Jumping the Gap: The Formation Conditions and Mass Function of “Pebble-Pile” Planetesimals

Philip F. Hopkins

Abstract

In a turbulent proto-planetary disk, dust grains undergo large density fluctuations and under the right circumstances, these grain overdensities can overcome shear, turbulent, and gas pressure support to collapse under self-gravity (forming a “pebble pile” planetesimal, or PPP). Using insights from simulations and a new analytic model for the fluctuations, we calculate the rate-of-formation and mass function of self-gravitating, collapsing planetesimal-mass bodies formed by this mechanism. The statistics of this process depend sensitively on the size/stopping time of the largest grains, disk surface density, and turbulent Mach numbers. However, when it occurs, we predict that the resulting planetesimal mass function is broad and quasi-universal, with a slope $dN/dM \propto M^{-1}$, spanning a size/mass range $\sim 10^{-10^6}$ km ($\sim 10^{-9} - 5 M_\oplus$). Collapse to planetesimal through super-Earth masses is possible. The key condition is that grain density fluctuations reach large amplitudes on large scales, where gravitational instability proceeds most easily (collapse of small grains is strongly suppressed by turbulent vorticity). We show this leads to a new criterion for “pebble-pile” formation: $\tau_e \gtrsim 0.05 \ln(Q^{1/2}/Z_d)/\ln(1 + 10^{1/4}) \sim 0.3 \psi(Q, Z, \alpha)$ where $\tau_e = t_c \Omega$ is the dimensionless particle stopping time. In a MMSN, this requires grains larger than $a = (50, 1, 0.1)$ cm at $r = (1, 30, 100)$ au. So at large radii, this can easily occur and seed core accretion. At small radii, it would depend on the existence of large “boulders.” However, because density fluctuations depend super-exponentially on $\tau_e$ (inversely proportional to disk surface density), lower-density disks are more unstable! In fact, we predict that cm-sized grains at $\sim 1$ au will form pebble piles in a disk with $\sim 10\%$ the MMSN density, so planet formation at $\sim$ au may generically occur “late” as disks are evaporating.

Keywords: planets and satellites: formation — protoplanetary discs — accretion, accretion disks — hydrodynamics — instabilities — turbulence

1 Introduction

Dust grains and aerodynamic particles are fundamental in astrophysics. These determine the attenuation and absorption of light in the interstellar medium (ISM), interaction with radiative forces and regulation of cooling, and form the building blocks of planetesimals. Of particular importance is the question of grain clustering and clumping — fluctuations in the local volume-average number/mass density of grains $\rho_f$ in turbulent gas.

Much attention has been paid to the specific question of grain density fluctuations and grain concentration in proto-planetary disks. In general, turbulence sets a “lower limit” to the degree to which grains can settle into a razor-thin sub-layer; and this has generally been regarded as a barrier to planetesimal formation (though see Lyra et al. 2009; Lee et al. 2010; Chiang & Youdin 2010, and references therein). However, it is also well-established that the number density of solid grains can fluctuate by multiple orders of magnitude when “stirred” by turbulence, even in media where the turbulence is highly sub-sonic and the gas is nearly incompressible (see e.g. Bracco et al. 1999; Cuzzi et al. 2001; Johansen & Youdin 2007; Carballido et al. 2008a; Bai & Stone 2010b,a;c; Pan et al. 2011). This can occur via self-excitation of turbulent motions in the “streaming” instability (Johansen & Youdin 2007), or in externally driven turbulence, such as that excited by the magnetorotational instability (MRI), global gravitational instabilities, or convection (Dittrich et al. 2013; Jalali 2013). Direct numerical experiments have shown that the magnitude of these fluctuations depends on the parameter $\tau_e = t_c \Omega$, the ratio of the gas “stopping” time (friction/drag timescale) $t_c$ to the orbital time $\Omega^{-1}$, with the most dramatic fluctuations around $\tau_e \sim 1$. These experiments have also demonstrated that the magnitude of clustering depends on the volume-averaged ratio of solids-to-gas ($\bar{\rho} \equiv \rho_f/\rho_g$), and basic properties of the turbulence (such as the Mach number). These have provided key insights and motivated considerable work studying these instabilities; however, the fraction of the relevant parameter space spanned by direct simulations is limited. Moreover, it is impossible to simulate anything close to the full dynamic range of turbulence in these systems: the “top scales” of the system are $\lambda_{\text{max}} \sim AU$, while the viscous/dissipation scales $\lambda_{\text{v}}$ of the turbulence are $\lambda_{\text{v}} \sim m$-km (Reynolds numbers $Re \sim 10^6 - 10^8$, under typical circumstances). Reliably modeling $Re \gtrsim 100$ remains challenging in state-of-the-art simulations. Clearly, some analytic understanding of these fluctuations would be tremendously helpful.

The question of “preferential concentration” of aerodynamic particles is actually much more well-studied in the terrestrial turbulence literature. There both laboratory experiments (Squires & Eaton 1991; Fessler et al. 1994; Rouson & Eaton 2001; Gualtieri et al. 2009; Monchaux et al. 2010) and numerical simulations (Cuzzi et al. 2001; Yoshimoto & Goto 2007; Hogan & Cuzzi 2007; Bec et al. 2009; Pan et al. 2011; Monchaux et al. 2012) have long observed that very small grains, with stokes numbers $St \equiv t_c/\lambda_{\text{v}} \sim 1$ (ratio of stopping time to eddy turnover time at the viscous scale) can experience order-of-magnitude density fluctuations at small scales (at/below the viscous scale). Considerable analytic progress has been made understanding this regime: demonstrating, for example, that even incompressible gas turbulence is unstable to the growth of inhomogeneities in grain density (Elperin et al. 1996; Elperin et al. 1998), and predicting the behavior of the small-scale grain-grain correlation function using simple models of gaussian random-field turbulence (Sigurgeirsson & Stu-
art 2002; Bec et al. 2007). But extrapolation to the astrophysically relevant regime is difficult for several reasons: the Reynolds numbers of interest are much larger, and as a result the Stokes numbers are also generally much larger (in the limit where grains do not cluster below the viscous/dissipation scale because $t_s \gg t_e(\lambda_{\text{max}})$), placing the interesting physics well in the inertial range of turbulence, and rotation/shear, external gravity, and coherent (non-random field) structures appear critical (at least on large scales). This parameter space has not been well-studied, and at least some predictions (e.g. those in Sigurjónsson & Stuart 2002; Bec et al. 2008; Zaičik & Allplchenkov (2009)) would naively lead one to estimate much smaller fluctuations than are recorded in the experiments above.

However, these studies still contribute some critical insights. They have repeatedly shown that grain density fluctuations are tightly coupled to the local vorticity field: grains are “flung out” of regions of high vorticity by centrifugal forces, and collect in the “interstices” (regions of high strain “between” vortices). Studies of the correlation functions and scaling behavior of higher Stokes-number particles suggest that, in the inertial range (ignoring gravity and shear), the same dynamics apply, but with the scale-free replacement of a “local Stokes number” $t_s/t_e$, i.e. what matters for the dynamics on a given scale are the vortices of that scale, and similar concentration effects can occur whenever the eddy turnover time is comparable to the stopping time (e.g. Yoshimoto & Goto 2007; Bec et al. 2008; Wilkinson et al. 2010; Gustavsson et al. 2012). Several authors have pointed out that this critically links grain density fluctuations to the phenomenon of intermittency and discrete, time-coherent structures (vortices) on scales larger than the Kolmogorov scale in turbulence (see Bec et al. 2009; Olla 2010, and references therein). In particular, Cuzzi et al. (2001) argue that grain density fluctuations behave in a multi-fractal manner: multi-fractal scaling is a key signature of well-tested, simple geometric models for intermittency (e.g. She & Leveque 1994). In these models, the statistics of turbulence are approximated by regarding the turbulent field as a hierarchical collection of “stretched” singular, coherent structures (e.g. vortices) on different scales (Dubrulle 1994; She & Waymire 1995; Chainais 2006). Such statistical models have been well-tested as a description of the gas turbulence statistics (including gas density fluctuations; see e.g. Burlaga 1992; Sorriso-Valvo et al. 1999; Budaev 2008; She & Zhang 2009; Hopkins 2012a).

However, only first steps have been taken to link them to grain density fluctuations: for example, in the phenomenological cascade model fit to simulations in Hogan & Cuzzi (2007).

In this paper, we use these theoretical and experimental insights to build a theory which “bridges” between the well-studied regime of small-scale turbulence and that of large, astrophysical particles in shearing, gravitating disks. The key concepts are based on the work above: we first assume that grain density fluctuations are driven by coherent eddies, for which we can calculate the perturbation owing to a single eddy with a given scale. Building on Cuzzi et al. (2001) and others, we then attach this calculation to a well-tested, simple, geometric cascade model for turbulence which predicts the statistics of intermittent eddies. This allows us to make predictions for a wide range of quantities, which we compare to simulations and experiments.

2 THE MODEL

Consider a grain-gas mixture in a Keplerian disk, at some (mid-plane) distance $r_s$ from the central star. The grains are in a disk with surface density $\Sigma_d$, exponential vertical scale-height $h_d$ and gravitational parameter $\lambda_d = h_d$.

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In the regime of interest in this paper, the Mach numbers of gas turbulence within the mid-plane dust layer are small ($Q \approx 10^4$). On most scales, larger grains require smaller fluctuations to collapse, because the initial dust disk is thinner (higher-density) and resistance from gas pressure is weaker. Both generally decrease with $\tau_d = \tau_e$, near the collapse overdensity $\tau_d$ is minimized; solid) or the characteristic scale where density fluctuations are maximized $\lambda_d(\tau_d = \tau_e)$; dotted). Both generally decrease with $\tau_d$ until $\tau_e \approx 1$. For small grains ($\tau_e \ll 1$), the critical overdensity near $\lambda_d(\tau_e = \tau_s) \ll h_d$ is very large because of turbulent support. We vary $Q$ and $\alpha$; the critical overdensities increase with $Q$, as expected, and with $\alpha$ (since turbulent support vs. gravity is larger), though the latter effect is weak.
internal density $\bar{\rho}_g \approx 2 \text{ g cm}^{-3}$ (Weingartner & Draine 2001), mass fraction $Z_d$ (that is not to say we assume there are no grains of different sizes; simply that we treat the dynamics of grains in "bins" of size and neglect interactions across bins). The mid-plane stopping time is

$$ t_s = \frac{\bar{\rho}_g R_d}{\rho_g(\bar{\rho}_g)} \times \begin{cases} \frac{1}{(4R_d)/(9\lambda_c)} & (R_d \leq 9\lambda_c/4) \\ \frac{(R_d > 9\lambda_c/4)}{(4R_d)/(9\lambda_c)} & \end{cases} $$

where $\lambda_c = 1/(n_g \sigma(H_2)) = \mu m_p/(\rho_g) \sigma(H_2)$ is the mean-free path in the gas. We can then define $\tau_\infty \equiv t_s \Omega$. This and $\sigma$ determine the dust scale height, $h_d = \sqrt{\alpha/(\alpha + \tau_\infty)} \approx \sqrt{1/\tau_h} h_d$, a general result that holds for both large and small $\tau_\infty$ (Carballido et al. 2006).

Now allow a fluctuation $\rho_\delta (k) = \delta \rho (\rho_\delta (z)) (\delta \rho \neq 1)$ of the mean grain density averaged on the scale $k$, where $k \equiv 1/\lambda$ is the wavenumber ($\lambda$ the wavelength) of the fluctuation. For the incompressible (Kolmogorov) turbulent cascade, we expect an rms turbulent velocity on each scale $\langle \delta \mathbf{v}^2 \rangle \equiv \alpha c_s^2 f_\lambda(\lambda/\lambda_{\max})$ with $f_\lambda \sim (\lambda/\lambda_{\max})^{7/3}$ where $\lambda_{\max}$ is the top/driving scale of the cascade (we take $\lambda_{\max} \approx H_d$). We can define the corresponding eddy turnover time $t_e(k) = \lambda/\langle \delta \mathbf{v}^2 \rangle^{1/2}$.

The grains will also have a scale-dependent velocity dispersion following the turbulent cascade, for which we can define $\langle \delta \mathbf{v}^2 \rangle \equiv \alpha c_s^2 f_\lambda(\lambda/\lambda_{\max})$. However, since they are partially-coupled, $g_r$ is in general a non-trivial function which we derive in Appendix A. On large scales (for small grains) where $t_e \ll t_s(k)$, the grains are well-entrained by the gas, so we expect $g_r \approx g_f$, but on small scales where $t_e \gg t_s(k)$, the grains are effectively collisionless, so have a constant (scale-independent) minimum velocity dispersion.

