DIVISIBILITY OF CERTAIN $\ell$-REGULAR PARTITIONS BY 2

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Abstract. For a positive integer $\ell$, let $b_\ell(n)$ denote the number of $\ell$-regular partitions of a nonnegative integer $n$. Motivated by some recent conjectures of Keith and Zanello, we establish infinite families of congruences modulo 2 for $b_3(n)$ and $b_{21}(n)$. We prove a specific case of a conjecture of Keith and Zanello on self-similarities of $b_3(n)$ modulo 2. We next prove that the series $\sum_{n=0}^{\infty} b_3(2n+1)q^n$ is lacunary modulo arbitrary powers of 2. We also prove that the series $\sum_{n=0}^{\infty} b_3(4n)q^n$ is lacunary modulo 2.

1. Introduction and statement of results

A partition of a positive integer $n$ is any non-increasing sequence of positive integers whose sum is $n$. The number of such partitions of $n$ is denoted by $p(n)$. The partition function has many congruence properties modulo primes and powers of primes. Ramanujan discovered beautiful congruences satisfied by $p(n)$ modulo 5, 7 and 11. Let $\ell$ be a fixed positive integer. An $\ell$-regular partition of a positive integer $n$ is a partition of $n$ such that none of its parts is divisible by $\ell$. Let $b_\ell(n)$ be the number of $\ell$-regular partitions of $n$. The generating function for $b_\ell(n)$ is given by

$$G_\ell(q) := \sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{f_\ell}{f_1},$$

where $f_k := (q^k; q^k)_\infty = \prod_{j=1}^{\infty} (1 - q^{jk})$ and $k$ is a positive integer.

In a very recent paper [3], Keith and Zanello studied $\ell$-regular partition for certain values of $\ell$. They proved various congruences for $b_3(n)$ and make the following conjecture regarding the self-similarity of $b_3(n)$.

**Conjecture 1.1.** For any prime $p > 3$, let $\gamma \equiv -24^{-1} \pmod{p^2}$, $0 < \gamma < p^2$. It holds for a positive proportion of primes $p$ that

$$\sum_{n=0}^{\infty} b_3(2(2n+\gamma))q^n \equiv \sum_{n=0}^{\infty} b_3(2n)q^{pn} \pmod{2}.$$  

In [3, Theorem 7], Keith and Zanello proved one specific case of Conjecture 1.1 corresponding to $p = 13$. In the following theorem, we prove another specific case of Conjecture 1.1 corresponding to $p = 17$.

**Theorem 1.2.** It holds that

$$\sum_{n=0}^{\infty} b_3(34n+24)q^n \equiv \sum_{n=0}^{\infty} b_3(2n)q^{17n} \pmod{2},$$

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and therefore
\[ b_3(2 \cdot 17^2 n + 58) \equiv 0 \pmod{2}, \]
and by iteration,
\[ b_k \left( 2 \cdot 17^{2k} n + 17^{2k-2} \cdot 58 + 24 \cdot \left( \frac{17^{2k-2} - 1}{288} \right) \right) \equiv 0 \pmod{2} \]
for all \( k \geq 1 \).

If we assume that Conjecture 1.1 is true for \( p = 29 \), then using the congruence
\[ \sum_{n=0}^{\infty} b_3(2(29n + 35)) q^n \equiv \sum_{n=0}^{\infty} b_3(2n) q^{29n} \pmod{2}, \]
one can deduce infinite families of congruences of the form
\[ b_3(2(29^{2k} n + 17^{2k-2} \cdot 58 + 24 \cdot \left( 17^{2k-2} - 1 \right) / 288)) \equiv 0 \pmod{2} \]
where \( 1 \leq k \leq 28 \). We do not know whether Conjecture 1.1 is true or not for \( p = 29 \). However, in the following theorem, we prove the congruences (1.3) without assuming (1.2) for \( p = 29 \).

**Theorem 1.3.** Let \( \alpha \in \{6, 64, 93, 122, 151, 180, 209, 238, 267, 296, 325, 354, 383, 412, 441, 470, 499, 528, 557, 586, 615, 644, 673, 702, 731, 760, 789, 818\} \). Then for all \( n \geq 0 \), we have
\[ b_3(2(29^{2n} n + \alpha)) \equiv 0 \pmod{2}. \]

Keith and Zanello \[3\] also studied 2-divisibility of \( b_21(n) \) and proved several congruences for primes \( p \equiv 13, 17, 19, 23 \pmod{24} \). To be specific, if \( p \equiv 13, 17, 19, 23 \pmod{24} \) is prime, then
\[ b_{21}(4(p^2 n + kp - 11 \cdot 24^{-1}) + 1) \equiv 0 \pmod{2} \]
for all \( 1 \leq k < p \), where \( 24^{-1} \) is taken modulo \( p^2 \). For example, if \( p = 13 \), then one obtains
\[ b_{21}(4 \cdot 13^2 n + 52k + 309) \equiv 0 \pmod{2} \]
for all \( k = 1, 2, \ldots, 12 \). In the following theorem we prove similar type of congruences for the prime \( p = 29 \).

