The Euler characteristic and the first Chern number in the covariant phase space formulation of string theory

R. Cartas-Fuentevilla

Instituto de Física, Universidad Autónoma de Puebla, Apartado postal J-48 72570, Puebla Pue., México (rcartas@sirio.ifuap.buap.mx).

Using a covariant description of the geometry of deformations for extendons, it is shown that the topological corrections for the string action associated with the Euler characteristic and the first Chern number of the normal bundle of the worldsheet, although do not give dynamics to the string, modify the symplectic properties of the covariant phase space of the theory. Future extensions of the present results are outlined.

Keywords: string theory, p-branes, covariant canonical formulation, topological invariants.

I. Introduction

As it is well known, string theory contains two natural topological invariants related with the different topologies of the two-dimensional world surface embedded in a background spacetime. Specifically, the Gauss-Bonnet action, which depends only on the purely intrinsic properties of the world surface, corresponds to the Euler characteristic, which counts the number of holes or handles of the world surface. Additionally, the first Chern number of the normal bundle of the world surface, which depends on the extrinsic properties of the world surface embedded in a (four-dimensional) background spacetime, gives us essentially the number of self-intersections of the world surface. Such topological invariants do not contribute effectively as Lagrangian terms to the string dynamics, although it is well known also that there is a global contribution in the path integral formulation of the theory, weighting the different topologies in the sum over world surfaces.
On the other hand, in the canonical formulations of the theory for quantization, based on the classical dynamics of the theory, the topological terms will have no any contribution, since the dynamics remains unmodified. The fact that such terms play a nontrivial role in the path integral formulation of the theory, and do not appear at all in the canonical scheme, is somewhat suspicious at first glance. Hence, the aim of this work is to show that, on the basis of a covariant description of the canonical formulation of the theory, the topological terms in the string action may have indeed a physically relevant contribution on the symplectic structure constructed on the corresponding covariant phase space. With these results, we give a new relevant role of the topological terms within a canonical formulation, which is completely unknown in the literature, levelling thus the roles of such terms in both approaches for quantization.

In the next section, we outline the covariant canonical formalism, and in Sec. III we give some remarks on the covariant canonical formulation of the Dirac-Nambu-Goto (DNG) action, in order to prepare the background for the subsequent sections, where the topological terms will be worked out. In Sec. VII we conclude with some discussions about our results and future extensions.

II. Covariant phase space and the exterior calculus

In this section we summarize the exterior calculus on the covariant phase space given in Ref. [1] but adjusted for the treatment of embeddings [2].

According to Ref. [1], in a given physical theory, the classical phase space is the space of solutions of the classical equations of motion, which corresponds to a manifestly covariant definition. The basic idea of the covariant description of the canonical formalism is to construct a symplectic structure on such a phase space, instead of choosing p’s and q’s.

In the present case, the phase space is the space of solutions of (12), and we shall call it Z. Any background quantity, will be associated with zero-forms on Z. The deformation operator δ (see Sec. III) acts as an exterior derivative on Z, taking k-forms into (k + 1)-forms, and it should satisfy

$$\delta^2 = 0,$$

(1)
and the Leibniz rule
\[ \delta(AB) = \delta A B + (-1)^A A \delta B. \] (2)

In particular, \( \delta X^\mu \) is the exterior derivative of the zero-form \( X^\mu \) [see Eq. (5)], and it will be closed,
\[ \delta^2 X^\mu = 0. \] (3)

Furthermore, since \( \phi^i = n^i_\mu \delta X^\mu \), and \( n^i_\mu \) corresponds to zero-forms on \( Z \), the scalar fields \( \phi^i \) are one-forms on \( Z \), and thus are anticommutating objects: \( \phi^i \phi^j = -\phi^j \phi^i \). This property allows us to verify that, being the vector field \( \delta = n^i_\mu \phi_i \), thus \( \delta^2 = n^i_\mu n^j_\lambda \phi_i \phi_j \), which vanishes because of the commutativity of the zero-forms \( n^i \) and the anticommutativity of the \( \phi^i \) on \( Z \), in full agreement with Eq. (1). It is important to mention, at this point, that the covariant deformation operator \( D_\delta \) (see Sec. III) also works as an exterior derivative on \( Z \), in the sense that maps \( k \)-forms into \((k+1)\)-forms; however \( D_\delta^2 \) does not vanish necessarily.

