Spherically symmetric Yang-Mills solutions in a \((4 + n)\)-
dimensional space-time

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Abstract

We consider the Einstein-Yang-Mills Lagrangian in a \((4 + n)\)-dimensional space-time. Assuming
the matter and metric fields to be independent of the \(n\) extra coordinates, a spherical symmetric
Ansatz for the fields leads to a set of coupled ordinary differential equations. We find that for
\(n > 1\) only solutions with either one non-zero Higgs field or with all Higgs fields constant exist. We
construct the analytic solutions which fulfill these conditions for arbitrary \(n\), namely the Einstein-
Maxwell-dilaton solutions.

We also present generic solutions of the effective 4-dimensional Einstein-Yang-Mills-Higgs-dilaton
model, which possesses \(n\) Higgs triplets coupled in a specific way to \(n\) independent dilaton fields.
These solutions are the abelian Einstein-Maxwell- dilaton solutions and analytic non-abelian solu-
tions, which have diverging Higgs fields. In addition, we construct numerically asymptotically flat
and finite energy solutions for \(n = 2\).

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I. INTRODUCTION

In an attempt to unify electrodynamics and general relativity, Kaluza introduced an extra - a fifth dimension and assumed all fields to be independent of the extra dimension. Klein followed this idea, however, he assumed the fifth dimension to be compactified on a circle of Planck length. The resulting theory describes 4-dimensional Einstein gravity plus Maxwell’s equations. One of the new fields appearing in this model is the dilaton, a scalar companion of the metric tensor. In an analogue way, this field arises in the low energy effective action of superstring theories and is associated with the classical scale invariance of these models.

When studying spherically symmetric solutions in higher dimensional systems, two possible approaches seem possible: a) to assume the solutions to be spherically symmetric in the full $d$ dimensions or b) to assume the solutions to be spherically symmetric only in 4 dimensions. The solutions obtained in the first approach are surely important at very high energies, i.e. very early stages of the universe. The second approach, which assumes the extra dimensions to be “spectator” is of importance for physics in the universe today.

Volkov recently followed the idea of extra dimensions and constructed a (4+1) dimensional Einstein-Yang-Mills (EYM) system. It turned out that the EYM particles are completely destroyed by gravity, but that, however, particle-like solutions (so-called “EYM vortices”) exist if one assumes the fields to be independent of the extra coordinate. The system then reduces to an effective 4-dimensional Einstein-Yang-Mills-Higgs-dilaton (EYMHD) model, which was studied in detail in [5]. Through the dimensional reduction, one Higgs triplet and a dilaton field appears in the model.

In this paper, we study the system of [4] in $(4 + n)$ dimensions, assuming all fields to be independent of the $n$ extra coordinates. We give the model and equations of motion in Section II. The solutions are spherically symmetric in 4 dimensions, while the extra dimensions (which are “spectator”) are associated with a Ricci flat manifold. It turns out that in $n > 1$ dimensions only solutions with one non-zero Higgs field or with all Higgs fields constant exist. We give the analytic solutions which fulfill this condition for generic $n$, namely the abelian Einstein-Maxwell-dilaton (EMD). In Section III, we present the 4-dimensional effective Einstein-Yang-Mills-Higgs-dilaton (EYMHD) model, which has $n$ independent Higgs and dilaton fields. Since in this model, no constraint on the Higgs
fields arises, we construct generic solutions. We give the analytic solutions available, namely the abelian Einstein-Maxwell-dilaton (EMD) solutions and “non-abelian” solutions with diverging Higgs fields. Finally, we present our numerical solutions of the effective model for \( n = 2 \). We give our conclusions in Section IV.

