A short proof of Host’s equidistribution theorem

Michael Hochman*

Abstract

This note contains a new proof of Host’s equidistribution theorem for multiplicatively independent endomorphisms of \( \mathbb{R}/\mathbb{Z} \). The method is a simplified version of our recent work on equidistribution under toral automorphisms [2] and is related to the argument in [3], but avoids the use of the scenery flow and of Marstrand’s projection theorem, using instead a direct Fourier argument to establish smoothness of the limit measure.

Contents

1 Introduction 1

2 General results on equidistribution 2
   2.1 The ergodic theorem for martingale differences . . . . . . . . . . 2
   2.2 Relating orbits to the local structure of \( \mu \) . . . . . . . . . . . . 3
   2.3 Equidistribution along the times \([\beta n]\) . . . . . . . . . . . . . . 4

3 The Fourier transform of scaled measures 5

4 Proof of Host’s theorem on \([0, 1]\) 7
   4.1 Natural extension, \( a \)-adic partition, conditional measures . . . 7
   4.2 Time change (matching the rates of \( T_a \) and \( T_b \)) . . . . . . . 8
   4.3 Applying Weyl’s criterion and Theorem 2.2 . . . . . . . . . . . . . . 9
   4.4 Identifying the limit using Proposition 2.7 . . . . . . . . . . . . . . 10
   4.5 Estimating the Fourier transform . . . . . . . . . . . . . . . . . . . . 11

5 Generalizations 11

1 Introduction

Furstenberg has famously conjectured that if \( a, b \in \mathbb{N} \) are multiplicatively independent integers, then the only Borel probability measures on \( \mathbb{R}/\mathbb{Z} \) that are invariant and ergodic under \( \times a \) and \( \times b \) are either atomic or Lebesgue. The conjecture has been partially verified by Rudolph and Johnson under an assumption of positive entropy; many generalizations exist.

*Supported by ISF grants 1702/17 and 3056/21
Closely related to Rudolph’s theorem is Host’s equidistribution theorem:\footnote{Host \cite{Host} proved this assuming $\gcd(a, b) = 1$. A more general statement, when $a \nmid b^k$ for all $k$, was proved by Lindenstrauss \cite{Lind}, and the general result by Hochman-Shmerkin \cite{Hochman-Shmerkin}.}

**Theorem 1.1** (\cite{Host} \cite{Lind} \cite{Hochman-Shmerkin}). If $\mu$ is a probability measure on $\mathbb{R}/\mathbb{Z}$ that is invariant, ergodic and has positive entropy under an endomorphism $\times a$, then $\mu$-a.e. point equidistributes for Lebesgue measure under $\times b$, provided $a$ and $b$ are multiplicatively independent.

In this note we give a new proof of Host’s theorem. The proof is a simplified version of the one in \cite{Lind}, but is also related to the proof from \cite{Hochman-Shmerkin}. It differs from the latter primarily in its “endgame”, where the use of Marstrand’s projection theorem is replaced by a more direct Fourier-theoretic argument, and it avoids the use of the scenery flow machinery found there. We also note that unlike Host’s original proof, the one here covers the general result for multiplicatively independent endomorphisms.

## 2 General results on equidistribution

In this section we establish some “soft” equidistribution results.

If $(X, \mathcal{B})$ is a standard Borel space and $A \subseteq \mathcal{B}$ is a measurable partition (or countably generated $\sigma$-algebra), we write $A(x)$ for the unique element of $A$ containing a point $x \in X$. If $\mu$ is a probability measure and $\mu(A) > 0$ then $\mu_A = \frac{1}{\mu(A)}\mu | A$ denotes the normalized restriction of $\mu$ to $A$. If $\mathcal{C} \subseteq \mathcal{B}$ is a countably generated $\sigma$-algebra, then $\mu^\mathcal{C}_x$ denotes the conditional measure of $\mu$ on $\mathcal{C}(x)$, which is defined $\mu$-a.e.

### 2.1 The ergodic theorem for martingale differences

We record a variant of the ergodic theorem for martingale differences, and a simple consequence.

