A numerical technique for solving multi-dimensional fractional optimal control problems

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ABSTRACT
In this article, we use the operation matrix (OM) of Riemann–Liouville fractional integral of the shifted Gegenbauer polynomials with the Lagrange multiplier method to provide efficient numerical solutions to the multi-dimensional fractional optimal control problems. The proposed technique transforms the under consideration problems into sets of nonlinear equations which are easy to solve. Numerical tests including numerical comparisons with some existing methods are introduced to demonstrate the accuracy and efficiency of the suggested technique.

1. Introduction
The mathematical topic of optimal control problems (OCPs) has received much attentions in the last years because it has a lot of important applications in physics, chemistry, engineering, etc. [1, 2]. Recently, fractional calculus has demonstrated its accuracy in modelling many applications in various scientific fields [3–5].

Fractional optimal control problem (FOCP) is an OCP in which the differential equations governing the dynamics of the system contain at least one fractional derivative operator. Due to the various applications of FOCPs in physics and engineering, they received much attentions from many researchers. These applications include the materials with memory and hereditary effects, dynamical processes containing gas diffusion and heat conduction in fractal porous media. Other applications of FOCPs are given in [1, 6]. Because of a lot of FOCPs does not have analytical exact solutions, numerous numerical methods are offered to overcome these problems [7–12]. In recent years, various OMs for different polynomials like Chebyshev polynomials [13, 14], Legendre polynomials [11, 15], Jacobi polynomials [10, 16, 17], Bernstein polynomials [18] and Laguerre polynomial [19] have been developed to cover the numerical solutions of different types of fractional differential equations [20–23]. Our technique depends on the OM of RL fractional integral of shifted Gegenbauer polynomial (SGPs). The Gegenbauer polynomials have many useful properties. They generalize Legendre polynomials \( L_j(t) \) and Chebyshev polynomials of the first \( T_j(t) \) and second \( U_j(t) \) kind, and they are special cases of Jacobi polynomials. The most important characteristic of SGPs is achieving rapid rates of convergence [24–27]. This encourages us to use these polynomials in the suggested technique as basis functions for finding the approximate solutions to a wide class of FOCPs in the following form:

\[
\text{Min.} J = \int_0^t f(t, x_j(t), u(t)) \, dt, \quad (1)
\]

subject to

\[
D^{(\nu)}x_j(t) = g_j(t, x_j(t)) + b_j(t)u(t), \quad (2)
\]

with

\[
D^{(\nu)}x_j(0) = x_j^{(i)}(0), \quad i = 0, 1, \ldots, m-1, \quad j = 1, 2, \ldots, n,
\]

where \( D^{(\nu)} \) is the Caputo fractional derivative of order \( \nu, \ m-1 < \nu \leq m \) and \( b_j(t) \neq 0, \ n \) is the number of variables. In the proposed technique; the state and the control variables are approximated by SGPs. By using the OM of Riemann–Liouville fractional integral (RLFI) of the SGPs with the Lagrange multiplier method (LMM), the FOCPs are reduced to simple sets of nonlinear equations (NEs). The efficiency of the proposed technique is demonstrated by some numerical examples including problems in one-, two- and three-dimensional spaces. Also numerical comparisons between the suggested technique and some published results are held.

The rest of the paper is systemized as follows: We begin with some preliminaries of fractional calculus and GPs. In Section 3, the shifted Gegenbauer operational matrix (SGOM) of RL fractional integral is derived. In Section 4, the convergence of the proposed method is discussed. In Section 5, the proposed technique of applying SGOM of fractional integration for solving...
FOCPs is presented. In Section 6, some illustrative examples are offered. Finally concluding remarks end the paper.

2. Preliminaries and definitions

2.1. Fractional calculus definitions

Definition 2.1: One of the popular definitions of the fractional integral is the Riemann–Liouville (RL), which get from the relation

\[ I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} f(\xi) \, d\xi, \]

\[ m - 1 < \alpha < m, m \in \mathbb{N}, \alpha > 0, t > 0, \]

(3)

The operator \( I^\alpha \) has properties said in [28], we just recall the next property

\[ I^\alpha t^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\nu + \beta + 1)} t^{\nu + \beta}. \]  

(4)

Definition 2.2: \( D^\nu \) is the RL fractional derivative of order \( \nu \) is defined by

\[ D^\nu f(t) = \frac{d^m}{dx^m}(t^{m-\nu} f(t)), \]

\[ m - 1 < \nu < m, m \in \mathbb{N}, \nu \in \mathbb{R}, \]  

(5)

where \( m \) is the smallest integer greater than \( \nu \).

