Chaos in the incompressible Euler equation on manifolds of high dimension

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Abstract We construct finite dimensional families of non-steady solutions to the Euler equations, existing for all time, and exhibiting all kinds of qualitative dynamics in the phase space, for example: strange attractors and chaos, invariant manifolds of arbitrary topology, and quasiperiodic invariant tori of any dimension. The main theorem of the paper, from which these families of solutions are obtained, states that for any given vector field $X$ on a closed manifold $N$, there is a Riemannian manifold $M$ on which the following holds: $N$ is diffeomorphic to a finite dimensional manifold in the phase space of fluid velocities (the space of divergence-free vector fields on $M$) that is invariant under the Euler evolution, and on which the Euler equation reduces to a finite dimensional ODE that is given by an arbitrarily small smooth perturbation of the vector field $X$ on $N$.

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1 Introduction

The motion of an incompressible inviscid fluid in a Riemannian manifold $(M, g)$ is described by a time-dependent vector field $u_t$ satisfying the Euler equations
\[ \partial_t u_t + \nabla_{u_t}^{LC} u_t = -\nabla p, \quad \text{div} u_t = 0. \]
Here \( \text{div}, \nabla, \text{and } \nabla^{LC} \) denote the divergence, gradient and covariant derivative on \( M \), defined with the metric \( g \); and \( p \) is a (time-dependent) function called the pressure, also an unknown in the equations.

These equations define a dynamical system of an infinite number of degrees of freedom: they can be interpreted as a first order ODE (which we will call the Euler system) in the linear space \( \mathcal{X}^m(M) \) of \( m \)-times differentiable divergence-free vector fields on the domain \( M \).

We will denote by \( \Phi_t(v) := u_t \) the flow of this ODE starting at the field \( v \in \mathcal{X}^m \). Small time existence and uniqueness hold for \( M \) closed (compact without boundary) and non-integer \( m > 1 \) \[8\]; moreover, the solution \( u_t \) is \( C^1 \) on the variable \( t \).

Global in time existence of solutions, however, is only known in 2 dimensions \[27\]; in higher dimensions, whether or not there are initial conditions for which the flow \( \Phi_t \) “blows-up” in finite time (i.e \( ||\Phi_t(v)||_{C^m(M)} \rightarrow \infty \) as \( t \rightarrow T < \infty \)) is a well-known open problem.

As with dynamical systems of a finite number of degrees of freedom, besides existence and uniqueness of solutions we would like to understand the qualitative properties of the flow \( \Phi_t \), when it exists. Examples of these properties are the number of equilibrium points and their stability \[3, 10\], the periodicity and almost periodicity of trajectories, and the geometry and dynamics of more complex invariant subsets (other qualitative properties of the Euler flow \( \Phi_t \) for which some results are known are mixing \[17, 18\] and wandering of solutions \[20, 21\], but our results will imply nothing about these).

So far, the only invariant sets whose existence has been established are the simplest: stationary solutions (i.e zeros, see e.g \[9, 11, 22\] and Chapter 2 of \[19\], for explicit constructions), heteroclinic and homoclinic trajectories between zeroes, periodic orbits, and quasiperiodic invariant tori \[7, 23\]; in these invariant sets, of course, the flow \( \Phi_t \) is well defined for all times.

Observe that, in the finite-dimensional invariant manifolds of \( \Phi_t \), the evolution of the fluid velocity in time can be completely described by the evolution of a finite set of parameters; in other words, the Euler equation reduces to a finite dimensional first order ODE on the invariant manifold. The previous paragraph implies that the Euler equation is only known to reduce to the simplest finite dimensional ODEs, conjugate to linear periodic or quasiperiodic flows on tori.

The goal of this paper is to show that, in fact, almost any finite dimensional smooth dynamics can be found in the phase space of the Euler system, in some Riemannian manifold.

Our results involve embedding manifolds into infinite-dimensional linear spaces. Before stating the main theorem, let us define what will be meant by smoothly embedding a manifold \( N \) into the space of divergence-free fields \( \mathcal{X}^\infty(M) \) of a Riemannian manifold \( M \). In our case, these embeddings will
always be contained inside finite dimensional linear subspaces, so the definition coincides with the usual one.

**Definition 1.1** *(Smooth embedding into $\mathcal{X}^\infty(M)$)* Let $N$ be a smooth manifold, and let $E \subset \mathcal{X}^\infty(M)$ be a finite-dimensional linear subspace of the space of smooth divergence-free fields on a Riemannian manifold $M$. A map $\Theta : N \to E$ is a $C^\infty$ embedding if, after identifying $E$ with $\mathbb{R}^d$ by choosing a basis, the corresponding map is a $C^\infty$ embedding in the usual sense. Notice that being a smooth embedding does not depend on the choice of basis, or on the topology we choose on $\mathcal{X}^\infty(M)$.

Our main result states:

**Theorem 1.2** Let $N$ be any closed (compact without boundary) manifold. Given any $C^\infty$ vector field $X$ on $N$, and two positive numbers $\epsilon$ and $m$, there is

(i) a $C^\infty$ vector field $Y$ in $N$ satisfying

$$\|X - Y\|_{C^m(N)} \leq \epsilon,$$

(ii) a compact Riemannian manifold $M$, together with a finite-dimensional linear subspace $E \subset \mathcal{X}^\infty(M)$

(iii) a $C^\infty$ embedding $\Theta : N \to E$

such that for any point $p \in N$, we have that the time-dependent divergence-free field $u_t = \Theta(\phi^t_Y(p))$ (here $\phi^t_Y$ stands for the flow of $Y$) is the solution to the Euler equation (1.1) on $M$ with initial velocity $u_0 = \Theta(p)$.

Furthermore, the derivatives of $\Theta$ at any point $p \in N$ are bounded independently of the $C^k$-norm on $\mathcal{X}^\infty(M)$. More precisely, let $(x_1, \ldots, x_n)$ be a coordinate chart on $N$ around $p$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ a multi-index. We have

$$\|\partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} \Theta(p)\|_{C^k(M)} \leq C(\alpha)$$

where $C(\alpha)$ is a constant that depends on $\alpha$, but not on $k$.

**Remark 1.3** Since $E \subset \mathcal{X}^\infty(M)$ is finite-dimensional, all the $C^k$-topologies on $\mathcal{X}^\infty(M)$ (including the $C^\infty$ one) induce the same topology on $E$, which is the standard euclidean topology. While the fact that $\Theta : N \to E$ is a smooth embedding does not depend on the $C^k$-topology we choose on $\mathcal{X}^\infty(M)$, the geometric properties of the embedding could a priori vary significantly. Indeed, the partial derivatives of $\Theta$ at a point $p \in N$ are elements of $\mathcal{X}^\infty(M)$; their size depends on the chosen norm $C^k$ on $\mathcal{X}^\infty(M)$, and could go to infinity with $k$. This would imply, for example, that the constant $C(k)$ in the inequality

$$\|\Theta(p) - \Theta(q)\|_{C^k(M)} \leq C(k)\text{dist}_N(p, q)$$

would go to infinity as $k \to \infty$.
satisfies $C(k) \to \infty$ as $k \to \infty$. The last statement in Theorem 1.2 ensures that this does not happen for our embedding.

We will deduce Theorems 1.2 from Theorem 1.4 below, whose proof will constitute the core of the paper:

**Theorem 1.4** Let $\mathbb{M}$ denote the $n$-sphere $S^n$ or the $n$-dimensional torus $T^n = (\mathbb{R}/\mathbb{Z})^n$, for any $n \geq 1$. Let $X$ be any vector field on $\mathbb{M}$ that can be expressed as a trigonometric polynomial (in the case of $T^n$) or as a polynomial vector field in $\mathbb{R}^{n+1}$ tangent to $S^n$. Then, there is:

(i) a compact Riemannian manifold $M$, together with a finite-dimensional linear subspace $E \subset \mathcal{X}^\infty(M)$

(ii) a smooth embedding $\Phi : \mathbb{M} \to E$

such that for any point $x \in \mathbb{M}$, the time-dependent divergence-free field $u_t = \Phi(\phi_X^t(x))$ is the solution to the Euler equation (1.1) on $M$ with initial condition $u_0 = \Phi(x)$. Moreover, the derivatives of $\Phi$ at any point in $\mathbb{M}$ are bounded independently of the $C^k$-norm on $\mathcal{X}^\infty(M)$.

**Remark 1.5** The dimension of the manifold $M$ in Theorem 1.4 can be computed exactly and depends only on $n$ and on the degree of the vector field $X$; we refer to Sect. 2.3 for more details. In the case of Theorem 1.2, the dependence of the dimension of $M$ is more complex, we discuss it and find an upper bound in Sect. 4.1.

Let us present rough sketches of the proofs.

To prove Theorem 1.4, the key is to construct a homogeneous quadratic ODE in $\mathbb{R}^d$, for $d$ big enough, having an invariant manifold diffeomorphic to $\mathbb{M}$ where the flow of the ODE is conjugate to the flow of the vector field $X$. This ODE, moreover, will be shown to preserve the standard euclidean inner product (in other words, its trajectories are always tangent to spheres), so we can embed it into the Euler equations on the manifold $SO(d) \times T^{d+1}$ (with a certain metric that depends on the ODE coefficients) applying a Theorem of Tao [23].

