WEIGHTED HARDY TYPE INEQUALITIES WITH ROBIN BOUNDARY CONDITIONS

ISMAIL KOMBE AND ABDULLAH YENER

Abstract. In this paper we establish a general weighted Hardy type inequality for the $p$-Laplace operator with Robin boundary condition. We provide various concrete examples to illustrate our results for different weights. Furthermore, we present some Heisenberg-Pauli-Weyl type inequalities with boundary terms on balls with radius $R$ at the origin in $\mathbb{R}^n$.

1. Introduction

The classical Hardy inequality states that
\[
\int_{\mathbb{R}^n} |\nabla \phi(x)|^p \, dx \geq \frac{|n-p|}{p} \int_{\mathbb{R}^n} |\phi(x)|^p \left|\frac{x_1}{x_n}\right|^p \, dx,
\]
and holds for all $\phi \in C_0^\infty(\mathbb{R}^n)$ if $1 < p < n$, and for all $\phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ if $p > n$. The constant $|(n-p)/p|^p$ in the right-hand side of (1.1) is sharp but not achieved. This inequality plays an important role in many areas such as analysis, geometry and mathematical physics. Owing to this, there is a vast amount of literature on the Hardy type inequalities together with their variations and generalizations.

For instance, V.G. Maz’ya [15] proved the following weighted integral inequality
\[
\int_{\mathbb{R}^n} |x_n|^{p-1} |\nabla \phi(x)|^p \, dx \geq \frac{1}{(2p)^p} \int_{\mathbb{R}^n} \frac{|\phi(x)|^p}{x_n^2 + x_n^2} \, dx,
\]
where $\phi \in C_0^\infty(\mathbb{R}^n)$.

In the paper [5], E.B. Davies proved the following extension of Hardy’s inequality for convex domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$,
\[
\int_{\Omega} |\nabla \phi(x)|^2 \, dx \geq \frac{1}{4} \int_{\Omega} \frac{|\phi(x)|^2}{\delta^2(x)} \, dx,
\]
where $\phi \in C_0^\infty(\Omega)$. Here

Date: August 8, 2022.
Key words and phrases. Hardy inequality, Boundary term, Robin boundary condition.
\[\delta(x) = \text{dist}(x, \partial \Omega) = \min_{y \in \partial \Omega} |x - y| \tag{1.4}\]

is the distance function to the boundary \(\partial \Omega\). Moreover the constant \(\frac{1}{4}\) is sharp and not achieved. We refer the interested reader to the monographs [13], [5], [2], [10], [17] and the paper [9].

On the other hand, very little work has been done on Hardy type inequalities with Robin boundary conditions. However, there has been some initiation in this area of interest. In an interesting paper, H. Kovářik and A. Laptev [12] proved, among other results, the following Hardy inequality for Laplace operators with Robin boundary conditions,

\[
\int_{\Omega} |\nabla \phi|^2 \, dx + \int_{\partial \Omega} \sigma |\phi|^2 \, d\nu \geq \frac{1}{4} \int_{\Omega} \left( \delta(x) + \frac{1}{2\sigma(p(x))} \right)^{-2} |\phi|^2 \, dx \\
+ \frac{1}{4} \int_{\Omega} \left( R_{in} + \frac{1}{2\sigma(p(x))} \right)^{-2} |\phi|^2 \, dx \\
+ \frac{1}{2} \int_{\partial \Omega} \left( R_{in} + \frac{1}{2\sigma(y)} \right)^{-1} |\phi|^2 \, d\nu.
\tag{1.5}\]

where \(\Omega \subseteq \mathbb{R}^n\) is a bounded convex domain with smooth boundary \(\partial \Omega\), \(\phi \in H^1(\Omega)\), \(0 \leq \sigma \in L^{\infty}(\partial \Omega)\), \(d\nu\) is the surface measure on \(\partial \Omega\) and \(R_{in} := \sup \{\delta(x) : x \in \Omega\}\). Here \(p(x)\) is the projection function \(p : \Omega \setminus S \rightarrow \partial \Omega\) defined by \(p(x) := y \in \partial \Omega : \delta(x) = |x - y|, \ x \in \Omega \setminus S\), where \(S \subseteq \Omega\) the subset of points in \(\Omega\) for which there exist at least two points \(y_1, y_2 \in \partial \Omega\) so that the minimum in (1.4) is achieved.

Later, T. Ekholm, H. Kovářik and A. Laptev [6] studied the best constant in a Hardy inequality for the \(p\)-Laplace operator on convex domains with Robin boundary conditions. Moreover, there has also been an interest regarding Hardy type inequalities for functions which does not vanish on the boundary of a given domain, see e.g. [1], [3], [4], [19] and [16].

