CRITICAL EXONENTS OF GRAPHS

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Abstract. The study of entrywise powers of matrices was originated by Loewner in the pursuit of the Bieberbach conjecture. Since the work of FitzGerald and Horn (1977), it is known that \( A^α := (a_{ij}^α) \) is positive semidefinite for every \( n \times n \) positive semidefinite matrix \( A = (a_{ij}) \) if and only if \( α \) is an integer or \( α ≥ n − 2 \). This surprising result naturally extends the Schur product theorem, and demonstrates the existence of a sharp phase transition in preserving positivity. In this paper, we study when entrywise powers preserve positivity for matrices with structure of zeros encoded by graphs. To each graph is associated an invariant called its critical exponent, beyond which every power preserves positivity. In our main result, we determine the critical exponents of all chordal/decomposable graphs, and relate them to the geometry of the underlying graphs. We then examine the critical exponent of important families of non-chordal graphs such as cycles and bipartite graphs. Surprisingly, large families of dense graphs have small critical exponents that do not depend on the number of vertices of the graphs.

1. Introduction and main result

Let \( f : \mathbb{R} \to \mathbb{R} \) be a real function and denote by \( \mathbb{P}_n \) the cone of \( n \times n \) real symmetric positive semidefinite matrices. The function \( f \) naturally operates entrywise on \( \mathbb{P}_n \) by defining \( f[A] := (f(a_{ij})) \). Whether or not the mapping \( A \mapsto f[A] \) preserves positivity (i.e., \( f[A] \in \mathbb{P}_n \) for all \( A \in \mathbb{P}_n \)) is an important problem that has been well-studied in the literature - see e.g. Schoenberg [37], Rudin [36], Herz [25], Horn [27], Christensen and Ressel [6], Vasudeva [40], and FitzGerald et al [14]. In one of their main results in the area, Schoenberg and Rudin (37, 36) have shown an important characterization of functions \( f \) that preserve positivity on \( \mathbb{P}_n \) for all \( n \). Their results show that such functions are precisely the absolutely monotonic functions (i.e., they are analytic with nonnegative Taylor coefficients).

When the dimension \( n \) is fixed, obtaining useful characterizations of functions preserving positivity is difficult, and very few results are known. In the pursuit of the Bieberbach conjecture (de Branges’s theorem), Loewner was led to study which real powers \( α > 0 \) preserve positivity (i.e., positive semidefiniteness) for all \( n \times n \) positive semidefinite matrices with positive entries. As a consequence of the Schur product theorem, every integer power trivially preserves positivity when applied entrywise. Identifying the other real powers that do so is a much more challenging task. The problem was solved by FitzGerald and Horn in 1977; in their landmark paper [13], they show that a real power \( α > 0 \) preserves positivity on \( n \times n \) matrices if and only if \( α \) is a positive integer or \( α ≥ n − 2 \). The work of FitzGerald and Horn was later extended in different directions by multiple authors including Bhatia and Elsner [4], Hiai [26], and Guillot, Khare and Rajaratnam [19] - see [19] for a history of the problem and recent developments.

In this paper, we significantly generalize the original problem by studying which entrywise powers preserve positivity when the matrices have an additional structure of zeros that is encoded by a graph. Motivation for this problem comes from its connection to the regularization of covariance/correlation matrices in high-dimensional probability and statistics. The study of entrywise
functions preserving positivity has recently received renewed attention in this area. For instance, powering up is a way to effectively and efficiently separate out signal from noise; see e.g. \[31, 12\]. More generally, it is common to use entrywise functions in high-dimensional probability and statistics to regularize covariance/correlation matrices and improve their properties (e.g., condition number, Markov random field structure, etc.) - see \[5, 23, 24, 21, 22\]. Preserving positivity is critical for such techniques to be useful in downstream applications. In recent work by the authors \[22, 18, 20\], classical results by Schoenberg and Rudin were extended in various settings motivated by modern-day applications. These include: 1) characterizing functions preserving positivity when applied only to the off-diagonal elements of matrices (as is often the case in applications), 2) preserving positivity under rank constraints, and 3) preserving positivity under sparsity constraints. Motivation for the second problem comes from the fact that the rank of covariance/correlation matrices generally corresponds to independence or conditional independence of the corresponding random variables. The problem of regularizing matrices with an original sparsity structure thus naturally occurs when there is prior knowledge available about these dependencies.

The present paper focuses on matrices with prescribed structure of zeros. Such matrices naturally occur in combinatorics (e.g. in spectral graph theory) and many other areas of mathematics - see e.g. \[1, 2, 12, 16\]. These matrices also occur naturally in multiple fields of the broader mathematical sciences such as optimization, network theory, and in modern high-dimensional probability and statistics.

The structure of zeros of a symmetric \(n \times n\) matrix \(A = (a_{ij})\) is encoded by an undirected graph \(G = (V, E)\), where \(V = \{1, \ldots, n\}\) and \((i, j) \notin E\) if and only if \(i = j\) or \(a_{ij} = 0\). Given \(n \in \mathbb{N}\) and \(I \subseteq \mathbb{R}\), let \(\mathbb{P}_n(I)\) denote the set of symmetric positive semidefinite \(n \times n\) matrices with entries in \(I\). Given a simple graph \(G = (V, E)\) with nodes \(V = \{1, 2, \ldots, n\}\), and a subset \(I \subseteq \mathbb{R}\), define

\[
\mathbb{P}_G(I) := \{ A \in \mathbb{P}_n(I) : a_{ij} = 0 \ \forall (i,j) \notin E, \ i \neq j \}.
\]

(1.1)

For simplicity, we denote \(\mathbb{P}_G(\mathbb{R})\) by \(\mathbb{P}_G\). All graphs in the remainder of the paper are assumed to be finite and simple.

An important family of graphs in mathematics as well as in applications is the family of chordal graphs (see e.g. \[8, \text{Chapter 5.5}, 15, \text{Chapter 4}\]). Recall that chordal graphs (also known as decomposable graphs, triangulated graphs, or rigid circuit graphs) are graphs in which all cycles of four or more vertices have a chord. Chordal graphs are perfect, and have a rich structure that has been well-studied in the literature. They also play a fundamental role in multiple areas including the matrix completion problem (see e.g. \[2, 16, 31\]), maximum likelihood estimation in the theory of Markov random fields \[30, \text{Section 5.3}\], and perfect Gaussian elimination \[15\]. For example, when solving sparse linear systems, it is important to preserve the structure of zeros of the original matrix for storage and computation efficiency purposes. By a result of Golub and other authors \[15, \text{Theorem 12.1}\], Gaussian elimination can be performed on a given sparse matrix without ever changing a zero entry to a nonzero entry if and only if the structure of zeros of the matrix forms a chordal graph.

In our main result, we characterize all the powers \(\alpha \geq 0\) that preserve positivity on \(\mathbb{P}_G([0, \infty))\) for each chordal graph \(G\). We also characterize all \(\alpha \in \mathbb{R}\) for which the odd and even extensions to \(\mathbb{R}\) of the power functions preserve positivity on \(\mathbb{P}_G(\mathbb{R})\). We demonstrate that for a given chordal graph \(G\), the set of powers preserving positivity on \(\mathbb{P}_G\) is driven by the existence of specific subgraphs of \(G\). Our results thus naturally connect the discrete structure of the graph \(G\) to the analytic properties of the cone of positive semidefinite matrices \(\mathbb{P}_G\). Our results also naturally extend the classical case where \(G = K_n\) (the complete graph on \(n\) vertices) that was studied by FitzGerald and Horn in \[13\] and many others \[4, 19, 26\]. Imposing the additional constraints on the structure of zeros, however,
leads to challenging problems. For example, many familiar constructions involved in preserving positivity (e.g., working with Schur complements) generally fail to preserve the underlying structure of zeros. As a consequence, many new techniques have to be developed to address such issues.

In the last section of the paper, we also determine the set of powers preserving positivity for many broad families of non-chordal graphs including cycles, bipartite graphs, and coalescences of graphs. In particular, we show that for some families of dense graphs (e.g. complete bipartite graphs), every power greater than 1 preserves positivity on \( P_G([0, \infty)) \). This result came as a surprise since, as shown by FitzGerald and Horn [13], non-integer powers smaller than \( n - 2 \) do not preserve positivity on \( n \times n \) matrices when no additional structure of zeros is imposed. The result also has important implications for the regularization of covariance/correlation matrices, by showing that small powers can be safely used to regularize covariance/correlation matrices having an appropriate original structure of zeros.

The remainder of the paper is structured as follows: the key definitions and the main theorem of the paper are introduced in the rest of Section 1. Useful preliminary results are discussed in Section 2. The main theorem is proved in Section 3, followed by a study of non-chordal graphs in Section 4. We conclude by discussing further questions and extensions.

1.1. Main result. In order to state our main theorem, we begin by introducing some notation. Given two \( n \times n \) matrices \( A = (a_{ij}) \) and \( B = (b_{ij}) \), their Hadamard (or Schur, or entrywise) product, denoted by \( A \odot B \), is defined by \( A \odot B := (a_{ij} b_{ij}) \). Note that \( A \odot B \) is a principal submatrix of the tensor product \( A \otimes B \). As a consequence, if \( A \) and \( B \) are positive (semi)definite, then so is \( A \odot B \). This result is known in the literature as the Schur product theorem [38]. Given \( \alpha \in \mathbb{R} \), we denote the entrywise \( \alpha \)th power of a matrix \( A \) with nonnegative entries by \( A^{\alpha} := (a_{ij}^\alpha) \), where we define \( 0^\alpha := 0 \) for all \( \alpha \). As a consequence of the Schur product theorem, it is clear that \( A^{\alpha k} \) is positive (semi)definite for all positive (semi)definite matrices \( A \) and all \( k \in \mathbb{N} \). Note that \( A^{\alpha} \) is not always well-defined if \( a_{ij} \in \mathbb{R} \). Following Hiai [26], it is natural to replace the power functions by their odd and even extensions to \( \mathbb{R} \) in order to deal with arbitrary positive semidefinite matrices. Given \( \alpha \in \mathbb{R} \), we define the odd and even extensions of the power functions as follows:

\[
\psi_\alpha(x) := \text{sgn}(x)|x|^{\alpha}, \quad \phi_\alpha(x) := |x|^{\alpha}, \quad \forall \ x \in \mathbb{R} \setminus \{0\},
\]

and \( \psi_\alpha(0) = \phi_\alpha(0) := 0 \). Given \( f : \mathbb{R} \to \mathbb{R} \), and \( A = (a_{ij}) \), define \( f[A] := (f(a_{ij})) \). We now introduce the main objects of study in this paper.

