Abstract. Every prism manifold can be parametrized by a pair of relatively prime integers $p > 1$ and $q$. In our earlier papers, we determined a complete list of prism manifolds $P(p, q)$ that can be realized by positive integral surgeries on knots in $S^3$ when $q < 0$ or $q > p$; in the present work, we solve the case when $0 < q < p$. This completes the solution of the realization problem for prism manifolds.

1. Introduction

Let $P(p, q)$ be an oriented prism manifold with Seifert invariants

$$(-1; (2, 1), (2, 1), (p, q)),$$

where $q$ and $p > 1$ are relatively prime integers. In [BHIM+16, BNOV17], we solved the Dehn surgery realization problem of prism manifolds for $q < 0$ and for $q > p$. The theme of the present work is to settle the remaining case $0 < q < p$. In [BHIM+16, Tables 1 and 2], the authors give a tabulation of prism manifolds that can be obtained by positive integral Dehn surgery on Berge–Kang knots [BK]. The tables conjecturally account for all realizable prism manifolds; in particular, [BHIM+16, Table 2] suggests that for a realizable $P(p, q)$ with $q > 0$, we must have $p \leq 2q + 1$. Indeed, this is the case:

**Theorem 1.1.** If $P(p, q)$ with $q > 0$ can be obtained by surgery on a knot $K \subset S^3$, then $p \leq 2q + 1$. If $p = 2q + 1$, then $K$ is the torus knot $T(2q + 1, 2)$.

Doig, in [Doi16, Conjecture 12], conjectured that if $P(p, q)$ is realizable, then $p \leq 2|q| + 1$. The main result of [BHIM+16] settles the conjecture for $q < 0$; Theorem 1.1 verifies it for $q > 0$.

Our second main result, Theorem 1.2 below, provides the solution of the realization problem for those $P(p, q)$ with $q < p < 2q$.

**Theorem 1.2.** The prism manifold $P(p, q)$ with $q < p < 2q$ can be obtained by $4q$–surgery on a knot $K \subset S^3$ if and only if $q = \frac{1}{r^2 - 2r - 1}(r^2 p - 1)$, with $r \leq -3$ odd and $p \equiv -2r + 5$ (mod $r^2 - 2r - 1$). Moreover, in this case, there exists a Berge–Kang knot $K_0$ such that $P(p, q) \cong S^3_{4q}(K_0)$, and that $K$ and $K_0$ have isomorphic knot Floer homology groups.

**Remark 1.3.** If we allow $r = -1$ in Theorem 1.2, we get $p = 2q + 1$: see Theorem 1.1.
1.1. The spherical manifold realization problem. The spherical manifold realization problem asks which spherical manifolds arise from positive integral surgery along a knot in $S^3$. Theorems 1.1 and 1.2 and our earlier results [BHM+16, BNOV17], combined with Gu’s work [Gu14] and Greene’s work [Gre13], provide a complete classification of realizable spherical manifolds. The interest is in finding a complete classification of knots in $S^3$ on which Dehn surgery produce spherical manifolds. In [Ber18], Berge proposed a complete list of knots in $S^3$ with lens space surgeries. Indeed, Berge’s conjecture states that the $P/P$ knots form a complete list of knots in $S^3$ that admit lens space surgeries. All the known examples of knots on which surgeries will result in non-lens space spherical manifolds are $P/SF$ knots.

We repeat the following conjecture from [BHM+16, Conjecture 1.7]: it is a generalization of Berge’s conjecture.

**Conjecture 1.4.** Let $K$ be a knot in $S^3$ that admits an integral surgery to a spherical manifold. Then $K$ is either a $P/SF$ or a $P/P$ knot.

1.2. Methodology. We first provide a brief overview of the methodology undertaken to solve the prism manifold realization problem in the cases $q < 0$ and $q > p$: the proof in both cases draws inspiration from that of Greene for lens spaces [Gre13]. We then discuss how (and why) the methodology is modified for the case of the present work.

We first require a combinatorial definition.

**Definition 1.5.** A vector $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_{n+1}) \in \mathbb{Z}^{n+2}$ that satisfies $0 \leq \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_{n+1}$ is a changemaker vector if for every $k$, with $0 \leq k \leq \sigma_0 + \sigma_1 + \cdots + \sigma_{n+1}$, there exists a subset $S \subset \{0, 1, \ldots, n+1\}$ such that $k = \sum_{i \in S} \sigma_i$.

The key idea is to use the correction terms in Heegaard Floer homology in tandem with Donaldson’s Theorem A. The following is immediate from [Gre13, Theorem 3.3].

**Theorem 1.6.** Suppose that $P(p, q)$ bounds a sharp four-manifold $X(p, q)$. If $P(p, q)$ arises from positive integer surgery on a knot $K$ in $S^3$, then the intersection lattice on $X(p, q)$ embeds as the orthogonal complement $\sigma^\perp$ of some changemaker vector $\sigma \in \mathbb{Z}^{n+2}$, with $n+1 = b_2(X)$.

See Section 5 for the definition of a sharp four-manifold, and see Subsection 1.3 for the definition of the intersection lattice. When $q < 0$ or $q > p$, it turns out that $P(p, q)$ bounds a sharp four-manifold $X(p, q)$. We then solved a combinatorial problem: we classified all lattices isomorphic to the intersection lattice of $X(p, q)$, whose complements are changemakers in $\mathbb{Z}^{n+2}$. There is a heavy analysis of lattices involved that forms the main body of [BHM+16, BNOV17]. Finally, we verified that for every $(p, q)$ corresponding to such a lattice, $P(p, q)$ is indeed realized by surgery on a $P/SF$ knot.

We now turn our attention to the case $0 < q < p$. In light of Theorem 1.1, it suffices to consider $q < p < 2q$. When $q < p < 2q$, $P(p, q)$ does not bound a sharp four–manifold. Thus, we cannot use the embedding restriction of Theorem 1.6 – an essential to the classification of realizable prism manifolds in the previous two cases. Our strategy to prove Theorem 1.2 is to replace Theorem 1.6 with another lattice theoretic obstruction for $P(p, q)$ to being realizable, as follows. The prism manifold $P'(2, 1)$ bounds a rational homology four-ball $Z_2$ (the left two...
components of Figure 2 where the 0–framed unknot is replaced by a dotted circle and $a_{-1} = 2$; and that there exists a negative definite cobordism $W$ from $P(2, 1)$ to $P(p, q)$ (the right $n + 1$ components of Figure 2). Suppose that $P(p, q)$ arises from surgery on a knot $K \subset S^3$, and let $W_{4q} = W_{4q}(K)$ be the corresponding two-handle cobordism obtained by attaching a two-handle to the four-ball along the knot $K$ with framing $4q$. Form $Z := Z_2 \cup P(2, 1) W$; it will be a smooth four-manifold with boundary $P(p, q)$. The intersection lattice on $Z$ is $\Lambda(q, -p)$, which is defined in Definition 3.1. Form $X := W \cup (-W_{4q})$. We prove that the intersection lattice on $X$ is isomorphic to $D_4 \oplus \mathbb{Z}^{n-2}$. Finally, form $\hat{X} := Z \cup (-W_{4q})$; see Figure 1. It follows that $\hat{X}$ is a smooth, closed, simply connected, negative definite four-manifold with $b_2(Z) = n + 2$ for some $n \geq 0$. Now, Donaldson’s Theorem A [Don83] implies that the intersection lattice on $\hat{X}$ is the Euclidean integer lattice $\mathbb{Z}^{n+2}$. This provides a necessary condition for $P(p, q)$ to be realizable: the lattice $\Lambda(q, -p)$ embeds as a codimension one sublattice of $\mathbb{Z}^{n+2}$. Our new obstruction now reads as follows:

**Theorem 1.7.** Suppose $P(p, q)$ with $q < p < 2q$ arises from positive integer surgery on a knot $K$ in $S^3$.

(a) The linear lattice $\Lambda(q, -p)$ embeds as the orthogonal complement to a changemaker $\sigma \in \mathbb{Z}^{n+2}, n + 1 = b_2(Z)$.

(b) There is an embedding of $D_4 \oplus \mathbb{Z}^{n-2}$ into $\mathbb{Z}^{n+2}$ such that there exists some short characteristic covector $\chi$ for $D_4 \oplus \mathbb{Z}^{n-2}$ with $\langle \chi, \sigma \rangle = i$ if and only if $-2q + g(K) \leq i \leq 2q - g(K)$.

The strategy is now apparent: determine the list of all pairs $(p, q)$ which pass the embedding restriction of Theorem 1.7. Finally, we verify that every manifold in our list is indeed realized by a knot surgery: we do so by comparing the list with the list of realizable manifolds tabulated in [BHM+16, Table 2]. It must be noted that Part (a) of Theorem 1.7 only provides a necessary condition for the prism manifold $P(p, q)$ to be realizable. Indeed, it is easy to find pairs $(p, q)$ that satisfy Part (a) of Theorem 1.7, but the corresponding prism manifolds are not realizable; for example $P(13, 9)$ and $P(16, 9)$. The 9–surgery on the torus knot $T(2, 5)$ is $L(9, 13) \cong L(9, 16)$, then work of Greene [Gre13] shows that the corresponding linear lattice satisfies Part (a) of Theorem 1.7. However, the manifold $P(16, 9)$ is not realizable because of the parity of 16 ($p$ is always odd for a realizable $P(p, q)$ [BHM+16]); and neither is $P(13, 9)$ by Theorem 1.2.
In the previous cases $q < 0$ and $q > p$ as well as in the lens space realization problem [Gre13], the first step was finding a sharp four-manifold bounded by $P(p, q)$ (respectively, the lens space $L(p, q)$): in each case a negative definite four-manifold was found; then it was almost immediate from the previous works of Ozsváth and Szabó [OS05b, OS03b] that the four-manifold is sharp. For the case at hand, however, $P(p, q)$ does not bound a sharp four-manifold. We need to carefully analyze the $d$–invariants of $P(p, q)$ in each Spin$^c$ structure in terms of the $d$–invariants of certain Spin$^c$ structures of $P(2, 1)$ and the grading shift of the cobordism $W$. In particular, we generalize the notion of sharpness to cobordisms between rational homology spheres, and show that the cobordism $W$ is sharp (Proposition 5.3): again, see Figure 1. Using that the intersection lattice on $X$ is isomorphic to $D^4 \oplus \mathbb{Z}^{n-2}$, it will be immediate that $X$ is a sharp four–manifold (Corollary 6.4). Using this finding, we are able to prove Theorem 1.7 and translate it into a more practical condition on the changemaker vector $\sigma$ (Proposition 6.11).

1.3. Notations. We use homology groups with integer coefficients throughout the paper. For a compact four–manifold $X$, regard $H_2(X)$ as an inner product space equipped with the intersection pairing $Q_X$ on $X$. Also, we refer to $(H_2(X), -Q_X)$ as the intersection lattice on $X$, where $-Q_X$ denotes the negation of the pairing of $Q_X$. Finally, we call an oriented three–manifold $Y$ a realizable manifold if it can be obtained by positive integral surgery on a knot in $S^3$.

1.4. Organization. This paper is organized as follows. In Section 2, we prove Theorem 1.1, thus solve the case of the realization problem when $2q < p$. In Section 3, we collect some basic results about linear lattices and changemaker lattices from [Gre13]. In Section 4, we study the topology of a certain type of cobordism between rational homology 3–spheres. In Section 5, we define sharp cobordisms, and prove that the cobordism $W$ between $P(2, 1)$ and $P(p, q)$ is sharp. In Section 6, we use the result in Section 5 to prove a strengthened changemaker condition in the case $q < p < 2q$. In Section 7 and Section 8, we use the strengthened changemaker condition to enumerate all the possible changemaker lattices we can have. In Section 9, we determine the pairs $(p, q)$ corresponding to the changemaker lattices, thus finish the proof of Theorem 1.2.

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2. Proof of Theorem 1.1

The goal of this section is to prove the following upper bound of $p$, and then to prove Theorem 1.1. Recall that we assume $q > 0$.

Proposition 2.1. If $P(p, q)$ is realizable, then $p \leq 2q + 1$. 

Remark 2.2. If $P(p, q)$ is realizable with $p = 2|q| \pm 1$, then $K$ must be a torus knot [NZ18, Theorem 1.6]. Recall that for a realizable $P(p, q)$, $p$ is odd [BHM+16]. In particular, if we restrict attention to hyperbolic knots on which surgeries will result in $P(p, q)$, then $p \leq 2|q| - 3$.

2.1. The Casson–Walker invariant of $P(p, q)$. Let

$$\Delta_K(T) = \alpha_0 + \sum_{i>0} \alpha_i(T^i + T^{-i})$$

be the normalized Alexander polynomial of $K$. If $K$ admits an L-space surgery, then $|\alpha_i| \leq 1$, $\alpha_q(K) = 1$, and +1 and −1 appear alternatingly among the nonzero $\alpha_i$ [OS05a, Theorem 1.2].

Given a real number $x$, let $\{x\} = x - [x]$ be the fractional part of $x$. Given a pair of coprime integers $n, m$ with $n > 0$, let $s(m, n)$ be the Dedekind sum

$$s(m, n) = \sum_{i=1}^{n-1} \left( \left( \frac{i}{n} \right) \left( \frac{im}{n} \right) \right),$$

where

$$((x)) = \begin{cases} \{x\} - \frac{1}{2}, & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0, & \text{if } x \in \mathbb{Z}. \end{cases}$$

Let $\lambda(\cdot)$ be the Casson–Walker invariant [Wal90], normalized so that

$$\lambda(S^3_1(T(3, 2))) = 2.$$ 

By [Les96, Proposition 6.1.1], the Casson–Walker invariant of $P(p, q)$ can be computed by the formula

$$\lambda(P(p, q)) = \frac{1}{12} \left( -\frac{p}{q} \left( \frac{1}{p^2} - \frac{1}{2} \right) - \frac{q}{p} + 3 + 12s(q, p) \right).$$

Since the Dedekind sum satisfies the reciprocity law

$$s(q, p) + s(p, q) = \frac{1}{12} \left( \frac{p}{q} + \frac{q}{p} + \frac{1}{pq} \right) - \frac{1}{4},$$

we get

$$\lambda(P(p, q)) = \frac{p}{8q} - s(p, q). \quad (2)$$

On the other hand, the surgery formula for the Casson–Walker invariant [BL90, Theorem 2.8] implies that

$$\lambda(S^3_{3q}(K)) = -s(1, 4q) + \frac{1}{4q} \Delta_K''(1)$$

$$= -\frac{(2q - 1)(4q - 1)}{24q} + \frac{1}{4q} \Delta_K''(1). \quad (3)$$

Lemma 2.3. For realizable $P(p, q)$ with $q$ odd, $p \equiv -1 \pmod{4}$. 

