QUATERNIONIC PROJECTIVE BUNDLE THEOREM AND GYSIN TRIANGLE IN MW-MOTIVIC COHOMOLOGY

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Abstract. In this paper, we show that the motive $H^m_{P^n}$ of the quaternionic Grassmannian (as defined by I. Panin and C. Walter) splits in the category of effective MW-motives (as defined by B. Calmès, F. Déglise and J. Fasel). Moreover, we extend this result to an arbitrary symplectic bundle, obtaining the so-called quaternionic projective bundle theorem. This enables us to define Pontryagin classes of symplectic bundles in the Chow-Witt ring.

As an application, we prove that there is a Gysin triangle in MW-motivic cohomology in case the normal bundle is symplectic.

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1. Introduction

The aim of this paper is to investigate the fundamental properties of MW-motivic cohomology, as defined by B. Calmès, F. Déglise and J. Fasel. This cohomology theory is a generalization of ordinary motivic cohomology as developed by V. Voevodsky. One of the basic properties of the latter is the existence of Chern classes associated to vector bundles. As usual, the first step is the construction of the first Chern class of a line bundle. Then, the higher Chern classes are defined by the Chern polynomial (see [Har77, A.3] in the case of the Chow ring), requiring the calculation of the cohomology of the projective bundle associated to a vector bundle (see [Har77, A.2]). In the motivic setting, the projective bundle theorem takes the following form (see [Deg12, 2.10], [MVW06, Theorem 15.12], [SV, Theorem 4.5] and [MVW06, Definition 14.1] for notations):

Proposition 1.1. Let $X$ be a smooth scheme over a perfect field and $\mathcal{E}$ be a vector bundle of rank $n$ over $X$, then the map

$$Z_{tr}(\mathbb{P}(\mathcal{E})) \xrightarrow{p \circ c_i(O_X(1))^i} \bigoplus_{i=0}^{n-1} Z_{tr}(X)(i)[2i]$$

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is an isomorphism in $\text{DM}^{\text{eff}}$, where $p : \mathbb{P}(\mathcal{E}) \to X$ is the structure map and $c_1$ is the first Chern class map.

One of the major differences between MW-motivic cohomology and its ordinary version is that the former doesn’t admit Chern classes, i.e. the MW-motivic cohomology ring of the projective space can’t be in general described in terms of the MW-motivic cohomology ring of the base scheme. Indeed, suppose that we have an isomorphism (for notations, see Section 2)

$$\tilde{Z}_{tr}(\mathbb{P}^2_k) \cong \bigoplus_{i=0}^2 \tilde{Z}(i)[2i]$$

Then, applying $\text{Hom}_{\text{DM}^{\text{eff}},-}(\mathbb{P}^2_k, \tilde{Z}(2)[4])$ on both sides, we find

$$\tilde{C}H^2(\mathbb{P}^2_k) \cong K_0^{MW}(k),$$

by Proposition 2.9 and Lemma 4.2 below, contradicting [Fas13, Corollary 11.8].

However, it is still possible to define interesting characteristic classes, following the method developed by I. Panin and C. Walter [PW10]. Our main result is the following (see Theorem 4.3):

**Theorem 1.1.** Let $X$ be a smooth scheme over an infinite perfect field of characteristic different from 2 and $(\mathcal{E}, m)$ be a symplectic vector bundle of rank $2n+2$ on $X$. Then, the map

$$\tilde{Z}_{tr}(HGr_X(\mathcal{E})) \xrightarrow{\pi \circ p_1(\mathcal{V}^\vee)} \bigoplus_{i=0}^n \tilde{Z}_{tr}(X)(2i)[4i]$$

is an isomorphism in $\tilde{DM}^{\text{eff},-}$, where $HGr_X(\mathcal{E})$ is the quaternionic projective bundle of $\mathcal{E}$ (see Definition 3.4), $\pi : HGr_X(\mathcal{E}) \to X$ is its structure map, $\mathcal{V}^\vee$ is the dual tautological bundle and $p_1$ is the first Pontryagin class map (see Definition 4.2).

In the statement of the theorem, $\tilde{DM}^{\text{eff},-}$ is the category of effective MW-motives as defined in [DF17, §3.2]. As a consequence of the above theorem, we can define (higher) Pontryagin classes for symplectic bundles which lie in the Chow-Witt ring (see Definition 4.4).

Moreover, we provide a Gysin triangle for MW-Motivic cohomology in some special case. With this in mind, recall that the Gysin triangle in Voevodsky’s category of effective motives is of the following form ([MVW06, Theorem 15.15] and [SV, Theorem 4.10]).

**Proposition 1.2.** Let $X$ be a smooth scheme over a perfect field and $Y \subseteq X$ be a smooth closed subscheme with codim$(Y) = n$. Then we have a distinguished triangle in $\text{DM}^{\text{eff}}$:

$$Z_{tr}(X \setminus Y) \to Z_{tr}(X) \to Z_{tr}(Y)/(n)[2n] \to Z_{tr}(X \setminus Y)[1].$$

It can’t be expected that such a triangle exists in $\tilde{DM}^{\text{eff},-}$. Indeed, it would yield the projective bundle formula as a corollary. Nevertheless, we are able to construct such a triangle when the normal bundle to $Y$ in $X$ is symplectic (Theorem 5.2) following the methods of [PW10] and [Deg12].

**Theorem 1.2.** Let $X$ be a smooth scheme over an infinite perfect field of characteristic different from 2 and $Y \subseteq X$ be a smooth closed subscheme with symplectic normal bundle and codim$(Y) = 2n$. Then we have a distinguished triangle

$$\tilde{Z}_{tr}(X \setminus Y) \to \tilde{Z}_{tr}(X) \to \tilde{Z}_{tr}(Y)/(2n)[4n] \to \tilde{Z}_{tr}(X \setminus Y)[1]$$

in $\tilde{DM}^{\text{eff},-}$. 
The organization of the paper is as follows. In Section 2 we briefly survey the properties of the category $\text{DM}_{eff}^{\omega}$ for the convenience of the reader. Here, our exposition slightly differs from the one in [DF17], avoiding altogether the notion of model category. We also recall the definition and basic properties of MW-motivic cohomology. In Section 3 we recall the definition of the quaternionian Grassmannian and set our conventions used in Section 4, where the proof of the main theorem takes place. We conclude this paper with the construction of the Gysin triangle in the last section.

**Conventions.**

(1) All the schemes are over an infinite perfect field $k$ of characteristic not 2 unless specified. The field $k$ is called the base field. The word 'smooth' always means 'smooth, separated and equidimensional' for convenience. Hence 'the category of smooth schemes over $k$' is then just 'the category of nonsingular, finite type, equidimensional and separated schemes over $k$'. We denote the category of smooth schemes by $\text{Sm}/k$.

(2) We always use the notation $S^c$ to denote the complement of the subset $S$ in some set.

2. MW-motivic complexes

In this section, we recall the basic definitions and facts about the category of MW-motives following [CF14], [DF17] (and sometimes [MVW06] and [SV] when we appeal to properties of the category of ordinary motives).

2.1. Sheaves with MW-transfers. Let $n \in \mathbb{Z}$, $F/k$ be a finitely generated field extension of the base field $k$ and $L$ be a one-dimensional $F$-vector space. One can define $K_n^{\text{MW}}(F,L)$ as in [Mor12] Remark 2.21. If $X$ is a smooth scheme, $\mathcal{L}$ is a line bundle over $X$ and $y \in X$, we set

$$\tilde{K}^n_{\omega}(k(y), \mathcal{L}) := K^n_{\omega}(k(y), \omega_{k(y)/k} \otimes_{O_{X,y}} \mathcal{L}_y),$$

where $k(y)$ is the residue field of $y$ and $\omega_{k(y)/k}$ is the determinant of the vector space $\Omega_{k(y)/k}$ of differentials. If $T \subset X$ is a closed set, $n, m \in \mathbb{Z}$, define

$$C^n_{RS,T}(X; \tilde{K}^n_{\omega}; \mathcal{L}) = \bigoplus_{z \in X^{(n)} \cap T} \tilde{K}^n_{m-n}(k(z), \omega_{z/k} \otimes \mathcal{L}),$$

where $X^{(n)}$ means the points of codimension $n$ in $X$. Then $C^n_{RS,T}(X; \tilde{K}^n_{\omega}; \mathcal{L})$ form a complex (see [Mor12] Definition 4.11, [Mor12] Remark 4.13, [Mor12] Theorem 4.31] and [Fas08, Définition 10.2.11]), which is called the Rost-Schmid complex with support on $T$. Define (see [CF14] Definition 3.1)

$$\overline{CH}^n_T(X, \mathcal{L}) = H^n(C^n_{RS,T}(X; \tilde{K}^n_{\omega}; \mathcal{L})).$$

For any $X, Y \in \text{Sm}/k$ (recall our conventions on smooth schemes), define $\omega(X, Y)$ to be the poset of closed subset in $X \times_k Y$ such that each of its component is finite over a connected component of $X$ and of dimension $\text{dim}X$. Let

$$\text{Cor}_k(X, Y) := \lim_{T} \overline{CH}^{\text{dim}Y}_{T}(X \times_k Y, \omega_{X \times_k Y/Y})$$

be the finite Chow-Witt correspondences between $X$ and $Y$ over $k$, where $T \in \omega(X, Y)$. For any $f \in \text{Cor}_k(X, Y)$ and $g \in \text{Cor}_k(Y, Z)$, we can define $g \circ f \in \text{Cor}_k(X, Z)$ as in [CF14] 4.2]. This produces an additive category $\text{Cor}_k$ whose objects are the same as in $\text{Sm}/k$ and whose morphisms are defined above. There is a functor $\gamma: \text{Sm}/k \rightarrow \text{Cor}_k$ sending a morphism to its graph (see [CF14] 4.3]).
We define a presheaf with MW-transfers to be a contravariant additive functor from $\text{Cor}_k$ to $\text{Ab}$ and call it a sheaf with MW-transfers if it’s a Nisnevich sheaf after restricting to $\text{Sm}/k$ via $\gamma$. For any smooth scheme $X$, let $\tilde{c}(X)$ be the presheaf with MW-transfers defined by

$$\tilde{c}(X)(Y) = \widetilde{\text{Cor}}_k(Y, X)$$

For obvious reasons, we call $\tilde{c}(X)$ the representable presheaf of $X$.

Let $\widetilde{PSh}(k)$ be the category of presheaves with MW-transfers and let $\widetilde{Sh}(k)$ be the full subcategory of sheaves with MW-transfers (see [DF17] Definition 1.2.1 and Definition 1.2.4]). Both categories are abelian and have enough injectives ([DF17 §1.1, Proposition 1.2.11]). There is an adjunction (see [DF17 Proposition 1.2.11])

$$\tilde{a} : \widetilde{PSh}(k) \rightleftarrows \widetilde{Sh}(k) : \tilde{0},$$

where $\tilde{a}$ is the sheafification functor and $\tilde{0}$ is the forgetful functor. We set

$$\tilde{\mathcal{Z}}_{tr}(X) = \tilde{a}(\tilde{c}(X)).$$

For any $F \in \widetilde{PSh}(k)$ and $T \in \text{Sm}/k$, we define a presheaf with MW-transfers $F^T$ by $F^T(X) = F(X \times T)$ following [MVW06 Exercise 2.9]. For any $f : T_1 \rightarrow T_2$ in $\text{Cor}_k$, we have a morphism $F^f : F^{T_2} \rightarrow F^{T_1}$ induced by the tensor product of correspondences (see [CF14 4.4]). It’s clear that if $F$ is a sheaf with MW-transfers, $F^T$ is a sheaf with MW-transfers as well. For any presheaf with MW-transfers $F$, we define a complex $C_*F$ with $(C_*F)_n = F^{\Delta^n}, n \geq 0$ with usual boundary maps for (co-)simplicial complexes, where $\Delta^n$ is the algebraic $n$-simplex (see [MVW06 Definition 2.14] for details).

For every $f \in \text{Cor}(X, Y)$, there is a natural map $\tilde{\mathcal{Z}}_{tr}(f) : \tilde{\mathcal{Z}}_{tr}(X) \rightarrow \tilde{\mathcal{Z}}_{tr}(Y)$ induced by $f$. For every $X \in \text{Sm}/k$ and $x : \text{Spec } k \rightarrow X$, we say that the pair $(X, x)$ is a pointed scheme. We define $\tilde{\mathcal{Z}}_{tr}((X_1, x_1) \wedge \ldots \wedge (X_n, x_n))$ for pointed schemes $(X_i, x_i)$ as the cokernel of the map

$$\theta_n : \oplus_i \tilde{\mathcal{Z}}_{tr}(X_1 \times \ldots \times X_i \times \ldots \times X_n) \xrightarrow{\sum (-1)^i \text{id}_{x_1 \times \ldots \times x_i \times \ldots \times x_n}} \tilde{\mathcal{Z}}_{tr}(X_1 \times \ldots \times X_n).$$

We denote $\tilde{\mathcal{Z}}_{tr}((X, x) \wedge \ldots \wedge (X, x))$ by $\tilde{\mathcal{Z}}_{tr}((X, x)^\wedge n)$, $\tilde{\mathcal{Z}}_{tr}((X, x))$ by $\tilde{\mathcal{Z}}_{tr}((X, x)^\wedge 1)$ and $\tilde{\mathcal{Z}}_{tr}(\text{Spec } k)$ by $\tilde{\mathcal{Z}}_{tr}((x, x)^\wedge 0)$.

