LIFTING MATROID DIVISORS ON TROPICAL CURVES

DUSTIN CARTWRIGHT

Abstract. Tropical geometry gives a bound on the ranks of divisors on curves in terms of the combinatorics of the dual graph of a degeneration. We show that for a family of examples, curves realizing this bound might only exist over certain characteristics or over certain fields of definition. Our examples also apply to the theory of metrized complexes and weighted graphs. These examples arise by relating the lifting problem to matroid realizability. We also give a proof of Mnev universality with explicit bounds on the size of the matroid, which may be of independent interest.

1. Introduction

The specialization inequality in tropical geometry gives an upper bound for the rank of a divisor on a curve in terms of a combinatorial quantity known as the rank of the specialization of the divisor on the dual graph of the special fiber of a degeneration [Bak08]. This bound can be sharpened by incorporating additional information about the components of the special fiber, giving augmented graphs [AC13] or metrized complexes [AB15]. All of these inequalities can be strict because there may be many algebraic curves and divisors with the same specialization. Thus, the natural question is whether, for a given graph and divisor on that graph, the inequality is sharp for some algebraic curve and divisor. If \( R \) is the discrete valuation ring over which the degeneration of the curve is defined, we will refer to such a curve and divisor as a lifting of the graph with its divisor over \( R \). In this paper, we show that the existence of a lifting can depend strongly on the characteristic of the field:

**Theorem 1.1.** Let \( P \) be any finite set of prime numbers. Then there exist graphs \( \Gamma \) and \( \Gamma' \) with rank 2 divisors \( D \) on \( \Gamma \) and \( D' \) on \( \Gamma' \) with the following property: For any infinite field \( k \), \( \Gamma \) and \( D \) lift over \( k[[t]] \) if and only if the characteristic of \( k \) is in \( P \), and \( \Gamma' \) and \( D' \) lift over \( k[[t]] \) if and only if the characteristic of \( k \) is not in \( P \).

We also show that the dependence on the field is not limited to the characteristic:

**Theorem 1.2.** Let \( k' \) be any number field. Then there exists a graph \( \Gamma \) with a rank 2 divisor \( D \) such that for any field \( k \) of characteristic 0, \( \Gamma \) and \( D \) lift over \( k[[t]] \) if and only if \( k \) contains \( k' \).
Both Theorem 1.1 and 1.2 are immediate consequences of the following:

**Theorem 1.3.** Let $X$ be a scheme of finite type over $\text{Spec } \mathbb{Z}$. Then there exists a graph $\Gamma$ with a rank 2 divisor $D$ such that, for any infinite field $k$, $\Gamma$ and $D$ lift over $k[[t]]$ if and only if $X$ has a $k$-point.

Theorems 1.1, 1.2, and 1.3 all apply equally well to divisors on weighted graphs [AC13] because the construction of a degeneration in Theorem 1.3 uses curves of genus 0 in the special fiber and for such components, the theory of weighted graphs agrees with unweighted graphs.

Moreover, these theorems also apply to the metrized complexes introduced in [AB15], which record the isomorphism types of the curves in the special fiber. Again, for rational components, the rank of the metrized complex will be the same as the rank for the underlying graph. For metrized complexes, there is a more refined notion of a limit $g_{d,r}^2$, which involves additionally specifying vector spaces of rational functions at each vertex. Not every divisor of degree $d$ and rank $r$ on a metrized complex lifts to a limit $g_{d,r}^2$, but the examples from the above theorems do:

**Proposition 1.4.** Let $\Gamma$ and $D$ be a graph and divisor constructed as in Theorem 1.3. Then for any lift of $\Gamma$ to a metrized complex with rational components, there also exists a lift of $D$ to a limit $g_{d,r}^2$.

If we were to consider divisors of rank 1 rather than rank 2, [ABBR15b] provides a general theory for lifting. They prove that if a rank 1 divisor can be lifted to a tame harmonic morphism with target a genus 0 metrized complex, then it lifts to a rank 1 divisor an algebraic curve. Moreover, the converse is true except for possibly some cases of wild ramification in positive characteristic. Using this, they give examples of rank 1 divisors which do not lift over any discrete valuation ring [ABBR15b, Sec. 5]. While the existence of a tame harmonic morphism depends on the characteristic, the dependence is only when the characteristic is at most the degree of the divisor [ABBR15b, Rmk. 3.9]. In contrast, lifting rank 2 divisors can depend on the characteristic even when the characteristic is bigger than the degree:

**Theorem 1.5.** If $P = \{p\}$ where $p \geq 443$ is prime, then the divisors $D$ and $D'$ in Theorem 1.1 can be taken to have degree less than $p$.

For simplicity, we’ve stated Theorems 1.1, 1.2, and 1.3 in terms of liftings over rings of formal power series, but some of our results also apply to other discrete valuation rings. In particular, these theorems apply verbatim with $k[[t]]$ replaced by any DVR which contains its residue field $k$. For other, possibly even mixed characteristic DVRs, we have separate necessary and sufficient conditions in Theorems 3.4 and 3.5 respectively.

The proof of Theorem 1.3 and its consequences use Mnëv’s universality theorem for matroids [Mnë88]. Matroids are combinatorial abstractions of vector configurations in linear algebra. However, not all matroids come from vector configurations and those that do are called realizable. Mnëv proved
that realizability problems for rank 3 matroids in characteristic 0 can encode arbitrary systems of integral polynomial equations and Lafforgue extended this to arbitrary characteristic [Laf03, Thm. 1.14]. Thus, Theorem 1.3 follows from universality for matroids together with a connection between matroid realizability and lifting problems, which is done in Theorems 3.4 and 3.5. We also give a proof of universality in arbitrary characteristic with explicit bounds on the size of the matroid in order to establish Theorem 1.5.

Matroids have appeared before in tropical geometry and especially as obstructions for lifting. For example, matroids yield examples of matrices whose Kapranov rank exceeds their tropical rank, showing that the minors do not form a tropical basis [DSS05, Sec. 7]. In addition, Ardila and Klivans defined the tropical linear space for any simple matroid, which generalizes the tropicalization of a linear space [AK06]. The tropical linear spaces are realizable as the tropicalization of an algebraic variety if and only if the matroid is realizable [KP11, Cor. 1.5]. This paper is only concerned with rank 3 matroids, which correspond to 2-dimensional fans and the graphs for which we construct lifting obstructions are the links of the fine subdivision of the tropical linear space (see [AK06, Sec. 3] for the definition of the fine subdivision).

Since rank 3 matroids give obstructions to lifting rank 2 divisors on graphs, it is natural to wonder if higher rank matroids give similar examples for lifting higher rank divisors. While we certainly expect there to be results similar to Theorems 1.1, 1.2, and 1.5 for divisors on graphs which have ranks greater than 2, it is not clear that higher rank matroids would provide such examples, or even what the right encoding of the matroid in a graph would be. From a combinatorial perspective, our graphs are just order complexes of the lattice of flats, but for higher rank matroids, the order complex is a simplicial complex but not a graph.

This paper is organized as follows. In Section 2, we introduce the matroid divisors which are our key class of examples and show that as combinatorial objects they behave as if they should have rank 2. In Section 3, we relate the lifting of matroid divisors to the realizability of the matroid. Section 4 looks at the applicability of our matroid to the question of lifting tropically Brill-Noether general divisors and shows that, with a few exceptions, matroid divisors are not Brill-Noether general. Finally, Section 5 provides a quantitative proof of Mnev universality as the basis for Theorem 1.5.

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2. Matroid divisors

In this section, we construct the divisors and graphs that are used in Theorems 1.3. As in [Bak08] and [BN07], we will refer to a finite formal sum
of the vertices of a graph as a \textit{divisor} on that graph. Divisors are related by so-called “chip-firing moves” in which the weight at one vertex is decreased by its degree and those of its neighbors are correspondingly each increased by 1. A reverse chip-firing move is the inverse operation.

As explained in the introduction, the starting point in our construction is a rank 3 simple matroid. A matroid is a combinatorial model for an arrangement of vectors, called elements, in a vector space. A rank 3 simple matroid corresponds to such an arrangement in a 3-dimensional vector space, for which no two vectors are multiples of each other. There are many equivalent descriptions of a matroid, but we will work with the flats, which correspond to vector spaces spanned by subsets of the arrangement, and are identified with the set of vectors that they contain. For a rank 3 simple matroid, there is only one rank 0 and one rank 3 flat, and the rank 1 flats correspond to the elements of the matroid, so our primary interest will be in rank 2 flats. Throughout this paper, \textit{flat} will always refer to a rank 2 flat.

