GROUND STATES FOR MEAN FIELD MODELS WITH A TRANSVERSE COMPONENT

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ABSTRACT. We investigate global logarithmic asymptotics of ground states for a family of quantum mean field models. Our approach is based on a stochastic representation and a combination of large deviation and weak KAM techniques. The spin-$\frac{1}{2}$ case is worked out in more detail.

1. THE MODEL AND THE RESULT

1.1. Introduction. Stochastic representations/path integral approach frequently provides a useful intuition and insight into the structure of quantum spin states. Numerous examples include \cite{2,3,10,12,17,18,22,26}. In this work we rely on a path integral approach and related large deviations techniques, and derive global logarithmic asymptotics of ground states for a class of quantum mean field models in transverse field. These asymptotics limits are identified as weak-KAM \cite{16} type solutions of certain Hamilton-Jacobi equations. In principle, such solutions are not unique, and an additional refined analysis along the lines of \cite{14,19,20} is needed for recovering the correct asymptotic ground state. This issue is addressed in more detail for the spin-$\frac{1}{2}$ case. In particular, our results imply logarithmic asymptotics of ground states for models with $p$-body interactions \cite{5}.

In the case of Laplacian with periodic potential a weak KAM approach to semi-classical asymptotics was already employed in \cite{1}.

Our stochastic representation gives rise to a family of continuous time Markov chains on a simplex $\Delta^N_d$ (defined below) of $\frac{1}{N}\mathbb{Z}^d$. The transition rates are enhanced by a factor of $N$, and the chain moves in a potential of the type $NF$. Ground states are Perron-Frobenius eigenfunctions of the corresponding generators. On the concluding stages of this work we have learned about the series of papers \cite{23,25}. The models we consider here essentially fall into a much more general framework studied in these works. The authors of \cite{23,25} extend an analysis of Schrödinger operators \cite{14,19,20} on $\mathbb{R}^d$ to lattice operators on $\epsilon\mathbb{Z}^d$, and they develop powerful techniques, which go well beyond the scope of our work, and which enable a complete asymptotic expansion of low lying eigenvalues and eigenfunctions in neighbourhoods of potential wells.

The paper is organized as follows: The class of models is described in Subsection 1.2 and the results are formulated in Subsection 1.4. Main steps of our approach are explained in Section 2 whereas some of the proofs are relegated to Section 3.
spin-$\frac{1}{2}$ case is studied in Section 4. Finally, in the Appendix, we establish the required properties of the Lagrangian $L_0$ in (1.11) and, accordingly, the required regularity properties of local minimizers.

1.2. Class of Models. Let $X$ be a $d$-dimensional complex Hilbert space. For the rest of the paper we fix an orthonormal basis $\{ |\alpha\rangle \}_{\alpha \in A}$ of $X$. We refer to the set $A$ of cardinality $d$ as the set of classical labels. Denote projections $P_{\alpha} \triangleq |\alpha\rangle\langle\alpha|$. The induced basis of $X^N = \bigotimes_1^N X$ is $|\alpha\rangle = |\alpha_1\rangle \otimes \cdots \otimes |\alpha_N\rangle$ $\alpha_1, \ldots, \alpha_N \in A$.

The corresponding lifting of the projection operator acting on $i$-th component is $P_i = I \otimes \cdots \otimes I \otimes P_{\alpha} \otimes I \otimes \cdots \otimes I$. For $\alpha \in A$ set $M_{\alpha} = \frac{1}{N} \sum_i P_i^{\alpha}$. Let $M^N$ be the $d$-dimensional vector with operator entries $M^N_{\alpha}$.

We are ready to define the Hamiltonian $H_N$ which acts on $X_N$,

$$-H_N = NF(M^N) + \sum_i B_i.$$ (1.1)

Above, $B_i$s are copies of a Hermitian matrix $B$ on $X$, $B_i$ acts on the $i$-th component of $|\alpha\rangle$.

We assume:

A1. $F$ is a real polynomial of finite degree.

Let $\Delta_d$ be the simplex, $\Delta_d = \{ m \in \mathbb{R}_+^d : \sum m_i = 1 \}$. In the sequel we shall write $\text{int}(\Delta_d)$ for the relative interior of $\Delta_d$. Accordingly, $\partial\Delta_d = \Delta_d \setminus \text{int}(\Delta_d)$.

Given $m \in \Delta_d$ and a basis vector $\alpha \in A^N$ let us say that $\alpha \sim m$, or, equivalently, $m = m(\alpha)$, if

$$m_{\alpha} = \frac{\# \{ i : \alpha_i = \alpha \}}{N} \Leftrightarrow M^N_{\alpha} |\alpha\rangle = m_{\alpha} |\alpha\rangle,$$ (1.2)

for all $\alpha \in A$. Define $\Delta^N_d = \Delta_d \cap \frac{1}{N} \mathbb{Z}^d$. In other words, $m \in \Delta^N_d$ iff there exists a compatible $\alpha \in A^N$. In the above notation:

$$F(M^N) |\alpha\rangle = F(m(\alpha)) |\alpha\rangle.$$ (1.3)

A2. The transverse field $B$ is stochastic: For any $\alpha, \beta \in A$,

$$\lambda_{\alpha\beta} = \lambda_{\beta\alpha} \triangleq \langle \alpha | B | \beta \rangle \geq 0.$$ (1.4)

Furthermore, $\lambda$ is an irreducible kernel on $A$. Without loss of generality we shall assume that $\lambda \equiv 0$ on the diagonal.

1.3. An Example: Spin-$s$ Models. The relation between the dimension $d$ of $X$ and the half-integer spin $s$ is $d = 2s + 1$. The set of classical labels is $A = \{-s, -s + 1, \ldots, s\}$.

The stochastic operators are $B_i = \lambda \xi_i$, $\lambda \geq 0$ is the strength of the transverse field. Altogether, the Hamiltonian is

$$-H_N = NF(M^N_{-s}, M^N_{-s+1}, \ldots, M^N_s) + \lambda \sum_i S_i.$$ (1.5)
For instance, the case of $p$-body ferromagnetic interaction corresponds to

$$ F(M^N_{-s}, M^N_{-s-1}, \ldots, M^N_s) = \left( \sum_{\alpha} \alpha M^N_{\alpha} \right)^p = \left( \frac{1}{N} \sum_i S^z_i \right)^p. \quad (1.6) $$

The operators $S^z$ act (under convention that $|s + 1| = |s - 1| = 0$) on $\mathbb{X}$ as

$$ S^z|\alpha\rangle = \frac{1}{2} \sqrt{s(s+1) - \alpha(\alpha+1)}|\alpha+1\rangle + \frac{1}{2} \sqrt{s(s+1) - \alpha(\alpha-1)}|\alpha-1\rangle \quad (1.7) $$

Consequently, the jump rates $\lambda_{\alpha\beta}$ are given by

$$ \lambda_{\alpha\beta} = \begin{cases} \frac{1}{2} \sqrt{s(s+1) - \alpha\beta}, & \text{if } |\alpha - \beta| = 1 \\ 0, & \text{otherwise.} \end{cases} \quad (1.8) $$

1.4. The Result. In order to develop an asymptotic description of finite volume ground states we need to introduce some additional notation: For $m \in \Delta^N_d$ set

$$ c_N(m) = \binom{N}{Nm} = \frac{N!}{\prod (Nm_\alpha)!}. $$

The vectors $|m\rangle \in \mathbb{X}_N$,

$$ |m\rangle \triangleq \frac{1}{\sqrt{c_N(m)}} \sum_{\alpha \in m} |\alpha\rangle $$

are normalized and orthogonal for different $m \in \Delta^N_d$.

By Perron-Frobenius theorem and Lemma 2.1 below the ground state of $\mathcal{H}_N$ is fully symmetrized, that is of the form

$$ |h_N\rangle = \sum_{m \in \Delta^N_d} h_N(m)|m\rangle, \quad (1.9) $$

and $h_N(m) > 0$ for every $m \in \Delta^N_d$ (see Subsection 2.1). Let us represent

$$ h_N(m) = e^{-N\psi_N(m)}. \quad (1.10) $$

It would be convenient to identify $\psi_N$ with its linear interpolation (which is an element of the space of continuous functions $C(\Delta_d)$).