### 2.1 Grain Density Fluctuations in Incompressible Gas

In Hopkins (2013b) (hereafter Paper I), we derive analytic expressions for the statistics of grain density fluctuations in a turbulent proto-planetary disk, and show that these accurately reproduce the results of full numerical simulations as well as laboratory turbulence experiments. We refer the interested reader to that paper for details, but briefly review the most important aspects of the model here.

Grain density fluctuations on different scales can be represented by a multiplicative random cascade. Consider a random point in space $x$ within the disk, and define the local density, averaged within a radius $\lambda$ as $\rho_\delta (x, \lambda) = M_d(x - x < \lambda)/(4\pi \lambda^3) = \rho_\langle \lambda > \rho_\delta \rangle$. Within the dynamic range of the turbulence ($\lambda_{\max} < \lambda < \lambda_{\infty}$), if we increment the "smoothing scale" $\lambda \rightarrow \lambda + \Delta \lambda$, then we expect the volume to include $m$ eddies of characteristic size $\lambda$, where $m$ is drawn from a Poisson distribution with $(m) = 2d \ln \lambda$ (the prefactor 2 follows from purely geometric considerations for vortices in three dimensions). Each eddy encountered imprints some density change on the local grain distribution, given by the function $\Delta \ln \delta$, which qualitatively behaves as described in § 1.

When $t_e$ is much smaller or much larger than the characteristic timescales on the scale $\lambda$ (shear and turbulent eddy turnover times $t_s$), $\Delta \ln \delta$ is small, because grains are either tightly coupled to the gas (the small $t_e$ limit) or effectively collisionless (the large $t_e$ limit), but when $t_e$ is similar to $t_s(\lambda)$, grains are flung out of regions of high vorticity ($\Delta \ln \delta < 0$) and concentrated in regions of low vorticity ($\Delta \ln \delta > 0$).

Quantitatively, the resulting change in $\delta$ from the eddy is given by $\Delta \ln \delta = c + 2 m \ln(1 + h^{-1}(\lambda))$, where $\mathcal{O} = \mathcal{O}(\tau_\infty, \tau_h)$ is a complicated function derived in Paper I (see Tables 1-2 there), but which depends only on the dimensionless timescale ratios $\tau_\infty \equiv t_s/\Omega^{-1}$ and $\tau_h \equiv t_s/t_e$: it declines linearly with $t_e$ or $t_e^{-1}$ when $t_e/t_h \ll 1$ or $t_e/t_h \gg 1$ (for either $t_e = \Omega^{-1}$ or $t_e = t_h$), and peaks at $\sim 0.3$ when $\tau_\infty \sim 1$. The function $h$ (also in Table 1 of Paper I) depends on $\lambda > \lambda_{\max}$ by taking the isotropic turbulent power spectrum $E(k) \propto k^{-5/3} (1 + |k/\lambda_{\max}|^{-2})^{-3(2-1/5)}$ (Bowman 1996). This gives

$$ \langle \delta \mathbf{v}^2 \rangle \equiv \alpha c_s^2 f_i(\lambda/\lambda_{\max}) $$

$$ f_i(k) = \frac{4\pi}{\sqrt{\pi} T(1/3)} \int_0^{+1/3} (1+i^2)^{-11/6} \mathrm{d} \tau $$

$$ \approx 1.89 \sqrt{2/3} (1 + 1.831 \sqrt{2/3})^{-2/7} $$

We use this (the approximate form is accurate to $\sim 1\%$ at all $k$) in all our numerical calculations. Some cutoff is necessary at large scales or else a power-law cascade contains a divergent kinetic energy (and we do not expect $\lambda_{\max} \gg H_d$). However, we do not include an explicit model for the dissipation range (scales below Kolmogorov $\lambda_{\infty}$ – i.e. we assume infinite Reynolds number – since all quantities in this paper are converged already on much larger scales. For the conditions of interest, $\lambda_0 \sim 0.1$ km.
in a more complicated fashion, but quite weakly, on the parameters \(\alpha \equiv (v_k^2)/c_s^2\) and \(b_2 \equiv \langle v_k^2(k)\rangle /\langle \delta \rangle^2\). Finally \(\kappa\) is simply given by mass conservation – i.e. the requirement that grains displaced from some regions end up in others.

For the model described above, we can sample the grain density PDF on all scales in a Monte Carlo fashion. Select some number (here \(\sim 10^5\)) random points in space, and begin by smoothing each on a scale \(\lambda > H\), where, by definition \(\rho_0(x, \lambda) = \langle \rho_0 \rangle\). Then, around each point, take differential steps in scale \(d \ln \lambda\), and integrate the effects of all eddies on the local grain density around each point as given above (looking up the appropriate functions in Paper I).

This defines a statistical distribution of grain densities on all scales. We can then define the mass function of “grain overdensities” which exceed some “interesting” minimum critical density \(\rho_{\text{crit}}(\lambda)\) on any given scale \(\lambda\). To avoid the ambiguity of “double-counting” or “clouds in clouds” (i.e. regions which exceed \(\rho_{\text{crit}}(\lambda)\)), which are embedded in some larger region \(\lambda_2 > \lambda_1\) which exceeds \(\rho_{\text{crit}}(\lambda_1)\), we specifically consider the “first crossing distribution” (see Bond et al. 1991; Hopkins 2012c);\(^2\) namely, the number of regions where \(\rho_{\text{crit}}(\lambda)\), defined uniquely by the largest size/mass scale on which \(\rho(x, \lambda) > \rho_{\text{crit}}(\lambda)\). In Hopkins (2013a), we show that this is uniquely given by

\[
\frac{dn}{d \ln \lambda} = \rho_{\text{crit}}(\lambda) \frac{df}{d \ln \lambda}
\]

(5)

where \(f\) is the fraction of “trajectories” (Monte-Carlo sampled points) which first exceed \(\rho_{\text{crit}}(\lambda)\) on the scale \(\lambda \rightarrow \lambda - d\lambda\) (without having exceeded the critical density on any larger scale), and the mapping between mass and scale is just given by the integral over volume in an exponential disk:

\[
M(R) \equiv 4 \pi \rho_{\text{crit}} h^3 \left[\frac{R^2}{2 \rho_k} + 1 + \frac{R}{\rho_k} \right] \exp \left[-\frac{R}{\rho_k}\right] - 1
\]

(6)

It is easy to see that on scales \(\lambda < H\), this is just \(M = (4\pi/3) \rho_{\text{crit}} \lambda^3\), on scales \(\lambda > H\), just \(M = \pi \Sigma \rho_{\text{crit}} \lambda^2\).

2.2 Criteria for Dynamical Gravitational Collapse

Now, to define the mass function of “interesting” grain density fluctuations, we need to define the critical density \(\rho_{\text{crit}}(\lambda = k^{-1})\). It is convenient to define

\[
\bar{\rho} = \rho(k) \equiv 1 + \frac{\rho_0(k)}{\rho_k}
\]

(7)

If we consider grains which are purely collisionless (no grain-gas interaction), then a Toomre analysis gives the following criterion for gravitational instability of a mid-plane perturbation of wavenumber \(k\):

\[
0 > \omega^2 = \kappa^2 + \langle v_k^2(k) \rangle k^2 - 4\pi G \rho_k \bar{\rho} \frac{|k h_a|}{1 + |k h_a|}
\]

(8)

Here the \(\kappa\) term represents the contribution of angular momentum resisting collapse, and \(\langle v_k^2(k) \rangle\) is the rms turbulent velocity of grains on the scale \(k\); for a derivation of the turbulent term here see Chandrasekhar (1951).\(^3\) The negative term in \(G\) represents self-gravity.

\(^2\) In Hopkins 2013a, we note that this aspect of the general calculation of mass functions of self-gravitating regions in turbulent fields is identical (up to variable changes) to the unique identification of dark matter halos (i.e. the resolution of the “halo within halo” problem).

\(^3\) More exactly, for grains on small scales – where they are locally collisionless – we should combine the turbulent velocity and density terms, taking instead \(\rho_0 \rightarrow \rho_0 F(\omega/n, k^2 \langle v_k^2(k) \rangle /\kappa^2)\) where \(F\) is the reduction factor and de-stabilizes the perturbation at sufficiently large \(\bar{\rho}\). The terms in \(|k h_a|\) on the right are the exact solution for an exponential vertical disk and simply interpolate between the two-dimensional (thin-disk) case on scales \(\gtrsim h_a\) and three-dimensional case on scales \(\lesssim h_a\) (see Elmegreen 1987; Kim et al. 2002, for derivations).

In the opposite, perfectly-coupled (\(\lambda \rightarrow 0\)) limit, we have a single fluid and obtain

\[
0 > \omega^2 = \kappa^2 + \frac{1}{\bar{\rho}} \left(\kappa^2 + \langle v_k^2(k) \rangle \right) k^2
\]

\[
+ \frac{\bar{\rho} - 1}{\bar{\rho}} \frac{\langle v_k^2(k) \rangle k^2 - 4\pi G \rho_k \bar{\rho}}{1 + |k h_a|}
\]

\[
= \kappa^2 + \frac{c_s^2 k^2}{\bar{\rho}} + \frac{\langle v_k^2(k) \rangle k^2 - 4\pi G \rho_k \bar{\rho}}{1 + |k h_a|}
\]

(9)

where \(c_s\) and \(\langle v_k^2(k) \rangle\) represent gas pressure and turbulent support (and we used \(\langle v_k^2(k) \rangle = \langle v_k^2 \rangle\) for the perfectly-coupled case). Note that the terms describing the gas pressure/kinetic energy density have a pre-factor \(1/\bar{\rho} = \rho_k/(\rho_k + \rho_0)\), since what we need for the mixed-grain-gas perturbation is the energy density per unit mass in the perturbation. Likewise, the kinetic energy density term has a pre-factor \((\bar{\rho} - 1)/\bar{\rho} = \rho_0/\rho_k\). Since both sit in the same external potential and self-gravitate identically, the \(\kappa\) and \(G\) terms need no pre-factor. In this limit we can think of the \(1/\bar{\rho}\) factor as simply an enhanced “mean molecular weight” from the perfectly-dragged gas grains (so the effective sound speed of the gas \(c_{s,eff}^2 = c_s/\sqrt{1 + \rho_k/\rho_0} = c_s/\bar{\rho}^{1/2}\)).

We can interpolate between these cases with

\[
0 > \omega^2 = \kappa^2 + \frac{\beta}{\bar{\rho}} \left(\kappa^2 + \langle v_k^2(k) \rangle \right) k^2
\]

\[
+ \frac{\bar{\rho} - 1}{\bar{\rho}} \frac{\langle v_k^2(k) \rangle k^2 - 4\pi G \rho_k \bar{\rho}}{1 + |k h_a|}
\]

(10)

The only important ambiguity in the above is the term \(\beta\), which we introduce to represent the strength of coupling between grains and gas (\(\beta = 0\) is un-coupled/collisionless; \(\beta = 1\) is perfectly-coupled). In general, \(\beta\) is some unknown, presumably complicated function of all the parameters above, which can only be approximated in the fully non-linear case by numerical simulations. However, the limits are straightforward: if a perturbation collapses on a free-fall time \(t_{\text{grav}} \ll t_s\), we expect \(\beta \rightarrow 0\) (since there is no time for gas to decelerate grains). Conversely if \(t_{\text{grav}} \gg t_s\), \(\beta \rightarrow 1\). Therefore in this paper we make the simple approximation\(^4\)

\[
\beta \approx \frac{t_{\text{grav}}}{t_s + t_{\text{grav}}} = 1 + t_s/t_{\text{grav}} \approx 1 + \left(1 + \frac{4\pi}{3\Omega} \frac{\bar{\rho}}{\pi} \right)^{-1}
\]

(11)

Alternative derivations of these scalings from the linear equations for coupled dust-gas fluid, and including the non-linear stochastic effects of turbulence, are presented in Appendices B-C, respectively. If anything, we have chosen to err on the side of caution and define a very strict criterion for collapse – almost all higher-order effects make collapse slightly easier, not harder. This determined by integration over the phase-space distribution. However, the relevant stability threshold comes from evaluating \(\mathcal{F}\) near \(\omega \approx 0\); in this regime we can Taylor expand \(\mathcal{F}\) (assuming a Maxwellian velocity distribution), and to leading order we recover the solution in Eq. 10. The exact solution can be determined for the purely collisional limit (again identical to Eq. 10) or the purely collisionless limit (identical to a stellar disk, where the minimum density for collapse \(\bar{\rho}\) is smaller by a factor \(= 0.935\). Given the other uncertainties in our calculation, this difference is negligible.