**Theorem 1.4.** Let \( \beta \in \{8, 37, 66, 95, 124, 153, 182, 211, 240, 269, 298, 327, 356, 414, 443, 472, 501, 530, 559, 588, 617, 646, 675, 704, 733, 762, 791, 820\} \). Then for all \( n \geq 0 \), we have
\[ b_{21}(4(29^{2n} n + \beta) + 1) \equiv 0 \pmod{2}. \]

In addition to the study of Ramanujan-type congruences, it is an interesting problem to study the distribution of the partition function modulo positive integers \( M \). To be precise, given an integral power series \( F(q) := \sum_{n=0}^{\infty} a(n) q^n \) and \( 0 \leq r < M \), we define
\[ \delta_r(F, M; X) := \frac{\# \{ n \leq X : a(n) \equiv r \pmod{M} \}}{X}. \]
An integral power series \( F \) is called *lacunary modulo* \( M \) if
\[ \lim_{X \to \infty} \delta_0(F, M; X) = 1, \]
that is, almost all of the coefficients of \( F \) are divisible by \( M \). In a recent paper \[1\], Cotron et al. proved lacunarity of certain eta-quotients modulo arbitrary powers of primes. We phrase their theorem as follows:
Theorem 1.5. \[ \text{[1, Theorem 1.1]} \] Let \( G(z) = \prod_{i=1}^{t} \frac{f_{i \alpha_i}}{f_{i \beta_i}}, \) and \( p \) is a prime such that \( p^a \) divides \( \gcd(\alpha_1, \alpha_2, \ldots, \alpha_u) \) and

\[
p^a \geq \sqrt{\frac{\sum_{i=1}^{t} \beta_i s_i}{\sum_{i=1}^{u} \alpha_i}},
\]

then \( G(z) \) is lacunary modulo \( p^j \) for any positive integer \( j \).

Keith and Zanello \[3\] studied lacunarity of the functions \( b_3(2n), b_{21}(4n), b_{21}(4n+1) \) and \( b_{21}(8n+3) \) modulo 2 using the technique developed by Landau \[5\]. In the following theorem, we prove that \( b_9(2n+1) \) is almost always divisible by arbitrary powers of 2.

Theorem 1.6. The series \( \sum_{n=0}^{\infty} b_9(2n+1)q^n \) is lacunary modulo \( 2^k \) for any positive integer \( k \).

Keith and Zanello \[3\] derived several congruences for the partition function \( b_9(n) \) modulo 2 using the theory of Hecke operators. In the following theorem we prove that \( b_9(4n) \) is almost always divisible by 2.

Theorem 1.7. The series \( \sum_{n=0}^{\infty} b_9(4n)q^n \) is lacunary modulo 2.

We prove Theorem 1.7 using the approach of Landau \[5\] and Theorem 1.5. However, we couldn’t find a similar proof for Theorem 1.6. We use a density result of Serre to prove Theorem 1.6.

2. Preliminaries

We recall some definitions and basic facts on modular forms. For more details, see for example \[4, 6\]. We first define the matrix groups

\[
\text{SL}_2(\mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},
\]

\[
\Gamma_\infty := \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\},
\]

\[
\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\},
\]

\[
\Gamma_1(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 \pmod{N} \right\},
\]

and

\[
\Gamma(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N}, \text{ and } b \equiv c \equiv 0 \pmod{N} \right\},
\]

where \( N \) is a positive integer. A subgroup \( \Gamma \) of \( \text{SL}_2(\mathbb{Z}) \) is called a congruence subgroup if \( \Gamma(N) \subseteq \Gamma \) for some \( N \). The smallest \( N \) such that \( \Gamma(N) \subseteq \Gamma \) is called the level of \( \Gamma \). For example, \( \Gamma_0(N) \) and \( \Gamma_1(N) \) are congruence subgroups of level \( N \).

Let \( \mathbb{H} := \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \) be the upper half of the complex plane. The group

\[
\text{GL}_2^+(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0 \right\}
\]
acts on $\mathbb{H}$ by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az + b}{cz + d}$. We identify $\infty$ with $\frac{1}{0}$ and define $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \frac{ar + bs}{cr + ds}$ where $\frac{r}{s} \in \mathbb{Q} \cup \{\infty\}$. This gives an action of $\text{GL}_2^+(\mathbb{R})$ on the extended upper half-plane $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. Suppose that $\Gamma$ is a congruence subgroup of $\text{SL}_2(\mathbb{Z})$. A cusp of $\Gamma$ is an equivalence class in $\mathbb{P}^1 = \mathbb{Q} \cup \{\infty\}$ under the action of $\Gamma$.

The group $\text{GL}_2^+(\mathbb{R})$ also acts on functions $f : \mathbb{H} \to \mathbb{C}$. In particular, suppose that $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}_2^+(\mathbb{R})$. If $f(z)$ is a meromorphic function on $\mathbb{H}$ and $\ell$ is an integer, then define the slash operator $|_{\ell}$ by $(f|_{\ell}\gamma)(z) := (\det \gamma)^{\ell/2}(cz + d)^{-\ell}f(\gamma z)$.

**Definition 2.1.** Let $\Gamma$ be a congruence subgroup of level $N$. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form with integer weight $\ell$ on $\Gamma$ if the following hold:

1. We have $f \left( \frac{az + b}{cz + d} \right) = (cz + d)^\ell f(z)$ for all $z \in \mathbb{H}$ and all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$.
2. If $\gamma \in \text{SL}_2(\mathbb{Z})$, then $(f|_{\ell}\gamma)(z)$ has a Fourier expansion of the form
   $$(f|_{\ell}\gamma)(z) = \sum_{n \geq 0} a_{n,\gamma}(q_N) q^n,$$
   where $q_N := e^{2\pi i z/N}$.

For a positive integer $\ell$, the complex vector space of modular forms of weight $\ell$ with respect to a congruence subgroup $\Gamma$ is denoted by $M_\ell(\Gamma)$.

