On ideal dynamic climbing ropes

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Abstract
We consider the rope climber fall problem in two different settings. The simplest formulation of the problem is when the climber falls from a given altitude and is attached to one end of the rope while the other end of the rope is attached to the rock at a given height. The problem is then finding the properties of the rope for which the peak force felt by the climber during the fall is minimal. The second problem of our consideration is again minimizing the same quantity in the presence of a carabiner. We will call such ropes mathematically ideal. Given the height of the carabiner, the initial height and the mass of the climber, the length of the unstretched rope and the distance between the belayer and the carabiner, we find the optimal (in the sense of minimized the peak force to a given elongation) dynamic rope in the framework of nonlinear elasticity. Wires of shape memory materials have some of the desired features of the tension–strain relation of a mathematically ideal dynamic rope, namely, a plateau in the tension over a range of strains. With a suitable hysteresis loop, they also absorb essentially all the energy from the fall, thus making them an ideal rope in this sense too.

Keywords
Dynamic climbing ropes, shape memory alloys, hysteresis, nonlinear elasticity

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Introduction
Climber fall is a central problem in rock climbing, and an important factor, which this article addresses, is minimizing the peak force felt by the climber as he or she falls, allowing a maximum elongation of the rope. Minimizing the peak force is important also for decreasing the likelihood that the anchor will be dislodged, and for minimizing the stress on the rope and on the bolts, carabiners and all kinds of other protections, thus increasing their lifetime. Some comprehensive analyses of the maximal forces and rope elongation and the design of optimal ropes, optimality meant in various senses, already exist. Leuthäusser1,2 studies the above general problem in the setting of viscoelasticity. When a climber falls, some of the energy is converted to heat during the fall, and it is assumed in the study by Leuthäusser1,2 that the coefficient of conversion, that is, the proportion of the energy converted to heat and the total energy, is equal to 0.5. Spörri3 carried out a numerical analysis for the forces acting on the climber, and the resultant motion of the climber, assuming Pavier’s4 three-parameter viscoelastic rope model.

There are other problems, too, that concern climber fall. One is the durability of the rope and a safe lower bound on the number of the falls the rope can handle due to tear during any fall, which has been studied numerically by Bedogni and Manes5 and in combination with an experiment by Pavier.4 A second problem concerns the stiffness or stretchiness of the rope (which controls the total extension of the rope during a fall) and the maximal handling force of bolts and carabiners.6,7 A third problem is mostly medical and concerns the most probable injuries of the climber.8

There are two kinds of ropes used in rope climbing: static and dynamic. A dynamic rope stretches and is the rope connecting the climber to the belayer or anchor, while a static rope has little stretch and is often used in ascending using jumars, hauling and for rappelling, equivalently called abseiling. In addition, static ropes or webbing are used to affix via carabiners the dynamic rope to the climbing wall. The problem we consider is to find and minimize the maximal force acting on the climber during a fall over a fixed distance. This peak

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force is the maximal value of the tension of the dynamic rope at the point attached to the climber. In other words, the first goal is to identify the properties of a "mathematically ideal dynamic rope" that is designed so the peak force on the climber is minimized during a fall with a specified elongation of the rope. Of course many factors influence whether a dynamic rope is ideal in practice (such as durability, cost, weight, knotability and redundancy so breaks of single filaments do not affect the overall strength), and for this reason, we use the term "mathematically ideal" to mean ideal in the restricted sense of minimizing the peak force for a given total elongation. A second goal is to obtain a rope which absorbs essentially all the energy of the fall, so that the climber does not rebound after the fall in contrast to the oscillations that bungy cords produce during bungy jumping. To achieve the first goal, we begin by assuming the rope is purely elastic, rather than viscoelastic. Later, we consider such ropes with suitable hysteresis loops which absorb all the energy from the fall, and thus which are ideal from the viewpoint of both goals.