**Definition 1.1.** Let \( n \geq 2 \) and let \( G = (V, E) \) be a simple graph on \( V = \{1, \ldots, n\} \). We define:

\[
\mathcal{H}_G := \{ \alpha \in \mathbb{R} : A^{\alpha} \in \mathcal{P}_G \text{ for all } A \in \mathcal{P}_G([0, \infty)) \},
\]

\[
\mathcal{H}_G^\psi := \{ \alpha \in \mathbb{R} : \psi_\alpha[A] \in \mathcal{P}_G \text{ for all } A \in \mathcal{P}_G(\mathbb{R}) \},
\]

\[
\mathcal{H}_G^\phi := \{ \alpha \in \mathbb{R} : \phi_\alpha[A] \in \mathcal{P}_G \text{ for all } A \in \mathcal{P}_G(\mathbb{R}) \}.
\]

Denote by \( K_n \) the complete graph on \( n \) vertices. The sets \( \mathcal{H}_{K_n}, \mathcal{H}_{K_n}^\psi, \) and \( \mathcal{H}_{K_n}^\phi \) were computed through several papers and the following theorem summarizes their results. The reader is referred to [19] for more details.

**Theorem 1.2** (FitzGerald–Horn [13], Bhatia–Elsner [1], Hiai [26], Guillot–Khare–Rajaratnam [19]). Let \( n \geq 2 \). The \( \mathcal{H} \)-sets of powers preserving positivity for \( G = K_n \) are:

\[
\mathcal{H}_{K_n} = \mathbb{N} \cup [n - 2, \infty),
\]

\[
\mathcal{H}_{K_n}^\psi = (-1 + 2\mathbb{N}) \cup [n - 2, \infty),
\]

\[
\mathcal{H}_{K_n}^\phi = 2\mathbb{N} \cup [n - 2, \infty).
\]
The above surprising result shows that there is a threshold value above which every power function \( x^\alpha, \psi_\alpha, \) or \( \phi_\alpha \) preserves positivity on \( \mathbb{P}_n([0, \infty)) \) or \( \mathbb{P}_n(\mathbb{R}) \), when applied entrywise. The threshold is commonly referred to as the critical exponent for preserving positivity. It is natural to extend the notion of critical exponents to arbitrary graphs.

**Definition 1.3.** Given a graph \( G \), define the Hadamard critical exponents of \( G \) to be
\[
CE_H(G) := \min\{\alpha \in \mathbb{R} : A \in \mathbb{P}_G([0, \infty)) \Rightarrow A^{\alpha} \in \mathbb{P}_G \text{ for every } \beta \geq \alpha\},
\]
\[
CE_H^\psi(G) := \min\{\alpha \in \mathbb{R} : A \in \mathbb{P}_G(\mathbb{R}) \Rightarrow \psi_\alpha[A] \in \mathbb{P}_G \text{ for every } \beta \geq \alpha\},
\]
\[
CE_H^\phi(G) := \min\{\alpha \in \mathbb{R} : A \in \mathbb{P}_G(\mathbb{R}) \Rightarrow \phi_\alpha[A] \in \mathbb{P}_G \text{ for every } \beta \geq \alpha\}.
\]

Note that since every graph \( G = (V, E) \) is contained in a complete graph, the critical exponents of \( G \) are well defined by Theorem 1.2 and bounded above by \(|V| - 2\).

We can now state our main result. Let \( K_n^{(1)} \) denote the complete graph on \( n \) vertices with one edge missing.

**Theorem 1.4 (Main result).** Let \( G \) be any chordal graph with at least 2 vertices and let \( r \) be the largest integer such that either \( K_r \) or \( K_r^{(1)} \) is an induced subgraph of \( G \). Then
\[
\mathcal{H}_G = \mathbb{N} \cup [r - 2, \infty),
\]
\[
\mathcal{H}_G^\psi = (-1 + 2\mathbb{N}) \cup [r - 2, \infty),
\]
\[
\mathcal{H}_G^\phi = 2\mathbb{N} \cup [r - 2, \infty).
\]

In particular, \( CE_H(G) = CE_H^\psi(G) = CE_H^\phi(G) = r - 2 \).

Theorem 1.4 thus demonstrates that an increase in sparsity generally reduces the Hadamard critical exponents. The precise way in which the critical exponents of chordal graphs are lowered is driven by the size of their largest maximal or nearly maximal cliques. This fact is especially important in applications, where covariance/correlation matrices need to be regularized by minimally modifying their entries while simultaneously preserving positive semidefiniteness. Theorem 1.4 shows that small powers can be used to achieve such a goal if the original matrices are sparse enough.

**Remark 1.5.** Theorem 1.4 shows that the critical exponent of a chordal graph \( G \) is bounded above by \( \max_{v \in V(G)} \deg(v) - 1 \). Note however that this bound is not sharp. For instance, for star graphs the critical exponent is always 1 (see Theorem 2.2).

The rest of the paper is devoted to proving Theorem 1.4. Most of the techniques and constructions that are traditionally used to study powers preserving positivity (e.g. spectral methods, Schur complements) do not preserve the structure of zeros of matrices. Studying powers preserving positivity under sparsity constraints is thus a challenging task that requires new ideas. In the rest of the paper, we develop multiple techniques for computing the \( \mathcal{H} \)-sets of graphs. In addition to proving Theorem 1.4, we use these techniques to compute the critical exponent of many non-chordal graphs as well. We demonstrate that the critical exponent does not always correspond to the order of the largest induced \( K_r \) or \( K_r^{(1)} \) minus 2 when \( G \) is non-chordal. We also show that many dense graphs (e.g. complete bipartite graphs) have a surprisingly small critical exponent that does not depend on their number of vertices. This is in stark contrast to the family of complete graphs \( K_n \), for which the critical exponent is \( n - 2 \).

2. Preliminary results: pendant edges and trees

We begin our analysis by recalling a general result that classifies entrywise functions preserving positivity for matrices with zeros according to a tree. Given a \( n \times n \) symmetric matrix \( A = (a_{ij}) \)
and a graph $G = (V, E)$ with vertex set $V = \{1, \ldots, n\}$ and edge set $E$, denote by

$$f_G[A] := \begin{cases} f(a_{ij}) & \text{if } (i, j) \in E \text{ or } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 2.1** (Guillot, Khare, and Rajaratnam, [22]). Suppose $I = [0, R)$ for some $0 < R \leq \infty$, and $f : I \to [0, \infty)$. Let $G$ be a tree with at least 3 vertices, and let $P_3$ denote the path graph on 3 vertices. Then the following are equivalent:

1. $f_G[A] \in \mathbb{P}_{G}$ for every $A \in \mathbb{P}_{G}(I)$;
2. $f_T[A] \in \mathbb{P}_{T}$ for all trees $T$ and all matrices $A \in \mathbb{P}_{T}(I)$;
3. $f_{P_3}[A] \in \mathbb{P}_{P_3}$ for every $A \in \mathbb{P}_{P_3}(I)$;
4. The function $f$ satisfies:

$$f(\sqrt{xy})^2 \leq f(x)f(y), \quad \forall x, y \in I \tag{2.1}$$

and is superadditive on $I$, i.e.,

$$f(x + y) \geq f(x) + f(y), \quad \forall x, y, x + y \in I. \tag{2.2}$$

It follows from Theorem 2.1 that $\mathcal{H}_G = [1, \infty)$ for any tree $G$. We now generalize this result to graphs obtained by pasting trees to vertices of graphs, and to the functions $\psi_\alpha$ and $\phi_\alpha$.

**Theorem 2.2.** Suppose $G'$ is not a disjoint union of copies of $K_2$. Construct a graph $G$ from $G'$ by attaching finitely many (possibly empty) trees to each node, at least one of which is not isolated. Then

$$\mathcal{H}_G = \mathcal{H}_{G'}, \quad \mathcal{H}^\psi_G = \mathcal{H}^\psi_{G'}, \quad \mathcal{H}^\phi_G = \mathcal{H}^\phi_{G'}.$$ 

In particular, if $G$ is a tree with at least 3 vertices, then $CE_H(G) = CE^\psi_H(G) = CE^\phi_H(G) = 1$.

In order to prove Theorem 2.2 we first introduce additional notation.

**Definition 2.3.** Let $A, B$ be two $n \times n$ matrices and $1 \leq i \leq n$. Write $A$ in block form:

$$A = \begin{pmatrix} A_{11} & u_1 & A_{12} \\ v_1^T & a_{ii} & v_2^T \\ A_{21} & u_2 & A_{22} \end{pmatrix}.$$ 

If $a_{ii} \neq 0$, then the Schur complement of $a_{ii}$ in $A$ is defined to be

$$A/a_{ii} := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} - a_{ii}^{-1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix}.$$ 

We also say that $A \geq B$ if $A - B \in \mathbb{P}_n$.

**Proof of Theorem 2.2.** We only prove the result for $\mathcal{H}^\psi_G$; the proofs are similar for $\mathcal{H}_G, \mathcal{H}^\phi_G$. The first step is to compute $\mathcal{H}^\psi_{P_3}$. Note by Theorem 1.2 that

$$[0, \infty) = \mathcal{H}^\psi_{K_2} \supset \mathcal{H}^\psi_{P_3} \supset \mathcal{H}^\psi_{K_3} = \mathbb{N} \cup [1, \infty),$$

so it suffices to show that no $\alpha \in (0, 1)$ preserves $\mathbb{P}_{P_3}$. Now fix $a \in [0, 1]$ and consider the matrix

$$A(a) := \begin{pmatrix} 1 & a & 0 \\ a & 1 & \sqrt{1 - a^2} \\ 0 & \sqrt{1 - a^2} & 1 \end{pmatrix} \in \mathbb{P}_{P_3}.$$ 

It is clear that $\psi_\alpha[A(a)] = \phi_\alpha[A(a)] = A(a)^{\alpha^2}$ has determinant $1 - (a^2)^\alpha - (1 - a^2)^\alpha$, and this is strictly negative if $a \in (0, 1)$ and $\alpha \in [0, 1)$, by the subadditivity of $x \mapsto x^\alpha$.

Now suppose $G'$ is a connected, nonempty graph, $v \in V(G')$, and $G$ is obtained by attaching a pendant edge to $v$ (i.e., adding a new vertex and connecting it by an edge to $v$). Also suppose $0 \in I \subset \mathbb{R}$ and $f : I \to \mathbb{R}$ is super-additive on $I \cap [0, \infty)$. Then we claim that $f[\cdot]$ preserves
positivity on \( P_G(I) \) if and only if it preserves positivity on \( P_{G'}(I) \). The proof uses arguments from the proof of [20, Theorem A].

