Avoiding defeat in a balls-in-bins process with feedback

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Abstract

Imagine that there are two bins to which balls are added sequentially, and each incoming ball joins a bin with probability proportional to the \(p\)th power of the number of balls already there. A general result says that if \(p > 1/2\), there almost surely is some bin that will have more balls than the other at all large enough times, a property that we call eventual leadership.

In this paper, we compute the asymptotics of the probability that bin 1 eventually leads when the total initial number of balls \(t\) is large and bin 1 has a fraction \(\alpha < 1/2\) of the balls; in fact, this probability is \(\exp(\mathcal{c}_p(\alpha) t + O(t^{2/3}))\) for some smooth, strictly negative function \(\mathcal{c}_p\). Moreover, we show that conditioned on this unlikely event, the fraction of balls in the first bin can be well-approximated by the solution to a certain ordinary differential equation.

1 Introduction

Consider a discrete-time process in which there are two bins, to which balls are added one at a time. Each incoming ball chooses probabilistically which bin to go to according to the following rule: if bin 1 currently has \(n_1\) balls and bin 2 has \(n_2\) balls, then the probability that bin 1 is chosen is

\[
\frac{f(n_1)}{f(n_1) + f(n_2)},
\]

where \(f\) is a fixed positive function. These so-called balls-in-bins processes with feedback function \(f\), which can be generalized to more than two bins (cf. Section 2 below) were introduced to the Discrete Mathematics community by Drinea, Frieze and Mitzenmacher.

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1The first author’s thesis [9] contains a longer background discussion of this and related processes.
This family of processes was intended as a model for competition that is mathematically similar to some so-called preferential attachment models for large networks [2, 1, 5].

The authors of [6] were especially interested in the case \( f(x) = x^p \) with \( p > 0 \) a parameter. In this case, there is a tendency that the rich get richer: since \( f \) is increasing, the more balls a bin has, the more likely it is to receive the next ball. One of the main questions addressed in [6] is whether this phenomenon results in effective preponderance by one of the bins in the long run. They proved that the answer is “yes” if \( p > 1 \) and “no” if \( p < 1 \). That is, if \( p > 1 \) then one of the two bins will obtain a \( 1 - o(1) \) fraction of all balls in the large-time limit, whereas if \( p < 1 \) the fractions of balls in the two bins both tend to \( 1/2 \). The case \( p = 1 \) is the well-known Pólya Urn, in which case the limiting number of balls in bin 1 has a non-degenerate distribution depending on the initial conditions, so the result in [6] seemingly completes the description of the family of processes given by the choices of \( p \).

However, stronger results are available. A paper by Khanin and Khanin introduced what amounts to the same process as a model for neuron growth, and proved that if \( p > 1 \), there almost surely is some bin that gets all but finitely many balls, an event that we call monopoly. They also show that for \( 1/2 < p \leq 1 \), monopoly has probability 0, but there almost surely will be some bin which will lead the process from some finite time on (we call this eventual leadership), whereas this cannot happen if \( 0 < p \leq 1/2 \). In fact, the result of [7] generalizes to any \( f \) with \( \min_{x \in \mathbb{N}} f(x) > 0 \), as shown e.g. in [14, 9, 10].

**Theorem 1 (From [7, 14, 9, 10])** If \( \{I_m\}_{m=0}^{+\infty} \) is a balls-in-bins process and feedback function \( f = f(x) \geq c \) for some \( c > 0 \), then there are three mutually exclusive possibilities, one of which happens almost surely irrespective of the initial conditions:

1. if \( \sum_{n \geq 1} f(n)^{-1} < +\infty \), of the bins receives all but finitely many balls (this is the monopolistic regime);

2. if \( \sum_{n \geq 1} f(n)^{-1} = +\infty \) but \( \sum_{n \geq 1} f(n)^{-2} < +\infty \), monopoly does not happen but one of the bins has more balls than the other at all large enough times (this is the eventual leadership regime);

3. if \( \sum_{n \geq 1} f(n)^{-2} = +\infty \), the balls alternate in leadership infinitely many times (this is the almost-balanced regime).

Notice that the three cases of the Theorem applied to the \( f(x) = x^p \) family correspond to \( p > 1 \), \( 1/2 < p \leq 1 \) and \( 0 \leq p < 1/2 \); in other words, this family of \( f \) has phase transitions at \( p = 1 \) and \( p = 1/2 \).

The present paper is part of a series of works by the two authors and by Michael Mitzenmacher in which several more quantitative aspects of the three regimes are elucidated. We are especially concerned with the eventual leadership and monopoly regimes, where there are initially \( t \) balls in the system and bin 1 has a fraction \( \alpha \in (0, 1/2) \) of those balls. It
is easy to show that bin 1 has a positive probability of eventually leading the process, but this probability should get smaller and smaller as $t$ increases. Thus we ask ourselves two simple questions:

1. How fast does the probability that bin 1 will escape its unfavorable initial conditions and eventually lead converge to 0 as $t \to +\infty$?

2. What is the typical behavior of the process, given that bin 1 does escape?

Our two main results apply to the case $f(x) = x^p$, $p > 1/2$. We show that the answer to the first question is “exponentially small” and compute the exact rate of decay. Below, let $[t, \alpha]$ denote the pair $(\lceil \alpha t \rceil, t - \lceil \alpha t \rceil)$.

**Theorem 2** Assume that we have a balls-in-bins process with feedback function $f(x) = x^p$, $p > 1/2$ (so that the strong eventual leadership condition holds), and let $E_{\text{Lead}}$ be the event that the first bin eventually leads the process. Then, for all fixed $\alpha \in (0, 1/2)$, the limit

$$c_p(\alpha) = \lim_{t \to +\infty} \frac{\ln \Pr_{[t, \alpha]}(E_{\text{Lead}})}{t} \in \mathbb{R}^{-}$$

exists, and is a smooth function of $\alpha$ satisfying $c_p'(\alpha) > 0$ on $(0, 1/2)$. Moreover, for any $\delta \in (0, 1/2)$ there exist $C_\delta \in \mathbb{R}^+$ and $T_\delta \in \mathbb{N}$ such that

$$\forall \alpha \in (\delta, 1/2 - \delta), \forall t \geq T_\delta, \ e^{c_p(\alpha) t - C_\delta t^{2/3}} \leq \Pr_{[t, \alpha]}(E_{\text{Lead}}) \leq e^{c_p(\alpha) t + C_\delta}. \quad (2)$$

The form of Theorem 2 should be compared with that of Crâmer’s Theorem [15], which estimates the exponential rate of decay of the probability of large deviations from the mean of sums of i.i.d. random variables. This analogy also applies to the proof of Theorem 2 contains computations with Laplace transforms that resemble those used to prove Crâmer’s Theorem. In our case, however, the random variables we consider, although not i.i.d., are of a very specific kind. Theorem 2 is proven in Section 4 below.

**Question 2.** It turns out to have a more surprising answer than 1. We will prove that conditioning on bin 1 escaping almost determines the behavior of the process, at least up to the time when bin 1 has half of the balls. To state this result precisely, define

$$g_p : (0, 1/2) \to \mathbb{R} \quad \alpha \mapsto -\alpha + \frac{p e^{c_p'(\alpha)}}{\alpha p e^{c_p'(\alpha)} + (1 - \alpha)^p}. \quad (3)$$

We will show below (cf. Remark 1) that $g_p(\alpha) > 0$ for all $\alpha \in (0, 1/2)$. This implies that the function $A = A_{\alpha, p}(\cdot)$ solving the following ODE is increasing.

$$\begin{cases}
\frac{dA}{ds}(s) = g_p(A(s)), & s > 0 \\
A(0) = \alpha
\end{cases} \quad (4)$$
Such a solution is only guaranteed to exist for \( s \in [0, T_{p,\alpha}) \), where \( T_{p,\alpha} \in \mathbb{R}^+ \cup \{+\infty\} \). Given that \( A_{p,\alpha} \) is increasing, \( T_{p,\alpha} \) is finite if and only if \( \lim_{s \to T_{p,\alpha}} A(s) = 1/2 \). In any case, there does exist some maximal \( T_{p,\alpha} \) as above, and \( A \) is uniquely defined as a function on \([0, T_{p,\alpha})\).

Our theorem can now be stated.

**Theorem 3** Consider a balls-in-bins process with feedback function \( f(x) = x^p \), \( 0 < p < 1/2 \) started from initial conditions \([t, \alpha]\). Let \( \hat{\alpha}_t(s) \) be the fraction of balls in bin 1 at time \([s t]\), and let \( \mathcal{E}_{\text{Lead}} \) be the event that bin 1 eventually leads. It then holds that for all \( K \in \mathbb{R}^+ \) satisfying \( K < T_{p,\alpha} \),

\[
\Pr_{[t,\alpha]} \left( \forall s \in [0, K], |\hat{\alpha}_t(s) - A_{p,\alpha}(s)| \leq W t^{-1/3} \mid \mathcal{E}_{\text{Lead}} \right) \geq 1 - e^{-\Omega(t^{1/3})}, \ t \gg 1. \tag{5}
\]

Here, \( W \) is a constant depending on \( \alpha \) and \( K \), but not on \( t \).

Notice that the two possibilities presented above – \( T_{p,\alpha} < +\infty \) or \( T_{p,\alpha} = +\infty \) – are both a priori legitimate. In the former case, one would be able to show that the random function \( \hat{\alpha}(\cdot) \) conditioned on \( \mathcal{E}_{\text{Lead}} \) converges weakly to \( A_{p,\alpha} \) in the space \( D_{[0, +\infty)} \). In the latter case, for all \( \epsilon > 0 \), there would be a value of \( K = K_\epsilon < T_{p,\alpha} \) such that, with probability tending to 1 \( \hat{\alpha}(K_\epsilon) > 1/2 - \epsilon \). It would be quite interesting to settle this matter: determining whether \( A(s) \to 1/2 \) as \( s \) converges to some finite \( T \) should only require a careful (but perhaps laborious) estimation of the RHS of \( g(\alpha) \) for \( \alpha \) near \( 1/2 \). The proof of Theorem 3 can be found in Section 5 below.

Before we proceed, let us briefly discuss our proof technique. This work employs the same fundamental tool as in the remaining papers in this series \([3, 11, 10]\), as well as in other references \([7, 14]\) (according to \([7]\), the technique originated in Davis’ work on reinforced random walks \([4]\)). We shall embed the discrete-time process we are interested in into a continuous-time process built from exponentially distributed random variables, so that inter-arrival times at different bins are independent and have an explicit distribution, which is very helpful in calculations. We call this the exponential embedding of the process. Our main conceptual contribution is to notice that the problems at hand lend themselves to proof via the exponential embedding method.

The rest of the paper is organized as follows. After preliminaries are discussed in Section 2, Section 3 describes the exponential embedding and its application to the eventual leadership event. The next two sections prove the main theorems, and Section 6 discusses some open questions.

