ON THE KERNEL OF THE \((\kappa,a)\)-GENERALIZED FOURIER TRANSFORM

D. V. GORBACHEV, V. I. IVANOV, AND S. YU. TIKHONOV

Abstract. For the kernel \(B_{\kappa,a}(x,y)\) of the \((\kappa,a)\)-generalized Fourier transform \(F_{\kappa,a}\), acting in \(L^2(\mathbb{R}^d)\) with the weight \(|x|^{a-2}v_{\kappa}(x)\), where \(v_{\kappa}\) is the Dunkl weight, we study the important question of when \(\|B_{\kappa,a}\|_\infty = B_{\kappa,a}(0,0) = 1\). The positive answer was known for \(d \geq 2\) and \(2 \frac{a}{\alpha} \in \mathbb{N}\). We investigate the case \(d = 1\) and \(2 \frac{a}{\alpha} \in \mathbb{N}\). Moreover, we give sufficient conditions on parameters for \(\|B_{\kappa,a}\|_\infty > 1\) to hold with \(d \geq 1\) and any \(a\).

We also study the image of the Schwartz space under the \(F_{\kappa,a}\) transform. In particular, we obtain that \(F_{\kappa,a}(\mathcal{S}(\mathbb{R}^d)) = \mathcal{S}(\mathbb{R}^d)\) only if \(a = 2\). Finally, extending the Dunkl transform, we introduce non-deformed transforms generated by \(F_{\kappa,a}\) and study their main properties.

1. Introduction

Let as usual \(\Delta\) be the Laplacian operator in \(\mathbb{R}^d\). For the Fourier transform

\[
\mathcal{F}(f)(y) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x)e^{-i\langle x,y \rangle} \, dx
\]

Howe [16] obtained the following spectral decomposition of \(\mathcal{F}\) using the harmonic oscillator \(-\Delta - |x|^2)/2\) and its eigenfunctions forming the basis in \(L^2(\mathbb{R}^d)\):

\[
\mathcal{F} = \exp\left(\frac{i\pi d}{4}\right) \exp\left(\frac{i\pi}{4}(\Delta - |x|^2)\right).
\]

Among other applications, this decomposition is useful to define the fractional power of Fourier transform; see [4] [19].

During last 30 years, a lot of attention has been given to various generalizations of the Fourier transform. As an important example, to develop harmonic analysis on weighted spaces, the Dunkl transform was introduced in [12]. The Dunkl transform \(F_{\kappa}\) is defined with the help of a root system \(\Omega \subset \mathbb{R}^d\), a reflection group \(G \subset O(d)\), and multiplicity function \(\kappa: \Omega \to \mathbb{R}_+\) such that \(\kappa\) is \(G\)-invariant. Here \(G\) is generated by reflections \(\{\sigma_\alpha: \alpha \in \Omega\}\), where \(\sigma_\alpha\) is a reflection with respect to hyperplane \((\alpha,x) = 0\).

The differential-difference Dunkl Laplacian operator \(\Delta_{\kappa}\) plays the role of the classical Laplacian [20]. If \(\kappa \equiv 0\), we have \(\Delta_{\kappa} = \Delta\). Dunkl Laplacian allows us to define the Dunkl harmonic oscillator \(\Delta_{\kappa} - |x|^2\) and the Dunkl transform

\[
\mathcal{F}_{\kappa} = \exp\left(\frac{i\pi}{2} \left(\frac{d}{2} + \langle \kappa \rangle\right)\right) \exp\left(\frac{i\pi}{4}(\Delta_{\kappa} - |x|^2)\right),
\]

where \(\langle \kappa \rangle = \frac{1}{2} \sum_{\alpha \in \Omega} \kappa(\alpha)\).

\textbf{Date:} November 17, 2022.
\textbf{2020 Mathematics Subject Classification.} 42B10, 33C45, 33C52.
\textbf{Key words and phrases.} \((\kappa,a)\)-generalized Fourier transform, Dunkl transform, Schwartz space, positive definiteness, unitary transform.

The research of D. Gorbachev and V. Ivanov was performed by a grant of RScF (project 18-11-00199), https://rscf.ru/project/18-11-00199. S. Tikhonov was partially supported by PID2020-114948GB-I00, 2017 SGR 358, the CERCA Programme of the Generalitat de Catalunya, Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M), and Ministry of Education and Science of the Republic of Kazakhstan (AP08856479).
Further extensions of Fourier and Dunkl transforms were obtained by Ben Saïd, Kobayashi, and Ørsted in [4]. They defined the \( a\)-deformed Dunkl harmonic oscillator
\[
\Delta_{\kappa,a} = |x|^{2-a}\Delta_{\kappa} - |x|^a, \quad a > 0,
\]
and the \((\kappa,a)\)-generalized Fourier transform
\[
\mathcal{F}_{\kappa,a} = \exp\left(\frac{i\pi}{2}(\lambda_{\kappa,a} + 1)\right)\exp\left(\frac{i\pi}{2a}\Delta_{\kappa,a}\right),
\]
which is a two-parameter family of unitary operators in \( L^2(D) \). V. Gorbachev, V. I. Ivanov, and S. Yu. Tikhonov

Throughout the paper, we assume that
d \( d \)-deformed Dunkl harmonic oscillator
\[
F_{\kappa,a}(\kappa,a) \equiv \frac{2\lambda_{\kappa}}{a}, \quad \lambda_{\kappa} = \langle \kappa \rangle + \frac{d - 2}{2}, \quad d_{\mu_{\kappa,a}}(x) = c_{\kappa,a}v_{\kappa,a}(x)dx, \quad v_{\kappa,a}(x) = |x|^a - |x|^a v_{\kappa}(x),
\]

Here
\[
\lambda_{\kappa,a} = \frac{2\lambda_{\kappa}}{a}, \quad \lambda_{\kappa} = \langle \kappa \rangle + \frac{d - 2}{2}, \quad d_{\mu_{\kappa,a}}(x) = c_{\kappa,a}v_{\kappa,a}(x)dx, \quad v_{\kappa,a}(x) = |x|^a - |x|^a v_{\kappa}(x),
\]

Throughout the paper, we assume that \( d + 2\langle \kappa \rangle + a - 2 = 2\lambda_{\kappa} + a > 0 \) or, equivalently, \( \lambda_{\kappa,a} > -1 \). Note that under this condition the weight function \( v_{\kappa,a} \) is locally integrable.

For \( a = 2 \), (11) reduces to the Dunkl transform, while if \( a = 2 \) and \( \kappa \equiv 0 \), then (11) is the classical Fourier transform. For \( a \neq 2 \), we arrive at deformed Dunkl and Fourier transforms, which have various applications. In particular, for \( a = 1 \) and \( \kappa \equiv 0 \) deformed Dunkl transform is the unitary inversion operator of the Schrödinger model of minimal representation of the group \( O(N + 1,2) \) [19].

The unitary operator \( \mathcal{F}_{\kappa,a} \) on \( L^2(D) \) can be written as the integral transform [4, (5.8)]
\[
\mathcal{F}_{\kappa,a}(f)(y) = \int_\mathbb{R}^d B_{\kappa,a}(x,y) f(x) d_{\mu_{\kappa,a}}(x)
\]
with the continuous symmetric kernel \( B_{\kappa,a}(x,y) \) satisfying \( B_{\kappa,a}(0,y) = 1 \). In particular,
\[
B_{0,2}(x,y) = e^{-i(x,y)}.
\]

One of the fundamental questions in the theory of deformed transforms is to investigate basic properties of the kernel \( B_{\kappa,a}(x,y) \), in particular, to know when it is uniformly bounded. To illustrate the importance of this property, note that the condition \( |B_{\kappa,a}(x,y)| \leq M \) implies the Hausdorff-Young inequality
\[
\|\mathcal{F}_{\kappa,a}(f)\|_{p',d_{\mu_{\kappa,a}}} \leq M^{2/p-1}\|f\|_{p,d_{\mu_{\kappa,a}}}, \quad 1 \leq p \leq 2, \quad \frac{1}{p} + \frac{1}{p'} = 1.
\]

A more important problem is to describe parameters so that there holds
\[
\|B_{\kappa,a}\|_{\infty} = \sup_{x,y \in \mathbb{R}^d} |B_{\kappa,a}(x,y)| = B_{\kappa,a}(0,0) = 1. \tag{2}
\]

In this case the Hausdorff-Young inequality holds with the constant 1 and one can define the generalized translation operator \( \tau^y f(x) \) in \( L^2(D) \) by
\[
\mathcal{F}_{\kappa,a}(\tau^y f)(z) = B_{\kappa,a}(y,z)\mathcal{F}_{\kappa,a}(f)(z)
\]
(see [15]), and moreover, its norm equals 1.

Let us list the known cases when (2) holds:
- for \( a = 2 \) [22];
- for \( a = 1 \) and either \( d = 1, \langle \kappa \rangle \geq 1/2 \) or \( d \geq 2, \langle \kappa \rangle \geq 0 \) [4, Propositions 5.10, 5.11], [15] Sect. 6];
• for $\frac{2}{a} \in \mathbb{N}$ and $d \geq 2$, $\langle \kappa \rangle \geq 0$ \cite{10}.

In this paper we continue to study the case

$$\frac{2}{a} \in \mathbb{N}.$$ 

Its importance was discussed in \cite{4, 7}, and \cite{10}. For $a = 2$, $\mathcal{F}_{\kappa,a}$ reduces to the Dunkl transform and (2) is valid.

Our first goal in this paper is, on the one hand, to extend the list of parameters for which (2) holds for $\frac{2}{a} \in \mathbb{N}$; on the other hand, to point out the cases when (2) does not hold. The following theorem describes positive results, where, for completeness, we include all known cases.

**Theorem 1.1** (see \cite{10} $d \geq 2$). Let $0 < a \leq 1$, $\frac{2}{a} \in \mathbb{N}$. If $d = 1$, $\langle \kappa \rangle \geq \frac{1}{2}$ or $d \geq 2$, $\langle \kappa \rangle \geq 0$, then equality (2) is true.

Our proof of Theorem 1.1 for $d = 1$ is based on an integral representation of $B_{\kappa,a}$ with the special kernel and a study of positiveness of this kernel. This approach is closely related to the theory of positive definite functions.

In the general case $d \geq 1$, we give the proof based on the approach developed in the papers \cite{6, 10, 11}, see Subsection 4.1, and the alternative proof based on representation with positive kernels, see Subsection 4.3.

With regard to negative results, we obtain the following theorem, where we specify parameters when Theorem 1.1 does not hold.

**Theorem 1.2.** In either of the following cases:

1. $d = 1$, $0 < a \leq 1$, and $\langle \kappa \rangle = \frac{1}{2} - \frac{a}{2}$, or
2. $d \geq 1$, $a \in (1, 2) \cup (2, \infty)$ and $\langle \kappa \rangle \geq 0$.

we have

$$\|B_{\kappa,a}\|_{\infty} > 1. \quad (3)$$

The rest of the paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.1 in the case $d = 1$. In Section 3 we study the properties of the one-dimensional kernel $B_{\kappa,a}$ for $\lambda_{\kappa,a} < 0$. In particular, in Subsection 3.1 we investigate positive definiteness of kernels of the integral transforms generated by $\mathcal{F}_{\kappa,a}$. In Section 4, we prove Theorem 1.1 in full generality as well as Theorem 1.2 (Subsection 4.3).

In Section 5 we study the question of how the $\mathcal{F}_{\kappa,a}$ transform acts on Schwartz functions. The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is invariant under the classical Fourier transform $\mathcal{F}_{0,2}$ and the Dunkl transform $\mathcal{F}_{\kappa,2}$ (see \cite{8}) but the case of deformed transforms is more complicated. In fact we show that $\mathcal{S}(\mathbb{R}^d)$ is not invariant under $\mathcal{F}_{\kappa,a}$ for $a \neq 2$, which contradicts a widely used statement in \cite{18} (see Remark 5.3). If $\frac{2}{a} \notin \mathbb{N}$, then the generalized Fourier transform may not be infinitely differentiable, and if $\frac{2}{a} \notin \mathbb{N}$, then it may not be rapidly decreasing at infinity. For $d = 1$ and $\frac{2}{a} \in \mathbb{N}$, the generalized Fourier transform of $f \in \mathcal{S}(\mathbb{R})$ is rapidly decreasing due to the representation $\mathcal{F}_{\kappa,a}(f)(y) = F_1(|y|^{\frac{a}{2}}) + yF_2(|y|^{\frac{a}{2}})$, where the even functions $F_1, F_2 \in \mathcal{S}(\mathbb{R})$ (see Proposition 5.4).