\(^4\) This approximation is motivated by expansion of the de-celeration of collapsing particles by molecular collisions, which is linear in time for \(t \ll t_s\).
criterion is sufficient to ensure that (at least in the initial collapse phase) a pebble pile is gravitationally bound (including the thermal pressure and turbulent kinetic energy), and gravitational collapse is sufficiently strong to overcome gas pressure forces, tidal forces/angular momentum/non-linear shearing of the overdensity, turbulent vorticity and “pumping” of the energy and momentum in the region, and ram-pressure forces from the “headwind” owing to radial drift. Similarly, when this criterion is met, the collapse timescale is faster than the orbital time, the grain drift timescale, the effective sound-crossing time of the clump, and the eddy turnover time. Using the definitions above, the criterion can be re-written:

\[
0 > 1 + \left( \frac{H_s}{h_d} \right)^2 \frac{1}{\lambda^2 \hat{\rho}} \left[ \beta \left( 1 + \alpha f_1 \frac{\lambda}{\lambda_{\text{max}}} \right) \right]
\]

\[+ \alpha \left( \hat{\rho} - 1 \right) g_{\text{f}1} \left( \frac{\lambda}{\lambda_{\text{max}}} \right) - \frac{2 Q^2}{1 + \lambda \hat{\rho}} \]

\[= 1 + \frac{\tau_s}{\lambda^2 \hat{\rho}} \left[ \beta \left( \alpha^{-1} + f_1 \right) + (\hat{\rho} - 1) g_{\text{f}1} \right] \left( 1 + \frac{2 Q^2}{1 + \lambda \hat{\rho}} \right) \]

where \( \hat{\lambda} \equiv \lambda/h_d \) and we abbreviate \( f_1 = f_1(\lambda/\lambda_{\text{max}}) \). This has the solution

\[
\hat{\rho} > \hat{\rho}_{\text{crit}}(\lambda) \equiv \psi_0 \left( 1 + \sqrt{1 + \psi_0^2 / \psi} \right)
\]

\[\psi_0 \equiv \frac{Q^2}{4} \left[ 1 + \frac{\lambda}{\lambda^2 \hat{\rho}} \right] \]

\[\psi_1 \equiv \frac{2 \pi}{\lambda^2} \left[ \beta (\alpha^{-1} + f_1) - g_{\text{f}1} \right] \left( 1 + \frac{\lambda}{\lambda^2 \hat{\rho}} \right) \]

(Note, if \( \beta \) itself is a function of \( \hat{\rho} \), then this is an implicit equation for \( \hat{\rho}_{\text{crit}} \) which must be solved numerically). Recall the dimensionless grain density fluctuation \( \delta_\rho = \rho_\text{d}/\langle \rho_\text{d} \rangle_0 \), so \( \hat{\rho} = 1 + \delta_\rho \langle \rho_\text{d} \rangle_0 / \langle \rho_\text{d} \rangle_0 = 1 + \delta_\rho \left( \Sigma_d / \Sigma_d \right) (H_s/h_d) = 1 + \delta_\rho Z_d \sqrt{\tau_s / \alpha} \). So in terms of \( \delta_\rho \), the criterion becomes

\[
\delta_\rho Z_d > \sqrt{\frac{\tau_s}{\alpha}} \left( \hat{\rho}_{\text{crit}}(\lambda) - 1 \right)
\]

### 2.3 Physical Disk Models

In order to attach physical values to the dimensionless quantities above, we require a disk model. We will adopt the following, motivated by the MMSN, for a disk of arbitrary surface density around a solar-type star (\( M_* \approx M_\odot, R_* \approx R_\odot, T_* \approx 6000 \text{K} \)):

\[
\Omega = \sqrt{\frac{GM_*}{r^3}} \approx 6.3 r_{\text{au}}^{-3/2} \text{yr}^{-1}
\]

\[\Sigma_d = \Sigma_0 = 1000 r_{\text{au}}^{-3/2} \text{g cm}^{-2}
\]

\[T_{\text{eff}, *} = \left( \frac{0.05 r_{\text{au}}}{4 r_{\text{au}}^2} \right)^{1/4} T_* \approx 140 r_{\text{au}}^{-3/7} \text{K}
\]

These choices determine the parameters

\[c_s = \sqrt{\frac{k_B T_{\text{mid}}}{\mu m_p}} \approx 0.64 r_{\text{au}}^{-3/14} \text{km s}^{-1}
\]

\[H_s / r_* \approx 0.022 r_{\text{au}}^{2/7}
\]

\[\langle \rho_d \rangle_0 \approx 1.5 \times 10^{-9} \Sigma_d r_{\text{au}}^{-39/14} \text{g cm}^{-3}
\]

\[Q = c_s \Omega / \pi G \Sigma_d \approx 61 \Sigma_d r_{\text{au}}^{-3/14}
\]

\[\Pi = \frac{1}{2} \rho_0 V K c_s \left( \frac{\partial (\rho_0)}{\partial r} \right) \approx 0.035 r_{\text{au}}^{2/7}
\]

\[\lambda_\alpha = \frac{1}{n_0} \sigma(H_2) \approx 1.2 \Sigma_d^{-1} r_{\text{ae}}^{-3/14} \text{cm}
\]

\[\tau_s \approx \text{MAX} \left\{ \frac{0.004 \Sigma_d r_{\text{au}}^{-3/2}}{R_e^2}, \frac{0.0014 R_e^4 r_{\text{ae}}^{-9/7}}{(1 + (r_{\text{au}}/3.2)^{3/7})} \right\}
\]

with \( \mu \approx 2.3 \) (appropriate for a solar mixture of molecular gas) and we take the molecular cross-section \( \sigma(H_2) \approx 2 \times 10^{-15} (1 + (T/700K)^{-1}) \) (Chapman & Cowling 1970).

The expression for \( T_{\text{mid},*} \) is the approximate expression for the case of a passive flared disk irradiated by a central solar-type star, assuming the disk is optically thick to the incident and re-radiated emission (in which case the external radiation produces a hot surface dust layer which re-radiates \( \sim 1/2 \) the absorbed light back into the disk, maintaining \( T_{\text{mid},*} \approx T_{\text{eff},*}^{2/7} \); see Chiang & Goldreich 1997).

#### 3 APPROXIMATE EXPECTATIONS

We now have everything needed to calculate the detailed statistics of collapsing regions. Before we do so, however, we can gain considerable intuition using the some simple approximations.

##### 3.1 Small Grains

For small grains (\( \tau_s \ll 1 \)), density fluctuations on scales \( \sim h_d \) are very weak (since the grains are well-coupled to gas on these scales). Large density fluctuations are, however, still possible on small scales, where \( t_s \sim t_c \). Consider this limit. In this regime, in a large Reynolds-number flow, the fluctuations are approximately self-similar, because all grains “see” a large, scale-free (power-law) turbulent cascade at both larger scales (\( t_s \gg t_c \)) and smaller scales (\( t_s \ll t_c \)). As shown in Paper I and Cuzzi et al. (2001); Yoshimoto & Goto (2007); Hogan & Cuzzi (2007); Pan et al. (2011), the

5 More accurately, we can take the effective temperature from illumination to be: \( T_{\text{eff},*} = \frac{3}{4} t_r T_r^4 / (4 r_{\text{au}}^4) \) with \( t_r \approx 0.005 r_{\text{au}}^{-1} + 0.05 r_{\text{au}}^{2/7} \) (Chiang & Goldreich 1997). Since we allow non-zero \( \alpha \), this implies an effective viscosity and accretion rate \( M \approx 3 \pi \alpha c_s^2 \Sigma_d / \Omega^{1/2} \) (Shakura & Sunyaev 1973), which produces an effective temperature \( T_{\text{eff},*} \approx 3 M \Omega^{1/2} (8 \pi \sigma T_r^4) \) (\( \sigma \) is the Boltzmann constant). Note this depends on the term \( c_s^2 \). A more accurate estimate of \( T_{\text{eff},*} \) is then given by solving the implicit equation \( T_{\text{eff},*} = \langle 3/4 \rangle [\nu_f + 4/3 + 2/(3 \nu_f)] T_{\text{eff},*}^{2/7} \). Here \( \nu_f \) is the Rosseland mean opacity, which we can take from the tabulated values in Semenov et al. (2003) (crudely, \( \nu_f \approx 5 \text{cm}^2 \text{s}^{-1} \) at \( T_{\text{mid}} > 160K \) and \( \nu_f \approx 2.4 \times 10^{-4} T^2 \text{cm}^2 \text{s}^{-1} \) at lower \( T_{\text{mid}} \). We use this more detailed estimate for our full numerical calculation, however it makes almost no difference for the parameter space we consider, compared to the simple scalings above.

\[c_s = \sqrt{\frac{k_B T_{\text{mid}}}{\mu m_p}} \approx 0.64 r_{\text{au}}^{-3/14} \text{km s}^{-1}
\]

\[H_s / r_* \approx 0.022 r_{\text{au}}^{2/7}
\]

\[\langle \rho_d \rangle_0 \approx 1.5 \times 10^{-9} \Sigma_d r_{\text{au}}^{-39/14} \text{g cm}^{-3}
\]

\[Q = c_s \Omega / \pi G \Sigma_d \approx 61 \Sigma_d r_{\text{au}}^{-3/14}
\]

\[\Pi = \frac{1}{2} \rho_0 V K c_s \left( \frac{\partial (\rho_0)}{\partial r} \right) \approx 0.035 r_{\text{au}}^{2/7}
\]

\[\lambda_\alpha = \frac{1}{n_0} \sigma(H_2) \approx 1.2 \Sigma_d^{-1} r_{\text{ae}}^{-3/14} \text{cm}
\]

\[\tau_s \approx \text{MAX} \left\{ \frac{0.004 \Sigma_d r_{\text{au}}^{-3/2}}{R_e^2}, \frac{0.0014 R_e^4 r_{\text{ae}}^{-9/7}}{(1 + (r_{\text{au}}/3.2)^{3/7})} \right\}
\]
maximum local density fluctuations in this limit saturate at values \( \delta_\rho^{\text{max}} \sim 300 \). Since \( t_c(\lambda < \lambda_{\text{max}}) \propto \lambda^{2/3} \), this “resonance” will occur at scales \( \lambda \approx \lambda_{\text{max}} \tau_s^{3/2} \alpha^{3/4} (H_0/\lambda_{\text{max}})^{1/2} \ll \lambda_{\text{max}} \) (so \( \lambda \sim \alpha^{1/4} \tau_s^2 \)).

We can, on these scales, also approximate \( f_\delta \approx g_\delta \approx (\lambda/\lambda_{\text{max}})^{2/3} \ll \alpha^{1/2} \tau_s \), and drop higher-order terms in \( \lambda/\lambda_{\text{max}} \) or \( \lambda \). If we take either the tightly coupled (\( \beta = 1 \)) or un-coupled (\( \beta = 0 \)) limits, we obtain

\[
\bar{\rho}_{\text{crit}} \sim \begin{cases} 
\frac{Q}{2} \alpha^{3/2} \tau_s & (\beta = 1) \\
\frac{Q}{2} (1 + \tau_s^{-2}) & (\beta = 0)
\end{cases}
\]

or

\[
\delta_\rho \gtrsim \begin{cases} 
Z_{\text{d}}^{-1} \tau_s^{-2} \alpha^{-1/4} (Q/2)^{1/2} & (\beta = 1) \\
Z_{\text{d}}^{-1} \tau_s^{-5/2} \alpha^{1/2} (Q/2) & (\beta = 0)
\end{cases}
\]

This is requires extremely large density fluctuations: for \( Z_{\text{d}} \sim Z_\odot \), and \( Q \sim 60 \) (MMSN at \( r_s \sim 1 \text{au} \)), this gives minimum \( \delta_\rho \) of \( \sim 3 \times 10^3 (\tau_s/0.1)^{-2} (\alpha/10^{-4})^{-1/4} \) and \( \sim 5000 (\tau_s/0.1)^{-3/2} (\alpha/10^{-4})^{1/2} \), respectively.