## 2 Preliminaries

**General notation.** Throughout the paper, \( \mathbb{N} = \{1, 2, 3, \ldots\} \) is the set of non-negative integers, \( \mathbb{R}^+ = [0, +\infty) \) is the set of non-negative reals, and for any \( k \in \mathbb{N} \setminus \{0\} \) \( [k] = \{1, \ldots, k\} \).
\( \chi_A \) is the indicator function of a set (or event) \( A \).

**Asymptotics.** We use the standard \( O/o/\Omega/\Theta \) notation. The expressions “\( a_n \sim b_n \) as \( n \to n_0 \)” and “\( a_n \ll b_n \) as \( n \to n_0 \)” mean that \( \lim_{n \to n_0} (a_n/b_n) = 1 \) and \( \lim_{n \to n_0} (a_n/b_n) = 0 \), respectively.

**Balls-in-bins.** Formally, a feedback function is a map \( f : \mathbb{N} \to (0, +\infty) \) with positive minimum. A balls-in-bins process with feedback function \( f \) and \( B \in \mathbb{N} \) bins is a discrete-time Markov chain \( \{(I_1(m), \ldots, I_B(m))\}_{m=0}^{+\infty} \) with state space \( \mathbb{N}^B \) and transitions given as follows. For every time \( m \geq 1 \) there exists an index \( i_m \in [B] \) such that \( I_m(i_m) = I_{m-1}(i_m) + 1 \) and \( I_m(i) = I_{m-1}(i) \) for \( i \in [B] \setminus \{i_m\} \). Moreover, the distribution of \( i_m \) is given by

\[
\Pr (i_m = i \mid \{I_m'(j) : 0 \leq m' < m, j \in [B]\}) = \frac{f(I_{m-1}(i))}{\sum_{j=1}^{B} f(I_{m-1}(j))}.
\]

We will usually refer to the index \( i_m \in [B] \) as the bin that receives a ball at time \( m \). For any \( B \), if \( E \) is an event of the process and \( u \in \mathbb{N}^B \), \( \Pr_u (E) \) is the probability of \( E \) when the initial conditions are set to \( u \). Finally, in the case \( B = 2 \), it will be convenient to use the notation \([t, \alpha] \ (t \in \mathbb{N}, 0 \leq \alpha \leq 1\) to denote the state \((\lceil \alpha t \rceil, t - \lceil \alpha t \rceil)\), i.e. there is a total of \( t \) balls in the bins, and the fraction of balls in bin 1 is (approximately) \( \alpha \).

**Exponential random variables.** \( X \overset{d}{=} \exp(\lambda) \) means that \( X \) is a random variable with exponential distribution with rate \( \lambda > 0 \), meaning that \( X \geq 0 \) and

\[
\Pr (X > t) = e^{-\lambda t} \ (t \geq 0).
\]

The shorthand \( \exp(\lambda) \) will also denote a generic random variable with that distribution. Some elementary but extremely useful properties of those random variables include

1. **Lack of memory.** Let \( X \overset{d}{=} \exp(\lambda) \) and \( Z \geq 0 \) be independent from \( X \). The distribution of \( X - Z \) conditioned on \( X > Z \) is still equal to \( \exp(\lambda) \).

2. **Minimum property.** Let \( \{X_i \overset{d}{=} \exp(\lambda_i)\}_{i=1}^{m} \) be independent. Then

\[
X_{\min} \equiv \min_{1 \leq i \leq m} X_i \overset{d}{=} \exp(\lambda_1 + \lambda_2 + \ldots \lambda_m)
\]

and for all \( 1 \leq i \leq m \)

\[
\Pr (X_i = X_{\min}) = \frac{\lambda_i}{\lambda_1 + \lambda_2 + \ldots \lambda_m} \quad (6)
\]
3. Multiplication property. If \( X =^d \exp(\lambda) \) and \( \eta > 0 \) is a fixed number, \( \eta X =^d \exp(\lambda/\eta) \).

4. Moments and transforms. If \( X =^d \exp(\lambda) \), \( r \in \mathbb{N} \) and \( t \in \mathbb{R} \),

\[
\text{Ex} [X^r] = \frac{r!}{\lambda^r}, \quad (7)
\]

\[
\text{Ex} [e^{tX}] = \begin{cases} 
\frac{1}{1-t} & (t < \lambda) \\
+\infty & (t \geq \lambda) 
\end{cases} \quad (8)
\]

3 The exponential embedding

3.1 Definition and key properties

Let \( f: \mathbb{N} \to (0, +\infty) \) be a feedback function, \( B \in \mathbb{N} \) and \( (a_1, \ldots, a_B) \in \mathbb{N}^B \). We define below a continuous-time process with state space \((\mathbb{N} \cup \{+\infty\})^B\) and initial state \((a_1, \ldots, a_B)\) as follows. Consider a set \( \{X(i, j) : i \in [B], j \in \mathbb{N}\} \) of independent random variables, with \( X(i, j) =^d \exp(f(j)) \) for all \((i, j) \in [B] \times \mathbb{N}\), and define

\[
N_i(t) \equiv \sup \left\{ n \in \mathbb{N} : \sum_{j=a_i}^{n-1} X(i, j) \leq t \right\} \quad (i \in [B], t \in \mathbb{R}^+ = [0, +\infty)),
\]

where by definition \( \sum_{j=i}^{k} (\ldots) = 0 \) if \( i > k \). Thus \( N_i(0) = a_i \) for each \( i \in [B] \), and one could well have \( N_i(T) = +\infty \) for some finite time \( T \) (indeed, that will happen for our cases of interest); but in any case, the above defines a continuous-time stochastic process, and in fact the \( \{N_i(\cdot)\}_{i=1}^B \) processes are independent. Each one of this processes is said to correspond to bin \( i \), and each one of the times

\[
X(i, a_i), X(i, a_i) + X(i, a_i + 1), X(i, a_i) + X(i, a_i + 1) + X(i, a_i + 2), \ldots
\]

is said to be an arrival time at bin \( i \). As in the balls-in-bins process, we imagine that each arrival correspond to a ball being placed in bin \( i \).

In fact, we claim that this process is related as follows to the balls-in-bins process with feedback function \( f \), \( B \) bins and initial conditions \((a_1, \ldots, a_B)\).

**Theorem 4 (Proven in [4, 7, 14, 9, 11])** Let the \( \{N_i(\cdot)\}_{i \in [B]} \) process be defined as above. One can order the arrival times of the \( B \) bins in increasing order (up to their first accumulation point, if they do accumulate) so that \( T_1 < T_2 < \ldots \) is the resulting sequence. The distribution of

\[
\{I_m = (N_1(T_m), N_2(T_m), \ldots, N_B(T_m))\}_{m \in \mathbb{N}}
\]

is the same as that of a balls-in-bins process with feedback function \( f \) and initial conditions \((a_1, a_2, \ldots, a_B)\).
One can prove this result\(^2\) as follows. First, notice that the first arrival time \(T_1\) is the minimum of \(X(j, a_j)\), \((1 \leq j \leq B)\). By the minimum property presented above, the probability that bin \(i\) is the one at which the arrival happens is like the first arrival probability in the corresponding balls-in-bins process with feedback:

\[
\Pr\left( X(i, a_i) = \min_{1 \leq j \leq B} X(j, a_j) \right) = \frac{f(a_i)}{\sum_{j=1}^{B} f(a_j)}.
\] (10)

More generally, let \(t \in \mathbb{R}^+\) and condition on \((N_i(t))_{i=1}^{B} = (b_i)_{i=1}^{B} \in \mathbb{N}^B\), with \(b_i \geq a_i\) for each \(i\) (in which case the process has not blown up). This amounts to conditioning on

\[
\forall i \in [B] \sum_{j=a_i}^{b_i-1} X(i, b_i) \leq t < \sum_{j=a_i}^{b_i} X(i, b_i).
\]

From the lack of memory property of exponentials, one can deduce that the first arrival after time \(t\) at a given bin \(i\) will happen at a \(\exp(f(b_i))\)-distributed time, independently for different bins. This almost takes us back to the situation of (10), with \(b_i\) replacing \(a_i\), and we can similarly deduce that bin \(i\) gets the next ball with the desired probability,

\[
\frac{f(b_i)}{\sum_{j=1}^{B} f(b_j)}.
\]

3.2 On the eventual leadership event

Before we move on to the main proofs, let us briefly discuss how the event \(\text{ELead}\) corresponding to eventual leadership by bin 1 can be expressed via the exponential embedding. We use the same notation and random variables introduced above, and in particular we use the embedded version of the balls-in-bins process defined above. However, we restrict ourselves to the \(B = 2\) case with

\[
\sum_{j=1}^{+\infty} f(j)^{-2} < +\infty.
\] (11)

Notice that this condition implies we are either in the monopolistic or in the eventual leadership regimes. Assume we start the process from state \((x, y) \in \mathbb{N}^2\) with \(x < y\) (i.e. bin 1 has less balls than bin 2). The event \(\text{ELead}\) is given by

\[
\text{ELead} \equiv \{ \exists m \geq 0 \forall M \geq m I_m(1) > I_m(2) \}.
\]

\(^2\)The exact attribution of this result is somewhat confusing. Ref. 7 cites the work of Davis 4 on reinforced random walks, where it is in turn attributed to Rubin.
This can be restated as follows. For $i \in \{1, 2\}$, let $U^{(i)}_r$ be the first $m \in \mathbb{N}$ such that $I_m(i) = r$, or set $U^{(i)}_r = +\infty$ if no such $m$ exists. Then

$$E_{\text{Lead}} \equiv \{ \exists r \geq 0 : \forall R \geq r \, U^{(1)}_R < U^{(2)}_R \}. $$

This carries over to the continuous-time process, in which the time it takes for bin 1 to reach level $R$ is $\sum_{j=x}^{R-1} X(1, j)$, and the analogous time for bin 2 is $\sum_{j=y}^{R-1} X(2, j)$. It is easy to show that

$$E_{\text{Lead}} = \left\{ \exists r \geq 0 : \forall R \geq r \left( \sum_{j=y}^{R-1} (X(1, j) - X(2, j)) - \sum_{j=x}^{y-1} X(2, j) < 0 \right) \right\}. \quad (12)$$

The key now is to show that $\sum_{j=y}^{R-1} (X(1, j) - X(2, j))$ converges as $R \to +\infty$. Indeed, the random variables in the sum,

$$X(i, j), i \in \{1, 2\}, j \geq y$$

are independent, and each term in the sum has zero mean (since $X(1, j) = d = X(2, j) = d \exp(f(j)))$ and variances that add up to (cf. (7))

$$\sum_{j=y}^{R-1} \text{Var}(X(1, j) - X(2, j)) = \sum_{j=y}^{R-1} \frac{2}{f(j)^2} \to +\infty \quad (by \ (11)).$$

Kolmogorov’s Three Series Theorem then implies that $\sum_{j=y}^{+\infty} (X(1, j) - X(2, j)) \in \mathbb{R}$ is a well-defined random variable, as stated. Moreover, the event in (12) holds if and only if $\sum_{j=y}^{+\infty} (X(1, j) - X(2, j)) - \sum_{j=x}^{y-1} X(2, j) < 0$, except for a null event, because $\sum_{j=y}^{+\infty} (X(1, j) - X(2, j))$ and $\sum_{j=x}^{y-1} X(2, j)$ are independent (by the definition of the exponential embeddings) and have no point masses in their distributions. It follows that

$$\Pr_{(x,y)}(E_{\text{Lead}}) = \Pr_{(x,y)} \left( \sum_{j=y}^{+\infty} (X(1, j) - X(2, j)) - \sum_{j=x}^{y-1} X(2, j) < 0 \right). \quad (14)$$

This equation is fundamental to our proofs.
4 Escaping a very likely defeat

In this section we present the proof of Theorem 2. For convenience, we have divided our argument into four parts. In Section 4.1 we outline our proof method, which consists of careful estimates of Laplace transforms. Such estimates are carried out in Section 4.2 and Section 4.3. Those results are collected and applied to the proof of the Theorem in Section 4.4.

