^{-1}_t) = C(M(\mathfrak{A}_Z), \mathfrak{A}^{-1}_t)$ and $O(M(\mathfrak{A}_Z), \mathfrak{A}^{-1}_t) = O(M(\mathfrak{A}_Z), \mathfrak{A}^{-1}_t)$.

Thus every connected component of $C(M(\mathfrak{A}_Z), \mathfrak{A}^{-1}_t)$ or $O(M(\mathfrak{A}_Z), \mathfrak{A}^{-1}_t)$ is a complex Banach homogeneous space under the group action of $C(M(\mathfrak{A}_Z), \mathfrak{A}^{-1}_t)$ or $O(M(\mathfrak{A}_Z), \mathfrak{A}^{-1}_t)$ by left multiplications, and embedding

$$O(M(\mathfrak{A}_Z), \mathfrak{A}^{-1}_t) \hookrightarrow C(M(\mathfrak{A}_Z), \mathfrak{A}^{-1}_t)$$

induces bijection between connected components of these complex Banach manifolds.

Remark 4.10. (1) If $\mathfrak{A}^{-1}_0$ is simply connected and the hypothesis of part (b) of Theorem 4.8 holds for all $p \in \mathfrak{A}^{-1}_t$, then embedding $\mathfrak{A}^{-1}_t \hookrightarrow O(M(\mathfrak{A}_Z), \mathfrak{A}^{-1}_t)$ assigning to each $p$ the map of the constant value $p$ induces bijection between sets of connected components of $\mathfrak{A}^{-1}_t$ and $O(M(\mathfrak{A}_Z), \mathfrak{A}^{-1}_t)$, (cf. Remark 4.1).

(2) While the first identity of Theorem 4.9 for algebra $C(M(\mathfrak{A}_Z), \mathfrak{A})$ is true in general with a compact Hausdorff space substituted instead of $M(\mathfrak{A}_Z)$, the second one was previously unknown even for $H^\infty$, see, e.g., [V]. For $H^\infty_{\text{comp}}$, see Remark 4.5(3), the second identity of Theorem 4.9 can be reformulated as follows:

($) A map $F \in H^\infty_{\text{comp}}(\mathfrak{A})$ has a left inverse $G \in H^\infty_{\text{comp}}(\mathfrak{A})$ if and only if for every $z \in \mathbb{D}$ there exists a left inverse $G_z$ of $F(z)$ such that $\sup_{z \in \mathbb{D}} \|G_z\|_a < \infty$.

This result is related to the classical Sz.-Nagy operator corona problem SN:

Sz.-Nagy Problem. Let $H_1, H_2$ be separable Hilbert spaces and $F \in H^\infty(L(H_1, H_2))$ be such that for some $\delta > 0$ and all $x \in H_1, z \in \mathbb{D}$, $\|F(z)x\| \geq \delta \|x\|$. Does there exist $G \in H^\infty(L(H_2, H_1))$ such that $G(z)F(z) = I_{H_1}$ for all $z \in \mathbb{D}$?

This problem is of great interest in operator theory (angles between invariant subspaces, unconditionally convergent spectral decompositions), as well as in control theory. It is also related to the study of submodules of $H^\infty$ and to many other subjects of analysis, see [N1], [N2], [T1], [T2], [V] and references therein. In general, the answer is known to be negative (see [T3], [T4], [T6]); it is positive as soon as $\dim H_1 < \infty$ or $F$ is a “small” perturbation of a left invertible function $F_0 \in H^\infty(L(H_1, H_2))$ (e.g., if $F - F_0$ belongs to $H^\infty(L(H_1, H_2))$ with values in the class of Hilbert Schmidt operators), see [T5], or $F \in H^\infty_{\text{comp}}(L(H_1, H_2))$, see [Br2] Th. 1.5 and ($\ast$) above.

It is worth noting that the proof of statement ($\ast$) for $H^\infty_{\text{comp}}(\mathfrak{A})$ would be shorter if we knew that $H^\infty$ has the Grothendieck approximation property, cf. [V] Th. 2.2. However, this long-standing problem remains unsolved (for some developments see, e.g. [BR] Th. 9 and [Br1] Th. 1.21).
Example 4.11. Let $L(X)$ be the Banach algebra of bounded linear operators on a complex Banach space $X$. Each $A \in L(X)^{-1}$ determines complemented subspace $X_1 := \text{ran} \ A \subset X$ isomorphic to $X$. Then the stabilizer $GL_0(X)(A) \subset GL_0(X)$ of $A$ consists of all operators $B \in GL_0(X)$ such that $B|_{X_1} = I_{X_1}$. If $X_2 \subset X$ is a complemented subspace to $X_1$, then each $B \in GL_0(X)(A)$ has a form

$$B = \begin{pmatrix} I_{X_1} & C \\ 0 & D \end{pmatrix}, \quad \text{where} \quad D \in GL(X_2) \quad \text{and} \quad C \in L(X_2, X_1).$$

Thus $GL_0(X)(A)$ is homotopy equivalent to the subgroup of $GL(X_2)$ consisting of all operators $D$ such that $\text{diag}(I_{X_1}, D) \in GL_0(X)$. In particular, this subgroup coincides with $GL(X_2)$ if the latter is connected.

Now, Theorem 4.8 leads to the following statement:

(1) Suppose $A \in L(X)^{-1}$ is such that group $GL(X/X_1)$ is connected. Then for each $F \in \mathcal{O}(M(\mathcal{A}_Z), L(X)^{-1})$ whose image belongs to the connected component containing $A$ there is $G \in \mathcal{O}(M(\mathcal{A}_Z), GL(X))$ such that $F = GA$.

Let us identify $X$ with $X_1$ by means of $A$ and regard $F(x), \ x \in M(\mathcal{A}_Z)$, and $A$ as operators in $L(X_1, X_1 \oplus X_2)$. Then we obtain

$$F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} I_{X_1} \\ 0 \end{pmatrix},$$

where $F_i \in \mathcal{O}(M(\mathcal{A}_Z), L(X_1, X_{i_1})), \ i = 1, 2,$ and $G_{ii} \in \mathcal{O}(M(\mathcal{A}_Z), L(X_i)), \ i = 1, 2,$ $G_{ij} \in \mathcal{O}(M(\mathcal{A}_Z), L(X_j, X_i)), \ i, j \in \{1, 2\}, \ i \neq j$. Now identity $F = GA$ implies that

$$F_1 = G_{11}, \quad F_2 = G_{21},$$

that is, $G$ extends $F$ to an invertible holomorphic operator-valued function.

Thus from (1) we obtain the following generalization of [B2, Th. 1.4].

(1') Suppose $F \in \mathcal{O}(M(\mathcal{A}_Z), L(Y_1, Y_2))$, where $Y_i, \ i = 1, 2,$ are complex Banach spaces such that for each $z \in \mathbb{D} \setminus \mathbb{Z}$ there exists a left inverse $G_z$ of $F(z)$ satisfying

$$\sup_{z \in \mathbb{D} \setminus \mathbb{Z}} \|G_z\| < \infty.$$
5. PROOFS OF PROPOSITIONS 2.1 AND 2.4

Proof of Proposition 2.1. (a) By $Q_Z : M(H^\infty) \to M(H^\infty_1)$ we denote the continuous map transposed to the embedding homomorphism $H^\infty_1 \hookrightarrow H^\infty$. Each $f \in H^\infty_1$ has a form $f = c + h$ for some $c \in C$ and $h \in I$ so that $\hat{f}(\xi) := \xi(f) = c$ for all $\xi \in Z := \text{hull}(I) \subset M(H^\infty)$. (Here and below $\hat{\ }$ is the Gelfand transform for $H^\infty$.) Hence, $Q_Z$ maps $Z$ to a point. Further, if $\xi_1, \xi_2 \in M(H^\infty) \setminus Z$ are distinct, there exists a function $f \in H^\infty$ such that $\hat{f}(\xi_1) = 1$ and $\hat{f}(\xi_2) = 0$. Also, by the definition of a hull, there exists $g \in I$ such that $\hat{g}(\xi_1) = 1$. Function $fg \in H^\infty_1$ and $Q_Z(\xi_1)(fg) := \hat{f}(\xi_1)\hat{g}(\xi_1) = 2 - i$. Thus functions in $H^\infty_1$ separate points of $M(H^\infty) \setminus Z$, i.e., $Q_Z$ is one-to-one outside $Z$. Finally, the corona theorem for $H^\infty_1$ implies that the image of $\mathbb{D}$ under $Q_Z$ is dense in $M(H^\infty_1)$ (cf. (2.1), (2.2)). Hence, $Q_Z$ is a surjection.

(b) Due to Suárez’s theorem [S1, Th. 4.5], $\dim M(H^\infty) = 2$; hence, $\dim K \leq 2$ for all compact $K \subset M(H^\infty) \setminus Z$. Since $Q_Z|_K$ is a homeomorphism, $\dim Q_Z(K) = \dim K \leq 2$ as well. Note that each compact subset of $M(H^\infty_1) \setminus \{z\}$, $z := Q_Z(\xi)$, has a form $Q_Z(K)$ for some $K$ as above. Thus $\dim K' \leq 2$ for all compact $K' \subset M(H^\infty_1) \setminus \{z\}$. Moreover, $\dim \{z\} = 0$. These two facts and the standard result of the dimension theory, see, e.g., [N, Ch. 2, Th. 9–11], imply that $\dim M(H^\infty_1) = 2$.

(c) According to the cohomology interpretation due to Suárez [S1, Th. 1.3] of Treil’s theorem [T6] stating that the Bass stable rank of algebra $H^\infty$ is one, the (Čech) cohomology exact sequence of the pair $(M(H^\infty), Z)$ acquires the form

$$
\to H^1(M(H^\infty), Z) \to H^1(Z, \mathbb{Z}) \to H^2(M(H^\infty), Z, \mathbb{Z}) \to H^2(M(H^\infty), \mathbb{Z}) \to H^2(Z, \mathbb{Z}) \to 0.
$$

From here by another Suárez’s result [S1, Cor. 3.9] asserting that $H^2(M(H^\infty), \mathbb{Z}) = 0$ we obtain that $H^2(M(H^\infty), Z, \mathbb{Z}) = 0$. Since $Q_Z : M(H^\infty) \to M(H^\infty_1)$ maps $Z$ to a point and is one-to-one outside of $Z$, due to the strong excision property for cohomology, see, e.g., [Sp, Ch. 6, Th. 5], the pullback map $Q^*_Z$ induces isomorphism of the Čech cohomology groups $H^2(M(H^\infty), \mathbb{Z}) \cong H^2(M(H^\infty), Z, \mathbb{Z})$. In particular, $H^2(M(H^\infty), Z) = 0$. □

Proof of Proposition 2.4. Suppose for $f \in \mathcal{O}(U \cap (\mathbb{D} \setminus Z), X)$ there exist open covers $(U_\alpha)_{\alpha \in A}$ of $U \subset M(\mathcal{O}Z)$ and $(V_\beta)_{\beta \in B}$ of $X$ and a map $\tau : A \to B$ such that $\mathcal{O}(V_\beta)$ (algebras of holomorphic functions on $V_\beta$, $\beta \in B$) separate points, $f(U_\alpha \cap (\mathbb{D} \setminus Z)) \subset V_{\tau(\alpha)}$,

$$
(5.1) \quad \bigcap_{W \subset Q_Z^{-1}(U) : W \cap Z \neq \emptyset} f(W \cap (\mathbb{D} \setminus Z)) \neq \emptyset.
$$

We require to show that $f$ extends to a map in $\mathcal{O}(U, X)$.

Lemma 5.1. For each $\alpha \in A$ map $Q_Z^*f : Q_Z^{-1}(U_\alpha) \cap (\mathbb{D} \setminus Z) \to V_{\tau(\alpha)}$ extends to a map in $C(Q_Z^{-1}(U_\alpha), V_{\tau(\alpha)})$.

Recall that $Q_Z|_{M(H^\infty) \setminus Z} = \text{id}$, see Convention in Section 2.1.

Proof. According to the hypotheses for each $g \in \mathcal{O}(V_{\tau(\alpha)})$ function $g \circ f \in H^\infty(U_\alpha \cap (\mathbb{D} \setminus Z))$. Hence, due to [S1, Th. 3.2],

(c) function $g \circ Q_Z^*f$ extends continuously to $Q_Z^{-1}(U_\alpha)$.

By $C_g \subset C$ we denote the range of $g \in \mathcal{O}(V_{\tau(\alpha)})$. Let us consider space

$$
T_\alpha := \prod_{g \in \mathcal{O}(V_{\tau(\alpha)})} C_g
$$
equipped with the product topology and continuous map \( \Psi_\alpha : V_{\tau(\alpha)} \to T_\alpha \),
\[
\Psi_\alpha(z)(g) := g(z), \quad z \in V_{\tau(\alpha)}, \quad g \in \mathcal{O}(V_{\tau(\alpha)}).
\]
Since \( \mathcal{O}(V_{\tau(\alpha)}) \) separates points, \( \Psi_\alpha \) is one-to-one. Therefore it maps compact set \( K := f(U_\alpha \cap (\mathbb{D} \setminus \mathbb{Z})) \) homeomorphically onto its image \( K_\alpha := \Psi_\alpha(K) \). Let \( \psi_\alpha^{-1} : K_\alpha \to K \) be the continuous inverse of \( \Psi_\alpha|_K \). Due to (o) map \( \Psi_\alpha \circ f \circ Q_x | \mathbb{D} \setminus \mathbb{Z} \) extends continuously to \( Q_x^{-1}(U_\alpha) \) and so has range in \( K_\alpha \). Thus \( Q_x f = \psi_\alpha^{-1} (\Psi_\alpha \circ f \circ Q_x | \mathbb{D} \setminus \mathbb{Z}) \) extends continuously to \( Q_x^{-1}(U_\alpha) \) and takes its values in \( K \subset V_{\tau(\alpha)} \), as required.

Since \( (U_\alpha)_{\alpha \in \Lambda} \) is an open cover of \( U \), Lemma \[5.1\] implies that \( Q_x f \) extends continuously to a map \( \tilde{F} \in \mathcal{O}(Q_x^{-1}(U), X) \).

Next, suppose \( Z \subset Q_x^{-1}(U) \). For points \( x_1, x_2 \in Z \), let \( B(x_i) \) be local bases of relatively compact open neighbourhoods of \( x_i \) in \( Q_x^{-1}(U) \), \( i = 1, 2 \). Then condition \[5.1\] and continuity of \( \tilde{F} \) imply that
\[
\{\tilde{F}(x_1)\} \cap \{\tilde{F}(x_2)\} = \bigcap_{i=1,2} \bigcap_{O_i \in B(x_i)} \tilde{F}(O_i) \neq \emptyset.
\]
Hence, \( \tilde{F}(x_1) = \tilde{F}(x_2) \) for all distinct \( x_1, x_2 \in Z \) and so \( \tilde{F}|_Z \) is constant. In turn, there is a map \( F \in \mathcal{O}(U, X) \) such that \( \tilde{F} = Q_x F \). By the definition of \( Q_x \), \( F|_{U \cap (\mathbb{D} \setminus \mathbb{Z})} = f \) as required.

The converse statement saying that \( f = F|_{U \cap (\mathbb{D} \setminus \mathbb{Z})} \) with \( F \in \mathcal{O}(U, X) \) satisfies conditions (a) and (b) of the proposition follows easily by continuity of \( F \).

6. Auxiliary Results

In this part we present some results used in the proofs of theorems of Section 2.2.

6.1. Maximal Ideal Space of \( H^\infty \). Recall that the pseudohyperbolic metric on \( \mathbb{D} \) is defined by
\[
\rho(z, w) := \left| \frac{z-w}{1-z \bar{w}} \right|, \quad z, w \in \mathbb{D}.
\]
For \( x, y \in M(H^\infty) \) the formula
\[
\rho(x, y) := \sup \{ |\hat{f}(y)| : f \in H^\infty, \hat{f}(x) = 0, \|f\|_{H^\infty} \leq 1 \}
\]
gives an extension of \( \rho \) to \( M(H^\infty) \). The Gleason part of \( x \in M(H^\infty) \) is then defined by \( \pi(x) := \{ y \in M(H^\infty) : \rho(x, y) < 1 \} \). For \( x, y \in M(H^\infty) \) we have \( \pi(x) = \pi(y) \) or \( \pi(x) \cap \pi(y) = \emptyset \). Hoffman’s classification of Gleason parts \[H\] shows that there are only two cases: either \( \pi(x) = \{ x \} \) or \( \pi(x) \) is an analytic disk. The former case means that there is a continuous one-to-one and onto map \( L_x : \mathbb{D} \to \pi(x) \) such that \( \hat{f} \circ L_x \in H^\infty \) for every \( f \in H^\infty \). Moreover, any analytic disk is contained in a Gleason part and any maximal (i.e., not contained in any other) analytic disk is a Gleason part. By \( M_a \) and \( M_s \) we denote sets of all nontrivial (analytic disks) and trivial (one-pointed) Gleason parts. It is known that \( M_a \subset M(H^\infty) \) is open. Hoffman proved that \( \pi(x) \subset M_a \) if and only if \( x \) belongs to the closure of an interpolating sequence in \( \mathbb{D} \). The base of topology on \( M_a \) consists of sets of the form \( \{ x \in M_a : |\hat{b}(x)| < \epsilon \} \), where \( b \) is an interpolating Blaschke product. This is because for a sufficiently small \( \epsilon \) set \( b^{-1}(\mathbb{D}_\epsilon) \subset \mathbb{D}, \mathbb{D}_\epsilon := \{ z \in \mathbb{C} : |z| < \epsilon \} \), is biholomorphic to \( \mathbb{D}_\epsilon \times b^{-1}(0) \), see \[Ga\] Ch. X, Lm. 1.4]. Hence, \( \{ x \in M_a : |\hat{b}(x)| < \epsilon \} \) is biholomorphic to \( \mathbb{D}_\epsilon \times b^{-1}(0) \).

Suárez \[S2\] proved that the set of trivial Gleason parts is totally disconnected, i.e., \( \dim M_s = 0 \) (as \( M_s \) is compact).
6.2. Holomorphic Banach vector bundles on $M(\mathcal{A})$. Let $E$ be a Banach vector bundle on $M(\mathcal{A})$ with fibre a complex Banach space $X$. We say that $E$ is holomorphic if it is defined on an open cover $(U_i)_{i \in I}$ of $M(\mathcal{A})$ by a holomorphic cocycle $q = \{g_{ij} \in O(U_i \cap U_j, GL(X))\}_{i,j \in I}$. In this case, $E|_{\mathcal{D} \setminus Z}$ is a holomorphic Banach vector bundle on $\mathbb{D} \setminus Z$ in the usual sense. Recall that $E$ is defined as the quotient space of disjoint union $\sqcup_{i \in I} U_i \times X$ by the equivalence relation:

\[(6.1)\quad U_i \times X \ni u \times x \sim u \times g_{ij}(u)x \in U_i \times X.\]

The projection $p : E \to M(\mathcal{A})$ is induced by natural projections $U_i \times X \to U_i$, $i \in I$.

A morphism $\varphi : (E_1, X_1, p_1) \to (E_2, X_2, p_2)$ of holomorphic Banach vector bundles on $M(\mathcal{A})$ is a continuous map which sends each vector space $p_1^{-1}(w) \cong X_1$ linearly to vector space $p_2^{-1}(w) \cong X_2$, $w \in M(\mathcal{A})$, and such that $\varphi|_{\mathcal{D} \setminus Z} : E_1|_{\mathcal{D} \setminus Z} \to E_2|_{\mathcal{D} \setminus Z}$ is a holomorphic map of complex Banach manifolds. If, in addition, $\varphi$ is bijective, it is called an isomorphism.

We say that a holomorphic Banach vector bundle on $M(\mathcal{A})$ is trivial if it is isomorphic to the bundle $E_X := M(\mathcal{A}) \times X$.

The following result was established in [Br3] by techniques developed in [Br1, Br2].

**Theorem 6.1 (Br3, Theorem 3.2).** For a complex Banach space $X$ there exists a complex Banach space $Y$ such that for each holomorphic Banach vector bundle $E$ on $M(H^\infty)$ with fibre $X$ the Whitney sum $E \oplus E_Y$ is trivial.

Let $E$ be a holomorphic Banach vector bundle on $M(\mathcal{A})$. For an open subset $U \subset M(\mathcal{A})$ by $\Gamma_0(U, E)$ we denote the complex vector space of holomorphic sections $s$ of $E$ over $U$ (i.e., such that $s|_{U \cap \mathcal{D} \setminus Z}$ are holomorphic sections of the bundle $E|_{U \cap \mathcal{D} \setminus Z}$ in the usual sense). We equip $\Gamma_0(U, E)$ with the topology $\tau_c$ of uniform convergence on compact subsets of $U$. It is generated by all subsets of the form $O_{K,V} := \{f \in \Gamma_0(U, E) : f(K) \subset V\}$, where $K \subset U$ is compact and $V \subset E$ is open subsets.

Using Theorem 6.1 we prove

**Theorem 6.2.** Let $E$ be a holomorphic Banach vector bundle on $M(H^\infty)$ and $U$ an open neighbourhood of $Z$. There is a continuous linear operator $L : \Gamma_0(U, E) \to \Gamma_0(M(\mathcal{A}), E)$ such that

$$L|_Z = f|_Z \quad \text{for all} \quad f \in \Gamma_0(U, E).$$

**Proof.** For $E$ a trivial bundle the result is established in [Br1, Th. 1.9]. In this case, we may assume without loss of generality that $E = E_X$. Then $(\Gamma_0(U, E), \tau_c)$ is isomorphic to the topological vector space $O(U, X)$ with the topology of uniform convergence on compact subsets of $U$. The corresponding continuous linear operator $L$ is constructed as in the proof of Theorem 1.7 in [Br1, pp. 213–214]. The general case is reduced to that one by means of Theorem 6.1.

Indeed, in the notation of the theorem, let $i : E \to E \oplus E_Y$ and $q : E \oplus E_Y \to E$, $q \circ i = \text{id}$, be continuous embedding and projection. They induce continuous linear maps $i_* : \Gamma_0(U, E) \to \Gamma_0(U, E \oplus E_Y)$, $i_* f := i \circ f$, and $q_* : \Gamma_0(M(\mathcal{A}), E \oplus E_Y) \to \Gamma_0(M(\mathcal{A}), E)$, $q_* g := q \circ g$. Since $E \oplus E_Y$ is trivial, by the above result of [Br1], there is a continuous linear operator $L' : \Gamma_0(U, E \oplus E_Y) \to \Gamma_0(M(\mathcal{A}), E \oplus E_Y)$ such that $L'|_Z = f|_Z$ for all $f \in \Gamma_0(U, E \oplus E_Y)$. We set

$$L := q_* L'|_* : \Gamma_0(U, E) \to \Gamma_0(M(\mathcal{A}), E).$$

Then $L$ is a continuous linear operator and for all $x \in Z$

$$(L f)(x) = q((L' (i_* f)))(x)) = q((i_* f)(x)) = (q \circ i)(f(x)) = f(x)$$

as required. \(\square\)
6.3. Runge-type approximation theorem. Recall that a compact subset $K \subset M(AZ)$ is called holomorphically convex if for any $x \notin K$ there is $f \in O(M(AZ))$ such that
$$\max_{K} |f| < |f(x)|.$$ 

**Theorem 6.3.** Each holomorphic section of a holomorphic Banach vector bundle $E$ on $M(AZ)$ defined on a neighbourhood of a holomorphically convex set $K \subset M(AZ)$ can be uniformly approximated on $K$ by sections from $\Gamma_{O}(M(AZ), E)$.

