DUAL COMPLEX OF LOG FANO PAIRS AND ITS APPLICATION TO WITT VECTOR COHOMOLOGY

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Abstract. We prove the contractibility of the dual complexes of weak log Fano pairs. As applications, we obtain a vanishing theorem of Witt vector cohomology of Ambro-Fujino type and a rational point formula in dimension three.

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1. Introduction

In the first half of this paper, we discuss the dual complex of Fano pairs. A dual complex is a combinatorial object which expresses how the components of $\Delta^{-1}$ intersect for a dlt pair $(X, \Delta)$. In [dFKX17], they study the dual complex of a dlt modification of a pair $(X, \Delta)$, and show that the dual complex is independent of the choice of a dlt modification (minimal dlt blow-up) up to PL homeomorphism. In [KX16] and [Mau], the dual complexes of log canonical dlt pairs $(X, \Delta)$ with $K_X + \Delta \sim_\mathbb{Q} 0$ are studied.

Our main theorem is the contractibility of the dual complexes of weak Fano pairs (see [KX16] 22 for a similar result).

Theorem 1.1 (= Theorem 3.8). Let $(X, \Delta)$ be a projective pair over an algebraic closed field of characteristic zero. Assume that $-(K_X + \Delta)$ is nef and big. Then for any dlt blow-up $g : (Y, \Delta_Y) \to (X, \Delta)$, the dual complex $D(\Delta_Y^{-1})$ is contractible, where we define $\Delta_Y$ by $K_Y + \Delta_Y = g^*(K_X + \Delta)$.
In this theorem, the coefficients of $\Delta$ might be larger than one contrary to the setting in [dFKX17]. We prove that the dual complex is independent of the choice of a dlt blow-up (which is possibly not minimal) up to homotopy equivalence in our setting (Proposition 2.13, see also Remark 2.14). The proof of Proposition 2.13 depends on the weak factorization theorem [AKMW02] and does not work in positive characteristic even in dimension three. Hence, we get the following weaker theorem in positive characteristic.

**Theorem 1.2** (= Theorem 3.7). Let $(X, \Delta)$ be a three dimensional projective pair over an algebraic closed field of characteristic larger than five. Assume that $-(K_X + \Delta)$ is nef and big. Then, there exists a dlt blow-up $g : (Y, \Delta_Y) \to (X, \Delta)$ such that the dual complex $D(\Delta_Y^{\geq 1})$ is contractible, where we define $\Delta_Y$ by $K_Y + \Delta_Y = g^*(K_X + \Delta)$.

In the latter half of this paper, we discuss an application of the above result on the dual complex to the study on Witt vector cohomology in positive characteristic. In [Esn03], it is shown that the vanishing $H^i(X, W^iO_X, \mathbb{Q}) = 0$ holds for $i > 0$ and a geometrically connected smooth Fano variety $X$ defined over an algebraic closed field $k$. This vanishing theorem is very impressive because it is not known whether $H^i(X, O_X) = 0$ holds or not by the lack of the Kodaira vanishing theorem in positive characteristic. In [GNT], Esnault’s result is generalized to klt pairs of dimension 3 and in char$(k) > 5$. In [NT], the result in [GNT] is generalized as a vanishing theorem of Nadel type (see Theorem 2.15). The following main theorem of this paper is another generalization of the result in [GNT] (we note that the result in [GNT] is Theorem 1.3 with the additional restriction that the pair $(X, \Delta)$ is klt).

**Theorem 1.3** (= Theorem 4.13). Let $k$ be a perfect field of characteristic $p > 5$. Let $(X, \Delta)$ be a three-dimensional projective $\mathbb{Q}$-factorial log canonical pair over $k$ with $-(K_X + \Delta)$ ample. Then $H^i(X, W^iO_X, \mathbb{Q}) = 0$ holds for $i > 0$.

By the vanishing theorem of Nadel type (Theorem 2.15), the proof of Theorem 1.3 is reduced to the topological study of the non-klt locus of the pair $(X, \Delta)$ (Proposition 4.5 and Proposition 4.12). For the proof of Proposition 4.5 and Proposition 4.12, we use the result on the dual complex (Theorem 1.2) to obtain the topological information.

An important application of Witt vector cohomology is the rational point formula on varieties defined over a finite field (cf. [Esn03, GNT, NT]). One of the motivation of the papers [Esn03, GNT, NT] is to generalize the Ax-Katz theorem [Ax64, Kat71], which states that any hypersurface $H \subset \mathbb{P}^n$ of degree $d \leq n$ defined over $\mathbb{F}_q$ has a rational point. Theorem 1.3 suggests that the Ax-Katz theorem might be generalized to singular ambient spaces, and we actually obtain the following theorem in dimension three.

**Theorem 1.4** (= Theorem 4.14). Let $k$ be a finite field of characteristic $p > 5$. Let $(X, \Delta)$ be a geometrically connected three-dimensional projective $\mathbb{Q}$-factorial log canonical pair over $k$ with $-(K_X + \Delta)$ ample. Then the number of the $k$-rational points on the non-klt locus of $(X, \Delta)$ satisfies

$$\#N_{klt}(X, \Delta)(k) \equiv 1 \mod |k|.$$ 

In particular, there exists a $k$-rational point on $N_{klt}(X, \Delta)$. 


If $X$ is klt and $\Delta$ is a reduced divisor, then $\text{Nklt}(X, \Delta) = \text{Supp}(\Delta)$ holds and Theorem 1.4 claims that there exists a $k$-rational point on $\text{Supp}(\Delta)$. This formulation can be seen as a generalization of the Ax-Katz theorem from the view point of birational geometry.

Acknowledgements. We would like to thank Professors Yoshinori Gongyo and Hiromu Tanaka for the discussions and Mirko Mauri for useful comments and suggestions. The author is partially supported by the Grant-in-Aid for Young Scientists (KAKENHI No. 18K13384).

2. Preliminaries

2.1. Notation.

- We basically follow the notations and the terminologies in [Har77] and [Kol13].
- For a field $k$, we say that $X$ is a variety over $k$ if $X$ is an integral separated scheme of finite type over $k$.
- A sub log pair $(X, \Delta)$ over a field $k$ consists of a normal variety $X$ over $k$ and an $R$-divisor $\Delta$ such that $K_X + \Delta$ is $R$-Cartier. A sub log pair is called log pair if $\Delta$ is effective. Note that the coefficient of $\Delta$ may be larger than one in this definition.
- Let $\Delta = \sum r_i D_i$ be an $R$-divisor where $D_i$ are distinct prime divisors. We define $\Delta \geq 1 := \sum r_i D_i$ and $\Delta ^{\leq 1} := \sum r'_i D_i$ where $r'_i := \min\{r_i, 1\}$. We also define $\Delta > 1, \Delta < 1, \text{ and } \Delta = 1$ similarly.
- Let $X$ be a variety over $k$. We denote by $(\ast)$ the condition that $k$ and $X$ satisfy one of the following conditions:
  (i) $\text{ch}(k) = 0$, or
  (ii) $\text{ch}(k) > 5$ and $\dim X = 3$.
This condition is necessary for running a certain MMP appeared in this paper [BCHM10][HX15][Bir16][BW17][Wal18][HNT].

2.2. Results on minimal model program. In this subsection we review results on minimal model program. In this subsection, $k$ is an algebraic closed field.

First, we review the definition of singularities of log pairs. In this paper, we treat the following definitions only under the condition $(\ast)$, because we do not know whether the definitions perform well also in $\dim X > 3$ and $\text{ch}(k) > 0$.

Definition 2.1. (1) Let $(X, \Delta)$ be a log pair over $k$. For a proper birational $k$-morphism $f : X' \to X$ from a normal variety $X'$ and a prime divisor $E$ on $X'$, the log discrepancy of $(X, \Delta)$ at $E$ is defined as
$$a_E(X, \Delta) := 1 + \text{coeff}_E(K_X - f^*(K_X + \Delta)).$$
(2) A log pair $(X, \Delta)$ is called klt (resp. log canonical) if $a_E(X, \Delta) > 0$ (resp. $\geq 0$) for any prime divisor $E$ over $X$.
(3) A log pair $(X, \Delta)$ is called dlt if the coefficients of $\Delta$ are at most one and there exists a log resolution $g : Y \to X$ of the pair $(X, \Delta)$ such that $a_E(X, \Delta) > 0$ holds for any $g$-exceptional prime divisor $E$ on $Y$. 
(4) Let \((X, \Delta)\) be a dlt pair and let \(\Delta = \sum_{i \in I} E_i\) be the irreducible decomposition. For any non-empty subset \(J \subset I\), an connected component of \(\bigcap_{i \in J} E_i\) is called a stratum of \(\Delta = 1\).

**Remark 2.2.** The above definition (3) is equivalent to the definition in [KM98, Definition 2.37]:

- The coefficients of \(\Delta\) are at most one. Moreover, there exists an open subset \(U \subset X\) such that \((U, \Delta|_U)\) is log smooth and no non-klt center of \((X, \Delta)\) is contained in \(X \setminus U\).

This equivalence is shown in [Sza94] in characteristic zero. The equivalence is also true in positive characteristic (in dimension three) since the Szabó’s resolution lemma ([Fuj17, Lemma 2.3.19]) also holds by [CP08, Proposition 4.1] (see also [Fuj17, Proposition 2.3.20] and [Bir16, 2.4, 2.5]). Hence, even in our definition, being dlt is preserved under the MMP ([KM98, Corollary 3.44]).

The following proposition is necessary for defining the dual complexes of dlt pairs.

**Proposition 2.3.** ([Kol13, Theorem 4.16], [Fuj07, Section 3.9], [DH16, Proposition 1]). Let \((X, \Delta)\) be a \(\mathbb{Q}\)-factorial dlt pair over \(k\) and \(\Delta = \sum_{i \in I} E_i\) be the irreducible decomposition. We assume the condition (*) (defined in Subsection 2.1). Then the following hold.

1. Let \(J \subset I\) be a subset. If \(\bigcap_{i \in J} E_i \neq \emptyset\), then each connected component of \(\bigcap_{i \in J} E_i\) is normal (hence irreducible) and has codimension \(\# J\).
2. Let \(J \subset I\) be a subset, and let \(j \in J\). Then each connected component of \(\bigcap_{i \in J} E_i\) is contained in the unique connected component of \(\bigcap_{i \in J \setminus \{j\}} E_i\).

**Proof.** See [Kol13] Theorem 4.16]. The assertion that each connected component of \(\bigcap_{i \in J} E_i\) is irreducible is not explicitly written in [Kol13, Theorem 4.16]. However it follows from the fact that the intersection of any two log canonical centers is also a union of log canonical centers (cf. [Fuj11, Theorem 9.1], [DH16, Lemma 1]).

(2) is trivial. 

In this paper, we will use the terminology “dlt blow-up” in the following sense.

**Definition 2.4.** Let \((X, \Delta)\) be a log pair over \(k\) and let \(g : Y \to X\) be a projective birational \(k\)-morphism. We call \(g\) a dlt blow-up of \((X, \Delta)\) if the following conditions hold:

1. \(a_E(X, \Delta) \leq 0\) holds for any \(g\)-exceptional prime divisor \(E\).
2. \((Y, \Delta|_Y)\) is a \(\mathbb{Q}\)-factorial dlt pair, where \(\Delta_Y\) is the \(\mathbb{R}\)-divisor defined by \(K_Y + \Delta_Y = g^*(K_X + \Delta)\).

