Braid Matrices and Quantum Gates for Ising Anyons Topological Quantum Computation

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Abstract

We study various aspects of the topological quantum computation scheme based on the non-Abelian anyons corresponding to fractional quantum hall effect states at filling fraction $5/2$ using the Temperley-Lieb recoupling theory. Unitary braiding matrices are obtained by a normalization of the degenerate ground states of a system of anyons, which is equivalent to a modification of the definition of the 3-vertices in the Temperley-Lieb recoupling theory as proposed by Kauffman and Lomonaco. With the braid matrices available, we discuss the problems of encoding of qubit states and construction of quantum gates from the elementary braiding operation matrices for the Ising anyons model. In the encoding scheme where 2 qubits are represented by 8 Ising anyons, we give an alternative proof of the no-entanglement theorem given by Bravyi and compare it to the case of Fibonacci anyons model. In the encoding scheme where 2 qubits are represented by 6 Ising anyons, we construct a set of quantum gates which is equivalent to the construction of Georgiev.

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I. INTRODUCTION

Quantum computers are expected to be able to perform calculations which are impossible for classical computers, due to quantum entanglement and quantum parallelism [1]. Unfortunately, quantum computers seem to be extremely difficult to build because of the unavoidable noise and decoherence caused by the coupling of the qubits and the ambient environment. It is a daunting task to construct a quantum computer that has a large number of qubits and has a low error rate. However, there is a promising approach, called topological quantum computation (TQC) [2,13], proposing to encode the qubit information into a topological quantum field. Kitaev [2] proposed that a system of anyons can be considered to be a quantum computer. Unitary matrices are related to moving the anyons around each other. Measurements are performed by joining anyons in pairs and observing the result of fusion. Interference experiments [14–19] are also proposed to initialize and read out quantum states. The computation is fault-tolerant by the topological nature of the system.

Different from fermions and bosons, which are the totally antisymmetric and the symmetric representations of the permutation group $S_n$, anyons carry fractional charges and have fractional statistics [20, 21] which result in nontrivial phases (for Abelian anyons) or matrices (for non-Abelian anyons) for permutations. In fact, the underlying symmetry of the system of anyons is the braid group $B_n$. Abelian anyons correspond to one-dimensional representations of $B_n$ and the quantum gates one can construct from them are very limited [22]. Non-Abelian anyons, on the contrary, are much more useful to TQC, since the braiding of non-Abelian anyons induces non-commuting (non-Abelian) representations of $B_n$, from which one can construct various quantum gates.

Physically, anyons are collective excitations in some condensed matter systems, such as the fractional quantum hall effect (FQHE) states of two dimensional electron liquids. For example, the effective theories of FQHE states with filling levels $\nu = 1/3$, $\nu = 5/2$ and $\nu = 12/5$ correspond to Abelian anyons, non-Abelian Moore-Read [23] and Read-Rezayi [24] anyons respectively. Mathematically, properties of anyons can be described by $SU(2)_k$ Chern-Simons effective field theories [25] and $Z_k$ parafermion conformal field theories (CFT) [26], the $k = 2$ and $k = 3$ cases corresponding to the $\nu = 5/2$ and the $\nu = 12/5$ FQHE states respectively.

The $SU(2)_k$ Chern-Simons theory is a topological quantum field theory [25] which has
a deep relationship to knot invariants, Jones Polynomial especially. Kauffman and Lomonaco studied unitary representations of braid groups in terms of $q$-deformed spin networks, or Temperley-Lieb recoupling theory.

It is one of our purpose to apply the method of Kauffman and Lomonaco to calculate explicitly the elementary braiding operation (EBO) matrices which govern the exchanges of Ising anyons, the first non-Abelian anyons model proposed by Moore and Read by constructing a wave function (the Pfaffian state) for the $\nu = 5/2$ FQHE state corresponding to the $SU(2)_n$ Chern-Simons theory. Direct experimental observation of fractional electron charge $e/4$ at the $\nu = 5/2$ FQHE state gives some evidences in support of the non-Abelian nature of this state. There are many works concerning the braiding properties of the Ising anyons using CFT method or quantum group method. Nayak and Wilczek suggested that the Pfaffian wave functions of $n$ (n even) Ising anyons form a $2^{n/2-1}$ dimensional spinor irreducible representation of the rotation group $SO(n)$, to which a rigorous treatment is given by Georgiev recently. Quantum group approach also gives equivalent results. As we will see, the EBO matrices for the Ising anyons can be elegantly derived by using the Temperley-Lieb recoupling theory.

One of the attractive properties of the Ising anyons TQC model is that the excitation gap at the corresponding filling fraction $\nu = 5/2$ is the highest one among all non-Abelian FQHE states, resulting in a very low (or even lower) error rate of $10^{-30}$. Although this Ising anyons model is not universal for TQC, i.e., the braid group representations are not dense in unitary groups, it receives extensive attention in the past few years. In fact, it is proved by Bravyi that no entangled states in the computational space can be obtained purely topologically and the Ising anyons TQC model is classically simulatable. We show that the same conclusion can be obtained from the Temperley-Lieb recoupling theory approach.

