Quantum parameter-estimation of frequency and damping of a harmonic-oscillator

Patrick Binder\textsuperscript{1,2,3} and Daniel Braun\textsuperscript{1}\textsuperscript{*}

\textsuperscript{1} Institute for Theoretical Physics, Tbingen University, 72076 Tbingen, Germany
\textsuperscript{2} BioQuant Center, Im Neuenheimer Feld 267, 69120 Heidelberg, Germany
\textsuperscript{3} Institute for Theoretical Physics, Heidelberg University, Philosophenweg 19, 69120 Heidelberg, Germany

(Dated: May 22, 2019)

We determine the quantum Cramér-Rao bound for the precision with which the oscillator frequency and damping constant of a damped quantum harmonic oscillator in an arbitrary Gaussian state can be estimated. This goes beyond standard quantum parameter estimation of a single mode Gaussian state for which typically a mode of fixed frequency is assumed. We present a scheme through which the frequency estimation can nevertheless be based on the known results for single-mode quantum parameter estimation with Gaussian states. Based on these results, we investigate the optimal measurement time. For measuring the oscillator frequency, our results unify previously known partial results and constitute an explicit solution for a general single-mode Gaussian state. Furthermore, we show that with existing carbon nanotube resonators (see J. Chaste et al. Nature Nanotechnology 7, 301 (2012)) it should be possible to achieve a mass sensitivity of the order of an electron mass $m_e^{-1/2}$.

\section{I. INTRODUCTION}

The harmonic oscillator is one of the most important model systems in all of physics. It is exactly solvable, both classically and quantum mechanically, and plays a fundamental role in quantum field theories, where its elementary excitations can be identified with e.g. photons or phonons. The harmonic oscillator arises as low-amplitude limit of a much wider class of non-harmonic oscillators, and its regular motion is at the basis of time- and frequency measurements. Indeed, the most precise measurements of a physical quantity are often achieved when transducing their variations into frequency changes. It is therefore of utmost importance to figure out how precisely the two characteristic quantities of a harmonic oscillator, namely its frequency and its damping can be measured in principle. A partial answer was provided in \cite{1}, where the quantum Cramér-Rao bound (QCRB) for the frequency measurement of an undamped harmonic oscillator in an arbitrary pure quantum state was calculated. The QCRB is the ultimate lower bound for the uncertainty with which a parameter can be estimated. It is optimized over all possible (POVM)-measurements (POVM=positive operator-valued measure, a class of measurements that includes but is more general than the usual projective von Neumann measurements), and over all data-analysis procedures (in the sense of unbiased estimator functions of the measurement results alone). It becomes relevant when all technical noise sources are eliminated, and only the noise inherent in the quantum state remains. Importantly, the QCRB can be saturated in the limit of a large number of measurements.

A damped harmonic oscillator leads, however, naturally to mixed quantum states, and for those the calculation of the QCRB is much more difficult than for pure states, owing to the need to diagonalize the density operator in an infinitely dimensional Hilbert space. In \cite{2} an attempt was made to obtain the QCRB for the frequency of a kicked and damped oscillator, by using the formulas for Gaussian states. Indeed, in \cite{4} the exact QCRB was found for any of the five parameters that uniquely fix an arbitrary Gaussian state of a harmonic oscillator. However, those formulas were derived for an oscillator of fixed frequency, and they cannot be directly applied for frequency estimation. Doing so would amount to considering the Hamiltonian $H = \hbar \omega a^\dagger a$ as a generator of a phase shift, i.e. the unknown parameter $\omega$ multiplies a hermitian generator, whose variance gives, up to a factor 4, the pure state quantum Fisher information (QFI). However, this ignores that the annihilation- and creation operators depend themselves on $\omega$. That they do so is most easily seen by writing them in the Fock-basis and realizing that the wave-functions corresponding to the energy eigenstates depend on $\omega$ through the oscillator length. Physically, ignoring the $\omega$-dependence of $a, a^\dagger$ hence implies that one neglects the $\omega$-dependence of the energy-eigenstates, which is particularly important at small times, i.e. much smaller than the period of the oscillator.

One might then think that calculating the QCRB for the damped harmonic oscillator is a hopeless endeavor if the formulas for the Gaussian states cannot be applied, and the state is not already diagonalized. Here we show, however, that there is a well-defined procedure that allows one to use those formulas nevertheless for the large and experimentally most relevant class of initial Gaussian states, by carefully incorporating the consequences of a change of frequency. This allows us to fully solve the problem of parameter estimation of a (weakly) damped harmonic oscillator, described by a Lindblad-master equation.
II. GENERAL FRAMEWORK

We start by briefly describing the dynamics of a damped harmonic oscillator. Afterwards we review the closed-form expression for the general quantum Fisher information (QFI) for single-mode Gaussian states [4].

A. Dynamics

We consider a quantum harmonic oscillator with bare frequency ω weakly coupled to a Markovian environment. Assuming the validity of the Born-Markov approximation and the rotating-wave approximation, the density matrix \( \rho \) of the oscillator evolves according to the master equation [5, 6]

\[
\frac{d\rho}{dt} = -i\omega [\hat{a}^\dagger \hat{a}, \rho] + \frac{\gamma}{2}\{2\hat{a}^\dagger \hat{a} - \hat{a}^\dagger \hat{a} - \rho + \rho \hat{a} \hat{a}^\dagger \}
\]

where we introduced the mean thermal photon number of the bath \( \bar{n} = (e^{\bar{x}^2} - 1)^{-1} \) at frequency \( \omega \), dimensionless inverse temperature \( \chi \equiv \hbar \omega / k_B T \), and the damping constant \( \gamma \).

By introducing the quadrature operator \( X = (\hat{q}, \hat{p})^T \), the three-dimensional vector \( S(t) = (M\omega \sigma_{qq}, \sigma_{qp}/M\omega, \sigma_{pp})^T \), where \( \sigma_{AB} = 1/2 \times \langle AB + BA \rangle - \langle A \rangle \langle B \rangle \) and by using the ME [1] one finds equations of motion [7]:

\[
\frac{d \langle X \rangle(t)}{dt} = G \langle X \rangle(t),
\]

\[
\frac{dS(t)}{dt} = KS(t) + S_{\text{inh}},
\]

where

\[
G = \begin{pmatrix}
-\gamma/2 & 1/M \\
-M\omega^2 & -\gamma/2
\end{pmatrix},
\]

\[
K = \begin{pmatrix}
-\gamma & 0 & 2\omega \\
0 & \omega - 2\gamma & \omega - \gamma
\end{pmatrix}
\]

and \( S_{\text{inh}} = \gamma \hbar (2\bar{n} + 1)/2 \begin{pmatrix} 1, 1, 0 \end{pmatrix}^T \). The solutions of the time evolution of the first order moments are given by \( \langle X \rangle(t) = \exp(Gt) \langle X \rangle(0) \). For the second order moments we get \( S(t) = \exp(Kt)S(0) + K^{-1}(\exp(Kt) - I)S_{\text{inh}} \), where \( I \) denotes the identity operator.

The two phase-space coordinates \( \hat{q} \) and \( \hat{p} \) are linked to the annihilation and creation operator \( \hat{a}_\omega \) and \( \hat{a}^\dagger_\omega \) of the mode by

\[
\hat{q} = \sqrt{\frac{\hbar}{2M\omega}}(\hat{a}^\dagger_\omega + \hat{a}_\omega),
\]

\[
\hat{p} = i\sqrt{\frac{\hbar}{2M\omega}}(\hat{a}^\dagger_\omega - \hat{a}_\omega).
\]

Summing up, \( \omega, \gamma, \) and \( \bar{n} \) are coded into a state by the dynamics [1], but in addition a state specified initially e.g. in the Fock basis acquires an \( \omega \)-dependence due to the \( \omega \)-dependence of the harmonic oscillator energy eigenstates (oscillator length).

