N=1 Worldsheet Boundary Couplings 
and Covariance of non-Abelian Worldvolume Theory

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Abstract
A systematic construction is given for N=1 open string boundary coupling to Abelian and non-Abelian Dp-brane worldvolume fields, in general curved backgrounds. The basic ingredient is a set of four “boundary vectors” that provide a unified description of boundary conditions and boundary couplings. We then turn to the problem of apparent inconsistency of non-Abelian worldvolume scalar couplings (obtained by T-duality), with general covariance. It means that the couplings cannot be obtained from a covariant action by gauge fixing ordinary general coordinate transformations (GCT). It is shown that the corresponding worldsheet theory has the same problem, but is also invariant under certain matrix-valued coordinate transformations (MCT) that can be used to restore its covariance. The same transformations act on the worldvolume, leading to a covariant action. Then the non-Abelian Dp-brane action obtained by T-duality corresponds to gauge fixing the MCT and not GCT, hence the apparent incompatibility with general covariance.

1 Introduction and Summary

The presence of non-Abelian scalar fields on the D-brane worldvolume gives rise to new interactions investigated in [1, 2, 3, 4], that have widely been used since and lead to interesting phenomena. In particular, in [3] the couplings of non-Abelian scalars were obtained by T-dualizing D9-branes to Dp-branes. While this is a consistent procedure, the resulting worldvolume action has a puzzling feature, which is that it cannot be obtained from a general covariant action on fixing a coordinate gauge, e.g., the static gauge. The reason is that some components of the non-Abelian scalars which appear in a covariant description and are non-zero even in the static gauge, do not show up in the action. Being matrices, they cannot be gauged away by ordinary coordinate transformations. This signals an apparent inconsistency of the theory with general covariance which is not acceptable.

This apparent inconsistency of the non-Abelian worldvolume action with general covariance cannot be attributed to a shortcoming in the derivation. In fact, it arises within the regime of validity of the procedure followed in [3]. Thus, while the missing terms needed to restore compatibility with covariance are easy to guess, we cannot simply introduce them into the action by hand as that amounts to tampering with the outcome of a consistent calculation. The resolution of the puzzle then requires understanding the origin of the problem and finding a mechanism to restore the missing terms.

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To achieve this, we study the string worldsheet boundary coupling to worldvolume fields, which is the microscopic origin of the worldvolume theory. We obtain the $N=1$ worldsheet boundary conditions and boundary couplings to Abelian and non-Abelian worldvolume fields, and in general backgrounds for general brane embeddings, which so far have not been investigated satisfactorily. The problem is formulated in terms of a set of four “boundary vectors”, $N^X$, $N^\psi$, $D_X$ and $D_\psi$, that live on the worldsheet boundary and which have components tangent and normal to the brane. The boundary conditions follow from setting to zero appropriate projections of the boundary vectors, while the surviving components describe the coupling of the worldsheet boundary to worldvolume fields. This leads to a natural description of worldsheet boundary couplings, consistent the general covariance, supersymmetry and T-duality. It is shown that all couplings to worldvolume scalars follow from shifts in the coordinates, appropriately supersymmetrized and with the non-Abelian features taken into account. The possibility of this interpretation in the supersymmetric theory is intimately related to the correct handling of the NS-NS 2-form gauge invariance on the boundary. Another feature is that the gauge field is always shifted by a combination of the scalars and the NS-NS 2-form field.

Having constructed the boundary action, we study how the boundary coupling to non-Abelian Dp-brane follows from the D9-brane on T-dualizing. The same procedure was applied in [3] to the worldvolume theory. The scalar couplings in the worldsheet boundary action obtained in this way share all features of the corresponding couplings in the worldvolume theory, including the apparent inconsistency with general covariance. However, the worldsheet theory also develops a symmetry which resolves the problem: Here covariance can be restored by the addition of terms that vanish due to the boundary conditions, and hence do not change the content of the theory. The vanishing terms are also generated by matrix-valued coordinate transformations. It then follows that the worldvolume theory should have a corresponding symmetry which involves shifting ordinary coordinates by matrix valued functions. Correctly implemented, this restores the covariance of the non-Abelian worldvolume action.

The apparent inconsistency of the non-Abelian worldvolume theory [3] with general covariance can now be understood based on the picture that emerges from the worldsheet considerations: I) Besides general coordinate transformations (GCT), the complete worldvolume action is also invariant under a group of matrix-valued coordinate transformations (MCT). Our worldsheet considerations probe only a part of the MCT. Some aspects of such transformations have been postulated and studied in [5, 6] in the worldvolume theory. II) Now, the larger symmetry allows us to fix a gauge using the MCT. This also fixes the GCT since they both act on the same objects. III) The Dp-brane action that emerges from T-duality is automatically in such a gauge which can thus be undone only by an element of MCT and not GCT. Therefore, if the existence of the MCT is not taken into account, one cannot relate this action to a covariant one through GCT alone, hence the apparent inconsistency between the two. IV) The worldsheet analysis provided us with the right element of MCT to undo the gauge fixing. The admissibility of this matrix-valued transformations follows from the worldsheet theory and is not obvious from the known structure of the worldvolume action in [3]. This summarizes the solution to the problem of covariance of the non-Abelian worldvolume theory.

The paper is organized as follows: In section 2 we introduce a set of four “boundary vectors” and express the boundary term in the variation of the $N=1$ supersymmetric worldsheet action in terms of them. In section 3, we briefly review the basics of the covariant description of brane geometry, with emphasis on the worldvolume scalars and their structure in the static
gauge. We then obtain the covariant form of Dp-brane boundary conditions in terms of the boundary vectors and examine their consistency with T-duality. In section 4, we construct worldsheet boundary couplings in terms of the boundary vectors and examine their behaviour under T-duality. The idea is first implemented in the bosonic theory and then generalized to N=1 Abelian and non-Abelian cases. The boundary couplings to scalars are interpreted as appropriate coordinate shifts in all the three cases. This crucially depends on the manifest NS 2-form gauge invariance of the couplings. The behaviour of the boundary action under T-duality suggests the possibility of coordinate shifts by non-Abelian matrices to restore compatibility with general covariance. In section 5, we turn to the issue of the compatibility of non-Abelian scalar couplings with general covariance in the worldvolume theory and its resolution based on our worldsheet considerations. As a test, the idea is first applied to the Abelian worldvolume theory and shown to work. We then consider the non-Abelian case and obtain the general covariant form of the scalar couplings. We also outline the general picture that emerges from the analysis and which puts our results in perspective. The implication for the formulation of D-brane interactions in terms of Clifford multiplication is also discussed. The conclusions are summarized in section 6. Appendix A explains our supersymmetry conventions and Appendix B contains the index conventions.

2 Boundary Variation and Boundary Vectors

In this section we consider NSR open strings with N=1 worldsheet supersymmetry in general metric $G_{MN}(X)$ and antisymmetric tensor field $B_{MN}(X)$ backgrounds. The boundary terms that arise on varying the action are written in terms of a set of “boundary vectors” which provide a natural realization of supersymmetry. They will play an important role in later sections.

2.1 Boundary variations

In the absence of boundary interactions, the open string worldsheet action with N=1 supersymmetry is (see the appendix for conventions),

$$S = \frac{1}{2} \int_\Sigma d^2 \sigma \int d\theta^+ \int d\theta^- E_{MN}(X) D_+ X^M D_- X^N + \frac{i}{4} \int_{\partial \Sigma} d\tau B_{MN}(\psi_+^M \psi_+^N + \psi_-^M \psi_-^N).$$

Here, $X$ is the N=1 superfield and $E = G + B$. This form of the action is known to be correct for constant backgrounds and no further modifications arise in space-time dependent backgrounds. The added boundary term in (1) has been noted in [8, 9, 10, 11] but its presence can be argued for on very general grounds: It is needed to cancel a similar term that is hidden in the superspace part so that, in terms of the component fields, the action $S$ does not contain boundary terms and has the same form as the closed string action. Such a boundary term, if present, would have meant that the bulk field $B_{MN}$ couples differently to open and closed strings, which should not be the case. Then, in terms of component fields and after integrating out the auxiliary field, one obtains,

$$S = \frac{1}{2} \int_\Sigma d^2 \sigma \left[ \partial_+ X^M E_{MN} \partial_- X^N + i \psi_+^M G_{MN} \nabla_- \psi_+^N + i \psi_-^M G_{MN} \nabla_+ \psi_-^N ight] + \frac{1}{2} \psi_+^M \psi_-^N \psi_-^P \psi_-^L R_{MNKL}(E).$$

(2)
The covariant derivatives are given by $\nabla_\pm \psi_\pm^N = \partial_\pm \psi_\pm^N + (\Gamma^N_{LK} \mp \frac{1}{2} H^N_{LK}) \partial_\pm X^K \psi_\pm^K$, and $R_{MNLK}(E)$ is the curvature tensor associated with $\nabla_+$. Boundary terms arise in the variation $\delta S$ of the action when deriving the equations of motion, as well as in its supersymmetry variation $\delta_{\text{susy}} S$. Both these should vanish by the same set of boundary conditions. The bulk term in $\delta S$ yields the equations of motion, hence we retain only the boundary terms,

$$\delta S = \frac{1}{2} \int_{\partial \Sigma} d\tau \left[ \delta X^L \left( E_{LM} \partial_- X^M - \partial_+ X^M E_{ML} + i\psi_-^M G_{MK} \Omega^K_{LN}(E) \psi_+^N - i\psi_+^M G_{MK} \Omega^K_{LN}(E^T) \psi_-^N \right) + i\psi_-^M G_{ML} \delta \psi_-^L - i\psi_+^M G_{ML} \delta \psi_+^L \right].$$

(3)

Naively one may regard $\delta X^L$ and $\delta \psi_\pm^L$ as independent variations and set the associated boundary terms to zero separately. For constant backgrounds this will lead to the correct boundary conditions. For $X$-dependent backgrounds, this turns out to be inconsistent with both worldsheet supersymmetry and with T-duality. To get the correct boundary conditions, we reorganize the terms in $\delta S$ writing them as a sum of bosonic and fermionic boundary terms, $(BBT)$ and $(FBT)$,

$$\delta S = \frac{1}{2} \int_{\partial \Sigma} d\tau \left[ (BBT) + i(FBT) \right],$$

(4)

where,

$$(BBT) = \delta X^L \left( E_{LM} \partial_- X^M - E^T_{LM} \partial_+ X^M - i\psi_-^M \partial_M E_{LN} \psi_+^N + i\psi_+^M \partial_M E^T_{LN} \psi_-^N - i\eta \psi_-^M \partial_L E_{MN} \psi_+^N \right) \bigg|_{\partial \Sigma},$$

(5)

$$(FBT) = \psi_-^M G_{MN} \delta \psi_+^N - \psi_+^M G_{MN} \delta \psi_-^N + \delta X^L \left( \frac{1}{2} \left[ \psi_-^M \partial_L B_{MN} \psi_+^N + \psi_+^M \partial_L B_{MN} \psi_-^N \right] + \eta \psi_-^M \partial_L E_{MN} \psi_+^N \right) \bigg|_{\partial \Sigma}.$$}

(6)

This decomposition may seem arbitrary but in fact it is fixed by worldsheet supersymmetry as well as by consistency with the constant background case. The parameter $\eta$, which is so far arbitrary (as it cancels between (5) and (4)), will be required to take values $\pm 1$. These can be assigned independently at both ends of the open string, $\sigma = 0, \pi$. From the example of constant backgrounds we know that if $\eta$ is assigned the same value at both ends, we are in the Ramond (R) sector while opposite $\eta$ assignments at $\sigma = 0$ and $\sigma = \pi$ lead to the Neveu-Schwarz (NS) sector.

In general under a supersymmetry transformation the action changes by a boundary term $\delta_{\text{susy}} S$. For boundary conditions to be consistent with $N=1$ worldsheet supersymmetry, they should also imply the vanishing of $\delta_{\text{susy}} S$. After some manipulations, one finds that the variation of (1) under a supersymmetry transformation is

$$\delta_{\text{susy}} S = \frac{1}{2} \int_{\partial \Sigma} d\tau \left[ (BBT) + (i + 2)(FBT) \right].$$

(7)

Here $(BBT)$ and $(FBT)$ are still given by (5) and (6) with $\delta X$ and $\delta \psi_\pm$ replaced by the corresponding supersymmetry variations $\delta_{\text{susy}} X$ and $\delta_{\text{susy}} \psi_\pm$ given in the appendix. We have also assumed that the left and the right supersymmetry transformation parameters are related by $\epsilon^- = \eta \epsilon^+$ as follows from restricting to constant backgrounds.
Comparing this with $\delta S$ in (11) it is now clear that boundary conditions should set $(BBT)$ and $(FBT)$ to zero independently. This is how worldsheet supersymmetry justifies the split in (12).

2.2 Boundary “vectors”

To develop a systematic description of the boundary conditions and boundary couplings, we now introduce two bosonic quantities $\mathcal{N}^{(X)}$, $\mathcal{D}_{(X)}$ and two fermionic ones $\mathcal{N}^{(\psi)}$, $\mathcal{D}_{(\psi)}$ that live on the worldsheet boundary,

$$\mathcal{N}^{(X)}_L = E_{LM} \partial_- X^M - E^T_{LM} \partial_+ X^M - i\psi^M_- \partial_M E_{LN} \psi^N_+ + i\psi^M_+ \partial_M E^T_{LN} \psi^N_+ - i\eta \psi^M_+ \partial_L E_{MN} \psi^N_+ \bigg|_{\partial \Sigma},$$  

$$\mathcal{N}^{(\psi)}_L = E_{LN} \psi^N_+ - \eta E^T_{LN} \psi^N_+ \bigg|_{\partial \Sigma},$$  

$$\mathcal{D}^L_{(X)} = \partial_- X^L \bigg|_{\partial \Sigma},$$  

$$\mathcal{D}^L_{(\psi)} = \psi^L_+ + \eta \psi^L_+ \bigg|_{\partial \Sigma}.$$  

We will refer to these as “boundary vectors” (even though $\mathcal{N}^{(X)}$ is not really a vector under general coordinate transformations). As the nomenclature (and the form) suggests, they will be associated with Neumann and Dirichlet boundary conditions.

