On the exact calculation of travelling wave solutions to nonlinear evolution equations

Francisco M. Fernández

INIFTA (UNLP, CCT La Plata-CONICET), División Química Teórica Blvd. 113 y 64 S/N, Sucursal 4, Casilla de Correo 16, 1900 La Plata, Argentina

We analyse and compare three methods for the exact calculation of travelling wave solutions to nonlinear partial differential equations. We simplify the so called \((G'/G)\)-expansion method and apply two of those methods to simple physical problems.

PACS numbers:

I. INTRODUCTION

There has recently been great interest in the application of the \((G'/G)\)-expansion method to obtaining travelling wave solutions of some nonlinear partial differential equations\[1, 2\]. The method was earlier applied to a variety of such problems\[3, 4\]. There are many mathematical recipes for that purpose\[1, 2\], two of them are the exp–function method\[5\] and the tanh–function method\[6\]. All those prescriptions look quite similar and have also been applied to similar toy problems. It is curious that there has not been much interest in comparing them.

In this paper we analyze those recipes. In Sec. \[II\] we outline the \((G'/G)\)-expansion method and in Sec. \[III\] propose a simplified version of it. In Sec. \[IV\] we compare the three prescriptions just mentioned and in Sec. \[V\] we apply them to two simple examples. Finally, in Sec. \[VI\] we draw conclusions from the results of the preceding sections.

II. THE \((G'/G)\)-EXPANSION METHOD

This method was proposed to obtain analytical travelling–wave solutions to nonlinear differential equations of the form\[2\]

\[
P(u, u_t, u_{tt}, u_x, u_{xx}, u_{tx}, \ldots) = 0 \tag{1}
\]

where the subscripts indicate differentiation of \(u(x, t)\) with respect to its arguments, and \(P\) is a polynomial function.

The \((G'/G)\)-expansion method has been applied to several examples of travelling waves that are particular solutions of the form \(u(x, t) = u(\xi)\), where \(\xi = x - ct\) so that \(u_t = -cu'\) and \(u_x = u'\)[1, 2, 3, 4].

One assumes that the travelling–wave solution to the differential equation \(1\) is of the form

\[
u(\xi) = \sum_{j=0}^{m} a_j \left[ \frac{G'(\xi)}{G(\xi)} \right]^j \tag{2}
\]

where the function \(G(\xi)\) is a solution to

\[
G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0 \tag{3}
\]

It follows from

\[
\frac{d}{d\xi} \left( \frac{G'}{G} \right) = - \left( \frac{G'}{G} \right)^2 - \lambda \frac{G'}{G} - \mu \tag{4}
\]

where the function \(G(\xi)\) is a solution to

\[
G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0 \tag{3}
\]

It follows from

\[
\frac{d}{d\xi} \left( \frac{G'}{G} \right) = - \left( \frac{G'}{G} \right)^2 - \lambda \frac{G'}{G} - \mu \tag{4}
\]
that \( d^k u/d\xi^k \) is a polynomial function of \( G'/G \) of degree \( m + k \). For this reason, substitution of Eq. (2) into Eq. (1) yields a polynomial function of \( G'/G \) and if we set its coefficients equal to zero we may obtain the unknown coefficients \( a_j \).

III. THE \((F'/F)\)–EXPANSION METHOD

The function \( F(\xi) = e^{\lambda \xi/2} G(\xi) \) is a solution of the differential equation \( F'' + (\mu - \lambda^2/4) F = 0 \) and satisfies

\[
\frac{G'}{G} = \frac{F'}{F} - \frac{\lambda}{2}
\]  

(5)

Therefore the series (2) becomes

\[
U(\xi) = \sum_{j=0}^{m} b_j \left[ \frac{F'(\xi)}{F(\xi)} \right]^j
\]  

(6)

where the coefficients \( b_j \) absorb the parameter \( \lambda \) that therefore dissapears from the equations. We may call \( \gamma = \mu - \lambda^2/4 \) and simply require that the function \( F(\xi) \) is a solution to

\[
F''(\xi) + \gamma F(\xi) = 0
\]  

(7)

For completeness and comparison we show the solutions to this trivial equation:

\[
F(\xi) = \begin{cases} 
  c_1 \cos \left( \sqrt{\gamma} \xi \right) + c_2 \sin \left( \sqrt{\gamma} \xi \right), & \gamma > 0 \\
  c_1 \cosh \left( \sqrt{-\gamma} \xi \right) + c_2 \sinh \left( \sqrt{-\gamma} \xi \right), & \gamma < 0 \\
  c_1 + c_2 \xi, & \gamma = 0 
\end{cases}
\]  

(8)

Notice that this expression leads to all the cases explicitly shown by Zayed and Gepreel [1]. Therefore, this method is entirely equivalent to the \((G'/G)\)–expansion method but the equation that defines the main function is simpler. In Sec. IV we show an application of the \((F'/F)\)–expansion method.

