Insight into the Dynamics of Fractional Maxwell Nano-Fluids Subject to Entropy Generation, Lorentz Force and Heat Source via Finite Difference Scheme

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Abstract: In recent times, the loss of useful energy and solutions to those energy challenges have a wide scope in different areas of engineering. This work focuses on entropy analysis for unsteady viscoelastic fluids. The momentum boundary layer and thermal boundary layer are described under the effects of a magnetic field in the absence of an induced magnetic field. The study of a fractional model of Maxwell nanofluid by partial differential equation using Caputo time differential operator can well address the memory effect. Using transformations, the fractional ordered partial differential equations (PDEs) are transfigured into dimensionless PDEs. Numerical results for fractional Maxwell nanofluids flow and heat transfer are driven graphically. The Bejan number is obtained following the suggested transformation of dimensionless quantities like entropy generation. A mathematical model of entropy generation, Bejan number, Nusselt number and skin friction are developed for nanofluids. Effects of different physical parameters like Brickman number, Prandtl number, Grashof number and Hartmann number are illustrated graphically by MAPLE. Results depict that the addition of nanoparticles in base-fluid controls the entropy generation that enhances the thermal conductivity and application of magnetic field has strong effects on the heat transfer of fractional Maxwell fluids. An increasing behavior in entropy generation is noticed in the presence of source term and thermal radiation parameter.

Keywords: fractional order; Maxwell fluids; nano-fluids; entropy generation (EG); Bejan number

1. Introduction

The best way to represent the natural phenomena is by using differential equations (DEs) with suitable boundary conditions. Recently, the fractional (non-integral) order of DEs has gained much interest because of their vast scope in many engineering fields like food engineering, oceanography, chemical reactions and chaos. All fractional derivatives like Riemann Levillie, Caputo, Caputo Fabrizio, Antangna Beleanu are used widely in practice [1–3], as a deep study can easily be handled using this approach. Natural convection along a vertical wall and cylinder has been explained using Caputo time-fractional derivatives and fractional derivatives [4,5]. The significant difference between fractional and ordinary fluid flow emerges at different times. Furthermore, with large time values, increasing the fractional parameter’s value increases velocity. Fractional calculus covers the complex structure of viscoelastic fluids in various research areas of glass fiber production, exotic lubricants, colloid solutions, extraction of polymer solutions and cooling processes [6]. A simplified Phan–Thien–Tanner model for viscoelastic fluids was investigated...
analytically in [7] with different physical parameters. It was established by Sheikh, N.A. et al. [8] that it is not enough to get experimental data by using a conventional derivative model for Maxwell fluids rather than fractional operators. Moreover, it was discovered that a link exists between Maxwell’s constitutive equation and molecular theory [9]. Few other investigations for analytical results have been done in many research articles. But in the rheological perspective, viscoelastic fractional order models have attracted much interest due to their wide range of applications. Fractional Maxwell models are derived by substituting the conventional derivative in the known Maxwell model of stress-strain expression by fractional-order derivatives. Analytical solutions of fractional Maxwell, fractional generalized Maxwell model, fractional second grade, third grade, Oldroyd-B model etc., via analytical techniques of Laplace transform, Fourier transform, Weber transform and Hankel transform are obtained in [10–14].

Many researchers focus on such physical processes involving entropy generation—EG. It is a well-known fact that all physical problems, especially heat transfer, involve entropy generation—EG. Entropy generation—EG plays a vital role in fluid dynamics. The first law of thermodynamics moves around heat transfer processes, whereas the second law of thermodynamics moves around the entropy generation of the system. Entropy generation—EG tells the feasibility and efficiency of the system. In other words, entropy describes the ways a process can control energy loss.

Since entropy generation—EG is important and happens in almost all thermo-dynamical processes, many researchers have worked in this direction. Among the various researchers who have done splendid work on entropy generation—EG, Adrian Bejan, has published many articles and books [15–21]. Considerable work is done by Pranab Kumar Mondal [22] for irreversibility analysis of Couette flow while applying weak and relatively strong pressure gradient. The fluid dynamics are studied by variation of volumetric entropy generation number and Bejan number.

To control the entropy generation—EG, addition of nanoparticles in base fluid gave a new direction to heat transfer problems (the formation of nanofluids). Nanofluids are nano-sized particles accomplished with base-fluids, i.e., Common nanoparticles Cu, Ag, Au and Fe. Water, engine oil and ethylene glycol are common base fluids. To enhance the thermal conductivity of base fluids, the idea of nanofluids is given by Choi et al. in [23] for the first time. Later on, Tiwari and Das discussed the effects of different shapes of nano-sized particles on thermal enhancement [24]. Using square shape cavity, different aspects of ball-shaped, cylindrical-shaped, and rod-shaped nanoparticles were studied [25]. Analytical results for the effects of nanofluids were driven by [26]. In [27], analytical results are investigated for temperature profile and Nusselt number under the effects of viscous dissipation and porous media. Using a traditional approach, numerical results were obtained, expressing the radial and tangential momentum across the disk decreases for higher Lorentz forces and slip factor [28].

Like many other key sources, entropy generation—EG includes viscous dissipation, chemical reaction, heat and mass transfer, heat convection and conduction and electrical conduction. Considerable work by B. Mahathish et al. has been done via the spectral quasi-linearization (SQL) method for entropy analysis and solving a Williamson model, which provides a base for fractional models. An analytical approach is applied to investigate the effects of viscous dissipation and limiting effects of Nusselt for temperature profile [29]. Bejan number-BN is the ratio of entropy generation—EG due to heat and total entropy generation—EG of system. But Awed in [30] gave a new definition to Bejan number-BN. Bejan number-BN describes the effects of magnetic field irreversibility and fluid friction irreversibility. An investigation has been conducted on laminar falling liquid film along an inclined heated surface in [31]. In [32–34] and many other articles, the exact and numerical solution of entropy generation—EG is published. B. Mahathish numerically investigates the effects of quadratic variation of density-temperature (quadratic convection) and the quadratic Rosseland thermal radiation using the modified Bongiorno Model (MBM) [35].
Similar analytical as well as numerical approaches can be seen in [36–39] for such different viscoelastic models.

Numerical investigations and many others were used in [40–43]; however, various research gaps are found in these articles, which are still not addressed. Such as how heat transfer can be enhanced by adding the nanoparticle to the base fluid? What is the effect of Lorentz forces on flow dynamics? What do numerical results predict about Skin friction and Nusselt number? What are the formulation of entropy generation and Bejan number in the presence of thermo-physical properties of nanoparticles? In reviewing these research gaps in literature surveys, the main task of this article is to develop a mathematical model of momentum and heat for fractional Maxwell nano-fluids for detailed insights into Lorentz force, heat source/sink, and Nusselt number and entropy generation—EG. A new definition of Bejan number—BN is introduced by adding the coefficients of thermo-physical properties of nanoparticles. Water is taken as base fluid, whereas Cu and Al₂O₃ are the nanoparticles used for graphical results of velocity and temperature profiles. The problem is first modeled fractionally by applying the definition of Caputo time derivative, then using transformation; a dimensionless analysis is done. The resulted model is solved by a numerical technique of finite difference scheme. Plots are drawn for Bejan number—BN against Br the Brickman number, \( H \) the square of Hartmann number and \( \Omega \) dimensionless temperature difference. Moreover, some graphical results are extended to evaluate the Nusselt number \( Nu \) and Skin friction \( S_f \). These results are computed via mathematical software MAPLE.