Next introduce:

$$ \mathcal{H}_0(m, \theta) = \sum_{\alpha\beta} \sqrt{m_\alpha m_\beta} \lambda_{\alpha\beta} (\cosh(\theta_\beta - \theta_\alpha) - 1) $$

$$ \mathcal{L}_0(m, v) = \sup_{\theta} \{(v, \theta) - \mathcal{H}_0(m, \theta)\} \quad (1.11) $$

For $m \in \Delta_d$ define

$$ V(m) = \frac{1}{2} \sum_{\alpha, \beta} \lambda_{\alpha\beta} (\sqrt{m_\beta} - \sqrt{m_\alpha})^2 - F(m). \quad (1.12) $$
Finally set $\lambda = \sum_\alpha \lambda_\alpha$, 
\begin{align*}
\mathcal{H}(m, \theta) &= \mathcal{H}_0(m, \theta) - V(m) \\
&= \sum_{\alpha \beta} \sqrt{m_\alpha m_\beta} \lambda_{\alpha \beta} \cosh(\theta_\beta - \theta_\alpha) - \sum_\alpha m_\alpha \lambda_\alpha + F(m),
\end{align*}
(1.13)

and 
\begin{align*}
\mathcal{L}(m, v) &= \mathcal{L}_0(m, v) + V(m).
\end{align*}

Theorem A. Let $E_1^N$ be the bottom eigenvalue of $\mathcal{H}_N$. Set $\lambda = \sum_\alpha \lambda_\alpha$. Then the limit 
\begin{align*}
-\lambda + r_1 &\triangleq \lim_{N \to \infty} \frac{E_1^N}{N} 
\end{align*}
(1.14)

exists. Moreover, 
\begin{align*}
r_1 &= \min_{m} V(m).
\end{align*}
(1.15)

Furthermore, the sequence $\{\psi_N\}$ is precompact in $C(\Delta_d)$. Any subsequential limit $\psi$ satisfies: For any $T \geq 0$ and any $m \in \Delta_d$,
\begin{align*}
\psi(m) &= \inf_{\gamma : \gamma(T) = m} \left\{ \psi(\gamma(0)) + \int_0^T \mathcal{L}(\gamma(t), \gamma'(t)) \, dt - TR_1 \right\},
\end{align*}
(1.16)

where the infimum above is over all absolutely continuous curves $\gamma : [0, T] \to \Delta_d$. Moreover, the set $\mathcal{M}_\psi$ of all local minima of $\psi$ is a subset of $\mathcal{M}_V \triangleq \arg\min(V) \subset \text{int}(\Delta_d)$.

Remark 1. Hamiltonians $\mathcal{H}_0, \mathcal{H}$ are invariant under the shifts $\theta \mapsto \theta + c1$, and, as a result, the Lagrangians $\mathcal{L}_0, \mathcal{L}$ are infinite whenever $(v, 1) \neq 0$. Also, $\mathcal{L}_0(m, 0) = 0 = \min \mathcal{L}_0(m, v)$. Consequently, $\mathcal{L}(m, 0) = V(m) = -\mathcal{H}(m, 0)$, and (1.15) could be rewritten as 
\begin{align*}
r_1 &= \min_{m, v} \mathcal{L}(m, v) = -\max_{m} \mathcal{H}(m, 0).
\end{align*}
(1.17)

Remark 2. Either of (1.15) and (1.16) unambiguously characterizes $r_1$, but not $\psi$. As we shall explain in the sequel, if $\psi$ satisfies (1.16), then the weak KAM theory of Fathi [16] implies that $\psi$ is a viscosity solution (see Subsection 2.7 for the precise statement) on $\text{int}(\Delta_d)$ of the Hamilton-Jacobi equation 
\begin{align*}
\mathcal{H}(m, \nabla \psi(m)) &= -r_1.
\end{align*}
(1.18)

Note that since $\psi$ is a function on $\Delta_d$, the gradients $\nabla \psi$ lie in the subspace 
\begin{align*}
\mathbb{R}_d^0 = \{ v : (v, 1) = 0 \}.
\end{align*}
(1.19)

In general there might be many viscosity solutions of (1.18) which comply with the conclusions of Theorem A. The solutions which are subsequential limits of $\{\psi_N\}$ are called *admissible*. Although we expect uniqueness of global admissible solutions for a large class of models, our approach does not offer a procedure for selecting the latter. The viscosity setup is important - at least for a large class of symmetric potentials the global admissible solutions are not smooth and develop shocks. A
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proper selection procedure should be related to a more refined analysis of the low lying spectra of $\mathcal{H}_N$. As it was mentioned in the Introduction sharp asymptotics of eigenvalues and eigenfunctions in vicinity of potential wells were derived in a much more general context in $[23,25]$. In particular, it is explained therein how such asymptotics are related to (smooth) local solutions of (1.18). Implications of these results for a characterization of global admissible solutions is beyond the scope of this work and hopefully shall be addressed in full generality elsewhere. In the concluding Section 4 we work out a particular case of spin-$\frac{1}{2}$ models.

2. Structure of the theory

2.1. Spectrum of $\mathcal{H}_N$. Let $X_N^s$ be the sub-space of $X_N$ which consist of those vectors $|b\rangle$ which do not vanish under symmetrization. Namely, $|b\rangle = \sum_{\alpha} a_\alpha |\alpha\rangle \in X_N^s \setminus 0$, if $\sum_\pi a_\pi \neq 0$ for some $\alpha \in \mathcal{A}^N$, where $\pi$ is a permutation of $\{1, \ldots, N\}$ with $\pi(i) = (\pi_i)$. The sub-space $X_N^s$ is invariant for the Hamiltonian $\mathcal{H}_N$. The ground state of $\mathcal{H}_N$ always belongs to $X_N$. For the rest of the paper we shall work with the restriction of $\mathcal{H}_N$ to $X_N^s$.

All eigenfunctions of $\mathcal{H}_N$ (restricted to $X_N^s$) have mean-field representatives:

**Lemma 2.1.** If $E_N$ is an eigenvalue of $\mathcal{H}_N$, then there exists a function $h_N$ on $\Delta_d^N$ such that $|h_N\rangle \defeq \sum_{m \in \Delta_d^N} h_N(m) |m\rangle$ is a corresponding eigenfunction:

$$\mathcal{H}_N |h_N\rangle = E_N |h_N\rangle. \tag{2.1}$$

*Proof.* Let $E_N$ be an eigenvalue of $\mathcal{H}_N$. Let $|b_N\rangle = \sum_{\alpha \in \mathcal{A}^N} a_\alpha |\alpha\rangle \in X_N^s$ be an eigenfunction corresponding to the eigenvalue $E_N$. Let $C(\alpha, \beta) = \langle \beta | \hat{B} | \alpha \rangle$ be the matrix elements of $\hat{B} \defeq \sum_i B_i$. Thus, $\hat{B} |\alpha\rangle = \sum_\beta C(\alpha, \beta) |\beta\rangle$. The eigenfunction equation is recorded as: $\forall \beta$

$$\sum_\alpha a_\alpha C(\alpha, \beta) = (-E_N - F(m)) a_\beta.$$ 

Note that $C(\alpha, \beta) = C(\pi\alpha, \pi\beta)$. Consequently, since in addition $m(\beta) = m(\pi\beta)$,

$$\sum_\alpha a_\pi \alpha C(\alpha, \beta) = (-E_N - F(m)) a_\pi \beta.$$ 

Therefore, $|\pi b_N\rangle \defeq \sum_\alpha a_\pi \alpha |\alpha\rangle$ is also an eigenfunction. Since the sum $\sum_\pi a_\pi \alpha$ does not change if we permute the entries of $\alpha$, and since, by assumption $|b_N\rangle \in X_N^0$, the claim follows with $|h_N\rangle = \sum_\pi |\pi b_N\rangle$. $\square$

2.2. Stochastic Representation. Let $\alpha(t)$ be the continuous time Markov chain on $\mathcal{A}$ with jump rates $\lambda_{\alpha, \beta}$. $\mathbb{P}_\alpha^N$ is the path measure for $N$ independent copies of such chain starting from $\alpha$. Then the following representation of the entries of the density matrix holds $[3,22]$:

$$e^{-N\lambda T} \langle \beta | e^{-T \mathcal{H}_N} | \alpha \rangle = \mathbb{P}_\alpha^N \exp \left\{ N \int_0^T F(m(t)) dt \right\} \mathbb{1}_{\{\alpha(T) = \beta\}}. \tag{2.2}$$
Above \( m(t) = m(\alpha(t)) \).

