A short Brownian motion proof of the Riemann hypothesis

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February 1, 2008

Abstract. We give a short probabilistic (a Brownian motion) proof of the Riemann hypothesis based on some surprising, unexpected and deep algebraic conjecture (MAC in short) concerning the relation between the Riemann zeta $\xi$ and a trivial zeta $\zeta_t$. That algebraic conjecture was firstly discovered and formulated in [MA].

1 Introduction.

Let $\mathbb{C}$ be the field of complex numbers. In the famous and historical Riemann’s paper [R] appears the Riemann zeta function $\zeta$. It is firstly defined ”locally” by the Dirichlet series as follows:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \text{ } Re(s) > 1. \quad (1.1.)$$

(Here and all in the sequel by $Re(s)$ and $Im(s)$ we denote the real and imaginary part of a complex number $s$, respectively).

Evidently, the most important (and principal in fact) property of the ”global” zeta $\zeta(s)$ (traditionally denoted by the same symbol), i.e. the unique meromorphic extension of (1.1), to whole complex plane $\mathbb{C}$, with the unique pole at $s = 1$ (with the residue 1) - is the Riemann Hypothesis (RH for short).

The mentioned above Riemann’s paper [R] contains its first formulation and it can be formally written as a simple logical statement (implication):

$$(RH) \quad (\zeta(s) = 0) \land (Im(s) \neq 0) \implies Re(s) = 1/2.$$
- (RH) figures as the Fourth Millennium Prize Problem (see e.g. [Ka] for a large review of RH).

The best known generalizations of (RH) are:
1. (gRH) for Dedekind zetas and Dirichlet L-functions (see e.g. [MD]).
2. (gRH) for L-functions associated with modular forms. For example, let us consider (the unique up to a constant) holomorphic modular form for $SL_2(\mathbb{Z})$ of weight 12:
   \[ \Delta(z) = q \prod_{n=0}^{\infty} (1 - q^n)^{24}, \]
   where $q = e^{2\pi i z}$. Let us denote by $\tau(n)$ its Fourier coefficients:
   \[ \Delta(z) = \sum_{n=1}^{\infty} \tau(n)q^n. \]
The sequence $\tau$ is well-known as the Ramanujan function (see e.g. [N, V.6]). The automorphy of $\Delta$ is expressed by the formula:
   \[ \Delta(\frac{a_{11}z + a_{12}}{a_{21}z + a_{22}}) = (a_{21}z + a_{22})^{12}\Delta(z), \quad [a_{ij}]_{2 \times 2} \in SL_2(\mathbb{Z}). \]
The corresponding Ramanujan L-function has the form
   \[ \zeta_r(s) = L(s, \Delta) = \sum_{n=1}^{\infty} \frac{\tau(n)n^{-11/2}}{n^s} = \prod_{p \in P} (1 - \tau(p)p^{-11/2}p^{-s} + p^{-2s})^{-1}. \]
   The generalized Riemann Hypothesis for $L(s, \Delta)$ says that all complex zeros of $L(s, \Delta)$ have real part 6 (see [MR], [B] and [G]).

3. The congruence Riemann hypothesis (cRH in short) as (gRH) for zetas in the algebraic geometry (see e.g. [Kob.]).

Given any sequence $N_r, r = 1, 2, 3, ..., $ we define the corresponding ”zeta-function” by the formal power series
   \[ Z(T) := \exp(\sum_{r=1}^{\infty} N_r \frac{T^r}{r}), \text{ where } \exp(u) := \sum_{k=0}^{\infty} \frac{u^k}{k!}. \]

Let $V$ be an affine or projective variety defined over the finite field $\mathbb{F}_q$ with $q$-elements. For any field $K \supset \mathbb{F}_q$, we let $V(K)$ denote the set of $K$-points of $V$. By the ”congruence zeta-function of $V$ over $\mathbb{F}_q$” we mean the zeta-function corresponding to the sequence $N_r = \#V(\mathbb{F}_{q^r})$. That is, we define
   \[ Z(V/\mathbb{F}_q; T) := \exp(\sum_{r=1}^{\infty} \#V(\mathbb{F}_{q^r}T^r/r)). \]

Andre Weil considering many special cases, formulated his famous Weil conjectures
(W_D)(Dwork theorem). The function $Z(V/F_q; T)$ is a rational function of the variable $T$:

$$Z(V/F_q; T) = \frac{\prod_{k=0}^{n} P_{2k+1}(T)}{\prod_{k=0}^{n} P_{2k}(T)}$$

, where $P_j \in \mathbb{Z}[T]$ and moreover $P_0(T) = 1 - T, P_{2n}(T) = 1 - q^n T$.

(W_B)(Betti numbers). Assuming that there exists an algebraic number field $K$ such that our variety $V$ is the reduction modulo a prime ideal of the integral ring $R_K$ of $K$, of some variety $W$, we obtain that

$$B_j(W) = \text{deg}(P_j),$$

where $B_j(W)$ is the j-th Betti number of the complex manifold $W_h = (W \times_{\mathbb{R}} \mathbb{C})$, i.e. $B_j(W) = \text{rank}_{\mathbb{Z}} H^j(W, \mathbb{Z})$ and $H^j(W, \mathbb{Z})$ is the j-th cohomology group of $W$ in the coefficients in $\mathbb{Z}$. (The exact construction of the functor $(\cdot)_h$ we omit - see e.g. [Ha] for details).

(W_F)(Functional equation and Euler-Poincare characteristic).

The function $Z(V/F_q; T)$ satisfies the following functional equation

$$Z(V/F_q; 1/q^n T) = \pm q^{\frac{\chi(W)}{2}} Z(V/F_q; T),$$

where $\chi(W) := \sum_{j=0}^{2n} (-1)^j B_j(W)$ is the Euler-Poincare characteristic of $W$ (equal also to the intersection index of the Cartesian product $V \times V$).

(W_R)(Congruence Riemann Hypothesis)

All zeros and poles of $Z(V/F_q; q^s)$ lies on the critical lines $Re(s) = j/2$. More exactly, for $0 \leq j \leq 2n$ we have ($n = \text{dim} V$)

$$P_j(T) = \prod_{k=1}^{B_j(W)} (1 - \alpha_{jk} T),$$

where $\alpha_{jk}$ are algebraic integers and $|\alpha_{jk}| = q^{j/2}$. In particular, we see a strict connection of (cRH) with the topological invariants $B_j(W)$ (Betti numbers) written directly as

$$(cRH) Z(V/F_q; q^{-s}) = \frac{\prod_{j=1}^{n} \prod_{k=1}^{B_{2j}(W)} (1 - \alpha_{jk} q^{-s})}{\prod_{j=1}^{n} \prod_{k=1}^{B_{2j+1}(W)} (1 - \alpha_{jk} q^{-s})}.$$

Finally, the classical Riemann zeta $\zeta(s)$, Dedekind zetas, Dirichlet L-functions as well as the congruence zetas $Z(V/F_q; q^{-s})$ are special examples of a more general zeta construction: let $X$ be a finite type schema over $\text{Spec} \, \mathbb{Z}$, then we can put

$$\zeta_X(s) = \prod (1 - N(x)^{-s})^{-1},$$

where the product is taken over all closed points $x \in X$ and $N(x)$ denotes the number of elements of the residual field $k(x)$.
(4) (gRH) for elliptic curves and algebraic varieties (see [Sh], [Kob]).

How far we can go with generalizations of the Riemann hypothesis? In fact the answer is unknown. Really, it is well-known that the congruence Weil zetas can be considered as counterparts of the local Euler components \((1 - p^{-s})^{-1}\) in the Euler product expansion of \(\zeta(s)\):

\[
\zeta(s) = \prod_{p \in P} (1 - p^{-s})^{-1}.
\]

Obviously \(P\) stands for the set of all prime numbers.

Let \(C\) be a non-singular projective curve of genus \(g\), defined over an algebraic number field \(k\). For each prime ideal \(p\) of its integral ring \(R_k\) we denote by \(p(C)\) a curve obtained from \(C\) by the reduction mod \(p\). As it is well-known, there exists only a finite set \(\mathbb{B}\) of prime ideals of \(k\), with the property that \(p(C)\) is non-singular (and multiplicity 1), if \(p\) does not belong to \(\mathbb{B}\).

One can show that the genus of such \(p(C)\) is equal \(g\) (see [S]). Let us consider the congruence Weil zeta \(\zeta_{p(C)}(s)\) of the curve \(p(C)\) over the residual field \(k_p := R_p/p\) of the form

\[
\zeta_{p(C)}(u) = \frac{P_p(u)}{(1 - u)(1 - N(p)u)}.
\]

Here \(N(p)\) is the number of elements of \(k_p\) (the absolute norm of \(p\)) and \(P_p\) is a polynomial of degree \(2g\) with the free term equal to 1.

The global Hasse-Weil zeta function of a curve \(C\) over \(k\) is defined as the infinite product (although it is also important to consider the "bad" prime ideals \(p\) and the Euler multiplicators for them) (see [S, Sect. 7] and [Kob, Chapter II])

\[
\zeta_{C/k}(s) := \prod_{p \in (P - \mathbb{B})} P_p(N(p)^{-s})^{-1}, \quad (1.2.)
\]

where \(s\) is the complex variable. In reality, one can define - in analogical way a zeta function of an arbitrary (non-singular and projective) algebraic variete \(V\) over \(k\).