To convince ourselves that these are the natural objects to work with, we look at the behaviour of boundary vectors under supersymmetry. Using the supersymmetry transformations of $X^M$ and $\psi^M_\pm$ and taking the left and right supersymmetry transformation parameters to be related by $\epsilon^- = \eta \epsilon^+$, one can check that the boundary vectors have very simple transformation properties under the N=1 worldsheet supersymmetry,

$$\delta_{susy} \mathcal{N}^{(\psi)}_M = -i\epsilon^- \mathcal{N}^{(X)}_M, \quad \delta_{susy} \mathcal{N}^{(X)}_M = -2\epsilon^- \partial_- \mathcal{N}^{(\psi)}_M, \quad \delta_{susy} \mathcal{D}^M_{(\psi)} = -\epsilon^- \partial_- \mathcal{D}^M_{(X)}.$$  

We can now express the boundary term in the variation of the action in terms of the boundary vectors. Note that under variations $\delta X^L$ and $\delta \psi^L_\pm$,  

$$\delta \mathcal{N}^{(\psi)}_L = E_{LN} \delta \psi^N_+ - \eta E^T_{LN} \delta \psi^N_+ + X^K (\partial_K E_{LN} \psi^N_+ - \eta \partial_K E^T_{LN} \psi^N_+) \bigg|_{\partial \Sigma},$$  

$$\delta \mathcal{D}^L_{(\psi)} = \delta \psi^L_+ + \eta \delta \psi^L_+ \bigg|_{\partial \Sigma}.$$  

Then, in terms of the boundary vectors, the expressions $(BBT)$ (15) and $(FBT)$ (16) take the simple forms,

$$\delta S = \frac{i}{2} \int_{\partial \Sigma} d\tau \left[ \delta X^L \mathcal{N}^{(X)}_L + \frac{i}{2} \mathcal{D}^L_{(\psi)} \delta \mathcal{N}^{(\psi)}_L - \frac{i}{2} \delta \mathcal{D}^L_{(\psi)} \mathcal{N}^{(\psi)}_L \right].$$  

The boundary conditions now follow in a straightforward way from the requirement that $(BBT)$ and $(FBT)$ vanish independently.
As an example, let us consider the simplest case of D9-branes. For constant backgrounds, the boundary conditions have been known for a long time [12]. In space-time dependent backgrounds, the problem was considered in [5]. It was later revisited in [11] [13] where a general parameterization of the boundary conditions was investigated by studying the classical N=1 superconformal algebra. In our approach, for open strings on a D9-brane, the Neumann boundary conditions follow directly from the the vanishing of the boundary variation (17) as \( N_L^{(T)} = 0 \) and \( N_L^{(C)} = 0 \), in agreement with [11]. We write these explicitly for later reference,

\[
E_{LM} \partial_\pm X^M - E_{LM}^T \partial_+ X^M - i \psi_+ \partial_M E_{LN} \psi^N + i \psi_- \partial_M E_{LN}^T \psi^N - i \eta \psi_+ \partial_L E_{MN} \psi^N |_{\partial \Sigma} = 0, \tag{18}
\]

\[
E_{LN} \psi_+ - \eta E_{LN}^T \psi^N |_{\partial \Sigma} = 0. \tag{19}
\]

As usual, the variations \( \delta \psi_\pm, \delta X \) are restricted to the class of functions that satisfy the boundary conditions so that \( N_L^{(C)} = 0 \) also implies \( \delta N_L^{(C)} = 0 \), leading to \( \delta S = 0 \).

### 3 Dp-brane Boundary Conditions

We start this section with a brief review of the covariant description of Dp-branes as embedded submanifolds in space-time. These geometrical notions are used in the rest of the paper. We then obtain the Dp-brane boundary conditions which take a particularly simple form in terms of the boundary vectors. The boundary conditions are then shown to be consistent with T-duality.

#### 3.1 Covariant description of Dp-branes

Let coordinates \( \xi^\alpha (\alpha = 0, 1, \ldots, p) \) parameterize the Dp-brane worldvolume. The embedding of the worldvolume as a hypersurface in space-time is then described by the functions \( X^M(\xi) \). The metric induced on the worldvolume is \( g_{\alpha\beta} = G_{MN} \partial_\alpha X^M \partial_\beta X^N \). The tangent and normal bundles to the brane are spanned by basis vectors \( \partial_\alpha X^M \partial_M \) and \( a^M_\alpha \partial_M \), respectively. \( \hat{a} = p+1, \ldots, 9 \) is a flat normal bundle index raised and lowered by the flat metric \( \delta_{\hat{a}\hat{b}} \). Worldvolume and space-time indices are raised and lowered by the corresponding metrics \( g_{\alpha\beta} \) and \( G_{MN} \), respectively. The orthogonality of tangent and normal vectors implies

\[
a^M_\alpha G_{MN} \partial_\alpha X^N = a_{\hat{a}N} \partial_\alpha X^N = 0. \tag{20}
\]

The three metrics \( g_{\alpha\beta}, G_{MN} \) and \( \delta_{\hat{a}\hat{b}} \) are related via

\[
G^{MN} = a^M_\alpha \delta_{\hat{a}\hat{b}} a^N_\beta + \partial_\alpha X^M g^{\alpha\beta} \partial_\beta X^N. \tag{21}
\]

Space-time vectors \( V^M \) and \( V_M \) have projections along the tangent and normal directions given by,

\[
V_\alpha = V_M \partial_\alpha X^M, \quad V_\hat{a} = V_M a^M_\alpha; \quad V^\alpha = V^M G_{MN} \partial_\beta X^N g^{\beta\alpha}, \quad V^{\hat{a}} = V^M a^M_{\hat{a}}. \tag{22}
\]

Using these one can verify that all indices can be consistently raised or lowered by the corresponding metrics \( G_{MN}, g_{\alpha\beta} \) and \( \delta_{\hat{a}\hat{b}} \). Conversely, a vector \( V^M \) on the hypersurface can be reconstructed in terms of its tangential and normal components,

\[
V^M = a_\alpha^M V^{\alpha} + \partial_\alpha X^M V^{\alpha}, \tag{23}
\]
 Similarly, for \( V_M = G_{MN} V^N \).

D-branes contain gauge fields and transverse scalars fields living on their worldvolumes. Since these are vector fields intrinsically tangent (for gauge fields) and normal (for scalars fields) to the brane, they naturally have components \( A_\alpha \) and \( \Phi^a \), respectively. However, one can also work in terms of the corresponding space-time components,

\[
A_M = G_{MN} \partial_\alpha X^N g^{\alpha\beta} A_\beta, \quad \phi^M = a^M_a \Phi^a.
\]  

Equation (20) insures that \( A_M \) is tangential and \( \phi^M \) is normal to the D-brane.

When performing T-dualities or computing scattering amplitudes, the manipulations involve components of \( A \) and \( \phi \) and often one does not directly deal with \( A \) and \( \Phi \). For this reason we will use different notations for these intrinsically worldvolume objects and their space-time projections. Throughout the paper, the indices on these fields and other quantities are raised and lowered by the corresponding metrics, \( G_{MN} \), \( g_{\alpha\beta} \) and \( \delta^a_{\hat{a}} \).

D-brane worldvolume actions are often computed in the static gauge to which the above description can be specialized. In static gauge, the embedding functions \( X^\mu(\xi^\alpha) \) are given by,

\[
X^\mu = \xi^\mu (\mu = 0, 1, \cdots, p), \quad X^i = \text{const} (i = p + 1, \cdots, 9).
\]  

One can verify from (20) that in this gauge, \( a^\mu_a = 0 \) and the normal frame is fully spanned by \( a^i_{\hat{a}} \). However, for a general metric \( G_{MN} \), generically all components of \( a^M_a = G^{Mi} a^i_{\hat{a}} \) are non-zero. Clearly the choice of static gauge affects only \( \phi_M \) but not \( \Phi^a \) which contains the actual degrees of freedom. In particular (24) implies that in the static gauge the only non-zero components of \( \phi_M \) are the \( \phi_i \), while generically, all components of \( \phi^M \) are non-zero. This simple fact is important in understanding the covariance of scalar field couplings in D-brane worldvolume actions and hence is emphasized here. To summarize, in the static gauge,

\[
a^M_a = \{ a^\mu_a, a^i_{\hat{a}} \} \quad \Rightarrow \quad \phi^M = \{ \phi^\mu, \phi^i \} \]  

All this remains unchanged if (25) is slightly generalized to

\[
X^\mu = X^\mu(\xi^\alpha) (\mu, \alpha = 0, 1, \cdots, p), \quad X^i = \text{const} (i = p + 1, \cdots, 9).
\]  

For us the difference between the two will be immaterial and both will be called the static gauge.

### 3.2 Dp-brane boundary conditions

We now turn to Dp-brane boundary conditions implied by the vanishing of \( (BBT) \) and \( (FBT) \) in (15) and (16), first discussing the general covariant case and then going to the static gauge.

Consider the brane embedding \( X^L(\xi) \). By definition, the boundary of the worldsheet parameterized by \( \tau \) is confined to the brane, \( X^L|_{\partial \Sigma} = X^L(\xi(\tau)) \). As a result, the variations \( \delta X^L|_{\partial \Sigma} \) are entirely tangent to the brane and have no components in directions normal to it. This is the Dirichlet boundary condition on the bosonic field \( X^L \) that can be expressed in various equivalent ways,

\[
\text{Dirichlet:} \quad \delta X^L|_{\partial \Sigma} = \partial_\alpha X^L \delta \xi^\alpha, \quad \text{or} \quad a^i_{\hat{a}} \delta X^L|_{\partial \Sigma} = 0, \quad \text{or} \quad a^\mu_a \partial_\tau X^L|_{\partial \Sigma} = 0.
\]  

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Here, the first equation along with (20) implies the second equation and the third follows from the fact that the worldsheet boundary confined to the brane is parameterized by $\tau$, so that $\partial_\tau X^L|_{\partial \Sigma} = \partial_\alpha X^\tau \partial_\tau \xi^\alpha$. Since the $\delta \xi^\alpha$ are arbitrary, the vanishing of the bosonic boundary term ($BBT$) in (15) amounts to $\partial_\alpha X^L \mathcal{N}_L^{(X)}|_{\partial \Sigma} = 0$, which is the bosonic Neumann boundary conditions for Dp-branes,

Neumann : \[ \partial_\alpha X^L \left( E_{LM} \partial_\tau X^M - E_T^{LM} \partial_\tau X^M \right) - i \psi_-^M \partial_M E_{LN} \psi_-^N + i \psi_+^M \partial_M E_T^{LN} \psi_+^N - i \eta \psi_+^M \partial_L E_{MN} \psi_-^N \right|_{\partial \Sigma} = 0. \tag{29} \]

To obtain the corresponding fermionic boundary conditions note that the N=1 supersymmetry variation of $X^L$ is $\delta_{\text{susy}} X^L = -(\epsilon^- \psi_-^L + \epsilon^+ \psi_+^L)$. On the boundary, $a^L_\alpha \delta_{\text{susy}} X^L|_{\partial \Sigma} = 0$ by (28) and $\epsilon^+ = \eta \epsilon^-$, leading to,

Dirichlet : \[ a^L_\alpha (\psi_-^L + \eta \psi_+^L)|_{\partial \Sigma} = a^L_\alpha D^{(\psi)} = 0. \tag{30} \]

Since, by (23), one can always write $\mathcal{D}^{(\psi)} = \mathcal{D}^{(\psi)}_a a^a + \partial_\alpha X^L$, then the vanishing of the fermionic boundary term ($FBT$) in (16) only requires the tangential component of $\mathcal{N}_L^{(\psi)}$ to vanish, $\partial_\alpha X^L \mathcal{N}_L^{(\psi)}|_{\partial \Sigma} = 0$. This is the fermionic Neumann boundary condition for Dp-branes,

Neumann : \[ \partial_\alpha X^L \left( E_{LM} \psi_-^N - \eta E_T^{LM} \psi_+^N \right)|_{\partial \Sigma} = 0. \tag{31} \]

Equations (28)-(31) complete the Dp-brane boundary conditions with N=1 worldsheet supersymmetry in general backgrounds and for general embedding.

We summarize the Dp-brane Neumann and Dirichlet boundary conditions in terms of the boundary vectors (8)-(11),

\[ \mathcal{N}_L^{(X)} \equiv \partial_\alpha X^M \mathcal{N}_M^{(X)} = 0, \quad \mathcal{N}_L^{(\psi)} \equiv \partial_\alpha X^M \mathcal{N}_M^{(\psi)} = 0, \]

\[ \mathcal{D}^{(X)}_a \equiv a^a_\alpha \mathcal{D}^{(X)}_a = 0, \quad \mathcal{D}^{(\psi)}_a \equiv a^a_\alpha \mathcal{D}^{(\psi)}_a = 0. \tag{32} \]

While the boundary vectors have enabled us to write elegant expressions for the boundary conditions, their usefulness extends beyond this. We will see in the next sections that the couplings of open strings to the worldvolume fields are naturally given in terms of the boundary vector components that are not set to zero by the boundary conditions.

Before proceeding further, we make some observations that contrast and compare boundary conditions in the N=1 theory to the bosonic theory, or the case with constant backgrounds:

1. Under open-closed string duality ($\sigma \leftrightarrow \tau$ accompanied by $E_{MN} \rightarrow -E_{MN}$ and $\psi_-^M \rightarrow i \psi_-^M$, to keep the action invariant), $\mathcal{N}_L^{(X)}$ does not go over to the closed string canonical momentum $P_\alpha$. The two differ by terms that are bilinear in worldsheet fermions and are proportional to $\partial_\alpha E_{MN}$.