For $E$ a trivial bundle on $M(H^\infty)$ the result was proved in [Br1] Th. 1.7.

**Proof.** Let $Q_{Z}: M(H^\infty) \to M(AZ)$ be the continuous surjection transposed to the embedding $H_{i}^{\infty}(Z) \hookrightarrow H^\infty$ (cf. Proposition 2.1). We set $z := Q_{Z}(Z)$.

**Lemma 6.4.** If $K \subset M(AZ)$ is holomorphically convex, $K \cup \{z\}$ is holomorphically convex.

**Proof.** Without loss of generality we may assume that $z \notin K$. Let $x \notin K \cup \{z\}$. Then there exists a function $f \in O(M(AZ))$ such that
$$m_{f} := \max_{K} |f| < |f(x)| = 1.$$ 

Let $g \in O(M(AZ))$ be such that $|g(x)| = 1$ and $g(z) = 0$. We set
$$m_{g} := \max_{K} |g|.$$ 

Replacing $f$ by $f^{n}$ with a sufficiently large $n \in \mathbb{N}$, if necessary, without loss of generality we may assume that
$$m_{f} m_{g} < 1.$$ 

Then for the function $fg \in O(M(AZ))$ we get
$$\max_{K \cup \{z\}} |fg| \leq m_{f} m_{g} < |f(x)g(x)| = 1.$$ 

This shows that set $K \cup \{z\}$ is holomorphically convex. □

Suppose that $s \in \Gamma_{O}(U, E)$, where $U$ is an open neighbourhood of a holomorphically convex set $K \subset M(AZ)$. Due to Lemma 6.4 without loss of generality we may assume that $z \in K$. (Indeed, for otherwise, there are open disjoint sets $U_{1} \subset U$ and $U_{2} \supset \{z\}$. Then we will prove Theorem 6.3 for $K$ replaced by $K \cup \{z\}$, $U$ by $U_{1} \cup U_{2}$ and $s$ by the section in $\Gamma_{O}(U_{1} \cup U_{2}, E)$ equals $s$ on $U_{1}$ and $0$ on $U_{2}$.) Thus $\tilde{K} := Q_{Z}^{-1}(K)$ is a holomorphically convex subset of $M(H^\infty)$ containing $Z$. In turn, $\tilde{E} := Q_{Z}^{-1}(E)$ is a holomorphic Banach vector bundle on $M(H^\infty)$ and $\tilde{s} := Q_{Z}^{-1}(s) \in \Gamma_{O}(U, \tilde{E})$ with $\tilde{U} := Q_{Z}^{-1}(U)$ an open neighbourhood of $\tilde{K}$. Note that $\tilde{s}$ is constant on $Z$.

Let $E_{V} := M(H^\infty) \times Y$ be such that bundle $\tilde{E} \oplus E_{Y}$ is trivial (see Theorem 6.1). By $i : \tilde{E} \to \tilde{E} \oplus E_{Y}$ and $q : \tilde{E} \oplus E_{Y} \to \tilde{E}$, $q \circ i = \text{id}$, we denote continuous embedding and projection. Then $i \circ \tilde{s} \in \Gamma_{O}(\tilde{U}, \tilde{E} \oplus E_{Y})$ and is constant on $Z$.

**Lemma 6.5.** Section $i \circ \tilde{s}$ can be uniformly approximated on $\tilde{K}$ by holomorphic sections from $\Gamma_{O}(M(H^\infty), \tilde{E} \oplus E_{Y})$ constant on $Z$.

**Proof.** By [Br1] Lm. 5.1 there exists a holomorphically convex set $V \subset \tilde{U}$ whose interior $\tilde{V}$ contains $\tilde{K}$. Since $\tilde{E} \oplus E_{Y}$ is trivial on $M(H^\infty)$, [Br1] Th. 1.7 implies that there exists a sequence $\{s_{j}\}_{j \in \mathbb{N}} \subset \Gamma_{O}(M(H^\infty), \tilde{E} \oplus E_{Y})$ converging to $i \circ \tilde{s}$ uniformly on $V$. Hence, the sequence $r_{j} := (i \circ \tilde{s} - s_{j})|_{\tilde{V}}$, $j \in \mathbb{N}$, of sections in $\Gamma_{O}(\tilde{V}, \tilde{E} \oplus E_{Y})$ converges to the zero section uniformly on $\tilde{V}$. Let $L : \Gamma_{O}(\tilde{V}, \tilde{E} \oplus E_{Y}) \to \Gamma_{O}(M(H^\infty), \tilde{E} \oplus E_{Y})$ be a
continuous linear operator such that \( Lf|_Z = f|_Z \) (see Theorem 5.2). We set \( \tilde{t}_j := Lt_i \in \Gamma_\mathcal{O}(M(H^\infty), \tilde{E} \oplus E_Y), \ i \in \mathbb{N} \). Then \( \tilde{t}_j|_Z = t_j|_Z, \ j \in \mathbb{N}, \) and \( \{\tilde{t}_j\}_{j \in \mathbb{N}} \) converges to the zero section uniformly on \( M(H^\infty) \). Finally, defining \( u_j := s_j + \tilde{t}_j, \ j \in \mathbb{N}, \) we obtain that \( \{u_j\}_{j \in \mathbb{N}} \subset \Gamma_\mathcal{O}(M(H^\infty), \tilde{E} \oplus E_Y) \) converges to \( i \circ \tilde{s} \) uniformly on \( \tilde{K} \) and \( u_j|_Z = (i \circ \tilde{s})|_Z \) (constant) for all \( j \).

The proof of the lemma is complete. \( \square \)

In the notation of the lemma sequence \( \{q \circ u_j\}_{j \in \mathbb{N}} \subset \Gamma_\mathcal{O}(M(H^\infty), \tilde{E}) \) converges to \( \tilde{s} \) uniformly on \( \tilde{K} \). Since \( (q \circ u_j)|_Z = \tilde{s}|_Z \) for all \( j \), there is a sequence \( \{v_j\}_{j \in \mathbb{N}} \subset \mathcal{S}_\mathcal{O}(M(\mathcal{A}Z), E) \) such that \( v_j \circ Q_Z = q \circ u_j, \ j \in \mathbb{N}, \) converging to \( s \) uniformly on \( K \).

This concludes the proof of the theorem. \( \square \)

6.4. Cousin-type lemma. Let \( E \) be a holomorphic Banach vector bundle with fibre \( X \) on \( M(\mathcal{A}Z) \) defined on a finite open cover \( \mathcal{U} = (U_i)_{i \in I} \) of \( M(\mathcal{A}Z) \) by a holomorphic cocycle \( g = \{g_{ij} \in \mathcal{O}(U_i \cap U_j, GL(X))\}_{i,j \in I} \), see [6.1]. A continuous section of \( E \) defined on a compact subset \( K \subset M(\mathcal{A}Z) \) is called holomorphic if it is the restriction of a holomorphic section of \( E \) defined in an open neighbourhood of \( K \). The space of such sections is denoted by \( \Gamma_\mathcal{O}(K, E) \). Let \( \Gamma_\mathcal{C}(K, E) \) be the topological vector space of continuous sections of \( E \) on \( K \) equipped with the topology of uniform convergence. This space is normable, a norm on \( \Gamma_\mathcal{C}(K, E) \) compatible with topology can be defined as follows.

Let us fix a finite refinement \( \mathcal{V} = (V_i)_{i \in L} \) of \( \mathcal{U} \) consisting of compact subsets. Let \( \tau : L \rightarrow I \) be the refinement map, i.e., \( V_i \subset U_{\tau(i)} \) for all \( l \in L \). Each \( s \in \Gamma_\mathcal{C}(K, E) \) in the local coordinates on \( V_i, l \in L_K := \{m \in L : V_m \cap K \neq \emptyset\} \), determined by (6.1), is represented by a pair \((v, s_l(v)), v \in K \cap V_i, s_l \in C(K \cap V_i, X)\), such that \( s_l(v) = g_{\tau(i)\tau(l)(v)} s_m(v) \) for all \( v \in K \cap V_i \cap V_m \neq \emptyset, l, m \in L_K \). We set

\[
\|s\|_{K; E}^\mathcal{O} := \max_{l \in L_K} \max_{v \in K \cap V_i} \|s_l(v)\|_X.
\]

It is readily seen that \( (\Gamma_\mathcal{C}(K, E), \|\cdot\|_{K; E}^\mathcal{O}) \) is a complex Banach space and the norm topology on \( \Gamma_\mathcal{C}(K, E) \) coincides with the topology of uniform convergence.

By \( \Gamma_\mathcal{A}(K, E) \) we denote the closure of \( \Gamma_\mathcal{O}(K, E) \) in the Banach space \( \Gamma_\mathcal{C}(K, E) \).

Suppose open \( U_1, U_2 \subset M(\mathcal{A}Z) \) are such that (a) \( z \not\in U_1 \cap U_i, i = 1, 2; \) (b) \( Q_{\mathcal{A}Z}^{-1}(\tilde{U}_1 \cap \tilde{U}_2) \subset \mathcal{M}_a \setminus \mathcal{Z} \). Let \( W_i \subset U_i, i = 1, 2, \) be compact subsets and \( W := W_1 \cup W_2 \).

**Theorem 6.6** (Cousin-type Lemma). The bounded linear operator of complex Banach spaces \( A : \Gamma_\mathcal{A}(\tilde{U}_1 \cap W, E) \oplus \Gamma_\mathcal{A}(\tilde{U}_2 \cap W, E) \rightarrow \Gamma_\mathcal{A}(\tilde{U}_1 \cap \tilde{U}_2 \cap W, E), \)

\[
A(f_1, f_2) := f_1|_{\tilde{U}_1 \cap \tilde{U}_2 \cap W} + f_2|_{\tilde{U}_1 \cap \tilde{U}_2 \cap W}, \quad f_i \in \Gamma_\mathcal{A}(U_i \cap W, E), \quad i = 1, 2,
\]

is surjective.

**Proof.** For \( E \) a trivial bundle on \( M(H^\infty) \) the result is established in [Br2, Th.3.1]. The general case is reduced to that one.

We must show that

(\circ) For each \( f \in \Gamma_\mathcal{A}(\tilde{U}_1 \cap \tilde{U}_2 \cap W, E) \) there exist \( f_i \in \Gamma_\mathcal{A}(\tilde{U}_i \cap W, E), \ i = 1, 2, \) such that \( f_1 + f_2 = f \) on \( \tilde{U}_1 \cap \tilde{U}_2 \cap W \).

Let \( \tilde{E} := Q_{\mathcal{A}Z}^E \) be the pullback of \( E \) to \( M(H^\infty) \) and \( E_Y := M(H^\infty) \times Y \) be such that \( \tilde{E} \oplus E_Y \) is trivial. As before, by \( i : \tilde{E} \rightarrow \tilde{E} \oplus E_Y \) and \( q : \tilde{E} \oplus E_Y \rightarrow \tilde{E}, \ q \circ i = id \), we denote continuous embedding and projection. We set

\[
\tilde{f} := f \circ Q_Z \in \Gamma_\mathcal{A}(Q_{\mathcal{A}Z}^{-1}(\tilde{U}_1 \cap \tilde{U}_2 \cap W), E), \quad V_i := Q_{\mathcal{A}Z}^{-1}(U_i), \quad \tilde{W}_i := Q_{\mathcal{A}Z}^{-1}(W_i), \quad i = 1, 2,
\]

\[
W := Q_{\mathcal{A}Z}^{-1}(W) (= \tilde{W}_1 \cup \tilde{W}_2).
\]
Since $Q_Z$ maps $Z$ to $z \in M(\mathcal{A}_Z)$ and is one-to-one on $M(H^\infty) \setminus S$, assumptions of the theorem imply that $\tilde{V}_i = Q_Z^{-1}(U_i)$, $i = 1, 2$. Thus, $i \circ \tilde{f} \in \Gamma_A(\tilde{V}_1 \cap \tilde{V}_2 \cap \tilde{W}, \tilde{E} \oplus E_Y)$. Since $\tilde{E} \oplus E_Y$ is trivial, [B72] Th. 3.1 yields sections $f'_i \in \Gamma_A(\tilde{V}_1 \cap \tilde{W}, \tilde{E} \oplus E_Y)$ such that

$$f'_1 + f'_2 = i \circ \tilde{f} \quad \text{on} \quad \tilde{V}_1 \cap \tilde{V}_2 \cap \tilde{W}.$$  

Hence, for $f''_i := q \circ f'_i \in \Gamma_A(\tilde{V}_i \cap \tilde{W}, \tilde{E})$, $i = 1, 2$,

$$f'' + f'' = \tilde{f} \quad \text{on} \quad \tilde{V}_1 \cap \tilde{V}_2 \cap \tilde{W}. $$

If $Z \cap ((\tilde{V}_1 \cup \tilde{V}_2) \cap \tilde{W}) = \emptyset$, then there exist some $f_i \in \Gamma_A(\tilde{U}_i \cap W, E)$ such that $f_i \circ Q_Z = f''_i$, $i = 1, 2$. Due to (6.4) these sections satisfy $(\circ)$. For otherwise, by the hypothesis of the theorem, $Z \subset V_{i_0} \cap \tilde{W}$ for some $i_0 \in \{1, 2\}$. Without loss of generality we may assume that $i_0 = 1$. Then according to Theorem 6.2 there is $g \in \Gamma_O(M(H^\infty), \tilde{E})$ such that $g|_Z = f''_1|_Z$. We set

$$\tilde{f}_1 := f''_1 - g, \quad \tilde{f}_2 := f''_2 + g.$$  

By the definition and due to (6.4),

$$\tilde{f}_1|_Z = 0 \quad \text{and} \quad \tilde{f}_2|_Z = \tilde{f}|_Z = \text{const.}$$

Hence, there are $f_i \in \Gamma_A(\tilde{U}_i \cap W, E)$ such that $f_i \circ Q_Z = \tilde{f}_i$, $i = 1, 2$. In view of (6.4), these sections satisfy $(\circ)$. \hfill \Box

6.5. Cartan-type Lemma. For basic facts of the Banach Lie group theory, see, e.g., [M].

Let $G$ be a complex Banach Lie group with Lie algebra $\mathfrak{g}$. The latter has the structure of a complex Banach space naturally identified with the tangent space of $G$ at unit $1_G$. By $\exp_\mathfrak{g} : \mathfrak{g} \to G$ we denote the corresponding holomorphic exponential map which maps a neighbourhood of 0 in $\mathfrak{g}$ biholomorphically onto a neighbourhood of $1_G \in G$.

Let $E$ be a topological bundle on $M(\mathcal{A}_Z)$ with fibre $G$ defined on a finite open cover $\mathcal{U} = (U_i)_{i \in I}$ of $M(\mathcal{A}_Z)$ by holomorphic transition functions $f_{ij} \in O(U_i \cap U_j \times G, G)$, $i, j \in I$ (i.e., restrictions of $f_{ij}$ to $(U_i \cap U_j \cap (\mathbb{D} \setminus Z)) \times G$ are holomorphic in the usual sense), such that

1. $f_{ij}(x, f_{j,k}(x, g)) = f_{ik}(x, g)$, $x \in U_i \cap U_j \cap U_k$, $g \in G$;
2. $f_{ij}(x, \cdot)$ is an automorphism of $G$ for each $x \in U_i \cap U_j$, $i, j \in I$.

Thus, $E$ is the quotient of disjoint union $\sqcup_{i \in I} U_i \times G$ by the equivalence relation:

$$U_j \times G \ni (x, g) \sim (x, f_{ij}(x, g)) \in U_i \times G.$$  

By $\Gamma_C(Y, E)$ we denote the set of continuous sections of $E$ over $Y \subset M(\mathcal{A}_Z)$. It has the natural group structure induced by the product on $G$ with the unit the restriction to $K$ of the section $1_E$ assigning to each $x \in M(\mathcal{A}_Z)$ the unit $1_G$ of the fibre of $E$ over $x$.

A section in $\Gamma_C(W, E)$, $W \subset M(\mathcal{A}_Z)$ is open, is called holomorphic if its restriction to $W \cap (\mathbb{D} \setminus Z)$ is the holomorphic section of the holomorphic fibre bundle $E|_{\mathbb{D} \setminus Z}$. The set of such sections is denoted by $\Gamma_O(W, E)$. For a compact set $K \subset M(\mathcal{A}_Z)$ a section of $E$ over $K$ is called holomorphic if it is the restriction of a holomorphic section of $E$ defined on an open neighbourhood of $K$. This set of sections is denoted by $\Gamma_O(K, E)$.

Associated with $E$ is the holomorphic Banach vector bundle $T(E)$ with fibre $\mathfrak{g}$ of complex tangent spaces to fibres of $E$ at $1_G$. It is defined on the cover $\mathcal{U}$ by cocycle $\{(d_2 f_{ij})_{1_G} \in O(U_i \cap U_j, \text{Aut}(\mathfrak{g}))\}_{i, j \in I}$, where $d_2$ is the differential with respect to the variable $g \in G$. Map $\exp_G$ induces a holomorphic map $\exp_E : T(E) \to E$ of holomorphic fibre bundles sending an open neighbourhood $U$ of the image of the zero section $0_{T(E)}$ of $T(E)$ biholomorphically onto a neighbourhood $V$ of the image of section $1_E$ in $E$. The complex vector space $\Gamma_C(K, T(E))$ of continuous sections of $T(E)$ on a compact
set $K \subset M(H^\infty_{\mathcal{U}})$ equipped with norm $\|K\|_\mathcal{U}$ defined with respect to a fixed refinement $\mathcal{U}$ of the cover $\mathcal{U}$ is a complex Banach space. It has the structure of a complex (Banach) Lie algebra with Lie product induced by the Lie product on $g$. In turn, map $\exp_E$ induces a map $(\exp_E)_* : \Gamma_C(K,T(E)) \to \Gamma_C(K,E)$, $s \mapsto \exp_E \circ s$, sending $\Gamma_C(K,T(E))$ to $\Gamma_C(K,E)$. Since $1_E|_K \in \Gamma_C(K,E)$, the latter is a subgroup of group $\Gamma_C(K,E)$. We equip $\Gamma_C(K,E)$ with the topology $\tau_u$ of uniform convergence. Then $(\Gamma_C(K,E),\tau_u)$ has the structure of a complex Banach Lie group with Lie algebra $\Gamma_C(K,T(E))$ and with exponential map $(\exp_E)_*$. Finally, by $\Gamma_A(K,E)$ we denote the closure of $\Gamma_C(K,E)$ in $\Gamma_C(K,E)$. It consists of continuous sections of $E$ on $K$ that can be uniformly approximated by sections of $\Gamma_C(K,E)$. Space $(\Gamma_A(K,E),\tau_u)$ is a complex Banach Lie subgroup of $\Gamma_C(K,E)$ with Lie algebra $\Gamma_A(K,T(E))$.

By $\tilde{U}_K \subset \Gamma_A(K,T(E))$ and $\tilde{V}_K \subset \Gamma_A(K,E)$ we denote open neighbourhoods of $0_{T(E)}|_K$ and $1_E|_K$ consisting of sections $s$ such that $s(K) \subset U$ and $s(K) \subset V$, respectively. Thus, $(\exp_E)_* : \tilde{U}_K \to \tilde{V}_K$ is a biholomorphic map.

It is readily seen (as $M(\partial Z)$ is compact and $\exp_E$ is continuous) that there is some $r \in (0,\infty)$ such that for each compact $K \subset M(\partial Z)$ the open ball $B_K(r) \subset \Gamma_A(K,T(E))$ of radius $r$ centred at $0_{T(E)}|_K$ is a subset of $\tilde{U}_K$ and

\[(6.5) \quad (\exp_E)_*(s_1) \cdot (\exp_E)_*(s_2) \in \tilde{V}_K \quad \text{for all} \quad s_i \in B_K(r), \quad i = 1,2.\]

In the next two results we retain notation of Theorem 6.6 that is, $(\circ) U_1, U_2 \subset M(\partial Z)$ are open such that (a) $z \notin \bar{U}_i \setminus U_i$, $i = 1,2$; (b) $Q^{-1}_Z(\bar{U}_1 \cap \bar{U}_2) \subset M_0 \setminus Z$, and $W_i \subset U_i$, $i = 1,2$, are compact, $W := W_1 \cup W_2$.

Our next result solves the analog of the “problème fondamental” of Cartan [Ca].

**Proposition 6.7** (Cartan-type Lemma). There is $0 < r_1 \leq r$ such that for every $F \in \Gamma_A(U_1 \cap U_2 \cap W,E)$ with $(\exp_E)^*_{-1}(F) \in B_{\bar{U}_1 \cap \bar{U}_2 \cap W}(r_1)$ there exist $F_j \in \Gamma_A(U_j \cap W,E)$, $j = 1,2$, such that $F_1 F_2 = F$ on $\bar{U}_1 \cap \bar{U}_2 \cap W$.

**Proof.** We set for brevity

\[Y_0 := \bar{U}_1 \cap \bar{U}_2 \cap W \quad \text{and} \quad Y_j := \bar{U}_j \cap W, \quad j = 1,2.\]

By (6.5) the holomorphic map $H : B_{Y_1}(r) \times B_{Y_2}(r) \to \Gamma_A(Y_0,T(E))$,

\[H(s_1,s_2) := ((\exp_E)^*_{-1}( (\exp_E)_*(s_1|_{Y_0}) \cdot (\exp_E)_*(s_2|_{Y_0}) )\right),

is well-defined. Its differential at $0_{T(E)}|Y_1 \times 0_{T(E)}|Y_2$ is the bounded linear operator $A : \Gamma_A(Y_1,T(E)) \times \Gamma_A(Y_2,T(E)) \to \Gamma_A(Y_0,T(E))$,

\[A(s_1,s_2) := s_1|_{Y_0} + s_2|_{Y_0}.

Due to Theorem 6.6, $A$ is surjective; hence by the implicit function theorem (see, e.g., [L]) there exists a continuous right inverse of $H$ defined on a ball $B_{Y_0}(r_1)$, $r_1 \in (0, r)$. \(\Box\)

Recall that a path in the complex Banach Lie group $\Gamma_A(K,E)$ is a continuous map $[0,1] \to \Gamma_A(K,E)$. The set of all sections in $\Gamma_A(K,E)$ that can be joined by paths in $\Gamma_A(K,E)$ with section $1_E|_K$ forms the connected component of the unit $1_E|_K$.

