STABILITY OF PLANAR RAREFACTION WAVES FOR SCALAR VISCOS CONSERVATION LAW UNDER PERIODIC PERTURBATIONS

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DEDICATED TO THE 80TH BIRTHDAY OF LING HSIAO

ABSTRACT. The large time behavior of the solutions to a multi-dimensional viscous conservation law is considered in this paper. It is shown that the solution time-asymptotically tends to the planar rarefaction wave if the initial perturbations are multi-dimensional periodic. The time-decay rate is also obtained. Moreover, a Gagliardo-Nirenberg type inequality is established in the domain $\mathbb{R} \times \mathbb{T}^{n-1} (n \geq 2)$, where $\mathbb{T}^{n-1}$ is the $n-1$-dimensional torus.

AMS Subject Classification: 35B65, 35L65, 35Q35
Keywords: Planar rarefaction wave, periodic perturbation, viscous conservation law

1. INTRODUCTION

We are concerned with a scalar viscous conservation law, which reads in $\mathbb{R}^n$ as

$$
\partial_t u(x,t) + \sum_{i=1}^{n} \partial_i (f_i(u(x,t))) = \Delta u(x,t), \quad t > 0, x \in \mathbb{R}^n,
$$

where $u(x,t) \in \mathbb{R}$, $\partial_i := \partial_{x_i}$ ($i = 1, 2, ..., n$) and $\Delta = \sum_{i=1}^{n} \partial_i^2$; the fluxes $f_i(u)$ ($i = 1, 2, ..., n$) are smooth and $f''_i(u) \geq a_0$ for some constant $a_0 > 0$.

It is well known \cite{11, 21} that the one-dimensional (1-d) conservation law, i.e. the right hand side of (1.1) is zero and $n = 1$, has rich wave phenomena including shocks and rarefaction waves. A centered rarefaction wave $u^R(x_1,t)$ is an entropy solution of the following Riemann problem

$$
\begin{cases}
\partial_t u^R(x_1,t) + \partial_1 f_1(u^R(x_1,t)) = 0, \\
u^R(x_1,0) = \begin{cases}
\bar{u}_l, & x_1 < 0, \\
\bar{u}_r, & x_1 > 0,
\end{cases}
\end{cases}
$$

with $\bar{u}_l < \bar{u}_r$, and has an explicit formula as

$$
u^R(x_1,t) = \begin{cases}
\bar{u}_l, & x_1 \leq f'_1(\bar{u}_l)t, \\
(f'_1)^{-1}(\frac{x_1}{t}), & f'_1(\bar{u}_l)t < x_1 \leq f'_1(\bar{u}_r)t, \\
\bar{u}_r, & x_1 > f'_1(\bar{u}_r)t.
\end{cases}
$$

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Feimin Huang is partially supported by NSFC Grant No. 11371349 and 11688101.
Qian Yuan is partially supported by the China Postdoctoral Science Foundation funded project (2019M660831).
It is shown in [7, 12, 24, 10] that the rarefaction wave given in (1.3) is asymptotically stable for both the inviscid and viscous conservation laws in $L^2$ framework provided that the initial perturbations are $L^2$ integrable. The analysis for the solutions of conservation laws with periodic initial data is important, cf. [11, 3, 1, 2, 25, 26, 27]. In particular, the large time behavior of the nonlinear waves under periodic perturbations for viscous conservation laws is firstly investigated in [25] with the aid of maximum principle. The 1-d rarefaction wave $u^R(x_1,t)$ becomes the planar rarefaction wave in the multi-dimensional (m-d) case. The main purpose of this paper is to use the energy method to extend the work of [25] to the m-d case, i.e., to show whether the planar rarefaction wave is stable or not under m-d periodic perturbations. Besides, we refer to [5, 7, 8, 9, 16, 17, 4, 13, 14, 6, 15] and the references therein for the other interesting works about the stability of shocks, rarefaction waves and contact discontinuities.

Now we formulate the main result. Since the centered rarefaction wave given in (1.3) is a weak solution and only Lipschitz continuous with respect to $x_1$, we need to construct a viscous version of rarefaction wave .temphasem to replace the original one.

Following [17, 24], $\tilde{u}^R(x_1,t)$ can be constructed as a smooth solution of the viscous conservation law (1.1), i.e.,

$$
\begin{align*}
\partial_t \tilde{u}^R + \partial_x (f_1(\tilde{u}^R)) &= \mathcal{F}^2 \tilde{u}^R, \\
\tilde{u}^R(x_1,0) &= \frac{\pi_l + \pi_r}{2} + \frac{\pi_r - \pi_l}{c_r + c_l} e^{c_l x_1 - c_r}. 
\end{align*}
$$

(1.4)

It is straightforward to check that

$$
\lim_{x_1 \to -\infty} \tilde{u}^R(x_1,0) = \overline{u}_l < \overline{u}_r = \lim_{x_1 \to +\infty} \tilde{u}^R(x_1,0).
$$

Consider the scalar equation (1.1) with the following initial data

$$
u(x,0) = u_0(x) = \tilde{u}^R(x_1,0) + w_0(x), \quad x \in \mathbb{R}^n, \tag{1.5}
$$

where $w_0(x)$ is a m-d periodic function defined on the n-d torus $\mathbb{T}^n = \prod_{i=1}^{n} [0,1]$. Without loss of generality (by adding the average constant onto $\overline{u}_l$ and $\overline{u}_r$, respectively), one can assume that

$$
\int_{\mathbb{T}^n} w_0(x) dx = 0. \tag{1.6}
$$

We are ready to state the main theorem.

**Theorem 1.1.** Assume that the periodic perturbation $w_0(x) \in H^{[\frac{d}{2}]+2}(\mathbb{T}^n)$ and satisfies (1.6).

Then there exists a unique global smooth solution $u$ of (1.1), (1.5) satisfying

$$
\|u(x,t) - \tilde{u}^R(x_1,t)\|_{L^\infty(\mathbb{R}^n)} \leq C (1 + t)^{-\frac{1}{2}}, \quad t > 0, \tag{1.7}
$$

where the constant $C > 0$ is independent of $t$.

**Remark 1.2.** It is shown in [10] that any viscous rarefaction waves connecting same end states are time-asymptotically equivalent in the $L^\infty(\mathbb{R})$ space with the rate $t^{-\frac{1}{2}}$. We remark that the stability result (1.7) also holds true for more general initial
values \( \tilde{u}^R(x_1,0) + v_0(x_1) + w_0(x) \) instead of (1.5), where \( v_0(x_1) \) is any 1-d function in \( L^1(\mathbb{R}) \cap H^1(\mathbb{R}) \).

**Remark 1.3.** Theorem 1.1 is the first work concerning the m-d periodic perturbations around the nonlinear waves for conservation laws, which shows that the oscillations in all directions around the planar rarefaction wave decay to zero with the rate \( t^{-\frac{1}{2}} \), even though the initial perturbations keep oscillating at infinity \( |x| \to +\infty \). In other words, the oscillations are eliminated due to the genuine nonlinearity of the equation.

Let us outline the proof of Theorem 1.1. Motivated by [24] and [10] in which the planar rarefaction waves are shown to be stable under m-d perturbations for the scalar equation (1.1), we want to use the energy method to prove Theorem 1.1. However, the initial periodic perturbation \( v_0(x) \) is not integrable on \( \mathbb{R}^n \), and has no any limit at far fields. Thus, the effective energy method developed in the previous articles can not be applied here directly. To overcome this difficulty, we construct a suitable ansatz \( \tilde{u}(x,t) \) which contains the oscillations in the \( x_1 \) direction so that the oscillations in the difference between the solution \( u \) and the ansatz \( \tilde{u} \) is eliminated in the \( x_1 \) direction, i.e., \( (u - \tilde{u})(x,t) \in L^1_{\text{loc}}(\mathbb{R}) \), This is the key point in our proof. Then the stability with time-decay rates (1.7) can be obtained by the \( L^p \) energy method developed in [10].

