BRST QUANTIZATION
OF THE MASSLESS MINIMALLY COUPLED SCALAR FIELD IN DE SITTER SPACE
(ZERO MODES, EUCLIDEANIZATION AND QUANTIZATION)

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ABSTRACT: We consider the massless scalar field on the four-dimensional sphere $S^4$. Its classical action $S = \frac{1}{2} \int_{S^4} dV (\nabla \phi)^2$ is degenerate under the global invariance $\phi \rightarrow \phi + \text{constant}$. We then quantize the massless scalar field as a gauge theory by constructing a BRST-invariant quantum action. The corresponding gauge-breaking term is a non-local one of the form $S_{GB} = \frac{1}{2\alpha V} (\int_{S^4} dV \phi)^2$ where $\alpha$ is a gauge parameter and $V$ is the volume of $S^4$. It allows us to correctly treat the zero mode problem. The quantum theory is invariant under $SO(5)$, the symmetry group of $S^4$, and the associated two-point functions have no infrared divergence. The well-known infrared divergence which appears by taking the massless limit of the massive scalar field propagator is therefore a gauge artifact. By contrast, the massless scalar field theory on de Sitter space $dS^4$ - the lorentzian version of $S^4$ - is not invariant under the symmetry group of that spacetime $SO(1,4)$. Here, the infrared divergence is real. Therefore, the massless scalar quantum field theories on $S^4$ and $dS^4$ cannot be linked by analytic continuation. In this case, because of zero modes, the euclidean approach to quantum field theory does not work. Similar considerations also apply to massive scalar field theories for exceptional values of the mass parameter (corresponding to the discrete series of the de Sitter group).
1 - INTRODUCTION

The euclidean approach (i.e. generalization of Wick rotation) to quantum field theory in curved spacetimes has been extensively used in particular i) in order to get Feynman propagators or anticommutator functions in an elegant way, ii) in connection with path integral quantization, and iii) in the context of quantum cosmology. (See [1,2,3,4,5] and references therein for more details.) Its principal advantages are the following: it permits one to deal with elliptic operators instead of hyperbolic ones and then to consider well-posed problems and mathematically well-defined objects and expansions. It also permits one to consider path integrals with well-defined measure on the space of paths and which are convergent rather than oscillating and divergent. In simple cases such as de Sitter, Anti-de Sitter or Schwarzschild spacetimes as well as globally static ones (and more generally for certain spacetimes which can be considered as sections of four-dimensional complex manifolds), it seems that the euclidean approach does not present any difficulties, at least if “boundary conditions” are considered with care.

In this paper, we study the quantization of the massless minimally coupled scalar field on the euclidean version $S^4$ of de Sitter space $dS^4$. We BRST-quantize that theory. It is characterized by a non-local gauge-breaking term. The corresponding ghost field $c$, antighost field $\overline{c}$ and auxiliary field $b$ are all constant. In path integrals, the non-propagating auxiliary field must be integrated in a complex direction (as the conformal factor of gravitation [2]). We calculate the two-point function $\langle \phi(x)\phi(x') \rangle$ associated with the scalar field. We show that its well-known infrared divergence is only a gauge artifact and that it exhibits an $SO(5)$-symmetry. ($SO(5)$ is the symmetry group of $S^4$.) We evaluate the associated renormalized stress-energy tensor; it is noted that the contributions of the fields $c, \overline{c}$ and $b$ cancel. The physical theory obtained by working on the ordinary version of de Sitter space does not possess a similar symmetry. Because of zero modes, a $SO(1,4)$-invariant propagator necessarily presents an infrared divergence [6]. (Let us recall that in that case, the breakdown of $SO(1,4)$-symmetry and the time-dependence of quantities such as $\langle \phi^2 \rangle$ has a great importance in the context of the cosmological inflation.) Thus, that physical theory cannot be obtained by analytic continuation from its euclidean counterpart. In that case, the euclidean approach cannot be used to understand the physical theory.