We refer the reader to [Oxl92] for a thorough reference on matroid theory, or [Kat14] for an introduction aimed at algebraic geometers. However, in the case of interest for this paper, we can give the following axiomatization:

\begin{definition}
A \textit{rank 3 simple matroid} \( M \) consists of a finite set \( E \) of \textit{elements} and a collection \( F \) of subsets of \( E \), called the \textit{flats} of \( M \), such that any pair of elements is contained in exactly one flat, and such that there are at least two flats. A \textit{basis} of such a matroid is a triple of elements which are not all contained in a single flat.
\end{definition}

By projectivizing the vector configurations above, a configuration of distinct \( k \)-points in the projective plane \( \mathbb{P}^2_k \) determines a matroid. The elements of this matroid are the points of the configuration and the flats correspond to lines in \( \mathbb{P}^2_k \), identified with the points contained in them. A matroid coming from a point configuration in this way is called \textit{realizable over} \( k \) and in Section 3, we will use the fact that matroid realizability can depend on the field.

Given a rank 3 simple matroid \( M \) with elements \( E \) and flats \( F \), we let \( \Gamma_M \) be the bipartite graph with vertex set \( E \cup F \), and an edge between \( e \in E \) and \( f \in F \) when \( e \) is contained in \( f \). The graph \( \Gamma_M \) is sometimes called the Levi graph of \( M \). We let \( D_M \) be the divisor on the graph \( \Gamma_M \) consisting of the sum of all vertices corresponding to elements of the ground set \( E \).

\begin{proposition}
The divisor \( D_M \) has rank 2.
\end{proposition}

\begin{proof}
To prove the theorem, we first need to show that for any degree 2 effective divisor \( E \), the difference \( D_M - E \) is linearly equivalent to an effective divisor. We build up a “toolkit” of divisors linearly equivalent to \( D_M \). First, for any flat \( f \), we can reverse fire \( f \). This moves a chip from each element contained in \( f \) to \( f \) itself. Thus, the result is an effective divisor whose multiplicity at \( f \) is the cardinality of \( f \), which is at least 2. Our second chip-firing move is to reverse fire a vertex \( e \) as well as all flats containing \( e \).
The net effect will be no change at \( e \) but all of its neighbors will end with \(|f| - 1 \geq 1\) chips. Third, we will use the second chip-firing move, after which all the flats which contain \( e \) have at least one chip, after which it is possible to reverse fire \( e \) again.

Now let \( E \) be any effective degree 2 divisor on \( \Gamma_M \). Thus, \( E \) is the sum of two vertices of \( \Gamma_M \). We consider the various combinations which are possible for these vertices. First, if \( E = [e] + [e'] \) for distinct elements \( e \) and \( e' \), then \( \Gamma_M - E \) is effective. Second, if \( E = [e] + [f] \), then we have two subcases. If \( e \) is in \( f \), then we reverse fire \( e \) and all flats containing it. If \( e \) is not in \( f \), then we can reverse fire just \( f \). Third, if \( E = [f] + [f'] \) for distinct flats \( f \) and \( f' \), then there are again two subcases. If \( f \) and \( f' \) have no elements in common, then we can reverse fire \( f \) and \( f' \). If \( f \) and \( f' \) have a common element, say \( e \), then we reverse fire \( e \) together with the flats which contain it. Fourth, if \( E = 2[e] \), then we use the third chip-firing move, which will move one chip onto \( e \) for each flat containing \( e \), of which there are at least 2. Fifth, if \( E = 2[f] \), then we reverse fire \( f \).

Finally, to show that the rank is at most 2, we give an effective degree 3 divisor \( E \) such that \( D_M - E \) is not linearly equivalent to any effective divisor. For this, let \( e_1, e_2, \) and \( e_3 \) form a basis for \( M \) and let \( f_{ij} \) be the unique flat containing \( e_i \) and \( e_j \) for \( 1 \leq i < j \leq 3 \). We set \( E = [f_{12}] + [f_{13}] + [f_{23}] \) and claim that \( D_M - E \) is not linearly equivalent to any effective divisor. We reverse fire \( e_1 \) together with all flats containing it and to get the following divisor linearly equivalent to \( D_M - E \):

\[
(1) \ [e_1] + (|f_{12}|-2)[f_{12}] + (|f_{13}|-2)[f_{13}] - [f_{23}] + \sum_{\substack{f_k \in e_1 \\ f_k \neq f_{12}, f_{13}}} (|f_k|-1)[f_k],
\]

which is effective except at \( f_{23} \).

We wish to show the divisor in (1) is not linearly equivalent to any effective divisor, which we will do by showing that it is \( f_{23} \)-reduced using Dhar’s burning algorithm [Dha90]. We claim that the burning procedure leads to \(|f_{1i}| - 1\) independent “fires” arriving at \(|f_{1i}|\) for \( i = 2, 3 \). Without loss of generality, it suffices to prove this for \( f_{12} \). Let \( e \) be any element in \( f_{12} \setminus e_1 \). Then \( e_1, e_3, \) and \( e \) form a basis in \( M \). Thus, the unique flat \( f \) containing both \( e_3 \) and \( e \) does not contain \( e_1 \). The path in \( \Gamma_M \) from \( f_{23} \) to \( e_3 \) to \( e \) to \( f_{12} \) does not encounter any chips until \( f_{12} \) and thus “burns” one of these chips. The last such path then continues on to \( e_1 \), whose sole chip is burned. The other path, via \( f_{13} \) does similarly, but then reaches all flats containing \( e_1 \). Therefore, the divisor is \( f_{23} \)-reduced and thus not linearly equivalent to an effective divisor.

Proposition 2.2 also shows that if \( \Gamma_M \) is made into a weighted graph by giving all vertices genus 0, then \( D_M \) has rank 2 on the weighted graph. The rank is, again, unchanged for any lifting of the weighted graph to a metrized complex. To show that \( D_M \) is also a limit \( g_d^2 \) as in Proposition 1.4, we
also need to choose 3-dimensional vector spaces of rational functions on the
variety attached to each vertex.

Proof of Proposition 1.4. We recall from [AB15] that a lift of \( \Gamma_M \) to a
metrized complex means associating a \( \mathbb{P}^1_k \) for each vertex \( v \) of the graph,
which we denote \( C_v \), and a point on \( C_v \) for each edge incident to \( v \). A lift of
the divisor \( D_M \) is a choice of a point on \( C_e \) for each element \( e \) of \( M \).

The data of a limit \( g^2_\delta \) is a 3-dimensional vector space \( H_v \) of rational
functions on each \( C_v \) [AB15, Sec. 5], which we choose as follows. For each
flat \( f \), we arbitrarily choose two elements from it and let \( p_{f,1} \) and \( p_{f,2} \) be
the points on \( C_f \) corresponding to the edges from \( f \) to each of the chosen
elements. Our vector space \( H_f \) consists of the rational functions which have
at worst simple poles at \( p_{f,1} \) and \( p_{f,2} \). For each element \( e \), we choose an
arbitrary flat containing \( e \) and let \( q_e \) be the point on \( C_e \) corresponding
to the edge to \( e \). Our vector space \( H_e \) consists of the rational functions which
have at worst poles at \( q_e \) and at the point of the lift of \( D_M \).

Now to check that these vector spaces form a limit \( g^2_\delta \), we need to show
that the refined rank is 2. For this, we use the same “toolkit” functions as in
the proof of Proposition 2.2, but we augment them with rational functions
from the prescribed vector spaces on the algebraic curves. The first item
from our toolkit was reverse firing a flat \( f \) to produce at least two points
on \( C_f \). We can use rational functions with poles at \( p_{f,1} \) and \( p_{f,2} \) to produce
any degree two effective divisor on \( C_f \). For each element \( e \), we choose an
arbitrary flat containing \( e \) and let \( q_e \) be the point on \( C_e \) corresponding
to the edge to \( e \). Our vector space \( H_e \) consists of the rational functions which
have at worst poles at \( q_e \) and at the point of the lift of \( D_M \).

The second item we needed in our toolkit was reverse firing an element \( e \)
together with all of the flats which contain it. Here, for each element \( e' \) other
than \( e \), we use the rational function with a pole at the divisor and a zero
at the point corresponding to the edge to the unique flat containing both \( e' \)
and \( e \). At each flat \( f \) containing \( e \), we can use any function with a pole
at \( p_{f,i} \), where \( i \in \{1, 2\} \) can be chosen to not be the edge leading to \( e \). This
produces a divisor at an arbitrary point of \( C_e \).