4.1 Our method of proof

As usual, our proof begins by writing down the event under consideration in terms of the exponential embedding random variables, using in this case (14) with \((x, y) = [t, \alpha]\).

\[
\Pr_{[t, \alpha]}(E\text{Lead}) = \Pr \left( \sum_{j=[\alpha t]}^{t-[\alpha t]-1} X(1, j) + \sum_{j=t-[\alpha t]}^{+\infty} (X(1, j) - X(2, j)) < 0 \right). \tag{15}
\]

Hence, if we define the following independent random variables

\[
A_t \equiv \sum_{j=[\alpha t]}^{t-[\alpha t]-1} X(1, j), \tag{16}
\]

\[
\Delta_t \equiv \sum_{j=t-[\alpha t]}^{+\infty} (X(1, j) - X(2, j)), \tag{17}
\]

and let \(Z_t \equiv A_t + \Delta_t\), we deduce that

\[
\Pr_{[t, \alpha]}(E\text{Lead}) = \Pr (Z_t < 0) \tag{18}
\]

and that, for all \(\lambda > 0\),

\[
\Pr_{[t, \alpha]}(E\text{Lead}) \leq \Ex [\exp(-\lambda Z_t)] = \Ex [\exp(-\lambda A_t)] \Ex [\exp(-\lambda \Delta_t)]. \tag{19}
\]

Thus the “standard trick” of employing the Laplace transform provides an upper bound on \(E\text{Lead}\). We now use a less standard trick for lower bounding this probability in terms of the same Laplace Transform. Our approach is essentially that of Spencer [13].

Let \(\lambda > 0\) and \(\eta_1 > \eta_2 > 0\) be given. Then

\[
\Pr (Z_t < 0) \geq \Pr (-\eta_1 < Z_t < -\eta_2) \geq e^{-\lambda \eta_1} \Ex \left[ e^{-\lambda Z_t} \chi_{\{-\eta_1 < Z_t < -\eta_2\}} \right], \tag{20}
\]

since \(-\eta_1 < Z_t < \eta_2\) implies that \(\lambda \eta_1 > -\lambda Z_t\). Now let \(\epsilon > 0\) be fixed. Then, if \(Z_t > -\eta_2\), then \(-\lambda Z_t < -(1 - \epsilon)\lambda Z_t - \eta_2\epsilon\), and if \(Z_t < -\eta_1\), then \(-\lambda Z_t < -\lambda(1 + \epsilon)Z_t - \epsilon \eta_1\). Thus

\[
e^{-\lambda Z_t} < e^{-(1+\epsilon)\lambda Z_t - \epsilon \eta_1} + e^{-(1-\epsilon)\lambda Z_t - \epsilon \eta_2}
\]

on the complement of \(\{-\eta_1 < Z_t < -\eta_2\}\). \(\tag{21}\)
Hence
\[ \mathbb{E} \left[ e^{-\lambda Z_t} \mathbb{1}_{Z_t \in (-\eta_1, -\eta_2)} \right] > \mathbb{E} \left[ e^{-\lambda Z_t} \right] - e^{\lambda \eta_1} \mathbb{E} \left[ e^{-(1+\epsilon)\lambda Z_t} \right] - e^{\lambda \eta_2} \mathbb{E} \left[ e^{-(1-\epsilon)\lambda Z_t - \epsilon \eta} \right] \]
whenever the Laplace transforms above are finite. Plugging this last inequality back into \(20\) yields the following general lower bound.

\[
\mathbb{P}(Z_t < 0) \geq e^{-\lambda \eta_1} \mathbb{E} \left[ e^{-\lambda Z_t} \right] \left( 1 - \frac{e^{\lambda \eta_1} \mathbb{E} \left[ e^{-(1+\epsilon)\lambda Z_t} \right] + e^{\lambda \eta_2} \mathbb{E} \left[ e^{-(1-\epsilon)\lambda Z_t} \right]}{\mathbb{E} \left[ e^{-\lambda Z_t} \right]} \right), \quad (22)
\]

How can one use the upper and lower bounds above? For the sake of understanding what follows, let us indicate how inequalities \(19\) and \(22\) are typically employed. Assume that there is a choice of \(\lambda^* = \lambda^*_t\) that minimizes or nearly minimizes the expression

\[
h_t(\lambda) \equiv \frac{1}{t} \ln \mathbb{E} \left[ e^{-\lambda Z_t} \right] \quad (23)
\]

Then one could hope that \(h_t'(\lambda^*) \approx 0, h_t''(\lambda^*) > 0\), and that there would exist a constant \(a\) not depending on \(t\) such that for all \(\delta > 0\) small enough

\[
h_t((1 \pm \delta)\lambda^*) \leq h_t(\lambda) + a\delta^2. \quad (24)
\]

Thus our main expectation is that \(h_t\) has an minimizer \(\lambda^*\) and that it behaves like a “nice” strictly convex function around \(\lambda^*\) in a way that does not depend on \(t\). Now assume that \(\epsilon\) is small enough (but fixed) and we set \(\eta_1 = \sqrt{\epsilon t}/\lambda, \eta_2 = \sqrt{\epsilon t}/\lambda\) in \(20\), then

\[
e^{-\lambda \eta_1 \epsilon} \mathbb{E} \left[ e^{-(1+\epsilon)\lambda Z_t} \right] \mathbb{E} \left[ e^{-\lambda^* Z_t} \right] = \exp \left\{ h_t((1 + \epsilon)\lambda^*) - h_t(\lambda^*) - \epsilon \right\} \quad (25)
\]

and similarly

\[
e^{-\lambda \eta_2 \epsilon} \mathbb{E} \left[ e^{-(1-\epsilon)\lambda Z_t} \right] \mathbb{E} \left[ e^{-\lambda^* Z_t} \right] = e^{-\Omega(t)}. \quad (27)
\]

Thus in this case, \(20\) and \(19\) (with the choice of \(\lambda = \lambda^*\)) would imply that

\[
(1 - e^{-\Omega(t)})e^{h_t(\lambda^*)t - \sqrt{\epsilon t}} \leq \mathbb{P}(Z_t < 0) \leq e^{h_t(\lambda^*)t}. \quad (28)
\]

This last expression would imply that

\[
h_t(\lambda^*) - \sqrt{\epsilon} - o(1) \leq \frac{\ln \mathbb{P}(Z_t < 0)}{t} \leq h_t(\lambda^*) + o(1) \quad \text{for } t \gg 1, \quad (29)
\]
for all small enough $\epsilon$, which shows that

$$\lim_{t \to +\infty} \frac{\ln \Pr \left( Z_t < 0 \right)}{t} - h_t(\lambda^*) = 0.$$  \hspace{1cm} (30)

The above exposition does not exactly correspond to our proof of Theorem 2. However, the spirit of the two proofs is the same. That is, we will show that the logarithms of our Laplace transforms are “strictly convex in the limit”, and use that to prove the desired result.

### 4.2 Analysis of the Laplace Transform

To apply the above method, we need to analyze the Laplace transform of $Z_t = \Delta_t + A_t$. We start with $\mathbb{E} \left[ \exp \left( -\lambda A_t \right) \right]$.

$$\mathbb{E} \left[ \exp \left( -\lambda A_t \right) \right] = \prod_{j=\lceil \alpha t \rceil}^{t-\lceil \alpha t \rceil - 1} \frac{1}{1 + \frac{\lambda}{j^p}}$$

$$= \exp \left\{ \sum_{j=\lceil \alpha t \rceil}^{t-\lceil \alpha t \rceil - 1} \ln \left( \frac{1}{1 + \frac{\lambda}{j^p}} \right) \right\}$$

With foresight, we parameterize $\lambda = \lambda(\rho) = \rho(1 - \alpha)t^p$, for some $\rho > 0$, and deduce that

$$\mathbb{E} \left[ \exp \left( -\lambda A_t \right) \right] = \exp \left\{ \sum_{j=\lceil \alpha t \rceil}^{t-\lceil \alpha t \rceil - 1} \ln \left( \frac{1}{1 + \frac{(1 - \alpha)t^p \rho}{(j/t)^p}} \right) \right\}$$

$$= \exp \left\{ t \times \left[ \frac{1}{t} \sum_{j=\lceil \alpha t \rceil}^{t-\lceil \alpha t \rceil - 1} \ln \left( \frac{1}{1 + \frac{(1 - \alpha)t^p \rho}{(j/t)^p}} \right) \right] \right\}.$$  \hspace{1cm} (35)

It is easy to see that the bracketed term is (close to) a Riemann sum. In fact, the function $u \mapsto \ln \left( \frac{1}{1 + \frac{(1 - \alpha)t^p \rho}{(u/t)^p}} \right)$ is monotone increasing, so for any $\lceil \alpha t \rceil \leq j \leq t - \lceil \alpha t \rceil$,

$$0 \leq \int_{\lceil \alpha t \rceil}^{t-\lceil \alpha t \rceil - 1} \ln \left( \frac{1}{1 + \frac{(1 - \alpha)t^p \rho}{v^p}} \right) dv - \frac{1}{t} \ln \left( \frac{1}{1 + \frac{(1 - \alpha)t^p \rho}{(t/t)^p}} \right)$$

$$= (1 - \alpha) \int_{\lceil \alpha t \rceil}^{t-\lceil \alpha t \rceil - 1} \ln \left( \frac{1}{1 + \frac{\rho}{u^p}} \right) du - \frac{1}{t} \ln \left( \frac{1}{1 + \frac{(1 - \alpha)t^p \rho}{(t/t)^p}} \right)$$

$$\frac{1}{t} \left[ \ln \left( \frac{1}{1 + \frac{(1 - \alpha)t^p \rho}{(t+1/t)^p}} \right) - \ln \left( \frac{1}{1 + \frac{(1 - \alpha)t^p \rho}{(t/t)^p}} \right) \right].$$  \hspace{1cm} (36)
Summing over \( j \) then yields

\[
0 \leq (1 - \alpha) \int_{\lceil \alpha t \rceil}^{\lceil \alpha(t-\lceil \alpha t \rceil) \rceil} \ln \left( \frac{1}{1 + \frac{2u}{\rho}} \right) du - \frac{1}{t} \sum_{j=\lceil \alpha t \rceil}^{t-\lceil \alpha t \rceil-1} \ln \left( \frac{1}{1 + \frac{1 - \alpha \rho}{(j/t)^p}} \right) \leq \frac{1}{t} \left[ \ln \left( \frac{1}{1 + \frac{1 - \alpha \rho}{(t-\lceil \alpha t \rceil)/t}} \right) - \ln \left( \frac{1}{1 + \frac{1 - \alpha \rho}{((\lceil \alpha t \rceil)/t)^p}} \right) \right].
\]

