ON THE CONLEY CONJECTURE FOR REEB FLOWS

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Abstract. In this paper we prove the existence of infinitely many closed Reeb orbits for a certain class of contact manifolds. This result can be viewed as a contact analogue of the Hamiltonian Conley conjecture. The manifolds for which the contact Conley conjecture is established are the pre-quantization circle bundles with aspherical base. As an application, we prove that for a surface of genus at least two with a non-vanishing magnetic field, the twisted geodesic flow has infinitely many periodic orbits on every low energy level.

Contents

1. Introduction 1
Acknowledgments 4
2. Main results 4
2.1. Conley conjecture for pre-quantization circle bundles 4
2.2. Discussion: topological conditions 5
2.3. Application: a charge in a magnetic field 7
3. Preliminaries: contact homology 8
3.1. Generalities: cylindrical and linearized contact homology 8
3.2. Local contact homology 10
3.3. Symplectically degenerate maxima 11
4. Proof of the contact Conley conjecture 13
4.1. The free homotopy class of the fiber 13
4.2. Cylindrical contact homology of a pre-quantization circle bundle 14
4.3. Proof of Theorem 2.1: the contact Conley conjecture 14
4.4. Proof of Theorem 2.4 15
References 17

1. Introduction

In this paper we establish a contact analogue of the Hamiltonian Conley conjecture – the existence of infinitely many periodic orbits – for Reeb flows on the pre-quantization circle bundles with aspherical base. As an application, we prove that for a surface of genus at least two with non-vanishing magnetic field, the twisted geodesic flow has infinitely many periodic orbits on every low energy level.
To put these results in perspective, recall that the Hamiltonian Conley conjecture asserts the existence of infinitely many periodic orbits for every Hamiltonian diffeomorphism of a closed symplectic manifold whenever the manifold meets some natural general requirements. This is the case for manifolds with spherically-vanishing first Chern class (of the tangent bundle) and for negative monotone manifolds; see [CGG, GG09, He12] and also [FH, Gi10, GG12, Hi09, LeC, Maz, SZ]. It is important to note, however, that the Conley conjecture, as stated, fails for some simple manifolds such as $S^2$: an irrational rotation of the sphere about the $z$-axis has only two periodic orbits, which are also the fixed points; these are the poles. In fact, any manifold that admits a Hamiltonian torus action with isolated fixed points also admits a Hamiltonian diffeomorphism with finitely many periodic orbits. Among these manifolds are $\mathbb{CP}^n$, the Grassmannians, and, more generally, most of the coadjoint orbits of compact Lie groups as well as symplectic toric manifolds. (There is also a variant of the Conley conjecture for such manifolds, considered in [Gü12, Gü13] and inspired by a celebrated theorem of Franks, [Fr92, Fr96], but this conjecture is not directly related to our discussion.)

To summarize, the collection of all closed symplectic manifolds naturally breaks down into two classes: those for which the Conley conjecture holds and those for which the Conley conjecture fails. The non-trivial assertion is then that the former class is non-empty and even quite large. At this stage, we are far from understanding where exactly the dividing line between the two classes is, but at least on the level of proofs there seems to be no connection between the Conley conjecture and the additive structure of the (ordinary or quantum) homology of the manifold.

The situation with closed contact manifolds looks more involved even if we leave aside such fundamental questions as the Weinstein conjecture.

First of all, there is a class of contact manifolds for which every Reeb flow has infinitely many closed orbits because the rank of contact or symplectic homology grows as a function of index or some other parameter connected with the order of iteration. This phenomenon, studied in [HM, McL], generalizes and is inspired by the results of [GM] establishing the existence of infinitely many closed geodesics for manifolds such that the homology of the free loop space grows. (A technical but important fact underpinning the proof is that the iterates of a given orbit can make only bounded contributions to the homology; see [GG10, GM, HM, McL] for various incarnations of this result.) By [VPS] and [AS, SW, Vi99], among contact manifolds in this class are the unit cotangent bundles $ST^*M$ whenever $\pi_1(M) = 0$ and the algebra $H^*(M; \mathbb{Q})$ is not generated by one element, and some others; [HM, McL]. This homologically forced existence of infinitely many Reeb orbits has very different nature from the Hamiltonian Conley conjecture where there is no growth of homology: the Floer homology of a Hamiltonian diffeomorphism of a closed manifold obviously does not change with iterations.

Then there are contact manifolds admitting Reeb flows with finitely many closed orbits. Among these are, of course, the standard contact spheres and, more generally, pre-quantization circle bundles over symplectic manifolds admitting torus actions with isolated fixed points (see [Gü14, Example 1.13]) including the Katok–Ziller flows. Another important group of examples, also containing the standard spheres, arises from contact toric manifolds; see [AM]. Note that these two classes overlap, but do not entirely coincide.
Finally, there is, as we show in this paper, a non-empty class of contact manifolds for which every Reeb flow (meeting certain natural index conditions) has infinitely many closed orbits, although there is no obvious homological growth— the rank of the relevant contact homology remains bounded. One can expect the class of manifolds for which the conjecture holds to be quite large, but at this point we can prove such unconditional existence of infinitely many closed Reeb orbits only for pre-quantization circle bundles of aspherical manifolds. (Moreover, for the sake of simplicity, we also make an additional assumption that the first Chern class of the contact structure is atoroidal. Note also that the index conditions mentioned above play a purely technical role, but are inherent in the construction of the cylindrical contact homology utilized in the proof.) This variant of the contact Conley conjecture is one of the main results of the paper (Theorem 2.1), and its proof, drawing from [GG09] and also [GH²M, HM], clearly shows the similarity of the phenomenon with the Hamiltonian Conley conjecture.

This picture is, of course, oversimplified and certainly not even close to covering all the range of possibilities, even on the homological level. (For instance, hypothetically, Reeb flows for overtwisted contact structures have infinitely many periodic orbits, but where should one place such contact structures in our “classification”? See [El, Yau] and also [BvK] for further details.)

It is also worth pointing out that our proof of the contact Conley conjecture heavily relies, in some instances beyond the formal level, on the machinery of cylindrical contact homology (see, e.g., [Bo09, Bo02, EGH] and references therein), which is yet to be fully put on a rigorous basis (see [HWZ10, HWZ11]).

As an application of our main result, we prove the existence of infinitely many periodic orbits for all low energy levels of twisted geodesic flows on surfaces with non-vanishing magnetic field (Theorem 2.4).

To be more precise, consider a closed Riemannian manifold $M$ and let $\sigma$ be a closed 2-form on $M$. Equip $T^*M$ with the twisted symplectic structure $\omega = \omega_0 + \pi^*\sigma$, where $\omega_0$ is the standard symplectic form on $T^*M$ and $\pi : T^*M \to M$ is the natural projection, and let $K$ be the standard kinetic energy Hamiltonian on $T^*M$ corresponding to a Riemannian metric on $M$. The Hamiltonian flow of $K$ on $T^*M$ describes the motion of a charge on $M$ in the magnetic field $\sigma$ and is referred to as a twisted geodesic or magnetic flow. In contrast with the geodesic flow (the case $\sigma = 0$), the dynamics of the twisted geodesic flow on an energy level depends on the level. In particular, when $M$ is a surface of genus $g \geq 2$, the example of the horocycle flow shows that a symplectic magnetic flow need not have periodic orbits on all energy levels. Note also that the dynamics of a twisted geodesic flow crucially depends on whether one considers low or high energy levels and, at least from a technical perspective, on whether $\sigma$ is assumed to be exact or symplectic.

The existence problem for periodic orbits of a charge in a magnetic field was first addressed in the context of symplectic geometry by V.I. Arnold in the early 80s; see [Ar86, Ar88]. Namely, Arnold proved, as a consequence of the Conley–Zehnder theorem, the existence of periodic orbits of a twisted geodesic flow on $\mathbb{T}^2$ with symplectic magnetic field for all energy levels when the metric is flat and all low energy levels for an arbitrary metric, [Ar88]. (It is still unknown if the second of these results can be extended to all energy levels.) Since Arnold’s work, the problem has been studied in a variety of settings. We refer the reader to, e.g., [Gi96a] for more details and further references prior to 1996 (see also [Ta]) and
to, e.g., [AMP, AM²P, CMP, GG04, GG07, Ke99, Scl, Scn12, Us] for by far an incomplete list of more recent results.

Here we focus on the case where the magnetic field form $\sigma$ is symplectic (e.g., non-vanishing when $\dim M = 2$), and we are interested in dynamics on low energy levels. In this setting, in all dimensions, the existence of at least one closed orbit on every sufficiently low energy level was proved in [GG07, Us]. It was also conjectured, and proved for $M = T^2$, in [GG07] that in fact every low energy level carries infinitely many periodic orbits when $M$ is symplectically aspherical. (Although this conjecture is merely one in a sequence of such hypothetical lower bounds (see, e.g., [Ar86, Ar88, Gi87, Gi96b, Ke99]), it differs from the previous ones in that it takes into account periodic orbits of arbitrarily large period, but not just the “short” orbits.) Thus the result of the present paper completes the proof of the conjecture in the case where $M$ is a surface. (See Remark 2.5 for a further discussion.)

