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**Lie symmetries and singularity analysis for generalized shallow-water equations**

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**Abstract:** We perform a complete study by using the theory of invariant point transformations and the singularity analysis for the generalized Camassa-Holm (CH) equation and the generalized Benjamin-Bono-Mahoney (BBM) equation. From the Lie theory we find that the two equations are invariant under the same three-dimensional Lie algebra which is the same Lie algebra admitted by the CH equation. We determine the one-dimensional optimal system for the admitted Lie symmetries and we perform a complete classification of the similarity solutions for the two equations of our study. The reduced equations are studied by using the point symmetries or the singularity analysis. Finally, the singularity analysis is directly applied on the partial differential equations from where we infer that the generalized equations of our study pass the singularity test and are integrable.

**Keywords:** Benjamin-Bono-Mahoney; Camassa-Holm; invariants; Lie symmetries; shallow water.

1 Introduction

The Lie symmetry analysis plays a significant role in the study of nonlinear differential equations. The existence of Lie symmetry for a given differential equation is equivalent with the existence of one-parameter point transformation which leaves the differential equation invariant. The later property can be used to reduce the number of independent variables on the case of partial differential equations (PDE), or reduce the order of an ordinary differential equation (ODE) [1], that is achieved thought the Lie invariants. In addition, Lie symmetries can be used for the determination of conservation laws.

One the most well-known applications of the latter are the two theorems of E. Noether [1]. However, there are also alternative methods to determine the conservation laws by using the Lie point symmetries without imposing a Lagrange function, some of these alternative approaches are described in [1–5] and references therein.

The are many applications of the Lie symmetries on the analysis of differential equations, for the determination of exact solutions, to determine conservation laws, study the integrability of dynamical systems or classify algebraic equivalent systems [6–13]. Integrability is a very important property of dynamical systems, hence it worth to investigate if a given dynamical system is integrable [14–21].

An alternative approach for the study of the integrability of nonlinear differential equations is the singularity analysis. In contrary with the symmetry analysis, singularity analysis is based on the existence of a pole for the differential equation. The first major result of the singularity analysis is the determination of the third integrable case of Euler’s equations for a spinning by Kowalevskaya [22]. Since then, the have been many contributions of the singularity analysis, mainly by the French school led by Painlevé [23–25] and many others [26–29]. Nowadays, the application of singularity analysis is summarized in the ARS algorithm [30–32] which has made the singularity analysis a routine tool for the practising applied mathematicians.

Singularity analysis and symmetry analysis have been applied in a wide range of differential equations arising from all areas of applied mathematics, for instance see [33–44] and references therein. The two methods are supplementary, on the study of integrability of differential equations. Usually, the symmetry method is applied to reduce the given differential equation into an algebraic equation, or into another well-known integrable differential equation. On the other hand, for differential equation which possess the Painlevé property, i.e. it pass the singularity test, its solution is written in terms of Laurent expansions, a recent comparison of the two methods is presented in [45].

In this work, we study the integrability of generalized Camassa-Holm (CH) and Benjamin-Bono-Mahoney (BBM)
equations [46, 47] by using the Lie point symmetries and the singularity analysis. These two equations describe shallow-water phenomena.

The CH equation is a well-known integrable equation. It was originally discovered by Fuchssteiner et al. in [48], however become popular a decade later by the study of CH where they proved the existence of peaked solutions, also known as peakons. On the other hand, BBM equation also known as regularized long-wave equation discovered in [49] and it is an extension of the KdV equation. The two equations are related, in the sense they have a common operator and partial common Hamiltonian structure. The modern treatment of the singularity analysis is included in Sections 3 and 4 where we study the existence of similarity solutions for the generalized CH and BBM equations, as also we prove the integrability of these equations [46, 47] by using the Lie point symmetries and the singularity analysis. These two equations describe shallow-water phenomena.

In Section 2 we present the basic elements on the mathematical tools of our consideration, that is, the Lie point symmetries and the singularity analysis. Our main analysis is included in Sections 3 and 4 where we study the existence of similarity solutions for the generalized CH and BBM equations, as also we prove the integrability of these two equations by using the singularity analysis. Finally, we discuss our results and draw our conclusions in Section 5.

2 Preliminaries

In this section we briefly discuss the application of Lie’s theory on differential equations as also the main steps of the singularity analysis.

