BTZ Black Hole with Higher Derivatives, the Second Law of Thermodynamics, and Statistical Entropy: A New Proposal

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ABSTRACT

I consider the thermodynamics of the BTZ black hole in the presence of the higher curvature and gravitational Chern-Simons terms, and its statistical entropy. I propose a new thermodynamical entropy, which being non-negative manifestly, such as the second law of thermodynamics is satisfied. I show that the new thermodynamical entropy agrees perfectly with the statistical entropy for all the values of the conformal factor of the higher curvature terms and the coupling constant of the gravitational Chern-Simons term, in contrast to some disagreements in the literatures. The agreement with both the higher curvature and gravitational Chern-Simons terms is possible because of an appropriate balancing of them, though it is not a trivial matter because of a conflict in the appropriate Hilbert space for a well-defined conformal field theory for each term.

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I. Introduction

Recently, the higher curvature corrections to the black hole entropies of the black holes in supergravity theories in diverse dimensions have been extensively studied, and it has been found that there are some good agreements with the statistical entropies from the microscopic counting of the number of states. (For a recent review, see Ref. [1].)

In all these analyses, the basic formalism for the thermodynamical entropies is known as the Wald’s formalism which provides a general entropy formula, based on the first law of thermodynamics, in the presence any covariant combinations of the curvatures [2, 3, 4]. But, there is a serious and well-known problem in this formalism: The second law of thermodynamics is not manifest, in contrast to the first law\(^2\). However, it does not seem that this question has been well explored in the recent studies of supergravity black holes. Actually, without the guarantee of the second law, there would be no justification for identifying the entropies, even though they satisfy the first law [5].

More recently, the corrections due to the gravitational Chern-Simons term [6, 7, 8] have been studied in several different approaches, and it has been found that there are good agreements between the thermodynamical entropies based on the first law, and the statistical entropies based on the boundary conformal field theories (CFT) [9, 10, 11, 12]. But, the agreements were not perfect and there were some discrepancies in a strong coupling regime, though not been well studied in the literatures.

In order to resolve the discrepancies I have re-considered the first law and argued that they can be resolved by considering a new entropy formulae such as the second law is guaranteed from some new re-arrangements of the usual form of the first law [13, 14]. In this paper, I study general higher curvature corrections as well and show that there is similar discrepancies for a “negative” conformal factor (\(\hat{\Omega} < 0\)), in which the thermodynamic entropy become negative, which has been claimed “unphysical” in the literatures [15] or speculated as an indication of some thermodynamic instability [16, 17]. But, I argue that this can be resolved also by considering appropriate new entropy formulae, which being manifestly non-negative and satisfying the second law, similarly to the case with the gravitational Chern-Simons. And, I show that the new thermodynamical entropy agrees perfectly with the statistical entropy for all values of the conformal factor of the higher curvature terms and the coupling constant of the gravitational Chern-Simons term. Here, the agreement with both the higher curvature and gravitational Chern-Simons corrections is not so trivial because the appropriate Hilbert spaces for well-defined CFT are in conflict, but I find that this is actually possible in our case, due to

\(^2\)For a class of higher curvature theories where the Lagrangian is a polynomial in the Ricci scalar \(R\), the second law can be proved with the null (matter) energy condition and cosmic censorship. But, for other cases, there has been no general proof.
a nice balancing of the two Hilbert spaces.

The plan of this paper is as follows.

In Sec. II, I consider the BTZ black hole in the presence of the generic, higher curvature corrections, and I identify new entropies, which being manifestly non-negative, such as the second law of thermodynamics can be satisfied. The obtained entropy agrees with the usual Wald’s formula for the positive conformal factor $\hat{\Omega} > 0$. However, it disagrees with the Wald’s formula for $\hat{\Omega} < 0$, which gives a negative entropy.

In Sec. III, I consider the statistical entropies and I find perfect agreements with the new thermodynamical entropies that have been found in Sec. II, even for the $\hat{\Omega} < 0$ case as well as the $\hat{\Omega} > 0$ case.

In Sec. IV and V, I consider the gravitational Chern-Simons correction term as well, in addition to the generic higher curvature terms. I find perfect agreements between the thermodynamical and statistical entropies for “all” values (either $\hat{\Omega} > 0$ or $\hat{\Omega} < 0$) of the conformal factor of the higher curvature terms and the coupling constant of the gravitational Chern-Simons. The agreement with both of the two corrections is possible because of an appropriate balancing of them, though it is not a trivial matter because of a conflict in the appropriate Hilbert space for a well-defined CFT for each correction.

In Sec. VI, I conclude with several open questions.

In this paper, I shall keep the Newton’s constant $G$ and the Planck’s constant $\hbar$ in order to clearly distinguish the quantum gravity effects with the classical ones. But, I shall use the units of $c = 1$, $k_B = 1$, for the speed of light $c$ and the Boltzman’s constant $k_B$ for convenience, as usual.

II. The BTZ black hole with higher curvatures

The (2+1)-dimensional gravity with higher curvature terms and a (bare) cosmological constant $\Lambda = -1/l_0^2$ can be generally described by the action on a manifold $\mathcal{M}$ [omitting some boundary terms]

$$I_g = \frac{1}{16\pi G} \int_{\mathcal{M}} d^3x \sqrt{-g} \left( f(g^{\mu\nu}, R_{\mu\nu}, \nabla_\mu) + \frac{2}{l_0^2} \right), \quad (2.1)$$

where $f(g^{\mu\nu}, R_{\mu\nu}, \nabla_\mu)$ is an arbitrary scalar function constructed from the metric $g^{\mu\nu}$, Ricci curvature tensor $R_{\mu\nu}$, and the covariant derivatives $\nabla_\mu$ [2, 3, 4]. This action is the most generic, diffeomorphically invariant form in three dimensions since there is no independent component of the Riemann tensor due to vanishing Weyl tensor. The equations of motion, by varying (2.1)
with respect to the metric, are

$$\frac{\partial f}{\partial g_{\mu\nu}} - \frac{1}{2} g^{\mu\nu} f - \frac{1}{l_0^2} g_{\mu\nu} = t_{\mu\nu},$$

(2.2)

where the pseudo-tensor $t_{\mu\nu}$ is given by

$$t_{\mu\nu} = \frac{1}{2}(\nabla^{\alpha} \nabla_{\beta} P^{\alpha}_{\mu} + \nabla_{\beta} \nabla_{\alpha} P^{\alpha}_{\mu} - \Box P^{\mu\nu} - g^{\mu\nu} \nabla_{\alpha} \nabla_{\beta} P_{\alpha\beta}).$$

(2.3)

In the absence of the higher curvature terms, there is a black hole solution, known as the BTZ (Banados-Teitelboim-Zanelli) solution, which is given by the metric [18]

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2(d\phi + N^2 dt)^2$$

(2.4)

with

$$N^2 = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{l_0^2 r^2}, \quad N^2 = \frac{r_+ + r_-}{l_0 r^2}.$$ (2.5)

Here, $r_+$ and $r_-$ denote the outer and inner horizons, respectively. The mass and angular momentum of the black hole are given by

$$m = \frac{r_+^2 + r_-^2}{8G l_0^2}, \quad j = \frac{2r_+ r_-}{8G l_0},$$ (2.6)

respectively. Note that these parameters satisfy the usual mass/angular momentum inequality $m^2 \geq j^2/l_0^2$ in order that the horizon exists or the conical singularity is not naked, with the equality for an extremal black hole having the overlapping inner and outer horizons. This satisfies the usual Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{l_0^2} g_{\mu\nu} = 0$$ (2.7)

with a constant curvature scalar $R = -6/l_0^2$.