Our next result generalizes [Br2 Th.4.1].

**Theorem 6.8.** Assume that $W_1$ is holomorphically convex and $W_1 \cap W_2 \neq \emptyset$. Let $F \in \Gamma_A(U_1 \cap U_2,E)$ belong to the connected component of $1_E|_{U_1 \cap U_2}$. Then there exist $F_i \in \Gamma_A(W_i,E)$, $i = 1,2$, such that $F_1 F_2 = F$ on $W_1 \cap W_2$. 


Proof. An open polyhedron $\Pi \subset M(\mathcal{A}_Z)$ is the set of the form

$$\Pi := \{ x \in M(\mathcal{A}_Z) : \max_{1 \leq j \leq l} |f_j(x)| < 1, \ f_j \in O(M(\mathcal{A}_Z)), \ 1 \leq j \leq l \}.$$

(Here $l$ can be any natural number.)

In turn, the closed polyhedron in $M(\mathcal{A}_Z)$ corresponding to $\Pi$ is defined as

$$\Pi^c := \{ x \in M(\mathcal{A}_Z) : \max_{1 \leq j \leq l} |f_j(x)| \leq 1, \ f_j \in O(M(\mathcal{A}_Z)), \ 1 \leq j \leq l \}.$$

(Note that the closure of $\Pi$ is not necessarily $\Pi^c$.)

The proof of the following result repeats literally the proof of Lemma 5.1 in [Br1].

**Lemma 6.9.** Let $N \subset M(\mathcal{A}_Z)$ be a neighbourhood of a holomorphically convex set $K$. Then there exists an open polyhedron $\Pi$ such that $K \subset \Pi \subset N$.

Now, since $W_1$ is holomorphically convex, according to Lemma 6.9 there exists a sequence of open polyhedrons $\{U_{1,i}\}_{i \in \mathbb{N}}$ containing $W_1$ such that $U_{1,i}^c \subset U_1$ and $U_{1,i+1}^c \subset U_{1,i}$, $i \in \mathbb{N}$. (Observe that by definition all $U_{1,i}^c$ are holomorphically convex.) Let us choose a sequence of open sets $U_{2,i} \subset U_2$ containing $W_2$ such that $U_{2,i+1} \subset U_{2,i}$ for all $i \in \mathbb{N}$. Then Proposition 6.7 is also valid with $U_j$ replaced by $U_{j,i}$, $j = 1, 2$, $W := W_1 \cup W_2$ by $W^i := U_{1,i+1}^c \cup U_{2,i+1}$ and $r_1$ by some $r_{1,i} \in (0, r)$.

Further, since $F$ in the statement of the theorem belongs to the connected component $C_0$ of the group $\Gamma_\mathcal{A}(\bar{U}_1 \cap \bar{U}_2, E)$ containing $1_E|_{\bar{U}_1 \cap \bar{U}_2}$, there exists a path $\gamma : [0, 1] \to C_0$ such that $\gamma(0) = 1_E|_{\bar{U}_1 \cap \bar{U}_2}$ and $\gamma(1) = F$. Continuity of $\gamma$ implies that there is a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ of $[0, 1]$ such that

$$\text{(exp}_E)^{-1} \left( \left( \gamma(t_j) \right)^{-1} \gamma(t_{j+1}) \right) \right|_{\bar{U}_{1,i}, \bar{U}_{2,i} \cap W_1^1} \subset B_{\bar{U}_{1,i}, \bar{U}_{2,i} \cap W_1^1}(r_{1,i}) \text{ for all } i.$$

Applying Proposition 6.7 to each $\gamma(t_i)^{-1} \gamma(t_{i+1})$, we obtain

$$\gamma(t_i)^{-1} \gamma(t_{i+1}) = F_i \ F_j \text{ on } \bar{U}_{1,i} \cap \bar{U}_{2,i} \cap W_i$$

for some $F_j \in \Gamma_\mathcal{A}(\bar{U}_{1,i} \cap W_i, E)$ such that $(\text{exp}_E)^{-1}(F_j) \in B_{\bar{U}_{1,i} \cap W_i}(r)$, $l = 1, 2$, cf. (6.5).

Next, we have

$$F = \prod_{i=0}^{k-1} \gamma(t_i)^{-1} \gamma(t_{i+1}).$$

To prove the theorem we use induction on $j$ for $1 \leq j \leq k$. The induction hypothesis is If

$$F^j := \prod_{i=0}^{j-1} \gamma(t_i)^{-1} \gamma(t_{i+1}),$$

then for some $F_{l,j} \in \Gamma_\mathcal{A}(\bar{U}_{l,j} \cap W^j, E)$, $l = 1, 2$,

$$F^j = F_{l,j} \ F_{j,l} \text{ on } \bar{U}_{l,j} \cap \bar{U}_{2,j} \cap W^j.$$
We write
\[ F_{2,j-1} F_1^{j-1} F_2^{j-1} = [F_{2,j-1}, F_1^{j-1} F_\epsilon^{j-1}] (F_1^{j-1} F_\epsilon^{j-1}) F_{2,j-1} F_\epsilon F_2^{j-1} \]
on the closure of \( \bar{U}_{1,j-1} \cap \bar{U}_{2,j-1} \cap W^{j-1} \). (Here \( [A_1, A_2] := A_1 A_2^{-1} A_2^{-1} \).)

According to (6.8) for a sufficiently small \( \epsilon \),
\[(\exp_E)^{-1} \left( \left( [F_{2,j-1}, F_1^{j-1} F_\epsilon^{j-1}] (F_1^{j-1} F_\epsilon^{j-1}) \right)_{\bar{U}_{1,j} \cap \bar{U}_{2,j} \cap W^j} \right) \in B_{\bar{U}_{1,j} \cap \bar{U}_{2,j} \cap W^j}(r_{1,j}).\]

Hence, by Proposition 6.7, there exist \( H_l \in \Gamma_A(\bar{U}_{l,j} \cap W^j, E) \), \( l = 1, 2 \), such that
\[ [F_{2,j-1}, F_1^{j-1} F_\epsilon^{j-1}] (F_1^{j-1} F_\epsilon^{j-1}) = H_1 H_2 \quad \text{on} \quad \bar{U}_{1,j} \cap \bar{U}_{2,j} \cap W^j. \]

In particular, we obtain
\[ F^j = F_{1,j-1} F_{2,j-1} F_1^{j-1} F_2^{j-1} = F_{1,j-1} H_1 H_2 F_{2,j-1} F_\epsilon F_2^{j-1} \quad \text{on} \quad \bar{U}_{1,j} \cap \bar{U}_{2,j} \cap W^j. \]

We set
\[ F_{1,j} := F_{1,j-1} H_1 \quad \text{on} \quad \bar{U}_{1,j} \cap W^j, \quad F_{2,j} := H_2 F_{2,j-1} F_\epsilon F_2^{j-1} \quad \text{on} \quad \bar{U}_{2,j} \cap W^j. \]

Then
\[ F^j = F_{1,j} F_{2,j} \quad \text{on} \quad \bar{U}_{1,j} \cap \bar{U}_{2,j} \cap W^j. \]

This completes the proof of the induction step.

Using this result for \( j := k - 1 \) together with the fact \( W_l \subset \bar{U}_{l,k+1} \), \( l = 1, 2 \), we obtain that there exist \( F_l \in \Gamma_A(W_l, E) \) such that \( F_1 F_2 = F \) on \( W_1 \cap W_2 \).

\[ \square \]

6.6. Blaschke Sets. Any open set of the form
\[ O_{b,\epsilon} := \{ x \in M(H^\infty) : |\hat{b}(x)| < \epsilon \}, \]
where \( b \) is an interpolating Blaschke product and \( \epsilon \) is so small that \( O_{b,\epsilon} \cap \bar{D} \) is biholomorphic to \( \bar{D} \times b^{-1}(0) \), cf. Section 6.1, will be called an open Blaschke set. For such \( \epsilon \) the closure \( \bar{O}_{b,\epsilon} \) is given by \( \{ x \in M(H^\infty) : |\hat{b}(x)| \leq \epsilon \} \) and so it is holomorphically convex.

Note that by the normality argument the biholomorphism in the definition of \( O_{b,\epsilon} \) extends (by means of the Gelfand transform) to a homeomorphism between \( O_{b,\epsilon} \) and \( \bar{D} \times b^{-1}(0) \).

Let \( G \) be a complex Banach Lie group with Lie algebra \( \mathfrak{g} \) and exponential map \( \exp_G \). By \( \mathcal{O}(K, G) \), \( K \subset M(\mathfrak{a}Z) \) compact, we denote the group with respect to the pointwise multiplication of restrictions to \( K \) of holomorphic maps into \( G \) defined on open neighbourhoods of \( K \) equipped the topology of uniform convergence. By \( \mathcal{A}(K, G) \) we denote the closure of \( \mathcal{O}(K, G) \) in the group of continuous maps \( K \to G \) equipped with the topology of uniform convergence. Then \( \mathcal{A}(K, G) \) is a complex Banach Lie group with Lie algebra \( \mathcal{A}(K, \mathfrak{g}) \) and exponential map \( \exp_G \circ f, f \in \mathcal{A}(K, \mathfrak{g}) \).

Theorem 6.10. Suppose \( G \) is simply connected and \( N \) is a compact subset of the Blaschke set \( O_{b,\epsilon} \). Then the complex Banach Lie group \( \mathcal{A}(K, G) \) is connected.

\[ \text{Proof.} \] Let \( 1_G \) be the constant map \( M(H^\infty) \to G \) whose value is the unit \( 1_G \) of \( G \). Then \( 1_G|_K \) is the unit of \( \mathcal{A}(K, G) \). We must show that each \( F \in \mathcal{A}(K, G) \) can be joined by a path with \( 1_G|_K \). Since \( \mathcal{O}(K, G) \) is dense in the complex Banach manifold \( \mathcal{A}(K, G) \), it suffices to prove the result for \( F \in \mathcal{O}(K, G) \). So assume that \( F \) is holomorphic in an open neighbourhood \( U \subset O_{b,\epsilon} \). Without loss of generality we may assume that \( b \) is not finite and identify \( O_{b,\epsilon} \) with \( \bar{D} \times bN \). (Here \( bN \) is the Stone-Cech compactification of \( \mathbb{N} \).)

Lemma 6.11. \( N \) can be covered by open sets \( S_j \times Y_j \subset U \), \( 1 \leq j \leq k \), such that \( Y_j \subset bN \) are clopen and pairwise disjoint, and each \( S_j \subset \bar{D} \) has finitely many connected components that are finitely connected domains.
Proof. Since $\beta N$ is totally disconnected, it is homeomorphic to the inverse limit of a family of finite sets, see, e.g., [N]. Hence, $O_{b,\epsilon}$ can be obtained as the inverse limit of a family of sets of the form $D_{\epsilon} \times F$, where $F$ is a finite set. In turn, $N$ can be obtained as the inverse limit of the family of compact subsets $K \times F \subset D_{\epsilon} \times F$. By the definition of the inverse limit topology, there exists an open neighbourhood $V \in D_{\epsilon} \times F$ of one of such $K \times F$ whose preimage under the continuous limit projection $\pi : D_{\epsilon} \times \beta N \to D_{\epsilon} \times F$ is simply connected, each connected component of $S_j$ is homotopic inside $O$, and $1\leq j \leq k$.

Lemma 6.12. Let $X = \bigcup_{j=1}^{k} S_j \times Y_j$ with $S_j$ and $Y_j$ as in Lemma 6.11. There exist refinements $X_n$ of $X$ and holomorphic maps $F_n \in \mathcal{O}_f(X_n, G)$, $n \in \mathbb{N}$, such that sequence $\{F_n\}_{n \in \mathbb{N}}$ converges uniformly to $F|_X$.

Proof. Let us consider maps $F^j := F|_{S_j \times Y_j}$, $1 \leq j \leq k$. Each $F^j$ can be regarded as a continuous map of the clopen set $Y_j \subset \beta N$ to the complex Banach Lie group $\mathcal{A}(S_j, G) \subset C(S_j, G)$ of continuous $G$-valued maps holomorphic on $S_j$ equipped with the topology of uniform convergence. We endow $\mathcal{A}(S_j, G)$ with a metric $d$ compatible with topology (existing by the Birkhoff-Kakutani theorem). Since $F^j$ is a continuous map of compact set $Y_j$ to metric space $(\mathcal{A}(S_j, G), d)$, it is uniformly continuous. Hence for each $n \in \mathbb{N}$, $1 \leq j \leq k$, there exists a partition $Y_j = \bigcup_{l=1}^{m_n} Y^j_{nl}$ into clopen subsets such that

\[ d(F^j(x), F^j(y)) \leq \frac{1}{n} \quad \text{for all} \quad x, y \in Y^j_{nl}, \quad 1 \leq l \leq m_n. \tag{6.9} \]

Let us fix points $x^j_{nl} \in Y^j_{nl}$, $n \in \mathbb{N}$, $1 \leq j \leq k$, $1 \leq l \leq m_n$, and define

$X_n := \bigcup_{j=1}^{k} \bigcup_{l=1}^{m_n} S_j \times Y^j_{nl}$

and

$F_n(z, x) := F(z, x^j_{nl})$, \quad $(z, x) \in S_j \times Y^j_{nl}, \quad 1 \leq j \leq k, \quad 1 \leq l \leq m_n$.

Due to (6.9), sequences $\{X_n\}_{n \in \mathbb{N}}$ and $\{F_n\}_{n \in \mathbb{N}}$ satisfy the requirements of the lemma. □

Lemma 6.12 shows that in order to prove Theorem 6.10 it suffices to assume that $F \in \mathcal{O}_f(X, G)$, where $X = \bigcup_{j=1}^{k} S_j \times Y_j$ is an elementary subset of $O_{b,\epsilon}$. In this case, $F|_{S_j \times Y_j}(s, y) = f_j(s), (s, y) \in S_j \times Y_j$, for a holomorphic map $f_j : S_j \to G$, $1 \leq j \leq k$. Since each connected component of $S_j$ is homotopic to a finite one-dimensional CW-complex (cf. Lemma 6.11) and $G$ is simply connected, each $f_j$ is homotopic to the constant map $S_j \ni s \mapsto 1_G \in G$. Then by [R] Th.2.1(b)] (the Banach-valued version of the Ramspsott theorem) each $f_j$ is homotopic to this map inside $\mathcal{O}(S_j, G)$. This implies that each $F|_{S_j \times Y_j}$ is homotopic inside $\mathcal{O}_f(S_j \times Y_j, G)$ to $1_{G|_{S_j \times Y_j}}$. Denote by $H_j : [0, 1] \to \mathcal{O}_f(S_j \times Y_j, G)$, $1 \leq j \leq k$, the corresponding homotopies. Then $H : [0, 1] \to \mathcal{A}(K, G)$, \[ H(t)(x) := H_j(t)(x) \quad \text{if} \quad x \in K \cap (S_j \times Y_j), \quad 1 \leq j \leq k, \]

is a path joining $F$ and $1_{G|_{K}}$.

The proof of the theorem is complete. □
Let $\pi : P \to M(\mathcal{A}_Z)$ be a holomorphic principal bundle with fibre a simply connected complex Banach Lie group $G$. For a compact set $K \subset M(\mathcal{A}_Z)$ we say that $P|_K$ is trivial if it is trivial in an open neighbourhood of $K$ (cf. Section 2.2).

**Corollary 6.13.** Suppose $V_i \subset U_i \subset M(\mathcal{A}_Z)$, $i = 1, 2$, are open such that (a) $z \notin \bar{U}_1 \setminus U_i$, $i = 1, 2$; (b) $Q_Z^{-1}(\bar{U}_1 \cap U_2) \subset M_n \setminus \mathbb{Z}$ is a subset of an open Blaschke set; (c) $V_1$ is holomorphically convex. We set $V := V_1 \cup V_2$. If bundles $P|_{\bar{U}_i}$, $i = 1, 2$, are trivial, then $P|_V$ is trivial.

**Proof.** Without loss of generality we may assume that $\bar{V}_1 \cap \bar{V}_2 \neq \emptyset$ and $\bar{U}_1 \cap \bar{U}_2$ is the proper subset of each $\bar{U}_i$, $i = 1, 2$ (for otherwise the statement is trivial). Then, by the hypothesis, $P|_{\bar{U}_1 \cup \bar{U}_2}$ is determined by a cocycle $c \in \mathcal{O}(\bar{U}_1 \cap \bar{U}_2, G)$. Since $Q_Z^{-1}(\bar{U}_1 \cap U_2) \subset M_n \setminus \mathbb{Z}$, pullback $Q_Z^*$ determines an isomorphism of complex Banach Lie groups $\mathcal{A}(\bar{U}_1 \cap \bar{U}_2, G)$ and $\mathcal{A}(Q_Z^{-1}(\bar{U}_1 \cap U_2), G)$. As $\bar{U}_1 \cap \bar{U}_2$ is a compact subset of an open Blaschke set, Theorem 6.10 implies that complex Banach Lie group $\mathcal{A}(\bar{U}_1 \cap U_2, G)$ is connected. Hence we can apply Theorem 6.8 to $c$ (with $E = M(\mathcal{A}_Z) \times G$ and $W_i := V_i$, $i = 1, 2$) to find some $c_1 \in \mathcal{A}(V_i, G)$, $i = 1, 2$, such that $c_1^{-1}c_2 = c$ on $\bar{V}_1 \cap \bar{V}_2$. This shows that $P|_V$ is trivial. □

In the sequel, we also require the following result.

**Proposition 6.14.** Suppose $K \subset M(\mathcal{A}_Z) \setminus \{z\}$ is compact such that $Q_Z^{-1}(K)$ is the closure of an open Blaschke product $O_{h, \varepsilon}$. Then $K$ is holomorphically convex.

**Proof.** Assume that $x' \in M(H^\infty)$ is such that $x := Q_Z(x') \notin K \cup \{z\}$, $\{z\} := Q_Z(Z)$. Since $Q_Z^{-1}(K) = O_{h, \varepsilon}$ is holomorphically convex, there is a function $f \in \mathcal{O}(M(H^\infty))$ such that $f(x') = 1$ and $\max_{Q_Z^{-1}(K)} |f| \leq \frac{1}{2}$. By definition $Z$ is the hull of ideal $I(Z) \neq (0)$. Hence, there exists a function $g \in \mathcal{O}(M(H^\infty))$ such that $g(x') = 1$ and $g|_Z = 0$. For a sufficiently large $n \in \mathbb{N}$ we obtain

$$(f^n g)(x') = 1 \quad \text{and} \quad \max_{Q_Z^{-1}(K) \cup \bar{Z}} |f^n g| < 1.$$ 

Since $(f^n g)|_Z = 0$, there exists a function $h \in \mathcal{O}(M(\mathcal{A}_Z))$ such that $Q_Z^* h = f^n g$. In particular,

$$|h(x)| = 1 \quad \text{and} \quad \max_{K \cup \{z\}} |h| < 1.$$ 

The latter condition shows that $K \cup \{z\} \subset M(\mathcal{A}_Z)$ is a holomorphically convex set. Let $U_1$ and $U_2$ be open neighbourhoods of $K$ and $z$ such that $\bar{U}_1 \cap \bar{U}_2 = \emptyset$. Consider the holomorphic function on $U_1 \sqcup U_2$ equals 0 on $U_1$ and 1 on $U_2$. By Theorem 6.3 (the Runge-type approximation theorem) it can be uniformly approximated on $K \cup \{z\}$ by holomorphic functions of $\mathcal{O}(M(\mathcal{A}_Z))$. In particular, there is a function $t \in \mathcal{O}(M(\mathcal{A}_Z))$ such that $|t(z)| > \frac{1}{2}$ and $\max_K |t| < \frac{1}{2}$. This shows that $K$ is holomorphically convex. □

7. Proofs of Theorem 2.6, Corollary 2.7 and Proposition 2.8

7.1. **Principal Bundles with Connected Fibres.** First, we prove a particular case of Theorem 2.6 for bundles with connected fibres.

Let $V \subset U \subset M(\mathcal{A}_Z)$ be open sets.

**Theorem 7.1.** Suppose $P$ is a holomorphic principal bundle on $U$ with fibre a connected complex Banach Lie group $G$. Then $P|_V$ is trivial.

**Proof.** By $\psi : G_u \to G$ we denote the universal covering of $G$. The complex Banach Lie group $G_u$ is simply connected and the kernel of $\psi$ is a discrete central subgroup $Z_G \subset G_u$.

Let $W$ be a relatively compact open subset of $U$ such that $V \subset W$. 

Proposition 7.2. There exist a principal $G_u$-bundle $P_u$ on $W$ and a morphism of principal bundles $Ψ : P_u → P|_{W}$.

Proof. Since $W$ is compact, $P|_{W}$ is defined on a finite open cover $\mathfrak{U} = (W_i)_{i ∈ I}$ of $W$ by a cocycle $c = \{c_{ij} ∈ C(W_i ∩ W_j, G)\}_{i,j ∈ I}$ such that $c_{ij}|_{W_i ∩ W_j} ∈ \mathcal{O}(W ∩ W_i ∩ W_j, G)$ for all $i, j ∈ I$. To prove the result one constructs a principal $G_u$-bundle $P_u$ on $W$ defined on a finite open cover $\mathfrak{U} = (V_i)_{i ∈ L}$ of $W$, a refinement of $\mathfrak{U}$, by cocycle $\hat{c} = \{\hat{c}_{lm} ∈ C(V_l ∩ V_m, G_u)\}_{l,m ∈ L}$ such that cocycle $ψ(\hat{c}) = (ψ(\hat{c}_{lm}) ∈ C(V_l ∩ V_m, G))_{l,m ∈ L}$ coincides with the pullback of $c$ to $\mathfrak{U}$ with respect to the refinement map $τ : L → I$. (Recall that $V_l ⊂ W_{τ(l)}$ for all $l ∈ L$ so that $ψ(\hat{c}_{lm}) = c_{τ(l)τ(m)}|_{V_l ∩ V_m}$, $l, m ∈ L$.) The required morphism $Ψ : P_u → P|_{W}$ in local coordinates on $\mathfrak{U}$ is defined as $Ψ(x, g) := (x, ψ(g))$, $(x, g) ∈ V_l × G_u$, $l ∈ L$. Note that since $ψ$ is locally biholomorphic, $P_u$ is holomorphic on $W$.

The construction of such bundle $P_u$ repeats word-for-word the one presented on pages 135–136 of the proof of [Br2] Th.6.4, where instead of Lemma 6.5 there one uses

Lemma 7.3. For each finitely generated subgroup $Z′ ⊂ Z_©$ the Čech cohomology group $H^2(W, Z)$ = 0.

The proof of this lemma is the same as that of Lemma 6.5 in [Br2] and relies upon the facts $dim W = 2$ and $H^2(W, Z) = 0$ which follow immediately from similar results for $M(\mathfrak{s}_2Z)$ established in Proposition 2.7 above. We leave the details to the readers. □

Proposition 7.2 implies that in order to prove the theorem it suffices to show that bundle $P_u|_{V}$ is trivial. (In this case there exists a holomorphic section $s : V → P_u$ and so $Ψ(s)$ is a holomorphic section of $P|_{V}$, i.e., $P|_{V}$ is trivial as well.) Let us prove this statement.

