Grothendieck Enriched Categories

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Abstract
In this paper, we introduce the notion of Grothendieck enriched categories for categories enriched over a sufficiently nice Grothendieck monoidal category \( \mathcal{V} \), generalizing the classical notion of Grothendieck categories. Then we establish the Gabriel-Popescu type theorem for Grothendieck enriched categories. We also prove that the property of being Grothendieck enriched categories is preserved under the change of the base monoidal categories by a monoidal right adjoint functor. In particular, if we take as \( \mathcal{V} \) the monoidal category of complexes of abelian groups, we obtain the notion of Grothendieck dg categories. As an application of the main results, we see that the dg category of complexes of quasi-coherent sheaves on a quasi-compact and quasi-separated scheme is an example of Grothendieck dg categories.

Keywords
Grothendieck category · The Gabriel-Popescu theorem · Enriched category · Dg category

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1 Introduction

A Grothendieck category is a cocomplete abelian category with a generator where filtered colimits are exact. The category of modules over a ring and the category of quasi-coherent sheaves on a scheme are examples of Grothendieck categories. A Grothendieck category is known to have various good properties: it admits an injective cogenerator; any object has an injective resolution; every continuous functor from it to the category of sets is representable; the adjoint functor theorem holds (see [23, §8.3, §9.6] for example).

While the definition above is given in terms of intrinsic properties of the category, a Grothendieck category is also characterized as a nice subcategory of the category of modules over a ring:

Theorem (Gabriel-Popescu [30]; see also Theorem 3.3 for a slightly generalized statement)
Let $A$ be a Grothendieck category and $G \in A$ be a generator. Let $R$ be the ring of endomorphisms $A(G, G)$. Then the following assertions hold.

(i) The additive functor $T = A(G, -): A \to \text{Mod}(R)$ has a left adjoint $S$.
(ii) $T$ is fully faithful.
(iii) $S$ is left exact.

We remark that there is a generalized result for algebraic categories [37].

The Gabriel-Popescu theorem, in a word, asserts that every Grothendieck category is realized as a reflective subcategory\(^1\) of the category of modules over a ring such that the left adjoint to the inclusion functor is left exact. An important point is that the good properties of Grothendieck categories stated above can be derived from this embedding theorem. Thus the essence of Grothendieck categories is the extrinsic characterization. Note that a similar characterization of Grothendieck topoi as a nice subcategory is known [5, Cor. 3.5.5], and Grothendieck categories can also be understood as the additive counterpart of Grothendieck topoi [7, 26].

On the other hand, the importance of dg categories has been recognized in many fields of mathematics [2, 22]. It is hence meaningful to study the analogs of Grothendieck categories and the Gabriel-Popescu theorem in the dg setting. In fact, there are Gabriel-Popescu type theorems for triangulated categories [31] and for stable infinity categories [28], both of which are relatives of dg categories. It is therefore natural to expect a similar theorem for dg categories.

Recall that dg categories are nothing but Ch-enriched categories, where Ch denotes the monoidal category of cochain complexes of abelian groups. On the other hand, an abelian category has the unique structure of a pre-additive category (a category enriched over the monoidal category Ab of abelian groups), and the Gabriel-Popescu theorem can be considered as a theorem in Ab-enriched category theory.

In this paper we define the notion of a Grothendieck category enriched over $V$, where $V$ is taken from a class of monoidal categories which includes both Ab and Ch, as a nice subcategory of the category of presheaves on a small $V$-category. We then establish the Gabriel-Popescu type theorem in this generality.

\(^1\) A subcategory is called reflective if its inclusion functor has a left adjoint.
We refer to a symmetric monoidal closed category that is complete and cocomplete as a *cosmos*. We call a cosmos that is also a Grothendieck category a *Grothendieck cosmos*. For a Grothendieck cosmos $\mathcal{V} = (\mathcal{V}, \otimes, I)$, let us consider the following conditions, which are adapted from [19]:

(C1) The unit object $I \in \mathcal{V}$ is finitely presentable.
(C2) $\mathcal{V}$ has a generating set of dualizable objects.

An object $C$ of a category $\mathcal{C}$ is said to be *finitely presentable* if the functor $\text{Hom}(C, -) : \mathcal{C} \rightarrow \text{Set}$ preserves filtered colimits. An object $X$ of a symmetric monoidal closed category is said to be *dualizable* if there is an isomorphism $[X, I] \otimes - \cong [X, -]$ of functors. We will show in Proposition 3.5 that a Grothendieck cosmos satisfying the conditions (C1) and (C2) is a *locally finitely presentable base*, a monoidal category in which we can discuss the finiteness of objects. In categories enriched over such a monoidal category, finite limits make sense as in usual categories.

Then, inspired by the Gabriel-Popescu theorem, we make the following definition.

**Definition A** (Definition 3.15) Let $\mathcal{V}$ be a locally finitely presentable base. A $\mathcal{V}$-category $\mathcal{A}$ is said to be a *Grothendieck $\mathcal{V}$-category* if there exist a small $\mathcal{V}$-category $\mathcal{C}$ and a $\mathcal{V}$-adjunction $S : [\mathcal{C}^{\text{op}}, \mathcal{V}] \xleftarrow{\perp} \mathcal{A} : T$ such that

(i) the right adjoint $T$ is fully faithful and
(ii) the left adjoint $S$ is left exact.

The Grothendieck cosmos $\text{Ch}(R)$ of cochain complexes of modules over a commutative ring $R$, which we have in mind in particular for applications, satisfies the conditions (C1) and (C2). Applying Definition A to $\mathcal{V} = \text{Ch}$, we get the notion of Grothendieck dg categories.

A remarkable property of Grothendieck $\mathcal{V}$-categories is that the following adjoint functor theorem holds. This indicates the usefulness of Grothendieck $\mathcal{V}$-categories.

**Proposition B** (Proposition 3.26) Let $\mathcal{V}$ be a Grothendieck cosmos with the conditions (C1) and (C2) and $F : \mathcal{A} \rightarrow \mathcal{B}$ a $\mathcal{V}$-functor.

(i) Suppose $\mathcal{A}$ is a Grothendieck $\mathcal{V}$-category. If it preserves all small conical colimits, then $F$ has a right adjoint.
(ii) Suppose $\mathcal{A}$ is a Grothendieck $\mathcal{V}$-category and $\mathcal{B}$ is cotensored. If the underlying ordinary functor $F_0$ is cocontinuous, then $F$ has a right adjoint.

The definition of Grothendieck enriched categories in Definition A is the generalization to enriched categories of the extrinsic characterization of Grothendieck categories given by the Gabriel-Popescu theorem. Hence it is natural to ask if we can characterize Grothendieck enriched categories in terms of intrinsic properties of enriched categories in such a way that for $\text{Ab}$-enriched categories the classical Gabriel-Popescu theorem is recovered. The main theorem of this paper, which is stated as Theorem C below, asserts that it is in fact possible in the case that the enriching monoidal category is a nice Grothendieck cosmos. This theorem is useful since, generally speaking, it is easier to confirm intrinsic properties of (enriched) categories rather than finding nice embeddings as in Definition A.

**Theorem C** (Theorem 3.18) Let $\mathcal{V}$ be a Grothendieck cosmos which satisfies the conditions (C1) and (C2). Then a $\mathcal{V}$-category $\mathcal{A}$ is a Grothendieck $\mathcal{V}$-category if and only if it fulfills the following conditions.
(1) $A$ is cocomplete.
(2) $A$ is finitely complete.
(3) The homomorphism theorem holds in $A$. That is, for any morphism $f$ in $A_0$ the canonical map $\text{Cok}(\text{Ker}(f)) \to \text{Ker}(\text{Cok}(f))$ is an isomorphism.
(4) Conical filtered colimits are left exact. Namely for any filtered category $\mathcal{J}$, the colimit $\mathcal{V}$-functor $\text{colim}_{\mathcal{J}} : [\mathcal{J}, A] \to A$ preserves finite limits.
(5) $A$ has a $\mathcal{V}$-generating set $\mathcal{C}$ of objects (see Definition 3.12 for the definition).

Let us give an outline of the Proof of Theorem C. It is easy to see that a Grothendieck $\mathcal{V}$-category as defined in Definition A satisfies the conditions (1)–(5) of Theorem C. To verify the converse, take a $\mathcal{V}$-category $A$ satisfying the conditions of Theorem C. Then it is known that we can associate with the inclusion functor $\mathcal{C} \hookrightarrow A$ an adjunction between $A$ and the presheaf category $[\mathcal{C}^{\text{op}}, \mathcal{V}]$. More generally, if $\mathcal{C}$ is a small $\mathcal{V}$-category and $A$ is a cocomplete $\mathcal{V}$-category, where $\mathcal{V}$ is a cosmos, then we can associate with any $\mathcal{V}$-functor $F : \mathcal{C} \to A$ an $\mathcal{V}$-adjunction $\text{Lan}_y F \dashv \text{Lan}_F y$ of left Kan extensions as follows:

Here $y$ denotes the Yoneda embedding. In this paper, we will refer to this adjunction as the nerve-and-realization ($\mathcal{V}$-)adjunction associated with $F$. For example, the adjunction in the Gabriel-Popescu theorem is the nerve-and-realization $\text{Ab}$-adjunction associated with the $\text{Ab}$-functor $R \to A$, where the ring $R$ is viewed as an $\text{Ab}$-category with one single object.

Once we obtain the nerve-and-realization adjunction $S \dashv T : [\mathcal{C}^{\text{op}}, \mathcal{V}] \to A$ associated with the inclusion functor $F : \mathcal{C} \hookrightarrow A$, we only need to show that $T$ is fully faithful and that $S$ is left exact. Since $T$ is a right adjoint, it is sufficient to check that its underlying functor $T_0$ is fully faithful. Likewise, by Lemma 3.21, it is sufficient to check that $S_0$ is left exact. Then, after we verify that $A_0$ is a Grothendieck category, we can use the Gabriel-Popescu theorem to prove that $T_0$ is fully faithful and $S_0$ is left exact.

As an application of Theorem C, we can easily verify that the change-of-base functor associated with a monoidal right adjoint preserves the property of being a Grothendieck enriched category.

**Proposition D** (Proposition 3.22) Let $\mathcal{V}$, $\mathcal{W}$ be Grothendieck cosmoi and $F \dashv G : \mathcal{V} \to \mathcal{W}$ a monoidal adjunction. Suppose that $\mathcal{V}$ satisfies the conditions (C1) and (C2). If a $\mathcal{W}$-category $\mathcal{B}$ is a Grothendieck $\mathcal{W}$-category, then the $\mathcal{V}$-category $G(\mathcal{B})$ is a Grothendieck $\mathcal{V}$-category.