In light of the results mentioned above, it is natural to investigate under what conditions the following general weighted Hardy-type inequalities with boundary terms are valid

\[
\int_{\Omega} a(x) |\nabla \phi|^p \, dx \geq \int_{\Omega} b(x) |\phi|^p \, dx + \int_{\partial \Omega} \beta(x) |\phi|^p \, d\nu.
\tag{1.6}\]

Within this framework, the main objective of this article is to study the general weighted Hardy type inequalities for the \(p\)-Laplace operators with Robin boundary conditions. We should emphasize that our unifying method is quite practical and constructive to obtain several weighted Hardy, Maz’ya and Heisenberg-Pauli-Weyl type inequalities with boundary terms.
2. Weighted Hardy Type Inequalities With Boundary Terms

In what follows, Ω represents smooth bounded domains contained in \( \mathbb{R}^n \), \( \partial_{\nu}u \) denotes the exterior normal derivative of \( u \) at the boundary of \( \Omega \) and \( d\nu \) is the surface measure on \( \partial \Omega \). The generic point is \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and \( r = |x| = \sqrt{x_1^2 + \cdots + x_n^2} \). We denote by \( B_R = \{ x \in \mathbb{R}^n : |x| < R \} \) the open ball at the origin with radius \( R \) in \( \mathbb{R}^n \). The boundary is the sphere \( \partial B_R = \{ x \in \mathbb{R}^n : |x| = R \} \).

We are now ready to state the main result of this paper.

**Theorem 2.1.** Let \( \Omega \) be a smooth bounded domain contained in \( \mathbb{R}^n \). Let \( a \in C^1(\Omega) \) be a nonnegative function, \( u \in C^\infty(\Omega) \) be a positive function, \( b \in L^1_{loc}(\Omega) \) and \( \beta \in L^1_{loc}(\partial \Omega) \) such that

\[
- \nabla \cdot (a(x)|\nabla u|^{p-2} \nabla u) \geq b(x) u^{p-1} \quad \text{a.e. in } \Omega \tag{2.1}
\]

and

\[
a(x)|\nabla u|^{p-2} \partial_{\nu}u = \beta(x) u^{p-1} \quad \text{a.e. at } \partial \Omega, \tag{2.2}
\]

where \( \partial_{\nu}u \) denotes the exterior normal derivative of \( u \) at the boundary of \( \Omega \). There exists a positive constant \( c_p = c(p) \) such that, if \( p \geq 2 \), then

\[
\int_\Omega a(x)|\nabla \phi|^p \, dx \geq \int_\Omega b(x)|\phi|^p \, dx + c_p \int_\Omega a(x)|\nabla \left( \frac{\phi}{u} \right)|^p u^p \, dx \\
+ \int_{\partial \Omega} \beta(x)|\phi|^p \, d\nu \tag{2.3}
\]

and if \( 1 < p < 2 \), then

\[
\int_\Omega a(x)|\nabla \phi|^p \, dx \geq \int_\Omega b(x)|\phi|^p \, dx + c_p \int_\Omega a(x) \frac{|\nabla \left( \frac{\phi}{u} \right)|^2 u^2}{(|\nabla \phi| + |\nabla \left( \frac{\phi}{u} \right)| u)^{2-p}} \, dx \\
+ \int_{\partial \Omega} \beta(x)|\phi|^p \, d\nu \tag{2.4}
\]

holds for all \( \phi \in C^\infty(\Omega) \). Here, \( d\nu \) is the surface measure on \( \partial \Omega \).

**Proof.** For any \( \phi \in C^\infty(\Omega) \) we now set \( \varphi := \frac{\phi}{u} \), where \( 0 < u \in C^\infty(\Omega) \). Thus,

\[
|\nabla \varphi|^p = |\varphi \nabla u + u \nabla \varphi|^p.
\]

We now use the following convexity inequality:

\[
|\xi + \eta|^p \geq |\xi|^p + p|\xi|^{p-2}\xi \cdot \eta + c_p |\eta|^p, \quad \forall \xi, \eta \in \mathbb{R}^n, \quad p \geq 2,
\]

where \( c_p \) is a positive constant depending only on \( p \), see [14]. As an immediate consequence of the above inequality with the vectors \( \xi = \varphi \nabla u \) and \( \eta = u \nabla \varphi \), we obtain

\[
|\nabla \phi|^p \geq |\nabla u|^p |\varphi|^p + u |\nabla u|^{p-2} \nabla u \cdot \nabla (|\varphi|^p) + c_p |\nabla \varphi|^p u^p. \tag{2.5}
\]
Multiplying the inequality (2.5) by \( a(x) \) on both sides, and then applying integration by parts over \( \Omega \) for the middle term, yields

\[
\int_{\Omega} a(x) |\nabla \phi|^p \, dx \geq \int_{\Omega} a(x) |\nabla u|^p |\varphi|^p \, dx - \int_{\Omega} \nabla \cdot (a(x) u |\nabla u|^{p-2} \nabla u) |\varphi|^p \, dx
\]