To prove Theorem 1.2, we embed the manifold $N$ in a sphere $S^n$ of high enough dimension, then extend the vector field $X$ on $N$ to $S^n$ in a way that guarantees structural stability of $N$ under sufficiently small perturbations of the extension of $X$ (the compactness of $N$ is used in this step). This ensured, we approximate the extension of $X$ by a polynomial vector field, and apply Theorem 1.4.

**Remark 1.6** The theorems above can be interpreted as a universality result for the (time-dependent) Euler equation, and as such it is interesting to compare them to the non-rigidity theorem due to Tao [24]. Our results are consistent
with the findings of Tao, both unveiling the flexible behaviour of the Euler equation, although it should be noted that they involve rather different facets of flexibility: while we embed a manifold $N$ into the phase space of divergence-free vector fields, so that the trajectories $x(t)$ of an autonomous vector field on $N$ get mapped mapped to solutions $u(t)$ of the Euler equation, Tao starts with a time-dependent divergence-free vector field $u_t(x)$ (defined for a finite time $t \in [0, T]$) on $N$, and adds a dependence on extra periodic parameters $y \in \mathbb{T}^m$ so that the extended vector field $u_t(x, y)$ becomes a solution to the Euler equations on the extended manifold $N \times \mathbb{T}^m$.

We also note that this notion of universality is different from the recent universality result obtained for steady Euler in [4]. There it is shown, using techniques from contact geometry, that any vector field on a compact manifold can be embedded, without perturbation, into a stationary solution of the Euler equations on a Riemannian manifold of higher dimension.

Remark 1.7 One may ask whether solutions of the Euler equation that blow-up in finite time could be constructed by finding a way to extend these results to vector fields on open manifolds. The fact is that, as long as we embed the open manifold inside invariant finite-dimensional linear subspaces of $\mathcal{X}_\infty(M)$ (as would be the case with these methods), blow-up cannot occur. Indeed, because of conservation of the $L^2$-norm, all trajectories of the Euler equation in a linear finite-dimensional invariant subspace $E \subset \mathcal{X}_\infty(M)$ are contained in spheres or ellipsoids in $E$, so their $C^k$ norms always remain bounded.

1.1 Applications of Theorems 1.2 and 1.4

The results above can be used to construct solutions to the Euler equations with dynamical structures that were previously unknown, or at best conjectural. In particular, there are finite dimensional invariant submanifolds in the phase space of the Euler equation where the evolution is chaotic:

Theorem 1.8 There is a Riemannian manifold $(M, g)$ of dimension 22, for which the Euler dynamical system is chaotic. More precisely, there is a 3-dimensional torus $\Sigma$ inside $\mathcal{X}_\infty(M, g)$ such that

(i) the solutions to the Euler equation with initial condition in $\Sigma$ exists for all time, and remain in $\Sigma$ (i.e, $\Sigma$ is an invariant torus of the Euler dynamical system)

(ii) the Euler dynamical system on $\Sigma$ has a compact invariant set (of volume zero in the 3-torus $\Sigma$) where the dynamic is chaotic (presence of transverse homoclinic intersections, horseshoes, and positive topological entropy); as well as invariant sets of positive volume covered by 2-dimensional invariant tori.
We will prove this Theorem in Sect. 5, as an application of Theorem 1.4.

Remark 1.9 Although one would expect that the manifold and the metric shouldn’t play a determinant role in the existence of chaos in the phase space, our results do not imply anything about the existence of chaos in the Euler equation on euclidean space, or other model constant curvature spaces. The manifold \( M \) in the theorem above is \( SO(6) \times \mathbb{T}^7 \), but the metric on it is not a product metric (see Sect. 6.4 for an example of how the metric looks in a simpler application of Theorem 1.4).

We can also use Theorem 1.2 to prove that some submanifolds of phase space are filled with strange attractors of hyperbolic type (for example, Smale-Williams solenoids). Indeed, let \( X \) be a vector field on some manifold \( N \) having a hyperbolic strange attractor \( A \). This means that any sufficiently small \( C^m \) perturbation of \( X \) also has a hyperbolic strange attractor \( A' \) close to \( A \), on which the flow is topologically conjugate to the flow of \( X \) in \( A \) (see e.g [15], Proposition 17.1.1 and Theorem 18.2.1). Thus we have

**Corollary 1.10** (Hyperbolic strange attractor) On the space of smooth divergence-free vector fields on some Riemannian manifolds \( M \), we can find invariant finite-dimensional subsets on which the Euler equation has solutions for all time that converge to an hyperbolic strange attractor.

**Remark 1.11** The Lorenz attractor is not hyperbolic, so the previous corollary does not apply to it. However, there is a sense in which we can still find it in the Euler system. We refer to Sect. 6.

Another consequence is that we can find finite dimensional families of trajectories that are Anosov, since Anosov flows are stable under small \( C^1 \) perturbations:

**Corollary 1.12** (Anosov flows) On the space of smooth divergence-free vector fields on some Riemannian manifolds \( M \), we can find finite-dimensional manifolds on which the Euler equation has solutions for all time and the dynamics is Anosov.

As a explicit example, we can consider a small perturbation of the geodesic flow on the unit tangent bundle of a genus \( g \geq 2 \) Riemannian surface with constant negative curvature.

More generally, Theorem 1.2 implies that any dynamics displayed by a vector field on a closed manifold that is structurally stable under small \( C^m \) perturbations is found in the Euler equations.

Finally, let us point out that, by virtue of Theorem 1.4, certain well-known types of steady state solutions of the Euler and Navier Stokes equations in \( \mathbb{T}^2 \),
\( \mathbb{T}^3 \) and \( \mathbb{S}^3 \) can be embedded exactly into the phase space of the Euler equations in a higher-dimensional manifold \( M \). That is, the lagrangian trajectories of the fluid particles in these steady solutions correspond exactly to eulerian trajectories of non-steady vector fields obeying the Euler equation on \( M \). More specifically, this can be done for Kolmogorov steady flows in \( \mathbb{T}^2 \) (steady solutions of the Navier-Stokes equations under forcing) and Beltrami fields in \( \mathbb{T}^3 \) and \( \mathbb{S}^3 \), as they have finite Fourier (or spherical harmonics) expansions.

The paper is organized as follows. The proof of Theorem 1.4 on embedding dynamics in \( \mathbb{S}^n \) and \( \mathbb{T}^n \) is given in Sect. 2, with the key Proposition (Proposition 1.2) proven in Sect. 3. Section 4 proves Theorem 1.2. We apply these results to prove Theorem 1.8 on chaotic dynamics in Sect. 5. We conclude with some further constructions and corollaries.

## 2 Proof of Theorem 1.4

The proof is broken down into two steps.

### 2.1 Step 1

In the following Proposition, we will call “polynomic” those vector fields on \( \mathbb{S}^n \) and \( \mathbb{T}^n \) that can be written as finite sums of sines and cosines (in the case of \( \mathbb{T}^n \)) or as polynomial vector fields in \( \mathbb{R}^{n+1} \) that are tangent to \( \mathbb{S}^n \). We define the degree of the polynomic vector field to be the modulus squared of the highest frequency (in \( \mathbb{T}^n \)) or the degree of the polynomial in \( \mathbb{R}^{n+1} \).

**Proposition 2.1** Let \( X \) be a polynomic vector field on \( M \) (=\( \mathbb{T}^n \) or \( \mathbb{S}^n \)). There is a smooth embedding \( \Psi : M \rightarrow \mathbb{R}^d \) (with \( d \) depending only on \( n \) and on the degree of the vector field) and a homogeneous quadratic ODE on \( \mathbb{R}^d \)

\[
\frac{dy_i}{dt} = B_{ijk} y_j y_k \text{ for } i = 1, \ldots, d,
\]

satisfying \( B_{ijk} = -B_{kji} \), such that for any point \( x \in M \), \( y(t) = \Psi(\phi^t_X(x)) \) is the solution of the quadratic ODE with initial condition \( y(0) = \Psi(x) \).

Moreover, if the vector field \( X \) is divergence-free in \( M \) (with respect to the round metric in \( \mathbb{S}^n \), or to the flat metric in \( \mathbb{T}^n \)), then the corresponding quadratic vector field

\[
V(y) = \sum_{ijk} B_{ijk} y_j y_k \frac{\partial}{\partial y_i}
\]

is also divergence-free in \( \mathbb{R}^d \) with the standard Euclidean metric.
We prove this proposition in Sect. 3. The fact that the vector field in $\mathbb{R}^d$ is divergence-free when $X$ is will not be needed in what follows, but we think it might be of interest.