(37)

Thus we deduce that for all \( \delta > 0 \) there exist \( C = C_\delta^{(1)} > 0, T_\delta \in \mathbb{N} \) such that if \( \alpha > \delta \) and \( t \geq T_\delta^{(1)} \), then

\[
-\frac{C_\delta^{(1)}}{t} \leq \frac{1}{t} \sum_{j=\lceil \alpha t \rceil}^{t-\lceil \alpha t \rceil-1} \ln \left( \frac{1}{1 + \frac{(1-\alpha)\rho}{((j-\lceil \alpha t \rceil)/t)^p}} \right) - (1 - \alpha) \int_{\lceil \alpha t \rceil}^{\lceil (1-\alpha) \rceil \lceil \alpha t \rceil \lceil \alpha t \rceil \lceil \alpha t \rceil} \ln \left( \frac{1}{1 + \frac{1 - \alpha \rho}{(j/t)^p}} \right) du \leq \frac{C_\delta^{(1)}}{t}.
\]

(38)

It follows that, for all \( \alpha > \delta \) and \( t \geq T_\delta^{(1)} \)

\[
(1 - \alpha) \int_{\lceil (1-\alpha) \rceil \lceil \alpha t \rceil \lceil \alpha t \rceil \lceil \alpha t \rceil}^{\lceil (1-\alpha) \rceil \lceil \alpha t \rceil \lceil \alpha t \rceil} \ln \left( \frac{1}{1 + \frac{1 - \alpha \rho}{(j/t)^p}} \right) \leq \frac{\ln \mathbb{E} \left[ \exp(-\lambda(\rho)A_t) \right]}{t} \leq (1 - \alpha) \int_{\lceil (1-\alpha) \rceil \lceil \alpha t \rceil \lceil \alpha t \rceil \lceil \alpha t \rceil}^{\lceil (1-\alpha) \rceil \lceil \alpha t \rceil \lceil \alpha t \rceil} \ln \left( \frac{1}{1 + \frac{1 - \alpha \rho}{(j/t)^p}} \right) + \frac{C_\delta^{(1)}}{t}.
\]

(39)

We now consider the Laplace transform of \( \Delta_t \), with the same parametrization \( \lambda = \lambda(\rho) \) as above.

\[
\mathbb{E} \left[ \exp(-\lambda\Delta_t) \right] = \prod_{j=t-\lceil \alpha t \rceil}^{+\infty} \frac{1}{1 - \frac{2u}{\rho}}
\]

(40)

\[
= \exp \left\{ \sum_{j=t-\lceil \alpha t \rceil}^{+\infty} \ln \left( \frac{1}{1 - \frac{(1-\alpha)\rho}{((j-\lceil \alpha t \rceil)/t)^p}} \right) \right\}
\]

(41)

\[
= \exp \left\{ \frac{1}{t} \sum_{j=t-\lceil \alpha t \rceil}^{+\infty} \ln \left( \frac{1}{1 - \frac{(1-\alpha)\rho}{((j-\lceil \alpha t \rceil)/t)^p}} \right) \right\}.
\]

(42)

Notice that this Laplace transform is infinite for \( \lambda \geq t - \lceil \alpha t \rceil \), and we therefore place the restriction \( \rho \in (0,1) \) to ensure that does not happen for all large enough \( t \). As above,
we have a something close to a Riemmann sum between brackets. Indeed, since the map \( u \mapsto \ln(1/(1 - (1 - \alpha)^{2p} \rho^2 / u^{2p})) \) is monotone decreasing, for all \( j \geq t - \lceil \alpha t \rceil \)

\[
0 \leq \frac{1}{t} \ln \left( \frac{1}{1 - (1 - \alpha)^{2p} \rho^2 / (j/t)^{2p}} \right) - \int_{t}^{j+1 \over t} \ln \left( \frac{1}{1 - (1 - \alpha)^{2p} \rho^2 / u^{2p}} \right) \, du
\]

\[
\leq \frac{1}{t} \ln \left( \frac{1}{1 - (1 - \alpha)^{2p} \rho^2 / (j/t)^{2p}} \right) - (1 - \alpha) \int_{j \over (1-\alpha)t}^{(j+1 \over t) \over 2} \ln \left( \frac{1}{1 - \rho^2 / u^{2p}} \right) \, du
\]

\[
\leq \left[ \frac{1}{t} \ln \left( \frac{1}{1 - (1 - \alpha)^{2p} \rho^2 / (j/t)^{2p}} \right) - \frac{1}{t} \ln \left( \frac{1}{1 - (1 - \alpha)^{2p} \rho^2 / ((j+1)/t)^{2p}} \right) \right]. \tag{43}
\]

Summing over \( j \), we conclude that

\[
0 \leq \frac{1}{t} \sum_{j=t-\lceil \alpha t \rceil}^{+\infty} \ln \left( \frac{1}{1 - (1 - \alpha)^{2p} \rho^2 / (j/t)^{2p}} \right) - (1 - \alpha) \int_{t-\lceil \alpha t \rceil}^{+\infty} \ln \left( \frac{1}{1 - \rho^2 / u^{2p}} \right) \, du
\]

\[
\leq \frac{1}{t} \ln \left( \frac{1}{1 - (1 - \alpha)^{2p} \rho^2 / ((t-\lceil \alpha t \rceil)/t)^{2p}} \right). \tag{44}
\]

This implies that for each \( \eta \in (0, 1) \) there exist \( C^{(2)}_{\eta} > 0 \) and \( T^{(2)}_{\eta} \in \mathbb{N} \) such that for all \( t \geq T^{(2)}_{\eta} \), if \( 0 < \rho < 1 - \eta \)

\[
- \frac{C^{(2)}_{\eta}}{t} \leq \frac{1}{t} \ln \mathbb{E}[\exp(-\lambda \Delta_t)] - (1 - \alpha) \int_{t-\lceil \alpha t \rceil}^{+\infty} \ln \left( \frac{1}{1 - \rho^2 / u^{2p}} \right) \, du
\]

\[
\leq \frac{C^{(2)}_{\eta}}{t}. \tag{45}
\]

To conclude the section, let

\[
g_t(\rho, \alpha) \equiv \ln \mathbb{E}[\exp\{-\rho(1 - \alpha)p/\rho^2[Z_t]\}] \tag{46}
\]

\[
= \sum_{j=\lceil \alpha t \rceil}^{t-\lceil \alpha t \rceil} \ln \left( \frac{1}{1 + (1 - \alpha)^{2p} \rho / (j/t)^{2p}} \right)
\]

\[
+ \sum_{j=t-\lceil \alpha t \rceil}^{+\infty} \ln \left( \frac{1}{1 - (1 - \alpha)^{2p} \rho^2 / (j/t)^{2p}} \right), \quad \alpha \in (0, 1/2), \rho \in (0, 1).
\]
Also define, for the same range of $\alpha, \rho$,

$$F_p(\rho, \alpha) \equiv (1-\alpha) \int_{\frac{1}{1-\alpha}}^{1} \ln \left( \frac{1}{1 + \frac{\rho}{u^p}} \right) du \quad (47)$$

$$+ (1-\alpha) \int_{1}^{+\infty} \ln \left( \frac{1}{1 - \frac{\rho}{u^p}} \right) du.$$ 

From the above, we deduce that for any $\delta > 0$, if we let $C_{\delta, \eta} \equiv C_{\delta}^{(1)} + C_{\eta}^{(2)}$ then $T_{\delta, \eta} \equiv T_{\delta}^{(1)} + T_{\eta}^{(2)}$, then

$$\forall \alpha \in (\delta, 1/2), \rho \in (0, 1), t \geq T_{\delta}, \quad \left| \frac{g_t(\rho, \alpha)}{t} - F_p(\rho, \alpha) \right| \leq \frac{C_{\delta, \eta}}{t}. \quad (48)$$

4.3 The asymptotic form of the Laplace transform

We now analyze the function $F_p(\rho, \alpha)$ introduced above, as well as its minimum over $\rho$, which we will prove to precisely the function in the statement of the Theorem.

$$c_p(\alpha) = \inf_{\rho \in (0, 1)} F_p(\rho, \alpha) \quad (\alpha \in (0, 1/2)). \quad (49)$$

We have the following formulae for all $\alpha \in (0, 1/2)$

$$\lim_{\rho \downarrow 0} F_p(\rho, \alpha) = 0, \quad (50)$$

$$\lim_{\rho \uparrow 1} F_p(\rho, \alpha) = +\infty, \quad (51)$$

$$\frac{1}{1-\alpha} \frac{\partial F_p}{\partial \rho}(\rho, \alpha) = - \int_{\frac{1}{1-\alpha}}^{1} \frac{1}{u^p + \rho} du + \int_{1}^{+\infty} \frac{2\rho}{u^{2p} - \rho^2} du, \quad (52)$$

$$\frac{1}{1-\alpha} \frac{\partial^2 F_p}{\partial \rho^2}(\rho, \alpha) = \int_{\frac{1}{1-\alpha}}^{1} \frac{1}{(u^p + \rho)^2} du + \int_{1}^{+\infty} \frac{2u^{2p} + 2\rho^2}{(u^{2p} - \rho^2)^2} du. \quad (53)$$

Notice, then, that

$$\forall \alpha > 0 \quad \inf_{\rho \in (0, 1)} \frac{\partial^2 F_p}{\partial \rho^2}(\rho, \alpha) \geq \inf_{\rho \in (0, 1)} (1-\alpha) \int_{\frac{1}{1-\alpha}}^{1} \frac{1}{(u^p + \rho)^2} du$$

$$= \inf_{\rho \in (0, 1)} (1-\alpha) \int_{\frac{1}{1-\alpha}}^{1} \frac{1}{(u^p + 1)^2} du \equiv a_\alpha > 0, \quad (54)$$

hence $F_p(\cdot, \alpha)$ is a strictly convex function of $\rho$, for any $\alpha \in (0, 1/2)$. Moreover,

$$\lim_{\rho \downarrow 0} \frac{\partial^2 F_p}{\partial \rho^2}(\rho, \alpha) = -(1-\alpha) \int_{\frac{1}{1-\alpha}}^{1} \frac{1}{u^p} du < 0, \quad (55)$$

