Modewise Entanglement of Gaussian States

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We address the decomposition of a multi-mode pure Gaussian state with respect to a bi-partite division of the modes. For any such division the state can always be expressed as a product state involving entangled two-mode squeezed states and single mode local states at each side. The character of entanglement of the state can therefore be understood modewise; that is, a given mode on one side is entangled with only one corresponding mode of the other, and therefore the total bi-partite entanglement is the sum of the modewise entanglement. This decomposition is generally not applicable to all mixed Gaussian states. However, the result can be extended to a special family of “isotropic” states, characterized by a phase space covariance matrix with a completely degenerate symplectic spectrum.

I. INTRODUCTION

While the full characterization of bi-partite entanglement for mixed states is still an open problem, much is known for the case of pure states. Under a bi-partite division, any pure state may be written in the Schmidt form

$$|\psi\rangle_{AB} = \sum_a \sqrt{p_a} |\phi_a\rangle_A |\chi_a\rangle_B$$ (1)

where $\langle\phi_a|\phi_b\rangle = \langle\chi_a|\chi_b\rangle = \delta_{ab}$, with unique Schmidt coefficients $\sqrt{p_a}$. The bi-partite entanglement (entanglement entropy) can then be fixed uniquely by the asymptotic yield of maximally entangled states and becomes a function of the Schmidt coefficients only. Moreover, the Schmidt decomposition appears to have an “irreducible” structure: generally speaking, equation (1) cannot be brought into a simpler form just by means of local transformations. For instance, a bi-partite system of $n \times n$ qubits cannot be generally brought to the form of a product of $n$ entangled pairs under local unitary transformations.

However, in the context of Bosonic Channel Capacity, Holevo and Werner have shown that a multi-mode Gaussian mixed state can always be purified by enlarging the system in such way that each normal mode is correlated with a corresponding single ancillary mode. This procedure achieves a pure Gaussian state between the system and ancilla in which the Schmidt decomposition takes the form of products of bi-partite two-mode Gaussian states. Implicit in these results is a general statement in the converse sense, which we believe is of considerable significance for the area of continuous variable entanglement. The statement is that the bi-partite entanglement of multi-mode Gaussian pure states is in fact reducible to the product of entangled pairs of single modes. In other words, bi-partite entanglement of a Gaussian pure state is essentially $1 \times 1$ mode Gaussian entanglement.

This result is directly applicable to various problems such as quantum-optical realizations of quantum information processing with Gaussian states, and the characterization of the entanglement content of harmonic oscillator chains and bosonic quantum-fields. Consider for instance the vacuum state of a free scalar field, which is Gaussian. While one would expect that in this state, the structure of entanglement between a given region of space and its complement would be of a rather complicated nature, such entanglement in fact occurs along separate “channels”, with each member of a set of collective modes in one region correlated with a corresponding single collective mode of the other.

In this paper we present two different frameworks from which the modewise decomposition of Gaussian states can be deduced, and discuss some of its implications. In the following section, we present this modewise decomposition in the form of a theorem applicable to arbitrary entangled pure Gaussian states and show how it follows from properties of the Schmidt decomposition. In section III we deal with the case of mixed states. Using the correspondence between correlation matrices and Gaussian states, the modewise decomposition implies a corresponding decomposition of covariance matrices. We therefore show how such a decomposition also holds for a certain class of “isotropic” Gaussian mixed states, defined from a corresponding symmetry of their covariance matrix.
II. MODewise Decomposition of Pure Gaussian States

To begin with, suppose a collection of $N$ canonical systems or “modes” is partitioned into two sets, i.e., Alice’s $A = \{ A_1, \ldots, A_m \}$ and Bob’s $B = \{ B_1, \ldots, B_n \}$, of sizes $m$ and $n$ respectively. If the quantum state of the modes is a pure Gaussian state $|\psi\rangle_{AB}$, the following theorem characterizes the entanglement between Alice and Bob:

**Theorem 1:** A Gaussian pure state $|\psi\rangle_{AB}$ for $m + n$ modes $A$ and $B$ may always be written as

$$|\psi\rangle_{AB} = |\bar{\psi}_1\rangle_{\bar{A}_1} |\bar{\psi}_2\rangle_{\bar{A}_2} |\bar{\psi}_3\rangle_{\bar{A}_3} \cdots |\bar{\psi}_s\rangle_{\bar{A}_s} |0\rangle_{\bar{A}_F} |0\rangle_{\bar{B}_F}$$  \hspace{1cm} (2)

for some $s \leq \min(m,n)$, where $\bar{A} = \{ \bar{A}_1, \ldots, \bar{A}_m \}$ and $\bar{B} = \{ \bar{B}_1, \ldots, \bar{B}_n \}$ are new sets of modes obtained from $A$ and $B$ respectively through local linear canonical transformations, the states $|\bar{\psi}_k\rangle$ are two-mode squeezed states of the form

$$|\bar{\psi}_k\rangle_{\bar{A}_k, \bar{B}_k} = \frac{1}{\sqrt{Z_k}} \sum_n e^{-\frac{1}{2} \bar{\beta}_k n^2} |n\rangle_{\bar{A}_k} |n\rangle_{\bar{B}_k}$$ \hspace{1cm} (3)

entangling the modes $\bar{A}_k$ and $\bar{B}_k$ for $k \leq s$, and $|0\rangle_{\bar{A}_F}$ and $|0\rangle_{\bar{B}_F}$ are products of oscillator ground states for the remaining modes in $\bar{A}$ and $\bar{B}$ respectively.

Before proving theorem 1, we first review some facts concerning the correspondence between Gaussian states and covariance matrices. Let us represent the canonical variables of a $k$-mode system by the vector

$$\eta = \eta_1 + \eta_2 + \cdots + \eta_k,$$ \hspace{1cm} (4)

where $\eta_i$ is the two component vector $\eta_i = (q_i, p_i)^T$, and assume throughout that $\langle \eta \rangle = 0$ for all states considered. A generally mixed Gaussian state $\rho$ describing a system of $k$-modes with $\langle \eta \rangle = 0$ is completely specified by its covariance matrix (CM), defined as

$$M = \text{Re} \langle \eta \eta^T \rangle.$$ \hspace{1cm} (5)

A unitary transformation on $\rho$ preserving the Gaussian character of the state implements a linear transformation of the modes $\vec{n} = S \eta$, known as a symplectic (or linear canonical) transformation $S \in Sp(2k, \mathbb{R})$. Such a transformation preserves the canonical structure of the commutation relations $[\eta, \eta^T] = iJ_{2k}$, where the $k$-mode symplectic matrix is given by

$$J_{2k} = \bigoplus_{i=1}^{k} J_2, \quad J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and satisfies $J_{2k}^2 = -I_{2k}$. Hence, a symplectic transformation preserves the symplectic structure under a similarity transformation, i.e.,

$$S J_{2k} S^T = J_{2k} \Rightarrow -(J_{2k} S)(J_{2k} S^T) = I_{2k}. \hspace{1cm} (6)$$

Under such a transformation, the state $\rho$ is brought to a new state $\tilde{\rho}$ with CM $\tilde{M} = SMST$. In particular, there exist symplectic transformations bringing the CM to the so-called Williamson normal form (WNF) \[ \tilde{M} = \lambda_1 I_2 + \lambda_2 I_2 + \cdots + \lambda_k I_2, \] (7)

where $\lambda_i$ are the non-negative eigenvalues of the matrix $iJ_k M$, also known as the symplectic eigenvalues. Expressed in the product Hilbert space corresponding to the new set of modes $\{ \vec{\eta} \}$ for $W = \text{Re} \langle \vec{\eta} \vec{\eta}^T \rangle$, the Gaussian state $\rho$ acquires the particularly simple form

$$\rho = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_k$$ \hspace{1cm} (8)

where $\rho_i$ is an oscillator thermal state for the $i$-th mode

$$\rho_i = \frac{1}{1 + \beta_i} \sum_n e^{-\beta_i n^2} |n\rangle_i \langle n|_i.$$ \hspace{1cm} (9)