Let \(A\) be an abelian variete over \(k\), i.e. an algebraic variete with the compatible group structure. It is well-known that there exists such a finite set \(\mathbb{B}'\) of prime ideals of \(k\) that for any \(p \in \mathbb{B}'\) the variety \(A\) has the good reduction modulo \(p\) in the Serre-Tate’s sense - or equivalently - is without a defect in \(p\) in the Shimura-Taniyama’s sense. Let \(p(A)\) be an abelian variete obtained from \(A\) by the reduction modulo \(p\), \(\pi_p\) - the Frobenius endomorphism on \(p(A)\) of the degree \(N(p)\) and \(R_l\) some \(l\)-adic representation of the endomorphism ring \(End(p(A))\) of \(p(A)\), where \(l\) is a prime number which is relatively prime with \(p\). Then the one-dimensional part of the zeta function of the variety \(p(A)\) over \(k_p\) is given by the formula

\[
P'_p(u) := det(1 - R_l(\pi_l)u).
\]

The zeta function of the abelian variete \(A\) over \(k\) is defined as the infinite product

\[
\zeta_{A/k}(s) := \prod_{p \in (P(A) - \mathbb{B}')} P'_p(N(p)^{-s})^{-1}. \quad (1.3.)
\]
If $A$ is the Jacobian variety of a curve $C$, then according to the Weil’s theorem the both zetas $\zeta_{A/k}(s)$ and $\zeta_{V/k}(s)$ coincides - in fact ( a part of the Langlands program, see e.g. [JL]).

According to the Hasse-Weil conjecture, each from the functions $\zeta_{C/k}(s)$ and $\zeta_{A/k}(s)$ has a holomorphic extension on the whole complex plane $\mathbb{C}$ and satisfies a functional equation (the invariance with respect to the modular map: $s \longrightarrow 2 - s$, see [S, (7.5.5) and Th.7.13]). In some special cases such global Hasse-Weil zetas coincides with some products of Hecke L-functions of Grossen-characters or some products of Dirichlet series. Reasuming, if $V$ is a variete of a dimension $\dim V = n$ over $k$, since as we see by our approach to the Riemann zeta - there unique thing which seems be important for the proof of (RH) is the form of a functional equation - then we can pose the following generalized Riemann hypothesis (gRH) concerning the global Hasse-Weil zeta $\zeta_{V/k}(s)$: all its complex zeros should lie on the critical line $\Re(s) = n$.

Let us also mention on the importance some Riemann hypothesis type conjectures and theorem which have some extremal values in maths (see e.g. [Kob, II.6]):

The Birch-Swinnerton-Dyer Conjecture: let $E$ be an elliptic curve and $L_E(s)$ its Hasse-Weil zeta. Then $L_E(1) = 0$ if and only if $E$ has infinitely many rational points, and

The Coates-Wiles theorem: let $E$ be an elliptic curve defined over $\mathbb{Q}$ and having complex multiplication. If $E$ has infinitely many $\mathbb{Q}$-points, then $L_E(1) = 0$.

Its glorious consequences is the Tunnel result.

(5) The Selberg Conjecture. Thus, a natural problem arises: how look a maximal class of zetas for which the generalized Riemann type hypothesis is true. A partial answer to this question gives the so called Selberg Conjecture (SC in short). It is the so called axiomatic theory of zeta functions. That very successful proposition came from A. Selberg [Se]. We say that a Dirichlet series

$$L_{\{a_n\}}(s) = L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

belongs to the Selberg class $Sel$ if it is absolutely convergent for $\Re(s) > 1$, has the meromorphic continuation on $\mathbb{C}$ (with the unique singularity - the pole of a finite order in $s = 1$). Moreover it is assumed that $L$ satisfies a functional equation (with many gamma factors) which connects values in the points $s$ and $1 - s$, it has a special type Euler product and finally satisfies the so called Ramanujan condition, which forces some restriction on the size of the coefficients $a_n$. The fundamental ”structural” conjecture associated with $Sel$ is a hypothesis, that all elements from the Selberg class come from the automorphic representations.

The supposition that the Riemann hypothesis is true for all functions $L$ from the Selberg class $Sel$ seems to be one of the most general formulations of that conjecture.

That can be zetas, for which the Riemann hypothesis is not true, shows the example of the Hurwitz zeta function $\zeta(s, a), a \in (0, 1]$. It is defined for $\Re(s) > 1$ as the
translated by a Dirichlet series (the generating function of $\mathbb{N}^* + a$):

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \ 0 < a \leq 1. \quad (1.4.)$$

The Hurwitz zeta $\zeta(s, a)$ has the analytic continuation for the whole complex plane. Let

$$\eta = \frac{\sqrt{10 - 2\sqrt{5}} - 2}{\sqrt{5} - 1} \quad (1.5.)$$

and let

$$\zeta_H(s) := \frac{1}{5^s}(\zeta(s, 1/5) + \eta\zeta(s, 2/5) - \eta\zeta(s, 3/5) - \zeta(s, 4/5)). \quad (1.5.)$$

Then $\zeta_H$ is an entire function of $s$ and satisfies the following functional equation

$$(\frac{5}{\pi})^{s/2}\Gamma(\frac{1+s}{2})\zeta_H(s) = (\frac{5}{\pi})^{(1-s)/2}\Gamma(1 - \frac{s}{2})\zeta_H(1 - s). \quad (1.6.)$$

Applying some methods from the Riemann zeta function, one can show that $\zeta_H$ has infinitely many zeros on the critical line (the Hardy-Littlewood type theorem for $\zeta_H$). However $\zeta_H$ has also infinitely many zeros in the half-plane $\Re(s) > 1$. The essential difference between $\zeta_H$ and $\zeta$ consists on the fact that $\zeta_H$ does not have the Euler product! Hence the corollary, that to have the full Riemann hypothesis, there existence only a functional equation is not suffices (at least in the area $\Re(s) > 1$) and the fact that $\zeta$ has the Euler product is important. More exactly, the (RH) for $\zeta_Q$ splits into two different kind of conjectures: the Right RH

$$(RH_{1^+}) \zeta_Q(s) \neq 0 \ for \ \Re(s) > 1$$

and the Left RH

$$(RH_{1^-}) \zeta_Q(s) \neq 0 \ for \ \Re(s) \in [0, 1].$$

(6) **The p-adic Riemann hypothesis** (see [WH]).

Researchers working in the theory of zetas are intrigued by the fact that the Riemann hypothesis arises as a conjecture for the quite another type of zetas : the zeta functions of divisors, introduced by Wan in [W1] (see also [W2]).

Let $\mathbb{F}_q$ be a finite field with $q$ elements, $q$ a power of a prime $p$. Let $X$ be a projective $n$-dimensional integral scheme defined over $\mathbb{F}_q$. Let $0 \leq r \leq n$ be an integer. A prime $r$-cycle of $X$ is an $r$-dimensional closed integral subscheme of $X$ defined over $\mathbb{F}_q$. An $r$-cycle on $X$ is a formal finite linear combination of prime $r$-cycles. An $r$-cycle is called effective, denoted $\sum n_i P_i \leq 0$, if each $n_i \leq 0$.

Each prime $r$-cycle $P$ has an associated graded coordinate ring $\oplus_{k=0}^{\infty} S_k(P)$ since $X$ is projective. By a theorem of Hilbert-Serre, for all $k$ sufficiently large, we have $\dim_{\mathbb{F}_q} S_k(P)$ equal to a polynomial $a_r k^r + (\text{lower terms})$. Define the degree of $P$, denoted $\deg(P)$, as $r!$
times the leading coefficient $a_r$. We extend the definition of degree to arbitrary r-cycles by $\deg(\sum n_i P_i) := \sum n_i \deg(P_i)$.

Defining the degree allows us to measure and compare the prime r-cycles. Define the zeta function of algebraic r-cycles on $X$ as

$$Z_r(X, T) := \prod_P (1 - T^{\deg(P)})^{-1},$$

where the product is taken over all prime r-cycles $P$ in $X$.

Denote the set of all effective r-cycles of degree $d$ on $X$ by $E_{r,d}(X)$. A theorem of Chow and van der Waerden states that this set has the structure of a projective variety. Since we are over a finite field, $E_{r,d}(X)$ is finite. This means that $Z_r(X, T)$ is a well-defined element of $1 + T\mathbb{Z}[[T]]$, and so, converges $p$-adically in the open unit disk $|T|_p < 1$.

Equivalent forms of this zeta function are

$$Z_r(X, T) = \sum_{d=0}^{\infty} \#E_{r,d}(X) T^d = \prod_{d=1}^{\infty} (1 - T^d)^{-N_d} = \exp(\sum_{k=1}^{\infty} \frac{T^k}{k} W_k),$$

where $N_d$ is the number of prime r-cycles of degree $d$ and $W_k := \sum_{d|k} dN_d$ is the weighted number of prime r-cycles of degree dividing $k$, each prime r-cycle of degree $d$ is counted $d$ times.

This zeta function will not be complex analytic in general, unlike the above classical zeta functions. However, the conjectural $p$-adic meromorphic continuation immediately yields a $p$-adic formula in terms of the $p$-adic zeros and poles of this zeta function as we shall see below.

The $p$-adic meromorphic continuation of $Z_r(X, T)$ would imply the complete $p$-adic factorization

$$Z_r(X, T) = \prod (1 - \alpha_i T) \prod (1 - \beta_j T),$$

where the products are now infinite with $\alpha_i \to 0$ and $\beta_j \to 0$ in $\mathbb{C}_p$. Taking the logarithmic derivative, we obtain a formula for $W_d$ in terms of infinite series:

$$W_d = \sum \beta_j^d - \sum \alpha_i^d.$$

Thus, the $p$-adic meromorphic continuation implies a well-structured formula for the sequence $W_k$. By Mobius inversion, this gives a well-structured formula for the sequence $N_k$ as well.

If $Z_r(X, T)$ is $p$-adic meromorphic, we can adjoin all the reciprocal zeros $\alpha_i's$ and all the reciprocal poles $\beta_j's$ to $\mathbb{Q}_p$. The resulting field extension of $\mathbb{Q}_p$ is called the splitting field of $Z_r(X, T)$ over $\mathbb{Q}_p$. This splitting field is automatically a Galois extension (possibly of infinite degree) over $\mathbb{Q}_p$ by the Weierstrass factorization of $Z_r(X, T)$ over $\mathbb{Q}_p$ and the fact that we are in characteristic zero.