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2. Main results

2.1. Conley conjecture for pre-quantization circle bundles. Consider a closed symplectic manifold $(M^{2n}, \omega)$ such that the form $\omega$, or more precisely its cohomology class $[\omega]$, is integral, i.e., $[\omega] \in H^2(M; \mathbb{Z})/\text{Tors}$. Let $\pi: P \to M$ be an $S^1$-bundle over $M$ with the first Chern class $-\omega$. The bundle $P$ admits an $S^1$-invariant 1-form $\alpha_0$ such that $d\alpha_0 = \pi^* \omega$ and $\alpha_0(R_0) = 1$, where $R_0$ is the vector field generating the $S^1$-action on $P$. In other words, when we set $S^1 = \mathbb{R}/\mathbb{Z}$ and identify the Lie algebra of $S^1$ with $\mathbb{R}$, the form $\alpha_0$ is a connection form on $P$ with curvature $\omega$; see, e.g., [GGK, Appendix A] for a detailed discussion of various sign and other conventions used in this setting.

Clearly, $\alpha_0$ is a contact form with Reeb vector field $R_0$, and the connection distribution $\xi = \ker \alpha_0$ is a contact structure on $P$. Up to a gauge transformation, $\xi$ is independent of the choice of $\alpha_0$. The circle bundle $P$ equipped with this contact structure or contact form is usually referred to as a pre-quantization circle bundle or a Boothby–Wang bundle. Also recall that a degree two (real) cohomology class on $P$ is said to be atoroidal if its integral over any smooth map $T^2 \to P$ is zero. (Note that such a class is necessarily aspherical.) Finally, in what follows we will denote by $\mathcal{f}$ the free homotopy class of the fiber of $\pi$.

The main tool used in this paper is the cylindrical contact homology. As is well known, to have this homology defined for a contact form $\alpha$ on any closed contact manifold $P$ one has to impose certain additional requirements on closed Reeb orbits of $\alpha$; [Bo09, EGH]. Namely, we say that a non-degenerate contact form $\alpha$ is index-admissible if its Reeb flow has no contractible closed orbits with Conley–Zehnder index $2-n$ or $2-n \pm 1$. (This not the standard term. Note also that $\dim P = 2n+1$.) In general, $\alpha$ or its Reeb flow is index-admissible when there exists a sequence of non-degenerate index-admissible forms $C^1$-converging to $\alpha$. 
This requirement is usually satisfied once \((P, \alpha)\) has some geometrical convexity properties. For instance, the Reeb flow on a strictly convex hypersurface in \(\mathbb{R}^{2n}\) is index-admissible; [HWZ98]. Likewise, as is observed in [Be], the twisted geodesic flow on a low energy level for a symplectic magnetic field on a surface of genus \(g \geq 2\) is index admissible; see the proof of Theorem 2.4 for more details.

Recall also that closed orbit is said to be weakly non-degenerate when at least one of its Floquet multipliers is different from 1; cf. [SZ]. A form \(\alpha\) or its Reeb flow is weakly non-degenerate when all closed Reeb orbits are weakly non-degenerate. Clearly, weak non-degeneracy is a \(C^\infty\)-generic condition.

The main result of the paper is the following theorem, which, as is pointed out in the introduction, can be viewed as an analogue of the Conley conjecture for contact forms \(\alpha\) on the pre-quantization bundle \(P\) over \(M\), supporting the (cooriented) contact structure \(\xi\), i.e., such that \(\ker \alpha = \xi\) and \(\alpha(R_0) > 0\).

**Theorem 2.1** (Contact Conley Conjecture). Assume that

\begin{enumerate}[(i)]
  \item \(M\) is aspherical, i.e., \(\pi_r(M) = 0\) for all \(r \geq 2\), and
  \item \(c_1(\xi) \in H^2(P; \mathbb{R})\) is atoroidal.
\end{enumerate}

Let \(\alpha\) be an index-admissible contact form on \(P\) supporting \(\xi\). Then the Reeb flow of \(\alpha\) has infinitely many closed orbits with contractible projections to \(M\). Assume furthermore that the Reeb flow has finitely many closed Reeb orbits in the free homotopy class \(f\) of the fiber and that these orbits are weakly non-degenerate. Then for every sufficiently large prime \(k\) the Reeb flow of \(\alpha\) has a simple closed orbit in the class \(f^k\).

Note that under the conditions of the theorem, all iterates \(f^k, k \in \mathbb{N}\), are distinct; see the discussion below and Lemma 4.1. Hence, the second part of the theorem implies the existence of infinitely many periodic orbits and is really a refinement of the first part, under the weak non-degeneracy condition.

**Remark 2.2** (Growth). It readily follows from Theorem 2.1 that, when the Reeb flow of \(\alpha\) is weakly non-degenerate, the number of simple periodic orbits of the Reeb flow of \(\alpha\) with period (or equivalently action) less than \(a \gg 0\) is bounded from below by \(C \cdot a/\ln a\), where \(C > 0\) depends only on \(\alpha\). In fact, we have the lower bound \(C_0 \cdot a/\ln a - C_1\), where \(C_0 = \inf \alpha(R_0)\) and \(C_1\) depends only on \(\alpha\). These growth lower bounds are typical for the Hamiltonian Conley conjecture type results mentioned in the introduction; see also [Hi93] for the case of closed geodesics on \(S^2\). (In dimension two, however, stronger growth results have been established in some cases; see, e.g., [LeC] and [Vi92, Prop. 4.13] and also [BH, FH, Ke12].) Note also that the weak non-degeneracy requirement here plays a technical role and probably can be eliminated; see Remark 3.6.

Finally, we would like to point out a similarity between Theorem 2.1 and the main result of [Gü13] where a variant of the Hamiltonian Conley conjecture is established for non-contractible orbits.

### 2.2. Discussion: topological conditions.

The conditions on \((P, \xi)\) imposed in the theorem and, in particular, condition (i), i.e., the requirement that \(M\) is an Eilenberg–MacLane space \(K(\pi, 1)\), deserve a detailed discussion.

First, however, let us introduce some notation to be used throughout the paper. Let \(\tilde{\pi}_1(P)\), where \(P\) is an arbitrary manifold, be the collection of the free homotopy
classes of maps $S^1 \to P$. We identify $\tilde{\pi}_1(P)$ with the set of conjugacy classes in $\pi_1(P)$. Although in general $\tilde{\pi}_1(P)$ is not a group, the powers of an element of this set are well defined. We denote by 1, the image of the unit in $\tilde{\pi}_1(P)$.

The role of condition (i) in the proof of the theorem is two-fold. Namely, it can be replaced by the following two conditions, which are both consequences of (i):

(i-a) the class $[\omega]$ is aspherical, i.e., $\omega|_{\pi_2(M)} = 0$,

(i-b) the fundamental group $\pi_1(M)$ has no torsion.

A possibly non-obvious point here is that (i) implies (i-b). (This fact, for any finite-dimensional CW-complex $M$, is sometimes attributed to P.A. Smith. Here is a proof taken from [Lü]: Recall that $H^*(\mathbb{Z}_k; \mathbb{Z}) = \mathbb{Z}_k$ for all even degrees $*$. Let $\mathbb{Z}_k \subset \pi_1(M)$ be a finite cyclic subgroup. We can take $M/\mathbb{Z}_k$, where $M$ is the universal covering of $M$, as the classifying space $B\mathbb{Z}_k$. If $M$ were finite-dimensional, we would have $H^*(\mathbb{Z}_k; \mathbb{Z}) = H^*(B\mathbb{Z}_k; \mathbb{Z}) = 0$ for $* > \dim M$. A contradiction.)

We will show in Section 4.1 that when (i-a) holds, all free homotopy classes $f^k$, where $k \in \mathbb{N}$ and $f \in \tilde{\pi}_1(P)$ is the class of a fiber, are distinct and none of these classes is trivial. This fact is used in a variety of ways, e.g., to ensure that the natural grading of the contact homology by the free homotopy classes is preserved in the filtered contact homology of $\alpha$ are simple. Condition (i-b) plays an absolutely crucial, albeit technical, role in the proof of Theorem 3.2.

On the other hand, condition (i-a) is clearly necessary for the theorem. This condition obviously fails for, say, $\mathbb{C}P^n$ or complex Grassmannians, as does the assertion of the theorem. In fact, both condition (i-a) and the assertion of the theorem fail whenever the base $M$ admits a Hamiltonian circle action with isolated fixed points; cf. [Gü14, Example 1.14]. As is shown in that example, this fact is essentially the reason for the existence of asymmetric Finsler metrics with finitely many closed geodesics on, say, the spheres; see [Ka] and also [Zi].

Finally, condition (ii) is imposed only for the sake of simplicity and can be eliminated once a suitably defined Novikov ring is incorporated into the contact homology. This condition is automatically met when $(M, \omega)$ is monotone or negative monotone, i.e., $c_1(TM) = \lambda \omega$ in $H^2(M; \mathbb{R})$ (but not only on $\pi_2(M)$) for some $\lambda \in \mathbb{R}$.

