Synchronisation in Invertible Random Dynamical Systems on the Circle

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Abstract

For a product of i.i.d. random orientation-preserving homeomorphisms or a Brownian flow on the circle, we show that if there is “sufficient random behaviour” then the trajectories of any two given initial conditions will almost surely mutually converge (“synchronisation”). Specifically, the following conditions guarantee that synchronisation occurs (and is “stable”): (i) no point is almost surely fixed under the flow, and (ii) every arc is able (with positive probability) to contract under the flow to a length less than its original length. Our result considerably generalises a result of Kleptsyn and Nalskii from 2004. We also provide a more general classification of synchronous behaviour.

1 Introduction

By “synchronisation” (in a deterministic or random dynamical system), we refer broadly to the phenomenon that the trajectories of different initial conditions approach each other over time in the long run. There already exist a few results concerning synchronisation in invertible random dynamical systems on the circle. Indeed, for an invertible differentiable RDS on the circle, the Lyapunov exponent associated to a given ergodic stationary distribution will generally be negative (e.g. for discrete time, [LeJan87, Proposition 1(b)] applied to the circle; or for continuous time, [Crau02, Corollary 4.4])—the argument for this is essentially based on the strict form of Jensen’s inequality. It then follows (by Propositions 2 and 3 of [LeJan87]) that with full probability, a full-measure set of initial conditions can be partitioned into finitely many classes, where the trajectories of any two initial conditions in the same class will synchronise. (This partition is not deterministic, although the number of classes in this partition is deterministic.) [Kai93] considers iterations of an orientation-preserving analytic diffeomorphism on the circle subject to a sequence of independent (but not necessarily identically distributed) random perturbations, and finds conditions under which some global-scale synchronous behaviour will almost surely occur.

However, the result that is most closely related to our present study is Theorem 1 of [KN04]. Here, no differentiability is assumed; invertible orientation-preserving iterated function systems on the circle are considered. It is shown that if both the system and its
inverse have minimal dynamics, and if at least one of the maps generated by the iterated
function system has an attracting fixed point whose basin of attraction is the whole
circle minus a singleton (namely, a repelling fixed point), then the distance between the
trajectories of any two given initial conditions will almost surely tend to 0 as time tends
to infinity; it is also shown that in this situation there is a random attractor-repeller pair.

In this paper, we shall present a much more general result: working with an invertible
statistically memoryless continuous RDS in either discrete or continuous time, we will
show that if there are no deterministic fixed points and every non-trivial arc has the
possibility of contracting to a shorter length under the flow, then (a) the distance between
the trajectories of any two given initial conditions will almost surely tend to 0 as time
tends to infinity, and (b) the trajectory of any given initial condition will almost surely
be asymptotically stable. (We will describe a system satisfying (a) as synchronising,
and we will describe a system satisfying both (a) and (b) as stably synchronising.) Our
proof is also significantly simpler than the one given in [KN04] (although both proofs are
ultimately based on the martingale convergence theorem).

It is worth pointing out that [New15] already gives necessary and sufficient conditions
for stable synchronisation in memoryless RDS on a compact metric space. However, one
of these conditions is the existence of locally asymptotically stable trajectories (which is
typically verified for differentiable systems through negativity of Lyapunov exponents);
due to the special structure of the circle, our present result does not require one to verify
such a condition.

As an example: we will see that for any Lipschitz vector field on the circle with no
non-trivial periodicity, the RDS generated by a Gaussian-white-noise perturbation of
this vector field will satisfy our conditions for stable synchronisation.

The structure of the paper will be as follows: In section 2, we will present our framework,
introduce some definitions, and present our main result (Theorem 2.13), together with
the example mentioned above (Example 2.14). In section 3, we will prove Theorem 2.12.
In section 4, we present the notion of a “crack point” (taken and adapted from [Kai93])—
that is, a point outside of which asymptotic contraction occurs—and classify synchronous
behaviour in terms of crack points (Theorem 4.5). In the same section, we then show that
in two-sided time, any stably synchronising system has a global heteroclinic connection
from a future-measurable repelling random fixed point to a past-measurable attracting
random fixed point (Proposition 4.27).

2 The basic setup and our result

Heuristically, we consider a non-deterministic flow on the circle, where the non-
determinism is due to some stationary and memoryless random process that affects the
flow. We formalise this as follows:

Let $\mathbb{T}$ be either $\mathbb{N}$ or $[0, \infty)$. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{P})$ be a filtered probability space,
and set $\mathcal{F}_s := \sigma(\mathcal{F}_t : t \in \mathbb{T}^+)$. (The elements of $\Omega$ are called “sample points”, and represent the different possible evolutions of the random process.) Let $(\theta^t)_{t \in \mathbb{T}^+}$ be a $\mathbb{T}^+$-indexed family of measurable functions $\Omega \to \Omega$ such that

(i) $\theta^0$ is the identity on $\Omega$, and $\theta^{s+t} = \theta^t \circ \theta^s$ for all $s, t \in \mathbb{T}^+$;

(ii) for all $s, t \in \mathbb{T}^+$, $\theta^{-t} \mathcal{F}_s \subset \mathcal{F}_{s+t}$;

(iii) for all $t \in \mathbb{T}^+$, $\theta^t \mathbb{P} = \mathbb{P}$;

(iv) for all $t \in \mathbb{T}^+$, $\mathcal{F}_t$ and $\theta^{-t} \mathcal{F}_s$ are independent $\sigma$-algebras (under $\mathbb{P}$).

(In our notation: $\theta^{-t}(E) := (\theta^t)^{-1}(E)$ for $E \in \mathcal{F}$; and $\theta^{-t} \mathcal{G} := \{\theta^{-t}(E) : E \in \mathcal{G}\}$ for $\mathcal{G} \subset \mathcal{F}$. If $\theta^t$ is bijective then we write $\theta^{-1}$ for the inverse map of $\theta^t$.)

($\theta^t$) is the time-shift dynamical system on the sample space $\Omega$: if we take the evolution represented by a sample point $\omega$, and then pretend that our “reference time $t = 0$” occurs at time $\tau$ later than it actually does, the result is precisely what $\theta^\tau \omega$ represents.

Example 2.1 (Gaussian white noise). Following sections A.2 and A.3 of [Arn98], an “eternal” one-dimensional Gaussian white noise process may be described according to the framework above as follows: Let $\Omega := \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$. For each $t \in [0, \infty)$, let $\mathcal{F}_t$ be the smallest $\sigma$-algebra on $\Omega$ with respect to which the projection $W_s : \omega \mapsto \omega(s)$ is measurable for every $s \in [0, t]$. Let $\mathcal{F}$ be the smallest $\sigma$-algebra on $\Omega$ with respect to which the projection $W_s : \omega \mapsto \omega(s)$ is measurable for every $s \in \mathbb{R}$. Let $\mathbb{P}$ be the Wiener measure on $(\Omega, \mathcal{F})$—that is, $\mathbb{P}$ is the unique probability measure under which the stochastic processes $(W_t)_{t \geq 0}$ and $(W_{-t})_{t \geq 0}$ are independent Wiener processes. Finally, for each $\tau \geq 0$ and $s \in \mathbb{R}$, set $\theta^\tau \omega(s) := \omega(\tau+s) - \omega(\tau)$; by taking inverses, this formula also holds for negative $\tau$. (Note that under our notation, whether we are working on one-sided time or two-sided time, $\mathcal{F}_t$ always represents the information available from time 0 to time $t$.)

Now let $\mathbb{S}^1$ be the unit circle, which we identify with $\mathbb{R}/\mathbb{Z}$ in the obvious manner, and let $l$ denote the Lebesgue measure on $\mathbb{S}^1$ (with $l(\mathbb{S}^1) = 1$). Let $\pi : \mathbb{R} \to \mathbb{S}^1$ denote the natural projection, i.e. $\pi(x) = x + \mathbb{Z} \in \mathbb{S}^1$; a lift of a point $x \in \mathbb{S}^1$ is a point $x' \in \mathbb{R}$ such that $\pi(x') = x$, and a lift of a set $A \subset \mathbb{S}^1$ is a set $B \subset \mathbb{R}$ such that $\pi(B) = A$. Define the metric $d$ on $\mathbb{S}^1$ by

$$d(x, y) = \min\{|x' - y'| : x' \text{ is a lift of } x, y' \text{ is a lift of } y\}.$$ 

Let $\varphi = (\varphi(t, \omega))_{t \in \mathbb{T}^+, \omega \in \Omega}$ be a $(\mathbb{T}^+ \times \Omega)$-indexed family of orientation-preserving homeomorphisms $\varphi(t, \omega) : \mathbb{S}^1 \to \mathbb{S}^1$ such that

(i) for all $\omega \in \Omega$, $\varphi(0, \omega)$ is the identity function on $\mathbb{S}^1$;

(ii) for all $s, t \in \mathbb{T}^+$ and $\omega \in \Omega$, $\varphi(s+t, \omega) = \varphi(t, \theta^\omega \omega) \circ \varphi(s, \omega)$;

(iii) for each $t \in \mathbb{T}^+$, the map $(\omega, x) \mapsto \varphi(t, \omega)x$ is $(\mathcal{F}_t \otimes \mathcal{B}(\mathbb{S}^1), \mathcal{B}(\mathbb{S}^1))$-measurable;

(iv) for each $\omega \in \Omega$, the map $(t, x) \mapsto \varphi(t, \omega)x$ is continuous.
(As in Theorem 1.1.6(ii) of [Arn98], we automatically have that for each \( \omega \in \Omega \) the map \((t, x) \mapsto \varphi(t, \omega)^{-1}(x)\) is continuous. It is also not hard to show that for each \( t \in \mathbb{T}^+ \) the map \((\omega, x) \mapsto \varphi(t, \omega)^{-1}(x)\) is \((\mathcal{F}_t \otimes \mathcal{B}^1), \mathcal{B}^1\))-measurable.)

In discrete time, \( \varphi \) represents a product of i.i.d. random homeomorphisms (namely, the homeomorphisms \( \varphi(1, \omega), \varphi(1, \theta^1 \omega), \varphi(1, \theta^2 \omega), \ldots \)). In continuous time, \( \varphi \) represents a “Brownian flow” of homeomorphisms. We refer to \( \varphi \) as a “random dynamical system”.

**Definition 2.2.** We say that \( \varphi \) is (two-point) **contractible** if for any distinct \( x, y \in \mathbb{S}^1 \),

\[
\mathbb{P}(\omega : \exists t \in \mathbb{T}^+ \text{ s.t. } d(\varphi(t, \omega)x, \varphi(t, \omega)y) < d(x, y)) > 0.
\]

**Proposition 2.3.** If \( \varphi \) is contractible then for any \( x, y \in \mathbb{S}^1 \), for \( \mathbb{P} \)-almost all \( \omega \in \Omega \) there exists an unbounded increasing sequence \((t_n)\) in \( \mathbb{T}^+ \) such that \( d(\varphi(t_n, \omega)x, \varphi(t_n, \omega)y) \to 0 \) as \( n \to \infty \).