But, even in the presence of the generic higher curvature terms, the BTZ solution can be still a solution since the \textit{local} structure would be “unchanged” by higher curvatures. The only effects would be some “re-normalization” of the bare parameters $l_0, r_+, \text{ and } r_- [9,11,12,15]$: The renormalized cosmological constant will be denoted by $\Lambda_{\text{ren}} = -1/l^2$ and the function \( l = l(l_0) \) depends on the details of the function $f$; however, I shall use the same notations $r_\pm$

3In the renormalized frame, one can also “construct” the Einstein equation $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{l_0^2} g_{\mu\nu} = 0$, as in (2.7), from the relations $R_{\mu\nu,\alpha\beta} = (R/6)(g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha})$ and $R = -6/l^2$.

4For $f = R + aR^2 + bR_{\mu\nu}R_{\mu\nu}$ with some appropriate coefficients $a, b [15]$, the function $l = l(l_0)$ is given by $-6l^{-2} = \left( -1 \pm \sqrt{1 - 24(b-a)l_0^2} \right)/(2(b-a))$. 

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in the renormalized frame also, for brevity. And, in this case one finds $t^{\mu \nu} = 0$ trivially from $P_{\alpha \beta} \propto g_{\alpha \beta}$ for any constant-curvature solution [15]. On the other hand, the original mass and angular momentum, computed from the standard Hamiltonian approach [19, 3], become

$$M = \hat{\Omega} m, \quad J = \hat{\Omega} j,$$

(2.8)

respectively, where the conformal factor $\hat{\Omega}$ is defined by

$$\hat{\Omega} \equiv \frac{1}{3} g_{\mu \nu} \frac{\partial f}{\partial R_{\mu \nu}},$$

(2.9)

which being constant for any constant-curvature solution [15]. Note that $\hat{\Omega}$ is “not” positive definite such as the usual inequality of the mass and angular momentum would not be valid generally\(^5\)

$$M - J/l = \hat{\Omega} (m - j/l),$$

(2.10)

but it depends on the sign of $\hat{\Omega}$: For $\hat{\Omega} > 0$ (case (i)) one has the usual inequality $M \geq J/l$; however, for $\hat{\Omega} < 0$ (case (ii)), one has an anomalous inequality $J/l \geq M$ with the “negative” $M$ and $J$, though their magnitudes still satisfy the usual bound $M^2 \geq J^2/l^2$.

Now, by considering the first law of thermodynamics

$$\delta M = \Omega_+ \delta J + T_+ \delta S_W,$$

(2.11)

with the Hawking temperature $T_+$ and the angular velocity $\Omega_+$ of the (outer) event horizon $r_+$

$$T_+ = \frac{\hbar \kappa}{2\pi} \bigg|_{r_+} = \frac{\hbar (r_+^2 - r_-^2)}{2\pi l^2 r_+}, \quad \Omega_+ = -N^\phi \bigg|_{r_+} = \frac{r_-}{lr_+},$$

(2.12)

for the surface gravity function

$$\kappa = \frac{1}{2} \frac{\partial N^2}{\partial r},$$

(2.13)

the black hole entropy can be identified as

$$S_W = \frac{\hat{\Omega} 2\pi r_+}{4G\hbar}.$$  

(2.14)

This agrees with the Wald’s entropy formula [15], and this should be the case since the entropy in the Wald’s formalism is basically defined by the first law of thermodynamics [4]. But, I

\(^5\)Here, $m$ and $j$ represent the usual mass and angular momentum for the metric (2.4) in the renormalized frame $m = \frac{r^2 + r^2}{8G l^2}$, $j = \frac{2r_+ r_-}{8G l}$, with the renormalized parameters $l, r_\pm$, such as one has the usual inequality $m - j/l \geq 0$ still.
must note that the first law can not be “proved” without knowing the form of the entropy and temperature, basically. Actually, in the case of black holes, we usually know the Hawking temperature, as in (2.12), from the Hawking radiation analysis with a given, Riemannian, metric and so we can identify the entropy, by “assuming” the first law. This is a basic process to compute the entropy in the general class of gravity theories (see, for example, Refs. [3, 4]).

However, a basic problem of this approach is that the second law is not guaranteed, in general. Actually, in the higher derivative gravity theories there would be deep changes in the entropy, and the second law or the Hawking’s area (increasing) theorem has to be revisited completely, generally. But, in regards to the area theorem, this is “not” true in our case: Our space is maximally symmetric, i.e., a constant curvature space, and so the higher curvature effects to the energy momentum tensor, $t_{\mu\nu}$ of (2.3) vanishes, as I have clearly noted in the paragraph below (2.7). Other higher curvature effects in the Einstein equation (2.2) from the arbitrary function $f$, give only some re-normalization of the bare parameters $l_0, r_+, r_-$, as I have explained in the same paragraph, such as the resulting equation be just the original vacuum Einstein equation (2.7) with the parameters’ re-normalization. So, as far as the area theorem for the outer horizons is concerned, the usual derivations via the Raychaudhuri’s equation with the null energy condition for matter’s energy-momentum tensor still works in our case since the vacuum Einstein equation satisfies the null-energy condition, trivially [20]. Moreover, there is another approach, called the “physical process” [21, 4]), to prove the area theorem which does not depends on the details of the gravity [2]. It depends only on the first law with an appropriate energy condition, and the area law is evident, in this approach, from the first law (2.11) also. (For the details, see Ref. [22].) Hence, it is clear that the entropy formula (2.14) satisfies the second law, for the case of $\hat{\Omega} > 0$, since it is already in the Bekenstein’s form, which being proportional to the area of the outer horizon $A_+ = 2\pi r_+$, such as the Hawking’s area (increasing) theorem implies the increasing entropy [23].

On the other hand, the situation is quite different for $\hat{\Omega} < 0$ since (2.14) would not guarantee the second law nor the positivity because it would “decrease” indefinitely, with the negative values, as the outer horizon $r_+$ be increased, from the area theorem. Actually, this seems to be a general feature of higher derivative gravities in arbitrary dimensions [24, 16, 25, 17] or Taub-Bolt spacetime with a cosmological constant [17, 26], and in the literatures it has been speculated...
as an indication of some thermodynamic instability (for example, see Refs. [16, 17]). But, this
seems to be physically nonsensical since the entropy is non-negative, “by its definition” as a
measure of disorderedness [27]; the positiveness of the entropy is a “minimum” requirement
that must be satisfied if the entropy has a statistical mechanical origin [4]. Moreover, without
the guarantee of the second law, there would be no justification for identifying the entropies,
even though they satisfy the first law [5]. So, in this paper I consider a different approach which
can resolve the two problems, simultaneously. The new resolution is to consider an entropy
\[ S_{W'} = |\hat{\Omega}| \frac{2\pi r_+}{4G\hbar}, \]  
which is non-negative manifestly and also satisfying the second law from the area theorem\(^9\),
as in the case of \( S_W \) in (2.14) for a positive \( \hat{\Omega} \). But, in this case I must pay the price, by
considering a new temperature
\[ T_+^{'} \equiv -T_+, \]  
instead of \( T_+ \), in order to satisfy the first law also.

In the following sections, I will show that the new approach is actually what favored by
the statistical entropy through a CFT analysis, which provides the new entropy formula (2.15)
directly: With the correct values of \( M \), \( J \), and the entropy \( S_{W'} \), which is non-negative and
proportional to the (outer) horizon area, there is no other choice in the temperature.