It is worth noticing that the simplification proposed here also applies to the generalized \((G'/G)\)–expansion method [7] because it is based on the same equation for \( G(\xi) \).

IV. COMPARISON OF DIFFERENT METHODS

The function \( w(\xi) = -F'(\xi)/F(\xi) \) is a solution to the Riccati equation

\[
w' = \gamma + w^2
\]  

(9)

proposed by Fan [6] to expand the traveling–wave solutions in the form

\[
U(\xi) = \sum_{j=0}^{m} a_j w^j
\]  

(10)

We appreciate that the extended tanh–function method is equivalent in principle to the \((F'/F)\)–expansion method and thereby to the \((G'/G)\)–expansion method. There is a difference, however, in that the latter two methods appear
to be more flexible because they have the two independent solutions of the generating equation built in the expansion function through the constants $c_1$ and $c_2$.

We can rewrite the solution to Eq. (7) in a different way

$$F(\xi) = c_1 e^{\alpha \xi} + c_2 e^{-\alpha \xi}, \alpha = \sqrt{-\gamma}$$

(11)

so that the expansion (3), and consequently also (2), reduces to a ratio of two polynomial functions of $e^{\alpha \xi}$ of degree $2m$. We thus conclude that the $(G'/G)$–expansion method is a particular case of the Exp–function method (9) that is based on a solution of the form

$$u(\eta) = \frac{\sum_{j=-c}^{d} a_j \exp(j\eta)}{\sum_{j=-c}^{d} b_j \exp(j\eta)}, \eta = kx + \omega t$$

(12)

Notice that $\alpha \xi = \eta$ if $\alpha = k$ and $c = -\omega/k$. It is clear that we can rewrite the rational approximation (12) in such a way that it only shows positive powers of $e^{\eta}$ in the numerator and denominator.

In the following section we apply the Exp–function method to a problem that apparently cannot be treated by means of the $(G'/G)$–expansion method.

V. APPLICATIONS

It is not the purpose of this paper to abound with arbitrary tailor–made toy model equations for the application of the methods just described. However, in what follows we consider two well–known exactly solvable problems that will enable us to compare the methods outlined above.

Our first example is Fisher’s equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1 - u)$$

(13)

that was originally derived for the simulation of propagation of a gene in a population and also arises in heat and mass transfer, biology, and ecology. This equation is so popular that appears in most on–line encyclopaedias of mathematics (8). For our present purposes it suffices to mention the application of the Exp–function method (9).

If we choose $\xi = x - ct$ (we may also choose $\xi = x + ct$) we obtain

$$u'' + cu' + u(1 - u)$$

(14)

The leading terms of $u''$ and $u^2$ are of degree $m + 2$ and $2m$, respectively, and therefore they do not cancel unless $m = 2$. If we substitute Eq. (11) into Eq. (14) we easily obtain $b_2 = 6, b_1 = -6c/5, b_0 = [25(8\gamma + 1) - c^2]/50, \gamma = -c^2/100$. The constant term gives us the only values of $c$ for which there are exact solutions: $c = \pm 5/\sqrt{6}$. For $c = 5/\sqrt{6}$ we obtain

$$u(\xi) = \frac{1}{\left[1 + C e^{\xi/\sqrt{6}}\right]^2}$$

(15)

By means of the Exp–function method Zhou (4) identified four cases that produced four solutions named $u_j(x, t)$, $j = 1, 2, 3, 4$. However, they are not essentially different as one can show that they are closely related by the symmetry of the problem: $u_1(-x, t) = u_2(x, t), u_3(-x, t) = u_4(x, t)$, and $u_1(x, -t)$ becomes $u_3(x, t)$ after substitution of $4/b_0$. 

for $b_0$ in the latter. On the other hand, Eq. (15) is identical to $u_2(x, t)$ if we simply substitute $2/b_0$ for $C$. The other solutions follow from the indicated symmetry or by choosing the other root $c = -5/\sqrt{6}$.

Our second example is the Bratu–Gelfand equation

$$u''(x) + \lambda e^{u(x)} = 0, \quad u'(0) = u(1) = 0$$

(16)

that appears in simple models for the stationary theory of the thermal explosion[10]. First, we have to convert this strongly nonlinear equation into a polynomial one, which we do by means of the transformation $u(x) = -n \ln v(x)$ that leads to $\lambda v^{2-n} + n (v'^2 - vv'') = 0$. In order to have the simplest equation we choose $n = 2$, so that

$$2 (v'^2 - vv'') + \lambda = 0, \quad v'(0) = 0, \quad v(1) = 1$$

(17)

If we substitute

$$v(x) = a_{-1} e^{-\alpha x} + a_0 + a_1 e^{\alpha x}$$

(18)

which is a particular case of Eq. (12), into Eq. (17) we easily obtain

$$a_0 = 0, \quad a_{-1} = \frac{\lambda}{8a_1 \alpha^2}$$

(19)

