Quantum coherence control of solid-state charge qubit by means of a suboptimal feedback algorithm

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The quantum coherence control of a solid-state charge qubit is studied by using a suboptimal continuous feedback algorithm within the Bayesian feedback scheme. For the coherent Rabi oscillation, the present algorithm suggests a simple bang-bang control protocol, in which the control parameter is modulated between two values. For the coherent protection of idle state, the present approach is applicable to arbitrary states, including those lying on the equator of the Bloch sphere which are out of control in the previous Markovian feedback scheme.

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Introduction.— Being stimulated by the interest of solid-state quantum computation, in recent years the measurement problem of solid-state qubit is becoming an intensive research subject. In particular, the setup of a charge qubit measured by a mesoscopic QPC, has received considerable attention. Very recently, the mesoscopic QPC was experimentally demonstrated as a charge-sensitive electro-meter. Strikingly, rather than the projective strong measurement, the continuous weak measurement can be employed to realize some marvellous tasks, such as quantum state initialization, stabilization, and coherence protection, etc.

The central idea is the measurement-result-based feedback control, which falls typically into two main categories, say, the Markovian scheme and the Bayesian scheme. In quantum optics, the study of quantum feedback control has been going on for more than a decade. However, in solid states it is a relatively new subject. In this work, based on a suboptimal feedback algorithm, we study the problem of quantum coherence protection of a solid-state charge qubit which is subject to the continuous weak measurement by a mesoscopic QPC.

Formalism.— To be specific, the charge qubit is assumed to be two coupled quantum dots (CQD’s), occupied by a single electron, whose Hamiltonian reads

\[ H_{qb} = \varepsilon (c_1^\dagger c_2 - c_1 c_1) + \Omega (c_1^\dagger c_2 + c_2^\dagger c_1). \]

Here \( c_1^\dagger \) (\( c_1 \)) is the electron creation (annihilation) operator in the inq dot, \( \varepsilon \) and \( \Omega \) are, respectively, the energy offset of the two local dot-states and their coupling strength. At zero-temperature and in the limit of large bias voltage across the QPC, the measurement back-action on the qubit is described by the following master equation (\( \hbar = 1 \))

\[ \dot{\rho}(t) = -i[H_{qb}, \rho(t)] + \mathcal{D}[\mathcal{T} + \mathcal{X}n_1] \rho(t), \]

where \( n_1 = c_1^\dagger c_1 \) is the occupation-number operator of the first dot, the parameters \( \mathcal{T} \) and \( \mathcal{X} \) are given by

\[ |\mathcal{T}|^2 = 2\pi|T|^2 g_C eV \equiv D_2, \quad |\mathcal{X}|^2 = 2\pi|T + \chi|^2 g_C eV \equiv D_1. \]

Here, the tunneling amplitudes through QPC, \( T \) and \( \chi \), are assumed to be independent of the reservoir states, and real for simplicity; \( g_C \) and \( g_R \) are the density of states for the left and right reservoirs; \( V = (\mu_L - \mu_R)/e \) is the bias voltage across the QPC. The superoperator in Eq. (1) is defined as \( \mathcal{D}[\rho] = \mathcal{J}[\rho] - \mathcal{A}[\rho], \) where \( \mathcal{J}[\rho] = r\rho r^\dagger \) and \( \mathcal{A}[\rho] = (r^\dagger r \rho + r^\dagger r \rho^\dagger)/2. \)

Intuitively, corresponding to the qubit electron jumps between the two dots, the continuous output current \( I(t) \) is a complicated stochastic random quantity. But, quite remarkably, by using \( I(t) \) one can design a proper feedback Hamiltonian \( H_{fb}(t) \) to maintain or improve the quantum coherence of the qubit. Conditioned on \( I(t) \) and involving \( H_{fb}(t) \), Eq. (1) is unravelled as

\[ \begin{align*}
\dot{\rho}_{cb}(t) &= -i[H_{qb}, \rho_{cb}(t)] dt + \sum_{j=x,y,z} \gamma_j \mathcal{D}[\mathcal{X} j] \rho_{cb}(t) dt \\
&+ \mathcal{D}[\mathcal{X} j] \rho_{cb}(t) dt + \mathcal{H}[\mathcal{X} j] \rho_{cb}(t) dW(t) \\
&- i[H_{fb}(t), \rho_{cb}(t)] dt.
\end{align*} \]

