Gravitational Wave Detection with Michelson Interferometers

S. Sivasubramanian, Y.N. Srivastava† and A. Widom
Physics Department, Northeastern University, Boston MA USA
†Physics Department & INFN, University of Perugia, Perugia IT

Electromagnetic methods recently proposed for detecting gravitational waves modify the Michelson phase shift analysis (historically employed for special relativity). We suggest that a frequency modulation analysis is more suited to general relativity. An incident photon in the presence of a very long wavelength gravitational wave will have a finite probability of being returned as a final photon with a frequency shift whose magnitude is equal to the gravitational wave frequency. The effect is due to the non-linear coupling between electromagnetic and gravitational waves. The frequency modulation is derived directly from the Maxwell-Einstein equations.

PACS numbers: 04.30.-w, 04.30.Nk, 04.40.Nr

I. INTRODUCTION

There has been considerable recent interest in the nature of gravitational and electromagnetic wave interactions, especially regarding optical techniques for detecting gravitational waves [1, 2, 3, 4, 5, 6, 7, 8]. Among the possible electromagnetic techniques for detecting gravitational waves is a modification of the methods which were historically important for the application of Michelson interferometers [9] to special relativity. A new technique can be employed as central for the detection of gravitational waves. It resides in the non-linear coupling between an incident traveling gravitational wave and almost standing Fabry-Perot cavity electromagnetic waves. Theoretically, such a non-linear wave coupling leads to a dynamic modulation and frequency shift of a finally detected electromagnetic signal. Such a dynamic (for general relativity) frequency shift measurement would supersede the earlier notions of Michelson who concentrated on static (for special relativity) phase shifts as measured from an interference pattern. Recall the Michelson interferometer as schematically pictured in Fig.1.

Our purpose is to provide a detailed derivation of the gravitational modulation of the electromagnetic cavity waves from the coupled Maxwell-Einstein theory. In Sec. II we derive the tree level Feynman diagram corresponding to a photon transition $\gamma_i \rightarrow \gamma_f$ in the presence of a gravitational wave. If, respectively, $\omega_g (\omega_i)$ denotes the frequency of the gravitational wave (initial photon), then the final photon frequency $\omega_f$ in the modulation sideband obeys

$$\omega_f = \omega_i \pm \omega_g.$$  \hspace{1cm} (1)

In Sec. III the electromagnetic modes of the cavity in the presence of a gravitational wave are explored. The frequency modulation follows from the nature of the coupling between the gravitational wave and the Maxwell electromagnetic pressure tensor. In Sec. IV the “Fermi-Golden-Rule” rates for the transitions $\gamma_i \rightarrow \gamma_f$ in Eq. (1) are computed in terms of the power spectrum of gravitational strain fluctuations, and both electromagnetic and gravitational polarizations will be discussed. In the concluding Sec. V the notion of gravitational wave induced modulation side bands will be discussed in virtue of the vacuum polarization response induced by the gravitational strain.

II. ELECTROMAGNETIC AND GRAVITATIONAL WAVE INTERACTIONS

The mathematical form of the electromagnetic-gravitational wave interaction arising from the gravita-
III. FREQUENCY MODULATION

For the purpose of describing the gravitational wave, we employ the metric expansion in which $g_{\mu\nu}$ differs only slightly from flat space-time; i.e.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \ldots,$$

where flat space time is described by $\eta_{\mu\nu}$ and $h_{\mu\nu}$ is described by a spatial transverse traceless strain $u$; i.e.

$$(h_{\mu\nu}) = 2 \begin{pmatrix} u_{xx} & u_{xy} & u_{xz} & 0 \\ u_{yx} & u_{yy} & u_{yz} & 0 \\ u_{zx} & u_{zy} & u_{zz} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (7)

From Eqs. (4) and (2) it follows (to lowest order in the gravitational strain $u$) that the Maxwell field Lagrangian is

$$L_{Maxwell} = \frac{1}{8\pi} \int \left( |E|^2 - |B|^2 \right) d^3 r + L_{int} \hspace{1cm} (8)$$

where the interaction between the electromagnetic field and the gravitational strain is given by

$$L_{int} = \int (u : P) d^3 r. \hspace{1cm} (9)$$

In Eq. (9), $P^{ij} = T^{ij}$ is the electromagnetic spatial pressure tensor, normally written as

$$P = \frac{1}{8\pi} \left\{ 1 \left( |E|^2 + |B|^2 \right) - 2 (E \cdot E + B \cdot B) \right\}. \hspace{1cm} (10)$$