We should stress that this no-entanglement theorem does not mean that there is no entanglement between Ising anyons at all. This rule only applies to the qubit encoding scheme (which is consistent with the quantum circuit model) where each qubit is encoded in 4 Ising anyons. Entangled quantum gates, such as controlled-Z and controlled-NOT (CNOT), can be realized purely by braiding 6 Ising anyons with definite topological charge (quantum spin). Using the EBO matrices obtained in this paper, we construct a set of useful 1-qubit and 2-qubit quantum gates, which is not the same as, but equivalent to the
construction by Georgiev \cite{41, 42}.

The outline of the remainder of this paper is as follows. Section 2 reviews the general models of non-Abelian anyons for TQC and the formalism of the Temperley-Lieb recoupling theory needed later. Unitary representations of the braid group in the Hilbert space of the degenerate ground states of non-Abelian anyons are obtained by a physical argument which requires that the fusion paths of the anyons form an orthonomal basis of the Hilbert space. In section 3, we derive the EBO matrices of the Ising anyons model. In section 4, we study some aspects of Ising anyons TQC using the results of section 3. Conclusions and discussions are presented in Section 5.

II. TEMPERLEY-LIEB RECOUPLING THEORY AND UNITARY REPRESENTATIONS OF BRAID GROUPS

In this section, we first review the definition of the quantum states of a system of anyons and the Temperley-Lieb recoupling theory \cite{30} and then discuss the method to produce unitary representations of braid groups.

A. Models of non-Abelian anyons for TQC

A model of non-Abelian anyons consists of the following three elements \cite{11}: a list of particle types, the fusion rules, and the braiding rules. In the formalism of $SU(2)_k$ Chern-Simons theory, anyons are quasi-particles having half-integer $q$-spins (spins for short) $s = 0, 1/2, 1, \cdots, k/2$ as their quantum numbers. The fusion rules of these particles are truncated versions of the rules of addition of ordinary angular momenta,

\[
s_1 \otimes s_2 = |s_1 - s_2| \oplus |s_1 - s_2| + 1 \oplus \cdots \oplus \min(s_1 + s_2, k - s_1 - s_2). \tag{1}
\]

When a number of non-Abelian anyons with definite spins fuse consecutively into a single anyon with some spin, the sequences of the intermediate spins of the fusion paths represent different quantum states of the Hilbert space.

Anyons commonly appear as collective excitations in 2 dimensional systems. When they move, their world lines propagate in a 3 dimensional space-time. Thus the exchange of a pair of anyons corresponds to the braiding of their world-lines. (We will call the braiding
of the world-lines of anyons shortly as the braiding of anyons, but it is important to keep in mind what it means actually.) In TQC, we perform quantum computations by braiding anyons to realize certain quantum gates. Any braiding can be expressed as a sequence of EBOs whose representations in the above Hilbert space are the EBO matrices we want to find. The essential task of deriving the EBO matrices is the determination of the so called R-matrix and F-matrix introduced first in the context of CFT [49]. The former is the unitary matrix inducing the exchange of neighboring anyons with definite total spin, and the latter accounts for the associativity of fusions of anyons. In the next subsection, we will give their diagrammatic definitions in terms of the Temperley-Lieb recoupling theory.

B. Temperley-Lieb recoupling theory

Temperley-Lieb recoupling theory [30] is based on the Kauffman bracket polynomial model for the Jones polynomial at roots of unity and the tangle-theoretic Temperley-Lieb algebra.

1. Braid group and Temperley-Lieb algebra

The Artin braid group $B_n$ can be presented as a set of generators $\sigma_1, \sigma_2, ..., \sigma_{n-1}$ that obey the following relations,

\begin{align}
\sigma_i \sigma_j &= \sigma_j \sigma_i \quad \text{for } |i - j| \geq 2; \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } i = 1, 2, ..., n-2.
\end{align}

The Temperley-Lieb algebra $TL_n$ can be presented similarly as a set of generators $U_1, U_2, ..., U_{n-1}$, whose representations are related to the representations $\rho(\sigma_i)$ of $B_n$ by

\[ \rho(\sigma_i) = A + A^{-1}U_i, \]

where the Kauffman variable $A$ is taken to be $A = ie^{i\pi/2r}$ for Jones polynomial at $4r$-th roots of unity such that the quantum dimension of the spin 1/2 anyon is $d = -A^2 - A^{-2} = 2\cos(\pi/r)$. 

