Gravitationally bound states of two neutral fermions and a Higgs field can be parametrically lighter than their constituents

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We analyze the localized states of a pair of neutral fermions interacting with a Higgs field and the metric of spacetime, extending the Einstein-Dirac formalism introduced by Finster, Smoller, and Yau [Phys. Rev. D 59, 104020 (1999)]. We demonstrate that, when the coupling between the fermions and the Higgs field is strong, there is a class of bound states in which the total (ADM) mass no longer increases proportionally to the mass of the constituent fermions. This phenomenon enables fermionic particles with much larger masses than in the Higgs-free case to form stable localized states.

The reconciliation of quantum mechanics with general relativity is one of the main outstanding questions of modern physics. Despite the current absence of a fully satisfactory theory of quantum gravity, much progress has been made by treating the gravitational field classically, most notably, perhaps, in the prediction of Hawking radiation from black holes [1]. Often, this ‘semiclassical’ approach focuses on the construction of quantum field theories in curved spacetime, but this is necessarily limited in its scope. In particular, modeling the full dynamics of general relativity poses challenges, and often the ‘back-action’ of the matter sector on the spacetime is neglected. For systems in which this is important, e.g. those with strong self-gravity, an alternative is therefore required. One such approach, employed here, is to treat the matter not as a quantum field, but as a first-quantized wavefunction.

This approach may be used to analyze gravitationally localized quantum states, where particles are bound by their gravitational interaction but the state is prevented from collapse by the effects of the uncertainty principle. In the context of scalar fields, such objects are known as “boson stars” [2–4], which have been proposed as candidates for dark matter [5] as well as black hole mimickers [6]. Their fermionic counterparts [7, 8] have received significantly less attention, however, due to the added complexity of spin considerations. Referred to variously as “fermion stars”, “Dirac stars”, and “Dirac solitons”, these objects could potentially prove useful as models of the microscopic structure of Standard-Model particles.

The major breakthrough in the study of fermionic localized states was provided by Finster et al. [9], who generated the first spherically symmetric, numerical solutions of the coupled Dirac and Einstein equations. The resulting “particlelike” states, comprising a pair of neutral fermions, are free from singularities, with a branch of solutions identified as stable. Subsequent extensions of their analysis include charged fermions [10], the coupling to an SU(2) Yang-Mills field [11], and the cases of one [12] and many [13, 14] fermions.

In this Letter we present a hitherto unexplored extension to this framework: the addition of a Higgs field. This allows the fermion mass, which in previous analyses was treated as an input parameter, to be generated dynamically via the Higgs mechanism, as is the case for Standard-Model fermions. By numerically solving the minimally-coupled Einstein, Dirac, and Higgs equations, we show that spherically symmetric particlelike solutions exist for a wide range of parameter values, and are similar in structure to the original Einstein-Dirac states found in ref. [9]. Intriguingly, however, we find a class of solutions in which the total energy (measured by the ADM mass) is no longer proportional to the mass of the constituent fermions.
fermions. Instead, these two mass scales decouple at strong fermion-Higgs coupling, allowing the fermion mass to rise well above the ADM mass of the resulting bound state. This constitutes the main result of this Letter, and is illustrated in Fig. 1.

Equations of motion. The action for the minimally-coupled Einstein-Dirac-Higgs system can be written, using the mostly positive metric signature convention $(-, +, +, +)$, as

$$S_{\text{EDH}} = \int \left( \frac{R}{16\pi G} + \mathcal{L}_m \right) \sqrt{-g} \; d^4x,$$

where $g = \det(g_{\mu\nu})$ is the determinant of the metric, $R$ is the Ricci scalar, and the Lagrangian density for the gravitational sector is of the usual Einstein-Hilbert form. Here and throughout this Letter we use natural units, in which $h = c = 1$. The Lagrangian density for the matter sector is

$$\mathcal{L}_m = \bar{\Psi}(\not{D} - \mu h)\Psi - \frac{1}{2}(\nabla^\nu h)(\nabla_\nu h) - V(h),$$

where $\not{D}$ is the Dirac operator in curved spacetime. The fermions are coupled to a Higgs field $h$, which we model as a real scalar field, with coupling strength $\mu$. Hence the fermion mass $\mu h$ becomes a locally varying quantity, set by the local value of $h$. The Higgs potential $V(h)$ is taken to be of the usual 'Mexican hat' form,

$$V(h) = \lambda(h^2 - v^2)^2,$$

where the constant $\lambda$ is an overall scaling factor, and the stable minima of the potential occur at the vacuum expectation values $h = \pm v$. The mass of the Higgs field is that associated with small displacements around $v$, and takes the value $m_H = 2v\sqrt{2}\lambda$.

Extremizing the action (1) with respect to the spinor $\Psi$, the metric $g_{\mu\nu}$, and the Higgs field $h$, gives respectively the Dirac, Einstein, and Higgs equations:

$$\left( \not{D} - \mu h \right) \Psi = 0; \quad (4)$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}; \quad (5)$$

$$\nabla_\nu \nabla^\nu h = \mu \bar{\Psi} \Psi + \frac{dV}{dh}. \quad (6)$$

We seek static, spherically symmetric solutions to this coupled system, corresponding to energy eigenstates. Using the usual spherical polar coordinate system $(t, r, \theta, \phi)$, the metric, viewed by an inertial observer at spatial infinity, can be written as

$$g_{\mu\nu} = \text{diag} \left( -\frac{1}{T(r)^2}, \frac{1}{A(r)}, r^2, r^2 \sin^2 \theta \right),$$

where the forms of the metric fields $T(r)$ and $A(r)$ are to be determined. For the fermionic sector, the simplest case compatible with spherical symmetry is that of two fermions arranged in a singlet state. The appropriate ansatz for the spinor wavefunction in this case is stated in [9], and takes the following form:

$$\Psi_a = \frac{\sqrt{T(r)}}{r} \left( \begin{array}{c} \alpha(r) e_a \\ -i\beta(r) \sigma^e e_a \end{array} \right) e^{-i\omega t}. \quad (8)$$

The two fermions, identified by their value of $a \in \{1, 2\}$, are assumed to have a common energy $\omega$, with their wavefunctions differing only via the two-component basis vectors $e_1 = (1, 0)^T$ and $e_2 = (0, 1)^T$. The radial dependence of the spinor is controlled by the unknown fermion fields $\alpha(r)$ and $\beta(r)$, which can be identified, in the non-relativistic limit, with the fermion and anti-fermion parts of the wavefunction respectively.

