THE BETTI SIDE OF THE DOUBLE SHUFFLE THEORY.
I. THE HARMONIC COPRODUCT

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Abstract. The double shuffle and regularization relations between multiple zeta values lead to the construction of the "double shuffle and regularization" scheme, which is a torsor under a "double shuffle" group scheme (Racinet). This group scheme is the target of a morphism from the Tannakian group of the category of unramified mixed Tate motives relative to the "de Rham" realization. The aim of this series is to construct a "Betti" counterpart of this group scheme.

The present paper is based on the following observations: (a) the torsor structure of the "double shuffle and regularization" scheme arises from restriction of the regular action of a group of automorphisms of a topological free Lie algebra on itself (Racinet); (b) the double shuffle group scheme is contained in the stabilizer of a particular element of a module over the group of automorphisms of the topological Lie algebra, which identifies with the harmonic coproduct (earlier work of the authors). These observations lead to the construction of a "Betti" analogue of the harmonic coproduct.

We explicitly compute this "Betti" coproduct by making use of the following tools: (i) an interpretation of the harmonic coproduct in terms of infinitesimal braid Lie algebras, inspired by the preprint of Deligne and Terasoma (2005); (ii) the similar interpretation of an explicit coproduct in terms of braid groups; (iii) the collection of morphisms relating braid groups and infinitesimal braid Lie algebras arising from associators (Drinfeld, Bar-Natan).

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0. Introduction

0.1. Double shuffle theory. The \( \mathbb{Q} \)-linear category \( \mathcal{T} := \text{MT}(\mathbb{Z})_{\mathbb{Q}} \) of mixed Tate motives over \( \mathbb{Z} \) has been constructed in \([\text{DeG}]\). This is a Tannakian category, equipped with fiber functors \( \omega^B, \omega^{\text{DR}} : \mathcal{T} \to \text{Vec}_{\mathbb{Q}} \) to the category of finite dimensional \( \mathbb{Q} \)-vector spaces, which are called the Betti and de Rham realizations. It gives rise to a \( \mathbb{Q} \)-scheme \( \text{Isom}^\otimes_{\mathcal{T}}(\omega_B, \omega_{\text{DR}}) \) given by \( k \mapsto \text{Isom}^\otimes_{\mathcal{T}}(\omega_B \otimes k, \omega_{\text{DR}} \otimes k) \), where \( k \) runs over \( \mathbb{Q} \)-algebras. It also gives rise to the "motivic Galois" \( \mathbb{Q} \)-group schemes \( \text{Aut}^\otimes_{\mathcal{T}}(\omega_X) \), for \( X \in \{ B, \text{DR} \} \) given by \( k \mapsto \text{Aut}^\otimes_{\mathcal{T}}(\omega_X \otimes k) \). Then \( \text{Isom}^\otimes_{\mathcal{T}}(\omega_B, \omega_{\text{DR}}) \) is equipped with free and transitive actions of \( \text{Aut}^\otimes_{\mathcal{T}}(\omega_{\text{DR}}) \) from the left, and of \( \text{Aut}^\otimes_{\mathcal{T}}(\omega_B) \) from the right, which commute with one another.
The category $\mathcal{T}$ is closely connected with the family of numbers known as multiple zeta values (MZVs), which are given by

$$\forall s \geq 1, k_1 \geq 1, \ldots, k_{s-1} \geq 1, k_s \geq 2, \quad \zeta(k_1, \ldots, k_s) = \sum_{0 < n_1 < \cdots < n_s} \frac{1}{n_1^{k_1} \cdots n_s^{k_s}};$$

it follows from their integral expressions that these numbers, as well as $2\pi i$, are periods of mixed Tate motives.

Combining [Dr] and [LeM], one can exhibit a family of algebraic relations satisfied by the MZVs and $2\pi i$, the "associator relations". The corresponding $\mathbb{Q}$-scheme is the scheme of associators; it is equipped with free and transitive actions of $\mathbb{Q}$-group schemes $k \mapsto \text{GT}(k)$ from the left, and $k \mapsto \text{GRT}(k)$ from the right (the Grothendieck-Teichmüller group scheme and its graded version, also denoted $\text{GT}(-)$ and $\text{GRT}(-)$), which commute with one another.

In [An], it is explained that there is a $\mathbb{Q}$-scheme morphism from $\text{Isom}_{\mathcal{T}}^\otimes (\omega_B, \omega_{\text{DR}})$ to the scheme of associators; there are also $\mathbb{Q}$-group scheme morphisms $\text{Aut}_{\mathcal{T}}^\otimes (\omega_B) \to \text{GT}(-)$ and $\text{Aut}_{\mathcal{T}}^\otimes (\omega_{\text{DR}}) \to \text{GRT}(-)$, which are compatible with the actions of each side.

In [IKZ, R], another family of algebraic relations satisfied by MZVs and $2\pi i$ was exhibited, the "double shuffle and regularization relations". The corresponding $\mathbb{Q}$-scheme was constructed in [R]: it is a functor $k \mapsto \sqcup_{\mu \in k^\times} \{\mu\} \times \text{DMR}_0(k)$, which is equipped with a free and transitive left action of a $\mathbb{Q}$-group scheme $k \mapsto k^\times \ltimes (\text{DMR}_0(k), \oplus)$ (see Proposition 1.3). Note that in contrast to the situation of associators, no commuting right action of a $\mathbb{Q}$-group scheme is given in [R]. Along with a better understanding of the relations between braids and double shuffle theory, the construction of such an action is the main objective of the present series of papers; it is obtained in [EFu2] based on the results of the present paper.

The relations between the "associator" and "double shuffle" schemes were studied in [Fu2], where an inclusion of the former into the latter scheme was constructed. It was also proved that this inclusion is compatible with a $\mathbb{Q}$-scheme morphism, whose specialization at $k$ is a group morphism $\text{GRT}(k) \to k^{\times} \ltimes (\text{DMR}_0(k), \oplus)$; combining it with the morphism $\text{Aut}_{\mathcal{T}}^\otimes (\omega_{\text{DR}} \otimes k) \to \text{GRT}(k)$, we obtain a morphism $\text{Aut}_{\mathcal{T}}^\otimes (\omega_{\text{DR}} \otimes k) \to k^{\times} \ltimes (\text{DMR}_0(k), \oplus)$, which leads one to view $k^{\times} \ltimes (\text{DMR}_0(k), \oplus)$ as a "de Rham" object. The preprint [DeT], based on the ideas of Deligne's letter to Racinet (April 21, 2001) also introduces several ideas which could potentially apply to the problem of the comparison of the "associator" and "double shuffle" schemes.

The group $(\text{DMR}_0(k), \oplus)$ is a subgroup of a group $(\exp(\hat{\mathfrak{Lie}}_k(X)), \oplus)$ of automorphisms of a topological Lie algebra (see Theorem 1.1). In [EFu0], this group inclusion is explained as follows: it is shown that the group $(\exp(\hat{\mathfrak{Lie}}_k(X)), \oplus)$ acts on a $k$-module $\text{Hom}_{k\text{-mod}}(k\langle Y \rangle, k\langle Y \rangle^{S_2})$; this $k$-module contains an explicit element $\hat{\Delta}_s$ (the harmonic coproduct, which is one of the key

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1 We set $i := \sqrt{-1}$.

2 DMR stands for the French "double mélange et régularisation".

3 We take the opportunity of this paper to correct a typo from [EFu0, §1, 2 lines after eq. (1.1): the coefficient of $x_0 x_1$ in the series $\mathcal{F}$ is not $2\pi \sqrt{-1}$, but $-(2\pi \sqrt{-1})^2/24$. 

---
ingredients of the expression of the double shuffle relations); \((\text{DMR}_0(k), \odot)\) is then contained in the stabilizer subgroup of this element (see Theorem 1.4). One can easily extend the action of \((\exp(\mathcal{L}ib_k(X)), \odot)\) to an action of \(k^\times \ltimes (\text{DMR}_0(k), \odot)\) on the same space, and show that \(k^\times \ltimes (\text{DMR}_0(k), \odot)\) is contained in the stabilizer subgroup of \(\hat{\Delta}_*\) for the extended action (see Corollary 1.7).

Explicitly, \(k\langle\langle Y \rangle\rangle\) is the degree completion of the free associative algebra over generators \(y_n\) of degree \(n\) for each \(n \geq 1\) (this algebra is non-commutative), equipped with the topological Hopf algebra structure with coproduct \(\hat{\Delta}_*\) such that

\[
\forall n \geq 1, \quad \hat{\Delta}_*(y_n) = y_n \otimes 1 + 1 \otimes y_n + \sum_{k=1}^{n-1} y_k \otimes y_{n-k}.
\]

0.2. The main result of this paper. As mentioned above, the main task of this series of papers is a description of a ”Betti” \(\mathbb{Q}\)-group scheme, acting from the right on the double shuffle scheme and commuting with the action of the ”de Rham” group scheme \(k \mapsto k^\times \ltimes (\text{DMR}_0(k), \odot)\).

As a first step in this direction, we devote the present paper to a construction that we now explain. We first present the following statement, whose proof is immediate.

**Principle 0.1.** Let \(G\) be a group, the \(V\) be a \(G\)-module, let \(v \in V\). Let \(H\) be a subgroup of \(G\) such that for any \(h \in H\), \(h \cdot v = v\) (we say that \(H\) is contained in the stabilizer of \(v\)). Let \(\tilde{H} \subset G\) be a subset such that \(\tilde{H}\) is stable under the action of \(H\) from the left, and moreover is a torsor under the action of this group. Then the set \(\{\tilde{h}^{-1} \cdot v \mid \tilde{h} \in \tilde{H}\}\) contains exactly one element \(\tilde{v}\).

Set

\[(0.2.1) \quad G := k^\times \ltimes (\exp(\mathcal{L}ib_k(X)), \odot), \quad V := \text{Hom}_{k\text{-mod}}(k\langle\langle Y \rangle\rangle, k\langle\langle Y \rangle\rangle \hat{\otimes} 2), \quad v = \Delta_*,
\]

\[
H := k^\times \ltimes (\text{DMR}_0(k), \odot), \quad \tilde{H} := \sqcup_{\mu \in k^\times} \{\mu\} \times \text{DMR}_\mu(k),
\]

the action of \(G\) on \(V\) being as in (0.1). The hypothesis \(h \cdot v = v\) for \(h \in H\) follows from the stabilizer result of [EFu0], while the torsor structure of \(\tilde{H}\) over \(H\) follows from the torsor results of [R] (see also Proposition 1.3).

One may therefore apply Principle 0.1 in this situation, which leads to the construction of an element \(\tilde{v}\). The main result of this paper is an explicit description of this element. Before stating this result, we introduce the following material:

- we define \(k\langle\langle X \rangle\rangle\) to be the free associative algebra over generators \(x_0, x_1\). Then the map \(y_n \mapsto x_0^{n-1} x_1\) enables one to view \(k\langle\langle Y \rangle\rangle\) as a topological subalgebra of \(k\langle\langle X \rangle\rangle\);

- we define \(F_2\) to be the free group over generators \(X_0, X_1\) and \(kF_2\) be its group algebra; then \(\mathcal{W}_1^B := k \oplus kF_2(X_1 - 1)\) is a subalgebra of \(kF_2\);
we define \( \text{inj}^t : kF_2 \rightarrow k \langle \langle X \rangle \rangle \) to be the algebra morphism defined by \( X_0 \mapsto e^{-x}, \ X_1 \mapsto e^{-x} \) (where \( e^t \) is the series \( \sum_{n \geq 0} t^n/n! \)); it is injective and restricts to an injective algebra morphism \( \text{inj}_1^t : W^B_1 \rightarrow k \langle \langle Y \rangle \rangle \).

**Theorem 0.2.**

1) The algebra \( W^B_1 = k \oplus kF_2(X_1 - 1) \) is equipped with a Hopf algebra structure with coproduct

\[
\Delta_{\hat{\Delta}} : W^B_1 \rightarrow (W^B_1)^{\otimes 2},
\]

given by

\[
\Delta_{\hat{\Delta}}(X_{1}^{\pm 1}) = X_{1}^{\pm 1} \otimes X_{1}^{\pm 1}, \quad \Delta_{\hat{\Delta}}(Y_{n}^{\pm}) = Y_{n}^{\pm} \otimes 1 + 1 \otimes Y_{n}^{\pm} + \sum_{k = 1}^{n-1} Y_{k}^{\pm} \otimes Y_{n-k}^{\pm}
\]

for any \( n \geq 1 \),

where

\[
Y_{n}^{\pm} := (X_{0}^{\pm 1} - 1)^{n-1} X_{0}^{\pm 1} (1 - X_{1}^{\pm 1}) \quad (n \geq 1).
\]

2) There is a unique morphism of topological algebras \( k \langle \langle Y \rangle \rangle \rightarrow k \langle \langle Y \rangle \rangle^{\hat{\otimes} 2} \), such that the diagram

\[
\begin{array}{ccc}
W^B_1 & \xrightarrow{\Delta_{\hat{\Delta}}} & (W^B_1)^{\otimes 2} \\
\text{inj}_1^t \downarrow & & \downarrow (\text{inj}_1^t)^{\otimes 2} \\
k \langle \langle Y \rangle \rangle & \xrightarrow{\hat{\Delta}} & k \langle \langle Y \rangle \rangle^{\hat{\otimes} 2}
\end{array}
\]

commutes.

3) This morphism coincides with the element \( \hat{\nu} \) arising from Principle 0.1 in the setting (0.2.1).

Moreover, \( \hat{\nu} \) and \( \nu \) are related in the following way: both maps \( k \langle \langle Y \rangle \rangle \rightarrow k \langle \langle Y \rangle \rangle^{\hat{\otimes} 2} \) are compatible with the degree filtration of each side and the associated graded maps of \( \hat{\nu} \) and \( \nu = \hat{\Delta} \), coincide.

**0.3. Idea of the proof.** The proof of Theorem 0.2 1) is based on a presentation of the algebra \( W^B_1 \) (§2.1,2.2).

In §1.3 one constructs a completion \( \hat{W}^B_1 \) of \( W^B_1 \), together with an isomorphism \( \text{iso}_1^t : \hat{W}^B_1 \rightarrow k \langle \langle Y \rangle \rangle \) of topological algebras; the isomorphism \( \text{iso}_1^t \) is then the unique continuous extension of \( \text{inj}_1^t \). In §2.3 one proves the existence of a morphism of topological algebras \( \Delta_{\hat{\Delta}} : \hat{W}^B_1 \rightarrow (\hat{W}^B_1)^{\otimes 2} \) (which is necessarily unique), extending \( \Delta_{\hat{\Delta}} : W^B_1 \rightarrow (W^B_1)^{\otimes 2} \). The morphism of topological algebras \( k \langle \langle Y \rangle \rangle \rightarrow k \langle \langle Y \rangle \rangle^{\hat{\otimes} 2} \) from Theorem 0.2 2) is then necessarily equal to \( (\text{iso}_1^t)^{\otimes 2} \circ \Delta_{\hat{\Delta}} \circ (\text{iso}_1^t)^{-1} \).

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Our notation policy with respect to underlining can be described as follows: the notation for "de Rham" objects are not underlined (e.g. \( \Delta_{\hat{\Delta}} \)); the notation for the similar "Betti" objects are underlined (e.g. \( \Delta_{\hat{\Delta}} \)); objects relating de Rham and Betti sides are usually also underlined (e.g. \( S^Y_{(k,G), \Phi} \)).
Define now a continuous \( k \)-linear map \( \hat{\Delta}_\ast : \hat{W}^B \to (\hat{W}^B)^{\hat{\otimes}_2} \) by \( \hat{\Delta}_\ast := ((\iso')^{-1})^{\otimes 2} \circ \tilde{\nu} \circ \iso' \) (see also Lemma-Definition 1.11). Theorem 0.2 (3) is then equivalent to the equality \( \hat{\Delta}_\ast = \hat{\Delta}_\ast \), which is stated in Theorem 3.1 and proved in the subsequent sections.

Let us explain the structure of the proof of this theorem. By the definition of \( \hat{\Delta}_\ast \), this map may be characterized by the commutativity of a diagram

\[
\begin{array}{ccc}
\hat{W}^B & \xrightarrow{\hat{\Delta}_\ast} & (\hat{W}^B)^{\hat{\otimes}_2} \\
\hat{\Delta}_\ast & \downarrow \ & \downarrow \zeta^{\otimes 2} \\
\kappa \langle \langle Y \rangle \rangle & \xrightarrow{\Delta_\ast} & \kappa \langle \langle Y \rangle \rangle^{\hat{\otimes}_2}
\end{array}
\]

for some pair \((\mu, \Phi)\), with \( \mu \in k^\times \) and \( \Phi \in \text{DMR}_\mu(k) \) (see (1.3.1), with notation as in (1.1)). By using a primitiveness property of \( \Phi \) (Lemma 1.15) and the properties of \( \Gamma \)-functions of elements of \( \text{DMR}_\mu(k) \), one shows that \( \hat{\Delta}_\ast \) also fits in a commutative diagram

\[
\begin{array}{ccc}
\hat{W}^B & \xrightarrow{\hat{\Delta}_\ast} & (\hat{W}^B)^{\hat{\otimes}_2} \\
\hat{\Delta}_\ast & \downarrow \ & \downarrow \zeta^{\otimes 2} \\
\kappa \langle \langle Y \rangle \rangle & \xrightarrow{\Delta_\ast} & \kappa \langle \langle Y \rangle \rangle^{\hat{\otimes}_2}
\end{array}
\]

(see Proposition 1.19). As a side result, this diagram implies that \( \hat{\Delta}_\ast \) is an algebra morphism, a fact that is also a consequence of Theorem 3.1 (see Theorem 1.16). Since \( \hat{\Delta}_\ast \) is characterized by the commutativity of this diagram, the equality \( \hat{\Delta}_\ast = \hat{\Delta}_\ast \) is equivalent to the commutativity of the diagram

\[
\begin{array}{ccc}
\hat{W}^B & \xrightarrow{\hat{\Delta}_\ast} & (\hat{W}^B)^{\hat{\otimes}_2} \\
\hat{\Delta}_\ast & \downarrow \ & \downarrow \zeta^{\otimes 2} \\
\kappa \langle \langle Y \rangle \rangle & \xrightarrow{\Delta_\ast} & \kappa \langle \langle Y \rangle \rangle^{\hat{\otimes}_2}
\end{array}
\]

for some pair \((\mu, \Phi)\), \( \mu \in k^\times \), \( \Phi \in \text{DMR}_\mu(k) \). In (1.3.1) we introduce the notation \( \hat{W}^\text{DR}_r := \kappa \langle \langle Y \rangle \rangle \), and in (3.1) we introduce 'right' versions \( \hat{W}^\text{DR}_r \), \( \hat{W}_r^B \) of \( \hat{W}^\text{DR}_r \), \( \hat{W}_r^B \) as well as 'left-right' variants \( \hat{\Delta}_\ast^{l,r} : \hat{W}^\text{DR}_r \to (\hat{W}^\text{DR}_r)^{\hat{\otimes}_2} \), \( \hat{\Delta}_\ast^{l,r} : \hat{W}_r^B \to (\hat{W}_r^B)^{\hat{\otimes}_2} \) of the maps \( \hat{\Delta}_\ast \), \( \hat{\Delta}_\ast \); the above commutative diagram is then equivalent to the commutativity of

\[
\begin{array}{ccc}
\hat{W}^B & \xrightarrow{\hat{\Delta}_\ast^{l,r}} & (\hat{W}^B)^{\hat{\otimes}_2} \\
\hat{\Delta}_\ast^{l,r} & \downarrow \ & \downarrow \zeta^{\otimes 2} \\
\hat{W}^\text{DR}_r & \xrightarrow{\hat{\Delta}_\ast^{l,r}} & (\hat{W}^\text{DR}_r)^{\hat{\otimes}_2}
\end{array}
\]

(0.3.1)
In order to do this, one first proves this commutativity for $\Phi$ being an associator with parameter $\mu$. This is done in several steps: in §5 based on the preprint [DeT], we insert $\hat{\Delta}^{l,r}_*$ in a commutative diagram (5.3.3) based on infinitesimal braid Lie algebras. In §6 we construct a similar commutative diagram (6.3.3) relating $\hat{\Delta}^{l,r}_*$ and braid groups. After recalling the properties of associators of relating infinitesimal braid Lie algebras and braid groups in §7 we construct in §8 commutative diagrams relating with one another various constituents of (5.3.3) and (6.3.3). As a result, one proves the commutativity of (0.3.1) for $\Phi$ being an associator with parameter $\mu$ (see Proposition 8.17).