5
2. R-matrix and F-matrix

The basic object of the Temperley-Lieb recoupling theory is the Jones-Wenzl projector \[30\]. The left graph in Fig. 1 shows the Jones-Wenzl projector constructed on the basis of the Kauffman bracket polynomial expansion. The \(n\)-strand projector corresponds to the world line of an anyon with spin \(n/2\). The middle graph in Fig. 1 shows the 3-vertex constructed from the projectors which corresponds to the interaction (fusing or splitting) of 3 anyons with spins \(a/2\), \(b/2\), and \(c/2\). The right graph in Fig. 1 is a simplified notation for the 3-vertex. Note that the \(q\)-admissible conditions \[30\] for the 3-vertex,

\[
\begin{align*}
  a + b + c &= \text{even}; \\
  a + b - c &\geq 0, \ b + c - a \geq 0, \ c + a - b \geq 0; \\
  a + b + c &\leq 2r - 4
\end{align*}
\]

say exactly the same thing as the fusion rules Eq. (1) do due to the relation \(r = k+2\) between \(SU(2)_k\) Chern-Simon theory at level \(k\) and Jones polynomial at \(4r\)-th roots of unity \[25\] and the fact that the projector with label \(n\) represents the world line of an anyon with spin \(n/2\). Various spin networks can be constructed from Jones-Wenzl projectors and 3-vertices. See Fig. 2.

As the world lines of anyons and their fusions being identified with projectors and 3-vertices, we now consider the braiding properties of anyons in the context of the Temperley-Lieb recoupling theory. Fig. 3 shows the braiding of two anyons with spins \(a/2\) and \(b/2\)
\[ \Delta_n = n \quad \Theta(a, b, c) = a \quad \Theta(a, b, c) = a \]

\[ T(a b i) = \begin{array}{ccc} a & b & \Theta(a, b, c) \\ \Delta_c & \delta_{cd} & c \end{array} \]

FIG. 2: The definitions of the delta net \( \Delta_n \), the theta net \( \Theta(a, b, c) \), and the tetrahedron net \( T(a b i) \). Formulae \[30\] for evaluating these spin-nets are presented in the appendix.

\[ R_{c}^{ab} = \begin{array}{ccc} a & b & c \\ b & \Theta(a, b, c) \end{array} \]

FIG. 3: The action of the R-matrix, whose matrix elements are given by Eq. \[5\]. Our convention is that seen from above, the braiding of \( a \) and \( b \) is counterclockwise.

fusing into a spin \( c/2 \) anyon. Since this operation does not change the total spin of the two fusing anyons, the corresponding matrix, the R-matrix is diagonal in the underlying Hilbert space. The matrix element \( R_{c}^{ab} \) is given by the following formula \[30\],

\[ R_{c}^{ab} = (-1)^{(a-b-c)/2} A^{-(a(a+2)+b(b+2)-(c(c+2))/2}. \quad \text{(5)} \]

Not all braids are of this case, in which the two braiding anyons fuse into a single anyon. To see this, it is sufficient to consider the case of 4 anyons with total spin 0, which is
FIG. 4: The configuration of 4 anyons $a$, $b$, $c$, and $d$ fusing into 0 (the vacuum) is equivalent to the configuration of 3 anyons $a$, $b$, and $c$ fusing into one anyon $d$. The first equality follows from the fact that one can get a 0 only by fusing an anyon with label $j = d$ with the anyon with label $d$.

FIG. 5: The use of the F-matrix.

equivalent to 3 anyons fusing into the 4-th anyon. This is shown in Fig. 4.

The braiding of $b$ and $c$ in Fig. 5 can not be accomplished via a single R-matrix and is realized only by a combination of the R-matrix and the F-matrix. Fig. 6 shows the definition of the matrix element $F_{ij}^{(ab)(cd)}$ of the F-matrix as well as the formula to calculate
FIG. 6: The upper diagram shows the definition of the matrix element $F^{(ab)}_{cd}ij$ of the F-matrix. A summation on $j$ is implied. The lower diagram shows the derivation of the expression Eq. (6) of the F-matrix in terms of the delta net, the theta net, and the tetrahedron net. Expressions in Fig. 2 are used in the derivation.

it in terms of the delta net $\Delta_n$, the theta net $\Theta(a,b,c)$, and the tetrahedron net $T^{(a\ b\ i)}_{c\ d\ j}$,

$$
F^{(ab)}_{cd}ij = \frac{T^{(a\ b\ i)}_{c\ d\ j}\Delta_j}{\Theta(a,b,j)\Theta(c,d,j)}.
$$

\begin{align}
\text{C. Unitary representations of Artin braid groups} \\

\text{However, the F-matrix defined above in the Temperley-Lieb recoupling theory is not unitary, resulting in a non-unitary representation of the braid group. Unitary (in fact real and orthogonal) F-matrix, and hence unitary representation of $B_n$ can be obtained by a redefinition for the basis states in the Hilbert space of the anyons. The guideline of the following argument is the requirement that the fusion paths should represent an orthonormal basis of the Hilbert space of the degenerate ground states of a system of anyons.}