In an appendix, we extend (on $S^4$) the results obtained for the massless scalar field theory: we BRST-quantize the massive scalar field theories corresponding to the exceptional values $m_p^2 = -p(p + 3)(R/12)$ of the mass parameter. (Here $R$ is the scalar curvature of $S^4$ and $p \in \mathbb{N}^\ast$.) All these theories present finite-dimensional gauge invariances.
2 - QUANTIZATION OF THE MASSLESS SCALAR FIELD ON $S^4$

In order to understand the so-called infrared divergence which appears in the massless minimally coupled scalar field theory in de Sitter space, let us first consider the massive scalar field. Its euclidean action is given by

$$S(\phi) = \frac{1}{2} \int_{S^4} dV \left[ (\nabla \phi)^2 + m^2 \phi^2 \right] = \frac{1}{2} \int_{S^4} dV \left( -\Box \phi + m^2 \phi^2 \right),$$  \hspace{1cm} (2.1)

where $dV = (g)^{1/2} d^4x$. The four-dimensional sphere is characterized by a radius $1/H$ and therefore by a scalar curvature $R = 12H^2$. Its volume is given by

$$V = \int_{S^4} dV = \frac{8\pi^2}{3H^4}.$$  \hspace{1cm} (2.2)

In order to evaluate path integrals over $\phi$, we shall decompose $\phi$ on the complete set of the eigenfunctions of the laplacian $\Box$. Because $S^4$ is a compact riemannian manifold, $\Box$ possesses a discrete spectrum of eigenvalues $\lambda_n$. The corresponding eigenfunctions $\phi^i_n$ (see for example [7]) are such that

$$\Box \phi^i_n = -\lambda_n \phi^i_n \hspace{1cm} i = 1, ..., d_n$$ \hspace{1cm} (2.3)

$$\lambda_n = H^2 n(n + 3) \hspace{1cm} n = 0, 1, 2, ...$$ \hspace{1cm} (2.4)

The degeneracy of each eigenvalue $\lambda_n$ is $d_n = \frac{1}{6}(n+1)(n+2)(2n+3)$. Moreover, without loss of generality, the $\phi^i_n$ may be taken real and orthonormalized. We then have

$$\int_{S^4} dV \phi^i_n \phi^j_m = \delta_{nm}\delta_{ij},$$ \hspace{1cm} (2.5)

$$\sum_n \sum_{i=1}^{d_n} \phi^i_n(x)\phi^i_n(x') = \delta^4(x, x').$$ \hspace{1cm} (2.6)

Now, in order to simplify our notation, we will suppress all degeneracy indices, but in the following, all the sums and products over $n$ must be understood as sums and products over $n$ and $i$. (In the appendix, it will be necessary to reintroduce the degeneracy indices.) It should be noted that the lowest eigenvalue of the laplacian $\Box$ is $\lambda_0 = 0$. Its unique associated eigenfunction $\phi_0$ (zero mode) is a constant given by the normalization relation (2.5):

$$\phi_0 = V^{-1/2} = \sqrt{\frac{3}{8\pi^2}H^2}.$$ \hspace{1cm} (2.7)

Moreover, in the following, we shall also use the relation

$$\int_{S^4} dV \phi_n = 0 \hspace{1cm} \text{if } n \neq 0$$ \hspace{1cm} (2.8)

which is a direct consequence of (2.6). By expanding the field $\phi$ on the complete set of the $\phi_n$ eigenfunctions as

$$\phi = \sum_n a_n \phi_n$$ \hspace{1cm} (2.9)

and by using the relations (2.3) and (2.5) we get for the action (2.1)

$$S(\phi) = \frac{1}{2} \sum_n (\lambda_n + m^2)a_n^2.$$ \hspace{1cm} (2.10)
The two-point function $G(x, x'; m^2) = \langle \phi(x) \phi(x') \rangle$ is obtained from

$$G(x, x'; m^2) = \frac{\int d[\phi] \phi(x) \phi(x') \exp(-S)}{\int d[\phi] \exp(-S)} \tag{2.11}$$

where the measure $d[\phi]$ on the space of fields is

$$d[\phi] = \prod_n da_n. \tag{2.12}$$

By inserting (2.9) and (2.10) into (2.11) and by using the relations

$$\int_{-\infty}^{+\infty} dx \exp(-\alpha x^2) = \sqrt{\frac{\pi}{\alpha}}, \quad \int_{-\infty}^{+\infty} dx x \exp(-\alpha x^2) = 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} dx x^2 \exp(-\alpha x^2) = \frac{1}{2\alpha^{2}} \sqrt{\frac{\pi}{\alpha}},$$

one finds that

$$G(x, x'; m^2) = \sum_n \frac{\phi_n(x) \phi_n(x')}{\lambda_n + m^2}. \tag{2.13}$$

It is possible to perform the sum in (2.13). One then finds the usual result [8] giving the euclidean two-point function $G(x, x'; m^2)$:

$$G(x, x'; m^2) = \frac{R}{192\pi^2} \Gamma(3/2 + \nu) \Gamma(3/2 - \nu) F[3/2 + \nu, 3/2 - \nu, 2; Z(x, x')] \tag{2.14}$$

where $\nu^2 = \frac{n^2}{4} - \frac{m^2}{4}$ and $Z(x, x') = \cos^2[H\mu(x, x')/2]$. Here $\mu(x, x')$ is the geodesic distance between the points $x$ and $x'$ on $S^4$. Because $\mu(x, x')$ and $Z(x, x')$ are invariant under the symmetry group $SO(5)$, $G(x, x'; m^2)$ is also $SO(5)$-invariant.