2. The “boundary vector” $\mathcal{N}_L^{(X)}$ does not transform covariantly under space-time general coordinate transformations due to the non-covariance of its fermionic content (so it is not really a vector). This is because the structure of $\mathcal{N}_L^{(X)}$ is constrained by worldsheet supersymmetry which combines vectors $\psi_\pm^M$ with the coordinate $X^M$ in the same superfield. But the final expressions are always covariant.
3. Contrary to some statements in the literature, mere consistency of Dp-brane boundary conditions with N=1 worldsheet supersymmetry does not constrain the background fields in any way. Such constraints have to come from separate stability considerations and space-time supersymmetry.

The boundary conditions can now be specialized to the often used case of static gauge. The resulting expressions will be needed in the remaining sections to understand the relation between T-duality and covariance of the worldvolume actions. Break up the space-time coordinates resulting expressions will be needed in the remaining sections to understand the relation between

\[ (28)-(31) \] reduce to,

\[ \text{string boundary couplings to worldvolume fields.} \]

indeed consistent with T-duality in a transparent way all this, we now show that the correct boundary conditions obtained in the previous section are boundary couplings under T-duality leads to an understanding of these couplings. Motivated by couplings in the worldvolume theory \[ [2, 3] \]. As we will see later, the behaviour of worldsheet unsatisfactory situation. Besides, T-duality has been used to determine non-Abelian scalar this may be regarded as an has remained unresolved and subsequent work has not shed more light on it. Considering the otherwise close connection between D-branes and T-duality, this may be regarded as an unsatisfactory situation. Besides, T-duality has been used to determine non-Abelian scalar couplings in the worldvolume theory \[ [2, 3] \]. As we will see later, the behaviour of worldsheet boundary couplings under T-duality leads to an understanding of these couplings. Motivated by all this, we now show that the correct boundary conditions obtained in the previous section are indeed consistent with T-duality in a transparent way\(^1\) and also write down the transformation of the boundary vectors. The results will be used in the following section to construct the open string boundary couplings to worldvolume fields.

Let us start with a D9-brane in background fields \( \tilde{G}_{MN} \) and \( \tilde{B}_{MN} \) which do not depend on the \( d = 9 - p \) coordinates \( \tilde{X}^i \), but may depend on the remaining \( p + 1 \) coordinates \( \tilde{X}^\mu \). The Neumann boundary conditions are \[ [18, 19] \],

\[ \tilde{E}_{LM} \partial_- \tilde{X}^M \tilde{E}_{LM}^T \partial_+ \tilde{X}^M - i \tilde{\psi}_\mu^T \partial_\mu \tilde{E}_{LN} \tilde{\psi}_N^\Sigma + i \tilde{\psi}_+ \partial_\mu \tilde{E}_{LM}^T \tilde{\psi}_+^N - i \eta \tilde{\psi}_+^M \delta_\mu \partial_\lambda \tilde{E}_{MN} \tilde{\psi}_+^N |_{\partial \Sigma} = 0, \]

\[ \tilde{E}_{LN} \tilde{\psi}_N^\Sigma - \eta \tilde{E}_{LN}^T \tilde{\psi}_+^N |_{\partial \Sigma} = 0. \] (34)

\(^1\)The consistency of T-duality with the boundary conditions has also been shown by the authors in \[ [19] \].

3.3 Consistency with T-duality

T-duality has played an important role in the discovery and understanding of D-branes \[ [14, 15, 16, 17, 18, 3] \]. This is because, in constant backgrounds, T-duality transformations are known to interchange D and N boundary conditions thus naturally giving rise to Dp-branes in open string theory. This is also the case for non-constant backgrounds in the absence of fermions. Naturally, one expects the same to hold in the more general case of N=1 supersymmetric worldsheet theory in X-dependent backgrounds. This issue was first addressed in \[ [17, 8] \], however the problem has remained unresolved and subsequent work has not shed more light on it. Considering the otherwise close connection between D-branes and T-duality, this may be regarded as an unsatisfactory situation. Besides, T-duality has been used to determine non-Abelian scalar couplings in the worldvolume theory \[ [2, 3] \]. As we will see later, the behaviour of worldsheet boundary couplings under T-duality leads to an understanding of these couplings. Motivated by all this, we now show that the correct boundary conditions obtained in the previous section are indeed consistent with T-duality in a transparent way\(^1\) and also write down the transformation of the boundary vectors. The results will be used in the following section to construct the open string boundary couplings to worldvolume fields.
We now perform T-duality transformations along the $d = 9 - p$ directions $X^i$. This should lead to the correct static gauge boundary conditions for Dp-branes in the T-dual background $G_{MN}, B_{MN}$ with $X^i$ as the transverse directions.

The effect of the dualities on the boundary vectors can be studied in a systematic way in terms of the matrices $Q^{M}_{-N}$ and $P^{M}_{MN}$ defined as,

$$Q^{M}_{-N} = \begin{pmatrix} \delta^{ik} E_{kj} & \delta^{ik} E_{k\nu} \\ 0_{j}^{\mu} & \delta^{\mu}_{\nu} \end{pmatrix}, \quad Q^{M}_{+N} = \begin{pmatrix} -\delta^{ik} E_{kj}^{T} & -\delta^{ik} E_{k\nu}^{T} \\ 0_{j}^{\mu} & \delta^{\mu}_{\nu} \end{pmatrix},$$

$$P^{M}_{-MN} = \begin{pmatrix} \delta_{ij} & 0_{\mu} \\ E_{\mu ij} & E_{\mu\nu} \end{pmatrix}, \quad P^{M}_{+MN} = \begin{pmatrix} -\delta_{ij} & 0_{\mu} \\ E_{\mu ij}^{T} & E_{\mu\nu}^{T} \end{pmatrix},$$

where, $E = G + B$ and $E^{T} = G - B$. It can be shown that $Q^{M}_{\pm}^{-1}$ are given by the same expressions as for $Q_{\pm}$ but with $E$ replaced by $\tilde{E}$. The T-dual quantities are then related by $Q^{M}_{\pm} = Q_{\mp}^{N} Q_{\pm}, P^{M}_{MN} = P_{-MN}$, respectively, and using the form of the matrices $Q^{M}_{\pm}$. The T-dual quantities are then related by $Q^{M}_{\pm} = Q_{\mp}^{N} Q_{\pm}^{L}, P^{M}_{MN} = P_{-MN}$.

It is easy to check that the transformation of $\partial_{\pm}X$ follows from that of $\psi_{\pm}$ under worldsheet supersymmetry transformations.

Note the presence of the flat metric $\delta^{ik}$ in $Q_{\pm}$ (see for example, [22, 23]). Often this is not explicitly shown when writing the T-duality transformations. But since it will play a role later, we will briefly describe its origin: T-duality transformations along coordinates $X^{i}$ commute only with an $O(d)$ subgroup of the general coordinate transformations involving $X^{i}$. Hence these $O(d)$ transformations of the original background are identified with those of its dual and T-duality should explicitly preserve this identification at all stages of the manipulation. To make this manifest, the T-duality transformation formulae contain the $O(d)$ invariant metric to raise or lower indices.

Equipped with the above transformation rules, it is easy to show that the D9-brane boundary conditions lead to the correct Dp-brane boundary conditions under T-duality. For this, one can first verify that

$$\tilde{\psi}_{\pm}^{M} \partial_{\lambda} E_{MN} \tilde{\psi}_{\mp}^{N} = \psi_{\pm}^{M} \partial_{\lambda} E_{MN} \psi_{\mp}^{N}.$$  

Then, using (37) and (38) in (32) one obtains,

$$P^{LM}_{-M} \partial_{\pm} X^{M} - P^{LM}_{+M} \partial_{+} X^{M} = 0,$$

$$P^{LM}_{-M} \psi^{N}_{\pm} - \eta P^{LM}_{+M} \psi^{N}_{\mp} = 0.$$

Now restricting to $L = \mu$ and $L = \nu$, respectively, and using the form of the matrices $P^{M}_{\pm}$ in (36), one recovers the correct static gauge Dp-brane boundary conditions (33). This shows the consistency of boundary conditions with T-duality.

In fact, one can go beyond boundary conditions and with equal ease obtain the T-duality action on the boundary vectors (33)-(34) in static gauge. For backgrounds independent of $X^{i}$ and on using (36), it is straightforward to show that the boundary vectors transform as,

$$\tilde{\mathcal{N}}^{i}_{\mu} = \mathcal{N}^{i}_{\mu}, \quad \tilde{\mathcal{N}}^{i}_{\psi} = 2\delta_{ij} \mathcal{D}^{j}_{(X)}, \quad \tilde{\mathcal{N}}^{(\psi)}_{\mu} = \mathcal{N}^{(\psi)}_{\mu}, \quad \tilde{\mathcal{N}}^{(\psi)}_{i} = \delta_{ij} \mathcal{D}^{j}_{(\psi)},$$

$$\tilde{\mathcal{D}}^{\mu}_{(X)} = \mathcal{D}^{\mu}_{(X)}, \quad \tilde{\mathcal{D}}^{i}_{(X)} = \frac{1}{2} \delta^{ij} \mathcal{N}^{j}_{(X)}, \quad \tilde{\mathcal{D}}^{i}_{(\psi)} = \mathcal{D}^{i}_{(\psi)}, \quad \tilde{\mathcal{D}}^{i}_{(\psi)} = \delta^{ij} \mathcal{N}^{j}_{(\psi)}.$$  

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4 Worldsheet Boundary Couplings

So far we have only considered open strings in metric and antisymmetric tensor field backgrounds. In this section we will describe a formalism for coupling the string worldsheet boundary to D-brane worldvolume scalars and gauge fields. To elucidate the idea, we start with the bosonic case and then move on to the N=1 supersymmetric theory with Abelian and non-Abelian worldvolume fields. The structure of the boundary couplings and their behaviour under T-duality points to an enlarged symmetry of the non-Abelian D-brane worldvolume theories allowing us to obtain a covariant description of the scalar field couplings. In this section, the worldvolume fields are regarded as perturbations so the boundary conditions obtained earlier remain unchanged. This allows us to deal with non-Abelian worldvolume fields and obtain the associated open string vertex operators.

4.1 Worldsheet boundary couplings in bosonic theory

Let us consider the coupling of the bosonic open string to the worldvolume gauge fields $A_M$ and transverse scalars $\phi^M$. In flat space, $G_{MN} = \eta_{MN}$, $B_{MN} = 0$, and in the static gauge, worldvolume fields couple to the worldsheet boundary through,

$$\int_{\partial \Sigma} d\tau A_\mu \partial_\tau X^\mu - \int_{\partial \Sigma} d\tau \phi_i \partial_\sigma X^i.$$

The scalar coupling follows from a vertex operator consideration \cite{14, 15} as well as from T-duality \cite{16}. This expression can be easily generalized to a covariant one valid for curved D-branes and, for many purposes, is adequate even in the presence of non-trivial backgrounds. However, generalizations are needed when $B_{\mu i} \neq 0$. To discover the general form of the couplings, let us first understand the flat space case from a different point of view: $\partial_\tau X^M|_{\partial \Sigma}$ and $\partial_\sigma X^M|_{\partial \Sigma}$ are two space-time vectors on the boundary of the worldsheet. For a Dp-brane in flat background and in static gauge, the Neumann and Dirichlet boundary conditions, $\partial_\sigma X^\mu |_{\partial \Sigma} = 0$ and $\partial_\tau X^i |_{\partial \Sigma} = 0$, project out some components of these vectors. The surviving components $\partial_\tau X^i |_{\partial \Sigma}$ and $\partial_\sigma X^\mu |_{\partial \Sigma}$ are precisely the operators on the worldsheet boundary to which the D-brane scalars and vectors couple.

Having understood the flat space boundary couplings in this way, it is straightforward to write the couplings in any general background. In the bosonic case, the open string boundary conditions involve the following two vectors on the worldsheet boundary:

$$N_L = E_{LN} \partial_+ X^N - E^T_{LN} \partial_+ X^N|_{\partial \Sigma}, \quad D^L = \partial_+ X^L|_{\partial \Sigma}. \quad (42)$$

The scalar coupling is then

$$S_{Dp} = \int d\tau \left[ A_M D^M + \frac{1}{2} \phi^M N_M \right]. \quad (43)$$
The scalar field vertex operator now also has a $\partial_r X^M$ contribution when the B-field has indices both along and transverse to the brane, $a^M a_{\bar M} \partial_\phi X^N \neq 0$, which can be combined with the gauge field part,

$$S_{0\Sigma}^{DP} = \int d\tau \left[ (A_M + \phi L B_{LM}) \partial_r X^M - \phi N \partial_\phi X^N \right].$$  \hspace{1cm} (44)

Later we will see that $A_M$ always appears in this combination, also in the presence of supersymmetry and non-Abelian interactions. It may seem appealing to get rid of the extra term by absorbing it in a redefinition of $A_M$. But that would mean that non-trivial gauge fields could be created simply by switching on transverse scalar fields which should not be the case.

To insure invariance of the action under $B_{MN}$ 2-form gauge transformations, the transformation of $A_M$ should be modified such that,

$$\delta B_{MN} = \partial_M \Lambda_N - \partial_N \Lambda_M, \quad \delta A_M = -\Lambda_M - \phi L (\partial_\Lambda \Lambda_M - \partial_M \Lambda_L).$$  \hspace{1cm} (45)

The extra term in $\delta A_M$ becomes relevant only when the 2-form gauge transformation is not entirely restricted to the worldvolume. The origin of the modification will be explained below.