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The two last assertions prove that for any fixed \( \alpha \):

1. \( F_p(\cdot, \alpha) \) is a strictly convex function of \( \rho \);
2. \( F_p(\rho, \alpha) < 0, \partial_\rho F_p(\rho, \alpha) < 0 \) for all small enough \( \rho \);
3. \( F_p(\cdot, \alpha) \) thus has an unique minimum over \((0, 1)\), and this minimum is achieved at the unique value \( \rho^* = \rho^*(\alpha) \) such that
   \[
   \frac{\partial F_p}{\partial \rho}(\rho^*(\alpha), \alpha) = 0; \tag{56}
   \]
4. by the definition of \( c_p(\alpha) \) and the above items,
   \[
   c_p(\alpha) = \inf_{\rho \in (0, 1)} F_p(\rho, \alpha) = F_p(\rho^*(\alpha), \alpha) < 0; \tag{57}
   \]
5. by strict convexity (and using the definition of \( a_\alpha > 0 \) above) and the fact that \( \partial_\rho F_p(\rho^*(\alpha), \alpha) = 0 \), then there exists a value \( b = b(\alpha) \) depending continuously on \( \alpha \) such that, for all \( \epsilon > 0 \) small enough,
   \[
   F_p((1 \pm \epsilon)\rho^*(\alpha), \alpha) \leq F_p(\rho^*(\alpha), \alpha) + b_\alpha \epsilon^2 = c_p(\alpha) + b_\alpha \epsilon^2; \tag{58}
   \]
6. in fact, we can strengthen the previous item and say that for all \( \delta \in (0, 1/2) \) there exists \( B_\delta, \epsilon_\delta > 0 \) such that
   \[
   \forall \alpha \in (\delta, 1/2 - \delta), \epsilon \in (0, \epsilon_\delta), F_p((1 \pm \epsilon)\rho^*(\alpha), \alpha) \leq c_p(\alpha) + B_\delta \epsilon^2; \tag{59}
   \]
We now prove that \( c_p \) is a smooth function of \( \alpha \) with a positive derivative. To prove smoothness, we only need to show that \( \rho^* \) is a smooth function of \( \alpha \), since \( c_p(\alpha) \) is given by the formula in 56. But recall that we have shown that \( \rho^* \) is uniquely defined by the equation
   \[
   G(\rho^*(\alpha), \alpha) \equiv \frac{\partial F_p}{\partial \alpha}(\rho^*(\alpha), \alpha) = 0, \tag{60}
   \]
and we know that
   \[
   \frac{\partial G}{\partial \rho}(\rho, \alpha) = \frac{\partial^2 F_p}{\partial^2 \rho}(\rho, \alpha) \geq a_\alpha > 0. \tag{61}
   \]
Hence the Implicit Function Theorem applies [12], and implies that \( \rho^* \) is indeed a smooth function of \( \alpha \).

To prove that \( c'_p(\alpha) > 0 \), we first differentiate \( F_p(\rho, \alpha) \) with respect to \( \alpha \).
   \[
   \frac{\partial F_p}{\partial \alpha}(\rho, \alpha) = (1 - 2\alpha) \log \left( 1 + \frac{\rho}{\alpha} \right) - \frac{1}{1 - \alpha} F_p(\rho, \alpha). \tag{62}
   \]
Now notice that, by the chain rule,
\[
c'_p(\alpha) = \frac{d}{d\alpha}(F_p(\rho^*(\alpha), \alpha)) = \mathcal{O}(\alpha) + \frac{\partial F_p}{\partial \rho}(\rho^*(\alpha), \alpha) = \mathcal{O}(\alpha) + \frac{\partial F_p}{\partial \rho}(\rho^*(\alpha), \alpha)
\]
(63)

\[
\text{(via } \frac{\partial F}{\partial \rho}(\rho^*(\alpha), \alpha) = 0 \text{)} = \frac{\partial F_p}{\partial \rho}(\rho^*(\alpha), \alpha)
\]
(64)

\[
(\text{by (62)}) = (1 - 2\alpha) \log \left(1 + \frac{\rho}{(1 - \alpha)^p}\right) - \frac{F_p(\rho, \alpha)}{1 - \alpha}
\]
(65)

\[
\text{(omitting a } > 0 \text{ term)} \geq -\frac{c_p(\alpha)}{1 - \alpha}
\]
(66)

Finally, we show that \(\rho^*(\alpha)\) is a non-increasing function of \(\alpha\). (70)

This is important because it implies that, for all \(\alpha > \delta > 0\) and all small enough \(\epsilon\),
\[
(1 + \epsilon)\rho^*(\alpha) \leq (1 + \epsilon)\rho^*(\delta) \leq \eta < 1
\]
for some \(\eta = \eta^\delta\) depending on \(\delta\) only. In conjunction with (48), this will imply that
\[
\forall \alpha \in (\delta, 1/2), \rho \in (0, 1), t \geq T_\delta, \epsilon \in [0, \epsilon^\delta] \quad \left| g_t((1 \pm \epsilon)\rho^*(\alpha), \alpha) - F_p((1 \pm \epsilon)\rho^*(\alpha), \alpha) \right| \leq \frac{C^\delta}{t},
\]
(71)

where \(C^\delta \equiv C^\delta_{\delta, \eta^\delta}\) depends on \(\delta\) only.

To prove (70), we notice that
\[
\frac{\partial^2 F_p}{\partial \rho^2}(\rho^*(\alpha), \alpha) = (1 - 2\alpha) \log \left(1 + \frac{\rho}{(1 - \alpha)^p}\right) - \frac{1}{1 - \alpha} F_p(\rho, \alpha)
\]
(72)

\[
\frac{\partial^2 F_p}{\partial \rho^2}(\rho^*(\alpha), \alpha)
\]
(73)
where (73) follows from
\[ \partial_{\rho} F_p(\rho^*(\alpha), \alpha) = 0. \]

Hence, if \( \beta > 0 \) is small enough
\[ \frac{\partial F_p(\rho^*(\alpha), \alpha + \beta)}{\partial \rho} > \frac{\partial F_p(\rho^*(\alpha), \alpha)}{\partial \rho}. \] (74)

But by the strict convexity of \( F_p(\cdot, \alpha + \beta) \),
\[ \frac{\partial F_p(\rho, \alpha + \beta)}{\partial \rho} < \frac{\partial F_p(\rho^*(\alpha + \beta), \alpha + \beta)}{\partial \rho} \]
for all \( \rho < \rho^*(\alpha + \beta) \). Hence \( \rho^*(\alpha + \beta) < \rho^*(\alpha) \) whenever \( \beta \) is small enough. This finishes the proof.

4.4 Proof of Theorem 2

We now have all the tools necessary to prove Theorem 2.

\[ \text{Proof: (of Theorem 2)} \]

Let us now apply the upper and lower bounds (18) and (22) presented above. We will assume that \( \delta \leq \alpha \leq 1/2 - \delta \) for some constant \( \delta > 0 \), and prove bounds on \( \Pr_{[t,\alpha]}(\text{ELead}) \) that are uniform on that range of \( \alpha \).

In the current setting, \( Z_t = \Delta_t + A_t \) and we have defined
\[ g_t(\rho, \alpha) = \ln \mathbb{E} \left[ e^{-\lambda Z_t} \right] |_{\lambda = \rho(1-\alpha)t^p}, \]
hence for all fixed \( \rho \in (0, 1), \alpha \in (\delta, 1/2 - \delta) \) and \( t \geq T_\delta \) (cf. (48))
\[ \Pr_{[t,\alpha]}(\text{ELead}) = \Pr(Z_t < 0) \leq \exp(g_t(\rho, \alpha)). \] (75)

In particular, setting \( \rho = \rho^*(\alpha) \) and \( t, \alpha \) as above, we can use (71) to bound
\[ \ln \Pr_{[t,\alpha]}(\text{ELead}) \leq \exp(c_p(\alpha) t + C_\delta). \] (76)

The above upper bound can be matched via the lower bound method in (22). Let \( \epsilon > 0 \). One can set \( \lambda = \rho^*(\alpha)[(1 - \alpha)t]^p, \eta_1 = \sqrt{\epsilon t}/\lambda \) and \( \eta_2 = \sqrt{\epsilon t}/2\lambda \) in (22) to deduce
\[ \Pr_{[t,\alpha]}(\text{ELead}) = \Pr(Z_t < 0) \]
\[ \geq e^{g_t(\rho^*, \alpha)-\sqrt{\eta_1}} \left( 1 - \frac{e^{g_t((1+\epsilon)\rho^*, \alpha) - e^{3/2t}}}{e^{g_t(\rho^*)}} - \frac{e^{g_t((1-\epsilon)\rho^*, \alpha) - e^{3/2t}/2}}{e^{g_t(\rho^*)}} \right). \] (77)
Now notice that and any $0 < \epsilon < \epsilon_\delta$ (cf. (59) and (71)), one has that

$$g_t(\rho^*, \alpha) = c_p(\alpha) t \pm C_\delta, \quad (78)$$

$$g_t((1 + \epsilon)\rho^*, \alpha) = F_p((1 + \epsilon)\rho^*, \alpha) t \pm C_\delta \quad (by \ (59) \ and \ (71))$$

(79)

(for small enough $\epsilon$) \n
$$\leq c_p(\alpha) t + \frac{\epsilon^{3/2}}{4} t + C_\delta,$$

(80)

$$g_t((1 - \epsilon)\rho^*, \alpha) = F_p((1 - \epsilon)\rho^*, \alpha) t + C_\delta \quad (by \ (59) \ and \ (71))$$

(81)

(for small enough $\epsilon$) \n
$$\leq c_p(\alpha) t + \frac{\epsilon^{3/2}}{4} t + C_\delta.$$

Substitution back into (77) yields

$$\Pr_{[t, \alpha]}(E_{\text{Lead}}) \geq (1 - 2e^{C_\delta - \epsilon^{3/2}t/4}) \exp\{c_p(\alpha)t - C_\delta - \sqrt{\epsilon}t\}, \ t \geq T_\delta, \quad (81)$$

for any small enough $\epsilon$ and any $t \geq T_\delta$. In particular, if we set $\epsilon \equiv \left[(C_\delta + \ln 4)(4t)^{-2/3}\right]$, then $\epsilon \searrow 0$ as $t \to +\infty$, so that for all large enough $t$ the above formulae apply and

$$\Pr_{[t, \alpha]}(E_{\text{Lead}}) \geq \frac{\exp(c_p(\alpha)t - C'_\delta t^{2/3})}{2}, \quad (82)$$

for some constant $C'_\delta \geq C_\delta$. Redefining $C_\delta$ and $T_\delta$ if necessary, we can then conclude (using (76) and (82)) that

$$\forall \alpha \in (\delta, 1/2 - \delta), \forall t \geq T_\delta, \ c_p(\alpha) t - C_\delta t^{2/3} \leq \ln \Pr_{[t, \alpha]}(E_{\text{Lead}}) \leq c_p(\alpha) t + C_\delta. \quad (83)$$

