Energy distribution of solutions to defocusing semi-linear wave equation in higher dimensional space *

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Abstract

The topic of this paper is a semi-linear, defocusing wave equation $u_{tt} - \Delta u = -|u|^{p-1}u$ in sub-conformal case in the higher dimensional space whose initial data are radical and come with a finite energy. We prove some decay estimates of the solutions if initial data decay at a certain rate as the spatial variable tends to infinity. A combination of this property with a method of characteristic lines give a scattering result if the initial data satisfy

$$E_\kappa (u_0, u_1) = \int_{\mathbb{R}^d} (|x|^{\kappa} + 1) \left( \frac{1}{2} |\nabla u_0(x)|^2 + \frac{1}{2} |u_1(x)|^2 + \frac{1}{p+1} |u_0(x)|^{p+1} \right) dx < +\infty.$$ 

Here $\kappa = \frac{(2-d)p+(d+2)}{p+1}$. 

1 Introduction

1.1 Background

In this work we consider the defocusing nonlinear wave equation in dimensions $d \geq 3$.

$$\begin{cases}
\partial_t^2 u - \Delta u = -|u|^{p-1}u, & (x, t) \in \mathbb{R}^d \times \mathbb{R}; \\
u(\cdot, 0) = u_0; \\
u_t(\cdot, 0) = u_1.
\end{cases}$$ (CP1)

The conserved energy is defined by

$$E (u, u_t) = \int_{\mathbb{R}^d} \left( \frac{1}{2} |\nabla u(x, t)|^2 + \frac{1}{2} |u_t(x, t)|^2 + \frac{1}{p+1} |u(x, t)|^{p+1} \right) dx.$$ 

Local theory Defocusing nonlinear wave equations

$$\partial_t^2 u - \Delta u = -|u|^{p-1}u, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}$$

have been extensively studied, especially in the 3 or higher dimensional space. The existence and uniqueness of solutions to semi-linear wave equation like (CP1) follows a combination of suitable Strichartz estimates and a fixed-point argument, Kapitanski [1] and Lindblad-Sogge [2] give more details.

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Conjecture 1.1. Any solution to (CP1) with initial data \((u_0, u_1) \in \dot{H}^{s_p} \times \dot{H}^{s_p-1}\) must exist for all time \(t \in \mathbb{R}\) and scatter in both two time directions.

This is still an open problem. Although there are many related results by different methods.

Scattering results with a priori estimates It has been proved that if a solution \(u\) with a maximal lifespan \(I\) satisfies an a priori estimate

\[
\sup_{t \in I} \|(u(\cdot, t), u_t(\cdot, t))\|_{\dot{H}^{s_p} \times \dot{H}^{s_p-1}(\mathbb{R}^d)} < +\infty,
\]

then \(u\) is defined for all time \(t\) and scatters. In fact, there are many works for different range of \(d\) and \(p\), sometimes with a radial assumption. Different methods were used for different range of \(d\) and \(p\). The details can be found in Kenig-Merle [4], Killip-Visan [14] (3 dimension), Killip-Visan [15] (all dimensions) for energy supercritical case and R.Shen [5], Dodson-Lawrie [6] (3 dimension), Rodriguez [16] (dimension 4 and 5) for energy subcritical case.

Strong Assumption on initial data There are multiple scattering results if we assume that the initial data satisfy stronger regularity and/or decay conditions. These results are usually proved via a suitable global space-time integral estimate. In \(d \geq 3\) case, if the initial data \((u_0, u_1)\) satisfy

\[
\int_{\mathbb{R}^d} (1 + |x|)^2 \left( \frac{1}{2} |\nabla u_0(x)|^2 + \frac{1}{2} |u_1(x)|^2 + \frac{1}{p+1} |u_0(x)|^{p+1}\right) < +\infty
\]

the conformal conservation law (see Ginibre-Velo [18] and Hidano [19]) leads to the scattering of solutions for \(1 + \frac{4}{d-1} \leq p < 1 + \frac{4}{d-2}\). In 3-dimension case, R.Shen [17] proved the scattering result for \(3 \leq p \leq 5\) if initial data \((u_0, u_1)\) are radial and satisfy

\[
\int_{\mathbb{R}^3} (|x|+1) \left( \frac{1}{2} |\nabla u_0|^2 + \frac{1}{2} |u_1|^2 + \frac{1}{p+1} |u_0|^{p+1}\right) dx < +\infty,
\]

here \(\kappa > \kappa_0(p) = \frac{5-p}{p+1}\). In \(d = 3, p = 3\) case, Dodson [10] gives a proof of the conjecture above for (CP1) with radial data.

But most of these results above are in the conformal case or super-conformal case.

1.2 Main tools

Next we introduce main tools of this paper. The first tool is still a Morawetz-type estimate, and the second tool is the method of characteristic lines.

Morawetz estimates This kind of estimates were first found by Morawetz [3] for wave/Klein-Gordon equations. Lin-Strauss [7] then generalized Morawetz estimates to Schrödinger equations. Colliander-Keel-Staffilani-Takaoka-Tao [8] introduced interaction Morawetz estimates for Schrödinger equations. Nowadays the Morawetz estimate has been one of the most important tools in the study of dispersive equations.

Method of characteristic lines R.Shen [9] generalizes the 3D method to higher dimensions. Let \(u\) be a radial solution to (CP1) with a finite energy, reduce the equation to a one-dimensional one by defining \(w(r, t) = r^{\frac{d-1}{2}} u(r, t)\), and considering the equation that \(w\) satisfies

\[
(\partial_t + \partial_r)(w_t - w_r) = \partial_r^2 w - \partial_t^2 w = -\frac{(d-1)(d-3)}{4} r^{\frac{d-1}{2}} u - r^{\frac{d-1}{2}} |u|^{p-1} u.
\]

This enable us to evaluate the variation of \(w_{\pm} w_r\) along characteristic lines \(t \mp r = Const\) and obtain plentiful information about the asymptotic behaviour of solutions.
1.3 Main Results

Now we give the main results of this work. Throughout this paper we always assume \( d \geq 3 \) and \( p < 1 + \frac{4}{d-1} \). In this case it is well known that if the initial data come with a finite energy, then the solution exist for all time. Please refer to Ginibre-Velo [21]. Our first result is about the energy distribution of solutions to \((CP1)\).

