CONIC MANIFOLDS UNDER THE YAMABE FLOW

NIKOLAOS ROIDOS

ABSTRACT. We consider the unnormalized Yamabe flow on manifolds with conical singularities. Under certain geometric assumption on the initial cross-section we show well posedness of the short time solution in the $L^q$-setting. Moreover, we give a picture of the deformation of the conical tips under the flow by providing an asymptotic expansion of the evolving metric close to the boundary in terms of the initial local geometry. Due to the blow up of the scalar curvature close to the singularities we use maximal $L^q$-regularity theory for conically degenerate operators.

1. Introduction

Let $B$ be a smooth compact $(n+1)$-dimensional manifold, $n \geq 2$, with closed (i.e. compact without boundary) possibly disconnected smooth boundary $\partial B$ of dimension $n$. We endow $B$ with a degenerate Riemannian metric $g_0 = \{g_{ij}\}_{i,j \in \{1, \ldots, n+1\}}$ which in a collar neighborhood $[0, 1) \times \partial B$ of the boundary is of the form

$$g_0 = dx^2 + x^2 h(x),$$

where $h(x) = \{h_{ij}(x)\}_{i,j \in \{1, \ldots, n\}}$, $x \in [0, 1)$, is a smooth up to $x = 0$ family of non-degenerate up to $x = 0$ Riemannian metrics on the cross section $\partial B$. We call $B = (B, g_0)$ manifold with conical singularities or conic manifold; the singularities, i.e. the conical tips, correspond to the boundary $\{0\} \times \partial B$ of $B$.

We are interested in the unnormalized Yamabe flow on $B$. Namely, we search for a family of Riemannian metrics $\{g(t)\}_{t \geq 0}$ on $B$ satisfying

\begin{align*}
g'(t) + R_{g(t)} g(t) &= 0, \quad t \in (0, T), \\
g(0) &= g_0,
\end{align*}

where $R_{g(t)}$ is the scalar curvature on $(B, g(t))$ and $T > 0$.

We look as usual for solutions of (1.1)–(1.2) in the conformal class of $g_0$. The parabolic equation obtained, see (4.8)–(4.9), has a quasilinear term that is described by the degenerate Laplacian on $B$ and a forcing term involving the scalar curvature of $B$, which blows up close to the conical tips. We regard the Laplacian as a cone differential operator acting on weighted Mellin-Sobolev spaces and employ the related maximal $L^q$-regularity theory. Then, existence, uniqueness and maximal $L^q$-regularity of the short time solution is obtained by a theorem of Clément and Li. Moreover, we show that this solution becomes instantaneously smooth in space. In order to include the scalar curvature of $B$ into our underline $L^p$-space we impose the following geometric assumption on the cross-section of $B$, namely

\begin{align*}
\text{(1.3) \hspace{1cm} Assume that the scalar curvature $R_{h(0)}$ of the cross-section $\partial B = (\partial B, h(0))$ satisfies:} \\
R_{h(0)} &= n(n-1).
\end{align*}

Denote by $R^+_\omega$ the set of smooth functions on $B$ that are locally strictly positive constants close to the boundary (see Definition 3.2). Although under assumption (1.3) the initial scalar curvature still blows up (see (4.10)) we show the following.

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Theorem 1.1. Let that \( (1.3) \) is satisfied and denote
\[
\gamma_0 = \min \left\{ -1 + \sqrt{\left(\frac{n-1}{2}\right)^2 - \lambda_1}, \frac{n-1}{2} \right\},
\]
where \( \lambda_1 \) is the greatest non-zero eigenvalue of the Laplacian on \( \partial B \) induced by the metric \( h(0) \). Then, for any
\[
s \geq 0, \quad \delta \in (0, \gamma_0 - \frac{n-3}{2} - \frac{2}{q}) \quad \text{and} \quad p, q \in (1, \infty) \quad \text{with} \quad \frac{n+1}{p} + \frac{2}{q} < 1,
\]
there exists a \( T > 0 \) and a unique
\[
f \in W^{1,q}(0, T; \mathbb{H}_p^{\nu, \gamma_0-\delta}(\mathbb{B})) \cap L^q(0, T; \mathbb{H}_p^{\nu+2, \gamma_0+2-\delta}(\mathbb{B}) \oplus \mathbb{R}_+^+) \\
\leftrightarrow \bigcap_{\epsilon > 0} C([0, T]; \mathbb{H}_p^{\nu+2-\frac{2}{q}-\epsilon, \gamma_0+2-\frac{2}{q}-\epsilon}(\mathbb{B}) \oplus \mathbb{R}_+^+) \hookrightarrow C([0, T]; C(\mathbb{B}))
\]
such that \( g(t) = f(t)_{|_0} \) solves \( (1.1)-(1.2) \). Moreover, for any \( \tau \in (0, T) \) we have that
\[
f \in \bigcap_{\nu \geq 0} W^{1,q}(\tau, T; \mathbb{H}_p^{\nu, \gamma_0-\delta}(\mathbb{B})) \cap L^q(\tau, T; \mathbb{H}_p^{\nu, \gamma_0+2-\delta}(\mathbb{B}) \oplus \mathbb{R}_+^+) \\
\leftrightarrow \bigcap_{\nu \geq 0, \epsilon > 0} C([\tau, T]; \mathbb{H}_p^{\nu, \gamma_0+2-\frac{2}{q}-\epsilon}(\mathbb{B}) \oplus \mathbb{R}_+^+).
\]

The above result together with standard properties of Mellin-Sobolev spaces provide a picture of the deformation of the conical tips under the flow as well as its relation to the initial local geometry.