**Theorem 2.5.** Let \((X, \Delta)\) be a log pair over \(k\) with the condition (*). Then a dlt blow-up of \((X, \Delta)\) exists. Further, we can take a dlt blow-up \(g : V \to X\) with the following additional condition:

3. \(g^{-1}(\text{Nklt}(X, \Delta)) = \text{Nklt}(V, \Delta_V)\) holds.
Proof. By Step 2 in the proof of [HNT] Proposition 3.5, in order to show the existence of a dlt blow-up with the condition (3), it is sufficient to show the existence of a usual dlt blow-up (that is with the only conditions (1) and (2)). For the existence of a dlt blow-up, the same proof as in [Fuj11] Theorem 10.4] works as follows.

Let \( f : Y \to X \) be a log resolution of \((X, \Delta)\). Let \( F = \sum F_i \) be the sum of the \( f \)-exceptional divisors \( F_i \) with \( a_{F_i}(X, \Delta) \leq 0 \), and let \( G = \sum G_i \) be the sum of the \( f \)-exceptional divisors \( G_i \) with \( a_{G_i}(X, \Delta) > 0 \). Let \( \Delta \) be the strict transform of \( \Delta \) on \( Y \). We may assume that there exists an effective \( \mathbb{R} \)-divisor \( H \) on \( Y \) such that \( \text{Supp } H = \text{Supp } (F + G) \) and that \(-H\) is \( f \)-ample.

We set an \( \mathbb{R} \)-divisor \( \Omega \) as
\[
\Omega = \Delta^{\lambda_1} + F + (1-\epsilon)G - \delta H
\]
for sufficiently small \( \epsilon, \delta > 0 \). Since \(-H\) is \( f \)-ample, there exists an effective ample \( \mathbb{R} \)-divisor \( A \) on \( Y \) such that \(-\delta H \sim_{X, \mathbb{R}} A \). We set
\[
\Omega = \Delta^{\lambda_1} + F + (1-\epsilon)G + A.
\]

We may assume that \((Y, \Omega)\) is dlt. Note that \( K_Y + \Omega \sim_{X, \mathbb{R}} K_Y + \Omega \). Since \( A \) is ample, there exists an \( \mathbb{R} \)-divisor \( \Omega' \) such that \( \Omega' \sim_{\mathbb{R}} \Omega \) and \((Y, \Omega')\) is klt. Hence, we may run a \((K_Y + \Omega)\)-MMP over \( X \) and it terminates. Let \( Y' \) be the end result and let \( h : Y' \to X \) be the induced morphism. We shall show that \( h : Y' \to X \) is a dlt blow-up of \((X, \Delta)\). We have
\[
K_Y + \Omega \sim_{X, \mathbb{R}} K_Y + \Omega - f^*(K_X + \Delta)
\]
\[
= -(\Delta - \Delta^{\lambda_1}) + \sum a_i F_i + \sum g_i G_i - \epsilon G - \delta H,
\]
where \( a_i = a_{F_i}(X, \Delta) \leq 0 \) and \( b_i = a_{G_i}(X, \Delta) > 0 \). Since \( b_i > 0 \) and \( \epsilon \) and \( \delta \) are sufficiently small, it follows that
\[
\text{coeff}_{G_j} \left( \sum a_i F_i + \sum b_i G_i - \epsilon G - \delta H \right) > 0
\]
for each \( G_j \). Hence by the negativity lemma, all the divisors \( G_i \)'s are contracted in this MMP. Therefore \( a_E(X, \Delta) \leq 0 \) holds for any \( h \)-exceptional prime divisor \( E \). Since the \((K_Y + \Omega)\)-MMP is also a \((K_Y + \Omega')\)-MMP, the pair \((Y', \Omega_{Y'})\) is still dlt where \( \Omega_{Y'} \) is the push forward of \( \Omega \). Define \( \Delta_{Y'} \) by \( K_{Y'} + \Delta_{Y'} = h^*(K_X + \Delta) \). Then \( \Delta_{Y'}^{\lambda_1} \) is the push forward of \( \Delta^{\lambda_1} + F \) on \( Y' \), and hence \( 0 \leq \Delta_{Y'}^{\lambda_1} \leq \Omega_{Y'} \) holds. Therefore \((Y', \Delta_{Y'}^{\lambda_1})\) is also dlt. We have proved that \( g \) is a dlt blow-up of \((X, \Delta)\). \( \square \)

Remark 2.6. (1) When \( X \) is \( \mathbb{Q} \)-factorial, any dlt blow-up of \((X, \Delta)\) satisfies condition (3) in Theorem 2.5.

(2) If \( V \to X \) is a log resolution of \((X, \Delta)\), then we can construct (by the proof above) a dlt blow-up \( Y \to X \) of \((X, \Delta)\) such that the induced birational map \( Y \to V \) does not contract any divisor on \( Y \).

2.3. Dual complexes. In this subsection, we explain how to define a CW complex from a dlt pair, and we also prove an invariant property (Proposition 2.13).

First, we briefly review the notion of \( \Delta \)-complexes following [Hat02].
Definition 2.7. (1) Let \( X = \cup_{\alpha} F_\alpha \) be a CW complex with the attaching maps \( \varphi_\alpha : F_\alpha \to X \). We call \( X \) a \( \Delta \)-complex when each cell \( F_\alpha \) is a simplex and the restriction of \( \varphi_\alpha \) to each face of \( F_\alpha \) is equal to the attaching map \( \varphi_\beta : F_\beta \to X \) for some \( \beta \).

(2) A \( \Delta \)-complex \( X \) is called regular if the attaching maps are injective, or equivalently, if every \( d \)-cell in \( X \) has \( d + 1 \) distinct vertices.

(3) A regular \( \Delta \)-complex \( X \) is called a simplicial complex if the intersection of any two cells in \( X \) is a face of both cells, or equivalently, every \( k + 1 \) vertices in \( X \) is incident to at most one \( k \)-cell.

For a dlt pair \( (X, \Delta) \), we define the dual complex \( D(\Delta^{=1}) \).

Definition 2.8. (1) Let \( (X, \Delta) \) be a \( \mathbb{Q} \)-factorial dlt pair over \( k \) with the condition (\( * \)) and let \( \Delta^{=1} = \sum_{i \in I} E_i \) be the irreducible decomposition. Then the dual complex \( D(\Delta^{=1}) \) is a CW complex obtained as follows. The vertices of \( D(\Delta^{=1}) \) are the set of \( \{ E_i \}_{i \in I} \). To each \( k \)-codimensional stratum \( S \) of \( \Delta^{=1} \) we associate a \( k \)-dimensional cell. The attaching map is uniquely defined by Proposition 2.9 (2).

(2) Let \( (X, \Delta) \) be a \( \mathbb{Q} \)-factorial pair over \( k \) with the condition (\( * \)). Suppose that \( (X, \Delta^{=1}) \) is dlt. Then we define \( D(\Delta^{=1}) = D((\Delta^{=1})^{=1}) \).

Proposition 2.9. Let \( (X, \Delta) \) be a \( \mathbb{Q} \)-factorial dlt pair over \( k \) with the condition (\( * \)). Then the dual complex \( D(\Delta^{=1}) \) is a regular \( \Delta \)-complex.

Proof. The assertion follows from Proposition 2.9 (1). See [dFKX17] for more detail.

The following theorem from [dFKX17] says that the dual complex is preserved under a certain MMP up to simple-homotopy equivalence.

Theorem 2.10 ([dFKX17, Proposition 19]). Let \( (X, \Delta) \) be a \( \mathbb{Q} \)-factorial dlt pair over \( k \) with condition (\( * \)) and let \( f : X \dashrightarrow Y \) be a divisorial contraction or flip corresponding to a \((K_X + \Delta)\)-negative extremal ray \( R \). Assume that there is a prime divisor \( D_0 \subseteq \Delta^{=1} \) such that \( D_0 \cdot R > 0 \). Then \( D(\Delta^{=1}) \) collapses to \( D(\Delta_Y^{=1}) \) where we set \( \Delta_Y := f_* \Delta \). In particular \( D(\Delta^{=1}) \) and \( D(\Delta_Y^{=1}) \) are simple-homotopy equivalent.

Lemma 2.11. Let \( (X, \Delta) \) be a \( \mathbb{Q} \)-factorial pair over \( k \) with the condition (\( * \)). Suppose that \( (X, \Delta^{=1}) \) is dlt. Let \( f : Y \to X \) be a log resolution of \( (X, \Delta^{=1}) \) such that \( a_E(X, \Delta^{=1}) > 0 \) holds for any \( f \)-exceptional divisor \( E \). Let \( \Delta_Y \) be the (not necessarily effective) \( \mathbb{R} \)-divisor defined by \( K_Y + \Delta_Y = f^*(K_X + \Delta) \). Then the dual complexes \( D(\Delta^{=1}) \) and \( D(\Delta_Y^{=1}) \) are simple-homotopy equivalent.

Proof. Let \( F = \sum F_i \) be the sum of the \( f \)-exceptional divisors \( F_i \) with \( a_{F_i}(X, \Delta) \leq 0 \), and let \( G = \sum G_i \) be the sum of the \( f \)-exceptional divisors \( G_i \) with \( a_{G_i}(X, \Delta) > 0 \). Let \( \Delta \) be the strict transform of \( \Delta \) on \( Y \). Let \( H \) be an effective \( \mathbb{R} \)-divisor on \( Y \) such that \( \text{Supp } H = \text{Supp } (F + G) \) and that \( -H \) is \( f \)-ample. We set an \( \mathbb{R} \)-divisor \( \Omega \) as

\[
\Omega = \tilde{\Delta}^{=1} + F + (1 - \epsilon)G - \delta H
\]
for sufficiently small $\epsilon, \delta > 0$. Since $-H$ is $f$-ample, there exists an effective ample $\mathbb{R}$-divisor $A$ on $Y$ such that $-\delta H \sim_{/ X, \mathbb{R}} A$. We set

$$\Omega = \Delta^{\leq 1} + F + (1 - \epsilon) G + A.$$  

We may assume that $(Y, \Omega)$ is dlt. Note that $K_Y + \Omega \sim_{/ X, \mathbb{R}} K_Y + \Omega$ and $D(\Omega^{0}) = D(\Delta_0)$.  

Since $A$ is ample, there exists an $\mathbb{R}$-divisor $\Omega'$ such that $\Omega' \sim_{\mathbb{R}} \Omega$ and $(Y, \Omega')$ is klt. Hence, we may run a $(K_Y + \Omega)$-MMP over $X$ and it terminates. First, we prove that this MMP ends with $X$.

$$K_Y + \Omega \sim_{/ X, \mathbb{R}} K_Y + \Omega - f^*(K_X + \Delta^{\leq 1})$$  

where $a_i = a_{F_i}(X, \Delta^{\leq 1})$ and $b_i = a_{G_i}(X, \Delta^{\leq 1})$. Since $a_i, b_i > 0$ and $\epsilon$ and $\delta$ are sufficiently small, it follows that

$$\text{coeff}_E \left( \sum a_i F_i + \sum b_i G_i - \epsilon G - \delta H \right) > 0$$  

for any $f$-exceptional divisor $E$. Hence by the negativity lemma, all the divisors $F_i$’s and $G_i$’s are contracted in this MMP. Since $X$ is $\mathbb{Q}$-factorial, this MMP ends with $X$.  