**Theorem 1.1.** Assume \( d \geq 3, 1 + \frac{4}{d-1} > p > 1 + \frac{2}{d-1} \). Let \( u \) be a solution to \((CP1)\) with a finite energy. Then

(a) The following limits hold as time tends to infinity

\[
\lim_{t \to \pm \infty} \int_{|x| < |t|} \frac{|t| - |x|}{|t|^2} e(x,t) \, dx = 0.
\]

(b) The inward/outward part of energy vanishes as time tends positive/negative infinity.

\[
\lim_{t \to \pm \infty} \int_{\mathbb{R}^d} \left( |u_r \pm u_t|^2 + |\nabla u|^2 + |u|^{p+1} \right) \, dx = 0.
\]

(c) Furthermore, if the initial data satisfy \( E_\kappa (u_0, u_1) < +\infty \) for a constant

\[
\kappa \in \left( 0, \left( \frac{d-1}{d} \right) - \frac{1}{2} \right), \quad \text{if } 1 + \frac{4}{d-1} > p > 1 + \frac{2}{d-1};
\]

then we have the following decay estimates

\[
\lim_{t \to \pm \infty} \int_{|x| < |t|} \frac{|t| - |x|}{|t|^{1-\kappa}} e(x,t) \, dx = 0;
\]

\[
\lim_{t \to \pm \infty} |t|^\kappa \int_{\mathbb{R}^d} \left( |u_r \pm u_t|^2 + |\nabla u|^2 + |u|^{p+1} \right) \, dx = 0.
\]

As an application of the theory on energy distribution, we also prove the following scattering result.

**Theorem 1.2.** Assume \( d \geq 3, 1 + \frac{4}{d-1} > p > \frac{3-d+2d-d+2}{d-1} \). Let \( u \) be a radial solution to \((CP1)\) with a finite energy and the initial data \((u_0, u_1)\) satisfy \( E_{\frac{2-d+2d-d+2}{p+1}} (u_0, u_1) < +\infty \), then the solution \( u \) scatters in both two time directions. More precisely, there exist two radial finite-energy free waves \( \tilde{u}^+, \tilde{u}^- \), so that

\[
\lim_{t \to \pm \infty} \left\| \left( \tilde{u}^+(\cdot, t) - u(\cdot, t) \right) - \tilde{u}^-(\cdot, t) \right\|_{\mathcal{H}^1 \times L^2(\mathbb{R}^d)} = 0.
\]

Here \( E_{\frac{2-d+2d-d+2}{p+1}} (u_0, u_1) = \int_{\mathbb{R}^d} \left( |x|^{\frac{(2-d+2d-d+2)}{p+1}} + 1 \right) \left( \frac{1}{2} |\nabla u_0(x)|^2 + \frac{1}{2} |u_1(x)|^2 + \frac{1}{p+1} |u_0(x)|^{p+1} \right) \, dx \).

1.4 The Structure of This Paper

This paper is organized as follows. We first give a few preliminary results in section 2. In section 3 we give a Morewetz identity and a Morawetz inequality, which is the main tool of this paper. Next in section 4 we prove the energy distribution properties of the solutions. Finally we prove the scattering theory of the solution \( u \) under an additional decay assumption in the last section.

2 Preliminary Results

**Notations** In this work we will use the notation \( e(x,t) \) for the energy density

\[
e(x,t) = \frac{1}{2} |\nabla u(x,t)|^2 + \frac{1}{2} |u_t(x,t)|^2 + \frac{1}{p+1} |u(x)|^{p+1}.
\]
We use \( u_r, \nabla u \) for the derivative in the radial direction and the covariant derivative on the sphere centred at the origin, respectively:

\[
  u_r(x, t) = \frac{x}{|x|} \cdot \nabla u(x, t); \quad \nabla u = \nabla u - u_r \frac{x}{|x|}; \quad |\nabla u|^2 = |u_r|^2 + |\nabla u|^2.
\]

We also define the weighted energy

\[
  E_A(u_0, u_1) = \int_{\mathbb{R}^d} \left( |x|^{s} + 1 \right) \left( \frac{1}{2} |\nabla u_0(x)|^2 + \frac{1}{2} |u_1(x)|^2 + \frac{1}{p+1} |u_0(x)|^{p+1} \right) dx.
\]

In this work \( \sigma_R \) represents the regular measure of the sphere \( \{ x \in \mathbb{R}^d : |x| = R \} \). We also define \( c_d \) to be the area of the united sphere \( S^{d-1} \). Thus we have the following identities for any radial function \( f(x) \)

\[
  \int_{|x|=r} f(x) r^{d-1} dr = c_d r^{d-1} f(0); \quad \int_{\mathbb{R}^d} f(x) dx = c_d \int_0^\infty f(r) r^{d-1} dr.
\]

The notation \( A \lesssim B \) means that there exists a constant \( c \), so that the inequality \( A \leq cB \) holds. We may also put subscript \( (s) \) to indicate that the constant \( c \) depends on the given subscript \( (s) \) but nothing else. In particular, the symbol \( \lesssim_1 \) is used if \( c \) is an absolute constant.

**Lemma 2.1.** *(Pointwise Estimate)* Assume \( d \geq 3 \). all radial \( \dot{H}^1(\mathbb{R}^d) \) functions \( u \) satisfy

\[
  |u(r)| \lesssim_d r^{-\frac{d-2}{2}} \|u\|_{\dot{H}^1}, \quad r > 0.
\]

If \( u \) also satisfy \( u \in L^{p+1}(\mathbb{R}^d) \), then its decay is stronger as \( r \to +\infty \).

\[
  |u(r)| \lesssim_d r^{-\frac{2(d-1)}{d+1}} \|u\|_{\dot{H}^1} \|u\|_{L^{p+1}}, \quad r > 0.
\]

This lemma has been known for many years, more details can be found, for example, in C.E. Kenig and F. Merle [4] for \( d = 3 \) and R. Shen [9] for \( d \geq 3 \).

**Lemma 2.2.** *(Radiation Field)* Assume that \( d \geq 3 \) and let \( u \) be a solution to the free wave equation \( \partial_t^2 u - \Delta u = 0 \) with initial data \((u_0, u_1) \in H^1 \times L^2(\mathbb{R}^d) \). Then

\[
  \lim_{t \to +\infty} \int_{\mathbb{R}^d} \left( |\nabla u(x, t)|^2 - |u_r(x, t)|^2 + \frac{|u(x, t)|^2}{|x|^2} \right) dx = 0,
\]

and there exists a unique function \( G_+ \in L^2(\mathbb{R} \times S^{d-1}) \) such that

\[
  \lim_{t \to +\infty} \int_0^\infty \int_{S^{d-1}} \left| \frac{d}{dt} \partial_r u(r \theta, t) - G_+(r - t, \theta) \right|^2 d\theta dr = 0;
\]

\[
  \lim_{t \to +\infty} \int_0^\infty \int_{S^{d-1}} \left| \frac{d}{dt} \partial_r u(r \theta, t) + G_+(r - t, \theta) \right|^2 d\theta dr = 0.
\]

In addition, the map

\[
  (u_0, u_1) \to \sqrt{2} G_+
\]

\[
  \dot{H}^1 \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R} \times S^{d-1})
\]

is a bijective isometry.

