Elliptic dihedral covers in dimension 2, geometry of sections of elliptic surfaces, and Zariski pairs for line-conic arrangements

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Introduction

In this article, all varieties are defined over the field of complex numbers, $\mathbb{C}$. Let $X$ and $Y$ be normal projective varieties. We call $X$ a dihedral cover of $Y$ if there exists a finite surjective morphism $\pi: X \to Y$ such that the induced field extension of the rational function fields $\mathbb{C}(X)/\mathbb{C}(Y)$ is a Galois extension whose Galois group is isomorphic to a dihedral group.

Let $D_{2n}$ be the dihedral group of order $2n$. In order to present $D_{2n}$, we use the notation

$$D_{2n} = \langle \sigma, \tau \mid \sigma^2 = \tau^n = (\sigma \tau)^2 = 1 \rangle.$$

By $D_{2n}$-covers, we mean a Galois cover whose Galois group is isomorphic to $D_{2n}$. Given a $D_{2n}$-cover, we obtain a double cover, $D(X/Y)$, canonically by considering the $\mathbb{C}(X)^\tau$-normalization of $Y$, where $\mathbb{C}(X)^\tau$ denotes the fixed field of the subgroup generated by $\tau$. We denote these covering morphisms by $\beta_1(\pi): D(X/Y) \to Y$ and $\beta_2(\pi): X \to D(X/Y)$, respectively.

In [19], we introduce a notion of an elliptic $D_{2n}$-cover, whose definition is as follows:

**Definition 0.1** A $D_{2n}$-cover $\pi: X \to Y$ is called an elliptic $D_{2n}$-cover if it satisfies the following condition:

- $D(X/Y)$ has a structure of an elliptic fiber space $\varphi: D(X/Y) \to S$ over a projective variety $S$ with a section $O: S \to D(X/Y)$.
- The covering transformation $\sigma_{\beta_1(\pi)}$ coincides with the inversion with respect to the group law on the generic fiber $D(X/Y)_n$. Here the group law on $D(X/Y)_n$ is given by regarding $O$ as the zero element.

In this article, as a continuation of [19], we study an elliptic $D_{2p}$-cover ($p$: odd prime) of a rational ruled surface $\Sigma_d$ ($d$: even). Our main results are Theorems 3.1 and 4.1. As an application, we study some Zariski pairs of degree 7 for line-conic arrangements. Let us recall the definition of a Zariski pair.

**Definition 0.2** A pair $(B_1, B_2)$ of reduced plane curves $B_i$ ($i = 1, 2$) of degree $n$ is called a Zariski pair of degree $n$ if it satisfies the following condition:

(i) $B_i$ ($i = 1, 2$) are curves of degree $n$. The combinatorial type (see Definition 0.3 below) of $B_1$ is the same as that of $B_2$

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(ii) \((\mathbb{P}^2, B_1)\) is not homeomorphic to \((\mathbb{P}^2, B_2)\).

**Definition 0.3** \((\text{II})\) The **combinatorial type** of a curve \(B\) is given by a 7-tuple

\[
(Irr(B), \deg, \Sing(B), \Sigma_{\top}(B), \sigma_{\top}, \{B(P)\}_{P \in \Sing(B)}, \{\beta_P\}_{P \in \Sing(B)}),
\]

where:

- \(\text{Irr}(B)\) is the set of irreducible components of \(B\) and \(\deg : \text{Irr}(B) \to \mathbb{Z}\) assigns to each irreducible component its degree.

- \(\Sing(B)\) is the set of singular points of \(B\), \(\Sigma_{\top}(B)\) is the set of topological types of \(\Sing(B)\), and \(\sigma_{\top} : \Sing(B) \to \Sigma_{\top}(B)\) assigns to each singular point its topological type.

- \(B(P)\) is the set of local branches of \(B\) at \(P \in \Sing(B)\), (a local branch can be seen as an arrow in the dual graph of the minimal resolution of \(B\) at \(P\), see [5 Chapter II.8] for details) and \(\beta_P : B(P) \to \text{Irr}(B)\) assigns to each local branch the global irreducible component containing it.

Two curves \(B_1\) and \(B_2\) are said to have the **same combinatorial type** (or simply the **same combinatorics**) if their data of combinatorial types

\[
(Irr(B_i), \deg_i, \Sing(B_i), \Sigma_{\top}(B_i), \sigma_{\top1}, \{\beta_{i,P}\}_{P \in \Sing(B_i)}, \{B_i(P)\}_{P \in \Sing(B_i)}), \quad i = 1, 2,
\]

are equivalent, that is, if \(\Sigma_{\top}(B_1) = \Sigma_{\top}(B_2)\), and there exist bijections \(\varphi_{\Sing} : \Sing(B_1) \to \Sing(B_2)\), \(\varphi_P : B_1(P) \to B_2(\varphi_{\Sing}(P))\) (restriction of a bijection of dual graphs) for each \(P \in \Sing(B_1)\), and \(\varphi_{\text{Irr}} : \text{Irr}(B_1) \to \text{Irr}(B_2)\) such that \(\deg_2 \circ \varphi_{\text{Irr}} = \deg_1\), \(\sigma_{\top2} \circ \varphi_{\Sing} = \sigma_{\top1}\), and \(\beta_{2, \varphi_{\Sing}(P)} \circ \varphi_P = \varphi_{\text{Irr}} \circ \beta_{1,P}\).

Note that when \(B_i\) \((i = 1, 2)\) are irreducible, \(B_1\) and \(B_2\) have the same combinatorics if they have the same degree and the same local topological types for singularities. On the other extreme, for line arrangements, \(B_1\) and \(B_2\) have the same combinatorial type if they have the same set of incidence relations. The first example of a Zariski pair is given by Zariski ([23, 24]), which is as follows:

**Example 0.1** Let \((B_1, B_2)\) be a pair of irreducible sextics such that (i) both of \(B_1\) and \(B_2\) have six cusps as their singularities, and (ii) the six cusp of \(B_1\) are on a conic, while no such conic for \(B_2\). Then \((B_1, B_2)\) is a Zariski pair.

For these twenty years, Zariski pairs have been studied by many mathematicians and many examples have been found (see [1] and its reference). Among them, Zariski pairs for line arrangements of degrees 9 and 11 are considered by Artal Bartolo, Carmonoa Ruber, Cogolludo Agustin and Marco Buzunariz ([2, 3]), Rybnikov ([13]) and those for conic arrangements of degree 8 are considered by Namba and Tsuchihashi ([10]). In this article, we study Zariski pairs for line-conic arrangement.

As we explain in [1], the study of Zariski pairs, in general, consists of two parts:

1. To give curves \(B_1\) and \(B_2\) having the same combinatorics, but some “different properties,” e.g., the location of singularities as in Example 0.1.
To show \((P^2, B_1)\) is not homeomorphic to \((P^2, B_2)\).

One of our goals in this article is to add another method to find two curves with the same combinatorics. Namely we make use of geometry of sections of the Mordell-Weil group of an elliptic surface as follows:

Let \(\varphi : S \to P^1\) be an elliptic surface with a section \(O\) and let

\[
\begin{array}{ccc}
S' & \xleftarrow{\mu} & S \\
\downarrow{f'} & & \downarrow{f} \\
\Sigma_d & \xleftarrow{q} & \hat{\Sigma}_d.
\end{array}
\]

be the double cover diagram for \(S\) (see 2.2). Let \(\Delta_1\) and \(\Delta_2\) be sections of \(\Sigma_d\) with \(\Delta_i^2 = d\) \((i = 1, 2)\). Suppose that \((q \circ f)^* (\Delta_i)\) consists of two sections \(s_{\Delta_i}^\pm\) for each \(i\) and \(\hat{\Sigma}_d\) can be blow down to \(P^2\), which we denote by \(\overline{\varphi} : \hat{\Sigma}_d \to P^2\). Let \([2]s_{\Delta_i}^\pm\) be the duplication of \(s_{\Delta_i}^\pm\) in \(\text{MW}(S)\). In order to produce two reduced curves \(B_1\) and \(B_2\) with the same combinatorics, we use \(\overline{\varphi} \circ f(s_{\Delta_i}^\pm)\) \((i = 1, 2)\), \(\overline{\varphi} \circ f([2]s_{\Delta_i}^\pm)\), and \(\overline{\varphi}(\Delta(S/\hat{\Sigma}_d))\), where \(\Delta(S/\hat{\Sigma}_d)\) is the branch locus of \(f\). The author hopes that this method add a new viewpoint to the study of elliptic surfaces and their Mordell-Weil groups.

As for (II), we also make use of theory of dihedral covers and elementary arithmetic on the Mordell-Weil group of an elliptic surface as in our previous papers (17, 18, 19).

Now let us explain line-conic arrangements of degree 7 considered in this article.

**Line-conic arrangement 1**

Let \(C_i\) \((i = 1, 2)\) be smooth conics and let \(L_j\) \((i = 1, 2, 3, 4)\) be lines as follows:

(i) Both \(L_1\) and \(L_2\) meet \(C_1\) transversely. We put \(C_1 \cap L_1 = \{P_1, P_2\}\), \(C_1 \cap L_2 = \{P_3, P_4\}\).

(ii) \(C_2\) is tangent to \(C_1\) at two distinct points \(\{Q_1, Q_2\}\) or at one point \(\{Q\}\). We call the former type \((a)\) and the latter type \((b)\).

(iii) The tangent lines at \(C_1 \cap C_2\) do not pass through \(L_1 \cap L_2\).

(iv) \(C_2\) is tangent to \(L_1\) and \(L_2\).

(v) \(L_3\) passes through \(P_1\) and \(P_3\).

