Vanishing theorems for abelian varieties over finite fields

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Abstract

Let $\kappa$ be a field, finitely generated over its prime field, and let $k$ denote an algebraically closed field containing $\kappa$. For a perverse $\mathbb{Q}_\ell$-adic sheaf $K_0$ on an abelian variety $X_0$ over $\kappa$, let $K$ and $X$ denote the base field extensions of $K_0$ and $X_0$ to $k$. Then, the aim of this note is to show that the Euler-Poincare characteristic of the perverse sheaf $K$ on $X$ is a non-negative integer, i.e.

$$\chi(X, K) = \sum_{\nu} (-1)^\nu \dim_{\mathbb{Q}_\ell}(H^\nu(X, K)) \geq 0.$$  

This generalizes an analogous result of Franecki and Kapranov [FK] over fields of characteristic zero.

The proof of [FK] for the above estimate for the Euler-Poincare characteristic of perverse sheaves on abelian varieties over fields of characteristic zero relies on methods from the theory of $D$-modules via the Dubson-Riemann-Roch formula for characteristic cycles. In fact, one should expect that there exists a similar Riemann-Roch theorem also over fields of positive characteristic, extending the results of [AS] and generalizing the Grothendieck-Ogg-Shafarevich formula for the Euler-Poincare characteristic of sheaves on curves. However, in the absence of such deep results on wild ramification we will follow a different approach using methods of Gabber and Loeser [GL], based on Ekedahl’s adic formalism.

Let $k$ denote the algebraic closure of a finite field $\kappa$ of characteristic $p$. For an abelian variety $X_0$ over $\kappa$, let $X$ be the base extension of $X_0$ from $\kappa$ to $k$ for a fixed embedding $\kappa \subset k$. Let $\Lambda$ denote $\mathbb{Q}_\ell$ for some prime $\ell \neq p$. We fix a suspended subcategory $D = D(X)$ of the derived category $D^b_c(X, \Lambda)$ of $\Lambda$-adic sheaves with bounded constructible cohomology sheaves. We assume that $D$ satisfies the properties formulated in [KrW,§5]. An example is the category $D$ of all $K$ in $D^b_c(X, \Lambda)$ obtained by base extension from some objects $K_0$ in $D^b_c(X_0, \Lambda)$ with the property that $K$ decomposes into a direct sum of complex shifts of irreducible perverse sheaves on $X$. Let $P = P(X)$ denote the full subcategory of objects in $D$ that are perverse sheaves. The convolution product $\ast$ on $D$, induced by the group law on $X$, makes $(D, \ast)$ into a rigid $\Lambda$-linear monoidal symmetric category. But in general, the convolution product does not preserve the subcategory $P$. 

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By definition, a character $\chi : \pi_1(X) \to \Lambda^*$ of the etale fundamental group $\pi_1(X)$ of $X$ is a continuous homomorphism with values in the group of units $\sigma_\Lambda$ of the ring of integers $\sigma_\Lambda$ of a finite extension field $E_\Lambda \subset \Lambda$ of $\mathbb{Q}_\ell$. Associated to a character $\chi$, there is a smooth $\Lambda$-adic sheaf $L_\chi$ on $X$. For $K \in D$ resp. $K \in P$, the twist $K_\chi := K \otimes_\Lambda L_\chi$ is in $D$ resp. in $P$. Let $\pi_1(X)_\ell$ denote the maximal pro-$\ell$ quotient of $\pi_1(X)$. Any character $\chi$ of $\pi_1(X)$ is the product of a character $\chi_f$ of finite order prime to $\ell$, and a character that factorizes over the pro-$\ell$ quotient $\pi_1(X)_\ell$ of $\pi_1(X)$.