2. Mathematical Model and Formulations

Considering magnetohydrodynamic (MHD) flow of incompressible and unsteady fluid along the infinite vertical plate. The induced magnetic field and pressure gradient are neglected. Initially, at time \( t = 0 \), fluid has velocity zero and has a constant temperature \( \theta_w \). With the passage of time temperature of the system rise to \( \theta_w \). The fluid flow is considered along \( x \) – direction. The magnetic field is applied in the \( y \) – direction as illustrated in Figure 1 below.

![Figure 1. Geometry of the problem.](image)

Taking into account the Boussinesq approximation, the assumptions of the system are,

- Flow is unsteady, incompressible and 1-dimensional.
- Pressure gradient is absent.
- Body force is significant.
- Magnetic field is applied (ignoring the induced magnetic field).

Then the equation of continuity is restricted and takes the following form (can be seen in [44]);

\[
\nabla \cdot \mathbf{V} = 0
\] (1)
But the Navier–Stoke equation \cite{45,46} takes the form,
\[
\rho_{nf} \left[ \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} \right] = \text{div} \mathbf{T} + g(\rho \beta)_{nf}(\theta - \theta_\infty) + \mathbf{J} \times \mathbf{B}
\]  
(2)
where $\rho_{nf}$, $\mathbf{T}$, $\mathbf{J}$, $\mathbf{B}$, $g$, $\beta_{nf}$, $\theta$ and $\theta_\infty$ are dynamic viscosity of nanofluid, Cauchy stress tensor, current density, total magnetic field, gravitational acceleration, thermal expansion coefficient, the temperature of nanofluid and ambient temperature, respectively.

The stress tensor for Maxwell fluids in \cite{47} as,
\[
\mathbf{T} = -p\mathbf{I} + \mathbf{S},
\]  
(3)
where
\[
\mathbf{S} + \lambda_1 \frac{\delta \mathbf{S}}{\delta t} = \mu \mathbf{A}_1,
\]  
(4)
In these expressions $\mathbf{S}$, $\mathbf{A}_1$, $\lambda_1$, $\mu$, $D/Dt$ represents the extra stress tensor, Rivline–Ericksen tensor, kinematic viscosity and material time derivative. Also $\mathbf{A}_1$ is expressed in \cite{48} as;
\[
\mathbf{A}_1 = (\nabla \mathbf{V}) + (\nabla \mathbf{V})^T,
\]  
(5)
And
\[
\frac{\delta \mathbf{S}}{\delta t} = \frac{D\mathbf{S}}{Dt} - \mathbf{LS} - \mathbf{SL}^T,
\]  
(6)
By following the assumption of the problem, the Maxwell equation can be written in \cite{49} as;
\[
\mathbf{J} \times \mathbf{B} = -\left(\sigma_{nf} B_0^2 u, 0, 0\right),
\]  
(7)
Keep in view that $\mathbf{B} = B_0 + b_0$ is the sum of applied and induced magnetic field (neglected).

Using Equations (3)–(7), the Equation (2) takes the form;
\[
\rho_{nf} \frac{\partial u}{\partial t} = \frac{\partial S_{xy}}{\partial y} + g(\rho \beta_\theta)_{nf}(\theta - \theta_\infty) - \sigma_{nf} B_0^2 u,
\]  
(8)
Multiplying both sides of Equation (8) by $(1 + \lambda_1^\alpha D_t^\alpha)$
\[
(1 + \lambda_1^\alpha D_t^\alpha)\rho_{nf} \frac{\partial u}{\partial t} = (1 + \lambda_1^\alpha D_t^\alpha) \frac{\partial S_{xy}}{\partial y} + g(\rho \beta_\theta)_{nf}(1 + \lambda_1^\alpha D_t^\alpha)(\theta - \theta_\infty) - \sigma_{nf} B_0^2 (1 + \lambda_1^\alpha D_t^\alpha) u.
\]  
(9)
But the fractional constitutive equation for Maxwell fluids is given by \cite{50},
\[
(1 + \lambda_1^\alpha D_t^\alpha)S_{xy} = \mu \frac{\partial u}{\partial y} \text{ with } 0 < \alpha < 1,
\]  
(10)
This constitutive relation contains $D_t^\alpha$, Caputo fractional operator is defined in \cite{51} as;
\[
\frac{\mathcal{C}_0}{\Gamma(1-\alpha)} D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\eta)^{-\alpha} \frac{\partial f(\eta)}{\partial \eta} d\eta, \text{ with } 0 < \alpha < 1,
\]  
(11)
With $\Gamma(\cdot)$ as the Gamma function defined in \cite{51} by;
\[
\Gamma(z) = \int \eta^{z-1} e^{-\eta} d\eta, \text{ } z \in \mathbb{C}, \text{Re}(z)>0.
\]  
(12)
Using Equation (10) into Equation (9)
\[
(1 + \lambda_1^\alpha D_t^\alpha)\rho_{nf} \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} + (1 + \lambda_1^\alpha D_t^\alpha) g(\rho \beta_\theta)_{nf}(\theta - \theta_\infty) - (1 + \lambda_1^\alpha D_t^\alpha) \sigma_{nf} B_0^2 u,
\]  
(13)
The first law of thermodynamics [46,52] is;

\[(\rho C_P)_{nf}\left(\frac{\partial \theta}{\partial t}\right) = K_{nf}\frac{\partial^2 \theta}{\partial y^2} - \frac{\partial q_r}{\partial y} + Q(\theta - \theta_\infty).\]  \hspace{1cm} (14)

In this equation \((\rho C_P)_{nf}\), \(K_{nf}\), \(q_r\) and \(Q\) are constant of heat capacity, coefficient of thermal conductivity, radiative heat flux, and nanofluid thermal conductivity, respectively. By using the Rosselands approximation for fluids that are considered optically thick, the radiative heat flux \(q_r\) expressed in [53] is given as;

\[q_r = -\frac{4\sigma^* \theta^4}{3k^*} \frac{\partial \theta}{\partial y}.\] \hspace{1cm} (15)

In this expression \(\sigma^*\) and \(k^*\) are the Stefan–Boltzmann constant and mean spectral absorption constant, respectively. Approximating \(\theta^4\) by a Taylor’s series expansion in the neighborhood of \(\theta_\infty\) and neglecting higher power, \(\theta^4 = 4\theta_\infty^3\theta - 3\theta_\infty^4\), (neglecting higher power), so radiative heat flux is

\[q_r = -16\sigma^*\theta^3_\infty \frac{\partial \theta}{\partial y}.\] \hspace{1cm} (16)

Then Equation (12) becomes

\[E_G = \frac{K_{nf}}{\theta^2_\infty} \left(1 + \frac{16\sigma^*\theta^3_\infty}{3k^*K_{nf}}\right) \left(\frac{\partial \theta}{\partial y}\right)^2 + \frac{\mu_{nf}}{\theta_\infty} \left(\frac{\partial u}{\partial y}\right)^2 + \frac{\sigma_{nf} \theta_\infty^3}{\theta_\infty} u^2.\] \hspace{1cm} (17)

The second law of thermodynamics is given by [46],

\[E_G = \frac{K_{nf}}{\theta^2_\infty} \left(1 + \frac{16\sigma^*\theta^3_\infty}{3k^*K_{nf}}\right) \left(\frac{\partial \theta}{\partial y}\right)^2 + \frac{\mu_{nf}}{\theta_\infty} \left(\frac{\partial u}{\partial y}\right)^2 + \frac{\sigma_{nf} \theta_\infty^3}{\theta_\infty} u^2.\] \hspace{1cm} (18)