Here, $\vec{n}_i = \vec{a}_i^T$, $\vec{a}_i$, is the number operator associated with $\eta_i = (\eta_{i1} + i\eta_{i2})/\sqrt{2}$, and $\beta_i$ is related to the symplectic eigenvalue $\lambda_i$ by $\beta_i = \ln [(\lambda_i + 1/2)/(\lambda_i - 1/2)]$. Note that as a consequence of the uncertainty principle, admissible Gaussian states satisfy the condition $\forall i, \lambda_i \geq \frac{1}{2}$, with pure Gaussian states when $\forall i, \lambda_i = \frac{1}{2}$. For $\lambda_i = 1/2$, $\rho_i = |0\rangle_i |0\rangle_i$ is obtained as the limit of [8] as $\beta_i \to \infty$.

We now proceed with the proof of theorem 1. The Schmidt decomposition [8] automatically yields the diagonal form of the partial density matrices for $A$ and $B$:

$$\rho_A = \sum_a p_a |\phi_a\rangle \langle \phi_a|, \quad \rho_B = \sum_a p_a |\chi_a\rangle \langle \chi_a|,$$ \hspace{1cm} (10)

which are seen to be of equal rank and spectrum thus showing that the $p_a$s are unique. The basis states $|\phi_a\rangle_A$ and $|\chi_a\rangle_B$ are also unique (up to phase factors) for non-degenerate $p_a$ and otherwise may be chosen to be elements of any orthonormal basis spanning the degenerate subspace. Now, if $|\psi\rangle_{AB}$ is Gaussian, then the reduced density matrices are also Gaussian. Thus, $\rho_A$ and $\rho_B$ can be written in the form [8] in terms of the set of modes bringing the local covariance matrices into WNF. Suppose that there are $s$ modes in $A$ and $t$ modes in $B$ with symplectic eigenvalue $\lambda \neq 1/2$. Since the remaining modes factor out from the respective density matrices as projection operators onto their ground state, we may factor $|\psi\rangle_{AB}$ as $|\bar{\psi}\rangle_{AB} |0\rangle_{\bar{A}_F} |0\rangle_{\bar{B}_F}$ where $|0\rangle_{\bar{A}_F}$ and $|0\rangle_{\bar{B}_F}$ are collective ground states onto the modes with $\lambda = \frac{1}{2}$ and $|\bar{\psi}\rangle_{AB}$ is the generally entangled state for the remaining modes, $\bar{A}_1, \ldots, \bar{A}_s$, and $\bar{B}_1, \ldots, \bar{B}_t$. Concentrate then on $|\bar{\psi}\rangle_{AB}$, the partial density matrices of which may be written as

$$\tilde{\rho}_A = \sum_{\vec{n}_A} e^{-\vec{\beta}_A \cdot \vec{n}_A} |\vec{n}_A\rangle \langle \vec{n}_A| \quad \text{and} \quad \tilde{\rho}_B = \sum_{\vec{n}_B} e^{-\vec{\beta}_B \cdot \vec{n}_B} |\vec{n}_B\rangle \langle \vec{n}_B|,$$
(β_1, ..., β_n)^T and \( \tilde{\beta}_B = (\beta_B, ..., \beta_B)^T \) represent the distribution of thermal parameters on each side. Now, by our previous discussion, both density matrices have the same rank and the same eigenvalues. This means that there must exist a one-to-one pairing between the occupation number distributions \( \tilde{n}_A \) and \( \tilde{n}_B \), and such that

\[
\tilde{\beta}_A \cdot \tilde{n}_A = \tilde{\beta}_B \cdot \tilde{n}_B .
\]