The boundary conditions $v'(0) = 0$ and $v(1) = 1$ give us two additional equations

$$a_1 \alpha - \frac{\lambda}{8a_1 \alpha} = 0$$

$$a_1 e^\alpha + \frac{\lambda e^{-\alpha}}{8a_1 \alpha^2} = 1$$

(20)

respectively, from which we obtain

$$\lambda = \frac{8a_1^2 e^{2\alpha}}{(e^{2\alpha} + 1)^2}$$

$$a_1 = \frac{e^\alpha}{e^{2\alpha} + 1}$$

(21)

The first equation shows a well–known bifurcation diagram from which one obtains the critical condition of ignition $\alpha_c = 1.19967864$ that is a root of $d\lambda/d\alpha = 0$[10]. In other words, explosion takes place in a self–ignition process for a plane–parallel vessel when $\lambda = \lambda_c = \lambda(\alpha_c) = 0.8784576797$[10].

Neither the $(G'/G)$–expansion method[1, 2, 3, 4] nor the tanh–function method[6] provide a function of the form (18) that is already given by the more flexible Exp–function method[5].

VI. CONCLUSIONS

There seems to be an ever increasing number of methods for the solution of more or less trivial nonlinear problems; many of them are cited in the papers by Zayed and Gepreel[1] and Ganji and Abdolahzadeh[2]. Our analysis of three such recipes was motivated by the interest of this journal in one of those methods. We have shown how to simplify the main equation of the $(G'/G)$–expansion method and have compared it with the tanh–method and Exp–function methods. Our simple and straightforward discussion shows that the first two ones are similar in essence and that the
third one is more general and flexible. The three of them lead to solutions in the form of quotients of polynomial functions of exponential ones.

We have illustrated the simplification of the \((G'/G)\)-expansion method, named \((F'/F)\)-expansion method, by its application to the simple and widely known Fisher’s equation \([8, 9]\). Finally, we have verified the greater flexibility of the Exp–function method with the aid of the Bratu–Gelfand equation \([10]\).

The papers of Zayed and Gepreel \([1]\) and Ganji and Abdollahzadeh \([2]\) do not exhibit the amazing errors of other articles that I discussed in the past \([11, 12, 13, 14, 15, 16]\). However, they are standard applications of a method developed earlier and widely applied in previous papers by several authors \([1, 2]\) (see also the references cited therein).

It is surprising that Journal of Mathematical Physics (supposedly devoted to original and more elaborate mathematical applications to physics) accepts such kind of contributions. As an experiment I submitted a comment to JMP (with the content shown above) and it was rejected on the grounds that it did not “contain sufficiently significant new results to warrant its publication in JMP”. Does it seem that the Elsevier policy of publishing poor papers for reasons other than purely scientific ones is beginning to contaminate also JMP?

[1] E. M. E. Zayed and K. A. Gepreel, J. Math. Phys. 50, 013502 (12 pp.) (2008).
[2] D. D. Ganji and M. Abdollahzadeh, J. Math. Phys. 50, 013519 (10 pp.) (2009).
[3] M. Wang, X. Li, and J. Zhang, Phys. Lett. A 372, 417 (2008).
[4] M. Wang, J. Zhang, and X. Li, Appl. Math. Comput. 206, 321 (2008).
[5] J-H. He and X-H Wu, Chaos, Solitons Fractals 30, 700 (2006).
[6] E. Fan, Phys. Lett. A 277, 212 (2000).
[7] J. Zhang, X. Wei, and X. Lu, Phys. Lett. A 372, 3653 (2008).
[8] http://en.wikipedia.org/wiki/Fisher%27s-equation
http://eqworld.ipmnet.ru/en/solutions/npde/npde1101.pdf
http://mathworld.wolfram.com/FishersEquation.html
http://www.nationmaster.com/encyclopedia/Fisher%27s-equation
[9] X-W Zhou, J. Phys.: Conf. Ser. 96, 012063 (5 pp.) (2008).
[10] D. A. Frank-Kamenetskii, Diffusion and heat exchange in chemical kinetics (Princeton University Press, Princeton, NJ, 1955).
[11] F. M. Fernández, Perturbation Theory for Population Dynamics, arXiv:0712.3376v1
[12] F. M. Fernández, On Some Perturbation Approaches to Population Dynamics, arXiv:0806.0263v1
[13] F. M. Fernández, On the application of homotopy-perturbation and Adomian decomposition methods to the linear and nonlinear Schrödinger equations, arXiv:0808.1515v1
[14] F. M. Fernández, On the application of the variational iteration method to a prey and predator model with variable coefficients, arXiv.0808.1875v2
[15] F. M. Fernández, On the application of homotopy perturbation method to differential equations, arXiv:0808.2078v2
[16] F. M. Fernández, Homotopy perturbation method: when infinity equals five, 0810.3318v1