The boundary interaction \textsuperscript{14} is consistent with (and in fact required by) the interpretation of $\phi^M$ as an infinitesimal shift $X^M \rightarrow X^M + \phi^M$ in the position of the brane. Indeed, the $\phi$-dependent part $S_{0\Sigma}^\phi$ of the action can be generated from the background part by shifting the coordinates to $X^M + \phi^M$ and retaining terms linear in the shift,

$$S_{\Sigma}[X + \phi] + S_{0\Sigma}^A[X + \phi] \sim S_{\Sigma}[X] + S_{0\Sigma}[X] + S_{0\Sigma}^A[X].$$

Here, $S_{\Sigma}$ is the bosonic part of the worldsheet bulk action \textsuperscript{2}, and its variation under the shift gives rise to $S_{0\Sigma}^\phi$ along with a bulk term that vanishes by virtue of the equation of motion. Hence adding $\frac{1}{2} \phi^M N_M^T$ to the boundary action is equivalent to the infinitesimal coordinate shift $X^M \rightarrow X^M + \phi^M$. As for the gauge field part $S_{0\Sigma}^A$, to linear order in the worldvolume fields $S_{0\Sigma}^A[X + \phi] \sim S_{0\Sigma}^A[X]$ and it could have been dropped. However, its inclusion above clarifies the origin of the modification in the transformation of $A_M$ in \textsuperscript{15}: The left hand side of the above equation is invariant under the usual NS-NS 2-form gauge transformation with parameter $\Lambda(X + \phi)$. For the right hand side, this implies the modified transformation \textsuperscript{15}.

So far we have implicitly assumed that the world volume fields are Abelian but the considerations can be generalized to the non-Abelian case. When $A_M$ and $\phi^M$ are non-Abelian, then the boundary action $S_{0\Sigma}^{DP}$ cannot be simply added to the bulk worldsheet action, but should be inserted in the path integral through a path-ordered Wilson line,

$$\text{tr} \mathcal{P} \exp(i S_{0\Sigma}^{DP})$$  \hspace{1cm} (46)

The boundary action is still given by \textsuperscript{13} and the discussion above, as well as the T-duality derivation to be described in the next subsection continue to hold. If we still want to interpret this as arising from the infinitesimal shift $X^M \rightarrow X^M + \phi^M$ (with a non-Abelian $\phi^M$) in the bulk worldsheet action, then the above path ordering along $\tau$ should be applied to the full action when regarded as a function of the shifted coordinate. The non-Abelian case will be discussed in more detail in the supersymmetric theory.

For later reference, we express the above boundary couplings in the static gauge $X^\mu = \xi^\mu$, $X^i = \text{const}$. Then, as we have seen, $\phi_i = \Phi^i a_{\bar i}$ and $\phi_\mu = \Phi^i a_{\bar \mu} = 0$, and the boundary action \textsuperscript{14} becomes,

$$S_{0\Sigma}^{DP} = \int d\tau \left[ (A_M + \phi_i G^{iL} B_{LM}) \partial_\tau X^\mu - \phi_i \partial_\phi X^i \right]$$  \hspace{1cm} (47)

In this gauge the boundary conditions are $N_\mu = 0$, $D^i = 0$. 

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4.2 T-duality and restoration of covariance

We can go a step further and derive the scalar couplings of the last section using T-duality. An offshoot of this is to clarify the relation between the expression that results from T-duality and the corresponding covariant expression (or its static gauge form). While this relation is derived on the worldsheet boundary, it also holds in the D-brane worldvolume theory where it enables us to promote expressions obtained by T-duality to covariant ones. These considerations are superfluous in the Abelian theory where they simply follow from general coordinate transformation (GCT) invariance. However, they have non-trivial consequences in the non-Abelian case where the scalar couplings obtained by T-duality are seemingly inconsistent with general covariance. For simplicity, we start with the Abelian theory to explain the ideas and gradually generalize to the N=1 non-Abelian case in the following subsections.

Let us start with a D9-brane with the boundary coupling,

\[ S_{D9} = \int d\tau \tilde{A}_M \partial_\tau \tilde{X}^M = \frac{1}{2} \int d\tau \tilde{A}_M \left( \partial_+ \tilde{X}^M + \partial_- \tilde{X}^M \right). \]

On general grounds, this is T-dual to the boundary coupling on a Dp-brane with \( p < 9 \). To obtain the scalar field vertex operator on the lower dimensional brane, we regard the gauge field as a perturbation\(^2\) so that T-duality is given purely in terms of the closed string backgrounds \( G \) and \( B \). Then, using the T-duality relation \( \partial_\pm \tilde{X}^M = Q^{\pm}_{\pm N} \partial_\pm X^N \) of the bosonic theory with \( Q_{\pm} \) given in (35) and making the usual identifications, \( A_\mu = \tilde{A}_\mu \), \( \phi^i = \tilde{A}_j \delta^{ji} \), one obtains,

\[ S_{Dp} = \int d\tau \left[ A_\mu \partial_\tau X^\mu - \phi^i \left( G_{iM} \partial_\sigma X^M - B_{iM} \partial_\tau X^M \right) \right] = \int d\tau \left[ A_\mu \mathcal{D}^\mu + \frac{1}{2} \phi^i \mathcal{N}_i \right]. \]

On the one hand, from subsection 3.3 we know that the above T-duality results in a Dp-brane in the static gauge in which both \( \phi^i \) and \( \phi^\mu \) are non-zero (26). On the other hand, the expression in (48), which is a direct outcome of T-duality, does not contain the components \( \phi^\mu \) of the scalar field. Indeed, although very similar to the static gauge expression (17) (which is obtained from the covariant action (44) on gauge fixing the GCT), they are not yet exactly the same due to these missing components of \( \phi \). The relation between the two is relevant to understanding the covariance of non-Abelian D-brane worldvolume actions and will be spelt out below:

To recover the static gauge action (17) from the outcome of T-duality, we have to lower the index on \( \phi \) in (48) to \( \phi_i = G_{iM} \phi^M \). For this one needs the missing scalar field components \( \phi^\mu \). Now, under T-duality the D9-brane boundary condition \( \tilde{N}_M = 0 \) goes over to static gauge Dp-brane boundary conditions \( \mathcal{D}^i = 0, \mathcal{N}_\mu = 0 \). So we can add \( \frac{1}{2} \phi^\mu \mathcal{N}_\mu \) to the action without affecting its content. This supplies the missing terms and insures that the outcome of T-duality is consistent with the covariant action (17) or its gauge fixed version (17). We have also seen that, for small \( \phi^\mu \), the addition of \( \frac{1}{2} \phi^\mu \mathcal{N}_\mu \) to the boundary action is equivalent to shifting \( X^\mu \) to \( X^\mu + \phi^\mu \). Hence the outcome of T-duality is related to the covariant expression (or its static gauge version) by a coordinate shift. Conversely, consider a Dp-brane described by the embedding \( X^M(\xi) \) with Abelian scalar fields \( \phi^M(\xi) \). This description is covariant under GCT.

\(^2\) If \( \tilde{A}_M \) is regarded as a large background field, it modifies the boundary conditions. Then following [17], T-duality results in a non-flat brane \( X^\mu = \xi^\mu, X^i = \tilde{\phi}^i(\xi) \) but without the boundary scalar couplings. For small \( \tilde{\phi}^i \), expanding around \( X^i \) generates these coupling.
If we fix the static gauge right away, we end up with (47). However, we can also use the GCT invariance to first make a transformation $X'\mu = X\mu + \phi\mu$ to eliminate $\phi\mu$ and then go to the static gauge. Now, the resulting expression coincides with the outcome of T-duality (48).

To summarize, we saw that the outcome of T-duality can be written in a GCT covariant form by adding to the boundary action terms that vanish by the boundary conditions. For Abelian $\phi\mu$ this is equivalent to an ordinary coordinate transformation $X\mu \rightarrow X\mu + \phi\mu$ which can also be carried out in the corresponding D-brane worldvolume theory to restore its manifest covariance. Since these are known symmetries, the identification of their worldsheet origin is redundant. However, when $\phi$ becomes a matrix in the non-Abelian case, the shift is no longer an ordinary coordinate transformation and its admissibility is not a priori evident in the worldvolume theory. In this case, the worldsheet origin of the transformation becomes crucial to insure its existence on the worldvolume. This will be made more precise in the following subsections.

4.3 N=1 Supersymmetric Abelian worldsheet boundary couplings

In the supersymmetric case, we start from the D9-brane boundary couplings and obtain the rest by T-duality. From this a covariant expression can be guessed leading to the N=1 boundary couplings in general backgrounds and for any embedding. Alternatively these can be obtained by supersymmetrizing the bosonic result. We will then discuss the interpretation of scalars as coordinate shifts which is now more subtle.

The N=1 supersymmetric D9-brane boundary coupling is given by (12)

$$S_{\partial\Sigma}^{D9} = \int d\tau \left\{ A_M \partial_\tau X^M - \frac{i}{4} \left( \psi_-^M + \eta \psi_+^M \right) F_{MN} \left( \psi_-^N + \eta \psi_+^N \right) \right\}$$

Here we consider Abelian gauge fields postponing the non-Abelian case to the next subsection. Under the NS-NS 2-form gauge transformation, $\delta B = d\Lambda$, $\delta A = -\Lambda$, the variation of the first term above is canceled by a boundary term that arises from the variation of the bulk action (2). Hence the term containing $F_{MN}$ should be invariant by itself. Although this may not seem to be the case at first sight, a closer examination shows that its invariance can be made manifest. To see this, note that the D9-brane Neumann boundary condition (19) implies

$$D_M^{\psi}(\psi) N_M^{\psi}(\psi) = D_M^{\psi}(\psi) B_{MN} D_N^{\psi}(\psi) - 2\eta G_{MN} \psi_-^M \psi_+^N = 0.$$  

Adding zero in this form to the boundary action one can write,

$$S_{\partial\Sigma}^{D9} = \int d\tau \left\{ A_M \partial_\tau X^M - \frac{i}{4} D_M^{\psi}(\psi) F_{MN} D_N^{\psi}(\psi) - \frac{i}{4} D_M^{\psi}(\psi) N_M^{\psi}(\psi) \right\},$$

without changing the dynamics. This contains the combination $F_{MN} + B_{MN}$ which is manifestly invariant under 2-form gauge transformations. In other words, the non-invariance of $F_{MN}$ conspires with that of the Neumann boundary condition (19) to produce a gauge invariant result. We will see below that, besides making the 2-form gauge invariance manifest, the

\[3\] If the worldvolume fields are promoted to background fields, then they will enter the boundary conditions rendering them invariant under 2-form gauge transformations. However, as stated before, we regard $A$ and $\phi$ as perturbations to obtain their vertex operators in closed string backgrounds and to avoid problems in the non-Abelian case.

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added term is also needed to understand the nature of N=1 worldsheet couplings to D-brane scalars.

To obtain the N=1 supersymmetric boundary couplings on Dp-branes, we apply T-duality to \( (49) \) along the \( d \) directions \( X^i \). Then, restricting to \( \partial_i A_M = 0 \), and using \( (11) \) with \( A_\mu = A_\mu^\prime \), \( \phi^i = \tilde{A}_j \delta^{ij} \), one gets,

\[
\int_{\partial\Sigma} d\tau \left\{ A_\mu D^\mu (X) - \frac{i}{4} D_\omega^\mu F_{\mu\nu} D^\nu (\omega) + \frac{1}{2} \phi^i \mathcal{N}^i (X) - \frac{i}{2} D_\omega (\omega) \partial_\mu \phi^i \mathcal{N}^i (\omega) \right\} .
\]

It is now easy to write the general covariant form of the couplings. This is achieved, as in the bosonic case described in the last subsection, by using the boundary conditions \( \mathcal{D}^i (X) = \mathcal{D} (\omega) = 0 \) and \( \mathcal{N}^i (X) = \mathcal{N}^i (\omega) = 0 \) to complete the above action to,

\[
S_{\partial\Sigma}^{Dp} = \int_{\partial\Sigma} d\tau \left\{ A_M D^M (X) - \frac{i}{4} D_\omega^M F_{MN} D^N (\omega) + \frac{1}{2} \phi^M \mathcal{N}^M (X) - \frac{i}{2} D_\omega (\omega) \partial_\mu \phi^M \mathcal{N}^M (\omega) \right\} .
\]

This expression is valid beyond T-duality and one can verify, using \( (12) \), that it is invariant under N=1 worldsheet supersymmetry. Therefore it is the correct supersymmetric completion of \( (13) \) and applies to any D-brane embedding provided the boundary vectors satisfy the appropriate Dp-brane boundary conditions \( (32) \). It is understood that \( \phi^M \) and \( A_M \) are respectively normal and tangent to the D-brane \( (24) \), so that \( \phi_M \partial_\omega X^M = 0 \) and \( A_M a_a^M = 0 \). Boundary conditions imply that all derivatives acting on \( A_M \) and \( \phi^M \) are of the form \( \mathcal{D}_L (\omega) \partial_L = \mathcal{D}_L (\omega) \partial_L \) consistent with the fact that they live on the worldvolume and are functions of \( \xi^\alpha \). The covariance of the couplings will be discussed later.

As in the case of D9-brane, the term containing \( F_{MN} \) is not manifestly invariant under NS 2-form gauge transformations. To make the invariance manifest, to this action we add

\[
S_{\partial\Sigma}^{D_0,0} = \int_{\partial\Sigma} d\tau \left\{ - \frac{i}{4} D_\omega^\alpha \mathcal{N}^\alpha (\omega) \right\} ,
\]

which is zero by virtue of the boundary conditions and does not modify the theory. The fact that this is the bare minimum required by 2-form gauge invariance is important to get the correct scalar couplings and hence is emphasized here. For example, \( \mathcal{D}_\omega^\alpha \mathcal{N}^\alpha (\omega) \) also vanishes but is not needed and so should not be included \(^4\).

