Restricted Cohomology of Restricted Lie Superalgebras

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Abstract
Suppose the ground field \( F \) is an algebraically closed field characteristic of \( p > 2 \). In this paper, we investigate the restricted cohomology theory of restricted Lie superalgebras. Algebraic interpretations of low dimensional restricted cohomology of restricted Lie superalgebras are given. We show that there is a family of restricted model filiform Lie superalgebra \( L_{p,p}^\lambda \) structures parameterized by elements \( \lambda \in F^p \). We explicitly describe both the 1-dimensional ordinary and restricted cohomology superspaces of \( L_{p,p}^\lambda \) with coefficients in the 1-dimensional trivial module and show that these superspaces are equal. We also describe the 2-dimensional ordinary and restricted cohomology superspaces of \( L_{p,p}^\lambda \) with coefficients in the 1-dimensional trivial module and show that these superspaces are unequal.

Keywords
Restricted Lie superalgebras, restricted cohomology, restricted model filiform Lie superalgebras

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1 Introduction

As a natural generalization of Lie algebras, Lie superalgebras play an important role in the theoretical physics and mathematics. Let \( L = L_0 \oplus L_1 \) be a restricted Lie superalgebra. Then \( L_0 \) is a restricted Lie algebra and \( L_1 \) is a restricted \( L_0 \)-module. Therefore, restricted Lie
superalgebras play a central role in the theory of modular Lie superalgebras, just as in the modular Lie algebra situation.

Cohomology is a very important tool in the research of topology, smooth vector fields, holomorphic functions, etc. Cohomology theory is from the study of the topology of Lie groups and vector fields on Lie groups by Cartan. The standard complex of Lie algebra was originally constructed by Chevalley and Eilenberg in [6]. Many conclusions on cohomology of Lie algebras were given in [15], etc. Hochschild first considered the cohomology theory of restricted Lie algebras in [16]. In dissertation [8], Evans improved on the results in [16] and obtained a cochain complex that is capable of making computations. In recent years, restricted cohomology theory of restricted Lie algebras have aroused the interest of a great many researchers, see [9–12]. Shortly after the birth of supersymmetry most basic constructions and results of the classical theory of Lie algebras are generalized to the case of Lie superalgebras. Fuks and Leites in [13] calculated the cohomology of the classical Lie superalgebras with trivial coefficients. In [24], Scheunert and Zhang introduced in detail the concepts of cohomology of Lie superalgebra. In [25], Su and Zhang explicitly computed the 1, 2-dimensional cohomology of the classical Lie superalgebras $\mathfrak{sl}_m|n$ and $\mathfrak{osp}_{2|2n}$ with coefficients in the finite dimensional irreducible modules and the Kac modules. In addition, the works of Boe, Kujawa, Nakano, Bagci, Lehrer, Poletaeva et al. also promoted the development of cohomology theory of Lie superalgebras, see [1, 5, 20]. The papers of Iwai and Shimada [18], May [22] and Priddy [23] from the 1960s address the problem of constructing a resolution for computing the cohomology of restricted graded Lie algebras. Strictly speaking, these earlier works were written in the context of $\mathbb{Z}$-graded Lie algebras, also the same methods may apply to Lie superalgebras by reducing all of the gradings modulo 2. For a discussion of this earlier work that is specifically in the context of Lie superalgebras, see [7]. The resolutions constructed in these earlier works are not entirely explicit (owing to the inductive nature of the constructions), but the resolutions are relatively explicit in low degrees. In recent years, cohomology theory of restricted Lie superalgebras have aroused the interest of some researchers, see [2].

In the study of the reducibility of the varieties of nilpotent Lie algebras, Vergne introduced the concept of filiform Lie algebras, see [26]. Since then, the study of the filiform Lie algebras, especially the model filiform Lie algebra, has become an important subject. As what happens in the Lie case, (model) filiform Lie superalgebra is an important subject of Lie superalgebra. Many conclusions on cohomology of the model filiform Lie superalgebra were given in [4, 14, 21].

In this paper, we are interested in the cohomology theory of restricted Lie superalgebras. We give algebraic interpretations of restricted cohomology of restricted Lie superalgebra $L$: The superspace of all restricted outer superderivations is equal to 1-dimensional restricted cohomology of $L$ with coefficients in the adjoint module; Let superspaces $M$, $N$ be two restricted $L$-modules, then equivalence classes of extensions of $N$ by $M$ are in one to one correspondence with 1-dimensional restricted cohomology of $L$ with coefficients in the module $\text{Hom}_F(N, M)$; Equivalence classes of restricted central extensions of $L$ are in one to one correspondence with 2-dimensional restricted cohomology of $L$ with coefficients in the trivial module. We show that there is a family $L^\lambda_{p,p}$ of restricted model filiform Lie superalgebra structures parameterized by elements $\lambda \in \mathbb{F}$. We explicitly describe both the 1, 2-dimensional ordinary and restricted
cohomology of $L^\lambda_{p,p}$ with coefficients in the 1-dimensional trivial module.

This paper is organized as follows. In Section 2, we give some necessary concepts and notations. In Section 3, we give algebraic interpretations of restricted cohomology of restricted Lie superalgebra. In Section 4, we determine both the 1, 2-dimensional ordinary and restricted cohomology of restricted model filiform Lie superalgebra with coefficients in the 1-dimensional trivial module.

2 Preliminaries

Let $\mathbb{F}$ be an algebraically closed field of characteristics $p > 2$. All unspecified vector superspaces are taken over $\mathbb{F}$. Suppose $\mathbb{Z}_2 := \{\bar{0}, \bar{1}\}$ is the additive group of order 2. Suppose $\mathfrak{A} := \mathfrak{A}_0 \oplus \mathfrak{A}_1$ is a superalgebra. We write $|x| := \theta$ for the parity of a $\mathbb{Z}_2$-homogeneous element $x \in \mathfrak{A}_\theta$, $\theta \in \mathbb{Z}_2$.

2.1 Restricted Lie Superalgebras

Let us recall the definitions of Lie superalgebras and restricted Lie superalgebras [19, 27].

**Definition 2.1** A Lie superalgebra is a vector superspace $L = L_0 \oplus L_1$ with an even bilinear mapping $\{\cdot, \cdot\} : L \times L \rightarrow L$ satisfying the following axioms:

\[
[x, y] = -(-1)^{|x||y|}[y, x],
\]

\[
[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]
\]

for all $x, y, z \in L$.

**Definition 2.2** A restricted Lie superalgebra is a Lie superalgebra $L = L_0 \oplus L_1$ together with a map $[p] : L_0 \rightarrow L_0$, denoted by $x \mapsto x^{[p]}$, that satisfies:

1. $(L_0, [p])$ is a restricted Lie algebra,
2. $L_1$ is a restricted $L_0$-module.

2.2 Ordinary and Restricted Cohomology of Restricted Lie Superalgebras

Let us give some definitions relative to cohomology of Lie superalgebras [24] and restricted cohomology of Lie superalgebras. For details of ordinary cohomologies we refer to [3, 13, 24], which are the foundational papers on cohomology of Lie superalgebras. By [7, 8, 18, 22, 23], we derive the types of 1, 2-dimensional restricted cohomology of restricted Lie superalgebras.