In which \(E_G\) and \(\sigma_{nf}\) are volumetric local entropy generation and electrical conductivity. The proposed boundary and initial conditions of this physical phenomenon are defined in [54] below;

\[\begin{align*}
u(y,0) = 0, & \quad u(t,y,0) = 0, \quad u(0,t) = u_0e^{at}, \quad u(\infty,t) = 0, \\
\theta(y,0) = \theta_\infty, & \quad \theta(0,t) = \theta_\infty, \quad \theta(\infty,t) = \theta_\infty
\end{align*}\] \hspace{1cm} (19)

Using the following transformation and thermophysical properties of nanoparticles (can be seen in) [55], the dimensionless governing equation for velocity and temperature profile are obtained;

\[u^* = \frac{u}{u_0}, \quad t^* = \frac{u_0^2}{v} t, \quad \theta^* = \frac{\theta - \theta_\infty}{\theta_\infty - \theta_\infty}, \quad \lambda_1^* = \frac{u_0^2}{v} \lambda_1, \quad y^* = \frac{u_0 y}{v},\] \hspace{1cm} (20)

\[\begin{align*}
\frac{\rho_{nf}}{\rho_f} & = a_1 = (1 - \phi) + \phi \frac{\rho_{nf}}{\rho_f}, \\
\frac{\sigma_{nf}}{\sigma_f} & = a_2 = (1 - \phi) + \phi \frac{(\sigma_{nf})}{(\sigma_f)}, \\
\frac{k_{nf}}{k_f} & = a_3 = \frac{1}{(1 - \phi)^2}, \\
\frac{\mu_{nf}}{\mu_f} & = a_4 = (1 - \phi) + \phi \frac{(\mu_{nf})}{(\mu_f)}.
\end{align*}\] \hspace{1cm} (21)
The dimensionless velocity and temperature profile of the problem is given, and after omitting (* ) notation for the sack of brevity of mathematical modeling.

\begin{equation}
(1 + \lambda_1^p D_t^\alpha (\partial u / \partial t)) = b_1 \left( \frac{\partial^2 u}{\partial y^2} \right) + b_2 Gr(1 + \lambda_2^p D_t^\alpha) \theta - b_3 H_0^2 (1 + \lambda_3^p D_t^\alpha)(u).
\end{equation}

(22)

Moreover, the temperature equation takes the following form

\begin{equation}
b_4 Pr \left( 1 + \lambda_4^p D_t^\alpha (\partial \theta / \partial t) \right) = (1 + Nr) \left( 1 + \lambda_5^p D_t^\alpha \right) \frac{\partial^2 \theta}{\partial y^2} + \left( 1 + \lambda_6^p D_t^\alpha \right) Q_0 \theta.
\end{equation}

(23)

In Equations (22) and (23), \( b_1, b_2, b_3 \) and \( b_4 \) are the ratio of thermophysical properties given by

\[
b_1 = \frac{a_3}{a_1}, \quad b_2 = \frac{a_2}{a_1}, \quad b_3 = \frac{a_6}{a_1} \quad \text{and} \quad b_4 = \frac{a_4}{a_5}.
\]

Additionally, \( Gr, \ Ha, \ Pr, \ Nr \) and \( Q_0 \) are the Grashof number, Hartmann number, Prandtl number, radiation parameter and heat generation parameter, respectively, defined in [45] as:

\[
Gr = \frac{\nu g (\beta_t) f (\theta_w - \theta_\infty)}{u_0^3}, \quad H_0^2 = \frac{\sigma_f B_0^2}{\rho_f u_0^3}, \quad Pr = \frac{\mu (C_p)_f}{K_f}, \quad Nr = \frac{16 \sigma^* \theta_\infty^3}{a_5^3 k^* K_f} \quad \text{and} \quad Q_0 = \frac{Q u^2}{a_5 K_f u_0^3}.
\]

Additionally, the non-dimensional initial and boundary conditions are

\[
u(y, 0) = 0, \quad u_t(y, 0) = 0, \quad u(0, t) = e^{\alpha t}, \quad u(\infty, t) = 0.
\]

(24)

\[
\theta(y, 0) = 0, \quad \theta_t(y, 0) = 1, \quad \theta(\infty, t) = 0.
\]

(25)

The non-dimensional governing equation for velocity and temperature profile in Equations (22) and (23), with dimensionless initial and boundary conditions in Equations (24) and (25), represents the unsteady, incompressible flow fractional Maxwell nanofluids phenomena under the influence of magnetic fields. Water is taken as base fluid, but \( Cu \) or \( Al_2O_3 \) are the nanoparticles considered for nanofluid preparation. For the numerical results the following Table 1 containing thermo-physical properties of nanoparticles and base fluid will be utilized.

**Table 1.** The thermophysical properties of different base fluids and nanoparticles at room 25 °C [56].

| Material         | \( H_2O \) | \( Cu \) | \( Al_2O_3 \) |
|------------------|------------|----------|--------------|
| \( \rho \) \( (kgm^{-3}) \) | 997        | 8933     | 3970         |
| \( C_p \) \( (J Kg^{-1}k^{-1}) \) | 4197       | 385      | 765          |
| \( k \) \( (Wm^{-1}k^{-1}) \) | 6.13       | 400      | 40           |
| \( \beta \times 10^{-5} \) \( (k^{-1}) \) | 21         | 1.67     | 0.85         |
| \( \sigma \) \( (\Omega m)^{-1} \) | 0.05       | 5.96 \times 10^7 | 2.6 \times 10^6 |

3. Numerical Procedure

The finite difference scheme is a very efficient and powerful tool to investigate the numerical solutions of the problem arising and mathematical physics and mechanics. In this context, this section is dedicated to extending the finite-difference scheme to tackle the obtained set of fractional-order fluid models and heat transfer. For this, the discretization of the derivative of fractional-order of \( u, u_t \) and \( u_{yy} \) are specified as,

\[
\frac{C}{0} D_{t+1}^{\alpha} u(y, t_{j+1}) = \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)} \left[ u_{i+1}^{t+1} - u_i^t \right] + \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^i \left[ u_{i-j+1}^{t+1} - u_{i-j}^t \right] d_t^\alpha,
\]

(26)
In the discussed problem, the conditions are given below, and the source term can be approximated by means of the following concept

\[ u^2(y_i, t_j) = u(y_i, t_j) \]

where, \( d^\alpha_l = -1^{1-l} + (1 + l) 1^{1-l} \) for \( l = 1, 2, 3, \ldots, j \). Now, the rectilinear grid is assumed to examine the solution of the governing set of fractional-order fluid problems and heat transfer having grid spacing \( \Delta y > 0 \), \( \Delta t > 0 \) in the direction of space and time separately, where \( \Delta t = \frac{T}{N_T} \), \( \Delta y = \frac{L}{N_y} \) for \( \Delta y \), \( \Delta t \) from \( \mathbb{Z}^+ \). The inner points \( \{y_i, t_j\} \) in the discussed domain \( \Omega = [0, T] \times [0, L] \) are given as \( i \Delta y = y_i \) and \( j \Delta t = t_j \). The discretization of the governing set of fractional-order fluid problems and heat transfer at \( \{y_i, t_j\} \) is given as,

\[
\frac{1}{2\Delta t} \left( u_i^{j+1} - u_i^{j-1} \right) + \frac{\lambda_1^\alpha \Delta H^1_2}{1^{1-p}} \left( u_i^{j+1} - 2u_i^j + u_i^{j-1} \right) + \frac{\lambda_2^\alpha \Delta H^2_2}{1^{1-p}} \left( u_i^{j+1} - 2u_i^j + u_i^{j-1} \right)
\]

\[
\times \sum_{l=1}^L \left( u_i^{j+1-l} - 2u_i^{j-l} + u_i^{j-l-1} \right) b^\alpha_l = \frac{b_1}{\Delta y} \left( u_i^{j+1} - 2u_i^j + u_i^{j-1} \right)
\]

