An Erdős-Ko-Rado theorem for multisets

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Abstract

Let $k$ and $m$ be positive integers. A collection of $k$-multisets from $\{1, \ldots, m\}$ is intersecting if every pair of multisets from the collection is intersecting. We prove that for $m \geq k+1$, the size of the largest such collection is $\binom{m+k-2}{k-1}$ and that when $m > k+1$, only a collection of all the $k$-multisets containing a fixed element will attain this bound. The size and structure of the largest intersecting collection of $k$-multisets for $m \leq k$ is also given.

1 Introduction

The Erdős-Ko-Rado Theorem [6] is an important result in extremal set theory that gives the size and structure of the largest pairwise intersecting $k$-subset system from $[n] = \{1, \ldots, n\}$. This theorem is commonly stated as follows:

**Theorem 1.1.** Let $k$ and $n$ be positive integers with $n \geq 2k$. If $\mathcal{F}$ is a collection of intersecting $k$-subsets of $[n]$, then

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

Moreover, if $n > 2k$, equality holds if and only if $\mathcal{F}$ is a collection of all the $k$-subsets from $[n]$ that contain a fixed element from $[n]$.

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Note that if \( n < 2k \), any pair of \( k \)-subsets will be intersecting and so the largest intersecting collection will have size \( \binom{n}{k} \).

A multiset is a generalization of a set in which an element may appear more than once. As with sets, the order of the elements is irrelevant. The cardinality of a multiset is the total number of elements including repetitions. A \( k \)-multiset system on an \( m \)-set is a collection of multisets of cardinality \( k \) containing elements from \([m]\). We say that two multisets are intersecting if they have at least one element in common and that a collection of multisets is intersecting if every pair of multisets in the collection is intersecting.

In this paper, we give a generalization of the Erdős-Ko-Rado Theorem to intersecting multiset systems. Specifically, we prove the following two theorems for the cases when \( m \geq k + 1 \) and \( m \leq k \) respectively.

**Theorem 1.2.** Let \( k \) and \( m \) be positive integers with \( m \geq k + 1 \). If \( \mathcal{A} \) is a collection of intersecting \( k \)-multisets of \([m]\), then

\[
|\mathcal{A}| \leq \binom{m + k - 2}{k - 1}.
\]

Moreover, if \( m > k + 1 \), equality holds if and only if \( \mathcal{A} \) is a collection of all the \( k \)-multisets from \([m]\) that contain a fixed element from \([m]\).

If \( m < k + 1 \), larger collections are possible. For example, if \( m = k = 3 \), the seven \( k \)-multisets containing either two or three distinct elements from \([m]\) will form an intersecting collection since each multiset contains more than half the elements from \([m]\). We will use \( \mathcal{M}(\frac{m}{2}) \) to denote the collection of all \( k \)-multisets that contain more than \( \frac{m}{2} \) distinct elements from \([m]\) and \( \mathcal{M}(\frac{m}{2}) \) to denote the collection of all \( k \)-multisets from \([m]\) containing exactly \( \frac{m}{2} \) distinct elements. Then

\[
|\mathcal{M}(\frac{m}{2})| = \binom{m}{\frac{m}{2}} \binom{k - 1}{k - \frac{m}{2}}
\]

and

\[
|\mathcal{M}(\frac{m}{2})| = \sum_{j=\left\lceil \frac{m+1}{2} \right\rceil}^{m} \binom{m}{j} \binom{k - 1}{k - j}.
\]

**Theorem 1.3.** Let \( k \) and \( m \) be positive integers with \( m \leq k \). If \( \mathcal{A} \) is a collection of intersecting \( k \)-multisets of \([m]\), then:

1. If \( m \) is odd, \( |\mathcal{A}| \leq |\mathcal{M}(\frac{m}{2})| \) and equality holds if and only if \( \mathcal{A} = \mathcal{M}(\frac{m}{2}) \).