B. QFI of single-mode Gaussian states

Gaussian state. The Wigner function for an arbitrary density matrix \( \rho \) of a continuous variable system with a single degree of freedom (such as a single harmonic oscillator) is defined by [8]

\[
W(q, p) = \frac{1}{\pi \hbar} \int \int e^{-2i(p'y - \gamma)\rho + y} \ dy .
\]

By definition, a Gaussian state is a state whose Wigner function is Gaussian. Thus, for a Gaussian state \( \rho \) of a single harmonic oscillator (such as a single mode of an electro-magnetic field) the Wigner function takes the general form [9]

\[
W(q, p) = \frac{P}{\pi} \exp \left[ \frac{1}{2} (X - \langle X \rangle)^T \Sigma^{-1} (X - \langle X \rangle) \right],
\]

where \( X = (\hat{q}, \hat{p})^T \) is the quadrature operator, \( \Sigma \) is the covariance matrix, \( \langle \ldots \rangle \equiv \text{tr}(\ldots) \) defines the expectation value and \( P = \text{tr} \rho^2 \) is the purity. For single-mode Gaussian states the purity is completely described by the covariance matrix and is given by [10]

\[
P = \frac{\hbar}{2 \sqrt{\text{det}(\Sigma)}}.
\]

Next, we recall that a general single-mode Gaussian state \( \rho \) can always be represented as a rotated squeezed displaced thermal state \( \nu \), i.e. [9, 11]

\[
\rho = R(\psi)D(\alpha)S(\nu)S^\dagger(\nu)D^\dagger(\alpha)R(\psi),
\]

where \( S(z) = \exp(1/2(z\hat{a}^2 - z^*\hat{a}^\dagger)) \) is the squeezing operator, \( R(\psi) = \exp(\sqrt{\nu}\hat{a}^\dagger \hat{a}) \) denotes the rotation operator and \( D(\alpha) = \exp(\alpha\hat{a} - \alpha^*\hat{a}^\dagger) \) introduces the displacement operator. By introducing \( N_{\text{th}} = \text{tr}(\nu\hat{a}^\dagger \hat{a}) \), which denotes the number of initial thermal photons, and \( z = r e^{i\chi} \) the general Gaussian state can be parametrized by five real parameters \( \alpha, \psi, r, \chi, N_{\text{th}} \in \mathbb{R} \). Note that we keep \( N_{\text{th}} \) and \( \bar{n} \) as independent parameters.

Quantum Fisher information. We start from a density operator \( \rho_0 \), which depends on an unknown real scalar parameter \( \theta \). To estimate this parameter, \( m \) independent measurements with the outcome \( \xi = (\xi_1, \xi_2, \ldots, \xi_M)^T \) are taken. From the outcome we construct an estimator \( \hat{\theta}_{\text{est}} \). For unbiased estimators the sensitivity with which a parameter \( \theta \) can be measured has a lower bound, the so-called quantum Cramér-Rao bound (QCRB), given by [12, 15]

\[
\text{Var}[\hat{\theta}_{\text{est}}] \geq \frac{1}{m I(\rho_0; \theta)},
\]

where \( I(\rho_0; \theta) \) denotes the QFI. The fidelity, defined by \( \mathcal{F}(\rho_1, \rho_2) = \text{tr}\left[\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}\right]^{1/2} \), for two arbitrary single-mode Gaussian states \( \rho_1 \) and \( \rho_2 \) is given by [16]
\[ F(\rho_1, \rho_2) = \frac{2 \exp \left[ -\frac{1}{2} \left( (\mathbf{X}_1 - \mathbf{X}_2)^T (\Sigma_1 + \Sigma_2)^{-1} (\mathbf{X}_1 - \mathbf{X}_2) \right) \right]}{\sqrt{\Sigma_1 + \Sigma_2 + (1 - |\Sigma_1|)(1 - |\Sigma_2|)} - \sqrt{(1 - |\Sigma_1|)(1 - |\Sigma_2|)}}. \]  \hspace{1cm} (9)

This formula is valid generally for two Gaussian Wigner functions, regardless of the underlying system. It remains therefore valid if the two Wigner functions represent states of two different harmonic oscillators, notably harmonic oscillators that can differ in frequency. Using further the fact that the fidelity is linked to the QFI for pure states, where one does not get time evolution. For this purpose, we use the known restate, and not the initial pure state \( \rho \) is not considered. Lastly, we justify that one can still neglect the \( \omega \)-dependence of \( a, a^\dagger \) the QFI is given by

\[ I(\rho_\omega; \theta) = -2 \frac{\partial^2 F(\rho_\theta, \rho_\theta + \varepsilon)}{\partial \varepsilon^2} \bigg|_{\varepsilon=0} \]  \hspace{1cm} (10)

one obtains the general QFI for Gaussian states of a single harmonic oscillator of fixed frequency \[ 4 \]

\[ I(\rho_\theta; \theta) = \frac{1}{2} \text{tr} \left[ (\Sigma^{-1} \partial_\theta \Sigma)^2 \right] + 2 \left( \partial_\theta P \right)^2 \frac{1}{1 - P^2} \]  
\[ + (\partial_\theta X)^T \Sigma^{-1} \partial_\theta X. \]  \hspace{1cm} (11)

By following the approach adopted by Jiang in Ref. \[ 17 \], the same result can be obtained \[ 18 \].

### III. UNDAMPED CASE

This section provides a scheme for the calculation of and results for the QFI relevant for estimating the frequency \( \omega \) in the case of no damping.

#### A. Scheme for the estimation of the quantum Fisher information

Firstly we will illustrate that by directly using Eq. \[ 11 \] for a frequency measurement one does not get the full QFI, instead one obtains just that part that corresponds to having \( a^\dagger, a \) as frequency-independent generator of the time evolution. For this purpose, we use the known results of the QFI for pure states, where one does not get the full QFI if taking \( a^\dagger, a \) independent of \( \omega \). In particular, this means that directly inserting the solution of the dynamics into equation Eq. \[ 11 \] will not provide the correct result, as the \( \omega \)-dependence of the Fock basis is not considered. Lastly, we justify that one can still use Eq. \[ 11 \] if one treats the squeezing due to frequency change correctly, which leads to the scheme we propose.

