Posetal Games: Efficiency, Existence, and Refinement of Equilibria in Games with Prioritized Metrics

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Abstract—Modern applications require robots to comply with multiple, often conflicting rules and to interact with the other agents. We present Posetal Games as a class of games in which each player expresses a preference over the outcomes via a partially ordered set of metrics. This allows one to combine hierarchical priorities of each player with the interactive nature of the environment. By contextualizing standard game theoretical notions, we provide two sufficient conditions on the preference of the players to prove existence of pure Nash Equilibria in finite action sets. Moreover, we define formal operations on the preference structures and link them to a refinement of the game solutions, showing how the set of equilibria can be systematically shrunk. The presented results are showcased in a driving game where autonomous vehicles select from a finite set of trajectories. The results demonstrate the interpretability of results in terms of minimum-rank-violation for each player.

Index Terms—Autonomous Agents; Game Theory; Motion and Path Planning; Optimization and Optimal Control

I. INTRODUCTION

It is well known that decision making is a stressful task for human beings [1]. While robots do not get stressed (yet), their prospective ubiquity in our society is faced with similar challenges, charged with the need to “make the right choice” in complex environments. Indeed, embodied intelligence has to cope with laws of different severity, unwritten rules, different local cultures, liability issues, and different types of agents. This poses an unmatched challenge for decision making. There are two aspects which make the problem difficult. First, robots’ behavior needs to be compliant with rules written by humans for humans [2]. Such rules are often subject to interpretation and need to be contextualized. Second, the designed systems need to be robust to the highly interactive nature of unconstrained environments, for which too conservative approaches simply fail [3].

A clear example featuring both the aspects is autonomous driving. The early prototypes were designed to blindly respect the rules. Quickly, one realized that this was not enough, as one needed robots to obey the unwritten rules of human interactions to blend in [4].

Accounting for these aspects singularly comes with strong limitations. For instance, it is unclear how a system relying only on learned behaviors and “common practices” of human interactions will react when faced with a rare unfortunate situation. This creates a serious threat for all the participants involved and for the ego-robot also in terms of liability. Indeed, the outcome could be catastrophic. Similarly, a system designed to merely obey the rules can be perceived as dull (who would get stuck behind a double parked vehicle?).

This work aims at combining these two aspects. The interactive nature is captured by a game theoretical formulation where the players express a preference on the outcomes. The multi-objective nature of decisions is captured by a partially ordered (posetal) preference. The preference on the outcomes is expressed via a hierarchy of metrics which each player can specify, allowing one to choose a clear prioritization between objectives that cannot be bargained (e.g., collision and comfort). At the same time, players can also also express indifference between equally good alternatives (e.g., comfort and trip duration). Borrowing the idea from the work on minimum-violation planning, each metric can be interpreted as a soft constraint which gets systematically violated only if inevitable.

A. Related Work

Game theoretic models aiming at describing the interactive nature of multi-robot scenarios have seen quite a revival in the last years, both looking at planning aspects of the problem [5]–[10] and at estimation and learning of others’ cost functions [11]–[13]. Notably, few examples have also been deployed on commercial AVs [14]. However, all the proposed studies have as common denominator the usage of a scalar cost function.
Additionally, motivated by the complexity of traffic rules and the need for a transparent and interpretable system, another body of literature looked at techniques to specify objectives in a prioritized fashion [2], [15]–[17], leading to practical results [18].

Combining these two aspects results in games where players’ preferences are expressed as a binary relation on the outcomes. This kind of games is almost as old as game theory itself. Motivated by the attempt to model human decision making more accurately, already in [19]–[22] one can find formulations where players’ preferences are non trivial binary relations (e.g., interval orders, lexicographic orders, and semiorders). However, since these works originate from different epochs and fields, they often fail to relate to modern applications in robotics.

### B. Statement of Contribution

We formulate games where each player expresses a preference as a poset of metrics, inducing a preorder on the decision space. As shown in [2], this is a practical and scalable approach for behavior specification of a robot which needs to satisfy multiple (often contrasting) objectives at the same time. We analyze the resulting game providing three major theoretical contributions:

- We enrich the connotation of admissible equilibria by showing how they guarantee efficiency in the partial order of preferences.
- We provide two sufficient conditions for the existence of a pure Nash Equilibrium (NE) in posetal games with finite action sets. Interestingly, this depends on properties of the single metrics, but also on the combined preference structure of the players. In addition, we motivate the conditions by introducing two examples in which a pure NE does not exist. Each example violates only one condition at a time.
- We show that the set of equilibria of such a game is intimately related to the operations one performs on the preference structures. Particularly, any refining operation of a player’s preference refines the set of equilibria.

Finally, we showcase the discovered properties in a finite trajectory driving game, instantiated on CommonRoad scenarios [23].