**Proof.** See section 3.1 of [New15].

We will now introduce a slight variant of “contractibility”, based on a different type of distance measurement. Define the **anticlockwise distance function** \( d^+: \mathbb{S}^1 \times \mathbb{S}^1 \to [0, 1) \) by

\[
d^+(x, y) = \min\{r \geq 0 : \pi(x' + r) = y\}
\]

where \( x' \) may denote any lift of \( x \). Obviously \( d^+ \) is not symmetric, but rather satisfies the relation

\[
d^+(y, x) = 1 - d^+(x, y).
\]

It is easy to see that \( d^+ \) is lower semicontinuous, and therefore measurable. For any interval \( I \subset \mathbb{R} \) of positive length less than 1, if we let \( x_1 = \pi(\inf I), x_2 = \pi(\sup I) \) and \( J = \pi(I) \), we have that

\[
l(\varphi(t, \omega)J) = d^+(\varphi(t, \omega)x_1, \varphi(t, \omega)x_2).
\]

**Definition 2.4.** We say that \( \varphi \) is **compressible** if for any distinct \( x, y \in \mathbb{S}^1 \),

\[
\mathbb{P}(\omega : \exists t \in \mathbb{T}^+ \text{ s.t. } d^+(\varphi(t, \omega)x, \varphi(t, \omega)y) < d^+(x, y)) > 0.
\]

By reversing the order of inputs, this is equivalent to saying that for any distinct \( x, y \in \mathbb{S}^1 \),

\[
\mathbb{P}(\omega : \exists t \in \mathbb{T}^+ \text{ s.t. } d^+(\varphi(t, \omega)x, \varphi(t, \omega)y) > d^+(x, y)) > 0.
\]

Perhaps more intuitively, we can also define compressibility in terms of connected subsets of \( \mathbb{S}^1 \): \( \varphi \) is compressible if and only if for every connected set \( J \subset \mathbb{S}^1 \) with \( 0 < l(J) < 1 \),

\[
\mathbb{P}(\omega : \exists t \in \mathbb{T}^+ \text{ s.t. } l(\varphi(t, \omega)J) < l(J)) > 0.
\]

Again, this is equivalent to saying that for every connected set \( J \subset \mathbb{S}^1 \) with \( 0 < l(J) < 1 \),

\[
\mathbb{P}(\omega : \exists t \in \mathbb{T}^+ \text{ s.t. } l(\varphi(t, \omega)J) > l(J)) > 0.
\]

Obviously, if \( \varphi \) is compressible then \( \varphi \) is contractible.

We will not actually need the following result, but we include it for the sake of completeness:
Proposition 2.5. If \( \varphi \) is compressible then for any connected \( J \subset \mathbb{S}^1 \) with \( l(J) < 1 \) and any \( \varepsilon > 0 \) there exists \( t \in \mathbb{T}^+ \) such that
\[
P(\omega : l(\varphi(t, \omega)J) < \varepsilon) > 0.
\]

Proof. Suppose we have a connected set \( J \subset \mathbb{S}^1 \) with \( l(J) < 1 \) and a value \( \varepsilon > 0 \) such that for all \( t \in \mathbb{T}^+ \),
\[
P(\omega : l(\varphi(t, \omega)J) < \varepsilon) = 0.
\]
Let \( [x_1', x_2'] \subset \mathbb{R} \) be a lift of \( J \), and let \( x_1 = \pi(x_1') \) and \( x_2 = \pi(x_2') \). Now let \( \Delta \) be the diagonal in \( \mathbb{S}^1 \times \mathbb{S}^1 \), i.e. \( \Delta = \{(x, x) : x \in \mathbb{S}^1 \} \). (Note that \( d_+ \) is continuous on \( (\mathbb{S}^1 \times \mathbb{S}^1) \setminus \Delta \).)

Following the terminology of section 1 of [New15]: Let \( G_{(x_1, x_2)} \subset \mathbb{S}^1 \times \mathbb{S}^1 \) be the smallest closed set containing \( (x_1, x_2) \) that is forward-invariant under the two-point motion \( \varphi \times \varphi \).

The open set \( \{(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1 : 0 < d_+(x, y) < \varepsilon \} \) is not accessible from \( (x_1, x_2) \), and therefore (by Lemma 1.2.3 of [New15]) \( G_{(x_1, x_2)} \) does not intersect this set. So, if we let \( (u, v) \) be a point from the compact set \( K := \{(x, y) \in G_{(x_1, x_2)} : \varepsilon \leq d_+(x, y) \leq d_+(x_1, x_2) \} \) which minimises \( d_+ \) on \( K \), then \( (u, v) \) will minimise \( d_+ \) on the whole of \( G_{(x_1, x_2)} \setminus \Delta \). Since \( G_{(x_1, x_2)} \) is forward-invariant, we have that
\[
P(\omega : \exists t \in \mathbb{T}^+ \text{ s.t. } 0 < d_+(\varphi(t, \omega)u, \varphi(t, \omega)v) < d_+(u, v)) = 0.
\]

Obviously, since \( \varphi(t, \omega) \) is invertible for all \( t \) and \( \omega \), \( d_+(\varphi(t, \omega)u, \varphi(t, \omega)v) \) cannot be 0. Hence we have that
\[
P(\omega : \exists t \in \mathbb{T}^+ \text{ s.t. } d_+(\varphi(t, \omega)u, \varphi(t, \omega)v) < d_+(u, v)) = 0.
\]

So \( \varphi \) is not compressible. \( \square \)

Definition 2.6. We say that a point \( p \in \mathbb{S}^1 \) is a deterministic fixed point (of \( \varphi \)) if for \( \mathbb{P} \)-almost every \( \omega \in \Omega \), for all \( t \in \mathbb{T}^+ \), \( \varphi(t, \omega)p = p \).

Definition 2.7. We say that a set \( A \subset \mathbb{S}^1 \) is forward-invariant (under \( \varphi \)) if for \( \mathbb{P} \)-almost every \( \omega \in \Omega \), for all \( t \in \mathbb{T}^+ \), \( \varphi(t, \omega)A \subset A \).

Definition 2.8. We say that \( \varphi \) has reverse-minimal dynamics if the only open forward-invariant sets are \( \emptyset \) and \( \mathbb{S}^1 \).

Note that if \( \varphi \) has a deterministic fixed point \( p \) then \( \mathbb{S}^1 \setminus \{p\} \) is forward-invariant, and so \( \varphi \) does not have reverse-minimal dynamics.

Proposition 2.9. If \( \mathbb{T}^+ = [0, \infty) \) then the following are equivalent:

- \( \varphi \) has reverse-minimal dynamics;
- the only closed forward-invariant sets are \( \emptyset \) and \( \mathbb{S}^1 \).

Remark 2.10. In general, when \( \emptyset \) and \( \mathbb{S}^1 \) are the only closed forward-invariant sets, we say that \( \varphi \) has minimal dynamics. So Proposition 2.9 says that in continuous time, \( \varphi \) has minimal dynamics if and only if it has reverse-minimal dynamics.
Proof of Proposition 2.9. We first show that (i)⇒(ii). Suppose we have a closed forward-invariant non-empty proper subset $G$ of $\mathbb{S}^1$; we need to show that there exists an open forward-invariant non-empty proper subset $U$ of $\mathbb{S}^1$. First consider the case that $G$ is a singleton $\{p\}$; then it is clear that $p$ is a deterministic fixed point, and so $U := \mathbb{S}^1 \setminus \{p\}$ is forward-invariant. Now consider the case that $G$ is not a singleton, and let $V$ be a connected component of $\mathbb{S}^1 \setminus G$; we will show that $U := \mathbb{S}^1 \setminus V$ is forward-invariant. Fix any $\omega$ with the property that $\varphi(t, \omega)G \subset G$ for all $t \in \mathbb{T}^+$. Since $\partial V \subset G$, we have that for all $t$, $\varphi(t, \omega)\partial V \subset G$ and therefore in particular $\varphi(t, \omega)\partial V \cap V = \emptyset$. Now let $a, b : [0, \infty) \to \mathbb{R}$ be continuous functions with $a < b$ such that $[a(t), b(t)]$ is a lift of $\varphi(t, \omega)V$ for all $t$. (So $\{a(t), b(t)\}$ projects onto $\varphi(t, \omega)\partial V$ for all $t$.) We know that for all $t$, $a(t) \notin [a(0), b(0)]$; therefore (due to the intermediate value theorem), $a(t) \leq b(0)$ for all $t$. Similarly, we have that $b(t) \geq b(0)$ for all $t$. Hence $\bar{V} \subset \varphi(t, \omega)V$ for all $t$. Since $\varphi(t, \omega)$ is bijective for all $t$, it follows that $\varphi(t, \omega)U \subset U$ for all $t$.

Now to show that (ii)⇒(i), first observe that a set $A \subset \mathbb{S}^1$ is forward-invariant if and only if for $\mathbb{P}$-almost every $\omega \in \Omega$, for all $t \in \mathbb{T}^+$,

\[ \varphi(t, \omega)^{-1}(X \setminus A) \subset X \setminus A. \]

Hence the fact that (ii)⇒(i) can be derived by exactly the same argument that we used to show (i)⇒(ii), except applied to the family of functions $(\varphi(t, \omega)^{-1})_{t \in \mathbb{T}^+, \omega \in \Omega}$ rather than $(\varphi(t, \omega))_{t \in \mathbb{T}^+, \omega \in \Omega}$. \hfill \Box

Definition 2.11. We say that $\varphi$ is synchronising if for all $x, y \in \mathbb{S}^1$,

\[ \mathbb{P}(\omega : d(\varphi(t, \omega)x, \varphi(t, \omega)y) \to 0 \text{ as } t \to \infty) = 1. \]

Now in general, two “synchronising” trajectories of a random dynamical system can converge towards the same unstable path, such that in practice, two processes being modelled by these two trajectories will never actually synchronise. To overcome this problem, we introduce the following definition (taken from [New15], where it is not specific to the circle but can be applied to a RDS on any compact metric space):

Definition 2.12. We say that $\varphi$ is stably synchronising if $\varphi$ is synchronising and for all $x \in \mathbb{S}^1$,

\[ \mathbb{P}(\omega : \exists \text{ open } U \ni x \text{ s.t. } \text{diam}(\varphi(t, \omega)U) \to 0 \text{ as } t \to \infty) = 1. \]

Due to $\mathbb{S}^1$ being one-dimensional, this is equivalent to saying that for all $x \in \mathbb{S}^1$,

\[ \mathbb{P}(\omega : \exists \text{ open } U \ni x \text{ s.t. } l(\varphi(t, \omega)U) \to 0 \text{ as } t \to \infty) = 1. \]

Theorem 2.13. The following are equivalent:

(i) $\varphi$ is compressible and has no deterministic fixed points;

(ii) $\varphi$ is contractible and has reverse-minimal dynamics;

and in these cases $\varphi$ is stably synchronising.

Observe in particular that if $\varphi$ has reverse-minimal dynamics then contractibility, compressibility, synchronisation and stable synchronisation are all equivalent.
Example 2.14. Large classes of ODEs on (unbounded) Euclidean space have been proved to exhibit large-scale synchronous behaviour after the addition of Gaussian white noise to the right-hand side of the ODE (see [CF98] for the one-dimensional case, and [FGS14] for a more general case). We shall now do the same for ODEs on $S^1$. Let $b : \mathbb{R} \to \mathbb{R}$ be a 1-periodic Lipschitz function, and let $\varphi$ be the RDS on $S^1$ whose trajectories $(\varphi(t, \omega)x)_{t \geq 0}$ are projections onto $S^1$ of solutions of the SDE

$$dX_t = b(X_t)dt + \sigma dW_t$$

where $(W_t)_{t \geq 0}$ is a Wiener process and $\sigma \in \mathbb{R} \setminus \{0\}$. It is clear that if the least period of $b$ is less than 1 then $\varphi$ is not synchronising, since any arc of length $\alpha$ will remain of length $\alpha$ under the flow. But in the converse direction, using Theorem 2.13 it is not hard to show the following: If the least period of $b$ is 1 then $\varphi$ is stably synchronising. To see this, we argue as follows: Firstly, $\varphi$ clearly has no deterministic fixed points. Now fix any connected $J \subset S^1$ with $0 < l(J) < 1$, and let $[c_1, c_2] \subset \mathbb{R}$ be a lift of $J$ (so $c_2 - c_1 = l(J)$). Since $b$ is continuous and periodic but not $l(J)$-periodic, there must exist $a \in \mathbb{R}$ such that $b(a + l(J)) < b(a)$. Now it is known (see e.g. [Art75, Theorem C(a)]) that for any $x \in S^1$ the path of the solution $(\varphi(t, \omega)x)_{t \geq 0}$ depends continuously (in the topology of uniform convergence on bounded intervals) on the path of the driving Wiener process $(W_t(\omega))_{t \geq 0}$; hence, if the path of the Wiener process $(W_t(\omega))_{0 \leq t \leq 1}$ on the time interval $[0, 1]$ is sufficiently close (uniformly on $[0, 1]$) to the path $(\gamma_t)_{0 \leq t \leq 1}$ given by

$$\gamma_t = \begin{cases} \eta t & t \in [0, \frac{a-c_1}{\eta}] \\ a - c_1 & t \in [\frac{a-c_1}{\eta}, 1] \end{cases}$$

(where $\eta > 0$ may be any sufficiently large value), then the length of $\varphi(t, \omega)J$ will have fallen below $l(J)$ by some time shortly after $\frac{a-c_1}{\eta}$. Thus $\varphi$ is compressible. Hence Theorem 2.13 gives that $\varphi$ is stably synchronising.