Under these ansatzes, the Dirac equation (4) reduces to the following two real-valued equations:

$$\sqrt{A} \alpha' = \frac{\alpha}{r} - (\omega T + \mu h)\beta; \quad (9)$$

$$\sqrt{A} \beta' = -\frac{\beta}{r} + (\omega T - \mu h)\alpha, \quad (10)$$

where $' \equiv \frac{d}{dr}$. These are identical to those valid in the Einstein-Dirac case, except that the fermion mass is replaced by $\mu h$. Only two of the components of the Einstein equation (5) turn out to be independent, with the most convenient choices being the $tt$ and $rr$ components:

$$\frac{1 - A}{r^2} - \frac{A'}{r} = 8\pi G \left( \frac{2\omega}{r^2} T^2 (\alpha^2 + \beta^2) + \frac{1}{2} A(h')^2 + V(h) \right); \quad (11)$$

$$\frac{1 - A}{r^2} + \frac{2AT'}{rT} = 8\pi G \left( \frac{2T\sqrt{A}}{r^2} (\beta\alpha' - \alpha\beta') - \frac{1}{2} A(h')^2 - V(h) \right). \quad (12)$$

The bracketed terms in these two equations are respectively the energy density and the radial pressure in the matter sector. Finally, the Higgs equation (6) becomes

$$Ah'' - A \left( \frac{T'}{T} - \frac{A'}{2A} - \frac{2}{r} \right) h' = \frac{2\mu}{r^2} T (\alpha^2 - \beta^2) + \frac{dV}{dh}. \quad (13)$$

For localized, particlelike solutions, we require the following boundary conditions. First, the metric should be asymptotically flat, implying $A(r), T(r) \to 1$ as $r \to \infty$. Second, the fermion wavefunction should be correctly normalized, i.e. the inner product $(\Psi|\Psi) = 1$. Using (8), this can be rewritten as

$$4\pi \int_0^\infty \frac{T}{\sqrt{A}} (\alpha^2 + \beta^2) \; dr = 1. \quad (14)$$

Third, the energy-density contribution from the Higgs field should vanish at large $r$. Considering (11) and (12),
FIG. 2. Left: The intrinsic Mexican-hat Higgs potential $V(h)$, with two stable minima at $h = \pm v$. Right: The (inverted) effective Higgs potential $V_{\text{eff}}(h)$, which exhibits a tilt inside the fermion source.

This is achieved when $h' = V(h) = 0$, implying $h \to \pm v$ as $r \to \infty$.

**Higgs field dynamics.** Before presenting numerical solutions to this set of equations, we outline the expected behavior of the Higgs field, following a similar rationale to Schrödinger et al. [15]. Consider first the situation outside the fermion source, where the fermion fields $\alpha$ and $\beta$ are negligible. Let us also temporarily introduce a time-dependence to the Higgs field. In this case, the Higgs equation can be written as

$$T^2 h'' - Ah'' + A \left( \frac{T'}{T} - \frac{A'}{2A} - \frac{2}{r} \right) h' = -\frac{dV}{dh},$$

where the dot denotes a time-derivative. Note the sign difference between the $h$ and $h''$ terms; this implies that the two dynamically stable minima in the Higgs potential $(h = \pm v)$ are unstable maxima from the point of view of spatial variations [16]. Hence, in the static case that we are considering here, the Higgs potential is effectively inverted compared to its usual form.

At positions within the fermion source, the coupling to the fermions introduces an additional term on the right-hand side of the Higgs equation. Combining this with the Higgs potential $V(h)$, we can rewrite the Higgs equation as $\nabla_h \nabla^\mu h = -\partial V_{\text{eff}} / \partial h$, where we have defined an effective potential that takes the form

$$V_{\text{eff}}(h) = -\lambda (h^2 - v^2)^2 - \frac{2\mu}{r^2} T (\alpha^2 - \beta^2) h.$$  

The fermionic term introduces a ‘tilt’ to the intrinsic Mexican-hat Higgs potential via a term that is linear in $h$; this is illustrated schematically in Fig. 2.

We shall consider here only states in which $\alpha$ and $\beta$ are nodeless (ground states). Consequently, the asymptotic fermion mass, $m_f = \mu v$, must be positive, implying that the Higgs field outside the fermion source should asymptote to $h = \pm v$. In addition, since $\alpha$ is the dominant fermion field, the cumulative tilt to the Higgs potential must always be in the direction indicated in Fig. 2, implying $h < v$ within the fermion source.

**Numerical results.** We now present numerical solutions of (9)–(13), representing particlelike states consisting of two neutral fermions. Solutions are generated using Mathematica’s built-in differential equation solver, NDSolve, to an accuracy of 8 digits. As with the Einstein-Dirac case, normalizable solutions occur only at discrete values of the fermion energy, requiring us to tune $\omega$ to its ground-state value. We must also tune the value of the Higgs field at $r = 0$ such that it approaches its asymptotic value, $v$, as $r \to \infty$. Generation of solutions thus requires a two-parameter shooting procedure, details of which can be found in the supplemental material.