In the massless limit, $G(x, x'; m^2)$ given by (2.14) is divergent. From [9], we obtain

$$G(x, x'; m^2) = \frac{R^2}{384\pi^2 m^2} + \frac{R}{96\pi^2} \left[ \frac{1}{2} - \ln(1 - Z(x, x')) \right] + \mathcal{O}(m^2). \tag{2.15}$$

It is obvious that this infrared divergence comes from the zero mode $\phi_0$. Indeed, by considering the massless limit of (2.13), we see that all the denominators in that expression are nonzero except those associated with $n = 0$. Therefore the expression $\sum_{n \neq 0} \frac{\phi_n(x) \phi_n(x')}{\lambda_n + m^2}$ is finite while $\frac{\phi_0(x) \phi_0(x')}{\lambda_0 + m^2}$ diverges like $\frac{1}{m^2}$ because $\lambda_0 = 0$.

It should be noted that from the expansion (2.15) and the relation (2.7) we get

$$\sum_{n \neq 0} \frac{\phi_n(x) \phi_n(x')}{\lambda_n} = \frac{R}{96\pi^2} \left[ \frac{1}{2} - \ln(1 - Z(x, x')) \right]. \tag{2.16}$$

The infrared divergence is also present in the partition function $\mathcal{Z} = \int d[\phi] \exp(-S)$: when $m^2 = 0$, the partition function $\mathcal{Z} = \int d[\phi] \exp(-S)$ diverge because in its expression the term $\int_{-\infty}^{+\infty} da_0 \exp[-\frac{1}{2}(\lambda_0 + m^2)a_0^2]$ reduces to $\int_{-\infty}^{+\infty} da_0$.

Now, we shall prove that this infrared divergence is a gauge artifact. For $m^2 = 0$, the action (2.1) becomes

$$S(\phi) = \frac{1}{2} \int_{S^4} dV \left( \nabla \phi \right)^2 = -\frac{1}{2} \int_{S^4} dV \phi \Box \phi \tag{2.17}$$

and is invariant under the one-dimensional gauge symmetry $\phi \rightarrow \phi + \text{constant}$. The quantization of that theory can be realized by using BRST methods in the spirit of [10,11,12]. We consider a fermionic operator $s$
constructed such that \( s^2 = 0 \) and defined by its action on the field \( \phi \) and on all the algebraic fields introduced at the quantum level: we have
\[
\begin{align*}
s \phi &= c, \quad sc = 0, \quad s\overline{c} = b, \quad sb = 0. \tag{2.18}
\end{align*}
\]
Here \( c \) is the anticommuting ghost field associated to the invariance \( \phi \rightarrow \phi + \text{constant} \) and it is constant. \( \overline{c} \) is a constant anticommutating antighost and \( b \) is a constant commuting auxiliary field. The relation \( sc = 0 \) arises on this simple form because the gauge transformation is an abelian one while the relations \( s\overline{c} = b \) and \( sb = 0 \) are usual in the BRST formalism. The operator \( s \) must be interpreted as a linear differential operator graded by the ghost number. (The ghost number is 0 for \( \phi \) and \( b \), +1 for \( c \) and −1 for \( \overline{c} \). The ghost number of a product of fields is the sum of the ghost numbers of the fields.) We then have \( s(AB) = s(A)B + (-1)^{n(A)} As(B) \) where \( n(A) \) denotes the ghost number of \( A \). Moreover, we suppose that \( s \) commutes with spacetime variables and spacetime derivatives. In order to quantize the massless scalar field theory, we add to the classical action (2.17) the following gauge-fixing term
\[
S^{GF} = s \int_{S^4} dV \left( \overline{c} \phi - \frac{1}{2} \alpha \overline{c} b^2 \right) \tag{2.19}
\]
where \( \alpha \) is a gauge parameter. \( S^{GF} \) is \( s \)-exact and therefore \( s \)-invariant. Moreover, the classical action (2.17) is also \( s \)-invariant and thus the total quantum action is \( s \)-invariant. Furthermore, the fact that in the total action the gauge parameter \( \alpha \) is the coefficient of a \( s \)-exact term which is analytic in the fields ensures the gauge independence of the quantum theory (at least in the tree approximation) [10,11]. We obtain a better interpretation of \( S^{GF} \) by using (2.18) to obtain
\[
S^{GF}(\phi, c, \overline{c}, b) = \int_{S^4} dV \left( b \phi - \overline{c} c - \frac{1}{2} \alpha b^2 \right) = b \left( \int_{S^4} dV \phi \right) - V \overline{c} c - \frac{1}{2} \alpha V b^2 \tag{2.20}
\]
and then by performing the shift \( b \rightarrow b + \frac{1}{\alpha V} \int_{S^4} dV \phi \). (It should be noted that the Jacobian of the variable change \( (\phi, c, \overline{c}, b) \rightarrow (\phi, c, \overline{c}, b + \frac{1}{\alpha V} \int_{S^4} dV \phi) \) is equal to one. Thus, the measure on the space of all the fields \( d[\phi] d\overline{c} dc db \) remains unchanged.) The total quantum action becomes
\[
S^Q(\phi, c, \overline{c}, b) = \frac{1}{2} \int_{S^4} dV (\nabla \phi)^2 + \frac{1}{2 \alpha V} \left( \int_{S^4} dV \phi \right)^2 - V \overline{c} c - \frac{1}{2} \alpha V b^2. \tag{2.21}
\]
The second term in the right-hand side of (2.21) clearly appears as a non-local gauge-breaking term. It breaks the invariance \( \phi \rightarrow \phi + \text{constant} \).