The boundary action \( (32) \) is manifestly supersymmetric although in this form its covariance is not manifest. This is evident from the presence of \( \partial_M \phi^N \) (instead of \( \nabla_M \phi^N \)) as well as \( \mathcal{N}^N (X) \) which, as noted earlier, does not transform as a vector. Also the invariance of the \( \phi \)-dependent terms under the 2-form gauge transformation is not manifest. To make both these symmetries manifest, we note that after some manipulations equations \( (8) \) and \( (9) \) can be written as,

\[
\begin{align*}
\mathcal{N}^i (X) &= G_{MN} (\psi^N - \eta \psi^N_+) + B_{MN} D^N (\omega), \\
\mathcal{N}^i (\omega) &= 2 (B_{MN} \partial_\tau X^N - G_{MN} \partial_\sigma X^N) - i D^N (\omega) \Gamma^I_{NM} G_{LK} (\psi^K - \eta \psi^K_+) \\
&\quad - i D^N (\omega) \partial_N B_{ML} D^L (\omega) - i \eta \psi^N_+ \psi^K_+ H_{NLK} .
\end{align*}
\]

\(^4\)Note that T-duality gives \( \mathcal{D}_\omega^\mu \mathcal{N}^\mu (\omega) = \mathcal{D}_\omega^\mu \mathcal{N}^\mu (\omega) + \mathcal{N}^i (\omega) \mathcal{D}_\omega^i \) both of which vanish separately by Dp-brane boundary conditions. Of these only the first term is needed and is retained. Also, since the added term vanishes, so does its supersymmetry variation. Hence we do not need to add its supersymmetric completion \( X^M \mathcal{N}^N (X) \) (for D9-brane). Thus we retain the minimum required to make the 2-form gauge invariance manifest.
Substituting in (52), one recovers a manifestly covariant expression for the action as,
\[
S_{\partial \Sigma}^{Dp} = \int_{\partial \Sigma} d\tau \left\{ A^{(\phi)}_M \partial_\alpha X^M - i/4 D^M_\alpha F^{(\phi)}_{MN} D^N \right. \\
- \phi N \partial_\alpha X^N - i/2 D^{\alpha}_{(\psi)} \nabla_\alpha \phi_M \left( \psi^- M - \eta \psi^+ M \right) - i/2 \eta \phi^M \psi^N \psi^L \eta H_{MNL} \right\}. 
\] (54)

As in the bosonic case, the gauge field always appears through this shifted combination,
\[
A^{(\phi)}_M = A_M + \phi^L B_{LM},
\] (55)
and \( F^{(\phi)}_{MN} \) is the corresponding Abelian field strength. Another feature of this boundary action is that the covariant derivative \( \nabla_\alpha \phi_M = \partial_\alpha \phi_M - \partial_\alpha X^N \Gamma^L_{KM} \phi_L \) is in terms of the ordinary Christoffel connection, without a torsion contribution. In fact, the torsion term in (54) cannot be absorbed in the covariant derivative term in this form. Thus, the boundary action (52) is invariant under both general coordinate transformations and NS 2-form gauge transformations (after the addition of (53) and with the gauge field transformation given by (45)).

Let us now understand the scalar couplings as coordinate shifts. In the bosonic theory the boundary couplings of \( \phi^M \) arose from the shift \( X^M \rightarrow X^M + \phi^M(\xi) \) in the worldsheet bulk action, \( i.e. \), with only closed string backgrounds. In the N=1 case, in order to reproduce the \( \phi \) terms in (52), one has to include the new features of the theory. First, consistency with the supersymmetry transformation \( \delta_{\text{susy}} X^M = -\epsilon^{-2} D^{(\psi)} \) implies that the shift in \( X^M \) should be accompanied by a corresponding shift in \( D^{(\psi)} = \psi^- M + \eta \psi^+ M \),
\[
\Delta X^M = \phi^M(\xi), \quad \Delta D^{(\psi)} = D^{\alpha}_{(\psi)} \partial_\alpha \phi^M(\xi). 
\] (56)

To first order, the change \( \Delta S \) of the worldsheet bulk action (2) under these variations is given by (17). This is no longer zero since the variations \( \Delta X^M \) and \( \Delta D^{(\psi)} \) do not respect the boundary conditions. Besides, now the action contains an extra purely background dependent piece \( S^{Dp,0}_{\partial \Sigma} \) given by (53). Although boundary conditions set this term to zero, its variation \( \Delta S^{Dp,0}_{\partial \Sigma} \) under the shifts is not zero (since we do not modify the boundary conditions under the shift). The total variation of the purely background dependent part of the action is then given by (for more details see the discussion in the non-Abelian case below),
\[
\Delta S + \Delta S^{Dp,0}_{\partial \Sigma} = 1/2 \int d\tau \left\{ \Delta X^L \mathcal{N}^{(X)}_L - i \Delta D^L_{(\psi)} \mathcal{N}^{(\psi)}_L \right\}. 
\] (57)

Substituting for \( \Delta X^M \) and \( \Delta D^{M}_{(\psi)} \) from (56), we recover the \( \phi \)-dependent terms of the boundary action (52). This shows that in the N=1 theory too scalar boundary couplings emerge from infinitesimal coordinate shifts, provided supersymmetry and the presence of the extra term (53) are taken into account.

The discussion in subsection (4.2) of the relation between T-duality and covariance of the scalar couplings applies with appropriate generalizations: T-duality of the boundary conditions leads to a Dp-brane in the static gauge, \( X^\alpha = \xi^\alpha \) and \( X^i = \text{const.} \). In general both \( \phi^i \) and \( \phi^\mu \) are non-zero. However, the scalar couplings in (51) resulting from T-duality do not contain \( \phi^\mu \) and are related to the manifestly covariant action (52) (or its static gauge form) by the addition of \( i/2 N_{\mu}^{(X)} \phi^\mu \) and \( -i/2 D^{\alpha}_{(\psi)} \partial_\alpha \phi^\nu \mathcal{N}^{(\psi)}_{\nu} \), both of which vanish by virtue of the static gauge boundary conditions. But the addition of these terms correspond to shifting \( X^\mu \) to \( X^\mu + \phi^\mu \),
along with the appropriate shifts in $\psi^{\alpha}_{\pm}$. In the world volume theory, this coordinate shift is the transformation needed to convert the outcome of T-duality to the static gauge action and the above discussion clarifies its worldsheet origin. In the next subsection we will generalize this to the non-Abelian theory where is acquires non-trivial consequences.

Let us now rewrite the action (51) in a form that highlights the geometry of the brane. This is not relevant to the rest of the paper and can be skipped. Geometrically, $\phi^{M}$ is normal to the brane but the covariant derivative $\nabla_{\alpha}\phi^{M}$ has components both normal and tangent to the brane and decomposes as

$$\nabla_{\alpha}\phi^{M} = (\hat{\nabla}_{\alpha}\Phi^{\alpha})a^{M}_{\alpha} - (\Phi_{\alpha}\Omega^{\beta}_{\alpha\beta})g^{\beta\gamma}\partial_{\gamma}X^{M}. \tag{58}$$

In the first term, $\hat{\nabla}_{\alpha}\Phi^{\alpha} = \partial_{\alpha}X^{M}(\nabla_{M}\Phi^{N})a^{N}_{\alpha}$ is by definition the covariant derivative on the normal bundle. Its expression can be easily read off from this definition as,

$$\hat{\nabla}_{\alpha}\Phi^{\alpha} = \partial_{\alpha}\Phi^{\alpha} + \omega^{\alpha}_{\beta\beta}b^{\alpha}$$

where, $\omega^{\alpha}_{\beta\beta} = a^{\alpha}_{\beta}\partial_{\beta}X^{N} + \partial_{\alpha}X^{M}\Gamma^{N}_{MK}a^{N}_{\beta}a^{K}_{\beta}$.

The second term contains the second fundamental form of the embedding which can also be read off from (58) as,

$$\Omega^{M}_{\alpha\beta} = \partial_{\alpha}\partial_{\beta}X^{M} + \Gamma^{M}_{NK}\partial_{\alpha}X^{N}\partial_{\beta}X^{K}.$$

It gives the deviation of $\Phi^{M}$ from the normal under parallel transport by the Christoffel connection. In string theory one often encounters the torsionful connections $\Gamma^{\pm N}_{MK} = \Gamma^{N}_{MK} \pm \frac{i}{2}H^{N}_{MK}$. It can be easily checked that adding torsion to the left hand side of (58) leads to torsion contributions to the normal bundle connection and to the second fundamental form,

$$\hat{\nabla}^{\pm\alpha}_{\alpha}\Phi^{\alpha} = \hat{\nabla}_{\alpha}\Phi^{\alpha} \pm \frac{i}{2}\phi^{M}H_{MK}a^{M}_{\alpha}, \quad \Omega^{\pm K}_{\alpha\beta} = \Omega^{K}_{\alpha\beta} \pm \frac{i}{2}H^{K}_{\alpha\beta}, \tag{59}$$

where, $H_{MK\alpha} = H_{MK\alpha}a^{K}_{\alpha}$ and $H^{K}_{\alpha\beta} = H^{K}_{MN}\partial_{\alpha}X^{M}\partial_{\beta}X^{N}$. Substituting (59) in (61) and using (59) along with the Dirichlet boundary condition $D^{\alpha}_{(\psi)} = \psi^{\alpha} + \eta\psi^{\alpha} = 0$, one gets an alternative expression for the boundary action as

$$S^{DP}_{\partial\Sigma} = \int_{\partial\Sigma} d\tau \left\{ A^{(\psi)}_{M\alpha}\partial_{\tau}X^{M} - \frac{i}{4}D^{M}_{(\psi)}F^{N}_{MN}D^{N}_{(\psi)} - \phi_{\alpha}\partial_{\tau}X^{N} + i\psi^{\alpha}_{\pm}\hat{\nabla}^{\alpha}_{\alpha}\Phi^{\alpha}_{\mp}\psi^{\alpha}_{\pm} \right.$$ 

$$- i\psi^{\alpha}_{-}\hat{\nabla}^{\alpha}_{\alpha}\Phi^{\alpha}_{-} + \frac{i}{2}\eta\psi^{\alpha}_{+}\psi^{\alpha}_{-}\Omega^{\pm\alpha\beta}_{\alpha\beta}\phi_{\mp} + \frac{i}{4}(\psi^{\alpha}_{+}\psi^{\alpha}_{+} + \psi^{\alpha}_{-}\psi^{\alpha}_{-})H_{abM}\phi^{M} \right\}. \tag{60}$$

As usual, all indices are raised and lowered using the corresponding metrics $G^{MN}_{\alpha\beta}$ and $\delta_{\alpha\beta}$.

### 4.4 N=1 Supersymmetric non-Abelian boundary couplings

The more interesting case from the point of view of D-brane worldvolume theories is the boundary couplings of non-Abelian worldvolume fields and their behaviour under T-duality. The procedure for constructing the couplings is similar to the Abelian case and will be carried out here. The interpretation of scalar couplings as coordinate shifts is more subtle and will be addressed in the next subsection.

The starting point is the path-ordered supersymmetric Wilson line for the D9-brane which is inserted in the path-integral measure [12],

$$\text{tr} \mathcal{P} \exp(iS^{\partial\Sigma}_{DP}) = \text{tr} \mathcal{P} \exp \left( i \int_{\partial\Sigma} d\tau \left\{ A_{M}D^{M}_{(X)} - \frac{i}{4}D^{M}_{(\psi)}F^{N}_{MN}D^{N}_{(\psi)} \right\} \right). \tag{61}$$
Here, $A_M = A_M^a \lambda^a$ are non-Abelian gauge fields with $\lambda^a$ denoting the gauge group generators and $F_{MN} = \partial_M A_N - \partial_N A_M + i [A_M, A_N]$. The invariance of $S^Dp_{\Omega\Sigma}$ under $N=1$ supersymmetry requires an unconventional transformation of $\lambda^a$. The combined supersymmetry transformations are [12],

$$\delta_{\text{susy}} X^M = -\epsilon^{-} D^M_{(\psi)}, \quad \delta_{\text{susy}} D^M_{(\psi)} = -2i \epsilon^{-} \partial_r X^M, \quad \delta_{\text{susy}} \lambda^a = -i \epsilon^{-} D^L_{(\psi)} [A_L, \lambda^a]. \quad (62)$$

To obtain the non-Abelian Dp-brane boundary couplings from this, we take $A_M$ to be independent of $d$ coordinates $X^i$ and apply T-duality along these directions. Then, using (11) and making the usual identifications $\tilde{A}_\mu = A_\mu, \tilde{A}_i = \delta_{ij} \phi^j$ (where the origin of $\delta_{ij}$ was explained in section 3.3), the action $S^Dp_{\Omega\Sigma}$ in (61) yields,

$$S^Dp_{\Omega\Sigma} = \int_{\partial\Sigma} d\tau \left\{ A_M D^M_{(X)} - \frac{i}{4} D^M_{(\psi)} F_M^N D^N_{(\psi)} \right. \right.$$ 

$$\left. \quad + \frac{1}{2} \phi^M \mathcal{N}^M_{(X)} - \frac{i}{2} D^M_{(\psi)} \left( \partial_M \phi^L + i [A_M, \phi^L] \right) \mathcal{N}^L_{(\psi)} + \frac{1}{4} \mathcal{N}^M_{(\psi)} \left[ \phi^M, \phi^N \right] \mathcal{N}^N_{(\psi)} \right\}. \quad (63)$$

As before, the missing components can be inserted by using the static gauge boundary conditions $D^i_{(X)} = D^i_{(\psi)} = 0, \mathcal{N}^i_{(X)} = \mathcal{N}^i_{(\psi)} = 0$, without affecting the content of the theory,

$$S^Dp_{\Omega\Sigma} = \int_{\partial\Sigma} d\tau \left\{ A_M D^M_{(X)} - \frac{i}{4} D^M_{(\psi)} F_M^N D^N_{(\psi)} \right. \right.$$ 

$$\left. \quad + \frac{1}{2} \phi^M \mathcal{N}^M_{(X)} - \frac{i}{2} D^M_{(\psi)} \left( \partial_M \phi^L + i [A_M, \phi^L] \right) \mathcal{N}^L_{(\psi)} + \frac{1}{4} \mathcal{N}^M_{(\psi)} \left[ \phi^M, \phi^N \right] \mathcal{N}^N_{(\psi)} \right\}. \quad (64)$$

This final expression can be reinterpreted as the general form of the boundary couplings which goes beyond T-duality and is valid for any background and any embedding. As a check, one can verify its invariance under $N=1$ worldsheet supersymmetry that acts on the boundary vectors denoting the gauge group generators $\lambda^a$ that can be surmised from the action of the T-duality on the last equation in (62),

$$\delta_{\text{susy}} \lambda^a = -i \epsilon^{-} D^L_{(\psi)} [A_L, \lambda^a] - i \epsilon^{-} \mathcal{N}^L_{(\psi)} \left[ \phi^L, \lambda^a \right]. \quad (65)$$

As in the Abelian case, to the action (64) we should further add the extra term (53) which vanishes by virtue of Dp-brane boundary conditions, but which makes the NS 2-form gauge invariance of the $F_{MN}$-term manifest.