As usual, we define $\tilde{Z}(q) = C_*\tilde{\mathcal{Z}}_{tr}(G_{\text{aff}}^n)[-q]$ for $q \geq 0$ and further we set $\tilde{Z} = \tilde{Z}(0)$.

**Proposition 2.1.** Let $X \in \text{Sm}/k$ and $U_1 \cup U_2 = X$ be a Zariski covering. Then, we have an exact sequence of sheaves with MW-transfers:

$$0 \rightarrow \tilde{\mathcal{Z}}_{tr}(U_1 \cap U_2) \rightarrow \tilde{\mathcal{Z}}_{tr}(U_1) \oplus \tilde{\mathcal{Z}}_{tr}(U_2) \rightarrow \tilde{\mathcal{Z}}_{tr}(X) \rightarrow 0.$$

**Proof.** The proof of [MVW06 Proposition 6.14] applies, replacing [MVW06 Proposition 6.12] by [DF17 Lemma 2.3.6].

Following the notation in [MVW06 Lemma 2.13], we let $[x_i]$ be the composition $X_i \rightarrow \text{Spec } k \xrightarrow{x_i} X_i$ and $e_i \in \text{Cor}_k(X_i, X_i)$ to be $\text{id}_{X_i} - \tilde{\mathcal{Z}}_{tr}([x_i])$.

**Lemma 2.1.** In the notations above, $\tilde{\mathcal{Z}}_{tr}((X_1, x_1) \wedge \ldots \wedge (X_n, x_n)), n \geq 2$, is just the image of the map

$$e_1 \times \ldots \times e_n : \tilde{\mathcal{Z}}_{tr}(X_1 \times \ldots \times X_n) \rightarrow \tilde{\mathcal{Z}}_{tr}(X_1 \times \ldots \times X_n).$$

Moreover, the inclusion of $\tilde{\mathcal{Z}}_{tr}((X_1, x_1) \wedge \ldots \wedge (X_n, x_n))$ into $\tilde{\mathcal{Z}}_{tr}(X_1 \times \ldots \times X_n)$ as an image is a section of $e_1 \times \ldots \times e_n$. 


Proof. We prove the same statements after replacing \( \tilde{\mathcal{Z}}_{tr} \) by \( \bar{e} \) and then sheafify.

The first statement is just to say that \( \text{Ker}(e_1 \times \ldots \times e_n) = \text{Im}(\theta_n) \setminus \text{Im}(\theta_1) \subset \text{Ker}(e_1 \times \ldots \times e_n) \) because \( e_i \circ [x_i] = 0 \). \( \text{Ker}(e_1 \times \ldots \times e_n) \subset \text{Im}(\theta_n) \) because \( e_\alpha \times \ldots \times e_\alpha = id_{X_1 \times \ldots \times X_n} + \sum f_{\alpha_1} \times \ldots \times f_{\alpha_n} \),

where for every n-tuple \((\alpha_1, \ldots, \alpha_n)\), there exists one \( \alpha_i \) such that \( f_{\alpha_i} = -[x_i] \). So \( f_{\alpha_1} \times \ldots \times f_{\alpha_n} \) factor through \( id \times \ldots \times x_1 \times \ldots \times id \) for that \( i \).

The second statement follows from the fact that \( e_i \) is idempotent. \( \square \)

So by the lemma above, we may regard \( \tilde{\mathcal{Z}}_{tr}((X_1, x_1) \times \ldots \times (X_n, x_n)) \) as a direct summand of \( \tilde{\mathcal{Z}}_{tr}(X_1 \times \ldots \times X_n) \).

Lemma 2.2. For any two pointed schemes \((X_1, x_1), (X_2, x_2)\), we have a split exact sequence

\[
0 \rightarrow \tilde{\mathcal{Z}}_{tr}(X_1 \times X_2) \rightarrow \tilde{\mathcal{Z}}_{tr}(X_1 \times X_2, (x_1, x_2)) \rightarrow \tilde{\mathcal{Z}}_{tr}(X_1, x_1) \oplus \tilde{\mathcal{Z}}_{tr}(X_2, x_2) \rightarrow 0.
\]

Proof. We have a split short exact sequence (by direct computation)

\[
\tilde{\mathcal{Z}}_{tr}(\text{Spec } k) \oplus \tilde{\mathcal{Z}}_{tr}(X_1 \times X_2) \rightarrow \tilde{\mathcal{Z}}_{tr}(X_1 \times X_2, (x_1, x_2)) \rightarrow \tilde{\mathcal{Z}}_{tr}(X_1, x_1) \oplus \tilde{\mathcal{Z}}_{tr}(X_2, x_2),
\]

where \( \pi : X_1 \times X_2 \rightarrow \text{Spec } k \) is the structure map. Quotienting the first two terms above by \( \tilde{\mathcal{Z}}_{tr}(\text{Spec } k) \), we get the result (this technique is called ‘killing one point’). \( \square \)

2.2. Motivic complexes. Let \( D^- \) be the derived category of the category \( C^-(\tilde{\mathcal{S}}h(k)) \) of bounded above complexes of sheaves with MW-transfers [MV, §10.4]. The following proposition summarizes what we need about (Nisnevich) hypercohomology.

Proposition 2.2. For any \( C \in D^- \) and any \( i \in \mathbb{N} \), we have an isomorphism of functors \( \text{Sin}_k \rightarrow \text{Ab} \)

\[
\text{Hom}_{D^-}(\tilde{\mathcal{Z}}_{tr}(-), C[i]) \cong H^i(-, C).
\]

Further, let \( X \) be a smooth scheme, \( Z \subset X \) be a closed subset and \( U = X \setminus Z \). Then, we have an isomorphism of functors \( D^- \rightarrow \text{Ab} \)

\[
\text{Hom}_{D^-}(\tilde{\mathcal{Z}}_{tr}(X)/\tilde{\mathcal{Z}}_{tr}(U), -[i]) \cong H^i_Z(X, -).
\]

Proof. The first statement can be seen from the universal property in [GM, page 188]. For the second statement, one first proves that

\[
\text{Hom}_{\tilde{\mathcal{S}}h}(\tilde{\mathcal{Z}}_{tr}(X)/\tilde{\mathcal{Z}}_{tr}(U), -) \cong -Z(X),
\]

where the right hand side denotes sections with support in \( Z \), defined by the left exact sequence

\[
0 \rightarrow F_Z(X) \rightarrow F(X) \rightarrow F(U).
\]

Consequently, both terms have the same hypercohomology functor. Additionally, we have \( \text{Ext}^i(F, -) \cong \text{Hom}_{D^-}(F, -[i]) \) for any sheaf with MW-transfers \( F \), yielding the second statement. \( \square \)

Following the procedure described in [MV, Definition 9.2] (or [MV, Definition 14.1]), we define the triangulated category \( \tilde{D}^eff \) to be \( D^-[W^-_A] \). Slightly abusing notation, we still denote by \( \tilde{\mathcal{Z}}_{tr}(X) \) the class of \( \tilde{\mathcal{Z}}_{tr}(X) \) (seen as a complex concentrated in degree 0) in this category.

Note that in [DF, Definition 3.2.1], the category \( \tilde{D}^eff \) is defined by using full complexes instead of bounded above complexes. Let then \( D := D(\tilde{\mathcal{S}}h_{Nis}(k)) \) be the derived category of unbounded complexes.
Proposition 2.3. The functor \( D^- \to D \)
induces a fully faithful functor \( \widetilde{DM}^{eff,-} \to \widetilde{DM}^{eff} \).

Proof. Firstly, by [MVW06, Lemma 9.20] and [CD07, Proposition 3.3], a complex in \( D^- \) is \( \mathbb{A}^1 \)-local if and only if it’s \( T_{\mathbb{A}^1} \)-local in \( D \) (see [DF17, Definition 1.2.4, Definition 3.2.1]). The functor \( C_* \) preserves quasi-isomorphisms hence it induces a functor \( C_* \): \( D^- \to D^- \) by the same procedure as the remark before [SV, Theorem 1.12]. For any morphism \( f: X \to Y \) in \( D^- \), we have a commutative diagram in \( D^- \) of the form

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
C_*X & \xrightarrow{C_*(f)} & C_*Y,
\end{array}
\]

If \( f \) is an \( \mathbb{A}^1 \)-equivalence, \( C_*(f) \) will be an isomorphism in \( D^- \) by [MVW06, Lemma 9.21]. The vertical arrows are \( \mathbb{A}^1 \)-equivalences in both \( D^- \) and \( \widetilde{DM}^{eff} \) by [SV, Lemma 9.15] and [CD13, Lemma 5.2.35]. So \( f \) will also be an \( \mathbb{A}^1 \)-equivalence in \( \widetilde{DM}^{eff} \). Hence there is a functor \( \widetilde{DM}^{eff,-} \to \widetilde{DM}^{eff} \) such that the following diagram commutes

\[
\begin{array}{ccc}
D^- & \to & D \\
\downarrow & & \downarrow \\
\widetilde{DM}^{eff,-} & \to & \widetilde{DM}^{eff}
\end{array}
\]

by the universal property of the Verdier localization (see [Kra09, Proposition 4.6.2]).

Now suppose \( X,Y \in \widetilde{DM}^{eff,-} \). We have a commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_{\widetilde{DM}^{eff,-}}(X,Y) & \xrightarrow{\sim} & \text{Hom}_{\widetilde{DM}^{eff,-}}(C_*X,C_*Y) \\
\downarrow & & \downarrow \\
\text{Hom}_{\widetilde{DM}^{eff}}(C_*X,C_*Y) & \xrightarrow{\sim} & \text{Hom}_D(C_*X,C_*Y)
\end{array}
\]

where \( u,v \) are induced by the natural morphisms \( X \to C_*X \) and \( Y \to C_*Y \) and \( \beta,\gamma \) are isomorphisms by [MVW06, Lemma 9.19]. It follows that \( \alpha \) is bijective. \( \square \)

We are now going to define tensor products in \( \widetilde{DM}^{eff,-} \).

Definition 2.1. We say that a presheaf with MW-transfers is free if it’s a direct sum of sheaves of the form \( \widetilde{Z}_{tr}(X_1,x_1) \wedge \ldots \wedge (X_n,x_n)) \). If a presheaf with MW-transfers is a direct summand of a free presheaf with MW-transfers, we say it’s projective. A sheaf with MW-transfers is called free (resp. projective) if it’s a sheafication of a free (resp. projective) presheaf with MW-transfers. A bounded above complex of sheaves with MW-transfers is called free (resp. projective) if all its term are free (resp. projective).

So, the sheaf with MW-transfers \( \widetilde{Z}_{tr}(X_1,x_1) \wedge \ldots \wedge (X_n,x_n)) \) is projective by Lemma [2.1].

For any \( F,G \in \widetilde{PSh}(k) \), we can define \( F \otimes_{tr}^p G \in \widetilde{PSh}(k) \) as in the discussion before [SV, Lemma 2.1]. It has the same universal property as in [SV, Lemma 2.1]. Moreover, we define \( \text{Hom}(F,G) \) to be the presheaf with MW-transfers which sends \( X \in \text{Sm}/S \) to \( \text{Hom}(F,G_X) \). And if they are sheaves with MW-transfers, we define \( F \otimes_{tr} G = \tilde{a}(F \otimes_{tr}^p G) \). If \( G \) is a sheaf with MW-transfers, it’s clear that \( \text{Hom}(F,G) \)
is also a sheaf with MW-transfers. Finally, it’s clear from [SV, Lemma 2.1] that $F \otimes_{tr}^{pt} G \cong G \otimes_{tr}^{pt} F$ and $F \otimes_{tr} G \cong G \otimes_{tr} F$.

**Proposition 2.4.** For any $F, G, H \in PSh(k)$, we have isomorphisms

$$\text{Hom}(F \otimes_{tr}^{pt} G, H) \cong \text{Hom}(F, \text{Hom}(G, H)),$$

$$\text{Hom}(F \otimes_{tr}^{pt} G, H) \cong \text{Hom}(G, \text{Hom}(F, H))$$

being functorial in three variables. Similarly, for any $F, G, H \in Sh(k)$, we have isomorphisms

$$\text{Hom}(F \otimes_{tr} G, H) \cong \text{Hom}(F, \text{Hom}(G, H)),$$

$$\text{Hom}(F \otimes_{tr} G, H) \cong \text{Hom}(G, \text{Hom}(F, H))$$

being functorial in three variables.

*Proof.* This is clear from the definition of the bilinear map. 

**Proposition 2.5.** If a morphism $f : F_1 \longrightarrow F_2$ of presheaves with MW-transfers becomes an isomorphism after sheafifying, then so does the morphism $f \otimes_{tr}^{pt} G$ for any presheaf with MW-transfers $G$.