2.1 Lie symmetries

Consider the vector field

\[ X = \xi(x^k, u) \partial_k + \eta(x^k, u) \partial_u, \]

(1)

to be the generator of the local infinitesimal one-parameter point transformation,

\[ x^k = x^k + \epsilon \xi^k(x^k, u), \]

(2)

\[ \eta = \eta + \epsilon \eta(x^k, u). \]

(3)

Then X is called a Lie symmetry for the differential equation, H(y', u, u_0, u_1, ..., u_{k-1}, u_k), if there exists a function \( \lambda \) such that the following condition to hold

\[ X^{[n]} H = \lambda H \]

(4)

where \( X^{[n]} \) is called the second prolongation/extension in the jet-space and is defined as

\[ X^{[n]} = X + (D \eta - u_1 D \xi^k) \partial_{u_1} + (D \eta^{[1]} - u_2 D \xi^k) \partial_{u_2} + \ldots + (D \eta^{[k]} - u_{k+1} D \xi^{k}) \partial_{u_{k+1}} \]

(5)

The novelty of Lie symmetries is that they can be used to determine similarity transformations, i.e. differential transformations where the number of independent variables is reduced [1]. The similarity transformation is calculated with the use of the associated Lagrange’s system,

\[ \frac{dx^i}{\xi^i} = \frac{du}{u} = \frac{du_i}{u_{[i]}} = \ldots = \frac{du_{[k-1]}}{u_{[k-1]}}. \]

(6)

The similarity transformation in the case of PDEs is used to reduce the number of independent variables. The solutions derived by the application of Lie invariants are called similarity solutions.

2.2 Singularity analysis

The modern treatment of the singularity analysis is summarized in the ARS algorithm, established by Ablovitz, Ramani and Segur in [30–32]. There are three basic steps which are summarized as follows: (a) determine the leading-order term which describes the behaviour of the solution near the singularity, (b) find the position of the resonances which shows the existence and the position of the integration constants and (c) write a Laurent expansion with leading-order term determined in step (a) and perform the consistency test. More details on the ARS algorithm as also on the conditions which should hold at every step we refer the reader in n the review of Ramani et al. [50], where illustrated applications are presented.

It is important to mention that when a differential equation passes the conditions and requirement of the ARS algorithm we can infer that the given differential equation is algebraically integrable.

3 Generalized Camassa-Holm equation

We work with the generalized CH equation defined in [46, 47]

\[ u_t - u_{txtt} + \frac{(k + 2)(k + 1)}{2} u^k u_x = \left( \frac{k}{2} u^{k-1} u_x^2 + u^k u_x \right)_x, \]

(7)

where \( k \geq 1 \) is a positive integer number, while when \( k = 1 \) CH equation is recovered. The Lie symmetry analysis for the CH equation presented before in [35]. It was found that the CH is invariant under a three dimensional Lie algebra.
For the generalized CH Eq. (6) the application of Lie’s theory provides us that the admitted Lie point symmetries are three, and more specifically they are

\[ X_1 = \partial_t, \quad X_2 = \partial_u, \quad \text{and} \quad X_3 = t \partial_t - ku \partial_u. \]

The commutators and the adjoint representation of the Lie point symmetries are presented in Tables 1 and 2 respectively.

The results presented in Tables 1 and 2 can be used to classify the admitted Lie algebra as also to determine the one-dimensional optimal systems [51]. A necessary analysis to perform a complete classification of the similarity solutions.

As far as the admitted Lie algebra is concerned, from Table 1 it was found to be the \( \{2A_1 \phi_x A_1\} \) in the Morozov-Mubarakzyanov Classification Scheme [52–55].

In order to find the one-dimensional optimal systems we consider the generic symmetry vector

\[ X = a_1 X_1 + a_2 X_2 = a_3 X_3, \]

from where we find the equivalent symmetry by considering the adjoint representation. We remark that the adjoint action admits two invariant functions \( \phi_1(a_1) = a_3 \) and \( \phi_2(a_1) = a_3 \) which are necessary to simplify the calculations on the derivation of the one-dimensional systems. More specifically there are four possible cases, \( \{\phi_1, \phi_2 \neq 0\}, \{\phi_1 \neq 0, \phi_2 = 0\}, \{\phi_1 = 0, \phi_2 \neq 0\} \) and \( \{\phi_1 = 0, \phi_2 = 0\} \).

