Construction of equienergetic and Randić equienergetic graphs

Jahfar T K * and Chithra A V †
Department of Mathematics, National Institute of Technology Calicut, Kerala, India-673601

Abstract
In this paper, we give several constructions for the pairs of graphs to be equienergetic and Randić equienergetic graphs. Also, some new families of integral and Randić integral graphs are obtained. As an application, a sequence of graphs established with reciprocal eigenvalue property and anti-reciprocal eigenvalue property.

AMS classification: 05C50, 05C76

Keywords: Subdivision graph, equienergetic graphs, Randić equienergetic graphs, integral graphs, reciprocal eigenvalue property, anti-reciprocal eigenvalue property.

1 Introduction
In this paper, we consider simple connected graphs. Let \( G = (V, E) \) be a simple graph of order \( p \) and size \( q \) with vertex set \( V(G) = \{v_1, v_2, ..., v_p\} \) and edge set \( E(G) = \{e_1, e_2, ..., e_q\} \). The degree of a vertex \( v_i \) in \( G \) is the number of edges incident to it and is denoted by \( d_i = d_G(v_i) \). A graph \( G \) is called a regular graph, if all the vertices have the same degree. The path, complete graph and star graph on \( p \) vertices are denoted by \( P_p, K_p \) and \( K_{1,p−1} \) respectively. The adjacency matrix \( A(G) = [a_{ij}] \) of \( G \) is a square symmetric matrix of order \( p \) whose \((i,j)\)th entry is equal to one if the vertices \( v_i \) and \( v_j \) are adjacent, and is equal to zero otherwise. The eigenvalues of a graph \( G \) are defined as the eigenvalues of its adjacency matrix \( A(G) \). Denote the eigenvalues of \( A(G) \) by \( \lambda_1, \lambda_2, ..., \lambda_p \). Let \( \lambda_1, \lambda_2, ..., \lambda_t \) be the distinct eigenvalues of \( G \) with multiplicities \( m_1, m_2, ..., m_t \) respectively, then the spectrum of \( G \) is denoted by

\[
\text{spec}(G) = \left( \begin{array}{ccc} 
\lambda_1 & \lambda_2 & \ldots & \lambda_t \\
m_1 & m_2 & \ldots & m_t
\end{array} \right).
\]

The energy of a graph \( G \) was first introduced by Gutman [10] in 1978 and is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix, \( \varepsilon(G) = \sum_{i=1}^{p} |\lambda_i| \). The concept of graph energy arose in chemistry, (see [8,10]). The Randić
matrix $R(G)$ of the graph $G$ is a square matrix of order $p$ whose $(i, j)^{th}$ entry is equal to $\frac{1}{\sqrt{d_i d_j}}$ if the vertices $v_i$ and $v_j$ are adjacent, and is equal to zero otherwise. The eigenvalues of $R(G)$ are called Randić eigenvalues of $G$ and it is denoted by $\rho_i, 1 \leq i \leq p$. Let $\rho_1, \rho_2, ..., \rho_s$ be the distinct Randić eigenvalues of $G$ with multiplicities $m_1, m_2, ..., m_s$ respectively, then the Randić spectrum of $G$ is denoted by

$$RS(G) = \left(\begin{array}{cccc}
\rho_1 & \rho_2 & \cdots & \rho_s \\
m_1 & m_2 & \cdots & m_s
\end{array}\right).$$

If $G$ has no isolated vertices, then $R(G) = D^{-1/2}A(G)D^{-1/2}$ where $D$ is the diagonal matrix of vertex degrees of $G$ \[5\]. Randić energy of $G$ is defined as $\varepsilon_R(G) = \sum_{i=1}^{p} |\rho_i|$ \[5, 7, 11\]. The incidence matrix of a graph $G$, $I(G)$ is the $p \times q$ matrix whose $(i, j)^{th}$ entry is 1 if $v_i$ is incident to $e_j$ and 0 otherwise. The rank of the incidence matrix $I(G)$ is $p - 1$ if $G$ is bipartite and $p$ otherwise \[3\]. A graph $G$ is said to be integral if the eigenvalues of its adjacency matrix are all integers. Two non-isomorphic graphs are said to be cospectral if they have the same spectra, otherwise, they are known as non cospectral. Two non-isomorphic graphs $G_1$ and $G_2$ of the same order are said to be equienergetic if $\varepsilon(G_1) = \varepsilon(G_2)$ \[14\]. In analogous to equienergetic graphs, two non-isomorphic graphs of same order are said to be Randić equienergetic if they have the same Randić energy.

A graph $G$ is said to be singular (respectively, nonsingular) if $A(G)$ is singular (respectively, nonsingular). A nonsingular graph $G$ is said to have the reciprocal eigenvalue property (R) if for each eigenvalue $\lambda$ of adjacency matrix $A(G)$, its reciprocal $\frac{1}{\lambda}$ is also an eigenvalue of $A(G)$ \[4\]. Moreover if $\lambda$ and $\frac{1}{\lambda}$ has the same multiplicity, then that property is known as strong reciprocal eigenvalue property (SR). A nonsingular graph $G$ is said to have the anti-reciprocal eigenvalue property (-R) if for each eigenvalue $\lambda$ of adjacency matrix $A(G)$, its negative reciprocal $-\frac{1}{\lambda}$ is also an eigenvalue of $A(G)$ \[2\]. In addition, if $\lambda$ and $-\frac{1}{\lambda}$ have the same multiplicity, then that property is known as strong anti-reciprocal eigenvalue property (-SR) \[1\].

The rest of the paper is organized as follows. In Section 2, we give a list of some previously known results. In Section 3, we determine the energy and Randić energy of graphs obtained from a graph by other unary operations. Also, new classes of integral graphs are obtained. In Section 4, we construct new families of equienergetic and Randić equienergetic non-cospectral graphs. In Section 5, we establish some family of graphs with reciprocal eigenvalue property and anti-reciprocal eigenvalue property.

## 2 Preliminaries

In this section, we start with some definitions and terminology required for the discussions in subsequent sections.

**Definition 2.1.** \[8\] The Kronecker product of two graphs $G_1$ and $G_2$ is a graph $G_1 \times G_2$ with vertex set $V(G_1) \times V(G_2)$ and the vertices $(x_1, x_2)$ and $(y_1, y_2)$ are adjacent if and only if $(x_1, y_1)$ and $(x_2, y_2)$ are edges in $G_1$ and $G_2$ respectively.
Definition 2.2. [8] Let $A \in R^{m \times n}, B \in R^{p \times q}$. Then the Kronecker product of $A$ and $B$ is defined as follows:

$$A \otimes B = \begin{bmatrix}
a_{11}B & a_{12}B & \ldots & a_{1n}B \\
a_{21}B & a_{22}B & \ldots & a_{2n}B \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1}B & a_{m2}B & \ldots & a_{mn}B
\end{bmatrix}.$$

Proposition 2.1. [8] Let $A, B \in R^{m \times n}$. Let $\lambda$ be an eigenvalue of matrix $A$ with corresponding eigenvector $x$ and $\mu$ be an eigenvalue of matrix $B$ with corresponding eigenvector $y$, then $\lambda \mu$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector $x \otimes y$.

Definition 2.3. [8] The line graph, $L(G)$ of a graph $G$ has $E(G)$ as its vertex set and two vertices are adjacent in $L(G)$ if and only if the corresponding edges in $G$ are incident to a common vertex.

Theorem 2.1. [13] Let $G_1$ and $G_2$ be two non-cospectral $r_1$ regular graphs, $r_1 \geq 3$ with $p$ vertices. Then for any $k \geq 2$, $L^k(G_1)$ and $L^k(G_2)$ are regular non-cospectral and equienergetic graphs.

Definition 2.4. [13] The $m$-shadow graph $D_m(G)$ of a connected graph $G$ is constructed by taking $m$ copies of $G$ say, $G_1, G_2, \ldots, G_m$, then join each vertex $u$ in $G_i$ to the neighbors of the corresponding vertex $v$ in $G_j$, $1 \leq i, j \leq m$.

The adjacency matrix of $m$-shadow graph of $G$ is

$$A(D_m(G)) = \begin{bmatrix}
A(G) & A(G) & \ldots & A(G) \\
A(G) & A(G) & \ldots & A(G) \\
\vdots & \vdots & \ddots & \vdots \\
A(G) & A(G) & \ldots & A(G)
\end{bmatrix}_{m \times p}.$$

Proposition 2.2. [15] The energy of $m$-shadow graph of $G$ is, $\varepsilon(D_m(G)) = m \varepsilon(G)$.

Definition 2.5. [8] The subdivision graph of a graph $G$ is obtained by inserting new vertices between every edges of graph $G$. It is denoted by $S(G)$.