While Proposition 8.17 states the commutativity of (0.3.1) for $\Phi$ being an associator with parameter $\mu$, the proof of Theorem 3.1 needs the same statement for $\Phi$ an element of $\text{DMR}_\mu(k)$. In order to prove this last statement, one may either: (a) combine the result of [Fu2] according to which there is an inclusion of the set of associators defined over $k$ with parameter $\mu$ to $\text{DMR}_\mu(k)$ with the result to [Dr] according to which this set of associators is not empty, or (b) assume that $k = \mathbb{C}$, choose for $(\mu, \Phi)$ the pair $(1, \varphi_{\text{KZ}})$ (see §1.1), derive from there the equality $\hat{\Delta}_C = (\Delta_C^\text{KZ})^\wedge$ of the tensor products of the coproduct with $\mathbb{C}$, and use base change to prove this equality over a general $\mathbb{Q}$-algebra $k$ (see §9).

0.4. Organization of the paper. This paper is divided into three parts. In Part 1, we recall some background, prove 1) and 2) in Theorem 0.2 and give a reformulation of 3) in this theorem, which is Theorem 3.1. In Part 2, we prove Theorem 3.1 according to the ideas explained above. Part 3 is devoted to a discussion of the categorical aspects of this proof.

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0.5. Convention. Throughout the paper, except in §9 we fix a commutative, associative and unital $\mathbb{Q}$-algebra, which will be denoted $k$.

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1. A COPRODUCT \( \hat{\Delta} \) ON \( \hat{W}^B \) ARISING FROM THE STABILIZER INTERPRETATION OF THE DOUBLE SHUFFLE GROUP

In §1.1 we first recall the construction \( (\text{RS}) \) of the following structure, originating in the formalism of double shuffle relations for MZVs: a family of subsets \( \text{DMR}_\mu(k) \) indexed by \( \mu \in k \) of a group of automorphisms of the topologically free Lie algebra over two generators, denoted \( (\exp(\text{Lib}_k(X)), \oplus) \), the fact that \( \text{DMR}_0(k), \oplus \) is a subgroup, and the fact that this group acts by left multiplication in a free and transitive way on \( \text{DMR}_\mu(k) \) for any \( \mu \in k \). We also explain the construction of an action of \( k^\times \) on the group \( (\exp(\text{Lib}_k(X)), \oplus) \) and on its subgroup \( \text{DMR}_0(k), \oplus \), and that a subset \( \cup_{\mu \in k^\times} \{ \mu \} \times \text{DMR}_\mu(k) \) of the semidirect product \( k^\times \ltimes (\exp(\text{Lib}_k(X)), \oplus) \) constructed out of the various \( \text{DMR}_\mu(k) \) is a torsor over the semidirect product \( k^\times \ltimes \text{DMR}_0(k) \) under action by left multiplications.

In §1.2 we recall the result from \( \text{EfFu0} \) relating the double shuffle group \( \text{DMR}_0(k), \oplus \) with a stabilizer group, namely the stabilizer of the harmonic coproduct \( \hat{\Delta}_c \) from the theory of \( \text{RS} \), viewed as an element of the \( k \)-module \( \text{Hom}_{k\text{-mod}}(k\langle Y \rangle, k\langle Y \rangle)^{\otimes 2} \), which is a module over \( (\exp(\text{Lib}_k(X)), \oplus) \). We also slightly extend the result from \( \text{EfFu0} \) to semidirect products, by relating the semidirect product \( k^\times \ltimes (\text{DMR}_0(k), \oplus) \) with the stabilizer of \( \hat{\Delta}_c \) for an action of the semidirect product \( k^\times \ltimes (\exp(\text{Lib}_k(X)), \oplus) \) on \( \text{Hom}_{k\text{-mod}}(k\langle Y \rangle, k\langle Y \rangle)^{\otimes 2} \) (Corollary 1.7).

In §1.3 we interpret the basic algebras \( k\langle X \rangle, k\langle Y \rangle \) of \( \text{RS} \) in terms of Lie algebras: \( k\langle X \rangle \) identifies with the enveloping algebra \( U(f_2) \) of a free Lie algebra \( f_2 \) with two generators; \( k\langle Y \rangle \) identifies with a subalgebra, denoted \( \hat{\mathcal{W}}^\text{DR}_1 \) (a notation borrowed from \( \text{DeT} \)). We also introduce the Betti counterpart of these objects, namely the group algebra \( kF_2 \) of the free group \( F_2 \) over two generators and its subalgebra \( \hat{\mathcal{W}}^B \), as well as various filtrations and completions. The completions of \( \hat{\mathcal{W}}^B \) (resp. \( \hat{\mathcal{W}}^\text{DR}_1 \)) are denoted \( \hat{\mathcal{W}}_1^B \) (resp. \( \hat{\mathcal{W}}_1^\text{DR} \)); there is a canonical isomorphism \( \hat{\mathcal{W}}_1^B \simeq k\langle Y \rangle \). We also construct a family of topological algebra isomorphisms \( \text{iso}_1^\mu : \hat{\mathcal{W}}_1^B \rightarrow \hat{\mathcal{W}}_1^\text{DR} \) indexed by \( \mu \in k^\times \).

In §1.4 we apply Principle 0.1 in the following particular case given by (0.2.1). One then obtains an element \( \tilde{v} \in V = \text{Hom}_{k\text{-mod}}(\hat{\mathcal{W}}_1^\text{DR}, (\hat{\mathcal{W}}_1^\text{DR})^{\otimes 2}) \). We denote the pull-back
of this element through the isomorphism $\text{iso}^i : \hat{W}_l^B \to \hat{W}_l^{DR}$ by $\hat{\Delta}_*$; this is an element of $\text{Hom}_{k-\text{mod}}(\hat{W}_l^B, (\hat{W}_l^B)^{\otimes 2})$, which will be our main object of study.

In §1.5, we show that $\hat{\Delta}_*$ equips $\hat{W}_l^B$ with a topological cocommutative Hopf algebra structure (Theorem 1.16).

In §1.6, we show that $\hat{\Delta}_*$ is compatible with the filtration of $\hat{W}_l^B$ and show that the associated graded map can be identified with $\Delta_*$ (Theorem 1.17).

Finally, in §1.7, we show that $\hat{\Delta}_*$ fits in a certain commutative diagram, a result which will be useful for its computation in the later parts of the paper (Proposition 1.19).

Before closing this introduction, we give the following diagram of the main Hopf algebra structures introduced in this and the next sections, and of their logical interrelations.

\[
\begin{array}{ccc}
(k\langle Y \rangle, \Delta_*) \ (\text{(4.1)} [\text{Rac}]) & \xrightarrow{\text{completion}} & (k\langle\langle Y \rangle\rangle, \hat{\Delta}_*) \ (\text{(4.1)} ) \\
\downarrow \text{isom.} & & \downarrow \text{isom.} \\
(W_l^{DR}, \Delta_*) \ (\text{(4.3)} ) & \xrightarrow{\text{completion}} & (\hat{W}_l^{DR}, \hat{\Delta}_*) \ (\text{(4.3)} ) \\
\downarrow \text{pull-back by } S^Y_{\partial(\mu, \Phi)} \mid & & \mid \downarrow \text{(Theorem 3.1)} \\
(W_l^B, \Delta_*) \ (\text{(2.2)} ) & \xrightarrow{\text{completion}} & (\hat{W}_l^B, \hat{\Delta}_*) \ (\text{(2.3)} )
\end{array}
\]

1.1. Construction of a torsor over $k^\times \ltimes \text{DMR}_0(k)$.

1.1.1. A group inclusion $(\exp(\hat{\mathfrak{Lib}_k}(X)), \oplus) \subset (k\langle\langle X \rangle\rangle, \oplus)$. Let $x_0, x_1$ be two letters and let $X := \{x_0, x_1\}$. Let $k\langle X \rangle$ (resp., $k\langle\langle X \rangle\rangle$) be the $k$-algebra of noncommutative polynomials (resp., formal power series) over $X$. The group $k\langle\langle X \rangle\rangle^\times$ of invertible elements of the latter algebra coincides with the set of formal power series with invertible constant term; the product in this group will be denoted $(G, H) \mapsto G \cdot H$. Define $\mathfrak{Lib}_k(X)$ as the Lie subalgebra of $k\langle X \rangle$ generated by $X$, and $\hat{\mathfrak{Lib}_k}(X)$ as its completion in $k\langle\langle X \rangle\rangle$.

For $G \in k\langle\langle X \rangle\rangle^\times$ and $H \in k\langle\langle X \rangle\rangle$, define $G \oplus H \in k\langle\langle X \rangle\rangle$ by

\[
G \oplus H := G \cdot \tilde{a}_G(H) = a_G(H) \cdot G,
\]

where $\tilde{a}_G$, $a_G$ are the continuous automorphisms of $k\langle\langle X \rangle\rangle$ given by $\tilde{a}_G : x_0 \mapsto x_0, \ x_1 \mapsto G^{-1} \cdot x_1 \cdot G$ and

\[
a_G : x_0 \mapsto G \cdot x_0 \cdot G^{-1}, \quad x_1 \mapsto x_1.
\]
Then $\oplus$ restricts to a map $(\mathbf{k}\langle\langle X \rangle\rangle)^2 \to \mathbf{k}\langle\langle X \rangle\rangle^\times$ and equips $\mathbf{k}\langle\langle X \rangle\rangle^\times$ with a group structure. Then there is a group inclusion
\[(\exp(\widehat{\mathbf{Li}}_h(X)), \oplus) \subset (\mathbf{k}\langle\langle X \rangle\rangle^\times, \oplus)\.
\]

1.1.2. A group morphism $\Theta : (\exp(\widehat{\mathbf{Li}}_h(X)), \oplus) \to (\mathbf{k}\langle\langle X \rangle\rangle^\times, \oplus)$. For $G \in \exp(\widehat{\mathbf{Li}}_h(X))$, set
\[
\Gamma_G(t) := \exp\left(\sum_{n \geq 1} \frac{(-1)^n}{n} (G|x_0^n-1) t^n\right) \in \mathbf{k}[t]^\times.
\]
There is a group morphism $\Theta : (\exp(\widehat{\mathbf{Li}}_h(X)), \oplus) \to (\mathbf{k}\langle\langle X \rangle\rangle^\times, \oplus)$,
\[G \mapsto \Gamma_G(x_1)^{-1} \cdot G \cdot \exp(-G|x_0 x_0).\]

1.1.3. A module $(\mathbf{k}\langle\langle Y \rangle\rangle, g \mapsto S^Y_g)$ over the group $(\mathbf{k}\langle\langle X \rangle\rangle^\times, \oplus)$. Let $Y := \{y_n | n \geq 1\}$ be a set of variables; let $\mathbf{k}(Y)$ be the $\mathbf{k}$-algebra of noncommutative polynomials over $Y$. This algebra is graded by $\deg(y_n) = n$. Let $\mathbf{k}\langle\langle Y \rangle\rangle$ be its completion for the degree. The assignment $y_n \mapsto x_0^n x_1$ for $n \geq 1$ gives rise to an injective algebra morphism $\mathbf{k}(Y) \to \mathbf{k}(X)$, which yields an algebra isomorphism $\mathbf{k}\langle\langle Y \rangle\rangle \cong \mathbf{k} \oplus \mathbf{k}(X) x_1$.

One then has $\mathbf{k}(X) \cong \mathbf{k}(Y) \oplus \mathbf{k}(X) x_0$, which defines a projection map $\pi_Y : \mathbf{k}\langle\langle Y \rangle\rangle \to \mathbf{k}\langle\langle Y \rangle\rangle$.

There is a unique action of $(\mathbf{k}\langle\langle X \rangle\rangle^\times, \oplus)$ on the $\mathbf{k}$-module $\mathbf{k}\langle\langle Y \rangle\rangle$, given by the map $(\mathbf{k}\langle\langle X \rangle\rangle^\times, \oplus) \times \mathbf{k}\langle\langle Y \rangle\rangle \to \mathbf{k}\langle\langle Y \rangle\rangle$, $(G, h) \mapsto S^Y_G(h) := G \oplus h$. It induces an action of the same group on the $\mathbf{k}$-module $\mathbf{k}\langle\langle Y \rangle\rangle$, denoted $(\mathbf{k}\langle\langle X \rangle\rangle^\times, \oplus) \times \mathbf{k}\langle\langle Y \rangle\rangle \to \mathbf{k}\langle\langle Y \rangle\rangle$, $(G, h) \mapsto S^Y_G(h)$, defined by the condition that the map $\pi_Y : \mathbf{k}\langle\langle X \rangle\rangle \to \mathbf{k}\langle\langle Y \rangle\rangle$ is equivariant.

1.1.4. A $\mathbf{k}$-linear map $\hat{\Delta}_* : \mathbf{k}\langle\langle Y \rangle\rangle \to \mathbf{k}\langle\langle Y \rangle\rangle^\otimes 2$. Define $\Delta_* : \mathbf{k}(Y) \to \mathbf{k}(Y)^\otimes 2$ to be the algebra morphism such that
\[
y_n \mapsto y_n \otimes 1 + 1 \otimes y_n + \sum_{n', n'' > 0, n' + n'' = n} y_{n'} \otimes y_{n''}.
\]
It extends to a continuous algebra morphism $\hat{\Delta}_* : \mathbf{k}\langle\langle Y \rangle\rangle \to \mathbf{k}\langle\langle Y \rangle\rangle^\otimes 2$ (where the completion refers to the degree topology).

1.1.5. Definition of DMR$^\mu_0(\mathbf{k})$ and torsor results. For $\mu \in \mathbf{k}$, set
\[
\text{DMR}_\mu(\mathbf{k}) := \left\{ \Phi \in \exp(\widehat{\mathbf{Li}}_h(X)) \mid \pi_Y(\Theta(\Phi)) \text{ is } \hat{\Delta}_*-\text{primitive and } (\Phi|x_0) = (\Phi|x_1) = 0, \quad (\Phi|x_0 x_1) = -\mu^2/24. \right\}
\]
Then one has:

**Theorem 1.1** (§3.2.3 in [R]). (1) DMR$^0_0(\mathbf{k})$ is a subgroup of $(\exp(\widehat{\mathbf{Li}}_h(X)), \oplus)$, and $\mathbf{k} \mapsto \text{DMR}_0(\mathbf{k})$ is a pronipotent $\mathbb{Q}$-group scheme.
(2) For any $\mu \in \mathbf{k}$, the set $\text{DMR}_\mu(\mathbf{k})$ is nonempty. If $g \in \text{DMR}_0(\mathbf{k})$ and $\Phi \in \text{DMR}_\mu(\mathbf{k})$, then $g \otimes \Phi \in \text{DMR}_\mu(\mathbf{k})$. This defines a left action of $\text{DMR}_0(\mathbf{k})$ on $\text{DMR}_\mu(\mathbf{k})$, which is free and transitive.

(3) Let $A, B$ be noncommutative variables and denote $\varphi_{\text{KZ}} \in \mathbb{C}\langle\langle A, B \rangle\rangle$ the series constructed in [Dr] §2. Identify $\varphi_{\text{KZ}}$ with its image under the morphism $\mathbb{C}\langle\langle A, B \rangle\rangle \to \mathbb{C}\langle\langle X \rangle\rangle$, $A \mapsto x_0$, $B \mapsto -x_1$. Then $\varphi_{\text{KZ}}$ belongs to $\text{DMR}_1(\mathbb{C})$.

**Lemma 1.2.** Let $\mu \in \mathbf{k}^\times$ and $\Phi \in \text{DMR}_\mu(\mathbf{k})$. Then one has the following equality in $1 + t\mathbf{k}[t]$:

$$
\Gamma_\Phi(t)\Gamma_\Phi(-t) = \frac{\mu t}{e^{\mu t/2} - e^{-\mu t/2}}.
$$

**Proof.** Set $\widehat{\text{Lie}}^0_k(X)$ be the subspace of $\widehat{\text{lie}}_k(X)$ of all Lie series with vanishing coefficient of $x_0$. Then $\exp(\widehat{\text{Lie}}^0_k(X))$ is a subgroup of $(\exp(\widehat{\text{lie}}_k(X)), \otimes)$. Moreover, one checks that the map $(\exp(\widehat{\text{Lie}}^0_k(X)), \otimes) \to 1 + t\mathbf{k}[t])$, $G \mapsto \Gamma_G(t)$ is a group morphism. Composing it with the group endomorphism of $1 + t\mathbf{k}[t]$ given by $f(t) \mapsto f(t)f(-t)$, one obtains a group morphism $\varpi : (\exp(\widehat{\text{Lie}}^0_k(X)), \otimes) \to 1 + t\mathbf{k}[t])$, $G \mapsto \Gamma_G(t)\Gamma_G(-t)$.

The corresponding Lie algebra morphism is a morphism $\widehat{\text{Lie}}^0_k(X) \to t\mathbf{k}[t]$. Its restriction to the Lie algebra of $\text{DMR}_0(k)$ was shown in [R], Proposition 3.3.3 to be zero. It follows that $\text{DMR}_0(\mathbf{k})$ is contained in the kernel of $\varpi$.

Together with the torsor result from Theorem 1.1 (2), and the nonemptiness of $\text{DMR}_\mu(\mathbf{k})$ ([R], Theorem 1), this implies the existence of a functorial assignment taking any pair $(\mathbf{k}, \mu)$, where $\mathbf{k}$ is a $\mathbb{Q}$-algebra and $\mu \in \mathbf{k}^\times$, to a series $f_{k,\mu} \in 1 + \mathbf{k}[t]$], such that for any $G \in \text{DMR}_\mu(\mathbf{k})$, one has $\Gamma_G(t)\Gamma_G(-t) = f_{k,\mu}(t)$.