This result was known many years ago, please see Friedlander [11] [12]. Duyckaerts-Kenig-Merle [13] gives a proof for all dimensions \( d \geq 3 \).
3 Morawetz identity and Moroetz inequality

3.1 Morawetz identity

Proposition 3.1. (Morawetz identity) Let $u$ be a solution to (CP1) with a finite energy $E$. Then the following identity holds for any $R > 0$ and time $t_1 < t_2$.

\[
\frac{1}{2R}\int_{t_1}^{t_2} \int_{|x| < R} (|\nabla u|^2 + |u|^2 + \frac{(d-1)(p-1) - 2}{p+1} |u|^{p+1}) \, dx \, dt + \frac{d-1}{4R^2} \int_{t_1}^{t_2} \int_{|x|=R} |u|^2 \, d\sigma(x) \, dt
\]

\[
+ \int_{t_1}^{t_2} \int_{|x| > R} \left| \frac{|\nabla u|^2}{|x|} + \frac{(d-1)(p-1)}{2(p+1)} \frac{|u|^{p+1}}{|x|} + \frac{(d-3)(d-1)}{4} \frac{|u|^2}{|x|^3} \right| \, dx \, dt
\]

\[
+ \sum_{i=1,2} \int_{|x| < R} \left| \frac{1}{2} |u_x + \frac{d-1}{2} \frac{u}{|x|} + (-1)^i u_t |^2 + \left| \frac{|\nabla u|^2}{2} + \frac{|u|^{p+1}}{p+1} + \frac{(d-1)(d-3)}{8|x|^2} u(x,t)^2 \right| \right| \bigg|_{t=t_i} \bigg| \bigg| \, dx = 2E.
\]

Proof. We follow a similar argument to the given by Perthame and Vega in the final section of their work [20]. Let us first consider solutions with compact support. Given a constant $R$, we define two radian functions $\Psi$ and $\varphi$ by

\[
\nabla \Psi = \begin{cases} \nu, & \text{if } |x| \leq R; \\
R \nu/|x|, & \text{if } |x| \geq R; \end{cases} \quad \varphi = \begin{cases} 1/2, & \text{if } |x| \leq R; \\
0, & \text{if } |x| > R. \end{cases}
\]

Since $u$ is defined for all time $t$, we may also define a function on $R$

\[
\mathcal{E}(t) = \int_{\mathbb{R}^d} u_t(x,t) \bigg( \nabla u(x,t) \cdot \nabla \Psi + u(x,t) \left( \frac{\Delta \Psi}{2} - \varphi \right) \bigg) \, dx.
\]

We may differentiate $\mathcal{E}$, utilize the equation

\[
u_{tt} - \Delta u = -|u|^{p-1} u,
\]

apply integration by parts and obtain

\[
-\mathcal{E}'(t) = \int_{\mathbb{R}^d} \left( \sum_{i,j=1}^{d} u_i \Psi_{ij} u_j - \varphi \nabla u_i^2 + \varphi |u_t|^2 \right) \, dx + \frac{1}{4} \int_{\mathbb{R}^d} \nabla (|u|^2) \cdot \nabla (\Delta \Psi - 2\varphi) \, dx
\]

\[
+ \int_{\mathbb{R}^d} |u|^{p+1} \left( \frac{p-1}{2(p+1)} \Delta \Psi - \varphi \right) \, dx
\]

\[
= I_1 + I_2 + I_3.
\]

Here we have

\[
\Psi_{ij} = \begin{cases} \frac{\delta_{ij}}{|x|}, & \text{if } |x| < R; \\
\frac{R \delta_{ij}}{|x|^2}, & \text{if } |x| > R; \end{cases} \quad \Delta \Psi = \begin{cases} d, & \text{if } |x| < R; \\
R(d-1)/|x|, & \text{if } |x| > R; \end{cases}
\]

\[
\Delta \Psi - 2\varphi = \begin{cases} d - 1, & \text{if } |x| \leq R; \\
R(d-1)/|x|, & \text{if } |x| \geq R; \in C(\mathbb{R}^d). \end{cases}
\]

When $|x| > R$, we may calculate

\[
\sum_{i,j=1}^{d} u_i \Psi_{ij} u_j = \sum_{i,j=1}^{d} u_i \left( \frac{R \delta_{ij}}{|x|} - \frac{R x_i x_j}{|x|^3} \right) u_j = \frac{R}{|x|} \nabla u_i^2 - \frac{R |\nabla u| \cdot x_i}{|x|^3} u_j = \frac{R}{|x|} |\nabla u|^2.
\]

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Thus we have

$$I_1 = \frac{1}{2} \int_{|x| < R} \left( |\nabla u|^2 + |u_t|^2 \right) dx + R \int_{|x| > R} \frac{|\nabla u|^2}{|x|} dx.$$  \hspace{1cm} (4)

The last term in the equality above, a basic computation shows

$$I_3 = \frac{(d - 1)(p - 1) - 2}{2(p + 1)} \int_{|x| < R} |u|^{p+1} dx + \frac{(p - 1)(d - 1)R}{2(p + 1)} \int_{|x| > R} \frac{|u|^{p+1}}{|x|} dx.$$  \hspace{1cm} (5)

Let us calculate the left hand carefully

$$\begin{align*}
J_2 &= \frac{1}{4} \int_{\mathbb{R}^d} \nabla \left( |u|^2 \right) \cdot \nabla (\Delta \Psi - 2\varphi) dx \\
&= \frac{1}{4} \int_{|x| > R} \nabla \left( |u|^2 \right) \cdot \frac{-R(d - 1)x}{|x|^3} dx \\
&= \frac{1}{4} \int_{|x| > R} \left[ \text{div} \left( |u|^2 \cdot \frac{-R(d - 1)x}{|x|^3} \right) + (d - 3)(d - 1) \frac{R}{|x|^3} |u|^2 \right] dx \\
&= \frac{d - 1}{4R} \int_{|x| = R} |u|^2 d\sigma_R(x) + \frac{(d - 3)(d - 1)}{4} \int_{|x| > R} \frac{|u|^2}{|x|^3} dx.
\end{align*}$$  \hspace{1cm} (6)

Since $-\mathcal{E}'(t) = I_1 + I_2 + I_3$, we have

$$\int_{t_1}^{t_2} (I_1 + I_2 + I_3) dt = \mathcal{E} (t_1) - \mathcal{E} (t_2).$$  \hspace{1cm} (7)