Proof. By combining (2) and (3), we have
\begin{align*}
\frac{(2q - 1)(4q - 1)}{24q} + \frac{1}{4q} \Delta''_K(1) &= \lambda(P(p, q)) \\
&= \sum_{i=1}^{q-1} \left( \frac{i}{q} - \frac{1}{2} \right) \left( \frac{pi}{q} - \frac{1}{2} \right) \pmod{1} \\
&= \frac{p}{8q} - \frac{p(q - 1)(2q - 1)}{6q} + \frac{p(q - 1)}{4}.
\end{align*}

Multiplying both sides by $24q$, we get
\[1 - 6q + 8q^2 + p(-1 + 6q - 2q^2) \equiv 6\Delta''_K(1) \pmod{24q}.
\]
Since $\Delta''_K(1)$ is even and $p, q$ are odd, we get
\[2q + 1 + p(2q + 1) \equiv 0 \pmod{4}.
\]
So $p \equiv -1 \pmod{4}$. □

2.2. The Spin$^c$ structures. The $i$-th torsion coefficient of a knot $K$ is defined to be
\[t_i(K) = \sum_{j \geq 1} j\alpha_{i+j},\]
for $i \geq 0$, where the $\alpha_i$ are as in (1). Let
\[\varepsilon_i = t_i - t_{i+1}.
\]
When $K$ admits an L-space surgery, it is proved in [Ras03, Proposition 7.6] that
\[\varepsilon_i \in \{0, 1\}.
\]
Suppose $4q$–surgery on $K$ is $P(p, q)$, then $4q \geq 2g(K) - 1$ [OS11]. So
\[g(K) \leq 2q. \tag{4}\]
Since $a_{g(K)} = 1$ and $a_i = 0$ when $i > g(K)$, it follows from the definition of $t_i$ that
\[t_i = 0 \quad \text{if and only if } i \geq g(K). \tag{5}\]
In particular, by (4), we get
\[t_{2q} = 0. \tag{6}\]
For $i > 0$,
\[\alpha_i = t_{i-1} - 2t_i + t_{i+1} = \varepsilon_{i-1} - \varepsilon_i.
\]
Since $1 = \Delta_K(1) = \alpha_0 + 2\sum_{i>0} \alpha_i$, we can also get
\[\alpha_0 = 1 - 2\sum_{i>0} \alpha_i.
Thus
\[
\Delta_K(-1) = \alpha_0 + 2\sum_{i>0}(-1)^i\alpha_i = 1 - 4\sum_{i>0}(-1)^i\varepsilon_i. \tag{7}
\]

Given a knot $K \subset S^3$ and an integer $n > 0$, there is an affine isomorphism [OS03a]
\[
\varphi : \mathbb{Z}/n\mathbb{Z} \to \text{Spin}^c(S^3_n(K)).
\]

For simplicity, let $d(S^3_n(K), i) = d(S^3_n(K), \varphi(i))$.

From [OS03a], we have
\[
d(L(n, 1), i) = -\frac{1}{4} + \frac{(2i - n)^2}{4n}. \tag{8}
\]

Using [OS11, Theorem 1.2], we get
\[
d(S^3_n(K), i) = d(L(n, 1), i) - 2t_{\min(i,n-i)}. \tag{9}
\]

**Lemma 2.4.** Suppose that $P(p, q)$ is obtained by the $4q$–surgery on $K$. Let $i$ be an integer with $0 \leq i \leq q$. If $i$ is even, we have
\[
d(S^3_{4q}(K), q - i) = d(S^3_{4q}(K), q + i),
\]
and
\[
t_{q-i} - t_{q+i} = \frac{i}{2}.
\]

If $i$ is odd, we have
\[
d(S^3_{4q}(K), q - i) = d(S^3_{4q}(K), q + i) \pm 1,
\]
and
\[
t_{q-i} - t_{q+i} = \frac{i \mp 1}{2}.
\]

**Proof.** Since $S^3_{4q}(K)$ is a prism manifold, it contains a Klein Bottle. So the order–2 element in $H_1(S^3_{4q}(K))$ is represented by a curve in the Klein Bottle, such that the complement of the curve in the Klein Bottle is an annulus. By [NW14, Theorem 1.1], for any $j \in \mathbb{Z}/4q\mathbb{Z}$, we have
\[
|d(S^3_{4q}(K), j) - d(S^3_{4q}(K), j + 2q)| \leq 1. \tag{10}
\]

Since the conjugate of $\varphi(j + 2q)$ is $\varphi(2q - j)$, we have
\[
d(S^3_{4q}(K), j + 2q) = d(S^3_{4q}(K), 2q - j). \tag{11}
\]

Let $j = q - i$. Using (8) and (9), we get
\[
d(S^3_{4q}(K), q - i) - d(S^3_{4q}(K), q + i)
= -\frac{1}{4} + \frac{(2q - 2i - 4q)^2}{16q} - 2t_{q-i} - \left( -\frac{1}{4} + \frac{(2q + 2i - 4q)^2}{16q} - 2t_{q+i} \right)
= i - 2t_{q-i} + 2t_{q+i} \in \mathbb{Z}.
\]

Using (10) and (11), we get our conclusion. \qed
2.3. The proof of Proposition 2.1.

Proof of Proposition 2.1. By Lemma 2.4 and (6),
\[ t_0 = t_0 - t_{2q} \leq \left\lfloor \frac{q + 1}{2} \right\rfloor. \]

By [NZ18, Lemma 6.1], \( p = |\Delta_K|(-1)|. \) Using (7), we get
\[ p \leq 1 + 4 \sum_{i \geq 0} \varepsilon_i = 1 + 4t_0 \leq 1 + 4 \left\lfloor \frac{q + 1}{2} \right\rfloor. \]

When \( q \) is even, \( p \leq 2q + 1. \) When \( q \) is odd, \( p \leq 2q + 3. \) By Lemma 2.3, \( p \neq 2q + 3, \) so we must have \( p \leq 2q + 1. \) □

Proof of Theorem 1.1. The first statement is Proposition 2.1. The second statement follows from combining [NZ18, Theorem 1.6] and [BHM+16, Lemma 2.1]. □

3. Input from lattice theory

This section assembles facts about lattices that will be used in the paper. We mainly follow the treatment of [Gre15, Gre13, BHM+16, BNOV17].

Recall that an integral lattice is a finitely generated free abelian group \( L \) endowed with a positive definite symmetric bilinear form \( \langle , \rangle : L \times L \to \mathbb{Z}. \) Given \( v \in L, \) let \( |v| = \langle v, v \rangle \) be the norm of \( v. \) We can extend \( \langle , \rangle \) to a \( \mathbb{Q} - \text{valued} \) pairing on \( L \otimes \mathbb{Q}; \) using it we define
\[ L^* = \{ x \in L \otimes \mathbb{Q} | \langle x, y \rangle \in \mathbb{Z}, \forall y \in L \}. \]

The pairing on \( L \) descends to a non-degenerate, symmetric bilinear form on the discriminant group \( \overline{L} = L^*/L \)
\[ b : \overline{L} \times \overline{L} \to \mathbb{Q}/\mathbb{Z} \]
\[ b(\overline{x}, \overline{y}) \equiv \langle x, y \rangle \pmod{1}, \]
the linking form, where \( \overline{x} \) denotes the class of \( x \in L \) in \( \overline{L}. \) The discriminant of \( L \) is the order of the finite group \( \overline{L}. \) Let \( \text{Char}(L) = \{ x \in L^* | \langle x, y \rangle \equiv \langle y, y \rangle \pmod{2}, \forall y \in L \} \)
denote the set of characteristic covectors for \( L. \) The set \( C(L) = \text{Char}(L)/2L \) forms a torsor over the discriminant group \( \overline{L}. \) Given \( \chi \in C(L), \) define
\[ d_L([\chi]) = \min \left\{ \frac{|\chi' - \text{rk}(L)|}{4} \right\}_{\chi' \in [\chi]}, \] (12)
and call an element \( \chi \in \text{Char}(L) \) short if its norm is minimal in \( [\chi]. \) We call the pair \( (C(L), d_L) \) the \( d-\text{invariant} \) of the lattice \( L; \) in particular it is an invariant of the stable isomorphism type
of the lattice $L$ [OS05b, Theorem 4.7]. We drop $L$ from the notation when the lattice $L$ is understood from the context.

### 3.1. Linear lattices.

Given a pair of relatively prime positive integers $p, q$, write $\frac{p}{q}$ in a Hirzebruch–Jung continued fraction

$$\frac{p}{q} = a_{-1} - \frac{1}{a_0 - \frac{1}{\ddots - \frac{1}{a_n}}} = [a_{-1}, a_0, \ldots, a_n]^{-},$$

with $a_i \geq 2$ when $i \geq 0$ in Equation (13).

**Definition 3.1.** The linear lattice $\Lambda(q, -p)$ has a basis

$$\{x_0, \ldots, x_n\},$$

and inner product given by

$$\langle x_i, x_j \rangle = \begin{cases} a_i, & i = j \\ -1, & |i - j| = 1 \\ 0, & |i - j| > 1, \end{cases}$$

where the coefficients $a_i$, for $i \in \{0, \ldots, n\}$, are defined by the continued fraction (13). We call (14) the **vertex basis** of $\Lambda(q, -p)$.

**Remark 3.2.** The reason that we use $\Lambda(q, -p)$ instead of $\Lambda(q, p)$ is that our convention for lens spaces is different from that of [Gre13]. In our paper, the lens space $L(q, p)$ is oriented as the $\frac{q}{p}$–surgery on the unknot, and $P(p, q)$ is the $\frac{q}{p}$–surgery on $\mathbb{R}P^1 \# \mathbb{R}P^1 \subset \mathbb{R}P^3 \# \mathbb{R}P^3$, so they both bound 4–manifolds with intersection lattice $\Lambda(q, -p)$.

An element $\ell \in L$ is **reducible** if $\ell = x + y$ for some nonzero $x, y \in L$, with $\langle x, y \rangle \geq 0$, and **irreducible** otherwise. An element $\ell \in L$ is **breakable** if $\ell = x + y$ with $|x|, |y| \geq 3$ and $\langle x, y \rangle = -1$, and **unbreakable** otherwise.

**Definition 3.3.** In a linear lattice, if $I$ is any subset of $\{x_0, x_1, \ldots, x_n\}$ then write $[I] = \sum_{x \in A} x$. An interval is an element of the form $[I]$ with $I = \{x_a, x_{a+1}, \ldots, x_b\}$ for $0 \leq a \leq b \leq n$. We say that $a$ is the left endpoint of the interval, and $b$ is the right endpoint of the interval. Say that $[I]$ contains $x_i$ if $I$ does: we often write $x_i \in [I]$ in this case.

**Proposition 3.4.** [Gre13, Proposition 3.3] If $v \in \Lambda(q, -p)$ is irreducible, $v = \epsilon[I]$ for some $\epsilon = \pm 1$ and $[I]$ an interval.

From now on, let $[v]$ be the interval corresponding to $v$ when $v$ is irreducible.

**Definition 3.5.** A vertex $x_i$ has **high weight** if $|x_i| = a_i > 2$.

**Proposition 3.6.** [Gre13, Corollary 3.5(4)] An element $\epsilon[I] \in \Lambda(q, -p)$ with $\epsilon \in \{\pm 1\}$ is **unbreakable** if and only if $[I]$ contains at most one element of high weight.
Definition 3.7. For two intervals $[I]$ and $[J]$ with left endpoints $i_0, j_0$ and right endpoints $i_1, j_1$, say that $[I]$ and $[J]$ are distant if either $i_1 + 1 < j_0$ or $j_1 + 1 < i_0$, that $[I]$ and $[J]$ share a common end if $i_0 = j_0$ or $i_1 = j_1$, and that $[I]$ and $[J]$ are consecutive if $i_1 + 1 = j_0$ or $j_1 + 1 = i_0$. Write $[I] < [J]$ if $I \subset J$ and $[I]$ and $[J]$ share a common end, and $[I] \uparrow [J]$ if they are consecutive. If $[I]$ and $[J]$ are either consecutive or share a common end, say that they abut. If $I \cap J$ is nonempty and $[I]$ and $[J]$ do not share a common end, write $[I] \cap [J]$.

Proposition 3.8. [Gre13, Corollary 3.5(2)] The lattice $\Lambda(q, -p)$ is indecomposable; that is, $\Lambda(q, -p)$ is not the direct sum of two nontrivial lattices.

Proposition 3.9 (Proposition 3.6 of [Gre13]). If $\Lambda(q, p) \equiv \Lambda(q', p')$, then $q = q'$ and either $p \equiv p'$ or $pp' \equiv 1 \pmod{q}$.

3.2. Changemaker lattices. When a lattice $L$ is isomorphic to $\sigma^\perp$, the orthogonal complement of a changemaker vector $\sigma \in \mathbb{Z}^{n+2}$, $L$ is called a changemaker lattice.

Definition 3.10. The standard basis of $\sigma^\perp$ is the collection $S = \{v_1, \ldots, v_{n+1}\}$, where

$$v_j = \left(2e_0 + \sum_{i=1}^{j-1} e_i\right) - e_j$$

whenever $\sigma_j = 1 + \sigma_0 + \cdots + \sigma_{j-1}$, and

$$v_j = \left(\sum_{i \in A} e_i\right) - e_j$$

whenever $\sigma_j = \sum_{i \in A} \sigma_i$, with $A \subset \{0, \ldots, j - 1\}$ chosen to maximize the quantity $\sum_{i \in A} 2^i$. A vector $v_j \in S$ is called tight in the first case, just right in the second case as long as $i < j - 1$ and $i \in A$ implies that $i + 1 \in A$, and gappy if there is some index $i$ with $i \in A$, $i < j - 1$, and $i + 1 \notin A$. Such an index, $i$, is a gappy index for $v_j$.