The algebra $\mathbf{k}\langle\langle X \rangle\rangle$ is equipped with an action of the multiplicative group $\mathbf{k}^\times$ (scaling action), the automorphism corresponding to $k \in \mathbf{k}^\times$ being $k \cdot x_i := kx_i$, $i = 0, 1$. Then if $\mu \in \mathbf{k}^\times$ and $\Phi \in \text{DMR}_\mu(\mathbf{k})$, one has $k \cdot \Phi \in \text{DMR}_\mu(k)$. One then checks that $f_{k\cdot\mu}(t) = f_{k,\mu}(kt)$ for $k, \mu \in \mathbf{k}^\times$. In particular, one has $f_{k,\mu}(t) = f_{k,1}(\mu t)$ for $\mu \in \mathbf{k}^\times$.

We have the following equality in $\mathbb{C}\langle\langle A, B \rangle\rangle$:

$$
\log \varphi_{\text{KZ}} = -\sum_{n \geq 2} \frac{\zeta(n)}{(2\pi i)^n} \left(\text{ad} \ A\right)^n(B) + \text{terms containing } B \text{ more than once}.
$$

Moreover,

$$
\Gamma_\varphi_{\text{KZ}}(t) = \exp\left(\sum_{n \geq 2} \frac{(-1)^n}{n} \frac{\zeta(n)}{(2\pi i)^n} t^n\right) = e^{\gamma t/(2\pi i)}\Gamma(1 + \frac{t}{2\pi i}),
$$

where $\gamma$ is the Euler-Mascheroni constant and where the last identity follows from the following identity on the classical $\Gamma$-function $\log\Gamma(1 - t) = \gamma t + \sum_{n \geq 2} \zeta(k)t^k/k$. The following identities (a) $\Gamma(t + 1) = t\Gamma(t)$, (b) $\Gamma(t)\Gamma(-t) = \frac{\pi}{\sin(\pi t)}$ imply that $\Gamma(1 + \frac{t}{2\pi i})\Gamma(1 - \frac{t}{2\pi i}) = \frac{t}{e^{\pi i/2} - e^{-\pi i/2}}$, therefore $\Gamma_\varphi_{\text{KZ}}(t)\Gamma_{\text{KZ}}(-t) = \Gamma(1 + \frac{t}{2\pi i})\Gamma(1 - \frac{t}{2\pi i}) = \frac{t}{e^{\pi i/2} - e^{-\pi i/2}}$, so

$$
f_{1,\mathcal{C}}(t) = \frac{t}{e^{t/2} - e^{-t/2}} + 1 \in 1 + t\mathbb{C}[t].
$$

The index stands for 'Knizhnik-Zamolodchikov'.
This series actually has rational coefficients and therefore makes sense also in $1 + t k[[t]]$ for any $\mathbb{Q}$-ring $k$. By functoriality, we have therefore $f_{1,k}(t) = \frac{t}{e^{t/2} - e^{-t/2}}$, which implies $f_{\mu,k}(t) = \frac{t}{e^{t/2} - e^{-t/2}}$ for any $\mu \in k^\times$. 

\[ (1.1.8) \]

1.1.6. Actions of $k^\times$. Recall the action of the group $k^\times$ on the algebra $k\langle\langle X\rangle\rangle$ (Lemma [1.2]). It induces an action of $k^\times$ on the group $(k\langle\langle X\rangle\rangle^\times, \otimes)$, which restricts to an action of $k^\times$ on its subgroup $(\exp(\widehat{\mathfrak{mf}}_k(X)), \otimes)$. For $G$ any of these two groups, we denote by $k^\times \ltimes G$ its semidirect product with $k^\times$; this is the set $k^\times \times G$, equipped with the product

\[ (1.1.5) \quad (\mu, g) \circ (\mu', g') := (\mu \mu', g \circ (\mu \cdot g')). \]

One checks that for $G \in \exp(\widehat{\mathfrak{mf}}_k(X))$ and $\mu \in k^\times$, one has $\Gamma_{\mu \ast G}(t) = \Gamma_G(\mu t)$; it follows that the group morphism $\Theta$ (see (1.1.2)) is $k^\times$-equivariant, and therefore that it extends to a group morphism

\[ (1.1.6) \quad \Theta : k^\times \ltimes (\exp(\widehat{\mathfrak{mf}}_k(X)), \otimes) \to k^\times \ltimes (k\langle\langle X\rangle\rangle^\times, \otimes), \quad (\mu, G) \mapsto (\mu, \Theta(G)). \]

One also checks that the map

\[ (k^\times \times k\langle\langle X\rangle\rangle^\times) \to k\langle\langle Y\rangle\rangle, \quad ((\mu, G), h) \mapsto S_{(\mu, G)}(h) := G \otimes (\mu \cdot h) \]

defines an action of the group $k^\times \ltimes (k\langle\langle X\rangle\rangle^\times, \otimes)$ on the $k$-module $k\langle\langle X\rangle\rangle$. One checks that the $k$-submodule $k\langle\langle X\rangle\rangle_{x_0}$ of $k\langle\langle X\rangle\rangle$ is preserved by this action. This induces an action of the group $k^\times \ltimes (k\langle\langle X\rangle\rangle^\times, \otimes)$ on the $k$-module $k\langle\langle X\rangle\rangle$, defined by the condition that the projection $\pi_Y : k\langle\langle X\rangle\rangle \to k\langle\langle Y\rangle\rangle$ is equivariant under the action of this group. This action is denoted

\[ (1.1.7) \quad (k^\times \times k\langle\langle X\rangle\rangle^\times) \to k\langle\langle Y\rangle\rangle, \quad ((\mu, G), h) \mapsto S^Y_{(\mu, G)}(h). \]

One checks that $S^Y_{(\mu, G)}(h) = S^Y_{G}(\mu \cdot h)$ for $G \in k\langle\langle X\rangle\rangle^\times$, $\mu \in k^\times$, $h \in k\langle\langle Y\rangle\rangle$, where $\cdot$ is the action of $k^\times$ on $k\langle\langle Y\rangle\rangle$ by the algebra automorphism induced by $\mu \cdot y_n := \mu^n y_n$ for $n \geq 1$.

1.1.7. A torsor over a semidirect product group. It follows from the compatibilities of $\pi_Y$, $\Theta$ and $\Delta_*$ with the action of the group $k^\times$ that the action of this group on $(\exp(\widehat{\mathfrak{mf}}_k(X)), \otimes)$ is such that

\[ \forall k \in k^\times, \quad \forall \mu \in k, \quad k \bullet DMR_{\mu}(k) = DMR_{k\mu}(k). \]

This implies that the subgroup $(DMR_0(k), \otimes)$ of $(\exp(\widehat{\mathfrak{mf}}_k(X)), \otimes)$ is preserved by the action of $k^\times$. Let $k^\times \ltimes (DMR_0(k), \otimes)$ be the corresponding semidirect product; this is a subgroup of $k^\times \ltimes (\exp(\widehat{\mathfrak{mf}}_k(X)), \otimes)$. The subset $\{(\mu, g) | \mu \in k^\times, g \in DMR_{\mu}(k)\}$ of $k^\times \ltimes \exp(\widehat{\mathfrak{mf}}_k(X))$ may be denoted

\[ (1.1.8) \quad \downarrow_{\mu \in k^\times} \{\mu\} \times DMR_{\mu}(k). \]

As $k^\times \ltimes \exp(\widehat{\mathfrak{mf}}_k(X))$ is the underlying set of the group $k^\times \ltimes (\exp(\widehat{\mathfrak{mf}}_k(X)), \otimes)$, (1.1.8) may be viewed as a subset of this group.
Proposition 1.3. The action of the group \( k^x \ltimes (\exp(\hat{\mathfrak{Liib}}_k(X)), \oplus) \) on itself by left multiplication restricts to an action of the group \( k^x \ltimes (\mathrm{DMR}_0(k), \oplus) \) on the set \( \cup_{\mu \in k^x} \{ \mu \} \times \mathrm{DMR}_\mu(k) \). This action is free and transitive.

Proof. In \([1.1.3]\), assume that \( g \in \mathrm{DMR}_0(k) \) and \( g' \in \mathrm{DMR}_\mu'(k) \). Then \( \mu \cdot g' \in \mathrm{DMR}_\mu''(k) \), and \( g \oplus (\mu \cdot g') \in \mathrm{DMR}_\mu''(k) \). This proves the first statement. As the action of \( k^x \ltimes (\mathrm{DMR}_0(k), \oplus) \) on \( \cup_{\mu \in k^x} \{ \mu \} \times \mathrm{DMR}_\mu(k) \) is the restriction of a free action, it is free as well. Let us show that it is transitive. Let \( \mu', \mu'' \in k^x \) and \( g' \in \mathrm{DMR}_\mu'(k) \), \( g'' \in \mathrm{DMR}_\mu''(k) \). Set \( \mu := \mu'' / \mu' \), then \( \mu \cdot g' \in \mathrm{DMR}_\mu''(k) \). As the action of \( (\exp(\hat{\mathfrak{Liib}}_k(X)), \oplus) \) on \( \mathrm{DMR}_\mu''(k) \) is transitive, there exists \( g \in \mathrm{DMR}_0(k) \) such that \( g \oplus (\mu \cdot g') = g'' \). Then \( (\mu, g) \in k^x \ltimes \mathrm{DMR}_0(k), \oplus) \) is such that \( (\mu, g) \oplus (\mu', g') = (\mu'', g'') \), which proves the announced transitivity. \( \square \)

1.2. Stabilizer interpretation of double shuffle groups.

1.2.1. Stabilizer interpretation of \( \mathrm{DMR}_0(k) \). The stabilizer of \( \hat{\Delta}_* \) under the action of \((\exp(\hat{\mathfrak{Liib}}_k(X)), \oplus) \) on \( \text{Hom}_{k\text{-mod}}(k\langle\langle Y\rangle\rangle, k\langle\langle Y\rangle\rangle)^{\hat{\otimes}2}) \) induced by \( g \mapsto S^Y_{\Theta(g)} \) is

\[
\text{Stab}(\hat{\Delta}_*) = \{ g \in \exp(\hat{\mathfrak{Liib}}_k(X)) \mid \hat{\Delta}_* \circ S^Y_{\Theta(g)} = (S^Y_{\Theta(g)})^{\hat{\otimes}2} \circ \hat{\Delta}_* \}.
\]

One has

Theorem 1.4 (EFall). The subset \( \{ e^{\alpha x_1} \cdot g \cdot e^{\alpha x_0} \mid \alpha, \beta \in k, g \in \mathrm{DMR}_0(k) \} \) of \( \exp(\hat{\mathfrak{Liib}}_k(X)) \) is a subgroup of \((\exp(\hat{\mathfrak{Liib}}_k(X)), \oplus)) \) and one has

\[
\text{Stab}(\hat{\Delta}_*) = \{ e^{\alpha x_1} \cdot g \cdot e^{\alpha x_0} \mid \alpha, \beta \in k, g \in \mathrm{DMR}_0(k) \}
\]

(equality of subgroups of \((\exp(\hat{\mathfrak{Liib}}_k(X)), \oplus)) \).  

1.2.2. Stabilizer interpretation of \( k^x \ltimes \mathrm{DMR}_0(k) \). Combining the group morphism \((1.1.6) \) and the action of the target group of this morphism on the k-module \( k\langle\langle Y\rangle\rangle \) (see \((1.1.7) \)), we obtain an action of the group \( k^x \ltimes (\exp(\hat{\mathfrak{Liib}}_k(X)), \oplus) \) on the k-module \( k\langle\langle Y\rangle\rangle \) and therefore on the k-module \( \text{Hom}_{k\text{-mod}}(k\langle\langle Y\rangle\rangle, k\langle\langle Y\rangle\rangle)^{\hat{\otimes}2}) \). The stabilizer of \( \hat{\Delta}_* \) under the action of \( k^x \ltimes (\exp(\hat{\mathfrak{Liib}}_k(X)), \oplus) \) on this space is

\[
\text{Stab}^x(\hat{\Delta}_*) = \{ (k, g) \in k^x \times \exp(\hat{\mathfrak{Liib}}_k(X)) \mid \hat{\Delta}_* \circ S^Y_{\Theta(k, g)} = (S^Y_{\Theta(k, g)})^{\hat{\otimes}2} \circ \hat{\Delta}_* \}.
\]

Lemma 1.5. The action of \( k^x \) on \((\exp(\hat{\mathfrak{Liib}}_k(X)), \oplus)) \) preserves the subgroup \((\text{Stab}(\hat{\Delta}_*), \oplus)) \).

Proof. Let \( k \in k^x \) and \( G \in k\langle\langle X\rangle\rangle^X \). Then \( S_{k \bullet G} = (k \bullet -) \circ S_{G} \circ (k \bullet -)^{-1} \) (equality of k-linear endomorphisms of \( k\langle\langle X\rangle\rangle \)), therefore \( S_{k \bullet G} = (k \bullet -) \circ S^Y_{\Theta} \circ (k \bullet -)^{-1} \) (equality of k-linear endomorphisms of \( k\langle\langle Y\rangle\rangle \)). Moreover, it \( g \in \exp(\hat{\mathfrak{Liib}}_k(X)) \), then \( \Theta(k \bullet g) = k \bullet \Theta(g) \). It follows that

\[
(1.2.1) \quad S^Y_{\Theta(k \bullet g)} = (k \bullet -) \circ S^Y_{\Theta(g)} \circ (k \bullet -)^{-1}.
\]

\( ^{6} \)We denote by \( k\text{-mod} \) (resp., \( k\text{-alg} \)) the category of k-modules (resp., k-algebras).
If now \( g \in \text{Stab}(\hat{\Delta}_*) \) and \( k \in k^\times \), then
\[
\hat{\Delta}_* \circ S^Y_{\Theta(k,g)} = \hat{\Delta}_* \circ (k \bullet -) \circ S^Y_{\Theta(g)} \circ (k \bullet -)^{-1} = (k \bullet -)^{\otimes 2} \circ \hat{\Delta}_* \circ S^Y_{\Theta(g)} \circ (k \bullet -)^{-1}
\]
\[
= (k \bullet -)^{\otimes 2} \circ (S^Y_{\Theta(g)})^{\otimes 2} \circ \hat{\Delta}_* \circ (k \bullet -)^{-1} = (k \bullet -)^{\otimes 2} \circ (S^Y_{\Theta(g)})^{\otimes 2} \circ ((k \bullet -)^{-1})^{\otimes 2} \circ \hat{\Delta}_*
\]
where the first and last equalities follow from (1.221), the second and fourth equalities follow from the compatibility of \( \hat{\Delta}_* \) with the action of \( k^\times \), the third equality follows from \( g \in \text{Stab}(\hat{\Delta}_*) \). All this implies that \( k \bullet g \in \text{Stab}(\hat{\Delta}_*) \). □

One can then construct the semidirect product of \((\text{Stab}(\hat{\Delta}_*), \circ)\) by \( k^\times \), and one then has a group inclusion
\[
k^\times \ltimes (\text{Stab}(\hat{\Delta}_*), \circ) \subset k^\times \ltimes (\exp(\hat{\mathfrak{lib}}_k(X)), \circ).
\]

**Proposition 1.6.** One has
\[
(\text{Stab}^k(\hat{\Delta}_*), \circ) = k^\times \ltimes (\text{Stab}(\hat{\Delta}_*), \circ)
\]
(equality of subgroups of \( k^\times \ltimes (\exp(\hat{\mathfrak{lib}}_k(X)), \circ) \)).

**Proof.** Let \((k, G) \in k^\times \times \exp(\hat{\mathfrak{lib}}_k(X)) \). Then \((k, G) \in \text{Stab}^k(\hat{\Delta}_*)\) is equivalent to \((S^Y_{\Theta(k,G)})^{\otimes 2} \circ \hat{\Delta}_* = \hat{\Delta}_* \circ S^Y_{\Theta(k,G)}\). As \( S^Y_{\Theta(k,G)} = S^Y_{\Theta(G)} \circ (k \bullet -)\), this is equivalent to \((S^Y_{\Theta(G)} \circ (k \bullet -))^{\otimes 2} \circ \hat{\Delta}_* = \hat{\Delta}_* \circ S^Y_{\Theta(G)}\). Since \((k \bullet -)^{\otimes 2} \circ \hat{\Delta}_* = \hat{\Delta}_* \circ (k \bullet -)\), and since \((k \bullet -)\) is invertible, this is equivalent to \((S^Y_{\Theta(G)})^{\otimes 2} \circ \hat{\Delta}_* = \hat{\Delta}_* \circ S^Y_{\Theta(G)}\), therefore \( G \in \text{Stab}(\hat{\Delta}_*) \). □

**Corollary 1.7.** The subgroup \( \{e^{\beta x_1} \cdot g \cdot e^{\alpha x_0} \mid \alpha, \beta \in k, g \in \text{DMR}_0(k)\}, \circ \) of \( (\exp(\hat{\mathfrak{lib}}_k(X)), \circ) \) is stable under the action of \( k^\times \) and one has
\[
k^\times \ltimes \{e^{\beta x_1} \cdot g \cdot e^{\alpha x_0} \mid \alpha, \beta \in k, g \in \text{DMR}_0(k)\}, \circ = (\text{Stab}^k(\hat{\Delta}_*), \circ)
\]
(equality of subgroups of \( k^\times \ltimes (\exp(\hat{\mathfrak{lib}}_k(X)), \circ) \)).

**Proof.** This follows from the conjunction of Theorem [1.3] and Proposition [1.6] □

### 1.3. The algebras \( \mathcal{W}^\text{DR}_I, \mathcal{W}^\text{B}_I \) and their completions.

#### 1.3.1. Filtered algebras.**

By a filtered \( k \)-algebra, we mean an associative \( k \)-algebra \( A \) equipped with a descending sequence \((F^n A)_{n \geq 0}\) of sub-\( k \)-modules of \( A \) such that \( F^0 A = A \) and \( F^n A \cdot F^m A \subset F^{n+m} A \) for \( n, m \geq 0 \). The associated graded algebra \( \text{gr} A \) is defined as \( \oplus_{n \geq 0} F^n A / F^{n+1} A \). The associated topological algebra \( \hat{A} \) is the inverse limit over \( n \geq 0 \) of the quotients \( A / F^n A \); there is an algebra morphism \( A \to \hat{A} \). The topological algebra \( \hat{A} \) is equipped with a descending filtration given by \( F^n \hat{A} = \varprojlim F^n A / F^{n+1} A \). The constructions of \( \text{gr}(A) \) and \( \hat{A} \) are functorial with respect to the algebra morphisms which are compatible with the filtrations.