We rewrite in the form of

$$\begin{align*}
\mathcal{RE} (t_1) &= \int_{\mathbb{R}^d} R u_t(x, t) \left( \nabla u(x, t) \cdot \nabla \Psi + u(x, t) \left( \frac{\Delta \Psi}{2} - \varphi \right) \right) dx \\
&= \frac{1}{2} \int_{\mathbb{R}^d} \left( R^2 |u_t|^2 + |\nabla u(x, t_1)|^2 + \left| \nabla u(x, t_1) \cdot \nabla \Psi + u(x, t_1) \left( \frac{\Delta \Psi}{2} - \varphi \right) \right|^2 \right) dx \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^d} \left| \nabla u(x, t_1) \cdot \nabla \Psi + u(x, t_1) \left( \frac{\Delta \Psi}{2} - \varphi \right) - R u_t(x, t_1) \right|^2 dx \\
&= J_1 - J_2.
\end{align*}$$  \hspace{1cm} (8)

Then we calculate $J_1, J_2$

$$\begin{align*}
J_1 &= \frac{1}{2} \int_{\mathbb{R}^d} \left( R^2 |u_t|^2 + |\nabla u \cdot \nabla \Psi|^2 + \left( \frac{\Delta \Psi}{2} - \varphi \right) \nabla (|u|^2) \cdot \nabla \Psi + \left( \frac{\Delta \Psi}{2} - \varphi \right)^2 |u|^2 \right) dx \\
&= \frac{1}{2} \int_{\mathbb{R}^d} \left( R^2 |u_t|^2 + |\nabla u \cdot \nabla \Psi|^2 - \text{div} \left( \left( \frac{\Delta \Psi}{2} - \varphi \right) \nabla \Psi \right) |u|^2 + \left( \frac{\Delta \Psi}{2} - \varphi \right)^2 |u|^2 \right) dx.
\end{align*}$$  \hspace{1cm} (9)

A basic calculation shows

$$\text{div} \left( \left( \frac{\Delta \Psi}{2} - \varphi \right) \nabla \Psi \right) = \begin{cases} 
\frac{d(d - 1)/2}{2|x|^2}, & \text{if } |x| < R; \\
\frac{(d - 1)(d - 2)R^2}{2|x|^2}, & \text{if } |x| > R.
\end{cases}$$  \hspace{1cm} (10)
Thus we have
\[ J_1 = \frac{1}{2} \int_{|x|<R} \left[ R^2 |u|^2 + |x \cdot \nabla u|^2 + \frac{1-d^2}{4} |u|^2 \right] dx 
+ \frac{1}{2} \int_{|x|>R} \left[ R^2 |u|^2 + R^2 |u_r|^2 + \frac{R^2(3-d)|u|^2}{4|x|^2} \right] dx 
- \frac{R^2}{2} \int_{|x|<R} \left[ \frac{R^2 - |x|^2}{2R^2} |u|^2 + \frac{1}{2} \left( R^2 - \frac{|x|^2}{2} \right) |u_r|^2 + \frac{R^2(3-d)|u|^2}{4|x|^2} \right] dx 
+ \frac{(d-1)(3-d)R^2}{8} \int_{|x|>R} \frac{|u|^2}{|x|^2} dx - \frac{R^2}{2} \int_{|x|>R} \left( \frac{1}{2} \frac{\nabla u}{|u|} + \frac{1}{p+1} |u_r|^2 \right) dx. \]

(11)

In addition we have
\[ J_2 = \frac{1}{2} \int_{|x|<R} \left| x \cdot \nabla u + \frac{d-1}{2} u - Ru_t \right|^2 dx + \frac{R^2}{2} \int_{|x|>R} \left| \frac{x}{|x|} \cdot \nabla u + \frac{d-1}{2} \frac{u}{|x|} - u_t \right|^2 dx. \]

(12)

Combining \( J_1, J_2 \), we obtain
\[ RE(t_1) = R^2 E - R^2 \int_{|x|>R} \left[ \frac{1}{2} \left( u_r + \frac{d-1}{2} \frac{u}{|x|} - u_t \right)^2 + \frac{\nabla u}{2} + \frac{|u_r|}{2} + \frac{|u|^{p+1}}{p+1} + \frac{(d-1)(d-3)|u|^2}{8|x|^2} \right] dx 
- R^2 \int_{|x|<R} \left[ \frac{R^2 - |x|^2}{2R^2} |u|^2 + \frac{1}{2} \frac{|x|}{R} u_r + \frac{d-1}{2} \frac{u}{2} - u_t \right]^2 + \frac{\nabla u}{2} + \frac{|u_r|}{2} + \frac{|u|^{p+1}}{p+1} + \frac{(d-1)|u|^2}{8R^2} dx. \]

Finally we find a similar expression of \(-RE(t_2)\)
\[ -RE(t_2) = R^2 E - R^2 \int_{|x|>R} \left[ \frac{1}{2} \left( u_r + \frac{(d-1)u}{2|x|} + u_t \right)^2 + \frac{\nabla u}{2} + \frac{|u_r|}{2} + \frac{|u|^{p+1}}{p+1} + \frac{(d-1)(d-3)|u|^2}{8|x|^2} \right] dx 
- R^2 \int_{|x|<R} \left[ \frac{R^2 - |x|^2}{2R^2} |u|^2 + \frac{1}{2} \frac{|x|}{R} u_r + \frac{(d-1)u}{2R} + u_t \right]^2 + \frac{\nabla u}{2} + \frac{|u_r|}{2} + \frac{|u|^{p+1}}{p+1} + \frac{(d-1)|u|^2}{8R^2} dx. \]

Then plug all the expressions of \( I_1, I_2, I_3 \) and \(-RE(t_2), -RE(t_2)\) into the integral identity to finish the proof if \( u \) is compactly supported. In order to deal with the general solution \( u \), we fix a smooth radial cut-off function \( \phi: \mathbb{R}^2 \to [0, 1] \) so that
\[ \phi(x) = \begin{cases} 1, & \text{if } |x| \leq 1; \\ 0, & \text{if } |x| > 2; \end{cases} \]
define initial data \((u_{0,R'}(x), u_{1,R'}(x)) = \phi(x/R') (u(x,t_1), u_t(x,t_1))\) and consider the corresponding solution \( u_{R'} \) to (CP1). The argument above shows that \( u_{R'} \) satisfies Morawetz identity. We observe
1) The identity \( u_{R'}(x,t) = u(x,t) \) holds if \(|x| < R' + t_1 - t\) by finite speed of propagation;
2) \( E(u_{0,R'}, u_{1,R'}) \to E \) as \( R' \to \infty \).
3) The energies of \( u_{R'} \) and \( u \) in the region where \( u_{R'} \neq u \) both converge to zero as \( R' \to +\infty \) by finite speed of propagation and energy conservation law.
These facts enable us to take the limit \( R' \to +\infty \) and prove Morawetz identity for general solutions \( u \).

