Formulation of general dynamical invariants and their unitary relations for time-dependent three coupled quantum oscillators

Jeong Ryeol Choi

Department of Nanoengineering, Kyonggi University,
Yeongtong-gu, Suwon, Gyeonggi-do 16227, Republic of Korea

Abstract

A general dynamical invariant operator for three coupled time-dependent oscillators is derived. Although the obtained invariant operator satisfies the Liouville-von Neumann equation, its mathematical formula is somewhat complicated due to arbitrariness of time variations of parameters. The parametric conditions required for formulating this invariant are definitely specified. By using the unitary transformation method, the invariant operator is transformed to the one that corresponds to three independent simple harmonic oscillators. Inverse transformation of the well-known quantum solutions associated with such a simplified invariant enables us to identify quantum solutions of the coupled original systems. These solutions are exact since we do not use approximations not only in formulating the invariant operator but in the unitary transformation as well. The invariant operator and its eigenfunctions provided here can be used to characterize quantum properties of the systems with various choices of the types of time-dependent parameters.

Keywords: coupled oscillators; invariant operator; unitary transformation; eigenfunction; diagonalization

* E-mail: choiardor@hanmail.net
1. Introduction

A large part of modern quantum technologies are based on the utilization of entanglement between identical or different quantum devices, such as qubits, micro cavities, quantum dot transistors, nano resonators, and tunnel diodes. Generation and control of entanglement with high precision are requisite in order to build large-scale quantum architectures in quantum information science [1–3]. Hence it is important to understand quantum entanglement in coupled devices from a quantum-mechanical point of view. Lots of quantum information devices are analyzed using a model of coupled oscillators [4–6]. As a next-generation technological resource along this line, Boolean computations organized by means of interacting oscillators in quantum circuits [4] are expected to play a potential role in solving challenging computational problems, such as prime factorizations of large numbers and NP-complete problems. A model of coupled oscillators can also be applied in the research of other scientific branches, such as electromagnetic induced transparency [7, 8], periodicity of solar activity [9], locomotion gaits of bio-inspired robots [10, 11], and coherence in coupled semiconductor lasers [12].

To analyze coupled oscillatory systems accurately, their exact quantum formalism established on the basis of Hamiltonian diagonalization is necessary. However, if we diagonalize the Hamiltonian directly, there may arise an additional term in the Hamiltonian in addition to the diagonalization term. For the details of such an additional term, see the second term of Eq. (48) in Ref. [13] or the second term of Eq. (3) in Ref. [14] for examples. Because we do not know how to manage such a term exactly in the diagonalization of the Hamiltonian, it may be better to adopt an alternative method.

We introduce an invariant operator as such an alternative [15, 16] and diagonalize it instead of directly diagonalizing the Hamiltonian. Because an additional term does not appear in the diagonalization of the invariant, we only need to diagonalize the invariant itself in this case. The diagonalization of a dynamical invariant for two coupled time-dependent oscillators has already been carried out by ours in Ref. [17] considering such an advantage. We extend it to three coupled time-dependent oscillators and decouple the couplings in the oscillators in this work.
However, the mathematical manage of the operator may not be so easy in this case because of the three coupling terms in the invariant in addition to the time variations of parameters.

We will organize this work as follows. In Sec. 2, we will establish an invariant operator for the Hamiltonian of time-dependent three coupled oscillators. Parametric conditions required for such an invariant formulation will be found and specified. By using the unitary transformation method, we will transform the invariant into a simple form in Sec. 3, which is identical to the collection of three Hamiltonians of the simple harmonic oscillators (SHOs). We can regard the Hamiltonian itself as an invariant operator intrinsically for the case of the SHO. The invariant operator will be transformed in two steps. At first, the invariant operator will be simplified by a preliminary transformation using an appropriate unitary operator. Through the next transformation, the invariant operator will be finally reduced to that of three independent SHOs that are much simpler: that is, the invariant operator will be diagonalized. The eigenfunctions and eigenvalues of such a simplified (i.e., transformed) invariant operator are well known from basic quantum mechanics. Eventually, in Sec. 4, the eigenfunctions in the original systems will be identified by inverse transformation of the ones in the transformed systems. Some concluding remarks will be given in the last section.

2. Formulation of the Invariant

We introduce the Hamiltonian of time-dependent three coupled oscillators as

\[
\hat{H}(t) = \frac{1}{2} \sum_{j=1}^{3} \left[ \frac{\hat{p}_j^2}{m_j(t)} + b_j(t)(\hat{x}_j\hat{p}_j + \hat{p}_j\hat{x}_j) + m_j(t)\omega_j^2(t)\hat{x}_j^2 \right] + d_{12}(t)\hat{x}_1\hat{x}_2 + d_{13}(t)\hat{x}_1\hat{x}_3 + d_{23}(t)\hat{x}_2\hat{x}_3, \tag{1}
\]

where the parameters \(m_j(t), b_j(t), \omega_j(t),\) and \(d_{jk}(t)\) vary over time, but in a fashion that they are differentiable with respect to time. The coordinates are coupled via \(d_{jk}(t)\) terms in this Hamiltonian as can be seen. Additionally, this Hamiltonian involves \(b_j(t)\) terms that are frequently appeared in the mathematical treatment of damped oscillatory systems \[18, 21\].

Before we start to formulate a quantum invariant, let us briefly study the classical behavior
of oscillators based on their classical equations of motion which are

\[ \ddot{x}_1 + \frac{\dot{m}_1}{m_1} \dot{x}_1 + \tilde{\omega}_1^2 x_1 + \frac{d_{12}}{m_1} x_2 + \frac{d_{13}}{m_1} x_3 = 0, \quad (2) \]

\[ \ddot{x}_2 + \frac{\dot{m}_2}{m_2} \dot{x}_2 + \tilde{\omega}_2^2 x_2 + \frac{d_{12}}{m_2} x_1 + \frac{d_{23}}{m_2} x_3 = 0, \quad (3) \]

\[ \ddot{x}_3 + \frac{\dot{m}_3}{m_3} \dot{x}_3 + \tilde{\omega}_3^2 x_3 + \frac{d_{13}}{m_3} x_1 + \frac{d_{23}}{m_3} x_2 = 0. \quad (4) \]

Here, \( \tilde{\omega}_j \) are modified angular frequencies of the form

\[ \tilde{\omega}_j = \left( \omega_j^2 - b_j^2 - \dot{b}_j - \frac{\dot{m}_j}{m_j} \right)^{1/2}. \quad (5) \]

Equation (2) [Eq. (3), Eq. (4)] is not represented in terms of the canonical variable \( x_1 \) [\( x_2, x_3 \)] only, owing to the fact that the three variables are coupled unless \( d_{jk} = 0 \). Thereby the motion of each oscillator is affected by that of other oscillators through couplings. For instance, Eq. (2) reveals that the effect of oscillator 2 (oscillator 3) on oscillator 1 is large when \( d_{12} \) (\( d_{13} \)) is great; but it is relatively small when \( m_1 \) is large. The classical motion of other oscillators can also be interpreted in the same way through the use of Eqs. (3) and (4).

