Research Article

Weighted Composition Operators between the Fractional Cauchy Spaces and the Bloch-Type Spaces

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We characterize boundedness and compactness of weighted composition operators mapping the families of fractional Cauchy transforms into the Bloch-type spaces. Corollaries are obtained about composition operators and multiplication operators.

1. Introduction

Let $D$ denote the open unit disc in the complex plane and let $H(D)$ denote the space of functions analytic in $D$. Let $M$ denote the Banach space of complex-valued Borel measures on $T = \{|\xi| : |\xi| = 1\}$, endowed with the total variation norm.

For $\alpha > 0$, the space $F_\alpha$ of fractional Cauchy transforms is the collection of functions of the form

$$ f(z) = \int_T \frac{1}{1 - \xi z^\alpha} d\mu(\xi) \quad (z \in D), $$

(1)

where $\mu \in M$. The principal branch of the logarithm is used here. The space $F_\alpha$ is a Banach space, with norm given by

$$ \|f\|_{F_\alpha} = \inf \|\mu\|, $$

(2)

where $\mu$ varies over all measures in $M$ for which (1) holds. The families $F_\alpha$ have been studied extensively [1, 2].

Let $\beta > 0$. The Bloch-type space $B^\beta$ is the Banach space of functions analytic in $D$ such that $\sup_{z \in D} (1 - |z|^2)^{\beta} |f'(z)| < \infty$, with norm

$$ \|f\|_{B^\beta} = |f(0)| + \sup_{z \in D} (1 - |z|^2)^{\beta} |f'(z)|. $$

(3)

The integral representation (1) implies that $F_\alpha \subset B^{\alpha+1}$ and there is a constant depending only on $\alpha$ such that

$$ \|f\|_{B^{\alpha+1}} \leq C \|f\|_{F_\alpha} $$

(4)

for $f \in F_\alpha$.

It is known that any univalent $f \in H(D)$ belongs to $F_\alpha$ for any $\alpha > 2$. MacGregor [3] constructed a univalent function $f$ such that $f \notin F_2$. Let $g$ denote the normalized function $g = (f - f(0))/f'(0)$. Then $g \notin F_2$. Since $g \in S$, the classical family of schlicht functions, the Distortion Theorem [4] yields $g \in B^3$.

Let $\Phi$ be an analytic self-map of $D$ and let $u \in H(D)$. The weighted composition operator $W_{u,\Phi}$ is defined for $f \in H(D)$ by

$$ (W_{u,\Phi}f)(z) = u(z)f(\Phi(z)) \quad (z \in D). $$

(5)

If $u = 1$, then the operator $W_{u,\Phi}$ reduces to the composition operator $C_\Phi$ defined by $C_\Phi(f) = f \circ \Phi$. If $\Phi$ is the identity function, then $W_{u,\Phi}$ is the multiplication operator $M_u$ defined by $M_u(f) = uf$.

This paper characterizes $\Phi$ and $u$ for which $W_{u,\Phi} : F_\alpha \rightarrow B^\beta$ is bounded or compact. Corollaries are obtained for the operators $C_\Phi$ and $M_u$.

2. Boundedness

We follow the convention that $C$ denotes a positive constant, which may vary from one appearance to the next.

**Theorem 1.** Fix $\alpha > 0$ and $\beta > 0$. Let $u \in H(D)$ and let $\Phi$ be an analytic self-map of $D$. If

$$ \sup_{w \in D} |u'(w)| \left( \frac{1 - |w|^2}{1 - |\Phi(w)|^2} \right)^{\alpha} < \infty, $$

then $W_{u,\Phi}$ is bounded on $F_\alpha$. 

Corollary 1. Let $u \in H(D)$ and let $\Phi$ be an analytic self-map of $D$. If

$$ \sup_{w \in D} |u'(w)| \left( \frac{1 - |w|^2}{1 - |\Phi(w)|^2} \right)^{\alpha} < \infty, $$

then $W_{u,\Phi}$ is compact on $F_\alpha$. 


Theorem 1. Several lemmas are needed. Proofs of Lemmas 2 and 3 appear in [2].

Lemma 2. Fix \(\alpha > 0\) and let \(w \in D\). Let \(f_w(z) = 1/(1 - wz)\alpha\) for \(z \in D\). Then \(f_w \in F_\alpha\) and \(\|f_w\|_{F_\alpha} = 1\).

