STABLE COMMUTATOR LENGTH IN SUBGROUPS OF PL⁺ (I)

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ABSTRACT. Let G be a subgroup of PL⁺ (I). Then the stable commutator length of every element of [G : G] is zero.

1. INTRODUCTION

The purpose of this note is to prove a vanishing theorem for stable commutator length in groups of PL homeomorphisms of the interval. For convenience, we restrict attention to subgroups of the group of orientation preserving PL homeomorphisms, denoted in the sequel by PL⁺ (I), where I denotes the unit interval [0; 1]. By a theorem of Bavard (see x2), vanishing of stable commutator length is equivalent to the injectivity of the map from bounded to ordinary cohomology in dimension 2.

Since the dimension of the second bounded cohomology of a nonabelian free group is uncountable, this gives a new proof of the celebrated result of Brin–Squier (2) that PL⁺ (I) does not contain a nonabelian free subgroup.

There are at least two other important classes of groups for which vanishing of stable commutator length is known to hold:

Irreducible lattices in semisimple Lie groups of rank at least 2; this follows from more general work of Burger and Monod (3).

Amenable groups; in this case, bounded cohomology with real coefficients vanishes in every dimension by Trauber’s theorem (see x2) and therefore the map is injective for trivial reasons.

It is considered an important open question whether or not Thompson’s group F < PL⁺ (I), which consists of homeomorphisms with dyadic rational slopes and break points, is amenable. More generally, no counterexamples are known to the conjecture that a finitely presented torsion free group with the property that every subgroup has vanishing stable commutator length is amenable (this should perhaps be thought of as a kind of “homological” version of von Neumann’s conjecture). These and related problems are one of the main motivations for this paper.

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2. BACKGROUND MATERIAL

**Definition 2.1.** Let $G$ be a group, and $C^*(G)$ (the bar) complex of integral $G$-chains.
Let $C^*(G)_R$ be the dual complex of real-valued cochains. For each $n$, let $C_n^b(G)_R$ denote the vector space of cochains $f$ for which $\sup f(j)$ is finite, where $j$ ranges over the generators of $C^*(G)$. The (real) bounded cohomology of $G$, denoted $H^b_b(G;R)$, is the cohomology of the complex $C^b(G)_R$.

Note that $H^b_n(G;R)$ carries an $L^1$ pseudo-norm for each $n$.

**Theorem 2.2 (Trauber).** Let

$$1! H^1 G ! A ! 1$$

be a short exact sequence of groups, where $A$ is amenable. Then the natural homomorphisms $H^b_b(G;R) ! H^b_b(H;R)$ are isometric injections.

See e.g. [4] page 39 for a proof.

**Definition 2.3.** Let $G$ be a group, and $[G;G]$ the commutator subgroup. For any $g \in [G;G]$, the commutator length of $g$, denoted $\ell_g$, is the minimal number of commutators whose product is equal to $g$. The stable commutator length, denoted $\ell^s_g$ is defined to be

$$\ell^s_g = \lim_{n \to \infty} \frac{\ell_g(n)}{n}$$

By including bounded cochains in all cochains, one obtains canonical homomorphisms from bounded cohomology to ordinary cohomology. There is a fundamental relationship between stable commutator length and bounded cohomology, discovered by Bavard:

**Theorem 2.4 (Bavard).** Let $G$ be a group. Then the canonical map from bounded cohomology to ordinary cohomology $H^b_b(G;R) ! H^b(G;R)$ is injective if and only if the stable commutator length vanishes on $[G;G]$.

See [1].

Bavard’s theorem makes use of the notion of quasimorphisms:

**Definition 2.5.** Let $G$ be a group. A (homogeneous) quasimorphism on $G$ is a map $f : G \to R$ for which there is some smallest $c(f)$ (called the error or defect of $f$) such that

$$f(a^n) = nf(a)$$

and

$$f(a) + f(b) \leq f(ab)$$

for all $a,b \in G$.

Note that a homogeneous quasimorphism is necessarily a class function.

The set of all homogeneous quasimorphisms on $G$, denoted $Q(G)$, has the structure of a vector space. Quasimorphisms with error 0 are homomorphisms. There is an exact sequence

$$1! H^1(G;R) ! Q(G) ! H^2_b(G;R) ! H^2(G;R)$$
where denotes the coboundary map. See [1] for a proof.

Thus Bavard’s theorem may be interpreted as saying that if \( G \) is a group, the quotient \( Q(G) = H^1(G; R) \) is zero exactly when the stable commutator length of every element of \( [G; G] \) vanishes.