II. THE \((4 + n)\)-DIMENSIONAL EINSTEIN-YANG-MILLS MODEL

The Einstein-Yang-Mills Lagrangian in \( d = (4 + n) \) dimensions is given by:

\[
S = \int \left( \frac{1}{16\pi G_{(4+n)}} R - \frac{1}{4e^2} F_{MN}^a F^{aMN} \right) \sqrt{g_{(4+n)}} d^{(4+n)} x
\]

with the SU(2) Yang-Mills field strengths \( F_{MN}^a = \partial_M A_N^a - \partial_N A_M^a + \epsilon^{abc} A_M^b A_N^c \), the gauge index \( a = 1, 2, 3 \) and the space-time index \( M = 0, ..., (4 + n) - 1 \). \( G_{(4+n)} \) and \( e \) denote respectively the \((4 + n)\)-dimensional Newton’s constant and the coupling constant of the gauge field theory. \( G_{(4+n)} \) is related to the Planck mass \( M_{pl} \) by \( G_{(4+n)} = M_{pl}^{-(2+n)} \) and \( e^2 \) has the dimension of \([\text{length}]^n\). In the following, we denote the coordinates \( x_{(3+k)} \) by \( y_k \) with \( k = 1, ..., n \).

If both the matter functions and the metric functions are independent on \( y_k \), the fields can be parametrized as follows:

\[
g^{(4+n)}_{MN} dx^M dx^N = e^{-\Xi} g^{(4)}_{\mu\nu} dx^\mu dx^\nu + \sum_{k=1}^n e^{2\zeta_k} (dy^k)^2 , \quad \mu, \nu = 0, 1, 2, 3
\]

with

\[
\Xi = \sum_{k=1}^n \zeta_k
\]

and

\[
A_M^a dx^M = A_\mu^a dx^\mu + \sum_{k=1}^n \Phi_k^a dy^k.
\]

\( g^{(4)} \) is the 4-dimensional metric tensor and the \( \zeta_j \) and \( \Phi_j^a, j = 1, ..., n \), play the role of dilatons and Higgs fields, respectively.

Note that in the case \( n = 1 \), the above parametrization coincides with the one in \([4]\).
A. Spherically symmetric Ansatz

For the metric the spherically symmetric Ansatz in Schwarzschild-like coordinates reads [6]:

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -A^2(r)N(r)dt^2 + N^{-1}(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 , \]  

with

\[ N(r) = 1 - \frac{2m(r)}{r} . \]  

(6)

In these coordinates, \( m(\infty) \) denotes the (dimensionful) mass of the field configuration.

For the gauge and Higgs fields, we use the purely magnetic hedgehog ansatz [7]:

\[ A_r^a = A_t^a = 0 , \]  

\[ A_\theta^a = (1 - K(r))e^a_\varphi , \quad A_\varphi^a = -(1 - K(r)) \sin \theta e^a_\theta , \]  

\[ \Phi_j^a = c_j v H_j(r)e^a_r , \quad j = 1, \ldots, n , \]  

(8)

(9)

where \( v \) is a mass scale, while \( c_j \) are dimensionless constants determining the vacuum expectation values of the Higgs fields \( \langle \Phi_j \rangle = c_j v \). In absence of a Higgs potential these have to be set by hand. Finally, the dilatons are scalar fields depending only on \( r \):

\[ \zeta_j = \zeta_j(r) , \quad k = 1, \ldots, n . \]  

(10)

With these conventions, the non-vanishing components of the \((4 + n)\)-dimensional energy momentum tensor read:

\[ T_{0}^0 = -e^{\Xi} \left( \sum_{k=1}^{n} (A_k + C_k) + B + D \right) , \]  

\[ T_{1}^1 = -e^{\Xi} \left( \sum_{k=1}^{n} (-A_k + C_k) - B + D \right) , \]  

\[ T_{2}^2 = T_{3}^3 = -e^{\Xi} \left( \sum_{k=1}^{n} A_k - D \right) , \]  

\[ T_{j}^j = -e^{\Xi} \left( -2(A_j + C_j) + \sum_{k=1}^{n} (A_k + C_k) + B + D \right) \quad \text{for} \quad j = 1, \ldots, n , \]  

\[ T_{j}^k = e^{\Xi} e^{-(\zeta_j + \zeta_k)} \left( \frac{N}{2} H_j'H_j' + \frac{K^2}{x^2} H_j H_k \right) \quad \text{for} \quad j \neq k , \]  

(11)

where we use the abbreviations

\[ A_j = \frac{1}{2} e^{-2\zeta_j} N(H_j')^2 , \quad C_j = e^{-2\zeta_j} K^2 H_j^2 \frac{1}{x^2} \]  

(12)
and

\[ B = e^{\Xi} \frac{x}{x^2} N(K')^2 \quad D = e^{\Xi} \frac{1}{2x^4} (K^2 - 1)^2 \]  

(13)

The radial variable \( x = evr \) was introduced and the prime denotes the derivative with respect to \( x \).