We consider the case that only the dynamics of the state, and not the initial pure state \( \rho_0 = \langle \psi_0 | \psi_0 \rangle \) itself, depends on the frequency \( \omega \) to be measured. For given Hamiltonian \( \mathcal{H} = \hbar \omega (a^\dagger a + 1/2) \), the dynamics of the system is described by \( \rho(t) = U(t) \rho(0) U^\dagger(t) \), where \( U(t) = \exp(-i t \mathcal{H} / \hbar) \) is the time evolution operator. By

\[ I(\rho_\omega(t); \omega) = 4 \text{Var} \left[ t (\hat{a}^\dagger \hat{a} + 1/2), |\psi_0 \rangle \right], \]  \hspace{1cm} (12)

where \( \text{Var}[A, |\psi_0 \rangle] = \langle \psi_0 | A^2 |\psi_0 \rangle - \langle \psi_0 | A |\psi_0 \rangle^2 \) denotes the variance. For a general pure Gaussian state in the form of Eq. \[ 7 \], i.e. \( |\psi_0 \rangle = R(\psi) D(\alpha) S(\sigma e^{i\chi}) |0 \rangle \), the QFI reads

\[ I(\rho_\omega(t); \omega) = 4 \alpha^2 \left[ \cosh(2r) + \cos(\chi) \sinh(2r) \right] + 2 \sigma^2 \sinh^2(2r). \]  \hspace{1cm} (13)

Next, we determine the same QFI by directly using equation \[ 11 \]. For this we first use that we can write the time-evolved density operator in the following way:

\[ \rho_\omega(t) = R(\zeta) D(\alpha) S(z) |0 \rangle \langle 0 | S^\dagger(z) D^\dagger(\alpha) R^\dagger(\zeta), \]  \hspace{1cm} (14)

where \( \zeta = \psi - \omega t \). Using \( \sigma = e^{-r} \), equation (16) from \[ 4 \] can be rewritten as

\[ I(\rho_\omega(t); \zeta) = 4 \alpha^2 [\cosh(2r) + \cos(\chi) \sinh(2r)] + 2 \sigma^2 \sinh^2(2r). \]  \hspace{1cm} (15)

Thus, with \( d/d\omega = -t d/d\zeta \) we get the same result as obtained in equation \[ 13 \], of which we have demonstrated that by directly using equation \[ 11 \] the \( \omega \)-dependence of the basis is not considered.

In order to consider all frequency dependencies correctly, we have developed the following scheme for the estimation of the QFI:

1. Start with an initial Gaussian state \( \rho_0 \) in the Fock basis \( \{ |n \rangle_{\omega_0} \} \).
2. Perform a sudden change of frequency \( \omega_0 \rightarrow \omega \), which corresponds to a squeezing, at time \( t = 0 \).
3. Evolve the quantum state with respect to the new frequency \( \omega \).
4. Estimate the QFI \( I(\rho_\omega(t); \omega) \) by using Eq. \[ 11 \]..  
5. Take the limit \( \omega \rightarrow \omega_0 \).

The sudden change of frequency \( \omega_0 \rightarrow \omega \) at time \( t = 0 \) ensures that also the frequency dependence of the basis is considered. Furthermore, it can be shown that the frequency jump corresponds to squeezing (see Appendix \[ A \]), i.e.

\[ |n \rangle_{\omega_0} = S_\omega(s) |n \rangle_{\omega_0}, \]  \hspace{1cm} (16)

where \( s = -\tanh^{-1}(y_1) \) and \( y_1 = (\omega_0 - \omega) / (\omega_0 + \omega) \).
It should be noted that the introduced scheme is only needed to determine the QFI for a frequency measurement using Eq. (11). For pure states, for example, the QFI can be determined directly from the overlaps of the states propagated with slightly different frequency [1], or, equivalently, from the variance of the local generator, taking into account the \( \omega \)-dependence of \( \hat{a}_\omega \), (see Appendix. B). Furthermore, it should be noted that since the Fock basis does not depend on the damping constant, the introduced scheme is not needed for calculating the QFI for the estimation of \( \gamma \).

### B. Result for QFI for vanishing damping

By using the introduced scheme we now determine the QFI for the estimation of \( \omega \) for the general Gaussian state given in Eq. (7). For a time evolution of the Gaussian state with the harmonic oscillator \( \mathcal{H} = \hbar \omega (\hat{a}^\dagger \hat{a}_\omega + 1/2) \) follows the result (see Appendix. C).

\[
\begin{align*}
\omega^2 I(\rho(\tau); \omega) &= C_3 + 2C_1 \sin^2(\tau) \left[ \sinh^2(2r) \cos(\chi + 2\psi - \tau) + 1 + 2C_2 \alpha^2 (\cosh(2r) + \cos(\chi + 4\psi - 2\tau) \sinh(2r)) \right] \\
&\quad + 2C_1 \tau \sin(\tau) \left[ 4C_2 \alpha^2 \cos(2\psi - \tau) \cosh(2r) + \cos(\chi + 2\psi - \tau) (4C_2 \alpha^2 \sinh(2r) + \sinh(4r)) \right] \\
&\quad + 2C_1 \tau^2 \left[ 2C_2 \alpha^2 \cos(2r) + \cos \chi \sinh(2r) \right] + \sinh^2(2r),
\end{align*}
\]

(17)

where \( \tau = \omega t \) and

\[
\begin{align*}
C_1 &= \frac{(1 + 2N_{th})^2}{1 + 2N_{th}(1 + N_{th})}, \quad \text{(18a)} \\
C_2 &= \frac{1}{C_1(1 + 2N_{th})}, \quad \text{(18b)} \\
C_3 &= N_{th}(1 + N_{th}) \left[ \ln \left( \frac{1 + N_{th}}{N_{th}} \right) \right]^2. \quad \text{(18c)}
\end{align*}
\]

The first term \( (C_3) \) of Eq. (17) results from the \( \omega \)-dependence of the initial photon number \( N_{th} \), the second term is due to the \( \omega \)-dependence of the Fock basis, and the term \( \propto t^2 \) arises from \( \hat{a}^\dagger \hat{a} \) as generator of the time evolution.

For an initial thermal state \( \rho(0) = \nu \), Eq. (17) reduces to

\[
I(\nu(\tau); \omega) = \frac{2C_1 \sin^2(\tau) + C_3}{\omega^2}. \quad \text{(19)}
\]

Thus, a measurement with \( t > \pi/2\omega \) does not provide any additional information regarding the frequency and the QFI has an upper bound \((2C_1 + C_3)/\omega^2 \)--where \( C_1 \) itself is bounded by \( C_1 \in [1, 2] \) \( \forall N_{th} \) and \( C_3 \) is bounded by \( C_3 \in [0, 1] \) \( \forall N_{th} \). Furthermore, the result demonstrates that one can measure the frequency of a mode of an e.m. field without any light at all, just from the vacuum fluctuations. The latter have been measured directly in [21].

While our results from Eq. (17) agree with the obtained QFI for a coherent state [1], our result in Eq. (19) contains an extra term \( C_3/\omega^2 \) due to the consideration of the \( \omega \)-dependence of \( N_{th} \) neglected in [1]. It should also be noted that our result agrees with the result by calculating the QFI via the variance in the case of a general pure Gaussian state (see Appendix. [9]).

**Optimal state.** The QFI can be drastically increased by displacing and/or squeezing the initial thermal state.

### IV. DAMPED CASE

In this section we will calculate the QFI for mixed Gaussian states for the damped harmonic oscillator for estimating the oscillator frequency and damping constant. Furthermore, we determine the optimal measuring scheme and the optimal measuring time and we demonstrate that with existing carbon nanotube resonators it should be possible to achieve a mass sensitivity of the order of an electron mass \( \text{Hz}^{-1/2} \).
sequence, a longer measurement does not necessarily yield the upper bound given by $2/\omega^2$ (see Fig. 1). The upper bound can be reached in the high temperature limit. As a consequence, a longer measurement does not necessarily yield a better result for the experiment. In other words, there is an optimal measurement (OMT) time in which the frequency can be measured best, which is in accordance with the physical expectations.