Let $L$ be a Lie superalgebra and $M = M_0 \oplus M_1$ be an $L$-module. For $q \geq 0$, we let $\bigwedge^q L$ be the $q$-th super-exterior product of $L$, that is $\bigwedge^q L$ is the $q$-fold tensor product of $L$ modulo the $L$-submodule generated by the elements of the form:

\[
x_1 \otimes \cdots \otimes x_k \otimes x_{k+1} \otimes \cdots \otimes x_q + (-1)^{|x_k||x_{k+1}|}x_1 \otimes \cdots \otimes x_{k+1} \otimes x_k \otimes \cdots \otimes x_q
\]

for $x_1, \ldots, x_q \in L$. Set

\[
C^q(L; M) = \text{Hom}_\mathbb{F}\left(\bigwedge^q L, M\right).
\]

There spaces $C^q(L; M)$ are naturally $\mathbb{Z}_2$-graded. Note that

\[
C^0(L; M) = \text{Hom}_\mathbb{F}(\mathbb{F}, M) \cong M.
\]

Let $d^q : C^q(L; M) \rightarrow C^{q+1}(L; M)$ be given by the formula:

\[
d^q(\varphi)(x_1, \ldots, x_{q+1}) = \sum_{1 \leq i < j \leq q+1} (-1)^{\sigma_{ij}(|x_1, \ldots, x_{q+1}|)} \varphi([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{q+1})
\]
where $x_1, \ldots, x_{q+1} \in L$, $\varphi \in C^q(L; M)$ and
\[
\sigma_{ij}(x_1, \ldots, x_{q+1}) = i + j + |x_i|(|x_1| + \cdots + |x_{i-1}|) + |x_j|(|x_1| + \cdots + |x_{j-1}| + |x_i|),
\]
\[
\gamma_i(x_1, \ldots, x_{q+1}, \varphi) = i + 1 + |x_i|(|x_1| + \cdots + |x_{i-1}| + |\varphi|).
\]
A direct verification shows that $d^q|d^{q-1} = 0$ and $|d^q| = 0$. The elements of kernel of $d^q$ are called $q$-dimensional cocycles and the elements of image of $d^{q-1}$ are called $q$-dimensional coboundaries. We will denote the $q$-dimensional cocycles and coboundaries by $Z^q(L; M)$ and $B^q(L; M)$, respectively.

**Definition 2.3** Let $L$ be a Lie superalgebra, $M$ be an $L$-module and $q \geq 0$. We call $H^q(L; M) = Z^q(L; M)/B^q(L; M)$ is an ordinary cohomology of $L$ with coefficients in the module $M$.

Let $q \leq 1$, define $C^2(L; M) = C^2(L; M)$. If $\varphi \in C^2(L; M)$ and $\omega : L_0 \to M$, we say that $\omega$ has the $*$-property with respect to $\varphi$ if for all $\lambda \in \mathbb{F}$ and all $x, y \in L_0$,

(i) $\omega(\lambda x) = \lambda^p \omega(x),$

(ii) $\omega(x + y) = \omega(x) + \omega(y) + \sum_{|x_1| = |x|, |x_2| = |y|} \frac{1}{g(x)} \sum_{k=0}^{p-2} (-1)^k x_p \cdots x_{p+k-1} \varphi([x_1, \ldots, x_{p+k-1}], x_{p+k})$,

where $g(x)$ is the number of factors $x_i$ equal to $x$ and

\[
[x_1, \ldots, x_{p+k-1}] = [[[x_1, x_2], x_3], \ldots, x_{p+k-2}, x_{p+k-1}].
\]

Set
\[
C^2_\ast(L; M) = \{ (\varphi, \omega) \mid \varphi \in C^2(L; M), \omega : L_0 \to M \text{ has the } \ast\text{-property w.r.t. } \varphi \}.
\]

If $\alpha \in C^3(L; M)$ and $\beta : L_0 \times L_0 \to M$, we say that $\beta$ has the $\ast\ast$-property w.r.t. $\alpha$ if for all $\lambda \in \mathbb{F}$ and all $x, y, y_1, y_2 \in L_0$,

(i) $\beta(x, y)$ is linear with respect to $x$,

(ii) $\beta(x, \lambda y) = \lambda^p \beta(x, y),$

(iii)
\[
\beta(x_1, y_1 + y_2) = \beta(x, y_2) + \beta(x, y_2) - \sum_{h_1 = y_1, h_2 = y_2} \frac{1}{\mathcal{g}(y_1)} \sum_{j=0}^{p-2} (-1)^j \sum_{k=1}^{j} C^j_k h_p \cdots h_{p+k-1}
\]
\[
\cdot \alpha([x, h_{p-k}, \ldots, h_{p-j+1}], [h_{j}, \ldots, h_{p-j-1}], h_{p-j}).
\]

Set
\[
C^3_\ast(L; M) = \{ (\alpha, \beta) \mid \alpha \in C^3(L; M), \beta : L_0 \times L_0 \to M \text{ has the } \ast\ast\text{-property w.r.t. } \alpha \}.
\]

Let $q = 2$ or 3. The $\mathbb{Z}_2$-grading of $C^q_\ast(L; M)$ is inherited by
\[
C^q_\ast(L; M) = C^q_\ast(L; M)_0 \oplus C^q_\ast(L; M)_{\theta},
\]
where
\[
C^q_\ast(L; M)_0 = \{ (\alpha, \beta) \in C^q_\ast(L; M) \mid |\alpha| = \theta \} \quad \text{for all } \theta \in \mathbb{Z}_2.
\]

Now, an element $\varphi \in C^1(L; M)$ induces a map $\text{ind}^1(\varphi) : L_0 \to M$ by the formula
\[
\text{ind}^1(\varphi)(x) = \varphi(x^{[p]}) - x^{p-1} \varphi(x) \quad \text{for all } x \in L_0.
\]
Since \( d^1 \varphi \mid_{L_0 \times L_0} \in C^2(L_0; M) \), then the [8, Lemma 3.7] shows that \( \text{ind}^1(\varphi) \) satisfies the \( * \)-property w.r.t. \( d^1 \varphi \). An element \((\alpha, \beta) \in C^2_*(L; M)\) induces a map \( \text{ind}^2(\alpha, \beta) : L_0 \times L_0 \to M \) by the formula
\[
\text{ind}^2(\alpha, \beta)(x, y) = \alpha(x, y)[y] - \sum_{i+j=p-1} (-1)^i \alpha \left( \left[ x, \underbrace{\ldots, y, \ldots,} \right]_j \right) + x \beta(y) \quad \text{for all } x, y \in L_0. \tag{2.3}
\]

Since \( d^2 \alpha \mid_{L_0 \times L_0 \times L_0} \in C^3(L_0; M) \), then the [8, Lemma 3.10] shows that \( \text{ind}^2(\alpha, \beta) \) satisfies the \( ** \)-property w.r.t. \( d^2 \alpha \). The restricted differentials are defined by
\[
\begin{align*}
  d^0_* & : C_0^0(L; M) \to C_1^0(L; M), \quad d^0_* = d^0, \\
  d^1_* & : C_1^0(L; M) \to C_2^0(L; M), \quad d^1_* = (d^1 \varphi, \text{ind}^1(\varphi)), \\
  d^2_* & : C_2^0(L; M) \to C_3^0(L; M), \quad d^2_* = (d^2 \alpha, \text{ind}^2(\alpha, \beta)).
\end{align*}
\tag{2.4}
\]

Let \( q \leq 2 \). We can show that \( d^2_* d^1_* = 0 \) and \( \|d^2_*\| = 0 \). The elements of kernel of \( d^2_* \) are called \( q \)-dimensional restricted cocycles and the elements of image of \( d^1_* \) are called \( q \)-dimensional restricted coboundaries. We will denote the \( q \)-dimensional restricted cocycles and restricted coboundaries by \( Z_q^0(L; M) \) and \( B_q^0(L; M) \), respectively.