The question is then how the nonlinear elastic response of the rope should be tailored to minimize the maximal force acting on the climber during a fall. Intuitively, it makes sense that the rope should be designed to provide a constant braking force on the falling climber. While it may be the case that this result is known, we were unable to find a reference which addressed this condition. So a mathematical proof of this fact is provided. We draw attention to the fact that shape memory material wires have the desired characteristic of a plateau in the stress as the strain is varied. Furthermore, they can exhibit large hysteresis loops, and we observe that this feature is exactly what is needed to achieve the second goal, that is, to absorb all the energy of the fall without oscillation.

A similar problem has been studied by Reali and Stefanini in the setting of linear elasticity, but we believe nonlinear elasticity is more appropriate to the analysis of dynamic climbing ropes. We also remark that the problem of minimizing the maximal force during deceleration over a fixed distance is also appropriate to pilots landing on an aircraft carrier, but there the desired response can be controlled through the hydraulic dampers attached to the braking cable.

**Rope without a carabiner**

**Notation and problem setting**

In this section, we set up the notation for the climber fall problem in its simplest formulation. We direct the $x$-axis toward the direction of gravity, see Figure 1. We shall choose the gravitational potential to be 0 at the position $x = 0$ for convenience. Let us mention that the variable $x$ does not show the position of the climber, but rather it is the coordinate variable used in the undeformed configuration. There will be a climber affixed to a rock wall by means of a rope and the climber will fall vertically downward from a point directly above or below where the rope is attached to the wall, so that the problem is one-dimensional. The given rope of length $L$ will be attached to the rock at the point $x = 0$ and the climber of mass $m$ will be attached to the other end at some arbitrary height $x = h_0$, where $|h_0| \leq |L|$. Note that $h_0$ denotes the distance below where the rope is anchored and is negative if the climber is above the anchoring point. The rope will be assumed to be massless.

We assume that the climber is allowed to fall without hitting the ground up to the point $L + \Delta L$, that is, the maximal admissible stretching of the rope is $\Delta L$. We are interested in finding the properties of the rope so that given the initial data $L, \Delta L, m$ and $h_0$, it minimizes the maximal force felt by the climber during a fall. Such a rope will be called mathematically ideal.

We reformulate the problem as follows: instead of considering what happens before the rope becomes taut, we can set our clock so $t = 0$ marks the instant in time when the rope becomes taut, and at that time, the gravitational potential energy $mg(L - h_0)$ lost by the climber in falling a distance $L - h_0$ will have been converted into kinetic energy so that the climber will have a velocity $v_0 = \sqrt{2g(L - h_0)}$ in the direction of increasing $x$ (downward) at time $t = 0$. Thus, the deformation of the rope begins at time $t = 0$. We denote by $y(x, t)$ the position of a point on the rope at time $t$ that was initially located at $x$ at time $t = 0$; thus, a point on the rope at $x$ at $t = 0$ gets displaced by a distance $u(x, t) = y(x, t) - x$ (here, $y(x, t)$ will be assumed to be continuous and differentiable in $x$).

Once the rope begins to deform, its deformation is assumed to be described by nonlinear elasticity theory (ignoring viscosity). The elastic properties of the rope are given by a function $W$, representing the elastic energy density, of the one-dimensional strain

$$\varepsilon(x) = \frac{\partial u(x, t)}{\partial x} = \frac{\partial y(x, t)}{\partial x} - 1$$

![Figure 1. Before and after the fall. The figure is schematic, in that the climber before the fall, as illustrated on the left, should be directly below or above the point where the rope is attached.](image-url)
We assume the elastic energy does not depend on higher order derivatives of the deformation $y(x, t)$, that there is no air resistance, and (to begin with, as it will be something we will reconsider later in section “Realizability of the mathematically ideal rope”) that there is no energy absorption or dissipation during a fall. Thus, a fall will be a periodic phenomenon as the climber oscillates between the heights $x = h_0$ and $x = L + \Delta L$. We denote the time the climber reaches the critical point $x = L + \Delta L$ by $t = T$. We denote furthermore by $E_{el}$ the elastic energy of the rope, and by $E_{total}$ the total energy of the system. The total elastic energy for the rope, treated as a one-dimensional body, is then

$$E_{el}(t) = \int_0^L W(\varepsilon(x))dx$$

(1)