\[
+ \int_{\partial \Omega} a(x) u |\nabla u|^{p-2} |\varphi|^p \partial_{\nu} u \, du + c_p \int_{\Omega} a(x) |\nabla \varphi|^p u^p \, dx
\]

\[
= - \int_{\Omega} \nabla \cdot (a(x) |\nabla u|^{p-2} \nabla u) u |\varphi|^p \, dx
\]

\[
+ c_p \int_{\Omega} a(x) |\nabla \varphi|^p u^p \, dx + \int_{\partial \Omega} a(x) u |\nabla u|^{p-2} |\varphi|^p \partial_{\nu} u \, du.
\]

It therefore follows from (2.11) and (2.2) that

\[
\int_{\Omega} a(x) |\nabla \phi|^p \, dx \geq \int_{\Omega} b(x) u^p |\varphi|^p \, dx + c_p \int_{\Omega} a(x) |\nabla \varphi|^p u^p \, dx + \int_{\partial \Omega} \beta(x) u^p |\varphi|^p \, dv.
\]

Finally, making the variable change \( \varphi = \frac{\phi}{u} \) in the above integrals, one has the desired inequality for \( p \geq 2 \):

\[
\int_{\Omega} a(x) |\nabla \phi|^p \, dx \geq \int_{\Omega} b(x) |\phi|^p \, dx + c_p \int_{\Omega} a(x) \left| \nabla \left( \frac{\phi}{u} \right) \right|^p u^p \, dx + \int_{\partial \Omega} \beta(x) |\phi|^p \, dv.
\]

The inequality (2.4) can be proved in the same spirit, and in this case we apply the following convexity inequality:

\[
|\xi + \eta|^p \geq |\xi|^p + p |\xi|^{p-2} \xi \cdot \eta + c_p \frac{|\eta|^2}{(|\xi| + |\eta|)^{2-p}}, \quad \forall \xi, \eta \in \mathbb{R}^n, \quad 1 < p < 2,
\]

where \( c_p = c(p) > 0 \), see [14]. We shall omit the details. \( \square \)

2.1. Applications of Theorem 2.1. As we stated earlier, our method is quite practical to construct various weighted Hardy type inequalities including boundary terms on some bounded domains in \( \mathbb{R}^n \). To do this, we need to identify suitable model functions \( a \) and \( u \) that satisfy the above hypotheses in the given domain. Let us begin by considering the model functions

\[
a = |x|^{\alpha} \quad \text{and} \quad u = |x|^{-\left( \frac{n+\alpha-\rho}{p} \right)}
\]

in Theorem 2.1. After some computations, we readily get the subsequent result:
Corollary 2.1. Let $\alpha \in \mathbb{R}$ and $n + \alpha > p > 1$. Then the following inequality holds:

$$\int_{B_R} |x|^\alpha |\nabla \phi|^p \, dx \geq \left( \frac{n + \alpha - p}{p} \right)^p \int_{B_R} |x|^\alpha \phi^p \, dx$$

$$- \left( \frac{n + \alpha - p}{p} \right)^{p-1} R^{\alpha - p} \int_{\partial B_R} |\phi|^p \, d\nu$$

for all $\phi \in C^\infty(B_R)$.

On the other hand, by applying Theorem 2.1 with the following pair

$$a = |x|^\alpha \sinh |x| \quad \text{and} \quad u = |x|^{-\left(\frac{n + \alpha - p}{p}\right)}$$

and noting that $|x| \coth |x| \geq 1$, we obtain the power hyperbolic sine $L^p$ Hardy-type inequality with a boundary term.

Corollary 2.2. Let $\alpha \in \mathbb{R}$, $\gamma \geq 0$ and $n + \alpha + \gamma > p > 1$. Then the following inequality holds:

$$\int_{B_R} |x|^\alpha \sinh \gamma |x| |\nabla \phi|^p \, dx \geq \left( \frac{n + \alpha + \gamma - p}{p} \right)^p \int_{B_R} |x|^\alpha \sinh \gamma |x| |\phi|^p \, dx$$

$$- \left( \frac{n + \alpha + \gamma - p}{p} \right)^{p-1} \sinh \gamma R \int_{\partial B_R} |\phi|^p \, d\nu$$

for all $\phi \in C^\infty(B_R)$.

Recall that the following weighted $L^2$ Hardy type inequalities in the Euclidean setting were proved by Ghoussoub and Moradifam [11]: Let $s, t > 0$ and $\alpha, \gamma, m$ be real numbers.