Let us now formulate a quadratic invariant for the time-dependent coupled oscillators, which is useful in developing quantum theory of the systems. We assume that the invariant operator is represented in the form

\[ \hat{\mathbf{I}}(t) = \frac{1}{2} \sum_{j=1}^3 \left[ \alpha_j(t) \hat{p}_j^2 + \beta_j(t) (\hat{x}_j \hat{p}_j + \hat{p}_j \hat{x}_j) + \gamma_j(t) \hat{x}_j^2 \right] + \delta_{12}(t) \hat{x}_1 \hat{x}_2 + \delta_{13}(t) \hat{x}_1 \hat{x}_3 + \delta_{23}(t) \hat{x}_2 \hat{x}_3, \quad (6) \]

where \( \alpha_j(t), \beta_j(t), \gamma_j(t), \) and \( \delta_{jk}(t) \) are time-dependent coefficients that will be derived now. We take the dimension of \( \hat{\mathbf{I}}(t) \) as energy in this case as in the two coupled oscillators managed in Ref. [17]. By using the Liouville-von Neumann equation,

\[ \frac{d\hat{\mathbf{I}}}{dt} = \frac{\partial \hat{\mathbf{I}}}{\partial t} + \frac{1}{i\hbar} [\hat{\mathbf{I}}, \hat{\mathbf{H}}] = 0, \quad (7) \]

we can confirm that the coefficients should satisfy the equations

\[ \dot{\alpha}_j(t) = 2b_j(t) \alpha_j(t) - \frac{2\beta_j(t)}{m_j(t)}, \quad (8) \]
\begin{align}
\dot{\beta}_j(t) &= m_j(t)\alpha_j(t)\omega_j^2(t) - \gamma_j(t) \frac{\gamma_j(t)}{m_j(t)}, \\
\dot{\gamma}_j(t) &= -2b_j(t)\gamma_j(t) + 2m_j(t)\beta_j(t)\omega_j^2(t),
\end{align}

\begin{align}
\dot{\delta}_{12}(t) &= -\delta_{12}(t)[b_1(t) + b_2(t)] + d_{12}(t)[\beta_1(t) + \beta_2(t)], \\
\dot{\delta}_{13}(t) &= -\delta_{13}(t)[b_1(t) + b_3(t)] + d_{13}(t)[\beta_1(t) + \beta_3(t)], \\
\dot{\delta}_{23}(t) &= -\delta_{23}(t)[b_2(t) + b_3(t)] + d_{23}(t)[\beta_2(t) + \beta_3(t)],
\end{align}

\begin{align}
\frac{\delta_{12}(t)}{d_{12}(t)} &= \frac{\delta_{13}(t)}{d_{13}(t)} = \frac{\delta_{23}(t)}{d_{23}(t)} = F(t),
\end{align}

where \( F(t) = \alpha_1(t)m_1(t) \) under the requirement

\begin{align}
\alpha_1(t)m_1(t) = \alpha_2(t)m_2(t) = \alpha_3(t)m_3(t).
\end{align}

To determine the coefficients, we first put \( \alpha_j(t) \) as the same formula as that we adopted in the case of two coupled oscillators [17]:

\begin{align}
\alpha_j(t) = \alpha_{0,j}\rho_j^2(t),
\end{align}

where \( \rho_j \) are solutions of the following auxiliary equation

\begin{align}
\ddot{\rho}_j + \frac{\dot{m}_j}{m_j} \dot{\rho}_j + \omega_j^2(t)\rho_j = \frac{\Omega_j^2}{4m_j^2\rho_j^2},
\end{align}

which are real, while \( \Omega_j \) are real constants. Then, \( \beta_j(t) \) and \( \gamma_j(t) \) are also determined like in the case of two coupled oscillators [17], such that

\begin{align}
\beta_j(t) &= \alpha_{0,j}m_j(t)[b_j(t)\rho_j^2(t) - \rho_j(t)\dot{\rho}_j(t)], \\
\gamma_j(t) &= \alpha_{0,j} \left[ \frac{\Omega_j^2}{4\rho_j^2(t)} + m_j^2(t)\left(b_j^2(t)\rho_j^2(t) - 2b_j(t)\rho_j(t)\dot{\rho}_j(t) + \dot{\rho}_j^2(t)\right) \right].
\end{align}

While, according to Eq. (14), we can put \( \delta_{jk}(t) \) as

\begin{align}
\delta_{12}(t) &= F(t)d_{12}(t), \\
\delta_{13}(t) &= F(t)d_{13}(t), \\
\delta_{23}(t) &= F(t)d_{23}(t),
\end{align}
Eqs. (11)-(13) give the requirements that $d_{jk}(t)$ should follow. Rigorous evaluations show that such requirements are the relations of the form

\[
\dot{d}_{12}(t) = -G_{12}(t)d_{12}(t),
\]

\[
\dot{d}_{13}(t) = -G_{13}(t)d_{13}(t),
\]

\[
\dot{d}_{23}(t) = -G_{23}(t)d_{23}(t),
\]

where

\[
G_{12}(t) = \frac{\dot{m}_3(t)}{m_3(t)} + \frac{\dot{\rho}_1(t)}{\rho_1(t)} + \frac{\dot{\rho}_2(t)}{\rho_2(t)} + \frac{2\dot{\rho}_3(t)}{\rho_3(t)},
\]

\[
G_{13}(t) = \frac{\dot{m}_2(t)}{m_2(t)} + \frac{\dot{\rho}_1(t)}{\rho_1(t)} + \frac{2\dot{\rho}_2(t)}{\rho_2(t)} + \frac{\dot{\rho}_3(t)}{\rho_3(t)},
\]

\[
G_{23}(t) = \frac{\dot{m}_1(t)}{m_1(t)} + \frac{2\dot{\rho}_1(t)}{\rho_1(t)} + \frac{\dot{\rho}_2(t)}{\rho_2(t)} + \frac{\dot{\rho}_3(t)}{\rho_3(t)}.
\]

The methodology of deriving $G_{jk}(t)$ is represented in Appendix A. Thus, Eq. (6) with Eqs. (16), (18), (19), (20), (21), and (22) is the invariant operator. This operator is valid under the two groups of conditions, where the first group is given by Eq. (15) and the second group by Eqs. (23)-(25). Complete quantum description of coupled oscillatory systems may be possible through the use of this dynamical invariant.