Definition 3.11. For $v \in \mathbb{Z}^{n+2}$, supp $v = \{i|\langle e_i, v \rangle \neq 0\}$, supp$^+ v = \{i|\langle e_i, v \rangle > 0\}$, and supp$^- v = \{i|\langle e_i, v \rangle < 0\}$.

Lemma 3.12 (Lemma 3.12 (3) in [Gre13]). If $|v_{k+1}| = 2$, then $k$ is not a gappy index for any $v_j$ with $j \in \{1, \ldots, n + 1\}$.

Lemma 3.13 (Lemma 3.13 in [Gre13]). Each $v_j \in S$ is irreducible. In fact, suppose $A \subset \{0, 1, \ldots, j - 1\}$, then the vector

$$-e_j + \sum_{i \in A} e_i$$

is irreducible.

Lemma 3.14. Let $v = \sum_{i \in A} b_i e_i \in L$, with $A \subset \{0, 1, \ldots, n + 1\}$ and each $b_i \in \{-1, 1\}$. If $v = x + y$ with $\langle x, y \rangle \geq 0$, then there exists a subset $B \subset A$ such that

$$x = \sum_{i \in B} b_i e_i, \quad y = \sum_{i \in A \setminus B} b_i e_i.$$
Proof. Let $x = \sum x_i e_i, y = \sum y_i e_i$. Since $x_i + y_i \in \{-1, 0, 1\}, x_i y_i \leq 0$. If $\langle x, y \rangle \geq 0$, then each $x_i y_i = 0$, namely, one of $x_i, y_i$ is 0. So our conclusion holds. \qed

Lemma 3.15 (Lemma 3.15 in [Gre13]). If $v_j \in S$ is breakable, then it is tight.

Lemma 3.16 (Lemma 4.2(1) in [Gre13]). If $\Lambda(q, -p)$ is a changemaker lattice, then it contains at most one tight vector.

Lemma 3.17 (Lemma 3.12(1) in [Gre13]). For any $v_j \in S$, we have $j - 1 \in \text{supp}(v_j)$.

Definition 3.18. If $T$ is a set of irreducible vectors in a linear lattice $\Lambda(q, -p)$, the intersection graph $G(T)$ has vertex set $T$, and an edge between $v$ and $w$ if the intervals corresponding to $v$ and $w$ abut. We write $v \sim w$ if $v$ and $w$ are connected in $G(T)$.

Lemma 3.19. If the intervals corresponding to $v$ and $w$ abut, then $\langle v, w \rangle \neq 0$.

Lemma 3.20 (Lemma 4.4 in [Gre13]). If $v_i$ and $v_j$ are distinct unbreakable vectors with $|v_i|, |v_j| \geq 3$, then $|\langle v_i, v_j \rangle| \leq 1$, with equality if and only if $\langle v_i \rangle \uparrow \langle v_j \rangle$.

Lemma 3.21 (Corollary 4.5 in [Gre13]). If $v_i$ and $v_j$ are distinct unbreakable vectors with $|v_i|, |v_j| \geq 3$, then the high weight vertices contained in $v_i, v_j$ are different.

Definition 3.22. A claw in a graph $G$ is a quadruple $(v; w_1, w_2, w_3)$ of vertices such that $v$ neighbors all the $w_i$, but no two of the $w_i$ neighbor each other.

Lemma 3.23 (Lemma 4.8 of [Gre13]). The intersection graph $G(T)$ has no claws.

Definition 3.24. Given a set $T$ of unbreakable elements in a linear lattice and $v_1, v_2, v_3 \in T$, $(v_1, v_2, v_3)$ is a heavy triple if $|v_i| \geq 3$, and if each pair among the $v_i$ is connected by a path in $G(T)$ disjoint from the third.

Lemma 3.25 (Based on Lemma 4.10 of [Gre13]). $G(T)$ has no heavy triples.

4. The topology of certain cobordisms

In this section, we will consider the topology of a certain cobordism $W : Y_0 \to Y_1$. We assume that $W$ is obtained by adding $n+1$ two-handles along a link $L \subset Y_0$, such that one component $L_0$ of $L$ represents a 2–torsion in $H_1(Y_0)$, and all other components of $L$ are null-homologous in $Y_0$. Moreover, we assume that $|H_1(Y_0)| = 4$ and $W$ is negative definite. So $Y_1$ is a rational homology sphere. Let $\iota_i : Y_i \to W$ be the inclusion map, $\iota_*^i : H^2(W) \to H^2(Y_i)$ be the induced maps on cohomology, and $\iota_*^i : \text{Spin}^c(W) \to \text{Spin}^c(Y_i)$ be the induced maps on $\text{Spin}^c$, $i = 0, 1$.

We make the further assumption that $Y_0$ is the boundary of a compact 4–manifold $Z_0$ with $H_1(Z_0) \cong \mathbb{Z}/2\mathbb{Z}$ and $H_2(Z_0) = 0$, and $L_0$ is null-homologous in $Z_0$. Let $Z = Z_0 \cup_{Y_0} W$.

From the handle structure of $W$, we can compute

$$H_1(W) \cong \mathbb{Z}/2\mathbb{Z}, H_2(W) \cong \mathbb{Z}^{n+1}, H_1(W, Y_i) = 0, H_2(W, Y_i) \cong \mathbb{Z}^{n+1}, i = 0, 1.$$  

By the Universal Coefficient Theorem,

$$H_2(W) \cong \mathbb{Z}^{n+1} \oplus \mathbb{Z}/2\mathbb{Z}.$$

In particular, there exists a unique torsion class $\alpha \in H^2(W)$. Let $\alpha_i = \iota_i^*(\alpha)$, $i = 0, 1$. 
Since $Z$ is obtained by adding two-handles to $Z_0$, such that all attaching curves are null-homologous in $Z_0$, we have

$$H_1(Z) \cong H_1(Z_0) \cong \mathbb{Z}/2\mathbb{Z},$$

and the map $H_2(Z) \to H_2(Z, Z_0)$ is an isomorphism.

**Lemma 4.1.** The map $\iota_{W,Z}^* : H^2(Z) \to H^2(W)$ is injective with image containing $\alpha$. The map $\iota_{Y_0,Z_0}^* : H^2(Z_0) \to H^2(Y_0)$ is injective with image generated by $\alpha_0$. Moreover, $[L_0] \in H_1(Y_0)$ is the Poincaré dual of $\alpha_0$.

**Proof.** Using the long exact sequences

$$H^2(Z, W) \to H^2(Z) \to H^2(W), \quad H^2(Z_0, Y_0) \to H^2(Z_0) \to H^2(Y_0),$$

and the fact that $0 = H^2(Z, Y_0) \cong H^2(Z, W)$, we get that $\iota_{W,Z}^*$ and $\iota_{Y_0,Z_0}^*$ are injective.

By the Universal Coefficient Theorem, $H^2(Z) \cong \text{Hom}(H_2(Z), \mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}$, so it has a unique 2-torsion $\overline{\alpha}$. Since $\iota_{W,Z}^*$ is injective, $\iota_{W,Z}^*(\overline{\alpha})$ is a 2-torsion in $H^2(W)$, which must be $\alpha$. Let $\alpha_0$ be the restriction of $\overline{\alpha}$ to $H^2(Z_0)$. Using the commutative diagram

$$\begin{array}{ccc}
H^2(Z) & \to & H^2(Z_0) \\
\downarrow & & \downarrow \\
H^2(W) & \to & H^2(Y_0)
\end{array}$$

we see that $\iota_{Y_0,Z_0}^*(\alpha_0) = \alpha_0$. Since $H^2(Z_0) \cong \mathbb{Z}/2\mathbb{Z}$, the image of $\iota_{Y_0,Z_0}^*$ is generated by $\alpha_0$.

Since $L_0$ is null-homologous in $Z_0$, there exists a properly embedded oriented surface $F_0 \subset Z_0$ such that $\partial F_0 = L_0$. Thus the image of the Poincaré dual of $[F_0]$ under $\iota_{Y_0,Z_0}^*$ is the Poincaré dual of $[L_0]$. Since both $[L_0]$ and $[\alpha_0]$ have order 2, and $\iota_{Y_0,Z_0}^*(\alpha_0) = \alpha_0$, we get that $[L_0]$ is the Poincaré dual of $\alpha_0$.

**Lemma 4.2.** (1) For $i = 0, 1$, we have $\ker \iota_i^* \cong H^2(W, Y_i)$, and $\iota_i^*$ is surjective. In particular, $\alpha_i \neq 0$ in $H^2(Y_i)$.

(2) The kernel of the restriction map $(\iota_0^*)^* : \ker \iota_1^* \to H^2(Y_0)$ is isomorphic to $H^2(W, \partial W)$, and its image is generated by $\alpha_0$.

**Proof.** (1) The first statement follows from the long exact sequence

$$0 = H^1(Y_i) \to H^2(W, Y_i) \to H^2(W) \xrightarrow{\iota_i^*} H^2(Y_i) \to H^3(W, Y_i) = 0.$$ 

It follows that $\ker \iota_i^*$ is torsion-free, so $\alpha \notin \ker \iota_i^*$. Thus $\alpha_i \neq 0$.

(2) By (1), the map $(\iota_0^*)^*$ can be identified with $H^2(W, Y_1) \to H^2(Y_0)$, which is part of the long exact sequence

$$0 = H^1(\partial W, Y_1) \to H^2(W, \partial W) \to H^2(W, Y_1) \to H^2(\partial W, Y_1) = H^2(Y_0).$$

Thus $\ker(\iota_0^*)^*$ is $H^2(W, \partial W)$. 


By Poincaré duality, $(ι^*_{0})^*$ can be identified with the boundary map $∂_0^* : H_2(W, Y_0) → H_1(Y_0)$. By the handle decomposition of $W$, we see that the image of $∂_0^*$ is generated by $[L_0]$. By Lemma 4.1, $\text{im}(ι^*_{0})^*$ is generated by $α_0$. □

**Corollary 4.3.** For each $t ∈ Spin^c(Y_1)$, there exists a subset

\[ \mathcal{R}(t) = \{ ι_0, ι_1 = ι_0 + α_0 \} \subset Spin^c(Y_0) \]

such that for each $r ∈ Spin^c(Y_0)$, the set

\[ (ι^*_{0}, ι^*_1)^*(r, t) := (ι^*_{0})^{-1}(r) ∩ (ι^*_1)^{-1}(t) \]

is nonempty if and only if $r ∈ \mathcal{R}(t)$. Moreover, the set (16) is an $H^2(W, ∂W)$-torsor when it is nonempty.

**Proof.** This follows from Lemma 4.2 and the fact that $Spin^c$ is an $H^2$-torsor. □

By the long exact sequence

\[ 0 = H_2(Y_0) → H_2(W) → H_2(W, Y_0) → H_1(Y_0), \]

$H_2(W)$ embeds as an index–2 subgroup of $H_2(W, Y_0) ≅ \mathbb{Z}^{n+1}$. Thus we can extend the intersection form on $H_2(W)$ to $H_2(W, Y_0)$, with value in $\frac{1}{2}\mathbb{Z}$. Let

\[ \mathcal{L} ≅ H_2(W, Y_0) ≅ H_2(Z, Z_0) ≅ H_2(Z) \]

be the intersection lattice on the pair $(W, Y_0)$. Suppose that the generators corresponding to the two-handles are $x_0, \ldots, x_n$, where $x_0$ corresponds to the two-handle attached along $L_0$. Let

\[ L_0 = \langle 2x_0, x_1, \ldots, x_n \rangle \]

be the sublattice of $\mathcal{L}$ generated by $2x_0, x_1, \ldots, x_n$; then $L_0$ can be identified with the intersection lattice $H_2(W)$. Let

\[ \mathcal{L}^* = Hom(\mathcal{L}, \mathbb{Z}), \mathcal{L}^*_0 = Hom(\mathcal{L}_0, \mathbb{Z}) ⊂ \mathcal{L}^*. \]

Using the inner product on $\mathcal{L}$, we can embed $\mathcal{L}^*$ and $\mathcal{L}^*_0$ as sublattices of $\mathcal{L} ⊗ \mathbb{Q}$.

Let

\[ \tilde{\mathcal{C}} = \{ y ∈ \mathcal{L}^*_0 | \langle y, 2x_0 \rangle \equiv \langle 2x_0, x_0 \rangle, \langle y, x_j \rangle \equiv \langle x_j, x_j \rangle \ (\text{mod } 2), \ j > 0 \}. \]

Let $H^2(W) = H^2(W)/Tors = \mathcal{L}_0^*$ and let $\tilde{c}_1 : Spin^c(W) → H^2(W)$ be the composition of the map $c_1 : Spin^c(W) → H^2(W)$ and the quotient map $H^2(W) → H^2(W)$. Then $\tilde{\mathcal{C}}$ is the image of $\tilde{c}_1$.