As in [GL, p. 509] consider the ring $\Omega_X := \sigma_\Lambda[[\pi_1(X)_\ell]]$, a complete noetherian local ring of Krull dimension $1 + 2 \dim(X)$. For generators $\gamma$ of $\pi_1(X)_\ell \cong (\mathbb{Z}_\ell)^{2 \dim(X)}$, this ring is isomorphic to the formal power series ring $\sigma_\Lambda[[t_1, \ldots, t_n]]$ in the variables $t_i = \gamma - 1$ for $n = 2 \dim(X)$. For $C(X)_\ell = Spec(\Lambda \otimes_{\sigma_\Lambda} \sigma_\Lambda[[\pi_1(X)_\ell]])$ as in [GL, 3.2], define the scheme $C(X)$ as the disjoint union $\bigcup_{\chi} \{\chi\} \times C(X)_\ell$, for $\chi$ running over the characters $\chi$ of $\pi_1(X)$ of finite order prime to $\ell$. By [GL,A.2.2.3] the closed points of $C(X)_\ell$ are the $\Lambda$-valued points of $C(X)_\ell$. The $\Lambda$-valued points of the scheme $C(X)$ can be identified with the ‘continuous’ characters $\chi : \pi_1(X) \to \Lambda^*$.

As in loc. cit. there exists a continuous character $can_X : \pi_1(X) \to \Omega_X$ and an associated local system $L_X$ on $X$, which is locally free of rank 1 over $\Omega_X$. For $K \in D^{b}_X(X, \sigma_\Lambda)$ we consider $K \otimes_{\sigma_\Lambda} L_X$ as an object in $D^{b}_X(X, \Omega_X)$. For the structure morphism $f : X \to Spec(k)$, following [GL, p.512 and A.1] we define the Fourier transform $\mathcal{F} : D^{b}_X(X, \sigma_\Lambda) \to D^{b}_{coh}(\Omega_X)$ by $\mathcal{F}(K) = Rf_{\ast}(K \otimes_{\sigma_\Lambda} \Omega_X)$ (analogous to the Mellin transform in loc. cit). By proposition A.1 of loc. cit. the functor defined by extension of scalars $- \otimes^{L}_{\Omega_X} \Omega_X$ commutes with direct images for arbitrary morphisms $f : X \to Y$ between varieties $X, Y$ over $k$. By inverting $\ell$ and passing to the direct limit over all $\sigma_\Lambda \subset \Lambda$, we easily see that $\mathcal{F}$ induces a functor from $D$ to the derived category $D^{b}_{coh}(C(X)_\ell)$ of $C(C(X)_\ell)$-module sheaf complexes with bounded coherent sheaf cohomology (see loc.cit. p. 521). The functor thus obtained

$$\mathcal{F} : (D, *) \to (D^{b}_{coh}(C(X)_\ell), \otimes^{L}_{\Omega_X})$$

is a tensor functor, since $\mathcal{F}$ commutes with the convolution product; this follows from the arguments on p. 518 of [GL]. Similarly $\mathcal{F} : (D, *) \to (D^{b}_{coh}(C(X)), \otimes^{L}_{\Omega_X})$ can be defined as in loc. cit. Furthermore as in [GL, cor. 3.3.2], the specialization $Li^{\ast}_X : D^{b}_{coh}(C(X)_\ell) \to D^{b}_{coh}(\Lambda)$, defined by the inclusion $i_X : \{\chi\} \hookrightarrow C(X)$ of the closed point that corresponds to the character $\chi \in C(X)$, has the property

$$Li^{\ast}_X(\mathcal{F}(K)) = R\Gamma(X, K_\chi).$$

For a complex $M$ of $R$-modules and a prime ideal $p$ of $R$ the small support $supp_R(M) = \{p|k(p) \otimes_{R} M \neq 0\}$ is contained in the support $Supp_R(M) = \{p|M_p \neq 0\}$. 

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The latter is Zariski closed in $\text{Spec}(R)$. For a noetherian ring $R$ and a complex $M$ of $R$-modules with bounded and coherent cohomology $H^\bullet(M)$ both supports coincide: $\text{supp}_R(M) = \text{Supp}_R(M)$. For the regular noetherian ring $R = \Lambda \otimes_{\Lambda} [\pi_1(X)_{\ell}]$ furthermore any object $M$ in $D^b_{\text{coh}}(R) \cong D^b_{\text{coh}}(\mathcal{C}(X)_{\ell})$ is represented by a perfect complex, i.e. a complex of finitely generated projective $R$-modules of finite length. Notice that $L^\chi_f(\mathcal{F}(K)) = k(p) \otimes_R^L \mathcal{F}(K)$ holds for the maximal ideal $p$ of $R$ with residue field $k(p) = R/p$, defined by $\chi$.