In the following, we shall evaluate functional integrals by summing over \( \phi, \overline{c}, c \) and \( b \) (with this order) expressions of the form \( f(\phi, c, \overline{c}, b) \exp(-S^Q) \). In order to get convergent integrals over \( \phi \), the gauge parameter \( \alpha \) has to be taken positive. But then the last term in (2.21) is problematic. If the integration over \( b \) is taken on the real axis, the path integral diverges. In order to get convergent path integrals, it is necessary to adopt the following prescription: the integration over \( b \) has to be taken in the imaginary complex direction. In an equivalent way, we must change the sign in front of the last term of (2.21). A similar problem exists in the path integral approach of quantum gravity [2]. In that case, in order to get convergent integrals, the integration over the conformal factor has to be taken also in a complex direction. It is important to understand that the proposed prescription is not an artificial way to eliminate the infrared divergence. The infrared divergence problem and the problem of the divergence of the integrals over \( b \) are totally different. We believe that the second problem arises because of the nature of the auxiliary field: like the conformal factor of gravitation, it is not a propagating field. Exactly for the same reasons, we change
the sign in front of the ghost term in the action. (See [13] for a complementary discussion on the integration over \( c, \pi \) and \( b \).) In conclusion, in the path integrals we shall consider the positive definite action

\[
S^Q(\phi, c, \pi, b) = \frac{1}{2} \int_{S^4} dV \left( \nabla \phi \right)^2 + \frac{1}{2\alpha V} \left( \int_{S^4} dV \phi \right)^2 + V \pi c + \frac{1}{2} \alpha V b^2
\]

(2.22)

and we shall integrate over the real values of \( \phi \) and \( b \), and over the grassmannian variables \( \pi \) and \( c \) (with this order) by using the usual rules \( \int dc = 0, \int d\pi = 0, \int dc \ c = 1 \) and \( \int d\pi \ \pi = 1 \).

Let us first consider the partition function \( Z \) of the massless scalar field theory. It is defined as

\[
Z = \int d[\phi] \ d\pi \ dc \ db \ \exp(-S^Q).
\]

(2.23)

By writing \( \phi = \sum_n a_n \phi_n \) and from (2.3), (2.5), (2.7) and (2.8), it is obvious that

\[
S^Q(\phi, c, \pi, b) = \frac{1}{2} \sum_{n \neq 0} \lambda_n a_n^2 + \frac{1}{2\alpha} a_0^2 + V \pi c + \frac{1}{2} \alpha V b^2
\]

(2.24)

and therefore we get

\[
Z = 2\pi V^{1/2} \prod_{n \neq 0} \left( \frac{2\pi}{\lambda_n} \right)^{1/2}.
\]

(2.25)

\( Z \) is independent of the gauge parameter \( \alpha \). Moreover, it is not infrared divergent. It needs only a regularization because of the usual ultraviolet divergence of the term \( \prod_{n \neq 0} \left( \frac{2\pi}{\lambda_n} \right)^{1/2} \).