We now observe that the pairing \( \tilde{n}_A \leftrightarrow \tilde{n}_B \) is a homogeneous linear map, since \( \tilde{n}_A = 0 \) and \( \tilde{n}_B = 0 \) are paired (all \( \beta \)'s \( \not= 0 \)) and \( (\tilde{n}_A + \tilde{n}'_A, \tilde{n}_B + \tilde{n}'_B) \) satisfies (11) if \( (\tilde{n}_A, \tilde{n}_B) \) and \( (\tilde{n}'_A, \tilde{n}'_B) \) satisfy (11). However, if a linear map is one-to-one then the domain and range have the same dimensions. Thus we see that \( s = t \), in other words, the number of modes in \( A \) and \( B \) with symplectic eigenvalue different from \( 1/2 \) are the same. Now, label the modes on each side in ascending order of \( \beta \), so that

\[0 < \beta_{A1} \leq \beta_{A2} \leq ... \leq \beta_{B1} \leq \beta_{B2} \leq ... \leq \beta_{B_s} ;\]

Consider first the case \( \tilde{n}_A = (1, 0, 0)^T \), yielding the smallest non-zero value of \( \tilde{\beta} \cdot \tilde{n}_A \). By construction, this distribution must be paired with the smallest non-zero value of \( \tilde{\beta}_B \cdot \tilde{n}_B \), which is (or can be taken to be in the case of degenerate \( \tilde{n}_B \)) \( \tilde{n}_B = (1, 0, 0)^T \). We thus find that (11) has a solution provided that \( \beta_{A1} = \beta_{B1} \) (hence \( \lambda_{A1} = \lambda_{B1} \)), and by the linearity property we find for any \( \tilde{n}_A \), the map \( n_{A1} \rightarrow \tilde{n}_A = n_{A1} \). At this point we can repeat the procedure but applied to the subspace of the remaining modes, in other words, solve for a map between \( \tilde{n}'_A = (0, n_{A2}, ..., n_{A_{s-1}}) \) and \( \tilde{n}'_B = (0, n_{B2}, ..., n_{B_{s-1}}) \) such that \( \tilde{\beta}_A \cdot \tilde{n}'_A = \tilde{\beta}_B \cdot \tilde{n}'_B \). By a similar argument we find that \( \beta'_{A2} = \beta'_{B2} \) and \( n'_{A2} = n'_{B2} \). Iterating the procedure until all the components are exhausted, we find that the admissible solutions to (11) are \( \tilde{n}_A = \tilde{n}_B \) (with a freedom of re-ordering the labels of degenerate modes), provided that \( \tilde{\beta}_A = \tilde{\beta}_B \). Reconstructing the Schmidt decomposition of \( |\tilde{\psi}\rangle_{AB} \) from \( \rho_A \) and \( \rho_B \) we see that

\[
|\tilde{\psi}\rangle_{AB} = \frac{1}{Z} \sum \left( \sum_{s} e^{-\frac{1}{2} \tilde{\beta}_n |n\rangle |n}\right) = \left( \sum_{i=1}^{s} \left( \sum_{n} e^{-\frac{1}{2} \tilde{\beta}_n |n\rangle |n}\right) |n\rangle_{A_i} |n\rangle_{B_i} \right) .
\]

Thus, \( |\psi\rangle_{AB} = |\tilde{\psi}\rangle_{AB} |0\rangle_{A_F} |0\rangle_{B_F} \) is of the form (2).

### III. ISOTROPIC GAUSSIAN MIXED STATES

Although the previous result may be proved directly from general features of the Schmidt decomposition, it may also be embedded in the more general framework of mixed entangled Gaussian states. Theorem 1 can thus be seen to be a special case of a more general modewise decomposition theorem for a certain family of Gaussian mixed states. To define this family, we shall say that a covariance matrix is isotropic if there exists a symplectic transformation of the modes \( W = S M S^T \) with \( S \in Sp(2k, \mathbb{R}) \) that brings \( M \) to the form

\[
W = \lambda_0 I_{2k} , \quad \lambda_0 \geq \frac{1}{2} .
\]

An isotropic Gaussian state may thus be defined as a Gaussian state with an isotropic CM (An example of such a state would be the thermal state of a set of oscillator modes with degenerate frequencies). Note that all pure Gaussian states are isotropic with \( \lambda_0 = \frac{1}{2} \) (\( h = 1 \)).