To prove Theorem 1.1, we establish a Gagliardo-Nirenberg type inequality in the domain \( \Omega := \mathbb{R} \times \mathbb{T}^{n-1} \), which is the second novelty in this paper. The Gagliardo-Nirenberg (GN) inequality is very useful in the field of PDEs and usually holds in the whole space \( \mathbb{R}^n \) or in the domain with zero Dirichlet boundary condition. Let us recall the GN inequality which reads as, for any \( 1 \leq p \leq +\infty \) and integer \( 0 \leq j < m \),

\[
\left\| \nabla^j_x u \right\|_{L^p(\mathbb{R}^n)} \leq C \left\| \nabla^m_x u \right\|_{L^\theta(\mathbb{R}^n)} \left\| u \right\|_{L^{1-\theta}(\mathbb{R}^n)},
\]

where \( \frac{1}{p} = \frac{1}{n} + \left( \frac{1}{r} - \frac{m}{n} \right) \theta + \frac{1}{q}(1-\theta) \), \( \frac{1}{m} \leq \theta \leq 1 \) and \( C \) is a constant independent of \( u \). However, the n-d GN inequality (1.8) does not hold in the domain \( \Omega \) generally owing to the following counterexample.

**Counterexample:** It is noted that any 1-d function \( 0 \neq f(x_1) \in C^\infty_c(\mathbb{R}) \) is periodic in the \( x_i \) direction for \( i = 2, \cdots, n \) (one cannot exclude the 1-d case since the initial perturbations satisfying (1.6) include the 1-d periodic functions). Then corresponding to the case \( j = 0, p = \frac{n}{n-1}, \theta = m = r = 1 \) in (1.8), which is exactly the n-d Sobolev inequality, one can let \( \zeta_d(x_1) = f(d^{-1}x_1), d > 0 \), and a direct computation implies that

\[
\left\| \zeta_d \right\|_{L^{\frac{n}{n-d}}(\Omega)} = C_{n,d} \left\| \nabla_x \zeta_d \right\|_{L^1(\Omega)},
\]

where \( C_{n,d} := d^{\frac{n-1}{n}} \left\| f' \right\|_{L^{\frac{n}{n-d}}(\mathbb{R})} / \left\| f' \right\|_{L^1(\mathbb{R})} \). Since \( C_{n,d} \to +\infty \) as \( d \to +\infty \), we conclude that in general the n-d GN inequality (1.8) is not true in \( \Omega \) without additional conditions.

Instead, based on a function-decomposition in Lemma 3.3 below, we establish a Gagliardo-Nirenberg type inequality on the domain \( \Omega \).
Theorem 1.4 (GN type inequality on $\Omega = \mathbb{R} \times \mathbb{T}^{n-1}$). Let $u \in L^q(\Omega)$ and $\nabla^m u \in L^r(\Omega)$ where $1 \leq q, r \leq +\infty$ and $m \geq 1$, and $u$ is periodic in the $x_i$ direction for $i = 2, \ldots, n$. Then there exists a decomposition $u(x) = \sum_{k=0}^{n-1} u^{(k)}(x)$ such that each $u^{(k)}$ satisfies the $k + 1$-dimensional GN inequality (1.8), i.e.,

$$
\| \nabla^j u^{(k)} \|_{L^p(\Omega)} \leq C \| \nabla^m u \|_{L^r(\Omega)} \| u \|_{L^q(\Omega)}^{1-\theta_k},
$$

(1.10)

for any $0 \leq j < m$ and $1 \leq p \leq +\infty$ satisfying $\frac{1}{p} = \frac{j}{k+1} + \left( \frac{1}{r} - \frac{m}{k+1} \right) \theta_k + \frac{1}{q} (1 - \theta_k)$ and $\frac{1}{m} \leq \theta_k \leq 1$, with the following exceptional cases,

1) if $j = 0, rm < k + 1$ and $q = +\infty$, additional assumption that $u \to 0$ as $|x_1| \to +\infty$ is required;

2) if $1 < r < +\infty$ and $m - j - \frac{k+1}{r}$ is a non-negative integer, additional assumption that $\theta_k < 1$ is required.

Hence, it holds that

$$
\| \nabla^j u \|_{L^p(\Omega)} \leq C \sum_{k=0}^{n-1} \| \nabla^m u \|_{L^r(\Omega)} \| u \|_{L^q(\Omega)}^{1-\theta_k},
$$

(1.11)

where the constant $C > 0$ is independent of $u$.

Remark 1.5. It is noted that the first term for $k = 0$ on the right-hand side (RHS) of (1.11) is necessary since $u$ can be a 1-d function defined on $\mathbb{R}$. For example, considering the same function $\zeta_d(x_1)$ as in the counterexample, when corresponding to the case $j = 0, p = r = 2$ and $m = q = 1$ in (1.11), a direct computation gives that

$$
\| \zeta_d \|_{L^2(\Omega)} = C d^{\frac{1}{2}(3^{\theta-1})} \| \nabla \zeta_d \|_{L^2(\Omega)}^{\theta} \| \zeta_d \|_{L^1(\Omega)}^{1-\theta}
$$

(1.12)

for any $0 \leq \theta \leq 1$, where $C > 0$ is independent of $d$. The estimate (1.11) can be true only for $\theta = \theta_0 = \frac{1}{3}$. Otherwise, one can let $d \to 0+$ if $\theta < \frac{1}{3}$ and $d \to +\infty$ if $\theta > \frac{1}{3}$, respectively to get the contradiction.

Remark 1.6. An interesting interpolation inequality involving $\nabla^2 u$ in the domain $\mathbb{R} \times \mathbb{T}^2$ was established in [23].

The rest of the paper is organized as follows. Some preliminaries on the viscous rarefaction waves and the construction of the ansatz are given in Section 2. Section 3 is devoted to the proof of Theorem 1.4 and an interpolation inequality, Corollary 3.5. In Section 4, we show the desired a priori estimates, Theorem 4.2; thereafter, the proof of Theorem 1.1 is completed.

2. Preliminaries and Ansatz

Some properties of the viscous rarefaction wave are listed as follows.

Lemma 2.1. The viscous rarefaction wave $\tilde{u}^R(x_1, t)$ solving (1.4) satisfies

$$
\lim_{t \to +\infty} \| (\tilde{u}^R - u^R) (\cdot, t) \|_{L^p(\mathbb{R})} = 0.
$$

(2.1)

Moreover, for any $t > 0$ it holds that

$$
0 < \partial_1 \tilde{u}^R(x_1, t) \leq \min \left\{ \frac{C}{t}, \max_{x_1 \in \mathbb{R}} \left( u_0^R \right) \right\} \quad \forall x_1 \in \mathbb{R},
$$

(2.2)
\[ \| (\tilde{u}^R(\cdot, t) - \overline{u}_l) (\tilde{u}^R(\cdot, t) - \overline{u}_r) \|_{L^1(\mathbb{R})} \leq C(1 + t), \]  

(2.3)

\[ \| \partial_1 \tilde{u}^R(\cdot, t) \|_{L^p(\mathbb{R})} \leq C t^{-1 + \frac{1}{p}}, \quad p \in [1, +\infty), \]  

(2.4)

where the constant \( C > 0 \) is independent of \( t \).

**Proof.** The \( L^\infty \) time-asymptotically equivalence (2.1) has been proved in [7, 10]. By the maximum principles for (1.4) and the equation of the derivative \( v = \partial_1 \tilde{u}^R \), 

\[ \partial_1 v - \partial_1^2 v + f''_1(\tilde{u}^R) v^2 + f'_1(\tilde{u}^R) \partial_1 v = 0, \]  

(2.5)

one can get that 

\[ \sup_{x_1 \in \mathbb{R}, t \geq 0} |\tilde{u}^R(x_1, t)| \leq \| u_0^R \|_{L^\infty(\mathbb{R})} \]  

and 0 < \( \partial_1 \tilde{u}^R(x_1, t) \leq \max_{x_1 \in \mathbb{R}} (u_0^R)'(x_1) \).

Besides, it was shown in [19] that the solution \( \tilde{u}^R \) of the 1-d convex conservation law (1.4) satisfies the well-known Ole\'nik entropy condition, i.e., \( \partial_1 \tilde{u}^R(x_1, t) \leq \frac{C}{T} \) for all \( x_1 \in \mathbb{R} \), where \( C > 0 \) is independent of \( t \).