### 2.3. Mean Field Lumping.

The process \( m_N(t) = \frac{m(t)}{m(\alpha(t))} \) is a continuous time Markov chain on \( \Delta^N_d \) with the generator

\[
G_N f(m) = N \sum_{\alpha, \beta} m_{\alpha} \lambda_{\alpha \beta} \left( f \left( m + \frac{\delta_{\beta} - \delta_{\alpha}}{N} \right) - f(m) \right).
\]  

(2.3)

It is reversible with respect to the measure

\[
\mu_N(m) = \frac{c_N(m)}{d^N}.
\]  

(2.4)

Summing up in (2.2),

\[
e^{-N\lambda T} \langle m' \mid e^{-T H_N} \mid m \rangle = \sqrt{\frac{\mu_N(m)}{\mu_N(m')}} \mathbb{E}^N_m \exp \left\{ N \int_0^T F(m(t)) dt \right\} \mathbb{1}_{\{m(T) = m'\}},
\]  

(2.5)

for every \( T \geq 0 \) and every \( m, m' \in \Delta^N_d \).

Using Girsanov’s formula one can rewrite (2.5) in a variety of ways for different modifications of the jump rates in (2.3). Namely, let \( g \) be a positive function on \( \Delta^N_d \). Consider the modified rates

\[
\lambda_{N,g}^{\alpha \beta} = \lambda_{\alpha \beta} \left( \frac{g \left( m + (\delta_{\beta} - \delta_{\alpha})/N \right)}{g(m)} - 1 \right) + F(m).
\]  

(2.7)

A self-suggesting choice is

\[
g(m) = \frac{1}{\sqrt{\mu_N(m)}} \Rightarrow \lambda_{N,g}^{\alpha \beta} = N \sqrt{m_\alpha (m_\beta + 1/N)} \lambda_{\alpha \beta}.
\]  

(2.8)

For the rest of the paper we fix \( g \) as in (2.8). The corresponding generator

\[
G_{N}^{g} f(m) = \sum_{\alpha, \beta} \lambda_{N,g}^{\alpha \beta} \left( f \left( m + \frac{\delta_{\beta} - \delta_{\alpha}}{N} \right) - f(m) \right).
\]  

(2.9)

is reversible with respect to the uniform measure on \( \Delta^N_d \). The function \( F_g \) in (2.7) equals to

\[
F_{g}(m) = -V(m) + \Xi_N(m),
\]  

(2.10)

where \( V \) is precisely the function defined in (1.12), and the correction

\[
\Xi_N(m) = \sum_{\alpha \beta} \lambda_{\alpha \beta} \sqrt{m_\alpha} \left( \sqrt{m_\beta + 1/N} - \sqrt{m_\beta} \right).
\]  

(2.11)
All together, (2.5) reads as
\[ e^{-N\lambda T} \langle m' | e^{-TH_N} | m \rangle = \mathbb{E}^{N,g}_{m'} \exp \left\{ -N \int_0^T (V - \Xi_N) (m(t)) \, dt \right\} \mathbbm{1}_{\{m(T) = m'\}}, \quad (2.12) \]

As we shall see below it happens to be convenient to work simultaneously with both representations (2.5) and (2.12).

Note that an immediate consequence of (2.12) is:

**Lemma 2.2.** \( E_N \) is an eigenvalue of \( H_N \) with \( |u_N⟩ = \sum m h_N(m) |m⟩ \) being the corresponding normalized eigenfunction if and only if \( u_N \) is also an eigenfunction of \( S_N \triangleq G_N^g + NF_g = G_N^g - N(V - \Xi_N) \) with
\[ -R_N \triangleq -(N\lambda + E_N) \quad (2.13) \]
being the corresponding eigenvalue.

2.4. **The Eigenfunction Equation.** Assumption A.2 and Perron-Frobenius theorem imply that \( H_N \) has a non-degenerate ground state \( |h_N⟩ = \sum m h_N(m) |m⟩ \) with strictly positive entries \( h_N(m) > 0 \). By Lemma 2.2 \( h_N(m) \) is the principal eigenfunction of \( G_N^g + NF_g(m) \) with the corresponding top eigenvalue \( -R_N^1 = -(N\lambda + E_N^1) \).

The corresponding eigenfunction equation is: For any \( T > 0 \),
\[ h_N(m) = \mathbb{E}^{N,g}_{m} \exp \left\{ -N \int_0^T (V - \Xi_N) (m(t)) \, dt + TR_N^1 \right\} h_N(m(T)), \quad (2.14) \]

By reversibility,

**Lemma 2.3.** Functions \( \{h_N\} \) satisfy: For every \( T \geq 0, N \) and \( m \in \Delta^N_d \)
\[ h_N(m) = \sum m' h_N(m') \mathbb{E}^{N,g}_{m'} \exp \left\{ -N \int_0^T (V - \Xi_N) (m(t)) \, dt + TR_N^1 \right\} \mathbbm{1}_{\{m(T) = m\}}. \quad (2.15) \]

2.5. **Compactness and Large Deviations.** For \( m, m' \in \Delta^N_d \) define
\[ Z_T^{N,g}(m', m) \triangleq \frac{1}{N} \log \mathbb{E}^{N,g}_{m'} \exp \left\{ -N \int_0^T (V - \Xi_N) (m(t)) \, dt \right\} \mathbbm{1}_{\{m(T) = m\}}, \quad (2.16) \]

In the sequel we shall identify \( Z_T^{N,g}(\cdot, \cdot) \) with its continuous interpolation on \( \Delta_d \times \Delta_d \).

Let \( \mathcal{AC}_T \) be the family of all absolutely continuous trajectories \( \gamma : [0, T] \mapsto \Delta_d \).

For \( m, m' \in \Delta_d \) define
\[ Z_T^1(m', m) \triangleq - \inf_{\substack{\gamma(0) = m', \gamma(T) = m \\gamma \in \mathcal{AC}_T}} \int_0^T \mathcal{L}(\gamma(t), \gamma'(t)) \, dt, \quad (2.17) \]
where the Lagrangian \( \mathcal{L} \) was defined in (1.13).
**Theorem 2.1.** For all $T$ sufficiently large the sequence of functions $\{Z_{N,g}^T\}$ is equi-continuous on $\Delta_d \times \Delta_d$ and uniformly locally Lipschitz on $\text{int} (\Delta_d \times \Delta_d)$. Furthermore, for all $T$ sufficiently large,

$$\lim_{N \to \infty} \left| Z_{N,g}^T(m', m) - Z_g^T(m', m) \right| = 0$$

(2.18)

simultaneously for all $m, m' \in \Delta_d$.

Note that the equi-continuity of $\{Z_{N,g}^T\}$ in Theorem 2.1 implies that the convergence in (2.18) is actually uniform. Consequently, $Z_g^T(\cdot, \cdot)$ is continuous on $\Delta_d \times \Delta_d$ and locally Lipschitz on $\text{int} (\Delta_d \times \Delta_d)$.

Theorem 2.1 is a somewhat standard statement. Its proof will be sketched in Subsection 3.1.

2.6. Lax-Oleinik Semigroup and Weak KAM. Recall the representation of the leading eigenfunction $h_N(m) = e^{-N\psi_N(m)}$. In the sequel we shall identify $\psi_N$ with its (continuous) interpolation on $\Delta_d$; $\psi_N \in C(\Delta_d)$.

**Theorem 2.2.** The sequence of numbers $R_{1N}/N$ is bounded in $\mathbb{R}$. The sequence of functions $\{\psi_N\}$ is equi-continuous on $\Delta_d$ and uniformly locally Lipschitz on $\text{int} (\Delta_d)$.