Let $G$ be a simple $(p, q)$ graph. Then the number of vertices and edges in $S(G)$ are $p + q$ and $2q$ respectively. The adjacency matrix of $S(G)$ is

$$A(S(G)) = \begin{bmatrix}
O_{p \times p} & I(G) \\
(I(G))^T & O_{q \times q}
\end{bmatrix},$$

where $O$ is null matrix and $I(G)$ is the incidence matrix of $G$. Let $r$ be the rank of $I(G)$, then the rank of $A(S(G))$ is $2r$. The non-zero eigenvalues of $S(G)$ are denoted by $\tau_1, \tau_2, \ldots, \tau_{2r}$. If $G$ is $r_1$-regular, then the eigenvalues of $S(G)$ are $\pm \sqrt{\lambda_i + r_1}, i = 1, 2, \ldots, p$ and $0$ with multiplicity $q - p$. [8] The Randić matrix of $S(G)$ is

$$R(S(G)) = \begin{bmatrix}
D^{-\frac{1}{2}} & O \\
O & (2I_q)^{-\frac{1}{2}}
\end{bmatrix} \begin{bmatrix}
O & I(G) \\
(I(G))^T & O
\end{bmatrix}\begin{bmatrix}
D^{-\frac{1}{2}} & O \\
O & (2I_q)^{-\frac{1}{2}}
\end{bmatrix}$$

$$R(S(G)) = \begin{bmatrix}
O & D^{-\frac{1}{2}}I(G)(2I_q)^{-\frac{1}{2}} \\
(2I_q)^{-\frac{1}{2}}(I(G))^TD^{-\frac{1}{2}} & O
\end{bmatrix}.$$
Note that the rank of \( R(S(G)) \) is \( 2r \), its non-zero Randić eigenvalues are denoted by \( \gamma_1, \gamma_2, \ldots, \gamma_{2r} \).

**Proposition 2.3.** \[9\] If the graph \( G \) is \( r_1 \)-regular, then \( \varepsilon_{R}(G) = \frac{\varepsilon(G)}{r_1} \).

**Definition 2.6.** \[12\] Let \( G_1 \) and \( G_2 \) be two graphs on disjoint sets of \( p_1 \) and \( p_2 \) vertices, \( q_1 \) and \( q_2 \) edges, respectively. The edge corona \( G_1 \diamond G_2 \) of \( G_1 \) and \( G_2 \) is defined as the graph obtained by taking one copy of \( G_1 \) and \( q_1 \) copies of \( G_2 \), and then joining two end vertices of the \( i^{th} \) edge of \( G_1 \) to every vertex in the \( i^{th} \) copy of \( G_2 \).

3 Randić energy of specific graphs

In this section, some new operations on \( G \) are defined, the spectrum and energy of the resultant graphs are determined. Also, we compute the Randić spectrum and Randić energy of these new graphs. Moreover, equienergetic and Randić equienergetic graphs are constructed. In addition, we provide some new families of integral graphs.

The following operation is obtained by taking \( G_2 = K_p \) in \( G_1 \diamond G_2 \) and removing the edges of \( G_1 \).

**Operation 3.1.** Let \( G \) be a simple \((p, q)\) graph with vertex set \( V(G) = \{v_1, v_2, \ldots, v_p\} \) and edge set \( E(G) = \{e_1, e_2, \ldots, e_q\} \). Corresponding to every edge \( e_i, 1 \leq i \leq q \) in \( G \), introduce a set \( U_i^k \) of \( k \) (positive integer) isolated vertices and make every vertex in \( U_i^k \) adjacent to the vertices incident with \( e_i, i = 1, 2, \ldots, q \) and remove edges of \( G \) only. The resultant graph is denoted by \( S(G)_k \).

The number of vertices and edges of the graph \( S(G)_k \) are \( p+kq \) and \( 2kq \) respectively. If \( k = 1 \), then \( S(G)_1 \) coincides with the subdivision graph \( S(G) \).

The following figure illustrates the above operation.

![Figure 1. S(K_4)_3](image)

Let \( G \) be a simple \((p, q)\) graph. Using the suitable labeling of the vertices of \( S(G)_k \), the adjacency matrix of \( S(G)_k \) is

\[
A(S(G)_k) = \begin{bmatrix}
O & I(G) & I(G) & \cdots & I(G) \\
(I(G))^T & O & O & \cdots & O \\
(I(G))^T & O & O & \cdots & O \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(I(G))^T & O & O & \cdots & O
\end{bmatrix}_{p+kq}.
\]
The degree matrix of $S(G)_k$ is
\[
D(S(G)_k) = \begin{bmatrix}
kD & O & O & \ldots & O \\
O & 2I_q & O & \ldots & O \\
O & 2I_q & O & \ldots & O \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
O & O & O & \ldots & 2I_q
\end{bmatrix}.
\]

The Randić matrix of $S(G)_k$ is
\[
R(S(G)_k) = \begin{bmatrix}
O \\
(kD)^{-\frac{1}{2}}I(G)(2I_q)^{-\frac{1}{2}} & O & \ldots & O \\
\vdots \\
(kD)^{-\frac{1}{2}}I(G)^T(kD)^{-\frac{1}{2}} & O & \ldots & O
\end{bmatrix}.
\]

The following theorem gives a relation between the eigenvalues of $S(G)$ and $S(G)_k$.

**Theorem 3.1.** Let $G$ be a simple $(p,q)$ graph and $S(G)$ be the subdivision graph of $G$. Then the spectrum of $S(G)_k$ is
\[
\text{spec}(S(G)_k) = \left( \frac{0}{p + kq - 2r} \frac{\sqrt{k}\tau_1}{1} \frac{\sqrt{k}\tau_2}{1} \ldots \frac{\sqrt{k}\tau_{2r}}{1} \right)
\]
where $\tau_i$'s, $1 \leq i \leq 2r$, are the non-zero eigenvalues of $S(G)$.

**Proof.** Let $X$ and $Y$ be column matrix with orders $p \times 1$ and $q \times 1$ respectively and let $U = \begin{bmatrix} X \\ Y \end{bmatrix}_{(p+q) \times 1}$ be the eigenvector corresponding to the non-zero eigenvalue $\tau_i$, $1 \leq i \leq 2r$, of $S(G)$. Then $A(S(G))U = \tau_i U$.

That is, \[
\begin{bmatrix} O \\ I(G)^T \end{bmatrix}_{(p+q) \times (p+q)} \begin{bmatrix} X \\ Y \end{bmatrix}_{(p+q) \times 1} = \tau_i \begin{bmatrix} X \\ Y \end{bmatrix}_{(p+q) \times 1}.
\]

This gives $I(G)Y = \tau_i X$ and $(I(G))^TX = \tau_i Y$.

Next to find the eigenvalues of $S(G)_k$.

Let $Z = \begin{bmatrix} \sqrt{k}X \\ Y \\ Y \\ \vdots \\ Y \end{bmatrix}_{(p+kq) \times 1}$ be an eigenvector corresponding to a non-zero eigenvalue
√kτ_i, 1 ≤ i ≤ 2r of S(G)_k. This is because

\[
A(S(G)_k)Z = \begin{bmatrix}
O & I(G) & I(G) & \ldots & I(G) \\
(I(G))^T & O & O & \ldots & O \\
(I(G))^T & O & O & \ldots & O \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(I(G))^T & O & O & \ldots & O \\
\end{bmatrix}
\begin{bmatrix}
\sqrt{k}X \\
Y \\
Y \\
\vdots \\
Y \\
\end{bmatrix}_{(p+kq) \times 1}
\]

= \sqrt{k}

\[
= \sqrt{k}
\]

\[
= \sqrt{k}\tau_i
\]

\[
= \sqrt{k}\tau_i Z.
\]

Thus √kτ_i’s, i = 1, 2, ..., 2r are non-zero eigenvalues of S(G)_k. Therefore p + kq – 2r eigenvalues of S(G)_k are zeros. Thus,

\[spec(S(G)_k) = \begin{pmatrix}
0 & \sqrt{k}\tau_1 & \sqrt{k}\tau_2 & \ldots & \sqrt{k}\tau_{2r} \\
p + kq & 1 & 1 & \ldots & 1 \\
\end{pmatrix}.
\]

□

Corollary 3.2. Let G be a simple (p, q) graph. Then ε(S(G)_k) = √kε(S(G)).

Corollary 3.3. Let G be r_1-regular graph. Then ε(S(G)_k) = 2√k \sum_{i=1}^{p} \sqrt{\lambda_i + r_1}.

In the following corollary, we give a method to construct a new family of integral graphs.

Corollary 3.4. Let G be a simple (p, q) graph such that S(G) is integral and k be a perfect square. Then S(G)_k is integral.

Example 3.1. Let G = K_{1,3}. Then spec(S(G)) = \begin{pmatrix}
-2 & -1 & 0 & 1 & 2 \\
1 & 2 & 1 & 2 & 1 \\
\end{pmatrix}. Thus S(G)_4, S(G)_9, S(G)_{16} etc. are integral graphs.
Theorem 3.5. Let $G$ be a simple $(p, q)$ graph and $S(G)$ be the subdivision graph of $G$. Then the Randić spectrum of $S(G)_k$ is,

$$RS(S(G)_k) = \begin{pmatrix} 0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{2r} \\ p + kq - 2r & 1 & 1 & \cdots & 1 \end{pmatrix}$$

where $\gamma_i$'s, $1 \leq i \leq 2r$ are the non-zero Randić eigenvalues of $S(G)$.