By definition, for $K \in \mathcal{P}$ the spectrum $\mathcal{I}(K) \subseteq \mathcal{C}(X)(\Lambda)$ is the set of characters $\chi$ such that $H^\bullet(X, K_\chi) \neq H^0(X, K_\chi)$. Since $\chi(X, K_\chi) = \chi(X, K)$, under the assumption $\chi(X, K) = 0$ the condition $\chi \in \mathcal{I}(K)$ is equivalent to $H^\bullet(X, K_\chi) \neq 0$, and hence equivalent to $\mathcal{R}(X, K_\chi) \neq 0$. Hence for $\chi(X, K) = 0$, $\chi \in \mathcal{C}(X)_{\ell}(\Lambda)$ is in $\mathcal{I}(K)$ if and only if $\mathcal{R}(X, K_\chi) \neq 0$, or equivalently $\chi \in \text{Supp}_R(\mathcal{F}(K))$ holds. This implies

**Lemma 1.** For $K \in \mathcal{P}$ with $\chi(X, K) = 0$, the set of characters $\mathcal{I}(K) \cap \mathcal{C}(X)_{\ell}(\Lambda)$ is the set of closed points of a Zariski closed subset of $\mathcal{C}(X)_{\ell}$.

For simple objects $K$ in $\mathcal{P}$ we defined in [W] an integer in $[0, \dim(X)]$, the degree $v_K$ of $K$, and an irreducible monoidal perverse sheaf $\mathcal{P}_K$ in $\mathcal{P}$. By [W, Lemma 1.4] the Euler-Poincare characteristic $\chi(X, K)$ of $K$ on $X$ is zero if and only if $v_K > 0$; furthermore $\mathcal{P}_K \cong 1$ (unit object) holds if and only if $v_K = 0$. $\mathcal{P}_K$ is called a *monoid* in case $v_K > 0$. If $\chi(X, K) = 0$, the condition $\chi \in \mathcal{I}(K)$ is equivalent to $\mathcal{R}(X, K_\chi) = 0$ and the characters in $\mathcal{I}(K)$ are the closed points of the support of the Fourier transform $\mathcal{F}(K) \in D^b_{\text{coh}}(\mathcal{C}(X))$, a Zariski closed subset of $\mathcal{C}(X)$, from $(A \ast B)_\chi \cong A_\chi \ast B_\chi$ and the split monomorphisms $K[\pm v_K] \hookrightarrow \mathcal{P}_K \ast K$ and $\mathcal{P}_K[\pm v_K] \hookrightarrow K \ast K^\vee$ defined in [W], we see that the assertions $H^\bullet(X, K_\chi) = 0$ and $H^\bullet(X, (\mathcal{P}_K)_\chi) = 0$ are equivalent. Hence

**Lemma 2.** If $v_K > 0$ holds for a simple object $K \in \mathcal{P}$, then $\mathcal{I}(K) = \mathcal{I}(\mathcal{P}_K)$.

If $v_K > 0$ for either $i = 1$ or $i = 2$, by [KrW] all simple constituents $K[n]$ of $K_1 \ast K_2 \cong \bigoplus K[n]$ satisfy $v_K > 0$. In general, the semisimple complexes with simple constituents of vanishing Euler-Poincare characteristic define a tensor ideal $\mathcal{N}_{\text{Euler}}$ in $\mathcal{D}$. All monoids are in this tensor ideal $\mathcal{N}_{\text{Euler}}$. For any semisimple complex $K$ in $\mathcal{N}_{\text{Euler}}$, let $\mathcal{I}(K)$ denote the set of $\chi \in \mathcal{C}(X)(\Lambda)$ for which $H^\bullet(X, K_\chi) \neq 0$. Then $\mathcal{I}(K \oplus K') = \mathcal{I}(K) \cup \mathcal{I}(K')$, and by the Künneth formula

$$\mathcal{I}(K \ast K') = \mathcal{I}(K) \cap \mathcal{I}(K')$$

holds for all semisimple complexes $K, K'$ in $\mathcal{N}_{\text{Euler}}$. 