- If $\alpha \gamma > 0$ and $m \leq \frac{n-2}{2}$, then for all $\varphi \in C^\infty_0(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \frac{(s + t |x|^\alpha)^\gamma}{|x|^{2m+2}} |\nabla \varphi|^2 \, dx \geq \left( \frac{n - 2m - 2}{2} \right)^2 \int_{\mathbb{R}^n} \frac{(s + t |x|^\alpha)^\gamma}{|x|^{2m+2}} \varphi^2 \, dx. \quad (2.6)$$

- If $\alpha \gamma < 0$ and $2m - \alpha \gamma \leq n - 2$, then for all $\varphi \in C^\infty_0(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \frac{(s + t |x|^\alpha)^\gamma}{|x|^{2m+2}} |\nabla \varphi|^2 \, dx \geq \left( \frac{n + \alpha \beta - 2m - 2}{2} \right)^2 \int_{\mathbb{R}^n} \frac{(s + t |x|^\alpha)^\gamma}{|x|^{2m+2}} \varphi^2 \, dx. \quad (2.7)$$

We now extend and improve the above inequalities (2.6) and (2.7) to the $L^p$ case with a boundary term on the $R-$ball in $\mathbb{R}^n$. In order to do so, we now take the pair as

$$a = \frac{(s + t |x|^\alpha)^\gamma}{|x|^{pm}} \quad \text{and} \quad u = |x|^{-\left(\frac{p \alpha - m - \gamma}{p}\right)}$$

in Theorem 2.1. This gives the following improvement of the inequality (2.6).
Corollary 2.3. Let \( s, t > 0 \) and \( \alpha, \gamma, m \in \mathbb{R} \). If \( \alpha \gamma > 0 \) and \( 1 < p \leq n - pm \), then for all \( \phi \in C^\infty(B_R) \) one has

\[
\int_{B_R} \frac{(s + t |x|^\alpha)^\gamma}{|x|^{pm}} |\nabla \phi|^p \, dx \geq C_{n,p,m}^p \int_{B_R} \frac{(s + t |x|^\alpha)^\gamma}{|x|^{pm+p}} |\phi|^p \, dx \\
+ C_{n,p,m}^{p-1} \alpha \gamma t \int_{B_R} \frac{(s + t |x|^\alpha)^{\gamma-1}}{|x|^{pm+p-\alpha}} |\phi|^p \, dx \\
- C_{n,p,m}^{p-1} \frac{(s + t R^\alpha)^\gamma}{R^{pm+p-1}} \int_{\partial B_R} |\phi|^p \, d\nu,
\]

where \( C_{n,p,m} = \left( \frac{n-pm-p}{p} \right) \).

If we consider the units

\[
a = \frac{(s + t |x|^\alpha)^\gamma}{|x|^{pm}} \quad \text{and} \quad u = |x|^{-\left( \frac{n+\alpha\gamma-pm-p}{p} \right)},
\]

then we immediately obtain the following improvement of the inequality \( (2.7) \).

Corollary 2.4. Let \( s, t > 0 \) and \( \alpha, \gamma, m \in \mathbb{R} \). If \( \alpha \gamma < 0 \) and \( 1 < p \leq n + \alpha \gamma - pm \), then for all \( \phi \in C^\infty(B_R) \) one has

\[
\int_{B_R} \frac{(s + t |x|^\alpha)^\gamma}{|x|^{pm}} |\nabla \phi|^p \, dx \geq C_{n,p,m,\alpha,\gamma}^p \int_{B_R} \frac{(s + t |x|^\alpha)^\gamma}{|x|^{pm+p}} |\phi|^p \, dx \\
- C_{n,p,m,\alpha,\gamma}^{p-1} \alpha \gamma s \int_{B_R} \frac{(s + t |x|^\alpha)^{\gamma-1}}{|x|^{pm+p}} |\phi|^p \, dx \\
- C_{n,p,m,\alpha,\gamma}^{p-1} \frac{(s + t R^\alpha)^\gamma}{R^{pm+p-1}} \int_{\partial B_R} |\phi|^p \, d\nu,
\]

where \( C_{n,p,m,\alpha,\gamma} = \left( \frac{n+\alpha\gamma-pm-p}{p} \right) \).

Remark 2.2. Note that if \( \alpha = 0 \) or \( \gamma = 0 \) in the above two inequalities, then they reduce to Hardy type inequalities with usual weights. Hence, we are interested in the case when \( \alpha \gamma \neq 0 \).