In other words, the total elastic energy is the sum (integral) of the elastic energies associated with the stretching of each rope segment. Note that we do not assume that $W$ is a quadratic function of $\varepsilon$, and hence, it is not appropriate to associate an elastic modulus with the rope: the elastic modulus is an appropriate descriptor when the tension in the rope is proportional to the overall strain (and it is the elastic modulus which gives this constant of proportionality). However, in our mathematically ideal rope, we will see it is best to have a rope where the tension is independent of the strain.

From the theory of nonlinear elasticity, for example, Gurtin,\(^\text{10}\) the elastic energy density $W$ has to satisfy the following properties:

(P1) $W(\varepsilon) \geq 0$ for all $\varepsilon \in \mathbb{R}$ (deformations store elastic energy);

(P2) $W(0) = 0$ (no deformation-no energy);

(P3) $W(\varepsilon)$ is a quasiconvex function of $\varepsilon + 1$, which is equivalent to being a convex function of $\varepsilon$ in the one-dimensional case under consideration. Recall, that a function $f(x) : [a, b] \rightarrow \mathbb{R}$ is convex if it satisfies the inequality

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in [a, b], \lambda \in [0, 1]$. Geometrically, convexity means that the entire graph of the function is above the tangent line at any point $x \in [a, b]$. Physically, convexity is important for stability: if $W$ is not convex, then microscopic oscillations in the deformation of the rope are energetically favorable, and macroscopically, the behavior of the rope will be given by an energy function which is again convex, being the convexification of the local nonconvex energy function. (Such oscillations do occur in shape memory wires, a point which we will return to later.)

We aim to find under which conditions on $W$ the rope is mathematically ideal. Our goal is to mitigate the force on the climber during the fall. In the more general version of the problem, the only new feature will be that the rope also passes through one carabiner, attached to the rock, that has a friction coefficient $k$. The single carabiner case will be sufficient to make a generalization with an arbitrary integer number of carabiners, with frictional coefficients $k_i$.

### Optimal bounds via energy conservation

Denote by $b(t)$ the rope tension at the point $x = L$ at time $t$, and by $a(x, t)$ the acceleration of the rope point $x$ at time $t$, for $x \in [0, L]$ and $t \in \mathbb{R}$. We have to solve the minimization problem

$$\min_{t \geq 0} \left(\max_{t \geq 0} |b(t)| \right)$$

(2)

It is clear, as the motion of the climber is periodic, we can consider the minimization problem on the restricted time interval $[0, T]$ rather than the whole time interval $[0, \infty)$. It is evident that the tension $b(t)$ imposes an upward force on the climber for all $t \in [0, T]$. The other force acting on the climber is the gravitation force $mg$ pointing downward, thus being positive. So, the net force on the climber is $mg - b(t)$, and by Newton’s second law, we have that the mass times the acceleration downward is given by

$$ma(L, t) = mg - b(t)$$

(3)

Next, we prove the key inequality

$$\max_{t \in [0, T]} |b(t)| \geq \frac{mg(L + \Delta L - h_0)}{\Delta L}$$

(4)

It is actually a direct consequence of energy conservation. Namely, as is well-known, we have that the work done by the climber in the time interval $[0, T]$ equals on one hand $\int_0^L \int_{x_0}^{x_0 + \Delta L} ma(L, t)dx$, and on the other hand, it is the change in kinetic energy, that is, $-(mv_0^2)/2$. Therefore, we get

$$\int_0^L \int_{x_0}^{x_0 + \Delta L} ma(L, t)dx = -\frac{mv_0^2}{2}$$

(5)

