Abstract

Two maximization problems of Rényi entropy rate are investigated: the maximization over all stochastic processes whose marginals satisfy a linear constraint, and the Burg-like maximization over all stochastic processes whose autocovariance function begins with some given values. The solutions are related to the solutions to the analogous maximization problems of Shannon entropy rate.

Keywords: Rényi entropy, Rényi entropy rate, entropy rate, maximization, Burg’s Theorem.

1 Introduction

Motivated by recent results providing an operational meaning to Rényi entropy \[1\], we study the maximization of the Rényi entropy rate (or “Rényi rate”) over the class of stochastic processes \(\{Z_k\}_{k \in \mathbb{Z}}\) that satisfy

\[
\Pr[Z_k \in \mathcal{S}] = 1, \quad \mathbb{E}[r(Z_k)] \leq \Gamma, \quad k \in \mathbb{Z},
\]  

where \(\mathcal{S} \subseteq \mathbb{R}\) is some given support set, \(r(\cdot)\) is some cost function, \(\Gamma \in \mathbb{R}\) is some maximal-allowed average cost, and \(\mathbb{R}\) and \(\mathbb{Z}\) denote the reals and the integers respectively.

If instead of Rényi rate we had maximized the Shannon rate, we could have limited ourselves to memoryless processes, because the Shannon entropy
of a random vector is upper-bounded by the sum of the Shannon entropies of its components, and this upper bound is tight when the components are independent.\footnote{Throughout this paper “Shannon entropy” refers to differential Shannon entropy.} But this bound does not hold for Rényi entropy: the Rényi entropy of a vector with dependent components can exceed the sum of the Rényi entropies of its components. Consequently, the solution to the maximization of the Rényi rate subject to (1) is typically not memoryless. This maximum and the structure of the stochastic processes that approach it is the subject of this paper.

Another class of stochastic processes that we shall consider is related to Burg’s work on spectral estimation \cite{2}, \cite[Theorem 12.6.1]{3}. It comprises all (one-sided) stochastic processes \( \{X_i\}_{i \in \mathbb{N}} \) that, for some given \( \alpha_0, \ldots, \alpha_p \in \mathbb{R} \), satisfy

\[
E[X_iX_{i+k}] = \alpha_k, \quad \left( i \in \mathbb{N}, \ k \in \{0, \ldots, p\} \right),
\]

where \( \mathbb{N} \) denotes the positive integers. While Burg studied the maximum over this class of the Shannon rate, we will study the maximum of the Rényi rate.

We emphasize that our focus here is on the maximization of Rényi rate and not entropy. The latter is studied in \cite{4}, \cite{5}, \cite{6}, and \cite{7}.

To describe our results we need some definitions. The order-\( \alpha \) Rényi entropy of a probability density function (PDF) \( f \) is defined as

\[
h_\alpha(f) = \frac{1}{1 - \alpha} \log \int_{-\infty}^{\infty} f^\alpha(x) \, dx,
\]

where \( \alpha \) can be any positive number other than one. The integrand is non-negative, so the integral on the RHS of (3) always exists, possibly taking on the value \(+\infty\), in which case we define \( h_\alpha(f) \) as \(+\infty\) if \( 0 < \alpha < 1 \) and as \(-\infty\) if \( \alpha > 1 \). With this convention the Rényi entropy always exists and

\[
h_\alpha(f) > -\infty, \quad 0 < \alpha < 1,
\]

\[
h_\alpha(f) < +\infty, \quad \alpha > 1.
\]

When a random variable (RV) \( X \) is of density \( f_X \) we sometimes write \( h_\alpha(X) \) instead of \( h_\alpha(f_X) \). The Rényi entropy of some multivariate densities are computed in \cite{8}.
If the support of \( f \) is contained in \( S \), then
\[
h_\alpha(f) \leq \log |S|, \quad (\alpha > 0, \; \alpha \neq 1),
\]
where \( |A| \) denotes the Lebesgue measure of the set \( A \), and where we interpret \( \log |S| \) as \(+\infty\) when \( |S| \) is infinite. (Throughout this paper we define \( \log \infty = \infty \) and \( \log 0 = -\infty \).)

The Rényi entropy is closely related to the Shannon entropy:
\[
h(f) = -\int_{-\infty}^{\infty} f(x) \log f(x) \, dx.
\]
(The integral on the RHS of (7) need not exist. If it does not, then we say that \( h(f) \) does not exist.) Depending on whether \( \alpha \) is smaller or larger than one, the Rényi entropy can be larger or smaller than the Shannon entropy. Indeed, if \( f \) is of Shannon entropy \( h(f) \) (possibly \(+\infty\)), then by [9, Lemma 5.1 (iv)]:
\[
\begin{align*}
h_\alpha(f) &\leq h(f), \quad \text{for } \alpha > 1; \\
h_\alpha(f) &\geq h(f), \quad \text{for } 0 < \alpha < 1.
\end{align*}
\]
Moreover, under some mild technical conditions [9, Lemma 5.1 (ii)]:
\[
\lim_{\alpha \to 1} h_\alpha(f) = h(f).
\]

The order-\( \alpha \) Rényi rate \( h_\alpha(\{X_k\}) \) of a stochastic process (SP) \( \{X_k\} \) is defined as
\[
h_\alpha(\{X_k\}) = \lim_{n \to \infty} \frac{1}{n} h_\alpha(X_1^n)
\]
whenever the limit exists.\(^2\) Here \( X_i^n \) denotes the tuple \( (X_i, \ldots, X_j) \).

Notice that if each \( X_k \) takes value in \( S \), then \( X_1^n \) takes value in \( S^n \), and it then follows from (6) that \( h_\alpha(X_1^n) \leq \log |S|^n \) and thus
\[
h_\alpha(\{X_k\}) \leq \log |S|.
\]

\(^2\)We say that the limit exists and is equal to \(+\infty\) if for every \( M > 0 \) there exists some \( n_0 \) such that for all \( n > n_0 \) the Rényi entropy \( h_\alpha(X_1, \ldots, X_n) \) exceeds \( nM \), possibly by being \(+\infty\).
Another upper bound on $h_\alpha(\{X_k\})$, one that is valid for $\alpha > 1$, can be obtained by noting that when $\alpha > 1$ we can use (8) to obtain

$$h_\alpha(X^n_t) \leq h(X^n_t) \leq \sum_{i=1}^{n} h(X_i),$$

and thus, by (13),

$$h_\alpha(\{X_k\}) \leq h(\{X_k\}), \quad \alpha > 1,$$

whenever both $h_\alpha(\{X_k\})$ and the Shannon rate $h(\{X_k\})$ exist.

The Rényi rate of finite-state Markov chains was computed by Rachied, Alajaji, and Campbell [10] with extensions to countable state space in [11]. The Rényi rate of stationary Gaussian processes was found by Golshani and Pasha in [12]. Extensions are explored in [13].

2 Main Results

We discuss the constraints (1) and (2) separately. The proofs pertaining to the former are in Section 4 and to the latter in Section 5.

2.1 Max Rényi Rate Subject to (1)

Let $h^*(\Gamma)$ denote the supremum of $h(f_X)$ over all densities $f_X$ under which

$$\Pr(X \in \mathcal{S}) = 1 \quad \text{and} \quad E[r(X)] \leq \Gamma. \quad (16)$$

Here and throughout the supremum should be interpreted as $-\infty$ whenever the maximization is over an empty set. Thus, if no distribution satisfies (16), then $h^*(\Gamma)$ is $-\infty$.

We shall assume that for some $\Gamma_0 \in \mathbb{R}$

$$h^*(\Gamma_0) > -\infty, \quad (17a)$$

and

$$h^*(\Gamma) < \infty \quad \text{for every } \Gamma \geq \Gamma_0. \quad (17b)$$

Under this assumption the function $h^*$ has the following properties:
Proposition 1. Let $\Gamma_0$ satisfy (17). Then over the interval $[\Gamma_0, \infty)$ the function $h^*(\cdot)$ is finite, nondecreasing, and concave. It is continuous over $(\Gamma_0, \infty)$, and

$$\lim_{\Gamma \to \infty} h^*(\Gamma) = \log |S|. \quad (18)$$

Proof. Monotonicity is immediate from the definition because increasing $\Gamma$ enlarges the set of densities that satisfy (16). Concavity follows from the concavity of Shannon entropy, and continuity follows from concavity. It remains to establish (18). To this end we first argue that for every $\Gamma$,

$$h^*(\Gamma) \leq \log |S|. \quad (19)$$

When $|S|$ is infinite this is trivial, and when $|S|$ is finite this follows by noting that $h^*(\Gamma)$ cannot exceed the maximum of the Shannon entropy in the absence of cost constraints, and the latter is achieved by a uniform distribution on $S$ and is equal to $\log |S|$. In view of (19), our claim (18) will follow once we establish that

$$\lim_{\Gamma \to \infty} h^*(\Gamma) \geq \log |S|, \quad (20)$$

which is what we set out to prove next.