Lemma 3. Fix \(\alpha > 0\) and let \(f \in H(D)\). If \(f' \in F_{\alpha + 1}\), then \(f \in F_\alpha\) and there is a positive constant \(C\) independent of \(f\) such that

\[
\|f\|_{F_\alpha} \leq C \|f'\|_{F_{\alpha + 1}} + |f(0)|.
\]

The first statement in Lemma 4 is due to MacGregor [3]. The norm inequality is due to Hilschweiler and Nordgren [6].

Lemma 4. Let \(\alpha, \beta > 0\). If \(f \in F_\alpha\) and \(g \in F_\beta\), then \(fg \in F_{\alpha\beta}\) and

\[
\|fg\|_{F_{\alpha\beta}} \leq \|f\|_{F_\alpha} \|g\|_{F_\beta}.
\]

Lemma 5 will be used to develop test functions needed for the proof of the converse.

Lemma 5. Fix \(\alpha > 0\). Let \(w \in D\) and define

\[
k_w(z) = \frac{1 - |w|^2}{(1 - wz)^{\alpha + 1}} \quad (z \in D).
\]

Then \(k_w \in F_\alpha\) and there is a constant \(C\) independent of \(w\) such that \(\|k_w\|_{F_\alpha} \leq C\) for all \(w \in D\).

Proof. First assume \(\alpha = 1\) and fix generic \(w \in D\). A calculation shows that \(k_w\) is in the Hardy space \(H^1\) and \(\|k_w\|_{H^1} \leq 1\). Since the inclusion \(H^1 \subset F_1\) is bounded, this case is complete.

Fix \(\alpha > 1\). Then

\[
k_w(z) = \frac{1 - |w|^2}{(1 - wz)^{\alpha + 1}} \quad (z \in D).
\]

By the case for \(\alpha = 1\) and Lemma 2, \(k_w\) is the product of a function in \(F_1\) and a function in \(F_{\alpha - 1}\). By Lemma 4, \(k_w \in F_\alpha\) and there is a constant \(C\) independent of \(w\) such that \(\|k_w\|_{F_\alpha} \leq C\) for all \(w\).

Finally, fix \(\alpha_0 < \alpha < 1\). By the previous case,

\[
k'_w(z) = \frac{(\alpha + 1) \pi (1 - |w|^2)}{(1 - wz)^{\alpha + 2}} \in F_{\alpha + 1}
\]

and \(\|k'_w\|_{F_{\alpha + 1}} \leq C\) for all \(w\). By Lemma 3, \(k_w \in F_\alpha\) and \(\|k_w\|_{F_\alpha} \leq C\). The proof is complete.

We now prove the converse of Theorem 1. The test functions used in the proof first appeared in [5], in the context of the spaces \(B^{\alpha+1}\).

Theorem 6. Fix \(\alpha > 0\) and \(\beta > 0\). Let \(u \in H(D)\) and let \(\Phi\) be an analytic self-map of \(D\). Assume that \(W_{u,\Phi} : F_\alpha \rightarrow B^\beta\) is bounded. Then

\[
C_1 = \sup_{w \in D} \frac{|u'(w)| (1 - |w|^2)^\beta}{(1 - |\Phi(w)|^2)^{\alpha}} < \infty,
\]

\[
C_2 = \sup_{w \in D} \frac{|u(w)| (1 - |w|^2)^\beta}{(1 - |\Phi(w)|^2)^{\alpha + 1}} < \infty.
\]

Proof. Fix \(\alpha, \beta, u,\) and \(\Phi\) as described. By assumption there is a constant \(C\) independent of \(f\) such that

\[
\|W_{u,\Phi} (f)\|_{B^\beta} \leq C \|f\|_{F_\alpha}
\]

for all \(f \in F_\alpha\). The argument will first establish that \(C_1 < \infty\). Let \(w \in D\) and define

\[
g_w(z) = \frac{\alpha + 1}{(1 - \Phi(w)z)^\alpha} \cdot \frac{(1 - |\Phi(w)|^2)}{(1 - |\Phi(w)z|)^{\alpha + 1}}\]

\[(z \in D).\]