Compared to Eq. (2), the superoperator “\( \mathcal{K} \)” is defined by \( \mathcal{H}[\rho] = r\rho r^\dagger - \rho \text{Tr}[r\rho r^\dagger], \) \( dW(t) \) is the Weiner increment, which is a Gaussian-white-noise stochastic variable, having the property of \( E[dW(t)] = 0 \) and \( E[dW(t)dW(\tau)] = \delta(t - \tau) dt, \) with \( E[\cdots] \) the ensemble average over a large number of stochastic realizations. The Weiner increment \( dW(t) \) is associated with the output current \( I(t) \) in terms of the relation:

\[ I(t) - \bar{T} = \Delta I(\rho^1_{cb} - \rho^2_{cb})/2 + \sqrt{\mathcal{S}_0/\mathcal{X}(t)}, \]

where \( \xi(t) dt = dW(t), \) \( \bar{T} = (I_1 + I_2)/2, \) \( \Delta I = I_1 - I_2, \) and \( \mathcal{S}_0 = 2e\bar{T}. \) \( \rho^j(t) = (\rho_T(t)) j \) is the density matrix.
element over the qubit basis states. Based on the output current \( I(t) \), there are two typical schemes to design the feedback Hamiltonian, i.e., the so-called Markovian “\( I(t) \)-based” feedback \([3]\), and the Bayesian “\( \rho_c(t) \)-based” feedback \([2, 10]\). In the following, the second one will be implemented to protect the qubit coherent evolution and its idle state, on the basis of a suboptimal feedback algorithm \([15]\).

**Feedback Control of Rabi Oscillation.**—For simplicity, consider the symmetric qubit (\( \varepsilon = 0 \)). The “desired” pure-state evolution to be protected is

\[
|\psi_d(t)\rangle = \cos(\Omega t)|1\rangle - i \sin(\Omega t)|2\rangle ,
\]

where \( |1\rangle \) and \( |2\rangle \) are the dot states. The basic idea of Bayesian feedback control is to carry out first an estimate of the qubit state \( \rho_c(t) \) based on the noisy current \( I(t) \), then compare it with the desired state \( \psi_d(t) \) and design the feedback Hamiltonian using the calculated difference. In Ref. \([1]\) the estimated quantity is the relative phase between the superposed states “\( |1\rangle \) and \( |2\rangle \)”, i.e., \( \phi(t) = \mathrm{arctan}(2\text{Im}\rho_{22}(t)/|\rho_{12}(t)|^2) \), which is compared with \( \phi_0 = 2\Omega t \) (mod \( 2\pi \)) defined from the desired state. Then, the phase “error” \( \Delta\phi(t) = \phi(t) - \phi_0 \) is used to design the feedback Hamiltonian, i.e., \( H_{fb}(t) \propto \Delta\phi(t) \). In certain sense, this scheme may be inconvenient to be implemented in practice, since the feedback Hamiltonian is a complicated stochastic function of time. In the following, we propose an alternative scheme based on a suboptimal algorithm \([13]\).

For real-time feedback control, each feedback step acts only for an infinitesimal time, “\( \Delta t \)”. The suboptimal algorithm is based on the principle that the state evolution in each infinitesimal time step will maximize the fidelity of the estimated state with the desired (target) state. The state evolution in the presence of feedback is governed by Eq. \((2)\). As far as the term related to the feedback Hamiltonian is concerned, the final state \( \rho_c(t + \Delta t) \) is given by

\[
\rho_c(t + \Delta t) = \rho_c(t) - i[H_{fb}, \rho_c(t)]\Delta t - \frac{1}{2}[H_{fb}, [H_{fb}, \rho_c(t)]](\Delta t)^2 + \cdots . \tag{5}
\]

The fidelity of this state with the target state is defined by

\[
\langle \psi_d|\rho_c(t + \Delta t)|\psi_d \rangle = \langle \psi_d|\rho_c(t)|\psi_d \rangle - i\langle \psi_d|[H_{fb}, \rho_c(t)]|\psi_d \rangle\Delta t - \frac{1}{2}\langle \psi_d|[H_{fb}, [H_{fb}, \rho_c(t)]]|\psi_d \rangle(\Delta t)^2 + \cdots . \tag{6}
\]

