Dynamical synchronization and the horizon problem

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Abstract

The cosmological horizon problem is the question why spatial domains that were never in causal contact with each other now appear in precise symphony. We propose a solution to the horizon problem in which a globally synchronized early state is reached as the $\omega$-limit point of a transient, inhomogeneous Mixmaster universe. We show that the $\alpha$-limit set of the latter is a Kasner circle which represents a synchronized initial state of minimal entropy. Accordingly, unless the evolution is disrupted by quantum gravitational effects so that the initial state is not attained, Planck size domains emerge as causally disconnected, albeit in complete synchrony, as the universe enters an ‘isotropic’ state to remain so in future unison.

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1 Introduction

The observable part of the universe is at any time comprised of a large number of spatial subregions causally connected to the observer, not all of which are so connected to each other. This, as is well known, creates the so-called horizon problem, why any two subregions in the observer’s past causal cone, sufficiently separated as to never had been in causal contact throughout their entire history, now appear synchronized in their physically measurable properties (we shall use the word ‘sync’ for synchronization hereafter). If causally disjoint regions were so delicately brought to sameness very early, then the universe was in some very special state of low entropy initially, and the question arises as to why this was so. This is of course an old problem, cf. e.g., [1], pp. 525-6, [2], p. 815, [3], p. 506.

Inflationary expansion, as is well known, either through a phase transition or more generally, allows for initially ‘unsynchronized’ regions to become homogeneous very early hence explaining the subtle differences in the measurements of observables of two such regions [4], Sec. 4.1B, [5], p. 54, [6], Sec. 5.1. This explanation of the horizon problem is the result of the causal mechanism of pushing the big bang hypersurface sufficiently earlier, so that the past light cones of the disjoint regions manage to form a non-empty intersection hence allowing them to ‘homogenize’ (cf. e.g., [7], Sec. 9.7.2, [8], Sec. 28.3).

What is the relation between causality and synchronization in the horizon problem? In inflation, two regions need only be able to communicate causally with each other at some specific time in their past history to homogenize and be considered as synced. In addition, it has recently become clear that a single causally connected region may bifurcate to future states of exchanged stability and thereby desynchronize in the course of its evolution, and similarly two evolving causally uncorrelated regions will not be kept synced unless they were initially identical [9].

Here we introduce dynamical synchronization of spatial domains as the temporal analogue of the phase transition used in inflationary models and show that although causality-induced homogenization may be a sufficient condition for sync, it is by no
means a necessary one. We prove that the sync mechanism introduced in Refs. [10, 11] drives two spatial domains to sync gradually in time and thereby proceed in perfect unison, without having to be in causal contact ever before. We shall show that an early chaotic Belinski-Khalatnikov-Lifshitz (‘BKL’) phase makes the two regions to gradually ‘absorb’ each other and then proceed in symphony (for an analogous effect observed in the context of biological oscillators, see [12, 13, 14]).

2 Sync in a special case

The basic situation associated with the horizon problem is described in Fig. 1 where we see a generic snapshot of an ‘observer’ at G overlooking two spatial points B, E in opposite directions, and the two causally disconnected regions $\mathcal{B} = AC, \mathcal{E} = DF$ placed on the spatial hypersurface $\Sigma$. 

Figure 1: According to the horizon problem, rays such as $CBG$ and $DEG$ transport to G the news that the two causally disjoint regions $AC$ and $DF$ are completely synced although never in causal contact.
We shall present below a generalization of the results of [10, 11] that allows to tackle the horizon problem. The present approach is based on the dynamical equations of the inhomogeneous Mixmaster universe [15], [16], where the standard expansion-normalized, dimensionless variables $X_B(t) = (N_1, N_2, N_3, \Sigma_+^B, \Sigma_-^B)$ and $X_E(t) = (M_1, M_2, M_3, \Pi_+^E, \Pi_-^E)$ used for the two spatial Mixmaster ‘points’ $B, E$ respectively, satisfy the Wainwright-Hsu equations [17]. Thus we have, for the region $B$,

\[
N_1' = (q - 4\Sigma_+)N_1, \quad (1)
\]
\[
N_2' = (q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-)N_2, \quad (2)
\]
\[
N_3' = (q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-)N_3, \quad (3)
\]
\[
\Sigma_+ = -(2 - q)\Sigma_+ - 3S_+, \quad (4)
\]
\[
\Sigma_- = -(2 - q)\Sigma_- - 3S_, \quad (5)
\]

with the constraint,

\[
\Sigma_+^2 + \Sigma_-^2 + \frac{3}{2}(N_1^2 + N_2^2 + N_3^2 - 2(N_1N_2 + N_2N_3 + N_3N_1)) = 1, \quad (6)
\]

