A QUANTUM ALGORITHM TO SOLVE NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper we describe a quantum algorithm to solve sparse systems of nonlinear differential equations whose nonlinear terms are polynomials. The algorithm is nondeterministic and its expected resource requirements are polylogarithmic in the number of variables and exponential in the integration time. The best classical algorithm runs in a time scaling linearly with the number of variables, so this provides an exponential improvement. The algorithm is built on two subroutines: (i) a quantum algorithm to implement a nonlinear transformation of the probability amplitudes of an unknown quantum state; and (ii) a quantum implementation of Euler’s method.

1. Introduction

Systems of nonlinear differential equations arise in an astounding number of applications across the sciences ranging from engineering, biological systems, and mathematics. Their theory occupies a large proportion of the scientific literature and many subtle and profound techniques have been developed to solve them. The numerical solution of nonlinear ODEs is now a mature and well-established topic (see, e.g., [17]) and there are many stable and efficient classical algorithms to numerically integrate these equations, ranging from the “workhorse” Runge-Kutta method, to predictor-corrector methods and implicit methods.

The discovery of quantum algorithms (see, e.g., [13] for an introduction to quantum computation) has ushered in a new era where previously intractable problems can now be theoretically solved efficiently on a quantum computer. Such was the optimism generated by the discovery of the methods such as Shor’s factoring algorithm that many felt it would be a matter of time before efficient quantum algorithms would be found for many natural problems. This optimism has been largely diminished by the realisation that efficient quantum algorithms exploit subtle – and still largely mysterious – properties of delicate quantum superposition. Despite these complications several flavours of quantum algorithm have been developed exploiting, variously, algebraic structures, symmetries, and geometry (see, e.g., [5] for a recent review). Arguably some of these algorithms have less practical utility than, say, Shor’s factoring algorithm, and constitute more of a proof of principle.

Recently an efficient quantum algorithm to solve systems of linear equations [8] was described. This promises to allow the solution of, e.g., vast engineering problems. This result is inspirational in many ways and suggests that quantum computers may be good at solving more than linear equations. In this paper we investigate this hope in the context of nonlinear ODEs and we present an efficient quantum algorithm to integrate large sparse systems of nonlinear ODEs.

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2. Preliminaries

In this paper we are concerned with the solutions of a set of $n$ first-order nonlinear ODEs whose nonlinear terms are given by $n$ polynomials $f_\alpha(z)$ in $n$ variables $z_j$, $j = 1, 2, \ldots, n$, over $\mathbb{C}$. That is, we look for solutions $z(t)$ of the simultaneous set

$$
\frac{dz_1(t)}{dt} = f_1(z_1(t), z_2(t), \ldots, z_n(t)) \\
\frac{dz_2(t)}{dt} = f_2(z_1(t), z_2(t), \ldots, z_n(t)) \\
\vdots \\
\frac{dz_n(t)}{dt} = f_n(z_1(t), z_2(t), \ldots, z_n(t)),
$$

subject to the boundary condition $z(0) = b$. Standard results ensure that a solution to this initial value problem exists and is unique (see, eg., [1]).

We'll mostly only describe the algorithm for quadratic systems (the extension to higher degrees is straightforward). Quadratically nonlinear equations can exhibit a wide variety of phenomena, including, chaos and anomalous diffusion and classical examples include the eponymous Lorenz system and the Orszag-McLaughlin dynamical system.

We encode the variables $z_j(t)$ as the probability amplitudes of a quantum state of an $(n + 1)$-level quantum system:

$$|\phi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}\sum_{j=1}^{n} z_j |j\rangle,$$

where, to ensure that the state is normalised, we require $\sum_{j=1}^{n} |z_j|^2 = 1$. (Our constructions ensure that $|\phi\rangle$ always remains normalised.) While it is convenient to regard $|\phi\rangle$ as the state of a single $(n + 1)$-level quantum system, when actually implementing the algorithm on a quantum computer we'll encode $|\phi\rangle$ as a state of $\log(n)$ qubits in the natural way.