To optimize the fidelity, one should maximize the coefficient of \( \Delta t \), which is the dominant term. Similar to other control theories, the maximization must be subject to certain constraints, such as the restriction on the maximum eigenvalue of \( H_{fb} \), the sum of the norms of the eigenvalues, or the sum of the squares of the eigenvalues, etc. Physically, these constraints stem from the limitation of the feedback strength or finite Hamiltonian resources. Here we adopt the last type of constraint, namely,

\[
\text{Tr}[H_{fb}^2] = \sum_{n=1}^{\infty} \lambda_n^2(H_{fb}) \leq \mu. \tag{7}
\]

Under this constraint, the feedback Hamiltonian can be constructed as \([13]\)

\[
H_{fb} = i c [\psi_d(t)|\psi_d(t), \rho_c(t)] , \tag{8}
\]

where \( c = \frac{\mu}{\sqrt{2(\mu - \lambda_{\max})}} \), with \( \lambda_{\max} = \langle \psi_d(t)|\rho_c(t)|\psi_d(t) \rangle \), and \( b = \langle \psi_d(t)|\rho_c(t)|\psi_d(t) \rangle \). Substituting Eq. \((4)\) into Eq. \((8)\) yields

\[
H_{fb} = F \sigma_x , \tag{9}
\]

with

\[
F = \begin{cases} \sqrt{\frac{\mu}{2}}, & \Delta\phi < 0 \\ -\sqrt{\frac{\mu}{2}}, & \Delta\phi > 0 \end{cases} . \tag{10}
\]

In this case, the suboptimal feedback control resembles a simple bang-bang control, where the feedback parameter

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**FIG. 1:** The ensemble evolution of \( \rho_{11}(t) = E[\rho_{11}(t)] \) for different feedback strengths \( \mu=0 \) (gray solid curve); 0.001 (black solid curve); 0.1 (dashed curve).

**FIG. 2:** Synchronization degree \( D \) as a function of feedback strength \( \mu \), for several magnitudes of dephasing due to environment: \( \gamma_x = 0 \) (solid curve); 0.05\( \Gamma_d \) (dashed curve); and 0.25\( \Gamma_d \) (dash-dotted curve).
of using the above suboptimal feedback to protect the idle state of a qubit. This is essential in information storage, while the Rabi oscillation discussed above corresponds to information processing. In general, an arbitrary pure state of qubit can be expressed as
\[
|\psi_T\rangle = \cos \theta |1\rangle + \sin \theta e^{i\phi}|2\rangle .
\] (11)

The goal of feedback is to make it immune against the influence of environmental noise.

Following the same procedure used above, say, maximizing the "\(\Delta t\)" term in Eq. (4), and substituting Eq. (11) into Eq. (3), we obtain
\[
H_{fb} = \lambda [c_1 \sigma_z + c_2 \sigma_x + c_3 \sigma_y] ,
\] (12)
with
\[
\lambda = \sqrt{\mu/2(c_1^2 + c_2^2 + c_3^2)} ,
\]
\[
c_1 = 2\alpha \beta (\text{Re} \rho_{11}^* \sin \phi + \text{Im} \rho_{12}^* \cos \phi) ,
\]
\[
c_2 = (\beta^2 - \alpha^2) \text{Im} \rho_{12}^* - \alpha \beta (\rho_{11}^* - \rho_{22}^*) \sin \phi ,
\]
\[
c_3 = \alpha \beta \cos \phi (\rho_{11}^* - \rho_{22}^*) - (\alpha^2 - \beta^2) \text{Re} \rho_{12}^* ,
\]
in which \(\alpha = \cos \frac{\theta}{2}\) and \(\beta = \sin \frac{\theta}{2}\).