*Proof.* The condition is equivalent to the map $\text{Hom}(f, H)$ is an isomorphism between abelian groups for any sheaf with MW-transfers $H$. And

$$\text{Hom}(f \otimes_{tr}^{pt} G, H) \cong \text{Hom}(f, \text{Hom}(G, H))$$

by the proposition above. 

**Proposition 2.6.**

1. For any $X, Y \in Sm/k$, we have

$$\mathbb{Z}_{tr}(X) \otimes_{tr} \mathbb{Z}_{tr}(Y) \cong \mathbb{Z}_{tr}(X \times Y)$$

as sheaves with MW-transfers.

2. For any two pointed schemes $(X_1, x_1)$ and $(X_2, x_2)$

$$\mathbb{Z}_{tr}((X_1, x_1)) \otimes_{tr} \mathbb{Z}_{tr}((X_2, x_2)) \cong \mathbb{Z}_{tr}((X_1, x_1) \wedge (X_2, x_2))$$

as sheaves with MW-transfers.

*Proof.* We have $\bar{c}(X) \otimes_{tr}^{pt} \bar{c}(Y) \cong \bar{c}(X \times Y)$ just by the tensor products of correspondences. Then the first statement follows by Proposition 2.5. The second statement follows by a similar method.

**Definition 2.2.** We say that a morphism $p : E \longrightarrow X$ in $Sm/k$ is an $\mathbb{A}^n$-bundle if there is an open covering $\{U_i\}$ of $X$ such that $p^{-1}(U_i) \cong U_i \times_k \mathbb{A}^n$.

**Proposition 2.7.** Let $p : E \longrightarrow X$ be an $\mathbb{A}^n$-bundle. Then, the map

$$\mathbb{Z}_{tr}(p) : \mathbb{Z}_{tr}(E) \longrightarrow \mathbb{Z}_{tr}(X)$$

is an isomorphism in $\overline{DM}^{eff, -}(k)$.

*Proof.* For any $X \in Sm/k$, the projection $\mathbb{Z}_{tr}(X \times_k \mathbb{A}^n) \longrightarrow \mathbb{Z}_{tr}(X)$ is an $\mathbb{A}^1$-weak equivalence by definition. Suppose that we have two open sets $U_1$ and $U_2$ of $X$ such that the statement is true over $U_1$, $U_2$ and $U_1 \cap U_2$. Set $E_i = p^{-1}(U_i)$. Then we have a commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}_{tr}(E_1 \cap E_2) & \longrightarrow & \mathbb{Z}_{tr}(E_1) \oplus \mathbb{Z}_{tr}(E_2) & \longrightarrow & \mathbb{Z}_{tr}(p^{-1}(E_1 \cup E_2)) & \longrightarrow & 0 \\
| & & | & & | & & | & & |
\end{array}
\]

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}_{tr}(U_1 \cap U_2) & \longrightarrow & \mathbb{Z}_{tr}(U_1) \oplus \mathbb{Z}_{tr}(U_2) & \longrightarrow & \mathbb{Z}_{tr}(U_1 \cup U_2) & \longrightarrow & 0 \\
\end{array}
\]
by Proposition \ref{prop:2.1}. So the statement is also true over \(U_1 \cup U_2\). We now observe that we can pick a finite open covering \(\{U_i\}\) of \(X\) such that \(p^{-1}(U_i) \cong U_i \times_A \mathbb{A}^n\) for every \(i\) and work by induction on the number of open sets. \(\square\)

**Proposition 2.8.** Let \(P,Q,R\) be bounded above projective complexes and \(f : P \rightarrow Q\) be a quasi-isomorphism. Then \(f \otimes_{tr} R : \text{Tot}(P \otimes_{tr} R_i) \rightarrow \text{Tot}(Q \otimes_{tr} R_i)\) is a quasi-isomorphism.

**Proof.** We’ll proceed step by step.

**Step 1.** \(P = 0\) and \(R\) is a sheaf with MW-transfers regarded as a complex in degree 0. In this case, \(R\) is a direct summand of a free sheaf with MW-transfers. We may apply [SV, Corollary 2.3].

**Step 2.** \(P = 0\). This follows from the spectral sequence of total complexes.

**Step 3.** General case. Since \(f\) is a quasi-isomorphism, the cone \(C(f)\) is acyclic. So \(\text{Tot}(C(f) \otimes_{tr} R_i)\) is acyclic by Step 2, hence \(f \otimes_{tr} R\) is a quasi-isomorphism since taking cone and total complex commute. \(\square\)

For any \(C,D \in C^{-}(\widetilde{SH}(k))\), we may pick bounded above projective complexes \(P,Q\) such that we have quasi-isomorphisms \(P \rightarrow C\) and \(Q \rightarrow D\) in \(C^{-}(\widetilde{SH}(k))\) (if \(C\) is already projective, take \(P = C\), same for \(D\)) and define \(C \otimes^{L} D = \text{Tot}(P \otimes_{tr} Q_i)\). The proposition above shows that \(\otimes^{L}\) is well-defined in \(D^{-}\). Then, following the same development as in [MVW06, pages 67-68], we see that \(\otimes^{L}\) is well-defined in \(\widetilde{DM}^{eff,-}\).

We can also define exterior products \(\boxtimes\) as in [Deg12, 2.7]. Namely, since
\[
\widetilde{Z}_{tr}(X) \otimes_{tr} \widetilde{Z}_{tr}(X) \cong \widetilde{Z}_{tr}(X \times X)
\]
as sheaves, we can define the diagonal map \(\Delta : \widetilde{Z}_{tr}(X) \rightarrow \widetilde{Z}_{tr}(X) \otimes^{L} \widetilde{Z}_{tr}(X)\). If we have two maps \(f_i : \widetilde{Z}_{tr}(X) \rightarrow C_i, i = 1, 2\) in \(\widetilde{DM}^{eff,-}\), we define \(f_1 \boxtimes f_2\) as the composition
\[
\widetilde{Z}_{tr}(X) \xrightarrow{\Delta} \widetilde{Z}_{tr}(X) \otimes^{L} \widetilde{Z}_{tr}(X) \xrightarrow{f_1 \otimes f_2} C_1 \otimes^{L} C_2.
\]

**Proposition 2.9.** Let \(X\) be a smooth scheme, \(Z \subseteq X\) be a closed subset and \(i \geq 0\). Then
\[
\text{H}^i_{\widetilde{Z}}(X, \widetilde{Z}(i)) \cong C\overline{H}^{i}(X),
\]
and in particular
\[
\text{H}^{2i}_{\widetilde{Z}}(X, \widetilde{Z}(i)) \cong C\overline{H}^{i}(\cdot)
\]
functorially in \(X\). Moreover, the following diagram commutes for any \(i, j \geq 0\)
\[
\begin{array}{ccc}
\text{Hom}_{\widetilde{DM}^{eff,-}}(\widetilde{Z}_{tr}(X), \widetilde{Z}(i)[2i]) \times \text{Hom}_{\widetilde{DM}^{eff,-}}(\widetilde{Z}_{tr}(X), \widetilde{Z}(j)[2j]) & \longrightarrow & \overline{H}^{i}(X) \times \overline{H}^{j}(X) \\
\otimes & & \\
\text{Hom}_{\widetilde{DM}^{eff,-}}(\widetilde{Z}_{tr}(X), \widetilde{Z}(i+j)[2(i+j)]) & \longrightarrow & C\overline{H}^{i+j}(X)
\end{array}
\]
where the right-hand map is the intersection product on Chow-Witt groups. Consequently, we have isomorphisms \(\text{Hom}_{\widetilde{DM}^{eff,-}}(\widetilde{Z}_{tr}(X), \widetilde{Z}(i)[2i]) \rightarrow \overline{H}^{i}(X)\) which send the natural embedding \(j : \widetilde{Z}_{tr}(\text{Spec} \, k) \rightarrow \widetilde{Z}\) to 1 when \(i = 0\) and \(X = \text{Spec} \, k\).

**Proof.** See [DF17] Corollary 4.2.6. \(\square\)

**Proposition 2.10.** Let \(X, Y\) be smooth schemes. The map
\[
\text{Hom}_{\widetilde{DM}^{eff,-}}(\widetilde{Z}_{tr}(X), \widetilde{Z}_{tr}(Y)) \otimes \overline{Z}^{(i)} \rightarrow \text{Hom}_{\widetilde{DM}^{eff,-}}(\widetilde{Z}_{tr}(X)(i), \widetilde{Z}_{tr}(Y)(i)) , i > 0
\]
is an isomorphism.

Proof. See [FO] Theorem 5.0.1.

3. Grassmannian Bundles and Quaternionic Projective Bundles

First of all, we recall the basics on Grassmannian bundles and quaternionic projective bundles. Although these are well-known objects, we include the definitions here for the sake of notations. The reader may refer to [KL72], [Sha94] for Grassmannians, [Kle69] for Grassmannian bundles and [PW10] for quaternionic projective bundles.

Definition 3.1. Let $k$ be a field, $r$ be an integer and $1 \leq n \leq r$. Consider the ring

$$A(n, r) = k[p_{i_1, \ldots, i_n} | 1 \leq i_1, \ldots, i_n \leq r]$$

and the ideal $I(n, r) \subseteq A(n, r)$ generated by

$$\sum_{t=1}^{n+1} (-1)^{t-1} p_{i_1, \ldots, i_{n-1}, j, j_{t-1}, j_{t+1}, \ldots, j_{n+1}}, \quad \text{with } 1 \leq i_1, \ldots, i_{n-1}, j, j_1, \ldots, j_{n+1} \leq r,$$

$$p_{i_1, \ldots, i_n} - \text{sgn}(\sigma) p_{\sigma(i_1), \ldots, \sigma(i_n)}, \quad \text{for } \sigma \in S_n.$$

The scheme

$$\text{Gr}(n, r) = \text{Proj}(A(n, r)/I(n, r))$$

is the Grassmannian of rank $n$ quotients of a $k$-vector space of rank $r$.

Definition 3.2. Let $X$ be a $k$-scheme, $E$ locally free of rank $r$ on $X$, $1 \leq n \leq r$. Define a functor

$$F : X - \text{Sch}^{op} \rightarrow \text{Set}$$

$$f : T \rightarrow X \mapsto \{\mathcal{F} \subseteq f^*E | f^*E/\mathcal{F} \text{ is locally free of rank } n\}$$

with functorial maps defined by pull-backs. If $F$ is representable, the representative is called the Grassmannian bundle of rank $n$ of $E$, denoted by $\text{Gr}_X(n, E)$.

Proposition 3.1. The functor $F$ is representable. Further, if $E \cong O_X^\oplus r$, then $\text{Gr}_X(n, E) \cong \text{Gr}(n, r) \times_k X$ over $X$.

Proof. See [Kle69] Proposition 1.2. 

Let $p : \text{Gr}_X(n, E) \rightarrow X$ be the structure map. There is a universal element $\mathcal{F} \subseteq p^*E$ with quotient of rank $n$. The vector bundle $(p^*E/\mathcal{F})^\vee$ is called the tautological bundle of $\text{Gr}_X(n, E)$, denoted by $\mathcal{U}$. Its dual is just called the dual tautological bundle, denoted by $\mathcal{U}^\vee$.

Definition 3.3. Let $E \neq 0$ be a locally free sheaf of rank $n$ over a scheme $X$. $E$ is called symplectic if it’s equipped with a skew-symmetric $(v \cdot v = 0)$ and non degenerate inner product $m : E \times E \rightarrow O_X$ (hence $n$ is always even).

Now let $f : X \rightarrow Y$ be a morphism of schemes and $(E, m)$ be a symplectic bundle on $Y$. Then $(f^*E, f^*(m))$ is also a symplectic bundle, where $f^*(m)$ is the pull back of the map $E \rightarrow E^\vee$ induced by $m$.

The following is a basic tool when dealing with non degeneracy of inner products.

Proposition 3.2. Let $f : X \rightarrow Y$ be a morphism between schemes and $E$ be a locally free sheaf of finite rank over $Y$ with an inner product $m : E \times E \rightarrow O_X$. Then for any $x \in X$, $m$ is non degenerate at $f(x)$ if and only if $f^*(m)$ is non degenerate at $x$.

Proof. This is basically because $f$ induces local homomorphisms between stalks. 

The following proposition can be seen from the case of vector spaces.
Proposition 3.3. Suppose we have an injection \( i : \mathcal{E}_1 \rightarrow \mathcal{E}_2, \) where \( \mathcal{E}_2 \) symplectic and \( m_{\mathcal{E}_1} \) \( \mathcal{E}_1 \) is non degenerate. Define \( \mathcal{E}_1^\perp (U) := \mathcal{E}_1(U)^\perp \) for every \( U \). Then \( \mathcal{E}_1^\perp \) is again a symplectic bundle with inner product inherited from \( \mathcal{E}_2 \) and there exists a unique \( p : \mathcal{E}_2 \rightarrow \mathcal{E}_1 \) with \( p \circ i = id_{\mathcal{E}_1} \) and \( \text{Im}(id_{\mathcal{E}_2} - i \circ p) \subseteq \mathcal{E}_1^\perp \).