We describe our procedure in two stages: (1) a method to effect nonlinear transformations of the probability amplitudes of $|\phi\rangle$; (2) a quantum algorithm to implement Euler’s method.

3. A Quantum Algorithm to Effect a Nonlinear Transformation of the Amplitudes

In this section we describe a nondeterministic algorithm to prepare quantum states whose amplitudes are nonlinear functions of those of some input quantum state.

Suppose we want to effect a quadratic transformation on the probability amplitudes $z_j$ of $|\phi\rangle$. (We assume throughout that the initial state $|\phi\rangle$ can be efficiently prepared on a quantum computer, eg., using a combination of the methods of [7] and [8].) Since the amplitudes are unknown this is, in general, impossible and, unfortunately, this is the generic setting when integrating ODEs as every point on the solution trajectory can be regarded as an initial condition for the system. But suppose we have two copies of $|\phi\rangle$. In this case the probability amplitudes of the tensor product are given by

$$|\phi\rangle|\phi\rangle = \frac{1}{2} \sum_{j,k=0}^{n} z_j z_k |j\rangle|k\rangle,$$
where, for convenience, we set \( z_0 = 1 \) from now on. Evidently every monomial \( z_{j_1}^{l_{j_1}} z_{k_1}^{l_{k_1}} \), \( j_1, k_1 \leq 1 \), of degree less than 2 appears (more than once) in this expansion. Suppose we want to iterate the transformation

\[
\begin{align*}
  z \mapsto F(z),
  \end{align*}
\]

where

\[
F(z) = \begin{pmatrix}
  f_1(z) \\
  f_2(z) \\
  \vdots \\
  f_n(z)
\end{pmatrix},
\]

and \( f_\alpha, \alpha = 1, 2, \ldots, n, \) are quadratic polynomials

\[
f_\alpha(z) = \sum_{k,l=0}^{n} a^{(\alpha)}_{kl} z_k z_l,
\]

with \( a^{(\alpha)}_{kl} = a^{(\alpha)}_{lk} \) and \( f_0(z) = 1 \). Thus we aim to prepare the quantum state

\[
|\phi'\rangle = \frac{1}{\sqrt{2}} \sum_{\alpha=0}^{n} f_\alpha(z) |\alpha\rangle.
\]

where, to simplify matters, we’ve assumed that the transformation is measure preserving, i.e.,

\[
1 = \sum_{j=1}^{n} |z_j|^2 = \sum_{\alpha=1}^{n} f_*^{(\alpha)}(z) f_\alpha(z).
\]

The measure-preservation assumption plays an important role in our constructions, however, if it is relaxed our algorithm still proceeds unchanged: only the success probability is modified.

To ensure that our method is efficient we need to make several extra assumptions beyond measure preservation. The first assumption is that \( |a^{(\alpha)}_{kl}| = O(1) \), \( k,l,\alpha = 1, 2, \ldots, n \). The second assumption is that the map \( F \) is sparse, which means that

\[
|\{(k,l) \mid a^{(\alpha)}_{kl} \neq 0\}| \leq s/2, \quad \alpha = 1, 2, \ldots, n,
\]

and

\[
|\{\alpha \mid a^{(\alpha)}_{kl} \neq 0\}| \leq s/2, \quad k,l = 1, 2, \ldots, n,
\]

where \( s = O(1) \). Note that the assumption of sparsity means that each \( f_\alpha(z) \) can only involve at most \( s/2 \) monomials and that each variable appears in at most \( s/2 \) polynomials \( f_\alpha(z) \). The final assumption is that the Lipschitz constant for our system, i.e., that number \( \lambda \) such that

\[
\|F(x - y)\| \leq \lambda \|x - y\|
\]

in the ball \( \|x\|^2 \leq 1 \) and \( \|y\|^2 \leq 1 \), is \( O(1) \). While these assumptions are rather restrictive, as we discuss, there are still many important systems which satisfy them. We also later describe how to relax these assumptions.