The third and final operation we used was the previous item followed by
a reverse firing of \( e \). Here, we use the same rational functions as before, but
we can choose any rational function on \( C_e \) which has poles at the point of
the divisor and \( q_e \), thus giving us two arbitrary points on \( C_e \). We conclude
that rational functions can be found from the prescribed vector spaces to
induce a linear equivalence between the lift of \( D_M \) and any two points on
the metrized complex. \( \square \)

In the case of rank 1 divisors, lifts can be constructed using the theory
of harmonic maps of metrized complexes, which gives a complete theory
for divisors defining tamely ramified maps to \( \mathbb{P}^1 \) [ABBR15b]. A sufficient
condition for lifting a rank 1 divisor is for it to be the underlying graph of
a metrized complex which has a tame harmonic morphism to a tree (see
[ABBR15a, Sec. 2] for precise definitions). These definitions are limited to
the rank 1 case, but for rank 2 divisors we can subtract points to obtain a divisor of rank at least 1. In particular, if $D_M$ lifts, then for any element $e$, $D_M - [e]$ will be the specialization of a rank 1 effective divisor. However, the lifting criterion of [ABBR15b] is satisfied for these subtractions, independent of the liftability of $D_M$.

**Proposition 2.3.** Let $M$ be any rank 3 simple matroid and $e$ any element of $M$. Also, let $k$ be an algebraically closed field of characteristic not 2. Then, $\Gamma_M$ has a tropical modification $\tilde{\Gamma}_M$ such that $\tilde{\Gamma}_M$ can be lifted to a totally degenerate metrized complex over $k$ with a tame harmonic morphism to a genus 0 metrized complex, such that one fiber is a lift of the divisor $D_M - [e]$.

**Proof.** We first construct a modification $\tilde{\Gamma}_M$ of $\Gamma_M$ which has a finite harmonic morphism from $\tilde{\Gamma}_M$ to a tree $T$. The tree $T$ will be a star tree with a central vertex $w$, together with an unbounded edge, denoted $r_f$, for each flat $f$ which does not contain $e$, and a single unbounded edge $r_e$ corresponding to $e$. Our modification of $\Gamma_M$ consists of adding the following unbounded edges: At $e$, we add one unbounded edge $s_{e,f}$ for each flat $f$ containing $e$. At each element $e'$ other than $e$, we add one unbounded edge $s_{e',f}$ for each flat $f$ which contains neither $e$ nor $e'$. At a flat $f$, we add unbounded edges $s_{f,i}$ where $i$ ranges from 1 to $|f|$ if $e \notin f$ and from 1 to $|f| - 2$ if $e \in f$.

We now construct a finite harmonic morphism $\phi$ from $\tilde{\Gamma}_M$ to $T$. Each element other than $e$ maps to the central vertex $w$ of $T$ and thus the fiber of $w$ will be $D_M - [e]$, as desired. Each flat $f$ not containing $e$ maps to a point one unit of distance along the corresponding ray $r_f$ of $T$. Then the rays $s_{e',f}$ and $s_{f,i}$ also map to the ray $r_f$, starting at $w$ and $\phi(f)$ respectively.

We map the vertex $e$ to its unbounded ray $r_e$, at a distance of 2 from $w$, which leaves all of the flats containing $e$ along the same ray at a distance
of 1. The rays $s_{e,f}$ and $s_{f,i}$, for flats $f$ containing $e$ also map to $r_e$, starting distances of 2 and 1 from $w$ respectively. The map $\phi$ is depicted in Figure 1.

To check that $\phi$ is harmonic, we need to verify that locally, around each vertex $v$ of $\tilde{\Gamma}_M$, the same number of edges map to each of the edges incident to $\phi(v)$, and this number is the degree of $\phi$ at $v$ [BN09, Sec. 2]. We do this for the case when $v$ is a flat $f$ and the case of elements is similar. If $f$ does not contain $e$, then we there are $|f|$ rays mapping to the unbounded side of $r_f$ and the same number of edges mapping to the bounded side, connecting $f$ to the elements it contains. If $f$ does contain $e$, then there are $|f| - 2$ rays mapping to the unbounded side of $r_e$ together with the edge connecting $f$ to $e$. On the bounded side of $r_e$, there are also $|f| - 1$ edges, connecting $f$ to the elements $f \setminus \{e\}$.

To lift $\phi$ to a harmonic morphism of totally degenerate metrized complexes, we need to choose a map $\phi_v : \mathbb{P}^1 \to \mathbb{P}^1$ for each vertex $v$ of $\tilde{\Gamma}_M$ and an identification of the outgoing directions with points on $\mathbb{P}^1$. Having assumed characteristic not 2, we can choose a tame homomorphism of degree equal to the degree of $\phi$ at $v$ as $\phi_v$. We identify the edges incident to $v$ with points of $\mathbb{P}^1$ at which $\phi_v$ is unramified, since these edges all have expansion factor equal to 1. Thus, we have our desired morphism of metrized complexes. □

3. LIFTING MATROID DIVISORS

In this section, we characterize the existence of lifts of matroid divisors in terms of realizability of the corresponding matroids. Recall from [Bak08, Sec. 2A], that given a regular semistable family $X$ of curves over a discrete valuation ring $R$, there is a homomorphism $\rho$ from the group of divisors on the general fiber to the group of divisors on the dual graph of the special fiber. If $D$ is an effective divisor on a graph $\Gamma$ with rank $r$, we will say that a lifting of $\Gamma$ and $D$ over $R$ is a regular semistable family $\tilde{X}$ over $R$ with an effective divisor $\tilde{D}$ on the general fiber such that $\rho(\tilde{D}) = D$ and $\tilde{D}$ has rank $r$.

For our lifting criteria, we will need the following slightly weaker variant of realizability for matroids:

**Definition 3.1.** Let $k$ be a field. We say that a matroid $M$ has a Galois-invariant realization over an extension of $k$ if there exists a finite scheme in $\mathbb{P}^2_k$ which becomes a union of distinct points over $\overline{k}$, and these points realize $M$.

Equivalently, a Galois-invariant realization is a realization over a finite Galois extension $k'$ of $k$ such that the Galois group $\text{Gal}(k'/k)$ permutes the points of the realization. Thus, the distinction between a realization and a Galois-invariant realization is only relevant for matroids which have non-trivial symmetries. Moreover, Lemma 3.10 will show that any matroid can be extended to one where these symmetries can be broken, without affecting realizability over infinite fields.
Example 3.2. Let \( M \) be the matroid determined by all 21 points of \( \mathbb{P}^2_{\mathbb{F}_2} \). Then \( M \) is not realizable over \( \mathbb{P}^2_{\mathbb{F}_2} \) because it contains more than 7 elements, and there are only 7 points in \( \mathbb{P}^2_{\mathbb{F}_2} \). However, \( M \) is clearly realizable over \( \mathbb{F}_4 \) and the Galois group \( \text{Gal}(\mathbb{F}_4/\mathbb{F}_2) \cong \mathbb{Z}/2 \) acts on these points by swapping pairs. Thus, \( M \) has a Galois-invariant realization over an extension of \( \mathbb{F}_2 \).

Example 3.3. Let \( M \) be the Hesse matroid of 9 elements and 12 flats. Then \( M \) is not realizable over \( \mathbb{R} \) by the Sylvester-Gallai theorem. However, the flex points of any elliptic curve are a realization of \( M \) over \( \mathbb{C} \). If the elliptic curve is defined over \( \mathbb{R} \), then the set of all flexes points is also defined over \( \mathbb{R} \), so \( M \) has a Galois-invariant realization over an extension of \( \mathbb{R} \).

Theorem 3.4. Let \( \Gamma_M \) and \( D_M \) be the graph and divisor obtained from a rank 3 simple matroid \( M \) as in Section 2. Also, let \( R \) be any discrete valuation ring with residue field \( k \). If \( D_M \) lifts over \( R \), then the matroid \( M \) has a Galois-invariant realization over an extension of \( k \).

Proof of Theorem 3.4. Let \( \mathcal{X} \) be the semistable family over \( R \) and \( \overline{D} \) a rank 2 divisor on the general fiber of \( \mathcal{X} \) with \( \rho(\overline{D}) = D_M \). First, we make the simplifying assumption that the components of the special fiber are geometrically irreducible. Let \( \overline{D} \) denote the closure in \( \mathcal{X} \) of \( D \). By assumption, \( H^0(\mathcal{X}, \mathcal{O}(\overline{D})) \) is isomorphic to the free \( R \)-module \( R^3 \). By restricting a basis of these sections to the special fiber \( \mathcal{X}_0 \), we have a rank 2 linear series on the reducible curve \( \mathcal{X}_0 \). Since \( \overline{D} \) doesn’t intersect the components of \( \mathcal{X}_0 \) corresponding to the flats of \( M \) and \( \Gamma_M \) is bipartite, \( \overline{D} \) doesn’t intersect any of the nodes of \( \mathcal{X}_0 \). Thus, the base locus of our linear series consists of a finite number of smooth points of \( \mathcal{X}_0 \). Since the base locus consists of smooth points, we can subtract the base points to get a regular, non-degenerate morphism \( \phi: \mathcal{X}_0 \to \mathbb{P}^2_k \).