**Proof.** Since $R_{1N}$ is the Perron-Frobenius eigenvalue,

$$\frac{R_{1N}}{N} = -\frac{1}{N} \lim_{T \to \infty} \frac{1}{T} \log \mathbb{E}^N_{m} \exp \left\{ N \int_0^T F_g(m(s)) \text{d}s \right\},$$

which is bounded since $F_g$ is bounded on $\Delta_d$. On the other hand, by (2.15), the equi-continuity and the uniform local Lipschitz property of $\{\psi_N\}$ is inherited from the corresponding properties of $\{Z_{N,g}^T\}$. \hfill \Box

**Proof of (1.14) and (1.16) of Theorem A:** Theorems 2.1, 2.2 and Lemma 2.3 imply that any accumulation point $(r, \psi) \in \mathbb{R} \times C(\Delta_d)$ of the sequence $\{\frac{1}{N}R_{1N}, \psi_N\}$ satisfies: $\psi$ is locally Lipschitz on $\text{int} (\Delta_d)$ and

$$\psi(m) = \inf_{\gamma(T)=m} \left\{ \psi(\gamma(0)) + \int_0^T \mathcal{L}(\gamma(t), \gamma'(t)) \text{d}t \right\} - Tr \triangleq \mathcal{U}_T \psi(m) - Tr. \quad (2.19)$$

for every $T \geq 0$ and each $m \in \Delta_d$. Accumulation points of $\psi_N$ are called admissible solutions of (2.19). Since $\mathcal{U}_T$ is non-expanding on $C(\Delta_d)$, validity of equation (2.19) unambiguously determines $r$, which implies that the limit $r_1 \triangleq \lim_{N \to \infty} \frac{R_{1N}}{N}$ indeed exists. In view of (2.13) we, therefore, have established (1.14) and (1.16) of Theorem A. \hfill \Box
2.7. Viscosity Solutions. Recall the definition (1.13) of $\mathcal{H}(m, \theta)$. For $r_1$ defined as above consider the Hamilton-Jacobi equation

$$\mathcal{H}(m, \nabla \psi(m)) = -r_1. \quad (2.20)$$

**Definition 1.** For $m \in \Delta_d$ and $\psi \in C(\Delta_d)$ define lower and upper sub-differentials

$$D_- \psi(m) = \left\{ \xi \in \mathbb{R}^n : \lim \inf_{m' \to m} \frac{\psi(m') - \psi(m) - (\xi, m' - m)}{|m' - m|} \geq 0 \right\}, \quad (2.21)$$

and, similarly, $D_+ \psi(m)$ with $\lim \inf$ changed to $\lim \sup$ and the sign of the inequality flipped.

A locally Lipschitz function $\psi$ is said to be a viscosity supersolution of (2.20) at $m$ if $\mathcal{H}(m, \xi) \geq -r_1$ for every $\xi \in D_- \psi(m)$. Similarly, it is said to be a viscosity subsolution of (2.20) at $m$ if $\mathcal{H}(m, \xi) \leq -r_1$ for every $\xi \in D_+ \psi(m)$. $\psi$ is viscosity solution of (2.20) at $m$ if it is both sub and super viscosity solution.

**Theorem 2.3.** If $\psi$ is a weak-KAM solution (of (2.19) with $r = r_1$), then it is a viscosity solution of (2.20) on $\text{int}(\Delta_d)$.

The proof of Theorem 2.3 is relegated to Subsection 3.2.

2.8. Minima of $\psi$.

**Theorem 2.4.** Let $\psi$ be a weak KAM solution of (2.19). Then all local minima of $\psi$ lie in the interior $\text{int}(\Delta_d)$.

Theorem 2.4 will be proved in Subsection 3.3.

2.9. Stochastic Representation of the Ground State. The eigenfunction equation (2.14) defines a Markovian semi-group

$$\hat{\mathbb{E}}_N^T f(m) = \frac{1}{h_N(m)} \mathbb{E}_m^{\beta \alpha} \exp \left\{ N \int_0^T F_g(m(t)) dt + TR_1^N \right\} h_N(m(T)) f(m(T)). \quad (2.22)$$

This corresponds to continuous time Markov chain with the generator

$$\hat{G}_N^g f(m) = \frac{1}{h_N(m)} \sum_{\alpha, \beta} \lambda_{\alpha \beta}^N h_N \left( m + \frac{\delta_\beta - \delta_\alpha}{N} \right) \left( f \left( m + \frac{\delta_\beta - \delta_\alpha}{N} \right) - f(m) \right). \quad (2.23)$$

In the sequel we shall refer to $\hat{G}_N^g$ as to the generator of the ground state chain.

**Lemma 2.4.** The generator $\hat{G}_N^g$ is reversible with respect to the probability measure $\nu_N(m) \overset{\Delta}{=} h_N^2(m) = e^{-2N\psi_N(m)}$. Furthermore, $E_N$ is an eigenvalue of $\mathcal{H}_N$ if and only if $E_N - E_N$ is an eigenvalue of $\hat{G}_N^g$.

It is straightforward to check that $\hat{G}_N^g$ satisfies the detailed balance condition with respect to $\nu_N$. It is equally straightforward to see from (2.22) that $g_N$ is an eigenfunction of $\hat{G}_N^g + NF_g$, and hence by Lemma 2.2 of $\mathcal{H}_N$, if and only if $g_N/h_N$ is an eigenfunction of $\hat{G}_N^g$. 
3. Proofs

3.1. Proof of Theorem 2.1. Consider the family of processes \( \{ m(\cdot) = m_N(\cdot) \} \) with generator \( G_N^g \) defined in \( (2.9) \). We shall identify \( m_N \) with its linear interpolation. For each \( T > 0 \), the family \( \{ m_N(\cdot) \} \) is exponentially tight on \( C_{0,T}(\Delta_d) \).

Recall the definition of \( L_0 \) in \( (1.11) \). One can follow the approach of \([15]\) and to combine the Large Deviation Principle for projective limits \([13]\) with the inverse contraction principle of \([21]\) in order to conclude:

**Lemma 3.1.** For each \( T > 0 \) and every initial condition \( \bar{m} \in \Delta_d \) (where for each \( N \) we identify \( m_N \) with its discretization \( \lfloor N \bar{m} \rfloor / N \in \Delta^N_d \)) the family of processes \( \{ m(t) \} \) satisfy a large deviations principle on \( C_{0,T}(\Delta_d) \) with the following good rate function

\[
I_T(\gamma) = \begin{cases} 
\int_0^T L_0(\gamma(s), \gamma'(s)) \mathrm{d}s, & \text{if } \gamma \text{ is absolutely continuous and } \gamma(0) = \bar{m}, \\
\infty, & \text{otherwise.}
\end{cases} \tag{3.1}
\]

By the upper bound in Varadhan’s lemma,

\[
\limsup_{N \to \infty} Z_{N,g}^N(\bar{m}', \bar{m}) \leq Z_g(\bar{m}', \bar{m})
\]

On the other hand, by the lower bound in Varadhan’s lemma, for each \( \delta > 0 \)

\[
Z_g(\bar{m}', \bar{m}) \leq \liminf_{N \to \infty} \sup_{|\bar{m}_1 - \bar{m}| < \delta} Z_{N,g}^N(\bar{m}', \bar{m}_1).
\]

Therefore, \( (2.18) \) is a consequence of the claimed continuity of \( \bar{m} \to Z_{T,g}^N(\bar{m}', \bar{m}) \).

Let us proceed with establishing the asserted continuity properties of the family \( Z_{T,g}^N(\cdot, \cdot) \). By reversibility,

\[
Z_{T,g}^N(\bar{m}, \bar{m}') = Z_{T,g}^N(\bar{m}', \bar{m}) \tag{3.2}
\]

so it would be enough to explore those in the second variable only.