Let $Y_j \to Y_{j+1}$ be the step of the $(K_Y + \Omega)$-MMP over $X$, and let $R$ be the corresponding extremal ray, and $Y_j \to Z_j$ be its contraction. For a divisor $D$ on $Y$, we denote $D_{Y_j}$ the strict transform of $D$ on $Y_j$. We also denote $g : Y_j \to X$ the induced morphism. Then,

$$K_{Y_j} + \Omega_{Y_j} \sim_{/ X, \mathbb{R}} K_{Y_j} + \Omega_{Y_j} + g^*(K_X + \Delta)$$  

$$= - \left( \Delta_{Y_j} - (\Delta_{Y_j})^{\leq 1} \right) + \sum a_i^{F_i} F_{i,Y_j} + \sum b_i^{G_i} G_{i,Y_j} - \epsilon G_{Y_j} - \delta H_{Y_j},$$  

where $a_i^{F_i} = a_{F_i}(X, \Delta)$ and $b_i^{G_i} = a_{G_i}(X, \Delta)$. Since $a_i^{F_i} \leq 0$, it follows that

$$\text{coeff}_{F_{i,Y_j}} \left( \sum a_i^{F_i} F_{i,Y_j} + \sum b_i^{G_i} G_{i,Y_j} - \epsilon G_{Y_j} - \delta H_{Y_j} \right) < 0.$$  

Since $b_i^{F_i} > 0$ and $\epsilon$ and $\delta$ are sufficiently small, it follows that

$$\text{coeff}_{G_{i,Y_j}} \left( \sum a_i^{F_i} F_{i,Y_j} + \sum b_i^{G_i} G_{i,Y_j} - \epsilon G_{Y_j} - \delta H_{Y_j} \right) > 0.$$  

Since $(K_{Y_j} + \Omega_{Y_j}) \cdot R < 0$, at least one of the following conditions hold:

1. $D \cdot R > 0$ holds for some component $D \subset \text{Supp}(\Delta_{Y_j})^{\leq 1}$.  
2. $F_{i,Y_j} \cdot R > 0$ for some $i$.  
3. $G_{i,Y_j} \cdot R < 0$ for some $i$.  

Here, we have $K_{Y_j} + \Omega_{Y_j} \sim_{/ X, \mathbb{R}} K_{Y_j} + \Omega_{Y_j}$ and that a component of $\Omega_{Y_j}^{0}$ is one of $F_{i,Y_j}$’s or a component of $(\Delta_{Y_j})^{\leq 1}$. Hence in the case (1) or (2), by Theorem 2.10, the dual complexes $D((\Omega_{Y_j})^{0})^{1}$ and $D((\Omega_{Y_{j+1}})^{0})^{1}$ are simple-homotopy equivalent. In the case (3), the exceptional locus $L$ of $Y_j \to Z_j$ is contained in $\text{Supp}(G_{i,Y_j})$. Since $(Y_j, \Omega_{Y_j})$ is dlt, any stratum of $\Omega_{Y_j}^{0}$ is not contained in $\text{Supp}(G_{i,Y_j})$ and neither in $L$. Hence in the case (3), by [dFKX17 Lemma 16], it follows that $D((\Omega_{Y_j})^{0})^{1} = D((\Omega_{Y_{j+1}})^{0})^{1}$.  

Hence by induction on \(j\), the dual complexes \(\mathcal{D}(\Omega^{-1})\) and \(\mathcal{D}((\Omega_X)^{-1})\) are simple-homotopy equivalent. Since \(\mathcal{D}(\Omega^{-1}) = \mathcal{D}(\Delta_Y^{>1})\) and \(\mathcal{D}((\Omega_X)^{-1}) = \mathcal{D}(\Delta_Y^{>1})\), it follows that \(\mathcal{D}(\Delta_Y^{>1})\) and \(\mathcal{D}(\Delta_Y^{>1})\) are homotopy equivalent. □

**Lemma 2.12.** Let \((X, \Delta)\) be a sub log pair over \(k\) with the condition (*) such that \((X, \text{Supp} \, \Delta)\) is log smooth. Let \(Z\) be a smooth irreducible subvariety of \(X\) which has only simple normal crossing with \(\text{Supp} \, \Delta\). Let \(f : Y \to X\) be the blow up along \(Z\), and let \(\Delta_Y\) be the \(\mathbb{R}\)-divisor defined by \(\bigoplus Y + \Delta_Y = f^* (K_X + \Delta)\). Then the dual complexes \(\mathcal{D}(\Delta_Y^{>1})\) and \(\mathcal{D}(\Delta_Y^{>1})\) are simple-homotopy equivalent.

**Proof.** Set \(F = (\Delta_Y^{>1})^{\wedge}1\) and \(F_Y = (\Delta_Y^{>1})^{\wedge}1\). Let \(E\) be the \(E\)-exceptional divisor and \(\tilde{F}\) be the strict transform of \(F\) on \(Y\). We divide the case as follows:

1. \(Z\) is a stratum of \(F\).
2. \(Z \subset \text{Supp} \, F\) but \(Z\) is not a stratum of \(F\).
3. \(Z \not\subset \text{Supp} \, F\) (in particular, \(Z\) is not a stratum of \(F\)).

Suppose (1). Then \(F_Y = \tilde{F} + E\) holds. On the other hand, \(\mathcal{D}(\tilde{F} + E)\) and \(\mathcal{D}(F)\) are PL homeomorphic each other by \([\text{dFKX}17, \text{9}]\).

Suppose (2). Then \(F_Y = \tilde{F}\) or \(F_Y = \tilde{F} + E\) holds. On the other hand, \(\mathcal{D}(\tilde{F}) = \mathcal{D}(F)\) holds and \(\mathcal{D}(\tilde{F} + E)\) and \(\mathcal{D}(F)\) are simple-homotopy equivalent by \([\text{dFKX}17, \text{9}]\).

Suppose (3). Then \(F_Y = \tilde{F}\) holds and \(\mathcal{D}(\tilde{F}) = \mathcal{D}(F)\). □

**Proposition 2.13.** Let \((X, \Delta)\) be a pair over \(k\) with condition (i) in (*) and let \(f_1 : Y_1 \to X\) and \(f_2 : Y_2 \to X\) be two dlt blow-ups of \((X, \Delta)\). Define \(\mathbb{R}\)-divisors \(\Delta_{Y_1}\) by \(\bigoplus Y_1 + \Delta_{Y_1} = f_1^* (K_X + \Delta)\). Then the dual complexes \(\mathcal{D}(\Delta_{Y_1}^{>1})\) and \(\mathcal{D}(\Delta_{Y_1}^{>1})\) are simple-homotopy equivalent.

**Proof.** The same proof of \([\text{dFKX}17, \text{Proposition 11}]\) works. For the reader’s convenience, we give a sketch of proof.

By definition of dlt pairs, we can take a log resolution \(g_i : W_i \to Y_i\) of \((Y_i, \Delta_{Y_i})\) such that \(a_E(Y_i, \Delta_{Y_i}^{>1}) > 0\) holds for any \(g_i\)-exceptional divisor \(E\). Define \(\mathbb{R}\)-divisors \(\Delta_{W_i}\) on \(W_i\) by \(\bigoplus W_i + \Delta_{W_i} = g_i^* (K_Y + \Delta_{Y_i})\). By Lemma \(2.11\) the dual complexes \(\mathcal{D}(\Delta_{W_i}^{>1})\) and \(\mathcal{D}(\Delta_{W_i}^{>1})\) are simple-homotopy equivalent. By the weak factorization theorem \([\text{AKMW}02]\), the pairs \((W_1, \Delta_{W_1})\) and \((W_2, \Delta_{W_2})\) can be connected by a sequence of blow ups as in Lemma \(2.12\). Hence \(\mathcal{D}(\Delta_{W_1}^{>1})\) and \(\mathcal{D}(\Delta_{W_2}^{>1})\) are simple-homotopy equivalent. □

**Remark 2.14.**

1. In the proof of this proposition, the weak factorization theorem \([\text{AKMW}02]\) is used. So, the proof does not work in positive characteristic even in dimension three.
2. If \((X, \Delta)\) is log canonical in this proposition, then \(\mathcal{D}(\Delta_{Y_1}^{>1})\) and \(\mathcal{D}(\Delta_{Y_2}^{>1})\) are PL homeomorphic each other \((\text{dFKX}17, \text{Proposition 11})\).

2.4. **Results on the Witt vector cohomologies.** For the definition of the Witt vector cohomology and its basic properties, we refer to \([\text{GNT}]\) and \([\text{CR}12]\). The following vanishing theorem of Nadel type will be used in this paper.
Theorem 2.15 ([NT] Theorem 4.10). Let \((X, \Delta)\) be a projective log pair over a perfect field \(k\) with condition (ii) in \((\ast)\). Then

\[ H^i(X, W_{\text{Nklt}(X,\Delta)} Q) = 0 \]

holds for \(i > 0\), where \(\text{Nklt}(X,\Delta)\) denotes the reduced closed subscheme of \(X\) consisting of the non-klt points of \((X, \Delta)\) and \(W_{\text{Nklt}(X,\Delta)}\) is the coherent ideal sheaf on \(X\) corresponding to \(\text{Nklt}(X,\Delta)\).

3. Dual complex of weak Fano varieties

3.1. Dual complex of dlt pairs with a Mori fiber space structure.

Lemma 3.1. Let \((X, \Delta)\) be a \(\mathbb{Q}\)-factorial dlt pair over \(k\) with condition \((\ast)\) and let \(f : X \to Z\) be a projective surjective \(k\)-morphism to a quasi-projective \(k\)-scheme \(Z\) such that

1. \(\dim X > \dim Z\),
2. \(f\) has connected fibers,
3. \(-(K_X + \Delta)\) is \(f\)-ample, and
4. there exists an irreducible component \(D_0\) of \(\Delta = 1\) such that \(D_0\) is \(f\)-ample.

Then \(\mathcal{D}(\Delta = 1)\) is contractible.

Proof. Let \(g : D_0 \to Z\) be the induced morphism. Let \(\Delta = 1 = \sum_{i=0}^m D_i\) be the irreducible decomposition.

What we want to show is the following:

1. Any stratum \(S\) of \(\sum_{i=0}^m D_i\) intersects with \(D_0\), and
2. \(S \cap D_0\) is connected.

These conditions imply that the dual complex of \(\sum_{i=0}^m D_i\) is the cone of the dual complex of \(\sum_{i=1}^m D_i\) with the vertex \(D_0\). Therefore \(\mathcal{D}(\Delta = 1)\) is contractible. We show (1) and (2) by induction on \(\dim X\).

First we prove the following claim.

Claim 3.2. For any \(i \in \{1, \ldots, m\}\), it holds that

- (a) \(\dim f(D_i) < \dim D_i\), and
- (b) \(f(D_i) = f(D_i \cap D_0)\) holds, in particular \(D_i \cap D_0 \neq \emptyset\).
- (c) \(D_i \cap D_0\) is connected and irreducible.

Proof. We prove (a). Since the assertion is clear if \(\dim Z \leq \dim X - 2\), we may assume that \(\dim Z = \dim X - 1\).