3. Unitary Relations

Because the formula of the invariant derived in the previous section is somewhat complicated, its direct use in unfolding the associated quantum theory is not favorable. Instead, developing quantum theory of the systems with the help of the invariant operator simplified by unitary or canonical transformations may be better. We will adopt the unitary transformation method among the two for that purpose in this section. We first transform the invariant operator using a procedure adopted in Ref. [17] as

\[
\hat{\mathcal{I}}_A(t) = \hat{U}_A^{-1}\hat{\mathcal{I}}(t)\hat{U}_A,
\]

where $\hat{\mathcal{I}}_A(t)$ is a transformed invariant operator and $\hat{U}_A$ is a unitary operator transforming the
invariant, which reads

$$\hat{U}_A = \hat{U}_{A1} \hat{U}_{A2},$$

(30)

whereas

$$\hat{U}_{A1} = \prod_{j=1}^{3} \exp \left( \frac{i}{2\hbar} (\hat{p}_j \hat{x}_j + \hat{x}_j \hat{p}_j) \ln \sqrt{\frac{1}{M\alpha_j(t)}} \right),$$

(31)

$$\hat{U}_{A2} = \exp \left( -\frac{i}{2\hbar} \sum_{j=1}^{3} M\beta_j(t) \hat{x}_j^2 \right).$$

(32)

Then, using a little bit of algebra, the invariant operator is transformed into

$$\hat{I}_A(t) = \frac{1}{2} \sum_{j=1}^{3} \left( \frac{\hat{p}_j^2}{M} + M\omega_{0,j}^2 \hat{x}_j^2 \right) + M\delta_{12}(t) \sqrt{\alpha_1(t)\alpha_2(t)} \hat{x}_1 \hat{x}_2$$

$$+ M\delta_{13}(t) \sqrt{\alpha_1(t)\alpha_3(t)} \hat{x}_1 \hat{x}_3 + M\delta_{23}(t) \sqrt{\alpha_2(t)\alpha_3(t)} \hat{x}_2 \hat{x}_3,$$

(33)

where

$$\omega_{0,j}^2 = \alpha_j(t)\gamma_j(t) - \beta_j^2(t) = \frac{\alpha_{0,j}\Omega_j^2}{4}.$$

(34)

We can also represent $$\hat{I}_A(t)$$ in a matrix form, namely

$$\hat{I}_A(t) = \frac{1}{2M} \mathbf{p}^T \mathbf{p} + \frac{1}{2} M \mathbf{x}^T \Gamma \mathbf{x},$$

(35)

where $$\mathbf{x}^T = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$$, $$\mathbf{p}^T = (\hat{p}_1, \hat{p}_2, \hat{p}_3)$$, and

$$\Gamma = \begin{pmatrix} \omega_{0,1}^2 & \Delta_{12} & \Delta_{13} \\ \Delta_{12} & \omega_{0,2}^2 & \Delta_{23} \\ \Delta_{13} & \Delta_{23} & \omega_{0,3}^2 \end{pmatrix},$$

(36)

while $$\Delta_{jk} = \delta_{jk}(t) \sqrt{\alpha_j(t)\alpha_k(t)}$$. From the straightforward evaluations of the time derivatives of $$\Delta_{jk}$$ using Eqs. (16) and (20)-(22) with Eqs. (23)-(25), we have $$d\Delta_{jk}/dt = 0$$. This means that $$\Delta_{jk}$$ are constants. Because $$\omega_{0,j}^2$$ are also constants as can be seen from Eq. (34), all elements of $$\Gamma$$ are constants.

If we denote the eigenvalues of $$\Gamma$$ by $$\varpi_{0,1}^2$$, $$\varpi_{0,2}^2$$, and $$\varpi_{0,3}^2$$, they are known in the literature (see Appendix B). However, we are rather interested in the diagonalization of $$\Gamma$$ and the resultant formulae of eigenvalues attained by a rotational unitary transformation with certain angles
instead of them. This procedure is necessary for the whole description of the unitary transformation that we have initially planed. Attaining complete knowledge for quantum characteristics of the original systems may be possible only through the full process of that transformation.

The transformed invariant operator, \( \hat{\mathcal{I}}_A(t) \), is simpler than the original operator, \( \hat{\mathcal{I}}(t) \), since the terms in the parenthesis in Eq. (33) are identical to the Hamiltonian of SHOs. However, \( \hat{\mathcal{I}}_A(t) \) involves three coupling terms that must be removed through a further transformation. In order to eliminate them, we consider the following transformation as the next step:

\[
\hat{\mathcal{I}}_B(t) = \hat{U}^{-1}_B \hat{\mathcal{I}}_A(t) \hat{U}_B,
\]

where the unitary operator \( \hat{U}_B \) is of the form

\[
\hat{U}_B = \hat{U}_B^1 \hat{U}_B^2 \hat{U}_B^3,
\]

whereas

\[
\hat{U}_B^1 = \exp \left( -\frac{i\phi}{\hbar} (\hat{p}_3 \hat{x}_2 - \hat{p}_2 \hat{x}_3) \right),
\]

\[
\hat{U}_B^2 = \exp \left( -\frac{i\theta}{\hbar} (\hat{p}_1 \hat{x}_3 - \hat{p}_3 \hat{x}_1) \right),
\]

\[
\hat{U}_B^3 = \exp \left( -\frac{i\varphi}{\hbar} (\hat{p}_2 \hat{x}_1 - \hat{p}_1 \hat{x}_2) \right).
\]

In fact this transformation corresponds to a rotation of the matrix formula of \( \hat{\mathcal{I}}_A(t) \) expressed in Eq. (35):

\[
\hat{\mathcal{I}}_B(t) = \frac{1}{2M} \hat{p}^T \hat{p} + \frac{1}{2} M \hat{x}^T \mathcal{R} \Gamma \mathcal{R} \hat{x},
\]

where \( \mathcal{R} \) is the rotation matrix that is given by (see Appendix C)

\[
\mathcal{R} = \begin{pmatrix}
\cos \theta \cos \varphi & -\cos \theta \sin \varphi & \sin \theta \\
\cos \phi \sin \varphi + \sin \phi \sin \theta \cos \varphi & \cos \phi \cos \varphi - \sin \phi \sin \theta \sin \varphi & -\sin \phi \cos \theta \\
\sin \phi \sin \varphi - \cos \phi \sin \theta \cos \varphi & \sin \phi \cos \varphi + \cos \phi \sin \theta \sin \varphi & \cos \phi \cos \theta
\end{pmatrix}.
\]

In what follows, the transformation in Eq. (37) results in

\[
\hat{\mathcal{I}}_B(t) = \frac{1}{2} \sum_{j=1}^3 \left( \frac{\hat{p}_j^2}{M} + M\omega_{0,j}^2 \hat{x}_j^2 \right) + M\delta_{12} \hat{x}_1 \hat{x}_2 + M\delta_{13} \hat{x}_1 \hat{x}_3 + M\delta_{23} \hat{x}_2 \hat{x}_3,
\]
where

\[
\bar{\omega}_{0,1}^2 = \omega_{0,1}^2 \cos^2 \theta \cos^2 \varphi + \omega_{0,2}^2 (\sin \phi \sin \theta \cos \varphi + \cos \phi \sin \varphi)^2 \\
+ \omega_{0,3}^2 (\cos \phi \sin \theta \cos \varphi - \sin \phi \sin \varphi)^2 + 2 \{ \Delta_{12} \cos \theta \cos \varphi \\
\times (\sin \phi \sin \theta \cos \varphi + \cos \phi \sin \varphi) + \Delta_{13} \cos \theta \cos \varphi \\
\times (\sin \phi \sin \varphi - \cos \phi \sin \theta \cos \varphi) + \Delta_{23} \sin \theta \cos \varphi \sin \varphi \\
\times (\sin^2 \phi - \cos^2 \phi) + \cos \phi \sin \phi (\sin^2 \varphi - \sin^2 \theta \cos^2 \varphi) \}\},
\]

