Wei–Norman equations for a unitary evolution

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Abstract
The Wei–Norman technique allows one to express the solution of a system of linear non-autonomous differential equations in terms of product of exponentials with time-dependent exponents being solutions of a system of nonlinear differential equations. We show that in the unitary case, i.e. when the solution of the linear system is given by a unitary evolution operator, the nonlinear system, by an appropriate choice of ordering, can be reduced to a hierarchy of matrix Riccati equations. To this end, we consider a general linear non-autonomous dynamical system on the special linear group $\text{SL}(N, \mathbb{C})$. The unitary case, of particular significance for quantum optimal control problems, is then obtained by restriction to anti-Hermitian generators. We also point to the connections of the obtained results with the theory of the so-called Lie systems.

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1. Introduction

In two papers [1] and [2] written nearly 50 years ago, Wei and Norman developed a method for solving systems of linear differential equations with variable coefficients based on Lie-group techniques. As is well known, solutions of non-autonomous linear systems of differential equations can be expressed in terms of time-ordered exponentials. From this point of view, one can treat Wei–Norman formulae as a way to calculate such exponentials.

Since then the Wei–Norman method has found numerous applications in control and system theory, see, e.g. [3–10], as well as a basis for numerical approximation methods of integrating linear non-autonomous systems [11–13].

In quantum mechanics in general (see e.g. [14–16]), and in recently rapidly developing theory of quantum optimal control in particular, (see e.g. [17–22]), of special importance is the case when the underlying Lie group is the special unitary group $\text{SU}(N)$, since the pure quantum evolution preserves the norm of a state vector.
We show that in the case of the special linear group, $\text{SL}(N, \mathbb{C})$, the Wei–Norman method leads to a hierarchy of matrix Riccati equations, provided that a proper ordering of generators (a basis of the Lie algebra) is chosen. Our findings hinge substantially on the structure of the corresponding Lie algebra, $\mathfrak{sl}(N, \mathbb{C})$, in particular on the properties of its commutative ideals. The case of the unitary group, $\text{SU}(N)$, relevant for quantum mechanical applications is recovered by imposing additional restrictions on generators of the evolution to make them anti-Hermitian, i.e. belonging to the Lie algebra $\mathfrak{su}(N)$, for which $\mathfrak{sl}(N, \mathbb{C})$ is the complexification.

In the following section, we present briefly the basics of the Wei–Norman method for general complex semisimple Lie group. In section 3, we summarize the properties of the $\mathfrak{sl}(N, \mathbb{C})$ relevant for our reasoning, in particular, the structure of its Abelian ideals. Results of this section are translated in sections 4 and 5 to the properties of the adjoint endomorphism and exponential function on $\mathfrak{sl}(N, \mathbb{C})$. The final result for the Wei–Norman formulae in the case of $\mathfrak{sl}(N, \mathbb{C})$ is given in section 6. The corresponding results for $\mathfrak{su}(N)$ are easy to recover by restriction to anti-Hermitian generators. In section 7, we give explicit formulae for low dimensions.

Riccati equations are, in general, not solvable by quadratures. From this point of view we do not pretend to give the general solution to the problem of the evolution on the special linear and special unitary groups generated by non-autonomous linear equations. On the other hand, (matrix) Riccati equations have been a subject of intensive studies for many years. These resulted in the development of various exact and approximate methods of treating them. In concrete applications, such methods can be more advantageous than other approaches to linear non-autonomous equations. Equally important are the connections of Riccati equations with the theory of the so-called Lie systems. These are systems of nonlinear non-autonomous equations admitting a superpositions rule expressing the general solution in terms of a set of independent particular solutions. In the conclusions section, we shortly comment on the possible contribution of our findings to the theory of the Lie systems.

2. General Wei–Norman method

Let $G$ be an $n$-dimensional Lie group and $\mathfrak{g}$ its Lie algebra. We assume in the following that $\mathfrak{g}$ is complex and simple. Let also $\mathbb{R} \ni t \mapsto M(t) \in \mathfrak{g}$ be a curve in $\mathfrak{g}$ and $K(t)$ a curve in $G$ given by the differential equation

$$\frac{d}{dt} K(t) = M(t) K(t), \quad K(0) = I. \quad (1)$$

In $\mathfrak{g}$, we choose some basis $X_k$, $k = 1, \ldots, n$ in which $M(t)$ takes the form

$$M(t) = \sum_{k=1}^{n} a_k(t) X_k. \quad (2)$$

Equations (1) and (2) can be treated as a classical control system on the Lie group $G$ with functions $a_k(t)$ as controls. In quantum mechanical applications, the Schrödinger equation for an $N$-level system governed by a time-dependent Hamiltonian $H(t) = iM(t)$, is

$$i \frac{d \psi}{dt} = H(t) \psi, \quad (3)$$

where $\psi(t)$ is an $N$-component wavefunction of the system and the time dependence of the Hamiltonian stems from the coupling of the system to time-dependent classical (e.g. electromagnetic) fields, can also be reduced to (1) by writing the solution of (3) with an initial condition $\psi(0)$ as $\psi(t) = K(t) \psi(0)$ and substituting into (3). As a less obvious application in quantum mechanics, let us cite a non-perturbative treatment of pair creation processes in a
constant electromagnetic field in quantum electrodynamics [23]. The role of a fictitious time
is played in this case by a momentum argument of the so-called Dirac–Heisenberg–Wigner
function. The resulting system of three equations can be reduced to a single Riccati equation
along the lines given in this paper.