**Definition 2.4** Let \( L \) be a restricted Lie superalgebra, \( M \) be a restricted \( L \)-module and \( q \geq 0 \). We called \( H_q^0(L; M) = Z_q^0(L; M)/B_q^0(L; M) \) is \( q \)-dimensional restricted cohomology of \( L \) with coefficients in the module \( M \).

Note that \( H^0(L; M) = H^0_0(L; M) \).

## 3 Algebraic Interpretations

In this section, we give the algebraic interpretations of low dimensional restricted cohomology of restricted Lie superalgebras and show that equivalence classes of these objects are naturally corresponding to the restricted cohomology of restricted Lie superalgebras defined above.

**Definition 3.1** Let \( L \) be a restricted Lie superalgebra. A homogeneous linear map \( D : L \to L \) is called a restricted superderivation of \( L \) with parity \( |D| \) if for all \( x, y \in L, \ z \in L_0 \),
\[
\begin{align*}
  D([x, y]) &= (-1)^{|D||x|}[x, D(y)] + [D(x), y], \tag{3.1} \\
  D(z^{[p]}) &= (\text{ad } z)^{p-1} D(z). \tag{3.2}
\end{align*}
\]

Write \( (\text{Der}_{\text{res.}})_0(L) \) (resp., \( (\text{Der}_{\text{res.}})_1(L) \)) for the set of all restricted superderivations with parity \( 0 \) (resp., \( 1 \)) of \( L \). Denote
\[
\text{Der}_{\text{res.}}(L) = (\text{Der}_{\text{res.}})_0(L) \oplus (\text{Der}_{\text{res.}})_1(L).
\]

For \( x \in L, \) we have \( \text{ad } x \in \text{Der}_{\text{res.}}(L) \) and then \( \text{ad } L < \text{Der}_{\text{res.}}(L) \). We will call restricted superderivations of the form \( \text{ad } x \) inner and elements of the \( \text{Der}_{\text{res.}}(L)/\text{ad } L \) outer.

**Theorem 3.2** Let \( L \) is a restricted Lie superalgebra, then
\[
\text{Der}_{\text{res.}}(L)/\text{ad } L = H^1_1(L; L).
\]

**Proof** (1) Let \( \varphi \in C^1_*(L; L) \). By the equation (2.5) we know that \( \varphi \) is a 1-dimensional restricted cocycle if and only if \( d^1 \varphi = 0 \) and \( \text{ind}^1(\varphi) = 0 \).
(1.1) If \( d^1 \varphi = 0 \), then by the equation (2.1) we have
\[
0 = d^1 \varphi(x, y) = -\varphi([x, y]) + (-1)^{|x||\varphi|}[x, \varphi(y)] - (-1)^{|y|(|\varphi|+|x|)}[y, \varphi(x)] = -\varphi([x, y]) + (-1)^{|x||\varphi|}[x, \varphi(y)] + [\varphi(x), y]
\]
for all \( x, y \in L \). So \( \varphi \) satisfies the equation (3.1).

(1.2) If \( \text{ind}^1(\varphi) = 0 \), then by the equation (2.2) we have
\[
0 = \text{ind}^1(\varphi)(z) = \varphi(z^{[p]}) - z^{p-1} \varphi(z)
\]
for all \( z \in L_0 \). So \( \varphi \) satisfies the equation (3.2).

In summary, \( \varphi \) is a 1-dimensional restricted cocycle if and only if \( \varphi \in \text{Der}_{\text{res}}(L) \), that is
\[
Z^1 \text{res}(L; L) = \text{Der}_{\text{res}}(L).
\]

(2) Let \( \psi \in B^1\text{res}(L; L) \), then there is \( x \in C^\text{res}_0(L; L) \cong L \) such that \( \psi = d^1(x) \). So by the equations (2.1) and (2.4) we have
\[
\psi(y) = d^1(x)(y) = (-1)^{|x||y|}[y, x] = -[x, y]
\]
for all \( y \in L \). Therefore, \( \psi = -\text{ad} \ x \). It follows that \( B^1\text{res}(L; L) = \text{ad} \ L \).

The proof is complete. \( \square \)

**Definition 3.3** Let \( L \) be a restricted Lie superalgebra and superspaces \( M, N \) be two restricted \( L \)-modules. A restricted extension of \( N \) by \( M \) is an exact sequence
\[
0 \rightarrow M \xrightarrow{i} E \xrightarrow{\pi} N \rightarrow 0 \tag{3.3}
\]
of restricted \( L \)-modules and homomorphisms. Two extensions of \( N \) by \( M \) are equivalent if there is an isomorphism of restricted \( L \)-modules \( \sigma : E_1 \rightarrow E_2 \) so that
\[
\begin{array}{cccccc}
0 & \rightarrow & M & \xrightarrow{i_1} & E_1 & \xrightarrow{\pi_1} & N & \rightarrow & 0 \\
\| & \circ & \downarrow \sigma & & \circ & & \| \\
0 & \rightarrow & M & \xrightarrow{i_2} & E_2 & \xrightarrow{\pi_2} & N & \rightarrow & 0
\end{array} \tag{3.4}
\]
commutes, and we ask for a description of the set \( \text{Ext}_{\text{res}}(N, M) \) of equivalence classes of extensions of \( N \) by \( M \).

Let \( L \) be a restricted Lie superalgebra and \( M, N \) be two restricted \( L \)-modules. For \( x \in L \), \( \phi \in \text{Hom}_F(N, M) \) and \( n \in N \) we define
\[
(x\phi)(n) = x\phi(n) - (-1)^{|\phi||x|}\phi(xn). \tag{3.5}
\]

Then \( \text{Hom}_F(N, M) \) is a restricted \( L \)-module and we have the following Theorem.

**Theorem 3.4** Let \( L \) be a restricted Lie superalgebra and \( M, N \) be two restricted \( L \)-modules. Then \( \text{Ext}_{\text{res}}(N, M) \) is in one to one correspondence \( H^1_*(L; \text{Hom}_F(N, M)) \). In particular, \( \text{Ext}_{\text{res}}(F, N) \) is in one to one correspondence \( H^1_*(L; M) \).

**Proof** (1) Given an extension (3.3) of \( N \) by \( M \) and choose an \( F \)-linear map \( \rho \) such that \( \pi \rho = \text{Id}_N \). We define an element \( \varphi_\rho \in C^1_*(L; \text{Hom}_F(N, M)) \) by the formula
\[
\varphi_\rho(x)(n) = (-1)^{|x||\rho|}x\rho(n) - \rho(xn) \quad \text{for all } x \in L, n \in N, \tag{3.6}
\]
that is \( \varphi_\rho(x) = (-1)^{|x||\rho|} x\rho \). We claim that \( d^1\varphi_\rho = (d^1\varphi_\rho, \text{ind}^1(\varphi_\rho)) = 0 \), that is \( \varphi_\rho \in Z^1_L(L; \text{Hom}_F(N, M)) \).

