Perturbative analysis of the colored Alexander polynomial and KP soliton $\tau$-functions

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Abstract

In this paper we elaborate on the statement given in [1]. Mainly, we study the relation between the colored Alexander polynomial and the famous KP hierarchy. We explain and prove this relation by exploring the fact that the dispersion equations of the one soliton $\tau$-function are equivalent to the system of equations arising due to the special representation dependence of the Alexander polynomial. This appears to be even more interesting considering than the other limit of the HOMFLY polynomial also produces a KP $\tau$-function. We also discuss some interesting properties of the appearing constructions, such as the formulation in terms of supersymmetric polynomials.

1 Introduction

Nowadays knot theory is of great interest in mathematical physics. This is due to the fact that knot invariants appear in various physical problems such as quantum field theories [2-4], quantum groups [5], lattice models [6], CFT [7], topological strings [8], quantum computing [9] etc. These correspondences lead to generalizations of some already known invariants and to discoveries of new ones.

Polynomial invariants are probably one of the most developed ones. Here we take a more careful and peculiar look at the colored Alexander polynomial, which we discussed in [1]. There we discussed the general structure of the invariant and establish an intriguing connection to the famous KP hierarchy. Let us remind the qualitative aspects of the statements.

One of the most important polynomial invariants is the colored HOMFLY polynomial. In the TQFT context it is defined as the vacuum expectation value of a Wilson loop along the knot in Chern-Simons gauge theory, with the gauge group $G = \text{SU}(N)$ and representation $R$ [2][10]:

$$ \mathcal{H}_R^f(q,a) = \frac{1}{Z} \int D A \ e^{-\frac{\kappa}{4\pi} S_{\text{CS}}[A]} \ W_R(K,A), $$

(1)

where the Wilson loop and the Chern-Simons action given by

$$ W_R(K,A) = \text{tr}_R \ P \exp \left( \oint A^a_i(x) T^a dx^i \right), \quad S_{\text{CS}}[A] = \frac{\kappa}{4\pi} \int_M \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A). $$

(2)

The variables $q, a$ in the HOMFLY polynomial are

$$ q = e^h, \quad a = e^{N h}, \quad h := \frac{2\pi i}{\kappa + N}. $$

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In such a parametrization, the polynomial may be represented as a series in the variable $\hbar$. Furthermore, there is a natural “quasiclassical” double scaling expansion given by $\hbar \to 0, N \to \infty$ such that $N\hbar$ stays fixed. In other words, this means taking $q = 1$ and keeping the variable $a$ arbitrary. The polynomials that emerge as the special value of the HOMFLY polynomials $\mathcal{H}_R^K(1, a) = \sigma_R^K(a)$ are called the special polynomials [13]. Their $R$ dependence has a simple power-like form, which expresses them through the special polynomial in the fundamental representation $[13, 16]$: 

$$\sigma_R^K(a) = \left(\sigma_{|1|}^K(a)\right)^{|R|}. \quad (3)$$

Hereafter, we identify the representation $R$ with the Young diagram associated with it: $R = \{R_i\}, R_1 \geq R_2 \geq \ldots \geq R_{|R|}, |R| := \sum_i R_i$.

These polynomials exhibit integrable properties [16, 17]: their $R$ dependence allows one to construct from them a KP $\tau$-function, which is the value of the Ooguri-Vafa partition function:

$$\mathcal{A}_R^K(q) = \mathcal{A}_{|1|}^K(q^{|R|}), \quad \text{where } R = [r, 1^k].$$

which holds only for the representations corresponding to 1-hook Young diagrams. In [1] we have stated that as just as the special polynomials, the Alexander polynomials are encode information about the KP hierarchy. In this paper we want to accomplish two tasks. First of all we will provide proofs to all the statements formulated in the previous paper. Second, we provide a clearer look at the statement. The demystification relies on noticing that the "dual" limit is also connected to $\tau$-functions. In this case, the soliton $\tau$-functions. Their dispersion relations have the same form as the equations of the KP hierarchy and thus they appear to be equivalent to the property of the Alexander polynomial. In this sense acting on the soliton $\tau$-function by the generating function for the KP equations, produces a series, which contains all the group factors of the Alexander polynomial. The following diagram summarizes the result:

This kind of duality hints that even more information about the KP hierarchy is encoded in the HOMFLY polynomials for arbitrary values of $q, A$. The key to understanding the correspondence between the two limits is the connection between the Alexander polynomial and the soliton $\tau$-functions.

Another well-known perturbative expansion of the HOMFLY polynomial is the loop expansion [10], which is based on the gauge invariance of Chern-Simons theory. Evaluating the Wilson loop correlators can be done in some fixed gauge. In the temporal gauge [11], $A_0 = 0$, the Wilson loop acquires the polynomial form of the coloured HOMFLY invariant. When calculated in the holomorphic gauge [12] $A_x + iA_y = 0$, it gives the Kontsevich integral [20, 21]. The theory is gauge invariant, therefore the two object are equal, however, the Kontsevich integral is a perturbative expansion and the HOMFLY polynomial is not. Therefore this construction gives a

\footnote{Let us note that our notion of coloured Alexander polynomial is totally different from that defined in [13].}
perturbative description of the HOMFLY polynomial with arbitrary variables $q, a$. It appears to have a nicely looking structure [22]:

$$\mathcal{H}_R^K = \sum_n \left( \sum_j v_{n,j}^R r_{n,j}^R \right) \hbar^n. \tag{6}$$

A remarkable fact is that the knot dependence and the group theoretic dependence split explicitly. The group one is represented by the so called group factors $r_{n,j}^R$. They are group invariants that appear in the Kontsevich integral as some trivalent diagrams, which are further expressed as traces of products of the algebra generators $T^a$ and the structure constants:

$$r_{n,j}^R \sim \text{tr}_R(T^{a_1} \ldots T^{a_n}). \tag{7}$$

The knot dependent part is $v_{n,j}^K$, which are some numerical invariants. Another point is that they appear to be exactly the famous Vassiliev invariants or invariants of finite type [22, 23]. These numerical invariant are considered as potential candidates for a complete set of invariant and are therefore important to study. In this paper, however, we mostly focus on the group theoretic part.

Let us formulate a brief list of results of the paper:

- The dispersion relations of the one soliton $\tau$-function are in a one-to-one correspondence with the solutions of the Alexander system
- In other words the soliton anzats for the $\tau$-function corresponds to the single-hook property of the Alexander polynomial
- The group factors of the Alexander polynomials are supersymmetric in certain variables and thus their connection to the KP equations is established.
- The group factors are the weights for Hopf algebra of trivalent diagrams, hence we find a certain ideal in this algebra.
- Various properties of the group factors are studied.