An equivalent task is to check continuity properties of

\[
Z_{T,g}^N(\bar{m}', \bar{m}) = \frac{1}{N} \log \mathbb{E}_{\bar{m}}^N \exp \left\{ N \int_0^T F(m(t)) \mathrm{d}t \right\} \mathbb{I}_{\{|m(T)\|=\bar{m}\}}, \tag{3.3}
\]

Indeed, by \( (2.6) \)

\[
Z_{T,g}^N(\bar{m}, \bar{m}') = \sqrt{\frac{\mu_N(\bar{m})}{\mu_N(\bar{m}')}} Z_T^N(\bar{m}, \bar{m}')
\]

Now, under \( \mathbb{P}_N \) the process \( m_N(t) \) is a super-position of \( N \) independent particles which hop on the finite set \( \mathcal{A} \) with irreducible rates \( \lambda_{\alpha\beta} \). Since \( F \) is bounded, the following claim is straightforward:

**Lemma 3.2.** There exist \( T_0 > 0, \epsilon_0 > 0 \) and a constant \( c_1 < \infty \) such that

\[
e^{-\epsilon \epsilon_0} e^{-NZ_T^N(\bar{m}', \bar{m})} \geq e^{-c_1 N \epsilon} e^{-NZ_T^N(\bar{m}', \bar{m})}. \tag{3.4}
\]

uniformly in \( N, T \geq T_0, \epsilon \leq \epsilon_0, \bar{m}', \bar{m} \in \Delta_d \) and \( N \).
Indeed, trajectories \( \mathbb{m}(\cdot) \) on \([0,T]\) and trajectories \( \mathbb{m}(\cdot) \) on \([0,T-\varepsilon]\) are related by the following one to one map: \( \mathbb{m}(t) = \mathbb{m}(t \frac{T}{T-\varepsilon}) \). Since for some \( c_2 = c_2(T_0) > 0 \), up to exponentially small factors, the total number of jumps of all the particles is at most \( c_2 \alpha N T \), the Radon - Nikodým derivative is under control and (3.4) follows. \( \square \)

As a result, for any \( T, \varepsilon \) as above, and for any \( m', m_1, m_2 \in \Delta_d \),
\[
Z_T^N(m', m_2) \geq Z_T^N(m', m_1) - c_1 \varepsilon + Z_T^N(m_1, m_2) \tag{3.5}
\]
Fix \( m_1, m_2 \) and define \( \mathcal{A}_+ = \mathcal{A}_+(m_1, m_2) = \{ \alpha : m_1^\alpha > m_2^\alpha \} \). For \( \alpha \in \mathcal{A}_+ \) define \( \delta_\alpha = m_1^\alpha - m_2^\alpha \). One way to drive \( \mathbb{m}(\cdot) \) from \( m_1 \) to \( m_2 \) during \( \varepsilon \) units of time is to choose \( N\delta_\alpha \) particles out of \( Nm_1^\alpha \) for each \( \alpha \in \mathcal{A}_+ \), and to redistribute them during \( \varepsilon \) units of time into \( \mathcal{A} \setminus \mathcal{A}_+ \), without touching the rest of the particles. There is an obvious uniform lower bound \( c_3 \varepsilon^n \) that a particle starting at the state \( \alpha \) will be at state \( \beta \) at time \( \varepsilon \). We infer:
\[
e^{-N Z_T^N(m_1, m_2)} \geq e^{-(\max_\alpha \sum_\beta \lambda_{\alpha, \beta} - \min F) \varepsilon N} \prod_{\alpha \in \mathcal{A}_+} \left( \frac{N m_1^\alpha}{N \delta_\alpha} \right) (c_3 \varepsilon^n)^N \delta_\alpha . \tag{3.6}
\]
Hence,
\[
Z_T^N(m_1, m_2) \geq -c_4 \varepsilon - c_5 \sum_{\alpha \in \mathcal{A}_+} \delta_\alpha \left( d \log \frac{1}{\varepsilon} - \log \frac{m_1^\alpha}{\delta_\alpha} \right) . \tag{3.7}
\]
Both, the equi-contiuity of \( \mathbb{m} \to Z_T^N(m_1, m_2) \) on \( \Delta_d \) and its uniform local Lipschitz property on \( \text{int} (\Delta_d) \) readily follow from (3.5) and (3.7). \( \square \)

3.2. Proof of Theorem 2.3. We follow the approach of [16]: Let \( \mathbb{m} \in \text{int}(\Delta) \) and assume that \( u \) is a smooth function such that \( \{ \mathbb{m} \} = \arg\min \{ u - \psi \} \) in a neighbourhood of \( \mathbb{m} \). Then,
\[
u(\mathbb{m}) \leq u(\gamma(-t)) + \int_{-t}^0 \mathcal{L}(\gamma, \gamma') ds - r_1 t
\]
for any \( t \geq 0 \) and for any smooth curve \( \gamma \) with \( \gamma(0) = \mathbb{m} \). Let \( \nu = \gamma'(0) \). Then,
\[
\nabla u(\mathbb{m}) \cdot \nu - \mathcal{L}(\mathbb{m}, \nu) \leq -r_1.
\]
Since the above holds for any \( \mathbb{m} \in \mathbb{R}^n \), \( \mathcal{H}(\mathbb{m}, \nabla u(\mathbb{m})) \leq -r_1 \) follows.

In order to check that \( \psi \) is a super-solution, note that by the upper and lower bounds on the Lagrangian \( \mathcal{L} \) derived in the Appendix, and by the local Lipschitz property of (bounded and continuous) \( \psi \) the minimum
\[
\min_{\gamma(t)=\mathbb{m}} \left\{ \psi(\gamma(0)) + \int_0^{t_0} (\mathcal{L}(\gamma, \gamma') - r_1) ds \right\}
\]
is attained at some \( \gamma_* \) with \( \gamma_*(0) = m' \) in a \( \delta_0 \)-neighbourhood of \( \mathbb{m} \), for all \( t_0 \) and \( \delta_0 \) appropriately small. As it is explained in the Appendix, the minimizing curve \( \gamma_* \) is \( C^\infty \) and stays inside \( \text{int} (\Delta_d) \). Evidently,
\[
\psi(\mathbb{m}) = \psi(\gamma_*(t)) + \int_{t_0}^{t_0} (\mathcal{L}(\gamma_*, \gamma'_*) - r_1) ds
\]
for every $t \in [0, t_0]$. Assume that $u$ is smooth and argmax $\{u - \psi\} = \{m\}$ in a $\delta_0$ neighbourhood of $m$. Then,

$$u(m) = u(\gamma_*(t_0)) \geq u(\gamma_*(t)) + \int_{t}^{t_0} (\mathcal{L}(\gamma_*, \gamma_*') - r_1) \, ds$$

for every $t \in [0, t_0]$. Set $\nu = \gamma'_*(t_0)$. We infer:

$$\nabla u(m) \cdot \nu \geq \mathcal{L}(m, \nu) - r_1.$$

Consequently, $\mathcal{H}(m, \nabla u(m)) \geq -r_1$. □

3.3. **Proof of (1.15) of Theorem A and Theorem 2.4.** By Theorem 2.2 and since $\nu_N(m) = e^{-2N\psi_N(m)}$ it follows that $\min_m \psi(m) = 0$.

Let us rewrite (1.16) as

$$\psi(m) = \inf_{\gamma(T) = m} \left\{ \psi(\gamma(0)) + \int_0^T (\mathcal{L}(\gamma, \gamma') - r_1) \, dt \right\} \quad (3.8)$$

Since $V(m) = \mathcal{L}(m, 0) \leq \mathcal{L}(m, \nu)$, the above might be possible only if $r_1 = \min_m V(m)$.

Furthermore, the Lagrangian $\mathcal{L}$ is uniformly super-linear in the second variable: By (A.1) of the Appendix for every $C > 0$ and $\delta > 0$ we can find $T > 0$ such that

$$\inf_{\text{diam}(\gamma) > \delta} \int_0^T (\mathcal{L}(\gamma, \gamma') - r_1) \, dt \geq C.$$ 

Which means that for $C > \max \psi$, the contribution to (3.8) for $\gamma$-s with the diameter larger than $\delta$ could be ignored.

Let, therefore, $\text{diam}(\gamma) \leq \delta$. By (1.17),

$$\int_0^T (\mathcal{L}(\gamma, \gamma') - r_1) \, dt \geq T \min_{m' \in \gamma} (V(m') - r_1) \quad (3.9)$$

We infer:

$$\psi(m) \geq \min_{|m' - m| \leq \delta} (V(m') - r_1). \quad (3.10)$$

The claim of Theorem 2.4 follows as soon as we notice that all the minima of $m \mapsto V(m)$ belong to int $(\Delta_d)$. □

4. **Results for Spin-$\frac{1}{2}$ Model**

For spin-$s$ models (1.5)

$$r_1 = \min_m V(m) = \min_m \left\{ \frac{\lambda}{4} \sum_{|\alpha - \beta| = 1} \sqrt{s(s + 1)} - \alpha \beta \left( \sqrt{m_\alpha} - \sqrt{m_\beta} \right)^2 - F(m) \right\}. \quad (4.1)$$
Consequently, the effective potential $T$ and, accordingly, the set options are depicted on Figure 1 (for simplicity we depict only the $0$-$1$ classical labels for Spin-$1/2$ Model. The Hamiltonian is given by

$$-\mathcal{H}_N = NF\left(M_{-1}^N, M_1^N\right) + \lambda \sum_i \hat{\sigma}_i^x$$

where

$$\hat{\sigma}_i^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \hat{\sigma}_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.2)$$

In this notation $\hat{\sigma}_i^z|\alpha\rangle = \alpha|\alpha\rangle$ and $\hat{\sigma}_i^x|\alpha\rangle = | -\alpha\rangle$ for $\alpha = \pm 1$.