Since \( \overline{u}_l < \tilde{u}^R < \overline{u}_r \) and both \( \tilde{u}^R \) and \( \partial_1 \tilde{u}^R \) are uniformly bounded, it follows from (1.4) that 

\[ \frac{d}{dt} \left[ \int_{-\infty}^{0} (\tilde{u}^R(x_1, t) - \overline{u}_l) \, dx_1 + \int_{0}^{+\infty} (\overline{u}_r - \tilde{u}^R(x_1, t)) \, dx_1 \right] \leq C, \quad t > 0, \]

which implies that 

\[ \int_{-\infty}^{0} (\tilde{u}^R(x_1, t) - \overline{u}_l) \, dx_1 + \int_{0}^{+\infty} (\overline{u}_r - \tilde{u}^R(x_1, t)) \, dx_1 \leq C(1 + t), \quad t > 0, \]

and thus (2.3) holds true. And (2.4) can follow from 

\[ \| \partial_1 \tilde{u}^R \|_{L^p(\mathbb{R})}^p \leq \| \partial_1 \tilde{u}^R \|_{L^\infty(\mathbb{R})}^{p-1} \int_{\mathbb{R}} \partial_1 \tilde{u}^R(x_1, t) \, dx_1 \leq C t^{-p+1}. \]

\[ \square \]

Now we construct the ansatz. Set 

\[ g(x_1, t) := \frac{\tilde{u}^R(x_1, t) - \overline{u}_l}{\overline{u}_r - \overline{u}_l}, \quad x_1 \in \mathbb{R}, \quad t \geq 0, \]  

(2.6)

which is smooth, belongs to the interval \((0, 1)\) and satisfies \( \partial_1 g(x_1, t) > 0 \) for any \( x_1 \in \mathbb{R}, t \geq 0 \).

Let \( u_l(x, t) \) and \( u_r(x, t) \) denote the two periodic solutions of (1.1) (see [20] for the global existence) with the respective periodic data 

\[ u_l(x, 0) = \overline{u}_l + w_0(x) \quad \text{and} \quad u_r(x, 0) = \overline{u}_r + w_0(x), \quad x \in \mathbb{R}^n. \]  

(2.7)

And define 

\[ w_i(x, t) := u_i(x, t) - \overline{u}_i, \quad i = l \text{ or } r, \]  

(2.8)

which is a periodic function with zero average for any \( t \geq 0 \). Then we have

**Lemma 2.2.** If \( w_0(x) \in H^2(\mathbb{T}^n) \) satisfies (1.6), then for \( i = l \text{ or } r \), it holds that 

\[ \| w_i(\cdot, t) \|_{W^{1,\infty}(\mathbb{R}^n)} \leq C \| w_0 \|_{H^2(\mathbb{T}^n)} e^{-2\alpha t}, \quad t \geq 0, \]  

(2.9)

where the constants \( \alpha > 0 \) and \( C > 0 \) are independent of \( t \).

**Proof.** The proof of Lemma 2.2 is based on basic \( L^2 \) energy estimates and the Poincaré inequality on \( \mathbb{T}^n \), which is standard and thus omitted. \[ \square \]
We are ready to construct the ansatz as follows:
\[
\hat{u}(x, t) := w(x, t) (1 - g(x_1, t)) + u_r(x, t) g(x_1, t),
\]
which is periodic in the \( x_i \) direction for \( i = 2, \cdots, n \). By direct calculations, the source term induced by the ansatz \( \hat{u}(x, t) \) is given as follows.
\[
h(x, t) := \partial_t \hat{u} + \sum_{i=1}^{n} \partial_i \left( f_i(\hat{u}(x, t)) \right) - \Delta \hat{u}(x, t)
\]
\[
= (u_r - u_l) g(1-g) \sum_{i=1}^{n} \left[ \sigma_i(u_l, \tilde{u}) \bar{\partial}_i w_i - \sigma_i(u_r, \tilde{u}) \bar{\partial}_i w_i \right]
\]
\[
+ (u_r - u_l) \sigma_1(\tilde{\hat{u}}^R, \tilde{u}) \left( \tilde{\hat{u}} - \tilde{\hat{u}}^R \right) \bar{\partial}_1 g - 2 \bar{\partial}_1 (w_r - w_l) \bar{\partial}_1 g,
\]
where \( \sigma_i(u, v) := \int_0^1 f_i''(v + \theta(u-v))d\theta, \ i = 1, \cdots, n \). Thus, the source term \( h(x, t) \) is also periodic in the \( x_i \) direction for \( i = 2, \cdots, n \).

**Lemma 2.3.** Under the assumptions of Lemma 2.2, it holds that for any \( p \in [1, +\infty], \)
\[
\|h(\cdot, t)\|_{L^p(\Omega)} \leq C \|w_0\|_{H^{\frac{3}{2}}(\mathbb{T}^n)} e^{-\alpha t}, \quad t \geq 0,
\]
where \( \alpha > 0 \) is the constant in Lemma 2.2, and \( C > 0 \) is independent of \( t \).

**Proof.** By (2.3), (2.4) and Lemma 2.2, it is straightforward to check that (2.12) holds for \( p = 1 \) and \( +\infty \). And for \( p \in (1, +\infty) \), (2.12) follows from the interpolation inequality on \( \Omega = \mathbb{R} \times \mathbb{T}^{n-1} \).

### 3. Interpolation Inequality on \( \Omega = \mathbb{R} \times \mathbb{T}^{n-1} \)

Although the \( n \)-d GN inequality (1.8) does not hold in the domain \( \Omega = \mathbb{R} \times \mathbb{T}^{n-1} \) generally, it is true with an additional condition, that is,

**Lemma 3.1.** Let \( u \in L^q(\Omega) \), and its derivatives of order \( m \), \( \nabla^m u \in L^r(\Omega) \), where \( m \geq 1 \) and \( 1 \leq r, q \leq +\infty \). And assume that \( u \) is periodic in the \( x_i \) direction and satisfies
\[
\int_{\mathbb{T}} u(x) dx_i = 0 \quad \text{for all} \ i = 2, 3, \cdots, n.
\]
Then the \( n \)-d GN inequality (1.8) holds true for \( u \).

**Proof.** Following [18, Lecture II], it suffices to prove the following two extreme cases.

1) For a.e. \( x \in \Omega \), it holds that
\[
|u(x)|^n \leq \int_{\mathbb{R}} |\partial_1 u| dx_1 \prod_{i=2}^{n} \int_{\mathbb{T}} |\partial_i u| dx_i.
\]
2) For any \( 1 \leq i \leq n \), and \( 1 \leq q < +\infty, 1 < r < +\infty \), it holds that
\[
\int_{\mathbb{T}} |\partial_i u|^p dx_i \leq C \left( \int_{\mathbb{R}} |\partial_i^2 u|^r dx_i \right)^{\frac{1}{r}} \left( \int_{\mathbb{R}} |u|^q dx_i \right)^{\frac{1}{q}},
\]
where \( \frac{2}{p} = \frac{1}{r} + \frac{1}{q} \), and “\( \int \)” indicates the integral over \( \mathbb{R} \) if \( i = 1 \) or \( \mathbb{T} \) if \( i > 1 \).
Thanks to (3.1), it is straightforward to show (3.2). And (3.3) for $i = 1$ has been verified in [18]. It remains to show this inequality on $\mathbb{T} = [0, 1]$ for $i > 1$. In fact, for any bounded interval $I \subset \mathbb{R}$, it follows from [18, (2.7)] that

$$\int_I |\partial_i u|^p \, dx_i \leq C |I|^{1 + \frac{p}{q}} \left( \int_I |\partial_i^2 u|^r \, dx_i \right)^{\frac{p}{r}} + C |I|^{-(1 + \frac{p}{q})} \left( \int_I |u|^q \, dx_i \right)^{\frac{p}{q}}, \quad (3.4)$$