A cursory glance at the equations of motion suggests that $\mu$, $\lambda$, and $v$ are all free parameters within the theory. This is not the case however: the normalization condition (14) removes one degree of freedom, and thus only two of these may be independently specified. From a computational perspective, it is significantly more efficient to choose the two parameters as $v$ and $\xi$, where $\xi$ is the Higgs to (asymptotic) fermion mass ratio:

$$\xi \equiv \frac{m_H}{m_f} = \frac{2\sqrt{2\lambda}}{\mu}. \quad (17)$$

Each choice of $\{v, \xi\}$ defines a one-parameter family of solutions where, as for the Einstein-Dirac case, the value of the central redshift, $z = T(0) - 1$, uniquely identifies each state. This redshift can be employed as a measure of how relativistic a solution is, with $z \approx 1$ marking the crossover from non-relativistic to relativistic.

We have been successful in generating solutions with parameter values ranging approximately from $\{v, \xi\} = \{0.07, 0.03\} \to \{10, 30\}$. The upper limits on this arise from issues concerning numerical precision, but the reason behind the lower limits is less clear. In particular, we have been unable to obtain solutions below $v = 0.07$ for any value of $\xi$, although this may be due to the failure of our numerical method rather than an indication of their non-existence.

Examples of three particlelike solutions are shown in Fig. 3, corresponding to the three labeled states in Fig. 1. Solution (a) is Higgs-dominated ($\xi > 1$), while solutions (b) and (c) are fermion-dominated ($\xi < 1$). Plotted on the left are the radial profiles of the fermion number density $n_f(r)$, defined as

$$n_f = \frac{2T}{r^2} (\alpha^2 + \beta^2), \quad (18)$$

which take considerably different forms for the three states. For solution (a), the fermion density peaks at $r = 0$ (as in the Einstein-Dirac case), whereas, for solutions (b) and (c), the peak is shifted outwards in radius. The metric fields are singularity free, with $T(r)$ decreasing monotonically from a central maximum, while $A(r) \leq 1$ throughout. Outside the fermion source, the metric fields approach the standard Schwarzschild form, for which $T_{\text{Sch}}^{-2} = A_{\text{Sch}} = 1 - 2GM/r$. This allows us to identify an Arnowitt-Deser-Misner (ADM) mass $M$,
which provides a measure of the total energy of the localized state. In all three solutions, the Higgs field rises from a constant central value before asymptoting towards \( v = 0.08 \). In solution (a), the Higgs field deviates only slightly from its vacuum expectation value, and hence it is similar to an Einstein-Dirac state. For solutions (b) and (c), however, the fermion tilt is large enough that the Higgs field becomes negative at small \( r \), resulting in a negative local fermion mass.

**Mass scales.** It is interesting to compare the mass scales of the three states in Fig. 3. Despite their similar ADM masses (1.08, 0.946 and 0.899 respectively), they have significantly different asymptotic fermion masses (0.558, 1.00 and 3.68 respectively), as a consequence of their differing \( \mu \) values. Why, one might wonder, has the ADM mass not increased in proportion with the fermion mass for the states in which the fermion-Higgs coupling is strong, i.e. when the value of \( \mu \) is large? Although the reason for this mass-scale separation is not entirely clear, we can make progress by expressing the ADM mass using the Komar integral [17], finding

\[
M = 4\omega - 8\pi \int_0^\infty \frac{dr}{\sqrt{A}} \left[ \mu h(\alpha^2 - \beta^2) + \frac{r^2}{T} V(h) \right].
\]  

For states with large \( \mu \), the Higgs field is initially negative (due to the strong fermion tilt), and hence the first term in the integral switches sign between the inner and outer regions of the fermion source. Its overall contribution is therefore negligible, and thus the large value of \( \mu \) does not directly affect the ADM mass.

This is not a complete explanation, however, since we would expect the fermion energy \( \omega \) also to scale with the fermion mass. Instead, we find the reverse is true: \( \omega \) tends to be lower for states in which the fermion mass is large. Without knowing what precisely affects the value of \( \omega \), it is difficult to put forward an explanation for this. One suggestion may be that the change in the fermion density profiles (itself a consequence of the strong fermion-Higgs coupling) prevents \( \omega \) from increasing. Also potentially related is an observed disparity between the radial decay scales of the fermion and Higgs fields, with these coinciding at precisely \( \xi = 2 \). The link between this and the value of \( \omega \), however, is not obvious.

Despite this uncertainty, it is clear that the effect of mass scale separation is a consequence of strong coupling between the fermion and Higgs fields. However, when we generate solutions, \( \mu \) is not an input parameter; instead we specify the values of \( v \) and the Higgs-to-fermion mass ratio \( \xi \). What ranges of \( v \) and \( \xi \), then, correspond to strong coupling? Recall that each choice of \( \{ v, \xi \} \) de-
FIG. 5. Binding energy as a function of (rms) radius $\bar{R}$ for the family of localized states with $v = 0.08$ and $\xi = 0.14$. In the Einstein-Dirac case (inset), only a portion of the curve contains solutions that are bound (those with negative binding energy), but with the addition of the Higgs field it is possible for all solutions in a family, even the highly-relativistic states located near the center of the spiral, to become bound.

finishes a family of solutions, parameterized by the central redshift $z$. The fermion mass-energy relations for a selection of these are shown in Fig. 4. At low redshift, the curves converge and approach the non-relativistic relation, $\omega = m_f$, but they deviate significantly as the redshift increases. Overall, the families with lower values of $v$ and $\xi$ show the largest deviations from the Einstein-Dirac case, both towards lower $\omega$ and larger $m_f$. In particular, if we consider only the states of maximum fermion mass in each family, we find that indeed there is a general increase in $\mu$ as both $v$ and $\xi$ are decreased, and that the ADM and fermion mass scales decouple once $\xi < 2$. The results of this analysis (for fixed $v = 0.08$) were shown in Fig. 1. We note that it is not surprising that small $\xi$ implies large $m_f$ (and $\mu$), since $\xi \approx 1/\mu$, but the requirement for small values of $v$ is less clear.