We can determine certain two-forms on \( Z \) that will be useful for our present purposes. Considering that \( \delta \equiv \delta X^\mu \partial_\mu \) and \( D_\delta \equiv \delta X^\mu D_\mu \), we can show that \( D_\delta (\delta X^\mu) \) vanishes
\[ D_\delta (\delta X^\mu) = \delta X^\alpha D_\alpha \delta X^\mu = \delta X^\alpha [\partial_\alpha \delta X^\mu + \Gamma^\mu_{\alpha \lambda} \delta X^\lambda] = \delta^2 X^\mu + \Gamma^\mu_{\alpha \lambda} \delta X^\alpha \delta X^\lambda = 0, \] (4)
where the first term vanishes according to Eq. (3), and the second one because of the symmetry of \( \Gamma^\mu_{\alpha \lambda} \) in the indices \( \alpha \) and \( \lambda \), and the anticommutativity of \( \delta X^\alpha \) and \( \delta X^\lambda \). Hence, Eq. (4) suggests that \( D_\delta \) is, as well as \( \delta \), a measure of the closeness of \( \delta X^\mu \) on \( Z \).

### III. Covariant canonical formulation for DNG \( p \)-branes in a curved background

It will be convenient to do the general treatment for \( p \)-branes, and then to consider the particular case of string theory (1-brane), which will show the particularities of string theory as opposed to the other higher-dimensional objects.

In Ref. [2], it is shown that there exists an identically closed two-form on the space of solutions of the classical equations of motion (modulo gauge transformations) for \( p \)-branes propagating in a curved background, endowing to the physical phase space \( \hat{Z} \) of a symplectic structure. However, a more detailed study [3] shows that such a closed two-form is even an exact two-form, obtained
by direct exterior derivative of a *symplectic potential*, a global one-form on the phase space. The strategy in Ref. [3] for obtaining the global symplectic potential directly from the variations of the corresponding Lagrangian is as follows.

In the geometry of deformations of $p$-branes developed in Ref. [4], it is assumed that an infinitesimal deformation tangent to the world surface is not physically relevant, since it can be identified always with the action of a world surface diffeomorphism. However, as claimed in Ref. [3], it is precisely such an infinitesimal diffeomorphism that plays the role of our global symplectic potential on the phase space (and it is the first example showing that a *spurious* quantity in a conventional sense, may be physically relevant on the phase space). Hence, we will maintain explicitly a world surface diffeomorphism from the beginning, modifying slightly the original deformation scheme given in Ref. [4].

In this manner, following Ref. [4], the deformation of the world surface swept out by a $p$-brane (propagating in a curved background) is given by the infinitesimal spacetime variation

$$\xi^\mu \equiv \delta X^\mu = n_i^\mu \phi^i + e_a^\mu \phi^a,$$

(5)

where $n_i$ correspond to vector fields normal to the world surface and, $e_a$ to the vector fields tangent to such a surface [4]. Hence, considering that $D_\mu$ is the background torsionless covariant derivative, in Ref. [4] the normal deformation operator is defined as

$$D_\delta \equiv \delta^\mu D_\mu, \quad \delta \equiv n_i \phi^i,$$

(6)

and it is found that [4]

$$D_\delta e_a = (K_{ab}^i \phi^b) e^b + (\bar{\nabla}_a \phi_i) n^i,$$

$$D_\delta \gamma_{ab} = 2K_{ab}^i \phi_i,$$

$$D_\delta \sqrt{-\gamma} = \sqrt{-\gamma} K^i \phi_i,$$

(7)

which will be useful below. We define here the tangential deformation operator as

$$D_\Delta \equiv \Delta^\mu D_\mu, \quad \Delta \equiv e_a \phi^a,$$

(8)

and using the generalized Gauss-Weingarten equations, we find that

$$D_\Delta e_a = (\nabla_a \phi^b) e_b - K_{ab}^i \phi^b n_i,$$
\[ D_\Delta \gamma_{ab} = \nabla_a \phi_b + \nabla_b \phi_a, \]
\[ D_\Delta \sqrt{-\gamma} = \sqrt{-\gamma} \nabla_a \phi^a. \]  
\( (9) \)

