On the robustness of acoustic black hole spectra

S Finazzi\textsuperscript{1} and R Parentani\textsuperscript{2}

\textsuperscript{1} SISSA, via Bonomea 265, Trieste 34151, Italy; INFN sezione di Trieste
\textsuperscript{2} Laboratoire de Physique Théorique, CNRS UMR 8627, Bât. 210, Université Paris-Sud 11, 91405 Orsay Cedex, France

E-mail: finazzi@sissa.it and renaud.parentani@th.u-psud.fr

Abstract.

We study the robustness of the spectrum emitted by an acoustic black hole by considering series of stationary flows that become either subsonic or supersonic, i.e. when the horizon disappears. We work with the superluminal Bogoliubov dispersion of Bose–Einstein condensates. We find that the spectrum remains remarkably Planckian until the horizon disappears. When the flow is everywhere supersonic, new pair creation channels open. This will be the subject of a forthcoming work.

1. Introduction

In the hydrodynamic approximation, i.e. for long wavelengths, the propagation of sound waves in a moving fluid is analogous to that of light in a curved spacetime \cite{1,2}. More precisely, a one dimensional flow defines the acoustic metric

\[ ds^2 = -c^2 dt^2 + (dx - v dt)^2, \]

where \( v \) is the flow velocity and \( c \) the speed of sound. Assuming that the fluid flows from right to left (\( v < 0 \)), a sonic horizon is present where \( w = c + v \) crosses 0. This situation describes a black (white) hole horizon when \( \kappa_K \equiv \partial_x w \) evaluated at \( w = 0 \) is positive (negative). One thus expects that such system would emit a Hawking flux, i.e. a thermal flux of phonons at a temperature given by \( T_H = \kappa_K / 2\pi \) (in units where \( k_B = \hbar = 1 \)).

However, since Hawking radiation relies on short wavelength modes \cite{3,4}, the dispersion of sound waves, which is neglected in the hydrodynamic approximation, must be taken into account. Following the original work \cite{5}, this was done using analytical \cite{6,7,8,9} and numerical \cite{10,11,12,13} methods. Provided the dispersive scale \( \xi \), the “healing length”, is much smaller than the surface gravity scale \( c/\kappa \), it was found that the thermal properties of the spectrum are extremely robust. In fact, so robust that it is very difficult to characterize and analyze the small deviations with respect to the standard flux.

Our aim is to complete the analysis of \cite{12,13} where it was shown that the spectrum remains Planckian whenever \( T_H \lesssim \omega_{\text{max}}/15 \), where \( \omega_{\text{max}} \) is a critical frequency which scales as \( 1/\xi \) but, more importantly, also depends on \( w(x = -\infty) \), the asymptotic value of \( c + v \). Since this result was obtained using symmetrical profiles like \( w \sim D \tanh(\kappa x/D) \), one may wonder whether the robustness of the spectrum was not partially due to the symmetry of the flow w.r.t. the horizon, namely \( w(-x) = -w(x) \). To investigate this question, we shall consider flows which are highly
Figure 1. Four velocity profiles \( w = c + v \) of Eq. (2) with the same height \( D \) but four values of \( D_1 \). For \( D_1 = 0 \) (solid line), one recovers the symmetric case of [12] with the Killing horizon (solid vertical line) at \( x = 0 \). For \( -D < D_1 < 0 \) (dashed line), the horizon (dashed vertical line) is shifted to \( x > 0 \). In the last two cases, there is no horizon: for \( D_1 < -D \) (dotted line), the flow is everywhere supersonic, and for \( D_1 > D \) (dotdashed line), it is subsonic.

asymmetric. In particular we shall study the spectrum in the interesting limiting cases where the horizon disappears because the flux becomes either sub or supersonic. In parallel with this work, we investigated in [14] other classes of profiles generalizing those of [12, 13].