By Lemmas 2 and 5, there is a constant \(C\) such that \(\|g_w\|_{F_\alpha} \leq C\) for all \(w \in D\). Therefore

\[
\sup_{z \in D} \left(1 - |z|^2\right)^\beta \cdot \left|u'(z) g_w(\Phi(z)) + u(z) g'_w(\Phi(z)) \Phi'(z)\right|
\]

\[
\leq \|W_{u,\Phi} (g_w)\|_{B^\beta} \leq C
\]

for all \(w \in D\). Since

\[
g_w(\Phi(w)) = \frac{1}{(1 - |\Phi(w)|^2)^\alpha},\]

\[
g'_w(\Phi(w)) = 0,
\]

it follows that

\[
C_1 = \sup_{w \in D} \frac{|u'(w)| (1 - |w|^2)^\beta}{(1 - |\Phi(w)|^2)^{\alpha}} < \infty.
\]

In particular, (17) yields \(u \in B^\beta\).

To obtain the second condition in the theorem, let \(w \in D\) and define

\[
h_w(z) = \frac{1 - |\Phi(w)|^2}{(1 - \Phi(w)z)^{\alpha + 1}}\]

\[(z \in D).\]
By Lemma 5, there is a constant $C$ independent of $w$ such that
$$\|h_w\|_{F_{\alpha}} \leq C.$$ Relation (13) yields
$$\sup_{z \in D} \left(1 - |z|^2\right)^{\alpha}$$
\begin{equation}
\cdot |u'(z)h_w(\Phi(z)) + u(z)h'_w(\Phi(z)) \Phi'(z)| \leq \|W_{u,\Phi}(h_w)\|_{B^\beta} \leq C
\end{equation}
for all $w \in D$. Therefore
$$\sup_{|\Phi(z)|^{1/2} \leq |\Phi(w)|} \frac{(1 - |w|^2)^{\beta} |u(w)| |\Phi'(w)|}{(1 - |\Phi(w)|^2)^{\alpha+1}} < \infty. \quad (26)$$

Relations (23) and (26) yield
$$C_2 = \sup_{w \in D} \frac{|u(w)| |\Phi'(w)| (1 - |w|^2)^{\beta}}{(1 - |\Phi(w)|^2)^{\alpha+1}} < \infty \quad (27)$$
and the proof is complete. \hfill \Box

Let $\gamma, \beta > 0$. Ohno et al. [5] characterized $u$ and $\Phi$ for which $W_{u,\Phi} : B^\gamma \to B^\beta$ is bounded. Theorems 1 and 6 and their result yield the following corollary.

**Corollary 7.** Fix $\alpha, \beta > 0$. Let $u \in H(D)$ and let $\Phi$ be an analytic self-map.

\[ W_{u,\Phi} : F_{\alpha,\beta} \to B^\beta \text{ is bounded} \iff W_{u,\Phi} : B^{\alpha+1} \to B^\beta \text{ is bounded}. \quad (28) \]

Xiao [7] characterized the self-maps $\Phi$ for which $C_\Phi : B^\gamma \to B^\beta$ is bounded for $\gamma, \beta > 0$.

**Corollary 8.** Fix $\alpha, \beta$, and $\Phi$ as above.

\[ C_\Phi : F_{\alpha,\beta} \to B^\beta \text{ is bounded} \iff C_\Phi : B^{\alpha+1} \to B^\beta \text{ is bounded} \iff \sup_{w \in D} \frac{(1 - |w|^2)^{\beta} |\Phi'(w)|}{(1 - |\Phi(w)|^2)^{\alpha+1}} < \infty. \quad (29) \]

**Proof.** The equivalence of the first two conditions follows from Corollary 7. The equivalence of the second and third conditions is due to Xiao. \hfill \Box

Let $\gamma, \beta > 0$ and let $u \in H(D)$. The function $u$ is a multiplier of $B^\gamma$ into $B^\beta$ if $M_u(f) = uf \in B^\beta$ for every $f \in B^\gamma$. By the Closed Graph Theorem, it follows that $M_u : B^\gamma \to B^\beta$ is bounded. The collection of all such multipliers is denoted $M(B^\gamma, B^\beta)$. In [5], Ohno et al. characterized $u \in M(B^\gamma, B^\beta)$.