Lemma 2.1: If \( m - 1 < \nu \leq m \), \( m \in \mathbb{N} \), then

\[ D^\nu I^\alpha f(t) = f(t), \]

\[ I^\nu D^\nu f(t) = f(t) - \sum_{i=0}^{m-1} \frac{f^{(i)}(0^+)}{i!} t^i, \quad t > 0. \]  

(6)

2.2. Shifted Gegenbauer polynomials and their properties

The shifted ultraspherical (Gegenbauer) polynomials \( C_{S,j}^{(a)}(t) \), of degree \( j \in \mathbb{Z}^+ \), with the associated parameter \( \alpha > \frac{1}{2} \), are a sequence of real polynomials in the finite domain \([0, L]\). They are a family of orthogonal polynomials which has many properties:

The analytical form of the SGP is given by

\[ C_{S,j}^{(a)}(t) = \sum_{k=0}^{j} (-1)^j \frac{\Gamma(\alpha + \frac{1}{2}) \Gamma(j + k + 2\alpha)}{\Gamma(k + \alpha + \frac{1}{2}) \Gamma(2\alpha)(j - k)!k!} k!, \]

\[ C_{S,j}^{(a)}(0) = (-1)^j \frac{\Gamma(j + 2\alpha)}{\Gamma(2\alpha)} \]  

(7)

and

\[ C_{S,j}^{(a)}(t) = C_{S,j}^{(a)} \left( \frac{2t}{L} - 1 \right), \quad C_{S,j}^{(a)}(0) = 1, \]

\[ C_{S,j}^{(a)}(t) = \frac{2t}{L} - 1. \]

The orthogonal relation of SGPs is:

\[ \langle C_{S,j}^{(a)}(t), C_{S,j}^{(a)}(t) \rangle = \int_0^L C_{S,j}^{(a)}(t) C_{S,j}^{(a)}(t) \omega_S(t) \, dt \]

\[ = \lambda_{S,j}^{(a)} \delta_{j,k} \]  

(8)

where \( \omega_S(t) \) is the weight function, and it is even function given from the relation

\[ \omega_S^{(a)}(t) = (tL - t^2)^{\alpha-1/2} \]

and

\[ \lambda_{S,j}^{(a)} = \left( \frac{L}{2} \right)^{2\alpha} \lambda_j^{(a)}. \]

This polynomial recover the shifted Chebyshev polynomial of the first kind \( T_{S,j}(t) \equiv C_{S,j}^{(0)}(t) \), the shifted Legendre polynomial \( L_{S,j}(t) \equiv C_{S,j}^{(1/2)}(t) \), and the shifted Chebyshev polynomial of the second kind \( C_{S,j}^{(1)}(t) \equiv (1/(j + 1))U_{S,j}(t) \).

The square integrable function \( y(t) \) in \([0, L]\) is approximated by SGPs as:

\[ y(t) = \sum_{j=0}^{N} \tilde{y}_j C_{S,j}^{(a)}(t), \]

where the coefficients \( \tilde{y}_j \) are getting from

\[ \tilde{y}_j = (\lambda_{S,j}^{(a)})^{-1} \int_0^L y(t) \omega_S^{(a)}(t) C_{S,j}^{(a)}(t) \, dt. \]  

(9)

The approximation of function \( y(t) \) in the vector form is defined by

\[ y(t) = Y^T \phi(t), \]

(10)

where \( Y^T = [\tilde{y}_0, \tilde{y}_1, \ldots, \tilde{y}_N] \) is the shifted Gegenbauer coefficient vector, and

\[ \phi(t) = [C_{S,0}^{(a)}(t), C_{S,1}^{(a)}(t), \ldots, C_{S,N}^{(a)}(t)]^T \]  

(11)

is the shifted Gegenbauer vector. The \( q \) times repeated integration of Gegenbauer vector is

\[ \int P^q(\phi(t)), \]

(12)

where \( P^{(a)} \) is called the OM of the integration of \( \phi(t) \).

3. Fractional SGOM of integration

In this section, the SGOM of RLFI will be derived.