(45)

\[
\bar{\omega}_{0,2}^2 = \omega_{0,1}^2 \cos^2 \theta \sin^2 \varphi + \omega_{0,2}^2 (\cos \phi \cos \varphi - \sin \phi \sin \theta \sin \varphi)^2 \\
+ \omega_{0,3}^2 (\sin \phi \cos \varphi + \cos \phi \sin \theta \sin \varphi)^2 + 2 \{ \Delta_{12} \cos \theta \sin \varphi \\
\times (\sin \phi \sin \theta \sin \varphi - \cos \phi \cos \varphi) - \Delta_{13} \cos \theta \sin \varphi \\
\times (\cos \phi \sin \theta \sin \varphi + \sin \phi \cos \varphi) + \Delta_{23} \sin \theta \cos \varphi \sin \varphi \\
\times (\cos^2 \phi - \sin^2 \phi) + \cos \phi \sin \phi (\cos^2 \varphi - \sin^2 \theta \sin^2 \varphi) \}\},
\]

(46)

\[
\bar{\omega}_{0,3}^2 = \omega_{0,1}^2 \sin^2 \theta + \omega_{0,2}^2 \sin^2 \phi \cos^2 \theta + \omega_{0,3}^2 \cos^2 \phi \cos^2 \theta \\
- 2 \{ \Delta_{12} \sin \phi \sin \theta \cos \theta - \Delta_{13} \cos \phi \cos \theta \sin \theta \\
+ \Delta_{23} \cos \phi \sin \phi \cos^2 \theta \}\).
\]

(47)

By the way, we have represented the formulae of \( \delta_{jk} \) appeared in Eq. (44), separately, in Appendix D, since they are much more complicated and not so essential. Now the matrix formula of \( \hat{I}_B(t) \) becomes

\[
\hat{I}_B(t) = \frac{1}{2M} \mathbf{p}^T \mathbf{p} + \frac{1}{2} \mathbf{m}^T \bar{\Gamma} \mathbf{m},
\]

(48)

where

\[
\bar{\Gamma} = \begin{pmatrix} \bar{\omega}_{0,1}^2 & \bar{\delta}_{12} & \bar{\delta}_{13} \\ \bar{\delta}_{12} & \bar{\omega}_{0,2}^2 & \bar{\delta}_{23} \\ \bar{\delta}_{13} & \bar{\delta}_{23} & \bar{\omega}_{0,3}^2 \end{pmatrix}.
\]

(49)

For the purpose of diagonalization of Eq. (49), we take angles as

\[
\phi = \text{atan}(u_{\phi},v_{\phi}),
\]

(50)

\[
\theta = \text{atan}(u_{\theta},v_{\theta}),
\]

(51)

\[
\varphi = \pm \text{atan}(u_{\varphi},v_{\varphi}),
\]

(52)
where \( \vartheta \equiv \text{atan}(z_1, z_2) \) is the two-variable arctangent function of \( \tan \vartheta = z_2/z_1 \), and

\[
\begin{align*}
    u_\phi &= 2(\varpi^2_{0,1} - \varpi^2_{0,3})(\varpi^2_{0,2} - \varpi^2_{0,3})(\varpi^2_{0,3} - \omega^2_{0,1})\Delta_{13}\sin \theta - (\varpi^2_{0,1} - \varpi^2_{0,2}) \\
    &\quad \times [(\varpi^2_{0,1} + \varpi^2_{0,2})(\varpi^2_{0,3} - \omega^2_{0,1}) - \varpi^4_{0,3} + \omega^4_{0,1} + \Delta^2_{12} + \Delta^2_{13}]\Delta_{12}\sin(2\varphi), \\
    v_\phi &= 2(\varpi^2_{0,1} - \varpi^2_{0,3})(\varpi^2_{0,2} - \varpi^2_{0,3})(\omega^2_{0,1} - \varpi^2_{0,3})\Delta_{12}\sin \theta - (\varpi^2_{0,1} - \varpi^2_{0,2}) \\
    &\quad \times [(\varpi^2_{0,1} + \varpi^2_{0,2})(\varpi^2_{0,3} - \omega^2_{0,1}) - \varpi^4_{0,3} + \omega^4_{0,1} + \Delta^2_{12} + \Delta^2_{13}]\Delta_{13}\sin(2\varphi), \\
    u_\theta &= [(\varpi^2_{0,3} - \omega^2_{0,1})(\varpi^2_{0,3} - \varpi^2_{0,1} - \varpi^2_{0,2} + \omega^2_{0,1}) - \Delta^2_{12} - \Delta^2_{13}]^{1/2}, \\
    v_\theta &= \{[(\varpi^2_{0,1} - \omega^2_{0,2})\omega^2_{0,1} + \varpi^2_{0,1}\varpi^2_{0,2} + \Delta^2_{12} + \Delta^2_{13}]^{1/2}, \\
    u_\varphi &= \{-[(\varpi^2_{0,3} - \omega^2_{0,1})\omega^2_{0,1} + \varpi^2_{0,1}(\varpi^2_{0,3} - \omega^2_{0,1}) - \varpi^2_{0,3}\omega^2_{0,1} + \Delta^2_{12} + \Delta^2_{13}]^{1/2}, \\
    v_\varphi &= \{[(\varpi^2_{0,2} - \omega^2_{0,3})\omega^2_{0,1} + \varpi^2_{0,2}(\varpi^2_{0,3} - \omega^2_{0,1}) - \varpi^2_{0,3}\omega^2_{0,1} + \Delta^2_{12} + \Delta^2_{13}]^{1/2},
\end{align*}
\]  

(53)

(54)

(55)

(56)

(57)

(58)

while \( \varpi^2_{0,j} \) are given in Appendix B. The function \( \text{atan}(z_1, z_2) \) is defined during one cycle: for instance, it is defined in the range \(-\pi < \vartheta \leq \pi \) in Mathematica program [26]. There also exist other diagonalization sets of angles instead of Eqs. (50)-(52), and we have represented them in Appendix E.