If \( M \) is any \( k \)-submodule of \( A \), then the filtration of \( \hat{A} \) induces a filtration on the closure \( \hat{M} \) of \( M \) in \( \hat{A} \). We denote by \( \text{gr}(\hat{M}) \) the associated graded of this filtration and by \( \text{gr}(M) \) the
associated graded of the filtration of $M$ induced by that of $A$. Then the natural map $M \rightarrow \hat{M}$ induces an isomorphism $\text{gr}(M) \rightarrow \text{gr}(\hat{M})$. When $M$ is an algebra, $\hat{M}$ is a topological subalgebra of $\hat{A}$.

1.3.2. The algebras $\mathcal{W}_i^{\text{DR}}, \hat{\mathcal{W}}_i^{\text{DR}}$. The objects introduced so far fit in the formalism of [DeT] according to the following convention, which agrees with [R], footnote p. 187 and will to be used from now on:

$$e_0 := x_0, \quad e_1 := -x_1, \quad f_2 := \mathfrak{Li}_k(X), \quad \hat{f}_2 := \hat{\mathfrak{Li}}_k(X).$$

We will view $f_2$ as a graded Lie algebra, where $e_0, e_1$ are of degree 1.

Let $U(f_2)$ be the enveloping algebra of $f_2$, then we have the following isomorphisms

$$U(f_2) \simeq k\langle X \rangle, \quad U(f_2)^\wedge \simeq k\langle \langle X \rangle \rangle,$$

where $U(f_2)^\wedge$ is the degree completion of $U(f_2)$. As the subspace $U(f_2)e_1$ (resp., $U(f_2)^\wedge e_1$) of $U(f_2)$ (resp., $U(f_2)^\wedge$) is in the kernel of the counit, its sum with the image $k1$ of the unit map is direct.

**Definition 1.8.** We set

$$\mathcal{W}_i^{\text{DR}} := k1 \oplus U(f_2)e_1 \subset U(f_2), \quad \hat{\mathcal{W}}_i^{\text{DR}} := k1 \oplus U(f_2)^\wedge e_1 \subset U(f_2)^\wedge.$$  

Then $\mathcal{W}_i^{\text{DR}}$ (resp., $\hat{\mathcal{W}}_i^{\text{DR}}$) is a subalgebra (resp. closed subalgebra) of $U(f_2)$ (resp., $U(f_2)^\wedge$).

We then have algebra isomorphisms

$$\mathcal{W}_i^{\text{DR}} \simeq k\langle Y \rangle, \quad \hat{\mathcal{W}}_i^{\text{DR}} \simeq k\langle \langle Y \rangle \rangle.$$  

1.3.3. The algebras $\mathcal{W}_i^{\text{B}}, \hat{\mathcal{W}}_i^{\text{B}}$. Let $F_2$ be the free group with generators $X_0, X_1$. We denote by $kF_2$ its group algebra over $k$. The left ideal $kF_2(X_1 - 1)$ is contained in the kernel of the counit map, therefore its sum with the image $k1$ of the unit map is direct.

**Definition 1.9.** We set

$$\mathcal{W}_i^{\text{B}} := k1 \oplus kF_2(X_1 - 1) \subset kF_2.$$  

Then $\mathcal{W}_i^{\text{B}}$ is a subalgebra of $kF_2$.

Let $I \subset kF_2$ be the augmentation ideal. The collection $(I^n)_{n \geq 0}$ of powers of $I$ defines a descending filtration on $kF_2$. We denote by $(kF_2)^\wedge$ the corresponding completed algebra.

We denote by $\hat{\mathcal{W}}_i^{\text{B}}$ the completion of $\mathcal{W}_i^{\text{B}}$ for the $I$-adic topology of $kF_2$; it is a topological subalgebra of $(kF_2)^\wedge$. The left ideal $(kF_2)^\wedge(X_1 - 1)$ of $(kF_2)^\wedge$ is contained in the kernel of the counit map, therefore its sum with the image $k1$ of the unit map is direct.

**Lemma 1.10.** One has $\hat{\mathcal{W}}_i^{\text{B}} = k1 \oplus (kF_2)^\wedge(X_1 - 1)$ (equivalence of closed subalgebras of $(kF_2)^\wedge$).

**Proof.** Since $k1 \oplus (kF_2)^\wedge(X_1 - 1)$ is a closed subspace of $(kF_2)^\wedge$ containing $\mathcal{W}_i^{\text{B}}$, one has $\hat{\mathcal{W}}_i^{\text{B}} \subset k1 \oplus (kF_2)^\wedge(X_1 - 1)$. On the other hand, the closure of $kF_2(X_1 - 1)$ in $(kF_2)^\wedge$ is a left
ideal of \((kF_2)^\wedge\) which contains \(X_1 - 1\), therefore it contains \((kF_2)^\wedge(X_1 - 1)\). It follows that \(\hat{W}^B_l\) contains \(k1 \oplus (kF_2)^\wedge(X_1 - 1)\).

1.3.4. Isomorphisms between \(\text{gr}(\hat{W}^B_l)\) and \(W^{DR}_l\). Let \(\mu \in k^\times\). There is a unique isomorphism of filtered algebras

\[
\text{iso}_\mu : (kF_2)^\wedge \rightarrow U(f_2)^\wedge, \quad X_i \mapsto \exp(\mu e_i) \quad \text{for} \quad i = 0, 1,
\]

so given by \(X_0 \mapsto \exp(\mu x_0), X_1 \mapsto \exp(-\mu x_1)\) (the 'exponential isomorphism').

The associated graded isomorphism \(\text{gr}(\text{iso}_\mu) : \text{gr}((kF_2)^\wedge) \rightarrow \text{gr}(U(f_2)^\wedge)\) is given, in degree 1, by \(X_i \mapsto \mu e_i\), for \(i = 0, 1\).

There is a unique algebra isomorphism \(\text{iso}_\mu^f : \hat{W}^B_l \rightarrow \hat{W}^{DR}_l\), such that the diagram

\[
\begin{array}{ccc}
\hat{W}^B_l & \xrightarrow{\text{iso}_\mu^f} & \hat{W}^{DR}_l \\
\downarrow \quad \text{iso} & & \downarrow \quad \text{iso} \\
(kF_2)^\wedge & \rightarrow & U(f_2)^\wedge
\end{array}
\]

commutes.

For \(\mu \in k^\times\), \(\text{iso}_\mu^f\) is strictly compatible with the filtrations on both sides and therefore induces an algebra isomorphism

\[
\text{gr}(\text{iso}_\mu^f) : \text{gr}(\hat{W}^B_l) \rightarrow \text{gr}(\hat{W}^{DR}_l) = W^{DR}_l.
\]

If \(k \in k^\times\) and \(\mu' := k\mu\), then \(\text{iso}_\mu^f = (k \bullet -) \circ \text{iso}_\mu\), therefore \(\text{gr}(\text{iso}_\mu^f) = (k \bullet -) \circ \text{gr}(\text{iso}_\mu)\); similarly, \(\text{gr}(\text{iso}_\mu^f) = (k \bullet -) \circ \text{gr}(\text{iso}_\mu)\).

1.4. Construction of the coproduct \(\hat{\Delta}_\star\) on \(\hat{W}^B_l\).

1.4.1. The coproduct \(\hat{\Delta}_\star\) on \(\hat{W}^{DR}_l\). Define

\[
\Delta_\star : W^{DR}_l \rightarrow (W^{DR}_l)^{\otimes 2}
\]

as the pull-back of \(\Delta_\star\) by the isomorphism \(W^{DR}_l \simeq k\langle Y\rangle\). Then \(\Delta_\star\) extends to a topological coproduct \(\hat{W}^{DR}_l \rightarrow (\hat{W}^{DR}_l)^{\otimes 2}\), denoted by \(\hat{\Delta}_\star\).

1.4.2. The action \((k, G) \mapsto S^Y_{(k,G)}\) on \(\hat{W}^{DR}_l\). There are algebra isomorphisms \(U(f_2)^\wedge \simeq k\langle X\rangle\) and \(\hat{W}^{DR}_l \simeq k\langle Y\rangle\) given by [1.3.1] and [1.3.2]. Under these isomorphisms, the \(k\)-linear map \(\pi_Y : k\langle X\rangle \rightarrow k\langle Y\rangle\) corresponds to the map \(\pi_Y : U(f_2)^\wedge \rightarrow \hat{W}^{DR}_l\) induced by the decomposition \(U(f_2)^\wedge \simeq k1 \oplus U(f_2)^{\wedge e_1} \oplus U(f_2)^{\wedge e_0} \simeq \hat{W}^{DR}_l \oplus U(f_2)^{\wedge e_0}\).

Under the isomorphisms [1.3.1] and [1.3.2], the actions of the group \(k^\times \times (\exp(\oplus_{k}k(X)), \oplus)\) on the \(k\)-modules \(k\langle X\rangle\) given by \((k, G) \mapsto S_{(k,G)}\) and on \(k\langle Y\rangle\) given by \((k, G) \mapsto S^Y_{(k,G)}\) correspond to actions of the same group on \(U(f_2)^\wedge\) and \(\hat{W}^{DR}_l\), denoted \((k, G) \mapsto S_{(k,G)}\) and \((k, G) \mapsto S^Y_{(k,G)}\); moreover, \(\pi_Y : U(f_2)^\wedge \rightarrow \hat{W}^{DR}_l\) intertwines the two module structures.
1.4.3. Construction of $\hat{\Delta}_\ast$. In addition to the group morphism

$$k^\times \times (\exp(\widehat{\mathcal{L}ib}(X)), \circledast) \to \text{Aut}_{k\text{-mod}}(\hat{\mathcal{V}}_l^{\text{DR}}), \quad (k, G) \mapsto S_Y^Y_{\Theta(k,G)} = S^Y_{\Theta(k,G)} \circ (k \bullet -),$$

define a map

$$k^\times \times (\exp(\widehat{\mathcal{L}ib}(X)), \circledast) \to \text{Iso}_{k\text{-mod}}(\hat{\mathcal{V}}_l^B, \hat{\mathcal{V}}_l^{\text{DR}}), \quad (k, G) \mapsto S_Y^Y_{\Theta(k,G)},$$

by

$$S^Y_{\Theta(k,G)} := S^Y_{\Theta(k,G)} \circ \text{iso}_k^l = S^Y_{\Theta(k,G)} \circ \text{iso}^k_l.$$

We then have

$$\forall (k, G), (k', G') \in k^\times \times (\exp(\widehat{\mathcal{L}ib}(X)), \circledast), \quad S^Y_{\Theta((k, G) \circledast (k', G'))} = S^Y_{\Theta(k, G)} \circ S^Y_{\Theta(k', G')}.$$ 

**Lemma-Definition 1.11.** The map $\amalg_{\mu \in k^\times} \{\mu\} \times \text{DMR}_\mu(k) \to \text{Hom}_{k\text{-mod}}(\hat{\mathcal{V}}_l^B, (\hat{\mathcal{V}}_l^{B(\circledast)}))^2,$

$$(\mu, \Phi) \mapsto ((S^Y_{\Theta(\mu, \Phi)})^{-1})^{\circledast} \circ \hat{\Delta}_\ast \circ S^Y_{\Theta(\mu, \Phi)}$$

is constant. We denote the unique element of the image of this map by $\hat{\Delta}_\ast$. 

**Proof.** For $(\mu, \Phi)$ and $(\mu', \Phi')$ elements of $\amalg_{\mu \in k^\times} \{\mu\} \times \text{DMR}_\mu(k)$, there exists $(k, G) \in k^\times \times (\text{DMR}_0(k), \circledast)$ such that $(\mu', \Phi') = (k, G) \circledast (\mu, \Phi)$. Then

$$(S^Y_{\Theta(\mu', \Phi')})^{-1} \circ \hat{\Delta}_\ast \circ S^Y_{\Theta(\mu, \Phi')} = ((S^Y_{\Theta(k, G)} \circ S^Y_{\Theta(\mu, \Phi)})^{-1})^{\circledast} \circ \hat{\Delta}_\ast \circ S^Y_{\Theta(k, G)} \circ S^Y_{\Theta(\mu, \Phi)}$$

$$= ((S^Y_{\Theta(\mu, \Phi)})^{-1})^{\circledast} \circ \hat{\Delta}_\ast \circ S^Y_{\Theta(\mu, \Phi)}$$

where the first equality follows from (1.4.1), and the last equality follows from the inclusion $k^\times \times (\text{DMR}_0(k), \circledast) \subset (\text{Stab}^k(\hat{\Delta}_\ast), \circledast)$ (see Corollary [1.7].)

**Remark 1.12.** One can check that Principle [0.1] may be applied in the context given by [0.2.1], and that the resulting vector $\check{v} \in \mathcal{V}$ coincides with $(\text{iso}_k^l)^{\circledast} \circ \hat{\Delta}_\ast \circ (\text{iso}^k_l)^{-1}.$

1.5. **Algebra morphism property of $\hat{\Delta}_\ast$.**

1.5.1. The automorphisms $a_g$ of the algebra $U(f_2)^\wedge$. In (1.1.1), we defined a map $k(\langle X \rangle)^\times \ni g \mapsto a_g \in \text{Aut}_{k\text{-alg}}(k(\langle X \rangle))$. Let us denote by $(U(f_2)^\wedge)^\times \ni g \mapsto a_g \in \text{Aut}_{k\text{-alg}}(U(f_2)^\wedge)$ the map corresponding to it under the isomorphism $k(\langle X \rangle) \simeq U(f_2)^\wedge$.

1.5.2. The automorphisms $a_g'$ of the algebra $\hat{\mathcal{V}}_l^{\text{DR}}$.

**Lemma 1.13.** For $g \in (U(f_2)^\wedge)^\times$, the continuous automorphism $a_g$ of $U(f_2)^\wedge$ restricts to a continuous automorphism of the subalgebra $\hat{\mathcal{V}}_l^{\text{DR}}$, which will be denoted $a_g'$.

**Proof.** Let $g \in (U(f_2)^\wedge)^\times$. Since $a_g$ is a vector space automorphism of $U(f_2)^\wedge$, and since $\hat{\mathcal{V}}_l^{\text{DR}} = k \oplus U(f_2)^\wedge e_1$, there is a unique automorphism $a_g'$ of $\hat{\mathcal{V}}_l^{\text{DR}}$ such that $1 \mapsto 1$ and $ae_1 \mapsto a_g(a)e_1$ for $a \in U(f_2)^\wedge$. As $a_g(e_1) = e_1$, one has then $a_g'(w) = a_g(w)$ for any $w \in \hat{\mathcal{V}}_l^{\text{DR}}$. 

$\square$
1.5.3. Algebra morphism property of $\hat{\Delta}_*$. 

**Lemma 1.14.** One has 
\[ \forall g \in (U(f_2)^\wedge)^x, \quad S_g^Y = \rho_{\pi_Y(g)} \circ a_g^l \] 
(equality in $\text{Aut}_{k\text{-mod}}(\hat{\mathcal{W}}_l^{\text{DR}})$), where $\rho_{\pi_Y(g)}$ is the linear automorphism of $\hat{\mathcal{W}}_l^{\text{DR}}$ induced by right multiplication by $\pi_Y(g)$.

**Proof.** In this proof, we denote by $w \mapsto w_X$ the isomorphism $\hat{\mathcal{W}}_l^{\text{DR}} \cong k \oplus U(f_2)^\wedge e_1$ and by $u \mapsto u_Y$ the inverse isomorphism.

Let $w \in \hat{\mathcal{W}}_l^{\text{DR}}$. Let us compute $S_g^Y(w)$. A lift of $w$ under $U(f_2)^\wedge \cong \hat{\mathcal{W}}_l^{\text{DR}}$ is $w_X$; it belongs to $k \oplus U(f_2)^\wedge e_1$. Then $S_g(w_X) = a_g(w_X)g$. According to Lemma 1.13, $a_g(w_X) \in k \oplus U(f_2)^\wedge e_1$. Moreover, the map $U(f_2)^\wedge \cong \hat{\mathcal{W}}_l^{\text{DR}}$ satisfies the identity $\pi_Y(ab) = a_Y \pi_Y(b)$ for $a \in k \oplus U(f_2)^\wedge e_1$ and $b \in U(f_2)^\wedge$. Then 
\[ S_g^Y(w) = \pi_Y(S_g(w_X)) = \pi_Y(a_g(w_X)g) = (a_g(w_X))Y \pi_Y(g) = a_g^l(w) \pi_Y(g), \] 
which proves the statement. \[ \square \]

**Lemma 1.15.** Let $(\mu, \Phi) \in \sqcup_{\mu \in k \times \{\mu\}} \times \text{DMR}_\mu(k)$. Then 
\[ \hat{\Delta}_* = (a_{\Theta(\Phi)}^l \circ \text{iso}^{l}_{\mu})^{-1} \circ \hat{\Delta}_* \circ a_{\Theta(\Phi)}^l \circ \text{iso}^{l}_{\mu}. \]

**Proof.** One has 
\[
\begin{align*}
\hat{\Delta}_* &= ((S_{\Theta(\Phi)}^Y \circ \text{iso}^{l}_{\mu})^{-1})^{\otimes 2} \circ \hat{\Delta}_* \circ S_{\Theta(\Phi)}^Y \circ \text{iso}^{l}_{\mu} \quad \text{(by definition of $\hat{\Delta}_*$)} \\
&= ((\rho_{\pi_Y(\Theta(\Phi))} \circ a_{\Theta(\Phi)}^l \circ \text{iso}^{l}_{\mu})^{-1})^{\otimes 2} \circ \hat{\Delta}_* \circ \rho_{\pi_Y(\Theta(\Phi))} \circ a_{\Theta(\Phi)}^l \circ \text{iso}^{l}_{\mu} \quad \text{(by Lemma 1.14)} \\
&= ((a_{\Theta(\Phi)}^l \circ \text{iso}^{l}_{\mu})^{-1})^{\otimes 2} \circ (\rho_{\pi_Y(\Theta(\Phi))})^{-1})^{\otimes 2} \circ \hat{\Delta}_* \circ \rho_{\pi_Y(\Theta(\Phi))} \circ a_{\Theta(\Phi)}^l \circ \text{iso}^{l}_{\mu} \\
&= (a_{\Theta(\Phi)}^l \circ \text{iso}^{l}_{\mu})^{-1})^{\otimes 2} \circ \hat{\Delta}_* \circ a_{\Theta(\Phi)}^l \circ \text{iso}^{l}_{\mu} \quad \text{(using the primitiveness of $\pi_Y(\Theta(\Phi))$ for $\hat{\Delta}_*$).}
\end{align*}
\]
\[ \square \]

**Theorem 1.16.** $\hat{\Delta}_*$ equips $\hat{\mathcal{W}}_l^{B}$ with a topological cocommutative Hopf algebra structure.