**Proposition 4.4.** (1) The quotient $Spin^c(Y_1)/⟨α_1⟩$ can be identified with $\tilde{\mathcal{C}}/2\mathcal{L}$.

(2) Under the previous identification, suppose that the $⟨α_1⟩$-orbit $\{ t, t + α_1 \}$ is identified with $y + 2\mathcal{L}$ for some $y ∈ \tilde{\mathcal{C}}$. Let $\mathcal{R}(t) = \{ ι_0, ι_1 \}$. Then there exist $y_0, y_1 ∈ y + 2\mathcal{L}$, such that

\[ \tilde{c}_1((ι^*_0, ι^*_1)^*(r_0, t)) = y_0 + 2\mathcal{L}_0, \quad \tilde{c}_1((ι^*_0, ι^*_1)^*(r_1, t)) = y_1 + 2\mathcal{L}_0, \]

and

\[ \tilde{c}_1((ι^*_0, ι^*_1)^*( r_0, t + α_1)) = y_0 + 2\mathcal{L}_0, \quad \tilde{c}_1((ι^*_0, ι^*_1)^*( r_1, t + α_1)) = y_1 + 2\mathcal{L}_0. \]
Proof. (1) By Lemma 4.2, every \( t \in \text{Spin}^c(Y_1) \) is in the image of \( \iota_0^* \), and \( s_1, s_2 \in \text{Spin}^c(W) \) restrict to the same \( t \in \text{Spin}^c(Y_1) \) if and only if \( s_1 - s_2 \in H^2(W, Y_1) \cong H_2(W, Y_0) = \mathcal{L} \). So \( \text{Spin}^c(Y_1) \cong \text{Spin}^c(W) / \mathcal{L} \). Consider the map \( \bar{c}_1 : \text{Spin}^c(W) \to \bar{C} \). It is surjective, and \( \bar{c}_1(s_1) = \bar{c}_1(s_2) \) if and only if \( s_1 - s_2 \in \langle \alpha \rangle \). Using the formula

\[
c_1(s_1) - c_1(s_2) = 2(s_1 - s_2)
\]

we get that \( \text{Spin}^c(Y_1) / \langle \alpha_1 \rangle \cong \text{Spin}^c(W) / (\mathcal{L} + \langle \alpha \rangle) \cong \bar{C} / 2\mathcal{L} \).

(2) By Corollary 4.3, there exist \( s_0, s_1 \in \text{Spin}^c(W) \), such that

\[
(\iota_0^*, \iota_1^*)^{-1}(r_0, t) = s_0 + \mathcal{L}_0, \quad (\iota_0^*, \iota_1^*)^{-1}(r_1, t) = s_1 + \mathcal{L}_0.
\]

Since

\[
\iota_0^*(s_0 + \alpha) = \iota_0^*(s_1) + \alpha_0 = r_0 + \alpha_0 = r_0, \quad \iota_0^*(s_0 + \alpha) = r_1,
\]

we also have

\[
(\iota_0^*, \iota_1^*)^{-1}(r_0, t + \alpha_0) = s_0 + \alpha + \mathcal{L}_0, \quad (\iota_0^*, \iota_1^*)^{-1}(r_1, t + \alpha_1) = s_0 + \alpha + \mathcal{L}_0.
\]

Applying \( \bar{c}_1 \) to the above equalities, we get our conclusion. \( \square \)

For any \( s \in \text{Spin}^c(W) \), let

\[
gr(W, s) = \frac{c_1^2(s) + b_2(W)}{4}.
\]

For any \( t \in \text{Spin}^c(Y_1) \), let

\[
D_W(Y_1, t) = \max_{s \in \text{Spin}^c(W)} (d(Y_0, s|Y_0) + gr(W, s)).
\]

**Lemma 4.5.** There are exactly two Spin\(^c\) structures \( c_0, c_1 \in \text{Spin}^c(Y_0) \) which can be extended over \( Z_0 \). Moreover,

\[
c_1 = c_0 + \alpha_0, \quad d(Y_0, c_i) = 0, \quad i = 0, 1.
\]

**Proof.** By Lemma 4.1, \( \alpha_0 \) is the restriction of a cohomology class in \( H^2(Z_0) \). Let \( c_0 \in \text{Spin}^c(Y_0) \) be a Spin\(^c\) structure which is the restriction of a Spin\(^c\) structure on \( Z_0 \), then \( c_1 := c_0 + \alpha_0 \) also extends over \( Z_0 \). Since \( H^2(Z_0) \cong \mathbb{Z} / 2\mathbb{Z} \), \( c_0, c_1 \) are the only two Spin\(^c\) structures which can be extended over \( Z_0 \). It follows from [OS03a, Proposition 9.9] that \( d(Y_0, c_i) = 0 \). \( \square \)

**Lemma 4.6.** The image of

\[
\bar{c}_1 : (\iota_0^*)^{-1}(\{c_0, c_1\}) \to \overline{\Pi^2}(W)
\]

is \( \mathcal{C} := \text{Char}(\mathcal{L}) \).

**Proof.** Let \( s_0 \) be the restriction of a Spin\(^c\) structure on \( Z \) to \( W \), then \( s_0 \in (\iota_0^*)^{-1}(\{c_0, c_1\}) \). Clearly, \( \bar{c}_1(s_0) \in \mathcal{C} \). By Lemma 4.1, \( \iota_0^*|_{W,Z} \) is injective, so the image of \( H^2(Z) \) in \( \overline{\Pi^2}(W) \) can be identified with \( Hom(H_2(Z), \mathbb{Z}) = Hom(H_2(W, Y_0), \mathbb{Z}) = \mathcal{L}^* \). Thus \( \bar{c}_1((\iota_0^*)^{-1}(\{c_0, c_1\})) \) is a \( 2\mathcal{L}^* \)-torsor. Since \( \mathcal{C} \) is the unique \( 2\mathcal{L}^* \)-torsor containing \( \bar{c}_1(s_0) \), our conclusion holds. \( \square \)
Corollary 4.7. The sum
\[ \sum_{t \in \text{Spin}^c(Y_1)} D_W(Y_1, t) \]  
only depends on the lattice \( L \) and the correction terms of \( Y_0 \).

In fact, if we write (19) as a function
\[ \mathcal{D}(L, \{d_0, d_1\}) \]
of \( L \) and the multiset \( \{d_0, d_1\} \) of the correction terms of the two Spin\(^c\) structures other than \( \varepsilon_0, \varepsilon_1 \), then
\[ \mathcal{D}(L, \{d_0 + c, d_1 + c\}) = \mathcal{D}(L, \{d_0, d_1\}) + c|L_0^\ast/L| \]  
for any \( c \in \mathbb{Q} \). Note that, by Proposition 4.4, \( |H_1(Y_1)| = 2|L_0^\ast/L| \).

Proof. We will give the procedure of computing (19) from \( L \) and the correction terms of \( Y_0 \).

Let \( \varepsilon_0, \varepsilon_1 \) be the two Spin\(^c\) structures other than \( \varepsilon_0, \varepsilon_1 \) on \( Y_0 \). We choose \([z] \in \tilde{C}/2L\). By Proposition 4.4, \([z]\) corresponds to a pair of Spin\(^c\) structures \( t_0, t_1 = t_0 + \alpha_1 \in \text{Spin}^c(Y_1) \).

There are exactly two \( 2L_0^\ast\)-torsors contained in \( z + 2L \), denoted by \( T_0, T_1 \).

Next we check whether \( z + 2L \) is contained in \( C \). If it is contained in \( C \), it follows from Lemma 4.6 that each \( t_i \) is cobordant to \( \varepsilon_0 \) and \( \varepsilon_1 \), \( i = 0, 1 \). Since \( d(Y_0, \varepsilon_0) = d(Y_0, \varepsilon_1) = 0 \), by Proposition 4.4,
\[ D_W(Y_1, t_0) = D_W(Y_1, t_1) = 0 + \max_{y \in z + 2L} \frac{-\langle y, y \rangle + b_2(W)}{4}. \]

If \( z + 2L \) is not contained in \( C \), then each \( t_i \) is cobordant to \( \varepsilon_0 \) and \( \varepsilon_1 \). By Proposition 4.4, the multiset \( \{D_W(Y_1, t_0), D_W(Y_1, t_1)\} \) is equal to
\[ \left\{ \max_{y \in T_0} \frac{-\langle y, y \rangle + b_2(W)}{4}, d(Y_0, \varepsilon_0) = 0, d(Y_0, \varepsilon_1) = 0 \right\}, \]
\[ \max_{y \in T} \frac{-\langle y, y \rangle + b_2(W)}{4} \right\}. \]

Finally, to get (19), we add all the \( D_W(Y_1, t_0) + D_W(Y_1, t_1) \) together, for all \([z] \in \tilde{C}/2L\).

The equality (20) follows from the above procedure, since exactly \( \frac{1}{2}|H_1(Y_1)| \) values of \( D_W(Y_1, t) \) are increased by \( c \) after increasing \( d(Y_0, \varepsilon_i) \) by \( c \), \( i = 0, 1 \). \( \square \)

5. Sharp cobordisms

In this section, we will generalize the notion of sharp 4–manifolds defined by Greene [Gre15] to 4–dimensional cobordisms, and prove that certain cobordisms between prism manifolds are sharp. Recall that a smooth, compact, negative definite 4–manifold \( X \) with \( \partial X = Y \) is sharp if for every \( t \in \text{Spin}^c(Y) \), there exists some \( s \in \text{Spin}^c(X) \) extending \( t \) such that
\[ c_1(s)^2 + b_2(X) = 4d(Y, t) \]
**Definition 5.1.** Let $W : Y_0 \to Y_1$ be a smooth, connected, negative definite cobordism between two rational homology spheres $Y_0$ and $Y_1$. We say $W$ is **sharp**, if for any $t \in \text{Spin}^c(Y_1)$ we have

$$d(Y_1, t) = D_W(Y_1, t).$$

Here $D_W$ is defined using the formula (18).

**Lemma 5.2.** Let $Y_1, Y_2, Y_3$ be rational homology spheres, $W_1 : Y_1 \to Y_2$ and $W_2 : Y_2 \to Y_3$ be two negative definite cobordisms. If $W = W_1 \cup_{Y_2} W_2$ is sharp, then $W_2$ is sharp.

**Proof.** Let $s \in \text{Spin}^c(W)$ and let $s_i = s|W_i$, $i = 1, 2$, then

$$c_1^2(s) = c_1^2(s_1) + c_1^2(s_2).$$

Our conclusion follows from the the above equality. □

5.1. **A Kirby diagram of $P(p, q)$.** Suppose that

$$\frac{p}{q} = [a_{-1}, a_0, \ldots, a_n]^-$$

as in (13), where each $a_i$ is $\geq 2$ when $i \geq 0$.

![Kirby Diagram](image)

**Figure 2.** A manifold bounded by $P(p, q)$. If we replace the leftmost component with a dotted circle, we get a negative definite 4–manifold $Z(p, q)$.

Figure 2 is a surgery diagram of $P(p, q)$. The leftmost two components give rise to a surgery diagram of $P(a_{-1}, 1)$, and other components give rise to a negative definite cobordism

$$W(p, q) : P(a_{-1}, 1) \to P(p, q).$$

If we replace the leftmost component, which is unknotted with slope 0, with a dotted circle representing a one-handle, we get a negative definite 4–manifold $Z(p, q)$ bounded by $P(p, q)$, and the two leftmost components give rise to a rational homology ball $Z_{a_{-1}}$ bounded by $P(a_{-1}, 1)$, with $H_1(Z_{a_{-1}}) = \mathbb{Z}/2\mathbb{Z}$.

The main result of this section is the following proposition.

**Proposition 5.3.** The cobordism $W(p, q)$ is sharp.

For simplicity, we only prove the case $q < p < 2q$. The proof of the general case is similar. From now on, let $W = W(p, q)$. 
5.2. More Kirby diagrams. We will consider 3 other cobordisms. When \( q < p < 2q \), \( a_{-1} = 2 \). We have
\[
\frac{2q - (p - q)}{q - (p - q)} = 1 + \frac{q}{2q - p} = [a_0 + 1, a_1, \ldots, a_n]^-,
\]
Consider the following surgery diagram of \( P(p - q, q) \). By [BNOV17], this diagram gives rise to a sharp 4–manifold bounded by \( P(p - q, q) \). The component with label \(-4\) gives rise to \( P(1, 1) = L(4, -1) \), and the other two-handles give rise to a cobordism
\[
W_1 : P(1, 1) \to P(p - q, q).
\]

![Figure 3. A sharp 4–manifold \( X(p - q, q) \) bounded by \( P(p - q, q) \).](image)

Let
\[
\frac{p + q}{p} = [a'_0, a'_1, \ldots, a'_m]^-.
\]
By [BHM+16], \( P(p, -q) \) has a surgery diagram as in Figure 4, which gives rise to a sharp 4–manifold bounded by \( P(p, -q) \). The two components with label \(-2\) give rise to \( P(0, 1) = \mathbb{RP}^3 \# \mathbb{RP}^3 \), and the other two-handles give rise to a cobordism
\[
W' : P(0, 1) \to P(p, -q).
\]
Using the continued fraction
\[
\frac{-2q - (p - q)}{-q - (p - q)} = \frac{p + q}{p} = [a'_0, a'_1, \ldots, a'_m]^-,
\]
by [BNOV17], we get a surgery diagram of \( P(p - q, -q) \) as in Figure 5, which gives rise to a sharp 4–manifold bounded by \( P(p - q, -q) \). The component with label \(-4\) gives rise to \( P(1, 1) = L(4, -1) \), and the other two-handles give rise to a cobordism
\[
W'_1 : P(1, 1) \to P(p - q, -q).
\]
By Lemma 5.2, \( W_1, W', W'_1 \) are all sharp cobordisms.

**Lemma 5.4.** The intersection lattices on \((W, P(2, 1))\) and \((W_1, P(1, 1))\) are isomorphic; also, the intersection lattices on \((W', P(0, 1))\) and \((W'_1, P(1, 1))\) are isomorphic.
Proof. In Figure 2, consider the knot $L_0$ with label $-a_0$. The canonical longitude on $L_0$ is clearly rationally null-homologous in $P(2, 1) \setminus L_0$. As a result, the square of the generator of $H_2(W, P(2, 1))$ corresponding to the two-handle attached along $L_0$ is $-a_0$. In Figure 3, consider the knot $K_0$ with label $-(a_0 + 1)$. If the framing on $K_0$ is $-1$, the manifold we get by doing surgery on the two leftmost components is $P(1, 0)$ which has $b_1 > 0$. Thus the slope $-1$ on $K_0$ is rationally null-homologous in $P(1, 1) \setminus K_0$. As a result, the square of the generator of $H_2(W_1, P(1, 1))$ corresponding to the two-handle attached along $K_0$ is $-a_0$. So the intersection lattices on $(W, P(2, 1))$ and $(W_1, P(1, 1))$ are isomorphic.