Definition 3.4. Let \( X \) be a \( k \)-scheme and let \( (\mathcal{E}, m) \) be a symplectic bundle over \( X \). Define a functor

\[
H : X \rightarrow \text{Sch}^\text{op}
\]

with functorial maps defined by pull-backs.

Definition 3.5. Let \( \mathcal{H} = \text{Gr}(2, 2n + 2) \).

where \( n_{i,n+1} \) means the class of \( p_{i,n+1} \) in the quotient.

Proposition 3.4. The functor \( H \) is representable by a scheme \( HGr_X(\mathcal{E}) \). Further, if \( (\mathcal{E}, m) \cong \left( O_X^{2n+2}, \begin{pmatrix} I & \cdot^T \\ \cdot & -I \end{pmatrix} \right) \), then \( HGr_X(\mathcal{E}) \cong \mathcal{H} \times_k X \) over \( X \).

Proof. We have the structure map \( \pi : G_F(2n, \mathcal{E}) \rightarrow X \) and the tautological exact sequence

\[
0 \rightarrow \mathcal{F} \rightarrow \pi^* \mathcal{E} \rightarrow \mathcal{U}^\vee \rightarrow 0.
\]

Define

\[
HGr_X(\mathcal{E}) = \{ x \in \text{Gr}_F(2n, \mathcal{E}) | \pi^* (m)|_x \text{ is non degenerate at } x \}.
\]

Now we prove that \( HGr_F(T, HGr_X(\mathcal{E})) \cong H(T) \) for any \( X \)-scheme \( f : T \rightarrow X \).

\( HGr_{X}(\mathcal{E}) \) is an open subset of \( \text{Gr}_F(2n, \mathcal{E}) \). Given an \( X \)-morphism \( a : T \rightarrow HGr_X(\mathcal{E}) \), it induces an \( X \)-morphism \( b : T \rightarrow G_{F}(2n, \mathcal{E}) \) and this gives an exact sequence

\[
0 \rightarrow K \rightarrow f^* \mathcal{E} \rightarrow C \rightarrow 0
\]

obtained by applying \( b^* \) on the exact sequence in the beginning. So by the definition of \( HGr_{X}(\mathcal{E}) \), \( f^*(m)|_K \) is non degenerate. Conversely, given a morphism \( b : T \rightarrow G_{F}(2n, \mathcal{E}) \) such that \( f^*(m)|_K \) is non degenerate as above, so \( \pi^*(m)|_x \) is non degenerate at every point in \( \text{Im}(b) \) by Proposition 3.2. So \( \text{Im}(b) \subseteq HGr_X(\mathcal{E}) \).

For the second statement, consider an \( X \)-scheme \( f : T \rightarrow X \) and an \( X \)-morphism \( b : T \rightarrow \text{Gr}(2, 2n + 2)_k \) \( X \). Then \( b \) factors through \( \mathcal{H} \times_k X \) if and only if the composition

\[
c : T \rightarrow \text{Gr}(2, 2n + 2)_k \times_k X \rightarrow \text{Gr}(2, 2n + 2)_k
\]

factors through \( \mathcal{H} \). Denote the structure map \( \text{Gr}(2, 2n + 2)_k \rightarrow \text{pt} \) by \( p \). Then we have the tautological exact sequence

\[
0 \rightarrow \mathcal{F} \rightarrow p^* O^{2n+2}_{\text{pt}} \rightarrow \mathcal{U}^\vee \rightarrow 0
\]

as in the beginning. Then one proves that \( c \) factor through \( \mathcal{H} \) if and only if the inner product \( \left( p^* O^{2n+2}_{\text{pt}}, \begin{pmatrix} I & \cdot^T \\ \cdot & -I \end{pmatrix} \right) \) is non degenerate after restricted to \( c^* \mathcal{U} \) (take dual of the exact sequence above). Considering morphisms \( \text{Spec} K \rightarrow T \) where \( K \) is a field, we can assume \( T = \text{Spec} K \). Then the non vanishing of the formula \( \sum_{i=1}^{n+1} p_{i,n+1} \) in the Definition 3.3 is just equivalent to the non degeneracy required above.

\( \square \)

Definition 3.6. We will call \( HGr_X(\mathcal{E}) \) the quaternionic projective bundle of \( \mathcal{E} \).
Let \( p : HGr_X(\mathcal{E}) \longrightarrow X \) be the structure map. Then, there is a universal element \( \mathcal{F} \subseteq p^*\mathcal{E} \) which is just obtained by the restriction of the universal element of the Grassmannian bundle to \( HGr_X(\mathcal{E}) \). The vector bundle \( \mathcal{F} \) itself is called the tautological bundle of \( HGr_X(\mathcal{E}) \), denoted by \( \mathcal{W} \). Its dual is just called the dual tautological bundle, denoted by \( \mathcal{W}^\vee \). We will use the same symbol \( \mathcal{W} \) for all tautological bundles defined above if there is no confusion. Note that both \( \mathcal{W} \) and \( \mathcal{W}^\vee \) are symplectic by Proposition 5.3.

4. Quaternionic Projective Bundle Theorem

The following proposition can also be found in [MVW06, Corollary 15.3] and [SV] Proposition 4.3.

**Proposition 4.1.** For any \( n \geq 1 \), we have an isomorphism

\[
\tilde{Z}_{tr}(A^n \setminus 0) \cong \tilde{Z} \oplus \tilde{Z}(n)[2n - 1]
\]

in \( \widetilde{DM}^{eff,-} \).

*Proof.* We denote the point \((1, \ldots, 1) \in A^n\) by 1 for any \( n \). Then it suffices to prove that

\[
\tilde{Z}_{tr}((A^n \setminus 0, 1)) \cong \tilde{Z}(n)[2n - 1]
\]

by induction. For \( n = 1 \) this is by definition.

In general, write \( x_1, \ldots, x_n \) for the coordinates of \( A^n \) and set \( U_1 = D(x_1), U_2 = \bigcup_{i=2}^n D(x_i) \). Note that \( U_1 = (A^1 \setminus 0) \times A^{n-1}, U_2 = A^1 \times (A^{n-1} \setminus 0) \) and \( U_1 \cap U_2 = (A^1 \setminus 0) \times (A^{n-1} \setminus 0) \).

We have a commutative diagram in the category of sheaves with MW-transfers:

\[
\begin{array}{ccc}
\tilde{Z}_{tr}(U_1 \cap U_2, 1) & \longrightarrow & \tilde{Z}_{tr}(U_1, 1) \oplus \tilde{Z}_{tr}(U_2, 1) \\
\end{array}
\]

\[
\begin{array}{ccc}
\tilde{Z}_{tr}(U_1 \cap U_2, 1) & \longrightarrow & \tilde{Z}_{tr}(A^1 \setminus 0, 1) \oplus \tilde{Z}_{tr}(A^{n-1} \setminus 0, 1) \\
\end{array}
\]

where the right-hand vertical map is the sum of the respective projections. Considering the relevant sheaves as complexes concentrated in degree 0 and taking cones, we obtain a commutative diagram of triangles in \( D^- \)

\[
\begin{array}{ccc}
\tilde{Z}_{tr}(U_1 \cap U_2, 1) & \longrightarrow & \tilde{Z}_{tr}(U_1, 1) \oplus \tilde{Z}_{tr}(U_2, 1) \\
\end{array}
\]

\[
\begin{array}{ccc}
\tilde{Z}_{tr}(U_1 \cap U_2, 1) & \longrightarrow & \tilde{Z}_{tr}(A^1 \setminus 0, 1) \oplus \tilde{Z}_{tr}(A^{n-1} \setminus 0, 1) \\
\end{array}
\]

It follows from Proposition 2.1 that the map \( \tilde{Z}_{tr}(U_1, 1) \oplus \tilde{Z}_{tr}(U_2, 1) \to \tilde{Z}_{tr}(A^n \setminus 0, 1) \) induces a quasi-isomorphism \( C \to \tilde{Z}_{tr}(A^n \setminus 0, 1) \). Using now Lemma 2.2 we obtain a morphism of complexes \( \tilde{Z}_{tr}((A^1 \setminus 0, 1) \land (A^{n-1} \setminus 0, 1))[1] \to C' \) which is a quasi-isomorphism.

Applying now the exact localization functor \( D^- \to \widetilde{DM}^{eff,-} \) to \( (1) \) and using Proposition 2.7 we see that the map \( C \to C' \) is an isomorphism in \( \widetilde{DM}^{eff,-} \).

Altogether, we have obtained an isomorphism in \( \widetilde{DM}^{eff,-} \) of the form

\[
\tilde{Z}_{tr}(A^n \setminus 0, 1) \to \tilde{Z}_{tr}((A^1 \setminus 0, 1) \land (A^{n-1} \setminus 0, 1))[1].
\]

Now, the wedge product on the right-hand side can be computed as

\[
\tilde{Z}_{tr}((1 \setminus 0, 1)) \otimes^L \tilde{Z}_{tr}((A^{n-1} \setminus 0, 1)) \cong \tilde{Z}(1)[1] \otimes^L \tilde{Z}(n-1)[2n - 3] \cong \tilde{Z}(n)[2n - 2]
\]

in \( \widetilde{DM}^{eff,-} \) by Proposition 2.6 and induction hypothesis. Hence we are done. \( \square \)
Now let’s discuss the notion of orientation, which is a new feature in Chow-Witt theory.

**Definition 4.1.** Let $X$ be a scheme and let $E$ be a vector bundle over $X$. A section $s \in (\det E^\vee)(X)$ is called an orientation of $E$ if $s$ trivializes $\det E^\vee$. A vector bundle with an orientation is called orientable.

**Definition 4.2.** Let $X$ be a smooth scheme and $E$ be an orientable vector bundle of rank $n$ over $X$ with an orientation $s$. Define $e(E)$ to be the map such that the following diagram commutes (see [Fas08, Définition 13.2.1]):

\[
\begin{array}{ccc}
\widetilde{CH}^0(X) & \xrightarrow{\tilde{e}_n(E)} & \widetilde{CH}^n(X, \det E^\vee) \\
\downarrow s & & \downarrow s \\
\widetilde{CH}^n(X) & \xrightarrow{\sim} & \widetilde{CH}^n(X, \det E^\vee)
\end{array}
\]

If $n = 2$, define the first Pontryagin class under the orientation $s$ of $E$ to be $-e(E)(1) \in \widetilde{CH}^2(X)$ (see [AF16, remark before Proposition 3.1.1]), which is denoted by $p_1(E)$.

The following lemma is obvious.

**Lemma 4.1.** Let $(E, m)$ be a vector bundle of rank 2 over a scheme $X$ with a skew-symmetric inner product. Then $m$ is non degenerate iff the induced map $\bigwedge^2 E \longrightarrow \mathcal{O}_X$ is an isomorphism.

Hence for any symplectic bundle of rank 2, there is a canonical orientation induced by the dual of the isomorphism in the above lemma.

**Definition 4.3.** Let $E_1, E_2$ be two orientable vector bundles over a scheme $X$ with orientations $s_1, s_2$, respectively. An isomorphism $f : E_1 \longrightarrow E_2$ is called orientation preserving if $\det(f)^\vee(s_2) = s_1$.

**Proposition 4.2.** Let $E_1, E_2$ be two orientable vector bundles of rank $n$ over a smooth scheme $X$ with orientations $s_1, s_2$, respectively. If there is an orientation preserving isomorphism $f : E_1 \longrightarrow E_2$, then $e(E_1) = e(E_2)$.

**Proof.** Let $E_j$ be the total space of $E_j$, $p_j : E_j \longrightarrow X$ be the structure maps and $z_j : X \longrightarrow E_j$ be the zero sections. We have a diagram

\[
\begin{array}{ccc}
\widetilde{CH}^0(X) & \xrightarrow{\tilde{e}_n(E_1)} & \widetilde{CH}^n(X, \det E_1^\vee) \\
\downarrow s_1 & & \downarrow s_1 \\
\widetilde{CH}^n(X, \det E_1^\vee) & \xrightarrow{\sim} & \widetilde{CH}^n(X)
\end{array}
\]

in which the right triangle commutes since $f$ is orientation preserving. Hence we only have to prove that the left triangle commutes. For this, use the following commutative diagrams which can be catenated:

\[
\begin{array}{ccc}
\widetilde{CH}^0(X) & \xrightarrow{s_1 \otimes s_1} & \widetilde{CH}^0(X, \det E_1 \otimes \det E_1^\vee) \\
\downarrow id \otimes \det(f)^\vee & & \downarrow (\det(f) \otimes id)^{-1} \\
\widetilde{CH}^0(X, \det E_1 \otimes \det E_1^\vee) & \xrightarrow{s_2 \otimes s_2} & \widetilde{CH}^0(X, \det E_2 \otimes \det E_2^\vee)
\end{array}
\]
The following results hold:

**Proposition 4.3.**

\[
\begin{align*}
\widetilde{CH}^0(X, \det \mathcal{E}_1 \otimes \det \mathcal{E}_1^\vee) & \xrightarrow{z_1} \widetilde{CH}^n(E_1, p_1^* \det \mathcal{E}_1^\vee) \xleftarrow{p_1^*} \widetilde{CH}^n(X, \det \mathcal{E}_1) \\
\widetilde{CH}^0(X, \det \mathcal{E}_2 \otimes \det \mathcal{E}_2^\vee) & \xrightarrow{z_2} \widetilde{CH}^n(E_2, p_2^* \det \mathcal{E}_2^\vee) \xleftarrow{p_2^*} \widetilde{CH}^n(X, \det \mathcal{E}_2) \\
\end{align*}
\]

As an application, if two symplectic bundles of rank 2 are isomorphic (including their inner products) then their first Pontryagin classes under the canonical orientations are equal. Note that if they are just isomorphic as vector bundles, the statement is not true any more, since we can use automorphisms of trivial bundles.