**Definition 2.2.** [6, Definition 1.15] If $\chi$ is a Dirichlet character modulo $N$, then we say that a modular form $f \in M_\ell(\Gamma_1(N))$ has Nebentypus character $\chi$ if

$$f \left( \frac{az + b}{cz + d} \right) = \chi(d)(cz + d)^\ell f(z)$$

for all $z \in \mathbb{H}$ and all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. The space of such modular forms is denoted by $M_\ell(\Gamma_0(N), \chi)$.

In this paper, the relevant modular forms are those that arise from eta-quotients. Recall that the Dedekind eta-function $\eta(z)$ is defined by

$$\eta(z) := q^{1/24}(q; q)_\infty = q^{1/24} \prod_{n=1}^\infty (1 - q^n),$$

where $q := e^{2\pi i z}$ and $z \in \mathbb{H}$. A function $f(z)$ is called an eta-quotient if it is of the form

$$f(z) = \prod_{\delta \mid N} \eta(\delta z)^{r_\delta},$$

where $N$ is a positive integer and $r_\delta$ is an integer. We now recall two theorems from [6, p. 18] which will be used to prove our results.
Theorem 2.3. \[6\] Theorem 1.64] If \( f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta} \) is an eta-quotient such that \( \ell = \frac{1}{2} \sum_{\delta|N} r_\delta \in \mathbb{Z} \),

\[
\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}
\]

and

\[
\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},
\]

then \( f(z) \) satisfies

\[
f \left( \frac{az+b}{cz+d} \right) = \chi(d)(cz+d)^\ell f(z)
\]

for every \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N) \). Here the character \( \chi \) is defined by \( \chi(d) := \left( \frac{-1}{d} \right) \),

where \( s := \prod_{\delta|N} \delta^{r_\delta} \).

Suppose that \( f \) is an eta-quotient satisfying the conditions of Theorem 2.3 and that the associated weight \( \ell \) is a positive integer. If \( f(z) \) is holomorphic at all of the cusps of \( \Gamma_0(N) \), then \( f(z) \in M_\ell(\Gamma_0(N), \chi) \). The following theorem gives the necessary criterion for determining orders of an eta-quotient at cusps.

Theorem 2.4. \[6\] Theorem 1.65] Let \( c, d \) and \( N \) be positive integers with \( d \mid N \) and \( \gcd(c, d) = 1 \). If \( f \) is an eta-quotient satisfying the conditions of Theorem 2.3 for \( N \), then the order of vanishing of \( f(z) \) at the cusp \( \frac{c}{d} \) is

\[
\frac{N}{24} \sum_{\delta|N} \frac{\gcd(d, \delta)^2 r_\delta}{\gcd(d, \frac{c}{d})} d\delta
\]

We now recall a result of Sturm \[10\] which gives a criterion to test whether two modular forms are congruent modulo a given prime.

Theorem 2.5. Let \( p \) be a prime number, and \( f(z) = \sum_{n=n_0}^{\infty} a(n)q^n \) and \( g(z) = \sum_{n=n_1}^{\infty} b(n)q^n \) be modular forms of weight \( k \) for \( \Gamma_0(N) \) of characters \( \chi \) and \( \psi \), respectively, where \( n_0, n_1 \geq 0 \). If either \( \chi = \psi \) and

\[
a(n) \equiv b(n) \pmod{p} \quad \text{for all } n \leq \frac{kN}{12} \prod_{d \text{ prime; } d|N} \left( 1 + \frac{1}{d} \right),
\]

or \( \chi \neq \psi \) and

\[
a(n) \equiv b(n) \pmod{p} \quad \text{for all } n \leq \frac{kN^2}{12} \prod_{d \text{ prime; } d|N} \left( 1 - \frac{1}{d^2} \right),
\]

then \( f(z) \equiv g(z) \pmod{p} \) (i.e., \( a(n) \equiv b(n) \pmod{p} \) for all \( n \)).

We next recall the definition of Hecke operators. Let \( m \) be a positive integer and \( f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_\ell(\Gamma_0(N), \chi) \). Then the action of Hecke operator \( T_m \) on \( f(z) \) is defined by

\[
f(z)|T_m := \sum_{n=0}^{\infty} \left( \sum_{\delta \mid \gcd(n, m)} \chi(d) d^{\ell-1} a \left( \frac{nm}{d^2} \right) \right) q^n.
\]
In particular, if \( m = p \) is prime, we have
\[
f(z)|T_p := \sum_{n=0}^{\infty} \left( a(pm) + \chi(p)p^{f-1}a\left(\frac{n}{p}\right) \right) q^n.
\]

We take by convention that \( a(n/p) = 0 \) whenever \( p \nmid n \). If \( f \) is an \( \eta \)-quotient with the properties listed in Theorem 2.3 and \( p \mid s \) (here \( s \) is as defined in Theorem 2.3), then \( \chi(p) = 0 \) so that the latter term vanishes. In this case, we have the factorization property that
\[
\left( f \cdot \sum_{n=0}^{\infty} g(n)q^n \right)|T_p = \left( \sum_{n=0}^{\infty} a(pm)q^n \right) \left( \sum_{n=0}^{\infty} g(n)q^n \right).
\]

We finally recall a density result of Serre [9] about the divisibility of Fourier coefficients of modular forms.