Suppose that \(f(D_i) = Z\) holds for some \(i \in \{1, \ldots, m\}\). For a general fiber \(F\) of \(f\), \(\dim F = 1\) holds and \((D_i \cup D_0) \cap F\) is not connected. This contradicts the Kollár-Shokurov connectedness lemma (cf. [NT] Theorem 1.2]). Therefore, \(D_i\) does not dominate \(Z\) for any \(i \in \{1, \ldots, m\}\).

We prove (b). Let \(x \in f(D_i)\) be a closed point. Since \(\dim f(D_i) < \dim D_i\), there exists a curve \(C\) on \(X\) contained in \(D_i \cap f^{-1}(x)\). Since \(D_0\) is ample over \(Z\), the contracted curve \(C\) intersects with \(D_0\). This implies \(x \in f(D_i \cap D_0)\).

Thus we get \(f(D_i) = f(D_i \cap D_0)\).

We prove (c). Suppose that \(D_i \cap D_0\) is not connected. Let \(S\) be a connected component of \(D_i \cap D_0\) which satisfies \(f(S) = f(D_i \cap D_0)\). Let \(G\) be another connected component of \(D_i \cap D_0\). Then for any closed point \(x \in f(G)\),...
\( D_1 \cap D_0 \) is not connected over \( x \). However, this contradicts the Kollár-Shokurov connectedness lemma. Therefore \( D_1 \cap D_0 \) is connected. Then the irreducibility follows from Proposition \([2,3]\) (1).

Let \( S \) be a stratum of \( \sum_{i=1}^{m} D_i \). Then \( S \) is a connected component of \( \bigcap_{i \in I} D_i \) for some \( I \subset \{1, \ldots, m\} \). We may assume that \( 1 \in I \) possibly changing the indices.

Let \( C := f(D_1), \) and let \( D_1 \xrightarrow{f'} C' \xrightarrow{\sim} C \) be the Stain factorisation of \( D_1 \to C \). Let \( \Delta_{D_1} \) be the effective \( \mathbb{R} \)-divisor on \( D_1 \) defined by adjunction \( (K_X + \Delta)|_{D_1} = K_{D_1} + \Delta_{D_1} \). Then the following properties hold.

(d) \( (D_1, \Delta_{D_1}) \) is dlt.
(e) \( -(K_{D_1} + \Delta_{D_1}) \) is \( f' \)-ample.
(f) \( -D_0|_{D_1} \) is \( f' \)-ample.

We prove (1) and (2) by induction on \# \( I \). If \# \( I = 1 \), then (1) and (2) follow from Claim \([5,2]\). Suppose \# \( I \geq 2 \). Then \( S \) is also a stratum of \( \Delta_{D_1} \). Hence (1) and (2) holds by induction on the dimension. \( \square \)

**Lemma 3.3.** Let \((X, \Delta)\) be a \( \mathbb{Q} \)-factorial dlt pair over \( k \) with the condition (*) and let \( f : X \to Z \) be a \((K_X + \Delta)\)-Mori fiber space to a quasi-projective \( k \)-variety \( Z \). Suppose that \( f(\text{Supp } \Delta = 1) = Z \). Then \( \mathcal{D}(\Delta = 1) \) is contractible.

**Proof.** Since \( f(\text{Supp } \Delta = 1) = Z \), some irreducible component \( D_0 \) of \( \Delta = 1 \) satisfies \( f(D_0) = Z \). Since \( \rho(X/Z) = 1 \), it follows that \( D_0 \) is \( f \)-ample. Hence the assertion follows from Lemma \([3,1]\). \( \square \)

### 3.2. Dual complex of weak Fano varieties.

**Theorem 3.4.** Let \((X, \Omega)\) be a \( \mathbb{Q} \)-factorial projective log pair. Assume that

1. \((X, \Omega^{\Lambda^1})\) is dlt but not klt.
2. \( K_X + \Omega \sim_{\mathbb{R}} 0 \).
3. \( \text{Supp } \Omega^{>1} = \text{Supp } \Omega^{\geq 1} \).

Then, the dual complex \( \mathcal{D}(\Omega^{\geq 1}) \) is contractible.

**Proof.** Since \((X, \Omega^{\Lambda^1})\) is not klt, it follows that

\[
\text{Supp } \Omega^{>1} = \text{Supp } \Omega^{\geq 1} \neq \emptyset.
\]

Therefore, \( K_X + \Omega^{\Lambda^1} \sim_{\mathbb{R}} -(\Omega - \Omega^{\Lambda^1}) \) is not pseudo-effective. Further, by (3), \((X, \Omega^{\Lambda^1} - \epsilon(\Omega - \Omega^{\Lambda^1}))\) is klt for some small \( \epsilon > 0 \). Hence we may run a \((K_X + \Omega^{\Lambda^1})\)-MMP and ends with a Mori fiber space \( f : X_\ell \to Z \):

\[
X = X_0 \to X_1 \to \cdots \to X_\ell.
\]

Let \( \Omega_i \) be the push-forward of \( \Omega \) on \( X_i \). Then \( \Omega_i \) also satisfies the conditions (1)–(3). Further, since \( \Omega_\ell - \Omega_\ell^{\Lambda^1} \sim_{\mathbb{R}} -(K_{X_\ell} + \Omega_\ell^{\Lambda^1}) \) is ample over \( Z \), it follows that \( f(\text{Supp } \Omega_\ell^{>1}) = Z \). Hence, by Lemma \([3,3]\) the dual complex \( \mathcal{D}(\Omega_\ell^{>1}) \) is contractible.

Let \( R_i \) be the extremal ray of \( \overline{\text{NE}}(X_i) \) corresponding to the step of the MMP \( X_i \to X_{i+1} \). Since \( -(\Omega_i - \Omega_i^{\Lambda^1}) \cdot R_i < 0 \), it follows that some component \( D_i \) of \( \text{Supp}(\Omega_i^{>1}) = \text{Supp}(\Omega_i^{\Lambda^1}) \) satisfies \( D_i \cdot R_i > 0 \). By Theorem \([2,10]\) \( \mathcal{D}(\Omega_i^{>1}) \) and \( \mathcal{D}(\Omega_{i+1}^{>1}) \) are homotopy equivalent. Hence, \( \mathcal{D}(\Omega^{\geq 1}) = \mathcal{D}(\Omega^{\Lambda^1}) \) is contractible. \( \square \)
Proposition 3.5. Let $(X, \Delta)$ be a $\mathbb{Q}$-factorial projective pair over $k$ with the condition $(\ast)$. Suppose that $- (K_X + \Delta)$ is ample. Then for any dlt blow-up $g : (Y, \Delta_Y) \to (X, \Delta)$, the dual complex $\mathcal{D}(\Delta_Y^{\geq 1})$ is contractible, where we define $\Delta_Y$ by $K_Y + \Delta_Y = g^*(K_X + \Delta)$.

Proof. By Proposition 3.4 it suffices to find an effective $\mathbb{R}$-divisor $\Omega_Y$ on $Y$ such that

1. $(Y, \Omega_Y^{\geq 1})$ is dlt,
2. $K_Y + \Omega_Y \sim_{\mathbb{R}} 0$,
3. $\text{Supp}(\Omega_Y^{\geq 1}) = \text{Supp}(\Omega_Y^{\geq 1})$, and
4. $\text{Supp}(\Omega_Y^{\geq 1}) = \text{Supp}(\Delta_Y^{\geq 1})$.

Since $X$ is $\mathbb{Q}$-factorial, there exists an effective $\mathbb{R}$-divisor $F$ on $Y$ such that $-F$ is $g$-ample and $\text{Supp} F = \text{Excep}(g)$. Since $-(K_Y + \Delta_Y)$ is the pullback of an ample $\mathbb{R}$-divisor $-(K_X + \Delta)$ on $X$, it follows that $-(K_Y + \Delta_Y) - \epsilon F$ is ample for any sufficiently small $\epsilon > 0$.

Note that $\text{Supp} F \subset \text{Supp}(\Delta_Y^{\geq 1})$. Thus, we can find an effective $\mathbb{R}$-divisor $B$ on $Y$ such that

- $B \geq \epsilon F$,
- $-(K_Y + \Delta_Y) - B$ is still ample, and
- $\text{Supp} B = \text{Supp}(\Delta_Y^{\geq 1})$.

Then there exists an effective $\mathbb{R}$-divisor $A$ on $Y$ such that

- $A \sim_{\mathbb{R}} -(K_Y + \Delta_Y) - B$, and
- $(Y, \Delta_Y^{\geq 1} + 2A)$ is dlt (cf. [NT, Lemma 2.8] in positive characteristic case).

In particular, it follows that

- $(Y, \Delta_Y^{\geq 1} + A)$ is dlt,
- $(\Delta_Y + A)^{\geq 1} = \Delta_Y^{\geq 1}$.

Set $\Omega_Y := \Delta_Y + A + B$. Then (2) holds. Since

- $\Omega_Y^{\geq 1} = (\Delta_Y + A)^{\geq 1} = \Delta_Y^{\geq 1} + A$,

(1) also holds. (3) and (4) hold by the way of taking $A$ and $B$. \hfill \Box

Lemma 3.6. Let $(X, \Delta)$ be a $\mathbb{Q}$-factorial projective pair over $k$ with the condition $(\ast)$. Assume that $-(K_X + \Delta)$ is nef and big. Then there exists a dlt blow-up $g : (Y, \Delta_Y) \to (X, \Delta)$ such that the dual complex $\mathcal{D}(\Delta_Y^{\geq 1})$ is contractible, where we define $\Delta_Y$ by $K_Y + \Delta_Y = g^*(K_X + \Delta)$.

Proof. Since $-(K_X + \Delta)$ is nef and big, there exists an effective $\mathbb{R}$-divisor $E$ such that $-(K_X + \Delta) - \epsilon E$ is ample for any real number $\epsilon$ satisfying $0 < \epsilon \leq 1$. Let $h : W \to X$ be a log resolution of $(X, \Delta + E)$. For sufficiently small $\epsilon > 0$, we can assume that

1. For any $h$-exceptional prime divisor $F$ with $a_F(X, \Delta) > 0$, it still holds that $a_F(X, \Delta + \epsilon E) > 0$.

Let $g : Y \to X$ be a dlt blow-up of $(X, \Delta + \epsilon E)$ such that the birational map $Y \dashrightarrow W$ does not contract any divisor on $Y$ (Remark 2.6 (2)). By Proposition 3.5 the dual complex $\mathcal{D}(\Delta_Y + \epsilon g^*E)^{\geq 1}$ is contractible. Hence it is sufficient to check the following three conditions (the conditions (2) and (3) below mean that $g$ is also a dlt blow-up of $(X, \Delta)$):

1. $(Y, \Omega_Y^{\geq 1})$ is dlt,
2. $K_Y + \Omega_Y \sim_{\mathbb{R}} 0$,
3. $\text{Supp}(\Omega_Y^{\geq 1}) = \text{Supp}(\Omega_Y^{\geq 1})$, and
4. $\text{Supp}(\Omega_Y^{\geq 1}) = \text{Supp}(\Delta_Y^{\geq 1})$.
(2) \( a_F(X, \Delta) \leq 0 \) for any \( g \)-exceptional divisor \( F \).
(3) \( (Y, \Delta_Y^{\Delta}) \) is dlt.
(4) \( \text{Supp}( (\Delta_Y + \epsilon g^* E)^{\geq 1}) = \text{Supp}(\Delta_Y^{\geq 1}) \).