Similarly, we see that the square of the generator of $H_2(W', P(0, 1))$ and $H_2(W'_1, P(1, 1))$ corresponding to the two-handle attached along the knot with label $-a'_0$ is $-(a'_0 - 1)$. So the intersection lattices are isomorphic. □

**Lemma 5.5.** All four cobordisms $W, W_1, W', W'_1$ satisfy the assumptions in the beginning of Section 4.

**Proof.** The cobordism $W$ satisfies the assumptions by its construction.
For $W_1, W'_1$, notice that $P(1, 1)$ bounds a rational homology ball $Z_1$ with $H_1(Z_1) \cong \mathbb{Z}/2\mathbb{Z}$. Since $H_1(P(1, 1))$ is cyclic, the kernel of the surjective map $H_1(P(1, 1)) \to H_1(Z_1)$ is $2H_1(P(1, 1))$. From Figures 3 and 5, we see that the knot with label $-(a_0 + 1)$ or $-a'_0$ represents an element in $2H_1(P(1, 1))$. So $W_1, W'_1$ satisfy the assumptions.

For $W'$, the rational ball bounded by $\mathbb{R}P^3 \# \mathbb{R}P^3$ is $Z_0 = (\mathbb{R}P^3 \setminus B^3) \times I$. Clearly, the knot labeled with $-a'_0$ in Figure 4 is null-homologous in $Z_0$. □

5.3. The proof of Proposition 5.3. Recall from Section 5.1 that $P(a, 1)$ bounds a rational homology ball $Z_a$ with $H_1(Z_a) \cong \mathbb{Z}/2\mathbb{Z}$. There are exactly two Spin$^c$ structures $\mathfrak{o}_0, \mathfrak{e}_1 \in \text{Spin}^c(P(a, 1))$ which extend over $Z_a$. Let $\mathfrak{o}_0, \mathfrak{o}_1 \in \text{Spin}^c(P(a, 1))$ be two other Spin$^c$ structures, such that $d(P(a, 1), \mathfrak{o}_0) \geq d(P(a, 1), \mathfrak{e}_0)$.

**Lemma 5.6.** The correction terms of $P(a, 1)$ are

$$d(P(a, 1), \mathfrak{o}_0) = d(P(a, 1), \mathfrak{e}_1) = 0,$$

$$d(P(a, 1), \mathfrak{o}_0) = -\frac{a + 2}{4}, \quad d(P(a, 1), \mathfrak{o}_1) = -\frac{a - 2}{4}.$$  

**Proof.** The correction terms of $P(a, 1)$ are computed in [Doi15, Example 15], and they are $\{0, 0, -\frac{a + 2}{4}, -\frac{a - 2}{4}\}$. It is a standard fact that $d(P(a, 1), \mathfrak{e}_i) = 0, i = 0, 1$ [OS03a, Proposition 9.9]. So we must have $d(P(a, 1), \mathfrak{o}_i) = -\frac{a + 2}{4} + i, i = 0, 1$, by our choice of $\mathfrak{o}_0, \mathfrak{o}_1$. □

**Proof of Proposition 5.3 in the case $a_1 = 2$.** By [OS03a, Theorem 9.6],

$$d(P(p, q), t) \geq D_W(P(p, q), t). \quad (21)$$

Also, since $W_1, W', W'_1$ are sharp, we have

$$d(P(p - q, q), t_1) = D_{W_1}(P(p - q, q), t_1),$$

$$d(P(p, -q), t) = D_{W'}(P(p, -q), t),$$

$$d(P(p - q, -q), t_1) = D_{W'_1}(P(p - q, -q), t_1).$$

By Corollary 4.7, Lemma 5.4 and Lemma 5.6,

$$\sum_{t \in \text{Spin}^c(P(p, q))} D_W(P(p, q), t) = -\frac{2q}{4} + \sum_{t_1 \in \text{Spin}^c(P(p - q, q))} D_{W_1}(P(p - q, q), t_1),$$

$$-\frac{2q}{4} + \sum_{t \in \text{Spin}^c(P(p, -q))} D_{W'}(P(p, -q), t) = \sum_{t_1 \in \text{Spin}^c(P(p - q, -q))} D_{W'_1}(P(p - q, -q), t_1).$$
Adding the above two equalities together, and using (21) and the three equalities after it, we get

\[
0 = \sum_{t \in \text{Spin}^r(P(p,q))} d(P(p,q), t) + \sum_{t \in \text{Spin}^r(P(p,-q))} d(P(p,-q), t) \\
\geq \sum_{t \in \text{Spin}^r(P(p,q))} D_W(P(p,q), t) + \sum_{t \in \text{Spin}^r(P(p,-q))} D_W(P(p,-q), t) \\
= \sum_{t_1 \in \text{Spin}^r(P(p-q,q))} D_{W_1}(P(p-q,q), t_1) + \sum_{t_1 \in \text{Spin}^r(P(p-q,-q))} D_{W_1}(P(p-q,-q), t_1) \\
= \sum_{t_1 \in \text{Spin}^r(P(p-q,q))} d(P(p-q,q), t_1) + \sum_{t_1 \in \text{Spin}^r(P(p-q,-q))} d(P(p-q,-q), t_1) \\
= 0.
\]

So the equality in (21) must hold. \(\square\)

6. The changemaker condition when \(q < p < 2q\)

6.1. Positive definite manifold with boundary \(P(2,1)\). The goal of this subsection is to prove the following proposition.

**Proposition 6.1.** If \(X\) is a positive definite, simply connected four-manifold with \(\partial X \cong P(2,1)\), then the intersection form of \(X\) is isomorphic to \(D_4 \oplus \mathbb{Z}^{n-4}\) for some \(n\).

**Lemma 6.2.** If \(L \subset \mathbb{Z}^n\) is an index-two sublattice, then \(L \cong D_k \oplus \mathbb{Z}^{n-k}\) for some \(k \geq 1\). (In fact, there are indices \(i_1, \ldots, i_k\) such that \(L\) contains exactly the elements of \(\mathbb{Z}^n\) that have even pairing with \(e_{i_1} + \cdots + e_{i_k}\).) There are always two elements \(x \in \mathcal{T}\) with \(b(x,x) = 0 \pmod{1}\), and the other two elements satisfy \(b(x,x) = k/4 \pmod{1}\).

**Proof.** Let \(L \subset \mathbb{Z}^n\) have index two, and let \(i_1, \ldots, i_k\) be an enumeration of the indices \(i\) for which \(e_i \notin L\). Since \(L\) has index two, the elements \(\pm e_{i_j} \pm e_{i_j}\), are all in \(L\). Since these elements generate \(D_k\), we have \(L \cong D_k \oplus \mathbb{Z}^{n-k}\).

The dual lattice \(L^*\) is the set of elements of \(\mathbb{Q}^n\) with integral inner product with each element of \(L\), and in this representation we have that \(L^*\) is the set of vectors with integer components in all entries other than \(i_1, \ldots, i_k\), and with the components in entries \(i_1, \ldots, i_k\) either all integers or all half integers. Therefore, the discriminant group \(\mathcal{T}\) can be represented by the four vectors \(0, z = e_{i_1}, \text{ and} \)

\[
a = \frac{1}{2} (e_{i_1} + e_{i_2} + \cdots + e_{i_k}),
\]

\[
b = \frac{1}{2} (-e_{i_1} + e_{i_2} + \cdots + e_{i_k}).
\]

We have \(\langle z, z \rangle = 1 \equiv 0 \pmod{1}\), and \(\langle a, a \rangle = \langle b, b \rangle = k/4\). \(\square\)

**Lemma 6.3.** The \(d\)-invariant of \(L = D_k \oplus \mathbb{Z}^{n-k}\) takes on the values \(0,0,-k/4,1-k/4\).
Proof. The \( d \)-invariant is invariant under stable isomorphisms, so we can assume \( L = D_k \). Then a set of short representatives of the classes of characteristic covectors is \((1,\ldots,1)\), \((-1,\ldots,1)\), \((0,\ldots,0)\), and \((2,0,\ldots,0)\). These have norms \( k \), \( k \), \( 0 \), and \( 4 \). The result now follows: see Equation (12).

Proof of Proposition 6.1. As in Section 5.1, \( P(2,1) \) bounds a rational homology ball \( Z_2 \) with

\[
H_1(Z_2) \cong \mathbb{Z}/2\mathbb{Z}, \quad H_2(Z_2) = 0.
\]

If \( X \) is any simply connected positive definite 4-manifold with boundary \( P(2,1) \), then \( \hat{X} := X \cup_{P(2,1)} (-Z_2) \) is a closed, positive definite 4-manifold. Since \( \hat{X} \) can be obtained from \( X \) by attaching a two-handle, a three-handle and a four-handle, \( \hat{X} \) is also simply connected. By [Don83], \( \hat{X} \) has intersection form \( \mathbb{Z}^n \).

In the long exact sequence for the pair \((\hat{X},X)\), we have

\[
H_3(\hat{X},X) \to H_2(X) \to H_2(\hat{X}) \to H_2(\hat{X},X) \to H_1(X).
\]

We have \( H_3(\hat{X},X) \cong H_3(Z_2,\partial Z_2) \cong H^1(Z_2) = 0 \), \( H_2(\hat{X},X) \cong H^2(Z_2) \cong \mathbb{Z}/2\mathbb{Z} \), \( H_1(X) = 0 \), and both \( H_2(X) \) and \( H_2(\hat{X}) \) are torsionfree. Therefore, we have a short exact sequence

\[
0 \to H_2(X) \to H_2(\hat{X}) \to \mathbb{Z}/2\mathbb{Z} \to 0,
\]

so \( H_2(X) \) is an index-two subgroup of \( H_2(\hat{X}) \) under the natural inclusion map. Since \( \hat{X} \) has intersection lattice \( \mathbb{Z}^n \), the intersection lattice of \( X \) is an index-two sublattice of \( \mathbb{Z}^n \), so, by Lemma 6.2, is isomorphic to \( D_k \oplus \mathbb{Z}^{n-k} \).

Let \( X_0 \) be the positive definite plumbing 4-manifold with intersection form \( D_4 \), then \( P(2,1) = \partial X_0 \). Since the discriminant group and linking pairing of the intersection form of a 4-manifold are invariants of its boundary, Lemma 6.2 implies that \( k \) must be divisible by 4. Since the \( d \)-invariant of the intersection form of a positive definite 4-manifold gives an upper bound on the \( d \)-invariant of its boundary [OS03a] and \(-X_0\) is sharp [OS03b], Lemma 6.3 implies that \( k \leq 4 \). Therefore, \( k = 4 \), and the result follows. \( \square \)

Corollary 6.4. Any negative definite, simply connected 4-manifold with boundary \(-P(2,1)\) is sharp.

Proof. The 4–manifold \(-X_0\) is sharp. By Proposition 6.1, any negative definite, simply connected 4-manifold with boundary \(-P(2,1)\) has the same intersection form as that of \(-X_0\#(n - 4)CP^2\). \( \square \)

6.2. The changemaker condition. Whenever \( q < p < 2q \), using Proposition 5.3, there is a sharp cobordism \( W \) from \( P(2,1) \) to \( P(p,q) \). Suppose \( P(p,q) \) is positive surgery on some knot \( K \subset S^3 \). Let \( X = W \cup_{P(p,q)} (-W_{4q}(K)) \), then \( X \) is a negative definite manifold with boundary \(-P(2,1)\). Since \( X \) is obtained from \( W_{4q} \) (which is simply connected) by adding two-handles, \( X \) is simply connected. By combining Corollary 6.4 and Proposition 6.1, \( X \) is sharp and has intersection lattice \(-D_1 \oplus \mathbb{Z}^{n-2} \). Also, for \( Z_2 \) the rational homology ball with boundary \( P(2,1) \), the manifold \( \tilde{X} = X \cup_{P(2,1)} (-Z_2) \) is closed, simply connected and negative definite,
so has intersection lattice $-\mathbb{Z}^{n+2}$. From Kirby diagrams for $W$ and $Z = W \cup P_{(2,1)} (-Z_2)$ (see Figure 2), we can also see that the intersection lattice of $Z$ is the linear lattice $\Lambda(q, -p)$ with vertex basis $x_0, \ldots, x_n$, and the intersection lattice of $W$ is (as a sublattice of $\Lambda(q, -p)$) spanned by $2x_0, x_1, \ldots, x_n$. Therefore, the following diagram of homology groups

$\begin{array}{ccc}
H_2(W) & \longrightarrow & H_2(Z) \\
\downarrow & & \downarrow \\
H_2(X) & \longrightarrow & H_2(\hat{X})
\end{array}$

with maps induced by inclusions is isomorphic to the diagram

$\begin{array}{ccc}
\langle 2x_0, x_1, \ldots, x_n \rangle & \longrightarrow & \langle x_0, x_1, \ldots, x_n \rangle = -\Lambda(q, -p) \\
\downarrow & & \downarrow \\
-(D_4 \oplus \mathbb{Z}^{n-2}) & \longrightarrow & -\mathbb{Z}^{n+2}.
\end{array}$

**Lemma 6.5.** Regarding $H_2(W)$ as subgroups of $H_2(Z)$ and $H_2(X)$, which are subgroups of $H_2(\hat{X})$, then

$H_2(W) = H_2(Z) \cap H_2(X).$

**Proof.** By the exact sequence $H_2(Z) \to H_2(\hat{X}) \to H_2(\hat{X}, Z)$, an element $\beta \in H_2(\hat{X})$ is contained in the image of $H_2(Z)$ if and only if the image of $\beta$ in $H_2(\hat{X}, Z) \cong H_2(W_{4q}(K), \partial W_{4q}(K))$ is zero. Similarly, $\beta$ is contained in the image of $H_2(X)$ if and only if the image of $\beta$ in $H_2(\hat{X}, X) \cong H_2(Z_2, \partial Z_2)$ is zero, and $\beta$ is contained in the image of $H_2(W)$ if and only if the image of $\beta$ in $H_2(\hat{X}, W) \cong H_2(Z_2, \partial Z_2) \oplus H_2(W_{4q}(K), \partial W_{4q}(K))$ is zero. Our conclusion follows easily. \qed

The last piece of data we need is the class $[\hat{F}] \in H_2(-W_{4q}(K)) \subset H_2(X)$, where $\hat{F}$ is obtained by smoothly gluing the core of the handle attachment to a copy of a minimal genus Seifert surface $F$ for $K$; its homology class generates the second homology. Note that $H_2(-W_{4q}(K))$ is orthogonal to all of $H_2(W)$ and satisfies $\langle [\hat{F}], [\hat{F}] \rangle = -4q$ since $-W_{4q}(K)$ is negative definite. Let

$\varphi : \mathbb{Z}/4q\mathbb{Z} \to \text{Spin}^c(P(p, q))$

be the correspondence with $\varphi(i)$ equal $s_0|_{p(q, q)}$ for $s_0$ any Spin$^c$ structure on $-W_{4q}(K)$ satisfying

$\langle c_1(s_0), [\hat{F}] \rangle \equiv -4q + 2i \pmod{8q}.