2. Optimal measurement time and maximal quantum Fisher information

Coherent state. We start by considering an initial coherent state $\rho_\alpha(0) = D(\alpha)|0\rangle\langle 0|D^\dagger(\alpha)$. Recall, displacing the initial state is one of the possibilities to strongly increase the QFI in the undamped case. Since displacing the ground state only affects the expectation values of the quadrature operators and not the covariance matrix, the QFI of the coherent state can be written as

$$I(\rho_\alpha(\tau);\omega) = I(\rho_0(\tau);\omega) + I_\alpha(\tau),$$

(22)

where $\rho_0(\tau)$ denotes the time-evolved ground state and $I_\alpha(\tau) = (\partial_\omega (X))^T \Sigma^{-1} \partial_\omega (X)$. The QFI of the ground state is bounded by $2.135/\omega^2$ (see Appendix. C). Thus, the upper bound of the QFI for the ground state is increased by introducing the system-bath coupling, which can be explained by the $\omega$-dependence of $\bar{n}$. I.e. also in the damped case, the frequency can be measured when the system is initially prepared in the ground state. Straightforward calculation leads to

$$I_\alpha(\tau) = \frac{4\alpha^2 \sin^2(\tau) + \tau \sin(2\tau) + \tau^2}{\omega^2 (2\bar{n} + 1)e^{3\tau} - 2\bar{n}},$$

(23)

where $g = \gamma/\omega$ introduces a dimensionless damping constant. Thus, for frequency measurements an as big as possible displacement is recommended.

Since the QFI of the ground state is bounded (and small), $I(\rho_\alpha(\tau);\omega) \approx I_\alpha(\tau)$ applies for $\alpha^2 \gg g^2 \bar{n}$ (by assuming $n \gg 1$ and $g \ll 1$). For high enough temperatures, $\bar{n} \gg \alpha^2/g^2$, $I_\alpha(\tau)$ becomes arbitrarily small.

A. Measuring the oscillator frequency

By sticking to the scheme explained in Sec. III A we obtain the exact expression for the QFI for a general initial Gaussian state by considering the time evolution given by the ME (1), which can be found in the Appendix, in Eq. (C7) to (C12). However, since the solution is too heavy to report here, we will first look at the long-term behavior and then limit ourselves to specific initial states—coherent state and squeezed state.

1. Long-term behavior

For longer periods, the solution of ME (1) relaxes to the thermal equilibrium state, i.e. for $t \gg \gamma^{-1}$,

$$\rho \xrightarrow{t \gg \gamma^{-1}} e^{-\hbar \omega/k_{\text{B}} T} / \text{tr}(e^{-\hbar \omega/k_{\text{B}} T}) \equiv \rho_\infty.$$ (20)

It should be remembered that the thermal equilibrium state as well as the mean thermal photon number $\bar{n}$ also depend on the oscillator frequency $\omega$ itself. It can therefore be expected that the QFI does not vanish due to the dependency of the final state on the frequency. Since both first order moments vanish, i.e. $\lim_{t \to \infty} \langle X \rangle = 0$, only the first two terms of Eq. (11) contribute to QFI and calculation yields

$$I(\rho_\infty;\omega) = \frac{1}{2\omega^2} \left[ 2\bar{n}(1 + \bar{n}) \ln^2 \left( \frac{1 + \bar{n}}{\bar{n}} \right) + \frac{1 + 4\bar{n}(1 + \bar{n})}{1 + 2n(1 + \bar{n})} \right].$$ (21)

This means that for large times, the QFI has an upper bound given by $2/\omega^2$ (see Fig. 1). The upper bound can be reached in the high temperature limit. As a consequence, a longer measurement does not necessarily yield the result is independent of the damping constant.
Thus, the QFI can be significantly increased by squeezing

\[ I(\rho_0(\tau); \omega) = \frac{2}{\omega^2} \left[ 1 - \frac{\cos^2(\tau)}{(e^{\omega \tau} - 1)n} \right] + \mathcal{O}(1/n^2). \]  

(26)

That means that for high temperatures \( n \gg 1 \) the QFI of the ground state decays faster (\( \sim e^{-\tau g} \)) than \( I_0(\tau) \) (\( \sim \tau^2 e^{-\tau g} \)) and varies only slightly close to the time \( \tau_{\text{max}} \). Consequently, the use of \( I_0(\tau) \) for estimating the optimal measurement time leads, even in this range, to a good result of the OMT (see Fig. 2). Furthermore, it should be noted that the smaller \( g \), the larger \( n \) can be, so that the OMT is still very well described by \( I_0(\tau) \).

By reducing the system-bath coupling, the maximal QFI increases proportionally to \( \propto g^{-2} \). However, it should be noted that the OMT also increases proportionally to \( \propto g^{-1} \).

Thus, it is a natural to consider time as a resource and to introduce the rescaled maximal QFI \( I_{\text{max}}(\rho, \theta) = \max_{\tau} I(\rho, \theta)/t \) and the optimal measurement time \( \tau_{\text{max}}^{(t)} \) that maximizes it. For an initial coherent state we get

\[ I_{\text{max}}^{(t)}(\rho_0(\tau), t) = \frac{2}{ng} \left[ 1 - \frac{2n}{e(1 + 2n)} \right] \]  

(27)

\[ \tau_{\text{max}}^{(t)} = \frac{1}{g} \left[ 1 + W\left(-\frac{2n}{e(1 + 2n)}\right)\right]. \]  

(28)

Taking time into account as a resource leads to a reduction of the OMT.

**Squeezed state.** Besides displacement, squeezing the initial state is another possibility to increase the QFI in the undamped case. Therefore, we determine the QFI for a squeezed state \( \rho_s(0) = S(r) |0\rangle \langle 0| S^\dagger(r) \). For the coherent state we have seen that reducing the temperature leads to an increase in the QFI. This behavior is reasonable, since increased temperature implies increased damping according to the master equation \([1]\). A similar behavior can be observed here with the squeezed state. The QFI for a vanishing bath temperature, i.e. \( n = 0 \), reads

\[ I(\rho_s(\tau), \omega) = \left[ 8\omega^2 (2e^{\omega \tau} \sin^2(\tau) + e^{2\omega \tau} - \cosh(2\tau) + 1) \right]^{-1} \]

\[ \times \left[ 16\tau \sinh(2\tau) \sin(2\tau) (e^{\omega \tau} + \cosh(2\tau) - 1) - 4(e^{\omega \tau} - 1) \cosh(2\tau)(2\cos(2\tau) - 3) \right. \]

\[ + 4e^{\omega \tau}(e^{2\omega \tau} - 1) + (8\tau^2 + 1) \cosh(4\tau) - 8\sinh^2(\tau) \cosh^2(\tau) \cos(4\tau) - 8\tau^2 - 8\cos(2\tau) + 7 \].

(29)

Alternatively, for high squeezing and low temperatures, i.e. \( r \gg 1 \) and \( n \ll 1 \), the QFI can be approximated as (see Fig. 3)

\[ I(\rho_s(\tau), \omega) \approx \frac{e^{2\omega \tau}(2\tau + \sin(2\tau))^2}{4\omega^2(e^{\omega \tau} - 1)(1 + 2n)}. \]

(30)

Thus, the QFI can be significantly increased by squeezing also for an initial thermal state. Neglecting the oscillations, the OMT can be determined to

\[ \tau_{\text{max}} = \frac{1}{g} \left[ 2 + W(-2/e^2) \right] \approx \frac{1.59}{g}. \]

(31)

For sufficiently high squeezing and low temperature, the OMT does not depend on the squeezing and temperature anymore.
B. Measuring the damping constant

Next we consider the QFI for the estimation of the damping constant. First of all, the QFI disappears for large times, i.e. $I(\rho_\infty, \gamma) = 0$. This can be seen directly from the fact that the final thermal state (for the master equation approach) itself no longer depends on the damping constant. In other words, there is again an OMT.