We first note that for every $\Gamma \in \mathbb{R}$

$$h^*(\Gamma) \geq \log |\{x \in S : r(x) \leq \Gamma\}| \quad (21)$$

because when the RHS is finite it can be achieve by a uniform distribution on the set $\{x \in S : r(x) \leq \Gamma\}$, a distribution under which (16) clearly holds, and when it is infinite, it can be approached by uniform distributions on ever-increasing compact subsets of this set. We next note that, by the Monotone Convergence Theorem (MCT),

$$\lim_{\Gamma \to \infty} |\{x \in S : r(x) \leq \Gamma\}| = |S|. \quad (22)$$

Combining (21) and (22) establishes (20) and hence completes the proof of (18). \qed

For $\alpha > 1$ we note that (11), (14), and the definition of $h^*(\Gamma)$ imply that for every SP $\{Z_k\}$ satisfying (11)

$$h_\alpha(\{Z_k\}) \leq h^*(\Gamma), \quad \alpha > 1, \quad (23)$$
and consequently,
\[
\sup h_\alpha(\{Z_k\}) \leq h^*(\Gamma), \quad \alpha > 1,
\]  
(24)

where the supremum is over all SPs satisfying (1). Perhaps surprisingly, this bound is tight:

**Theorem 2** (Max Rényi Rate for \(\alpha > 1\)). Suppose that \(\alpha > 1\), and that \(\Gamma > \Gamma_0\), where \(\Gamma_0\) satisfies (17). Then for every \(\bar{\varepsilon} > 0\) there exists a stationary SP \(\{Z_k\}\) satisfying (1) whose Rényi rate is defined and exceeds \(h^*(\Gamma) - \bar{\varepsilon}\).

For \(0 < \alpha < 1\) we can use (12) to obtain for the same supremum
\[
\sup h_\alpha(\{Z_k\}) \leq \log|\mathcal{S}|, \quad 0 < \alpha < 1.
\]  
(25)

This seemingly crude bound is tight:

**Theorem 3** (Max Rényi Rate for \(0 < \alpha < 1\)). Suppose that \(0 < \alpha < 1\) and that \(\Gamma > \Gamma_0\), where \(\Gamma_0\) satisfies (17).

- If \(|\mathcal{S}| = \infty\), then for every \(M \in \mathbb{R}\) there exists a stationary SP \(\{Z_k\}\) satisfying (1) whose Rényi rate is defined and exceeds \(M\).

- If \(|\mathcal{S}| < \infty\), then for every \(\bar{\varepsilon} > 0\) there exists a stationary SP \(\{Z_k\}\) satisfying (1) whose Rényi rate is defined and exceeds \(\log|\mathcal{S}| - \bar{\varepsilon}\).

**Remark 4.** Theorems 2 and 3 can be generalized in a straightforward fashion to account for multiple constraints:

\[
E[r_i(Z_k)] \leq \Gamma_i, \quad i = 1, \ldots, m.
\]  
(26)

However, for ease of presentation we focus on the case of a single constraint.

A special case of Theorems 2 and 3 is when the cost is quadratic, i.e., \(r(x) = x^2\) and where there are no restrictions on the support, i.e., \(\mathcal{S} = \mathbb{R}\). In this case we can slightly strengthen the results of the above theorems: When we consider the proofs of these theorems for this case, we see that the proposed distributions are isotropic. We can thus establish that the constructed SP is centered and uncorrelated:

**Proposition 5** (Rényi Rate under a Second-Moment Constraint).
1. For every $\alpha > 1$, every $\sigma > 0$, and every $\bar{\varepsilon} > 0$ there exists a centered stationary SP $\{Y_k\}$ whose Rényi rate exceeds $\frac{1}{2} \log(2\pi e \sigma^2) - \bar{\varepsilon}$ and that satisfies

$$E[Y_k Y_{k'}] = \sigma^2 1\{k = k'\}. \quad (27)$$

2. For every $0 < \alpha < 1$, every $\sigma > 0$, and every $M \in \mathbb{R}$ there exists a centered stationary SP $\{Y_k\}$ whose Rényi rate exceeds $M$ and that satisfies (27).

This proposition will be the key to the proof of Theorem 6 ahead.

2.2 Max Rényi Rate Subject to (2)

Given $\alpha_0, \ldots, \alpha_p \in \mathbb{R}$, consider the family of all stochastic processes $X_1, X_2, \ldots$ satisfying (2). Assume that the $(p+1) \times (p+1)$ matrix whose Row-$\ell$ Column-$m$ element is $\alpha_{|\ell-m|}$ is positive definite. Under this assumption we have:

**Theorem 6.** The supremum of the order-$\alpha$ Rényi rate over all stochastic processes satisfying (2) is $+\infty$ for $0 < \alpha < 1$ and is equal to the Shannon rate of the $p$-th order Gauss-Markov process for $\alpha > 1$.

3 Preliminaries

3.1 Weak Typicality

Given a density $f$ on $S$ of finite Shannon entropy

$$-\infty < h(f) < \infty, \quad (28)$$

a positive integer $n$, and some $\varepsilon > 0$, we follow [3, Section 8.2] and denote by $T_n^\varepsilon(f)$ the set of $\varepsilon$-weakly-typical sequences of length $n$ with respect to $f$:

$$T_n^\varepsilon(f) = \left\{ x_1^n \in S^n : 2^{-n(h(f)+\varepsilon)} \leq \prod_{k=1}^{n} f(x_k) \leq 2^{-n(h(f) - \varepsilon)} \right\}. \quad (29)$$

By the AEP, if $X_1, \ldots, X_n$ are drawn IID according to some such $f$, then the probability of $(X_1, \ldots, X_n)$ being in $T_n^\varepsilon(f)$ tends to 1 as $n \to \infty$ (with $\varepsilon$ held fixed) [3, Theorem 8.2.2].
Given some measurable function \( r: S \to \mathbb{R} \), some density \( f \) that is supported on \( S \) and that satisfies
\[
\int_S f(x) |r(x)| \, dx < \infty, \tag{30}
\]
and given some \( n \in \mathbb{N} \) and \( \varepsilon > 0 \), we define
\[
G^\varepsilon_n(f) = \left\{ x^n_1 \in S^n : \left| \frac{1}{n} \sum_{k=1}^n r(x_k) - \int_S f(x) \, r(x) \, dx \right| < \varepsilon \right\}. \tag{31}
\]
By the Law of Large Numbers (LLN), if \( X_1, \ldots, X_n \) are drawn IID according to some density \( f \) that satisfies the above conditions, then the probability of \((X_1, \ldots, X_n)\) being in \( G^\varepsilon_n(f) \) tends to 1 as \( n \to \infty \) (with \( \varepsilon \) held fixed).

From the above observations on \( T^\varepsilon_n(f) \) and \( G^\varepsilon_n(f) \) we conclude that if \( X_1, \ldots, X_n \) are drawn IID according to some density \( f \) that is supported by \( S \) and that satisfies (28) and (30), then the probability of \((X_1, \ldots, X_n)\) being in the intersection \( T^\varepsilon_n(f) \cap G^\varepsilon_n(f) \) tends to 1 as \( n \to \infty \). Thus, for all sufficiently large \( n \),
\[
1 - \varepsilon \leq \int_{T^\varepsilon_n(f) \cap G^\varepsilon_n(f)} \prod_{k=1}^n f(x_k) \, dx^n 
\leq |T^\varepsilon_n(f) \cap G^\varepsilon_n(f)| 2^{-n(h(f) - \varepsilon)},
\]
where the second inequality holds by (29).

We thus conclude that if the support of \( f \) is contained in \( S \), the expectation of \(|r(X)|\) under \( f \) is finite, and \( h(f) \) is defined and is finite, then
\[
|T^\varepsilon_n(f) \cap G^\varepsilon_n(f)| \geq (1 - \varepsilon) 2^{n(h(f) - \varepsilon)}, \quad n \text{ large}. \tag{32}
\]

### 3.2 On the Rényi Entropy of Mixtures

The following lemma provides a lower bound on the Rényi entropy of a mixture of densities in terms of the Rényi entropy of the individual densities.