The decoupling of mass scales also affects the binding energy of states, defined as $E_b = M - 2m_f$, i.e. the difference between the energy of the state and that of two individual delocalized fermions. With this definition, a state is considered bound if it has a negative value of $E_b$ (energy is required to break it apart). As we have shown, it is possible, at strong coupling, for the fermion mass to far outweigh the ADM mass, suggesting that such states should be highly bound. Indeed, in contrast to the Einstein-Dirac case, it is even possible for entire families of solutions to become bound. An example of this is shown in Fig. 5.

Negative binding energy does not necessarily imply stability, however. In the Einstein-Dirac case, the only stable states are those with redshift lower than that of the most bound state. We expect the same criterion to apply here, implying that only a portion of the curve in Fig. 5 is stable (as indicated). Nonetheless, this stable branch does include states in which the constituent fermions are significantly more massive than allowed in the Einstein-Dirac system.

Summary. We have constructed localized solutions of the minimally coupled Einstein-Dirac-Higgs system, and have shown that the resulting partecular states are well behaved and free from singularities. Somewhat unexpectedly, at strong fermion-Higgs coupling, we find that the ADM mass appears to become parametrically smaller than the masses of the constituent fermions, allowing fermions of much larger mass than anticipated to form bound states. Although our analysis is restricted to a semi-classical approximation, these results may nonetheless be an indication of how fermionic objects with a finite extent may be expected to interact with a Higgs field, within the framework of general relativity.

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[1] S. W. Hawking, Particle creation by black holes, Commun. Math. Phys 43, 199 (1975).
[2] D. J. Kaup, Klein-Gordon Geon, Phys. Rev. 172, 1331 (1968).
[3] D. A. Feinblum and W. A. McKinley, Stable states of a scalar particle in its own gravitational field, Phys. Rev. 168, 1445 (1968).
[4] S. L. Liebling and C. Palenzuela, Dynamical boson stars, Living Rev. Relativ. 20, 5 (2017).
[5] J. Lee and I. Koh, Galactic halos as boson stars, Phys. Rev. D 53, 2236 (1996).
[6] D. F. Torres, S. Capozziello, and G. Lambiase, Supermassive boson star at the galactic center?, Phys. Rev. D 62, 104012 (2000).
[7] R. Ruffini and S. Bonazzola, Systems of self-gravitating particles in general relativity and the concept of an equation of state, Phys. Rev. 187, 1767 (1969).
[8] T. D. Lee and Y. Pang, Fermion soliton stars and black holes, Phys. Rev. D 35, 3678 (1987).
[9] F. Finster, J. Smoller, and S.-T. Yau, Particlelike solutions of the Einstein-Dirac equations, Phys. Rev. D 59, 104020 (1999).
[10] F. Finster, J. Smoller, and S.-T. Yau, Particle-like solutions of the Einstein–Dirac–Maxwell equations, Phys. Lett. A 259, 431 (1999).
[11] F. Finster, J. Smoller, and S.-T. Yau, The interaction of Dirac particles with non-Abelian gauge fields and gravity–bound states, Nucl. Phys. B 584, 387 (2000).
[12] C. Herdeiro, P. Perapechka, E. Radu, and Y. Shnir, Asymptotically flat spinning scalar, Dirac and Proca stars, Phys. Lett. B 797, 134845 (2019).
[13] P. E. D. Leith, C. A. Hooley, K. Horne, and D. G. Dritschel, Fermion self-trapping in the optical geometry of Einstein-Dirac solitons, Phys. Rev. D 101, 106012 (2020).
[14] P. E. D. Leith, C. A. Hooley, K. Horne, and D. G. Dritschel, Nonlinear effects in the excited states of
many-fermion Einstein-Dirac solitons, Phys. Rev. D 104, 046024 (2021).

[15] S. Schlögel, M. Rinaldi, F. Staelens, and A. Füzfa, Particlelike solutions in modified gravity: the Higgs monopole, Phys. Rev. D 90, 044056 (2014).

[16] S. Coleman, Aspects of symmetry (Cambridge University Press, 1985).

[17] R. M. Wald, General relativity (University of Chicago press, 2010).
**EQUATIONS OF MOTION — FURTHER DETAILS**

In this section, we expand upon the basic derivation of the equations of motion given in the main text of the Letter. Much of this is similar to the Einstein-Dirac case, so we refer the reader to the derivation given in [1]. Note however that some of the expressions differ due to sign conventions.

We begin by restating the matter Lagrangian for the Einstein-Dirac-Higgs system, along with the (spherically symmetric) metric and spinor ansatzes valid for a pair of fermions arranged in a singlet state:

\[
\mathcal{L}_m = \Psi (\hat{D} - \mu h) \Psi - \frac{1}{2} (\nabla \nu h)(\nabla_\nu h) - V(h); \quad (1)
\]

\[
g_{\mu \nu} = \text{diag} \left( -\frac{1}{T(r)^2}, \frac{1}{A(r)}, r^2, r^2 \sin^2 \theta \right); \quad (2)
\]

\[
\Psi_\alpha = \frac{\sqrt{T(r)}}{r} \left( \alpha(r)e_\alpha - i \sigma \tau \beta(r)e_\alpha \right) e^{-i \omega t}. \quad (3)
\]

The rationale behind this last expression is discussed in detail in [2]. Recall also the ‘Mexican-hat’ form taken for the Higgs potential:

\[
V(h) = \lambda (h^2 - v^2)^2. \quad (4)
\]

In general, the Dirac operator in curved spacetime can be written as \( \hat{D} = i \gamma^\mu \partial_\mu + \Gamma_\mu \), where \( \Gamma_\mu \) is the spin-connection, and \( \gamma^\mu \) are curved-space generalizations of the Dirac gamma matrices, chosen to obey the anticommutation relations \( \{ \gamma^\mu, \gamma^\nu \} = -2g^{\mu \nu} \). Note that the sign here is chosen such that the gamma matrices are identical to those generally used in the mostly negative metric signature, i.e. no additional factors of \( i \) are required.