Our paper is organized as follows. The colored Alexander polynomial is shortly introduced as a generalization of the usual topologically defined Alexander polynomial. In section 3 we study the perturbative expansion of the polynomial and define the Alexander system of equations. We are interested in studying the space of it’s solutions. Next the properties of the solutions are derived and the supersymmetric formulation is briefly discussed. Section 4 is devoted to the general properties of the KP equations and $\tau$-functions required for the formulation of our statement. Finally, in section 5 we will prove the main theorem stating the correspondence between the solutions of the Alexander equations and the soliton dispersion relations. In conclusion we will state some follow up questions and generalizations.

## 2 The Alexander polynomial

**Definition 2.1.** Let $K$ be a knot in $S^3$ and $X_\infty$ be the infinite cyclic cover of the knot complement. The homology group $H_1(X_\infty)$ is a module over $\mathbb{Z}[q,q^{-1}]$, which is called the Alexander module. The Alexander polynomial $A^K(q)$ is usually defined as a generator of a certain ideal in this module.

One may notice that the Alexander polynomial is a special value of the fundamental HOMFLY polynomial:

$$A^K(q) = \mathcal{H}_R^K(q,1). \tag{8}$$

HOMFLY polynomials can be defined for higher representations of the gauge group, therefore we have the following generalization:

**Definition 2.2.** The Colored Alexander polynomial is defined as a special value of the HOMFLY polynomial:

$$A^R_K(q) = \mathcal{H}_R^K(q,1) \quad \text{or} \quad A^R_K(e^h) = \lim_{N \to 0} \mathcal{H}_R^K(e^h, e^{Nh}). \tag{9}$$

Representations of $SU(N)$ are labelled by Young diagrams. We identify a representation $R$ with it’s diagram $[R_1, R_2, \ldots]$.
Definition 2.3. A single hook diagram is a diagram of the form:
\[ R = [r, 1, \ldots, 1] \] (10)

\[ \begin{array}{c}
\text{[5,3,2,1] diagram,} \\
\end{array} \quad \begin{array}{c}
\text{Single hook diagram [5,1,1,1]} \\
\end{array} \]

Theorem 2.1. For any knot \( K \) and any single hook diagram \( R = [r, 1, \ldots, 1] \):
\[ A^K_R(q) = A^K_{[1]}(q^{|R|}), \quad \text{where} \quad |R| = r + L, \] (11)

where \([1]\) is the fundamental representation.

Another important property is the symmetry of the HOMFLY (and, in particular, Alexander) polynomials with respect to the transposition of the Young diagram of the representation. This property holds for arbitrary diagrams \( R \) and comes from the corresponding property of quantum groups and WZW theories \([25,26]\), in the latter case, it is called the rank-level duality (see also \([27]\)):
\[ H^K_R(q, a) = H^K_R(q^{-1}, a). \] (12)

This property is immediately inherited by the Alexander polynomials,
\[ A^K_R(q) = A^K_{R^T}(q^{-1}). \] (13)

3 Alexander equations

3.1 The equations

As we have mentioned the HOMFLY polynomial admits a series expansion in the formal variable \( h \):
\[ H^K_R = \sum_n \left( \sum_j v^K_{n,j} r^R_{n,j} \right) h^n. \] (14)

According to definition (9) the Alexander polynomial is a special value of the latter, therefore it inherits the structure of the expansion. Let us substitute (14) in (72), put \( N = 0 \) and denote the resulting group factors as \( A^K_{n,m} \):
\[ A^K_R(q) - A^K_{[1]}(q^{|R|}) = \sum_n h^n \sum_m v^K_{n,m} \left( r^R_{n,m} - |R| \cdot r^{|1|}_{n,m} \right) \bigg|_{N=0} = \sum_n h^n \sum_m v^K_{n,m} A^K_{n,m} = 0. \] (15)

Since \( h \) is an arbitrary formal variable and this equality hold for all \( h \) we see that:
\[ A^K_{n,m} = 0. \] (16)

The group factors \( r^R_{n,j} \) appear in a form of traces of group generators and are group invariants. They are commonly studied as algebras of trivalent graphs with weights associated to them. In \([22]\) these trivalent graphs are the Feynman diagrams appearing in the perturbative expansion of the Wilson loop holonomy. An elaborate discussion is given in \([23]\) (see sections 5,6). There the trivalent graphs that satisfy certain relations form an algebra of closed Jacobi diagrams.

To construct knot invariants one associates Lie algebra weight systems to any Jacobi diagram in a manner which is clear from the following examples:

\[ D_1 = \begin{array}{c}
\end{array}, \quad \varphi_\mathfrak{g}(D_1) = \sum_{a=1}^{\dim \mathfrak{g}} T^a T^{a^*}; D_2 = \begin{array}{c}
\end{array}, \quad \varphi_\mathfrak{g}(D_2) = \sum_{a,b,c=1}^{\dim \mathfrak{g}} T^a T^b T^c T^{a^*} T^{b^*} T^{c^*} \]

Where \( T^a \) are the generators of the Lie algebra \( \mathfrak{g} \). This construction is naturally extended to trivalent diagrams,
which are also called Jacobi diagrams. By theorem 6.1.2 in [23] \( \varphi \) gives a homomorphism to the center of the universal enveloping algebra \( U(\mathfrak{g}) \). The group factors are constructed in the same manner for a general representation by composing the representation with the homomorphism \( \varphi \) and taking the trace. As they are traces of the representation of the center \( ZU(\mathfrak{g}) \) they they can be expanded into the basis of the Casimir invariants of the algebra [22]. Obviously exactly the same holds for the group factors of the Alexander polynomial.

\[
A_{n,m}^R = \sum_{|\Delta| \leq n} \alpha_{\Delta,m} C_\Delta(R),
\]

(17)

where we label the monomials of \( C_k \) by the Young diagrams in accordance with

\[
C_\Delta = \prod_{i=1}^{l(\Delta)} C_{\Delta_i}.
\]

(18)

Then, \( A_{n,m}^R \) can be considered as functions of Casimir invariants only, all the dependence on the representation entering through these latter ones:

\[
A_{n,m}^R = A_{n,m}(C)
\]

(19)

Note that one can also re-expand the difference

\[
A_{n,m}^R(q) - A_{1,0}^R(q^{|R|}) = \sum_n h^n \sum_{|\Delta| \leq n} C_\Delta(R) \sum_m v_{n,m}^K \alpha_{\Delta,m} = \sum_n h^n \sum_{|\Delta| \leq n} \alpha^K_\Delta C_\Delta(R),
\]

(20)

into the Casimir invariants instead of the group factors.

Equality (16) constitutes a property of the group factors of the Alexander polynomial. Let us look at generic polynomials of Casimir operators, that obey the desired property.

**Definition 3.1.** The Alexander system of equations is a linear system of equations on the coefficients \( x_{\Delta}^{(m)} \) given by

\[
X_{n,m}(C) := \sum_{|\Delta| = n} x_{\Delta}^{(m)} C_\Delta(R) = 0
\]

(21)

for any \( n \) and any single hook diagram with \( |R| = n \).