**Theorem 2.1.** Let $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \mathcal{B}_3 \subseteq \ldots \subseteq \mathcal{B}$ be an increasing sequence of $\sigma$-algebras in a probability space $(X, \mathcal{B}, \mu)$. Let $f_n \in L_\infty(\mu, \mathcal{B}_{n+1})$ be a uniformly bounded sequence. Then with probability one,

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (f_n - \mathbb{E}(f_n | \mathcal{B}_n)) = 0
$$

(2.1)

More generally, suppose $k \in \mathbb{N}$ and $f_n \in L_\infty(\mu, \mathcal{B}_{n+k})$. Then (2.1) still holds.

**Proof.** The first part (when $k = 1$) is the standard version \cite{Host} Chapter 7, Theorem 3. Now suppose $k \neq 1$. For each $p = 0, 1, 2, \ldots, k-1$, apply the standard version of the theorem to the increasing sequence of $\sigma$-algebras $(\mathcal{B}_{kn+p})_{n=1}^{\infty}$, obtaining the a.s. limit

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (f_{kn+p} - \mathbb{E}(f_{kn+p} | \mathcal{B}_{kn+p})) = 0
$$
Averaging these limits over \( p \) gives (2.1).

### 2.2 Relating orbits to the local structure of \( \mu \)

Let \( T \) be a measurable transformation of a compact metric space and \( \mu \) a Borel probability measure on \( X \). We do not assume that \( \mu \) is preserved by \( T \). Our goal is to describe the statistical behavior of the orbit of a \( \mu \)-typical point \( x \).

In [3, Theorem 2.1], we showed that if \( \mathcal{A} \) is a generating partition for \( T \) and \( \mathcal{A}^n(x) = (\bigvee_{i=0}^{n-1} T^{-i} \mathcal{A})(x) \), then any measure \( \nu \) for which the orbit equidistributes (possibly along a subsequence) can be described as a limit of averages of the measures \( T^n(\mu P^n(x)) \).

In the present work we use a different version in which \( \mathcal{A} \) is adapted to the dynamics of a different map, or from some hierarchical structure of \( \mu \). We require the atoms of \( \mathcal{A} \) must have some compatibility with the expansion of \( T^n \).

**Theorem 2.2.** Let \( T : X \to X \) be a continuous map of a compact metric space. Let \( A_1, A_2, A_3, \ldots \) be a refining sequence of Borel partitions. Let \( \mu \) be a Borel probability measure on \( X \) and assume that

\[
\sup_{n \in \mathbb{N}} \{\text{diam } T^n A : A \in \mathcal{A}_{n+k}, \mu(A) > 0\} \to 0 \quad \text{as } k \to \infty \tag{2.2}
\]

Then for \( \mu \)-a.e. \( x \),

\[
\lim_{N \to \infty} \left( \frac{1}{N} \sum_{n=1}^{N} \delta_{T^n x} - \frac{1}{N} \sum_{n=1}^{N} T^n \mu(A_n(x)) \right) = 0
\]

in the weak-* sense.

**Proof.** Let \( F \subseteq C(X) \) be a dense countable set. It is enough to prove that for every \( f \in F \), for \( \mu \)-a.e. \( x \),

\[
\lim_{N \to \infty} \left( \frac{1}{N} \sum_{n=1}^{N} f(T^n x) - \frac{1}{N} \sum_{n=1}^{N} \int f dT^n \mu(A_n(x)) \right) = 0 \tag{2.3}
\]

Fix \( f \in F \). Our assumption (2.2) implies the \( f \circ T^n - E(f \circ T^n | \mathcal{A}_{n+k}) \to 0 \) as \( k \to \infty \), uniformly in \( n \in \mathbb{N} \) and \( x \in \text{supp } \mu \). Therefore it suffices, for each \( k \), to prove (2.3) with \( E(f \circ T^n | \mathcal{A}_{n+k}) \) in place of \( f \circ T^n \). We can re-write the other term (2.3) as

\[
\int f dT^n \mu(A_n(x)) = \int f \circ T^n d\mu(A_n(x)) = E(\mu(f \circ T^n | \mathcal{A}_n))(x)
\]

With these modifications, (2.3) follows directly from the results of the previous section applied to the functions \( f_n = E(f \circ T^n | \mathcal{A}_{n+k}) \). \( \square \)
2.3 Equidistribution along the times $[\beta n]$

We will need an equidistribution result for pairs of orbits of the form $(\theta_n, T^{[\beta n]}x)$ where $\theta \in \mathbb{R}/\mathbb{Z}$ and $x$ is a typical point for the measure preserving map $T$. The argument is rather standard but we record the proof for completeness. We generically write $R_0$ to denote translation by $\theta$.