Proof. Let $M$ and $N$ be column matrix with orders $p \times 1$ and $q \times 1$ respectively and let $V = \begin{bmatrix} M \\ N \end{bmatrix}_{(p+q) \times 1}$ be the Randić eigenvector corresponding to the non-zero Randić eigenvalue $\gamma_i, 1 \leq i \leq 2r$ of $S(G)$. Then $R(S(G))V = \gamma_i V$. That is,

$$\begin{pmatrix} O \\ (2I_q)^{-\frac{1}{2}}(I(G))^T D^{-\frac{1}{2}} & O \end{pmatrix} \begin{bmatrix} M \\ N \end{bmatrix} = \gamma_i \begin{bmatrix} M \\ N \end{bmatrix}.$$

This gives $D^{-\frac{1}{2}}(I(G))^T (2I_q)^{-\frac{1}{2}} N = \gamma_i M$ and $(2I_q)^{-\frac{1}{2}} (I(G))^T D^{-\frac{1}{2}} M = \gamma_i N$.

Next to find the Randić eigenvalues of $S(G)_k$. If $V = \begin{bmatrix} M \\ N \end{bmatrix}_{(p+q) \times 1}$ is the Randić eigenvector of $S(G)$ corresponding to non-zero Randić eigenvalue $\gamma_i$, $1 \leq i \leq 2r$, then $W = \begin{bmatrix} \sqrt{k} M \\ N \\ N \\ \vdots \\ N \end{bmatrix}_{(p+kq) \times 1}$ is an eigenvector corresponding
to non-zero Randić eigenvalue $\gamma_i$ of $S(G)_k$. This is because

$$R(S(G)_k)W = \begin{bmatrix} (kD)^{-\frac{1}{2}} & 0 & 0 & \cdots & 0 \\
(2I_q)^{-\frac{1}{2}} & 0 & 0 & \cdots & 0 \\
0 & (2I_q)^{-\frac{1}{2}} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & (2I_q)^{-\frac{1}{2}}\end{bmatrix} \begin{bmatrix} O & I(G) & I(G) & \cdots & I(G) \\
(I(G))^T & O & O & \cdots & O \\
(I(G))^T & O & O & \cdots & O \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(I(G))^T & O & O & \cdots & O\end{bmatrix} \begin{bmatrix} \sqrt{kM} \\
N \\
N \\
\vdots \\
N \end{bmatrix} = \gamma_iW.$$

Thus $\gamma_i$, $i = 1, 2, \ldots, 2r$ are non-zero Randić eigenvalues of $S(G)_k$. Therefore $p+kq-2r$ Randić eigenvalues of $R(S(G)_k)$ are zeros. Thus

$$RS(S(G)_k) = \begin{bmatrix} 0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{2r} \\
p + kq - 2r & 1 & 1 & \cdots & 1 \end{bmatrix}.$$ 

\[\Box\]

In the following corollary, we give a method to construct a pair of graphs having the same Randić energy:

**Corollary 3.6.** Let $G$ be a simple $(p, q)$ graph. Then $\varepsilon_R(S(G)_k) = \varepsilon_R(S(G)).$

We shall introduce the following operations on $G$.

**Operation 3.2.** Let $G$ be a simple $(p, q)$ graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$, edge set $E(G) = \{e_1, e_2, \ldots, e_q\}$ and $S(G)$ be the subdivision graph of $G$ with vertex set $V(G) \cup E(G)$. Corresponding to each vertex $v_j, 1 \leq j \leq p$ in $S(G)$, introduce a set $V_j^n$ of $n$ isolated vertices and join each vertex of $V_j^n$ to the neighbors of $v_j$ in $S(G)$. Then corresponding to each vertex $e_j$ in $S(G)$ introduce a set of $t$ isolated vertices $E_j^t$, $1 \leq j \leq q$, where $t = n$ or $t = n - 1$, and join each vertex in $E_j^t$ to the neighbors of $e_j$, $1 \leq j \leq q$. The resultant graph is denoted by $S(G)_t$.
Note that, in the graph $S(G)_t^n$ has $(n + 1)p + (t + 1)q$ vertices and $(t + 1)(n + 1)2q$ edges.

The following figures illustrates the above operation

Figure 2. $S(P_3)_2^3$  
Figure 3. $S(K_2)_3^3$  
Figure 4. $S(P_3)_1^2$

Case 1. If $t = n$, then by proper labeling of the vertices of $S(G)_t^n$, it can be easily seen that

$$A(S(G)_t^n) = \begin{bmatrix}
O & I(G) & O & \ldots & O & I(G) \\
(I(G))^T & O & (I(G))^T & \ldots & (I(G))^T & O \\
O & I(G) & O & \ldots & O & I(G) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
O & I(G) & O & \ldots & O & I(G) \\
(I(G))^T & O & (I(G))^T & \ldots & (I(G))^T & O \\
\end{bmatrix}.$$ 

Case 2. If $t = n - 1$, then by proper labeling of the vertices of $S(G)_t^n$, it can be easily seen that

$$A(S(G)_t^n) = \begin{bmatrix}
O & I(G) & O & \ldots & I(G) & O \\
(I(G))^T & O & (I(G))^T & \ldots & O & (I(G))^T \\
O & I(G) & O & \ldots & I(G) & O \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
O & I(G) & O & \ldots & I(G) & O \\
(I(G))^T & O & (I(G))^T & \ldots & O & (I(G))^T \\
O & I(G) & O & \ldots & I(G) & O \\
\end{bmatrix}.$$ 

The degree matrix of $S(G)_t^n$ is as follows
If $t = n$, then
\[ D(S(G)^n_t) = \begin{bmatrix} tD & O & O & \ldots & O \\ O & 2nI_q & O & \ldots & O \\ O & O & tD & \ldots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \ldots & 2nI_q \end{bmatrix}. \]

If $t = n - 1$, then
\[ D(S(G)^n_t) = \begin{bmatrix} tD & O & O & \ldots & O \\ O & 2nI_q & O & \ldots & O \\ O & O & tD & \ldots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \ldots & tD \end{bmatrix}. \]

The Randić matrix of $S(G)^n_t$ is as

If $t = n$, then
\[ R(S(G)^n_t) = \begin{bmatrix} O & (tD)^{-\frac{1}{2}} I(G)^{(2nI_q)^{-\frac{1}{2}}} & O & \ldots & (tD)^{-\frac{1}{2}} I(G)^{(2nI_q)^{-\frac{1}{2}}} \\ (2nI_q)^{-\frac{1}{2}} (I(G))^T (tD)^{-\frac{1}{2}} O & (2nI_q)^{-\frac{1}{2}} (I(G))^T (tD)^{-\frac{1}{2}} O & \ldots & (2nI_q)^{-\frac{1}{2}} (I(G))^T (tD)^{-\frac{1}{2}} O \\ (2nI_q)^{-\frac{1}{2}} (I(G))^T (tD)^{-\frac{1}{2}} (tD)^{-\frac{1}{2}} I(G)^{(2nI_q)^{-\frac{1}{2}}} \ldots & \ldots & \ldots & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (2nI_q)^{-\frac{1}{2}} (I(G))^T (tD)^{-\frac{1}{2}} (tD)^{-\frac{1}{2}} I(G)^{(2nI_q)^{-\frac{1}{2}}} & \ldots & \ldots & \ldots & O \\ (2nI_q)^{-\frac{1}{2}} (I(G))^T (tD)^{-\frac{1}{2}} & \ldots & \ldots & \ldots & O \end{bmatrix}. \]

If $t = n - 1$, then
\[ R(S(G)^n_t) = \begin{bmatrix} O & (tD)^{-\frac{1}{2}} I(G)^{(2nI_q)^{-\frac{1}{2}}} & O & \ldots & O \\ (2nI_q)^{-\frac{1}{2}} (I(G))^T (tD)^{-\frac{1}{2}} O & (2nI_q)^{-\frac{1}{2}} (I(G))^T (tD)^{-\frac{1}{2}} O & \ldots & (2nI_q)^{-\frac{1}{2}} (I(G))^T (tD)^{-\frac{1}{2}} O \\ (2nI_q)^{-\frac{1}{2}} (I(G))^T (tD)^{-\frac{1}{2}} (tD)^{-\frac{1}{2}} I(G)^{(2nI_q)^{-\frac{1}{2}}} \ldots & \ldots & \ldots & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (2nI_q)^{-\frac{1}{2}} (I(G))^T (tD)^{-\frac{1}{2}} (tD)^{-\frac{1}{2}} I(G)^{(2nI_q)^{-\frac{1}{2}}} & \ldots & \ldots & \ldots & O \\ (2nI_q)^{-\frac{1}{2}} (I(G))^T (tD)^{-\frac{1}{2}} & \ldots & \ldots & \ldots & O \end{bmatrix}. \]

**Theorem 3.7.** Let $G$ be a simple $(p, q)$ graph and $S(G)$ be the subdivision graph of $G$. Then the spectrum of $S(G)^n_t$ is
\[ \text{spec}(S(G)^n_t) = \left( \begin{array}{c} 0 \\ \frac{1}{(n+1)p+(t+1)q-2r} \sqrt{(t+1)(n+1)r_1} \\ \vdots \\ 0 \end{array} \right) \]
where $\tau_i$, $1 \leq i \leq 2r$, are non-zero eigenvalues of $S(G)$.

**Proof.** Let $X$ and $Y$ be column matrix with orders $p \times 1$ and $q \times 1$ respectively and $U = \begin{bmatrix} X \\ Y \end{bmatrix}_{(p+q)\times 1}$ be the eigenvector corresponding to the non-zero eigenvalue $\tau_i$, $1 \leq i \leq 2r$, of $S(G)$. Then $A(S(G))U = \tau_i U$.