\text{We need only to consider the orthonormal problem of the states of four anyons fusing into the vacuum. The definition of the state } |i\rangle \text{ and the calculation of the inner product } \langle i|j\rangle \text{ of the two states } |i\rangle \text{ and } |j\rangle \text{ are shown in Fig. 7. We see that the orthogonal property is already satisfied (} \langle i|j\rangle \propto \delta_{ij} \text{), but the state vectors are not normalized (} \langle i|i\rangle \neq 1 \text{). It follows that an orthonormal basis } \{|\tilde{i}\}\rangle \text{ (} \langle \tilde{i}|\tilde{j}\rangle = \delta_{ij} \text{) of the Hilbert space is obtained by normalizing}
\end{align}
\begin{align*}
\langle i | j \rangle &= \frac{\Theta(a, c, i) \Theta(b, d, i)}{\Delta_i} \delta_{ij} \\
\end{align*}

FIG. 7: The upper diagram shows the graphic definition of the state vector $|i\rangle$ for 4 anyons with total spin 0. The graphic representation of the dual vector $\langle i |$ is defined to be the upside-down version of that of the vector $|i\rangle$. The lower diagram shows the calculation of inner product $\langle i | j \rangle$ of two states $|i\rangle$ and $|j\rangle$. Expressions in Fig. 2 are needed in the calculation.

\begin{align*}
\tilde{|i\rangle} &= \sqrt{\frac{\Delta_i}{\Theta(a, c, i) \Theta(b, d, i)}} \\
\end{align*}

FIG. 8: The normalized basis \{\tilde{|i\rangle}\} is obtained from the original one \{|i\rangle\} by multiplying an appropriate factor to each state $|i\rangle$.

each of the states, as depicted in Fig. 8. In the orthonormal basis \{\tilde{|i\rangle}\}, the new F-matrix can be derived to be

\begin{equation}
F^{(ab)}_{(cd)ij} = \frac{\sqrt{\Delta_i \Delta_j}}{\sqrt{\Theta(a, b, j) \Theta(c, d, j) \Theta(a, c, i) \Theta(b, d, i)}} T^{(a b \hat{i})}_{(c d \hat{j})}.
\end{equation}

This new F-matrix is real and orthogonal (hence unitary), as we will see in explicit calculations latter. We note that Kauffman and Lomonaco \[28\] obtained the unitary F-matrix by
FIG. 9: The fusion diagram for $n + 3$ ($n$ odd) Ising anyons with total spin 0. The first (the left most one) and the second Ising anyons fuse into an anyon with spin $x_1 = 0$ or 1, and then $x_1$ fuse with the third Ising anyon into an Ising anyon with spin $1/2$, · · · , and finally, $x_n$ fuse with the $(n + 2)$-th Ising anyon into the $(n + 3)$-th Ising anyon.

multiplying each 3-vertex with the following factor,

$$f(a, b, c) = \left(\Delta_a \Delta_b \Delta_c\right)^{1/4} / (\Theta(a, b, c))^{1/2}.$$  \hspace{1cm} (8)

One can check that this modification to 3-vertices results in the same F-matrix given by Eq. (7).

III. EBO MATRICES FOR ISING ANYONS

In this section, we apply the results of section 2 to obtain explicitly the representations of the generators of the braid group governing the exchanges of Ising anyons. Our presentation follows closely to Kauffman and Lomonaco [28, 29].

For the Ising anyons model, the Kauffman variable is $A = i e^{i \pi/8}$, and the quantum dimension of the spin 1/2 Ising anyons is $d = -A^2 - A^{-2} = \sqrt{2}$. The allowed spins of anyons in this model are 0, 1/2 and 1 and the fusion rules for these anyons can be deduced from Eq. (1) to be,

$$0 \otimes j = j \quad \text{for} \quad j = 0, 1/2, 1;$$
$$1/2 \otimes 1/2 = 0 \oplus 1;$$
$$1 \otimes j = 1 - j \quad \text{for} \quad j = 0, 1/2, 1.$$  \hspace{1cm} (9)
The dimension of the Hilbert space of \( n \) spin 1/2 Ising anyons with total spin 0 is \( 2^{n/2-1} \), approaching \( d^n \) in the limit \( n \to \infty \). According to the above fusion rules, the fusion diagram of \( n + 3 \) (\( n \) must be odd) Ising anyons with total spin 0 takes the form as shown in Fig. 9 [50].

We now calculate the unitary representation of \( B_{n+3} \) for the braiding of the \( n + 3 \) Ising anyons. We denote the elementary braiding operation (EBO) of the first and the second Ising anyons as \( \sigma_1 \), the EBO of the second and the third Ising anyons as \( \sigma_2 \), \( \cdots \). The corresponding EBO matrices are denoted by \( \rho(\sigma_i) \) where \( 1 \leq i \leq n + 2 \).