Since any \( g \)-exceptional divisor is \( h \)-exceptional, the conditions (2) and (4) follow from (1). By (2), it follows that \( \Delta_Y \geq 0 \) and hence (3) is obvious. \( \square \)

**Theorem 3.7.** Let \((X, \Delta)\) be a projective pair over \( k \) with the condition \((\ast)\). Assume that \(-(K_X + \Delta)\) is nef and big. Then there exists a dlt blow-up \( g : (Y, \Delta_Y) \to (X, \Delta) \) such that the dual complex \( D(\Delta_Y^{\geq 1}) \) is contractible, where we define \( \Delta_Y \) by \( K_Y + \Delta_Y = g^*(K_X + \Delta) \). Further, we can take such \( g \) with the additional condition (3) in Theorem 2.5.

**Proof.** Let \( h : (X', \Delta_{X'}) \to (X, \Delta) \) be a dlt blow-up of \((X, \Delta)\) with the condition (3) in Theorem 2.5. Then, \( X' \) is \( \mathbb{Q} \)-factorial and \(-(K_{X'} + \Delta_{X'})\) is still nef and big.

By Lemma 3.6 there exists a dlt blow-up \((X'', \Delta_{X''}) \to (X', \Delta_{X'})\) such that the dual complex \( D(\Delta_{X''}^{\geq 1}) \) is contractible. Since the composition \( X'' \to X \) is again a dlt blow-up of \((X, \Delta)\) with condition (3) in Theorem 2.5 we complete the proof. \( \square \)

In characteristic zero, we can conclude the following stronger statement.

**Theorem 3.8.** Let \((X, \Delta)\) be a projective pair over \( k \) with the condition (i) in \((\ast)\). Assume that \(-(K_X + \Delta)\) is nef and big. Then for any dlt blow-up \( g : (Y, \Delta_Y) \to (X, \Delta) \), the dual complex \( D(\Delta_Y^{\geq 1}) \) is contractible, where we define \( \Delta_Y \) by \( K_Y + \Delta_Y = g^*(K_X + \Delta) \).

**Proof.** The assertion follows from Theorem 3.7 and Proposition 2.13. \( \square \)

4. Vanishing theorem on log canonical Fano varieties

In this section, we prove a vanishing theorem of Witt vector cohomology of Fano varieties of Ambro-Fujino type (Theorem 4.13).

4.1. Non-klt locus of log Fano three-folds. In this subsection, we prove Lemma 4.1 Proposition 4.5 and Proposition 4.12 which will be used in the proof of Theorem 4.13.

**Lemma 4.1.** Let \( k \) be an algebraic closed field of characteristic \( p > 5 \). Let \((X, \Delta)\) be a three-dimensional projective log canonical pair over \( k \) with \(-(K_X + \Delta)\) ample. Suppose that \( \text{Nklt}(X, \Delta) \) is pure dimension one. Then each irreducible component of \( \text{Nklt}(X, \Delta) \) is a rational curve.

**Proof.** Let \( C_0 \) be an irreducible component of \( \text{Nklt}(X, \Delta) \). Let \( f : Y \to X \) be a dlt blow-up of \((X, \Delta)\) (Theorem 2.5). We define \( \Delta_Y \) by \( K_Y + \Delta_Y = f^*(K_X + \Delta) \). Let \( E \subset \text{Supp} \Delta_Y^{\geq 1} \) be a component which dominates \( C_0 \). We define \( \Delta_E \) by \( K_E + \Delta_E = (K_Y + \Delta_Y)|_E \). Then \( K_E + \Delta_E \) is not nef since \(-(K_X + \Delta)\) is ample. By the cone theorem for surfaces (cf. [Tan14 Proposition 3.15]), there exists a \((K_E + \Delta_E)\)-negative rational curve \( B \). Since \((K_E + \Delta_E) \cdot B < 0 \) and \(-(K_E + \Delta_E)\) is the pulled back of a divisor on \( C_0 \), it follows that \( B \) dominates \( C_0 \), which proves the rationality of \( C_0 \). \( \square \)
We prove two properties (Proposition 4.3, Proposition 4.4) on the dual complex $\mathcal{D}(\Delta_Y^{\geq 1})$ of a dlt blow-up. They will be used in the proof of Proposition 4.5.

First, we introduce some notations. When we write that $G$ is a $\Delta$-complex, we regard $G$ as the set of its simplices. Hence, when we write $S \in G$, then $S$ is a simplices of $G$.

**Definition 4.2.**

1. For a $\Delta$-complex $G$, we denote by $|G|$ the topological space of $G$. Hence, $|G| = \bigcup_{S \in G} S$ as set.
2. For a $\Delta$-complex $G$ and its subset $G' \subset G$, we define the star $\text{st}(G', G)$ by

$$\text{st}(G', G) = \{ S \in G \mid S \cap S' \neq \emptyset \text{ for some } S' \in G' \}.$$

3. For a $\Delta$-complex $G$, and $S, S' \in G$, we write $S < S'$ when $S$ is a face of $S'$.

**Proposition 4.3.** Let $k$ be an algebraic closed field of characteristic $p > 5$. Let $(X, \Delta)$ be a three-dimensional projective log canonical pair over $k$ with $-(K_X + \Delta)$ nef and big. Suppose that $C := \text{Nklt}(X, \Delta)$ is of pure dimension one. Let $f : (Y, \Delta_Y) \to (X, \Delta)$ be a dlt blow-up such that $\text{Supp} \Delta_Y^{\geq 1} = f^{-1}(C)$ and that $G := \mathcal{D}(\Delta_Y^{\geq 1})$ is contractible. Let $C_0$ be an irreducible component of $C = \text{Nklt}(X, \Delta)$.

Let $G'$ be the subcomplex of $G$ which consists of the strata of $\Delta_Y^{\geq 1}$ which dominate $C_0$. Let $U := |G| \setminus |G'|$ be the open subset of $|G|$, and let $V \subset |G|$ be a sufficiently small open neighborhood of $|G'|$. Then following hold.

1. $G'$ is a connected subcomplex of $G$ of dimension at most one.
2. For each connected component $U'$ of $U$, it follows that $V \cap U'$ is also connected.

**Proof.** We prove (1). Only the connectedness of $G'$ is non-trivial. Let $q \in C_0$ be a general closed point. Since $q$ is general, we may assume that

- for a strata $S$ of $\Delta_Y^{\geq 1}$, $q \in f(S)$ holds if and only if $f(S) = C_0$ holds.

Then the connectedness of the fiber $f^{-1}(q)$ shows that the connectedness of $G'$.

For (2), consider the Mayer–Vietoris sequence

$$H_1(|G|, \mathbb{Q}) \to H_0(U \cap V, \mathbb{Q}) \to H_0(U, \mathbb{Q}) \oplus H_0(V, \mathbb{Q}) \to H_0(|G|, \mathbb{Q}) \to 0.$$  

Here, $H_1(|G|, \mathbb{Q}) = 0$ follows because $|G|$ is contractible, $H_0(|G|, \mathbb{Q}) = H_0(V, \mathbb{Q}) = 0$ follows because $|G|$ and $V$ are connected. Therefore, it follows that $H_0(U \cap V, \mathbb{Q}) \cong H_0(U, \mathbb{Q})$, which proves the claim. $\square$

**Proposition 4.4.** Let $(X, \Delta), (Y, \Delta_Y), C, C_0, G, G', U, V$ be as in the Proposition 4.3. Let $U'$ be a connected component of $U$. Consider a pair $(B, S)$ with the following conditions:

1. $B \in \text{st}(G', G) \setminus G'$ is an edge, and $S \in G'$ is a vertex.
2. $S < B$ and $B \cap U' \neq \emptyset$.

For any two pairs $(B, S)$ and $(B', S')$ with the conditions (a) and (b) above, the assertion is that there exists a sequence of pairs

$$(B, S) = (B_0, S_0), (B_1, S_1), \ldots, (B_k, S_k) = (B', S')$$
with \( k \geq 0 \) and the following conditions:

(c) Each \((B_j, S_j)\) satisfies the conditions (a) and (b).

(d) For each \( 0 \leq i \leq k - 1 \), the pairs \((B_i, S_i), (B_{i+1}, S_{i+1})\) satisfy one of the following conditions:

(d-1) \( S_i = S_{i+1} \) holds, and \( B_i, B_{i+1} < F \) for some 2-simplex \( F \in G \).

(d-2) \( S_i \neq S_{i+1} \) holds, and \( S_i, S_{i+1} < E \) for some edge \( E \in G' \). Further, \( B_i, B_{i+1}, E < F \) for some 2-simplex \( F \in G \).

Proof. Note that \( U' \cap V \) is a connected component of \( U \cap V \) by Proposition 4.3 and that \( U \cap V \subset \bigcup_{A \in \text{st}(G', G) \setminus G'} \text{int} \ A \).

We also note that giving a pair \((B, S)\) with conditions (a) and (b) is equivalent to giving \( B'\) in the following set.

\[
\Gamma := \{ B' \mid B' \text{ is a connected component of } B \cap (U' \cap V) \text{ for some edge } B \in \text{st}(G, G') \setminus G' \}
\]

Indeed, for a pair \((B, S)\) with conditions (a) and (b), there exists the unique connected component of \( B \cap (U' \cap V) \) which is around \( S \). Inversely, for \( B' \) in the set above, the corresponding \( B \) and \( S \) are uniquely determined.

Let \( B' \) (resp. \( B'\)) be the connected component of \( B \cap (U' \cap V) \) (resp. \( B' \cap (U' \cap V) \)) which is around \( S \) (resp. \( S' \)).

Since \( B', B'\subset U' \cap V \) and \( U' \cap V \) is connected, we can take the following sequence

\[
B(0) := B', F'(0), B(1), F'(1), \ldots, B^{(k-1)}, F^{(k-1)}, B(k) := B'^\circ
\]

with the following conditions:

- Each \( B(i) \) is a connected component of \( B_i \cap (U' \cap V) \) for some edge \( B_i \in \text{st}(G', G) \setminus G' \) (equivalently \( B(i) \in \Gamma \)).
- Each \( F(i) \) is a connected component of \( F_i \cap (U' \cap V) \) for some 2-simplex \( F_i \in \text{st}(G', G) \setminus G' \).
- \( B(i), B(i+1) \subset F(i) \)

Possibly passing to a subsequence, we may also assume that

- \( B(i) \neq B(i+1) \) for each \( 0 \leq i \leq k - 1 \).

We denote by \( S_i \) the unique vertex of \( B_i \) which is around \( B(i) \). Obviously, \( S_i \) is a vertex of \( G' \).

For each \( i \), there are two possibilities.

(e-1) The connected component of \( F_i \cap G' \) which is around \( F(i) \) is zero dimensional.

(e-2) The connected component of \( F_i \cap G' \) which is around \( F(i) \) is one dimensional.
Suppose $i$ satisfies (e-1). Then $S_i = S_{i+1}$ and $S_i$ is the common vertex of $B_i$ and $B_{i+1}$. Therefore $(B_i, S_i)$ and $(B_{i+1}, S_{i+1})$ satisfy (d-1).