We look for the solution $K(t)$ in the form

$$K(t) = \prod_{k=1}^{n} \exp(u_k(t)X_k), \quad (4)$$

involving $n$ unknown functions $u_k$. Differentiating (4) we obtain

$$K' = \sum_{l=1}^{n} u'_l \prod_{k=1}^{l-1} \exp(u_kX_k)X_l \prod_{k=l}^{n} \exp(u_kX_k)$$

$$= \sum_{l=1}^{n} u'_l \prod_{k=1}^{l-1} \exp(u_kX_k)X_l \prod_{k=l}^{n} \exp(-u_kX_k) \prod_{k=1}^{l} \exp(u_kX_k)$$

$$= \sum_{l=1}^{n} u'_l \prod_{k=1}^{l-1} \text{Ad}_{\exp(u_kX_k)} \cdot X_l K = \sum_{l=1}^{n} u'_l \prod_{k\geq l} \exp(u_k \text{ad}_X) \cdot X_l K, \quad (5)$$

where by $'$ we denote the differentiation with respect to $t$ and $\text{Ad}$ is the adjoint action of $G$
on $\mathfrak{g}$,

$$\text{Ad}_g \cdot X := gXg^{-1}, \quad g \in G, \quad X \in \mathfrak{g}. \quad$$

In the last equality in (5), we used $\text{Ad}_{\exp(bX)} = \exp(b \text{ad}_X)$, where $\text{ad}_X = [X, \cdot]$ is the adjacent action of $\mathfrak{g}$ on itself.

Comparing (1) and (5), we obtain

$$M(t) = \sum_{l=1}^{n} u'_l \prod_{k=1}^{l-1} \exp(u_k \text{ad}_X) \cdot X_l. \quad (6)$$

Both sides of (6) are elements of $\mathfrak{g}$ and expanding both in the basis $X_k$ we arrive at a system of
coupled differential equations for $u_k$ in terms of $a_l, k, l = 1, \ldots, n$.

It is worthwhile to rewrite (6) in a more compact form. Let us denote

$$A^{(l)} = \prod_{k\neq l} \exp(u_k \text{ad}_X), \quad (7)$$

and, consequently,

$$A^{(l)} \cdot X_l = \sum_{j=1}^{n} A^{(l)}_{j} X_j, \quad (8)$$

Now (6) can be written as

$$\sum_{j=1}^{n} a_j X_j = \sum_{l=1}^{n} u'_l \sum_{j=1}^{n} A^{(l)}_{j} X_j = \sum_{j=1}^{n} \left( \sum_{l=1}^{n} A^{(l)}_{jl} u'_l \right) X_j,$$

hence,

$$a_j = \sum_{l=1}^{n} A^{(l)}_{jl} u'_l,$$

or in a compact form,

$$a = Au', \quad (9)$$
where $A$ is an $n \times n$ matrix with elements $A_{jl} = A_{lj}$, i.e., its $l$th column is equal to the $l$th column of the matrix $A^{(t)}$, cf (8), and $a$ and $u$ are the vectors of the coefficients of $M$ in (2) and the unknowns $u_k$. If $A$ is invertible, we obtain thus a system of (nonlinear) differential equations solved for the first derivatives

$$u' = A^{-1} a.$$  \hspace{1cm} (10)

It can be shown that $A$ is invertible at least locally.

**Lemma 2.1.** There exists an interval $I_0 = (-\epsilon, \epsilon) \subseteq \mathbb{R}$ such that $A(t)$ defined above is invertible for $t \in I_0$.

**Proof.** Since $K(0) = I$ and using the fact that every element of a Lie group $G$ can be written in a unique way as a product of exponentials of the elements of a basis of its Lie algebra $\mathfrak{g}$, we obtain that $u_k(0) = 0$ with $k = 1, \ldots, n$. From this and (7), we obtain that $A^{(t)} = I$ for every $t$. Since $A_{jl} = A^{(t)}_{jl} = \delta_{jl}$, we obtain that $A(0) = I$. As $\det(A(t))$ is continuous and $\det(A(0)) = 1$, it follows that $\det(A(t))$ is different from zero for values of $t$ close enough to $0$. Hence, $A(t)^{-1}$ exists close to $t = 0$. \hfill \Box

3. The structure of $\mathfrak{sl}(N, \mathbb{C})$ algebra

In the following, we are going to apply the above-described scheme to the case of the unitary group. From the previously mentioned quantum-mechanical point of view, this is quite natural, since the quantum mechanical Hamiltonian $H$ is a Hermitian matrix. Hence, a fortiori, $M(t) = -iH(t)$ is anti-Hermitian and, as such, belongs to the Lie algebra of an appropriate unitary group (we can restrict our considerations to special unitary groups since the overall phase factor is irrelevant from the quantum-mechanical point of view).

Let us start with a brief summary of the structure of simple Lie algebras [24].

Each complex simple Lie algebra $\mathfrak{g}$ can be decomposed into the root spaces with respect to a chosen Cartan subalgebra $\mathfrak{h}$ (a maximal commutative subalgebra of $\mathfrak{g}$),

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$  \hspace{1cm} (11)

where the one-dimensional root spaces are defined as

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \forall H \in \mathfrak{h}\}.$$  

The linear forms $\alpha \in \mathfrak{h}^*$ (the dual space to the algebra $\mathfrak{h}$) are called roots and the element $X_\alpha$ spanning the subspace $\mathfrak{g}_\alpha$ is called the root vector. By fixing a basis $\{H_1, \ldots, H_r\}$ in $\mathfrak{h}$ we can identify the set of roots $\Delta$ with a set of $r$-dimensional vectors $\alpha = (\alpha_1, \ldots, \alpha_r)$, where $\alpha^* = \alpha(H_k)$. The number $r = \dim \mathfrak{h}$ is called the rank of $\mathfrak{g}$. Among the roots we can find a set $\Pi$ of $r$ positive simple roots, $\Pi = \{\alpha_1, \ldots, \alpha_r\}$, such that for each $\alpha \in \Delta$,

$$\alpha = \sum_{i=1}^r m_i \alpha_i,$$

where either all $m_i$ are non-negative (such roots $\alpha$ forming the set of positive roots $\Delta_+$) or all $m_i$ are non-positive (such roots $\alpha$ forming the set $\Delta_-$ of negative roots). There is a one-to-one correspondence between the positive and negative roots: for each positive root $\alpha$ there is a negative one $-\alpha$.