It’s obvious that the group-factors \( A_{m,n} \) of the Alexander polynomial are linear combination of the basis solutions to these equations:

\[
A_{n,m}(C) \in \text{Span}\left( \oplus_{k \leq n} X_{k,m}(C) \right)
\]

(22)

Therefore we consider (21) as equations defining the general structure of the polynomial. We denote by \( A_1 \) the vector space of solutions of the Alexander equations (21) at order \( n \), where by a solution we mean polynomials in Casimir invariants satisfying (21). We call a solution even, if it’s an even polynomial in \( C_k(R) \) and odd otherwise. The space of all solutions is the following graded vector space:

\[
A_1 = \bigoplus_n A_n
\]

(23)

The discussion about the trivalent graphs above allows to formulate the problem in other terms. We are studying Jacobi diagrams, which are weighted with an \( sl_N \) weight system, with \( N \) then set to zero. Hence the Alexander equations describe an ideal in the algebra of Jacobi diagrams with weights vanishing for single single hook representations of \( sl_N \). We leave a more detailed analysis of this side of the problem for future studies.

### 3.2 Properties of the Alexander equations

Now let us explore some properties of the defined system (21).

**Statement 3.1.** The Casimir invariants as functions of the Young diagram \( R \) can be represented as the following shifted symmetric functions [28]:

\[
C_k(R) = \sum_{i=1} \left[ (R_i - i + 1/2)^k - (-i + 1/2)^k \right].
\]

(24)
Restricted to 1-hook diagrams $R = [r, 1, \ldots, 1]$, this formula reduces to:

$$C_k(R) = (r - 1/2)^k - (-L - 1/2)^k,$$

or, in a more symmetric form, with $l = l(R) = L + 1$ being the length of partition:

$$C_k(R) = (r - 1/2)^k + (-1)^k + 1 (l - 1/2)^k.$$

(25)

Corollary 3.1. As a corollary of the explicit definition (21), the symmetry with respect to transposition of the diagram reads

$$C_\Delta(R^T) = (-1)^{|\Delta| + l(\Delta)} C_\Delta(R). \quad R = [r, 1^L]$$

(27)

Proof. Simply notice, that the transposition of a single hook diagrams exchanges $r$ and $l$ introducing a factor of $(-1)^{l-1}$ for the $i$’th multiplier.

Generally one would think that solutions of the Alexander equations may contain even and odd polynomials. However, it appears that such solutions always factor into an even and odd part, which vanish separately.

Lemma 3.1. The graded space $\mathcal{A}$ of solutions of the Alexander equations decomposes into a direct sum of graded spaces of even and odd solutions:

$$\mathcal{A} = \mathcal{A}^e \oplus \mathcal{A}^o$$

(28)

Proof. Equation (21) should hold for any 1-hook diagrams $R$, therefore it should be invariant under a transposition of the diagrams. Now suppose we have some solution $X_n(C)$ to (21), which means that for any 1-hook Young diagram $R$:

$$X_n = \sum_{|\Delta| = n} \beta_\Delta C_\Delta(R) = 0$$

(29)

where some $\beta_\Delta$ may vanish. Under transposition of $R$ each monomial behaves according to (27). Since $|\Delta|$ is fixed after the transposition we get:

$$\sum_{|\Delta| = n} \beta_\Delta C_\Delta(R^T) = \sum_{|\Delta| = n} (-1)^{l(\Delta)} \beta_\Delta C_\Delta(R) = 0$$

(30)

Since $l(\Delta)$ is either odd or even, denote the even one’s as $\Delta_1$ and the odd ones as $\Delta_2$, and split the sum:

$$0 = \sum_{|\Delta_1| = n} \beta_{\Delta_1} C_{\Delta_1}(R) + \sum_{|\Delta_2| = n} \beta_{\Delta_2} C_{\Delta_2}(R) = \sum_{|\Delta_1| = n} \beta_{\Delta_1} C_{\Delta_1}(R) - \sum_{|\Delta_2| = n} \beta_{\Delta_2} C_{\Delta_2}(R) \Rightarrow$$

$$\Rightarrow X_n^e = \sum_{|\Delta_2| = n} \beta_{\Delta_2} C_{\Delta_2}(R) = 0, \quad X_n^o = \sum_{|\Delta_1| = n} \beta_{\Delta_1} C_{\Delta_1}(R) = 0$$

This means that any solution $X_n \in \mathcal{A}$ splits into the sum of $X_n^e \in \mathcal{A}^e$ and $X_n^o \in \mathcal{A}^o$ which vanish separately, therefore are solutions (21).

Some examples of the solutions to the system are:

$$h^4 : X_{1,1}^e(C) = C_1^4 - 4C_1C_3 + 3C_2^2;$$

$$h^5 : X_{5,1}^e(C) = 2C_2C_1^3 - 3C_2C_1 + 2C_2C_3,$$

$$X_{5,1}^o(C) = C_1 \left( C_1^4 - 4C_1C_3 + 3C_2^2 \right);$$

$$h^6 : X_{6,1}^e(C) = 4C_2^2 \left( C_1^4 - 4C_3C_1 + 3C_2^2 \right), \quad X_{6,1}^o(C) = C_2 \left( C_1^4 - 4C_1C_3 + 3C_2^2 \right);$$

$$X_{6,2}^e(C) = 2C_3C_1^3 - 3C_2C_1^2 - 8C_2^2 + 9C_2C_4,$$

$$X_{6,2}^o(C) = C_2 \left( C_1^4 - 4C_1C_3 + 3C_2^2 \right);$$

$$X_{6,3}^e(C) = C_3C_1^3 + 3C_2^2C_1^2 - 9C_2C_1 + 5C_2^3;$$

$$X_{6,3}^o(C) = 2C_2C_1^3 - 3C_2C_1^2 - 8C_2^2 + 9C_2C_4.$$

(31)

(32)

Theorem 3.1. The dimensions of the homogeneous components of $\mathcal{A}$ are given by:

$$\dim \mathcal{A}_n^e = p_e(n) - \left\lfloor \frac{n}{2} \right\rfloor, \quad l(\Delta) \in 2\mathbb{Z}$$

(33)

$$\dim \mathcal{A}_n^o = p_o(n) - \left\lfloor \frac{n + 1}{2} \right\rfloor, \quad l(\Delta) \in 2\mathbb{Z} + 1,$$

(34)

where $p_e(n)$ and $p_o(n)$ is the number of partitions of $n$ into even and odd number of integers respectively.
**Proof.** Denote \( x = r - 1/2, y = -L - 1/2 \), then the Casimir invariants \((26)\) take the form:

\[
C_k(R) = x^k - y^k
\]  

Equation \((21)\) should hold for all \( r, L \), therefore it should also hold for \( x, y \). Therefore it is equivalent to the vanishing of the coefficients of powers of \( x, y \). It may happen that not all of these coefficients are independent. Therefore our goal is to find the number of independent coefficients, which will give us the number of equations on the initial variables \( \xi_\Delta \).