(1.1) For all \( x, y \in L \) and \( n \in N \), by the equations (2.1), (3.5) and (3.6) we have
\[
d^1\varphi_\rho(x, y)(n) = -\varphi_\rho([x, y])(n) + (-1)^{|x||\varphi_\rho|}(x\varphi_\rho(y))(n) - (-1)^{|y||\varphi_\rho|}(y\varphi_\rho(x))(n)
\]
\[= -(-1)^{|x|+|y|}|\rho|[x, y]\rho(n) + \rho([x, y]n)
+ (-1)^{|x||\varphi_\rho|}x\varphi_\rho(y)(n) - (-1)^{|y||\varphi_\rho|}y\varphi_\rho(x)(yn)
- (-1)^{|y||x|}y\varphi_\rho(x)(n) + \varphi_\rho(x)(yn)
= -(-1)^{|x|+|y|}|\rho|[x, y]\rho(n) + \rho([x, y]n)
+ (-1)^{|x||\varphi_\rho|}(-1)^{|y||\rho|}x\rho(y)(n) - (-1)^{|x||\varphi_\rho|}y\rho(x)(yn)
- (-1)^{|y||x|}y\rho(x)(n) + \rho(xyn)
+ (-1)^{|x||\rho|}x\rho(yn) - \rho(xyn)
= [-(-1)^{|x|+|y|}|\rho|[x, y]\rho(n) + (-1)^{|x||\varphi_\rho|}(-1)^{|y||\rho|}x\rho(y)(n)
- (-1)^{|y||x|}y\rho(x)(n)]
+ [\rho([x, y]n) + (-1)^{|y||x|}\rho(xyn) - \rho(xyn)]
+ [(-1)^{|x||\varphi_\rho|}x\rho(yn) + (-1)^{|x||\rho|}x\rho(yn)]
+ [-(-1)^{|y||x|}(-1)^{|y||\rho|}y\rho(xn) + (-1)^{|y||x|+|\varphi_\rho|}y\rho(xn)].
\]
Note that \( |\varphi_\rho| = |\rho| \). Then \( d^1\varphi_\rho(x, y)(n) = 0 \), that is \( d^1\varphi_\rho = 0 \).

(1.2) For all \( x \in L_0 \), by the equations (2.2) and (3.6) we have
\[
\text{ind}^1(\varphi_\rho)(x) = \varphi_\rho(x^{[p]}) - x^{p-1}\varphi_\rho(x) = x^{|p|}\rho - x^p\rho = 0.
\]
Then \( \text{ind}^1(\varphi_\rho) = 0 \).

(2) Conversely, if \( \varphi \in Z^1_L(L; \text{Hom}_F(N, M)) \), we construct an extension of \( N \) by \( M \) as follows.
We set \( E = N \oplus M \) and
\[
x(n, m) = (xn, xm + (-1)^{|x||\varphi|}x\varphi(y)(n)),
\]
where \( x \in L \), \( n \in N \) and \( m \in M \). We claim that \( E \) is a restricted \( L \)-module and
\[
0 \longrightarrow M \hookrightarrow E \twoheadrightarrow N \longrightarrow 0
\]
is a restricted extension of \( N \) by \( M \).

(2.1) Since \( \varphi \in Z^1_L(L; \text{Hom}_F(N, M)) \), then by the equation (2.1) we have
\[
\varphi([x, y]) = (-1)^{|x||\varphi|}x\varphi(y) - (-1)^{|y||x|+|\varphi|}y\varphi(x)
\]
(3.8) for all \( x, y \in L \). By the equations (3.7) and (3.8) we have
\[
(xy - (-1)^{|x||y|}yx)(n, m)
= x(yn, ym + (-1)^{|\varphi|(|y|+|\varphi|)}y\varphi(y)(n))
- (-1)^{|x||y|}y(xn, xm + (-1)^{|\varphi|(|x|+|\varphi|)}x\varphi(x)(n))
= (xyn, xym + (-1)^{|\varphi|(|y|+|\varphi|)}x\varphi(y)(n) + (-1)^{|\varphi|(|x|+|y|+|\varphi|)}x\varphi(x)(yn))
\]
for all \( n \in N \) and \( m \in M \).

(2.2) Since \( \varphi \in Z^1(L; \text{Hom}_F(N, M)) \), then by the equation (2.2)
\[
\varphi(x[p]) = x^{p-1}\varphi(x)
\]
for all \( x \in L_0 \). By the equations (3.7) and (3.9) we have
\[
x[p](n, m) = (x[p]n, x[p]m + (-1)^{|x||n|}\varphi(x[p])(n)) = (x^p n, x^p m + (-1)^{|x||n|}x^{p-1}\varphi(x)(n)) = x^p(n, m)
\]
for all \( n \in N \) and \( m \in M \).

(3) We claim that \( \text{Ext}_{res}(N, M) \) is in one to one correspondence \( H^1_*(L; \text{Hom}_F(N, M)) \).

(3.1) Suppose that \( \varphi_1, \varphi_2 \in Z^1_*(L; \text{Hom}_F(N, M)) \), then by the equation (3.1)
\[
\varphi = \varphi_2 - \varphi_1
\]

(3.9)

(3.10) for all \( n \in N \) and \( m \in M \). Clearly this map is an isomorphism of vector spaces making the diagram (3.4) commutes.

(3.11) Let \( x \in L \), \( n \in N \) and \( m \in M \). Then by the equations (2.4), (3.5), (3.7) and (3.10) we have
\[
\sigma(x(n, m)) = x(n, m) - (-1)^{\theta|n|}f(n)
\]

(3.12) for all \( n \in N \) and \( m \in M \). Then by the equations (2.4), (3.5), (3.7) and (3.10) we have
\[
\sigma(x[p](n, m)) = x[p](n, m) - (-1)^{\theta|n|}x[p-1]\varphi_1(x[p])(n)
\]
So $\sigma$ is a homomorphism of restricted $L$-modules.

In summary, $\sigma$ is an isomorphism of restricted $L$-modules. Therefore, $E_1$ and $E_2$ are equivalent.

(3.2) Since $d^0 = d^0$ and $C^1(L; \text{Hom}_\mathbb{F}(N, M)) = C^1(L; \text{Hom}_\mathbb{F}(N, M))$, it follows from the classical cohomology theory that restricted cocycle whose cohomology class depends only on the equivalence class of the extension.

The proof is complete. \hfill $\Box$

**Definition 3.5** We say that a restricted Lie superalgebra $K$ is strongly abelian if in addition to $[K, K] = 0$, we also have $K^{[p]}_0 = 0$.