**Manuscript organization:** Sec. II provides the necessary preliminaries, formally defining the concept of metrics, preferences, and refinement. Sec. III introduces Posetal Games providing results on fairness of admissible NE, existence of pure NE, and equilibria refinement. These concepts are showcased in trajectory driving games in Sec. IV.

### II. Preliminaries

The ensuing formalization builds on [2], [15], and in general on the related work on minimum-violation planning, where the objectives and constraints of an agent are prioritized according to a hierarchy. First, we define the concept of metric and model a player’s preference as a prioritized order of metrics. Second, we consider an order among preferences which will naturally induce a refinement in the space of solutions. Finally, we recall desirable operations that can be applied to a preference structure, with a focus on those resulting in a refinement. We assume the reader is familiar with basic facts of order theory [24].

#### A. Preferences over metrics

**Definition 1 (Metric).** Consider a compact set \( \Gamma \), representing a decision space. The \( k \)-th metric is a map \( m^k : \Gamma \to O^k \), where \( O^k \) is the corresponding outcome set. We assume \( m^k \) to be lower semi-continuous.

We will omit the superscript to indicate the union of all the metrics, that is, \( m = [m^1, \ldots, m^n] \). For the sake of simplicity, throughout this manuscript we will consider \( O^k = \mathbb{R}_{\geq 0} \). Nevertheless, with the due precautions, the presented results hold for any preorderd outcome set.

**Definition 2 (Preference).** Given a set of metrics \( \mathcal{M} = \{m^1, \ldots, m^n\} \). A preference \( P \) is specified as a partial order over \( \mathcal{M} \), i.e., \( P = (\mathcal{M}, \succeq) \).

A simple preference from the driving domain is shown in Fig. 2. As observed in [2], such a preference (Def. 2) naturally induces a pre-order on the outcome set \( O = \prod_k O^k \), denoted by \((O, \preceq)\). More importantly, a preorder is induced also over the decision space.

**Lemma 3 (Preorder induced on decision space).** A preference \( P \) (Def. 2) induces a preorder on the decision space \((\Gamma, \preceq_P)\), where

\[
\gamma \preceq_P \gamma' \iff m(\gamma) \preceq_O m(\gamma').
\]

**Proof.** Reflexivity is clear. Suppose \( \gamma \preceq_P \gamma' \) and \( \gamma' \preceq_P \gamma'' \), i.e., \( m(\gamma) \preceq_O m(\gamma') \) and \( m(\gamma') \preceq_O m(\gamma'') \). Since \((O, \preceq_O)\) is a preorder, \( m(\gamma) \preceq_O m(\gamma'') \), and hence \( \gamma \preceq_P \gamma'' \). \( \blacksquare \)

Importantly, when comparing two elements in a preordered set, one can have four possible outcomes: the first is preferred, the second is preferred, the two are uncomparable, or, they are indifferent. This concept is exemplified when comparing trajectories of an AV in Fig. 2.

#### B. Order on preferences and refining operations

Given the set of all possible metrics \( \mathcal{M} \), we denote by \( pr(\mathcal{M}) \) the set of all preorders over \( \mathcal{M} \). Note that a preference could be in general defined only on a subset \( \mathcal{M} \subseteq \mathcal{M} \) of all the metrics. Technically, this collapses the outcome set for the metrics which do not appear in the preference to an equivalence class: the player is indifferent to any value such metrics might assume.

1 We represent preferences via Hasse diagrams. In a Hasse diagram for poset \( P \), one writes \( m^1 \) below \( m^2 \) and connect them with a line if \( m^1 \preceq_P m^2 \). Relations arising from reflexivity and transitivity are omitted.
Preferences themselves can be ordered via a preorder.

**Definition 4 (Preorder of Preferences).** We define a preorder of preferences \( \langle \text{pr}(M), \preceq \rangle \) as follows. Given any two preferences \( P = \langle M, \preceq \rangle, P' = \langle M', \preceq' \rangle \in \text{pr}(M) \), one has:
\[
P \preceq P' \iff \langle O, \prec \rangle \subseteq \langle O, \prec' \rangle,
\]
meaning that \( P \) relates to \( P' \) iff, when a pair of outcomes \( \langle o_1, o_2 \rangle \) is in strict relation (either first- or second-preferred) according to \( P \), the same relation must hold for \( P' \).

**Lemma 5.** Def. 4 indeed defines a preorder.

**Proof.** Clearly, one has \( \langle O, \prec \rangle \subseteq \langle O, \prec \rangle \) (reflexivity). Furthermore, if \( \langle O, \prec \rangle \subseteq \langle O, \prec' \rangle \) and \( \langle O, \prec' \rangle \subseteq \langle O, \prec'' \rangle \), then \( \langle O, \prec \rangle \subseteq \langle O, \prec'' \rangle \) (transitivity). \( \blacksquare \)

We can now leverage Def. 4 to define the concept of preference refinement.