Finally, the result for trees follows immediately by applying the previous two steps to \( I = [0, \infty) \) and \( f(x) = x^\alpha \) for \( \alpha \geq 1 \), and to \( I = \mathbb{R} \) and \( f(x) = \psi_\alpha(x), \phi_\alpha(x) \) for \( \alpha \geq 1 \).

As a consequence, we characterize all graphs with Hadamard critical exponent 0.

**Corollary 2.4.** Given a graph \( G \), the following are equivalent:

1. \( G \) is a disjoint union of copies of \( K_2 \).
2. \( G \) does not contain a connected subgraph with three vertices.
3. \( H_G = [0, \infty) \).
4. \( 0 \in H_G \).
5. \( H_G \not\subset [1, \infty) \).
6. \( CE_H(G) = 0 \).

**Proof.** That the first two conditions are equivalent is obvious. That \((1) \implies (3) \implies (4) \implies (5)\) is also clear. Now if (2) fails to hold, then this connected subgraph of \( G \) is either the path graph \( P_3 \) or the complete graph \( K_3 \). In both cases, (5) also fails to hold, by Theorems 1.2 and 2.2. This shows that \((5) \implies (2)\), and hence that (1)–(5) are equivalent. Next, clearly (6) \implies (3). Conversely, if (1) holds then it is easy to show that \( H_G = [0, \infty) \), so that (6) also holds. \( \square \)

**Remark 2.5.** We remark that Corollary 2.4 also holds if \( H_G \) is replaced by \( H_G^\psi \) or \( H_G^\phi \), and \( CE_H(G) \) is replaced by the corresponding critical exponent. The proof is similar.

**Remark 2.6.** Corollary 2.4 shows that for all graphs that are not disjoint unions of copies of \( K_2 \), the set of powers preserving positivity are all contained in \([1, \infty)\). For this reason, and without further reference, we do not consider non-positive entrywise power functions in the remainder of the paper. Similarly, the Schur product theorem implies that \( \mathbb{N} \subset H_G, -1 + 2\mathbb{N} \subset H_G^\psi, 2\mathbb{N} \subset H_G^\phi \) for all graphs \( G \), and these facts are used below without further reference.

### 3. Proof of the Main result

In this section we develop all the tools that are required to compute the \( \mathcal{H} \)-sets for chordal graphs. We begin by recalling some important properties of chordal graphs (see e.g. [3, Chapter 5.5], [13, Chapter 4] for more details).

Let \( G = (V, E) \) be an undirected graph. Given \( C \subset V \), denote by \( G_C \) the subgraph of \( G \) induced by \( C \). A **clique** in \( G \) is a complete induced subgraph of \( G \). A subset \( C \subset V \) is said to separate \( A \subset V \) from \( B \subset V \) if every path from a vertex in \( A \) to a vertex in \( B \) intersects \( C \). A partition \((A, C, B)\) of subsets of \( V \) is said to be a **decomposition** of \( G \) if

1. \( C \) separates \( A \) from \( B \); and
2. \( G_C \) is complete.

A graph \( G \) is said to be **decomposable** if either \( G \) is complete, or if there exists a decomposition \((A, C, B)\) of \( G \) such that \( G_{A \cup C} \) and \( G_{B \cup C} \) are decomposable.

Let \( G \) be a graph and let \( B_1, \ldots, B_k \) be a sequence of subsets of vertices of \( G \). Define:

\[
H_j := B_1 \cup \cdots \cup B_j, \quad R_j = B_j \setminus H_{j-1}, \quad S_j = H_{j-1} \cap B_j, \quad 1 \leq j \leq k, \quad (3.1)
\]

and \( H_0 := \emptyset \). The sets \( H_j, R_j, \) and \( S_j \) are respectively called the **histories**, **residuals**, and **separators** of the sequence. The sequence \( B_1, \ldots, B_k \) is said to be a **perfect ordering** if:

1. For all \( 1 \leq i \leq k \), there exists \( 1 \leq j < i \) such that \( S_i \subset B_j \); and
2. The sets \( S_i \) are complete for all \( 1 \leq i \leq k \).

Decompositions and perfect orderings provide important characterizations of chordal graphs, as summarized in Theorem 3.1.
Theorem 3.1 ([30] Propositions 2.5 and 2.17). Let \( G = (V, E) \) be an undirected graph. Then the following are equivalent:

1. \( G \) is chordal (i.e., each cycle with 4 vertices or more in \( G \) has a chord).
2. \( G \) is decomposable.
3. The maximal cliques of \( G \) admit a perfect ordering.

We now relate the decomposition of a chordal graph \( G \) to properties of functions preserving positivity on \( \mathbb{P}_G \). Given a graph \( G \) and a function \( f : \mathbb{R} \to \mathbb{R} \) with \( f(0) = 0 \), we say that \( f[-] \) is Loewner super-additive on \( \mathbb{P}_G(\mathbb{R}) \) if \( f[A + B] - f[A] - f[B] \in \mathbb{P}_G(\mathbb{R}) \) whenever \( A, B \in \mathbb{P}_G(\mathbb{R}) \). Note that this notion coincides with the usual notion of super-additivity on \([0, \infty)\) when \( G \) has only one vertex.

**Theorem 3.2.** Let \( G = (V, E) \) be a graph with a decomposition \((A, C, B)\). Also let \( f : \mathbb{R} \to \mathbb{R} \).

1. If \( f[-] \) preserves positivity on \( \mathbb{P}_{G_{A,C}} \) and on \( \mathbb{P}_{G_{B,C}} \), and is Loewner super-additive on \( \mathbb{P}_{G_C} \), then \( f[-] \) preserves positivity on \( \mathbb{P}_G \).
2. Conversely, if \( f = \psi_\alpha \) or \( f = \phi_\alpha \) and \( f[-] \) preserves positivity on \( \mathbb{P}_G \), then \( f[-] \) is Loewner super-additive on \( \mathbb{P}_{G_{C'}} \), for every clique \( C' \subset C \) for which there exist vertices \( v_1 \in A, v_2 \in B \) that are adjacent to every \( v \in C' \).

In particular, when \( f = \psi_\alpha \) or \( f = \phi_\alpha \) and \( |C| = 1 \), \( f[-] \) preserves positivity on \( \mathbb{P}_G \), if and only if \( f[-] \) preserves positivity on \( \mathbb{P}_{G_{A,C}} \) and \( \mathbb{P}_{G_{B,C}} \) and is Loewner super-additive on \([0, \infty)\).

Theorem 3.2 immediately implies that if a superadditive function preserves positivity on \( \mathbb{P}_2 \), then it does so on \( \mathbb{P}_G \) for all trees \( G \). The result thus extends [20, Theorem A]. (See [20, Theorem 2.6] for a characterization of entrywise functions preserving positivity on \( \mathbb{P}_2 \).

The proof of Theorem 3.2 requires some preliminary results. We first recall previous work on Loewner superadditive functions. The powers that are Loewner superadditive on \( \mathbb{P}_n(\mathbb{R}) = \mathbb{P}_{K_n}(\mathbb{R}) \) have been classified in [19].

**Theorem 3.3** (Guillot, Khare, and Rajaratnam [19, Theorem 5.1]). Given an integer \( n \geq 2 \), the sets of entrywise power functions \( x^\alpha, \psi_\alpha, \phi_\alpha \) (with \( \alpha \in \mathbb{R} \)) which are Loewner super-additive maps on \( \mathbb{P}_n \) are, respectively,

\[
\mathbb{N} \cup [n, \infty), \quad (1 + 2\mathbb{N}) \cup [n, \infty), \quad 2\mathbb{N} \cup [n, \infty).
\]

Moreover, for all \( \alpha \in (0, n) \) \( \setminus \mathbb{N} \), there exist \( u, v \in [0, \infty)^n \) such that \( (uu^T + vv^T)^\alpha \not\in \mathbb{P}_n \). Similarly, if \( f \equiv \psi_\alpha \) with \( \alpha = 2k \) for \( 1 \leq k \leq n/2 - 1 \), or \( f \equiv \phi_\alpha \) with \( \alpha = 2k - 1 \) for \( 1 \leq k \leq n/2 \), then there exist \( u, v \in \mathbb{R}^n \) such that \( f[uu^T + vv^T] \not\in \mathbb{P}_n \).

The following corollary is an immediate consequence of Theorem 3.2 and Theorem 3.3.

**Corollary 3.4.** Let \( G = (V, E) \) be a graph with a decomposition \((A, C, B)\). Suppose there exist vertices \( v_1 \in A \) and \( v_2 \in B \) that are adjacent to every \( v \in C \). Let \( f = \psi_\alpha \) or \( f = \phi_\alpha \) for some \( \alpha \in \mathbb{R} \). Then \( f[-] \) preserves positivity on \( \mathbb{P}_G \) if and only if either

1. \( \alpha \in -1 + 2\mathbb{N} \) if \( f = \psi_\alpha \) or \( \alpha \in 2\mathbb{N} \) if \( f = \phi_\alpha \), or
2. \( f[-] \) preserves positivity on \( \mathbb{P}_{G_{A,C}} \) and \( \mathbb{P}_{G_{B,C}} \) and \( |\alpha| \geq |C| \).

**Lemma 3.5.** In the statement of the result and the remainder of the paper, we adopt the following convention to simplify notation: given a graph \( G \) and an induced subgraph \( G' \), we identify \( \mathbb{P}_G(I) \) with a subset of \( \mathbb{P}_G(I) \) when convenient, via the assignment \( M \mapsto M \oplus \mathbf{0}_{(V(G) \setminus V(G')) \times (V(G) \setminus V(G'))} \).

**Lemma 3.5.** Let \( G = (V, E) \) be a graph with a decomposition \((A, C, B)\) of \( V \), and let \( M \) be a symmetric matrix. Assume the principal submatrices \( M_{AA} \) and \( M_{BB} \) of \( M \) are invertible. Then the following are equivalent:

1. \( M \in \mathbb{P}_G \).
(2) $M = M_1 + M_2$ for some matrices $M_1 \in \mathbb{P}_{G_{A,U}}$ and $M_2 \in \mathbb{P}_{G_{B,U}}$.