As an immediate application of Proposition D, we see that for a quasi-compact and quasi-separated scheme $X$ over a commutative ring $R$, the dg category of complexes of quasi-coherent sheaves on $X$ is a Grothendieck $\text{Ch}(R)$-category (Example 3.24). The Gabriel-Popescu theorem for triangulated categories is shown in [31]. It is proved that any algebraic well-generated triangulated category is a localization of the derived category of some small dg category with respect to a localizing subcategory generated by a set of objects. On the other hand, a Grothendieck dg category is pretriangulated so that its homotopy category has a natural triangulated structure. The homotopy category of a Grothendieck dg category is not well-generated, but locally well-generated in the sense of [38].
However, in view of the relations of dg categories to triangulated categories and stable infinity categories, the notion of Grothendieck dg categories seems too naive. We should rather work in the localization $\text{HodgCat}$ of $\text{dgCat}$ with respect to quasi-equivalences and use derived dg categories in place of the dg categories of dg modules. Unfortunately, our methods do not apply to $\text{HodgCat}$ directly. It is a future task to solve this issue.

After the first draft of this paper appeared on the arXiv, it was pointed out by Ivan Di Liberti that Definition 3.15 also appears in [18] under the name of $V$-topoi.

Outline of this paper. Section 2 is devoted to preliminaries on enriched categories. In Sect. 2.1 we review the basic concepts of enriched category theory, referring the reader to [4, 20] for further details. In Sect. 2.2 we discuss about changing enrichments of enriched categories. In Sect. 2.3 we recall the definition of a generator and a strong generator of an ordinary category and an abelian category. In Sect. 2.4 we give the definition of a locally finitely presentable base and describe what finite limits are like in enriched categories, following [8, 21]. In Sect. 2.5 we recall the notion of dualizable objects and the relations with weighted (co)limits (Proposition 2.20).

In Sect. 3.1 we state the slightly generalized version of the classical Gabriel-Popescu theorem, which will be used in the proof of the main theorem (Theorem 3.18).

In Sect. 3.2 we introduce Grothendieck cosmoi satisfying the finiteness conditions (C1) and (C2). We show the fact that they are locally finitely presentable bases. This fact is due to [19]. We also explain their examples.

Main results are given in Sect. 3.3. There we define the notions of enriched generators and enriched Grothendieck categories. Then we prove the Gabriel-Popescu type theorem for Grothendieck enriched categories (Theorem 3.18). After that, as an application of the main theorem, we show in Proposition 3.22 that the property of being Grothendieck enriched categories is preserved under changing the enrichments by a monoidal right adjoint functor. We use this proposition to give examples of Grothendieck enriched categories. Finally we observe that the adjoint functor theorem holds for Grothendieck enriched categories.

Relations of the results of this paper to [18] will be discussed in Sect. 3.4.

2 Preliminaries on Enriched Categories

We tacitly assume that all categories discussed in this paper are locally small.

2.1 Enriched Categories

We use the framework of enriched categories. The general references of this subject are Kelly [20] and Borceux [4]. We adopt the terminology of Kelly’s book.

Let $V = (V, \otimes, I, [-, -])$ be a symmetric monoidal closed category whose underlying category $V$ is complete and cocomplete, with the unit object $I$ and the internal hom $[-, -]$. For a $V$-category $C$, the underlying category $C_0$ has the same objects as $C$ and hom sets $\text{Hom}_V(I, C(D, D))$. This construction forms a functor $(\cdot)_0 : V\text{-Cat} \to \text{Cat}$ from the category of $V$-categories to the category of ordinary categories which has a left adjoint $(\cdot)_V : \text{Cat} \to V\text{-Cat}$.

We recall the notion of weighted (co)limits; note that in [20, Ch. 3], they are called indexed (co)limits. Given two $V$-functors $F : \mathcal{J} \to C$ and $W : \mathcal{J} \to V$, the limit of $F$ weighted by $W$ is an object $\{W, F\} \in C$ together with an isomorphism
\[ C(C, \{W, F\}) \cong [\mathcal{J}, \mathcal{V}](W, C(C, F-)) \]

natural in \( C \in C \). Given \( F : \mathcal{J} \to C \) and \( W : \mathcal{J}^{\text{op}} \to \mathcal{V} \), the colimit of \( F \) weighted by \( W \) is an object \( W \star F \in C \) together with an isomorphism

\[ C(W \star F, C) \cong [\mathcal{J}^{\text{op}}, \mathcal{V}](W, C(F-, C)) \]

natural in \( C \in C \). The (co)limit is called small if the domain \( \mathcal{J} \) of its weight \( W \) is small. We call the \( \mathcal{V} \)-category \( C \) (co)complete if it admits all small (co)limits, respectively.

As a special case of weighted limits where \( \mathcal{J} \) is the unit category \( \mathcal{I} \), we have the notion of cotensor products: that is, for \( X \in \mathcal{V} \) and \( D \in C \), the cotensor product of \( X \) and \( D \) is an object \( X \star D \in C \) together with an isomorphism

\[ C(C, X \star D) \cong [X, C(C, D)] \]

natural in \( C \in C \). Dually, the tensor product of \( X \) and \( D \) is an object \( X \otimes D \in C \) together with an isomorphism

\[ C(X \otimes D, C) \cong [X, C(D, C)] \]

natural in \( C \in C \). We call the \( \mathcal{V} \)-category \( C \) (co)tensored if it admits all (co)tensor products, respectively.

We also recall the notion of left Kan extensions.

**Definition 2.1** Let \( F : C \to M \) and \( K : C \to D \) be \( \mathcal{V} \)-functors. A \( \mathcal{V} \)-functor \( \text{Lan}_K F : D \to M \) is called the left Kan extension of \( F \) along \( K \) if for any \( \mathcal{V} \)-functor \( S : D \to M \), there is a natural bijection

\[ \text{Hom}_{\text{Fun}(D, M)}(\text{Lan}_K F, S) \cong \text{Hom}_{\text{Fun}(C, M)}(F, SK). \]

**Definition 2.2** Let \( F : C \to M \) and \( K : C \to D \) be \( \mathcal{V} \)-functors. A \( \mathcal{V} \)-functor \( T : D \to M \) is called the pointwise left Kan extension of \( F \) along \( K \) if for any \( d \in D \) and \( m \in M \), there is a natural isomorphism

\[ M(Td, m) \cong [C^{\text{op}}, \mathcal{V}](D(K-, d), M(F-, m)). \]

Note that in Kelly’s book [20], Kan extensions in the sense of Definition 2.1 are referred to as weak Kan extensions and pointwise Kan extensions in the sense of Definition 2.2 as Kan extensions. By [20, Thm. 4.43], a pointwise left Kan extension, if it exists, is a left Kan extension. As in [20, Prop. 4.33], pointwise left Kan extensions do exist if \( M \) is cocomplete and \( C \) is small.

We take a look at left Kan extensions concerning the Yoneda embedding.

**Proposition 2.3** Let \( C \) be a small \( \mathcal{V} \)-category and \( F : C \to D \) a \( \mathcal{V} \)-functor. Then the left Kan extension \( \text{Lan}_F y : D \to [C^{\text{op}}, \mathcal{V}] \) of the Yoneda embedding \( y : C \to [C^{\text{op}}, \mathcal{V}] \) along \( F \) satisfies \( \text{Lan}_F y(d) \cong D(F-, d) \) for any \( d \in D \).

**Proof** Since the presheaf \( \mathcal{V} \)-category \( [C^{\text{op}}, \mathcal{V}] \) is cocomplete, the pointwise left Kan extension \( \text{Lan}_F y \) exists. Then using the (enriched) Yoneda lemma, we have natural isomorphisms

\[
[C^{\text{op}}, \mathcal{V}](\text{Lan}_F y(d), P) 
\cong [C^{\text{op}}, \mathcal{V}](D(F-, d), [C^{\text{op}}, \mathcal{V}](y-, P))
\cong [C^{\text{op}}, \mathcal{V}](D(F-, d), P)
\]

for \( P \in [C^{\text{op}}, \mathcal{V}] \). Thus the Yoneda lemma implies \( \text{Lan}_F y(d) \cong D(F-, d) \). \( \square \)
Theorem 2.4 Let $\mathcal{C}$ be a small $\mathcal{V}$-category and $F : \mathcal{C} \to \mathcal{D}$ a $\mathcal{V}$-functor. If the pointwise left Kan extension $\text{Lan}_y F$ exists, then there is a $\mathcal{V}$-adjunction $\text{Lan}_y F \dashv \text{Lan}_F y$:

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{y} & \mathcal{D} \\
\downarrow \text{Lan}_F y & & \downarrow \text{Lan}_y F \\
\mathcal{D} & \xleftarrow{F} & \mathcal{C}^{\text{op}}, \mathcal{V}
\end{array}
\]

In particular, this is the case if $\mathcal{D}$ is cocomplete by [20, Prop. 4.33].

Proof By the Yoneda lemma and Proposition 2.3, for any $d \in \mathcal{D}$ and $P \in [\mathcal{C}^{\text{op}}, \mathcal{V}]$ we have natural isomorphisms

\[
\mathcal{D}(\text{Lan}_y F(P), d) \cong [\mathcal{C}^{\text{op}}, \mathcal{V}](\mathcal{D}(-, d), \mathcal{D}(\text{Lan}_y F(-), d)) \\
\cong [\mathcal{C}^{\text{op}}, \mathcal{V}](\mathcal{D}(-, d), \mathcal{D}(F(-, d))) \\
\cong [\mathcal{C}^{\text{op}}, \mathcal{V}](P, \text{Lan}_F y(d)).
\]

Hence $\text{Lan}_y F$ is a left adjoint to $\text{Lan}_F y$. \qed

In this paper, we will call the $\mathcal{V}$-adjunction of Theorem 2.4 the nerve-and-realization adjunction associated with $F$, inspired by [29, “nerve and realization"], though it is somewhat lengthy.

Proposition 2.5 ([20, Prop. 4.23]) Let $F : \mathcal{C} \to \mathcal{M}$ and $K : \mathcal{C} \to \mathcal{D}$ be $\mathcal{V}$-functors. Suppose that the pointwise left Kan extension $\text{Lan}_K F$ of $F$ along $K$ exists. If $K$ is fully faithful, then we have $\text{Lan}_K F \circ K \cong F$.

Theorem 2.6 ([20, Thm. 4.51]) Let $\mathcal{C}$ be a small $\mathcal{V}$-category and $\mathcal{D}$ be a cocomplete $\mathcal{V}$-category. Then there are one-to-one correspondences between the following sets:

(i) the set of isomorphic classes of $\mathcal{V}$-functors $F : \mathcal{C} \to \mathcal{D}$,
(ii) the set of isomorphic classes of cocontinuous $\mathcal{V}$-functors $S : [\mathcal{C}^{\text{op}}, \mathcal{V}] \to \mathcal{D}$,
(iii) the set of isomorphic classes of $\mathcal{V}$-adjunctions $S \dashv T : [\mathcal{C}^{\text{op}}, \mathcal{V}] \to \mathcal{D}$.