Remark 2.15 (Comparison of our result with [KN04]). Let us say that an orientation-preserving homeomorphism $f : S^1 \to S^1$ is simple if there exist points $r, a \in S^1$ (which we respectively call the repeller and attractor of $f$) such that $f(r) = r$ and for every $x \in S^1 \setminus \{r\}$, $f^n(x) \to a$ as $n \to \infty$. (It follows that $f(a) = a$.) Let $S$ be a finite set of orientation-preserving homeomorphisms of $S^1$, and let $\nu$ be a probability measure on $S$ (equipped with the discrete $\sigma$-algebra $2^S$) assigning strictly positive mass to each singleton in $S$. Let $(\Omega, \mathcal{F}, \mathbb{P}) = (S^N, (\sigma S)^{\otimes N}, \nu^{\otimes N})$. For each $n \in \mathbb{N}$ and $\omega = (f_i)_{i \in \mathbb{N}} \in \Omega$, let $\varphi(n, \omega) = f_n \circ \ldots \circ f_1$ and let $\bar{\varphi}(n, \omega) = f_n^{-1} \circ \ldots \circ f_1^{-1}$; and let $\varphi(0, \omega)$ and $\bar{\varphi}(0, \omega)$ be the identity function for all $\omega$. So $\varphi$ and $\bar{\varphi}$ are random dynamical systems on $S^1$. Theorem 1 of [KN04] states that if

(i) there exist $f_1, \ldots, f_k \in S$ such that $F := f_k \circ \ldots \circ f_1$ is simple;

(ii) $\varphi$ has minimal dynamics;

(iii) $\bar{\varphi}$ has minimal dynamics;

then $\varphi$ is synchronising\footnote{Theorem 1 of [KN04] actually states that if these three conditions are satisfied then there is a function $z : \Omega \to S^1$ such that $z^{-1}(\{x\})$ is a $\mathbb{P}$-null set for all $x \in S^1$ and for $\mathbb{P}$-almost every $\omega$, for all $x, y \in S^1 \setminus \{z(\omega)\}$, $d(\varphi(n, \omega)x, \varphi(n, \omega)y) \to 0$ as $n \to \infty$. It is then stated as a corollary that $\varphi$ is synchronising. However, as follows from the material that we will present in section 4, the conclusion of Theorem 1 of [KN04] is actually equivalent to the corollary that $\varphi$ is synchronising (assuming $|S| > 1$).}.

Note that the statement that $\bar{\varphi}$ has minimal dynamics is
equivalent to the statement that \( \varphi \) has reverse-minimal dynamics. Now as pointed out in Remark 3 of [KN04], \( \varphi \) has minimal dynamics if and only if for all \( x \in S^1 \) and non-empty open \( U \subset S^1 \) there exist \( h_1, \ldots, h_m \in S \) such that \( h_m \circ \cdots \circ h_1(x) \in U \). (See also section 1.2 of [New15].) Likewise, \( \tilde{\varphi} \) has reverse-minimal dynamics if and only if for all \( x \in S^1 \) and non-empty open \( U \subset S^1 \) there exist \( h_1, \ldots, h_m \in S \) such that \( x \in h_m \circ \cdots \circ h_1(U) \). With this, it is easy to see that Theorem 1 of [KN04] is just a special case of our result, and indeed that condition (ii) of this theorem can even be dropped: Suppose conditions (i) and (iii) are satisfied; since \( \varphi \) has reverse-minimal dynamics, we just need to show that \( \varphi \) is contractible, and then Theorem 2.13 will give that \( \varphi \) is stably synchronising. So fix any distinct points \( x, y \in S^1 \). Let \( r \) be the repeller of \( F \). Since \( \varphi \) has reverse-minimal dynamics, there must exist \( h_1, \ldots, h_m \in S \) such that \( r \in h_m \circ \cdots \circ h_1(S^1 \setminus \{x, y\}) \), i.e. such that \( h_m \circ \cdots \circ h_1(x) \) and \( h_m \circ \cdots \circ h_1(y) \) are both distinct from \( r \). So, letting \( H := h_m \circ \cdots \circ h_1 \), we clearly have that

\[
d(F^m \circ H(x), F^m \circ H(y)) < d(x, y)
\]

for sufficiently large \( n \). Hence \( \varphi \) is contractible, and therefore \( \varphi \) is stably synchronising.

### 3 Proof of Theorem 2.13

We will first prove that (i)⇒(ii)⇒stable synchronisation, and then, using material developed along the way, we will show that (ii)⇒(i). (However, the author expects that there will exist a more elementary proof of the fact that (ii)⇒(i).) Proving that (i)⇒(ii) is fairly straightforward. The main tools in the proof that (ii)⇒stable synchronisation are the Krylov-Bogolyubov theorem and the martingale convergence theorem.

**Proof that (i)⇒(ii).** Suppose \( \varphi \) is compressible and has no deterministic fixed points. Obviously \( \varphi \) is contractible; so we just need to show that \( \varphi \) has reverse-minimal dynamics. Suppose for a contradiction that we have an open forward-invariant non-empty proper subset \( U \) of \( S^1 \). \( S^1 \setminus U \) cannot be a singleton, since there are no deterministic fixed points. So let \( V \) be a maximal-length connected component of \( U \); since \( \varphi \) is compressible, there is a positive-measure set of sample points \( \omega \in \Omega \) such that for some \( t \in \mathbb{T}^+ \), \( l(\varphi(t, \omega)V) > l(V) \). However, \( \varphi(t, \omega)V \) is connected for all \( t \) and \( \omega \), and so if \( l(\varphi(t, \omega)V) > l(V) \) then \( \varphi(t, \omega)V \) cannot be a subset of \( U \). This contradicts the fact that \( U \) is forward-invariant. \( \square \)

**Definition 3.1.** We will say that a probability measure \( \rho \) on \( S^1 \) is reverse-stationary if for all \( t \in \mathbb{T}^+ \) and \( A \in \mathcal{B}(S^1) \),

\[
\rho(A) = \int_{\Omega} \rho(\varphi(t, \omega)A) \mathbb{P}(d\omega).
\]

Note that for any \( s \in \mathbb{T}^+ \), since \( \mathbb{P} \) is \( \theta^s \)-invariant this is equivalent to saying that for all \( t \in \mathbb{T}^+ \) and \( A \in \mathcal{B}(S^1) \),

\[
\rho(A) = \int_{\Omega} \rho(\varphi(t, \theta^s \omega)A) \mathbb{P}(d\omega).
\]

Now we will say that a probability measure \( \rho \) on \( S^1 \) is atomless if for all \( x \in S^1 \), \( \rho(\{x\}) = 0 \).

---

2 If \( \theta^t \) is a measurable self-isomorphism of \( \Omega \) for all \( t \in \mathbb{T}^+ \) then it would make sense to define \( \varphi(t, \omega) \) for negative \( t \), by \( \varphi(t, \omega) = \varphi(-t, \theta^{\omega} \omega)^{-1} \); however, we emphasise that even in this case, for the definition of reverse-stationarity (and likewise for the definition of stationarity in section 4) we must restrict to nonnegative \( t \).
Lemma 3.2. Suppose we have an atomless reverse-stationary probability measure $\rho$. Then for any connected $J \subset S^1$, for $\mathbb{P}$-almost all $\omega \in \Omega$, $\rho(\varphi(t, \omega), J)$ converges as $t \to \infty$.

The main idea of the proof is the same as in Lemma 1 of [LeJan87].

Proof. Fix a connected $J \subset S^1$, and for each $t$ and $\omega$ let $h_t(\omega) = \rho(\varphi(t, \omega), J)$. Since $\rho$ is atomless, the map $t \mapsto h_t(\omega)$ is continuous for all $\omega$. So if we can show that $(h_t)_{t \in \mathbb{T}^+}$ is an $(\mathcal{F}_t)_{t \in \mathbb{T}^+}$-adapted martingale, then the martingale convergence theorem will give the desired result. Fix any $s, t \in \mathbb{T}^+$, and define $g : \Omega \times \Omega \to [0, 1]$ by

$$g(\omega, \tilde{\omega}) = \rho(\varphi(t, \theta^s \tilde{\omega}) \circ \varphi(s, \omega)(J)).$$

Note that $g$ is $(\mathcal{F}_s \otimes \theta^{-s} \mathcal{F}_t, \mathcal{B}([0, 1]))$-measurable. Now for all $\omega$,

$$h_{s+t}(\omega) = g(\omega, \omega),$$

and

$$h_s(\omega) = \int_{\Omega} g(\omega, \tilde{\omega}) \mathbb{P}(d\tilde{\omega}).$$

(The latter equality is due to the reverse-stationarity of $\rho$.) So then, since $\mathcal{F}_s$ and $\theta^{-s} \mathcal{F}_t$ are independent $\sigma$-algebras, it follows (e.g. by Exercise 124(B) of [New15a]) that $h_s$ is a version of the conditional expectation of $h_{s+t}$ given $\mathcal{F}_s$. So we are done.

Lemma 3.3. Suppose $\varphi$ is contractible and admits a reverse-stationary probability measure $\rho$ that is atomless and has full support. Then $\varphi$ is synchronising.

We will soon prove that under these conditions, $\varphi$ is actually stably synchronising.

Proof of Lemma 3.3. Fix any distinct $x, y \in S^1$. Let $J \subset S^1$ be a connected set with $\partial J = \{x, y\}$. By Proposition 2.3 and Lemma 3.2, there is a $\mathbb{P}$-full set of sample points $\omega$ with the properties that

(a) there exists an unbounded increasing sequence $(t_n)$ in $\mathbb{T}^+$ such that

$$d(\varphi(t_n, \omega)x, \varphi(t_n, \omega)y) \to 0 \quad \text{as } n \to \infty;$$

(b) $\rho(\varphi(t, \omega), J)$ converges as $t \to \infty$.

Fix any $\omega$ with both these properties, and let $(t_n)$ be as in (a). For any $n$, $d(\varphi(t_n, \omega)x, \varphi(t_n, \omega)y)$ is precisely the smaller of $l(\varphi(t_n, \omega), J)$ and $1 - l(\varphi(t_n, \omega), J)$. Hence there must exist a subsequence $(t_{m_n})$ of $(t_n)$ such that either $l(\varphi(t_{m_n}, \omega), J) \to 0$ as $n \to \infty$ or $l(\varphi(t_{m_n}, \omega), J) \to 1$ as $n \to \infty$. Since $\rho$ is atomless, it follows that either $\rho(\varphi(t_{m_n}, \omega), J) \to 0$ as $n \to \infty$ or $\rho(\varphi(t_{m_n}, \omega), J) \to 1$ as $n \to \infty$. Since $\rho(\varphi(t, \omega), J)$ is convergent as $t \to \infty$, it follows that either $\rho(\varphi(t, \omega), J) \to 0$ as $t \to \infty$ or $\rho(\varphi(t, \omega), J) \to 1$ as $t \to \infty$. Since $\rho$ has full support, it follows that either $l(\varphi(t, \omega), J) \to 0$ as $t \to \infty$ or $l(\varphi(t, \omega), J) \to 1$ as $t \to \infty$. Hence $d(\varphi(t, \omega)x, \varphi(t, \omega)y) \to 0$ as $t \to \infty$. ☐

Lemma 3.4. Under the hypothesis of Lemma 3.3, for any connected $J \subset S^1$,

$$\mathbb{P}(\omega : l(\varphi(t, \omega), J) \to 0 \text{ as } t \to \infty) = 1 - \rho(J).$$
Proof. Fix any connected $J \subset \mathbb{S}^1$. As in the proof of Lemma 3.3, we have that for $\mathbb{P}$-almost every $\omega \in \Omega$, either
\[
\rho(\varphi(t, \omega)J) \to 0 \quad \text{and} \quad l(\varphi(t, \omega)J) \to 0 \quad \text{as} \quad t \to \infty,
\]
or
\[
\rho(\varphi(t, \omega)J) \to 1 \quad \text{and} \quad l(\varphi(t, \omega)J) \to 1 \quad \text{as} \quad t \to \infty.
\]
So then, if we let $E$ denote the set of sample points $\omega$ for which we are in the latter scenario, the dominated convergence theorem gives that
\[
\int_{\Omega} \rho(\varphi(t, \omega)J) \mathbb{P}(d\omega) \to \int_{\Omega} \mathbb{1}_E(\omega) \mathbb{P}(d\omega) = \mathbb{P}(E).
\]
But we also know that for any $t$,
\[
\int_{\Omega} \rho(\varphi(t, \omega)J) \mathbb{P}(d\omega) = \rho(J).
\]
Hence the probability of the latter scenario is $\rho(J)$, and the probability of the former scenario is $1 - \rho(J)$.