2. If \( m \) is even, \( |\mathcal{A}| \leq \left| \mathcal{M}(\frac{m}{2}) \right| + \frac{1}{2} \left| \mathcal{M}(\frac{m}{2}) \right| \) and equality holds if and only if \( \mathcal{A} \) consists of \( \mathcal{M}(\frac{m}{2}) \) and a maximal intersecting collection of \( k \)-multisets from \( \mathcal{M}(\frac{m}{2}) \).

A \( k \)-multiset on an \( m \)-set can be represented as an integer sequence of length \( m \) with the integer in each position representing the number of repetitions of the corresponding element from \([m]\). For example, if \( m = 6 \), the multiset \( \{1, 2, 2, 4\} \) can
be represented by the integer sequence \((1, 2, 0, 1, 0, 0)\). For a \(k\)-multiset, the sum of the integers in the corresponding integer sequence will equal \(k\).

Erdős-Ko-Rado type results for intersecting families of integer sequences are known (e.g. \([9]\), \([10]\), \([11]\)). In these, the sum of the entries in the integer sequence is not restricted to \(k\) and the definition of intersection is different from our definition for multisets. In \([2]\), Anderson proves an Erdős-Ko-Rado type result for multisets but uses yet another definition of intersection. A definition of intersection equivalent to ours is used in several theorems for intersecting collections of vectors presented by Anderson in \([3]\). These theorems were originally written in terms of sets of noncoprime divisors of a number by Erdős et al. in \([5]\) and \([7]\), and again the sum of the entries is not restricted to \(k\).

More recently, Brockman and Kay \([4]\) proved the result in Theorem 1.2 provided that \(m \geq 2k\). Mahdian \([13]\) proved the bound on the size of a collection of intersecting \(k\)-multisets when \(m > k\) using a method similar to Katona’s cycle proof for sets \([12]\). Our results improve the bound on \(m\) given in \([4]\) and give the size and structure of the largest possible intersecting collection for all values of \(m\) and \(k\).

## 2 Proof of Theorem 1.2

Our proof of this theorem uses a homomorphism from a Kneser graph to a graph whose vertices are the \(k\)-multisets of \([m]\).

A Kneser graph, \(K(n, k)\), is a graph whose vertices are all of the \(k\)-sets from an \(n\)-set, denoted by \(\binom{[n]}{k}\), and where two vertices are adjacent if and only if the corresponding \(k\)-sets are disjoint. Thus an independent set of vertices in the Kneser graph is an intersecting \(k\)-set system. We will use \(\alpha(K(n, k))\) to denote the size of the largest independent set in \(K(n, k)\).

We now define a multiset analogue of the Kneser graph. For positive integers \(k\) and \(m\), let \(M(m, k)\) be the graph whose vertices are the \(k\)-multisets from the set \([m]\), denoted by \(\binom{[m]}{k}\), and where two vertices are adjacent if and only if the corresponding multisets are disjoint. For this graph, the number of vertices is equal to \(\binom{m}{k} = \binom{m+k-1}{k}\) and an independent set is an intersecting \(k\)-multiset system.

Let \(n = m + k - 1\). Then \(K(n, k)\) has the same number of vertices as \(M(m, k)\) and \(B \cap [m] \neq \emptyset\) for any \(B \in \binom{[n]}{k}\).

For a set \(A \subseteq [m]\) of cardinality \(a\) where \(1 \leq a \leq k\), the number of \(k\)-sets, \(B\), from \([n]\) such that \(B \cap [m] = A\) will be equal to

\[
\binom{n-m}{k-a} = \binom{k-1}{k-a}.
\]

Similarly, the number of \(k\)-multisets from \([m]\) that contain all of the elements of \(A\) and no others will be equal to

\[
\binom{a}{k-a} = \binom{a+(k-a)-1}{k-a} = \binom{k-1}{k-a}.
\]

