The Feynman Variational Principle in the Worldline Representation of Field Theory

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Abstract

Following Feynman’s treatment of the non-relativistic polaron problem, similar techniques are used to study relativistic field theories: after integrating out the bosonic degrees of freedom the resulting effective action is formulated in terms of particle trajectories (worldlines) instead of field operators. The Green functions of the theory are then approximated variationally on the pole of the external particles by using a retarded quadratic trial action. Application to a scalar theory gives non-perturbative, covariant results for vertex functions and scattering processes. Recent progress in dealing with the spin degrees of freedom in fermionic systems, in particular Quantum Electrodynamics, is discussed. We evaluate the averages needed in the Feynman variational principle for a general quadratic trial action and study the structure of the dressed fermion propagator.

I. INTRODUCTION: THE POLARON PROBLEM

Variational principles are widely used in classical mechanics, atomic, molecular and nuclear physics but rarely in relativistic field theory. However, there exists a successful example in a non-relativistic field theory, the so-called polaron problem. This describes an electron which slowly moves through an ionic crystal and polarizes its surroundings: the lattice vibrations (phonons) act back on the electron and change its properties. Feynman [1] realized that the phonons can be integrated out exactly in the path integral giving rise to an effective two-time action for the electron only. He then applied a variational principle based on Jensen’s inequality

\[ \int \mathcal{D}x \, e^{-S} \geq \int \mathcal{D}x \, e^{-S_t} \cdot \exp\left(-\langle S - S_t \rangle_{S_t}\right) \]

where \( \langle \ldots \rangle_{S_t} \) denotes an average with respect to a suitable trial action \( S_t \). Feynman choose a retarded quadratic action and obtained excellent results for the ground state energy. Actually his is the best analytic method which works for all coupling constants: for weak coupling one recovers the 1-loop result whereas at large coupling the maximal deviation is only 2.2 %.
II. SCALAR THEORY: THE RELATIVISTIC POLARON

We have extended Feynman’s polaron approach to a simple relativistic theory, the scalar Wick-Cutkosky model which describes heavy particles (“nucleons”) interacting by the exchange of light particles (“mesons”) via a Yukawa coupling $\mathcal{L}_{\text{int}} = -g \Phi^2 \varphi$. Starting from the generating functional for the Green functions of the theory the following steps lead to a “relativistic polaron”

(a) Integrate out the heavy particle $\Phi$ and neglect the determinant, i.e. pair production (“quenched approximation”).

(b) Use the proper time representation to exponentiate

$$\frac{1}{\hat{p}^2 - M^2 - 2g\varphi(\hat{x}) + i0} = \int_0^\infty \frac{dT}{2i\kappa_0} \exp \left\{ \frac{iT}{2\kappa_0} \left[ \hat{p}^2 - M^2 - 2g\varphi(\hat{x}) \right] \right\}$$

(2)

where $\kappa_0 > 0$ reparametrizes the proper time without changing the physics. We note that $\kappa_0 \to i$ and $p^2 \to -p^2$ gives the corresponding expression in Euclidean space.

(c) Write the $x$-space matrix elements of Eq. (2) as a path integral over the particle trajectory $x_{\mu}(t)$ (“particle or worldline representation”).

(d) Perform the integration over the meson field $\varphi$ to get an effective, retarded two-time action in terms of the nucleon trajectory only.

(e) Make a variational approximation using the Feynman-Jensen variational principle with a retarded, quadratic trial action $S_t$. For $T \to \infty$ the 2-point function $G_2(p)$ develops a pole at $p^2 = M^2$ and one finds the following relation between bare mass $M_0$ and physical mass $M$

$$-M_0^2 = (\lambda^2 - 2\lambda)M^2 + 2\left\{ \Omega[A] + V_{\text{WC}}[\mu^2] \right\}.$$  

(3)

This we call Mano’s equation because K. Mano was the first to study the scalar field theory with polaron methods [3].

In Eq. (3) $\lambda$ is a variational parameter and $\Omega[A]$ and $V_{\text{WC}}[\mu^2]$ denote the kinetic and potential term, respectively. The first one arises from the quadratic fluctuations while the latter one is obtained by averaging the interaction term with the trial action. $A(E)$ denotes the variational “profile function” and $\mu^2(\sigma)$ the “pseudotime” which is related to $A(E)$ via a nonlinear integral transformation. For the explicit expressions and for more details see Refs. [4]. The Minkowski space version of Eq. (1) only guarantees stationarity but one can show that the r.h.s. of Eq. (3) is also minimal. In either case one may now vary Mano’s equation with respect to $\lambda$ and $A(E)$ to find the optimal values. This has been done numerically for the 2-point function and extended to the case of an arbitrary number of mesons in the initial/final state so that form factors, meson-nucleon scattering and deep inelastic scattering from the nucleon could be calculated in a non-perturbative manner [4].
III. SPINOR THEORY: QED