There are two categories of the matrix \( \Gamma \) in this context, where the plus sign in Eq. (52) is applied to the first category (class 1) whereas the minus sign to the second category (class 2). To see the details of the two classes of \( \Gamma \), let us look the transformation \( \mathbb{R}^T \Gamma \mathbb{R} \) in Eq. (42), which can be fulfilled in relation with \( \phi, \theta, \) and \( \varphi \) in turn using Eq. (C1) in Appendix C. We consider the transformation up to \( \theta \) in this process:

\[
\Gamma_\theta = \mathbb{R}^T_{x_2}(\theta) \mathbb{R}^T_{x_1}(\phi) \Gamma_\phi \mathbb{R}_{x_1}(\phi) \mathbb{R}_{x_2}(\theta),
\]

(59)

where \( \phi \) is being expressed in terms of \( \varphi \) using Eq. (52) with the plus sign. Then \( \Gamma \) is the first category if and only if this procedure yields \( \delta'_{13} = \delta'_{23} = 0 \), where \( \delta'_{13} (\delta'_{23}) \) is an element of \( \Gamma_\theta \), which corresponds to the first row and third column (the second row and third column); \( \Gamma \) is the second category otherwise. This manifestation is the definition of the two classes of \( \Gamma \) or the rule for distinguishing them from each other. In fact, for the case of class 2, the transformation, Eq. (59), with the choice of \( \phi \) represented in terms of \( \varphi \) in Eq. (52) with the minus sign gives \( \delta'_{13} = \delta'_{23} = 0 \).
We see that all $\bar{\delta}_{jk}$ in Appendix D reduce to zero by choosing angles as Eqs. (50)-(52), leading to attaining the diagonalization of $\bar{\Gamma}$. Meanwhile, the momentum parts in Eq. (44) do not altered by this transformation. Thus the finally transformed invariant is just written as

$$\hat{I}_B = \frac{1}{2} \sum_{j=1}^{3} \left( \frac{\hat{p}_j^2}{M} + M\bar{\omega}_{0,j}^2 \hat{x}_j^2 \right).$$

Although the formulae of $\bar{\omega}_{0,j}^2$ in this equation are somewhat complicated as can be seen from Eqs. (45)-(47), they are constants over time because they are represented in terms of $\omega_{0,j}^2$ and $\Delta_{jk}$ only, which are already proved to be constants. $\bar{\omega}_{0,j}^2$ are mathematically equivalent to $\bar{\omega}_{0,j}^2$ given in Appendix B respectively, since the considered rotational angles, Eqs. (50)-(52), are eigenangles. For actual cases, it is better to treat the transformed systems by replacing $\bar{\omega}_{0,j}^2$ with $\bar{\omega}_{0,j}^2$ in Eq. (60) because $\bar{\omega}_{0,j}^2$ are much simpler in a relative sense. The matrix, Eq. (36), is positive-definite when and only when all of the leading principal minors are positive according to the Sylvester’s criterion [27]. For more detailed descriptions of the condition for the existence of such positive-definite eigenvalues, refer for example to Ref. [28].

4. Quantum Solutions

We will show in this section that the previous formulation of the invariant and the related unitary relations can be utilized to derive quantum solutions of the systems. We introduce annihilation operators associated with SHOs for that purpose, such that

$$\hat{a}_{0,j} = \sqrt{\frac{M \bar{\omega}_{0,j}}{2\hbar}} \hat{x}_j + \frac{i}{\sqrt{2M\bar{\omega}_{0,j}\hbar}} \hat{p}_j,$$

and the corresponding creation operators as the Hermitian adjoint of Eq. (61), $\hat{a}_{0,j}^\dagger$. Then, it is possible to represent Eq. (60) in the form

$$\hat{I}_B = \sum_{j=1}^{3} \hbar \bar{\omega}_{0,j} \left( \hat{a}_{0,j}^\dagger \hat{a}_{0,j} + \frac{1}{2} \right).$$

The annihilation operators, $\hat{a}_j$, in the original systems are related to $\hat{a}_{0,j}$ by

$$\hat{a}_j = \hat{U}_A \hat{U}_B \hat{a}_{0,j} \hat{U}_B^{-1} \hat{U}_A^{-1}.$$
The complete formulae of $\hat{a}_j$ are shown in Appendix F. Now we can express the invariant operator in the original systems in terms of $\hat{a}_j$ and their Hermitian adjoints $\hat{a}_j^\dagger$ (creation operators):

$$\hat{I}(t) = \sum_{j=1}^{3} \hbar \tilde{\omega}_{0,j} \left( \hat{a}_j^\dagger \hat{a}_j + \frac{1}{2} \right).$$  \hspace{1cm} (64)

Let us write the eigenvalue equations for the lastly transformed invariant operator as

$$\hat{I}_B u_{0,n_1,n_2,n_3}(x_1, x_2, x_3) = \lambda_{n_1,n_2,n_3} u_{0,n_1,n_2,n_3}(x_1, x_2, x_3),$$  \hspace{1cm} (65)

where $\lambda_{n_1,n_2,n_3}$ are eigenvalues and $u_{0,n_1,n_2,n_3}(x_1, x_2, x_3)$ are eigenfunctions. $\lambda_{n_1,n_2,n_3}$ are constants because $\hat{I}_B$ is independent of time. By solving Eq. (65), we have the familiar eigenfunctions and eigenvalues which are

$$u_{0,n_1,n_2,n_3}(x_1, x_2, x_3) = \prod_{j=1}^{3} 4^{\frac{1}{\sqrt{2n_j N_j}}} \sqrt{\frac{M}{\hbar}} H_{n_j} \left( \sqrt{\frac{M}{\hbar}} x_j \right) \exp \left[ -\frac{M}{2\hbar} \frac{\tilde{\omega}_{0,j}}{x_j^2} \right],$$  \hspace{1cm} (66)

$$\lambda_{n_1,n_2,n_3} = \sum_{j=1}^{3} \hbar \tilde{\omega}_{0,j} \left( n_j + \frac{1}{2} \right).$$  \hspace{1cm} (67)

The eigenvalue equations in the original systems can also be written as

$$\hat{I}(t) u_{n_1,n_2,n_3}(x_1, x_2, x_3, t) = \lambda_{n_1,n_2,n_3} u_{n_1,n_2,n_3}(x_1, x_2, x_3, t),$$  \hspace{1cm} (68)

where $u_{n_1,n_2,n_3}(x_1, x_2, x_3, t)$ are eigenfunctions, while the eigenvalues are the same as Eq. (67).

The mathematical relation between the eigenfunctions in the original systems and those in the transformed systems is given by

$$u_{n_1,n_2,n_3}(x_1, x_2, x_3, t) = \hat{U} u_{0,n_1,n_2,n_3}(x_1, x_2, x_3),$$  \hspace{1cm} (69)

where

$$\hat{U} = \hat{U}_A \hat{U}_B.$$  \hspace{1cm} (70)

Note that this relation is inverse of the previous transformation. By evaluating Eq. (69), we easily have

$$u_{n_1,n_2,n_3}(x_1, x_2, x_3, t) = \prod_{j=1}^{3} 4^{\frac{1}{\sqrt{2n_j N_j}}} \sqrt{\frac{\omega_{0,j}}{\pi \hbar \alpha_j(t)}} \sqrt{\frac{1}{2n_j N_j}} H_{n_j} \left( \sqrt{\frac{\omega_{0,j}}{\hbar}} X_j \right) \exp \left[ -\frac{1}{2\hbar} \left( \tilde{\omega}_{0,j} X_j^2 + i \beta_j(t) \alpha_j(t) X_j^2 \right) \right],$$  \hspace{1cm} (71)
where

\[
\begin{pmatrix}
X_1 \\
X_2 \\
X_3
\end{pmatrix}
= \mathbb{R}^T \begin{pmatrix}
\alpha_1^{-1/2}(t)x_1 \\
\alpha_2^{-1/2}(t)x_2 \\
\alpha_3^{-1/2}(t)x_3
\end{pmatrix}.
\]  

(72)

Hence, the eigenfunctions of the complicated original invariant operator are derived by inversely transforming the simple eigenfunctions associated with the transformed invariant operator. These eigenfunctions are basic in the study of quantum features of the systems.