(vi) \(L_4\) passes through \(P_1\) and \(P_4\).

(vii) Both \(L_3\) and \(L_4\) meet \(C_2\) transversely.

We put \(B_1 := C_1 + C_2 + L_1 + L_2 + L_3\) and \(B_2 := C_1 + C_2 + L_1 + L_2 + L_4\). Then \(B_1\) and \(B_2\) have the same combinatorics.
Let $C_1, C_2$ and $C_3$ be smooth conics and $L_1$ and $L_2$ be lines as follows:

(i) $C_1$ and $C_2$ meet transversely. We put $C_1 \cap C_2 = \{P_1, P_2, P_3, P_4\}$.

(ii) $C_3$ is tangent to both $C_1$ and $C_2$ such that the intersection multiplicities at intersection points are all even. By exchanging $C_1$ and $C_2$ if necessary, we may assume that there are three possibilities:

(a) $C_3 \cap C_1 = \{Q_1, Q_2\}, C_3 \cap C_2 = \{Q_3, Q_4\}$,

(b) $C_3 \cap C_1 = \{Q_1\}, C_3 \cap C_2 = \{Q_2, Q_3\}$ or

(c) $C_3 \cap C_1 = \{Q_1\}, C_3 \cap C_2 = \{Q_2\}$.

(iii) No tangent line at $Q_i$ is bitangent of $C_1 + C_2$.

(iv) $L_1$ passes through $P_1$ and $P_3$.

(v) $L_2$ passes through $P_1$ and $P_4$. 

\[ \text{Line-conic arrangement 1 of type (a)} \]

\[ \text{Line-conic arrangement 2 of type (a)} \]
(vi) Both of \( L_1 \) and \( L_2 \) meet \( C_3 \) transversely.

We put \( B_1 := C_1 + C_2 + C_3 + L_1 \) \( B_2 := C_1 + C_2 + C_3 + L_2 \). Then \( B_1 \) and \( B_2 \) have the same combinatorics.

**Theorem 0.1**

(i) Let \((B_1, B_2)\) be the pair of Line-conic arrangement 1. Then \((B_1, B_2)\) is a Zariski pair.

(ii) Let \( C_1 \) and \( C_2 \) be conics intersecting four distinct points, \( P_1, P_2, P_3 \) and \( P_4 \) and let \( L_0, L_1 \) and \( L_2 \) be lines through \( \{P_1, P_2\}, \{P_1, P_3\} \) and \( \{P_1, P_4\} \), respectively. Choose a point \( z_0 \) on \( C_1 \) such that the tangent line at \( z_0 \) to \( C_1 \) is not tangent to \( C_2 \). Then there exist just three conics \( C^{(0)}_3, C^{(1)}_3 \) and \( C^{(2)}_3 \) satisfying the following conditions:

- \( z_0 \in C^{(i)}_3 \) for each \( i \),
- For each \( i, C^{(i)}_3 \) is tangent to both \( C_1 \) and \( C_2 \) such that the intersection multiplicities \( I_x(C^{(i)}_3, C_j) \) \((j = 1, 2)\) at \( \forall x \in C^{(i)}_3 \cap C_j \) \((j = 1, 2)\) are all even.
- For \( i = 1, 2 \), if both of \( C_1 + C_2 + C^{(i)}_3 + L_1 \) and \( C_1 + C_2 + C^{(i)}_3 + L_2 \) have the combinatoric for Line-conic arrangement 2 of the same type, then \((C_1 + C_2 + C^{(i)}_3 + L_1, C_1 + C_2 + C^{(i)}_3 + L_2)\) is a Zariski pair.

**Remark 0.1**
The triple \((C_1 + C_2 + C^{(i)}_3 + L_0, C_1 + C_2 + C^{(i)}_3 + L_1, C_1 + C_2 + C^{(i)}_3 + L_2)\) may be a candidate for a Zariski triple. Our method in this article, however, does not work to see whether it is or not.

This article consists of 5 sections. In §1 and §2, we summarize some facts and results for theory of elliptic surfaces and \( D_{2n} \)-covers, which we need to prove our theorem. We prove Theorem 5.1 in §3 and Theorem 4.1 in §4. In §5, we prove Theorem 0.1 and give another example of a Zariski pair.

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## 1 \( D_{2n} \)-covers

In this section, we summarize some facts on Galois covers. We refer to [15] and [11 §3] for details.

We start with terminology on Galois covers. Let \( X \) and \( Y \) be normal projective varieties with finite morphism \( \pi : X \to Y \). We say \( X \) is a Galois cover of \( Y \) if the induced field extension \( \mathbb{C}(X)/\mathbb{C}(Y) \) by \( \pi^* \) is Galois, where \( \mathbb{C}(\bullet) \) means the rational function field of \( \bullet \). Note that the Galois group acts on \( X \) such that \( Y \) is obtained as the quotient space with respect to this action (cf. [16 §1]). If the Galois group \( \text{Gal}(\mathbb{C}(X)/\mathbb{C}(Y)) \) is isomorphic to a finite group \( G \), we call \( X \) a \( G \)-cover of \( Y \). The branch locus of \( \pi : X \to Y \), which we denote by \( \Delta_\pi \) or \( \Delta(X/Y) \), is the subset of \( Y \) consisting of points \( y \) of \( Y \), over which \( \pi \) is not locally
isomorphic. It is well-known that $\Delta_f$ is an algebraic subset of pure codimension 1 if $Y$ is smooth ([25]).

Suppose that $Y$ is smooth. Let $B$ be a reduced divisor on $Y$ with irreducible decomposition $B = \sum_{i=1}^{r} B_i$. A G-cover $\pi : X \rightarrow Y$ is said to be branched at $\sum_{i=1}^{r} e_i B_i$ if (i) $\Delta_f = B$ (here we identify $B$ with its support) and (ii) the ramification index along $B_i$ is $e_i$ for each $i$, where the ramification index mean the one along the smooth part of $B_i$ for each $i$. Note that the study of G-covers is related to that of Zariski pairs, since we have the following proposition (see [1] for details):

**Proposition 1.1** ([2, Proposition 3.6]) Let $\gamma_i$ be a meridian around $B_i$, and $[\gamma_i]$ denote its class in the topological fundamental group $\pi_1(Y \setminus B, p_0)$. Let $Y$ be a smooth projective variety and let $B = B_1 + \cdots + B_r$ be the decomposition into irreducible components of a reduced divisor $B$ on $Y$. If there exists a G-cover $\pi : X \rightarrow Y$ branched at $e_1 B_1 + \cdots + e_r B_r$, then there exists a normal subgroup $H_\pi$ of $\pi_1(Y \setminus B, p_0)$ such that:

(i) $[\gamma_i]^{e_i} \in H_\pi, [\gamma_i]^k \notin H_\pi, (1 \leq k \leq e_i - 1)$, and

(ii) $\pi_1(Y \setminus B, p_0)/H_\pi \cong G$.

Conversely, if there exists a normal subgroup $H$ of $\pi_1(Y \setminus B, p_0)$ satisfying the above two conditions for $H_\pi$, then there exists a G-cover $\pi_H : X_H \rightarrow Y$ branched at $e_1 B_1 + \cdots + e_r B_r$.

We keep our notation for $D_{2n}$-covers in Introduction. Here are two propositions for later use.

**Proposition 1.2** Let $n$ be an odd integer $\geq 3$. Let $Z$ be a smooth double cover of a smooth projective variety $Y$. We denote its covering morphism and covering transformation by $f$ and $\sigma_f$, respectively. Let $D$ be an effective divisor on $Z$ satisfying the following condition:

(i) $D$ and $\sigma_f^* D$ have no common component.

(ii) If $D = \sum a_i D_i$ denotes its irreducible decomposition, then $\gcd(a_i, n) = 1$ for every $i$.

(iii) $D - \sigma_f^* D$ is $n$-divisible in $\text{Pic}(Z)$.

Then there exists a $D_{2n}$-cover $\pi : X \rightarrow Y$ such that

(a) $\beta_2(\pi)$ is branched at $n((D + \sigma_f^* D)_\text{red})$.

(b) $D(X/Y) = Z$ and $f = \beta_2(\pi)$.

**Proof.** By [15, Proposition 0.4], our statements except the ramification indices are straightforward. As for the ramification indices, it follows from the last line of the proof of [15, Proposition 0.4].

**Proposition 1.3** Let $n$ be an odd integer $\geq 3$. Let $\pi : X \rightarrow Y$ be a $D_{2n}$-cover such that both $Y$ and $D(X/Y)$ are smooth. Let $\sigma_{\beta_1}$ be the covering transform of $\beta_1(\pi)$. If $\beta_2(\pi)$ is branched at $nD$ for some non-zero reduced divisor $D$ on $D(X/Y)$, then there exists an effective divisor $D$, whose irreducible decomposition is $\sum a_i D_i$ satisfying the following conditions:

(i) $D$ and $\sigma_{\beta_1}^* D$ have no common component.
(ii) $D - \sigma_{\beta_1}^* D$ is $n$-divisible in $\text{Pic}(D(X/Y))$.

(iii) For every $i$, $\gcd(a_i, n) = 1$.

(iv) $D = (D + \sigma_{\beta_1}^* D)_{\text{red}}$.