An application to a more realistic theory like Quantum Electrodynamics (QED) requires the description of the spin degrees of freedom of the electron in the path integral. It is well known that this can be done by using anticommuting Grassmann variables and we have developed a particular 4-dimensional formalism for massive Dirac particles [5]. We then followed essentially the same steps as in the scalar case with a few modifications. For example, Eq. (2) is replaced by the (super) proper time representation

\[
\frac{1}{\bar{\Pi} - M + i0} = \int_0^\infty dT \int d\chi \exp \left[ -\frac{i}{2\kappa_0} (M^2 T + M\chi) \right] \exp \left[ \frac{i}{2\kappa_0} \left( \bar{\Pi} T + \bar{\Pi} \chi \right) \right]
\]

(\bar{\Pi}_\mu = \hat{p}_\mu - eA_\mu(\hat{x})) where we use the Berezin integrals \(\int d\chi = 0\), \(\int d\chi \chi = 1\). Near the pole the electron propagator takes the form

\[
G_2(p) \rightarrow Z \frac{\slashed{p} + M_{up}}{\slashed{p}^2 - M^2 + i0}
\]

where \(M_{up}\) may be different from \(M\) (see below) and Mano’s equation reads

\[
M^2(2\lambda - \lambda^2) - M_0^2 = 2 \left\{ \Omega[A] - \Omega[A_f] + V_{\text{QED}} [\mu_f^2, \mu^2] \right\}.
\]

Here \(A(E)\) is the bosonic and \(A_f(E)\) the fermionic profile function. Using methods described in Ref. [6] we have calculated recently the average of the QED interaction term with respect to a general trial action which is quadratic in orbital and spin variables. The result is

\[
V_{\text{QED}} = -4\pi\alpha \frac{\nu^2 e}{\kappa_0} \int_0^\infty d\sigma \int \frac{d^Dk}{(2\pi)^D} \frac{1}{k^2 + i0} \left\{ \frac{D-1}{4} k^2 \left[ \left( \mu_2^2(\sigma) \right)^2 - \left( \bar{\mu}_2^2(\sigma) \right)^2 \right] 
+ \lambda^2 M^2 + \lambda^2 (D-2) \frac{(k \cdot p)^2}{k^2} \right\} \exp \left\{ \frac{i}{2\kappa_0} \left[ k^2 \mu_2^2(\sigma) + 2\lambda p \cdot k \right] \right\}
\]

where \(\alpha = e^2/(4\pi)\) is the fine structure constant and \(\nu\) an arbitrary mass parameter needed in \(D = 4 - 2\epsilon\) dimensions. Examination of Eq. (7) reveals that it is more singular at small proper times than \(V_{\text{WC}}\) in the (super-renormalizable) Wick-Cutkosky model making the renormalization procedure more difficult.

The mass appearing in the numerator of Eq. (5) is determined by

\[
M_{up} = M_0 \left[ \lambda + \lambda' - \lambda \lambda' + \lambda' V_\chi \right]^{-1}
\]

where \(\lambda'\) is a spin variational parameter analogous to \(\lambda\). Varying with respect to this parameter gives \(\lambda = 1 + V_\chi \Rightarrow M_{up} = M_0/(1 + V_\chi) = M_0/\lambda\) if inserted into Eq. (8). We also have evaluated the spin part of the interaction average and – for the special case \(A_f(E) = A(E)\) – found that \(V_\chi = -V_{\text{QED}}/(\lambda M^2)\). This is a manifestation of the exact supersymmetry between bosonic and fermionic variables in the worldline description of a spinning particle [5]. The kinetic terms \(\Omega\) then cancel and one finds from Eq. (8) that \(M = M_0/\lambda \Rightarrow M_{up} = M\). Thus supersymmetry guarantees that there is only one pole at \(\slashed{p} = M\) in the dressed electron propagator as it should be. We are presently deriving and analyzing the corresponding variational equations.
IV. SUMMARY

The worldline representation of field theory leads to a big reduction in the number of degrees of freedom, e.g. \( \Phi(x_\mu), \varphi(x_\mu) \rightarrow x_\mu(t) \). This is essential for a successful application of the Feynman variational principle which (for technical reasons) is restricted to quadratic, albeit retarded trial actions. In this way encouraging nonperturbative results have been obtained for a simple scalar field theory. Describing the spin degrees of freedom by Grassmann variables allows the extension to fermionic theories. Applications to problems like chiral symmetry breaking in strong coupling QED or the anomalous magnetic moment of the electron may be envisioned but still a number of open problems have to be addressed. Among these the consistent renormalization to all orders in our variational scheme seems to be the most urgent one.
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