In this manner, the action for DNG \( p \)-branes,

\[ S_0 = -\sigma_0 \int d^D \xi \sqrt{-\gamma}, \]
\( (10) \)

considering world surface diffeomorphisms, has as first variation

\[ 0 = (D_\delta + D_\Delta) S_0 = -\sigma_0 \int d^D \xi \sqrt{-\gamma} K^i \phi_i - \sigma_0 \int d^D \xi \sqrt{-\gamma} \nabla_a \phi^a, \]  
\( (11) \)

where the last of Eqs. (7) and (9) have been considered; from Eq. (11) we can see that \( D_\Delta S_0 \) is associated with a total divergence that can be indeed negligible, since it does not contribute locally to the dynamics.

Therefore, from Eq. (11), the equations of motion for extremal surfaces are

\[ K^i = 0, \]
\( (12) \)

whose set of solutions defines, in fact, the covariant phase space \( Z \) of the theory.

On the other hand, from the deformation dynamics [obtained linearizing Eq. (12)], and using the scheme of (self-)adjoint operators, it is found that the current \( j^a = \phi^i \tilde{\nabla}^a \phi^i \) is world surface covariantly conserved (\( \nabla_a j^a = 0 \)), and corresponds to a closed two-form on the phase space \( D_\delta j^a = 0 \) [2]. Therefore, one can finally construct a covariant and gauge invariant symplectic structure \( \omega \) for the theory [2],

\[ \omega \equiv \int_\Sigma \sqrt{-\gamma} j^a d\Sigma_a, \]
\( (13) \)

independent on the choice of \( \Sigma \) (a spacelike section of the world surface corresponding to a Cauchy \( p \)-surface for the configuration of the \( p \)-brane). However, the symplectic current \( j^a \) is even an exact two-form [3], since from Eqs. (4), (5), and the first of Eqs. (7) one finds that

\[ \delta \phi^a = D_\delta (e^a_\mu \delta X^\mu) = -j^a, \]  
\( (14) \)

and \( j^a \) is in particular a closed form, \( \delta j^a = 0 \), because of the nilpotency of \( \delta \) [see Eq. (1)]. Therefore \( \phi^a \) (the tangential projection of the deformation of the embedding), coming directly from the pure
divergence term in Eq. (11), plays the role of a global symplectic potential on the phase space. As pointed out above, $\phi^a$ gives no dynamics to the string, but it is physically relevant on the phase space, in accordance with Eq. (14).

It is important to emphasize here, the significance of a covariant and gauge invariant symplectic structure for the theory. $\omega$ in Eq. (13) represents a complete Hamiltonian description of the covariant phase space, preserving manifestly all relevant symmetries of the theory. Hence, $\omega$ represents a starting point for the study of the symmetry aspects and also a covariant description of the canonical formulation of the theory for quantization. For the general features of the covariant phase space formulation, see for example Ref. [1]. Note, however, that the concept of symplectic potential does not appear at all in Ref. [1].

In the next section we will follow the same procedure employed in this section for determining the symplectic potential $\phi^a$, in order to find the contribution of the Gauss-Bonnet topological term in the action on the covariant phase space of the theory.