**Definition 6 (Preference refinement).** Let \( P, P' \in \text{pr}(M) \). Preference \( P' \) is a refinement of \( P \) if \( P \preceq P' \) as by Def. 4.

This notion of refinement is analogous to [2, Def. 18].

### C. Refining operations on preferences

We now look at preference-refining operations. Without loss of generality, we define such operations directly on preferences. Due to the induced preorders on outcomes and decision space (Lemma 3), the refinement propagates.

a) **Priority refinement.** This operation corresponds to “adding an edge” to the graph representing the preference.

**Definition 7 (Priority refinement operation).** Consider a preference \( P = \langle M, \preceq \rangle \). A priority refinement operation on \( P \) is any operation \( z: \text{pr}(M) \rightarrow \text{pr}(M) \) such that \( z(P) \) refines \( P \).

b) **Aggregation.** This operation allows one to condense two metrics of the preference into a single one.

**Definition 8 (Aggregation).** Given a preference \( P = \langle M, \preceq \rangle \) and two uncomparable metrics \( m_1, m_2 \in M \), an aggregation for \( m_1, m_2 \) is given by \( \text{agg}_{\alpha}(M, M) \rightarrow M \times M, \text{agg}_{\alpha} (P) := P' \), with:
\[
P' = S_1 \cup S_2 \cup \{ (m, m') \in P \mid m, m' \in M \setminus \{m_1, m_2\} \},
\]
\[
S_1 = \bigcup_{\langle m, m' \rangle \in \text{P}} \langle \alpha(m_1, m_2), m \rangle \cup \bigcup_{\langle m, m' \rangle \in \text{P}} \langle m, \alpha(m_1, m_2) \rangle,
\]
where \( \alpha \) is an embedding of the product poset over \( m_1 \times m_2 \) into \( \mathbb{R}_{\geq 0} \).

**Remark 9.** \( \alpha \) needs to be a strictly monotone map in both arguments. Allowed choices include linear combinations with positive coefficients: \( \alpha(m_1, m_2) = am_1 + bm_2 \), \( a, b \in \mathbb{R}_{\geq 0} \). This construction is similar in purpose to [2, Def. 16], but we aggregate uncomparable (and not indifferent) metrics.

c) **Augmentation.** This operation allows one to add a new metric at the lowest level of priority. Note that as discussed in [2], adding a rule can be a more general operation. Yet, adding a rule at the “bottom” results in a refinement; completely different preferences could otherwise arise. Therefore, in the following, we refer to augmentation in this sense only.

**Definition 10 (Metric augmentation operation).** Consider a preference \( P = \langle M, \preceq \rangle \). Metric augmentation consists of augmenting \( M \) to \( M' = M \cup \{m' \} \subseteq M \) and defining a preference \( P' = \langle M', \preceq' \rangle \) such that \( m' \preceq' m \) for all \( m \in M \).

Fig. 3 exemplifies the three operations (Defs. 7, 8 and 10).

**Lemma 11.** Consider a preference \( P = \langle M, \preceq \rangle \). Applying any of the operations (priority refinement, aggregation, and augmentation) to \( P \) results in a preference \( P' \) which refines \( P \).

Technically, Lemma 11 states that refinement operations are monotone maps \( \text{pr}(M) \rightarrow \text{pr}(M) \).

### III. Games With Posetal Preferences

In the last decades, games where players express a general preference on outcomes have been investigated [19], [21], [22]. Yet, these studies do not instantiate these concepts in engineering frameworks where, for instance, expressing the preference directly on outcomes or on decision spaces is impractical.

We now formally consider games where players express a preference over the outcomes via a prioritized hierarchy of objectives and constraints (the posetal preference).

**Definition 12 (Game with Posetal Preferences).** A Game with Partially Ordered Preferences (GPOP) (in short, posetal game) is specified as follows.

- There is a finite set of players \( \mathcal{A} \).
- Each player possesses a compact decision space, denoted by \( \Gamma_i \). The joint decision space is \( \Gamma = \prod \Gamma_i \), thus \( \gamma = (\gamma_1, \cdots, \gamma_i, \cdots) \) denotes a joint action profile.
- Given the action profile for the players, an outcome of the game for each player \( i \in \mathcal{A} \) is obtained via a deterministic metric function, mapping joint decisions to the outcome space \( m_i: \Gamma \rightarrow O_i \) (Def. 1).
- Each player \( i \in \mathcal{A} \) specifies a preference \( P_i \) (Def. 2).

It follows (Lemma 3) that the induced preorder on decisions \( (\Gamma_i, \preceq_{\Gamma_i}) \) is:
\[
\gamma_i \preceq_{\Gamma_i} \gamma'_i \iff m_i(\gamma_i, \gamma_i) \preceq_{O_i} m_i(\gamma'_i, \gamma_i) \forall \gamma_i \in \Gamma_i.
\]

Given the setup of a GPOP, standard game theory concepts follow naturally. To characterize rational players and the solutions of a game, we first consider Best Response (BR).