Hence there exists a bijection, \(f : \binom{[n]}{k} \rightarrow \binom{[m]}{k}\), such that for any \(B \in \binom{[n]}{k}\), the set of distinct elements in \(f(B)\) will be equal to \(B \cap [m]\).
If \( A, B \in \binom{[n]}{k} \) are two adjacent vertices in the Kneser graph, then \((A \cap [m]) \cap (B \cap [m]) = \emptyset\) and hence \( f(A) \cap f(B) = \emptyset\). Therefore \( f(A) \) is adjacent to \( f(B) \) if \( A \) is adjacent to \( B \) and so the bijection \( f : \binom{[n]}{k} \to \binom{[m]}{k} \) is a graph homomorphism. In fact, \( K(n, k) \) is isomorphic to a spanning subgraph of \( M(m, k) \). Thus

\[
\alpha(M(m, k)) \leq \alpha(K(n, k)).
\]

From the Erdős-Ko-Rado Theorem, we have that if \( n \geq 2k \),

\[
\alpha(K(n, k)) = \binom{n-1}{k-1}.
\]

Thus, for \( m \geq k + 1 \),

\[
\alpha(M(m, k)) \leq \binom{n-1}{k-1} = \binom{m+k-2}{k-1}.
\]

An intersecting collection of \( k \)-multisets from \([m] \) consisting of all \( k \)-multisets containing a fixed element from \([m] \) will have size \( \binom{m+(k-1)-1}{k-1} = \binom{m-k-2}{k-1} \). Therefore

\[
\alpha(M(m, k)) = \binom{m+k-2}{k-1}
\]

which gives the upper bound on \( \mathcal{A} \) in Theorem 1.2

To prove the uniqueness statement in the theorem, let \( m > k + 1 \) and let \( \mathcal{A} \) be an intersecting multiset system of size \( \binom{m+k-2}{k-1} \). With the homomorphism defined above, the pre-image of \( \mathcal{A} \) will be an independent set in \( K(n, k) \) of size \( \binom{n-1}{k-1} \). Since \( m > k + 1 \) and \( n = m + k - 1 \), it follows that \( n > 2k \) so, by the Erdős-Ko-Rado theorem, \( f^{-1}(\mathcal{A}) \) will be a collection of all the \( k \)-subsets of \([n] \) that contain a fixed element from \([n] \). If the fixed element, \( x \), is an element of \([m] \), then it follows from the definition of \( f \) that every multiset in \( \mathcal{A} \) will contain \( x \). Thus \( \mathcal{A} \) will be a collection of all the \( k \)-multisets from \([m] \) that contain a fixed element from \([m] \) as required. If \( x \notin [m] \), then \( f^{-1}(\mathcal{A}) \) will include the sets \( A = \{1, m+1, \ldots, n\} \) and \( B = \{2, m+1, \ldots, n\} \) since \( m > k + 1 \) implies that \( m > 2 \). But \( f(A) \cap f(B) = \emptyset \) which contradicts our assumption that \( \mathcal{A} \) is an intersecting collection of multisets. Therefore, when \( m > k + 1 \), if \( \mathcal{A} \) is an intersecting collection of multisets of the maximum possible size, then \( \mathcal{A} \) is the collections of all \( k \)-multisets containing a fixed element from \([m] \).

The case when \( m = k + 1 \) is analogous to the case when \( n = 2k \) in the Erdős-Ko-Rado theorem. The size of the largest possible intersecting collection is equal to \( \binom{m+k-2}{k-1} \) but collections attaining this bound are not limited to those having a common element in all \( k \)-multisets.

## 3 Proof of Theorem 1.3

Although Theorem 1.2 is restricted to \( m \geq k + 1 \), the inequality \( \alpha(M(m, k)) \leq \alpha(K(n, k)) \) still holds when \( m \leq k \). However, the resulting inequality

\[
\alpha(M(m, k)) \leq \binom{n}{k} = \binom{m+k-1}{k}
\]
is not particularly useful since for \( m > 1 \) this bound is not attainable. Clearly, two multisets consisting of \( k \) copies of different elements from \([m]\) will not intersect.