Now let’s start to calculate the motive of \(HP^n\). Let \(x_1, \ldots, x_{2n+2}\) be the coordinates of the underlying vector space of \(HP^n\). For any \(a = 1, \ldots, n + 1\), set \(V_a = \sum_i x_{a+n+i} k \cdot x_i\). We have a diagram:

\[
\begin{array}{ccc}
\text{Spec } k & \xrightarrow{u} & HP^{n-1} \\
\downarrow \pi & & \downarrow \text{w} \\
(X^{n+1})^c \xrightarrow{j} & HP^n & (s) \\
\end{array}
\]

where \(u, v, w\) are structure maps,

\[
k \left( \begin{array}{c} x_1, \ldots, x_{2n} \\ y_1, \ldots, y_{2n} \end{array} \right) = \left( \begin{array}{c} x_1, \ldots, x_n, 0, x_{n+1}, \ldots, x_{2n}, 0 \\ y_1, \ldots, y_n, 0, y_{n+1}, \ldots, y_{2n}, 0 \end{array} \right),
\]

\(j\) is the inclusion and

\[
\pi \left( \begin{array}{c} x_1, \ldots, x_{2n+1}, 0 \\ y_1, \ldots, y_{2n+1}, 0 \end{array} \right) = \left( \begin{array}{c} x_1, \ldots, x_n, x_{n+2}, \ldots, x_{2n+1} \\ y_1, \ldots, y_n, y_{n+3}, \ldots, y_{2n+1} \end{array} \right)
\]

(here, \(v_1, v_2\) means a two-dimensional subspace written in its coordinates spanned by \(v_1, v_2\) in a \(k\)-vector space). Note that the lower diagram doesn’t commute, i.e. \(k \circ \pi \neq j\).

**Proposition 4.3.** The following results hold:

1. \(\pi^* (\mathcal{W}^\vee_{HP^{n-1}}) \cong j^* (\mathcal{W}^\vee_{HP^n})\) as symplectic bundles.

2. Let \(z : HP^n \to \mathcal{W}^\vee\) be the zero section of \(\mathcal{W}^\vee\) then there is a section \(s\) of \(\mathcal{W}^\vee\) such that we have a transversal cartesian square (see [AF16, Theorem 2.4.1]):

\[
\begin{array}{ccc}
(X^{n+1})^c & \xrightarrow{j} & HP^n \\
\downarrow j & & \downarrow z \\
HP^n & \xrightarrow{s} & \mathcal{W}^\vee_{HP^n} \\
\end{array}
\]

**Proof.** See [PW10, Theorem 4.1, (d), (v)].
Theorem 4.1. For any \( n \geq 0 \), we have\[
\widetilde{Z}_{tr}(H^{P^n}) \cong \oplus_{i=0}^{n} \widetilde{Z}(2i)[4i]
\]
in \( \overline{DM}^{eff} \).

Proof. Set \( U_n^n = \bigcup_{i=1}^{n} X_i \subseteq H^{P^n} \). The normal bundle \( N_1(X_i)_{/H^{P^n}} \) is symplectic by similar results in Proposition 4.3 for \( X_0^1 \) instead of \( X_0^{n+1} \). So the normal bundle \( N_a := N_1(u_{a} \setminus X_i)_{/u_{a}} \) is also symplectic. Thus it’s trivialized by a section \( s_a \) by the canonical orientation of its dual. Moreover, \( U_n^n \setminus X_i \) is of codimension 2 in \( U_n^n \).

We have an \( \mathbb{A}^2 \)-bundle \( \pi : (X_i^1)^c \to H^{P^{n-1}} \) by [PW10] Theorem 3.2, and then \( U_n^{n-1} \setminus X_i^1 \) is also an \( \mathbb{A}^2 \)-bundle over \( U_n^{n-1} \).

Now we prove by induction that\[
\widetilde{Z}_{tr}(U_n^n) \cong \oplus_{i=0}^{n-1} \widetilde{Z}(2i)[4i].
\]
This is true for \( a = 1 \) by [PW10] Theorem 3.4(a) and Proposition 2.7. We thus suppose it’s true for some \( a \geq 1 \) and prove the result for \( a + 1 \). Let then\[
\theta : \widetilde{Z}_{tr}(U_n^n) \to \oplus_{i=0}^{a-1} \widetilde{Z}(2i)[4i]
\]
be such an isomorphism.

We claim that the inclusion \( j : \widetilde{Z}_{tr}(U_n^n) \to \widetilde{Z}_{tr}(U_n^{a+1}) \) splits in \( \overline{DM}^{eff} \). Indeed, Proposition 2.3 yield a commutative diagram in which the vertical homomorphisms are isomorphisms.

\[
\begin{array}{ccc}
\text{Hom}_{\overline{DM}^{eff}}(\widetilde{Z}_{tr}(U_n^{a+1}), \widetilde{Z}_{tr}(U_n^n)) & \xrightarrow{j} & \text{Hom}_{\overline{DM}^{eff}}(\widetilde{Z}_{tr}(U_n^n), \widetilde{Z}_{tr}(U_n^n)) \\
\downarrow \theta \quad & & \downarrow \theta \\
\text{Hom}_{\overline{DM}^{eff}}(\oplus_{i=0}^{a-1} CH^{2i}(U_n^{a+1}), \oplus_{i=0}^{a-1} \widetilde{Z}(2i)[4i]) & \xrightarrow{j^*} & \oplus_{i=0}^{a-1} CH^{2i}(U_n^{a+1}) \\
\end{array}
\]

It suffices then to prove that for any \( i = 0, 2, \ldots, 2a - 2 \), the pull-back
\[
j^* : CH^i(U_n^{a+1}) \to CH^i(U_n^n)
\]
is an isomorphism since the first horizontal arrow in the above diagram will be an isomorphism.

We use induction on \( a \) again to prove the claim on \( j^* \). The cases for \( i = 0 \) are easy. Hence we suppose \( i > 0 \), which implies \( a, n > 1 \). The result now follows from the following two commutative diagrams (see [Fas08] Remarque 10.4.8, [Fas08] Corollaire 10.4.10] and [Fas08] Corollaire 11.3.2]) with splitting exact rows in the first one (following from [PW10] Theorem 3.4(a)):

\[
\begin{array}{cccc}
0 & \xrightarrow{} & CH^i_{U_n^{n+1} \setminus X_{i}^1(U_{n}^{a+1})} & \xrightarrow{} & CH^i(U_{n}^{a+1}) & \xrightarrow{} & CH^i(X_{i}^1) & \xrightarrow{} & 0 \\
0 & \xrightarrow{} & CH^i_{U_n^{n} \setminus X_{i}^1(U_{n}^{a})} & \xrightarrow{} & CH^i(U_{n}^{a}) & \xrightarrow{} & CH^i(X_{i}^1) & \xrightarrow{} & 0
\end{array}
\]
by Proposition 4.1. So by killing one point, we get a distinguished triangle in \( DM \). We have proved that

\[ \text{if} \quad n = 0, \ldots, n \]

Proof. Let \( \tilde{Z}_{tr}(U^a_n \cap X^a_0) \) be the structure map. Then the map

\[ Z_{tr}(U^a_n) \oplus \tilde{Z}_{tr}(X^a_0) \]

yielding an exact triangle in \( DM \). Moreover, we have an \( \mathbb{A}^1 \)-bundle \( p : \mathbb{A}^{n+1} \to X^a_0 \) (see [PW10, Theorem 3.4(a)]) and it follows that

\[ \tilde{Z}_{tr}(U^a_n \cap X^a_0) \cong \tilde{Z}_{tr}(\mathbb{A}^{2a} \setminus 0 \times \mathbb{A}^{n+2a+1}) \cong \tilde{Z} \oplus \tilde{Z}(2a)[4a - 1] \]

by Proposition 4.4. So by killing one point, we get a distinguished triangle in \( DM \). We have proved that \( j \) splits and therefore

\[ \tilde{Z}_{tr}(U^a_n) \cong \tilde{Z}_{tr}(U^a_0) \oplus \tilde{Z}(2a)[4a] \]

completing the induction process.

Now we want to improve Theorem 4.1 and find an explicit isomorphism using the first Pontryagin class of the dual tautological bundle on \( HP^n \).

The following proposition has a very similar version in [PW10, Theorem 8.1], but the twists are considered here.

**Proposition 4.4.** Let \( w : HP^n \to \text{Spec}(k) \) be the structure map. Then the map

\[ f_{0,i} : CH^0(\text{Spec}(k)) \to CH^0(HP^n) \]

is an isomorphism between abelian groups, where \( i = 0, \ldots, n \). Here, \( \mathcal{U}^\vee \) is endowed with its canonical orientation.

**Proof.** We prove the result by induction on \( n \) and use the notation of Diagram (*). If \( n = 0 \), there is nothing to prove.

We note that \( j^*(\mathcal{U}^\vee_{HP^n}) \cong N(\mathcal{X}_{U^a_1})_{HP^n} \) by Proposition 4.3.
We have a commutative diagram with split exact row for any $i \geq 0$ (as the one in Theorem 4.1):

$$
0 \to \widetilde{CH}^{2i-2}((X_0^{n+1})^c, \det(j^*(\mathcal{U}_{HP^n}))) \to j^* \to \widetilde{CH}^{2i}(HP^n) \to \widetilde{CH}^{2i}(X_0^{n+1}) \to 0
$$

Here we first pick the canonical orientation $s \in \det(\mathcal{U}_{HP^n})$ of $\mathcal{U}_{HP^n}$ and identify $\widetilde{CH}^{2i-2}((X_0^{n+1})^c)$ with $\widetilde{CH}^{2i-2}((X_0^{n+1})^c, j^*(\mathcal{U}_{HP^n}) \otimes j^*(\det(\mathcal{U}_{HP^n})))$. Then $t$ and $t'$ are just the inverse of the isomorphism induced by $id \otimes j^*$ and $s$. On the other hand, we have an $A^1$-bundle $p : A^{4n+1} \to X_0^n$ by [PW10, Theorem 3.4(a)]. Then, the statement is true for $i = 0$. Moreover, it follows that $\widetilde{CH}^{2i}(X_0^{n+1}) = 0$ if $i > 0$. Thus $j_*$ is an isomorphism if $i > 0$. In this case, the map $-j_* \circ \pi \circ f_{n,i-1}$ will also be an isomorphism. It suffices to show that it is equal to $f_{n,i}$ to conclude.