**Lemma 7.** Let \( f_1, \ldots, f_p \) be probability density functions on \( \mathbb{R}^n \) and \( q_1, \ldots, q_p \geq 0 \) nonnegative numbers that sum to one. Let \( f \) be the mixture density
\[
f(x) = \sum_{\ell=1}^p q_\ell f_\ell(x), \quad x \in \mathbb{R}^n.
\]
Then

\[ h_\alpha(f) \geq \min_{1 \leq \ell \leq p} h_\alpha(f_\ell). \]

**Proof.** For 0 < \( \alpha < 1 \) this follows by the concavity of Rényi entropy. Consider now \( \alpha > 1 \):

\[
\log \int f^\alpha(x) \, dx = \log \int \left( \sum_{\ell=1}^{p} q_\ell f_\ell(x) \right)^\alpha \, dx \\
\leq \log \int \sum_{\ell=1}^{p} q_\ell f_\ell(x)^\alpha \, dx \\
= \log \left( \sum_{\ell=1}^{p} q_\ell \int f_\ell(x)^\alpha \, dx \right) \\
\leq \log \max_{1 \leq \ell \leq p} \int f_\ell(x)^\alpha \, dx \\
= \max_{1 \leq \ell \leq p} \log \int f_\ell(x)^\alpha \, dx,
\]

from which the claim follows because \( 1/(1 - \alpha) \) is negative. Here the first inequality follows from the convexity of the mapping \( \xi \mapsto \xi^\alpha \) (for \( \alpha > 1 \)), and the second inequality follows by upper-bounding the average by the maximum. \( \blacksquare \)

We next turn to upper bounds.

**Lemma 8.** Consider the setup of Lemma 7.

1. If \( \alpha > 1 \) then

\[
h_\alpha(f) \leq \min_{1 \leq \ell \leq p} \left\{ \frac{\alpha}{1 - \alpha} \log q_\ell + h_\alpha(f_\ell) \right\}.
\]

2. If \( 0 < \alpha < 1 \) then

\[
h_\alpha(f) \leq \frac{1}{1 - \alpha} \log p + \max_{1 \leq \ell \leq p} h_\alpha(f_\ell).
\]
Proof. We begin with the case where $\alpha > 1$. Since the densities and weights are nonnegative,
\[
\left( \sum_{\ell=1}^{p} q_{\ell} f_{\ell}(x) \right)^{\alpha} \geq \left( q_{\ell'} f_{\ell'}(x) \right)^{\alpha}, \quad \ell' \in \{1, \ldots, p\}.
\]
Integrating this inequality; taking logarithms, and dividing by $1 - \alpha$ (which is negative) we obtain
\[
h_{\alpha}(f) \leq \frac{\alpha}{1 - \alpha} \log q_{\ell'} + h_{\alpha}(f_{\ell'}), \quad \ell' \in \{1, \ldots, p\}.
\]
Since this holds for every $\ell' \in \{1, \ldots, p\}$, we can minimize over $\ell'$ to obtain (33).

We next turn to the case where $0 < \alpha < 1$.
\[
\log \int \left( \sum_{\ell=1}^{p} q_{\ell} f_{\ell}(x) \right)^{\alpha} \, dx \leq \log \int \max_{1 \leq \ell \leq p} f_{\ell}^{\alpha}(x) \, dx
\]
\[
\leq \log \int \sum_{\ell=1}^{p} f_{\ell}^{\alpha}(x) \, dx
\]
\[
= \log \sum_{\ell=1}^{p} \int f_{\ell}^{\alpha}(x) \, dx
\]
\[
\leq \log \left( p \max_{1 \leq \ell \leq p} \int f_{\ell}^{\alpha}(x) \, dx \right)
\]
\[
= \log p + \log \max_{1 \leq \ell \leq p} \int f_{\ell}^{\alpha}(x) \, dx
\]
\[
= \log p + \max_{1 \leq \ell \leq p} \log \int f_{\ell}^{\alpha}(x) \, dx.
\]
Dividing this inequality by $1 - \alpha$ (positive) yields (34).

3.3 Bounded Densities

Proposition 9. If a density $f$ is bounded, and if $\alpha > 1$, then $h_{\alpha}(f) > -\infty$.

Proof. Let $f$ be a density that is upper-bounded by the constant $M$ (which must therefore be positive), and suppose that $\alpha > 1$. In this case
\[
f^{\alpha}(x) = f^{\alpha-1}(x) f(x)
\]
\[
\leq M^{\alpha-1} f(x),
\]
because $\xi \mapsto \xi^{\alpha - 1}$ is monotonically increasing when $\alpha > 1$. Integrating over $x$ we obtain

$$\int f^\alpha(x) \, dx \leq M^{\alpha - 1} < \infty.$$  

Since $\alpha > 1$, this implies that

$$\frac{1}{1 - \alpha} \log \int_{-\infty}^{\infty} f^\alpha(x) \, dx > -\infty. \quad \square$$

The following proposition, which is proved in Appendix A, demonstrates that $h^*$ can be approached by bounded densities.

**Proposition 10.** Suppose that $\Gamma \in (\Gamma_0, \infty)$, where $\Gamma_0$ satisfies (17). Then for every $\delta > 0$ there exists some bounded density $f^*$ supported by $S$ such that

$$\int f^*(x) r(x) \, dx < \Gamma + \delta, \quad (37a)$$

$$h(f^*) > h^*(\Gamma) - \delta. \quad (37b)$$

### 3.4 The Marginals of the Uniform Density on $T^\varepsilon_n(f) \cap G^\varepsilon_n(f)$

**Lemma 11.** Let $f^*$ be a density on $S$ having finite order-$\alpha$ Rényi entropy

$$h_\alpha(f^*) > -\infty \quad (38)$$

for some

$$\alpha > 1 \quad (39)$$

and satisfying (28) and (30). For every $n \in \mathbb{N}$, let $(X_1, \ldots, X_n)$ be drawn uniformly from the set $T^\varepsilon_n(f^*) \cap G^\varepsilon_n(f^*)$, where $\varepsilon$ is some fixed positive number. Then for every sufficiently large $n$ the following holds: for any $\rho \in \{1, \ldots, n\}$ the $\rho$-tuple $(X_1, \ldots, X_\rho)$ has finite order-$\alpha$ Rényi entropy

$$h_\alpha(X_1, \ldots, X_\rho) > -\infty, \quad (\rho \in \{1, \ldots, n\}, \ \alpha > 1). \quad (40)$$

**Proof.** Denote the uniform density over $T^\varepsilon_n(f^*) \cap G^\varepsilon_n(f^*)$ by $f_n$, and let $q_n$ be the product density

$$q_n(x) = \prod_{k=1}^{n} f^*(x_k), \quad x \in S^n. \quad (41)$$
Henceforth let \( n \) be sufficiently large for (32) to hold. Consequently,

\[
f_n(x) \leq \frac{1}{1 - \varepsilon} 2^{-n(h(f^*) - \varepsilon)}, \quad x \in \mathcal{S}^n.
\]

Using this inequality and the definition in (29) of \( \mathcal{T}_n^\varepsilon(f^*) \), we can upper-bound \( f_n \) in terms of \( q_n \) for tuples in \( \mathcal{T}_n^\varepsilon(f^*) \):

\[
f_n(x) \leq \frac{1}{1 - \varepsilon} 2^{2n\epsilon} q_n(x), \quad x \in \mathcal{T}_n^\varepsilon(f^*). \quad (43)
\]

For every \( \rho \in \{1, \ldots, n\} \) we can obtain the density \( f_n(x_1, \ldots, x_\rho) \) of \( (X_1, \ldots, X_\rho) \) by integrating \( f_n(x_1, \ldots, x_n) \) over \( x_{\rho + 1}, \ldots, x_n \):

\[
f_n(x_1, \ldots, x_\rho) = \int_{x_{\rho + 1}, \ldots, x_n} f_n(x) I\{x \in \mathcal{T}_n^\varepsilon(f^*) \cap \mathcal{G}_n^\varepsilon(f^*)\} \, dx_{\rho + 1} \cdots dx_n
\]

\[
\leq \frac{1}{1 - \varepsilon} 2^{2n\epsilon} \int q_n(x) I\{x \in \mathcal{T}_n^\varepsilon(f^*) \cap \mathcal{G}_n^\varepsilon(f^*)\} \, dx_{\rho + 1} \cdots dx_n
\]

\[
\leq \frac{1}{1 - \varepsilon} 2^{2n\epsilon} \int q_n(x) \, dx_{\rho + 1} \cdots dx_n
\]

\[
= \frac{1}{1 - \varepsilon} 2^{2n\epsilon} f^*(x_1) \cdots f^*(x_\rho), \quad x_1, \ldots, x_\rho \in \mathcal{S}, \quad (44)
\]

where \( I\{\cdot\} \) denotes the indicator function, and the first inequality follows from (43); the second by increasing the range of integration; and the final equality follows from (41).