The root spaces have the following property:

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta},$$  \hspace{1cm} (12)

(for $\beta = -\alpha$ we have $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{h}$).
In terms of the positive and negative roots, the decomposition (11) can be rewritten as
\[ g = n_+ \oplus h \oplus n_-, \quad n_\pm = \bigoplus_{\alpha \in \Delta_\pm} g_\alpha. \]

The subalgebras
\[ b_\pm = h \oplus n_\pm \]
are called the Borel subalgebras relative to the Cartan subalgebra \( h \).

In the following, we take \( G = SL(N, \mathbb{C}) \) and \( g = \mathfrak{sl}(N, \mathbb{C}) \). In this case, \( n = N^2 - 1 \). To construct a basis in \( g \), we define
\[ S_{kl} = e_{k} e_{l}^\dagger, \]
where \( e_k, k = 1, \ldots, N \), is the standard basis in \( \mathbb{C}^N \), i.e., \( (e_k)_l = \delta_{kl} \). The commutation relations for \( S_{kl} \) read
\[ [S_{ij}, S_{kl}] = \delta_{kj} S_{il} - \delta_{il} S_{kj}. \]

For \( p \neq q \), the elements \( S_{pq} \) are the root vectors. The root corresponding to the root space spanned by \( S_{pq} \) will be denoted by \( \alpha_{pq} \). From \( S_{kl} \), we construct a basis \( X_m \) by the following renumbering:
\[ X_m = S_{pq}, \quad m = \frac{(N + q - 1)(N - q)}{2} + p, \quad \text{for } p < q, \]
\[ X_m = S_{kl} - S_{l+1,l+1}, \quad m = \frac{1}{2} N(N - 1) + l, \quad l = 1, \ldots, N - 1, \]
\[ X_m = S_{q,l}, \quad m = N^2 - \frac{(N + q - 1)(N - q)}{2} + p, \quad \text{for } p < q. \]

One may choose roots \( \alpha_{pq} \) corresponding to root vectors \( S_{pq} \) for \( p < q \) to be the positive roots. Then, the matrices (13) and (15) generate maximal nilpotent subalgebras \( n_+ \) and \( n_- \), respectively (conjugated with respect to the standard Hermitian structure on \( \mathbb{C}^N \)), whereas (14) generate the Cartan subalgebra of \( \mathfrak{sl}(N, \mathbb{C}) \). In this basis, \( n_+ \) is the set of strictly upper triangular matrices and \( n_- \) is the set of strictly lower triangular matrices.

The set \( \Pi \) of positive simple roots consists of elements of \( \alpha_{l,l+1} \) for \( l = 1, \ldots, N - 1 \).

In what follows, we will show many useful features of this basis. First observe that according to (13), \( n_+ := \text{span}\{X_1, \ldots, X_{N(N-1)/2}\} \) and the order of root vectors spanning \( n_+ \) corresponds to the following order of roots:
\[ \alpha_{i,j} > \alpha_{k,l} \iff j < l \text{ or } (j = l \text{ and } i > k). \]

Analogously, \( n_- := \text{span}\{X_{N(N+1)/2}, \ldots, X_{N(N-1)}\} \) and the root vectors spanning \( n_- \) are ordered by the corresponding roots in the following way:
\[ \alpha_{i,j} > \alpha_{k,l} \iff i > k \text{ or } (i = k \text{ and } j < l). \]

Recall, that by the theorem of Lie [24], if \( L \) is a nilpotent Lie algebra of dimension \( r \), then there exist a sequence of ideals \( I_k \) fulfilling
\[ 0 = I_0 \subset I_1 \subset \ldots \subset I_{r-1} \subset I_r = L \]
and a basis \( Y_1, \ldots, Y_r \) in \( L \), such that
\[ I_k := \text{span}\{Y_1, \ldots, Y_k\}. \]
The basis \([X_1, \ldots, X_{N(N-1)/2}]\) of \(n_+\) defined in (13) has this property. It follows from (12) and the fact that \(a_{i,j} + a_{k,l} \in \Delta\), provided \(i = l\) or \(j = k\). In both cases, \(a_{i,j} + a_{k,l} < a_{i,j}\), as can be easily checked. Thus, \(I_k := \text{span}\{X_1, \ldots, X_k\}\) is an ideal in \(n_+\) for any \(k = 1, \ldots, \frac{1}{2}N(N-1)\).

A direct consequence of the Lie theorem is that in this basis the matrix of endomorphism \(\text{ad}_{X_k} : n_+ \to n_+\) is strictly upper triangular [24]. Moreover, according to (15) the matrices generating \(n_-\) are the transpositions of the matrices generating \(n_+\) numbered in the reverse order. This implies that the matrices of endomorphisms \(\text{ad}_{X_k} : n_- \to n_-\) for \(k = \frac{1}{2}N(N+1), \ldots, N^2-1\) are strictly lower triangular.

We may decompose \(n_+\) in a particular direct sum of subspaces to modify slightly the original Wei–Norman expansion (4). To this end let us consider the family \([\alpha_k], k = 1, \ldots, N-1\), of subspaces (the numbering of the root vectors \(X_i\) is the same as in (13)),

\[
a_k = \text{span}\{X_{i_k}, X_{i_k+1}, \ldots, X_{i_k+N-k-1}\}, \quad i_k = N(k-1) - \frac{k(k-1)}{2} + 1.
\]

In the standard matrix representation of \(\mathfrak{sl}(N, \mathbb{C})\), the subspace \(a_k\) consists of matrices with only non-vanishing entries above the diagonal in the \((N-k+1)\)th column.