Corollary 1.2. Let \( s, \delta, p, q, \lambda_1 \) be as in \( (1.4) \), \( \epsilon > 0 \) and \( T \), \( f \) be as in Theorem 1.1. Then, there exist \( f_1 \in C([0, T]; \mathbb{R}_+^+) \) and \( f_2 \in C([0, T]; \mathbb{H}_p^{\nu, \gamma_0-\delta}(\mathbb{B})) \leftrightarrow C([0, T]; C(\mathbb{B})) \)
with \( f_1(0) = 1 \) and \( f_2(0) = 0 \) such that in local coordinates \( (x, y) \) on the collar part \( (0, 1) \times \partial B \) we have
\[
g(t) = (f_1(t) + f_2(t, x, y))(dx^2 + x^2 h(x)), \quad t \in [0, T], \quad \text{with} \quad |f_2(t, x, y)| \leq C x^{\beta-\epsilon},
\]
where
\[
\beta = \min \left\{ -\frac{n-1}{2} + \sqrt{\left(\frac{n-1}{2}\right)^2 - \lambda_1}, 1 \right\} - \frac{2}{q} - \delta > 0
\]
and the constant \( C > 0 \) depends only on the data \( \{\mathbb{B}, s, \delta, p, q, \epsilon, T\} \).

In the case of a closed manifold the Yamabe flow as well as its normalized version (see, e.g., [3] (5)) have been extensively studied in many aspects (existence, regularity, convergence, etc.). Concerning this situation, and in order to avoid the large amount of literature, for a complete introduction to the problem we only refer to the survey [9] and to the references therein.

The main difficulty of \( (1.1)-(1.2) \) compared to the classical case is the degeneracy of the associated Laplacian and at the same time the blow up of the scalar curvature close to the conical tips. In general, the Yamabe flow on manifolds with singularities is less studied. We close this section by pointing out some important contributions to this direction; these results agree with ours in the case of conic manifolds.

In [25] and [27] the problem is considered on \textit{singular manifolds} (including conic manifolds) in the sense of Amann (see, e.g., [7], [14], [27]) by using maximal continuous and maximal \( L^q \)-regularity theory, respectively. By showing maximal regularity results for certain classes of degenerate differential operators (see [25] Theorem 3.5, [28] Theorem 3.8, [27] Theorem 4.9) as well as [24] and [26]), in [25] Theorem 4.4 and [27] Theorem 5.1 it is shown under some assumptions that the type of the singularities is preserved under the evolution for short time.

In [7] and [8] the Yamabe flow is studied on manifolds with edge type singularities. By obtaining heat kernel estimates and then establishing mapping properties of the heat operator on appropriate Hölder spaces, under certain geometric assumptions in [8] Theorem 1.7 it is shown short time existence of a smooth solution which is improved in [7] Theorem 1.3] to a long time existence and convergence result for the normalized Yamabe flow. For similar problems on conic manifolds we also refer to [11], [2], [13], [14], [15] and [31].
2. Maximal $L^q$-regularity for quasilinear parabolic equations

Let $X_1 \overset{d}{\rightarrow} X_0$ be a continuously and densely injected complex Banach couple.

**Definition 2.1** (Sectorial operators). Let $\mathcal{P}(K, \theta) \in [0, \pi)$, $K \geq 1$, be the class of all closed densely defined linear operators $A$ in $X_0$ such that

$$S_0 = \{ \lambda \in \mathbb{C} \mid |\arg(\lambda)| \leq \theta \} \cup \{0\} \subset \rho(-A) \text{ and } (1 + |\lambda|)^||(A + \lambda)^{-1}\|_{\mathcal{L}(X_0)} \leq K \text{ when } \lambda \in S_0.$$  

The elements in $\mathcal{P}(\theta) = \cup_{K \geq 1} \mathcal{P}(K, \theta)$ are called (invertible) sectorial operators of angle $\theta$.

Consider the following abstract parabolic first order Cauchy problem

$$u'(t) + Au(t) = w(t), \quad t \in (0, T),$$

$$u(0) = 0,$$

where $-A : X_1 \to X_0$ is the infinitesimal generator of an analytic semigroup on $X_0$ and $w \in L_q^q(0, T; X_0)$, $q \in (1, \infty)$, $T > 0$. The operator $A$ has maximal $L^q$-regularity if for any $w \in L_q^q(0, T; X_0)$ there exists a unique $u \in W^1^q(0, T; X_0) \cap L^q(0, T; X_1)$ solving (2.1) - (2.2). In this case $u$ depends continuously on $w$. Furthermore, the above property is independent of $q$ and $T$.

We recall next the following boundedness condition for the resolvent of an operator.