The first EBO matrix \( \rho(\sigma_1) \) is easy to calculate. It depends only on the label \( x_1 \). In the basis \( \{ x_1 = |0\rangle, |1\rangle \} \), \( \rho(\sigma_1) \) is simply given by the following R-matrix,

\[
R = \begin{pmatrix} R_{01}^{11} & 0 \\ 0 & R_{21}^{11} \end{pmatrix} = \begin{pmatrix} -A^{-3} & 0 \\ 0 & A \end{pmatrix},
\]

which corresponds to the following Temperley-Leib generator,

\[
U = AR - A^2 = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{d} & 0 \\ 0 & 0 \end{pmatrix}.
\]

To calculate \( \rho(\sigma_2) \), we need to calculate the following F-matrix in the same basis as above (using Eq. (7) and the formulae in the appendix),

\[
F = \begin{pmatrix} F_{11}^{(11)} & F_{11}^{(11)} & 0 \\ F_{11}^{(11)} & F_{11}^{(11)} & 0 \\ 0 & 0 & F_{11}^{(11)} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},
\]

corresponding to the following Temperley-Leib generator,

\[
V = AS - A^2 = \begin{pmatrix} 1/d & 1/d \\ 1/d & 1/d \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},
\]

where \( S = \rho(\sigma_2) = FRF^{-1} \).

Now consider the case of \( \rho(\sigma_3) \). When either \( x_1 \) or \( x_3 \) or both of them equal to 0, the situation is similar to the case of \( \rho(\sigma_1) \) where only R-matrix elements are needed to be calculated. The case in which both \( x_1 \) and \( x_3 \) are 1 deserves special consideration. The EBO \( \sigma_3 \) does not change the value of \( x_2 \) when both \( x_1 \) and \( x_3 \) are 1, and the matrix element of \( \rho(\sigma_3) \) in this case is found to be the same as in the case where both \( x_1 \) and \( x_3 \) are 0 by doing some graphic calculations.
The other EBO matrices can be calculated in the same way. By choosing the basis of the $n + 3$ Ising anyons with total spin 0 ($n$ odd) as $\{|x_1, x_2, \ldots x_n\rangle\}$ where $x_i$ equals to 0, 1/2, or 1 such that the fusion rules are met at each fusion vertex along the whole fusion path, as shown in Fig. 9 we can find a representation of the Temperley-Lieb algebra $T_{L_{n+3}}$. For the first and the last generators of $T_{L_{n+3}}$, we have,

$$U_1 |0, x_2, \ldots, x_n\rangle = \sqrt{2} |0, x_2, \ldots, x_n\rangle;$$
$$U_1 |1, x_2, \ldots, x_n\rangle = 0;$$
$$U_{n+2} |x_1, \ldots, x_{n-1}, 0\rangle = \sqrt{2} |x_1, \ldots, x_{n-1}, 0\rangle;$$
$$U_{n+2} |x_1, \ldots, x_{n-1}, 1\rangle = 0.$$ (14)

For $U_2$ and $U_{n+1}$, we have,

$$U_2 |0, 1/2, x_3, \ldots, x_n\rangle =$$
$$\frac{1}{\sqrt{2}} |0, 1/2, x_3, \ldots, x_n\rangle + \frac{1}{\sqrt{2}} |1, 1/2, x_3, \ldots, x_n\rangle;$$
$$U_2 |1, 1/2, x_3, \ldots, x_n\rangle =$$
$$\frac{1}{\sqrt{2}} |0, 1/2, x_3, \ldots, x_n\rangle + \frac{1}{\sqrt{2}} |1, 1/2, x_3, \ldots, x_n\rangle;$$
$$U_{n+1} |x_1, \ldots, x_{n-2}, 1/2, 0\rangle =$$
$$\frac{1}{\sqrt{2}} |x_1, \ldots, x_{n-2}, 1/2, 0\rangle + \frac{1}{\sqrt{2}} |x_1, \ldots, x_{n-2}, 1/2, 1\rangle;$$
$$U_{n+1} |x_1, \ldots, x_{n-2}, 1/2, 1\rangle =$$
$$\frac{1}{\sqrt{2}} |x_1, \ldots, x_{n-2}, 1/2, 0\rangle + \frac{1}{\sqrt{2}} |x_1, \ldots, x_{n-2}, 1/2, 1\rangle.$$ (15)

For the middle ones, we have,

$$U_i |x_1, \ldots, x_{i-3}, 0, 1/2, 0, x_{i+1}, \ldots, x_n\rangle =$$
$$\sqrt{2} |x_1, \ldots, x_{i-3}, 0, 1/2, 0, x_{i+1}, \ldots, x_n\rangle;$$
$$U_i |x_1, \ldots, x_{i-3}, 0, 1/2, 1, x_{i+1}, \ldots, x_n\rangle = 0;$$
$$U_i |x_1, \ldots, x_{i-3}, 1/2, 0, 0, x_{i+1}, \ldots, x_n\rangle =$$
$$\frac{1}{\sqrt{2}} |x_1, \ldots, x_{i-3}, 1/2, 0, 0, x_{i+1}, \ldots, x_n\rangle +$$
$$\frac{1}{\sqrt{2}} |x_1, \ldots, x_{i-3}, 1/2, 1, x_{i+1}, \ldots, x_n\rangle;$$
$$U_i |x_1, \ldots, x_{i-3}, 1/2, 1, 0, x_{i+1}, \ldots, x_n\rangle =$$
$$\frac{1}{\sqrt{2}} |x_1, \ldots, x_{i-3}, 1/2, 0, 0, x_{i+1}, \ldots, x_n\rangle +$$
$$\frac{1}{\sqrt{2}} |x_1, \ldots, x_{i-3}, 1/2, 1, x_{i+1}, \ldots, x_n\rangle;$$
$$U_i |x_1, \ldots, x_{i-3}, 1, 1/2, 0, 0, x_{i+1}, \ldots, x_n\rangle = 0;$$
$$U_i |x_1, \ldots, x_{i-3}, 1, 1/2, 1, x_{i+1}, \ldots, x_n\rangle =$$
$$\sqrt{2} |x_1, \ldots, x_{i-3}, 1, 1/2, 1, x_{i+1}, \ldots, x_n\rangle.$$ (16)
The EBO matrices can be obtained immediately from the above representations of theTemperley-Lieb algebra by using Eq. (3). One can check that these representations of thebraid group indeed satisfy the Artin relations in Eq. (2).