The two-point function \( G(x, x') = \langle \phi(x)\phi(x') \rangle \) is now given by

\[
G(x, x') = \frac{\int d[\phi] \ d\pi \ dc \ db \ \phi(x)\phi(x') \exp(-S^Q)}{\int d[\phi] \ d\pi \ dc \ db \ \exp(-S^Q)}.
\]

(2.26)

From (2.24) and by inserting \( \phi = \sum_n a_n \phi_n \) in (2.26) we obtain

\[
G(x, x') = \sum_{n \neq 0} \frac{\phi_n(x)\phi_n(x')}{\lambda_n} + \alpha \phi_0(x)\phi_0(x')
\]

(2.27)

and from (2.7) and (2.16) one finds

\[
G(x, x') = \frac{R}{96\pi^2} \left[ \frac{1/2}{1 - Z(x, x')} - \ln(1 - Z(x, x')) \right] + \alpha \left( \frac{3H^4}{8\pi^3} \right).
\]

(2.28)

Clearly, \( G(x, x') \) is finite and is \( SO(5) \)-invariant. The so-called infrared divergence is nothing but a gauge artefact. It occurs when the gauge parameter \( \alpha \) goes to \( \infty \). Similarly, the Feynman propagator and the anticommutator function are also finite and \( SO(5) \)-invariant. Moreover, these two last Green functions possess Hadamard expansions. It should be noted that all these two-point functions depend on the gauge parameter \( \alpha \) but that the physical quantities calculated from them must be gauge parameter independent (even in the limit \( \alpha \rightarrow \infty \)). The choice (2.20) for the gauge-fixing action ensures it.

An important example of a physical quantity is provided by \( \langle T_{\mu\nu} \rangle \), the vacuum expectation value of the stress-energy tensor. The stress-energy operator is formally constructed from \( S^Q \) by

\[
T_{\mu\nu} = -\frac{2}{g^{1/2} \delta g^{\mu\nu}} \delta S^Q
\]

(2.29)
and by using the fact that in the transformation \( g_{\mu \nu} \rightarrow g_{\mu \nu} + \delta g_{\mu \nu} \) we have \( g^{1/2} \rightarrow g^{1/2} + \frac{1}{2} g^{1/2} g_{\mu \nu} \delta g_{\mu \nu} \), we get

\[
T_{\mu \nu} = T_{\mu \nu}^{\text{Cl}} + T_{\mu \nu}^{\text{GB}} + T_{\mu \nu}^{b} + T_{\mu \nu}^{b}
\]  

(2.30a)

where

\[
T_{\mu \nu}^{\text{Cl}} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu \nu} (\nabla \phi)^2,
\]

(2.30b)

\[
T_{\mu \nu}^{\text{GB}} = \frac{1}{2 \alpha V} \left( \int_{S^4} dV \phi \right)^2 g_{\mu \nu} - \frac{1}{\alpha V} \left( \int_{S^4} dV \phi \right) \phi g_{\mu \nu},
\]

(2.30c)

\[
T_{\mu \nu}^{b} = -\tau c g_{\mu \nu},
\]

(2.30d)

\[
T_{\mu \nu}^{b} = -\frac{1}{2} \alpha b^2 g_{\mu \nu}.
\]

(2.30e)

At the quantum level \( \langle T_{\mu \nu} \rangle \) is formally given by

\[
\langle T_{\mu \nu} \rangle = \frac{\int d[\phi] \int dc \int db \ T_{\mu \nu} \exp(-S^Q)}{\int d[\phi] \int dc \int db \ \exp(-S^Q)},
\]

(2.31)

The calculation of the contributions to \( \langle T_{\mu \nu} \rangle \) of the ghost and antighost fields \( c \) and \( \tau \) and of the auxiliary field \( b \) are trivial. We get (with obvious notations)

\[
\langle T_{\mu \nu}^{c} \rangle = \frac{1}{V} g_{\mu \nu},
\]

(2.32)

\[
\langle T_{\mu \nu}^{b} \rangle = -\frac{1}{2V} g_{\mu \nu}.
\]

(2.33)

With regard to the contribution of \( \phi \), the situation is a little more complicated. Let us remarks that if we expand \( \phi \) on the form \( \phi = \sum_n a_n \phi_n \), the coefficient \( a_0 \) does not appear in the expression (2.30b). Moreover, because of (2.8), only terms of type \( a_0 a_n \) appear in the expression (2.30c). As a consequence, we show that (with obvious notations)

\[
\langle T_{\mu \nu}^{\text{Cl}} \rangle = \frac{\prod_{n \neq 0} \int da_n T_{\mu \nu}^{\text{Cl}} \exp(-\frac{1}{2} \sum_{n \neq 0} \lambda_n a_n^2)}{\prod_{n \neq 0} \int da_n \exp(-\frac{1}{2} \sum_{n \neq 0} \lambda_n a_n^2)},
\]