III. Statistical entropy

It is well known that, in the absence of the higher curvature terms, the statistical entropy
of the BTZ black hole can be computed from a two-dimensional CFT, which is described
by Virasoro algebras, on the asymptotic Anti-de Sitter (\( AdS \)) boundary with the help of the
Cardy formula, and there is a complete agreement with the thermodynamical Bekenstein-
Hawking entropy. There are basically two approaches to compute the CFT, i.e., the Virasoro
algebras. One approach is a quantum approach which identifies the central charges of the
CFT, in the context of the conjectured AdS/CFT correspondence [28], by evaluating the
anomalies of the CFT effective action on the AdS boundary, from the regularized bulk gravity
action [29, 30, 31]. The other approach is a classical one which directly computes (classical)
Virasoro algebras based on the classical symmetry algebras of the asymptotic isometry of \( AdS_3 \)
[32, 33, 34, 35, 36]. It is widely known that these two approaches agree completely, and this
provides an explicit check of the AdS/CFT correspondence.

\(^9\)The physical process version of the second law for this definition or the area theorem is related to the same
geometric effect as that of the \( \hat{\Omega} > 0 \) case. (See Ref. [22] for the details.)
Recently, these analyses have been generalized to the theories with higher curvature terms, and some good agreements were known between the thermodynamical and statistical entropies, as well as the agreements between the holographic anomaly approach and classical symmetry algebra approach. But, in contrast to the usual claims in the literatures, these analyses have some problems which “might” invalidate the AdS/CFT correspondence. First, there are some discrepancies in the usual thermodynamical and statistical entropies, though this has not been well explored in the literatures [15, 9, 12]. Second, the computation about the classical symmetry algebra, by transforming a gravity action with the higher curvature terms into the usual Einstein-Hilbert action with some auxiliary tensor matter fields, “might” have some problems since there would be some non-trivial, boundary contributions in the Virasoro generators $\hat{L}_m^\pm$ and central charges from the matter fields “in general” [37, 38, 39], in contrast to the work [15], though the agreements seems to be plausible in the context of AdS/CFT. In this paper, the second problem will not be discussed further and left as an open problem. In the remainder of this paper, I will concentrate only the first problem and in the context of the quantum approach of the “holographic anomalies”, which has been computed rigorously recently, such as the second problem does not occur, in contrast to Ref. [15].

Now, in order to discuss the first problem in detail, I start by noting the holographic (conformal) anomalies in the expectation values of the boundary stress tensor [9], for a boundary metric $ds^2 \simeq -r^2 dx^+ dx^-$ with $r$ taken to infinity,

$$\langle T_{++}(x^+) \rangle = -\frac{\hbar \hat{c}^+}{24\pi}, \quad \langle T_{--}(x^+) \rangle = -\frac{\hbar \hat{c}^-}{24\pi}$$

with the central charges [40, 9] [I follow the conventions of Ref. [31] ]

$$\hat{c}^\pm = \hat{\Omega} \frac{3l}{2G\hbar}.$$  

(3.2)

Note that the obtained central charges have a quantum origin, which would has been introduced via the regularization procedure.

By considering (3.1) as the anomalous transformations of the boundary stress tensors under the diffeomorphism $\delta x^\pm = -\xi^\pm(x^\pm),$

$$\delta_{\xi^+} T_{++} = 2\partial_+ \xi^+ T_{++} + \xi^+ \partial_+ T_{++} - \frac{\hbar \hat{c}^+}{24\pi} \partial^3 \xi^+$$

$$= \frac{1}{i} [T_{++}, \hat{L}^+ [\xi^+]],$$

$$\delta_{\xi^-} T_{--} = 2\partial_- \xi^- T_{--} + \xi^- \partial_- T_{--} - \frac{\hbar \hat{c}^-}{24\pi} \partial^3 \xi^-$$

$$= \frac{1}{i} [T_{--}, \hat{L}^- [\xi^-]].$$

(3.3)
with the generators
\[
\hat{L}^\pm[\xi^\pm] = \frac{1}{\hbar} \oint dx^\pm T_{\pm \pm} \xi^\pm(x^\pm) + \frac{\hat{c}^\pm}{24},
\] (3.4)

one can obtain a pair of quantum Virasoro algebras
\[
[\hat{L}_m^\pm, \hat{L}_n^\pm] = (m - n)\hat{L}_{m+n}^\pm + \frac{\hat{c}^\pm}{12}m(m^2 - 1)\delta_{m+n,0}
\] (3.5)

with the central charges \(\hat{c}^\pm\) for the right(+) / left(-)-moving sectors and for a monochromatic basis \(\xi^\pm = e^{imx^\pm}\) with the integer numbers \(m\) and \(n\). In the absence of the higher curvature terms, this reduces to the usual result for the holographic conformal anomaly of \(AdS_3\) from \(\hat{\Omega} = 1\) [29, 30, 31], whereas higher curvature terms produce the departures from the unity, i.e., \(\hat{\Omega} - 1\), which can be either positive or negative, depending on the coupling constants of the higher curvature terms.

Then, let me consider the ground state Virasoro generators, expressed in terms of the black hole’s mass and angular momentum:
\[
\hat{L}_0^\pm = \frac{lM \pm J}{2\hbar} + \frac{\hat{c}^\pm}{24}
= \hat{\Omega}^\pm \frac{(lm \pm j)}{2\hbar} + \frac{\hat{c}^\pm}{24}.
\] (3.6)

With the Virasoro algebras of \(\hat{L}_m^\pm\) in the standard form, which are defined on the plane, one can use the Cardy formula for the asymptotic states [41, 42, 43, 44, 45]
\[
\log \rho(\hat{\Delta}^\pm) \simeq 2\pi \sqrt{\frac{1}{6} \left( \hat{c}^\pm - 24\hat{\Delta}_{\min}^\pm \right) \left( \hat{\Delta}^\pm - \frac{\hat{c}^\pm}{24} \right)},
\] (3.7)

where \(\hat{\Delta}^\pm\) are the eigenvalues, called conformal weights, of the operator \(\hat{L}_0\) for black-hole quantum states \(|\hat{\Delta}^\pm\rangle\) and \(\hat{\Delta}_{\min}^\pm\) are their minimum values. Here, I note that the above Cardy formula, which comes from the saddle-point approximation of the CFT partition function on a torus, is valid only if the following two conditions are satisfied:
\[
\frac{24\hat{\Delta}_{\eff}^\pm}{\hat{c}_{\eff}^\pm} \gg 1, \quad \frac{\hat{c}_{\eff}^\pm \hat{\Delta}_{\eff}^\pm}{\hat{c}_{\eff}^\pm} \gg 1,
\] (3.8)
(3.9)

where \(\hat{\Delta}_{\eff}^\pm = \hat{\Delta}^\pm - \hat{c}^\pm/24\), \(\hat{c}_{\eff}^\pm = \hat{c}^\pm - 24\hat{\Delta}_{\min}^\pm\) are the effective conformal weights and central charges, respectively; from the first condition, the higher-order correction terms are exponentially suppressed as \(e^{-2\pi\epsilon^\pm(\hat{\Delta}^\pm-\hat{\Delta}_{\min}^\pm)}\) with \(\epsilon^\pm \equiv 24\hat{\Delta}_{\eff}^\pm/\hat{c}_{\eff}^\pm\); from the second condition, the usual
saddle-point approximation is reliable, i.e., $\rho(\hat{\Delta}^\pm)$ dominates in the partition function (see Refs. [45, 14] for the details).