**Theorem 2.6 (Serre).** Let \( f(z) \) be a modular form of positive integer weight \( k \) on some congruence subgroup of \( SL_2(\mathbb{Z}) \) with Fourier expansion
\[
f(z) = \sum_{n=0}^{\infty} a(n)q^n,
\]
where \( a(n) \) are algebraic integers in some number field. If \( m \) is a positive integer, then there exists a constant \( c > 0 \) such that there are
\[
O\left( \frac{X}{(\log X)^c} \right) \text{ integers } n \leq X \text{ such that } a(n) \text{ is not divisible by } m.
\]

### 3. Proof of Theorems 1.2

**Proof.** We first recall the following even-odd dissection of the 3-regular partitions [3, (6)]:
\[
\sum_{n=0}^{\infty} b_3(n)q^n = \frac{f_3^8}{f_1^3} = f_3^8 + q f_3^{10} (\text{mod } 2). \tag{3.1}
\]

Thus, extracting the terms with even powers of \( q \), we obtain
\[
\sum_{n=0}^{\infty} b_3(2n)q^n = \frac{f_3^4}{f_3} (\text{mod } 2). \tag{3.2}
\]

Let
\[
G_{3,1}(z) := \frac{\eta^4(z)\eta^2(51z)\eta(17z)}{\eta(3z)}
\]
and
\[
G_{3,2}(z) := \frac{\eta^4(17z)\eta^2(3z)\eta(z)}{\eta(51z)}.
\]

By Theorems 2.3 and 2.4 we find that \( G_{3,1}(z) \) and \( G_{3,2}(z) \) are modular forms of weight 3, level 51 and character \( \chi_0 = (-3:17^2) \). By (3.2) the Fourier expansions of our forms satisfy
\[
G_{3,1}(z) = \left( \sum_{n=0}^{\infty} b_3(2n)q^{n+5} \right) f_{51}^2 f_{17}.
\]
and

\[ G_{3,2}(z) = \left( \sum_{n=0}^{\infty} b_3(2n)q^{17n+1} \right) f_3^2 f_1. \]

We then calculate that

\[ G_{3,1}(z)T_{17} \equiv \left( \sum_{n=0}^{\infty} b_3(34n + 24)q^{n+1} \right) f_3^2 f_1 \pmod{2}. \]

Since the Hecke operator is an endomorphism on \( M_3(\Gamma_0(51), \chi_0) \), we have that \( G_{3,1}(z)T_{17} \in M_3(\Gamma_0(51), \chi_0) \). By Theorem 2.5 the Sturm bound for this space of forms is 18. We wish to verify the congruence

\[ q \left( \sum_{n=0}^{\infty} b_3(34n + 24)q^n \right) f_3^2 f_1 \equiv q f_{17} f_3^2 f_1 \pmod{2}. \]

The coefficient of \( q^{18} \) on the left side involves the value \( b_3(636) \); thus, \( f_3/f_1 \) must be expanded at least that far, and the product on the right side must be constructed up to the \( q^{18} \) terms. Finally, expansion with a calculation package such as Sage confirms that all coefficients up to the desired bound are congruent modulo 2, and the first part of the theorem is established.

Since only powers for which \( 17 \mid n \) can be nonzero on the right side of the statement, we obtain:

\[ b_3(34(17n+1) + 24) = b_3(2 \cdot 17^2 n + 58) \equiv 0 \pmod{2}. \]

Finally, recursively applying the relation

\[ b_3(2n) \equiv b_3(34 \cdot 17n + 24) \pmod{2}, \]

we obtain

\[ b_3(2 \cdot 17^2 n + 58) \equiv b_3(2 \cdot 17^2(17^2 n + 29) + 24) \pmod{2} \]
\[ = b_3(2 \cdot 17^4 n + 17^2 \cdot 58 + 24) \]
\[ \equiv b_3(2 \cdot 17^6 n + 17^4 \cdot 58 + 17^2 \cdot 24 + 24) \pmod{2} \]
\[ \equiv \ldots \]
\[ \equiv b_3 \left( 2 \cdot 17^{2k} n + 17^{2k-2} \cdot 58 + 24 \left( \frac{17^{2k-2} - 1}{288} \right) \right) \equiv 0 \pmod{2}, \]

where the last line is given by a finite geometric summation. This completes the proof of the theorem. \( \square \)

4. Proof of Theorems 1.3 and 1.4

We prove Theorems 1.3 and 1.4 using the approach developed in [7, 8]. Throughout this section, \( \Gamma \) denotes the full modular group \( \text{SL}_2(\mathbb{Z}) \). We recall that the index of \( \Gamma_0(N) \) in \( \Gamma \) is

\[ [\Gamma : \Gamma_0(N)] = N \prod_{p \mid N} (1 + p^{-1}), \]

where \( p \) denotes a prime.
For a positive integer $M$, let $R(M)$ be the set of integer sequences $r = (r_\delta)_{\delta|M}$ indexed by the positive divisors of $M$. If $r \in R(M)$ and $1 = \delta_1 < \delta_2 < \cdots < \delta_k = M$ are the positive divisors of $M$, we write $r = (r_\delta_1, \ldots, r_\delta_k)$. Define $c_r(n)$ by
\[
\sum_{n=0}^{\infty} c_r(n)q^n := \prod_{\delta|M} (q^{\delta}; q^{\delta})_{\infty}^{r_\delta} = \prod_{\delta|M} \prod_{n=1}^{\infty} (1 - q^{n\delta})^{r_\delta}.
\] (4.1)
The approach to proving congruences for $c_r(n)$ developed by Radu [7, 8] reduces the number of coefficients that one must check as compared with the classical method which uses Sturm’s bound alone.