Let $M(F_{\alpha,\beta})$ denote the set of analytic functions $u$ for which $M_u : F_{\alpha,\beta} \to B^\beta$ is bounded. Corollary 9 follows from Corollary 7 and the characterization in [5] for the case $\gamma = \alpha + 1 > 1$.

**Corollary 9.** Fix $\alpha, \beta > 0$ and let $u \in H(D)$.

\[ M \left( F_{\alpha,\beta}^\gamma, B^\beta \right) = \begin{cases} B^{\beta-\alpha}, & \text{if } \beta > \alpha + 1, \\ H^\alpha, & \text{if } \beta = \alpha + 1, \\ \{0\}, & \text{if } \beta < \alpha + 1. \end{cases} \quad (30) \]
3. Compactness

A characterization is given for functions \( u, \Phi \) for which \( W_{u, \Phi} : F_{\alpha} \to B^\beta \) is compact.

**Lemma 10.** Fix \( \alpha > 0 \) and let \( w \in D \). Define \( L_w \) by

\[
L_w(z) = \frac{(1 - |w|^2)^2}{(1 - \overline{w}z)^\alpha + 2} \quad (z \in D).
\]

(31)

Then \( L_w \in F_{\alpha} \) and there is a constant \( C \) such that \( \|L_w\|_{F_{\alpha}} \leq C \) for all \( w \in D \).

**Proof.** First fix \( \alpha = 2 \) and let \( w \in D \). A particular case of Lemma 5 provides a constant \( C \) independent of \( w \in D \) such that

\[
\left\| \frac{1 - |w|^2}{(1 - \overline{w}z)^\alpha} \right\|_{F_{\alpha}} \leq C
\]

(32)

for all \( w \in D \). Since

\[
L_w(z) = \frac{(1 - |w|^2)^2}{(1 - \overline{w}z)^\alpha} \quad (z \in D),
\]

(33)

Lemma 4 now implies that \( L_w \in F_2 \) and \( \|L_w\|_{F_2} \leq C \) for all \( w \).

When \( \alpha > 2 \),

\[
L_w(z) = \frac{(1 - |w|^2)^2}{(1 - \overline{w}z)^\alpha} \frac{1}{(1 - |w|^2)^{\alpha - 2}}.
\]

(34)

By the previous case and Lemma 2, \( L_w \) is the product of a function in \( F_2 \) and a function in \( F_{\alpha - 2} \). By Lemma 4, \( L_w \in F_{\alpha} \) and \( \|L_w\|_{F_{\alpha}} \leq C \) for all \( w \in D \).

Fix \( \alpha, 1 \leq \alpha < 2 \). By the previous cases \( L_w \in F_{\alpha + 1} \) and

\[
\|L_w\|_{F_{\alpha + 1}} \leq C \quad \text{for all} \quad w \in D.
\]

Lemma 3 shows that \( L_w \in F_{\alpha} \) and \( \|L_w\|_{F_\alpha} \leq C \). A similar argument applies when \( 0 < \alpha < 1 \). The proof is complete. \( \square \)

Lemma 11 is the standard sequential criterion for compactness.

**Lemma 11.** Fix \( \alpha, \beta > 0 \). The operator \( W_{u, \Phi} : F_{\alpha} \to B^\beta \) is compact if and only if \( \|W_{u, \Phi}(f_n)\|_{B^\beta} \to 0 \) as \( n \to \infty \) for any sequence \( (f_n) \) in \( F_{\alpha} \) with \( \|f_n\|_{F_{\alpha}} \leq C \) and \( f_n \to 0 \) uniformly on compact subsets of \( D \) as \( n \to \infty \).