We have \(\dim a_k = N-k\) and \(n_+ = \bigoplus_{k=1}^{N-1} a_k\). Each \(a_k\) is an Abelian subalgebra of \(b_k\), and \(a_1\) is an Abelian ideal of \(b_1\). Let us also define

\[
b_k = \bigoplus_{l=k}^{N-1} a_l.
\]

Obviously, \(a_k \subset b_k \subset b_{k-1}\). Moreover, each \(b_k\) is a subalgebra of \(n_+\) and \(a_k\) is an Abelian ideal of \(b_k\), i.e.,

\[
[b_k, b_k] \subset b_k, \quad [a_k, a_k] = 0, \quad [a_k, b_k] \subset a_k.
\]

The subalgebras of \(n_-\) will be denoted by \(\tilde{a}_k\). Each \(\tilde{a}_k\) is the Hermitian conjugate of \(a_k\) and is generated by the basis elements (15) in the following way:

\[
\tilde{a}_k = \text{span}\{X_{j_k}, X_{j_k+1}, \ldots, X_{j_k+N-k-1}\}, \quad j_k = N(N-k) - \frac{k(k+1)}{2}.
\]

The subalgebras \(\tilde{a}_k\) are defined in analogy to (19) and together with \(a_k\) they follow relations analogous to (20). We have the following decomposition of \(\mathfrak{g} = \mathfrak{sl}(N, \mathbb{C})\) into semidirect sum of \(2N-1\) commuting subalgebras

\[
\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i=1}^{N-1} a_i \oplus \bigoplus_{j=1}^{N-1} \tilde{a}_j.
\]

### 4. Properties of the adjoint endomorphism

Expression (6) consists of products of exponents of \(\text{ad}_{X_\alpha}\). In this section, we present some fundamental properties of the adjoint endomorphisms corresponding to root vectors in \(\mathfrak{sl}(N, \mathbb{C})\) which are crucial for usefulness of the Wei–Norman method in the case of the \(\text{SL}(N, \mathbb{C})\) group.

**Lemma 4.1.** Let \(\alpha\) be the root and \(X_\alpha \in \mathfrak{sl}(N, \mathbb{C})\) be the corresponding root vector. Then:

(i) the image of \((\text{ad}_{X_\alpha})^2\) is equal to \(g_\alpha\) and

\[\ker((\text{ad}_{X_\alpha})^2) = \text{span}\{X_1, \ldots, X_{N^2-1}\} \setminus \{X_{-\alpha}\}.\]

(ii) \((\text{ad}_{X_\alpha})^3 = 0\).
Proof. First observe that statement 2 follows from 1. To prove the latter recall that [24],

\[
\text{ad}_{X_{\alpha}} (X_{\beta}) = [X_{\alpha}, X_{\beta}] = \begin{cases} 
H_{\alpha}, & \alpha + \beta = 0 \\
N_{\alpha, \beta} X_{\alpha+\beta}, & \alpha + \beta \in \Delta \\
0, & \text{otherwise},
\end{cases}
\]

where \(H_{\alpha} \in \mathfrak{h}\) corresponds to a root \(\alpha\) and \(N_{\alpha, \beta}\) is some constant. We have

\[
(\text{ad}_{X_{\alpha}})^3(X_{\beta}) = [X_{\alpha}, [X_{\alpha}, X_{\beta}]] = \begin{cases} 
-\alpha(H_{\alpha})X_{\alpha}, & \alpha + \beta = 0, \\
N_{\alpha, \beta} N_{\alpha, \alpha+\beta} X_{\alpha+\beta}, & \alpha + \alpha + \beta \in \Delta, \\
0, & \text{otherwise}.
\end{cases} \tag{23}
\]

Since the condition \(\alpha + \alpha + \beta \in \Delta\) is never fulfilled for \(\mathfrak{sl}(N, \mathbb{C})\), equation (23) implies that \((\text{ad}_{X_{\alpha}})^3\) sends any element of \(n_+ \cup n_-\) into \(\mathfrak{g}_\alpha\). And since for \(H \in \mathfrak{h}\),

\[
(\text{ad}_{X_{\alpha}})^3(H) = [X_{\alpha}, [X_{\alpha}, H]] = -\alpha(H)[X_{\alpha}, X_{\alpha}] = 0,
\]

the only element of basis (13)–(15) on which \((\text{ad}_{X_{\alpha}})^2\) takes a non-zero value is \(X_{-\alpha}\), which ends the proof of statement 1.

Observe that if we have two commuting matrices of a given nilpotency order \(r\), then the sum of the matrices is also nilpotent of order \(r\). It follows from the Jordan theorem [24] that the matrices can be expressed in the Jordan form in the same basis and are block diagonal with the same blocks of maximal size \(r - 1\). Since \(\mathfrak{a}_k\) and \(\tilde{\mathfrak{a}}_k\) are commuting subalgebras, lemma 4.1 yields

**Corollary 4.2.** If \(X \in \mathfrak{a}_k\) or \(X \in \tilde{\mathfrak{a}}_k\), then \((\text{ad}_{X})^3 = 0\).

**Lemma 4.3.** Let \(\alpha\) be the root and \(X_{\alpha} \in \mathfrak{sl}(N, \mathbb{C})\) be the corresponding root vector. In the basis defined by (13)–(15), the matrices of \(\text{ad}_{X_{\alpha}}\) for \(\alpha \in \Delta_+\) are strictly upper triangular and for \(\alpha \in \Delta_-\) are strictly lower triangular.