Combining Lemmas 3.3 and 3.4, we have:

**Corollary 3.5.** Under the hypotheses of Lemma 3.3, $\varphi$ is stably synchronising.

**Proof.** We already know (from Lemma 3.3) that $\varphi$ is synchronising. Now fix any $x \in X$. Let $(U_n)_{n \in \mathbb{N}}$ be a nested sequence of connected neighbourhoods of $x$ such that $\cap_n U_n = \{x\}$. For each $n$,
\[
\mathbb{P}(\omega : \exists \text{ open } U \ni x \text{ s.t. } l(\varphi(t, \omega)U) \to 0 \text{ as } t \to \infty) \\
\geq \mathbb{P}(\omega : l(\varphi(t, \omega)U_n) \to 0 \text{ as } t \to \infty) \\
= 1 - \rho(U_n).
\]
But since $\rho$ is atomless, $\rho(U_n) \to 0$ as $n \to \infty$. Hence
\[
\mathbb{P}(\omega : \exists \text{ open } U \ni x \text{ s.t. } l(\varphi(t, \omega)U) \to 0 \text{ as } t \to \infty) = 1.
\]
So we are done.

So then, to complete the proof that (ii) implies stable synchronisation, we just need to show that (ii) implies the existence of a reverse-stationary probability measure that is atomless and has full support. (Contractibility will not actually play a part in this; we will show that if $\varphi$ has reverse-minimal dynamics then there exists a reverse-stationary probability measure that is atomless and has full support.)

**Lemma 3.6.** $\varphi$ admits at least one reverse-stationary probability measure.

**Proof.** For each $x \in \mathbb{S}^1$ and $t \in \mathbb{T}^+$, define the probability measure $\bar{\varphi}^t_x$ on $\mathbb{S}^1$ by
\[
\bar{\varphi}^t_x(A) := \mathbb{P}(\omega : x \in \varphi(t, \omega)A) = \mathbb{P}(\omega : x \in \varphi(t, \theta^s\omega)A)
\]
for all $A \in \mathcal{B}(\mathbb{S}^1)$ and any $s \in \mathbb{T}^+$. For any $s, t \in \mathbb{T}^+$, we have that
\begin{itemize}
\item \((\varphi^{s+t})_{x\in\mathbb{S}^1}\) is the family of transition probabilities associated to the random map 
\(\omega \mapsto \varphi(s+t, \omega)^{-1}\);
\item \((\varphi^t)_{x\in\mathbb{S}^1}\) is the family of transition probabilities associated to the random map 
\(\omega \mapsto \varphi(s, \omega)^{-1}\);
\item \((\varphi^t)_{x\in\mathbb{S}^1}\) is the family of transition probabilities associated to the random map 
\(\omega \mapsto \varphi(t, \theta^n \omega)^{-1}\);
\end{itemize}

and therefore, since \(\mathcal{F}_s\) and \(\theta^{-s}\mathcal{F}_t\) are independent, it follows (e.g. as in [New15a, Proposition 127]) that the Chapman-Kolmogorov equation
\[
\varphi^{s+t}(A) = \int_{\mathbb{S}^1} \varphi^s_y(A) \varphi^t_x(dy)
\]
is satisfied (for any \(x \in \mathbb{S}^1\) and \(A \in \mathcal{B}(\mathbb{S}^1)\)). Moreover, since the map \((t, x) \mapsto \varphi(t, \omega)^{-1}(x)\) is continuous for each \(\omega \in \Omega\), the dominated convergence theorem gives that the map 
\((t, x) \mapsto \varphi^t_x\) is continuous (with respect to the topology of weak convergence). Therefore, the Krylov-Bogolyubov theorem ([Kif86, Lemma 5.2.1] or [New15a, Theorem 114]) gives that there exists a probability measure \(\rho\) on \(\mathbb{S}^1\) with the property that for all \(t \in \mathbb{T}^+\) and 
\(A \in \mathcal{B}(\mathbb{S}^1)\),
\[
\rho(A) = \int_{\mathbb{S}^1} \varphi^t_x(A) \rho(dx).
\]
But this is precisely equivalent to saying that \(\rho\) is reverse-stationary.

\begin{proposition}
An open set \(U \subset \mathbb{S}^1\) is forward-invariant if and only if for each \(t \in \mathbb{T}^+\), for \(\mathbb{P}\)-almost all \(\omega \in \Omega\), \(\varphi(t, \omega)U \subset U\).
\end{proposition}

\begin{proof}
The “only if” direction is clear. Now suppose that for each \(t \in \mathbb{T}^+\), for \(\mathbb{P}\)-almost all 
\(\omega \in \Omega\), \(\varphi(t, \omega)U \subset U\). Let \(G := \mathbb{S}^1 \setminus U\). Then for each \(t\), for \(\mathbb{P}\)-almost all \(\omega\), \(\varphi(t, \omega)^{-1}(G) \subset G\). Hence in particular, letting \(D\) be a countable dense subset of \(\mathbb{T}^+\), we have that for \(\mathbb{P}\)-almost every \(\omega\), for all \(t \in D\), \(\varphi(t, \omega)^{-1}(G) \subset G\). But since the map 
\(t \mapsto \varphi(t, \omega)^{-1}(x)\) is continuous for each \(x\) and \(\omega\), it follows that for \(\mathbb{P}\)-almost every \(\omega\), for all \(t \in \mathbb{T}^+\), \(\varphi(t, \omega)^{-1}(G) \subset G\) and so \(\varphi(t, \omega)U \subset U\).
\end{proof}

\begin{remark}
We likewise have that a closed set \(G \subset \mathbb{S}^1\) is forward-invariant if and only if for each \(t \in \mathbb{T}^+\), for \(\mathbb{P}\)-almost all \(\omega \in \Omega\), \(\varphi(t, \omega)G \subset G\).
\end{remark}

\begin{lemma}
For any reverse-stationary probability measure \(\rho\), \(\mathbb{S}^1 \setminus \text{supp } \rho\) is forward-invariant.
\end{lemma}

\begin{proof}
Let \(\rho\) be a reverse-stationary probability measure, and let \(U := \mathbb{S}^1 \setminus \text{supp } \rho\). For each \(t \in \mathbb{T}^+\),
\[
0 = \rho(U) = \int_{\Omega} \rho(\varphi(t, \omega)U) \mathbb{P}(d\omega).
\]
Therefore, for each \(t\), for \(\mathbb{P}\)-almost all \(\omega\), \(\rho(\varphi(t, \omega)U) = 0\) and so \(\varphi(t, \omega)U \subset U\). Hence, by Proposition 3.7, \(U\) is forward-invariant.
\end{proof}

\begin{lemma}
If \(\varphi\) has reverse-minimal dynamics then every reverse-stationary probability measure is atomless and has full support.
\end{lemma}
Proof. Lemma 3.9 gives that if \( \phi \) has reverse-minimal dynamics then every reverse-stationary probability measure has full support. Now suppose that \( \phi \) has reverse-minimal dynamics and let \( \rho \) be a probability measure on \( S^1 \) that is not atomless; we will show that \( \rho \) is not reverse-stationary. Let \( m := \max\{\rho(\{x\}) : x \in S^1\} \) and let \( P = \{x \in S^1 : \rho(\{x\}) = m\} \). (So \( \rho(P) = m|P| \).) Since \( \phi \) has reverse-minimal dynamics, \( S^1 \setminus P \) is not forward-invariant and so (by Proposition 3.7) there must exist \( t_0 \in \mathbb{T}^+ \) such that

\[
\mathbb{P}(\omega : P \neq \phi(t_0, \omega)P) > 0.
\]

Obviously, for any \( \omega \), if \( P \neq \phi(t_0, \omega)P \) then \( \rho(\phi(t_0, \omega)P) < \rho(P) \). Hence we have that

\[
\int_{\Omega} \rho(\phi(t_0, \omega)P) \mathbb{P}(d\omega) < \rho(P).
\]

So \( \rho \) is not reverse-stationary.

Combining Lemmas 3.6 and 3.10, we have:

**Corollary 3.11.** If \( \phi \) has reverse-minimal dynamics then \( \phi \) admits a reverse-stationary probability measure that is atomless and has full support.

This completes the proof that (ii) \( \Rightarrow \) stable synchronisation. Finally, to show that (ii) \( \Rightarrow \) (i), we simply combine Corollary 3.11 above with the following corollary of Lemma 3.4:

**Corollary 3.12.** Under the hypotheses of Lemma 3.3, \( \phi \) is compressible.

Proof. For any connected \( J \subset S^1 \) with \( 0 < l(J) < 1 \), since \( \rho \) has full support, \( \rho(J) < 1 \). Lemma 3.4 then gives the required result.

### 4 Crack points and synchronisation

In this section, we will give a geometric characterisation of synchronisation and stable synchronisation. From there, we will go on to consider random fixed points.

First, observe that for any \( \omega \in \Omega \), the binary relation \( \sim_\omega \) on \( S^1 \) defined by

\[
x \sim_\omega y \iff d(\phi(t, \omega)x, \phi(t, \omega)y) \to 0 \text{ as } t \to \infty
\]

is an equivalence relation. It is easy to show (by considering only rational times) that the set \( \{(\omega, x, y) \in \Omega \times S^1 \times S^1 : x \sim_\omega y\} \) is an \((\mathcal{F}_* \otimes \mathcal{B}(S^1 \times S^1))\)-measurable set.

**Definition 4.1.** We will say that a sample point \( \omega \in \Omega \) is **contractive** if there exists \( z \in S^1 \) such that for all \( x, y \in S^1 \setminus \{z\} \), \( x \sim_\omega y \). We will say that a sample point \( \omega \in \Omega \) is **synchronising** if for all \( x, y \in S^1 \), \( x \sim_\omega y \).

Obviously, any synchronising sample point is also contractive: we can just take the point \( z \) in Definition 4.1 to be any point in \( S^1 \). However, if a sample point is contractive and not synchronising, then there is clearly only one possible choice for the point \( z \).

**Definition 4.2** (adapted from [Kai93]). We will say that a point \( r \in S^1 \) is a **crack point** of a sample point \( \omega \in \Omega \) if for every \( A \subset S^1 \) with \( r \notin \bar{A} \), \( \text{diam}(\phi(t, \omega)A) \to 0 \) as \( t \to \infty \).
Equivalently, \( r \) is a crack point of \( \omega \) if and only if for every open \( U \ni r \), \( l(\varphi(t,\omega)U) \to 1 \) as \( t \to \infty \). Obviously, any sample point can only admit at most one crack point.