**IV. CONSTRUCT QUANTUM GATES FROM THE EBO MATRICES FOR THEISING ANYONS MODEL**

After obtaining the EBO matrices of the Ising anyons model, we study in this sectionsome aspects of Ising anyons TQC.

The qubit encoding scheme which is consistent to the quantum circuit model is to useeach group of 4 Ising anyons with total spin 0 for each qubit such that an \( n \)-qubit systemuses \( 4n \) Ising anyons. This is the encoding scheme used by Bravyi [39] who proved a no-entanglement theorem which states that entangled 2-qubit states can never be prepared bypure topological braiding operations. The proof by Bravyi uses the stabilizer constrains andthe no-leakage error conditions. In the following, we give a graphical demonstration of thisresult from the Temperley-Lieb recoupling approach.

In this qubit encoding scheme, 2-qubit states are encoded in 8 Ising anyons, 4 Ising anyonsfor each qubit. See Fig. 10. The two groups of EBOs \( \{ \sigma_1, \sigma_2 \} \) and \( \{ \sigma_6, \sigma_7 \} \) apply completelywithin the first and the second qubits respectively and can not generate entanglement be-tween the two qubits. Since \( \sigma_3 \) depends on \( x_1 \) and \( x_3 \), \( \sigma_5 \) depends on \( x_3 \) and \( x_5 \), and \( \sigma_4 \) canchange \( x_3 \) from 0 to a superposition of 0 and 1, it is possible to create an entangled state bya sequence of these EBOs, such as \( \sigma_3^{-1} \sigma_4^{-1} \sigma_5 \sigma_4 \sigma_3 \). However, it is impossible to avoid leakageerrors by braiding this way. To see this, it is convenient to change the fusion paths in Fig. 10 toanother basis as shown in Fig. 11, where the total spin of the 2-qubit system can be either0 or 1. The EBO sequence \( \sigma_3^{-1} \sigma_4^{-1} \sigma_5 \sigma_4 \sigma_3 \) in Fig. 10 is equivalent to the single EBO \( \sigma_3 \) inFig. 11. Since the braid matrices of \( \sigma_3 \) in Fig. 11 for the two sectors (total spin 0 and 1) arenot equivalent, and entanglement can not be created without using \( \sigma_3 \) in Fig. 11 leakageerror from the computational space (labeled by \( x_1 \) and \( x_5 \) in both Fig. 10 and Fig. 11) tothe uncomputational space (labeled by \( x_3 \) in Fig. 10 and \( x_{\text{total}} \) in Fig. 11) is unavoidable.

It is instructive to compare the above situation with the case of the Fibonacci anyonsmodel which is universal for TQC [4–6]. See Fig. 10 too. For Fibonacci anyons, eachintermediate spin \( x_i \) can be either 0 or 1 (as long as no two 0s appear consecutively) and
FIG. 10: Part (a) shows the initial state of the two qubits. The 4 anyons $a$, $b$, $c$, and $d$ form the first qubit and the other 4 anyons $e$, $f$, $g$, and $h$ form the second qubit. Part (b) shows the full quantum labels needed when we braid the world lines of the anyons. The computational space is spanned by the states $\{ |x_1, x_5\rangle = |0, 0\rangle, |0, 1\rangle, |1, 0\rangle, |1, 1\rangle \}$. In this notation, the initial state in part (a) is $|0, 0\rangle$. For Ising anyons, $x_2$ and $x_4$ can only be $1/2$; for Fibonacci anyons, $x_i$ can be either 0 or 1 such that $x_i + x_{i+1} > 0$.

FIG. 11: The formation of the spin 0 ($x_{\text{total}} = 0$) and spin 1 ($x_{\text{total}} = 1$) sectors for 8 Ising anyons. Each sector is 4 dimensional and a basis $\{ |x_1, x_5\rangle = |0, 0\rangle, |0, 1\rangle, |1, 0\rangle, |1, 1\rangle \}$ can be chosen for both sectors. The rightmost diagram in this figure is related to part (b) of Fig. 10 by a change of basis using a F-matrix.
entangled states can be generated by braiding the 8 Fibonacci anyons. Bonesteel et al. \cite{46-48} constructed some entangled 2-qubit gates such as the controlled-$iX$ gate by weaving two Fibonacci anyons ($c$ and $d$) from the control qubit into the target qubit which approximates the identity matrix, followed by a braiding within the target qubit which approximates the $iX$ gate, and then weaving them back to their original positions. The controlling operation is realized by virtue of the fact that the whole braiding does nothing when the total spin of $c$ and $d$ is 0 and acts as the $iX$ gate on the target qubit when the total spin of $c$ and $d$ is 1. The nonuniversality of the Ising anyons model prevents us from realizing entangled gate in this way.