Suppose $i$ satisfies (e-2). Then there exists an edge $E_i \in G'$ such that $E_i, B_i, B_{i+1} < F_i$. Then $S_i$ (resp. $S_{i+1}$) is the common vertex of $E_i$ and $B_i$ (resp. $E_i$ and $B_{i+1}$). Then $(B_i, S_i)$ and $(B_{i+1}, S_{i+1})$ satisfy (d-2). □

**Proposition 4.5.** Let $k$ be an algebraic closed field of characteristic $p > 5$. Let $(X, \Delta)$ be a three-dimensional projective log pair over $k$ with $-(K_X + \Delta)$ nef and big. Suppose that $\text{Nklt}(X, \Delta)$ is of pure dimension one. Then for each irreducible component $C_0$ of $\text{Nklt}(X, \Delta)$, its normalization $\overline{C_0} \to C_0$ is a universal homeomorphism.

**Proof.** Set $C := \text{Nklt}(X, \Delta)$. By contradiction, suppose that $p \in C_0$ is a singular point such that the normalization $\overline{C_0} \to C_0$ is not a universal homeomorphism around $p$. Let $p^{(1)}, \ldots, p^{(m)}$ be the inverse image of $p$. By the assumption, $m \geq 2$.

Let $f : (Y, \Delta_Y) \to (X, \Delta)$ be a dlt blow-up such that $\text{Supp} \Delta_Y^{\geq 1} = f^{-1}(C)$ (Theorem 3.7). Let $G = D(\Delta_Y^{\geq 1})$. We may assume that $G$ is contractible by Theorem 3.7.

We define $\{S_i\}_{i \in I}$ and $\{T_j\}_{j \in J}$ as follows:

- Let $\{S_i\}_{i \in I}$ be the set of the irreducible components of $\Delta_Y^{\geq 1}$ which dominate $C_0$.
- Let $\{T_j\}_{j \in J}$ be the set of the irreducible components $T_j$ of $\Delta_Y^{\geq 1}$ which do not dominate $C_0$ but $p \in f(T_j)$.

For each $S \in \{S_i\}_{i \in I}$, since $S$ is normal and dominates $C_0$, $S \to C_0$ factors through $S \to \overline{C_0} \to C_0$. We denote by $S_{p^{(k)}}$ the fiber $S \to \overline{C_0}$ over $p^{(k)}$. Obviously, we have

1. $S_{p^{(k)}} \cap S_{p^{(j)}} = \emptyset$ holds for each $k \neq j$.

On the other hand, $f^{-1}(p)$ is connected. Since $f^{-1}(p) \cap T \neq \emptyset$ for each $T \in \{T_j\}_{j \in J}$,

$$\left( \bigcup_{i \in I, k} S_{i, p^{(k)}} \right) \cup \left( \bigcup_{j \in J} T_j \right) = f^{-1}(p) \cup \bigcup_{j \in J} T_j$$

is also connected. Hence, by (1), possibly changing the indices of $p^{(1)}, \ldots, p^{(m)}$, we can conclude the following:

2. There exist $S, S' \in \{S_i\}_{i \in I}$ and a sequence $T_1, \ldots, T_\ell \in \{T_j\}_{j \in J}$ with $\ell \geq 0$ such that

$$S_{p^{(1)}} \cap T_1 \neq \emptyset, \ T_1 \cap T_2 \neq \emptyset, \ldots, \ T_{\ell-1} \cap T_\ell \neq \emptyset, \ T_\ell \cap S'_{p^{(2)}} \neq \emptyset.$$
We define $G'$ be the subcomplex of $G$ which consists of the strata of $\Delta_{p}^{1}$ which dominate $C_{0}$. Let $U := |G| \setminus |G'|$ be the open subset of $|G|$, and let $V$ be a sufficiently small open neighborhood of $G'$.

**STEP 1.** In this step, we prove the following statement from the condition (2).

(3) There exist a connected component $U'$ of $U$, and pairs $(B, S)$ and $(B', S')$ with the condition (a) and (b) in Proposition 4.3 such that $S_{p(1)}' \cap B \neq \emptyset$ and $S_{p(2)}' \cap B' \neq \emptyset$ hold. Here we allow $B = B'$ and $S = S'$.

Suppose $\ell = 0$ in (2), that is, $S_{p(1)} \cap S_{p(2)}' \neq \emptyset$. Then, we can take a one dimensional stratum $B \subset S \cap S'$ such that $S_{p(1)} \cap S_{p(2)}' \cap B \neq \emptyset$ holds. If $B$ dominates $C_{0}$, then $B \rightarrow C_{0}$ also factors through $C_{0}$ since $B$ is normal. We define $B_{p(1)}$ and $B_{p(2)}$ as the fibers of $B \rightarrow C_{0}$ over $p(1)$ and $p(2)$. Then $S_{p(1)} \cap B = B_{p(1)}$ and $S_{p(2)}' \cap B = B_{p(2)}$ hold, and they contradict the fact that $B_{p(1)} \cap B_{p(2)} = \emptyset$. Therefore, $B$ does not dominate $C_{0}$, that is, $B \notin G'$.

Hence the condition (3) holds when we set $B' = B$.

Suppose $\ell \geq 1$ in (2). Then we can take a one dimensional stratum $B \subset S \cap T_{1}'$ (resp. $B' \subset T_{1}' \cap S'$) such that $S_{p(1)} \cap B \neq \emptyset$ (resp. $S_{p(2)}' \cap B' \neq \emptyset$). Since $T_{1}$ and $T_{\ell}$ do not dominate $C_{0}$, neither do $B$ and $B'$, hence $B, B' \notin G'$. Moreover, $T_{1} \cup \cdots \cup T_{\ell}$ is connected, $B$ and $B'$ have intersection with a same connected component $U'$ of $U$.

We have proved (3). Here, we note that the condition $S_{p(1)} \cap B \neq \emptyset$ (resp. $S_{p(2)}' \cap B' \neq \emptyset$) in (3) actually implies that $B \subset S_{p(1)}$ (resp. $B' \subset S_{p(2)}'$). Indeed, it follows from the facts that $B \subset S$ (resp. $B' \subset S'$) and that $S$ (resp. $S'$) dominates $C_{0}$, but $B$ (resp. $B'$) does not dominate $C_{0}$. Therefore we conclude the following.

(4) There exist a connected component $U'$ of $U$, and pairs $(B, S)$ and $(B', S')$ with the condition (a) and (b) in Proposition 4.3 such that $B \subset S_{p(1)}$ and $B' \subset S_{p(2)}'$ hold. Here we allow $B = B'$ and $S = S'$.

**STEP 2.** Let $(B, S), (B', S')$ be pairs which satisfy (a) and (b) in Proposition 4.3. Suppose that one of the following conditions holds:

(d-1) $S = S'$ holds, and $B, B' < F$ for some 2-simplex $F \in G$.
(d-2) $S \neq S'$ holds, and $S, S' < E$ for some edge $E \in G'$. Further, $B, B', E < F$ for some 2-simplex $F \in G$.

In this step, we prove the condition $B \subset S_{p(1)}$ implies $B' \subset S_{p(1)}'$.
Suppose (d-1). \( S = S' \) in this case. Since \( B, B' \subset F \) for some 2-simplex \( F \subset G \), it follows that \( B \cap B' \neq \emptyset \). Since \( B \subset S_{p(1)} \), it holds that \( B' \cap S_{p(1)} \neq \emptyset \). Then \( B' \subset S_{p(1)} \) holds because of the facts that \( B' \subset S \) and that \( S \) dominates \( C_0 \), but \( B' \) does not dominate \( C_0 \).

Suppose (d-2). Since \( B, B', E \subset F \) for some 2-simplex \( F \subset G \), \( B \cap B' \cap E \neq \emptyset \) holds. Since \( B \subset S_{p(1)} \), it follows that

\[
S_{p(1)} \cap B' \cap E \neq \emptyset.
\]

Here we note that \( E \) dominates \( C_0 \) and hence \( E \to C_0 \) factors through \( C_0 \) because \( E \) is normal. We denote by \( E_{p(1)} \) the fiber of \( E \to C_0 \) over \( p(1) \). Since \( E \subset S, S' \), it holds that

\[
S_{p(1)} \cap E = E_{p(1)} \subset S'_{p(1)}.
\]

Hence we obtain that \( B' \cap S'_{p(1)} \neq \emptyset \). It implies that \( B' \subset S'_{p(1)} \) by the same reason as before.

**STEP 3.** In this step, we assume the condition (4) in STEP 1, and lead a contradiction.

Let \( (B, S), (B', S') \) be the pairs in (4) in STEP 1. Then \( (B, S), (B', S') \) satisfy the condition (a), (b) in Proposition 4.3. Hence by Proposition 4.3, we can take a sequence \( (B, S) = (B_0, S_0), (B_1, S_1), \ldots, (B_k, S_k) = (B', S') \) as in Proposition 4.3.

By STEP 2 and the assumption that \( B \subset S_{p(1)} \), it follows that \( B_k \subset S_{k,p(1)} \) for each \( k \) by induction. Therefore \( B' \subset S'_{p(1)} \) holds and it contradicts the assumption that \( B' \subset S'_{p(2)} \) and the fact that \( S'_{p(1)} \cap S'_{p(2)} = \emptyset \).

In order to state Proposition 4.4, we introduce a notation.

**Definition 4.6.** Let \( C \) be a scheme of finite type of pure dimension one over an algebraic closed field \( k \). Let \( C = C_1 \cup C_2 \cup \cdots \cup C_\ell \) be the irreducible decomposition. We define whether \( C \) forms a tree or not by induction on the number \( \ell \) of the irreducible components.

Any irreducible curve \( C \) forms a tree. We call that the union of irreducible curves \( C = C_1 \cup C_2 \cup \cdots \cup C_\ell \) forms a tree, if there exists \( i \) such that \( C' = \bigcup_{j \neq i} C_j \) forms a tree and \( \#(C_i \cap C_j) = 1 \).

First, we prepare some notation in combinatorics.

**Definition 4.7.** For a sequence \( (a_1, \ldots, a_n) \) of length \( n \), we define the operations (a), (b) as follows. Here we define \( a_{n+1} := a_1 \) and \( a_{n+2} := a_2 \) by convention.

(a) If \( a_i = a_{i+1} \) for some \( 1 \leq i \leq n \), we remove \( a_i \) and get a new sequence \( (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \) of length \( n - 1 \).

(b) If \( a_i = a_{i+2} \) for some \( 1 \leq i \leq n \), then we remove \( a_i \) and \( a_{i+1} \), and get a new sequence \( (a_1, \ldots, a_{i-1}, a_i = a_{i+2}, a_{i+3}, \ldots, a_n) \) of length \( n - 2 \).

For a sequence \( (a_1, \ldots, a_n) \) of length \( n \), applying the operation (a) (resp. the operations (a) and (b)) repeatedly, we get a sequence \( (b_1, \ldots, b_m) \) with the condition that \( b_i \neq b_{i+1} \) for each \( i \) (resp. the condition that \( b_i \neq b_{i+1} \) and \( b_i \neq b_{i+2} \)). We call such \( (b_1, \ldots, b_m) \) the \((a)\)-reduction (resp. \((a,b)\)-reduction) of \((a_1, \ldots, a_n)\).
We say that a sequence \((a_1, \ldots, a_n)\) has the trivial \((a)\)-reduction (resp. trivial \((a,b)\)-reduction) if the \((a)\)-reduction (resp. the \((a,b)\)-reduction) has length 0. We note here that the \((a)\)-reduction of a sequence of length 1 has length 0 by definition.