2. Phonon spectra in asymmetric flows

We consider elongated condensates, stationary flowing along the longitudinal direction \( x \). We assume that the transverse dimensions are small enough that the relevant phonon excitations are longitudinal. The system is thus effectively described by the 1 + 1 dimension metric \((1)\) where \( v \) and \( c \) depend only on \( x \). We work with flows \( w = v + c \) given by

\[
\frac{w(x)}{c_0} = D_1 + D \tanh \left( \frac{\kappa x}{Dc_0} \right),
\]

\( D \) fixes the gap \( [w(\infty) - w(-\infty)]/2c_0 \), while \( D_1 \), which was taken to be 0 in [12, 13], fixes the asymmetry between sub and supersonic regimes. When \( D_1 > D \), the flow is everywhere subsonic, and when \( D_1 < -D \), it is supersonic, see Fig. 1. Instead, when \( |D_1| < D \), there is a sonic (Killing) horizon where \( w = 0 \), localized at

\[
x_H = -\frac{Dc_0}{\kappa} \arctanh \left( \frac{D_1}{D} \right),
\]

It separates the sub \((x > x_H)\) from the supersonic region \((x < x_H)\). In such a flow, when ignoring dispersion, the spectrum of upstream phonons spontaneously emitted from the horizon would be very simple, and would strictly correspond to the Hawking radiation \([1, 5]\). It would follow a Planck law at Hawking temperature \( T_H = \kappa_K/2\pi \). The surface gravity \( \kappa_K \) is here

\[
\kappa_K \equiv \partial_x (c + v)|_{x=x_H} = \kappa \left[ 1 - \left( \frac{D_1}{D} \right)^2 \right].
\]

Taking dispersion into account, the spectrum becomes much more complicated. In this paper, we consider phonons in Bose–Einstein condensates (BEC) with the Bogoliubov dispersion

\[
(\omega - vk)^2 = \Omega^2(k) = c^2 k^2 + \frac{\hbar^2 k^4}{4m^2} = c^2 k^2 \left( 1 + \frac{\xi^2 k^2}{2} \right),
\]
where $\omega$ is the lab conserved frequency. In the sequel, following the numerical procedure of [13], we study the deviations of $n_\omega$, the phonon spectrum, w.r.t. the standard Planck one, by varying the offset parameter $D_1$ while keeping fix all the other parameters.

To characterize these deviations, we use the temperature function $T_\omega$ defined by

$$n_\omega \equiv \frac{1}{\exp(\omega/T_\omega) - 1},$$

which is constant when the spectrum is Planckian. This quantity thus allows a direct comparison with the Hawking temperature $T_H = \kappa_K/2\pi$ of Eq. (4). In Fig. 2 $T_\omega$ and $T_H$ (horizontal lines) are plotted for various values of $D_1$. We distinguish four regimes:

- For $|D_1|$ sufficiently smaller than $D$, the spectrum is Planckian until $\omega$ approaches $\omega_{\text{max}}$, where it vanishes. For $D_1 > 0$, $\omega_{\text{max}}$ decreases since $|w(x = -\infty)|$ does so, and conversely for $D_1 < 0$. However for both signs, at low frequency $T_\omega$ closely follows $\kappa_K/2\pi$ of [11].

- When $|D_1| \rightarrow D$, $\kappa_K$ of [11] drops down to 0 and the sonic horizon disappears. In this critical regime, deviations from Planckianity appears even at low frequency.

- For $D_1 > D$, the flow is everywhere subsonic. There is no particle production because there are no negative norm modes with $\omega > 0$.

- For $D_1 < -D$, the flow is supersonic. A new critical frequency $\omega_{\text{min}} < \omega_{\text{max}}$ appears. For $0 < \omega < \omega_{\text{min}}$, there are now 4 asymptotic in and out modes, and the scattering matrix is $4 \times 4$. Since the code of [13] is designed to the $3 \times 3$ case, this regime requires a new code [13]. However, for $\omega_{\text{min}} < \omega < \omega_{\text{max}}$ there are only 3 modes, and our code can handle this frequency band. In this range, as can be seen in the left panel, $n_\omega$ is close to the case where $D_1 > -D$, even though the spectrum is not at all Planckian.