Then, the statistical entropy for the asymptotic states becomes

$$S_{\text{stat}} = \log \rho(\hat{\Delta}^+) + \log \rho(\hat{\Delta}^-)$$

$$= \frac{\pi}{4G\hbar} |\hat{\Omega}(r_+ + r_-)| + \frac{\pi}{4G\hbar} |\hat{\Omega}(r_+ - r_-)|$$

$$= \frac{|\hat{\Omega}|}{4G\hbar} \frac{2\pi r_+}{4G\hbar} (3.10)$$

where I have chosen $\hat{\Delta}^\pm_{\text{min}} = 0$ as usual [46, 37]; from (3.6), this corresponds to the AdS vacuum solution with $m = -1/(8G)$ and $j = 0$ in the usual context, but one has

$$M = -\frac{\hat{\Omega}}{8G}, \quad J = 0$$

(3.11)

in the new context. Note that the correct “$1/\hbar$” factor for the semiclassical black hole entropy comes from the appropriate recovering of $\hbar$ in (3.2) and (3.6). According to the conditions of validity (3.8) and (3.9), this entropy formula is valid only when both of the two conditions

$$(r_+ \pm r_-) \gg l, \quad (3.12)$$

$$(r_+ \pm r_-) \gg \hbar G \quad (3.13)$$

are satisfied. The usual semiclassical limit of large black hole (area), in which the back-reaction of the emitted radiation from the black hole is neglected [47, 48] and so the thermodynamical entropy formula (2.14) and (2.15) from the first law can be reliable, agrees with the condition (3.13). The condition (3.12) provides one more restriction on the black hole systems, though this does not seem to be needed, in general. But, at this stage, the condition of large central charges $\hat{c}^\pm \gg 1$, i.e., $l \gg \hbar G$ [46], which would be related to the leading supergravity approximation of AdS/CFT correspondence [28], is not needed yet. It is interesting to note also that the statistical entropy (3.10) from the Cardy formula (3.7) has basically the same form for both the Einstein-Hilbert action and the higher curvature corrected action; the only changes are some correction terms in the central charges and the conformal weights themselves, rather than considering the higher order corrections to the Cardy formula itself! This is because the higher curvature terms do not necessarily imply the quantum corrections, such as even the higher curvature terms can be treated semiclassically by neglecting the back reaction effects, which are quantum effects, and so (3.9) or (3.13) can be satisfied [14].

Now, let me consider the following two cases, depending on the signs of $\hat{\Omega}$: (i). $\hat{\Omega} \geq 0$ and (ii). $\hat{\Omega} < 0$. 

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(i). $\Omega \geq 0$: In this case, I have $|\Omega| = \Omega$ and the statistical entropy (3.10) becomes

$$S_{\text{stat}} = \Omega \frac{2\pi r}{4G\hbar}. \quad (3.14)$$

This agrees exactly with the usual Wald’s entropy formula (2.14), as was known also in the literatures [9, 12, 15]. And, this is the case where $\hat{c}^\pm$ and $\hat{\Delta}^\pm - \hat{c}^\pm/24$ are positive definite such as the Cardy formula (3.7) has a well-defined meaning. In the gravity side also it shows the usual behavior with the “positive” mass and angular momentum satisfying the normal inequality $M \geq J/l$, as well as $M^2 \geq J^2/l^2$.

(ii). $\Omega < 0$: In this case, I have $|\Omega| = -\hat{\Omega}$ and so the statistical entropy (3.10) becomes

$$S_{\text{stat}} = -\hat{\Omega} \frac{2\pi r}{4G\hbar}. \quad (3.15)$$

This agrees exactly with the modified entropy formula (2.15), which is manifestly positive and guarantees the second law of thermodynamics, even in this case. But, this can not agree with the usual Wald’s formula (2.14), giving a negative entropy, though this has not been well-explored in the literatures\(^{10}\); this discrepancy can be only resolved in the modified entropy formula (2.15). And, this is the case where there is an abnormal mass bound due to $M \leq J/l$ with negative $M$ and $J$. Moreover, in the CFT side also, this is not the usual system either because $\hat{c}^\pm = \hat{\Omega}(3l/2G\hbar)$ and $\hat{\Delta}^\pm - \hat{c}^\pm/24 = \hat{\Omega}(ml - j)/2\hbar$ are negative valued, but the CFT is perfectly well defined. The application of the Cardy formula to the case of negative $\hat{c}^\pm$ and $\hat{\Delta}^\pm - \hat{c}^\pm/24$ might be questioned due to the existence of negatives-norm states with the usual condition $\hat{L}_n^\pm |\hat{\Delta}^\pm \rangle = 0$ ($n > 0$) for the highest-weight state $|\hat{\Delta}^\pm \rangle$. However, this problem can be easily cured, though not quite well-known, by considering another representation of the Virasoro algebras with $\hat{L}_n^\pm \equiv -\hat{L}_n^\pm$, $\hat{c}^\pm \equiv -\hat{c}^\pm$ and $\hat{L}_n^\pm |\hat{\Delta}^\pm \rangle = 0$ ($n > 0$) for the new highest-weight state $|\hat{\Delta}^\pm \rangle$ [49, 13, 14]; this implies that the Hilbert space need to be “twisted” in which the whole states vectors be constructed from the doubly-twisted highest-weight state $|\hat{\Delta}^+ \rangle \otimes |\hat{\Delta}^- \rangle$. The formula (3.10), which is invariant under this substitution—actually their self-compensations of the negative signs produce the real and positive statistical entropy, should be understood in this context.

In summary, I have found exact agreements between the new thermodynamic black hole entropies which are manifestly non-negative, satisfying the second law, and have been evaluated in the bulk (AdS) gravity side and the CFT entropies in the asymptotic boundary, for any value of the conformal factor $\hat{\Omega}$ of higher curvature gravities. So, the modified entropy formula

\(^{10}\)The errors in Ref. [15] came from the missing of “absolute values” in the computation; for example, in (32) or (39), $\Omega$ should be corrected due to $\sqrt{\Omega^2} = |\Omega|$. In other literatures [9, 10, 12], $\hat{\Omega} > 0$ has been implicitly, or explicitly assumed, instead.
for a negative conformal factor $\hat{\Omega} < 0$ seems to be supported by the sub-leading order with generic higher curvature terms, as well as in the leading order with the Einstein-Hilbert action.

**IV. Inclusion of a gravitational Chern-Simons term (I): Thermodynamics**

In three (or odd in general) dimensions, the gravitational Chern-Simons term can also be included as a higher derivative correction, as well as higher curvature corrections. The total action with a gravitational Chern-Simons term as well as generic higher curvature terms is described by the action

$$I_{g(tot)} = I_g + I_{GCS},$$

where the gravitational Chern-Simons term is given by

$$I_{GCS} = \frac{\hat{\beta} l_0}{64\pi G} \int_M d^3x \epsilon^{\mu\nu\alpha} \left( R_{ab\mu\nu} \omega^{ab\alpha} + \frac{2}{3} \omega^{b\mu}_{c\nu} \omega^{c\alpha}_{a\nu} \omega^{a\beta}_{\alpha} \right).$$

Here, the spin-connection 1-form $\omega^{ab\mu} = \omega^{a\mu}_{b\nu} dx^\nu$, $\omega^{a\mu}_{b\nu} = -\omega^{b\nu}_{a\mu}$ is determined by the torsion-free condition $de^a + \omega^{a\mu}_b dx^\mu$, $\omega^{a\mu}_{b\nu} = -\omega^{b\nu}_{a\mu}$ is determined by the torsion-free condition $de^a + \omega^{a\mu}_b dx^\mu$, and the curvature is then $R_{ab\mu\nu} = \partial_{\mu} \omega_{ab\nu} + \omega^{c\mu}_{a\nu} \omega_{cb\nu} - \omega^{c\mu}_{b\nu} \omega_{ca\mu}$. I take the same definitions as in Ref. [10] for the curvature 2-form $R_{ab} = (1/2)R_{ab\mu\nu} dx^\mu \wedge dx^\nu$ and the spin-connection 1-form $\omega^{ab\mu}$. Note that $I_{GCS}$ is of third derivative order and it does not have the diffeomorphism symmetry in the “bulk”.