Proof. The statement essentially follows from Proposition 0.5 and its proof in [15]. We, however, give another simple proof based on the idea of versal $D_{2n}$-covers (see [20], [22] for versal Galois covers). By [22], there exists an element $\xi \in \mathbb{C}(X)$ such that the action of $D_{2n}$ on $\xi$ is given in such a way that:

$$
\begin{cases}
\xi^\sigma = \frac{1}{\xi} \\
\xi^\tau = \zeta_n \xi, \quad \zeta_n = \exp\left(\frac{2\pi i}{n}\right)
\end{cases}
$$

By using $\xi$, we have $\mathbb{C}(D(X/Y)) = \mathbb{C}(Y)(\xi^n), \mathbb{C}(X) = \mathbb{C}(Y)(\xi)$. Put $\theta = \xi^n \in \mathbb{C}(D(X/Y))$. Let $(\theta), (\theta)_0$ and $(\theta)_\infty$ be the divisor of $\theta$, the zero and polar divisors of $\theta$, respectively. Write $(\theta)_0$ in such a way that:

$$(\theta)_0 = \sum_i a_i D_i + nD',$$

where $D_i$’s are irreducible divisor on $D(X/Y)$ with $1 \leq a_i < n$ and $D'$ is an effective divisor on $D(X/Y)$. Since $\sigma$ induces $\sigma_{\beta_1}$ on $D(X/Y)$ and $\theta^\sigma (= \theta^{\sigma_{\beta_1}}) = 1/\theta$, we have equalities of divisors:

$$(\theta)_\infty = \sum_i a_i \sigma_{\beta_1}^* D_i + n\sigma_{\beta_1}^* D'$$

$$(\theta) = (\varphi)_0 - (\varphi)_\infty = \sum_i a_i (D_i - \sigma_{\beta_1}^* D_i) + n(D' - \sigma_{\beta_1}^* D').$$

Now we put $D = \sum_i a_i D_i$. Since we may assume that $(\theta)_0$ and $(\theta)_\infty$ have no common components, our statements (i) and (ii) follow. As $X$ is the $\mathbb{C}(D(X/Y))(\sqrt[n]{\theta})$-normalization of $D(X/Y)$ and the ramification index along $D_i$ is $n/\gcd(a_i, n)$, our statements (iii) and (iv) follow.

□

Corollary 1.1 Under the same assumption of Proposition 1.3, if $D$ is an irreducible divisor on $Y$ such that $(\beta_1(\pi))^{-1}(D) \subset \Delta_{\beta_2(\pi)}$, then $\beta_1(\pi)^* D$ consists of two irreducible components. In particular, in the case of $\dim Y = 2$, $I_x(D, \Delta_{\beta_1(\pi)})$ is even for $\forall x \in D \cap \Delta_{\beta_1(\pi)}$

Proof. The first statement is immediate from Proposition 1.3. For the second statement, let $\tilde{D}$ be the normalization of $D$. If there exists $x \in D \cap \Delta_{\beta_1(\pi)}$ such that $I_x(D, \Delta_{\beta_1(\pi)})$ is odd, $\beta_1(\pi)$ induces a branched double cover of $\tilde{D}$. This means $\beta_1(\pi)^* D$ is irreducible. □
2 Elliptic surfaces

2.1 General Facts

We first summarize some facts on the theory of elliptic surfaces. As for details, we refer to [7], [8], [9] and [14].

In this article, the term, an elliptic surface, always means a smooth projective surface $S$ equipped with a structure of a fiber space $\varphi: S \to C$ over a smooth projective curve, $C$, as follows:

(i) There exists a non-empty finite subset, $\text{Sing}(\varphi)$, of $C$ such that $\varphi^{-1}(v)$ is a smooth curve of genus 1 for $v \in C \setminus \text{Sing}(\varphi)$, while $\varphi^{-1}(v)$ is not a smooth curve of genus 1 for $v \in \text{Sing}(\varphi)$.

(ii) $\varphi$ has a section $O : C \to S$ (we identify $O$ with its image).

(iii) There is no exceptional curve of the first kind in any fiber.

For $v \in \text{Sing}(\varphi)$, we put $F_v = \varphi^{-1}(v)$. We denote its irreducible decomposition by

$$F_v = \Theta_{v,0} + \sum_{i=1}^{m_v-1} a_{v,i} \Theta_{v,i}$$

where $m_v$ is the number of irreducible components of $F_v$ and $\Theta_{v,0}$ is the irreducible component with $\Theta_{v,0}O = 1$. We call $\Theta_{v,0}$ the identity component. The types of singular fibers are classified as follows ([7]):

- Type $I_b$
- Type $I_b^* (b: \text{even})$
- Type $I_b^* (b: \text{odd})$
Note that every smooth irreducible component is a rational curve with self-intersection number $-2$.

We also define a subset of $\text{Sing}(\varphi)$ by $\text{Red}(\varphi) := \{ v \in \text{Sing}(\varphi) \mid F_v \text{ is reducible} \}$. Let $\text{MW}(S)$ be the set of sections of $\varphi : S \to C$. From our assumption, $\text{MW}(S) \neq \emptyset$. By regarding $O$ as the zero element of $\text{MW}(S)$ and considering fiberwise addition (see [7, §9] or [21, §1] for the addition on singular fibers), $\text{MW}(S)$ becomes an abelian group. We denote its addition by $\dot{+}$.

Also for $k \in \mathbb{Z}$ and $s \in \text{MW}(S)$, we write

$$[k]s := \begin{cases} 
\text{k-times addition of } s \text{ if } k \geq 0 \\
\text{k-times addition of the inverse of } s \text{ if } k < 0.
\end{cases}$$

Let $\text{NS}(S)$ be the Néron-Severi group of $S$ and let $T_\varphi$ be the subgroup of $\text{NS}(S)$ generated by $O, F$ and $\Theta_{v,i}$ ($v \in \text{Red}(\varphi), 1 \leq i \leq m_v - 1$). Then we have the following theorems:

**Theorem 2.1** ([14, Theorem 1.2]) Under our assumption, $\text{NS}(S)$ is torsion free.
Theorem 2.2 ([14, Theorem 1.3]) Under our assumption, there is a natural map \( \tilde{\psi} : \text{NS}(S) \to \text{MW}(S) \) which induces an isomorphism of groups

\[ \psi : \text{NS}(S)/T_\varphi \cong \text{MW}(S). \]

In particular, \( \text{MW}(S) \) is a finitely generated abelian group.

In the following, by the rank, \( \text{rank } \text{MW}(S) \), we mean that of the free part of \( \text{MW}(S) \). By Theorem 2.2, we may regard the addition of two sections in \( \text{NS}(S) \) as that in \( \text{MW}(S) \). We, however, use the notation \( + \) in order to distinguish them. For a divisor on \( S \), we put \( s(D) = \psi(D) \). Then we have

Lemma 2.1 ([14] Lemma 5.1) \( D \) is uniquely written in the form:

\[ D \approx s(D) + (d - 1)O + nF + \sum_{v \in \text{Red}(\varphi)} \sum_{i=1}^{m_v-1} b_{v,i} \Theta_{v,i}, \]

where \( \approx \) denotes the algebraic equivalence of divisors, and \( d, n \) and \( b_{v,i} \) are integers defined as follows:

\[ d = DF \quad n = (d - 1)\chi(\mathcal{O}_S) + OD - s(D)O, \]

and

\[ \begin{pmatrix} b_{v,1} \\ \vdots \\ b_{v,m_v-1} \end{pmatrix} = A_v^{-1} \begin{pmatrix} D\Theta_{v,1} - s_D \Theta_{v,1} \\ \vdots \\ D\Theta_{v,m_v-1} - s_D \Theta_{v,m_v-1} \end{pmatrix} \]

Here \( A_v \) is the intersection matrix \( (\Theta_{v,i}, \Theta_{v,j})_{1 \leq i,j \leq m_v - 1} \).

For a proof, see [14].

Put \( \text{NS}_Q := \text{NS}(S) \otimes \mathbb{Q} \) and \( T_{\varphi,Q} := T_{\varphi} \otimes \mathbb{Q} \). Since \( \text{NS}(S) \) is torsion free under our setting, there is no harm in considering \( \text{NS}_Q \). By using the intersection pairing, we have the orthogonal decomposition \( \text{NS}_Q = T_{\varphi,Q} \oplus (T_{\varphi,Q})^\perp \). In [14], the homomorphism \( \phi : \text{MW}(S) \to (T_{\varphi,Q})^\perp \subset \text{NS}_Q \) is defined as follows:

\[ \phi : \text{MW}(S) \ni s \mapsto s - O - (sO + \chi(\mathcal{O}_S))F + \sum_{v \in \text{Red}(\varphi)} (\Theta_{v,1}, \ldots, \Theta_{v,m_v-1})(-A_v)^{-1} \begin{pmatrix} s\Theta_{v,1} \\ \vdots \\ s\Theta_{v,m_v-1} \end{pmatrix} \in (T_{\varphi,Q})^\perp. \]

Also, in [14], a \( \mathbb{Q} \)-valued bilinear form \( \langle \cdot, \cdot \rangle \) on \( \text{MW}(S) \) is defined by \( \langle s_1, s_2 \rangle := -\phi(s_1)\phi(s_2) \), where the right hand side means the intersection pairing in \( \text{NS}_Q \). Here are two basic properties of \( \langle \cdot, \cdot \rangle \):

- \( \langle s, s \rangle \geq 0 \) for \( \forall s \in \text{MW}(S) \) and the equality holds if and only if \( s \) is an element of finite order in \( \text{MW}(S) \).
• An explicit formula for $\langle s_1, s_2 \rangle$ ($s_1, s_2 \in \text{MW}(S)$) is given as follows:

$$\langle s_1, s_2 \rangle = \chi(O_S) + s_1O + s_2O - s_1s_2 - \sum_{v \in \text{Red}(\varphi)} \text{Contr}_v(s_1, s_2),$$

where $\text{Contr}_v(s_1, s_2)$ is given by

$$\text{Contr}_v(s_1, s_2) = \left( s_1 \Theta_{v, 1}, \ldots, s_1 \Theta_{v, m_v - 1} \right) \left( -A_v \right)^{-1} \left( \begin{array}{c} s_2 \Theta_{v, 1} \\ \vdots \\ s_2 \Theta_{v, m_v - 1} \end{array} \right).$$

As for explicit values of $\text{Contr}_v(s_1, s_2)$, we refer to [14, (8.16)].