**Lemma 4.3.** A sample point \( \omega \) admits a crack point if and only if it is contractive.

**Proof.** It is obvious that if \( \omega \) admits a crack point \( r \) then for every \( x, y \in S^1 \setminus \{r\} \), \( x \sim_\omega y \). Conversely, suppose we have a point \( z \in S^1 \) such that for all \( x, y \in S^1 \setminus \{z\} \), \( x \sim_\omega y \). Fix any arbitrary \( x_0 \in \mathbb{R} \setminus \pi^{-1}(\{z\}) \), and let \( v_0 \in (0,1) \) be such that \( \pi(x_0 + v_0) = z \). For each \( v \in [0,1] \), let \( J_v = \pi([x_0, x_0 + v]) \). Note that for each \( v \in [0,1] \setminus \{v_0\} \), \( \pi(x_0) \sim_\omega \pi(x_0 + v) \) and therefore \( l(\varphi(t,\omega)J_v) \) either converges to 0 or converges to 1 as \( t \to \infty \). So let

\[
\begin{align*}
v_c := \sup\{v \in [0,1] : l(\varphi(t,\omega)J_v) \to 0 \text{ as } t \to \infty\} \\
= \inf\{v \in [0,1] : l(\varphi(t,\omega)J_v) \to 1 \text{ as } t \to \infty\}.
\end{align*}
\]

Then it is clear that \( \pi(x_0 + v_c) \) is a crack point.

Of course, if a sample point \( \omega \) is contractive and not synchronising, then the unique point \( z \) described in Definition 4.1 is precisely the crack point of \( \omega \).

We will write \( \Omega_c \subset \Omega \) for the set of contractive sample points, and \( \Omega_s \subset \Omega_c \) for the set of synchronising sample points. We will write \( \tilde{\varphi} : \Omega_c \to S^1 \) to denote the function mapping a contractive sample point to its crack point. It is easy to show that for every \( t \in \mathbb{T}^* \), \( \theta^{t}(\Omega_c) \subset \Omega_c \), \( \theta^{-t}(\Omega_c) \subset \Omega_c \), \( \theta^{t}(\Omega_s) \subset \Omega_s \), \( \theta^{-t}(\Omega_s) \subset \Omega_s \), and \( \tilde{\varphi}(\theta^t \omega) = \varphi(t,\omega) \tilde{\varphi}(\omega) \) for all \( \omega \in \Omega_c \).

**Lemma 4.4.** \( \Omega_c \) and \( \Omega_s \) are \( \mathcal{F}_c \)-measurable sets, and \( \tilde{\varphi} \) is measurable with respect to the \( \sigma \)-algebra \( \mathcal{F}_c \) of \( \mathcal{F}_c \)-measurable subsets of \( \Omega_c \).

**Proof.** Observe that a sample point \( \omega \) admits a crack point if and only if for every \( n \in \mathbb{N} \) there exists a closed connected set \( J_n \subset S^1 \) with rational endpoints such that \( l(J_n) \in (\frac{1}{n},1) \) and \( l(\varphi(t,\omega)J_n) \to 0 \) as \( t \to \infty \). the complement of \( \bigcup_{n=1}^{\infty} J_n \) is precisely \( \{\tilde{\varphi}(\omega)\} \). Hence \( \Omega_c \) is \( \mathcal{F}_c \)-measurable. Now for any \( \omega \in \Omega_c \) and any non-empty closed connected \( K \subset S^1 \), observe that \( \tilde{\varphi}(\omega) \in K \) if and only if for every closed connected \( J \subset S^1 \setminus K \) with rational endpoints, \( l(\varphi(t,\omega)J) \to 0 \) as \( t \to \infty \). Hence \( \tilde{\varphi} \) is \( \mathcal{F}_c \)-measurable. Finally, writing \( x + \frac{1}{2} \) to mean the point diametrically opposite \( x \), it is clear that a contractive sample point \( \omega \) is synchronising if and only if both \( \tilde{\varphi}(\omega) \sim \tilde{\varphi}(\omega) + \frac{1}{2} \). Hence \( \Omega_s \) is \( \mathcal{F}_c \)-measurable.

**Theorem 4.5** (Classification theorem for synchronisation). \( \mathbb{P}(\Omega_c) \in \{0,1\} \). If \( \mathbb{P}(\Omega_c) = 0 \) then \( \varphi \) is not synchronising. If \( \mathbb{P}(\Omega_c) = 1 \), exactly one of the following scenarios occurs:

1. There exists a unique deterministic fixed point \( p \), and for \( \mathbb{P} \)-almost every \( \omega \in \Omega_c \), \( \tilde{\varphi}(\omega) = p \) and \( \varphi(t,\omega)x \to p \) for all \( x \in S^1 \) as \( t \to \infty \); in this case, \( \mathbb{P}(\Omega_s) = 1 \) and \( \varphi \) is synchronising, but \( \varphi \) is not stably synchronising;

2. \( \mathbb{P}(\Omega_s) = 0 \), there is at most one deterministic fixed point, \( \mathbb{P}(\omega \in \Omega_c : \tilde{\varphi}(\omega) = x) = 0 \) for every \( x \in S^1 \), and \( \varphi \) is stably synchronising;

3. \( \mathbb{P}(\Omega_s) = 0 \), there is a deterministic fixed point \( p \) such that \( \tilde{\varphi}(\omega) = p \) for \( \mathbb{P} \)-almost all \( \omega \in \Omega_c \), there is at most one other deterministic fixed point besides \( p \), and \( \varphi \) is not synchronising.
Our proof of Theorem 4.5 will extend from Lemma 4.6 to Corollary 4.15. (It is worth saying that Lemma 4.7, Lemma 4.8 and Proposition 4.9 are not specific to the circle, but hold for more general continuously invertible memoryless RDS on separable metric spaces. Lemma 4.6 is a general statement about memoryless filtered dynamical systems.)

**Lemma 4.6.** For any $t \in T^+ \setminus \{0\}$, the restricted probability measure $\mathbb{P}|_{\mathcal{F}_t}$ is ergodic with respect to $\theta^t$ (viewed as a measurable self-mapping of $(\Omega, \mathcal{F}_+)$).

For a proof, see e.g. Corollary 133 of [New15a]. Note that it follows from Lemma 4.6 that $\mathbb{P}(\Omega_c)$ and $\mathbb{P}(\Omega_s)$ are equal to 0 or 1.

Now recall from the proof of Lemma 3.6 that for each $x \in S^1$ and $t \in T^+$, we define the probability measure $\varphi^t_x$ on $S^1$ by

$$\varphi^t_x(A) := \mathbb{P}(\omega : x \in \varphi(t, \omega)A) = \mathbb{P}(\omega : \varphi(t, \omega)^{-1}(x) \in A).$$

So for each $t$, $(\varphi^t_x)_{x \in S^1}$ is the family of transition probabilities associated to the inverse of the time-$t$ mapping of $\varphi$. As was implicitly seen in the proof of Lemma 3.6, Definition 3.1 can equivalently be formulated as follows: we say that a probability measure $\rho$ on $S^1$ is reverse-stationary if for every $t \in T^+$, $\rho$ is a stationary distribution for the family of transition probabilities $(\varphi^t_x)_{x \in S^1}$.

For each $x \in S^1$ and $t \in T^+$, we define the probability measure $\varphi^t_x$ on $S^1$ by

$$\varphi^t_x(A) := \mathbb{P}(\omega : \varphi(t, \omega)x \in A).$$

For each $t$, $(\varphi^t_x)_{x \in S^1}$ is the family of transition probabilities associated to the time-$t$ mapping of $\varphi$. We will say that a probability measure $\rho$ on $S^1$ is stationary if for every $t \in T^+$, $\rho$ is a stationary distribution for the family of transition probabilities $(\varphi^t_x)_{x \in S^1}$. As in the proof of Lemma 3.6, $(\varphi^t_x)_{x \in S^1, t \in T^+}$ satisfies the Chapman-Kolmogorov equations and therefore the Krylov-Bogolyubov theorem gives the existence of at least one stationary probability measure.

**Lemma 4.7.** Suppose we have an $\mathcal{F}_r$-measurable function $q : \Omega \to S^1$ and a time $\tau \in T^+$ such that for $\mathbb{P}$-almost all $\omega \in \Omega$, $\varphi(\tau, \omega)q(\omega) = q(\theta^\tau \omega)$. Then $q_*\mathbb{P}$ is an ergodic measure for the family of transition probabilities $(\varphi^{\tau}_x)_{x \in S^1}$.

**Proof.** First we show that $\rho$ is stationary with respect to $(\varphi^{\tau}_x)_{x \in S^1}$. Note that the map $\omega \mapsto q(\theta^\tau \omega)$ is $(\theta^{-\tau}\mathcal{F}_+)$-measurable. For any $A \in \mathcal{B}(S^1)$,

$$\int_{S^1} \varphi^{\tau}\rho(A) q_*\mathbb{P}(dx) = \int_{S^1} \varphi^{\tau}(A) (q \circ \theta^\tau)_*\mathbb{P}(dx) \quad \text{(since $\mathbb{P}$ is $\theta^\tau$-invariant)},$$

$$= \int_{\Omega} \varphi^{\tau}(\theta^\tau \omega)(A) \mathbb{P}(d\omega),$$

$$= \int_{\Omega} \mathbb{P}(\tilde{\omega} : \varphi(\tau, \tilde{\omega})^{-1}(q(\theta^\tau \omega)) \in A) \mathbb{P}(d\omega),$$

$$= \mathbb{P}(\omega : \varphi(\tau, \omega)^{-1}(q(\theta^\tau \omega)) \in A) \quad \text{since $\mathcal{F}_r$ and $\theta^{-\tau}\mathcal{F}_+$ are independent},$$

$$= \mathbb{P}(\omega : q(\omega) \in A)$$

$$= q_*\mathbb{P}(A).$$

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Suppose \( \phi \) is ergodic with respect to the family of transition probabilities \( \rho \). Proposition 4.9. Suppose we have a probability measure \( \rho \). Hence\( \rho \) is a deterministic fixed point. Applying this to \( \tilde{\rho} \), we have \( q \circ \theta^{-1} \) and \( \tilde{\rho} \) is stationary with \( \tilde{\rho} \) and \( \tilde{\rho} \) are independent since \( \mathcal{F}_t \) and \( \theta^{-1} \mathcal{F}_t \) are independent.

\[
(\tau, \omega) \in \mathcal{F}_t \text{ and } \theta^{-1} \mathcal{F}_t \text{ are independent}
\]

\[
\int_{(q \circ \theta^{-1})^{-1}(A)} \mathbb{P}(q^{-1}(A) \cap (q \circ \theta^{-1})(A)) = \int_{(q \circ \theta^{-1})^{-1}(A)} \mathbb{P}(q^{-1}(A) | \theta^{-1} \mathcal{F}_t)(\omega) \mathbb{P}(d\omega)
\]

(For a justification of the antepenultimate equality, see e.g. [New15a, Exercise 124(A)].) Now let \( A \in \mathcal{B}(S^1) \) be a set such that for \( \rho \)-almost every \( x \in A \), \( \tilde{\rho}^x(A) = 1 \); we need to show that \( q_\ast \mathbb{P}(A) \in \{0, 1\} \). We have

\[
\mathbb{P}(q^{-1}(A) \cap (q \circ \theta^{-1})^{-1}(A)) = \int_{(q \circ \theta^{-1})^{-1}(A)} \mathbb{P}(q^{-1}(A) | \theta^{-1} \mathcal{F}_t)(\omega) \mathbb{P}(d\omega)
\]

\[
= \int_{(q \circ \theta^{-1})^{-1}(A)} \mathbb{P}(\hat{\omega}: \phi(\tau, \hat{\omega})^{-1}(q(\theta^{-1} \omega)) \in A) \mathbb{P}(d\omega)
\]

(For a justification of the second equality, see e.g. [New15a, Exercise 124(B)].) So then, letting \( E := q^{-1}(A) \), we have that \( \mathbb{P}(\theta^{-1}(E) \setminus E) = 0 \). Since \( E \in \mathcal{F}_t \), it follows by Lemma 4.6 that \( \mathbb{P}(E) \in \{0, 1\} \), i.e. \( q_\ast \mathbb{P}(A) \in \{0, 1\} \) as required.