**Definition 13 (BR for GPOPs).** The BR for Player \( i \in \mathcal{A} \) is a set-valued map \( \text{BR}_i: \Gamma_i \rightarrow \Gamma_i \). A strict and weak version, respectively \( \text{BR}_i^s, \text{BR}_i^w \), are given by
\[
\text{BR}_i^s: \gamma_i \mapsto \{ \gamma_i \in \Gamma_i \mid m_i(\gamma_i) <_{O_i} m_i(\gamma'_i, \gamma_i) \forall \gamma'_i \in \Gamma_i \},
\]
\[
\text{BR}_i^w: \gamma_i \mapsto \{ \gamma_i \in \Gamma_i \mid m_i(\gamma_i, \gamma_i) \neq_{O_i} m_i(\gamma_i, \gamma_i) \forall \gamma_i \in \Gamma_i \}.
\]

While extensions to stochastic frameworks have been proposed in [21], in this work we consider pure actions and deterministic metrics.
Note that given a non-empty set of alternatives, $\text{BR}^*_i(\gamma_i)$ is always non-empty. Indeed, for every player $i \in A$, the weak BR map corresponds to finding the minimal elements of the preorder induced on the decision space (Lemma 3).

The BR map definition allows one to formally define the notion of NE of GPOPs.

**Definition 14** (NE of GPOPs). Let $\mathcal{G}$ be a GPOP. A strategy profile $\gamma \in \Gamma = \prod_{i \in A} \Gamma_i$ is a NE if it is a BR for all players. From the best-responses we distinguish strict and weak NE:

$$\gamma_i \in \text{BR}^*_i(\gamma_{-i}) \quad \forall i \in A \quad \text{strict [weak],}$$

and denote by $\text{NE}^*_i(\gamma)$ the set of strict [weak] NE of $\mathcal{G}$.

**Remark 15** (Semantics of NE). A strong NE consists of a strategy profile where the choice of every player is first-preferred to all the alternative ones. The notion of weak NE captures the cases in which for some players a strategy can be uncomparable or indifferent to some of the alternatives, yet there is none which is strictly better. Clearly, $\text{NE}^*_i(\gamma) \subseteq \text{NE}^{\leq}_i(\gamma)$.

Finally, as in standard game-theory, one can consider dominating NE, called admissible.

**Definition 16** (Admissible NE). Consider a game $\mathcal{G}$ with a set of NE $\text{NE}^{\leq}_i(\gamma)$. The set of admissible NE is given by

$$\text{NE}^{\leq}_A(\mathcal{G}) := \{ \gamma \in \text{NE}^{\leq}_i(\gamma) \mid m(\gamma') \not\preceq_O m(\gamma) \forall \gamma' \in \text{NE}^{\leq}_i(\gamma) \}.$$

**Remark 17.** Clearly, $\text{NE}^{\leq}_A(\mathcal{G}) \subseteq \text{NE}^{\leq}_i(\gamma)$.

### A. Rank of Solutions

In the setting of GPOPs, admissible NE not only are a proxy for NE efficiency, but also guarantee efficiency on the hierarchy of priorities (Lemma 21). To formalize this concept, we first define the notion of rank.

**Definition 18** (Rank of a metric). Given a preference $P$, each metric $m$ has rank $r(m) := \text{height}(Q), \ Q = \uparrow \{m\}$, where height$(Q)$ represents the maximum cardinality of a chain in $Q$ and $\uparrow$ represents the upper closure operator.

Loosely speaking, we can see the Hasse diagram of a preference as a downward-directed graph. Then, the rank of a metric corresponds to the length of the longest path (chain) from any root (top) to the metric.

We can now extend the notion of rank to players’ actions. Here, the rank of an action describes the critical priority that a player can reach without violating any metric of higher order.\(^3\)

**Definition 19** (Rank of an action). Consider a GPOP $\mathcal{G}$. The rank of action $\gamma \in \Gamma$ for player $i \in A$ with preference $P_i = (\mathcal{M}, \preceq)$ is denoted $R_i(\gamma)$, and is given by

$$\begin{align*}
    r(m^k_i(\gamma)),
    & \quad m^k_i(\gamma) \neq \min \langle O^k, \preceq \rangle \wedge (\forall \gamma' \in \text{NE} \gamma \geq m^k_i(\gamma)) = \min \langle O^k, \preceq \rangle, \\
    & \text{otherwise}
\end{align*}$$

Def. 19 can be interpreted as follows. The rank of a joint action from the viewpoint of a specific player is a proxy for the satisfaction level of the outcomes achieved through the action (measured as the rank of highest non-minimized metrics in the player’s preference poset). If all metrics in the preference structure are minimized, the rank is simply the rank of the minima of the preference poset.

Def. 19 can be extended to account for multiple players.