Before proceeding with our proof of Theorem 1.3 we define the support of a multiset. If \( A \) is a \( k \)-multiset from \([m]\), the support of \( A \), denoted by \( S_A \), is the set of distinct integers from \([m]\) in \( A \). Note that two \( k \)-multisets, \( A, B \in \binom{[m]}{k} \), will be intersecting if and only if \( S_A \cap S_B \neq \emptyset \) and that each \( S_A \) will have a unique complement, \( S_A^c = [m] \setminus S_A, \) in \([m]\).

Let \( \mathcal{A} \) be an intersecting family of \( k \)-multisets of \([m]\) of maximum size and let \( M \in \mathcal{A} \) be a \( k \)-multiset such that \( |S_M| = \min \{|S_A| : A \in \mathcal{A} \} \). If \( m = 2 \), it is easy to see that the theorem holds, so we will assume that \( m > 2 \).

Suppose that \( |S_M| < \frac{m}{2} \). Let \( B_1 = \{ A \in \mathcal{A} : S_A = S_M \} \) and let \( B_2 = \{ B \in \binom{[m]}{k} : S_B = S_M \} \). Then \( B_1 \subseteq \mathcal{A} \) and \( B_2 \cap \mathcal{A} = \emptyset \).

We will now show that \( \mathcal{A}' := (\mathcal{A} \setminus B_1) \cup B_2 \) is an intersecting family of \( k \)-multisets from \([m]\) that is larger than \( \mathcal{A} \). By construction, every multiset in \( \mathcal{A} \setminus B_1 \) contains at least one element from \([m]\) \( S_M \), and \([m]\) \( S_M = S_B \) for all \( B \in B_2 \). Thus \( \mathcal{A}' \) is an intersecting collection of \( k \)-multisets.

Let \( |S_M| = i \). Then

\[
|B_1| = \binom{i}{k-i} = \binom{k-1}{k-i}.
\]

Since \( |S_M| = m - i \), it follows that

\[
|B_2| = \binom{m-i}{k-(m-i)} = \binom{k-1}{k-m+i}.
\]

To show that \( |\mathcal{A}'| > |\mathcal{A}| \), it is sufficient to show that

\[
\frac{k-1}{k-m+i} > \frac{k-1}{k-i},
\]

or equivalently, that

\[
(k-i)! (i-1)! > (k-m+i)! (m-i-1)!.
\]

Since \( i < \frac{m}{2} \) and \( m \leq k \), we have that \( k-i > k-\frac{m}{2} > k-m+i \geq 1 \). Therefore,

\[
(k-i)! (i-1)! = (k-i) (k-i-1) \ldots (k-m+i+1) (k-m+i)! (i-1)! \\
\quad \geq (m-i) (m-i-1) \ldots (i+1) (k-m+i)! (i-1)! \\
\quad = \frac{m-i}{i} (m-i-1)! (k-m+i)! \\
\quad > (m-i-1)! (k-m+i)!
\]

as required. Thus if \( \mathcal{A} \) is of maximum size, it cannot contain a multiset with less than \( \frac{m}{2} \) distinct elements from \([m]\).

It is easy to see that any \( k \)-multiset containing more than \( \frac{m}{2} \) distinct elements from \([m]\) will intersect with any other such \( k \)-multiset. This completes the proof of the theorem for the case when \( m \) is odd. When \( m \) is even, it is necessary to consider
the $k$-multisets which contain exactly $\frac{m}{2}$ distinct elements, that is, the $k$-multisets in $\mathcal{M}(\frac{m}{2})$. These multisets will intersect with any multiset containing more than $\frac{m}{2}$ distinct elements. However, $\mathcal{M}(\frac{m}{2})$ is not an intersecting collection. For any $A \in \mathcal{M}(\frac{m}{2})$, all of the $k$-multisets, $B$, where $S_B = S_A$ will be in $\mathcal{M}(\frac{m}{2})$ and will not intersect with $A$. Since the size of a maximal intersecting collection of $\frac{m}{2}$-subsets of $[m]$ is $\frac{1}{2}\left(\begin{array}{c}m+1\end{array}\right)$ and each $\frac{m}{2}$-subset is the support for the same number of multisets in $\mathcal{M}(\frac{m}{2})$, an intersecting collection of $k$-multisets will contain at most half of the $k$-multisets in $\mathcal{M}(\frac{m}{2})$.  