Let us observe that the congruence Weil zeta $Z(X, s)$ is exactly the case $Z_0(X, T)$. Thus $Z_0(X, T) \in \mathbb{Q}(T)$ is rational and it satisfies the Riemann hypothesis by the Weil
Conjectures. If \( r = n = \dim(X) \), then \( Z_n(X, T) \) is trivially rational, its zeros and poles are roots of unity, and thus satisfies the Riemann hypothesis as well. In particular, \( Z_r(X, T) \) is well understood if \( n \leq 1 \). So, we will assume \( n = \dim(X) \geq 2 \) and \( 1 \leq r \leq n - 1 \).

Let \( CH_r(X) \) be the Chow group of r-cycles on \( X \); that is, the free abelian group generated by the prime r-cycles on \( X \) modulo the rational equivalence. Let \( EffCone_r(X) \) be the set of effective r-cycle classes in \( CH_r(X) \). Assume that \( EffCone_r(X) \) is a finitely generated monoid. Then is postulated the following \( p \)-adic Riemann hypothesis: the splitting field of \( Z_r(X, T) \) over \( \mathbb{Q}_p \) is a finite extension of \( \mathbb{Q}_p \). Moreover, all zeros and poles, except for finitely many, are simple.

(8) **Dynamical zeta functions.**

Let \( f : X \to X \) be such a homeomorphism of a topological space \( X \) that the number \( N_n(f) \) of the points of the period \( n \), i.e. the number of solutions of the equation \( f^n(x) = x \) (obviously \( f^n \) denotes the n-th iteration of \( f \)) is finite for each \( n \geq 1 \). The function (or rather a formal series)

\[
\zeta_f(t) = \exp \left( \sum_{n=1}^{\infty} \frac{N_n(f)t^n}{n} \right)
\]

is called the **zeta function of a homeomorphism** \( f \) (see [AI]). It is an extremally surprising fact, that similarly as in the algebraic geometry case of congruence Weil zeta , such dynamical zetas are often rational. More exactly, let \( (\Sigma_A, f) \) be an irreducible Topological Markov Chain (TMC in short, see [AI]) with the periodicity index \( h \). Then (see [AI, Th.4.2])

\[
(1) \quad N_n(f) = h\lambda^n(A) + \sum_{|\lambda_i| < \lambda(A)} \lambda_i^n \quad \text{for } n = ph,
\]

\[
(2) \quad N_n(f) = 0 \quad \text{if} \quad n \equiv 0(\text{mod} h),
\]

The zeta function \( \zeta_f(t) \) of an arbitrary TMC \( (\Sigma_A, f) \) is rational and the Riemann hypothesis is trivial in his "zero part" :

\[
\zeta_f(t) = \frac{1}{\det(E - tA)},
\]

and the defining series is convergent for \(| t | < 1/\lambda(A) \) (here \( \lambda(A) \) denotes the largest positive eigenvalue of the matrix \( A \)).

**P. Bowen** has also showed that zeta functions of \( A^\# \)-homeomorphism are rational (see Th.12.1 and Th.13.1 of [AI]). Extremally exciting is also an application of dynamical zetas for Lorentz attractors (see [W]). They appear in the proof of the fundamental Guckenheimer theorem, which explains the structure of the Lorentz attractors (see[W, Theorem]) : there exists only two different topological types of the Lorentz attractors.

More about the dynamical-topological zeta functions a reader can find in a beautiful reviewed article [Ru] by D. Ruelle.
Let us observe that dynamical zetas are strictly connected with congruence Weil zetas: let $F_{r_q}: V \to V$ be the Frobenius map (acting by $z \mapsto z^q$ on coordinates), $|Fix f^n|$ is the number of fixed points of the $n$-th iterate of $f$. Then

$$Z(V/F_q)(s) = \zeta_{Fr_q}(s) = \exp(\sum_{n=1}^{\infty} \frac{s^n}{n} |Fix f^n_q|)$$

Finally, the last interesting future of the dynamical zetas $\zeta_f(s)$ is the triviality in zeros (in general) of the Riemann hypothesis in this case (although they have ”non-trivial” functional equations and Euler products in this case). However in the much general zetas-like zeta functions associated with the weighted dynamical systems, for example the Lefschets zeta function $\zeta_L(s)$, the Selberg zeta function $\zeta_S(s) := \prod_{k=0}^{\infty} \zeta(s+k)^{-1}$ and the currently popular Ihara-Selberg zeta function $\zeta_I(s)$ associated with a finite unoriented graph $G$ - the Riemann hypothesis problem is actual (well-posed) (see [Ru]). It is also well-known that $1/\zeta_I$ is a polynomial and that $\zeta_I$ satisfies the Riemann hypothesis precisely when $G$ is Ramanujan (Ramanujan graphs were named by Lubotzky, Phillips, and Sarnak; examples are not easy to construct).

In [MA], [MH], [AM] and [ML] we initiated and developed the method (technique) of the proving of the Riemann Hypothesis (RH in short) by using some methods from the measure and integration theory both on LCA groups (see [MA] and [MH]) and infinite-dimensional linear topological spaces (see [AM] and [MR]).

The mentioned above meromorphic extension $\zeta(s)$ of the Dirichlet series (1.1) is given explicite by the following classical Riemann functional analytic continuation equation (Rface in short)

$$\zeta(s) = \frac{\pi^{s/2}}{\int_0^{\infty} s^{s/2-1} e^{-x} dx [\frac{1}{s(s-1)} + \int_1^{\infty} (x^{s/2-1} + x^{-(s+1)/2})(\sum_{n=1}^{\infty} e^{-\pi n^2 x}) dx]. \quad (1.10.)$$

Based on (1.10) and the existence of the Hodge measure $H_2^*$ in [ML] we give a Hodgian proof of (RH).

The whole our approach to (RH) (also in this paper) is based on a subsequence extensions of that principal (Rface) to the more general class of functions than the gaussian canonical function $G(x) := e^{-\pi x^2}$.

In [MA] we proposed to extend (1.10) and consider the class of fixed points $\omega^+$ of the canonical Fourier transform $F(f)(x) := \int_{\mathbb{R}} e^{2\pi ixy} f(y)dy$ (let us observe that $G$ is only one special example 0f $\omega^+$) of the form (ffp-equation):

$$(M(\omega^+)\zeta)(s) = \frac{\omega^+(0)}{s(s-1)} + \int_1^{\infty} \theta(\omega^+)(x)(x^{s-1} + x^{-s}) dx, \quad (1.11.)$$

where $M(f)(s) := \int_0^{\infty} x^{s-1} f(x) dx$ stands for the Mellin transform and $\theta(f)(x) := \sum_{n=1}^{\infty} f(nx)$ is the theta Jacobi transform. Beside (1.11), the second important technical tool is the notion of the RH-fixed point of $F : \omega^+_A - G = A$ (associated with an amplitude $A$) (see [MA] for details).
To prove the existence of $\omega^+$ we used an expanded aparatus of the measure theory - in particular Haar measures, Riesz measures, Bogoluboff-Kriloff measures and Herbrandt distributions - to give an algebraic proof of (RH).

That direction of investigations was next directly continued in [MH]. The ffp-equation (1.11) is subsequently extended for classes of eigenvectors of the whole family $(F^p : p > 0)$ of Fourier transforms $(F^p f)(x) = \int_{\mathbb{R}} e^{2\pi i x y} f(y) dy$, corresponding to the real valued eigenvalues $\lambda : F^p \omega_\lambda^p = \lambda \omega_\lambda^p$. Thus, in [MH] we derived the $F$-eigenvalue analytic continuation equations for eigenvectors $\omega_\lambda^p$ of $F^p$, i.e. $F^p \omega_\lambda^p = \lambda \omega_\lambda^p$ (although for the purposes of [MH] we need only some very special eigenvectors - the Herbrandt distributions $\text{Herbrandt distributions}$). Subsequently, in 

\begin{equation}
\sum_{g \in \{id, \ldots, m\}} |\lambda|^{-2s(g)} (M(\omega_\lambda^p))(g(s)) = \frac{(\lambda + 1)\omega_\lambda^p(0)}{2s(s - 1)} + (|\lambda|^{-1} + sgn(\lambda)) \int_1^\infty \theta(\omega_\lambda^p)(\lambda^2)(x^{s-1} + sgn(\lambda)x^{-s})dx.
\end{equation}

In some sense, in this paper we consider a ”maximal” extension of (1.10) - the so called Muntz’s relations- for the functions $p$ from the cylinder of the Poisson space $\mathbb{P}_1$ (see Th.2), of the form

\begin{equation}
(M(p)\zeta)(s) = \frac{1}{s(s - 1)} + \int_1^\infty [x^{s-1}\theta(p)(x) + x^{-s}\theta((F^p))(x)]dx.
\end{equation}

A big Poisson space $\mathbb{P}_1(\mathbb{R}_+)$ has such an adventage that the famous Poisson Summation Formula (PSF in short) holds for the functions from that one. Moreover, on the Borel $\sigma$-field $\mathcal{P}$ of $\mathbb{P}_1$, we will be able to define - the principal for this short proof of (RH) - the family of the Wiener-Riemann measures $\{r_m\}$ - indexed by some simple class of moment functions $m$.

The family $\{r_m\}$ is in fact induced by the standard Brownian motion $B^0$ (see Section 2) - and hence follows the huge role of the Brownian motion (or rather the Wiener measure $w$) for that proof of (RH).

In the above three types of the functional equations written (mentioned) above, appears always the simple algebraic meromorphic function

$$\zeta_{\mathbb{P}_1(\mathbb{C})}(s) := \frac{1}{s(s - 1)},$$

(the congruence Weil zeta of the complex projective space $\mathbb{P}_1(\mathbb{C})$ - see the end of Section 3).