That is, 
\[ \begin{bmatrix} O & (I(G))^T \\ (I(G)) \end{bmatrix}_{(p+q)\times (p+q)} \begin{bmatrix} X \\ Y \end{bmatrix}_{(p+q)\times 1} = \tau_i \begin{bmatrix} X \\ Y \end{bmatrix}_{(p+q)\times 1}. \]

This gives $I(G)Y = \tau_i X$ and $(I(G))^T X = \tau_i Y$.

Next to find the eigenvalues of $S(G)^n_t$.

Case 1. $t = n$. 

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Let $Z_1 = \begin{bmatrix} X \\ Y \\ \vdots \\ X \\ Y \end{bmatrix}^{((n+1)p+(t+1)q) \times 1}$ be an eigenvector corresponding to the non-zero eigenvalue $(n+1)\tau_i$, $1 \leq i \leq 2r$ of $S(G)^n_t$. This is because

$$A(S(G)^n_t)Z_1 = \begin{bmatrix} O & I(G) & O & \cdots & I(G) \\ (I(G))^T & O & (I(G))^T & \cdots & O \\ O & I(G) & O & \cdots & I(G) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (I(G))^T & O & (I(G))^T & \cdots & O \end{bmatrix} \begin{bmatrix} X \\ Y \\ \vdots \\ X \\ Y \end{bmatrix} = \begin{bmatrix} (n+1)\tau_i X \\ (n+1)\tau_i Y \\ \vdots \\ (n+1)\tau_i X \\ (n+1)\tau_i Y \end{bmatrix} = (n+1)\tau_i Z_1.$$ 

Case 2. $t = n - 1$.

Let $Z_2 = \begin{bmatrix} \sqrt{(t+1)}X \\ \sqrt{(n+1)}Y \\ \vdots \\ \sqrt{(t+1)}X \\ \sqrt{(n+1)}Y \\ \sqrt{(t+1)}X \end{bmatrix}^{((n+1)p+(t+1)q) \times 1}$ be an eigenvector corresponding to the non-zero eigenvalue $(n+1)\tau_i$, $1 \leq i \leq 2r$ of $S(G)^n_t$. This is because

$$A(S(G)^n_t)Z_2 = \begin{bmatrix} O & I(G) & O & \cdots & I(G) \\ (I(G))^T & O & (I(G))^T & \cdots & O \\ O & I(G) & O & \cdots & I(G) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (I(G))^T & O & (I(G))^T & \cdots & O \end{bmatrix} \begin{bmatrix} \sqrt{(t+1)}X \\ \sqrt{(n+1)}Y \\ \vdots \\ \sqrt{(t+1)}X \\ \sqrt{(n+1)}Y \\ \sqrt{(t+1)}X \end{bmatrix} = \begin{bmatrix} (n+1)\tau_i \sqrt{(t+1)}X \\ \sqrt{(n+1)}Y \\ \vdots \\ \sqrt{(t+1)}X \\ \sqrt{(n+1)}Y \\ \sqrt{(t+1)}X \end{bmatrix} = (n+1)\tau_i Z_2.$$
zero eigenvalue $\sqrt{(n+1)(t+1)} \tau_i$, $i \leq i \leq 2r$ of $S(G)_i^n$. This is because

$$A(S(G)_i^n)Z_2 = \begin{bmatrix}
O & I(G) & O & \ldots & I(G) & O \\
(I(G))^T & O & (I(G))^T & \ldots & O & (I(G))^T \\
O & I(G) & O & \ldots & I(G) & O \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(I(G))^T & O & (I(G))^T & \ldots & O & (I(G))^T \\
O & I(G) & O & \ldots & I(G) & O
\end{bmatrix}
\begin{bmatrix}
\sqrt{(t+1)}X \\
\sqrt{(n+1)}Y \\
\vdots \\
\sqrt{(t+1)}X \\
\sqrt{(n+1)}Y \\
\sqrt{(t+1)}X
\end{bmatrix}$$

$$= \begin{bmatrix}
\sqrt{(n+1)(t+1)} X \\
\sqrt{(n+1)} Y \\
\vdots \\
\sqrt{(n+1)(t+1)} X \\
\sqrt{(n+1)} Y \\
\sqrt{(n+1)(t+1)} X
\end{bmatrix}$$

$$= \sqrt{(n+1)(t+1)} \tau_i Z_2.$$

Thus, if $\tau_i$’s, $1 \leq i \leq 2r$, are non-zero eigenvalue of $S(G)$, then $\sqrt{(n+1)(t+1)} \tau_i$ is non-zero eigenvalue of $S(G)_i^n$. Therefore $(n+1)p + (t+1)q - 2r$ eigenvalues of $S(G)_i^n$ are zeros. Thus,

$$\text{spec}(S(G)_i^n) = \begin{pmatrix}
0 \\
\sqrt{(t+1)(n+1)} \tau_1 \\
\sqrt{(t+1)(n+1)} \tau_2 \\
\vdots \\
\sqrt{(t+1)(n+1)} \tau_2r
\end{pmatrix}. \quad \Box$$

**Corollary 3.8.** The energy of $S(G)_i^n$ is, $\varepsilon(S(G)_i^n) = \sqrt{(n+1)(t+1)} \varepsilon(S(G))$.

**Example 3.2.** Let $G = K_2$. Then

$$\text{spec}(S(K_2)) = \begin{pmatrix}
-\sqrt{2} & 0 & \sqrt{2} \\
1 & 1 & 1
\end{pmatrix}.$$

If $n = 2$ and $t = 1$, then the adjacency matrix of $S(K_2)_i^n$ is,

$$A(S(K_2)_i^n) = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}.$$
\[
spec(A(S(K_2))^2_1) = \begin{pmatrix} -2\sqrt{3} & 0 & 2\sqrt{3} \\ 1 & 6 & 1 \end{pmatrix}.
\]

Then \(\varepsilon(S(K_2)^2_1) = \sqrt{6}\varepsilon(S(K_2)).\)

**Example 3.3.** Let \(G = K_{1,3}\), then
\[
spec(S(K_{1,3})) = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 2 & 1 \end{pmatrix}.
\]
\[
spec(S(K_{1,3})^1_1) = \begin{pmatrix} -4 & -2 & 0 & 2 & 4 \\ 1 & 2 & 8 & 2 & 1 \end{pmatrix}.
\]

Hence \(\varepsilon(S(K_{1,3})^1_1) = 2\varepsilon(S(K_{1,3})).\)

**Remark 3.1.** The spectrum of \(S(G)_{(t+1)(n+1)}\) is
\[
spec(S(G)_{(t+1)(n+1)}) = \begin{pmatrix} 0 \\ p+(n+1)(t+1)q-2r \\ \sqrt{(t+1)(n+1)r_1} \\ \sqrt{(t+1)(n+1)r_2} \\ \ldots \\ \sqrt{(t+1)(n+1)r_{2r}} \end{pmatrix}.
\]

The spectrum of \(S(G)^n_t\) is
\[
spec(S(G)^n_t) = \begin{pmatrix} 0 \\ (n+1)p+(t+1)q-2r \\ \sqrt{(t+1)(n+1)r_1} \\ \sqrt{(t+1)(n+1)r_2} \\ \ldots \\ \sqrt{(t+1)(n+1)r_{2r}} \end{pmatrix}.
\]

The energy of the graphs \(S(G)_{(t+1)(n+1)}\) and \(S(G)^n_t\) are same but their orders are different.

Now we obtain a new family of integral graphs as follows.

**Corollary 3.9.** If \(t = n\) and \(S(G)\) is integral, then \(S(G)^n_t\) is integral.

**Example 3.4.** Let \(G = K_{1,3}\), \(spec(S(G)) = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 2 & 1 \end{pmatrix}\). Then \(S(G)^n_t\) is integral for every \(n\).

In the following Proposition gives some pairs of cospectral graphs.

**Proposition 3.1.** Let \(G\) be a simple \((p, q)\) graph with \(p = q\), then the graphs \(S(G)^2_2\) and \(S(G)^1_1\) are cospectral graphs.

**Proof.** Proof follows from Theorems \ref{thm:cospectral1} and \ref{thm:cospectral2}.

**Example 3.5.**

Consider the graphs \(G\) (see Figure 5). Then the spectrum of \(S(G), S(G)^2_2\) and \(S(G)^1_1\) are

![Figure 5. Graph G with p = q](image)

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\[
\text{spec}(S(G)) = \begin{pmatrix}
\sqrt{\frac{5+\sqrt{17}}{2}} & \sqrt{\frac{5+\sqrt{17}}{2}} & \sqrt{2} & -\sqrt{2} & -1 & 1 & 1 \sqrt{\frac{5-\sqrt{17}}{2}} & -\sqrt{\frac{5-\sqrt{17}}{2}} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}.
\]
\[
\text{spec}(S(G)_2) = \begin{pmatrix}
\frac{0}{4} & \sqrt{\frac{5+\sqrt{17}}{2}} & -\sqrt{\frac{5+\sqrt{17}}{2}} & 2 & -2 & -\sqrt{2} & \sqrt{2} & \sqrt{5-\sqrt{17}} & -\sqrt{5-\sqrt{17}} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}.
\]
\[
\text{spec}(S(G)_1) = \begin{pmatrix}
\frac{0}{4} & \sqrt{\frac{5+\sqrt{17}}{2}} & -\sqrt{\frac{5+\sqrt{17}}{2}} & 2 & -2 & -\sqrt{2} & \sqrt{2} & \sqrt{5-\sqrt{17}} & -\sqrt{5-\sqrt{17}} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}.
\]

Remark 3.2. Let \( G \) be a simple \((p, q)\) graph and \( S(G) \) be the subdivision graph of \( G \). Then \( \varepsilon (S(G)_1) = \varepsilon (S(G)_2) \).