**B. Equations of motion**

Defining the coupling constant \( \alpha = \sqrt{\frac{g}{N}} \) and the mass function \( \mu = evm \) we obtain two first order equations for the two metric functions

\[ \mu' = \alpha^2 x^2 \left( \sum_{k=1}^{n} (A_k + C_k) + B + D \right) + \frac{1}{2} N x^2 \Theta , \]  

(14)

\[ A' = \alpha^2 Ax \left( \sum_{k=1}^{n} e^{-2\zeta_k} (H_k')^2 + 2 e^{\Xi} K' \right) + x A \Theta \]  

(15)

with

\[ \Theta = \frac{3}{4} \left( \sum_{k=1}^{n} (\zeta_k')^2 + \frac{2}{3} \sum_{k>k'} \zeta_k \zeta_k' \right) . \]  

(16)

The components of the Einstein equations related to the extra dimensions lead (after suitable linear combinations) to equations for the \( n \) dilaton fields:

\[ (x^2 AN \zeta_j')' = \alpha^2 A x^2 e^{2\Xi} \left[ \frac{2}{n+2} \sum_{M=0}^{d-1} T_{M}^{j} - 2T_j \right] , \quad j = 1, ..., n . \]  

(17)

The field equations for the gauge field and the \( n \) Higgs fields read respectively

\[ (e^{2\Xi} AN K')' = A \left( \frac{1}{x^2} e^{2\Xi} K(K^2 - 1) + \sum_{k=1}^{n} e^{-2\zeta_k} K H_k' \right) , \]  

(18)

\[ (e^{-2\zeta_j} x^2 AN H_j')' = 2A e^{-2\zeta_j} K^2 H_j , \quad j = 1, ..., n . \]  

(19)

Finally, since the off-diagonal components of the Einstein tenor vanish, we obtain an extra constraint on the fields from the \( jk \)-components of the energy-momentum tensor:

\[ e^{2\Xi} e^{-(\zeta_j + \zeta_k)} \left( \frac{N}{2} H_j H_k' + \frac{K^2}{x^2} H_j H_k \right) = 0 \]  

(20)

Clearly, these equations are symmetric under the simultaneous exchange \( \zeta_j, H_j \leftrightarrow \zeta_k, H_k \).

Further, they are invariant under the rescaling

\[ H_j \to \lambda^{-1} H_j , \quad x \to \lambda x , \quad \alpha \to \lambda^2 \alpha , \quad \lambda \in IR . \]  

(21)
The constraint (20) will only be fulfilled for specific cases. With our parametrisation, there are two possibilities: (i) only one non-zero Higgs field and arbitrary gauge field $K(x)$ or (ii) constant Higgs fields and $K(x) = 0$.

Since we are interested in generic solutions, we go back to the 4-dimensional effective action motivated by this model in which, of course, off-diagonal terms of the energy-momentum tensor don’t appear. Before we discuss this effective model, we give the solutions with constant Higgs fields available in the “full” model, namely the Einstein-Maxwell-dilaton (EMD) solutions.

Let us remark that there are several possibilities of modification of the “full” model considered here, which could eventually lead to solutions with generic Higgs and gauge fields, namely we could

(i) add a non-diagonal term of the form $f_{56} dx_5 dx_6$ in the metric, this leads to one extra equation for the $f_{56}$ function, but it has to be checked that this function stays regular,

(ii) start with an SU(N) gauge group and choose the different Higgs fields in orthogonal SU(2) subalgebras of the Lie algebra of SU(N).