First, let’s treat the case of the even solutions. The number of variables in the case is just \( p_e(n) \), the number of partitions of even length. Therefore monomials \( C_\Delta \) always contain at least two Casimir operators. The product of two Casimir operators expands as:

\[
C_{k_1}C_{k_2} = (x^{k_1} - y^{k_1})(x^{k_2} - y^{k_2})
\]  

This means that the whole expression is divisible by \( (x - y)^2 \), therefore we are left with a polynomial in \( x, y \) of order \( n - 2 \). Now, notice that even though \( C_k(R) \) is antisymmetric in \( x, y \) the even solution is symmetric.

Hence, we have:

\[
X_{n,m}(C) = (x - y)^2 \hat{X}_{n,m}
\]  

\( \hat{X}_{n,m} \) is also symmetric, therefore it belongs to the ring symmetric polynomials of order \( n - 2 \) in two variables \( \Lambda(x, y) \). Moreover, it lies in the homogeneous component \( \Lambda_{n-2}(x, y) \). To find the independent equations on \( \xi_\Delta \) we must a basis in this ring. One basis in the ring of symmetric functions is given by the complete symmetric polynomials:

\[
h_k(x, y) = \sum_{l_1 + l_2 = k} x^{l_1}y^{l_2}.
\]  

The basis in \( \Lambda_{n-2}(x, y) \) is given by the polynomials:

\[
h_\lambda(x, y) = h_{\lambda_1}h_{\lambda_2}, \quad \lambda_1 \geq \lambda_2 \geq 0, \quad \lambda_1 + \lambda_2 = n - 2
\]  

Now, the coefficients of each \( h_\lambda(x, y) \) give rise to independent equations on \( \xi_\Delta \). Hence, the dimension of the space in question is

\[
\dim \mathcal{A}_n = p_e - \dim \Lambda_{n-2}(x, y), \quad (40)
\]

with

\[
\dim \Lambda_{n-2}(x, y) = \left\lfloor \frac{n}{2} \right\rfloor
\]

The case of odd solutions is treated in the same manner. This time the shortest monomial is of length 1, therefore the equation is divisible by \( (x - y) \). Dividing an expression, antisymmetric in \( x, y \), by another antisymmetric expression we get once again a symmetric polynomial in \( x, y \) this time of degree \( n - 1 \). Carrying out a similar analysis once again, we obtain for the dimension of the space of odd solutions:

\[
\dim \mathcal{A}_n = p_o - \dim \Lambda_{n-1}(x, y), \quad (42)
\]

with

\[
\dim \Lambda_{n-1}(x, y) = \left\lfloor \frac{n + 1}{2} \right\rfloor
\]

**Remark.** The generating function for \( p_e(n) \) and \( p_o(n) \) are:

\[
\frac{1 + \varphi_4(0, q)}{2(q; q)_\infty} = \sum_{n=0}^{\infty} p_e(n)q^n
\]

\[
\frac{1 - \varphi_4(0, q)}{2(q; q)_\infty} = \sum_{n=0}^{\infty} p_o(n)q^n
\]
**Lemma 3.2.** The group factors in the expansion of the Alexander equation lie in a certain subspace of $\mathcal{A}l$:

$$A_{2n,m}(C) \in \bigoplus_{k=2}^{n} (A_{2k}^n \oplus A_{2k-1}^n) \subset$$  \hspace{1cm} (45)

$$A_{2n+1,m}(C) \in \bigoplus_{k=2}^{n+1} (A_{2k}^n \oplus A_{2k-1}^n) \subset \mathcal{A}l$$  \hspace{1cm} (46)

In other words in every order $n$ of $\mathcal{A}$, the terms of degree $k = n \mod 2$ consist of even monomials $C_{\Delta}$, and the terms of degree $k = n + 1 \mod 2$ consist of odd monomials.

**Proof.** Let us apply the rank-level duality to the expansion components. Since the expansion corresponds to letting $q = e^{b}$, for the group factor the property reads:

$$A_{n,m}(C(R^{T})) = (-1)^{n} A_{n,m}(C(R)).$$  \hspace{1cm} (47)

At the same due to (27), we acquire a factor of $(-1)^{|\Delta|+l(\Delta)}$ for each summand of order $|\Delta| = k$. For the two properties to be consistent we should have:

$$(-1)^{n} = (-1)^{k+l(\Delta)}.$$  \hspace{1cm} (48)

\[\square\]

Let us state another property, which motivates the later discussion:

**Lemma 3.3.** The odd solutions can be generated by multiplying the polynomials from $\mathcal{A}l^c$ by suitable odd monomials $C_{\Delta}$ with $l(\Delta)$ - odd.

We will return to the proof of this lemma after proving the main theorem in the next sections. However it’s useful to hold in mind, since it explains our concentration on the even solutions $\mathcal{A}l_{e}$ in the forthcoming section. For example:

$$X_{5,1}^{o}(C) = C_{1}X_{4,1}^{c}(C)$$

$$X_{6,1}^{o}(C) = \frac{1}{3}C_{2}X_{4,1}^{c}(C) - \frac{1}{3}C_{1}X_{5,1}^{c}(C)$$

$$X_{6,2}^{o}(C) = C_{2}X_{4,1}^{c}(C)$$  \hspace{1cm} (49)

### 3.3 Group factors of the Alexander polynomial

Let us illustrate how these lemmas work for the group factors of the Alexander polynomial:

$$A_{4,1}^{K}(C) = X_{4,1}^{c}(C)$$

$$A_{5,m}(C) = 0$$

$$A_{6,1}(C) = X_{4,1}^{c}(C), \quad A_{6,3}(C) = X_{6,1}^{c}(C) - \frac{5}{3}X_{6,2}^{c}(C) - \frac{2}{3}X_{6,3}^{c}(C),$$

$$A_{7,1}(C) = X_{6,1}^{c}(C)$$

$$A_{8,1}(C) = X_{6,1}^{c}(C), \quad A_{8,2}(C) = A_{6,3}(C)$$

$$A_{8,3}(C) = C_{1}^{2}X_{4,1}^{c}(C) + 7X_{8,6}^{c}(C) - 7X_{8,7}^{c}(C) + X_{8,8}^{c}(C), \quad A_{8,4}(C) = C_{1}^{2}X_{4,1}^{c}(C)$$  \hspace{1cm} (50)
with the solutions of the Alexander equations at order 8 being:

\[
X_{k,1}^x = -C_2^2 \left( C_1^4 - 4C_3 C_1 + 3C_2^2 \right), \quad X_{k,2}^x = - \left( C_1^4 - 4C_3 C_1 + 3C_2^2 \right)(3C_2^2 + 4C_1 C_3), \\
X_{k,3}^x = -C_2 \left( C_2 C_1^4 - 2C_4 C_1^2 + C_2^4 \right), \quad X_{k,4}^x = -4C_3 C_1^5 - 7C_2^2 C_1 + 16C_5 C_1^3 - 5C_2^4.
\]

\[
X_{k,5}^x = 5C_1^4 \left( C_1^4 - 4C_3 C_1 + 3C_2^2 \right), \quad X_{k,6}^x = (4C_3 C_1^5 - 5C_2^2 C_1^4 + 5C_2^4 + 60C_4^3 - 64C_3 C_5), \\
X_{k,7}^x = (C_2^4 - 4C_6 C_2 + 3C_2^4), \quad X_{k,8}^x = 4C_3 C_1^5 + 11C_2^2 C_1^4 - 64C_7 C_1 + 21C_2^4 + 28C_2^4.
\]