Finally, in Section 6 we study one-dimensional non-deformed unitary transforms generated by $\mathcal{F}_{\kappa,a}$:

$$\mathcal{F}_{\kappa,a}^\lambda(g)(v) = \int_{-\infty}^{\infty} e_{2r+1}(uv, \lambda)g(u) \frac{|u|^{2\lambda+1}}{2^{\lambda+1}\Gamma(\lambda+1)} du,$$

where $r \in \mathbb{Z}^+$, $\lambda \geq -1/2$, and the kernel

$$e_{2r+1}(uv, \lambda) = j_\lambda(uv) + i(-1)^r u^{r+1} \frac{(uv)^{2r+1}}{2^{2r+1}(\lambda+1)_{2r+1}} j_{\lambda+2r+1}(uv)$$

\footnote{The case $\langle \kappa \rangle > 0$ was announced in \cite{10} Remark 3. The proof is similar to the one of \cite{10} Theorem 9.}
is an eigenfunction of the differential-difference operator
\[ \delta_{\lambda} g(u) = \Delta_{\lambda+1/2} g(u) - 2r(\lambda + r + 1) \frac{g(u) - g(-u)}{u^2}. \]

Here \( \Delta_{\lambda+1/2} \) is the one-dimensional Dunkl Laplacian for \( (\kappa) = \lambda + \frac{1}{2} \). Note that such unitary transforms give new examples of an important class of Bessel-Hankel type transforms with the kernel \( k(uv) \), see, e.g., [21], Chap. VIII. In particular, they generalize the one-dimensional Dunkl transform \( (r = 0) \).

2. PROOF OF THEOREM 1.1 IN THE ONE-DIMENSIONAL CASE

In what follows, we assume that
\( d = 1, \quad a > 0, \quad \kappa = (\kappa) \geq 0, \quad \lambda_{\kappa} = \kappa - 1/2, \quad 2\lambda_{\kappa} + a > 0, \quad \lambda = \lambda_{\kappa, a} = 2\lambda_{\kappa}/a, \)

\[ v_{\kappa, a}(x) = |x|^{2a} e^{-\lambda x}, \quad d\mu_{\kappa, a}(x) = c_{\kappa, a} v_{\kappa, a}(x) dx, \quad c_{\kappa, a}^{-1} = 2^{\lambda} \Gamma(\lambda + 1), \]

and \( F_{\kappa, a}(f)(y) \) is the \((\kappa, a)\)-generalized Fourier transform (1) on the real line. Firstly, let us investigate when the kernel of \( F_{\kappa, a} \) is uniformly bounded. Using [4, Sect. 5], we can write the kernel as
\[ B_{\kappa, a}(x, y) = j_\lambda \left( \frac{2}{a} |xy|^{a/2} \right) + \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + 1/2)} \frac{x}{a^{2/2^a}} j_{\lambda + \frac{1}{2}} \left( \frac{2}{a} |xy|^{a/2} \right), \quad (4) \]

where \( j_\lambda(x) = 2^\lambda \Gamma(\lambda + 1)x^{-\lambda} J_\lambda(x) \) is the normalized Bessel function and \( J_\lambda(x) \) is the classical Bessel function. Then the asymptotic behavior of \( J_\lambda(x) \) (see [25], Chapt. VII, 7.1]) immediately allows us to derive the following

Proposition 2.1. The conditions
\[ 0 < a \leq 2, \quad \kappa \geq 1 - \frac{a}{4}, \quad \text{or} \quad a \geq 2, \quad \kappa \geq 0, \quad (5) \]

are necessary and sufficient for boundedness of the kernel \( B_{\kappa, a}(x, y) \).

The main goal of this section is to prove Theorem 1.1 for \( d = 1 \).

Proof. Note that \( B_{\kappa, a}(x, y) = b_{\kappa, a}(xy) \), where
\[ b_{\kappa, a}(x) = j_\lambda \left( \frac{2}{a} |x|^{a/2} \right) + \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + 1/2)} \frac{x}{a^{2/2^a}} j_{\lambda + \frac{1}{2}} \left( \frac{2}{a} |x|^{a/2} \right). \quad (6) \]

Therefore, under the conditions of Theorem 1.1, it suffices to establish the inequality \( |b_{\kappa, a}(x)| \leq 1 \) for \( x \in \mathbb{R} \).

Let \( a = \frac{2}{R}, \quad R \in \mathbb{N} \). Equality (6) can be written as
\[ b_{\kappa, a}(x) = j_\lambda \left( R|x|^{1/R} \right) + \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + 1/2)} \frac{R}{2} (-i)^R x j_{\lambda + 1/R} \left( R|x|^{1/R} \right). \]

Let \( x \in \mathbb{R} \). In the case \( R = 2r + 1, \quad a = \frac{2}{2r+1}, \quad R \in \mathbb{Z}_+, \) and \( v = (2r + 1)x^{1/2r+1}, \quad x = \left( \frac{v}{2r+1} \right)^{2r+1}, \) there holds
\[ e_{2r+1}(v, \lambda) = b_{\kappa, a} \left( \left( \frac{v}{2r+1} \right)^{2r+1} \right) = j_\lambda(v) + i(-1)^{r+1} \frac{v^{2r+1}}{2^{2r+1}(\lambda + 1)_{2r+1}} j_{\lambda + 2r+1}(v), \quad (7) \]

where
\[ (a)_n = \frac{\Gamma(a + n)}{\Gamma(a)} = a(a + 1) \cdots (a + n - 1) \]

is the Pochhammer symbol.
In the case \( R = 2r, a = \frac{1}{r}, r \in \mathbb{N}, \) and \( v = 2r|x|^{2r} \) sign \( x, x = \left( \frac{r}{x} \right)^{2r} \) sign \( v, \) we have

\[
e_{2r}(v, \lambda) = b_{\kappa,a} \left( \left( \frac{v}{2r} \right)^{2r} \text{ sign } v \right) = j_{\lambda}(v) + (-1)^r \frac{v^{2r}}{2^{2r}(\lambda + 1)^{2r}} j_{\lambda+2r}(v) \text{ sign } v. \tag{8}
\]

In order to see that \(|e_{2r+1}(v, \lambda)|, |e_{2r}(v, \lambda)| \leq 1,\) we will need several auxiliary results. We start with the following identity

\[
\frac{v^2}{4(\lambda + 1)(\lambda + 2)} j_{\lambda+2r}(v) = j_{\lambda+1}(v) - j_{\lambda}(v), \tag{9}
\]

which follows from the recurrence relation for the Bessel function \( J_\lambda(v) \) (see [25, Chapter III, 3.2]). Then by induction we establish

**Lemma 2.2.** If \( r \in \mathbb{N}, \) then

\[
\frac{v^{2r}}{2^{2r}(\lambda + 1)^{2r}} j_{\lambda+2r}(v) = (-1)^r j_{\lambda}(v) + \sum_{s=1}^{r-1} (-1)^{s+r} \frac{r}{s} \left( \frac{\lambda + r}{\lambda + 1} \right)_s j_{\lambda+s}(v) + \frac{(\lambda + r + 1)(\lambda + 1)^{r-1}}{(\lambda + 1)^{r-1}} j_{\lambda+r}(v). \tag{10}
\]

**Proof.** For \( r = 1, \) the needed formula coincides with (9). Assume that (10) is valid for every \( k \leq r-1 \) and \( \lambda.\) Denote by \( a_s^r(\lambda), s = 0, 1, \ldots, r, \) the coefficients by \( j_{\lambda+s}(v) \) in the decomposition (10). Taking into account (9) and the inductive assumption, we derive that

\[
\frac{v^{2r}}{2^{2r}(\lambda + 1)^{2r}} j_{\lambda+2r}(v) = \frac{v^{2r-2}}{2^{2r-2}(\lambda + 1)^{2r-2}} \frac{v^2}{4(\lambda + 2r - 1)(\lambda + 2r)} j_{\lambda+2r-2}(v) = \frac{v^{2r-2}}{2^{2r-2}(\lambda + 1)^{2r-2}} \{j_{\lambda+1} j_{\lambda+2r-2} - j_{\lambda+2r-2}(v)\}
\]

\[
= \frac{\lambda + 2r - 1}{\lambda + 1} \sum_{s=1}^{r} a_{s-1}^{r-1}(\lambda + 1) j_{\lambda+s}(v) - \sum_{s=0}^{r-1} a_{s}^{r-1}(\lambda) j_{\lambda+s}(v).
\]

It is enough to show that

\[
a_0^r(\lambda) = -a_0^{r-1}(\lambda), \quad a_s^r(\lambda) = \frac{\lambda + 2r - 1}{\lambda + 1} a_{s-1}^{r-1}(\lambda + 1),
\]

\[
a_s^r(\lambda) = \frac{\lambda + 2r - 1}{\lambda + 1} a_{s-1}^{r-1}(\lambda + 1) - a_s^{r-1}(\lambda), \quad s = 1, \ldots, r - 1.
\]

Indeed, using the induction step, we have that

\[
a_0^r(\lambda) = (-1)^r, \quad a_s^r(\lambda) = \frac{\lambda + 2r - 1}{\lambda + 1} \left( \frac{(\lambda + r + 1)(\lambda + 1)^{r-1}}{(\lambda + 1)^{r-1}} \right),
\]

and, for \( s = 1, \ldots, r - 1,\)

\[
a_s^r(\lambda) = (-1)^{s+r} \frac{\lambda + 2r - 1}{\lambda + 1} \left( \frac{(\lambda + r + 1)(\lambda + 1)^{r-1}}{(\lambda + 1)^{r-1}} \right) = (-1)^{s+r} \frac{\lambda + 2r - 1}{\lambda + 1} \frac{(\lambda + r + 1)(\lambda + 1)^{r-1}}{(\lambda + 1)^{r-1}}
\]

which completes the proof. \(\square\)

**Lemma 2.3.** For \( r \in \mathbb{Z}_+, \) we have

\[
\frac{v^{2r+1}}{2^{2r+1}(\lambda + 1)^{2r+1}} j_{\lambda+2r+1}(v) = (-1)^{r+1} \sum_{s=0}^{r} (-1)^s \frac{r}{s} \left( \frac{(\lambda + r + 1)(\lambda + 1)^{r-1}}{(\lambda + 1)^{r-1}} \right) j_{\lambda+s}(v).
\]
where $q$ represents the functions $e_{2r+1}(v,\lambda)$ and $e_{2r}(v,\lambda)$.

**Proof.** Using Lemma 2.2 and the equality

$$j'_\lambda(v) = -\frac{v}{2(\lambda + 1)} j_{\lambda+1}(v),$$

we derive that

$$2^{2r+1}(\lambda + 1) j_{\lambda+2r+1}(v) = \frac{v}{2(\lambda + 1)} 2^r(\lambda + 2)^r j_{\lambda+1+2r}(v)$$

$$= \frac{v}{2(\lambda + 1)} \left\{ (-1)^r j_{\lambda+1}(v) + \sum_{s=1}^{r-1} (-1)^{s+r} \binom{r}{s} \frac{(\lambda + r + 1)s}{(\lambda + 2)s} j_{\lambda+1+s}(v) + \frac{(-1)^{s+r+1}}{2^r(\lambda + 1)} j_{\lambda+1+s}(v) - \frac{(-1)^r}{(\lambda + 1)} j_{\lambda+r+1}(v) \right\}$$

$$= (-1)^{r+1} j'_\lambda(v) + \sum_{s=1}^{r-1} (-1)^{s+r+1} \binom{r}{s} \frac{(\lambda + r + 1)s}{(\lambda + 1)s} j'_s(v) - \frac{(-1)^r}{(\lambda + 1)} j'_r(v).$$

Taking into account (7), (8), Lemmas 2.2, 2.3 and

$$j_\lambda(v) = c_\lambda \int_{-1}^{1} (1 - t^2)^{\lambda-1/2} e^{-int} dt, \quad j'_\lambda(v) = -ic_\lambda \int_{-1}^{1} (1 - t^2)^{\lambda-1/2} te^{-int} dt$$

with $c_\lambda = \frac{\Gamma(\lambda + 1)}{\sqrt{\pi} \Gamma(\lambda + 1/2)}$, $\lambda > -1/2$ (see [25] Chapt. III, 3.3), we arrive at the following integral representations of the functions $e_{2r+1}(v,\lambda)$, and $e_{2r}(v,\lambda)$.