where,

\[
q = 2(\Sigma_+^2 + \Sigma_-^2), \quad (7)
\]
\[
S_+ = \frac{1}{2}((N_2 - N_3)^2 - N_1(2N_1 - N_2 - N_3)), \quad (8)
\]
\[
S_- = \frac{\sqrt{3}}{2}(N_3 - N_2)(N_1 - N_2 - N_3), \quad (9)
\]

and a prime denotes differentiation with respect to the $\tau$-time. Similarly for the second Mixmaster region $E$ described by the variables $X_E(t)$: the subset $M = (M_1, M_2, M_3)$ satisfy similar equations to (1)-(3), while the shear variables $\Pi = (\Pi_+, \Pi_-)$ satisfy the system,

\[
\Pi_+ = -(2 - p)\Pi_+ - 3Q_+, \quad (10)
\]
\[
\Pi_- = -(2 - p)\Pi_- - 3Q_-, \quad (11)
\]
with \( p = 2(\Pi_+^2 + \Pi_-^2) \), the spatial curvatures \( Q \)'s like the \( S \)'s in (8), (9) but with the \( M \)'s in the corresponding places of the \( N \)'s, and the constraint identical to (6) but with the \((M, \Pi)\)’s in the places of the \((N, \Sigma)\)’s.

To examine the two regions \( B, E \) for sync, we set the \( E \)-system variables \( M \) equal to the \( N \)'s respectively, and introduce the synchronization function \( \omega = (\omega_+, \omega_-) \), with \( \omega_\pm = \Sigma_\pm - \Pi_\pm \). Then complete synchronization of the two Mixmaster oscillating spatial points arises provided,

\[
\omega \to (0, 0), \quad \text{as} \quad \tau \to -\infty, \quad (12)
\]

otherwise the two oscillating spatial points evolve so that syncing them together becomes impossible. The \( E \)-system equations become,

\[
\begin{align*}
M_i &= N_i, \quad i = 1, 2, 3 \\
\Pi_+^\prime &= -(2 - p)\Pi_+ - 3S_+ \quad (14) \\
\Pi_-^\prime &= -(2 - p)\Pi_- - 3S_- \quad (15)
\end{align*}
\]

From the two constraint equations, namely Eq. (6) and the analogous one for the \((M, \Pi)\) variables, it then follows that \( q = p \), and in this case there is a Liapunov function for the system (4), (5), (14), (15), namely,

\[
V(\omega_+, \omega_-) = \frac{1}{2} (\omega_+^2 + \omega_-^2), \quad (16)
\]

which makes the state \( \omega = 0 \) globally asymptotically stable, so that the system syncs exponentially fast [11].

### 3 Early-time synchronization

Below we shall treat the more general case when \( q \neq p \). The \( \omega \)-behaviour can be deduced from the infinitesimal limit, that is the variational equation (cf. e.g., [18, 19, 20]), which for the dynamical equations (4), (5), (14), (15) gives the following linear system,

\[
\dot{\omega} = (D_\Sigma X_B)\omega = 0, \quad (17)
\]
where $D_\Sigma X_B$ is the Jacobian of the vector field appearing on the right-hand-side of $\Sigma$-subsystem (4), (5) with respect to $\Sigma$ only. Explicitly,

$$\begin{pmatrix}
\dot{\omega}_+ \\
\dot{\omega}_-
\end{pmatrix} = \begin{pmatrix}
-2 + 2(\Sigma_+^2 + \Sigma_-^2) + 4\Sigma_+^2 \\
4\Sigma_+ \Sigma_- \\
4\Sigma_+ \Sigma_- \\
-2 + 2(\Sigma_+^2 + \Sigma_-^2) + 4\Sigma_-^2
\end{pmatrix} \begin{pmatrix}
\omega_+ \\
\omega_-
\end{pmatrix}, \quad (18)
$$

and the system (4), (5), (14), (15) will sync (that is $\omega \rightarrow (0,0)$) provided all eigenvalues of the matrix $D_\Sigma X_B$ are negative.

The examination of the equilibria of the system (4), (5) is a well-studied problem in mathematical cosmology [17, 21]. Restricted to the present problem, we shall be interested in the behaviour of solutions of the system (18) near the following two vacuum equilibria of the system (4), (5):

- **EQ-1**: Friedmann-Lemaître point $\mathcal{F}$.
  \[ \Sigma_+ = \Sigma_- = 0, \quad N_1 = N_2 = N_3 = 0. \quad (19) \]

- **EQ-2**: Kasner circle $\mathcal{K}$.
  \[ \Sigma_+^2 + \Sigma_-^2 = 1, \quad N_1 = N_2 = N_3 = 0, \quad \Sigma_+, \Sigma_- : \text{constants.} \quad (20) \]

The equilibrium **EQ-1** at the origin is a stable sink node for (18) because of the presence of the double eigenvalue $-2$. In this case the system *syncs* in the future direction leading to the state described above as the Friedmann-Lemaître point for initial conditions satisfying $\Sigma_+^2 + \Sigma_-^2 < 1$.