### 2.2 Double cover construction of an elliptic surface

For details about this subsection, see [8, Lectures III and IV]. Let $\varphi : S \rightarrow C$ be an elliptic surface. By our assumption, the generic fiber of $\varphi$ can be considered as an elliptic curve over $\text{C}(C)$, the rational function field of $C$. The inverse morphism with respect to the group law induces an involution $[-1]_{\varphi}$ on $S$. Let $S/\langle [-1]_{\varphi} \rangle$ be the quotient by $[-1]_{\varphi}$. The quotient surface $S/\langle [-1]_{\varphi} \rangle$ is known to be smooth and $S/\langle [-1]_{\varphi} \rangle$ can be blown down to its relatively minimal model $W$ over $C$ satisfying the following condition:

Let us denote

• $f : S \rightarrow S/\langle [-1]_{\varphi} \rangle$: the quotient morphism,

• $q : S/\langle [-1]_{\varphi} \rangle \rightarrow W$: the blow down, and

• $S \xrightarrow{\mu} S' \xrightarrow{f'} W$: the Stein factorization of $q \circ f$.

Then we have:

1. The branch locus $\Delta_{f'}$ of $f'$ consists of a section $\Delta_0$ and the trisection $T$ such that its singularities are at most simple singularities (see [4, Chapter II, §8] for simple singularities and their notation) and $\Delta_0 \cap T = \emptyset$.

2. $\Delta_0 + T$ is 2-divisible in $\text{Pic}(W)$.

3. The morphism $\mu$ is obtained by contracting all the irreducible components of singular fibers not meeting $O$.

Conversely, if $\Delta_0$ and $T$ on $W$ satisfy the above condition, we obtain an elliptic surface $\varphi : S \rightarrow \mathbb{P}^1$, as the canonical resolution of a double cover $f' : S' \rightarrow W$ with $\Delta_{f'} = \Delta_0 + T$, and the diagram (see [6] for the canonical resolution):

$$\begin{array}{ccc}
S' & \xrightarrow{\mu} & S \\
\downarrow f' & & \downarrow f \\
W & \xleftarrow{q} & \hat{W}.
\end{array}$$
Here \( q \) is a composition of blowing-ups so that \( \hat{W} = S/([-1]_q) \). Hence any elliptic surface is obtained as above. In the following, we call the diagram above the double cover diagram for \( S \).

In the case of \( C = \mathbb{P}^1 \), \( W \) is the Hirzebruch surface, \( \Sigma_d \), of degree \( d = 2\chi(O_S) \) and \( \Delta_f \) is of the form \( \Delta_0 + T \), where \( \Delta_0 \) is a section with \( \Delta_0^2 = -d \) and \( T \sim 3(\Delta_0 + df) \), \( f \) being a fiber of the ruling \( \Sigma_d \to \mathbb{P}^1 \).

**Remark 2.1**

- For each \( v \in \text{Sing}(\varphi) \), the type of \( \varphi^{-1}(v) \) is determined by the type of singularity of \( T \) on \( f_v \) and the relative position between \( f_v \) and \( T \) (see [9, Table 6.2]).

- Note that the covering transformation, \( \sigma_f \), of \( f \) coincides with \([-1]_\varphi \). Also, by [7, Theorem 9.1], the action of \( \sigma_f \) on irreducible components of singular fibers is described as follows:

  | Type of a singular fiber | The action on irreducible component |
  |-------------------------|-----------------------------------|
  | \( I_n \)               | \( \Theta_0 \mapsto \Theta_0 \)   |
  |                         | \( \Theta_i \mapsto \Theta_{n-i} \ |
  |                         | \( i = 1, \ldots, n - 1 \)       |
  | \( I_n \) (n : even)    | \( \Theta_i \mapsto \Theta_i \)   |
  |                         | \( \forall i \)                   |
  | \( I_n \) (n : odd)     | \( \Theta_i \mapsto \Theta_i \)   |
  |                         | \( \forall i, j \)                |
  | \( \Pi, \Pi^*, \Pi^* \) | \( \Theta_i \mapsto \Theta_1 \)   |
  | \( IV \)                | \( \Theta_0 \mapsto \Theta_0 \)   |
  |                         | \( \Theta_1 \mapsto \Theta_2 \)   |
  |                         | \( \Theta_2 \mapsto \Theta_1 \)   |
  | \( IV^* \)              | \( \Theta_i \mapsto \Theta_i \)   |
  |                         | \( i = 0, 3, 6 \)                 |
  |                         | \( \Theta_1 \mapsto \Theta_2 \)   |
  |                         | \( \Theta_2 \mapsto \Theta_1 \)   |
  |                         | \( \Theta_4 \mapsto \Theta_5 \)   |
  |                         | \( \Theta_5 \mapsto \Theta_4 \)   |

### 3 Elliptic \( D_{2p} \)-cover over rational ruled surface

Let \( \varphi : S \to \mathbb{P}^1 \) be an elliptic surface over \( \mathbb{P}^1 \). Let \( S \to \hat{\Sigma}_d \) be the double cover appearing in the double cover diagram for \( S \) in [2.2]

We first note that any elliptic \( D_{2p} \)-cover (\( p \): odd prime) \( \pi_p : X_p \to \hat{\Sigma}_d \) satisfies the following conditions:

- \( S = D(X_p/\hat{\Sigma}_d) \) and \( \beta_1(\pi_p) = f \).
- The branch locus of \( \beta_2(\pi_p) \) is of the form
  \[ D + \sigma_f^*D + \Xi + \sigma_f^*\Xi \]

  where

  1. all irreducible components of \( D \) are horizontal and there is no common component between \( D \) and \( \sigma_f^*D \), and
  2. all irreducible component of \( \Xi \) are vertical and there is no common component between \( \Xi \) and \( \sigma_f^*\Xi \).
Remark 3.1  Under the above notation, the case when $D = \emptyset$ (resp. = a section) is considered in the author’s previous work ([15, 16, 17, 18]) (resp. [19]).

In the following, we always assume that

\((*)\ D \neq \emptyset.\)

The proposition below, which is a generalization of [19, Propositions 4.1 and 4.2], plays an important role in this article:

**Theorem 3.1** Let $p$ be an odd prime. Let $C_1, \ldots, C_r$ be irreducible horizontal divisors on $S$ such that $\sum_{i=1}^{r} C_i$ and $\sum_{j=1}^{r} \sigma_j^* C_i$ have no common component. Then (I) and (II) in the below are equivalent:

(I) Put $C = \sum_{i=1}^{r} C_i$. There exists an elliptic $D_{2p}$-cover $\pi_p : X_p \to \hat{\Sigma}_d$ such that

- $D(X_p/\hat{\Sigma}_d) = S$ and $\beta_1(\pi_p) = f$.
- $\Delta_{\beta_2(\pi_p)} = \text{Supp}(C + \sigma_j^* C + \Xi + \sigma_j^* \Xi)$
  where all irreducible components of $\Xi$ are vertical and there is no common component between $\Xi$ and $\sigma_j^* \Xi$.

(II) Let $s(C_i) = \tilde{\psi}(C_i)$ $(i = 1, \ldots, r)$. There exist integers $a_i$ $(i = 1, \ldots, r)$ such that

- $1 \leq a_i < p$ $(i = 1, \ldots, r)$ and
- $\sum_{i=1}^{r} [a_i] s(C_i) \in [p] \text{MW}(S) := \{ [p] s \mid s \in \text{MW}(S) \}$.

**Proof.** (I) $\Rightarrow$ (II) Let $D$ be the effective divisor in Proposition [13]. We put $D = D_{\text{hor}} + D_{\text{ver}}$, where the irreducible components of $D_{\text{hor}}$ are all horizontal, while those of $D_{\text{ver}}$ are all in fibers of $\varphi$. By Proposition [13] (iv), $(D_{\text{hor}} + \sigma_j^* D_{\text{hor}})_{\text{red}} = \sum_{i=1}^{r} C_i + \sum_{i=1}^{r} \sigma_j^* C_i$.

**Claim.** We may assume that $D_{\text{hor}} = \sum_{i=1}^{r} a_i C_i$.

**Proof of Claim.** If $\sigma_j^* C_i$ is an irreducible component of $D_{\text{hor}}$, then we consider

\[ D'_\text{hor} := D_{\text{hor}} + (p - a_i) C_i - a_i \sigma_j^* C_i, \]

and put $D' = D'_{\text{hor}} + D_{\text{ver}}$. Then we infer that $D'$ also satisfies all four conditions in Proposition [13]. After repeating this process finitely many times, we may assume that $D_{\text{hor}} = \sum_{i=1}^{r} a_i C_i$.