**Proof.** By Lemma 1.15 for any $(\mu, \Phi) \in \sqcup_{\mu \in k \times \{\mu\}} \times \text{DMR}_\mu(k)$, there is a commutative diagram
\[
\begin{array}{ccc}
\hat{\mathcal{W}}_l^{B} & \xrightarrow{\hat{\Delta}_*} & (\hat{\mathcal{W}}_l^{B})^{\otimes 2} \\
\downarrow a_{\Theta(\Phi)}^l \circ \text{iso}^{l}_{\mu} & & \downarrow (a_{\Theta(\Phi)}^l \circ \text{iso}^{l}_{\mu})^{\otimes 2} \\
\hat{\mathcal{W}}_l^{\text{DR}} & \xrightarrow{\hat{\Delta}_*} & (\hat{\mathcal{W}}_l^{\text{DR}})^{\otimes 2}
\end{array}
\]

The result then follows from the facts that $a_{\Theta(\Phi)}^l \circ \text{iso}^{l}_{\mu}$ is an algebra isomorphism $\hat{\mathcal{W}}_l^{B} \to \hat{\mathcal{W}}_l^{\text{DR}}$ and that $\hat{\Delta}_*$ equips $\hat{\mathcal{W}}_l^{\text{DR}}$ with a topological cocommutative Hopf algebra structure. \[ \square \]
1.6. Filtration properties of $\hat{\Delta}_*$. 

**Theorem 1.17.**

1. $\hat{\Delta}_*$ is compatible with the filtration of $\hat{\mathcal{W}}^B_l$.
2. The associated graded coproduct $\text{gr}(\hat{\Delta}_*) : \text{gr}(\hat{\mathcal{W}}^B_l) \to (\text{gr}(\hat{\mathcal{W}}^B_l))^\otimes 2$ gives the coproduct $\Delta_* : \mathcal{W}^\text{DR}_l \to (\mathcal{W}^\text{DR}_l)^\otimes 2$ under the isomorphism $\text{gr}(\text{iso}^f_\mu) : \text{gr}(\hat{\mathcal{W}}^B_l) \to \mathcal{W}^\text{DR}_l$ for any $\mu \in k^\times$ (see (1.3.4)), namely

$$\Delta_* = \text{gr}(\text{iso}^f_\mu)^\otimes 2 \circ \text{gr}(\hat{\Delta}_*) \circ \text{gr}(\text{iso}^f_\mu)^{-1}.$$

**Proof.** Let us prove (1). Let $(\mu, \Phi) \in \sqcup \mu \in k^\times \{\mu\} \times \text{DMR}_\mu(k)$.

The composed isomorphism $\hat{\mathcal{W}}^B_l \xrightarrow{\text{iso}^f_\mu} \hat{\mathcal{W}}^\text{DR}_l \xrightarrow{a_{\Theta(\Phi)}} \hat{\mathcal{W}}^\text{DR}_l$ fits in the commutative diagram

$$
\begin{array}{ccc}
\hat{\mathcal{W}}^B_l & \xrightarrow{\text{iso}^f_\mu} & \hat{\mathcal{W}}^\text{DR}_l \\
\downarrow & & \downarrow \\
(kF_2)^\wedge & \xrightarrow{\text{iso}} & U(f_2)^\wedge \\
\end{array}
$$

Since the bottom isomorphisms are compatible with filtrations, and since the filtrations of the algebras of the top row are induced by those of the algebras of the bottom row, the top isomorphisms, and therefore also the composed isomorphism $a_{\Theta(\Phi)} \circ \text{iso}^f_\mu$, are strictly compatible with filtrations.

This fact, together with the fact $\hat{\Delta}_*$ is compatible with filtrations, and that $\hat{\Delta}_* = a_{\Theta(\Phi)} \circ \text{iso}^f_\mu$, implies that $\hat{\Delta}_*$ is compatible with filtrations. This proves (1).

Let us prove (2). The fact that $\hat{\Delta}_*$ is the pullback of $\Delta_*$ under the isomorphism $a_{\Theta(\Phi)} \circ \text{iso}^f_\mu$ (see Lemma 1.15), then implies that $\hat{\Delta}_*$ is compatible with filtrations. This proves (1).

1.7. A commutative diagram involving $\hat{\Delta}_*$. For $\mu \in k^\times$ and $\Phi \in (U(f_2)^\wedge)^\times$, we define

$$a^l_{(\mu, \Phi)} : \hat{\mathcal{W}}^B_l \to \hat{\mathcal{W}}^\text{DR}_l, \quad a_{(\mu, \Phi)} : (kF_2)^\wedge \to U(f_2)^\wedge$$

by

$$a^l_{(\mu, \Phi)} := a^l_\Phi \circ \text{iso}^f_\mu, \quad a_{(\mu, \Phi)} := a_\Phi \circ \text{iso}^f_\mu,$$

where $a^l_\Phi$ is as in Lemma 1.13, $a_\Phi$ is as in 1.5.1, $\text{iso}^f_\mu$ is as in 1.3.4 and $\text{iso}^f_\mu$ as in 1.3.4.
Combining the commutativity of diagram (1.3.3) with the compatibility of $a_\Phi$ with $a_{\Phi}$, we get the commutative diagram

\[
\begin{array}{ccc}
\hat{\mathcal{W}}^B & \xrightarrow{a^l_{(\mu, \Phi)}} & \hat{\mathcal{W}}^{\text{DR}} \\
\downarrow \quad & \quad & \downarrow \quad \\
(kF_2) \wedge & \xrightarrow{a^l_{(\mu, \Phi)}} & U(f_2) \wedge
\end{array}
\]

Lemma 1.18. The diagram

\[
\begin{array}{ccc}
\hat{\mathcal{W}}^B & \xrightarrow{a^l_{(\mu, \Phi)}} & \hat{\mathcal{W}}^{\text{DR}} \\
\downarrow \quad & \quad & \downarrow \quad \\
\hat{\mathcal{W}}^B_{\text{Ad}(\Gamma_\Phi (-e_1))} & \xrightarrow{a^l_{(\mu, \Phi)}} & \hat{\mathcal{W}}^{\text{DR}}_{\text{Ad}(\Gamma_\Phi (-e_1))}
\end{array}
\]

commutes. \[\blacksquare\]

Proof. For $\Phi \in \exp(\hat{\mathfrak{sl}}_k(X))$, one has

\[a_{\Phi(\Phi)} = \text{Ad}(\Gamma_\Phi (-e_1)) \circ a_{\Phi},\]

(equality in $\text{Aut}_{k,\text{alg}}(k\langle\langle X \rangle\rangle)$). One also has $a_{(\mu, \Phi)} = a_{\Phi} \circ \text{iso}_{\mu}$. This implies that the two triangles in the following diagram

\[
\begin{array}{ccc}
(kF_2) \wedge & \xrightarrow{\text{iso}} & k\langle\langle X \rangle\rangle \\
\downarrow \quad & \quad & \downarrow \quad \\
\hat{\mathcal{W}}^B \wedge & \xrightarrow{a_{\Phi}} & \hat{\mathcal{W}}^{\text{DR}} \\
\downarrow \quad & \quad & \downarrow \quad \\
k\langle\langle X \rangle\rangle & \xrightarrow{a_{\Phi(\Phi)}} & k\langle\langle X \rangle\rangle
\end{array}
\]

commute, therefore that the square commutes. The result follows from restricting this diagram to $\hat{\mathcal{W}}^B_I$ and $\hat{\mathcal{W}}^{\text{DR}}_I$.

For $i = 0, 1$, set by abuse of notation

\[
e_i := e_i \otimes 1 \in (U(f_2)^\wedge)\hat{\otimes}^2, \quad f_i := 1 \otimes e_i \in (U(f_2)^\wedge)\hat{\otimes}^2;
\]

when $i = 1$, we will view these elements as elements of $(\hat{\mathcal{W}}^{\text{DR}}_I)\hat{\otimes}^2$.

Proposition 1.19. If $\mu \in k^\times$ and $\Phi \in \text{DMR}_\mu(k)$, then the diagram

\[
\begin{array}{ccc}
\hat{\mathcal{W}}^B_I & \xrightarrow{\hat{\Delta}_I} & (\hat{\mathcal{W}}^B_I)^{\hat{\otimes}^2} \\
\downarrow \quad & \quad & \downarrow \quad \\
\hat{\mathcal{W}}^{\text{DR}}_I & \xrightarrow{\hat{\Delta}_I} & (\hat{\mathcal{W}}^{\text{DR}}_I)^{\hat{\otimes}^2}
\end{array}
\]

commutes.

\[\text{Throughout the paper, we will use the following notation: if } A \text{ is an algebra and if } a \in A^\times \text{ (the group of its invertible elements), then } \text{Ad}(u) \text{ is the automorphism of } A \text{ given by } a \mapsto uau^{-1}.\]
Proof. Together with Lemma 1.18, diagram (1.5.1) implies that the following diagram
\begin{equation}
\begin{array}{ccc}
\hat{W}_B^I & \xrightarrow{\hat{\Delta}_*} & (\hat{W}_B^I)^{\otimes 2} \\
\downarrow \mathcal{L}_{(\mu, \Phi)} & \quad & \downarrow \mathcal{L}_{(\mu, \Phi)}^{\otimes 2} \\
\hat{W}_l^{DR} & \xrightarrow{\text{Ad}(\Gamma_\Phi(-e_1)^{-1})} & (\hat{W}_l^{DR})^{\otimes 2}
\end{array}
\end{equation}
commutes.

The identity \( \hat{\Delta}_* \circ \text{Ad}(h(e_1)) = \text{Ad}(h(e_1 + f_1)) \circ \hat{\Delta}_* \) in \( \text{Hom}_{k\text{-alg}}(\hat{W}_l^{DR}, (\hat{W}_l^{DR})^{\otimes 2}) \), where \( h \) is an invertible formal series, then implies the result. \( \square \)

2. A coproduct \( \hat{\Delta}_* \) on \( \hat{W}_B^I \) with an explicit expression

The purpose of this section is to construct an explicit element \( \hat{\Delta}_* \) of \( \text{Hom}_{k\text{-mod}}(\hat{W}_B^I, (\hat{W}_B^I)^{\otimes 2}) \), to be compared in the sequel of the paper with the element \( \hat{\Delta}_* \) obtained in \( \S \). This construction proceeds as follows. In \( \S \) we give a presentation of the algebra \( \hat{W}_B^I \) (Proposition 2.2). We use this presentation in \( \S \) to construct an algebra morphism \( \hat{\Delta}_* : \hat{W}_B^I \to (\hat{W}_B^I)^{\otimes 2} \) (Proposition 2.3). In \( \S \) we show that \( \hat{\Delta}_* \) is compatible with the filtration of \( \hat{W}_B^I \) introduced in \( \S \) (Proposition 2.14), and define \( \hat{\Delta}_* \) to be the element of \( \text{Hom}_{k\text{-mod}}(\hat{W}_B^I, (\hat{W}_B^I)^{\otimes 2}) \) obtained from \( \hat{\Delta}_* \) by completion. We also show that the associated graded morphism of \( \hat{\Delta}_* \) can be identified with \( \Delta_* \) (Proposition 2.17).

2.1. Presentation of \( \hat{W}_B^I \). Set for \( k \in \mathbb{Z} \),

\[ \xi_k^+ := X_k^0(X_1 - 1), \quad \text{and} \quad \xi_k^- := X_k^0(X_1^{-1} - 1). \]

These are elements of \( \hat{W}_B^I = \mathbb{k} \oplus \mathbb{k}F_2(X_1 - 1) \subset \mathbb{k}F_2. \)

Lemma 2.1. The algebra \( \hat{W}_B^I \) is the algebra whose generators are \( (\xi_k^+)_k \in \mathbb{Z}; \ (\xi_k^-)_k \in \mathbb{Z}; \) and relations are

\begin{equation}
\forall k \in \mathbb{Z}, \quad \xi_k^- \xi_0^- = \xi_k^- \xi_0^+ = -\xi_k^+ \xi_k^- - \xi_k^-.
\end{equation}

Proof. Let \( \mathcal{A} \) be the associative non-commutative algebra generated by \( (\xi_k^+)_k \in \mathbb{Z}; \ (\xi_k^-)_k \in \mathbb{Z} \) divided by the two-sided ideal generated by the relation given in (2.1.1).

One checks that there is an algebra morphism

\[ f : \mathcal{A} \to \mathbb{k} \oplus \mathbb{k}F_2(X_1 - 1), \]

uniquely determined by

\[ \xi_k^+ \mapsto X_k^0(X_1 - 1) \quad \text{and} \quad \xi_k^- \mapsto X_k^0(X_1^{-1} - 1). \]

We will prove that \( f \) is an algebra isomorphism.
Let us first prove that $f$ is surjective. For this, we will prove that for any word $w$ in $X_0^{±1}, X_1^{±1}$, there exists a polynomial without constant term $P_w$ over a set of noncommutative variables $\{(\tilde{\xi}_k^+)_{k \in \mathbb{Z}}, \tilde{\xi}_0^−\}$ indexed by $\mathbb{Z} \cup \{0\}$, such that
\[
w \cdot (X_1 - 1) = P_w((\tilde{\xi}_k^+)_{k \in \mathbb{Z}}, \tilde{\xi}_0^−).
\]
We argue by induction on the length $|w|$ of $w$. If $w = 1$, then $P_w = \tilde{\xi}_0^+$. Assume the statement for $|w| < n$ and let $w$ be a word of length $n$. Then $w = gw'$, where $g \in \{X_0^{±1}, X_1^{±1}\}$ and $w'$ has length $n - 1$. If $g = X_1^{±1}$, then
\[
w \cdot (X_1 - 1) = X_1^{±1}w'(X_1 - 1) = P_w((\tilde{\xi}_k^+)_{k \in \mathbb{Z}}, \tilde{\xi}_0^−),
\]
where $P_w := (1 + \tilde{\xi}_0^+)P_{w'}$. If $g = X_0^{±1}$, then
\[
w \cdot (X_1 - 1) = X_0^{±1}w'(X_1 - 1) = P_w((\tilde{\xi}_k^+)_{k \in \mathbb{Z}}, \tilde{\xi}_0^−),
\]
where we set
\[
P_w := \sum_{k \in \mathbb{Z}} (\tilde{\xi}_k^+)P_{w_0}^+ - \tilde{\xi}_k^+\tilde{\xi}_0^−(P_{w_0})^−1,
\]
in which the expressions $(P_{w_0})^+$ and $(P_{w_0})^−$ have the following meaning: for $P$ a polynomial in the noncommutative variables $(\tilde{\xi}_k^+)_{k \in \mathbb{Z}}, \tilde{\xi}_0^−$ without constant term, $(P)_k^+$ and $(P)_0^−$ are the polynomials in the same variables such that
\[
P = \sum_{k \in \mathbb{Z}} \tilde{\xi}_k^+(P)_k^+ + \tilde{\xi}_0^−(P)_0^−.
\]
This surely proves that $f : \mathcal{A} \to \mathbb{k} \oplus \mathbb{k}F_2(X_1 - 1)$ is surjective.

To prove that $f$ is injective, we will:

a) equip the vector space $\mathcal{A}[t^{±1}]$ with an algebra structure $\ast$, such that $\mathcal{A} \to \mathcal{A}[t^{±1}]$, $a \mapsto a \otimes 1$ is an algebra morphism;

b) construct an algebra isomorphism $F : \mathcal{A}[t^{±1}] \to \mathbb{k}F_2$;

c) check that $F$ extends $f$, in the sense that the following diagram commutes
\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{f} & \mathbb{k} \oplus \mathbb{k}F_2(X_1 - 1) \\
\downarrow{\otimes 1} & & \downarrow{F} \\
\mathcal{A}[t^{±1}] & \xrightarrow{F} & \mathbb{k}F_2
\end{array}
\]

The above diagram implies that $f$ is injective, which finally implies that it is an isomorphism.

(a) Construction of an algebra structure on $\mathcal{A}[t^{±1}]$. Let $\mathbb{k}((\xi_k^+)_{k \in \mathbb{Z}})_+$ be the ideal of $\mathbb{k}((\xi_k^+)_{k \in \mathbb{Z}})$ generated by the generators $(\xi_k^+)_{k \in \mathbb{Z}}$. We have a direct sum decomposition $\mathbb{k}((\xi_k^+)_{k \in \mathbb{Z}}) = \mathbb{k} \oplus \mathbb{k}((\xi_k^+)_{k \in \mathbb{Z}})_+$. Left multiplication induces a linear isomorphism
\[
(\oplus_{k \in \mathbb{Z}}(\mathbb{k}_{\xi_k^+} \oplus \mathbb{k}_{\xi_k^−})) \otimes \mathbb{k}((\xi_k^+)_{k \in \mathbb{Z}}) \xrightarrow{\sim} \mathbb{k}((\xi_k^+)_{k \in \mathbb{Z}})_+,
\]
therefore there is a linear automorphism $T$ of $\mathbb{k}((\xi_k^+)_{k \in \mathbb{Z}})_+$, uniquely determined by
\[
\forall k \in \mathbb{Z}, \forall a \in \mathbb{k}((\xi_k^+)_{k \in \mathbb{Z}}), \quad T(\xi_k^+ a) = \xi_{k+1}^+ a.
\]
Let $I$ be the two-sided ideal of $k\langle (\xi_k^\pm)_{k\in\mathbb{Z}} \rangle$ generated by (2.1.1). Then $I \subset k\langle (\xi_k^\pm)_{k\in\mathbb{Z}} \rangle_+$. Set $A^+ := k\langle (\xi_k^\pm)_{k\in\mathbb{Z}} \rangle_+/I$, then

$$A = k \oplus A^+.$$ 

Let $V \subset k\langle (\xi_k^\pm)_{k\in\mathbb{Z}} \rangle_+$ be the linear span of all $\xi_k^+ \xi_0^- + \xi_k^- + \xi_k^+$ and $\xi_k^- \xi_0^+ + \xi_k^+ + \xi_k^-$, for $k \in \mathbb{Z}$. Then

$$I = V \cdot k\langle (\xi_k^\pm)_{k\in\mathbb{Z}} \rangle + k\langle (\xi_k^\pm)_{k\in\mathbb{Z}} \rangle \cdot V \cdot k\langle (\xi_k^\pm)_{k\in\mathbb{Z}} \rangle.$$ 

It follows from the definition of $T$ that $T^{\pm1}$ maps the subspace $k\langle (\xi_k^\pm)_{k\in\mathbb{Z}} \rangle_+ \cdot V \cdot k\langle (\xi_k^\pm)_{k\in\mathbb{Z}} \rangle$ to itself. Moreover, for any $a \in k\langle (\xi_k^\pm)_{k\in\mathbb{Z}} \rangle$, one has

$$T^{\pm1}((\xi_{k \pm 1}^+ \xi_0^- + \xi_{k \pm 1}^- + \xi_{k \pm 1}^+)a) = (\xi_{k \pm 1}^+ \xi_0^- + \xi_{k \pm 1}^- + \xi_{k \pm 1}^+)a, \quad T^{\pm1}((\xi_{k \pm 1}^- \xi_0^+ + \xi_{k \pm 1}^+ + \xi_{k \pm 1}^-)a) = (\xi_{k \pm 1}^- \xi_0^+ + \xi_{k \pm 1}^+ + \xi_{k \pm 1}^-)a,$$

which implies that $T^{\pm1}$ maps $V \cdot k\langle (\xi_k^\pm)_{k\in\mathbb{Z}} \rangle$ to itself. All this implies that $T^{\pm1}(I) \subset I$, so $T(I) = I$. It follows that $T$ induces a linear automorphism of $A^+$. Relation (2.1.3) then implies:

(2.1.4) for $P, P' \in A^+$ and $a \in \mathbb{Z}$, we have $T^a(PP') = T^a(P)P'$.