The more general theorem is a consequence of the following Lemma concerning isotropic CMs:

**Lemma 1:** Let \( M \) be an isotropic CM for \( m + n \) modes

\[\eta = \eta_A \oplus \eta_B \text{ with symplectic eigenvalue } \lambda_0 .\]

Then there exist local symplectic transformations \( \tilde{\eta}_A = S_A^T \eta_A \) and \( \tilde{\eta}_B = S_B^T \eta_B \) such that upon appropriate pairing of the modes, the covariance matrix takes the form

\[
\tilde{M} = \tilde{M}_{\lambda_1 \lambda_1} \oplus \tilde{M}_{\lambda_2 \lambda_2} \oplus \ldots \oplus \tilde{M}_{\lambda_s \lambda_s} \oplus \lambda_0 I_{2(n+m-s)}
\]

for some \( s \leq \min(m, n) \), where \( \tilde{M}_{\lambda_i \lambda_i} \) is an isotropic correlation matrix for the two-mode sector \( \eta_{\lambda_i \lambda_i} = \eta_{\lambda_i} \oplus \eta_{\lambda_i} \) of the form

\[
\tilde{M}_{\lambda_i \lambda_i} = \begin{pmatrix}
\lambda_i & 0 & \kappa_i & 0 \\
0 & \lambda_i & 0 & -\kappa_i \\
\kappa_i & 0 & \lambda_i & 0 \\
0 & -\kappa_i & 0 & \lambda_i
\end{pmatrix} , \quad \kappa_i^2 = \lambda_i^2 - \lambda_0^2 .
\]

The diagonal elements \( \lambda_i \) in (13) are at the same time the symplectic eigenvalues of the local CMs \( M_A = \text{Re}(\eta_A \eta_A^T) \) and \( M_B = \text{Re}(\eta_B \eta_B^T) \) differing from \( \lambda_0 \), and the last block in (13) gives the CM for the remaining modes.

Given the correspondence between CMs and Gaussian mixed states, the following extension of Theorem 1 immediately follows:

**Theorem 2:** An isotropic Gaussian state \( \rho^{(0)}_{AB} \) of symplectic eigenvalue \( \lambda_0 \) for the \( m + n \) modes \( A \) and \( B \) may always be written in the form

\[
\rho^{(0)}_{AB} = \tilde{\rho}_{\lambda_1 \lambda_1} \otimes \tilde{\rho}_{\lambda_2 \lambda_2} \otimes \ldots \otimes \tilde{\rho}_{\lambda_s \lambda_s} \otimes \tilde{\rho}_{\lambda_0 \lambda_0} \otimes \tilde{\rho}_{\lambda_0 \lambda_0}
\]

where the new modes \( \{A_i\} \) and \( \{B_i\} \) are obtained by local symplectic transformations from \( A \) and \( B \), \( \tilde{\rho}_{\lambda_i \lambda_i} \) and \( \tilde{\rho}_{\lambda_0 \lambda_0} \) are mixed Gaussian two-mode states with CM of the form (13), and \( \tilde{\rho}_{\lambda_i \lambda_i} \) and \( \tilde{\rho}_{\lambda_0 \lambda_0} \) are mixed states for the remaining modes in \( A \) and \( B \) respectively with diagonal isotropic CM of symplectic eigenvalue \( \lambda_0 \).

