INTERACTION OF LARGE PILES WITH A MULTILAYER SOIL MASS, TAKING INTO ACCOUNT HARDENING AND SOFTENING

Zaven G. Ter-Martirosyan, Armen Z. Ter-Martirosyan, Aleksandr S. Akuletsky
National Research Moscow State University of Civil Engineering, Moscow, RUSSIA

Abstract: This article discusses the formulation and solution of the problem of the interaction of a long pile with the surrounding multilayer and underlying soils, taking into account the rheological properties of the surrounding soil mass. The creep process is considered taking into account hardening and softening. The problem was considered in a linear setting. The solution is presented by analytical method. To describe the creep process, the rheological parameters of hardening and softening were used. An expression is obtained for finding the reduced shear modulus for a multilayer soil mass. A dependence is obtained for determining the force on the pile heel on time, taking into account the rheological parameters of hardening and softening. Analytical solutions in the article are supported by a graphical part. The graphs of the dependence of the settlement of the pile, the force on the heel of the pile cutting through alternating layers, on time for various parameters of viscosity, as well as for variable parameters of hardening and softening are given. The solutions obtained can be used for preliminary determination of the movement of long piles with the surrounding multilayer and underlying soils.

Keywords: pile interaction, multilayer and underlying soils, analytical method, reduced modulus, rheological properties, settlement rate, hardening factor, softening factor.
INTRODUCTION

Construction sites are characterized mainly by difficult engineering and geological conditions, represented by the presence of several layers at the base, including weak clayey water-saturated soils. Under these conditions, as a rule, it is used: soil consolidation [1–5], soil reinforcement [6–7], significant deepening of the underground part of buildings, etc. But the pile foundation is considered as the main type of foundation on such sites [8–14]. When the basement of weak clayey soils occurs, the settlement of the building can continue for a long period of time. There are cases when the settlements of buildings and structures did not subside for several decades. The most famous example of the Leaning Tower of Pisa, the slope of which developed over several centuries. In the design of foundations on such soils, the forecast of settlement over time is of great importance. The strength and stability of structures will depend both on the rate of development of the settlement over time and on the final settlement of the structure. Therefore, the approach to the description of the process of foundation settlement should be considered as rheological [15–22]. It is known that when a long pile interacts with the surrounding multilayer and underlying soils, a complex inhomogeneous stress-strain state arises. In this paper, we consider the problem of the interaction of a long pile with a multilayer soil massif in a linear formulation, which has rheological properties, as well as the problem of determining the reduced shear modulus for a soil massif.

Studies of the operation of a long pile show that the effect of the length of the pile on the surrounding soil mass extends to a distance of no more than 6–7 pile diameters, and of the same order in depth under its lower end [23]. The distance between the piles less than six diameters ensures the displacement of the pile and the soil in the inter-pile space at the same time. These studies allow us to consider the displacement of the pile foundation and soil as a single massif, and also allow the problem of the interaction of a long pile with a soil massif to be considered as the problem of the interaction of a pile with a soil massif of limited dimensions in the form of a cylinder with a diameter $2b$ and height $L > l$, where $l$ is the length of the pile (Fig. 1).

![Figure 1. Design model of interaction of pile with multilayer soil column.](image)

MATERIALS AND METHODS

Analysis of the stress-strain state of soils around the pile and under its end showed that shear deformations prevail when the pile interacts with the soil, volumetric deformations can be ignored [24]. The solution to the problem will be considered for a round pile. We also assume that the stiffness of the pile significantly exceeds the stiffness of the soil.

$$E_\text{os.} \gg E_\text{sp}.$$ 

Let us write down the equilibrium equation for the considered case (Fig. 1):

$$N = T + R,$$  \hspace{1cm} (1)

where

$$N = \pi a^2 p_1,$$  \hspace{1cm} (2)
\[ T = 2\pi a l \tau , \quad R = \pi a^2 p_2. \]  
\[ (3) \]

Substituting equations (2), (3), (4) into equation (1), we obtain expressions for \( \tau \):

\[ \tau = (p_1 - p_2) \frac{a}{2l}. \]  
\[ (5) \]

Since \( E_a \gg E_{op} \), the settlement of the pile of each layer under consideration is equal, i.e.

\[ S_1 = S_2 = S_3 = S, \]  
\[ (6) \]

where \( S_i \) is a pile settlement for \( i \)-th layer; \( S \) is a total pile settlement.