In the past direction the situation is different. For **EQ-2**, on any point lying on the Kasner circle, $\Sigma_+^2 + \Sigma_-^2 = 1$, the Jacobian $D_\Sigma X_A$ is given by the matrix $4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. This means that every point on the $\Sigma_+$-axis is an unstable equilibrium and all orbits are

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1Strictly speaking, the vanishing of the velocity vector field to yield these equilibria does not require the vanishing of the $N_i$’s, only the two conditions for **EQ-1** and **EQ-2** on the shear variables respectively. We shall still call the resulting equilibria with the names they acquire when $N_1 = N_2 = N_3 = 0$. 

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repelled from it (on lines parallel to the $\Sigma_-$-axis). Therefore all orbits are unstable and move away from the Kasner circle in the future direction ($\tau \to \infty$).

In other words, a common Kasner circle will represent the \textit{past} attractor of the inhomogeneous system in the direction $\tau \to -\infty$, where the system will sync as $\omega \to (0, 0)$. As the two arbitrary spatial points $A, B$ gradually approach their common Kasner circle, they become trapped in a state that satisfies $|p^A_\alpha - p^B_\alpha| \sim 0$, or, in terms of the two BKL maps, $|u^A - u^B| \sim 0$, where $p^{A,B}_\alpha, u^{A,B}$ are the Kasner exponents and BKL parameters of the two spatial points respectively. This is like having two oscillators with phases $\psi^{A,B}$ and $p^{A,B}_1 = (1 - 2 \cos \psi^{A,B})/3$, $p^{A,B}_{2,3} = (1 + \cos \psi^{A,B} \pm \sqrt{3} \sin \psi^{A,B})/3$, where $\Sigma_+ = \cos \psi, \Sigma_- = \sin \psi$. The synchronization of such a system may also be interpreted as two Kuramoto-like oscillators with time-dependent couplings all moving on a circle representing the phase space of the problem, cf. \cite{11, 10}. Initially, the phases are distributed uniformly and randomly on the circle and the points are moving counterclockwise around the circle as their phases increase. However, they acquire common phases exponentially fast and are grouped together to synchronize to a common phase.

4 Discussion

These results allow for some comments on the entropy of the initial state as well as the horizon problem. The effect on sync on the BKL oscillatory behaviour in the generic inhomogeneous case is to gradually drive the system to past simpler states characterized by the resetting of the values of the BKL parameter $u$ for different spatial points to approach a stable chaotically oscillating homogeneous and synced Kasner circle state at the origin. This describes precisely the notion of `absorption' mentioned earlier.

What is the entropy of the resulting initial state? Each time a spatial point syncs with another one, it loses information and the system finds itself in a state of smaller entropy. As more and more points absorb each other and acquire similar properties through sync, the volume of synced states increases and the effects of sync become more pronounced.
The system approaches an initial asymptotic state with ‘minimal’ entropy, smaller than in all previous states but higher than in any conceivable isotropic state (because of the nature of the Kasner circle).

A universe starting in a minimal entropy initial Kasner circle state and evolving in the future direction contains spatial points each one of which organizes itself exponentially soon to a BKL chaotically oscillating transient period, before arriving at the \((\Sigma_+, \Sigma_-) = (0, 0)\) (Friedmannian) synced state of higher entropy. The latter is a future stable attractor and, baring quantum effects, may have existed very early. As such, it also contains many causally disjoint regions formed during the BKL period of the transiently evolving (homogeneous) Mixmaster points (because of the formation of horizons in the way first described in \[23\]), all of which however being synced with each other and proceeding in unison.

This is an essential difference between the model introduced presently and the usual Friedmannian evolution described in Ref. \[9\] because it implies that an observer sitting at the point \(G\) of Fig. \[1\] sees causally disconnected but synced domains. The observed sync between two causally unrelated domains arises dynamically before the universe entered a Friedmann stage but quite independently of any causal influences. We may say that sync does not violate relativistic causality but transcends it. We have therefore shown that in a generic inhomogeneous universe that includes a chaotic BKL period it becomes possible for two causally disconnected domains to synchronize with each other and proceed in perfect unison for their future evolution.

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