Using the vierbein formalism, one can relate the curved-space gamma matrices to their flat-space counterparts, \( \gamma^\alpha \), by the relation \( \gamma^\mu = e^\mu_\alpha \gamma^\alpha \). Using the metric ansatz (2), and working in spherical polar coordinates \((t, r, \theta, \phi)\), the only non-zero vierbein components are \( e^t_i = T, e^r_i = \sqrt{A} \) and \( e^\theta_i = e^\phi_i = 1 \). Thus the curved-space gamma matrices take the explicit forms:

\[
\gamma^t = T \gamma^0; \quad (5)
\]

\[
\gamma^r = \sqrt{A} (\gamma^1 \sin \theta \cos \phi + \gamma^2 \sin \theta \sin \phi + \gamma^3 \cos \theta); \quad (6)
\]

\[
\gamma^\theta = \frac{1}{r} (\gamma^1 \cos \theta \cos \phi + \gamma^2 \cos \theta \sin \phi - \gamma^3 \sin \theta); \quad (7)
\]

\[
\gamma^\phi = \frac{1}{r \sin \theta} (\gamma^1 \sin \phi + \gamma^2 \cos \phi), \quad (8)
\]

where the flat-space gamma matrices can be written in terms of the usual Pauli matrices, as follows:

\[
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \quad (9)
\]

Within this formalism, it can be shown [1] that the Dirac operator in curved spacetime takes the form:

\[
\hat{D} = i \gamma^\mu \partial_\mu + \frac{i}{2} \nabla_\mu \gamma^\mu \quad (10)
\]

\[
= i \gamma^t \frac{\partial}{\partial t} + i \gamma^r \left( \frac{\partial}{\partial r} + \frac{1}{r} \left( 1 - \frac{1}{\sqrt{A}} \right) - \frac{T}{2T} \right) + i \gamma^\theta \frac{\partial}{\partial \theta} + i \gamma^\phi \frac{\partial}{\partial \phi}. \quad (11)
\]

By applying this to the spinor ansatz (3), it is then straightforward to obtain the explicit expressions for the Dirac equations quoted in the Letter.

To derive the Einstein equations, we calculate the stress-energy tensor by varying the matter Lagrangian as per the definition:

\[
T_{\mu \nu} = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu \nu}} (\sqrt{-g} \mathcal{L}_m). \quad (12)
\]

For the fermionic sector, it was shown in [1] that, for a singlet state, the only contribution to the variation of the Dirac operator is from the first term in (10). To evaluate this, the following identities prove useful:

\[
\delta \gamma^\mu = \frac{1}{2} g_{\sigma \mu} \gamma^\sigma \delta g^{\mu \nu}; \quad (13)
\]

\[
\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu \nu} \delta g^{\mu \nu}; \quad (14)
\]

\[
\delta g_{\sigma \nu} = -g_{\sigma \rho} g_{\mu \nu} \delta g^{\mu \rho}. \quad (15)
\]

The variation of the Higgs terms in the matter Lagrangian is straightforward, and thus we arrive at the
final form of the stress-energy tensor:
\[
T_{\mu\nu} = -\sum_{a=1}^{2} \Re \left\{ \overline{\Psi}_a (i\gamma_\mu \partial_\nu) \Psi_a \right\} + (\partial_\mu h)(\partial_\nu h) \\
- g^{\mu\nu} \left[ \frac{1}{2} (\partial^\sigma h)(\partial_\sigma h) + V(h) \right],
\]
(16)
where, as in the Einstein-Dirac case, the contribution from the two fermions can simply be added. Explicitly, the non-zero components of the (mixed) stress-energy tensor are therefore
\[
T^t_t = \frac{2aT^2}{r^2}(\alpha^2 + \beta^2) - \frac{1}{2} A(h')^2 - V(h);
\]
(17)
\[
T^r_r = \frac{2T\sqrt{\lambda}}{r^2} (\alpha' - \beta') + \frac{1}{2} A(h')^2 - V(h);
\]
(18)
\[
T^\theta_\theta = T^\phi_\phi = \frac{2T}{r^3} (\alpha^2 - \frac{1}{2} A(h')^2 - V(h)).
\]
(19)
Note that only two of these are independent, since the stress-energy tensor is divergenceless, i.e. \(\nabla_\mu T^\mu_\nu = 0\).