**Definition 3.6** Let $L$ be a restricted Lie superalgebra and $K$ be a strongly abelian restricted Lie superalgebra. A restricted extension of $L$ by $K$ is an exact sequence

$$0 \longrightarrow K \overset{\iota_1}{\longrightarrow} E \overset{\pi}{\longrightarrow} L \longrightarrow 0$$

(3.11)

of restricted Lie superalgebras and homomorphisms. Two extensions of $L$ by $K$ being equivalent if there is an isomorphism of restricted Lie superalgebras $\sigma : E_1 \rightarrow E_2$ so that

$$0 \longrightarrow K \overset{\iota_1}{\longrightarrow} E_1 \overset{\pi_1}{\longrightarrow} L \longrightarrow 0$$

$$\text{commutes.}$$

Since $K$ is a strongly abelian restricted Lie superalgebra and $\iota(K) \vartriangleleft E$, then a restricted extension of $L$ by $K$ gives the structure of an $L$-module by the action $x \cdot k = [\widehat{x}, \iota(k)]$, where $x \in L$, $k \in K$ and $\widehat{x} \in E$ is any element satisfying $\pi(\hat{x}) = x$. Since $\pi(\hat{x}^{[p]}) = \pi([\hat{x}]^{[p]}) = x^{[p]}$, so $x^{[p]} \cdot k = [\hat{x}^{[p]}, k] = [\hat{x}^{[p]}, k] = (\text{ad} \hat{x})^p(k) = x^{[p]} \cdot k$

for all $x \in L_0$ and $k \in K$. Then $K$ is a restricted $L$-module. We remark here that if $\iota(K)$ is contained in the center of $E$, then $K$ is a trivial $L$-module. Such an extension is called central.

**Theorem 3.7** Let $L$ be a restricted Lie superalgebra and $K$ be strongly abelian restricted Lie superalgebra. Then the set of equivalence classes of restricted center extensions of $L$ by $K$ is in one to one correspondence with $H^2(L; K)$. In particular, $H^2(L; \mathbb{F})$ is in one to one correspondence with equivalence classes of 1-dimensional restricted center extensions of $L$.

**Proof** (1) Given an extension (3.11) of $L$ by $K$ and choose an $\mathbb{F}$-linear map $\rho$ such that $\pi \rho = \text{Id}_L$. Since $\pi$ is a homomorphism of restricted Lie superalgebras, so

$$[\rho(x), \rho(y)] - \rho([x, y]), \quad \rho(x)^{[p]} - \rho(x^{[p]}) \in \text{Ker}(\pi) = \text{Im}(\iota).$$

Then we can define $\alpha_\rho : L \times L \rightarrow K$ and $\beta_\rho : L_0 \rightarrow K$ with the formula

$$\alpha_\rho(x, y) = \iota^{-1}([\rho(x), \rho(y)] - \rho([x, y])), \quad (3.12)$$

Restricted Cohomology
\[ \beta_\rho(z) = \iota^{-1}(\rho(z)[p] - \rho(z[p])), \]

where \( x, y \in L \) and \( z \in L_0 \). Similar to [8, Theorem 3.7] we can prove that \( \beta_\rho \) has the \( * \)-property w.r.t. \( \alpha_\rho \). Then \( (\alpha_\rho, \beta_\rho) \in C^2_s(L; K) \). We claim that \( (\alpha_\rho, \beta_\rho) \in Z^2_s(L; K) \).

(1.1) Let \( x_1, x_2, x_3 \in L \). By the equations (2.1) and (3.12) we have

\[
\begin{align*}
\partial^2 \alpha_\rho(x_1, x_2, x_3) &= -\alpha_\rho([x_1, x_2], x_3) + (-1)^{|x_1||x_3|} \alpha_\rho([x_1, x_3], x_2) \\
&- (-1)^{|x_1||x_2|+|x_3|} \alpha_\rho([x_2, x_3], x_1) \\
&= \iota^{-1}(-[\rho([x_1, x_2]), \rho(x_3)] + (-1)^{|x_2||x_3|}[\rho([x_1, x_3]), \rho(x_2)]) \\
&- (-1)^{|x_1||x_2|+|x_3|}[\rho([x_2, x_3]), \rho(x_1)] \\
&+ \rho([x_1, x_2], x_3) - (-1)^{|x_2||x_3|}\rho([x_1, x_3], x_2) \\
&+ (-1)^{|x_1||x_2|+|x_3|}\rho([x_2, x_3], x_1)).
\end{align*}
\]

Since \( \iota^{-1}([\rho(x_1), \rho(x_2)] - \rho([x_1, x_2])) \in K \) and \( [\iota(K), E] = 0 \), so

\[
\iota^{-1}([[\rho(x_1), \rho(x_2)], \rho(x_3)]) = \iota^{-1}([\rho([x_1, x_2]), \rho(x_3)]).
\]

Then

\[
\begin{align*}
\partial^2 \alpha_\rho(x_1, x_2, x_3) &= \iota^{-1}(-[[\rho(x_1), \rho(x_2)], \rho(x_3)] + (-1)^{|x_2||x_3|}[[\rho(x_1), \rho(x_3)], \rho(x_2)]) \\
&- (-1)^{|x_1||x_2|+|x_3|}[[\rho(x_2), \rho(x_3)], \rho(x_1)] \\
&+ \rho([x_1, x_2], x_3) - (-1)^{|x_2||x_3|}[[x_1, x_3], x_2]) + (-1)^{|x_1||x_2|+|x_3|}[[x_2, x_3], x_1]) \\
&= 0,
\end{align*}
\]

that is \( \partial^2 \alpha_\rho = 0 \).

(1.2) Since \( K \) is a trivial \( L \)-module, then by the equations (2.3) and (3.12) we have

\[
\begin{align*}
\text{ind}^2(\alpha_\rho, \beta_\rho)(x, y) &= \alpha_\rho(x, y[p]) - \sum_{i+j=p-1} (-1)^{i}y^{i}\alpha_\rho\left(x, \underbrace{y, \ldots, y}_{j}\right) + x\beta_\rho(y) \\
&= \iota^{-1}([\rho(x), \rho(y[p])] - \rho([x, y[p]]) - \alpha_\rho\left(x, \underbrace{y, \ldots, y}_{p-1}\right), y) \\
&= \iota^{-1}([\rho(x), \rho(y[p])] - \rho([x, y[p]]) - \alpha_\rho\left(x, \underbrace{y, \ldots, y}_{p-1}\right), y) \\
&= \iota^{-1}([\rho(x), \rho(y)] - \rho([x, y[p]]) - \alpha_\rho\left(x, \underbrace{y, \ldots, y}_{p-1}\right), y) \\
&= \iota^{-1}([\rho(x), \rho(y)] - \rho([x, y[p]]) - \alpha_\rho\left(x, \underbrace{y, \ldots, y}_{p-1}\right), y) \\
&- \iota^{-1}\left(\rho\left(x, \underbrace{y, \ldots, y}_{p-1}\right), \rho(y) + \rho\left(x, \underbrace{y, \ldots, y}_{p-1}\right)\right) = 0
\end{align*}
\]

for all \( x, y \in L_0 \).

By the equation (2.6) we have \( \partial^2(\alpha_\rho, \beta_\rho) = 0 \).

(2) For each \( (\alpha, \beta) \in Z^2_s(L; K) \), one can construct a restricted central extension \( L \) by \( K \) as follows. We define \( L_{\alpha, \beta} = K \oplus L \) as a vector space and we define the Lie bracket and \( p \)-operator
in $L_{\alpha,\beta}$ by the formula

\[(k_1, l_1), (k_2, l_2)] = (\alpha(l_1, l_2), [l_1, l_2]), \tag{3.13}\]
\[(k, l)^{[p]} = (\beta(l), [l]^p), \tag{3.14}\]

where $k \in K_0, k_1, k_2 \in K$ and $l \in L_0, l_1, l_2 \in L$.