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Lemma 3. If for a simple perverse sheaf $K$ in $\mathcal{N}_{\text{Euler}} \subset \mathcal{D}$ and a character $\chi_f$ of order prime to $\ell$ the Krull dimension of $\{\chi_f\} \times \mathcal{C}(K)$ is zero, then $K$ is a character twist of the perverse sheaf $\delta_X := \Lambda_X[\dim(X)].$

Proof. We assume $\chi_f = 1$ by twisting $K$. $\mathcal{F}(K)$ is represented by a perfect complex $P$ in $D^b_c(R)$. By assumption the Krull dimension of the support $Y$ of $\mathcal{F}(K)$ in $\mathcal{C}(K)_\ell$ is zero, hence $Y$ is a finite union of closed points. For $\chi$ corresponding to a closed point $y \in Y$, let $m_y$ be the associated maximal ideal of $R$ with residue field $\Lambda_y$. Then $\mathcal{R}^\dagger(X, K_{\chi}) \cong \delta_X^* (\mathcal{F}(K)) \cong P \otimes^L_{\Lambda_y} \Lambda_y$. We claim: $H^i(X, K_{\chi}) \neq 0$ holds for some $i$ with $|i| \geq \dim(X)$; hence $K_{\chi} \cong \delta_X$ and so the lemma follows.

To prove our claim, we replace $R$ by its localization at $m_y$, a regular local ring of dimension $d = 2\dim(X)$. We may assume $P = (0 \to P_2 \to \cdots \to P_b \to 0)$ is minimal, so all $P_i$ are finite free $R$-modules and $d_i \otimes_R \Lambda_y = 0$ holds for the differentials $d_i$. Since $\Lambda_y$ is the only simple module of the local ring $R$, $H^i(P \otimes^L_{\Lambda_y} \Lambda_y) \in D^b_c(|a - 2\dim(X)|, \Lambda_y)$ holds for $P \in D^b_c(|a|, R)$ (use Koszul complexes). Now assume $P_a \neq 0$. Then $H^a(P) \neq 0$ by minimality, and the cone $C$ of $H^a(P) \to P$ has zero cohomology in degrees $\leq a$. Thus $H^i(C \otimes^L_{\Lambda_y} \Lambda_y) = 0$ holds for $i \leq a - 2\dim(R)$ and $H^{a - 2\dim(X)}(P \otimes^L_{\Lambda_y} \Lambda_y) \cong H^{a - 2\dim(X)}(H^a(P) \otimes^L_{\Lambda_y} \Lambda_y)$. By the left exactness of $\text{Tor}_2^{\mathcal{D}}(\mathcal{A}, \Lambda_y)$ then $H^{a - 2\dim(X)}(H^a(P) \otimes^L_{\Lambda_y} \Lambda_y)$ contains $H^{a - 2\dim(X)}(U \otimes^L_{\Lambda_y} \Lambda_y)$, for the socle $U$ of the $R$-module $H^a(P)$. Notice $U$ is nontrivial and a direct sum of simple modules $\Lambda_y$ by our assumptions. Since $\text{Tor}_2^{\mathcal{D}}(\mathcal{A}, \Lambda_y) \cong \Lambda_y$, hence $H^{a - 2\dim(X)}(U \otimes^L_{\Lambda_y} \Lambda_y) \neq 0$. This proves $H^{a - 2\dim(X)}(X, K_{\chi}) \neq 0$. Then similarly $H^b(X, K_{\chi}) \neq 0$ if $P_b \neq 0$. So, our claim follows from $b - (a - 2\dim(X)) \geq 2\dim(X)$.

Lemma 4. For an irreducible perverse sheaf $K$ on $X$, the group $\Delta_K = \{\chi \mid K \cong K_{\chi}\}$ is a subgroup of the group $\mathcal{C}(X) \langle \Lambda \rangle$ of all characters $\chi$ of $\pi_1(X)$. It is a proper subgroup unless $K$ is a skyscraper sheaf. More precisely, let $A$ be the abelian subvariety generated by the support of the perverse sheaf $K$ on $X$ and let $K(A)$ denote the subgroup of characters in $\mathcal{C}(X) \langle \Lambda \rangle$ whose restriction to $A$ becomes trivial. Then $K(A)$ is a subgroup of $\Delta_K$ and the quotient $\Delta_K/K(A)$ is a finite group.