2.2 Step 2

The next and final step consists in embedding the quadratic ODE obtained in step 1 into the phase space of the Euler equation in some manifold $M$. We apply the following special case of Theorem 1.1 in [23], which we restate here in a slightly different wording, adapted to our setting:

**Theorem 2.2** (T. Tao [23]) Let

$$\frac{dy_i}{dt} = V_i(y) = \sum_{j,k=1}^d \tilde{B}_{ijk} y_j y_k \text{ for } i = 1, \ldots, d$$

be a homogeneous quadratic ODE on $\mathbb{R}^d$, with $\tilde{B}_{ijk} = \tilde{B}_{ikj}$ and

$$\sum_{i,j,k=1}^d \tilde{B}_{ijk} y_i y_j y_k = 0. \quad (2.1)$$

Then there is a Riemannian manifold $M$ (that can be explicitly taken to be $SO(d) \times T^{d+1}$ with a metric depending on the coefficients $\tilde{B}_{ijk}$) and a linear injective map $T : \mathbb{R}^d \to \mathcal{X}^\infty(M)$ such that, for any $y \in \mathbb{R}^d$, the time-dependent vector field $u_t := T(\phi^t(y))$ is a smooth solution to the Euler equation on $M$ with initial condition $u_0 = T(y)$.

Observe that the coefficients $B_{ijk}$ in Proposition 2.1 do not directly satisfy the hypothesis in Theorem 2.2, because they are not symmetric in the $j, k$ indices. Nevertheless, setting

$$\tilde{B}_{ijk} := \frac{1}{2}(B_{ijk} + B_{ikj}) \quad (2.2)$$

we see that the coefficients $\tilde{B}_{ijk}$ and $B_{ijk}$ define the same ODE. Moreover, by virtue of Proposition 2.1 we have that $B_{ijk} = -B_{kji}$, so

$$\sum_{i,j,k=1}^d \tilde{B}_{ijk} y_i y_j y_k = 0.$$

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Thus we can apply Theorem 2.2 to the ODE given by the coefficients \( \tilde{B}_{ijk} \). The embedding \( \Phi \) in the statement of Theorem 1.4 is then given by \( \Phi = T \circ \Psi \).

To conclude the proof of Theorem 1.4, it remains to be shown, on the one hand, that \( \Phi = T \circ \Psi \) is a smooth embedding, and on the other hand that its derivatives are bounded independently of the norm \( C^k \) in the space of smooth divergence-free fields \( \mathcal{X}^\infty(M) \).

Let \( \{ e_1, \ldots, e_d \} \) denote the standard basis of \( \mathbb{R}^d \); the images \( \{ w_\mu := T(e_\mu) \} \) are smooth divergence-free fields and constitute a basis of the finite dimensional linear subspace \( E \subset \mathcal{X}^\infty \) where we embed \( \mathbb{M} \). Observe that the map \( \Phi \) simply takes the form

\[
\Phi(p) = \Psi_1(p)w_1 + \cdots + \Psi_d(p)w_d .
\]

Thus, according to Definition 1.1, the fact that \( \Phi = T \circ \Psi \) is a smooth embedding follows directly from the fact that \( \Psi: \mathbb{M} \to \mathbb{R}^d \) is a smooth embedding.

Now, the derivatives of \( \Phi \) of arbitrary order \( r \) at a point \( p \in \mathbb{M} \) have the schematic form

\[
D_p^r \Phi = T \circ D_p^r \Psi = (D_p^r \Psi_1)w_1 + \cdots + (D_p^r \Psi_d)w_d
\]

where by \( D_p^r \) we mean any combination of partial derivatives (defined with a smooth chart around the point \( p \in \mathbb{M} \)) of order up to \( r \). Notice that \( D_p^r \Phi \) is simply a smooth divergence-free vector field contained in \( E \subset \mathcal{X}^\infty(M) \): our goal is to show that its \( C^k \) norm \( ||D_p^r \Phi||_{C^k(M)} \) can be bounded by a constant that does not depend on \( k \). This will be the case if the \( C^k \)-norms of the basis vector fields \( w_\mu = T(e_\mu) \) are bounded independently of \( k \), since

\[
||D_p^r \Phi||_{C^k(M)} \leq ||D_p^r \Psi_1|| ||w_1||_{C^k(M)} + \cdots + ||D_p^r \Psi_d|| ||w_d||_{C^k(M)} .
\]

Proving the later requires a more detailed discussion of the map \( T \) in Tao’s Theorem.

The map \( T \) has the form

\[
T(e_\mu) = \Pi^* U_\mu + \sum_{v=1}^d \Pi^* F_{\mu v} \frac{\partial}{\partial \tau_v} \tag{2.3}
\]

where \( \Pi \) is the projection map \( \Pi: SO(d) \times \mathbb{T}^{d+1} \to SO(d) \), \( \{ \frac{\partial}{\partial \tau_v} \}_{v=1}^{d+1} \) is the standard basis of \( T\mathbb{T}^{d+1} \), and \( U_\mu \) and \( F_{\mu v} \) are vector fields and functions on \( SO(d) \), respectively, that we will describe below. Our goal is to show that the \( C^k \) norms of the \( U_\mu \) and \( F_{\mu v} \) are bounded uniformly in \( k \), which will imply that the vector fields \( w_\mu = T(e_\mu) \) have the same property.
Remark 2.3 Observe that the vector fields in the image of the map $T$ do not depend on the last coordinate $t_{d+1}$ in $\mathbb{T}^{d+1}$ in any way. This extra dimension might seem redundant at first sight, but it is needed in Tao’s construction to ensure that the Riemannian metric in $SO(d) \times \mathbb{T}^{d+1}$ that makes the fields $u_t := T(y(t))$ satisfy the first Euler equation

$$\partial_t u_t + \nabla_{u_t}^L u_t = -\nabla p$$

induces a volume form that is preserved by the fields $u_t$.

Consider $SO(d)$ as a submanifold in the linear space $\text{Mat}(d)$ of $d \times d$ matrices, so that for any orthogonal matrix $Q$ we can identify the tangent space $T_Q SO(d)$ with a linear subspace of $\text{Mat}(d)$. As defined in [23], Sect. 5, for $\mu = 1, \ldots, d$, $U\mu$ is the vector field on $SO(d)$ whose value at any $Q \in SO(d)$ is given by

$$U_{\mu}(Q) = S_{\mu} Q \quad (2.4)$$

where $S_{\mu}$ is a matrix in $\mathfrak{so}(d)$ and $S_{\mu} Q$ denotes matrix multiplication. The matrices $S_{\mu}$ depend only on the coefficients $B_{ijk}$ of the quadratic ODE, and are defined in the following way ([23], Eq. (25)):

$$(S_{\mu})_{ij} := \frac{2}{3}(\tilde{B}_{i\mu j} - \tilde{B}_{j\mu i}) = \frac{1}{3}(2B_{i\mu j} + B_{ij\mu} - B_{ji\mu})$$

where in the last equality we have used the definition of $\tilde{B}$ in Eq. 2.2 and the fact that $B_{i\mu j} = -B_{j\mu i}$.

Notice that, by construction, the vector field $U_{\mu}$ is invariant under the right action on $SO(d)$. In other words, if for any element $P \in SO(d)$ we denote by $R_P : SO(d) \to SO(d)$ the diffeomorphism $R_P(Q) = QP$, then

$$d_Q R_P(U_{\mu}(Q)) = U_{\mu}(QP).$$

Indeed,

$$d_Q R_P(U_{\mu}(Q)) = U_{\mu}(Q)P = S_{\mu} Q P = U_{\mu}(QP).$$

As for the functions $F_{\mu\nu} : SO(d) \to \mathbb{R}$, they are defined as

$$F_{\mu\nu}(Q) = e_{\mu} \cdot Qe_{\nu} = Q_{\mu\nu}, \quad (2.5)$$

where by $Qe_{\nu}$ we denote the action of the matrix $Q$ on the vector $e_{\nu}$, and $a \cdot b$ is the dot product.

To see that the $C^k$-norms of the functions and the vector fields are bounded independently of $k$, consider the standard basis $\{E_\alpha\}_{\alpha=1}^{\frac{d(d-1)}{2}}$ of the Lie algebra
\(\mathfrak{s}\mathfrak{o}(d)\), consisting of antisymmetric matrices having only two non-zero entries, one equal to 1 and the other to \(-1\). The derivative of the function \(F_{\mu\nu}\) at a point \(Q \in SO(d)\) in the direction given by the tangent vector \(QE_\alpha \in T_Q SO(d)\) is given by

\[
\frac{d}{dt} \bigg|_{t=0} F_{\mu\nu}(Qe^tE_\alpha) = e_\mu \cdot QE_\alpha e_\nu
\]

Iterating this, we see that the derivatives of order \(k\) at the point \(Q\) of the function \(F_{\mu\nu}\) have the form

\[e_\mu \cdot QE_{\alpha k} \ldots E_\alpha e_\nu\]

The products of the matrices \(E_\alpha\) have at most one non-zero element in any row or column, equal to 1 or \(-1\). Thus the \(k\)-derivatives are always equal to some entry \(Q_{\rho\lambda}\) of the matrix \(Q\), or to zero, which means that the \(C^k\) norms of the functions \(F_{\mu\nu}\) are bounded independently of \(k\).