Consequently, with the use of the invariant functions \( \phi_1 \) and \( \phi_2 \) and Table 2 we find that the possible one-dimensional optimal systems are

\[ X_1, X_2, X_3, cX_1 + X_2 \text{ and } X_2 + aC_3. \]

### Table 1: Commutators of the admitted Lie point symmetries by the differential Eq. (6).

| \( \left[ \right] \) | \( X_1 \) | \( X_2 \) | \( X_3 \) |
|-----------------|---------|---------|---------|
| \( X_1 \)   | 0       | 0       | \( kX_1 \) |
| \( X_2 \)   | 0       | 0       | 0       |
| \( X_3 \)   | \( -kX_1 \) | 0       | 0       |

### Table 2: Adjoint representation for the Lie point symmetries of the differential Eq. (6).

| \( \text{Ad} (\exp(cX)) \) \( X_j \) | \( X_1 \) | \( X_2 \) | \( X_3 \) |
|-------------------------------|---------|---------|---------|
| \( X_1 \) | \( X_1 \) | \( X_2 \) | \( X_3 - kX_1 \) |
| \( X_2 \) | \( X_1 \) | \( X_2 \) | \( X_1 \) |
| \( X_3 \) | \( e^{kX_1} \) | \( X_2 \) | \( X_3 \) |

We proceed with the application of the latter one-dimensional system in order to reduce the PDE (6) into an ODE.

### 3.1 Analytic solutions

In this section we proceed with the application of the Lie point symmetries to the nonlinear generalized CH equation. In order to solve the reduced equation we apply the Lie point symmetries and when it is no possible to proceed the reduction process, we consider the singularity analysis by applying the ARS algorithm.

#### 3.1.1 Reduction with \( X_1 \): Static solution

The application of the Lie point symmetry vector \( X_1 \) indicates that the solution is static, i.e. \( u = U(x) \) where now function \( U(x) \) satisfies the third-order ODE

\[ \frac{(k + 2)(k + 1)}{2} U'' U_x - \left( \frac{k}{2} U^{k-1} U_x^2 + U^{k} U_{xx} \right) = 0. \]  

The latter equation can be easily integrated, and be written in the equivalent form

\[ U_{xx} + \frac{k}{2U} U_x^2 = \frac{U_0}{U^k} + \frac{(k + 2)}{2} U = 0, \]

where \( U_0 \) is a constant of integration. Equation (11) admits the following conservation law

\[ \frac{1}{2} U^k \left( U_x \right)^2 - U_0 U - \frac{U^{k+2}}{2} = U_1, \]

in which \( U_1 \) is a second constant of integration. Equation (12) can be integrated by quadratures.

Conservation law (12) is nothing else than the Hamiltonian function of the second-order ODE (11). Before we proceed with another reduction let us now apply the singularity analysis to determine the analytic solution of Eq. (10).

### Singularity analysis

We substitute \( U(x) = U_0 \chi^p, \chi = x - x_0 \), in (10) and we find the polynomial expression

\[ (k + 1)(k + 2) \chi^{(k+1)-1} + (p(k + 1) - 2)(p(k + 2) - 2) \chi^{(k+1)-3} = 0, \]

Hence we can infer that the only possible leading-order behaviour to the terms with \( \chi^{(k+1)-3} \), where from the requirement
\[ (p (k + 1) - 2) (p (k + 2) - 2) = 0, \] (14)

provides
\[ p_1 = \frac{2}{k + 2} \quad \text{or} \quad p_2 = \frac{2}{k + 1}, \] (15)

while constant \( U_0 \) is undetermined.

Consider now the leading order term \( p_1 \). In order to find the resonances we replace \( U(x) = U_0 \chi^p + \mu \chi^{p,rs} \) in (14) and we linearize around the \( \mu = 0 \). From the linear terms of \( \mu \), the coefficient of the leading order terms \( \chi^{\frac{s+1}{k+1}} \) are \( s (s + 1) (s(k + 2) - 2) \), where the requirement the latter expression to be zero provides the three resonances
\[ s_1 = -1, \quad s_2 = 0, \quad \text{and} \quad s_3 = \frac{2}{k + 2}, \] (16)

We remark that because \( k \) is always a positive integer number, \( p_1 \) and \( s_1 \) are always rational numbers. Resonance \( s_1 \) indicates that the singularity is movable the position of the singularity is one of the three integration constants. Resonances \( s_2 \) shows that the coefficient constant of the leading-order term should be arbitrary since it is also one of the integration constants of the problem. The third constant it is given at the position of the resonance \( s_3 \) and depends on the value \( k \). Moreover, because all the resonances are positive the solution will be given by a Right Painlevé Series. In order to complete the ARS algorithm we should perform the consistency test. For that we select a special value of \( k \).