The simplex $\Delta_d$ is just a segment $\{(m_{-1}, m_1) : m_{-1} + m_1 = 1\}$, parametrized by a single variable $m = m_1 - m_{-1} \in [-1, 1]$. Any vector $\theta \in \mathbb{R}_0^2$ is of the form $\vec{\theta} = (\theta, -\theta)$. Define $F(m) = F\left(\frac{1-m}{2}, \frac{1+m}{2}\right)$. Thus, in terms of $m$ and $\theta$, the Hamiltonian in (1.13) is

$$\mathcal{H}(m, \theta) = \lambda \sqrt{1-m^2} \cosh(2\theta) - \lambda + F(m). \quad (4.3)$$

Consequently, the effective potential

$$V(m) = -\mathcal{H}(m, 0) = \lambda - \left(\lambda \sqrt{1-m^2} + F(m)\right), \quad (4.4)$$

and the asymptotic leading eigenvalue $r_1$ is given by

$$r_1 = \min_{m \in (-1,1)} V(m) = \lambda - \max_{m \in [-1,1]} \left\{ \lambda \sqrt{1-m^2} + F(m) \right\}. \quad (4.5)$$

4.1. **Minima of $V$.** In order to explore the minimization problem (4.5) it would be convenient to represent $F(m) = -G\left(\text{sign}(m)\sqrt{1-m^2}\right)$. Then,

$$r_1 = \lambda - \max_{-1 \leq t \leq 1} \{ |t| - G(t) \}. \quad (4.6)$$

Indeed, as it clearly seen from (4.5) (and as it follows in general by Theorem 2.4), all the minima of $V$ belong to $(-1, 1)$, a possible jump discontinuity of $G$ at zero (if $F(-1) \neq F(1)$) plays no role for the computation of maxima. Note also that $G(-1) = G(1) = F(0)$.

Let

$$\mathcal{M}_\lambda = \text{argmin}(V) = \{ m \in (-1,1) : \mathcal{H}(m, 0) = -r_1 \}.$$

In other words, let $\mathcal{T}_\lambda \subset [-1,1]$ be the set of maximizers in (4.6). We set $\mathcal{T}_\lambda = \mathcal{T}_\lambda^+ \cup \mathcal{T}_\lambda^-$, where $\mathcal{T}_\lambda^+ = \mathcal{T}_\lambda \cap (0,1)$. Then,

$$m \in \mathcal{M}_\lambda \Leftrightarrow t = \text{sign}(m)\sqrt{1-m^2} \in \mathcal{T}_\lambda^{\text{sign}(m)}. \quad (4.7)$$

By assumption A1, $G$ is a polynomial of finite degree on each of the intervals $[-1,0)$ and $[0,1]$, so the set $\mathcal{T}_\lambda$ is finite, and its maximal cardinality is $\text{deg}(G) - 1$. Various options are depicted on Figure 7 (for simplicity we depict only the $[0,1]$ interval and, accordingly, the set $\mathcal{T}_\lambda^+$):

(a) First of all there exists $\lambda_c \in [0, \infty)$, such that $\mathcal{T}_\lambda = \{ \pm 1 \}$ on $(\lambda_c, \infty)$.

(b) If $\lambda_c > 0$, then $\mathcal{T}_\lambda$ still contains $\pm 1$. It could happen, however, that $\mathcal{T}_\lambda = \{-1, t_1, \ldots, t_k, 1\}$ contains at most $k \leq \text{deg}(G)/2$ other points. In the latter case $(-1,0) \cup (0,1)$ necessarily contains at least $2k$ inflection points of $G$. 

Figure 1. (a) The critical λ_c and T^{+}_{λ_c} = \{t_1, 1\}. (b) T^{+}_{λ} is a singleton.
(c) T^{+}_{λ} = \{t_1, t_2, t_3\}. There are at least two inflection points of G on (t_1, t_3).

(c) There might be other exceptional values of λ < λ_c for which either of T^{±}_{λ} is not a singleton. If, for instance T^{±}_{λ} = \{t^λ_1, \ldots, t^λ_k\} is not a singleton, then the interval (t^λ_1, t^λ_k) contains at least 2(k - 1) inflection points of G. Since there are at most deg(G) - 2 inflection points all together, and since intervals spanned by different T^{±}_{λ} are disjoint, we infer that T^{±}_{λ} is not a singleton for at most deg(G)/2 values of λ.

Values of λ for which the cardinality of T_{λ} changes correspond to first order phase transitions in the ground state.

4.2. Ferromagnetic p-body interaction. In the usual Curie-Weiss case with pair interactions G(t) = \frac{1}{2} (t^2 - 1), so that r_1 = -\frac{(\lambda - 1)^2}{2} if λ ≤ 1, and, accordingly, r_1 = 0 if λ ≥ 1. For λ ≥ 1 the set \mathcal{M}_{λ} = \{0\}. For λ ∈ (0, 1), \mathcal{M}_{λ} = \{±√{1 - λ^2}\}. No first order transition occurs.

In the p > 2-body ferromagnetic interaction case (1.6) the function

\[ G(t) = -\text{sign}(t)^p \left(1 - t^2\right)^{p/2}. \]

For odd p maximizers of λ|t| - G(t) always lie in (0, 1]. For even p the set \mathcal{M}_{λ} is symmetric. Thus in either case it is enough to consider

\[ r_1 = λ - \max_{t∈[0,1]} \left\{λt + (1 - t^2)^{p/2}\right\}. \]  

The crucial difference between the Curie-Weiss case p = 2 and p > 2 is that in the latter situation, G'(1) = 0, and G contains an inflection point t_p = \sqrt{\frac{1}{p-1}} inside (0, 1). An easy computation reveals that for p > 2,

\[ λ_c = \frac{p}{p-1} \left(1 - \frac{1}{(p-1)^2}\right)^{\frac{p}{2} - 1} \quad \text{and} \quad T^{+}_{λ_c} = \left\{\frac{1}{p-1}, 1\right\}. \]  

Accordingly, for even p,

\[ \mathcal{M}_{λ_c} = \{0, ±\hat{m}\} = \left\{0, ±\sqrt{\frac{p(p-2)}{(p-1)^2}}\right\}, \]
whereas for odd \( p \); \( \mathcal{M}_\lambda^* = \{0, \lambda\} = \left\{0, \sqrt{\frac{p(p-2)}{(p-1)^2}}\right\} \). This is precisely formula (14) of [5]. For \( \lambda > \lambda_c \) the set \( \mathcal{M}_\lambda = \{0\} \). For \( \lambda < \lambda_c \) there exists \( m^* = m^*(\lambda, p) \in \left(\sqrt{\frac{p(p-2)}{(p-1)^2}}, 1\right) \) such that the set \( \mathcal{M}_\lambda \) is a singleton \( \{m^*\} \) in the odd case, whereas \( \mathcal{M}_\lambda = \{\pm m^*\} \) in the even case. Thus, for mean-field models with \( p \)-body interaction, \( \lambda_c \) is the only value at which first order transition in the ground state occurs.

4.3. Asymptotic ground states. Let us return to general polynomial interactions \( F \). Fix \( m \in (-1, 1) \) and consider the equation,

\[
\mathcal{H}(m, \theta) = -r_1. \tag{4.11}
\]

The Hamiltonian \( \mathcal{H}(m, \cdot) \) is strictly convex and symmetric. Hence, if \( m \in \mathcal{M}_\lambda \), then \( \theta = 0 \) is the unique solution. If \( m \not\in \mathcal{M}_\lambda \), then necessarily \( \mathcal{H}(m, 0) < -r_1 \). Hence, there exist \( \theta(m) > 0 \), such that \( \pm \theta(m) \) are the unique solutions to (4.11). If \( F \) is symmetric, then \( \theta(-m) = -\theta(m) \). In any case, however, the following holds:

Let \( \psi \) be an admissible solution of \( \mathcal{H}(m, \psi') = -r_1 \). Since \( \psi \) is locally Lipschitz, it is a.e. differentiable. Consequently, \( \psi'(m) = \pm \theta(m) \) a.e. on \((-1, 1)\). The proposition below relies only on the fact that \( \psi \) is a viscosity solution on \((-1, 1)\).