Since we work in a traceless gauge $tr(u) = 0$, the Lagrangian coupling of the Fabry-Perot cavity fields ($E$ and $B$) to the gravitational strain $u$ is simply written as

$$L_{int} = - \left( \frac{1}{4\pi} \right) \int (E \cdot u \cdot E + B \cdot u \cdot B) d^3 r. \hspace{1cm} (11)$$

We note that an “effective” electromagnetic Lagrangian $L[E,B]$ may be used to define the Maxwell displacement field $D$ and the magnetic intensity $H$ via the functional derivatives

$$D = 4\pi \frac{\delta L}{\delta E} \quad \text{and} \quad H = -4\pi \frac{\delta L}{\delta B}. \hspace{1cm} (12)$$

Thus, Eqs. (8), (11) and (12) imply a tensor dielectric response $D = \epsilon \cdot E$ and a tensor magnetic permeability $B = \mu \cdot H$ determined by the gravitational wave; Explicitly, the response functions are given to linear order in $u$ by

$$\epsilon_{jk}(r,t) = \delta_{jk} - 2u_{jk}(r,t), \hspace{1cm} \mu^{-1}j_{jk}(r,t) = \delta_{jk} + 2u_{jk}(r,t). \hspace{1cm} (13)$$

A gravitational wave thereby acts (via $\epsilon$ and $\mu$) as a weak “moving grating” in the vacuum modulating the frequency of traveling electromagnetic waves.

---

FIG. 2: Shown is a process $\gamma_i \rightarrow \gamma_f$, in the presence of an external gravitational metric disturbance $\delta g_{\mu\nu}$. The Feynman diagram describes the scattering of light in a gravitational field. For a gravitational wave disturbance at frequency $\omega_g$, the frequency shift in the photon obeys $\omega_f - \omega_i = \pm \omega_g$. Such a frequency shift should appear experimentally as modulation sidebands about the initial photon frequency $\omega_i$.

---

\[ \text{tional metric} \]
\[ -c^2 dt^2 = g_{\mu\nu} dx^\mu dx^\nu, \hspace{1cm} (2) \]
\[ \text{and the cavity electromagnetic field} \]
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \hspace{1cm} (3) \]
\[ \text{can be described by the action} \]
\[ S_{\text{Maxwell}} = \frac{1}{16\pi c} \int g^{\mu\nu} g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} d\Omega, \hspace{1cm} (4) \]
\[ \text{wherein} \ d\Omega = \sqrt{-g} \ d^4 x. \hspace{1cm} \text{For a small change in the gravitational metric} \delta g_{\mu\nu} \text{representing a weak incident gravitational wave, the coupling into the electromagnetic field may be written in terms of the Maxwell stress tensor} \ T^{\mu\nu}; \]
\[ \delta S_{\text{Maxwell}} = \frac{1}{2c} \int T^{\mu\nu} \delta g_{\mu\nu} d\Omega, \]
\[ T^{\mu\nu} = \frac{1}{4\pi} \left( F^{\mu\beta} F_{\beta\nu} - \frac{\eta^{\mu\nu}}{4} (F^{\alpha\beta} F_{\alpha\beta}) \right). \hspace{1cm} (5) \]
In the limit of long wavelength gravitational waves, the strain \( u \) is uniform in space over the length scale of the interferometer. One thereby can apply the gravitational wave quadrupole approximation and consider that \( u \) depends only on time. Eq. (10) then reads
\[
L_{\text{int}} = u(t) : \int \mathcal{P}(r, t) d^3 r. \tag{14}
\]
To lowest order perturbation theory in \( u \), the interaction Lagrangian in Eq. (14) may be replaced by the Hamiltonian [10]
\[
H_{\text{int}} = -u(t) : \int \mathcal{P}(r, t) d^3 r. \tag{15}
\]
It is theoretically fortunate (and experimentally unfortunate) that the gravitational wave induced photon transitions are so weak that the lowest order in \( u \) perturbation theory is virtually exact.