**Lemma 4.8.** Suppose we have a probability measure \( \rho \) on \( S^1 \) and a time \( \tau \in \mathbb{T}^+ \) such that \( \rho \) is ergodic with respect to the family of transition probabilities \( (\tilde{\rho}_x^\tau)_{x \in S^1} \). Then either \( \rho \) is atomless, or \( \rho = \frac{1}{n} (\delta_{x_1} + \ldots + \delta_{x_n}) \) for some \( n \in \mathbb{N} \) and distinct points \( x_1, \ldots, x_n \in S^1 \) satisfying \( \phi(\tau, \omega) \{x_1, \ldots, x_n\} = \{x_1, \ldots, x_n\} \) for \( \mathbb{P} \)-almost all \( \omega \in \Omega \).

**Proof.** Suppose \( \rho \) is not atomless. Let \( m := \max \{\rho(\{x\}) : x \in S^1\} \), and let \( \{x_1, \ldots, x_n\} =: P \) be the set of all those points \( x \in S^1 \) for which \( \rho(\{x\}) = m \). As we have already seen (in the contrapositive direction) within the proof of Lemma 3.10, since \( \rho \) is stationary with respect to \( (\tilde{\rho}_x^\tau)_{x \in S^1} \), \( \phi(\tau, \omega)P = P \) for \( \mathbb{P} \)-almost all \( \omega \in \Omega \). Hence in particular, \( \tilde{\rho}_x^\tau(P) = 1 \) for each \( x \in P \). Since \( \rho \) is ergodic with respect to \( (\tilde{\rho}_x^\tau)_{x \in S^1} \), it follows that \( \rho(P) \in \{0, 1\} \). But \( \rho(P) > 0 \), and therefore \( \rho(P) = 1 \). So we are done.

**Proposition 4.9.** Suppose we have an \( \mathcal{F}_t \)-measurable function \( q : \Omega \to S^1 \) and a time \( \tau \in \mathbb{T}^+ \) such that for \( \mathbb{P} \)-almost all \( \omega \in \Omega \), \( \phi(\tau, \omega)q(\omega) = q(\theta^\tau \omega) \). Then \( q_\ast \mathbb{P} \) is either atomless or a Dirac mass.

**Remark 4.10.** In the case that \( q_\ast \mathbb{P} \) is a Dirac mass \( \delta_p \), we have that for \( \mathbb{P} \)-almost all \( \omega \in \Omega \), \( \phi(\tau, \omega)p = p \). Hence, if every \( t \in \mathbb{T}^+ \) has the property that for \( \mathbb{P} \)-almost all \( \omega \in \Omega \), \( \phi(t, \omega)q(\omega) = q(\theta^t \omega) \), and if \( q_\ast \mathbb{P} \) is a Dirac mass \( \delta_p \), then (by Proposition 3.7) \( p \) is a deterministic fixed point. Applying this to \( \tilde{\tau} : \mathbb{P}(\Omega_{\ast}) = 1 \) then \( \tilde{\tau}_\ast(\mathbb{P}|_{\mathcal{F}_t}) \) is either atomless or supported on a deterministic fixed point.

**Remark 4.11.** Suppose \( \theta^t \) is a measurable self-isomorphism of \( (\Omega, \mathcal{F}) \) for each \( t \), and suppose we have a \( \sigma(\theta^t \mathcal{F}_t : t \in \mathbb{T}^+) \)-measurable function \( q : \Omega \to S^1 \) and a time \( \tau \in \mathbb{T}^+ \) such that for \( \mathbb{P} \)-almost all \( \omega \in \Omega \), \( \phi(\tau, \omega)q(\omega) = q(\theta^\tau \omega) \). Then (by considering the time-reversal of \( \phi \)) we once again have that \( q_\ast \mathbb{P} \) is either atomless or a Dirac mass.

**Proof of Proposition 4.9.** Suppose \( q_\ast \mathbb{P} \) is not atomless. By Lemmas 4.7 and 4.8, there exist distinct points \( x_1, \ldots, x_n \in S^1 \) such that \( q_\ast \mathbb{P} = \frac{1}{n}(\delta_{x_1} + \ldots + \delta_{x_n}) \) and \( \{x_1, \ldots, x_n\} \) is
almost surely invariant under the time-$\tau$ map of $\varphi$. Let $E := q^{-1}(\{x_1\})$. For each $m \in \mathbb{N}$, for any $F \in \mathcal{F}_m$, we have that

$$
\mathbb{P}(E \cap F) = \int_F \mathbb{P}(E|\mathcal{F}_m)(\omega) \mathbb{P}(d\omega)
$$

$$
= \int_F \mathbb{P}(\tilde{\omega} : \varphi(\tau, \omega)^{-1}(q(\theta^\tau \tilde{\omega})) = x_1) \mathbb{P}(d\omega)
$$

since $\mathcal{F}_\tau$ and $\theta^{-\tau} \mathcal{F}_+$ are independent

$$
= \int_F \mathbb{P}(\tilde{\omega} : q(\theta^\tau \tilde{\omega}) = \varphi(\tau, \omega)x_1) \mathbb{P}(d\omega)
$$

$$
= \int_F q_* \mathbb{P}(\{\varphi(\tau, \omega)x_1\}) \mathbb{P}(d\omega)
$$

$$
= \int_F \frac{1}{n} \mathbb{P}(d\omega)
$$

$$
= \frac{1}{n} \mathbb{P}(F)
$$

$$
= \mathbb{P}(E) \mathbb{P}(F).
$$

(As in the second half of the proof of Lemma 4.7, the second equality can be justified using [New15a, Exercise 124(B)].) So $E$ is independent of $\mathcal{F}_m$ for each $m \in \mathbb{N}$, and is therefore independent of $\mathcal{F}_+$. In particular, $E$ is independent of itself, and so $\mathbb{P}(E) = 1$; hence $n = 1$ and $q_* \mathbb{P} = \delta_{x_1}$.

**Lemma 4.12.** If $\mathbb{P}(\Omega_c) = 1$, then $\varphi$ is stably synchronising if and only if $\tilde{\varphi}_*(\mathbb{P}|_{\mathcal{F}_c})$ is atomless.

**Proof.** Suppose that $\mathbb{P}(\Omega_c) = 1$ and $\tilde{\varphi}_*(\mathbb{P}|_{\mathcal{F}_c})$ is atomless. Given any $x, y \in \mathbb{S}^1$, for $\mathbb{P}$-almost every $\omega \in \Omega_c$, $\tilde{\varphi}(\omega) \notin \{x, y\}$ and so $x \sim_\omega y$. Furthermore, given any $x \in \mathbb{S}^1$, for $\mathbb{P}$-almost every $\omega \in \Omega_c$, $\tilde{\varphi}(\omega) \neq x$ and so there exists a neighbourhood $U$ of $x$ such that $\text{diam}(\varphi(t, \omega)U) \to 0$ as $t \to \infty$. Hence $\varphi$ is stably synchronising. Conversely: for any $\omega \in \Omega_c$ there does not exist a neighbourhood $U$ of $\tilde{\varphi}(\omega)$ such that $\text{diam}(\varphi(t, \omega)U) \to 0$ as $t \to \infty$; hence if there exists a value $x \in \mathbb{S}^1$ which $\tilde{\varphi}$ can take with strictly positive probability, then $\varphi$ cannot be stably synchronising.

**Lemma 4.13.** If $\mathbb{P}(\Omega_c) = 1$ and $\tilde{\varphi}_*(\mathbb{P}|_{\mathcal{F}_c})$ is atomless then $\mathbb{P}(\Omega_s) = 0$.

**Proof.** Suppose $\mathbb{P}(\Omega_c) = 1$ and $\tilde{\varphi}_*(\mathbb{P}|_{\mathcal{F}_c})$ is atomless. Define the functions

$$
\Theta : \Omega \times \mathbb{S}^1 \to \Omega \times \mathbb{S}^1
$$

$$
\Theta_2 : \Omega \times \mathbb{S}^1 \times \mathbb{S}^1 \to \Omega \times \mathbb{S}^1 \times \mathbb{S}^1
$$

by

$$
\Theta(\omega, x) = (\theta^1 \omega, \varphi(1, \omega)x)
$$

$$
\Theta_2(\omega, x, y) = (\theta^1 \omega, \varphi(1, \omega)x, \varphi(1, \omega)y).
$$

Fix a stationary probability measure $\rho$. It is not hard to show (e.g. as in [New15a, Theorem 143(i)]) that the probability measure $\mathbb{P}|_{\mathcal{F}_c} \otimes \rho$ on $(\Omega \times \mathbb{S}^1, \mathcal{F}_c \otimes \mathcal{B}(\mathbb{S}^1))$ is invariant under $\Theta$. Since $\tilde{\varphi}(\theta^1 \omega) = \varphi(1, \omega)\tilde{\varphi}(\omega)$ for all $\omega \in \Omega_c$, it then follows that the probability measure $w$ on $(\Omega \times \mathbb{S}^1 \times \mathbb{S}^1, \mathcal{F}_c \otimes \mathcal{B}(\mathbb{S}^1 \times \mathbb{S}^1))$ given by

$$
w(A) = \mathbb{P} \otimes \rho( (\omega, x) \in \Omega_c \times \mathbb{S}^1 : (\omega, x, \tilde{\varphi}(\omega)) \in A )
$$

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is invariant under $\Theta_2$. Now let $\Delta := \{(x, x) : x \in \mathbb{S}^1\}$. Since $\tilde{r}_*(\mathbb{P}|_{\mathbb{S}^1})$ is atomless, we have that
\[
w(\Omega \times \Delta) = \mathbb{P} \otimes \rho((\omega, x) \in \Omega_c \times \mathbb{S}^1 : \tilde{r}(\omega) = x) = \int_{\mathbb{S}^1} \tilde{r}_*(\mathbb{P}|_{\mathbb{S}^1})(x) \rho(dx) \quad \text{(by Fubini’s theorem)} = 0.
\]
Therefore, there must exist $\varepsilon > 0$ such that $w(\Omega \times U_\varepsilon) > 0$, where
\[
U_\varepsilon := \{(x, y) \in \mathbb{S}^1 : d(x, y) > \varepsilon\}.
\]
The Poincaré recurrence theorem then gives that there is a $w$-positive measure set of points in $\Omega \times \mathbb{S}^1 \times \mathbb{S}^1$ which return infinitely often to $\Omega \times U_\varepsilon$ under iterations of the map $\Theta_2$. In other words,
\[
\mathbb{P} \otimes \rho((\omega, x) \in \Omega_c \times \mathbb{S}^1 : \limsup_{n \to \infty} d(\varphi(n, \omega)x, \varphi(n, \omega)\tilde{r}(\omega)) > \varepsilon) > 0.
\]
By Fubini’s theorem, it follows that there is a $\mathbb{P}$-positive measure set of sample points that are not synchronising. \hfill \Box

**Lemma 4.14.** Suppose we have a sample point $\omega \in \Omega$ and a dense set $S \subset \mathbb{S}^1$ such that for all $x, y \in S$, $x \sim_\omega y$. Then $\omega$ admits a crack point.

**Proof.** This is the same as the proof of the “if” direction in Lemma 4.3, just replacing $\{z\}$ with $\mathbb{S}^1 \setminus S$ and $\{v_0\}$ with $\{v \in (0, 1) : \pi(x_0 + v) \in \mathbb{S}^1 \setminus S\}$. \hfill \Box

**Corollary 4.15.** If $\varphi$ is synchronising then $\mathbb{P}(\Omega_c) = 1$.

**Proof.** Suppose $\varphi$ is synchronising, and fix a countable dense set $S \subset \mathbb{S}^1$. Since $S$ is countable, we have that for $\mathbb{P}$-almost every $\omega \in \Omega$, for all $x, y \in S$, $x \sim_\omega y$ and therefore $\omega \in \Omega_c$ (by Lemma 4.14). So $\mathbb{P}(\Omega_c) = 1$. \hfill \Box

We make one final remark before being in a position to prove Theorem 4.5: Given a sample point $\omega \in \Omega$ and a point $x \in \mathbb{S}^1$, let us say that $x$ is a fixed point of $\omega$ if $\varphi(t, \omega)x = x$ for all $t \in \mathbb{T}^+$; it is clear that any contractive sample point $\omega \in \Omega_c$ can admit at most two fixed points. Therefore, if $\mathbb{P}(\Omega_c) = 1$ then $\varphi$ has at most two deterministic fixed points. It is also clear that if $\varphi$ is synchronising then $\varphi$ has at most one deterministic fixed point.