**Proof.** Let \(X_{\alpha} \in n_+\). In the previous section, we have mentioned the theorem of Lie, which has as a direct consequence the fact that the matrix of \(\text{ad}_{X_{\alpha}} : n_+ \to n_+\) is strictly upper triangular. So the sector of the matrix of endomorphism \(\text{ad}_{X_{\alpha}} : \mathfrak{g} \to \mathfrak{g}\) corresponding to \(n_+\) is strictly upper triangular. Since \([n_+, \mathfrak{h}] \subset n_+\), the only non-zero matrix elements of \(\text{ad}_{X_{\alpha}}\) in the sector corresponding to \(\mathfrak{h}\), lie above the diagonal. Finally, we consider the action \(\text{ad}_{X_{\alpha}} (X_{\beta})\) for \(X_{\beta} \in n_-\). The result is nonzero in two cases: \(\alpha + \beta = 0\) or \(\alpha + \beta \in \Delta\). In the first case, \(\text{ad}_{X_{\alpha}} (X_{\beta}) \in \mathfrak{h}\) and the corresponding matrix element is above the diagonal. In the second case, it may be easily checked directly from the definition (16)–(17) that \(\alpha + \beta < \beta\). This finishes the proof for \(X_{\alpha} \in n_+\). The proof for \(X_{\alpha} \in n_-\) is analogous.

**Lemma 4.4.** Let \(X_{\alpha} \in \mathfrak{a}_k \subset \mathfrak{sl}(N, \mathbb{C})\) or \(X_{\alpha} \in \tilde{\mathfrak{a}}_k \subset \mathfrak{sl}(N, \mathbb{C})\), where \(\alpha\) is the corresponding root. The subalgebras \(\mathfrak{a}_l\), \(\tilde{\mathfrak{a}}_l\) for \(l < k\) and the subalgebra \(\mathfrak{b}_k \oplus \mathfrak{h} \oplus \tilde{\mathfrak{b}}_k\) are the invariant subspaces of \(\text{ad}_{X_{\alpha}}\).

**Proof.** Let \(X_{\alpha} \in \mathfrak{a}_k\). We consider three cases.

(i) \(X_{\beta} \in \mathfrak{a}_l, l < k\). In this case, \(X_{\beta} \in \mathfrak{b}_l\) and \(X_{\beta} \in \mathfrak{h}\), because \(\mathfrak{a}_k \subset \mathfrak{b}_k \subset \mathfrak{b}_l\). Thus, \(\text{ad}_{X_{\alpha}} (X_{\beta}) = [X_{\alpha}, X_{\beta}] \in \mathfrak{b}_l\). On the other hand, \(X_{\beta} \in \mathfrak{a}_l\) and \(\mathfrak{a}_l\) is an ideal in \(\mathfrak{b}_l\), so \(\text{ad}_{X_{\alpha}} (X_{\beta}) \in \mathfrak{a}_l\).

(ii) \(Y \in \mathfrak{b}_l \oplus \mathfrak{h} \oplus \tilde{\mathfrak{b}}_k\). We have also \(X_{\beta} \in \mathfrak{b}_k \oplus \mathfrak{h} \oplus \tilde{\mathfrak{b}}_k\) and the property \(\text{ad}_{X_{\alpha}} (Y) \in \mathfrak{b}_l \oplus \mathfrak{h} \oplus \tilde{\mathfrak{b}}_k\) follows from the fact that \(\mathfrak{b}_l \oplus \mathfrak{h} \oplus \tilde{\mathfrak{b}}_k\) is a subalgebra of \(\mathfrak{g}\) and this follows directly from the definition of \(\mathfrak{b}_l\) and \(\tilde{\mathfrak{b}}_k\) (see (19)).
(iii) $X_\beta \in \tilde{a}_l, l < k$. According to definitions (18) and (21), the roots are
\[
\alpha = (i, N - k + 1), \quad i < N - k + 1, \\
\beta = (N - l + 1, j), \quad N - l + 1 > j.
\]
We have
\[
\text{ad}_{X_\beta}(X_\alpha) = [X_\alpha, X_\beta] \neq 0 \iff \alpha + \beta \in \Delta \iff i = j \text{ or } k = l.
\]
Condition $k = l$ contradicts the assumption $k > l$. So the only assumption is $i = j$, which implies $\alpha + \beta = (N - l + 1, N - k + l)$. Since $k > l$, we have $N - l + 1 > N - k + 1$, so $\text{ad}_{X_\beta}(X_\alpha) = X_{\alpha + \beta} \in \tilde{a}_l$.

The same reasoning holds for $X_\alpha \in \tilde{a}_l$.

Since for a given $k$ we have the following decomposition of $g$:
\[
g = a_1 \oplus \ldots \oplus a_{k-1} \oplus \left(b_1 \oplus \mathfrak{h} \oplus \tilde{b}_l\right) \oplus \tilde{a}_{k-1} \oplus \ldots \oplus \tilde{a}_1,
\]
lemmas 4.1, 4.3 and 4.4 yield

**Corollary 4.5.** Let $X_\alpha \in a_k \subset \mathfrak{sl}(N, \mathbb{C})$ (or $X_\alpha \in \tilde{a}_k \subset \mathfrak{sl}(N, \mathbb{C})$) be the root vector. In the basis (13)–(15), the matrix of endomorphism $\text{ad}_{X_\alpha}$ is nilpotent of order 3, strictly upper triangular (lower triangular) and block diagonal with respect to decomposition (24).

5. **Exponential function on commuting nilpotent subalgebras of $\mathfrak{sl}(N, \mathbb{C})$**

Corollary 4.5 implies the following.

**Corollary 5.1.** For $X \in a_k$ or $X \in \tilde{a}_k$, the matrix of $\exp(\text{ad}_X)$ is a quadratic polynomial in $\text{ad}_X$.

Moreover, in the basis (13)–(15), the matrix of $\exp(\text{ad}_X)$ is upper triangular (lower triangular for $X \in \tilde{a}_k$) and block diagonal with respect to decomposition (24).