Since we have already shown that $c_p$ is smooth and monotone-increasing in $\alpha$ (cf. Section 4.3), the Theorem follows. \hspace{1cm} \Box

5 The most likely escape path

This section is dedicated to Theorem 3. After some preliminaries are considered in Section 5.1, we then estimate (in Section 5.2) the transition probabilities of the balls-in-bins process conditioned on $E_{\text{Lead}}$. Those estimates are used to show in Section 5.3 that for short enough times, the conditioned process evolves from a state $[t, \alpha]$ to a state $\approx [(1 + \eta)t, \alpha + gp(\alpha)\eta]$, thus staying close to the tangent of the ODE. The final steps of the proof are presented in Section 5.4, and a Lemma used in Section 5.3 is proven in Section 5.5.
5.1 Preliminaries

According to Theorem 2, the map
\[ c_p : (0, \frac{1}{2}) \rightarrow \mathbb{R}^{-} \quad \alpha \mapsto \lim_{t \to +\infty} \ln \frac{\Pr_{[t, \alpha]}(E_{\text{Lead}})}{t} \]  
(84)
is infinitely differentiable. In particular, this means that, for all \( \delta \in (0, 1/2) \), the suprema
\[ D_{\delta}^{(r)} \equiv \sup_{\delta \leq \alpha \leq \frac{1}{2} - \delta} \left| \frac{1}{r!} c_p' \right| (\alpha) \quad (r \in \mathbb{N} \cup \{0\}) \]  
(85)
are all finite. Moreover, \( c_p' (\alpha) > 0 \) on \((0, 1/2)\), and
\[ d_{\delta}^{(1)} \equiv \min_{\delta \leq \alpha \leq \frac{1}{2} - \delta} c_p' (\alpha) > 0. \]  
(86)
Theorem 2 also tells us that, for \( \delta \) as above, there exist \( T_{\delta} \in \mathbb{N} \) and \( C_{\delta} \in \mathbb{R}^{+} \) such that for all \( t \geq T_{\delta} \) and all \( \alpha \in (\delta, 1/2 - \delta) \),
\[-C_{\delta} t^{2/3} \leq \ln \Pr_{[t, \alpha]}(E_{\text{Lead}}) - c_p (\alpha) t \leq C_{\delta}. \]  
(87)
Therefore, if \( \alpha, \alpha' \in (\delta, 1/2 - \delta) \) and \( t, t' \geq T_{\delta} \)
\[ e^{c_p(\alpha') (t' - t) - C_{\delta}(1 + t'^{2/3})} \leq \frac{\Pr_{[t', \alpha']} (E_{\text{Lead}})}{\Pr_{[t, \alpha]} (E_{\text{Lead}})} \leq e^{c_p(\alpha') (t' - t) - C_{\delta}(1 + t^{2/3})} \]  
(88)
Moreover, if we also have that \( \alpha - \alpha' = \epsilon, t' - t = \eta t \) for \( \epsilon, \eta < 1/4 \) (say), then
\[ \left| \frac{1}{t} \ln \frac{\Pr_{[t', \alpha']} (E_{\text{Lead}})}{\Pr_{[t, \alpha]} (E_{\text{Lead}})} - \eta c_p (\alpha) + \epsilon c_p' (\alpha)(1 + \eta) \right| \leq D_{\delta}^{(2)} \epsilon^2 + D_{\delta}^{(1)} \epsilon \eta + C_{\delta} \left( \frac{1}{t} + \frac{1}{t^{1/3}} \right). \]  
(89)
This last equation will be repeatedly used in what follows.

5.2 Transitions conditioned on ELead

Define
\[ [t, \alpha] \mapsto [t^*, \alpha^*] \quad \alpha, \alpha^* \in (0, 1), t, t^* \in \mathbb{N}, t < t^*, \]  
(90)
to be the event that the initial state of the process is \([t, \alpha]\), and that at time \( t^* - t \) the state of the process is \([t^*, \alpha^*]\). The goal of this section is to estimate the probability of the transition
\[ \Pr_{[t, \alpha]} ([t, \alpha] \mapsto [t^*, \alpha^*] | E_{\text{Lead}}) \]  
(91)
for the case when \( t \leq t^* \leq (1 + \eta)t \) for some \( \eta > 0 \) and \( t \) is large, and we assume (for simplicity) that \( \alpha t, \alpha^*t^* \) are integers. We will require that \( \alpha \pm \eta \in (\delta, 1/2 - \delta) \) for some \( 0 < \delta < 1/2 \), so that for all \( c \in (\alpha - \eta, \alpha + \eta) \) the bounds on \( \Pr_{[t,c]}(\text{ELead}) \) coming from the previous section apply with the same value of \( \delta \) (and thus the same \( C_\delta, T_\delta \)). Notice that we can assume that

\[
\delta \leq \alpha - \eta \leq \frac{\alpha}{1 + \eta} \leq \alpha^* \leq \alpha + \eta \leq 1/2 - \delta, \tag{92}
\]

otherwise the given probability is 0. One has that

\[
\Pr_{[t*,\alpha^*]}(\text{ELead}) = \exp(c_p(\alpha)(t^* - t) + c'_p(\alpha)(\alpha^* - \alpha)t^* \pm D_\delta^{(2)} + D_\delta^{(1)}\eta^2 t). \tag{94}
\]

It will be convenient to have the above equation in a slightly different form,

\[
\frac{\Pr_{[t*,\alpha^*]}(\text{ELead})}{\Pr_{[t,\alpha]}(\text{ELead})} = \exp(c_p(\alpha)(t^* - t) - \alpha c'_p(\alpha)(t^* - t) \pm (D_\delta^{(2)} + D_\delta^{(1)}\eta^2 t)(e^{c'_p(\alpha)}\alpha^*t^* - \alpha t) \tag{95}
\]

As for

\[
\Pr_{[t,\alpha]}([t, \alpha] \mapsto [t^*, \alpha^*]) \tag{96}
\]

notice that there \( t^* - t = \eta t \) and that \( \alpha^*t^* - \alpha t \), hence there exist

\[
\begin{pmatrix}
  t^* - t \\
  \alpha^*t^* - \alpha t
\end{pmatrix}
\]

ways of moving from state \([t, \alpha]\) to state \([t^*, \alpha^*]\). For each one of those ways, each step in which a ball is added to bin 1 has probability

\[
\frac{(\alpha t + a)^p}{(\alpha t + a)^p + ((1 - \alpha)t + b)^p}
\]

of occurring (for some \( 0 \leq a, b \leq t^* - t \leq \eta t \)), whereas steps in which a ball is added to bin 2 have probability

\[
\frac{((1 - \alpha)t + b)^p}{(\alpha t + a)^p + ((1 - \alpha)t + b)^p}.
\]
for \(a, b\) as above. There exist absolute constants \(R_\delta\) and \(\eta_\delta\) only depending on \(\delta\) such that, if \(0 < \eta < \eta_\delta\):

\[
e^{-R_\delta \eta} \frac{\alpha^p}{\alpha^p + (1 - \alpha)^p} \leq \frac{(\alpha t + a)^p}{(\alpha t + a)^p + ((1 - \alpha)t + b)^p} \leq e^{R_\delta \eta} \frac{\alpha^p}{\alpha^p + (1 - \alpha)^p},
\]

and

\[
e^{-R_\delta \eta} \frac{(1 - \alpha)^p}{\alpha^p + (1 - \alpha)^p} \leq \frac{((1 - \alpha)t + b)^p}{(\alpha t + a)^p + ((1 - \alpha)t + b)^p} \leq e^{R_\delta \eta} \frac{(1 - \alpha)^p}{\alpha^p + (1 - \alpha)^p},
\]

and therefore, any path moving connecting state \([t, \alpha]\) to state \([t^*, \alpha^*]\) has probability

\[
e^{-R_\delta \eta(t^* - t)} \left(\frac{\alpha^p}{\alpha^p + (1 - \alpha)^p}\right)^{\alpha^* t^* - \alpha^* t} \left(1 - \frac{\alpha^p}{\alpha^p + (1 - \alpha)^p}\right)^{t^* - t - (\alpha^* t^* - \alpha^* t)}, \quad (97)
\]

(any such path must have \(\alpha^* t^* - \alpha t^*\) steps bin 1 receives a ball, out of a total of \(t^* - t\) steps).

Letting

\[
\rho(\alpha) \equiv \frac{\alpha^p}{\alpha^p + (1 - \alpha)^p},
\]

we conclude that

\[
\text{Pr}_{[t, \alpha]}([t, \alpha] \mapsto [t^*, \alpha^*]) = e^{R_\delta \eta^2 t^*} \times \left(\frac{t^* - t}{\alpha^* t^* - \alpha t}\right) \rho(\alpha)^{\alpha^* t^* - \alpha t} (1 - \rho(\alpha))^{t^* - t - (\alpha^* t^* - \alpha^* t)}, \quad (98)
\]

and that (cf. (95))

\[
\text{Pr}_{[t, \alpha]}([t, \alpha] \mapsto [t^*, \alpha^*] \mid \text{ELead}) = \exp\{c_p(\alpha)(t^* - t) - \alpha c'_p(\alpha)(t^* - t) \pm (R_\delta + D^{(2)}_\delta + D^{(1)}_\delta) \eta^2 t\}
\]

\[
\times \left(\frac{t^* - t}{\alpha^* t^* - \alpha t}\right) \left|\rho(\alpha)\right| c'_p(\alpha)^{\alpha^* t^* - \alpha t} (1 - \rho(\alpha))^{t^* - t - (\alpha^* t^* - \alpha^* t)}. \quad (99)
\]

for all \(t \geq T_\delta, 0 < \eta < \eta_\delta, t \leq t^* \leq (1 + \eta)t\) and \(\alpha^*\) as above.

### 5.3 The most likely transitions

We continue with the same setup as above, and state a useful lemma that we prove in Section 5.5.

**Lemma 1** Let \(a > 0\), \(0 < \rho < 1\) and an integer \(m \in \mathbb{N}\), \(m \geq 2\) be given. Define, for \(n \in [m] \cup \{0\},

\[
b(n) \equiv \binom{m}{n} \rho^n a^{m-n}
\]

Then
1. the sequence \( \{b(n) \mid 0 \leq n \leq m\} \) is unimodal;

2. \( \max_{0 \leq n \leq m} b(n) \) is achieved at \( n_0 = \left\lfloor \frac{\rho a m + (1 - \rho)}{\rho a + (1 - \rho)} \right\rfloor \);

3. for all \( K > 0 \), \( \sum_{|n - n_0| > K \sqrt{m}} b(n) \leq m b(n_0) e^{-\frac{K^2}{2U_\delta}} \).