\[ \square \]

3.2 Morawetz inequalities

A combination of Morawetz identity and finite speed of propagation gives a few useful inequalities, which is the main tool of this paper. The key observation here is that if \( R \) is large, the first term in the Morawetz identity is almost \( 2E \) when \( t_1 \leq -R \) and \( t_2 \leq R \), thus all other terms must be small.
Corollary 3.2. Let $u$ be a solution to (CP1) with initial data $(u_0, u_1) \in \left( \dot{H}^1 (\mathbb{R}^d) \cap L^{p+1} (\mathbb{R}^d) \right) \times L^2 (\mathbb{R}^d)$. Given any $R > 0, r \geq 0$ we have

$$\sum_{j=1}^{6} M_j \leq \int_{\mathbb{R}^d} \min \{|x|/R, 1\} \left( |\nabla u_0|^2 + |u_1|^2 + \frac{2}{p+1} |u_0|^{p+1} \right) dx.$$

The notations $M_j$ are defined by

$$M_1 = \frac{1}{2R} \int_{|x|<|R+r+1|} \int_{|x|<R} \left( |\nabla u|^2 + |u_t|^2 + \frac{(d-1)(p-1) - 2}{p+1} |u|^{p+1} \right) dxdt;$$

$$M_2 = \frac{d-1}{2(p+1)R} \int_{|x|=R} |u_t|^2 dxdt;$$

$$M_3 = \frac{d-1}{4R^2} \int_{-R-r}^{R+r} \int_{|x|=R} |u|^2 d\sigma(x) dt;$$

$$M_4 = \int_{-R-r}^{R+r} \int_{|x|>R} \left( \frac{\nabla u^2}{|x|} + \frac{(d-1)(p-1) |u|^{p+1}}{2(p+1)} |x| + \frac{(d-3)(d-1) |u|^2}{4} \right) dxdt;$$

$$M_5 = \sum_{\pm} \int_{|x|<R} \left( \frac{R^2 - |x|^2}{2R^2} |u_t|^2 + \frac{1}{2} \left| \frac{|x|}{R} u_t + \frac{(d-1)u}{2R} \pm u_t \right|^2 + \frac{(d-1)u_t^2}{2} + \frac{|u_t|^2}{8} + \frac{|u_t|^{p+1}}{2(p+1)} \right)_{t=\pm(R+r)} dx;$$

$$M_6 = \sum_{\pm} \int_{|x|>R} \left( \frac{1}{2} |u_t + \frac{(d-1)u}{2x} \pm u_t|^2 + \frac{|\nabla u|^2}{2} + \frac{1}{p+1} |u|^{p+1} + \frac{(d-1)(d-3) |u|^2}{8} \right)_{t=\pm(R+r)} dx.$$

Proof. We first choose $t_1 = -R - r, t_2 = R + r$ in the Morawetz identity, the first term above can be written as a sum of three terms

$$\frac{1}{2R} \int_{-R-r}^{R+r} \int_{|x|<R} \left( |\nabla u|^2 + |u_t|^2 + \frac{(d-1)(p-1) - 2}{p+1} |u|^{p+1} \right) dx dt$$

$$= M_1 + \frac{1}{2R} \int_{-R}^{R} \int_{|x|<R} \left( |\nabla u|^2 + |u_t|^2 + \frac{(d-1)(p-1) - 2}{p+1} |u|^{p+1} \right) dx dt$$

$$= M_1 + M_2 + \frac{1}{2R} \int_{-R}^{R} \int_{|x|<R} \left( |\nabla u|^2 + |u_t|^2 + \frac{2}{p+1} |u|^{p+1} \right) dx dt.$$

Thus

$$\frac{1}{2R} \int_{-R}^{R} \int_{|x|<R} \left( |\nabla u|^2 + |u_t|^2 + \frac{2}{p+1} |u|^{p+1} \right) dx dt + \sum_{j=1}^{6} M_j = 2E. \quad (13)$$

In order to prove the first inequality we only need to show

$$I = 2E - \frac{1}{2R} \int_{-R}^{R} \int_{|x|<R} \left( |\nabla u|^2 + |u_t|^2 + \frac{2}{p+1} |u|^{p+1} \right) dx dt$$

$$\leq \int_{\mathbb{R}^d} \min \{|x|/R, 1\} \left( |\nabla u_0|^2 + |u_1|^2 + \frac{2}{p+1} |u_0|^{p+1} \right) dx.$$
This follows energy conservation law and finite speed of propagation of energy

\[
I = \frac{1}{R} \int_{-R}^{R} \int_{|x| > R} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |u_t|^2 + \frac{1}{p+1} |u|^{p+1} \right) \, dx \, dt
\]

\[
\leq \frac{1}{R} \int_{-R}^{R} \int_{|x| > R - |t|} \left( \frac{1}{2} |\nabla u_0|^2 + \frac{1}{2} |u_1|^2 + \frac{1}{p+1} |u_0|^{p+1} \right) \, dx \, dt
\]

\[
= \frac{1}{R} \int_{\mathbb{R}^d} \min\{||x|, R\} \left( |\nabla u_0|^2 + |u_1|^2 + \frac{2}{p+1} |u_0|^{p+1} \right) \, dx.
\]

\[\square\]

4 Energy Distribution

In this section we prove Theorem 1.1. It suffices to consider the positive time direction \( t > 0 \), since the wave equation is time-reversible.