Pick $s \in \widetilde{CH}^0(Spec \, k)$. Then

$$
-j_*(t(\pi^*(f_{n-1,i-1}(s)))) = -j_*(t(\pi^*(v^*(s) \cdot p_1(\mathcal{U}_{HP^{n-1}})^{-1})))
$$

where we have used [AF16, Proposition 3.1.1], [AF16, Lemma 2.1.2], [AF16, Theorem 2.1.3], [CF14, Corollary 3.4], [Fas07, Proposition 6.6] and [Fas07, Proposition 7.2]. By Proposition 4.3 and [AF16, Proposition 2.4.1], we see that

$$
t'(j_*(1)) = t'(s^*(z_1(1))) = t'((p^*)^{-1}(z_1(1))) = e(\mathcal{U}_{HP^n}(1)),
$$

giving the result. \qed

**Lemma 4.2.** Let $X$ be a smooth scheme and let $i, j \geq 0$. Then

$$
\text{Hom}_{\mathcal{DM}^{eff,-}}(\widetilde{Z}_{tr}(X)(i)[2i], \widetilde{Z}(j)[2j]) = \begin{cases} 
0 & \text{if } i > j, \\
\widetilde{CH}^{j-i}(X) & \text{if } i \leq j.
\end{cases}
$$

**Proof.** If $i \leq j$, the lemma follows from Propositions 2.9 and 2.10. Suppose then that $i > j$. The exact sequence of sheaves with MW-transfers

$$
0 \to \widetilde{Z}_{tr}(A^i \setminus 0) \to \widetilde{Z}_{tr}(A^i) \to \widetilde{Z}_{tr}(A^i)/\widetilde{Z}_{tr}(A^i \setminus 0) \to 0
$$

yields an exact triangle in $\mathcal{DM}^{eff,-}$ of the form

$$
\widetilde{Z}_{tr}(A^i \setminus 0) \to \widetilde{Z}_{tr}(A^i) \to \widetilde{Z}_{tr}(A^i)/\widetilde{Z}_{tr}(A^i \setminus 0) \to \widetilde{Z}_{tr}(A^i \setminus 0)[1]
$$

As $\widetilde{Z}_{tr}(A^i) \simeq \widetilde{Z}_{tr}(Spec(k))$ by Proposition 2.7, we see that the first map is split. Consequently, we get an isomorphism

$$
\widetilde{Z}_{tr}(A^i)/\widetilde{Z}_{tr}(A^i \setminus 0) \simeq \widetilde{Z}_{tr}(A^i \setminus 0, 1)[1]
$$


and it follows from Proposition 2.11 that
\[ \widetilde{Z}_{tr}(H^i)/\widetilde{Z}_{tr}(A^i) \simeq \widetilde{Z}_{tr}(i)[2i] \]
in \( DM^{eff, \cdot} \).
Therefore,
\[ \widetilde{Z}_{tr}(X)(i)[2i] \simeq \widetilde{Z}_{tr}(X \times A)/\widetilde{Z}_{tr}(X \times (A^i \setminus 0)) \]
and it follows from Propositions 2.7 and 2.9 that
\[ \text{Hom}_{DM^{eff, \cdot}}(\widetilde{Z}_{tr}(X)(i)[2i], \widetilde{Z}(j)[2j]) \simeq \mathbb{C}^H_{X \times A}(X \times A^i) = 0. \]
\( \square \)

**Corollary 4.1.** For any \( i, j \geq 0 \), we have
\[ \text{Hom}_{DM^{eff, \cdot}}(\widetilde{Z}(i)[2i], \widetilde{Z}(j)[2j]) = \begin{cases} 0 & \text{if } i \neq j. \\ \mathbb{C}^H_{X}(k) & \text{if } i = j. \end{cases} \]
In other terms, the motives \( \widetilde{Z}(i)[2i] \) are mutually orthogonal in the triangulated category \( DM^{eff, \cdot} \).

**Lemma 4.3.** Let \( \mathcal{C} \) be an additive category. Let \( M, M_i, i = 1, \ldots, n \) be objects in \( \mathcal{C} \) such that \( \text{Hom}_{\mathcal{C}}(M, M_j) = 0 \) if \( i \neq j \). Suppose that there is an isomorphism \( \varphi : M \longrightarrow \oplus_i M_i \). Then for any morphism \( \varphi' : M \longrightarrow \oplus_i M_i \), \( \varphi' \) is an isomorphism if and only if \( \varphi'_i \) is a free generator of \( \text{Hom}_{\mathcal{C}}(M, M_i) \) as left \( \text{End}_{\mathcal{C}}(M) \)-module for any \( i \), where \( \varphi'_i \) is composition of \( \varphi' \) and the \( i \)th projection.

**Proof.** Suppose that \( \varphi' \) is an isomorphism. We prove that \( \varphi'_i \) a free generator of \( \text{Hom}_{\mathcal{C}}(M, M_i) \) as a left \( \text{End}_{\mathcal{C}}(M) \)-module.

The action is free since \( \varphi'_i \) is surjective. Now suppose \( \psi \in \text{Hom}_{\mathcal{C}}(M, M_i) \). Since \( \text{Hom}_{\mathcal{C}}(M, M_j) = 0 \) if \( i \neq j \), we see that \( \psi = (\psi \circ \varphi'^{-1}) \circ i\varphi_i \) where \( i\varphi_i \) is the natural map as direct sum. Hence \( \psi \) can be generated by \( \varphi'_i \), so \( \varphi'_i \) is indeed a free generator.

Conversely, if we have a morphism \( \varphi' : M \longrightarrow \oplus_i M_i \) such that \( \varphi'_i \) is a free generator of \( \text{Hom}_{\mathcal{C}}(M, M_i) \), then \( \varphi'_i = f_i \circ \varphi_i \) for some isomorphism \( f_i \). Hence \( \varphi' \) is also an isomorphism.

**Theorem 4.2.** The map
\[ \widetilde{Z}_{tr}(H^p) \xrightarrow{p_1(\mathcal{B}^\vee)^t} \bigoplus_{i=0}^n \widetilde{Z}(2i)[4i] \]
is an isomorphism in \( DM^{eff, \cdot} \). Here, \( \mathcal{B}^\vee \) is endowed with its canonical orientation.

**Proof.** By Theorem 4.3, Corollary 4.2, and Lemma 4.3, it remains to prove that \( p_1(\mathcal{B}^\vee)^t \) is a free generator of \( \text{Hom}_{DM^{eff, \cdot}}(\widetilde{Z}_{tr}(H^p), \mathbb{Z}(2i)[4i]) \). By Proposition 2.10, \( \text{End}_{DM^{eff, \cdot}}(\widetilde{Z}(2i)[4i]) \) is commutative, so we only have to prove that it generates \( \text{Hom}_{DM^{eff, \cdot}}(\widetilde{Z}_{tr}(H^p), \mathbb{Z}(2i)[4i]) \).

Using the notation of Diagram (*), we see that the composition
\[ \text{Hom}_{DM^{eff, \cdot}}(\widetilde{Z}_{tr}(\text{Spec } k), \mathbb{Z}) \xrightarrow{w} \text{Hom}_{DM^{eff, \cdot}}(\widetilde{Z}_{tr}(H^p), \mathbb{Z}) \xrightarrow{p^t} \text{Hom}_{DM^{eff, \cdot}}(\widetilde{Z}_{tr}(H^p), \mathbb{Z}(2i)[4i]) \]
is an isomorphism by Proposition 4.3 where \( p := p_1(\mathcal{B}^\vee) \). Now given a map \( \psi \in \text{Hom}_{DM^{eff, \cdot}}(\widetilde{Z}_{tr}(H^p), \mathbb{Z}(2i)[4i]) \), we can find its preimage \( \lambda \) under the map
above. So we have a commutative diagram:

\[ \xymatrix{ \tilde{Z}_{tr}(H^P) \ar[r]^\Delta \ar[d]_{p'} & \tilde{Z}_{tr}(H^P) \otimes L \tilde{Z}_{tr}(H^P) \ar[r]^{id \otimes w} \ar[d]_{p'} & \tilde{Z}_{tr}(H^P) \otimes L \tilde{Z}_{tr}(Spec k) \ar[r]^{id \otimes \lambda} \ar[d]_{id \otimes j} & \tilde{Z}(2i)[4i] \otimes L \tilde{Z} \ar[d] \ar[r] & \\ \tilde{Z}(2i)[4i] \otimes L \tilde{Z}_{tr}(Spec k) \ar[r] & \tilde{Z}(2i)[4i] \otimes L \tilde{Z} } \]

showing that \( \psi \) is generated by \( p^i \). We are done. \( \square \)

**Theorem 4.3.** Let \( X \) be a smooth scheme and let \((E, m)\) be a symplectic vector bundle of rank \( 2n + 2 \) on \( X \). Let \( \pi : HGr_X(E) \to X \) be the projection. Then, the map

\[ \tilde{Z}_{tr}(HGr_X(E)) \xrightarrow{\pi \otimes (\mathcal{W}^\vee)^i} \oplus_{i=0}^n \tilde{Z}_{tr}(X)(2i)[4i] \]

is an isomorphism in \( \overline{DM}^{eff,-} \), functorial for \( X \) in \( Sm/k \). Here, \( \mathcal{W}^\vee \) is endowed with its canonical orientation.

**Proof.** We first prove that the map

\[ \tilde{Z}_{tr}(HGr_X(E)) \xrightarrow{\pi \otimes (\mathcal{W}^\vee)^i} \oplus_{i=0}^n \tilde{Z}_{tr}(X)(2i)[4i] \]

is functorial in \( X \). Let then \( f : Y \to X \) be a morphism of schemes. We have a commutative diagram

\[ \xymatrix{ HGr_Y(\ast E) \ar[r] \ar[d]_{\pi} & HGr_X(E) \ar[d]_{\pi} \\
Y \ar[r]_f & X } \]

yielding a commutative diagram in \( \overline{DM}^{eff,-} \)

\[ \xymatrix{ \tilde{Z}_{tr}(HGr_Y(\ast E)) \ar[r] \ar[d]_{\pi} & \tilde{Z}_{tr}(HGr_X(E)) \ar[d]_{\pi} \\
\tilde{Z}_{tr}(Y) \ar[r]_f & \tilde{Z}_{tr}(X). } \]

On the other hand, we have a commutative diagram

\[ \xymatrix{ \tilde{Z}_{tr}(HGr_X(f \ast E)) \ar[r] \ar[d]_{p_1(\mathcal{W}^\vee)^i} & \tilde{Z}_{tr}(HGr_X(E)) \ar[d]_{p_1(\mathcal{W}^\vee)^i} \\
\tilde{Z}(2i)[4i] \ar[r] & \tilde{Z}(2i)[4i] } \]
for any i by Proposition \(2.1\) and naturality of the first Pontryagin class (Proposition \(4.2\)). Consequently, we get a commutative diagram

\[
\begin{array}{ccc}
\mathbb{Z}_{atr}(HGr(f^*\mathcal{E})) & \longrightarrow & \mathbb{Z}_{atr}(HGr(X(\mathcal{E}))) \\
\pi_{\otimes \mathbb{P}_1(\mathcal{Y}')} & & \pi_{\otimes \mathbb{P}_1(\mathcal{Y}')} \\
\oplus \mathbb{Z}_{atr}(Y)(2i)[4i] & \longrightarrow & \oplus \mathbb{Z}_{atr}(X)(2i)[4i]
\end{array}
\]

proving that the isomorphism is natural.

Let’s now prove the first statement. We pick a finite open covering \(\{U_\alpha\}\) of \(X\) such that

\[
(\mathcal{E}, m)|_{U_\alpha} \cong \left( \mathcal{O}_{\mathbb{P}_2}^{2n+2}, \left( \begin{array}{cc} 0 & I \\ -I & 0 \end{array} \right) \right)
\]

for every \(\alpha\) and we work by induction on the number of the open sets. If there is just one open set, \(HGr_X(\mathcal{E}) \cong H^P X\) and we conclude tensoring the isomorphism of Theorem \(4.2\) with \(\mathbb{Z}_{atr}(X)\).

Suppose next that \(X = U_1 \cup U_2\) and the argument holds for \((\mathcal{E}, m)|_{U_1}\), \((\mathcal{E}, m)|_{U_2}\) and \((\mathcal{E}, m)|_{U_1 \cap U_2}\). Set \(\mathcal{E}_1\) for the restrictions of \(\mathcal{E}\) to \(U_1\) and \(\mathcal{E}_1\) for its restriction to the intersection. Using Proposition \(2.1\) we obtain exact triangles

\[(2) \quad \mathbb{Z}_{atr}(U_1 \cap U_2) \to \mathbb{Z}_{atr}(U_1) \oplus \mathbb{Z}_{atr}(U_2) \to \mathbb{Z}_{atr}(X) \to \mathbb{Z}_{atr}(U_1 \cap U_2)[1] \]

and

\[(3) \quad \mathbb{Z}_{atr}(HGr(\mathcal{E}_1)) \to \mathbb{Z}_{atr}(HGr(\mathcal{E}_1)) \oplus \mathbb{Z}_{atr}(HGr(\mathcal{E}_2)) \to \mathbb{Z}_{atr}(HGr(\mathcal{E})) \to (\ldots)[1].\]

Tensoring with \(\mathbb{Z}(2i)[4i]\) being exact, we obtain shifted versions of \((2)\) and a diagram

\[(4) \quad \mathbb{Z}_{atr}(HGr(\mathcal{E}_1)) \longrightarrow \mathbb{Z}_{atr}(HGr(\mathcal{E}_1)) \oplus \mathbb{Z}_{atr}(HGr(\mathcal{E}_2)) \longrightarrow \mathbb{Z}_{atr}(HGr(\mathcal{E})) \longrightarrow (\ldots)[1]\]

\[
\begin{array}{ccc}
\pi \otimes \mathbb{P}_1(\mathcal{Y}') & & \pi \otimes \mathbb{P}_1(\mathcal{Y}') \\
\oplus \mathbb{Z}_{atr}(U_1 \cap U_2)(2i)[4i] & \longrightarrow & \oplus \mathbb{Z}_{atr}(U_1)(2i)[4i] \rightarrow \oplus \mathbb{Z}_{atr}(X)(2i)[4i] \rightarrow (\ldots)[1].
\end{array}
\]

The two left-hand squares commute by naturality, and we now prove that the third also commutes. We have a commutative diagram

\[
\begin{array}{ccc}
\mathbb{Z}_{atr}(HGr(\mathcal{E})) & \longrightarrow & \mathbb{Z}_{atr}(HGr(\mathcal{E}_1))[1] \\
\pi & & \pi[1] \\
\mathbb{Z}_{atr}(X) & \longrightarrow & \mathbb{Z}_{atr}(U_1 \cap U_2)[1].
\end{array}
\]