Subsequently, in $\zeta_{\mathbb{P}_1(\mathbb{C})}(s)$ is written the fundamental polynomial of two variables with integer coefficients and degree 2:

$$\zeta(s) = \text{Im}(\frac{1}{s(s - 1)}) = \text{Im}(s)(2\text{Re}(s) - 1) = y(2x - 1).$$
According to the below surprising property of $\zeta_t$ and for some another important algebraic reason, we call the function $\zeta_t(s)$ - the **trivial zeta** - since its formally satisfies the following **Trivial Riemann Hypothesis** (TRH in short) - according to its formal reasonblance (similarity) with the true (RH).

$$(TRH) \ (\zeta_t(s) = 0) \land (Im(s) \neq 0) \implies Re(s) = 1/2.$$

As a consequence, in [MA] we formulated the following **Main Algebraic Conjecture** (MAC in short) that:

**MAC** TRH implies RH.

More exactly, we can formulate (MAC) in the following algebraic geometry language:

Let $\zeta(C) := \{s \in \mathbb{C} : \zeta(s) = 0\}$ be the **zero-dimensional complex analytic manifold** and $\zeta_t(C) := \{(x, y) \in \mathbb{R}^2 : \zeta_t(x, y) = 0\}$ be the **1-dimensional algebraic (affine) variete over \mathbb{R}**. Then (MAC) means that

$$(MAC) \ \zeta(C) \subset \zeta_t(C), \quad (1.15.)$$
i.e. $\zeta(C)$ is a subvariete of $\zeta_t(C)$.

The Muntz’s relations considered in this paper, are modeled on the following extension of the (Rface) considered in [AM]: let $\theta M A$ be the $\mathbb{R}_+-$cone of **theta-Mellin admissible** real valued functions on $\mathbb{R}_+$, i.e. the smallest space of functions on which the Mellin transform $M$ does not vanish and the (PSF) holds. Moreover, each $f \in \theta M A$ satisfies the initial condition : $\hat{f}(0) = f(0) = 0$. As a consequence of the above conditions on $\theta M A$ we get the following family of (Rface) indexed by $\theta M A$:

$$(M(f)\zeta)(s) = \int_1^\infty (x^{s-1}\theta(f)(x) + x^{-s}\theta(\hat{f})(x))dx, \quad (1.16.)$$
i.e. in (1.16) does not appear the congruence zeta $\frac{1}{s(x-1)}$.

In opposite to (MAC) we thus also have the **Main Oscilatory Conjecture** (MOC in short) : in the right-hand sides of (1.10), (1.11), (1.12) and (1.13) appears another very important analytic expressions - the **Fresnel integrals** $F(A)(\nu) := \int_0^\infty \sin \nu x dA(x)$, associated with a positive measure $A$ and a periodicity $\nu > 0$. **Arnold** and others devoted the whole book [AWG-Z] to the asymptotics of the so called oscilatory integrals of the form (see [AWG-Z, Sect.2 and 3])

$$\int_{\mathbb{R}^n} e^{i\phi(x))} A(x)dx,$$

where the functions $\phi$ and $A$ are called the **phase** and **amplitude**, respectively.

The main property of $F(A)(\nu)$ is its positivity (the Fresnel lemma - Oscilatory lemma or finally - the Nakayama lemma) - for the positive, continuous, integrable and decreasing densities ($\frac{dA}{dx}$).

The (MOC) is also a simple logical statement that:

**The Fresnel lemma** implies RH.
(It is mainly proved in [AM]).

Thus, the whole effort of the papers [MA], [MH] and [ML] was concentrated on the proving of (MAC)+(MOC). Moreover - in fact - we only proved and used (MOC) in [AM] and only (MAC) - in this paper.

In the all mentioned papers devoted to the proofs of (RH), the proof was always attained by the writing of a suitable Riemann hypothesis functional equations. Exactly the same is in this paper, the Riemann hypothesis is attained by writing the following Brownian motion Riemann hypothesis equation:

\[
\lim_{k} \text{Im}(m_s(P_1,P,r_m)\zeta(s)) = \frac{\zeta_{t}(s)}{|\zeta_{\mathbb{P}^{1}(\mathbb{C})}(s)|^2},
\]

with \(\text{Re}(s) \in (0,1/2)\).

In such a way, the measure characteristic numbers \(m_s\) (see Sect.4) gives the direct relation between the analytic properties of \(\zeta\) and the algebraic nature of the pair \((\zeta_t,\zeta_{\mathbb{P}^{1}(\mathbb{C})})\). One can say that in the family \((m_s,b_s)\) (see Sect.2) is written the whole information concerning the Riemann hypothesis - like in the Betti numbers \(B^i(X) := \dim_{\mathbb{C}}H^i(X,\mathbb{C})\) (where \(H^i(X,\mathbb{C})\) is the \(i\)-th cohomology space of a manifold \(X\) with coefficients in \(\mathbb{C}\), \(i = 0,1,...,2n; n = \dim(X)\)) and in the Euler-Poincare characteristic \(\chi(X) := \sum_{i=0}^{2n}(-1)^iB^i(X)\) (i.e. in the pair \((B(X),\chi(X))\)) is written the whole information concerning the congruence Riemann hypothesis (see \((W_R)\) and \((cRH)\)).

\section{The Poisson space \(\mathbb{P}^{1}(\mathbb{R}^{+})\) and the Wiener-Riemann measures \(r_m\).}

Let \(C = C(\mathbb{R}^{+})\) be the real vector space of all real valued continuous functions on \(\mathbb{R}^{+} = [0, +\infty)\). Let \(L^1 = L^1(\mathbb{R}^{+}, dx)\) be the real Banach space of all absolutely integrable real valued functions. If \(f\) belongs to \(C \cup L^1\) then by \(f^+\) we denote its symmetrisation, i.e.

\[
f^+(x) := \begin{cases}
  f(x) & \text{if } x \geq 0 \\
  f(-x) & \text{if } x < 0
\end{cases}.
\]

By \(\mathcal{F}\) we denote the canonical Fourier (cosine) transform, i.e.

\[
(\mathcal{F}f^+)(x) := \int_{-\infty}^{+\infty} e^{2\pi ixy}f^+(y)dy =: \hat{f}^+(x) = 2 \int_{0}^{\infty} \cos(2\pi xy)f(y)dy;
\]

\(x \in \mathbb{R}, f \in L^1\).

Moreover, we denote

\[
\hat{C} := \{ f : \mathbb{R}^{+} \longrightarrow \mathbb{R} : \hat{f}^+ | \mathbb{R}^{+} \in C\},
\]

\(12\)}
and
\[ \hat{L}^1 := \{ f : \mathbb{R}_+ \rightarrow \mathbb{R} : \hat{f}^+ \in \mathbb{R}_+ \}. \]

Finally, we denote the real vector space being the intersection of the all above Banach and Frechet spaces by \( \text{Inv}(\mathbb{R}_+) \), i.e.
\[ \text{Inv}(\mathbb{R}_+) := C \cap \hat{C} \cap L^1 \cap \hat{L}^1. \]

As in the all previous our papers devoted to \((gR)\) (see [MA, MH, AM, ML] and [MR]), it will be very convenient to consider the canonical Jacobi theta transform \( \theta \).

It is very convenient to consider \( \theta \) on the Schwartz space \( \mathcal{S}(\mathbb{R}_+) \) of all rapidly decreasing and smooth functions on \( \mathbb{R}_+ \). Thus
\[ \theta(f)(x) := \int_{\mathbb{N}^*}^{} f(nx) dc_{\mathbb{Z}}(n) = \sum_{n=1}^{\infty} f(nx), \]
where \( f \in \mathcal{S}(\mathbb{R}_+), x > 0 \) and \( c_{\mathbb{Z}} \) denotes the calculating measure on \( \mathbb{N}^* \), i.e. the unique Haar measure on the LCA group \((\mathbb{Z}, +)\) normalized by \( c_{\mathbb{Z}}(\{0\}) = 1 \).

As it was communicated to us by S. Albeverio (by a private communication), the Schwartz space is to big for the below Mellin transform \( M \). It follows from the fact that \( M \) has also a "singularity" at zero (beside the singularity at the plus infinity).

Thus by \( \mathcal{M}(\mathbb{R}_+) \) we denote the sub-vector space of \( \mathcal{S}(\mathbb{R}_+) \) consisting with such \( f \in \mathcal{S}(\mathbb{R}_+) \) that
\[ (MT) \int_0^1 x^{Re(s)-1} \vert f(x) \vert \; dx < +\infty, \]
for all \( Re(s) \in I := (0, 1) \).

**Remark 1** Let us observe that \( \mathcal{M}(\mathbb{R}_+) \) is essentially smaller than \( \mathcal{S}(\mathbb{R}_+) \). Really, for example the n-th Hermitite functions \( H_n(x) \) being the n-th derivatives of the canonical gaussian functions \( G(x) \), i.e. \( H_n(x) := G^{(n)}(x) \) obviously belong to the Schwartz space. But their Mellin transform
\[ M(H_n)(s) := \int_0^{\infty} x^{s-1} H_n(x) dx = \int_0^{\infty} x^{s-1} G^{(n)}(x) dx = s(s-1)...(s-n) \Gamma(s-n) \]
is convergent for \( s \) with \( Re(s) > n! \) (see also [MH, Prop.7])

Thus we have deal with rather subtle problem of the convergence of non-proper integrals. They also have the "singularity" at zero (besides at infinity) and obviously the Schwartz space \( \mathcal{S}(\mathbb{R}) \) is a "good" condition on the behaviour of functions in infinity.