After a straightforward but long and tedious calculation we find for the QFI of a general Gaussian state

$$I(\rho; \gamma) = \frac{P^2(\tau)g^2\tau^2}{\gamma^2 e^{2g\tau}} \left\{ a^2 e^{2g\tau} [A_1(\cosh(2r) - \cos(\chi) \sinh(2r)) + a_{1,\tau}] + \frac{2P^4(\tau)}{1 - P^4(\tau)} [A_1^2 + A_1(a_{1,\tau} - a_1) \cosh(2r) - a_1 a_{1,\tau}]^2 ight\} + \frac{P^2(\tau)}{1 + P^2(\tau)} \left\{ A_1^2 + A_1^2 (a_1^2 + a_{1,\tau}) \cosh(4r) + 2A_1(a_{1,\tau} - a_1) (A_1^2 - a_1 a_{1,\tau}) \cosh(2r) - 4a_1 a_{1,\tau} A_1^2 + a_1^2 a_{1,\tau} \right\},$$

(32)

where $a_1 = 1 + 2\bar{n}$, $a_{1,\tau} = (e^{gr} - 1)a_1$, $A_1 = 1 + 2N_{th}$ and

$$P(\tau) = e^{gr} [A_1^2 + a_1^2 + 2a_{1,\tau}A_1 \cosh(2r)]^{-1/2}. \quad (33)$$

The result does not depend on the rotation angle $\psi$, but only on the squeezing angle $\chi$. In contrast to frequency measurement, the QFI for measuring $\gamma$ is maximized for $\chi = \pi$. This is in agreement with the physical expectation, as the relevant dynamic here is the relaxation of $\langle X \rangle$. To illustrate the result, we again consider specific initial states — thermal state, displaced thermal state and squeezed state.

**Thermal state.** The QFI of a thermal state $\nu$ can be written as

$$I(\nu(\tau), \gamma) = \frac{(\bar{n} - N_{th})^2 g^2 \tau^2}{\gamma^2 [(e^{gr} - 1)\bar{n} + N_{th}](e^{gr} (1 + \bar{n}) + N_{th} - \bar{n})}. \quad (34)$$

The greater the deviation of the initial temperature from the bath temperature, the better $\gamma$ can be measured. In particular, for a vanishing deviation, i.e. $N_{th} = \bar{n}$, the QFI vanishes, since in this case the state has no dynamics at all. For $\bar{n} = 0$, the OMT is given by

$$\tau_{\text{max}} = \frac{2 + W(2N_{th} e^{-2})}{g}. \quad (35)$$

**Displaced thermal state.** For an initial displaced thermal state $\rho_{\alpha, N_{th}}(0) = D(\alpha)\nu D(I(\alpha))$ the QFI for measuring $\gamma$ reads

$$I(\rho_{\alpha, N_{th}}(\tau), \gamma) = I(\nu(\tau), \gamma) + \frac{\alpha^2 g^2 \tau^2}{\gamma^2 [2N_{th} - 2\bar{n} + e^{gr} (1 + 2\bar{n})]}. \quad (36)$$

Particularly for $N_{th} = \bar{n}$, the QFI simplifies to

$$I(\rho_{\alpha, \bar{n}}(\tau), \gamma) = \frac{\alpha^2 g^2 \tau^2}{\gamma^2 e^{gr} (1 + 2\bar{n})}. \quad (37)$$

and the OMT is given by $\tau_{\text{max}} = 2/g$. For $N_{th} = \bar{n}$ only the 3rd part of equation (36) contributes to the QFI, i.e. the QFI results solely from the relaxation of $\langle p \rangle$, $\langle q \rangle$. By considering the rescaled QFI the OMT reduces to $\tau_{\text{max}} = 1/g$.

**Squeezed state.** The low temperature limit behavior, i.e. $\bar{n} = 0$, of the QFI for an initial squeezed state $\rho_\tau$ is given by

$$I(\rho_\tau(\tau), \gamma) = \frac{[e^{2gr} - 2(e^{gr} - 1)] g^2 \tau^2 \sinh^2(r)}{\gamma^2 (e^{gr} - 1) [2(e^{gr} - 1) \sinh^2(r) + e^{2gr}]}.$$

(38)

The sensitivity with which the damping parameter can be measured improves by squeezing, displacing and/or a temperature deviation (see Fig. 4).

C. Nano-mechanical resonators

In the following we apply the results obtained to nano-mechanical resonators, which function as precision mass

![FIG. 4](image-url)
sensors as their resonance frequency changes when additional mass is adsorbed. More precisely, we consider carbon nanotube resonators. Using the QCRB \( \delta M_{\text{min}} = 2M / \sqrt{\max \{m, \omega \}} \).

Assuming a coherent state with oscillation amplitude of about 10 nm for the carbon nanotube resonator in [22] \((M = 3 \times 10^{-22} \text{ kg}, \omega = 2\pi \times 1.865 \text{ GHz}, T = 4 \text{ K and } Q \sim 10^3)\), \(\delta M_{\text{min}}\) according to [39] is slightly below one proton mass. Using the OMT given by \(t_{\text{max}} = 270 \text{ ns}\), the sensitivity corresponds to \(\delta M_{\text{min}}\sqrt{t_{\text{max}}} = 0.8 m_0 \text{ Hz}^{-1/2}\), which is less than 1/4000 of the experimentally determined mass sensitivity of slightly more than one proton mass after 2 s averaging time.

In [1] the theoretically achievable \(\delta M_{\text{min}}\) for the carbon nanotube resonator in [23] \((M = 10^{-21} \text{ kg}, \omega = 2\pi \times 328.5 \text{ MHz}, T = 300 \text{ K and } Q \sim 10^3)\) was determined to the order of a thousandth of an electron mass. Including the system-bath coupling, \(\delta M_{\text{min}}\) increases to about 74 proton masses, where the OMT is given by \(t_{\text{max}} = 1.5 \text{ ms}\). This result is equivalent to 0.8 u/\(\sqrt{\text{Hz}}\), which approximately corresponds to one hundredth of the 78 u/\(\sqrt{\text{Hz}}\) achieved in the experiment.

\[ \delta M_{\text{min}} = \frac{2M}{\sqrt{m_{\text{max}}}}. \quad (39) \]

V. CONCLUSIONS

In summary, we have derived the quantum Cram-Rao bound for measuring the oscillator frequency and damping constant encoded in the dynamics of a general mixed single-mode Gaussian state of light, including damping through photon loss described by a Lindblad master equation. We first demonstrated that the QFI for Gaussian states of a single harmonic oscillator of fixed frequency cannot be directly applied to frequency measurements. Next, we presented a scheme through which the frequency estimation can nevertheless be based on the results of Pinel et al. [4].

Furthermore, we showed that displacing and/or squeezing the initial state significantly increases the precision with which \(\omega\) and \(\gamma\) can be estimated. For measuring \(\omega\) and \(r \neq 0\), \(\chi = 0\) is optimal, whereas for measuring \(\gamma\), \(\chi = 0\) maximizes the QFI.

Our results can serve as important benchmarks for the precision of frequency measurements of any harmonic oscillator with given damping. In particular, we found optimal measurement times that limit the sensitivity per \(\sqrt{\text{Hz}}\) with which frequencies can be measured, in contrast to the undamped case, where e.g. coherent states lead to growing QFI for arbitrarily large times.