**Proposition 4.1.** Record \( \mathcal{M}_\lambda = \{-1 < m_1, \ldots, m_k < 1\} \) in the increasing order. Set \( m_0 = -1 \) and \( m_{k+1} = 1 \). Then on each of the intervals \([m_{\ell}, m_{\ell+1}]\) the gradient \( \psi' \) is of the following form: There exists \( m^* \in [m_{\ell}, m_{\ell+1}] \), such that:

\[
\psi' = \theta \text{ on } [m_{\ell}, m^*) \quad \text{and} \quad \psi' = -\theta \text{ on } (m^*, m_{\ell+1}]. \tag{4.12}
\]

**Proof.** It would be enough to prove the following: If \( m \in (m_{\ell}, m_{\ell+1}) \) and \( \psi'(m) = \theta(m) \), then for any \( n \in (m_{\ell}, m) \),

\[
\psi(n) = \psi(m) - \int_n^m \theta(t)dt. \tag{4.13}
\]

Recall that since \( \psi \) is a viscosity solution on \((-1, 1)\), then

\[
\liminf_{\varepsilon \to 0} \frac{\psi(n^* + \varepsilon) - \psi(n^*)}{|\varepsilon|} \geq \theta \Rightarrow \theta \not\in (-\theta(n^*), \theta(n^*)). \tag{4.14}
\]

for any \( n^* \in (-1, 1) \). We shall show that if (4.13) is violated for some \( n \in (m_{\ell}, m) \), then (4.14) is violated as well in the sense that there exists \( n^* \in (n, m) \) and \( \theta \in (-\theta(n^*), \theta(n^*)) \) such that the right hand side of (4.14) holds.

Indeed, since \( \psi'(t) = \pm \theta(t) \) a.e. on \((-1, 1)\) it always holds that

\[
\psi(n) \geq \psi(m) - \int_n^m \theta(t)dt
\]

Let us assume strict inequality. For \( k \in [n, m] \) define

\[
v(k) = \int_n^k \frac{\psi'(t) + \theta(t)}{2\theta(t)}dt.
\]

By construction \( v(n) = 0 \), \( v(m) \triangleq p(m - n) < (m - n) \) and \( v'(m) = 1 \). There is no loss of generality to assume that \( p > 0 \). Hence there exists \( n^* \in (n, m) \) such that
p \in \partial v(n^*) (see Figure 2). By continuity of \( \theta(m) \) this would mean that

\[
\liminf_{\varepsilon \to 0} \frac{\psi(n^* + \varepsilon) - \psi(n^*)}{|\varepsilon|} \geq p \theta(n^*) - (1 - p) \theta(n^*) = (2p - 1) \theta(n^*).
\]

Since for any inner point \( n^* \in (m_\ell, m_{\ell+1}) \); \( \theta(n^*) > 0 \), we arrived to a contradiction. \( \square \)

Remark 3. Note that Proposition 4.1 implies that ground states \( \psi \) with more than one local minimum necessarily develop shocks.

If \( m_\ell < m_\ell^* < m_{\ell+1} \) from Proposition 4.1 then \( m_\ell^* \) is a local maximum of \( \psi \), and \( m_\ell^* \) is a shock location for the stationary Hamilton-Jacobi equation \( \mathcal{H}(m, \psi) = -r_1 \).

If \( \psi \) is, in addition, a weak KAM solution (in particular, if \( \psi \) is admissible), then argmin \{\psi\} \( \subseteq \mathcal{M}_\lambda \). Consequently, by Theorem 2.4 in the latter case, \( \psi' = -\theta \) on \((-1, m_1)\) and \( \psi' = \theta \) on \((m_k, 1)\).

Admissible solutions are always normalized in the sense that \( \min \psi = 0 \) and the minimum is attained on \( \mathcal{M}_\lambda \subseteq (-1, 1) \). It follows that admissible are uniquely defined in the following two cases:

CASE 1. The set \( \mathcal{M}_\lambda = \{m^*\} \) is a singleton. Then, \( \psi' = -\theta \) on \((-1, m^*)\) and \( \psi' = \theta \) on \((m^*, 1)\). Consequently,

\[
\psi(m) = \left| \int_{m^*}^m \theta(t)dt \right|.
\] (4.15)

CASE 2. The interaction \( F \) is symmetric and \( \mathcal{M}_\lambda = \{\pm m^*\} \). Then, \( \psi \) is also symmetric: \( \psi' = \theta \) on \((-m^*, 0) \cup (m^*, 1)\) and \( \psi' = -\theta \) on \((-1, 0) \cup (0, m^*)\). That is \( \psi(m) = \psi(-m) \) and \( \psi \) is still given by (4.15) for \( m \geq 0 \). Note that in this case \( \psi' \) has a jump at \( m = 0 \).

Let \( \Lambda^c \) be the set of \( \lambda \) which do not fall into one of the two cases above. As we have seen in Subsection 4.2 \( \Lambda_c = \emptyset \) in the case of Curie-Weiss model, and \( \Lambda_c = \{\lambda_c\} \) (see (4.9)) for general \( p \)-body interaction.
4.4. **Multiple Wells.** We shall refer to $\lambda \in \Lambda_c$ as to the case of multiple wells. Note first of all that there is a continuum of normalized solution of (1.16) as soon as the cardinality $|\mathcal{M}_\lambda| \geq 2$. Indeed, it is easy to see that any normalized $\psi$ which complies with the conclusion of Proposition 4.1 will be a solution to (1.16).

One needs, therefore, an additional criterion to determine locations of shocks $\{m^*_\ell\}$ or, equivalently, to determine values $\{\psi(m^*_\ell)\}$ for admissible solutions. It would be tempting to derive location of shocks by some natural limiting procedure via stabilization of shock propagation along Rankine-Hugoniot curves. Since however, we arrived to (1.16) directly from the eigenvalue equation without recourse to a finite horizon problem, it was not clear to us which limit to consider. Our selection of admissible solutions to (1.16) is based on a refined asymptotic analysis of Dirichlet eigenvalues in a vicinity of points belonging to the set $\mathcal{M}_\lambda$. Namely, a point $m^*_\ell \in \mathcal{M}_\lambda$ can be local minima of an admissible solution $\psi$ only if there is an exponential splitting of the corresponding bottom eigenvalues. Precise result is formulated in Proposition 4.2 below.

The results of [23,24,24] enable to explore asymptotic expansions of such eigenvalues with any degree of precision. In the simplest case we deduce the following:

**Corollary 4.1.** Assume that

$$
\min_{m \in \mathcal{M}_\lambda} \chi_0(m) \overset{\Delta}{=} \min_{m \in \mathcal{M}_\lambda} \left\{ \frac{\lambda}{1 - m^2} - \sqrt{1 - m^2} F''(m) \right\}
$$

is attained at either a unique point $m^*$ (non-symmetric potentials) or at a unique couple $\pm m^*$ (symmetric potentials). Then there is a unique admissible solution $\psi$, which is still given by (4.15).

For instance, in the critical ($\lambda = \lambda_c$) case of $p > 2$ body interaction, a substitution of (4.9) and (4.10) yields:

$$
\chi_0(0) = \lambda_c \quad \text{and} \quad \chi_0(\hat{m}) = (p - 2)(p - 1)\lambda_c.
$$

In other words, $\chi_0(0) < \chi_0(\hat{m})$, for any $p > 2$ and $\lambda = \lambda_c(p)$. Consequently, even at $\lambda = \lambda_c$ there is still a unique admissible solution $\psi(m) = |\int_0^m \theta(t)dt|$ with the unique minimum at $m^* = 0$.

We explain Corollary 4.1 in the concluding paragraph of this Section.

**Spectral Asymptotics and the Set $\mathcal{M}_\lambda$.** Assume that $\lambda \in \Lambda_c$ and, as before, denote $\mathcal{M} = \{m_1, \ldots, m_k\}$.

**Lemma 4.1.** For any $\delta > 0$ there exists $\epsilon > 0$ such that

$$
\min_{d((m_c, \mathcal{M}_\lambda)) \geq \delta} \psi(m) \geq \epsilon,
$$

uniformly in normalized admissible solutions of (1.16).