Proof The correspondence between the first set and the second is given by $F \mapsto \text{Lan}_y F$ and $S \mapsto S \circ y$. The correspondence between the second set and the third is given by $S \mapsto (S \dashv \text{Lan}_y S F)$ and $(S \dashv T) \mapsto S$. Details are left to the reader. \qed

2.2 Change of Enrichments

When we have two monoidal categories and a functor between them that preserves monoidal structures in a sense, we can change enrichments of enriched categories.

Definition 2.7 Let $\mathcal{V} = (V, \otimes, I)$ and $\mathcal{W} = (W, \otimes, I')$ be monoidal categories. A lax monoidal functor between them consists of

- a functor $F : \mathcal{V} \to \mathcal{W}$;
- for every pair $X, Y \in \mathcal{V}$ of objects, a natural morphism $\tau_{XY} : F(X) \otimes F(Y) \to F(X \otimes Y)$ of $\mathcal{W}$;
- a morphism $\sigma : I' \to F(I)$ of $\mathcal{W}$.
so that these morphisms are compatible with the monoidal structures (see [4, Def. 6.4.1]). If $\tau_{XY}$ and $\sigma$ all are isomorphisms, then $F$ is said to be just a \textit{monoidal functor}.

For example, the representable functor $\text{Hom}_V(I, -) : V \to \text{Set}$ is lax monoidal.

For an adjunction $F \dashv G : V \to W$ between monoidal categories, the right adjoint $G$ is lax monoidal if the left adjoint $F$ is monoidal (see [29, “monoidal adjunction”]). We will refer to an adjunction whose left adjoint is monoidal as a \textit{monoidal adjunction}.

\textbf{Proposition 2.8} (Change of base [4, Prop. 6.4.3]) Let $F : V \to W$ be a lax monoidal functor between monoidal categories. Then $F$ induces a functor $F : V\text{-Cat} \to W\text{-Cat}$ between the categories of enriched categories which sends a $V$-category $C$ to the $W$-category $F(C)$ such that

- its objects are the same as those of $C$;
- its $\text{Hom}$ objects are $\text{Hom}(C(C, D)) \in W$;
- its composition maps are
  \[ F(C(B, C)) \otimes F(C(A, B)) \xrightarrow{\tau} F(C(B \otimes A, C)) \xrightarrow{F(m)} F(C(A, C)) \]
- its identity maps are
  \[ I \xrightarrow{\sigma} F(I) \xrightarrow{F(j_C)} F(C(C, C)) \].

If we take the lax monoidal functor $\text{Hom}_V(I, -) : V \to \text{Set}$ for $F$ in Proposition 2.8, then we get the functor $(-)_0 : V\text{-Cat} \to \text{Cat}$ taking underlying categories.

Now we consider a lax monoidal functor $G$ that is the right adjoint of a monoidal adjunction.

\textbf{Proposition 2.9} Let $F \dashv G : V \to W$ be a monoidal adjunction. Then for any $W$-category $D$, there is an isomorphism $(G(D))_0 \cong D_0$ of ordinary categories.

\textbf{Proof} Since $F$ is a monoidal functor, we have

\[ \text{Hom}_V(I, G(-)) \cong \text{Hom}_W(F(I), -) \cong \text{Hom}_W(I', -,) \],

which induces the isomorphism. \hfill \Box

\textbf{Proposition 2.10} Let $F \dashv G : V \to W$ be a monoidal adjunction. Then for $X \in V$ and $Y \in W$, there is a natural isomorphism

\[ G(W(F(X), Y)) \cong V(X, G(Y)) \].

\textbf{Proof} For any $Z \in V$, we have

\[ \text{Hom}_V(Z, G(W(F(X), Y))) \cong \text{Hom}_W(F(Z), W(F(X), Y)) \]

\[ \cong \text{Hom}_W(F(Z) \otimes F(X), Y) \]

\[ \cong \text{Hom}_W(F(Z \otimes X), Y) \]

\[ \cong \text{Hom}_V(Z \otimes X, G(Y)) \]

\[ \cong \text{Hom}_V(Z, V(X, G(Y))) \].

Hence we obtain

\[ G(W(F(X), Y)) \cong V(X, G(Y)) \]

by the Yoneda lemma. \hfill \Box

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Proposition 2.10 states that a monoidal adjunction \( F \dashv G \) becomes a \( \mathcal{V} \)-adjunction between \( \mathcal{V} \) and \( G(\mathcal{W}) \).

**Proposition 2.11** Let \( F \dashv G : \mathcal{V} \to \mathcal{W} \) be a monoidal adjunction. If a \( \mathcal{W} \)-category \( \mathcal{D} \) is cotensored (or tensored), then so is the \( \mathcal{V} \)-category \( G(\mathcal{D}) \).

**Proof** In the case where \( \mathcal{D} \) is tensored, by Proposition 2.10 we have
\[
\mathcal{V}(X, G(\mathcal{D})(C, D)) = \mathcal{V}(X, G(\mathcal{D}(C, D))) \\
\cong G(\mathcal{W}(F(X), \mathcal{D}(C, D))) \\
\cong G(\mathcal{D}(F(X) \otimes_{\mathcal{W}} C, D)) \\
= G(\mathcal{D})(F(X) \otimes_{\mathcal{W}} C, D)
\]
for \( X \in \mathcal{V} \) and \( C, D \in G(\mathcal{D}) \), where \( \otimes_{\mathcal{W}} \) denotes tensor product in the \( \mathcal{W} \)-category. Therefore the \( \mathcal{V} \)-category \( G(\mathcal{D}) \) admits the tensor product as \( X \otimes_{\mathcal{V}} C = F(X) \otimes_{\mathcal{W}} C \).

In a similar way, when \( \mathcal{D} \) is cotensored, we have
\[
\mathcal{V}(X, G(\mathcal{D})(C, D)) = \mathcal{V}(X, G(\mathcal{D}(C, D))) \\
\cong G(\mathcal{W}(F(X), \mathcal{D}(C, D))) \\
\cong G(\mathcal{D}(C, F(X) \triangleright_{\mathcal{D}} D)) \\
= G(\mathcal{D})(C, F(X) \triangleright_{\mathcal{D}} D).
\]
Therefore \( G(\mathcal{D}) \) admits the cotensor product as \( X \triangleright_{\mathcal{V}} C = F(X) \triangleright_{\mathcal{W}} C \).

**Corollary 2.12** Let \( F \dashv G : \mathcal{V} \to \mathcal{W} \) be a monoidal adjunction. If a \( \mathcal{W} \)-category \( \mathcal{D} \) is complete and cocomplete, then the \( \mathcal{V} \)-category \( G(\mathcal{D}) \) is so.

**Proof** If a \( \mathcal{W} \)-category \( \mathcal{D} \) is complete and cocomplete, then the underlying category \( \mathcal{D}_0 \), and hence \( (G(\mathcal{D}))_0 \) by Proposition 2.9, is complete and cocomplete as an ordinary category. On the other hand, Proposition 2.11 shows that \( G(\mathcal{D}) \) is cotensored and tensored. Therefore it follows from [20, Thm. 3.73] that \( G(\mathcal{D}) \) is complete and cocomplete as an enriched category.

**2.3 Generators and Strong Generators of Categories**

In this subsection, we recall the definition of a generator and a strong generator of an ordinary category, i.e. a Set-category. See for instance [3], though we use slightly different terminology.

Write \( \text{Psh}(\mathcal{S}) = [\mathcal{S}^{\text{op}}, \text{Set}] \) for the presheaf category on a small ordinary category \( \mathcal{S} \). We will refer to a nonempty set of objects \( \mathcal{S} \) of a category \( \mathcal{C} \) as a *family of objects*. We identify a family of objects \( \mathcal{S} \) with a small full subcategory spanned by \( \mathcal{S} \).

For a family of objects \( \mathcal{S} \) of an ordinary category \( \mathcal{C} \), let \( F : \mathcal{S} \to \mathcal{C} \) be the inclusion functor. We call \( \mathcal{S} \) a *generating set of objects* if the left Kan extension \( \text{Lan}_F y : \mathcal{C} \to \text{Psh}(\mathcal{S}) ; \quad c \mapsto \text{Hom}(- , c)|_{\mathcal{S}^{\text{op}}} \) is faithful. We also call \( \mathcal{S} \) a *strongly generating set of objects* if the left Kan extension \( \text{Lan}_F y : \mathcal{C} \to \text{Psh}(\mathcal{S}) \) is faithful and conservative. Recall that a functor is said to be *conservative* if it reflects isomorphisms. An object \( G \) is called a *(strong) generator* if \( \mathcal{S} = \{G\} \) is a (strongly) generating set of objects, respectively.

For cocomplete abelian categories, various characterizations of a generating set of objects are known as follows.

**Proposition 2.13** For a family of objects \( \mathcal{S} \) of a cocomplete abelian category \( \mathcal{A} \), the following conditions are equivalent.
(i) $S$ is a generating set of objects.
(ii) $S$ is a strongly generating set of objects.
(iii) The left Kan extension $\mathrm{Lan}_F y : A \to \mathcal{Psh}(S)$ is conservative, where $F : S \hookrightarrow A$ denotes the inclusion functor.
(iv) For any $A \in \mathcal{A}$, there is an exact sequence
\[
\bigoplus_{j \in J} s_j \to \bigoplus_{i \in I} s_i \to A \to 0
\]
such that each $s_i, s_j$ is in $S$.

2.4 Locally Finitely Presentable Bases and Finite Limits in Enriched Categories

In this subsection, we review the adaptation of the notion of finite limits from ordinary categories to enriched categories. Main references are [8, 21].

Recall that for a category $\mathcal{C}$ with filtered colimits, an object $C \in \mathcal{C}$ is said to be finitely presentable if the representable functor $\mathrm{Hom}_\mathcal{C}(C, -) : \mathcal{C} \to \text{Set}$ preserves filtered colimits. For example, finitely presentable objects of the category $\text{Set}$ of sets are nothing but finite sets; finitely presentable objects of the category $\text{Mod}(R)$ of modules over a commutative ring $R$ are nothing but finitely presented modules.

Definition 2.14 ([21], [8]) A symmetric monoidal closed category $\mathcal{V} = (V, \otimes, I)$ is called a locally finitely presentable base if it satisfies the following conditions.

(i) $\mathcal{V}$ is locally finitely presentable, which means that it is cocomplete and has a strongly generating set of finitely presentable objects.
(ii) The unit object $I$ is finitely presentable.
(iii) The tensor product $X \otimes Y$ of finitely presentable objects $X, Y \in \mathcal{V}$ is finitely presentable.

We write $\mathcal{V}_{fp}$ for the full subcategory of finitely presentable objects.