Integrating equation (3) in $x$ over the interval $[L, L + \Delta L]$, we obtain

$$-\int_{L}^{L + \Delta L} ma(L, t)dx = \int_{L}^{L + \Delta L} b(t)dx - mg\Delta L$$

and thus taking into account equation (5) and the formula for the initial velocity, $v_0 = \sqrt{2g(L - h_0)}$, we get
Finally, applying the inequality
\[ \int_{L}^{L + \Delta L} b(t) \, dx \leq \Delta L \max_{t \in [0, T]} |b(t)| \]
to equation (6), we arrive at equation (4). Observe, that equality in equation (4) holds if and only if the tension \( b(t) \) and thus the acceleration \( a(L, t) \) are constant in the interval \([0, T]\), which is the scenario that a mathematically ideal rope must develop. Thus, in this case, we get
\[
\begin{align*}
  a(L, t) &= a_0 = \frac{g(h_0 - L)}{\Delta L} \\
  b(t) &= b_0 = \frac{mg(L + \Delta L - h_0)}{\Delta L}
\end{align*}
\]  

This mathematically ideal rope causes the force where the rope is attached to the climber to step from a zero force to a constant value as the rope becomes taut. In practice, one would want a more gradual transition to allow the climber’s body time to respond to the force. Certainly, the effects of the force cannot propagate through the climber’s body faster than the speed of elastic waves; but more significantly, if we consider the climber’s body as a viscoelastic object, then one would not want the transition time to be faster than the typical viscoelastic relaxation time. The undesirable instantaneous step function in the force will be mollified if we replace the mathematically ideal rope by an approximation to it, such as the shape memory material rope, suggested later, which has a transition region before the stress plateau.

Optimal elastic energy density function

In this section, we find a formula for the elastic energy density function \( W \) associated with a mathematically ideal rope. Under our assumption that the rope is massless, equilibrium of forces implies the tension \( b \) in the rope must be constant along the rope. Also, if the rope is optimal in the sense that the maximal possible force felt by the climber is minimized, then \( b \) must be independent of the strain and given by equation (7). Suppose a small segment of rope extending from \( x = x_0 \) to \( x = x_0 + \ell \) in the undeformed state at \( t = 0 \) gets extended under the deformation at some time \( t > 0 \) less than \( T \) to the length \( \ell + \delta \ell \). The work done on the rope \( b_0 \delta \ell \) (being the force times the distance) must go into the elastic energy stored in this rope segment, implying

\[
W(\ell, t) = \int_{x_0}^{x_0 + \ell} b(t) \, dx = mg\Delta L + \frac{mg^2}{2} = mg(L + \Delta L - h_0) \tag{6}
\]

Using the fact that the energy density is 0 in the undeformed state at \( t = 0 \) when \( y(x, t) = x \), this will hold if
\[
W(\varepsilon) = b_0 \varepsilon = \frac{mg(L + \Delta L - h_0)}{\Delta L} \varepsilon \tag{9}
\]

This defines \( W \) for values \( \varepsilon \geq 0 \). (Equivalently, one could use the fact that the local tension \( b = b_0 \), which is the one-dimensional stress, is \( \partial W(\varepsilon)/\partial \varepsilon \) to deduce that \( W(\varepsilon) \) necessarily has the form (9)). Moreover, it is clear that \( W \) satisfies the properties (P1) and (P2) for \( \varepsilon \geq 0 \). The case \( \varepsilon < 0 \) corresponds to rope compression, which is assumed to require no energy; thus, we must take \( W(\varepsilon) = 0 \) for \( \varepsilon < 0 \). In conclusion, we get a final formula for the energy density function
\[
W(\varepsilon) = \frac{mg(L + \Delta L - h_0)}{\Delta L} \varepsilon \quad \text{if } \varepsilon \geq 0, \quad 0 \quad \text{if } \varepsilon < 0 \tag{10}
\]

which clearly satisfies all properties (P1–P3).