The space \( \mathcal{M}(\mathbb{R}_+) \) we call the space of Mellin admissible functions. The Mellin transform \( M : \mathcal{M}(\mathbb{R}_+) \rightarrow \mathbb{C} \) is defined as
\[ M(f)(s) := \int_0^{+\infty} x^s f(x) \frac{dx}{x} ; \; Re(s) > 0, f \in \mathcal{M}(\mathbb{R}_+). \]

We propose the following fundamental for the purposes of this paper
Definition 2.1. By the Poisson space $\mathbb{P}_1(\mathbb{R}_+)$ ( $\mathbb{P}_1$ for short) we mean (understand) the set of all functions $p: \mathbb{R}_+ \rightarrow \mathbb{R}$ which satisfies the following five conditions:

- $(P_0)$ $p(0) = 1$,
- $(P_1)$ $p^+ \in \text{Inv}(\mathbb{R})$,
- $(P_2)$ for all $x, y \in \mathbb{R}_+$
  \[ \sum_{n=1}^{\infty} x^2 | p(nx) | =: \theta(x^2 p_+)(y) < \infty, \]
- $(P_3)$ for all $x, y \in \mathbb{R}_+$
  \[ \sum_{n=1}^{\infty} x^2 | p^+(nx) | =: \theta(x^2 \hat{p}_+)(y) < \infty, \]
- $(P_4)$ $p \in \mathcal{M}(\mathbb{R}_+)$. The importance of the Poisson space $\mathbb{P}_1$ follows from the two sources:

1. for the functions from $\mathbb{P}_1$ hold the below Poisson Summation Formula (fundamental when we work with zetas) (PSF in short) and the Mellin transform is well-defined on its.

2. On $\mathbb{P}_1$ there exist probabilities $r_m$ - with properties, which are sufficient to prove the Riemann Hypothesis. More exactly, we have the following technical proposition and theorem:

Proposition 1 For each $p \in \mathbb{P}_1$ holds

(i) the Poisson Summation Formula (PSF in short), i.e.

\[ (PSF) \quad \theta(p)(1) + p(0) = \theta(p^+)(1) + \hat{p}^+(0), \]

and

(ii) $M(p)(s)$ is well-defined (exists) for all $s$ with $\text{Re}(s) > 0$ and moreover if $p = \exp^{-1} \in \mathbb{P}_1$ then

\[ M(\exp^{-1})(s) = \int_0^{\infty} x^{s-1}e^{-x}dx = \Gamma(s), \]

is the classical gamma function which does not vanish everywhere.

All the above notions and facts, as well as their proofs, in a large context and deepest can be find in the Weil’s book [We].

For the purposes of this paper we fix an arbitrary continuous moment function $m: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, which satisfies the following three conditions

- $(m_0)$ $m(0) = 1$,
- $(m_1)$ $0 \leq m(x) \leq 1$ for all $x \in \mathbb{R}_+$
- $(m_2)$ $m(x) = 0$ for $x \geq 1$. 

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A stochastic process $B^m = (B^m_t : t \geq 0)$ defined on a probability space $(\Omega, \mathcal{A}, P)$ is said to be the Brownian motion with a moment function $m$ if it is gaussian (i.e. its all 1-dimensional distributions are gaussian), its medium function is $m$:

$$EB^m_t = m(t),$$

and its covariance function (or correlation function)

$$E(B^m_t - m(t))(B^m_s - m(s)) = \min(t, s).$$

Here and all in the sequel $EX$ denotes the expected value of a random variable $X$ (rv in short).

For there existence of such $B^m$ see e.g. [Wong, II.3]. In particular, $B^m$ has got the continuous paths (trajectories), since it satisfies the well-known Kolmogorov condition (see e.g. [Wong, II.4, Th.4.2]).

Let us observe at once that $B^m$ has the direct representation: let $B^0 = (B^0_t : t \geq 0)$ be the standard Brownian motion. Then

$$B^m_t = B^0_t + m(t).$$

The Brownian motion $B^m$ can be considered as a random element (re in short) $B^m : (\Omega, \mathcal{A}, P) \rightarrow (C, \mathcal{C})$, where $C$ is the $\sigma$-field generated by cylinders in $C$, i.e. the sets of the form

$$C = C(t_1, ..., t_n; A) := \{c \in C : (c(t_1), ..., c(t_n)) \in A\},$$

where $A$ is a Borel set from $\mathbb{R}^n$ and $0 < t_1 < ... < t_n$.

It is well-known that $C$ coincides with the $\sigma$-field of all Borel sets $\mathcal{B}$ of $C$ endowed with the Frechet topology of the almost uniform convergence on compacta. Normally, $(C, \mathcal{C})$ is called the phase space of $B^m$.

Finally, $B^m$ (as a re) determines the probability $w^m$ on the phase space $(C, \mathcal{C})$ by the formula (the distribution of $B^m$):

$$w^m(C) := P((B^m)^{-1}(C)) , C \in \mathcal{C}.$$ \hspace{1cm} (2.28.)

In the sequel we call $w^m$ - the m-Wiener measure. The m-Wiener measure $w^m$ (similarly like p-stable measures $s^p$ with $p \in (0, 1)$ considered in [AM]) permits us to define principial for this short proof of (RH) - the probability $r_m$.

To do this, we first of all define the sequence of the Wiener-Riemann processes $\{R^n : n = 0, 1, ...\}$ by the simple formula

$$R^n_t := G(t)B^m_{\sqrt{t}}, \ t \in [n, n + 1].$$ \hspace{1cm} (2.29.)

Let $\mathcal{P}$ be the $\sigma$-field of $\mathbb{P}_1$ generated by cylinders of $\mathbb{P}_1$.  

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**Theorem 1** For each moment function $m$ which satisfies the conditions $(m_0) - (m_2)$, there exists a $\sigma$-additive measure $r_m$ on the phase space $(\mathbb{P}_1(\mathbb{R}_+), \mathcal{P})$ with the following four - let us say - RH properties:

(RH$_0$)(Non-triviality, probabilisticity): $r_m(\mathbb{P}_1) = 1$.
(RH$_1$)(Starting point): $\int_{\mathbb{P}_1} p(0) dr_m(p) = EB_0^m = m(0) = 1$.
(RH$_2$)(Vanishing of moments): for all $t \geq 1$ holds

$$\int_{\mathbb{P}_1} p(t) dr_m(p) = E(G(t)B^{m}_{\sqrt{t}}) = 0.$$ 

(RH$_3$)(The Fubini obstacle - Hardy-Littlewood theorem obstacle - Existence of moments of 1/2-stable Levy distributions):

the Brownian characteristic numbers $b_s$ of $\mathbb{P}_1$ or double Wiener integrals of $\mathbb{P}_1$

$$b_s = b_s(\mathbb{P}_1, \mathcal{P}, r_m; \mathbb{R}_+) := \int_{\mathbb{P}_1 \times \mathbb{R}_+} |x^{-s}p(x)| dr_m(p)dx \quad (2.30.)$$

are

(i) finite (convergent) if $Re(s) \in (0, 1/2)$, and
(ii) infinite (divergent) if $Re(s) \geq 1/2$.

In the sequel the measure $r_m$ we call the **Wiener-Riemann measure** (by analogy with the Levy-Riemann measures $\{r_p: p \in I := (0, 1)\}$ considered in [AM]).

**Proof.** We define the measure $r_m$ by the following simple formula:

$$WRm) \ r_m(A) := \sum_{n=1}^{\infty} \frac{1}{2^n} P(R^n_{(i)} \in A \cap C[n-1, n]) \ , \ A \in \mathcal{P},$$

where by $C[n-1, n]$ we denoted the space of all real valued continuous functions defined on the segment $[n-1, n]$, considered as a subspace of $C$ through the embedding $e_n : C[n-1, n] \rightarrow C(\mathbb{R}_+)$ by the formula: $e_n(f)(t) = f(t)$ if $n-1 \leq t \leq n; e_n(f)(t) = f(n-1)$ if $t \leq n-1$ and $e_n(f)(t) = f(n)$ if $t \geq n$ and moreover we have

$$P(R^n_{(i)} \in A \cap C[n-1, n]) = P(f \in C(\mathbb{R}_+) : G(t)(f\sqrt{t}) = m(t) \in A \cap C[n-1, n]) =$$

$$= \ P(f \in C(\mathbb{R}_+) : f(t) \in (A \cap C[\sqrt{n-1}, \sqrt{n}-m)G^{-1}) = w_0((A \cap C[\sqrt{n-1}, \sqrt{n}-m)G^{-1}).$$

Now the conditions (RH$_0$ - RH$_3$) are trivially satisfy by $r_m$. Reely, the conditions $(P_0 - P_4)$ on the Poisson space (as the conditions on $p$ at zero and infinity) are quite ”independent” from the condition $p \in C[n-1, n]$ for $n \geq 2$, i.e. $\mathbb{P}_1 \cap C[n-1, n] = C[n-1, n]$ and $w_0(C[n-1, n]) = 1$.

Moreover

$$\int_{\mathbb{P}_1} p(t) dr_m(p) = \frac{1}{2^n} \int_{C[n-1, n]} p(t) dw_m(p) = G(t)m(\sqrt{t})/2^n = 0,$$

if $t \in [n-1, n], n \geq 1.$
Thus, we concentrate only on the unique non-trivial property (RH$_4$) of $r_m$. The idea of the proof is taken from (based on) the Proposition 3 of [AM]. In fact, in the Brownian characteristic numbers $a_n$ are written all what is important for our Brownian proof of (RH).

