Wellposedness for stochastic continuity equations with Ladyzhenskaya-Prodi-Serrin condition

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Abstract

We consider the stochastic divergence-free continuity equations with Ladyzhenskaya-Prodi-Serrin condition. Wellposedness is proved meanwhile uniqueness may fail for the deterministic PDE. The main issue of uniqueness relies on stochastic characteristic method and the generalized Itô-Ventzel-Kunita formula. The stability property for the unique solution is proved with respect to the initial data. Moreover, a persistence result is established by a representation formula.

1 Introduction

In this paper we establish wellposedness for stochastic divergence-free continuity equations. Namely, we consider the following Cauchy problem: Given an initial-data $u_0$, find $u(t,x;\omega) \in \mathbb{R}$, satisfying

$$
\begin{align*}
\partial_t u(t,x;\omega) + \text{div} \left( u(t,x;\omega) \left( b(t,x) + \frac{dB_t(\omega)}{dt} \right) \right) &= 0, \\
\left. u \right|_{t=0} &= u_0,
\end{align*}
$$

(1.1)

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\((t, x) \in U_T, \omega \in \Omega\), where \(U_T = [0, T] \times \mathbb{R}^d\), for \(T > 0\) be any fixed real number, \((d \in \mathbb{N})\), \(b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d\) is a given vector field, with \(\text{div} b(t, x) = 0\), \(B_t = (B^1_t, ..., B^d_t)\) is a standard Brownian motion in \(\mathbb{R}^d\) and the stochastic integration is taken (unless otherwise mentioned) in the Stratonovich sense. In fact, through of this paper, we fix a stochastic basis with a \(d\)-dimensional Brownian motion \((\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0, T]\}, \mathbb{P}, (B_t))\).

The Cauchy problem for the stochastic transport equation has taken great attention recently, see for instance [3], [4], [14], [18], [19], [20], and more recently the initial-boundary value problem in [23]. Concerning the deterministic case of the problem (1.1), also in a non-regular framework, the reader is mostly addressed to [9] and [2]. Those papers deal respectively with the Sobolev and the BV spatial regularity case, where the uniqueness proof relies on commutators, see DeLellis [8] for a nice review on that. The reader is direct towards the following references at the cited papers.

The main issue in this paper is to prove uniqueness of weak \(L^\infty\)-solution (see Definition 1.1) of the Cauchy problem (1.1) for vector fields

\[
 b \in L^q([0, T], (L^p(\mathbb{R}^d))^d), \quad p, q < \infty,
\]

\[
 p \geq 2, \quad q > 2, \quad \text{and} \quad \frac{d}{p} + \frac{2}{q} < 1.
\]

The last condition (1.2) is known in the fluid dynamic’s literature as the Ladyzhenskaya-Prodi-Serrin condition, with \(\leq\) in place of \(<\). Here, we do not assume any differentiability (one of the main assumptions in [3]), nor boundedness (also important in [14]) of the vector field \(b\). The uniqueness result, see Theorem 2.1, is established using the transportation property of the continuity equation for divergence free vector fields. Therefore, we have sharpened the answer of the following question: Why noise improves the deterministic theory for transport/continuity equations?

In fact, that noise could improve the theory of transport equations was first discovered by [14]. More precisely, the condition assumed in [14] is Hölder continuity and boundedness of \(b\), and an integrability condition on the divergence. Others results appear in [3] where no \(L^\infty\)-control on the divergence is required, however, weak differentiability is assumed. Our result is more advanced in the sense that we work with integrable and not differentiability coefficients. Further, no boundedness of \(b\) is assumed.
We recall that the Ladyzhenskaya-Prodi-Serrin condition (1.2) (with local integrability, and \( p, q \leq \infty \)) was first considered by Krylov, Röckner [16]. In that paper, they proved the existence and uniqueness of strong solutions for SDE

\[
X_{s,t}(x) = x + \int_s^t b(r, X_{s,r}(x)) \, dr + B_t - B_s,
\]

(1.3)

where given \( t \in [0, T] \) and \( x \in \mathbb{R}^d \), it was shown that

\[
\mathbb{P} \left( \int_0^T |b(t, X_t)| \, dt = \infty \right) = 0.
\]

More recently, Fedrizzi, Flandoli see [10, 11] proved the \( \alpha \)-Hölder continuity of the stochastic flow \( x \to X_{s,t} \) for any \( \alpha \in (0, 1) \). Moreover, they prove that it is a stochastic flow of homeomorphism.