Using (44) we can now lower-bound \( h_\alpha (X_1, \ldots, X_\rho) \) as follows. If a density \( f \) is upper-bounded by \( Kg \), where \( g \) is some other density and \( K \) is some positive constant, and if \( \alpha > 1 \), then

\[
h_\alpha(f) = \frac{1}{1 - \alpha} \log \int f^\alpha(x) \, dx
\]

\[
\geq \frac{1}{1 - \alpha} \log \int K^\alpha g^\alpha(x) \, dx
\]

\[
= \frac{\alpha}{1 - \alpha} \log K + h_\alpha(g), \quad (45)
\]

where the inequality holds because \( \alpha > 1 \) so the pre-log is negative. Using this and (44) we obtain

\[
h_\alpha(X_1, \ldots, X_\rho) \geq \frac{\alpha}{1 - \alpha} \log \left( \frac{1}{1 - \varepsilon} 2^{2n\epsilon} \right) + \rho h_\alpha(f^*)
\]

\[
> -\infty. \quad \square
\]
4 Proofs of Theorems 2 and 3

The following proposition is useful for stationarization.

**Proposition 12.** Let \( f_n \) be some density on \( S^n \) having order-\( \alpha \) Rényi entropy \( h_\alpha(f_n) \) and satisfying

\[
\sum_{k=1}^{n} E[r(X_k)] \leq n \Gamma, \quad (X_1, \ldots, X_n) \sim f_n. \tag{46}
\]

Then there exists a stationary SP \( \{Z_k\} \) satisfying (\( \Box \)) for which the following holds:

- If

  \[
  h_\alpha(X_1, \ldots, X_\rho), h_\alpha(X_{n-\rho'+1}, \ldots, X_n) > -\infty,
  \]
  \[
  \rho, \rho' \in \{1, \ldots, n-1\}, \quad \tag{47}
  \]
  whenever \((X_1, \ldots, X_n) \sim f_n \) and \( \rho, \rho' \in \{1, \ldots, n-1\}\), then

  \[
  \lim_{m \to \infty} \frac{1}{m} h_\alpha(Z_1, \ldots, Z_m) \geq \frac{1}{n} h_\alpha(f_n). \tag{48}
  \]

- If

  \[
  h_\alpha(X_1, \ldots, X_\rho), h_\alpha(X_{n-\rho'+1}, \ldots, X_n) < +\infty,
  \]
  \[
  \rho, \rho' \in \{1, \ldots, n-1\}, \quad \tag{49}
  \]
  whenever \((X_1, \ldots, X_n) \sim f_n \) and \( \rho, \rho' \in \{1, \ldots, n-1\}\), then

  \[
  \lim_{m \to \infty} \frac{1}{m} h_\alpha(Z_1, \ldots, Z_m) \leq \frac{1}{n} h_\alpha(f_n). \tag{50}
  \]

- And if both (47) and (49) hold, then

  \[
  \lim_{m \to \infty} \frac{1}{m} h_\alpha(Z_1, \ldots, Z_m) = \frac{1}{n} h_\alpha(f_n). \tag{51}
  \]
Proof. Consider first the (nonstationary) SP \( \{Y_k\} \) that we construct by drawing
\[
\ldots, Y_{-n+1}^0, Y_1^n, Y_{n+1}^{2n}, \ldots \sim \text{IID } f_n.
\]
To stationarize it, let \( T \) be drawn uniformly over \( \{0, \ldots, n-1\} \) independently of \( \{Y_k\} \), and define the stationary SP
\[
Z_k = Y_{k+T}, \quad k \in \mathbb{Z}. \tag{52}
\]
It satisfies (II). Consider now any \( m \) larger than \( 2n \), and express \( Z_m^1 \) in one of two different ways depending on whether \( T \) is zero or not. For \( T = 0 \)
\[
Z_m^1 = Y_1^n, \ldots, Y_{\tilde{\nu}n}^{\nu n+1}, Y_{\tilde{\nu}n+1}^{\nu n+1}, \ldots, Y_m \tag{53}
\]
where
\[
\tilde{\nu} = \left\lfloor \frac{m}{n} \right\rfloor, \tag{54a}
\]
\[\tilde{\rho} = m - n \left\lfloor \frac{m}{n} \right\rfloor \in \{0, \ldots, n-1\}. \tag{54b}\]
And for \( T \in \{1, \ldots, n-1\} \)
\[
Z_m^1 =
Y_{T+1}^{\nu n+1}, \ldots, Y_n, Y_{n+1}^{2n}, \ldots, Y_{\nu n+1}^{\nu n}, Y_{\rho+1}^{\nu n+1}, \ldots, Y_m^{\rho+1} \tag{55}
\]
where
\[
\nu = \left\lfloor \frac{m - n + T}{n} \right\rfloor, \tag{56b}
\]
\[\rho = m - n + T - n \left\lfloor \frac{m - n + T}{n} \right\rfloor \in \{0, \ldots, n-1\}. \tag{56c}\]
Denote the density of \( Z_m^1 \) by \( f_Z \) and its conditional density given \( T = t \) by \( f_{Z \mid T = t} \).
To establish (48) we use Lemma 7, which implies that
\[
h_\alpha(f_Z) \geq \min_{0 \leq t \leq n-1} h_\alpha(f_{Z \mid T = t}). \tag{57}
\]
To compute $h_\alpha(f_{Z|T=0})$ we use (53) to obtain
\begin{equation}
 h_\alpha(f_{Z|T=0}) = \frac{m}{n} h_\alpha(f_n) + h_\alpha(X_1, \ldots, X_\rho) \tag{58}
 \end{equation}
\begin{equation}
 \geq \frac{m}{n} h_\alpha(f_n) + 0 \wedge \min_{1 \leq \rho \leq n-1} \{ h_\alpha(X_1, \ldots, X_\rho) \}. \tag{59}
 \end{equation}
where the second term on the RHS of (58) should be interpreted as zero when $\rho$ is zero, and where $a \wedge b$ denotes the minimum of $a$ and $b$.

And to compute $h_\alpha(f_{Z|T=t})$ for $t \in \{1, \ldots, n-1\}$ we use (55) to obtain
\begin{equation}
 h_\alpha(f_{Z|T=t}) = h_\alpha(X_{n-\rho'+1}, \ldots, X_n) \\
 + \left[ \frac{m-n+t}{n} \right] h_\alpha(f_n) + h_\alpha(X_1, \ldots, X_\rho), \tag{60}
 \end{equation}
where $\rho, \rho'$ are obtained from (56) by substituting $t$ for $T$, and the last term on the RHS should be interpreted as zero when $\rho$ is zero.

It thus follows from (57), (59), (60), and the above interpretation that
\begin{equation}
 h_\alpha(f_Z) \geq \min_{1 \leq \rho \leq n-1} \{ h_\alpha(X_{n-\rho'+1}, \ldots, X_n) \} \\
 + 0 \wedge \min_{1 \leq \rho \leq n-1} \{ h_\alpha(X_1, \ldots, X_\rho) \} \\
 + \min_{0 \leq t \leq n-1} \left\{ \left[ \frac{m-n+t}{n} \right] h_\alpha(f_n) \right\}. \tag{61}
 \end{equation}

The first two terms do not depend on $m$ and are greater than $-\infty$ whenever (47) holds. Dividing (61) by $m$ and letting $m$ tend to infinity (with $n$ held fixed), establishes (48).

To establish (50) we need an upper bound on $h_\alpha(f_Z)$. Such a bound can be obtained from Lemma 8. The exact form of the bound depends on whether $\alpha$ exceeds 1 or not. But either form leads to (50) upon dividing by $m$ and letting it tend to infinity.