By the assumption that \( \overline{D} \) specializes to \( D_M \), we have an upper bound on the degree of \( \phi \) restricted to each component of \( \mathcal{X}_0 \). For a flat \( f \) of \( M \), the corresponding component \( C_f \) has degree 0 under \( \phi \), so \( \phi(C_f) \) consists of a single point. For an element \( e \), the corresponding component \( C_e \) has either degree 1 or 0 depending on whether the intersection of \( \overline{D} \) with \( C_e \) is contained in the base locus. If the intersection is in the base locus, then \( C_e \) again maps to a point, and if not, \( C_e \) maps isomorphically to a line in \( \mathbb{P}^2_k \). Thus, the image \( \phi(\mathcal{X}_0) \) is a union of lines in \( \mathbb{P}^2_k \), which we will show to be a dual realization of the matroid \( M \). Let \( f \) be a flat of \( M \). Since the component of \( \mathcal{X}_0 \) corresponding to \( f \) maps to a point, the images of the components corresponding to the elements in \( f \) all have a common point of intersection.

Now let \( e_1 \) be an element of \( M \) and suppose that the component \( C_{e_1} \) maps to a point \( \phi(C_{e_1}) \). Since every other element \( e' \) is in a flat with \( e_1 \),
that means that $\phi(C_e')$, the image of the corresponding component must contain the point $\phi(C_{e_1})$. Since $\phi$ is non-degenerate, there must be at least one component $C_{e_2}$ which maps to a line. Let $e_3$ be an element of $M$ which completes $\{e_1, e_2\}$ to a basis. Thus, the flat containing $e_2$ and $e_1$ is distinct from the flat containing $e_2$ and $e_3$. Since $\phi$ maps $C_{e_2}$ isomorphically onto its image, this means that $\phi(C_{e_3})$ must meet $\phi(C_{e_2})$ at a point distinct from the point $\phi(C_{e_1})$. Thus, $\phi(C_{e_3})$ must equal $\phi(C_{e_2})$. Any other element $e''$ in $M$ forms a basis with $e_1$ and either $e_2$ or $e_3$ (or both). In either case, the same argument again shows that $C_{e''}$ must map to the same line as $C_{e_2}$ and $C_{e_3}$. Again, since $\phi$ is non-degenerate, this is impossible. Thus, we conclude that $\phi$ maps each component $C_e$ corresponding to an element $e$ isomorphically onto a line in $\mathbb{P}_k^2$. We’ve already shown that for any set of elements in a flat, the corresponding lines intersect at the same point. Moreover, because each component $C_e$ maps isomorphically onto its image, distinct flats must correspond to distinct points in $\mathbb{P}_k^2$. Thus, $\phi(X_0)$ is a dual realization of the matroid $M$.

If the components of the special fiber are not geometrically irreducible, then we can find a finite étale extension $R'$ of $R$ over which they are. In our construction of a realization of $M$ over the residue field of $R'$, we can assume that we’ve chosen a basis of $H^0(\mathcal{X} \times_R R', \mathcal{O})$ that’s defined over $R$. Then, the matroid realization will be the base extension of a map of $k$-schemes $X_0 \to \mathbb{P}_k^2$. We let $k'$ be the Galois closure of the residue field of $R'$. Then $\text{Gal}(k'/k)$ acts on the realization of $M$ over $k'$, but the total collection of lines is defined over $k$, and thus invariant. Thus, $M$ has a Galois-invariant realization over an extension of $k$ as desired.

For the converse of Theorem 3.4, we need to consider realizations of matroids over discrete valuations ring $R$, by which we mean $R$-points in $\mathbb{P}^2$ whose images in both the residue field and the fraction field realize $M$. For example, if $R$ contains a field over which $M$ is realizable, then $M$ is realizable over $R$. We say that $M$ has a Galois-invariant realization over an extension of $R$ if there exists a finite, flat scheme in $\mathbb{P}^2_R$ whose special and general fiber are Galois-invariant realizations of $M$ over extensions of the residue field and fraction field of $R$, respectively.

In the following theorem, a complete flag refers to the pair of an element $e$ and a flat $f$ such that $e$ is contained in $f$.

**Theorem 3.5.** Let $R$ be a discrete valuation ring with residue field $k$. Let $M$ be a simple rank 3 matroid with a Galois-invariant realization over an extension of $R$. Assume that $|k| > m - 2n + 1$, where $n$ is the number of elements of $M$ and $m$ is the number of complete flags. Then $\Gamma_M$ and $D_M$ lift over $R$.

Note that Theorem 3.5 does not make any completeness or other assumptions on the DVR beyond the cardinality of the residue field. In contrast, ignoring $D_M$ and its rank, a semistable model $\mathcal{X}$ is only known to exist for an arbitrary graph when the valuation ring is complete [Bak08, Thm. B.2].
We construct the semistable family in Theorem 3.5 using a blow-up of projective space. We begin with a computation of the Euler for this blow-up.

**Lemma 3.6.** Let $S$ be the blow-up of $\mathbb{P}^2_K$ at the points of intersection of an arrangement of $n$ lines. If $A$ is the union of the strict transforms of the lines and the exceptional divisors, then the dimension of $H^0(S, \mathcal{O}(A))$ is at least $2n + 1$.

**Proof.** We first use Riemann-Roch to compute that $\chi(\mathcal{O}(A))$ is $2n + 1$. Let $m$ be the number of complete flags of $M$. We let $H$ denote the pullback of the class of a line on $\mathbb{P}^2$ and $C_f$ to denote the exceptional lines. Then, we have the following linear equivalences

$$A \sim nH - \sum_f (|f| - 1)C_f$$
$$K_S \sim -3H + \sum_f C_f$$

Now, Riemann-Roch for surfaces tells us that

$$\chi(\mathcal{O}(A)) = \frac{A^2 - A \cdot K_S}{2} + 1 = \frac{n^2 - \sum_f(|f| - 1)^2 + 3n - \sum_f(|f| - 1)}{2} + 1$$

(2)

We can think of the summation $\sum_f |f|(|f| - 1)$ as an enumeration of all triples of a flat and two distinct elements of the flat. Since two distinct elements uniquely determine a flat, we have the identity that $\sum_f |f|^2 = n(n - 1)$, so (2) simplifies to $\chi(\mathcal{O}(A)) = 2n + 1$.

It now suffices to prove that $H^2(S, \mathcal{O}(A))$ is zero, which is equivalent, by Serre duality, to showing that $K_S - A$ is not linearly equivalent to an effective divisor. The push-forward of $K_S - A$ to $\mathbb{P}^2$ is $-(n + 3)H$, which is not linearly equivalent to an effective divisor, and thus $H^2(S, \mathcal{O}(A))$ must be zero. Therefore,

$$\chi(\mathcal{O}(A)) = H^0(S, \mathcal{O}(A)) - H^1(S, \mathcal{O}(A)) \leq H^0(S, \mathcal{O}(A)),$$

which together with the computation above yields the desired inequality. □

**Proof of Theorem 3.5.** We first assume that $M$ is realizable over $R$, and then at the end, we’ll handle Galois-invariant realizations over extensions. Thus, we can fix a dual realization of $M$ as a set of lines in $\mathbb{P}^2_R$, and let $S$ be the blow-up of $\mathbb{P}^2_R$ at all the points of intersections of the lines. We let the divisor $A \subset S$ be the sum of the strict transforms of the lines and the exceptional divisors. Note that $A$ is a simple normal crossing divisor whose dual complex is $\Gamma_M$. As in the proof of Theorem 3.4, we denote the components of $A$ as $C_f$ and $C_e$ corresponding to a flat $f$ and an element $e$ of $M$ respectively.
We claim that $A$ is a base-point-free divisor on $S$. Any two lines of the matroid configuration are linearly equivalent in $\mathbb{P}^2_R$. The preimage of a linear equivalence between lines corresponding to elements $e$ and $e'$ is the divisor:

$$[C_{e'}] - [C_e] + \sum_{f : e' \in f, e \notin f} [C_f] - \sum_{f : e \in f, e' \notin f} [C_f].$$

Thus, we have a linear equivalence between $A$ and a divisor which doesn’t contain $C_e$, nor $C_f$ for any of the flats containing $e$ but not $e'$. By varying $e$ and $e'$, we get linearly equivalent divisors whose common intersection is empty.