Proof. Clearly (2) $\implies$ (1). Now let $M \in \mathbb{P}_G$. The matrix $M$ can be written in block form as

$$M = \begin{pmatrix} M_{AA} & M_{AC} & 0 \\ M_{AC}^T & M_{CC} & M_{CB} \\ 0 & M_{CB}^T & M_{BB} \end{pmatrix}. $$

It is not difficult to verify that

$$M = \begin{pmatrix} M_{AA} & 0 & 0 \\ M_{AC}^T & I_{|C|} & M_{CB} \\ 0 & 0 & M_{BB} \end{pmatrix} \begin{pmatrix} M_{AA}^{-1} & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & M_{BB}^{-1} \end{pmatrix} \begin{pmatrix} M_{AA} & 0 & 0 \\ M_{AC}^T & I_{|C|} & M_{CB} \\ 0 & 0 & M_{BB} \end{pmatrix}^T, \quad (3.2)$$

where $S := M_{CC} - M_{AC}^TM_{AA}^{-1}M_{AC} - M_{CB}M_{BB}^{-1}M_{CB}^T$. It follows that $S$ is positive semidefinite. Now let

$$M_1 := \begin{pmatrix} M_{AA} & M_{AC} & 0 \\ M_{AC}^T & M_{AC}M_{AA}^{-1}M_{AC} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & M_{CC} - M_{AC}^TM_{AA}^{-1}M_{AC} & M_{CB} \\ 0 & M_{CB}^T & M_{BB} \end{pmatrix}. $$

Clearly, $M = M_1 + M_2$. Computing the Schur complement of $M_{AA}$ in the upper left $2 \times 2$ blocks of $M_1$, we conclude that $M_1 \in \mathbb{P}_{G_{A,U}}$. Similarly, the Schur complement of the lower right $2 \times 2$ blocks of $M_2$ is equal to $S$ and therefore $M_2 \in \mathbb{P}_{G_{B,U}}$. This proves the desired decomposition of $M$. \hfill \Box

Using the above results, we now prove Theorem 3.2.

Proof of Theorem 3.2. Suppose $f$ satisfies the conditions in (1), and $M \in \mathbb{P}_G$. Then, in particular, $f$ preserves positivity on $\mathbb{P}_{K_2}(\mathbb{R}) = \mathbb{P}_2(\mathbb{R})$, whence $f$ is continuous on $(0, \infty)$ by [27, Theorem 1.2]. Moreover, $f$ is right-continuous at 0 as shown at the beginning of the proof of [20, Theorem C]. Also note that $f(0) = 0$ because $f[-]$ is super-additive on $[0, \infty) = \mathbb{P}_1 \subset \mathbb{P}_G$ and preserves positivity on $\mathbb{P}_1 \subset \mathbb{P}_{G_{A,U}}$. Now given $\epsilon > 0$, let $\tilde{M}_\epsilon := M + \epsilon \cdot I_{|G|}$, where $I_{|K|}$ denotes the $k \times k$ identity matrix. By Lemma 3.5, $\tilde{M}_\epsilon = M_1 + M_2$ with $M_1 \in \mathbb{P}_{G_{A,U}}$ and $M_2 \in \mathbb{P}_{G_{B,U}}$. By assumption $f[M_1]$ and $f[M_2]$ are positive semidefinite. Moreover, $D := f[M_\epsilon] - f[M_1] - f[M_2]$ belongs to $\mathbb{P}_G$ by the assumption of superadditivity on $f$ on $\mathbb{P}_G$. It follows that $f[M_\epsilon] = f[M_1] + f[M_2] + D$ is positive semidefinite for every $\epsilon > 0$. We conclude by continuity that $f[M]$ is positive semidefinite, proving (2).

Next, suppose $f = \psi_\alpha$ or $\phi_\alpha$ for $\alpha \in \mathbb{R}$, and $f[-]$ preserves positivity on $\mathbb{P}_G$. Then clearly $f[-]$ preserves positivity on $\mathbb{P}_{G_{A,U}}$ and $\mathbb{P}_{G_{B,U}}$. Moreover, suppose there exist $v_1 \in A$, $v_2 \in B$, and a clique $C' \subset C$ of size $m$ such that $v_1$ and $v_2$ are adjacent to every vertex in $C'$. Assume, without loss of generality, that the vertices of $C$ as labelled in the following order: $v_1$, the $m$ vertices in $C'$, $v_2$, and the remaining vertices of $G$. Now given vectors $u, v \in \mathbb{R}^m$ and a $m \times m$ symmetric matrix $M$, define the matrix

$$W(u, v, M) := \begin{pmatrix} 1 & u^T & 0 \\ u & M & v \\ 0 & v^T & 1 \end{pmatrix}. \quad (3.3)$$

Then $W(u, v, uu^T + vv^T) \oplus 0_{V - (m+2)} \in \mathbb{P}_G(\mathbb{R})$, so by the assumptions on $f$, we conclude that $f[W(u, v, uu^T + vv^T)] = W(f[u], f[v], f[uu^T + vv^T]) \in \mathbb{P}_{m+2}(\mathbb{R})$. Now using the same decomposition as in Equation (3.2), we conclude that

$$f[uu^T + vv^T] - f[u]f[u^T] - f[v]f[v^T] = f[uu^T + vv^T] - f[uu^T] - f[vv^T] \geq 0. \quad (3.4)$$

Thus $f = \psi_\alpha, \phi_\alpha$ is Loewner super-additive on rank one matrices in $\mathbb{P}_m$. By Theorem 3.3, the Loewner super-additive powers preserving positivity on rank 1 matrices are the same as the Loewner super-additive powers. We therefore conclude that $f$ is Loewner super-additive on all of $\mathbb{P}_m$. \hfill \Box
We now have all the ingredients to prove the main result of the paper.

**Proof of Theorem 4.1.** Before proving the result for all chordal graphs, let us prove it for the “nearly complete” graphs $K^{(1)}_r$. The result is obvious for $r = 2$. Now suppose $r \geq 3$. First note that $K_{r-1} \subset K_r^{(1)} \subset K_r$, so

$$2N \cup \{r - 2, \infty\} = H^\phi_{K_r} \subset H^\psi_{K_r^{(1)}} \subset H^\phi_{K_r-1} = 2N \cup \{r - 3, \infty\}.$$  

Similarly, we have $(-1 + 2N) \cup \{r - 2, \infty\} \subset H^\psi_{K_r^{(1)}} \subset (-1 + 2N) \cup \{r - 3, \infty\}$. Now label the vertices from 1 to $r$ such that $(1, r) \notin E(K_1^{(1)})$, and apply Corollary 3.4 with $A = \{1\}$, $B = \{r\}$, and $S = \{2, \ldots, r - 1\}$. It follows immediately that

$$H^\phi_{K_r^{(1)}} = 2N \cup \{r - 2, \infty\}, \quad H^\psi_{K_r^{(1)}} = (-1 + 2N) \cup \{r - 2, \infty\}.$$  

Finally, $H_{K_r} = N \cup \{r - 2, \infty\} \subset H_{K_r^{(1)}}$. To show the reverse inclusion, suppose $x^\alpha$ preserves $P^{(1)}_r([0, \infty))$. Given $u, v \in [0, \infty)^{r-2}$ and $M \in P_{r-2}([0, \infty))$, define $W(u, v, M)$ as in Equation (3.3). Then $W(u, v, uu^T + vv^T) \in P^{(1)}_r([0, \infty))$, so

$$W(u, v, uu^T + vv^T)^{\alpha \alpha} \in P^{(1)}_r([0, \infty)), \quad \forall u, v \in [0, \infty)^{r-2}.$$  

Proceeding as in Equation (3.4), we conclude that the entrywise function $x \mapsto x^\alpha$ is Loewner super-additive on rank one matrices in $P_{r-2}([0, \infty))$. Thus $\alpha \in \mathbb{N}$ or $\alpha \geq r - 2$ by Theorem 3.3. It follows that $H_{K_r^{(1)}} = N \cup \{r - 2, \infty\}$. This proves the theorem for $G = K_r^{(1)}$.

Now suppose $G$ is an arbitrary chordal graph, which without loss of generality we assume to be connected. Denote by $r$ the largest integer such that $G$ contains $K_r$ or $K_r^{(1)}$ as an induced subgraph. By the above calculation,

$$H_G \subset N \cup \{r - 2, \infty\}, \quad H^\psi_G \subset (-1 + 2N) \cup \{r - 2, \infty\}, \quad H^\phi_G = 2N \cup \{r - 2, \infty\}. \quad (3.5)$$  

We now prove the reverse inclusions. By Theorem 3.1, the maximal cliques of $G$ admit a perfect ordering $\{C_1, \ldots, C_k\}$. We will prove the reverse inclusions in (3.5) by induction on $k$. If $k = 1$, then $G$ is complete and the inclusions clearly hold by Theorem 1.2. Suppose the result holds for all chordal graphs with $k = l$ maximal cliques, and let $G$ be a graph with $k = l + 1$ maximal cliques. For $1 \leq j \leq k$, define

$$H_j := C_1 \cup \cdots \cup C_j, \quad C_j = C_j \setminus H_{j-1}, \quad S_j = H_{j-1} \cap C_j \quad (3.6)$$  

as in Equation (3.1). By [30] Lemma 2.11, the triplet $(H_{k-1}, S_k, R_k)$ is a decomposition of $G$. Let $r$ be the largest integer such that $G$ contains $K_r$ or $K_r^{(1)}$ as an induced subgraph, and let $\alpha \in \{r - 2, \infty\}$. By the induction hypothesis, the three $\alpha$-th power functions preserve positivity on $P_{G_{H_{k-1}}} = P_{G_{H_{k-1}}}$, and preserve positivity on $P_{G_{S_k}} = P_{G_{S_k}}$. We now claim that $\alpha \geq |S_k| + 2$. Clearly, $|S_k| \leq r$ since $S_k$ is complete. If $|S_k| = r$, then $C_k$ is contained in one of the previous cliques, which is a contradiction. Suppose instead that $|S_k| = r - 1$. Since $\{C_1, \ldots, C_k\}$ is a perfect sequence, $S_i \subset C_i$ for some $i < k$. Let $v \in C_i \setminus S_k$ and let $w \in R_k$. Note that both $v$ and $w$ are adjacent to every $s \in S_k$. Thus, the subgraph of $G$ induced by $S_k \cup \{v, w\}$ is isomorphic to $K^{(1)}_{r+1}$, which contradicts the definition of $r$. We therefore conclude that $\alpha \geq |S_k| + 2$, as claimed. As a consequence, the $\alpha$-th power functions are Loewner super-additive on $P_{S_k}$ by Theorem 3.3. Applying Theorem 3.2, we conclude that $\alpha \in H^\psi_G \cup H^\phi_G$. Since $H^\psi_G \cup H^\phi_G \subset H_G$, we obtain that $\alpha \in H_G$ as well. This concludes the proof of the theorem.