To conclude the proof we note that (51) follows from (50) and (48). \hfill \square

**Proof of Theorem 2.** Since $h^*(\cdot)$ is continuous on the ray $(\Gamma_0, \infty)$, and since $\Gamma > \Gamma_0$ by the theorem’s hypotheses, $h^*(\cdot)$ is continuous at $\Gamma$. Consequently, we can find some $\Gamma'$ for which
\begin{equation}
 \Gamma' < \Gamma \tag{62a}
 \end{equation}
\begin{equation}
 h^*(\Gamma') > h^*(\Gamma) - \epsilon. \tag{62b}
 \end{equation}
These inequalities imply that we can find some $\delta > 0$ small enough so that
\begin{align}
\Gamma' + \delta &< \Gamma 
\tag{63a}
\end{align}
\begin{align}
h^*(\Gamma') - \delta &> h^*(\Gamma) - \bar{\varepsilon}.
\tag{63b}
\end{align}
By Proposition 10 there exists some bounded density $f^*$ supported by $S$ such that
\begin{align}
\int f^*(x) r(x) \, dx &< \Gamma' + \delta, 
\tag{64a}
\end{align}
\begin{align}
h(f^*) &> h^*(\Gamma') - \delta.
\tag{64b}
\end{align}
Moreover, the boundedness of $f^*$, the hypothesis that $\alpha > 1$, and Proposition 9 imply that
\begin{align}
h_\alpha(f^*) &> -\infty.
\tag{64c}
\end{align}
These inequalities combine with (63) to imply
\begin{align}
\int f^*(x) r(x) \, dx &< \Gamma 
\tag{65a}
\end{align}
\begin{align}
h(f^*) &> h^*(\Gamma) - \bar{\varepsilon}.
\tag{65b}
\end{align}
We can hence choose $\varepsilon > 0$ small enough so that
\begin{align}
\int f^*(x) r(x) \, dx &< \Gamma - \varepsilon
\tag{66a}
\end{align}
\begin{align}
h(f^*) &> h^*(\Gamma) - \bar{\varepsilon} + \varepsilon.
\tag{66b}
\end{align}
Let $f_n$ be the uniform density over
\[
\mathcal{T}_n^\varepsilon(f^*) \cap \mathcal{G}_n^\varepsilon(f^*).
\]
The cost of $f_n$ can be bounded by noting that its support is contained in $\mathcal{G}_n^\varepsilon(f^*)$, and
\begin{align}
x_1^n \in \mathcal{G}_n^\varepsilon(f^*) &\implies \frac{1}{n} \sum_{k=1}^{n} r(x_k) < \int f^*(x) r(x) \, dx + \varepsilon \\
&\implies \frac{1}{n} \sum_{k=1}^{n} r(x_k) < \Gamma,
\end{align}
where the second implication follows from (66a). Thus,

$$\int_{\mathcal{S}} f_n(x) \sum_{i=1}^{n} r(x_i) \, dx \leq n \Gamma. \quad (67)$$

To lower-bound its Rényi entropy, we note that by the LLN (in combination with (66a)) and the AEP (see Section 3.1)

$$|\mathcal{T}_n^\varepsilon(f^*) \cap \mathcal{G}_n^\varepsilon(f^*)| \geq (1 - \varepsilon) 2^{n(h(f^*) - \varepsilon)}, \quad n \text{ large.} \quad (68)$$

Consequently,

$$h_\alpha(f_n) \geq n(h(f^*) - \varepsilon) + \log(1 - \varepsilon) \quad n \text{ large,}$$
or, upon dividing by $n$,

$$\frac{1}{n} h_\alpha(f_n) \geq h(f^*) - \varepsilon + \frac{1}{n} \log(1 - \varepsilon) \quad (69)$$

for all sufficiently large $n$. We now choose $n$ large enough so that not only will (69) hold but also its RHS satisfy

$$h(f^*) - \varepsilon + \frac{1}{n} \log(1 - \varepsilon) > h^*(\Gamma) - \bar{\varepsilon}. \quad (70)$$

(This is possible by (66b).) For this $n$ we thus have

$$\frac{1}{n} h_\alpha(f_n) > h^*(\Gamma) - \bar{\varepsilon}. \quad (71)$$

The inequalities (70) and (67) indicate that $f_n$ is a good candidate for the application of Proposition 12. We hence proceed to check its hypotheses.

By Lemma 11 and (63c), if $X_1, \ldots, X_n \sim f_n$ then

$$h_\alpha(X_1, \ldots, X_\rho) > -\infty, \quad \rho \in \{1, \ldots, n-1\}, \quad (71)$$

and, since $f_n$ is permutation invariant, we also infer

$$h_\alpha(X_{n-\rho'+1}, \ldots, X_n) > -\infty, \quad \rho' \in \{1, \ldots, n-1\} \quad (72)$$

so (47) holds. And, since $\alpha > 1$, it follows from (5) that (49) also holds. We can thus apply Proposition 12 to conclude the proof. \qed
Proof of Theorem [\textcircled{3}]. We first prove the theorem when \(|S| = \infty\). We distinguish between two cases. The first case, which is the case with which we begin, is when there exists some \(n \in \mathbb{N}\) and a density \(f_n^\ast\) on \(X_1, \ldots, X_n\) such that
\[
\Pr[X_i \in S] = 1, \quad E[r(X_i)] \leq \Gamma, \quad i \in \{1, \ldots, n\}
\] (73)
and
\[
h_\alpha(X_1, \ldots, X_n) = +\infty.
\] (74)
To apply Proposition [\textcircled{12}] to this density, we note that, since \(0 < \alpha < 1\), Inequality (4) implies (47), and the proposition thus guarantees the existence of a stationary SP \(\{Z_k\}\) satisfying (1) and (48) so
\[
\lim_{m \to \infty} \frac{1}{m} h_\alpha(Z_1, \ldots, Z_m) = +\infty.
\] (75)
This concludes the proof for the case at hand.

We next turn to the second case where \(|S|\) is still infinite, but any tuple whose components satisfy the constraints has Rényi entropy smaller than \(\infty\):
\[
\left(\Pr[X_i \in S] = 1, \quad E[r(X_i)] \leq \Gamma, \quad i \in \{\nu_1, \ldots, \nu_2\}\right)
\Rightarrow \left(h_\alpha(X_{\nu_1}, \ldots, X_{\nu_2}) < \infty\right).
\] (76)
Since \(|S|\) is infinite, it follows from Proposition [\textcircled{1}] that \(h^\ast(\Gamma) \to \infty\) as \(\Gamma \to \infty\). Consequently, there exists some \(\Gamma_1\) such that
\[
h^\ast(\Gamma_1) > M.
\] (77)
Since \(h^\ast\) is monotonic, there is no loss in generality in assuming, as we shall, that
\[
\Gamma_1 > \Gamma.
\] (78)
Let \(\varepsilon \in (0, 1)\) be small enough so that
\[
h^\ast(\Gamma_1) > M + 3\varepsilon
\] (79)
\[
\Gamma_0 + \varepsilon < \Gamma < \Gamma_1 - \varepsilon.
\] (80)
Let the densities \( f^{(0)} \) and \( f^{(1)} \) be within \( \varepsilon \) of achieving \( h^*(\Gamma_0) \) and \( h^*(\Gamma_1) \) in the sense that their support is contained in \( S \) and

\[
\left( \int_S f^{(\ell)}(x) \cdot r(x) \, dx \leq \Gamma_\ell, \quad h(f^{(\ell)}) > h^*(\Gamma_\ell) - \varepsilon \right),
\]

\( \ell \in \{0, 1\} \). (81)

For every \( n \in \mathbb{N} \), define

\[
S_\ell = T^\varepsilon_n(f^{(\ell)}) \cap G^\varepsilon_n(f^{(\ell)}), \quad \ell \in \{0, 1\}.
\]

It follows from the LLN and AEP that, for all sufficiently large \( n \),

\[
|S_\ell| \geq (1 - \varepsilon) 2^{n(h(f^{(\ell)}) - \varepsilon)}, \quad \ell \in \{0, 1\}.
\]

(82)

Assume now that \( n \) is large enough for this to hold. Let \( \delta > 0 \) be small enough so that

\[
(1 - \delta) (\Gamma_0 + \varepsilon) + \delta (\Gamma_1 + \varepsilon) \leq \Gamma.
\]

(84)

(Such a \( \delta \) can be found in view of (80).)

Consider now the mixture density

\[
f_n(x^n_1) = (1 - \delta) \frac{1}{|S_0|} \mathbf{1}\{x^n_1 \in S_0\} + \delta \frac{1}{|S_1|} \mathbf{1}\{x^n_1 \in S_1\}.
\]

(85)

Let \( X^n_1 \) be of density \( f_n \). Using (84) and an argument similar to the one leading to (67) we obtain

\[
\sum_{k=1}^{n} \mathbb{E}[r(X_k)] \leq n\Gamma.
\]

(86)

In fact, the permutation invariance of \( f_n \) implies the stronger statement

\[
\mathbb{E}[r(X_k)] \leq \Gamma, \quad k = 1, \ldots, n.
\]

(87)

We next lower-bound \( h_\alpha(X^n_1) \). To this end, we first argue that the sets \( S_0 \) and \( S_1 \) are disjoint. To see this, note that by the definition of the sets \( G^\varepsilon_n(f^{(0)}) \), \( G^\varepsilon_n(f^{(1)}) \) and by (81)

\[
x^n_1 \in G^\varepsilon_n(f^{(0)}) \implies \frac{1}{n} \sum_{k=1}^{n} r(x_k) < \int f^{(0)}(x) \cdot r(x) \, dx + \varepsilon
\]

\[
\implies \frac{1}{n} \sum_{k=1}^{n} r(x_k) < \Gamma_0 + \varepsilon,
\]

(88)
and

\[ x^n_1 \in G_n^\varepsilon(f^{(1)}) \implies \frac{1}{n} \sum_{k=1}^{n} r(x_k) > \int f^{(1)}(x) r(x) \, dx - \varepsilon \]

\[ \implies \frac{1}{n} \sum_{k=1}^{n} r(x_k) > \Gamma_1 - \varepsilon, \]  \hspace{1cm} (89)

From (80), (88), and (89) we now conclude that \( G_n^\varepsilon(f^{(0)}) \) and \( G_n^\varepsilon(f^{(1)}) \) are disjoint and hence also \( S_0 \) and \( S_1 \).