Let $m$ be a positive integer. For any integer $s$, let $[s]_m$ denote the residue class of $s$ in $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$. Let $\mathbb{Z}_m^*$ be the set of all invertible elements in $\mathbb{Z}_m$. Let $S_m \subseteq \mathbb{Z}_m$ be the set of all squares in $\mathbb{Z}_m$. For $t \in \{0, 1, \ldots, m - 1\}$ and $r \in R(M)$, we define a subset $P_{m,r}(t) \subseteq \{0, 1, \ldots, m - 1\}$ by
\[
P_{m,r}(t) := \left\{ t' : \exists \delta[s]_{24m} \in S_{24m} \text{ such that } t' \equiv ts + \frac{s - 1}{24} \sum_{\delta|M} \delta r_\delta \pmod{m} \right\}.
\]

**Definition 4.1.** Suppose $m, M$ and $N$ are positive integers, $r = (r_\delta) \in R(M)$ and $t \in \{0, 1, \ldots, m - 1\}$. Let $k = k(m) := \gcd(m^2 - 1, 24)$ and write
\[
\prod_{\delta|M} \delta^{v_\delta} = 2^s \cdot j,
\] where $s$ and $j$ are nonnegative integers with $j$ odd. The set $\Delta^*$ consists of all tuples $(m, M, N, (r_\delta), t)$ satisfying these conditions and all of the following.

1. Each prime divisor of $m$ is also a divisor of $N$.
2. $\delta|M$ implies $\delta|mN$ for every $\delta \geq 1$ such that $r_\delta \neq 0$.
3. $kN \sum_{\delta|M} r_\delta mN/\delta \equiv 0 \pmod{24}$.
4. $kN \sum_{\delta|M} r_\delta \equiv 0 \pmod{8}$.
5. $\gcd(-24r, 24m - 2\sum_{\delta|M} \delta r_\delta) \equiv 0 \pmod{24}$.
6. If $2|m$, then either $4|kN$ and $8|sN$ or $2|s$ and $8|(1 - j)N$.

Let $m, M, N$ be positive integers. For $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma, r \in R(M)$ and $r' \in R(N)$, set
\[
p_{m,r}(\gamma) := \min_{\lambda \in \{0, 1, \ldots, m - 1\}} \frac{1}{24} \sum_{\delta|M} r_\delta \frac{\gcd^2(\delta a + \delta k c, m c)}{\delta m}
\] and
\[
p_{r'}(\gamma) := \frac{1}{24} \sum_{\delta|N} r'_\delta \frac{\gcd^2(\delta, c)}{\delta}.
\]

**Lemma 4.2.** [7 Lemma 4.5] Let $u$ be a positive integer, $(m, M, N, r = (r_\delta), t) \in \Delta^*$ and $r' = (r'_\delta) \in R(N)$. Let $\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \subseteq \Gamma$ be a complete set of representatives of the double cosets of $\Gamma_0(N)\backslash \Gamma/\Gamma_{\infty}$. Assume that $p_{m,r}(\gamma_i) + p_{r'}(\gamma_i) \geq 0$ for all $1 \leq i \leq n$. Let $t_{\min} = \min_{\nu \in \{p_{m,r}(t)\}} \nu$ and
\[
\nu := \frac{1}{24} \left\{ \left( \sum_{\delta|M} r_\delta + \sum_{\delta|N} r'_\delta \right) \left[ \Gamma : \Gamma_0(N) \right] - \sum_{\delta|N} \delta r'_\delta \right\} - \frac{1}{24m} \sum_{\delta|M} \delta r_\delta - t_{\min} \frac{m}{m}.
\]
If the congruence \( c_r(mn + t') \equiv 0 \pmod{u} \) holds for all \( t' \in P_{m,r}(t) \) and \( 0 \leq n \leq \lfloor \nu \rfloor \), then it holds for all \( t' \in P_{m,r}(t) \) and \( n \geq 0 \).

To apply Lemma 4.2 we utilize the following result, which gives us a complete set of representatives of the double coset in \( \Gamma_0(N)\backslash \Gamma/\Gamma_\infty \).

**Lemma 4.3.** \([11, \text{Lemma 4.3}]\) If \( N \) or \( \frac{1}{2}N \) is a square-free integer, then

\[
\bigcup_{\delta \mid N} \Gamma_0(N) \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix} \Gamma_\infty = \Gamma.
\]

**Proof of Theorem 1.3.** From \([6, \text{Lemma 4.3}]\), we have

\[
\sum_{n=0}^{\infty} b_3(n)q^n \equiv \frac{f_3^4}{f_3} + q^{\frac{f_3^{10}}{f_3}} \pmod{2}.
\]

Extracting the terms with even powers of \( q \), we obtain

\[
\sum_{n=0}^{\infty} b_3(2n)q^n \equiv \frac{f_3^4}{f_3} \pmod{2}.
\]

Let \( (m,M,N,r,t) = (841,3,87,4,-1,64) \). It is easy to verify that \( (m,M,N,r,t) \in \Delta^* \) and \( P_{m,r}(t) = \{6,64,151,180,209,238,296,412,499,615,673,702,731,760\} \). By Lemma 4.3, we know that \( \{\delta \in \mathbb{Z} : \delta|87\} \) forms a complete set of double coset representatives of \( \Gamma_0(N)\backslash \Gamma/\Gamma_\infty \). Let \( \gamma_\delta = \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix} \). Let \( r' = (0,0,0,0) \in R(87) \) and we use Sage to verify that \( p_{m,r}(\gamma_\delta) + p_{m,r}(\gamma_\delta^*) \geq 0 \) for each \( \delta|N \). We compute that the upper bound in Lemma 1.2 is \( \lfloor \nu \rfloor = 14 \). Using Sage we verify that \( b_3(1682n + 2t') \equiv 0 \pmod{2} \) for all \( t' \in P_{m,r}(t) \) and for \( n \leq 14 \). By Lemma 4.2 we conclude that \( b_3(1682n + 2t') \equiv 0 \pmod{2} \) for all \( t' \in P_{m,r}(t) \) and for all \( n \geq 0 \). To prove the remaining congruences, we take \( (m,M,N,r,t) = (841,3,87,4,-1,93) \). It is easy to verify that \( (m,M,N,r,t) \in \Delta^* \) and \( P_{m,r}(t) = \{93,122,267,325,354,383,441,470,528,557,586,644,789,818\} \). Following similar steps as shown before, we find that \( b_3(1682n + 2t') \equiv 0 \pmod{2} \) for all \( t' \in P_{m,r}(t) \) and for all \( n \geq 0 \). This completes the proof of the theorem. \( \square \)