The components of the Einstein tensor can be obtained from the metric ansatz (2) via the standard procedure of calculating the Christoffel symbols, the Riemann and Ricci tensors, and the Ricci scalar. In the sign conventions used here, the expressions for these are as follows:
\[
\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\alpha\tau} (\partial_\mu g_{\nu\tau} + \partial_\nu g_{\mu\tau} - \partial_\tau g_{\mu\nu});
\]
\[
R^\mu_{\nu\alpha\beta} = \partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\lambda\alpha} \Gamma^\lambda_{\nu\beta} - \Gamma^\mu_{\lambda\beta} \Gamma^\lambda_{\nu\alpha};
\]
\[
R_{\mu\nu} = R^\rho_{\rho\mu\nu}; \quad R = R^\mu_{\mu}; \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}.
\]
(20)
Explicitly, the only non-zero (mixed) Einstein tensor components are:
\[
G^t_t = \frac{1}{r^2} \left( -1 + A + rA' \right);
\]
(21)
\[
G^r_r = \frac{1}{r^2} \left( -1 + A - \frac{2rAT'}{T} \right);
\]
(22)
\[
G^\theta_\theta = G^\phi_\phi = \frac{A'}{2r} - \frac{A'T'}{2T^2} - \frac{2A(T')^2}{T^2} - \frac{AT'}{rT} - \frac{AT''}{T}.
\]
(23)
Combining these with the stress-energy tensor components above, we obtain the final expressions for the Einstein equations, the \(tt\) and \(rr\) instances of which are stated in the Letter.

**NUMERICAL METHOD**

In this section, we shall outline the numerical method used to generate ‘particlelike’ states of the Einstein-Dirac-Higgs system. First, recall that localized solutions require the following boundary conditions: the metric must be asymptotically flat; the fermions must be correctly normalized; and the Higgs field must asymptote to its vacuum expectation value. In addition to these, we desire states to be free from singularities, which can be achieved by imposing regularity of all fields at \(r = 0\). This leads to the following small-\(r\) expansion:
\[
\alpha(r) = \alpha_1 r + \ldots
\]
(24)
\[
\beta(r) = \frac{1}{3} \alpha_1 (\omega T_0 - \mu h_0) r^2 + \ldots
\]
(25)
\[
T(r) = T_0 + \frac{4\pi G}{3} T_0 \left( V(h_0) + \alpha_1^2 T_0 (\mu h_0 - \omega T_0) \right) r^2 + \ldots
\]
(26)
\[
A(r) = 1 - \frac{8\pi G}{3} \left( V(h_0) + 2\omega \alpha_1^2 T_0^2 \right) r^2 + \ldots
\]
(27)
\[
h(r) = h_0 + \frac{1}{3} \left( 2\lambda h_0 (h_0^2 - v^2) + \mu \alpha_1^2 T_0 \right) r^2 + \ldots
\]
(28)
where the coefficients \(\alpha_1, T_0\) and \(h_0\) are unconstrained. This expansion is used to initialize the numerical solver at a small but non-zero radius (\(r_\text{in} = 10^{-8}\)).

In order to deal with the conditions of asymptotic flatness and normalization, we implement a ‘rescaling’ procedure similar to that outlined in [1]. This relies on the fact that, if \(\alpha, \beta \to 0\) and \(h \to v\) at large \(r\), the equations of motion imply \(A \to 1\) and \(T \to \text{const.}\) as \(r \to \infty\), and the normalization integral will evaluate to a constant. Thus, it is sufficient to generate an ‘unscaled’ solution (denoted by a tilde), for which we set \(T_0 = 1\) and \(\tilde{\mu} = 1/v\), and ensure that the fermion fields decay at large \(r\) and the Higgs field asymptotes to \(v\). The true, physical solution can then be obtained by simply rescaling the fields such that both \(T(\infty)\) and the normalization integral are equal to 1.

More formally, having generated an unscaled solution, we define
\[
\tau = \lim_{r \to \infty} \tilde{T}(r);
\]
(29)
\[
\chi^2 = 4\pi \int_0^\infty \left( \tilde{\alpha}^2 + \tilde{\beta}^2 \right) \tilde{T} \tilde{A}^{-1/2} dr,
\]
(30)
and then rescale the fields and parameters as follows, noting that \(v\) needs no rescaling:
\[
\alpha(r) = \sqrt{\frac{\tau}{\chi}} \tilde{\alpha}(\chi r);
\]
\[
\beta(r) = \sqrt{\frac{\tau}{\chi}} \tilde{\beta}(\chi r);
\]
\[
T(r) = \frac{1}{\tau} \tilde{T}(\chi r);
\]
\[
A(r) = \tilde{A}(\chi r);
\]
\[
h(r) = \tilde{h}(\chi r);
\]
\[
\omega = \chi \tau \tilde{\omega};
\]
\[
\mu = \tilde{\mu};
\]
\[
\lambda = \frac{\lambda}{\chi}.
\]
(31)
The above procedure fixes values for both \(T_0\) and \(\mu\) (\(T_0 = 1/\tau\) and \(\mu = \chi/v\)), leaving just \(\alpha_1, \omega, h_0, \lambda\) and \(v\) to consider. Ideally, we would like to treat \(\lambda\) and \(v\) as free parameters within the theory, but complications arise since any input value of \(\lambda\) will be changed by
Solve, with an explicit Runge-Kutta method. In the majority of cases, an accuracy of 8 digits suffices to obtain the solution, but this must be increased for states in which the Higgs field is slow in approaching its asymptotic value \( v \) (e.g. at low redshift). As mentioned in the Letter, the method fails to find solutions below \( v \approx 0.07 \) even at high accuracy. It is unclear whether this implies that solutions do not exist, or simply that an alternative method is required to obtain them.

**BINDING ENERGY IN TERMS OF \( \xi \) AND \( v \)**

We stated in the Letter that it is when both \( \xi \) and \( v \) are small that the separation of mass scales is most prominent, and that this drives an overall decrease in binding energy and thus more strongly bound states. As a result, entire families of solutions can become bound, an example of which was shown in Fig. 5 of the Letter.

Here, we present a more detailed analysis of this phenomenon, in which we vary the values of \( \xi \) and \( v \) and analyze the impact on the least bound state in each family. The results of this are summarized in Fig. S1. Overall, there is a general decrease in the binding energy, \( E_b = M - 2m_f \), as both \( \xi \) and \( v \) are decreased, with the regions in which \( E_b \) is negative indicating that the entire family of states with those parameter values is bound. The accompanying plots of the fermion and ADM masses reveal that this is indeed driven by an increase in the fermion mass of the states that is not accompanied by a corresponding increase in the ADM mass.