**Lemma 2.4.** If $r \in \mathbb{Z}_+, \lambda > -1/2$, then

$$e_{2r+1}(v,\lambda) = c_\lambda \int_{-1}^{1} (1 - t^2)^{\lambda-1/2} q_{2r+1}(t,\lambda)e^{-int} dt,$$

where $q_{2r+1}(t,\lambda)$ is a polynomial of degree $2r + 1$ with respect to $t$ given by

$$q_{2r+1}(t,\lambda) = 1 + t \sum_{s=0}^{r} (-1)^{s} \binom{r}{s} \frac{(\lambda + r + 1)s}{(\lambda + 1/2)s} (1 - t^2)^s.$$

**Lemma 2.5.** If $r \in \mathbb{N}, \lambda > -1/2$, then

$$e_{2r}(v,\lambda) = c_\lambda \int_{-1}^{1} (1 - t^2)^{\lambda-1/2} q_{2r}(t,\lambda)e^{-int} dt,$$

where $q_{2r}(t,\lambda)$ is a polynomial of degree $2r$ with respect to $t$ given by

$$q_{2r}(t,\lambda) = q_{2r}(t,\lambda) = 1 + \text{sign } v \left\{ \sum_{s=0}^{r} (-1)^{s} \binom{r}{s} \frac{(\lambda + r)s}{(\lambda + 1/2)s} (1 - t^2)^s \right\}.$$

For our further analysis, it is important to know for which $\lambda$ the polynomials $q_{2r}(t,\lambda)$, $q_{2r+1}(t,\lambda)$ are nonnegative on $[-1,1]$. If $r = 0$, $q_1(t,\lambda) = 1 - t \geq 0$ on $[-1,1]$, that is, $q_1$ does not depend on $\lambda$. This special case corresponds to parameters $a = 2$, $\kappa \geq 0$ and hence (11) is a well-known integral representation of the kernel of the one-dimensional Dunkl transform. In other cases, positivity conditions for $q_{2r}(t,\lambda)$ and $q_{2r+1}(t,\lambda)$ depend on $\lambda$. We will see that there holds

$$q_{2r}(t,\lambda) = 1 + \text{sign } v \frac{P_n^{(\alpha-1/2)}}{P_n^{(\alpha)}(1)}(t), \quad q_{2r+1}(t,\lambda) = 1 + \frac{P_n^{(\alpha-1/2)}}{P_n^{(\alpha)}(1)}(t),$$

where \(\{P_n^{(\alpha)}(t)\}_{n=0}^{\infty}\) is the system of Gegenbauer (ultraspherical) polynomials, that is, the family of polynomials orthogonal on $[-1,1]$ with respect to the weight function $(1 - t^2)^\alpha$, $\alpha > -1$, normalized by $P_n^{(\alpha)}(1) = 1$. Note that

$$\frac{1}{\lambda} C_n^\lambda(t) = \frac{2\Gamma(2\lambda + n)}{n! \Gamma(2\lambda + 1)} P_n^{(\lambda-1/2)}(t), \quad \lambda > -1/2,$$
where $C_n^\lambda(t)$ are the Gegenbauer polynomials given in [1, Chapt. X, 10.9].

**Lemma 2.6.** For $\lambda > -1/2$ and $r \geq 0$, there hold
\[
P_{2r}^{(\lambda-1/2)}(t) = \sum_{s=0}^r (-1)^s \binom{r}{s} \frac{(\lambda + r)_s}{(\lambda + 1/2)_s} (1 - t^2)^s
\]
and
\[
P_{2r+1}^{(\lambda-1/2)}(t) = t \sum_{s=0}^r (-1)^s \binom{r}{s} \frac{(\lambda + r + 1)_s}{(\lambda + 1/2)_s} (1 - t^2)^s.
\]
Therefore, (13) holds.

**Proof.** Since $\frac{(-r)_s}{s!} = (-1)^s \binom{r}{s}$, taking into account (14), [1, Chapt. X, 10.9(21)], we get
\[
\sum_{s=0}^r (-1)^s \binom{r}{s} \frac{(\lambda + r)_s}{(\lambda + 1/2)_s} (1 - t^2)^s = \sum_{s=0}^r (-1)^s \frac{(\lambda + r)_s}{s! (\lambda + 1/2)_s} (1 - t^2)^s = _2F_1(-r, \lambda + r; \lambda + 1/2; 1 - t^2) = \frac{(2r)! \Gamma(2\lambda + 1)}{2\Gamma(2\lambda + 2r)} \frac{1}{\lambda} C_{2r}^\lambda(t) = P_{2r}^{(\lambda-1/2)}(t).
\]
Similarly, using (14), [1, Chapt. X, 10.9(22)], we arrive at
\[
t \sum_{s=0}^r (-1)^s \binom{r}{s} \frac{(\lambda + r + 1)_s}{(\lambda + 1/2)_s} (1 - t^2)^s = t \sum_{s=0}^r (-1)^s \frac{(\lambda + r + 1)_s}{s! (\lambda + 1/2)_s} (1 - t^2)^s = _2F_1(-r, \lambda + r + 1; \lambda + 1/2; 1 - t^2) = \frac{(2r + 1)! \Gamma(2\lambda + 1)}{2\Gamma(2\lambda + 2r + 1)} \frac{1}{\lambda} C_{2r+1}^\lambda(t) = P_{2r+1}^{(\lambda-1/2)}(t).
\]

We complete the proof of Theorem 1.1 noting that $|P_n^{(\lambda-1/2)}(t)| \leq 1$ for $t \in [-1, 1]$ under the condition $\lambda \geq 0$ or, equivalently, $\kappa \geq 1/2$ (see [23, Chapt. VII, 7.32.2]). Then in light of Remark 3.2 and Lemma 2.5, the polynomials $q_{2r}(t, \lambda)$ and $q_{2r+1}(t, \lambda)$ are nonnegative on $[-1, 1]$ and therefore Lemmas 2.4 and 2.5, together with (7) and (8), yield the statement of Theorem 1.1.

To illustrate this, for $a = \frac{2}{2r+1}$, $v = (2r+1)x^{2r+1}$, we have
\[
|b_{\kappa, a}(x)| = |e_{2r+1}(v)| \leq c_{\lambda} \int_{-1}^{1} (1 - t^2)^{\lambda-1/2} q_{2r+1}(t, \lambda) \, dt = 1.
\]

The following integral representations of $b_{\kappa, a}(x)$ follow from the lemmas above.

**Corollary 2.7.** If $\lambda = (2\kappa - 1)/a$ and $\kappa > \frac{1}{2} - \frac{a}{4}$, then for $x \in \mathbb{R}$
\[
b_{\kappa, a}(x) = c_{\lambda} \int_{-1}^{1} (1 - t^2)^{\lambda-1/2} (1 + P_{2r+1}^{(\lambda-1/2)}(t)) e^{-i(2r+1)\pi r x^{2r+1}} \, dt, \quad a = \frac{2}{2r+1}, \quad r \in \mathbb{Z}_+, \quad
\]
and
\[
b_{\kappa, a}(x) = c_{\lambda} \int_{-1}^{1} (1 - t^2)^{\lambda-1/2} (1 + \text{sign} \, x \, P_{2r}^{(\lambda-1/2)}(t)) e^{-i(2r|r|/\pi) \text{sign} \, x} \, dt, \quad a = \frac{1}{r}, \quad r \in \mathbb{N}.
\]

**Remark 2.8.** The representations of $b_{\kappa, a}(x)$ given in Corollary 2.7 can be also obtained from
\[
n! \int_0^\pi e^{i \lambda \cos \theta} C_n^\lambda(\cos \theta) (\sin \theta)^{2\lambda} \, d\theta = 2^\lambda \sqrt{\pi} \Gamma(\lambda + 1/2) (2\lambda)_n i^n z^{-\lambda} J_{\lambda+n}(z)
\]
(see [1, Chapt. X, 10.9(38)]) without applying Lemmas 2.2 and 2.3. These lemmas are of independent interest.
Remark 2.9. The positiveness of the polynomials $q_{2r}(t, \lambda)$, $q_{2r+1}(t, \lambda)$ is sufficient, but not necessary for the estimate $|B_{\kappa,a}(x,y)| \leq 1$ to hold. In [15], it was mentioned that for $a = 1$ this estimate holds also for $1/4 < \kappa_0 \leq \kappa < 1/2$.

Note that the case $\kappa < 1/2$ corresponds to $\lambda < 0$, which we study in detail in the next section.

3. THE CASE $d = 1$ AND $\lambda_{\kappa,a} < 0$

Let us investigate whether the polynomials $q_{2r}(t, \lambda)$ and $q_{2r+1}(t, \lambda)$ are nonnegative for $\lambda = \lambda_{\kappa,a} = (2\kappa - 1)/a < 0$. In order to do this, we decompose them by polynomials $q_{2r}(t,0)$ and $q_{2r+1}(t,0)$.

First, we decompose the Gegenbauer polynomials in terms of the Chebyshev polynomials using the well-known result (see [10.9(17)])

$$C_n^\lambda(\cos \theta) = \sum_{s=0}^{n} \frac{(\lambda)_s (\lambda)_{n-s}}{s! (n-s)!} \cos(n-2m)\theta.$$ (15)

Let us start with the case $R = 2r + 1$.

Lemma 3.1. For any $r \in \mathbb{Z}_+$ and $\lambda > -1/2$, there holds

$$P_{2r+1}^{(\lambda-1/2)}(t) = \sum_{s=0}^{r} b_s^r(\lambda) P_{2r+1-2s}^{(-1/2)}(t),$$ (16)

where

$$b_s^r(\lambda) = \frac{2(2r + 2 - s)_s (\lambda)_s (\lambda + r + 1)_{r-s}}{4^r (\lambda + 1/2)_s s!}, \quad s = 0, 1, \ldots, r.$$

Proof. Taking into account (14) and (15), we obtain

$$\frac{\Gamma(2\lambda + 2r + 1)}{(2r + 1)! \Gamma(2\lambda)} P_{2r+1}^{(\lambda-1/2)}(t) = C_{2r+1}^\lambda(t) = 2 \sum_{s=0}^{r} \frac{(\lambda)_s (\lambda)_{2r+1-s}}{s! (2r + 1 - s)!} P_{2r+1-2s}^{(-1/2)}(t).$$

Hence, the duplication formula for gamma function implies

$$b_s^r(\lambda) = \frac{2(2r + 1)! (\lambda)_s \Gamma(2\lambda) \Gamma(\lambda + 2r + 1 - s)}{(2r + 1 - s)! \Gamma(\lambda) \Gamma(2\lambda + 2r + 1)} = \frac{2(2r + 2 - s)_s (\lambda)_s (\lambda + r + 1)_{r-s}}{4^r (\lambda + 1/2)_s s!}.$$

Remark 3.2. We see that in the decomposition (14) the zero coefficient is positive, and all other coefficients are also positive for $\lambda > 0$ and negative for $\lambda < 0$. Note that the normalization of the Gegenbauer polynomials $P_n^{(1)}(1) = 1$ implies the equality

$$\sum_{s=0}^{r} b_s^r(\lambda) = 1$$

and for $\lambda > 0$ the Gegenbauer polynomials $P_{2r+1}^{(\lambda-1/2)}(t)$ are the convex hull of the Chebyshev polynomials $P_{2r+1-2s}^{(-1/2)}(t)$, $s = 0, 1, \ldots, r$. In particular, taking into account that $P_{2r+1-2s}^{(-1/2)}(t) = \cos((2r + 1 - 2s) \arccos t)$, we easily obtain for the Gegenbauer polynomials the estimate

$$|P_{2r+1}^{(\lambda-1/2)}(t)| \leq 1, \quad t \in [-1, 1] \quad \text{and} \quad \lambda > 0.$$

Thus, we are in a position to state the required result on the decomposition for the polynomial $q_{2r+1}(t, \lambda)$.
Corollary 3.3. For any \( r \in \mathbb{Z}_+ \) and \( \lambda > -1/2 \), there holds
\[
q_{2r+1}(t, \lambda) = \sum_{s=0}^{r} b_s^r(\lambda) q_{2r-2s+1}(t, 0).
\]

Using (13) and the properties of the coefficients \( b_s^r(\lambda) \), \( s = 0, 1, \ldots, r \), from Remark 3.2, we establish the following result.

Corollary 3.4. For any \( r \in \mathbb{N} \) and \( \lambda \in (-1/2, 0) \), the polynomial \( q_{2r+1}(t, \lambda) \) is negative at the points of local minimum of the Chebyshev polynomial \( P_{2r+1}(\lambda) = \cos ((2r + 1) \arccos t) \).

In the case \( R = 2r \), similarly, one can obtain the decomposition of the polynomial \( q_{2r}(t, \lambda) \).

Lemma 3.5. For any \( r \in \mathbb{N} \) and \( \lambda > -1/2 \), there holds
\[
P_{2r}^{(\lambda-1/2)}(t) = \sum_{s=0}^{r} d_s^r(\lambda) P_{2r-2s}^{(-1/2)}(t),
\]
where
\[
d_s^r(\lambda) = \frac{2(2r + 1 - s)_s(\lambda)_s(\lambda + r)_r}{4^r(\lambda + 1/2)_r s!}, \quad s = 0, 1, \ldots, r - 1, \quad d_r^r(\lambda) = \frac{(r + 1)_r(\lambda)_r}{4^r(\lambda + 1/2)_r r!}.
\]

Proof. In light of (14) and (15), we have
\[
\frac{\Gamma(2\lambda + 2r)(2\lambda)}{(2r)! \Gamma(2\lambda)} P_{2r}^{(\lambda-1/2)}(t) = C_{2r}(t) = 2 \sum_{s=0}^{r-1} \frac{(\lambda)_s(\lambda + r)_r}{s!(2r - s)!} P_{2r-2s}^{(-1/2)}(t) + \frac{(\lambda)_r^2}{(r!)^2} P_0^{(-1/2)}(t).
\]

Then, using the duplication formula for gamma function, we obtain for \( s = 0, 1, \ldots, r - 1 \)
\[
d_s^r(\lambda) = \frac{2(2r)! (\lambda)_s(\lambda + 2r - s)}{(2r - s)! \Gamma(\lambda) \Gamma(2\lambda + 2r)} = \frac{2(2r + 1 - s)_s(\lambda)_s(\lambda + r)_r}{4^r(\lambda + 1/2)_r s!},
\]
while for \( s = r \),
\[
d_r^r(\lambda) = \frac{(2r)! (\lambda)_r \Gamma(2\lambda) \Gamma(\lambda + 2r)}{(r!)^2 \Gamma(2\lambda + 2r)} = \frac{(r + 1)_r(\lambda)_r}{4^r(\lambda + 1/2)_r r!}.
\]

\[\square\]

Corollary 3.6. For any \( r \in \mathbb{N} \) and \( \lambda > -1/2 \), the following decomposition holds
\[
q_{2r}(t, \lambda) = \sum_{s=0}^{r} d_s^r(\lambda) q_{2r-2s}(t, 0).
\]

Since \( d_0^r(\lambda) > 0 \) and \( \sign d_s^r(\lambda) = \sign \lambda, s = 1, \ldots, r, \) for \( \lambda > -1/2 \), taking into account (13), we arrive at the following result.