**Remark 3.6.** The critical exponent of a chordal graph $G$ can also be defined as $\max(c - 2, s)$, where $c$ is the largest clique size of $G$ and $s$ is the size of the largest separator associated to a perfect clique ordering of $G$ (see Equation (3.6)). This follows from the proof of Theorem 1.4 where
it was shown that if such a separator has size \( s \), then either \( s \leq c - 2 \) or \( G \) contains \( K_{s+2}^{(1)} \) as an induced subgraph. The critical exponent can also be computed by replacing \( s \) by the size of the largest intersection of two maximal cliques, as shown in Corollary 3.7 below.

We now mention several consequences of the above analysis in this section. The following corollary provides a formula that can be used to systematically compute the critical exponent of a chordal graph.

**Corollary 3.7.** Suppose \( G = (V, E) \) is chordal with \( V = \{v_1, \ldots, v_m\} \), and let \( C_1, \ldots, C_n \) denote the maximal cliques in \( G \). Define the \( m \times n \) “maximal clique matrix” \( M(G) \) of \( G \) to be \( M(G) := (1(v_i \in C_j)) \), i.e.,

\[
M(G)_{ij} = \begin{cases} 
1 & \text{if } v_i \in C_j, \\
0 & \text{otherwise}.
\end{cases}
\]

Let \( u_1, \ldots, u_n \in \{0, 1\}^m \) denote the columns of \( M(G) \). Then the critical exponent of \( G \) is given by

\[
CE_H(G) = CE_H^\psi(G) = CE_H^\phi(G) = \max_{i,j} (u_i^T u_j - 2\delta_{i,j}),
\]

(3.7)

i.e., the largest entry of \( (M(G))^T M(G) - 2I_{|V|} \).

**Proof.** Let \( c \) and \( s \) denote the maximum of the diagonal and off-diagonal entries of \((M(G))^T M(G) - 2I_{|V|}\) respectively. Clearly, \( c \) is the size of the maximal cliques of \( G \) minus 2 and \( s = \max_{i \neq j} |C_i \cap C_j| \). By Theorem 3.1, the cliques of \( G \) admit a perfect ordering, say, \( C_1, \ldots, C_n \). For \( i \neq j \), let \( k, l \) be such that \( C_i = C_k \) and \( C_j = C_l \). Without loss of generality, assume \( i_k < i_l \). Then \( C_i \cap C_j = C_k \cap C_l \cap H_{i_k-1} \cap C_i = S_i \), where our notation is as in Equation (3.6). Thus, \( s \leq \max_{j=1,\ldots,n} |S_j| \). Conversely, since \( C_1, \ldots, C_n \) is a perfect ordering, for every \( 1 \leq j \leq n \), we have \( S_j \subset C_{i_j} \cap C_{i_j'} \) for some \( i_j < i_j' \). Thus, \( S_j \subset C_{i_j} \cap C_{i_j'} \), and so \( s \geq \max_{j=1,\ldots,n} |S_j| \). It follows that \( s \) corresponds to the order of the largest separator in the perfect ordering of the cliques of \( G \). We conclude by Theorem 1.4 and Remark 3.6 that the critical exponents of \( G \) correspond to the maximal entry of \( (M(G))^T M(G) - 2I_{|V|} \).

For completeness, we remark that Theorem 3.2 also has the following consequence for general entrywise maps. The proof is similar to that of Theorem 1.4.

**Corollary 3.8.** Let \( G \) be a chordal graph with a perfect maximal clique ordering \( \{C_1, \ldots, C_k\} \). Define

\[
c := \max_{i=1,\ldots,k} |C_i|,
\]

\[
s := \max_{i=1,\ldots,k} |S_i|,
\]

where \( S_i \) is defined as in Equation (3.6). If \( f : \mathbb{R} \to \mathbb{R} \) is such that \( f[-] \) preserves positivity on \( \mathbb{P}_{K_c} \) and is Loewner super-additive on \( \mathbb{P}_{K_s} \), then \( f[-] \) preserves positivity on \( \mathbb{P}_G \).

Note that Corollary 3.8 uses a clique ordering of the vertices of a chordal graph \( G \). A natural parallel approach in studying functions preserving positivity is to build the graph \( G \) step by step by using a perfect ordering of the vertices. The following proposition formalizes this procedure.

**Definition 3.9.** Given a graph \( G \) on a vertex set \( V \), denote by \( N(v) \) the neighborhood of a vertex \( v \in V \), i.e., \( N(v) = \{w \in V : (v, w) \in E\} \). A vertex \( v \in V \) is said to be simplicial if \( N(v) \cup \{v\} \) is complete. An ordering \( \{v_1, \ldots, v_n\} \) of the vertices of \( V \) is said to be a perfect elimination ordering if for all \( i = 1, \ldots, n \), \( v_i \) is simplicial in the subgraph of \( G \) induced by \( \{v_1, \ldots, v_i\} \).

**Proposition 3.10.** Let \( G \) be a chordal graph with a perfect elimination ordering of its vertices \( \{v_1, \ldots, v_n\} \). For all \( 1 \leq k \leq n \), denote by \( G_k \) the induced subgraph on \( G \) formed by \( \{v_1, \ldots, v_k\} \), so that the neighbors of \( v_k \) in \( G_k \) form a clique. Define \( c \) to be the size of a maximal clique in \( G \), and

\[
d := \max_{k=1,\ldots,n} \deg_{G_k}(v_k).
\]
If \( f : \mathbb{R} \to \mathbb{R} \) is any function such that \( f[-] \) preserves positivity on \( \mathbb{P}^1_{K_c} (\mathbb{R}) \) and \( f[M + N] \geq f[M] + f[N] \) for all \( M \in \mathbb{P}_d (\mathbb{R}) \) and \( N \in \mathbb{P}_d (\mathbb{R}) \), then \( f[-] \) preserves positivity on \( \mathbb{P}_G (\mathbb{R}) \).

As an illustration, if \( G \) is a tree, then \( c = 2 \) and \( d = 1 \). Thus the result extends [20] Theorem A to arbitrary chordal graphs, with weakened hypotheses.

**Proof.** First note that \( f(0) = 0 \) since \( f \) is nonnegative and super-additive on \([0, \infty)\) by assumption. We now prove the result for \( G_k \) by induction on \( k \). Clearly the result holds for \( k = 1 \). Now suppose the result holds for \( k \). Assume without loss of generality that the neighbors of \( v_{k+1} \in V(G) \) are \( v_1, \ldots, v_l \) for some \( 1 \leq l \leq k \), which are all adjacent to one another. Now write a matrix \( A \in \mathbb{P}_{G_{k+1}} (\mathbb{R}) \) in the following block form, and also define an associated matrix \( U(A) \):

\[
A = \begin{pmatrix} P & Q & u \\ Q^T & R & 0 \\ u^T & 0^T & a \end{pmatrix}, \quad U(A) := \begin{pmatrix} a^{-1}uu^T & u \\ u^T & a \end{pmatrix},
\]

where \( Q \) is \( l \times (k-l) \), and we may assume that \( a > 0 \). Note that if \( f(a) = 0 \), then applying \( f \) entrywise to the submatrix \( U(A) \oplus 0_{(k-l) \times (k-l)} \in \mathbb{P}_{G_k} (\mathbb{R}) \) (by abuse of notation) shows that \( f[u] = 0 \). Hence \( f[A] \in \mathbb{P}_{G_{k+1}} (\mathbb{R}) \) by the induction hypothesis for \( G_k \).

Now suppose \( f(a) > 0 \). It suffices to show that the Schur complement \( S_{f[A]} \) of \( f[A] \) with respect to \( f(a) \) is also positive semidefinite. Note that the Schur complement \( S_A \) of \( A \) with respect to \( a \) belongs to \( \mathbb{P}_{G_{k+1}} (\mathbb{R}) \). Therefore by the induction hypothesis, \( f[S_A] \) is also positive semidefinite. Thus it suffices to show that \( S_{f[A]} - f[S_A] \geq 0 \). Now compute:

\[
S_{f[A]} - f[S_A] = \begin{pmatrix} f[P] - f(a)^{-1}f[u]f[u]^T - f[P - a^{-1}uu^T] & 0 \\ 0^T & 0 \end{pmatrix}.
\]

Next, note that \( c \geq l+1 \) since the subgraph of \( G \) induced by \( \{v_1, \ldots, v_l, v_{k+1}\} \) is complete. Moreover, since \( U(A) \in \mathbb{P}_{G_k} (\mathbb{R}) \), it follows by the assumptions on \( f \) that the Schur complement of the last entry in \( f[U(A)] \) is positive semidefinite, i.e.,

\[
f[U(A)] = \begin{pmatrix} f[a^{-1}uu^T] & f[u] \\ f[u]^T & f(a) \end{pmatrix} \geq 0 \implies f[a^{-1}uu^T] \geq \frac{f[u]f[u]^T}{f(a)}.
\]

Furthermore, \( f(0) = 0 \) and \( l = \deg_{G_k} v_k \leq d \). Hence \( f[M + N] \geq f[M] + f[N] \) for all \( M \in \mathbb{P}_l (\mathbb{R}) \) and \( N \in \mathbb{P}_l (\mathbb{R}) \). Set \( N := a^{-1}uu^T \) and \( M := P - N \), and compute using the above analysis:

\[
f[P] - f(a)^{-1}f[u]f[u]^T - f[P - a^{-1}uu^T] \geq f[P] - f[a^{-1}uu^T] - f[P - a^{-1}uu^T] \geq 0.
\]

It follows that \( S_{f[A]} \geq f[S_A] \geq 0 \), whence \( f[A] \in \mathbb{P}_{G_{k+1}} (\mathbb{R}) \) as claimed. This completes the induction step. The result now follows by setting \( k = n \).

We now study how the set of powers preserving positivity on \( \mathbb{P}_G \) can be related to the corresponding set of powers for \( \mathbb{P}_{G/\nu} \), for arbitrary graphs \( G \).

As an illustration of Theorem 1.3, we compute in Corollary 3.11 the critical exponents of well-known chordal graphs explicitly. Recall that an Apollonian graph is a planar graph formed from a triangle graph by iteratively adding an interior point as vertex, and connecting it to all three vertices of the smallest triangle subgraph in whose interior it lies. A graph is outerplanar if every vertex of the graph lies in the unbounded face of the graph in a planar drawing. An outerplanar graph is maximal if adding an edge makes it non-outerplanar. A graph \( G = (V,E) \) is split if its vertices can be partitioned into a clique \( C \) and an independent subset \( V \setminus C \). Finally, the band graph with \( n \) vertices \( \{1, \ldots, n\} \) and bandwidth \( d \) is the graph where \((i,j) \in E \) if and only if \( i \neq j \) and \( |i-j| \leq d \). For references, see e.g. [3 11 15 22].