It is worth stressing here that, by considering the model functions

\[
a = \left( 1 + |x|^{p-r} \right)^{\alpha(p-1)} \quad \text{and} \quad u = \left( 1 + |x|^{p-r} \right)^{1-\alpha}
\]

in Theorem \( 2.1 \), we obtain an improved version of one of the Skrzypczak’s result in \( [18] \) with a boundary term on the \( R \)-ball in \( \mathbb{R}^n \).
Corollary 2.5. Let $1 < p < n$ and $\alpha > 1$. Then for all $\phi \in C^\infty(B_R)$, there holds
\[
\int_{B_R} \left(1 + \frac{|x|^p}{p-1}\right)^{\alpha(p-1)} |\nabla \phi|^p \, dx \geq n \left( \frac{p(\alpha - 1)}{p-1} \right)^{p-1} \int_{B_R} \left(1 + \frac{|x|^p}{p-1}\right)^{(\alpha-1)(p-1)} |\phi|^p \, dx \\
- \left( \frac{p(\alpha - 1)}{p-1} \right)^{p-1} \frac{1}{R} \int_{\partial B_R} \left(1 + \frac{|x|^p}{p-1}\right)^{(\alpha-1)(p-1)} |\phi|^p \, d\nu.
\]

Another consequence of Theorem 2.1 with the functions $a = |x|^\alpha$ and $u = (1 + |x|^p)^{-\left(\frac{\alpha + \alpha - p}{p}\right)}$ leads us to the following linearized Sobolev inequality:

Corollary 2.6. Let $\alpha \in \mathbb{R}$ and $n + \alpha > p > 1$. Then the inequality
\[
\int_{B_R} |x|^\alpha |\nabla \phi|^p \, dx \geq \left( \frac{n + \alpha - p}{p-1} \right)^{p-1} \left( n + \alpha \right) \int_{B_R} \left(1 + \frac{|x|^p}{p-1}\right)^p |\phi|^p \, dx \\
- \left( \frac{n + \alpha - p}{Rp - R} \right)^{p-1} \int_{\partial B_R} |\phi|^p \, d\nu
\]
holds for every $\phi \in C^\infty(B_R)$.

Even though the literature has mostly focused on power radial weights, we now establish $L^p$ Hardy-type inequalities with nonradial weights with respect to the inclusion of boundary terms on smooth or piecewisely smooth bounded domains in $\mathbb{R}^n$.

In order to do so, we now take the pair as
\[
a = \frac{e^{\alpha x_1}}{\alpha^{p-1}} \quad \text{and} \quad u = e^{\alpha x_1}.
\]
This gives the following result.

Corollary 2.7. Let $\alpha > 0$ and $p > 1$. Then, for all $\phi \in C^\infty(B_R)$, we have
\[
\int_{B_R} \frac{e^{\alpha x_1}}{\alpha^{p-1}} |\nabla \phi|^p \, dx \geq -\alpha p \int_{B_R} e^{\alpha x_1} |\phi|^p \, dx + \frac{1}{R} \int_{\partial B_R} x e^{\alpha x_1} |\phi|^p \, d\nu.
\]

On the other hand, by specializing the functions as
\[
a = e^{\alpha(1-p)x_1} \quad \text{and} \quad u = e^{\alpha x_1},
\]
we have another weighted $L^p$ Hardy type inequality with a boundary term.

Corollary 2.8. Let $\alpha \in \mathbb{R}$ and $p > 1$. Then, for all $\phi \in C^\infty(B_R)$, we have
\[
\int_{B_R} e^{\alpha(1-p)x_1} |\nabla \phi|^p \, dx \geq \frac{\alpha |\alpha|^{p-2}}{R} \int_{\partial B_R} x e^{\alpha(1-p)x_1} |\phi|^p \, d\nu.
\]
We now set the non-symmetric functions
\[ a = e^{-px_1} \quad \text{and} \quad u = e^{x_1}. \]
This yields the following \( L^p \) Hardy type inequality with non-symmetric weights and a boundary remainder term.

**Corollary 2.9.** For any \( \phi \in C^\infty (B_R) \) and \( p > 1 \), one has
\[
\int_{B_R} \frac{|\nabla \phi|^p}{e^{px_1}} \, dx \geq \int_{B_R} \frac{\phi^p}{e^{px_1}} \, dx + \frac{1}{R} \int_{\partial B_R} \frac{x_1}{e^{px_1}} |\phi|^p \, d\nu.
\]

Another immediate application of Theorem 2.1 with the following functions
\[ a \equiv 1 \quad \text{and} \quad u = e^{x_1 + \cdots + x_n} \]
is the following inequality:

**Corollary 2.10.** For any \( \phi \in C^\infty (B_R) \) and \( p > 1 \), one has
\[
\int_{B_R} |\nabla \phi|^p \, dx \geq (1 - p) n^{p/2} \int_{B_R} |\phi|^p \, dx + \frac{n^{p/2}}{R} \int_{\partial B_R} (x_1 + \cdots + x_n) |\phi|^p \, d\nu.
\]

Let us mention that when one consider Theorem 2.1 with the pair
\[ a = e^{x_1 + \cdots + x_n} \quad \text{and} \quad u = e^{x_1 + \cdots + x_n}, \]
one can obtain the following result including an exponential weight:

**Corollary 2.11.** For any \( \phi \in C^\infty (B_R) \) and \( p > 1 \), one has
\[
\int_{B_R} e^{x_1 + \cdots + x_n} |\nabla \phi|^p \, dx \geq -pn^{p/2} \int_{B_R} e^{x_1 + \cdots + x_n} |\phi|^p \, dx
+ \frac{n^{p/2}}{R} \int_{\partial B_R} (x_1 + \cdots + x_n) e^{x_1 + \cdots + x_n} |\phi|^p \, d\nu.
\]

**Remark 2.3.** Note that one can also apply Theorem 2.1 on some domains \( \Omega \) in \( \mathbb{R}^n \) with piecewisely smooth boundary \( \partial \Omega \). The following is the first corollary in this direction.

Suppose that
\[ a \equiv 1 \quad \text{and} \quad u = x_1 r, \]
then we derive the following inequality:
**Corollary 2.12.** Let $\Omega = \{x \in B_{R}: x_1 > 0\}$. For all $\phi \in C^{\infty}(\Omega)$, there holds
\[
\int_{\Omega} |\nabla \phi|^2 \, dx \geq - (n + 1) \int_{\Omega} \frac{\phi^2}{|x|} \, dx + \int_{\partial \Omega} \left( \frac{1}{R} - \frac{1}{x_1} \right) \phi^2 \, d\nu.
\]

When we take the following pair
\[
a = \left( \frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_n^2} \right)^{\frac{2}{p}} \quad \text{and} \quad u = (x_1 x_2 \ldots x_n)^{1/p}
\]
in Theorem 2.1, we have the subsequent weighted $L^p$ Hardy type inequality with boundary terms.

**Corollary 2.13.** Let $1 < p < \infty$ and $\Omega = \{x \in B_{R}: x_i > 0, \ i = 1, 2, \ldots, n\}$. Then, for all $\phi \in C^{\infty}(\Omega)$, we have
\[
\int_{\Omega} \left( \frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_n^2} \right)^{\frac{2}{p}} |\nabla \phi|^p \, dx \geq \frac{1}{p} \int_{\Omega} \left( \frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_n^2} \right) |\phi|^p \, dx
- \frac{1}{p^{p-1}} \int_{\partial \Omega} \left( \frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_n^2} \right) |\phi|^p \, d\nu
+ \frac{n}{R p^{p-1}} \int_{\partial \Omega} |\phi|^p \, d\nu.
\]

We now set the pair as
\[
a = x_1^{p-1} \quad \text{and} \quad u = \left( \log \frac{R}{x_1} \right)^{\frac{p-1}{p}}.
\]
Hence, we derive the power logarithmic $L^p$ Hardy type inequality.

**Corollary 2.14.** Let $p > 1$ and $\Omega = \{x \in B_{R}: x_1 > 0\}$. Then, for all $\phi \in C^{\infty}(\Omega)$, we have
\[
\int_{\Omega} x_1^{p-1} |\nabla \phi|^p \, dx \geq \left( \frac{p-1}{p} \right)^p \int_{\Omega} \frac{|\phi|^p}{\log \frac{R}{x_1}} \, dx
+ \left( \frac{p-1}{p} \right)^{p-1} \frac{1}{R} \int_{\partial \Omega} \frac{R - x_1}{\log \frac{R}{x_1}}^{p-1} |\phi|^p \, d\nu.
\]

Inspired by the inequality of Maz’ya (1.2) and considering the functions
\[
a = x_n^{p-1} \quad \text{and} \quad u = (x_{n-1}^2 + x_n^2)^{-\frac{1}{p}},
\]
we obtain the following Maz’ya type inequality with a boundary term.
Corollary 2.15. Let $p > 1$ and $\Omega = \{x \in B_R : x_n > 0\}$. Then, for all $\phi \in C^\infty(\Omega)$, we have

$$\int_\Omega x_n^{p-1} |\nabla \phi|^p \, dx \geq \frac{1}{p^p} \int_\Omega \frac{|\phi|^p}{(x_{n-1}^2 + x_n^2)^{p/2}} \, dx - \frac{1}{R p^{p-1}} \int_{\partial \Omega} \frac{x_n^{p-1}(x_{n-1}^2 + x_n^2 - R x_n)}{(x_{n-1}^2 + x_n^2)^{p/2}} |\phi|^p \, d\nu.$$

Another consequence of the Theorem 2.1 with the special functions

$$a = e^{ax_1^2 x_2^2 \cdots x_n^2} \quad \text{and} \quad u = e^{-ax_1^2 x_2^2 \cdots x_n^2}$$

is the following inequality.