(2.34)

\[
\langle T_{\mu \nu}^{\text{GB}} \rangle = \frac{\int da_0 T_{\mu \nu}^{\text{GB}} \exp(-\frac{a_0^2}{2a})}{\int da_0 \exp(-\frac{a_0^2}{2a})}.
\]

(2.35)

The calculation of \( \langle T_{\mu \nu}^{GB} \rangle \) then gives

\[
\langle T_{\mu \nu}^{GB} \rangle = -\frac{1}{2V} g_{\mu \nu}.
\]

(2.36)

At this level, it should be noted that the contributions of the gauge-breaking term (2.36), of the ghost-antighost term (2.32) and of the \( b \) term (2.33) cancel. Therefore \( \langle T_{\mu \nu} \rangle \) reduces to \( \langle T_{\mu \nu}^{\text{Cl}} \rangle \). It remains for us to calculate (2.34). This term needs a regularization. By noting that \( \langle T_{\mu \nu}^{\text{Cl}} \rangle \) is also obtained by the point-splitting of a quantity which possesses a symmetric Hadamard expansion, one finds the regularized vacuum expectation value of the stress-energy tensor [14,15]. Indeed, we have

\[
\langle T_{\mu \nu} \rangle = \langle T_{\mu \nu}^{\text{Cl}} \rangle = \frac{1}{2} \lim_{x \rightarrow x} \left( \nabla_\mu \nabla_\nu - \frac{1}{2} g_{\mu \nu} g^{\rho \sigma} \nabla_\rho \nabla_\sigma \right) \left( \sum_{n \neq 0} \frac{\phi_n(x) \phi_n(x')}{\lambda_n} + (x \leftrightarrow x') \right).
\]

(2.37)

Now it should be noted that \( \sum_{n \neq 0} \frac{\phi_n(x) \phi_n(x')}{\lambda_n} + (x \leftrightarrow x') \) which is given by

\[
\sum_{n \neq 0} \frac{\phi_n(x) \phi_n(x')}{\lambda_n} + (x \leftrightarrow x') = \frac{R}{48\pi^2} \left[ \frac{1}{2} \ln \left( \frac{1}{1 - Z(x, x')} - \ln(1 - Z(x, x')) \right) \right]
\]

(2.38)
possesses a symmetric Hadamard expansion. We have

\[
\sum_{n \neq 0} \frac{\phi_n(x)\phi_n(x')}{\lambda_n} + (x \leftrightarrow x') = \frac{1}{(2\pi)^2} \left[ \frac{\Delta^{1/2}(x, x')}{\sigma(x, x')} + V(x, x') \ln \sigma(x, x') + W(x, x') \right]
\]  

(2.39)

where \(\sigma(x, x')\) is linked to the geodesic distance between \(x\) and \(x'\) by \(2\sigma(x, x') = \mu^2(x, x')\), \(V(x, x')\) is a smooth geometrical function while \(W(x, x')\) is a smooth state-dependent function and \(\Delta(x, x')\) is the Van Vleck determinant. (See for example [14,15] and references therein for more details on the notation.) In the present case, because of the maximal symmetry of (2.38), all the coefficients of the expansion of \(W(x, x')\) in powers of \(\sigma(x, x')\) are constant. Therefore, from [14,15] (see for example (3.7) of [15]), one obviously finds that [16]

\[
\langle T_{\mu\nu} \rangle_{\text{ren}} = \frac{29R^2}{138240\pi^2} g_{\mu\nu}.
\]

(2.40)

Of course, \(\langle T_{\mu\nu} \rangle_{\text{ren}}\) is independent of the gauge parameter \(\alpha\) and is maximally symmetric.
3 - REMARKS AND CONCLUSION

It was possible to correctly treat on $S^4$ the zero mode problem arising in the massless minimally coupled scalar field theory and to get a $SO(5)$-invariant quantum theory by considering it as a gauge theory. The compactness of the background manifold has played a crucial role. The absence of an infrared divergence in the massless scalar field theory on $S^4$ and the $SO(5)$-invariance of the quantum theory is easy to understand: the gauge-breaking term added to the classical action of the theory has allowed us to replace in the expression of the two-point function $\langle \phi(x)\phi(x') \rangle$ the infinite and constant term $(\phi_0)^2/\lambda_0$ by the regular and constant one $\alpha(\phi_0)^2$.