\[
+ b_2 \Delta H_2 \left( \theta_i^{j+1} - \theta_i^j \right) + \frac{b_2 \Delta H_2}{1^{1-p}} \sum_{l=1}^L \left( \theta_i^{j+1-l} - \theta_i^{j-l} \right) b^\alpha_l - b_3 \Delta H_2 \left( \theta_i^{j+1} - \theta_i^j \right)
\]

\[
+ \frac{b_3 \Delta H_2}{1^{1-p}} \sum_{l=1}^L \left( \theta_i^{j+1-l} - \theta_i^{j-l} \right) b^\alpha_l + b_4 \left( \theta_i^{j+1} - 2\theta_i^j + \theta_i^{j-1} \right)
\]

\[
+ \frac{b_5 \Delta H_2}{1^{1-p}} \sum_{l=1}^L \left( \theta_i^{j+1-l} - \theta_i^{j-l} \right) b^\alpha_l = \frac{1}{\Delta y} \left( \theta_i^{j+1} - 2\theta_i^j + \theta_i^{j-1} \right)
\]

\[
+ \frac{1}{\Delta y} \sum_{l=1}^L \left( \theta_i^{j+1-l} - 2\theta_i^{j-l} + \theta_i^{j-l-1} \right) b^\alpha_l = \frac{1}{\Delta y} \sum_{l=1}^L \left( \theta_i^{j+1-l} - 2\theta_i^{j-l} + \theta_i^{j-l-1} \right) b^\alpha_l
\]

\[
+ \frac{Q_0 \Delta H_2}{1^{1-p}} \left( \theta_i^{j+1} - \theta_i^j \right) + \frac{Q_0 \Delta H_2}{1^{1-p}} \sum_{l=1}^L \left( \theta_i^{j+1-l} - \theta_i^{j-l} \right) b^\alpha_l + Q_0 \theta_i^{j+1}.
\]

for \( j = 1, 2, 3, \ldots, N-1 \), \( i = 1, 2, 3, \ldots, N-1 \), with the following initial and boundary conditions,

\[
u_i^0 = 0, u_i^1 = u_i^{-1}, \theta_i^0 = 0, \theta_i^1 = \theta_i^{-1}, \text{ for } i = 0, 1, 2, 3, \ldots, M,
\]

\[
u_i^0 = \exp(aj \Delta t), u_i^M = 0, \theta_i^0 = 1, \theta_i^M = 0, \text{ for } j = 1, 2, 3, \ldots, N-1.
\]

3.1. Numerical Analysis and Discussion

**Test Problem.** For the validation of the applied scheme, a test problem is considered as

\[
\frac{\partial}{\partial y} D^\alpha u(y, t) = \frac{\partial^2}{\partial y^2} u(y, t) - \frac{\partial}{\partial y} u(y, t) + h(y, t)
\]

In the discussed problem, the conditions are given below, and the source term can be selected against the choice of fractional-order derivative.

\[
u(0) = u_t(0) = u(\infty, t) = 0 \text{ and } u(0, t) = e^{at}
\]
Since such a physical problem contains \( u(y, t) = (y - t)^2 \) as the exact solution. Its accuracy has been checked by a number of simulations for the proposed scheme. The plots in Figure 2a,b are drawn for the maximum absolute error (MAE) and computational order of convergence (COC) for different ranges of \( N \), which is \( N = 10, 20, 40, 80, 160, 320, 640 \).

\[
\text{MAE} = \max_{1 \leq i \leq M, 1 \leq j \leq N} |u(y_i, t_j) - u'_i|, \quad \text{COC} = \log \left( \frac{\text{MAE}(k)}{\text{MAE}(k+1)} \right) / \log \left( \frac{N(k+1)}{N(k)} \right).
\]

![Figure 2](image_url)

Figure 2. Code validation of proposed scheme and varying time mesh size against (a) computational order of convergence (COC) (b) maximum absolute error (MAE), and varying mesh size for (c) time and (d) space against \( L_\infty \)-norm between consecutive solutions.

The convergence of the applied scheme is observed against the selection of each fractional-order derivative, and its convergence order enhances for \( \alpha \to 1 \). Figure 2c,d contains the \( L_\infty \)-norm between consecutive solutions that is \( |u^{i+1} - u'|_\infty \) and \( |u_{i+1} - u_i|_\infty \) when \( 0 \leq i, j \leq N, M = 500 \). Again, it is found that the proposed scheme is very efficient, accurate and reliable for this problem. It also demonstrates that the solution is stable against the selection of fractional order and mesh parameters.

3.2. Entropy Generation

For viscous fluid flow in a magnetic field, the volumetric rate of local entropy generation \( E_G \) is defined in [57] as;

\[
E_G = E_\theta + E_f + E_m, \quad (30)
\]
That is the sum of entropy generation due to heat transfer, due to fluids friction and due to the magnetic field effect, separately mentioned here (can be seen in [33])

\[ E_\theta = \frac{K_{nf}}{\theta_\infty} \left( 1 + \frac{16 \sigma^* \theta_\infty^3}{3k^* K_{nf}} \right) \left( \frac{\partial \theta}{\partial y} \right)^2 \]

\[ E_f = \frac{\mu_{nf}}{\theta_\infty} \left( \frac{\partial u}{\partial y} \right)^2 \]

\[ E_m = \frac{\sigma_{nf} B_0^2}{\theta_\infty} u^2 \]

Combining all the results

\[ E_G = \frac{K_{nf}}{\theta_\infty^2} \left( 1 + \frac{16 \sigma^* \theta_\infty^3}{3k^* K_{nf}} \right) \left( \frac{\partial \theta}{\partial y} \right)^2 + \frac{\mu_{nf}}{\theta_\infty} \left( \frac{\partial u}{\partial y} \right)^2 + \frac{\sigma_{nf} B_0^2}{\theta_\infty} u^2 \]

(31)

The dimensionless entropy generation calculated is;

\[ N_s = \frac{E_G}{E_0} = \left[ a_5(1 + N_r) \left( \frac{\partial \theta}{\partial y} \right)^2 + a_3 \frac{Br}{\theta_\infty} \left( \frac{\partial u}{\partial y} \right)^2 + a_6 \frac{Br}{\theta_\infty} H_a u^2 \right] \]

(32)

where \( E_0 = \frac{K_f u_0^2 (\theta_\infty - \theta_0)^2}{\theta_\infty^2} \), \( \Omega = \frac{(\theta_\infty - \theta_0)}{\theta_\infty}, Br = \frac{\mu_{nf} \theta_\infty^2}{K_f (\theta_\infty - \theta_0)}, H_a = M = \frac{\sigma_{nf} B_0^2}{\rho/n_0^2} \) and \( N_r = \frac{16 \sigma^* \theta_\infty^3}{3k^* K_{nf}} \).

where \( E_0 \), is the characteristic entropy generation rate, \( Br \) is the Brickman number, \( H_a^2 = M \) is the square of the Hartmann Number and \( \Omega \) is the dimensionless temperature difference.