Corollary 3.7. If \( r \in \mathbb{N} \) and \( \lambda \in (-1/2, 0) \), then the polynomial \( q_{2r}(t, \lambda) \) is negative at the points of local extremum of the Chebyshev polynomial \( P_{2r}^{(-1/2)}(t) = \cos (2r \arccos t) \).

Corollaries 3.4 and 3.7 show that the polynomials \( q_{2r}(t, \lambda) \) and \( q_{2r+1}(t, \lambda) \) for \( r \geq 1 \) change sign and, therefore, the method of the proof of the estimate \( |B_{\kappa,\sigma}(x, y)| \leq 1 \) used in Theorem 1.4 cannot be applied when \( \lambda < 0 \). In this case the problem remains open.
3.1. On positive definiteness. Recall that a continuous function \( f \) on \( \mathbb{R} \) is positive definite if for any \( \{v_1, \ldots, v_n\} \subset \mathbb{R} \) and \( \{z_1, \ldots, z_n\} \subset \mathbb{C} \) there holds
\[
\sum_{s,l=1}^{n} z_s \overline{v_l} f(v_s - v_l) \geq 0.
\]
Bochner’s theorem states that any continuous positive definite function \( f(v), f(0) = 1 \), is the Fourier transform of a probability measure.

Recall that the function \( e_{2r+1}(v, \lambda) \) is given in (7) and (11), (13). Taking into account (8), (12) and (13), let us define also the functions
\[
e_{2r,+}(v, \lambda) = c_{\lambda} \int_{-1}^{1} (1 - t^2)^{\lambda-1/2}(1 + P_{2r}(\lambda-1/2)(t)) e^{-i\lambda t} dt,
\]
\[
e_{2r,-}(v, \lambda) = c_{\lambda} \int_{-1}^{1} (1 - t^2)^{\lambda-1/2}(1 - P_{2r}(\lambda-1/2)(t)) e^{-i\lambda t} dt.
\]
Combining Theorem 1.1 for \( d = 1 \), Lemma 2.6, Corollaries 3.3, 3.4, 3.6, and 3.7, we deduce the following

**Corollary 3.8.** The functions \( e_{2r+1}(v, \lambda) \) and \( e_{2r,+}(v, \lambda) \) are positive definite if and only if \( \lambda \geq 0 \).

It is worth mentioning that our proof of Theorem 1.1 for \( d = 1 \), in fact, was based on the positive definiteness of these functions.

4. The kernel of the generalized Fourier transform in the multivariate case

4.1. When the kernel is bounded by one. In this section, for the sake of completeness, we give the proof of Theorem 1.1 in the general case, following ideas from the paper [10]. We stress that the proof below also allows one to deal with the case \( d = 1 \).

Let \( d \in \mathbb{N}, a > 0, \langle \kappa \rangle \geq 0, \) and \( \eta = \lambda - \langle \kappa \rangle + \frac{d-2}{2} > 0 \). For \( w \geq 0 \) and \( \tau \in [-1, 1] \) define
\[
\Psi_a^d(w, \tau) = 2^{2\eta/a} \Gamma \left( \frac{2\eta + a}{a} \right) \sum_{j=0}^{\infty} e^{-i\tau j} \frac{\eta + j}{\eta} w^{-2\eta/a} J_{2(\eta+j)}(a) C_j^\eta(\tau).
\]
(17)

Putting in (17)
\[
x, y \in \mathbb{R}^d, \quad x = |x| x', \quad y = |y| y', \quad x', y' \in \mathbb{S}^{d-1}, \quad w = \frac{2}{a} (|x||y|)^{a/2}, \quad \tau = \langle x', y' \rangle,
\]
we obtain the function
\[
K_a^d(x, y) = a^{2\eta/a} \Gamma \left( \frac{2\eta + a}{a} \right) (|x||y|)^{-\eta} \sum_{j=0}^{\infty} e^{-i\tau j} \frac{\eta + j}{\eta} J_{2(\eta+j)}(a) C_j^\eta(\langle x', y' \rangle).
\]

Note that \( K_a^d(x, y) = K_a^d(y, x) \) and \( \Psi_a^d(0, \tau) = K_a^d(0, y) = 1 \).

Let \( V_x f(x) \) be the intertwining operator in the Dunkl harmonic analysis [20], which is a positive operator satisfying
\[
V_x f(x) = \int_{\mathbb{R}^d} f(\xi) d\mu_\varepsilon^x(\xi).
\]
The representing measures \( \mu_\varepsilon^x(\xi) \) are compactly supported probability measures with \( \text{supp} \mu_\varepsilon^x(\xi) \subset \text{co}\{g x: g \in G\} [22]\).
The kernel of the generalized Fourier transform $\mathcal{F}_{\kappa,a}$ is given by [4, Chaps. 4, 5]

$$B_{\kappa,a}(x, y) = V_\kappa K^d_{\kappa,a}(x, |y|)(y')$$

$$= a^{2\eta/a} \Gamma \left( \frac{2\eta + a}{a} \right) (|x||y|)^{-\eta} \sum_{j=0}^{\infty} e^{-\frac{\eta + j}{2} \frac{2}{a} (|x||y|)^{\alpha/2}} V_\kappa C^\eta_j (\langle x', \cdot \rangle)(y'). \tag{18}$$

If $\langle \kappa \rangle = 0$, that is, $\eta = (d - 2)/2$, then $V_\kappa = I$ is the identity operator and $K^d_{\kappa,a}(x, y) = B_{0,a}(x, y)$.

Since the operator $V_\kappa$ is positive and $V_\kappa 1 = 1$, the proof of equality (2) can proceed as follows: if $|\Psi^d_{a}(w, \tau)| \leq 1$, then $|B_{\kappa,a}(x, y)| \leq 1$.

Let further $\frac{a}{2} \in \mathbb{N}$, $a = \frac{2}{\eta}$. The inequality $|\Psi^d_{a}(w, \tau)| \leq 1$ was proved in [10] under the conditions $\eta > 0$ and $\eta = (d - 2)/2$. We note that the proof of this estimate in [10] also holds for $\eta = \langle \kappa \rangle + (d - 2)/2 > 0$. Hence, (2) is valid for $\langle \kappa \rangle + (d - 2)/2 > 0$. If $\eta = 0$, then $d = 1$, $\kappa = 1/2$ or $d = 2$, $\kappa \equiv 0$ and in these cases (2) holds as well (see Theorem 1.1 and [5]). This completes the proof of Theorem 1.1.

4.2. Positive definiteness of $\Psi^d_{a}$. Another proof of Theorem 1.1. In [10], to evaluate the kernel $B_{0,a}(x, y)$, the authors used the Laplace transform of the function $\Psi^d_{a}$ given by (17). In the general case $d \geq 1$, we give another proof of the estimate $|\Psi^d_{a}| \leq 1$ based on the approach from the papers [6, 11], which allows one to show positive definiteness of the function $\Psi^d_{a}(w, \tau)$ with respect to $w$. Note also that for $a = 1, 2$ the functions

$$\Psi^d_{1}(w, \tau) = j_{\lambda_1,1/2}(w\sqrt{(1 + \tau)/2}), \quad \Psi^d_{2}(w, \tau) = e^{-iw\tau},$$

are positive definite (see [4, Example 4.18], [10]).

Let

$$I_n(b) = \left( \frac{b}{2} \right)^{\eta} \sum_{j=0}^{\infty} \frac{1}{j! \Gamma(j + \eta + 1)} \left( \frac{b^2}{4} \right)^j$$

be the modified Bessel function of the first kind (see [11, Chapt. 7, 7.2.2]) and

$$\Phi^m_2(\beta_1, \ldots, \beta_m; \gamma; x_1, \ldots, x_m) = \sum_{j_1, \ldots, j_m \geq 0} \frac{(\beta_1)_{j_1} \cdots (\beta_m)_{j_m}}{(\gamma)_{j_1 + \cdots + j_m}} \frac{x_1^{j_1}}{j_1!} \cdots \frac{x_m^{j_m}}{j_m!}$$

be the second Humbert function of $m$ variables (see [13, Chapt. 2, 2.1.1.2]). In the case when $\gamma - \sum_{j=1}^{m} \beta_j > 0$ and $\beta_j > 0$, $j = 1, \ldots, m$, the hypergeometric function $\Phi^m_2(\beta_1, \ldots, \beta_m)$ admits the following integral representation

$$\Phi^m_2(\beta_1, \ldots, \beta_m; \gamma; x_1, \ldots, x_m) = C^m_\beta(\gamma) \int_{T^m} e^{\sum_{j=1}^{m} x_j t_j} (1 - \sum_{j=1}^{m} t_j) \gamma - \sum_{j=1}^{m} \beta_j - 1 \prod_{j=1}^{m} t_j^{\beta_j - 1} dt_1 \cdots dt_m, \tag{19}$$

where

$$C^m_\beta(\gamma) = \frac{\Gamma(\gamma)}{\Gamma(\gamma - \sum_{j=1}^{m} \beta_j) \prod_{j=1}^{m} \Gamma(\beta_j)}$$

and

$$T^m = \left\{ (t_1, \ldots, t_m) : t_j \geq 0, j = 1, \ldots, m, \sum_{j=1}^{m} t_j \leq 1 \right\}$$

is the unit simplex in $\mathbb{R}^m$ ([9, 17]).

Taking into account the results from [6, 11], we obtain the following proposition, which gives an alternative proof of Theorem 1.1.
Proposition 4.1. Let \( d \in \mathbb{N}, a = \frac{2}{R}, R \in \mathbb{N}, R \geq 2, \eta = \langle \kappa \rangle + \frac{d-2}{2} > 0, \tau \in [-1, 1] \) and \( q = \arccos \tau \). The function \( w \mapsto \Psi^d_a(w, \tau) \) is a positive definite entire function of exponential type

\[
\theta(a, \tau) = \begin{cases} 
\cos \frac{\pi}{R}, & R = 2r \text{ and } 0 \leq q \leq \pi \text{ or } R = 2r + 1 \text{ and } 0 \leq q \leq \pi/2, \\
\cos \frac{\pi q - q}{R}, & R = 2r + 1 \text{ and } \pi/2 \leq q \leq \pi.
\end{cases}
\]

(20)

Proof. Let us consider the function \( f_{R, \eta}(b, \tau) = \Gamma(R\eta + 1) \left( \frac{b}{2} \right)^\eta \sum_{j=0}^\infty \frac{j + \eta}{\eta} I_{R(j+\eta)}(b) C_j^\eta(\tau), \quad R \in \mathbb{N}. \)

Let \( \tau = \langle x', y' \rangle, b = -iw, R = \frac{2}{R}, R \geq 2 \). Since \( I_{\eta}(-iw) = e^{-i\eta t} J_{\eta}(w) \) [1 Chapt. 7, 7.2.2], then \( f_{R, \eta}(b, \tau) = \Psi^d_a(w, \tau) \). It is known (see [6, 11]) that

\[
f_{R, \eta}(b, \tau) = b_0 \Phi^{(R-1)}(\eta, \ldots, \eta; R\eta; b_1 - b_0, \ldots, b_{R-1} - b_0),
\]

where \( b_j = b \cos \left( \frac{q - 2\pi j}{R} \right), j = 0, \ldots, R - 1 \). In light of [19] and using \( \eta > 0 \), we get

\[
\Psi^d_a(w, \tau) = f_{R, \eta}(-iw, \tau) = C_\eta \int_{T^{R-1}} e^{-iw\{\cos \left( \frac{q - 2\pi j}{R} \right) + \sum_{j=1}^{R-1} \cos \left( \frac{q - 2\pi j}{R} \right) t_j \}} \left( 1 - \sum_{j=1}^{R-1} t_j \right)^\eta \prod_{j=1}^{R-1} t_j^{\eta-1} \ dt_1 \cdots dt_{R-1},
\]

(21)

where \( C_\eta = \Gamma(R\eta)/\Gamma(\eta)^R \). For \( \alpha > 0 \) and \( \beta \in \mathbb{R} \), the function \( \alpha e^{i\beta w} \) is positive definite, hence the function \( \Psi^d_a(w, \tau) \) is also positive definite with respect to \( w \).