Physically, even if we ignore gas pressure, and the density fluctuation is small-scale (so shear can be neglected), grains must still overcome their turbulent velocity dispersion in order to collapse. A simple energy argument requires \( GM_{\odot}(\lambda/\lambda_{\text{max}}) \gtrsim M_d(\lambda/\lambda_{\text{max}}) (\rho_{\text{crit}}(\lambda))^2 \); using \( M_d(\lambda) \sim \rho_d \lambda^3 \) and \( (\rho_{\text{crit}}(\lambda))^2 \sim (\lambda/\lambda_{\text{max}})^{2/3} \), this is just \( G\rho_d \gtrsim (t_c^2)^{-1} \). In other words, the collapse time \( t_{\text{grav}} \sim (G\rho_d)^{-1/2} \) must be shorter than the eddy turnover time (within the grains) on the same scale. But recall, the clustering occurs characteristically on a scale where for the gas, \( t_c \sim t_s \). Thus, the grains are at least marginally coupled, and the grain \( t_s^2 \sim t_s \sim t_c \) – the same eddies that induce strong grain clustering necessarily induce turbulent grain motions with eddy turnover time on the same scale \( \sim t_s \) (see Bec et al. 2009). So collapse of even a collisionless grain population requires \( t_{\text{grav}} \lesssim t_s \). Using \( Q \sim \Sigma_0^2 (G\rho_d) \) and \( \rho_{\text{crit}} \sim \rho_{\text{grav}} \) (for \( \rho \gg 1 \)), we see this is equivalent to the \( \beta = 0 \) criterion above.

Since, in this limit, \( t_{\text{grav}} < t_s \), taking \( \beta = 0 \) is in fact a good approximation (and since the \( \beta = 1 \) criterion requires a still higher density, so \( t_{\text{grav}} \ll t_s \), it is not the relevant case limit here).

Thus even with no gas pressure effects (\( \beta = 0 \), collapse \( \delta_\rho^{\text{max}} \gtrsim \delta_\rho^{\text{collapse}} \)) requires \( \tau_s \gtrsim 0.2 (\alpha/10^{-4})^{1/3} (\delta_\rho^{\text{max}} Z_{\text{d}}/1000Z_\odot)^{-2/3} (Q/60)^{2/5} \) – unless

![Figure 3. Predicted mass function of collapsing (self-gravitating) pebble-pile planetesimals formed by turbulent grain density concentrations. We plot the cumulative number formed at various radial distances from the star (per unit orbital distance: dV/dlnr\text{au}), as a function of mass (in Earth masses). The disk is our standard MMSN model (Z = Z_\odot, \Sigma_0 = 1; see § 2.3), with \( \alpha = 10^{-4} \). Different line types assume the grain mass is concentrated in grains of different sizes (as labeled). If the grains are very large (10cm), then pebble piles can collapse directly to masses from \( \sim 10^{-5} - 1M_\oplus \) over a range of orbital radii \( \sim 0.1 - 20 \text{au} \). If grains only reach 1 cm, the lower \( \tau_s \) super-exponentially suppresses this process at smaller radii, and it can only occur at large radii \( \gtrsim 20 - 30 \text{au} \), where \( \tau_s \gtrsim 0.1 \) (however the range of masses at these radii is large, from \( \sim 10^{-4} - 10M_\oplus \)). For maximum grain sizes = 1 mm, this is pushed out to \( \gtrsim 100 \text{au} \).](image-url)
the disks are extremely quiescent \((\alpha \ll 10^{-7})\), we are forced to consider large grains (where \(\tau_i \ll 1\) is not true).

3.2 Large Grains

For large grains, fluctuations are possible on large scales. For a flat perturbation spectrum, the most unstable scale is \(\lambda \sim h_d\), so take this limit now. In this case \(f_1 \approx g_1 \approx (\alpha/\tau_i)^{1/3}\) and \(\lambda \approx 1\), giving

\[
\tilde{\rho}_{\text{crit}} \sim \begin{cases} 
\left(\frac{Q \tau_i}{\alpha}\right)^{1/2} & (\beta = 1) \\
Q(1 + \tau_i^{2/3} \alpha^{1/3}) & (\beta = 0) 
\end{cases}
\]  \(30\)

or

\[
\delta_\rho \gtrsim \begin{cases} 
Z_{\alpha}^{-1} Q^{1/2} & (\beta = 1) \\
Z_{\alpha}^{-1} Q \sqrt{\alpha/\tau_i} & (\beta = 0) 
\end{cases}
\]  \(31\)

Even at \(Z_{\alpha} \sim Z_{\odot}\) and \(Q \sim 60\), this gives a minimum \(\delta_\rho\) of \(\sim 400\) and \(\sim 100(\tau_i/0.1)^{1/3}(\alpha/10^{-4})^{1/3}\), respectively. Collapse is far “easier” when grains can induce fluctuations on large scales.

In this limit, the \(\beta = 0\) criterion is just the a Roche criterion, \(t_{\text{grav}} \lesssim \Omega^{-1}\) (the turbulence is sub-sonic, so its support is not dominant on large scales). The \(\beta = 1\) criterion is more subtle: recall that the “effective” sound speed of the coupled fluid is \(c_{\text{eff}} = c_{\Omega} = \Omega H_d\), and \(Q = \Omega Z_{\alpha}^{-1}/G\rho_0\). Then we see this criterion is equivalent to \(t_{\text{grav}} \lesssim t_{\text{cross}} \equiv h_d/c_{\text{eff}}\), i.e. that the collapse time is shorter than the effective sound-crossing time on the scale \(h_d\). For \(\tau_i \lesssim 1\), these generally do allow \(t_{\text{grav}} \gtrsim t_s\), so \(\beta = 1\) is the more relevant limit – but importantly, collapse of the two-fluid medium even on timescales \(\gg t_s\) is allowed, providing a large overdensity can form on sufficiently large scales.

In Paper I we derive approximate expressions for the maximum density fluctuations of large grains on large scales (Table 2 therein), \(\ln \delta_{\text{max}}^0 \sim 6 \delta_0 (1 + \delta_0)^{-1}[1 + b_2(1 + \delta_0) + b_2^{1/2}(\sqrt{1 + \delta_0^2} - 1)]\) where \(\delta_0 \approx 3.2 \tau_i/(1 + \tau_i^2)\) and \(b_2 \sim 1\) depends on the ratio of drift to turbulent velocities. For \(\tau_i \lesssim 1\), this becomes \(\delta_{\text{max}}^0 \approx \exp[20 \ln (1 + b_2) \tau_i]\); comparing this to the above (\(\beta = 1\)) criterion requires \(\tau_i \gtrsim 0.05 \ln (Q^{1/2} Z_{\alpha})/[\ln (1 + b_2)] \sim 0.3\). So sufficiently large grains can indeed achieve these fluctuations.

4 NUMERICAL RESULTS

Now we perform this calculation in detail.

4.1 The Collapse Threshold

4.1.1 Dependence on Spatial Scale: Large Scales are Favored

In Fig. 1 we illustrate how the threshold for self-gravity derived in § 2.2 scales as a function of various properties. Recall, the combination \(\delta_\rho Z_{\alpha}\) must exceed some value (Eq. 17) which is a function only of \((\tau_i, Q, \alpha)\) in order for an over-density to collapse. So the collapse threshold in dimensionless units of grain-density fluctuations scales inversely with the dust-to-gas mass ratio \(Z_{\alpha}\). We see that, as is generic for Jeans/Toomre collapse and expected from the arguments in § 3, higher over-densities are required for collapse on small scales, with a minimum in \(\delta_\rho\) around \(\lambda \sim h_d\). On small scales the thermal pressure term \((\propto \lambda^{-2}/\beta)\) dominates the support vs. gravity \((\propto \beta)\), giving \(\tilde{\rho}_{\text{grav}} \propto \lambda^{-1}\). On very large scales \(\lambda \gg h_d\) angular momentum dominates and, just as in the Toomre problem, \(\tilde{\rho}_{\text{crit}} \propto \lambda\).

4.1.2 Dependence on Grain Properties

We also see that, generically, larger grains (larger \(\tau_i\)) require smaller \(\delta_\rho\) for collapse. This is because (with other disk properties fixed) the initial dust disk settles to a smaller scale height (larger density), and because the resistance by gas pressure is weaker. The change in this behavior for large grains \(\tau_i \gtrsim 1\) on small scales owes to the fact that the velocity dispersions of large grains decouple from the gas and become scale-independent (do not decrease with \(\lambda\)) on small scales.

If we focus on \(\delta_\rho\) around scales \(\lambda \sim h_d\) or \(\lambda \sim \lambda_e(t_e = t_s)\), as in § 3, we confirm our approximate scalings above. Near \(h_d\), collapse requires modest over-densities \(\sim 100 - 1000\), very weakly dependent on \(\tau_i\) or \(\alpha\) (for small \(\alpha \lesssim 10^{-5}\)) and \(\propto Q^{1/2}\), confirming our approximate scaling for \(\beta = 1\) (since in this limit, \(t_{\text{grav}} \gtrsim t_s\), \(\beta \sim 1\)). Around \(\lambda \sim \lambda_e(t_e = t_s)\), we see, as expected, a very strong scaling \(\delta_\rho \propto \tau_i^{5/2}\) with weak residual dependence on \(\alpha\) (and also \(\propto Q\)), as expected from our derivation above.

4.1.3 Importance of Gas Pressure

In Fig. 2 we repeat this exercise but simply force \(\beta = 1\) or \(\beta = 0\). We can see that either approximation fails, at some range of
scales, by about an order of magnitude. Assuming $\beta = 1$ does not much change the criteria for large-scale collapse, however, and the change at very small scales is large but well into the regime where the values of $\delta_s$ must be extremely high no matter what choices we make. But assuming $\beta = 0$ under-predicts the collapse thresholds by an order-of-magnitude or more on large scales (the most interesting range for our calculation). In this regime the collapse thresholds are such that the collapse time is longer than the stopping time, so it is not a good approximation to neglect gas pressure.

4.2 The Mass Function of Resulting Pebble-Pile Planetesimals

Given these thresholds, we can now calculate the mass function of collapsing dust density fluctuations. Fig. 3 shows the results for our “default” MMSN model ($\Sigma_0 = 1$, $Z_d = Z_\odot$, $\alpha = 10^{-4}$), at various radii, assuming different grain sizes.

4.2.1 Dependence on Orbital Distance and Grain Size

As expected, if the grains are sufficiently large ($\tau_s \sim 1$), self-gravitating pebble piles can indeed form over a range of orbital radii, with a wide range of self-gravitating masses. For $R_d \sim 10$ au, all radii $r_w \sim 0.1 - 10$ have $\tau_s \sim 1$ and form pebble piles. At still smaller radii, $Q$ is very large and suppresses collapse; at larger radii, $\tau_s \gg 1$, and so grain-density fluctuations are actually suppressed because the grains are approximately collisionless (the cutoff we see occurs at approximately $\tau_s \gtrsim 3 - 5$). For smaller grains, we must go to larger radii before $\tau_s \sim 1$, and collapse becomes possible. For $R_d \sim 1$ au, pebble pile formation at $t < 10$ au is completely suppressed — we stress that because the density fluctuations depend exponentially on $\tau_s$, the predicted number density is $< 10^{-10}$ here! We see this rapid threshold behavior set in between $\tau_s \sim 0.1 - 0.3$, a parameter space we explore further below.

4.2.2 The Minimum and Maximum Masses

Where possible, these collapse events form objects with a range of masses $\sim 10^{-8} - 10 M_\oplus$. The maximum mass is given by the behavior of the largest eddies with scales $\lambda \sim H_d$. Recall, in this model, the grains are essentially passive, so if turbulent eddies exist with $\lambda_c \gtrsim H_d$ (with the appropriate $t_s \sim t_e$), they still drive grain density fluctuations in the midplane dust layer (so long as the eddy intersects the midplane somewhere) on scales $\sim \lambda_c$, even if we take

$$M_{\text{coll}}^{\text{max}}(\tau_s \sim 1) \sim \sqrt{2\pi} \frac{\sqrt{Q}}{\tau_s} \frac{\beta}{1/2} \frac{\rho_0}{\alpha \Sigma_0} \frac{1/2}{1/4} \frac{H^3}{H^3}$$

$$\sim 0.03 \left( \frac{\alpha}{10^{-4}} \right) \frac{1/4}{1/4} \frac{\Sigma_0}{\rho_0} \frac{1/2}{1/2} \frac{M_\oplus}{M_\odot}$$

So the maximum mass is only weakly-dependent on $\alpha$ (since the grain layer has essentially zero thickness relative to these eddies), while it increases with disk surface density and is nearly proportional to radius (because the disk mass increases with $r_w$).