Theorem 3.1: Let \( \phi(t) \) be the shifted Gegenbauer vector and \( \alpha > 0 \) then

\[ I^\alpha \phi(t) \simeq P^{(\nu)}(\phi(t)), \]

(13)
where $t \in [0, 1]$ and $P^{(v)}$ is called OMFI of order $v$ in the RL sense, it is an $(N+1) \times (N+1)$ and is written as

$$
P^{(v)} = \begin{pmatrix}
\sum_{k=0}^{i} \xi_{0,0,k} & \sum_{k=0}^{i} \xi_{0,1,k} & \cdots & \sum_{k=0}^{i} \xi_{0,N,k} \\
\sum_{k=0}^{i} \xi_{1,0,k} & \sum_{k=0}^{i} \xi_{1,1,k} & \cdots & \sum_{k=0}^{i} \xi_{1,N,k} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{k=0}^{i} \xi_{N,0,k} & \sum_{k=0}^{i} \xi_{N,1,k} & \cdots & \sum_{k=0}^{i} \xi_{N,N,k}
\end{pmatrix}, \quad (14)
$$

where $\xi_{ij,k}$ is given by

$$
\xi_{ij,k} = \mathcal{Z} \times \mathcal{Y},
$$

where

$$
\mathcal{Z} = \sum_{k=0}^{i} (-1)^{i-k} \frac{\Gamma(\alpha + \frac{1}{2}) \Gamma(i + k + 2\alpha)}{\Gamma(k + \alpha + \frac{1}{2}) \Gamma(2\alpha)(i-k)!},
$$

$$
\mathcal{Y} = \sum_{t=0}^{i} (-1)^{i-t} \frac{\Gamma(j + \alpha)^2 \Gamma^2(\alpha + \frac{1}{2})}{2^{(1-4\alpha)} \Gamma(2\alpha + j) \Gamma(2\alpha)(\Gamma(v + k + f + \alpha + \frac{1}{2}) \Gamma(v + f + 1)}.
$$

Proof: From relation (7) and by using Equations (3) and (4), we can write

$$
C_{S,t}^{(v)}(t) = \sum_{k=0}^{i} (-1)^{i-k} \frac{\Gamma(\alpha + \frac{1}{2}) \Gamma(i + k + 2\alpha)}{\Gamma(k + \alpha + \frac{1}{2}) \Gamma(2\alpha)(i-k)!} f^v(t^k),
$$

$t \in [0, 1]$

$$
= \sum_{k=0}^{i} (-1)^{i-k} \frac{\Gamma(\alpha + \frac{1}{2}) \Gamma(i + k + 2\alpha)}{\Gamma(k + \alpha + \frac{1}{2}) \Gamma(2\alpha)(i-k)!} t^{k+v},
$$

$i = 0, 1, 2, \ldots, N.

The function $t^{k+v}$ can be written as a series of $N+1$ terms of Gegenbauer polynomial,

$$
t^{k+v} = \sum_{j=0}^{N} \tilde{\xi} C_{S,j}^{(v)}(t),
$$

where

$$
\tilde{\xi} = \sum_{j=0}^{i} (-1)^{j-t} \frac{\Gamma(2\alpha + j + f) \Gamma(2\alpha + j) \Gamma(v + k + f + \alpha + \frac{1}{2}) \Gamma(v + f + 1)}{2^{(1-4\alpha)} \Gamma(2\alpha + j) \Gamma(2\alpha)(\Gamma(v + k + f + \alpha + \frac{1}{2}) \Gamma(v + f + 1))}.
$$

Now, by employing Equations (16)-(18) we obtain:

$$
l^v C_{S,j}^{(v)}(t) = \sum_{k=0}^{i} \sum_{j=0}^{N} (-1)^{i-k} \frac{\Gamma(\alpha + \frac{1}{2}) \Gamma(i + k + 2\alpha)}{\Gamma(\alpha + \frac{1}{2}) \Gamma(2\alpha)(i-k)!} \xi_{ij,k} C_{S,j}^{(v)}(t),
$$

$i = 0, 1, \ldots, N.

(19)

where $\xi_{ij,k}$ is given in Equation (15).

Writing the last equation in a vector form gives

$$
l^v C_{S,j}^{(v)}(t) \simeq \left[ \sum_{k=0}^{i} \sum_{j=0}^{N} \xi_{ij,k} \right] \phi(t),
$$

$i = 0, 1, \ldots, N,

(20)

which finishes the proof. $\blacksquare$

4. Error estimation and convergence analysis

4.1. Error estimation

In the following theorem, the error estimation for the approximated functions will be expressed in terms of Gram determinant [29].