The resulting equations of motion are

$$\frac{\partial f}{\partial g^{\mu\nu}} - \frac{1}{2} g^{\mu\nu} f - \frac{1}{l_0^2} g^{\mu\nu} = t^{\mu\nu} + \hat{\beta} l_0 C^{\mu\nu},$$

where the Cotton tensor $C^{\mu\nu}$ is defined by

$$C^{\mu\nu} = \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu\sigma} \nabla_\rho (R^\rho_{\sigma} - \frac{1}{4} \delta^\rho_{\sigma} R),$$

which is traceless and covariantly conserved [6]. The BTZ solution (2.4), (2.5) satisfies the same equations of motion as (2.7) from $C^{\mu\nu} = 0$, like as $t^{\mu\nu} = 0$, for any constant curvature solution, and of course with the renormalized parameters $l, r_+, and r_-.$

11Note that the dimensionless coupling constant $\hat{\beta}$ is related to the one used in Refs. [6, 7, 8] as $\hat{\beta} = -1/(\mu l_0)$, in Ref. [11] as $\hat{\beta} = -\beta/\mu_0$, and in Ref. [10] as $\hat{\beta} = -32\pi G\beta_{KL}/l_0$. 

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In the absence of the higher curvature terms, the mass and angular momentum are found to be\(^{12}\)

\[
M_{GCS} = m + \hat{\beta}j/l_0, \quad J_{GCS} = j + \hat{\beta}l_0m.
\]  

(4.5)

So, in the presence of the higher curvature terms as well as the gravitational Chern-Simons term, one can evaluate the total mass and angular momentum as

\[
M_{\text{tot}} = m + (\hat{\Omega} - 1)m + \hat{\beta}j/l, \\
J_{\text{tot}} = j + (\hat{\Omega} - 1)j + \hat{\beta}lm,
\]

(4.6)

by summing the two contributions with the appropriate renormalization of the parameters \(l_0, r_+, r_-\) \(^{12}\). Here, \((\hat{\Omega} - 1)\)-factors came from the higher curvature corrections of (2.8) and \(\hat{\beta}\)-factors came from the gravitational Chern-Simons corrections of (4.5). In contrast to the case with the higher curvature corrections only, the usual inequalities of the mass and angular momentum are not generally valid,

\[
M_{\text{tot}}^2 - J_{\text{tot}}^2/l^2 = \hat{\Omega}^2 \left(1 - \frac{\hat{\beta}^2}{\hat{\Omega}^2}\right) (m^2 - j^2/l^2).
\]

(4.8)

but depends on the values of the ratio, \(\hat{\eta} \equiv \hat{\beta}/\hat{\Omega}\): For small values of ratio, \(|\eta| < 1\), the usual inequality in magnitudes is preserved, i.e., \(M_{\text{tot}}^2 \geq J_{\text{tot}}^2/l^2\); however, for the large values of ratio, \(|\eta| > 1\), one has an anomalous inequality with an exchanged role of the mass and angular momentum as \(J_{\text{tot}}^2/l^2 \geq M_{\text{tot}}^2\); also, at the critical value \(|\eta| = 1\), the modified mass and angular momentum are “always” saturated, i.e., \(M_{\text{tot}}^2 = J_{\text{tot}}^2/l^2\), regardless of inequality of the bare parameters \(m, j\) and the signs of \(\hat{\Omega}\). But, the inequality for \(M_{\text{tot}}\) and \(J_{\text{tot}}\) depends on the sign of \(\hat{\Omega}\), also.

Now, by considering the first law of thermodynamics as

\[
\delta M_{\text{tot}} = \Omega_+ \delta J_{\text{tot}} + T_+ \delta S_{W(\text{tot})}
\]

(4.9)

with the temperature \(T_+\) and angular velocity \(\Omega_+\) of the outer horizon \(r_+\), the total entropy can be identified as

\[
S_{W(\text{tot})} = \hat{\Omega} \left(\frac{2\pi r_+}{4Gh} + \frac{\hat{\eta}^2 \pi r_-}{4Gh}\right).
\]

(4.10)

\(^{12}\)This has been checked in several different approaches, e.g., the quasi-local method’s in Ref. [50], the super-angular momentum’s in Ref. [51], the ADM’s in Refs. [52, 53], the holography’s in Refs. [10, 11].
This agrees with the Wald’s entropy formula [12], as it should be. But, as I have argued for the gravitational Chern-Simons corrected case in the recent works [13, 14] and in Sec. II for the higher-curvature corrected case, the positiveness nor the second law of thermodynamics would not be guaranteed by the entropy (4.10), generally. Especially for the gravitational Chern-Simons correction term in (4.10), being proportional to the inner-horizon area $A_-=2\pi r_-$, there is no guarantee of the second law due to lack of area (increasing) theorem for the inner horizon; rather, it seems like that this would rather decrease due to the instability of the inner horizon [54, 55]. The only way of guaranteeing the second law from the entropy (4.10) is to consider an appropriate balancing of $\dot{\Omega}$ and $\dot{\eta}$ such as the contributions from the area of the outer-horizon area $A_+=2\pi r_+$ dominate those from $A_-$: Since $r_+ \geq r_-$ is always satisfied, this condition is equivalent to $|\dot{\eta}| < 1$ with $\dot{\Omega} > 0$. Actually, this is the case where the usual mass/angular momentum inequalities hold from (4.6)∼(4.8) and the system behaves as an ordinary BTZ black hole, though there are some shifts and conformal factor corrections in the mass, angular momentum, and entropy [12].

On the other hand, for the other values of $\dot{\Omega}$ and $\dot{\eta}$, the entropy of (4.10) does not guarantee the positiveness nor the second law, and I need some different forms of the entropy. There are, including the above ordinary case, totally $2 \times 3 = 6$ possible cases from 2 possibilities for $\dot{\Omega}$ ($\dot{\Omega} > 0$, $\dot{\Omega} < 0$) and 3 possibilities for $\dot{\eta}$ ($|\dot{\eta}| \leq 1$, $\eta > 1$, $\eta < -1$). Let me consider the following five cases, in addition to the case of (a). $\dot{\Omega} > 0$, $|\dot{\eta}| \leq 1$ for the ordinary black holes above, depending on the values of $\dot{\Omega}$ and $\dot{\eta}$: (b). $\dot{\Omega} > 0$, $\dot{\eta} > 1$, (c). $\dot{\Omega} > 0$, $\dot{\eta} < -1$, (d). $\dot{\Omega} < 0$, $|\dot{\eta}| \leq 1$, (e). $\dot{\Omega} < 0$, $\dot{\eta} > 1$, and (f). $\dot{\Omega} < 0$, $\dot{\eta} < -1$.