C. Einstein-Maxwell-Dilaton (EMD) solutions

Assuming the matter fields to be constant:

$$K(x) = 0 , \ H_j(x) = c_j , \ j = 1, \ldots, n$$

and

$$\zeta_j(x) = \zeta(x) \ \ \forall \ j ,$$

we find that the above equations (14), (15), (17) admit exact solutions which are related to the Einstein-Maxwell-Dilaton (EMD) solutions [3]. Here, we will only discuss the extremal case, which corresponds to a solution with horizon at the origin.

The extremal EMD solutions have unit magnetic charge and mass

$$\frac{\mu_\infty}{\alpha^2} = \frac{1}{\alpha} \sqrt{\frac{n+2}{2(n+1)}} .$$

The value of the metric component $N(x)$ at the origin $x = 0$ reads:

$$N_{EMD}(0) = \left( \frac{n}{2(n+1)} \right)^2$$
and the dilaton field is:

\[ \zeta_{\text{EMD}}(x) = \frac{1}{n + 1} \ln \left( 1 - \frac{X_-}{X} \right) \]  

(26)

with

\[ X_- = \left( \frac{2(n + 1)}{n + 2} \right)^{1/4}, \quad \frac{x}{\alpha} = X \left( 1 - \frac{X_-}{X} \right)^{n/(2n+2)} \]  

(27)

### III. THE EFFECTIVE 4-DIMENSIONAL EINSTEIN-YANG-MILLS-HIGGS-DILATON (EYMHD) MODEL

As in the 5-dimensional case [4] (in our notation \( n = 1 \)) the equations given in the previous section (apart from the constraint (20)) can equally well be derived from an effective 4-dimensional Einstein-Yang-Mills-Higgs-Dilaton (EYMHD) Lagrangian. The Lagrangian density for the matter fields then reads:

\[
L_M = -\frac{1}{4} e^{2\Gamma} F_{\mu\nu}^a F^{a,\mu\nu} - \sum_{k=1}^n \frac{1}{2} e^{-4\kappa \Psi_k} D_\mu \Phi^a_1 D^{\mu} \Phi^a_1 \\
- \frac{1}{2} \left( \sum_{k=1}^n \partial_\mu \Psi_k \partial^\mu \Psi_k + \frac{2}{3} \sum_{k>k'} \partial_\mu \Psi_k \partial^\mu \Psi_{k'} \right)
\]

(28)

with

\[ \Gamma = \sum_{k=1}^n \Psi_k \]  

(29)

The kinetic part in the dilaton fields could be diagonalized, however, we find it more convenient to leave it in the form above which reveals the symmetry \( \Psi_j, \Phi_j \leftrightarrow \Psi_{j'}, \Phi_{j'} \).

The Lagrangian (28) is then coupled minimally to Einstein gravity according to the full action

\[
S = S_G + S_M \\
= \int \sqrt{-g^{(4)}} \left( L_G + L_M \right) d^4 x
\]

(30)

where \( L_G = R / (16\pi G_4) \), \( R \) is the Ricci scalar and \( G_4 \) is the 4-dimensional Newton’s constant.

Note that the dilaton fields are coupled by an independent coupling constant \( \kappa \) to the gauge and Higgs fields. In this respect, the dilatons here are treated as independent scalar fields, while in the action (11) they appear as parts of the metric tensor.

After the rescaling

\[ \Psi_j = v \psi_j, \quad \kappa = \frac{\gamma}{v}, \quad \alpha = v \sqrt{G_4} \]

(31)
the resulting set of equations only depends on the coupling constants $\alpha$ and $\gamma$. We refrain from giving the explicit form of the equations here, but refer the reader to [5] for the case $n = 1$.

Note that the equations (14), (15), (17), (18), (19) become equivalent to the field equations associated to (30) by using the same Ansätze for the 4-dimensional metric, the gauge and Higgs fields, but by identifying

$$
\zeta_j = 2\gamma\psi_j = 2\kappa\Psi_j, \quad \alpha^2 = 3\gamma^2.
$$

Remarkably, this identification turns out to be independent on $n$.

This model now has solutions for which all $n$ Higgs fields can be non-constant. First, we will present the 4-dimensional effective counterparts of the abelian Einstein-Maxwell-dilaton solutions. Then, we will present analytic non-abelian solutions with non-constant but diverging Higgs fields, which are not available in the full model since they don’t fulfill the constraint (20). Finally, we will give the asymptotically flat, finite energy solutions for $n = 2$, which we construct numerically.