We shall now discuss the problem of constructing pairs of graphs having the same Randić energy.

**Theorem 3.10.** Let \( G \) be a simple \((p, q)\) graph. Then \( \varepsilon_R(S(G)_1^n) = \varepsilon_R(S(G)) \).

**Proof.** Let \( M \) and \( N \) be column matrix with orders \( p \times 1 \) and \( q \times 1 \) respectively and let \( V = \begin{bmatrix} M \\ N \end{bmatrix} \) be the Randić eigenvector corresponding to the non-zero Randić eigenvalue \( \gamma_i \), \( 1 \leq i \leq 2r \), of \( S(G) \). Then \( R(S(G)_1^n) V = \gamma_i V \).

That is,
\[
\begin{pmatrix}
O \\
(2I_q)^{-\frac{1}{2}}(I(G))^T D^{-\frac{1}{2}} \\
(2I_q)^{-\frac{1}{2}}(I(G))^T D^{-\frac{1}{2}} M & O
\end{pmatrix} = \gamma_i \begin{bmatrix} M \\ N \end{bmatrix}.
\]

This gives \( D^{-\frac{1}{2}} I(G)(2I_q)^{-\frac{1}{2}} N = \gamma_i M \) and \( (2I_q)^{-\frac{1}{2}}(I(G))^T D^{-\frac{1}{2}} M = \gamma_i N \).

Next, to find the eigenvalues of \( R(S(G)_1^n) \).

Case 1. \( t = n \).  

If \( V \) is the Randić eigenvector of \( S(G) \) corresponding to non-zero Randić eigenvalue \( \gamma_i \), \( 1 \leq i \leq 2r \), then \( W_1 = \begin{bmatrix} M \\ N \\ M \\ \vdots \\ M \\ N \end{bmatrix} \) is a Randić eigenvector corresponding to non-zero Randić eigenvalue \( \gamma_i \) of \( S(G)_1^n \). This is because
\[
R(S(G)^{n})W_{1} = \begin{bmatrix}
(nD)^{-\frac{1}{2}} & O & O & \ldots & O \\
O & (2nI_{q})^{-\frac{1}{2}} & O & \ldots & O \\
O & O & (2nI_{q})^{-\frac{1}{2}} & \ldots & O \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
O & O & O & \ldots & (2nI_{q})^{-\frac{1}{2}}
\end{bmatrix}
\cdot
\begin{bmatrix}
O & I(G) & O & \ldots & I(G) \\
(I(G))^T & O & I(G)^T & \ldots & O \\
O & I(G) & O & \ldots & I(G) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(I(G))^T & O & (I(G))^T & \ldots & O
\end{bmatrix}
\cdot
\begin{bmatrix}
M \\
N \\
M \\
N \\
\vdots \\
M \\
N
\end{bmatrix}
\]

\[
= \begin{bmatrix}
O & (nD)^{-\frac{1}{2}}I(G)(2nI_{q})^{-\frac{1}{2}} & O & \ldots & O \\
(nD)^{-\frac{1}{2}}I(G)(2nI_{q})^{-\frac{1}{2}} & O & (nD)^{-\frac{1}{2}}I(G)(2nI_{q})^{-\frac{1}{2}} & \ldots & O \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(2nI_{q})^{-\frac{1}{2}}I(G)^T(nD)^{-\frac{1}{2}} & O & (2nI_{q})^{-\frac{1}{2}}I(G)^T(nD)^{-\frac{1}{2}} & \ldots & O
\end{bmatrix}
\cdot
\begin{bmatrix}
M \\
N \\
M \\
N \\
\vdots \\
M \\
N
\end{bmatrix}
\]

\[
= \begin{bmatrix}
n(nD)^{-\frac{1}{2}}I(G)(2nI_{q})^{-\frac{1}{2}}N \\
n(2nI_{q})^{-\frac{1}{2}}I(G)^T(nD)^{-\frac{1}{2}}M \\
n(2nI_{q})^{-\frac{1}{2}}I(G)(nD)^{-\frac{1}{2}}N \\
\vdots \\
n(2nI_{q})^{-\frac{1}{2}}I(G)^T(nD)^{-\frac{1}{2}}M
\end{bmatrix}
\cdot
\begin{bmatrix}
\gamma_{i}M \\
\gamma_{i}N \\
\gamma_{i}M \\
\vdots \\
\gamma_{i}N
\end{bmatrix}
= \gamma_{i}
\begin{bmatrix}
M \\
N \\
M \\
N \\
\vdots \\
M \\
N
\end{bmatrix}
= \gamma_{i}W_{1}.
\]

Case 2. \( t = n - 1. \)
If \( V \) is the Randić eigenvector of \( S(G) \) corresponding to non-zero Randić eigenvalue \( \gamma_{i}, \)

\[
1 \leq i \leq 2r, \text{ then } W_{2} = \begin{bmatrix}
\sqrt{\gamma_{i}M} \\
\sqrt{\gamma_{i}N} \\
\sqrt{\gamma_{i}M} \\
\vdots \\
\sqrt{\gamma_{i}M} \\
\sqrt{\gamma_{i}N} \\
\sqrt{\gamma_{i}M}
\end{bmatrix}_{(n+1)p+(t+1)q\times 1}
\]
to non-zero Randić eigenvalue $\gamma_i$ of $S(G)_{t+}^n$. This is because

$$R(S(G)^n_{t+})W_2 = \begin{bmatrix} (n-1)D^{-\frac{1}{2}} & 0 & 0 & \ldots & 0 \\ 0 & (2nI_q)^{-\frac{1}{2}} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & (n-1)D^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} O & (I(G))^T & \ldots & (I(G))^T \\ O & I(G) & \ldots & I(G) \\ \vdots & \vdots & \ddots & \vdots \\ O & I(G) & \ldots & I(G) \end{bmatrix}$$

$$= \begin{bmatrix} t(tD)^{-\frac{1}{2}}I(G)(2nI_q)^{-\frac{1}{2}}N \\ n(2nI_q)^{-\frac{1}{2}}(I(G))^T(tD)^{-\frac{1}{2}}M \\ t(tD)^{-\frac{1}{2}}I(G)(2nI_q)^{-\frac{1}{2}}N \\ n(2nI_q)^{-\frac{1}{2}}(I(G))^T(tD)^{-\frac{1}{2}}M \end{bmatrix} = \begin{bmatrix} \gamma_i\sqrt{tM} \\ \gamma_i\sqrt{nN} \\ \gamma_i\sqrt{tM} \\ \gamma_i\sqrt{nN} \end{bmatrix} = \gamma_iW_2.$$

Thus, if $\gamma_i$'s, $1 \leq i \leq 2r$, are non-zero Randić eigenvalue of $S(G)$, then $\gamma_i$ is also non-zero Randić eigenvalue of $S(G)_{t+}^n$. Therefore $(n+1)p + (t+1)q - 2r$ Randić eigenvalues $S(G)_{t+}^n$ are zeros. Thus

$$RS(S(G)_{t+}^n) = \begin{pmatrix} 0 & \gamma_1 & \gamma_2 & \ldots & \gamma_{2r} \\ (n+1)p + (t+1)q - 2r & 1 & 1 & \ldots & 1 \end{pmatrix}.$$

Next we give the construction of Randić equienergetic graphs by means of graphs $S(G)_k$ and $S(G)_{t+}^n$.

**Theorem 3.11.** Let $G$ be a simple $(p,q)$ graph and $S(G)$ be the subdivision graph of $G$. Then $S(G)_k$ and $S(G)_{t+}^n$ are Randić equienergetic whenever the order of $S(G)_k$ and $S(G)_{t+}^n$ are equal.

**Proof.** Proof follows from Corollary 3.6 and Theorem 3.10. ☐
Note. Let $G$ be a simple $(p, q)$ graph.

(1) If $p = q$, then the order of $S(G)_k$ and $S(G)_t^n$ are equal if and only if $k = n + t + 1$.
(2) Order of $S(G)_k$ and $S(G)_t^n$ are equal if and only if $p + kq = (t + 1)q + (n + 1)p$ if and only if either $k = n\left(\frac{p+q}{2}\right) + 1$ or $k = n\left(\frac{p+q}{2}\right)$.

From the following theorem, we construct an infinite family of Randi{c} cospectral graphs.

**Proposition 3.2.** Let $G$ be a simple $(p, q)$ graph, $G_1 = S(G)_k$ and $G_2 = S(G)_t^n$ with $k = n^2 + t + 1$. Then $G_1$ and $G_2$ are Randi{c} cospectral.

**Proof.** By Theorem 3.5 and proof of Theorem 3.10, we get $RS(G_1) = RS(G_2)$.

**Example 3.6.** Let $G = P_3$, then

$$RS(S(P_3)) = \begin{pmatrix} -1 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$RS(S(P_3)_2^2) = \begin{pmatrix} -1 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

and

$$RS(S(P_3)_6) = \begin{pmatrix} -1 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$}

Thus $RS(S(P_3)_2^2) = RS(S(P_3)_6)$. Note that $S(P_3)_2^2$ and $S(P_3)_6$ are non-isomorphic. Thus the graphs $S(P_3)_2^2$ and $S(P_3)_6$ are non-isomorphic Randi{c} cospectral graphs.