(b). $\dot{\Omega} > 0$, $\dot{\eta} > 1$: In order to study this case, I first note the following identity in the BTZ system [13, 14]

\[
\delta m = \Omega_+ \delta j + T_+ \delta S_{BH}
\]

(4.11)

\[
= \Omega_- \delta j + T_- \delta S_{-}
\]

(4.12)

with the temperature $T_-$ and angular velocity $\Omega_-$ of the inner horizon $r_-\n$

\[
T_- = \left. \frac{\hbar \kappa}{2\pi} \right|_{r_-} = \left. \frac{\hbar (r_+^2 - r_-^2)}{2\pi l^2 r_-} \right|, \quad \Omega_- = \left. -N^\phi \right|_{r_-} = \frac{r_+}{l r_-}
\]

(4.13)

and the usual Bekenstein-Hawking entropy

\[
S_{BH} = \frac{2\pi r_+}{4G\hbar}
\]

(4.14)

Here, the physical relevances of the parameters $T_-$ and $\Omega_-$ are not clear. But, here and below, I use $T_-$, $\Omega_-$ just for convenience in identifying the new entropy, from the “assumed” first law.
of thermodynamics (4.12).\textsuperscript{13}

Now, let me consider, from (4.6) and (4.7),

$$\delta M_{\text{tot}} - \Omega_\text{-} \delta J_{\text{tot}} = \hat{\Omega} [\delta m - \Omega_\text{-} \delta j + \hat{\eta}(\delta j/l - \Omega_\text{-} \delta m)],$$  \hspace{1cm} (4.15)

instead of $\delta M_{\text{tot}} - \Omega_+ \delta J_{\text{tot}}$ in (4.9). Then, it is easy to see that the first two terms in the right hand side become $T_- \delta S_-$ by using the second identity (4.12). And also, the final two terms in the bracket become $T_- \delta S_{BH}$ by using the first identity (4.11) and another identity

$$\Omega_- = \Omega_+^{-1} l^{-2}. \hspace{1cm} (4.16)$$

So, finally I find that (4.15) becomes a new re-arrangement of the first law as

$$\delta M_{\text{tot}} = \Omega_\text{-} \delta J_{\text{tot}} + T_- \delta S_{\text{new}}$$ \hspace{1cm} (4.17)

with a new black hole entropy

$$S_{\text{new}} = \hat{\Omega} \left( \frac{2\pi r_-}{4G\hbar} + \hat{\eta} \frac{2\pi r_+}{4G\hbar} \right). \hspace{1cm} (4.18)$$

With the new entropy formula, it is easy to see the previous argument for the second law of thermodynamics of (4.10) in the small values of coupling as $|\hat{\eta}| < 1$, with $\hat{\Omega} > 0$, can now be applied to that of (4.18) in the large values of coupling as $|\hat{\eta}| > 1$.

(c). $\hat{\Omega} > 0, \hat{\eta} < -1$: In this case, the entropy formula (4.18) would not guarantee the second law of thermodynamics nor the positiveness of the entropy: The entropy would “decrease” indefinitely, with the negative values, as the outer horizon $r_+$ be increased from the Hawking’s area theorem [23]. But, there is a simple way of resolution from the new form of the first law (4.17), as in the case of $\hat{\Omega} < 0$ with the higher curvature terms only in Sec. II. It is to consider

$$S_{\text{new}}' = -S_{\text{new}} = -\hat{\Omega} \left( \frac{2\pi r_-}{4G\hbar} + \hat{\eta} \frac{2\pi r_+}{4G\hbar} \right), \hspace{1cm} (4.19)$$

$$T_- ' = -T_- = \frac{\hbar (r_+^2 - r_-^2)}{2\pi l^2 r_-}, \hspace{1cm} (4.20)$$

instead of $S_{\text{new}}, T_-$ and actually this choice seems to be unique: One might consider $S_{\text{new}}'' = \hat{\Omega} \left[ \frac{2\pi r_-}{4G\hbar} - \hat{\eta} \frac{2\pi r_+}{4G\hbar} \right]$, but then the first law (4.17) is not satisfied.

(d). $\hat{\Omega} < 0, |\hat{\eta}| \leq 1$: This is similar to the case (ii) of Sec. II and III such as the appropriate entropy formula is

$$S_{W(\text{tot})}' = -S_{W(\text{tot})} = -\hat{\Omega} \left( \frac{2\pi r_+}{4G\hbar} + \hat{\eta} \frac{2\pi r_-}{4G\hbar} \right), \hspace{1cm} (4.21)$$

\textsuperscript{13}I have used the definition of $\kappa$ as $\nabla^\nu (\chi^\mu \chi_\mu) = -\kappa \chi^\nu$ for the horizon Killing vector $\chi^\mu$ in order to determine its sign, as well as its magnitude.
with the characteristic temperature $T_+ \equiv -T_+$.

With this form of the entropy the second law is guaranteed. The entropy $S_{W(tot)}'$ is an increasing function of the area of the outer horizon, consistently with the Bekenstein’s argument [5].

(e). $\hat{\Omega} < 0, \hat{\eta} > 1$: This system is effectively the same as the case (c), and the same entropy $S_{new}'$ of (4.19) and $T_-'$ of (4.20) apply.

(f). $\hat{\Omega} < 0, \hat{\eta} < -1$: This is effectively the same system as that of the case (b), and so the same entropy $S_{new}$ of (4.18) and $T_-$ of (4.13) apply.

V. Inclusion of a gravitational Chern-Simons term (II): Statistical entropy

In the absence of the higher curvature terms, the central charges of the holographic anomalies (3.1) are obtained as

$$\hat{c}_{GCS}^\pm = \frac{3l}{2G\hbar},$$

(5.1)

with $\gamma^\pm = 1 \pm \hat{\beta}$ for the right/left-moving sectors, respectively [10, 11]. On the other hand, in the absence of the gravitational Chern-Simons term, the central charges are given by (3.2). Now when both the higher curvature and gravitational Chern-Simons terms present, their contributions are summed to obtain the total central charges as follows [12]:

$$\hat{c}_{tot}^\pm = \frac{3l}{2G\hbar} + (\hat{\Omega} - 1)\frac{3l}{2G\hbar} + (\gamma^\pm - 1)\frac{3l}{2G\hbar}$$

$$= \hat{\Omega}(1 \pm \hat{\eta})\frac{3l}{2G\hbar},$$

(5.2)

And also, regarding the Virasoro generators and their ground state generators $\hat{L}_{0(GCS)}^\pm$, they are also summed to get the total Virasoro generators: In the absence of the higher curvature terms, the ground state Virasoro generators, in the standard form of a CFT on the plane, are given by

$$\hat{L}_{0(GCS)}^\pm = \frac{lM_{GCS} \pm J_{GCS}}{2\hbar} + \frac{\hat{c}_{GCS}^\pm}{24}$$

$$= \gamma^\pm \frac{(lm \pm j)}{2\hbar} + \frac{\hat{c}_{GCS}^\pm}{24},$$

(5.3)

whereas in the absence of the gravitational Chern-Simons term, $\hat{L}_0^\pm$ are given by $\hat{L}_0^\pm = \hat{\Omega}\frac{(lm \pm j)}{2\hbar} + \hat{c}_{tot}^\pm$ as in (3.6). So, the total ground state generators are given by

$$\hat{L}_{0(tot)}^\pm = \left[1 + (\hat{\Omega} - 1) + (\gamma^\pm - 1)\right]\frac{(lm \pm j)}{2\hbar} + \frac{\hat{c}_{tot}^\pm}{24}$$
\[
\begin{align*}
\hat{\Omega}(1 \pm \hat{\eta}) \frac{(|lm \pm j|)}{2}\hat{c}_\text{tot}^{\pm} = \\
\hat{c}_\text{tot}^{\pm} = \frac{LM_\text{tot} \pm J_\text{tot}^{\pm}}{2}\hat{c}_\text{tot}^{\pm}. \quad (5.4)
\end{align*}
\]

Now, with the above CFT data \(\hat{c}_\text{tot}^{\pm}, \hat{L}_0^{\pm}(\text{tot})\), one can compute the statistical entropy for the asymptotic states, from the Cardy formula (3.7) with the appropriate conditions (3.8) and (3.9), as follows [\(\hat{\gamma}^{\pm} = \hat{\Omega}(1 \pm \hat{\eta})\)]

\[
S_{\text{stat(tot)}} = \log \rho(\hat{\Delta}_\text{tot}^{+}) + \log \rho(\hat{\Delta}_\text{tot}^{-}) = \frac{\pi}{4G\hat{h}}(\hat{\gamma}^{+} + |\hat{\gamma}^{-}|)r_{+} + \frac{\pi}{4G\hat{h}}(\hat{\gamma}^{-} - |\hat{\gamma}^{+}|)r_{-}, \quad (5.5)
\]

where \(\hat{\Delta}_\text{tot}^{\pm}\) are the eigenvalues of the operators \(\hat{L}_0^{\pm}(\text{tot})\) for black-hole quantum states \(|\hat{\Delta}_\text{tot}^{\pm}\rangle\) and I have chosen their minimum values as \(\hat{\Delta}_\text{min(tot)}^{\pm} = 0\); from (5.4), this corresponds to the \(AdS_3\) vacuum having \(m = -1/(8G)\) and \(j = 0\) in the usual context as usual, but it has a permanent rotation, as well as the conformal-factor corrections,

\[
M_\text{tot} = -\frac{\hat{\Omega}}{8G}, \quad J_\text{tot} = -\frac{l\hat{\Omega}\hat{\eta}}{8G} \quad (5.6)
\]

in the new context [10].