Lemma 2.3. Let $X, Y$ be compact metric spaces, and let $S : Y \to Y$ be a continuous map with an invariant measure $\mu$. Let $\{x_k\} \subseteq X$ be a fixed sequence, let $n_k \to \infty$, and suppose that the sequence $(x_k, S^{n_k}y)_{k=1}^{\infty}$ equidistributes to a measure $\nu_y$ on $X \times Y$ for $\mu$-a.e. $y$. Let $\nu = \int \nu_y d\mu(y)$ and $\tau = \pi_1\nu$, where $\pi_1(x, y) = x$. Then $\nu = \tau \times \mu$.

Proof. The averages

$$\nu_{y, N} = \frac{1}{N} \sum_{k=1}^{N} \delta_{x_k} \times \delta_{S^{n_k}y}$$

converge to $\nu_y$ for $\mu$-a.e. $y$, therefore $\int \nu_{y, N} d\mu(y) \to \int \nu_y d\mu(y)$. On the other hand, by $S$-invariance of $\mu$ we have

$$\int \nu_{y, N} d\mu(y) = \left( \frac{1}{N} \sum_{k=1}^{N} \delta_{x_k} \right) \times \mu$$

Thus, the measure $\int \nu_{y, N} d\mu(y)$ is a product measure whose second marginal is $\mu$, so their limit $\int \nu_y d\mu(y)$ has this form as well. \qed

Applying this with $X$ the trivial one-point system, we get:

Corollary 2.4. If $Y$ is a compact metric space, $S : Y \to Y$ is continuous, $\mu$ is an invariant probability measure on $Y$ and $n_k \to \infty$ is such that $(S^{n_k}y)_{k=1}^{\infty}$ equidistributes for a measure $\nu_y$ for $\mu$-a.e. $y$, then $\int \nu_y d\mu(y) = \mu$.

Proposition 2.5. Let $(X_1, \mu_1, T_1)$ and $(X_2, \mu_2, T_2)$ be measure preserving systems on standard measure spaces, and fix real numbers $\beta_1, \beta_2 > 0$. Then for $\mu_1 \times \mu_2$-a.e. $(x_1, x_2)$, the orbit $(T_1^{[\beta_1 n]}x_1, T_2^{[\beta_2 n]}x_2)$ equidistributes for a measure $\nu_{x_1, x_2}$ on $X_1 \times X_2$ satisfying

$$\int \nu_{x_1, x_2} d\mu_1 \times \mu_2(x_1, x_2) = \mu_1 \times \mu_2$$

Remark 2.6. It is important to note that in general $\nu_{x_1, x_2}$ may not be $T_1 \times T_2$-invariant. Indeed if $(X, \mu, T)$ has a rational multiple of $1/\beta_i$ in its pure point spectrum, it won’t be.

Proof. We prove this first when $X_2$ is the trivial (one point system), so we are dealing with a single transformation $(X, \mu, T)$ and parameter $\beta > 0$. Form the suspension of $(X, \mu, T)$ by the constant function of height 1, obtaining the flow $\{\tilde{T}_t\}_{t \in \mathbb{R}}$ on $\tilde{X} = X \times [0, 1]/\sim$ where $\sim$ is the relation $(x, 1) \sim (Tx, 0)$, so $\tilde{T}_t$
preserves $\tilde{\mu} = \mu \times \text{Lebesgue}$. Let $F \subseteq C(X)$ be a dense countable set and for $f \in F$ let $\tilde{f} \in C(\tilde{X})$ be given by $\tilde{f}(x,t) = f(x)$. Apply the ergodic theorem to the time-$\beta$ map $T_\beta = \tilde{T}_\beta$ and the maps $\tilde{f}$ ($f \in F$) to conclude that the averages

$$
\frac{1}{N} \sum_{n=1}^{N} \tilde{f}(T^\beta (x,t)) = \frac{1}{N} \sum_{n=1}^{N} f(T^{[\beta n + t]} x)
$$