We select \( k = 2 \), and we consider the right Laurent expansion
\[ U(x) = U_0 \chi + U_1 \chi^2 + \sum_{k = 3}^{\infty} U_k \chi^{k+1}, \] (17)

we find that \( U_0 \) is the third integration constant while the first coefficient constants are
\[ U_2(U_0, U_1) = \frac{7}{8} \frac{(U_1)^2}{U_0}, \quad U_3 = \frac{5}{4} \frac{(U_1)^3}{(U_0)^2} U_0. \]

Hence, the consistency test is satisfied and expression (16) is one solution of the third-order ODE (9).

We work similar and for the second leading-order behaviour \( p_2 = \frac{2}{k+2} \), the resonances are derived to be
\[ s_1 = -1, \quad s_2 = 0, \quad s_3 = \frac{2}{k + 1}, \] (19)

from where we infer that the solution is given by a Mixed Painlevé Series. However, we perform the consistency test for various values of the integer number \( k \), and we conclude that for this leading-order behaviour the differential equation does not pass the singularity test.

In order to understand better why only the leading-order behaviour \( p_1 \) passes the singularity test, let us perform the same analysis for the second-order ODE (10). By replacing in \( U(x) = U_0 \chi^p \) in (10) we find that the unique leading-order behaviour is that with \( p = \frac{2}{k+2} \) with arbitrary \( U_0 \). The two resonances now are calculated to be \( s_1 = -1 \) and \( s_2 = 0 \), from where we can infer that the solution is given by a Right Painlevé series and the two integration constants is the \( U_0 \) and the position of the singularity. In that case, since we know the two integration constants it is not necessary to perform the consistency test.

3.1.2 Reduction with \( X_2 \): Stationary solution

Reduction with the vector field \( X_2 \) provides the stationary solution \( u(t, x) = U(t) \), where \( U_0 = 0 \), that is \( u(t, x) = u_0 \). This is the trivial solution.

3.1.3 Reduction with \( X_3 \): Scaling solution I

From the symmetry vector \( X_3 \) we derive the Lie invariants
\[ u = U(x) t^\frac{p}{2}, \quad x \] (20)

hence, by replacing in (6) we end up with the following third-order ODE
\[ 2k U^k U_{xxx} - 2(1 - 2k^2 U^{k-1} U_x) U_{xx} + k^2 (1 - k) k U^{k-2} (U_x)^2 \]
\[ - (k + 2) (k + 1) U^k U_x + 2U = 0, \] (21)

The latter equation admits only one Lie point symmetry, the vector field \( X_2 \). The latter vector field can be used to reduce Eq. (20) into a second-order nonautonomous ODE, with no symmetries. Hence, we proceed with the application of the singularity analysis for Eq. (21).

We replace \( U(x) = U_0 \chi^p \) in (20) where we find the following expression
\[ - 2U_0 \chi^p + 2U_0 p (p - 1) \chi^{p-2} + U_0^{k+1} k (k + 2) (k + 1) \chi^{p(k+1)-1} \]
\[ - U_0^{k+1} p (p (k + 2) - 2) (p (k + 1) - 2) \chi^{p(k+1)-3} = 0. \] (22)

From the latter term we find that the only possible leading terms with \( k \) positive integer number are \( p - 2 = (pk + 1) - 3 \) from where we find that \( p = \frac{1}{k} \) while \( U_0 \) is given by the following expression
\[ U_0^k = 2(2 - k), \]  
from where we infer that there is a leading-order behaviour only for \( k = 2 \).

The resonances are calculated to be
\[ s_1 = -1, \quad s_2 = \frac{k - 1}{k}, \quad s_2 = \frac{k - 2}{2k}, \]  
from where we can infer that for \( k > 2 \), the solution is given by a Right Painlevé Series. We perform the consistency test by choosing \( k = 3 \). Hence the Laurent expansion is written as
\[ U(x) = U_0 x^\frac{1}{k} + U_1 x^\frac{2}{k} + \sum_{i=2}^{\infty} U_i x^{\frac{i+2}{k}}, \]  
where \( U_1 \) and \( U_0 \) are two integration constants of the solution while the rest coefficient constants are \( U_i = U_i(U_1, U_0) \).