**Theorem 12.** Fix \( \alpha, \beta > 0 \). Assume that \( W_{u, \Phi} : F_{\alpha} \to B^\beta \) is bounded. The operator \( W_{u, \Phi} : F_{\alpha} \to B^\beta \) is compact if and only if

\[
\lim_{|\Phi(w)| \to 1} \frac{|u'(w)| (1 - |w|^2)^\beta}{(1 - |\Phi(w)|^2)^\alpha} = 0,
\]

(35)

and

\[
\lim_{|\Phi(w)| \to 1} \frac{|u(w)| |\Phi'(w)| (1 - |w|^2)^\beta}{(1 - |\Phi(w)|^2)^{\alpha + 1}} = 0.
\]

(36)

**Proof.** Fix \( \alpha, \beta > 0 \) and assume that \( W_{u, \Phi} : F_{\alpha} \to B^\beta \) is bounded.

First assume the limit conditions (35) and (36). Corollary 7 implies that \( W_{u, \Phi} : B^{\alpha+1} \to B^\beta \) is bounded and it now follows as in [5] that \( W_{u, \Phi} : B^{\alpha+1} \to B^\beta \) is compact. Suppose that \( (f_n) \) is a sequence in \( F_{\alpha} \) such that \( \|f_n\|_{F_{\alpha}} \leq C \) for all \( n \) and \( f_n \to 0 \) uniformly on compact subsets. By relation (4), \( \|f_n\|_{B^{\alpha+1}} \leq C \) and thus \( \|W_{u, \Phi}(f_n)\|_{B^\beta} \to 0 \) as \( n \to \infty \). By Lemma 11, \( W_{u, \Phi} : F_{\alpha} \to B^\beta \) is compact.

Now assume that \( W_{u, \Phi} : F_{\alpha} \to B^\beta \) is compact. We may assume that \( \|\Phi(w_n)\| \to 1 \) as \( n \to \infty \). For \( n = 1, 2, \ldots \) define

\[
h_n(z) = \frac{1 - |\Phi(w_n)|^2}{(1 - \overline{\Phi(w_n)}z)^{\alpha + 1}} \quad (z \in D).
\]

(37)

By Lemma 5, \( \|h_n\|_{F_\alpha} \leq C \) for all \( n \). Also \( h_n \to 0 \) uniformly on compact subsets of \( D \) as \( n \to \infty \). Thus \( \|W_{u, \Phi}(h_n)\|_{B^\beta} \to 0 \) as \( n \to \infty \) and

\[
\sup_{w \in D} |1 - |w|^2|^\beta \cdot \left| u'(w) h_n(\Phi(w)) + u(w) h_n'(\Phi(w)) \Phi'(w) \right| \to 0
\]

(38)

as \( n \to \infty \). Calculations yield

\[
(1 - |w_n|^2)^\beta \frac{|u'(w_n)|}{(1 - |\Phi(w_n)|^2)^\alpha} + (\alpha + 1) |u(w_n)| |\Phi'(w_n)| \left( \frac{|\Phi(w_n)|^2}{1 - |\Phi(w_n)|^2} \right)^{\alpha + 1} \to 0
\]

(39)

as \( n \to \infty \).

The argument will first establish that

\[
\frac{(1 - |w_n|^2)^\beta}{(1 - |\Phi(w_n)|^2)^\alpha} \to 0
\]

(40)

as \( n \to \infty \). As in [5], define the test functions

\[
f_n(z) = \frac{(\alpha + 2) (1 - |\Phi(w_n)|^2)}{(1 - \overline{\Phi(w_n)}z)^{\alpha + 1}} + \frac{(\alpha + 1) (1 - |\Phi(w_n)|^2)^2}{(1 - \overline{\Phi(w_n)}z)^{\alpha + 2}},
\]

(41)

where \( z \in D \) and \( n = 1, 2, \ldots \). Then \( f_n \to 0 \) uniformly on compact subsets as \( n \to \infty \). By Lemmas 10 and 5, there
is a constant $C$ with $\|f_n\|_{F_\alpha} \leq C$ for all $n$. It now follows that
\[
\sup_{z \in D} \left(1 - |z|^2\right)^\beta 
\cdot |u'(z) f_n(\Phi(z)) + u(z) f_n'(\Phi(z))\Phi'(z)|
\leq \|W_{u,\Phi}(f_n)\|_{B^\beta} \to 0 \quad \text{as } n \to \infty.
\] (42)
In particular,
\[
(1 - |w_n|^2)^\beta |u'(w_n) f_n(\Phi(w_n)) + u(w_n) f_n'(\Phi(w_n))\Phi'(w_n)| \to 0
\] as $n \to \infty$.
Since $f_n(\Phi(w_n)) = \frac{1}{(1 - |\Phi(w_n)|^2)\alpha}$, (44)
since $f_n'(\Phi(w_n)) = 0$, relation (40) is established. Since $(w_n)$ is a generic sequence with $|\Phi(w_n)| \to 1$ as $n \to \infty$, relation (35) holds.