To apply this lemma, we give new names to familiar quantities.

\[
\begin{align*}
m &= m(t^*, t) = t^* - t \leq \eta t \\
n &= n(t, t^* \alpha, \alpha^*) = \alpha^* t^* - \alpha t \\
a &= a(\alpha) = e^{c_p(\alpha)} \\
\rho &= \rho(\alpha) \\
n_0 &= n_0(\alpha, t^*, t) = \left\lfloor \frac{\rho a m + (1 - \rho)}{\rho a + (1 - \rho)} \right\rfloor.
\end{align*}
\]

With those definitions,

\[
\frac{n_0}{m} \leq \frac{\rho a m + (1 - \rho)}{(\rho a + 1 - \rho)m} + \frac{1}{m} \leq 1 - \frac{(1 - \rho)(m - 1)}{(\rho a + 1 - \rho)m}.
\]

For \( \alpha \in (\delta, 1/2 - \delta) \), \( a \leq e^{d(2)} \). Moreover, \( \rho(\alpha) \) is a continuous function of \( \alpha \) that is between \( \rho(\delta) > 0 \) and \( 1/2 \) for \( \alpha \) as above. Thus there exists a constant \( U = U_\delta \in (0, 1) \) such that

\[
\frac{n_0}{m} \leq 1 - U_\delta
\]

and therefore the Lemma implies that, for \( K > 0 \)

\[
\sum_{|n - n_0| > K \sqrt{m}} b(n) \leq m b(n_0) e^{-\frac{K^2}{2U_\delta}}.
\]

Now notice that, in the present case, (99) implies that

\[
b(n) = \Pr_{[t, \alpha]} ([t, \alpha] \mapsto [t^*, \alpha] \mid \text{ELead}) \frac{\exp\{c_p(\alpha)(t^* - t) - \alpha c_p'(\alpha)(t^* - t) \pm (R_\delta + D^{(2)}_\delta + D^{(1)}_\delta) \eta^2 t\}}{\exp\{c_p(\alpha)(t^* - t) - \alpha c_p'(\alpha)(t^* - t) \pm (R_\delta + D^{(2)}_\delta + D^{(1)}_\delta) \eta^2 t\}}.
\]

It follows that for all \( K > 0 \), \( \alpha \in (\delta, 1/2 - \delta) \), \( t \geq T_\delta \), \( 0 < \eta < \eta_\delta \), \( t \leq t^* \leq (1 + \eta)t \) and \( \alpha^* \) as above

\[
\Pr_{[t, \alpha]} ([t, \alpha] \mapsto [t^*, \alpha] \mid \text{ELead}) \leq (t^* - t) \exp \left\{ 2(R_\delta + D^{(2)}_\delta + D^{(1)}_\delta) \eta^2 t - \frac{K^2}{2U_\delta} \right\}.
\]
To use this formula, we will assume that \( t^* - t = \eta t \) (i.e., equality instead of the above inequality), and then set \( K = \sqrt{\eta t} \). Then

\[
\Pr_{[t, \alpha]} \left( [t, \alpha] \mapsto [t^*, \alpha] \text{ with } |n(t, t^*, \alpha, \alpha^*) - n_0(\alpha, t^*, t)| > \eta^{3/2} t \mid \text{ELead} \right) \\
\leq \eta t \exp \left\{ 2(R_\delta + D_\delta^{(2)} + D_\delta^{(1)}) \eta^2 t - \frac{\eta t}{2U_\delta} \right\}, \tag{108}
\]

and (by making \( \eta_\delta \) smaller if necessary) we can ensure that there exists \( V_\delta > 0 \) such that, with \( 0 < \eta < \eta_\delta \),

\[
\Pr_{[t, \alpha]} \left( [t, \alpha] \mapsto [t^*, \alpha] \text{ with } |n(t, t^*, \alpha, \alpha^*) - n_0(\alpha, t^*, t)| > \eta^{3/2} t \mid \text{ELead} \right) \\
\leq e^{-V_\delta \eta t}. \tag{109}
\]

To conclude this part, we look at how \( \alpha^* \) behaves when

\[
|n(t, t^*, \alpha, \alpha^*) - n_0(\alpha, t^*, t)| \leq \eta^{3/2} t.
\]

In that case,

\[
\alpha^* = \frac{\alpha t + n}{t^*} = \frac{\alpha}{1 + \eta} + \frac{1}{1 + \eta} \frac{n}{t}.
\]

As defined above,

\[
n_0 \frac{t}{t} = \frac{1}{t} \left[ \frac{\rho a \eta t + (1 - \rho)}{(\rho a + 1 - \rho)} \right] = \frac{\rho a \eta}{\rho a + 1 - \rho} \pm \frac{1 - \rho}{(\rho a + 1 - \rho)t}.
\]

Hence, by making \( \eta_\delta \) even smaller if necessary, we can guarantee that

\[
\alpha^* = \alpha (1 - \eta) + \frac{\rho a \eta}{\rho a + 1 - \rho} \pm \left( 2\eta^2 + \frac{1 - \rho}{(\rho a + 1 - \rho)t} + \eta^{3/2} \right) \tag{110}
\]

Hence, recalling that

\[
g_p(\alpha) \equiv -\alpha + \frac{\rho a}{\rho a + 1 - \rho} = -\alpha + \frac{\alpha \rho e^{\rho e^p(\alpha)} - \alpha}{\alpha \rho e^{\rho e^p(\alpha)} + (1 - \alpha)^p}, \tag{111}
\]

and noticing that there exists a constant \( Q_\delta \) such that (for the above range of \( \alpha, \eta, \) etc)

\[
\frac{1 - \rho}{(\rho a + 1 - \rho)t} \leq Q_\delta \frac{t}{t},
\]

we deduce the following bound:

\[
\Pr_{[t, \alpha]} \left( [t, \alpha] \mapsto [(1 + \eta)t, \alpha^*] \right) : \frac{\alpha^* - \alpha}{\eta} = g_p(\alpha) \pm (2\eta + \sqrt{\eta} + Q_\eta/\eta t) \mid \text{ELead} \]

\[
\geq 1 - e^{-V_\delta \eta t}. \tag{112}
\]
This holds for all $\delta < \alpha < 1/2 - \delta$, $0 < \eta < \eta_\delta$ and $t \geq T_\delta$. Notice also that we had assumed that $\alpha t \in \mathbb{N}$, but this restriction is unnecessary if we make $Q_\delta$ larger. To summarize our conclusions, we state them in slightly modified form as the following Lemma.

**Lemma 2** For each $\delta \in (\delta, 1/2 - \delta)$, there exist constants $\eta_\delta \in \mathbb{R}^+$, $V_\delta, Q_\delta \in \mathbb{R}^+$ and $T_\delta \in \mathbb{N}$ such that for all integers $t^* > t \geq T_\delta$ such that $0 < \eta = \frac{t}{t^*} - 1 < \eta_\delta$,

$$\Pr_{[t, \alpha]} \left( [t, \alpha] \mapsto [(1 + \eta)t, \alpha^*] : \frac{\alpha^* - \alpha}{\eta} = g_\alpha(\alpha) \pm Q_\eta \left( \sqrt{\eta} + \frac{1}{t^* - t} \right) \mid \text{ELead} \right) \geq 1 - e^{-V_\delta (t^* - t)}.$$  \hspace{1cm} (113)

Notice that the constants appearing in the Lemma might be slightly different than those appearing before it, but this is nothing but a slight abuse of notation.

**Remark 1** Let us now show why $g_\alpha(\alpha) \geq 0$ always, as stated in the introduction to this chapter. Choose $\delta < \alpha^* < \alpha < 1/2 - \delta$ be fixed, but also close enough to $\alpha$. Then $c_\alpha(\alpha) \geq \alpha$ always (by Theorem 2), and one can easily deduce from the reasoning proving (89) and Lemma 2 that for all $\eta > 0$ fixed (but small enough)

$$\Pr_{[t, \alpha]} \left( [t, \alpha] \mapsto [(1 + \eta)t, \alpha^*] \mid \text{ELead} \right) \leq e^{-V_\delta \eta t}.$$  \hspace{1cm} (113)

There are $\eta t$ choices for the number of balls in the first bin at time $\eta t$, and one can easily deduce via an union bound that

$$\Pr_{[t, \alpha]} \left( [t, \alpha] \mapsto [(1 + \eta)t, \alpha^*] \mid \text{ELead} \right) \leq e^{-\Omega(t)},$$

(Notice that, as shown above, only $\alpha^* > \alpha/(1 + \eta) > \alpha - \eta$ need to be considered, so one can pick $\eta$ so that $\alpha^* > \delta$ for all relevant $\alpha^*$.) On the other hand, we know from Lemma 2 that with very high probability $[t, \alpha]$ evolves into a state $[(1 + \eta)t, \alpha^*]$ with $\alpha^* - \alpha = g(\alpha)\eta + O(\eta^{3/2} + 1/t)$. If $g(\alpha) < 0$, we could pick a small enough $\eta$ and a large enough $t$ such that $\alpha^* < \alpha$ with overwhelming probability, but this was shown to be impossible above.

### 5.4 Proof of Theorem 3

We now have what we need to prove Theorem 3.

**Proof:** of Theorem 3 The idea of the proof is to iterate uses of Lemma 2 which shows that, typically speaking, for any small $\eta$, the fraction $\alpha^*$ of balls after $\eta t$ time steps in bin 1 stays close to the straight line passing through $\alpha$ with slope $g_\alpha(\alpha)$. Since the solution $A(\cdot) = A_{p, \alpha}(\cdot)$ of the ODE also stays close to those straight lines (at least locally), this technique will give us the desired result. Let $\alpha \in (0, 1/2)$ and $0 < K < T_{p, \alpha}$ be as in the statement of the theorem. Choose some $L \in (K, T_{p, \alpha})$ and let $\delta$ be such that

$$\delta \leq \min \left\{ \alpha - \epsilon, \frac{1}{2} - A(L) - \epsilon \right\}$$  \hspace{1cm} (114)
for some $0 < \epsilon \min\{\alpha, 1/2 - A(L)\}$ (as discussed in the introduction, $A(s) < 1/2$ for $s < T_{p, \alpha}$, so such an $\epsilon$ exists).

Now recall the notation from Lemma 2 and assume (as we might) that $t$ satisfies

$$t \geq T_\delta \eta = \eta_t \equiv \left[\frac{t^{1/3}}{\epsilon}\right] < \min\{\eta_\delta, L - K\}$$

(115)

Clearly, all that (115) requires is that $t$ is large enough. Assuming it holds, there exists an integer $N_t \in \mathbb{N}$ such that

$$K \leq \eta_t N_t < L$$

(116)

and we will assume, without loss of generality, that in fact $\eta_t N_t = K < L$. We also define, for convenience

$$G_{\delta}^{(r)} \equiv \sup_{\delta \leq z \leq 1/2 - \delta} \frac{1}{r!} \left| d^r g_\mu(z) \right| \quad (r \in \mathbb{N}),$$

(117)

which is a finite quantity since $g$ is infinitely differentiable on $(0, 1/2)$.