To prove Lemma 1, first perform local symplectic transformations \( \tilde{\eta}_A \oplus \tilde{\eta}_B = (S_A \oplus S_B) \eta_A \oplus \eta_B \) bringing the local CMs \( M_A = \text{Re}(\eta_A \eta_A^T) \), \( M_B = \text{Re}(\eta_B \eta_B^T) \) into WNF. The total CM thus obtained may be written as

\[
\tilde{M} = \text{Re}(\eta \eta^T) = \begin{pmatrix}\tilde{W} & \tilde{K} \\ \tilde{K}^T & \tilde{W}_B \end{pmatrix} ,
\]
with \( W_A = \bigoplus_{i=1}^{m} \chi_{\lambda_i} \mathbb{I}_2 \) and \( W_B = \bigoplus_{i=1}^{n} \chi_{\eta_i} \mathbb{I}_2 \). We next note a useful fact regarding isotropic CMs. If (18) is satisfied, then \( M = \lambda_0 S S^T \) for some symplectic transformation \( S \). However, as is easily verified, \( S' = S^T \) is a symmetric symplectic transformation. Consequently, (19) implies that an isotropic \( k \)-mode CM satisfies:

\[
-(JM)^2 = \lambda_0^2 \mathbb{I}_{2k}.
\]

Replacing (17) into (18), and using the fact that \([W, J] = 0\), the following equations are obtained:

\[
W_A^2 - (J_n K)(J_m K^T) = \lambda_0^2 \mathbb{I}_{2m} \tag{19a}
\]

\[
W_B^2 - (J_n K^T)(J_m K) = \lambda_0^2 \mathbb{I}_{2m} \tag{19b}
\]

\[-W_A K + J_n K J_m W_B = 0. \tag{19c}
\]

Consider then a \( 2 \times 2 \) sub-block \( \tilde{K}_{ij} \equiv \langle \eta_i, \eta_j^T \rangle \) of \( K \).

Since \( W_A \) and \( W_B \) are diagonal, from equation (19c) we find that

\[
\lambda_{\lambda_i, \lambda_j} = \lambda_{\eta_i, \eta_j} J_2 \tilde{K}_{ij} J_2.
\]  

(20)

It is not hard to verify that unless \( \lambda_{\lambda_i, \lambda_j} = \lambda_{\eta_i, \eta_j} \), this equation has no solution for \( \tilde{K}_{ij} \) other than \( \tilde{K}_{ij} = 0 \). Thus we find that modes in \( A \) and modes in \( B \) with different symplectic eigenvalues are uncorrelated.

Next, let \( \tilde{\eta}_{\lambda_i} \) and \( \tilde{\eta}_{\eta_j} \) stand for the modes in \( A \) and \( B \) with the same local symplectic eigenvalue \( \lambda \), and group the modes according to their eigenvalues so that \( \tilde{M} \) takes the Jordan form \( \tilde{M} = \bigoplus_{\lambda} \tilde{M}_\lambda \) where each block \( \tilde{M}_\lambda \) is the CM for the degenerate eigenmodes \( \tilde{\eta}_{\lambda_i} \oplus \tilde{\eta}_{\eta_j} \). Concentrating on a given \( \lambda \), assume that \( g_A \) and \( g_B \) are the degeneracies of \( \lambda \) in the symplectic spectra of \( W_A \) and \( W_B \) respectively, so that \( \tilde{M}_\lambda \) may be written as

\[
\tilde{M}_\lambda = \left( \begin{array}{cc}
\mathbb{I}_{2 g_A} & \tilde{K}_{\lambda} \\
\tilde{K}_{\lambda}^T & \lambda \mathbb{I}_{2 g_B}
\end{array} \right).
\]  

(21)

Now note that \( \tilde{M}_\lambda \) is also an isotropic CM with symplectic eigenvalue \( \lambda_\sigma \). Substituting (21) into (18) therefore yields

\[
J_{2g_A} \tilde{K}_{\lambda} J_{2g_B} \tilde{K}_{\lambda}^T = (\lambda^2 - \lambda_\sigma^2) \mathbb{I}_{2g_A} \tag{22a}
\]

\[
J_{2g_B} \tilde{K}_{\lambda}^T J_{2g_A} \tilde{K}_{\lambda} = (\lambda^2 - \lambda_\sigma^2) \mathbb{I}_{2g_B} \tag{22b}
\]

\[
J_{2g_B} \tilde{K}_{\lambda} J_{2g_A} \tilde{K}_{\lambda}^T = \tilde{K}_{\lambda}. \tag{22c}
\]