We now define a bilinear map

$$*: A[t^{\pm1}] \otimes 2 \to A[t^{\pm1}]$$

as follows:

1) for $a, b \in \mathbb{Z}$, $(1 \otimes t^a) * (1 \otimes t^b) = 1 \otimes t^{a+b}$;

2) for $a \in \mathbb{Z}$, $P \in A^+$,

$$(1 \otimes t^a) * (P \otimes t^b) := T^a(P) \otimes t^b, \quad (P \otimes t^a) * (1 \otimes t^b) := P \otimes t^{a+b},$$

3) for $a, b \in \mathbb{Z}$, and $P, Q \in A^+$,

$$(P \otimes t^a) * (Q \otimes t^b) := (P \cdot T^a(Q)) \otimes t^b,$$

where $\cdot$ is the product in $A^+$.

It then follows from a case analysis, using (2.1.4), that

(2.1.5) the product $*$ defines an associative algebra structure on $A[t^{\pm1}]$.

(b) Construction of an algebra isomorphism $(A[t^{\pm1}], *) \simeq kF_2$. The elements $1 \otimes t$ and $1 \otimes t^{-1}$ of $(A[t^{\pm1}], *)$ are mutually inverse; so are the elements $1 + \xi_0^+$ and $1 + \xi_0^-$ of $A$, and therefore the elements $(1 + \xi_0^+) \otimes 1$ and $(1 + \xi_0^-) \otimes 1$ of $A[t^{\pm1}]$. All this implies that there is an algebra morphism

$$G: kF_2 \to (A[t^{\pm1}], *)$$

uniquely determined by

$$X_0^{\pm1} \mapsto 1 \otimes t^{\pm1}, \quad X_1^{\pm1} \mapsto (1 + \xi_0^+) \otimes 1.$$
One checks that the algebra homomorphism $f : \mathcal{A} \to k \otimes k F_2(X_1 - 1)$ restricts to a morphism of algebras without unit

$$f^+ : \mathcal{A}^+ \to k F_2(X_1 - 1),$$

and that the following diagram commutes

$$\begin{array}{ccc}
\mathcal{A}^+ & \xrightarrow{f^+} & k F_2(X_1 - 1) \\
\downarrow & & \downarrow \circ \psi^+ \\
\mathcal{A}^+ & \xrightarrow{f^+} & k F_2(X_1 - 1)
\end{array}$$

Define now a linear map

$$F : \mathcal{A}[t^{\pm 1}] \to k F_2$$

by

$$1 \otimes t^a \mapsto X_0^a, \quad P \otimes t^b \mapsto f^+(P) X_0^b$$

for $a, b \in \mathbb{Z}$, $P \in \mathcal{A}^+$. Then:

1) for $a, b \in \mathbb{Z}$, $F((1 \otimes t^a)(1 \otimes t^b)) = X_0^a X_0^b = X_0^{a+b} = F((1 \otimes t^a) * (1 \otimes t^b));$

2) for $a \in \mathbb{Z}$, $P \in \mathcal{A}^+$,

$$F((1 \otimes t^a)(P \otimes 1)) = X_0^a f^+(P) = f^+((T^a)(P)) = F(T^a(P) \otimes 1) = F(((1 \otimes t^a) * (P \otimes 1));$$

3) for $a, b \in \mathbb{Z}$, $P, Q \in \mathcal{A}^+$,

$$F(P \otimes t^a) F(Q \otimes t^b) = f^+(P) X_0^a f^+(Q) X_0^b = f^+(P) f^+(T^a(Q)) X_0^b = f^+((P T^a(Q)) X_0^b) = F((P \otimes t^a) * (Q \otimes t^b)).$$

All this implies that $F$ is an algebra morphism $\mathcal{A}[t^{\pm 1}] \to k F_2$. Let us now show that the algebra morphisms $F, G$ are mutually inverse.

We have

$$F \circ G(X_0^{\pm 1}) = F((1 \otimes t^{\pm 1}) = X_0^{\pm 1},$$

$$F \circ G(X_1^{\pm 1}) = F(((1 + \xi_0^{\pm}) \otimes 1) = 1 + F((\xi_0^{\pm} \otimes 1) = 1 + (X_1^{\pm 1} - 1) = X_1^{\pm 1},$$

so $F \circ G = \text{id}_{k F_2}$.

Similarly, $G \circ F$ is an algebra endomorphism of $(\mathcal{A}[t^{\pm 1}], *)$.

Given the structure of $*$, a collection of generators of this algebra is given by $\{g \otimes 1, 1 \otimes t^{\pm 1}\}$, where $g$ runs over a collection of algebraic generators of $\mathcal{A}$, so such a collection of algebraic generators is $\{\xi_k \otimes 1, 1 \otimes t^{\pm 1}\}$. One has

$$G \circ F(\xi_k \otimes 1) = F(f^+(\xi_k)) = F(G(X_0^{\pm} X_1^{\pm 1} - 1)) = G(X_0^{\pm} G(X_1^{\pm 1} - 1) = (1 \otimes t^{\pm 1})^* \xi_0^{\pm} \otimes 1) = \xi_k^{\pm} \otimes 1, $$

so $G \circ F = \text{id}_{\mathcal{A}[t^{\pm 1}]}$.

(c) Diagram involving $f$ and $F$. The commutation of (2.1.2) follows from the definition of $F$. \qed
Lemma 2.1 leads to the following presentation:

**Proposition 2.2.** The algebra $W_B^*$ is generated by the elements

\begin{align}
Y_n^+ &:= (X_0 - 1)^{n-1}X_0(1 - X_1) 
&= (n > 0), \nonumber \\
Y_n^- &:= (X_0 - 1)^{n-1}X_0^{-1}(1 - X_1^{-1}) 
&= (n > 0), \nonumber \\
X_1 \text{ and } X_1^{-1}. \nonumber
\end{align}

with defining equations

\begin{equation}
X_1X_1^{-1} = X_1^{-1}X_1 = 1. \tag{2.1.7}
\end{equation}

**Proof.** By the presentations in the previous lemma, we have $\xi_k^- = -\xi_k^+(1 + \xi_0^-)$ and $\xi_k^- = -\xi_k^+(1 + \xi_0^+)$, from which we can deduce that $A$ is generated by $\xi_k^+$ and $\xi_k^-$ for $k \geq 0$ with defining equation (2.1.1) only for $k = 0$.

It is immediate to see that $(Y_n^+)^{n>0}$ (resp. $(Y_n^-)^{n>0}$) can be expressed by linear combinations of $(\xi_n^-)^{n>0}$ (resp. $(\xi_n^+)^{n>0}$) and vice versa and that the equation (2.1.1) for $k = 0$ is equivalent to (2.1.7), from which we can deduce that $\Delta$ is well-defined. One directly checks the cocommutativity and coidassociativity of $\Delta$. \hfill \Box

2.2. An algebra morphism $\Delta_\xi : W_B^* \to (W_B^*)^\otimes 2$.

**Proposition 2.3.** There is a unique algebra morphism

\begin{equation}
\Delta_\xi : W_B^* \to (W_B^*)^\otimes 2
\end{equation}

such that

\begin{equation}
\Delta_\xi(X_1^{\pm 1}) = X_1^{\pm 1} \otimes X_1^{\pm 1}, \quad \Delta_\xi(Y_k^\pm) = Y_k^\pm \otimes 1 + 1 \otimes Y_k^\pm + \sum_{k',k'' > 0, k' + k'' = k} Y_{k'}^\pm \otimes Y_{k''}^\pm \quad \text{for any } k \geq 1. \tag{2.1.7}
\end{equation}

It equips $W_B^*$ with a cocommutative Hopf algebra structure.

**Proof.** The relations between the generators (see Proposition 2.2) are obviously preserved, therefore $\Delta_\xi$ is well-defined. One directly checks the cocommutativity and coidassociativity of $\Delta_\xi$. One also checks that for $t$ a formal parameter, the series $1 + \sum_{k\geq 1} t^k Y_k^\pm$ are group-like. It follows that $\Delta_\xi$ admits an antipode given by $X_1^{\pm 1} \mapsto X_1^{\mp 1}$, $Y_k^\pm \mapsto \sum_{a \geq 1} (-1)^a \sum_{k_1 + \ldots + k_a = k} Y_{k_1}^\pm \ldots Y_{k_a}^\pm$. \hfill \Box

**Remark 2.4.** If $g$ is a Lie algebra and $\Gamma$ is a group acting on $g$ by Lie algebra automorphisms, then the semidirect product $U(g) \rtimes k\Gamma$ is equipped with a cocommutative Hopf algebra structure, defined by the conditions that the elements of $g$ are primitive and the elements of $\Gamma$ are group-like. The Hopf algebra $(W_B^*, \Delta_\xi)$ is isomorphic to the one obtained through this construction, with $g$ being the free Lie algebra over generators $(Y_{k^\pm(l)})_{k \geq 1, \ell \in \mathbb{Z}}$ and $\Gamma$ being $\mathbb{Z}$, acting on $g$ by $1 \cdot Y_k^{\pm(l)} = Y_k^{\pm(l+1)}$. The isomorphism between $U(g) \rtimes k\Gamma$ and $W_B^*$ is given by $Y_k^{\pm(l)} \mapsto \text{Ad}(X_1^l)(Y_k^\pm)$, $\Gamma \ni \pm 1 \mapsto X_1^{\pm 1} \in W_B^*$. 

Introduce the following convention. Let $A$ be an abelian group and let $f : \mathbb{Z} \to A$ be a function. For $a, b \in \mathbb{Z}$, we set

$$
\sum_{k=a}^{b} f(k) := \begin{cases} 
  f(a) + \cdots + f(b) & \text{if } b \geq a, \\
  0 & \text{if } b = a - 1, \\
  -f(a - 1) - \cdots - f(b + 1) & \text{if } b < a - 1
\end{cases}
$$

(2.2.1)

Lemma 2.5. For any $k \in \mathbb{Z}$, one has

$$
\Delta_k(X^k_0(1 - X_1)) = X^k_0(1 - X_1) \otimes 1 + 1 \otimes X^k_0(1 - X_1) + \sum_{i=1}^{k-1} X^i_0(1 - X_1) \otimes X^{k-i}_0(1 - X_1),
$$

(2.2.2)

$$
\Delta_k(X^k_0(1 - X^{-1}_i)) = X^k_0(1 - X^{-1}_i) \otimes 1 + 1 \otimes X^k_0(1 - X^{-1}_i) - \sum_{i=0}^{k} X^i_0(1 - X^{-1}_i) \otimes X^{k-i}_0(1 - X^{-1}_i).
$$

(2.2.3)

Proof. Let $t$ be a formal parameter. As we have seen, the series $s_\pm(t) := 1 + \sum_{k \geq 1} t^k Y^\pm_k$ are group-like for $\Delta_k$. One computes these series as follows

$$
s_\pm(t) = 1 + t X_0^{\pm 1} \{ 1 - t(X_0^{\pm 1} - 1) \}^{-1} (1 - X_1^{\pm 1}) = (1 + t - t X_0^{\pm 1})^{-1} (1 + t - t X_0^{\pm 1} X_1^{\pm 1}) = \hat{s}_\pm(u),
$$

where $u := t/(1 + t)$ and $\hat{s}_\pm(u) := (1 - u X_0^{\pm 1})^{-1} (1 - u X_0^{\pm 1} X_1^{\pm 1})$. It follows that the series $\hat{s}_\pm(u)$ are group-like. Together with their expansions $\hat{s}_\pm(u) = 1 + \sum_{k \geq 1} u^k X_0^{\pm k} (1 - X_1^{\pm 1})$, this implies (2.2.2) for $k \geq 1$ and (2.2.3) for $k \leq -1$. Since $X_0^{\pm 1}$ are group-like, (2.2.2) for $k$ is equivalent to (2.2.3) for $-k$, which implies (2.2.2) for $k \leq -1$ and (2.2.2) for $k \geq 1$. Finally, (2.2.2) and (2.2.3) for $k = 0$ are direct consequences of the fact that $X_1^{\pm 1}$ are group-like.

Remark 2.6. There is a unique Hopf algebra automorphism $\theta$ of $(\mathcal{W}^B_i, \Delta_i)$ such that $\theta(X_1^{\pm}) := X^\mp_1$ and $\theta(Y_1^{\pm}) := Y^\mp_1$ for $k > 0$. This automorphism is compatible with the algebra inclusion $\mathcal{W}_i^B \subset \mathbf{k} F_2$ and with the automorphism of $\mathbf{k} F_2$ induced by $X_i \mapsto X_i^{-1}$, $i = 0, 1$.

2.3. Construction and properties of $\Delta_i$. Recall from (1.3.3) that for $n \geq 0$, $F^n \mathcal{W}_i^B$ is the intersection $\mathcal{W}_i^B \cap F^n$, where $I \subset \mathbf{k} F_2$ is the augmentation ideal. The collection of these spaces defines a filtration of the algebra $\mathcal{W}_i^B$, and $\mathcal{W}_i^B$ is then its completion. The algebra $\mathcal{W}_i^B$ is naturally filtered and the natural morphism $\text{gr}(\mathcal{W}_i^B) \to \text{gr}(\mathcal{W}_i^B)$ is an algebra isomorphism.

2.3.1. Computation of $F^n \mathcal{W}_i^B$. For $(\epsilon, s) \in \{+, -\} \times \mathbb{Z}$ and $n \geq 1$, set

$$
\xi(\epsilon, s|n) := X_0^n(X_0 - 1)^{n-1}(X_1^{\epsilon} - 1)
$$

(1.3.3)

(where $X_1^{\pm} := X_1$ and $X_1^{-1} := X_1^{-1}$); note that $\xi(\pm, s|1) = \xi^\pm_s$ for $s \in \mathbb{Z}$ (see (2.2)). For $n \geq 0$, define the subspace $I_i(n)$ of $\mathcal{W}_i^B$ as follows:

- if $n = 0$, then $I_i(0) := \mathcal{W}_i^B$;
- if $n \geq 1$, then $I_i(n)$ is the subspace of $\mathcal{W}_i^B$ linearly spanned by the products

$$
\xi(\epsilon_1, s_1|n_1) \cdots \xi(\epsilon_k, s_k|n_k),
$$

where $k \geq 1$, $(\epsilon_1, s_1), \ldots, (\epsilon_k, s_k) \in \{+, -\} \times \mathbb{Z}$ and $n_1 + \cdots + n_k \geq n$. 
The algebra $\mathcal{W}^B_i$ is equipped with a morphism $\epsilon_i : \mathcal{W}^B_i \to k$, obtained by restriction of the augmentation morphism $kF_2 \to k$. As the family $\{\xi(\epsilon, s|n)|\epsilon, s \in \{+, -\} \times \mathbb{Z}, n \geq 1\}$ algebraically generates $\mathcal{W}^B_i$, one has the equality

$$I_i(1) = \text{Ker}(\epsilon_i).$$

**Lemma 2.7.**

1. For any $n \geq 0$, one has $I_i(n) = F^n\mathcal{W}^B_i$.

2. The composition of the morphism $\oplus_{n \geq 0}I_i(n)/I_i(n+1) \to \text{gr}(\mathcal{W}^B_i)$ induced by the isomorphism from (1), of the isomorphism $\text{gr}(\text{iso}^1) : \text{gr}(\mathcal{W}^B_i) \to \mathcal{W}^B_i$ for $\mu = 1$ from \[1\] and of the isomorphism $\mathcal{W}^{\text{DR}}_i \simeq k(Y)$ from \[2\] is the direct sum over $n \geq 0$ of the maps

$$I_i(n)/I_i(n+1) \to (\text{degree } n \text{ part of } k(Y)), $$

the class of $\xi(\epsilon_1, s_1|n_1) \cdots \xi(\epsilon_k, s_k|n_k) \mapsto (-\epsilon_1 y_{n_1}) \cdots (-\epsilon_k y_{n_k})$.  

for $\epsilon_1, \ldots, \epsilon_k \in \{+, -\}$ and $n_1 + \cdots + n_k = n$.

**Proof.** The statements are obvious for $n = 0$.

For any $(\epsilon, s) \in \{+, -\} \times \mathbb{Z}$ and any $k \geq 1$, one has $\xi(\epsilon, s|k) \in F^k\mathcal{W}^B_i$, therefore for any $n \geq 1$, one has $I_i(n) \subseteq F^n\mathcal{W}_i$.

This collection of inclusions induces, for any $n \geq 0$, a linear map $I_i(n)/I_i(n+1) \to F^n\mathcal{W}^B_i/F^{n+1}\mathcal{W}^B_i$. The collection of inclusions $F^n\mathcal{W}^B_i \subset I^n$ induces, for any $n \geq 0$, an injection $F^n\mathcal{W}^B_i/F^{n+1}\mathcal{W}^B_i \hookrightarrow F^n/I^n$. Finally, according to \[\text{B}, \text{chap. 2, } \S 5, \text{ no. 4, Theorem 2,}\] there is a graded algebra isomorphism between $\text{gr}(kF_2) = \oplus_{n \geq 0}I^n/I^{n+1}$ and the free algebra in two generators $k\langle X \rangle = k\langle x_0, x_1 \rangle$; this isomorphism takes $x_0, x_1$ to the classes of $X_0 - 1, X_1 - 1$ in $I/I^2$. Summarizing, we have a diagram

$$\frac{I_i(n)}{I_i(n+1)} \to \frac{F^n\mathcal{W}^B_i}{F^{n+1}\mathcal{W}^B_i} \subset \frac{I^n}{I^{n+1}} = k\langle x_0, x_1 \rangle [n]$$

where $[n]$ indicates the part of degree $n$.

Recall that $I_i(n)/I_i(n+1)$ is linearly spanned by the $\xi(\epsilon_1, s_1|n_1) \cdots \xi(\epsilon_k, s_k|n_k)$, where $(\epsilon_1, s_1), \ldots, (\epsilon_k, s_k) \in \{+, -\} \times \mathbb{Z}$ and $n_1 + \cdots + n_k = n$.

For $s \in \mathbb{Z}$, one has $\xi(+, s|n) - \xi(+, 0|n) = \xi(+, 0|n+1) + \cdots + \xi(+, s-1|n+1)$, therefore $\xi(+, s|n) \equiv \xi(+, 0|n) \text{ mod } I_i(n+1)$. Also $\xi(-, s|n) + \xi(+, s|n) = -\xi(+, s|n) \xi(-, 0|1)$, therefore $\xi(-, s|n) \equiv -\xi(+, 0|n) \text{ mod } I_i(n+1)$. All this implies that for any $(\epsilon, s) \in \{+, -\} \times \mathbb{Z}$, $\xi(\epsilon, s|n) \equiv c_\epsilon \xi(+, 0|n) \text{ mod } I_i(n+1)$, therefore a generating family of $I_i(n)/I_i(n+1)$ reduces to $(\xi(+, 0|n_1) \cdots \xi(+, 0|n_k))_{n_1 + \cdots + n_k = n}$.