Having established that \( S_0 \) and \( S_1 \) are disjoint, we can now compute \( h_\alpha(f_n) \) directly to obtain:

\[ \frac{h_\alpha(X^n_1)}{n} = \frac{1}{n(1 - \alpha)} \log \left( (1 - \delta)^\alpha |S_0|^{1-\alpha} + \delta^\alpha |S_1|^{1-\alpha} \right) \]

\[ \geq \frac{1}{n(1 - \alpha)} \log \left( \delta^\alpha |S_1|^{1-\alpha} \right). \]  \hspace{1cm} (90)

From this, (83), (81), and (79) it now follows that we can find some sufficiently large \( n \) for which

\[ \frac{h_\alpha(X^n_1)}{n} > M. \]  \hspace{1cm} (91)

To apply Proposition 12 we note that (87) and (76) imply that (49) holds. And the fact that \( \alpha \in (0, 1) \) implies by (11) that (17) holds. Hence, by the proposition, there exists a stationary SP satisfying the constraints and whose Rény rate is \( n^{-1}h_\alpha(X^n_1) \) and thus exceeds \( M \). This concludes the proof when \( |S| = \infty \).

The proof when \( |S| < \infty \) is very similar. In fact, it is a bit simpler because \( |S| < \infty \) implies (76). We begin the proof by noting that, since \( |S| < \infty \), Proposition 1 implies that \( h^*(\Gamma) \to \log |S| \) as \( \Gamma \to \infty \). Consequently, there exists some \( \Gamma_1 \) such that

\[ h^*(\Gamma_1) > \log |S| - \tilde{\varepsilon}. \]  \hspace{1cm} (92)

Replacing \( M \) with \( \log |S| - \tilde{\varepsilon} \) in the derivation that leads from (77) to (91), we obtain a density \( f_n \) for which

\[ \frac{h_\alpha(X^n_1)}{n} > \log |S| - \tilde{\varepsilon}. \]  \hspace{1cm} (93)

The result then follows from Proposition 12 by noting that the LHS of (49) is upper bounded by \( n \log |S| \) and by noting that (17) holds by (1) because \( 0 < \alpha < 1 \). \( \square \)
5 Proof of Theorem 6

Proof of Theorem 6. Recall the assumption that the \((p + 1) \times (p + 1)\) matrix whose Row-\(\ell\) Column-\(m\) element is \(\alpha_{|\ell-m|}\) is positive definite. This implies\,\,[14] that there exist constants \(a_1, \ldots, a_p, \sigma^2\) and a \(p \times p\) positive definite matrix \(K_p\) such that the following holds\,\,[3] if the random \(p\)-vector \((W_{1-p}, \ldots, W_0)\) is of second-moment matrix \(K_p\) (not necessarily centered) and if \(\{Z_i\}_{i=1}^\infty\) are independent of \((W_{1-p}, \ldots, W_0)\) with

\[
E[Z_i] = 0, \quad i \in \mathbb{N}, \tag{94a}
\]

\[
E[Z_i Z_j] = \sigma^2 I\{i = j\}, \quad i, j \in \mathbb{N}, \tag{94b}
\]

then the process defined inductively via

\[
X_i = \sum_{k=1}^p a_i X_{i-k} + Z_i, \quad i \in \mathbb{N} \tag{95}
\]

with the initialization

\[
(X_{1-p}, \ldots, X_0) = (W_{1-p}, \ldots, W_0) \tag{96}
\]

satisfies the constraints \([2]\).

(By Burg’s maximum entropy theorem \,[3] Theorem 12.6.1, of all stochastic processes satisfying \([2]\) the one of highest Shannon rate is the \(p\)-th order Gauss-Markov process. It is obtained when \((W_{1-p}, \ldots, W_0)\) is a centered Gaussian and \(\{Z_i\}\) are IID \(\sim \mathcal{N}(0, \sigma^2)\). Its Shannon entropy rate is \((1/2) \log(2\pi e \sigma^2)\).

We first consider the case where \(\alpha > 1\). Let \(a_1, \ldots, a_p, \sigma^2\) and \(K_p\) be as above, and let \(\varepsilon > 0\) be arbitrarily small. By Proposition 5 there exists a SP \(\{Z_i\}\) such that \([91]\) holds and such that

\[
\lim_{n \to \infty} \frac{1}{n} h_\alpha(Z_1, \ldots, Z_n) \geq \frac{1}{2} \log(2\pi e \sigma^2) - \varepsilon. \tag{97}
\]

The matrix \(K_p\) is positive definite, so by the spectral representation theorem we can find vectors \(w_1, \ldots, w_p \in \mathbb{R}^p\) and constants \(q_1, \ldots, q_p > 0\) with \(q_1 + \cdots + q_p = 1\) such that

\[
K_p = \sum_{\ell=1}^p q_\ell w_\ell w_\ell^T. \tag{98}
\]

\[3\text{The Row-}\ell\text{ Column-}\,m\text{ element of the matrix } K_p \text{ is } \alpha_{|\ell-m|}. \text{ This matrix is thus the result of deleting the last column and last row of the } (p + 1) \times (p + 1) \text{ matrix that we assumed was positive definite.} \]
(The vectors are eigenvectors of \( K_p \), and the constants \( q_1, \ldots, q_p \) are the scaled eigenvalues of \( K_p \).) Draw the random vector \( W \) independently of \( \{Z_i\} \) with

\[
\Pr[W = w_\ell] = q_\ell,
\]

so that, by (98),

\[
E[WW^\top] = K_p.
\]

Construct now the stochastic process \( \{X_i\} \) using (95) initialized with \((X_{1-p}, \ldots, X_0)^\top\) being set to \( W \).

The resulting SP thus satisfies (2). We next study its Rényi rate. To that end, we study the Rényi entropy of the vector \( X_1^n \). Let \( f_X \) denote its density, and let \( f_{X|w_\ell} \) denote its conditional density given \( W = w_\ell \), so

\[
f_X(x) = \sum_{\ell=1}^p q_\ell f_{X|w_\ell}(x), \quad x \in \mathbb{R}^n.
\]

Consequently, by Lemma 7,

\[
h_\alpha(f_X) \geq \min_{1 \leq \ell \leq p} h_\alpha(f_{X|w_\ell}), \quad (99)
\]

and by Lemma 8

\[
h_\alpha(f_X) \leq \min_{1 \leq \ell \leq p} \left\{ \frac{\alpha}{1 - \alpha} \log q_\ell + h_\alpha(f_{X|w_\ell}) \right\}. \quad (100)
\]

We next study \( h_\alpha(f_{X|w_\ell}) \) for any given \( \ell \in \{1, \ldots, p\} \). Recalling that \( W \) and \( \{Z_i\} \) are independent, we conclude that, conditional on \( W = w_\ell \), the random variables \( X_1, \ldots, X_n \) are generated inductively via (95) with the initialization

\[
(X_{1-p}, \ldots, X_0)^\top = w_\ell.
\]

Conditionally on \( W = w_\ell \), the random variables \( X_1, \ldots, X_n \) are thus an affine transformation of \( Z_1, \ldots, Z_n \). The transformation is of unit Jacobian (because the partial-derivatives matrix has 1’s on the diagonal and 0’s on the upper triangle), and thus

\[
h_\alpha(f_{X|w_\ell}) = h_\alpha(Z_1, \ldots, Z_n), \quad \ell \in \{1, \ldots, p\}. \quad (101)
\]

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From this, (99), and (100) it follows that

\[ h_\alpha(Z_1^n) \leq h_\alpha(f_X) \leq \min_{1 \leq \ell \leq p} \left\{ \frac{\alpha}{1-\alpha} \log q_\ell \right\} + h_\alpha(Z_1^n). \]

Dividing by \( n \) and using (97) establishes the result.