As for the \(C^k\)-norms of the vector fields \(U_\mu\), it is enough to check them at the identity, because of the right-invariance of the vector fields. In other words, it is enough to prove that any \(k\)-iterated commutators

\[[E_{\alpha_1}, [E_{\alpha_2}, \ldots, [E_{\alpha_k}, S_\mu] \ldots]]\]

have norms (for any metric on \(\mathfrak{s}\mathfrak{o}(d)\)) that can be bounded independently of \(k\). This follows from the fact that the commutators of the basis elements \(\{E_\alpha\}\) are all of the type \([E_\alpha, E_\beta] = \pm E_\gamma\) for \(\alpha \neq \beta\). Indeed, notice that, since \(S_\mu \in \mathfrak{s}\mathfrak{o}(d)\), we can write

\[S_\mu = \sum_{\beta=1}^{\frac{d(d-1)}{2}} a_\beta E_\beta\]

for some coefficients \(a_\beta \in \mathbb{R}\). Thus,

\[[E_{\alpha_1}, [E_{\alpha_2}, \ldots, [E_{\alpha_k}, S_\mu] \ldots]] = \sum_{\beta=1}^{\frac{d(d-1)}{2}} a_\beta [E_{\alpha_1}, [E_{\alpha_2}, \ldots, [E_{\alpha_k}, E_\beta] \ldots]]\]

and the claim follows because for any \(\beta = 1, \ldots, \frac{d(d-1)}{2}\), the iterated commutator \([E_{\alpha_1}, [E_{\alpha_2}, \ldots, [E_{\alpha_k}, E_\beta] \ldots]]\) is either some other basis element \(\pm E_\gamma\), or zero.
2.3 The dimension of the manifold $M$

As the proof of the Proposition 2.1, given in Sect. 3 below, will make manifest, the parameter $d$ in $SO(d) \times \mathbb{T}^{d+1}$ is at most the dimension of the linear space of trigonometric polynomials of degrees up to the degree, say $D$, of $X$ (when $M = \mathbb{T}^n$); or, in the case of $\mathbb{S}^n$, at most the dimension of the space of harmonic polynomials of degrees up to $D + 1$.

More explicitly, in the case of the torus, this dimension is equal to the number of points in $\mathbb{Z}^n$ that lie in the (closed) ball of radius $\sqrt{D}$. This number grows like the volume of the ball, $C(n)D^n$, plus a lower order error term (the estimation of this error is the $n$-dimensional generalization of the famous Gauss circle problem in number theory, see e.g [5]). In the case of $\mathbb{S}^n$, this dimension is equal to

$$d = 1 + \sum_{j=1}^{D+1} \binom{j+n-2}{j-1} \frac{2j+n-1}{j} = \binom{D+n}{D+1} \frac{2D+2+n}{n}$$

for any $n \geq 1$ and $D \geq 0$.

The dimension of $M = SO(d) \times \mathbb{T}^{d+1}$ grows then as the square of these numbers.

3 Proof of Proposition 2.1

3.1 Proof for $M = \mathbb{T}^n$

The field $X$ can be written as a sum of the form

$$X(x) = \sum_{k \in \mathbb{Z}^n, |k| \leq \Lambda} a_k \sin(2\pi k \cdot x) + b_k \cos(2\pi k \cdot x)$$

with $a_k, b_k \in \mathbb{R}^n$ and $a_k = -a_{-k}, b_k = b_{-k}$.

Let $d(\Lambda)$ be the number of integer lattice points contained in the $n$-ball of radius $\Lambda$. Consider the map $\Psi : \mathbb{T}^n \to \mathbb{R}^{2d(\Lambda)}$ given by

$$\Psi(x) = \{\sin(2\pi k \cdot x), \cos(2\pi k \cdot x)\}_{k \in \mathbb{Z}^n \cap B^n(0, \Lambda)}$$

where we represent the points $(q, p) \in \mathbb{R}^{2d(\Lambda)}$ as $(q, p) := \{q_k, p_k\}_{k \in \mathbb{Z}^n \cap B^n(0, \Lambda)}$.

**Lemma 3.1** For $\Lambda \geq 1$, the map $\Psi$ is an embedding.

**Proof of Lemma 3.1** It suffices to prove the claim for $\Lambda = 1$. Denote by $B$ the $n$-dimensional ball of radius 1. The frequencies $k \in \mathbb{Z}^n \cap B$ are the ones of the form
\[ k = \pm (0, 0, \ldots, 1, 0, \ldots, 0) \]

plus the zero vector, so \( d(1) = 2n + 1 \).

First we show that \( d\Psi \) is injective. Suppose that for some point \( x \) there is a vector \( v \) in the kernel of the differential:

\[
d_x \Psi(v) = \sum_k 2\pi v \cdot k \cos(2\pi k \cdot x) \frac{\partial}{\partial q_k} - 2\pi v \cdot k \sin(2\pi k \cdot x) \frac{\partial}{\partial p_k} = 0
\]

this means that \( v \cdot k = 0 \) for all \( k \in \mathbb{Z}^n \cap B \). But the frequencies \( k \) span the whole \( \mathbb{R}^n \), so we must have \( v = 0 \).

It remains to be shown that for any two distinct points \( x, y \in \mathbb{T}^n \) we must have \( \Psi(x) \neq \Psi(y) \). This is easy to see, for if \( \Psi(x) = \Psi(y) \),

\[
\cos(2\pi k \cdot x) = \cos(2\pi k \cdot y), \quad \sin(2\pi k \cdot x) = \sin(2\pi k \cdot y)
\]

for integer frequencies \( k \in \mathbb{Z}^n \cap B \). This is only possible if \( x = y + 2\pi m \) for some \( m \in \mathbb{Z}^n \), that is, if \( x \) and \( y \) label the same point in \( \mathbb{T}^n \).

**Remark 3.2** If the Fourier expansion of \( X \) does not have a constant \((k = 0)\) component, we can define the map \( \Psi \) as

\[
\Psi(x) = \{ \sin(2\pi k \cdot x), \cos(2\pi k \cdot x) \}_{k \in \mathbb{Z}^n \cap B \cap (0, \Lambda) \setminus \{0\}}.
\]

It can be readily checked that this does not affect the assertion in Lemma 3.1, nor any further step in the proof.

Consider now the vector field \( \Psi_\ast(X) \) in \( \Psi(\mathbb{T}^n) \). It has the form

\[
d_x \Psi(X) = \sum_{k, k' \in \mathbb{Z}^n \cap B} 2\pi \left( a_{k'} \cdot k \sin(2\pi k' \cdot x) \cos(2\pi k \cdot x) + \right.
\]

\[
+ b_{k'} \cdot k \cos(2\pi k' \cdot x) \cos(2\pi k \cdot x) \left. \right) \frac{\partial}{\partial q_k} - 2\pi \left( a_{k'} \cdot k \sin(2\pi k' \cdot x) \sin(2\pi k \cdot x) + \right.
\]

\[
+ b_{k'} \cdot k \cos(2\pi k' \cdot x) \sin(2\pi k \cdot x) \left. \right) \frac{\partial}{\partial p_k}
\]

Consider as well the following vector field on the whole space \( \mathbb{R}^{2d(\lambda)} \)

\[
V(q, p) = 2\pi \sum_k \sum_{k'} \left( (a_{k'} \cdot k q_{k'} + b_{k'} \cdot k p_{k'}) p_k \frac{\partial}{\partial q_k} - (a_{k'} \cdot k q_{k'} + b_{k'} \cdot k p_{k'}) q_k \frac{\partial}{\partial p_k} \right).
\]
We see that \( d_x \Psi(X) = V(q, p) \) when \((q, p) = \Psi(x)\), that is, \( V \) is tangent to \( \Psi(\mathbb{T}^n) \) and there, it is equal to \( \Psi_* X \).

The vector field \( V \) defines the homogeneous quadratic ODE:

\[
\frac{dq_k}{dt} = \sum_{k'} (a_{k'} \cdot k \ q_{k'} + b_{k'} \cdot k \ p_{k'}) \ p_k \\
\frac{dp_k}{dt} = -\sum_{k'} (a_{k'} \cdot k \ q_{k'} + b_{k'} \cdot k \ p_{k'}) \ q_k
\]

To check that this ODE satisfies the antisymmetry condition in Proposition 2.1, let us relabel the coordinates in \( \mathbb{R}^{2d(\Lambda)} \) as \( x_\alpha \), with \( \alpha \in \{1, \ldots, 2d(\Lambda)\} \). We see that the coefficients \( B_{\alpha\beta\gamma} = 0 \) unless \( x_\alpha = q_k \) and \( x_\gamma = p_k \) or viceversa. In that case, we have either

\[
B_{q_k,q_{k'},p_k} = a_{k'} \cdot k = -B_{p_k,q_{k'},q_k}
\]

or

\[
B_{q_k,p_{k'},p_k} = b_{k'} \cdot k = -B_{p_k,p_{k'},q_k}.
\]

Thus \( B_{\alpha\beta\gamma} = -B_{\gamma\beta\alpha} \), that is, the coefficients of the ODE are always antisymmetric under exchange of the first and last index, as we wanted to show.