Since $W$ is compact, $P_u|_{W}$ is defined by a holomorphic $G_u$-valued cocycle on a finite open cover $\mathfrak{U} = (V_i)_{i ∈ L}$ of $W$; in particular, $P_u|_{V_i} = V_i × G_u$ for all $i ∈ L$. Since $M_s$ is totally disconnected and $Q_Z$ maps $Z$ to $z ∈ M(\mathfrak{s}_2Z)$ and is one-to-one outside, $Q_Z(M_s) ∪ \{z\}$ is totally disconnected as well (see, e.g., [N] Ch.2, Th.9-11). In particular, there exists a finite open cover of $(Q_Z(M_s) ∪ \{z\}) ∩ W$ by pairwise disjoint open subsets of $W$ such that $P_u$ is trivial over each of them. Thus $P_u$ is trivial over the union of these sets forming an open neighbourhood of $(Q_Z(M_s) ∪ \{z\}) ∩ W$. Without loss of generality we may assume that this neighbourhood coincides with $V_1$.

Let us choose some open neighbourhood $V_0 ∋ V_1$ of $(Q_Z(M_s) ∪ \{z\}) ∩ W$ and a sequence of open sets $W^i \subseteq V$ containing $V_0$ such that $W^i+1 ⊆ W^i$, $i ∈ N$. Since $Z^{-1}(V \setminus V_0)$ is a compact subset of $M_a \setminus Z$, it can be covered by open Blaschke sets $O_{b_1,e_1}, ..., O_{b_N,e_N}$ so that all $O_{b_j,e_j} ∋ (M_a \setminus Z) ∩ W$ and each $Q_Z(O_{b_j,e_j})$ is a subset of one of $V_l$, $l ∈ L$. For every $i ∈ N$ we set $W^i_{j+1} := Q_Z(O_{b_j,e_j})$, $λ_i := 1 + \frac{1}{i}$, $1 ≤ j ≤ N$.

Then $(W^i_j)_{1≤j≤N+1}$, $i ∈ N$, are open covers of $V$ such that (a) $P_u|_{W^i_j}$ are trivial for all $i, j$; (b) all $Q^1_Z(W^i_j)$, $2 ≤ j ≤ N + 1$, $i ∈ N$, are open Blaschke sets relatively compact in $M_a \setminus Z$; (c) $W^i_{j+1} ⊆ W^i_j$ for all $i, j$; (d) all $W^i_j$, $2 ≤ j ≤ N + 1$, $i ∈ N$, are holomorphically convex (see Proposition 6.14).

Next, for $1 ≤ k ≤ N + 1$ and $i ∈ N$ we set $Z^k_i := \bigcup_{j=1}^{k} W^i_j$.

Using induction on $k$, $1 ≤ k ≤ N + 1$, we prove that each bundle $P_u|_{Z^k_i}$ is trivial.
Indeed, $P_u|_{Z_l^1}$ is trivial by the definition of $V_1$. Assuming that the statement is valid for $k - 1$ let us prove it for $k$.

To this end, we apply Corollary 6.13 with $U_1$ being an open relatively compact subset of $W_{k-1}^k$ containing $W_k^k$, $U_2$ an open relatively compact subset of $Z_{k-1}^{k-1}$ containing $Z_k^{k-1}$, $V_1 := W_k^k$, $V_2 := Z_k^{k-1}$. Since $V_1 \cup V_2 = Z_k^k$, the required statement follows from the corollary.

For $k = N + 1$ we obtain $V \subset Z_{N+1}^{N+1}$. Thus $P_u|_{V}$ is trivial as required.

The proof of the theorem is complete. \[ \Box \]

7.2. Proof of Theorem 2.6. Suppose $\pi : P \to U$ is a holomorphic principal $G$-bundle trivial on an open neighbourhood $V$ of a compact set $K \subset U$. If $P|_{V}$ is defined on an open cover $(V_l)_{l \in L}$ of $V$ by cocycle $g = \{g_{lm} \in \mathcal{O}(V_l \cap V_m, G)\}_{l, m \in L}$, the associated principal $C(G)$-bundle $P_{C(G)}|_{V}$ is defined by cocycle $q(g) = \{q(g_{lm}) \in C(V_l \cap V_m, C(G))\}_{l, m \in L}$.

(Recall that $C(G) := G/G_0$ is the discrete group of connected components of $G$ and $q : G \to C(G)$ is the corresponding quotient homomorphism.)

Since $P|_{V}$ is trivial, there exist $g_l \in \mathcal{O}(V_l, G)$, $l \in L$, such that

$$ q^{-1}(g_l g_m) = q(g_{lm}) \quad \text{on} \quad V_l \cap V_m. $$

This implies that

$$ q(g_l)^{-1}q(g_m) = q(g_{lm}) \quad \text{on} \quad V_l \cap V_m, $$

i.e., bundle $P_{C(G)}|_{V}$ is topologically trivial and so $P_{C(G)}|_{K}$ as well.

Conversely, suppose $P_{C(G)}|_{K}$ is trivial. Then there is an open neighbourhood $V \subset U$ of $K$ such that $P_{C(G)}|_{V}$ is trivial as well (see, e.g., [L] Lm. 4 for the proof of this well-known fact). Thus if $P|_{V}$ is defined on a finite open cover $\mathfrak{U} = (V_l)_{l \in L}$ of $V$ by a cocycle $g = \{g_{lm} \in \mathcal{O}(V_l \cap V_m, G)\}_{l, m \in L}$, there exist $h_l \in C(V_l, C(G))$, $l \in L$, such that

$$ h_l^{-1}h_m = q(g_{lm}) \quad \text{on} \quad U_l \cap U_m. $$

Replacing $V$ by a relatively compact open subset containing $K$, if necessary, without loss of generality we may assume that each $h_l$ admits a continuous extension to compact set $V_l$, $l \in L$. Since $h_l : V_l \to C(G)$ is continuous and $C(G)$ is discrete, the image of $h_l$ is finite. In particular, there exists a continuous locally constant function $h_l : V_l \to G$ such that $q \circ h_l = h_l$. By definition each $h_l \in \mathcal{O}(V_l, G)$. Let us define cocycle $\tilde{g} = \{\tilde{g}_{lm} \in \mathcal{O}(V_l \cap V_m, G)\}_{l, m \in L}$ by the formulas

$$ \tilde{g}_{lm} := \tilde{h}_l g_{lm} \tilde{h}_m^{-1} \quad \text{on} \quad V_l \cap V_m. $$

Then $\tilde{g}$ determines a holomorphic principal $G$-bundle $\tilde{P}$ on $V$ isomorphic to $P|_{V}$. Also, (7.1) implies that for all $l, m \in L$

$$ q(\tilde{g}_{lm}) = 1_{C(G)}. $$

Thus each $\tilde{g}_{lm}$ maps $V_l \cap V_m$ into $G_0$. In particular, $\tilde{g}$ determines also a subbundle of $\tilde{P}$ with fibre $G_0$. According to Theorem 7.1 this subbundle is trivial over an open neighbourhood $W \subset V$ of $K$ (because $G_0$ is connected). Thus there exist $\tilde{g}_l \in \mathcal{O}(W \cap V_l, G_0)$, $l \in L_W := \{k \in L : W \cap V_k \neq \emptyset\}$, such that

$$ \tilde{g}_l^{-1}\tilde{g}_m = \tilde{g}_{lm} \quad \text{on} \quad (W \cap V_l) \cap (W \cap V_m). $$

From here and (7.2) we obtain that for all $l, m \in L_W$

$$ (\tilde{g}_l h_l)^{-1}(\tilde{g}_m h_m) = g_{lm} \quad \text{on} \quad (W \cap V_l) \cap (W \cap V_m). $$

This shows that holomorphic principal $G$-bundle $P|_{W}$ is trivial.

The proof of the theorem is complete.
7.3. Proof of Corollary 2.7

(1) In this case \( C(G) \) is trivial so the statement follows directly from Theorem 2.6.

(2) By the hypothesis the associated bundle \( P_{C(G)} \) on \( M(H^\infty) \) has a finite fibre. Then it is trivial by \([Br3\ Lm.\ 8.1]\) and the result follows from Theorem 2.6.

(3) Let \( P|_K \) be defined on a finite open cover \( (U_i)_{i \in I} \) of \( K \) by a cocycle \( g = \{g_{ij} \in C(U_i \cap U_j, G)\}_{i,j \in I} \). Since \( P|_K \) is topologically trivial, there exist \( g_i \in C(U_i, G) \), \( i \in I \), such that

\[
 g_i^{-1} g_j = g_{ij} \quad \text{on} \quad U_i \cap U_j.
\]

This implies that

\[
 q(g_i)^{-1} q(g_j) = q(g_{ij}) \quad \text{on} \quad U_i \cap U_j,
\]

i.e., \( P_{C(G)}|_K \) is topologically trivial. Thus to get the result we apply Theorem 2.6.

7.4. Proof of Proposition 2.8. We require

**Lemma 7.4.** The Čech cohomology group \( H^1(M(\mathfrak{g}_Z), \mathbb{Z}) \) is nontrivial.

**Proof.** Due to the Arens-Royden theorem (see \([A2], [Ro]\)) it suffices to show that there exists a function in \( \mathcal{O}(M(\mathfrak{g}_Z), \mathbb{C}^*) \) whose pullback to \( M(H^\infty) \) by \( Q_Z \) does not have a bounded logarithm.

To this end let us take a function \( f \in I(Z) \) such that \( \|f\|_{H^\infty} = 1 \). Then \( \hat{f}(x) = 1 \) for some \( x \in M(H^\infty) \). By the definition \( K := \hat{f}(M(H^\infty)) \) is a compact subset of the closed unit disk \( \mathbb{D} \) containing 0 and 1, and \( f(\mathbb{D}) \) is the open dense subset of \( K \). In particular, \( z = 1 \) is the limit point of \( f(\mathbb{D}) \). Consider holomorphic function \( h(z) := i \cdot \Log \left( \frac{1}{z+1} \right) \), \( z \in \mathbb{D} \); here \( \Log \) is the principal value of the logarithmic function. It is unbounded on a neighbourhood of \( z = 1 \) in \( \mathbb{D} \) and therefore \( h|_{f(\mathbb{D})} \) is unbounded as well. Moreover, \( \Re h = -\Arg \left( \frac{1}{z+1} \right) \) is bounded (taking values in \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \)). Thus \( g := h \circ f \) is an unbounded holomorphic function on \( \mathbb{D} \) such that \( e^{\pm g} \in H^\infty \). Since \( h \) is holomorphic at \( z = 0 \), function \( g \) can be presented as a uniformly convergent series in variable \( f \) on each set \( N_f(\varepsilon) := \{ z \in \mathbb{D} : |f(z)| < \varepsilon \}, \varepsilon \in (0, 1) \). Then due to \([S1\ Th.\ 3.2]\) \( \hat{g}|_{N_f(\varepsilon)} \) admits a continuous extension \( \hat{g} \in \mathcal{O}(N_f(\varepsilon)) \), \( N_f(\varepsilon) := \{ z \in \mathbb{D} : |\hat{f}(x)| < \varepsilon \} \), such that \( \hat{g}|_{\mathbb{D}} = h(0) = 0 \). By uniqueness of the extension \( \hat{g} \in \mathcal{O}(M(H^\infty), \mathbb{C}^*) \) and equals 1 on \( Z \). In particular, there exists \( \hat{g} \in \mathcal{O}(M(\mathfrak{g}_Z), \mathbb{C}^*) \) such that \( Q_Z \hat{g} = \hat{e}^\vartheta \). Since by our construction function \( \hat{e}^\vartheta \) does not have a bounded logarithm, \( \hat{g} \) is as required.

Let \( G \) be a complex Banach Lie group such that group \( C(G) \) has a nontorsion element. The construction of a nontrivial holomorphic principal \( G \)-bundle on \( M(\mathfrak{g}_Z) \) is similar to that of \([Br3\ Th.\ 3.1]\).

Consider an integer-valued continuous cocycle \( \{c_{ij}\}_{i,j \in I} \) defined on an open cover \( (U_i)_{i \in I} \) of \( M(\mathfrak{g}_Z) \) representing a nonzero element of \( H^1(M(\mathfrak{g}_Z), \mathbb{Z}) \), see Lemma 7.4. Let \( \langle a \rangle := \{a^n : n \in \mathbb{Z}\} \subset C(G) \) be the subgroup generated by a nontorsion element \( a \). Then \( \langle a \rangle \cong \mathbb{Z} \). Let \( p : B \to M(\mathfrak{g}_Z) \) be the principal bundle with (discrete) fibre \( C(G) \) defined on the cover \( (U_i)_{i \in I} \) by \( C(G) \)-valued cocycle \( \{a^{c_{ij}}\}_{i,j \in I} \).

**Lemma 7.5.** \( B \) is a nontrivial bundle on \( M(\mathfrak{g}_Z) \).

**Proof.** Suppose, on the contrary, that \( B \) is trivial. Then there are \( g_i \in C(U_i, C(G)), i \in I \), such that

\[
 g_j(x) = g_i(x) a^{c_{ij}(x)} \quad \text{for all} \quad x \in U_i \cap U_j, \quad i, j \in I.
\]

Next, the restriction of \( \{Q_Z^{c_{ij}}\}_{i,j \in I} \) to the open cover \( (\mathbb{D} \cap Q_Z^{-1}(U_i))_{i \in I} \) of \( \mathbb{D} \) represents an element of the cohomology group \( H^1(\mathbb{D}, \mathbb{Z}) \). Since \( \mathbb{D} \) is contractible, this group is trivial.
In particular, there exist continuous functions $c_i : \mathbb{D} \cap Q_Z^{-1}(U_i) \to \mathbb{Z}$ such that
\[
c_j(x) = c_i(x) + (Q_Z^i c_j(x)) \quad \text{for all } x \in (\mathbb{D} \cap Q_Z^{-1}(U_i)) \cap (\mathbb{D} \cap Q_Z^{-1}(U_j)), \ i, j \in I.
\]
This and (7.3) imply that
\[
g(x) := (Q_Z^i g_i)(x) a^{-c_i(x)}, \quad x \in \mathbb{D} \cap Q_Z^{-1}(U_i), \ i \in I,
\]
is a continuous map on $\mathbb{D}$ with values in $C(G)$. Since $C(G)$ is a discrete space and $\mathbb{D}$ is connected, $g(x) = g(0)$ for all $x \in \mathbb{D}$. Hence each map $g^{-1}(0) Q_Z^i g_i : \mathbb{D} \cap Q_Z^{-1}(U_i) \to C(G)$ has range in $\langle a \rangle$. Further, since $\mathbb{D}$ is an open dense subset of $M(H^\infty)$ and $Q_Z^{-1}(U_i) \subset M(H^\infty)$ is open, $\mathbb{D} \cap Q_Z^{-1}(U_i)$ contains $Q_Z^{-1}(U_i)$. Since $Q_Z^i g_i$ is continuous on $Q_Z^{-1}(U_i)$ and $\langle a \rangle$ is a closed subset of $C(G)$, extending $(g^{-1}(0) Q_Z^i g_i)|_{\mathbb{D} \cap Q_Z^{-1}(U_i)}$ by continuity we get that each $g^{-1}(0) Q_Z^i g_i : Q_Z^{-1}(U_i) \to C(G)$ has range in $\langle a \rangle$. Therefore each $g^{-1}(0) g_i : U_i \to C(G)$ has range in $\langle a \rangle$ as well.

Then equations (cf. (7.3))
\[
g^{-1}(0) g_j(x) = g^{-1}(0) g_i(x) a^{c_{ij}(x)} \quad \text{for all } x \in U_i \cap U_j, \ i, j \in I,
\]
show that cocycle $\{ a^{c_{ij}} \}_{i,j \in I}$ determines the trivial bundle in the category of principal bundles on $M(\mathcal{A}_Z)$ with fibre $\langle a \rangle \cong \mathbb{Z}$. Equivalently, cocycle $\{ c_{ij} \}_{i,j \in I}$ represents 0 in $H^1(M(\mathcal{A}_Z), \mathbb{Z})$, a contradiction. \hfill $\square$

Now, consider an element $b \in G$ such that $q(b) = a$. We set $g_{ij} := b^{c_{ij}}$. Then $\{ g_{ij} \}_{i,j \in I}$ is a holomorphic $G$-valued 1-cocycle on cover $(U_i)_{i \in I}$ of $M(\mathcal{A}_Z)$. It determines a holomorphic principal bundle $P$ on $M(\mathcal{A}_Z)$ with fibre $G$. By the definition the associated bundle $P_{c(G)}$ coincides with $B$ and therefore by Lemma 7.5 it is nontrivial. Hence, $P$ is a nontrivial holomorphic $G$-bundle by Theorem 2.6.

The proof of Proposition 2.8 is complete.

8. Proof of Theorem 2.9

We will establish a more general result required in the proofs of theorems of Section 3.2.

Let $P_1$ be a holomorphic principal bundle on $M(\mathcal{A}_Z)$ with fibre a complex Banach Lie group $G$ and $P_2$ be a holomorphic principal $G$-bundle on an open set $U \subset M(\mathcal{A}_Z)$. Let $V \subset U$ be an open subset.

**Theorem 8.1.** If $P_1|_U$ and $P_2$ are topologically isomorphic, then $P_1|_V$ and $P_2|_V$ are holomorphically isomorphic.

**Proof.** Passing to suitable refinements of open covers of $M(\mathcal{A}_Z)$ and $U$ over which bundles $P_1$ and $P_2$ are trivialized and diminishing $U$, if necessary, without loss of generality we may assume that $P_1$ is defined on a finite open cover $\mathcal{U}_1 = (U_i)_{i \in I_1}$ of $M(\mathcal{A}_Z)$ by a holomorphic $G$-valued cocycle $g = \{ g_{ij} \in \mathcal{O}(U_i \cap U_j, G) \}_{i,j \in I_1}$ and there is a subset $I_2 \subset I_1$ such that $P_2$ is defined on the finite open cover $\mathcal{U}_2 = (U_i)_{i \in I_2}$ of $U$ by a holomorphic $G$-valued cocycle $h = \{ h_{ij} \in \mathcal{O}(U_i \cap U_j, G) \}_{i,j \in I_2}$. By the hypothesis of the theorem $P_1|_U$ and $P_2$ are topologically isomorphic, i.e., there exist $f_i \in \mathcal{C}(U_i, G)$, $i \in I_2$, such that for all $i,j$
\[
f_i^{-1} g_{ij} f_j = h_{ij} \quad \text{on } U_i \cap U_j.
\]

Applying to equations (8.1) the quotient homomorphism $q : G \to C(G) := G/G_0$ we get for all $i,j \in I_2$
\[
q(f_i)^{-1} q(g_{ij}) q(f_j) = q(h_{ij}) \quad \text{on } U_i \cap U_j.
\]

This shows that the associated principal $C(G)$-bundles $P_{c(G)}|_U$ and $P_{c(G)}$ are isomorphic.

Diminishing $U$, if necessary, without loss of generality we may assume that each $f_i$ admits a continuous extension to $\bar{U}_i$. Then the image of each $q(f_i)$ is finite. In particular,
there exist continuous locally constant maps $c_i : U_i \to G$ such that $q \circ c_i = q(f_i), i \in I_2$.
We set $c_i := 1_G$ for all $i \in I_1 \setminus I_2$ and define a holomorphic principal $G$-bundle $\tilde{P}_1$ on
$M(\mathcal{A}_Z)$ by cocycle $\tilde{g} = \{g_{ij} := c_i^{-1}g_{ij}c_j \in O(U_i \cap U_j, G)\}_{i,j \in I_1}$. Then $\tilde{P}_1$ is holomorphically
isomorphic to $P_1$ and $q(\tilde{g})|_{U_i} = q(h)$. To avoid abuse of notation in what follows we assume
that cocycle $g$ initially has this property.
Let $A(P_1)$ be a holomorphic fibre bundle on $M(\mathcal{A}_Z)$ with fibre $G_0$ defined on the
cover $\mathfrak{U}_1$ by holomorphic cocycle $\text{Ad}(g) = \{\text{Ad}(g_{ij}) \in O(U_i \cap U_j, \text{Aut}(G_0))\}_{i,j \in I_1}$ (here
$\text{Ad}(u)(v) := u^{-1}vu, u \in G, v \in G_0$.) Consider the family of holomorphic sections $s = \{s_{ij} \in \Gamma_O(U_i \cap U_j, A(P_1))\}_{i,j \in I_2}$ of $A(P_1)$ such that in the trivialization of $A(P_1)$ on $U_i$,
$$s_{ij}(x) = (x, h_{ij}(x)g_{ji}(x)), \quad x \in U_i.$$ (Note that $s_{ij}$ is well-defined as $q(h_{ij}g_{ji}) = 1_{C(G)}$.)

**Lemma 8.2.** Family $s$ forms a holomorphic $A(P_1)$-valued 1-cocycle on cover $\mathfrak{U}_2$.

**Proof.** We must show that $s_{ij} s_{jk} s_{ki} = 1_{A(P_1)}$ on $U_i \cap U_j \cap U_k \neq \emptyset, i, j, k \in I_2$. It suffices
to check this in local coordinates of $A(P_1)$ on $U_i$. Then, for $x \in U_i \cap U_j \cap U_k$,
$$s_{ij}(x) = (x, h_{ij}(x)g_{ji}(x)), \quad s_{jk}(x) = (x, \text{Ad}(g_{ij}^{-1}(x))(h_{jk}(x)g_{kj}(x))),$$
$$s_{ki}(x) = (x, \text{Ad}(g_{ik}^{-1}(x))(h_{ki}(x)g_{ik}(x))).$$
Using these and that $(g_{ij})_{i,j \in I_1}$ and $(h_{ij})_{i,j \in I_2}$ are cocycles we obtain
$$s_{ij}(x)s_{jk}(x)s_{ki}(x) = (x, (h_{ij}(x)g_{ji}(x)) \cdot (g_{ij}(x)h_{jk}(x)g_{kj}(x))g_{ki}(x)) \cdot (g_{ik}(x)h_{ki}(x)g_{ik}(x)g_{ki}(x))) = (x, h_{ij}(x)h_{jk}(x)h_{ki}(x)) = (x, 1_G) = 1_{A(P_1)}(x).$$

Let $\psi : G_u \to G_0$ be the universal covering of $G_0$. By the covering homotopy theorem,
for each automorphism $\text{Ad}(g), g \in G_0$, there is an automorphism $A(g)$ of $G_u$ such
that $\psi \circ A(g) = \text{Ad}(g) \circ \psi$. Also, since the fundamental group of $G_0$ is naturally isomorphic
to a subgroup of the center of $G_u$, it is readily seen that $A(g) = \text{Ad}(\tilde{g})$ for all $\tilde{g} \in G_u$ such
that $\psi(\tilde{g}) = g$. By definition, $A(g_1g_2) = A(g_1)A(g_2)$ for all $g_1, g_2 \in G$. Hence, $A : G \to \text{Inn}(G_u), g \mapsto A(g)$, is a homomorphism from $G$ to the group of inner automorphisms of
$G_u$. Since $\psi$ is locally biholomorphic, the covering homotopy theorem implies that map
$A(\cdot)h : G \to G_u$ is holomorphic for each $h \in G_u$.