It turns out that implementing the desired transformation will, in general, require that we make some destructive nonunitary transformation of the system’s state. To understand what is required we now set up the operator

\[
A = \sum_{\alpha,k,l=0}^{n} a^{(\alpha)}_{kl} |\alpha0\rangle \langle kl|.
\]
We now initialise our system in the state

\[ H = -iA \otimes |1\rangle_P \langle 0| + iA^\dagger \otimes |0\rangle_P \langle 1|. \]

We now evolve our system according to \( H \) essentially the von Neumann measurement prescription [16]):

\[ |\phi\rangle |\phi\rangle |0\rangle_P, \]

and evolve according to \( H \) for a time \( t = \epsilon \). The time we can evolve depends crucially on the sparsity of \( H \) and the desired error [3]; roughly speaking, when the sparsity of \( A \) is some constant \( s \) we can efficiently simulate (in terms of \( \log(n) \)), up to some prespecified precision, the evolution for any constant time \( t \).

After the evolution the system ends up in the state

\[ |\Psi\rangle = e^{i\epsilon H} |\phi\rangle |\phi\rangle |0\rangle = \sum_{j=0}^{\infty} \frac{(i\epsilon H)^j}{j!} |\phi\rangle |\phi\rangle |0\rangle = |\phi\rangle |\phi\rangle |0\rangle + \epsilon A |\phi\rangle |\phi\rangle |1\rangle + \cdots . \]

Now noting that

\[ A |\phi\rangle |\phi\rangle = \frac{1}{2} \sum_{\alpha,k,l=0}^{n} a_{kl}^{(\alpha)} z_k z_l |\phi\rangle |\phi\rangle = \frac{1}{\sqrt{2}} |\phi'\rangle |0\rangle \]

we now measure \(|\Psi\rangle\) on the ancilla qubit and postselect on \( |1\rangle \); we succeed with probability \( \approx \frac{1}{2} e^2 \) (thanks to the measure preservation property; if our polynomial map is not measure preserving then this success probability will be proportionally lower) and our posterior state is

\[ \frac{\sqrt{2}}{\epsilon} (I \otimes I \otimes \langle 1|) |\Psi\rangle = \frac{1}{\sqrt{2}} \sum_{\alpha,k,l=0}^{n} a_{kl}^{(\alpha)} z_k z_l |\phi\rangle |\phi\rangle = |\phi'\rangle |0\rangle. \]

If we fail then we end up with some “poisoned” state \(|\Psi'\rangle\) (this occurs with probability \( \approx 1 - \frac{1}{2} e^2 \)), which we discard. In order to ensure, with high probability, that we end up with at least one copy of \(|\phi'\rangle\) we need to repeat this process on roughly \( 16/\epsilon^2 \) fresh pairs \(|\phi\rangle |\phi\rangle\). It is an interesting question whether one can design \( H \) so that the poisoned state can be recovered and used again. Such a possibility would allow one to substantially reduce the resource requirements of our algorithm.

(Obviously, because the expansion (15) is truncated to first order, we don’t exactly produce \(|\phi'\rangle\), but rather some approximation to \(|\phi'\rangle\). To correct this we actually use the method described in [8] to implement the transformation

\[ |\phi\rangle |\phi\rangle |0\rangle \mapsto \sqrt{1 - \epsilon^2 H^2} |\phi\rangle |\phi\rangle |0\rangle + i \epsilon H |\phi\rangle |\phi\rangle |0\rangle, \]

where \( |\epsilon| \leq 1/\|H\| \). Since \( H \) is sparse we have, by Geršgorin’s theorem [10], that \( \|H\| \leq cs \), where \( c \) is a constant. This allows us to assume that when we succeed we will obtain precisely a copy of \(|\phi'\rangle\), modulo only imperfections in the simulation of \( e^{i\epsilon H} \). We discuss these errors in the appendix.)