It remains to be shown that, if \( X \) is divergence-free, \( V \) is also divergence-free, i.e

\[
\text{div} V = \sum_k \frac{\partial V_{q_k}}{\partial q_k} + \frac{\partial V_{p_k}}{\partial p_k} = 0.
\]

Indeed, we have

\[
\frac{\partial V_{q_k}}{\partial q_k} = a_k \cdot k \ p_k \\
\frac{\partial V_{p_k}}{\partial p_k} = b_k \cdot k \ q_k
\]

and if \( X \) is divergence-free, the coefficients in its Fourier expansion satisfy \( a_k \cdot k = b_k \cdot k = 0 \).

### 3.2 Proof for \( \mathbb{M} = S^n \)

Let \( \{A_\mu\}_{\mu=1}^m \) be the standard basis of the Lie algebra \( \mathfrak{so}(n + 1) \) of \((n + 1) \times (n + 1)\) traceless antisymmetric matrices (so here \( m = \frac{n(n+1)}{2} \)), consisting of
antisymmetric matrices having only two non-zero entries, one equal to 1 and
the other to \(-1\). More precisely, if the index \(\mu\) labels the pair \((i, j)\), \(i < j\), we have 
\((A_\mu)_{ij} = 1\) and \((A_\mu)_{ji} = -1\). This basis verifies, for every fixed pair 
\((i, j), (k, l)\), the relation

\[
\sum_{\mu=1}^{m} (A_\mu)_{ij} (A_\mu)_{kl} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}, \tag{3.1}
\]

which will be useful below.

It is easy to check that the vector fields in \(\mathbb{R}^{n+1}\) given by

\[
h_\mu(x) = A_\mu \cdot x \tag{3.2}
\]

(where \(A_\mu \cdot x\) denotes matrix multiplication of \(A_\mu\) and the vector \(x \in \mathbb{R}^{n+1}\))
are tangent to \(S^n \subset \mathbb{R}^{n+1}\) and, moreover, for any \(x \in S^n\),

\[
\text{span} \{h_1(x), \ldots, h_m(x)\} = T_x S^n.
\]

The vector field \(X\) in \(S^n\) can be written as

\[
X = \sum_{\mu=1}^{m} f_\mu h_\mu \tag{3.3}
\]

where \(f_\mu : S^n \rightarrow \mathbb{R}\) are smooth functions, the restrictions of the scalar product
\(X \cdot h_\mu\) to \(S^n\). Indeed, for any vector field \(V\) in \(\mathbb{R}^{n+1}\), the following identity holds

\[
|x|^2 V = \sum_{\mu=1}^{m} (V \cdot h_\mu(x)) h_\mu(x) + (V \cdot x) x,
\]

as a straightforward consequence of Eqs. (3.1) and (3.2). Since \(X\) is tangent
to \(S^n\), for \(|x| = 1\) we have that \(X(x) \cdot x = 0\) and Eq. (3.3) follows.

For any given \(N \in \mathbb{N}\), let \(\{Y_\alpha\}_{\alpha=1}^{d(N)}\) be a \(L^2\)-orthonormal basis of the space
of spherical harmonics in \(S^n\) of degree up to \(N\). We recall that this is the linear
space spanned by the eigenfunctions of the Laplace-Beltrami operator (defined
with the round metric) of eigenvalues up to \(N(N+n-1)\). Equivalently, they are
obtained as the restrictions to \(S^n\) of the homogeneous harmonic polynomials
in \(\mathbb{R}^{n+1}\) of degrees up to \(N\). We label the elements of the basis in increasing
degree, so that if the degree of \(Y_\alpha\) is less than the degree of \(Y_\beta, \alpha < \beta\).

Note that the components of the vector fields \(\{h_\mu\}\) are homogeneous poly-
nomials of degree 1. Since the vector field \(X\) is the restriction to \(S^n\) of a
polynomial vector field in $\mathbb{R}^{n+1}$, its components $f_\mu$ on $S^n$ must be given by a finite sum of spherical harmonics

$$f_\mu = \sum_{\alpha=1}^{d(N)} c_\mu^\alpha Y_\alpha$$

up to degree $N$, where $N$ is the degree of $X$ plus one (recall that by Eq. (3.3), the $f_\mu$ are the restrictions of the scalar product $X \cdot h_\mu$ to $S^n$, and the components of the vector fields $h_\mu$ are polynomials of degree one).

Define a map $\Psi : S^n \rightarrow \mathbb{R}^{d(N)}$ as

$$\Psi(x) = (Y_1(x), \ldots, Y_{d(N)}(x)).$$

As in the case of the torus, our first goal is to prove that the map $\Psi$ is an embedding. It suffices to do so for $N = 1$; in this case the spherical harmonics are just restrictions to $S^n$ of affine functions on $\mathbb{R}^{n+1}$, so the map is simply

$$\Psi(x) = (1, x_1(x), x_2(x), \ldots, x_{n+1}(x))$$

or some coordinate permutation and rotation of the above, depending on our choice of basis of spherical harmonics. This is clearly an embedding.

Remark 3.3 As in the case of the torus, when the functions $f_\mu$ have zero mean (i.e., when they do not have projection into the constant functions on $S^n$), we can define the map $\Psi$ as

$$\Psi(x) = (Y_2(x), Y_3(x), \ldots, Y_{d(N)}(x)).$$

Denote by $y_\beta$, with $\beta = 1, \ldots, d(N)$, the euclidean coordinates in the target space $\mathbb{R}^{d(N)}$. The vector field $\Psi_*(X)$ in $\Psi(S^n)$ has the form:

$$d_x \Psi(X) = \sum_{\mu=1}^{m} \sum_{\beta=1}^{d(N)} f_\mu(x) h_\mu(Y_\beta)(x) \frac{\partial}{\partial y_\beta} = \sum_{\mu=1}^{m} \sum_{\alpha, \beta=1}^{d(N)} c_\mu^\alpha Y_\alpha(x) h_\mu(Y_\beta)(x) \frac{\partial}{\partial y_\beta}.$$

Here $h_\mu(Y_\beta)(x)$ represents the derivative of the eigenfunction $Y_\beta$ at the point $x$ in the direction of the vector field $h_\mu$. We now claim that we can express these derivatives as linear combinations of the original spherical harmonics $\{Y_\alpha\}$, with $\alpha \leq d(N)$. In other words, there are explicit coefficients $\theta_\mu^\nu$, with $\mu = 1, \ldots, m$ and $\nu, \beta = 1, \ldots, d(N)$, so that

$$h_\mu(Y_\beta)(x) = \sum_{\gamma=1}^{d(N)} \theta_\mu^\nu \gamma Y_\gamma(x).$$
These coefficients will depend on the basis of harmonics \( \{Y_\alpha\} \), but not on the point \( x \) (and actually, they will be zero unless the degree of \( Y_\gamma \) matches that of \( Y_\beta \), but we will not use this property).

The existence of such coefficients is a straightforward consequence of the relationship between the spherical harmonics in \( S^n \) and the representations of the group \( SO(n + 1) \), but in order to keep the article as elementary and self-contained as possible, we will give here a simple proof.

Recall that the vector fields \( h_\mu \) are given by

\[
h_\mu(x) = A_\mu \cdot x.
\]

For any \( t \in \mathbb{R} \), the matrices \( \Lambda^I_\mu := \exp(tA_\mu) \) are elements of \( SO(n + 1) \). Thus, if \( P(x) \) is an harmonic polynomial of degree \( \alpha \), \( Q(x) := P(\Lambda^I_\mu x) \) is also an harmonic polynomial of the same degree.

The spherical harmonics, being restrictions to \( S^n \) of the homogeneous harmonic polynomials, inherit this invariance under the action of \( \Lambda^I_\mu \), so that for any basis element \( Y_\beta \) we have

\[
Y_\beta(\Lambda^I_\mu x) = \sum_{\gamma=1}^{d(N)} \Theta^\gamma_\beta(\Lambda^I_\mu) Y_\gamma(x)
\]

where the coefficients \( \Theta^\gamma_\beta(\Lambda^I_\mu) \) do not depend on the point.

Now observe that

\[
\left. \frac{d}{dt} \right|_{t=0} Y_\beta(\exp(t A_\mu)x) = h_\mu(Y_\beta)(x),
\]

so defining

\[
\theta^\gamma_{\mu\beta} = \left. \frac{d}{dt} \right|_{t=0} \Theta^\gamma_\beta(\exp(t A_\mu))
\]

our claim follows.

Hence

\[
d_x \Psi(X) = \sum_{\mu=1}^{m} \sum_{\alpha,\beta=1}^{d(N)} e^\alpha_\mu Y_\alpha(x) h_\mu(Y_\beta)(x) \frac{\partial}{\partial y_\beta} = \sum_{\mu=1}^{m} \sum_{\alpha,\beta,\gamma=1}^{d(N)} c^\alpha_\mu Y_\alpha(x) \theta^\gamma_{\mu\beta} Y_\gamma(x) \frac{\partial}{\partial y_\beta}.
\]
This means that the quadratic vector field in $\mathbb{R}^{d(N)}$ defined as

$$V = \sum_{\mu=1}^{m} \sum_{\alpha,\beta,\gamma=1}^{d(N)} \theta_{\mu\beta}^{\gamma} c_{\mu}^{\alpha} y_{\alpha} \frac{\partial}{\partial y_{\beta}}$$

is tangent to $\Psi(S^n)$, and coincides there with $d\Psi(X)$.