Taking the trace of equations (22a) and (22b) and using the cyclic property of the trace \( \text{Tr}[J_{2g_B} \tilde{K}_{\lambda} J_{2g_A} \tilde{K}_{\lambda}^T] = \text{Tr}[J_{2g_B} \tilde{K}_{\lambda}^T J_{2g_A} \tilde{K}_{\lambda}] \), we obtain that \( (\lambda^2 - \lambda_\sigma^2)(g_A - g_B) = 0 \). Thus we see that the respective degeneracies of the symplectic eigenvalue \( \lambda \) in the local covariance matrices \( M_A \) and \( M_B \) must be the same for \( \lambda \neq \lambda_\sigma \). In such a case, let \( g_A = g_B = g \) and from equations (22) deduce that

\[
\tilde{K}_{\lambda} = J_{2g} \tilde{K}_{\lambda} J_{2g} \tilde{K}_{\lambda}^T = (\lambda^2 - \lambda_\sigma^2) J_{2g} \tag{23a}
\]

\[
J_{2g} \tilde{K}_{\lambda} J_{2g} \tilde{K}_{\lambda}^T = \tilde{K}_{\lambda}. \tag{23b}
\]

Next, define a matrix \( \beta = \oplus^3 \sigma_3 \), where \( \sigma_3 \) is the standard Pauli matrix, and hence satisfying \( \beta^2 = \mathbb{I}_{2g} \), \( \beta^T = \beta \), \( \{\beta, J_{2g}\} = 0 \), and \( \beta J_{2g} \beta = -J_{2g} \). Re-expressing \( K \) in terms of some other \( 2g \times 2g \) matrix \( \tilde{K}_\lambda \) as

\[
\tilde{K}_\lambda = \sqrt{\lambda^2 - \lambda_\sigma^2} \mathbb{I}_{2g} \tag{24}
\]

and substituting into (24) we find that \( \tilde{K}_\lambda \) must satisfy

\[
O_{\lambda} J_{2g} O_{\lambda}^T = J_{2g}, \quad O_{\lambda}^T O_{\lambda} = \mathbb{I}_{2g}; \tag{25}
\]

in other words, \( O_{\lambda} \) must be an orthogonal symplectic transformation. One can then perform a one-sided symplectic transformation, say \( O_{\lambda}^T \) on the \( A \)-modes, leaving the local CMs invariant in \( \tilde{M}_\lambda \) and bringing \( \tilde{K}_\lambda \) into the block diagonal form, i.e.,

\[
\tilde{K}_\lambda = O_{\lambda}^T \tilde{K}_\lambda = \sqrt{\lambda^2 - \lambda_\sigma^2} \bigoplus_i \sigma_3. \tag{26}
\]

In this way, we achieve a pair-wise decomposition of the degenerate subspace itself, where each pair has a covariance matrix of the form (8). Finally, we note that for the degenerate subspace associated with \( \lambda = \lambda_\sigma \), in which the degeneracies on each side are not restricted to be the same, equations (25) imply that \( \tilde{K}_\lambda \tilde{K}_\lambda^T = 0 \Rightarrow \tilde{K}_\sigma = 0 \). Therefore, local modes with symplectic eigenvalue \( \lambda_\sigma \) decouple, as expected from the pure case \( \lambda_\sigma = \frac{1}{2} \).

To conclude with, we note that it is known that for a \( 1 \times 1 \)-mode Gaussian mixed state, the Peres-Horodecki partial transpose criterion (11) is both necessary and sufficient (12) for entanglement and hence for distillability (13). In the present case of a two mode CM of the form (15), the partial transpose criterion implies that the state is entangled iff \( \lambda_i > \lambda_\sigma^2 + \frac{1}{2} \). Consequently, an isotropic Gaussian state is entangled and distillable iff at least one of the symplectic eigenvalues of its local CMs satisfies this condition.

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