Then, let me consider the following four cases, depending on the values of \(\hat{\Omega}\) and \(\hat{\eta}\): (a). \(\hat{\Omega} > 0, |\hat{\eta}| \leq 1\), (b). \(\hat{\Omega} > 0, \hat{\eta} > 1\) or \(\hat{\Omega} < 0, \hat{\eta} < -1\), (c). \(\hat{\Omega} > 0, \hat{\eta} < -1\) or \(\hat{\Omega} < 0, \hat{\eta} > 1\), (d). \(\hat{\Omega} < 0, |\hat{\eta}| \leq 1\).

(a). \(\hat{\Omega} > 0, |\hat{\eta}| \leq 1\): In this case, I have \(|\hat{\gamma}^{\pm}| = \hat{\gamma}^{\pm}\) and the statistical entropy (5.5) becomes

\[
S_{\text{stat(tot)}} = \hat{\Omega} \left( \frac{2\pi r_{+}}{4G\hat{h}} + \hat{\eta}\frac{2\pi r_{-}}{4G\hat{h}} \right) \quad (5.7)
\]

from \(\hat{\gamma}^{+} + \hat{\gamma}^{-} = 2\hat{\Omega}, \gamma^{+} - \gamma^{-} = 2\hat{\Omega}\hat{\eta}\). This agrees exactly with the usual entropy formula (4.10), which agrees with the Wald’s formula also [12]. And, this is the case where \(\hat{c}_\text{tot}^{\pm}\) and \(\hat{\Delta}_\text{tot}^{\pm} - \hat{c}_\text{tot}^{\pm}/24\) are positive definite such as the Cardy formula (3.7) has a well-defined meaning. In the gravity side also it shows the usual behavior with the “positive” mass and angular momentum, satisfying the normal inequality \(M_\text{tot}^2 \geq J_\text{tot}^2/\ell^2\).

(b). \(\hat{\Omega} > 0, \hat{\eta} > 1\) or \(\hat{\Omega} < 0, \hat{\eta} < -1\): In this case, I have \(|\hat{\gamma}^{\pm}| = \hat{\gamma}^{\pm}, |\hat{\gamma}^{-}| = -\hat{\gamma}^{-}\) and so the statistical entropy (5.5) becomes

\[
S_{\text{stat(tot)}} = \hat{\Omega} \left( \frac{2\pi r_{-}}{4G\hat{h}} + \hat{\eta}\frac{2\pi r_{+}}{4G\hat{h}} \right). \quad (5.8)
\]
This agrees exactly with the new entropy formula (4.18), which guarantees the second law of thermodynamics in this case. And, this is the case where there is an abnormal change of the role of the mass and angular momentum due to $J_{\text{tot}}^2/L^2 \geq M_{\text{tot}}^2$, even though $M_{\text{tot}}$ and $J_{\text{tot}}$ both are positive definite, as usual. Moreover, in the CFT side also, this is not the usual system either because $c_{\text{tot}}^- = \gamma^- 3l/(2Gh)$ and $\Delta_{\text{tot}}^- - c_{\text{tot}}^-/24 = \gamma_{\text{tot}} (ml - j)/2h$ are negative valued, though their self-compensations of the negative signs produce the real and positive statistical entropy. However, the CFT is perfectly well defined, as I have discussed in the case (ii) of Sec. III, by considering another representation of the Virasoro algebra with $\tilde{c}_{\text{tot}}^- = \gamma^- 3l/(2Gh)$ and $\tilde{\Delta}_{\text{tot}}^- - \tilde{c}_{\text{tot}}^-/24 = \gamma_{\text{tot}} (ml - j)/2h$ are negative valued, though their self-compensations of the negative signs produce the real and positive statistical entropy. Moreover, in the CFT side also, this is not the usual system either because $\tilde{c}_{\text{tot}}^- = \gamma^- 3l/(2Gh)$ and $\tilde{\Delta}_{\text{tot}}^- - \tilde{c}_{\text{tot}}^-/24 = \gamma_{\text{tot}} (ml - j)/2h$ are negative valued, though their self-compensations of the negative signs produce the real and positive statistical entropy. However, the CFT is perfectly well defined, as I have discussed in the case (ii) of Sec. III, by considering another representation of the Virasoro algebra with $\tilde{L}_{n(\text{tot})}^- \equiv -\tilde{L}_{-n(\text{tot})}^-$ and $\tilde{L}_{n(\text{tot})}|\tilde{\Delta}_{\text{tot}}^-\rangle = 0$ for a new highest-weight state $|\tilde{\Delta}_{\text{tot}}^-\rangle$ [13, 14]; this implies that the Hilbert space need to be “twisted” in which the whole states vectors be constructed from the twisted highest-weight state $|\tilde{\Delta}_{\text{tot}}^+\rangle \otimes |\tilde{\Delta}_{\text{tot}}^-\rangle$, in contrast to the double twistings for the case (ii) of Sec. III. The formula (5.5), which is invariant under this substitution, should be understood in this context. On the other hand, if I take the limit $\hat{\eta} \to \infty$, this becomes the “exotic” black hole system that occurs in several different contexts [56, 57, 58, 59, 13]. Interestingly, this includes the limiting case of $\hat{\Omega} \to 0$ with any finite, non-vanishing $\hat{\beta} = \hat{\Omega} \hat{\eta}$, as well as the case of $\hat{\beta} \to \infty$, with the finite $\hat{\Omega}$, in which there is only the gravitational Chern-Simons term, without the Einstein-Hilbert and its higher curvature corrections.

(c). $\hat{\Omega} > 0$, $\hat{\eta} < -1$ or $\hat{\Omega} < 0$, $\hat{\eta} > 1$: In this case, I have $|\hat{\gamma}^+| = -\hat{\gamma}^+$, $|\hat{\gamma}^-| = \hat{\gamma}^-$ and the statistical entropy (5.5) becomes

$$S_{\text{stat(tot)}} = -\hat{\Omega} \left( \frac{2\pi r_-}{4Gh} + \hat{\eta} \frac{2\pi r_+}{4Gh} \right).$$

Note that this is positive definite and this should be the case due to the definition $S_{\text{stat(tot)}} = \log (\rho(\hat{\Delta}_{\text{tot}}^+) \rho(\hat{\Delta}_{\text{tot}}^-)) \geq 0$ for the number of states $\rho(\hat{\Delta}_{\text{tot}}^+) \geq 1$. This agrees exactly with the modified new entropy formula (4.19), which guarantees the second law. And this is the case where $M_{\text{tot}}$ can be negative and $J_{\text{tot}}$ has the opposite direction to the bare one $j$, in contrast to the positive definite $M_{\text{tot}}$ and $J_{\text{tot}}$ in the cases of (a) and (b), as well as the anomalous inequality $J_{\text{tot}}^2/L^2 \geq M_{\text{tot}}^2$. In the CFT side, $c_{\text{tot}}^+$ and $\Delta_{\text{tot}}^+ - c_{\text{tot}}^+ / 24$ become negative-valued now, and I need to twist this right-moving sector, rather than the left-moving one as in the case (b), $\tilde{L}_{n(\text{tot})}^+ \equiv -\tilde{L}_{-n(\text{tot})}^+$, $\tilde{c}_{\text{tot}}^+ \equiv -\tilde{c}_{\text{tot}}^+$ and $\tilde{L}_{n(\text{tot})}|\tilde{\Delta}_{\text{tot}}^+\rangle = 0$ for the twisted highest-weight state $|\tilde{\Delta}_{\text{tot}}^+\rangle \otimes |\tilde{\Delta}_{\text{tot}}^-\rangle$.