With all of this, it is now straightforward to complete the proof of Theorem 4.5:

**Proof of Theorem 4.5.** We have established that $\mathbb{P}(\Omega_c)$ is either 0 or 1; by Corollary 4.15, if $\mathbb{P}(\Omega_c) = 0$ then $\varphi$ is not synchronising. If $\mathbb{P}(\Omega_c) = 1$ then, as we have established, $\mathbb{P}(\Omega_c)$ is either 0 or 1. In the case that $\mathbb{P}(\Omega_c) = 1$, $\varphi$ is obviously synchronising and we know, by Lemma 4.13 combined with Remark 4.10, that $\tilde{r}_*(\mathbb{P}|_{\mathbb{S}^1})$ is a Dirac mass supported on a deterministic fixed point $p$ (which must be the only deterministic fixed point since $\varphi$ is synchronising); Lemma 4.12 also gives that in this case, $\varphi$ is not stably synchronising. In the case that $\mathbb{P}(\Omega_c) = 1$ and $\mathbb{P}(\Omega_s) = 0$, we have the following: if $\tilde{r}_*(\mathbb{P}|_{\mathbb{S}^1})$ is atomless then Lemma 4.12 gives that $\varphi$ is stably synchronising (and therefore has at most one
We will refer to the pair \((f^t)_{t \in \mathbb{T}^*}\) of homeomorphisms \(f^t : \mathbb{S}^1 \to \mathbb{S}^1\) such that \(f^{s+t} = f^s \circ f^t\) for all \(s, t \in \mathbb{T}^*\). We will say that an autonomous flow \((f^t)\) is simple if there exist distinct points \(r, a \in \mathbb{S}^1\) (called the repeller and the attractor of \((f^t)\)) such that \(f^t(r) = r\) for all \(t \in \mathbb{T}^+\), and \(f^t(x) \to a\) as \(t \to \infty\) for all \(x \in \mathbb{S}^1 \setminus \{r\}\). (It follows that \(f^t(a) = a\) for all \(t \in \mathbb{T}^+\).) If \((f^t)\) is simple, it is easy to show that the set of invariant probability measures for \((f^t)\) is given by \(\{\lambda \delta_r + (1 - \lambda) \delta_a : \lambda \in [0, 1]\}\).

We will refer to the pair \((a, r)\) as a global attractor-repeller pair for \((f^t)\), on the heuristic grounds that “\(r\) pushes all points in \(\mathbb{S}^1\) (other than itself) towards \(a\)”. We now go on to generalise the notion of a “global attractor-repeller pair” to the random setting. We will need a couple of assumptions.

**Assumption A:** For all \(t \in \mathbb{T}^+\), \(\theta^t\) is a measurable isomorphism of \((\Omega, \mathcal{F})\) into itself.

The heuristic interpretation of this assumption is that the underlying random process has been going on since eternity past. We refer to \(\mathcal{F}_+\) as the future \(\sigma\)-algebra, and we define the past \(\sigma\)-algebra as \(\mathcal{F}_- := \sigma(\theta^t \mathcal{F}_t : t \in \mathbb{T}^+) \subset \mathcal{F}\). It is not hard to show that \(\mathcal{F}_+\) and \(\mathcal{F}_-\) are independent (under \(\mathbb{P}\)).

Let \(\tilde{\mathcal{F}} := \sigma(\mathcal{F}_+ \cup \mathcal{F}_-)\). It is easy to check that for all \(t \in \mathbb{T}^+\), \(\theta^t \tilde{\mathcal{F}} = \theta^{-t} \tilde{\mathcal{F}} = \tilde{\mathcal{F}}\). Moreover, we have the following.

**Lemma 4.16.** For all \(t \in \mathbb{T}^+ \setminus \{0\}\), \(\mathbb{P}|_{\tilde{\mathcal{F}}}\) is ergodic with respect to \(\theta^t\) (viewed as a measurable self-mapping of \((\Omega, \tilde{\mathcal{F}})\)).

**Proof.** Fix \(t \in \mathbb{T}^+ \setminus \{0\}\). For each \(n \in \mathbb{Z}\), let \(G_n = \theta^{nt} \mathcal{F}_+\). Let \(G_{\infty} := \bigcap_{n \in \mathbb{Z}} G_n\), and observe that \(\tilde{\mathcal{F}} = \sigma(\bigcup_{n \in \mathbb{Z}} G_n)\). Let \(E \in \mathcal{F}\) be a set with \(\theta^{-t}(E) = E\), and let \(g : \Omega \to \mathbb{S}^1\) be a version of the conditional probability of \(E\) given \(\mathcal{F}_+\) (which is equal to \(G_0\)). By the transformation-of-conditional-expectations formula (e.g. [New15a, Exercise 13(C)]), for every \(n \in \mathbb{Z}\), \(g \circ \theta^{nt}\) is a version of the conditional probability of \(E\) given \(G_n\). By a version of the Kolmogorov 0-1 law (e.g. [New15a, Proposition 132]), \(G_{\infty}\) is a \(\mathbb{P}\)-trivial \(\sigma\)-algebra; hence the constant map \(\omega \to \mathbb{P}(E)\) is a version of the conditional probability of \(E\) given \(G_{\infty}\). Therefore, by Lévy’s downward theorem ([Will91, Theorem 14.4]), \(g \circ \theta^{nt}(\omega) \to \mathbb{P}(E)\) as \(n \to \infty\) for \(\mathbb{P}\)-almost all \(\omega \in \Omega\). But since \(\theta^t\) is itself \(\mathbb{P}\)-preserving, it follows (e.g. since almost-sure convergence implies convergence in distribution, or alternatively by the Poincaré recurrence theorem) that \(g(\omega) = \mathbb{P}(E)\) for \(\mathbb{P}\)-almost all \(\omega \in \Omega\). Obviously we therefore have that for each \(n \in \mathbb{N}\), \(g \circ \theta^{-nt}\) is equal to \(\mathbb{P}(E)\) almost everywhere, i.e. the constant map \(\omega \to \mathbb{P}(E)\) is a version of the conditional probability of \(E\) given \(G_n\), and so \(E\) is

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\(^3\)see also the section “Invariant and Tail \(\sigma\)-Algebra” on p547 of [Arn98].
independent of $\mathcal{G}_n$. Since this is true for every $n$, it follows that $E$ is independent of $\tilde{\mathcal{F}}$; therefore, in particular, $E$ is independent of itself, and so $\mathbb{P}(E) \in \{0, 1\}$. \hfill \qed

Hence (by restricting $\mathcal{F}$ to $\tilde{\mathcal{F}}$ if necessary) we may add the following assumption without loss of generality:

**Assumption B:** $\mathbb{P}$ is an ergodic measure of the dynamical system $(\theta^t)_{t \in \mathbb{T}^1}$ on $(\Omega, \mathcal{F})$.

With these assumptions, we start by introducing some general theory of invariant measures for random dynamical systems.

Let $\mathcal{M}_1$ be the set of probability measures on $\mathbb{S}^1$, equipped with the “evaluation $\sigma$-algebra” $\sigma(\rho \mapsto \rho(A) : A \in \mathcal{B}(\mathbb{S}^1))$; so for any measurable space $(X, \Sigma)$, a function $f : X \to \mathcal{M}_1$ is measurable if and only if the map $\xi \mapsto f(\xi)(A)$ from $X$ to $[0, 1]$ is measurable for all $A \in \mathcal{B}(\mathbb{S}^1)$. It is not hard to show that the map $x \mapsto \delta_x$ is a measurable embedding of $\mathbb{S}^1$ into $\mathcal{M}_1$.

A random probability measure is an $\Omega$-indexed family $(\mu_\omega)_{\omega \in \Omega}$ of probability measures $\mu_\omega$ on $\mathbb{S}^1$ such that the map $\omega \mapsto \mu_\omega$ from $\Omega$ to $\mathcal{M}_1$ is measurable (which is equivalent to saying that for each $A \in \mathcal{B}(\mathbb{S}^1)$, the map $\omega \mapsto \mu_\omega(A)$ from $\Omega$ to $[0, 1]$ is measurable).

**Definition 4.17.** We say that a probability measure $\mu$ on $(\Omega \times \mathbb{S}^1, \mathcal{F} \otimes \mathcal{B}(\mathbb{S}^1))$ is compatible if $\mu(E \times \mathbb{S}^1) = \mathbb{P}(E)$ for all $E \in \mathcal{F}$. We write $\mathcal{M}_1^\mathbb{P}$ for the set of compatible probability measures.

The **disintegration theorem** (e.g. [Crau02a, Proposition 3.6]) gives that for every compatible probability measure $\mu$ there exists a random probability measure $(\mu_\omega)_{\omega \in \Omega}$ (unique up to $\mathbb{P}$-almost everywhere equality) such that

$$\mu(A) = \int_{\Omega} \mu_\omega(A_{\omega}) \mathbb{P}(d\omega)$$

for all $A \in \mathcal{F} \otimes \mathcal{B}(\mathbb{S}^1)$, where $A_{\omega}$ denotes the $\omega$-section of $A$. The random probability measure $(\mu_\omega)$ is called a (version of the) disintegration of $\mu$, and we will refer to $\mu$ as the integrated form of $(\mu_\omega)$.

**Lemma 4.18.** Let $\mu^1$ and $\mu^2$ be compatible probability measures, with $(\mu^1_\omega)$ and $(\mu^2_\omega)$ being disintegrations of $\mu^1$ and $\mu^2$ respectively. If $\mu^1$ and $\mu^2$ are mutually singular then for $\mathbb{P}$-almost every $\omega \in \Omega$, $\mu^1_\omega$ and $\mu^2_\omega$ are mutually singular.

**Proof.** Suppose we have a set $A \in \mathcal{F} \otimes \mathcal{B}(\mathbb{S}^1)$ such that $\mu^1(A) = 1$ and $\mu^2(A) = 0$. Then it is clear that for $\mathbb{P}$-almost all $\omega \in \Omega$, $\mu^1_\omega(A_{\omega}) = 1$ and $\mu^2_\omega(A_{\omega}) = 0$. So we are done. \hfill \qed

Heuristically, the map $x \mapsto \delta_x$ serves as a natural way of identifying points in $\mathbb{S}^1$ with measures on $\mathbb{S}^1$. We can give a “randomised” version of this same concept:

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4It is clear that $x \mapsto \delta_x$ is measurable. For the converse: for any $A \in \mathcal{B}(\mathbb{S}^1)$, $\{\delta_x : x \in A\}$ is precisely the set of probability measures $\rho$ for which $\rho \otimes \rho(\{(x, x) : x \in A\}) = 1$; it is easy to show that the set $\{(x, x) : x \in A\}$ is measurable and that the map $\rho \mapsto \rho \otimes \rho$ is measurable (according to the respective evaluation $\sigma$-algebras), and therefore $\{\delta_x : x \in A\}$ is measurable.
**Definition 4.19.** For any measurable function \( q : \Omega \to \mathbb{S}^1 \), we define the probability measure \( G_q \) on \((\Omega \times \mathbb{S}^1, \mathcal{F} \otimes \mathcal{B}(\mathbb{S}^1))\) by \( G_q(A) = \mathbb{P}(\omega \in \Omega : (\omega, q(\omega)) \in A) \).

Note that \( G_q \) is a compatible probability measure, and admits the disintegration \((\delta_{q(\omega)})_{\omega \in \Omega}\).