**Corollary 3.11.** The critical exponents of some important chordal graphs are given in Table 1.
Table 1. Critical exponents of important families of chordal graphs with \( n \) vertices.

| Graph \( G \)                                              | \( CE_H(G), CE_H^{\psi}(G), CE_H^{\phi}(G) \) |
|-----------------------------------------------------------|--------------------------------------------------|
| Tree                                                     | \( {1} \)                                         |
| Complete graph \( K_n \)                                 | \( n - 2 \)                                       |
| Minimal planar triangulation of \( C_n \) for \( n \geq 4 \) | \( 2 \)                                           |
| Apollonian graph, \( n \geq 3 \)                         | \( \min(3, n - 2) \)                             |
| Maximal outerplanar graph, \( n \geq 3 \)                | \( \min(2, n - 2) \)                             |
| Band graph with bandwidth \( d \leq n \)                 | \( \min(d, n - 2) \)                             |
| Split graph with maximal clique \( C \)                   | \( \max(|C| - 2, \max \deg(V \setminus C)) \) |

Proof. We will prove the result only for band graphs. First, if \( n = d, d + 1 \), then \( G = K_d, K^{(1)}_{d+1} \) respectively, and so the critical exponents are \( d - 2 \) and \( d - 1 \), which shows the result. Now suppose \( n \geq d + 2 \). The maximal cliques of \( G \) are

\[
C_l := \{l, l + 1, \ldots, l + d\} \quad 1 \leq l \leq n - d.
\]

It is not difficult to verify that this enumeration of the maximal cliques of \( G \) is perfect. The largest clique has size \( d + 1 \) and the largest separator (as defined in Equation (3.6)) has size \( d \). It follows from Theorem 1.4 (see Remark 3.6) that the three critical exponents of \( G \) are equal to \( d \). \( \square \)

Remark 3.12. Another important family of chordal graphs that is widely used in applications is the family of interval graphs [15, Chapter 8]. Given a family \( V \) of intervals in the real line, the corresponding interval graph has vertex set equal to \( V \), and two vertices are adjacent if the corresponding intervals intersect. Interval graphs are known to be chordal; moreover, to compute their critical exponents we define the height function at \( x \in \mathbb{R} \) to be the number of intervals containing \( x \). It is standard that the maximal cliques correspond precisely to the intervals containing the local maxima of the height function; see e.g. [32, Section 2]. The critical exponent of interval graphs can then be easily computed by using Corollary 3.7.

Note that every non-chordal graph \( G \) is contained in a minimal triangulation \( G_{\Delta} \). This triangulation immediately provides an upper bound on the critical exponents for preserving positivity for \( G \). A lower bound is provided by \( r - 2 \), where \( r \) is the size of the largest clique in \( G \). The critical exponents of some non-chordal graphs are studied in more detail in Section 4.

4. Non-chordal graphs

In the remainder of the paper, we discuss power functions preserving positivity on \( \mathbb{P}_G \) for graphs \( G \) that are not chordal. We begin by extending Corollary 3.8 to general graphs. Recall that a decomposition of a graph \( G = (V, E) \) is a partition \( (A, C, B) \) of \( V \), where \( C \) separates \( A \) from \( B \) (i.e., every path from a vertex \( a \in A \) to a vertex \( b \in B \) contains a vertex in \( C \)), and \( G_C \) is complete. A graph is said to be prime if it admits no such decomposition. For example, every cycle is prime. A decomposition separates a graph into two components \( G_{A \cup C} \) and \( G_{B \cup C} \). Iterating this process until it cannot be performed anymore produces prime components of the graph \( G \). The resulting prime components can be ordered to form a perfect sequence (as defined after Equation (3.1)) - see [7, 35]. When \( G \) is chordal, its prime components are all complete. Conversely, if the prime components of a graph are complete, the graph is chordal by [9] (see also [7, Theorem 3.1]).

Theorem 4.1. Let \( G \) be a graph with a perfect ordering \( \{B_1, \ldots, B_k\} \) of its prime components, and let \( f : \mathbb{R} \to \mathbb{R} \) be such that \( f(0) = 0 \). Define

\[
s := \max_{i=1, \ldots, k} |S_i|,
\]
where $S_i$ is defined as in Equation 3.1. If $f[-]$ preserves positivity on $\mathbb{P}_{B_i}$ for all $1 \leq i \leq k$ and is Loewner super-additive on $\mathbb{P}_K$, then $f[-]$ preserves positivity on $\mathbb{P}_G$.

Proof. The proof is the same as the proof of Theorem 1.4.

As a consequence, we immediately obtain the following corollary.

**Corollary 4.2.** In the notation of Theorem 4.4, let $\alpha > 0$ and let $f = \psi_\alpha$ or $f = \phi_\alpha$. Suppose $f[-]$ preserves positivity on $\mathbb{P}_{B_i}(\mathbb{R})$ for every prime component $B_i$ of $G$, and $\alpha \geq |S_i|$ for all $i$. Then $f$ preserves Loewner positivity on $\mathbb{P}_G(\mathbb{R})$.

A natural question of interest is thus to determine the critical exponents of prime graphs, and other simple non-chordal graphs. In the next two subsections, we examine the case of cycles and of bipartite graphs. Along the way, we develop general techniques to compute critical exponents of graphs, including studying the Schur complement of a graph, and pasting paths to graphs. We conclude the paper by constructing many more examples of non-chordal graphs for which the critical exponent can be obtained explicitly by forming coalescences of graphs.

### 4.1. Cycles, Schur complements, and path addition

We begin by proving that for even cycles, the critical exponents $CE_H(G), CE_H^\psi(G), CE_H^\phi(G)$ are not all equal, which is unlike the case of chordal graphs.

**Proposition 4.3.** For all $n \geq 3$,

$$H_{C_n} = H_{C_n}^\psi = [1, \infty), \text{ and } H_{C_n}^\phi = [2, \infty).$$

Moreover, for $n > 4$, $[2, \infty) \subset H_{C_n}^\phi \subset [1, \infty)$, with $1 \notin H_{C_n}^\phi$ for $n$ even.

We prove Proposition 4.3 in this section. Along the way, we describe various constructions on a graph under which the Hadamard critical exponents can be controlled. The first of these constructions is termed the Schur complement, and generalizes the pendant edge construction in Theorem 2.2, which shows that the $H$-sets do not change when a tree is pasted on a vertex of a graph.

**Definition 4.4.** Let $G = (V, E)$ be a graph and let $v \in V$. Define the Schur complement graph of $G$ with respect to $v$, denoted $G/v$, to be the graph $G = (V \setminus \{v\}, E')$, where $(i, j) \in E'$ if and only if one of the following condition holds:

1. $(i, j) \in E \cap (V \setminus \{v\}) \times (V \setminus \{v\});$
2. $(i, v) \in E$ and $(j, v) \in E$.

For instance, the Schur complement of any vertex in a path $P_n$ for $n > 2$, cycle $C_n$ for $n > 3$, or complete graph $K_n$ for $n > 2$, is $P_{n-1}, C_{n-1}, K_{n-1}$ respectively.

The definition of the graph Schur complement is designed to be compatible with the Schur complement of a matrix in $\mathbb{P}_G$. Namely, if we take the Schur complement of $A \in \mathbb{P}_G$ with respect to its $(v, v)$-th entry which is positive, then the resulting matrix is in $\mathbb{P}_{G/v}$. This makes the construction a very relevant one, as Schur complements provide a crucial tool for computing Hadamard critical exponents. For example, the following result relates $H_{G}^\psi$ and $H_{G/v}^\psi$ for vertices $v \in V(G)$ with independent neighbors.

**Theorem 4.5.** Suppose $G = (V, E)$ is not a disjoint union of copies of $K_2$, and $v \in V$. Then $1 + H_{G/v} \subset H_{G}$. Suppose $v$ has $k > 1$ neighbors in $G$, and they are independent. Then $H_{G/v}^\psi \subset H_{G}^\psi \subset H_{G} \subset H_{G}^\phi$. 

Indeed, this difference is a diagonal matrix with diagonal entries \( \alpha \) complement of \( p \) for suitable \( \psi \).

**Definition 4.6.** Fix a graph \( G \) connecting two non-adjacent vertices in \( G \).

The following useful result is a consequence of Theorem 4.5.

**Corollary 4.7.** Suppose \( G = (V, E) \) is not a disjoint union of copies of \( K_2 \), and \( v_1, v_2 \in V \). Then \( \mathcal{H}_{G_2(v_1, v_2)} \subset \mathcal{H}_{G_3(v_1, v_2)} \subset \mathcal{H}_{G_4(v_1, v_2)} \subset \cdots \subset \mathcal{H}_{G} \); moreover, \( \mathcal{H}_{G_1(v_1, v_2)} \subset \mathcal{H}_{G_2(v_1, v_2)} \) if \( v_1, v_2 \) are not adjacent.

**Proof.** First note that \( \mathcal{H}_{G_m(v_1, v_2)} \subset \mathcal{H}_{G} \) since \( G \subset G_m(v_1, v_2) \) for all \( m \geq 1 \). We first show that \( \mathcal{H}_{G_1(v_1, v_2)} \subset \mathcal{H}_{G_2(v_1, v_2)} \) if \( v_1, v_2 \) are not adjacent in \( G \). Suppose \( G \) has \( n+2 \) vertices, with \( v_i = n + i \) for \( i = 1, 2 \). Let the additional path of edge-length \( m \) connecting \( v_1, v_2 \). Then by Theorem 4.5

\[
\mathcal{H}_{G_1(v_1, v_2)} = \mathcal{H}_{G_2(v_1, v_2)}/v \subset \mathcal{H}_{G_2(v_1, v_2)}. \tag{4.3}
\]
Now fix $m \in \mathbb{N}$ and let $G'_m(v_1)$ be the graph obtained by attaching a path of edge-length $m \in \mathbb{N}$ at one end to $v_1$, and leaving the other end free/pendant. Let $w_m$ be the free vertex with $w_0 := v_1$; then $G_{m+1}(v_1,v_2) = G_1(G'_m(v_1), w_m, v_2)$. Hence by Equation (4.3),
\[
\mathcal{H}^\psi_{G_{m+1}(v_1,v_2)} = \mathcal{H}^\psi_{G_1(G'_m(v_1), w_m, v_2)} \subset \mathcal{H}^\psi_{G_2(G'_m(v_1), w_m, v_2)} = \mathcal{H}^\psi_{G_{m+2}(v_1,v_2)}.
\]

It is now possible to obtain information about the critical exponents for cycle graphs.