Similarly, we may consider for convenience the inverse \( Y_{s,t} := X_{s,t}^{-1} \), which satisfies the following backward stochastic differential equations,

\[
Y_{s,t} = y - \int_s^t b(r, Y_{r,t}) \, dr - (B_t - B_s),
\]

(1.4)

for \( 0 \leq s \leq t \). Usually \( Y \) is called the time reversed process of \( X \).

One of the main motivations to consider problem (1.1) comes from the study of stochastic partial differential equations in fluid dynamics. In this direction, our ansatz is based on rational Continuum Mechanics. Let \( \psi \) be a physical quantity, which is a tensor field of order \( m \). Also, we consider the supply and the flux of \( \psi \), denoted respectively \( \sigma_\psi, \phi_\psi \), which are tensor fields of order \( m \) and \( m + 1 \). Then, the general stochastic balance equation has (at least formally) the form

\[
\partial_t \psi + \text{div} \left( \psi \otimes \frac{dX}{dt} - \phi_\psi \right) = \sigma_\psi,
\]

with \( X_t \) given for instance by

\[
X_t(x) = x + \int_0^t v(s, X_s(x)) \, ds + V_t,
\]

where \( v \) is the velocity field, and \( V_t \) is a stochastic process, which is not due necessarily from a Brownian motion, but posses for instance a Markov
property. Therefore, our main assumption is to randomly perturb the motion of the physical quantity $\psi$. In particular, taking $\psi = \rho$, and $\psi = \rho \mathbf{v}$, the Cauchy problem for the incompressible non-homogeneous stochastic Navier-Stokes equations may be written as

\[
\begin{aligned}
\partial_t \rho + \text{div} \left( \rho \left( \mathbf{v} + \frac{dV_t}{dt} \right) \right) &= 0, \\
\text{div} \mathbf{v} &= 0, \\
\partial_t (\rho \mathbf{v}) + \text{div} \left( \rho \mathbf{v} \otimes (\mathbf{v} + \frac{dV_t}{dt}) - T \right) &= \rho f,
\end{aligned}
\]

(1.5)

where $\rho$ is the density, and $T$ is the stress tensor field given by

\[
T = 2 \mu(\rho) D(\mathbf{v}) - p I_d
\]

with the scalar function $p$ called pressure. Moreover, $D(\mathbf{v})$ is the symmetric part of the gradient of the velocity field, $\mu$ is the dynamic viscosity, and $f$ is an external body force. The above problem (1.5) seems to us an onset of turbulence, which is a challenge phenomena to understand in the (incompressible) fluid dynamics theory. The reader is further addressed to Flandoli [12], Mikulevicius, Rozovskii [22], and references therein.

Moreover, the uniqueness result obtained by the authors for the stochastic continuity equation here in this paper, have to open new directions to establish existence of solutions to stochastic conservation laws, for non-homogenous flux functions with very-low regularity. In this direction, we recall the stochastic averaging lemmas for kinetic equations studied recently by Lions, Perthame, Souganidis [21].

The plan of exposition is as follows: In the rest of this section, we shall prove existence of weak $L^\infty$-solutions via the Itô-Ventzel-Kunita formula. In Section 2, we prove the uniqueness of weak $L^\infty$-solutions. Moreover, we show that the unique solution is given by a representation formula, in terms of the initial data and the stochastic flow associated to equation (1.1). In Section 3, we present stability results for the solution with respect to the initial datum. Finally, we discuss in Section 4 some extensions, regularity results and interesting open problems are pointed out.
1.1 Existence of weak solutions

Hereupon, we assume
\[ b \in L^1_{\text{loc}}(U_T). \]  
(1.6)

Also, we consider that \( u_0 \in L^\infty(\mathbb{R}^d) \). The next definition tell us in which sense a stochastic process is a weak solution of (1.1). Hereafter the usual summation convention is used.