Bejan Number

The Bejan number is the irreversibility distribution parameter which is expressed mathematically as (can be seen in [58]);

\[ Be = \frac{\text{Entropy generation due heat transfer}}{\text{Total Entropy generation}} \]

That is

\[ Be = \frac{\frac{K_f u_0^2 (\theta_\infty - \theta_0)^2}{\theta_\infty^2} \left[ a_5(1 + N_r) \left( \frac{\partial \theta}{\partial y} \right)^2 \right]}{\left[ a_5(1 + N_r) \left( \frac{\partial \theta}{\partial y} \right)^2 + a_3 \frac{Br}{\theta_\infty} \left( \frac{\partial u}{\partial y} \right)^2 + a_6 \frac{Br}{\theta_\infty} H_a^2 u^2 \right]} \]

(33)

Then the reduced expression for the Bejan number is given as;

\[ Be = \frac{\left( 1 + N_r \right) \left( \frac{\partial u}{\partial y} \right)^2}{\left[ a_5(1 + N_r) \left( \frac{\partial \theta}{\partial y} \right)^2 + a_6 \frac{Br}{\theta_\infty} H_a^2 u^2 \right]} \]

(34)

With \( b_5 = \frac{a_5}{\theta_\infty}, b_6 = \frac{a_6}{\theta_\infty} \).

3.3. Skin Friction and Nusselt Number

For measuring shear stress and heat transfer effects in an ordinary integer order system, local skin friction and Nusselt number are defined in [59] as;

\[ S_f = -\mu_{nf} \left( \frac{\partial u}{\partial y} \right)_{y=0} \]

(35)

and

\[ Nu = -K_{nf} \left( 1 + \frac{16 \sigma^* \theta_\infty^3}{3k^* K_f} \right) \left( \frac{\partial \theta}{\partial y} \right)_{y=0} \]

(36)
The skin friction coefficient and local Nusselt number for (FMF) can be written by using Equation (5), that is, the fractional stress tensor for Maxwell fluid on the plate with fractional time Caputo derivative (detail can be seen in [60]).

\[ S_f + \lambda_s^\beta \frac{\partial^\beta S_f}{\partial t^\beta} = -\mu_f \left( \frac{\partial u}{\partial y} \right)_{y=0}. \] (37)

\[ Nu + \lambda_s^\beta \frac{\partial^\beta Nu}{\partial t^\beta} = -K_{nf} \left( 1 + \frac{16\sigma^2 \theta^2}{3k^2 K_f} \right) \left( \frac{\partial \theta}{\partial y} \right)_{y=0}. \] (38)

The non-dimensional form of Equations (38) and (39) is given as

\[ S_f + \lambda_s^\beta \frac{\partial^\beta S_f}{\partial t^\beta} = -a_3 \mu_f \left( \frac{\partial u}{\partial y} \right)_{y=0}. \] (39)

\[ Nu + \lambda_s^\beta \frac{\partial^\beta Nu}{\partial t^\beta} = -a_5 (1 + Nr) \left( \frac{\partial \theta}{\partial y} \right)_{y=0}. \] (40)

4. Results and Discussion

This section of the article deliberates the detailed results and discusses the plots driven against different physical parameters like $H_a$, $Pr$, $Nr$, $Gr$, $\phi$, $\alpha$, $\beta$ and $Q_0$ representing the magnetic field parameter (the square of Hartmann Number), Prandtl number, radiation parameter, Grashof Number, volumetric fraction of nanoparticle, fractional order parameter, and heat generation parameter, respectively. The behavior of these aforementioned physical parameters on dimensionless velocity profile $u(y, t)$, temperature profile $\theta(y, t)$, Bejan number—BN $Be$, Skin friction $S_f$ and Nusselt number $Nu$ are drawn by mathematical software MAPLE. The mathematical fractional model of Maxwell nanofluid is developed by using the Caputo time fractional operator. After transforming the governing equations to a dimensionless governing model, the finite difference method (FDM) is used for the discretization of the model. FDM is a strong tool for dealing with such kinds of problems. The simulation is performed by developing and executing codes.

Results are obtained by solving Equations (22), (23), (32), (39) and (40) with initial and boundary conditions illustrated in Equations (24) and (25) and physical properties of nanoparticles in Equation (21) and Table 1. Various suitable ranges of physical parameter ($H_a = 1, 2, 5$), ($Pr = 3.5, 6.2, 15$), ($Gr = 0, 1, 2$), ($\phi = 0.01, 0.1, 0.2$), ($\alpha = 0.2, 0.4, 0.6, 0.8, 1$), ($Nr = 0, 2, 5$), ($Q_0 = 0, 2, 5$) for heat transfer, velocity analysis, skin friction, Nusselt number, entropy generation and Bejan number—BN are considered, and also particular exertion has been done on the effects of these parameters for heat enhancement.

Figure 3 shows the effect of the magnetic field parameter on the velocity profile of fractional Maxwell fluids. It can be seen that velocity decreased by increasing the value of $H_a$ because of the Lorentz’s force. The rise in $H_a$ caused strengthening in the Lorentz force, which increases the internal resistance to flow particles; consequently, fluid velocity decreased. Whereas an increase in Skin friction occurs, as shown in Figure 3b. Since Bejan number $Be$ is the ratio of total entropy generation to entropy generation due to heat transfer. Thus, advancement in $H_a$ boosts the Bejan number—BN $Be$ and can be noticed in Figure 3c. An opposite behavior of total entropy generation $N_s$ relative to Bejan number—BN $Be$ appears against the increasing value of $H_a$, as can be seen in Figure 3d.

Figure 4 depicts the effects of the Grashof number $Gr$ on velocity profile $u(y, t)$. The enhancement in value of $Gr$ results in increasing the fluid velocity, which can be physically justified as the increasing value of $Gr$ means lower the viscous forces and hence increasing the velocity of fractional Maxwell fluids. While Skin friction $S_f$ increases, the increasing the value of $Gr$ can be noticed in Figure 4b. A slight effect of $Gr$ on Bejan number—BN $Be$ and entropy generation $N_s$ can be seen in Figure 4c,d, respectively. Similar results are reported by Sarojamma, G., et al. in [61] for comparison. $Gr$ is the ratio of buoyancy forces to viscous...
force, increasing value of Gr results in laminar boundary layer and vice versa. Because high value of Gr give rise to the temperature of molecules, consequently, the intermolecular forces become weak. Thus, velocity profile \( u(y, t) \) also rises. On the other hand, fluid particles collectively gain momentum as Gr increases, so additional heat is lost nearby, that is why skin friction reduces as shown in Figure 4b. Since Bejan number \( Be \) is ratio of entropy generation due to heat to the total entropy generation of the system, that is why \( Gr \) reduces the value of Bejan number—BN deliberated in Figure 4c.

Figure 3. Impact of the Hartmann number on (a) dimensionless velocity profile, (b) coefficient of skin friction, (c) Bejan number and (d) entropy generation when \( \Lambda_1 = 0.6, \Lambda_2 = 0.5, Gr = 5, Ha = 10, Pr = 6, Q_0 = 5, Nr = 3.5, a = -1, Br = 0.5, \Omega = 10, \phi = 0.1 \).

Figure 5 shows the effects of the volume fraction parameter \( \phi \) on fluid velocity, and it is examined that advancement in controlled volume fraction parameter \( \phi \) lowers the velocity profile due to the effect; that is, the viscosity of fractional Maxwell fluids increases by increasing \( \phi \). On the other hand, the addition of nanoparticles in the base fluid causes an improvement in the heat transfer rate at the boundary layer. As the thermal conductivity of base fluid is enhanced, consequently increasing the fluid’s internal temperature can be seen in Figure 5b. Since Nusselt number \( Nu \) is the ratio of convective heat transfer to conductive heat transfer, increasing \( \phi \) has a decreasing relation with Nusselt number \( Nu \). This is because the skin friction decreases with the passage of time against the volume fraction parameter \( \phi \). These results for skin friction and Nusselt number \( Nu \) are viewed in Figure 5c,d, respectively. An increasing trend is noticed for the Bejan number \( Be \) and can be seen in Figure 5e as the heat transfer rate becoming better by increasing \( \phi \). Whereas in Figure 4f entropy generation \( N_s \) enhanced by increasing the value of \( \phi \). Because the viscosity of fluid increased by increasing \( \phi \).
Figure 4. Influence of Grashof number $Gr$ on (a) dimensionless velocity profile, (b) coefficient of skin friction, (c) Bejan number and (d) entropy generation when $\Lambda_1 = 0.6$, $\Lambda_2 = 0.5$, $Gr = 5$, $Ha = 10$, $Pr = 6$, $Q_0 = 5, Nr = 3.5, a = -1, Br = 0.5, \Omega = 10, \phi = 0.1$.