In [21], the category of Definition 2.14 is called a locally finitely presentable category as a closed category. We use the terminology of [8, Def. 1.1].

The cartesian closed category $\text{Set}$ of sets is an example of locally finitely presentable bases. Later in Sect. 3.2 we will observe that the monoidal categories $\text{Ab}$ of abelian groups and $\text{Ch}$ of complexes of abelian groups also serve as examples.

Definition 2.15 Let $\mathcal{V}$ be a locally finitely presentable base.

A $\mathcal{V}$-category $\mathcal{J}$ is said to be finite if $\mathrm{ob}(\mathcal{J})$ is a finite set and if for any pair $j, k \in \mathcal{J}$ of objects, the Hom object $\mathcal{J}(j, k) \in \mathcal{V}$ is finitely presentable. A $\mathcal{V}$-functor $W : \mathcal{J} \to \mathcal{V}$ is said to be finite if $\mathcal{J}$ is a finite $\mathcal{V}$-category and if for any object $j \in \mathcal{J}$, the image $W(j) \in \mathcal{V}$ is finitely presentable. We define a finite limit as a limit weighted by a finite $\mathcal{V}$-functor. We say that a $\mathcal{V}$-category $\mathcal{C}$ is finitely complete if it admits all finite limits.

A $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{D}$ is said to be left exact (or finitely continuous) if the domain $\mathcal{C}$ admits finite limits and $F$ preserves them.

Note that the cotensor product $X \triangleleft -$ with a finitely presentable object $X \in \mathcal{V}_{fp}$ is a finite limit. Finite colimits are defined dually.

Proposition 2.16 ([21, Prop. 4.3]) Let $\mathcal{V}$ be a locally finitely presentable base. A $\mathcal{V}$-category $\mathcal{C}$ admits all finite limits if and only if it admits both finite conical limits on ordinary finite categories and cotensor products with finitely presentable objects $X \in \mathcal{V}_{fp}$.
Proposition 2.17 Let \( V \) be a locally finitely presentable base and \( F : C \to D \) a \( V \)-functor between \( V \)-categories. Suppose that \( C \) admits finite limits. Then \( F \) is left exact if and only if it preserves both finite conical limits on ordinary finite categories and cotensor products with finitely presentable objects \( X \in V_{fp} \).

**Proof** This follows from Proposition 2.16. \( \Box \)

### 2.5 Dualizable Objects

In this subsection, we recall an important notion of finiteness for objects of monoidal categories. Let \( V = (V, \otimes, I, \ [-, -]) \) be a symmetric monoidal closed category.

**Definition 2.18** An object \( X \) of a symmetric monoidal closed category \( V \) is said to be dualizable if there is an isomorphism of functors

\[
[X, I] \otimes - \cong [X, -].
\]

We write \( V_{du} \) for the full subcategory of dualizable objects.

For example, dualizable objects of the symmetric monoidal closed category \( \text{Mod}(R) \) of modules over a commutative ring \( R \) are nothing but finitely generated projective modules.

**Remark 2.19** There are several equivalent definitions of dualizable objects; see [25, III§1, Thm. 1.6], [13, Thm. 1.3], and [15, Prop. 2.10.8]. One of them is the following: \( X \) is dualizable if there are an object \( Y \) and two morphisms \( \eta : I \to Y \otimes X, \epsilon : X \otimes Y \to I \) such that the compositions

\[
X \cong X \otimes I \xrightarrow{X \otimes \eta} X \otimes Y \otimes X \xrightarrow{\epsilon \otimes X} I \otimes X \cong X,
\]

\[
Y \cong I \otimes Y \xrightarrow{\eta \otimes Y} Y \otimes X \otimes Y \xrightarrow{Y \otimes \epsilon} Y \otimes I \cong Y
\]

are the identity morphisms. This shows that \( X \), viewed as a profunctor \( I \otimes I \cong I \xrightarrow{X} V \), is a left adjoint in the bicategory of profunctors. This observation is related to the following proposition.

**Proposition 2.20** Let \( V \) be a symmetric monoidal closed category. For \( X \in V \), the following conditions are equivalent.

1. \( X \) is dualizable.
2. For any \( V \)-category \( C \), the cotensor product \( X \triangleleft C \) with an object \( C \in C \) is an absolute limit, that is, preserved by arbitrary \( V \)-functors out of \( C \).
3. For any \( V \)-category \( C \), the tensor product \( X \otimes C \) with an object \( C \in C \) is an absolute colimit, that is, preserved by arbitrary \( V \)-functors out of \( C \).

**Proof** This assertion is a special case of [36]. Here we only verify the implication (i) \( \Rightarrow \) (ii). Before that, we recall from [20, (3.71)] that any \( V \)-functor \( F : \mathcal{J} \to C \) has the coend formula

\[
\int_{A \in \mathcal{J}} \mathcal{J}(A, J) \otimes FA \cong FJ,
\]

if \( C \) is cocomplete.
Let us consider a \( \mathcal{V} \)-functor \( S : C \to D \). For any \( D \in D \), we have by the above coend formula

\[
\mathcal{D}(D, S(X \dashv C)) \cong \int^{C' \in C} \mathcal{C}(C', X \dashv C) \otimes \mathcal{D}(D, SC')
\]
\[
\cong \int^{C' \in C} \mathcal{V}(X, C(C', C)) \otimes \mathcal{D}(D, SC')
\]
\[
\cong \int^{C' \in C} [X, I] \otimes C(C', C) \otimes \mathcal{D}(D, SC')
\]
\[
\cong [X, I] \otimes \int^{C' \in C} C(C', C) \otimes \mathcal{D}(D, SC')
\]
\[
\cong [X, I] \otimes \mathcal{D}(D, SC)
\]
\[
\cong \mathcal{V}(X, \mathcal{D}(D, SC))
\]
\[
\cong \mathcal{D}(D, X \dashv SC).
\]

These isomorphisms are natural in \( D \), and hence we have \( S(X \dashv C) \cong X \dashv SC \) by the Yoneda lemma.

\[\square\]

3 Grothendieck Enriched Categories

3.1 The Gabriel-Popescu Theorem

We first recall the definition of Grothendieck categories.

**Definition 3.1** A *Grothendieck category* \( \mathcal{A} \) is an abelian category that possesses the following properties:

(i) it admits all small colimits;

(ii) filtered colimits are exact;

(iii) it has a generator \( G \), which means that the functor \( \mathcal{A}(G, -) : \mathcal{A} \to \text{Ab} \) is faithful.

Remember that an abelian category has the unique Ab-enrichment. Examples of Grothendieck categories are the category \( \text{Mod}(R) \) of modules over a ring \( R \) and the category \( \text{Qcoh}(X) \) of quasi-coherent sheaves on a scheme \( X \) (see [34, Tag 077K]).

**Remark 3.2** Note that replacing a generator with a generating set of objects in the definition of Grothendieck categories makes no difference. More precisely, in a cocomplete abelian category, the existence of a generator is equivalent to that of a generating set of objects. Indeed, given a generating set \( S = \{s_i\}_{i \in I} \) of objects, we can verify that \( G := \bigoplus_{i \in I} s_i \) is a generator.

Before stating a slightly generalized version of the Gabriel-Popescu theorem [30], we remark that a cocomplete abelian category \( \mathcal{A} \) is also cocomplete as an Ab-enriched category (see [20, Prop. 3.76]), so that an additive functor \( F : C \to \mathcal{A} \) yields the nerve-and-realization adjunction

\[
\text{Lan}_y F \dashv \text{Lan}_F y : \text{Mod}(C) \to \mathcal{A}
\]

by Theorem 2.4.
**Theorem 3.3** (The Gabriel-Popescu theorem [24, 30]) Let \( A \) be a Grothendieck category that has a generating set \( C \) of objects. We regard \( C \) as a full preadditive subcategory and write \( F : C \hookrightarrow A \) for the inclusion additive functor. Then the following assertions on the nerve-and-realization adjunction associated with \( F \) hold.

(i) The right adjoint \( \text{Lan}_F y \) is fully faithful.

(ii) The left adjoint \( \text{Lan}_y F \) is left exact.

This form of the Gabriel-Popescu theorem is given in [24, Thm. 2.1]; see also [26]. There is an analogous result for algebraic categories ([37]).

The Gabriel-Popescu theorem is an extrinsic characterization of Grothendieck categories. We generalize this theorem to enriched categories.

### 3.2 Grothendieck Cosmoi

Sometimes a complete and cocomplete symmetric monoidal closed category \( V = (V, \otimes, I) \) is called a *cosmos*. If moreover \( V \) is a Grothendieck category, then we will call it a *Grothendieck cosmos*. For a Grothendieck cosmos \( V \), let us consider a kind of finiteness conditions as follows, which are adapted from [19]:

(C1) the unit object \( I \) is finitely presentable;

(C2) \( V \) has a generating set \( \{g_j\}_{j \in J} \) of dualizable objects.

Note that \( \{g_j\}_{j \in J} \) is strongly generating by Proposition 2.13.

**Lemma 3.4** ([19, Lem. 6.7]) Let \( V \) be a Grothendieck cosmos. If \( X, Y \in V \) are dualizable, then so are \( X \oplus Y, X \otimes Y, \) and \( [X, Y] \).

**Proposition 3.5** ([19, Prop. 6.9 (a)]) If a Grothendieck cosmos \( V \) satisfies the conditions (C1) and (C2), then it is a locally finitely presentable base.

**Proof** We begin by proving that dualizable objects of \( V \) are finitely presentable. Take a dualizable object \( X \in V \). Then we have composites of isomorphisms

\[
\text{Hom}_V(X, -) \cong \text{Hom}_V(I \otimes X, -) \cong \text{Hom}_V(I, [X, -]) \cong \text{Hom}_V(I, [X, I] \otimes -).
\]

Since \( I \) is finitely presentable, \( \text{Hom}_V(I, -) \) preserves filtered colimits. Hence \( \text{Hom}_V(X, -) \) also does, which shows that \( X \) is finitely presentable. From this, we can conclude that \( V \) has a strongly generating set of finitely presentable objects and hence is locally finitely presentable.

Second, we claim that \( X \in V \) is finitely presentable if and only if there is an exact sequence

\[
P_1 \to P_0 \to X \to 0
\]

with \( P_1, P_0 \) dualizable. Indeed, if \( X \) is finitely presentable, it follows from [35, Prop. V.3.4] that there is an exact sequence in Proposition 2.13 (iv) where the indexing sets \( I, J \) are finite. Since by Lemma 3.4 finite direct products of dualizable objects are dualizable again, we have a desired sequence. On the other hand, if \( X \) has a presentation like \((*) \), dualizable objects \( P_0, P_1, \) and hence \( X \), are finitely presentable.