**Definition 2.2** (R-sectorial operators). Denote by $\mathcal{R}(K, \theta) \in [0, \pi)$, $K \geq 1$, the class of all operators $A \in \mathcal{P}(\theta)$ in $X_0$ such that for any choice of $\lambda_1, \ldots, \lambda_N \in S_0 \setminus \{0\}$ and $x_1, \ldots, x_N \in X_0$, $N \in \mathbb{N} \setminus \{0\}$, we have

$$\left\| \sum_{k=1}^N \epsilon_k \lambda_k (A + \lambda_k)^{-1} x_k \right\|_{L^2(0,1,X_0)} \leq K \left\| \sum_{k=1}^N \epsilon_k x_k \right\|_{L^2(0,1,X_0)},$$

where $\{\epsilon_k\}_{k=1}^\infty$ is the sequence of the Rademacher functions. The elements in $\mathcal{R}(\theta) = \cup_{K \geq 1} \mathcal{R}(K, \theta)$ are called $R$-sectorial operators of angle $\theta$.

If we restrict to the class of UMD (unconditionality of martingale differences property, see, e.g., Section III.4.4]) Banach spaces, then the following result holds.

**Theorem 2.3** (Kalton and Weis, [18 Theorem 6.5]). In a UMD Banach space any $R$-sectorial operator of angle greater than $\frac{\pi}{2}$ has maximal $L^q$-regularity.

Let $q \in (1, \infty)$, $U$ be an open subset of $(X_1, X_0)_{\frac{1}{q}, q}$, $A(\cdot) : U \to \mathcal{L}(X_1, X_0)$ and $F(\cdot, \cdot) : U \times [0, T_0] \to X_0$, for some $T_0 > 0$. Consider the problem

$$u'(t) + A(u(t))u(t) = F(u(t), t) + G(t), \quad t \in (0, T),$$

$$u(0) = u_0,$$

where $T \in (0, T_0)$, $u_0 \in U$ and $G \in L^q(0, T_0; X_0)$. Maximal $L^q$-regularity for the linearization $A(u_0)$ together with appropriate Lipschitz continuity conditions imply existence of a short time solution for the above equation.

**Theorem 2.4** (Clément and Li, [10 Theorem 2.1]). Assume that:

1. **(H1)** $A(\cdot) \in C^1(U; \mathcal{L}(X_1, X_0)).$
2. **(H2)** $F(\cdot, \cdot) \in C^{1, -1}(U \times [0, T_0]; X_0).$
3. **(H3)** $A(u_0)$ has maximal $L^q$-regularity.

Then, there exists a $T \in (0, T_0)$ and a unique

$$u \in W^{1, q}(0, T; X_0) \cap L^q(0, T; X_1)$$

solving (2.3) - (2.4).

Finally, recall the following embedding of the maximal $L^q$-regularity space, namely

$$W^{1, q}(0, T; X_0) \cap L^q(0, T; X_1) \hookrightarrow C([0, T]; (X_1, X_0)_{\frac{1}{q}, q}), \quad T > 0, \ q \in (1, \infty),$$

see e.g. [8] Theorem III.4.10.2.
3. The Laplacian as a cone differential operator

When the Laplacian $\Delta$ on $\mathbb{B}$ is restricted to the collar part $(0, 1) \times \partial B$ it takes the degenerate form

$$\Delta = \frac{1}{x^n} \left( (dx^2)^2 + (n - 1 - \frac{x \partial_x \log |h(x)|}{2 \log |h(x)|}) (dx^2) + \Delta_h(x) \right),$$

where $\Delta_h(x)$ is the Laplacian on $\partial B$ induced by the metric $h(x)$. We regard $\Delta$ as an element in the class of cone differential operators or Fuchs type operators, i.e. the naturally appearing degenerate differential operators on $\mathbb{B}$. In this section we recall some basic facts and results from the related underline pseudodifferential theory, i.e. the cone calculus, towards the direction of the study of nonlinear partial differential equations. For more details we refer to [16], [17], [20], [21], [22], [27], [29] and [30].

Furthermore, take a covering $\kappa_i : U_i \subseteq \partial B \to \mathbb{R}^n$, $i \in \{1, ..., N\}$, $N \in \mathbb{N} \setminus \{0\}$, of $\partial B$ by coordinate charts and let $\{\phi_i\}_{i \in \{1, ..., N\}}$ be a subordinated partition of unity. For any $s \in \mathbb{R}$ and $p \in (1, \infty)$ let $H^s_p(\mathbb{B})$ be the space of all distributions $u$ on $\mathbb{B}^n$ such that

$$\|u\|_{H^s_p(\mathbb{B})} = \sum_{i=1}^N \| M_{\gamma_i} (1 \otimes \kappa_i)_* (\omega \phi_i u) \|_{H_p^s(\mathbb{R}^{n+1})} + \|(1 - \omega) u\|_{H_p^s(\mathbb{B})},$$

is defined and finite, where $\ast$ refers to the push-forward of distributions. The space $H^s_p(\mathbb{B})$, called (weighted) Mellin-Sobolev space, is independent of the choice of the cut-off function $\omega$, the covering $\{\kappa_i\}_{i \in \{1, ..., N\}}$ and the partition $\{\phi_i\}_{i \in \{1, ..., N\}}$: if $\Delta$ is as in (3.3), then it induces a bounded map

$$A : H^s_{p+\gamma}(\mathbb{B}) \to H^s_{p+\gamma}(\mathbb{B}),$$

Finally, if $s \in \mathbb{N}$, then equivalently, $H^s_{p, \gamma}(\mathbb{B})$ is the space of all functions $u$ in $H^s_{p, \text{loc}}(\mathbb{B}^n)$ such that near the boundary

$$x^{\frac{n+1}{2}} (x \partial_x^2) (\omega(x) u(x, y)) \in L^p_{\text{loc}}([0, 1) \times \partial B, \sqrt{\log |h(x)|} \frac{dx}{x}) \quad \text{for all } \gamma \in \mathbb{R}, \quad 1 < p < \infty.$$