Appendix A: Change of basis

By presenting the scheme for the estimation of the QFI for measuring \(\omega\) we made use of the fact that the frequency jump corresponds to squeezing. Next we prove the statement, i.e. the following formula

\[ |n\rangle_{\omega_0} = S_{\omega}(s) |n\rangle_{\omega_1}, \quad (A1) \]

where \(s = -\tanh^{-1}(y_1)\). For the sake of simplicity the two parameters

\[ y_1 = \frac{\omega_0 - \omega}{\omega_0 + \omega}, \quad y_2 = \frac{2\sqrt{\omega_0 \omega}}{\omega_0 + \omega} \quad (A2) \]

are introduced. A squeezed number state is given by [24]

\[ \omega \langle m | S_{\omega}(s) | n \rangle_{\omega} = \frac{\sqrt{n!}}{\cosh n^{1/2} |s|} \sum_{j=0}^{[n/2]} \frac{(-d)^j \cosh 2j |s|}{(n - 2j)!j!} \]

\[ \times \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{d^k \sqrt{(n - 2j + 2k)}}{k!} |n| \langle m | n - 2j + 2k \rangle_{\omega}, \quad (A3) \]

where \(d \equiv (s/2|s|) \tanh |s|\) and \([n/2]\) denotes the floor function. With \(m = n - 2j + 2k\) and \(k \in \mathbb{N}\) we get \(k = j + \frac{m-n}{2} \in \mathbb{N}\). This means in particular that \(m\) and \(n\) must be both even or both odd numbers, otherwise the overlap disappears. If \(m\) and \(n\) satisfy this condition and by using \(\cosh |s| = 1/y_2\) and \(d = -y_1/2\), the expression can be further simplified as follows

\[ \omega \langle m | S_{\omega}(s) | n \rangle_{\omega} = \sqrt{y_2 m! n!} \sum_{j=0}^{[n/2]} \frac{(-1)^j + \frac{n-m}{2} \left(\frac{y_2}{y_2}\right)^{2j + \frac{n-m}{2}}}{(n - 2j)!j!(j + \frac{m-n}{2})!} y_2^{-2j}. \quad (A4) \]

By changing the index of summation to \(l = n - 2j\) we get the new upper bound of \(\min(m, n)\), where \(\min(m, n)\) denotes the smaller of the two integers \(m, n\). \(l\) is also bounded by \(m\), since \(k = j + \frac{m-n}{2} \in \mathbb{N}\) and thus \(l \leq m\). Using the new index of summation we get [24]

\[ \omega \langle m | S_{\omega}(s) | n \rangle_{\omega} = \sqrt{y_2 m! n!} \sum_{l=0,1} \frac{(2y_2)!}{l!} |y_2^{(m+n-2l)/2} (-1)^{(m-l)/2} (\frac{m-1}{2})! (\frac{m-l}{2})! | \]

\[ = R_{\omega_0}(m, n) \quad = \omega \langle m | n \rangle_{\omega_0}, \quad (A5) \]

where \(R_{\omega_0}(m, n)\) denotes the overlap matrix element between energy eigenstates of the two oscillators with frequency \(\omega\) and \(\omega_0\). Since this is true for all \(m\), we have proven the formula. Thus, for any density operator follows

\[ \rho_{\omega_0} = S_{\omega}(s) \rho_{\omega} S_{\omega}^*(s), \quad (A6) \]
where \( s = -\tanh^{-1}(y_1) \) and \( \tilde{\rho}_\omega \) corresponds to the initial state \( \rho_{\omega_0} \) by replacing the frequency \( \omega_0 \) of the basis with the new frequency \( \omega \). Thus, we have shown that the initial frequency change corresponds to a squeezing. It should be noted that in the case of a vanishing frequency change, i.e. \( \omega_0 = \omega \), \( y_1 = 0 \), \( s = 0 \) and \( S(s = 0) = I \) follow and thus \( \rho_{\omega_0} = \rho_{\omega} \) is ensured.

**Appendix B: QFI for pure states**

In the following it will be shown that the introduced scheme provides the correct QFI for an undamped pure Gaussian state. Therefore, the QFI is calculated analogously to chapter [II.A](#) but this time also the \( \omega \)-dependence of \( \hat{a}^\dagger, \hat{a} \) are taken into account.

This means, we consider the case that only the dynamics of the state, and not the initial state

\[
\rho_0 = |\psi_0\rangle\langle\psi_0|, \tag{B1}
\]

where \( |\psi_0\rangle = R(\psi)D(\alpha)S(r e^{i\chi})|0\rangle \), depends on the frequency \( \omega \) to be measured. For given Hamiltonian \( \hat{H}_\omega = \hbar \omega (\hat{a}_\omega^\dagger \hat{a}_\omega + 1/2) \), the dynamics of the system is described by \( \rho_\omega = U_\omega \rho_0 U_\omega^\dagger \), where \( U_\omega = \exp[ -i \omega t (\hat{a}_\omega^\dagger \hat{a}_\omega + 1/2) ] \) is the time evolution operator. With the help of the local generator

\[
\mathcal{K} = iU_\omega^\dagger \frac{\partial U_\omega(t)}{\partial \omega}, \tag{B2}
\]

the QFI can be rewritten as follows [26]

\[
I(\rho_\omega; \omega) = 4 \text{Var}[\mathcal{K}^k|\psi_0\rangle]. \tag{B3}
\]

If \( A \) is a Matrix depending on the parameter \( x \), \( A = A(x) \), then [27]

\[
\frac{\partial}{\partial x} e^{A(x)} = \int_0^1 e^{\alpha A(x)} \frac{\partial A(x)}{\partial x} e^{-\alpha A(x)} d\alpha \cdot e^{A(x)}, \tag{B4}
\]

Using this formula we can rewrite the local generator \( \mathcal{K} \) as [28]

\[
\mathcal{K} = \frac{t}{\hbar} \int_{-1}^1 V(\alpha) \frac{\partial H_\omega}{\partial \omega} V^\dagger(\alpha) d\alpha, \tag{B5}
\]

where \( V(\alpha) = \exp(-i\alpha t H_\omega / \hbar) \). The derivative of the Hamiltonian \( H_\omega \) with respect to the oscillator frequency \( \omega \) reads

\[
\frac{\partial H_\omega}{\partial \omega} = \hbar \alpha_\omega^\dagger \hat{a}_\omega + \frac{\hbar}{2} \left( (\hat{a}_\omega^\dagger)^2 + \hat{a}_\omega^2 + 1 \right), \tag{B6}
\]

where we made use of \( \partial_\omega \hat{a}_\omega^\dagger = \hat{a}_\omega/2\omega \) and \( \partial_\omega \hat{a}_\omega = \hat{a}_\omega^\dagger/2\omega \), which can be seen from their representation in the Fock state basis \( |n\rangle_\omega \). With the help of

\[
e^{-i\psi \hat{a}_\omega^\dagger} \hat{a}_\omega e^{i\psi \hat{a}_\omega^\dagger} = e^{i\psi} \hat{a}_\omega \tag{B7}
\]

we get

\[
V(\alpha) \frac{\partial H_\omega}{\partial \omega} V^\dagger(\alpha) = \hbar \alpha_\omega^\dagger \hat{a}_\omega + \frac{\hbar}{2} \left[ e^{-2i\omega t} (\hat{a}_\omega^\dagger)^2 + e^{2i\omega t} \hat{a}_\omega^2 + 1 \right]. \tag{B8}
\]