where $C > 0$ is independent of $I$. Since $u$ is periodic in the $x_i$ direction with period 1, the first term on the RHS of (3.4) must be greater than the second one, as long as the length of $I$ is equal to a large integer $N$. Similar to [18, (2.8)], one can cover $[0, N]$ by finite intervals $I_j \subset [0, 2N]$, at each of which, the first term on the RHS of (3.4) (where $I = I_j$) is either greater than or equal to the second one. The proof is to repeat that of L. Nirenberg’s in [18]. However, to make this paper complete, we still give the details here. First let $k > 0$ be large enough and fixed, and consider $I = (0, \frac{N}{k})$ in (3.4). If the first term on the RHS is greater than the second one, set $I_1 = (0, \frac{N}{k})$, and it holds that

$$\int_{I_1} |\partial_i u|^p \, dx_i \leq C \left( \frac{N}{k} \right)^{1 + \frac{p}{q}} \left( \int_{I_1} |\partial_i^2 u|^r \, dx_i \right)^{\frac{p}{r}} \left( \int_{I_1} |u|^q \, dx_i \right)^{\frac{p}{q}}, \quad (3.5)$$

where the constant $C > 0$ is independent of either $k$ or $N$. Otherwise, if the second term on the RHS of (3.4) is greater, then due to the choice of $N$, one can extend the interval $(0, \frac{N}{k})$ to a larger one, $I_1 \subset [0, 2N]$, until the two terms become equal. Then one has that

$$\int_{I_1} |\partial_i u|^p \, dx_i \leq C \left( \int_{I_1} |\partial_i^2 u|^r \, dx_i \right)^{\frac{p}{r}} \left( \int_{I_1} |u|^q \, dx_i \right)^{\frac{p}{q}}, \quad (3.6)$$

where the constant $C > 0$ is independent of either $k$ or $N$. Starting at the end point of $I_1$, one can repeat this process to set $I_2, I_3, \cdots$, until $[0, N]$ is covered. For the fixed $k$, there must be finitely many such intervals $I_j$, each of which is contained in $[0, 2N]$. Then summing the estimates (3.5) and (3.6) together yields

$$\int_0^N |\partial_i u|^p \, dx_i \leq C k \left( \frac{N}{k} \right)^{1 + \frac{p}{q}} \left( \int_0^N |\partial_i^2 u|^r \, dx_i \right)^{\frac{p}{r}} \left( \int_0^{2N} |u|^q \, dx_i \right)^{\frac{p}{q}} + C \left( \int_0^{2N} |\partial_i^2 u|^r \, dx_i \right)^{\frac{p}{r}} \left( \int_0^{2N} |u|^q \, dx_i \right)^{\frac{p}{q}}.$$

Letting $k \to +\infty$, one has that

$$\int_0^N |\partial_i u|^p \, dx_i \leq C \left( \int_0^{2N} |\partial_i^2 u|^r \, dx_i \right)^{\frac{p}{r}} \left( \int_0^{2N} |u|^q \, dx_i \right)^{\frac{p}{q}},$$

which yields (3.3), since $u$ is periodic in the $x_i$ direction and $\frac{p}{r} + \frac{p}{q} = 1$. \hfill \Box

**Remark 3.2.** Lemma 3.1 means that the GN inequality (1.8) holds under the additional condition (3.1). Nevertheless, it is difficult to verify (3.1) for the perturbation $u(x, t) - \tilde{u}(x, t)$ for any $x_i$ direction ($i = 2, \cdots, n$) even though it holds initially, i.e., $\int_\mathbb{T} w_0(x) \, dx_i = 0 \ \forall i = 2, \cdots, n$, which is much stronger than the original condition (1.6).
To prove Theorem 1.1, we shall establish a GN type inequality on the domain \( \mathbb{R} \times \mathbb{T}^{n-1} \) without any additional condition by applying Lemma 3.1 and the following decomposition lemma.

For any \( k \geq 1 \), set
\[
\mathcal{A}_k = \left\{ u(x) \text{ is measurable on } \mathbb{R} \times \mathbb{T}^k \text{ with } x_1 \in \mathbb{R}, \text{ and periodic in } x_i \in \mathbb{T}, \right. \\
\left. \text{satisfying } \int_{\mathbb{T}} u(x) dx_i = 0 \text{ for all } i = 2, \ldots, k+1 \right\}. \tag{3.7}
\]
Then for any function \( u \in \mathcal{A}_k \), the \( k+1 \)-dimensional GN inequality (1.8) holds true. We are ready to decompose general function \( u \) in terms of \( \mathcal{A}_k \), i.e.,

**Lemma 3.3 (Decomposition Lemma).** One can decompose \( u(x) \) as follows,
\[
u(x) = \sum_{k=0}^{n-1} u^{(k)}(x) \text{ a.e. } x \in \Omega, \tag{3.8}\]
where
\[
u^{(0)}(x) = u^{(0)}(x_1) = \int_{\mathbb{T}^{n-1}} u(x_1, x_2, \ldots, x_n) dx_2 \cdots dx_n, \tag{3.9}\]
\[
u^{(k)}(x) = \sum_{2 \leq i_1 < \cdots < i_k \leq n} u_{i_1, \ldots, i_k}(x_1, x_{i_1}, \ldots, x_{i_k}), \quad k = 1, \ldots, n-1,
\]
where each \( u_{i_1, \ldots, i_k}(x_1, x_{i_1}, \ldots, x_{i_k}) \in \mathcal{A}_k \). Moreover, for any \( m \geq 0 \) and \( 1 \leq p \leq +\infty \), it holds that
\[
\| \nabla_x^m u^{(0)} \|_{L^p(\Omega)} + \sum_{k=1}^{n-1} \sum_{2 \leq i_1 < \cdots < i_k \leq n} \| \nabla_x^m u_{i_1, \ldots, i_k} \|_{L^p(\Omega)} \leq C \| \nabla_x^m u \|_{L^p(\Omega)}, \tag{3.10}\]
where \( C > 0 \) is independent of \( u \).

**Remark 3.4.** To avoid excessive words, we omit the assumptions in Lemma 3.3 that the integrals (3.9) and
\[
\int_{\mathbb{T}^{n-1-k}} u(x) \frac{dx_2 \cdots dx_n}{dx_{i_1} \cdots dx_{i_k}}, \quad 1 \leq k \leq n-1, \quad 2 \leq i_1 < \cdots < i_k \leq n,
\]
should exist, here and hereafter we use \( \int_{\mathbb{T}^{n-1-k}} (\cdot) \frac{dx_2 \cdots dx_n}{dx_{i_1} \cdots dx_{i_k}} \) to denote the integral that is integrated with respect to \( x_2, \ldots, x_n \) except for \( x_{i_1}, \ldots, x_{i_k} \).

**Proof.** Starting with \( u^{(0)} \) defined in (3.9), set
\[
u_i(x_1, x_i) = \int_{\mathbb{T}^{n-2}} (u - u^{(0)}) \frac{dx_2 \cdots dx_n}{dx_i}, \quad i = 2, \ldots, n. \tag{3.11}\]
It is straightforward to check that \( \nu_i(x_1, x_i) \in \mathcal{A}_1 \). Define \( u^{(1)}(x) := \sum_{i=2}^{n} u_i \). Assume that for all \( l = 1, 2, \ldots, k-1 \) with \( k \geq 2 \) and \( 2 \leq j_1 < \cdots < j_l \leq n \), all the functions \( u_{j_1, \ldots, j_l} = u_{j_1, \ldots, j_l}(x_1, x_{j_1}, \ldots, x_{j_l}) \in \mathcal{A}_l \) and thus \( u^{(l)} := \sum_{2 \leq j_1 < \cdots < j_l \leq n} u_{j_1, \ldots, j_l} \) have been well-defined. Then for any fixed sequence \( 2 \leq i_1 < \cdots < i_k \leq n \), set
\[
u_{i_1, \ldots, i_k}(x_1, x_{i_1}, \ldots, x_{i_k}) := \int_{\mathbb{T}^{n-1-k}} (u - \sum_{l=0}^{k-1} u^{(l)}) \frac{dx_2 \cdots dx_n}{dx_{i_1} \cdots dx_{i_k}} \tag{3.12}\]
where the indices remains to show (3.10). It follows from (3.9) and Minkowski inequality that

\[ \int_T u_{j_1, \ldots, j_{k-1}} \, dx_{i_r} = \begin{cases} 0 & \text{if } i_r \in \{j_1, \ldots, j_{k-1}\}, \\ u_{j_1, \ldots, j_{k-1}} & \text{if } i_r \notin \{j_1, \ldots, j_{k-1}\}. \end{cases} \]  