We now look for a function $g \in H^0(S, \mathcal{O}(A)) \otimes_R k$ which does not vanish at the nodes of $A$. For each of the $m$ nodes, the condition of vanishing at that node amounts to one linear condition on $H^0(S, \mathcal{O}(A)) \otimes_R k$. Since $A$ is base-point-free, this is a non-trivial linear condition, defining a hyperplane. Moreover, because of the degrees of the intersection of $A$ with its components, the only functions vanishing on all of the nodes are multiples of the defining equation of $A$. If the residue field is sufficiently large, then we can find an element $g \in H^0(S, \mathcal{O}(A)) \otimes_R k$ avoiding these hyperplanes, and $|k| > m - 2n + 1$ is sufficient by Lemmas 3.6 and 3.7. Now we lift $g$ to $\tilde{g} \in H^0(S, \mathcal{O}(A))$, and set $X$ to be the scheme defined by $h + \pi \tilde{g}$, where $h$ is the defining equation of $A$ and $\pi$ is a uniformizer of $R$. It is clear that $X$ is a flat family of curves over $R$ whose special fiber is $A$ and thus has dual graph $\Gamma_M$. It remains to check that $X$ is regular and for this it is sufficient to check the nodes of $A_k$. In the local ring of a node, $h$ is in the square of the maximal ideal, but by construction $\pi \tilde{g}$ is not, and thus, at this point $X$ is regular.

Finally, we can take $D$ to be the preimage of any line in $\mathbb{P}^2_R$ which misses the points of intersection. Again, by Lemma 3.7 below, it is sufficient that that $|k| > \ell - 2$, where $\ell$ is the number of flats. We claim that $m - 2n + 1 \geq \ell - 2$, and we’ve assumed that $|k| > m - 2n + 1$. This claimed inequality can be proved using induction similar to the proof of Theorem 4.1, but it also follows from Riemann-Roch for graphs [BN07, Thm. 1.12]. Since $\Gamma_M$ has genus $m - \ell - n + 1$, then the Riemann-Roch inequality tells us that

$$2 = r(D_M) \geq n - (m - \ell - n + 1) = \ell - m + 2n - 1,$$

which is equivalent to the claimed inequality.

Now, we assume that $M$ may only have a Galois-invariant realization over an extension of $R$. We can construct the blow-up $S$ in the same way, since the singular locus of the line configuration is defined over $R$. Again, the divisor $A$ is base-point-free, because we’ve already checked that it is base point free after passing to an extension where the lines are defined. Finally, we need to choose the function $g$ and the line which pulls back to $D$ by avoiding certain linear conditions defined over an extension of $k$. However, when restricted to $k$, these remain linear conditions, possibly of higher codimension, so we can again avoid them under our hypothesis on $|k|$.
Lemma 3.7. Let $k$ be the field of cardinality $q$ and let $H_1, \ldots, H_m$ be hyperplanes in the vector space $k^N$. Let $c$ be the codimension of the intersection $H_1 \cap \cdots \cap H_m$. If $q > m - c + 1$, then there exists a point in $k^N$ not contained in any hyperplane.

Proof. We first quotient out by the intersection $H_1 \cap \cdots \cap H_m$, so we’re working in a vector space of dimension $c$ and we know that no non-zero vector is contained in all hyperplanes. This means that the vectors defining the hyperplanes span the dual vector space, so we can choose a subset as a basis. Thus, we assume that the first $c$ hyperplanes are the coordinate hyperplanes. The complement of these consists of all vectors with non-zero coordinates, of which there are $(q - 1)^c$. Each of the remaining $m - c$ hyperplanes contains at most $(q - 1)^{c-1}$ of these. Our assumption is that $q - 1 > m - c$, so there must be at least one point not contained in any of the hyperplanes. □

We illustrate Theorems 3.4 and 3.5 and highlight the difference between their conditions with the following two examples.

Example 3.8. Let $M$ be the Fano matroid, which whose realization in $\mathbb{P}^2_{\mathbb{F}_2}$ consists of all 7 $\mathbb{F}_2$-points. Then $M$ is realizable over a field if and only if the field has equicharacteristic 2. Thus, by Theorem 3.4, a necessary condition for $\Gamma_M$ and $D_M$ to lift over a valuation ring $R$ is that the residue field of $R$ has characteristic 2. On the other hand, $M$ has 7 elements and 21 complete flags, so Theorem 3.5 says that if $R$ has equicharacteristic 2 and the residue field of $R$ has more than 8 elements, then $\Gamma_M$ and $D_M$ lift over $R$. We do not know if there exists a lift of $\Gamma_M$ and $D_M$ over any valuation ring of mixed characteristic 2.

Example 3.9. One the other hand, let $M$ be the non-Fano matroid, which is realizable over $k$ if and only $k$ has characteristic not equal to 2. Moreover, $M$ is realizable over any valuation ring $R$ in which 2 is invertible. Thus, $\Gamma_M$ and $D_M$ lift over a valuation ring $R$, only if the residue field of $R$ has characteristic different than 2 by Theorem 3.4. The converse is true, so long as the residue field has more than 11 elements by Theorem 3.5.

Since Theorems 3.4 and 3.5 refer to Galois-invariant realizations, we will need the following lemma to relate such realizations with ordinary matroid realizations.

Lemma 3.10. Let $M$ be a matroid. Then there exists a matroid $M'$ such that for any infinite field $k$, the following are equivalent:

1. $M$ has a realization over $k$.
2. $M'$ has a realization over $k$.
3. $M'$ has a Galois-invariant realization over an extension of $k$.

Proof. We use the following construction of an extension of a matroid. Suppose that $M$ is a rank 2 matroid and $f$ is a flat of $M$. We construct a
matroid $M''$ which contains the elements of $M$, together with an additional element $x$. The flats of $M''$ are those of $M$, except that $f$ is replaced by $f \cup \{x\}$, and two-element flats for $x$ and every element not in $f$. By repeating this construction, we can construct a matroid $M'$ such every flat which comes from one of the flats of $M$ has a different number of elements.

Now we prove that the conditions in the lemma statement are equivalent for this choice of $M'$. First, assume that $M$ has a realization over an infinite field $k$. We can inductively extend this to a realization of $M'$. At each step, when adding an element $x$ as above, it is sufficient to place $x$ at a point along the line corresponding to $f$ such that it doesn’t coincide with any of the other points, and it is not contained in any of the lines spanned by two points not in $f$. We can choose such a point for $x$ since $k$ is infinite. Second, if $M'$ has a realization over $k$, then by definition, it has a Galois-invariant realization over an extension of $k$.

Finally, we suppose that $M'$ has a Galois-invariant realization over an extension of $k$ and we want to show that $M$ has a realization over $k$. Suppose we have a realization over a Galois extension $k'$ of $k$. Since all the flats from the original matroid contain different numbers of points, the Galois group does not permute the corresponding lines in the realization. Therefore, the lines and thus also the points from the original matroid $M$ must be defined over $k$. Therefore, the restriction of this realization gives a realization of $M$ over $k$, which completes the proof of the lemma. □

**Proof of Theorem 1.3.** As in the theorem statement, let $X$ be a scheme of finite type over $\mathbb{Z}$. We choose an affine open cover of $X$ and let $\tilde{X}$ be the disjoint union of these affine schemes. By the scheme-theoretic version of Mnëv’s universality theorem, either Theorem 1.14 in [Laf03] or our Theorem 5.3, there is a matroid $M$ whose realization space is isomorphic to an open subset $U$ of $\tilde{X} \times \mathbb{A}^N$ and $U$ maps surjectively onto $X$. Now let $M'$ be the matroid as in Lemma 3.10 and we claim that $\Gamma_{M'}$ and $D_{M'}$ have the desired properties for the theorem.

Let $k$ be any infinite field, and then $X$ clearly has a $k$-point if and only if $\tilde{X}$ has a $k$-point. Likewise, since $k$ is infinite, any non-empty subset of $\mathbb{A}^N_k$ has a $k$-point, so $U$ also has a $k$-point if and only if $X$ has a $k$-point. By Lemma 3.10, these conditions are equivalent to $M'$ having a Galois-invariant realization over an extension of $k$. Supposing that $X$ has a $k$-point and thus $M'$ has a realization over $k$, then $\Gamma_{M'}$ and $D_{M'}$ have a lifting over $k[[t]]$ by Theorem 3.5. Conversely, if $D_{M'}$ has a lifting over $k[[t]]$, then $M'$ has a Galois-invariant realization over an extension of $k$ by Theorem 3.4, and thus $M$ has a realization over $k$ by Lemma 3.10, so $X$ has a $k$-point. □

4. **Brill-Noether theory**

In this section, we take a detour and look at connections to Brill-Noether theory and the analogy between limit linear series and tropical divisors. In the theory of limit linear series, a key technique is the observation that that
if the moduli space of limit linear series on the degenerate curve has the expected dimension then it lifts to a linear series \[\text{[EH86, Thm. 3.4]}\]. Here, the expected dimension of limit linear series of degree \(d\) and rank \(r\) on a curve of genus \(g\) is \(\rho(g, r, d) = g - (r + 1)(g + r - d)\). It is natural to ask if a tropical analogue of this result is true: if the dimension of the moduli space of divisor classes on a tropical curve of degree \(d\) and rank at least \(r\) has (local) dimension \(\rho(g, r, d)\), then does every such divisor lift? See \[\text{[CJP15]}\] for further discussion and one case with an affirmative answer and \[\text{[Cop15]}\] for a negative answer to a related question. The main result of this section is that the matroid divisors and graphs we’ve constructed do not provide a negative answer to the above question.