It remains to check that monoidal products of finitely presentable objects are finitely presentable. If \( X, Y \in V \) are finitely presentable, there exist two exact sequences

\[
P_1 \to P_0 \to X \to 0, \quad Q_1 \to Q_0 \to Y \to 0
\]
with \( P_1, P_0, Q_1, Q_0 \) dualizable. Then it follows from [6, Chapter II §3.6 Prop. 6] that
\[
(P_1 \otimes Q_0) \oplus (P_0 \otimes Q_1) \to P_0 \otimes Q_0 \to X \otimes Y \to 0
\]
is exact. Because both \((P_1 \otimes Q_0) \oplus (P_0 \otimes Q_1)\) and \(P_0 \otimes Q_0\) are dualizable, \(X \otimes Y\) is finitely presentable. \( \square \)

**Example 3.6** The category \( \text{Mod}(R) \) of modules over a commutative ring \( R \) is a Grothendieck category with \( R \) a generator and becomes a symmetric monoidal closed category with the ordinary tensor product of modules. This Grothendieck cosmos \( \text{Mod}(R) \) satisfies the conditions (C1) and (C2).

**Example 3.7** For a ringed space \((X, O_X)\), let \( \text{Mod}(X) \) denote the category of sheaves of \( O_X \)-modules on \( X \). For an open subset \( U \subseteq X \), let a sheaf \( S_U \) be the extension by zero of \( O_U = O_X|_U \). Then \( \text{Mod}(X) \) is a Grothendieck category with \( \{ S_U \mid U \subseteq X \text{ is open} \} \) a generating set of objects ([26]) and becomes a symmetric monoidal closed category with the tensor product \( \otimes_X \) of \( O_X \)-modules and Hom sheaf \( \mathcal{H}om_X \).

If \( X \) has a base of compact open subsets, then \( \text{Mod}(X) \) is a locally finitely presentable category with \( \{ S_U \mid U \text{ is compact and open} \} \) a generating set of finitely presentable objects. However, unless \( X \) itself is compact, \( O_X \) is not finitely presentable and hence \( \text{Mod}(X) \) does not satisfy the condition (C1) ([32]).

**Example 3.8** The category \( \text{Qcoh}(X) \) of quasi-coherent sheaves on a scheme \( X \) is a Grothendieck category ([34, Tag 077K]). The inclusion functor \( \text{Qcoh}(X) \to \text{Mod}(X) \) has a right adjoint \( Q_X \), which is called the coherator ([34, Tag 08D6]). Then the tensor product of \( O_X \)-modules together with \( \mathcal{H}om_X^{\text{qc}} := Q_X \mathcal{H}om_X \) makes \( \text{Qcoh}(X) \) into a symmetric monoidal closed category.

If \( X \) is quasi-compact and quasi-separated, then \( \text{Qcoh}(X) \) satisfies the condition (C1). Furthermore \( \text{Qcoh}(X) \) also fulfills the condition (C2) if \( X \) is projective (see [19, Example 6.3]).

**Example 3.9** For a commutative ring \( R \), let \( \text{Ch}(R) \) be the category of cochain complexes of \( R \)-modules. Define the *sphere complex* \( S(R) \) and the *disk complex* \( D(R) \) to be complexes concentrated in degree 0 and in degrees \(-1\) and 0, respectively, as follows:
\[
S(R) : \quad \cdots \to 0 \to 0 \to R \to 0 \to \cdots,
\]
\[
D(R) : \quad \cdots \to 0 \to R \xrightarrow{id} R \to 0 \to \cdots.
\]
Then \( \text{Ch}(R) \) is a Grothendieck category with \( \{ D(R)[n] \mid n \in \mathbb{Z} \} \) a generating set of objects. It also forms a symmetric monoidal closed category \( \langle \text{Ch}(R), \otimes^R, S(R), \text{Hom}_R^\bullet \rangle \), where \( \otimes^R \) is the total tensor product of complexes.

It is not hard to see that
\[
\text{Ch}(R)_{\text{fp}} = \{ X \in \text{Ch}(R) \mid X \text{ is bounded and each term } X^n \text{ is finitely presentable} \}
\]
and [13, Prop. 1.6] shows that
\[
\text{Ch}(R)_{\text{du}} = \{ X \in \text{Ch}(R) \mid X \text{ is bounded and each term } X^n \text{ is finitely generated and projective} \}.
\]
In particular, \( S(R) \) and \( D(R) \) are finitely presentable and dualizable. Therefore the Grothendieck cosmos \( \langle \text{Ch}(R), \otimes^R \rangle \) satisfies the conditions (C1) and (C2).
Example 3.10 More generally, for any Grothendieck cosmos $\mathcal{V}$, the category $\text{Ch}(\mathcal{V})$ of cochain complexes in $\mathcal{V}$ becomes a Grothendieck cosmos with the total tensor product and the total Hom complex whose unit object is the sphere complex $S(I)$ of the unit object $I$ of $\mathcal{V}$ (See [17, Thm. 3.2]). If $\mathcal{V}$ has a generating set $\{g_j\}_{j \in J}$ of objects, then $\{D(g_j)[n] \mid j \in J, n \in \mathbb{Z}\}$ forms a generating set of objects in $\text{Ch}(\mathcal{V})$.

If $\mathcal{V}$ satisfies the conditions (C1) and (C2), then $\text{Ch}(\mathcal{V})$ also does. In fact, $S(X) \in \text{Ch}(\mathcal{V})$ is finitely presentable if $X \in \mathcal{V}$ is so, and $D(X)[n] \in \text{Ch}(\mathcal{V})$ is dualizable if $X \in \mathcal{V}$ is so.

Example 3.11 For a commutative ring $R$, we can define another tensor product on $\text{Ch}(R)$ ([16]). For two complexes $M$ and $N$, the modified tensor product $M \otimes^\bullet_R N$ is defined as the complex such that

$$(M \otimes^\bullet_R N)^n \rightarrow (M \otimes^\bullet_R N)^{n+1}, \quad x \otimes y \mapsto d_M(x) \otimes y.$$ 

The modified Hom complex $\underline{\text{Hom}}^\bullet_R(M, N)$ is the complex such that

$$\text{Hom}_{\text{Ch}(R)}(M, N[n]) \rightarrow \text{Hom}_{\text{Ch}(R)}(M, N[n+1]), \quad f \mapsto ((-1)^n d_N \circ f^m)_{m \in \mathbb{Z}}.$$ 

Then $\text{Ch}(R)$ becomes a symmetric monoidal closed category $(\text{Ch}(R), \otimes^\bullet_R, D(R), \underline{\text{Hom}}^\bullet_R)$. Each $D(R)[n]$ is dualizable also in this monoidal category, and the Grothendieck cosmos $(\text{Ch}(R), \otimes^\bullet_R)$ satisfies the conditions (C1) and (C2).

3.3 Grothendieck $\mathcal{V}$-enriched Categories

Let $\mathcal{V}$ be a cosmos, i.e., a complete and cocomplete symmetric monoidal closed category.

We now introduce the notion of generators in the enriched setting.

Definition 3.12 Let $\mathcal{A}$ be a $\mathcal{V}$-category. We say that a full subcategory $\mathcal{C}$ is a $\mathcal{V}$-generating set of objects if it is small and the underlying functor $(\text{Lan}_F y)_0$ of the left Kan extension of the Yoneda embedding $y: \mathcal{C} \rightarrow [\mathcal{C}^{\text{op}}, \mathcal{V}]$ along the inclusion $\mathcal{V}$-functor $F: \mathcal{C} \leftarrow \mathcal{A}$ is faithful.

Proposition 3.13 Let $F: \mathcal{C} \rightarrow \mathcal{A}$ be a $\mathcal{V}$-functor and suppose $\mathcal{C}$ is small. Write the full subcategory of images of $F$ as $\mathcal{A}' = \{Fc \mid c \in \mathcal{C}\}$. If $(\text{Lan}_F y_\mathcal{C})_0: \mathcal{A}_0 \rightarrow [\mathcal{C}^{\text{op}}, \mathcal{V}]_0$ is faithful, then $\mathcal{A}'$ is a $\mathcal{V}$-generating set of objects in $\mathcal{A}$.

Proof Since $\mathcal{C}$ is small, $\mathcal{A}'$ is also small. If $i: \mathcal{A}' \leftarrow \mathcal{A}$ denotes the inclusion functor, then we have

$$\text{Lan}_F y_\mathcal{C} \cong F^* \circ \text{Lan}_i y_{\mathcal{A}'}: \mathcal{A} \rightarrow [\mathcal{A}_0^{\text{op}}, \mathcal{V}] \rightarrow [\mathcal{C}^{\text{op}}, \mathcal{V}].$$

By assumption, $(\text{Lan}_F y_\mathcal{C})_0$, and hence $(\text{Lan}_i y_{\mathcal{A}'})_0$, is faithful. This shows that $\mathcal{A}'$ is a $\mathcal{V}$-generating set of objects in $\mathcal{A}$.

Proposition 3.14 Let $\mathcal{A}$ be a $\mathcal{V}$-category and $\mathcal{C}$ a full subcategory. If $\mathcal{C}_0$ is a generating set of objects in $\mathcal{A}_0$, then $\mathcal{C}$ is a $\mathcal{V}$-generating set of objects in $\mathcal{A}$.

Proof Let $F: \mathcal{C} \leftarrow \mathcal{A}$ be the inclusion functor. By Proposition 2.3, the underlying functor $(\text{Lan}_F y)_0: \mathcal{A}_0 \rightarrow \text{Fun}(\mathcal{C}_0^{\text{op}}, \mathcal{V})$ of the left Kan extension is given by the following correspondences:

$$\mathcal{A}_0 \ni A \mapsto (\text{Lan}_F y)_0(A) = \mathcal{A}(-, A)|_{\mathcal{C}},$$

$$\mathcal{A}_0(A, B) \ni f \mapsto (\text{Lan}_F y)_0(f) = f \circ -: \mathcal{A}(-, A)|_{\mathcal{C}} \rightarrow \mathcal{A}(-, B)|_{\mathcal{C}}.$$
For $c \in C_0$, we have $A_0(c, f) = U(A(c, f)) = U(f \circ -)$. In order to prove $(\Lan_F y)_0$ is faithful, we assume $(\Lan_F y)_0(f) = (\Lan_F y)_0(g)$ for any pair $f, g \in A_0(A, B)$. Then we have $A_0(c, f) = A_0(c, g)$ for all $c \in C_0$, which yields $f = g$ since $C_0$ is a generating set of objects of $A_0$. Thus $C$ is a $\mathcal{V}$-generating set of objects.

Inspired by the Gabriel-Popescu theorem, we define an enriched version of Grothendieck categories as follows.