We choose \( R = t, r = 0 \) in corollary 3.3. The following inequalities hold for large time \( t > 0 \).

\[
\sum_{\pm} \int_{|x| < t} \left( \frac{t^2 - |x|^2}{2t^2} |u_r|^2 + \frac{1}{2} \frac{|x|}{t} u_r + \frac{(d-1)u}{2t} \pm u_t \right)^2 + \frac{(d^2 - 1) |u_t|^2}{8t^2} + \frac{\nabla u_t^2}{2} + \frac{|u|^{p+1}}{p+1} \, dx
\]

\[
+ \sum_{\pm} \int_{|x| > t} \left( \frac{1}{2} u_r + \frac{(d-1)u}{2|x|} \pm u_t \right)^2 + \frac{|\nabla u|^2}{2} + \frac{1}{p+1} |u|^{p+1} + \frac{(d-1)(d-3) |u_t|^2}{8} \, dx
\]

\[
\leq \frac{4 - (d-1)(p-1)}{2(p+1)t} \int_{-t}^{t} \int_{|x| < t} |u(x, t')|^{p+1} \, dx \, dt' + \int_{\mathbb{R}^d} \min\{|x|/R, 1\} \left( |\nabla u_0|^2 + |u_1|^2 + \frac{2}{p+1} |u_0|^{p+1} \right) \, dx.
\]

We observe that

\[
|\nabla u|^2 + |u_t|^2 \lesssim_1 |u_r|^2 + \frac{|x|}{t} u_r + \frac{(d-1)u}{2t} + u_t \, dx
\]

and

\[
|u_r + u_t|^2 \lesssim_1 \frac{(t - |x|)^2}{t^2} |u_r|^2 + \frac{|x|}{t} u_r + \frac{(d-1)u}{2t} + u_t \, dx, \quad \text{if } |x| < t;
\]

\[
|u_r + u_t|^2 \lesssim_1 \frac{1}{2} u_r + \frac{(d-1)u}{2|x|} + u_t \, dx + \frac{(d-1)(d-3) |u_t|^2}{8 |x|^2}, \quad \text{if } |x| > t.
\]

Then obtain

\[
\frac{1}{p+1} \int_{\mathbb{R}^d} (|u(x, t)|^{p+1} + |u(x, t^+)|^{p+1}) \, dx + c_1 \sum_{\pm} \int_{|x| < t} \frac{t - |x|}{t} \left( |\nabla u(x, \pm t)|^2 + |u_t(x, \pm t)|^2 \right) \, dx
\]

\[
+ c_2 \sum_{\pm} \int_{\mathbb{R}^d} \left( (u_r \pm u_t)(x, \pm t)^2 + |\nabla u(x, \pm t)|^2 \right) \, dx
\]

\[
\leq \frac{4 - (d-1)(p-1)}{2(p+1)t} \int_{-t}^{t} \int_{|x| < t} |u(x, t')|^{p+1} \, dx \, dt' + 2 \int_{\mathbb{R}^d} \min\{|x|/t, 1\} c(x, 0) \, dx.
\]

Here \( c_1, c_2 > 0 \) are absolute constants. For convenience we introduce the notation

\[
Q(t) = \frac{1}{p+1} \int_{\mathbb{R}^d} (|u(x, t)|^{p+1} + |u(x, t^+)|^{p+1}) \, dx + c_1 \sum_{\pm} \int_{|x| < t} \frac{t - |x|}{t} \left( |\nabla u(x, \pm t)|^2 + |u_t(x, \pm t)|^2 \right) \, dx
\]

\[
+ c_2 \sum_{\pm} \int_{\mathbb{R}^d} \left( (u_r \pm u_t)(x, \pm t)^2 + |\nabla u(x, \pm t)|^2 \right) \, dx.
\]
Then the inequality above implies that \( Q(t) \) satisfies the recurrence formula

\[
Q(t) \leq \frac{\lambda}{t} \int_0^t Q(t') dt' + 2 \int_{\mathbb{R}^d} \min\{|x|/t, 1\} e(x, 0) dx.
\]

Here

\[0 < \lambda = \frac{4 - (d - 1)(p - 1)}{2} < 1.\]

**Proof of part (a)(b)** We may rewrite the recurrence formula as

\[
Q(t) \leq \frac{\lambda}{t} \int_0^t Q(t') dt' + o(1).
\]

We may take upper limits of both sides and obtain an inequality

\[
\limsup_{t \to +\infty} Q(t) \leq \limsup_{t \to +\infty} \frac{\lambda}{t} \int_0^t Q(t') dt' \leq \lambda \limsup_{t \to +\infty} Q(t).
\]

We recall the fact \( \lambda \in (0, 1) \) and observe that \( Q(t) \lesssim E \) is uniformly bounded, therefore we have

\[
\limsup_{t \to +\infty} Q(t) = 0.
\]

This verifies (a)(b).

**Proof of part (c)** Now we assume that initial data satisfy additional decay assumption. We start by multiplying both sides by \( t^{k-1} \) and integrate from \( t = 1 \) to \( t = T \), utilize our assumption on initial data, then obtain

\[
\int_1^T t^{k-1} Q(t) dt \leq \int_1^T t^{k-1} \left( \frac{\lambda}{t} \int_0^t Q(t') dt' \right) dt + C_{\kappa} \int_{\mathbb{R}^d} \min\{|x|, |x|^\kappa\} e(x, 0) dx
\]

\[
\leq \frac{\lambda}{1 - \kappa} \int_0^T \min\{(t')^{k-1}, 1\} Q(t') dt' + C_{\kappa} \int_{\mathbb{R}^d} \min\{|x|, |x|^\kappa\} e(x, 0) dx
\]

\[
\leq \frac{\lambda}{1 - \kappa} \int_1^T (t')^{k-1} Q(t') dt' + C_{\kappa} \int_{\mathbb{R}^d} \min\{|x|, |x|^\kappa\} e(x, 0) dx + C_k E.
\]

Here

\[
\frac{\lambda}{1 - \kappa} = \frac{4 - (d - 1)(p - 1)}{2(1 - \kappa)} < 1,
\]

since we have assumed \( \kappa < \frac{(d-1)(p-1)}{2} \). Therefore we have

\[
\int_1^T t^{k-1} Q(t) dt \lesssim_{\rho, \kappa} \int_{\mathbb{R}^d} \min\{|x|, |x|^\kappa\} e(x, 0) dx + E.
\]

Because neither the right hand side nor the implicit constant here depends on \( T \), we make \( T \to +\infty \) to conclude

\[
\int_1^\infty t^{k-1} Q(t) dt < +\infty.
\]

Combining this with the fact \( Q(t) \lesssim E \), we have

\[
\int_0^\infty t^{k-1} Q(t) dt < +\infty.
\]

We may multiply both sides of the recurrence formula by \( t^\kappa \):

\[
t^\kappa Q(t) \leq \lambda \int_0^t t^{k-1} Q(t') dt' + 2 \int_{\mathbb{R}^d} \min\{|x|/t^{\kappa-1}, t^\kappa\} e(x, 0) dx.
\]