Furthermore, the coefficients $\theta_{\mu\beta}^{\gamma}$ are antisymmetric in $\gamma, \beta$:

$$\theta_{\mu\beta}^{\gamma} = \int_{S^n} h_{\mu}(Y_{\beta})(x) Y_{\gamma}(x) d\Omega(x) = - \int_{S^n} h_{\mu}(Y_{\gamma})(x) Y_{\beta}(x) d\Omega(x) = -\theta_{\mu\gamma}^{\beta}$$

(here we have integrated by parts and used the fact that the vector fields $h_{\mu}$, being infinitesimal generators of isometries, are divergence-free); so the coefficients

$$B_{\beta\alpha\gamma} := \sum_{\mu=1}^{m} \theta_{\mu\beta}^{\gamma} c_{\mu}^{\alpha}$$

satisfy $B_{\beta\alpha\gamma} = -B_{\alpha\beta\gamma}$, as we wanted to show.

Finally, we are left to prove that, assuming the vector field $X$ in $S^n$ is divergence-free with respect to the round metric, $V$ in $\mathbb{R}^{d(N)}$ is divergence-free with respect to the Euclidean metric.

In terms of the expansion in spherical harmonics, the divergence-free condition reads

$$\text{div}_{S^n} X = \sum_{\mu=1}^{m} \sum_{\alpha=1}^{d(N)} c_{\mu}^{\alpha} h_{\mu}(Y_{\alpha}) = \sum_{\mu=1}^{m} \sum_{\alpha,\beta=1}^{d(N)} c_{\mu}^{\alpha} \theta_{\mu\alpha}^{\beta} Y_{\beta} = 0 .$$

Thus we conclude that, if $X$ is divergence-free, the coefficients $c_{\mu}^{\alpha}$ satisfy, for any $\beta$:

$$\sum_{\mu=1}^{m} \sum_{\alpha=1}^{d(N)} c_{\mu}^{\alpha} \theta_{\mu\alpha}^{\beta} = 0 ,$$

so

$$\text{div}_{\mathbb{R}^{d(N)}} V = \sum_{\mu=1}^{m} \sum_{\alpha,\beta,\gamma=1}^{d(N)} \theta_{\mu\beta}^{\gamma} c_{\mu}^{\alpha} y_{\alpha} + \theta_{\mu\beta}^{\gamma} c_{\mu}^{\alpha} y_{\alpha} = 0$$

(here we have also used the fact that $\theta_{\mu\beta}^{\beta} = 0$ because of antisymmetry).
4 Proof of Theorem 1.2

The idea of the proof is as follows: first the manifold $N$ is embedded into a sphere $S^n$ of suitable dimension, and the push-forward of the vector field $X$ is extended to the whole $S^n$. By constructing this extension in a suitable way, we can ensure that any other vector field close enough to the extended vector field has an invariant manifold diffeomorphic to $N$, on which it is very close to $X$. We then take a polynomial approximation of the extension of $X$ and apply Theorem 1.4 to conclude. We note that the proof can be carried out analogously starting with an embedding of $N$ into $T^n$.

Let $F : N \rightarrow S^n$ be an embedding of $N$ into $S^n$. Provided we take the dimension of the sphere high enough, such an embedding always exists.

Our aim now is to extend the vector field $F^*(X)$ to the whole $S^n$, so that $F(N)$ is an $r$-normally hyperbolic invariant manifold:

**Definition 4.1** (Normally hyperbolic invariant manifold, [14] Section 1) Let $Y$ be a smooth vector field on a manifold $M$, and denote by $\phi^t_Y : M \rightarrow M$ its flow at time $t \in \mathbb{R}$. We will say that a submanifold $V \subset M$ is an $r$-normally hyperbolic invariant manifold of $Y$ if $Y$ is tangent to $V$ and, moreover, there is a continuous splitting $TM|_V = E^u \oplus TV \oplus E^s$ and constants $c > 0$, $0 \leq \mu < \lambda$ such that, for any $x \in V$ we have:

(i) For any $t \in \mathbb{R}$, $d_x\phi^t_Y(E^u_x) = E^u_{\phi^t_Y(x)}$, and analogously for $E^s$.

(ii) For any $v \in E^s_x$ and $t \geq 0$, $\|d_x\phi^t_Y(v)\| \leq ce^{-\lambda t}\|v\|$.

(iii) For any $v \in E^u_x$ and $t \geq 0$, $\|d_x\phi^{-t}_Y(v)\| \leq ce^{-\lambda t}\|v\|$.

(iv) For any $v \in T_xV$ and $t \in \mathbb{R}$, $\|d_x\phi^t_Y(v)\| \leq ce^{\mu |t|}\|v\|$.

Let $(x, z_1, \ldots, z_k), x \in N, (z_1, \ldots, z_k) \in NF(x) F(N), k = n - \text{dim } N$, be local coordinates parametrizing a tubular neighbourhood $V$ of $F(N) \subset S^n$. Define a vector field $Z$ that is given in the local coordinates by

$$Z(x, z_1, \ldots, z_k) = X(x) - C \left( z_1 \frac{\partial}{\partial z_1} + \cdots + z_k \frac{\partial}{\partial z_k} \right).$$

Because $N$ is compact, if we take the constant $C$ large enough the field $Z$ is $r$-normally hyperbolic on $F(N)$ for any a priori chosen $r$. Indeed, at any point $F(x) \in N$, define $E^s_{F(x)} := \text{span} \{ \partial_{z_1}, \ldots, \partial_{z_k} \}$, and $E^u_{F(x)} := \{0\}$. We have a continuous splitting $TS^n|_{F(N)} = E^u \oplus TF(N) \oplus E^s$, automatically satisfying items (i) and (iii) in Definition 4.1. As for item (ii), it is satisfied taking $\lambda := C$, since we have $d_{F(x)}\phi^t_Z(\partial_{z_i}) = e^{-Ct}\partial_{z_i}$. 

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Finally, let $\Lambda > 0$ be a constant such that, for any $x \in N$, $v \in T_x N$ and $t > 0$, we have $e^{-\Lambda t}||v|| \leq ||d_x \phi^t(v)|| \leq e^{\Lambda t}||v||$. Such a constant must always exist by compactness of $N$ (we can choose for example the supremum over all points $x \in N$ of the Lyapunov exponent of $X$ at $x$ with greatest absolute value). By choosing $C > \Lambda r$, we can take $\mu := \Lambda r + \epsilon < C = \lambda$ such that (iv) holds.

We now extend $Z$ smoothly in an arbitrary way to the rest of $\mathbb{S}^n$, still denoting this extension by $Z$.

A crucial property of $r$-normally hyperbolic flows is their structural stability (Theorem 4.1 in [14]), that is, any other vector field $Z'$ close enough to $Z$ in the $C^r$ norm, $r \geq 1$, has the following property: there is an embedding $F' : N \to \mathbb{S}^n$, close to $F$ in the $C^r$ norm, such that $Z'$ is tangent to $F'(N)$. This also implies, in particular, that the vector field $dF'^{-1}(Z'|_{F'(N)})$ on $N$ is close to $X$ in the $C^{r-1}$ norm.

We now take $Z'$ to be a polynomic vector field on $\mathbb{S}^n$, approximating $Z$ to any desired degree of accuracy (more precisely, we approximate by polynomials the components of the vector field $Z$ with respect to the vector fields $h_\mu$ defined in Sect. 3). Theorem 1.2 follows by applying Theorem 1.4 to $Z'$.

4.1 The dimension of $M$

The dimension of the Riemannian manifold $M$ in Theorem 1.2 will depend only on the dimension $n$ of the sphere in which we embed $N$, and on the degree of the polynomic vector field $Z'$ approximating the extension of $X$. More precisely, $M = SO(d) \times \mathbb{T}^{d+1}$, where $d$ is the dimension of the space of spherical harmonics in $\mathbb{S}^n$ whose degrees are at most one plus that of $Z'$ (see Sect. 2.3). The degree of $Z'$ will in turn depend on the acceptable error $\epsilon$ in the approximation of $Z$ by $Z'$, which in applications will be determined by the robustness of the dynamical feature of $X$ we are interested in.

To get a quantitative bound on the degree, and hence on the dimension, as a function of $\epsilon$, we can use a multidimensional Jackson-type theorem. For example, by virtue of Theorem 2 in [2], we have that, for any integer $k \geq 0$, there are polynomials $Z'$ of any degree satisfying:

$$||Z - Z'||_{C^m(V)} \leq \frac{c}{(\text{degree } Z')^k}||Z||_{C^{m+k}(V)}$$

where $c$ is a constant depending on the dimension $n$, on the desired norm of approximation $m$, and on the tubular neighbourhood $V$ of $F(N)$. Observe that, by how the extension $Z$ was constructed, the $C^{m+k}$ norms of $Z$ can be bounded by those of $X$ in $N$, modulo a constant depending on the Lyapunov exponents of $X$ and on the derivatives of the embedding of $N$. Thus,
degree $Z' \leq \left( \frac{C'}{\epsilon} ||X||_{C^{m+k}(\mathbb{N})} \right)^{\frac{1}{\kappa}}$.