**Note.** The family of graphs $S(K_2)^{n}$ is Randi{c} integral for every $t$ and $n$ with $t = n$ or $t = n - 1$.

## 4 Equienergetic and Randi{c} equienergetic graphs

In this section, we obtain an infinite family of equienergetic and Randi{c} equienergetic non-cospectral graphs.

**Operation 4.1.** Let $G$ be a simple $(p, q)$ graph, $D_m(G), m > 3$ be the $m$-shadow graph of $G$ and $G_1, G_2, ..., G_m$ be the $m$ copies of $G$ in $D_m(G)$. The graph $F_1^m(G)$ is obtained by deleting all edges connecting the vertices of $G_i$ and $G_{m-(i-1)}, 2 \leq i \leq m - 1$ and removing the edges of $G_1$.

The number of vertices and edges in $F_1^m(G)$ are $pm$ and $(m^2 - m + 1)q$ respectively.

**Theorem 4.1.** The energy of the graph $F_1^m(G)$ is, $\varepsilon(F_1^m(G)) = \left[m - 2 + \sqrt{m^2 - 2m + 5}\right] \varepsilon(G)$.

**Proof.** With the suitable labeling of the vertices, the adjacency matrix of $F_1^m(G)$ is

$$A(F_1^m(G)) = \begin{bmatrix} O & A(G) & A(G) & \ldots & A(G) & A(G) \\ A(G) & A(G) & A(G) & \ldots & O & A(G) \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ A(G) & A(G) & O & \ldots & A(G) & A(G) \\ A(G) & O & A(G) & \ldots & A(G) & A(G) \\ A(G) & A(G) & A(G) & \ldots & A(G) & A(G) \end{bmatrix}_{pm}$$
Let $X^* = \begin{bmatrix} (m-3) & \sqrt{m^2-2m+5} \\ (m-3) & \sqrt{m^2-2m+5} \\ \vdots \\ (m-3) & \sqrt{m^2-2m+5} \\ 1 \end{bmatrix}^{m \times 1}$, then $BX^* = \begin{bmatrix} (m-1)+\sqrt{m^2-2m+5} \\ 1 \end{bmatrix} \cdot X^*$ and 

$Y^* = \begin{bmatrix} (m-3)-\sqrt{m^2-2m+5} \\ (m-3)-\sqrt{m^2-2m+5} \\ \vdots \\ (m-3)-\sqrt{m^2-2m+5} \\ 1 \end{bmatrix}^{m \times 1}$, then $BY^* = \begin{bmatrix} (m-1)-\sqrt{m^2-2m+5} \\ 1 \end{bmatrix} \cdot Y^*$.

Case 1. $m$ is even.

Let $E_i = \begin{bmatrix} -2 \\ e_i \end{bmatrix}_{m \times 1}$, $1 \leq i \leq \frac{m-2}{2}$, where $e_i$ is the column vector having $i^{th}$ entry and $(m-1-i)^{th}$ entry one and all other entries zeros. Then $BE_i = -E_i$. Let $E_i^* = \begin{bmatrix} 0 \\ e_i^* \end{bmatrix}_{m \times 1}$, $1 \leq i \leq \frac{m-2}{2}$ where $e_i^*$ is the column vector having $i^{th}$ entry $-1$, $(m-1-i)^{th}$ entry 1 and all other entries zeros. Then $BE_i^* = E_i^*$.

$$\text{spec}(B) = \begin{bmatrix} (m-1)+\sqrt{m^2-2m+5} & (m-1)-\sqrt{m^2-2m+5} \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} m-2 \\ m-2 \end{bmatrix}$$.

Hence

$$\text{spec}(F_1^m(G)) = \begin{bmatrix} (m-1)+\sqrt{m^2-2m+5} & (m-1)-\sqrt{m^2-2m+5} & -\lambda_i & \lambda_i \\ 2 & 1 & m-2 & m-2 \end{bmatrix} \begin{bmatrix} \lambda_i \\ \lambda_i \\ m-2 \\ m-2 \end{bmatrix}, 1 \leq i \leq p.$$

Case 2. $m$ is odd.

Let $T = \begin{bmatrix} -1 \\ e_{\frac{m-1}{2}} \end{bmatrix}_{m \times 1}$, where $e_{\frac{m-1}{2}}$ is the column vector having $\left(\frac{m-1}{2}\right)^{th}$ entry one and
all other entries zeros. Then $BT = -T$.

Let $T^*_i = \begin{bmatrix} -2 \\ e_i \end{bmatrix}_{m \times 1}$, $1 \leq i \leq \frac{m-3}{2}$ where $e_i$ as in the case 1. Then $BT^*_i = -T^*_i$.

Let $T^{**}_i = \begin{bmatrix} 0 \\ e^*_i \end{bmatrix}_{m \times 1}$, $1 \leq i \leq \frac{m-3}{2}$ where $e^*_i$ as in the case 1. Then $BT^{**}_i = T^{**}_i$.

So the spectrum of $B$ is,

$$\text{spec}(B) = \begin{pmatrix} \frac{(m-1) + \sqrt{m^2 - 2m + 5}}{2} & \frac{(m-1) - \sqrt{m^2 - 2m + 5}}{2} & -1 \\ \frac{(m-1) + \sqrt{m^2 - 2m + 5}}{2} & \frac{(m-1) - \sqrt{m^2 - 2m + 5}}{2} & \frac{1}{m-1} \frac{1}{m-3} \end{pmatrix}.$$ 

Hence

$$\text{spec}(F^m_1(G)) = \begin{pmatrix} \frac{(m-1) + \sqrt{m^2 - 2m + 5}}{2} \lambda_i & \frac{(m-1) - \sqrt{m^2 - 2m + 5}}{2} \lambda_i & -\lambda_i \\ \frac{(m-1) + \sqrt{m^2 - 2m + 5}}{2} \lambda_i & \frac{(m-1) - \sqrt{m^2 - 2m + 5}}{2} \lambda_i & \frac{1}{m-1} \frac{1}{m-3} \lambda_i \end{pmatrix}, 1 \leq i \leq p.$$ 

Thus the energy of $F^m_1(G)$ is,

$$\varepsilon(F^m_1(G)) = \left[ m - 2 + \sqrt{m^2 - 2m + 5} \right] \varepsilon(G).$$

The next theorem gives a relation between the Randić energy of $F^m_1(G)$ and Randić energy of $G$.

**Theorem 4.2.** The Randić energy of the graph $F^m_1(G)$ is, $\varepsilon_R(F^m_1(G)) = \varepsilon_R(G) + \frac{(m-1)\varepsilon_R(G)}{m}$.

**Proof.** The Randić matrix of $F^m_1(G)$ is

$$R(F^m_1(G)) = \begin{pmatrix} \left((m-1)D\right)^{-\frac{1}{2}} & O & \ldots & O & O \\ O & \left((m-1)D\right)^{-\frac{1}{2}} & \ldots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \ldots & \left((m-1)D\right)^{-\frac{1}{2}} & O \\ O & O & \ldots & O & \left((m-1)D\right)^{-\frac{1}{2}} \end{pmatrix}_{pm} \begin{pmatrix} O & A(G) & A(G) & \ldots & A(G) & A(G) \\ A(G) & A(G) & A(G) & \ldots & O & A(G) \\ A(G) & A(G) & A(G) & \ldots & \vdots & \vdots \\ A(G) & A(G) & A(G) & \ldots & \vdots & \vdots \\ A(G) & A(G) & A(G) & \ldots & A(G) & A(G) \end{pmatrix}_{pm}$$

$$= \begin{pmatrix} \left((m-1)D\right)^{-\frac{1}{2}} & O & \ldots & O & O \\ O & \left((m-1)D\right)^{-\frac{1}{2}} & \ldots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \ldots & \left((m-1)D\right)^{-\frac{1}{2}} & O \\ O & O & \ldots & O & \left((m-1)D\right)^{-\frac{1}{2}} \end{pmatrix}_{pm}$$
Let \( \text{Case 2.} \)

Then \( \text{Operation 4.2.} \)
The number of vertices and edges in $F_m^2(G)$ are $pm$ and $(m^2 - m + 1)q$ respectively.

**Theorem 4.3.** The energy of the graph $F_m^2(G)$ is, $\varepsilon(F_m^2(G)) = \left[ m - 2 + \sqrt{m^2 - 2m + 5} \right] \varepsilon(G)$.

**Proof.** With the suitable labeling of the vertices, the adjacency matrix of $F_m^2(G)$ is

$$A(F_m^2(G)) = \begin{bmatrix} A(G) & A(G) & \ldots & A(G) \\ A(G) & O & \ldots & A(G) \\ \vdots & \vdots & \ddots & \vdots \\ A(G) & A(G) & \ldots & O \end{bmatrix}_{pm}$$

$$= \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 1 \end{bmatrix} \otimes A(G)$$

$$= H \otimes A(G), \text{ where } H = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 1 \end{bmatrix}_m$$

Let $P = \begin{bmatrix} (3-m)+\sqrt{m^2-2m+5} \\ 2 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{m \times 1}$ and $Q = \begin{bmatrix} (3-m)-\sqrt{m^2-2m+5} \\ 2 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{m \times 1}$, then $HP = \begin{bmatrix} (m-1)+\sqrt{m^2-2m+5} \end{bmatrix} P$

and $HQ = \begin{bmatrix} (m-1)-\sqrt{m^2-2m+5} \end{bmatrix} Q$.