**Definition 1.1.** A stochastic process \( u \in L^\infty(U_T \times \Omega) \) is called a weak \( L^\infty \)-solution of the Cauchy problem (1.1), when for any \( \varphi \in C^\infty_c(\mathbb{R}^d) \), the real value process \( \int u(t,x)\varphi(x)dx \) has a continuous modification which is a \( \mathcal{F}_t \)-semimartingale, and for all \( t \in [0,T] \), we have \( \mathbb{P} \)-almost sure

\[
\int_{\mathbb{R}^d} u(t,x)\varphi(x)dx = \int_{\mathbb{R}^d} u_0(x)\varphi(x)dx + \int_0^t \int_{\mathbb{R}^d} u(s,x)b(s,x)\partial_i\varphi(x)dxds \\
+ \int_0^t \int_{\mathbb{R}^d} u(s,x)\partial_i\varphi(x)dx dB^i_s.
\]  
(1.7)

**Lemma 1.2.** Under condition (1.6), there exits a week \( L^\infty \)-solution \( u \) of the Cauchy problem (1.1).

**Proof.** 1. First, let us considerer the following auxiliary Cauchy problem for the continuity equation, that is to say

\[
\begin{aligned}
\begin{cases}
\partial_t v(t,x) + \text{div} \left( v(t,x) b(t,x + B_t) \right) = 0, \\
v(0,x) = u_0(x).
\end{cases}
\end{aligned}
\]  
(1.8)

According to a minor modification of the arguments in DiPerna, Lions [9], see Proposition II.1 (taking only test functions defined on \( \mathbb{R}^d \)), it follows that, there exists a function \( v \in L^\infty(U_T \times \Omega) \), which is a solution of the auxiliary problem (1.8) in the sense that, it satisfies for each test function \( \varphi \in C^\infty_c(\mathbb{R}^d) \)

\[
\int_{\mathbb{R}^d} v(t,x)\varphi(x)dx = \int_{\mathbb{R}^d} u_0(x)\varphi(x)dx \\
+ \int_0^t \int_{\mathbb{R}^d} v(s,x)b(s,x + B_s) \cdot \nabla\varphi(x)dxds.
\]  
(1.9)
One observes that, the process $\int v(t, x)\varphi(x)dx$ is adapted, since it is the weak limit in $L^2([0, T] \times \Omega)$ of adapted processes, see [24] Chapter III for details.

2. Now, let us define for each $y \in \mathbb{R}^d$,

$$F(y) := \int_{\mathbb{R}^d} v(t, x) \varphi(x + y) \, dx.$$ 

Then, applying the Itô-Ventzel-Kunita Formula, see Theorem 8.3 of [17], to $F(B_t)$, it follows from (1.9)

$$\int_{\mathbb{R}^d} v(t, x) \varphi(x + B_t) \, dx = \int_{\mathbb{R}^d} u_0(x) \varphi(x) \, dx$$

$$+ \int_0^t \int_{\mathbb{R}^d} b(s, x + B_s) \cdot \nabla \varphi(x + B_s)v(s, x) \, dxds$$

$$+ \int_0^t \int_{\mathbb{R}^d} v(s, x) \partial_i \varphi(x + B_s)dx \circ dB_s^i, \quad (1.10)$$

where we have used that

$$\frac{\partial}{\partial y_i} \varphi(x + y) = \frac{\partial}{\partial x_i} \varphi(x + y).$$

3. Finally, defining $u(t, x) := v(t, x - B_t)$ we obtain from equation (1.10) that, $u(t, x)$ is a weak $L^\infty$—solution of the stochastic Cauchy problem (1.1).

2 Uniqueness

We prove the uniqueness result in this section, where it will be considered the divergence-free condition, that is

$$\text{div } b = 0 \quad (2.11)$$

(understood in the sense of distributions), and also the Ladyzhenskaya-Prodi-Serrin condition (1.2).

As mentioned in the introduction, under condition (1.2) we have suitable regularity of the stochastic characteristics. Indeed, under the divergence-free
condition the continuity equation turns to transport equation. Therefore, the main feature of the transport equation, which is the transport property, it is used by the authors to show uniqueness, completely different from the renormalization (due commutators) idea used in [3], [4], [14] and [20]. Then, we have the following

**Theorem 2.1.** Assume conditions (1.2) and (2.11). If \( u, v \in L^\infty(U_T \times \Omega) \) are two weak \( L^\infty \)-solutions for the Cauchy problem (1.1), with the same initial data \( u_0 \in L^\infty(\mathbb{R}^d) \), then for each \( t \in [0, T] \), \( u(t) = v(t) \) almost everywhere in \( \mathbb{R}^d \times \Omega \).