By Claim and Proposition [13] (iii), there exists $L \in \text{Pic}(S)$ such that

\[ \sum_{i=1}^{r} a_i (C_i - \sigma_j^* C_i) + D_{\text{ver}} - \sigma_j^* D_{\text{ver}} \sim pL, \]
where $\sim$ means linear equivalence of divisors (Note that linear equivalence coincides with algebraic equivalence on $S$). This implies
\[
\tilde{\psi}(\sum_{i=1}^{r} a_i(C_i - \sigma_j^* C_i)) = [p] \tilde{\psi}(L) \quad \text{in } \text{MW}(S).
\]
As $\tilde{\psi}(\sigma_j^* C_i) = [-1]s(C_i)$, we have
\[
\tilde{\psi}(\sum_{i=1}^{r} a_i(C_i - \sigma_j^* C_i)) = [2]([a_1]s(C_1) + \ldots + [a_r]s(C_r)).
\]
Since $p$ is an odd prime, we infer that $[a_1]s(C_1) + \ldots + [a_r]s(C_r) \in [p] \text{MW}(S)$.

$(II) \Rightarrow (I)$ Our proof is similar to that of [19, Proposition 4.2]. By Lemma 2.1, we have
\[
C_i \sim s(C_i) + (d_i - 1)O + n_i F + \sum_{v \in \text{Red}(\varphi)} \sum_{j=1}^{m_v} b_{v,j} \Theta_{v,j}.
\]
This implies
\[
\sum_{i=1}^{r} a_i C_i \sim \sum_{i=1}^{r} a_i s(C_i) + \sum_{i=1}^{r} a_i \left((d_i - 1)O + n_i F + \sum_{v \in \text{Red}(\varphi)} \sum_{j=1}^{m_v} b_{v,j} \Theta_{v,j}\right).
\]
By our assumption, there exists $s_o$ such that $\sum_{i=1}^{r} [a_i] s(C_i) = [p] s_o$ in $\text{MW}(S)$. By Theorem 2.2 this implies that
\[
\sum_{i=1}^{r} a_i s(C_i) \sim p s_o + \left(-p + \sum_{i=1}^{r} a_i\right) O + n_o F + \sum_{v \in \text{Red}(\varphi)} \sum_{j=1}^{m_v} c_{v,j} \Theta_{v,j}
\]
for some integers $n_o, c_{v,j}$. Hence we have
\[
\sum_{i=1}^{r} a_i C_i \sim p s_o + \left(-p + \sum_{i=1}^{r} a_i d_i\right) O + \left(n_o + \sum_{i=1}^{r} a_i n_i\right) F
\]
\[
+ \sum_{v \in \text{Red}(\varphi)} \sum_{j=1}^{m_v - 1} \left(c_{v,j} + \sum_{i=1}^{r} a_i b_{v,j}^{(i)}\right) \Theta_{v,j},
\]
and put
\[
D' := \sum_{i=1}^{r} a_i C_i + \sum_{v \in \text{Red}(\varphi)} \sum_{j=1}^{m_v - 1} \left(c_{v,j} + \sum_{i=1}^{r} a_i b_{v,j}^{(i)}\right) \sigma_j^* \Theta_{v,j}.
\]
Then we have
\[
D' - \sigma_j^* D' \sim p (s_o - \sigma_j^* s_o).
\]
The left hand side of the above equivalence contains some redundancy in the sum for $\Theta_{v,i}$ and $\sigma_j^* \Theta_{v,i}$. By taking the action of $\sigma_j$ on $\Theta_{v,i}$'s (see Remark 2.1) into account, we can find a divisor $D = \sum_{i=1}^{r} a_i C_i + \sum_j k_j \Xi_j$ and $\Xi'$ on $S$ such that
(i) all $\Xi_j$ and all irreducible components of $\Xi'$ are those in fibers not meeting $O$,
(ii) $D$ and $\sigma_j^*D$ have no common component,
(iii) $1 \leq k_j < p$, and
(iv) $D' - \sigma_j^*D' = D - \sigma_j^*D + p\Xi'$.

Now we easily infer that $D$ satisfies the three conditions in Proposition 1.2 for $p$. □

Fix an isomorphism $\text{MW}(S) = M_o \oplus \text{MW}_{\text{tor}}, M_o \cong \mathbb{Z}^r$, $r = \text{rank} \text{MW}(S)$. By Theorem 3.1 we have the following proposition:

**Proposition 3.1** Choose $s \in M_o$ such that $M_o/\mathbb{Z}s$ is free. For any finite number of odd prime numbers $p_1, \ldots, p_l$, there exists a section $s_{p_1,\ldots,p_l}$ satisfying the following conditions:

(i) $(s_{p_1,\ldots,p_l}, s_{p_1,\ldots,p_l}) = (p_1 \cdots p_l)^2(s, s)$.

(ii) For any prime $p \not\in \{p_1, \ldots, p_l\}$, there exists an elliptic $D_{2p}$-cover $\pi_p : X_p \to \hat{\Sigma}_d$ such that

- $D(X_p/\hat{\Sigma}_d) = S$, $\beta_1(\pi_p) = f$, and
- $\beta_2(\pi_p)$ is branched at $p(s + s_{p_1,\ldots,p_l} + \sigma_j^*(s + s_{p_1,\ldots,p_l}) + \Xi)$, where all irreducible components of $\Xi$ are those in fibers not meeting $O$.

(iii) For $p \in \{p_1, \ldots, p_l\}$, there exists no elliptic $D_{2p}$-cover $\pi_p : X_p \to \hat{\Sigma}_d$ as in (ii)

(iv) $\{s_{p_1,\ldots,p_l}, [-1]s_{p_1,\ldots,p_l}\}$ is unique up to torsion elements.

**Proof.** Define $s_{p_1,\ldots,p_l} := [\Pi_{i=1}^l p_i]s$. By Theorem 3.1 our statements (i), (ii) and (iii) are immediate. Suppose that $s' \in \text{MW}(S)$ satisfies the statements (i), (ii) and (iii). Put $s' = s'_o + t'_o, s'_o \in M_o, t'_o \in \text{MW}_{\text{tor}}$. By Theorem 3.1 for $p \not\in \{p_1, \ldots, p_l\}$, there exists an integer $k (1 \leq k < p)$ such that

$$s + [k]s'_o = 0$$

in $M_o/pM_o$. This implies that there exist integers $l$ and $l'$ with $\gcd(l, l') = 1$ such that $[l]s + [l']s'_o = 0$ in $\text{MW}(S)$. Choose integers $m, m'$ with $ml + m'l' = 1$. Then

$$0 = [m](l)s + [l']s'_o = s + [-m'l']s + [ml']s'_o = s + [l']([-m']s + [m]s'_o).$$

Since $M_o/\mathbb{Z}s$ is free, we infer that $l' = 1$ and $s'_o = [-l]s$. Thus

$$\langle s'_o, s'_o \rangle = l^2(s_o, s_o) = (p_1 \cdots p_l)^2(s, s).$$

Since $\langle s, s \rangle \neq 0$ by the basic properties of $\langle, \rangle$ (see §1), we have $l = \pm p_1 \cdots p_l$. Hence $s'$ is equal to $[\pm 1]s_{p_1,\ldots,p_l}$ up to torsion elements. □
4 Applications

Let \( \varphi : S \to \mathbb{P}^1 \) be an elliptic surface and we keep our notation for the double cover construction for \( S \) in \( \S 2.2 \). We fix an isomorphism \( \text{MW}(S) \cong M_o \oplus \text{MW}_{\text{tor}} \cong \mathbb{Z}^{\oplus r}, \) \( r = \text{rank} \text{MW}(S) \). Choose \( s_1, s_2 \in M_o \) so that \( s_1 \) and \( s_2 \) are a part of a basis of \( \mathbb{Z}^{\oplus r} \), i.e., \( M_o/\mathbb{Z}s_1 + \mathbb{Z}s_2 \) is free of rank \( r - 2 \). Put \( s_3 := [2]s_1 \).

**Theorem 4.1** For any odd prime \( p \), there exists an elliptic \( D_{2p} \)-cover \( \pi_p : X_p \to \hat{\Sigma}_d \) such that the horizontal part of the branch locus, \( \Delta_{\beta_2(\pi_p)} \), of \( \beta_2(\pi_p) \) is of the form

\[
s_1 + s_3 + \sigma_2^*(s_1 + s_3),
\]

while there exists no elliptic \( D_{2p} \)-cover \( \pi_p : X_p \to \hat{\Sigma}_d \) such that the horizontal part of the branch locus of \( \beta_2(\pi_p) \) is \( s_2 + s_3 + \sigma_2^*(s_2 + s_3) \).

**Proof.** Since \( [p - 2]s_1 + s_3 \in [p] \text{MW}(S) \), we have the first statement by Theorem 3.1. Since \( s_1 \) and \( s_2 \) are a part of a basis, their image in \( M_o/\mathbb{Z}s_1 + \mathbb{Z}s_2 \) are linearly independent over \( \mathbb{Z}/p\mathbb{Z} \). Hence our second statement follows from Theorem 3.1. \( \square \)

By Proposition 1.1 and Theorem 3.1, we have

**Corollary 4.1** Let \( T \) be the trisection on \( \Sigma_d \) appearing in the double cover diagram for \( S \). Put \( \Delta_i := q \circ f(s_i) \) (\( i = 1, 2, 3 \)). Then there exists a \( D_{2p} \)-cover of \( \Sigma_d \) branched at \( 2(\Delta_0 + T) + p(\Delta_1 + \Delta_3) \), while there exists no \( D_{2p} \)-cover of \( \Sigma_d \) branched at \( 2(\Delta_0 + T) + p(\Delta_2 + \Delta_3) \).