We first give the upper approximation of the iterated integral $I_{dxdr}(s):=\int_0^\infty dx|p(x)|dr_m(p)$ for $s$ with $u = Re(s) \in (0, 1/2)$.
Let us observe that

$$\int_{\mathbb{P}_1} |p(x)| dr_m(p) = E(G(x) | B_{\sqrt{\pi}^m})).$$

Let $u = Re(s)$. Hence

$$\int_0^\infty x^{u-1}(\int_{\mathbb{P}_1} |p(x)| dr_m(p))dx \leq \int_0^\infty x^{u-1}G(x)m(x)dx + \int_0^\infty x^{u-1}G(x)E | B_{\sqrt{\pi}^0}^0 | dx.\tag{2.31.}$$

Since in the above inequality the first integral obviously exists, thus the problem of the convergence of $I_{dxdr}(s)$ is reduced to the convergence of the integral

$$\int_0^\infty x^{u-1}G(x)E | B_{\sqrt{\pi}^0}^0 | dx = \int_0^\infty x^{u-1}G(x)\left(\int_\mathbb{R} |y + y_0| \frac{e^{-(y+y_0)^2/2x}}{\sqrt{2\pi x}}dy\right)dx,\tag{2.32.}$$

for any $y_0 > 0$, according to the facts that the distribution of $B_{\sqrt{\pi}^0}$ is gaussian with mean zero and variance $\sqrt{x}$ and the translation invariance of the Lebesgue measure.

Now, let us observe (what was firstly observed in [AM]), that the iterated integral in the right-hand side of (2.32) - according to the approximation : $e^{-y^2/2x}e^{-2|y|y_0/2x} \leq e^{-|y|y_0/x}$ and a suitable substitution can be approximated as follows:

$$\leq \left(\int_0^\infty x^u \frac{e^{-y_0^2/2x}}{\sqrt{2\pi x^{3/2}}}dx\right)(y_0^{-2}\max_{x \geq 0}(x^2G(x)))\int_\mathbb{R}(|t|e^{-|t|}dt + \max_{x \geq 0}(xG(x)))\int_\mathbb{R}e^{-|t|})dt).\tag{2.33.}$$

But now, the function $d_{y_0}(x):= \frac{y_0e^{-y_0^2/2x}}{\sqrt{2\pi x^{3/2}}}$ is exactly the density of a random variable $L_{y_0}$ with the $1/2$-stable Levy distribution (with a parameter $y_0$). It is well-known that the distribution of $L_{y_0}$ is concentrated on $\mathbb{R}_+$ and that it is the unique p-stable distribution with $p \in (0, 1)$, which has an elementary analytic and simple formula for density of power-exponential form (see e.g. also [AM]).

The most important fact concerning $1/2$-stable Levy distributions (i.e. some very specific probability measures on $\mathbb{R}_+$) what we use for this proof, is the problem of the existence of moments of $L_{y_0}$. It is well-known that (see [AM]):

$$E(L_{y_0}^u) = \int_0^\infty x^u y_0e^{-y_0^2/2x}dx \leq +\infty \text{ if } 0 < u < 1/2,\tag{2.34.}$$

and
(ii) \( E(L_{y_0})^u = +\infty \) if \( u \geq 1/2 \).

Combining (3.33) with (3.34) we finally obtain that the iterated integral \( I_{dxdr}(s) \) is finite if \( \text{Re}(s) \in (0, 1/2) \). Since obviously the measures \( r_0 \) and \( dx \) are \( \sigma \)-finite, then according to the Tonelli theorem (see e.g. [M, Th.XIII.18.4]), the Brownian characteristic numbers \( b_s \) are finite in this case.

We refer a reader to [M, Sect. XIII] for a very detailed and deep discussion of the Fubini theorem theory.

It is a surprising fact as a huge role plays Fubini theorem (FT in short) in this Brownian motion approach to (RH) as well as in the \( p \)-stable strategy of the proof of (RH) in [AM]. With such a huge role of (FT) we first have met when we had worked over the problem of there existence of Sazonov topologies in [Sa].

Looking only at the above upper approximations (evaluations) of \( I_{dxdr}(s) \) for \( \text{Re}(s) \in (0, 1) \), it is not clear why \( I_{dxdr}(s) = +\infty \) for \( \text{Re}(s) \geq 1/2 \). Even worse, at first glance it seems that also by that method, we can obtain the finitness of \( I_{dxdr}(s) \) in the mentioned above halfplane.

The below lower approximation - also modeled on the previous one - however shows that (FT) is violated in the case of the triplet:

\[
(\int_\mathbb{P}_1 \times \mathbb{R}_+ | x^{s-1} p(x) | dr_0(p) dx, I_{dxdr}(s), I_{drdx}(s)) \text{ for } \text{Re}(s) \geq 1/2.
\]

Thus, the Fubini-Tonelli theorem stands for (is) a true barrier (obstacle) for the extension of the below (\( R_{fe_B} \)) for \( \text{Re}(s) \geq 1/2 \) and is mainly responsible for the non-triviality of the Riemann hypothesis.

It is easy to see that (let assume that \( y_0 > 1 \))

\[
b_s = \int_\mathbb{P}_1 \times \mathbb{R}_+ | x^{s-1} p(x) | dr_m(p) dx \geq \int_1^\infty x^{u-1} G(x) E | B_{\sqrt{x}}^0 | dx \geq (2.35.)
\]

\[
\geq (\min_{x \geq 1} (x^2 G(x))) \int_{\mathbb{R}} t \left| e^{-|t|} dt + \min_{x \geq 1} (x G(x)) \int_{\mathbb{R}} e^{-|t|} dt \right) E(L_{y_0}^u).
\]

Thus really \( I_{drdx}(s) = +\infty \) for \( u = \text{Re}(s) \geq 1/2 \), since for such \( s \) there is: \( E(L_{y_0}^u) = \infty \) (see above (ii)).

3 The Muntz relations for the quintet \((\zeta, \zeta_{p1} (\mathbb{C}), M, \mathcal{F}, \theta)\).

Muntz (see [Ti, p.]) was probably the first, who considered the below extension of the classical Riemann functional continuation equation, for a large class of functions than \( \exp^{-1} \).

The importance and a huge role of this extension was observed subsequently in [MA], [MH], [MD] and [AM] - in the different contexts, for different class of functions. It also seems that it can be compared only with the ingenious idea of Grothendieck - during his work on the congruence Riemann hypothesis - of the extension of the notion of the set topology to the category topologies and its main consequence - there existence of good Weil cohomologies - e.g. l-adic etale Grothendieck cohomologies.
Theorem 2 (Family of Riemann functional analytic continuation equations for $\zeta$ and the class $\mathbb{P}_1$).

For each $p \in \mathbb{P}_1$ and $s$ with $\operatorname{Re}(s) > 0$ the functional equation holds

$$(M(p)\zeta)(s) = \frac{1}{s(s-1)} + \int_1^\infty [x^{s-1}\theta(p)(x) + x^{-s}(\mathcal{F}p)(x)]dx. \quad (3.36.)$$

Proof. Let $p \in \mathbb{P}_1(\mathbb{R}_+)$ be arbitrary. By $(P_5)$, the Mellin transform $M(p)(s)$ is well-defined for $\operatorname{Re}(s) > 0$. From the definition of the Mellin transform $M$ as the integral, on substituting $nx$ for $x$ under the integral, we have

$$\frac{M(p)(s)}{n^s} = \int_0^\infty p(nx)x^{s-1}dx, \quad \operatorname{Re}(s) > 0. \quad (3.37.)$$

Hence, for $\operatorname{Re}(s) > 1$ we obtain that beautiful relation between $M, \zeta$ and $\theta$:

$$(M(p)\zeta)(s) = \int_0^\infty \theta(p)(x)x^{s-1}dx = M(\theta(p))(s), \quad (3.38.)$$

since the above series is absolutely convergent (i.e. $p$ satisfies the condition $(P_2)$) and we can interchange the order of summation and integration.

Since $p \in \mathbb{P}_1$, then $p \in \text{Inv}(\mathbb{R}_+)$ and satisfies $(P_3)$ and $(P_4)$, i.e. the following two series:

$$\sum_{n \in \mathbb{Z}} p(x + ny) \quad \text{and} \quad \sum_{n \in \mathbb{Z}} \hat{p}(x + ny)$$

are absolutely and almost uniformly convergent for all $x, y \in \mathbb{R}_+$. In particular, such $p$ is $(\mathbb{R}, \mathbb{Z})$ - admissible, in the sense of Weil’s general assumption (see [We, VII.2]). As the consequence the (PSF) holds. Using its and the initial condition $(P_0): p(0) = 1$, changing variables, we can write

$$(M(p)\zeta)(s) = \int_0^1 x^{s-2}\theta(p)(\frac{1}{x})dx + \int_1^\infty x^{s-1}\theta(p)(x)dx = \quad (3.39.)$$

$$= \int_1^\infty [x^{-s}\theta(p)(x) + x^{s-1}\theta(p)(x)]dx =: I(\theta(p))(s).$$

The integral on the right-hand side of (3.39) converges uniformly for $-\infty < a \leq \operatorname{Re}(s) \leq b < +\infty$, since for $x \geq 1$, we have

$$|x^{-s}| \leq x^{-a} \quad \text{and} \quad |x^{s-1}| \leq x^{b-1},$$

while, for $l > 1$ (obviously it suffices to take $l = 2$ - like in $(P_2$ and $(P_3$)) holds

$$\theta(p)(x) \leq \sum_{n=1}^\infty |p(nx)| \leq \sup_{x \in \mathbb{R}_+} |p(x)| \frac{x^l \zeta(l)}{x^l},$$

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and
\[ \theta(\hat{p})(x) \leq \sup_{x \in \mathbb{R}_+} |\hat{h}tp(x)| \frac{x^l}{x^l} \cdot \theta(l) \]

Therefore, for each \( p \in \mathbb{P}_1 \), the integral in (3.39) represents an entire function of \( s \). Moreover - as it is well-known - \( M(exp^{-1})(s) = \Gamma(s) \) does not vanishes anywhere (obviously \( exp^{-1} \in \mathbb{P}_1 \)).

Thus the \( \theta/G \)-quotient
\[ \zeta(s) := \zeta(exp^{-1})(s) := \frac{1}{s(s-1)\Gamma(s)} + \frac{I(\theta(exp^{-1}))(s)}{\Gamma(s)}, \ s \in \mathbb{C}, \quad (3.40.) \]
gives the meromorphic continuation of ”local” zeta to the whole complex plane.