(3.13)

Then it holds that

\[
\int_T u_{i_1, \ldots, i_k} \, dx_{i_r} = \int_{T^{n-k}} \left( u - \sum_{l=0}^{k-2} u^{(l)} \right) \frac{dx_2 \cdots dx_n}{dx_{i_1} \cdots dx_{i_{r-1}} dx_{i_{r+1}} \cdots dx_i} \\
- \sum_{2 \leq j_1 < \cdots < j_{k-1} \leq n} \int_{T^{n-k}} u_{j_1, \ldots, j_{k-1}} \frac{dx_2 \cdots dx_n}{dx_{i_1} \cdots dx_{i_{r-1}} dx_{i_{r+1}} \cdots dx_i} \\
= u_{i_1, \ldots, i_{r-1}, i_{r+1}, \ldots, i_k} - u_{i_1, \ldots, i_{r-1}, i_r, i_{r+1}, \ldots, i_k} \\
= 0,
\]

which means \( u_{i_1, \ldots, i_k} \in A_k \) and then the decomposition (3.8) for function \( u \) holds. It remains to show (3.10). It follows from (3.9) and Minkowski inequality that

\[
\| \nabla^m u(0) \|_{L^p(\mathbb{R})} \leq \left\| \nabla^m u \right\|_{L^1(T^{n-1}; dx_2 \cdots dx_n)} \left\| u - u^{(0)} \right\|_{L^p(\mathbb{R}; dx_1)} \\
\leq \left\| \nabla^m u \right\|_{L^p(\mathbb{R}; dx_1)} \left\| u - u^{(0)} \right\|_{L^p(\mathbb{R})} \\
\leq \left\| \nabla^m u \right\|_{L^p(\Omega)},
\]

where the last inequality is derived from the Hölder inequality. Similarly, one can obtain from (3.11) that for any \( 2 \leq i \leq n \),

\[
\| \nabla^m u_i \| \leq \left\| \nabla^m u \right\| \left\| u - u^{(0)} \right\|_{L^p(\mathbb{R})} \leq 2 \left\| \nabla^m u \right\|_{L^p(\Omega)}.
\]

The remaining functions \( u_{i_1, \ldots, i_k} \) defined in (3.12) can be proved in the same way, and then the proof of Lemma 3.3 is completed. \( \square \)

**Proof of Theorem 1.4.** We first decompose \( u(x) = \sum_{k=0}^{n-1} u^{(k)}(x) \) as in Lemma 3.3. Then it follows from Theorem 3.1 and Lemma 3.3 that \( u^{(0)} \in L^q(\mathbb{R}), \nabla^m u^{(0)} \in L^r(\mathbb{R}), \) and it satisfies the 1-d GN inequality (1.8); each \( u_{i_1, \ldots, i_k} \in L^q(\Omega), \nabla^m u_{i_1, \ldots, i_k} \in L^r(\Omega) \) and it satisfies the \((k + 1)-d\) GN inequality (1.8). Hence,

\[
\| \nabla^j u^{(0)} \|_{L^p(\mathbb{R})} \leq C \left\| \nabla^m u^{(0)} \right\|^{\theta_0}_{L^r(\mathbb{R})} \left\| u^{(0)} \right\|^{1-\theta_0}_{L^r(\Omega)} \leq C \left\| \nabla^m u \right\|^{\theta_0}_{L^r(\Omega)} \left\| u \right\|^{1-\theta_0}_{L^r(\Omega)}; \\
\| \nabla^j u^{(k)} \|_{L^p(\Omega)} \leq \sum_{2 \leq i_1 < \cdots < i_k \leq n} \| \nabla^j u_{i_1, \ldots, i_k} \|_{L^p(\Omega)} \\
\leq C \left\| \nabla^m u \right\|^{\theta_k}_{L^r(\Omega)} \left\| u \right\|^{1-\theta_k}_{L^r(\Omega)} \leq C \left\| \nabla^m u \right\|^{\theta_k}_{L^r(\Omega)} \left\| u \right\|^{1-\theta_k}_{L^r(\Omega)},
\]

where the indices \( \theta_0, \ldots, \theta_{n-1} \) are that introduced in Theorem 1.4. The proof is finished. \( \square \)
In order to use the $L^p$ energy method developed in [10], we shall establish the following interpolation inequality by Theorem 1.4.

**Corollary 3.5 (Interpolation Inequality in $\Omega$).** For any $2 \leq p < \infty$ and $1 \leq q \leq p$, it holds that

$$
\|u\|_{L^p(\Omega)} \leq C \sum_{k=0}^{n-1} \left\| \nabla \left( |u|^\frac{p}{2} \right) \right\|_{L^2(\Omega)}^{2\gamma_k \frac{1}{1+\gamma_k p}} \left\| u \right\|_{L^q(\Omega)}^{\frac{1}{1+\gamma_k p}},
$$

where $\gamma_k = \frac{k+1}{2} \left( \frac{1}{q} - \frac{1}{p} \right)$ and the constant $C = C(p, q, n) > 0$ is independent of $u$.

**Proof.** For the function $v(x) = |u(x)|^\frac{p}{2}$, it follows from Theorem 1.4 that

$$
\|v\|_{L^p(\Omega)} \leq C \sum_{k=0}^{n-1} \left\| \nabla v \right\|_{L^2(\Omega)}^{\theta_k} \left\| u \right\|_{L^{\frac{p}{1-\theta_k}}(\Omega)}^{1-\theta_k},
$$

where $\frac{1}{2} = \left( \frac{1}{q} - \frac{1}{k+1} \right) \theta_k + \frac{1}{2k} (1 - \theta_k)$ with $\frac{1}{2} \leq r_k \leq 1$ and $0 \leq \theta_k \leq 1$ for $k = 0, 1, \cdots, n - 1$. This yields that

$$
\|u\|_{L^p(\Omega)} \leq C \sum_{k=0}^{n-1} \left\| \nabla \right\|_{L^2(\Omega)}^{\frac{p}{2} \theta_k} \left\| u \right\|_{L^\frac{p}{1-\theta_k}(\Omega)}^{1-\theta_k}.
$$

If $2 \leq p \leq 2q$, choosing $r_k = \frac{q}{p} \geq \frac{1}{2}$ for $k = 0, \cdots, n - 1$ can finish the proof. If $p > 2q$, let $\frac{1}{2} < r_k < 1$, which will be determined later. Since $q < p r_k < p$, it follows from interpolation that

$$
\|u\|_{L^{p r_k}(\Omega)} \leq \|u\|_{L^q(\Omega)}^{\frac{1-p r_k}{p}} \|u\|_{L^p(\Omega)}^{\frac{p r_k}{p}},
$$

where $\rho_k = \frac{p r_k - q}{(p-q) r_k} \in (0, 1)$. Plugging (3.16) into (3.15) yields that

$$
\|u\|_{L^p(\Omega)} \leq C \sum_{k=0}^{n-1} \left\| \nabla \left( |u|^\frac{p}{2} \right) \right\|_{L^2(\Omega)}^{\frac{p}{2} \theta_k} \left\| u \right\|_{L^{\frac{p}{1-\theta_k}}(\Omega)}^{(1-\rho_k)(1-\theta_k)} \|u\|_{L^p(\Omega)}^{\rho_k(1-\theta_k)}.
$$

For $k = 0, 1, \cdots, n - 1$ with $n \geq 2$ and $r_k \in \left( \frac{1}{2}, 1 \right)$, one can obtain by simple calculations that

$$
\rho_k(1-\theta_k) = \frac{p}{p-q} \times \frac{r_k - \frac{q}{p}}{\frac{k+1}{2} - \frac{k-1}{2} r_k} = \left( \frac{2(p-q)}{(k+3)(p-q)}, 1 \right).
$$