**Definition 3.15** Let $\mathcal{V}$ be a locally finitely presentable base. A $\mathcal{V}$-category $\mathcal{A}$ is said to be a Grothendieck $\mathcal{V}$-category if there exist a small $\mathcal{V}$-category $\mathcal{C}$ and a $\mathcal{V}$-adjunction

$$
S : [\mathcal{C}^{\text{op}}, \mathcal{V}] \dashv \dashv A : T
$$

such that

(i) the right adjoint $T$ is fully faithful and

(ii) the left adjoint $S$ is left exact.

Recall from Proposition 3.5 that Grothendieck cosmoi with the conditions (C1) and (C2) are locally finitely presentable bases.

**Remark 3.16** Definition 3.15 also appears in [18] under the name of $\mathcal{V}$-topoi. In [18] a full subcategory of a ($\mathcal{V}$-)category whose inclusion functor has a left exact left adjoint, as in Definition 3.15, is called being lex-reflective or a localization. The study of localizations of a category has a long history; see the answers [11, 12] to questions on mathoverflow for a list of references.

**Remark 3.17** If we take $\mathcal{V} = \text{Ab}$, then the Grothendieck $\text{Ab}$-categories are ordinary Grothendieck categories and if $\mathcal{V} = \text{Set}$, then the Grothendieck $\text{Set}$-categories are Grothendieck topoi, the categories of sheaves on sites. This indicates that Grothendieck categories can be seen as the categories of linear sheaves on linear sites. In fact, Gabriel topologies are precisely the $\mathbb{Z}$-linear version of Grothendieck topologies. Lowen [27] and Ramos González [33] study Grothendieck categories from the viewpoint of the theory of linear sheaves. See also [14, Sect. 2.1]. The additive analogue of pretopology is considered in [9].

For a locally finitely presentable base $\mathcal{V}$, the $\mathcal{V}$-category $\mathcal{V}$ itself and the presheaf category $[\mathcal{C}^{\text{op}}, \mathcal{V}]$ on a small $\mathcal{V}$-category $\mathcal{C}$ are Grothendieck $\mathcal{V}$-categories. Note that since Grothendieck $\mathcal{V}$-categories are reflective full subcategories of presheaf categories, they are complete and cocomplete by [20, Prop. 3.75].

Definition 3.15 of Grothendieck $\mathcal{V}$-categories is extrinsic. On the other hand, under the assumption that $\mathcal{V}$ is a nice Grothendieck cosmos, the following theorem gives an intrinsic characterization.

**Theorem 3.18** Let $\mathcal{V}$ be a Grothendieck cosmos with the conditions (C1) and (C2). Then a $\mathcal{V}$-category $\mathcal{A}$ is a Grothendieck $\mathcal{V}$-category if and only if it fulfills the following conditions.

1. $\mathcal{A}$ is cocomplete.
2. $\mathcal{A}$ is finitely complete.
3. The homomorphism theorem holds in $\mathcal{A}$, which means that for any morphism $f$ in $A_0$ the canonical map $\text{Cok}(\text{Ker}(f)) \to \text{Ker}(\text{Cok}(f))$ is an isomorphism.
4. Conical filtered colimits are left exact, which means that for any ordinary filtered category $\mathcal{J}$, the colimit $\mathcal{V}$-functor $\text{colim}_F : [\mathcal{J}, \mathcal{A}] \to \mathcal{A}$ preserves finite limits.

\( \odot \) Springer
(5) $\mathcal{A}$ has a $\mathcal{V}$-generating set $C$ of objects.

Note that $\mathcal{A}_0$ has the natural $\text{Ab}$-enriched structure. Hence we can make sense of the (co)kernel $\text{Ker}$ (resp. $\text{Cok}$) of a morphism as the (co)equalizer with the zero morphism, whose existence is guaranteed by the finite (co)completeness of $\mathcal{A}$.

**Remark 3.19** In [18, Prop. 2.6], Garner and Lack give a characterization of $\mathcal{V}$-topoi as small-exact $\mathcal{V}$-categories with a small dense subcategory, where $\mathcal{V}$ is a locally finitely presentable base. Relations of Theorem 3.18 to Garner–Lack’s result will be discussed in Sect. 3.4.

**Proposition 3.20** ([1, Thm. 4.2]) Let $\mathcal{V}$ be a Grothendieck cosmos with $\{g_j\}_{j \in J}$ a generating set of objects. For a small $\mathcal{V}$-category $\mathcal{C}$, the ordinary category $\text{Fun}(\mathcal{C}, \mathcal{V})$ of $\mathcal{V}$-functors is a Grothendieck category with $\{g_j \otimes \mathcal{C}(c, -) \mid j \in J, c \in \mathcal{C}\}$ a generating set of objects.

**Lemma 3.21** Let $\mathcal{V}$ be a Grothendieck cosmos with the conditions (C1) and (C2). For a finitely complete $\mathcal{V}$-category $\mathcal{C}$, a $\mathcal{V}$-functor $S: \mathcal{C} \to \mathcal{D}$ is left exact if and only if its underlying functor $S_0: \mathcal{C}_0 \to \mathcal{D}_0$ is left exact.

**Proof** It is obvious that if $S$ is left exact, then so is $S_0$.

If $S_0$ is left exact, then $S$ preserves conical finite limits on ordinary finite categories. Hence, it is sufficient by Proposition 2.17 to show that $S$ preserves cotensor products $X \otimes c$ with $X \in V_{fp}$ and $c \in C$. From the proof of Proposition 3.5, for any $X \in V_{fp}$ there is an exact sequence

$$P_1 \to P_0 \to X \to 0$$

with $P_1, P_0$ dualizable in $\mathcal{V}$. Since the cotensor product forms the $\mathcal{V}$-adjunction $\mathcal{C}(-, c) \dashv - \otimes c$, the right adjoint $- \otimes c: \mathcal{V}^{\text{op}} \to \mathcal{C}$ preserves limits, and so we have the exact sequence

$$0 \to X \otimes c \to P_0 \otimes c \to P_1 \otimes c.$$

From the left exactness of $S_0$, we also obtain the exact sequence

$$0 \to S(X \otimes c) \to S(P_0 \otimes c) \to S(P_1 \otimes c).$$

Now, by Proposition 2.20, the cotensor products $P_0 \otimes c, P_1 \otimes c$ are preserved by $S$ because $P_0, P_1$ are dualizable. Thus

$$0 \to S(X \otimes c) \to P_0 \otimes S(c) \to P_1 \otimes S(c)$$

is also exact, yielding $S(X \otimes c) \cong X \otimes S(c)$. This proves that $S$ is left exact. \qed

Now we are ready to give the Proof of Theorem 3.18.

**Proof of Theorem 3.18** In order to prove the necessity, let $\mathcal{A}$ be a Grothendieck $\mathcal{V}$-category and $S \dashv T: [C^{\text{op}}, \mathcal{V}] \to \mathcal{A}$ the $\mathcal{V}$-adjunction of Definition 3.15. Note that $\mathcal{A}$ is complete and cocomplete since it is a reflective full subcategory of the presheaf category $[C^{\text{op}}, \mathcal{V}]$. Theorem 2.6 leads to $T \cong \text{Lan}_F y$, where $F:=S \circ y: \mathcal{C} \to \mathcal{A}$. In particular, $T_0 \cong (\text{Lan}_F y)_0$ is faithful. Thus, using Proposition 3.13, we see that $\mathcal{A}$ has $\mathcal{A}' = \{Fc \mid c \in \mathcal{C}\}$ as a $\mathcal{V}$-generating set of objects.

The underlying category $\mathcal{A}_0$ is a Giraud subcategory of the Grothendieck category $\text{Fun}(C^{\text{op}}, \mathcal{V})$, so it follows from [35, Prop. X.1.3] that $\mathcal{A}_0$ is also a Grothendieck category. Note that the homomorphism theorem holds in $\mathcal{A}$ and filtered colimits are left exact. Lemma 3.21 shows that conical filtered colimits in $\mathcal{A}$ are left exact as well.
It remains to prove the sufficiency. Let \( F: \mathcal{C} \hookrightarrow \mathcal{A} \) denote the inclusion functor of the \( \mathcal{V} \)-generating set \( \mathcal{C} \) of objects. The cocompleteness of \( \mathcal{A} \) leads by Theorem 2.4 to the \( \mathcal{V} \)-adjunction, the nerve-and-realization adjunction associated with \( F \), as in the following diagram

\[
\begin{array}{ccc}
[C^{\text{op}}, \mathcal{V}] & \xrightarrow{\text{Lan}_F} & \mathcal{A} \\
\text{Lan}_y F & & \\
\text{Lan}_F y & \xleftarrow{F} & \\
\mathcal{C} & \xrightarrow{\text{Lan}_F} & \mathcal{A}.
\end{array}
\]

If we write \( S = \text{Lan}_y F \) and \( T = \text{Lan}_F y \), then we want to show that \( S \) is left exact and that \( T \) is fully faithful. For this purpose, it is sufficient by Lemma 3.21 and the fact\(^2\) in [20, §1.11] to observe that \( S_0 \) left exact and \( T_0 \) is fully faithful. Note that \( T_0 \) is faithful since \( \mathcal{C} \) is a \( \mathcal{V} \)-generating set of objects.

We recall from Proposition 3.20 that \( \text{Fun}(C^{\text{op}}, \mathcal{V}) \) is a Grothendieck category that has \( \{g_j \otimes \mathcal{C}(-, c) \mid j \in J, c \in \mathcal{C}\} \) as a generating set of objects. Let \( G \) be the full \( \mathcal{V} \)-subcategory of \( [C^{\text{op}}, \mathcal{V}] \) whose objects are \( \{g_j \otimes \mathcal{C}(-, c) \mid j \in J, c \in \mathcal{C}\} \), and \( G: G_0 \hookrightarrow \text{Fun}(C^{\text{op}}, \mathcal{V}) \) be its inclusion \( \mathcal{V} \)-functor. Then the underlying functor \( G_0: G_0 \hookrightarrow \text{Fun}(C^{\text{op}}, \mathcal{V}) \) becomes an Ab-functor, with which we get the associated nerve-and-realization Ab-adjunction \( S' \dashv T' \).