Theorem 4.1: Assume that $H = L^2[0, 1]$ is the Hilbert space, and let $Y$ be a closed subspace of $H$ such that $Y = \text{Span}(c_{S,j}^{(v)}(t), c_{S,j}^{(v)}(t), \ldots, c_{S,j}^{(v)}(t))$. Let $y(t)$ be an arbitrary element of $H$ and $y^*(t)$ be the unique best approximation of $y(t)$ out of $Y$, then

$$
\| y(t) - y^*(t) \|^2 = \frac{\text{Gram}(y(t), c_{S,j}^{(v)}(t), c_{S,j}^{(v)}(t), \ldots, c_{S,j}^{(v)}(t)))}{\text{Gram}(c_{S,j}^{(v)}(t), c_{S,j}^{(v)}(t), \ldots, c_{S,j}^{(v)}(t)))},
$$

(21)

where

$$
\text{Gram}(y(t), c_{S,j}^{(v)}(t), c_{S,j}^{(v)}(t), \ldots, c_{S,j}^{(v)}(t))) =

\begin{vmatrix}
\langle y(t), y(t) \rangle & \langle y(t), c_{S,j}^{(v)}(t) \rangle & \cdots & \langle y(t), c_{S,j}^{(v)}(t) \rangle \\
\langle c_{S,j}^{(v)}(t), y(t) \rangle & \langle c_{S,j}^{(v)}(t), c_{S,j}^{(v)}(t) \rangle & \cdots & \langle c_{S,j}^{(v)}(t), c_{S,j}^{(v)}(t) \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle c_{S,j}^{(v)}(t), c_{S,j}^{(v)}(t) \rangle & \langle c_{S,j}^{(v)}(t), c_{S,j}^{(v)}(t) \rangle & \cdots & \langle c_{S,j}^{(v)}(t), c_{S,j}^{(v)}(t) \rangle
\end{vmatrix}
$$

4.2. Convergence analysis

Consider the error, $E_p$, of the operational matrix of RLFI as

$$
E_p = P^v \Phi(t) - l^v \Phi(t),
$$
where
\[ E_N = [E_{P,0}, E_{P,1}, \ldots, E_{P,N}]^T \]
is an error vector.

From Equation (17), we had approximated \( t^{k+v} \) as
\[ \sum_{j=0}^{N} \tilde{t}C_{S_j}^g(t). \] From the above theorem, we have
\[
\left\| t^{k+v} - \sum_{j=0}^{N} \tilde{t}C_{S_j}^g(t) \right\|_2 = \left( \frac{1}{\text{Gram}(C_{S_0}^g(t), C_{S_1}^g(t), \ldots, C_{S_N}^g(t)))} \right)^{1/2}.
\] (22)

By using Equation (19), the upper bound of the operational matrix of integration will be
\[
\|E_{P,j}\|_2 = \left\| I^j C_{S_j}^g(t) - \sum_{j=0}^{N} \sum_{k=0}^{j} \xi_{i,j,k} C_{S_k}^g(t) \right\|_2, \quad i = 0, \ldots, N,
\] (23)
\[
\leq \sum_{k=0}^{j} \frac{\Gamma(\alpha + 1/2) \Gamma(j + k + 2\alpha)}{\Gamma(\alpha + 2/2) \Gamma(k + \alpha + 2\alpha)(j + k + 2\alpha)} \frac{1}{\text{Gram}(C_{S_0}^g(t), C_{S_1}^g(t), \ldots, C_{S_N}^g(t)))} \right)^{1/2}.
\] (24)

The following theorem illustrates that by increasing the number of SGPs the error tends to zero.

**Theorem 4.2:** Suppose that function \( y(t) \in L^2[0,1] \) is approximated by \( g_N(t) \) as follows
\[
g_N(t) = \mu_0 C_{S_0}^g(t) + \mu_1 C_{S_1}^g(t) + \cdots + \mu_N C_{S_N}^g(t),
\]
where
\[
\mu_i = \int_0^1 C_{S_i}^g(t)y(t) \, dt, \quad i = 0, \ldots, N.
\]
Consider
\[
s_N(y) = \int_0^1 [y(t) - g_N(t)]^2 \, dt,
\]
then we have
\[
\lim_{N \to \infty} s_N(y) = 0.
\]
For the proof [30].