### 3.1.4 Reduction with \( cX_1 + X_2 \): Travel-wave solution

The travel-wave similarity solution is determined by the application of the Lie invariants of the symmetry vector \( cX_1 + X_2 \) where \( c \) is the travel-wave speed. The invariant functions for that vector field are determined to be
\[ u(t, x) = U(\xi), \quad \xi = x - c^{-1}t, \]  
where \( U(\xi) \) satisfies the following third-order ODE
\[ 2(1 - cU^k)U_{\xi\xi\xi} - 4ckU^{k-1}U_xU_{\xi\xi} - (c(k-1)U^{k-1}U_{\xi\xi} + 2c(k+2)(k+1)U^k)U_\xi + 2U_\xi = 0. \]  
Equation (26) is autonomous and admit only one symmetry vector \( \partial_x \). It can easily integrated as follows
\[ 2(2 - cU^k)U_{\xi\xi} - c(2U^{k-2} - U^{k-1})(U_\xi)^2 + c(k+2)U^{k-3} - 2U_\xi + U_0 = 0, \]  
which can be solved by quadratures.

Let us now apply the singularity analysis to write the analytic solution of equation in (28) by using Laurent expansions. We apply the ARS algorithm and we find the leading order term \( U(\xi) = U_0(\xi - \xi_0)^p \) with \( p = \frac{2}{k-2} \) and \( U_0 \) arbitrary. The resonances are calculated to be \( s_1 = 0 \) and \( s_2 = 0 \), which means that the solution is given by a Right Painlevé Series with integration constants the position of the singularity \( \xi_0 \) and the coefficient constant of the leading order term \( U_0 \). The step of the Painlevé Series depends on the value of \( k \), for instance for \( k = 2 \), \( p = \frac{1}{2} \) and the step is \( \frac{1}{2} \), while for \( k = 3 \), \( p = \frac{3}{2} \) and the step is \( \frac{3}{2} \).

### 3.1.5 Reduction with \( X_2 + aX_3 \): Scaling solution II

We complete our analysis by determine the similarity solution given by the symmetry vector \( X_2 + aX_3 \). The that specific symmetry the Lie invariants are calculated
\[ u(t, x) = U(\xi)e^{i\alpha}, \quad \xi = x + \frac{1}{ak}\ln t. \]  
Therefore, by selecting \( \xi \) to be the new independent variable and \( U(\xi) \) the new dependent variable we end up with the third-order ODE
\[ 2(1 + \alpha k)U_{\xi\xi\xi} - 2a(1 - 2k^2U^{k-1}U_\xi)U_{\xi\xi} + (a(k-1)U^{k-2}(U_\xi)^2 - 2 - akU^k(k+2)(k+1))U_\xi + 2aU = 0. \]  
The latter equation is autonomous and admit only one point symmetry, the vector field \( \partial_\xi \), which can be used to reduce by one the order of the ODE. The resulting second-order ODE has no symmetries. Hence, the singularity analysis is applied to study the integrability of (29).

In order to perform the singularity analysis we do the change of variable \( V = U^\alpha \). Hence by replacing \( V(\xi) = V_0(\xi - \xi_0)^p \) in (29) we find the leading-order terms
\[ p_1 = -1 \quad \text{and} \quad p_2 = 2 \quad \text{for} \quad k > 1, \]  
while \( V_0 \) is arbitrary.

The resonances are calculated to be
\[ p_1 : s_1 = -1, \quad s_2 = 0 \quad \text{and} \quad s_3 = 1; \]  
\[ p_2 : s_1 = -1, \quad s_2 = 0 \quad \text{and} \quad s_3 = -2. \]

We apply the consistency test where we find that only only the leading-order term \( p_1 \) provides a solution, which is given by the following Right Painlevé Series
\[ V(\xi) = V_0(\xi - \xi_0)^{-1} + \sum_{i=1}^{\infty} V_i(\xi - \xi_0)^{-i}. \]

### 3.2 Singularity analysis

Until now we applied the singularity analysis to study the integrability of the ODEs which follow by the similarity reduction for the generalized CH equation. However, it is possible to apply the singularity analysis directly in the PDE. We follow the steps presented in [29].