**Proof of Proposition 4.3.** For $n = 3$ the result is clear from Theorem 1.2, since $C_3 = K_3$. We compute for $n \geq 4$ and any vertex $v_m \in C_m$ (for $m \leq n$), using Theorems 1.2, 2.2, and 4.5:
\[
[1, \infty) = \mathcal{H}_{P_3} \supset \mathcal{H}_{C_n} \supset \mathcal{H}^\psi_{C_n} \supset \mathcal{H}^\psi_{C_{n-1}} \supset \cdots \supset \mathcal{H}^\psi_{C_3} = [1, \infty).
\]
It follows that $\mathcal{H}_{C_n} = \mathcal{H}^\psi_{C_n} = [1, \infty)$ for $n \geq 3$.

We now compute $\mathcal{H}^\psi_{C_4}$. Note that the matrix $A := (\cos((j-k)\pi/4))_{j,k=1}^4$, which was well-studied in [4, 19], lies in $\mathbb{P}_{C_4}(\mathbb{R})$. It was further shown in [4] that $\phi_\alpha(A) \notin \mathbb{P}(\mathbb{R})$ for $\alpha \in (0,2)$. Thus $\mathcal{H}^\psi_{C_4} \subset [2, \infty)$. On the other hand, $\mathcal{H}^\psi_{C_4} \supset \mathcal{H}^\psi_{K_4} = [2, \infty)$ by Theorem 1.2, which shows the result for $C_4$. Next for general $n$, $\mathcal{H}^\psi_{C_n} \subset [1, \infty)$ by Theorem 2.2, since $C_n \supset P_3$. On the other hand, observe that
\[
|CE^\psi_{H}(G) - CE^\psi_{H}(G)| \leq 1 \tag{4.4}
\]
for all graphs $G$, because $\psi_\alpha(x) = x\phi_{\alpha-1}(x)$ and $\phi_\alpha(x) = x\psi_{\alpha-1}(x)$ for $\alpha \in \mathbb{R}$, so that $1 + \mathcal{H}^\psi_{G} \subset \mathcal{H}^\psi_{G}$ by the Schur product theorem, and similarly for $\mathcal{H}^\psi_{G}$. Applying Equation (4.4) for $G = C_n$, it follows that $[2, \infty) \subset \mathcal{H}^\psi_{C_n}$ by the Schur product theorem, since $\phi_{\alpha+1}(x) = x \cdot \psi_\alpha(x)$ and $[1, \infty) \subset \mathcal{H}^\psi_{C_n}$. Finally, observe using [10, Example 5.2] that $1 \notin \mathcal{H}^\psi_{C_2}$, for $n \geq 2$, whence $CE^\psi_{H}(C_2) > 1$.

**Remark 4.8.** The same analysis as above leads us to conclude that if $G_n$ is the graph on $2n$ vertices with only the “diameter” edges $(1, n+1), \ldots, (n, 2n)$ missing, then $\mathcal{H}^\psi_{G_n} = 2\mathbb{N} \cup [2n-2, \infty)$. (Note here that $G_n$ is not chordal - e.g., the induced subgraph on the vertices $\{1, 2, n, n+1\}$ is $C_4$.) This assertion is proved using the properties of the matrix $A_n := (\cos((j-k)\pi/2n))_{j,k=1}^{2n} \in \mathbb{P}_{G_n}$, which were explored in [4, 19]. In particular, $\mathcal{H}_{G_n} = \mathbb{N} \cup [2n-2, \infty)$ and $CE^\psi_{H}(G_n) \in [2n-3, 2n-2]$. For completeness we also point out that the matrices with entries $\cos(\alpha_j - \alpha_k)$ are rank 2 correlation matrices, and in fact, are the set of extreme points of rank 2 in the space of all correlation matrices, by [11].

Proposition 4.3 allows us to strengthen Corollary 4.7 in the particular case where $G_0 = K_4^{(1)}$ or $K_4$.

**Proposition 4.9.** Suppose $H_0 = K_4^{(1)}$ or $K_4$. Now given a graph $H_m$ for $m \geq 0$ and an integer $n_{m+1} \geq 3$, create a new graph $H_{m+1}$ by attaching a cycle $C_{n_{m+1}}$ to $H_m$ along any common edge. Then for all $m \geq 0$,
\[
\mathcal{H}_{G_m} = \mathbb{N} \cup [2, \infty), \quad \mathcal{H}^\psi_{G_m} = (-1 + 2\mathbb{N}) \cup [2, \infty), \quad \mathcal{H}^\phi_{G_m} = 2\mathbb{N} \cup [2, \infty).
\]
In particular, $CE^\psi_{H}(H_m) = CE^\psi_{H}(H_m) = CE^\phi_{H}(H_m) = 2$ for all $m \geq 0$.

**Proof.** First note using Theorems 1.2 and 1.4 that $\mathcal{H}_{H_m} \setminus \{1\}, \mathcal{H}^\psi_{H_m} \setminus \{1\}, \mathcal{H}^\phi_{H_m} \subset [2, \infty)$ for all $m \geq 0$. To show that $[2, \infty)$ is contained in the three $\mathcal{H}$-sets we use induction on $m \geq 0$. The result clearly holds for $H_0$ by Theorems 1.2 and 1.4. Now assume the result holds for $H_m$, and suppose $C_{n_{m+1}}$ is attached to $H_m$ along the common edge $(1, 2)$ (without loss of generality). Let $A := V(H_m) \setminus \{1, 2\}$, $C := \{1, 2\}$, and $B := V(C_{n_{m+1}}) \setminus \{1, 2\}$. where $V(H_m), V(C_{n_{m+1}})$ denote the vertex sets of $H_m$ and $C_{n_{m+1}}$ respectively. For every $\alpha \geq 2$, the maps $\psi_\alpha, \phi_\alpha$ preserve positivity on $\mathbb{P}_{H_m}(\mathbb{R})$ by the
Proof. Step 1: Complete bipartite graphs. We begin by proving that the complete bipartite graph $K_{n,n}$ satisfies: $\mathcal{H}_{K_{n,n}} = [1, \infty)$ for all $n \geq 2$. Indeed, $P_3 \subset K_{n,n}$ since $n \geq 2$, so we conclude via Theorem 2.2 that $\mathcal{H}_{K_{n,n}} \subset \mathcal{H}_{P_3} = [1, \infty)$. To show the reverse inclusion, let $\alpha > 0$, $m, n \in \mathbb{N}$, and let

$$A = \begin{pmatrix} D_{m \times m} & X_{m \times n} \\ X^T & D_{n \times n} \end{pmatrix} \in \mathbb{P}_{K_{m,n}}([0, \infty]),$$

with $\max(m, n) > 1$, and where $D, D'$ are diagonal matrices. Given $\epsilon > 0$, define the matrix

$$X_{D, D'}(\epsilon, \alpha) := (D + \epsilon \text{Id}_m)^{\alpha(-\alpha/2)} \cdot X^{\alpha} \cdot (D' + \epsilon \text{Id}_n)^{\alpha(-\alpha/2)}.$$

Also observe that for all block diagonal matrices $A$ of the above form and all $\epsilon, \alpha > 0$,

$$(A + \epsilon \text{Id}_{m+n})^{\alpha} = \text{Diag} \left( \begin{pmatrix} \text{Id}_m \\ X_{D, D'}(\epsilon, \alpha)^T \\ \text{Id}_n \end{pmatrix}, \begin{pmatrix} X_{D, D'}(\epsilon, \alpha) \\ \text{Id}_n \\ (D' + \epsilon \text{Id}_n)^{\alpha/2} \end{pmatrix} \right).$$

We now compute for $\alpha, \epsilon > 0$:

$$(A + \epsilon \text{Id}_{m+n})^{\alpha} \in \mathbb{P}_{K_{m,n}}([0, \infty))$$

$$\iff \begin{pmatrix} \text{Id}_m \\ X_{D, D'}(\epsilon, \alpha)^T \\ \text{Id}_n \end{pmatrix} \in \mathbb{P}_{K_{m,n}}([0, \infty))$$

$$\iff \text{Id}_m - X_{D, D'}(\epsilon, \alpha)X_{D, D'}(\epsilon, \alpha)^T \in \mathbb{P}_m(\mathbb{R})$$

$$\iff \|u\| \geq \|X_{D, D'}(\epsilon, \alpha)^T u\|, \forall u \in \mathbb{R}^n$$

$$\iff \sigma_{\max}(X_{D, D'}(\epsilon, \alpha)) \leq 1,$$

where $\sigma_{\max}$ denotes the largest singular value. Now note that if $m = n$, then the above calculation shows that $(A + \epsilon \text{Id}_m)^{\alpha} \in \mathbb{P}_{K_{m,n}}([0, \infty))$ if and only if $\rho(X_{D, D'}(\epsilon, \alpha)) \leq 1$, where $\rho$ denotes the spectral radius.

4.2. Bipartite graphs. Another commonly encountered family of non-chordal graphs are the bipartite graphs. We now examine the critical exponents of these graphs.

Theorem 4.10. Suppose $G$ is a connected bipartite graph with at least 3 vertices. Then,

$$\mathcal{H}_G = [1, \infty), \quad [2, \infty) \subset \mathcal{H}^\psi_G \subset [1, \infty), \quad \{1\} \cup [3, \infty) \subset \mathcal{H}^\psi_G \subset [1, \infty).$$

If moreover $K_{2,2} \subset G \subset K_{2,m}$ for some $m \geq 2$, then $\mathcal{H}_G^\psi = [2, \infty)$ and $\{1\} \cup [2, \infty) \subset \mathcal{H}^\psi_G \subset [1, \infty)$.

Theorem 4.10 has a very surprising conclusion: it shows that broad families of dense graphs such as complete bipartite graphs have small critical exponents that do not grow with the number of vertices of the graphs. As a consequence, small entrywise powers of a positive semidefinite matrix with such a structure of zeros preserves positivity. This is important since such procedures are often used to regularize positive definite matrices (e.g. covariance/correlation matrices), where the goal is to minimally modify the entries of the original matrix. Note that such a result is in sharp contrast to the general case where there is no underlying structure of zeros.
To finish this first step of the proof, now suppose \( \alpha \geq 1 \) and \( A \in \mathbb{P}_{K_{n,n}}([0, \infty)) \). Then \( A + \epsilon \text{Id} \in \mathbb{P}_{K_{n,n}}((0, \infty)) \) for all \( 0 < \epsilon \ll 1 \), so by the above analysis with \( \alpha = 1 \), \( \rho(X_{D,D'}(\epsilon, 1)) \leq 1 \) for all \( 0 < \epsilon \ll 1 \). Applying \cite[Lemma 5.7.8]{28} implies that

\[
\rho(X_{D,D'}(\epsilon, \alpha)) \leq \rho(X_{D,D'}(\epsilon, 1))^\alpha \leq 1.
\]

It follows from the above analysis and the continuity of entrywise powers that \( A^{\alpha} \in \mathbb{P}_{K_{n,n}}([0, \infty)) \). Thus \([1, \infty) \subseteq \mathcal{H}_{K_{n,n}}\).