**5. Application of SGOM for FOCPs**

In this section, the SGPs are used with the OM of the RLFI for solving the following FOCPs (1) and (2)

\[
J[X, U] = \int_0^1 f(R^T, X_j^T P^v + B_j^T, X_j^T) \phi(t) \, dt.
\] (31)
Employing Equations (25), (28) and (29), the dynamic constraints (2) can be approximated as
\[ X^T \phi(t) - g(R^T, X^T U^T + B^T)\phi(t) - Q^T_\theta \phi(t)\phi^T(t)U = 0. \]
(32)
Let
\[ Q^T_\theta (\theta^T H)^T = \phi^T (t)H^T_j, \]
where \( H_j \) is an \( N \times N \) matrix.

The elements of the matrix \( H_j \) are getting from the following relation
\[ H_{j,k} = \frac{1}{\lambda_{\delta j}} \sum_{\mu=0}^{\infty} q_{\mu} \int_0^1 \omega_{\delta j}^\mu(t)C_{\delta j}^\mu(t)C_{\delta k}^\nu(t)dt, \]
where \( l = 0, 1, \ldots, N \) and \( k = 0, 1, \ldots, N \).

By using Equation (33) into Equation (32), we obtain
\[
\begin{align*}
(X_j^T - g_j(R^T, X_j^T U^T + B_j^T) - \phi^T(t)H_j^T R_j \phi(t) = 0, \\
(X_j^T - g_j(R^T, X_j^T U^T + B_j^T) - \phi^T(t)H_j^T R_j \phi(t) = 0.
\end{align*}
\]
(35)

Secondly, assume that
\[ J^*(X_j, U, \Lambda) = J(X_j, U) + (X^T - g_j(R^T, X_j^T U^T + B_j^T) - \phi^T(t)H_j^T R_j \phi(t))\Lambda, \]
where \( \Lambda \) is the unknown Lagrange multiplier vector,
\[
\begin{pmatrix}
\lambda_0 \\
\lambda_1 \\
\vdots \\
\lambda_N
\end{pmatrix}.
\]
(36)

By applying the necessary conditions of optimality to Equation (36), we have
\[ \frac{\partial J^*}{\partial X_j} = 0, \quad \frac{\partial J^*}{\partial U} = 0, \quad \frac{\partial J^*}{\partial \Lambda} = 0. \]
(37)

By using Newton iterative method, this system of NEs can be solved for the unknown coefficients of the vectors \( X, U \) and \( \Lambda \).

### 5.2. Approximation of our problem

In our case, the set of Gegenbauer polynomials, \( C_{\delta j}^\nu(t) \) is used as a basis which form the space \( D_1[0, 1] = \{y(t) : y \text{ is continuously differentiable on interval } [0, 1]\} \), with uniform norm \( \|y\| = \|y\|_\infty + \|y\|_\infty \). Let us consider \( M_n = \theta_0 C_{\delta 0}^\nu(t) + \theta_1 C_{\delta 1}^\nu(t) + \ldots + \theta_n C_{\delta n}^\nu(t) \), where \( M_n \) is the \( n \)-dimensional subspace of \( D_1[0, 1] \) and \( \theta_0, \theta_1, \ldots, \theta_n \) are arbitrary real numbers. If we choose \( \theta_0, \theta_1, \ldots, \theta_n \) in such a way that \( M_n \) minimizes \( J \) denoting the minimum by \( \sigma_n \). Then, we should have \( M_n \subset M_{n+1} \), this implies \( \sigma_n \geq \sigma_{n+1} \).  

**Theorem 5.1:** Consider the functional \( J \) then \( \lim_{n \to \infty} \sigma_n = \sigma \) where \( \sigma = \inf_{x \in D_1[0, 1]} J \). For the proof, see [21, 31].

### 6. Illustrative problems

**One-Dimensional Problems**

**Problem 6.1:** Consider the following FOCP [21],
\[ \min J = \frac{1}{2} \int_0^1 (x^2(t) + u^2(t)) \, dt, \]
with the dynamic constraint
\[ D^\nu x(t) = -x(t) + u(t), 0 \leq \nu \leq 1, \]
x(0) = 1.