Before we proceed with the application of the ARS algorithm we make the change of transformation \( u(t, x) = v(t, x)^{-1} \) in (6). For the new variable we search for a
singular behaviour of the form \( v(t, x) = v_0(t, x)\phi(t, x)p \), where \( v_0(t, x) \) is the coefficient function and \( \phi(t, x)p \) is the leading-order term which describe the singularity.

The first step of the ARS algorithm provides two values of \( p, p_1 = -1 \) and \( p_2 = -2 \), where \( v_0(t, x) \) is arbitrary. A necessary and sufficient condition in order these two leading-order terms to exists is \( \phi_3\phi_2 \neq 0 \). Otherwise other leading-order terms follow, however these possible cases studied before. The resonances for these two leading-order terms are those given in (31) and (32).

Consequently, the following two Painlevé Series should be studied for the consistency test

\[
v(t, x) = v_0(t, x)\phi(t, x)^{-1} + \sum_{i=1}^{\infty} v_i(t, x)\phi(t, x)^{-1+i}, \tag{34}
\]

\[
v(t, x) = v_0(t, x)\phi(t, x)^{-2} + \sum_{i=1}^{\infty} v_i(t, x)\phi(t, x)^{-2+i}. \tag{35}
\]

By replacing (32) we find that the second integration constant is \( v_1(t, x) \). On the other hand, the series (33) does not pass the consistency test. We conclude that the generalized CH equation passes the singularity test and it is an integrable equation.

We proceed our analysis with the BBM equation.

4 Generalized Benjamin-Bono-Mahoney equation

The generalized BBM equation is

\[
u_t - u_{xx} + \beta u^4 u_x = 0, \tag{36}
\]

where \( k \) is a positive integer number. Equation (34) can been seen as the lhs of (6) when \( \beta = \frac{(k+1)(k+3)}{2} \) and reduce to the BBM equation when \( k = 1 \). For the case of \( k = 1 \) the Lie symmetry analysis for the BBM equation presented recently in [56, 57].

We apply the Lie theory in order to determine the point transformations which leave Eq. (34) invariant. We found that Eq. (34) admits three point symmetries which are the vector fields \( X_1, X_2, X_3 \) presented in Section 3. Hence, the admitted Lie algebra is the \( 2A_1 \phi_3 \phi_1 \) and there are five one-dimensional optimal systems as presented in (8). We proceed with the application of the Lie point symmetries for the determination of similarity solutions.

4.1 Analytic solutions

For Eq. (34) the application of the Lie symmetries \( X_1 \) and \( X_2 \) provide the trivial solution \( u(t, x) = u_0 \) for both cases.

4.1.1 Reduction with \( X_3 \): Scaling solution I

The application of the Lie invariants which given by the symmetry vector \( X_3 \) gives \( u(t, x) = U(x)t^s \) where \( U(x) \) satisfies the second-order ODE

\[
U_{xx} + \beta k U^k U_x - U = 0. \tag{37}
\]

The latter equation is autonomous and admit the point symmetry \( \delta_s \) which can be used to reduce Eq. (35) into the following first-order ODE

\[
y_x = \beta k x^2 y^2 - 3y^3, \tag{38}
\]

where \( y(z) = (U_x)^{-1} \) and \( z = U(x) \).

However, Eq. (35) can be easily solved analytical by using the singularity analysis. Indeed from the ARS algorithm we find the leading-order behaviour

\[
U(x) = \left( \frac{k + 1}{\beta k^2} \right)^{\frac{1}{k}} x^k, \tag{39}
\]

with resonances

\[
s_1 = -1 \quad \text{and} \quad s_2 = \frac{1 + k}{k}. \tag{40}
\]

In order to perform the consistency test we have to select specific value for the parameter \( k \). Indeed for \( k = 2 \) we write the Laurent expansion

\[
U(x) = \left( \frac{3}{4\beta} \right)^{\frac{1}{3}} x^{\frac{2}{3}} + \sum_{i=1}^{\infty} U_i x^{-\frac{2i}{3}}, \tag{41}
\]

and by replacing in (35) we find that

\[
U_1 = 0, U_2 = 0, U_3 = -\frac{1}{3\beta}, U_5 = 0, U_6 = -\frac{\sqrt{3}(U_3)}{4}, \quad \ldots . \tag{42}
\]

where \( U_3 \) is the second integration constant. We conclude that Eq. (35) passes the Painlevé test.