Let us restrict to the case of the Laplacian $\Delta$ and regard it as an unbounded operator in $H^s_{p, \gamma}(\mathbb{B})$, $s, \gamma \in \mathbb{R}$, $p \in (1, \infty)$, with domain $C^\infty_c(\mathbb{B})$. The domain of its minimal extension (i.e. its closure) $\Delta_{\ast, \text{min}}$ is given by

$$\mathcal{D}(\Delta) = \left\{ u \in \bigcap_{\gamma > 0} H^{s+2, \gamma+2-\epsilon}_{p, \gamma}(\mathbb{B}) \mid \Delta u \in H^s_{p, \gamma}(\mathbb{B}) \right\}.$$

If in addition the conormal symbol of $\Delta$, i.e. the following family of differential operators

$$x^2 - (n - 1) \lambda \Delta (0) : \mathcal{C} \mapsto \mathcal{L}(H^s_p(\partial \mathbb{B}), H^s_{p, \gamma}(\mathbb{B})).$$
is invertible on the line \( \{ \lambda \in \mathbb{C} \mid \text{Re}(\lambda) = \frac{n-3}{2} - \gamma \} \), then we have precisely
\[
D(\Delta_{\text{max}}) = \mathcal{H}_p^{n+2, \gamma+2}(\mathbb{B}),
\]
The domain of the maximal extension \( \Delta_{\text{max}} \) of \( \Delta \), defined by
\[
D(\Delta_{\text{max}}) = \left\{ u \in \mathcal{H}_p^{n, \gamma}(\mathbb{B}) \mid \Delta u \in \mathcal{H}_p^{n, \gamma}(\mathbb{B}) \right\},
\]
is expressed as
\[
(3.7) \quad D(\Delta_{\text{max}}) = D(\Delta_{\text{min}}) \oplus E_{\Delta, \gamma},
\]
where \( E_{\Delta, \gamma} \) is a finite-dimensional space called asymptotics space. \( E_{\Delta, \gamma} \) is independent of \( s \); it consists of linear combinations of \( C^\infty(\mathbb{B}^\circ) \) functions that vanish on \( \partial \mathbb{B} \setminus \{ (0,1) \times \partial \mathbb{B} \} \) and in local coordinates \( (x,y) \in (0,1) \times \partial \mathbb{B} \) they are of the form \( \omega(x)c(y)x^{-\rho} \log^m(x) \) where \( c \in C^\infty(\partial \mathbb{B}), \rho \in \mathbb{C} \) with
\[
\frac{n-3}{2} - \gamma \leq \text{Re}(\rho) < \frac{n-3}{2},
\]
and \( m \in \{0,1\} \). The exponents \( \rho \) are determined explicitly by the metric \( h(\cdot) \). Due to (3.7), there are several closed extensions of \( \Delta \) in \( \mathcal{H}_p^{n, \gamma}(\mathbb{B}) \) each one corresponding to a subspace of \( E_{\Delta, \gamma} \). For an overview on the domain structure of a general \( \mathbb{B} \)-elliptic cone differential operator we refer to [17, Section 3] or alternatively to [23, Section 2.2]-[28, Section 2.3].

**Definition 3.2.** Recall that \( \partial \mathbb{B} = \cup_{i=1}^{k_s} \partial \mathbb{B}_i \), for certain \( k_s \in \mathbb{N} \setminus \{0\} \), where \( \partial \mathbb{B}_i \) are closed, smooth and connected. Denote by \( \mathcal{C}_\omega \) the space of all \( C^\infty(\mathbb{B}^\circ) \) functions \( c \) that vanish on \( \partial \mathbb{B} \setminus \{ (0,1) \times \partial \mathbb{B} \} \) and on each component \( (0,1) \times \partial \mathbb{B}_i, i \in \{1, \ldots, k_s\} \), they are of the form \( c \omega \), where \( c_i \in \mathbb{C}, \mathcal{C}_\omega \) consists of smooth functions that are locally constant close to the boundary. Endow \( \mathcal{C}_\omega \) with the norm \( \| \cdot \|_{\mathcal{C}_\omega} \) given by \( c \mapsto \| c \|_{\mathcal{C}_\omega} = (\sum_{i=1}^{k_s} |c_i|^2)^{\frac{1}{2}} \). Moreover, denote by \( \mathbb{R}_+^k \) the subspace of \( \mathcal{C}_\omega \) consisting of functions \( c \) such that \( c_i \in \mathbb{R} \) and by \( \mathbb{R}_+^k \) its subset consisting of functions \( c \) such that \( c_i > 0 \).