To complete the proof note that relations (39) and (40) yield
\[
(1 - |w_n|^2)^\beta |u(w_n)| |\Phi'(w_n)| |\Phi(w_n)|
\leq (1 - |\Phi(w_n)|^2)^{\alpha+1} \to 0
\] as $n \to \infty$. Since $|\Phi(w_n)| \to 0$,
\[
(1 - |w_n|^2)^\beta |u(w_n)||\Phi'(w_n)|
\leq (1 - |\Phi(w_n)|^2)^{\alpha+1} \to 0
\] as $n \to \infty$. Condition (36) follows and the proof is complete. $\square$

**Corollary 13.** Fix $\alpha, \beta > 0$ and assume that $W_{u,\Phi} : F_\alpha \to B^\beta$ is bounded.
\[W_{u,\Phi} : F_\alpha \to B^\beta \text{ is compact} \iff W_{u,\Phi} : B^{\alpha+1} \to B^\beta \text{ is compact}.\]
(47)

*Proof.* The hypothesis and Corollary 7 yield that $W_{u,\Phi} : B^{\alpha+1} \to B^\beta$ is bounded.

Assume that $W_{u,\Phi} : B^{\alpha+1} \to B^\beta$ is compact. Since the inclusion $F_\alpha \subset B^{\alpha+1}$ is bounded, it follows that $W_{u,\Phi} : F_\alpha \to B^\beta$ is compact.

Assume that $W_{u,\Phi} : F_\alpha \to B^\beta$ is compact. By Theorem 12, conditions (35) and (36) hold. These conditions are sufficient to imply that the bounded operator $W_{u,\Phi} : B^{\alpha+1} \to B^\beta$ is compact [5]. $\square$

Let $\gamma, \beta > 0$ and assume that $C_{\Phi} : B^\gamma \to B^\beta$ is bounded. In [7], Xiao provided additional conditions on $\Phi$ necessary and sufficient for $C_{\Phi} : B^\gamma \to B^\beta$ to be compact.

**Corollary 14.** Fix $\alpha, \beta > 0$ and assume that $C_{\Phi} : F_\alpha \to B^\beta$ is bounded. The following are equivalent:

1. $C_{\Phi} : F_\alpha \to B^\beta$ is compact.
2. $C_{\Phi} : B^{\alpha+1} \to B^\beta$ is compact.
3. $\lim_{w_{\Phi}(w) \to 1} |(1 - |w|^2)^\beta| |\Phi'(w)|/(1 - |\Phi(w)|^2)^{\alpha+1} = 0$.

*Proof.* Corollary 13 yields the equivalence of the first and second conditions.

Since $C_{\Phi} : F_\alpha \to B^\beta$ is bounded, Corollary 8 yields that $C_{\Phi} : B^{\alpha+1} \to B^\beta$ is bounded. Under this hypothesis, Xiao [7] proved the equivalence of the second and third conditions. $\square$

Fix $\gamma, \beta > 0$. In [5], Ohno et al. characterized $u$ for which the bounded operator $M_u : B^\gamma \to B^\beta$ is compact.

Let $u \in H(D)$ and let $\beta > 0$. Recall that $u$ is in the little Bloch space $B^\beta_0$ if
\[
\lim_{|z| \to 1} (1 - |z|^2)^\beta |u'(z)| = 0.
\] (48)

Corollary 13 and the characterization in [5] for $\gamma = \alpha + 1 > 1$ yield the following result.

**Corollary 15.** Fix $\alpha, \beta > 0$ and assume $M_u : F_\alpha \to B^\beta$ is bounded.

1. Assume $\beta > \alpha + 1$. $M_u : F_\alpha \to B^\beta$ is compact $\iff u \in B^{\beta-\alpha}_0$.
2. Assume $\beta \leq \alpha + 1$. $M_u : F_\alpha \to B^\beta$ is compact $\iff u \equiv 0$.

**Conflicts of Interest**

The author declares that there are no conflicts of interest regarding the publication of this paper.

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