converge for $\mu$-a.e. $x$ and Lebesgue-a.e. $t \in [0,1]$. This means that for $\mu$-a.e. $x$ and a.e. $t \in [0,1]$, the point $(x,t)$ equidistributes for a measure $\tilde{\nu}_{x,t}$. Next, note that, endowing $\tilde{X}$ with the metric induced by the product metric on $X \times [0,1]$, the orbits of $(x,t)$ and $(x,t')$ are within distance $|t - t'|$ of each other outside a set of density $|t - t'|$ provided that $|t - t'| < 1/2$, and we conclude that $\tilde{\nu}_{x,0}$ is well-defined for $\mu$-a.e. $x$ (we see this by approximating the orbit of $(x,0)$ by orbits of $(x,t_n)$ for Lebesgue-typical $t_n \searrow 0$). Thus, $\nu_x = \tilde{\nu}_{x,0}$ is well defined. The fact that $\int \nu_x d\mu(x) = \mu$ follows from the previous lemma and its corollary.

The generalization to two maps (or more generally, $k$ maps) is proved in the same way, considering the suspension $\mathbb{R}^2$-action $\{T_{1,s} \times T_{2,t}\}$ on $\tilde{X}_1 \times \tilde{X}_2$, and applying the ergodic theorem to the map $T_{1,\beta_1} \times T_{2,\beta_2}$.

**Corollary 2.7.** Let $(X,\mu,T)$ be an ergodic measure preserving system on a compact metric space. Let $\beta > 0$ and $\theta \in \mathbb{R}$. Then for $\mu$-a.e. $x$ the sequence $(n\theta, T^{[n\beta]}x)$ equidistributes for a measure $\nu_x$ on $[0,1] \times X$ that satisfies $\int \nu_x d\mu(x) = \tau \times \mu$, where $\tau$ is the invariant measure on $([0,1], R_\theta)$ supported on the orbit closure of $0$.

**Proof.** Apply Proposition 2.3 to $X_1 = ([0,1], R_\theta, \tau)$ and $X_2 = (X,\mu,T)$, and with $\beta_1 = 1$ and $\beta_2 = \beta$. We conclude that for Lebesgue-a.e. $u \in [0,1)$ and $\mu$-a.e. $x$ (chosen independently),

$$(n\theta + u, T^{[n\beta]} x)$$

equidistributes for a measure $\nu_{u,x}$, and these measures integrate to $m \times \mu$, where $m$ denotes Lebesgue measure on $[0,1)$. Since translation in the first coordinate is a continuous action commuting with the other dynamics, we conclude, by translating the first coordinate by $-u$ that for $\mu$-a.e. $x$ the sequence

$$(n\theta, T^{[n\beta]} x)$$

equidistributes for $\nu_{0,x} = \nu_x$. Since translation of the first coordinate does not affect the projection to the second coordinate, we conclude also that $\nu_x$ projects to $\mu$ on the last coordinate. Of course, the first coordinate equidistributes for $\tau$. Applying now Lemma 2.3 we find that $\int \nu_x d\mu(x) = \tau \times \mu$. \qed

### 3 The Fourier transform of scaled measures

We establish some elementary estimates on the Fourier coefficients of well spread-out measures on the line, when they are scaled by a random amount.
Let \( e(x) = \exp(2\pi ix) \), and for \( m \in \mathbb{R} \) we write \( e_m(x) = e(mx) = \exp(2\pi imx) \). We write \( \hat{f} \) and \( \hat{\nu} \) for the Fourier transform of a function or measure, respectively; we also sometimes write it as

\[
\mathcal{F}_m(\nu) = \hat{\nu}(m)
\]

We define translation and scaling maps of the real line:

\[
R_\theta x = x + \theta \\
S_t x = tx
\]

We note that \(|\hat{R}_\theta \nu| = |\hat{\nu}|\).

**Lemma 3.1.** Let \( f \in C^1([a,b]) \) and suppose that \( \int_a^b f(x)dx = 1 \). Then for \( \xi \neq 0 \),

\[
|\hat{f}(\xi)| < \frac{1}{\pi \xi} (\|f\|_{\infty} + (b-a)\|f'\|_{\infty})
\]

**Proof.** Using integration by parts,

\[
|\hat{f}(\xi)| = |\int_a^b f(x)e_{\xi}(x)dx| \\
= |\frac{1}{2\pi i \xi}f(b)e_{\xi}(b) - \frac{1}{2\pi i \xi}f(a)e_{\xi}(a) - \frac{1}{2\pi i \xi} \int_a^b f'(x)e_{\xi}(x)dx| \\
\leq \frac{2}{2\pi \xi} \|f\|_{\infty} + \frac{1}{2\pi \xi} \|f'\|_{\infty} (b-a) \\
\leq \frac{1}{\pi \xi} (\|f\|_{\infty} + (b-a)\|f'\|_{\infty})
\]