Note, however, that \( W(\varepsilon) \) is not a strictly convex function of \( \varepsilon \). If it was strictly convex, we could use Jensen’s inequality
\[
\frac{1}{L} \int_{0}^{L} W\left(\frac{\partial y(x, t)}{\partial x} - 1\right) \geq \frac{1}{L} W\left(\frac{y(L, t)}{L} - 1\right) \tag{11}
\]

with equality holding only when \( y(x, t) \) is independent of \( x \), to deduce that the deformation of the rope which minimizes the elastic energy is necessarily homogeneous (A function \( f(x, t) : \mathbb{R}^2 \rightarrow \mathbb{R} \) is homogeneous in \( x \) for each \( t \in \mathbb{R} \) if \( f(\lambda x, t) = \lambda f(x, t) \) for all \( \lambda, x, t \in \mathbb{R} \). In other words, its derivative in the \( x \) variable does not depend on \( x \) at each time \( t \). Recall, that Jensen’s inequality asserts the following: if a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is convex, then for any interval \([a, b] \subset \mathbb{R}\) and any continuous function \( g : [a, b] \rightarrow \mathbb{R} \) the inequality holds
\[
\frac{1}{b-a} \int_{a}^{b} f(g(x)) \, dx \geq f\left(\frac{1}{b-a} \int_{a}^{b} g(x) \, dx\right) \tag{12}
\]

In the absence of this strict convexity, there is no reason to assume homogeneity of the deformation: the deformation could be any function \( y(x, t) \) with the prescribed values of \( y(0, t) = 0 \) and \( y(L, t) \) (given below in equation (15)) and with \( \partial y(x, t)/\partial x > 1 \), where this last condition is required by constancy of the tension \( b \) along the rope. The actual deformation which is
selected could depend on higher order gradient terms in the energy function which we have neglected to include and could be very sensitive to slight inhomogeneities in the rope.

If we assume the selected deformation is homogeneous, then we have

\[ y(x, t) = \frac{x}{L} y(L, t) \]  

(13)

We can integrate \( a(L, t) \) in \( t \) to get

\[ v(L, t) = \int_0^t a(L, t)dt + v_0 = a_0 t + v_0 \]  

(14)

Further integration gives

\[ y(L, t) = \int_0^t v(L, t)dt = \frac{a_0}{2} t^2 + v_0 t + L \]  

(15)

Thus, due to equation (13), we finally arrive at

\[ y(x, t) = \frac{x}{L} \left( \frac{a_0}{2} t^2 + v_0 t + L \right) \]  

(16)

**Rope with a carabiner**

In this section, we consider the climber fall problem in the presence of a carabiner with a friction coefficient \( k \). Namely, we assume that the rope passes through a carabiner attached to the rock wall, with one end of the rope attached to the climber, while the other end is fastened to the belayer. The rope line from the belayer to the carabiner forms a given angle \( \alpha \) with the vertical axis \( x \), see Figure 2. Then we are again seeking a mathematically ideal rope. As was done in section “Rope without a carabiner,” we reformulate the problem as given in Figure 3.

Assume that the rope is vertical and is attached to the rock at the origin \( x = 0 \) and the carabiner is attached to the rock at the point \( x = L_1 \), while the rope length between the carabiner and the climber is \( L_2 \) and the climber has an initial velocity \( v_0 \), see Figure 3. In addition, due to the carabiner, the rope tension jumps from \( b(t) \) below the carabiner to \( mb(t) \) above the carabiner, whereby the capstan equation (also known as the Euler–Eytelwein formula) \( \mu = e^{\frac{\pi - \alpha}{2k}} \) and \( k \) is the coefficient of friction between the metal surface of the carabiner and the rope, see Figure 4.