We close this section with a particular close extension of the Laplacian. Under certain choice of the weight \( \gamma, \mathcal{C}_\omega \) becomes a subspace of \( \mathcal{E}_{\Delta, \gamma} \). By recalling that the Mellin-Sobolev spaces are UMD spaces, the corespondent realization of the Laplacian enjoys the property of maximal \( L^p \)-regularity as we can see from the following.

**Theorem 3.3.** ([19 Theorem 4.1] or [20 Theorem 5.6]) Let \( s \geq 0, p \in (1,\infty) \) and
\[
\gamma \in \left( \frac{n-3}{2}, \min \left\{ -1 + \sqrt{\left( \frac{n-1}{2} \right)^2 - \lambda_1}, \frac{n+1}{2} \right\} \right),
\]
where \( \lambda_1 \) is the greatest non-zero eigenvalue of the boundary Laplacian \( \Delta_{h(0)} \). Consider the closed extension \( \Delta_s \) of \( \Delta \) in
\[
X_0^s = \mathcal{H}_p^{n, \gamma}(\mathbb{B})
\]
with domain
\[
D(\Delta_s) = X_1^s = D(\Delta_{\text{min}}) \oplus \mathcal{C}_\omega = \mathcal{H}_p^{n+2, \gamma+2}(\mathbb{B}) \oplus \mathcal{C}_\omega.
\]
Then, for any \( \theta \in [0,\pi) \) there exists some \( c > 0 \) such that \( c - \Delta_s \) is \( R \)-sectorial of angle \( \theta \).

4. Conic manifolds under the Yamabe flow

After setting \( g(t) = u^{-\frac{2}{n-4}}(t)g_0, t \geq 0, \) to [11]-[12] we obtain the following parabolic quasilinear equation over \( u \), namely
\[
(4.8) \quad u'(t) - nu^{-\frac{2}{n-4}}(t)\Delta u(t) = -\frac{n-1}{4} u^{-\frac{4}{n-4}}(t)R_{g_0}, \quad t \in (0,T),
\]
\[
(4.9) \quad u(0) = 1,
\]
where \( R_{g_0} \) is the scalar curvature of \( \mathbb{B} \). When \( R_{g_0} \) is restricted on \( (0,1) \times \partial \mathbb{B} \) it satisfies
\[
(4.10) \quad xR_{g_0} = \frac{1}{x}(R_{h(0)} - n(n-1)) + \text{non-singular terms},
\]
where \( R_{h(0)} \) is the scalar curvature of \( \partial \mathbb{B} \) (see also [12 Theorem 2.1] or [13 (2.8)]).
Proof of Theorem 1.1. Let $\gamma = \gamma_0 - \delta$ and $\nu \geq 0$. By [21] Lemma 3.2 and [21] Lemma 5.2 we have the embedding

\begin{equation}
H^{\nu+2-\frac{2}{n}+\varepsilon,\gamma+2-\frac{2}{n}+\varepsilon}_p(\mathbb{B}) \oplus C_{\omega} \hookrightarrow (X^\nu_1, X^\nu_2)_{\frac{1}{2}, \varphi} \hookrightarrow H^{\nu+2-\frac{2}{n}+\varepsilon,\gamma+2-\frac{2}{n}+\varepsilon}_p(\mathbb{B}) \oplus C_{\omega} \hookrightarrow C(\mathbb{B}),
\end{equation}

valid for all $\varepsilon > 0$ sufficiently small. Let $U_\nu$ be an open bounded subset of $(X^\nu_1, X^\nu_2)_{\frac{1}{2}, \varphi}$ containing 1. Due to (4.11), we choose $U_\nu$ in such a way that there exists a finite closed path $\Gamma_\nu$ in $\{ z \in \mathbb{C} \mid \Re(z) < 0 \}$ surrounding $\bigcup_{\nu \in U_\nu} \text{Ran}(v)$ with $\Gamma_\nu \cap \bigcup_{\nu \in U_\nu} \text{Ran}(v) = \emptyset$. For any $u_1, u_2 \in U_\nu$ and $\eta \in \mathbb{R}$ we have

\begin{equation}
u^\eta - u_2^\eta = (u_2 - u_1) \frac{1}{2\pi i} \int_{\Gamma_\nu} (-z)^{\eta}(u_1 + z)^{-1}(u_2 + z)^{-1} dz.
\end{equation}

Moreover, the choice of the data, [21] Lemma 3.2 and [21] Lemma 6.2 imply the following

\begin{equation}
\left\{ \begin{array}{l}
\text{For all } \varepsilon > 0 \text{ sufficiently small, the space } H^{\nu+2-\frac{2}{n}+\varepsilon,\gamma+2-\frac{2}{n}+\varepsilon}_p(\mathbb{B}) \oplus C_{\omega} \text{ is a Banach algebra (up to norm equivalence) and also closed under holomorphic functional calculus.} \\
\text{Furthermore, by (4.11) and (4.13)} \\
\text{For all } \varepsilon > 0 \text{ sufficiently small, the set } \{ (v + z)^{-1} | v \in U_\nu, z \in \Gamma_\nu \} \\
\text{is bounded in } H^{\nu+2-\frac{2}{n}+\varepsilon,\gamma+2-\frac{2}{n}+\varepsilon}_p(\mathbb{B}) \oplus C_{\omega}. \\
\text{Finally, due to [21] Corollary 3.3 we have}
\end{array} \right.
\end{equation}

\begin{equation}
\left\{ \begin{array}{l}
\text{For all } \varepsilon > 0 \text{ sufficiently small, elements in } H^{\nu+2-\frac{2}{n}+\varepsilon,\gamma+2-\frac{2}{n}+\varepsilon}_p(\mathbb{B}) \oplus C_{\omega} \\
\text{act by multiplication as bounded operators on } X^\nu_0.
\end{array} \right.
\end{equation}

We split the proof in several steps.