In these following plots in Figure 6, results are drawn for velocity profile $u(y,t)$, temperature profile $\theta(y,t)$, skin friction $S_f$, Nusselt number $Nu$ and Bejan number-BN $Be$ against fractional parameter $a$.

The reason is that the gradual increase in fractional parameter $a$ gives rise to the viscosity of the nanofluid. This means intermolecular forces between the nanoparticles and base-fluid particles increase; consequently, Brownian motion of particles reduces, that is, a decrease in velocity profile $u(y,t)$ occurs. The consequences of fractional order parameter $a$ for velocity profile $u(y,t)$ are inverse as depicted in Figure 6a. On the other hand, $a$ have direct relation for $\theta(y,t)$, as shown in Figure 6b. Figure 6c shows that for an increasing value of $a$ the entropy generation $N_s$ decreases, whereas a rise in $a$ varies $Be$ directly, deliberated in Figure 6d.

Figure 7 shows the effects of thermal radiation parameter $Nr$ on temperature $\theta(y,t)$ of the fluid. Applying thermal radiations gives rise to the temperature of particles of nanofluids. Hence, the particles’ kinetic energy increases, and the rate of collision between the particles of the nanofluids becomes high, which is why a rising increase in temperature profile occurs. Therefore it is concluded that increasing the value of $Nr$ causes an increase in fluid temperature. $Nu$ is the ratio of convective to conductive heat transfer across the boundary, but enhancement in conduction occurs with the addition of nanoparticle therefore, a decrease occurs in $Nu$, depicted in Figure 7a,b, respectively. A decrease in Bejan number—BN $Be$ can be noticed in Figure 7c, whereas entropy decreases initially and then increases gradually, as can be seen in Figure 7d. Similar results are reported in [61].

Prandtl number $Pr$ is the dimensionless number and is the ratio of momentum to thermal diffusivity. It is a fluid property but does not have any dependence on flow type. Thus, an increase in $Pr$ means heat transfer is favored to occur by momentum, not conduction. This parameter controls the relative thickness and thermal boundary layer in
heat transfer problems. Lowering the value of $Pr$ means the heat diffuses spontaneously as compared to momentum, which thickens the thermal boundary layer rather than the momentum boundary layer. Therefore, an increase in $Pr$ decreases the temperature profile $\theta(y, t)$ of fractional Maxwell fluids as expressed in Figure 8a, which expectedly decreases the Nusselt number $Nu$. Since Bejan number $Be$ has an inverse relation with entropy generation $Ns$, due to heat transfer, that is, $Be$ decreases with the increase in value of $Pr$ as illustrated in Figure 8.

Figure 9 shows the effects of the heat source term $Q_0$ on temperature profile $\theta(y, t)$, since heat source gives rise to temperature profile as shown in Figure 9a, but with the passage of time, it is noticed that the Nusselt number $Nu$ decrease is dependent inversely on the conduction of heat. Thus, entropy generation $Ns$ expectedly decreases as shown in Figure 9d. This phenomenon gives rise to the Bejan number $Be$ because entropy generation due to heat transfer has an inverse relation with the Bejan number $Be$, as shown in Figure 9c.

![Figure 5](image-url)  
**Figure 5.** Effect of $\phi$ on (a) dimensionless velocity profile, (b) non-dimensional temperature profile, (c) coefficient of skin friction, (d) local Nusselt number $Nu$, (e) Bejan number and (f) entropy generation when $\Lambda_1 = 0.6$, $\Lambda_2 = 0.5$, $Gr = 5$, $Ha = 10$, $Pr = 6$, $Q_0 = 5$, $Nr = 3.5$, $a = -1$, $Br = 0.5$, $\Omega = 10$, $\phi = 0.1$. 


and then increases gradually, as can be seen in Figure 7d. Similar results are reported in Bejan number—BN illustrated in Figure 8.

Thus, an increase in thermal diffusivity. It is a fluid property but does not have any dependence on flow type. An increase in momentum boundary layer. Therefore, an increase in conduction occurs with the addition of nanoparticles across the boundary, but enhancement in conduction is expectedly decreases with the increase in value of Prandtl number (Pr). Since Bejan number has an inverse relation with entropy generation when (α = β) non-dimensional temperature profile, (c) Bejan number and (d) Entropy generation when (Λ₁ = 0.6, Λ₂ = 0.5, Gr = 5, Ha = 10, Pr = 6, Q₀ = 5, Nr = 3.5, a = −1, Br = 0.5, Ω = 10, φ = 0.1).

Figure 7. Influence of thermal radiation parameter Nr on (a) dimensionless velocity profile, (b) non-dimensional temperature profile, (c) Bejan number and (d) entropy generation when Λ₁ = 0.6, Λ₂ = 0.5, Gr = 5, Ha = 10, Pr = 6, Q₀ = 5, Nr = 3.5, a = −1, Br = 0.5, Ω = 10, φ = 0.1.
The solution obtained via the finite difference method is excellent in agreement with the test problem and existing results which shows that the finite difference method is a powerful approach in solving such problems.

**Figure 8.** Effect of Pr on (a) dimensionless velocity profile, (b) coefficient of skin friction, (c) Bejan number and (d) entropy generation, when $\Lambda_1 = 0.1$, $\Lambda_2 = 0.2$, $Gr = 5$, $Ha = 2$, $Pr = 6.2$, $Q_0 = 2.5$, $Nr = 5$, $Br = 2$, $\Omega = 10$, $\phi = 0.1$.

**Figure 9.** Influence of heat source $Q_0$ on (a) dimensionless velocity profile (b) non-dimensional temperature profile (c) Bejan number and (d) entropy generation when $\Lambda_1 = 0.6$, $\Lambda_2 = 0.5$, $Gr = 5$, $Ha = 10$, $Pr = 6$, $Q_0 = 5$, $Nr = 3.5$, $a = -1$, $Br = 0.5$, $\Omega = 10$, $\phi = 0.1$. 
5. Conclusions

The graphical analysis of fractional Maxwell nanofluids is made in this article under the influence of a magnetic field (ignoring induced magnetic field). The pressure gradient is supposed to be absent. Effects of different physical parameters are drawn by using the mathematical software MAPLE. The model is formulated by applying the Caputo time derivative. Using suitable transformations, governing equations are made dimensionless.

MHD fractional Maxwell nanofluids are studied numerically; quantities like entropy generation, Bejan number, Skin friction, and Nusselt number are investigated using the finite difference method.

Hence key findings of this study are given below;

a- It is noted that for increasing the value of fractional order parameter $\alpha$, the velocity profile $u(y,t)$ decreases, whereas the temperature profile $\theta(y,t)$ increases.

b- The addition of nanoparticles to base fluid enhances the thermal conductivity of fractional Maxwell nanofluids, increasing the value of volume fraction of nanoparticles $\phi$ and decreasing entropy generation $N_s$.

c- The magnetic field effect influences the temperature $\theta(y,t)$ and velocity $u(y,t)$ profile with inverse and direct behavior, respectively.

d- Nusselt number increases with the variation in $Pr$, and a decrease occurs in $Nu$ with the increase in thermal radiation parameter.

e- The temperature profile varies directly with the thermal radiation parameter $Nr$, and increasing the value of $Nr$ decreases the Nusselt number. Whereas entropy generation $N_s$ increases, and the Bejan number decrease with a rising value of $Nr$.