The parameter $d$ in $SO(d) \times \mathbb{T}^{d+1}$ is bounded by the dimension of the space of harmonic polynomials of degrees up to one plus the degree $D$ of $Z'$ (see the discussion above Eq. (3.4))

$$d \leq \left( \frac{D + n}{D + 1} \right) \frac{2D + 2 + n}{n} \sim D^n,$$

so finally

$$\dim M < \left( \frac{C'}{\epsilon} ||X||_{C^{m+k}(\mathbb{N})} \right)^{\frac{2n}{\kappa}}.$$

5 Proof of Theorem 1.8

The family of vector fields on $\mathbb{T}^3$ defined as

$$u_{ABC}(x_1, x_2, x_3) = (A \sin x_3 + C \cos x_2) \frac{\partial}{\partial x_1} + (B \sin x_1 + A \cos x_3) \frac{\partial}{\partial x_2} + (C \sin x_2 + B \cos x_1) \frac{\partial}{\partial x_3}$$

for parameters $A, B, C \in \mathbb{R}$, are called the ABC (Arnold-Beltrami-Childress) flows. They have been extensively studied (see e.g [1] and references therein), and are known to have chaotic invariant sets for certain values of the parameters $A, B, C [6, 28]$.

An ABC flow is given by sines and cosines with integer frequencies in the unit sphere. Thus, arguing as in the proof of Theorem 1.4 for the torus, we see that ABC vector fields are embedded exactly in the Euler dynamics on $M = SO(6) \times \mathbb{T}^7$ (there are seven integer points in the ball of radius 1, but we can exclude the zero vector by virtue of Remark 3.2). Theorem 1.8 follows.

6 Additional comments

Here we give some additional constructions of interesting dynamics inside the Euler system which do not follow immediately from Theorems 1.2 and 1.4, but rather need the concert of other results:
6.1 The Lorenz attractor in the Euler equations

The Lorenz attractor is a paradigmatic example of attractor and a popular emblem of chaos. It arises in an ODE in \( \mathbb{R}^3 \) that is obtained from a Galerkin truncation of the Boussinesq equation (itself a PDE approximating the Navier-Stokes equations).

The goal of this subsection is to embed 3-dimensional geometric Lorenz flows (vector fields introduced in [26] having attractors with the same qualitative dynamics as the Lorenz system) into the Euler dynamical system on some Riemannian manifold \( M \), so that the Euler equations reduce to the Lorenz dynamics in a finite dimensional subset of the phase space. In other words, in that manifold, the Lorenz dynamics are not a toy model of the Navier-Stokes equations, but an exact description of ideal fluid motion. Notice that here, by contrast with Theorem 1.8, we are embedding a system exhibiting dissipative chaos (the Lorenz flows do not preserve volume; a finite dimensional volume-preserving ODE cannot have an attractor).

Lorenz attractors are not stable under perturbation of the flow [26], which precludes the direct application of Theorem 1.2.

Nevertheless, by a theorem of Guckenheimer and Williams, the set of geometric Lorenz flows with support in the unit ball \( B \subset \mathbb{R}^3 \) contains an open set in the \( C^1 \) topology (see the main Theorem in [13]; this is not how the theorem is stated, but it stems from the proof). In other words, there are vector fields \( v \) in \( B \) that have a geometric Lorenz attractor, and such that any small enough \( C^1 \) perturbation of them also has a geometric Lorenz attractor.

An analogous result holds in any other 3-dimensional compact manifold, in particular, in \( S^3 \). Indeed, consider one such vector field \( v \) in \( B \), and embed the ball into \( S^3 \), extending \( v \) smoothly to some vector field \( w \). Any small enough \( C^1 \) perturbation of \( w \) will have a geometric Lorenz attractor inside \( B \).

Therefore, there are polynomial vector fields in \( S^3 \) having a geometric Lorenz attractor, because polynomial vector fields are dense in the \( C^1 \) topology. Applying Theorem 1.4 to one of these vector fields, that we denote by \( X \), we embed \( S^3 \) into the space of smooth divergence-free fields on some Riemannian manifold \( M \), so that the trajectories of \( X \) are mapped to solutions of the Euler equation on \( M \). Thus we have a finite-dimensional family of initial conditions (parametrized by points in a subset of \( S^3 \) ) that converge, under Euler evolution, to a geometric Lorenz attractor inside the space of divergence-free vector fields on \( M \) (the image under the embedding given by Theorem 1.4 of the Lorenz attractor of \( X \) inside \( S^3 \)).
6.2 The universal template

There is a Riemannian manifold $M$, and a 3-dimensional family of divergence-free vector fields $\Sigma \subset \mathcal{X}^\infty(M)$, with the following properties:

(i) $\Sigma$ is diffeomorphic to a 3-sphere, and invariant under the Euler dynamical system.

(ii) The Euler dynamics on $\Sigma$ contains sets of periodic orbits representing every isotopy class of knots and links.

Indeed, there is a vector field in $S^3$ containing periodic orbits of every isotopy class of knots and links, which are moreover stable under sufficiently small $C^m$ perturbations (Corollary 3.2.19 in [12]). Applying Theorem 1.2, the claim follows.

6.3 Euler trajectories and translation surfaces of triangular billiards

Let $P \subset \mathbb{R}^2$ be a triangle. We will henceforth assume that $P$ comprises the interior and the sides of the triangle, but not the vertices.

The billiard on $P$ is the dynamical system defined thus: a particle at point $q \in P$, with initial velocity $v$ of modulus 1, moves with constant velocity while in the interior of $P$, and is reflected every time the trajectory hits a side of the triangle. If the trajectory hits a vertex, the particle stops. Any billiard trajectory is thus completely determined by the initial position $q \in P$ and the angle $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ of the initial velocity.

By means of the unfolding construction of Katok-Zemljakov [16], to any triangle $P$ we can associate an open, smooth Riemann surface $S_P$ with a flat metric, so that the geodesic flow on the unit tangent bundle of $S_P$ is equivalent to the billiard on $P$.

The surface $S_P$ can be endowed with an atlas whose transition maps are euclidean translations, and so it is called the translation surface of $P$. In the coordinates given by this atlas, the geodesics on $S_P$ are straight lines, and their slope is globally well-defined because the transition functions are translations. Thus for any given angle $\theta$, we can define a foliation on $S_P$ whose leaves are the geodesics of slope $\tan(\theta)$.

**Corollary 6.1** For any triangle $P$, there is a metric on $M = SO(91) \times \mathbb{T}^{92}$ so that the Euler equation on $M$ has the following property: for any angle $\theta$, there is a compact surface $\Sigma_{P,\theta} \subset \mathcal{X}^\infty(M)$, invariant under the Euler evolution, and such that

(i) The surface $\Sigma_{P,\theta}$ minus a finite number of points is diffeomorphic to the translation surface $S_P$. These finite points in $\Sigma_{P,\theta}$ are stationary solutions of the Euler equation.
The one-dimensional foliation on $\Sigma_{P, \theta}$ whose leaves are given by the Euler trajectories is diffeomorphic to the foliation on $S_P$ whose leaves are given by geodesics of slope $\tan(\theta)$.

Proof They key ingredient in the proof is the dictionary between polygonal billiards and homogeneous foliations in $\mathbb{C}^2$ due to F. Valdez [25]. Let $\lambda_1, \lambda_2, \lambda_3$ be the angles of the triangle $P$. In $\mathbb{C}^2$ we define the following homogeneous holomorphic vector field:

$$X = z_1(\lambda_3 z_2 + \lambda_2 (z_2 - z_1)) \partial z_1 + z_2(\lambda_3 z_1 + \lambda_1 (z_1 - z_2))$$

The integral curves of $\Re(X)$ and $\Im(X)$ generate a 2-dimensional homogeneous foliation $\mathcal{F}$ in $\mathbb{R}^4$.

By virtue of Theorem 1.1 in [25], for any angle $\theta$ we can find a leaf $L$ of $\mathcal{F}$ that is diffeomorphic to $S_P$ through a diffeomorphism that maps the integral curves of $\Re(X)$ in $L$ to the leaves of the geodesic foliation of slope $\tan(\theta)$ in $S_P$.

As argued in Section 1.1 of [25], the homogeneity of the construction above allows us to deduce the analogous result in $\mathbb{RP}^3$ and $S^3$. In particular, the foliation in $S^3$ defined by the integral curves of the vector fields

$$U = \Re(X) - (\Re(X) \cdot \partial_r) \partial_r$$
$$V = \Im(X) - (\Im(X) \cdot \partial_r) \partial_r$$

(where $\partial_r$ denotes the radial unit vector field) has a leaf which, minus the zeros of $U$, is diffeomorphic to $S_P$, and on which $U$ defines a foliation diffeomorphic to the geodesic foliation of any given slope.

The vector fields $U$ and $V$ in $S^3$ are the restrictions of polynomial vector fields of degree 4 on $\mathbb{R}^4$. Arguing as in the proof of Theorem 1.4 in the case of the $S^3$, we see that we can embed the vector field $U$ into the phase space of the Euler equation in $SO(91) \times T^{92}$ (91 being the dimension of the linear space of spherical harmonics in $S^3$ of degree up to 5 (recall that, in our construction, the top degree of the spherical harmonics is one plus the degree of the polynomial vector field, see Sect. 2.3 and Eq. 3.4)). Corollary 6.1 follows.