**IV. The Gauss-Bonnet topological term**

The Gauss-Bonnet term for an arbitrary closed $p$-brane without physical boundaries is proportional to the Ricci scalar $\mathcal{R}$ constructed from the world surface metric $\gamma_{ab}$,

$$\chi \equiv \sigma_1 \int d^D \xi \sqrt{-\gamma} \mathcal{R},$$

(15)

whose first variation, according to the geometry of deformations of Ref.[4], is given by

$$D_\delta (\sqrt{-\gamma} \mathcal{R}) = -2\sqrt{-\gamma} G_{ab} K_{i}^{ab} \phi^i + \sqrt{-\gamma} \nabla_a \psi^a_{\mathcal{N}},$$

(16)

where $G_{ab}$ is the world surface Einstein tensor

$$G_{ab} = \mathcal{R}_{ab} - \frac{1}{2} \gamma_{ab} \mathcal{R},$$

(17)

and

$$\psi^a_{\mathcal{N}} = \gamma^{ab} D_\delta \gamma^c_{bc} - \gamma^{bc} D_\delta \gamma^a_{bc},$$

(18)
is the analogous to $\phi^a$ in Section III. Note that in Ref. [4], the pure divergence term involving $\psi_N^a$ is completely ignored (which is correct in a conventional analysis of the brane dynamics), without suspecting the relevant role that such a term will play in a phase space formulation of the theory.

In this manner, Eq. (16) gives the universal contribution of the Gauss-Bonnet term on the brane dynamics (the first term on the right-hand side), and on the covariant phase space through $\psi_N^a$. In this sense, there is no surprise for an arbitrary $p$-brane, since $\chi$ changes the dynamics and correspondingly the symplectic structure on the phase space. Nevertheless, as it is well known, in a two-dimensional world surface (and only for such a case), swept out for a string, the world surface Einstein tensor vanishes (for an arbitrary embedding background dimension),

$$G_{ab} = 0,$$

and there is not effect on the string dynamics. However, there is a nontrivial contribution of the topological term on the phase space through symplectic potential (18), independently on the null contribution at the level of the string dynamics. For example, in the case considered in Sec. III, if Eqs. (12) are the equations of motion for the closed string dynamics described by external surfaces, the inclusion of the topological term $\chi$ in the action leaves Eqs. (12) unchanged (and thus the phase space itself, defined as the space of solutions of the equations of motion, is unaltered), but the corresponding symplectic potential on the phase space is no longer $-\sigma_0 \phi^a$, but $-\sigma_0 \phi^a + \sigma_1 \psi_N^a$, where $\psi_N^a$ is given in Eq. (18).

Although we have considered only the DNG closed strings as the reference Lagrangian term for the inclusion of the GB term, the main idea is to show that $\sigma_1 \psi_N^a$ is the universal contribution of the latter on the phase space, and similarly for any action describing strings, for example, some action including terms with curvature corrections [3]. Therefore, if $\Phi^a$ is the symplectic potential for such a general action, we can construct the symplectic structure $\omega$ as

$$\omega = \int_{\Sigma} D\delta \sqrt{-\gamma}(\Phi^a + \sigma_1 \psi_N^a)d\Sigma_a,$$

with the wanted properties of closeness ($\delta \omega = 0$). The closeness of $\omega$ is equivalent to the Jacobi identity that the Poisson brackets satisfy, in the usual Hamiltonian scheme [5].

Considering the deformation formulas of Ref. [4], we can determine explicitly the universal contribution of $\psi_N^a$ to the symplectic current of the theory, in terms of $\phi^i$, the only measure of the
deformation that cannot be gauged away,

\[ D_\beta \psi_N^a = -2\phi_i \{ K^{aib} \nabla_b (K^{ij}_c \phi_j) + K^{bcj} \nabla_b (K^{aj}_c \phi_j) + \nabla_c (K^{aj}_b \phi_j) - \nabla^a (K^{ij}_b \phi_j) \}; \]  

which will constitute the universal integral kernel of the Euler characteristic on the symplectic geometry of string theory. Note that even in the simplest case of a DNG closed string dynamics described by Eq. (12), \( D_\beta \psi_N^a \) in Eq. (21) does not vanish. Therefore, the topological term modifies drastically the symplectic properties of the phase space of the theory, without changing the dynamics and the phase space itself.