5. Conclusion

A general dynamical invariant operator of time-dependent three coupled oscillators was formulated based on its mathematical definition. The parameters of the oscillatory systems that we have considered vary in the most general way in time so long as the restrictions raised in the invariant formulation allow. The invariant operator was diagonalized by its unitary transformations. From such a procedure, the unitary relation between the original invariant operator and the one for SHOs was elucidated.

The transformation of the invariant was performed in two steps. The invariant operator was simplified by the first transformation as can be seen from Eq. (33), but it still involves three cross terms. Through the second transformation, the cross terms have been removed and, as a consequence, the invariant operator has been represented in terms of constant parameters only.

Using the fact that the eigenfunctions of the transformed (or diagonalized) invariant operator are well known, we obtained the eigenfunctions associated with the original systems via the inverse transformation of such known ones. Our analysis in this work is exact provided that the two groups of conditions in parameteric variations given in the text hold. In contrast to this, approximations have been usually employed in the previous analyses of time-dependent coupled oscillators, under the assumption of the adiabatic evolution of the systems [24, 25] or sudden quenches of parameters [14, 29–31]. Some authors otherwise considered particular cases where the additional term does not appear in the transformation of the Hamiltonian or at least it can be neglected [13, 32].
It may be possible to use our analysis of the dynamical invariant in characterizing quantum properties of coupled oscillatory systems, such as nano-optomechanical systems [25, 33, 34], arrays of electromechanical devices [35], and biological/neural oscillator networks [36]. According to the recent trend that the size of optomechanical and electromechanical devices becomes smaller and smaller towards nanoscale, the quantum features in such systems become prominent. Our dynamical invariant developed in this work is crucial as a tool of quantum analyses of them, because a large part of such devices are described by using a model of coupled oscillators.

Appendix A: About the formula of $G_{jk}(t)$

Equations (23)-(25) which involve $G_{jk}(t)$ are evaluated using Eqs. (11)-(13), respectively. We see from Eq. (15) that $F(t)$ can be represented in three other forms. Among them, $F(t) = \alpha_3(t)m_3(t)$ is used when we derive Eq. (23) from Eq. (11), $F(t) = \alpha_2(t)m_2(t)$ is used when we derive Eq. (24) from Eq. (12), and $F(t) = \alpha_1(t)m_1(t)$ is used when we derive Eq. (25) from Eq. (13).

However, other combinations of the formulae of $F(t)$ are equally allowed in the derivations of the three $G_{jk}(t)$. For instance, if we use $F(t) = \alpha_1(t)m_1(t)$ for all three derivations of Eqs. (23)-(25), we have.

$$G_{12}(t) = \frac{\dot{m}_1(t)}{m_1(t)} + \frac{3\dot{\rho}_1(t)}{\rho_1(t)} + \frac{\dot{\rho}_2(t)}{\rho_2(t)}, \quad \text{(A1)}$$

$$G_{13}(t) = \frac{\dot{m}_1(t)}{m_1(t)} + \frac{3\dot{\rho}_1(t)}{\rho_1(t)} + \frac{\dot{\rho}_3(t)}{\rho_3(t)}, \quad \text{(A2)}$$

whereas $G_{23}(t)$ is already given in Eq. (25) in the text (i.e., it is not altered). The formulae of $G_{jk}(t)$ obtained with the use of other combinations of the formulae of $F(t)$ can now be easily conjectured through the expressions given in Eqs. (A1) and (A2), and other expressions along this line in the text.
Appendix B: The eigenvalues of $\Gamma$

The eigenvalues $\varpi^2_{0,j}$ of $\Gamma$ appeared in Eq. (36) have been reported in previous literature \[37, 38\]. They are given by

$$\varpi^2_{0,1} = \frac{1}{3} \left[ \omega_0^2 + \frac{J}{\sqrt{2}} \cos \Theta \right],$$  
(B1)

$$\varpi^2_{0,2} = \frac{1}{3} \left[ \omega_0^2 + \frac{J}{\sqrt{2}} \cos \left( \Theta - \frac{2\pi}{3} \right) \right],$$  
(B2)

$$\varpi^2_{0,3} = \frac{1}{3} \left[ \omega_0^2 + \frac{J}{\sqrt{2}} \cos \left( \Theta + \frac{2\pi}{3} \right) \right],$$  
(B3)

where $\omega_0 = (\omega^2_{0,1} + \omega^2_{0,2} + \omega^2_{0,3})^{1/2}$ and

$$J = 2 \left[ (\omega^2_{0,1} - \omega^2_{0,2})^2 + (\omega^2_{0,1} - \omega^2_{0,3})^2 + (\omega^2_{0,2} - \omega^2_{0,3})^2 + 6\Delta^2 \right]^{1/2},$$  
(B4)

$$\Theta = \frac{1}{3} \arccos \left( \frac{A}{2B^{3/2}} \right),$$  
(B5)

with

$$\Delta = \left( \Delta^2_{12} + \Delta^2_{13} + \Delta^2_{23} \right)^{1/2},$$  
(B6)

$$A = -3(\omega^2_{0,1} + \omega^2_{0,2})(\omega^2_{0,1} + \omega^2_{0,3})(\omega^2_{0,2} + \omega^2_{0,3})$$

$$-27(\omega^2_{0,1}\Delta^2_{23} + \omega^2_{0,2}\Delta^2_{13} + \omega^2_{0,3}\Delta^2_{12}) + 9\omega^4_0 \Delta^2$$

$$+2(\omega^6_{0,1} + \omega^6_{0,2} + \omega^6_{0,3}) + 18(\omega^2_{0,1}\omega^2_{0,2}\omega^2_{0,3} + 3\Delta_{12}\Delta_{13}\Delta_{23}),$$  
(B7)

$$B = \frac{1}{2} \left[ (\omega^2_{0,1} - \omega^2_{0,2})^2 + (\omega^2_{0,1} - \omega^2_{0,3})^2 + (\omega^2_{0,2} - \omega^2_{0,3})^2 \right] + 3\Delta^2.$$  
(B8)

Note that the magnitudes of $\varpi^2_{0,i}$ are in decreasing order: $\varpi^2_{0,1} \geq \varpi^2_{0,2} \geq \varpi^2_{0,3}$.