**IV. PHOTON TRANSITION RATES**

The amplitude for a transition to take place in a long time period \( \tau \) is given in lowest order perturbation theory as
\[
\mathcal{A}(I \rightarrow F; \tau) = -\left( \frac{i}{\hbar} \right) \int \tau \langle F | H_{\text{int}}(t) | I \rangle dt = \left( \frac{i}{\hbar} \right) \int u(t) : Q_{FI} e^{i\omega_{FI} t} dt, \tag{16}
\]
where the Bohr transition frequency \( \omega_{FI} = (E_F - E_I)/\hbar \) and
\[
Q_{FI} = \int \langle F | \mathcal{P}(r) | I \rangle d^3 r. \tag{17}
\]
The transition rate per unit time for a radiation transition,
\[
\Gamma_{I \rightarrow F} = \lim_{\tau \rightarrow \infty} \frac{|\mathcal{A}(I \rightarrow F; \tau)|^2}{\tau}, \tag{18}
\]
follows from Eqs. (10) and (16) to have the form
\[
\Gamma_{I \rightarrow F} = \frac{2\pi}{\hbar} (Q_{jk})^*_{FI} (Q_{lm})_{FI} S_{jklm}(\omega_{FI}). \tag{19}
\]
Apart from the matrix elements squared of the photon field operators \( Q_{FI} \) in Eqs. (17), the effective (fourth rank tensor) “density of states” in the Fermi golden rule Eq. (19) is determined by the power spectrum of gravitational wave strains via
\[
S_{jklm}(\omega) = \lim_{\tau \rightarrow \infty} \langle \tilde{u}_{jk}(\omega) \tilde{u}_{lm}(\omega) \rangle \tag{20}
\]
where
\[
\tilde{u}(\omega) = \frac{1}{\sqrt{2\pi} \tau} \int e^{i\omega t} u(t) dt. \tag{21}
\]
The power spectrum may equally well be written in terms of the two time correlation functions
\[
u_{lm}(t_2) u_{jk}(t_1) = \int_{-\infty}^{\infty} S_{jklm}(\omega) e^{-i\omega(t_2-t_1)} d\omega. \tag{22}
\]
Let \( \xi \) be the polarization tensor for a gravitational wave described by the strain \( u \). Furthermore let
\[
S_\xi(\omega) = \xi^*_{jk} S_{jklm}(\omega) \xi_{lm} = S_\xi(-\omega) \tag{23}
\]
be the power spectrum of gravitational strain fluctuations with polarization \( \xi \). Such a power spectrum is easily related to the gravitational wave energy per unit time per unit area \( dP_\xi(\omega) \) incident on the interferometer in a bandwidth \( d\omega \); It is (for \( \omega > 0 \))
\[
dP_\xi(\omega) = \left( \frac{c^3 \omega^2}{4\pi G} \right) S_\xi(\omega) d\omega, \tag{24}
\]
where \( G \) is the gravitational coupling strength.

For a gravitational wave emission source with a frequency \( \omega_g \) and a Lorentzian width \( \gamma \), the resulting power spectrum is plotted in Fig. 3. The incident gravitational wave acts as an effective density of final states for the induced photon transition rate \( \gamma_i \rightarrow \gamma_f \). If the gravitational wave intensity distribution \( dP_\xi(\omega)/d\omega \) is singularly peaked, then the “density of final states” will have two symmetrically located peaks. These will give rise to the modulation sidebands in the spectrum of detected photons emerging from the interferometer.

The Fabry-Perot detector efficiency may be computed from Eq. (19) by averaging over initial radiation states.
and summing over final radiation states; i.e.
\[
\frac{d\Gamma(\omega)}{d\omega} = \sum_I \sum_F p_I \Gamma(I \rightarrow F) \delta(\omega - \omega_{FI}),
\]
\[
\frac{d\Gamma(\omega)}{d\omega} = D_{jklm}(\omega) S_{jklm}(\omega), \tag{25}
\]
where the radiation (detector) correlation functions are defined as
\[
G_{jklm}(t) = \int_{-\infty}^{\infty} D_{jklm}(\omega) e^{-i\omega t} d\omega,
\]
\[
G_{jklm}(t_2 - t_1) = \frac{2\pi}{\hbar^2} (Q_{jk}(t_1)Q_{lm}(t_2)) \tag{26}
\]
Since \(Q(t) = \int P(r,t) d^3r\), the detection efficiency is, in virtue of Eqs. (25) and (26), proportional to the power spectrum of the integrated radiation pressure in the Fabry-Perot cavities. If the bandwidth of the input laser is large on the scale of the gravitational wave frequency, then the detection efficiency is roughly frequency independent and \(\propto N_e^2/Hz\), where \(N_e\) is the number of photons stored in the Fabry-Perot cavities. The signal to noise ratio for detecting the modulation side bands is given by
\[
\frac{\text{signal}}{\text{noise}} = \frac{\sqrt{4\pi S(\omega)}}{u_\omega} \tag{27}
\]
where \(u_\omega \sim 10^{-22/\sqrt{Hz}}\) is the strain noise level with present technologies for kilohertz gravitational wave signals.