All conditions of the theorem on $(P, \xi)$ are obviously satisfied when $M$ is a surface of genus greater than or equal to one or when $M = \mathbb{T}^{2n}$, or for negative monotone (or monotone, if they exist) hyperbolic Kähler manifolds. Furthermore, the requirements of the theorem are met by the product $M_1 \times M_2$ whenever they are met by $M_1$ and $M_2$.

Remark 2.3. It is worth pointing out that the $S^1$-bundle $\pi: P \to M$ is not unique and is not quite determined by $[\omega]$. Consider the “duality” exact sequence

$$0 \to \text{Ext}_\mathbb{Z}(H_1(M; \mathbb{Z}); \mathbb{Z}) \to H^2(M; \mathbb{Z}) \to \text{Hom}_\mathbb{Z}(H_2(M; \mathbb{Z}); \mathbb{Z}) \to 0.$$ 

We can identify the last term in this sequence with the integral de Rham cohomology $H^2(M; \mathbb{Z})/\text{Tors}$, which the class $[\omega]$ belongs to, and the first term with
$T = \text{Tors}(H_1(M;\mathbb{Z}))$. Thus we have

$$0 \to T \to H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{Z})/\text{Tors} \to 0.$$  

The $S^1$-bundle $\pi: P \to M$ is uniquely determined by its first Chern class $u$ which is a lift of $-\omega$ to $H^2(M;\mathbb{Z})$, but not just by $-\omega$. (Of course, there is no ambiguity when $T = 0$.) Finally, it is not hard to see from the proof of Theorem 2.1 that condition (i-b) can be replaced by the requirement that for any cyclic subgroup $G \subset \pi_1(M)$, the pull-back of $u$ to $H^2(G;\mathbb{Z})$ is zero.

2.3. **Application: a charge in a magnetic field.** Let now $M$ be a closed orientable surface equipped with a Riemannian metric and $\sigma$ be a closed two-form (a magnetic field) on $M$. The two-form $\omega = \omega_0 + \pi^*\sigma$, where $\pi$ is the natural projection $T^*M \to M$ and $\omega_0$ is the standard symplectic form on $T^*M$, is symplectic. Here we will assume that $\sigma$ is also symplectic (i.e., non-vanishing, since $M$ is a surface). As is mentioned in the introduction, the motion of a unit charge on $M$ is governed by the twisted geodesic flow, i.e., the Hamiltonian flow with respect to $\omega$ of the standard kinetic energy Hamiltonian $K: T^*M \to \mathbb{R}$, given by $K(p) = \|p\|^2/2$ in self-explanatory notation.

Let us now focus on the levels $P_\epsilon = \{K = \epsilon\}$ for small values of $\epsilon > 0$.

**Theorem 2.4.** Assume that $M$ has genus $g \geq 2$. Then for every small $\epsilon > 0$, the flow of $K$ has infinitely many simple periodic orbits on $P_\epsilon$ with contractible projections to $M$. Moreover, assume that the flow has finitely many periodic orbits in the free homotopy class $\tilde{f}$ of the fiber. Then, without any non-degeneracy assumptions, for every sufficiently large prime $k$ there is a simple periodic orbit in the class $\tilde{f}^k$.

We will prove this theorem in Section 4.4.

**Remark 2.5.** As was observed in [GG07, Prop. 1.5], an analogous result also holds when $M = T^2$. This is an immediate application of the Conley conjecture proved in this case in [FH]. Furthermore, it was conjectured in [GG07] that in all dimensions the twisted geodesic flow has infinitely many periodic orbits on every low energy level whenever $\sigma$ is symplectic and $(M,\sigma)$ is symplectically aspherical. Thus Theorem 2.4 settles the two-dimensional case of this conjecture. We see no reason why an analogue of Theorem 2.4 should hold when $M = S^2$. Although this is not entirely obvious, we tend to think that an example can be found by applying a variant of the Katok-Ziller construction, [Ka, Zi], to a constant magnetic field on $S^2$ to obtain a twisted geodesic flow with symplectic $\sigma$ and finitely many simple orbits on (some) arbitrarily low energy levels; cf. [Scn11, Theorem 1.3] and [GG04, Section 7].

**Remark 2.6.** When, in the setting of Theorem 2.4, a twisted geodesic flow has finitely many periodic orbits in the free homotopy class of the fiber, the growth lower bounds from Remark 2.2 apply without any non-degeneracy assumptions. Furthermore, we can replace the contact action $a$ by the period of the Hamiltonian flow, but not in general by the Hamiltonian action. Finally, note that when $M$ is a surface of genus $g \geq 1$, condition (i-b) is easy to verify geometrically. Indeed, for $M = T^2$, condition (i-b) obviously holds. For $g \geq 2$, $\pi_1(M)$ is a subgroup of the group of isometries $\text{PSL}(2;\mathbb{R})$ of the hyperbolic plane. As is well known, the elements of $\pi_1(M)$ are necessarily hyperbolic isometries (or hyperbolic or parabolic, when $M$ is not compact), and hence have infinite order.
3. Preliminaries: contact homology

Our goal in this section is to review the definitions and results concerning contact homology necessary for the proof and to set our conventions.

3.1. Generalities: cylindrical and linearized contact homology. We start our discussion by briefly recalling the definition and basic properties of the cylindrical and linearized contact homology in the setting we are interested in. Our goal here is to mainly set our conventions and notation. We refer the reader to, e.g., [Bo09, EGH] and references therein, for a much more detailed account.

Let \((P^{2n+1}, \xi)\) be a closed contact manifold with atoroidal first Chern class \(c_1(\xi)\). Fix a free homotopy class \(f \in \tilde{\pi}_1(P)\).

Let \(\alpha\) be a non-degenerate, index–admissible (i.e., without contractible closed Reeb orbits of index \(2-n\) or \(2-n \pm 1\)), contact form supporting \(\xi\). The cylindrical contact homology \(HC_*(\xi; f)\) of \(\xi\) for the class \(f\) is the homology of a certain complex \(CC_*(\alpha; f)\) generated by the (good) closed Reeb orbits of \(\alpha\) in the homotopy class \(f\) with the differential counting rigid holomorphic cylinders in the symplectization of \(P\) asymptotic to periodic orbits. Although the complex obviously depends on \(\alpha\) (and some auxiliary structures), its homology is well-defined and, in particular, independent of the form.

Likewise, for \(a < b\) outside the action spectrum \(S(\alpha)\), the filtered complex \(CC_*(\alpha; f)\) is generated by the orbits \(x\) with action \(A(x) := \int x\alpha\) in the interval \(I := (a, b)\). The resulting homology \(HC_*(\alpha; f)\) depends on \(\alpha\), but not on the auxiliary structures. Furthermore, it is invariant under deformations of the end points \(a\) and \(b\) and the form \(\alpha\) as long as the end points remain outside the action spectrum; see [GH2M, Proposition 5]. In particular, the homology is also defined “by continuity” for degenerate index–admissible contact forms. (Recall from Section 2.1 that a degenerate form is index–admissible if there exists a sequence of non-degenerate index–admissible forms \(C^1\)-converging to \(\alpha\).) When \(a = -\infty\) and \(b = +\infty\), we recover the total cylindrical contact homology \(HC_*(\xi; f)\).

We note a minor difference of this definition from the standard one, where \(CC_*(\alpha; f)\), for \(I = (a, b)\), is set to be the quotient \(CC_*(\alpha; f) / CC_*(\alpha; f)\) of the complex \(CC_*(\alpha; f)\). The advantage of this approach is that, similarly to the Hamiltonian case (see, e.g., [GG04, Section 4.2.1]) the complex \(CC_*(\alpha; f)\) is defined and, as is easy to see, its homology is equal the standard \(HC_*(\alpha; f)\) even when only the closed Reeb orbits with action in the window \(I\) are non-degenerate and, when contractible, have index different from \(2-n\) or \(2-n \pm 1\), provided that the regularity requirements are met.

The grading of the cylindrical contact complex and the homology deserves a special discussion. First, note that to have the Conley–Zehnder index \(\mu_{\text{CZ}}(x)\) of a closed non-degenerate Reeb orbit \(x\) in the class \(f\) defined, we need to have a trivialization of \(\xi_x\). The standard recipe calls for fixing a trivialization (up to homotopy) of \(\xi\) along a reference loop in \(f\). Connecting \(x\) to the reference loop by a cylinder and extending the trivialization along the cylinder, we obtain a well-defined (up to homotopy) trivialization of \(\xi\) along \(x\). Then the condition that \(c_1(\xi)\) is atoroidal guarantees that the resulting trivialization is independent of the cylinder. In what follows we will work with the collection of classes \(f^k, k \in \mathbb{N}\),
assuming, unless \( f = 1 \), that all classes \( f^k \) are distinct, and none of these classes is trivial. For our purposes it is crucial to choose trivializations compatible with iterations. In other words, we fix a trivialization of \( \xi \) along a loop in the class \( f \) and the trivialization for the class \( f^k \) is then obtained by taking the \( k \)-th iteration, in the obvious sense, of this trivialization. (It is essential at this point that all classes \( f^k \) are distinct and none of them is trivial.)