**Lemma 3.2.** Let \( \nu \in \mathcal{P}(\mathbb{R}) \) and \( b > 1 \). Then for every \( r > 0 \) and \( m \neq 0 \),

\[
\int_0^1 |\hat{S}_t \nu(m)|^2dt \leq \frac{1}{r \cdot m \cdot \ln b} + \int \nu(B_r(y))d\nu(y)
\]

**Proof.** Let \( Y, Y' \) be independent random variables with distribution \( \nu \), so \( tY \) has distribution \( S_t \nu \). Using the definition of the Fourier transform and Fubini,

\[
\int_0^1 |\hat{S}_t \nu(m)|^2dt = \int_0^1 |\mathbb{E}(e(mb'Y))|^2dt \\
= \int_0^1 \mathbb{E}(e(mb'Y) \cdot e(mb'Y'))dt \\
= \int_0^1 \mathbb{E}(e(mb'(Y - Y')))|dt \\
= \mathbb{E}(\int_0^1 e(mb'(Y - Y'))dt)
\]

(3.1)
For each pair of values \(y, y'\) of \(Y, Y'\), the integral \(\int_0^1 e(mb'(Y - Y'))dt\) is just the \(m(Y - Y')\)-th Fourier coefficient of the random variable \(Z\), where \(Z\) is the push-forward of the uniform measure on \([0, 1]\) by the map \(t \mapsto b'\). The density function \(f : [1, b] \to \mathbb{R}\) of \(Z\) satisfies \(f(z) = (z \ln b)^{-1}\) so \(f'(z) = -(z^2 \cdot \ln b)^{-1}\), and we get the bounds

\[
\|f\|_\infty, \|f'\|_\infty \leq \frac{1}{\ln b}
\]

which by the previous lemma (using \(m(y - y')\) in place of \(\xi\)) gives us

\[
|\int_0^1 e_m(b'(y - y'))dt| \leq \frac{2}{\pi m \ln(b)(y - y')}
\]

\[
< \frac{1}{m \ln(b)(y - y')}
\]

We can now evaluate (3.1) as follows:

\[
\int_0^1 |\hat{S}_b \nu(m)^2 dt| \leq \int_0^1 e(mb(Y - Y'))dt
\]

\[
= \int \int \int_0^1 e(mb' (y - y'))dt \, d\nu(y') \, d\nu(y)
\]

\[
= \int \left( \int_{\mathbb{R}\setminus B_r(y)} \int_0^1 e(mb' (y - y'))dt \, d\nu(y') \right) \, d\nu(y)
\]

\[
+ \int_{B_r(y)} \int_0^1 e(mb' (y - y'))dt \, d\nu(y') \, d\nu(y)
\]

\[
\leq \int \left( \frac{1}{m \ln(b)(y - y')} \right) \, d\nu(y') + \int_{B_r(y)} 1 \, d\nu(y') \, d\nu(y)
\]

\[
\leq \frac{1}{m \ln(b)r} + \int \nu(B_r(y)) \, d\nu(y)
\]

as claimed.

\[\square\]

4 Proof of Host’s theorem on \([0, 1]\)

We write \(T_a x = ax \mod 1\). Let \(a, b \geq 2\) be multiplicatively independent integers and \(\mu\) a \(T_a\)-invariant and ergodic measure on \([0, 1]\) with positive entropy.

4.1 Natural extension, \(a\)-adic partition, conditional measures

Let \(A\) denote the partition of \([0, 1]\) into intervals \([k/a, (k+1)/a)\), \(k = 0, \ldots, a-1\), and \(A_n = \bigvee_{i=0}^{n-1} T_a^{-i} A\) the \(a\)-adic partition of generation-\(n\) whose atoms are of
the form \([k/a^n, (k+1)/a^n]\) for \(k = 0, \ldots, a^n - 1\). We also write \(A_n(x)\) for the atoms of \(A_n\) containing \(x\), sometimes identified with the word of the first \(n\) \(a\)-adic digits in the representation of \(x \in [0,1]\). Let \(*\) denote concatenation of sequences.