The Gabriel-Popescu Theorem 3.3 states that the left adjoint \( S' \) is left exact and the right adjoint \( T' \) is fully faithful:

\[
\begin{array}{ccc}
\text{Mod}(G_0) & \xleftarrow{S'} & \text{Fun}(C^{\text{op}}, \mathcal{V}) \\\n\text{G}_0 & \xrightarrow{G_0} & \text{Fun}(C^{\text{op}}, \mathcal{V}) \\
\end{array}
\]

Now the composite \( S \circ G: G \to [C^{\text{op}}, \mathcal{V}] \to \mathcal{A} \) of \( \mathcal{V} \)-functors is fully faithful. Indeed, for each object \( g_j \otimes c \) of \( G \) we have

\[ S \circ G(g_j \otimes \mathcal{C}(-, c)) \cong g_j \otimes S(\mathcal{C}(-, c)) \cong g_j \otimes c. \]

Thus

\[
\begin{align*}
\mathcal{G}(g_j \otimes \mathcal{C}(-, c), g_j' \otimes \mathcal{C}(-, c')) \\
= [C^{\text{op}}, \mathcal{V}](g_j \otimes \mathcal{C}(-, c), g_j' \otimes \mathcal{C}(-, c')) \\
\cong \mathcal{V}(g_j, [C^{\text{op}}, \mathcal{V}](\mathcal{C}(-, c), g_j' \otimes \mathcal{C}(-, c'))) \\
\cong \mathcal{V}(g_j, g_j' \otimes \mathcal{C}(c, c')) \quad & \text{by the definition of } g_j \otimes \mathcal{C}(-, c), \\
\cong \mathcal{V}(g_j, g_j' \otimes \mathcal{A}(c, c')) \\
\cong \mathcal{V}(g_j, \mathcal{A}(c, g_j' \otimes c')) \quad & \text{by Proposition 2.20,} \\
\cong \mathcal{A}(g_j \otimes c, g_j' \otimes c') \quad & \text{by the definition of } g_j \otimes c, \\
\cong \mathcal{A}(S \circ G(g_j \otimes \mathcal{C}(-, c)), S \circ G(g_j' \otimes \mathcal{C}(-, c'))).
\end{align*}
\]

\(^2\) A right enriched adjoint is fully faithful if and only if its underlying functor is so.
Hence the underlying functor $H = S_0 \circ G_0$ is also fully faithful, so $G_0$ can be regarded as a full subcategory $\{g_j \otimes c \mid j \in J, c \in C\}$ of $A_0$ via $H$. Under this identification, $G_0$ is a generating set of objects of $A_0$, because in the diagram

$$
\begin{array}{ccc}
    & \text{Mod}(G_0) & \\
    S' & \downarrow & \downarrow \\
    T' & \downarrow & \downarrow \\
    \text{Fun}(C^{\text{op}}, V) & \rightarrow & A_0 \\
    \text{H} & \rightarrow & \text{H} \\
\end{array}
$$

the composite $S_0 \circ S' \circ y$ is isomorphic to $H : G_0 \rightarrow A_0$ by Proposition 2.5 and hence we have $\text{Lan}_H y \cong T' \circ T_0$, from which we see $\text{Lan}_H y$ is faithful. By assumption it follows that $A_0$ becomes an abelian category with Grothendieck’s condition (AB5). Thus it is a Grothendieck category with $G_0$ a generating set of objects. Using the Gabriel-Popescu theorem again, we conclude that $S_0 \circ S' \cong \text{Lan}_y H$ is left exact and that $T' \circ T_0 \cong \text{Lan}_H y$ is fully faithful.

Now we are able to observe that $S_0$ is left exact and $T_0$ is fully faithful. Since $T'$ and $T' \circ T_0$ are fully faithful, so is $T_0$. Next, consider a finite limit $\lim_k P_k$ in $\text{Fun}(C^{\text{op}}, V)$. Now that $\text{Fun}(C^{\text{op}}, V)$ is a reflective full subcategory of $\text{Mod}(G)$, we have $\lim_k P_k \cong S'(\lim_k T'(P_k))$. Then by the left exactness of $S_0 \circ S'$ it follows that

$$
S_0(\lim_k P_k) \cong S_0 \circ S'(\lim_k T'(P_k)) \cong \lim_k S_0 \circ S'(T'(P_k)) \cong \lim_k S_0(P_k),
$$

which proves $S_0$ is left exact. Therefore we obtain the desired conclusion.

As an application of the intrinsic characterization of Grothendieck enriched categories we obtained, we show that the property of being a Grothendieck enriched category is preserved by the change-of-base functor associated to the right adjoint of a monoidal adjunction.

**Proposition 3.22** Let $\mathcal{V}, \mathcal{W}$ be Grothendieck cosmoi and $F \dashv G : \mathcal{V} \rightarrow \mathcal{W}$ a monoidal adjunction. Suppose that $\mathcal{V}$ satisfies the conditions (C1) and (C2). If a $\mathcal{W}$-category $B$ is a Grothendieck $\mathcal{W}$-category, then the $\mathcal{V}$-category $G(B)$ is a Grothendieck $\mathcal{V}$-category.

**Proof** From the proof of the necessity of Theorem 3.18, the Grothendieck $\mathcal{W}$-category $B$ is complete and cocomplete and its underlying category $B_0$ is a Grothendieck category.

Let us check that the $\mathcal{V}$-category $G(B)$ satisfies the conditions (1)–(5) in Theorem 3.18. Since $B$ is complete and cocomplete, Corollary 2.12 yields that $G(B)$ is also complete and cocomplete. As is seen in Proposition 2.9, there is an isomorphism $(G(B))_0 \cong B_0$ of categories. Thus the homomorphism theorem holds, and by Lemma 3.21 conical filtered colimits are left exact in $G(B)$. Moreover $(G(B))_0$ has a generating set of objects, and hence $G(B)$ has a $\mathcal{V}$-generating set of objects by Proposition 3.14. Therefore it follows from Theorem 3.18 that $G(B)$ is a Grothendieck $\mathcal{V}$-category.

**Example 3.23** Let $X, Y$ be schemes and $f : X \rightarrow Y$ a quasi-compact and quasi-separated morphism between them. We also suppose that the Grothendieck cosmos $\text{Qcoh}(Y)$ satisfies the conditions (C1) and (C2); this is the case if $Y$ is projective by Example 3.8. Since $f$
is quasi-compact and quasi-separated, the pushout functor \( f_*: \text{Mod}(\mathcal{O}_X) \to \text{Mod}(\mathcal{O}_Y) \) preserves the quasi-coherence, inducing a functor \( f_*: \text{Qcoh}(X) \to \text{Qcoh}(Y) \). Thus we have an adjunction

\[
 f^*: \text{Qcoh}(Y) \rightleftarrows \text{Qcoh}(X): f_*
\]

which turns out to be a monoidal adjunction. If \( \text{Ch}(\text{Qcoh}(X)) \) denotes the category of complexes of objects of \( \text{Qcoh}(X) \), then it is also a Grothendieck cosmos with the total tensor product and the total internal Hom (Example 3.10).

If we view \( \text{Qcoh}(X) \) itself as the \( \text{Qcoh}(X) \)-category \( \mathcal{B} = \text{Qcoh}(X) \), then it is clearly a Grothendieck \( \text{Qcoh}(X) \)-category. Hence, by Proposition 3.22, the \( \text{Qcoh}(Y) \)-category \( \mathcal{A} := f_*(\mathcal{B}) \) is a Grothendieck \( \text{Qcoh}(Y) \)-category. Recall that \( \mathcal{A} \) here is the \( \text{Qcoh}(Y) \)-category whose objects are quasi-coherent sheaves on \( X \) and whose Hom objects are \( f_*\text{Hom}_{\mathcal{X}}^{\text{qc}}(\mathcal{F}, \mathcal{G}) \in \text{Qcoh}(Y) \).

Furthermore, the monoidal adjunction \( f^*: \text{Qcoh}(Y) \to \text{Qcoh}(X) \) induces the monoidal adjunction \( f^*: \text{Ch}(\text{Qcoh}(Y)) \to \text{Ch}(\text{Qcoh}(X)) \) between categories of complexes, and the Grothendieck cosmos \( \text{Ch}(\text{Qcoh}(Y)) \) also satisfies the conditions (C1) and (C2) by Example 3.10.

Regarding \( \text{Ch}(\text{Qcoh}(X)) \) itself as a \( \text{Ch}(\text{Qcoh}(X)) \)-category \( \mathcal{B}' = \text{Ch}(\text{Qcoh}(X)) \), we also see that it is a Grothendieck \( \text{Ch}(\text{Qcoh}(X)) \)-category, and hence by Proposition 3.22 the \( \text{Ch}(\text{Qcoh}(Y)) \)-category \( \mathcal{A}' := f_*(\mathcal{B}') \) is a Grothendieck \( \text{Ch}(\text{Qcoh}(Y)) \)-category. Note that \( \mathcal{A}' \) has complexes of quasi-coherent sheaves on \( X \) as objects and \( f_*\text{Hom}_{\mathcal{X}}^{\text{qc}}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \in \text{Ch}(\text{Qcoh}(Y)) \) as Hom objects.

**Example 3.24** As a special case of Example 3.23, consider a quasi-compact and quasi-separated scheme \( f: X \to \text{Spec}(R) \) over a commutative ring \( R \) (for example, separated schemes of finite type over a field). Via the equivalence \( \text{Mod}(R) \simeq \text{Qcoh}(\text{Spec}(R)) \) of categories, the pushout functor \( f_* \) is naturally isomorphic to the global section functor \( \Gamma(X, -): \text{Qcoh}(X) \to \text{Mod}(R) \). Then we can verify \( f_*\text{Hom}_{\mathcal{X}}^{\text{qc}}(\mathcal{F}, \mathcal{G}) \simeq \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \).

Define a \( \text{Ch}(R) \)-category (i.e., a dg category over \( R \)) \( \mathcal{A} \) as follows: objects of \( \mathcal{A} \) are complexes of quasi-coherent sheaves on \( X \) and Hom complexes of \( \mathcal{A} \) are \( \mathcal{A}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \in \text{Ch}(R) \) such that

\[
\mathcal{A}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)^n = \prod_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{O}_X}(\mathcal{F}^i, \mathcal{G}^{i+n}),
\]

\[
d^n(h) = \{d_G^{i+n} \circ h^i - (-1)^n h^{i+1} \circ d_{\mathcal{F}^\bullet}\}_{i \in \mathbb{Z}}.
\]

In other words,

\[
\mathcal{A}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq f_*(\text{Hom}_{\mathcal{X}}^{\text{qc}}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)).
\]

Then \( \mathcal{A} \simeq f_*(\text{Ch}(\text{Qcoh}(X))) \) holds and it follows from Proposition 3.22 that \( \mathcal{A} \) is a Grothendieck dg category over \( R \).

We can prove that the adjoint functor theorem holds for Grothendieck \( \mathcal{V} \)-categories. This indicates the usefulness of Grothendieck \( \mathcal{V} \)-categories.

**Proposition 3.25** Let \( \mathcal{V} \) be a locally finitely presentable base and \( F: \mathcal{A} \to \mathcal{B} \) a \( \mathcal{V} \)-functor. Suppose \( \mathcal{A} \) is a Grothendieck \( \mathcal{V} \)-category. If it preserves all small weighted colimits, then \( F \) has a right adjoint.

\( \square \) Springer
From Definition 3.15, we observe that the \( V \)-functor \( \langle S \circ y \rangle : C \to A \) is dense in the sense of \([20, \S 5.1]\) and \( A \) has a small dense full subcategory (\([20, \text{Sect. 5.3}]\)). Therefore it follows from \([20, \text{Thm. 5.33}]\) that \( F \) has a right adjoint. \( \square \)

**Proposition 3.26** Let \( V \) be a Grothendieck cosmos with the conditions (C1) and (C2), and \( F : A \to B \) a \( V \)-functor.