**Definition 4.8.** Let \(C = C_1 \cup C_2 \cup \cdots \cup C_\ell\) be in Definition 4.6. We call that a sequence \((a_1, \ldots, a_n)\) is a cycle sequence if the following conditions (1), (2) hold.

1. Each \(a_i\) is an irreducible component of \(C\) or a closed point on \(C\).
2. The \((a)\)-reduction \((b_1, \ldots, b_m)\) of \((a_1, \ldots, a_n)\) satisfies the following conditions:
   1. \(\dim b_i \neq \dim b_{i+1}\) for each \(1 \leq i \leq m\). We set \(b_{m+1} = b_1\) by convention.
   2. If \(\dim b_i = 0\), then \(b_i \in b_{i-1} \cap b_{i+1}\). We set \(b_0 = b_m\) by convention.

Moreover, we say that a cycle sequence \((a_1, \ldots, a_n)\) is trivial if it has the trivial \((a,b)\)-reduction.

**Example 4.9.** Consider curves \(C = C_1 \cup \cdots \cup C_5\) in the following figure.

Set sequences \(Q_1, Q_2\) as follows:
\[
Q_1 = (P_1, C_1, P_2, C_2, P_3, C_3, P_4, C_4, P_5, C_5, P_3, C_2, P_2, C_1),
\]
\[
Q_2 = (P_1, C_1, P_2, C_2, P_3, C_3, P_4, C_4, P_5, C_5, P_3, C_4, P_3, C_2, P_2, C_1).
\]
Their \((a)\)-reductions are themselves. These two sequences satisfy the condition (2-1), (2-2), hence they are cycle sequences. \(Q_1\) is not trivial, but \(Q_2\) is trivial. Indeed the \((a,b)\)-reduction of \(Q_1\) is \((C_3, P_3, C_4, P_4, C_5, P_3)\).

A typical example of cycle sequences we will see in the proof of Proposition 4.12 is as follows.

**Example 4.10.** Let \(k\) be an algebraic closed field of characteristic \(p > 5\). Let \((X, \Delta)\) be a three-dimensional projective log pair over \(k\) with \(-\left(K_X + \Delta\right)\) nef and big. Suppose that \(\text{Nklt}(X, \Delta)\) is pure dimension one. Let \(f : (Y, \Delta_Y) \to (X, \Delta)\) be a dlt blow-up, and let \(G := D(\Delta_Y^{\geq 1})\) be the dual complex of \(\Delta_Y^{\geq 1}\). Then \(G\) is a regular \(\Delta\)-complex by Proposition 2.4.

For an edge \(C\) in \(G\), its two vertices \(S\) and \(S'\) are distinct because \(G\) is regular. We denote by \(C(S, S')\) the oriented 1-cell corresponding to \(C\) with initial point \(S\) and final point \(S'\).

When we write
\[
P : S_1 \xrightarrow{C_1} S_2 \xrightarrow{C_2} \cdots \xrightarrow{C_{n-1}} S_n \xrightarrow{C_n} S_{n+1},
\]
we assume the following condition.
• For each $1 \leq i \leq n$, $S_i \neq S_{i+1}$ holds and $S_i$ and $S_{i+1}$ are the two vertices of $C_i$.

We denote by $P$ the edge path obtained by joining the oriented 1-cell $C_1(S_1, S_2), \ldots, C_n(S_n, S_{n+1})$. The edge path $P$ is called an edge loop when $S_1 = S_{n+1}$.

Let $P$ as above be an edge loop in $G$. Then $(f(S_1), f(C_1), \ldots, f(S_n), f(C_n))$ is a cycle sequence because $f(S_{n-1}) \subset f(C_n) \subset f(S_n)$ holds. We say that the sequence $(f(S_1), f(C_1), \ldots, f(S_n), f(C_n))$ is the image of the edge loop $P$ for simplicity.

**Lemma 4.11.** Let $C = C_1 \cup C_2 \cup \cdots \cup C_\ell$ be in Definition 4.9. Suppose that $C$ is connected but does not form a tree. Then there exists a non-trivial cycle sequence $(a_1, \ldots, a_n)$ such that $a_i$’s are distinct.

**Proof.** Since $C$ does not form a tree, there exist irreducible components $B_1, \ldots, B_k$ of $C$ with the condition

$$(3) \ # \left( B_i \cap (\bigcup_{j \neq i} B_j) \right) \geq 2 \text{ for each } 1 \leq i \leq k.$$ 

We set $b_1 = B_1$ and take a point $b_2$ in $B_1 \cap (\bigcup_{j \neq 1} B_j)$. Inductively, we set $b_{2i+1}$ and $b_{2i+2}$ for $i \geq 1$ as follows:

- We take an arbitrary $B_m$ among $\{B_1, \ldots, B_k\} \setminus \{b_{2i-1}\}$ such that $b_{2i} \in b_{2i-1} \cap B_m$.
- We take an arbitrary point $p$ in $(B_m \cap (\bigcup_{j \neq m} B_j)) \setminus \{b_{2i}\}$.
- We set $b_{2i+1} = B_m$ and $b_{2i+2} = p$.

We can repeat this process by the condition (3). Since $\{B_1, \ldots, B_k\}$ is a finite set, there exist $m_1, m_2$ with $m_2 \geq m_1 + 4$ such that

- $b_{m_1} = b_{m_2}$ but $b_{m_1}, \ldots, b_{m_2-1}$ are distinct.

Then the sequence $(b_{m_1}, \ldots, b_{m_2-1})$ satisfies the condition (1), (2) in Definition 4.8. Since $b_{m_1}, \ldots, b_{m_2-1}$ are distinct, the sequence has the non-trivial $(a,b)$-reduction. \qed

**Proposition 4.12.** Let $k$ be an algebraic closed field of characteristic $p > 5$.

Let $(X, \Delta)$ be a three-dimensional projective log pair over $k$ with $-(K_X + \Delta)$ nef and big. Suppose that Nklt$(X, \Delta)$ is pure dimension one. Then Nklt$(X, \Delta)$ forms a tree.

**Proof.** Note that $C := \text{Nklt}(X, \Delta)$ is connected by [NT Theorem 1.2]. Suppose that $C = \text{Nklt}(X, \Delta)$ does not form a tree. Let $C = C_1 \cup \cdots \cup C_\ell$ be the irreducible decomposition.

**STEP 1.** Let $f : (Y, \Delta_Y) \to (X, \Delta)$ be a dlt blow-up such that $\text{Supp} \Delta_Y^{\geq 1} = f^{-1}(C)$. Let $G := \mathcal{D}(\Delta_Y^{\geq 1})$ be the dual complex. We may assume that $G$ is contractible by Theorem 3.7.

The assertion in this step is that there exists an edge loop (see Example 4.10 for the notation)

$S_1 \xleftarrow{C_1} S_2 \xleftarrow{C_2} \cdots \xleftarrow{C_n} S_n \xleftarrow{C_n} S_1$

in $G$ such that

- its image $(f(S_1), f(C_1), \ldots, f(S_n), f(C_n))$ is a non-trivial cycle sequence.
By Lemma 4.11 there exists a non-trivial cycle sequence \((a_1, \ldots, a_n)\) such that \(a_i\)'s are distinct. Since \(a_i\)'s are distinct, the (a)-reduction of this sequence is itself. Therefore, this sequence itself satisfies the conditions (2-1) and (2-2) in Definition 4.8. We may assume that \(\dim a_1 = 0\).

Suppose that \(i\) is odd.

Then, we can take an edge path \(P_1:\)

\[
P_1: S_1^{(i)} \xrightarrow{C_1^{(i)}} S_2^{(i)} \xrightarrow{C_2^{(i)}} \cdots \xrightarrow{C_{m_i-1}^{(i)}} S_{m_i}^{(i)}
\]

in \(G\) with the following conditions:

(4) \(f(S_1^{(i)}) = a_{i-1}\) and \(f(S_{m_i}^{(i)}) = a_{i+1}\).

(5) \(a_i \in f(S_j^{(i)})\) but \(f(S_j^{(i)}) \neq a_{i-1}, a_{i+1}\) for \(2 \leq j \leq m_i - 1\).

Such \(P_1\) can be taken by the connectedness of the subcomplex of \(G\) which consists of the simplices corresponding to the stratum \(S\) of \(\Delta_Y^{\geq 1}\) which satisfies \(a_i \in f(S)\).

Suppose that \(i\) is even.

Then, we can take an edge path \(P_1:\)

\[
P_1: S_1^{(i)} \xrightarrow{C_1^{(i)}} S_2^{(i)} \xrightarrow{C_2^{(i)}} \cdots \xrightarrow{C_{m_i-1}^{(i)}} S_{m_i}^{(i)}
\]

in \(G\) with the following conditions:

(6) \(S_1^{(i)} = S_{m_i-1}^{(i-1)}\) and \(S_{m_i}^{(i)} = S_1^{(i+1)}\) (here, \(S_{m_i-1}^{(i-1)}\) and \(S_1^{(i+1)}\) were already taken).

(7) \(f(S_j^{(i)}) = a_i\) for \(1 \leq j \leq m_i\) and \(f(C_j^{(i)}) = a_i\) for \(1 \leq j \leq m_i - 1\).

Such \(P_1\) can be taken by the connectedness of the subcomplex of \(G\) which consists of the simplices corresponding to the stratum \(S\) of \(\Delta_Y^{\geq 1}\) which satisfies \(a_i = f(S)\) (Proposition 4.3 (1)).

Connecting the edge paths \(P_1, P_2, \ldots, P_n\), we get an edge loop \(P\), which is possibly not simple.

We prove that the image of \(P\) is a non-trivial cycle sequence. Note that the image of any edge loop in \(G\) is a cycle sequence (Example 4.10). Therefore, it is sufficient to show that the (a,b)-reduction of the image of \(P\) is non-trivial.

For an even number \(i\), it follows that

\[
f(S_1^{(i)}) = f(C_1^{(i)}) = \cdots = f(C_{m_i-1}^{(i)}) = f(S_{m_i}^{(i)}) = a_i
\]

by the condition (7). Hence, applying the operation (a) to the image of \(P\)

\[
(\ldots, f(C_{m_i-1}^{(i-1)}), f(S_{m_i-1}^{(i-1)}), f(S_1^{(i)}), \ldots, f(S_{m_i}^{(i)}), f(S_1^{(i+1)}), f(C_1^{(i+1)}), \ldots)
\]
it can be reduced to
\[ \left( \ldots , f(C_{m-1}^{i-1}), f(S_{m-1}^{i-1}) = a_i = f(C_1^{i+1}), f(C_1^{i+1}), \ldots \right). \]

For an odd number \( i \), we set
\[ b_1^{(i)} = f(C_1^{i}), \quad b_2^{(i)} = f(C_1^{i+1}), \quad \ldots, \quad b_{2m_{i-1}}^{(i)} = f(C_{m_{i-1}}^{i}), \quad b_{2m_{i-1}}^{(i)} = f(S_{m_{i}}^{i}). \]

Then, they satisfy the following conditions:
- \( b_1^{(i)} = a_{i-1} \) and \( b_{2m_{i-1}}^{(i)} = a_{i+1} \) by the condition (4).
- \( a_i \in b_j^{(i)} \) but \( b_j^{(i)} \neq a_{i-1}, a_{i+1} \) for \( 2 \leq j \leq 2m_{i} - 2 \) by the condition (5).
- If \( b_j^{(i)} \neq b_{j+1}^{(i)} \), then either \( b_j^{(i)} = a_i \) or \( b_{j+1}^{(i)} = a_i \) (This is because \( a_i \) is a point and we have an inclusion either \( a_i \in b_j^{(i)} \subset b_{j+1}^{(i)} \) or \( a_i \in b_{j+1}^{(i)} \subset b_j^{(i)} \)).