Proof. Suppose $K$ is not a skyscraper sheaf. Then the support $Y$ of $K$ generates an abelian subvariety $A \neq 0$ of $X$. We may replace $X$ by this subvariety $A$. Then the natural morphism $H^1(X, A) \to H^1(Y, A)$ is injective, and hence $\pi_1(Y, y_0) \to \pi_1(X, y_0)$ has finite cokernel [S, lemma VI.13.3, prop. VI.17.14], say of index $C$. There exists a Zariski open dense subset $U$ of $Y$ and a smooth $\Lambda$-adic sheaf $E$ on $U$, defining a $\Lambda$-adic representation $\rho$, such that $K|_U \cong E[\dim(Y)]$. Since $\rho \otimes \chi \cong \rho$
for all $\chi \in \Delta_K$, viewed as characters $\chi$ of $\pi_1(Y, y_0)$, we obtain the following bound 
\[ \# \Delta_K \leq C \cdot \dim_{\Lambda}(\rho) \] 
from the next lemma.

**Lemma 5.** Let $\rho$ be an irreducible representation of a group $\Gamma$ on a finite-dimensional vectorspace over $\Lambda$, and let $\Delta$ be a finite group of abelian characters $\chi : \Gamma \to \Lambda^*$, defining a normal subgroup $\Gamma' = \text{Ker}(\Delta)$ such that $\Gamma/\Gamma' \cong \Delta^*$. Then $\rho \otimes \chi \cong \rho$ for all $\chi \in \Delta$ implies $\rho \cong \text{Ind}_{\Gamma'}^{\Gamma}(\rho')$ for some irreducible representation $\rho'$ of $\Gamma'$. In particular

\[ \# \Delta \leq \# \Delta \cdot \dim_{\Lambda}(\rho') = \dim_{\Lambda}(\rho). \]

**Proof.** For the convenience of the reader we give the proof. If $\rho \cong \text{Ind}_{\Gamma'}^{\Gamma}(\rho_0)$ for some subgroup $\Gamma' \leq \Gamma_0 \leq \Gamma$, we may replace the pair $(\Gamma, \rho)$ by $(\Gamma_0, \rho_0)$. Indeed, $\rho_0 \otimes (\chi|_{\Gamma_0}) \cong \rho_0$ for $\chi \in \Delta$ holds. To show this: $\rho_0$ is a constituent of $\text{Ind}_{\Gamma_0}^{\Gamma}(\rho_0)|_{\Gamma_0} \cong \rho|_{\Gamma_0}$, and therefore also a constituent of $(\rho \otimes \chi)|_{\Gamma_0}$. Hence $\rho_0 \otimes (\chi|_{\Gamma_0}) \cong \rho_0^0$ by Mackey’s lemma for some $s \in \Gamma$, with $s$ a priori depending on $\chi \in \Delta$. But $s \in \Gamma_0$, since otherwise $\rho_0$ could be extended to a projective representation of $\langle \Gamma_0, s \rangle \leq \Gamma$, and this is easily seen to contradict the irreducibility of $\rho \cong \text{Ind}_{\Gamma_0}^{\Gamma}(\rho_0)$. Therefore $s \in \Gamma_0$, and this implies our claim: $\rho_0 \otimes (\chi|_{\Gamma_0}) \cong \rho_0$ for all $\chi \in \Delta$.

Using induction in steps, without loss of generality we can therefore assume that $\rho \not\cong \text{Ind}_{\Gamma_0}^{\Gamma}(\rho_0)$ holds for any $\Gamma_0$ in $\Gamma$ such that $\Gamma' \leq \Gamma_0 \neq \Gamma$. We then have to show $\Gamma = \Gamma'$. If $\Gamma' \neq \Gamma$, we may now also replace the group $\Gamma'$ by some larger group $\Gamma_0$ with prime index in $\Gamma$. Then there exists a character $\chi \in \Delta$ with kernel $\Gamma_0$. By Mackey’s theorem and $\rho \not\cong \text{Ind}_{\Gamma_0}^{\Gamma}(\rho_0)$, the restriction $\rho|_{\Gamma_0}$ is an isotypic multiple $m \cdot \rho_0$ of some irreducible representation $\rho_0$ of $\Gamma_0$. Therefore $(\rho_0)^s \cong \rho_0$ holds for all $s \in \Gamma$. Hence $\rho_0$ can be extended to a representation of $\Gamma$ on the representation space of $\rho_0$ (there is no obstruction for extending the representation since $\Gamma/\Gamma_0$ is a cyclic group). By Frobenius reciprocity, this extension is then isomorphic to $\rho$; so $m = 1$. In other words, the restriction of $\rho$ to $\Gamma_0$ is an irreducible representation of $\Gamma_0$, hence equal to $\rho_0$.