(2.1) The equation (3.13) is well known that the Jacobi identity is equivalent to $d^2 \alpha = 0$.

(2.2) The equation (3.14) is a $p$-operator precisely because $\beta$ has the $*$-property with respect to $\alpha$ and $\text{ind}^2(\alpha, \beta) = 0$.

(3) Next we show that for $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in Z^2(L; K)$, such that the central extensions $L_{\alpha_1, \beta_1}$ and $L_{\alpha_2, \beta_2}$ are equivalent, we have $(\alpha_1 - \alpha_2, \beta_1 - \beta_2) \in B^2(L; K)$. Let $\sigma$ be an isomorphism of restricted Lie superalgebras and $\rho_1, \rho_2$ be two $F$-linear maps such that

\[
\begin{array}{cccc}
0 & \longrightarrow & K & \stackrel{i_1}{\longrightarrow} & L_{\alpha_1, \beta_1} & \stackrel{\pi_1}{\longrightarrow} & L & \longrightarrow & 0 \\
\| & \circ & \| & \circ & \| & \| & \| & \| & \|
\end{array}
\tag{3.15}
\]

commutes and $\pi_1 \rho_1 = \pi_2 \rho_2 = \text{Id}_L$. Since

\[
\pi_2(\sigma \rho_1 - \rho_2) = \pi_2 \sigma \rho_1 - \pi_2 \rho_2 = \pi_1 \rho_1 - \pi_2 \rho_2 = 0,
\]

so $(\sigma \rho_1 - \rho_2)(L) \subseteq \text{Ker}(\pi_2) = \text{Im}(\iota_2)$. Then $\varphi = \iota_2^{-1}(\sigma \rho_1 - \rho_2) \in C^1_\varphi(L; K)$. We claim that

\[
d^1(\varphi) = (\alpha_2 - \alpha_1, \beta_2 - \beta_1).
\]

(3.1) By proof of [17, Lemma 4.1], we have $d^1 \varphi = \alpha_2 - \alpha_1$.

(3.2) For all $x \in L_0$, the same methods were given in the proof of [8, Lemma 3.25] show that

\[
\text{ind}^1(\varphi)(x) = (\beta_2 - \beta_1)(x).
\]

(4) Let $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in Z^2(L; K)$, such that $(\alpha_2, \beta_2) - (\alpha_1, \beta_1) \in B^2(L; K)$, that is there exists $\varphi \in C^1_\varphi(L; K)$ satisfying $(\alpha_2, \beta_2) - (\alpha_1, \beta_1) = d^1(\varphi)$. Now we prove that the extensions $L_{\alpha_1, \beta_1}$ and $L_{\alpha_2, \beta_2}$ are equivalent. Let us define $\sigma : L_{\alpha_1, \beta_1} \to L_{\alpha_2, \beta_2}$ by

\[
\sigma(k, l) = (k - \varphi(l), l) \tag{3.16}
\]

for all $k \in K$ and $l \in L$. It is clear that $\sigma$ is bijective and such that diagram (3.15) commutes. We claim $\sigma$ is a homomorphism of restricted Lie superalgebras.

(4.1) Let us check that $\sigma$ is a homomorphism of Lie superalgebras. By the equations (3.13) and (3.16) we have

\[
[\sigma(k_1, l_1), \sigma(k_2, l_2)] = [(k_1 - \varphi(l_1), l_1), (k_2 - \varphi(l_2), l_2)]
\]
\[
= (\alpha_2(l_1, l_2), [l_1, l_2]) = (\alpha_1(l_1, l_2) + d^1(\varphi)(l_1, l_2), [l_1, l_2])
\]
\[
= (\alpha_1(l_1, l_2) - \varphi([l_1, l_2]), [l_1, l_2]) = \sigma(\alpha_1(l_1, l_2), [l_1, l_2])
\]
\[
= \sigma([(k_1, l_1), (k_2, l_2)]
\]

for all $k_1, k_2 \in K$ and $l_1, l_2 \in L$.

(4.2) Next we show that $\sigma$ is restricted. By the equations (3.14) and (3.16) we have

\[
\sigma((k, l)^{[p]}) = \sigma(\beta_1(l), [l]^p) = (\beta_1(l) - \varphi([l]^p), [l]^p)
\]
where $\lambda$ restricted Lie superalgebra. If we let

\[ (\beta_2(l), l)[p] = (k - \varphi(l), l)[p] = (\sigma(k, l))[p] \]

for all $k \in K_0$ and $l \in L_0$.

The proof is complete. 

\[ \square \]

4 Ordinary and Restricted Cohomology of the Restricted Model Filiform Lie Superalgebras

In this Section, we determine both the 1,2-dimensional ordinary and restricted cohomology of restricted model filiform Lie superalgebra with coefficients in the 1-dimensional trivial module.

4.1 Restricted Model Filiform Lie Superalgebras

Fix a pair of positive integers $n, m$, let

\[ L_{n,m} = \text{span}_\mathbb{F}\{X_1, \ldots, X_n \mid Y_1, \ldots, Y_m\} \]

and Lie super-brackets are given by

\[ [X_1, X_i] = X_{i+1}, \quad 2 \leq i \leq n - 1, \quad [X_1, Y_j] = Y_{j+1}, \quad 1 \leq j \leq m - 1 \]

with all other brackets are zero. We call $L_{n,m}$ the model filiform Lie superalgebra and $\{X_1, \ldots, X_n \mid Y_1, \ldots, Y_m\}$ the standard basis of $L_{n,m}$. Next we show that $L_{p,p}$ admits the structure of a restricted Lie superalgebra. If we let $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathbb{F}^p$, and set

\[ \left( \sum_{k=1}^{p} a_k X_k \right)[p] = \left( \sum_{k=1}^{p} a_k^p \lambda_k \right) X_p, \]

where $a_k \in \mathbb{F}$, then by [10], we know that $(L_{p,p})_0$ is a restricted Lie algebra. Since

\[ \left[ \left( \sum_{k=1}^{p} a_k X_k \right)[p], Y \right] = \left[ \left( \sum_{k=1}^{p} a_k^p \lambda_k \right) X_p, Y \right] = 0 = \text{ad} \left( \sum_{k=1}^{p} a_k X_k \right)^p(Y), \]

where $Y \in (L_{p,p})_1$, then $L_{p,p}$ is a restricted Lie superalgebra, which we denote by $L_{p,p}^\lambda$.