We end this section by considering the case when \( S \) is a rational elliptic surface. In this case, we have the double cover diagram for \( S \) as follows:

\[
\begin{array}{c}
S' \leftarrow^\mu S \\
\Sigma_2 \leftarrow^q \hat{\Sigma}_2.
\end{array}
\]

Write \( q := q_1 \circ \cdots \circ q_r : \hat{\Sigma}_2 = \Sigma_2^{(r)} \to \cdots \to \Sigma_2^{(1)} \to \Sigma_2^{(0)} = \Sigma_2 \), where \( q_i \) is a blowing up at a point at \( \Sigma_2^{(i-1)} \). Put \( \Delta_{f'} = \Delta_0 + T \). In the following, we assume that

\( T \) has a node \( x_o \).

Note that this is equivalent to the fact that \( S \) has a singular fiber of type I_2 or II by [9, Table 6.2]. We may assume that \( q_1 \) is a blowing up at \( x_o \). Let \( E_1 \) be the exceptional divisor of \( q_1 \) and let \( f_o \) and \( T \) be the proper transforms of a fiber, \( f_o \), through \( x_o \) and \( T \), respectively. Then we have the following picture:
Note that if $f_o$ meets both of the local branches of $T$ at $x_o$ transversely, we have the case (a), while if $f_o$ is tangent to one of the local branches of $T$ at $x_o$, we have the case (b).

Blow down $f_o$ and $\Delta_0$ in this order. Then the resulting surface is $\mathbb{P}^2$. We denote this composition of blowing downs by $\overline{\psi}_1 : \Sigma_2^{(1)} \to \mathbb{P}^2$ and put $Q := \overline{\psi}_1(T)$. Then $Q$ is a reduced quartic with the distinguished point $z_o := \overline{\psi}_1(f_o \cup \Delta_0)$. Note that $\overline{\psi}_1(E_1)$ is the tangent line $L_{z_o}$ of $Q$ at $z_o$. Put $\overline{\psi} := \overline{\psi}_1 \circ q_2 \circ \cdots \circ q_r$ and we have the following diagram:

$$
\begin{array}{c}
S'' \leftarrow \overline{\psi} & S \\
f'' \downarrow & \downarrow f \\
\mathbb{P}^2 \leftarrow \overline{\psi}_2.
\end{array}
$$
Here $S''$ is the Stein factorization of $\varphi \circ f$. Note that $S''$ is a double cover with branch locus $Q$ and that the pencil of lines through $z_0$ gives rise to the elliptic fibration of $S$. Now we have the following proposition.

**Proposition 4.1** Let $s_1, s_2$ and $s_3$ be sections as in the beginning of this section and put $C_i := \varphi(s_i)(i = 1, 2, 3)$. If $(Q + C_1 + C_3)$ and $(Q + C_2 + C_3)$ have the same combinatorics, then $(Q + C_1 + C_3, Q + C_2 + C_3)$ is a Zariski pair.

**Proof.** Our statement is immediate from Proposition \[11\] Theorem \[11\] and the following lemma. \hfill $\square$

**Lemma 4.1** Let $p$ be an odd prime. For $i = 1, 2$, there exists a $D_{2p}$-cover $\varpi_p : \mathcal{X}_p \to \mathbb{P}^2$ of $\mathbb{P}^2$ branched at $2Q + p(C_i + C_3)$ if and only if there exists an elliptic $D_{2p}$-cover $\pi_p : \mathcal{X}_p \to \hat{\Sigma}_2$ of $\hat{\Sigma}_2$ such that the horizontal part of $\Delta_{\beta_2(\pi_p)}$ is $s_i + s_3 + \sigma^*_f(s_i + s_3)$.

**Proof.** Suppose that there exists a $D_{2p}$-cover $\varpi_p : \mathcal{X}_p \to \mathbb{P}^2$ branched at $2Q + p(C_i + C_3)$. Let $\varpi_p^{(i)} : \mathcal{X}_p^{(i)} \to \Sigma_2^{(i)}$ be the induced $D_{2p}$-cover, i.e., $\mathcal{X}_p^{(i)}$ is the $\mathbb{C}(\mathcal{X}_p)$-normalization of $\Sigma_2^{(i)}$. Since $D(\mathcal{X}_p^{(i)}/\mathbb{P}^2) = S''$ and $\beta_1(\varpi_p) = f''$, $D(\mathcal{X}_p^{(i)}/\Sigma_2^{(i)})$ is the $\mathbb{C}(S'')$-normalization of $\Sigma_2^{(i)}$. Hence $\Delta_{\beta_1(\varpi_p^{(i)})} = \Delta_0 + \Gamma$ as $\varphi^*Q = \Delta_0 + \Gamma + 2\Gamma_0$. This implies that $D(\mathcal{X}_p^{(r)}/\hat{\Sigma}_2) = S$ and $\beta_1(\varpi_p^{(r)}) = f$. As $C_i = \varphi \circ f(s_i)(i = 1, 2, 3)$, $\varpi_p^{(r)} : \mathcal{X}_p^{(r)} \to \hat{\Sigma}_2$ is an elliptic $D_{2p}$-cover such that the horizontal part of $\Delta_{\beta_2(\varpi_p^{(r)})}$ is $s_i + s_3 + \sigma^*_f(s_i + s_3)$.

Conversely, suppose that there exists an elliptic $D_{2p}$-cover $\pi_p : \mathcal{X}_p \to \hat{\Sigma}_2$ such that the horizontal part of $\Delta_{\beta_2(\pi_p)}$ is $s_i + s_3 + \sigma^*_f(s_i + s_3)$. Since $E_1$ gives rise to an irreducible component “$\Theta_1$” of singular fiber of type I$_2$ or I, the preimage of $E_1$ in $\hat{\Sigma}_2$ is not contained in the branch locus of $\pi_p$ by Corollary \[11\] and Remark \[2.1\]. Now let $\mathcal{X}_p$ be the Stein factorization of $\varphi \circ \pi_p$. Then the induced $D_{2p}$-cover $\pi_p : \mathcal{X}_p \to \mathbb{P}^2$ is branched at $2Q + p(C_i + C_3)$. \hfill $\square$

**5 Proof of Theorem 0.1**

**Proof of Theorem 0.1 (i).** Put $Q = C_1 + L_1 + L_2$ and choose a point $z_0 \in C_1 \cap C_2$ as the distinguished point. Let $f^\circ : S_Q^\circ \to \mathbb{P}^1$ be a double cover with branch locus $Q$ and let $\varphi_{z_0} : S(Q, z_0) \to \mathbb{P}^1$ be the rational elliptic surface as in § 4. By our construction of $S_{Q, z_0}$, both $L_3$ and $L_4$ give rise to sections, which we denote by $s_{L_i}^+$ and $s_{L_i}^-(i = \sigma_f s_{L_i}^+ = [-1]s_{L_i}(i = 3, 4)$, respectively. By labeling singular fibers suitably, we may assume that $s_{L_i}^+(i = 3, 4)$ and reducible singular fibers meet as in the following picture:
Here we assume that $\Theta_{1,0}$ and $O$ come from $z_0$. By the explicit formula of $\langle , \rangle$, we have

$$
\langle s^\pm_L, s^\pm_L \rangle = \frac{1}{2} \quad (i = 3, 4) \quad \langle s^+_L, s^+_L \rangle = 0.
$$

By [11], $\text{MW}(S(\mathbb{Q}, z_0)) \cong (\mathbb{A}^1)^{\oplus 2} \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ and we may assume that

$$(A_1^1)^{\oplus 2} \cong \mathbb{Z}s^+_L \oplus \mathbb{Z}s^+_L,$$

and that the 2-torsion sections arise from $C_1, L_1$ and $L_2$.

As for $(\mathbb{C} \circ \mathbb{f})(C_2)$, it also gives rise to two sections $s^\pm_{C_2}$. Since $C_2$ does not pass through any singularities of $C_1 + L_1 + L_2$ and $s^\pm_{C_2} O = 0$, we have $\langle s^\pm_{C_2}, s^\pm_{C_2} \rangle = 2$.

On the other hand, any element $s \in \text{MW}(S(\mathbb{Q}, z_0))$ with $\langle s, s \rangle = 2$ is of the form

$$
[2] s^\pm_L + \tau, \quad (i = 3, 4) \quad \tau \in \text{MW}(S(\mathbb{Q}, z_0))_{\text{tor}}.
$$

If $\tau \neq 0$, then $s^\pm_{C_2}$ meets $\Theta_{i,1}$ for some $i$ by considering the addition on singular fibers (see [7, Theorem 9.1] or [21, §1]). Hence, by the explicit formula for $\langle , \rangle$, we have $s^\pm_{C_2} O = 0$. On the other hand, $s^\pm_{C_2} O = 0$ by our construction. Thus we infer $\tau = 0$ and we may assume that $s^\pm_{C_2} = [2] s^\pm_{L_3}$ after relabeling $\pm, L_3$ and $L_4$, if necessary. Therefore

$$
[s^\pm_{C_2} + [p - 2] s^+_L] \in [p] \text{MW}(S(\mathbb{Q}, z_0))
$$

for any odd prime $p$, while

$$
[s^\pm_{C_2} + [k] s^+_L] \notin [p] \text{MW}(S(\mathbb{Q}, z_0))
$$

for any odd prime $p$ and $1 \leq k \leq p - 1$. By Proposition 4.1, we infer that $(C_1 + C_2 + L_1 + L_2 + L_3, C_1 + C_2 + L_1 + L_2 + L_4)$ is a Zariski pair.