We notice that there are less independent group factors at each order, than there are solutions of the Alexander equations. Obviously, we have a restriction given by lemma 3.2. However, there are certainly more: for example the components \( A_{l,2k-1} \) and \( A_{l,2k-1}^c \) from [15, 16] are absent. Moreover, even from what is left not all of the solutions enter independently. This hints that there are more symmetries of the Alexander polynomial, that would exactly fix its group factors.

### 3.4 Supersymmetric polynomials

The formula (35) can be extended to generic Young diagrams by parametrize them in terms of hooks, i.e. by the diagonal boxes. In the same manner one can introduce the variables \( x_i = r_i - 1/2, \ y_i = -L_i - 1/2 \) for the \( i \)-th hook. Then the Casimir invariants read:

\[
C_k(x_1, \ldots, y_1, \ldots) = \sum_{i=1}^n (x_i^k - y_i^k) \quad (52)
\]

This expression is symmetric in the shifted variables \( x_i, y_i \) separately. Formally this means that the Casimir invariants are supersymmetric in the variables \( x, y \).

**Definition 3.2.** A polynomial \( P(x, y) \) is supersymmetric if it is invariant under separate permutations of \( x \)-s and \( y \)-s and for \( x_i = y_i = z \) doesn’t depend on \( z \).

As in the usual case one can define the ring of supersymmetric polynomials and look for various basis in it. One of those is given by the supersymmetric generalization of power sums [35]:

\[
\Psi(x_1, \ldots, y_1, \ldots) = \sum_{i=1}^n x_i^k - \sum_{j=1}^m y_j^k \quad (53)
\]

Therefore the Casimir invariants are special cases of these supersymmetric power sums. In these terms we can give another formulation of our problem: \( A_{l,n} \) is an ideal in the ring of supersymmetric functions that vanishes when all but one pair of variables are set equal. The connection of supersymmetric polynomials and the KP hierarchy was studied in [30].

**Remark.** The supersymmetric polynomials are usually discussed in the context of Lie superalgebras, namely the space generated by Casimir invariant of \( gl_{n|m} \) isomorphic to the space of supersymmetric \((n, m)\) polynomials [37].

### 4 The KP hierarchy

Now we need a few standard results about the KP hierarchy [32, 33].

**Definition 4.1.** The operator

\[
D^n : (f, g) \rightarrow (D^n f \cdot g) \quad (54)
\]

defined by

\[
f(x + y)g(x - y) = \sum_{n=0}^\infty \frac{1}{n!} (D^n f \cdot g) y^n
\]

is called a Hirota derivative.

The multivariate Hirota derivative is defined in a similar fashion via the multivariate Taylor expansion:

\[
f(x + y)g(x - y) = \sum_{l_1, \ldots, l_m} \left( D^{l_1} \cdots D^{l_m} f \cdot g \right) y_1^{l_1} y_2^{l_2} \cdots y_m^{l_m} \quad (56)
\]
Here we notice an important point. If \( g = f \), then odd powers of Hirota derivatives act trivially for any \( f \). The KP Hierarchy is defined by the following bilinear identity:

\[
\int \frac{dz}{2\pi i} e^{zT} \prod_i \frac{1}{e^{T_i} - e^{-T_i} + \epsilon_i} \cdot \tau = 0,
\]

where \( D_{T_i} \) are the Hirota derivatives, \( \tilde{D} \equiv (D_{T_1}, \frac{1}{2}D_{T_2}, \ldots) \), and \( x = \{x_1, x_2, \ldots\} \) are formal parameters. Consequently, it can be shown that the KP hierarchy is defined by the following generating function:

\[
\sum_{i=0}^{\infty} \chi_i(-2t)\chi_{i+1}(\tilde{D}t)e^{[\sum_j t_jD_{T_j}]} \cdot \tau = 0,
\]

where \( \chi_i \) are the Schur functions in symmetric representations. The KP equations in the Hirota form appear as coefficients of expansion of \( (58) \) in \( t_i \) in a polynomial form \( P_n(D_1, D_2, \ldots, D_{n-1}) \). The first few equations are:

- The equation of order 4, which is the KP equation itself:

\[
(4D_1D_2 - 3D_2^2 - D_1^3)\tau \cdot \tau = 0
\]

- Higher KP equations:

\[
\begin{align*}
P_3 &= 3D_1D_4 - 2D_2D_3 - D_1^3D_2; \\
P_{6,1} &= D_1^5 (D_1^4 - 4D_3D_1 + 3D_2^2), \\
P_{6,2} &= D_1^5 - 20D_3D_1^3 - 45D_2^2D_1^2 + 144D_3D_1 - 80D_2^3, \\
P_{6,3} &= D_1^5 + 10D_3D_1^3 - 36D_3D_1 - 20D_2^3 + 45D_2D_4;
\end{align*}
\]

The equations of the hierarchy follow from expanding the polynomial. It appears that some of these equations are not independent. Moreover some of the differ only by terms odd in powers of Hirota operators. Therefore we propose to look at a certain set of equations which do not contain operators that act trivially for any \( f \). It’s naturally graded by the power of the polynomial \( P^e(x) = \bigoplus_n P_n^e(x) \).

We want to establish a connection of the group factors with soliton \( \tau \)-functions. Let us remind how the soliton \( \tau \)-functions of the KP hierarchy are constructed. One considers a Hirota equation:

\[
P(D_1, D_2, \ldots)\tau \cdot \tau = 0
\]

Look for a solution as a formal series:

\[
\tau = 1 + \epsilon f_1 + \epsilon^2 f_2 + \ldots
\]

In the first order one gets the following dispersion relation for \( f_1 = e^{\sum_j k_j t_j} \):

\[
\frac{1}{2}(P(k_1, k_2, \ldots) + P(-k_1, -k_2, \ldots)) = 0
\]

Where the symmetrization appears as a consequence of the trivial action of odd Hirota derivatives. The action of the KP hierarchy on the soliton \( \tau \)-function produces a set of dispersion relations. As these are just polynomials in the momentum variables we are free to generate an ideal in the ring of even polynomial by these dispersion relations. Denote the ring of even polynomials in variables \( x_1, x_2, \ldots \) with integer coefficients by \( P^e(x) \). It’s naturally graded by the power of the polynomial \( P^e(x) = \bigoplus_n P_n^e(x) \).