The representation (21) implies that \( \Psi^d_a(w, \tau) \) is an entire function of exponential type with respect to \( w \). Let us calculate its type. Since for \( j = 1, \ldots, R - 1, q = \arccos \tau, \tau \in [-1, 1], \cos \frac{q}{R} - \cos \frac{q - 2\pi j}{R} = 2 \cos \frac{q}{R} \cos \frac{\pi j}{R} \geq 0 \), then for any \((t_1, \ldots, t_{R-1}) \in T^{R-1},

\[
\min_{1 \leq j \leq R-1} \cos \frac{q - 2\pi j}{R} \leq \cos \frac{q}{R} + \sum_{j=1}^{R-1} \left( \cos \frac{q - 2\pi j}{R} - \cos \frac{q}{R} \right) t_j \leq \cos \frac{q}{R}.
\]

Since

\[
\min_{1 \leq j \leq R-1} \cos \frac{q - 2\pi j}{R} = \begin{cases} 
- \cos \frac{q}{R}, & R = 2r \text{ or } R = 2r + 1 \text{ and } 0 \leq q \leq \pi/2, \\
- \cos \frac{\pi q - q}{R}, & R = 2r + 1 \text{ and } \pi/2 \leq q \leq \pi,
\end{cases}
\]

the type of the function \( \Psi^d_a(w, \tau) \) is given by (20).

\( \square \)

Remark 4.2. Using (17) and Lemmas 2.2, 2.3 we get for \( t, \tau \in (-1, 1), \lambda = 2\eta/a \),

\[
\Psi^d_a(w, \tau) = c_\lambda \int_{-1}^{1} (1 - t^2)^{\lambda - 1/2} \sum_{j=0}^\infty \frac{\eta + j}{\eta} P_{2j/a}^{(\lambda - 1/2)}(t) C_j^\eta(\tau) e^{-iwt} \ dt.
\]

It is clear that the condition

\[
\psi^d_a(t, \tau) = \sum_{j=0}^\infty \frac{\eta + j}{\eta} P_{2j/a}^{(\lambda - 1/2)}(t) C_j^\eta(\tau) \geq 0, \quad t, \tau \in (-1, 1),
\]

is equivalent to positive definiteness of the function \( \Psi^d_a(w, \tau) \). Proposition 4.1 shows that the function \( \psi^d_a(w, \tau) \) is positive and its support as a function of \( t \) lies on the interval...
Using [2, Chapt. VIII, 8.7(31)], we obtain

\[ j_{\eta,\lambda} \]

and

\[ 1 \]

which completes the proof of (3). Note that similar arguments also implies the proof of (3) for

\[ d \]

Proof of Theorem 1.2.

Proposition 2.1).

If \( 0 < a < 2 \), then \( \| B_{\kappa,a} \|_\infty = \infty \) (see Proposition 2.1).

4.3. Parameters when the kernel is not bounded by one. We will show that in some cases the uniform norm of the kernel \( B_{\kappa,a}(x,y) \) is not bounded by 1 and can be either finite or infinite. We mention that the conditions when the norm is not finite are known only in the one-dimensional case. In particular, if \( d = 1, 0 < a < 2, \) and \( \kappa < \frac{1}{2} - \frac{a}{4} \), then \( \| B_{\kappa,a} \|_\infty = \infty \) (see Proposition 2.1).

Proof of Theorem 1.2. Let first \( d = 1 \) and \( \lambda = \lambda_{\kappa,a} = \frac{2\kappa-1}{a} \). Changing variables \( x = \left( \frac{a}{2} | v \right)^{2/a} \text{sign} \ v \) in (6), we get

\[ e_{\kappa,a}(v) = b_{\kappa,a} \left( \left( \frac{a}{2} | v \right)^{2/a} \text{sign} \ v \right) = j_\lambda(v) + \frac{\Gamma(\lambda + 1) \cos \frac{\pi}{a} | v |^{2/a} j_{\lambda+\frac{a}{2}}(v) \text{sign} \ v}{2^{2/a} \Gamma(\lambda + 1 + 2/a)} - i \frac{\Gamma(\lambda + 1) \sin \frac{\pi}{a} | v |^{2/a} j_{\lambda+\frac{a}{2}}(v) \text{sign} \ v}{2^{2/a} \Gamma(\lambda + 1 + 2/a)} \]  

If \( 0 < a \leq 1 \) and \( \kappa = \frac{1}{2} - \frac{a}{4} \), then \( \lambda = -1/2, j_{-1/2}(v) = \cos v, j_{1/2}(v) = \sin v \).

Assume first that \( \cos \frac{\pi}{a} = 0 \) or \( a = \frac{2r+1}{r} \), \( r \in \mathbb{N} \). Since \( \sin \frac{\pi}{a} \neq 0 \), \( j_{-1/2}(2\pi) = 1, j_{1/2}(2\pi) = 0 \), and \( j_{2\pi+1/2}(2\pi) \neq 0 \) (see [25, Chapt. 15, 15.28]), then from (22) we get \( |e_{\kappa,a}(2\pi) = |e_{2\pi+1}(2\pi, -1/2) > 1 \).

Let now \( \cos \left( \frac{\pi}{a} \right) \neq 0 \). In light of (22), there holds

\[ |e_{\kappa,a}(v)| \geq \left| \cos v + \frac{\Gamma(\frac{1}{2}) \cos \frac{\pi}{a} | v |^{2/a} j_{\frac{a}{2} - \frac{1}{2}}(v) \text{sign} \ v}{2^{2/a} \Gamma(\frac{a}{2} + \frac{3}{2})} \right| \]

Since

\[ (2\pi s)^{2/a} j_{\frac{a}{2} - \frac{1}{2}}(2\pi s) = \frac{2^{2/a} \Gamma(\frac{3}{2} + \frac{1}{2})}{\Gamma(\frac{1}{2})} \left\{ \cos \frac{\pi}{a} + O\left( \frac{1}{s} \right) \right\} \]

as \( s \to +\infty \) (see [25, Chapt. VII, 7.1]), we deduce that for sufficiently large \( s \)

\[ |e_{\kappa,a}(2\pi s)| \geq 1 + \cos^2 \frac{\pi}{a} + O\left( \frac{1}{s} \right) > 1, \]

which completes the proof of (3). Note that similar arguments also implies the proof of (3) for \( 1 < a < 2 \) and \( \kappa = \frac{1}{2} - \frac{a}{4} \).

Now we consider the multivariate case. Suppose that \( d \geq 1, a \in (1, 2) \cup (2, \infty), \langle \kappa \rangle \geq 0 \), and \( \eta = \lambda_{\kappa}, \lambda = \frac{2\eta}{\lambda} = \frac{2(\kappa)+d-2}{2} \). In view of (5) and using estimate (3), we may assume that \( \eta, \lambda > -1/2 \) and \( \cos \frac{\pi}{a} \neq 0 \) for \( d \geq 1 \).

Note that (18) can be equivalently written as

\[
B_{\kappa,a}(x,y) = \sum_{j=0}^{\infty} e^{-\frac{\pi i x y}{\eta}} \frac{\eta + j}{\eta} 2^{2j/a} \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + 1 + 2j/a)} w^{2j/a} j_{\lambda+\frac{a}{2}}(w) V_{\kappa} C_j^\eta(\langle x', \cdot \rangle)(y').
\]
Since
\[ |j_\lambda(w)| \leq 1, \quad \left| \frac{\eta + j}{\eta} V_\kappa C_\lambda^\eta((x', \cdot))(y') \right| \leq c(\eta)(j^{2\eta} + 1), \quad \frac{1}{\eta} C_1^\eta(t) = 2t, \]
then for \(0 \leq w \leq 1\) we have
\[
\left| \sum_{j=2}^{\infty} e^{-\frac{iz_i}{\eta}} \frac{\eta + j}{\eta} \frac{\Gamma(\lambda + 1)}{2^{2j/a}\Gamma(\lambda + 1 + 2j/a)} w^{2j/a} j_{\lambda+\frac{z}{a}}(w) V_\kappa((x', \cdot))(y') \right| \\
\leq c(\eta)w^{4/a} \sum_{j=2}^{\infty} \frac{\Gamma(\lambda + 1)(j^{2\eta} + 1)}{2^{2j/a}\Gamma(\lambda + 1 + 2j/a)} \leq c(\kappa, \eta)w^{4/a}
\]
and
\[
B_{\kappa,a}(x, y) = j_\lambda(w) + \frac{2(\eta + 1)\Gamma(\lambda + 1)\cos \frac{\pi}{a}}{2^{2/a}\Gamma(\lambda + 1 + 2/a)} w^{2/a} j_{\lambda+\frac{z}{a}}(w) V_\kappa((x', \cdot))(y') \\
- i \frac{2(\eta + 1)\Gamma(\lambda + 1)\sin \frac{\pi}{a}}{2^{2/a}\Gamma(\lambda + 1 + 2/a)} w^{2/a} j_{\lambda+\frac{z}{a}}(w) V_\kappa((x', \cdot))(y') + O(w^{4/a}), \quad w \to 0.
\]
Therefore,
\[
|B_{\kappa,a}(x, y)| \geq \left| j_\lambda(w) + \frac{2(\eta + 1)\Gamma(\lambda + 1)\cos \frac{\pi}{a}}{2^{2/a}\Gamma(\lambda + 1 + 2/a)} w^{2/a} j_{\lambda+\frac{z}{a}}(w) V_\kappa((x', \cdot))(y') + O(w^{4/a}) \right|. \tag{23}
\]
Since the operator \(V_\kappa\) is isomorphism on the space of homogeneous polynomials of degree 1, then for some \(x', y' \in S^{d-1}\) we deduce
\[
V_\kappa((x', \cdot))(y') = \int_{\mathbb{R}^d} \langle x', \xi \rangle d\mu_\kappa^\kappa(\xi) \neq 0, \quad \text{sign}(V_\kappa((x', \cdot))(y')) = \text{sign}(\cos(\pi/a)). \tag{24}
\]
We have
\[
j_\lambda(w) = 1 + O(w^2), \quad j_{\lambda+\frac{z}{a}}(w) = 1 + O(w^2)
\]
as \(w \to 0\). This, \(23\) and \(24\) imply that, for given \(x', y' \in S^{d-1}\) and sufficiently small positive \(w\),
\[
|B_{\kappa,a}(x, y)| \geq 1 + \frac{2(\eta + 1)\Gamma(\lambda + 1)\cos \frac{\pi}{a}}{2^{2/a}\Gamma(\lambda + 1 + 2/a)} w^{2/a} |V_\kappa((x', \cdot))(y')| + O(w^{4/a}) + O(w^2) > 1.
\]
\[\square\]

Remark 4.3. (i) For \(0 < a \leq 1, \frac{2}{d} \in \mathbb{N}\), equality \(2\) holds provided \(\kappa \geq \kappa_0(a)\), where \(\frac{1}{2} - \frac{2}{d} \leq \kappa_0(a) \leq 1/2\). The problem of determining \(\kappa_0(a)\) is open.

(ii) Theorem 1.2 shows that our conjecture in \([15]\) asserting that \(3\) holds under the condition \(2(\kappa) + d + a \geq 3\) is not valid for \(d \geq 1\) and \(a \in (1, 2) \cup (2, \infty)\).

5. Image of the Schwartz space

In this section, for the \((\kappa, a)\)-generalized Fourier transform, we study the image of the Schwartz space \(S(\mathbb{R}^d)\).
5.1. The case of $a > 0$. Let $d\nu_\eta(u) = b_\eta u^{-\eta+1} du$, $b_\eta^{-1} = 2^\eta \Gamma(\eta + 1)$, $d\nu_{\eta,a}(u) = b_{\eta,a} u^{2\eta+a-1} du$, $b_{\eta,a}^{-1} = a^{2\eta/a} \Gamma(2\eta/a + 1)$,
\begin{equation*}
H_\eta(f_0)(v) = \int_0^\infty f_0(u) j_\eta(uv) \; d\nu_\eta(u), \quad \eta \geq -1/2,
\end{equation*}
be the Hankel transform and
\begin{equation*}
H_{\eta,a}(f_0)(v) = \int_0^\infty f_0(u) j_{\eta,a}(\frac{2}{a} (uv)^{a/2}) \; d\nu_{\eta,a}(u), \quad 2\eta + a \geq 1,
\end{equation*}
be $a$-deformed Hankel transform (see [4, 15]).

Recall that $\lambda_\kappa = \langle \kappa \rangle + (d-2)/2$, $\lambda = \lambda_{\kappa,a} = 2\lambda_\kappa/a$.