Then we also have the mean index \( \Delta(x) \) well-defined regardless of whether \( x \) is degenerate or not. (See, e.g., [Lo, SZ] for the definition of the mean index and its properties.) Moreover, our convention guarantees that the mean index is homogeneous:

\[
\Delta(x^r) = r\Delta(x)
\]

for all \( r \in \mathbb{N} \), where \( x \) is, of course, in one of the free homotopy classes \( f^k \). Furthermore, recall that for any choice of trivializations we have

\[
|\mu_{cz}(\tilde{x}) - \Delta(x)| \leq n
\]

for every sufficiently small non-degenerate perturbation \( \tilde{x} \) of \( x \), and that the inequality is strict when \( x \) is weakly non-degenerate.

When \( f = 1 \), the condition that \( c_1(\xi) \) is atoroidal can be relaxed, and it suffices to require this class to be aspherical: \( c_1(\xi)|_{\pi_2(P)} = 0 \). Then, as is well known, every contractible closed Reeb orbit carries a canonical (up to homotopy) trivialization and (3.2) holds automatically.

Finally, in the context of this paper it is much more convenient to depart from the standard convention and have the contact homology graded by the Conley–Zehnder index of the orbit without the shift of degree by \((n + 1) - 3\). (Note that the dimension of \( P \) is \( 2n + 1 \).) Thus, throughout the paper, the contact homology is graded by the Conley–Zehnder index.

Remark 3.1. If instead of assuming that \( c_1(\xi) \) is atoroidal we imposed a stronger condition that \( c_1(\xi) = 0 \) in \( H^2(P; \mathbb{Z}) \), we could have obtained a trivialization of \( \xi \) along every loop, compatible with iterations, by fixing a non-vanishing section, up to homotopy, of the determinant bundle \( \wedge^2 \xi \); cf. [Es, GGo]. We also note that, when \( \alpha \) is non-degenerate, our definition of the filtered contact homology still makes sense even if the end-points of \( I \) are in \( S(\alpha) \). Of course, in this case the homology is very sensitive to the deformations of \( I \) and \( \alpha \).

Finally, for \( \mathfrak{F} \subset \tilde{\pi}_1(P) \) set

\[
HC^I_*(\alpha; \mathfrak{F}) = \bigoplus_{\mathfrak{g} \in \mathfrak{F}} HC^I_*(\alpha; \mathfrak{g}).
\]

As is pointed out above, we will usually have \( \mathfrak{F} = \{ f^k \mid k \in \mathbb{N} \} \) where all classes \( f^k \) are distinct and none of these classes is trivial.

In several instances we will also need to work with linearized contact homology. Below we only briefly specify our conventions. For a detailed discussion of the subject, we refer the reader to, e.g., [Bo09, EGH] and, in particular, to [BO, Section 3.1].

In this case we start with a strong symplectic filling \( W \) of \((P, \xi = \ker \alpha)\), i.e., a compact symplectic manifold \( (W, \omega) \) such that \( \partial W = P \) and \( \omega|_P = d\alpha \) for some \( \alpha \) and that a natural orientation compatibility condition is satisfied. The form \( \alpha \) need not be index–admissible, but the linearized contact homology is defined only when a filling exists and the homology depends on the filling. (However, a filling for \( \alpha \)
can be adjusted and turned into a filling for any other form supporting $\xi$ without changing the total linearized contact homology.) When working with linearized contact homology, we need to replace $\tilde{\pi}_1(P)$ by $\tilde{\pi}_1(W)$ everywhere in the above discussion. We use the notation $HC^I_\mu(\alpha; W, \zeta)$ for the filtered linearized contact homology, where now $\zeta \in \tilde{\pi}_1(W)$. This is a vector space over $Q$ which depends on $I$ and $\alpha$ and $\zeta$ (as in the cylindrical case) and also on $W$. Furthermore, when $\zeta \neq 1$, we assume for the sake of simplicity that the filling is exact (i.e., $[\omega] = 0$) and that $c_1(TW) = 0$ in $H^2(P; Z)$. (This condition can be relaxed.)

If $\zeta = 1$, it suffices to require that $W$ is symplectically aspherical, i.e., $[\omega]|_{\pi_2(W)} = 0 = c_1(TW)|_{\pi_2(W)}$. Note that in this case the contact action given by (3.1) is, in general, different from the symplectic area bounded by an orbit in $W$, unless the orbits is contractible in $P$ or the filling is exact. In what follows, the action is always taken to be the contact action as defined by (3.1).

3.2. Local contact homology. Next, let us review the construction of the local contact homology and the relevant results. Here we follow [GHFM, HM] and we refer the reader to these two papers for proofs and details.

Consider an isolated closed orbit $x$, not necessarily simple, of the Reeb flow of a contact form $\alpha$. The local contact homology $HC_\ast(x)$ of $x$ is the homology of the complex $CC_\ast(x, \alpha)$ generated by the (good) periodic orbits which $x$ splits into under a non-degenerate perturbation $\alpha$ of $\alpha$ with the differential defined again by counting rigid holomorphic cylinders in the symplectization of a tubular neighborhood of $x$. The resulting complex depends on $\alpha$, but its homology is well-defined. (Note that here it is essential that $x$ is isolated.) The absolute grading of the local contact homology is defined once we fix a trivialization of $\xi|_x$. In the setting of Section 3.1, we can use for instance the trivialization arising from our global trivialization convention. As in the global case, throughout the paper the local contact homology is graded by the Conley–Zehnder index.

For instance, when $x$ is non-degenerate and good, $HC_\ast(x)$ is $Q$ concentrated in degree, equal to $\mu(x)$. When $x$ is non-degenerate and bad, $HC_\ast(x) = 0$.

The local contact homology of periodic orbits of $\alpha$ are building blocks of $HC_\ast(\xi; \hat{\mathfrak{g}})$. Namely, there exists a spectral sequence with $E^1 = \bigoplus_x HC_\ast(x)$, where the sum is over all (not necessarily simple) closed Reeb orbits of $\alpha$ in $\hat{\mathfrak{g}}$, converging to $HC_\ast(\xi; \hat{\mathfrak{g}})$. As a result, $HC_m(\xi; \hat{\mathfrak{g}}) = 0$ if there exists a form $\alpha$ such that $HC_m(\xi) = 0$ for all $x$ in $\hat{\mathfrak{g}}$.

To be more precise, assume that all closed Reeb orbits of $\alpha$ are isolated and, as a consequence, the action spectrum $S(\alpha)$ is discrete: $S(\alpha) = \{c_1, c_2, \ldots\}$, where $c_1 < c_2 < \cdots$. Pick arbitrary positive real numbers $a_i$ separating the points of the spectrum:

$$0 < a_0 < c_1 < a_1 < c_2 < a_2 < c_3 < \cdots$$

It is easy to see that

$$HC_\ast^{(a_p, a_{p+1})}(\alpha; \hat{\mathfrak{g}}) = \bigoplus_x HC_\ast(x),$$

where the sum is taken over all $x$ in $\hat{\mathfrak{g}}$ with $A(x) = c_p$. Hence, the increasing filtration $CC_\ast^{(a_p, a_{p+1})}(\alpha; \hat{\mathfrak{g}})$ of $CC_\ast(\alpha)$ gives rise to a spectral sequence with $E^1_{p,q} = \bigoplus_x HC_{p+q}(x)$ converging to $HC_\ast(\xi; \hat{\mathfrak{g}})$.

In general, $HC_\ast(x)$ is supported in the interval $[\Delta(x) - n, \Delta(x) + n]$ or, in other words, $HC_m(x)$ can be non-zero only for $m$ in this interval. (This readily follows
from (3.3).) Thus we can write, using self-explanatory notation,

$$\text{supp } \text{HC}_*(x) \subset [\Delta(x) - n, \Delta(x) + n],$$

(3.4)

and, when $x$ is weakly non-degenerate, the inclusion is strict at both end-points of the interval, i.e., $\Delta(x) \pm n$ are not in the support.

For our purposes it is essential to understand how the local contact homology behaves under iterations. Let now $x$ be a simple closed Reeb orbit. A positive integer $k$, the order of iteration, is said to be admissible if the Floquet multiplier 1 occurs with the same the multiplicity for $x$ and $x^k$, i.e., $k$ is not divisible by the order of any root of unity among the Floquet multipliers of $x$. For instance, $k$ is admissible when both $x$ and $x^k$ are non-degenerate or, as the opposite extreme, any $k$ is admissible when $x$ is totally degenerate (i.e., all Floquet multipliers are equal to 1). Furthermore, every sufficiently large prime is admissible regardless of the Floquet multipliers of $x$. Note also that an admissible iteration of an isolated periodic orbit is automatically isolated; see [CMPY, GG10].