In terms of quasimorphisms, Bavard proves the following sharper statement:

**Theorem 2.6 (Bavard).** Let \( G \) be a group, and \( g \in [G; G] \). Then

\[
\hat{\lambda}(g) = \frac{1}{2} \sup_{f \in \mathcal{Q}(G)} \frac{\langle f \rangle(g)}{\langle f \rangle}
\]

Theorem 2.2 and Theorem 2.4 together imply that \( \hat{\lambda}(g) = 0 \) for any \( g \in [G; G] \) whenever \( G \) is an amenable group.

3. **SUBGROUPS OF PL**

Given a subgroup \( G < \text{PL}^+(\mathcal{I}) \) we denote by \( \text{fix}(G) \) the set of common fixed points of all elements of \( G \).

**Definition 3.1.** The *endpoint homomorphism* is the homomorphism

\[ :\text{PL}^+(\mathcal{I}) ! \mathbb{R} \mathbb{R} \]

defined by

\[ (g) = (\log g^0(0); \log g^0(1)) \]

Given \( G < \text{PL}^+(\mathcal{I}) \), denote by \( G_0 \) the kernel of \( \circ \) restricted to \( G \).

Observe that every \( g \in G_0 \) fixes a neighborhood of both 0 and 1.

**Theorem A.** Let \( G \) be a subgroup of \( \text{PL}^+(\mathcal{I}) \). Then the stable commutator length of every element of \( [G; G] \) is zero.

**Proof.** **Case 1:** \( \text{fix}(G) = \{0, 1\} \).

Let \( G_0 \) be the kernel of \( :G ! \mathbb{R} \mathbb{R} \). Let \( K = [G_0; G_0] \) and let \( g \in K \). Then we can write

\[ g = [a_1; b_1] [a_2; b_2] \cdots [a_m; b_m] \]

for some integer \( m \) and \( a_i; b_i \) in \( G_0 \). Let \( J \) be the smallest interval which contains the support of all the \( a_i; b_i \) and \( g \). Then \( J \) is properly contained in \( (0; 1) \). Since \( \text{fix}(G) \) contains no interior points, there is some \( j \in G \) with \( 3J \setminus J = \) ; and therefore \( j^3(J) \setminus J = \) ; for all nonzero \( n \).

Let \( G_0(J) \) be the subgroup of \( G_0 \) consisting of elements with support contained in \( J \). For each \( n \) we define a diagonal monomorphism

\[ n : G_0(J) ! G_0 \]

by

\[ n(g) = \sum_{i=0}^{\infty} g^i \]

where the superscript notation denotes conjugation. Define

\[ g^0 = \left( (g^0)^n \right)^{n+1} \]

Then

\[ [g^0, j] = n(g) (g^n 1)^{n+1} \]
On the other hand,
\[ n \ (g) = \ n ( [a_1; b_2]) = [ [a_1; b_2] ] = [ \ n (a_1); \ n (b_2) ] = [ \ n (a_1); \ n (b_2) ] \]
and therefore \( g^n = 1 \) can be written as a product of at most \( n + 1 \) commutators in elements of \( G \). Since \( m \) is fixed but \( n \) is arbitrary, it follows that the stable commutator length of \( g \) is zero, and hence \( \mu (G \ast G_0) = 0 \) for every quasimorphism \( \mu : G \to G_0 \).

Now, let \( g \ast G \). Observe that \( G_0 \ast G \) is normal in \( G \), so we can form the quotient \( H = G/G_0 \) which is two-step solvable, and therefore amenable. Let \( \phi : G \to H \) be the quotient homomorphism. By Trauber’s Theorem 2.4 and Bavard’s Theorem 2.4, this proves the theorem when \( \phi (g) = 0 \) in \( H \). This means that we can write
\[ \phi (g) = [a_1; b_2] = \phi _n \]
where \( c \ast G \), where \( n \) is as small as we like, and \( m = n \) as small as we like. Let \( \mu \) be a quasimorphism of defect at most \( 1 \). By the above, we have \( \mu (e) = 0 \), and therefore \( \mu (g^n) = 2m + 1 \), and \( \mu (g) \to \). Since \( n \) is arbitrarily big, and \( m = n \) as small as we like, \( \mu (g) = 0 \). Since \( \mu \) and \( h \) were arbitrary, \( \mu (H) = H \ast H = 0 \).

Applying Theorem 2.4 proved the theorem when \( \mu (G) = f \ast G \).

Case 2: fix \( H \) is arbitrary.

Suppose \( \mu (G \ast G) \) has defect at most \( 1 \), and suppose \( \mu (g) \neq 0 \) where \( g \ast G \). Let \( H \) be a finitely generated subgroup of \( G \) such that \( g \ast H \). Then \( \mu (H) \) is equal to the intersection of \( \mu (H_i) \) for the generators \( H_i \). The fixed set of any element of \( GL^+ (1) \) is a union of finitely many points and intervals, so the same is true for \( \mu (H) \). Hence \( \mu (H) \) consists of finitely many open intervals, whose closures we denote by \( \mu _1; \mu _2; \ldots ; \mu _n \).