**Proof of Theorem 1.4.** We begin our proof by recalling the following even-odd distribution formula of the 21-regular partitions \([8, \text{(9)}]\):

\[
\sum_{n=0}^{\infty} b_{21}(n)q^n = \frac{f_{21}}{f_1} \equiv f_8^3 f_3^8 f_{21}^4 + q f_8^3 f_{21}^4 + q^6 f_8^3 f_{21}^4 + q^{\frac{f_3^{16}}{f_1}} + q^4 f_8^3 f_{21}^4 + q^7 f_8^3 f_{21}^4 \pmod{2}.
\]

Extracting the terms with odd powers of \( q \), we obtain

\[
\sum_{n=0}^{\infty} b_{21}(2n + 1)q^n \equiv q f_1^4 f_{21}^2 + q^3 f_1^4 f_{21}^2 + q^7 f_1^4 f_{21}^2 \pmod{2}.
\]

Finally, extracting the terms with even powers of \( q \), we obtain

\[
\sum_{n=0}^{\infty} b_{21}(4n + 1)q^n \equiv \frac{f_1^4}{f_1} \pmod{2}.
\]
Let \((m, M, N, r, t) = (841, 3, 87, (-1, 4), 414)\). We verify that \((m, M, N, r, t) \in \Delta^* \) and \(P_{m,r}(t) = \{8, 124, 182, 211, 240, 269, 356, 414, 501, 530, 559, 588, 646, 762\}\). By Lemma \ref{lem:4.2} we conclude that the upper bound in Lemma \ref{lem:4.2} is computed that the upper bound in Lemma \ref{lem:4.2} is \(S_{\delta} = \left\lfloor \left[\frac{1}{\delta} \right] : \delta | 87 \right\rfloor\) forms a complete set of double coset representatives of \(\Gamma_0(N) \backslash \Gamma / \Gamma_\infty\). Let \(\gamma_\delta = \left[\frac{1}{\delta} 0 \right] \). Let \(r' = (0, 0, 0, 0) \in R(87)\) and we use \textit{Sage} to verify that \(p_{m,r}(\gamma_\delta) + p_{r'}(\gamma_\delta) \geq 0\) for each \(\delta | N\). We compute that the upper bound in Lemma \ref{lem:4.2} is \(\left\lfloor \frac{1}{\nu} \right\rfloor = 14\). Using \textit{Sage} we verify that \(b_{21}(4(841n + t') + 1) \equiv 0 \pmod{2}\) for all \(t' \in P_{m,r}(t)\) and for \(n \leq 14\). By Lemma \ref{lem:4.2} we conclude that \(b_{21}(4(841n + t') + 1) \equiv 0 \pmod{2}\) for all \(t' \in P_{m,r}(t)\) and for all \(n \geq 0\). To prove the remaining congruences, we take \((m, M, N, r, t) = (841, 3, 87, (-1, 4), 443)\). It is easy to verify that \((m, M, N, r, t) \in \Delta^* \) and \(P_{m,r}(t) = \{37, 66, 95, 153, 298, 327, 443, 472, 617, 675, 704, 733, 791, 820\}\). Following similar steps as shown before, we find that \(b_{21}(4(841n + t') + 1) \equiv 0 \pmod{2}\) for all \(t' \in P_{m,r}(t)\) and for all \(n \geq 0\). This completes the proof of the theorem.

5. Proof of Theorems \ref{thm:1.6} and \ref{thm:1.7}

In order to prove Theorems \ref{thm:1.6} and \ref{thm:1.7} we first prove the following lemma.

**Lemma 5.1.** We have

\[
\sum_{n=0}^{\infty} b_9(2n + 1)q^n = \frac{f_2^2 f_3 f_{18}}{f_1^3 f_6}; \tag{5.1}
\]

\[
\sum_{n=0}^{\infty} b_9(4n)q^n \equiv \frac{f_3}{f_1 f_\delta} \pmod{2}. \tag{5.2}
\]

**Proof.** Letting \(\ell = 9\) in \ref{eq:1.1} we have

\[
\sum_{n=0}^{\infty} b_9(n)q^n = \frac{f_9}{f_1}. \tag{5.3}
\]

From Lemma 3.5 in \ref{lem:3.5} we have

\[
\frac{f_9}{f_1} = \frac{f_1^3 f_{18}}{f_2^2 f_{12} f_{36}} + q \frac{f_2^2 f_6 f_{36}}{f_2^2 f_{12}}. \tag{5.4}
\]

Extracting the terms with odd powers of \(q\) and then using \ref{eq:5.1}, we obtain

\[
\sum_{n=0}^{\infty} b_9(2n + 1)q^n = \frac{f_2^2 f_3 f_{18}}{f_1^3 f_6}. \tag{5.5}
\]