Example 5.1. Consider $f(x) = e^{-|x|^2} \in S(\mathbb{R}^d)$, $f(x) = f_0(\rho)$, $\rho = |x|$. If $|y| = v$, $\rho = (a/2)^{1/a} u^{2/a}$, then (see [4, 15])
\begin{equation*}
\mathcal{F}_{\kappa,a}(f)(y) = H_{\lambda_{\kappa,a}}(f_0)(v) = \int_0^\infty f_0(\rho) j_{\lambda_{\kappa,a}}(\frac{2}{a} (\rho v)^{a/2}) \; d\nu_{\lambda_{\kappa,a}}(\rho) = \int_0^\infty \exp\left( -\left(\frac{a}{2}\right)^{2/a} u^{4/a}\right) j_{\lambda}(\left(\frac{2}{a}\right)^{1/2} v^{a/2} u) \; d\nu_{\lambda}(u).
\end{equation*}

Let
\begin{equation*}
g_a(u) = \exp\left( -\left(\frac{a}{2}\right)^{2/a} u^{4/a}\right).
\end{equation*}

Assuming that $\mathcal{F}_{\kappa,a}(f)(y)$ is rapidly decreasing at infinity, the same is true for the function
\begin{equation*}
G_a(v) = H_\lambda(g_a)(v) = \int_0^\infty g_a(u) j_\lambda(uv) \; d\nu_\lambda(u)
\end{equation*}
as $v \to \infty$. If $a = 4$, then in light of [2, Chap. VIII, 8.6(4)] the function
\begin{equation*}
G_a(v) = \frac{c_{\lambda_{\kappa,a}}}{(1 + v^2)^{\lambda+3/2}}
\end{equation*}
decreases at infinity not faster than a power function. If $2/a$ is a non-integer and $a \neq 4$, then $4/a$ is also non-integer, and therefore $g_a(u)$ has finite smoothness at the origin. On the other hand, since $g_e, G_a \in L^1(\mathbb{R}_+, d\nu_\lambda)$, we obtain
\begin{equation*}
g_e(u) = H_\lambda(G_a)(u) = \int_0^\infty G_a(v) j_\lambda(uv) \; d\nu_\lambda(v)
\end{equation*}
and the right-hand side is infinitely differentiable at the origin due to fast decreasing of $G_a(v)$. This contradiction shows us that the generalized Fourier transform cannot rapidly decrease at infinity. Moreover, let $\partial f = f'$. Since
\begin{equation*}
(\partial^2_{uv} j_\lambda(uv))|_{v=0} = -\frac{u^2}{2(\lambda + 1)}, \quad (\partial^4_{uv} j_\lambda(uv))|_{v=0} = \frac{3u^4}{4(\lambda + 1)(\lambda + 2)},
\end{equation*}
then
\begin{equation*}
G_a(v) = \xi_0 + \xi_1 v^2 + O(v^4), \quad v \to 0, \quad \xi_1 \neq 0,
\end{equation*}
and
\begin{equation*}
\mathcal{F}_{\kappa,a}(f)(y) = \xi_0 + \tilde{\xi}_1 |y|^a + O(|y|^{2a}), \quad y \to 0, \quad \tilde{\xi}_1 \neq 0.
\end{equation*}
If $a$ is not even, then $\mathcal{F}_{\kappa,a}(f)(y)$ has finite smoothness at the origin.

The following statement follows from Example 5.1.

Proposition 5.2. Let $d \in \mathbb{N}$.

(i) The condition $\frac{a}{2} \in \mathbb{N}$ is necessary for the embedding $\mathcal{F}_{\kappa,a}(S(\mathbb{R}^d)) \subset C^\infty(\mathbb{R}^d)$ to hold.
Indeed, suppose $\lambda \neq \text{false}$ for Lemma 2.12 that the Schwartz space is invariant under the generalized Fourier transform is the even functions

Putting in (25)

Further, if $a \in \mathbb{N}$, then $g_e(u), u^{-2/a}g_o(u)$ decrease rapidly at infinity and there exist

Then the one-dimensional generalized Fourier transform can be written as

Putting in (25)

and taking into account (4), we deduce

If $f \in \mathcal{S}(\mathbb{R})$, then $g_e(u), u^{-2/a}g_o(u)$ decrease rapidly at infinity and there exist

Therefore, the representation (27) shows that, for even $a$ we have the embedding $\mathcal{F}_{\kappa,a}(\mathcal{S}(\mathbb{R})) \subset C^\infty(\mathbb{R})$.

Further, if $a \in \mathbb{N}$, $f \in \mathcal{S}(\mathbb{R})$, then the functions $g_e(u), u^{-2/a}g_o(u) \in \mathcal{S}(\mathbb{R}_+)$ as well as (see [20]) the even functions

Hence, we prove the following

**Proposition 5.4.** Suppose $a \in \mathbb{N}$, $\lambda \geq -1/2$, $f \in \mathcal{S}(\mathbb{R})$, then

where the even functions $F_1, F_2 \in \mathcal{S}(\mathbb{R})$.

Thus, for $2/a \in \mathbb{N}$ the generalized Fourier transform decreases rapidly at infinity and we arrive at the following result.

**Proposition 5.5.** (i) The embedding $\mathcal{F}_{\kappa,a}(\mathcal{S}(\mathbb{R})) \subset C^\infty(\mathbb{R})$ is valid if and only if $\frac{2}{a} \in \mathbb{N}$.

(ii) The set $\mathcal{F}_{\kappa,a}(\mathcal{S}(\mathbb{R}))$ consists of rapidly decreasing functions at infinity if and only if $\frac{2}{a} \in \mathbb{N}$.
Let
\[ S_{\text{org}}(\mathbb{R}) = \{ f \in \mathcal{S}(\mathbb{R}) : \partial^k f(0) = 0, \ k \in \mathbb{N} \}. \]

The class \( S_{\text{org}}(\mathbb{R}) \) is dense in \( L^2(\mathbb{R}, d\mu_{\kappa,a}) \). Suppose \( a > 0, f(x) \in S_{\text{org}}(\mathbb{R}) \), then the functions \( g_\kappa(u), u^{-1/2}g_\kappa(u) \in S(\mathbb{R}_+) \), cf. [27]. Therefore, the even functions \( H_\lambda(g_\kappa)(v), H_{\lambda+a}(u^{-1/2}g_\kappa)(v) \) also belong to \( \mathcal{S}(\mathbb{R}) \).

Thus, we obtain the following

**Proposition 5.6.** Suppose \( a > 0, \lambda \geq -1/2, \) and \( f \in S_{\text{org}}(\mathbb{R}) \); then the generalized Fourier transform \( \mathcal{F}_{\kappa,a}(f) \) enjoys the representation [28].

### 5.2. The case of irrational \( a \).
We will show that if \( a \) is irrational, any nontrivial Schwartz function possesses similar properties to the Gaussian function in Example 5.1. To see this, we need auxiliary properties of the kernel of the generalized Fourier transform.

Let \( S^{d-1} \) be the unit sphere in \( \mathbb{R}^d, x = \rho x', \rho = |x| \in \mathbb{R}_+, x' \in S^{d-1}, \) and \( dx' \) be the Lebesgue measure on the sphere. If \( a^{-1}_\kappa = \int_{S^{d-1}} v_\kappa(x') dx' \), \( d\sigma_\kappa(x') = a_\kappa v_\kappa(x') dx', \) then \( d\mu_{\kappa,a}(x) = dv_{\kappa,a}(x') \) and \( c_{\kappa,a} = b_{\kappa,a}a_\kappa \).

Denote by \( \mathcal{H}_n^d(v_\kappa) \) the subspace of \( \kappa \)-spherical harmonics of degree \( n \in \mathbb{Z}_+ \) in \( L^2(S^{d-1}, d\sigma_\kappa) \) (see [13] Chap. 5). Let \( \mathcal{P}_n^d \) be the space of homogeneous polynomials of degree \( n \) in \( \mathbb{R}^d \). Then \( \mathcal{H}_n^d(v_\kappa) \) is the restriction of ker \( \Delta_\kappa \cap \mathcal{P}_n^d \) to the sphere \( S^{d-1} \).

If \( l_n \) is the dimension of \( \mathcal{H}_n^d(v_\kappa) \), we denote by \( \{ Y_j^n : j = 1, \ldots, l_n \} \) the real-valued orthonormal basis \( \mathcal{H}_n^d(v_\kappa) \) in \( L^2(S^{d-1}, d\sigma_\kappa) \). A union of these bases forms orthonormal basis in \( L^2(S^{d-1}, d\sigma_\kappa) \) consisting of \( \kappa \)-spherical harmonics.

Let us rewrite [13] as follows
\[
B_{\kappa,a}(x,y) = \sum_{j=0}^\infty e^{-\frac{i\pi n}{2}} a_\kappa + j \frac{\Gamma(2\lambda_\kappa/a + 1)/\Gamma(2\lambda_\kappa + j)/a + 1)}{a^{2j/a} \Gamma(2\lambda_\kappa + j)/a + 1)} j_{2(\lambda_\kappa + j)}(2a) Y_j^n(x).
\]

Integrating this and using orthogonality of \( \kappa \)-spherical harmonics, as in the case of Dunkl kernel (see [13 Theorem 5.3.4], [21 Corollary 2.5]), we obtain the following crucial property of the kernel of the generalized Fourier transform.

**Proposition 5.7.** If \( x, y \in \mathbb{R}^d, x = \rho x', y = vy', \) then
\[
\int_{S^{d-1}} B_{\kappa,a}(x,vy')Y_j^n(y') d\sigma_\kappa(y') = e^{-\frac{i\pi n}{2}} a_\kappa \frac{\Gamma(2\lambda_\kappa/a + 1)}{\Gamma(2\lambda_\kappa + j)/a + 1)} j_{2(\lambda_\kappa + j)}(2a) Y_j^n(x).
\]

Denote by \( \mathcal{S}(\mathbb{R}_+) \) the subspace of even functions from \( \mathcal{S}(\mathbb{R}) \).

**Proposition 5.8.** For irrational \( a \) and a nontrivial function \( f \in \mathcal{S}(\mathbb{R}^d) \), we have \( \mathcal{F}_{\kappa,a}(f) \notin \mathcal{S}(\mathbb{R}^d) \).

**Proof.** If. Assume that \( f(x) = \rho^n \psi(\rho)Y_j^n(x') \), where \( n \in \mathbb{Z}_+, x = \rho x', \) and \( \psi \in \mathcal{S}(\mathbb{R}_+) \). Since \( \rho^n Y_j^n(x') = Y_j^n(x) \) is the homogeneous polynomial of degree \( n \), then \( f(x) = \psi(|x|)Y_j^n(x) \in \mathcal{S}(\mathbb{R}^d) \). If \( y = vy' \), \( \rho = (a/2)^{1/2} u^{2/a} \), then by [13]
\[
\mathcal{F}_{\kappa,a}(f)(y) = e^{-\frac{i\pi n}{2}} a_\kappa \psi(\rho) \int_0^\infty \psi(|x|) Y_j^n(x) \left( \frac{2}{a} \right)^{1/2} u^{a/2} d\nu_{\lambda,a}(\rho)
\]
\[
= e^{-\frac{i\pi n}{2}} a_\kappa \psi(\rho) \int_0^\infty \psi \left( \left( \frac{a}{2} \right)^{1/2} u^{2/a} \right) j_{2(\lambda_\kappa + j)}(2a) u^{a/2} d\nu_{2(\lambda_\kappa + j)}(u).
\]

Moreover, following the ideas used in Example 5.1, we obtain that the function
\[
g_\kappa(v) = \int_0^\infty \psi \left( \left( \frac{a}{2} \right)^{1/2} u^{2/a} \right) j_{2(\lambda_\kappa + j)}(2a) u^{a/2} d\nu_{2(\lambda_\kappa + j)}(u)
\]
with irrational \( a \) cannot decrease rapidly as \( v \to \infty \) and it has finite smoothness at the origin. Therefore, it follows that \( \mathcal{F}_{\kappa,a}(f) \notin \mathcal{S}(\mathbb{R}^d) \).
2. Let now \( f \in \mathcal{S}(\mathbb{R}^d) \) be any non-zero function. Then its spherical \( \kappa \)-harmonic expansion is given by

\[
f(px') = \sum_{n=0}^{\infty} \sum_{j=1}^{l_n} f_{nj}(\rho) Y_n^j(x'), \quad f_{nj}(\rho) = \int_{\mathbb{S}^{d-1}} f(px') Y_n^j(x') d\sigma_n(x')
\]

(see [14]). Since the subspaces \( \mathcal{H}_n^d(v_n) \) are orthogonal, then \( f_{nj}(0) = 0, \ j = 0, 1, \ldots, n - 1. \) Changing variables \( x' \to -x' \) implies \( f_{nj}(-\rho) = (-1)^n f_{nj}(\rho). \) Hence, setting a non-zero function \( g_{nj}(f)(x) = f_{nj}(\rho) Y_n^j(x') \in \mathcal{S}(\mathbb{R}^d), \) we have \( g_{nj}(x) = \rho^n \psi(\rho) Y_n^j(x') \) with some \( \psi \in \mathcal{S}(\mathbb{R}_+). \)

In order to apply the results obtained in case 1, we need to show that

\[
\mathcal{F}_{\kappa,a}(g_{nj}(f))(y) = g_{nj}(\mathcal{F}_{\kappa,a}(f))(y).
\]

Indeed, we note that

\[
\mathcal{F}_{\kappa,a}(g_{nj}(f))(y) = e^{-i\pi n/a} Y_n^j(y) H_{\lambda+n,a}(\psi(v)).
\]

On the other hand, in view of Proposition 5.7 we deduce that

\[
g_{nj}(\mathcal{F}_{\kappa,a}(f))(y) = Y_n^j(y') \int_{\mathbb{S}^{d-1}} \mathcal{F}_{\kappa,a}(f)(vy') Y_n^j(y') d\sigma_{\kappa,a}(y')
\]

\[
= e^{-i\pi n/a} Y_n^j(y) \int_{\mathbb{R}^d} \frac{f(x) \Gamma(2\lambda_n/a + 1)}{a^{2n/a} \Gamma(2(\lambda_n + n)/a + 1)} j_2^{i(\lambda+n)} \left( \frac{2}{a} (\rho v)^{a/2} \right) Y_n^j(x) d\mu_{\kappa,a}(x)
\]

\[
= e^{-i\pi n/a} Y_n^j(y) \int_{0}^{\infty} \psi(\rho) j_2^{i(\lambda+n)} \left( \frac{2}{a} (\rho v)^{a/2} \right) d\nu_{\lambda+n,a}(\rho)
\]

\[
= e^{-i\pi n/a} Y_n^j(y) H_{\lambda+n,a}(\psi(v)).
\]

Assuming here that \( f, \mathcal{F}_{\kappa,a}(f) \in \mathcal{S}(\mathbb{R}^d) \) yields \( g_{nj}(f), \mathcal{F}_{\kappa,a}(g_{nj}(f)) \in \mathcal{S}(\mathbb{R}^d), \) which contradicts the first case.