If we want to iterate the polynomial map \( z \mapsto F(z) \) a constant (in \( n \)) number \( m \) times to produce the state \(|\phi^{(m)}\rangle\) we need to start with \( (\frac{\sqrt{2}}{\epsilon})^m \) initial states. Thus the total spatial resources required by this algorithm scale as \( (\frac{\sqrt{2}}{\epsilon})^m \log(n) \) because the cost of storing \( n \) variables via encoding in \(|\phi\rangle\) scales linearly with \( \log(n) \). (See Proposition 1 for a precise statement of the spatial resource requirements of our algorithm.)

The simulation of the evolution \( e^{i\epsilon H} \) on a quantum computer cannot be done perfectly; we quantify, in proposition 2, the errors that accumulate throughout the running of the polynomial iteration algorithm: if the evolution \( e^{i\epsilon H} \) can be simulated up to error \( \delta \), then after \( m \) steps the final state will have accumulated
an error no worse that $\delta(3\gamma)^{m+1}$. Thus, choosing the simulation error to satisfy $\delta < (3\gamma)^{-m}$ will ensure that the $m$th iterate is exponentially close the desired state. The costs [3] of simulation to this level of precision imply that the total running time $T$ of our algorithm scales as $T \sim m \text{poly}(\log(n) \log^*(n))s^{2\kappa\sqrt{m}}$, where $\kappa$ is some $O(1)$ constant and

$$\log^*(n) \equiv \min\{r \mid \log^{(r)}(n) < 2\}$$

with $\log^{(r)}$ denoting the $r$th iterated logarithm, and we’ve assumed that the transformation (18) is carried out in parallel on each of the remaining pairs in each iteration. Thus the parallelised temporal scaling is subexponential in $m$ and polynomial in $\log(n)$.

To complete the description of our iteration algorithm we need to describe how to read out information about the solution. After $m$ iterations the system will be, up to some prespecified error, in the quantum state $|\phi(m)\rangle$ encoding the $m$th iterate of $F$ in the probability amplitudes. To access information about the solution we need to make measurements of the system. In principle, any hermitian observable $M = \sum_{j,k=0}^{n} M_{j,k} |j\rangle\langle k|$ may be measured to extract information. Via Hoeffding’s inequality we learn that we can estimate, using repeated measurements, the quantity

$$\langle M \rangle \equiv \sum_{j,k=0}^{n} z_j M_{j,k} z_k$$

to within any desired additive error. In practice the observable $M$ is measured using (a discretisation of) von Neumann’s measurement prescription [4, 16], so the evolution $e^{iM}$ must be efficiently simulable on a quantum computer. Many natural such operators fall into this class, including – via a quantum fourier transform – the operator whose measurement statistics provide information on the sums

$$S_k = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} x_j e^{\frac{2\pi i j k}{n}}.$$  

As noted in [8], and evident from our construction here, by measurement of multiple copies of $|\phi(m)\rangle$ it is also possible to extract information about polynomial functions of the solution.

The flexibility to measure efficiently implementable hermitian operators to extract information about the solution provides the key to the exponential separation between our method and the best classical method. Indeed, if we only wanted to learn one element $z_j$, for some $j$, of the $m$th iterate then there is actually an efficient classical algorithm with the same resource scaling. (This is similar to the situation with linear equations [6, 2] where if we only want to learn about one element of the solution vector of a well-conditioned sparse set of linear equations we can do this efficiently classically. Indeed, even if the system is badly conditioned, we can still learn about parts of the solution in the well-conditioned subspace.)

4. Solving nonlinear differential equations

In this section we show how to use the method we’ve just described to integrate a sparse set of simultaneous nonlinear differential equations for any constant time. Suppose we want to integrate the system\(^1\) (1) with the initial condition $z(0) = b$, where $f_j$ are sparse polynomials. The simplest approach is to use Euler’s method

\(^1\)We again assume that our system is measure preserving which means that $\sum_{j=0}^{n} z_j f_j(z(t)) + z_j f_j^*(z(t)) = 0$ and that the Lipschitz constant is $O(1)$.\)
[17]: we pick some small step size \( h \) and iterate the map

\[
(22) \quad z_j \mapsto z_j + h z_j' = z_j + h f_j(z).
\]