**Proof.** By linearity, it is enough to show that a weak \( L^\infty \)-solution \( u \) with initial condition \( u_0(x) = 0 \) vanishes identically. Let \( \phi_\varepsilon, \phi_\delta \) be standard symmetric mollifiers. Thus \( u_\varepsilon(t, \cdot) = u(t, \cdot) * \phi_\varepsilon \) verifies

\[
\int_{\mathbb{R}^d} u(t, z) \phi_\varepsilon(y - z) dz = \int_0^t \int_{\mathbb{R}^d} u(s, z) b^i(s, z) \partial_i \phi_\varepsilon(y - z) dz ds
\]

\[
+ \int_0^t \int_{\mathbb{R}^d} u(s, z) \partial_i \phi_\varepsilon(y - z) dz dB_i^s.
\]  

(2.12)

Now, we denote by \( b^\delta \) the standard mollification of \( b \) by \( \phi_\delta \), and let \( X^\delta_t \) be the associated flow given by the SDE (1.3) replacing \( b \) by \( b^\delta \). Similarly, we consider \( Y^\delta_t \), which satisfies the backward SDE (1.4).

Since \( \text{div } b^\delta = 0 \) (in other words, the Jacobian of the stochastic flow is identically one), for each \( \varphi \in C^\infty_c(\mathbb{R}^d) \), it follows that

\[
\int_{\mathbb{R}^d} (u * \phi_\varepsilon)(X^\delta_t) \varphi(y) dy = \int_{\mathbb{R}^d} (u * \phi_\varepsilon)(y) \varphi(Y^\delta_t) dy, 
\]  

(2.13)

for each \( t \in [0, T] \). On the other hand, applying Itô’s formula to the product

\[
(u * \phi_\varepsilon)(y) \varphi(Y^\delta_t),
\]

we obtain that
\[
\int_{\mathbb{R}^d} (u \ast \phi_\varepsilon)(y) \varphi(Y^\delta_t) \, dy = \int_0^t \int_{\mathbb{R}^d} u_\varepsilon(s, y) \, b^\delta(s, y) \cdot \nabla \varphi(Y^\delta_s) \, dy \, ds
\]
\[
- \int_0^t \int_{\mathbb{R}^d} u_\varepsilon(s, y) \, \partial_i \varphi(Y^\delta_s) \, dy \, dB_s^i
\]
\[
+ \int_0^t \int_{\mathbb{R}^d} \varphi(Y^\delta_s) \int_{\mathbb{R}^d} u(s, z) \, b(s, z) \cdot \nabla \phi_\varepsilon(y - z) \, dz \, dy \, ds
\]
\[
+ \int_0^t \int_{\mathbb{R}^d} \varphi(Y^\delta_s) \int_{\mathbb{R}^d} u(s, z) \, \partial_i \phi_\varepsilon(y - z) \, dz \, dy \, dB_s^i.
\] (2.14)

Therefore, from (2.13), (2.14), we may write
\[
\int_{\mathbb{R}^d} (u \ast \phi_\varepsilon)(x) \varphi(x) \, dx = - \int_0^t \int_{\mathbb{R}^d} u_\varepsilon(s, y) \, b^\delta(s, y) \cdot \nabla \varphi(Y^\delta_s) \, dy \, ds
\]
\[
- \int_0^t \int_{\mathbb{R}^d} u_\varepsilon(s, y) \, \partial_i \varphi(Y^\delta_s) \, dy \, dB_s^i
\]
\[
+ \int_0^t \int_{\mathbb{R}^d} \varphi(Y^\delta_s) \int_{\mathbb{R}^d} u(s, z) \phi_\varepsilon(y - z) \, b(s, z) \cdot \nabla \varphi(Y^\delta_s) \, dz \, dy \, ds
\]
\[
+ \int_0^t \int_{\mathbb{R}^d} \varphi(Y^\delta_s) \int_{\mathbb{R}^d} u(s, z) \phi_\varepsilon(y - z) \, \partial_i \varphi(Y^\delta_s) \, dz \, dy \, dB_s^i,
\]
where we have used that, \( \phi_\varepsilon \) is symmetric.