**Proof for Theorem 0.1 (ii).** Put $Q = C_1 + C_2$ and choose a point $z_0 \in C_1 \cap C_3$ as the distinguished point. Let $f'_Q : S'_Q \to \mathbb{P}^2$ be a double cover with branch locus $Q$ and let
\( \varphi_{z_0} : S_{(\mathbb{Q}, z_0)} \to \mathbb{P}^1 \) be the rational elliptic surface as in \( \S 4 \). By our construction of \( S_{\mathbb{Q}, z_0} \), \( L_0, L_1 \) and \( L_2 \) give rise to sections, which we denote by \( s_{L_i}^+ \) and \( s_{L_i}^- (= \sigma_i^* s_{L_i}^+ = [-1]s_{L_i}) \) \( (i = 0, 1, 2) \), respectively. By labeling singular fibers suitably, we may assume that \( s_{L_i}^+ \) \( (i = 0, 1, 2) \) and reducible singular fibers meet as in the following picture:

![Diagram of singular fibers meeting]

Here we assume that \( \Theta_{1,0} \) and \( O \) come from \( z_0 \). By the explicit formula of \( \langle \cdot, \cdot \rangle \), we have

\[ \langle s_{L_i}^+, s_{L_j}^+ \rangle = 1/2, \quad (i = 0, 1, 2) \quad \langle s_{L_i}^+, s_{L_j}^- \rangle = 0, \quad (i, j = 0, 1, 2, i \neq j) \]

By [11], \( \text{MW}(S_{(\mathbb{Q}, z_0)}) \cong (A_1)^{\oplus 3} \oplus (\mathbb{Z}/2\mathbb{Z}) \) and we may assume that

\[ (A_1)^{\oplus 2} \cong \mathbb{Z}s_{L_0}^+ \oplus \mathbb{Z}s_{L_1}^+ \oplus \mathbb{Z}s_{L_2}^+ \]

and that the unique 2-torsion section arises from \( C_1 \).

By [7 Theorem 9.1], \( 2|s_{L_i}^\pm \) \( (i = 0, 1, 2) \) meet the identity component at each singular fiber. Hence by the explicit formula for \( \langle \cdot, \cdot \rangle \), we have \( [2|s_{L_i}^\pm O = 0 \) for each \( i \). This implies that, for each \( i \), \( C_{L_i} := \overline{\tau \circ f([2|s_{L_i}^\pm}) \) is a conic not passing through \( P_j \) \( (j = 1, 2, 3, 4) \). If \( C_{L_i} \) and \( C_1 + C_3 \) has an intersection point at which intersection multiplicity is odd, then we easily see that the closure of \( (\tau \circ f)^{-1}(C_{L_i} \setminus z_0) \) is irreducible. This is impossible, as \( C_{L_i} \) is the image of \( [2|s_{L_i}^\pm \). Hence we have three conic satisfying the first two conditions.

Conversely, suppose that there exists a conic \( C_0 \) satisfying the first two conditions. We infer that \( C_0 \) gives rise to two sections \( s_{C_0}^\pm \). Since \( C_0 \) does not pass through any singularities of \( C_1 + C_2 \) and \( s_{C_0}^\pm O = 0 \), we have \( \langle s_{C_0}^\pm, s_{C_0}^\pm \rangle = 2 \). On the other hand, any element \( s \in \text{MW}(S_{(\mathbb{Q}, z_0)}) \) with \( \langle s, s \rangle = 2 \) is of the form

\[ [2|s_{L_i}^\pm + \tau, \quad (i = 0, 1, 2) \quad \tau \in \text{MW}(S_{(\mathbb{Q}, z_0)})_{\text{tor}}. \]

By a similar argument to that in the case of Line-conic arrangement 1, we infer \( \tau = 0 \). Hence we infer that \( C_{L_i} \) are all conics satisfying the first two conditions. Now put \( C_{3}^{(i)} := C_{L_i}, s_{C_{3}^{(i)}} := [2|s_{L_i}^\pm \) \( (i = 1, 2) \). For \( (i, j) = (1, 2), (2, 1) \), we have

\[ s_{C_{3}^{(i)}} + |p - 2|s_{L_i}^\pm \in [p] \text{MW}(S_{(\mathbb{Q}, z_0)}) \]
for any odd prime $p$, while
\[ s_{C^3_i}^+ + k s^+_i \notin [p] \text{MW}(S_{(Q, z_0)}) \]
for any odd prime $p$ and $1 \leq k \leq p - 1$.

By Proposition 1.1 if both of $C_1 + C_2 + C^3_i + L_1$ and $C_1 + C_2 + C^3_i + L_2$ have the combinatorics for Line-conic arrangement 2 of the same type, then $(C_1 + C_2 + C^3_i + L_1, C_1 + C_2 + C^3_i + L_2)$ is a Zariski pair for each $i = 1, 2$. \( \square \)

**Remark 5.1** Let $B$ be the line-conic arrangement as in Theorem 0.1. By Corollary 1.1 if there exists a $D_{2p}$-cover $\pi : X \rightarrow \mathbb{P}^2$ with branch locus $B$, then $\Delta_{\frac{1}{p}}(\pi) = L_1 + L_2 + C_1$ (resp. $C_1 + C_2$) for Line-conic arrangement 1 (resp. 2). This means that the $D_{2p}$-covers in our proof of Theorem 1.1 are only possible ones. Therefore for the fundamental group $\pi_1(\mathbb{P}^2 \setminus B, \ast)$, we infer that
\[
\pi_1(\mathbb{P}^2 \setminus (L_1 + L_2 + L_3 + C_1 + C_2), \ast) \not\cong \pi_1(\mathbb{P}^2 \setminus (L_1 + L_2 + L_4 + C_1 + C_2), \ast)
\]
for Line-conic arrangement 1, and
\[
\pi_1(\mathbb{P}^2 \setminus (L_1 + C_1 + C_2 + C^3_i), \ast) \not\cong \pi_1(\mathbb{P}^2 \setminus (L_2 + C_1 + C_2 + C^3_i), \ast)
\]
for Line-conic arrangement 2.

**Example 5.1** Let $(T, X, Z)$ be homogeneous coordinates of $\mathbb{P}^2$ and let $(t, x) := (T/Z, X/Z)$ be affine coordinates of $\mathbb{P}^2$ and consider a conic and four lines as follows:
\[
C_1 : x - t^2 = 0, \quad L_1 : x - 3t + 2 = 0, \quad L_2 : x + 3t + 2 = 0,
\]
\[
L_3 : x - t - 2 = 0, \quad L_4 : x - 1 = 0.
\]

Note that $C_1 \cap (L_1 \cup L_2) = \{[\pm 1, 1, 1], [\pm 2, 4, 1]\}$. Put $Q = C_1 + L_1 + L_2$ and choose $[0, 1, 0]$ as the distinguished point $z_0$. Let $S_{(Q, z_0)}$ be the rational elliptic surface obtained as in §4. Then its generic fiber is an elliptic curve over $\mathbb{C}(t)$ given by the Weierstrass equation:
\[
y^2 = (x - t^2)(x - 3t + 2)(x + 3t + 2).
\]

Under this setting, we may assume that the sections $s^+_L, (i = 3, 4)$ are as follows:
\[
s^+_L = (t + 2, \pm 2\sqrt{2}(t - 2)(t + 1)) \quad s^+_L = (1, \pm 3(t + 1)(t - 1)).
\]

Hence we have
\[
[2]s^+_L = \left(\frac{9}{8}t^2, \frac{1}{32}t\sqrt{2}(9t^2 - 16)\right), \quad [2]s^+_L = (t^2 + \frac{1}{4}, \frac{1}{2}t^2 - \frac{9}{8})
\]
Now put
\[
C_2 : x - \frac{4}{9}t^2 = 0, \quad C'_2 : x - t^2 - \frac{1}{4} = 0.
\]

Then $(Q + C_2 + L_3, Q + C_2 + L_4)$ is a Zariski pair for Line-conic arrangement 1 of type $(a)$, and $(Q + C'_2 + L_3, Q + C'_2 + L_4)$ is a Zariski pair for Line-conic arrangement 1 of type $(b)$. \( \Box \)
Example 5.2 We keep the same coordinates as Example 5.1. Consider two conics and two lines:

\[ C_1 : x - t^2 + 2 = 0, \quad C_2 : x^2 - 2x + t^2 - 4 = 0, \]
\[ L_1 : x - t = 0, \quad L_2 : x - 3t + 4 = 0. \]

Note that \( C_1 \cap C_2 = \{[\pm 2, 2, 1], [\pm 1, -1, 1]\} \). Put \( Q = C_1 + C_2 \) and choose \([0,1,0]\) as the distinguished point \( z_0 \). Let \( S_{(Q,z_0)} \) be the rational elliptic surface obtained as before. Then its generic fiber is an elliptic curve over \( \mathbb{C}(t) \) given by the Weierstrass equation:

\[ y^2 = (x - t^2 + 2)(x^2 - 2x + t^2 - 4). \]

Then we may assume that the sections \( s_{L_i}^\pm (i = 1, 2) \) are as follows:

\[ s_{L_1}^\pm = (t, \pm \sqrt{-2}(t + 1)(t - 2)), \quad s_{L_2}^\pm = (3t - 4, \pm \sqrt{-10}(t - 1)(t - 2)). \]

Thus we have

\[ [2]s_{L_1}^+ = \left( \frac{1}{2}t^2 - 2, -\frac{1}{2}\sqrt{-2}t(t^2 - 4) \right), \quad [2]s_{L_2}^+ = \left( \frac{1}{10}t^2 - 2, -\frac{3}{100}\sqrt{-10}t(t^2 + 20) \right). \]

Now we put

\[ C_3 : x - \frac{1}{2}t^2 + 2 = 0, \quad C'_3 : x - \frac{1}{10}t^2 + 2 = 0 \]

Then both \((Q + C_3 + L_1, Q + C_3 + L_2)\) and \((Q + C'_3 + L_1, Q + C'_3 + L_2)\) are Zariski pairs for Line-conic arrangement 2 of type (a).