### A. Einstein-Maxwell-Dilaton (EMD) solutions

Assuming the matter fields to be constant:

$$
K(x) = 0, \quad H_k(x) = c_k, \quad k = 1, \ldots, n
$$

and

$$
\psi_k(x) = \psi(x) \quad \forall \ k,
$$

we find that the equations associated to the effective action (30) admit exact solutions which are related to the Einstein-Maxwell-Dilaton (EMD) solutions [3]. As in the previous section, we will only discuss the extremal case. Note that now, of course, the solution depends on both $\alpha$ and $\gamma$.

The extremal EMD have unit magnetic charge and mass

$$
\frac{\mu_\infty}{\alpha^2} = \frac{1}{\sqrt{\alpha^2 + \tilde{\gamma}^2}},
$$

where

$$
\tilde{\gamma}^2 = \frac{3n}{2 + n}\gamma^2.
$$
The value of the metric component $N(x)$ at the origin $x = 0$ reads:

$$N_{EMD}(0) = \left(\frac{\tilde{\gamma}^2}{\alpha^2 + \tilde{\gamma}^2}\right)^2 \quad (37)$$

and the dilaton field is:

$$\psi_{EMD}(x) = \left(\frac{3}{2 + n}\right) \frac{1}{\alpha^2 + \tilde{\gamma}^2} \ln \left(1 - \frac{X}{X^-}\right) \quad (38)$$

with

$$X^- = \left(\frac{\alpha^2 + \tilde{\gamma}^2}{\alpha^2}\right)^{1/4}, \quad \frac{x}{\alpha} = X \left(1 - \frac{X}{X^-}\right)^{\tilde{\gamma}^2/(\alpha^2 + \tilde{\gamma}^2)} \quad (39)$$

### B. Analytic, non-abelian solutions

Very similar to the $n = 1$ case, non-abelian, analytic solutions exist in this system. If we assume the limit $\alpha^2 = 3\tilde{\gamma}^2$ (which, of course, is the most interesting one), we find solutions of the form:

$$N(x) = c_1, \quad A = x^{(1+c_2)}, \quad \psi_1(x) = \psi_2(x) = ... \equiv \psi(x) = \frac{4\alpha}{\sqrt{3n}} \ln \left(\frac{x}{c_3}\right)$$

$$H_1(x) = H_2(x) = ... = H(x) = \sqrt{c_4} \left(\frac{x}{c_3}\right)^{2/n}, \quad K(x) = \sqrt{q} \quad (40)$$

where $c_1, c_2, c_3$ and $c_4$ are constants depending only on $n$, $q$ and $\alpha$ and are given by:

$$c_1 = \frac{-2q^2n^3(n + 1) + qn^3(2n^2 + 5n + 6) - n^4}{8qn(n + 1) + 4(2 - n)}$$

$$c_2 = \frac{2}{n} \left(1 + 2\alpha^2 c_4\right)$$

$$c_3 = \frac{-4q^3\alpha^2(n + 1) + 2q^2\alpha^2(2n^2 + 7n + 8) + 4q\alpha^2(-n^2 - 3n - 3) + 2\alpha^2 n}{2q^2n(-n^2 - 3n - 2) + q(2n^4 + 5n^3 + 6n^2 + 2n - 4) - n^3}$$

$$c_4 = \frac{1 - q}{nc_3^2} \quad (41)$$

and $q$ satisfies the equation

$$q^3(2n^4 + 4n^3 + 2n^2) + q^2(-2n^5 - 3n^4 + n^3 + 10n^2 + 8n) + q(3n^4 + 2n^3 + 3n^2 + 6n + 8) - n^3 + n^2 - 2n = 0 \quad (42)$$