Shear deformation of the elementary soil layer around the pile can be determined by the following relationship:

\[ \gamma_i(r) = -\frac{\tau_i(r)}{G_i}, \]  
\[ (7) \]

where \( G_i \) is shear modulus for \( i \)-th layer; \( i = 1, 2, ..., n \) is layer number.

Based on condition (6), we can write an expression for the tangential stresses of the \( i \)-th layer:

\[ \tau_i = \frac{\tau}{G} G_i, \]  
\[ (8) \]

where \( G \) is a reduced shear modulus for multilayer soil mass.

Based on the condition for the distribution of shear stresses along the length of the pile, we obtain:

\[ \tau = \tau_1 l_1 + \tau_2 l_2 + \tau_3 l_3, \]  
\[ (9) \]

Considering (9) and (8) together, we obtain an expression for determining the reduced shear modulus for a multilayer soil mass:

\[ G = \frac{l_1 G_1 + l_2 G_2 + l_3 G_3}{l}, \]  
\[ (10) \]

where \( l \) is a pile length; \( i = 1, 2, ..., n \) is layer number.

Let us write the dependence for the rate of change in shear deformation around the pile, taking into account the rheological properties of the surrounding soil mass:

\[ \dot{\gamma} = -\frac{\dot{\tau}_a}{G} + \frac{\tau_a}{\eta(t)}, \]  
\[ (11) \]

where \( \dot{\tau}_a \) is shear stress rate; \( \tau_a = T / 2\pi a l \); \( \eta(t) \) – weighted average viscosity index.

Because the forces transferred to the pile are constant (\( p_1 = const \)), the pressure rate at the pile head does not change (\( p_1 = 0 \)). Based on this, we determine the rate of change in shear stresses:

\[ \dot{\tau}_a = -\frac{\dot{p}_2}{2l}. \]  
\[ (12) \]

The rate of settlement of the pile from the action of shear stresses on the lateral surface, taking into account the elastic-viscous characteristics of the surrounding soil mass:

\[ \dot{V}_r = \frac{a \tau a}{\eta(t)} \ln \left( \frac{b}{a} \right) + \frac{a \dot{\tau}_a}{G} \ln \left( \frac{b}{a} \right), \]  
\[ (13) \]

where \( G \) was obtained from (10).

Let us determine the rate of settlement of the pile due to the deformation of the soil under the lower end of the pile, assuming that the pile acts as a flat round stamp. The equation is:

\[ \dot{V}_r = \dot{p}_2 \frac{\pi a (1 - 0)}{4G_0}, \]  
\[ (14) \]

where \( \dot{p}_2 \) is pressure rate under the pile heel; \( 0 \) \( G_0 \) and \( G_0 \) is deformation parameters of the soil.
under the lower end of the pile; $K \leq 1$ – coefficient taking into account the depth of application of the load on the pile heel.

Based on the fact that $E_{cr} \gg E$, the rate of settlement from forces on the lateral surface is equal to the rate of settlement from the action of forces at the level of the lower end of the pile. Equating (13) and (14), as well as taking into account (5) and (12), we obtain:

$$p_2 = \frac{(p_1 - p_2) \frac{a^2}{2l} \ln \left(\frac{b}{a}\right) - p_2 \frac{a^2}{2l} \ln \left(\frac{b}{a}\right)}{\pi a (1 - \nu_0) K} = \frac{\pi a (1 - \nu_0) K}{4G_0} + \frac{1}{G},$$

$$\text{(15)}$$

After performing certain transformations, we get the following differential equation:

$$\dot{p}_2 + p_2 \frac{1}{\eta(t) A} = \frac{p_1}{\eta(t) A},$$

$$\text{где}$$

$$A = \frac{\pi (1 - \nu_0) K l}{2G_0 a \ln \left(\frac{b}{a}\right)} + \frac{1}{G},$$