On de Sitter space $dS^4$, the situation is in fact less simple. A BRST-treatment along the lines of Section 2 is not possible because of the infinite volume of that spacetime. Furthermore, the existence of the infrared divergence of the two-point functions is also the consequence of the presence of zero modes, but now the zero modes are infinite and time-dependent. When $dS^4$ is described by a coordinate system whose corresponding spatial sections are compact, the complete set of mode solutions of the wave equation is discreet. It is then possible to replace the two infinite and time-dependent zero modes by two regular but also time-dependent zero modes. The resulting two-point functions are then time-dependent and therefore break the $SO(1,4)$-invariance of the spacetime [17]. Moreover, the renormalized vacuum expectation of the stress-energy tensor is time-dependent [18]. At the contrary, when $dS^4$ is described by a coordinate system whose corresponding spatial sections are non compact, the complete set of mode solutions of the wave equation is continuous and no regularization procedure (at the level of the two-point functions) applies without destroying the structure of the Fock space of quantum states.

Because the quantum theory on $dS^4$ is not $SO(1,4)$-invariant, it cannot be linked to the $SO(5)$-invariant quantum theory construct on $S^4$. In particular, the Green functions of the two theories cannot be linked by analytic continuation. Euclideanization of spacetime is a powerful method in quantum field theory, but it must be used with lot of care. By changing the topology of the background manifold, it may completely change the nature of a problem and its solution, especially when zero modes are involved.

A similar conclusion to ours has been obtained by Mazur and Mottola in [22], where the problem of the conformal mode of quantum gravity is extensively discussed. The authors question the validity of the procedure of euclideanization in quantum gravity and advocate the necessity to consider the lorentzian form of the path integrals.

Recently, many calculations have been performed involving the graviton propagator in de Sitter space [19,20,21]. All that calculations are done less or more explicitly on the euclidean version of de Sitter space. They provide a graviton propagator which is $SO(1,4)$-invariant (by analytic continuation from $S^4$ to $dS^4$) but which presents a pathological behaviour at large distance leading to divergences in certain physical quantities. We believe that it could be the consequence of the treatment on $S^4$. A study on $dS^4$ might provide a true physical graviton propagator which could break de Sitter invariance but which is not so pathological.

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APPENDIX

Let us consider the massive scalar field theory defined by (2.1) for the value $m_p^2 = -p(p + 3)H^2$ with $p \in \mathbb{N}^*$. (It should be recalled that in $m_p^2$ there is included a coupling term with the scalar curvature of the sphere.) For such a value, the parameter $\nu$ in (2.14) is equal to $p + 3/2$ and therefore the two-point function (2.14) diverges. From (2.13), it is obvious that such a divergence occurs because $m_p^2 = -p(p + 3)H^2$ is the opposite of an eigenvalue of the laplacian. Therefore, it is a consequence of the fact that the operator $\Box + p(p+3)H^2$ possesses zero modes. As in the massless case, this divergence in the massive theory is a gauge artifact: indeed, for $m_p^2 = -p(p + 3)H^2$, the euclidean action (2.1) is invariant under the $d_p$-dimensional gauge transformation

$$\phi(x) \rightarrow \phi(x) + \sum_{i=1}^{d_p} C_i \phi_i^1(x),$$

where the $C_i$ are arbitrary constants; it is then necessary to quantize it as a gauge theory.

Thus, as in the massless case, let us add to (2.1) the $s$-invariant term (2.19). The difference with the massless case is the following: in (2.18) and (2.19) the ghost field $c$, the antighost field $\bar{c}$ and the auxiliary field $b$ are now space-dependent; they live in the zero mode subspace spanned by the $d_p$ functions $\phi_i^1$. Therefore, we will write

$$c(x) = \sum_{i=1}^{d_p} c_i \phi_i^1(x) \quad \bar{c}(x) = \sum_{i=1}^{d_p} \bar{c}_i \phi_i^1(x) \quad b(x) = \sum_{i=1}^{d_p} b_i \phi_i^1(x).$$

Then, by expanding the scalar field $\phi$ as

$$\phi(x) = \sum_{n} \sum_{i=1}^{d_n} a_{n,i} \phi_n^i(x),$$

and by using the normalization relation (2.5) and by performing the shifts $b_i \rightarrow b_i + \frac{1}{\alpha}(a_{p,i})$, we obtain for the quantum action

$$S^Q(\phi, c, \bar{c}, b) = \frac{1}{2} \sum_{n \neq p} \sum_{i=1}^{d_n} (\lambda_n + m_p^2)(a_{n,i})^2 + \frac{1}{2\alpha} \sum_{i=1}^{d_p} (a_{p,i})^2 - \sum_{i=1}^{d_p} \bar{c}_i c_i - \frac{1}{2\alpha} \sum_{i=1}^{d_p} b_i^2.$$  \hspace{1cm} (A.4)