\section{Further work}

An obvious open problem is determining the size and structure of the largest collection of $t$-intersecting $k$-multisets, i.e. collections of multisets where the size of the intersection for every pair of multisets is at least $t$. (We define the intersection of two multisets to be the multiset containing all elements common to both multisets with repetitions.) The following conjecture is a version of Conjecture 5.1 from [4].

\textbf{Conjecture 4.1.} Let $k$, $m$ and $t$ be positive integers with $t \leq k$ and $m \geq t(k-t)+2$. If $A$ is a collection of intersecting $k$-multisets of $[m]$, then

$$|A| \leq \left(\begin{array}{c}m + k - t - 1 \\ k - t \end{array}\right).$$

Moreover, if $m > t(k-t)+2$, equality holds if and only if $A$ is a collection of all the $k$-multisets from $[m]$ that contain a fixed $t$-multiset from $[m]$.

The lower bound on $m$ in this conjecture was obtained by substituting $m+k-1$ for $n$ in the corresponding bound for sets given by Frankl [8] and Wilson [14]. The conjecture is supported by the fact that when $m > t(k-t)+2$, a collection consisting of all $k$-multisets containing a fixed $t$-multiset is larger than a collection consisting of all $k$-multisets containing $t+1$ elements from a set of $t+2$ distinct elements of $[m]$ and that these two collections are equal in size when $m = t(k-t)+2$. Furthermore, when $m = t(k-t)+1$, collections larger than $\left(\begin{array}{c}m+k-t-1 \\ k-t \end{array}\right)$ are possible. For example, if $t = 2$, $k = 5$ and $m = 7$, the cardinality of the collection of all $k$-multisets containing three or more elements from $\{1, 2, 3, 4\}$ is 91 while $\left(\begin{array}{c}m+k-t-1 \\ k-t \end{array}\right) = 84$.

The existence of a graph homomorphism from the Kneser graph $K(n,k)$ to its multiset analogue $M(m,k)$ in the proof of Theorem 12 gave a simple and straightforward way to show that the size of the largest independent set in $M(m,k)$ is no larger than the size of the largest independent set in $K(n,k)$. These graphs can be generalized as follows: let $K(n,k,t)$ be the graph whose vertices are the $k$-subsets of $[n]$ and where two vertices, $A, B$, are adjacent if $|A \cap B| < t$ and let $M(m,k,t)$ be the graph whose vertices are the $k$-multisets of $[m]$ and where two vertices, $C, D$ are adjacent if $|C \cap D| < t$.

If a bijective homomorphism from $K(n,k,t)$ to $M(m,k,t)$ exists, it could be used to prove a bound not only on the maximum size of a $t$-intersecting collection as given in Conjecture 4.1 but also on the maximum size when $k-t \leq m \leq t(k-t)+2$ using the Complete Erdős-Ko-Rado theorem of Ahlswede and Khachatrian [1]. However,
it is not clear that such a homomorphism exists. The conditions placed on the bijection in the proof of Theorem 1.2 are not sufficient to ensure that the bijection is a homomorphism since for two $k$-multisets, $A$ and $B$, having $|S_A \cap S_B| < t$ does not imply that $|A \cap B| < t$.

The simple fact that if a graph $G$ is isomorphic to a spanning subgraph of a graph $H$, then $\alpha(H) \leq \alpha(G)$ may be useful in proving Erdős-Ko-Rado theorems for different objects. It would be interesting to determine if there are combinatorial objects other than multisets which have this relationship to an object for which an Erdős-Ko-Rado type result is known.

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