**Definition 20** (Common rank of a strategy profile). Consider a GPOP $\mathcal{G}$ with $n$ players. The common rank of a strategy profile $\gamma \in \Gamma$ is: $R(\gamma) := \min \{R_1(\gamma), \ldots, R_n(\gamma)\}$.

**Lemma 21.** Consider a GPOP $\mathcal{G}$ with a set of players $A$, admitting a set of NE $\text{NE}^{\leq}_i(\gamma)$. Consider $\gamma^* \in \text{NE}^{\leq}_i(\gamma) \setminus \text{NE}^{\leq}_i(\gamma)$ and $\gamma^{*'} \in \text{NE}^{\leq}_i(\gamma)$. It holds

$$R_i(\gamma^*) \leq R_i(\gamma^{*'}).$$

Note that Eq. (3) implies $R(\gamma) \leq R(\gamma^*)$. \[proof\]

**Remark 22.** In other words, admissible NE not only guarantee dominating payoffs among NE as per Def. 16, but also guarantee a same or higher rank (i.e., that the critical metric has same or lower priority) with respect to any non-admissible NE.

### B. On the existence of NE in GPOPs

The seminal work of Nash [25] that proved the existence of equilibria for finite games in mixed strategies has been extended to games with ordered preferences. A review is provided by [21]. Nevertheless, games where the existence of NE can be guaranteed in pure actions are often desirable for computational reasons in engineering applications.

In the following, we provide sufficient conditions which guarantee the existence of pure NE for GPOPs with a discrete action space. We translate the idea of the “players not having contrasting coupled objectives” to conditions on single metrics and combined preferences of the players. These allow to define a surrogate potential for the game. Then, the existence of pure NE follows via standard game theory results [26].

#### a) Jointly Communal Objectives: We consider the idea of players not having contrasting objectives by extending the idea of communal metrics [17, Def. 4] to arbitrary pairs of metrics. The idea is that if a player improves on a metric through a unilateral deviation in strategy, and the others do not, then the player has lower priority. This allows us to define the notion of jointly communal for the actions of the players.

**Definition 23** (Jointly communal metrics). Consider a GPOP with players $A$. We say that metric $m^k, m^l$ are jointly communal if and only if, for all $i \in A, \gamma, \gamma' \in \Gamma$ it holds

$$m^k_i(\gamma') - m^k_i(\gamma) < 0 \Rightarrow \sum_{j \in A^l} (m^l_j(\gamma') - m^l_j(\gamma)) \leq 0,$$

where $A^l \subseteq A$ represents the subset of agents featuring the $l$-th metric in their preference structure.

Note that Eq. (4) is not restrictive for many common cost functions used in mobile robotics, including, among others, collision costs, clearance objectives (i.e., keeping a minimum safety distance from obstacles) [17]. Moreover, any pair of personal metrics (i.e., metrics whose outcome depends on the single player strategy only) is trivially jointly communal.

To guarantee existence of NE, Def. 23 needs to hold for all incomparable metrics in a game. Formally, we require that:

\(^3\)We see this as an analogy to active constraints in constrained optimization.
We need to show Theorem 26. visual example is provided in Fig. 4. Clearly, the corresponding component in other components relative to joint metrics of higher priority do aggregation of the metric, i.e., a) If Player \(i\) player example is reported in Fig. 5, where we aggregate via sum. aggregation is well-defined given Cond. 24. An illustrative example is reported in Fig. 5b, which clearly violates Cond. 25. Let’s assume that the preferences satisfy Cond. 24. In this case, one can create a game which satisfies Cond. 24, but for which no pure strategy NE exists (Fig. 6d). Technically, the preferences as given in Fig. 6b prevent \(P^n\) from being a poset, and hence prevent the construction of the potential as described in Thm. 26.

**Condition 24.** Let \(P^n := \bigcup_{i \in A} P_i\). Then, all incomparable metrics in \(P^n\) are pairwise jointly communal.

The relation \(P^n\) is the union of the players’ preferences. A visual example is provided in Fig. 4.

Two comments are in order. First, if all \(M_i\) are disjoint, \(P^n\) is clearly a poset over \(\bigcup_{i \in A} M_i\). Second, in case \(M_i = M_j\) for all \(i, j \in A\), \(P^n\) corresponds to the join (least upper bound) \(\bigvee_{i \in A} P_i\) in \(\mathcal{P}(M)\).

**b) Consistent preferences:** The complexity of allowing arbitrarily prioritized metrics requires the introduction of a second condition, now on the preferences of the players.

To ease the reading, in the following we denote with \(\bar{m} \in \bar{M}\) the joint metrics, opposed to the personal ones, as the metrics that depend on the actions and states of many players [17]. For instance, the time taken by an agent to cross an intersection can be computed as a function of its own trajectory only. Instead, a joint metric such as minimum clearance or collision energy is by definition a function of two or more trajectories.