The lower limit is also predicted (not a resolution effect) because on sufficiently small scales, $H_{\text{cross}} = \lambda_c / \sqrt{Q} \ll h_s$ (so grains do not have time to interact with eddies) and/or $t_e \gg t_s$ (so turbulent eddies do not significantly perturb the gas). For $\tau_s \lesssim 1$, $t_e \ll t_s$ occurs on scales $\lambda / H \lesssim \alpha^{1/4} \tau_s^{1/4} \ll h_s$ (so $M \approx (4\pi/3) \rho_0 \lambda^3$), where a combination of gas pressure and turbulence form the dust layer to be infinitely thin. Indeed, this is just one of the toy-model cases considered in Paper I: a large in-plane vortex in a disk perturbing the razor-thin (two-dimensional) dust distribution in the midplane; for which we show the identical scalings apply as the three-dimensional case. It simply becomes dust surface density fluctuations that are driven by the large vortices trapping or expelling dust, rather than three-dimensional density fluctuations. So surface density fluctuations can form over a wide range of scales $h_d \lesssim h_s \lesssim h_d$; for a large eddy with $\lambda_c \gtrsim h_d$, the enclosed grain mass in the perturbation becomes $M \approx \pi \Sigma_0 \lambda^2 = 2\pi (\rho_0) H^2 \lambda^2$. Based on the arguments in § 3, we expect $t_g \gtrsim t_e$ (so $\beta \sim 1$) on these scales, so the dominant term resisting collapse is gas pressure and $\beta_{\text{crit}} \sim (Q \tau_s / 2 \alpha \lambda)^{1/2}$. If we assume $\tau_s \sim 1$ and that the largest eddies reach $H_d$, then we obtain

$$M_{\text{coll}}^{\text{max}}(\tau_s \sim 1) \sim \sqrt{2\pi} \frac{\sqrt{Q}}{\tau_s} \frac{\beta}{1/2} \frac{\rho_0}{\alpha \Sigma_0} \frac{1/2}{1/4} \frac{H^3}{H^3}$$

$$\sim 0.03 \left( \frac{\alpha}{10^{-4}} \right) \frac{1/4}{1/4} \frac{\Sigma_0}{\rho_0} \frac{1/2}{1/2} \frac{M_\oplus}{M_\odot}$$

6 Of course, if the grains themselves drive the turbulence, as in the streaming instability case, then such eddies will not exist. But this is accounted for in our model, in the parameters $\alpha$ and $\beta_{\text{crit}}$. 7 Interestingly, if we had very large-scale fluctuations, $\lambda \gg H$, then the shear/angular momentum term would be the dominant term resisting collapse and we would obtain $M_{\text{coll}}^{\text{max}} \sim \pi Q (\rho_0) H^3$, i.e. just the standard Hill mass.
invariant source of support ($\rho_{\text{crit}} \propto \lambda^{-1}$). After some algebra we obtain

$$M_{\text{min}}^{\text{collapse}}(\tau_s \sim 1) \sim \frac{2\sqrt{2\pi}}{3} \frac{Q^{1/2}}{\tau_s^2} (\rho_0) n H^2 \sim 2 \times 10^{-7} \left(\frac{\alpha}{10^{-4}}\right)^{3/2} \Sigma_0^{1/2} \tau_s^{27/28} M_{\odot} \tag{35}$$

Note that the “dynamic range” of the mass function is just

$$\frac{M_{\text{min}}^{\text{collapse}}}{M_{\text{max}}^{\text{collapse}}} \sim \alpha \tag{36}$$

4.2.3 Dependence on Turbulence Strength

Fig. 4 shows how the MF depends on $\alpha$. As expected from our simple calculation above, the “maximum” masses and top-end of the MF is nearly independent of $\alpha$, but the “minimum” mass and low-mass end depends strongly. Increasing $\alpha$ truncates the MF at higher minimum masses, because collapse is more difficult both owing to the thicker grain disk (so it is harder to collapse on scales $\lesssim h_0$) and increased local turbulent kinetic energy resisting collapse. At very high $\alpha \gtrsim 10^{-2}$, this eliminates entirely collapse at some orbital radii $\lesssim 1$ au (though for the most part the criteria for collapse at high masses are unchanged). At very low $\alpha \lesssim 10^{-8}$, turbulence is so weak that significant density perturbations are not generated and the MF goes to zero; however, such a low $\alpha$ would actually imply that the grain disk is razor-thin and would directly gravitationally fragment (without any grain clustering).

4.2.4 The Mass Function Slope

We also see the MF becomes flatter as $\alpha$ increases. Qualitatively, this follows from the same argument, that higher $\alpha$ suppresses small-scale collapse. Quantitatively, we can understand the slope as follows. The MF is given by Eq. 5; the exact solution must be evaluated numerically for the non-Gaussian statistics and complications are distributed approximately as a log-normal, and the dependence on scale is weak (logarithmic), then we can approximate the MF by the Press & Schechter (1974) solution for the mass function of density fluctuations above a fixed threshold in a Gaussian random field.

$$\frac{dn}{d\ln M} \sim \frac{\rho_{\text{crit}}(M)}{M} \frac{B_0}{\sqrt{2\pi} \Sigma^2} \frac{dS}{d\ln M} \exp\left(-\frac{S^2}{2}\right) \tag{37}$$

$$M \frac{dn}{d\ln M} \propto \delta_\rho \left|\ln \delta_\rho + S/2\right| \lambda_{\text{max}}^{1/2} \delta_\rho^{1/2} \frac{d\ln M}{S^{1/2}} \exp(-S/8) \tag{38}$$

where in the latter equality we have used $B \equiv \ln \delta_\rho + S/2$ and $\rho_{\text{crit}} \propto \delta_\rho \propto \delta_\rho$, and dropped all constant prefactors (since we want to isolate just the logarithmic slope).

In Paper I, we show that for large grains ($\tau_s \sim 1$), the power in logarithmic density fluctuations on large scales ($\zeta chapter 1 $ is approximately scale-free: $d\delta/d\ln \lambda \approx S_0$ constant, with $S_0 \approx C_{\text{os}} |\delta_\rho| \approx 2/(\sqrt{2}) \tau_s^2 / (1 + \tau_s^2)^{3/2}$. This comes simply from the fact that the centrifugal force in large eddies is dominated by $\Omega$, which is scale-independent. Over most of the dynamic range of interest, $M_{\text{collapse}} \propto \rho_{\text{crit}} \lambda^{3/2} \delta \lambda^{1/2}$, with gas pressure providing the dominant support on scales $\lesssim h_0$, so $\delta \propto \lambda^{-1}$ (see § 3). Combining these power-law approximations with the above, and dropping terms in the pre-factor that are slowly varying in $\ln \lambda$ (such as the $S^{3/2}$ term), we obtain

$$M \frac{dn}{d\ln M} = M^2 \frac{dn}{dM} \times f(\ln M) M^{q/2} \tag{39}$$

$$q \sim \frac{\ln \delta_\rho}{4 S_0} + \left(\frac{S_0}{16}\right) \left(\frac{1}{4}\right) \sim 1.1$$

$$\sim \frac{1}{1 - 0.2 \ln(\alpha/10^{-2})} \tag{41}$$

where the latter equality comes from noting that the $(S_0/16 - 1/4)$ term is small for all $\tau_s \sim 1$ of interest, and evaluating $\delta_\rho$ as in § 3 for $\tau_s \sim 1/3$ (the approximate threshold where we see the MF rise, though variations $\tau_s \sim 1/3 - 3$ have weak effects here).

Qualitatively, we can understand this as follows. Since the density fluctuations are approximately scale-free over some range, if the “collapse threshold” were also scale-free, then the entire system would be scale-free and we would expect self-similar structure, or $q \approx 0$ (equal mass over each logarithmic interval in mass). This is a very generic consequence of e.g. supersonic gas turbulence (Hopkins 2012b). However, the threshold is not scale free; collapse is “more difficult” (requires larger $\delta_\rho$) on small scales, so the MF is biased towards higher-mass objects (larger-scale fluctuations). To leading order, a threshold which grows “steeply” below $h_0$ leads to $q \sim 1$; the logarithmic correction for $\alpha$ reflect the fact that as $\alpha$ is lowered, collapse on small scales becomes “easier” (for the reasons discussed above), so the MF is less biased towards higher-masses.

4.2.5 Dependence on Metallicity & Disk Densities

Fig. 5 repeats our MF calculation, this time varying the nebula properties (surface density $\Sigma_0$ and metallicity $Z_0$). At otherwise fixed conditions, increasing the metallicity does not have much effect on the predicted mass function (as expected from our simple derivation above). However, it does weaken the range of orbital radii where pebble piles can form at all – we discuss this further below, but note that the effect is only logarithmic in the metallicity.

Varying the disk surface density – with otherwise fixed properties – has a more dramatic effect on pebble pile formation. Once again though, most of this effect is in controlling whether piles form at all, not changing the mass function when they do form. Increasing $\Sigma_0$ does (weakly) shift the maximum in the MF to higher masses, in line with our expectation for $M_{\text{max}}^{\text{collapse}}$.

4.3 General Conditions for Collapse

Our numerical calculation allows us to map the parameter space in which dynamical grain collapse can occur. As we noted above, the solutions are essentially Boolean: depending on the parameters of a given disk, either pebble pile formation is common, or it is exceptionally rare/impossible. Therefore we treat this as a binary process and ask under which parameters we recover an interesting probability of pebble pile formation. At a given radius, for a solar-type star, the key parameters are the grain size $R_0$ and the disk parameters: $\alpha$, $\Sigma_0$, and $Z_0$.

Figs. 6-8 map the minimum grain size $R_3$ needed for the formation of collapsing pebble piles, as a function of $\alpha$, $\Sigma_0$, and $Z_0$, at

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8 In Hopkins 2013a, we derive this for more generic random fields, and include a detailed discussion of the accuracy of the approximation for different collapse thresholds and statistics. For our purposes here, it is adequate to approximate the slope of the MF over regions where it is locally power-law like.

9 Formally we define “an interesting probability of pebble pile formation” as a mean predicted $\langle N(> M) \rangle > 1$ in the mass function, integrated down to a mass $M_{\text{max}} = 10^{-8} M_0$. But because the scaling of the predicted MF is super-exponential in the important quantities above, modest changes to the exact threshold we choose makes only a tiny (sub-logarithmic) difference to our calculation.
Figure 6. Minimum grain size needed for pebble-pile formation, as a function of orbital distance from a solar-type star and disk surface density. Distance is in au, and surface density $\Sigma_0 \equiv \Sigma(r)/(1000\rho_m^{3/2})$ is the density normalized to the MMSN. In all cases we take $Z_d = Z_\odot$ and $\alpha = 10^{-4}$. Color encodes the minimum grain size above which formation and collapse of pebble pile planetesimals will occur, increasing from red-green-blue (lines show the contours for specific values of $R_{d,\,cm} = 0.1, 1, 10, 30$). Dotted lines of the corresponding color show our simple analytic threshold estimate for the same grain size. In the MMSN ($\log \Sigma_0 = 0$), small grains with $\gtrsim 1 \text{ cm}$ ($0.1 \text{ cm}$) can form pebble piles at $r \gtrsim 30 \text{ au}$ ($\gtrsim 100 \text{ au}$), but large $\sim 10 - 30 \text{ cm}$ “boulders” are required to trigger the process at $\sim 1 - 3 \text{ au}$. However, the process is strongly sensitive to surface density, and lower density disks will, at the same $R_{d,\,cm}$, form pebble piles more easily. At $\Sigma \sim 0.1 \Sigma_{\text{MMSN}}$, $\sim 1 \text{ cm}$ grains can trigger pile formation at $\sim 3 \text{ au}$.