Solving this condition for $q$ numerically for various values of $n$, we find one real positive solution for $n = 1, 2, 3$ and three positive solutions for $n = 4, 5, 6, 7$. However, when more than one solution for $q$ are available, it turns out that only one of the solutions is acceptable since for the others some of the remaining parameters (e.g. $c_4$ or $c_3$) become negative. In Table 1, we give the acceptable solutions and their numerical values for the parameters $c_1$, $c_2$, $c_3$, $c_4$.
Table 1: Values of $q$ and constants $c_1$, $c_2$, $c_3$, $c_4$ for different $n$

| $n$ | $K(x) = \sqrt{q}$ | $c_1$ | $c_2$ | $\frac{c_3^2}{\alpha^2}$ | $\alpha^2 c_4$ |
|-----|-------------------|-------|-------|---------------------------|----------------|
| 1   | 0.293             | 0.016 | 5.246 | 1.126                     | 0.811          |
| 2   | 0.294             | 0.064 | 1.702 | 1.301                     | 0.351          |
| 3   | 0.280             | 0.102 | 0.982 | 1.297                     | 0.237          |
| 4   | no real solution  |       |       |                           |                |
| 5   | 0.249             | 0.149 | 0.495 | 1.581                     | 0.118          |
| 6   | 0.236             | 0.151 | 0.554 | 0.476                     | 0.331          |
| 7   | 0.225             | 0.086 | 2.402 | 0.036                     | 3.704          |

C. Numerical results for $n = 2$

The 4-dimensional effective theory for $n = 1$ was studied in [5].

Here, we will put the emphasis on the case $n = 2$ and again discuss the solutions in the limit $\alpha^2 = 3\gamma^2$. In this model, two Higgs fields naturally occur. Theories involving two Higgs fields are also interesting from the viewpoint of the supersymmetric extension of the Standard model.

The case $n = 2$ involves two parameters only, namely $\alpha$ and $\rho \equiv c_2/c_1$. As discussed previously in this paper, it is sufficient to consider $\rho \geq 1$.

The solutions can then be characterized by their mass:

$$M = \frac{1}{\sqrt{c_1^2 + c_2^2}} \frac{\mu(\infty)}{\alpha^2},$$

where the first factor is extracted in such a way that the mass of the solution in the $\alpha = 0$ limit is normalized to 1, corresponding to the BPS monopole.

The values $\psi_{1,2}(0)$ will appear to be useful to further characterize the solutions. The deviation of the solutions from the flat space solutions can be “measured” by the values of $A(x)$ at the origin, $A(0)$, and by the minimum of $N(x)$, say $N_m$. Since $A(x)$ always increases monotonically, $A(0)$ also represents the minimum of the function.

In the following, we will discuss our numerical results [8] for two different and fixed values of $\rho$ and varying $\alpha$.

We again discuss the solutions in the limit of $\alpha^2 = 3\gamma^2$. 

10
1. Boundary conditions

We will study globally regular, asymptotically flat solutions of the system above. This implies the following boundary conditions:

\[ K(0) = 1, \quad H_j(0) = 0, \quad \partial_x \psi_j|_{x=0} = 0, \quad \mu(0) = 0 \]  \hspace{1cm} (44)

at the origin and

\[ K(\infty) = 0, \quad H_j(\infty) = c_j, \quad \psi_j(\infty) = 0, \quad A(\infty) = 1, \]  \hspace{1cm} (45)

at infinity. We assume \( c_j \neq 0 \) in order for the solutions to have a magnetic charge. Using the symmetries discussed above, we see that we can set \( \min\{c_j, j = 1, ..., n\} = c_1 = 1 \) without losing generality and study the equations with varying \( \alpha \) and \( c_2, c_3, ..., c_n > 1 \). Of course, in the case \( c_1 = c_2 = ... = c_n = 1 \) the equations are symmetric and we expect \( H_1(x) = H_2(x) = ... H_n(x) \equiv H(x), \quad \psi_1(x) = \psi_2(x) = ... = \psi_n(x) \equiv \psi(x) \).

2. \( \rho = 1 \)

In this case, the two Higgs functions \( H_1(x) \) and \( H_2(x) \) are equal. This, of course, implies immediately that also \( \psi_1(x) = \psi_2(x) \).