Line-conic arrangement 2 of type (b). Consider two conics and two lines:

\[ C_1 : x - t^2 + 2 = 0, \quad C_2 : x^2 - 2x + t^2 - 4 = 0, \]
\[ L_1 : x - t = 0, \quad L_2 : x + 1 = 0. \]

Put \( Q = C_1 + C_2 \) and choose \([0,1,0]\) as the distinguished point \( z_0 \). Let \( S_{(Q,z_0)} \) be the rational elliptic surface obtained as before. Then its generic fiber is an elliptic curve over \( \mathbb{C}(t) \) given by the Weierstrass equation:

\[ y^2 = (x - t^2 + 2)(x^2 - 2x + t^2 - 4). \]

Then we may assume that the sections \( s_{L_i}^\pm (i = 1, 2) \) is as follows:

\[ s_{L_2}^\pm = (-1, \pm \sqrt{-1}(t - 1)(t + 1)). \]

Thus we have

\[ [2]s_{L_2}^+ = \left( t^2 - \frac{17}{4}, \frac{3}{8}\sqrt{10}(4t^2 - 19) \right). \]

Now we put

\[ C_3 : x - t^2 + \frac{17}{4} = 0. \]

As \( C_3 \) is tangent to \( C_1 \) (resp. \( C_2 \)) at one point (resp. two distinct points), we infer that \((Q + C_3 + L_1, Q + C_3 + L_2)\) is a Zariski pair for Line-conic arrangement 2 of type (b).
Line-conic arrangement 2 of type (c). Consider two conics and two lines:

\[ C_1 : x - t^2 + \frac{1}{2} = 0, \quad C_2 : x^2 - x + t^2 = 0, \]
\[ L_1 : x = \frac{1}{\sqrt{2}}, \quad L_2 : \sqrt{x^2} (\sqrt{c_1} - c_2) x + t - \frac{1}{4} (\sqrt{1 - c_1} + c_2) = 0, \]
where \( c_1 = \sqrt{2 + 2\sqrt{2}}, c_2 = \sqrt{-2 + 2\sqrt{2}}. \) Note that
\[ C_1 \cap C_2 = \left\{ \left[ \pm \sqrt{-1/2 + 1/\sqrt{2}}, 1/\sqrt{2}, 1 \right], \left[ \pm \sqrt{-1/2 - 1/\sqrt{2}}, -1/\sqrt{2}, 1 \right] \right\}. \]

Put \( Q = C_1 + C_2 \) and choose \([0, 1, 0] \) as the distinguished point \( z_o. \) Let \( S(Q, z_o) \) be the rational elliptic surface obtained as before. Then its generic fiber is an elliptic curve over \( \mathbb{C}(t) \) given by the Weierstrass equation:
\[ y^2 = \left( x - t^2 - \frac{1}{2} \right) (x^2 - x + t^2). \]
Then we may assume that the sections \( s_{L_i}^\pm \) are as follows:
\[ s_{L_i}^\pm = \left( \frac{1}{\sqrt{2}} \pm \frac{1}{2} \sqrt{-1/2 - 1/\sqrt{2}} \right). \]
Thus we have
\[ [2]s_{L_i}^\pm = \left( t^2, \sqrt{-1/2} \right). \]
Now we put
\[ C_3 : x - t^2 = 0 \]
Then \( (Q + C_3 + L_1, Q + C_3 + L_2) \) is a Zariski pair for Line-conic arrangement 2 of type (c).

We end this section by giving another example:

**Proposition 5.1** Let \( Q \) be an irreducible quartic with a \( \mathbb{D}_4 \) singularity, \( P. \) Let \( z_o \) be a point on \( Q \) such that the tangent line \( L_{z_o} \) at \( z_o \) meets \( Q \) with two other distinct points. Let \( L_1, L_2 \) and \( L_3 \) be three tangent lines which meet \( Q \) at \( P \) with multiplicity 4. Then the following statements hold:

(i) For each \( L_i, \) there exists a unique conic \( C_i \) such that (a) \( z_o \in C_i, \) (b) \( P \notin C_i \) and (c) for \( \forall x \in C_i \cap Q, I_x(C_i, Q) \) is even.

(ii) For any odd prime \( p, \) there exists a \( D_{2p} \)-cover of \( \mathbb{P}^2 \) branched at \( 2Q + p(C_i + L_i) \) for each \( i = 1, 2, 3, \) while there exists no \( D_{2p} \)-cover of \( \mathbb{P}^2 \) branched at \( 2Q + p(C_i + L_j) \) for any \( i, j \) (\( i \neq j \)).

**Proof.** (i) Let \( f_Q'' : \mathbb{P}^2 \) be a double cover with branch locus \( Q \) and let \( \varphi_{z_o} : S(Q, z_o) \to \mathbb{P}^1 \) be the rational elliptic surface obtained as in \( \S 4. \) By our assumption on \( Q \) and \( z_o, \) the configuration of reducible singular fiber of \( \varphi_{z_o} \) is \( 10^6, 12 \) and three lines \( L_i (i = 1, 2, 3) \) give rise to sections \( s_{L_i}^\pm (i = 1, 2, 3) \), respectively. By labeling irreducible components of singular fibers suitably, we have the following picture for \( s_{L_i}^\pm (i = 1, 2, 3) \):
By the explicit formula for $(\cdot, \cdot)$, we have

$$\langle s_{1\text{L}}, s_{2\text{L}} \rangle = \frac{1}{2} (i = 1, 2, 3), \quad \langle s_{i\text{L}}, s_{j\text{L}} \rangle = 0 \ (i \neq j).$$

By [11], we have $\text{MW}(S_{(Q, z_o)}) \cong (A^*)^{\otimes 3}$. Hence we may assume that

$$\text{MW}(S_{Q, z_o}) \cong Zs_{1\text{L}} \oplus Zs_{2\text{L}} \oplus Zs_{3\text{L}}.$$

By the lattice structure of $\text{MW}((S_{(Q, z_o)}), all elements $s \in \text{MW}(S_{Q, z_o})$ with $\langle s, s \rangle = 2$ given by $[2]s_{i\text{L}}, (i = 1, 2, 3)$. By [7, Theorem 9.1], $[2]s_{i\text{L}} (i = 1, 2, 3)$ meet the identity component at each singular fiber. Hence, $[2]s_{i\text{L}}, O = 0 (i = 1, 2, 3)$ by the explicit formula for $(\cdot, \cdot)$. By our construction of $S_{(Q, z_o)}$, $C_i := q_0 f([2]s_{i\text{L}}) \sim \Delta_i + 2\mathfrak{f} (i = 1, 2, 3)$. Hence $C_i := q_0 f([2]s_{i\text{L}})$ $(i = 1, 2, 3)$ are all conic and $z_o \in C_i, P \not\in C_i$. Moreover as $[2]s_{i\text{L}} \neq [2]s_{j\text{L}} (i = 1, 2, 3)$, our assertion for the intersection multiplicities follows.

(ii) By Theorem 4.4 and Lemma 4.1, our statement follows.

**Corollary 5.1** If $L_i$ and $L_j$ $(i \neq j)$ meet $C_i$ transversely, then $(Q + L_i + C_i, Q + L_j + C_j)$ is a Zariski pair.

**Proof.** Since the combinatorics of $Q + L_i + C_i$ and $Q + L_j + C_j$ are the same, our assertion follows from Proposition 5.1.

**Example 5.3** We keep the same coordinates as in Examples 5.1 and 5.2. Consider $Q, L_1$ and $L_2$ as follows:

$$Q : f_Q(t, x) := x^3 + \frac{343}{64} \left( \frac{121}{49} t^2 + \frac{768}{2401} t \right) x^2 + \frac{343}{64} \left( \frac{384}{2401} t^2 + \frac{92}{49} t + \frac{1}{7} \right) x + \frac{35}{16} t^4 + \frac{1}{7} t^3 = 0$$

$L_1 : x + t = 0, \quad L_2 : x - \frac{\zeta_3 - 2}{7} t = 0, \quad \zeta_3 = \exp(2\pi i/3)$
Q is irreducible and has a $\mathbb{D}_4$ singularity at $(0,0)$. Both $L_1$ and $L_2$ meet $Q$ at $(0,0)$ with multiplicity 4. Choose $[0,1,0]$ as the distinguished point $z_0$. Let $S_{(Q,z_0)}$ be the rational elliptic surface obtained as before. Then its generic fiber is an elliptic curve over $\mathbb{C}(t)$ given by the Weierstrass equation $y^2 = f_Q(t,x)$. Under these circumstances, we have

$$s_{L_1}^\pm = \left(-t, \pm \frac{\sqrt{343}}{8} t^2\right), \quad s_{L_2}^\pm = \left(\frac{\zeta_3 - 2}{7} t, \pm \frac{\sqrt{71 + 39\sqrt{-3}}}{8\sqrt{14}} t^2\right)$$

Then we have

$$[2]s_{L_1}^+ = \left(\frac{144}{16807} - \frac{127}{343} t - \frac{19}{28} t^2, \frac{-\sqrt{7}(55296 + 1947456t + 1450204t^2 + 167649825t^3)}{184473632}\right).$$

Now put

$$C : x - \frac{144}{16807} + \frac{127}{343} t + \frac{19}{28} t^2.$$ 

Since one can see that both of $L_1$ and $L_2$ meet $C$ with two distinct points, $Q + C + L_1$ and $Q + C + L_2$ have the same combinatorics. By Corollary 5.1 $(Q + C + L_1, Q + C + L_2)$ is a Zariski pair.

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