The solution obtained via the finite difference method is excellent in agreement with the test problem and existing results which shows that the finite difference method is a strong and reliable technique to deal with such kind of complex models and it gives a key direction for further study.

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References
1. Saqib, M.; Ali, F.; Khan, I.; Sheikh, N.A.; Alam Jan, S.A. Exact solutions for free convection flow of generalized Jeffrey fluid: A Caputo-Fabrizio fractional model. Alex. Eng. J. 2018, 57, 1849–1858. [CrossRef]
2. Saqib, M.; Hanif, H.; Abdeljawad, T.; Khan, I.; Shafie, S.; Nisar, K.S. Heat transfer in mhd flow of maxwell fluid via fractional cattaneo-friedrich model: A finite difference approach. Comput. Mater. Contin. 2020, 65, 1959–1973. [CrossRef]
3. Sheikh, N.A.; Ali, F.; Saqib, M.; Khan, I.; Alam Jan, S.A.; Alshomrani, A.S.; Alghamdi, M.S. Comparison and analysis of the Atangana–Baleanu and Caputo–Fabrizio fractional derivatives for generalized Casson fluid model with heat generation and chemical reaction. Results Phys. 2017, 7, 789–800. [CrossRef]
4. Shah, N.; Hajizadeh, A.; Zeb, M.; Ahmad, S.; Mahsud, Y.; Animasaun, I.L. Effect of magnetic field on double convection flow of viscous fluid over a moving vertical plate with constant temperature and general concentration by using new trend of fractional derivative. Open J. Math. Sci. 2018, 2, 253–265. [CrossRef]
5. Shah, N.A.; Elhaqeeb, T.; Animasaun, I.L.; Mahsud, Y. Insight into the natural convection flow through a vertical cylinder using caputo time-fractional derivatives. Int. J. Appl. Comput. Math. 2018, 4, 80. [CrossRef]
6. Raza, N.; Abdullah, M.; Butt, A.R.; Awam, A.U.; Haque, E.U. Flow of a second grade fluid with fractional derivatives due to a quadratic time dependent shear stress. Alex. Eng. J. 2018, 57, 1963–1969. [CrossRef]
7. Sarma, R.; Jain, M.; Mondal, P.K. Towards the minimization of thermodynamic irreversibility in an electrically actuated microflow of a viscoelastic fluid under electrical double layer phenomenon. Phys. Fluids 2017, 29, 103102. [CrossRef]
8. Makris, N.; Constantiniou, M. Fractional-derivative Maxwell model for viscous dampers. J. Struct. Eng. 1991, 117, 2708–2724. [CrossRef]
9. Friedrich, B.; Herschbach, D.R. Spatial orientation of molecules in strong electric fields and evidence for pendular states. Nature 1991, 353, 412–414. [CrossRef]
10. Hayat, T.; Nadeem, S.; Asghar, S. Periodic unidirectional flows of a viscoelastic fluid with the fractional Maxwell model. Appl. Math. Comput. 2004, 151, 153–161. [CrossRef]
11. Qi, H.; Jin, H. Unsteady rotating flows of a viscoelastic fluid with the fractional Maxwell model between coaxial cylinders. Acta Mech. Sin. 2006, 22, 301–305. [CrossRef]
12. Qi, H.; Xu, M. Unsteady flow of viscoelastic fluid with fractional Maxwell model in a channel. Mech. Res. Commun. 2007, 34, 210–212. [CrossRef]
13. Wenchang, T.; Mingyu, X. Plane surface suddenly set in motion of a viscoelastic fluid with fractional Maxwell model. Acta Mech. Sin. 2002, 18, 342–349. [CrossRef]
14. Wenchang, T.; Wenzhao, P.; Mingyu, X. A note on unsteady flow of a viscoelastic fluid with the fractional Maxwell model between two parallel plates. Int. J. Non-Linear Mech. 2003, 38, 645–650. [CrossRef]
15. Bejan, A. A study of entropy generation in fundamental convective heat transfer. J. Heat Transf. 1979, 101, 718–725. [CrossRef]
16. Bejan, A. Second law analysis in heat transfer. Energy 1980, 5, 720–732. [CrossRef]
17. Bejan, A. Second-law analysis in heat transfer and thermal design. In Advances in Heat Transfer; Elsevier: Amsterdam, The Netherlands, 1982; pp. 1–58.
18. Bejan, A. Convection Heat Transfer; John Wiley & Sons: Hoboken, NJ, USA, 2013.
19. Bejan, A. Entropy Generation Minimization: The Method of Thermodynamic Optimization of Finite-Size Systems and Finite-Time Processes; CRC Press: Boca Raton, FL, USA, 2013.
20. Bejan, A. Advanced Engineering Thermodynamics; John Wiley & Sons: Hoboken, NJ, USA, 2016.
21. Bejan, A.; Tsatsaronis, G.; Moran, M.J. Thermal Design and Optimization; John Wiley & Sons: Hoboken, NJ, USA, 1995.
22. Mondal, P.K. Entropy analysis for the Couette flow of non-Newtonian fluids between asymmetrically heated parallel plates: Effect of applied pressure gradient. Phys. Scr. 2014, 89, 125003. [CrossRef]
23. Choi, S.U.; Eastman, J.A. Enhancing Thermal Conductivity of Fluids with Nanoparticles; Argonne National Lab.: Lemont, IL, USA, 1995.
24. Tiwari, R.K.; Das, M.K. Heat transfer augmentation in a two-sided lid-driven differentially heated square cavity utilizing nanofluids. Int. J. Heat Mass Transf. 2007, 50, 2002–2018. [CrossRef]
25. Zaraki, A.; Ghalambaz, M.; Chamkha, A.J.; Ghalambaz, M.; De Rossi, D. Theoretical analysis of natural convection boundary layer heat and mass transfer of nanofluid: Effects of size, shape and type of nanoparticles, type of base fluid and working temperature. Adv. Powder Technol. 2015, 26, 935–946. [CrossRef]
26. Aman, S.; Khan, I.; Ismail, Z.; Salleh, M.Z.; Al-Mdallal, Q.M. Heat transfer enhancement in free convection flow of CNTs Maxwell nanofluids with four different types of molecular liquids. Sci. Rep. 2017, 7, 2445. [CrossRef]
27. Mondal, P. Thermodynamically consistent limiting forced convection heat transfer in a symmetrically heated porous channel: An analytical study. Transp. Porous Media 2013, 100, 17–37. [CrossRef]
28. Kumar, M.; Mondal, P.K. Irreversibility analysis of hybrid nanofluid flow over a rotating disk: Effect of thermal radiation and magnetic field. Colloids Surf. A Physicochem. Eng. Asp. 2022, 635, 128077. [CrossRef]
29. Mondal, P.; Mukherjee, S. Viscous dissipation effects on the limiting value of Nusselt numbers for a shear driven flow between two asymmetrically heated parallel plates. Front. Heat Mass Transf. (FHMT) 2012, 3, 033004. [CrossRef]
30. Awad, M.M. A new definition of Bejan number. Therm. Sci. 2012, 16, 1251–1253. [CrossRef]
31. Saouli, S.; Aïboud-Saouli, S. Second law analysis of laminar falling liquid film along an inclined heated plate. Int. Commun. Heat Mass Transf. 2004, 31, 879–886. [CrossRef]