The map

$$\frac{I_i(n)}{I_i(n+1)} \to k\langle x_0, x_1 \rangle [n]$$

then takes $\xi(+, 0|n_1) \cdots \xi(+, 0|n_k)$ to $(-x_0^{n_1-1}x_1) \cdots (-x_0^{n_k-1}x_1)$, and therefore maps the generating family

$$(\xi(+, 0|n_1) \cdots \xi(+, 0|n_k))_{n_1 + \cdots + n_k = n}$$
of $I_l(n)/I_l(n+1)$ to a linearly independent family of $k\langle x_0, x_1 \rangle[n]$. It follows that this family of $I_l(n)/I_l(n+1)$ is also linearly independent, therefore that it is a basis of $I_l(n)/I_l(n+1)$. It also follows that the map (2.3.1) is injective.

Using the injectivity of (2.3.1) and composing back the map (2.3.1) with the isomorphism $\frac{I^n}{I_n} \cong k\langle x_0, x_1 \rangle[n]$, we obtain that the map

$$\frac{I_l(n)}{I_l(n+1)} \to \frac{I^n}{I_n^{n+1}}$$

is injective for any $n \geq 0$. By an induction argument, one then obtains that for any $n \geq 0$, the map

$$\frac{I_l(0)}{I_l(n)} \to \frac{I^0}{I^n}$$

is injective. As this is the map

$$\frac{k \oplus kF_2(X_1 - 1)}{I_l(n)} \to \frac{kF_2}{I^n},$$

it follows that $(k \oplus kF_2(X_1 - 1)) \cap I^n \subset I_l(n)$, so $F^nW^B_l \subset I_l(n)$.

All this proves that

$$F^nW^B_l = I_l(n),$$

i.e., the first statement, as well as the second statement. □

**Lemma 2.8.** The subspace $I_l(n)$ is a two-sided ideal of $W^B_l$. In particular, $I_l(n)$ is stable under right multiplication by $X_1^{\pm 1}$.

**Proof.** This follows from Lemma 2.7, 1) as each $F^nW^B_l$ is a two-sided ideal of $W^B_l$. □

2.3.2. **Definition of $\hat{\Delta}_d$.** Define a linear map

$$\xi : k[t^{\pm 1}] \to W^B_l$$

by

$$\xi(f) := f(X_0)(X_1 - 1)$$

for $f \in k[t^{\pm 1}]$.

**Lemma-Definition 2.9.** The assignment $f \mapsto \frac{t'f(t'') - t''f(t')}{t'-t''}$ defines a linear map $k[t^{\pm 1}] \to k[t^{\pm 1}, t'^{\pm 1}]$ We denote by

$$\text{Op} : k[t^{\pm 1}] \to k[t^{\pm 1}] \otimes^2$$

the composition of this linear map with the isomorphism $k[t^{\pm 1}, t'^{\pm 1}] \to k[t^{\pm 1}] \otimes^2$ given by $t' \mapsto t \otimes 1$, $t'' \mapsto 1 \otimes t$.

**Proof.** For $a \in \mathbb{Z}$, the image of $t^a$ is $-t' t''(t'^{a-2} + \cdots + t''^{a-2})$ if $a \geq 1$, and $(t' t'')(t''-a + \cdots + t'^{-a})$ if $a \leq 0$. □
Proof. Assume that Lemma 2.13.

Let Lemma 2.11.

For takes \((((C \text{ family of rational numbers})) \smallfrown \text{Lemma 2.10, Lemma 2.11 and Lemma 2.12, one obtains the existence of a family of rational numbers } C(n, a, s, s', s'')\) indexed by families \((n, a, s, s', s'')\), where: \(n \geq 1, a \in [1, n - 1], s, s', s'' \in \mathbb{Z}\), such that the map \(\Delta_3 : \mathcal{W}_1^B \to (\mathcal{W}_1^B)^{\otimes 2}\) is such that

\[
\xi(f) \mapsto \xi(f) \otimes 1 + 1 \otimes \xi(f) + \xi^\otimes 2(\text{Op}(f))
\]

for any \(f \in k^{(\pm 1)}\).

For \(a \geq 0\), let us denote by \(((t - 1)^a)\) the ideal of \(k^{(\pm 1)}\) generated by \((t - 1)^a\).

Lemma 2.11. For \(n \geq 2\), the map \(\text{Op} : k^{(\pm 1)} \to k^{(\pm 1)^{\otimes 2}}\) defined in Lemma-Definition 2.9 takes \(((t - 1)^{n-1})\) to \(\sum_{a=1}^{n-1}((t - 1)^{a-1}) \otimes ((t - 1)^{n-a-1})\).

Proof. Assume that \(f \in ((t - 1)^{n-1})\), that is \(f(t) = (t - 1)^{n-1}a(t)\), with \(a(t) \in k^{(\pm 1)}\). Then

\[
\frac{t'f(t'') - t''f(t')}{t' - t''} = \frac{t'(t'' - 1)^{n-1}a(t''') - t''(t' - 1)^{n-1}a(t')}{t' - t''}
\]

\[
= (t'' - 1)^{n-1}t'a(t'') - t''a(t') + t'a(t')(t'' - 1)^{n-1} - (t' - 1)^{n-1}
\]

The second term of the last line belongs to the announced space \(\sum_{a=1}^{n-1}((t - 1)^{n-1}) \otimes ((t - 1)^{n-a-1})\), while the first term belongs to \(k^{(\pm 1)} \otimes ((t - 1)^{n-1})\), which is contained in this space. This proves the result. \(\square\)

For \(n \geq 1\), the ideal \(((t - 1)^{n-1})\) of \(k^{(\pm 1)}\) is linearly spanned over \(k\) by \(\{t^s(t - 1)^{n-1}|s \in \mathbb{Z}\}\).

By \(\xi(t^s(t - 1)^{n-1}) = \xi(+, s|n)\), this implies:

Lemma 2.12. Let \(n \geq 1\). The vector subspaces \(\{\xi(f)|f \in ((t - 1)^{n-1})\}\) and \(\text{Span}_k\{\xi(+, s|n)|s \in \mathbb{Z}\}\) of \(\mathcal{W}_1^B\) coincide.

Lemma 2.13. Let \(n \geq 1\) and \((\epsilon, s) \in \{+, -\} \times \mathbb{Z}\). The image under \(\Delta_3 : \mathcal{W}_1^B \to (\mathcal{W}_1^B)^{\otimes 2}\) of \(\xi(\epsilon, s|n)\) belongs to \(\sum_{a,b \geq 0|a+b=n}I(a) \otimes I(b)\).

Proof. Combining Lemma 2.10, Lemma 2.11 and Lemma 2.12 one obtains the existence of a family of rational numbers \(C(n, a, s, s', s'')\) indexed by families \((n, a, s, s', s'')\), where: \(n \geq 1, a \in [1, n - 1], s, s', s'' \in \mathbb{Z}\), such that the map \(\Delta_3 : \mathcal{W}_1^B \to (\mathcal{W}_1^B)^{\otimes 2}\) is such that

\[
\xi(+, s|n) \mapsto \xi(+, s|n) \otimes 1 + 1 \otimes \xi(+, s|n) + \sum_{a=1}^{n-1} \sum_{s', s'' \in \mathbb{Z}} C(n, a, s, s', s'') \xi(+, s'|a) \otimes \xi(+, s''|n-a)
\]
for any $n \geq 1$ and $s \in \mathbb{Z}$. The right-hand side of (2.3.2) obviously belongs to $\sum_{a,b \geq 0|a+b=n} I_i(a) \otimes I_i(b)$ (a subspace of $\mathcal{W}_t^B \otimes 2$), so $\Delta_\varnothing(\xi(+, s|n))$ belongs to this space.

Proposition 2.13 implies that $\Delta_\varnothing(\xi(-, s|n)) = -\Delta_\varnothing(\xi(+, s|n))(X_1^{-1} \otimes X_1^{-1})$. This equality, together with Lemma 2.8, implies that the images of $\xi(\pm, s|n)$ by $\Delta_\varnothing$ belong to the same space.

\[ \square \]

**Proposition 2.14.** The morphism $\Delta_\varnothing$ is compatible with the filtration of $\mathcal{W}_t^B$, in other words, if $n \geq 0$, then $\Delta_\varnothing(I_i(n)) \subset \sum_{a,b \geq 0|a+b=n} I_i(a) \otimes I_i(b)$.

**Proof.** For $n = 0$, the result is trivial as $I_i(0) = \mathcal{W}_t^B$. For $n \geq 1$, the result follows from Lemma 2.13 and from $I_i(a_1) \cdots I_i(a_k) \subset I_i(a_1 + \cdots + a_k)$ for $a_1, \ldots, a_k \geq 0$. (When $n = 1$, the result may also be viewed as a consequence of $\xi_l^{\otimes 2} \circ \Delta_\varnothing = \epsilon_l$.)

Proposition 2.14 implies that for any $n \geq 0$, $\Delta_\varnothing$ induces an algebra morphism

$$\mathcal{W}_t^B/I_i(n) \rightarrow (\mathcal{W}_t^B)^{\otimes 2}/ \sum_{a,b \geq 0|a+b=n} I_i(a) \otimes I_i(b).$$

**Definition 2.15.** The continuous algebra morphism obtained by taking inverse limits with respect to $n$ will be denoted

$$\hat{\Delta}_\varnothing : \mathcal{W}_t^B \rightarrow (\mathcal{W}_t^B)^{\otimes 2}.$$

2.3.3. Associated graded morphism of $\Delta_\varnothing$.

**Lemma 2.16.** The map $\hat{\Delta}_\varnothing : \mathcal{W}_t^B \rightarrow (\mathcal{W}_t^B)^{\otimes 2}$ is such that

$$\xi(+, 1|n) = \xi(+, 1|n) \otimes 1 + 1 \otimes \xi(+, 1|n) - \sum_{a=1}^{n-1} \xi(+, 1|a) \otimes \xi(+, 1|n-a)$$

for any $n \geq 1$.

**Proof.** This follows from Lemma 2.10 and from

$$\text{Op}(t(t-1)^n) = \frac{t't''(t''-1)^{n-1} - t't''(t'-1)^{n-1}}{t'-t''} = -\sum_{a=1}^{n-1} t'(t'-1)^{a-1}t''(t''-1)^{n-a-1}.$$
Proof. The two sides of (2.3.3) belong respectively to $I(l(n))$ and to $\sum_{a=0}^{n} I_l(a) \otimes I_l(n-a)$. The image of $\xi(1,n)$ in the degree $n$ part of the associated graded algebra of $W^B_l$ is $-x_0^{n-1}x_1 \in k(Y)$. It follows that the associated graded morphism of $\Delta_*$ is the morphism

$$k(Y) \to k(Y) \otimes^2, \quad -x_0^{n-1}x_1 \mapsto -x_0^{n-1}x_1 \otimes 1 - 1 \otimes x_0^{n-1}x_1 - \sum_{a=1}^{n-1} (-x_0^{a-1}x_1) \otimes (-x_0^{n-a-1}x_1),$$

that is $\Delta_*$. □

3. THEOREM 3.1 AND ITS REFORMULATION

As we explained, 3) in Theorem 0.2 is equivalent to the following statement.

**Theorem 3.1.** The coproducts $\hat{\Delta}_*$ from Theorem 1.16 and $\hat{\Delta}_d$ from Definition 2.15 coincide, namely

$$\hat{\Delta}_* = \hat{\Delta}_d.$$  

The sequel of this section is devoted to the statement of an equivalent formulation of this result. This formulation is based on "right" variants $W^B_{l,r}, W^B_{l,r}$ of the algebras $W^B_l, W^B_{l}$ of §1. In §3.1 we introduce these "right" variants and the related material. In §3.2 (Proposition 3.13), we state the announced equivalent formulation and prove its equivalence with Theorem 3.1.

3.1. **Ingredients of diagram (3.2.1): "right" counterpart of the material from §§1 and 2**

3.1.1. **Algebras $W^B_{l,r}, W^B_{l,r}, W^B_{l,r}, W^B_{l,r}$**. In §1.3 we introduced the algebra $U(f_2)$, its completion $U(f_2)^\wedge$ and the subalgebras $W^B_{l}, W^B_{l}$.

**Definition 3.2.** We set

$$W^B_{l,r} := k1 \oplus e_1U(f_2) \subset U(f_2), \quad \hat{W}^B_{l,r} := k1 \oplus e_1U(f_2)^\wedge \subset U(f_2)^\wedge.$$  

Then $W^B_{l,r}$ (resp., $\hat{W}^B_{l,r}$) is a subalgebra (resp., closed subalgebra) of $U(f_2)$ (resp., $U(f_2)^\wedge$).

In §1.3 we also introduced the algebra $kF_2$, its completion $(kF_2)^\wedge$ and the subalgebras $W^B_{l}, W^B_{l}$. We now set:

**Definition 3.3.** We set

$$W^B_{l,r} := k1 \oplus (X_1 - 1) \cdot kF_2 \subset kF_2, \quad \hat{W}^B_{l,r} := k1 \oplus (X_1 - 1) \cdot (kF_2)^\wedge \subset (kF_2)^\wedge.$$  

Then $W^B_{l,r}$ (resp., $\hat{W}^B_{l,r}$) is a subalgebra (resp., closed subalgebra) of $kF_2$ (resp., $(kF_2)^\wedge$).
3.1.2. Isomorphisms $\text{Ad}(e_1)$, $\text{Ad}(X_1 - 1)$, $\text{Ad}(f(X_1 - 1))$.

**Lemma 3.4.** There is a unique isomorphism of graded algebras

$$\text{Ad}(e_1) : \hat{W}^\text{DR}_l \to \hat{W}^\text{DR}_r,$$

given by $1 \mapsto 1$ and $ae_1 \mapsto e_1a$ for $a \in U(f_2)$. It extends to an isomorphism $\hat{W}^\text{DR}_l \to \hat{W}^\text{DR}_r$, also denoted $\text{Ad}(e_1)$.

**Proof.** Immediate; note that $\text{Ad}(e_1) : \hat{W}^\text{DR}_l \to \hat{W}^\text{DR}_r$ is a restriction of the automorphism $\text{Ad}(e_1)$ of the localization of $U(f_2)$ with respect to its invertible element $e_1$. \(\square\)

**Lemma 3.5.** There is a unique algebra isomorphism

$$\text{Ad}(X_1 - 1) : \hat{W}^\text{B}_l \to \hat{W}^\text{B}_r,$$

given by $1 \mapsto 1$ and $a \cdot (X_1 - 1) \mapsto (X_1 - 1) \cdot a$ for $a \in U(f_2)$. It extends to an isomorphism $\hat{W}^\text{B}_l \to \hat{W}^\text{B}_r$, also denoted $\text{Ad}(e_1)$.

**Proof.** Immediate; similarly to the remark of the proof of Lemma 3.4, $\text{Ad}(X_1 - 1)$ is a restriction of a conjugation automorphism. \(\square\)

Let $f \in t \cdot k[[t]]^\times$, so $f = tg$, with $g \in k[[t]]^\times$. Define $\text{Ad}(f(X_1 - 1)) : \hat{W}^\text{B}_l \to \hat{W}^\text{B}_r$ by $1 \mapsto 1$, $a \cdot (X_1 - 1) \mapsto f(X_1 - 1) \cdot a \cdot g(X_1 - 1)^{-1}$. One shows:

**Lemma 3.6.** $\text{Ad}(f(X_1 - 1)) : \hat{W}^\text{B}_l \to \hat{W}^\text{B}_r$ is an isomorphism of filtered topological algebras. The associated graded morphism identifies with $\text{Ad}(e_1) : \hat{W}^\text{DR}_l \to \hat{W}^\text{DR}_r$.

3.1.3. The isomorphisms $\text{iso}_r^\mu$. Let $\mu \in k^\times$. In 1.3.4 we defined an isomorphism $\text{iso}_r : (kF_2)^^\wedge \to U(f_2)^^\wedge$ of filtered topological algebras.

There is a unique algebra isomorphism $\text{iso}_r^\mu : \hat{W}^\text{B}_l \to \hat{W}^\text{DR}_r$, such that the diagram

$$
\begin{CD}
\hat{W}^\text{B}_l @>{\text{iso}_r^\mu}>> \hat{W}^\text{DR}_r \\
@VVV @VVV \\
(kF_2)^^\wedge @>{\text{iso}_r}>> U(f_2)^^\wedge
\end{CD}
$$

commutes.

For $\mu \in k^\times$, $\text{iso}_r^\mu$ is strictly compatible with the filtrations on both sides and therefore induces an algebra isomorphism $\text{gr}(\text{iso}_r^\mu) : \text{gr}(\hat{W}^\text{B}_l) \to \text{gr}(\hat{W}^\text{DR}_r) = W^\text{DR}_r$.

3.1.4. The automorphisms $a_g^\mu$. In 1.7 we attached to $g \in (U(f_2)^^\wedge)^\times$ an automorphism $a_g$ of the topological $k$-algebra $U(f_2)^^\wedge$.

**Lemma 3.7.** If $g \in (U(f_2)^^\wedge)^\times$, then $a_g$ restricts to an automorphism of the topological algebra $\hat{W}^\text{DR}_r$, which will be denoted as $a_g^\mu$. 
Proof. Since $a_g$ is a vector space automorphism of $U(f_2)^\wedge$, and as $\hat{W}_r^{DR}$ decomposes as $kI \oplus e_1U(f_2)^\wedge$, there is a unique vector space automorphism $a_g^r$ of $\hat{W}_r^{DR}$, such that $a_g^r(1) = 1$ and $a_g^r(e_1a) = e_1a_g(a)$ for any $a \in U(f_2)^\wedge$. As $a_g(e_1) = e_1$, one has $a_g^r(y) = a_g(y)$ for $y \in \hat{W}_r^{DR}$.

3.1.5. The isomorphisms $a_{(\mu,g)}^r$. Let $\mu \in k^\times$ and $g \in (U(f_2)^\wedge)^\times$. We define an isomorphism of filtered topological algebras

$$a_{(\mu,g)}^r : \hat{W}_r^B \rightarrow \hat{W}_r^{DR}$$

by $a_{(\mu,g)}^r := a_g^r \circ \text{iso}_r^\mu$.

Lemma 3.8. For $\mu \in k^\times$ and $g \in (U(f_2)^\wedge)^\times$, the diagram

\[
\begin{array}{ccc}
\hat{W}_r^B & \rightarrow (kF_2)^\wedge \\
\downarrow a_{(\mu,g)}^r & & \downarrow \text{iso}_r^\mu \\
\hat{W}_r^{DR} & \rightarrow (U(f_2)^\wedge)
\end{array}
\]

is commutative.