4.3.1 Dependence on Disk Densities: Lower-Density Disks Promote Collapse

Given this, we see that at fixed $r_{\text{au}}$, varying the disk surface density – with otherwise fixed properties – has a dramatic effect on pebble pile formation. First recall that since $\langle \rho_b \rangle_{\text{eff}} = \langle \rho_b \rangle_{\text{eff}} H_L \Omega$, $\tau \propto R_{d,\,cm}/\Sigma_{\text{eff}}(r)$ depends only on the grain size and disk surface density for any equilibrium disk. Combining that with the simple analytic criterion on $\tau$, we derived above for large fluctuations, we require a minimum $R_{d,\,cm} \propto \Sigma_{\text{eff}}(r)$ for pebble pile formation (or in more detail, $R_{d,\,cm} \gtrsim 100 \psi(Q, Z_d, \alpha)/(\Sigma_{\text{eff}}/1000 \text{ g cm}^{-2})$ where $\psi$ collects the logarithmic corrections; see § 5).

This means that for otherwise fixed grain sizes, lower surface density disks are more prone to pebble pile formation! Physically, if we keep $R_{d,\,cm}$ fixed and decrease $\Sigma, \tau$ increases. But the maximum amplitude of grain fluctuations then grows super-exponentially in $\tau$ (for $\tau \lesssim 1$, because the ability of grains to concentrate particles is very sensitive to this number, and there is a large “multiplier” effect from all turbulent eddies in the cascade; see Hogan & Cuzzi 2007; Bec et al. 2007). The threshold for a density fluctuation to collapse does increase also, but this scales only linearly $\propto Q \propto \Sigma^{-1}$. So the increased clustering “wins.”

Specifically, if we assume maximum sizes $R_{d,\,cm} \sim 1$, then pebble pile formation is only possible at $\gtrsim 30 \text{ au}$ in a MMSN, but this radius moves in to $\gtrsim 3 \text{ au}$ in a $\Sigma_0 = 0.1 \text{ disk} (10x$ lower-density), and $\gtrsim 1 \text{ au}$ in a $\Sigma_0 = 0.01 \text{ disk}$.

Such low-density disks may be very common. Andrews et al. (2013) recently compiled a large sample of protoplanetary disks; they found $M_{\text{disk}} \propto M_\star$, with a median disk-to-stellar mass ratio of $\approx 0.003$; for the MMSN profile out to $\sim 100 \text{ au}$, this would give...
$\Sigma_0 \sim 0.2$; these are consistent with direct measurements of surface density profiles at large radii (Isella et al. 2009). So at least $\sim 50\%$ of disks may be in this regime! If we interpret some of the observational scatter in $M_{\text{disk}}/M_*$ or $\Sigma_0$ as an evolutionary effect, then most disks must spend a significant fraction of their lifetime in this lower-density state — more than sufficient for pebble pile formation to occur. Indeed, at some point, disks must evaporate, so all disks pass through such a phase — and because the collapse is dynamical (occurs on timescale $\sim \Omega^{-1}$), all disks should experience a phase where cm-sized grains have $\tau_s \sim 1$ even at small radii.

The question is whether such grains would cluster — the simulations modeling clustering can be freely scaled to this parameter space and show large-amplitude fluctuations (see Bai & Stone 2010a; Johansen et al. 2012; Dittrich et al. 2013; Jalali 2013). The question is whether such low-density disks could contain or support cm-sized grains. Some models suggest the maximum grain size scales $\propto M_{\text{disk}}$; so the existence of large-grains in a low-density disk would depend on their surviving from an earlier phase (which they can only do for the shorter of either the drift or shattering timescales). This question is outside the scope of this paper, but is of major importance for future study.

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4.3.2 Dependence on Metallicity: Higher-Metallicity Helps, But Only Weakly

As noted above, the metallicity $Z_d$ has a weak effect on the conditions where pebble piles can form. In agreement with the threshold we estimated $\tau_s \sim 0.05 \ln (Q/\zeta/\Omega H) \ln (1 + b_d)$, the minimum $\tau_s$ (hence minimum grain size) needed to trigger collapse decreases with increasing metallicity. But this dependence is only logarithmic; so for $R_\text{d} \sim 10 \text{ cm}$ the range of pebble-pile forming radii in e.g. a $\Sigma_0 \sim 0.1 \text{ disk shrinks from } \sim 0.05 \text{ to } 6 \text{ au when } Z_d \sim 20 Z_\odot \text{ to } \sim 0.2 \text{ to } 3 \text{ au when } Z_d \sim 1 Z_\odot \text{ and } \sim 0.3 \text{ to } 3 \text{ au when } Z_d \sim 0.1 Z_\odot$. For a higher-density disk the effects are slightly weaker; for a lower-density disk ($\Sigma_0 \sim 1$), pebble pile formation ceases even for large grains below $Z_d \lesssim 0.1 Z_\odot$.

4.3.3 Dependence on Turbulent $\alpha$

We can also examine the dependence on the turbulent $\alpha$ parameter. Higher-$\alpha$ increases the clustering amplitude of grains, because it implies a larger dynamic range of the turbulent cascade; but the effect is weak because so long as any eddies exist with $t_e \sim \Omega^{-1}$, the “added” dynamic range is outside the resonant range. Lower-$\alpha$ im-
plies a more-dense grain disk, hence a lower threshold for pebble pile formation \((\propto Q\sqrt{\tau})\); this enters logarithmically in the critical \(\tau\). Together, these effects mean that the net dependence of the minimum grain size on \(\alpha\) is quite weak.

However, we stress that some of this weak dependence stems from the assumption in our model that the characteristic timescale of large eddies is \(\sim \Omega^{-1}\). Depending on the details of the mechanism driving the turbulence, long-lived “zonal flows” with coherence time \(\gg \Omega^{-1}\) can form (see Dittrich et al. 2013). As shown in Paper I, these can individually strongly alter the local grain clustering (see Fig. 9 therein).

5 DISCUSSION

We use the recently-developed analytic model from Paper I, which describes the statistics of grain density fluctuations in a turbulent proto-planetary disk, to calculate the rate and probability of formation of “pebble-pile” planetesimals – self-gravitating collections of (relatively large) grains, which should collapse rapidly (on a dynamical timescale) into >km-size planetesimals. The analytic model provides an excellent fit to the grain density statistics independent of the source of the turbulence – so this model is equally applicable to regions where the streaming instability is active (Youdin & Goodman 2005), or where the turbulence is driven by the MRI (Dittrich et al. 2013), or simply shear and/or Kelvin–Helmholtz instabilities (Bai & Stone 2010c; Jalali 2013). The key differences between these sources of turbulence enter our calculations as the parameters describing turbulence (e.g. the traditional \(\alpha\)-parameter).

5.1 Dynamical Collapse is Possible for Large Grains:

- The most important parameter determining the collapse of grains is the ratio of stopping to orbital time, \(\tau_{\text{p}} \equiv t_{\text{s}}/\Omega\). Large grain density fluctuations occur on large scales in the disk when \(\tau_{\text{p}} \sim 1\). We derive the criterion for the largest of these fluctuations to overcome tidal/centrifugal/coriolis forces, shear, gas pressure, and turbulent kinetic energy. This will occur when

\[
\tau_{\text{p}} \gtrsim 0.05 \frac{\ln(Q^{1/2}/Z_0)}{\ln(1+b_d)}
\]

which we can write as

\[
\tau_{\text{p}} \approx 0.004 \frac{R_d}{1000 \text{g cm}^{-2}} \left(\frac{\Sigma_{\text{gas}}(r)}{1000 \text{g cm}^{-2}}\right)^{-1} \gtrsim 0.4 \psi
\]

\[
\psi \equiv \frac{1 + 0.08 \ln\left[\frac{(Q(r)/60)(Z_d/Z_0)}{1 + (\alpha/10^{-4})^{1/4}}\right]}{\ln\left[1 + (\alpha/10^{-4})^{1/4}\right]} \sim 1
\]

or

\[
R_d,cm \gtrsim 100 \psi \left(\frac{\Sigma_{\text{gas}}(r)}{1000 \text{g cm}^{-2}}\right)
\]

For a MMSN with plausible turbulent \(\alpha\) values, this criterion translates to very large “boulders” with \(R_d \gtrsim 30\) cm at 1 au; but more plausible “large grains” or “pebbles” with \(R_d \sim 1\) cm at \(\sim 30\) au (or \(\sim 1\) mm at \(\sim 100\) au). For the MMSN regime, which is well-sampled by simulations, this analytically-calculated threshold is in excellent agreement with the results of full numerical simulations (see e.g. Bai & Stone 2010a; Johansen et al. 2012).

- Dynamical Collapse is Not Possible for Small Grains:

Small grains also cluster strongly – in fact they can, under the right circumstances, cluster just as strongly as large grains (see Squires & Eaton 1991; Cuzzi et al. 2001; Pan et al. 2011). However, this clustering occurs on very small scales, where \(t_s \sim t_c\) (the small-scale eddy turnover time). On these scales, even if we ignore gas drag, the local turbulent velocity dispersion (induced by the same eddies that generate the density fluctuations) means that the grain free-fall time must be shorter than the stopping time in order for dynamical collapse to proceed \((G\rho_d \gtrsim t_c^{-1})\). For small grains, this requires an enormous overdensity which is not achieved in any calculations. However, we stress that this conclusion applies only to dynamical (not secular) grain collapse.

- Lower-Surface Density Disks Are More Prone to Grain-Pile Collapse!

Lower-surface density disks are “more stable” in the Toomre sense, and require larger relative overdensities to overcome the Roche and other criteria and collapse. However, the parameter \(\tau_{\text{p}} \propto R_d/\Sigma_{\text{gas}}\) is inversely proportional to the disk surface density, and the relative magnitude of the maximum grain density fluctuations scales super-exponentially with \(\tau_{\text{p}}\) (for \(\tau_{\text{p}} \lesssim 1\)). So for reasonable densities \(\Sigma_{\text{gas}} \sim 0.01 - 1\), the enhanced grain clustering “wins,” and the minimum grain size needed for fluctuations decreases with \(\Sigma_{\text{gas}}\) (although the maximum planetesimal size will also decrease).

For a disk which begins as a MMSN at \(\sim 1 - 3\) au, if the maximum grain size can reach \(\sim 5\) cm, then the grains are too-well coupled to collapse “initially.” But, as the gaseous disk is eventually dissipated, when more than \(\sim 90\%\) of its mass has been removed, then the grains will suddenly cross the threshold above, and the density fluctuations will increase super-exponentially until collapse occurs. The key question is whether such large grains could survive or still be newly-made at this very late stage in proto-planetary disk evolution.

- We Predict a General “Initial Mass Function” of Planetesimals:

When this instability occurs, it leads to a mass function of collapsing grain overdensities with a quasi-universal form, which we approximate as a power-law with a lognormal-like cutoff above/below some maximum/minimum mass:

\[
\frac{dN}{dM} \propto M^{q-2} \exp\left(-\ln\left[1 + \frac{M}{M_{\text{max}}} + \frac{M_{\text{min}}}{M}\right]\right)
\]

\[
q \approx \frac{1.1}{1 - 0.2 \ln(\alpha/0.01)} \sim 1
\]

\[
M_{\text{max}} \sim 0.03 \left(\frac{\alpha}{10^{-4}}\right)^{1/4} \Sigma_{\text{min}}^{1/2} r_{\text{au}}^{27/28} M_\odot
\]

\[
M_{\text{min}} \sim \alpha M_{\text{max}}
\]

Since \(q \sim 1 > 0\), this means that most of the mass in the new collapsing planetesimals is in the largest objects, with mass \(\sim M_{\text{max}}\).

- Direct-Collapse to Earth Masses is Possible:

This characteristic maximum mass increases with disk surface density and distance from the star (approximately linearly), in the same qualitative manner as a Jeans mass, although they are not identical. At sufficiently large radii in dense disks – e.g. \(r_{\text{au}} \gtrsim 30 - 100\) au in a MMSN, direct collapse to Earth and super-Earth masses becomes possible! Super-earth masses appear to constitute the maximum masses that can be achieved under realistic circumstances.

As the turbulence becomes weaker (\(\alpha\) decreases), the characteristic mass decreases as well, and the mass function becomes more concentrated towards the low-mass end. These lower masses are still more than large enough to provide self-gravitating, >km-size planetesimal seeds. However, capturing this low mass behavior is potentially a problem for direct numerical simulations, given both the small mass and size resolution required to capture the relevant scales.