4.1.2 Reduction with \( cX_1 + X_2 \): Travel-wave solution

The travel-wave solution of the generalized BBM equation is \( u = U(\xi) \), where \( \xi = x - c^{-1}t \) and \( U(\xi) \) satisfies the differential equation

\[
U_{\xi\xi} + (\gamma U^k - 1)U_\xi = 0. \tag{43}
\]

The latter equation can be integrated easily

\[
U_\xi + \left( \frac{c_0}{k + 1} U^{k+1} - U \right) + U_0 = 0, \tag{44}
\]

that is
where \( U_0, U_1 \) are two integration constants. The latter differential equation can be solved easily by quadratures.

As far as the singularity analysis is concerned for Eq. (44), the ARS algorithm provides the leading-order behaviour

\[
U(\xi) = U_0(\xi - \xi_0)^{\frac{1}{2}}, \quad U_0^{\frac{1}{2}} = -2\frac{(k + 1)(k + 2)}{\beta ck^2}, \quad (46)
\]

with resonances

\[
s_1 = -1 \text{ and } s_2 = \frac{2(k + 2)}{k}.
\]

The consistency test has been applied for various values of the positive integer \( k \), and we can infer that Eq. (42) is integrable according to the singularity analysis.

### 4.1.3 Reduction with \( X_2 + \alpha X_3 \): Scaling solution II

From the Lie symmetry \( X_2 + \alpha X_3 \) we find the similarity reduction \( u(t, x) = U(\xi)x^\frac{1}{2} \), \( \xi = x + \frac{1}{\alpha k} \ln t \) where \( U(\xi) \) is a solution of the following differential equation

\[
U_{\xi \xi \xi} - aU_{\xi \xi} - (a\beta U^k + 1)U_\xi + aU = 0. \quad (47)
\]

Equation (45) can be reduced to the following second-order ODE by use of the point symmetry vector \( \partial_\xi \),

\[
z^2 y_{xx} + z(y_x)^2 - az y_x - (a\beta z^k + 1)y + az = 0, \quad (48)
\]

where \( z = U(x) \) and \( y(z) = U_x \).

We apply the ARS algorithm for Eq. (45) and we find that it passes the singularity test for the leading order behaviour

\[
U(\xi) = U_0(\xi - \xi_0)^{\frac{1}{2}}, \quad U_0^{\frac{1}{2}} = 2\frac{(k + 1)(k + 1)}{a\beta k^2}, \quad (49)
\]

with resonances

\[
s_1 = -1, \quad s_2 = \frac{2(k + 1)}{k}, \quad s_3 = \frac{2(k + 2)}{k}. \quad (50)
\]

### 4.2 Singularity analysis

We complete our analysis by applying the singularity test in the generalized BBM equation in a similar way as we did in Section 3.2 for the generalized CH equation. Indeed we find the leading order term

\[
\frac{1}{2}(U_t)^2 + \left( \frac{c\beta}{(k + 1)(k + 2)} U^{k+2} - \frac{U_t^2}{2} \right) + U_0 U - U_1 = 0, \quad (45)
\]

and resonances those given in (48). We performed the consistency test and we infer that the generalized BBM equation passes the singularity test for any value of the positive integer parameter \( k \).

### 5 Conclusion

In this work we studied the existence of similarity solutions of the generalized CH and generalized BBM equations. The approach that we used is that of the Lie point symmetries. We determined the admitted invariant point transformations for the differential equations of our consideration and we determined the one-dimensional optimal systems by using the adjoint representation of the admitted Lie algebra. The two differential equations of our consideration are invariant under the same Lie symmetry vectors which form the same Lie algebra with the CH and the BBM equations.

For each of the equations we perform five different similarity reductions where the PDEs are reduced to third-order ODEs. The integrability of the resulting equations is studied by using symmetries and/or the singularity analysis. In the case of the generalized CH equation most of the reduced ODEs cannot be solved by using Lie symmetries, hence the application of the singularity analysis was necessary to determine the analytic solutions of the reduced equations.

Finally, we study the integrability of the PDEs of our consideration by applying the singularity analysis directly on the PDEs and not on the reduced equations. From the latter analysis we found that the generalized CH and BBM equations pass the singularity analysis and their solutions are given by Right Painlevé Series.

This work contribute to the subject of the integrability of generalized equations describe shallow-water phenomena. The physical implication of the new analytic solutions will be presented in a future communication.

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