Let us consider a topological bundle $A_u(P_1)$ on $M(\mathcal{A}_Z)$ with fibre $G_u$ defined on open
cover $\mathfrak{U}_1$ by holomorphic transitions functions $\{A(g_{ij}) \in O(U_i \cap U_j \times G_u, G_u)\}_{i,j \in I_1}$ (cf.
Section 6.5 above). The quotient map $\psi : G_u \to G_0$ induces a holomorphic bundle
map $\Psi : A_u(P_1) \to A(P_1)$ defined in local coordinates on $U_i$ as $\Psi(x, g) := (x, \psi(g))$, $(x, g) \in U_i \times G_u, i \in I$.

**Proposition 8.3.** There exists a holomorphic $A_u(P_1)$-valued 1-cocycle $\tilde{s} = \{\tilde{s}_{ij}\}$ on $\mathfrak{U}_2$
such that $\Psi \circ \tilde{s}_{ij} = s_{ij}$ for all $i, j \in I_2$.

**Proof.** Since $q(g_{ij}) = q(h_{ij})$ for all $i, j \in I_2$, the hypothesis of the theorem implies that there exists $f_i \in C(U_i, G_0), i \in I_2$, such that for all $i, j$
$$f_i^{-1}g_{ij}f_j = h_{ij} \quad \text{on} \quad U_i \cap U_j.$$ Let $s_k$ be a continuous section of $A(P_1)|_{U_k}$ such that $s_k(x) = (x, f_k(x)), x \in U_k$, in local
coordinates on $U_k$. Then for $x \in U_i \cap U_j$ we obtain (using local coordinates on $U_i$)
$$s_i(x)s_{ij}(x)s_j^{-1}(x) = (x, f_i(x)(h_{ij}(x)g_{ji}(x))(\text{Ad}(g_{ij}^{-1}(x))(f_j^{-1}(x)))) = (x, f_i(x)h_{ij}(x)f_j^{-1}(x)g_{ij}^{-1}(x)) = (x, 1_G) = 1_{A(P_1)}(x).$$
Thus for all $i, j \in I_2$ 
\[ s_{ij} = s_i^{-1}s_j \quad \text{on} \quad U_i \cap U_j. \]

Passing to a refinement of $\Omega_1$ and then diminishing $U$, if necessary, without loss of generality we may assume that for each $i \in I_2$ the closure of $f_i(U_i)$ belongs to a simply connected open subset $V_i \subset G_0$. In particular, the holomorphic map $\psi^{-1} : V_i \to G_u$ inverse to $\psi|_{V_i}$ is defined. Let $\tilde{s}_k$ be a continuous section of $A_u(P_1)|_{U_k}$ such that $\tilde{s}_k(x) = (x, \psi^{-1}_k(f_k(x)))$, $x \in U_k$, in local coordinates on $U_k$. We set for all $i, j \in I_2$
\[ \tilde{s}_{ij} := \tilde{s}_i^{-1}\tilde{s}_j \quad \text{on} \quad U_i \cap U_j. \]

Then 
\[ \Psi \circ \tilde{s}_{ij} = s_i^{-1}s_j = s_{ij} \quad \text{on} \quad U_i \cap U_j. \]

Since map $\psi$ is locally biholomorphic, each $\tilde{s}_{ij} \in \Gamma_C(U_i \cap U_j, A_u(P_1))$. By definition, family $\tilde{s} = \{\tilde{s}_{ij}, i, j \in I_2\}$ is an $A_u(P_1)$-valued 1-cocycle on $\Omega_2$. \qed 

We set $I := \{i \in I_2 : V \cap U_i \neq \emptyset\}$ and $V_i := V \cap U_i$, $i \in I$.

**Proposition 8.4.** There exist sections $t_i \in \Gamma_C(V_i, A_u(P_1))$, $i \in I$, such that for all $i, j \in I$
\[ t_i^{-1}t_j = \tilde{s}_{ij} \quad \text{on} \quad V_i \cap V_j. \]

**Proof.** As in the proof of Theorem 8.3 let us construct open covers $\mathfrak{W}^i = (W_j)^{1 \leq j \leq N + 1}$, $i \in \mathbb{N}$, of $\tilde{V}$ consisting of relatively compact open subsets of sets of cover $\Omega_2$ of $U$ satisfying
(a) all $QZ^{-1}(W_j)$, $2 \leq j \leq N + 1$, are open Blaschke sets relatively compact in $M_a \setminus Z$,
(b) $W_{j+1}^i \subset W_j^i$ for all $i, j$; (c) all sets $\tilde{W}_j^i$, $2 \leq j \leq N + 1$, $i \in \mathbb{N}$, are holomorphically convex.

For $1 \leq k \leq N + 1$ and $i \in \mathbb{N}$ we set
\[ Z_k^i := \bigcup_{j=1}^{k} W_j^i. \]

By $\{\tilde{s}_{ij}^1\}_{1 \leq i, j \leq N + 1}$ we denote the pullback of cocycle $\tilde{s} = \{\tilde{s}_{ij}, i, j \in I_2\}$ to cover $\mathfrak{W}^1$ by the refinement map. Using induction on $k \in \{1, \ldots, N + 1\}$ we prove the following statement: (c) There exist sections $t_i^k \in \Gamma_C(W_i^k, A_u(P_1))$, $1 \leq i \leq k$, such that for all $1 \leq i, j \leq k$
\[ (t_i^k)^{-1}t_j^k = \tilde{s}_{ij}^1 \quad \text{on} \quad W_i^k \cap W_j^k. \]

If $k = 1$, the statement is trivial (we just set $t_i^1 := 1_{A_u(P_1)|_{W_i^1}}$).

Assuming that the statement is valid for $k \in \{1, \ldots, N\}$ let us prove it for $k + 1$.

We set $t_{ij}^k := 1_{A_u(P_1)|_{W_k^i}}$ for $k + 1 \leq j \leq N + 1$ and define a new holomorphic $A_u(P_1)$-valued 1-cocycle $\tilde{s}^k = \{\tilde{s}_{ij}^k\}_{1 \leq i, j \leq N + 1}$ on $\mathfrak{W}^k$ by the formulas
\[ \tilde{s}_{ij}^k := t_i^k t_j^{-1} \tilde{s}_{ij} \quad \text{on} \quad W_i^k \cap W_j^k, \quad 1 \leq i, j \leq N + 1. \]

Then for all $1 \leq i, j \leq k$ we have $\tilde{s}_{ij}^k = 1_{A_u(P_1)}$ on $W_i^k \cap W_j^k$. Moreover, restrictions of $\tilde{s}_{i,k+1}^k$, $1 \leq i \leq k$, to $Z_k^i \cap W_{k+1}^k$ glue together to define a section $t_{k+1}^i \in \Gamma_C(Z_k^i \cap W_{k+1}^k, A_u(P_1))$.

Indeed, if $x \in (W_i^k \cap W_j^k) \cap W_{k+1}^k$, $1 \leq i, j \leq k$, then since $\tilde{s}^k$ is a cocycle,
\[ \tilde{s}_{i,k+1}^k(x)(\tilde{s}_{j,k+1}^k(x))^{-1} = \tilde{s}_{ij}^k(x) = 1_{A_u(P_1)}(x). \]

Hence,
\[ \tilde{s}_{i,k+1}^k = \tilde{s}_{j,k+1}^k \quad \text{on} \quad (W_i^k \cap W_j^k) \cap W_{k+1}^k \]
as required.

Next, consider the open cover $\{Z_k^i, W_{k+1}^i\}$ of $Z_k^i$. By our definition, $Z_k^i \cap W_{k+1}^i \subset Z_k^i \cap W_{k+1}^j$ and so $QZ^{-1}(Z_k^i \cap W_{k+1}^j)$ is a compact subset of open Blaschke set $W_{k+1}^j$. 


Moreover, $W_{k+1}^k$ is holomorphically convex. Choosing some open sets $U_i$, $i = 1, 2$, such that $Z_{k+1}^k \Subset U_1 \Subset Z_k^k$ and $W_{k+1}^k \Subset U_2 \Subset W_{k+1}^k$ and using that $t_{k+1}|U_1 \cap U_2$ belongs to the connected component of $1_{\mathcal{A}_u(P_1)}U_1 \cap U_2$ (this follows from Theorem 6.10 if we write $t_{k+1}$ in local coordinates on $W_{k+1}^k$) we obtain by Theorem 6.10 that there exist some sections $t_{k+1} \in \Gamma_{\mathcal{O}}(Z_{k+1}^k, \mathcal{A}_u(P_1))$ and $t_{k+1} \in \Gamma_{\mathcal{O}}(W_{k+1}^k, \mathcal{A}_u(P_1))$ such that

$$(t_{k+1})^{-1}t_{k+1} = t_{k+1} \quad \text{on} \quad Z_{k+1}^k \cap W_{k+1}^k.$$

Let us define

$$t_{k+1}(x) := \begin{cases} t_{k+1}(x), & x \in W_{k+1}^k, \quad 1 \leq i \leq k, \\ t_{k+1}(x), & x \in W_{k+1}^k. \end{cases}$$

Then by the induction hypothesis for $x \in W_{k+1}^k \cap W_{k+1}^j$, $1 \leq i, j \leq k$,

$$(t_{k+1}(x))^{-1}t_{k+1}(x) = (t_{k+1}(x))^{-1}(t_{k+1}(x))^{-1}t_{k+1}(x) = s_{ij}^1.$$

Also, for $x \in W_{k+1}^k \cap W_{k+1}^j$, $1 \leq i, k \leq k$,

$$(t_{k+1}(x))^{-1}t_{k+1}(x) = (t_{k+1}(x))^{-1}(t_{k+1}(x))^{-1}t_{k+1}(x) = (t_{k+1}(x))^{-1}t_{k+1}(x) = s_{ij}^1.$$
and arguing as in the proof of Proposition 8.3 (but in the reverse order) for all $i, j \in I$ we obtain

$$g_i^{-1}g_ig_j = h_{ij} \quad \text{on} \quad V_i \cap V_j.$$ 

These identities show that bundles $P_1|_V$ and $P_2|_V$ are holomorphically isomorphic.

The proof of Theorem 8.11 (and therefore of Theorem 2.9) is complete. \qed

9. Proofs of Theorems 2.10 and 2.11

9.1. Auxiliary result. Let $\pi : P \to M(\mathcal{A}_Z)$ be a topological principal bundle with fibre a complex Banach Lie group $G$.

Proposition 9.1. There exist a compact polyhedron $Q$ of dimension $\leq 2$, a surjective continuous map $M(\mathcal{A}_Z) \to Q$ and a topological principal $G$-bundle $p : E \to Q$ such that pullback $f^*E$ is isomorphic to $P$.

The proof of this technical result would be much shorter if we knew that the classifying space of group $G$ is homotopy equivalent to an absolute neighbourhood retract. (In general, it is unknown.)

Proof. Let $\exp_G : \mathfrak{g} \to G_0$ be the exponential map of the Lie algebra $\mathfrak{g}$ of $G$ and $U' \subset \mathfrak{g}$ be an open ball centered at $0$ such that $\exp_G : U' \to V'$ is biholomorphic. We fix a smaller open ball $U \subset U'$ centered at $0$ such that $V \cdot V \subset V'$, where $V := \exp_G(U)$.

Passing to a refinement, if necessary, without loss of generality we may assume that $P$ is defined on a finite open cover $\mathcal{U} = (U_i)_{i \in I}$, $I := \{1, 2, \ldots, N\} \subset \mathbb{N}$, of $M(\mathcal{A}_Z)$ by a cocycle $c = \{c_{ij} \in \mathcal{C}(U_i \cap U_j, G)\}_{i,j \in I}$ such that each $c_{ij}$ with $i < j$ has a form $c_{ij}(x) = g_{ij}^{-1}c_{ji}(x)$, $x \in U_i \cap U_j$, $g_{ij} \in G$, and $\tilde{c}_{ij} \in \mathcal{C}(U_i \cap U_j, V)$. (Then for all $i > j$ we have $c_{ij} = c_{ji}^{-1} = \tilde{c}_{ji}^{-1}g_{ji}^{-1}$.)

Next, since $\dim M(\mathcal{A}_Z) = 2$, by the Mardešić theorem [Ma, Th.1] $M(\mathcal{A}_Z)$ can be presented as the inverse limit of the inverse system of metrizable compacta $\{Q_b, p_{bb'}\}$ with $\dim Q_b \leq 2$; here $b$ ranges over a directed set $B$ and $p^b : M(\mathcal{A}_Z) \to Q_b$ are the corresponding inverse limit projections. In turn, by the Freudenthal theorem, each $Q_b$ can be presented as the inverse limit of a sequence of compact polyhedra $\{Q_{bn}, p_{bmn}\}$ with $\dim Q_{bn} \leq 2$; here $p^b_{bn} : Q_b \to Q_{bn}$ are the corresponding inverse limit projections. Then by the Stone-Weierstrass theorem the union of pullback algebras $(p^b \circ p^b_{bn})^*\mathcal{C}(Q_{bn})$, $b \in B$, $n \in \mathbb{N}$, is dense in $C(M(\mathcal{A}_Z))$.

Let us consider $\mathfrak{g}$-valued functions $c_{ij}' := \exp_G^{-1}(\tilde{c}_{ij}) \in \mathcal{C}(U_i \cap U_j, \mathfrak{g})$, $i, j \in I$, $i < j$. Since the algebra of complex-valued continuous functions on a compact Hausdorff spaces has the Grothendieck approximation property [G], each $c_{ij}'$ can be uniformly approximated on $U_i \cap U_j$ by continuous functions from the algebraic tensor product $C(U_i \cap U_j) \otimes \mathfrak{g}$. In turn, extending functions in $C(U_i \cap U_j) \otimes \mathfrak{g}$ to continuous functions in $C(M(\mathcal{A}_Z)) \otimes \mathfrak{g}$ by the Tietze-Urysohn extension theorem, we obtain that each $c_{ij}'$ can be uniformly approximated on $U_i \cap U_j$ by functions in algebras $((p^b \circ p^b_{bn})^*\mathcal{C}(Q_{bn})) \otimes \mathfrak{g}$, $b \in B$, $n \in \mathbb{N}$.

From here, using that $\{c_{ij}\}_{i,j \in I}$ is a cocycle, we conclude that there exist some $b_0 \in B$, $n_0 \in \mathbb{N}$ and functions $d_{ij}' \in ((p^b_0 \circ p^b_{0n})^*\mathcal{C}(Q_{b_0n})) \otimes \mathfrak{g}$ sufficiently close to $c_{ij}'$ (with $i, j \in I$, $i < j$) such that for $x \in U_i \cap U_j$, $t \in [0, 1]$ and

$$d_{ij}'(x) := \begin{cases} g_{ij} \exp_G(t d_{ij}'(x) + (1 - t)c_{ij}'(x)) & \text{if } i < j \\ \exp_G(-t d_{ij}'(x) - (1 - t)c_{ij}'(x))g_{ji}^{-1} & \text{if } i > j \\ 1_G & \text{if } i = j, \end{cases}$$

where $G = \exp_G(U)$.
Further, by the definition of the inverse limit topology and since dim $M(\mathcal{A}Z) = 2$, there exist $b_l \leq b_0$ and $n_1 \in \mathbb{N}$, $n_1 \geq n_0$ and a finite open cover $\mathcal{U} = \{U_l\}_{l \in L}$ of $Q_{b_l}n_1$ of order at most 3 such that cover $((p^{h_0} \circ p^{n_0})^{-1}(V_l))_{l \in L}$ is a refinement of $\mathcal{U}$. Since by the definition of the inverse limit, $(p^{h_0} \circ p^{n_0})^*C(Q_{b_0n_0}) \subset (p^{h_1} \circ p^{n_1})^*C(Q_{b_1n_1})$, there exist functions $e'_{ij} \in C(Q_{b_1n_1}) \otimes g$ such that $(p^{h_1} \circ p^{n_1})^*(e'_{ij}) = d'_{ij}$ for all $i,j \in I$ with $i < j$.

We will assume without loss of generality that $Q_{b_1n_1}$ is a simplicial complex in some $\mathbb{R}^d$. By $\| \cdot \|_2$ we denote the Euclidean norm on $\mathbb{R}^d$ Then uniform continuity of $e'_{ij}$ implies that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in Q_{b_1n_1}$ with $\| x - y \|_2 < \delta$ and all $i, j \in I$ with $i < j$,

$$
\| e'_{ij}(x) - e'_{ij}(y) \|_g < \varepsilon.
$$

**Lemma 9.2.** There exist a compact polyhedron $Q$ of dimension $\leq 2$, a continuous surjective map $f : M(\mathcal{A}Z) \to Q$ and a continuous map $g : Q \to Q_{b_1n_1}$ such that the finite open cover $((g \circ f)^{-1}(V_l))_{l \in L}$ of $M(\mathcal{A}Z)$ is a refinement of $\mathcal{U}$ and the analog of (9.1) is valid with $d'_{ij}(x)$ replaced by

$$
e'_{ij}(x) := \begin{cases} 
g_{ij} \exp_G\left(t((g \circ f)^*e'_{ij})(x) + (1-t)c'_{ij}(x)\right) & \text{if } i < j \\
\exp_G\left(-t((g \circ f)^*e'_{ij})(x) - (1-t)c'_{ij}(x)\right)g_{ji}^{-1} & \text{if } i > j \\
1_G & \text{if } i = j,
\end{cases}
$$

$x \in \tilde{U}_i \cap \tilde{U}_j$, $t \in [0, 1]$, $i, j \in I$.

**Proof.** By [Ma, Lm. 1] for every $\delta > 0$ there exist a compact polyhedron $Q_\delta$ of dimension $\leq 2$, a continuous surjective map $f_\delta : M(\mathcal{A}Z) \to Q$ and a continuous map $g_\delta : Q \to Q_{b_1n_1}$ such that for all $x \in M(\mathcal{A}Z)$

$$
\| (g_\delta \circ f_\delta)(x) - (p^{b_0} \circ p^{n_0})(x) \|_2 < \delta.
$$

This and (9.2) imply that for all $x \in M(\mathcal{A}Z)$ and $i, j \in I$ with $i < j$

$$
\| d'_{ij}(x) - ((g_\delta \circ f_\delta)^*e'_{ij})(x) \|_g < \varepsilon.
$$

Choosing here sufficiently small $\varepsilon$ and $\delta$ we obtain by continuity of $\exp_G$ and operations on $G$ that the analog of (9.1) is valid with $d'_{ij}$ replaced by $e'_{ij}$ for all $i, j \in I$.

Next, let $\tau : L \to I$ be the refinement map for the refinement $((p^{h_0} \circ p^{n_0})^{-1}(V_l))_{l \in L}$ of $\mathcal{U}$. Since $(p^{h_0} \circ p^{n_0})^{-1}(V_l)$ is a compact subset of $U_{\tau(l)}$, inequality (9.3) implies easily that for a sufficiently small $\delta$ and all $l \in L$, $(g_\delta \circ f_\delta)^{-1}(V_l) \subset U_{\tau(l)}$ as well. For such $\delta$ we set $Q := Q_\delta$, $f := f_\delta$ and $g := g_\delta$. \hfill $\square$

We set $\mathcal{W} := (\tilde{V}_l)_{l \in L}$, $\tilde{V}_l := g^{-1}(V_l)$. Then $\mathcal{W}$ is a finite open cover of $Q$ of order $\leq 3$. Since dim $Q \leq 2$, according to the Ostrand theorem on colored dimension [O, Th. 3] there exists a finite open cover $\mathcal{W}$ of $Q$ which can be represented as the union of families $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$, where $\mathcal{W}_i = \{W_{i,l}\}_{l \in L}$ are such that closures of their subsets are pairwise disjoint and $\overline{W}_{i,l} \subset \tilde{V}_l$ for each $i \in \{1, 2, 3\}$ and $l \in L$. 


c_{ij}(x)(d_{ij}^{-1}(x))^{-1} \in V, \quad x \in U_i \cap U_j, \quad \text{for all } i, j \in I \quad \text{and}

d_{ij} \in V, \quad x \in U_i \cap U_j \cap U_k, \quad \text{for all } i, j, k \in I.

(9.1)
As before, $\tau : L \to I$ stands for the refinement map for the refinement $((p^{bo} \circ p_{bo})^{-1}(V_i))_{i \in L}$ of $\Omega$. Lemma 9.2 implies that for all $i, j \in I$ maps $\tilde{e}_{ij} \in C(Q, G)$,
\[
\tilde{e}_{ij} := \begin{cases} 
   g_{ij}(\exp_G \circ (g^*e_{ij})) & \text{if } i < j \\
   (\exp_G \circ (-g^*e_{ji}))g^{-1}_{ji} & \text{if } i > j \\
   1_G & \text{if } i = j,
\end{cases}
\]
satisfy
\[
\tilde{e}_{\tau((l)\tau(m))}(x)\tilde{e}_{\tau(n)\tau(l)}(x)\tilde{e}_{\tau(n)\tau(l)}(x) \in V, \quad x \in \hat{V}_i \cap \hat{V}_m \cap \hat{V}_n, \quad \text{ for all } l, m, n \in L.
\]
We set $W_i = \bigcup_{l \in L} W_i,l$, $i = 1, 2, 3$, and consider the closed cover $(W_i)_{i=1}^3$ of $Q$. By definition,
\[
W_i \cap W_j = \bigcup_{k \in L} \left( \bigcup_{l \in L} W_{i,k} \cap W_{j,l} \right);
\]
here each nonempty $W_{i,k} \cap W_{j,l}$ is a compact subset of $\hat{V}_k \cap \hat{V}_l$.

For $i, j \in \{1, 2, 3\}$ we define maps $\tilde{e}_{ij} \in C(W_i \cap W_j, G)$ by the formulas
\[
(9.4) \quad \tilde{e}_{ij}(x) := \tilde{e}_{\tau(k)\tau(l)}(x), \quad x \in W_{i,k} \cap W_{j,l}, \quad k, l \in L.
\]

Lemma 9.3. Suppose $\tilde{W}_1 \cap \tilde{W}_2 \cap \tilde{W}_3 \neq \emptyset$. Then there exists a map $\tilde{e} \in C(\tilde{W}_1 \cap \tilde{W}_3, V)$ such that
\[
\tilde{e}_{12}(x)\tilde{e}_{23}(x)(\tilde{e}_{31}(x)\tilde{e}(x)) = 1_G \quad \text{for all } x \in \tilde{W}_1 \cap \tilde{W}_2 \cap \tilde{W}_3.
\]

Proof. We set $\tilde{e}_{123}(x) := \tilde{e}_{12}(x)\tilde{e}_{23}(x)\tilde{e}_{31}(x)$, $x \in \tilde{W}_1 \cap \tilde{W}_2 \cap \tilde{W}_3$. By definition, $\tilde{e}_{123} \in C(W_1 \cap W_2 \cap W_3)$; hence map $\exp_G^{-1}\tilde{e}_{123} \in C(W_1 \cap W_2 \cap W_3; U)$ is well defined. Since $U \subset \mathfrak{g}$ is a ball, by the Dugundji extension theorem there exists some $e' \in C(W_1 \cap W_3, U)$ such that $e' = -\exp_G^{-1}\tilde{e}_{123}$ on $W_1 \cap W_2 \cap W_3$. We set $\tilde{e} := \exp_G \circ e'$. Then the required identity is fulfilled. \hfill $\Box$

For $\tilde{W}_1 \cap \tilde{W}_2 \cap \tilde{W}_3 = \emptyset$, we define $\tilde{e} \in C(\tilde{W}_1 \cap \tilde{W}_3, V)$ to be the constant map with constant value $1_G$.