Now for \( \delta > 0 \) fixed, passing to the limit as \( \varepsilon \) goes to \( 0^+ \), we obtain from the above equation
\[
\int_{\mathbb{R}^d} u(X^\delta_t) \varphi(x) \, dx = - \int_0^t \int_{\mathbb{R}^d} u(s, y) \, b^\delta(s, y) \cdot \nabla \varphi(Y^\delta_s) \, dy \, ds
\]
\[
+ \int_0^t \int_{\mathbb{R}^d} u(s, y) \, b(s, z) \cdot \nabla \varphi(Y^\delta_s) \, dy \, ds.
\] (2.15)

At this point, we use an important and recent result obtained by Fredizzia and Flandolli, see [10]. More precisely, applying Lemma 3 and Lemma 5 in
that paper, we can pass to the limit in (2.15) as \(\delta\) goes to \(0^+\), to conclude that
\[
\int_{\mathbb{R}^d} u(X_t)\varphi(x) = 0
\]
for each \(\varphi \in C_c^\infty(\mathbb{R}^d)\), and \(t \in [0, T]\).

Finally, let \(K\) be any compact set in \(\mathbb{R}^d\). Then, we have
\[
\int_K \mathbb{E}|u(t, x)| \, dx = \lim_{\delta \to 0} \int_K \mathbb{E}|u(t, X_t^\delta) - u_0(X_t^\delta)| \, dx
\]
\[
= \lim_{\delta \to 0} \mathbb{E}\int_{Y_t^\delta(K)} |u(t, X_t)| \, dx
\]
\[
= \mathbb{E}\int_{Y_t(K)} |u(t, X_t)| \, dx = 0,
\]
where we have used (2.16) and the regularity of the stochastic flow. Consequently, the thesis of our theorem is proved.

We also have a representation formula in terms of the initial condition \(u_0\) and the (inverse) stochastic flow associated to SDE (1.3). Then, we have the following

**Proposition 2.2.** Assume conditions (1.2), and (2.11). Given \(u_0 \in L^\infty(\mathbb{R}^d)\), the stochastic process \(u(t, x) := u_0(X_t^{-1}(x))\) is the unique weak \(L^\infty\)-solution of the Cauchy problem (1.1).

**Proof.** 1. First, let us assume that \(b\) is regular, and we denote by \(u_0^\delta\) the standard mollification of \(u_0\). It is well known, see for instance [19], that \(u^\delta(t, x) := u_0^\delta(X_t^{-1}(x))\) is the unique classical solution of the associated transport equation, thus a weak \(L^\infty\)-solution of (1.1) with \(u^\delta\) and \(u_0^\delta\) in place of \(u\) and \(u_0\). For each test function \(\varphi \in C_c^\infty(\mathbb{R}^d)\), it follows that
\[
\int_{\mathbb{R}^d} u^\delta(t, x) \varphi(x) \, dx = \int_{\mathbb{R}^d} u_0^\delta(X_t^{-1}) \varphi(x) \, dx
\]
converges strongly in \(L^2([0, T] \times \Omega)\) to
\[
\int_{\mathbb{R}^d} u_0(X_t^{-1}) \varphi(x) \, dx = \int_{\mathbb{R}^d} u(t, x) \varphi(x) \, dx.
\]
Then, \( u_0(X^{-1}_t) \) is a weak \( L^\infty \)-solution of the Cauchy problem (1.1). By Theorem 2.1, uniqueness theorem, it is the only one.