We begin with the following classification:

**Theorem 4.1.** Let \(M\) be a rank 3 simple matroid, with \(g\) and \(d\) equal to the genus of \(\Gamma_M\) and degree of \(D_M\) respectively. If \(\rho = \rho(g, 2, d) \geq 0\), then \(M\) is one of the following matroids:

1. The one element extension of the uniform matroid \(U_{2,n-1}\), with \(\rho = n - 2\).
2. The uniform matroid \(U_{3,4}\), with \(\rho = 0\).
3. The matroid defined by the vectors: \((1,0,0)\), \((1,0,1)\), \((0,0,1)\), \((0,1,1)\), \((0,1,0)\), with \(\rho = 1\).
4. The matroid in the previous example together with \((1,0,\lambda)\) for any element \(\lambda\) of the field other than 1 and 0, with \(\rho = 0\).
5. The matroid consisting of the point of intersection between any pair in a collection of 4 generic lines, for which we can take the coordinates to be the vectors from (3) together with \((1,1,1)\), with \(\rho = 0\).

The last three cases of Theorem 4.1 are illustrated in Figure 2.

**Proof.** We first compute the invariants for the graph \(\Gamma_D\) and divisor \(D_M\) constructed in Section 2. As before, we let \(n\) be the number of elements of \(M\), \(\ell\) the number of flats, and \(m\) the number of complete flags. Since \(\Gamma_D\) consists of \(m\) edges and \(n + \ell\) vertices, it has genus \(m - n - \ell + 1\). It is also immediate from its definition that \(D_M\) has degree \(n\). Thus, the expected dimension of rank 2 divisors is

\[
(3) \quad \rho = m - n - \ell + 1 - 3((m - n - \ell + 1) + 2 - n) = 5n + 2\ell - 2m - 8
\]
Now, assume that $\rho$ is non-negative for $M$ and we consider what happens to $\rho$ when we remove a single element $e$ from a matroid, where $e$ is not contained in all bases. For every flat containing $e$, we decrease the number of complete flags by 1 if that flat contains at least 3 elements, and if it contains 2 elements, then we decrease the number of flags by 2 and the number of flats by 1. Thus, by (3), $\rho$ drops by $5 - 2s$, where $s$ is the number of flats in $M$ which contain $e$. Since $e$ must be contained in at least 2 flats, either $M \setminus e$, the matroid formed by removing $e$ has positive $\rho$ or $e$ is contained in exactly 2 flats.

We first consider the latter case, in which $e$ is contained in exactly two flats, which we assume to have cardinality $a + 1$ and $b + 1$ respectively. The integers $a$ and $b$ completely determine the matroid because all the other flats consist of a pair of elements, one from each of these sets. Thus, there are $ab + 2$ flats and $2ab + a + b + 2$ complete flags. By using (3), we get $\rho = 5(a + b + 1) + 2(ab + 2) - 2(2ab + a + b + 2) - 8 = -2ab + 3a + 3b - 3.$

One can check that, up to swapping $a$ and $b$, the only non-negative values of this expression are when $a = 1$ and $b$ is arbitrary or $a = 2$ and $b$ is 2 or 3. These correspond to cases (1), (3), and (4) respectively from the theorem statement.

Now we consider the case that $e$ contained in more than two flats, in which case $M \setminus e$ satisfies $\rho > 0$. By induction on the number of elements, we can assume that $M \setminus e$ is on our list, in which case the possibilities with $\rho > 0$ are (1) and case (3). For the former matroid, if $e$ is contained in a flat of $M \setminus e$, then $M$ is a matroid of the type from the previous paragraph, with $a$ equal to 1 or 2. On the other hand, if $e$ contained only in 2-element flats, then $e$ is contained in $n - 1$ flats, so $\rho(M) = \rho(M \setminus e) + 5 - 2(n - 1) = (n - 3) + 7 - 2n = 4 - n.$

The only possibility is $n = 4$, for which we get (2), the uniform matroid. Finally, if $M \setminus e$ is the matroid in case (3), then the only relevant possibilities are those for which $e$ is contained in at most 3 flats, for which the possible matroids are (4) or (5). □

**Proposition 4.2.** If $R$ is a DVR and $M$ is one of the matroids in Theorem 4.1, then $M$ has a Galois-invariant realization over an extension of $R$.

**Proof.** The matroids (2), (3) and (5) are regular matroids, i.e. realizable over $\mathbb{Z}$, so they are *a fortiori* realizable over any DVR. Moreover, the other matroids in case (1) and (4) are realizable over $R$ so long as the residue field has at least $n - 2$ and 3 elements respectively. We will show that if the residue field is finite, then the one-element extension of $U_{2,n-1}$ has a Galois-invariant realization over $R$. The other case is similar.

Let $M$ be the one-element extension of $U_{2,n-1}$ and suppose the residue field $k$ is finite. We choose a polynomial with coefficients of degree $n - 1$ in $R$ whose reduction to $k$ is square-free. Adjoining the roots of this polynomial
defines an unramified extension $R'$ of $R$, and we write $a_1, \ldots, a_{n-1}$ for its roots in $R$. Then, the vectors $(1, a_1, 0), \ldots, (1, a_{n-1}, 0), (0, 0, 1)$ give a Galois-invariant realization of $M$ over $R'$, which is what we wanted to show. □

5. Quantitative Mnëv universality

In this section, we prove a quantitative version of Mnëv universality over $\text{Spec} \mathbb{Z}$ with Theorem 1.5 as our desired application. We follow the strategy of [Laf03, Thm. 1.14], but use the more efficient building blocks used in, for example, [LV13]. We pay close attention to the number of points used in our construction in order to get effective bounds on the degree of the corresponding matroid divisor. These bounds are expressed in terms of the following representation.

**Definition 5.1.** Let $S_n$ denote the polynomial ring $\mathbb{Z}[y_1, \ldots, y_n]$. In the extension $S_n[t]$, we also introduce the coordinates $x_i$ defined by $x_0 = t$ and $x_i = y_i + t$ for $1 \leq i \leq n$. In addition, for $n < i \leq m$, suppose we have elements $x_i \in S_n[t]$ such that:

1. Each $x_i$ is defined as one of $x_i = x_j + x_k$, $x_i = x_j x_k$, or $x_i = x_j + 1$, where $j, k < i$.
2. Each $x_i$ is monic as a polynomial in $t$ with coefficients in $S_n$.

The coordinates $x_i$ for $1 \leq i \leq n$ will be called free variables and the three operations for defining new variables in (1) will be called addition, multiplication, and incrementing, respectively.

Moreover, fix finite sets of equalities $E$ and inequalities $I$ consisting of pairs $(i, j)$ such that $x_i - x_j$ is in $S_n \subset S_n[t]$. We then say that the algebra:

$S_n[(x_i - x_j)^{-1}]_{(i, j) \in I}/\langle x_i - x_j \mid (i, j) \in E \rangle$

has an elementary monic representation consisting of the above data, namely, the integers $n$ and $m$, the expression of each $x_i$ as an addition, multiplication, or increment for $n < i \leq m$, and the sets of equalities and inequalities.

The inequalities $I$ in Definition 5.1 are not strictly necessary because an inverse to $x_i - x_j$ can always be introduced as a new variable, but the direct use of inequalities in the representation can be more efficient for some algebras.