At \( \nu = 1 \) the exact solution is
\[ x(t) = \cosh(\sqrt{2}t) + \beta \sinh(\sqrt{2}t), \]
\[ u(t) = (1 + \sqrt{2}\beta) \cosh(\sqrt{2}t) + (\sqrt{2} + \beta) \sinh(\sqrt{2}t), \]
where \( \beta = -\frac{\cosh(\sqrt{2}) + \sinh(\sqrt{2})}{\sqrt{2} \cosh(\sqrt{2}) + \sinh(\sqrt{2})} \).

By applying the suggested technique to Problem 6.1, the resultant numerical results for the state \( x(t) \) and the control \( u(t) \) variables are displayed through Figure 1(a,b), respectively, at \( \nu = 0.75, 0.85, 0.95, 1 \) with the exact solutions for \( N = 8 \). We noted that the obtained solutions cover the classical results when the value of the fractional order tends to unity. Also in Tables 1 and 2, the absolute errors of the state \( x(t) \) and the control \( u(t) \) variables for Problem 6.1 are calculated at various choices of \( N \). It is observed that the efficiency of the proposed method is increased by increasing \( N \).

**Problem 6.2:** Consider the following FOCP [21, 32]
\[ J = \frac{1}{2} \int_0^1 (x^2(t) + u^2(t)) \, dt, \]
such that
\[ D^\nu x(t) = tx(t) + u(t), 0 \leq \nu \leq 1, \quad x(0) = 1. \]

In Figure 2(a,b), the obtained results of the variables \( x(t) \) and \( u(t) \) of Problem 6.2 are plotted for different values of \( \nu \). We noted that the control \( u(t) \) variables for Problem 6.2 with different values of \( \nu \) at \( N = 8 \) together with the results obtained in [13] and [33] are tabulated. Obviously, our estimated results are in a good agreement with the results in [13, 33].

**Problem 6.3:** Consider the following FOCP [14]
\[ \min J = \int_0^1 \left( (x(t) - t^2)^2 + \left( u(t) + t^4 - \frac{20t^{9/10}}{9t^{1/10}} \right)^2 \right) \, dt, \]
with the dynamic
\[ D^\nu x(t) = t^2 x(t) + u(t), 1 \leq \nu \leq 2, \]
x(0) = \dot{x}(0) = 0.
Figure 1. The behaviour of the approximate solutions of problem 6.1 for $N = 5$ and $\nu = 0.75, 0.85, 0.95, 1$ with the exact solution. (a) $x(t)$. (b) $u(t)$.

Table 1. The absolute errors of $x(t)$ for Problem 6.1 at various choices of $N$.

| $t$  | $N = 3$           | $N = 5$           | $N = 8$           | $N = 10$          |
|------|-------------------|-------------------|-------------------|-------------------|
| 0    | $1.25437 \times 10^{-3}$ | $6.25467 \times 10^{-6}$ | $6.26213 \times 10^{-10}$ | $1.98861 \times 10^{-9}$ |
| 0.1  | $3.30159 \times 10^{-4}$ | $2.39179 \times 10^{-6}$ | $1.22351 \times 10^{-10}$ | $6.29444 \times 10^{-10}$ |
| 0.2  | $4.86069 \times 10^{-5}$ | $1.21248 \times 10^{-6}$ | $3.57127 \times 10^{-11}$ | $5.40695 \times 10^{-10}$ |
| 0.3  | $7.8748 \times 10^{-5}$  | $1.7249 \times 10^{-6}$  | $1.1152 \times 10^{-10}$  | $2.81709 \times 10^{-10}$  |
| 0.4  | $3.34676 \times 10^{-4}$ | $6.82411 \times 10^{-7}$ | $1.53137 \times 10^{-10}$ | $2.13957 \times 10^{-10}$ |
| 0.5  | $4.57932 \times 10^{-4}$ | $1.93055 \times 10^{-7}$ | $6.82487 \times 10^{-12}$ | $5.58041 \times 10^{-10}$ |
| 0.6  | $2.30996 \times 10^{-4}$ | $3.10922 \times 10^{-7}$ | $1.46338 \times 10^{-10}$ | $3.82721 \times 10^{-10}$ |
| 0.7  | $2.02962 \times 10^{-4}$ | $1.9004 \times 10^{-7}$  | $1.17524 \times 10^{-10}$ | $1.53133 \times 10^{-10}$ |
| 0.8  | $5.2129 \times 10^{-4}$  | $9.16645 \times 10^{-7}$  | $2.17533 \times 10^{-11}$ | $5.698 \times 10^{-10}$    |
| 0.9  | $2.42861 \times 10^{-4}$ | $2.49026 \times 10^{-6}$ | $1.0693 \times 10^{-10}$ | $7.5132 \times 10^{-10}$    |
| 1    | $1.25411 \times 10^{-3}$ | $6.25466 \times 10^{-6}$ | $5.86238 \times 10^{-10}$ | $2.68463 \times 10^{-9}$   |
Table 2. The absolute errors of $u(t)$ for Problem 6.1 at various choices of $N$. 