So far the feedback Hamiltonian is constructed on the basis of maximizing the coefficient of \(\Delta t\) in Eq. (4). However, this algorithm does not work if the target state is the eigenstate of \(\rho_e\), namely, \(\rho_e|\psi_T\rangle = \lambda_T|\psi_T\rangle\). For instance, such situation occurs when \(\theta = 0\) and \(\pi\). In this case, the dominant "\(\Delta t\)"-term in Eq. (10) vanishes, and one is forced to maximize the coefficient of \((\Delta t)^2\). Under the constraint on the Hamiltonian as described by Eq. (11), the explicit construction of the optimal \(H_{fb}\) reads
\[
H_{fb} = \sqrt{\mu/2} (|v_1\rangle \langle \psi_T| + |\psi_T\rangle \langle v_1|),
\] (13)
where \(|v_1\rangle\) is the eigenvector with the largest eigenvalue of \(\rho_e(t)\). Notice that for the idle target state \(|1\rangle\) (or \(|2\rangle\), the conditional evolution described by Eq. (2) will not generate quantum coherence between \(|1\rangle\) and \(|2\rangle\), because there is no coherent driving source. Thus the states \(|1\rangle\) and \(|2\rangle\) must be the eigenstates of \(\rho_e(t)\), with eigenvalues \(\rho_{11}^*\) and \(\rho_{22}^*\), respectively. From Eq. (13), the concrete form of feedback Hamiltonian is detailed as
\[
H_{fb} = \left\{ \begin{array}{ll}
\sqrt{\frac{\mu}{2}} \sigma_x , & \rho_{11}^* < \rho_{22}^* , \\
0 , & \rho_{11}^* > \rho_{22}^* ,
\end{array} \right.
\] (14)
for \(\theta = 0\) (\(|\psi_T\rangle = |1\rangle\)), and
\[
H_{fb} = \left\{ \begin{array}{ll}
\rho_{11}^* < \rho_{22}^* , & \sqrt{\frac{\mu}{2}} \sigma_x , \\
0 , & \rho_{11}^* > \rho_{22}^* ,
\end{array} \right.
\] (15)
for \(\theta = \pi\) (\(|\psi_T\rangle = |2\rangle\)). Here we have used the property that if \(|v_1\rangle\) in Eq. (13) is the target state, we should set \(H_{fb} = 0\) for that time step, since in this case the fidelity will not increase under the feedback action.

In the following numerical simulations, we first consider the problem of merely eliminating the back-action

![Diagram](image_url)
of an ideal detector (i.e. \( \gamma_j = 0 \)), in the absence of any other external influence of environment. The result is shown in Fig. 3(a). It is observed that the synchronization degree is independent of the relative phase “\( \phi \)” between the two superposed components [c.f. Eq. (11)], but depends on “\( \theta \)” symmetrically around “\( \theta = \pi/2 \)” (the minimal point). Since observing “\( \sigma_z \)” will induce dephasing of the target state, this leads to the smaller synchronization degree for the more coherently superposed state. For \( \theta = 0 \) and \( \pi \), the corresponding target states are the eigenstates of the observable \( \sigma_z \); they are thus immune against the back-action of the detector, leading to \( D = 1 \) and \( H_{fb} = 0 \) according to the proposed feedback scheme. On the other hand, for \( \theta = \pi/2 \), the corresponding target state is \( |\psi_T \rangle = \frac{1}{\sqrt{2}} (|1\rangle + e^{i\phi}|2\rangle) \), which is the most coherently superposed state. This state is destroyed most seriously by the detector’s back-action, whose synchronization degree is thus minimal.

Now we consider the influence of external environment. In reality, a possible error is the \( \sigma_x \)-type, which causes similar pure dephasing as the detector’s back-action does. As a consequence, the (“\( \theta, \phi \)”)-dependence behavior of the synchronization degree shown in Fig. 3 (b) is similar to Fig. 3(a), but it decreases with the dephasing strength. Another possible error is the \( \sigma_x \)-type, which causes relaxation between the states \( |1\rangle \) and \( |2\rangle \). The feedback result is shown by Fig. 4, where the synchronization degree depends on both “\( \theta \)” and “\( \phi \)”.

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**Conclusion.**— To summarize, based on a suboptimal feedback algorithm we have presented a study on quantum coherence control of a solid-state charge qubit, for both its quantum evolution and idle state. For the coherent Rabi oscillation, our study leads to a simple bang-bang control feedback scheme, in terms of modulating a single control parameter by only two values, which differs from the existing Bayesian scheme [11] and may be easier to implement in some cases. For the coherence protection of idle state, the present approach is applicable to arbitrary states, including those lying on the equator of the Bloch sphere which are out of control in the previous Markovian feedback scheme [24]. In addition to eliminating the measurement back-action, the proposed feedback scheme can also protect the qubit from environmental influence to some extent. However, as many other approaches, it cannot protect the qubit from drastic influence of external environment. This drawback may be partially overcome by developing other optimal feedback algorithms, which is a promising but challenging task in this field.

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