$$\text{для}$$

$$\text{(17)}$$

The general solution of the differential equation (16) is found by the formula [25]:

$$p_2(t) = e^{-\frac{dt}{\eta(t) A}} \left( \frac{p_1}{\eta(t) A} \right) e^{\frac{dt}{\eta(t) A}} dt + C,$$

$$\text{для}$$

$$\text{(18)}$$

To describe the creep process, we use the rheological parameters of hardening and softening. Let us consider the solution of Eq. (18) taking into account hardening, when $\eta(t) = \eta_0 e^{\alpha \tau}$. In this case:

$$p_2(t) = e^{-\frac{dt}{\eta(t) A}} \left( \frac{p_1}{\eta(t) A} \right) e^{\frac{dt}{\eta(t) A}} dt + C,$$

$$\text{для}$$

$$\text{(19)}$$

where $\eta_0$ is initial coefficient of soil viscosity; $\alpha$ is soil hardening factor.

$$p_2(t) = e^{\alpha \eta_0 A} \left( \frac{p_1}{\eta_0 A} e^{\alpha \eta_0 A} dt + C \right) = e^{\alpha \eta_0 A} \left( p_1 e^{\alpha \eta_0 A} dt + C \right) = p_1 + C e^{\alpha \eta_0 A}$$

Integration constant $C$ is determined from the initial condition at $t=0$. Then:

$$C = (p_2(0) - p_1) e^{-1}$$

$$\text{для}$$

$$\text{(21)}$$

Finally, we obtain:

$$p_2(t) = p_1 + (p_2(0) - p_1) e^{\alpha \eta_0 A}.$$  

$$\text{для}$$

$$\text{(22)}$$

The settlement of the pile at a certain point in time $t$ can be determined by the formula:

$$V(t) = p_2(t) \frac{\pi a (1 - \nu_0) K}{4G_0},$$

$$\text{для}$$

$$\text{(23)}$$

where $p_2(t)$ we find by the formula (22).

Consider the solution to Eqs. (22) and (23) with the initial condition $p_2(0) = 0$ with variable values $\eta_1 = 1 \cdot 10^{11} \text{ Pa}$, $\eta_2 = 5 \cdot 10^{11} \text{ Pa}$, $\eta_3 = 1 \cdot 10^{12} \text{ Pa}$, $\eta_4 = 5 \cdot 10^{12} \text{ Pa}$, $\alpha = 0.05$; $l = 30 \text{ m}$; $a = 0.5 \text{ m}$; $b = 6.5 \cdot a$; $E_1 = 30 \text{ MPa}$; $E_2 = 10 \text{ MPa}$; $E_3 = 25 \text{ MPa}$; $E_0 = 50 \text{ MPa}$; $\nu_1 = \nu_2 = \nu_3 = \nu_0 = 0.35$; $K = 0.7$.
The general solution of the differential equation (16) is found by the formula (25): 

$$ p_2(t) = e^{-\pi t} \left( \int p_1 e^{-\beta t} \, dt + C \right) $$

where $\eta_0$ is the initial coefficient of soil viscosity; $\beta$ is the soil softening factor.

Integration constant C is determined from the initial condition at. Then:

Consider the solution to Eqs. (22) and (23) for variable values $\alpha_1 = 0.10, \alpha_2 = 0.15, \alpha_3 = 0.20, \alpha_4 = 0.25$, as well as $\eta_1 = 1.10^{12}/s$; $l = 30$ m; $a = 0.5$ m; $b = 6.5 \cdot a$; $E_1 = 30$ MPa, $E_2 = 10$ MPa, $E_3 = 25$ MPa, $E_0 = 50$ MPa; $\nu_1 = \nu_2 = \nu_3 = \nu_0 = 0.35$; $K = 0.7$.