Let \( L^0(\mathbb{P}, \mathbb{S}^1) \) denote the set of equivalence classes of measurable functions from \( \Omega \) to \( \mathbb{S}^1 \) (where “equivalence” means agreement on a \( \mathbb{P} \)-full set). For any measurable \( q : \Omega \to \mathbb{S}^1 \), let \( \hat{q} \in L^0(\mathbb{P}, \mathbb{S}^1) \) be the equivalence class represented by \( q \). By uniqueness of disintegration, the map \( \hat{q} \mapsto G_q \) from \( L^0(\mathbb{P}, \mathbb{S}^1) \) to \( \mathcal{M}_1^\mathbb{P} \) is injective; thus, heuristically, we have a natural way of identifying “random points in \( \mathbb{S}^1 \)” (defined up to almost sure equality) with compatible probability measures.

Now for each \( t \in \mathbb{T}^+ \), define the map \( \Theta^t : \Omega \times \mathbb{S}^1 \to \Omega \times \mathbb{S}^1 \) by

\[
\Theta^t(\omega, x) = (\theta^t \omega, \varphi(t, \omega) x).
\]

Note that \((\Theta^t)_{t \in \mathbb{T}^+}\) is a dynamical system on \( \Omega \times \mathbb{S}^1 \)—that is, \( \Theta^0 \) is the identity function and \( \Theta^{s+t} = \Theta^t \circ \Theta^s \) for all \( s, t \in \mathbb{T}^+ \).

**Definition 4.20.** We say that a probability measure \( \mu \) on \( \Omega \times \mathbb{S}^1 \) is \( \varphi \)-invariant if \( \mu \) is both compatible and invariant under \((\Theta^t)_{t \in \mathbb{T}^+}\) (that is, \( \Theta^t \mu = \mu \) for all \( t \in \mathbb{T}^+ \)).

**Lemma 4.21.** Let \((\mu_\omega)\) be a random probability measure. The integrated form of \((\mu_\omega)\) is \( \varphi \)-invariant if and only if for each \( t \in \mathbb{T}^+ \), for \( \mathbb{P} \)-almost all \( \omega \in \Omega \), \( \varphi(t, \omega)_* \mu_\omega = \mu_{\theta^t \omega} \).

The proof is a straightforward exercise (see e.g. [Arn98, Theorem 1.4.5(ii)]).

**Definition 4.22.** A random fixed point is a measurable function \( q : \Omega \to \mathbb{S}^1 \) with the property that for each \( t \in \mathbb{T}^+ \), for \( \mathbb{P} \)-almost every \( \omega \in \Omega \), \( \varphi(t, \omega) q(\omega) = q(\theta^t \omega) \).

So then (e.g. as a special case of Lemma 4.21) a measurable function \( q : \Omega \to \mathbb{S}^1 \) is a random fixed point if and only if \( G_q \) is \( \varphi \)-invariant. Moreover, as a consequence of Assumption B, we have the following:

**Lemma 4.23.** For any random fixed point \( q : \Omega \to \mathbb{S}^1 \), \( G_q \) is ergodic with respect to \((\Theta^t)_{t \in \mathbb{T}^+}\).

**Proof.** Fix any \( A \in \mathcal{F} \otimes \mathcal{B}(\mathbb{S}^1) \), and for each \( t \in \mathbb{T}^+ \), let

\[
E_t := \{ \omega \in \Omega : \Theta^t(\omega, q(\omega)) \in A \}.
\]

Note that for every \( t \in \mathbb{T}^+ \), \( G_q(\Theta^{-t}(A) \Delta A) = \mathbb{P}(E_t \Delta E_0) \). Now since \( q \) is a random fixed point, we have that for each \( t \in \mathbb{T}^+ \), \( \mathbb{P}(\theta^{-t}(E_0) \Delta E_t) = 0 \).

So, if \( G_q(\Theta^{-t}(A) \Delta A) = 0 \) for all \( t \in \mathbb{T}^+ \), then \( \mathbb{P}(\theta^{-t}(E_0) \Delta E_0) = 0 \) for all \( t \in \mathbb{T}^+ \), so \( \mathbb{P}(E) \in \{0, 1\} \) (since \( \mathbb{P} \) is \((\Theta^t)\)-ergodic), so \( G_q(A) \in \{0, 1\} \). Hence \( G_q \) is \((\Theta^t)\)-ergodic.

Having developed the above general theory, we can now describe the set of \( \varphi \)-invariant probability measures when \( \mathbb{P}(\Omega_c) = 1 \).
Proposition 4.24. Suppose $\mathbb{P}(\Omega_c) = 1$, and let $r : \Omega \to \mathbb{S}^1$ be a measurable function with $r(\omega) = \hat{r}(\omega)$ for all $\omega \in \Omega_c$. (Obviously $r$ is a random fixed point, and can be chosen to be $\mathcal{F}_s$-measurable.) Then either

(A) $G_r$ is the only $\varphi$-invariant probability measure; or

(B) there exists an $(\mathcal{F}_s, \mathcal{B}(\mathbb{S}^1))$-measurable random fixed point $\alpha : \Omega \to \mathbb{S}^1$, with $\alpha(\omega) \neq \hat{r}(\omega)$ for $\mathbb{P}$-almost every $\omega \in \Omega_c$, such that the set of $\varphi$-invariant probability measures is given by $\{\lambda G_r + (1 - \lambda)G_\alpha : \lambda \in [0, 1]\}$.

Definition 4.25. If $\mathbb{P}(\Omega_c) = 1$ and case (B) of Proposition 4.24 holds, then we will say that $\varphi$ is simple. In this case, letting $r$ and $\alpha$ be as in Proposition 4.24, we refer to the pair $(\alpha, r)$ as a global random attractor-repeller pair for $\varphi$.

Before proving Proposition 4.24, it will be useful to introduce the following definition (taken from [KN04]):

Definition 4.26. The spread $D(\rho)$ of a probability measure $\rho$ on $\mathbb{S}^1$ is defined as

$$D(\rho) := \inf\{\varepsilon > 0 : \exists \text{closed connected } J \subset \mathbb{S}^1 \text{ with } l(J) < \varepsilon \text{ and } \rho(J) > 1 - \varepsilon\}.$$ 

It is not hard to show (by considering connected sets with rational endpoints) that the map $\rho \mapsto D(\rho)$ from $\mathcal{M}_1$ to $[0, \frac{1}{2}]$ is measurable.

Proof of Proposition 4.24. Suppose $G_r$ is not the only $\varphi$-invariant probability measure, and let $\mu$ be a $\varphi$-invariant probability measure distinct from $G_r$. Let $\mu^a$ and $\mu^s$ denote respectively the absolutely continuous and singular parts of the Radon-Nikodym decomposition of $\mu$ with respect to $G_r$. By Lemma 4.23, $G_r$ is ergodic and therefore $\mu^a$ must be a scalar multiple of $G_r$. Therefore the probability measure $\nu$ on $\Omega \times \mathbb{S}^1$ given by $\nu(A) = \frac{\mu^a(A)}{\mu^s(\Omega \times \mathbb{S}^1)} \mu^s(A)$ is itself $\varphi$-invariant. Let $(\nu_\omega)_{\omega \in \Omega}$ be a disintegration of $\nu$. By Lemma 4.18, $\nu_\omega(\{r(\omega)\}) = 0$ for $\mathbb{P}$-almost all $\omega \in \Omega$. Hence $D(\varphi(t, \omega), \nu_\omega) \to 0$ as $t \to \infty$ for $\mathbb{P}$-almost every $\omega \in \Omega_c$. By Lemma 4.21, this implies that for $\mathbb{P}$-almost all $\omega \in \Omega$, $D(\nu_{\varphi(t, \omega)}) \to 0$ as $n \to \infty$ (in the integers). By the Poincaré recurrence theorem and the measurability of $D(\cdot)$, it follows that $D(\nu_\omega) = 0$ for $\mathbb{P}$-almost all $\omega \in \Omega$, i.e. $\nu_\omega$ is a Dirac mass for $\mathbb{P}$-almost all $\omega \in \Omega$. So there exists a measurable function $\tilde{a} : \Omega \to \mathbb{S}^1$ such that $\nu_\omega = \delta_{\tilde{a}(\omega)}$ for $\mathbb{P}$-almost all $\omega \in \Omega$ (and so $\nu = G_{\tilde{a}}$). Since $\nu$ is $\varphi$-invariant, it follows that $\tilde{a}$ is a random fixed point.

So far, then, we have seen that $\mu$ is a convex combination of $G_r$ and $G_{\tilde{a}}$ for some random fixed point $\tilde{a} : \Omega \to \mathbb{S}^1$ such that $\tilde{a}(\omega) \neq r(\omega)$ for $\mathbb{P}$-almost all $\omega \in \Omega$; and it is obvious that any convex combination of $G_r$ and $G_{\tilde{a}}$ is $\varphi$-invariant. We will next show that it is possible to modify $\tilde{a}$ on a $\mathbb{P}$-null set to obtain an $\mathcal{F}_s$-measurable function: Fix any point $y \in \mathbb{S}^1$ such that $\mathbb{P}_s r(\{y\}) = 0$. For $\mathbb{P}$-almost every $\omega \in \Omega$, $d(\varphi(n, \omega)y, \tilde{a}(\theta^n \omega)) \to 0$ as $n \to \infty$ (in the integers). Since almost sure convergence implies convergence in probability and $\mathbb{P}$ is $\theta^n$-invariant for all $n$, it follows that the sequence $(a_n)_{n \in \mathbb{N}}$ of random variables $a_n : \omega \mapsto \varphi(n, \theta^{-n}\omega)y$ converges in probability to $\tilde{a}$. Now $a_n$ is clearly $\mathcal{F}_s$-measurable for

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5 See e.g. Proposition 29(A) and Theorem 34 of [New15a] for a justification for this reasoning.
each $n$, and so it follows that there exists an $F_-$-measurable function $a: \Omega \to S^1$ such that $a(\omega) = \tilde{a}(\omega)$ for $\mathbb{P}$-almost all $\omega \in \Omega$.

Finally, to complete the proof, we only need to show that up to $\mathbb{P}$-almost everywhere equality, $a$ is the only random fixed point that is $\mathbb{P}$-almost everywhere distinct from $r$. Let $b: \Omega \to S^1$ be any random fixed point that is $\mathbb{P}$-almost everywhere distinct from $r$; then it is clear that for $\mathbb{P}$-almost every $\omega \in \Omega$, $d(b(\theta^n\omega), a(\theta^n\omega)) \to 0$ as $n \to \infty$ (in the integers). Hence the Poincaré recurrence theorem gives that $d(b(\omega), a(\omega)) = 0$, i.e. $b(\omega) = a(\omega)$, for $\mathbb{P}$-almost all $\omega \in \Omega$. So we are done.

Proposition 4.27. If $\varphi$ is stably synchronising then $\varphi$ is simple.

For the proof of Proposition 4.27, we will need the following: Let us say that a compatible probability measure $\mu$ is past-measurable if there is a version $(\mu_{\omega})$ of the disintegration of $\mu$ such that the map $\omega \mapsto \mu_{\omega}$ is $F_-$-measurable. As part of [KS12, Theorem 4.2.9(ii)], the following holds:

Lemma 4.28. For every stationary probability measure $\rho$, there is a past-measurable $\varphi$-invariant probability measure $\mu^\rho$ such that $\rho(A) = \mu^\rho(\Omega \times A)$ for all $A \in \mathcal{B}(S^1)$.

(This fact is not specific to the circle, but holds for more general Polish spaces.)

Proof of Proposition 4.27. Suppose $\mathbb{P}(\Omega_c) = 1$ and $\varphi$ is not simple. We have established that (due to the Krylov-Bogolyubov theorem) there exists a stationary probability measure $\rho$. Since $G_r$ is the only $\varphi$-invariant measure, Lemma 4.28 then implies that $G_r$ is past-measurable. So $r$ has an $F_-$-measurable modification. But we also know that $r$ has an $F_+$-measurable modification. Since $F_-$ and $F_+$ are independent, it follows that $\mathbb{P}_r$ is a Dirac mass, and so by Lemma 4.12, $\varphi$ is not stably synchronising.

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6e.g. by [New15a, Lemma 4(B)], using the fact that convergence in probability implies the existence of a subsequence on which we have almost sure convergence.
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