Finally we apply dominated convergence theorem to finish the proof of Theorem 1.1.
Remark 4.1. When \( d > 3 \), we have
\[
|u_r + u_t|^2 \lesssim \left| u_r + \frac{(d-1)u}{2|x|} + u_t \right|^2 + \frac{(d-1)(d-3)}{8} \frac{|u|^2}{|x|^2}, \quad \text{if } |x| > t.
\]
But the term \( \frac{(d-1)(d-3)}{8} \frac{|u|^2}{|x|^2} = 0 \) when \( d = 3 \), so the inequality above does not hold. We can redefine
\[
Q(t) = \frac{1}{p+1} \int_{\mathbb{R}^d} \left( |u(x,t)|^{p+1} + |u(x,-t)|^{p+1} \right) dx + c_1 \sum_{\pm} \int_{|x|<t} \frac{t-|x|}{t} \left( |\nabla u(x, \pm t)|^2 + |u_t(x, \pm t)|^2 \right) dx
+ c_2 \sum_{\pm} \int_{|x|\geq t} \left( |u_r + (d-1)u| \pm u_t \right)^2 + |\nabla u(x, \pm t)|^2 \right) dx.
\]
We also have
\[
\limsup_{t \to +\infty} Q(t) = 0,
\]
and
\[
t^\kappa Q(t) \leq \lambda \int_0^t t^{\kappa-1} Q(t') dt' + 2 \int_{\mathbb{R}^d} \min\{|x|/t^{\kappa-1}, t^\kappa\} e(x,0) dx.
\]
Because
\[
|u_r + u_t|^2 \lesssim \left| u_r + \frac{(d-1)u}{2|x|} + u_t \right|^2 + c \frac{|u|^2}{|x|^2}, \quad \text{if } |x| > t.
\]
We can finish the proof of Theorem 1.1 by an estimate.
\[
\int_{|x|>t} \frac{|u|^2}{|x|^2} dx \lesssim \left( \int_{|x|>t} (|u|^2)^{\frac{\kappa+5}{\kappa+1}} dx \right)^{\frac{\kappa+1}{\kappa+5}} \left( \int_{|x|>t} (|x|^{-2})^{\frac{\kappa+5}{\kappa+1}} dx \right)^{\frac{\kappa+5}{\kappa+1}}.
\]
When \( d = 3 \), we have \( p \in (2,3) \) and the inequality
\[
t^{\frac{\kappa+5}{\kappa+1}} \ll t^{-\kappa},
\]
thus
\[
\int_{|x|>t} \frac{|u|^2}{|x|^2} dx \ll t^{-\kappa}.
\]

5 Scattering Theory

5.1 Transformation to 1D

In order to take full advantage of our radial assumption, we use the following transformation: if \( u \) is a radial solution to (CP1), then \( w(r,t) = r^{\frac{d-4}{4}} u(r,t) \), where \( |x| = r \), is a solution to one-dimensional wave equation
\[
(\partial_t^2 + \partial_r^2) (w_t - w_r) = \partial_r^2 w - \partial_r^2 w = -\frac{(d-1)(d-3)}{4} \frac{r^{d-4}}{r^4} u - r^{d-4} |u|^{p-1} u.
\]
We define
\[
v_+(r,t) = w_t(r,t) - w_r(r,t); \quad v_-(r,t) = w_t(r,t) + w_r(r,t).
\]
A simple calculation shows that \( v_\pm \) satisfy the equation
\[
(\partial_t \pm \partial_r) v_\pm(r, t) = \partial_r^2 w - \partial_r^2 w = -\frac{(d - 1)(d - 3)}{4} r^{\frac{d - 5}{2}} u - r^{\frac{d - 1}{2}} |u|^{p - 1} u.
\]
This gives variation of \( v_\pm \) along characteristic lines \( t \pm r = \text{Const.} \)
\[
v_+ (t_2 - \eta, t_2) - v_+ (t_1 - \eta, t_1) = \int_{t_1}^{t_2} f(t - \eta, t) dt, \quad t_2 > t_1 > \eta; \quad (15)
v_- (s - t_2, t_2) - v_- (s - t_1, t_1) = \int_{t_1}^{t_2} f(s - t, t) dt, \quad t_1 < t_2 < s. \quad (16)
\]
Here the function \( f(r, t) \) is defined by
\[
f(r, t) = -\frac{(d - 1)(d - 3)}{4} r^{\frac{d - 5}{2}} u(r, t) - r^{\frac{d - 1}{2}} |u|^{p - 1} u(r, t).
\]
Then we give the upper bounds of the integral above. According to Lemma 2.1 we have
\[
\int_{t_1}^{t_2} (t - \eta)^{\frac{d - 5}{2}} |u(t - \eta, t)| dt
\leq \left\{ \int_{t_1}^{t_2} \left[ (t - \eta)^{\frac{d - 3}{2}} |u(t - \eta, t)| \right]^2 dt \right\}^{1/2} \left\{ \int_{t_1}^{t_2} \left[ (t - \eta)^{-1/2} \right]^2 dt \right\}^{1/2}
\leq \left\{ \int_{\eta}^{t_2} (t - \eta)^{d - 3} |u(t - \eta, t)|^2 dt \right\}^{1/2} (t_2 - t_1)^{-1/2}
\lesssim_2 E^{1/2} (t_1 - \eta)^{-1/2}.
\]
In addition we have
\[
\int_{t_1}^{t_2} (t - \eta)^{\frac{d - 5}{2}} |u(t - \eta, t)|^p dt
\leq \left\{ \int_{t_1}^{t_2} \left[ (t - \eta)^{\frac{(d - 1)p}{p + 1}} |u(t - \eta, t)|^p \right]^\frac{p + 1}{p} dt \right\}^{\frac{p}{p + 1}} \left\{ \int_{t_1}^{t_2} \left[ (t - \eta)^{\frac{(d - 1)(p - 1)}{2(p + 1)}} \right]^{p + 1} dt \right\}^{\frac{1}{p + 1}}
\leq \left\{ \int_{\eta}^{t_2} (t - \eta)^{d - 1} |u(t - \eta, t)|^{p + 1} dt \right\}^{\frac{p}{p + 1}} \left\{ \int_{t_1}^{t_2} (t - \eta)^{-\frac{(d - 1)(p - 1)}{2(p + 1)}} dt \right\}^{\frac{1}{p + 1}}
\lesssim_2 E^{\frac{d - 1}{2}} (t_1 - \eta)^{-\frac{(d - 1)(p - 1) - 2}{2(p + 1)}}.
\]
we combine these estimates above with (15) and (16) to obtain

**Lemma 5.1.** Let \( u \) be a radial solution wave equation with a finite energy \( E \). Then we have
\[
|v_+ (t_2 - \eta, t_2) - v_+ (t_1 - \eta, t_1)| \lesssim_2 E^{1/2} (t_1 - \eta)^{-1/2} + E^{\frac{p}{p + 1}} (t_1 - \eta)^{-\beta(d, p)/2};
\]
\[
|v_- (s - t_2, t_2) - v_- (s - t_1, t_1)| \lesssim_2 E^{1/2} (s - t_2)^{-1/2} + E^{\frac{p}{p + 1}} (s - t_2)^{-\beta(d, p)/2}.
\]
Here \( \beta(d, p) = \frac{(d - 1)(p - 1) - 2}{p + 1} \).