**Proposition 6.6.** There is an extension $r \in \text{Spin}^c(X)$ of $\varphi(i)$ over $X$ with $c_1(r)$ a short characteristic covector of $D_4 \oplus \mathbb{Z}^{n-2}$ if and only if $g(K) \leq i \leq 4q - g(K)$.

**Proof.** Since $X$ has boundary $-P(2, 1)$ and $b_2(X) = n + 2$, we have that for any $r \in \text{Spin}^c(X)$,

$d(-P(2, 1), r|_{P(2, 1)}) \geq \frac{(c_1(r))^2 + (n + 2)}{4}, \quad (22)$
and since $X$ is sharp this is an equality if and only if $c_1(\tau)$ is a short characteristic covector of $-H_2(X) = D_4 \oplus \mathbb{Z}^{n-2}$. Similarly, for any $s_1 \in \text{Spin}^c(W)$,
\[
d(P(p, q), s_1|_{P(p, q)}) \geq d(P(2, 1), s_1|_{P(2, 1)}) + \frac{(c_1(s_1))^2 + (n + 1)}{4}
\] (23)
and since $W$ is sharp as a cobordism, for each $t \in \text{Spin}^c(P(p, q))$ there is some $s_1 \in \text{Spin}^c(W)$ such that this is an equality and $s_1|_{P(p, q)} = t$.

For $s_0 \in \text{Spin}^c(-W_{4q}(K))$ with
\[
\langle c_1(s_0), [\hat{F}] \rangle = -4q + 2i
\]
(so that in particular $\varphi(i) = s_0|_{P(p, q)}$), we have
\[
(c_1(s_0))^2 = -\frac{(4q - 2i)^2}{4q}.
\]
Using (8) and (9), we have
\[
d(P(p, q), s_0|_{P(p, q)}) = \frac{-(c_1(s_0))^2 - 1}{4} - 2t_{\min(i, 4q - i)}(K).
\]
Since $t_i(K) \geq 0$ and (5),
\[
d(P(p, q), s_0|_{P(p, q)}) \leq \frac{-(c_1(s_0))^2 - 1}{4}
\] (24)
with equality if and only if $\langle c_1(s_0), [\hat{F}] \rangle = -4q + 2i$ for some $i$ with $g(K) \leq i \leq 4q - g(K)$. Note that inequality (22) is the difference of inequalities (24) and (23) if $s_0|_{P(p, q)} = s_1|_{P(p, q)}$. If $g(K) \leq i \leq 4q - g(K)$, then there is some extension $s_0$ of $\varphi(i)$ over $-W_{4q}(K)$ that achieves equality in (24), and there is always some extension $s_1$ of $\varphi(i)$ over $W$ achieving equality in (23). These two Spin$^c$ structures glue to a Spin$^c$ structure $\tau$ on $X = W \cup (-W_{4q}(K))$ that will achieve equality in (22), so $c_1(\tau)$ is short and $\tau|_{P(p, q)} = \varphi(i)$.

Conversely, if $\tau \in \text{Spin}^c(X)$ has $c_1(\tau)$ short, then $\tau$ achieves equality in (22), so $s_0 = \tau|_{-W_{4q}(K)}$ and $s_1 = \tau|_W$ will achieve equality in (23) and (24), respectively. Therefore, $s_0|_{P(p, q)} = \tau|_{P(p, q)}$ will equal $\varphi(i)$ for some $g(K) \leq i \leq 4q - g(K)$. □

Putting all of these together, we have a Euclidean lattice $\mathbb{Z}^{n+2} = -H_2(\hat{X})$, with a corank–1, linear sublattice
\[
-H_2(W) \cong \Lambda(q, -p) = \langle x_0, \ldots, x_n \rangle
\]
and a sublattice $D_4 \oplus \mathbb{Z}^{n-2} = -H_2(X)$ such that
\[
\langle 2x_0, \ldots, x_n \rangle = \langle x_0, \ldots, x_n \rangle \cap (D_4 \oplus \mathbb{Z}^{n-2}).
\] (25)
Since $\Lambda(q, -p)$ has discriminant $q$ and corank $1$ and is embedded primitively in $\mathbb{Z}^{n+2}$ (this follows from the long exact sequence of the pair $(X \cup Z_0, W \cup Z_0)$), the orthogonal complement of $\Lambda(q, -p)$ has discriminant $q$ and rank 1, so is generated by a vector $\sigma$ with $\langle \sigma, \sigma \rangle = q$. Since $|(|[\hat{F}], [\hat{F}]|) = 4q$ and $[\hat{F}]$ is contained in the orthogonal complement of $\Lambda(q, -p)$, we must have $[\hat{F}] = 2\sigma$. Therefore, Proposition 6.6 gives the following:
Proposition 6.7. If \( P(p,q) \) is the result of 4q surgery on some knot \( K \subset S^3 \) and \( q < p < 2q \), then there is an embedding of \( \Lambda(q,-p) \) into \( \mathbb{Z}^{n-2} \) as the orthogonal complement of a vector \( \sigma \) and an embedding \( D_4 \oplus \mathbb{Z}^{n-2} \hookrightarrow \mathbb{Z}^{n+2} \) such that there exists some short characteristic covector \( \chi \) for \( D_4 \oplus \mathbb{Z}^{n-2} \) with \( \langle \chi, \sigma \rangle = 2q - i \) if and only if \(-2q + g(K) \leq i \leq 2q - g(K)\).

Pushing the logic of Proposition 6.6 a little further, the Alexander polynomial of \( K \) can be recovered from \( \sigma \):

**Proposition 6.8.** For \( 0 \leq i \leq 2q \), the torsion coefficient \( t_i(K) \) satisfies

\[
t_i(K) = \min_{\chi \in \text{Char}(D_4 \oplus \mathbb{Z}^{n-2})} \left[ \frac{\langle \chi, \chi \rangle - n - 2}{8} \right].
\]

**Proof.** Since \( \hat{F} = 2\sigma \) and the intersection lattice on \( X \) is \( D_4 \oplus \mathbb{Z}^{n-2} \), any characteristic covector \( \chi \) for \( D_4 \oplus \mathbb{Z}^{n-2} \) with \( \langle \chi, \sigma \rangle = 2q - i \) is the first Chern class of a Spin\(^c\) structure \( \tau \) on \( X \) with

\[
\langle c_1(\tau), [\hat{F}] \rangle = -4q + 2i.
\]

(Note that we need to change the sign of the inner product.) Then, exactly as in the proof of Proposition 6.6, the restriction of \( \tau \) to \(-W_{4q} = -W_{4q}(K)\) satisfies

\[
d(P(p,q), \tau|_{P(p,q)}) = \frac{-(c_1(\tau|_{-W_{4q}}))^2 - 1}{4} - 2t_i(K).
\]

Let \( s_1 \) be the restriction of \( \tau \) to \( W \), then \( s_1 \) satisfies

\[
d(P(p,q), s_1|_{P(p,q)}) \geq d(P(2,1), s_1|_{P(2,1)}) + \frac{(c_1(s_1))^2 + (n + 1)}{4}
\]

Combining (27) and (28) together,

\[
t_i(K) \leq \frac{-(c_1(\tau))^2 - (n + 2)}{8} - \frac{d(P(2,1), \tau|_{P(2,1)})}{2}.
\]

Using Proposition 5.3, some \( s_1 \in \text{Spin}^c(W) \) achieves equality in (28) with \( s_1|_{P(p,q)} = \varphi(i) \). Let \( \tau \in \text{Spin}^c(X) \) be the extension of \( s_1 \) with (26), then \( \tau \) achieves equality in (29). Therefore,

\[
t_i(K) = \min_{\tau \in \text{Spin}^c(X)} \left[ \frac{-(c_1(\tau))^2 - (n + 2)}{8} - \frac{d(P(2,1), \tau|_{P(2,1)})}{2} \right].
\]

Since \( t_i(K) \) is an integer and \( d(P(2,1), \tau|_{P(2,1)}) \) will always be either 0 or -1, we get

\[
t_i(K) = \min_{\tau \in \text{Spin}^c(X)} \left[ \frac{-(c_1(\tau))^2 - (n + 2)}{8} \right].
\]

Finally, Spin\(^c\) structures \( \tau \) on \( X \) with (26) correspond (under the first Chern class and a change in the sign of the inner product) with characteristic covectors \( \chi \) of \( D_4 \oplus \mathbb{Z}^{n-2} \) with \( \langle \chi, \sigma \rangle = 2q - i \), and \(- (c_1(\tau))^2 = \langle \chi, \chi \rangle \), so the desired formula follows. \( \square \)
By Proposition 6.1, specifying a sublattice $D_4 \oplus \mathbb{Z}^{n-2} \subset \mathbb{Z}^{n+2}$ is equivalent to choosing 4 indices $a > b > c > d$ such that for $v \in \mathbb{Z}^{n+2}$, $v \in D_4 \oplus \mathbb{Z}^{n-2}$ if and only if \( \langle v, e_a + e_b + e_c + e_d \rangle \) is even. The characteristic covectors for $D_4 \oplus \mathbb{Z}^{n-2}$ come in two types: those that are the restrictions of characteristic covectors of $\mathbb{Z}^{n+2}$, which can be represented by elements of $\mathbb{Z}^{n+2}$ with all entries odd, and those that are not, which can be represented by elements of $\mathbb{Z}^{n+2}$ with the entries in positions $a, b, c$, and $d$ even and all other entries odd. Call these two types of covectors even and odd, respectively. The short characteristic covectors are exactly the ones with all odd entries equal to $\pm 1$, and the even entries (if any) equal to $\pm 2$, 0, 0, and 0 in some order.

As in [Gre13], we will assume $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_{n+1})$ with

\[ 0 \leq \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_{n+1}. \]

Moreover, we can assume that for any two indices $i, j \in \{0, 1, \ldots, n+1\}$, we always have

\[ i > j, \quad \text{if } \sigma_i = \sigma_j, i \in \{a, b, c, d\}, \text{ and } j \notin \{a, b, c, d\}. \quad (32) \]

**Definition 6.9.** Let $\text{Short}(D_4 \oplus \mathbb{Z}^{n-2}) = \text{Short}_0 \cup \text{Short}_1$, with $\text{Short}_0 = \text{Short}(\mathbb{Z}^n)$ the set of even short characteristic covectors and $\text{Short}_1 = \text{Short}(D_4 \oplus \mathbb{Z}^{n-2}) - \text{Short}_0$ the set of odd characteristic covectors. Let

\[ \chi^0 = -\sum_{i=0}^{n+3} e_i \]

and

\[ \chi^1 = -2e_a - \sum_{i \notin \{a, b, c, d\}} e_i \]

be the elements of $\text{Short}_0$ and $\text{Short}_1$, respectively, minimizing $\langle \chi, \sigma \rangle$. Let

\[ T_0 = \left\{ \frac{1}{2}(\chi - \chi^0) \mid \chi \in \text{Short}_0 \right\} \]

and

\[ T_1 = \left\{ \frac{1}{2}(\chi - \chi^1) \mid \chi \in \text{Short}_1 \right\} \]

be called the sets of even and odd test vectors, respectively.

For $\chi \in \mathbb{Z}^{n+2}$, let $\chi_i$ denote the component of $\chi$ corresponding to the index $i$. The following result is easy to see.

**Proposition 6.10.** For $\chi \in T_1$, $(\chi_d, \chi_c, \chi_b, \chi_a) = (\pm 1, 0, 0, 1)$ or $(0, \pm 1, 0, 1)$ or $(0, 0, \pm 1, 1)$ or $(0, 0, 0, 2)$ or $(0, 0, 0, 0)$.

**Proposition 6.11.** The sets $\{\langle \chi, \sigma \rangle \mid \chi \in T_0\}$ and $\{\langle \chi, \sigma \rangle \mid \chi \in T_1\}$ are both intervals of integers beginning at 0. Also,

\[ \sum_{i=0}^{n+1} \sigma_i = \max\{\langle \chi, \sigma \rangle \mid \chi \in T_0\} = \max\{\langle \chi, \sigma \rangle \mid \chi \in T_1\} \pm 1. \quad (33) \]
Proof. By Proposition 6.7, the set \( \{ \langle \chi, \sigma \rangle \mid \chi \in \text{Short}(D_4 \oplus \mathbb{Z}^{n-2}) \} \) is an interval of integers. For each \( i \in \{0, 1\} \), the set \( \{ \langle \chi, \sigma \rangle \mid \chi \in \text{Short}_i \} \) contains the elements of this interval with the same parity. So the parities are different for \( i = 0 \) and \( i = 1 \). In particular, both sets are arithmetic progressions of step size 2, so subtracting off the smallest element and dividing by 2 gives intervals beginning at 0. \( \square \)

Corollary 6.12. \( \sigma \) is a changemaker.

Proof. The set \( T_0 \) consists of just vectors with all entries 0 or 1. \( \square \)

Proof of Theorem 1.7. This follows from the combination of Corollary 6.12 and Proposition 6.7. \( \square \)

Corollary 6.13. \( \sigma_a = \sigma_b + \sigma_c + \sigma_d + \theta \), where \( \theta \in \{-1, 1\} \).

Proof. Using (33), we see that
\[
\sum_{i=0}^{n+1} \sigma_i = 2e_a + \left( \sum_{j \notin \{a, b, c, d\}} \sigma_j \right) \pm 1.
\]
The result is now immediate. \( \square \)

Lemma 6.14. An irreducible vector \( v \in \sigma^\perp \) has an odd pairing with the vector \( e_a + e_b + e_c + e_d \) if and only if \( [v] \) contains \( x_0 \).