Appendix C: About the formula of $R$

The matrix $R$ given in Eq. (13) can be rewritten by rotation matrices of each angle as \[13\]

$$R = R_{x_1}(\phi)R_{x_2}(\theta)R_{x_3}(\varphi)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$  
(C1)
Appendix D: The representation of $\bar{\delta}_{jk}$

The mathematical representations of $\bar{\delta}_{jk}$ appeared in Eq. (44) are

$$
\bar{\delta}_{12} = -\omega_{0,1}^{2} \cos^{2} \theta \cos \varphi \sin \varphi + \omega_{0,2}^{2} [\cos \phi \sin \phi \sin \theta \cos(2\varphi) \\
+ \cos \varphi \sin \varphi (\cos^{2} \phi - \sin^{2} \phi \sin^{2} \theta)] - \omega_{0,3}^{2} [\cos \phi \sin \phi \sin \theta \cos(2\varphi) \\
+ \cos \varphi \sin \varphi (\cos^{2} \phi \sin^{2} \theta - \sin^{2} \phi)] + \Delta_{12} \cos \theta [\cos \phi \cos(2\varphi) \\
- 2 \sin \phi \sin \theta \cos \varphi \sin \varphi] + \Delta_{13} \cos \theta [\sin \phi (\cos^{2} \varphi - \sin^{2} \varphi) \\
+ \cos \phi \sin \theta \sin(2\varphi)] + \Delta_{23} \{\sin \theta \cos(2\varphi)(\sin^{2} \phi - \cos^{2} \phi) \\
+ (1/4)[3 - \cos(2\theta)] \sin(2\phi) \sin(2\varphi)],
$$

(D1)

$$
\bar{\delta}_{13} = \omega_{0,1}^{2} \cos \theta \sin \theta \cos \varphi - \omega_{0,2}^{2} \sin \phi \cos \theta (\sin \phi \sin \theta \cos \varphi \\
+ \cos \phi \sin \varphi) + \omega_{0,3}^{2} \cos \phi \cos \theta (\sin \phi \sin \varphi - \cos \phi \sin \theta \cos \varphi) \\
- \Delta_{12} [\sin \phi \cos^{2} \theta \cos \varphi - \sin \theta (\sin \phi \sin \theta \cos \varphi + \cos \phi \sin \varphi)] \\
+ \Delta_{13} [\cos \phi \cos^{2} \theta \cos \varphi + \sin \theta (\sin \phi \sin \varphi - \cos \phi \sin \theta \cos \varphi)] \\
+ \Delta_{23} \cos \theta [\sin(2\phi) \sin \theta \cos \varphi + \cos(2\phi) \sin \varphi],
$$

(D2)

$$
\bar{\delta}_{23} = -\omega_{0,1}^{2} \cos \theta \sin \theta \sin \varphi + \omega_{0,2}^{2} \sin \phi \cos \theta (\sin \phi \sin \theta \sin \varphi \\
- \cos \phi \cos \varphi) + \omega_{0,3}^{2} \cos \phi \cos \theta (\sin \phi \cos \varphi + \cos \phi \sin \theta \sin \varphi) \\
+ \Delta_{12} [\cos \phi \sin \theta \cos \varphi + \sin \phi \cos(2\theta) \sin \varphi] \\
+ \Delta_{13} [\sin \phi \sin \theta \cos \varphi - \cos \phi \cos(2\theta) \sin \varphi] \\
+ \Delta_{23} \cos \theta [\cos \varphi (\cos^{2} \phi - \sin^{2} \phi) - \sin(2\phi) \sin \theta \sin \varphi].
$$

(D3)

Appendix E: Diagonalization angles

The matrix $\Gamma$ can also be diagonalized by other angle sets instead of Eqs. (50)-(52). For instance, it is diagonalized by the unitary or matrix transformation using the following sets of angles $[\phi, \theta, \varphi]$:

$$
\text{atan}(u_{\phi}, v_{\phi}), \mp \text{atan}(u_{\theta}, v_{\theta}), \text{atan}(u_{\varphi}, -v_{\varphi}),
$$

(E1)
\[
\begin{align*}
[\tan(u\phi, v\phi), \pm \tan(u\theta, v\theta), -\tan(u\varphi, -v\varphi)], \quad (E2) \\
[\tan(u\phi, v\phi), \tan(u\theta, v\theta), \pm \tan(-u\varphi, -v\varphi)], \quad (E3) \\
[\tan(u\phi, v\phi), \tan(u\theta, -v\theta), \tan(u\varphi, \mp v\varphi)], \quad (E4) \\
[\tan(u\phi, v\phi), \tan(-u\theta, v\theta), \tan(\mp u\varphi, -v\varphi)], \quad (E5)
\end{align*}
\]

where upper signs are for the class 1 of \(\Gamma\) and lower signs for class 2. One can diagonalize \(\Gamma\) using a set of angles among six sets (the one in Eqs. (50)-(52) and the above five sets) or another set unknown yet, depending on one’s taste. However, there is no set of angles that can diagonalize both classes of \(\Gamma\). One should be careful that the first transformation for class 1 (class 2) of \(\Gamma\) must be carried out with respect to \(\phi\) represented in terms of \(\theta\) and \(\varphi\) with upper (lower) signs for all six sets.

**Appendix F: Full representation of \(\hat{a}_j\)**

The straightforward evaluation of Eq. (63) using Eqs. (30) and (38) gives

\[
\hat{a}_j = \sqrt{\frac{\bar{\omega}_{0,j}}{2\hbar}} \hat{X}_j + \frac{i}{\sqrt{2\bar{\omega}_{0,j}\hbar}} \hat{P}_j, \quad (F1)
\]

where

\[
\begin{align*}
\begin{pmatrix}
\hat{X}_1 \\
\hat{X}_2 \\
\hat{X}_3
\end{pmatrix} &= \mathbb{R}^T \begin{pmatrix}
\alpha_1^{-1/2}(t) \hat{x}_1 \\
\alpha_2^{-1/2}(t) \hat{x}_2 \\
\alpha_3^{-1/2}(t) \hat{x}_3
\end{pmatrix}, \quad (F2) \\
\begin{pmatrix}
\hat{P}_1 \\
\hat{P}_2 \\
\hat{P}_3
\end{pmatrix} &= \mathbb{R}^T \begin{pmatrix}
\alpha_1^{1/2}(t) \{\hat{p}_1 + [\beta_1(t)/\alpha_1(t)] \hat{x}_1\} \\
\alpha_2^{1/2}(t) \{\hat{p}_2 + [\beta_2(t)/\alpha_2(t)] \hat{x}_2\} \\
\alpha_3^{1/2}(t) \{\hat{p}_3 + [\beta_3(t)/\alpha_3(t)] \hat{x}_3\}
\end{pmatrix}. \quad (F3)
\end{align*}
\]
[1] E. Takou, E. Barnes, and S. E. Economou, Precise control of entanglement in multinuclear spin registers coupled to defects. arXiv:2203.09459v2 [quant-ph] (2022).

[2] W. Dong, F. A. Calderon-Vargas, and S. E. Economou, Precise high-fidelity electron-nuclear spin entangling gates in NV centers via hybrid dynamical decoupling sequences. New J. Phys. 22, 073059 (2020).

[3] C. E. Bradley, J. Randall, M. H. Abobeih, R. C. Berrevoets, M. J. Degen, M. A. Bakker, M. Markham, D. J. Twitchen, and T. H. Taminiau, A ten-qubit solid-state spin register with quantum memory up to one minute. Phys. Rev. X 9(3), 031045 (2019).