**Definition 4.2.** Define the Hirota ideal \( \tilde{KP} \subset P^e(k) \) as the ring of polynomial generated by the dispersion relations of the KP hierarchy

\[
\tilde{KP} = \text{span} \{ \text{Dispersion relations of the soliton } \tau \text{-function} \}
\]

Different order KP equations produce different order dispersion relation, i.e. the Hirota ideal is naturally graded: \( \tilde{KP} = \bigoplus_n \tilde{KP}_n \) As it was mentioned, \( (57) \) generates all Hirota equations, hence it also generates the dispersion
relations when $\tau$ is specified to be of the form $\text{[63]}$. Therefore, the generating function of the dispersion relations looks like:

$$
\oint \frac{dz}{2\pi i} e^{2\sum z^i t^i e^{\sum_1^\infty k_i e^{\sum t_i k_i}}} + \oint \frac{dz}{2\pi i} e^{2\sum z^i t^i e^{\sum_1^\infty k_i e^{\sum t_i k_i}}}.
$$

(66)

While having a generating function, there is no convenient way to enumerate all of the dispersion relations in such way that their linear independence would be evident. However we can find a subset, which has this property.

For $Y = [i, j], |Y| = i + j, i \geq j \geq 2$ define the polynomials:

$$
h_Y(k) = \begin{vmatrix}
\chi_i(-\hat{k}/2) & \chi_i(\hat{k}/2) & \chi_{i+1}(\hat{k}/2) \\
\chi_{j-1}(\hat{k}/2) & \chi_{j-1}(\hat{k}/2) & \chi_{j}(\hat{k}/2) \\
1 & 1 & \frac{1}{2}k_1
\end{vmatrix} + \begin{vmatrix}
\chi_i(\hat{k}/2) & \chi_i(-\hat{k}/2) & \chi_{i+1}(\hat{k}/2) \\
\chi_{j-1}(\hat{k}/2) & \chi_{j-1}(\hat{k}/2) & \chi_{j}(\hat{k}/2) \\
1 & 1 & \frac{1}{2}k_1
\end{vmatrix}.
$$

(67)

**Proposition 4.1.** The polynomials (67) are some of the dispersion relations of the KP hierarchy and are linearly independent.

**Proof.** By [63] the equations:

$$
\begin{vmatrix}
\chi_i(-\hat{D}/2) & \chi_i(\hat{D}/2) & \chi_{i+1}(\hat{D}/2) \\
\chi_{j-1}(\hat{D}/2) & \chi_{j-1}(\hat{D}/2) & \chi_{j}(\hat{D}/2) \\
1 & 1 & \frac{1}{2}D_1
\end{vmatrix} \tau \cdot \tau = 0
$$

(68)

are a subset of the equations of the hierarchy. Hence as in (64) the polynomials (67) form dispersion relations. To see their linear independence one expands the first terms of Schur polynomials and computes the determinant. It appears that:

$$
h_Y(k) = \frac{k_i k_j}{ij} + \sum_{q=2}^{\infty} \sum_{r_1 + \cdots + r_2 = |Y|} P_{r_1 \cdots r_2}^{ijk} k_{r_1} \cdots k_{r_2q}
$$

(69)

These polynomials are uniquely defined by their second order terms and are linearly independent. \(\square\)

Let us denote the ideal generated by (67) by $\mathcal{KP} = \bigoplus_n \mathcal{KP}_n$. Then we have the obvious inclusion:

$$
\mathcal{KP} \subset \widehat{\mathcal{KP}}
$$

(70)

### 5 Main result

This section will be devoted to the announced proof of our statement given in [1] and discussed above. It consists of two parts, which are formulated as separate lemmas: the first basically repeats the simple calculation in [1] and the second is a combinatorial construction, that completes the proof.

**Theorem 5.1.** The Hirota ideal and the even part of the ring of solutions of the Alexander equation are in fact the same graded ideal, expressed in different variables.

**Remark.** The single hook property of the Alexander polynomial is equivalent to the system of dispersion equations for the KP one soliton $\tau$-function.

**Remark.** As mentioned in lemma 5.3 the odd part of the ring of solutions of the Alexander equations is generated by the even part (see Corollary 5.1)

**Proof.**

**Lemma 5.1.** Any generator of $\widehat{\mathcal{KP}}$ satisfies the of the Alexander equation. In other words the Hirota ideal is an ideal in the ring of even solutions of the Alexander system:

$$
\widehat{\mathcal{KP}} \subset \mathcal{A}\ell^e
$$

(71)
Proof of Lemma 5.1. The Hirota equations are fully specified the bilinear identity (recall, that it gives rise to the generating function). To prove that every polynomial in the Hirota ideal is an even solution of the Alexander equations substitute \( D_k \) with \( C_j(R) \) in the symmetrized bilinear identity (57). The resulting integral:
\[
\oint \frac{dz}{2\pi i} e^{2\sum z^i t_i e^{\sum \frac{1}{\lambda_i} C_i e^{\sum \lambda_i C_i}}} + \oint \frac{dz}{2\pi i} e^{2\sum z^i t_i e^{\sum \frac{1}{\lambda_i} C_i e^{\sum \lambda_i C_i}}} = 0.
\]
(72)
vanzishes when \( R \) is a 1-hook diagram. Indeed, the value of the Casimir invariant on 1-hook diagrams is given by (35). Simplify one of the exponential in the integrand:
\[
e^{-\sum t_i (x^i + y^i)} e^{\sum \lambda_i C_i} = e^{-\sum t_i (x^i + y^i)} (x^i + y^i) = \frac{z - x}{z - y}.
\]
(73)
while the same exponential in the second integral is the reciprocal of this expression. This allows us to compute the integrals taking the residues at \( z_1 = x, z_2 = y \). Therefore, restricted to 1-hook diagrams, (72) vanishes:
\[
(y - x)e^{-\sum t_i (x^i + y^i)} + (x - y)e^{-\sum t_i (x^i + y^i)} = 0.
\]
(74)
We see that any polynomial arising from the bilinear identity vanishes, when Hirota derivatives are replaced with 1-hook values of Casimir invariants (24). This proves the lemma.