Let

\[
\Omega^- = \{ (\omega_n)_{n \leq 0} : \omega_n \in \{0, \ldots, a-1\} \}
\]

and take the natural extension of \(([0,1], \mu, T_a)\), which we realize as the space \(\Omega = \Omega^- \times [0,1]\) with the map \(T_a(\omega, x) = (\omega * A(x), T_a x)\). The measure \(\mu\) extends uniquely to an ergodic \(\tilde{T}_a\)-invariant measure \(\tilde{\mu}\) on \(\tilde{\Omega}\) such that the projection \((\omega, x) \mapsto x\) is a factor map of \((\tilde{\Omega}, \tilde{\mu}, \tilde{T}_a) \to ([0,1], \mu, T_a)\). We denote the elements of \(\tilde{\Omega}\) by \(\tilde{\omega}\). We also sometimes identify \(\tilde{\mu}\) with its projection to \(\Omega^-\).

Lift the partition \(A\) of \([0,1]\) via the projection \((\omega, x) \mapsto x\) to a partition \(\tilde{A}\) in \(\tilde{\Omega}\) so that \(\tilde{A}_n = \bigcup_{n-1 \leq i \leq 0} \tilde{T}_a^{-i} A\) is the lift of \(A_n\).

From now on we do not distinguish between \(A\) and \(\tilde{A}\) and similarly for other partitions.

Let \(C = \bigcup_{i=-\infty}^{0} \tilde{T}_a^{-i} \tilde{A}\) denote the \(\sigma\)-algebra in \(\tilde{\Omega}\) generated by projection “to the past”, \((\omega, x) \mapsto \omega \in \Omega^-\).

Let \(\{\tilde{\mu}_\omega\}_{\omega \in \Omega^-}\) denote the corresponding disintegration. We abbreviate \(\tilde{\mu}_\omega\) or \(\tilde{\mu}_{\omega,x}\) for \(\tilde{\mu}_{(\omega,x)}\), which does not introduce any ambiguity since \(\tilde{\mu}_{(\omega,x)}\) depends only on the \(\Omega^-\)-component \(\omega\) of \((\omega, x)\). The atoms \(C(\omega, x) = \{\omega\} \times [0,1]\) of \(C\) are naturally identified with \([0,1]\), giving an identification of \(\tilde{\mu}_\omega\) with a measure on \([0,1]\), which we denote \(\mu_\omega\). We thus have

\[
\mu = \int \mu_\omega d\tilde{\mu}(\omega)
\]

By \(C = \bigcup_{i=-\infty}^{0} \tilde{T}_a^{-i} A\), we have \(C \vee A_n = \tilde{T}_a^n C\), which gives us the equivariance relation

\[
T_a^n \left( (\mu_\omega)_ {A_n(x)} \right) = \mu_{T_a^n(\omega,x)}
\]

4.2 Time change (matching the rates of \(T_a\) and \(T_b\))

Define

\[
\alpha = \frac{\log b}{\log a}
\]

Independence of \(a, b\) implies that \(\alpha\) is irrational. Set

\[
n' = \lfloor \alpha n \rfloor
\]

so that \(b^{n'} \approx a^{n'}\); more precisely, we can write

\[
b^{n'} a^{-n'} = a^{\alpha n - \lfloor \alpha n \rfloor} = a^z
\]

where

\[
z = \alpha n \mod 1
\]

is the orbit of 0 under the irrational rotation \(R_\alpha : z \mapsto z + \alpha \mod 1\) of the compact group \(\mathbb{R}/\mathbb{Z}\).
4.3 Applying Weyl’s criterion and Theorem 2.2

We wish to show that the sequence \( \{ T_b^n x \}_{n=1}^{\infty} \) equidistributes for Lebesgue measure for \( \mu \)-a.e. \( x \). By Weyl’s equidistribution criterion, we must show for \( \mu \)-a.e. \( x \) and for every \( m \in \mathbb{Z} \setminus \{0\} \) that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e_m(T_b^n x) = 0
\]

where \( e_m(t) = \exp(2\pi it) \). It is enough to show that for every \( \varepsilon > 0 \),

\[
\mu \left( x : \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e_m(T_b^n x) < \varepsilon \right) > 1 - \varepsilon
\]