V. CONCLUSION

We have shown that the electromagnetic waves in the Fabry-Perot cavities of a Michelson interferometer may be described by the Lagrangian density
\[
\mathcal{L} = \frac{1}{8\pi} \left( E \cdot \epsilon \cdot E - B \cdot \mu^{-1} \cdot B \right), \tag{28}
\]
wherein the vacuum dielectric response and magnetic permeability are related to the gravitational wave transverse traceless tensor via Eq. (13); i.e.
\[
\epsilon_{jk}(r,t) = \delta_{jk} - h_{jk}(r,t)
\]
\[
(\mu^{-1})_{jk}(r,t) = \delta_{jk} + h_{jk}(r,t). \tag{29}
\]
The coordinates \(r\) being used are such that the ends of the cavities have fixed positions in a gravitational “floating” situation. Since the \(\epsilon\) and \(\mu\) tensors formally change light velocity in the arms of the interferometer, the distances between the ends of the arms (as measured by travel times of light signals) will be changing when a gravitational wave is present. An oscillating \(\epsilon\) and \(\mu\) will then produce frequency modulation side bands. Such modulations constitute a very well known effect in quantum optics.

Acknowledgments

We would be pleased to thank Professor Rainer Weiss for an enlightening discussion on the nature of Michelson interferometer measurements of gravitational waves.

[1] J. Weber, Phys. Rev. 117, 306 (1960).
[2] R.L. Forward, Phys. Rev., D17 379 (1978).
[3] LIGO: B. Barish and R. Weiss, Phys. Today, 52 44 (1999); A. Abramovici et al., Science 256, 325 (1992); B. Barish, AIP Conf. Proc. 575, 3 (2001).
[4] GEO: J. Hough, et al., Proceedings of TAMA Workshop, Saitama, Japan, Edited by K. Tsubono, M.-K. Fujimoto, K. Kuroda, Universal Academy Press, Tokyo, Japan, pp. 175 (1997); K. Danzmann, “Current Topics in Astrofundamental Physics”, Edited by N. Sanchez and A. Zichichi, World Scientific, Singapore 349, (1993).
[5] VIRGO: B. Caron, et al., Proc. of: “Le rencontre de Marigny”, France, (1996); Proc. of Conference on Mathematical Aspects of Theories of Gravitation, Poland, (1996); “Gravitational Wave Experiments”, Proceedings of the Edoardo Amaldi Conference, Edited by E. Coccia, G. Pizzella, and F. Ronga, World Scientific, Singapore, pp. 86 (1995).
[6] LISA: P. Bender, K. Danzmann, et al., “Laser Interferometer Space Antenna for the Detection of Gravitational Waves”, Pre-Phase A Rep. MPQ233, Max-Planck-Institut fr Quantenoptik, Garching, (1998); M. Tinto, F.B. Estabrook and J.W. Armstrong Phys. Rev. D65, 082003 (2002).
[7] TAMA: M. Ando and K. Tsubono, AIP Conf. Proc. 523, 128 (2000).
[8] AIGO: D. E. McClelland et al., AIP Conf. Proc. 523, 140 (2000).
[9] A.A. Michelson, “Studies in Optics”, Dover Publications, New York, (2002).
[10] Y.N. Srivastava, A. Widom and G. Pizzella, gr-qc/0302024 (2003).
[11] L. Mandel and E. Wolf, “Optical Coherence and Quantum Optics”, Cambridge University Press, New York, pp. 396 (1995).