V. The first Chern number of the normal bundle of the worldsheet

The self-intersection number of the worldsheet embedded in a four-dimensional background space-time, given essentially by the first Chen number of the normal bundle of the worldsheet, has the analytic expression [6, 7]

\[ \nu = \sigma_2 \int d^2 \xi \Omega, \]  

in terms of the extrinsic twist curvature:

\[ \Omega = \frac{1}{2} \epsilon_{ij} \epsilon^{ab} \Omega_{ab}^{ij}, \]  

\[ \Omega_{ab}^{ij} = \partial_b \omega_a^{ij} - \partial_a \omega_b^{ij}, \]

where \( \omega_a^{ij} \) corresponds to the extrinsic twist potential [4, 7]. Using Eqs. (23), \( \nu \) can be rewritten explicitly as a topological invariant in terms of a total divergence,

\[ \nu = \sigma_2 \int d^2 \xi \sqrt{-\gamma} \nabla_a (\epsilon_{ij} \epsilon^{ab} \omega_b^{ij}), \]  

where \( \epsilon^{ab} = \sqrt{-\gamma} \epsilon^{ab} \) [7]. In order to determine the variation of \( \nu \), we exploit the frame gauge dependence of the potential \( \omega_b^{ij} \), which means that it can always be set equal to zero at any single chosen point by an appropriate choice of the normal frame,

\[ \delta \nu = \sigma_2 \int d^2 \xi \sqrt{-\gamma} \nabla_a (\epsilon_{ij} \epsilon^{ab} \delta \omega_b^{ij}), \]
where the (normal) variation of the twist potential is given in terms of $\phi^i$ by [4, 7]

$$\delta \omega_{ij}^b = -2K_{cb}^i \nabla^c \phi^j + R_{\mu
u\alpha\beta} n^{\mu i} n^{\nu j} n^{\alpha k} e^\beta_b \phi_k,$$

where $R_{\mu
u\alpha\beta}$ is the Riemann tensor of the (four-dimensional) background spacetime. In this manner, following the ideas of the present work, from Eq. (25) we can identify $\Theta^a = \sigma_2 \epsilon_{ij} e^{ab} \delta \omega_{ij}^b$ as a symplectic potential for $\nu$. Thus, the contribution of $\nu$ on the integral Kernel of the symplectic structure of the theory is given by the deformation (exterior derivative) of $\Theta^a$,

$$\delta \Theta^a = -K_i \phi^i \Theta^a,$$

where we have considered that $\delta \sqrt{-\gamma} = \sqrt{-\gamma} K_i \phi^i$ [4]. Note that the effect of adding $\nu$ to the DNG string action is, in addition to leave unaltered the dynamics governed by $K^i = 0$ [Eq. (12)], to leave unchanged the symplectic structure for the theory (unlike the case of $\chi$ in Sec. IV), since in this case $\delta \Theta^a = 0$ in accordance with Eq. (27); of course the situation is different in a more general case than that described by the DNG action, where in general $K^i \neq 0$. Therefore, if $\Psi^a$ is the symplectic potential for such a general action (which may include, for example, the Gauss-Bonnet term $\chi$ considered in Sec. IV), the symplectic structure $\omega$ of string theory (in four b dimensions) including the term $\nu$ will take the form

$$\omega = \int_{\Sigma} D_\delta (\sqrt{-\gamma} \Psi^a) d\Sigma_a + \int_{\Sigma} D_\delta \Theta^a d\Sigma_a,$$

which is evidently closed.

The same argument employed in Ref. [2] for demonstrating the nondegeneracy of the symplectic structure for DNG branes works for the contributions of the topological terms on the symplectic structure of the theory. In this manner the $\omega$’s in (20) and (28) are nondegenerate and are defined on the reduced phase space $Z/G$, with $G$ being the volume of the group of infinitesimal spacetime diffeomorphisms [2].