While Euler’s method is pretty terrible in practice, especially for stiff ODEs, it does provide a basic proof of principle; more sophisticated methods such as 4th order Runge-Kutta and predictor-corrector methods are essentially only polynomially more efficient. As our algorithm is precisely an implementation of Euler’s method on the probability amplitudes, we can appeal to the standard theory (see, e.g., [11]) concerning its correctness and complexity; thus our algorithm suffers from all the standard drawbacks of Euler’s method, including a 1st-order decrease in error in terms of the step size \( h \). Nonetheless, it will provide us with an algorithm scaling polynomially with \( \log(n) \), where \( n \) is the number of variables. However, it should be noted that our method scales exponentially with the inverse step size. Thus, without modification, our algorithm is really only suited to well-conditioned systems.

As we’ve indicated, the idea behind our approach is very simple: we integrate the system using Euler’s method. Thus, given \( |\phi(t)\rangle \) we aim to prepare

\[
(23) \quad |\phi(t + h)\rangle = |\phi(t)\rangle + h|\phi'(t)\rangle + O(h^2),
\]

where now

\[
(24) \quad |\phi'(t)\rangle = \frac{1}{\sqrt{2}} \sum_{\alpha=0}^{n} f_\alpha(z(t))|j\rangle = \frac{1}{\sqrt{2}} \sum_{\alpha,k,l=0}^{n} a_{k,l}^{(\alpha)} z_k(t)z_l(t)|\alpha\rangle.
\]

To implement this transformation we suppose we have two copies of \( |\phi(t)\rangle \) and apply the method of the previous section to implement the polynomial transformation \( z_\alpha \mapsto z_\alpha + h f_\alpha(z(t)) \). Note that this transformation is only measure preserving to \( O(h) \); the success probability will be diminished by a factor of \( O(h^2) \), which can be made negligible by reducing \( h \) in the standard way.

So, to integrate the system \( (1) \) forward in time to \( t = O(1) \) we begin by discretising time into \( m \) steps. (Thus our step size is \( h = t/m \).) We then prepare \( (\frac{16}{\pi})^m \) copies of \( |\phi(0)\rangle \), where \( \epsilon \) is as in §3, and apply the method of §3 to produce approximately \( (\frac{16}{\pi})^m \) copies of \( |\phi(t/m)\rangle \) in expected time \( \text{poly}(\log(n)) \). We then iterate until we produce at least one copy of \( |\phi(t)\rangle \) in expected time \( \text{poly}(m, \log(n)) \) with probability greater than \( 1/3 \). The resources required by this approach scale polynomially with \( \log(n) \) and exponentially with \( t \) and \( 1/h \). 

5. Extensions and applications

In this section we briefly describe several extensions and applications of our quantum Euler’s method.

What sort of systems will be tractable with our approach? We only sketch a couple of examples here, leaving the wider application of our approach to more detailed investigations. Because the sparsity and the measure-preserving properties play a key role in the resource scaling of our algorithm it is desirable to focus on those sparse systems preserving the “hamiltonian” \( \sum_{j=1}^{n} |z_j|^2 \). One example of such a system is the Orszag-McLaughlin dynamical system [14, 15]:

\[
(25) \quad \frac{dx_j}{dt} = x_{j+1}x_{j+2} + x_{j-1}x_{j-2} - 2x_{j+1}x_{j-1}, \quad j = 1, \ldots, n,
\]

with periodic boundary conditions \( x_{n+1} \equiv x_1 \). The variables \( x_j \) are real and preserve \( \sum_{j=1}^{n} x_j \). The dynamics generated by this system are extremely complicated.

Another example of a system which can be studied using our algorithm is the (discrete) nonlinear Schrödinger equation on any finite graph \( G = (V,E) \) of bounded
degree:

\[ -i \frac{dz_v}{dt} = 2 \deg(v) z_v - \sum_{w \sim v} z_w + |z_v|^k z_v, \quad v \in V, \]

where \( k \in \mathbb{N} \).