Since \( \mu \) is invariant under \( T_a^k \) for all \( k \), the last property follows if we show that for every \( \varepsilon > 0 \) there is a \( k \) such that

\[
\mu \left( x : \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e_m(T_b^n \circ T_a^k x) < \varepsilon \right) > 1 - \varepsilon \quad (4.2)
\]

If \( \omega \) is chosen according to \( \tilde{\mu} \) and \( x \) conditionally independently according to \( \mu_\omega \), then \( x \) has distribution \( \mu \). Thus, for \( \tilde{\mu} \)-typical \( \omega \), we want to apply Theorem 2.2 to the limit in the last event for \( \mu_\omega \)-typical \( x \) and \( A_n' \); we can do so because \( T_b^n A_{n'+k}(x) \) has diameter \( O(a^{-k}) \), by choice of \( n' \). Recalling that \( S_t \) denotes scaling by \( t \), there exists \( c = c(x, n) \) so that \( T_a^n |_{A_{n'}(x)} = S_{a^{n'}} + c \). Writing \( \tau_c \) for translation by \( c \), by (4.1) we have \( T_a^n ( (\mu_\omega)_{A_{n'}(x)} ) = \mu_{\tilde{T}_{a}^{n'}(\omega, x)} \) and hence \( (\mu_\omega)_{A_n(x)} = S_{a^{-n}} \tau_{-c} \left( \mu_{\tilde{T}_{a}^{n'}(\omega, x)} \right) \). This gives

\[
T_b^n \left( (\mu_\omega)_{A_{n'}(x)} \right) = S_{a^n} \circ S_{b^n} \circ \left( S_{a^{-n}} \tau_{-c} (\mu_{\tilde{T}_{a}^{n'}(\omega, x)}) \right) \mod 1
\]

\[
= \tau_{-b^n a^n c} \left( S_{a^n} \circ S_{b^n} \left( \mu_{\tilde{T}_{a}^{n'}(\omega, x)} \right) \right) \mod 1 \quad (4.3)
\]

Applying Theorem 2.2 to the limit in (4.2), inserting the expression above for
we identify \( R \tilde{\varepsilon} \) since we are concerned with the system where \( \varepsilon > 0 \) for every \( n \) changing \( \varepsilon \) add another coordinate in the shift space \( \tilde{\varepsilon} \) for every \( n \n T \) Thus, it is enough to show that for all \( m \) such that \( n \n T \) can just pass to a model of the system representing \( z \), \( \eta \) meaning \( z \tilde{\varepsilon} \) and using the triangle inequality, we have

\[
\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e_m(T^n_b T^n_k x) = \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \int e_m d(T^n_b (T^n_k (\mu_\omega) A, (x)))
\]

\[
= \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} F_m (T^n_b (T^n_k (\mu_\omega) A, (x)))
\]

\[
\leq \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left| F_m (T^n_b (T^n_k (\mu_\omega) A, (x))) \right|
\]

\[
\leq \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left| F_m (S^{\alpha\tau} (S^\alpha \mu_{T^n \tilde{\varepsilon} (\omega, x)}) \right|
\]

\[
\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left| F_m (S^{\alpha\tau} (S^\alpha \mu_{T^n \tilde{\varepsilon} (\omega, x)}) \right|
\]

Thus, it is enough to show that for all \( m \in \mathbb{Z} \setminus \{0\} \), for every \( \varepsilon > 0 \), there exists a \( k \) such that

\[
\tilde{\mu} \left( (\omega, x) : \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} F_m \left( S^{\alpha\tau} (S^\alpha \mu_{T^n \tilde{\varepsilon} (\omega, x)}) \right) < \varepsilon \right) > 1 - \varepsilon \quad (4.5)
\]

### 4.4 Identifying the limit using Proposition 2.7

The limit in (4.5) can be identified using Corollary 2.7 with \( \tau = \text{Lebesgue measure} \), there is a decomposition \( \tau \times \tilde{\mu} = \int \nu_{\omega, x} d \tilde{\mu}(\omega, x) \) where \( \nu_{\omega, x} \) are measures on \( (\mathbb{R}/\mathbb{Z}) \times \tilde{\Omega} \) such that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left| F_m (S^{\alpha\tau} (S^\alpha \mu_{T^n \tilde{\varepsilon} (\omega, x)}) \right| = \int \left| F_m (S^{\alpha\tau} (S^\alpha \mu_\eta)) \right| d\nu_{\omega, x}(z, \eta)
\]