[4] G. Csaba and W. Porod, Coupled oscillators for computing: a review and perspective. Appl. Phys. Rev. 7(1), 011302 (2020).

[5] K. Komarova, H. Gattuso, R. D. Levine, and F. Remacle, Quantum device emulates the dynamics of two coupled oscillators. J. Phys. Chem. Lett. 11(17), 6990–6995 (2020).

[6] A. Mallick, M. K. Bashar, D. S. Truesdell, B. H. Calhoun, S. Joshi, and N. Shukla, Using synchronized oscillators to compute the maximum independent set. Nat Commun. 11, 4689 (2020).

[7] A. G. Litvak and M. D. Tokman, Electromagnetically induced transparency in ensembles of classical oscillators. Phys. Rev. Lett. 88(9), 095003 (2002).

[8] C. L. G. Alzar, M. A. G. Martinez, and P. Nussenzveig, Classical analog of electromagnetically induced transparency. Am. J. Phys. 70(1), 37–41 (2002).

[9] Y. Muraki, Application of a coupled harmonic oscillator model to solar activity and El Niño phenomena. J. Astron. Space Sci. 35(2), 75–81 (2018).

[10] S. Dutta, A. Parihar, A. Khanna, J. Gomez, W. Chakraborty, M. Jerry, B. Grisafe, A. Raychowdhury, and S. Datta, Programmable coupled oscillators for synchronized locomotion. Nat. Commun. 10, 3299 (2019).

[11] P. S. Stein, Application of the mathematics of coupled oscillator systems to the analysis of the
neural control of locomotion. *Fed. Proc.* **36**(7), 2056–2059 (1977).

[12] G. C. Dente, C. E. Moeller, and P. S. Durkin, Coupled oscillators at a distance: applications to coupled semiconductor lasers. *IEEE J. Quantum Electron.* **26**(6), 1014–1022 (1990).

[13] S. Hassoul, S. Menouar, H. Benseridi, and J. R. Choi, Quantum dynamics for general time-dependent three coupled oscillators based on an exact decoupling. *Physica A* **604**, 127755 (2022).

[14] R. Habarrih, A. Jellal, and A. Merdaci, Dynamics and redistribution of entanglement and coherence in three time-dependent coupled harmonic oscillators. *Int. J. Geom. Methods Mod. Phys.* **18**(8), 2150120 (2021).

[15] H. R. Lewis, Jr., Class of exact invariants for classical and quantum time-dependent harmonic oscillators. *J. Math. Phys.* **9**(11), 1976–1986 (1968).

[16] H. R. Lewis, Jr. and W. B. Riesenfeld, An exact quantum theory of the time-dependent harmonic oscillator and of a charged particle in a time-dependent electromagnetic field. *J. Math. Phys.* **10**(8), 1458–1473 (1969).

[17] J. R. Choi, Formulation of general dynamical invariants and their unitary relations for time-dependent coupled quantum oscillators. [arXiv:2210.07551v1 [quant-ph]] (2022).

[18] T. J. Li, A concise quantum mechanical treatment of the forced damped harmonic oscillator. *Cent. Eur. J. Phys.* **6**(4), 891–894 (2008).

[19] R. Daneshmand and M. K. Tavassoly, Description of atom-field interaction via quantized Caldirola-Kanai Hamiltonian. *Int. J. Theor. Phys.* **56**(4), 1218–1232 (2017).

[20] D. Chruściński and J. Jurkowski, Quantum damped oscillator I: Dissipation and resonances. *Ann. Phys.* **321**(4), 854–874 (2006).

[21] D. Chruściński, Quantum damped oscillator II: Bateman’s Hamiltonian vs. 2D parabolic potential barrier. *Ann. Phys.* **321**(4), 840–853 (2006).

[22] K. H. Yeon, C. I. Um, S.-K. Hong, and T. F. George, Quantum unitary transformation corresponding to the classical square canonical transformation and its connected quantum systems. *J. Korean Phys. Soc.* **46**(3), 591–596 (2005).

[23] Z.-Z. Li, W.-H. Han, and Z.-Y. Li, Unitary transformation of general nonoverlapping-image multimode interference couplers with any input and output ports. *Chin. Phys. B* **29**(1), 014206
[24] J. R. Choi and S. Ju, Quantum characteristics of a nanomechanical resonator coupled to a superconducting LC resonator in quantum computing systems. *Nanomaterials* **9**(1), 20 (2019).

[25] J. R. Choi, Entropic analysis of optomechanical entanglement for a nanomechanical resonator coupled to an optical cavity field. *SciPost Phys. Core* **4**(3), 024 (2021).

[26] S. Wolfram, The Mathematica Book (Wolfram Media, Champaign, 2003), 5th ed.

[27] G. T. Gilbert, Positive definite matrices and Sylvester’s criterion. *Am. Math. Mon.* **98**(1), 44–46 (1991).

[28] F. M. Fernández, Comment on: “Entanglement in three coupled oscillators” [Phys. Lett. A 384 (2020) 126134]. *Phys. Lett. A* **384**, 126577 (2020).

[29] D.-K. Park, Dynamics of entanglement and uncertainty relation in coupled harmonic oscillator system: exact results. *Quantum Inf. Process.* **17**(6), 147 (2018).

[30] S. Ghosh, K. S. Gupta, and S. C. L. Srivastava, Entanglement dynamics following a sudden quench: An exact solution. *Europhys. Lett.* **120**(5), 50005 (2017).

[31] D.-K. Park, Dynamics of entanglement in three coupled harmonic oscillator system with arbitrary time-dependent frequency and coupling constants. *Quantum Inf. Process.* **18**(9), 282 (2019).

[32] D. X. Macedo and I. Guedes, Time-dependent coupled harmonic oscillators. *J. Math. Phys.* **53**(5), 052101 (2012).

[33] S. Chakraborty and A. K. Sarma, Entanglement dynamics of two coupled mechanical oscillators in modulated optomechanics. *Phys. Rev. A* **97**(2), 022336 (2018).

[34] M. H. Nadiki and M. K. Tavassoly, The amplitude of the cavity pump field and dissipation effects on the entanglement dynamics and statistical properties of an optomechanical system. *Opt. Commun.* **452**(5), 31–39 (2019).

[35] I. Mahboob, M. Mounaix, K. Nishiguchi, A. Fujiwara, and H. Yamaguchi, A multimode electromechanical parametric resonator array. *Sci. Rep.* **4**, 4448 (2014).

[36] C. Bick, M. Goodfellow, C. R. Laing, and E. A. Martens, Understanding the dynamics of biological and neural oscillator networks through exact mean-field reductions: a review. *J. Math. Neurosci.* **10**, 9 (2020).
[37] M. J. Kronenburg, A method for fast diagonalization of a $2 \times 2$ or $3 \times 3$ real symmetric matrix. arXiv:1306.6291v4 [math.NA] (2015).

[38] P. B. Denton, S. J. Parke, T. Tao, and X. Zhang, Eigenvectors from eigenvalues: A survey of a basic identity in linear algebra. Bull. Am. Math. Soc. 59(1), 31–58 (2022).