32. Ji, Y.; Zhang, H.-C.; Yang, X.; Shi, L. Entropy generation analysis and performance evaluation of turbulent forced convective heat transfer to nanoﬂuids. Entropy 2017, 19, 108. [CrossRef]
33. Qing, J.; Bhatti, M.M.; Abbas, M.A.; Rashidi, M.M.; Ali, M.E.-S. Entropy generation on MHD Casson nanofluid flow over a porous stretching/shrinking surface. Entropy 2016, 18, 123. [CrossRef]
34. Sheikholeslami, M.; Arabkoohsar, A.; Khan, I.; Shafaei, A.; Li, Z. Impact of Lorentz forces on Fe3O4-water ferrofluid entropy and exergy treatment within a permeable semi annulus. J. Clean. Prod. 2019, 221, 885–898. [CrossRef]
35. Mahanthesh, B. Flow and heat transport of nanomaterial with quadratic radiative heat flux and aggregation kinematics of nanoparticles. Int. Commun. Heat Mass Transf. 2021, 127, 105521. [CrossRef]
36. Gaikwad, H.S.; Basu, D.N.; Mondal, P.K. Non-linear drag induced irreversibility minimization in a viscous dissipative flow through a micro-porous channel. Energy 2017, 119, 588–600. [CrossRef]
37. Kumar, M.; Mondal, P.K. Bejan’s flow visualization of buoyancy-driven flow of a hydromagnetic Casson fluid from an isothermal wavy surface. Phys. Fluids 2021, 33, 093113. [CrossRef]
38. Sarma, R.; Gaikwad, H.; Mondal, P.K. Effect of conjugate heat transfer on entropy generation in slip-driven microflow of power law fluids. Nanoscale Microscale Thermophys. Eng. 2017, 21, 26–44. [CrossRef]
39. Tyagi, P.K.; Kumar, R.; Mondal, P.K. A review of the state-of-the-art nanofluid spray and jet impingement cooling. Phys. Fluids 2020, 32, 121301. [CrossRef]
40. Hayat, T.; Khan, M.I.; Farooq, M.; Yasseen, T.; Alsaedi, A. Water-carbon nanofluid flow with variable heat flux by a thin needle. J. Mol. Liq. 2016, 224, 786–791. [CrossRef]
41. Hayat, T.; Waqas, M.; Shehzad, S.; Alsaedi, A. On 2D stratified flow of an Oldroyd-B fluid with chemical reaction: An application of non-Fourier heat flux theory. J. Mol. Liq. 2016, 223, 566–571. [CrossRef]
42. Khan, M.I. Transportation of hybrid nanoparticles in forced convective Darcy-Forchheimer flow by a rotating disk. Int. Commun. Heat Mass Transf. 2021, 122, 105177. [CrossRef]
43. Khan, M.I.; Waqas, M.; Hayat, T.; Alsaedi, A. Chemically reactive flow of Maxwell liquid due to variable thicked surface. Int. Commun. Heat Mass Transf. 2017, 86, 231–238. [CrossRef]
44. Wang, C.-C. Mathematical Principles of Mechanics and Electromagnetism: Part A: Analytical and Continuum Mechanics; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2013; Volume 16.
45. Saqib, M.; Khan, I.; Shafie, S. Application of fractional differential equations to heat transfer in hybrid nanofluid: Modeling and solution via integral transforms. Adv. Differ. Equ. 2019, 2019, 52. [CrossRef]
46. Khan, A.; Karim, F.U.; Khan, I.; Alkanhal, T.A.; Ali, F.; Khan, D.; Nisar, K.S. Entropy generation in MHD conjugate flow with wall shear stress over an infinite plate: Exact analysis. Entropy 2019, 21, 359. [CrossRef]
47. Anwar, T.; Kumam, P.; Khan, I.; Watthayu, W. Heat transfer enhancement in unsteady MHD natural convective flow of CNTs Oldroyd-B nanofluid under ramped wall velocity and ramped wall temperature. Entropy 2020, 22, 401. [CrossRef]
48. Salah, F.; Azziz, Z.A.; Ayem, M.; Ching, D.L.C. MHD accelerated flow of Maxwell fluid in a porous medium and rotating frame. Int. Sch. Res. Not. 2013, 2013, 485805. [CrossRef]
49. Saqib, M.; Shafie, S.; Khan, I.; Chu, Y.-M.; Nisar, K.S. Symmetric MHD channel flow of nonlocal fractional model of BTF containing hybrid nanoparticles. Symmetry 2020, 12, 663. [CrossRef]
50. Friedrich, C. Relaxation and retardation functions of the Maxwell model with fractional derivatives. Rheol. Acta 1991, 30, 151–158. [CrossRef]
51. Sheikh, N.A.; Ali, F.; Khan, I.; Gohar, M.; Saqib, M. On the applications of nanofluids to enhance the performance of solar collectors: A comparative analysis of Atangana-Baleanu and Caputo-Fabrizio fractional models. Eur. Phys. J. Plus 2017, 132, 1–11. [CrossRef]
52. Jan, S.A.A.; Ali, F.; Sheikh, N.A.; Khan, I.; Saqib, M.; Gohar, M. Engine oil based generalized brinkman-type nano-liquid with molybdenum disulphide nanoparticles of spherical shape: Atangana-Baleanu fractional model. Numer. Methods Partial. Differ. Equ. 2018, 34, 1472–1488. [CrossRef]
53. Raptis, A. Flow of a micropolar fluid past a continuously moving plate by the presence of radiation. Int. J. Heat Mass Transf. 1998, 18, 2865–2866. [CrossRef]
54. Saqib, M.; Khan, I.; Shafie, S. Application of Atangana–Baleanu fractional derivative to MHD channel flow of CMC-based-CNT’s nanofluid through a porous medium. Chaos Solitons Fractals 2018, 116, 79–85. [CrossRef]
55. Yusuf, T.; Maboob, F.; Khan, W.A.; Gbadeyan, J.A. Irreversibility analysis of Cu-TiO2-H2O hybrid-nanofluid impinging on a 3-D stretching sheet in a porous medium with nonlinear radiation: Darcy-Forchhiemer’s model. Alex. Eng. J. 2020, 59, 5247–5261. [CrossRef]
56. Usman, M.; Hamid, M.; Zubair, T.; Haq, R.U.; Wang, W. Cu-Al2O3/Water hybrid nanofluid through a permeable surface in the presence of nonlinear radiation and variable thermal conductivity via LSM. Int. J. Heat Mass Transf. 2018, 126, 1347–1356. [CrossRef]
57. Atmaca, M.; Gumus, M.; Inan, A.T.; Yilmaz, T. Optimization of irreversible cogeneneration systems under alternative performance criteria. Int. J. Thermophys. 2009, 30, 1724–1732. [CrossRef]
58. Saqib, M.; Ali, F.; Khan, I.; Sheikh, N.A.; Khan, A. Entropy generation in different types of fractionalized nanofluids. Arab. J. Sci. Eng. 2019, 44, 531–540. [CrossRef]
59. Zhao, J.; Zheng, L.; Zhang, X.; Liu, F. Unsteady natural convection boundary layer heat transfer of fractional Maxwell viscoelastic fluid over a vertical plate. Int. J. Heat Mass Transf. 2016, 97, 760–766. [CrossRef]
60. Khan, A.Q.; Rasheed, A. Mixed convection magnetohydrodynamics flow of a nanofluid with heat transfer: A numerical study. Math. Probl. Eng. 2019, 2019, 8129564. [CrossRef]
61. Sarojamma, G.; Vajravelu, K.; Sreelakshmi, K. A study on entropy generation on thin film flow over an unsteady stretching sheet under the influence of magnetic field, thermostability, thermal radiation and internal heat generation/absorption. Commun. Numer. Anal. 2017, 2, 141–156. [CrossRef]