(we identify \( \mathbb{R}/\mathbb{Z} \) with \( [0, 1) \), which makes the term \( \alpha^2 \) meaningful. One should note that \( \eta \mapsto \mu_\eta \) is not continuous as a function on \( (\tilde{\Omega}^+, \tilde{\mu}) \), so the equidistribution established in Corollary 2.7 does not formally imply the last equation. But since we are concerned with \( \tilde{\mu} \)-typical \( (\omega, x) \), we can just pass to a model of the system where \( \eta \mapsto \mu_\eta \) is continuous and apply the corollary there. For example, add another coordinate in the shift space \( \tilde{\Omega} \) representing \( \mu_{(\omega, x)} \). Thus, up to changing \( \varepsilon \), (4.7) will follow if we show that for every \( \varepsilon > 0 \), for some \( k \)

\[
\int \left( \int |F_m (S^{\alpha\tau} (S^\alpha \mu_\eta))| \right) d\nu_{\omega, x}(z, \eta) < \varepsilon
\]

Using \( \int \nu_{\omega, x} d \tilde{\mu}(\omega, x) = \tau \times \tilde{\mu} \) and Cauchy-Schwartz, it is enough to show that for every \( \varepsilon > 0 \), for some \( k \)

\[
\int \int_0^1 |F_m (S^{\alpha\tau} (S^\alpha \mu_\omega))|^2 dz d \tilde{\mu}(\omega) < \varepsilon
\]
4.5 Estimating the Fourier transform

The last inequality follows directly from Lemma 3.2 applied to $\nu = S_{a^k \mu \eta}$ and $r = a^{k/2}$, provided $k$ satisfies $1/(a^{k/2} \cdot m \cdot \ln a) < \varepsilon/2$ and

$$\int \nu(B_r(y))\,d\nu(y) < \varepsilon/2$$

The first inequality holds for all large enough $k$, and the second inequality, involving $\nu = S_{a^k \mu \eta}$, can be re-written as

$$\int \mu_\eta(B_{a^{-k/2}}(y))\,d\mu_\eta(y) < \varepsilon/2$$

This holds for a fixed non-atomic $\mu_\eta$ for all large enough $k$, and therefore for large $k$ it holds with probability arbitrarily close to one over the choice of $\eta$. Because $\mu$ has positive entropy, $\mu_\eta$ are a.s. non-atomic; this concludes the proof.

5 Generalizations

Our argument can be generalized in many ways. For example the same proof applies if $\mu$ is a strongly separated self-similar measure on $[0,1]$ defined by contractions $f_i(x) = r_i x + t_i$, with $\log r_i / \log b \notin \mathbb{Q}$ for at least one $i$. One chooses $A_n$ to be the partition of $\mu$ into cylinder measures of diameter comparable to $b^{-n}$. If all $r_i$ are equal, we find that $T^n \mu_{A_n}(z)$ is, up to a translation, simply $\mu$ scaled by $b^n$, with $(z_n)$ an irrational rotation, and the analysis continue as before. If there are distinct $r_i$, then $z_n$ is a random sequence coming from the symbolic coding, and a small additional argument is needed in place of Proposition 2.5. The method extends beyond self-similar measures to positive entropy ergodic measures on the symbolic coding of the attractor. It is likely that one can show that the methods recovers the main results of [3] concerning so-called uniformly scaling measures generating a scale-invariant distribution under suitable spectral assumptions.

In the forthcoming paper [2], this method is extended to groups of endomorphisms of $\mathbb{T}^d$. The precise statements can be found there.

References

[1] William Feller. *An introduction to probability theory and its applications. Vol. II.* Second edition. John Wiley & Sons Inc., New York, 1971.

[2] Michael Hochman. Toral endomorphisms and equidistribution. *preprint*, 2021.

[3] Michael Hochman and Pablo Shmerkin. Equidistribution from fractal measures. *Inventiones mathematicae*, pages 1–53, 2015.
[4] Bernard Host. Nombres normaux, entropie, translations. *Israel J. Math.*, 91(1-3):419–428, 1995.

[5] Elon Lindenstrauss. $p$-adic foliation and equidistribution. *Israel J. Math.*, 122:29–42, 2001.