As in the massless case, it is necessary in the calculations to rotate the integration contours for the $b_i$, $c_i$ and $\bar{c}_i$, or equivalently to integrate over the real values of these variables but with the quantum action

$$S^Q(\phi, c, \bar{c}, b) = \frac{1}{2} \sum_{n \neq p} \sum_{i=1}^{d_n} (\lambda_n + m_p^2)(a_{n,i})^2 + \frac{1}{2\alpha} \sum_{i=1}^{d_p} (a_{p,i})^2 + \sum_{i=1}^{d_p} \bar{c}_i c_i + \frac{1}{2\alpha} \sum_{i=1}^{d_p} b_i^2.$$  \hspace{1cm} (A.5)

The true two-point function $G(x, x'; m_p^2) = \langle \phi(x)\phi(x') \rangle$ can then be easily obtained. By inserting (A.3) and (A.5) in (2.26), one finds

$$G(x, x'; m_p^2) = \sum_{n \neq p} \sum_{i=1}^{d_n} \frac{\phi_n^i(x)\phi_n^i(x')}{\lambda_n + m_p^2} + \alpha \sum_{i=1}^{d_p} \phi_p^i(x)\phi_p^i(x').$$  \hspace{1cm} (A.6)
All the sums $\sum_{i=1}^{d_n} \phi^i_n(x) \phi^i_n(x')$ are $SO(5)$-invariant [the $(\phi^i_n)_{i=1,\ldots,d_n}$ form a basis of the $d_n$-dimensional representation of $SO(5)$] and are given by [20]

$$\sum_{i=1}^{d_n} \phi^i_n(x) \phi^i_n(x') = \frac{6d_n}{16\pi^2} H^4 F[-n, n + 3; 2; 1 - Z(x, x')] \quad (A.7)$$

(Here the expansion of the hypergeometric function $F$ terminates and in fact that sum reduces to a Gegenbauer polynomial.) Therefore, the two-point function $G(x, x'; m^2)$ is also $SO(5)$-invariant and the quantum theory possesses this invariance. Moreover, as in the massless case, and because of the choice of the gauge-fixing term, the quantum theory is independent of the gauge parameter $\alpha$.

To conclude this appendix let us note i) that the massless case can be considered as the particular case $p = 0$ in the previous calculations, ii) that on $dS^4$ the divergences appearing in the quantum theories for the mass values $m_p^2 = -p(p + 3)H^2$ with $p \in \mathbb{N}^*$ are not gauge artifacts: they are real and correspond to the impossibility to construct $SO(1,4)$-invariant theories. That last point has been studied in [23] in the case $p = 1$ which described the scalar part of the metric fluctuation.
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\[ S = \frac{1}{4} \int d^4 x \, F^2 \]
is invariant under the gauge transformation \( A_\mu \rightarrow A_\mu + V_\mu \). The BRST-quantization of the theory is realized by introducing a ghost field \( c \), an antighost field \( \overline{c} \), an auxiliary field \( b \) and a fermionic operator \( s \). (Here \( c, \overline{c}, b \) and \( s \) are non-constant.) The operator \( s \) is such that \( s A_\mu = c A_\mu, s = 0, \overline{s} = b \) and \( s b = 0 \). The gauge-fixing term is now
\[ S_{GF} = s \int d^4 x \, (\overline{c} A_\mu \overline{\gamma}^\mu c - \frac{1}{2} \overline{c} b c) \]
and by shifting \( b \) we get for the quantum action
\[ S_Q(A_\mu, c, \overline{c}, b) = \frac{1}{4} \int d^4 x \, F^2 + \frac{1}{12} \int d^4 x \, (A_\mu \overline{\gamma}^\mu)^2 - \int d^4 x \, \overline{c} \alpha \int d^4 x \, b^2. \]
The b term in the action is still problematic. If we decompose \( b \) on the \( \phi_n \) as \( b = \sum_n b_n \phi_n \), we get
\[ -\frac{1}{2} \alpha \int d^4 x \, b^2 = -\frac{1}{2} \alpha \sum_n b_n^2. \]
Therefore, in order to obtain well-defined path integrals, it is necessary to integrate all the \( b_n \) from \(-\infty \) to \( +\infty \). On the contrary, the ghost and antighost fields are now propagating fields and no change of sign of the corresponding action is necessary.
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