Several extensions of our algorithm are possible. The first obvious extension is to systems whose nonlinearity is cubic or higher. This can be done in the natural way by consuming 3 (or more, for higher degrees of nonlinearity) copies of \( |\phi(t)\rangle \) at each step to implement the desired nonlinear transformation. A second extension is to certain densely defined systems, i.e., those for which the operator is dense: because, following [8], we can actually implement any efficiently computable function \( g \) of \( A \) in the step, we can access some dense operators \( g(A) \). A third extension allows the efficient computation of the equal-time statistics of deterministic dynamical systems. Here the idea is to apply our method not to \( |\phi\rangle = \sum_{\alpha} z_{\alpha}(t)|\alpha\rangle \), but rather to one half of an initial state which is an entangled pair:

\[ |\Phi\rangle = \int d\mu(z)|\phi(z)\rangle|z\rangle_P, \]

where \( \langle z|z'\rangle = \delta(z - z') \) and \( d\mu(z) \) is an efficiently implementable probability measure, e.g., uniform or gaussian [7]. Applying our algorithm to this initial state allows us to efficiently sample equal-time statistics via measurements on \( P \). This should be contrasted with the classical Hopf functional approach [9] to solving this problem which introduces a Fokker-Planck type partial differential equation to study distributions of solution trajectories. (The application of the Hopf functional approach to the Orszag-McLaughlin system is considered in [12].)

Finally, it is not implausible that there is a trade-off between time and space; perhaps there is a quantum algorithm which integrates a constant number of variables which scales polynomially with \( \log(t) \) and \(-\log(h)\)?

6. Conclusions and future directions

We have presented a quantum algorithm to iterate large systems of sparse polynomial maps. We’ve also described an implementation of Euler’s method to integrate a system of ODEs. As long as the system is sparse the resources required by the method are polynomial in \( \log(n) \), where \( n \) is the number of variables. However, the resources consumed by the method scale exponentially with the inverse step size and the integration time, as well as with the degree of the nonlinearity.

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Appendix A. Proofs of the claims

In this appendix we present the technical proofs of the claims made in the text.

A.1. Description of the algorithm. In this subsection we provide the formal specification of our iteration algorithm.

Algorithm 1

Set \( N = (\frac{\gamma}{2})^{-m} \), according to Proposition 1;
Initialise the system in the state \( |\Phi\rangle := (|\phi\rangle |\phi\rangle |0\rangle)^\otimes N/2 \), where \( |\phi\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} \sum_{j=1}^{n} z_j |j\rangle \);
Set \( H := -iA \otimes |1\rangle\langle 0| + iA^\dagger \otimes |0\rangle\langle 1| \);
for \( i = m \) to 1 in steps of \(-1\) do
    \( S = 0 \);
    for \( j = 1 \) to \( N \) in steps of \( 1 \) do
        Evolve the \( j \)th pair \( |\phi_{2j-1}\rangle |\phi_{2j}\rangle |0\rangle \) according to (18), measure the ancilla, and postselect on “1”.
        \( |\Phi'_j\rangle := -i\sqrt{2} (1 \otimes 1 \otimes |1\rangle) H |\phi_{2j-1}\rangle |\phi_{2j}\rangle |0\rangle \);
        if success (i.e. \( |\Phi'_j\rangle = |\phi_{2j-1}\rangle |0\rangle \)) then \( S := S + 1 \);
    end for
    \( N := 2 \lfloor S/2 \rfloor \);
if \( S < 2^{i-1} \) then Exception: algorithm failed;
Set \( |\Phi\rangle := (|\phi^n\rangle |\phi^n\rangle |0\rangle)^\otimes N/2 \);
end for

A.2. Bounding the expected running time. We now prove the following

Proposition 1. Let \( m \geq 6 \). If the number \( N \) of initial states for Algorithm 1 with \( p = e^2/2 \) is \( (\frac{\gamma}{10})^{-m} \), then it succeeds in producing at least one state \( |\phi^{(m)}\rangle \) with probability at least \( 1/3 \).