**ALTERNATIVE FAMILIES OF SOLUTIONS**

Recall that, of the three physical parameters \( \{\mu, \lambda, v\} \), only two can be freely specified as inputs; the other is fixed by imposing normalization. As previously mentioned, the computationally efficient choice for the two

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**FIG. S1.** Contour plots showing the binding energy \( E_b \), fermion mass \( m_f \), and ADM mass \( M \) of the least bound state, as a function of the parameters \( \xi \) and \( v \) that define each family. A negative value for the binding energy indicates that the entire family of solutions is bound. Note that the contours become less smooth at smaller values of \( \xi \) where the errors arising from our numerics are largest.

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the rescaling procedure. Instead, we therefore choose \( v \) and \( \xi = m_H/m_f = 2\sqrt{2\lambda}/\mu \) as our input parameters, since both of these quantities remain unchanged under the rescaling. For each choice of \( \{\xi, v\} \), the value of \( \alpha_1 \) can be freely specified, defining a 1-parameter family of solutions that can be parameterized either by \( \alpha_1 \) or (after rescaling) by the central redshift \( z = T(0) - 1 \). Note that, however, in contrast to the Einstein-Dirac case, this correspondence is not necessarily one-to-one. Indeed, for small values of \( \xi \) and \( v \), multiple (ground-state) solutions can be found with the same value of \( \alpha_1 \), but upon rescaling their values of \( z \) are found to differ. Nonetheless, we find that each solution can still be uniquely identified by its central redshift, but not necessarily by its value of \( \alpha_1 \). This property further complicates the solution-finding procedure, since we must ensure that all solutions for a particular value of \( \alpha_1 \) have been identified.

Finally, the values of \( \omega \) and \( h_0 \) must be determined such that the fermion fields become normalizable (decay sufficiently rapidly at large \( r \)) and the Higgs field asymptotes to \( v \). Due to the coupling between the fields, these cannot be sought independently, and thus a two-parameter shooting procedure is required. Fortunately, it is possible to implement this sequentially by first choosing a value for \( h_0 \), performing a simple binary chop to find the value of \( \omega \) for which the fermions become normalizable, and then noting the behavior of the Higgs field at large \( r \). If \( h(\infty) < v \), then \( h_0 \) should be increased, and likewise if \( h(\infty) > v \) then \( h_0 \) should be decreased. By iterating this procedure, it is possible to force \( h(\infty) \) to its vacuum expectation value, while ensuring that the fermions remain normalizable. Note that this works well only for solutions that are Higgs dominated (\( \xi < 2 \)); for fermion-dominated states, the shooting order should be reversed such that a binary chop in \( h_0 \) is performed at chosen values of \( \omega \).

With regard to the numerical solver itself, we use Mathematica’s built-in differential equation solver, ND-
input parameters is \( v \) and \( \xi = 2\sqrt{2\lambda/\mu} \). As such, the families of solutions presented in the Letter are defined by these values.

It is important to note that this is not a unique choice. It should in principle be possible to choose any two parameters from \( \{\mu, \lambda, v\} \) (or two independent combinations), with each separate choice defining a distinct family of states. As an example of this, we have generated the family of solutions defined by \( \{\lambda = 0.053, v = 0.3\} \), and the resulting fermion mass-energy curve is shown in Fig. S2. Note that this contains only a relatively small number of points, since it is significantly more difficult to obtain computationally. Nonetheless, it is clear that this family of states exhibits the expected spiraling behavior, and we expect other parameter choices to produce similar curves. It is important to point out, however, that choosing a new pair of parameters to use does not produce a new set of states; it only reparamerises the 2-dimensional manifold of solutions.

FIG. S2. The fermion mass-energy relation for the family of states with parameter values \( v = 0.30 \) and \( \lambda = 0.053 \), along with the corresponding Einstein-Dirac curve for comparison.

EXCITED STATES

The analysis presented in the Letter is limited to ground-state solutions of the Einstein-Dirac-Higgs system. We are also able to obtain excited states, and have found that these are similar in structure to those for the Einstein-Dirac case. In particular, for each value of the central redshift \( z \), there exists a (presumably infinite) tower of excited states, where the \( n^{\text{th}} \) excited state contains a total of \( n \) nodes in the fields \( \alpha \) and \( \beta \).

An example of an 8th excited state is shown in Fig. S3. Note that the additional oscillations in the fermion fields affect not only the metric but also the Higgs field, since the tilt in the effective Higgs potential is proportional to \( \alpha^2 - \beta^2 \). Note also that the term 'excited' only implies higher fermion energy when the system is non-relativistic. For example, it is possible at high redshift for an excited state to have a lower value of \( \omega \) than the ground state, and in such cases only the nodal structure can be used to categorize the states.

FIG. S3. An example of an 8th excited state \( (n = 8) \), showing the radial profiles of the fermion, metric and Higgs fields. There are a total of eight nodes in \( \alpha \) and \( \beta \) (four in each). The parameter values for this solution are \( \{n = 8, \xi = 0.283, v = 0.100, z = 1.172, \mu = 11.61, \lambda = 1.347, \omega = 0.986, m_f = 1.161, m_H = 0.328, M = 2.302, R = 13.84, E_b = -0.019\} \).