| $t$  | $N = 3$            | $N = 5$            | $N = 8$            | $N = 10$            |
|------|--------------------|--------------------|--------------------|--------------------|
| 0    | $3.77566 \times 10^{-4}$ | $1.88239 \times 10^{-6}$ | $3.27507 \times 10^{-10}$ | $8.1688 \times 10^{-11}$ |
| 0.1  | $1.09654 \times 10^{-4}$ | $7.0819 \times 10^{-7}$  | $6.36999 \times 10^{-11}$ | $2.70723 \times 10^{-11}$ |
| 0.2  | $1.42091 \times 10^{-4}$ | $3.9987 \times 10^{-7}$  | $1.83641 \times 10^{-11}$ | $1.54725 \times 10^{-11}$ |
| 0.3  | $8.61549 \times 10^{-4}$ | $4.98381 \times 10^{-7}$ | $5.90761 \times 10^{-11}$ | $9.72863 \times 10^{-12}$ |
| 0.4  | $1.13021 \times 10^{-4}$ | $2.49281 \times 10^{-7}$ | $8.13922 \times 10^{-11}$ | $2.06191 \times 10^{-11}$ |
| 0.5  | $1.3787 \times 10^{-4}$  | $5.81012 \times 10^{-7}$ | $3.49132 \times 10^{-12}$ | $2.08388 \times 10^{-11}$ |
| 0.6  | $5.73272 \times 10^{-5}$ | $4.96686 \times 10^{-8}$ | $7.88284 \times 10^{-11}$ | $1.56014 \times 10^{-11}$ |
| 0.7  | $7.57957 \times 10^{-5}$ | $5.92684 \times 10^{-8}$ | $6.40287 \times 10^{-11}$ | $3.26667 \times 10^{-11}$ |
| 0.8  | $1.60967 \times 10^{-5}$ | $2.40909 \times 10^{-7}$ | $1.19774 \times 10^{-11}$ | $8.78333 \times 10^{-11}$ |
| 0.9  | $6.26788 \times 10^{-5}$ | $7.611 \times 10^{-7}$   | $5.92166 \times 10^{-11}$ | $1.10134 \times 10^{-11}$ |
| 1    | $3.77513 \times 10^{-4}$ | $1.88239 \times 10^{-6}$ | $3.25376 \times 10^{-10}$ | $1.70672 \times 10^{-10}$ |

Figure 2. The behaviour of the approximate solutions of problem 6.2 for $N = 3$ and $\nu = 0.75, 0.85, 0.95, 1$. (a) $x(t)$. (b) $u(t)$.

Figure 3(a, b) show the approximated results of state $x(t)$ and the control $u(t)$ variables of Problem 6.3 at $N = 3$ and $\nu = 1.85, 1.95$ and 2. In Table 4 comparisons of obtained results for the minimum values of $J$ of Problem 6.3 at different choices of $N$ together with the results obtained in [14] are tabulated. It is noted that the approximated results obtained by the suggested technique are more accurate than the results in [14].