**Short time existence.** We apply Theorem [2.4] to (4.14)-(4.15) with $X_0 = X^\nu_0$, $X_1 = X^\nu_1$, $A(\cdot) = -n(-\frac{1}{4} + \Delta)$, $F(\cdot) = \frac{n-1}{4}(-\frac{1}{4} - n\Delta)R_{\delta_0}$, $G = 0$, $u_0 = 1$ and $U = U_\nu$. Concerning (H1), by (4.14)-(4.15) with $\nu = s$ we have that

\begin{equation}
\|A(u_1) - A(u_2)\|_{L^s(X^\nu_1, X^\nu_2)} \\
= \|u_1 \frac{\Delta u_1}{\Delta t} - u_2 \frac{\Delta u_2}{\Delta t}\|_{L^s(X^\nu_1, X^\nu_2)} \\
\leq C_1\|u_1 \frac{\Delta u_1}{\Delta t} - u_2 \frac{\Delta u_2}{\Delta t}\|_{L^s(X^\nu_2)} \\
\leq C_2\|u_1 - u_2\|_{C(X^\nu_1, X^\nu_2)}
\end{equation}

for certain constants $C_1, C_2 > 0$.

Similarly, concerning (H2), by noting that $R_{\delta_0} \in X^\nu_0$ due to (4.13)-(4.14), for $\varepsilon > 0$ sufficiently small we estimate

\begin{equation}
\|F(u_1) - F(u_2)\|_{X^\nu_0} \\
\leq C_3\|u_1 \frac{\Delta u_1}{\Delta t} - u_2 \frac{\Delta u_2}{\Delta t}\|_{H^{\nu+2-\frac{2}{n}+\varepsilon,\gamma+2-\frac{2}{n}+\varepsilon}} \|R_{\delta_0}\|_{X^\nu_0} \\
\leq C_4\|u_1 - u_2\|_{C(X^\nu_0, X^\nu_0)}
\end{equation}

for some constants $C_3, C_4 > 0$.

The property (H3) follows immediately from Theorem [3.3]. Therefore, there exists a $T > 0$ and a unique

\begin{equation}
\begin{array}{l}
\int_{\varepsilon > 0} \bigcup_{(0, T]} (0, T; X^\nu_0) \cap L^q(0, T; X^\nu_1) \hookrightarrow C([0, T]; (X^\nu_1, X^\nu_0)_{\frac{1}{2}, q}) \\
\hookrightarrow \int_{\varepsilon > 0} C([0, T]; H^{\nu+2-\frac{2}{n}+\varepsilon,\gamma+2-\frac{2}{n}+\varepsilon}(\mathbb{B}) \oplus C_{\omega}) \hookrightarrow C([0, T]; C(\mathbb{B}))
\end{array}
\end{equation}
We solve (4.13)–(4.14), where for the last embedding we have used (2.3) and (4.11). By taking the complex conjugate in (4.13)–(4.14) and using the above uniqueness, we conclude that we can replace $C_\omega$ in (4.17) by $\mathbb{R}_+$. 

**Smoothness in space.** We apply [23] Theorem 3.1] to (4.8)–(4.9) with $Y_t^j = X_t^{i+\frac{1}{4}}$, where $A$, $F$ as above. According to (4.17), we choose $T > 0$ small enough such that $u(t) \in U$, $t \in [0, T]$; in particular, we can replace $C_\omega$ in (4.17) by $\mathbb{R}_+$. 

Concerning the condition (i) of [23, Theorem 3.1], by (4.13), (4.15) and (4.17) we have that $A(u(t)) : Y_t^j \rightarrow Y_t^0$, $t \in [0, T]$, is a well defined map, and moreover by [21] Theorem 6.1] it has maximal $L^q$-regularity. Furthermore, by (4.10), we have

$$
\|A(u(t_1)) - A(u(t_2))\|_{L(Y_t^j, Y_t^0)} \leq C_5 \|u(t_1) - u(t_2)\|_{Y_t^j, Y_t^0}^{1 \frac{1}{4}},
$$

with some constant $C_5$ depending only on $u$ and $T$. Thus, $A(u(\cdot)) \in C([0, T]; L(Y_t^0, Y_t^0))$ due to (4.17).