Hence, applying the operation (a) to the image of \( P \)
\[ \left( \ldots , f(C_{m_{i-1}}^{i-1}), b_1^{(i)}, \ldots, b_{2m_{i-1}}^{(i)} = f(C_1^{i+1}) \ldots \right), \]
it can be reduced to
\[ \left( \ldots , f(C_{m_{i-1}}^{i-1}), a_{i-1}, c_1, a_i, c_1, \ldots, a_i, c_1, a_{i+1}, f(C_1^{i+1}) \ldots \right), \]
for some \( c_1, \ldots, c_{n_i} \). Applying the operation (b), it can be reduced to
\[ \left( \ldots, f(C_{m_{i-1}}^{i-1}), f(S_{m_{i}}^{i-1}) = a_{i-1}, a_i, a_{i+1} = f(S_1^{i+1}), f(C_1^{i+1}) \ldots \right). \]

Therefore, the (a,b)-reduction of the image of \( P \) is \( (a_1, a_2, \ldots, a_n) \), which is non-trivial since \( a_i \)'s are distinct. We have proved that the image of \( P \) is a non-trivial cycle sequence.

**STEP 2.** Let \( Q \) be an edge loop
\[ Q : S'_1 \xrightarrow{C'_1} S'_2 \xrightarrow{C'_2} \cdots \xrightarrow{C'_{n-1}} S'_{n} \xrightarrow{C'_{n}} S'_1 \]
in \( G \). Suppose that \( C'_i \neq C'_{i+1} \) there exist \( i \) and a 2-simplex \( F \) in \( G \) such that \( C'_i, C'_{i+1} < F \). Let \( C' < F \) be the edge which is different from \( C'_i \) and \( C'_{i+1} \).

Then we have a new edge loop \( Q' \):
\[ Q' : S'_1 \xrightarrow{C'_1} S'_2 \xrightarrow{C'_2} \cdots \xrightarrow{C'_{i-1}} S'_i \xrightarrow{C'_i} S'_{i+2} \xrightarrow{C'_{i+2}} \cdots \xrightarrow{C'_{n-1}} S'_{n} \xrightarrow{C'_{n}} S'_1. \]

We claim in this step that
- the image of \( Q \) and the image of \( Q' \) have the same (a,b)-reduction.
We have four cases.

(i) \( f(S'_i) = f(S'_{i+1}) = f(S'_{i+2}) \).

(ii) \( f(S'_i) = f(S'_{i+2}) \neq f(S'_{i+1}) \).

(iii) \( f(S'_i) = f(S'_{i+1}) \neq f(S'_{i+2}) \) or \( f(S'_i) \neq f(S'_{i+1}) = f(S'_{i+2}) \).

(iv) \( f(S'_i), f(S'_{i+1}), f(S'_{i+2}) \) are distinct.

Suppose (i). Applying the operation (b) twice to \( R \), and once to \( R' \), we get the same sequence

\[
(f(S'_1), f(C'_1), \ldots, f(C'_{i-1}), f(S'_i), f(S'_{i+2}), f(C'_{i+2}), \ldots, f(C'_n)).
\]

Suppose (ii). Since \( f(C'_i) \subset f(S'_i) \cap f(S'_{i+1}) \) and \( f(S'_i) \neq f(S'_{i+1}) \), it follows that \( \dim f(C'_i) = 0 \). By the same reason, it follows that \( \dim f(C'_{i+1}) = 0 \).

Suppose (iii). \( f(S'_i) = f(S'_{i+1}) \neq f(S'_{i+2}) \). By the same reason as in the case (ii), \( f(C'_{i+1}) = f(F) = f(C'_i) \). Applying the operation (b) once to \( R \), and once to \( R' \), we get the same sequence

\[
(f(S'_1), f(C'_1), \ldots, f(C'_{i-1}), f(S'_i), f(S'_{i+2}), f(C'_{i+2}), \ldots, f(C'_n)).
\]

Suppose (iv). In this case, \( f(C'_i) = f(C'_{i+1}) = f(C'_i) \). Applying the operation (b) once to \( R \), we get the sequence \( R' \)

\[
(f(S'_1), f(C'_1), \ldots, f(C'_{i-1}), f(S'_i), f(S'_i), f(S'_{i+2}), (C'_{i+2}), \ldots, f(C'_n)).
\]

In any case, \( R \) and \( R' \) have the same (a,b)-reduction.

**STEP 3.** Let \( Q \) be an edge loop

\[
Q : S'_1 \xrightarrow{C'_1} S'_2 \xrightarrow{C'_2} \cdots \xrightarrow{C'_{i-1}} S'_i \xrightarrow{C'_i} S'_{i+1} \xrightarrow{C'_{i+1}} S'_{i+2} \xrightarrow{C'_{i+2}} \cdots \xrightarrow{C'_{i+3}} S'_n \xrightarrow{C'_n} S'_1
\]

in \( G \). Suppose that there exist \( i \) such that \( C'_i = C''_{i+1} \). Then \( S'_i = S'_{i+2} \) holds.

Then we have a new edge loop \( Q' \):

\[
Q' : S'_1 \xrightarrow{C'_1} S'_2 \xrightarrow{C'_2} \cdots \xrightarrow{C'_{i-1}} S'_i \xrightarrow{C'_{i+2}} S'_{i+3} \xrightarrow{C'_{i+3}} \cdots \xrightarrow{C'_{i-1}} S'_n \xrightarrow{C'_n} S'_1.
\]

We claim in this step that
• the image of $Q$ and the image of $Q'$ have the same (a,b)-reduction.

The image of $Q$ is

$$R : (f(S'_1), f(C'_1), \ldots, f(S'_i), f(C'_i), f(S'_{i+1}), f(C'_{i+1}), f(S'_{i+2}), f(C'_{i+2}), \ldots, f(C'_n)),$$

and the image of $Q'$ is

$$R' : (f(S'_1), f(C'_1), \ldots, f(S'_i), f(C'_i), f(S'_{i+2}), f(C'_{i+2}), \ldots, f(C'_n)).$$

Since $C'_i = C'_{i+1}$ and $S'_i = S'_{i+2}$, applying the operation (b) twice to $R$, we get $R'$. Hence $R$ and $R'$ have the same (a,b)-reduction.

**STEP 4.** By STEP 1, there exists an edge loop $Q$

$$Q : S_1 \xrightarrow{C_1} S_2 \xrightarrow{C_2} \ldots \xrightarrow{C_{n-1}} S_n \xrightarrow{C_n} S_1$$

in $G$ such that

• its image $(f(E_1), f(C_1), \ldots, f(E_n), f(C_n))$ is a non-trivial cycle sequence.

Since $G$ is simply connected, applying the operation in STEP 2 and STEP 3 (and the reversing operation) repeatedly, we get a trivial path

$$Q' : S'$$

for some vertex $S'$ [Geo08, Theorem 3.4.1]. The (a,b)-reduction of the image of $Q$ is not trivial but that of $Q'$ is trivial. This contradicts STEP 2 and STEP 3.

□

4.2. Vanishing theorem of Witt vector cohomology of Ambro-Fujino type.

**Theorem 4.13.** Let $k$ be a perfect field of characteristic $p > 5$. Let $(X, \Delta)$ be a three-dimensional projective $\mathbb{Q}$-factorial log canonical pair over $k$ with $-(K_X + \Delta)$ ample. Then $H^i(X, W\mathcal{O}_X, \mathbb{Q}) = 0$ holds for $i > 0$.

**Proof.** By replacing $\Delta$ smaller, we may assume that dim $\text{Nklt}(X, \Delta) \leq 1$.

By the exact sequence

$$0 \to W_{\text{Nklt}(X, \Delta), \mathbb{Q}} \to W\mathcal{O}_X, \mathbb{Q} \to W\mathcal{O}_{\text{Nklt}(X, \Delta), \mathbb{Q}} \to 0,$$

and the Nadel type vanishing $H^i(X, W_{\text{Nklt}(X, \Delta), \mathbb{Q}}) = 0$ for $i > 0$ (Theorem 2.15), it is sufficient to show that

$$H^1(\text{Nklt}(X, \Delta), W\mathcal{O}_{\text{Nklt}(X, \Delta), \mathbb{Q}}) = 0.$$

Here, we may assume that $k$ is algebraically closed by [NT, Lemma 2.15]. Since dim $\text{Nklt}(X, \Delta) \leq 1$ and $\text{Nklt}(X, \Delta)$ is connected ([NT, Theorem 1.2]), we may assume that $\text{Nklt}(X, \Delta)$ is a union of curves.

Let $C := \text{Nklt}(X, \Delta) = C_1 \cup C_2 \cup \ldots \cup C_l$ be the irreducible decomposition. By Lemma 4.12, Proposition 4.20 and Proposition 4.12, the curve $C$ satisfies the following conditions.

1. Each $C_i$ is a rational curve.
2. Each normalization of $C_i$ is a universal homeomorphism.
3. $C = C_1 \cup \ldots \cup C_l$ forms a tree (see Definition 1.6).

Then $H^1(C_i, W\mathcal{O}_{C_i, \mathbb{Q}}) = 0$ follows from (1) and (2) (cf. [GNT, Lemma 2.21, 2.22]). Hence the desired vanishing $H^1(C, W\mathcal{O}_{C, \mathbb{Q}}) = 0$ follows from (3). □
4.3. Rational point formula. As an application of Theorem 4.13, we obtain the following rational point formula.

**Theorem 4.14.** Let $k$ be a finite field of characteristic $p > 5$. Let $(X, \Delta)$ be a geometrically connected three-dimensional projective $\mathbb{Q}$-factorial log canonical pair over $k$ with $-(K_X + \Delta)$ ample. Then the number of the $k$-rational points on the non-klt locus on $(X, \Delta)$ satisfies

$$\# \text{Nklt}(X, \Delta)(k) \equiv 1 \mod |k|.$$ 

In particular, there exists a $k$-rational point on $	ext{Nklt}(X, \Delta)$.

**Proof.** Let $Z = \text{Nklt}(X, \Delta)$ and let $I_Z$ be the corresponding coherent ideal sheaf. By Theorem 2.15 and Theorem 4.13, $H^i(X, W_{I_Z, \mathbb{Q}}) = 0$, and $H^i(X, W_{O_X, \mathbb{Q}}) = 0$ hold for $i > 0$. By the exact sequence

$$0 \to W_{I_Z} \to W_{O_X, \mathbb{Q}} \to W_{O_Z, \mathbb{Q}} \to 0,$$

$H^i(Z, W_{O_Z, \mathbb{Q}}) = 0$ holds for $i > 0$. By [BBE07, Proposition 6.9 (i)], it follows that $\#Z(k) \equiv 1 \mod |k|$. □

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