Corollary 2.16. For all $\phi \in C^\infty(B_R)$ and $\alpha > 0$, we have

$$\int_{B_R} e^{\alpha x_1^2 x_2^2 \cdots x_n^2} |\nabla \phi|^2 \, dx \geq 2\alpha \int_{B_R} e^{\alpha x_1^2 x_2^2 \cdots x_n^2} \left( \frac{1}{x_1^2} + \cdots + \frac{1}{x_n^2} \right) |\phi|^2 \, dx - \frac{2\alpha n}{R} \int_{\partial B_R} e^{\alpha x_1^2 x_2^2 \cdots x_n^2} \left( x_1^2 \cdots x_n^2 \right) |\phi|^2 \, d\nu.$$

Heisenberg-Pauli-Weyl Inequality. Finally, we turn our attention to the classical Heisenberg-Pauli-Weyl inequality which states that

$$\left( \int_{\mathbb{R}^n} |\nabla f(x)|^2 \, dx \right) \left( \int_{\mathbb{R}^n} |x|^2 |f(x)|^2 \, dx \right) \geq \frac{n^2}{4} \left( \int_{\mathbb{R}^n} |f(x)|^2 \, dx \right)^2 \tag{2.8}$$

for all $f \in L^2(\mathbb{R}^n)$. It has been studied in various contexts and we refer to [8] and [7] for an overview of the history and the relevance of this inequality.

It is worth mentioning here that Theorem 2.1 can be applied to obtain the local Heisenberg-Pauli-Weyl-type inequalities with boundary terms. In order to be more precise, we now take $p = 2$ in Theorem 2.1 and choose the following functions

$$a \equiv 1 \quad \text{and} \quad u = e^{-\alpha |x|^2}, \quad \alpha > 0.$$

This allows the derivation of the inequality

$$\int_{B_R} |\nabla \phi|^2 \, dx \geq -4\alpha^2 \int_{B_R} |x|^2 \phi^2 \, dx + 2\alpha n \int_{B_R} \phi^2 \, dx - 2\alpha R^2 \int_{\partial B_R} \phi^2 \, d\nu.$$

Optimizing in $\alpha$, we find the local Heisenberg-Pauli-Weyl type inequality with a boundary term.
Corollary 2.17. For all $\phi \in C^\infty(B_R)$, we have
\[
\left( \int_{B_R} |\nabla \phi|^2 \, dx \right) \left( \int_{B_R} |x|^2 \phi^2 \, dx \right) \geq \frac{1}{4} \left( n \int_{B_R} \phi^2 \, dx - R^2 \int_{\partial B_R} \phi^2 \, d\nu \right)^2.
\]

On the other hand, by setting the functions $a \equiv 1$ and $u = e^{-\alpha|x|}$, $\alpha > 0$,
we shall deduce from Theorem 2.1 that
\[
\int_{B_R} |\nabla \phi|^2 \, dx \geq -\alpha^2 \int_{B_R} \phi^2 \, dx + \alpha \left( (n - 1) \int_{B_R} \frac{\phi^2}{|x|} \, dx - \int_{\partial B_R} \phi^2 \, d\nu \right).
\]

Hence, a very similar argument applies to obtain the following version of the Heisenberg-Pauli-Weyl type inequality.

Corollary 2.18. For every $\phi \in C^\infty(B_R)$, one has
\[
\left( \int_{B_R} |\nabla \phi|^2 \, dx \right) \left( \int_{B_R} \phi^2 \, dx \right) \geq \frac{1}{4} \left( (n - 1) \int_{B_R} \frac{\phi^2}{|x|} \, dx - \int_{\partial B_R} \phi^2 \, d\nu \right)^2.
\]

Considering the functions
\[
a \equiv 1 \quad \text{and} \quad u = e^{-\alpha x^2_n}, \quad \alpha > 0,
\]
as a consequence of Theorem 2.1, we readily have
\[
\int_{B_R} |\nabla \phi|^2 \, dx \geq -4\alpha^2 \int_{B_R} x^2_n |\phi|^2 \, dx + 2\alpha \left( \int_{B_R} |\phi|^2 \, dx - \int_{\partial B_R} x^2_n |\phi|^2 \, d\nu \right).
\]

Arguing as above, we obtain the following inequality.