In the flat limit (\( \alpha = 0 \)) the solution is the BPS monopole \([10, 11]\) with \( A(x) = N(x) = 1, \psi_1(x) = \psi_2(x) = 0 \) and the matter functions \( K(x), H_1(x) = H_2(x) \) have the well known BPS profiles.

Increasing \( \alpha \), our numerical results reveal that the solution gets progressively deformed by gravity. The solutions form a branch on which the mass \( M \) diminishes with increasing \( \alpha \). The same holds true for the values \( A(0), N_m \).

The function \( N(x) \) indeed develops a minimum which becomes deeper while gravity increases. At the same time the dilaton functions \( \psi_{1,2}(x) \) are non trivial for \( \alpha \neq 0 \). The value of \( \psi_{1,2}(x = 0) \) is negative and decreases with increasing \( \alpha \).

Our results are illustrated in Fig. [10]. The figure also demonstrates that the branch of gravitating solutions does not exist for arbitrarily large values of the parameter \( \alpha \). Indeed, we find that solutions cease to exist for \( \alpha > \alpha_{\text{max}}^{(1)} \) and \( \alpha_{\text{max}}^{(1)} \) thus constitutes a maximal value of \( \alpha \). We find that this critical value depends on the parameter \( \rho \) and is given here by:

\[ \alpha_{\text{max}}^{(1)} \approx 0.75 \quad \text{for} \quad \rho = 1 \]  \hspace{1cm} (46)
Our numerical analysis further reveals that for $\alpha \leq \alpha_{max}^{(1)}$ another branch of solution exists. The mass of the solutions on this second branch is higher than the mass of the corresponding solutions on the first branch. This is demonstrated in Fig. 2.

Our numerical results further suggest as illustrated by the figures that the solutions on the second branch stop to exist at some local minimal value $\alpha = \alpha_{min}^{(1)}$.

To be more precise, the further evolution of the branches is involved: indeed several small branches exist for $\alpha \epsilon [\alpha_{min}^{(k)}, \alpha_{max}^{(k+1)}]$ with a smaller and smaller extent in $\alpha$. Nevertheless, it appears clearly from the figures that for $\alpha \rightarrow \alpha_{cr}$, where $\alpha_{cr} := \alpha_{min}^{(k)} = \alpha_{max}^{(k+1)}$ for some $k$, the value $A(0)$ tends to zero, while the values $|\psi_{1,2}(0)|$ increase considerably and likely become infinite. Note, however, that the additional branches are not visible in the mass plot (see Fig. 2), because the numerical values are very close to those of the second branch. The existence of several branches was also noticed for the $n = 1$ system in [4] and in another Einstein-Yang-Mills model in 5 dimensions [12].

We find numerically:

$$\alpha_{min}^{(1)} \approx 0.240 , \quad \alpha_{max}^{(2)} \approx 0.294 , \quad \alpha_{cr} \approx 0.285 \quad (47)$$

At the same time, the value $N_m$ stays strictly positive and tends to the value given by (40) in the critical limit. We find numerically that $N_m \approx 0.065$ for $\alpha \approx \alpha_{cr}$, while $c_1 \approx 0.064$ for $n = 2$. The function $N(x)$ reaches this minimum at a value of $x$, which tends to zero for $\alpha \rightarrow \alpha_{cr}$. At the same time, the function $N(x)$ becomes nearly flat on a plateau surrounding this minimum.

The function $K(x)$ starts to develop oscillations around the value $\sqrt{q} = 0.294$. Processing on the branches, the number of oscillations increases and becomes infinite in the critical limit.

Very similar to what was observed in [4, 6], the Yang-Mills domain walls get disconnected from the outside world by an infinitely long throat in the strong gravity limit.

3. $\rho = 2$

In order to understand the influence of two Higgs fields with different expectation values, i.e. $\rho \neq 1$, on the domain of existence of the solutions, we studied in detail the case $\rho = 2$.