4.2 Cochain Complexes with Trivial Coefficients

For ordinary cohomology of $L_{p,p}^\lambda$ with coefficients in the 1-dimensional trivial module, the relevant cochain superspaces (with dual bases) are:

\[ C^0(L_{p,p}^\lambda; \mathbb{F}) : \quad \Omega^0 = \{1\}, \]
\[ C^1(L_{p,p}^\lambda; \mathbb{F}) : \quad \Omega^1 = \{X^i, Y^j \mid 1 \leq i, j \leq p\}, \]
\[ C^2(L_{p,p}^\lambda; \mathbb{F}) : \quad \Omega^2 = \{X^{i,j}, X^kY^l, Y^s,t \mid 1 \leq i < j \leq p, 1 \leq k, l \leq p, 1 \leq s \leq t \leq p\}, \]
\[ C^3(L_{p,p}^\lambda; \mathbb{F}) : \quad \Omega^3 = \{X^{i,j,k}, X^r,sY^t, X^u,vw,Y^r,x,y,z \mid 1 \leq i < j < k \leq p, 1 \leq r < s \leq p, \]
\[ 1 \leq t, u \leq p, 1 \leq v \leq w \leq p, 1 \leq x \leq y \leq z \leq p\}. \]

By the equation (2.1) we have

\[ d^1(X^i) = -X^{1,i-1}, d^1(Y^j) = -X^1Y^{j-1}, d^1(X^1) = d^1(X^2) = d^1(Y^1) = 0, \quad i \geq 3, j \geq 2, \quad (4.1) \]
\[ d^2(X^{i,i}) = 0, d^2(X^{2,3}) = 0, d^2(X^{2,j}) = -X^{1,2,j-1}, \quad i \geq 2, j \geq 4, \quad (4.2) \]
\[ d^2(X^{i,i+1}) = -X^{1,i-1,i+1}, d^2(X^{j,k}) = -X^{1,j-1,k} - X^{1,j,k-1}, \quad i, j \geq 3, k - 1 > j, \quad (4.3) \]
\[ d^2(X^{1,Y^i}) = 0, d^2(X^2Y^1) = 0, d^2(X^2Y^j) = -X^{1,2}Y^{j-1}, \quad i \geq 1, j \geq 2. \quad (4.4) \]
For restricted cohomology of $L^\lambda_{p,p}$ with coefficients in the 1-dimensional trivial module, the relevant cochain superspaces (with bases) are:

\[
\begin{align*}
C^0_*(L^\lambda_{p,p}; F) &= \Omega^0, \\
C^1_*(L^\lambda_{p,p}; F) &= \Omega^1, \\
C^2_*(L^\lambda_{p,p}; F) &= \{ (0, X^i), (\varphi, \varphi) \mid 1 \leq i \leq p, \varphi \in \Omega^2 \},
\end{align*}
\]

where $X^k\{ \sum_{i=1}^p a_i X_i \} = a^p_k, a_k \in F, \varphi : (L^\lambda_{p,p})_0 \rightarrow F$ that vanishes on the basis and has the $*$-property w.r.t. $\varphi$.

\[\text{4.3 The Cohomology } H^1(L^\lambda_{p,p}; F) \text{ and } H^2(L^\lambda_{p,p}; F)\]

In this subsection we study 1-dimensional restricted cohomology of $L^\lambda_{p,p}$ with coefficients in the 1-dimensional trivial module.

**Theorem 4.1** Let $\lambda \in \mathbb{F}^p$. Then we have

\[H^1(L^\lambda_{p,p}; F) = H^*_*(L^\lambda_{p,p}; F)\]

and the classes of $\{ X^1, X^2, Y^1 \}$ form a basis.

**Proof** It follows easily from (4.1) that $\dim(\text{Ker } d^1) = 3$ and $\{ X^1, X^2, Y^1 \}$ is a basis for this kernel. So that

\[H^1(L^\lambda_{p,p}; F) = \text{Ker } d^1 = \text{span}_{F} \{ X^1, X^2, Y^1 \}.\]

Now, let $\varphi \in H^1(L^\lambda_{p,p}; F)$ and $\varphi = a_1 X^1 + a_2 X^2 + a_3 Y^1$, where $a_1, a_2, a_3 \in F$. Then by the equation (2.2) we have

\[
\text{ind}^1(\varphi) \left( \sum_{i=1}^p b_i X_i \right) = \varphi \left( \sum_{i=1}^p (b_i X_i)^p \right) = (a_1 X^1 + a_2 X^2 + a_3 Y^1) \left( \sum_{k=1}^p b_k^p \lambda^k X_p \right) = 0,
\]

where $b_i \in F$. So $H^1(L^\lambda_{p,p}; F) \subseteq H^*_*(L^\lambda_{p,p}; F)$. The proof is complete. \qed

\[\text{4.4 The Cohomology } H^2(L^\lambda_{p,p}; F) \text{ and } H^2(L^\lambda_{p,p}; F)\]

In this subsection we study 2-dimensional restricted cohomology of $L^\lambda_{p,p}$ with coefficients in the 1-dimensional trivial module. For $i \in \{ 5, 7, \ldots, p + 2 \}$, $2 \leq j \leq p$ and $k \in \{ 2, 4, \ldots, p + 1 \}$ we put

\[
\varphi_i = \sum_{r=2}^{\lceil \frac{i}{2} \rceil} (-1)^r X^{r,i-r}, \quad \psi_j = \sum_{s=2}^{j} (-1)^s X^s Y^{j-s+1}, \quad \phi_k = \sum_{l=1}^{k-1} (-1)^l Y^l, k - t + (-1)^{\frac{k-1}{2}} Y^{\frac{k-1}{2}},
\]

where $\lfloor \frac{i}{2} \rfloor$ is the largest integer $\leq \frac{i}{2}$. Then we have the following theorem.

**Theorem 4.2** Let $\lambda \in \mathbb{F}^p$. Then we have $\dim H^2(L^\lambda_{p,p}; F) = 2p + 1$ and the cohomology classes of cocycles

\[
\Omega^4 = \{ X^1p, X^4 Y^p, \varphi_5, \varphi_7, \ldots, \varphi_{p+2}, \psi_2, \ldots, \psi_p, \phi_2, \phi_4, \ldots, \phi_{p+1} \}
\]

form a basis.
Proof. It follows easily from the equation (4.1) that \(\dim(\text{Im}\, d^2) = 2p - 3\) and

\[
\Omega^5 = \{X^{1,i}, X^{1}Y^{j} \mid 2 \leq i \leq p - 1, 1 \leq j \leq p - 1\}
\]

is a basis for this image. By the equations (4.2)–(4.7) we have

\[
d^2\varphi_i = -X^{1,2,i-3} + \sum_{r=3}^{\lfloor \frac{i}{2}\rfloor - 1} (-1)^r(-X^{1,r-1,i-r} - X^{1,r,i-r-1}) - (-1)^{\lfloor \frac{i}{2}\rfloor}X^{1,\lfloor \frac{i}{2}\rfloor - 1,i-\lfloor \frac{i}{2}\rfloor} = 0,
\]

\[
d^2\psi_j = -X^{1,2}Y^{j-2} + \sum_{s=3}^{j-1} (-1)^s(-X^{1,s-1}Y^{j-s+1} - X^{1,s}Y^{j-s}) - (-1)^{j}X^{1,j-1}Y^1 = 0,
\]

\[
d^2\phi_k = X^1Y^{1,k-2} + \sum_{t=2}^{\lfloor \frac{k-1}{2}\rfloor} (-1)^t(-X^{1}Y^{t-1,k-t} - X^{1}Y^{t,k-t-1}) - (-1)^{\lfloor \frac{k}{2}\rfloor}X^{1}Y^{\lfloor \frac{k}{2}\rfloor - 1,\lfloor \frac{k}{2}\rfloor} = 0,
\]

for \(i \in \{5,7,\ldots,p + 2\}, 2 \leq j \leq p\) and \(k \in \{2,4,\ldots,p + 1\}\). Then

\[
\Omega^6 = \{X^{1,i}, X^{1}Y^{j}, \varphi_5, \varphi_7, \ldots, \varphi_{p+2}, \psi_i, \phi_2, \phi_4, \ldots, \phi_{p+1} \mid 2 \leq i \leq p, 1 \leq j \leq p\} \subseteq \text{Ker} d^2.
\]

Let \(\varphi \in \text{Ker} d^2\). By the equations (4.2)–(4.7), we can suppose that

\[
\varphi \in \text{span}_F \{X^{i,j} \mid 1 \leq i < j \leq p\} \cup \text{span}_F \{X^kY^l \mid 1 \leq k, l \leq p\} \cup \text{span}_F \{Y^{s,t} \mid 1 \leq s \leq t \leq p\}.
\]

(1) Suppose \(\varphi \in \text{span}_F \{X^{i,j} \mid 1 \leq i < j \leq p\}\). Then it is clear that any cocycle element has to include either the basis element \(X^{1,i}\) (\(2 \leq i \leq p\)), and in this case this is a cocycle element, or it has to have one element of type \(X^{2,j}\) (\(3 \leq j \leq p\)) in the combination, and all those are a combination of \(\varphi_5, \varphi_7, \ldots, \varphi_{p+2}\).