From \ref{eq:5.1}, extracting the terms with even powers of \(q\) and then using \ref{eq:5.3}, we obtain

\[
\sum_{n=0}^{\infty} b_9(2n)q^n = \frac{f_3^5 f_9}{f_1^3 f_3 f_{18}} \equiv \frac{f_3^5}{f_1^3 f_9} \pmod{2}. \tag{5.6}
\]

From \ref{lem:2.5} \ref{eq:2.5}, we have

\[
\frac{f_3^5}{f_3} = \frac{f_3^3}{f_{12}} - 3q \frac{f_2^2 f_{12}}{f_4 f_6}. \tag{5.6}
\]
Magnifying equation (6.6) by \( q \rightarrow q^3 \) and combining with (6.5), we obtain
\[
\sum_{n=0}^{\infty} b_9(2n)q^n = \frac{f_3}{f_1} \left( \frac{f_{12}^1}{f_{36}^1} + q^3 \frac{f_{54}^1}{f_{18}^1} \right) \pmod{2}.
\]
Extracting the terms with even powers of \( q \), we obtain
\[
\sum_{n=0}^{\infty} b_9(4n)q^n = \frac{f_7}{f_1 f_5^1} \pmod{2}.
\]
This completes the proof of the lemma.

\[\square\]

**Proof of Theorem 1.6.** Let
\[
A(z) := \prod_{n=1}^{\infty} \left( \frac{1 - q^{54n}}{1 - q^{108n}} \right) = \frac{\eta^2(54z)}{\eta(108z)^2}.
\]
Then using the binomial theorem we have
\[
A^{2^k}(z) = \frac{\eta^{2^{k+1}}(54z)}{\eta^{2^k}(108z)} \equiv 1 \pmod{2^{k+1}}.
\]
Define \( B_k(z) \) by
\[
B_k(z) := \left( \frac{\eta^2(6z)\eta(9z)\eta(54z)}{\eta^3(3z)\eta(18z)} \right) A^{2^k}(z) = \frac{\eta^2(6z)\eta(9z)\eta^{2^{k+1}+1}(54z)}{\eta^3(3z)\eta(18z)\eta^{2^k}(108z)}.
\]
Modulo \( 2^{k+1} \), we have
\[
B_k(z) \equiv \frac{\eta^2(6z)\eta(9z)\eta(54z)}{\eta^3(3z)\eta(18z)} = q^2 \left( \frac{(q^6; q^6)_\infty^2 (q^9; q^9)_\infty (q^{54}; q^{54})_\infty}{(q^3; q^3)_\infty^3 (q^{18}; q^{18})_\infty} \right) \pmod{2^{k+1}}.
\]
Combining (5.1) and (5.7), we obtain
\[
B_k(z) \equiv \sum_{n=0}^{\infty} b_9(2n + 1)q^{3n+2} \pmod{2^{k+1}}.
\]
Now, \( B_k(z) \) is an eta-quotient with \( N = 324 \). We next prove that \( B_k(z) \) is a modular form for all \( k \geq 6 \). We know that the cusps of \( \Gamma_0(324) \) are represented by fractions \( \frac{c}{d} \), where \( d \mid 324 \) and \( \gcd(c, d) = 1 \). By Theorem 2.3, we find that \( B_k(z) \) is holomorphic at a cusp \( \frac{c}{d} \) if and only if
\[
(2^{k+1} + 1) \frac{\gcd(d, 54)^2}{54} + 2 \frac{\gcd(d, 6)^2}{6} + 3 \frac{\gcd(d, 3)^2}{3} - \frac{\gcd(d, 18)^2}{18} \geq 0.
\]
Equivalently, if and only if
\[
L := (2^{k+2} + 2)G_1 + 36G_2 + 12G_3 - 108G_4 - 6G_5 - 2^k \geq 0,
\]
where \( G_1 = \frac{\gcd(d, 54)^2}{\gcd(d, 108)^2}, G_2 = \frac{\gcd(d, 6)^2}{\gcd(d, 108)^2}, G_3 = \frac{\gcd(d, 9)^2}{\gcd(d, 108)^2}, G_4 = \frac{\gcd(d, 3)^2}{\gcd(d, 108)^2}, \) and \( G_5 = \frac{\gcd(d, 18)^2}{\gcd(d, 108)^2} \) respectively.

We now consider the following four cases according to the divisors of 324 and find the values of \( G_i \) for \( i = 1, 2, \ldots, 5 \). Let \( d \) be a divisor of \( N = 324 \).
Lemma 5.2. Let $r(n)$ and $s(n)$ be quadratic polynomials. Then
\[
\left( \sum_{n \in \mathbb{Z}} q^{r(n)} \right) \left( \sum_{n \in \mathbb{Z}} q^{s(n)} \right)
\]
is lacunary modulo 2.