We next turn to the case \( 0 < \alpha < 1 \). For every \( M > 0 \) arbitrarily large, we use Proposition 5 to construct \( \{Z_i\} \) as above but with

\[ \lim_{n \to \infty} \frac{1}{n} h_\alpha(Z_1, \ldots, Z_n) \geq M. \]

The proof continues as for the case where \( \alpha \) exceeds one.

6 Discussion

6.1 On Theorem 2

As the following heuristic argument demonstrates, one has to walk a fine line in order to achieve the supremum promised in Theorem 2. To see why, let us focus on the case where \( h^*(\cdot) \) is strictly increasing and where there exist real constants \( \lambda_0, \lambda_1 \in \mathbb{R} \) for which the function \( f^*(x) = \exp (\lambda_0 + \lambda_1 r(x)) 1 \{x \in S\} \) is a density achieving \( h^*(\Gamma) \). For any other density \( g \) supported on \( S \) and satisfying

\[ \int_S g(x) r(x) \, dx = \Gamma \]  

we then have (as in the proof of [3, Theorem 12.1.1])

\[ h(g) = h(f^*) - D(g\|f^*) \]
\[ = h^*(\Gamma) - D(g\|f^*). \]

Using this and (14) we thus obtain that if \( \{Z_k\} \) is a stationary SP and if \( f_Z \) is the density of \( Z_1 \) and

\[ \int_S f_Z(x) r(x) \, dx = \Gamma, \]

then

\[ h_\alpha(\{Z_k\}) \leq h^*(\Gamma) - D(f_Z\|f^*), \quad \alpha > 1. \]
Thus, for $h_\alpha(\{Z_k\})$ to be close to $h^*(\Gamma)$, the density of $Z_1$ must be “close” (in relative-entropy) to $f^*$. We can repeat this argument for the joint density of $Z_1, Z_2$ to infer that $Z_1$ and $Z_2$ must be “nearly independent” with each being of density “nearly” $f^*$. More generally, for every fixed $m \in \mathbb{N}$ the joint density of $Z_1, \ldots, Z_m$ must be nearly of a product form. But, of course choosing $\{Z_k\}$ IID will not work, because this choice would lead to a Rényi rate equal to $h_\alpha(f_{Z_1})$, which is typically smaller than $h(Z_1)$ (see (8)).

6.2 On Theorem 6

Theorem 6 has bearing on the spectral estimation problem, i.e., the problem of extrapolating the values of the autocovariance sequence from its first $p + 1$ values. One approach is to choose the extrapolated sequence to be the autocovariance sequence of the stochastic process that—among all stochastic processes that have an autocovariance sequence that starts with these $p + 1$ values—maximizes the Shannon rate, namely the $p$-th order Gauss-Markov process (Burg’s theorem).

A different approach might be to choose some $\alpha > 1$ and to replace the maximization of the Shannon rate with that of the order-$\alpha$ Rényi rate. As we next argue, Theorem 6 shows that this would result in the same extrapolated sequence. Indeed, inspecting the proof of the theorem we see that the stochastic process $\{X_i\}$ that we constructed, while not a Gauss-Markov process, has the same autocovariance sequence as the $p$-th order Gauss-Markov process that satisfies the constraints. And, for $\alpha > 1$ the supremum can only be achieved by a stochastic process of this autocovariance sequence: for any other autocovariance function the Rényi rate is upper bounded by the Shannon rate (because $\alpha > 1$), and the latter is upper bounded by the Shannon rate of the Gaussian process, which, unless the autocovariance sequence is that of the $p$-th order Gauss-Markov process, is strictly smaller than the supremum (Burg’s theorem).

A Proof of Proposition 10

In this appendix we present two lemmas, which we then use to prove Proposition 10 on approaching $h^*(\Gamma)$ using bounded densities.

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4We are ignoring here the fact that one might consider approaching the supremum with (105) only being an inequality.
Lemma 13. Let $f$ be a density supported by $S$ for which $h(f)$ is defined;

$$\int f(x) |r(x)| \, dx < \infty;$$

(107)

and for which

$$\int f(x) r(x) \, dx \leq \Gamma$$

(108)

for some $\Gamma \in \mathbb{R}$. Then for every $\delta > 0$ there exists a density $\tilde{f}$ that is bounded, supported by $S$, and that satisfies

$$\int \tilde{f}(x) r(x) \, dx \leq \Gamma + \delta$$

(109)

and

$$h(\tilde{f}) \geq h(f) - \delta.$$  

(110)

Proof. Let $0 < \varepsilon < 1$ be fixed (small), with its choice specified later. It follows from (107) and the MCT that there exists some $M_1$ sufficiently large so that

$$\int \left( f(x) - \left( f(x) \wedge M_1 \right) \right) |r(x)| \, dx < \varepsilon,$$

where we recall that $a \wedge b$ stands for $\min\{a, b\}$. Since the density $f$ integrates to 1, we can find some $M_2$ sufficiently large so that

$$\int \left( f(x) \wedge M_2 \right) \, dx > 1 - \varepsilon.$$

Define now

$$M = \max\{1, M_1, M_2\}.$$  

(111)

For this $M$ we have:

$$\int \left( f(x) \wedge M \right) \, dx > 1 - \varepsilon,$$

(112a)

$$\int \left( f(x) - \left( f(x) \wedge M \right) \right) |r(x)| \, dx < \varepsilon,$$

(112b)

$$\left(f(x) \geq 1\right) \implies \left(f(x) \wedge M \geq 1\right).$$  

(112c)

Consider now the bounded density

$$\tilde{f}(x) = \frac{1}{\beta} \left(f(x) \wedge M\right)$$

(113a)
where

\[ \beta = \int (f(\tilde{x}) \land M) \, d\tilde{x}. \]  \hspace{1cm} (113b)

Note that because \( f(x) \land M \) is upper-bounded by \( f(x) \), which integrates to one, and because of (112a)

\[ 1 - \varepsilon \leq \beta \leq 1, \]  \hspace{1cm} (114)

so

\[ (f(x) \land M) \leq \tilde{f}(x) \leq \frac{1}{1 - \varepsilon} (f(x) \land M). \]  \hspace{1cm} (115)

Moreover, \( \tilde{f} \) is supported by \( \mathcal{S} \).

Given \( \delta > 0 \) we next show that by choosing \( \varepsilon \) sufficiently small we can guarantee that both (109) and (110) hold. Be begin with the former. Starting with (113a) we have

\[
\int \tilde{f}(x) \, r(x) \, dx \\
= \frac{1}{\beta} \int (f(x) \land M) \, r(x) \, dx \\
= \frac{1}{\beta} \int (f(x) - (f(x) - f(x) \land M)) \, r(x) \, dx \\
= \frac{1}{\beta} \int f(x) \, r(x) \, dx \\
+ \frac{1}{\beta} \int (f(x) - (f(x) \land M)) (-r(x)) \, dx \\
\leq \frac{1}{\beta} \Gamma + \frac{1}{\beta} \int (f(x) - (f(x) \land M)) |r(x)| \, dx \\
\leq \frac{1}{\beta} \Gamma + \frac{1}{\beta} \varepsilon \\
\leq \Gamma + \frac{\varepsilon}{1 - \varepsilon} |\Gamma| + \frac{\varepsilon}{1 - \varepsilon}, \]

where the first inequality follows from (108); the second from (112b); and the last from (114).
We next study $h(\tilde{f})$. Starting with the definition of $\tilde{f}$,

$$
\begin{align*}
    h(\tilde{f}) &= \int \frac{1}{\beta} (f(x) \land M) \log \frac{\beta}{f(x) \land M} \, dx \\
    &= \log \beta + \frac{1}{\beta} \int (f(x) \land M) \log \frac{1}{f(x) \land M} \, dx \\
    &= \log \beta + \frac{1}{\beta} \int_{x: f(x) \leq 1} (f(x) \land M) \log \frac{1}{f(x) \land M} \, dx \\
    &\quad + \frac{1}{\beta} \int_{x: f(x) > 1} (f(x) \land M) \log \frac{1}{f(x) \land M} \, dx.
\end{align*}
$$

(117)

By (111), $f(x) \land M = f(x)$ whenever $f(x) \leq 1$, so

$$
\int_{x: f(x) \leq 1} (f(x) \land M) \log \frac{1}{f(x) \land M} \, dx = \int_{x: f(x) \leq 1} f(x) \log \frac{1}{f(x)} \, dx.
$$

(118)

Since $\xi \log \xi^{-1}$ is decreasing for $\xi > 1$, and since $f(x) > 1$ implies $f(x) \land M > 1$ (by (112c)),

$$(f(x) \land M) \log \frac{1}{f(x) \land M} \geq f(x) \log \frac{1}{f(x)}, \quad \left( f(x) > 1 \right)$$

and hence

$$
\begin{align*}
    \int_{x: f(x) > 1} (f(x) \land M) \log \frac{1}{f(x) \land M} \, dx \\
    &\geq \int_{x: f(x) > 1} f(x) \log \frac{1}{f(x)} \, dx.
\end{align*}
$$

(119)

Summing (118) and (119) we obtain

$$
\int (f(x) \land M) \log \frac{1}{f(x) \land M} \, dx \geq h(f).
$$

(120)

Using this, (117), and (114) we conclude that

$$
h(\tilde{f}) = h(f), \quad \text{whenever } h(f) = \infty
$$
and
\[ h(\tilde{f}) \geq \log(1 - \varepsilon) + h(f) - \frac{\varepsilon}{1 - \varepsilon}|h(f)|, \]
whenever \( |h(f)| < \infty \). (121)

And obviously \( h(\tilde{f}) \geq h(f) \) whenever \( h(f) = -\infty \).