2. Now, we denote by \( b^\delta \) the standard mollification of \( b \), and let \( X^\delta_t \) be the associated flow given by the SDE (1.3), i.e. replacing \( b \) by \( b^\delta \). From item 1, we have that \( u^\delta(t,x) = u_0(X^{\delta,-1}_t) \) is the unique weak \( L^\infty \)-solution of (1.1) with \( u^\delta \) and \( b^\delta \) in place of \( u \) and \( b \). Then, applying Lemma 3 of [10], we have for each test function \( \varphi \in C_c^\infty (\mathbb{R}^d) \) that

\[
\int_{\mathbb{R}^d} u^\delta(t,x) \varphi(x) \, dx = \int_{\mathbb{R}^d} u_0(X^{\delta,-1}_t) \varphi(x) \, dx
\]

converges strong in \( L^2([0,T] \times \Omega) \) to

\[
\int_{\mathbb{R}^d} u_0(X^{\delta,-1}_t) \varphi(x) \, dx = \int_{\mathbb{R}^d} u(t,x) \varphi(x) \, dx.
\]

Therefore, \( u_0(X^{\delta,-1}_t) \) is the representative formula, which is the unique weak \( L^\infty \)-solution of the Cauchy problem (1.1).

3 Stability

To end up the well-posedness for the Cauchy problem (1.1), it remains to show the stability property for the solution with respect to the initial datum. First, we establish a weak-stability result, then we show a strong one, assuming the strong convergence of the initial data.

**Theorem 3.1.** Assume conditions (1.2), and (2.11). Let \( \{u^n_0\} \) be any sequence, with \( u^n_0 \in L^\infty(\mathbb{R}^d) \) \((n \geq 1)\), converging weakly-star to \( u_0 \in L^\infty(\mathbb{R}^d) \). Let \( u(t,x) \), \( u^n(t,x) \) be the unique weak \( L^\infty \)-solution of the Cauchy problem (1.1), for respectively the initial data \( u_0 \) and \( u^n_0 \). Then, for all \( t \in [0,T] \), and for each function \( \varphi \in C^0_c(\mathbb{R}^d) \) \( \mathbb{P} \)-a.s.

\[
\int_{\mathbb{R}^d} u^n(t,x) \varphi(x) \, dx \text{ converges to } \int_{\mathbb{R}^d} u(t,x) \varphi(x) \, dx \quad \mathbb{P} \text{- a.s.}
\]

Moreover, if \( u^n_0 \) converges to \( u_0 \) in \( L^\infty(\mathbb{R}^d) \), then

\[ u^n(t,x) \text{ converge to } u(t,x) \text{ in } L^\infty(U_T \times \Omega). \]
Proof. 1. From Proposition 2.2 we may write

\[ u^n(t, x) = u^n_0(X^{-1}_t), \quad \text{and} \quad u(t, x) = u_0(X^{-1}_t). \]

Since \( \text{div } b = 0 \) (in other words, the Jacobian of the stochastic flow is identically one), for each \( \varphi \in C_c^\infty(\mathbb{R}^d) \), it follows that

\[ \int_{\mathbb{R}^d} u^n(t, x) \varphi(x) \, dx = \int_{\mathbb{R}^d} u^n_0(X^{-1}_t) \varphi(x) \, dx = \int_{\mathbb{R}^d} u^n_0(x) \varphi(X_t) \, dx \quad (3.17) \]

and analogously

\[ \int_{\mathbb{R}^d} u(t, x) \varphi(x) \, dx = \int_{\mathbb{R}^d} u_0(X^{-1}_t) \varphi(x) \, dx = \int_{\mathbb{R}^d} u_0(x) \varphi(X_t) \, dx. \quad (3.18) \]

Now, by hypothesis \( u^n_0 \) converges weak* \( L^\infty(\mathbb{R}^d) \) to \( u_0 \), thus

\[ \int_{\mathbb{R}^d} u^n_0(x) \varphi(X_t) \, dx \text{ converge to } \int_{\mathbb{R}^d} u_0(x) \varphi(X_t) \, dx, \]

for all \( t \in [0, T] \) and \( \mathbb{P} \)-a.s. From equations (3.17) and (3.18) we deduce that \( u^n(t, x) \) converge to \( u(t, x) \) weak* \( L^\infty(\mathbb{R}^d) \) for all \( t \in [0, T] \) and \( \mathbb{P} \)-a.s., which finish the proof of weak-stability.