Concerning the condition (ii) of [23] Theorem 3.1], first note that (4.11) implies

$$
(Y_t^j, Y_t^0 \rightarrow_{s>0} \int_{\mathbb{R}^n} \mathcal{H}_{p+2, \frac{1}{2}, \gamma+2, \frac{1}{2}}(\mathcal{B}) \oplus C_\omega, j \in \mathbb{N}.
$$

Therefore, if $v \in \mathcal{Z} \cap (Y_t^j, Y_t^0 \rightarrow_{s>0}, j \in \mathbb{N}$, then by (4.13), (4.15) and [23] Remark 2.8 (b)] the operator $A(v) : Y_t^{j+1} \rightarrow Y_t^{j+1}$ is well defined and moreover has maximal $L^q$-regularity. In addition, similarly to (4.10), for any $w \in C([0, T]; \mathcal{Z} \cap (Y_t^j, Y_t^0 \rightarrow_{s>0}), j \in \mathbb{N}$, and $t_1, t_2 \in [0, T]$ we have that

$$
\|A(w(t_1)) - A(w(t_2))\|_{L(Y_t^{j+1}, Y_t^{j+1})} \leq C_6 \|w - w\|_{L(Y_t^j, Y_t^{j+1})} \leq C_7 \|w(t_1) - w(t_2)\|_{L(Y_t^j, Y_t^{j+1})}^{1 \frac{1}{4}}(\mathcal{B}) \oplus C_\omega
$$

with some constants $C_6, C_7 > 0$ depending only on $w$ and $T$, i.e. $A(w(\cdot)) \in C([0, T]; L(Y_t^{j+1}, Y_t^{j+1}))$ due to (4.18).

Finally, concerning the condition (iii) of [23] Theorem 3.1], first note that $R_{g_0} \cap \mathcal{Z}$ in $Y_{g_0}^j$. Moreover, by (4.12)–(4.14) and (4.18) we have that

$$
\mathcal{Z} \cap (Y_t^j, Y_t^0 \rightarrow_{s>0}, v \in \mathcal{C}([0, T]; \mathcal{H}_{p+2, \frac{1}{2}, \gamma+2, \frac{1}{2}}(\mathcal{B}) \oplus C_\omega)
$$

for all $\varepsilon > 0$. Therefore, by [21] Corollary 3.3] we deduce that $F(w(\cdot)) \in L^q(\mathcal{Z} \cap (Y_t^j, Y_t^0 \rightarrow_{s>0}), j \in \mathbb{N}$. We conclude that for each $\tau \in (0, T]$, we have

$$
u \geq 0 \quad \mathcal{C}([\tau, T]; X_t^{n+\nu}) \cap L^q(\mathcal{Z} \cap (Y_t^j, Y_t^0 \rightarrow_{s>0}, \nu \geq 0, \varepsilon > 0)
$$

for all $\varepsilon > 0$. Therefore, by (4.14)–(4.19) for each $\tau \in \mathbb{R}$ we have

$$
u \geq 0 \quad \mathcal{C}([\tau, T]; \mathcal{H}_{p+2, \frac{1}{2}, \gamma+2, \frac{1}{2}}(\mathcal{B}) \oplus C_\omega)
$$

and

$$
u \geq 0 \quad \mathcal{C}([\tau, T]; \mathcal{H}_{p+2, \frac{1}{2}, \gamma+2, \frac{1}{2}}(\mathcal{B}) \oplus C_\omega).
$$

Therefore, by (4.11), (4.17) and (4.19) we conclude that

$$
\partial_t u = \frac{4}{n-1} u^{\frac{n-1}{n}} \partial_t u \in L^q(\mathcal{Z} \cap (Y_t^j, Y_t^0 \rightarrow_{s>0}, \nu \geq 0, \varepsilon > 0),
$$

\[n \geq 0 \quad \mathcal{C}([\tau, T]; \mathcal{H}_{p+2, \frac{1}{2}, \gamma+2, \frac{1}{2}}(\mathcal{B}) \oplus C_\omega).
\]
and hence
\[ f \in W^{1,q}(0, T; X_0^\prime) \cap \bigcap_{\nu \geq 0} W^{1,q}(\tau, T; X_0^\prime). \]

Let \( \mu_j : \Omega_j \subset \mathbb{B} \rightarrow \mathbb{R}^{n+1}, \ j \in \{1, \ldots, N\}, \ N \in \mathbb{N}\setminus\{0\}, \) be a covering of \( \mathcal{B} \) by coordinate charts, let \( \{\omega_j\}_{j \in \{1, \ldots, N\}} \) be a subordinated partition of unity and let \( \{y^1, \ldots, y^{n+1}\} \) be local coordinates in \( \Omega_j. \) On the collar part \( [0, 1) \times \partial \mathcal{B} \) we have \( \mu_j : \Omega_j \subset \mathcal{B} \rightarrow [0, \infty) \times \mathbb{R}^n \) and we choose \( y^1 = x. \) Denote \( \{g^{ij}\}_{i,j \in \{1, \ldots, n+1\}} = \{g_{ij}\}_{i,j \in \{1, \ldots, n+1\}} \) and recall the identity
\[ (4.21) \quad \Delta u^m = mu^{m-1} \Delta u + m(m-1)u^{m-2}\langle \nabla u, \nabla u \rangle_{g_0}, \quad m \in \mathbb{R}, \]
where in local coordinates
\[ \nabla u = \sum_{i,j=1}^{n+1} g^{ij} \frac{\partial u}{\partial y^i} \frac{\partial}{\partial y^j} \]
and \( \langle \cdot, \cdot \rangle_{g_0} \) is the Riemannian scalar product induced by \( g_0. \) In particular, on \( (0, 1) \times \partial \mathcal{B} \) we have
\[ (4.22) \quad \langle \nabla u, \nabla v \rangle_{g_0} = \frac{1}{x^2} (x \partial_x u)(x \partial_x v) + \sum_{i,j=1}^{n} h^{ij}(x) \frac{\partial u}{\partial y^i+1} \frac{\partial v}{\partial y^j+1}, \]
where \( \{h^{ij}\}_{i,j \in \{1, \ldots, n\}} = \{h_{ij}\}_{i,j \in \{1, \ldots, n\}}. \)