The result now follows by choosing \( \varepsilon \) small enough to guarantee that the RHS of (116) does not exceed \( \Gamma + \delta \) and—if \( h(f) \) is finite—that the RHS of (121) exceeds \( h(f) - \delta \). \( \square \)

The following lemma addresses the case where (107) does not hold.

**Lemma 14.** Let the density \( f \) supported by \( S \) be such that
\[ \int f(x) \, r(x) \, dx = -\infty \] (122)
and \( h(f) \) is defined and exceeds \(-\infty\)
\[ h(f) > -\infty. \] (123)

Then there exists a sequence of densities \( \{\tilde{f}_k\} \) supported by \( S \) for which
\[ \int \tilde{f}_k(x) \, |r(x)| \, dx < \infty, \]
\[ \lim_{k \to \infty} h(\tilde{f}_k) = h(f), \]
and
\[ \lim_{k \to \infty} \int \tilde{f}_k(x) \, r(x) \, dx = -\infty. \]

**Proof.** Define \( r^+ \triangleq \max\{r, 0\} \) and \( r^- \triangleq \max\{-r, 0\} \), so \( r = r^+ - r^- \) with \( r^+(x), r^-(x) \geq 0 \). By (122),
\[ \int f(x) \, r^-(x) \, dx = \infty, \] (124a)
\[ \int f(x) \, r^+(x) \, dx < \infty. \] (124b)

Define for every \( k \in \mathbb{N} \)
\[ \mathcal{D}_k \triangleq \{x : r^-(x) \leq k\}. \] (125)
By the MCT
\[
\lim_{k \to \infty} \int_{\mathcal{D}_k} f(x) r^+(x) \, dx = \int f(x) r^+(x) \, dx < \infty
\]  
(126a)
and
\[
\lim_{k \to \infty} \int f(x) r^-(x) I\{x \in \mathcal{D}_k\} \, dx = \infty.
\]  
(126b)
Consequently,
\[
\lim_{k \to \infty} \int_{\mathcal{D}_k} f(x) r(x) \, dx = -\infty.
\]  
(127)

The lemma’s hypotheses guarantee that \( h(f) \) is defined and exceeds \(-\infty\). Consequently,
\[ h(f) = h^+(f) - h^-(f), \]
with
\[ h^-(f) < \infty, \quad h^+(f) \leq \infty, \]  
(128)
where,
\[
h^+(f) \triangleq \int f(x) \log \frac{1}{f(x)} I\{f(x) \leq 1\} \, dx,
\]
\[
h^-(f) \triangleq \int f(x) \log f(x) I\{f(x) > 1\} \, dx.
\]
By the MCT
\[
\int_{\mathcal{D}_k} f(x) \log \frac{1}{f(x)} I\{f(x) \leq 1\} \, dx \uparrow h^+(f)
\]
and
\[
\int_{\mathcal{D}_k} f(x) \log f(x) I\{f(x) > 1\} \, dx \uparrow h^-(f)
\]
so, upon subtracting (and recalling \( h^-(f) < \infty \))
\[
\lim_{k \to \infty} \int_{\mathcal{D}_k} f(x) \log \frac{1}{f(x)} \, dx = h(f).
\]  
(129)
Define
\[ \beta_k \triangleq \int_{D_k} f(x) \, dx. \]

Note that since \( f \) is a density, \( \beta_k \leq 1 \)
and (by the MCT)
\[ \beta_k \uparrow 1. \quad (130) \]
Consequently,
\[ 0 < \beta_k \leq 1, \quad k \text{ large.} \quad (131) \]
For every such sufficiently large \( k \), define the density
\[ \tilde{f}_k(x) \triangleq \beta_k^{-1} f(x) I\{x \in D_k\}. \]
It is supported by \( S \), and its entropy \( h(\tilde{f}_k) \) can be expressed as
\[
\begin{align*}
    h(\tilde{f}_k) &= \int \tilde{f}_k(x) \log \frac{1}{\tilde{f}_k(x)} \, dx \\
    &= \int_{D_k} \tilde{f}_k(x) \log \frac{1}{f(x)} \, dx \\
    &= \int_{D_k} \frac{1}{\beta_k} f(x) \log \frac{\beta_k}{f(x)} \, dx \\
    &= \log \beta_k + \frac{1}{\beta_k} \int_{D_k} f(x) \log \frac{1}{f(x)} \, dx.
\end{align*}
\]
From this, (129), and (130) we obtain
\[ \lim_{k \to \infty} h(\tilde{f}_k) = h(f). \quad (132) \]
And as to the expectation of \( r(x) \) under \( \tilde{f}_k \):
\[
\begin{align*}
    \int \tilde{f}_k(x) \, r(x) \, dx &= \frac{1}{\beta_k} \int_{D_k} f(x) \, r(x) \, dx \\
    &= \frac{1}{\beta_k} \int_{D_k} f(x) \, r^+(x) \, dx - \frac{1}{\beta_k} \int_{D_k} f(x) \, r^-(x) \, dx.
\end{align*}
\]
The first term on the LHS is finite by (131) and (124b). The second tends to \(-\infty\) by (130) and (127). Hence,
\[
\lim_{k \to \infty} \int \tilde{f}_k(x) r(x) \, dx = -\infty.
\] (133)

Moreover,
\[
\int \tilde{f}_k(x) |r(x)| \, dx
= \frac{1}{\beta_k} \int_{D_k} f(x) r^+(x) \, dx + \frac{1}{\beta_k} \int_{D_k} f(x) r^-(x) \, dx
\leq \frac{1}{\beta_k} \int f(x) r^+(x) \, dx + k
< \infty,
\] (134)
where the first inequality follows from the nonnegativity of \(r^+\) and from the definition of the set \(D_k\) (125), and the second inequality follows from (124b) and (131).

The lemma now follows from (134), (132), and (133). \(\square\)

**Proof of Proposition 10.** Since \(\Gamma\) exceeds \(\Gamma_0\), it follows from (17) that
\[
-\infty < h^*(\Gamma) < \infty.
\] (135)

Let the density \(f\) nearly achieve \(h^*(\Gamma)\) in the sense that it is supported by \(S\) and that
\[
\int f(x) r(x) \, dx \leq \Gamma, \quad \text{and} \quad h(f) > h^*(\Gamma) - \frac{\delta}{2}.
\] (136)

By (135), (136), and the definition of \(h^*(\Gamma)\),
\[
-\infty < h(f) < \infty.
\] (137)

If \(\int f(x)|r(x)| \, dx\) is finite, then the result follows directly from Lemma 13. It remains to prove the result when this integral is infinite. In this case \(\int f(x) r(x) \, dx = -\infty\) by (136) (because \(\Gamma < \infty\)). Using this, the finiteness of \(h(f)\) (137), and Lemma 14, we infer the existence of a density \(\tilde{f}\) that supported by \(S\) and for which
\[
\int \tilde{f}(x) |r(x)| \, dx < \infty,
\] (138a)
\[ h(\tilde{f}) > h(f) - \frac{\delta}{2}, \quad (138b) \]
\[ \int \tilde{f}(x) r(x) \, dx < \Gamma. \quad (138c) \]

Applying Lemma 13 to the density \( \tilde{f} \), we conclude that there exists a bounded density \( f^* \) that is supported by \( S \) and that satisfies
\[ h(f^*) > h(\tilde{f}) - \delta \quad \text{and} \quad \int f^*(x) r(x) \, dx \leq \Gamma + \delta \quad (139) \]
and hence, in view of (138) and (136),
\[ h(f^*) > h^*(\Gamma) - \delta \quad \text{and} \quad \int f^*(x) r(x) \, dx \leq \Gamma + \delta. \quad (140) \]

The existence of \( f^* \) concludes the proof of the proposition for the case where \( \int f(x) |r(x)| \, dx \) is infinite.

\[ \square \]

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