Remark 6.2 Corollary 6.1 can be generalized to more general polygonal billiards using the construction in Section 5 of [25].

6.4 An explicit example

In this section, we will show explicitly the result of applying Theorem 1.4 to the vector field $X = \cos(\theta) \partial_\theta$ defined on $S^1$. This vector field has two heteroclinic trajectories, going from $\theta = \frac{3\pi}{2}$ to $\theta = \frac{\pi}{2}$. 
Following Sect. 2, we embed $X$ into a quadratic vector field in $\mathbb{R}^2$ with the map $\Psi(\theta) = (\cos(\theta), \sin(\theta))$. The resulting quadratic ODE is

$$\frac{dy_1}{dt} = -y_1 y_2, \quad \frac{dy_2}{dt} = y_1^2,$$

that is, we have, in the notation introduced in Sect. 2, $B_{112} = -1$, $B_{211} = 1$, and the rest of the coefficients $B_{ijk} = 0$.

From these coefficients we construct, following Tao [23], the vector fields $U_\mu$, $\mu = 1, 2$, and the functions $F_{\mu\nu}$ in $SO(2) \cong S^1$, defined by Eqs. (2.4) and (2.5), respectively. Then, using Eq. (2.3), we construct the map $T : \mathbb{R}^2 \to \mathcal{X}^\infty(S^1 \times \mathbb{T}^3)$. The result is the following: for a vector $y = (y_1, y_2) \in \mathbb{R}^2$, we have:

$$T(y) = -y_1 \partial_\varphi + \left( y_1 \cos(\varphi) + y_2 \sin(\varphi) \right) \partial_{t_1} +$$

$$+ \left( y_2 \cos(\varphi) - y_1 \sin(\varphi) \right) \partial_{t_2}$$

where $\varphi$ is the coordinate in the first $SO(2) \cong S^1$ component, and $t_1, t_2$ are the first two coordinates in $\mathbb{T}^3$. Notice that the vector fields $T(y)$ are actually defined in $S^1 \times \mathbb{T}^2 = \mathbb{T}^3$, since they do not depend in any way on the $t_3$ coordinate. We will explain later why this coordinate is introduced, and how, in fact, it is redundant in this particular case (see also Remark 2.3).

The embedding $\Phi : S^1 \to \mathcal{X}^\infty(\mathbb{T}^4)$ is defined as $\Phi = T \circ \Psi$. It reads:

$$\Phi(\theta) = -\cos(\theta) \partial_\varphi + \left( \cos(\theta) \cos(\varphi) + \sin(\theta) \sin(\varphi) \right) \partial_{t_1} +$$

$$+ \left( \sin(\theta) \cos(\varphi) - \cos(\theta) \sin(\varphi) \right) \partial_{t_2}.$$

It is easy to check that every member of this $S^1$-family of vector fields in $\mathbb{T}^4$ is divergence-free with respect to the standard the volume form

$$\Omega = d\varphi \wedge dt_1 \wedge dt_2 \wedge dt_3,$$

in other words, the 3-form $\Omega(\Phi(\theta), \cdot)$ is closed. The same family, considered as defined in $\mathbb{T}^3$, is also divergence-free with respect to the volume form $\Omega_0 = d\varphi \wedge dt_1 \wedge dt_2$.

We now have to determine the metric $g$ in $\mathbb{T}^4$ with respect to which the images $\Phi(\theta(t))$ of the orbits of $X$ are solutions of the Euler equation. Following Tao’s construction in [23], we first define, from the functions $F_{\mu\nu}(\theta)$, the following family of one forms:
\[ \alpha(\theta) := \left( \cos(\theta) \cos(\varphi) + \sin(\theta) \sin(\varphi) \right) dt_1 + \\
\quad \left( \sin(\theta) \cos(\varphi) - \cos(\theta) \sin(\varphi) \right) dt_2. \]

The goal is to find a metric that makes \( \alpha(\theta) \) the dual of the vector field \( \Phi(\theta) \). Following Sects. 3 and 4 in [23], we introduce two metrics: one in \( S^1 \times T^2 \), given by

\[
g_0 = \frac{1}{4} d\varphi \otimes d\varphi + \left( 1 + \frac{1}{4} \cos^2(\varphi) \right) dt_1 \otimes dt_1 + \left( 1 + \frac{1}{4} \sin^2(\varphi) \right) dt_2 \otimes dt_2 + \\
\quad + \frac{1}{4} \cos(\varphi) \left( \left( d\varphi \otimes dt_1 + dt_1 \otimes d\varphi \right) - \frac{1}{4} \sin(\varphi) \left( d\varphi \otimes dt_2 + dt_2 \otimes d\varphi \right) - \\
\quad \quad - \frac{1}{4} \cos(\varphi) \sin(\varphi) \left( dt_1 \otimes dt_2 + dt_2 \otimes dt_1 \right) \]

and an extension of the above to \( T^4 \)

\[ g = g_0 + (\det g_0)^{-1} dt_3 \otimes dt_3. \]

It can be checked that we have indeed

\[ \alpha(\theta) = i_{\Phi(\theta)} g = g(\Phi(\theta), \cdot) \]

and the same is true with respect to \( g_0 \), considering \( \alpha(\theta) \) as defined in \( T^3 \).

To see that \( g_0 \) is a metric, consider it as a symmetric 3-by-3 matrix with respect to the basis \( (d\varphi, dt_1, dt_2) \). It can be checked that this matrix defines a positive definite quadratic form, for example using Sylvester’s criterion: the determinants of the upper left 1-by-1, 2-by-2 and 3-by-3 minors are all equal to \( \frac{1}{4} \). By construction, \( g \) is clearly a metric and its determinant is always 1, that is, the volume form induced by \( g \) coincides with \( \Omega \).

Before checking that this metric gives indeed solutions to the Euler equations, let us explain why, in this particular example, the introduction of the \( t_3 \) coordinate in Tao’s construction is in fact not needed, and it is enough to consider the embedding to be \( \Phi : S^1 \rightarrow \mathcal{X}^\infty(S^1 \times T^2) \) and the metric simply \( g_0 \). The \( t_3 \) coordinate is introduced to ensure that the volume form induced by \( g \) is the same as \( \Omega \) (which is the volume form preserved by construction by the vector fields \( \Phi(\theta) \)). But in this case, the volume form induced by \( g_0 \) in \( T^3 \)
is already preserved by the vector fields $\Phi(\theta)$. Indeed, the determinant of $g_0$ is a constant, namely $\frac{1}{4}$, so the volume form induced by $g_0$ is $\frac{1}{2}\Omega_0$.

Now let us check that $\Phi(\theta): S^1 \to \mathbb{T}^3$ yields solutions to the Euler equations with the metric $g_0$. Since we already know that the $\Phi(\theta)$ are divergence-free with respect to the volume $\Omega_0$, we just have to check that, if $\theta(t)$ is a solution of the ODE

$$\frac{d\theta(t)}{dt} = \cos(\theta(t)),$$

then

$$\partial_t \Phi(\theta(t)) + \nabla^{LC}_{\Phi(\theta(t))} \Phi(\theta(t)) = -\nabla p \tag{6.1}$$

where $\nabla^{LC}$ and $\nabla$ are the Levi-Civitta connection and the gradient associated with the metric $g_0$.

At this stage, the purpose of the previous constructions becomes apparent if we work with the Euler equation in its co-tangent bundle formulation, rather than with Eq. (6.1). Defining the dual 1-form

$$\alpha_t := \alpha(\theta(t)) = i_{\Phi(\theta(t))} g_0$$

we have (see e.g [1]) that Eq. 6.1 is equivalent to:

$$\partial_t \alpha_t + i_{\Phi(\theta(t))} d\alpha_t = -dB$$

with $B = p + \frac{1}{2}g(\Phi(\theta(t)), \Phi(\theta(t)))$.

Since $\frac{d}{dt}\theta = \cos(\theta)$, we have,

$$\partial_t \alpha_t = \cos(\theta(t))\left(-\sin(\theta(t)) \cos(\varphi) + \cos(\theta(t)) \sin(\varphi)\right)dt_1 +$$

$$+ \sin(\theta(t))\left(\cos(\theta(t)) \cos(\varphi) + \sin(\theta(t)) \sin(\varphi)\right)dt_2.$$

On the other hand,

$$i_{\Phi(\theta(t))} d\alpha_t = -\cos(\theta(t))\left(-\sin(\theta(t)) \cos(\varphi) + \cos(\theta(t)) \sin(\varphi)\right)dt_1 -$$

$$- \sin(\theta(t))\left(\cos(\theta(t)) \cos(\varphi) + \sin(\theta(t)) \sin(\varphi)\right)dt_2,$$
which means that

\[ \partial_t \alpha_t + i \Phi(\theta(t)) \wedge \alpha_t = 0 , \]

so \( \alpha_t \), and thus its dual vector field \( \Phi(\theta(t)) \), are a solution of the Euler equations on \( (\mathbb{P}^3, g_0) \).

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