POWER-LAW SOLUTIONS

For the Einstein-Dirac system, it was shown in [3] that the structure of localized states can be understood in terms of distinct zones. In particular, for high-redshift solutions, there exists a ‘power-law’ zone where the solution approximates that of the massless Einstein-Dirac equations, for which all fields have a simple power-law dependence on \( r \). It was also demonstrated that the infinite-redshift solution located at the center of the spiraling curves contains a power-law zone that extends all the way to \( r = 0 \). Here, we show that similar properties exist for the states in the Einstein-Dirac-Higgs system.
First, we derive the analog of the massless power-law solution. In the context of the Einstein-Dirac-Higgs system, this requires $\omega T \gg \mu h$, i.e. that the local fermion mass is negligible compared to the local fermion energy. Since this occurs in regions in which the solution is highly relativistic, we expect the energy density from the fermions to dominate over the contribution from the Higgs field. Hence the Dirac and Einstein equations can be reduced to:

\[
\sqrt{A} \alpha' = -1 \frac{\alpha}{r} - \omega T \beta; \quad (32)
\]

\[
\sqrt{A} \beta' = -1 \frac{\beta}{r} + \omega T \alpha; \quad (33)
\]

\[
1 - A - r A' = 16\pi G \omega T^2 (\alpha^2 + \beta^2); \quad (34)
\]

\[
1 - A + 2r AT' \frac{T}{T} = 16\pi G T \sqrt{A} (\beta \alpha' - \alpha \beta'). \quad (35)
\]

This is precisely the massless Einstein-Dirac system; thus the power-law zone is a region in which the Higgs field has no effect on either the metric or the distribution of the fermion source. From [3], the solution to the above system is

\[
\alpha(r) = \alpha_p r; \quad \beta(r) = \beta_p r; \quad (36)
\]

where

\[
\alpha_p = 3^{1/4} \sqrt{\frac{\omega}{48\pi G (\sqrt{3} - 1)}}; \quad (37)
\]

\[
\beta_p = 3^{1/4} \sqrt{\frac{\omega}{48\pi G (\sqrt{3} + 1)}}. \quad (38)
\]

The behavior of the Higgs field in the power-law zone is governed by the Higgs equation, which, upon substituting the solution above, reduces to

\[
\frac{1}{3} h'' + r h' = \frac{\mu}{12\sqrt{2\pi G r}} + 4\lambda h (h^2 - v^2). \quad (39)
\]

To proceed further, we assume that the power-law zone occurs at small $r$ (this should certainly be the case at high redshift), and that $h(r)$ also has a simple power-law dependence. Thus, in order for the Higgs energy density not to contribute to the Einstein equations, we require $h'(r)$ to lead with a power greater than $-1$ at small $r$. This implies that the second term on the right hand side of (39) must be negligible. We can then solve for $h(r)$ to give

\[
h(r) = \frac{\mu}{12\sqrt{2\pi G}} r - \frac{c_1}{2r^2} + c_2, \quad (40)
\]

where $c_1$ and $c_2$ are constants. From the argument above we are forced to set $c_1 = 0$, and therefore, in the power law zone, the Higgs field must be approximately linear, i.e.

\[
h(r) = \frac{\mu}{12\sqrt{2\pi G}} r + c, \quad (41)
\]

where $c$ is a constant.

These expressions can be readily checked by analyzing the structure of high-redshift states, an example of which is shown in Fig. S4. This clearly illustrates the separation of the solution into three distinct zones: the core (in which the fields follow the small-$r$ expansion), the power-law zone (where all fields have approximately power-law dependence on $r$), and the evanescent zone (in which the fermion fields decay exponentially). As predicted, the Higgs field is indeed approximately linear in the power-law zone ($h' = \text{const}$.), and the precise numerical values agree well with those derived in the expressions above. Note that the oscillations within the power-law zone are caused by a fermion self-trapping effect, details of which can be found in [4].

As the redshift is increased further, the spatial extent of the core shrinks towards zero, and it is therefore possible to generate infinite-redshift states numerically by replacing the small-$r$ expansion (24)–(28) with the power-law expressions (36) and (41). An example of such an infinite-redshift solution is shown in Fig. S5. Note that the metric field $T$ diverges at $r = 0$, and hence the state contains a central singularity. The input values for this solution are $\{\xi, v\} = \{0.28, 0.10\}$, and thus we expect the state to lie at the center of the orange spiral in Fig. 4 of the Letter. Noting the output parameter values of $m_f = 0.468$ and $\omega = 0.260$, this is indeed confirmed to be the case.
FIG. S5. An example of an infinite-redshift state, showing the radial profiles of the fermion, metric and Higgs fields. The power-law zone here extends to \( r = 0 \), with \( \alpha \sim r \), \( \beta \sim r \), \( T \sim 1/r \), \( A = 1/3 \) and \( h \sim r + h_0 \) at small \( r \), as evident. The parameter values for this solution are \( \{ \xi = 0.283, v = 0.100, \mu = 4.677, \lambda = 0.219, \omega = 0.260, m_f = 0.468, m_H = 0.132, M = 0.942, R = 2.908, E_b = 0.006 \} \).

[1] F. Finster, J. Smoller, and S.-T. Yau, Particlelike solutions of the Einstein-Dirac equations, Phys. Rev. D 59, 104020 (1999).
[2] J. L. Blázquez-Salcedo and C. Knoll, Constructing spherically symmetric Einstein–Dirac systems with multiple spinors: Ansatz, wormholes and other analytical solutions, Phys. Rev. D 80, 1 (2020).
[3] D. Bakucz Canário, S. Lloyd, K. Horne, and C. A. Hooley, Infinite-redshift localized states of Dirac fermions under Einsteinian gravity, Phys. Rev. D 102, 084049 (2020).
[4] P. E. D. Leith, C. A. Hooley, K. Horne, and D. G. Dritschel, Fermion self-trapping in the optical geometry of Einstein-Dirac solitons, Phys. Rev. D 101, 106012 (2020).

Eur. Phys. J. C. 80, 1 (2020).