Corollary 2.19. For every $\phi \in C^\infty(B_R)$, one has
\[
\left( \int_{B_R} |\nabla \phi|^2 \, dx \right) \left( \int_{B_R} x^2_n |\phi|^2 \, dx \right) \geq \frac{1}{4} \left( \int_{B_R} |\phi|^2 \, dx - \int_{\partial B_R} x^2_n |\phi|^2 \, d\nu \right)^2.
\]

Finally, when we take the pair as
\[
a \equiv 1 \quad \text{and} \quad u = e^{-\alpha(x^2_{n-1} + x^2_n)}, \quad \alpha > 0,
\]
we get
\[
\int_{B_R} |\nabla \phi|^2 \, dx \geq -4\alpha^2 \int_{B_R} \left( x^2_{n-1} + x^2_n \right) |\phi|^2 \, dx + 4\alpha \int_{B_R} |\phi|^2 \, dx - \frac{2\alpha}{R} \int_{\partial B_R} \left( x^2_{n-1} + x^2_n \right) |\phi|^2 \, d\nu.
\]

Hence, with a similar approach, we achieve the following inequality.
Corollary 2.20. For all \( \phi \in C^\infty (B_R) \), we have
\[
\left( \int_{B_R} |\nabla \phi|^2 \, dx \right) \left( \int_{B_R} \left( x_{n-1}^2 + x_n^2 \right) |\phi|^2 \, dx \right) \geq \frac{A}{4},
\]
where \( A = \left( 2 \int_{B_R} |\phi|^2 \, dx - \frac{1}{R} \int_{\partial B_R} \left( x_{n-1}^2 + x_n^2 \right) |\phi|^2 \, d\nu \right)^2 \).

References

[1] Adimurthi, *Hardy-Sobolev inequality in \( H^1 (\Omega) \) and its applications*, Comm. Contem. Math., 4 (2002), 409–434.

[2] A. A. Balinsky, W. D. Evans and R. T. Lewis, *The Analysis and Geometry of Hardy’s Inequality*, Springer, Heidelberg, 2015.

[3] E. Berchio, D. Cassani and F. Gazzola, *Hardy-Rellich inequalities with boundary remainder terms and applications*, Manuscripta Math. 131 (2010), no. 3-4, 427–458.

[4] C. Cowan, *Optimal Hardy inequalities for general elliptic operators with improvements*, Commun. Pure Appl. Anal. 9 (2010), no. 1, 109–140.

[5] E.B. Davies, *A review of Hardy inequalities, in: The Maz’ya Anniversary Collection*, vol. 2, in: Oper. Theory Adv. Appl., vol. 110, Birkhäuser, Basel, 1999.

[6] T. Ekholm, H. Kovářík and A. Laptev, *Hardy inequalities for \( p \)-Laplacians with Robin boundary conditions*, Nonlinear Anal. 128 (2015), 365–379.

[7] W. Erb, *Uncertainty principles on Riemannian manifolds*, Logos Berlin, 2011.

[8] G. B. Folland and A. Sitaram, *The uncertainty principle: A mathematical survey*, J. Fourier Anal. Appl., 3 (1997), 207–238.

[9] J.A. Goldstein, I. Kombe and A. Yener, *A unified approach to weighted Hardy type inequalities on Carnot Groups*, Discrete Contin. Dyn. Syst. 37 (2017), no.4, 2009–2021.

[10] N. Ghoussoub and A. Moradifam, *Functional Inequalities: New Perspectives and New Applications*, Mathematical Survey and Monographs series, AMS, 2013.

[11] N. Ghoussoub and A. Moradifam, *Bessel potentials and optimal Hardy and Hardy-Rellich inequalities*, Math. Ann. 349 (2011), 1-57.

[12] H. Kovářík and A. Laptev, *Hardy inequalities for Robin Laplacians*, J. Funct. Anal. 262 (2012), 4972–4985.

[13] A A. Kufner and B. Opic, *Hardy-type inequalities*, Pitman Reserach Notes in Math. Series 219, Longman Sci. and Tech, Harlow, 1990.

[14] P. Lindqvist, *On the equation \( \text{div}(|\nabla u|^{p-2} \nabla u) + \lambda |u|^{p-2} u = 0 \)*, Proc. Amer. Math. Soc. 109 (1990), 157–164.

[15] V.G. Maz’ya, *Sobolev Spaces*, Springer, Berlin, 1985.

[16] Y. Pinchover and I. Versano, *Optimal Hardy-weights for elliptic operators with mixed boundary conditions*, arXiv:2103.13979.

[17] M. Ruzhansky and D. Suragan, *Hardy inequalities on homogeneous groups*, Progress in Math. Vol. 327, Birkhäuser, 2019.

[18] I. Skrzypczak, *Hardy type inequalities derived from \( p \)-harmonic problems*, Nonlinear Anal. 93 (2013), 30-50.

[19] Z.Q. Wang and M. Zhu, *Hardy inequalities with boundary terms*, Electron. J. Differential Equations 2003, No. 43.
Ismail Kombe, Department of Mechatronics Engineering, Faculty of Engineering, Istanbul Commerce University, Kucukyalı, 34840, İstanbul, Türkiye
Email address: ikombe@ticaret.edu.tr

Abdullah Yener, Department of Mathematics, Faculty of Humanities and Social Sciences, Istanbul Commerce University, Beyoğlu, 34445, İstanbul, Türkiye
Email address: ayener@ticaret.edu.tr