**Condition 25.** Given a GPOP it holds, for all \(i \in A:\)

\[
\bar{m}_i^k \preceq_{P} \bar{m}_i^l \iff \exists j \in A \setminus \{i\} : m_j^k \preceq_{P} m_j^l.
\]

(c) Existence of pure NE for GPOPs:

**Theorem 26.** A GPOP with finite action sets satisfying Conds. 24 and 25 admits a pure NE.

**Proof.** We organize the proof as follows. First, we show how the satisfaction of Conds. 24 and 25 allows one to construct a posetal potential function \(Pot\). Second, we show that the minimum of \(Pot\) is a pure NE. Consider a GPOP with agents \(A\), each characterized by a preference \(P_i, i \in A\).

1) The satisfaction of Cond. 25 guarantees that \(P^n\) is a poset.\(^4\) We now construct a partially ordered potential \(Pot\) as follows. Leveraging Def. 8, starting from \(P^n\), we sequentially aggregate all joint metrics \(\bar{m}_i^k\) for all players \(i \in A\). Note that sequential aggregation is well-defined given Cond. 24. An illustrative example is reported in Fig. 5, where we aggregate via sum.

Given Conds. 24 and 25, \(Pot\) is a poset. In the following, we denote its carrier set by \(\bar{M}\) and \(\bar{M} \setminus \bar{M}\) by \(\bar{M}^c\).

Now, consider a unilateral deviation in strategy \(\gamma_i \rightarrow \gamma_i’\) for player \(i \in A\) (i.e., a strategy switch \(\gamma \rightarrow \gamma’\), where \(\gamma = (\gamma_i, \gamma_{-i})\) and \(\gamma’ = (\gamma_i’, \gamma_{-i})\) such that \(m_i(\gamma) \prec_{P} m_i(\gamma’).\) We need to show \(Pot(\gamma’) < Pot(\gamma).\) We have two cases.

a) If Player \(i\) improved due to an improvement on a joint metric, i.e., \(m_i(\gamma) < m_i(\gamma’), \gamma’ \in \bar{M}\), Conds. 24 and 25 impose improvement of \(Pot\) in the component relative to the aggregation of the l-th joint metric. Furthermore, they guarantee other components relative to joint metrics of higher priority do not deteriorate \(Pot\). Clearly, such an improvement does not deteriorate any personal metric in \(Pot\) for the other players. Hence, in this case \(Pot(\gamma’) < Pot(\gamma).\) b) If, instead, Player \(i\) improved due to a personal metric, i.e., \(m_i(\gamma’) < m_i(\gamma), \gamma’ \in \bar{M}^c\), clearly the corresponding component in \(Pot\) will improve. Such an improvement could deteriorate a joint metric for Player \(i\) and other players, but the form of \(Pot\) and Conds. 24 and 25 guarantee that the interested joint components of \(Pot\) will be dominated by \(m_i^k\) in priority for Player \(i\) (the case in which they are higher/uncomparable in priority was covered in a)). Furthermore, improvement on a personal metric for Player \(i\) cannot deteriorate personal metrics of other players (Cond. 24). Finally, note that improvements happening both due to joint and personal metrics simultaneously are captured by the composition of the above cases. Therefore, \(Pot(\gamma’) < Pot(\gamma).\) Pot is a valid potential function, and a GPOP satisfying Conds. 24 and 25 is a posetal potential game.

2) As for standard game theory [27], let \(Pot(\gamma^*)\) be a minimal element of the potential, by definition of potential nobody can unilaterally change strategy receiving a strictly better payoff otherwise \(Pot(\gamma^*)\) would not be a minimal element of the poset. Thus \(\gamma^*\) is a weak NE. The same reasoning applies for the strict counterpart assuming \(Pot(\gamma^*)\) is a minimal element. Note that the existence of such potential allows one to additionally prove the convergence of widely used algorithms, such as iterated strictly-better response schemes as in [17].

We further motivate Conds. 24 and 25 providing in Fig. 6 two counterexamples. Each violates only one condition, yet we can construct games where a pure equilibrium does not exist.

C. Refining Equilibria via Preferences

In this section we leverage the notion of refinement, introduced in Sec. II, and show how preference refinements

\(^4\)Without loss of generality we can consider players with preferences on disjoint metric sets.
allow one to refine NE of GPOPs. Lemma 27 and Thm. 28 encapsulate this concept.