Next, we define $h_{ij} := \tilde{e}_{ij}$ for $i, j \in \{1, 2, 3\}$ with $(i, j) \notin \{(1, 3), (3, 1)\}$ and $h_{31} := \tilde{e}_{31} \tilde{e}$, $h_{13} = \tilde{e}^{-1}\tilde{e}_{13}$.

Lemma 9.4. Family $\{h_{ij} \in C(\tilde{W}_i \cap \tilde{W}_j, G)\}_{1 \leq i, j \leq 3}$ is a $G$-valued 1-cocycle on the cover $(\tilde{W}_i)_{i=1}^3$ of $Q$.

Proof. By the definition of $\tilde{e}_{ij}$, cf (9.4), $h_{ij} = h_{ji}^{-1}$ for all $i, j \in \{1, 2, 3\}$. Clearly, it suffices to prove the result for $\tilde{W}_1 \cap \tilde{W}_2 \cap \tilde{W}_3 \neq \emptyset$. In this case it suffices to check only that
\[
h_{12} h_{23} = 1_G \quad \text{on } \tilde{W}_1 \cap \tilde{W}_2 \cap \tilde{W}_3.
\]
This follows straightforwardly from Lemma 9.3 and the definition of $h_{ij}$. \hfill $\Box$

Let us define the required principal $G$-bundle $p : E \to Q$ of the proposition by cocycle $h = \{h_{ij}\}$ on the cover $(\tilde{W}_i)_{i=1}^3$ of $Q$ (see (2.3) above). Then we must prove the following result.

Lemma 9.5. Topological principal $G$-bundles $f^*E$ and $P$ on $M(\mathfrak{gZ})$ are isomorphic.

Proof. We set $\mathfrak{M}^* = (W_{i,l}^*)_{l \in L, 1 \leq i \leq 3}$, where $W_{i,l}^* := f^{-1}(W_{i,l})$. The finite open cover $\mathfrak{M}^*$ of $M(\mathfrak{gZ})$ is a refinement of the cover $((g \circ f)^{-1}(V_i))_{l \in L}$ which, in turn, is a refinement of the cover $\Omega$ (see Lemma 9.2). Also, we consider the closed cover $(W_i^*)_{i=1}^3$ of $M(\mathfrak{gZ})$ where $W_i^* := f^{-1}(W_i)$, $i = 1, 2, 3$. \hfill $\Box$
For $i, j \in \{1, 2, 3\}$ we define maps $\ell_{ij} \in C(\hat{W}_1^* \cap \hat{W}_j^*, C([0, 1], G))$ by the formulas

$$(\ell_{ij}(x))(t) := e^{t(\varphi(k) \tau(t))}(x), \quad x \in \hat{W}_{i,k}^* \cap \hat{W}_{j,l}^*, \quad k, l \in L.$$ 

By definition, see Lemma 9.2 and the text before (9.4),

$$(9.5) \quad (\ell_{ij}(x))(0) = c_{\varphi(k) \tau(t)}(x), \quad (\ell_{ij}(x))(1) = (f^* \varphi(k) \tau(t))(x), \quad x \in \hat{W}_{i,k}^* \cap \hat{W}_{j,l}^*, \quad k, l \in L.$$ 

Note that $C([0, 1], G)$ is a complex Banach Lie group with respect to the pointwise multiplication of maps whose Lie algebra is $C([0, 1], \mathfrak{g})$.

For $e$ as in Lemma 9.3 we define a map $\ell \in C(\hat{W}_1^* \cap \hat{W}_3^*, C([0, 1], V))$ by the formula

$$(9.6) \quad (\ell(x))(t) := \exp_G(t \exp^{-1}(\ell(e(f(x))))) , \quad x \in \hat{W}_1^* \cap \hat{W}_3^*, \quad t \in [0, 1].$$

Next, we consider a continuous $C([0, 1], G)$-valued map

$$\ell_{123}(x) := \ell_{12}(x) \ell_{23}(x) (\ell_{31}(x) t(x)), \quad x \in \hat{W}_1^* \cap \hat{W}_2^* \cap \hat{W}_3^*.$$ 

due to Lemmas 9.2, 9.3 and since $\{c_{ij}\}_{i,j \in I}$ is a cocycle on $\mathfrak{U}$, the image of $\ell_{123}$ consists of continuous maps $F \in C([0, 1], V')$ (recall that $V \cdot V \subset V'$) such that $F(0) = F(1) = 1_G$. Therefore map $\ell_{123} \in C(\hat{W}_1^* \cap \hat{W}_2^* \cap \hat{W}_3^*, C([0, 1], U'), C([0, 1], U'), C([0, 1], U'))$, $(\ell_{123}'(x))(t) := \exp_{G}^{-1}(\ell_{123}(x))(t)) , \quad x \in \hat{W}_1^* \cap \hat{W}_2^* \cap \hat{W}_3^*, \quad t \in [0, 1]$, is well defined and its image consists of maps $F \in C([0, 1], U')$ such that $F(0) = F(1) = 0$.

The space $C_0([0, 1], \mathfrak{g})$ of continuous maps $F : [0, 1] \rightarrow \mathfrak{g}$ with $F(0) = F(1) = 0$ is a closed subspace of the Banach space $C([0, 1], \mathfrak{g})$ and for a closed ball $B \subset \mathfrak{g}$ centered at 0 set $C_0([0, 1], \mathfrak{g}) \cap C([0, 1], B)$ is a closed ball centered at zero map in $C_0([0, 1], \mathfrak{g})$. In particular, we can apply to $\ell_{123}'$ the Arens extension theorem, see, e.g., [AI] Th. 4.1, to find some $\ell' \in C(\hat{W}_1^* \cap \hat{W}_3^*, C([0, 1], \mathfrak{g}) \cap C([0, 1], U'))$ which extends $\ell_{123}'$. We define $\ell' \in C(\hat{W}_1^* \cap \hat{W}_3^*, C([0, 1], U'))$ by the formula

$$(9.7) \quad \ell_{12}(x) \ell_{23}(x) (\ell_{31}(x) \ell(x) \ell(x)) = 1_G \quad \text{for all} \quad x \in \hat{W}_1^* \cap \hat{W}_2^* \cap \hat{W}_3^*,$$

where $1_G(t) = 1_G$ for all $t \in [0, 1]$.

Next, let us define $\kappa_{ij} := \ell_{ij}$ for $i, j \in \{1, 2, 3\}$ with $(i, j) \notin \{(1, 3), (3, 1)\}$ and $\kappa_{31} := \ell_{31} \ell' \ell$. Then similar to Lemma 9.3 we obtain that family $\kappa := \{ \kappa_{ij} \in C(\hat{W}_1^* \cap \hat{W}_j^*, C([0, 1], G)) \}$ is a continuous $C([0, 1], G)$-valued 1-cocycle on the cover $(\hat{W}_i^*), i=1,2,3$ of $M(\mathcal{A}_Z)$. Since $2W^*$ is the natural refinement of the cover $(\hat{W}_i^*), i=1,2,3$, the pullback of $\kappa$ to $2W^*$ by the refinement map determines a family of continuous $G$-valued 1-cocycles $\kappa(t)$ on $2W^*$ depending continuously on $t \in [0, 1]$. By definitions of $\kappa_{ij}$ and $\ell'$, see (9.5), (9.6), $\kappa(0)$ is the pullback to $2W^*$ by means of the refinement map $\tau$ of the cocycle $c$ determining bundle $P$. On the other hand, $c(1)$ is the pullback to $2W^*$ of the cocycle $f^*h_{ij}$ of the cocycle $f^*E$. Thus, the general result of the fibre bundles theory [Hus] Ch. 4, Cor. 9.7 implies that $P$ and $f^*E$ are isomorphic. 

The proof of the proposition is complete.

\[ \square \]

9.2. Proof of Theorem 2.10. Let $A \in \mathcal{A}_Z$ and $D$ be the set of all finite subsets of $A$ directed by inclusion $\subseteq$. Then $M(\mathcal{A}_Z)$ is naturally identified with $M(A)$ and is presented as the inverse limit of the inverse system of compacta $M(A_\alpha), F_\beta, \alpha$, where $\alpha$ ranges over $D$ and $A_\alpha$ is the unital closed subalgebra of $A$ generated by $\alpha$, see Section 2.2. Recall that if $\alpha = \{f_1, \ldots, f_n\} \subset D$, then $F_\alpha = (f_1, \ldots, f_n) : M(\mathcal{A}_Z) \rightarrow \mathbb{C}^n$ is the inverse limit projection and $M(A_\alpha)$ is the polynomially convex hull of its image. Also, if $\alpha, \beta \in D$ with $\alpha \supseteq \beta$, then $F_\beta : \mathbb{C}^# \rightarrow \mathbb{C}^#$, $\mathbb{C}^# \supseteq (z_1, \ldots, z_#) \mapsto (z_1, \ldots, z_#) \in \mathbb{C}^#$. 


On the other hand, $M(\mathcal{A}_Z)$ is the inverse limit of inverse system $\{K_\alpha, F_\alpha^\alpha\}$, where $K_\alpha := F_\alpha(M(\mathcal{A}_Z)) \subset M(A_\alpha)$. Therefore by the definition of the inverse limit topology, for each $\beta \in D$ and an open neighbourhood $O \subset \mathbb{C}^#\beta$ of $K_\beta$ there exists $\alpha \supseteq \beta$ such that

\begin{equation}
M(A_\alpha) \subset (F_\beta^\alpha)^{-1}(O).
\end{equation}

Let $\pi : P \to M(\mathcal{A}_Z)$, $Q$ and $f \in C(M(\mathcal{A}_Z), Q)$ be as in Proposition 9.1.

**Proposition 9.6.** There exists $\beta \in D$ and a continuous map $g : K_\beta \to Q$ such that maps $f, g \circ F_\beta \in C(M(\mathcal{A}_Z), Q)$ are homotopic.

**Proof.** Without loss of generality we may assume that $Q$ is a (finite) simplicial complex in some $\mathbb{R}^d$ (recall that $Q$ is a compact polyhedron). Then there exists an open neighbourhood $U \subset \mathbb{R}^d$ and a continuous retraction $r : U \to Q$. Let $A_\beta^\alpha \subset C(M(\mathcal{A}_Z))$ be the (nonclosed) subalgebra generated by functions in $A_\beta$ and their complex conjugate. By the Stone-Weierstrass theorem algebra $\cup_{\beta \in D} A_\beta^\alpha$ is dense in $C(M(\mathcal{A}_Z))$. In particular, regarding $f$ as a map in $C(M(\mathcal{A}_Z), \mathbb{R}^d)$ we can find $\beta \in D$ and a map $f_\beta \in C(M(\mathcal{A}_Z), \mathbb{R}^d)$ with coordinates in $A_\beta^\alpha$ such that

\begin{equation}
tf_\beta(x) + (1 - t)f(x) \in U \quad \text{for all} \quad x \in M(\mathcal{A}_Z), \quad t \in [0, 1].
\end{equation}

Next, by definitions of $K_\beta$ and $F_\beta$, there exists a map $g_\beta \in C(K_\beta, \mathbb{R}^d)$ such that $f_\beta = g_\beta \circ F_\beta$. We set $g := r \circ g_\beta \in C(K_\beta, Q)$. Then by (9.9)

\[H(x, t) := r(tf_\beta(x) + (1 - t)f(x)), \quad x \in M(\mathcal{A}_Z), \quad t \in [0, 1],\]

is a homotopy between $f$ and $g \circ F_\beta$. \hfill $\square$

Using the Tietze-Urysohn extension theorem composed with retraction $r \in C(U, Q)$ we extend the map $g$ of the proposition to a map $\tilde{g} \in C(O, Q)$ for some open neighbourhood $O$ of $K_\beta$. Let $\alpha \supseteq \beta$ be such that $M(A_\alpha) \subset (F_\beta^\alpha)^{-1}(O)$ (cf. (9.8)). Since $M(A_\alpha)$ is polynomially convex, there is a Stein neighbourhood $N$ of $M(A_\alpha)$ contained in $(F_\beta^\alpha)^{-1}(O)$.

Next, let $p : E \to Q$ be the topological principal $G$-bundle of Proposition 9.1. Consider principal $G$-bundle $(\tilde{g} \circ F_\beta^\alpha)|_\alpha E$ on $N$. Since $N$ is Stein, by the Bungart theorem [Bu Th. 8.1] this bundle is isomorphic to a holomorphic principal $G$-bundle $\tilde{E}$ on $N$. Further, since by Proposition 9.6 maps $f$ and $g \circ F_\beta$ are homotopic, bundles $f^*E$ and $(g \circ F_\beta)^*E = F_\alpha^\beta((g \circ F_\beta^\alpha)|_\alpha E)$ on $M(\mathcal{A}_Z)$ are isomorphic (see, e.g., [Hus Ch. 4]). These imply that the topological principal $G$-bundle $P(= f^*E)$ is isomorphic to the holomorphic principal $G$-bundle $F_\alpha^\beta \tilde{E}$.

The proof of the theorem is complete.

9.3. **Proof of Theorem 2.11.** First, we show that for each holomorphic principal $G$-bundle $P$ on $M(\mathcal{A}_Z)$ there exist $\alpha \in D$ and a holomorphic principal $G$-bundle $\tilde{P}$ defined on a neighbourhood of $M(A_\alpha)$ such that $P$ and $F_\alpha^* \tilde{P}$ are holomorphically isomorphic.

Indeed, as in the proof of Theorem 2.10 we find such $\tilde{P}$ and $\alpha \in D$ that $F_\alpha^* \tilde{P}$ and $P$ are isomorphic as topological bundles. Then due to Theorem 2.9 they are holomorphically isomorphic as required.

Second, we show that if holomorphic principal $G$-bundles $P_1, P_2$ defined on a neighbourhood of $M(A_\beta)$ are such that $F_\beta^\alpha P_1$ and $F_\beta^\alpha P_2$ are holomorphically isomorphic, then there exists $\alpha \supseteq \beta$ such that the holomorphic principal $G$-bundles $(F_\beta^\alpha)^* P_1$ and $(F_\beta^\alpha)^* P_2$ defined on a neighbourhood of $M(A_\alpha)$ are isomorphic.

This will complete the proof of the theorem.

In the proof of the preceding statement with use the following general result.
Let a compact Hausdorff space $X$ be the inverse limit of the inverse limit system of compacta $\{X_\alpha, p^\alpha_\beta\}$ where $\alpha$ ranges over a directed set $\Lambda$ and $p_\alpha : X \to X_\alpha$ are the corresponding inverse limit projections. Let $E_1$ and $E_2$ be topological principal $G$-bundles on some $X_\beta$ with fibre $G$ a complex Banach Lie group.

**Proposition 9.7.** Suppose principal $G$-bundles $p^*E_1$ and $p^*E_2$ on $X$ are isomorphic. Then there exists some $\alpha \geq \beta$ such that principal $G$-bundles $(p^\alpha_\beta)^*E_1$ and $(p^\alpha_\beta)^*E_2$ on $X_\alpha$ are isomorphic.

As in the case of Proposition 9.1, the proof would be much shorter if we knew that the classifying space of group $G$ is homotopy equivalent to an absolute neighbourhood retract.

**Proof.** Let $U \subset \mathfrak{g}$ be an open ball centered at 0 of the Lie algebra $\mathfrak{g}$ of $G$ such that the exponential map $\exp_G : \mathfrak{g} \to G$ is biholomorphic on $U$. We set $V := \exp_G(U)$.

Without loss of generality we may assume that $E_1$ and $E_2$ are defined on a finite open cover $\mathcal{U} = (U_i)_{i \in I}$ of $X_\beta$ by cocycles $(c_{ij}^k \in C(U_i \cap U_j, G))_{i,j \in I}$, $k = 1, 2$. By the hypothesis, there exist some $c_i \in C(p^{-1}_\beta(U_i), G)$, $i \in I$, such that

$$
(9.10) \quad c_i^{-1} \cdot p^*c_{ij} \cdot c_j = p^*_\beta c_{ij} \quad \text{on} \quad p^{-1}_\beta(U_i) \cap p^{-1}_\beta(U_j), \quad i,j \in I.
$$

Then there exist a finite open refinement $\mathcal{V} = (V_i)_{i \in L}$ of the cover $(p^{-1}_\beta(U_i))_{i \in I}$ of $X$ with refinement map $\tau : L \to I$, maps $e_i \in C(V_i, V)$ and elements $d_i \in G$ such that

$$
(9.11) \quad c_{\tau(i)} = d_i \cdot e_i \quad \text{on} \quad V_i, \quad l \in L.
$$

By the definition of the inverse limit topology, there exist some $\tilde{\beta} \in \Lambda$ and a finite open cover $\mathcal{W} = (W_k)_{k \in K}$ of $X_{\tilde{\beta}}$ such that $(p^{-1}_{\tilde{\beta}}(W_k))_{k \in K}$ is a refinement of $\mathcal{V}$ with refinement map $\sigma : K \to L$ and $p^{-1}_{\tilde{\beta}}(\tilde{W}_k) \subset C_{\sigma(k)}(V_i)$ for all $k \in K$.

We set for all $k, m \in K$, $i = 1, 2$,

$$
(9.12) \quad f_{km}^i := c_{(\tau \circ \sigma)(k)\sigma(m)}, \quad f_k := c_{(\tau \circ \sigma)(k)}, \quad g_k := d_{\sigma(k)}, \quad h_k := e_{\sigma(k)}.
$$

Then (9.10), (9.11) imply

$$
(9.13) \quad f_k^{-1} \cdot p^*_\beta f_{km} \cdot f_m = p^*_\beta f_{km} \quad \text{on} \quad p^{-1}_\beta(\tilde{W}_k) \cap p^{-1}_\beta(\tilde{W}_m), \quad k,m \in K.
$$

$$
(9.14) \quad f_k = g_k \cdot h_k \quad \text{on} \quad p^{-1}_\beta(\tilde{W}_k), \quad k \in K.
$$

Here $h_k \in C(p^{-1}_\beta(\tilde{W}_k), V)$, $k \in K$.

Since by the Stone-Weierstrass theorem algebra $\bigcup_{\alpha \in \Lambda} p^*_\alpha C(X_\alpha)$ is dense in $C(X)$, by the Tietze-Urysohn extension theorem each $\exp^{-1}_G(h_k)$ can be uniformly approximated by maps in algebras $p^*_\alpha C(X_\alpha) \otimes \mathfrak{g}$.

Let us construct a sequence of open balls $U_s \subset U$, $1 \leq s \leq k := \# K$, centered at 0 such that for $V_s := \exp_G(U_s)$

$$
(9.15) \quad V_s \cdot f_{km}^2(x) \cdot V_s \cdot V_s \subset f_{km}^2(x) \cdot V_{s+1} \quad \text{for all} \quad x \in p^{-1}_\beta(\tilde{W}_k) \cap p^{-1}_\beta(\tilde{W}_m), \quad k,m \in K.
$$

We set $U_k := U$. Since each $f_{km}^2 \cdot V$ is an open neighbourhood of $f_{km}^2$ in complex Banach Lie group $C(p^{-1}_\beta(\tilde{W}_k) \cap p^{-1}_\beta(\tilde{W}_m), G)$, set $K$ is finite and each $p^{-1}_\beta(\tilde{W}_k) \cap p^{-1}_\beta(\tilde{W}_m)$ is compact, due to continuity of the product on $G$ there is a ball $U_{k-1} \subset U_k$ such that $V_{k-1} \cdot f_{km}^2 \cdot V_{k-1} \cdot V_{k-1} \subset f_{km}^2 \cdot V_{k-1}$ for all $k,m \in K$. Applying the same argument to $f_{km}^2 \cdot V_{k-1}$ we construct the required ball $U_{k-2}$, etc.
Next, let us choose \( \tilde{\alpha} \geq \tilde{\beta} \) and elements \( \ell'_k \in C(X_{\tilde{\alpha}}) \otimes g \) such that \( p_{\tilde{\alpha}}^* \ell'_k|_{p_{\tilde{\beta}}^{-1}(W_k)} \) with \( \ell_k := g_k \cdot \exp_G(\ell'_k) \) are so close to \( f_k \) that (cf. (9.13))

\[
(p_{\tilde{\alpha}}^* \ell_k)^{-1} \cdot p_{\tilde{\beta}}^* \ell_k \subset p_{\tilde{\beta}}^* \ell \quad \text{on} \quad p_{\tilde{\beta}}^{-1}(W_k) \cap p_{\tilde{\beta}}^{-1}(W_m), \quad k, m \in K.
\]

Lemma 9.8. There exist \( \alpha \geq \tilde{\alpha} \) and maps \( r_k \in C((p_{\tilde{\beta}}^* \ell^{-1}(W_k), V), k \in K, \) such that for all \( k, m \in K \)

\[
r_k^{-1} \cdot (p_{\tilde{\alpha}}^* \ell_k)^{-1} (p_{\tilde{\beta}}^* f_{km}^1 \cdot (p_{\tilde{\alpha}}^*)^* \ell_m \cdot r_m = (p_{\tilde{\beta}}^* f_{km}^2 \quad \text{on} \quad (p_{\tilde{\beta}}^* \ell^{-1}(W_k) \cap (p_{\tilde{\beta}}^* \ell^{-1}(W_m)).
\]

Proof. Due to (9.16) for all \( k, m \in K, \)

\[
\ell_k^{-1} \cdot (p_{\tilde{\alpha}}^* f_{km}^1 \cdot (p_{\tilde{\beta}}^*)^* \ell_k \subset (p_{\tilde{\beta}}^* f_{km}^2 \quad \text{on} \quad ((p_{\tilde{\beta}}^* \ell^{-1}(W_k) \cap (p_{\tilde{\beta}}^* \ell^{-1}(W_m)) \cap p_{\tilde{\beta}}(X).
\]

By continuity, there is an open neighbourhood \( N \subset X_{\tilde{\alpha}} \) of \( p_{\tilde{\beta}}(X) \) such that analogous implications are still valid on all \( (p_{\tilde{\beta}}^* \ell^{-1}(W_k) \cap (p_{\tilde{\beta}}^* \ell^{-1}(W_m)) \cap N \). By the definition of the inverse limit topology, there exists \( \alpha \geq \tilde{\alpha} \) such that \( X_{\alpha} = (p_{\tilde{\alpha}}^* \ell^{-1}(N) \). Hence, for all \( k, m \in K, \)

\[
(p_{\tilde{\alpha}}^* \ell_k)^{-1} \cdot (p_{\tilde{\beta}}^* f_{km}^1 \cdot (p_{\tilde{\alpha}}^*)^* \ell_m \subset (p_{\tilde{\beta}}^* f_{km}^2 \quad \text{on} \quad ((p_{\tilde{\beta}}^* \ell^{-1}(W_k) \cap (p_{\tilde{\beta}}^* \ell^{-1}(W_m)) \cap N.
\]

In particular, there exist some \( h_{km} \in C(((p_{\tilde{\beta}}^* \ell^{-1}(W_k) \cap (p_{\tilde{\beta}}^* \ell^{-1}(W_m)), V) \) such that

\[
(p_{\tilde{\alpha}}^* \ell_k)^{-1} \cdot (p_{\tilde{\beta}}^* f_{km}^1 \cdot (p_{\tilde{\beta}}^* f_{km}^2 \cdot h_{km}.
\]

Without loss of generality we may assume that \( K = \{1, \ldots, \tilde{k}\} \subset \mathbb{N} \). We set

\[
Z_l = \bigcup_{k=1}^{l} (p_{\tilde{\beta}}^* \ell^{-1}(W_k).
\]

Using induction on \( l \in \{1, \ldots, \tilde{k}\} \) we prove the following statement:

(9.17) There exist maps \( r_k \in C((p_{\tilde{\beta}}^* \ell^{-1}(W_k), V), 1 \leq l \leq \tilde{k}, \) such that for all \( 1 \leq k, m \leq l \)

\[
(r_k^l)^{-1} \cdot (p_{\tilde{\beta}}^* f_{km}^1 \cdot h_{l} \cdot r_m = (p_{\tilde{\beta}}^* f_{km}^2 \quad \text{on} \quad (p_{\tilde{\beta}}^* \ell^{-1}(W_k) \cap (p_{\tilde{\beta}}^* \ell^{-1}(W_m).
\]

If \( l = 1 \), the statement trivially holds with \( r_1^1 = 1_G \).