Furthermore, we checked that the shape of the spectrum is hardly changed when varying the healing length $\xi$. Namely, spectra calculated with different values of $\xi$ almost coincide when $T_\omega$ is plotted versus the rescaled quantity $\omega/\omega_{\text{max}}(\xi)$. To identify the parameters governing the deviations from Planckianity we define

$$\Delta_H \equiv \frac{f(\omega = T_H) - f_{TH}(\omega = T_H)}{f_{TH}(\omega = T_H)},$$

**Figure 2.** Left panel: $T_\omega$ for 6 values of $D_1/D$, with $D = 0.5$ and $\kappa/\sqrt{2c} = 0.1$ fixed. The horizontal lines represent $T_H(D_1)$ of Eq. (4), respectively for $|D_1| = 0, 0.2, 0.4$. For $D_1 < -D$ the curve is truncated at $\omega_{\text{min}}$ as explained in the text. Right panel: $\Delta_H$ as a function of $\omega_{\text{max}}/\kappa$ for two values of $D_1/D$ and various $D_1$. When $\omega_{\text{max}}/\kappa$ is large enough, deviations from Planckianity are small and, more importantly, governed by the ratio $D_1/D$ only.
where \( \tilde{f}(\omega) \equiv \omega n_\omega \) is the actual energy flux and \( f_{Th}(\omega) \) is the thermal flux at temperature \( T_H \). In Fig. 2 right panel, \( \Delta H \) is plotted as a function of \( \omega_{\text{max}}(\xi)/\kappa \). Various values of \( D_1 \) are used, while the ratio \( D_1/D \) is kept constant respectively for the three lower curves (\( D_1/D = -1/5 \)), and for the three upper curves (\( D_1/D = 4/5 \)). As expected, \( \Delta H \) goes to 0 for large values of \( \omega_{\text{max}}/\kappa \). What we learn here is that the leading deviations from Planckianity do not depend separately on \( D_1 \) or \( D \) but only on their ratio.

3. Comments and conclusions

In this work, we have studied the robustness of the properties of black hole radiation in BEC by considering flows characterized by a high asymmetry between the sub and the supersonic region. We found that the spectrum remains Planckian until one approaches the critical cases (\( |D_1| \rightarrow D \)) where the surface gravity (11) vanishes.

In fact, for \( 0 < D_1 \rightarrow D \), since the fluid flows in the supersonic region with a velocity just above that of sound, the cutoff frequency \( \omega_{\text{max}} \) goes to zero. Hence the range of frequencies where the temperature \( T_\omega \) is nearly constant shrinks and disappears in the limit. As expected, no emission of particles is found when the flow remains everywhere subsonic.

The situation is much more interesting in the opposite case, for \( 0 > D_1 \rightarrow -D \), when pushing the velocity of the subsonic region towards the speed of sound. In this regime, even though \( \kappa_K \) of (4) goes again to 0, \( \omega_{\text{max}} \) now increases. As a result, the low frequency plateau of \( T_\omega \) at low temperature again shrinks, but the spectrum becomes more and more blue. This can be understood from the fact that more dispersive modes, i.e. modes with higher \( \omega \), have still their turning point near \( x = 0 \) where the gradient of \( w \) is \( \kappa \gg \kappa_K \).

When \( D_1 < -D \), there is no sonic horizon since the flow is everywhere supersonic. Yet the spectrum is continuously deformed even though the Planck character is completely lost in the frequency range \( \omega_{\text{min}} < \omega < \omega_{\text{max}} \), where \( \omega_{\text{min}} \) is a new critical frequency. For \( \omega < \omega_{\text{min}} \), new scattering channels open because there is a fourth asymptotic mode. Investigations of this case \( [15] \) are in progress.

To conclude, we stress that the thermal spectrum at the naïve surface gravity (4) provides a reliable approximation of the actual phonon spectrum in all cases but the critical ones where the sonic horizon disappears. In these extreme cases, the spectrum is no longer thermal and its properties are governed by the dispersive properties of field.

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