Proof. Let \( S \) be the random variable which counts the number of successes out of the \( N/2 \) trials in each round and let \( N_j \) denote the number of successfully produced states in round \( j \). Hoeffding’s inequality provides a bound on the cumulative
distribution function of a binomial random variable $S$ with parameters $(N, p)$:

$$F(k; N, p) = \mathbb{P}(S \leq k) \leq e^{-\frac{2(Np-k)^2}{N}}$$

for $0 \leq k \leq \mathbb{E}[S] = \frac{Np}{2}$. We declare failure if $S \leq k$, so that, defining $k = \lambda N$, we have

$$\mathbb{P}(\text{failure}) \leq e^{-\frac{2(Np-\lambda N)^2}{N}} = e^{-2N(p-\lambda)^2}.$$

Therefore

$$\mathbb{P}(\text{success}) \geq 1 - e^{-2N(p-\lambda)^2}.$$ 

We want the $m$-step algorithm to produce at least one state $|\phi^{(m)}\rangle$ encoding the $m$th iterate, i.e. we want $N_m \geq 1$ with probability at least $1/3$. That is, we require

$$\mathbb{P}(m \text{ successes}) = \mathbb{P}(\text{Success}_1)\mathbb{P}(\text{Success}_2) \cdots \mathbb{P}(\text{Success}_m) \geq (1 - e^{-2N_1(p-\lambda)^2})(1 - e^{-2N_2(p-\lambda)^2}) \cdots (1 - e^{-2N_m(p-\lambda)^2}) \geq 1/3.$$ 

If the algorithm succeeds, after one step we produce at least $\frac{k}{2} = \frac{\lambda N}{2}$ states ($k$ successes) with probability at least $(1 - e^{-2N_1(p-\lambda)^2})$, and so after $j$ steps we have that

$$N_j \geq \left(\frac{\lambda}{2}\right)^{j-1} N$$

with probability at least $(1 - e^{-2N_1(p-\lambda)^2})(1 - e^{-2N_2(p-\lambda)^2}) \cdots (1 - e^{-2N_j(p-\lambda)^2})$.

To ensure that the final success probability is greater than $1/3$ we demand that

$$\mathbb{P}(\text{Success}_j) \geq 1 - e^{-2N_j(p-\lambda)^2} \geq 1 - \frac{1}{m}$$

so that

$$\mathbb{P}(m \text{ successes}) \geq \left(1 - \frac{1}{m}\right)^m = e^{-1} + O(1/m) \geq 1/3,$$

for $m \geq 6$. Therefore, for all $j$ we need that

$$1 - e^{-2N_j(p-\lambda)^2} \geq 1 - \frac{1}{m}$$

that is,

$$\frac{1}{m} \geq e^{-2N_j(p-\lambda)^2}.$$ 

The RHS of (29) is decreasing in $N_j$ and so is maximum when $j = m$. From (28) we have

$$N_m \geq \left(\frac{\lambda}{2}\right)^{m-1} N$$

substitution into (29) gives

$$\frac{1}{m} \geq e^{-2\left(\frac{\lambda}{2}\right)^{m-1}N(p-\lambda)^2},$$

that is, we require that

$$\log m \leq 2\left(\frac{\lambda}{2}\right)^{m-1} N(p-\lambda)^2.$$
If we choose \( \lambda = \rho/2 \) and \( N = (\lambda/4)^{-m} = (\rho/4)^{-m} \) then from (28) we have

\[
N_m \geq \left(\frac{\lambda}{2}\right)^{m-1} N = \left(\frac{\lambda}{2}\right)^{m-1} \left(\frac{\lambda}{4}\right)^{-m} = 2^{m-2}\frac{\lambda^m}{\lambda},
\]

and the result follows.

\[\square\]

A.3. **Bounding the accumulated error.** We now bound the accumulated error that builds up throughout the running of the quantum iteration algorithm.