Varying $\alpha$, the existence of several branches of solutions is qualitatively very similar to the case $\rho = 1$. The numerical analysis, however, reveals that for $\rho > 1$, the solutions exist
on a smaller interval of the coupling constant $\alpha$ in comparison to the case $\rho = 1$. Namely, we find:

$$\alpha_{max}^{(1)} \approx 0.496, \quad \alpha_{min}^{(1)} \approx 0.150, \quad \alpha_{max}^{(2)} = 0.185, \quad \alpha_{cr} \approx 0.179 \quad \text{for} \quad \rho = 2. \quad (48)$$

In the critical limit for $\alpha \approx \alpha_{cr}$, we find

$$M \approx 2.05, \quad N_m \approx 0.068 \quad (49)$$

One of the apparent differences is that the two dilaton fields are non-equal. The values of $\psi_1(0)$ and $\psi_2(0)$ are superposed on Fig. 3. The difference $\psi_2(0) - \psi_1(0)$ is also shown and indicates that in the critical limit, the two dilaton profiles remain significantly different close to the origin.

The profiles of the function $K(x), N(x), \psi_1(x)$ and $\psi_2(x)$ are shown in Fig. 4 for $\alpha = 0.179$. This solution is already close to the critical solution. Note that the function $K(x)$ starts to develop oscillations completely analogue to the case $\rho = 1$.

The numerical results suggest that in the critical limit a solution of the analytic, non-abelian type described in the previous subsection is reached. The question how this solution looks like for $\rho \neq 1$ will be addressed in a future publication.

IV. CONCLUSIONS

Both string theories [13] as well as so-called “brane worlds” [14], which assume the Standard model fields to be confined on a 3-brane (that is embedded in a higher dimensional space-time) have enhanced the idea that space-time possesses more than four dimensions. In the former, the extra dimensions are compactified on a scale of the Planck length, while in the latter they are non-compact. Non-perturbative, classical solutions of field theory models certainly play a major role in these theories.

It is therefore natural to investigate the classical solutions of higher dimensional Einstein-Yang-Mills theory. In this paper, we have investigated Einstein-Yang-Mills theories in $1+3+n$ dimensions, in which the matter and metric fields are chosen to be independent of the extra $n$ coordinates. Dimensional reduction then leads to $n$ dilaton fields coupled individually to $n$ Higgs fields. These $n$ Higgs fields can have independent vacuum expectation values.
The corresponding spherically symmetric equations admit several types of solutions: (a) vacuum solutions, (b) embedded abelian Einstein-Maxwell-dilaton solutions, (c) non-abelian solutions with diverging Higgs fields and (d) fully non-abelian magnetically charged solutions. The vacuum solutions (a) are trivial, while (b) and (c) were constructed analytically for the case when all Higgs fields have identical expectation values. The solutions of type (d) had to be constructed numerically.

We believe that type (b) and (c) solutions can also be constructed analytically for generic values of the Higgs’s expectation values and we plan to reconsider this problem in a future publication.

It would also be interesting to study this model for larger gauge groups or to adopt a different compactification scheme of the codimension space, which here was chosen to be $S^1 \times S^1 \times \ldots S^1$. Following the investigation for $n = 1$ [15], the model including a cosmological constant is presently under investigation. In [15] the introduction of a cosmological constant in the (4+1)-dimensional model led to a Liouville type potential in the effective 4-dimensional theory.

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FIG. 1: The values $A(0)$, $N_m$, $-\psi(0)$ are shown as functions of $\alpha$ for $\rho = 1$. The indices “1”, “2”, “3”, “4”, respectively correspond to the 1., 2., 3. and 4. branch of solutions (see also Fig. 2).
FIG. 2: The mass $M$ of the solutions is shown as function of $\alpha$ for $\rho = 1$ and $\rho = 2$. “1”, “2” and “3”, respectively denote the 1., 2. and 3. branch of solutions.
FIG. 3: The values of $-\psi_1(0)$ and $-\psi_2(0)$ as well as the difference $\psi_2(0) - \psi_1(0)$ are shown as functions of $\alpha$ for $\rho = 2$. "1", "2", "3" and "4", respectively denote the 1., 2., 3. and 4. branch of solutions.
FIG. 4: The profiles of the functions $\psi_1(x)$, $\psi_2(x)$, $K(x)$, $A(x)$ and $N(x)$ are shown for $\rho = 2$ and $\alpha$ close to the critical value $\alpha_{cr} \approx 0.179$. 