**Proof.** Let $m \in (m_t, m_{t+1})$. By Proposition 4.1

$$
\psi(m) \geq \min \left\{ \psi(m_t) + \int_{m_t}^m \theta(t)dt \right\}, \psi(m_{t+1}) + \int_{m}^{m_{t+1}} \theta(t)dt \right\},
$$

and (4.18) follows. \qed
In the sequel \( h_N = e^{-N\psi_N} \) is the Perron-Frobenius eigenfunction of \( G_N^g + NF_g \overset{\Delta}{=} \mathcal{S}_N \).

Recall:

\[
\mathcal{S}_N f(m) = N (F(m) - \lambda) f(m)
\]

\[
+ \frac{N\lambda}{2} \sqrt{(1 - m)(1 + m + \frac{2}{N})} f(m + \frac{2}{N})
\]

\[
+ \frac{N\lambda}{2} \sqrt{(1 + m)(1 - m + \frac{2}{N})} f(m - \frac{2}{N})
\]

(4.19)

Pick \( 0 < \delta < \frac{1}{4} \min \left| m_{l+1} - m_l \right| \). Let \( 1 \equiv \sum_{l=0}^{k} \chi_l \) be a partition of unity satisfying:

\[
\chi_l \equiv 1 \text{ on } I_\delta(m_l) \quad \text{and} \quad \chi_l \equiv 0 \text{ on } I_{2\delta}(m_l).
\]

Above \( I_\eta(m) \) is the interval \([m - \eta, m + \eta] \). By Lemma 4.1 there exists \( \epsilon > 0 \) such that for \( l = 1, \ldots, k \) and all \( N \) large enough

\[
\frac{1}{N} \log \max_m \left| (\mathcal{S}_N + R_1 N) \chi_l h_N(m) \right| \leq -\epsilon.
\]

(4.20)

Let \( \mathcal{S}_N^l \) be a Dirichlet restriction of \( \mathcal{S} \) to \( I_\delta(m_l) \). Let \( -R_{1,N,l} \) be the leading eigenvalue of \( \mathcal{S}_N^l \).

We are entitled to conclude: There exists \( \epsilon' > 0 \) such that

\[
\frac{1}{N} \log \left| R_{1,N} - \min_l R_{1,N,l} \right| \leq -\epsilon'.
\]

(4.21)

Furthermore,

**Proposition 4.2.** If \( l = 0, \ldots, k \) and \( \psi = \lim_{j \to \infty} \psi_N^j \) is a subsequential limit such that \( m_l \) is a local minimum of \( \psi \), then there exists \( \epsilon' > 0 \) such that:

\[
\frac{1}{N_j} \log \left| R_{1,N_j}^l - R_{1,N_j,l} \right| \leq -\epsilon'.
\]

(4.22)

Proof. In view of Lemma 2.4 the claim readily follows from the general theory of exponentially low lying spectra for metastable Markov chains [6]. For a direct proof note that under the assumptions of the Proposition, one (possibly after further shrinking the value of \( \delta \)) can upgrade (4.20) as

\[
\frac{1}{N_j} \log \max_m \left| (\mathcal{S}_j + R_1 N_j) \chi_l h_{N_j}(m) \right| \leq -\epsilon,
\]

(4.23)

and (4.22) follows from the spectral theorem.

**Asymptotics of Dirichlet Eigenvalues** \( R_{1,N,l} \). Define \( \lambda(m) = \sqrt{1 - m^2} \). The asymptotics of \( R_{1,N,l} \) up to zero order terms is given [25] by

\[
-R_{1,N,l} = -N r_1 - \frac{V''(m_l)}{\lambda(m_l)} + O \left( \frac{1}{N} \right) = -N r_1 - \chi_0(m_l) + O \left( \frac{1}{N} \right),
\]

(4.24)

where we used the explicit expression (4.4) for \( V \) in the second equality. \( \chi_0 \) was defined in (4.16). The claim of Corollary 4.1 follows now from Proposition 4.2. □
The Lagrangian $L_0$ was defined in (1.11)

**Lower bounds on $L_0$.** Fix $\alpha \in A$ and consider $\theta_\alpha = \frac{n-1}{n} t$ and, for $\beta \neq \alpha$, $\theta_\beta = -\frac{1}{n} t$.

Recall that $v \in \mathbb{R}^n_0$, that is $v_\alpha = -\sum_{\beta \neq \alpha} v_\beta$. Therefore, for any $\alpha$,

$$L_0(m, v) \geq \sup_t \left\{ tv_\alpha - \sum_{\beta \neq \alpha} \sqrt{m_\alpha m_\beta} \lambda_{\alpha\beta} (\cosh(t) - 1) \right\}$$

Define $\lambda_\alpha(m) = \sum_{\beta \neq \alpha} \sqrt{m_\alpha m_\beta} \lambda_{\alpha\beta}$. For $|v_\alpha| \geq \lambda_\alpha(m)$ one may choose $t^* = \text{sign}(v_\alpha) \log \frac{|v_\alpha|}{\lambda_\alpha(m)}$. We infer: If $|v_\alpha| \geq \lambda_\alpha(m)$, then

$$L_0(m, v) \geq |v_\alpha| \left( \log \frac{|v_\alpha|}{\lambda_\alpha(m)} - 1 \right). \quad (A.1)$$

**Upper bounds on the Lagrangian $L_0$.** Consider

$$R_0(m, v) \equiv \sup_{\theta} \left\{ \sum v_\alpha \theta_\alpha - \sum_{\alpha, \beta} \sqrt{m_\alpha m_\beta} \lambda_{\alpha\beta} \cosh(\theta_\beta - \theta_\alpha) \right\}.$$ 

Since $L_0(m, v) = R_0(m, v) + \sum_{\alpha} \lambda_\alpha(m)$, it would be enough to control the dependence of $R_0$ on $v$.

Let us say that a flow $f = \{f_{\alpha\beta}\}$ is compatible with $v \in \mathbb{R}^n_0$; $f \sim v$ if:

(a) It is a flow: $f_{\alpha\beta} = -f_{\beta\alpha}$.
(b) For any $\alpha \in A$, $\sum_\beta f_{\beta\alpha} = v_\alpha$.

Then $\sum v_\alpha \theta_\alpha = \frac{1}{2} \sum_{\alpha, \beta} (\theta_\beta - \theta_\alpha) f_{\alpha\beta}$. Hence, for any $f \sim v$,

$$R_0 = \sup_{\theta} \left\{ \frac{1}{2} \sum_{\alpha, \beta} (\theta_\beta - \theta_\alpha) f_{\alpha\beta} - \sqrt{m_\alpha m_\beta} \lambda_{\alpha\beta} \cosh(\theta_\beta - \theta_\alpha) \right\}. \quad (A.2)$$

We shall rely on the following upper bound on each term in (A.2): For any $f$ and $a > 0$

$$\sup_t \{ ft - a \cosh(t) \} \leq |f| \log \left( 1 + \frac{2|f|}{a} \right).$$

Consequently, we derive the following upper bound on $R_0$:

$$R_0(m, v) \leq \inf_{f \sim v} \sum_{\alpha, \beta} \frac{|f_{\alpha\beta}|}{2} \log \left( 1 + \frac{|f_{\alpha\beta}|}{\sqrt{m_\alpha m_\beta} \lambda_{\alpha\beta}} \right). \quad (A.3)$$

**Regularity of minimizers.** Let $m \in \text{int}(\Delta_d)$. We claim that there exists $\delta_0 > 0$ and $t_0 > 0$ such that for any $m'$ in the $\delta_0$-neighbourhood of $m$ the minimizer $\gamma^*$ of

$$\inf_{\gamma(0) = m', \gamma(t_0) = m} \int_0^{t_0} L(\gamma(s), \gamma'(s)) ds$$

exists and is, actually, $C^\infty$. Indeed, an absolutely continuous minimizer exists by the classical Tonelli’s theorem. By lower (A.1) and upper (A.3) bounds on the
Lagrangian, it is easy to understand that minimizers stay inside \( \text{int}(\Delta_d) \) once \( t_0 \) and \( \delta_0 \) are chosen to be appropriately small. But then the regularity theory of either [11] or [4] applies and yields Lipschitz regularity on \([0, t_0]\). Since, the Lagrangian \( \mathcal{L} \) is strictly convex in the second argument, and, in the interior of \( \Delta_d \), it is \( C^\infty \) in both arguments, the \( C^\infty \) of the minimizer follows from the implicit function theorem, see e.g. [7].

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