6. Remarks on open and closed strings

In Ref. [7] open strings with topologically inspired boundary conditions are considered, specifically the topological terms considered here for closed strings. In both cases, such topological terms
do not affect the equations of motion; however, in the case of open strings such terms lead to boundary conditions to be implemented in addition to the equations of motion, unlike the case treated here of closed strings, where such complementary conditions do not appear at all, because of the absence of physical boundaries. Thus, it is opportune to emphasize the results presented in this work: the topological terms for closed strings do not modify the classical dynamics, neither imposing any additional condition, but contributing explicitly to the symplectic structure of the theory, and thus they may have possible quantum effects.

It is important to remark also that the symplectic potentials for the topological terms $\chi$ and $\nu$ are the same for both closed and open strings; however for the former the symplectic structure will be constructed on the phase space defined by the (unmodified) equations of motion, and for the latter on a restriction of the same phase space defined by the complementary conditions mentioned above.

7. Remarks and prospects

In Ref. [8], it is proved that the topological terms modify drastically the deformation dynamics of string theory, in such a way that the symplectic current obtained as a consequence of the self-adjointness of that deformation dynamics, is in full agreement with the currents obtained in the present treatment calculating the variations of the symplectic potentials. Specifically it is proved that the symplectic currents of the topological terms in Eqs. (20) and (28) are worldsheet covariantly conserved: \( \nabla_a (D_b \sqrt{-\gamma} \psi^a_N) = 0 = \nabla_a (D_b \Theta^a) \), which ensures that the $\omega$’s in (20) and (28) are independent on the choice of $\Sigma$. It is important to mention that such worldsheet conserved currents that play the role of integral kernels for the symplectic structures on $Z$ in the present approach, can be considered, in a more ordinary sense, as Noetherian currents for the topological invariants considered, which will allow us in this sense to obtain conserved currents associated with any continuous symmetries of the background. In the context of brane dynamics in the current literature, these conserved currents only have been considered in this conventional sense [9].

Since a symplectic structure $\omega$ governs the transition between the classical and quantum do-
mains, and allows us to consider also the aspects of symmetry of the theory, it may be interesting to study the possible contribution of the topological terms on the Poincaré charges, Poincaré algebra, and the relevant commutation relations of the theory. In this sense, because of the presence of the topological terms, the quantum version of string theory obtained from the unmodified classical dynamics, may be radically different to that obtained from the global description of the phase space given in terms of $\omega$. All these questions will be the subject of forthcoming communications.

ACKNOWLEDGMENTS

The author acknowledges the financial support by the Sistema Nacional de Investigadores (México), and wishes to thank A. Escalante for conversations.

References

[1] C. Crncović and E. Witten, in Three Hundred Years of Gravitation, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1987).
[2] R. Cartas-Fuentevilla, Class. Quantum Grav., 19, 3571 (2002).
[3] A. Escalante-Hernandez, “Basic symplectic geometry for p-branes with thickness in a curved background”, in preparation (2004).
[4] R. Capovilla, and J. Guven, Phys. Rev. D, 51, 6736 (1995).
[5] N. M. J. Woodhouse, Geometric Quantization, (Oxford University Press, New York, 1990).
[6] A. M. Polyakov, Gauge fields and strings, (Hardwood Academic Press, 1987).
[7] R. Capovilla, and J. Guven, Class. Quant. Grav., 15, 1111 (1998).
[8] A. Escalante-Hernandez, “Deformation dynamics and the topological terms in the covariant phase space formulation for string theory”, in preparation (2004).
[9] G. Arreaga, R. Capovilla, and J. Guven, Annals Phys. 279, 126(2000); R. Battye, and B. Carter, Class. Quantum Grav., 17, 3325 (2000); B. Carter, 1997 *Brane dynamics for treatment of cosmic strings and vortons*, in *Recent Developments in Gravitation and Mathematics, Proc. 2nd Mexican School on Gravitation and Mathematical Physics (Tlaxcala, 1996)* (http://kaluza.physik.unicolmb.de/2MS) ed. A. Garcia, C. Lammerzahl, A. Macias and D. Nuñez (Konstanz: Science Network).