Assuming that it is valid for \( l \in \{1, \ldots, \tilde{k} - 1\} \) let us prove it for \( l + 1 \).

We set \( r_k^l := 1_G \) for \( l + 1 \leq k \leq \tilde{k} \) and define a new \( G \)-valued 1-cocycle \( c' = \{c_{km}^l\}_{1 \leq k, l \leq \tilde{k} - 1 + 1} \) on \( \mathcal{M} := ((p_{\tilde{\beta}}^* \ell^{-1}(W_k))_{k \in K} \) by the formulas

\[
c_{km}^l := (r_k^l)^{-1} \cdot (p_{\tilde{\beta}}^* f_{km}^1 \cdot h_{km} \cdot r_m^l \quad \text{on} \quad (p_{\tilde{\beta}}^* \ell^{-1}(W_k) \cap (p_{\tilde{\beta}}^* \ell^{-1}(W_m), 1 \leq k, m \leq \tilde{k}.
\]

Then for all \( 1 \leq k, m, l \leq \tilde{k} \) we have

\[
c_{km}^l = (p_{\tilde{\beta}}^* f_{km}^2 \quad \text{on} \quad (p_{\tilde{\beta}}^* \ell^{-1}(W_k) \cap (p_{\tilde{\beta}}^* \ell^{-1}(W_m).
\]

Due to (9.15), \( c_{km}^l (x) \in f_{km}^2 (x) \cdot V_{l+1} \) for all \( x \) and \( k, m \in K \). In particular, restrictions of \( c_{kl+1} \) to \( k \leq l \), to \( (p_{\tilde{\beta}}^* \ell^{-1}(W_k) \cap (p_{\tilde{\beta}}^* \ell^{-1}(W_{l+1}) \) have the form

\[
c_{kl+1} = (p_{\tilde{\beta}}^* f_{kl+1}^2 \cdot t_{kl+1} ^{-1} \text{ for some } t_{kl+1} \in C((p_{\tilde{\beta}}^* \ell^{-1}(W_k) \cap (p_{\tilde{\beta}}^* \ell^{-1}(W_{l+1}), V_{l+1}).
\]

Note that maps \( t_{kl+1} \) glue together over all nonempty intersections to give a map \( t_{l+1} \in C(Z_l \cap (p_{\tilde{\beta}}^* \ell^{-1}(W_{l+1}), V_{l+1}) \). Indeed, if \( x \in (p_{\tilde{\beta}}^* \ell^{-1}(W_k) \cap (p_{\tilde{\beta}}^* \ell^{-1}(W_{l+1})) \cap (p_{\tilde{\beta}}^* \ell^{-1}(W_{l+1}), 1 \leq k, m \leq l, \) then since \( c' \) and \( (p_{\tilde{\beta}}^* f_{km}^2 \}_{k,m \in K} \) are cocycles on \( \mathcal{M};
\]

\[
(p_{\tilde{\beta}}^* f_{kl+1}^2 (x) \cdot (p_{\tilde{\beta}}^* f_{kl+1}^2 (x))^{-1} = (p_{\tilde{\beta}}^* f_{km}^2 (x) = c_{km}^l (x) = c_{kl+1} (x) \cdot (c_{ml+1} (x))^{-1}
\]

\[
= (p_{\tilde{\beta}}^* f_{kl+1}^2 (x) \cdot t_{kl+1} (x) \cdot (t_{ml+1} (x))^{-1} \cdot ((p_{\tilde{\beta}}^* f_{kl+1}^2 (x))^{-1}.
\]
This implies that \( t_{kl+1}(x) = t_{ml+1}(x) \) as required.

Using the Arens extension theorem we extend \( t_{l+1} \) to a map \( \tilde{t}_{l+1} \in C((p_\beta^2)^{-1}(\tilde{W}_{l+1}), V_{l+1}) \) and define

\[
\tilde{t}_{l+1}^{l+1} := \tilde{t}_{l+1}^{-1} \quad \text{and} \quad r_{l+1}^{l+1} := r_{l+1}^{-1}, \quad 1 \leq l.
\]

Then for \( 1 \leq k \leq l \),

\[
(r_k^{l+1})^{-1} \cdot (p_\beta^2)^{-1} \cdot f_{kl+1} \cdot h_{kl+1} \cdot r_{l+1}^{l+1} = c_{kl+1} \cdot t_{kl+1}^{-1} = (p_\beta^2)^{-1} \cdot f_{kl+1}
\]
on \( (p_\beta^2)^{-1}(\tilde{W}_k) \cap (p_\beta^2)^{-1}(\tilde{W}_{l+1}) \), and for others \( k, m \in \{1, \ldots, l + 1\} \) analogous identities on \((p_\beta^2)^{-1}(\tilde{W}_k) \cap (p_\beta^2)^{-1}(\tilde{W}_m)\) are valid by the induction hypothesis and the definition of cocycle.

This completes the proof of the induction step.

Choosing in the statement \( l = \tilde{k} \) and defining \( r_k := r_{k}^{-1}, k \in K \), we obtain the assertion of the lemma. \( \square \)

To finish the proof of the proposition observe that bundles \((p_\beta^2)^*E_i\) can be defined by cocycles \(\{(p_\beta^2)^*f_{km}\}_{k,m \in K}, i = 1, 2\), on cover \(\widetilde{\Omega} \), see (9.12). Then Lemma 9.8 shows that these cocycles determine the same element of cohomology set \( H^1(X_\alpha, G) \). This implies that topological principal \( G \)-bundles \((p_\beta^2)^*E_1\) and \((p_\beta^2)^*E_2\) on \( X_\alpha \) are isomorphic.

The proof of the proposition is complete. \( \square \)

To complete the proof of Theorem 2.11 we apply Proposition 9.7 to \( X = M(A_\beta) \) and \( X_\alpha \) the closed \( \frac{1}{\#_\alpha} \)-neighbourhood of \( M(A_\alpha) \) in \( \mathbb{C}^{\#_\alpha} \), i.e., \( X_\alpha := \cup_{z \in M(A_\alpha)} \overline{B}_z(\frac{1}{\#_\alpha}) \subseteq \mathbb{C}^{\#_\alpha} \), where \( B_z(\frac{1}{\#_\alpha}) \subseteq \mathbb{C}^{\#_\alpha} \) is the closed Euclidean ball of radius \( \frac{1}{\#_\alpha} \) centered at \( z \). Then the inverse limit of the inverse limit system of compacta \( \{X_\alpha, F_\alpha^\beta\} \), where \( \alpha \) ranges over the directed set \( D \), coincides with \( M(A_\beta) \) as well. Suppose bundles \( P_1, P_2 \) of the second statement of Theorem 2.11 are defined on an open neighbourhood \( N \subseteq \mathbb{C}^{\#_\beta} \) of \( M(A_\beta) \). By the definition of the inverse limit topology there exists some \( \beta \supseteq \beta \) such that \( X_\beta \subseteq (F_\beta^\beta)^{-1}(N) \). In particular, bundles \((F_\beta^\beta)^*P_i, i = 1, 2\), are defined on an open neighbourhood of \( X_\beta \subseteq \mathbb{C}^{\#_\beta} \). Since pullbacks of these bundles to \( M(A_\beta) \) are isomorphic to topological principal \( G \)-bundles, Proposition 9.7 is applicable. Due to the proposition, there is \( \alpha \supseteq \beta \) such that bundles \((F_\alpha^\alpha)^*E_i, i = 1, 2\), on \( X_\alpha \) are isomorphic. As \( X_\alpha \) is an open neighbourhood of the polynomially convex set \( M(A_\alpha) \), there exists a Stein neighbourhood \( O \subseteq X_\alpha \) of \( M(A_\alpha) \). Then holomorphic bundles \((F_\beta^\beta)^*E_i\) restricted to \( O \) are topologically isomorphic and so by Bungart’s theorem [Bu, Th. 8.1] they are holomorphically isomorphic as well. This completes the proof of the second statement of Theorem 2.11 and therefore of the theorem.

10. Proofs of Theorems 3.1, 3.6 and Corollary 3.3

10.1. Proof of Theorem 3.1. Since \( O(K, X) \) is dense in the complex Banach Lie group \( \mathcal{A}(K, X) \), it suffices to show that if \( f \in O(K, X) \), then \( f|_K \) can be joined by a path in \( O(K, X) \) with the constant map of the value \( 1_G \) (the unit of group \( \mathcal{A}(K, X) \)).

To this end, let \( U \subset \mathfrak{g} \) be an open ball centered at \( 0 \) such that \( \exp_G |_U : U \to G \) is biholomorphic. Suppose \( f \) is holomorphic on an open neighbourhood \( W \) of \( K \). We choose a finite cover \( \mathcal{V} = (V_l)_{l \in L} \) of \( K \) by relatively compact open subsets of \( W \) so that

\[
f|_{V_l} = g_l f_l \quad \text{for some} \quad g_l \in G, \quad f_l \in O(V_l, \exp_G(U)), \quad l \in L.
\]
By definitions of $g_l$ and $f_l$ there are paths $u_l : [0, 1] \to G$ and $v_l : [0, 1] \to \mathcal{O}(V_l, \exp_G(U))$ such that $u_l(0) = v_l(0)(x) = 1_G$ for all $x \in V_l$, and $u_l(1) = g_l, v_l(1) = f_l$. We set

$$z_l(t) = u_l(t) v_l(t), \quad t \in [0, 1].$$

Then $z_l$ can be regarded as a map in $\mathcal{O}(V_l, C([0, 1], G)), l \in L$. Let us consider cocycle $\{c_{lm}\}_{l,m \in L}$ on $\mathfrak{U}$ defined by the formulas

$$c_{lm} := z_l^{-1} z_m \quad \text{on } V_l \cap V_m.$$

Each $c_{lm}$ has range in the complex Lie group $C_0([0, 1], G)$ of maps in $C([0, 1], G)$ equal to $1_G$ at the endpoints. By definition, each element of $C_0([0, 1], G)$ is a loop in $G$ with the basepoint $1_G$. Since $G$ is simply connected, such a loop can be joined in $C_0([0, 1], G)$ with the constant loop of the value $1_G$. This shows that group $C_0([0, 1], G)$ is connected.

Next, cocycle $\{c_{lm}\}_{l,m \in L}$ determines a holomorphic principal $C_0([0, 1], G)$-bundle $P$ on $\bigcup_{l \in L} V_l$. Since the fibre of $P$ is connected, Corollary 2.7 implies that $P|_{W'}$ is trivial on an open neighbourhood $W' \subseteq W$ of $K$. Hence, there exist $c_l \in \mathcal{O}(W' \cap V_l, C_0([0, 1], G))$ (we assume that all $K \cap V_l \neq \emptyset$ so that all $W' \cap V_l \neq \emptyset$ as well) such that

$$c_l^{-1} c_m = c_{lm} \quad \text{on } (W' \cap V_l) \cap (W' \cap V_m), \quad l, m \in L.$$

Comparing (10.1) and (10.2) we get a map $H \in \mathcal{O}(W', C([0, 1], G))$ such that

$$H|_{W' \cap V_l} := z_l c_l^{-1}, \quad l \in L.$$

Clearly, $H(x, 0) = 1_G$ and $H(x, 1) = f(x)$ for all $x \in W'$. Thus $H|_K$ determines a path in $\mathcal{O}(K, G)$ joining $f$ with the constant map of the value $1_G$. This shows that $A(K, G)$ is connected.

The proof of the theorem is complete.

10.2. **Proof of Theorem 3.2.** (1) Let $X$ be a complex Banach homogeneous space under the action of a complex Banach Lie group $G$. We use the following result see, e.g., [R, Prop. 1.4]:

1. For each $p \in X$, $g \in G$ there exist neighbourhoods $U$ of $g$ in $G$ and $V$ of $g \cdot p$ in $X$ and a holomorphic map $f_{g,p} : V \to U$ such that $\pi_U \circ f_{g,p}$ is the identity on $V$.

Let $G_0$ be the connected component of the unit of $G$ and $p$ be a point in a connected component $X'$ of $X$. Then (1) implies that $O_p := \{g \cdot p\}_{g \in G_0}$ is an open connected subset of $X'$. Clearly, for distinct $p_1, p_2 \in X'$, sets $O_{p_k}, k = 1, 2$, either coincide or disjoint. Therefore if $X' \neq O_p$, it can be presented as disjoint union of open connected subsets which contradicts connectedness of $X'$. Hence, each connected component $X'$ of $X$ is a complex Banach homogeneous space under the action of $G_0$.

Further, let $\psi : G_u \to G_0$ be the universal covering of $G_0$. Since $\psi$ is a locally biholomorphic epimorphism of groups, each connected component $X'$ of $X$ can be considered as a complex Banach homogeneous space under the action $G_u \times X' \to X'$, $(g, p) \mapsto \psi(g) \cdot p$.

Finally, since space $M(\mathfrak{a}_Z)$ is connected, images of homotopic maps $M(\mathfrak{a}_Z) \to X$ belong to the same connected component. These arguments show that it suffices to prove the following statement:

2. **Suppose** $X$ is a connected complex Banach homogeneous space under the action of a simply connected complex Banach Lie group $G$. If $f_1, f_2 \in \mathcal{O}(M(\mathfrak{a}_Z), X)$ are homotopic, then they are homotopic in $\mathcal{O}(M(\mathfrak{a}_Z), X)$.

To this end, we fix $p \in X$ and consider an open cover $\mathcal{U} = (U_i)_{i \in I}$ such that there exist maps $s_i \in \mathcal{O}(U_i, G)$ with $s_i \cdot p = \text{id}_{U_i}$, $i \in I$, cf. (1). Then $s_i(x) \cdot p = s_j(x) \cdot p = x$ for all $x \in U_i \cap U_j$. This implies that $c_{ij}(x) := s_i^{-1}(x)s_j(x)$ satisfy $c_{ij}(x) \cdot p = p$ for all $x \in U_i \cap U_j$, i.e., $c_{ij} \in \mathcal{O}(U_i \cap U_j, G(p))$. Thus, cocycle $c = \{c_{ij}\}_{i,j \in I}$ on $\mathcal{U}$ determines a holomorphic principal $G(p)$-bundle $P$ on $X$. Since $f_1$ and $f_2$ are homotopic, pullbacks $f_1^* P$ and $f_2^* P$ are
topologically isomorphic holomorphic principal \( G(p) \)-bundles on \( M(\mathcal{A}_Z) \), see, e.g., [Hus Ch. 4]. Thus, by Theorem 2.10 they are holomorphically isomorphic. Hence, there exists a common finite open refinement \( \mathcal{W} = (V_l)_{l \in I} \) of covers \( (f^{-1}_k(U_l))_{l \in I} \) with refinements maps \( \tau_k : L \to I \), \( k = 1, 2 \), and maps \( c_l \in \mathcal{O}(V_l \cap V_m, G(p)) \), \( l \in L \), such that
\[
c_l^{-1}(f_1^*c_{r_1(l)}r_1(m))c_m = f_2^*c_{r_2(l)}r_2(m) \quad \text{on} \quad V_l \cap V_m, \quad l, m \in L.
\]
From here, using the definition of cocycle \( c \), we get
\[
(10.3) \quad c_l^{-1}(f_1^*s_{r_1(l)})^{-1}(f_1^*s_{r_1(m)})c_m = (f_2^*s_{r_2(l)})^{-1}(f_2^*s_{r_2(m)}) \quad \text{on} \quad V_l \cap V_m, \quad l, m \in L.
\]
This determines a map \( g \in \mathcal{O}(M(\mathcal{A}_Z), G) \) such that
\[
g|_{V_l} = (f_2^*s_{r_2(l)})^{-1}(f_1^*s_{r_1(l)})^{-1}, \quad l \in L.
\]
Then from (10.3) we obtain
\[
f_2 = f_2^*s_{r_2(l)} \cdot p = (g f_1^*s_{r_1(l)} c_l) \cdot p = (g f_1^*s_{r_1(l)}) \cdot p = g \cdot f_1 \quad \text{on} \quad V_l, \quad l \in L.
\]
Since \( G \) is simply connected \( g \) can be joined by a path \( g_t \), \( t \in [0, 1] \), with the constant map \( g_0 \) of value \( 1_G \), see Theorem 3.1. Then \( g_t \cdot f_1 \) determines homotopy in \( \mathcal{O}(M(\mathcal{A}_Z), X) \) joining \( f_1 \) and \( f_2 \).

This proves injectivity of map \( \Theta : [M(\mathcal{A}_Z), X]_\mathcal{O} \to [M(\mathcal{A}_Z), X] \) of the theorem.

(2) Suppose that \( f \in C(M(\mathcal{A}_Z)) \). For \( A \in \mathcal{A}_Z \), consider \( M(\mathcal{A}_Z) \) as the inverse limit of the inverse limiting system \( \{ M(A_\alpha), F_\alpha \} \), where a runs over directed set \( D \), see Section 2.2. Then by the Stone-Weierstrass theorem and since \( X \) is an absolute neighbourhood retract (see [P]) there are \( \alpha \in D \) and a map \( g \in C(U, X) \) defined on a neighbourhood of \( M(A_\alpha) \) such that \( f \) and \( F_\alpha g \) are homotopic. Further, as \( M(A_\alpha) \subset \mathbb{C}^\alpha \) is polynomially convex, there exists an open Stein neighbourhood \( N \) of \( M(A_\alpha) \) containing in \( U \). Then due to Ranssott’s “Oka principle” [Râ, R Th. 2.1] there is a map \( \bar{g} \in \mathcal{O}(N, X) \) homotopic to \( g|_N \). Hence, map \( f \) is homotopic to map \( F_\alpha \bar{g} \in \mathcal{O}(M(\mathcal{A}_Z), X) \).

This proves surjectivity of map \( \Theta : [M(\mathcal{A}_Z), X]_\mathcal{O} \to [M(\mathcal{A}_Z), X] \) and completes the proof of the theorem.

10.3. Proof of Corollary 3.3. Since \( \dim(M(\mathcal{A}_Z)) = 2 \) and \( H^2(M(\mathcal{A}_Z), \pi_2(X)) = 0 \), see Lemma 7.3, under the hypotheses of the corollary there is a one-to-one correspondence between elements of \( [M(\mathcal{A}_Z), X] \) and the Čech cohomology group \( H^1(M(\mathcal{A}_Z), \pi_1(X)) \), see [Hn2] Th. (11.4]). Since by Theorem 3.2 \( \Theta : [M(\mathcal{A}_Z), X]_\mathcal{O} \to [M(\mathcal{A}_Z), X] \) is a bijection, this gives the required statement.

Remark 10.1. The correspondence of the corollary is defined as follows.

Let \( r : X_u \to X \) be the universal covering of \( X \). It can be considered as a principal bundle \( P \) on \( X \) with fibre \( \pi_1(X) \) defined by a cocycle \( c \) with values in \( \pi_1(X) \) on an open cover of \( X \). By definition, \( c \) determines an element \( \{ c \} \in H^1(X, \pi_1(X)) \). Now, if \( f \in C(M(\mathcal{A}_Z), X) \) then pullback \( f^*P \) is a principal \( \pi_1(X) \)-bundle on \( M(\mathcal{A}_Z) \) defined by cocycle \( f^*c \) representing class \( \{ f^*c \} \in H^1(M(\mathcal{A}_Z), \pi_1(X)) \). If \( f, g \in C(M(\mathcal{A}_Z), X) \) are homotopic, then \( \{ f^*c \} = \{ g^*c \} \). Map \( \mathcal{O}(M(\mathcal{A}_Z), X) \ni f \mapsto \{ f^*c \} \in H^1(M(\mathcal{A}_Z), \pi_1(X)) \) determines the required correspondence.

10.4. Proof of Theorem 3.5. (1) Let \( f \in \mathcal{O}(M(\mathcal{A}_Z), X) \). Due to the arguments of the proof of Theorem 3.2(2), there exist \( \alpha \in D \) and a holomorphic map \( h \in \mathcal{O}(N, X) \) defined in a neighbourhood \( N \) of \( M(A_\beta) \) such that holomorphic maps \( f \) and \( F_\alpha h \) are homotopic. Then by Theorem 3.2 these maps are homotopic in \( \mathcal{O}(M(\mathcal{A}_Z), X) \). This shows that map \( \mathcal{F}_A : \lim [M(A_\alpha), X]_\mathcal{O} \to [M(\mathcal{A}_Z), X]_\mathcal{O} \) is surjective.