Thus, for any $\delta \in \left( \frac{2(p-2q)}{3(p-q)}, 1 \right)$, there exist $r_0, r_1, \cdots, r_{n-1} \in \left( \frac{1}{2}, 1 \right)$ such that $\rho_k(1-\theta_k) = \delta$ for any $k = 0, 1, \cdots, n - 1$. Then (3.17) yields that

$$
\|u\|_{L^p(\Omega)} \leq C \sum_{k=0}^{n-1} \left\| \nabla \right\|_{L^2(\Omega)}^{\frac{p}{2} \theta_k} \left\| u \right\|_{L^{\frac{p}{1-\theta_k}}(\Omega)}^{(1-\rho_k)(1-\theta_k)}.
$$

A direct calculation implies that $\frac{2\gamma_k}{p(1-\delta)} = \frac{2\gamma_k}{1+\gamma_k p}$ and $\frac{2\gamma_k}{1-\delta} = \frac{1}{1+\gamma_k p}$ for all $k = 0, \cdots, n - 1$, where $\gamma_k = \frac{k+1}{2} \left( \frac{1}{q} - \frac{1}{p} \right)$, and the proof is completed. \qed

**Lemma 3.6 ([10] Lemma 2.2).** For any $2 \leq p < \infty$, it holds that

$$
\| \partial_i u \|_{L^p(\Omega)} \leq C \left\| \partial_i \left( |u|^\frac{p}{2} \right) \right\|_{L^2(\Omega)}^{\frac{2}{p}} \| u \|_{L^p(\Omega)}^{\frac{2}{p}},
$$

where the constant $C > 0$ is independent of $u$. (3.19)
4. A PRIORI ESTIMATES AND PROOF

Denote the perturbation by \( \phi(x, t) := u(x, t) - \hat{u}(x, t) \), which is periodic in the \( x_i \) direction for \( i = 2, \ldots, n \), and satisfies

\[
\partial_t \phi + \sum_{i=1}^n \partial_i \left[ f_i(\hat{u} + \phi) - f_i(\hat{u}) \right] = \Delta \phi - h, \tag{4.1}
\]

\[
\phi(x, 0) = 0. \tag{4.2}
\]

**Remark 4.1.** In fact, given the initial data \( u_0(x) = \hat{u}^R(x_1, 0) + v_0(x_1) + w_0(x) \) stated in Remark 1.2, the initial condition (4.2) turns to \( \phi(x, 0) = v_0(x_1) \in L^1(\mathbb{R}) \cap H^1(\mathbb{R}) \) instead, which makes no difference in the following proof.

We shall prove the global existence and large time behavior of solution \( \phi(x, t) \) to the Cauchy problem (4.1) and (4.2). The global existence can be established by obtaining the a priori estimates (4.3) and (4.4) below, since the local existence of the solution \( \phi \) to (4.1) with the initial data in \( L^1(\Omega) \cap H^1(\Omega) \) is standard.

**Theorem 4.2** (A priori estimates). Assume that \( \phi(x, t) \) is the unique smooth solution to (4.1), (4.2) for any \( t \in [0, T] \), then it holds that

\[
\|\phi(t)\|_{L^p(\Omega)} \leq C_p(1 + t)^{-\frac{1}{2} + \frac{1}{p}}, \quad \forall p \in [1, +\infty), \tag{4.3}
\]

\[
\|\nabla \phi(t)\|_{L^p(\Omega)} \leq C_p(1 + t)^{-1 + \frac{1}{p}}, \quad \forall p \in [2, +\infty), \tag{4.4}
\]

where the constants \( C > 0 \) and \( C_p > 0 \) are independent of \( t \).

**Proof.** The uniform bound of \( \|\phi(t)\|_{L^p(\Omega)} \) follows from the maximum principle easily. And following the \( L^p \) energy method as in [10], we first prove (4.3).

**Step 1.** We first show the \( L^1 \) estimates. Given \( \delta > 0 \), let \( S_\delta(\eta) \) be a \( C^2 \) convex approximation to the function \( |\eta| \), e.g.

\[
S_\delta(\eta) = \begin{cases} 
-\eta, & \eta \leq -\delta; \\
-\frac{\eta^4}{8\delta^4} + \frac{3\eta^2}{4\delta} + \frac{3\delta}{8}, & -\delta < \eta \leq \delta; \\
\eta, & \eta > \delta.
\end{cases}
\]

Multiplying \( S_\delta'(\phi) \) on both sides of (4.1) yields that

\[
\partial_t S_\delta(\phi) + S_\delta''(\phi) |\nabla \phi|^2 + \int_0^\infty S_\delta''(\eta) (f_1'(\hat{u} + \eta) - f_1'(\hat{u})) \, d\eta \, \partial_1 \hat{u}^R \\
= \sum_{i=1}^n \partial_i \left[ \cdots \right] - \int_0^\infty S_\delta''(\eta) (f_1'(\hat{u} + \eta) - f_1'(\hat{u})) \, d\eta \, \partial_1 (\hat{u} - \hat{u}^R) \tag{4.5}
\]

\[
- \sum_{i=2}^n \int_0^\phi S_\delta''(\eta) (f_1'(\hat{u} + \eta) - f_1'(\hat{u})) \, d\eta \, \partial_1 \hat{u} - S_\delta(\phi) h,
\]
where
\[
\{ \ldots \} = S''_\delta(\phi) \partial_t \phi - S''_\delta(\phi) \left( f_i(\bar{u} + \phi) - f_i(\bar{u}) \right) + \int_0^\phi S''_\delta(\eta) \left( f_i(\bar{u} + \phi) - f_i(\bar{u}) \right) d\eta.
\]
Since \( S''_\delta \geq 0 \), \( f''_i > 0 \), \( \partial_t \bar{u}^R > 0 \) and \( |\phi| \leq S_\delta(\phi) \), integrating (4.5) over \( \Omega \), together with Lemmas 2.2 and 2.3, gives that
\[
\frac{d}{dt} \int_\Omega S_\delta(\phi) dx \leq C e^{-\alpha t} \int_\Omega \sum_{i=1}^n \int_0^\phi S''_\delta(\eta) \left( f'_i(\bar{u} + \eta) - f'_i(\bar{u}) \right) d\eta \right| dx + C \| h \|_{L^1(\Omega)}
\]
\[
\leq C e^{-\alpha t} \int_\Omega \left| \int_0^\phi S''_\delta(\eta) \right| d\eta \right| dx + C e^{-\alpha t}
\]
\[
\leq C e^{-\alpha t} \| \phi \|_{L^1(\Omega)} + C e^{-\alpha t}
\]
\[
\leq C e^{-\alpha t} \int_\Omega S_\delta(\phi) dx + C e^{-\alpha t},
\]
where \( C > 0 \) is independent of \( \delta \). Then by the Gronwall inequality and letting \( \delta \to 0^+ \), one has that
\[
\| \phi(t) \|_{L^1(\Omega)} \leq C.
\] (4.6)

For \( p \in [2, +\infty) \), multiplying (4.1) by \( |\phi|^{p-2} \phi \) yields that
\[
\frac{1}{p} \partial_t |\phi|^p + \frac{4(p-1)}{p^2} \left| \nabla |\phi|^\frac{p}{2} \right|^2 + (p-1) \int_0^\phi (f'_i(\bar{u} + \eta) - f'_i(\bar{u})) |\eta|^{p-2} d\eta \partial_t \bar{u}^R
\]
\[
= \sum_{i=1}^n \partial_i \{ \ldots \} - (p-1) \int_0^\phi (f'_i(\bar{u} + \eta) - f'_i(\bar{u})) |\eta|^{p-2} d\eta \partial_t \bar{u}^R
\]
\[
- (p-1) \sum_{i=2}^n \int_0^\phi (f'_i(\bar{u} + \eta) - f'_i(\bar{u})) |\eta|^{p-2} d\eta \partial_i \bar{u}^R - |\phi|^{p-2} \phi h,
\] (4.7)

where
\[
\{ \ldots \} = |\phi|^{p-2} \phi \partial_t \phi - (f_i(\bar{u} + \phi) - f_i(\bar{u})) |\phi|^{p-2} \phi
\]
\[
+ (p-1) \int_0^\phi (f_i(\bar{u} + \eta) - f_i(\bar{u})) |\eta|^{p-2} d\eta.
\]