(d). $\hat{\Omega} < 0$, $|\hat{\eta}| \leq 1$: In this case, I have $|\hat{\gamma}^\pm| = -\hat{\gamma}^\pm$ and the statistical entropy (5.5) becomes

$$S_{\text{stat(tot)}} = -\hat{\Omega} \left( \frac{2\pi r_+}{4Gh} + \hat{\eta} \frac{2\pi r_-}{4Gh} \right).$$

(5.10)
This agrees exactly with the modified entropy formula (4.21), which is positive definite and guarantees the second law as well as the first law. This case is exactly the same situation as in the case (ii) of Sec. III, and there is the upper bound for the mass, i.e., \( M_{\text{tot}} \leq J_{\text{tot}}/l \leq 0 \), though one has the usual inequality in the magnitudes \( M_{\text{tot}}^2 \geq J_{\text{tot}}^2/l^2 \). And, in contrast to the above (b), (c) cases, I need the doubly-twisted highest-weight state \( |\tilde{\Delta}_{\text{tot}}^+\rangle \otimes |\tilde{\Delta}_{\text{tot}}^+\rangle \) with the new presentation of the Virasoro algebras for \( \tilde{L}_n^{\pm,(\text{tot})} \equiv -\hat{L}_n^{\pm,(\text{tot})} \), \( \hat{c}_{\text{tot}}^{\pm} \equiv -\hat{c}_{\text{tot}}^{\pm} \), and \( \tilde{L}_n^{\pm,(\text{tot})}|\tilde{\Delta}_{\text{tot}}^\pm\rangle = 0 \) (\( n > 0 \)).

It is interesting to note that I have perfectly well-defined CFT for all the possible cases and there are no conflicts, in the general theory of (4.1), between the “single” twisting of the Hilbert space with the gravitational Chern-Simons of Sec.III and the “double” twistings with the higher curvatures [13, 14]: They are well self-organized such as the negative norm states do not occur.

In summary, I have found exact agreements between the new thermodynamical black hole entropies for the bulk (AdS) gravity with the gravitational Chern-Simons as well as the general higher curvature terms and the CFT entropies in the asymptotic boundary, for all values of the coupling \( \hat{\eta} \) and the conformal factor \( \hat{\Omega} \). It is remarkable that CFT has no conflict, for the general theory with both the higher curvature and gravitational Chern-Simons terms, between the single twisting of the Hilbert space for the gravitational Chern-Simons term and double twisting for the higher curvatures such as the CFT can be well-defined for all cases, though this is not so clear at first. So, the new entropy formulae for the strong coupling \( |\hat{\eta}| > 1 \), either \( \hat{\Omega} > 0 \) or \( \hat{\Omega} < 0 \) and the modified entropy formula for \( \hat{\Omega} < 0 \), \( |\hat{\eta}| \leq 1 \) seem to be supported by the CFT approach also. This reveals the AdS/CFT correspondence in the sub-leading orders with the “all” higher curvature terms and the higher derivative term of the gravitational Chern-Simons, as well as in the leading order with the Einstein-Hilbert action.

VI. Summary and open problems

I have studied the thermodynamics of the BTZ black hole in the presence of all the higher curvature terms and a gravitational Chern-Simons term, and its solid connection with a statistical approach, based on the holographic anomalies.

The main results are as follows:

First, for the case of a large coupling \( |\hat{\eta}| > 1 \), with any value of \( \hat{\Omega} \), the new entropy formulae are proposed, from the purely thermodynamic point of view such as the second law of thermodynamics be guaranteed.

Second, for the case of \( \hat{\Omega} < 0 \), \( |\hat{\eta}| \leq 1 \), the modified (Wald’s) formula is proposed from purely the second law.

Third, I have found supports for the proposals from a CFT based approach which reproduces
the new entropy formulae for $|\hat{\eta}| > 1$ and the modified formulae for $\hat{\Omega} < 0$, $|\hat{\eta}| \leq 1$, as well as the usual entropy formula for a small coupling $|\hat{\eta}| \leq 1$, $\hat{\Omega} > 0$. This would provide a non-trivial check of the AdS/CFT correspondence, in the presence of higher curvature/derivative terms in the gravity theory. I have also found that there is no conflict, for the general theory with both the higher curvature and gravitational Chern-Simons terms, from the different Hilbert space for each term.

Some open problems would be the followings:

1. A difficult problem of the new entropy formulae is that they require rather unusual characteristic temperature $T_\pm = \kappa/(2\pi)|\nu_\pm$ or temperature $T_\pm' = -T_\pm$, being negative-valued, or $T_\pm' = -T_\pm$, and angular velocity $\Omega_\pm$, being the inner-horizon angular velocity in the BTZ black hole. The “negative” temperature is quite well defined in the statistical mechanics when there is an upper bound of the energy levels [27]. This situation is is quite similar to our case where the entropy is a function of the mass and there is upper bound of mass. So, the negative-valued temperature might not be so strange in this context. But this is contrast to the Hawking temperature in the usual Hawking radiation whose radiation spectrum is determined by the metric alone [60].

2. Can we compute the “classical” Virasoro algebra “directly” in the higher curvature frame, without recourse to the frame transformation to the theories without the higher curvatures ?

2’. Can we explicitly prove that the auxiliary tensor matters which appear after the frame transformation have “no” contributions to the Virasoro algebras, such as there are perfect agreements between the holographic anomalies and the classical Virasoro algebra approaches ? This would provide a more “direct” check of the AdS/CFT correspondence.

3. Can we find the gauge theoretic formulations of the higher curvature gravities ? This would provide a more “explicit” computation of the classical Virasoro algebras [35, 37, 14].

Complete answers to these problems are still missing. But, as far as the first open problem is concerned, I have recently proposed how this might be circumvented by noting some possible limitations of the standard approach initiated by Hawking [60] in our unusual circumstances [13, 20]. Here, the important point would be that a dynamical geometry responds differently under the emission of Hawking radiation, even though the formal metric is the same [13]. For the case of negative conformal factor ($\hat{\Omega} < 0$) in the higher curvature black hole in Sec. II, for example, the emission of a particle with a (positive) energy $\omega$ would reduce the black holes’s mass $M$, which being negative valued, from the conservation of energy, but this corresponds to the “increasing” of the (positive) mass $m$ in the ordinary BTZ black hole context, due to the negative factor in (2.8). So, this implies that the horizon, in the ordinary BTZ black hole context, expands as it emits Hawking radiation with a positive energy, in contrast to the case of positive mass black hole.
Here, the conservations of energy and angular momentum, which are not well enforced in the standard computation, would have a crucial role. In this respect, the Parikh and Wilczek’s approach [61], which provides a direct derivation of Hawking radiation as a quantum tunneling by considering the global conservation law naturally, would be an appropriate framework to study this problem. This is currently under study.

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