For an isolated simple periodic orbit $x$, there exists a sequence $s_k \in \mathbb{Z}$ indexed by all admissible iterations of $x$ such that

$$\dim \text{HC}_*(x^k) \leq \dim \text{HC}_*-s_k(x) \text{ where } \lim_{k \to \infty} \frac{s_k}{k} = \Delta(x).$$

(3.5)

Moreover, $s_k = \Delta(x)k$ when $x$ is totally degenerate. We refer the reader to [HM] for a proof of this fact; see also [GHFM]. In particular, when all iterations of $x$ are isolated, the sequence $\dim \text{HC}_*(x^k)$ is bounded as a function of $k$. These results can be thought of as generalizations of the classical Gromoll–Meyer theorem (see [GM]) to contact homology; see also [McL] for an analogue of the Gromoll–Meyer theorem for symplectic homology.

The proof of (3.5) is based on the relation between the local contact homology of an isolated orbit, say $y$, and the local Floer homology $\text{HF}_*(\psi)$ of its Poincaré return map $\psi$. Namely, for a simple orbit $y$ we just have $\text{HC}_*(y) \cong \text{HF}_*(\psi)$ with our grading conventions. When the orbit is iterated, i.e., $y = x^k$ and $\psi = \varphi^k$ where $x$ is simple, $\varphi$ is the Poincaré return map of $x$ and $k$ is admissible, the relation is more involved. However, even in this case, we still have $\dim \text{HC}_*(y) \leq \dim \text{HF}_*(\varphi)$. (See [GHFM, HM] for the proofs.) For a simple orbit, the result can also be established by repeating word-for-word the proof of [EKP, Proposition 4.30]. The example where $x$ and $y$ are both non-degenerate and $y$ is bad shows that a strict inequality does occur.) Finally, a version of (3.5) holds for the local Floer homology (see [GG10] for a proof), i.e., to be more precise,

$$\dim \text{HF}_*(\varphi^k) = \dim \text{HF}_*-s_k(\varphi) \text{ with } \lim_{k \to \infty} \frac{s_k}{k} = \Delta(x),$$

where again $s_k = \Delta(x)k$ when $x$ is totally degenerate, and (3.5) follows.

### 3.3. Symplectically degenerate maxima.

A closed Reeb orbit $x$ is said to be a symplectically degenerate maximum or SDM if $\text{HC}_{\Delta(x) + n}(x) \neq 0$, i.e., the local contact homology of $x$ is non-trivial at the right end-point of the maximal support interval. Such orbits are necessarily totally degenerate, and $\text{HC}_*(x)$ is $\mathbb{Q}$ when $* = \Delta(x) + n$ and zero otherwise. An iteration of an SDM orbit is again an SDM. We refer the reader to [GHFM, HM] for the proofs of these facts. Note also that, although both the mean index and the grading of $\text{HC}_*(x)$ depend on the trivialization of $\xi|_x$, the notion of an SDM is independent of the trivialization.
The role of SDM Reeb orbits in our proof of Theorem 2.1 is similar to that of SDM orbits for Hamiltonian diffeomorphisms in the proof of the Hamiltonian Conley conjecture; [Gi10, GG09, Hi09]. We have the following result where, for technical reasons, we need to use the linearized contact homology; see Remark 3.6.

**Theorem 3.2 ([GH²M]).** Let \((W, \omega)\) be a strong symplectic filling of a closed contact manifold \((P, \xi)\) and let \(c \in \pi_1(W)\). Assume furthermore that at least one of the following two conditions is satisfied:

- \(W\) is symplectically aspherical, i.e., \([\omega]|_{\pi_2(W)} = 0 = c_1(TW)|_{\pi_2(W)}\), and \(c = 1\), or
- \([\omega] = 0\) in \(H^2(P; \mathbb{R})\) and \(c_1(TW) = 0\) in \(H^2(P; \mathbb{Z})\).

Let \(x\) be a simple isolated closed Reeb orbit of a contact form \(\alpha\) on \((P, \xi)\). Assume that \(x\) is an SDM with mean index \(\Delta = \Delta(x)\) and action \(c = \mathcal{A}(x)\) and that \(x\) is in the class \(c\). Then for any \(\epsilon > 0\) there exists \(k_\epsilon \in \mathbb{N}\) such that

\[
HC_{\mathcal{A}(\epsilon)}(\mathcal{A}(\epsilon); W) \neq 0 \text{ for all } k > k_\epsilon.
\]

This theorem is proved in [GH²M], although it is stated slightly differently in that paper. We emphasize that in Theorem 3.2 and Corollary 3.4 below the manifold \(P\) need not be a pre-quantization circle bundle, and hence these results apply to a broader class of manifolds than Theorem 2.1.

**Remark 3.3.** Note also that in Theorem 3.2, as in similar results in the Hamiltonian setting (see, e.g., [Gi10, Proposition 4.7], [GG09, Theorem 1.7] and [He12, Theorem 1.5]), one should, strictly speaking, replace the action interval by \((kc - \delta, kc + \epsilon)\) for some arbitrarily small \(\delta \in (0, \epsilon)\) and require \(\epsilon\) to be outside a certain zero measure set to make sure that the end points of the interval are not in the action spectrum.

As a consequence of Theorem 3.2, we obtain

**Corollary 3.4 ([GH²M]).** In the setting of Theorem 3.2, the Reeb flow of \(\alpha\) has infinitely many simple periodic orbits.

Note however that this result, in such a general setting, affords no control on the free homotopy classes of the simple orbits or their growth rate. In contrast with its Hamiltonian counterpart, the corollary is not entirely obvious. For the sake of completeness and because the argument is used in the proof of Theorem 2.1, we include a detailed proof of the corollary; cf. [GG09, Section 3.2].

**Proof.** By Theorem 3.2, given \(\epsilon > 0\), for every sufficiently large \(k\), there exists a closed Reeb orbit \(y_k\) such that \(kc < \mathcal{A}(y_k) < kc + \epsilon\).

Arguing by contradiction, assume that \(\alpha\) has only finitely many simple closed Reeb orbits \(z_1, \ldots, z_r\). Then, for every large \(k\), we have \(y_k = z^{m_k}\) for at least one orbit \(z_i\). Set \(a_i = \mathcal{A}(z_i)\). We have \(kc < m_k a_i < kc + \epsilon\). Once \(\epsilon < a_i\) for all \(i\), it follows that

\[
0 < \|kc\|_{a_i} < \epsilon \text{ for all } k \text{ and } i,
\]

where \(\|f\|_a\) stands for the distance from \(t \in \mathbb{R}\) to the nearest point in \(a\mathbb{Z}\).

We will show that this is impossible: when \(\epsilon > 0\) is sufficiently small there is a sequence \(k_j \to \infty\) such that either \(\|k_j c\|_{a_i} = 0\) or \(\|k_j c\|_{a_i} > \epsilon\) for every \(k_j\) and \(i\).

We consider two cases: \(c \in a_i \mathbb{Q}\) and \(c \notin a_i \mathbb{Q}\). In the former case, there exists \(\delta_i > 0\) such that for every \(k\) either \(\|kc\|_{a_i} = 0\) or \(\|kc\|_{a_i} > \delta_i\). In other words, when \(c \in a_i \mathbb{Q}\) and \(\epsilon < \delta_i\), (3.6) fails for all \(k\).
In the latter case, the sequence $ck$ is equidistributed in the circle $\mathbb{R}/a_i\mathbb{Z}$. Thus, for every $\delta < a_i/2$, we have $\|kc\|_{a_i} > \delta$ with probability $1 - 2\delta/a_i$. It follows that, when $\epsilon > 0$ is sufficiently small, for all $i$ such that $c \notin a_i\mathbb{Q}$ the condition $\|kc\|_{a_i} > \epsilon$ is satisfied with positive probability, i.e., for a positive density sequence $k_j$.

**Remark 3.5 (Symplectically Degenerate Minima).** A sister notion of an SDM is that of a symplectically degenerate minimum (SDMin) obtained by replacing the right end point of the maximal support interval by the left end point. In other words, an isolated Reeb orbit $x$ is said to be an SDMin if $HC_{\Delta(x)-n}(x) \neq 0$. This notion is also of interest in Hamiltonian and contact dynamics; see [GH2M, Remark 1.3] and [He11]. Symplectically degenerate maxima and minima have very similar properties, and variants of Theorem 3.2 and Corollary 3.4 hold when $x$ is an SDMin with now $HC^{[kc-\epsilon,kc]}_{\Delta-n-1}(\alpha; W, c^k) \neq 0$.

**Remark 3.6.** We expect an analogue of Theorem 3.2 to hold in the context of cylindrical contact homology. However, the proof from [GH2M] does not readily translate to this setting without extra assumptions on $\alpha$ along the lines of strong index positivity/negativity in addition to $\alpha$ being index-admissible. The difficulty is that it is not clear how to make the forms $\alpha_{\pm}$, used in the proof to “estimate” the contact homology of $\alpha$, index-admissible without an extra condition of this type. This analogue of Theorem 3.2 is of interest because, for instance, it would allow one to eliminate the weak non-degeneracy requirement in the second part of Theorem 2.1 and in the growth lower bounds from Remark 2.2.

## 4. Proof of the contact Conley conjecture

In this section we prove Theorems 2.1 and 2.4, starting with some elementary preliminary observations.