Let us consider the solution of Eq. (18) taking into account the softening, when $\eta(t) = \eta_0 e^{-\beta t}$. In this case:

$$ p_2(t) = e^{-\pi t} \eta_0 e^{-\beta t} \left( \int p_1 e^{-\beta t} \, dt + C \right), $$

where $\eta_0$ is the initial coefficient of soil viscosity; $\beta$ is the soil softening factor.

$$ p_2(t) = e^{-\pi t} \eta_0 e^{-\beta t} \left( p_1 + C \right). $$

Figure 2. Dependency graphs $p_2(t)$ (up) and $V_R(t)$ (down) at various parameters of the viscosity of the surrounding soil.
Consider the solution to Eqs. (27) and (23) with the initial condition \( p_2(0) = 0 \) with variable values \( \eta_1 = 1 \cdot 10^{11} \), \( \eta_2 = 5 \cdot 10^{11} \), \( \eta_3 = 1 \cdot 10^{12} \), \( \eta_4 = 5 \cdot 10^{12} \), as well as \( \beta = 0.05; l = 30 \text{ m}; a = 0.5 \text{ m}; b = 6.5 \cdot a; E_1 = 30 \text{ MPa}, E_2 = 10 \text{ MPa}, E_3 = 25 \text{ MPa}, E_0 = 50 \text{ MPa}; v_1 = v_2 = v_3 = v_0 = 0.35; K = 0.7 \)

Consider the solution to Eqs. (27) and (23) for variable values \( \beta_1 = 0.10, \beta_2 = 0.15, \beta_3 = 0.20 \), \( \beta_4 = 0.25 \), as well as \( \eta_1 = 1 \cdot 10^{12} \); \( l = 30 \text{ m}; a = 0.5 \text{ m}; b = 6.5 \cdot a; E_1 = 30 \text{ MPa}, E_2 = 10 \text{ MPa}, E_3 = 25 \text{ MPa}, E_0 = 50 \text{ MPa}; v_1 = v_2 = v_3 = v_0 = 0.35; K = 0.7 \)

\[ C = \frac{(p_2(0) - p_1)}{\eta_0^2} \]  

Finally, we obtain:

\[ p_2(t) = p_1 + (p_2(0) - p_1) e^{-\frac{2(e^{0.05} - 1)}{\eta_0}}. \]  

\[ \text{(27)} \]  

**Figure 4.** Dependency graphs \( p_2(t) \) (up) and \( V_R(t) \) (down) at various parameters of the viscosity of the surrounding soil

**Figure 5.** Dependency graphs \( p_2(t) \) (up) and \( V_R(t) \) (down) at different values of the hardening coefficient
RESULTS

Analysis of the obtained dependences shows that over time, the stress under the pile base and the pile settlement change at different rates and tend to a constant value (for \( t \to \infty \), \( p_{2e}(t) \to p_{2e} = \text{const} \), \( V_{p}(t) \to V_{p,\infty} = \text{const} \)). Therefore, based on condition (5), the shear stresses on the lateral surface of the pile decrease with time. The obtained dependencies make it possible to predict the development of precipitation in time. According to the data obtained, with an increase in the viscosity coefficient, a decrease in the rate of stress change under the pile base, as well as pile settlement, is observed. According to the obtained dependences (Fig. 3, 5), with an increase in the hardening coefficient, the final pressure under the pile base and the pile settlement decreases. Taking into account the rheological coefficient of softening, a complex nature of the time redistribution of efforts between the head and the fifth pile is observed.

CONCLUSION AND DISCUSSION

When the pile interacts with the surrounding multilayer soil massif, which has elastic-viscous characteristics, a complex stress-strain state arises, in which the stress under the pile base changes over time. According to the obtained dependences, over time, the stresses on the heel of the pile increase, while the shear stresses decrease. The obtained dependences allow predicting the development of pile settlement in time. The rheological properties of a multilayer soil massif have a significant impact on the nature of the redistribution of forces on the pile between the lateral surface and the lower end. The results obtained indicate that the formation of stress-strain state around the pile in time depends on the form of the function of changing the viscosity parameters. Analysis of the graphs obtained shows that the time of stabilization of the sediment, as well as the time of stabilization of the pressure under the heel of the pile, significantly depends on the rheological coefficients of hardening and softening.

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