It is thus convenient to rewrite equation (6) as a sum of terms corresponding to the commuting subalgebras of decomposition (22). We define
\[
U_k := \prod_{i \in J_k} \exp(u_i \cdot \text{ad}_X), \quad \tilde{U}_k := \prod_{i \in \tilde{J}_k} \exp(u_i \cdot \text{ad}_X),
\]
where $J_k = \{i_k, \ldots, i_k + N - k - 1\}$ is the index range defined in (18), so for a given $k$ the index $i \in J_k$ numerates the generators of the subalgebra $a_k$, and $\tilde{J}_k = \{j_k, \ldots, j_k + N - k - 1\}$ is the index range defined in (21), so for a given $k$ the index $i \in \tilde{J}_k$ numerates the generators of the subalgebra $\tilde{a}_k$. For $l < N - k - 1$, we also define
\[
V_{kl} := \prod_{i = i_k}^{i_k + l} \exp(u_i \cdot \text{ad}_X), \quad \tilde{V}_{kl} := \prod_{i = j_k}^{j_k + l} \exp(u_i \cdot \text{ad}_X),
\]
where for a given $k$ the index $l$ runs over the first $l - 1$ elements of $J_k$ or $\tilde{J}_k$. We set
\[
U_0 = \tilde{U}_N = V_{00} = \tilde{V}_{00} = 1.
\]
Observe that the commutativity of the subalgebras $a_k$ implies that
\[
U_k = \prod_{i \in J_k} \exp(u_i \cdot \text{ad}_X) = \exp \left( \sum_{i \in J_k} u_i \cdot \text{ad}_X \right) = \exp \left( \text{ad} \left( \sum_{i \in J_k} u_i X_i \right) \right),
\]
so $U_k$ equals $\exp(\text{ad}_X)$ for some $X \in a_k$ (analogously $\tilde{U}_k$ equals $\exp(\text{ad}_X)$ for some $X \in \tilde{a}_k$), so $U_k$ and $\tilde{U}_k$ have the triangularity and block diagonality properties implied by corollary 5.1 and the same statement holds for $V_{kl}$ and $\tilde{V}_{kl}$. From lemma 4.1, we have also the following crucial property.
Corollary 5.2. The only matrix elements of operators \( U_k \) or \( V_{kl} \) with quadratic dependence on parameters \( u_i \) appear in columns corresponding to \( \tilde{a}_k \) for the same index \( k \) and the only matrix elements of operators \( \tilde{U}_k \) or \( \tilde{V}_{kl} \) which depend quadratically on parameters \( u_i \) are the elements appearing in columns corresponding to \( a_k \) for the same index \( k \). All other matrix elements of \( U_k, V_{kl}, \tilde{U}_k \) or \( \tilde{V}_{kl} \) are linear polynomials in parameters \( u_i \).

Let \( i_0 = \frac{1}{2}N(N-1)+1 \) be the number of the first element of Cartan subalgebra according to the ordering (13)–(15). We define also

\[
H := \prod_{i=i_0}^{i_0+N-1} \exp(u_i\mathbf{ad}_{X_i}), \quad H_l := \prod_{i=i_0}^{i_0+l} \exp(u_i\mathbf{ad}_{X_i}),
\]

where \( l \in \{1, \ldots, N-2\} \) and \( X_i \) generate Cartan subalgebra \( \mathfrak{h} \) (see (14)). \( H \) and \( H_l \) are diagonal matrices and \( H_0 = 1 \).

6. Wei–Norman method in a properly chosen basis

Equation (6) can be now rewritten in terms of operators defined in (25), (26) and (27) in the following simplified way:

\[
M(t) = \sum_{k=1}^{N-1} \prod_{l=0}^{N-k-2} U_l \cdot \sum_{i=0}^{N-2} V_{kl} u'_i X_{k+i} + \sum_{j=1}^{N-1} H_j u'_j X_{j+i} \tag{28}
\]

\[
+ \prod_{j=1}^{N-1} U_j \cdot \sum_{i=0}^{N-2} H_i u'_i X_{i+n+i} \tag{29}
\]

\[
+ \prod_{j=1}^{N-1} U_j \cdot H \cdot \prod_{k=1}^{N-1} \tilde{U}_l \cdot \sum_{i=0}^{N-2} \tilde{V}_{kl} u'_i X_{k+i}. \tag{30}
\]

On the other hand, the matrix \( M(t) \) by definition (2) is a linear combination of generators \( X_k \) with given coefficients \( a_0(t) \). The first step in order to solve the system of differential equations is to solve linear algebraic equations for \( u'_i \). This is done by matrix inversion (9)–(10). We will show now that expression (28)–(30) allows for the separation of the full set of equations into sectors corresponding to decomposition (22).

The first term in (28) corresponding to \( k = 1 \) is an element of \( \alpha_1 \). It follows from corollary 5.1 and the fact that it is a sum of terms of the form \( \mathbf{ad}_X Y \) for \( X \in \alpha_1 \) and \( Y \in \alpha_1 \). We move this term to the left-hand side of equation (28)–(30). The rest of the sum remaining on the right-hand side has a common factor \( U_j \), so we multiply both sides by \( U_j^{-1} \). After these operations the right-hand side does not contain terms proportional to elements of \( \alpha_1 \), because it stems from a composition of actions of operators which are block diagonal with respect to decomposition (24) on generators of \( \alpha_k \) with \( k > 1 \) or \( \mathfrak{h} \) or \( \tilde{a}_k \). The left-hand side reads

\[
U_1^{-1} \left( M(t) - \sum_{i=1}^{N-1} V_{kl} u'_i X_i \right) = U_1^{-1} \left( \sum_{j=1}^{N-1} a_j(t) X_j + \sum_{i=1}^{N-1} V_{kl} u'_i X_i \right), \tag{31}
\]