If now \( p \in \mathbb{P}_1 - \{exp^{-1}\} \) (let us observe that in this case \( M(p) \) can have zeros!), then again according to (3.39) we have the identity
\[ (M(p)\zeta)(s) = \frac{1}{s(s-1)} + I(\theta(p))(s), \ Re(s) > 1. \quad (3.41.) \]

But now, the left-hand and right-hand sides of the above equation (3.41) are the analytic functions in \( D := Re(s) > 0 - \{1\} \). Thus, they must be equal in \( D \), according to the uniqueness of the analytic continuation of a holomorphic function in a domain (see e.g. [M, XV.2]).

In the equations (3.36) - beside \( \zeta, M(p), F(p) \) and \( \theta(p), p \in \mathbb{P}_1 \) - appears the simple algebraic but very important (deciding) meromorphic function \( \frac{1}{s(s-1)} \) - from the point of view of the Riemann hypothesis.

It is very convenient to look at this moment on \( \frac{1}{s(s-1)} \) (since \( \zeta(s) \) has only the unique pole in \( s = 1 \), so \( s = 0 \) is only an ”apparent pole” of \( \zeta \)) as on the trivial zeta of \( \mathbb{P}^1(\mathbb{C}) \), since its form is surprisingly close to the form of the congruence Weil zeta \( Z(\mathbb{P}^1(\mathbb{F}_q)/\mathbb{F}_q; s) \) of the projective line \( \mathbb{P}^1(\mathbb{F}_q) \) over the algebraic closure \( \mathbb{F}_q \) of the finite \( \mathbb{q} \)-elements field \( \mathbb{F}_q \) (cf. e. g. [Kob, V.1]) : \( Z(\mathbb{P}^1(\mathbb{F}_q)/\mathbb{F}_q; s) = \frac{1}{(1-s)(1-q^s)} \). Since \( q = p^d \) for some positive integer \( d \), where \( p = char(\mathbb{F}_q) \) is the characteristic of the field \( \mathbb{F}_q \), and -as it is well-known, the characteristic of the field \( \mathbb{C} : char(\mathbb{C}) = q \) is zero : \( q = 0 \). Thus formally : \( \frac{1}{s(s-1)} - Z(\mathbb{P}^1(\mathbb{F}_0)/\mathbb{F}_0; s) \), i.e. we formally put \( \mathbb{F}_0 = \mathbb{F}_0 := \mathbb{C} \).

We define also the trivial Riemann zeta \( \zeta_t \) by the trivial formula :
\[ \zeta_t(s) := Im(s)(2Re(s) - 1) = |s(s-1)|^2 Im\left(\frac{1}{s(s-1)}\right), \ s \in \mathbb{C}. \quad (3.42.) \]

4 The Brownian motion and Fubini theorem as a sufficient and efficient tools for a proof of the Riemann hypothesis.

Let \( \mathcal{M} \) be the category of all functional measure spaces. Thus an object of \( \mathcal{M} \) is a triple \((M, A, \mu)\), where \( M = F(\mathbb{R}_+) \) is a non-empty set of functions \( f : \mathbb{R}_+ \rightarrow \mathbb{C} \), endowed
with a \( \sigma \)-field \( \mathcal{A} \) of subsets of \( M \) and \( \mu : \mathcal{A} \to \mathbb{R}_+ \) is a positive \( \sigma \)-additive measure. Then, we can say on the **measure characteristic numbers** \( m_s : \mathcal{M} \to \mathbb{C} \cup \{\infty\} \) as the family of functors on the category \( \mathcal{M} \) indexed by \( s \in \mathbb{C} \) and defined by the formula: let \((M, \mathcal{A}, \mu) \in \mathcal{M} \) be arbitrary. Then

\[
m_s(M, \mathcal{A}, \mu) := \int_{M \times \mathbb{R}_+} x^{s-1} f(x) d\mu(f) dx.
\]

(4.43.)

Let us observe at once three elementary properties of the measure characteristic numbers:

1. \( m_s \) is a **measure invariant**, i.e. if two measure spaces \((X, \mathcal{A}, \mu)\) and \((Y, \mathcal{B}, \nu)\) are **isomorphic** in the category \( \mathcal{M} \) (we write then \((X, \mathcal{A}, \mu) \simeq_{M} (Y, \mathcal{B}, \nu)\)), i.e. there exists a set isomorphism \( f : X \to Y \) with the property that \( \mathcal{A} = f^{-1}(\mathcal{B}) \) and \( \nu = f^*(\mu) \) (\( \nu \) is the transport of \( \mu \) by \( f \)), then

\[
m_s(X, \mathcal{A}, \mu) = m_s(Y, \mathcal{B}, \nu), \quad s \in \mathbb{C}.
\]

(4.44.)

2. \( |m_s(\mathbb{P}_1, \mathcal{P}, r_m)| \leq b_s(\mathbb{P}_1, \mathcal{P}, r_m), \)

(4.45.)

and finally \( m_s \) are **submultiplicative**, i.e.

3. \( m_s(X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mu_1 \times \mu_2) \leq m_s(X_1, \mathcal{A}_1, \mu_1)m_s(X_2, \mathcal{A}_2, \mu_2). \)

(4.46.)

**Remark 2** Let us remark that the above defined measure and Brownian characteristic numbers defined in the category \( \mathcal{M} \), are some analogs of the Betti numbers and the Euler-Poincare characteristic in the category \( \mathcal{T} \) of the topological spaces. Like \( m_s \) are measure invariants (see (1)) the Euler-Poincare characteristic \( \chi(X) \) of a topological space \( X \) is obviously the **topological invariant** (even a **homotopy invariant**). Let us mention here its three fundamental properties:

(i) let \( p : E \to B \) be a locally trivial vector bundle with the fibre \( F \). Then (under some restrictions on the spaces \( E, B, F \)) their characteristics are associated by the formula: \( \chi(E) = \chi(B)\chi(F) \).

(ii) In particular, an Euler-Poincare characteristic of a direct product of two topological spaces is equal their product, i.e. \( \chi(X \times Y) = \chi(X)\chi(Y) \).

(iii) With the help of the relation: \( \chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B) \), which is true for any cutting triple \((A \cup B, A, B)\), one can calculate the Euler-Poincare characteristic of any 2-dimensional compact manifold. In particular \( \chi(\cdot) \) is a finitely additive measure.

**Theorem 3** The Riemann hypothesis is true.

**Proof.** According to Th.2, we have two real valued equations

\[
\text{Im}[\zeta(s)(p)] = \frac{\zeta(s)}{s(s-1)^2} + \int_1^\infty [x^{Re(s)-1} \theta(p)(x) - x^{-Re(s)} \theta(p)(x)] \sin(Im(s)x) dx.
\]

(4.47.)
We integrate (4.48) with respect to the **Wiener-Riemann measure** $r_m$ and obtain:

\[
Im\left[ \int_{\mathbb{P}_1} (M(p)\zeta(s))dr_m(p) \right] = Z(\mathbb{P}^1(\mathbb{C})/\mathbb{C}; s)\zeta(s) + (4.48.)
\]

\[
+ Im\left( \int_{\mathbb{P}_1} dr_m(p) \int_0^\infty dx \sum_{n=1}^\infty [x^{s-1}p(nx) + x^{-s}\hat{p}(nx)] \right).
\]

According to the Th.1 ($RH_4$) we also have

\[
\left( \int_{\mathbb{P}_1} dr_m(p) \int_0^\infty x^{s-1}p(x)dx \right)\zeta(s) = m_s(\mathbb{P}_1, \mathbb{P}, r_m)\zeta(s),
\]

where $m_s(\mathbb{P}_1, \mathbb{P}, r_m)$ is the measure characteristic number of that measure space and obviously: $0 < Re(s) < 1/2$. For the right-hand side of (4.50) we have an ”easy” Fubini-Tonelli theorem.

Reelly, let us consider the triple iterated integral

\[
\int_{\mathbb{P}_1} \int_0^\infty dx \int_{\mathbb{P}_1} |p(nx)| dr_m(p).
\]

(4.49.)

Then, making the substitution : $nx = t$ and using the property ($R_4$) of the Theorem 1 we get

\[
I_{cxy}(\beta) := \sum_{n=1}^\infty \int_1^\infty x^{\alpha-1}dx \int_{\mathbb{P}_1} |p(nx)| dx \leq \zeta(2-\alpha)b_\alpha + \infty, \quad (4.50.)
\]

whereas $0 < \alpha < 1/2$.

Analogously, let us denote

\[
I_{cxy}(\beta) := \sum_{n=1}^\infty \int_1^\infty x^{-\beta}dx \int_{\mathbb{P}_1} |p(nx)| dy \int_{\mathbb{P}_1} |dr(p) = \sum_{n=1}^\infty \int_1^\infty x^{-\beta}dx \int_{\mathbb{P}_1} G(nxy)dy \int_{\mathbb{P}_1} |dr| \leq \zeta(2+1/4) \int_1^\infty dx \int_{\mathbb{P}_1} G(nxy)(nxy)^{1/4}dy.
\]

(4.51.)

Making the substitution : $n^2xy = z, dy = z/n^2x$ in the inner integral on $\mathbb{R}$, we get

\[
I_{cxy}(\beta) \leq \sum_{n=1}^\infty \int_1^\infty \frac{dx}{n^2x^{1+\beta}} \int_{\mathbb{P}_1} G(z/n)(z/n)^{1/4}dz = \zeta(2+1/4) \int_1^\infty \frac{dx}{x^{1+\beta}} \int_{\mathbb{P}_1} G(z)z^{1/4}dz < +\infty, \quad (4.52.)
\]

for all $\beta > 0$. According to the **Tonelli theorem** all above triple iterated integrals coincides with $I_{cxy}(\alpha)$ and they are **finite** for $\alpha \in (0, 1/2)$ whereas all changings of iteration in $I_{cxy}(\beta)$ coincides and are **finite** for all $\beta > 0$. 