Finally, $\rho \otimes \chi \cong \rho$ implies $\chi \hookrightarrow \rho^\vee \otimes \rho$ (as a one dimensional constituent). Therefore $\bigoplus_{\chi \in \Delta_0} \chi \hookrightarrow \rho^\vee \otimes \rho$, as representations of $\Gamma$. Restricted to $\Gamma_0$, this implies $\# \Delta_0 \cdot 1 \hookrightarrow \rho_0^\vee \otimes \rho_0$, since $\rho|_{\Gamma_0} \cong \rho_0$. But $\text{Hom}_{\Gamma_0}(1, \rho_0^\vee \otimes \rho_0) \cong \text{Hom}_{\Gamma_0}(\rho_0, \rho_0) \cong \Lambda$ since $\rho_0$ is irreducible. Hence $\# \Delta_0 = [\Gamma : \Gamma_0] = 1$. This implies $\Gamma = \Gamma_0$, and hence $\Gamma = \Gamma'$. □

**Proposition 1.** Suppose $\dim(X) > 0$. Then for any finite set $\{P_1, \ldots, P_m\}$ of monoids in $P$, there exist characters $\chi \in \mathcal{C}(X)$ such that $\chi \not\in \bigcup_{i=1}^{m} \mathcal{I}(P_i)$. 

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Proof. Since the spectrum of $R = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}[[x_1, \ldots, x_n]]$ is not the union of finitely many Zariski closed proper subsets for $n = 2\dim(X) > 0$, it suffices that the spectrum $\mathcal{I}(\mathcal{P}) = \mathcal{I}(\mathcal{P}) \cap \mathcal{C}(X)_{\ell}(\Lambda)$ of each monoid $\mathcal{P}$ is the set of closed points of some proper Zariski closed subset of $\mathcal{C}(X)_{\ell}$. We prove this by descending induction on the degree $\nu_{\mathcal{P}}$. For $\nu_{\mathcal{P}} = \dim(X)$ this is clear, since in this case $\mathcal{I}(\mathcal{P})$ is a single point ([W, lemma 1]). For a given monoid $\mathcal{P}$ and fixed $\nu = \nu_{\mathcal{P}} < \dim(X)$, assume our assertion is true for all monoids $\mathcal{P}$ of degree $\nu_{\mathcal{P}} > \nu$. By lemma 4 there exists a character $\chi \in \mathcal{C}(X)_{\ell}$ such that $\mathcal{P}_\chi \not\in \mathcal{P}$. Since $\mathcal{P}$ and $\mathcal{P}_\chi$ have the same degree $\nu = \nu_{\mathcal{P}}$, this implies that all constituents $K[m], K \in \mathcal{P}$ of $\mathcal{P}_\chi$ have associated monoids $\mathcal{P}_k$ of degree $> (\nu_{\mathcal{P}} + \nu_{\mathcal{P}_\chi})/2 = \nu$ by [W, cor. 4, lemma 1]. Hence $\mathcal{I}(\mathcal{P} \ast \mathcal{P}_\chi)$ is contained in a proper Zariski closed subset of the spectrum $\mathcal{C}(X)_{\ell}$, by lemma 2 and the induction assumption. Suppose $\mathcal{I}(\mathcal{P})$ were not contained in a proper Zariski closed subset of $\mathcal{C}(X)_{\ell}$. Then $\mathcal{I}(\mathcal{P}) = \mathcal{I}(\mathcal{P}) \cap \mathcal{C}(X)_{\ell}(\Lambda)$, and therefore $\mathcal{I}(\mathcal{P}) = \mathcal{I}(\mathcal{P}) \cap \mathcal{I}(\mathcal{P}_\chi)$. Hence $\mathcal{I}(\mathcal{P}_\chi)$ would be contained in a proper Zariski closed subset of $\mathcal{C}(X)_{\ell}$. Indeed, this would follow from $\mathcal{I}(\mathcal{P}_\chi) = \mathcal{I}(\mathcal{P}) \cap \mathcal{I}(\mathcal{P}_\chi) = \mathcal{I}(\mathcal{P} \ast \mathcal{P}_\chi)$ and the induction assumption. On the other hand, $\mathcal{I}(\mathcal{P}_\chi) = \chi^{-1} \cdot \mathcal{I}(\mathcal{P}) = \mathcal{C}(X)_{\ell}(\Lambda)$. This gives a contradiction, and proves our claim for the fixed degree $\nu$. Now proceed by induction. \qed