Since
\[
(p-1) \int_0^\phi (f'_i(\bar{u} + \eta) - f'_i(\bar{u})) |\eta|^{p-2} d\eta \partial_t \bar{u}^R \geq C \partial_t \bar{u}^R |\phi|^p \geq 0,
\]
then integrating (4.7) over \( \Omega \), together with Lemma 2.3, gives that
\[
\frac{d}{dt} \| \phi \|_{L^p(\Omega)} + \left\| \nabla |\phi|^\frac{p}{2} \right\|_{L^2(\Omega)}^2 + \left\| \left( \partial_t \bar{u}^R \right)^\frac{1}{p} \phi \right\|_{L^p(\Omega)}
\]
\[
\leq C e^{-\alpha t} \| \phi \|_{L^p(\Omega)} + C \left( \| \phi \|_{L^p(\Omega)} \right)^{1-\frac{1}{p}} \| h \|_{L^p(\Omega)}
\]
\[
\leq C e^{-\alpha t} \| \phi \|_{L^p(\Omega)} + C e^{-\alpha t}. \] (4.8)

Multiplying (4.8) by \( (1+t)^\beta \), where \( \beta > \frac{n}{2} (p-1) \) is a constant, then integrating the resulting equation over \( (0, T) \) yields that
\[
(1+T)^\beta \| \phi(T) \|_{L^p(\Omega)} + \int_0^T (1+t)^\beta \left\| \nabla |\phi|^\frac{p}{2} \right\|_{L^2(\Omega)}^2 dt
\]
\[ \leq C \int_0^T e^{-\alpha t} (1 + t)^\beta \| \phi \|^p_{L^p(\Omega)} \, dt + C \int_0^T (1 + t)^{\beta - 1} \| \phi \|^p_{L^p(\Omega)} \, dt \]

\[ \leq C + C \int_0^T (1 + t)^{\beta - 1} \| \phi \|^p_{L^p(\Omega)} \, dt. \quad (4.9) \]

It follows from Corollary 3.5 and (4.6) that

\[ C \int_0^T (1 + t)^{\beta - 1} \| \phi \|^p_{L^p(\Omega)} \, dt \]

\[ \leq C \sum_{k=0}^{n-1} \int_0^T (1 + t)^{\beta - 1} \left\| \nabla |\phi| \right\|^2_{L^2(\Omega)} \| \phi \|^p_{L^p(\Omega)} \, dt \]

\[ \leq \frac{1}{2} \int_0^T (1 + t)^\beta \left\| \nabla |\phi| \right\|^2_{L^2(\Omega)} \, dt + C (1 + T)^{\beta - \gamma_k p}, \quad (4.10) \]

where \( \gamma_k = \frac{k+1}{2} \left( 1 - \frac{1}{p} \right) \). This, together with (4.9), yields that

\[ (1 + T)^\beta \| \phi(T) \|^p_{L^p(\Omega)} + \int_0^T (1 + t)^\beta \left\| \nabla |\phi| \right\|^2_{L^2(\Omega)} \, dt \leq C + C (1 + T)^{\beta - \gamma_k p}. \]

Thus (4.3) holds for \( p \in [2, +\infty) \). The case for \( p \in (1, 2) \) follows from the interpolation.

**Step 2.** We now prove (4.4). Let \( \psi_i := \partial_i \phi \). Taking the derivative on (4.1) with respect to \( x_i \) yields that

\[ \partial_i \psi_i + \sum_{j=1}^n \partial_j \left( f_j'(\bar{\psi}) \psi_i \right) + \sum_{j=1}^n \partial_j \left[ (f_j'(\bar{\psi} + \phi) - f_j'(\bar{\psi})) (\partial_i \bar{\psi} + \psi_i) \right] = \Delta \psi_i - \partial_i \bar{h}. \]

Multiplying the result by \( |\psi_i|^{p-2} \psi_i \), we arrive at

\[ \frac{1}{p} \partial_t |\psi_i|^p + \frac{4(p-1)}{p^2} \left\| \nabla |\psi_i| \right\|^2 + \frac{p-1}{p} f_i''(\bar{\psi}) \partial_i \bar{h} |\psi_i|^p \]

\[ = \sum_{j=1}^n \partial_j \{ \cdots \} - \frac{p-1}{p} f_i''(\bar{\psi}) \partial_i (\bar{u} - \bar{u}^R) |\psi_i|^p - \frac{p-1}{p} \sum_{j=2}^n f_j''(\bar{\psi}) \partial_j \bar{h} |\psi_i|^p \]

\[ + (p - 1) \sum_{j=1}^n \left( f_j'(\bar{\psi} + \phi) - f_j'(\bar{\psi}) \right) (\partial_i \bar{\psi} + \psi_i) |\psi_i|^{p-2} \partial_j \psi_i \]

\[ - \partial_i \left( |\psi_i|^{p-2} \psi_i \partial_i \bar{h} \right) + (p - 1) |\psi_i|^{p-2} \partial_i \psi_i \bar{h}, \quad (4.11) \]

where

\[ \{ \cdots \} = |\psi_i|^{p-2} \psi_i \partial_j \psi_i - \frac{1}{p} f_j''(\bar{\psi}) |\psi_i|^p - \left( f_j'(\bar{\psi} + \phi) - f_j'(\bar{\psi}) \right) (\partial_i \bar{\psi} + \psi_i) |\psi_i|^{p-2} \psi_i. \]
Then integrating (4.11) over $\Omega$ and using Lemma 2.3 yield that

$$
\frac{d}{dt} \|\psi_i\|_{L^p(\Omega)}^p + \left\| \nabla |\psi_i|^{\frac{p}{2}} \right\|_{L^2(\Omega)}^2 + \left\| \left( \partial_t u R \right)^{\frac{1}{2}} \psi_i \right\|_{L^p(\Omega)}^p \\
\leq C e^{-at} \|\psi_i\|_{L^p(\Omega)}^p + C \int_{\Omega} |\partial_t u| |\phi| |\psi_i|^{p-2} |\nabla \psi_i| \, dx \\
+ C \int_{\Omega} |\phi| |\psi_i|^{p-1} |\nabla \psi_i| \, dx + C \int_{\Omega} |\psi_i|^{p-2} |\nabla \psi_i| |h| \, dx. 
$$

(4.12)

First, it follows from Lemma 2.3 that

$$
I_3 = C \int_{\Omega} \left\| \nabla |\psi_i|^{\frac{p}{2}} \right\|_{L^2(\Omega)}^2 \left( e^{-at} |\psi_i|^p \right)^{\frac{p-2}{p}} e^{\frac{p-2}{2p}at} |h| \, dx \\
\leq \frac{1}{8} \left\| \nabla |\psi_i|^{\frac{p}{2}} \right\|_{L^2(\Omega)}^2 + C e^{-at} \|\psi_i\|_{L^p(\Omega)}^p + C e^{-\frac{1}{2}(\frac{p}{2}+1)t},
$$

(4.13)

where the Hölder inequality for $\frac{1}{2} + \frac{p-2}{2p} + \frac{1}{p} = 1$ is used. Following [10], one can claim that

$$
\int_{\Omega} |\phi| |\psi_i|^{\frac{p}{2}} \left\| \nabla |\psi_i|^{\frac{p}{2}} \right\|_{L^2(\Omega)}^2 \, dx \leq C \sum_{k=0}^{n-1} \|\phi\|_{L^{2(k+1)}(\Omega)} \|\psi_i\|_{L^p(\Omega)}^\frac{p}{2} \left\| \nabla |\psi_i|^{\frac{p}{2}} \right\|_{L^2(\Omega)}^\frac{3}{2}. 
$$

(4.14)