**Proposition 2.** The error \( \delta_m = \|\phi^{(m)} - \psi^{(m)}\| \) that accumulates after \( m \) iterations of the quantum iteration algorithm 1 is bounded by

\[
\delta_m \leq \frac{\eta}{3} \left( \frac{(3\gamma)^{m+1} - 1}{3\gamma - 1} - 1 \right)
\]

where \( \eta \) is the error in simulating \( e^{i\varepsilon H} \), \( \psi^{(m)} \) is the actual state produced by the algorithm, and \( \gamma \) is an \( O(1) \) constant which depends on the sparsity \( s \).

**Proof.** Our proof works by analysing the errors that accumulate in pairs \( |\phi_j\rangle|\phi_j\rangle \) of states during the \( j \)th round. Suppose we have an error in the starting pair, so instead of \( |\phi_0\rangle \) we have \( |\phi_0\rangle + |\Delta\phi_0\rangle = |\psi_0\rangle \), say, where the initial error \( |\Delta\phi_0\rangle \) has magnitude \( \delta_0 \), i.e. \( \delta_0 = \|\Delta\phi_0\| \). Suppose also that our simulation of \( U = e^{i\varepsilon H} \) is imperfect, i.e. \( V \) is the operator that is actually applied and \( \|U - V\|_{\infty} \leq \eta \).

We initialise our system into the state \( |\psi_0\rangle|\psi_0\rangle|0\rangle \) and evolve according to \( V \). This particular pair will then be in the state

\[V|\psi_0\rangle|\psi_0\rangle|0\rangle\]

To measure this state on the ancilla qubit and postselect on “1” we apply the measurement operator \( P_1 = \mathbb{I} \otimes \mathbb{I} \otimes |1\rangle\langle 1| \). The (subnormalised) posterior state of the pair is then

\[|\bar{\psi}_1\rangle = P_1V(|\phi_0\rangle + |\Delta\phi_0\rangle)(|\phi_0\rangle + |\Delta\phi_0\rangle)|0\rangle.\]

Recall that if there were no errors in the starting state and the algorithm was perfect we would have

\[|\phi_1\rangle = \frac{\sqrt{2}}{\epsilon} P_1U|\phi_0\rangle|\phi_0\rangle|0\rangle.\]

Let the error between the subnormalised posterior states \( |\bar{\psi}_1\rangle \) and \( |\bar{\phi}_1\rangle \) after one step have magnitude \( \overline{\delta}_1 = \|\bar{\psi}_1\rangle - |\bar{\phi}_1\rangle\| \). We bound this as follows

\[
\overline{\delta}_1 \leq \|P_1V(|\Delta\phi_0\rangle|\phi_0\rangle + |\phi_0\rangle|\Delta\phi_0\rangle + |\Delta\phi_0\rangle|\Delta\phi_0\rangle) + P_1(V - U)|\phi_0\rangle|\phi_0\rangle\|
\leq \|\Delta\phi_0\rangle|\phi_0\rangle + |\phi_0\rangle|\Delta\phi_0\rangle + |\Delta\phi_0\rangle|\Delta\phi_0\rangle\| + \|V - U\|_{\infty}
\leq \delta_0 + \delta_0 + \delta_0^2 + \eta \leq 3\delta_0 + \eta.
\]

We bound the error \( \delta_1 \) between the normalised posterior states by

\[
\delta_1 = \|\psi_1\rangle - |\phi_1\rangle\| \leq \frac{2}{\|\bar{\phi}_1\rangle\|} \overline{\delta}_1
= \frac{2\sqrt{2}}{\epsilon} (3\delta_0 + \eta) = \gamma(3\delta_0 + \eta),
\]

where \( \gamma \) is, by assumption, a constant of order \( s \).
Repeating this argument allows us to set up the recurrence
\begin{equation}
\delta_j \leq \gamma(3\delta_{j-1} + \eta), \quad j = 1, 2, \ldots, m.
\end{equation}
Solving the recurrence, and assuming that the initial states are constructed perfectly (i.e., $\delta_0 = 0$) gives us
\begin{equation}
\delta_m \leq \eta \left( \frac{(3\gamma)^m - 1}{3\gamma - 1} - 1 \right).
\end{equation}
\hfill \Box

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