Proof of Theorem 1.7. We first recall the following identity [3, (7)]:
\[
f_3^3 \equiv f_3 + qf_9^3 \pmod{2}.
\]

We rewrite the above identity as
\[
\frac{f_3}{f_1} \equiv f_1^2 + q^3 \frac{f_9}{f_1} \pmod{2}. \tag{5.9}
\]

Combining (5.2) and (5.9), we obtain
\[
\sum_{n=0}^{\infty} b_9(4n)q^n \equiv \frac{f_3^6}{f_9^2} f_1^2 + q^6 \frac{f_9^6}{f_3} f_1^2 \pmod{2}. \tag{5.10}
\]

We note that the second term of (5.10) is lacunary modulo 2, by Theorem 1.6. For the first term of (5.10), we again recall the following identity [3, p. 12]
\[
\frac{f_3^3}{f_3} \equiv 1 + \sum_{n \in \mathbb{Z}} q^{(3n-1)^2} \pmod{2},
\]
which is quadratic. Hence, the same holds true for $\left( \frac{f_3^3}{f_9} \right)^2$, by substituting $q$ with $q^6$. More precisely, we obtain
\[
\left( \frac{f_3^3}{f_9} \right)^2 \equiv 1 + \sum_{n \in \mathbb{Z}} q^{6(3n-1)^2} \pmod{2}. \tag{5.11}
\]

Case (i). For $d \in \{4, 12, 36, 108, 324\}$, we find that $G_1 = 1$, $1/81 \leq G_2 \leq 1$, $1/36 \leq G_4 \leq 1$ and $1/9 \leq G_5 \leq 1$. Hence,
\[
L \geq 2^{k+2} + 2 + 36/81 + 12/36 - 108 - 6 - 2^k = 3 \cdot 2^k - 112 - 7/9.
\]

Since $k \geq 6$, we have $L \geq 0$.

Case (ii). For $d = 4, 12$, we find that $G_1 = G_2 = G_5 = 1/4$ and $G_3 = G_4 = 1/16$. Hence, $L = 2$.

Case (iii). For $d = 36$, we find that $G_1 = G_5 = 1/4$, $G_2 = 1/36$, $G_3 = 1/16$ and $G_4 = 1/144$. Hence, the value of $L$ is equal to 0.

Case (iv). For $d = 108, 324$, we find that $G_1 = 1/4$, $G_2 = 1/324$, $G_3 = 1/144$, $G_4 = 1/1296$ and $G_5 = 1/36$. Hence, we have value of $L$ equal to 4/9.

Hence, $B_k(z)$ is holomorphic at every cusp $\frac{a}{d}$ for all $k \geq 6$. Using Theorem 2.3, we find that the weight of $B_k(z)$ is equal to $2^{k-1}$. Also, the associated character for $B_k(z)$ is given by $\chi_1 = (\frac{4 \cdot 3^{2k-2} \cdot 2^k}{d})$. This proves that $B_k(z) \in M_{2k-1}(\Gamma_0(324), \chi_1)$ for all $k \geq 6$. Also, the Fourier coefficients of $B_k(z)$ are all integers. Hence by Theorem 2.6, the Fourier coefficients of $B_k(z)$ are almost always divisible by $m = 2^k$, for any positive integer $k$. Due to (5.8), the same holds for $b_9(2n + 1)$. This completes the proof of the theorem.

We now prove Theorem 1.7. We recall the following classical result due to Landau [3].

Lemma 5.2. Let $r(n)$ and $s(n)$ be quadratic polynomials. Then
\[
\left( \sum_{n \in \mathbb{Z}} q^{r(n)} \right) \left( \sum_{n \in \mathbb{Z}} q^{s(n)} \right)
\]
is lacunary modulo 2.

Proof of Theorem 1.7. We first recall the following identity [3, (7)]:
\[
f_3^3 \equiv f_3 + qf_9^3 \pmod{2}.
\]

We rewrite the above identity as
\[
\frac{f_3}{f_1} \equiv f_1^2 + q^3 \frac{f_9}{f_1} \pmod{2}. \tag{5.9}
\]

Combining (5.2) and (5.9), we obtain
\[
\sum_{n=0}^{\infty} b_9(4n)q^n \equiv \frac{f_3^3}{f_9^2} f_1^2 + q^6 \frac{f_9^6}{f_3} f_1^2 \pmod{2}. \tag{5.10}
\]

We note that the second term of (5.10) is lacunary modulo 2, by Theorem 1.6. For the first term of (5.10), we again recall the following identity [3, p. 12]
\[
\frac{f_3^3}{f_3} \equiv 1 + \sum_{n \in \mathbb{Z}} q^{(3n-1)^2} \pmod{2},
\]
which is quadratic. Hence, the same holds true for $\left( \frac{f_3^3}{f_9} \right)^2$, by substituting $q$ with $q^6$. More precisely, we obtain
\[
\left( \frac{f_3^3}{f_9} \right)^2 \equiv 1 + \sum_{n \in \mathbb{Z}} q^{6(3n-1)^2} \pmod{2}. \tag{5.11}
\]
Now, squaring the Euler’s Pentagonal Number formula, we have
\[ f_1^2 = \sum_{n \in \mathbb{Z}} q^{n(3n-1)} \pmod{2}. \quad (5.12) \]
Finally combining (5.11) and (5.12), and then applying Lemma 5.2 we conclude that the first term of (5.10) is also lacunary modulo 2. This completes the proof of the theorem. \(\square\)

Remark 5.3. Theorem 1.7 can also be proved using the Serre’s density result as shown in the proof of Theorem 1.6. For this, we rewrite (5.2) in terms of \(\eta\)-quotients and obtain
\[ \sum_{n=0}^{\infty} b_9(4n)q^{12n+1} \equiv \frac{\eta^7(36z)}{\eta(12z)\eta^2(108z)} \pmod{2}. \quad (5.13) \]
Let \( F(z) = \frac{\eta^7(36z)}{\eta(12z)\eta^2(108z)} \). As shown in the proof of Theorem 1.6 one can prove that \( F(z) \in M_2(\Gamma_0(1296), (\frac{28, x^7}{})). \) By Theorem 2.6, the Fourier coefficients of \( F(z) \) are almost always divisible by \( m = 2 \). Due to (5.13), the same holds for \( b_9(4n) \).

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