By \( (4.8)-(4.9) \) and \( (4.21) \) the conformal factor \( f \) satisfies
\[ (4.23) \quad u^{\frac{n-4}{4}}(t)u'(t) - \frac{n(n-1)}{4} \Delta f(t) = \frac{n(n-5)}{n-1} u^{\frac{n-4}{4}}(t)\langle \nabla u(t), \nabla u(t) \rangle_{g_0} - \frac{n-1}{4} u^{\frac{n-4}{4}}(t)R_{g_0}, \quad t \in (0, T), \]
\[ (4.24) \quad f(0) = 1. \]
Due to \( (4.15), (4.17), (4.19) \) and \( (4.20), \) we have that
\[ (4.25) \quad u^{\frac{n-4}{4}} \partial_t u \in L^q(0, T; H^{s+1-\gamma}) \cap \bigcap_{\nu \geq 0} L^2(\tau, T; H^{s+\gamma} \cap \mathbb{B}). \]

Let \( \tilde{x} \) be a \( C^\infty(\mathbb{B}^\circ) \) function such that \( \tilde{x} = x \) in local coordinates on \( (0, 1) \times \partial \mathcal{B} \) and \( \tilde{x} \geq \frac{1}{2} \) on \( B_1((0, 1) \times \partial \mathcal{B}). \) In each \( \Omega_j, \ j \in \{1, \ldots, N\}, \) denote by \( \partial_i \) the local derivative \( \partial_{y^i} \) with the convention that when we are on the collar part \( (0, 1) \times \partial \mathcal{B} \) we denote \( \partial_{z_1} = \partial_x \) and \( \partial_{z_i} = \frac{1}{\tilde{x}} \partial_{y^i}, \ i \in \{2, \ldots, n+1\}. \) We have that
\[ (4.26) \quad \omega_x^\tau \tilde{x}^{-1} \partial_z u \in L^q(0, T; H^{s+1-\gamma})(\mathbb{B}) \cap \bigcap_{\nu \geq 0} L^2(\tau, T; H^{s+\gamma} \cap \mathbb{B}), \]
and by \( (4.20) \) that
\[ (4.27) \quad \omega_x^\tau \tilde{x} \partial_z u \in \bigcap_{\nu \geq 0} C([0, T] \cap \partial \mathcal{B}) \cap \bigcap_{\nu \geq 0} C(\mathbb{B}), \]
\[ (4.28) \quad u^{\frac{n-4}{4}}(\nabla u, \nabla u)_{g_0} \in L^q(0, T; H^{s+\gamma} \cap \mathbb{B}) \cap \bigcap_{\nu \geq 0} L^2(\tau, T; H^{s+\gamma} \cap \mathbb{B}). \]

Finally, by \( (4.15) \) and \( (4.20) \) we have that
\[ (4.29) \quad u^{\frac{n-4}{4}} R_{g_0} \in L^q(0, T; H^{s+\gamma} \cap \mathbb{B}) \cap \bigcap_{\nu \geq 0} L^2(\tau, T; H^{s+\gamma} \cap \mathbb{B}). \]
Hence, from (4.28), (4.29) and (4.30) we deduce that

\[
\Delta f \in L^q(0, T; H^{s, \gamma}_p(B)) \cap \bigcap_{\nu > 0} L^q(\tau, T; H^{\nu, \gamma}_p(B)).
\]

Since \(X^\nu_T, \nu \geq s\), is a Banach algebra due to (4.13), by (4.17) and (4.19) we have that \(f(t) \in X^\nu_T\) for almost all \(t \in I_\nu\), where \(I_\nu = (0, T)\) when \(\nu = s\) and \(I_\nu = (\tau, T)\) when \(\nu > s\). Moreover, for each \(\nu \geq s\) we have that

\[
\|f(t)\|_{X^\nu_T} \leq C_\text{f}(\|f(t)\|_{X^s_T} + \|\Delta f(t)\|_{X^s_T}) \quad \text{for almost all} \quad t \in I_\nu,
\]

and for certain \(C_\text{f} > 0\). Therefore, by (4.30), (4.31) and Minkowski inequality we conclude that

\[
f \in L^q((0, T; H^{s+\gamma+2}_p(B) \oplus C_\omega)) \cap \bigcap_{\nu > 0} L^q(\tau, T; H^{\nu, \gamma+2}_p(B) \oplus C_\omega).
\]

The continuity follows from (4.29) and (4.31).

\[\square\]

**Proof of Corollary 1.2.** The result follows by Theorem 1.1 and [20, Corollary 2.9].

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Institut für Analysis, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany

E-mail address: roidos@math.uni-hannover.de