**Step 2: General bipartite graphs.** We now prove the result for a general bipartite graph. Suppose \( G = (V, E) \) is any connected bipartite graph on \( m, n \) vertices, with \( m + n = |V| \geq 3 \) and \( n \geq m \). Then by the previous step,

\[
P_3 = K_{2,1} \subseteq G \subseteq K_{n,n} \implies [1, \infty) \subseteq \mathcal{H}_G \subseteq \mathcal{H}_{K_{n,n}} = [1, \infty),
\]

which shows that \( \mathcal{H}_G = [1, \infty) \). Next, suppose \( \alpha \geq 2 \) and \( A \in \mathbb{P}_{G}(\mathbb{R}) \). Then \( A^{\alpha} = A \circ A \in \mathbb{P}_{G}([0, \infty)) \) by the Schur product theorem, so by the previous assertion, \( \phi_A[A] = (A \circ A)^{\alpha/2} \in \mathbb{P}_{G}([0, \infty)) \). It follows immediately that \([2, \infty) \subseteq \mathcal{H}_G^\phi \subseteq [1, \infty)\). In turn, this implies via Equation (4.4) that \( \{1\} \cup [3, \infty) \subseteq \mathcal{H}^\phi_{G} \subseteq [1, \infty)\).

To conclude the proof, suppose further that \( C_4 = K_{2,2} \subseteq G \subseteq K_{2,m} \). To study \( \mathcal{H}_G \) we will use the family of split graphs \( K_{K_2,m} \) for \( m \geq 2 \). These are chordal graphs with \( m + 2 \) vertices, with vertices \( m + 1, m + 2 \) connected to every other vertex (and to each other). By Theorem 1.4, Proposition 4.3 and the definition of \( G \), we obtain

\[
\{1\} \cup [2, \infty) = \mathcal{H}^\phi_{K_{K_2,m}} \subseteq \mathcal{H}_G^\phi \subseteq \mathcal{H}^\phi_{C_4} = [1, \infty), \quad [2, \infty) = \mathcal{H}^\phi_{K_{2,m}} \subseteq \mathcal{H}_G^\phi \subseteq \mathcal{H}^\phi_{C_4} = [2, \infty).
\]

This concludes the proof. \( \square \)

### 4.3. Coalescences

In this concluding section, we show how many more examples of non-chordal graphs can be constructed by forming coalescence of graphs. Recall that the coalescences of two graphs \( G_1, G_2 \) is the graph obtained from their disjoint union \( G_1 \bigcup G_2 \) by identifying a vertex from both of them \cite{17, 39}. We now discuss how to extend the proof-strategy of Theorem 3.2 to such graphs.

**Proposition 4.11** (Coalescence graphs). Suppose \( G_1, \ldots, G_k \) are connected graphs with at least one edge each, and \( G \) is any coalescence of \( G_1, \ldots, G_k \) for some \( k > 1 \). Also suppose \( f : \mathbb{R} \to \mathbb{R} \) with \( f(0) = 0 \). Then \( f[-] \) preserves positivity on \( \mathbb{P}_G(\mathbb{R}) \) if and only if:

1. \( f[-] \) preserves positivity on each \( \mathbb{P}_{G_i}(\mathbb{R}) \), and
2. \( f \) is continuous and super-additive on \([0, \infty)\).

In particular for any \( \alpha \in \mathbb{R} \), the power function \( \psi_\alpha \) or \( \phi_\alpha \) preserves positivity on \( \mathbb{P}_G(\mathbb{R}) \) if and only if it does so on \( \mathbb{P}_{G_i}(\mathbb{R}) \) for all \( 1 \leq i \leq k \) and \( \alpha \geq 1 \). In other words,

\[
\mathcal{H}_G^\psi = [1, \infty) \cap \bigcap_{i=1}^{k} \mathcal{H}_{G_i}^\psi, \quad \mathcal{H}_G^\phi = [1, \infty) \cap \bigcap_{i=1}^{k} \mathcal{H}_{G_i}^\phi.
\]

**Remark 4.12.**

1. Note that the characterization provided by Proposition 4.11 is independent of which nodes in the graphs \( G_i \) are identified with one another.
2. Also observe that when \( k = 2 \) and \( G_2 = K_2 \), the resulting graph \( G \) in Proposition 4.11 is the graph \( G_1 \) with one pendant edge added. Proposition 4.11 therefore implies the conclusion of Theorem 2.2.

The proof of Proposition 4.11 relies on a stronger result that is akin to Theorem 3.2 but holds for arbitrary graphs:
Theorem 4.13. Let $G = (V, E)$ be a nonempty graph and let $(A, C, B)$ be a partition of $V$ where $C$ separates $A$ from $B$. Also let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(0) = 0$. Suppose $f[\cdot]$ preserves positivity on $\mathbb{P}_G$. Then

1. $f[\cdot]$ preserves positivity on $\mathbb{P}_{A \cup C}$ and on $\mathbb{P}_{B \cup C}$;
2. $f$ is continuous on $[0, \infty)$, and
3. $f[\cdot]$ is Loewner super-additive on $\mathbb{P}_m(\mathbb{R})$, whenever there exist $A' \subset A, C' \subset C, B' \subset B$ such that $G_{A'}, G_{C'}, G_{B'}$ are cliques of size $m$, and every vertex of $A'$ and $B'$ is connected to every vertex in $C'$.

Note that the converse to Theorem 4.13 was proved in Theorem 3.2(1).

Proof. Clearly, $f[\cdot]$ preserves positivity on $\mathbb{P}_{A \cup C}$ and on $\mathbb{P}_{B \cup C}$ since they are induced subgraphs of $G$. Also $f$ preserves positivity on $\mathbb{P}_{K_2}(\mathbb{R}) = \mathbb{P}_2(\mathbb{R})$, whence $f$ is continuous on $(0, \infty)$ by [27, Theorem 1.2]. Moreover, $f$ is right-continuous at $0$ as shown at the beginning of the proof of [20, Theorem C]. Now, write the vertices of $G$ in the order $A', C', B', V \setminus (A' \cup C' \cup B')$, and consider the matrices

$$
\mathcal{M}_1(N) := \begin{pmatrix}
N & N & 0 & 0 \\
N & N & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
\mathcal{M}_2(N) := \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & N & N & 0 \\
0 & N & N & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
N \in \mathbb{P}_m(\mathbb{R}).
$$

Clearly $\mathcal{M}_1(N) \in \mathbb{P}_{G_{A \cup C}}$ and $\mathcal{M}_2(N) \in \mathbb{P}_{G_{B \cup C}}$ for all $N \in \mathbb{P}_m(\mathbb{R})$. Thus given any $N_1, N_2 \in \mathbb{P}_m(\mathbb{R})$ and $\epsilon > 0$, it follows that $f[\mathcal{M}_1(N_1) + \mathcal{M}_2(N_2)] + \mathcal{M}_1[\epsilon \text{Id}_m] + \mathcal{M}_2[\epsilon \text{Id}_m] \in \mathbb{P}_G(\mathbb{R})$, i.e.,

$$
\begin{pmatrix}
f[N_1] + \epsilon \text{Id}_m \\
f[N_1] + \epsilon \text{Id}_m \\
0_{m \times m}
\end{pmatrix}, \\
\begin{pmatrix}
f[N_1] + \epsilon \text{Id}_m \\
f[N_1] + N_2 + 2\epsilon \text{Id}_m \\
0_{m \times m}
\end{pmatrix} \in \mathbb{P}_G(\mathbb{R}).
$$

Proceeding as in the proof of Theorem 3.2 (see Equation (3.4)), it follows that $f[N_1 + N_2] - f[N_1] - f[N_2] \in \mathbb{P}_m(\mathbb{R})$, i.e., $f[\cdot]$ is Loewner super-additive on $\mathbb{P}_m(\mathbb{R})$. 

Having proved Theorem 4.13, it is now possible to prove Proposition 4.11 about coalescences of graphs.

Proof of Proposition 4.11. We prove the result by induction on $k$, with the base case of $i = 2$ and the higher cases proved similarly. Let $G_i'$ denote the coalescence of the graphs $G_1, \ldots, G_i$, for each $i = 1, \ldots, k$. Applying Theorem 4.13 with $C_1 = \emptyset$ corresponding to the vertex along which $G_{i-1}'$ and $G_i$ are glued together, we conclude that $f[\cdot]$ preserves positivity on $\mathbb{P}_{G_i}$ for all $i$, $f$ is continuous on $[0, \infty)$, and $f$ is super-additive on $[0, \infty)$. Similarly, if $f[\cdot]$ preserves positivity on $\mathbb{P}_{G_i}$ for all $i$, and is super-additive on $[0, \infty)$, then $f[\cdot]$ preserves positivity on $\mathbb{P}_G$ by Theorem 3.2.

As a consequence of Propositions 4.11 and 4.3, we determine the critical exponents of coalescences of cycles. Such graphs, often called cactus graphs or cactus trees, are useful in applications and have recently been used to compare sets of related genomes [33].

Corollary 4.14. Suppose $G$ is a connected cactus graph with at least 3 vertices. Then,

$$
\mathcal{H}_G = \mathcal{H}_G^{\emptyset} = [1, \infty), \quad [2, \infty) \subset \mathcal{H}_G^{\emptyset} \subset [1, \infty).
$$

Proof. This follows immediately from Propositions 4.3 and 4.11 and Equation (4.4). \qed
Concluding remarks and questions. The set of powers preserving positivity was determined for many graphs in the paper, including chordal graphs, cycles, and complete bipartite graphs. Apart from computing the $H$-sets for every graph, two natural questions arise:

1. In all of the examples of graphs studied in this paper, it has been shown that $CE_H(G) = CE^\psi_H(G) = r - 2$, where $r$ is the largest integer such that $G$ contains either $K_r$ or $K^{(1)}_r$ as an induced subgraph. Does the same result hold for all graphs?

2. Are the critical exponents of a graph always integers? Can this be shown without computing the critical exponents explicitly?

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