The problem is again finding a rope that solves the problem

\[ \min \left( \max_{t \in [0, T]} |b(t)| \right) \]  

(17)

The setting of the reformulated problem makes it clear that problem (17) is solved when \( b(t) = b_0 \) as well as \( a(L, t) = a_0 \) for \( t \in [0, T] \), where \( a_0 \) and \( b_0 \) are given.
by equation (7). We have to now find a rope that develops a constant resultant force acting on the climber during a fall cycle. It turns out that the type of rope found in the previous section works also for this case. The idea is to take $L = L_2$ in formula (10) and take $y(x, t) = x$ for $x < L_1$ so that the rope develops no deformation between the rock and the carabiner and thus fulfills the desired conditions. To understand that, let us plot the tension–strain diagram corresponding to the rope given by equation (10) with $L$ replaced by $L_2$, see Figure 5. Because the tension above the carabiner is strictly smaller than $b_0$, it is then clear from the diagram that the rope develops no stretching in the segment $[0, L_1]$, as we wanted to establish. Note that this conclusion remains valid even if the Euler–Eytelwein formula is questioned, as it may be because the rope diameter is comparable to that of the metal diameter in the carabiner as observed by Weber and Ehrmann\textsuperscript{11}; all we require for the analysis is that the tension in the rope between the carabiner and the belayer be less than that between the carabiner and the climber (i.e. $\mu < 1$). The necessity of having the carabiner is to lessen the force on the rope attached to the belayer, to allow the belayer to be positioned in various places, and to prevent the climber from falling off the wall.

**Realizability of the mathematically ideal rope**

Here, we provide some arguments which suggest that a rope approximately realizing the condition (10) is not beyond the realm of possibility. The characteristic feature of the tension–strain diagram of Figure 5 is the plateau in the tension as the strain is varied. Wires of shape memory materials such as Nitinol (an alloy of nickel and titanium, see https://en.wikipedia.org/wiki/Nitinol) have such plateaus, although in the currently available shape memory wires the plateau in the tension occurs only for strains less than about 8% (see Šittner et al.\textsuperscript{12} and references therein). As Figure 14 in Attaway\textsuperscript{6} shows, normal dynamic climbing ropes can have strains of up to 15%. The reason for the plateau is illustrated in Figure 6. At a microscopic scale, the elastic energy might not be a convex function of $\varepsilon$ but given by the function $W_{\text{mic}}(\varepsilon)$, shown in black in Figure 6. A strain having the value $\varepsilon$ in the region of non-convexity is energetically unstable, and on a macroscopic scale, the material *phase separates* into a collection of segments having microscopic strains $\varepsilon_1$ or $\varepsilon_2$ in proportions $\theta$ and $1 - \theta$ where $\theta = (\varepsilon - \varepsilon_1)/(\varepsilon_2 - \varepsilon_1)$. The elastic energy density of this mixture is

\[
W(\varepsilon) = \theta W_{\text{mic}}(\varepsilon_1) + (1 - \theta) W_{\text{mic}}(\varepsilon_2) = W_{\text{mic}}(\varepsilon_2) + \frac{\varepsilon - \varepsilon_1}{\varepsilon_2 - \varepsilon_1} [W_{\text{mic}}(\varepsilon_1) - W_{\text{mic}}(\varepsilon_2)]
\]

(18)

which depends linearly on $\varepsilon$ for strains between $\varepsilon_1$ and $\varepsilon_2$: thus, the wire tension is independent of the macroscopic strain in this interval. (We remark in passing that the treatment for the deformation of two- or three-dimensional shape memory materials, rather than wires, has also been developed but is more complicated and involves quasiconvexification rather than convexification, see, for example, Ball and James.\textsuperscript{13})

In reality, the tension–strain diagram of shape memory wires has a hysteresis loop, which is not encompassed by our purely elastic formulation. Although hysteresis can be minimized in shape memory materials,\textsuperscript{14} a hysteresis loop in the response of a rope would actually be a necessity: it would be good if the tension in the rope was just slightly more than $mg$ when the rope retracts after it reaches its maximum extension, as then the climber would slowly rebound from the fall, returning close to $x = L$. This close-to-ideal tension–strain relation with a hysteresis loop is sketched in Figure 7. In some circumstances, it might be best if the rope remained close to the maximum extension $x = L + \Delta L$ after the fall: this will be the case if the return path in the hysteresis loop was below the line $b = mg$.