(i) Suppose \( A \) is a Grothendieck \( V \)-category. If it preserves all small conical colimits, then \( F \) has a right adjoint.

(ii) Suppose \( A \) is a Grothendieck \( V \)-category and \( B \) is cotensored. If the underlying ordinary functor \( F_0 \) is cocontinuous, then \( F \) has a right adjoint.

**Proof** Note that \( V \) is a locally finitely presentable base by Proposition 3.5. Since \( V \) has a generating set of dualizable objects, Proposition 2.13 shows that any object \( X \in V \) has an exact sequence of the form

\[
\bigoplus_{i} P_i \to \bigoplus_{j} P_j \to X \to 0
\]

(\( \diamond \))

where \( P_i, P_j \) are dualizable.

(i) It follows from \([20, \S 3.8]\) and \([20, (3.69), (3.70)]\) that all colimits can be written as conical colimits of tensor products. Hence we only need to show that \( F \) preserves tensor products, from which Proposition 3.25 deduces that \( F \) has a right adjoint.

For \( X \in V \) and \( A \in A \), consider the tensor product \( X \otimes A \). Since the \( V \)-functor \( - \otimes A : V \to A \) preserves colimits, \( \diamond \) induces the exact sequence

\[
\bigoplus_{i} (P_i \otimes A) \to \bigoplus_{j} (P_j \otimes A) \to X \otimes A \to 0.
\]

Using the assumption that \( F \) preserves conical colimits and the fact that \( P_i \otimes A, P_j \otimes A \) are absolute colimits, we obtain the exact sequence

\[
\bigoplus_{i} (P_i \otimes F(A)) \to \bigoplus_{j} (P_j \otimes F(A)) \to F(X \otimes A) \to 0.
\]

Thus the left exactness of \( B(-, B) \) gives the exact sequence

\[
0 \to B(F(X \otimes A), B) \to B \left( \bigoplus_{j} (P_j \otimes F(A)), B \right) \to B \left( \bigoplus_{i} (P_i \otimes F(A)), B \right).
\]

On the other hand, by the left exactness of \( V(-, B(F(A), B)) \), we get from \( \diamond \) the exact sequence

\[
0 \to V(X, B(F(A), B)) \to V \left( \bigoplus_{i} P_i, B(F(A), B) \right) \to V \left( \bigoplus_{j} P_j, B(F(A), B) \right).
\]

Then, considering isomorphisms

\[
B \left( \bigoplus_{i} (P_i \otimes F(A)), B \right) \cong \prod_{i} B(P_i \otimes F(A), B) \cong \prod_{i} V(P_i, B(F(A), B)) \cong V \left( \bigoplus_{i} P_i, B(F(A), B) \right),
\]

we have \( B(F(X \otimes A), B) \cong V(X, B(F(A), B)) \). Therefore \( F(X \otimes A) \) forms the tensor product \( X \otimes F(A) \), and hence \( F \) preserves the tensor product \( X \otimes A \).

(ii) Because \( B \) is cotensored, \([20, \text{Sect. 3.8}]\) shows that ordinary colimits existing in \( B_0 \) become conical colimits in \( B \). The cocontinuity of \( F_0 \) implies that \( F \) preserves conical colimits, so the assertion follows from (i). \( \square \)
3.4 Relations to Garner–Lack’s Result [18]

In [18], Garner and Lack introduce the notion of *small-exactness* for categories enriched over a locally finitely presentable base $\mathcal{V}$.

Let $\mathcal{P}D$ denote the free cocompletion of a $\mathcal{V}$-category $D$. We do not give its definition here, and refer the reader to [18] or [20, §5.7]. If $D$ is small, then $\mathcal{P}D$ is precisely the same as the presheaf category $[D^{op}, \mathcal{V}]$.

**Definition 3.27** ([18]) Let $\mathcal{V}$ be a locally finitely presentable base. A $\mathcal{V}$-category $C$ is said to be *small-exact* if it is finitely complete, cocomplete, and satisfies the following equivalent conditions.

(a) $C$ is *lex-reflective* in the cocompletion $\mathcal{P}D$ of some finitely complete $\mathcal{V}$-category $D$. Namely, there is a fully faithful $\mathcal{V}$-functor $C \rightarrow \mathcal{P}D$ which admits a left exact left adjoint.

(b) For any small finitely complete $\mathcal{V}$-category $K$ and any left exact $\mathcal{V}$-functor $D: K \rightarrow C$, the left Kan extension $\text{Lan}_{\gamma} D: \mathcal{P}K \rightarrow C$ along the Yoneda embedding is also left exact.

Note that $\mathcal{P}K = [K^{op}, \mathcal{V}]$ in (b), since $K$ is small. These conditions are extracted from [18, Prop. 2.4 (9) and (4)].

They also introduce the notion of $\mathcal{V}$-topoi, which we call Grothendieck $\mathcal{V}$-categories in this paper, and characterize them in [18, Prop. 2.6] as small-exact $\mathcal{V}$-categories with a small dense subcategory. Note that a small full subcategory $C$ of a $\mathcal{V}$-category $D$ is said to be *dense* if the left Kan extension $\text{Lan}_{\gamma} J$ of the Yoneda embedding along the inclusion $J: C \hookrightarrow D$ is fully faithful. The condition that a category has a small dense subcategory is a kind of generating condition for a category. Hence the notion of small-exactness can be considered to capture $\mathcal{V}$-topoi without generators in purely categorical, extrinsic terms.

It is shown in [18, Prop. 2.5] that for a $\text{Set}$-category, namely a category in the usual sense, being small-exact is equivalent to being an infinitary pretopos. The latter notion is defined solely by intrinsic properties of categories. The following proposition gives a similar intrinsic characterization of small-exactness for $\text{Ab}$-categories.

**Proposition 3.28** A preadditive category is small-exact if and only if it is (AB5)-abelian, that is a cocomplete abelian category with exact filtered colimits.

**Proof** The proof is essentially the same as [18, Prop. 2.5], which is about $\text{Set}$-categories. For the ‘if’ part, we use [10, Cor. 2.3.5 (1)] to see that (AB5)-abelian categories satisfy the condition (b) of Definition 3.27. \(\square\)

When $\mathcal{V}$ is a Grothendieck cosmos with conditions (C1) and (C2) we can prove, by reduction to the case of preadditive categories, that a $\mathcal{V}$-category is small-exact if and only if it satisfies the conditions of Theorem 3.18 except for the generating condition (5).

**Proposition 3.29** Let $\mathcal{V}$ be a Grothendieck cosmos with the conditions (C1) and (C2). Then a $\mathcal{V}$-category $A$ is small-exact if and only if it satisfies the conditions (1)–(4) of Theorem 3.18.

**Proof** The idea of the proof is quite similar to that of Theorem 3.18. It is immediate to check that if a $\mathcal{V}$-category $A$ is small-exact, then $A$ satisfies the conditions (1)–(4).

In order to prove the converse, suppose that $A$ satisfies the conditions (1)–(4) of Theorem 3.18. To observe that the condition (b) of Definition 3.27 holds for $A$, take a small finitely
complete \( \mathcal{V} \)-category \( \mathcal{C} \) and a left exact \( \mathcal{V} \)-functor \( D : \mathcal{C} \to \mathcal{A} \). Since \( \mathcal{A} \) is cocomplete, we can consider the left Kan extension \( \text{Lan}_y D : [\mathcal{C}^{\text{op}}, \mathcal{V}] \to \mathcal{A} \):

\[
\begin{array}{ccc}
[\mathcal{C}^{\text{op}}, \mathcal{V}] & \xrightarrow{\text{Lan}_y D} & \mathcal{A} \\
\gamma_C & \downarrow & \downarrow \text{D} \\
\mathcal{C} & \xrightarrow{\gamma_C} & \mathcal{A}.
\end{array}
\]

We want to show that \( S := \text{Lan}_y D \) is left exact, and to do so, by Lemma 3.21 we only need to verify that \( S_0 = (\text{Lan}_y D)_0 \) is left exact.

As shown in the proof of Proposition 3.5, dualizable objects are finitely presentable, so the tensor product with a dualizable object is a finite limit. Hence \( \mathcal{C} \) is closed under tensoring with dualizable objects in \( [\mathcal{C}^{\text{op}}, \mathcal{V}] \). Therefore we see by Proposition 3.20 that \( \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{V}) = [\mathcal{C}^{\text{op}}, \mathcal{V}]_0 \) is a Grothendieck category that has \( C_0 \) as a generating set of objects via \( (\gamma_C)_0 : C_0 \to \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{V}) \). Now by the Gabriel-Popescu Theorem 3.3 we have as the left Kan extension of \( (\gamma_C)_0 \) the left exact left adjoint \( S' : \text{Mod}(C_0) \to \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{V}) \) whose right adjoint is fully faithful:

\[
\begin{array}{ccc}
\text{Mod}(C_0) & \xrightarrow{S'} & \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{V}) \\
\gamma_{C_0} & \downarrow & \downarrow \text{S'} \\
C_0 & \xrightarrow{(\gamma_C)_0} & \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{V}).
\end{array}
\]

Then

\[ S_0 \circ S' \circ (\gamma_C)_0 \cong S_0 \circ (\gamma_C)_0 \cong D_0 \]

and hence \( S_0 \circ S' \) is isomorphic to \( \text{Lan}_y D_0 \) by Theorem 2.6. Since we can easily observe that \( \mathcal{A}_0 \) is (AB5)-abelian and \( D_0 \) is left exact, it follows from Proposition 3.28 and the condition (b) for \( \mathcal{A}_0 \) that \( S_0 \circ S' \) is also left exact. By the same argument as in the last paragraph of the proof of Theorem 3.18, we conclude that \( S_0 \) is left exact, which completes the proof. \( \square \)

Thus we find from [18, Prop. 2.6] that, under the same assumption as in Proposition 3.29, a \( \mathcal{V} \)-category satisfying the conditions (1)–(4) of Theorem 3.18 is a Grothendieck \( \mathcal{V} \)-category if it has a dense subcategory. Nevertheless we can prove more; in view of Proposition 3.29, Theorem 3.18 actually shows the following.

**Proposition 3.30** Let \( \mathcal{V} \) be a Grothendieck cosmos with the conditions (C1) and (C2). Then any \( \mathcal{V} \)-generating set of objects in a small-exact \( \mathcal{V} \)-category forms a small dense subcategory.

That is to say, for a small-exact \( \mathcal{V} \)-category over such a \( \mathcal{V} \), the notion of a \( \mathcal{V} \)-generating set of objects is equivalent to the a priori stronger notion of a small dense subcategory.

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