**5.2 Scattering by energy decay**

In this section we prove Theorem 1.2. Let us recall the lemma 5.1, we obtain there exists a function \( g_+(\eta) \in L^2(\mathbb{R}) \) with \( \|g_+\|^2_{L^2(\mathbb{R})} \leq E/c_d, \) so that
\[
v_+ (t - \eta, t) \to 2g_+(\eta) \quad \text{in} \quad L^2_{loc}(\mathbb{R}), \quad \text{as} \quad t \to +\infty.
\]
The asymptotic behaviour of $v_-$ is similar as $t \to -\infty$

$$v_-(s - t, t) \to 2g_-(s) \quad \text{in } L^2_{\text{loc}}(\mathbb{R}), \quad \text{as } t \to -\infty,$$

let $t_2 \to +\infty$ in the first inequality of lemma 5.1, we have

$$|2g_+(\eta) - v_+(t - \eta, t)| \lesssim_{d,p,E} (t - \eta)^{-\beta(d,p)/2}, \quad \eta < t - 1.$$

We apply a change of variable $r = t - \eta$ and rewrite this in the form

$$|v_+(r, t) - 2g_+(t - r)| \lesssim_{d,p,E} r^{-\beta(d,p)/2}, \quad r > 1.$$

Similarly we have

$$|v_-(r, t) - 2g_-(t + r)| \lesssim_{d,p,E} r^{-\beta(d,p)/2}, \quad r > 1.$$

These gives the following upper limits

$$\limsup_{t \to +\infty} \int_{t - c|\beta(d,p)|}^{t + R} \left( |v_+(r, t) - 2g_+(t - r)|^2 + |v_-(r, t) - 2g_-(t + r)|^2 \right) dr \lesssim_{d,p,E} c.$$  \hfill (17)

We ignore $g_-(t + r)$ in the upper limits above because

$$\lim_{t \to +\infty} \int_0^\infty |g_-(t + r)|^2 dr = \lim_{t \to +\infty} \int_0^\infty |g_-(s)|^2 ds = 0.$$

We recall $v_{\pm} = w_t \mp w_r$ and rewrite the upper limits above in term of $w$

$$\limsup_{t \to +\infty} \int_{t - c|\beta(d,p)|}^{t + R} \left( |w_r(r, t) + g_+(t - r)|^2 + |w_t(r, t) - g_+(t - r)|^2 \right) dr \lesssim_{d,p,E} c.$$  \hfill (18)

Next we utilize the identities $r^{d-1} u_r = w_r - (d - 1)r^{d-2} u/2, r^{d-1} u_t = w_t$ and a direct consequence of the pointwise estimate $|u(r,t)| \lesssim_{d,E} r^{-\frac{2(d-1)}{d}}$ (lemma 2.1) to conclude

$$\limsup_{t \to +\infty} \int_{t - c|\beta(d,p)|}^{t + R} \left( |r^{d-1} u_r(r, t) + g_+(t - r)|^2 + |r^{d-1} u_t(r, t) - g_+(t - r)|^2 \right) dr \lesssim_{d,p,E} c.$$  \hfill (22)

By lemma 1.2 (radiation fields), there exists a radial free wave $\tilde{u}_r^+$, so that

$$\lim_{t \to +\infty} \int_0^\infty \left( |r^{d-1} \tilde{u}_r^+(r, t) + g_+(t - r)|^2 + |r^{d-1} \tilde{u}_t^+(r, t) - g_+(t - r)|^2 \right) dr = 0.$$  \hfill (19)

Therefore we have

$$\limsup_{t \to +\infty} \int_{t - c|\beta(d,p)|}^{t + R} r^{d-1} \left( |u_t(r, t) - \tilde{u}_t^+(r, t)|^2 + |u_r(r, t) - \tilde{u}_r^+(r, t)|^2 \right) dr \lesssim_{d,p,E} c.$$  \hfill (20)

Finite speed of propagation of energy implies

$$\limsup_{R \to \infty} \int_0^\infty \left( |u_t(r, t) - \tilde{u}_t^+(r, t)|^2 + |u_r(r, t) - \tilde{u}_r^+(r, t)|^2 \right) dr = 0.$$  \hfill (21)

We combine (20) with (21) and obtain

$$\limsup_{t \to +\infty} \int_{t - c|\beta(d,p)|}^{t + R} r^{d-1} \left( |u_t(r, t) - \tilde{u}_t^+(r, t)|^2 + |u_r(r, t) - \tilde{u}_r^+(r, t)|^2 \right) dr \lesssim_{d,p,E} c.$$  \hfill (22)
Finally we consider the region \( \{ x : |x| < t - c \cdot t^{\beta(p)} \} \). We utilize the conclusion of Theorem 1.1.(c), and obtain

\[
\lim_{t \to +\infty} \int_{|x| < t - c \cdot t^{\beta(p)}} e(x, t)dx \lesssim_{c} \lim_{t \to +\infty} t^{\frac{(2-p)(d+2)}{p+1}} \int_{|x| < t} \frac{|x|}{t} e(x, t)dx = 0.
\]

Please note that in the sub-conformal range our assumption \( p > \frac{3-d+\sqrt{d^2-d+1}}{d-1} \) guarantees that \( \frac{(2-dp+(d+2)}{p+1} < \frac{(d-1)(p-1)−2}{2} \).

We also have

\[
\lim_{t \to +\infty} \int_{|x| < t - ct^{\beta(p)}} \left( |\nabla \tilde{u}^+(x, t)|^2 + |\tilde{u}^+_t(x, t)|^2 \right) dx = 0.
\]

Combining these two limits we obtain

\[
\lim_{t \to +\infty} \int_{|x| < t - ct^{\beta(p)}} \left( |\nabla \tilde{u}^-(x, t) - \nabla u(x, t)|^2 + |\tilde{u}^-_t(x, t) - u_t(x, t)|^2 \right) dx = 0.
\]

We combine this with stronger exterior scattering to conclude

\[
\limsup_{t \to +\infty} \int_{\mathbb{R}^3} \left( |\nabla \tilde{u}^-(x, t) - \nabla u(x, t)|^2 + |\tilde{u}^-_t(x, t) - u_t(x, t)|^2 \right) dx \lesssim_{d, p, E} c.
\]

We make \( c \to 0^+ \) and finish the proof.

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