We note that the no-entanglement theorem only applies to the above qubit encoding scheme where 2 qubits are represented by 8 Ising anyons. Entangled quantum gates can be constructed in a different qubit encoding scheme, as studied by Georgiev \cite{41, 42}. In this scheme, 1-qubit and 2-qubit states are encoded in 4 and 6 Ising anyons with total spin 0 respectively.

Consider 1-qubit gates first. Taking the basis of the Hilbert space to be (see Fig. 9) \{ $|x_1\rangle = |0\rangle, |1\rangle$ \}, the two dimensional EBO matrices of $B_4$ for four Ising anyons are found to be

$$\rho^{(2)}(\sigma_1) = \rho^{(2)}(\sigma_3) = e^{i\pi/8} \begin{pmatrix} -1 & 0 \\ 0 & i \end{pmatrix};$$

$$\rho^{(2)}(\sigma_2) = -\frac{e^{-i\pi/8}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \quad (18)$$

One can construct the Hadamard gate $H$, the phase gate $S$, and the three Pauli gates

\[ \begin{aligned}
\rho^{(2)}(\sigma_1) &= \begin{pmatrix} -1 & 0 \\ 0 & i \end{pmatrix}; \\
\rho^{(2)}(\sigma_2) &= -\frac{e^{-i\pi/8}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.
\end{aligned} \]

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\rho^{(2)}(\sigma_2) &= -\frac{e^{-i\pi/8}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.
\end{aligned} \]
X, Y, and Z using the two dimensional EBO matrices given above (∼ means equal up to an unimportant global phase),

\[ H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \sim \rho^{(2)}(\sigma_1 \sigma_2 \sigma_1); \quad (19) \]

\[ S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \sim \rho^{(2)}(\sigma_1^{-1}); \quad (20) \]

\[ X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \rho^{(2)}(\sigma_2 \sigma_2); \quad (21) \]

\[ Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sim \rho^{(2)}(\sigma_1 \sigma_1 \sigma_2^{-1} \sigma_2^{-1}); \quad (22) \]

\[ Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sim \rho^{(2)}(\sigma_1 \sigma_1). \quad (23) \]

Fig. 12 shows the encoding of the 1-qubit states as well as the Hadamard gate constructed by three braids.

However, it fails to construct the π/8 gate,

\[ T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \sim \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix}, \quad (24) \]

reflecting the fact that the Ising anyons model is not universal for quantum computation. To remedy this, we have to supplement braiding with some non-topological operations [39, 40].

Now consider the 2-qubit case. Taking the basis of the Hilbert space to be (see Fig. 9) \{ |x_1, x_2, x_3 \rangle = |0, 1/2, 0 \rangle, |0, 1/2, 1 \rangle, |1, 1/2, 0 \rangle, |1, 1/2, 1 \rangle \}, the four dimensional EBO matrices of \( B_6 \) for 6 Ising anyons with total spin 0 read

\[ \rho^{(4)}(\sigma_1) = e^{i\pi/8} \text{diag} (-1, -1, i, i); \quad (25) \]

\[ \rho^{(4)}(\sigma_2) = -\frac{e^{-i\pi/8}}{\sqrt{2}} \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ i & 0 & 1 \\ 0 & i & 0 \end{pmatrix}; \quad (26) \]

\[ \rho^{(4)}(\sigma_3) = e^{i\pi/8} \text{diag} (-1, i, i, -1); \quad (27) \]
\[ \rho^{(4)}(\sigma_4) = -\frac{e^{-i\pi/8}}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & i & 1 \end{pmatrix}; \]  
\[ \rho^{(4)}(\sigma_5) = e^{i\pi/8} \text{diag} (-1, i, -1, i). \]  

From the four dimensional EBO matrices given above, one can construct useful 2-qubit quantum gates, such as CNOT (up to an unimportant global phase),

\[ \text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \sim \rho^{(4)}(\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_3\sigma_4\sigma_1), \]  
and controlled-\( Z \),

\[ \text{controlled-}Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \sim \rho^{(4)}(\sigma_1\sigma_3^{-1}\sigma_5). \]  

Fig. 13 shows the encoding of the 2-qubit states as well as the braiding diagram of CNOT. Note that the braid sequences for CNOT and controlled-\( Z \) are not unique, which is a consequence of the Artin relations Eq. (2) for the generators of the braid group. Our constructions are different from, but equivalent to the ones given by Georgiev [41, 42].