One may notice that this calculation resembles the general proof of the bilinear identity [32]. What basically happens here is that \( k_j = C_j(R) \) for any single hook diagram \( R \) solves the dispersion relation. Therefore the values of the Casimir operators on single hook diagrams are the momentum variables in the soliton \( \tau \)-function of the KP hierarchy. To obtain an \( n \)-soliton \( \tau \)-function we need to have \( n \) different solutions to the dispersion relation, which in the terms of Casimir invariants would just mean taking \( n \) different hook diagrams:
\[
\tau(t_1, t_2, \ldots) = \sum_{H' \subset H_n} c_{H'} \exp \left( \sum_{R \in H'} \xi(R) \right)
\]
(75)
where \( H \) is a set of exactly \( n \) single hook diagrams, \( c_{H'} \) is a constant, the sum runs over all subsets of \( H \) and
\[
\xi(R) = \sum_{k=1}^{\infty} C_k(R) t_k
\]
(76)
Remark. In these terms the reduction to the KdV \( \tau \)-function look quite natural, mainly it means taking hook diagrams that are symmetric with respect to transposition.

Now since we have that every polynomial (dispersion relation) from the Hirota ideal satisfies the Alexander system of equations we will prove by comparing dimensions that they are equal. Lemma 5.1 obviously guarantees, that each for homogenous components \( \tilde{K}\mathcal{P}_n \subset A\ell^e \).

Lemma 5.2. The vectors space dimension of the homogeneous components of \( K\mathcal{P} \) are given by (33).

Hence by this lemma we will have the following chain of inclusions:
\[
\tilde{K}\mathcal{P} \hookrightarrow K\mathcal{P} \hookrightarrow A\ell^e
\]
We see that all of the inclusions are in fact equalities.

Proof of Lemma 5.2. First, let us interpret the quantities (33). Let us call the Young diagrams having \( n \) diagonal boxes \( n \) hook diagrams in concordance with one hook diagrams. Then (33) is the number of Young diagrams with more than 2 hooks and of even length. We will denote this set of diagrams as \( \mathcal{Y}_e \).

Now, as one can notice the polynomials, that generate the Hirota ideal (67), are naturally numbered by 2-hook diagrams with two rows \( \mathcal{Y}_2 \), which is nice since it’s a subset of \( \mathcal{Y}_e \).
Solutions of Alexander equation

Diagrams with more than 2-hooks:

Young diagrams with 2 rows and 2 hooks:

\[ Y_2 \subset Y \]

First of all let us perform a triangular transformation in \( KP_n \), to get for \( Y = [y_1, y_2] \):

\[
h_Y(k) = \frac{1}{y_1y_2}k_{y_1}k_{y_2} - \frac{1}{|Y| - 1}k_{|Y|-1}k_1 + \ldots \tag{77}
\]

We abuse that notation and use \( h \) for these new polynomials. Now the first term reflects the Young diagram associated to the polynomial, while the second only depends on the order. From now on we are not interested in carrying the coefficients since they exactly correspond to the monomial. We will symbolically denote the polynomials as follows:

\[
h_Y(k) = [y_1, y_2] - [|Y| - 1, 1] \tag{78}
\]

Any element from \( \mathcal{A}_{n} \) is represented by a Young diagram. However, contrary to the \( KP_n \) case, there is no direct correspondence yet, i.e. we can’t tell the structure of a polynomial, which is represented by a diagram \( T \). Take some element from \( \mathcal{A}_{n} \) and the corresponding diagram \( T \) and cut it into two halves: the first two rows and everything else.

\[
T = [Y, Y'], \quad T \in Y_e, \quad Y \in Y_2 \tag{79}
\]

Since \( T \) is at least a 2-hook diagram \( Y \) is also a 2-hook diagram. Therefore to \( Y \) we can assign a polynomial \( h_Y(k) \) and to \( Y' \) a monomial \( k_{Y'} \). Vice versa, each \( T \) can be constructed by putting together two partitions \( Y \) and \( Y' \) subject to necessary conditions. That way we can assign a polynomial to the resulting diagrams \( T \):

\[
\hat{h}_T(k) = h_Y(k)k_{Y'} \tag{80}
\]

Let us illustrate the procedure. Take \( Y \) - a partition of \( n - 2 \), say \( Y = [n - 4, 2] \), then the only possibility to form a partition of \( n \) of even length is to add a \( Y' = [1, 1] \). This way we construct:

\[
\hat{h}_{[n-4,2,1,1]} = h_{[n-4,2]}k_1^2 \tag{81}
\]

We want to inductively build the polynomials corresponding to the diagrams following the steps above. Our concern is the linear independence of the resulting polynomials, since our initial goal was counting the dimensions of the Hirota ideal. Since we know the generating set \( KP \), start with taking a 2 row 2 hook partition \( Y \). The first non-trivial case is \( |Y| = n - 2 \), which we have already described. The case \( |Y| = n - 3 \) is treated exactly in the same manner, however, let us right it down for clearance. The partitions \( Y \) have the form:

\[
[n - 5, 2] - [n - 4, 1]
\]

\[
[n - 5 - i, 2 + i] - [n - 4, 1]
\]
We can only multiply by a monomial $k_2k_1$, denoted by $[2, 1]$. Therefore, by combining these partitions we get the general form:

$$
\hat{h}_{[n-5-i,2+i,2,1]} = [n-5-i, 2+i, 2, 1] - [n-4, 2, 1, 1]
$$

(82)

Now on the next step additional actions are needed. A general polynomial in $\mathcal{Y}_e$ for $|Y| = n - 4$ will have a form:

$$
[n-6-i, 2+i] - [n-5, 1]
$$

(83)

Here we have two diagrams, that we can attach to the bottom:

$$
Y' = [3, 1] \leftrightarrow k_3k_1, \quad Y' = [1, 1, 1, 1] \leftrightarrow k_4^1.
$$

(84)

While following our construction we encounter a polynomial:

$$
[n-6, 3, 2, 1] - [n-5, 3, 1, 1],
$$

(85)

which we get by attaching $[3, 1]$ to $[n-6, 2] - [n-5, 1]$. We notice that the first summand has actually already appeared in (82), for $i = 1$. Even though the second terms are different this hints us that the polynomials built on this step might not be linearly independent with the ones constructed in the first step.

We want to construct a set of polynomial which are explicitly independent.

To do this, notice that we can find a polynomial $\hat{h}_T$ for $|Y| = n - 2$, whose first term will be $[n-4, 2, 1, 1]$ and one with $[n-5, 3, 1, 1]$. These are the polynomials $\hat{h}_{[n-4,2,1,1]}$ and $\hat{h}_{[n-5,3,1,1]}$. This leads to a linear transformation and a redefinition:

For $|Y| = n - 3$:

$$
\hat{h}_{[n-5-i,2+i,2,1]} \rightarrow \hat{h}_{[n-5-i,2+i,2,1]} + \hat{h}_{[n-4,2,1,1]} = [n-5-i, 2+i, 2, 1] - [n-3, 1, 1, 1]
$$

(86)

For $|Y| = n - 4$:

$$
\hat{h}_{[n-6-i,2+i,2,1]} \rightarrow \hat{h}_{[n-6-i,2+i,2,1]} + \hat{h}_{[n-5,3,1,1]} = [n-6-i, 2+i, 2, 1] - [n-3, 1, 1, 1]
$$

(87)

This transformation makes the constructed polynomials explicitly independent since the all have different monomials as their first term.