**Proposition 5.2.** There exists an elementary monic representation of any finitely generated $\mathbb{Z}$-algebra.

**Proof.** We begin by writing the $\mathbb{Z}$-algebra as

$R = \mathbb{Z}[y_1, \ldots, y_n]/\langle f_1 - g_1, \ldots, f_m - g_m \rangle$,

where each polynomial $f_k$ and $g_k$ has positive integral coefficients. Then $f_k$ and $g_k$ can be constructed by a sequence of multiplication and addition operations applied to the variables $y_i$ and the constant $1$. Obviously, we can assume that our multiplication never involves the constant $1$. To get an elementary monic representation, we first replace the variables $y_i$ with
We eliminate terms of Theorem 5.3 (Mnëv universality). Z will be an affine scheme over Spec with 

\[ \text{for } y \text{ where the ellipses denote terms with lower degree in the } x \text{ variables. Both the matroid and its potential realization will be built up equalities, then } a \text{ exists a rank } \ell \text{ which involve } t \text{ in the order of descending degree with respect to the variables } y. \text{ In particular, if } ct^s y^{a_1}_1 \cdots y^{a_m}_m \text{ is a term in } x_i - x_j, \text{ then, by swapping } i \text{ and } j \text{ if necessary, we can assume that } c \text{ is positive. We then use multiplication to construct}

\[ x_\ell = t^s x_1^{a_1} \cdots x_n^{a_n} = t^s (t + y_1)^{a_1} \cdots (t + y_n)^{a_n} = t^s y_1^{a_1} \cdots y_m^{a_m} + \ldots, \]

where the ellipses denote terms with lower degree in the y variables. We use c addition operations to construct } x'_{j'} = x_j + cx_\ell, \text{ if } x_j \text{ and } x_\ell \text{ have different degrees in } t. \text{ However, if not, then these additions would not be monic, so we instead compute } x'_{j'} = x_j + nx_\ell + t^d \text{ and also } x_{j'} = x_i + t^d, \text{ where } d \text{ is chosen larger than the } t\text{-degree of } x_i, x_j, \text{ and } x_\ell. \text{ Now, in the expression of } x_i - x_{j'} \text{ or } x_{j'} - x_j \text{ as a polynomial in the } y \text{ variables and } t, \text{ we’ve eliminated the term } ct^s y_1^{a_1} \cdots y_m^{a_m} \text{ while only introducing new terms which all have lower degree in the } y \text{ variables. By induction, we can construct variables } x_{i''} \text{ and } x_{j''} \text{ such that } x_{i''} - x_{j''} = f_k - g_k. \]

Given a matroid } M, \text{ its possible realizations form a scheme, called the realization space of the matroid [Kat14, Sec. 9.5]. Explicitly, given a matroid with } n \text{ elements, each flat of the matroid defines a closed, determinantal condition in } (\mathbb{P}_k^n) \text{ and each triple of elements which is not in any flat defines an open condition by not being collinear. The realization space is the quotient by } PGL_3(\mathbb{Z}) \text{ of the scheme-theoretic intersection of these conditions. We will only consider the case when this action is free, in which case the quotient will be an affine scheme over Spec } \mathbb{Z}.

**Theorem 5.3** (Mnëv universality). For any finite-type } \mathbb{Z} \text{-algebra } R, \text{ there exists a rank 3 matroid } M \text{ whose realization space is an open subset } U \subset \mathbb{A}^N \times \text{Spec } R \text{ such that } U \text{ projects surjectively onto } \text{Spec } R.

Moreover, if } R \text{ has an elementary monic representation with } n \text{ free variables, } a \text{ additions, } m \text{ multiplications, } o \text{ increments, } e \text{ equalities, and } i \text{ inequalities, then } M \text{ has}

\[ 3n + 7a + 7o + 6m + 5e + 6i + 6 \]

\[ N = 3(n + a + o + m + e + i) + 1 \]

**Proof.** By Proposition 5.2, we can assume that } R \text{ has an elementary monic representation. Both the matroid and its potential realization will be built up
from the elementary monic representation, beginning with the free variables and then applying the addition, multiplication, and increment operations. We describe the constructions of both the matroid and the realization in parallel for ease of explaining their relationship.

We begin with the free variables. For \( x_0 = t \) and for each free variable \( x_i \) of the representation, we have a line, realized generically, passing through a common fixed point. In the figures below, we will draw these horizontally so that the common point is at infinity. On each of these lines we have 3 additional points, whose positions along the line are generic. Our convention will always be that points whose relative position is not specified are generic. In other words, unless otherwise specified to lie on a line, each pair of points correspond to a 2-element flat.

From each set of 4 points on one of these free variable lines, we can take the cross-ratio, which is invariant under the action of \( \text{PGL}_3(\mathbb{Z}) \). Therefore, by taking the cross-ratio on each line as the value for the corresponding coordinate \( x_i = t + y_i \) or \( x_0 = t \), we define a morphism from the realization space of the matroid defined thus far to \( \mathbb{A}^{n+1} = \text{Spec} \ S_n[t] \), where \( S_n = \mathbb{Z}[y_1, \ldots, y_n] \) as in Definition 5.1. Our goal with the remainder of the construction is to constrain the realization such that the projection to \( \text{Spec} \ S_n \) is surjective onto \( \text{Spec} \ R \subset \text{Spec} \ S_n \).

Concretely, the cross-ratio is the position of one point on the line in coordinates where the other points are at 0, 1, and \( \infty \). For us, the point common to all variable lines will be at \( \infty \), so we will refer to the other points along the line as the “0” point, the “1” point, and the variable point. We will next embed the operations of addition, multiplication, and incrementing from the elementary monic representation. The result of each of these operations will be encoded as the cross-ratio of 4 points on a generic horizontal line, in the same way as with the free variables.
First, multiplication of distinct variables $x_i = x_j x_k$ is constructed as in Figure 3, where the $x_j$ and $x_k$ lines refer to the lines previously constructed for those variables and the other points are new. We can choose the horizontal line for $x_i$ as well as the additional points generically so that there none are collinear with previously constructed points. Then, one can check that the cross-ratio of the solid points on the central line is the product of the cross-ratios on the other two lines. If $j$ equals $k$, the diagram may be altered by moving the corresponding lines so that they coincide. In either case, the construction uses 6 additional points.

Second, the addition of variables $x_i = x_j + x_k$ can be constructed as in Figure 4. In this configuration, there will be an additional coincidence if $x_j = x_k$ in that the empty circle on the $x_i$ line, the middle point on the $x_k$ line and a point on the bottom line will be collinear. However, since $x_j + x_k$, $x_j$, and $x_k$ are all monic polynomials in the variable $t$, $x_j - x_k$ is also monic in $t$ and so a sufficiently generic choice of $t$ will ensure that $x_j - x_k$ is non-zero. Similarly if $x_j = -1$, then the empty point on the $x_i$ line, the rightmost point on the $x_k$ line and a marked point on the bottom line will be collinear, but this can again be avoided by adjusting $t$. For the addition operation, we’ve used 7 additional points.

Third, for incrementing a single variable, $x_i = x_j + 1$, we specialize the configuration in Figure 4 so that $x_k = 1$, giving Figure 5. The line labeled with 1 can be chosen once and used in common for all increment operations, since it functions as a representative of the constant 1. We’ve used 7 additional points for each increment operation, together with 2 points common to all such operations.

At this point, the realization space still surjects onto Spec $S_n$, and so we still need to impose the equalities and inequalities. Each inequality $x_i \neq x_j$ can be imposed using the diagram in Figure 6, which works by projecting the two variable points to the same line and getting different points. By replacing the projections of the two variables to the central line with the
\[ x_i = x_j + 1 \]

**Figure 5.** Configuration for incrementing. The solid and empty circles represent the variables and the auxiliary points respectively, as in Figure 3, with the exception that, on the \( x_i \) line, the variable point is the rightmost solid point (since \( x_i > 1 \)).

\[ x_i \neq x_j \]

**Figure 6.** Configuration for imposing inequality. The solid and empty circles represent the variables and auxiliary points respectively, as in Figure 3.

same point, we can use a similar figure to assert equality \( x_i = x_j \). These use 6 and 5 additional points respectively.

To summarize, we’ve agglomerated the configurations in Figures 3, 4, 5, and 6 to give a matroid whose realization space projects to Spec \( R \subset \text{Spec} \, S_n \). The realization of this matroid is determined by the values of the \( y_i \), together with a number of parameters, such as the height of the horizontal lines, which are allowed to be generic, and thus the realization space is an open subset of \( A^N \times \text{Spec} \, R \).