Proof. Suppose \( v \in \sigma^\perp \) is irreducible. The pairing \( \langle v, e_a + e_b + e_c + e_d \rangle \) is even if and only if \( v \in D_4 \oplus \mathbb{Z}^{n-2} \), which is equivalent to \( v \in \langle 2x_0, \ldots, x_n \rangle \) by (25). Since \( v \) is irreducible, \( v \notin \langle 2x_0, \ldots, x_n \rangle \) if and only if \( [v] \) contains \( x_0 \). \( \square \)

Let
\[
G = 1 + \sigma_0 + \sigma_1 + \cdots + \sigma_{d-1}.
\]

(34)

Lemma 6.15. There exists \( \chi \in T_1 \) with \( \langle \chi, \sigma \rangle = G \). Let \( f \) be the minimal index such that \( f > d \) and \( f \notin \{a, b, c\} \).

If \( \chi_a = 0 \), then
\[
G \geq \sigma_f.
\]

If \( \chi_a \neq 0 \), then
\[
G \geq \sigma_a - \sigma_b = \sigma_c + \sigma_d + \theta.
\]

Proof. Using Proposition 6.11, there exists \( \chi \in T_1 \) with \( \langle \chi, \sigma \rangle = G \). If \( \chi_a = 0 \), by Proposition 6.10 we have \( \chi_b = \chi_c = \chi_d = 0 \), then there must be an index \( i > d, i \notin \{a, b, c\} \), with \( \chi_i \neq 0 \) as otherwise \( \langle \chi, \sigma \rangle < G \). So
\[
G = \langle \chi, \sigma \rangle \geq \sigma_i \geq \sigma_f.
\]

If \( \chi_a \neq 0 \), by Proposition 6.10 we have
\[
G = \langle \chi, \sigma \rangle \geq \sigma_a - \sigma_b = \sigma_c + \sigma_d + \theta.
\] \( \square \)
7. Bounding $d$

In this section, we will prove that $d = 0$. We assume that $d > 0$ for contradiction.

Recall that we write $(e_0, e_1, \ldots, e_{n+1})$ for the orthonormal basis of $\mathbb{Z}^{n+2}$, and $\sigma = \sum \sigma_i e_i$. Since $\Lambda(q, -p)$ is indecomposable (Proposition 3.8), $\sigma_0 \neq 0$, otherwise $\sigma^\perp$ would have a direct summand $\mathbb{Z}$. So $\sigma_0 = 1$. By Lemma 6.14, we have that $[v_d]$ contains $x_0$. Set

$$w = \theta e_0 + e_d + e_c + e_b - e_a,$$

where $\theta \in \{-1, 1\}$ is as in Corollary 6.13.

**Lemma 7.1.** $w$ is an irreducible vector of $\sigma^\perp$. Also, $x_0 \notin [w]$.

**Proof.** Corollary 6.13 shows that $w$ is in $\sigma^\perp$. Suppose $w = x + y$ with $x, y \in \sigma^\perp$ and $\langle x, y \rangle \geq 0$. If both $x, y$ are nonzero, by Lemma 3.14 we may assume that one of the vectors is $e_d - e_0$ and the other is $-e_a + e_b + e_c$. Both vectors will then be irreducible and $x_0 \in [x], [y]$. That implies $\langle x, y \rangle \neq 0$, which is a contradiction. The second statement is immediate from Lemma 6.14. \[\square\]

**Corollary 7.2.** If one of the following two conditions holds, then $\theta = 1$:

1. $\sigma_d = 1$;
2. there exists a vector $v$ with $\langle v, e_0 \rangle = -\langle v, e_d \rangle = 1$, $\max \text{supp}(v) = d$ and $|\langle v, w \rangle| \leq 1$.

**Proof.** If $\sigma_d = 1$ and $\theta = -1$, then $w = (-e_0 + e_d) + (e_c + e_b - e_a)$ is reducible, a contradiction to Lemma 7.1.

If there exists a vector $v$ as in the statement, then since $\langle v, e_0 \rangle = -\langle v, e_d \rangle = 1$ and $\max \text{supp}(v) = d$, we have $\langle v, w \rangle = \theta - 1$. Using $|\langle v, w \rangle| \leq 1$, we have $\theta = 1$. \[\square\]

**Remark 7.3.** When $d > 0$, we have $[v_d]$ contains $x_0$. For any $0 < i < d$, $[v_i]$ does not contain $x_0$. Also, $\text{supp}(v_i) \cap \text{supp}(w) = \emptyset$ or $\{0\}$, so $|\langle w, v_i \rangle| \leq 2$.

**Lemma 7.4.** Suppose that $0 \notin \text{supp}(v_d)$, then $[v_d] \nmid [w]$.

**Proof.** We can compute $\langle w, v_d \rangle = -1$. Assume that $[v_d] \nmid [w]$ does not happen, then either $[v_d] \prec [w]$ or $[v_d] \nmid [w]$. Note that $x_0 \in [v_d]$ and $x_0 \notin [w]$.

If $[v_d] \prec [w]$, then $|v_d| = 2$, and $[w]$ and $[v_d]$ share their right end. This is not possible since $|w| > |v_d|$.

If $[v_d] \nmid [w]$, then $|[v_d] \cap [w]| = 3$, and there exists $\epsilon \in \{-1, 1\}$ such that $w = \epsilon [w]$ and $v_d = -\epsilon [v_d]$. So $w + v_d = x + y$ with $[x]$ and $[y]$ being distant, and we may assume $x_0 \in [x]$.

Since $v_d$ is not tight, $v_d$ is unbreakable. So $|v_d| = |[w] \cap [v_d]| = 3$, and $|x| = 2$. We get $v_d = e_i + e_{d-1} - e_d$ for some $0 < i < d - 1$, and

$$w + v_d = \theta e_0 + e_i + e_{d-1} + e_c + e_b - e_a.$$

Using Lemma 3.14 and the fact that $x_0 \in [x]$, we have either $x = e_j - e_0$ for some $j \in \{0, i, d-1\}$ or $x = -e_0 + e_k$ for some $k \in \{c, b\}$. If $x = e_j - e_0$, then $\sigma_j = \sigma_a = \sigma_b$, contradicting Corollary 6.13. If $x = -e_0 + e_k$, then $\theta = -1$ and $\sigma_d = \sigma_k = 1$, contradicting Corollary 7.2. \[\square\]
Lemma 7.5. Suppose that $0 \notin \text{supp}(v_d)$ and $|(v_i, v_d)| = 1$ for some $i$ with $0 < i < d$. Then $i = 1$.

Proof. Since $i < d$, $x_0 \notin [v_i]$ by Lemma 6.14. We have $[v_d] \uparrow [w]$ by Lemma 7.4.

If $[v_i] \uparrow [v_d]$, then $[v_i]$ and $[w]$ share their left end. If $|v_i| > 2$, we have $2 \leq |(v_i, w)|$, hence $(v_i, e_0) = 2$ and $v_i$ is tight. If we also have $i > 1$, then $|v_i| \geq 6 > |w|$, so $|(v_i, w)| = |w| - 1 = 4$, which is not possible. So in order to prove $i = 1$, we only need to assume $|v_i| = 2$ in this case.

If $[v_i]$ and $[v_d]$ share their right end, then we must have $|v_i| = 2$.

In the above two cases we have $|v_i| = 2$ and $[v_i]$ abuts the right end of $[v_d]$, so $|(v_i, w)| = 1$, which implies $i = 1$.

If $[v_i] \cap [v_d]$, then $|v_i \cap [v_d]| = |v_d| = 3$. By Lemma 3.21, $v_i$ is tight. If $i > 1$, $|v_i| \geq 6 = |w| + |v_d| - 2$. Since $[v_d] \uparrow [w]$, the interval $[v_i]$ must contain all high weight vertices of $[w]$. Thus $|(w, v_i)| \geq |w| - 2 = 3$, a contradiction (Remark 7.3).

Lemma 7.6. $v_d$ is not gappy.

Proof. Suppose for contradiction that $v_d$ is gappy. Take the index $i$ to be the smallest gappy index of $v_d$. First suppose that $i = 0$. Then, using Lemma 3.12, $v_1$ will be tight with $|v_1| = 5$.

Note that $\langle w, v_1 \rangle = 2\theta$, $|v_1| = |w| = 5$, so $[w] \cap [v_1]$ with $|[v_1] \cap [w]| = 4$, and there exists $\epsilon \in \{-1, 1\}$ such that $w = \epsilon|w|$ and $v_1 = \theta \epsilon[v_1]$. It follows that $w - \theta v_1 = x + y$ with $|x|$ and $|y|$ being distant, $|x| = |y| = 3$.

Now

$$w - \theta v_1 = -\theta e_0 + \theta e_1 + e_d + e_c + e_b - e_a.$$  

Since $x_0 \notin [w], [v_1]$, we have $x_0 \notin [x], [y]$. Using Lemma 3.14, one of $x, y$ has the form $\pm e_j + e_k + e_l$, where $j \in \{0, 1\}, \{k, l\} \subseteq \{d, c, b\}$, but this vector is not in $\sigma^\perp$, a contradiction.

Suppose $i > 0$. Then $i = \min \text{supp}(v_d)$ by [Gre13, Paragraph 2 in Section 6, and Propositions 8.6, 8.7, 8.8]. Since $\langle v_{i+1}, v_d \rangle = 1$, by Lemma 7.5 we have $i + 1 = 1$, a contradiction.

Proposition 7.7. $\min \text{supp}(v_d) \leq 1$.

Proof. Set $i = \min \text{supp}(v_d)$. If $i > 0$, since $\langle v_i, v_d \rangle = -1$, by Lemma 7.5 we have $i = 1$.

Let $G$ be defined as in (34). Our strategy is to first find a bound for $G$, and then find a bound for the integer $d$. Next, we do a case-by-case analysis to find that indeed $d = 0$.

Lemma 7.8. $v_d$ is not tight.

Proof. Suppose for contradiction that $v_d$ is tight. Using Lemma 6.15, we get

$$\sigma_d = G \geq \min \{\sigma_f, \sigma_d + \sigma_c + \theta\} \geq \min \{\sigma_f, 2\sigma_d - 1\},$$

which is not possible by (32) and Corollary 7.2.

Combining Proposition 7.7 and Lemmas 7.6 and 7.8, we have:
Corollary 7.9. \( v_d = v_{d,0} e_0 + e_1 + \cdots - e_d \) with \( v_{d,0} \in \{0,1\} \).

With the notation of Corollary 7.9 in place, we start the analysis to deduce \( d = 0 \). The following identity will be useful to keep in mind:

\[
\sigma_d = G - 2 + v_{d,0}.
\] (36)

Lemma 7.10. If either \( |v_d| > 2 \) or \( d = 1 \), then

\[
G \geq \sigma_d + \sigma_c + \theta.
\]

Proof. Let \( \chi \) be the vector as in Lemma 6.15. By that lemma, it will suffice to show \( \chi_a \neq 0 \). Assume that \( \chi_a = 0 \), then Lemma 6.15 implies that \( G \geq \sigma_f > \sigma_d \). Using (36), we have that \( G \leq \sigma_d + 2 \), so \( \sigma_f \in \{\sigma_d + 1, \sigma_d + 2\} \).

If \( \sigma_f = \sigma_d + 1 \), set \( v'_f = -e_f + e_d + e_0 \). If \( \sigma_f = \sigma_d + 2 \), set \( v'_f = -e_f + e_d + e_1 + e_0 \). (Note that \( d \neq 1 \) in this case, otherwise \( G = 2 \neq \sigma_d + 2 \).) In either case, \( v'_f \) is irreducible and also in \( \sigma^\perp \). Since \( \langle v'_f, e_a + e_b + e_c + e_d \rangle = 1 \), we get that \( x_0 \in [v'_f] \). So \( [v_d] \) and \( [v'_f] \) share their left endpoint. If \( |v_d| > 2 \), then \( |\langle v_d, v'_f \rangle| \geq 2 \), which contradicts the direct computation \( |\langle v_d, v'_f \rangle| \leq 1 \). If \( d = 1 \), using Lemma 7.8, we get \( \langle v_d, v'_f \rangle = 0 \): this is still giving a contradiction since the intervals \([v_d]\) and \([v'_f]\) share their left endpoints, and so \( \langle v_d, v'_f \rangle \neq 0 \).

Proposition 7.11. If \( |v_d| = 2 \), then either \( d = 1, G = 2 \), or else \( d = 2, G \in \{3, 4\} \).

If \( |v_d| > 2 \), then \( d \in \{3, 4\}, \theta = -1, v_{d,0} = 0 \), and \( 1 + d \leq G \leq 5 \).

Proof. If \( |v_d| = 2 \), our conclusion follows from Corollary 7.9.

Now we assume that \( |v_d| > 2 \). Using Lemma 7.10, we have

\[
G \geq \sigma_d + \sigma_c + \theta \geq 2\sigma_d + \theta = 2(G - 2 + v_{d,0}) + \theta,
\]

thus

\[
G \leq 4 - \theta - 2v_{d,0}.
\] (37)

If \( d \leq 2 \), by Corollary 7.9 we have \( v_{d,0} = 1 \) and \( d = 2 \). We have \( x_0 \in [v_2] \) while \( x_0 \notin [w] \). Since \( |v_2| = 3 < |w| \), we must have \( |\langle v_2, w \rangle| \leq 1 \). Then \( \theta = 1 \) by Corollary 7.2. So \( G \leq 1 \) by (37), which is not possible.

If \( d \geq 3 \), it follows from (37) that

\[
4 - \theta - 2v_{d,0} \geq G \geq d + 1 \geq 4,
\]

so \( \theta = -1, v_{d,0} = 0, d \leq 4 \) and \( G \leq 5 \).

Proposition 7.11 implies that \( d \in \{0, 1, 2, 3, 4\} \). We now argue that \( d = 0 \).

Proposition 7.12. \( d = 0 \).