Summarizing, the generalized Fourier transform for irrational \( a \) drastically deforms even very smooth functions. It was mentioned in [1] Chap. 5] that the generalized Fourier transform has a finite order only for rational \( a. \) Therefore, the case of irrational \( a \) is of little interest in harmonic analysis.

6. NON-DEFORMED UNITARY TRANSFORMS GENERATED BY \( \mathcal{F}_{\kappa,a} \)

Let us study the case \( a = \frac{2}{2r+1} \) and \( \lambda = \lambda_{\kappa,a} = (2\kappa - 1)/a \geq -1/2 \) in more detail. Recall that \( A \) is given by (24) and we can also assume that \( x, u \in \mathbb{R}. \)

Since

\[
\int_{-\infty}^{\infty} |f(x)|^2 d\mu_{\kappa,a}(x) = \int_{-\infty}^{\infty} |Af(u)|^2 d\nu_{\lambda}(u), \quad d\nu_{\lambda}(u) = \frac{|u|^{2\lambda+1} du}{2^{\lambda+1} \Gamma(\lambda + 1)},
\]

the linear operator \( A: L^2(\mathbb{R}, d\mu_{\kappa,a}) \to L^2(\mathbb{R}, d\nu_{\lambda}) \) is an isometric isomorphism. The inverse operator is given by \( A^{-1} g(x) = g((2r + 1)^{1/2} x^{1/(2r+1)}). \)

In view of (7) and (26), \( B_{\kappa,a}(x,y) = e_{2r+1}(uv, \lambda) \) and

\[
A \mathcal{F}_{\kappa,a}(f)(v) = \int_{\mathbb{R}} e_{2r+1}(uv, \lambda) Af(u) \ d\nu_{\lambda}(u).
\]
This formula defines the non-deformed transform $\mathcal{F}_r^\lambda$, for $\lambda > -1/2$ and $r \in \mathbb{Z}_+$,

$$\mathcal{F}_r^\lambda(g)(v) = \int_{-\infty}^{\infty} e_{2r+1}(uv, \lambda)g(u) \, d\nu_\lambda(u)$$

$$= \int_{-\infty}^{\infty} (j_\lambda(uv) + i(1 - t)^{r+1}) (uv)^{2r+1}(\lambda + 1)_{2r+1} j_{2r+1}(uv) \, d\nu_\lambda(u)$$

$$= c_\lambda \int_{-\infty}^{\infty} \int_{-1}^{1} (1 - t^2)^{-1/2}(1 + P_{2r+1}(\lambda - 1/2)(t)) e^{-iuv\lambda} dt \, d\nu_\lambda(u).$$

Moreover, its kernel satisfies the estimate $|e_{2r+1}(uv, \lambda)| \leq M_\lambda < \infty$ and, importantly, $M_\lambda = 1$ for $\lambda \geq 0$. If $r = 0$, we recover the one-dimensional Dunkl transform.

Below we study an invariant subspaces ($\subset C^\infty$) of the $\mathcal{F}_r^\lambda$ transform. The Plancherel theorem for $\mathcal{F}_{\kappa,a}$ given by

$$\int_{-\infty}^{\infty} |\mathcal{F}_{\kappa,a}(f)(y)|^2 \, d\mu_{\kappa,a}(y) = \int_{-\infty}^{\infty} |f(x)|^2 \, d\mu_{\kappa,a}(x), \quad f \in L^2(\mathbb{R}, d\mu_{\kappa,a}),$$

implies that $\mathcal{F}_r^\lambda$ is a unitary operator in $L^2(\mathbb{R}, d\nu_\lambda)$, i.e.,

$$\int_{-\infty}^{\infty} |\mathcal{F}_r^\lambda(g)(v)|^2 \, d\nu_\lambda(v) = \int_{-\infty}^{\infty} |g(u)|^2 \, d\nu_\lambda(u).$$

Since the reverse operator satisfies $(\mathcal{F}_{\kappa,a})^{-1}(f)(x) = \mathcal{F}_{\kappa,a}(f)(-x)$ [4, Theorem 5.3], we have

$$(\mathcal{F}_r^\lambda)^{-1}(f)(u) = \int_{-\infty}^{\infty} e_{2r+1}(uv, \lambda)f(v) \, d\nu_\lambda(v).$$

If $g, \mathcal{F}_r^\lambda(g) \in L^1(\mathbb{R}, d\nu_\lambda)$, then one may assume that $g, \mathcal{F}_r^\lambda(g) \in C_{\nu}(\mathbb{R})$. Moreover, the inversion formula

$$g(u) = \int_{-\infty}^{\infty} e_{2r+1}(uv, \lambda)\mathcal{F}_r^\lambda(g)(v) \, d\nu_\lambda(v)$$

holds not only in $L_2$ sense but also pointwise.

Considering the derivatives of the kernel $e_{2r+1}(uv, \lambda)$, we note that

$$\partial^n e_{2r+1}(uv, \lambda) = (iu)^n c_\lambda \int_{-1}^{1} t^n (1 - t^2)^{-1/2}(1 + P_{2r+1}(\lambda - 1/2)(t)) e^{-iuv\lambda} dt,$$

and so,

$$|\partial^n e_{2r+1}(uv, \lambda)| \leq M_\lambda |u|^n.$$ 

Then, for $g \in \mathcal{S}(\mathbb{R})$, we have

$$|\partial^n \mathcal{F}_r^\lambda(g)(v)| = \left| \int_{-\infty}^{\infty} g(u)\partial^n e_{2r+1}(uv, \lambda) \, d\nu_\lambda(u) \right| \leq M_\lambda \int_{-\infty}^{\infty} |u|^n |g(u)| \, d\nu_\lambda(u) < \infty. \quad (30)$$

Therefore, $\mathcal{F}_r^\lambda(\mathcal{S}(\mathbb{R})) \subset C^\infty(\mathbb{R})$. However, $\mathcal{F}_r^\lambda(\mathcal{S}(\mathbb{R})) \not\subset \mathcal{S}(\mathbb{R})$. Indeed, assuming $g, \mathcal{F}_r^\lambda(g) \in \mathcal{S}(\mathbb{R})$, by orthogonality of the Gegenbauer polynomials for $s = 0, 1, \ldots, r - 1$,

$$\int_{-1}^{1} t^{2s+1}(1 - t^2)^{-1/2}(1 + P_{2r+1}(\lambda - 1/2)(t)) \, dt = 0$$

and

$$\partial^{2s+1} \mathcal{F}_r^\lambda(g)(0) = 0, \quad \partial^{2s+1} g(0) = 0, \quad (31)$$

which is not true for arbitrary $g$.

Put for $n \in \mathbb{Z}_+$

$$\mathcal{S}_n(\mathbb{R}) = \{ g \in \mathcal{S}(\mathbb{R}): \partial^{2s+1} g(0) = 0, \quad s = 0, 1, \ldots, n - 1 \}, \quad \mathcal{S}_0(\mathbb{R}) = \mathcal{S}(\mathbb{R}).$$

The set $\mathcal{S}_n(\mathbb{R})$ is dense in $L^2(\mathbb{R}, d\mu_{\kappa,a})$ and in $L^2(\mathbb{R}, d\nu_\lambda)$. 
Example 6.1. Consider the function \( g_{2s+1}(u) = u^{2s+1}e^{-u^2} \in \mathcal{S}_s(\mathbb{R}), \ s \in \mathbb{Z}_+ \). By means of [4, 7, 2, Chap. VIII, 8.6(14)], and [1, Chap. VI, 6.1], we get

\[
\mathcal{F}^\lambda_r(g_{2s+1})(v) = i(-1)^{r+1}v^{-\lambda} \int_0^\infty u^{\lambda+2s+2}e^{-u^2} J_{\lambda+2r+1}(uv) \, du
\]

\[
= ic_{r,\lambda,s} v^{2r+1} \sum_{l=0}^\infty \frac{(\lambda + s + r + 2)l}{(\lambda + 2r + 2)l} \left( -\frac{v^2}{4} \right)^l
\]

\[
= ic_{r,\lambda,s} v^{2r+1} \Phi(\lambda + s + r + 2, \lambda + 2r + 2, -\frac{v^2}{4}), \quad c_{r,\lambda,s} > 0.
\]

We consider two cases. If \( s = 0, 1, \ldots, r - 1 \), then asymptotics as \( v \to \infty \) [3, Chap. VI.6.13.1] implies

\[
\Phi(\lambda + s + r + 2, \lambda + 2r + 2, -\frac{v^2}{4}) = \Gamma(\lambda + 2r + 2)\left(\frac{v^2}{4}\right)^{-\lambda+2r+2} \left(1 + O\left(\frac{1}{v^2}\right)\right),
\]

that is, \( \mathcal{F}^\lambda_r(g_{2s+1}) \notin \mathcal{S}(\mathbb{R}) \). If \( s \geq r \), then applying the Kummer transform [3, Chap. VI.6.3.7] gives us

\[
\mathcal{F}^\lambda_r(g_{2s+1})(v) = ic_{r,\lambda,s} v^{2r+1} e^{-2v^2/4} \sum_{l=0}^{s-r} \left( \frac{v^2}{4} \right)^l,
\]

that is, \( \mathcal{F}^\lambda_r(g_{2s+1})(v) \in \mathcal{S}_r(\mathbb{R}) \).

Since \( g_{2r+1} \in \mathcal{S}_r(\mathbb{R}) \) and \( \mathcal{F}^\lambda_r(g_{2r+1})(v) \in \mathcal{S}_r(\mathbb{R}) \), one can conjecture that \( \mathcal{F}^\lambda_r(\mathcal{S}_r(\mathbb{R})) = \mathcal{S}_r(\mathbb{R}) \). In order to show this (see Proposition 6.3), we will need some auxiliary results.

Recall that for the weight function \( |x|^{2\lambda+1} \) the differential-difference Dunkl operator of the first and second order are given by

\[
T_{\lambda+1/2} g(u) = \partial g(u) + (\lambda + 1/2) \frac{g(u) - g(-u)}{u},
\]

\[
\Delta_{\lambda+1/2} g(u) = T_{\lambda+1/2}^2 g(u) = \partial^2 g(u) + \frac{2\lambda + 1}{u} \partial g(u) - (\lambda + 1/2) \frac{g(u) - g(-u)}{u^2}
\]

(see [1, 20]). Let us define the operator

\[
\delta_{\lambda} g(u) = T_{\lambda+1/2}^2 g(u) - 2r(\lambda + r + 1) \frac{g(u) - g(-u)}{u^2},
\]

which is obtained by changing variables \( x = x(u) \) as in (20) in the Dunkl Laplacian

\[
\Delta_{\alpha} f(x) = \frac{(2r + 1)^{2r-1}}{u^{4r-1}} \delta_{\lambda} g(u).
\]

By direct calculations we verify that the kernel \( e_{2r+1}(uv, \lambda) \) is the eigenfunction of \( \delta_{\lambda} \):

\[
(\delta_{\lambda})_u e_{2r+1}(uv, \lambda) = -|v|^2 e_{2r+1}(uv, \lambda).
\]

Using [20, Proposition 2.18] for \( g \in \mathcal{S}(\mathbb{R}) \), we have

\[
\int_{-\infty}^{\infty} T_{\lambda+1/2}^2 g(u)e_{2r+1}(uv, \lambda) \, d\nu_l(u) = \int_{-\infty}^{\infty} g(u)(T_{\lambda+1/2}^2)_u e_{2r+1}(uv, \lambda) \, d\nu_l(u).
\]

Suppose \( g \in \mathcal{S}_1(\mathbb{R}) \), then

\[
\frac{g(u) - g(-u)}{u^2} = \frac{2g_0(u)}{u^2} \in \mathcal{S}(\mathbb{R})
\]
and
\[
\int_{-\infty}^{\infty} \frac{g(u) - g(-u)}{u^2} e_{2r+1}(uv, \lambda) \, d\nu_{\lambda}(u) = \int_{-\infty}^{\infty} \frac{g(u) e_{2r+1}(uv, \lambda) - e_{2r+1}(-uv, \lambda)}{u^2} \, d\nu_{\lambda}(u).
\]

This, \((33)\) and \((34)\) for any \(g \in S_1(\mathbb{R})\) yield
\[
\int_{-\infty}^{\infty} \delta_{\lambda} g(u) e_{2r+1}(uv, \lambda) \, d\nu_{\lambda}(u) = \int_{-\infty}^{\infty} g(u) (\delta_{\lambda}) u e_{2r+1}(uv, \lambda) \, d\nu_{\lambda}(u)
= -|v|^2 \int_{-\infty}^{\infty} g(u) e_{2r+1}(uv, \lambda) \, d\nu_{\lambda}(u). \tag{35}
\]

Applying \((35)\) for \(g \in S_n(\mathbb{R})\), we get
\[
\int_{-\infty}^{\infty} \delta_{\lambda}^n g(u) e_{2r+1}(uv, \lambda) \, d\nu_{\lambda}(u) = (-1)^n |v|^{2n} \int_{-\infty}^{\infty} g(u) e_{2r+1}(uv, \lambda) \, d\nu_{\lambda}(u). \tag{36}
\]

Lemma 6.2. Suppose \(g \in S(\mathbb{R})\) and
\[
a_n(g)(v) = \sum_{k=0}^{n} \sum_{l=0}^{k} \frac{1}{(k-l)! (2l+1)!} \frac{\partial^{2l+1} g(0)}{v^{2k+1} e^{-v^2}},
\]
then \(g - a_n(g) \in S_{n+1}(\mathbb{R})\).