where we used definition (2). Since there are no elements spanning \( \alpha_1 \) on the right-hand side, the first \( N-1 \) components of (31) have to vanish. These \( N-1 \) equations depend only on \( u_i \) for \( i < N \) (since \( U_1 \) depends only on them) and \( u'_i \) for \( i < N \) (since \( V_{kl} \) are upper triangular, which follows from corollary 5.1). Moreover, \( U_1 \) is a quadratic function of function \( u_i \) so we end up with a matrix Riccati equation for functions \( u_i \) for \( i < N \). Once this system of
The first sum corresponds to the Cartan subalgebra $\mathfrak{h}$. Since $\mathfrak{h}$ is commutative, we have $\exp(\text{ad}_X) Y = Y$, for $X, Y \in \mathfrak{h}$, so in (32) $H X_{i+1} = X_{i+1}$, which follows from definition (27). The second sum in (32) is a combination of the generators of subalgebras $\tilde{\mathfrak{a}}_k$ only, because it results from an action of the diagonal operator $H$ and the lower triangular operators $\tilde{U}_j$ and $\tilde{V}_k$ on elements of $\tilde{\mathfrak{a}}_k$. Thus, the functions $u_i$ corresponding to the generators of $\mathfrak{h}$ can be found by a simple integration of the solutions to the previously found Riccati equations. After substitution of these solutions and multiplication of both sides of (32) by $H^{-1}$, we are left with an equation where on both sides there are only terms proportional to the generators $X_j$ spanning $\tilde{\mathfrak{a}}_k$, because all other terms have canceled. In order to find the solutions for the remaining functions $u_i$, we proceed as follows. First, we multiply both sides by $\tilde{U}_j^{-1}$ which depends on functions $u_i$ corresponding to $\tilde{a}_1$ only. The derivatives $u'_i$ on the right-hand side appear only in the first term of the sum, because of the block diagonality of $\tilde{V}_k$ operators. Thus, this term separates from the rest and has to be equal to the action of $\tilde{U}_j^{-1}$ on the left-hand side. The fact that there are no generators of $\tilde{a}_q$ in the equation and corollary 5.2 implies that this set of equations will be linear in the functions $u_i$. Once we solve it, the solutions may be substituted into the equation and the terms proportional to generators of $\tilde{a}_1$ will cancel. Then, we multiply both sides by $\tilde{U}_1$ and proceed in the same way obtaining a set of linear equations for the functions $u_i$ corresponding to $\tilde{a}_2$. We keep repeating the described procedure until we find linear equations for all remaining functions $u_i$.

The above-described procedure enables the conversion of highly nonlinear differential equation (1) into the hierarchy of matrix Riccati equations and linear matrix differential equations. This procedure is effective in the sense that it provides an algorithm that may be applied directly. The authors have written a program in Maple which performs this procedure for any given $N$ and tested its successful performance up to $N = 10$. The program is provided as supplementary material (available at stacks.iop.org/JPhysA/46/265208/mmedia).

It is worth mentioning that the crucial ingredient for realizing the described algorithm in practice is the order of the generators (13)–(15). Once this basis is used for computations and the inverse in (10) is successfully computed, the separation of the system of equations comes up automatically. For large $N$, the computation of the inverse is the part of the algorithm with the largest computational complexity, which scales with $N$ as $N!$. It follows from the fact that the matrix $A$ in (9) which is in principal of dimension $(N^2 - 1) \times (N^2 - 1)$ has a special block upper triangular form with the largest block to invert of the size $(N - 1) \times (N - 1)$. The inversion has to be realized by Cramer’s rule, which has the mentioned computational complexity.
7. Examples

7.1. SL\((2, \mathbb{C})\)

For \(N = 2\), the system (10) reduces to one Riccati equation,
\[
\dot{u}_1 = a_1 + 2a_2 u_1 - a_3 u_1^2, \tag{33}
\]
and two equations for the remaining unknowns,
\[
\begin{align*}
\dot{u}_2 &= a_2 - a_3 u_1, \\
\dot{u}_3 &= a_3 e^{2u_2},
\end{align*}
\]
which give \(u_2\) and \(u_3\) by simple integrations, once solutions of (33) are known.

7.2. SL\((3, \mathbb{C})\)

For \(N = 3\), we obtain from (10):

(i) A system of two coupled Riccati equations:
\[
\begin{align*}
\dot{u}_1 &= a_1 + (2a_5 - a_4) u_1 + a_6 u_2 - a_8 u_1^2 - a_7 u_1 u_2, \\
\dot{u}_2 &= a_2 + a_3 u_1 + (a_4 + a_5) u_2 - a_9 u_1 u_2 - a_7 u_2^2,
\end{align*}
\]
which for further reference we will rewrite in the form
\[
\dot{u}_{(1)} = c_{(1)} + C_{(1)} u_{(1)} + u_{(1)} u_{(1)}^T b_{(1)}, \tag{34}
\]
with
\[
\begin{align*}
u_{(1)} &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, & c_{(1)} &= \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, & b_{(1)} &= \begin{bmatrix} -a_8 \\ -a_7 \end{bmatrix}, \\
C_{(1)} &= \begin{bmatrix} 2a_5 - a_4 & a_6 \\ a_3 & a_4 + a_5 \end{bmatrix}.
\end{align*}
\]

(ii) An equation for \(u_3\) which reduces to a scalar Riccati equation upon substituting the solution of (34):
\[
\dot{u}_3 = (a_3 - a_9 u_2) + (2a_4 - a_5 + a_8 u_1) u_3 + (a_7 u_1 - a_6) u_3^2. \tag{35}
\]

(iii) Equations for the coefficients of the Cartan algebra generators which are solved by single integrations once the solutions of (34) and (35) are known,
\[
\begin{align*}
\dot{u}_4 &= a_4 - a_9 u_3 + a_7 (u_1 u_3 - u_2), \\
\dot{u}_5 &= a_5 - a_8 u_1 - a_7 u_2.
\end{align*}
\]