Tensoring with the morphism corresponding to the \(i\)-th power of the first Pontryagin class \(\mathbb{Z}_{atr}(HGr(\mathcal{E})) \to \mathbb{Z}(2i)[4i]\), we obtain a commutative diagram

\[(5) \quad \mathbb{Z}_{atr}(HGr(\mathcal{E})) \otimes \mathbb{Z}_{atr}(HGr(\mathcal{E})) \longrightarrow \mathbb{Z}_{atr}(HGr(\mathcal{E}_1)) \otimes \mathbb{Z}_{atr}(HGr(\mathcal{E}))[1]
\]

\[
\begin{array}{ccc}
\pi \otimes \mathbb{P}_1(\mathcal{Y}') & & \pi \otimes \mathbb{P}_1(\mathcal{Y}') \\
\mathbb{Z}_{atr}(X) \otimes \mathbb{Z}(2i)[4i] & \longrightarrow & \mathbb{Z}_{atr}(U_1 \cap U_2) \otimes \mathbb{Z}(2i)[4i][1]
\end{array}
\]

On the other hand, the open cover

\[(HGr(\mathcal{E}_1) \times HGr(\mathcal{E})) \cup (HGr(\mathcal{E}_2) \times HGr(\mathcal{E})) = HGr(\mathcal{E}) \times HGr(\mathcal{E})\]
yields a Mayer-Vietoris triangle, and the commutative diagrams

\[
\begin{array}{ccc}
\text{HGr}(E_i) & \longrightarrow & \text{HGr}(E) \\
\downarrow & & \downarrow \\
\text{HGr}(E_i) \times \text{HGr}(E) & \longrightarrow & \text{HGr}(E) \times \text{HGr}(E),
\end{array}
\]

in which the first vertical arrow is the product of the identity and the inclusion and the second vertical arrow is the diagonal map, induce a morphism of Mayer-Vietoris triangles and in particular a commutative diagram

\[
\begin{array}{ccc}
\tilde{\text{Z}}_{tr}(\text{HGr}(E)) & \longrightarrow & \tilde{\text{Z}}_{tr}(\text{HGr}(E)) \\
\downarrow & & \downarrow \\
\tilde{\text{Z}}_{tr}(\text{HGr}(E)) \otimes \tilde{\text{Z}}_{tr}(\text{HGr}(E)) & \longrightarrow & \tilde{\text{Z}}_{tr}(\text{HGr}(E)) \otimes \tilde{\text{Z}}_{tr}(\text{HGr}(E)),
\end{array}
\]

where the right-hand vertical map is the tensor of the identity with the morphism \(\tilde{\text{Z}}_{tr}(\text{HGr}(E)) \longrightarrow \tilde{\text{Z}}_{tr}(\text{HGr}(E))\).

Concatenating Diagrams (5) and (6), we obtain that the third triangle in (4) also commutes. Moreover, our induction hypothesis and the five lemma imply that the third morphism in (4) is an isomorphism as well.

We conclude the proof of the theorem by observing that we may reduce the case of a general covering \(\{U_\alpha\}\) of \(X\) to the case of a covering by two open subschemes using induction again. \(\square\)

Arguing as in [PW10, Theorem 8.2], we can deduce a similar version of Pontryagin classes for Chow-Witt rings.

**Proposition 4.5.** Let \(X\) be a smooth scheme, \(E\) be a symplectic bundle of rank \(2n + 2\) over \(X\) and \(k = \min\{\lfloor \frac{j}{2} \rfloor, n\}\). Then the map

\[
\theta_j : \oplus_{i=0}^{k} \mathcal{C}H^{j-2i}(X) \overset{p^* p_1(\mathcal{W})^i}{\longrightarrow} \mathcal{C}H^j(\text{HGr}_X(E))
\]

is an isomorphism, where \(j \geq 0\), \(p : \text{HGr}_X(E) \longrightarrow X\) is the structure map, \(\mathcal{W}^i\) is the dual tautological bundle endowed with its canonical orientation.

**Proof.** We apply \(\text{Hom}_{\text{DM}^{tr}}(-, \tilde{\text{Z}}_j(2j))\) to both sides of the isomorphism in Theorem 4.3. Note that we have an isomorphism for \(i \leq \lfloor \frac{j}{2} \rfloor\)

\[
\text{Hom}_{\text{DM}^{tr}}(-, \tilde{\text{Z}}_{tr}(X)(2i)[4i], \tilde{\text{Z}}_j(2j)) \longrightarrow \mathcal{C}H^{j-2i}(X)
\]

by Proposition 4.2.

Now suppose that we have an element \(s \in \mathcal{C}H^{j-2i}(X), i \leq k\), which corresponds to a morphism \(\varphi : \tilde{\text{Z}}_{tr}(X) \longrightarrow \tilde{\text{Z}}(j-2i)[2j-4i]\). We conclude the proof using the commutative diagrams

\[
\begin{array}{ccc}
\tilde{\text{Z}}_{tr}(\text{HGr}_X(E)) & \longrightarrow & \tilde{\text{Z}}_{tr}(\text{HGr}_X(E)) \\
\downarrow & & \downarrow \\
\tilde{\text{Z}}_{tr}(X) \otimes L \tilde{\text{Z}}(2i)[4i] & \longrightarrow & \tilde{\text{Z}}_{tr}(X) \otimes L \tilde{\text{Z}}(2i)[4i] \longrightarrow \tilde{\text{Z}}(j)[2j]
\end{array}
\]

\[
\begin{array}{ccc}
\tilde{\text{Z}}_{tr}(\text{HGr}_X(E)) & \longrightarrow & \tilde{\text{Z}}_{tr}(\text{HGr}_X(E)) \\
\downarrow & & \downarrow \\
\tilde{\text{Z}}_{tr}(X) \otimes L \tilde{\text{Z}}(2i)[4i] & \longrightarrow & \tilde{\text{Z}}_{tr}(X) \otimes L \tilde{\text{Z}}(2i)[4i] \longrightarrow \tilde{\text{Z}}(j)[2j]
\end{array}
\]
and
\[ \bigoplus_{i=0}^{k} H_{DM^{eff,-}}(\widetilde{\mathcal{Z}}_{tr}(X)(2i)[4i], \mathbb{Z}(j)(2j)) \overset{\cong}{\longrightarrow} H_{DM^{eff,-}}(\widetilde{\mathcal{Z}}_{tr}(HGr_{X}(\mathcal{E})), \mathbb{Z}(j)(2j)) \]

\[ \bigoplus_{i=0}^{k} \mathbb{H}^{j-2i}(X) \overset{p^* p_1(\mathcal{Y})^i}{\longrightarrow} \mathbb{H}^{j}(HGr_{X}(\mathcal{E})). \]

**Definition 4.4.** In the above proposition, set \( \zeta := p_1(\mathcal{Y}) \) and \( \theta_{n+1} := (\zeta_i) \in \bigoplus_{i=1}^{n+1} \mathbb{H}^{2i}(X) \). Define \( p_0(\mathcal{E}) = 1 \in \mathbb{H}^{0}(X) \), \( p_a(\mathcal{E}) = (-1)^{a-1} \zeta_i, 1 \leq a \leq n+1 \). \( p_a(\mathcal{E}) \) is called the \( a \)th Pontryagin classes of \( \mathcal{E} \). They are uniquely characterized by the Pontryagin polynomial
\[ \zeta^{n+1} - p^*(p_1(\mathcal{E}))\zeta^n + \ldots + (-1)^{n+1} p^*(p_{n+1}(\mathcal{E})) = 0. \]

**5. Gysin Triangle**

**Definition 5.1.** Let \( X \) be a smooth scheme and \( Y \subseteq X \) be a closed subset. Consider the quotient sheaf with MW-transfers
\[ \widetilde{M}_{Y}(X) := \widetilde{\mathcal{Z}}_{tr}(X)/\widetilde{\mathcal{Z}}_{tr}(X \setminus Y). \]

Its image in \( \widetilde{M}^{eff,-}_{DM} \) will be called the relative motive of \( X \) with support in \( Y \) (see [Dég12, Definition 2.2] and the remark before [SV, Corollary 5.3]). By abuse of notation, we still denote it by \( \widetilde{M}_{Y}(X) \).

The aim of the Gysin Triangle is precisely to compute those relative motives, and we now show how to perform this computation in our case.

First of all, we have étale excision, in the sense that the following result holds (see [SV, Lemma 4.11]).

**Proposition 5.1.** Let \( f : X \longrightarrow Y \) be an étale morphism between smooth schemes, \( Z \subseteq Y \) be a closed subset of \( Y \) such that the map \( f^{-1}(Z) \longrightarrow Z \) is an isomorphism (the schemes are endowed with their reduced structure), then the map \( \widetilde{M}_{f^{-1}(Z)}(X) \longrightarrow \widetilde{M}_{2}(Y) \) is an isomorphism of sheaves with MW-transfers.

**Proof.** By the condition given, we get a Nisnevich covering \( f \Pi_{id} : \Pi_{II}(Y \setminus Z) \longrightarrow Y \) of \( Y \). So we have a commutative diagram with exact (after sheafications) rows and columns by [DF17, Lemma 1.2.6]:

\[ \begin{array}{cccc}
0 & 0 & 0 & 0 \\
\tilde{c}(Y \setminus Z) & \tilde{c}(Y \setminus Z) & & \\
\tilde{c}((X \amalg (Y \setminus Z)) \times_{Y} (X \amalg (Y \setminus Z))) & \tilde{c}(X \amalg (Y \setminus Z)) & p & \tilde{c}(Y) \\
\tilde{c}(X) & r & \tilde{c}(Y) & 0 \\
0 & 0 & 0 & 0
\end{array} \]
We want to show that \( \ker(q) = \overline{c}(X \setminus f^{-1}(Z)) \) after sheafification yielding the statement.

We clearly have \( \overline{c}(X \setminus f^{-1}(Z)) \subseteq \ker(q) \) and \( r \) maps onto \( \ker(q) \) after sheafification. So it suffices to show that \( \Im(r) \subseteq \overline{c}(X \setminus f^{-1}(Z)) \). The sheaf \( \overline{c}((X \amalg (Y \setminus Z)) \times_Y (Y \amalg (Y \setminus Z))) \) is decomposed into four direct components

\[
\overline{c}(X \times_Y X), \overline{c}(X \times_Y (Y \setminus Z)), \overline{c}((Y \setminus Z) \times_Y X), \overline{c}(Y \setminus Z) \times_Y (Y \setminus Z)
\]

via disjoint unions so we just have to calculate their images under \( r \) respectively. The calculations for last three components are easy and we only explain the computation of the first one.

We have a Cartesian square

\[
\begin{array}{ccc}
X \times_Y X & \xrightarrow{p_1} & X \\
\downarrow & \swarrow_{\pi} & \downarrow \quad f \\
X & \xrightarrow{f} & Y.
\end{array}
\]

Then for any \( x \in \pi^{-1}(Z) \), \( p_1(x) = p_2(x) \) and the morphisms \( k(p_1(x)) \rightarrow k(x) \) induced by \( p_1 \) and \( p_2 \) are equal since \( f^{-1}(Z) \cong Z \). So by [Mil80, Corollary 3.13], \( p_1 = p_2 \) on the connected component containing \( x \). Hence \( p_1 = p_2 \) on a closed and open set \( U \) containing \( \pi^{-1}(Z) \). Again, \( \overline{c}(X \times_Y X) = \overline{c}(U) \oplus \overline{c}(U') \). So we have \( \Im(r|_{\overline{c}(U)}) = 0 \) and \( \Im(r|_{\overline{c}(U')}) \subseteq \overline{c}(X \setminus f^{-1}(Z)) \). So we have proved that \( \Im(r) \subseteq \overline{c}(X \setminus f^{-1}(Z)) \).

Next, recall the following result (sometimes called homotopy purity).

**Proposition 5.2.** Let \( X \) be a smooth scheme and \( Y \subseteq X \) be a smooth closed subscheme. Then

\[
\widetilde{M}_Y(X) \cong \widetilde{M}_Y(N_{Y/X})
\]

in \( \widetilde{DM}_{eff,f,-} \). Here, the embedding \( Y \subseteq N_{Y/X} \) is the zero section.

**Proof.** See [Pan, Theorem 2.2.8]. Alternatively, one may use [MV98, §3, Theorem 2.23] and the sequence of functors of [DF17, §3.2.4.a].

Now let \( X \) be a smooth scheme and \( (\mathcal{E}, m) \) be a symplectic vector bundle of rank \( 2n \) over \( X \) with total space \( E \).

Recall that, as in the discussion before [PWT0, Theorem 4.1], \( O_X \oplus \mathcal{E} \oplus O_X \) is also a symplectic vector bundle with inner product

\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & m & 0 \\
-1 & 0 & 0
\end{pmatrix}.
\]

**Definition 5.2.** (1) Define \( N^- \) by the cartesian square

\[
\begin{array}{ccc}
\Gr_X(2n, \mathcal{E} \oplus O_X) & \xrightarrow{i} & \Gr_X(2n, O_X \oplus \mathcal{E} \oplus O_X) \\
\downarrow & & \downarrow j \\
N^- & \xrightarrow{} & \Gr_X(O_X \oplus \mathcal{E} \oplus O_X)
\end{array}
\]

where \( i \) comes from the projection \( p_{23} : O_X \oplus \mathcal{E} \oplus O_X \rightarrow \mathcal{E} \oplus O_X \) and \( j \) is the inclusion (see Proposition 3.4).