After some manipulations very similar to the N=1 Abelian case, the action (64) can be written in a manifestly general coordinate invariant form,

$$S^Dp_{\Omega\Sigma} = \int_{\partial\Sigma} d\tau \left\{ A_M^{(\phi)} \partial_r X^M - \frac{i}{4} D^M_{(\psi)} F_M^N D^N_{(\psi)} - \phi_N \partial_\sigma X^N - \frac{i}{2} D^\alpha_{(\psi)} \nabla^{(\phi)}_{\alpha} \phi_M \left( \psi^M_+ - \eta \psi^M_+ \right) \right. \right.$$ 

$$\left. \quad + i \eta \psi^N_+ \psi^L_+ \left( [\phi_N, \phi_L] - \frac{1}{2} \phi^M H_{MNL} \right) \right\}. \quad (66)$$

The covariant derivative is now,

$$\nabla^{(\phi)}_{\alpha} \phi_M = \partial_\alpha X^L \left( \partial_L \phi_M - \Gamma^N_{LM} \phi_N + i [A^{(\phi)}_L, \phi_M] \right), \quad (67)$$

and throughout the gauge field appears through the shifted combination,

$$A^{(\phi)}_M = A_M + \phi^L B_{LM} ,$$

including in the field strength, $F^{(\phi)}_{MN} = \partial_M A^{(\phi)}_N - \partial_N A^{(\phi)}_M + i [A^{(\phi)}_M, A^{(\phi)}_N]$. To get the $[\phi_N, \phi_L]$ term in this form, we have used $D^M_{(\psi)} \phi_M = 0$. The action can also be rewritten in the form (60).
4.5 Non-Abelian scalar couplings as coordinate shifts

The open string boundary coupling to non-Abelian scalar fields also follow from a shift of the coordinates $X^M$ by the matrix valued fields $\phi^M$, although with some subtleties. The implication is however more drastic as it leads to the notion of non-Abelian coordinate shifts as a symmetry of the D-brane worldvolume theory. This in particular allows us to relate worldvolume scalar couplings obtained by T-duality to the corresponding covariant expressions. The shift is no longer a part of general coordinate transformations.

Consider the coordinate shift $\Delta X^M = \phi^M$ performed in the purely background dependent part of the action. This is an unusual shift as it involves adding a non-Abelian matrix $\phi^M = \phi^a M^a$ to the ordinary coordinate $X^M$. To carry this out, the path-ordering in the Wilson line \[61\] should be extended to include the full worldsheet action factor $\exp(i S)$ in the path integral. In practice, path ordering will not cause much complication since we are interested only in first order variations. To preserve supersymmetry, the shift in $X^M$ should be accompanied by a shift $\Delta D^M_{(\psi)} = \Delta (\psi^M_+ + \eta \psi^M_-)$ of the fermions. This is determined by extending the supersymmetry transformation of $X^M$ to its variation, $\delta_{\text{susy}} \Delta X^M = -\epsilon^- \Delta D^M_{(\psi)}$. Using \[65\], the supersymmetry variation of $\phi^M = \phi^a M^a$ is

$$\delta_{\text{susy}} \phi^N = -\epsilon^- (D^L_{(\psi)} \partial_L \phi^N + i D^L_{(\psi)} [A_L, \phi^N] + i N^N_{(\psi)} [\phi^L, \phi^N]) .$$

From this one can read off $\Delta D^M_{(\psi)}$, leading to the bosonic and fermionic shifts,

$$\Delta X^M = \phi^M, \quad \Delta D^M_{(\psi)} = D^L_{(\psi)} (\partial_L \phi^M + i [A_L, \phi^M]) + i N^N_{(\psi)} [\phi^L, \phi^M].$$

(68)

To first order in the shifts, the variation of the worldsheet bulk action is given by \[17\],

$$\Delta S = \frac{1}{2} \int_{\partial \Sigma} d\tau \left[ \Delta X^L N^N_L + \frac{i}{2} D^L_{(\psi)} \Delta N^N_L - \frac{i}{2} \Delta D^L_{(\psi)} N^N_L \right].$$

(69)

This is non-zero since only the boundary vectors satisfy boundary conditions but not the shifts $\Delta X$ and $\Delta D_{(\psi)}$. To this we have to add the variation of the purely background term \[53\] required by NS 2-form gauge invariance. While this term and part of its variation vanish due to the boundary condition $N^N_\alpha = 0$, the non-zero piece in the variation is given by,

$$\Delta S_{\delta \Sigma,0} = -\frac{i}{4} \int_{\delta \Sigma} d\tau \left\{ D^\alpha_{(\psi)} \left( \partial_\alpha X^L N^N_L \right) \right\} = -\frac{i}{4} \int_{\delta \Sigma} d\tau \left\{ D^L_{(\psi)} \left( N^N_L \right) + D^\alpha_{(\psi)} \left( \partial_\alpha X^L \right) N^N_L \right\},$$

(70)

where we have used $D^\alpha_{(\psi)} \partial_\alpha X^L = D^L_{(\psi)}$. Then, the expected scalar field boundary couplings $S^{(\phi)} = \Delta S + \Delta S_{\delta \Sigma,0}$ are

$$S^{(\phi)} = \frac{1}{2} \int_{\partial \Sigma} d\tau \left\{ \Delta X^L N^N_L - \frac{i}{2} \Delta D^L_{(\psi)} N^N_L - \frac{i}{2} D^\alpha_{(\psi)} \left( \partial_\alpha X^L \right) N^N_L \right\} .$$

(71)

If the shift is Abelian, then $\Delta (\partial_\alpha X^L) = \partial_\alpha (\Delta X^L) = \partial_\alpha \phi^L$, as can be verified by noting that the supersymmetry transformations \[12\] also hold for $\Delta D_{(\psi)}$ and $\Delta D_{(X)}$. In this case we end up with the expression \[57\] which was used in the N=1 Abelian case.
In the non-Abelian theory $\triangle$ and $\partial_\alpha$ do not commute and the above identification is not consistent with supersymmetry. One can attempt to find the correct expression for $\triangle \left( \partial_\alpha X^L \right)$ by considering instead $\triangle \left( \partial_\tau X \right) \equiv \triangle D_{(X)}$ and taking the right hand side to be defined by the supersymmetry transformation $\delta_{\text{susy}}(\triangle D_{(\psi)}) = -2i\epsilon^\tau \triangle D_{(X)}$. Then using (68) one finds that $\triangle D_{(X)} = \partial_\tau X^M (\partial_M \phi^L + i [A_M, \phi^L]) + \cdots$. If we ignore the extra terms this suggests, \footnote{This approach is not fully consistent since a straightforward generalization of (12) to non-Abelian quantities does not lead to a closed set of equations, although we can still get some information from it. The complete expression for $\triangle D_{(X)}$ obtained in this way is:

$$\partial_\tau X^M D_M \phi^L + \frac{i}{2} N^{(X)}_{(\psi)} [\phi^P, \phi^L] + \frac{1}{4} D^{(\psi)}_{(\delta)} D^M_{(\psi)} [F_{KM}, \phi^L] - \frac{1}{2} N^{(\psi)}_{(\delta)} D^M_{(\psi)} [D_M \phi^P, \phi^L] - \frac{1}{2} N^{(\psi)}_{(\delta)} N^{(\psi)}_{(K)} [\phi^K, [\phi^P, \phi^L]],$$

where $D_M = \partial_M + i [A_M, \cdot]$.}

$$\triangle (\partial_\alpha X^L) = \partial_\alpha X^M \left( \partial_M \phi^L + i [A_M, \phi^L] \right).$$

Substituting this along with (68) in (71) one recovers the correct $\phi$-dependent terms in the non-Abelian boundary action. While this leads to the correct result, the above derivation is not fully satisfactory in view of footnote 5. One way of making this rigorous would be to find a closed form of the supersymmetry transformations (12) when applied to non-Abelian quantities. We will not follow this approach but rather describe an alternative derivation which, although formal, is more illuminating:

The problem has its origin in the addition of $\phi^M$, which transform in the adjoint representation of the gauge group, to gauge singlets $X^M$. A way out would be to make $\phi^M$ behave more like gauge singlets. This can be achieved at least formally, albeit at the expense of locality. Consider a gauge transformation $U$ in the non-Abelian theory,

$$A'_\alpha = U^{-1} A_\alpha U - i U^{-1} \partial_\alpha U,$$

$$\phi^M = U^{-1} \phi^M U.$$ \hspace{1cm} (72)

On the worldsheet boundary, $A_\alpha$ and $\phi$ depend on $\tau$ through $\xi^\alpha(\tau)$. We can always choose $U$ such that at a given point $\tau$ the gauge field vanishes (although the field strength is non-zero). This is the analogue of Riemann normal coordinates for the gauge bundle. For this, consider a path $\xi(\tau)$ from some $\tau_0$ to $\tau$ and choose $U$ as the Wilson line,

$$U(\tau, \tau_0) = \mathcal{P} \exp \left( -i \int_{\tau_0}^\tau A_\tau d\tau \right),$$ \hspace{1cm} (73)

where $A_\tau = A_\alpha \partial_\alpha \xi^\alpha = A_M \partial_\tau X^M$. Keeping the ordering in mind, one can see from (72) that at the point $\tau$, $A'_\alpha (\tau) = 0$ and furthermore, $\phi^M(\tau) = U(\tau_0, \tau) \phi^M(\tau_0) U(\tau, \tau_0)$ is invariant under local gauge transformations that are localized in the neighbourhood of $\tau$ but which vanish at $\tau_0$ (in particular one can choose $\tau_0$ at infinity and ignore its effect). Now the $\phi^M$, although non-local, are singlets at $\tau$ and are the natural objects by which the coordinates can be shifted. Thus in this gauge the shifts (68) become,

$$\triangle X^M = \phi^M,$$

$$\triangle D^M_{(\psi)} = D^L_{(\psi)} \partial_\tau \phi^M + i N^{(\psi)}_L \left[ \phi^L, \phi^M \right].$$ \hspace{1cm} (74)

Treating $\phi^L$ as a gauge singlet at $\tau$, we write $\triangle \left( \partial_\alpha X^L \right) = \partial_\alpha \left( \triangle X^L \right) = \partial_\alpha \phi^L$ as in the Abelian case. Then substitution in (71) reproduces the boundary couplings (64) to $\phi^M$ in the gauge
where $A'_M = 0$. The gauge field can be reinstated by the inverse gauge transformations $U^{-1}$ on noting that,

$$\partial_\alpha \phi^M = U^{-1}(\partial_\alpha \phi^M + i [A_\alpha, \phi^M]) U.$$  \hfill (75)

To summarize, we have seen that the N=1 worldsheet boundary couplings to non-Abelian worldvolume scalars are reproduced by shifting the $X^M$ by the matrices $\phi^M$ in the background part of the action. Carried out consistently, this also implies the shift $\partial_\alpha X^M \to \partial_\alpha X^M + \partial_\alpha \phi^M + i [A_\alpha, \phi^M]$.

We can now discuss the issue of T-duality and covariance which is very similar to the bosonic theory with Abelian fields, although now the implications are non-trivial: T-duality on D9-branes gives rise to Dp-brane boundary conditions in static gauge $X^\mu = \xi^\mu, X^i = \text{const}$. While the scalar field components $\phi^i$ and $\phi^\mu$ are generically non-zero in this gauge (as summarized in (26)), T-duality is capable of generating only $\phi^i$. In particular, it produces the Dp-brane boundary action (63) in which the $\phi^\mu$ do not appear. These missing components are needed if we are to reproduce the static gauge version of the covariant action (66), insuring consistency with general covariance. We have seen that the missing components $\phi^\mu$ can be included in the action through the addition of terms like $\phi^\mu N^{(X)}(X)$ that vanish by virtue of the boundary conditions. This has no effect on the physics and in this sense is a symmetry operation. Finally, we have seen that the same operation can also be implemented by shifting the coordinates by the matrix valued field $X^\mu \to X^\mu + \phi^\mu$, which when performed correctly, also results in $\partial_\alpha X^\mu \to \partial_\alpha X^\mu + \partial_\alpha \phi^\mu + i [A_\alpha, \phi^\mu]$.

D-brane worldvolume action is determined by the string worldsheet theory and therefore shares its behaviour under T-duality and inherits the above symmetry. Since worldsheet boundary conditions do not have a direct counterpart in the worldvolume theory, it is the implementation of the symmetry by the matrix-valued coordinate shift that provides its worldvolume realization. Unlike on the worldsheet, the extra terms generated by the shift on the worldvolume do not vanish identically, indicating that the shift is part of a non-trivially realized group of matrix-valued coordinate transformations, MCT. These generalize ordinary general coordinate transformations, GCT. The worldsheet considerations above uncover only a part of the MCT, enough to address the issue of covariance, but do not clarify its general group structure. The implications of this will be discussed in the next section.