(2) Suppose \(\varphi \in \text{span}_F \{X^kY^l \mid 1 \leq k, l \leq p\}\). Then it is clear that any cocycle element has to include either the basis element \(X^1Y^i\) (\(1 \leq i \leq p\)), and in this case this is a cocycle element, or it has to have one element of type \(X^2Y^j\) (\(1 \leq j \leq p\)) in the combination, and all those are a combination of \(\psi_2, \psi_4, \ldots, \psi_p\).

(3) Suppose \(\varphi \in \text{span}_F \{Y^{s,t} \mid 1 \leq s \leq t \leq p\}\). Then it is clear that any cocycle element has to have one element of type \(Y^{1,i}\) (\(1 \leq i \leq p\)) in the combination, and all those are a combination of \(\phi_2, \phi_4, \ldots, \phi_{p+1}\).

The linear independence of the cocycle elements in \(\Omega^6\) are clear. Then \(\text{Ker} d^2 = \text{span}_F \Omega^6\) and \(\text{dim} \text{Ker} d^2 = 4p - 2\). The proof is complete. \(\square\)

**Theorem 4.3** Let \(\lambda = 0\). Then we have \(\text{dim} H^2_\ast(L^\lambda_{p,p}, \mathbb{F}) = 3p + 1\) and the cohomology classes of cocycles

\[
\{(\varphi, \varphi_\ast), (0, X^i) \mid \varphi \in \Omega^4, 1 \leq i \leq p\}
\]

form a basis.

Proof. (1) If \(\lambda = 0\), then \(\text{ind}^2(\alpha, \beta) = 0\) for all \((\alpha, \beta) \in C^2_\ast(L^\lambda_{p,p}, \mathbb{F})\). So for that every 2-cocycle \(\varphi \in C^2(L^\lambda_{p,p}, \mathbb{F})\), we have \(d^2_\ast(\varphi, \varphi_\ast) = (d^2_\ast \varphi, \text{ind}^2(\varphi, \varphi_\ast)) = 0\), that is \((\varphi, \varphi_\ast) \in \text{Ker} d^2_\ast\), where \(\varphi_\ast : (L^\lambda_{p,p})_0 \to \mathbb{F}\) that vanishes on the basis and has the \(\ast\)-property with respect to \(\varphi\). The proof of Theorem 4.2 shows that the linearly independent subset

\[
\{(\varphi, \varphi_\ast), (0, X^i) \mid \varphi \in \Omega^6, 1 \leq i \leq p\}
\]

is a basis of \(\text{Ker} d^2_\ast\).
(2) Let \( \psi \in C^1_\lambda(L_{p,p}^\lambda, \mathbb{F}) \). Then \( d^1_\psi(\psi) = (d^1\psi, \text{ind}^1(\psi)) \), where \( \text{ind}^1(\psi)(x) = \psi(x^{[p]}) \) for all \( x \in (L_{p,p}^\lambda)_0 \). Since \( \lambda = 0 \), so \( \text{ind}^1(\psi) = \psi_* \). It follows easily from the proof of Theorem 4.2

\[
\text{Im}d^1_* = \text{span}_\mathbb{F}\{(\varphi, \varphi_*) \mid \varphi \in \Omega^5\}.
\]

The proof is complete. \( \square \)

**Theorem 4.4** Let \( \lambda \neq 0 \). Then we have \( \dim H^2_\lambda(L_{p,p}^\lambda, \mathbb{F}) = 3p - 1 \) and the cohomology classes of cocycles

\[
\{(\varphi, \varphi_*), (0, X^i_s) \mid \varphi \in \Omega^4 - \{X^{1,p}, \varphi_{p+2}\}, 1 \leq i \leq p\}
\]

form a basis.

**Proof**

1. Let \( (\varphi, \omega) \in C^2_\lambda(L_{p,p}^\lambda, \mathbb{F}) \) and

\[
\varphi = \sum_{1 \leq i < j \leq p} a_{i,j} X^{i,j} + \sum_{1 \leq k, l \leq p} b_{k,l} X^k Y^l + \sum_{1 \leq s \leq t \leq p} c_{s,t} Y^s t, \in C^2_\lambda(L_{p,p}^\lambda, \mathbb{F}),
\]

where \( a_{i,j}, b_{k,l}, c_{s,t} \in \mathbb{F} \). By the equation (2.3) we have

\[
\text{ind}^2(\varphi, \omega)(X_i, X_j) = \varphi(X_i, X_j^{[p]}) = \lambda_j a_{i,p}
\]

for all \( 1 \leq i, j \leq p \). Since \( \lambda \neq 0 \), so \( d^2_\lambda(\varphi, \omega) = (d^2 \varphi, \text{ind}^2(\varphi, \omega)) = 0 \) if and only if \( \varphi \in \text{Ker} d^2 \) and \( a_{1,p} = \cdots = a_{p-1,p} = 0 \). This observation, together with the proof of Theorem 4.2, proves the following

\[
\{(\varphi, \varphi_*), (0, X^i_s) \mid \varphi \in \Omega^6 - \{X^{1,p}, \varphi_{p+2}\}, 1 \leq i \leq p\}
\]

is a basis of \( \text{Ker} d^2 \).

2. By the equation (2.5) we have

\[
\begin{align*}
d^1_\psi(X^1) &= d^1_\psi(X^2) = (0, 0), & & d^1_\psi(X^i) = (-X^{1,i-1}, X^{(i-1)}), & & 3 \leq i \leq p - 1, \\
d^1_\psi(X^p) &= \left(-X^{1,p-1}, (X^{p-1})_s + \sum_{k=1}^p \lambda_k X^k_\lambda\right), \\
d^1_\psi(Y^1) &= (0, 0), & & d^1_\psi(Y^j) = (-X^1 Y^j, (-X^1 Y^j)_s), & & 2 \leq j \leq p.
\end{align*}
\]

Then

\[
\text{Im}d^1_* = \text{span}_\mathbb{F}\left\{(\varphi, \varphi_*), \left(X^{1,p-1}, (X^{1,p-1})_s - \sum_{k=1}^p \lambda_k X^k_{\lambda}\right) \mid \varphi \in \Omega^5 - \{X^{1,p-1}\}\right\}.
\]

The proof is complete. \( \square \)

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