Let \( \mu : H \to \mathbb{R}^n \) denote the product of the endpoint homomorphisms for each \( \mu _i \) and let \( H_0 \) denote the kernel. We will show that \( \mu \) vanishes on \( H \ast H_0 \) contrary to the fact that \( \mu (g) \neq 0 \) and \( g \ast H \). Suppose \( r \ast H \). Let \( r = [a_1; b_2] = \phi _n \]
where all the \( a_i; b_2 \) have support in the union \( \mu (J_i) \) where \( J_i \) is an interval for each \( i \). For each \( i \) there is \( \mu _i (J_i) \) with \( \mu _i (J_i) \) \( J_i \) = \( \mu _i \). Note that this implies \( \mu _i (J_i) \) \( J_i \) = \( \mu _i \) for all nonzero \( n \).

However, we claim that we can construct a single element \( j \ast H \) such that \( j (J_i) \) \( J_i \) = \( \mu _i \) for all \( i \) simultaneously.

The case \( n = 1 \) is trivial; in the interests of exposition we describe the situation \( n = 2 \) in detail before moving on to the general case.

Without loss of generality, we may assume \( \mu _1 \) moves \( J_1 \) to the right. Now, let \( J_2 \) be the smallest interval which contains both \( J_2 \) and \( \mu _1 (J_2) \) and let \( \mu _2 \) be such that \( \mu _1 (J_2) \) \( J_2 \) = \( \mu _1 \). After replacing \( \mu _2 \) by \( \mu _2 \) if necessary, we may also assume that \( \mu _2 \) moves the leftmost point of \( J_1 \), which we denote by \( J_1 \), to the right. We also use the notation \( J_i \) to denote the rightmost point of \( J_i \), i.e.
\[ \mu _1 (J_1) \) \( J_1 \) \( \mu _2 (J_1) \) \( J_1 \]

Then
\[ \mu _1 (J_1) \) \( J_1 \) \( \mu _2 (J_1) \) \( J_1 \]
and therefore
\[ \hat{j}_1 \hat{j}_2 (J_1) \setminus J_1 = \hat{j} \]
Moreover,
\[ \hat{j}_1 \hat{j}_2 (J_2) \setminus J_2 = \hat{j}_1 (\hat{j}_2 (J_2) \setminus \hat{j}_1 (J_2)) \hat{j}_1 (\hat{j}_2 (J_2) \setminus J_2) = \hat{j} \]

Now we treat the general case. As before, without loss of generality, we assume \( \hat{j}_1 \) moves \( J_1 \) to the right. For all \( i > 1 \) we let \( J_i \) denote the smallest interval which contains both \( J_i \) and \( \hat{j}_1 (J_i) \). By induction, we assume that there is some \( j \) with \( \hat{j}_1 (J_i) \setminus J_i = \hat{j} \) for all \( i > 1 \) simultaneously. After replacing \( j \) with \( \hat{j}_1 \) if necessary, we may assume that \( j \) moves the leftmost point of \( J_1 \) to the right. Then the argument above shows that
\[ \hat{j}_1 \hat{j}_1 (J_1) \setminus J_1 = \hat{j} \]
for all \( i \). Therefore we have proved the claim.

But now the proof that \( \epsilon (e) = 0 \) follows exactly as in Case 1, since for any \( m \) there is a diagonal monomorphism
\[ m : G_0 (\prod J_i) \rightarrow G_0 \]
defined by
\[ m (c) = \prod_{i=0}^{\infty} c^j \]
where now \( j \) moves every \( J_i \) off itself simultaneously. Since \( e \) was arbitrary, it follows that stable commutator length vanishes on \( [H_0 ; H_0] \) and since \( H = [H_0 ; H_0] \) is amenable, \( f \) must vanish on all of \( H \) by Theorem 2.4, contrary to the definition of \( H \). This contradiction implies that \( Q (G) = H^1 (G) = 0 \), and the theorem follows.

Remark 3.2. The “diagonal trick” is a variation on Mather’s argument (5) to prove the acyclicity of \( \text{Homeo}_0 (\mathbb{R}^n) \). This argument was modified by Matsumoto-Morita (6) to prove vanishing of all the bounded cohomology of \( \text{Homeo}_0 (\mathbb{R}^n) \). One significant difference between \( \text{PL}^+ (I) \) and \( \text{Homeo}_0 (\mathbb{R}^n) \) is that every finitely generated subgroup \( G \) of \( \text{Homeo}_0 (\mathbb{R}^n) \) is contained in an \textit{unrestricted} wreath product with \( \mathbb{Z} \) (i.e. a product of the form \( \hat{G} \circ G \)), whereas in a PL group, only \textit{restricted} wreath products (i.e. products of the form \( \hat{G} \circ Z \)) with infinite groups are possible.
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