## 5 Covariant Coupling of Scalars in Worldvolume Theories

This section addresses the issue of the apparent incompatibility of non-Abelian scalar couplings with general covariance which arises in the known form of the worldvolume theory. As advertised, the puzzle is resolved by the extra symmetry observed on the worldsheet boundary. We first consider the Abelian case to demonstrate the method and then turn to the non-Abelian theory. At the end, we present the general picture which puts everything in context.

### 5.1 Review of scalar couplings in Abelian worldvolume theory

Here we review the structure of the Abelian D-brane worldvolume theory with emphasis on the relation between T-duality and covariance, in order to set the stage for the discussion of the
non-Abelian case in the next subsection. We will also comment on the nature of the scalar field as a coordinate difference on the worldvolume.

The Abelian theory on the worldvolume of a single Dp-brane is given by a sum of the Dirac-Born-Infeld (DBI) \[15\] and Chern-Simons (CS) actions \[24\]
\[
\int d^{p+1} \xi e^{-\phi} \sqrt{\text{det}(P[i] + F)} + \int P[C'] \wedge e^F,
\]
where higher order curvature corrections have been ignored. The structure of the action depends on the brane geometry given by the embedding \(X^M(\xi^\alpha)\) through \(P[\cdots]\) which denotes the pull-back of space-time tensors to the brane worldvolume. For a space-time tensor \(V_{M_1 \cdots M_n}\) (which stands for \(E = G + B\) in the DBI part and for the Ramond-Ramond potentials \(C^{(n)}\) in the CS part), it denotes,
\[
P[V] = \partial_{\alpha_1} X^{M_1} \cdots \partial_{\alpha_n} X^{M_n} V_{M_1 \cdots M_n}.
\]
For the RR potentials we use the notation, \(C' = C \wedge e^F\), where \(C\) are invariant under NS 2-form gauge transformations. \(F = dA\) is the gauge field strength on the worldvolume (see subsection \[31\] for conventions). In the Abelian theory, the coupling of D-brane charges \(e^F\) to background RR potentials is described by exterior multiplication. This action is manifestly invariant under space-time general coordinate transformations (GCT).

Although the scalar fields do not explicitly appear in the above description, string worldsheet considerations suggest that at least to first order, they could be hidden as coordinate shifts in the embedding functions \(X^M(\xi)\) \[15\]. To see this explicitly, one starts with the D9-brane worldvolume action with gauge fields \(\tilde{A}_M\) and obtains the Dp-brane action by T-dualizing along the coordinates \(X^i\), leaving \(X^\mu\) unchanged. After identifying \(\tilde{A}_\mu = A_\mu\), \(\tilde{A}_i = \delta_{ij} \tilde{\phi}^j\), a small generalization of the procedure in \[17\] \[18\] yields the Dp-brane with the scalar couplings (see footnote \[2\]). This is still given by \[76\] but now in the semi-static gauge, \(X^\mu = X^\mu(\xi^\alpha)\), \(X^i = \tilde{\phi}^i(\xi^\alpha)\) where the pull-backs take the form,
\[
P[V]_{T\text{- dual}} = \cdots \partial_\alpha X^\mu V_\mu \cdots + \cdots \partial_\alpha \tilde{\phi}^i V_i \cdots.
\]
Note that in the above, the scalar fields \(\tilde{\phi}\) naturally appear as coordinate differences and hence are distinguished from the scalar fields \(\phi^M\) (or \(\Phi^i\)) of the worldsheet boundary theory which are vectors normal to the brane. The relation between the two will be discussed later.

We now make the connection between covariance, T-duality and scalar couplings even more explicit: \[78\] gives the coupling of scalar fields in a specific GCT gauge, one that followed from T-duality. One may ask how the scalars couple in general? From worldsheet considerations it is clear that on a Dp-brane defined by the embedding \(X^M(\xi)\) there will exist scalar fields \(\phi^M\) (or \(\tilde{\phi}^M\), to be specified below). Since both \(X^M(\xi)\) and the scalars correspond to brane shape and position, there is some freedom in how much of this information one encodes in each one the two. It is then evident that the general coupling of scalars in the Abelian worldvolume action should be through \(X^M + \phi^M\) and the expression for the pull-backs to be used in \[76\] is
\[
P[V]_{X+\phi} = \partial_{\alpha_1} (X^{M_1} + \phi^{M_1}) \cdots \partial_{\alpha_n} (X^{M_n} + \phi^{M_n}) V_{M_1 \cdots M_n}(X + \phi).
\]
This is a covariant expression for the coupling of scalars as long as \(\phi^M\) are regarded as coordinate differences. The outcome of T-duality \[78\] is a gauge fixed version of the above for which we have to fix the static gauge and also get rid of \(\phi^\mu\) by the coordinate transformation \(X^\mu \rightarrow X^\mu - \phi^\mu\). Conversely, the \(\phi^\mu\) can be reinstated in the T-duality expression \[78\] by the coordinate
shift $X^\mu \rightarrow X^\mu + \phi^\mu$, partially undoing the gauge fixing. It is in this sense that the Abelian Dp-brane action obtained by T-duality is compatible with covariance.

The coordinate shift $\delta X^\mu = \phi^\mu$ that insured the compatibility of T-duality and general covariance is a part of GCT, which is a symmetry of the Abelian worldvolume action [(76)]. In section 4 the same shift appeared as an invariance of the worldsheet boundary action obtained by T-duality and served the same purpose as on the worldvolume. The obvious conclusion is that even if we did not know about the general covariance of the worldvolume action, this property of the boundary theory would have been enough to indicate the existence of such a shift symmetry on the worldvolume enabling us to promote the outcome of T-duality in (78) to a covariant expression. This is an example of how our worldsheet considerations can be applied to the worldvolume theory. It also shows that the shift symmetry observed on the worldsheet is part of a larger symmetry of the worldvolume theory, in this case the GCT. The generalization to non-Abelian theory would imply the existence of a group of matrix-valued coordinate transformations, MCT, only a part of which is observed in the worldsheet theory.

We will now remark on the relation between the scalars $\phi^M$ that appear in the worldsheet boundary action and the $\tilde{\phi}^M$ of the worldvolume action. $\phi^M$ transforms as a vector under GCT as is evident from the structure of the worldsheet operator to which it coupled. On the other hand, $\tilde{\phi}^M$ appears as a coordinate difference and transforms accordingly. A manifestation of the difference is that we encounter ordinary derivatives of $\tilde{\phi}^M$ but only covariant derivatives of $\phi^M$. As suggested in [5] it is natural to think of the relation in terms of the Riemann normal coordinates [28, 29] which also appears in other contexts in the physics literature [26, 27]. This allows one to express a coordinate difference $\Delta X^M$ in terms of vectors $u^M$ (that acquire the interpretation of tangent vectors at $X$ to geodesics from $X$ to $X + \Delta X$),

$$\Delta X^M = u^M - \sum_{n=2}^{\infty} \frac{1}{n!} \Gamma^M_{L_1 \cdots L_n} u^{L_1} \cdots u^{L_n}. \quad (80)$$

Here, $\Gamma^M_{L_1 \cdots L_n} = \nabla_{(L_1} \cdots \nabla_{L_{n-1}} \Gamma^M_{L_n)}$, with the covariant derivatives acting only on the lower indices, are evaluated at a point $X^M(\xi)$ on the D-brane and we regard $\Delta X$ as a displacement from this point. The manipulations involving Riemann normal coordinates require that $u^M$ spans all directions, so we write $u^M = u^\alpha \partial_\alpha X^M + \phi^M$. Only at the end of the day can we restrict the results to the normal directions, $u^M = \phi^M$, in which case $\Delta X^M = \tilde{\phi}^M$ becomes a displacement away from the brane. Using the Riemann normal coordinate formalism, one can see that [26, 27],

$$\partial_\alpha \tilde{\phi}^M = \nabla_\alpha \phi^M - \frac{1}{3} \partial_\alpha X^K R^M_{L_1 L_2 K} \phi^{L_1} \phi^{L_2} + \cdots \quad (81)$$

where $\nabla_\alpha \phi^M = \partial_\alpha \phi^M + \partial_\alpha X^N \Gamma^M_{NL} \phi^L$, and the ellipses denote terms with higher powers and derivative of the curvature tensor. It is the right hand side of this equation that should emerge from an appropriate worldsheet calculation of the worldvolume action. Note that the covariant derivative above is the same as the one in (54); not restricted to the normal bundle and without a torsion contribution. To lowest order $\phi$ and $\tilde{\phi}$ are the same and it was only to this order that the worldsheet manipulations were carried out.

5.2 Covariance of non-Abelian worldvolume theory

In this subsection we finally address the issue of covariance of the scalar couplings in non-Abelian worldvolume theory. Geometrically, a stack coincident D-branes is still described by
the embedding functions $X^M(\xi)$, and the non-Abelian worldvolume fields $A_M$ and $\phi^M$ are, respectively, tangent and normal to the stack. The covariance of this description under general coordinate transformations (GCT) is reflected in the covariance of the worldsheet boundary couplings (86) which, in turn, determine the D-brane worldvolume action (say, through a manifestly covariant background field computation as in [26, 15]. The obvious implication is that the worldvolume theory should at least exhibit ordinary general coordinate invariance, besides other larger non-Abelian symmetries that it may also possess.

In practice, the couplings of non-Abelian scalars in the worldvolume action are obtained by T-duality or computations around flat background [1, 2, 3, 4], where covariance is not manifest. Even then, one normally expects the final results to be consistent with general covariance in the sense that they are obtainable from a GCT invariant action on going to a specific coordinate gauge, as happens in the Abelian theory. This however is not the case in the non-Abelian theory. To see this, let us consider the couplings of non-Abelian scalars obtained in [3]. Instead of writing the full action, we concentrate on the general structure of the couplings involving the scalars. These are determined in large part by T-duality and, in the static gauge, where $X^\mu = \xi^\mu$ along the brane and $X^i = \text{const}$ in the “transverse” directions, they appear as

$$V_{\mu...} + D_\mu \phi^i V_{i...} = V_{\mu...} + (\partial_\mu \phi^i + i [A_\mu, \phi^i]) V_{i...},$$

$$\cdots [\phi^i, \phi^j] V_{ij...},$$

$$V(X^\mu, X^i + \phi^i) = e^{\phi^i \partial / \partial X^i} V(X^\mu, X^i).$$

The tensor $V_{\cdots}$ stands for $C_{M_1...M_n}^\prime$ in the Chern-Simons action and is given in terms of $E_{MN} = G_{MN} + B_{MN}$ or the dilaton $\varphi$ in the DBI action. The first expression takes the place of ordinary pull-back and is interpreted as its non-Abelian generalization. Expressions of the second type appear in the BDI as well as in the CS action where they result in a D$p$-brane carrying charges corresponding to larger branes. The third expression indicates that the closed string backgrounds in the non-Abelian worldvolume action should be regarded as functions of the non-Abelian scalars [24] (these cannot be seen via T-duality due to the need for isometries).

It is easy to see that the above structures cannot follow from covariant expressions on choosing a coordinate gauge: A covariant action will contain all components of $\phi^M$, including $\phi^\mu$, which are generically non-zero even in the static gauge, as discussed in section 3.1. Also, being matrix valued, the $\phi^\mu$ cannot be gauged away or reintroduced into the action by ordinary coordinate transformations, unlike Abelian scalars. On the other hand, these components of the non-Abelian scalars do not appear in (82)-(84) which shows that they cannot follow from a covariant expression on fixing a GCT gauge. Besides this, the expressions (82)-(84) also contain other sources of non-covariance involving the Christoffel connection. These contain derivatives of the metric and arise at higher orders in perturbation theory to which the derivation in [3] is not sensitive.

The puzzle is that the above apparent inconsistency with general covariance is not entirely a result of overlooking terms in the calculation. Rather, the T-duality used in [3] is certainly valid for terms not containing derivatives of closed string backgrounds (the $V_{M_1...M_n}$ above). But the incompatibility with general covariance already shows up at this level, within the domain of validity of the derivation. This also prevents us from adding the missing components $\phi^\mu$ by hand, without a deeper understanding of their absence, as this would amount to changing by hand the outcome of a valid derivation. However, terms involving derivatives of the background fields (for example, the connection) are missed by T-duality and can be added by hand if required.
for covariance. One should then be able to reproduce them by a microscopic calculation.

An understanding of the problem and a mechanism for its resolution emerge from our worldsheet considerations. Recall that the worldsheet boundary action obtained by T-duality in (63) involves the same set of scalar field couplings as appear in (82)-(84), and therefore exhibits the same inconsistency with covariance. On the worldsheet boundary the missing components \(\phi^\mu\) could be reinserted into the action by adding to it terms that vanish by virtue of the boundary conditions. This restores covariance without affecting the dynamics. The implication is that the complete worldvolume theory should also have a corresponding symmetry (irrespective on how it is implemented) that goes beyond GCT and allows us to insert into the action the missing terms needed for covariance. The actual implementation of this symmetry also follows from the worldsheet theory. It involves a shift of the coordinates \(X^\mu\) by matrix valued fields the admissibility of which is not evident from the known structure of the worldvolume action.

The realization of this idea in the Abelian theory was verified in the previous subsection. The details and some fine tuning required in the non-Abelian case are discussed below.

An aspect not determined by the worldsheet considerations is when to regard the scalar field as a normal vector \(\phi^M\) and when to regard it as a coordinate difference \(\bar{\phi}^M\). Since the two differ by higher derivatives of the metric, the T-duality used here and in [3] (which has no \(\alpha'\) corrections) does not distinguish between them. In fact, in (82)-(84), \(\phi^i\) seems to correspond to a vector whenever it appears within a commutator and to a coordinate difference otherwise. On the worldsheet, the difference does not show up since the shift is taken to be infinitesimal, whereas \(\bar{\phi}\) differs from \(\phi\) at higher orders. The choice between the two has to be made depending on what is consistent with covariance. In principle, all this can be verified by microscopic calculations although that will not be attempted here.