Let $F_i = \begin{bmatrix} 0 \\ -1 \\ f_i \end{bmatrix}_{m \times 1}, 1 \leq i \leq m-2$, where $f_i$ is the column vector having $i^{th}$ entry one and all other entries zeros. Then $HF_i = -F_i$. So the simple eigenvalues of $H$ are $\frac{(m-1)+\sqrt{m^2-2m+5}}{2}, \frac{(m-1)-\sqrt{m^2-2m+5}}{2}, \text{ and } -1$ with multiplicity $m-2$ times. Hence

$$\text{spec}(F_m^2(G)) = \begin{bmatrix} \frac{(m-1)+\sqrt{m^2-2m+5}}{2} \\ \lambda_i \\ \frac{(m-1)-\sqrt{m^2-2m+5}}{2} \end{bmatrix}_{m-2}, 1 \leq i \leq p.$$ 

Thus we have $\varepsilon(F_m^2(G)) = \left[ m - 2 + \sqrt{m^2 - 2m + 5} \right] \varepsilon(G)$. \hfill $\square$

**Theorem 4.4.** The Randić energy of the graph $F_m^2(G)$ is, $\varepsilon_R(F_m^2(G)) = \varepsilon_R(G) + \frac{(m-1)\varepsilon_R(G)}{m}$.
Proof. The Randić matrix of $F_2^m(G)$ is

$$R(F_2^m(G)) = \begin{bmatrix} (tD)^{-\frac{1}{2}} & O & \cdots & O \\ O & ((m-1)D)^{-\frac{1}{2}} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & ((m-1)D)^{-\frac{1}{2}} \end{bmatrix}_{pm} \begin{bmatrix} A(G) & A(G) & \cdots & A(G) \\ A(G) & A(G) & \cdots & A(G) \\ \vdots & \vdots & \ddots & \vdots \\ A(G) & A(G) & \cdots & O \end{bmatrix}_{pm}$$

$$= \begin{bmatrix} \frac{1}{m} & \sqrt{\frac{1}{m(m-1)}} & \cdots & \sqrt{\frac{1}{m(m-1)}} \\ \frac{1}{m} & \frac{1}{m-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m} & \frac{1}{m-1} & \cdots & 0 \end{bmatrix}_m \otimes D^{-\frac{1}{2}}A(G)D^{-\frac{1}{2}} \quad (4.0.1)$$

$$= L \otimes D^{-\frac{1}{2}}A(G)D^{-\frac{1}{2}}, \text{ where } L = \begin{bmatrix} \frac{1}{m} & \sqrt{\frac{1}{m(m-1)}} & \cdots & \sqrt{\frac{1}{m(m-1)}} \\ \frac{1}{m} & \frac{1}{m-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m} & \frac{1}{m-1} & \cdots & 0 \end{bmatrix}_m.$$

Let $P^* = \begin{bmatrix} \sqrt{\frac{m}{m-1}} \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{m \times 1}$ and $Q^* = \begin{bmatrix} -(m-1)\sqrt{\frac{m-1}{m}} \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{m \times 1}$, then $LP^* = 1P^*$ and $LQ^* = -\frac{1}{m(m-1)}Q^*$. Let $F_i$ as in Theorem 4.3. Then $LF_i = -\frac{1}{m-1}F_i$. So the simple eigenvalues of $L$ are $1, -\frac{1}{m-1}$ and $-\frac{1}{m-1}$ with multiplicity $m-2$ times. Thus

$$RS(F_2^m(G)) = \begin{pmatrix} \rho_i & \frac{1}{m(m-1)} \rho_i & \frac{1}{m-1} \rho_i \\ 1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 1 \end{pmatrix}_{m \times 1}, 1 \leq i \leq p.$$ 

Hence we get $\varepsilon_R(F_2^m(G)) = \varepsilon_R(G) + \frac{(m-1)\varepsilon_R(G)}{m}$.

From the following Propositions we can construct a family of graphs which are both equienergetic and Randić equienergetic non-cospectral graphs.

**Proposition 4.1.** Let $G$ be a simple $(p,q)$ graph. Then the graphs $F_1^m(G)$ and $F_2^m(G)$ are equienergetic non-cospectral graphs for every $m$.

*Proof. Proof follows from Theorems 4.1 and 4.3.*

**Proposition 4.2.** Let $G$ be a simple $(p,q)$ graph. Then the graphs $F_1^m(G)$ and $F_2^m(G)$ are Randić equienergetic non-cospectral graphs for every $m$.

*Proof. Proof follows from Theorems 4.1 and 4.3.*
Proof. Proof follows from Theorems 4.2 and 4.4.

The following Proposition gives an infinite family of Randić equienergetic graphs from a given pair of Randić equienergetic regular graphs.

**Proposition 4.3.** Let $G_1$ and $G_2$ be two $r_1$-regular Randić equienergetic graphs non-cospectral graphs, then the $k^{th}$ iterated line graphs $L^k(G_1)$ and $L^k(G_2)$ are Randić equienergetic non-cospectral graphs.

**Proof.** Proof follows from Theorem 2.1 and Proposition 2.3.

Proposition 4.4 gives an infinite family of Randić equienergetic graphs from a given pair of Randić equienergetic graphs.

**Proposition 4.4.** Let $G_1$ and $G_2$ be two Randić equienergetic non-cospectral graphs, and $H$ be any graph, then the graphs $H \times G_1$ and $H \times G_2$ are Randić equienergetic non-cospectral graphs.

**Proof.** If $G_1$ and $G_2$ are Randić equienergetic, so are $H \times G_1$ and $H \times G_2$ for any graph $H$, since $\varepsilon_R(H \times G_1) = \varepsilon_R(H).\varepsilon_R(G_1) = \varepsilon_R(H).\varepsilon_R(G_2) = \varepsilon_R(H \times G_2)$ and the resulting graphs have the same order. Also if $G_1$ and $G_2$ are non-cospectral, then $H \times G_1$ and $H \times G_2$ are non-cospectral.

5 Application

In this section, we construct some sequence of graphs satisfying the property (R) (respectively, (SR), (-R), (-SR)). Also, we obtain some class of graphs with property (R) but not (SR).

The following theorem helps us to construct graphs which satisfies the property (-R) but not (-SR).

**Theorem 5.1.** Let $G$ be a graph satisfying the property (SR) and $m$ be an odd positive integer. Then $F^{m}_1(G)$ satisfies the property (-R) but not (-SR).

**Proof.** Let $\lambda \in \text{spec}(G)$ and $m$ be an odd positive integer. Then

$$\text{spec}(F^{m}_1(G)) = \left(\frac{(m-1)+\sqrt{m^2-2m+5}}{2} \lambda, \frac{(m-1)-\sqrt{m^2-2m+5}}{2} \lambda, -\frac{\lambda}{2}, -\frac{\lambda}{2} \right).$$

Since $G$ satisfies the property (SR), corresponding to each eigenvalue $\lambda$ of $G$, $\frac{1}{\lambda}$ is also an eigenvalue of $G$ with the same multiplicity. Let $\alpha = \frac{(m-1)+\sqrt{m^2-2m+5}}{2} \lambda$, then

$$-\frac{2}{\alpha} \left(\frac{2}{(m-1)+\sqrt{m^2-2m+5}} \right) = \frac{(m-1)-\sqrt{m^2-2m+5}}{2} \left(\frac{1}{\lambda}\right) \in \text{spec}(F^{m}_1(G))$$

as $G$ satisfies the property (SR).

Similarly for $\beta = \frac{(m-1)-\sqrt{m^2-2m+5}}{2} \lambda$, $-\frac{1}{\beta} \left(\frac{2}{(m-1)-\sqrt{m^2-2m+5}} \right) = \frac{(m-1)+\sqrt{m^2-2m+5}}{2} \left(\frac{1}{\lambda}\right) \in \text{spec}(F^{m}_1(G))$. Also $\frac{1}{\lambda} \in \text{spec}(F^{m}_1(G))$ and $-\frac{1}{\lambda} \in \text{spec}(F^{m}_1(G))$ as $G$ satisfies the property (SR). Thus $F^{m}_1(G)$ satisfies the property (-R). Since $\lambda$ and $-\lambda$ has different multiplicity in $F^{m}_1(G)$, we have $\lambda$ and $-\frac{1}{\lambda}$ has different multiplicity in $F^{m}_1(G)$. Thus $F^{m}_1(G)$ satisfies the property (-R) but not (-SR).
The eigenvalues given in each examples are decimal approximations calculated by the help of mathlab.