### 4.1. The free homotopy class of the fiber.

Let, as in Section 2.1, $\pi: P \to M$ be a principle $S^1$-bundle with the first Chern class $-[\omega]$, and let $f$ be the free homotopy class of the fiber of $\pi$. In this section we show that requirements (i-a) and (i-b) guarantee that the classes $f^k, k \in \mathbb{N}$, satisfy the conditions mentioned in Section 2.2 and needed for the proof of Theorem 2.1. For instance, we will show that $f$ is primitive and all classes $f^k$ are distinct. Our first result is

**Lemma 4.1.** Assume that condition (i-a) holds: $\omega|_{\pi_2(M)} = 0$. Then $f^k = f^l$ in $\pi_1(P)$ only when $k = l$ and, in particular, $f^k \neq 1$ for $k \neq 0$.

Note that the lemma would be absolutely obvious if the class of the fiber were non-zero in $H_1(P; \mathbb{Z})/\text{Tors}$. However, clearly the image of $f$ in the homology is a torsion class when $[\omega] \neq 0$, and a proof is due.

**Proof.** Consider the homotopy long exact sequence of $\pi: P \to M$. We claim that, since $\omega|_{\pi_2(M)} = 0$, the connecting map $\partial: \pi_2(M) \to \pi_1(S^1)$ is trivial. Indeed, the image $\partial(s)$ of a class $s \in \pi_2(M)$ is equal to $\langle \omega, s \rangle \cdot f$, where $f \in \pi_1(S^1)$ is the class of the fiber oriented by $R_0$. By the assumption, $\omega|_{\pi_2(M)} = 0$, and we have $\partial = 0$.

Thus the $\pi_1$-part of the long exact sequence turns into the short exact sequence

$$1 \to \pi_1(S^1) \to \pi_1(P) \to \pi_1(M) \to 1,$$

and hence $\pi_1(S^1) = \mathbb{Z}$ is a normal subgroup of $\pi_1(P)$. In other words, for any $g \in \pi_1(P)$, the conjugation by $g$ is an automorphism of $\pi_1(S^1) = \mathbb{Z}$. Then $gfg^{-1} = f^{-1}$.
because $f^{k+1}$ are the only generators. (We are using here multiplicative notation, for, in general, $\pi_1(P)$ is not commutative.) Moreover, in fact, $gf^{-1} = f$ since $\pi : P \to M$ is a principle $S^1$-bundle and hence orientable. Now it follows that $f^k$ is conjugate to $f^l$ only when $k = l$. In particular, $f^k$ is (conjugate to) $1$ only when $k = 0$.

It also follows from the exact sequence (4.1) that the only elements in $\hat{\pi}_1(P)$ which project to $1 \in \hat{\pi}_1(M)$ are $f^k$, $k \in \mathbb{Z}$, i.e., the elements of $\pi_1(S)$. Next, we have

**Lemma 4.2.** Assume that condition (i-b) is met: $\pi_1(M)$ is torsion free. Then for every $k \in \mathbb{N}$ the only solutions $h \in \hat{\pi}_1(P)$ and $l \geq 0$ of the equation $h^l = f^k$ are $h = f^r$, for some $r \in \mathbb{N}$, and $l = k/r$. (In particular, $f$ is primitive.)

**Proof.** Clearly, it is sufficient to show that the equation $h^l = f^k$ in $\pi_1(P)$ has no other solutions than $h = f^r$ and $l = k/r$. Projecting to $M$ and denoting the image of $h$ in $\pi_1(M)$ by $\tilde{h}$, we arrive at $\tilde{h}^l = 1$. By (i-b), $\tilde{h} = 1$. Hence, $h \in \pi_1(S) = \mathbb{Z}$ and the result follows. \hfill \Box

#### 4.2. Cylindrical contact homology of a pre-quantization circle bundle

Although our proof of Theorem 2.1 would go through with any choice of trivializations described in Section 3.1, it is more convenient to specialize this choice further. Namely, we take the fiber of $\pi$ over a point $p \in M$ as a reference loop in the class $\mathfrak{f}$ and the pull-back of a frame in $T_p\mathcal{M}$ as the reference trivialization. Note that then the $k$-th iteration of the fiber is the reference loop for $f^k$, and the reference trivialization for $f^k$ is still the pull-back of a frame in $T_p\mathcal{M}$.

With this choice of trivializations, for all $k \in \mathbb{N}$ we have

$$HC_n(\xi; f^k) = H_{*+n}(M; \mathbb{Q}).$$

(4.2)

In particular, $HC_n(\xi; f^k) = \mathbb{Q}$ for all $k \in \mathbb{N}$. The isomorphism (4.2) immediately follows from the Morse-Bott description of contact homology; see [Bo02, Bo09]. Indeed, up to a shift of degree equal to the mean index (for the Reeb flow of $\alpha_0$) of the $k$th iteration of the fiber, the contact homology $HC_n(\xi; f^k)$ is the homology of the base $M$ with the Floer homology grading, i.e., the grading shifted down by $n$. (We recall again that throughout the paper the contact homology is graded by the Conley–Zehnder index.) Finally, for our choice of trivializations, the mean index of the (iterated) fiber is zero, and we arrive at (4.2).

#### 4.3. Proof of Theorem 2.1: the contact Conley conjecture

We may assume that $a$ has finitely many closed Reeb orbits in the class $\mathfrak{f}$. (Otherwise there is nothing to prove.) Denote these orbits by $x_1, \ldots, x_r$. Note that all of these orbits are simple. Indeed, by Lemma 4.2, the class $\mathfrak{f}$ is primitive, but if one of the orbits $x_i$ were iterated so would be the class $\mathfrak{f}$. As in the Hamiltonian setting, the proof splits into two cases.

Assume first that one of the orbits $x_i$, say $x = x_1$, is an SDM. This is the “degenerate case” of the theorem. Consider the unit disk bundle $E = P \times D^2/S^1$ over $M$ equipped with symplectic form $\omega_E = [\pi^*\omega + d(p\alpha_0)]/2$, where $p = |z|^2$, $z \in D^2$, and $\alpha_0$ is a connection 1-form. (See Section 2.1.) The symplectic manifold $(E, \omega_E)$ is a strong symplectic filling of $(P, \alpha_0)$. Rescaling the form and applying a contact isotopy, we obtain a strong filling, which we simply denote by $W$, of $(P, \alpha)$. Clearly, $M \hookrightarrow W$ is a homotopy equivalence, and the inclusion $P \hookrightarrow W$
is homotopic, in the obvious sense, to the projection $P \to M$. The fiber of $P$ is contractible in $W$, i.e., the image of the homotopy class $f \in \tilde{\pi}_1(W)$ of the fiber is $c = 1 \in \tilde{\pi}_1(W)$. In general, the filling $W$ is neither exact nor does it have zero first Chern class. However, $\pi_2(W) = 0$ since $M$ is aspherical, and, in particular, $W$ is symplectically aspherical. Hence, Theorem 3.2 and Corollary 3.4 apply with $c = 1$, and the Reeb flow of $\alpha$ has infinitely many periodic orbits. However, Corollary 3.4 provides no information on the homotopy classes of these orbits and an extra argument is needed to show that the orbits have contractible projections to $M$ or, equivalently, are contractible in $W$.

Arguing by contradiction, assume that the flow has only finitely many simple orbits $\{z_i\}$ contractible in $W$. The orbits $x_i$ are among these orbits, but there can be other orbits with contractible projections to $M$. By Theorem 3.2, given $\epsilon > 0$, for every sufficiently large $k$, the Reeb flow has a closed Reeb orbit $y_k$ contractible in $W$ with action in the range $(kc, kc + \epsilon)$. The orbit $y_k$ need not be simple.

As is pointed out in Section 4.1, an element $\eta \in \tilde{\pi}_1(P)$ (e.g., $[y_k]$ or $[z_i]$) is trivial in $\tilde{\pi}_1(W) = \tilde{\pi}_1(M)$ if and only if $\eta = f^l$ for some $l \in \mathbb{Z}$. Thus $y_k$ can only be an iteration of one of the orbits $z_i$, i.e., $y_k = z_i^{m_k}$. Now exactly the same argument as the proof of Corollary 3.4 shows that this is impossible.

The second case is when none of the orbits $x_i$ is an SDM. This is the so-called “non-degenerate case” of the theorem since, for instance, none of these orbits is an SDM when all $x_i$ are weakly non-degenerate. In spirit, the proof of this case goes back to [SZ]. Let $k$ be a sufficiently large prime. To prove the theorem, it suffices to show that the class $f^k$ contains a simple periodic orbit. Then, by Lemma 4.1, all iterates $f^k$, $k \in \mathbb{N}$, are distinct and hence so are the orbits.

Set $\Delta_i = \Delta(x_i)$. We require $k$ to be admissible for all orbits with $\Delta_i = 0$ and large enough to ensure that $k|\Delta_i| > 2n$ for all orbits with $\Delta_i \neq 0$.