**Conclusion**

We do not expect this article to have an immediate effect on the climbing community, but by providing a
al.15 and Xie and Rousseau,16 although these do not link alloys, but polymers, for example, Eisenhaure et al. recently developed shape memory materials that may be more suitable for use in ropes than others which may help guide the development of new ropes. Also, it may motivate research into alternate shape memory materials which may be too fast (see, for example, Figure 14 in Heller et al.17). In order for the plateaus to be retained at a desired rate of deformation, the phase separation must occur on an appropriately fast time scale. Another unwanted characteristic is that the shape and position of the hysteresis loop in general vary according to the number of deformation cycles the wire has undergone.

While shape memory wires only have stress plateau extending up to 8% strain, while dynamic ropes can have strains of 15%, this may actually be an advantage as pointed out to us by Wendy Crone at the University of Wisconsin. The reason is that with a mathematically ideal rope, one can have the same peak force with less overall elongation than in a standard dynamic rope, and less elongation minimizes the chance of collision with a rock outcrop or another climber.

Of course a “rope” built from shape memory material wires may have disadvantages not considered here, such as being too heavy (although titanium is comparatively light among metals), too expensive, not easily coiled, not easily knotted or having properties which are temperature-dependent. Cables have been built from shape memory wires and their properties have been studied.18,19 Significantly, Reedlunn et al.19 find that the hysteresis loop can extend up to a strain of about 12% (see their Figure 6(a)), depending on the placement of the wires in the cable, but in this case the stress plateau is lost: an alternate cable design has (as shown in the same figure) a hysteresis loop up to a strain of about 8% while retaining the stress plateau. Cables, like wires, have a hysteresis loop that varies according to the number of deformation cycles, see Figure 9 of Ozbulut et al.,18 where the hysteresis loop after 100 cycles is much narrower than after one cycle. It may be an advantage to combine fibers or wires with shape memory characteristics with other rope elements, but we have not studied this possibility.

We remark that recently developed shape memory alloys can reliably go through 10 million transformation cycles without fatigue,20 although in this case the hysteresis loop is quite small. There may be other applications where this analysis is useful. For example, suppose one wants to drop cargo from a plane or a helicopter, without deploying parachutes (which may easily be seen by an enemy). Then, to lessen the impact of the fall on the cargo, it may be useful to attach a tether to the cargo which becomes taut at a certain distance above the ground: the distance to the ground sets the maximal elongation. One would again want to minimize the peak tension in the tether, thus minimizing the forces on the plane or the helicopter and on the cargo. In such applications to achieve the desired mathematically ideal tether which produces a constant braking force, one could use a tether with sacrificial elements which tear and dissipate energy as it is stretched. A one-dimensional model with these sacrificial elements was studied by Cherkaev et al.21 and shows an approximate plateau in the tension versus elongation (with oscillations), see their Figure 3. Such sacrificial elements are believed to be a mechanism giving bone its strength22 and account for resilience of double-network hydrogels23 (see also http://imechanica.org/node/13088).

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Appendix I

Notation

- $a_0$: constant acceleration/deceleration of the climber
- $a(x, t)$: acceleration of the rope point at time $t$ with initial (at $t = 0$) coordinate $x$
- $b(t)$: rope tension at the point $x = L$ at time $t$
- $b_0$: constant rope tension at the point $x = L$ at constant acceleration
- $E_{el}$: stored elastic energy of the rope
- $E_{total}$: total energy of the system
- $g$: gravitational acceleration
- $k$: friction coefficient between the rope and the carabiner
- $L$: length of the undeformed rope
- $m$: mass of the climber
- $T$: time
- $t$: time moment when the rope reaches maximal stretch for the first time
- $u(x, t)$: rope displacement function
- $v(x, t)$: velocity of the rope point $x$ at time $t$
- $v_0$: initial velocity of the climber in the reformulated problem
- $W$: rope elastic energy density
- $x$: coordinate variable of the system
- $y(x, t)$: rope deformation function
- $\Delta L$: maximal stretch of the rope in a fall cycle
- $\varepsilon(x, \varepsilon_i)$: elastic strain