Recall that we are starting the balls-in-bins process from state $[t, \alpha]$. We will first look at the differences

$$\Delta_j \equiv |\hat{\alpha}(j \eta) - A(j \eta)|, \quad j \in [N_t] \cup \{0\},$$

(118)

and show that these differences remain small with high probability. At $j = 0$,

$$\hat{\alpha}(0) = \left[\frac{\alpha t}{\epsilon}\right] = A(0) \pm \frac{1}{t}$$

so the differences are small at the start. Now assume that, for some $j \in [N_t] \cup \{0\}$, after conditioning on $\text{ELead}$,

With probability $\geq 1 - P_j, \forall i \in [j] \cup \{0\}, \Delta_i \leq \gamma_i \leq \epsilon$.

(119)

where

$$0 \leq \frac{1}{t} = \gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_j < \epsilon$$

and $P_j \in \mathbb{R}^+$. We will show that (again conditioning on $\text{ELead}$)

With probability $\geq 1 - P_j - e^{-W_\delta}^{1/3}$, $\forall i \in [j + 1] \cup \{0\}, \Delta_i \leq \gamma_i$, where

$$\gamma_{j+1} \equiv \gamma_j + (G_{\delta}^{(1)} + G_{\delta}^{(2)}) \eta^2 + Q_\delta \left(\eta^{3/2} + \frac{1 + K}{\eta t}\right).$$

(Here, $W_\delta > 0$ is a constant depending only on $\delta$). To prove this, let us condition on a value of $\hat{\alpha}_t(\eta_j)$ that is compatible with the event described in (119). This means that

$$\hat{\alpha}_t(\eta_j) = \alpha_j \text{ with } |A(\eta_j) - \alpha_j| \leq \gamma_j.$$
In this case, since \( \gamma_j < \epsilon \) and \( A \) is increasing,

\[
\delta < \alpha - \epsilon = A(0) - \epsilon \leq \alpha_j \leq A(L) + \epsilon < \frac{1}{2} - \delta. \tag{122}
\]

Using the Markov Property of the balls-in-bins process shows that

\[
\Pr_{[t,\alpha]}(\Delta_{j+1} \leq \gamma_{j+1} \mid \text{ELead}, \hat{\alpha}_t(\eta_j) = \alpha_j) = \Pr_{[(1+j\eta)t,\alpha_j]}(\Delta_{j+1} \leq \gamma_{j+1} \mid \text{ELead}) \tag{123}
\]

since \( \hat{\alpha}_t(s) \) is the number of balls in bin 1 at time \([st]\) (i.e. when there are \(t + [st]\) balls in the system) and in the present case \( \eta t \in \mathbb{N} \). To evaluate the latter probability, notice first that

\[
|A((j+1)\eta) - A(j\eta) - \eta A'(j\eta)| = |A((j+1)\eta) - A(j\eta) - \eta g_p(A(j\eta))| \leq G_\delta^{(2)} \eta^2. \tag{126}
\]

Moreover, by (121) and the choice of \( t \geq T_\delta \) one can apply Lemma 2 with \((1 + j\eta)t\) replacing \( t \) and \( \eta/(1 + j\eta) \) replacing \( \eta \) to deduce that, conditioned on \( \hat{\alpha}_t(j\eta) = \alpha_j \) as above, the probability that

\[
|\hat{\alpha}_t((j+1)\eta) - \hat{\alpha}_t(j\eta) - \eta g_p(\hat{\alpha}_t(j\eta))| \leq Q_\delta \left( \eta^{3/2} + \frac{1 + j\eta}{\eta t} \right) \tag{127}
\]

is at least \( 1 - e^{-V_\delta \eta t/(1 + j\eta)} \). When the two previous equations hold,

\[
|\hat{\alpha}_t(s_0 + \eta) - A(s_0 + \eta)| \leq |\hat{\alpha}_t(s) - A(s_0)| + \eta|g_p(\hat{\alpha}_t(s_0)) - g_p(A(s_0))| + G_\delta^{(2)} \eta^2 + Q_\delta \left( \eta^{3/2} + \frac{1}{\eta t} \right) \leq \gamma_j + (G_\delta^{(1)} + G_\delta^{(2)}) \eta^2 + Q_\delta \left( \eta^{3/2} + \frac{1 + K}{\eta t} \right). \tag{128}
\]

Thus, for any \( \alpha_j \) compatible with \( \Delta_j \leq \gamma_j \), one has that

\[
\Pr_{[t,\alpha]}(\Delta_{j+1} \leq \gamma_{j+1} \mid \text{ELead}, \hat{\alpha}_t(\eta j) = \alpha_j) \geq 1 - e^{-V_\delta \eta t/(1 + K)}, \tag{129}
\]

from which (120) immediately follows.

Now notice that if

\[
\gamma_N \equiv N \left\{ (G_\delta^{(1)} + G_\delta^{(2)}) \eta^2 + Q_\delta \left( \eta^{3/2} + \frac{1 + K}{\eta t} + \frac{1}{t} \right) \right\} \leq \epsilon
\]

then one can use (119) and (120) repeatedly to deduce that

\[
\Pr_{[t,\alpha]}(\forall j \in [N] \cup \{0\}, \Delta_j \leq \gamma_N \mid \text{ELead}) \geq 1 - N e^{-V_\delta \eta t/(1 + K)}. \tag{130}
\]
But notice that \( N = K/\eta \), so a simple calculation shows that

for all large enough \( t, \gamma_N \leq W_\delta t^{-1/3} \)

with \( W_\delta \in \mathbb{R}^+ \) depending only on \( \delta \). Hence \( \gamma_N \leq \epsilon \) for all large enough \( t \), and for such \( t \) \[130\] holds. Finally, one can easily show that in the event described by \[130\],

\[
\forall j \in [N - 1] \cup \{0\}, \forall s \in (0, \eta), |\hat{\alpha}_t(j \eta + s) - A(s)| \leq 2\eta + \gamma_N = O \left( t^{-2/3} \right), t \gg 1. \tag{131}
\]

Hence, \[130\] actually implies that for all large enough \( t \),

\[
\Pr_{[\ell,\alpha]} \left( \sup_{s \in [0,K]} |\hat{\alpha}_t(s) - A(s)| \leq W_\delta t^{-1/3} \mid \text{ELead} \right) \geq 1 - O \left( t^{2/3} \right) e^{-V_\delta t^{1/3}/(1+K)}, \tag{132}
\]

for a possibly larger \( W_\delta \). Since \( \delta \) is ultimately defined in terms of \( \alpha \) and \( K \), \[132\] implies the Theorem. □

5.5 Proof of Lemma 1

To conclude the chapter, we prove Lemma 1.

Proof: [of Lemma 1] Notice first that, if \( 0 < n < m \)

\[
\frac{b(n)}{b(n+1)} = \frac{n+1}{m-n} \frac{1-p}{ap} \tag{133}
\]

Notice that \( x \mapsto (x+1/m)/(1-x) = (x+1/m) \sum_{\ell \geq 1} x^\ell \) is an increasing function of \( x \) that is equal to \( 1/m < 1 \) at \( x = 0 \) and goes to \( +\infty \) as \( x \to 1 \). Hence, if

\[
x_0 = \frac{pa}{1+p} - \frac{1}{m} = \frac{pa - 1-p}{m} \frac{1}{pa + (1-p)}
\]

then

\[
\forall x \in [0,1) \begin{cases} \frac{x+1}{1-x} \frac{1-p}{ap} > 1 & \iff x > x_0 \\ \frac{x+1}{1-x} \frac{1-p}{ap} = 1 & \iff x = x_0 \\ \frac{x+1}{1-x} \frac{1-p}{ap} < 1 & \iff x < x_0 \end{cases} \tag{134}
\]

As a result, if we let \( n_0 \equiv \lfloor x_0 m \rfloor \) (which is the same definition as in the statement of the lemma), we have that (using \[133\])

\[
\forall 0 < j \leq m - n_0 \frac{b(n_0)}{b(n_0 + j)} = \prod_{i=1}^{j} \frac{b(n_0 + i - 1)}{b(n_0 + i)} > 1
\]
and similarly
\[ \forall 0 < j \leq n_0 \quad \frac{b(n_0 - j)}{b(n_0 - j)} = \prod_{i=1}^{j} \frac{b(n_0 - i + 1)}{b(n_0 - i)} < 1 \]

This proves the first two items in the lemma. As for the last one, it suffices to show that for all \( j > K \sqrt{m} \)
\[ b(n_0 + j), b(n_0 - j) \leq b(n_0) e^{-\frac{K}{2(1-x)}} \]
We only prove the first inequality; the proof of the second is almost identical. As before, we write (using (134))
\[ b(n_0 + j) = b(n_0) \prod_{i=1}^{j} \frac{b(n_0 + i)}{b(n_0 + i - 1)} = b(n_0) \left( \prod_{i=1}^{j} \frac{m - (n_0 + i)}{n_0 + i + 1} \right) \left( \frac{pa}{1-p} \right)^j \]
Now notice that (using the definition of \( x_0 \) and \( n_0 \))
\[ \left( \prod_{i=1}^{j} \frac{m - (n_0 + i)}{n_0 + i + 1} \right) < \left( \frac{m - n_0}{n_0 + 1} \right)^j \prod_{i=1}^{j} \left( 1 - \frac{i}{m - n_0} \right) \]
\[ \leq \left( \frac{m - n_0}{n_0 + 1} \right)^j \exp \left( - \sum_{i=1}^{j} \frac{i}{m - n_0} \right) \]
\[ = \left( \frac{1 - x}{x + 1/m} \right)^j \exp \left( - \frac{j(j + 1)}{2(1-x)m} \right) \]
where \( x \equiv n_0/m \) is bigger than \( x_0 \). As a result of (134)
\[ \frac{1 - x}{x + 1/m} \frac{ap}{1-p} < 1 \]
and putting this together with the previous inequalities
\[ \frac{b(n_0 + j)}{b(n_0)} < \left( \frac{pa}{1-p} \right)^j \left( \frac{1 - x}{x + 1/m} \right)^j \exp \left( - \frac{j(j + 1)}{2(1-x)m} \right) \leq \exp \left( - \frac{j(j + 1)}{2(1-x)m} \right) \]
which, together with the fact that \( j > K \sqrt{m} \), finishes the proof. □
6 Open problems

The results proven here only apply to feedback functions $f(x) = x^p$. However, we have been able to prove other results for more general functions; see [8, 10, 11] for several examples. It would be interesting to see extensions of the present work to those other feedback functions as well.

Another open problem is to determine the asymptotic behaviour of the ordinary differential equation in Theorem 3, especially whether the solution blows up in finite time (i.e. $T_{p,\alpha} < +\infty$). We conjecture that this is not the case, but a proof would require a careful analysis of the ODE.

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