- High Metallicities Are Not Required:

Very large local gas-to-dust ratios \(Z_d \sim 1\) are required for the streaming instability to
grow and self-excite turbulence (Youdin & Goodman 2005). This has often been incorrectly interpreted to mean that large metallicities are required for any large grain density fluctuations (and has led to a large body of work studying how regions with order-of-magnitude “enhanced” metallicities may form). But that is only true if there is nothing else, other than the streaming instability, to provide a source of turbulence (Bai & Stone 2010c). Laboratory experiments (Squires & Eaton 1991; Rouson & Eaton 2001; Gualtieri et al. 2009; Monchaux et al. 2010; Monchaux et al. 2012), simulations (Hogan et al. 1999; Yoshimoto & Goto 2007; Carballido et al. 2008b; Pan et al. 2011; Dittrich et al. 2013), and analytic calculations (Sigurdsson & Stuart 2002; Bec et al. 2008; Zaichik & Alipchenkov 2009; Hopkins 2013b) actually all find that very large grain density fluctuations occur even when $Z_d = 0$ (i.e. there is zero back-reaction of grains on gas), *provided there is some external source of turbulence.* This may come from the MRI, from Kelvin-Helmholtz or shear instabilities, from gravito-turbulent instabilities if the disk is sufficiently massive, or other objects in the disk.

Of course (all else being equal), dynamical collapse of pebble piles is easier if the “initial” dust-to-gas ratio is larger, since smaller density fluctuations are required. However, we again emphasize that since the fluctuation amplitudes can be very large under the right conditions, the “threshold” for sufficiently large fluctuations depends only weakly (logarithmically) on $Z_d$, and can very well occur for at solar metallicities.

### 5.2 Future Work

This is only a first attempt at combining an analytic model for grain density fluctuations with simple criteria for self-gravity, to estimate the conditions for pebble-pile planetesimal formation. As such we have made a number of approximations, which can be improved in future work. We have, for example, considered only monolithic, collisionless grain populations: the grain clustering statistics can be modified by non-linear interactions between grains of different sizes (see Bai & Stone 2010a), or by collisions between grains during the collapse process (Johansen et al. 2012). These simulations suggest the effects are not enough to qualitatively change our conclusions, but can quantitatively make collapse easier or more difficult depending on the exact parameters. Ultimately, direct numerical simulations should be used to refine and quantitatively improve the predictions here, incorporating these and other high-order effects (such as intermittency in turbulence, and local correlation structures between vorticity and grain densities). However, it remains extremely challenging to resolve the Reynolds numbers required to accurately follow the grain dynamics. Eventually though, our hope is that the models here will provide an analytic framework to interpret and generalize these results.

Probably the biggest approximation we make is the interpolation between the tightly-coupling and loosely-coupled cases, i.e. the regime where grain-gas back reaction is important. This is discussed in Paper I, but the exact structure of turbulence in this regime is uncertain, and should be further explored.

Finally, the model here allows us to identify “candidate regions” for the formation of self-gravitating pebble pile planetesimals: regions which accumulate sufficient grain density to be self-gravitating, linearly unstable, and simultaneously exceed the Roche, Jeans, and Toomre criteria. However, we do not attempt to follow the non-linear evolution of these regions. Simulating in detail the collapse of these pebble piles would be extremely interesting: it is not obvious how the grains will stick or shatter as they collapse (which may modify subsequent collapse). A single region may fragment into a sub-spectrum of masses: what we identify here is an upper limit (the “parent region” mass, not necessarily the mass of a single solid object that will form from the above). Many questions need to be explored to fully link this to planet formation.

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APPENDIX A: GRAIN VELOCITY DISPERsions

Various authors have modeled the statistics of grain velocity dispersions in gas turbulence (see Voelk et al. 1980; Markiewicz et al. 1991; Ormel & Cuzzi 2007; Pan & Padoan 2013). We follow these works to derive an approximate expression for the grain-grain velocity dispersion \( \langle v_{ij}^2 \rangle \equiv \alpha \cdot g_i(\lambda_i) \equiv \langle v_{i}^2(k \rightarrow 0) \rangle g_i(\lambda_i) \).

First consider the contribution of eddies larger than the \( \lambda_i = k_i^{-1} \) of interest. Voelk et al. (1980) argue that the relative velocity dispersion induced on grains with separation \( \lambda_i \), by eddies with size scale \( \lambda > \lambda_i \), can be written

\[
\langle \Delta V_{i \lambda}^2 \rangle = \frac{1}{2} \left( V_{\rho i}^2 + V_{\rho j}^2 - 2 V_{\rho i} V_{\rho j} \right) = V_{\rho}^2 - V_{c}^2
\]

where \( V_{\rho} \) is the inertial-space rms velocity to which all particles are accelerated, and the \( V_{\rho i} V_{\rho j} \) term is the “cross term” – the component of the velocity imparted on the grains which is coherent across the scale (since well-coupled grains in large eddies may be accelerated to large absolute velocities by those eddies, but the relative velocity between grains on small scales will be small). The second simplification comes from our adopting a mono-population of grains (so \( V_{\rho i}^2 = V_{\rho j}^2 = V_{\rho}^2 \)). From Ormel & Cuzzi (2007), we have

\[
V_{\rho i}^2 \approx \int_{K_i}^{K} \text{d}k E(k) \left( 1 - K^2 \right)
\]

\[
V_{\rho j}^2 = \int_{K_i}^{K} \text{d}k E(k) \phi(k, k^*) \left( 1 - K^2 \right)
\]

with \( K \equiv 1/(1 + t_i/t) \), where \( t_i \equiv t_0(k) \). The \( K^2 \) term in the first integral comes from the “n=1” gas velocity autocorrelation function used in Markiewicz et al. (1991) and Ormel & Cuzzi (2007).

Here \( k^* \) is the boundary between “Class I” eddies (where particles are trapped) and “Class II” eddies (where eddies decay before providing more than small perturbations to the particle); formally \( k^* \) is defined by \( t_i = t_0^* + k^* V_{\rho(a)}(k^*) \) (Voelk et al. 1980). The function \( \phi \) is any function which interpolates between 1 for eddies with \( k < k^* \) and 0 for eddies with \( k > k^* \). Voelk et al. (1980) approximate this with a step function at \( k = k^* \); for numerical convenience and slightly improved accuracy, we adopt the simple linear interpolation \( \phi = t_i/(t_i + t_i^*) \). We have checked, though, that the difference between this choice and a step function is negligible in our calculations in the text.

More generally, we can use

\[
V_{\rho i}^2 = \int_{K_i}^{\text{max}(k^*, K_i)} \text{d}k E(k) \left( 1 - K^2 \right)
\]

\[
+ \int_{\text{max}(k^*, K_i)}^{k_i} \text{d}k E(k) \left( 1 - K^2 \right) [g(\chi) + K h(\chi)]
\]

where \( g(\chi) = \chi^{-1} \tan^{-1}(\chi) \) and \( h(\chi) = 1/(1 + \chi^2) \) with \( \chi = K_{i,k} V_{\rho(a)}(k) \). \( V_{\rho(a)}(k) = \int_{K_i}^{k} \text{d}k E(k) K^2 \) from Voelk et al. (1980), which must be solved numerically. However as shown in Ormel & Cuzzi (2007), the approximate expression above (which assumes \( h(\chi) \approx g(\chi) \approx 1 \)) introduces a negligible error for all particle sizes of interest.
Combining these approximations we have

$$\langle \Delta V^2_{\lambda<\lambda_k} \rangle \approx \int_{t_k}^{t_h} \frac{dk}{k} \cdot 2E(k) \left(1 - K^2 \right) \left(1 + \frac{1}{1 + \frac{t_k}{t_h}} \right) \tag{A5}$$

At finite scale $\lambda > 0$, we also need to consider the contribution to grain motion from eddies with smaller sizes. As in the derivation of the above relations, we assume that eddy structure on successive scales is uncorrelated. Thus, the contribution from eddies with $\lambda < \lambda_h$ is just

$$\langle \Delta V^2_{\lambda<\lambda_h} \rangle = \lambda^2 \langle \lambda < \lambda_h \rangle \equiv \int_{t_k}^{t_h} \frac{dk}{k} \cdot 2E(k) \left(1 - K^2 \right) \tag{A6}$$
i.e. eddies with internal scale $\lambda < \lambda_h$ do not contribute to the coherent component $V_c$ on scales $\lambda \geq \lambda_h$. For the cases we study here, we can also take the Kolmogorov scale $k_h \to \infty$ with negligible error. Thus we obtain

$$\Delta V^2(k) = \int_{t_k}^{t_h} \frac{dk}{k} \cdot 2E(k) \left(1 - K^2 \right) \left(1 + \frac{1}{1 + \frac{t}{t_h}} \right)$$

Determining $k^*$ is, in general, non-trivial, but Ormel &uzzi (2007) note that $t_h$ can be well-approximated by $t_h \approx \text{MIN}(\phi_h / t_k)$ ($\phi^c = (1 + \sqrt{5})/2$ or $t^c_k \sim (8t_k/5)^{1/2} + t_k$).

The upper limit $k_1$ here represents the driving scale. In Ormel &uzzi (2007), this is taken as a fixed value, with $E(k)$ a pure power-law ($\propto k^{-5/3}$) for $k > k_1$, so $t_1 = t_1(k/k_1)^{-2/3}$. In this case, using the definition $\int_{t_k}^{t} \frac{dk}{k} \cdot 2E(k) \left(1 - K^2 \right) = \alpha c^2/2$, we obtain

$$\frac{\Delta V^2(\lambda)}{\alpha c^2} = \mathcal{g}_\epsilon(\lambda / \lambda_{\max}) \equiv \frac{\left(\frac{1}{1 + \frac{1}{\lambda}} \right)}{\left(\frac{1 + \frac{1}{\lambda}}{1 + \frac{1}{\epsilon}} \right)}$$

$$\phi(\lambda / \lambda_{\max}) \equiv \frac{\left(\frac{\epsilon / \lambda + 1}{\epsilon / \lambda - 1} \right) \ln \left(\frac{1 + \frac{1}{\lambda}}{1 + \frac{1}{\epsilon}} \right)}{\left(\frac{\epsilon / \lambda + 1}{\epsilon / \lambda - 1} \right) \ln \left(\frac{1 + \frac{1}{\lambda}}{1 + \frac{1}{\epsilon}} \right)}$$

where $\lambda_1 \equiv t_k / t_h$ and $\lambda_2 \equiv t_1 / t_h = \lambda / (\lambda / \lambda_{\max})$. This is a tedious expression, but its relevant scalings are clear if we approximate $\phi(k, x)$ as a step function and $t_x \sim \text{MIN}(\phi(t_x / t_h))$; then

$$\frac{\Delta V^2(\lambda)}{\alpha c^2} = \mathcal{g}_\epsilon(\lambda / \lambda_{\max}) \equiv \frac{\left(\frac{\lambda}{\epsilon / \lambda} \right)^2}{1 + \frac{1}{\lambda}} \tag{A9}$$

$$\phi(\lambda / \lambda_{\max}) \equiv \frac{\left(\frac{\epsilon / \lambda + 1}{\epsilon / \lambda - 1} \right) \ln \left(\frac{1 + \frac{1}{\lambda}}{1 + \frac{1}{\epsilon}} \right)}{\left(\frac{\epsilon / \lambda + 1}{\epsilon / \lambda - 1} \right) \ln \left(\frac{1 + \frac{1}{\lambda}}{1 + \frac{1}{\epsilon}} \right)}$$

For $t \ll t_1$, $t \ll t_2$, this scales as $\frac{\lambda}{\lambda_{\max}} = (\lambda / \lambda_{\max})^{2/3}$, i.e. $\psi^c(\lambda)$ (or $\psi^c(\lambda)$) in the grain and gas velocities are well-coupled. But on sufficiently small scales where $t \gg t_1$, this goes to the constant ($\lambda$-independent) value $\psi^{-1} = t_1 / t_h$ (the turbulent dispersion imparted by eddies with $t_1 \sim t_1$).

As noted in the text we can more accurately include the driving-range ($\lambda > \lambda_{max}$) using a full expression for $E(k)$, and taking $k_1 \to \infty$. In this case $\Delta V^2(\lambda)$ can only be evaluated numerically. However, motivated by the form for the turnover of $E(k)$ at $k \gg k_1$, we can approximate the full numerical solution at all $\lambda$ by simply inserting

$$\lambda_1 = \frac{t_0 (\lambda / \lambda_{\max})}{t_h}$$

$$\lambda_2 = \frac{t_0 (\lambda / \lambda_{\max})}{t_h}$$

$$\lambda \to \lambda_{\max}^{2/3} \left(1 + (\lambda / \lambda_{\max})^{2/3} \right)^{1/3}$$

into the expressions above (derived for a sharp cutoff at $\lambda_{\max}$). This approximation is accurate to $\sim 10\%$, well within the range of uncertainties in our earlier approximations.