Insertion and subsequent integration provides the local generator

\[
\mathcal{K} = t \left( \hat{a}_\omega^\dagger \hat{a}_\omega + 1/2 \right) - \frac{i}{4\omega} \left[ (1 - e^{-2i\omega t}) \hat{a}_\omega^2 + (1 - e^{2i\omega t}) \hat{a}_\omega^{\dagger 2} \right]. \tag{B9}
\]

Next, the QFI is calculated. The annihilation and creation operator \( \hat{c}_\omega \) and \( \hat{c}^\dagger_\omega \), defined by

\[
\hat{c}_\omega = S_\omega r e^{i\chi} D_\omega^\dagger(\alpha) R_\omega^\dagger(\psi) \hat{a}_\omega R_\omega(\psi) D_\omega(\alpha) S_\omega(r e^{i\chi}) \tag{B10}
\]

and \( (\hat{c}_\omega)^\dagger = \hat{c}^\dagger_\omega \), can be used to rewrite the expectation values of \( \mathcal{K} \) as follows

\[
\langle \psi_0 | \mathcal{K}^k | \psi_0 \rangle = \langle 0 | \mathcal{K}^k | \hat{a}_\omega^\dagger \hat{a}_\omega - \hat{c}_\omega^\dagger \hat{c}_\omega | 0 \rangle. \tag{B11}
\]

Using the formulæ \( \hat{a} | n \rangle = \sqrt{n} | n - 1 \rangle \) and \( \hat{a}^\dagger | n \rangle = \sqrt{n+1} | n + 1 \rangle \), we obtain the QFI after a straightforward calculation:

\[
I(\rho(t); \omega) = \frac{2t}{\omega} \sin \omega t \left[ 4\alpha^2 \cos(2\psi - \omega t) \cosh 2r + \cos(\chi + 2\psi - \omega t) \left( 4\alpha^2 \sinh 2r + \sinh 4r \right) \right]
+ \frac{2}{\omega^2} \sin^2 \omega t \left[ \sinh^2 2r \cos^2(\chi + 2\psi - \omega t) + 1 + 2\alpha^2( \cosh 2r \cos(\chi + 4\psi - 2\omega t) \sinh 2r) \right]
+ 2t^2 \left[ 2\alpha^2( \cosh 2r + \cos \chi \sinh 2r) + \sinh^2 2r \right]. \tag{B12}
\]

Comparison with Eq. [17] for a pure Gaussian state, i.e. \( N_{\text{th}} = 0 \), shows that the results are identical.
Appendix C: Calculation of QFI

Here we report the calculation of the QFI for measuring $\omega$. First, the dynamics resulting from ME (1) is determined. Then the QFI for the undamped case is calculated. Finally, the exact QFI for the damped case is given.

The solutions of ME (1) are given by [7]

\[ \langle q \rangle_t = e^{-\gamma t} \left[ \cos(\omega t) \langle q \rangle_0 + \frac{1}{M\omega} \sin(\omega t) \langle p \rangle_0 \right], \quad (C1a) \]

and

\[ \sigma_{qq}(t) = \frac{\hbar}{2M\omega}(1 + 2n)(1 - e^{-\gamma t}) \]
\[ + e^{-\gamma t} \left[ \cos^2(\omega t) \sigma_{qq}(0) + \frac{\sin^2(\omega t)}{M^2\omega^2} \sigma_{pp}(0) + \frac{\sin(2\omega t)}{M\omega} \sigma_{pq}(0) \right], \quad (C2a) \]

\[ \sigma_{pp}(t) = \frac{\hbar M\omega}{2}(1 + 2n)(1 - e^{-\gamma t}) \]
\[ + e^{-\gamma t} \left[ \cos^2(\omega t) \sigma_{pp}(0) + M^2\omega^2 \sin^2(\omega t) \sigma_{qq}(0) - M\omega \sin(2\omega t) \sigma_{pq}(0) \right], \quad (C2b) \]

\[ \sigma_{pq}(t) = e^{-\gamma t} \left[ \cos(2\omega t) \sigma_{pq}(0) + \frac{1}{M\omega} \sin(\omega t) \cos(\omega t) \left( \sigma_{pp}(0) - M^2\omega^2 \sigma_{qq}(0) \right) \right], \quad (C2c) \]

Indeed, the second equation (C1b) is an immediate consequence of $p = M\partial_\omega q$. For the general single-mode Gaussian state in Eq. (7), the initial expectation values are given by

\[ \langle q \rangle_0 = \alpha \sqrt{\frac{2\hbar}{M\omega_0}} \cos(\psi), \quad (C3a) \]
\[ \langle p \rangle_0 = \alpha \sqrt{\frac{2\hbar M\omega_0}{\omega_0}} \sin(\psi), \quad (C3b) \]

\[ \sigma_{qq}(0) = \frac{\hbar}{2M\omega_0} (2N_{\text{th}} + 1) \left[ \cosh(2r) + \cos(\chi + 2\psi) \sinh(2r) \right], \quad (C3c) \]

\[ \sigma_{pp}(0) = \frac{\hbar M\omega_0}{2} (2N_{\text{th}} + 1) \left[ \cosh(2r) - \cos(\chi + 2\psi) \sinh(2r) \right], \quad (C3d) \]

\[ \sigma_{pq}(0) = \frac{\hbar}{2} (2N_{\text{th}} + 1) \sin(\chi + 2\psi) \sinh(2r). \quad (C3e) \]

Here we give the expectation values with respect to the initial frequency $\omega_0$. The time evolution of the ME (1), on the other hand, is with respect to the new frequency $\omega$, as described in the scheme.

1. Undamped case

We start with the calculation for the QFI of the undamped case of Eq. (17). The undamped dynamic correlation function responds to the expectation values from Eq. (C1) and Eq. (C2) for $\gamma \to 0$. By using these results we calculated the five parameters of interest $\Sigma^{-1}$, $\partial_\omega \Sigma$, $P$, $\partial_\omega P$, $\partial_\omega \langle X \rangle$.

For the sake of clarity we give the results after executing the limit $\omega_0 \to \omega$ and additionally use the dimensionless time $\tau = \omega t$. The derivative of quadrature operator is given by

\[ \partial_\omega \langle X \rangle = \alpha \sqrt{\frac{2\hbar M\omega}{\omega}} \left[ \frac{\tau \sin(\psi - \tau) - \sin(\psi) \sin(\tau)}{\omega M\omega} \right], \quad (C4) \]

The purity and its derivative read

\[ P = \frac{1}{1 + 2N_{\text{th}}}, \quad (C5a) \]
\[ \partial_\omega P = \frac{2N_{\text{th}}(1 + N_{\text{th}}) \ln(1 + 1/N_{\text{th}})}{\omega(1 + 2N_{\text{th}})^2}, \quad (C5b) \]

whereas the derivative of the covariance matrix is described by the following equations:

\[ \frac{2M\omega}{\hbar} \partial_\omega \sigma_{qq}(t) = \frac{1}{\omega P} \left[ -2 \cosh(2r) \sin^2(\tau) + (\cos(\chi + 2\psi) - \cos(2\tau - \chi - 2\psi) - 2\tau \sin(2\tau - \chi - 2\psi)) \sinh(2r) \right] \]
By inserting Eq. (C4)-(C6) into Eq. (11) one obtains

\[
\begin{align*}
\frac{2}{\hbar} \partial_\omega \sigma_{pp}(t) &= \frac{1}{\omega P} [2 \cos(2\tau) \sin^2(\tau) + (\cos(\chi + 2\psi) - \cos(2\tau - \chi - 2\psi) + 2\tau \sin(2\tau - \chi - 2\psi)) \sinh(2\tau)] \\
&\quad - \frac{\partial_\omega P}{P^2} [\cosh(2\tau) - \cos(2\tau - \chi - 2\psi) \sinh(2\tau)],
\end{align*}
\]

\[
\begin{align*}
\frac{2}{\hbar} \partial_\omega \sigma_{pq}(t) &= \frac{\partial_\omega P}{P^2} \sin(2\tau - \chi - 2\psi) \sinh(2\tau) - \frac{1}{\omega P} (\sin(2\tau) \cosh(2\tau) + 2\cos(2\tau - \chi - 2\psi) \sinh(2\tau)).
\end{align*}
\]

By inserting Eqs. (C4)-(C6) into Eq. (11) one obtain Eq. (17).