**Lemma 27.** Consider $P = \langle M, \preceq\rangle$ and $P' = \langle M', \preceq'\rangle$, and the respective induced preorders on the set of outcomes $\langle O, \preceq\rangle$, $\langle O, \preceq'\rangle$. It holds:

$$P \preceq_{\text{pr}(M)} P' \implies \min \langle O, \preceq\rangle \succeq \min \langle O, \preceq'\rangle.$$  

**Proof.** Let $p$ be a minimal element of $\langle O, \preceq'\rangle$, implying that there is no $q \in O$ such that $q \preceq' p$. Per absurdum, let’s assume that $p$ is not a minimal element of $\langle O, \preceq\rangle$, meaning that there is a $q \in O$ such that $q \preceq p$. However, we know from Def. 4 that $\langle O, \preceq\rangle \subseteq \langle O, \preceq'\rangle$, which implies that $(q, p) \in \langle O, \preceq\rangle$. This contradicts the starting assumption. \hfill\qed

What is more, via Lemma 3 one can extend the result of Lemma 27 to minima of $\langle \Gamma, \preceq\rangle$.

**Theorem 28.** Consider two GPOPs $G_1, G_2$ sharing the same constituents, exception made for players’ preferences. We denote the preference of Player $i$ in $G_j$ by $P_i^j$. It holds:

$$P_i^j \preceq_{\text{pr}(M)} P_i^j \forall i \in \mathcal{A} \implies \text{NE}^{\preceq}(G_1) \succeq \text{NE}^{\preceq}(G_2).$$

In other words, Thm. 28 states that refining players’ preferences in the sense of Def. 6 filters the NE of the game.

**Proof.** From Lemma 27 we know that $P_i \preceq P_i^j \forall i \in \mathcal{A} \implies \min \langle \Gamma, \preceq\rangle \succeq \min \langle \Gamma, \preceq'\rangle$. Therefore:

$$\text{NE}^{\preceq}(G) = \{ \gamma \in \Gamma | \gamma_i \in \text{BR}_{Pr}^{\preceq}(\gamma_i) \forall i \in \mathcal{A} \}$$

$$\succeq \{ \gamma \in \Gamma | \gamma_i \in \min \langle \Gamma, \preceq'\rangle \forall i \in \mathcal{A} \} = \text{NE}^{\preceq}(G').$$

\hfill\qed

**IV. TRAJECTORY DRIVING GAME WITH POSETEL PREFERENCES**

In this section, we evaluate the properties of GPOPs in an urban driving context. The players are AVs solving the motion planning problem by choosing from a finite set of generated trajectories. The proposed setup is integrated on reproducible, publicly available CommonRoad scenarios [23].

Given a scenario, we formulate a discrete trajectory game [17] as follows. Each player comes with initial states and a pre-specified task to accomplish (e.g., navigate the intersection exiting WEST). The corresponding action set consists of many trajectories, which are generated as described in Par. a). The choice of the trajectory for the AV is guided by the preference expressed as a poset of cost functions commonly used in motion planning. These include collision energy in case of an impact, drivable area violation, minimum clearance, time needed to reach the goal, progress along the prescribed reference, longitudinal and lateral comfort (function of the acceleration), lateral and heading deviation from the reference path.

**a) Trajectory generation:** In light of minimum-violation planning principles [15] we do not constrain the generation of trajectories to be already compliant with the constraints and objectives of the players, for instance by considering only trajectories strictly compliant with the rules of the road. We consider a single-track model for the vehicles and generate kinematically feasible trajectories around one or multiple reference paths assigned as goal directions. We promote heterogeneity of inputs and dispersion of states to not rule out a priori physically possible alternatives. For the sake of the example, we consider only trajectories which terminate the game, i.e., which reach the goal region. This is not a limitation of the method, which could compare arbitrary trajectories, but rather a simplification to compare NE of the same game. Promoting heterogeneity of trajectories is key for a minimum-violation planning approach to not miss out on possible solutions. We envision similar heuristics to be devised for game theoretic settings as well; yet, this is out of scope for this work.

**A. Equilibria of the game**

Given a scenario, vehicles, tasks, and preferences, we compute the equilibria of the game performing an exhaustive search over the space of joint actions (trajectories) for different refinements of preferences. As per Def. 14, equilibria can either be strong or weak, and a subset of them could be admissible.

In the following, we characterize and analyze equilibria for the setting reported in Fig. 7. Specifically, the setting includes three players, i.e., $\mathcal{A} = \{P_1, P_2, P_3\}$ with respective goals and possible trajectories. The preferences are matched with Fig. 9a. Intuitively, the preferences of $P_1, P_2, P_3$ can be interpreted as follows. All players prioritize the minimization of collision energy, but while $P_2$ and $P_3$ care more about preventing drivable area violation than preserving clearance and quickly progressing along a prescribed path, $P_1$ cares equally about drivable area violation and clearance, and prioritizes both over time.
Equilibrium \(P_3\) is weak and admissible, and \(P_2\) is weak and non-admissible. Clearly, \(P_2\) dominates \(P_3\) in any way: indeed, all agents are better off in drivable area violation (Fig. 8).

Finally, we are able to see Lemma 21 at work. By comparing equilibrium \(2\) with equilibrium \(3\), we notice \(r_i(3) \leq r_i(2)\) for all \(i \in A\) (indeed, \(r_i(3) = 2\) and \(r_i(2) = 3\)).