**APPENDIX B: STABILITY CONDITIONS FOR A PARTIALLY-COUPLED GRAIN-GAS FLUID**

Here we briefly describe an alternative derivation of a gravitational collapse criterion for grains in a thin disk. In cylindrical coordinates, the continuity and Euler equations take the form

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} (R \Sigma v_R) + \frac{1}{\rho} \frac{\partial}{\partial \phi} (\Sigma v_\phi) = 0$$

$$\frac{\partial v_R}{\partial t} + \frac{v_R}{R} \frac{\partial}{\partial R} + \frac{v_\phi}{\rho} \frac{\partial}{\partial \phi} - \frac{\partial \Phi}{\partial R} = - \frac{\partial \Phi}{\partial R} - v_R \frac{\partial \Phi}{\partial \phi}$$

$$\frac{\partial v_\phi}{\partial t} + \frac{v_R}{R} \frac{\partial}{\partial R} + \frac{v_\phi}{\rho} \frac{\partial}{\partial \phi} + \frac{\partial \Phi}{\partial \phi} = - \frac{\partial \Phi}{\partial \phi} - v_\phi \frac{\partial \Phi}{\partial \phi}$$

Now assume we perturb a form of $\Sigma_\alpha \propto v_\phi \propto v_R \propto \exp (i/m \phi + kR - \omega t)$ (to the background equilibrium solution $\Sigma_0, v_\phi, v_R$, etc.) and linearize the above equations. We also invoke the WKB (local) approximation for the perturbation potential $\Phi_l \approx -2\pi G \langle k \rangle^{-1} \Sigma_1$; however, for now we retain all terms in the “unperturbed” background flow (i.e. retain all terms to $O(\langle k \rangle^{-1})$). After some lengthy algebra we obtain the dispersion relation:

$$\left(\omega^2 + G_0 k + i \omega \left[ \frac{1}{t_h} - \frac{v_\phi}{\rho} \right] \right) \times$$

$$\left(\omega^2 + G_0 m^2 k + i \omega \left[ \frac{1}{t_h} - \frac{v_\phi}{\rho} \right] \right) =$$

$$\left[ \frac{v_\phi}{\rho} \omega + 2 \frac{v_R}{R} \right] \left(1 + \eta_\Sigma \right) \left[ \frac{v_\phi}{\rho} \omega - i G_0 m^2 k \right]$$

$$G_0 \equiv 2\pi G \Sigma_0 \text{SIGN}(k)$$

$$\eta_\Sigma \equiv \frac{\partial \ln X}{\partial \ln R}$$

$$\kappa \equiv \frac{m}{R}$$

$$\omega \equiv \omega + \frac{m}{\rho} \left(1 + \eta_\Sigma \right) \left( \frac{v_\phi}{\rho} \omega - i G_0 m^2 k \right)$$

This forms a quartic equation for $\omega$, but one solution is the trivial $\omega = 0$, so there are three interesting solution branches. Since the interesting parameter space is $t_h \sim \Omega^{-1}$, and the drift velocity is $\sim \eta_\Sigma \propto \eta \propto 1$, we can drop the higher-order terms in the drift, and restrict to purely radial modes, to simplify this substantially with negligible effect on the character of the solution. This gives

$$\Phi^2 \tau_t^2 + 2 \Phi \tau_t \Omega - \Phi - (1 - \tau_t \rho \tau_t) \Omega = 0$$

where $\Phi \equiv \omega / \Omega$ and $\rho \tau_t \equiv (2\pi G \Sigma_0 \langle k \rangle^{-2} c^2 \langle k \rangle^2 \Omega^{-2}$. $\Phi^2 \tau_t^2 + 2 \Phi \tau_t \Omega - \Phi - (1 - \tau_t \rho \tau_t) \Omega = 0$

First note that if $\rho \tau_t \leq 0$, then all solutions for $\Phi$ have imaginary part $\text{Im}(\Phi) \leq 0$, i.e. are decaying or stable – there can be no instability. However, if $\rho \tau_t \geq 0$, there is always a growing mode. If $0 \ll \rho \tau_t \ll 1$, this mode has $\Phi = i \rho \tau_t \Omega$, so grows on a timescale $\gamma^{-1} = 1/(\rho \tau_t \Omega)$, $\gg \tau_t^{-1}$. This is the “secular” sedimentation instability, which grows very slowly. While this may be an important channel for grain growth or object formation, it is not the focus of this paper (and there are serious difficulties involved in collapse of objects on much longer than the disk dynamical time, which must be considered over much longer evolutionary timescales as opposed to the simple “threshold model” we consider here). Therefore we do not consider that limit here, but will examine it explicitly in a future paper.
On the other hand, when $\rho_R \gtrsim 1$, then we obtain $\text{Im}(\Theta) = \rho_R^{1/2} - 1/(2\tau_s)$. So growth on the dynamical timescale requires $\rho_R > 1/(2\tau_s)$, i.e.

$$0 > \Omega^2 \text{MAX}[1, (2\tau_s)^{-1}] + \frac{\beta}{\rho} c_i^2 k^2 - 2\pi G \Sigma_0 |k| \tag{B11}$$

So, for the $\tau_s \sim 1$ of interest, can take $\text{MAX}[1, (2\tau_s)^{-1}] \sim 1$ and arrive at the dispersion relation used in the text, up to the turbulent terms.

It is trivial to see that this satisfies the traditional Toomre, Roche, and Jeans criteria simultaneously. Shear (even the fully non-linear terms) forces are overcome when $\rho > \rho_{\text{Roche}}$, and gas pressure and angular momentum are explicitly included. A velocity dispersion term can be added using the approximate methods in Chandrasekhar (1951); Vandervoort (1970); Bonazzola et al. (1987); but to leading order in any of these approaches this is identical to the addition of the $v_i(k)$ term in the same manner as $c_i$, as in the text. In any case because we are interested in the limit where $v_i \ll c_i$, its contribution at the scales of interest is not a significant uncertainty.

As noted in Cuzzi et al. (2008), another non-linear term which can suppress collapse is ram pressure from the “headwind” encountered by a grain group as it moves through the disk. The relevant criterion for whether the pebble-pile can resist instability in the ram pressure shredding the distribution is the Weber number, the ratio of surface gravity (effectively, “surface tension” of the collapsing cloud) $G^{2/3} \Sigma_0$ to the ram pressure force per unit area ($\rho v_{\text{disk}}^2$, where $v_{\text{disk}} = f(\tau_s) \eta \nu \sim (\tau_s/(1 + \tau_s^2)) (c_i/v_k)^2 v_k$) is easily calculated following Nakagawa et al. (1986). At a radius $r_s$ in the disk, with Keplerian velocity $v_k$, this is satisfied for all $\tau_s$ if $\Sigma_0 \gtrsim (c_i/v_k)^2 f(\tau_s) Q^{-1/2} \Omega^2 r_s$. But it is straightforward to verify that this is automatically satisfied if Eq. B11 is already satisfied.

APPENDIX C: ACCOUNTING FOR TURBULENT VELOCITY FLUCTUATIONS DURING COLLAPSE

We now present a derivation of the role of turbulent velocity fluctuations in dynamical collapse, which is simplified but accounts for the fully non-linear turbulent fluctuations (not just their rms value) during collapse.

First assume a grain overdensity exceeds the criterion above – i.e. can collapse dynamically despite shear and gas pressure effects. In order to avoid being doomed, it must survive for a time $t_f = t_{\text{collapse}} \approx (\rho_R^{1/2} \Omega)^{-1}$, without encountering a turbulent gas structure or eddy which induces a shear velocity $> v_{\text{max}} \sim v_{\text{collapse}} = k^{-1} t_{\text{collapse}}$. This is always less than the “escape velocity,” since that is defined by free-fall from infinite distance; but it is still sufficient to “reset” collapse. Define time and velocity in units of the rms eddy turnover time and velocity dispersion (for the scale of interest): $\tau \equiv t/(t_{\text{collapse}})$ and $x \equiv v/(v_{\text{runout}})^{1/2}$. Let $t_f/(t_{\text{collapse}}) = B = v_{\text{max}}/(v_{\text{runout}})^{1/2}$. Moreover, recall that $(t_{\text{collapse}}) = \lambda/(v_{\text{runout}})^{1/2}$ (where $\lambda \equiv k^{-1}$), and, for $v_{\text{max}} = \lambda/(c_i t_{\text{collapse}}), \tau_f = 1/B$.

In fully-developed turbulence, to lowest order, the distribution of one-dimensional velocities ($v_i, v_j, v_k$) on a given scale is Gaussian

\[ P_0(x|S_0) = \frac{dP(<x|S_0)}{dx} = \frac{1}{\sqrt{2\pi} S_0} \exp \left( -\frac{x^2}{2S_0^2} \right) \tag{C1} \]

with variance $S_{0,v} = \langle v_{\text{runout}}^2(k) \rangle /3$, or in the units above, $S_0 \equiv S_{0,v} = 1/3$.

The correlation timescale for $x$ is $\approx t_c(k)$ – this is measured in experiments and simulations (Yakhot 2008; Pan & Scannapieco 2010; Konstandin et al. 2012), and often is, in fact, how $t_c(k)$ is defined. So to lowest order, we can think of the turbulent field as “refreshed” or “resampled” on a timescale $\Delta t \sim t_c(k)$ (or $\Delta t \sim 1$). For $\tau_f \gg 1$, this means we “draw” from the distribution in Eq. C1

\[ N \approx t_f / \Delta t = t_f \text{ times. We require, for each draw, that } |x| < B, \text{ which has probability } P(|x| < B) = \text{erf}(B\sqrt{3/2}) \].

The probability that all draws are successful is $P(|x| < B \cap x < \tau_f \sim \text{erf}(B\sqrt{3/2})^\tau_f / \tau_f$. Finally, we note that this was just for one velocity component; we must consider each of three components independently. This gives the probability of survival

\[ P(|x| < B \cap x < \tau_f) \sim \text{erf}(B\sqrt{3/2})^\tau_f / \tau_f \]

This is an extremely steep function of $B$ for $B < 1$, approximately $\approx \exp[3B^{-1} (\ln B + (1/2) \ln(6/\pi))]$, and $P \ll 1$ for $B < 1$. So we do not expect this to be “common” for small $B$. Since turbulence is inherently stochastic process, we cannot deterministically say whether a given region will or will not encounter a large turbulent eddy during its collapse. Lacking that, we want our “collapse criterion” to identify regions where there is a large (order-units) probability of “successful” collapse (i.e. not encountering a too-large turbulent shear/vorticity). We therefore require $P > 0.5$ (i.e. the probability of survival is larger than that of disruption), which requires $B_{\text{min}} > 0.8$; this choice of $P$ is arbitrary but because it is a steep function of $B$, changing the “threshold” has weak effects on $B$ (at $B_{\text{min}} = 0.4, P \sim 0.5$), and $B_{\text{min}} = 1, P \sim 0.8$.

Now recall $v_{\text{max}} \approx v_{\text{collapse}} = k^{-1} t_{\text{collapse}} = k^{-1} \Omega^{1/2} r_s$, so this requirement becomes $(\rho_R \Omega)^2 > B_{\text{min}} (v_{\text{runout}})^2$, or

\[ 0 > \tau^2 + \frac{\beta}{\rho} c_i^2 k^2 + B_{\text{min}} (v_{\text{runout}})^2 - 2\pi G \Sigma_0 |k| \tag{C3} \]

Since $B_{\text{min}} \sim 1$ is somewhat uncertain, we simply adopt $B_{\text{min}} = 1$ in the text (corresponding to the linear derivation for a gas fluid in Chandrasekhar 1951); however, the difference between this and $B_{\text{min}} = 0.8$ is negligible for all of our results. We simply note that the choice ($B_{\text{min}} = 1$) in the text also applies to non-linear, fluctuating turbulent velocity fields during collapse, and corresponds to a probability $P > 0.8$ that the region will “successfully” collapse in the limit where turbulence is the dominant source of support (compared to rotation and shear).

\[ \text{11} \]

In the presence of intermittency, this is not exactly true; however, the effects on the second-order correlation function (which is what matters here) is very weak. We can, for example, repeat our derivation using the distribution function predicted by She & Leveque (1994), and find it gives only a $\sim 5\%$-level correction to our calculation.

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