Let us prove the general statement.

**Proposition 5.1.** For all $T$ - $k$-hook Young diagrams, such that $|T| = n$, with $l(T)$ - even and $k > 1$, there exists such transformation of $\hat{h}_T$ such that:

$$
\hat{h}_T = h_Yk_Y, \quad T = [Y, Y']
$$

(88)

are linear independent.

**Proof.** Along the lines of the given examples we want to prove that for any given $Y \vdash n-k$ and $Y' \vdash k$ such that $T = [Y, Y']$ is a partition too, the corresponding polynomial $\hat{h}_T = h_Yk_Y$, is defined only by the first summand, which is precisely $k_T$.

Let us start by noticing that in case, when $Y'$ have different lengths, the resulting polynomials will for sure be linearly independent. Therefore we can consider the cases of $l(T) = 4, 6, 8, \ldots$ separately.

1) Let’s first construct the case $l(T) = 4 \leftrightarrow l(Y') = 2$. Denote $Y' = [b_1, b_2], b_1 \geq b_2 \geq 1$, then

$$
\hat{h}_T = [n-k-2-i, 2+i, b_1, b_2] - [n-k-1, b_1, b_2, 1], \quad \text{if } 2+i \geq b_1
$$

(89)

We have demonstrated the cases $k = 2, k = 3$ before. Now suppose all polynomials for $k < k_0$ are brought in the form:

$$
\hat{h}_T = [T] - [n-3, 1, 1, 1]
$$

(90)

Then to do the same for $k = k_0$ we must find the term $Y = [n-k_0-1, b_1, b_2, 1]$ in the already constructed polynomials simply by solving a simple equation

$$
n-k-2-i = n-k_0-1, \quad b_1 = 2+i.
$$

(91)
Therefore we find the needed term for \( i = b_1 - 2 \geq 0 \) and \( k = k_0 - b_1 + 1 \), with \( k_0 > k \geq 2 \) since \( b_1 \leq k_0 - 1 \). Finally we set the corresponding \( b'_1 = b_2, b'_2 = 1 \). That way by induction we get all the polynomials in the desired form and their independence becomes explicit. The inequalities guarantees the existence of such term.

Let us illustrate the last step by another example. Take, for instance, \( k_0 = 5 \) and \( Y' = [3,2] \) and assume for \( k < 5 \) everything is sorted out. We begin with:

\[
\tilde{h}_T = [T] - [n - 6,3,2,1]
\]

We have \( i = 1, k = 3, b'_1 = 2, b'_2 = 1 \). This means we have to look for a polynomial of the form:

\[
\tilde{h}_T = [n - 3 - 2 - 1,2 + 1,2,1] - [n - 3,1,1,1] = [n - 6,3,2,1] - [n - 3,1,1,1],
\]

which we should have built in the previous steps of the induction.

2) The cases \( l(Y') > 2 \) are treated in the same way. Mainly for each such case \( l(Y') = l \) we will have a minimal \( k = t \), where such \( |Y'| = k \) is possible. The corresponding monomial will be \( k_{Y', k} = k'_l \). A general polynomial will look like:

\[
h_T = [n - k - 2 - i,2 + i,b_1,\ldots,b_l] - [n - k - 1,b_1,\ldots,b_l,1]
\]

We want to bring them all in the form:

\[
h_T = [n - k - 2 - i,2 + i,b_1,\ldots,b_l] - [n - k - 1,1^l,1] = [T] - [n - k - 1,1^l,1]
\]

Again at each level \( k_0 \) we want to cancel the second term by a first term of some polynomial with \( k < k_0 \). Then by induction we can bring all the polynomials in the desired form. Let us see how this is realized.

At level \( k_0 \) we want to cancel the term:

\[
[n - k_0 - 1,b_1,\ldots,b_l,1] \text{ by } [n - k - 2 - i,2 + i,b'_1,\ldots,b'_l]
\]

We see that we need to solve the same equations \((91)\) to see that such cancellation is indeed possible.

\[\square\]

As announced lemma \((3.3)\) is a corollary of the theorem.

**Corollary 5.1.** The space of odd solutions \( AY^n \) is generated by even solutions of lower orders.

**Proof.** We have naturally labelled all the even solutions of the Alexander equations by Young diagrams of even length having more than 1 hook - \( Y_e \). A solution, labelled by \( T \in Y_e \) contains a unique term \( C_T \). Now, let’s interpret the quantity \((3.3)\) for the odd case. Again it is exactly the number of diagrams of odd length and having more than one hook - \( Y_o \).

This means we may use the same idea. Take any diagram \( T \in Y_o \), and cut the lowest row \( T_{l(T)} \), this way we get a diagram \( T' \) from \( Y_o \). This means \( h_{T'}(C) \) is a even solution of the Alexander equations. Hence, the product:

\[
h_T(C) = h_{T'}(C)C_{T_{l(T)}}
\]

is a odd solution to the Alexander system. Of course all this may be done with any diagram from \( Y_o \). Moreover, these odd solutions are also independent since they have a unique term, corresponding to the diagram, they are labelled by.

\[
T = \begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
= T'
\]

\[
\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
= T_{l(T)}
\]

\[\square\]

The fact that we only need to study the even part of the solutions is in tact with the triviality of the odd Hirota equations. Hence really there is a correspondence between the whole space \( AY \) and the dispersion relations of the soliton \( \tau \)-functions.
6 Discussion

In this paper we have proved and elaborated on the statement made in [1]. The theorem connects the solutions of the Alexander equations with the dispersion relations of them. The special form of the group factor of the Alexander equations is the dispersion relation of the KP soliton. In these terms the property of the polynomial holds because the special form of the Casimir invariants is exactly the one needed to solve these relations. From our point of view, this idea brings a clearer understanding to the initially mysterious coincidence between the Alexander polynomial group factors and the KP Hirota equations/τ-functions.

Our results open various new perspectives:

- Does the property of the colored Alexander polynomial generalize in some way to diagrams with two and more hooks? If it does it should reflect on the KP side of the story and modify the construction.

- Let us consider an n-soliton τ-function. This would mean finding n different solutions to the dispersion relation, i.e. considering several different values of the Casimir invariants. This might be a way to invariants of links.

- As mentioned not all of the solutions of the Alexander equations enter the Alexander polynomial itself. Therefore there should be a set of symmetry conditions, which sort out some of the solutions (See section 3.3).

- The supersymmetric nature of these solutions may also open an interesting way of looking at the Alexander polynomial.

- We found that the two limits of the HOMFLY polynomials are connected to two types of τ-functions of the KP hierarchy and, hence, we are closer to understating the integrable properties of the HOMFLY polynomial for generic variables.

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