To show that there exists a closed Reeb orbit in the class $f^k$, we again argue by contradiction. Indeed, assume the contrary. Then, by Lemma 4.2, every orbit in $f^k$ is necessarily of the form $x_i^k$. We claim that $HC_n(\alpha; f^k) = 0$, which contradicts (4.2). It suffices to show that $HC_n(x_i^k) = 0$ for all orbits $x_i$. To prove this, note that

$$\text{supp} \, HC_n(x_i^k) \subset [k\Delta_i - n, k\Delta_i + n],$$

by (3.2) and (3.4), and hence $HC_n(x_i^k) = 0$ when $\Delta_i \neq 0$. When $\Delta_i = 0$, there are further two cases to consider. The first one is when $x_i$ is weakly non-degenerate. Then so is $x_i^k$, since $k$ is admissible, and hence $HC_n(x_i^k) = 0$. (For the end-points of the interval in (3.4) are not in the support.) The second case is when $x_i$ is totally degenerate. Then $\dim HC_*(x_i^k) \leq \dim HC_*(x_i)$ by (3.5) together with the “moreover” part and, in particular, $HC_n(x_i^k) = HC_n(x_i) = 0$ because $x_i$ is not an SDM. \hfill \Box

Remark 4.3. As is clear from the proof above, one can relax the assumption that the orbits are weakly non-degenerate in the second part of the theorem and replace it by the requirement that none of the orbits is an SDM.

4.4. Proof of Theorem 2.4. The level $P_\epsilon$ is a circle bundle over $M$, isomorphic to the unit circle bundle $P \subset T^*M$. As is well known, when $g \neq 1$, the form $\omega$ has contact type on $P_\epsilon$ for all small $\epsilon > 0$. Indeed, the first Chern class of the $S^1$-bundle $\pi: P \to M$ is equal to the Euler class $e(M)$. Since $g \neq 1$, there exists $\kappa \in \mathbb{R}$ such that $[\sigma] = \kappa e(M)$. Fix a connection form $\alpha_0$ on $P$ with curvature $\sigma / \kappa$
as in Section 2.1 and set $\lambda_0 = \kappa \alpha_0$. Clearly, $d\lambda_0 = \pi^* \sigma$ and $\lambda_0$ is a contact form with Reeb vector field $\kappa^{-1} R_0$. Next denote by $\lambda_1$ the restriction of the standard Liouville form $pdq$ to $P$. Let us identify $P_t$ with $P$ via the fiberwise dilation by $\sqrt{2\varepsilon}$. Then the pull back of $\omega|_{P_t}$ to $P$ is $d(\lambda_0 + \sqrt{2\varepsilon} \lambda_1)$. When $\varepsilon > 0$ is small, this is a contact form.

Denote by $\alpha$ the resulting contact primitive of $\omega|_{P_t}$. In other words, $\alpha$ is the push-forward of $\lambda_0 + \sqrt{2\varepsilon} \lambda_1$ to $P$. The underlying contact structure $\ker \alpha$ is isotopic, for all small $\varepsilon > 0$, to the pre-quantization contact structure $\ker \lambda_0 = \ker \alpha_0$. It readily follows now that when $g \geq 2$ conditions (i) and (ii) of Theorem 2.1 are satisfied.

Furthermore, the Reeb flow on $P_t$ is index-admissible. Indeed, first note that every closed, homologous to zero Reeb orbit $x$ on $P_t$ has mean index $\Delta(x) < 2\chi(M) + o(1)$ as $\varepsilon \to 0$; see [Be]. Fix now a sufficiently large $T$, depending on the geometry of the magnetic field and the metric. Then for any small non-degenerate perturbation $\tilde{\alpha}$ of $\alpha$, every closed Reeb orbit $\tilde{x}$ of $\tilde{\alpha}$ of period less than $T$ is close to an orbit of $\alpha$, and hence also has mean index $\Delta(\tilde{x}) < 2\chi(M) + o(1)$ when $\tilde{x}$ is contractible in $P$. Long orbits $\tilde{x}$ of $\tilde{\alpha}$, i.e., the orbits with period greater than or equal to $T$, do not necessarily arise from the orbits of $\alpha$. However, such orbits automatically have large negative mean index $-O(T) < 2\chi(M)$ (as $T \to \infty$); see [GG07]. In either case, the Conley-Zehnder index of $\tilde{x}$ is no greater than $2\chi(M) + 1 + o(1) < 0$ by (3.3).

To finish the proof of the first part of the theorem, the existence of infinitely many periodic orbits with contractible projections to $M$, it remains to apply Theorem 2.1.

To show that there is indeed a simple orbit in the class $\mathfrak{f}^k$ for every large prime $k$, we can focus on the case where one of the simple orbits $x_1, \ldots, x_r$ in the class $\mathfrak{f}$ is an SDM. For the “non-degenerate case” is also covered by (the proof of) Theorem 2.1; see Remark 4.3. We do this by applying Theorem 3.2 to a conveniently chosen filling of $P_t$ with an extra component added to it. Namely, let us fix a metric with constant negative curvature on $M$ and let $P' \subset T^* M$ be a high energy level with respect to this metric. Clearly, $P'$ is a contact type hypersurface in $T^* M$, and the Reeb flow on $P'$ is topologically conjugate to the hyperbolic geodesic flow on $M$; see, e.g., [Gi96a]. (Moreover, one can find such a hyperbolic metric with area form proportional, with constant factor, to the magnetic field $\sigma$. Then the twisted geodesic flow on $P'$ is smoothly conjugate to the geodesic flow; cf. [Ar61].) As a consequence, all closed orbits of the flow on $P'$ have non-contractible projections to $M$.

Consider now the subset $W$ of $T^* M$ bounded by $P'$ and $P_t$. This is a strong filling of $P' \cup P_t$; see [McD] and also [Ge]. The filling $W$ is exact and $c_1(TW) = 0$. Hence, Theorem 3.2 applies with $\epsilon = \mathfrak{f}$, the homotopy class of the fiber. The inclusion $P_t \hookrightarrow W$ is a homotopy equivalence, and the Reeb flow on $P'$ has no closed orbits in any class $f^k$. Let $x$ be an SDM in the class $\mathfrak{f}$. Note that now, as in Section 4.2, we can assume without loss of generality that $\Delta(x) = 0$. By Theorem 3.2 applied to $W$, for every sufficiently large $k$, there exists a periodic orbit $y$ of the flow on $P_t$ such that $[y] \in f^k$ and $0 < \Delta(y) < 2n + 1$. By Lemma 4.1, when $k$ is prime, either $y$ is simple or it is an iteration of one of the orbits $x_i$ with $\Delta(x_i) > 0$. The latter is clearly impossible when $k$ is so large that $k\Delta(x_i) > 2n + 1$ for all $x_i$ with $\Delta(x_i) > 0$. This completes the proof of the theorem. $\square$
References

[AMP] A. Abbondandolo, L. Macarini, G.P. Paternain, On the existence of three closed magnetic geodesics for subcritical energies, Preprint arXiv:1305.1871; to appear in Comm. Math. Helv.

[AM^2P] A. Abbondandolo, L. Macarini, M. Mazzucchelli, G.P. Paternain, Infinitely many periodic orbits of exact magnetic flows on surfaces for almost every subcritical energy level, Preprint arXiv:1404.7641.

[AS] A. Abbondandolo, M. Schwarz, On the Floer homology of cotangent bundles, Comm. Pure Appl. Math., 59 (2006), 254–316.

[AM] M. Abreu, L. Macarini, Contact homology of good toric contact manifolds, Compos. Math., 148 (2012), 304–334.

[Ar61] V.I. Arnold, Some remarks on flows of line elements and frames, Soviet Math. Dokl., 2 (1961), 562–564.

[Ar86] V.I. Arnold, First steps of symplectic topology, Russ. Math. Surveys, 41 (1986), 1–21.

[Ar88] V.I. Arnold, On some problems in symplectic topology, in Topology and Geometry – Rochlin Seminar, O.Ya. Viro (ed.), Lect. Notes in Math., vol. 1346, Springer, 1988.

[Be] G. Benedetti, The contact property for nowhere vanishing magnetic fields on the two-sphere, Preprint arXiv:1308.2128.

[Bo02] F. Bourgeois, Morse–Bott approach to contact homology, Ph.D. thesis, Stanford, 2002.

[Bo09] F. Bourgeois, A survey of contact homology, CRM Proceedings and Lecture Notes, 49 (2009), 45–60.

[BO] F. Bourgeois, A. Oancea, An exact sequence for contact- and symplectic homology Invent. Math., 175 (2009), 611–680.

[BvK] F. Bourgeois, O. van Koert, Contact homology of left-handed stabilizations and plumbing of open books, Commun. Contemp. Math., 12 (2010), 223–263.

[BH] B. Bramham, H. Hofer, First steps towards a symplectic dynamics, Surveys in Differential Geometry, 17 (2012), 127–178.

[CGG] M. Chance, V.L. Ginzburg, B.Z. Gürel, Action-index relations for perfect Hamiltonian diffeomorphisms, J. Symplectic Geom., 11 (2013), 449–474.

[CMPY] S.-N. Chow, J. Mallet-Paret, J.A. Yorke, A periodic orbit index which is a bifurcation invariant. Geometric Dynamics (Rio de Janeiro, 1981), 109–131, Lecture Notes in Math., 1007, Springer, Berlin, 1983.