Note that the 4 dimensional Hilbert space of 6 Ising anyons with total spin 0 is a subspace of the 8 dimensional space of 8 Ising anyons. The other subspace, which is also 4 dimensional, corresponds to 6 anyons with total spin 1. See Fig. 11. Using the method in section 3, we can also find the EBO matrices for the spin 1 sector. It turns out that the EBO matrices \( \rho^{(4)}(\sigma_1) \), \( \rho^{(4)}(\sigma_2) \), \( \rho^{(4)}(\sigma_4) \), and \( \rho^{(4)}(\sigma_5) \) in the spin 1 sector take the same form as in the spin 0 sector, but \( \rho^{(4)}(\sigma_3) \) has a different form, \( \rho^{(4)}(\sigma_3) = e^{i\pi/8} \text{diag} (i, -1, -1, i) \). Therefore, entangled 2-qubit quantum gates will have different braid sequences in the spin 1 sector. For example, one possible braid sequence in the spin 1 sector for CNOT is \( \sigma_5^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5\sigma_3\sigma_4\sigma_1 \), which is not topologically equivalent to the one given in Eq. (30). Despite of the non-equivalence of the braid sequences for a given quantum gate in these two sectors, the computational power of the two sectors are equivalent [44].
FIG. 13: The 2-qubit states with 6 Ising anyons with total spin 0 and the braiding diagram for the CNOT gate. $x_1$ and $x_3$ can be 0 or 1 and $x_2$ can only be 1/2. Note that the braid sequence for CNOT is not unique, a consequence of the Artin relations for the generators of the braid group.

V. CONCLUSION AND DISCUSSION

As demonstrated in previous sections, the Temperley-Lieb recoupling theory provides a natural language for describing the braiding properties of non-Abelian anyons. We have applied this theory to derive the EBO matrices of the Ising anyons model. We paid a special attention to the normalization of the degenerate ground states corresponding to the fusion paths of the anyons. This normalization results in the correct unitary F-matrices and is equivalent to the redefinition of the 3-vertices proposed by Kauffman and Lomonaco [28].

One important feature for the construction of the two-qubit gates is that we can not construct them without the use of $\sigma_3$, the EBO acting between the two qubits. This is because that the EBOs $\sigma_1$ and $\sigma_2$ act only on the first qubit and $\sigma_4$ and $\sigma_5$ act only on the second qubit. Indeed, the first two and the last two EBO matrices can be expressed as a tensor product of two matrices, and the middle EBO matrix $\rho^{(4)}(\sigma_3)$ can not, reflecting the (topological) entanglement of the 2 qubits. This entanglement is crucial for the construction of the 2-qubit entangled gates. However, to get this entanglement, we need to project the $B_8$ representation to either the spin 0 or the spin 1 $B_6$ representations. Alternatively, entangled quantum gates can be constructed by parity measurement as well as braiding operations.
The construction of the 2-qubit gates in each sector can be easily achieved by brute force search, since the braid lengths of controlled-Z and CNOT are very short (3 and 7 respectively). However, there is a more heuristic approach, namely, the genetic algorithm (GA) approach. A possible braid sequence for the CNOT gate can be found within a minute using GA, while it takes a much longer time using the brute force approach. The superiority of GA over brute force search is not significant for Ising anyons TQC, but we expect that there is a potential application of GA to Fibonacci anyons topological quantum compiling [46–48].

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Appendix A: Formulae for evaluating the spin-nets

In this appendix, we present the formulae for the evaluations of the ∆-net, the Θ-net, and the tetrahedral net [30].

The ∆-net evaluation is

\[ \Delta_n = (-1)^n [n + 1], \]  

where \([n]\) is the \(q\)-deformed integer defined as \([n] = (A^{2n} - A^{-2n})/(A^2 - A^{-2})\). The Θ-net evaluation is

\[ \Theta(a, b, c) = (-1)^{i + j + k} \frac{[i + j + k + 1][i][j][k]!}{[i + j][j + k][k + i]!} \]  

where the \(q\)-deformed fractional \([n]!\) is defined as \([n]! = [n][n-1]...[2][1]\), and the integers \(i\), \(j\), and \(k\) are determined by the relations \(a = i + j\), \(b = j + k\), and \(c = k + i\). The bracket evaluation of the tetrahedral net is

\[ T^{(a \ b \ i \ c \ d \ j)} = \prod_{m,n} [b_m - a_m]! \sum_{\max(a_m) \leq s \leq \min(b_n)} \frac{(-1)^s [s+1]!}{\prod_{m} (s-a_m)! \prod_{n} (b_n-s)!}, \]  

where \(\prod_{m,n} [b_m - a_m]!\) represents the product of factorials of the differences between \(b_m\) and \(a_m\) for all \(m\) and \(n\).
where $a_m$ and $b_n$ are given by $a_1 = (a + d + i)/2$, $a_2 = (b + c + i)/2$, $a_3 = (a + b + j)/2$, $a_4 = (c + d + j)/2$, $b_1 = (b + d + i + j)/2$, $b_2 = (a + c + i + j)/2$, and $b_3 = (a + b + c + d)/2.$

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