To show that the projection to Spec \( R \) is surjective, we take any point of Spec \( R \), which we can assume to be defined over an infinite field. Our construction of the realization required us to avoid certain coincidences, such as any \( x_i \) being 0 or 1 or an equality \( x_j = x_k \) in any addition step. Each such coincidence only occurs for a finite number of possible values for \( t \) we choose \( t \) outside the union of all coincidences, and we can construct a realization of the matroid.
\begin{align*}
x_0 &= t \\
x_1 &= x_0 x_0 = t^2 \\
x_2 &= x_0 x_1 = t^3 \\
x_3 &= x_0 + 1 = t + 1 \\
\vdots \\
x_{\ell+2} &= x_{\ell+1} + 1 = t + \ell \\
x_{\ell+3} &= x_1 + x_{\ell+1} = t^2 + t + \ell \\
x_{\ell+4} &= x_{\ell+3} + x_{\ell+1} = t^2 + 2t + 2 \ell \\
x_{\ell+5} &= x_0 x_{\ell+4} = t^3 + 2t^2 + 2t \ell \\
x_{\ell+6} &= x_{\ell+2} x_{\ell+2} = t^2 + 2t \ell + \ell^2 \\
x_{\ell+7} &= x_{\ell+6} + 1 = t^2 + 2t \ell + \ell^2 + 1 \\
\vdots \\
x_{\ell+p-\ell^2+6} &= x_{\ell+p-\ell^2+5} + 1 = t^2 + 2t \ell + p \\
x_{\ell+p-\ell^2+7} &= x_2 + x_{\ell+p-\ell^2+6} = t^3 + t^2 + 2t \ell + p \\
x_{\ell+p-\ell^2+8} &= x_1 + x_{\ell+p-\ell^2+7} = t^3 + 2t^2 + 2t \ell + p
\end{align*}

Figure 7. System of equations used in the elementary monic representations of $\mathbb{Z}/p$ and $\mathbb{Z}[p^{-1}]$, where $p$ is a prime and $\ell$ denotes the smallest integer less than $\sqrt{p}$.

Finally, we justify the quantitative parts of the theorem statement. The number of elements of $M$ is computed by summing the number of elements for each of the building blocks together with 1 element for the common point on the horizontal lines, 3 elements for the variable $x_0 = t$, and 2 elements for the horizontal line representing 1 in Figure 5. For the computation of $N$, we can assume that the coordinates on $\mathbb{P}^2$ are such that the common point of the horizontal lines is $(1 : 0 : 0)$, the points representing 1 are $(0 : 0 : 1)$ and $(1 : 0 : 1)$, and the “0” and “1” points of the $x_0 = t$ line are $(0 : 1 : 0)$ and $(0 : 1 : 1)$ respectively. These fix the automorphisms of $\mathbb{P}^2$. Then, one can check that each additional free variable and each of the building blocks adds 3 additional generic parameters. Finally, the value of $t$ one more parameter, which gives the expression for $N$. \hfill \Box

Proof of Theorem 1.5. By Theorems 3.4, 3.5, and 5.3, it will be enough to construct sufficiently parsimonious elementary monic representations of the algebras $\mathbb{Z}/p$ and $\mathbb{Z}[p^{-1}]$ and thus matroids $M$ and $M'$, respectively,
representing these equations. Let \( \ell \) be the smallest integer less than \( \sqrt{p} \).

The elementary monic representations for both \( M \) and \( M' \) use the equations shown in Figure 7. Then, \( \mathbb{Z}/p \) can be represented by adding an equality between \( x_{\ell+5} \) and \( x_{\ell+p-\ell^2+8} \) and \( \mathbb{Z}[p^{-1}] \) can be represented by an inequality between the same pair of variables.

In either case, this representation uses no free variables, \( \ell+p-\ell^2 \) increments, 4 additions, and 4 multiplications. Thus, by Theorem 5.3, \( M \) and \( M' \) have \( 7(\ell+p-\ell^2) + 64 \) and \( 7(\ell+p-\ell^2) + 63 \) elements respectively. We’ll bound the former since it is larger. We first rewrite the number of elements as

\[
7(\ell+p-\ell^2) + 64 = p - \ell^2 + 7\ell + 6(p - \ell^2) + 64
\]

To show that (4) is smaller than \( p \), we note that since \( p \geq 443 \), then \( \ell \geq 21 \).

We now have two cases. First, if \( \ell = 21 \), then the largest prime number less than \( 22^2 \) is 479, so \( p - \ell^2 \leq 38 \). Using this, we can bound (4) as

\[
p - 21^2 + 7 \cdot 21 + 6(38) + 64 = p - 2 < p
\]

On the other hand, if \( \ell \geq 22 \), then the choice of \( \ell \) means that \( p < (\ell + 1)^2 \), so \( p - \ell^2 \leq 2\ell \). Therefore, we can bound (4) as follows:

\[
p - \ell^2 + 7\ell + 6(2\ell) + 64 = p - \ell^2 + 19\ell + 64
\]

\[
= p - (\ell - 19)\ell + 64
\]

\[
\leq p - 2 < p
\]

Thus, the number of elements of \( M \) and \( M' \) is less than \( p \).

We take the graphs \( \Gamma \) and \( \Gamma' \) and the divisors \( D \) and \( D' \) for the theorem statement to be the matroid divisors of \( M \) and \( M' \) respectively. Since \( M \) and \( M' \) have fewer than \( p \) elements, \( D \) and \( D' \) have degree less than \( p \). Moreover, since \( k \) is an infinite field, by Theorems 3.4 and 3.5, \( \Gamma \) and \( D \) lift over \( k[[t]] \) if and only if \( M \) is representable over \( k \), which means that the characteristic of \( k \) equals \( p \). Similarly, \( \Gamma' \) and \( D' \) lift if and only if the characteristic of \( k \) is not \( p \). \( \square \)

**Remark 5.4.** The threshold for \( p \) in Theorem 1.5 is not optimal. For example, by using a different construction when \( p \) is closer to a larger square number than to a smaller square, it is possible to reduce the bound to 331.

**References**

[AB15] Omid Amini and Matthew Baker. Linear series on metrized complexes of algebraic curves. *Math. Ann.*, to appear, 2015.

[ABBR15a] Omid Amini, Matthew Baker, Erwan Brugallé, and Joseph Rabinoff. Lifting harmonic morphisms I: metrized complexes and Berkovich skeleta. *Res. Math. Sci.*, to appear, 2015.

[ABBR15b] Omid Amini, Matthew Baker, Erwan Brugallé, and Joseph Rabinoff. Lifting harmonic morphisms II: tropical curves and metrized complexes. *Algebra Number Theory*, to appear, 2015.

[AC13] Omid Amini and Lucia Caporaso. Riemann-Roch theory for weighted graphs and tropical curves. *Adv. Math.*, 240:1–23, 2013.
Federico Ardila and Carly Klivans. The Bergman complex of a matroid and phylogenetic trees. *J. Combin. Theory Ser. B.*, 96(1):38–49, 2006.

Matthew Baker. Specialization of linear systems from curves to graphs. *Algebra Number Theory*, 2(6):613–653, 2008. With an appendix by Brian Conrad.

Matthew Baker and Serguei Norine. Riemann-Roch and Abel-Jacobi theory on a finite graph. *Adv. Math.*, 215:766–788, 2007.

Matthew Baker and Serguei Norine. Harmonic morphisms and hyperelliptic graphs. *Int. Math. Res. Not.*, 15:2914–2955, 2009.

Dustin Cartwright, David Jensen, and Sam Payne. Lifting divisors on a generic chain of loops. *Canad. Math. Bull.*, to appear, 2015.

Marc Coppens. A metric graph satisfying $w_1^4 = 1$ that cannot be lifted to a curve satisfying $\dim(w_1^4) = 1$. preprint, arXiv:1501.03740, 2015.

Deepak Dhar. Self-organized critical state of the sandpile automaton models. *Phys. Rev. Lett.*, 64:1613–1616, 1990.

Mike Develin, Francisco Santos, and Bernd Sturmfels. On the rank of a tropical matrix. In *Combinatorial and computational geometry*, volume 52 of *Math. Sci. Res. Inst. Publ.*, pages 213–242. Cambridge Univ. Press, Cambridge, 2005.

David Eisenbud and Joe Harris. Limit linear series: Basic theory. *Invent. math.*, 85:337–371, 1986.

Eric Katz. Matroid theory for algebraic geometers. preprint, arXiv:1409.3503, 2014.

Eric Katz and Sam Payne. Realization spaces for tropical fans. In *Combinatorial aspects of commutative algebra and algebraic geometry*, volume 6, pages 73–88. Abel Symp., Springer, 2011.

Laurent Lafforgue. *Chirurgie des grassmanniennes*, volume 19 of *CRM Monograph Series*. American Mathematical Society, 2003.

Seok Hyeong Lee and Ravi Vakil. Mnev-Sturmfels universality for schemes. In *A celebration of algebraic geometry*, volume 18 of *Clay Math. Proc.*, pages 457–468. Amer. Math. Soc., Providence, RI, 2013.

Nikolai E. Mniev. *The universality theorems on the classification problem of configuration varieties and convex polytopes varieties*, volume 1346 of *Lect. Notes in Math.*, pages 527–543. Springer, 1988.

James Oxley. *Matroid Theory*, volume 21 of *Oxford graduate texts in mathematics*. Oxford Univ. Press, 1992.

Department of Mathematics, University of Tennessee, 227 Ayres Hall, Knoxville, TN 37996-1320

E-mail address: cartwright@utk.edu