B. Refinement in a driving game with posetal preferences

We now showcase the importance of Thm. 28, instantiating it in the case study reported in Sec. IV. We report all the preferences used within the case study, as well as the refinement operations relating them in Fig. 9a. We start with the setting depicted in Fig. 9b, in which players \(P_2\) and \(P_3\) share the same preference structure \(\preceq\). The specific game possesses 28 weak NE, no strong NE, and 17 admissible NE. We depict strong NE in red and weak NE in yellow. From here, we refine the preference of \(P_1\) from \(\preceq\) to \(\succeq\), by repeatedly applying a priority refinement operation (Def. 7). The resulting game possesses 16 weak NE, 1 strong NE, and 10 admissible NE, highlighting the validity of Thm. 28 (indeed, by refining \(P_1\)’s preference we are shrinking the set of weak NE of the game). We further refine the preferences, now both for \(P_1\) and \(P_2, P_3\). For \(P_1\), we aggregate area violation with clearance and longitudinal comfort with lateral comfort via Def. 8, and get \(\succeq\). Similarly, for \(P_2, P_3\), we aggregate clearance with time and progress along a given reference, and longitudinal with lateral comfort, obtaining \(\succeq\). The resulting game possesses no weak NE, 7 strong NE, and 3 admissible NE, again agreeing with Thm. 28.

C. Discussion

The presented case study has pedagogical value and practically showcases all the theoretical results. Notably, we show how the admissible NE represent the “fairest” minimum violation for the agents’ preferences and how the preference refinement shrinks the set of solutions. The richness of posetal preferences allows one to rethink standard game-theoretical tools. For instance, the concept of security strategies can be considered in relation to an arbitrary rank of the others’ preferences. That is, a player could assume that the others play rational only up to rank \(k\).

The presented case has practical limitations due to the combinatorial explosion of joint actions. We envision two ways to address this. The first is to consider a game played in a receding horizon fashion, over a shorter time horizon; this will curb significantly the action space. The second option is to solve a scalarized version of the game (e.g., enumerating the homotopy classes of who goes first) with different techniques and then check how the solutions are ranked with respect to the original preferences. In this regard, we highlight that the posetal preferences provide a formidable checker for the “quality” of equilibria, which could be provided by different systems: an oracle, approximate solution methods, or learning methods.

V. Conclusions

In this work, we presented games in which players express a preference on outcomes via a prioritized hierarchy of metrics. This aims at modeling the interactive aspect of multi-robot settings in combination with the multi-objective nature of mobile robots. Particularly, for modern applications which often requires non-trivial behavior specification.

We extended classic game theoretic constructions to this kind of games, and provided sufficient conditions for the existence of pure NE for finite actions. Furthermore, we demonstrated that one can systematically refine the set of equilibria performing interpretable operations on the preferences (e.g., aggregating two metrics). Finally, we instantiated the developed methodologies in the practical case of trajectory selection for AVs in urban driving scenarios.

This work opens various avenues for future research. First, posetal preferences offer a formidable tool that generalizes relevant aspects of decision making problems for robots. For instance, the proposed framework could include objectives specified in linear temporal logic [28]. Second, the framework could be extended to include a bit of “slackness” in the preference structures, e.g., considering lexicographic semiorders [29]. The addition of a indifference threshold prevents a small noise in high priority metrics to be detrimental for lower metrics. For example, if lane keeping is more important than comfort, still one can allow to trade off a small lane violation if it helps avoiding a hard braking.

Third, while this work develops a thorough understanding of posetal games, it poses new challenges on the computational side of things. Particularly, this entails the extension of common optimization tools (e.g., optimization on continuous decision spaces) to the case of preordered sets. Finally, the monotonicity properties of posetal games in terms of equilibria refinement provide an ideal setting for the development of dynamic, estimation-based techniques. Each preference can be seen as a specific type of player, laying the foundations for Bayesian frameworks. In particular, for the application of self-driving cars, we recognize that high rank objectives are common to all rational players, while the specific preference of players can be learned and refined dynamically via the single interactions.

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(a) Preferences used in the case study and their refinement via Defs. 7 and 8.

(b) Preference in Fig. 9a yields 28 weak NE, no strong NE, and 17 admissible NE.

(c) Preference in Fig. 9a yields 16 weak NE, 1 strong NE, and 10 admissible NE.

(d) Preference in Fig. 9a yields no weak NE, 7 strong NE, and 3 admissible NE.

Fig. 9. From left to right, we observe the same game played with refined preferences. Strong NE are depicted in red, and weak NE in yellow. From (a) to (b) P1 refines her preferences via Def. 7, and from (b) to (c) P1 refines her preferences via Def. 7 and Def. 8. Further refining the preference shrinks the set of NE of the game.

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