Two-Dimensional Problems

Problem 6.4: Consider the following FOCS [34]

$$
\min J = \frac{1}{2} \int_0^1 (x_1^2(t) + x_2^2(t) + u^2(t)) \, dt,
$$

where $x_1(t)$ and $x_2(t)$ are the state variables and $u(t)$ is the control variable.
under the constraint
\[
D^\nu x_1(t) = -x_1(t) + x_2(t) + u(t),
\]
\[
D^\nu x_2(t) = -2x_2(t),
\]
\[
x_1(0) = x_2(0) = 1.
\]

Table 3. The estimated values of $J$ for different values of $\nu$ and $N = 8$ for Problem 6.2.

| $\nu$ | Present method | Method of [13] | Present of [33] |
|-------|----------------|----------------|-----------------|
| 1     | 0.484268       | 0.48426        | 0.48427         |
| 0.99  | 0.483463       | 0.48346        | 0.48347         |
| 0.90  | 0.475883       | 0.47588        | 0.47605         |
| 0.80  | 0.466978       | 0.46697        | 0.46722         |

Table 4. The estimated values of $J$ at various choices of $N$ and $\nu = 1.1$ for Problem 6.3.

| $N$ | Present method | Method of [14] |
|-----|----------------|----------------|
| 4   | $2.23277 \times 10^{-6}$ | $4.76932 \times 10^{-6}$ |
| 5   | $8.24619 \times 10^{-7}$  | $1.47243 \times 10^{-6}$ |
| 6   | $3.56358 \times 10^{-7}$  | $5.37825 \times 10^{-7}$  |
| 8   | $9.08978 \times 10^{-8}$  | $1.06099 \times 10^{-7}$  |
| 9   | $5.12433 \times 10^{-8}$  | $5.44304 \times 10^{-8}$  |

At $\nu = 1$ the exact solution is
\[
x_1(t) = 0.018352 e^{\sqrt{2}t} + 2.48165 e^{-\sqrt{2}t} - \frac{3e^{-2t}}{2},
\]
\[
x_2(t) = e^{-2t},
\]

Figure 3. The behaviour of the approximate solutions of problem 6.3 for $N = 3$ with $\nu = 1.85, 1.95, 2$. (a) $x(t)$. (b) $u(t)$. 
Figure 4. The behaviour of the approximate solutions of problem 6.4 for $N = 8$ and $\nu = 0.65, 0.75, 0.85, 0.95, 1$, with exact solution. (a) $x_1(t)$. (b) $x_2(t)$. (c) $u(t)$. 
Figure 4(a–c) illustrate the behaviour of state variables $x_1(t), x_2(t)$ and control variable $u(t)$, respectively, for $N=8$ and $\nu = 0.5, 0.75, 0.85, 0.95$ and 1 with the exact solutions. Tables 5–7 display the comparison of the absolute errors by using our mechanism and the method mention in [35] of the variables $x_1(t), x_2(t)$ and $u(t)$ for Problem 6.4 at $\nu = 1$ and $N = 3, 4, 6$. These results show that the suggested method is more accurate than the results of [35]. This problem was solved in [34] by a different technique. The results shown in Figure 4(a–c) are in a good agreement with the results established in [34]. It is remarkable that we achieved satisfactory numerical results with at least 8 numbers of the SGP while in [34], number of approximations starts in 8 and increases up to 128 are used to obtain satisfactory results. So we can deduce that our numerical technique is less computational than that in [34].

### Three-Dimensional Problem

**Problem 6.5:** Consider the following FOCS ([36])

$$
\min J = \frac{1}{2} \int_0^1 (x_1^2(t) + x_2^2(t) - x_3^2(t) + u^2(t)) \, dt
$$

with the dynamic constraints

$$
\begin{align*}
D^\nu x_1(t) &= -2x_1(t) - tx_2(t) + u(t), \\
D^\nu x_2(t) &= 3x_1(t) + x_3(t) - u(t), \\
D^\nu x_3(t) &= tx_1(t) + x_2(t), \\
x_1(0) &= x_2(0) = x_3(0) = 1.
\end{align*}
$$

Figure 5(a–d) illustrate the behaviour of approximate solution of the variables $x_1(t), x_2(t), x_3(t)$ and $u(t)$, respectively, for $N = 5$ at various choices of $\nu \ (\nu = 0.65, 0.75, 0.85, 0.95$ and 1).
7. Conclusions

In this paper, a new numerical mechanism has been derived to find the approximate solutions of the multi-dimensional FOCPs, this numerical mechanism depends on the SGOM of RLFI. The SGOM of fractional integration reduced the FOCP into an equivalent integral problem. The properties of the SGPs together with the LMM are used to transform the equivalent functional integral equation problem into an algebraic system of equations, which is easily to solve. The applicability, accuracy and rapidity by using few terms of the SGPs of the proposed mechanism are illustrated by numerical applications and the numerical comparisons with some existing methods in the literatures.

Disclosure statement

No potential conflict of interest was reported by the author.

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