(iv) Equations for the coefficients of the generators of the second ('lower-triangular') nilpotent subalgebra:
\[
\begin{align*}
\dot{u}_6 &= (a_6 - a_7 u_1) e^{2u_1} u_5, \\
\dot{u}_7 &= (a_7 u_3 + a_8) u_6 e^{-u_1+2u_2} + a_7 e^{u_1+2u_2}, \\
\dot{u}_8 &= (a_8 + a_7 u_3) e^{-u_1+2u_2},
\end{align*}
\]
which are solved by simple consecutive integrations.
7.3. \( \text{SL}(4, \mathbb{C}) \)

A similar structure emerges for \( N = 4 \). The system of 15 equations (10) separates into:

(i) A system of three coupled Riccati equations,

\[
\mathbf{u}'_1 = \mathbf{c}_1 + \mathbf{C}_1 \mathbf{u}_1 + \mathbf{u}_1 \mathbf{u}_1^T \mathbf{b}_1, \tag{36}
\]

where

\[
\mathbf{u}_1 = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} -a_{15} \\ -a_{14} \\ -a_{13} \end{bmatrix},
\]

and

\[
\mathbf{C}_1 = \begin{bmatrix} -a_8 + 2a_9 & a_{12} & a_{11} \\ a_4 & -a_7 + a_8 + a_9 & a_{10} \\ a_5 & a_6 & a_7 + a_9 \end{bmatrix}.
\]

(ii) A system of two coupled Riccati equations,

\[
\mathbf{u}'_2 = \mathbf{c}_2 + \mathbf{C}_2 \mathbf{u}_2 + \mathbf{u}_2 \mathbf{u}_2^T \mathbf{b}_2, \tag{37}
\]

where

\[
\mathbf{u}_2 = \begin{bmatrix} u_4 \\ u_5 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} a_4 - a_{15}u_2 \\ a_5 - a_{15}u_3 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -a_{12} + a_{14}u_1 \\ -a_{11} + a_{13}u_1 \end{bmatrix},
\]

and

\[
\mathbf{C}_2 = \begin{bmatrix} -a_7 + 2a_8 - a_9 - a_{14}u_2 + a_{15}u_4 & a_{10} - u_2a_{13} - a_{15} \\ a_6 - a_{14}u_3 & a_7 + a_8 - a_9 - a_{13}u_3 + a_{15}u_4 \end{bmatrix}.
\]

Observe that once a solution of (36) is known, system (37) is closed since (38) and (39) are given in terms of some known functions.

(iii) A scalar Riccati equation,

\[
u'_6 = a_6 - a_{12}u_5 + a_{14}u_4u_5 - a_{14}u_3 - 2(a_7 - a_8 - u_6a_{11} + u_6a_{12} - u_5a_{13} + a_{13}u_1u_5 - a_{14}u_4u_4 + a_{14}u_4u_4 + (-a_{10} + u_4a_{11} - a_{13}u_1u_4 + a_{13}u_2)u_6^2, \tag{40}\]

with coefficients depending on solutions of (36) and (37).

(iv) The remaining nine equations for \( u_7, \ldots, u_{15} \) which are solved by single interactions of functions constructed from the initial coefficients, \( a_7, \ldots, a_{15} \), and solutions of (36), (37) and (40).

7.4. Relation to \( SU(N) \) group

As already mentioned, for quantum-mechanical applications a relevant group is the special unitary group \( SU(N) \). In order to obtain the equations for evolution on \( SU(N) \) from the examples for \( SL(N, \mathbb{C}) \) listed above, one has to restrict to an anti-Hermitian matrix \( M \) in (2). This restriction corresponds to the following conditions on complex parameters \( a_k(t) \):

\[
a_k(t) = -a_{k+N}(t) \quad \text{for} \quad k = 1, \ldots, \frac{1}{2}N(N-1),
\]

\[
a_k(t) = -\overline{a_k(t)} \quad \text{for} \quad k = \frac{1}{2}N(N-1) + 1, \ldots, \frac{1}{2}N(N-1) + N - 1.
\]
Thus, $N^2 - 1$ complex parameters $a_k$ can be replaced by $N^2 - 1$ real parameters $b_k$, $c_k$ and $d_k$ in the following way:

\[
\begin{align*}
    a_k(t) &= b_k(t) + ic_k(t), & \text{for } k = 1, \ldots, \frac{1}{2}N(N-1), \\
    a_{N^2-k}(t) &= -b_k(t) + ic_k(t), & \text{for } k = 1, \ldots, \frac{1}{2}N(N-1), \\
    a_{N(N-1)+k}(t) &= id_k(t), & \text{for } k = 1, \ldots, N-1.
\end{align*}
\]

These can be easily substituted into examples presented above. Functions $u_k$ will remain complex.

8. Conclusions and discussion

The reduction of the system (1)–(2) to the set of (matrix) Riccati and linear equations does not provide the general solution of the system, since the general method for solving the Riccati equation is not known. Nevertheless, there are many methods to study matrix Riccati equations, e.g. superposition rules, piecewise linearized methods and others (see [25] and references therein).

Scalar and matrix Riccati equations and system of the form (1)–(2) are examples of the so-called Lie systems. Properties of such systems have been studied for more than century, see for example [26] and the most recent review paper [25] with an exhaustive list of references therein. It is known that every system of Riccati equations is related to a Lie group action and the solution of this system is equivalent to the solution of the system of the form (1)–(2), but not every system of the form (1)–(2) is equivalent to a Riccati equation system. In this work, we have shown that for the groups under consideration, namely $SU(N)$ and $SL(N, \mathbb{C})$, the system (1)–(2) is equivalent to the hierarchy of Riccati equations and linear equations. This is a new contribution to the theory of Lie systems. The question arises if an analogous procedure can be applied to other groups, and this will be investigated in our future work.

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