Proof. Suppose that \( d = 1 \). Using Lemma 7.8, we get that \( v_1 = -e_1 + e_0 \). We have that \( G = 2 \) and \( \sigma_1 = 1 \). By Corollary 7.2 and Lemma 7.10, we get that

\[
2 = G \geq \sigma_c + \sigma_1 + 1 \geq 3,
\]
which is a contradiction.

Suppose that \( d = 2 \). It follows from Proposition 7.11 that \(|v_2| = 2\). We separate the cases to whether \( \sigma_1 (= \sigma_2) \) is 1 or 2.

First assume that \( \sigma_1 = \sigma_2 = 1 \). If \( c \neq 3 \), then \( x_0 \in [v_3] \), thus \([v_2]\) and \([v_3]\) share their left end. So \( \langle v_3, v_2 \rangle \neq 0 \). In particular, \( 1 \notin \text{supp}(v_3) \). Since \( \sigma_0 = \sigma_1 = 1 \), \( 0 \notin \text{supp}(v_3) \), so \(|v_3| = 2\), which is impossible as \( \sigma_3 > 1 \) by (32). If \( c = 3 \), note that \( \theta = 1 \) by Corollary 7.2, by Lemma 6.15 we have

\[
3 = G \geq \min\{\sigma_f, \sigma_3 + 2\}.
\]

By (32), \( \sigma_f > \sigma_3 \), so we have \( \sigma_3 \leq 2 \). If \( \sigma_3 = 1 \), then \( \langle v_3, w \rangle = 0 \) and \( \langle v_3, v_2 \rangle = -1 \). Since \( x_0 \notin [v_3] \), \([v_3]\) abuts the right endpoint of \([v_2]\). Since \( [v_2] \nmid [w] \) by Lemma 7.4, we get \( \langle v_3, w \rangle \neq 0 \), a contradiction. If \( \sigma_3 = 2 \), then \( v_3 = -e_3 + e_2 + e_1 \). We have \( v_3 \sim v_1 \sim v_2 \), \(|v_1| = |v_2| = 2\), \([v_2] \nmid [w] \), so \([v_3]\) contains the leftmost high weight vertex of \([w]\), which contradicts the fact that \( \langle v_3, w \rangle = 0 \).

Next we suppose that \( (d = 2 \) and \( \sigma_1 = \sigma_2 = 2 \). Then \( v_1 = 2e_0 - e_1, v_2 = e_1 - e_2 \). We have \( x_0 \in [v_2], x_0 \notin [v_1], [w] \), and \([v_2]\) abuts both \([v_1]\) and \([w]\). So \([v_1]\) and \([w]\) share their left endpoint. It follows that \( \langle v_1, w \rangle = 4 \), which is not possible by Remark 7.3.

Suppose \( d \geq 3 \). Proposition 7.11 implies that \( v_d = -e_d + e_{d-1} + \cdots + e_1 \). Also, since \( 5 \geq G \geq 2 + \sigma_1 + \sigma_2 \), we find that \( \sigma_1 = 1 \). Consider the vector \( v_d' = v_d - e_1 + e_0 \). Since \( \theta = -1 \) by Proposition 7.11, \( \langle v_d', w \rangle = 0 \). Using Corollary 7.2, we get \( \theta = 1 \), a contradiction. \( \square \)

8. The Case \( d = 0 \)

We now turn our attention to the classification in the case \( d = 0 \): in what follows, we classify all changemaker linear lattices of this sort.

**Lemma 8.1.** \( c = 1, \sigma_c = 1 \), and \( \sigma_a = \sigma_b + 1 \).

**Proof.** By Lemma 6.15, we have

\[
1 = G \geq \min\{\sigma_f, \sigma_a - \sigma_b\} \geq \min\{\sigma_f, \sigma_c + \sigma_0 - 1\} = \min\{\sigma_f, \sigma_c\}.
\]

Using (32), we get \( \sigma_c = 1, c = 1 \), and \( \sigma_a = \sigma_b + 1 \). \( \square \)

For the rest of the section, we will replace \( w \) in (35) with

\[
w' = -e_a + e_b + e_c.
\]

The following is an immediate corollary of Lemma 8.1.

**Corollary 8.2.** The vector \( w' \) is an irreducible, unbreakable vector in \( \sigma^\perp \), and \( x_0 \in [w'] \).

**Lemma 8.3.** \( b = 2, \sigma_b = 1 \), and \( \sigma_a = 2 \). Hence \( (\sigma_0, \ldots, \sigma_a) = (1, 1, 1, 2^{[s]}, 2) \) for some \( s \geq 0 \).

**Proof.** Suppose towards a contradiction that \( b > 2 \). Since \( \sigma_0 = \sigma_1 = 1 \) and \( b > 2, \sigma_2 \in \{2, 3\} \).

If \( \sigma_2 = 2 \), then \( \langle v_2, v_1 \rangle = 0, \langle v_2, w' \rangle = 1 \) and \( \langle v_1, w' \rangle = -1 \). Since \(|v_1| = 2 \) and \( x_0 \notin [v_1] \), \([v_1]\) abuts the right end of \([w']\). If \([v_2]\) also abuts \([w']\), noting that \( x_0 \notin [v_2] \), it abuts the right
end of $[w']$, so $[v_2]$ abuts $[v_1]$, contradicting the fact that $\langle v_2, v_1 \rangle = 0$. Thus we must have $[v_2] \cap [w'] = \epsilon [v_2]$ and $w' = \epsilon [w']$ for some $\epsilon \in \{1, -1\}$. It follows that $w' - v_2$ is reducible. However, $w' - v_2 = -e_a + e_b + e_2 - e_0$ is irreducible by Lemma 3.14 and the fact that $\sigma_a = \sigma_b + 1$, a contradiction.

If $\sigma_2 = 3$, then $[v_2]$ contains $x_0$, so $[w'] < [v_2]$. However, since $|w'| = 3$, this can happen only if $|\langle v_2, w' \rangle| = 2$, contradicting the fact that $\langle v_2, w' \rangle = 1$.

Having proved $b = 2$, we must have $\sigma_2 \in \{1, 2, 3\}$. If $\sigma_2 = 2$, the interval $[v_2]$ contains $x_0$, so $[v_2]$ and $[w']$ share their left end, a contradiction to the direct computation $\langle v_2, w' \rangle = 0$. If $\sigma_2 = 3$, using Proposition 6.11, there must be some $\chi \in T_1$ with $\langle \chi, \sigma \rangle = 2$. Moreover, since $\{0, 1, 2\} = \{d, c, b\}$, $\sigma_f > \sigma_2 = 3$. Therefore, $\chi_a \neq 0$ by Proposition 6.10. Using Lemma 8.1, $\sigma_a = 4$. It must be the case that for some $i \in \{b, c, d\}$, $\chi_i = -1$ and $\chi_j = 0$ for $j \neq i, a$. Then $\langle \chi, \sigma \rangle$ is either 1 or 3, a contradiction.

Therefore, $b = 2$, $\sigma_2 = 1$, and $\sigma_a = \sigma_b + 1 = 2$.

**Lemma 8.4.** $\sigma_i = 2s + 3$ for $i > a$. That is, $\sigma = (1, 1, 1, 2|s|, 2, 2s + 3|d|)$ with $s, t \geq 0$.

**Proof.** First, consider $v_{a+1}$. Since $\sigma_{a+1} > 2$, $m := \min \supp(v_{a+1}) < a$, so if $m \geq 3$ then $s := a - 3 > 0$ and there would be a claw centered at $v_m$, a contradiction to Lemma 3.23. Therefore, $\supp(v_{a+1}) \cap \{0, 1, 2\}$ is nonempty, thus is one of $\{0, 1, 2\}$, $\{1, 2\}$, or $\{2\}$ by Lemma 3.12.

We note that $x_0 \in [v_a]$ no matter $s = 0$ or $s > 0$.

We claim that there is no index $j$ such that $v_j$ is tight. Otherwise, we have $j > a$ and $[v_j]$ contains $x_0$, so $[v_a] < [v_j]$. If $s > 0$, $\langle v_a, v_j \rangle = 0$, a contradiction to $[v_a] < [v_j]$. If $s = 0$, then $|v_a| = 3$ hence $|\langle v_a, v_j \rangle| = 2$, contradicting the direct computation $\langle v_a, v_j \rangle = 1$.

If $m = 0$, then $3 \in \supp(v_{a+1})$ since otherwise $\langle v_3, v_{a+1} \rangle = 2$, a contradiction to Lemma 3.20. Then since $|v_i| = 2$ for $3 < i \leq a$, $v_{a+1}$ is just right by the claim in the last paragraph. However, if $s > 0$, then $\langle v_3, v_4, v_1, v_{a+1} \rangle$ will give a claw, a contradiction (Lemma 3.23). If $s = 0$ then $[v_3]$ contains $x_0$ so $[v_1]$ and $[v_{a+1}]$ must both abut the right endpoint of $[v_3]$, contradicting the fact that they are orthogonal.

If $m = 1$, then again we must have $3 \in \supp(v_{a+1})$ and $v_{a+1}$ just right. Since $\{a, b, c, d\} \cap \supp(v_{a+1}) = 3$, $x_0 \in [v_{a+1}]$, so $[v_a] < [v_{a+1}]$ and $|\langle v_{a+1}, v_a \rangle| = |v_a| - 1$. This contradicts the direct computation of $\langle v_a, v_{a+1} \rangle$ no matter $s = 0$ or $s > 0$.

If $m = 2$, then $v_{a+1} = e_2 + e_k + \cdots + e_a - e_{a+1}$ for some $3 \leq k \leq a$. If $3 < k < a$, there is a claw $\langle v_k, v_{k-1}, v_{k+1}, v_{a+1} \rangle$ (Lemma 3.23). If $k = a$ and $a > 3$, then $x_0 \subset [v_a]$ but $x_0 \notin [v_{a+1}]$, and so $[v_a] \notin [v_{a+1}]$ since $|v_a| = 2 < |v_{a+1}|$. If $s = 1$, then since $x_0 \notin [v_3]$, $\langle v_3, v_a \rangle = -1$, $[v_3] \in [v_{a+1}]$ will share a right weight vertex, which is not possible. If $s > 1$, then both $[v_{a+1}]$ and $[v_{a-1}]$ abut the right endpoint of $[v_a]$, hence $\langle v_{a+1}, v_{a-1} \rangle = \pm 1$, a contradiction to the direct computation $\langle v_{a+1}, v_{a-1} \rangle = 0$. Therefore, $k = 3$, so $v_{a+1}$ is just right and $\sigma_{a+1} = 2s + 3$.

Finally, suppose that for some $j > a + 1$, $|v_j| > 2$. Take $j$ to be the smallest such index. Then $v_j$ is unbreakable by our earlier claim. Let $\ell = \min \supp(v_j)$. If either $\ell \geq a + 1$ or $3 \leq \ell < a$, there will be a claw centered at $v_j$, contradicting Lemma 3.23. If $\ell = a$, then $[v_j]$ contains $x_0$, so $[v_a] < [v_j]$. If $s = 0$, $|v_a| = 3$, thus $[v_j]$ contains the high weight vertex of $[v_a]$.
a contradiction. If \( s > 0 \), \([v_3]\) is connected to \([v_a]\) via a (possibly empty) sequence of norm 2 vectors, so the intervals \([v_3]\) and \([v_j]\) will share a high weight vertex, a contradiction. If \( \ell < 3 \), then there is a heavy triple \((v_3, v_{a+1}, v_j)\), contradicting Lemma 3.25.

9. Proof of Theorem 1.2

Lemma 8.4 specifies a changemaker vector in \(\mathbb{Z}^{n+2}\) whose orthogonal complement is the linear changemaker lattice \(\Lambda(q, -p)\). From the integers \(a_0, a_1, \cdots a_n\) in (15), we can recover \(p\) and \(q\) using (13). Since \(q < p < 2q\), we have

\[
\frac{p}{q} = [2, a_0, a_1, \ldots, a_n]^-.
\]

We use the following facts:

**Lemma 9.1.** \([\text{Gre13, Lemma 9.5 (2) and (3)}]\) For integers \(s, t, b\) with \(b \geq 2\) and \(s, t \geq 0\),

1. \([\cdots, b, 2^{[t-1]}]^- = [\cdots, b - 1, -t]^-.
2. If \([2^{[s+1]}, b, \cdots]^- = \frac{p}{q}\), then \([- (s + 2), b - 1, \cdots]^- = \frac{p}{q - p}\).

We have

\[
\sigma = (1, 1, 1, 2^s, 2, 2s + 3^t),
\]

with \(s, t \geq 0\). One can check that the standard basis of the linear changemaker lattice

\[
S = \{v_{s+3}, \cdots, v_3, v_1, v_2, v_{s+4}, \cdots, v_{s+t+3}\}
\]

coinsides with its vertex basis with norms given by

\[
\{2^s, 3, 2, 2, s + 3, 2^{[t-1]}\}.
\]

By Lemma 6.14, \([v_{s+3}]\) contains \(x_0\), so \(v_{s+3} = x_0\). Hence we have

\[
\frac{p}{q} = [2^{[s+1]}, 3, 2, 2, s + 3, 2^{[t-1]}]^-.
\]

Using Lemma 9.1, we see that

\[
q = 7 + 4s + 9t + 12st + 4s^2t, \quad \text{and}
\]

\[
p = 11 + 4s + 14t + 16st + 4s^2t.
\]

It is straightforward to check that

\[
q = \frac{1}{r^2 - 2r - 1}(r^2 p - 1),
\]

with \(r = -2s - 3\) and \(p \equiv -2r + 5 \pmod{r^2 - 2r - 1}\).

**Proof of Theorem 1.2.** Suppose \(P(p, q) \cong S^3_{4q}(K)\), the above computation shows that \((p, q)\) must be as in the statement. On the other hand, if \((p, q)\) is as in the statement, it follows from \([\text{BHM}+16, \text{Table 2}]\) that there exists a Berge–Kang knot \(K_0\) such that \(P(p, q) \cong S^3_{4q}(K_0)\). For the second statement, we note that \(K\) and \(K_0\) correspond to the same changemaker vector. Using Proposition 6.8, we know that \(\Delta_K = \Delta_{K_0}\), so \(HF(K) \cong HF(K_0)\) by \([\text{OS05a, Theorem 1.2}]\).
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