In fact, we first decompose $v := |\psi_i|^{\frac{p}{2}} = \sum_{k=0}^{n-1} v^{(k)}$ as in Theorem 1.4, then

$$
\int_{\Omega} |\phi| |\psi_i|^{\frac{p}{2}} \left\| \nabla |\psi_i|^{\frac{p}{2}} \right\|_{L^2(\Omega)}^2 \, dx \leq \sum_{k=0}^{n-1} \int_{\Omega} |\phi| |v^{(k)}| \left\| \nabla v \right\| \, dx \\
\leq \|\phi\|_{L^2(\Omega)} \|v^{(0)}\|_{L^{\frac{2p}{p-2}}(\Omega)} \left\| \nabla v \right\|_{L^2(\Omega)} + \|\phi\|_{L^4(\Omega)} \|v^{(1)}\|_{L^4(\Omega)} \left\| \nabla v \right\|_{L^2(\Omega)} \\
+ \sum_{k=2}^{n-1} \|\phi\|_{L^{2(k+1)}(\Omega)} \left\| v^{(k)} \right\|_{L^4(\Omega)} \left\| \nabla v \right\|_{L^2(\Omega)}^{\frac{4(k+1)}{4(k+1)-p}} \left\| \nabla v \right\|_{L^2(\Omega)}^{\frac{3}{2}} \\
\leq \sum_{k=0}^{n-1} \|\phi\|_{L^{2(k+1)}(\Omega)} \|v\|_{L^2(\Omega)}^{\frac{1}{2}} \left\| \nabla v \right\|_{L^2(\Omega)}^{\frac{3}{2}},
$$

which yields (4.14). Then combining (4.3) and (4.14), the term $I_2$ in (4.12) satisfies that

$$
I_2 \leq C(1 + t)^{-\frac{1}{4}} \|\psi_i\|_{L^p(\Omega)} \left\| \nabla |\psi_i|^{\frac{p}{2}} \right\|_{L^2(\Omega)}^{\frac{3}{2}} \\
\leq C(1 + t)^{-1} \|\psi_i\|_{L^p(\Omega)}^p + \frac{1}{8} \left\| \nabla |\psi_i|^{\frac{p}{2}} \right\|_{L^2(\Omega)}^2.
$$

(4.15)
For $I_1$, if $i \neq 1$, it follows from the Hölder inequality and (4.3) that

$$I_1 \leq Ce^{-at} \int_\Omega |\phi| |\psi_i|^{\frac{p-2}{2}} |\nabla \psi_i| \, dx$$

$$\leq Ce^{-at} \|\phi\|_{L^p(\Omega)} \|\psi_i\|_{L^p(\Omega)} \left\|\nabla |\psi_i|^\frac{p}{2}\right\|_{L^2(\Omega)}$$

$$\leq Ce^{-at} + Ce^{-at} \|\psi_i\|_{L^p(\Omega)}^p + \frac{1}{8} \left\|\nabla |\psi_i|^\frac{p}{2}\right\|_{L^2(\Omega)}^2. \hspace{1cm} (4.16)$$

If $i = 1$, then

$$I_1 \leq C \int_\Omega \partial_1 \tilde{u}^R |\phi| |\psi_1|^{p-2} |\nabla \psi_1| \, dx + Ce^{-at} \int_\Omega |\phi| |\psi_1|^{p-2} |\nabla \psi_1| \, dx,$$  \hspace{1cm} (4.17)

where the second term on the RHS of (4.17) can be estimated in the same way as in (4.16). For the first term, one can use Lemma 3.6 to obtain that

$$\int_\Omega \partial_1 \tilde{u}^R |\phi| |\psi_1|^{p-2} |\nabla \psi_1| \, dx \leq C \left\|\partial_1 \tilde{u}^R\right\|_{L^\infty(\Omega)} \left\|\phi\right\|_{L^p(\Omega)} \left\|\nabla |\psi_1|^\frac{p}{2}\right\|_{L^2(\Omega)}^2$$

$$\leq C \left\|\partial_1 \tilde{u}^R\right\|_{L^\infty(\Omega)} |\phi|_{L^p(\Omega)}^p + \frac{1}{8} \left\|\nabla |\psi_1|^\frac{p}{2}\right\|_{L^2(\Omega)}^2. \hspace{1cm} (4.18)$$

Thus, collecting Equations (4.13) and (4.15) to (4.18) and applying (2.2), one has that

$$\frac{d}{dt} \|\psi_i\|_{L^p(\Omega)}^p + \left\|\nabla |\psi_i|^\frac{p}{2}\right\|_{L^2(\Omega)}^2 \leq Ce^{-at} + C(1+t)^{-1} \|\psi_i\|_{L^p(\Omega)}^p + C(1+t)^{-\frac{p+2}{2}} \|\phi\|_{L^p(\Omega)}^p. \hspace{1cm} (4.19)$$

Multiplying (4.19) by $(1+t)^\beta$ with $\beta > p + 1$ and then integrating the result over $[0,T]$, one has that

$$(1+t)^\beta \|\psi_i(t)\|_{L^p(\Omega)}^p + \int_0^T (1+t)^\beta \left\|\nabla |\psi_i|^\frac{p}{2}\right\|_{L^2(\Omega)}^2 \, dt$$

$$\leq C + C \int_0^T (1+t)^{\beta-1} \|\psi_i(t)\|_{L^p(\Omega)}^p \, dt + C \int_0^T (1+t)^{\beta-\frac{p+2}{2}} \|\phi\|_{L^p(\Omega)}^p \, dt. \hspace{1cm} (4.20)$$

It follows from Lemma 3.6 that

$$\int_0^T (1+t)^{\beta-1} \|\psi_i(t)\|_{L^p(\Omega)}^p \, dt$$

$$\leq C \int_0^T (1+t)^{\beta-1} \left\|\nabla |\psi_i|^\frac{p}{2}\right\|_{L^2(\Omega)}^2 \|\phi\|_{L^p(\Omega)}^p \, dt$$

$$\leq \frac{1}{2} \int_0^T (1+t)^\beta \left\|\nabla |\psi_i|^\frac{p}{2}\right\|_{L^2(\Omega)}^2 \, dt + C \int_0^T (1+t)^{\beta-\frac{p+2}{2}} \|\phi\|_{L^p(\Omega)}^p \, dt.$$  

From (4.3), one can get that

$$\int_0^T (1+t)^{\beta-\frac{p+2}{2}} \|\phi(t)\|_{L^p(\Omega)}^p \, dt \leq C \int_0^T (1+t)^{\beta-\frac{p+2}{2}} (1+t)^{-\frac{p-1}{2}} \, dt$$

$$\leq C(1+T)^{\beta-p+\frac{1}{2}}.$$  

The proof of Theorem 4.2 is finished. \hfill \Box
Proof of Theorem 1.1. It follows from Theorems 1.4 and 4.2 that
\[
\|\phi\|_{L^\infty(\Omega)} \leq C \sum_{k=0}^{n-1} \|\nabla \phi\|_{L^p_k(\Omega)}^{\theta_k} \|\phi\|_{L^q_k(\Omega)}^{1-\theta_k} \\
\leq C \sum_{k=0}^{n-1} (1+t)^{-\left[(1-\frac{1}{q_k})(\theta_k + \frac{1}{2} - \frac{1}{2q_k}) (1-\theta_k)\right]},
\]
where \(0 = \left(\frac{1}{p_k} - \frac{1}{k+1}\right) \theta_k + \frac{1}{q_k} (1-\theta_k), \max\{k+1, 2\} < p_k < +\infty\) and \(1 \leq q_k < +\infty\) for \(k = 0, 1, \cdots, n-1\). A direct calculation yields that
\[
\left(1 - \frac{1}{2p_k}\right) \theta_k + \left(\frac{1}{2} - \frac{1}{2q_k}\right) (1-\theta_k) = \frac{1}{2} + \frac{k\theta_k}{2(k+1)} \geq \frac{1}{2},
\]
which implies that
\[
\|\phi\|_{L^\infty(\mathbb{R}^n)} = \|\phi\|_{L^\infty(\Omega)} \leq C (1+t)^{-\frac{1}{2}}.
\]
Thus,
\[
\|u - \tilde{u}^R\|_{L^\infty(\mathbb{R}^n)} \leq \|\phi\|_{L^\infty(\mathbb{R}^n)} + \|\tilde{u} - \tilde{u}^R\|_{L^\infty(\mathbb{R}^n)} \leq C (1+t)^{-\frac{1}{2}}.
\]
\(\square\)

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