Now we can demonstrate explicitly how the implementation of this enlarged symmetry promotes (82)-(84) to covariant expressions. First consider the expression \(V(X^\mu, X^i + \phi^i)\) in (84). We concentrate on the argument of \(V\) since its tensor index structure is part of (82). Clearly the symmetry observed on the worldsheet boundary allows us to shift \(X^\mu\) infinitesimally by \(\phi^\mu\), resulting in \(V(X^M + \phi^M)\). In analogy with the Abelian case, covariance demands that the vector \(\phi^M\) in the argument should be replaced by the non-Abelian “coordinate difference” \(\bar{\phi}^M\), given by the Riemann normal coordinate relation (80). On restricting to transverse scalars,

\[
\bar{\phi}^M = \phi^M - \sum_{n=2}^{\infty} \frac{1}{n!} \Gamma^M_{L_1 \ldots L_n} \phi^{L_1} \cdots \phi^{L_n}.
\]

(85)

Here we have simply generalized the Abelian expression to the non-Abelian \(\phi^M\) without bothering about the attendant subtleties. The issue of normal coordinates involving matrices has been considered in detail in [5] and will not be discussed here further. The function \(V(X^M + \bar{\phi}^M)\) can be expanded by the generalization of the Taylor expansion using the normal coordinate formalism [26, 27, 28],

\[
V(X^M + \bar{\phi}^M) = e^{\phi^M \nabla_M} V(X^M) + \cdots.
\]

(86)

where the covariant derivative contains the Christoffel connection \(\Gamma^K_{MN}\) and the ellipses represent corrections involving the curvature tensor. This is the covariant generalization of (84).

Next we consider the expression \(V_{\mu \ldots} + D_\mu \phi^i V_{\ldots}\) (82). The static gauge used can be slightly generalized to \(X^\mu = X^\mu(\xi^\alpha),\) \(X^i = const\) for which the derivation in [3] still goes through. We can also regard the trace of \(\phi^i\) (or part of it) as contributing to \(X^i\). This takes us out of the static gauge and leads to \(\partial_\alpha X^M V_M \ldots + D_\alpha \phi^i V_{\ldots}\). Simply shifting \(X^\mu\) is not enough to
render this covariant. But recall that the correct implementation of the shift transformation on the worldsheet resulted in \( X^\mu \to X^\mu + \phi^\mu \) and \( \partial_\alpha X^\mu \to \partial_\alpha X^\mu + D_\alpha \phi^\mu \). This was achieved by the formal trick described around equation (72) and justified there: We first make a gauge transformation \( \phi' = U^{-1} \phi U \) with \( U \) given by the Wilson line (73) evaluated along some fictitious path on the worldvolume ending at \( \xi^\alpha \). This sets \( A'_\alpha = 0 \) at \( \xi^\alpha \), leading to

\[
\partial_\alpha X^M V_M \cdots = \partial_\alpha (X^M + \phi'^M) V_M \cdots.
\]

For this to be sensible on the worldvolume, the vector \( \phi^M \) should be replaced by the coordinate difference \( \bar{\phi}^M \) (85). In practice, the difference should emerge from higher order corrections in the microscopic computation of the worldvolume action, as argued in [5]. Now we have the covariant generalization of (82),

\[
U \partial_\alpha (X^M + \bar{\phi}^M) V_M \cdots U^{-1} = U (\partial_\alpha X^M + \nabla_\alpha \phi^M + \cdots) V_M \cdots U^{-1} = (\partial_\alpha X^M + \nabla_\alpha (A^M) \phi^M + \cdots) V_M \cdots
\]

where \( \nabla_\alpha (A^M) \phi^M = \nabla_\alpha \phi^M + i [A_\alpha, \phi^M] \) is gauge and GCT covariant, and we have used (75) and (81). The ellipses denote curvature dependent terms. The tensor \( V \cdots \) itself has a structure as in (86).

Last we consider (83). The shift in \( X^\mu \) cannot be used directly to covariantize this and even on the worldsheet the corresponding commutator term in (83) was rendered covariant by a shift in the fermionic worldsheet coordinates in (68) and not \( X^\mu \) itself. As yet it is not clear how the coordinate shift should be generalized to also implement this aspect of the worldsheet symmetry on the brane worldvolume. Nevertheless worldsheet considerations have shown that such a generalization of the shift should exist and convert (83) into the corresponding covariant expression,

\[
\cdots [\phi^M, \phi^N] V_{MN} \cdots.
\]

The covariant version of the non-Abelian scalar couplings in [3] can thus be obtained by replacing the structures (82)-(84) by (87), (88) and (86), respectively.

As such, the covariant expressions above could be easily guessed and the worldvolume action modified accordingly without invoking the worldsheet theory and it matrix-valued shift symmetry. However, since the action in [3] is obtained following a consistent procedure, there is no room for introducing into it new terms by hand (except for those which fall beyond the scope of the derivation, like terms involving the Christoffel connection). Therefore, to address the issue or covariance of scalar couplings in [3], an approach like the one followed here becomes indispensable. It not only yields a covariant expression, but also highlights the existence and importance of matrix-valued coordinate transformations.

To summarize, the above discussion leads to the following picture of the relation between GCT covariance, matrix-valued transformations and T-duality in the worldvolume theory:

- The complete non-Abelian worldvolume theory has an enlarged symmetry group consisting of matrix-valued coordinate transformations (MCT) which contains ordinary general coordinate transformations (GCT) as a subgroup. The general structure of MCT and how it includes GCT has not been specified. Issues pertaining to this type of enlarged symmetry has been considered in [5, 6].

- One can now fix a gauge using MCT. The Dp-brane action obtained by T-duality in [3] is in fact in such a gauge (explicitly, this involves fixing the static gauge for \( X^M \) and
then using MCT to transform away the non-Abelian scalars $\phi^\mu$ keeping $\phi^i$). Since MCT contains GCT, gauge fixing the former also fixes the latter.

- A gauge that is fixed using MCT can only be undone by a matrix valued transformation, and not by a GCT alone. Therefore, if we disregard the possibility of matrix valued transformations, then expressions in this gauge seem inconsistent with general covariance since they cannot be obtained from covariant expressions by choosing a GCT gauge.

- This is the case with the non-Abelian worldvolume action of [3]. The possibility of matrix valued coordinate transformations is not evident from the action. However, worldsheet considerations indicated the existence of a specific matrix-valued transformation which is the one needed to undo the MCT gauge fixing and hence render the expressions GCT invariant.

One may regard the combination $X^M + \bar{\phi}^M$ or $X^M + \bar{\phi}^iM$ as a matrix valued coordinate and formulate MCT in terms of this. This is the approach followed in [5] in the context of D0-branes. Also see [6] with emphasis on non-Abelian branes within branes. In this approach the understanding of ordinary general covariance becomes more involved. Here we follow a more conservative approach of treating $X^M$ and $\phi^M$ separately to make the GCT manifest.

### 5.3 Coupling to RR potentials through Clifford multiplication

Before ending we will briefly comment on the coupling of charges carried by D-branes to background RR-potentials, as contained in the Chern-Simons part of the worldvolume action. It is well known that in the absence of the scalar field excitations a Dp-brane carries charges, besides its own charge, corresponding to smaller branes on its worldvolume [24]. These charges couple to background RR-potentials through exterior multiplication of forms which provides an elegant description of the couplings.

The couplings get modified in the presence of non-Abelian scalars as found in [1,2,3]. Physically this implies that Dp-branes can also carry charges corresponding to larger branes. The couplings of these new charges to RR-backgrounds now also involve contractions and are no longer described by an elegant exterior multiplication. It is desirable to find a unified description of these couplings that replaces the exterior product. In [30] it was shown that such a unified description is provided by the Clifford multiplication of forms. However, the details of the formalism were not fully satisfactory: Clifford multiplication can be regarded as a multiplication of Dirac gamma matrices. Since the results in [3] were based on T-duality, it suggested the use of gamma matrices associated with the T-duality group which do not have a very natural meaning in the theory. Thus, while this correctly reproduces the couplings, the formalism is tied to a specific gauge.

It is much more appealing to base the Clifford multiplication on the usual space-time gamma matrices that naturally occur in the theory. These are also suggested by worldsheet considerations. However, their use leads to the presence of extra terms in the action, not included in [3]. One can check that these extra terms are precisely the ones needed to complete the gauge fixed action (involving couplings of the form (82)-(84)) to the one consistent with covariance. As we have seen these terms in fact do exist. Therefore a covariant description of the D-brane coupling to RR-backgrounds is provided by Clifford multiplication as in [30], but associated with the space-time gamma matrices.
6 Conclusions

In the first part of this paper (section 2 to section 4) we consider the N=1 open string worldsheet theory in general non-constant backgrounds and give a unified description of D-brane boundary conditions and boundary couplings in terms of a set of four boundary “vectors”, $\mathcal{N}^{(x)}$, $\mathcal{N}^{(\psi)}$, $\mathcal{D}_{(x)}$ and $\mathcal{D}_{(\psi)}$. The complete set of boundary couplings are constructed for the bosonic and N=1 Abelian and non-Abelian theories, consistent with supersymmetry, T-duality and general covariance. One aspect of these couplings is that, when written in a manifestly covariant form, the gauge field always appears in the combination $A_M + \phi^L B_{LM}$. In the N=1 case, the invariance of the boundary action under the NS 2-form gauge transformation can be made manifest by adding to it a term $-\frac{i}{4} \mathcal{D}_{(\psi)}^\alpha \mathcal{N}_\alpha$ that vanishes by virtue of the boundary conditions. The presence of this term is crucial to insure that the scalars $\phi^M$ can still be interpreted as infinitesimal coordinate shifts. This also holds in the non-Abelian theory provided the shift is correctly interpreted, for example in a gauge that sets the gauge field to zero at the point under consideration.

One obvious use of these couplings is in the calculation of terms in the worldvolume action, which is not the aim here. Rather we investigate the behaviour of the scalar couplings under T-duality and note that they behave exactly as in the Dp-brane worldvolume theory. In particular, the non-Abelian scalars exhibit the same apparent incompatibility with general covariance. However, on the worldsheet boundary this has a simple resolution due to an invariance that allows us to shift coordinates by appropriate matrices. It is then clear that the same symmetry should also operate in the D-brane worldvolume theory, rendering it consistent with general covariance, although its existence is not evident from the known form of the action.

The resolution of the apparent inconsistency of non-Abelian scalar couplings (as obtained by T-duality in [2, 3]) with general covariance is then based on the following picture: The fully covariant non-Abelian worldvolume action should also be invariant under a set of matrix-valued coordinate transformations (MCT). The action obtained by T-duality appears in a fixed MCT gauge, in which the components $\phi^\mu$ of the scalars are gauges away. This looks incompatible with covariance since this gauge fixing cannot be achieved or undone by a general coordinate transformation. The resolution therefore lies in the existence of MCT and the particular transformation required to undo the gauge emerges from the worldsheet considerations.

In this paper we have taken the conservative approach of not combining coordinates $X^M$ and scalars $\phi^M$ into non-Abelian coordinates, as that would complicate the understanding of general covariance which was our main concern. Once the minimum requirement of general covariance is insured, one can take this approach and study the problem on the lines of [5, 6].

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Appendices

A Supersymmetry conventions

We use the following conventions for the N=1 supersymmetry on the worldsheet parameterized by $\sigma^\pm = \frac{1}{2}(\tau \pm \sigma)$. The bosonic field $X^M$ and the fermionic fields $\phi^M_\pm$ are combined into the superfield $X^M + \theta^+ \phi_+ + \theta^- \psi_- + \theta^- \theta^+ F$. The supercovariant derivatives are $D_\pm = -\partial/\partial \theta^\pm + i \theta^\pm \partial_\pm$. The supersymmetry variations take the form,

$$\delta_{\text{susy}} X^M = -\epsilon^+ \psi^M_+ - \epsilon^- \psi^M_-,$$
$$\delta_{\text{susy}} \psi^M_\pm = -i \epsilon^\pm \partial_\pm X^M \pm \epsilon^\mp F^M,$$
$$\delta_{\text{susy}} F^M = -i \epsilon^+ \partial_+ \psi^M_- + i \epsilon^- \partial_- \psi^M_+.$$

Now consider the superfield, $L = E_{MN}(X)X^MX^N = L_X + \theta^+ L_+ + \theta^- L_- \theta^+ L_F$. From the supersymmetry transformation of the F-term, we have

$$\delta_{\text{susy}} L_F = -i \epsilon^+ \partial_+ L_- + i \epsilon^- \partial_- L_+.$$

The worldsheet action (1) is written as a sum of two parts $S = S_\Sigma + S_{\partial \Sigma}$. The first part involves the Lagrangian density $\int d\theta^+ d\theta^- L = L_F$ and therefore its supersymmetry variation is,

$$\delta_{\text{susy}} S_\Sigma = -i \int d\tau (\epsilon^+ L_- + \epsilon^- L_+) |_{\partial \Sigma}.$$

Adding this to the supersymmetry variation of $S_{\partial \Sigma}$ leads to $\delta_{\text{susy}} S$ in (7).

B Index Conventions

Capital letters from the middle of the alphabet $K, L, M, N$ label 10-dimensional space-time indices. $\alpha, \beta$ denote worldvolume indices and $\hat{a}, \hat{b}$ correspond to flat normal frame indices. In the static gauge, the space-time coordinates identified with the brane worldvolume coordinates are labeled by $\mu, \nu, \cdots$, while the coordinate not along the worldvolume are labeled by $i, j, \cdots$. The letters $a, b$ denote gauge group generators.

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