Example 5.1. Let $G$ be a graph in Figure 6.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6}
\caption{G}
\end{figure}

Then

$$\text{spec}(G) = \left( \frac{1}{2} \begin{array}{cccccc}
1 & -3 \pm \sqrt{3} \\
1+\sqrt{33} \pm \sqrt{18+2\sqrt{33}} & 4 \\
1-\sqrt{33} \pm \sqrt{18-2\sqrt{33}} & 4
\end{array} \right).$$

$$\text{spec}(F_5^1(G))$$

\[
\begin{pmatrix}
1 & -1 & 12.8935 & -0.0776 & -11.0902 & 0.0902 & -7.7268 & 0.1294 \\
2 & 4 & 1 & 1 & 1 & 1 & 1 & 1 \\
4.2361 & -0.2361 & -2.3223 & 0.4306 & -3.0437 & 0.3285 & 3.0437 & -0.3285 \\
2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 \\
-1.6180 & 0.6180 & -2.6180 & 0.3820 & 2.6180 & -0.3820 & 1.3917 & -0.7185 \\
1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 \\
-1.8241 & 0.5482 & -0.5482 & 1.8241 \\
1 & 2 & 1 & 2
\end{pmatrix}
\]

Note that $G$ in Figure 6 satisfy the property (SR). The graph $F_5^1(G)$ satisfies the property (-R) but not (-SR).

Theorem 5.2. Let $G$ be a graph satisfying the property (-SR) and $m$ be an odd positive integer. Then the graph $F_m^1(G)$ satisfies the property (R) but not (SR).

Proof. Proof is similar to Theorem 5.1.

The following theorem helps us to construct a new family of graphs which satisfies the property (-SR).
Theorem 5.3. Let $m$ be an even positive integer. Then $G$ has the property (SR) if and only if $F_1^m(G)$ has the property (-SR).

Proof. Let $\lambda \in \text{spec}(G)$ and $m$ be an even positive integer. Then

$$\text{spec}(F_1^m(G)) = \left(\frac{(m-1)+\sqrt{m^2-2m+5}}{2} \lambda, \frac{(m-1)-\sqrt{m^2-2m+5}}{2} \lambda, -\frac{m-2}{2} \frac{\lambda}{m-2}\right).$$

Since $G$ satisfies the property (SR), corresponding to each eigenvalue $\lambda$ of $G$, $\lambda$ is also an eigenvalue of $G$ with the same multiplicity. Let $\alpha = \frac{(m-1)+\sqrt{m^2-2m+5}}{2} \lambda$, then

$$-\frac{1}{\alpha} = \frac{(m-1)-\sqrt{m^2-2m+5}}{(m-1)+\sqrt{m^2-2m+5}} \frac{1}{\lambda} = \frac{(m-1)-\sqrt{m^2-2m+5}}{2} \lambda \in \text{spec}(F_1^m(G))$$

as $G$ satisfies the property (SR). Similarly for $\beta = \frac{(m-1)-\sqrt{m^2-2m+5}}{2} \lambda$, $\frac{1}{\beta} = \frac{(m-1)+\sqrt{m^2-2m+5}}{2} \lambda = \frac{(m-1)-\sqrt{m^2-2m+5}}{2} \lambda \in \text{spec}(F_1^m(G))$. Also $\frac{1}{\lambda} \in \text{spec}(F_1^m(G))$ and $-\frac{1}{\lambda} \in \text{spec}(F_1^m(G))$ as $G$ satisfies the property (SR).

Thus $F_1^m(G)$ satisfies the property (-R). Note that multiplicity of each eigenvalue and its negative reciprocal in $F_1^m(G)$ are same. Thus $F_1^m(G)$ satisfies the property (-SR). Converse also follows by similar arguments.

Example 5.2. Let $G$ be a graph in Figure 6. Then $\text{spec}(F_1^4(G))$

$$= \left(\begin{array}{cccccc}
1 & -1 & 10.0528 & -0.0995 & -8.6468 & 0.1156 & -6.0244 & 0.166 \\
2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\
3.0437 & -0.3285 & -3.0437 & 0.3285 & 2.6180 & -0.3820 & 1.8241 & -0.5482 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1.8241 & 0.5482 & 3.3028 & -0.3028 & 2.6180 & -0.3820 & -1.2615 & 0.7927 \\
1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 \\
1.0851 & -0.92157 & -1.8241 & 0.5482 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right).$$

Note that $F_1^4(G)$ satisfies property (-SR).

Theorem 5.4. Let $m$ be an even positive integer. Then $G$ has the property (-SR) if and only if $F_1^m(G)$ has the property (SR).

Proof. Proof is similar to Theorem 5.3.

Example 5.3.

$$\text{spec}(K_3 \circ K_1) = \left(\begin{array}{cccc}
2.4142 & -0.4142 & -1.6180 & 0.6180 \\
1 & 1 & 2 & 2 \\
\end{array}\right).$$
Figure 7. \(K_3 \circ K_1\)

Figure 8. \(F_1^4(K_3 \circ K_1)\)

\[
\text{spec}(F_1^4(K_3 \circ K_1)) = \begin{pmatrix}
2.4142 & 0.4142 & -5.3440 & -0.1871 & -2.4142 & -0.4142 \\
1 & 1 & 2 & 2 & 1 & 1 \\
-1.3681 & -0.7310 & 0.1254 & 7.9736 & 2.0412 & 0.4899 \\
1 & 1 & 1 & 1 & 2 & 2 \\
1.6180 & 0.6180 & -1.6180 & -0.6180 & & & \\
2 & 2 & 2 & 2 & & &
\end{pmatrix}.
\]

Note that \(K_3 \circ K_1\) satisfy the property (-SR), but the graph \(F_1^4(K_3 \circ K_1)\) satisfies the property (SR).

The following theorem helps us to construct a sequence of graphs satisfies the property (R).

**Theorem 5.5.** Let \(G\) be a graph satisfying the property (R). Then \(F_2^m(G)\) satisfies the property (R) if and only if \(G\) is bipartite.

**Proof.** Let \(\lambda \in \text{spec}(G)\). Then

\[
\text{spec}(F_2^m(G)) = \begin{pmatrix}
\frac{(m-1)+\sqrt{m^2-2m+5}}{2} & \frac{(m-1)-\sqrt{m^2-2m+5}}{2} & -\lambda \\
\frac{(m-1)+\sqrt{m^2-2m+5}}{2} & \frac{(m-1)-\sqrt{m^2-2m+5}}{2} & -\lambda \\
\end{pmatrix}.
\]

Since \(G\) is bipartite and satisfies the property (R), corresponding to each eigenvalue \(\lambda\) of \(G\), \(-\lambda, \frac{1}{\lambda}, -\frac{1}{\lambda}\) are also an eigenvalue of \(G\). Let \(\alpha = \frac{(m-1)+\sqrt{m^2-2m+5}}{2}\), then \(\frac{1}{\alpha} = \frac{2}{(m-1)+\sqrt{m^2-2m+5}}\lambda\) and \(\frac{1}{\beta} = \frac{2}{(m-1)-\sqrt{m^2-2m+5}}\lambda\) are also eigenvalues of \(G\). Thus \(F_2^m(G)\) satisfies the property (R).
Conversely assume that $F_m^2(G)$ satisfies the property (R). Then $\frac{(m-1)+\sqrt{m^2-2m+5}}{2} \lambda \in \text{spec}(F_m^2(G))$ implies that $\frac{(m-1)-\sqrt{m^2-2m+5}}{2} (-\frac{1}{\lambda}) \in \text{spec}(F_m^2(G))$. From $\text{spec}(F_m^2(G))$, we get $\frac{1}{\lambda} \in \text{spec}(G)$. By property (R) of $G$, $-\lambda \in \text{spec}(G)$. Thus $\lambda$ and $-\lambda$ are eigenvalues of $G$. Therefore, $G$ is bipartite. 

**Example 5.4.**

$$\text{spec}(P_4) = \begin{pmatrix} -1.6180 & -0.6180 & 1.6180 & 0.6180 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\text{spec}(F_2^3(P_4)) = \begin{pmatrix} -3.9063 & -0.2560 & 3.9063 & 0.2560 & 1.4921 & 0.6702 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ -1.4921 & -0.6702 & 1.6180 & 0.6180 & -1.6180 & -0.6180 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$ 

Note that, $P_4$ is bipartite and satisfying property(R) and $F_2^3(P_4)$ satisfying the property (R).

The following example illustrates that if $G$ is non-bipartite, then $F_m^2(G)$ need not satisfies the property (R).

**Example 5.5.** Let $H$ be a graph in Figure 9.

![Figure 9. H](image)

Then

$$\text{spec}(H) = \begin{pmatrix} 3.4081 & 0.2934 & -2.2589 & -0.4427 & 1.6180 & 0.6180 & -1.6180 & -0.6180 \\ 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 \end{pmatrix}$$

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Note that $H$ is non-bipartite and satisfies the property (R) but $F^3_2(H)$ doesn’t satisfy the property (R).

The following theorem helps us to construct a sequence of graphs satisfies the property (-SR).

**Theorem 5.6.** Let $G$ be a graph satisfying the property (-SR). Then $F^m_2(G)$ satisfies the property (-SR) if and only if $G$ is bipartite.

**Proof.** By the same argument as in Theorem 5.5. \qed

# 6 Conclusion

The concept of Randić equienergetic graphs is analogous to the concept of equienergetic graphs. We provide some new methods for constructing families of equienergetic and Randić equienergetic graphs based on $S(G)_k$ and $S(G)_n$. Based on these graphs, we also construct some new families of integral graphs. In addition, some new families of equienergetic and Randić equienergetic graphs are obtained by using the graphs $F^m_1(G)$ and $F^m_2(G)$. Moreover, a sequence of graphs with reciprocal eigenvalue property and anti-reciprocal eigenvalue property are established.

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