By inserting Eq. (C4)-(C6) into Eq. (11) we specify the three terms of Eq. (11) separately, i.e.

\[
I(\rho(t); \omega) = I_{1,\omega} + I_{2,\omega} + I_{3,\omega},
\]

where

\[
\begin{align*}
I_{1,\omega} &= (2(1 + P^2))^{-1} \text{tr} \left[ (\Sigma^{-1} \partial_\theta \Sigma)^2 \right], \\
I_{2,\omega} &= 2(\partial_\theta P)^2 (1 - P^4)^{-1}, \\
I_{3,\omega} &= (\partial_\theta (\chi))^T \Sigma^{-1} \partial_\theta (\chi).
\end{align*}
\]

2. Damped case

By repeating the previous calculations with a non-vanishing \(\gamma\), we estimate the QFI for damped Gaussian states for measuring \(\omega\). Since the solution is too heavy, repeating the previous calculation for a non-vanishing damping leads to the following exact result of the QFI:

\[
I_{1,\omega} = \frac{P^4(\chi)}{2\omega^2 e^{4\gamma t^2} (1 + P^2(\chi))} \left\{ 8A^2_1 (A^2_1 + a^2_{1,\tau} + 2a_{1,\tau} A_1 C_\tau) S^2_\tau \right. \\
+ 8A_1 (A^2_1 + a^2_{1,\tau} + 2a_{1,\tau} A_1 C_\tau) [A_1 \sin(\chi) C_\tau + (a_{1,\tau} + A_1 C_\tau) \sin(2\tau - \xi)] S_\tau \tau \\
+ \left. \frac{A^2_1}{2} (4A^2_1 (S^2_\tau + 2) + A^2_2 A^2_3 + 2A_2 A_3 a_{2,\tau} a_3 C_\tau + a^2_{2,\tau} a^2_3 (2S^2_\tau + 1)] \\
- 2A_1 [4A_1 \cos(2\tau) + A_1 (\cos(2\chi) + \cos(4\tau - 2\chi) + 4\sin(\xi) \sin(2\tau - \xi))] S^2_\tau - 4a_{2,\tau} a_3 \sin(\tau) \sin(\tau - \xi) S_\tau \\
+ A_1 a_{1,\tau} \left( 2A_1 [4A_1 C_\tau \sin^2(\tau) (3 + 2 \cos(2\tau - \xi) S^2_\tau) + S_\tau \cos(2\tau - \xi) (A_2 A_3 - 2a_{2,\tau} a_3 C_\tau) - 2A_2 A_3 S_\tau \cos(\xi)] \\
+ A^2_2 A^2_3 C_\tau + 2A_2 A_3 a_{2,\tau} a_3 + a^2_{2,\tau} a^2_3 C_\tau \right) \\
+ \frac{a^2_{1,\tau}}{2} \left( 4A^2_1 (7 + 6S^2_\tau) + a^2_{2,\tau} a^2_3 + 2A_2 A_3 [2a_{2,\tau} a_3 C_\tau + A_2 A_3 (1 + 2S^2_\tau)] \\
+ 2A_1 \left( A_1 S^2_\tau [2(\cos(2\tau - 2\chi) - \cos(2\tau)) - \cos(4\tau - 2\xi) + \cos(2\xi)] - 12A_1 \cos(2\tau) \\
+ 8A_2 A_3 C_\tau S_\tau \sin(\tau) \sin(\tau - \xi) - 4a_{2,\tau} a_3 S_\tau \cos(\tau) \cos(\tau - \xi)] \right) \\
+ a_1^3 [2a_3 A_3 S_\tau \cos(2\tau - \xi) - 4A_1 C_\tau (\cos(2\tau - 2\xi)) + 2a^2_{1,\tau} \right\},
\]

\[
I_{2,\omega} = \frac{e^{-4\gamma t} P^6(\chi)}{2\omega^2 (1 - P^4(\chi))} \left[ A_1 A_2 A_3 + a_{1,\tau} a_2 a_3 (A_1 a_{2,\tau} a_3 + a_{1,\tau} A_2 A_3) C_\tau - 2a_{1,\tau} A_1 \cos(\xi) S_\tau \right]^2,
\]

\[
I_{3,\omega} = \frac{4 e^{-2\gamma t} A^2 \partial_\omega P^2}{\omega^2} \left\{ [a_{1,\tau} + A_1 (\cos(\chi) S_\tau)] \tau^2 + [\cos(\tau - 2\psi) (a_{1,\tau} + A_1 C_\tau) + A_1 \cos(\tau - \xi) S_\tau ] \tau \sin(\tau) \\
+ [a_{1,\tau} + A_1 (C_\tau + \cos(2\tau - \xi) S_\tau)] \sin(\tau) \right\},
\]

where we introduced a new angle \(\xi = \chi + 2\psi\), \(C_\tau = a_2 = 4\bar{n}(1 + \bar{n})\), \(A_2 = 4N_{th}(1 + N_{th})\), \(\cosh(2\tau)\), \(S_\tau = \sinh(2\tau)\) and

\[
\begin{align*}
a_1 &= 1 + 2\bar{n}, \\
A_1 &= 1 + 2N_{th}.
\end{align*}
\]
\[ a_3 = \ln(1 + 1/\bar{n}), \quad A_3 = \ln(1 + 1/N_{1b}), \quad (C12c) \]

\[ a_{1,2,\tau} = (e^{\sigma\tau} - 1)a_1, \quad (C12d) \]

Finally the maximum of the QFI of an initial ground state, which was used to approximate the QFI of the coherent state, is determined. For an initial ground state, the QFI simplifies to

\[ \omega^2 I(\rho; \omega) = \frac{1 + [e^{\sigma\tau} (1 + 2\bar{n}) - 2\bar{n}]^2 - 2[e^{\sigma\tau} (1 + 2\bar{n}) - 2\bar{n}] \cos(2\tau)}{2[2\bar{n}^2 - 2e^{\sigma\tau} \bar{n}(1 + 2\bar{n}) + e^{2\sigma\tau} (1 + 2\bar{n} + 2\bar{n}^2)]} + \frac{(e^{\sigma\tau} - 1)\bar{n}(1 + \bar{n})^2 \ln^2(1 + 1/\bar{n})}{e^{\sigma\tau}(1 + \bar{n}) - \bar{n}}. \quad (C13) \]

Numerical maximization of \( I(\rho; \omega) \) with respect to the three parameters \( \tau, g, \bar{n} \) returns the value \( 2.135/\omega^2 \).