For $K \in \mathcal{P}$ the $\ell$-spectra $\mathcal{I}(K)_{\ell} := \mathcal{I}(K) \cap \{\chi_f\} \times \mathcal{C}(X)_{\ell}(\Lambda) \subset \mathcal{I}(K)$ at some given point $\chi_f$ of $\mathcal{I}(K)$ are the $\Lambda$-valued points of a Zariski closed subset of $\{\chi_f\} \times \mathcal{C}(X)_{\ell}$ by lemma 4. Replacing $K$ by $K_{\chi_f}$, we may always assume $\chi_f = 1$.

**Corollary 1.** For any semisimple complex $K \in \mathcal{D}$ contained in $N_{\text{Euler}}$, there exists in $\mathcal{C}(X)_{\ell}(\Lambda)$ a character $\chi \notin \mathcal{I}(K)$.

**Proof.** Since $\mathcal{I}(K) = \mathcal{I}(\mathcal{P}_K)$ for simple $K$ and $\mathcal{I}(\bigoplus_{i=1}^m K_i) \subseteq \bigcup_{i=1}^m \mathcal{I}(K_i)$, this is an immediate consequence of lemma 2 and proposition 1. \qed

**Theorem 1.** For arbitrary $K \in \mathcal{P}$, the Euler-Poincaré characteristic $\chi(X, K)$ is non-negative. Hence, in particular, the reductive supergroup $G(K)$ attached to $K$ in [KrW, §7] is a reductive algebraic group over $\Lambda$.

**Proof.** We may assume that $K$ is irreducible. Then, to show $\chi(X, K) \geq 0$, it is enough to show the existence of a character $\chi$ such that $H^v(X, K_\chi) = 0$ holds for all $v \neq 0$. Then $\chi(X, K) = \chi(X, K_\chi) = \dim_\Lambda(H^0(X, K_\chi))$, and the claim obviously follows from $\dim_\Lambda(H^0(X, K_\chi)) \geq 0$. So, we have to find a character $\chi \notin \mathcal{I}(K)$. By [KrW, §9], for all irreducible perverse sheaves $K$ there exists a perverse sheaf $T$ in $N_{\text{Euler}}$, depending on $K$, such that $H^\bullet(X, K_\chi) \neq H^0(X, K_\chi)$ holds if and only if $\chi \notin \mathcal{I}(T)$. Hence, by corollary 1 there exists a character $\chi \notin \mathcal{I}(T) = \mathcal{I}(K)$. \qed
The crucial fact that $\mathcal{S}(K)$ is the spectrum $\mathcal{S}(T)$ for an object $T$ in $N_{Euler}$, already exploited in the proof of the last theorem, furthermore implies

**Theorem 2.** For any $K \in P$ on $X$ and any character $\chi_f$ of $\pi_1(X)$ of order prime to $\ell$, the set of characters $\chi \in \mathcal{C}(X)(\Lambda)$ for which $\chi_f^*\chi$ is in $\mathcal{S}(K)$ is the set of closed points of a proper Zariski closed subset of $\mathcal{C}(X)_\ell$.

For base fields $F$ of characteristic $p > 0$, the following corollary now easily follows from theorem 1 by a specialization argument. For the case of fields $F$ of characteristic zero see [FK]; but our argument could also be extended to the characteristic zero case.

**Corollary 2.** For $\mathcal{K}_\ell$-adic perverse sheaves $K_0$ on abelian varieties $X_0$ defined over a field $F$ finitely generated over its prime field, with base extensions $K$ resp. $X$ to an algebraic closure of $F$, the Euler-Poincare characteristic $\chi(X,K)$ is non-negative.

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