Proof. We have \(g - a_n(g) \in S(\mathbb{R})\). Applying the Leibniz rule for \(s = 0, 1, \ldots, n\),
\[
\frac{\partial^{2s+1} a_n(g)(0)}{(2s+1)!} = \frac{1}{(2s+1)!} \sum_{k=0}^{s} \sum_{l=0}^{k} \frac{\partial^{2l+1} g(0)}{(k-l)! (2l+1)!} \frac{(2s+1)}{2k+1} (2k+1)! \partial^{2s-2k} (e^{-u^2})(0)
= \sum_{k=0}^{s} \sum_{l=0}^{k} \frac{\partial^{2l+1} g(0)}{(k-l)! (2l+1)!} \frac{(2s+1)}{2k+1} \partial^{s-l} (e^{-u^2})(0)
= \frac{1}{(s-l)!} \sum_{l=0}^{s-l} \frac{(-1)^{s-l} \partial^{2l+1} g(0)}{(2l+1)!} (1-1)^{s-l}
\]
\[
= \frac{\partial^{2s+1} g(0)}{(2s+1)!}.
\]

\[\square\]

Proposition 6.3. Let \(\lambda > -1/2\) and \(r \in \mathbb{Z}_+\). We have \(\mathcal{F}_r^\lambda(S_r(\mathbb{R})) = S_r(\mathbb{R})\).

If \(r = 0\) we recover the result by de Jeu \cite{deJe} for the Dunkl transform.

Proof. Let \(r \in \mathbb{N}\) and \(g \in S_r(\mathbb{R})\). It is enough to show that \(g\) satisfies the condition:
\[
\partial^m (v^{2n} \mathcal{F}_r^\lambda(g)(v)) \text{ is bounded for any } m \in \mathbb{Z}_+, ~ n \geq r + 1. \tag{37}
\]

We have
\[
\partial^m (v^{2n} \mathcal{F}_r^\lambda(g)(v)) = \partial^m (v^{2n} \mathcal{F}_r^\lambda(g - a_{n-1}(g))(v)) + \partial^m (v^{2n} \mathcal{F}_r^\lambda(a_{n-1}(g))(v)).
\]
Since \(\partial^{2l+1} g(0) = 0, l = 0, 1, \ldots, r - 1, \) then
\[
a_{n-1}(g)(v) = \sum_{k=r}^{n} \sum_{l=r}^{k} \frac{1}{(k-l)! (2l+1)!} \frac{\partial^{2l+1} g(0)}{v^{2k+1} e^{-v^2}} \in S_r(\mathbb{R}).
\]

By \((32)\), condition \((37)\) is valid for \(a_{n-1}(g)\). By Lemma 6.2 \(g - a_{n-1}(g) \in S_n(\mathbb{R})\). In light of \((36)\), \(v^{2n} \mathcal{F}_r^\lambda(g - a_{n-1}(g))(v)\) is the \(\mathcal{F}_r^\lambda\)-transform of the function \((-1)^n \delta_{\lambda}^n (g - a_{n-1}(g)) \in S(\mathbb{R})\). Applying for this transform inequality \((30)\), we get the property \((37)\) for \(g - a_{n-1}(g)\). Therefore, \(\mathcal{F}_r^\lambda(g) \in S(\mathbb{R})\). By virtue of \((29)\) and \((31)\), \(\mathcal{F}_r^\lambda(g) \in S_r(\mathbb{R})\). \[\square\]
Remark 6.4. An alternative proof of Proposition 6.3 reads as follows. Let \( g \in S_r(\mathbb{R}) \). In light of (25)–(27), we have the following representation

\[
F_\lambda^r(g)(v) = \int_{-\infty}^{\infty} j_\lambda(uv)g(u) \, d\nu_\lambda(u) + \frac{i(-1)^{r+1}v^{2r+1}}{2^{2r+1}(\lambda + 1)_{2r+1}} \int_{-\infty}^{\infty} u^{2r+1} j_{\lambda + 2r+1}(uv)g(u) \, d\nu_\lambda(u) \\
= \int_{0}^{\infty} j_\lambda(uv)g_\epsilon(u) \, d\nu_\lambda(u) + i(-1)^{r+1}v^{2r+1} \int_{0}^{\infty} j_{\lambda + 2r+1}(uv)u^{-(2r+1)}g_\epsilon(u) \, d\nu_{\lambda + 2r+1}(u) \\
= H_\lambda(g_\epsilon)(v) + i(-1)^{r+1}v^{2r+1} H_{\lambda + 2r+1}(u^{-(2r+1)}g_\epsilon)(v) = F_1(v) + v^{2r+1}F_2(v),
\]

where even functions \( F_1, F_2 \in S(\mathbb{R}) \). Therefore, \( F_\lambda^r(g) \in S_r(\mathbb{R}) \). The first proof was given to underline the important properties of \( F_\lambda^r \)-transform and its kernel.

Since \( f \in S(\mathbb{R}) \) implies \( Af \in S_r(\mathbb{R}) \), Proposition 6.3 yields the following result.

Corollary 6.5. Let \( a = \frac{2}{2r+1} \) and \( \kappa \geq \frac{r}{2r+1} \). If \( f \in S(\mathbb{R}) \), then \( F_{\kappa,a}(f)((2r + 1)^{-1/2}v^{2r+1}) \in S_r(\mathbb{R}) \) or, equivalently, \( F_{\kappa,a}(f)(v) = g((2r + 1)^{1/2}v^{1/(2r+1)}), g \in S_r(\mathbb{R}) \); cf. (28).

Now we discuss the case \( \lambda = -1/2 \).

Remark 6.6. If \( \lambda = -1/2, r \in \mathbb{Z}_+ \), then

\[
\delta_{-1/2}g(u) = \partial^2 g(u) - r(2r + 1) \frac{g(u) - g(-u)}{u^2}, \quad e_1(uv, -1/2) = e^{-iuv} (r = 0).
\]

Taking into account (7) and passing to the limit in (7) as \( \lambda \to -1/2 \), we deduce that for \( r \geq 1 \)

\[
e_{2r+1}(uv, -1/2) = \cos(uv) + i(-1)^{r+1} \frac{(uv)^{2r+1}}{2^{2r+1}(1/2)_{2r+1}} j_{2r+1/2}(uv) \\
= e^{-iuv} - (r + 1/2) \int_{-1}^{r-1} \sum_{s=0}^{r-1} (-1)^s \binom{r}{s+1} \frac{r+3/2}{s!} (1-t^2) e^{-iuv} dt.
\]

Taking this into account and analyzing the proofs above, we note that all mentioned results in this section for the transform \( F_\lambda^r \) in the case \( \lambda > -1/2 \) are also valid for \( \lambda = -1/2 \). In particular, \( F_{-1/2}^r, r \in \mathbb{Z}_+ \), are the unitary transforms in the non-weighted \( L^2(\mathbb{R}, dx) \), where \( F_{0}^{-1/2} \) corresponds to the classical Fourier transform.

References

[1] H. Bateman, A. Erdélyi, et al., Higher Transcendental Functions, II, McGraw Hill Book Company, New York, 1953.
[2] H. Bateman, A. Erdélyi, et al., Tables of Integral Transforms, II, McGraw-Hill Book Company, New York, 1954.
[3] H. Bateman, A. Erdélyi, et al., Higher Transcendental Functions, I, McGraw Hill Book Company, New York, 1953.
[4] Ben Saïd S., T. Kobayashi, and B. Orsted, Laguerre semigroup and Dunkl operators, Compos. Math., 148 (2012), no. 4, 1265–1336.
[5] H. De Bie, The kernel of the radially deformed Fourier transform, Integral Transforms Spec. Funct., 24 (2013), 1000–1008.
[6] H. De Bie and P. Lian, The Dunkl kernel and intertwining operator for dihedral groups, J. Funct. Anal. 280 (2021), no. (7), 109392.
[7] M. Bouabata, S. Negzaoui, M. Sifi, A new product formula involving Bessel functions, Int. Transf. Spec. Funct., 33 (2022), no. (3), 247–263.
[8] M.F.E. de Jeu, The Dunkl transform, Invent. Math. 113 (1993), 147–162.
[9] J.F. Chamayou and J. Wesolowski, Lauricella and Humbert functions through probabilistic tools, Integral Transf. Spec. Funct. 20 (2009), 529–538.
ON THE KERNEL OF THE \((\kappa, a)\)-GENERALIZED FOURIER TRANSFORM

[10] D. Constales, H. De Bie, and P. Lian, *Explicit formulas for the Dunkl dihedral kernel and the \((\kappa, a)\)-generalized Fourier kernel*, J. Math. Anal. Appl. 460 (2018), no. 2, 900–926.

[11] L. Deleaval and N. Demni, *Generalized Bessel functions of dihedral-type: expression as a series of confluent Horn functions and Laplace-type integral representation*, The Ramanujan Journal 54 (2021), 197–217.

[12] C.F. Dunkl, *Hankel transforms associated to finite reflections groups*, Contemp. Math. 138 (1992), 123–138.

[13] C. F. Dunkl and Y. Xu, *Orthogonal Polynomials of Several Variables*, Cambridge Univ. Press, 2001.

[14] H. Exton, *Multiple Hypergeometric Functions and Applications*, Ellis Horwood, 1983.

[15] D.V. Gorbachev, V.I. Ivanov, and S.Yu. Tikhonov, *Pitt’s inequalities and uncertainty principle for generalized Fourier transform*, Int. Math. Res. Notices 23 (2016), 7179–7200.

[16] R. Howe, *The oscillator semigroup, in the mathematical heritage of Hermann Weyl*, Proc. Symp. Pure Math. 48, R.O. Wells, Ed. AMS Providence, 1988.

[17] P. Humbert, *The confluent hypergeometric functions of two variables*, Proc. Roy. Soc. Edinburgh 41 (1920), 73–82.

[18] T.R. Johansen, *Weighted inequalities and uncertainty principles for the \((k; a)\)-generalized Fourier transform*, Internat. J. Math. 27 (2016), no. 3, 1650019.

[19] T. Kobayashi and G.Man, *The Schrödinger model for the minimal representation of the indefinite orthogonal group \(O(p, q)\)*, Memoirs of the American Mathematical Societies. Providence, RI: Amer. Math. Soc. 212 (2011), no. 1000.

[20] M. Rösler, *Dunkl operators. Theory and applications: in Orthogonal Polynomials and Special Functions*, Lecture Notes in Math. Springer-Verlag, 1817 (2002), 93–135.

[21] M. Rösler, *A positive radial product formula for the Dunkl kernel*, Trans. Amer. Math. Soc. 355 (2003), no. 6, 2413–2438.

[22] M. Rösler, *Positivity of Dunkl’s intertwining operator*, Duke Math. J. 98 (1999), 445–463.

[23] G. Szegö, *Orthogonal polynomials*, Third Edition, American Mathematical Society, 1974.

[24] E.C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, 2nd ed., Clarendon Press, Oxford, 1948.

[25] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, 1966.

D. V. GORBACHEV, TULA STATE UNIVERSITY, DEPARTMENT OF APPLIED MATHEMATICS AND COMPUTER SCIENCE, 300012 TULA, RUSSIA

Email address: dvvgmail@mail.ru

V. I. IVANOV, TULA STATE UNIVERSITY, DEPARTMENT OF APPLIED MATHEMATICS AND COMPUTER SCIENCE, 300012 TULA, RUSSIA

Email address: ivaleryi@mail.ru

S. YU. TIKHONOV, CENTRE DE RECERCA MATÈMATICA, CAMPUS DE BELLATERRA, EDIFICI C 08193 BELLATERRA (BARCELONA), SPAIN; ICREA, PG. Lluís Companys 23, 08010 BARCELONA, SPAIN, AND UNIVERSITAT AUTÒNOMA DE BARCELONA.

Email address: stikhonov@crm.cat