2. Now, let us consider the strong-stability. Again, from Proposition 2.2 we have

\[ u^n(t, x) = u^n_0(X^{-1}_t) \quad \text{and} \quad u(t, x) = u_0(X^{-1}_t). \]

Therefore, we have

\[ \sup_{U_T \times \Omega} |u^n(t, x) - u(t, x)| = \sup_{U_T \times \Omega} |u^n_0(X^{-1}_t) - u_0(X^{-1}_t)| \]

\[ \leq \sup_{\mathbb{R}^d} |u^n_0(x) - u_0(x)|. \]

Then, the thesis follows by the hypothesis of \( u^n_0 \) converges to \( u_0 \) in \( L^\infty(\mathbb{R}^d) \).

\[ \square \]

Remark 3.2. Clearly, a much stronger stability result is: Under conditions of Theorem 3.1, if \( u^n_0 \) converges weakly-star to \( u_0 \in L^\infty(\mathbb{R}^d) \), then \( u^n(t, x) \) converges to \( u(t, x) \) in \( L^\infty(U_T \times \Omega) \). Although, even with the Brownian perturbation, it is not difficult to see that, the stochastic partial differential equations behaves exactly like the linear deterministic case.
Remark 3.3. It remains open the stability result with respect to $b$ under the Ladyzhenskaya-Prodi-Serrin condition (1.2). If additionally, we assume that $b(t)$ belongs to $W_{loc}^{1,1}$ or $BV_{loc}$, we can show the stability property for the solution with respect to $b$. In fact, the notion of renormalized solutions is valid and can be extended quite easily to a stochastic framework. For interesting remarks on renormalized solutions in the stochastic case see [3].

4 Final comments

1. Following the same arguments in the proof of Theorem 2.1, we may also treat the case in which $b$ is Hölder not necessarily bounded, with $\text{div} b = 0$. Indeed, from Theorem 7 in [13], if $b$ is Hölder not necessarily bounded, then $X_t$ is a Hölder continuous stochastic flow of diffeomorphisms. Then, a similar result of Theorem 2.1 and Proposition 2.2 holds.

2. As pointed out by Colombini, Luo and Rauch in [7], there exists an important example of $b \in L^\infty \cap W^{1,p}, (\forall p < \infty)$, such that the propagation of the continuity in the deterministic transport equation is missing. That is to say, besides the uniqueness is established in this case, the persistence condition is not, one may start with a continuous initial data, but the deterministic solution of the transport equation is not continuous. However, in the stochastic case we have the persistence property. In fact, let $u(t, x)$ be the unique weak $L^\infty$-solution of the Cauchy problem (1.1), with $u_0 \in C_b(\mathbb{R}^d)$ (i.e. a continuous bounded function). By Proposition 2.2 we have

$$u(t, x) = u_0(X_t^{-1}),$$

and recall that under condition (1.2), $X_t$ is a Hölder continuous stochastic flow of homeomorphisms (see Section 5 of [11]). Therefore, the continuity of $u$ follows. Analogously, we have the persistence property when $b$ is Hölder, not necessarily bounded, with $\text{div} b = 0$. Indeed, see item 1 above.

Also concerning the persistence property, we recall from [10] that, a certain Sobolev regularity is maintained under the Ladyzhenskaya-Prodi-Serrin condition, that is,

$$u_0 \in \bigcap_{r \geq 1} W^{1,r} \Rightarrow u(t, \cdot) \in \bigcap_{r \geq 1} W^{1,r}_{loc}.$$
3. It seems to us a very interesting question if \((1.2)\) is sharp, which is to say if we can consider

\[
\frac{d}{p} + \frac{2}{q} \leq 1 \quad \text{instead of} \quad \frac{d}{p} + \frac{2}{q} < 1. \tag{4.19}
\]

This question is posed in particular to SDE, and personal communications from Krylov and Röckner tell us that it remains open.

If we assume that \(b\) does not depend on \(t\), \(\text{div}b = 0\) (i.e. autonomous divergence free vector fields), consider dimension \(d = 2\), and suppose that \((4.19)\) is true, then we may have a uniqueness result for the stochastic transport equation without any geometrical condition as required in the deterministic case, see in particular Hauray \[15\], and also Alberti, Bianchini, Crippa \[1\]. Moreover, these results have to open new ideas to solve the Muskat Problem (at least in dimension 2), see Chemetov, Neves \[5, 6\].

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