(2) Define

\[
N = \{ x \in \Gr_X(2n, O_X \oplus \mathcal{E} \oplus O_X) | \mathcal{E}' \rightarrow p^*(O_X \oplus \mathcal{E} \oplus O_X) \rightarrow p^*(O_X \oplus O_X) \text{ iso. at } x \},
\]

where \( p : \Gr_X(2n, O_X \oplus \mathcal{E} \oplus O_X) \rightarrow X \) is the structure map and

\[
0 \rightarrow \mathcal{E}' \rightarrow p^*(O_X \oplus \mathcal{E} \oplus O_X) \rightarrow \mathcal{E}' \rightarrow 0
\]
is the tautological exact sequence. Note that $N$ is an open set of the Grassmannian $Gr_X(2n, O_X \oplus \mathcal{E} \oplus O_X)$.

(3) Define
\[ V = \{ x \in Gr_X(2n, \mathcal{E} \oplus O_X) | \mathcal{F}' \longrightarrow q^*(\mathcal{E} \oplus O_X) \longrightarrow q^*O_X \text{ is an isomorphism at } x \}, \]
where $q : Gr_X(2n, \mathcal{E} \oplus O_X) \longrightarrow X$ is the structure map and
\[ 0 \longrightarrow \mathcal{F}' \longrightarrow q^*(\mathcal{E} \oplus O_X) \longrightarrow \mathcal{F}'' \longrightarrow 0 \]
is the tautological exact sequence. As above, note that $V$ is an open set of $Gr_X(2n, \mathcal{E} \oplus O_X)$.

The notations of $N^-$ and $N$ come from [PW10 Theorem 4.1], but our treatment is slightly different.

**Lemma 5.1.** 1) Let $T$ be an $X$-scheme and $f : T \longrightarrow Gr_X(2n, O_X \oplus \mathcal{E} \oplus O_X)$ be an $X$-morphism. Then
\[ \text{Im}(f) \subseteq N \iff f^*(\mathcal{E}') \longrightarrow (p \circ f)^*(O_X \oplus \mathcal{E} \oplus O_X) \longrightarrow (p \circ f)^*(O_X \oplus O_X) \text{ is an isomorphism.} \]
Consequently, $N^- \subseteq N \cap HGr_X(O_X \oplus \mathcal{E} \oplus O_X)$.

2) Let $T$ be an $X$-scheme and $f : T \longrightarrow Gr_X(2n, \mathcal{E} \oplus O_X)$ be an $X$-morphism. Then
\[ \text{Im}(f) \subseteq V \iff f^*(\mathcal{F}') \longrightarrow (q \circ f)^*(\mathcal{E} \oplus O_X) \longrightarrow (q \circ f)^*O_X \text{ is an isomorphism.} \]
Furthermore, $N^- = V$.

**Proof.** 1) $\iff$ Easy. For the $\subseteq$ part, set $C = \text{Coker}(\mathcal{E}' \longrightarrow p^*(O_X \oplus \mathcal{E} \oplus O_X) \longrightarrow p^*(O_X \oplus O_X))$.

We see that $N = \text{Supp}(C)^c$. Since $f^{-1}(\text{Supp}(C)) = \text{Supp}(f^*C)$, $f^{-1}(\text{Supp}(C)) = \emptyset$ hence $f^{-1}(N) = T$. So $\text{Im}(f) \subseteq N$.

For the second statement, let $v : N^- \longrightarrow X$ be the structure map. The bundle $N^-$ has a map $\varphi$ towards $Gr_X(2n, O_X \oplus \mathcal{E} \oplus O_X)$ hence we have a subbundle $K \subseteq v^*(O_X \oplus \mathcal{E} \oplus O_X)$. Since $\varphi$ factors through $Gr_X(2n, \mathcal{E} \oplus O_X)$, the first inclusion $v^*O_X \longrightarrow v^*(O_X \oplus \mathcal{E} \oplus O_X)$ factors through $K$, which makes $v^*O_X$ a subbundle of $K$. Since $\varphi$ also factors through $HGr_X(O_X \oplus \mathcal{E} \oplus O_X)$, the inner product is non degenerate on $K$. So for every $x \in N^-$, there is an affine neighborhood $U$ of $x$ such that $K(U)$ is a free $O_{N^-}(U)$-module with a basis $(1, 0, 0)$ and $(x_1, x_2, x_3)$. Hence $x_3 \in O_{N^-}(U)^*$ by non degeneracy. Hence the map $K \longrightarrow v^*(O_X \oplus \mathcal{E} \oplus O_X) \longrightarrow v^*(O_X \oplus O_X)$ is surjective on $U$. So we see that $N^- \subseteq N$ by the first statement.

2) The first statement can be proved in the same way as in 1). For the second statement, we have a commutative diagram with exact rows:
\[
\begin{array}{cccccc}
0 & \longrightarrow & K' & \longrightarrow & v^*(O_X \oplus \mathcal{E}) & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \\
& & \downarrow{p_1} & & \downarrow{p_{12}} & & \downarrow{\nu_1} & & \\
0 & \longrightarrow & K' \oplus O_X & \longrightarrow & v^*(O_X \oplus \mathcal{E} \oplus O_X) & \longrightarrow & \mathcal{E} & \longrightarrow & 0
\end{array}
\]

Hence there is a section in $K'(N^-)$ which maps to $(1, s, 0)$ in $v^*(O_X \oplus \mathcal{E} \oplus O_X)$. This section turns the map $K' \longrightarrow v^*(O_X \oplus \mathcal{E}) \longrightarrow v^*O_X$ into an isomorphism. So $N^- \subseteq V$. The inclusion $V \subseteq N^-$ can be proved using a similar method. □

**Lemma 5.2.** Let $T$ be an $X$-scheme and $f : T \longrightarrow Gr_X(2n, O_X \oplus \mathcal{E} \oplus O_X)$ be an $X$-morphism. Let $\varphi$ be the composite
\[
(p \circ f)^*O_X \xrightarrow{f_1} (p \circ f)^*(O_X \oplus \mathcal{E} \oplus O_X) \longrightarrow f^*\mathcal{E}''.
\]
Then
\[ \text{Im}(f) \subseteq \text{Gr}_X(2n, \mathcal{E} \oplus O_X)^c \iff \varphi \text{ is injective and has a locally free cokernel.} \]

**Proof.**

\[ \text{Im}(f) \subseteq \text{Gr}_X(2n, \mathcal{E} \oplus O_X)^c \iff \forall g : \text{Spec } K \to T, \text{Im}(f \circ g) \subseteq \text{Gr}_X(2n, \mathcal{E} \oplus O_X)^c, \]
where \( K \) is a field. So let’s assume \( T = \text{Spec } K \). In this case,

\[ \text{Im}(f) \subseteq \text{Gr}_X(2n, \mathcal{E} \oplus O_X)^c \iff f \text{ does not factor through } \text{Gr}_X(2n, \mathcal{E} \oplus O_X), \]
and the latter condition is equivalent to \( \varphi \neq 0 \). Hence

\[ \text{Im}(f) \subseteq \text{Gr}_X(2n, \mathcal{E} \oplus O_X)^c \iff \forall g : \text{Spec } K \to T, g^*(\varphi) \neq 0. \]

Now we may assume that \( T \) is affine and use the residue fields of \( T \). Locally, the map \( \varphi \) is like \((a_i) : A \to A^{2n}\) and the condition just says that the ideal \((a_i)\) is the unit ideal, which is equivalent to \((a_i)\) being injective and \(\text{Coker}((a_i))\) being projective. This just says that \( \varphi \) is injective and has a locally free cokernel. \(\square\)

Consider next the following square

\[
\begin{array}{ccc}
N^- & \xrightarrow{l} & N \\
\downarrow v & & \downarrow u \\
X & \xrightarrow{z} & E
\end{array}
\]

where \( l \) is given by \( N^- \subseteq N \) and \( v \) is just the structure map (of \( N^- \)). Let \( r : N \to X \) be the structure map of \( N \). We have the tautological exact sequence

\[
0 \to r^*(O_X \oplus O_X) \to r^*(O_X \oplus \mathcal{E} \oplus O_X) \to r^*\mathcal{E} \to 0
\]

\[
(1,0) \iff (1,s_1,0) \\
(0,1) \iff (0,s_2,1)
\]

and \( u \) is induced by \( s_1 \). Finally, \( z \) is the zero section of \( E \).

**Proposition 5.3.** The above square is a Cartesian square.

**Proof.** The map \( l \) induces an exact sequence

\[
0 \to v^*(O_X \oplus O_X) \to v^*(O_X \oplus \mathcal{E} \oplus O_X) \to v^*\mathcal{E} \to 0
\]

\[
(1,0) \iff (1,s,0)
\]

But \((1,0,0)\) belongs to the kernel, so \( s = 0 \). Hence the square commutes and is Cartesian. \(\square\)

Now, we use the square

\[
\begin{array}{ccc}
N & \xrightarrow{w} & E \\
\downarrow u & & \downarrow \pi \\
E & \xrightarrow{\pi} & X
\end{array}
\]

where \( w \) is induced by \( s_2 \) in (**) . We see that it’s a Cartesian square just by that diagram, since there are two (arbitrary) sections there. So \( u \) is a \( \mathbb{A}^{2n} \)-bundle.

The third step of the calculation is the following theorem. It has a similar version in [PW10, Proposition 4.3], but we are not considering the same embedding as there.

**Proposition 5.4.**

\[
\widetilde{M}_X(E) \cong \widetilde{M}_N(N) \cong \widetilde{M}_N(H\text{Gr}_X(O_X \oplus \mathcal{E} \oplus O_X))
\]

in \( \widetilde{DM}^{eff,-} \).
Proof. The first isomorphism comes from Proposition 5.3 and the fact that $u : N \to E$ is an $A^n$-bundle. Then second isomorphism is because $N^- \subseteq N \cap HGr_X(O_X \oplus \mathcal{E} \oplus O_X)$ by Lemma 5.1 and Proposition 5.1. □

Now by Lemma 5.2, the natural embedding $HGr_X(O_X \oplus \mathcal{E} \oplus O_X) \to HGr_X(E)$ factor through $(N^-)^c$, thus we have a map $i : HGr_X(O_X \oplus \mathcal{E} \oplus O_X) \to (N^-)^c$.

**Proposition 5.5.**

\[
\widetilde{Z}_{\text{tr}}(i) : \widetilde{Z}_{\text{tr}}(HGr_X(O_X \oplus \mathcal{E} \oplus O_X)) \to \widetilde{Z}_{\text{tr}}((N^-)^c)
\]

is an isomorphism in $\widetilde{DM}^{eff}$. □

Proof. Follows from the proof of [PW10, Theorem 5.2].

Finally, the following theorem completes the calculation. Its proof is similar to that of [Deg12, Lemma 2.12].

**Theorem 5.1.**

\[
\widetilde{M}_X(E) \cong \widetilde{Z}_{\text{tr}}(X)(2n)[4n]
\]

in $\widetilde{DM}^{eff}$. □

Proof. By Proposition 5.5, $\widetilde{M}_X = HGr_X(O_X \oplus \mathcal{E} \oplus O_X))$ is just the cone of the embedding $i : HGr_X(O_X \oplus \mathcal{E} \oplus O_X) \to HGr_X(O_X \oplus \mathcal{E} \oplus O_X)$. By Theorem 4.3, we have a commutative diagram where the vertical arrows are isomorphisms

\[
\begin{array}{cccc}
\widetilde{Z}_{\text{tr}}(HGr_X(O_X \oplus \mathcal{E} \oplus O_X)) & \to & \widetilde{Z}_{\text{tr}}(HGr_X(O_X \oplus \mathcal{E} \oplus O_X)) \\
\oplus_{i=0}^{n-1} \widetilde{Z}_{\text{tr}}(X)(2i)[4i] & \to & \oplus_{i=0}^{n} \widetilde{Z}_{\text{tr}}(X)(2i)[4i].
\end{array}
\]

Now, $i$ pulls back the tautological bundle to the tautological bundle, giving the result. □

Thus we have the following theorem.

**Theorem 5.2.** Let $X$ be a smooth scheme and let $Y \subseteq X$ be a smooth closed subscheme with a symplectic normal bundle with codim$(Y) = 2n$. Then we have a distinguished triangle

\[
\widetilde{Z}_{\text{tr}}(X \setminus Y) \to \widetilde{Z}_{\text{tr}}(X) \to \widetilde{Z}_{\text{tr}}(Y)(2n)[4n] \to \widetilde{Z}_{\text{tr}}(X \setminus Y)[1]
\]

in $\widetilde{DM}^{eff}$. □

Proof. Follows from Theorem 5.1 and Proposition 5.2.

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