(2) Suppose holomorphic maps \( f_1, f_2 \) into \( X \) defined on a neighbourhood of \( M(A_\beta) \) are such that \( F_\alpha f_1 \) and \( F_\beta f_2 \) are homotopic in \( \mathcal{O}(M(\mathcal{A}_Z), X) \). Then by the definition of the
inverse limit topology and since $X$ is an absolute neighbourhood retract, there exists $\alpha \supseteq \beta$ such that maps $(F_\beta^g)^*f_1$ and $(F_\alpha^g)^*f_2$ are defined and homotopic on a neighbourhood $U$ of $M(A_\alpha)$, see, e.g., \[\text{[Li, Lm. 1]}\] and the reference therein. As $M(A_\alpha)$ is polynomially convex, there is a Stein neighbourhood $N$ of $M(A_\alpha)$ containing in $U$. Then maps $(F_\beta^g)^*f_1|_N$ and $(F_\alpha^g)^*f_2|_N$ are homotopic and so the Ramsport "Oka principle" \[\text{[Ra, R Th. 2.1]}\] implies that they are homotopic in $O(N,G)$.

This shows that map $\mathcal{H}_A : \lim M(A_\alpha),X|_O \rightarrow [M(\mathcal{A}_Z),X]|_O$ is injective and completes the proof of the theorem.

10.5. **Proof of Theorem 3.6** Let $f \in O(M(\mathcal{A}_Z),X)$. Using $(\ast)$ as in the proof of Theorem 3.2 we construct a holomorphic principal $G(p)$-bundle $P$ on the connected component of $X$ containing the image of $f$. Since the fibre $G(p)$ of holomorphic principal bundle $f^*P$ on $M(\mathcal{A}_Z)$ is connected, $f^*P$ is trivial by Corollary 2.7(1). Thus it has a holomorphic section. Repeating literally the arguments of the proof of Theorem 3.2 we construct by means of this section a map $\tilde{f} \in O(M(\mathcal{A}_Z),X)$ such that $f(x) = \tilde{f}(x) \cdot p$ for all $x \in M(\mathcal{A}_Z)$. We leave the details to the reader.

The proof of the theorem is complete.

11. **Proofs of Theorems 3.7 and 3.8**

**Proof of Theorem 3.7** (1) Suppose $f \in O(U,X)$ can be uniformly approximated on the compact subset $K \subset U$ by maps in $C(M(\mathcal{A}_Z),X)$. Since $X$ is an absolute neighbourhood retract, see \[\text{[P]},\] the latter implies that $f$ is homotopic to the restriction to $U$ of a map in $C(M(\mathcal{A}_Z),X)$. Then according to Theorem 3.2 it is homotopic to the restriction to $U$ of a map $g \in O(M(\mathcal{A}_Z),X)$. In particular, since $M(\mathcal{A}_Z)$ is connected, images of $g$ and $f$ belong to a connected component $X_0$ of $X$. As in the proof of Theorem 3.2 $X_0$ is a complex Banach homogeneous space under the action of a simply connected complex Banach Lie group $G$. Let $H \subset G$ be the stationary subgroup of a point in $X_0$ under this action. Then $G$ is biholomorphic to the holomorphic principal bundle $P$ on $X_0$ with fibre $H$ (cf. $(\ast)$ in the proof of Theorem 3.2). Since $f$ and $g|_U$ are homotopic, the holomorphic principal bundles $f^*P$ and $g^*P|_U$ on $U$ are topologically isomorphic. Then by Theorem 3.1 they are holomorphically isomorphic on an open neighbourhood $V \subset U$ of $K$. From here as in the proof of Theorem 3.2(1) we obtain that there is a map $h \in O(V,G)$ such that $f = h \cdot g|_V$. Since $G$ is simply connected, for an open neighbourhood $W \subset V$ of $K$ map $h|_W$ can be joined by a path in $O(W,G)$ with the constant map of the value $1_G$; see the proof of Theorem 3.1. Since $K$ is holomorphically convex, the latter and the Runge approximation theorem (see Theorem 6.8) imply, by a standard argument, that $h|_K$ can be uniformly approximated on $K$ by maps $h_n \in O(M(\mathcal{A}_Z),G)$, $n \in \mathbb{N}$. In particular, maps $h_n \cdot g$ converges to $f$ uniformly on $K$.

This completes the proof of part (1) of the theorem.

(2) Suppose $X$ is simply connected. Then it is the complex Banach homogeneous space under the action of a simply connected complex Banach Lie group $G$ and the stationary subgroup $H \subset G$ of a point $p \in X$ under this action is connected. (This follows, e.g., from the long exact sequence of homotopy groups of fibration $G \rightarrow X$ with fibre $H$.) In particular, in the notation of the first part of the proof, holomorphic bundle $f^*P$ on $U$, $f \in O(U,X)$, has connected fibre $H$. Thus, by Corollary 2.7(1), $f^*P|_V$ is trivial on an open neighbourhood $V \subset U$ of $K$. From here as in the proof of Theorem 3.2 (cf. also Theorem 3.6) we obtain that $f|_V = \tilde{f} \cdot p$ for some $\tilde{f} \in O(V,G)$. Since $G$ is simply connected, as in the proof of the first part of the theorem we get that $\tilde{f}|_K$ can be uniformly approximated
on $K$ by maps $\tilde{f}_n \in \mathcal{O}(M(\mathcal{A}_Z), G)$, $n \in \mathbb{N}$. In particular, maps $\tilde{f}_n \cdot p$ converges to $f$ uniformly on $K$.

The proof of the theorem is complete. \hfill $\Box$

Proof of Theorem 11.8. First, we prove the particular case of part (2) of the theorem for $X$ being a connected complex Banach Lie group $G$.

Suppose that $f \in \mathcal{O}(U, G)$ for an open neighbourhood $U$ of a hull $Z$. Let $Q_Z : M(H^\infty) \to M(\mathcal{A}_Z)$ be the quotient map of Proposition 11.1 sending $Z$ to a point $z$. Then $Q_Z(U) \subset M(\mathcal{A}_Z)$ is an open neighbourhood of $z$. Since $Q_Z$ is one-to-one outside $Z$, there is a map $h \in \mathcal{O}(M(\mathcal{A}_Z) \setminus \{z\}, G)$ such that $Q_Z^* h = f$ on $M(H^\infty) \setminus Z$. Consider the open cover $\mathcal{U} := \{M(\mathcal{A}_Z) \setminus \{z\}, Q_Z(U)\}$ of $M(\mathcal{A}_Z)$. Let $P$ be the holomorphic principal $G$-bundle on $M(\mathcal{A}_Z)$ defined on $\mathcal{U}$ by cocycle $c := h|_{Q_Z(U) \setminus \{z\}} \in \mathcal{O}(Q_Z(U) \setminus \{z\}, G)$. Since group $G$ is connected, Corollary 2.7(1) implies that bundle $P$ is trivial. Equivalently, there exists a global holomorphic section of $P$ which in local coordinates on $\mathcal{U}$ is given by maps $c_1 \in \mathcal{O}(M(\mathcal{A}_Z) \setminus \{z\}, G)$ and $c_2 \in \mathcal{O}(Q_Z(U), G)$ such that

$$c_1 c_2^{-1} = c \quad \text{on} \quad Q_Z(U) \setminus \{z\}.$$  

Replacing $c_i$ by $c_i c_2(z)^{-1}$, $i = 1, 2$, if necessary, without loss of generality we may assume that $c_2(z) = 1_G$. Then (11.1) implies that maps $Q_Z^* c_1 \in \mathcal{O}(M(H^\infty) \setminus Z, G)$ and $g Q_Z^* c_2 \in \mathcal{O}(U, G)$ are equal on $U \setminus Z$ and therefore they glue together to determine a map $\tilde{f} \in \mathcal{O}(M(H^\infty), G)$. By definition, $\tilde{f}|_Z = f$, as required.

Next, we establish the following result.

**Lemma 11.1.** Under hypotheses (1), (2) of the theorem there is an open neighbourhood $V \subseteq U$ of $Z$ and a map $g \in \mathcal{O}(M(H^\infty), X)$ such that maps $f|_V$ and $g|_V$ are homotopic.

**Proof.** For part (1), there is a map in $C(M(H^\infty), X)$ extending $f|_Z$. Such a map is homotopic to a map $g \in \mathcal{O}(M(H^\infty), X)$ by Theorem 3.2. This gives the required result with $V := U$.

Now suppose that $f$ satisfies conditions of part (2) of the theorem. Since due to Proposition 2.1 $\dim Z \leq 2$ and $H^2(Z, \mathbb{Z}) = 0$, there is a one-to-one correspondence between homotopy classes of continuous maps $Z \to X$ and elements of $H^1(Z, \pi_1(X))$, see [Hu2, Th. (11.4)] , described similarly to that of Remark 10.1. On the other hand, by Treil’s theorem 16 and its corollary 31 , Th 1.3 inclusion $Z \hookrightarrow M(H^\infty)$ induces an epimorphism of groups $H^1(M(H^\infty), \mathbb{Z}) \to H^1(Z, \mathbb{Z})$. Thus Corollary 3.3 and Remark 10.1 imply that there is a map $g \in \mathcal{O}(M(H^\infty), X)$ such that maps $f|_Z$ and $g|_Z$ are homotopic. Further, since $Z$ is the inverse limit of the inverse system of compacta $\{V, \subseteq \}$ , where $V$ runs over the directed set (with the order given by inclusions of sets) of open neighbourhoods of $Z$, the previous statement implies that $f|_V$ and $g|_V$ are homotopic on an open neighbourhood $V \subseteq U$ of $Z$, see, e.g., [Li, Lm. 1] and the reference there. \hfill $\Box$

Using this lemma, as in the proof of Theorem 3.7 we obtain that for some open neighbourhood $W \subseteq V$ of $Z$ there exists $h \in \mathcal{O}(W, G)$ such that $f|_W = h \cdot g|_W$. Here $G$ is a simply connected complex Banach Lie group such that $X$ is the complex Banach homogeneous space with respect to the action of $G$. Further, according to the particular case of part (2) of the theorem established above, there is a map $\tilde{h} \in \mathcal{O}(M(H^\infty), G)$ such that $\tilde{h}|_Z = h|_Z$. We set $f := \tilde{h} \cdot g$. Then $\tilde{f} \in \mathcal{O}(M(H^\infty), X)$ and $\tilde{f}|_Z = h|_Z \cdot g|_Z = f|_Z$.

The proof of the theorem is complete. \hfill $\Box$

12. Proofs of Theorems 4.2, 4.4, 4.8, 4.9 and Proposition 4.7

**Proof of Theorem 4.2.** Let $\mathcal{U} = (U_i)_{i \in I}$ be a finite open cover of $\mathbb{C}^n$ by simply connected sets. Then there are maps $\varphi_i \in \mathcal{O}(U_i, \mathbb{C}^n)$ such that $\pi \circ \varphi_i = \text{id}_{U_i}$, $i \in I$. In particular,
Suppose \( f \in \mathcal{O}(\mathbb{D}, \mathbb{C}^n) \) is the restriction of a continuous map \( \tilde{f} \in C(M(H^\infty), \mathbb{C}^n) \). We set \( \mathfrak{U} = (V_i)_{i \in I}, \) \( V_i := f^{-1}(U_i), \) \( i \in I, \) and consider additive \( \Gamma \)-valued 1-cocycle \( \tilde{f}^* = \{ \tilde{f}^* c_{ij} \}_{i,j \in I} \) on cover \( \mathfrak{U} \) of \( M(H^\infty). \) Since \( \Gamma \) is isomorphic to \( \mathbb{Z}^{2n} \), there are linearly independent over \( \mathbb{R} \) vectors \( v_1, \ldots, v_{2n} \in \mathbb{C}^n \) such that \( \Gamma = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \oplus \cdots \oplus \mathbb{Z}v_{2n} \). Hence, each \( \tilde{f}^* c_{ij} = \sum_{k=1}^{2n} c^k_{ij} v_k \) for some \( c^k_{ij} \in C(V_i \cap V_j, \mathbb{Z}) \) and \( c^k := \{ c^k_{ij} \}_{i,j \in I} \) are integer-valued 1-cocycles on \( \mathfrak{U}, 1 \leq k \leq 2n. \) Cocycles \( c^k \) determine holomorphic principal \( \mathbb{C} \)-bundles on \( M(H^\infty) \) which are trivial by Corollary 2.7(1). Thus there exist functions \( g^k \in \mathcal{O}(V_i) \) such that

\[
(12.1) \quad g^k_i - g^k_j = c^k_{ij} \quad \text{on} \quad V_i \cap V_j, \quad i, j \in I, \quad 1 \leq k \leq 2n.
\]

On the other hand, since \( \mathbb{D} \) is contractible, restrictions \( c^k|_{\mathbb{D}} \) are trivial cocycles, i.e., there exist functions \( n^k_i \in C(V_i \cap \mathbb{D}, \mathbb{Z}) \) such that

\[
(12.2) \quad n^k_i - n^k_j = c^k_{ij} \quad \text{on} \quad (V_i \cap \mathbb{D}) \cap (V_j \cap \mathbb{D}), \quad i, j \in I, \quad 1 \leq k \leq 2n.
\]

Equations (12.1) and (12.2) show that functions \( g^k_i - n^k_i \in \mathcal{O}(V_i \cap \mathbb{D}), \) \( i \in I, \) glue together over all nonempty intersections \( (V_i \cap \mathbb{D}) \cap (V_j \cap \mathbb{D}) \) to determine functions \( h^k \in \mathcal{O}(\mathbb{D}), \) \( k \in \{1, \ldots, 2n\}. \) By definition, each \( e^{2\pi \sqrt{-1} h^k} \in \mathcal{O}(\mathbb{D}, \mathbb{C}^*) \) extends to a function in \( \mathcal{O}(M(H^\infty), \mathbb{C}^*) \). In particular, the imaginary part of each \( h^k \) is bounded and so each \( h^k \in \text{BMOA}, \) see, e.g., [Ga, Ch. VI, Th. 1.5].

Next, let us consider maps \( \tilde{f}^* \psi_i = \sum_{k=1}^{2n} g^k_i v_k \in \mathcal{O}(V_i, \mathbb{C}^n), \) \( i \in I. \) By the definition of cocycle \( c \) and due to equation (12.1), these maps glue together over all nonempty intersections \( V_i \cap V_j \) to determine a map \( g \in \mathcal{O}(M(H^\infty), \mathbb{C}^n). \) These imply that map \( \tilde{f} \in \mathcal{O}(\mathbb{D}, \mathbb{C}^n), \)

\[
(12.3) \quad \tilde{f}(x) = (\tilde{f}^* \psi_i)(x) - \sum_{k=1}^{2n} n^k_i v_k, \quad x \in V_i \cap \mathbb{D}, \quad i \in I,
\]

is well-defined and \( \tilde{f} = g + \sum_{k=1}^{2n} h^k v_k \) on \( \mathbb{D}. \) Therefore, since all \( h^k \in \text{BMOA}, \) all coordinates \( \tilde{f}_k \) of map \( \tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_n) \) belong to \( \text{BMOA}. \) Also, according to (12.3),

\[
\tilde{f} = \pi \circ \tilde{f}.
\]

This completes the proof of the first part of the theorem.

Conversely, suppose that \( f = \pi \circ \tilde{f} \) for some \( \tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_n) \in \mathcal{O}(\mathbb{D}, \mathbb{C}^n) \) with all \( \tilde{f}_k \in \text{BMOA}. \) We must show that \( f \) extends to a map in \( \mathcal{O}(M(H^\infty), \mathbb{C}^n). \)

In fact, as follows from [Br3, Th.1.11] there exist an open finite cover \( \mathcal{U} = (U_i)_{i \in I} \) of \( M(H^\infty) \) and locally constant continuous maps \( c_i \in C(U_i \cap \mathbb{D}, \mathbb{C}^n) \) such that

\[
(12.4) \quad \sup_{x \in U_i \cap \mathbb{D}, i \in I} \| \tilde{f}(x) - c_i(x) \|_{\mathbb{C}^n} < \infty.
\]

Let \( \Pi \subseteq \mathbb{C}^n \) be the symmetric convex hull of vectors \( v_1, \ldots, v_{2n} \) generating \( \Gamma. \) Then \( \Pi \) is the fundamental compact under the action of \( \Gamma \) on \( \mathbb{C}^n, \) i.e.,

\[
\mathbb{C}^n = \bigcup_{z \in \Gamma} (z + \Pi).
\]

Let \( V \) be a connected component of \( U_i \cap \mathbb{D}. \) By definition, \( c_i|_V \) is a constant vector. Thus there is a vector \( d_{i,V} \in \Gamma \) such that \( c_i|_V - d_{i,V} \in \Pi. \) In this way we construct locally
constant maps $d_i \in \mathcal{C}(U_i \cap \mathbb{D}, \Gamma)$, $d_i|_V := d_i|_V$, $V \subset U_i \cap \mathbb{D}$ is clopen, such maps $c_i - d_i$ have ranges in $\Pi$, $i \in I$. From here and (12.3) we get
\begin{equation}
\sup_{x \in U_i \cap \mathbb{D}, i \in I} \| \hat{f}(x) - d_i(x) \|_{C^\alpha} < \infty.
\end{equation}

Next, since each map $\hat{f}|_{U_i \cap \mathbb{D}} - d_i$ is bounded holomorphic, according to Suárez theorem [11 Th.3.2], it admits an extension $g_i \in \mathcal{O}(U_i, \mathbb{C}^n)$. Then continuous maps $\pi \circ g_i \in \mathcal{O}(U_i, \mathbb{C}^n)$ satisfies $\pi \circ g_i = f$ on $U_i \cap \mathbb{D}$, $i \in I$. In particular, \begin{equation}
\pi \circ g_i - \pi \circ g_j = 0 \quad \text{on} \quad (U_i \cap U_j) \cap \mathbb{D}, \ i, j \in I.
\end{equation}
Since, due to the Carleson corona theorem, each $(U_i \cap U_j) \cap \mathbb{D}$ is dense in $U_i \cap U_j$, equation (12.6) implies that maps $\pi \circ g_i$ glue together over all nonempty intersections $U_i \cap U_j$ to determine a map $\hat{f} \in (M(H^\infty), \mathbb{C}^n)$ such that $\hat{f}|_{\mathbb{D}} = f$.

This completes the proof of the theorem. □

Proof of Theorem 4.4. Parts (a) and (b) follow straightforwardly from Theorems 3.2 and 3.6. For part (c) we apply Theorem 3.5 to find for each $f \in \mathcal{O}(M(\mathcal{A}Z), \mathcal{A})$ some $\alpha \in D$, a map $h \in \mathcal{O}(U, \mathcal{A})$ defined on a neighbourhood $U$ of $M(A_\alpha)$ and a map $g \in \mathcal{O}(M(\mathcal{A}Z), \mathcal{A}_{01})$ such that $gfg^{-1} = F_\alpha^h h$. Using that $M(A_\alpha)$ is polynomially convex, we choose a Weil polynomial polyhedron $\Pi$ such that $M(A_\alpha) \subset \Pi \subset U$. Then the application of the Weil integral representation formula \begin{equation}
M(\mathcal{A}Z) \to \mathcal{A} \subset \mathbb{C}
\end{equation}
is uniform approximation by a sequence of maps in $A_\alpha \otimes \mathcal{A}$ as required. □

Proof of Proposition 4.7. For a fixed $p \in \mathcal{A}_{-1}$ consider holomorphic map $\pi^p : \mathcal{A}_{-1} \to \mathcal{A}_{-1}$, $\pi^p(g) := gp$. Its differential $d\pi^p_{\mathcal{A}_{-1}} \subset L(\mathcal{A})$ at $1_{\mathcal{A}}$ is given by the formula $d\pi^p_{\mathcal{A}_{-1}}(a) := ap$. Let us show that $d\pi^p_{\mathcal{A}_{-1}}$ is surjective and its kernel is a complemented subspace of $\mathcal{A}$.

Indeed, if $q \in \mathcal{A}$ is such that $qp = 1_{\mathcal{A}}$, then linear map $r : \mathcal{A} \to \mathcal{A}$, $r(a) := aq$, is a continuous right inverse of $d\pi^p_{\mathcal{A}_{-1}}$ which gives the required statement.

This and the implicit function theorem (see, e.g., [11 Prop. 1.2]) imply that there exist open neighbourhoods $U_p$ of $0 \in \mathcal{A}$ and $V_p$ of $p$ such that $\pi^p(U_p) = V_p$. From here one deduces easily that $\mathcal{A}_{-1}$ acts transitively on each connected component of $\mathcal{A}_{-1}$.

In fact, suppose $\gamma : [0, 1] \to \mathcal{A}_{-1}$ is a path joining left-invertible elements $p$ and $q$. Then for each $t \in [0, 1]$ there is $\varepsilon_t > 0$ such that $\gamma([0, 1] \cap (t - \varepsilon_t, t + \varepsilon_t)) \subset V_\gamma(t)$. This and compactness of $[0, 1]$ show that there exist points $0 = t_1 < t_2 < \cdots < t_{k+1} = 1$ such that $\gamma([t_i, t_{i+1}]) \subset V_\gamma(t_i)$ for all $1 \leq i \leq k$. In turn, there are elements $g_i \in U_\gamma(t_i) \subset \mathcal{A}_{-1}$ such that $\gamma(t_{i+1}) = g_i \gamma(t_i)$ for all $1 \leq i \leq k$. This yields $q = (g_k \cdots g_1) p$ completing the proof of the claim.

Thus according to the definition of Section 3.2, each connected component of $\mathcal{A}_{-1}$ is a complex Banach homogeneous space under the action of $\mathcal{A}_{-1}$. □

Proof of Theorem 4.8. The results follow from Theorems 3.5 and 3.6 and the Weil integral representation formula as in the proof of Theorem 4.3 above. □

Proof of Theorem 4.9. Let us show that if $f \in \mathcal{O}(M(\mathcal{A}Z), \mathcal{A}_{-1})$, then $f \in \mathcal{O}(M(\mathcal{A}Z), \mathcal{A}_{-1})$. Indeed, according to Theorem 4.8 (b) it suffices to assume also that $f \in \cup_{a \in \mathbb{D}} A_a \otimes \mathcal{A} (\subset \mathcal{O}(M(\mathcal{A}Z)) \otimes \mathcal{A})$. Then $f \in \mathcal{O}(M(\mathcal{A}Z), \mathcal{A}_{-1})$ by a special case of the Bochner-Phillips-Allan-Markus-Semetsul theory, see [11 Th. 2.2] and references therein.

Thus, $\mathcal{O}(M(\mathcal{A}Z), \mathcal{A}_{-1}) \subset \mathcal{O}(M(\mathcal{A}Z), \mathcal{A}_{-1})$.

The converse implication $\mathcal{O}(M(\mathcal{A}Z), \mathcal{A}_{-1}) \subset \mathcal{O}(M(\mathcal{A}Z), \mathcal{A}_{-1})$ is obvious.
The statement of the theorem for algebra $C(M(\mathcal{A}), \mathfrak{A})$ follows from [V, Th. 2.2] as $C(M(\mathcal{A}), \mathfrak{A}) = C(M(\mathcal{A})) \otimes \varepsilon \mathfrak{A}$ by the approximation property of $C(M(\mathcal{A}))$.

Finally, the statement of the second part of theorem follows from Theorem 3.2. □

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