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Thus finally, the averaging of the Muntz’s relations - given in Th.2 - with respect to the Wiener-Riemann measure $r_m$ leads to the following functional equation: let $s = u + iv$. Then

$$
Im[\zeta(s) \int_0^\infty x^{s-1} dx \int_{\mathbb{P}_1} p(x) dr_m(p)] = \frac{\int_{\mathbb{P}_1} p(0) dr_m(p) \zeta(s)}{|\zeta_{\mathbb{P}_1}(s)|^2} + (4.53.)
$$

$$
+ \sum_{n=1}^\infty \int_1^\infty x^{-u} \sin(v \log x) dx \int_{\mathbb{P}_1} p(nx) dr_m(p) = \sum_{n=1}^\infty x^{-u} \sin(v \log x) \int_{\mathbb{R}} \cos(2\pi nxy) dy \int_{\mathbb{P}_1} p'(y) dr_m(p).
$$

According to the properties $(RH_0) - (RH_3)$ of Th.1, we have

$$
\int_{\mathbb{P}_1} p(0) dr_m(p) = 1, \int_{\mathbb{P}_1} p(nx) dr_m(p) = 0 \text{ if } x \geq 1, \quad (4.54.)
$$

and

$$
\int_{\mathbb{P}_1} p'(y) dr_m(p) = m(|y|) \text{ if } |y| \leq 1. \quad (4.55.)
$$

Combining (4.55), (4.56) with (4.57) we can reduce the right-hand side of (4.55) to the equation:

$$
Im[m_s(\mathbb{P}_1, \mathcal{P}, r_m)\zeta(s)] = \frac{\zeta_t(s)}{|\zeta_{\mathbb{P}_1}(s)|^2} - \sum_{n=1}^\infty \int_1^\infty x^{-u} \sin(v \log x) \hat{m}(nx) dx. \quad (4.56.)
$$

We have the following easy approximation for the triple integral which appears in the right-hand side of (4.58):

$$
|\sum_{n=1}^\infty \int_1^\infty x^{-u} \sin(v \log x) dx \int_{\mathbb{R}} \cos(2\pi nxy) m(y) dy| \leq \sum_{n=1}^\infty \int_1^\infty x^{-u} |\hat{m}(nx)| dx. \quad (4.57.)
$$

Since obviously $\hat{m} \in \mathcal{S}(\mathbb{R})$, then the right-hand side of (4.59) is less then

$$
(\sum_{n=1}^\infty \frac{1}{n^2}) \int_1^\infty \frac{x^{-u} dx}{x^2} \max_{n \in \mathbb{N}, x \geq 1} |(nx)^2 \hat{m}(nx)| \leq \zeta(2) \int_1^\infty \frac{dx}{x^{2+m} x_0^2} |m(x_0)|. \quad (4.58.)
$$

Let us take any sequence $\{m_k\}$ of moment functions which converges pointwisely to the characteristic function $\chi_0(x)$ of $\{0\}$ in $\mathbb{R}$ (not to the Dirac delta distribution $\delta_0(x)$!). Since $|m_k(x)| \leq \chi_{[0,1]}(x)$, using the Lebesgue’s dominated convergence theorem, for $Re(s) \in (0, 1/2)$ we finally get

$$
\lim_{k \to \infty} Im[m_s(\mathbb{P}_1, \mathcal{P}, r_{m_k})\zeta(s)] = \zeta_t(s) |\zeta_{\mathbb{P}_1(\mathbb{C})}(s)|^2. \quad (4.59.)
$$

Since the non-trivial Riemann zeta $\zeta$ zeros lies symmetricaly with respect to the lines: (i) critical $Re(s) = 1/2$ and (ii) $Im(s) = 0$, then the equation (4.61) gives the most direct and an integral form proof of the Main Algebraic Conjecture (MAC in short):

$$
(MAC) \quad \zeta(\mathbb{C}) \subset \zeta_t(\mathbb{C}).
$$

Thus (RH) is proved.
Remark 3 In [MA, Remark.20] we observed a strange violation of symmetry with respect to the functions $\text{Re}(s)$ and $\text{Im}(s)$ in the Riemann hypothesis problem. Roughly speaking it consists on the fact that the function $\text{Im}(s)$ is much most important than $\text{Re}(s)$ (and in fact fundamental) for (RH). That strategy also is explored in [MH], [MR], [MD] and [AM] (as well as in the prepared now paper [MRam]). In this paper we ”broken” that violation of the symmetry, i.e. in this paper $\text{Im}(s)$ and $\text{Re}(s)$ are ”equally important”. But in our opinion, the mentioned above violation of the symmetry was one from reasons that (RH) was an open problem for 150 years. A trial of a partial explanation of that phenomena, was presented in [MA]. Here we give - maybe - a better explanation of that one : let us observe, that instead of the case of $\text{Re}(s)$, the function $\text{Im}(s)$ seems to be much more interested from the another point of view: it is a simple example of the non-holomorphic modular form of weight 2.

On the other hand - for some time it has been known - that (RH) is strictly connected with the fundamental properties of the theta modular forms.

References

[1] [AI] Aleksjejev V.M. and Iacobson M.B., Appendix, Symbolic dynamics and hyperbolic dynamical systems (in Russian), in Matematika series 13 (Novoje v zarubieznj naukije, editors : A.N. Kolomogorov and S.P. Novikov), Mir, Moskou 1979, p.196-244.

[2] [AM] Albeverio S. and Mądrecki A., Probability theory and the Riemann hypothesis (preprint 2004)p.1-40.

[3] [AWG-Z] Arnold W.I., Varchenko A.N. and Hussein-Zade S.M., Singularities of differentials maps ( Monodromy and asymptotic integrals) (in Russian), Moskou, Nauka, 1984.

[4] [B] Bump D., Automorphic forms and representations

[5] [G] Gelbart S., An elelmentary introduction to the Langlands program ,BAMS10(1984), 177-219.

[6] [H] Hartshorne R., Algebraic Geometry , Graduate Texts in Mathematics 52, Springer-Verlag NY-H-B, 1977.

[7] [JL] Jacquet H. and Langlands R.P., Automorphic Forms on GL(2) , Lecture Notes in Mathematics ( A collection of informal reports and seminars Edited by A. Dold and B.Eckmann), Springer-Verlag, B-H-NY, 1970.

[8] [Ka] Kaczorowski J., Czwarty problem milenijny : Hipoteza Riemanna, Wiadomosci Matematyczne XXXVIII(2002), p.91-120.

[9] [Kob] Kobliitz N., Introduction to Elliptic Curves and Modular Forms ,Springer-Verlag, NY-B-H-T, Graduate Texts in Mathematics 97,1984.
[10] [MA] Mądrecki A., *Algebraic proof of Riemann hypothesis* (preprint 2002, partially referred in TAMS during 2003), 1-62.

[11] [MD] Mądrecki A., *Proof of generalized Riemann hypothesis for Dedekind zetas and Dirichlet L-functions* (preprint 2003 Math. Ann.), 1-48.

[12] [MH] Mądrecki A., *Hermitian proof of Riemann hypothesis* (preprint 2004 Math.Z), 1-32.

[13] [MR] Mądrecki A., *The Riemann hypothesis and some stochastic Laplace representation* (preprint 2004, evaluated in Dissertationes Mathematicae), 1-46.

[14] [MRam] Mądrecki A., *A proof of the Riemann-Ramanujan conjecture* (preprint 2004 prepared to Reports on Mathematical Physics.).

[15] [M] Maurin K., *Analiza II (Wstęp do analizy globalnej)* (in Polish), BM41, PWN Warszawa 1971.

[16] [N] Narkiewicz W. *Uniform Distribution of Sequences of Integers in Residue Classes*, Lecture Notes in Math.1087 Springer-Verlag, B-H-NY-T, 1984.

[17] [Ni] Nitecki Z., *Differentiable Dynamics (An Introduction to the Orbit Structure of Diffeomorphisms)*, The MIT Presss, Cambridge, Massachusetts, and London, England, 1971.

[18] [R] Riemann B., *Ueber die Anzahl der Primzahlen unter einer gegebener Grosse*, Monatsberichte Akad. Berlin, November 1859=Gesammelte math. Werke, Berlin 1892,145-153.

[19] [Ru] Ruelle D., *Dynamical zeta functions and transfer operators*, Notices of the AMS, N.8, Vol.49, September 2002.

[20] [WH] Wan D. and Haessig C.D., *On the p-adic Riemann hypothesis for the zeta function of divisors*, JNT 104(2004),335-352.

[21] [W1] Wan D., D. Jungnickel, H. Niederreiter(Eds.), *Finite Fields and Applications*, Springer, Berlin,2001,pp.437-461; Manuscripta Math.74(1992)pp.413-444.

[22] [W2] Wan D., *On the Riemann hypothesis for the characteristic p zeta function*, JNT 58(1996), pp.196-212.

[23] [We] Weil A., *Basic Number Theory*, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, Band 144, Springer 1967.

[24] [Wi] Williams R.F., *The structure of Lorentz Attractors*, Lecture Notes in Math. 615, Springer-Verlag, BHNY 1977, p.94-112.
[25] [Wo] Wong E., *Stochastic Processes in Information and Dynamical Systems*, McGraw-Hill, Inc., 1971.

[26] [Sa] Mądrecki A., *On Sazonov type topology in p-adic Banach space*, Math. Z. 30(1985), 225-236.

[27] [Se] Selberg A., *Contributions to the theory of the Riemann zeta-function*, Archiv f. Mathematik og Naturvidenskab.48(1946), p.89-155.

[28] [S] Shimura G., *Introduction to the arithmetic theory of automorphic functions*, Iwanami Shoten, Publishers and Princeton University Press, 1971.

[29] [T] Titchmarsh E., *The theory of the Riemann Zeta-function*, Oxford University Press, London, 1951.

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