[CMP] G. Contreras, L. Macarini, G.P. Paternain, Periodic orbits for exact magnetic flows on surfaces, Int. Math. Res. Not. IMRN, 2004, no. 8, 361–387.

[El] Y. Eliashberg, Invariants in contact topology, in Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), Doc. Math. 1998, Extra Vol. II, 327–338.

[EGH] Y. Eliashberg, A. Givental, H. Hofer, Introduction to symplectic field theory, Geom. Funct. Anal., 2000, Special Volume, Part II, 560–673.

[EKP] Y. Eliashberg, S.S. Kim, L. Polterovich, Geometry of contact transformations and domains: orderability vs. squeezing, Geom. Topol., 10 (2006), 1635–1747.

[Es] J. Espina, On the mean Euler characteristic of contact structures, Preprint arXiv:1011.4364; to appear in Internat. J. Math.

[Fr92] J. Franks, Geodesics on $S^2$ and periodic points of annulus homeomorphisms, Invent. Math., 108 (1992), 403–418.

[Fr96] J. Franks, Area preserving homeomorphisms of open surfaces of genus zero, New York Jour. of Math., 2 (1996) 1–19.

[FH] J. Franks, M. Handel, Periodic points of Hamiltonian surface diffeomorphisms, Geom. Topol., 7 (2003), 713–756.

[Ge] H. Geiges, Examples of symplectic 4-manifolds with disconnected boundary of contact type, Bull. London Math. Soc., 27 (1995), 278–280.

[Gi87] V.L. Ginzburg, New generalizations of Poincaré’s geometric theorem, Funct. Anal. Appl., 21 (1987), 100–106.

[Gi96a] V.L. Ginzburg, On closed trajectories of a charge in a magnetic field. An application of symplectic geometry in Contact and symplectic geometry (Cambridge, 1994), 131–148, Publ. Newton Inst., 8, Cambridge Univ. Press, Cambridge, 1996.
[Gi96b] V.L. Ginzburg, On the existence and non-existence of closed trajectories for some Hamiltonian flows, *Math. Z.*, **223** (1996), 397–409.

[Gi10] V.L. Ginzburg, The Conley conjecture, *Ann. of Math.*, **172** (2010), 1127–1180.

[Go] V.L. Ginzburg, Y. Gören, Iterated index and the mean Euler characteristic, Preprint arXiv:1311.0547.

[GG04] V.L. Ginzburg, B.Z. Gürel, Relative Hofer-Zehnder capacity and periodic orbits in twisted cotangent bundles, *Duke Math. J.*, **123** (2004), 1–47.

[GG07] V.L. Ginzburg, B.Z. Gürel, Periodic orbits of twisted geodesic flows and the Weinstein–Moser theorem, *Comment. Math. Helv.*, **84** (2009), 865–907.

[GG09] V.L. Ginzburg, B.Z. Gürel, Action and index spectra and periodic orbits in Hamiltonian dynamics, *Geom. Topol.*, **13** (2009), 2745–2805.

[GG10] V.L. Ginzburg, B.Z. Gürel, Local Floer homology and the action gap, *J. Sympl. Geom.*, **8** (2010), 323–357.

[GG12] V.L. Ginzburg, B.Z. Gürel, Conley conjecture for negative monotone symplectic manifolds, *Int. Math. Res. Not. IMRN*, 2012, no. 8, 1748–1767.

[GHFM] V.L. Ginzburg, D. Hein, U.L. Hryniewicz, L. Macarini, Closed Reeb orbits on the sphere and symplectically degenerate maxima, *Acta Math. Vietnam.*, **38** (2013), 55–78

[GM] D. Gromoll, W. Meyer, Periodic geodesics on compact Riemannian manifolds, *J. Differential Geom.*, **3** (1969), 493–510.

[GGK] V. Guillemin, V. Ginzburg, Y. Karshon, *Cobordisms and Hamiltonian Group Actions*, Mathematical Surveys and Monographs, 98; American Mathematical Society, Providence, RI, 2002.

[Gü12] B.Z. Gürel, Periodic orbits of Hamiltonian systems linear and hyperbolic at infinity, Preprint arXiv:1209.3529; to appear in *Pacific J. Math.*

[Gü13] B.Z. Gürel, On non-contractible periodic orbits of Hamiltonian diffeomorphisms, *Bull. Lond. Math. Soc.*, 2013, doi: 10.1112/blms/bdt051.

[Gü14] B.Z. Gürel, Perfect Reeb flows and action-index relations, Preprint arXiv:1401.2665.

[He11] D. Hein, The Conley conjecture for the cotangent bundle, *Arch. Math.*, **96** (2011), 85–100.

[He12] D. Hein, The Conley conjecture for irrational symplectic manifolds, *J. Sympl. Geom.*, **10** (2012), 183–202.

[Hf93] N. Hingston, On the growth of the number of closed geodesics on the two-sphere, *Int. Math. Res. Not. IMRN*, 1993, no. 9, 253–262.

[Hf09] N. Hingston, Subharmonic solutions of Hamiltonian equations on tori, *Ann. of Math.*, **170** (2009), 525–560.

[HWZ98] H. Hofer, K. Wysocki, E. Zehnder, The dynamics on three-dimensional strictly convex energy surfaces, *Ann. of Math.*, **148** (1998), 197–289.

[HWZ10] H. Hofer, K. Wysocki, E. Zehnder, SC-smoothness, retractions and new models for smooth spaces, *Discrete Contin. Dyn. Syst.*, **28** (2010), 665–788.

[HWZ11] H. Hofer, K. Wysocki, E. Zehnder, Applications of polyfold theory I: The Polyfolds of Gromov–Witten Theory, Preprint arXiv:1107.2097.

[HM] U. Hryniewicz, L. Macarini, Local contact homology and applications, Preprint arXiv:1202.3122.

[Ka] A.B. Katok, Ergodic perturbations of degenerate integrable Hamiltonian systems, *Izv. Akad. Nauk SSSR Ser. Mat.*, **37** (1973), 539–576.

[Ke99] E. Kerman, Periodic orbits of Hamiltonian flows near symplectic critical submanifolds, *Int. Math. Res. Not. IMRN*, 1999, no. 17, 953–969.

[Ke12] E. Kerman, On primes and period growth for Hamiltonian diffeomorphisms, *J. Mod. Dyn.*, **6** (2012), 41–58.

[LeC] P. Le Calvez, Periodic orbits of Hamiltonian homeomorphisms of surfaces, *Duke Math. J.*, **133** (2006), 125–184.

[Lo] Y. Long, *Index Theory for Symplectic Paths with Applications*, Progress in Mathematics, 207, Birkhäuser Verlag, Basel, 2002.

[Lü] W. Lück, Aspherical manifolds, *Bulletin of the Manifold Atlas* (2012), 1–17, at http://www.map.mpim-bonn.mpg.de/aspherical_manifolds.

[Maz] M. Mazzucchelli, Symplectically degenerate maxima via generating functions, *Math. Z.*, **275** (2013), 715–739.
D. McDuff, Symplectic manifolds with contact type boundaries, *Invent. Math.*, **103** (1991), 651–671.

M. McLean, Local Floer homology and infinitely many simple Reeb orbits, *Algebr. Geom. Topol.*, **12** (2012), 1901–1923.

D. Salamon, J. Weber, Floer homology and the heat flow, *Geom. Funct. Anal.*, **16** (2006), 1050–1138.

D. Salamon, E. Zehnder, Morse theory for periodic solutions of Hamiltonian systems and the Maslov index, *Comm. Pure Appl. Math.*, **45** (1992), 1303–1360.

F. Schlenk, Applications of Hofer’s geometry to Hamiltonian dynamics, *Comment. Math. Helv.*, **81** (2006), 105–121.

M. Schneider, Closed magnetic geodesics on $S^2$, *J. Differential Geom.*, **87** (2011), 343–388.

M. Schneider, Closed magnetic geodesics on closed hyperbolic Riemann surfaces, *Proc. Lond. Math. Soc. (3)*, **105** (2012), 424–446.

I.A. Taimanov, Closed extremals on two-dimensional manifolds, *Russian Math. Surveys* **47** (1992), no. 2, 163–211.

M. Usher, Floer homology in disc bundles and symplectically twisted geodesic flows, *J. Mod. Dyn.*, **3** (2009), 61–101.

M. Vigué-Poirrier, D. Sullivan, The homology theory of the closed geodesic problem, *J. Differential Geometry*, **11** (1976), 633–644.

C. Viterbo, Symplectic topology as the geometry of generating functions, *Math. Ann.*, **292** (1992), 685–710.

C. Viterbo, Functors and computations in Floer cohomology, I, *Geom. Funct. Anal.*, **9** (1999), 985–1033.

M.-L. Yau, Vanishing of the contact homology of overtwisted contact 3–manifolds, *Bull. Inst. Math. Acad. Sin. (N.S.)*, **1** (2006), 211–229.

W. Ziller, Geometry of the Katok examples, *Ergodic Theory Dynam. Systems*, **3** (1983), 135–157.

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