Proof. Consider the diagram

\[
\begin{array}{ccc}
\hat{W}_r^B & \rightarrow (kF_2)^\wedge \\
\downarrow a_{(\mu,g)}^r & & \downarrow \text{iso}_r^\mu \\
\hat{W}_r^{DR} & \rightarrow U(f_2)^\wedge \\
\downarrow a_g^r & & \downarrow a_g \\
\hat{W}_r^{DR} & \rightarrow U(f_2)^\wedge \\
\end{array}
\]

It follows from the definition of $a_{(\mu,g)}$ (resp. of $a_{(\mu,g)}^r$) in (1.7.1) (resp. in (3.1.1)) that the right (resp. left) triangle commutes. It follows from the definition of $\text{iso}_r^\mu$ (resp. $a_g^r$) in (3.1.3) (resp. in Lemma 3.7) that the upper (resp. lower) trapezoid is commutative. This implies that the outer rectangle is commutative.

Lemma 3.9. For $\mu \in k^\times$ and $g \in (U(f_2)^\wedge)^\times$, the diagram

\[
\begin{array}{ccc}
\hat{W}_l^B & \rightarrow \hat{W}_r^B \\
\downarrow a_{(\mu,g)}^l & & \downarrow a_{(\mu,g)}^r \\
\hat{W}_l^{DR} & \rightarrow \hat{W}_r^{DR} \\
\end{array}
\]

is commutative.
Proof. This follows from the commutativity of the diagrams

\[
\begin{array}{ccc}
\hat{W}_l^B & \xrightarrow{\text{Ad}(\log(X_1))} & \hat{V}_r^B \\
\hat{W}_l^{DR} & \xrightarrow{\text{Ad}(e_1)} & \hat{V}_r^{DR}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\hat{W}_r^B & \xrightarrow{\text{Ad}(e_1)} & \hat{V}_r^B \\
\hat{W}_r^{DR} & \xrightarrow{\text{Ad}(e_1)} & \hat{V}_r^{DR}
\end{array}
\]

which we now prove.

If \( f \in (k F_2)^\wedge \), then

\[
\text{iso}_\mu \circ \text{Ad}(\log(X_1))(f \cdot (X_1 - 1)) = \text{iso}_\mu(\log(X_1) \cdot f) \cdot \frac{X_1 - 1}{\log(X_1)} = \mu e_1 \cdot \text{iso}_\mu(f) \cdot \frac{e^{\mu e_1} - 1}{\mu e_1}
\]

which proves that the restriction of the two maps of the left diagram agree on \((k F_2)^\wedge(X_1 - 1) \subset \hat{W}_l^B\). Since they also agree on \(k 1 \subset \hat{W}_l^B\), the left diagram is commutative.

If \( f \in (U(k F_2))^\wedge \), then

\[
a_g \circ \text{Ad}(e_1)(f \cdot e_1) = a_g(e_1 \cdot f) = e_1 \cdot a_g(f) = \text{Ad}(e_1)(a_g(f) \cdot e_1) = \text{Ad}(e_1) \circ a_g(f \cdot e_1),
\]

which proves that the restriction of the two maps of the right diagram agree on \((U(k F_2))^\wedge e_1 \subset \hat{W}_l^{DR}\). Since they also agree on \(k 1 \subset \hat{W}_l^{DR}\), the right diagram is commutative. \(\square\)

3.1.6. *The algebra morphisms \(\Delta^l_{r}^{\wedge}\) and \(\Delta^r_{r}^{\wedge}\).* Recall that \(\text{Ad}(e_1)\) is an isomorphism of graded algebras \(\hat{W}_l^{DR} \to \hat{W}_r^{DR}\). We set:

**Definition 3.10.** \(\Delta^l_{r}^{\wedge} : \hat{W}_l^{DR} \to (\hat{W}_r^{DR})^{\wedge 2}\) is the composition \(\text{Ad}(e_1)^{\wedge 2} \circ \Delta_*\).

Then \(\Delta^l_{r}^{\wedge}\) is morphism of graded algebras. It therefore induces a morphism between the degree completions of these algebras, which we denote by

\[
(3.1.2) \quad \hat{\Delta}^l_{r} : \hat{W}_l^{DR} \to (\hat{W}_r^{DR})^{\wedge 2}.
\]

Recall that \(\text{Ad}(X_1 - 1)\) is an isomorphism of algebras \(\hat{W}_l^B \to \hat{W}_r^B\). We set:

**Definition 3.11.** \(\Delta^l_{r}^{B} : \hat{W}_l^B \to (\hat{W}_r^B)^{\wedge 2}\) is the composition \(\text{Ad}(X_1 - 1)^{\wedge 2} \circ \Delta_*\).

Since the algebra morphisms \(\Delta_*\) and \(\text{Ad}(X_1 - 1) : \hat{W}_l^B \to \hat{W}_r^B\) are both compatible with filtrations, so is the morphism \(\Delta^l_{r}^{B}\). We define

\[
(3.1.3) \quad \hat{\Delta}^l_{r} : \hat{W}_l^B \to (\hat{W}_r^B)^{\wedge 2}
\]

to be the completion of this morphism with respect to these filtrations. Then \(\hat{\Delta}^l_{r}\) coincides with the composition \(\hat{W}_l^B \xrightarrow{\Delta_*} (\hat{W}_r^B)^{\wedge 2} \xrightarrow{\text{Ad}(X_1 - 1)} (\hat{W}_r^B)^{\wedge 2}\).

Using Proposition 2.3.4 and Lemma 3.3.6, one proves:

**Lemma 3.12.** The map \(\hat{\Delta}^l_{r}\) is compatible with the filtrations of its source and target, and the associated graded map coincides with \(\Delta^l_{r}\), so \(\text{gr}(\hat{\Delta}^l_{r}) = \Delta^l_{r}\).
3.2. Reformulation of the main theorem in terms of diagram \((3.2.1)\).

**Proposition 3.13.** Let \(\mu \in k^\times\) and \(\Phi \in DMR_\mu(k)\).

The equality \(\hat{\Delta}_x = \hat{\Delta}_y\) is equivalent to the commutativity of

\[
(3.2.1) \quad \begin{array}{ccc}
\hat{\Delta}_x & \xrightarrow{\hat{\Delta}_x} & (\hat{\Delta}_x)^{\otimes 2} \\
\hat{\Delta}_y & \xrightarrow{\hat{\Delta}_y} & (\hat{\Delta}_y)^{\otimes 2}
\end{array}
\]

where \(\hat{\Delta}_x, \hat{\Delta}_y\) are commutative because

\[
\begin{align*}
\tag{3.1.1} \hat{\Delta}_x & = \hat{\Delta}_y, \\
\tag{3.2.1} \hat{\Delta}_x & = \hat{\Delta}_y.
\end{align*}
\]

Proof. Consider the diagram

\[
\begin{array}{ccc}
\hat{\Delta}_x & \xrightarrow{\hat{\Delta}_x} & (\hat{\Delta}_x)^{\otimes 2} \\
\hat{\Delta}_y & \xrightarrow{\hat{\Delta}_y} & (\hat{\Delta}_y)^{\otimes 2}
\end{array}
\]

This diagram is split into four squares, which we call TL, TR, BL, BR (where T, B stand for top, bottom and R, L stand for right, left). The commutativity of TL is equivalent to \(\hat{\Delta}_x = \hat{\Delta}_y\). Moreover, BL is the commutative square from Proposition 1.19, TR is trivial, and BR is commutative because \(\hat{\Delta}_x(\hat{\Delta}_y) = e^{\mu_1}\). Moreover, the rows of TR, BR and the columns of BL, BR are invertible. It follows that \(\hat{\Delta}_x = \hat{\Delta}_y\) is equivalent to the commutativity of the external diagram.

According to Definition 3.11 the top arrow of the external diagram coincides with that of \((3.2.1)\). The left and right arrows of the external diagram trivially coincide with those of \((3.2.1)\).

Let us show that the bottom arrow of the external diagram coincides with that of \((3.2.1)\). One checks that the diagram

\[
\begin{array}{ccc}
\hat{\Delta}_x & \xrightarrow{\hat{\Delta}_x} & (\hat{\Delta}_x)^{\otimes 2} \\
\hat{\Delta}_y & \xrightarrow{\hat{\Delta}_y} & (\hat{\Delta}_y)^{\otimes 2}
\end{array}
\]
is commutative.

Together with Definition 3.10, this implies that the bottom map of the external diagram is equal to the composed map \( \hat{\Delta}_{p}^{\prime} (\hat{W}_{DR}^{l}) \otimes^{2} \hat{\text{Ad}}(e^{\mu_{e} - 1}f_{e} - \epsilon_{e}f_{e}) \Gamma_{\Phi} (e^{\epsilon_{e}f_{e} - 1}f_{e}) \rightarrow (\hat{W}_{DR}^{r}) \otimes^{2}. \) The functional equation (1.1.4) then implies that this map coincides with the bottom map of (3.2.1).

All this implies that the external diagram coincides with (3.2.1), and therefore that \( \hat{\Delta}_{p}^{\prime} = \hat{\Delta}_{r} \) is equivalent to the commutativity of (3.2.1). \( \square \)
Part 2. Proof of Theorem 3.1

§§ 4–9 are devoted to the proof of Theorem 3.1. By Proposition 3.13, this proof can be reduced to the commutativity of the diagram (3.2.1) relating $\hat{\Delta}_{l,r}^*$ and $\hat{\Delta}_{l,r}^\#$ for some pair $(\mu, \Phi)$, $\mu \in k^\times$, $\Phi \in DMR_\mu(k)$.

§ 4 contains preliminary material to be used in §§ 5 and 6. In § 5, we construct a commutative diagram relating $\hat{\Delta}_{l,r}^*$ with infinitesimal braid Lie algebras (diagram (5.3.3)). In § 6, we similarly relate $\hat{\Delta}_{l,r}^\#$ with braid groups (diagram (6.3.3)). In § 7, we recall some facts on associators, and in particular how these objects relate braid groups to infinitesimal braid Lie algebras. These properties are used in § 8 for proving the commutativity of various diagrams involving constituents of (5.3.3) and (6.3.3), which enables us to prove the commutativity of (3.2.1) for any $(\mu, \Phi)$ with $\mu \in k^\times$ and $\Phi$ in the set $M_\mu(k)$ of associators with the parameter $\mu$. In § 9, we then obtain two proofs of Theorem 3.1: the first one uses the inclusion $M_\mu(k) \subset DMR_\mu(k)$ (Fu2), and the second one uses base change and the properties of the associator $\varphi_{KZ}$ (see § 1.1.5).

4. Algebraic results to be used in §§ 5 and 6

In this section, we prove two algebraic results that will be used both in § 5 and in § 6. The two results allow for the construction of algebra morphisms in different contexts. The context of the first result is that of an algebra containing an ideal with freeness properties (§ 4.1, Lemma 4.1). The context of the second result is an algebra morphism to a matrix algebra (§ 4.2, Lemma 4.3).

4.1. Construction of algebra morphisms based on ideals with freeness properties.

Lemma 4.1. Let $R$ be an associative algebra and let $J \subset R$ be a two-sided ideal. Assume that $(j_a)_{a \in \{1, \ldots, d\}}$ is a family of elements of $J$, which constitutes a basis of $J$ for its left $R$-module structure. For $r \in R$, let $(m_{ab}(r))_{a, b \in \{1, \ldots, d\}}$ be the collection of elements of $R$ defined by $j_a r = \sum_{b=1}^d m_{ab}(r) j_b$.

Then the map $R \to M_d(R)$, $r \mapsto (m_{ab}(r))_{a, b \in \{1, \ldots, d\}}$ is an algebra morphism.

Proof. Let $R^{\oplus d} \to J$ be the map $(r_1, \ldots, r_d) \mapsto \sum_{a=1}^d r_a j_a$. This is an isomorphism of left $R$-modules. It sets up an isomorphism of algebras $\text{End}_{R, \text{left}}(J) \cong \text{End}_{R, \text{left}}(R^{\oplus d})$, where the index "$R$-left" means endomorphisms of left $R$-modules. The map $R \to \text{End}_{R, \text{left}}(J)$, $r \mapsto (j \mapsto j r)$ is an algebra morphism when $R$ is equipped with the opposite algebra structure. Similarly, the map $M_d(R) \to \text{End}_{R, \text{left}}(R^{\oplus d})$ taking $M \in M_d(R)$ to the image under the canonical isomorphism $R^{\oplus d} \cong M_{1 \times d}(R)$ of the endomorphism $X \mapsto XM$ is an algebra isomorphism when $M_d(R)$ is equipped with the opposite algebra structure. We then obtain a sequence of algebra morphisms

$$R^{\text{op}} \to \text{End}_{R, \text{left}}(J) \cong \text{End}_{R, \text{left}}(R^{\oplus d}) \cong M_d(R)^{\text{op}},$$
which yields an algebra morphism \( R \rightarrow M_d(R) \). One checks that this morphism is given by the announced formula. \( \square \)

**Lemma 4.2.** Let \( R \) be an associative algebra, let \( J \subset R \) be a two-sided ideal, free as a left \( R \)-module with basis \( (j_a)_{a \in [1,d]} \). Let \( f : R \rightarrow R' \) be an algebra isomorphism, and let \( (j'_a)_{a \in [1,d]} \) be a basis of \( f(J) \subset R' \) as a left \( R' \)-module.

Then there exists a unique element \( P \in \text{GL}_d(R') \), such that \( \begin{pmatrix} f(j_1) \\ \vdots \\ f(j_d) \end{pmatrix} = P \begin{pmatrix} j'_1 \\ \vdots \\ j'_d \end{pmatrix} \) (equality in \( M_{d \times 1}(R') \)). The morphisms \( \varpi : R \rightarrow M_d(R) \) and \( \varpi' : R' \rightarrow M_d(R') \) respectively attached to the data \( (J \subset R, (j_a)_{a \in [1,d]}) \) and \( (J' \subset R', (j'_a)_{a \in [1,d]}) \) as in Lemma 4.1 are related by the following diagram

\[
\begin{array}{ccc}
R & \xrightarrow{\varpi} & M_d(R) \\
\downarrow f & & \downarrow M_d(f) \\
R' & \xrightarrow{\varpi'} & M_d(R') \\
\end{array}
\]

Proof. For \( r \in R \), one has \( \begin{pmatrix} j_1 \\ \vdots \\ j_d \end{pmatrix} r = \varpi(r) \begin{pmatrix} j_1 \\ \vdots \\ j_d \end{pmatrix} \), therefore \( \begin{pmatrix} f(j_1) \\ \vdots \\ f(j_d) \end{pmatrix} f(r) = f(\varpi(r)) \begin{pmatrix} f(j_1) \\ \vdots \\ f(j_d) \end{pmatrix} \), that is \( P \begin{pmatrix} j'_1 \\ \vdots \\ j'_d \end{pmatrix} f(r) = f(\varpi(r))P \begin{pmatrix} j'_1 \\ \vdots \\ j'_d \end{pmatrix} \).

Comparing this with the identity \( \begin{pmatrix} j'_1 \\ \vdots \\ j'_d \end{pmatrix} r' = \varpi'(r') \begin{pmatrix} j'_1 \\ \vdots \\ j'_d \end{pmatrix} \) for \( r' \in R' \), one obtains \( \varpi'(r') = P^{-1} \cdot f(\varpi(f^{-1}(r'))) \cdot P \). \( \square \)

### 4.2. Construction of algebra morphisms based on morphisms to matrix algebras.

Let \( R \) be an associative algebra and let \( e \in R \). Define \( \cdot_e : R \times R \rightarrow R \) by \( r \cdot_e r' := rer' \). Then \( (R, \cdot_e) \) is an associative algebra. The subspaces \( Re \) and \( eR \) of \( R \) are subalgebras. There are two algebra morphisms \( \text{mor}_{R,e}^1 : (R, \cdot_e) \rightarrow Re \) and \( \text{mor}_{R,e}^\bot : (R, \cdot_e) \rightarrow eR \), given by \( r \mapsto re \) and by \( r \mapsto er \).

**Lemma 4.3.** Let \( R, S \) be associative algebras, let \( e \in R \), let \( n \geq 1 \), and let \( f : R \rightarrow M_n(S) \) be an algebra morphism. Assume that there exist elements \( \text{row} \in M_{1 \times n}(S) \) and \( \text{col} \in M_{n \times 1}(S) \), such that \( f(e) = \text{col} \cdot \text{row} \).

Then the map \( \tilde{f} : (R, \cdot_e) \rightarrow S \), defined by \( r \mapsto \text{row} \cdot f(r) \cdot \text{col} \) is an algebra morphism. One then has a commutative diagram

\[
\begin{array}{ccc}
R & \xrightarrow{f} & M_n(S) \\
\cong & \downarrow \text{row} \cdot (\cdot) \cdot \text{col} & \\
(R, \cdot_e) & \xrightarrow{\tilde{f}} & S \\
\end{array}
\]

where

\[(4.2.2) \quad \text{row} \cdot (-) \cdot \text{col} : M_n(S) \to S\]

is the linear map defined by \(m \mapsto \text{row} \cdot m \cdot \text{col}\); in diagram (1.2.1), the horizontal arrows are algebra morphisms and the vertical arrows are vector space morphisms.

Proof. For \(r, r' \in R\), one has

\[
\tilde{f}(r \cdot r') = \tilde{f}(r r') = \text{row} \cdot f(r) f(r') \cdot \text{col} = \text{row} \cdot f(r) \cdot \text{col} \cdot \text{row} \cdot f(r') \cdot \text{col} = \tilde{f}(r) \tilde{f}(r').
\]

\[\square\]

5. Relationship between \(\Delta_*\) and infinitesimal sphere braid Lie algebras

The purpose of this section is to construct a commutative diagram relating \(\Delta_*^{lr}\) with infinitesimal braid Lie algebras (diagram (5.3.3)); it is inspired by [DeT], more specifically by §6.3 and Proposition 6.2 in that paper.

In §5.1 we recall the definition of the infinitesimal braid Lie algebras \(t_n, p_n (n \geq 1)\), as well as the morphisms \(\ell : p_4 \to p_5, \text{pr}_4 : p_5 \to p_4 (i \in [1, 5])\) and \(\text{pr}_{12} : p_5 \to (p_4)^{\otimes 2}\) relating them.

In §5.2, we introduce an ideal \(J(\text{pr}_5)\) of the universal enveloping algebra \(U(p_5)\) arising from \(\text{pr}_5\), and show its freeness as a left \(U(p_5)\)-module (Lemma 5.3). Lemma 4.1 then gives rise to an algebra morphism \(\varpi : U(p_5) \to M_3(U(p_5))\). By composing \(\varpi\) with the morphisms \(\ell\) and \(\text{pr}_{12}\), we construct a morphism \(\rho : U(f_2) \to M_3(U(f_2)^{\otimes 2})\). In Lemma 5.6, by introducing suitable elements \(r \in M_{1 \times 3}(U(f_2)^{\otimes 2})\) and \(\text{col} \in M_{3 \times 1}(U(f_2)^{\otimes 2})\), we show that the morphism \(\rho\) satisfies the hypothesis of Lemma 4.3. Applying this lemma, we obtain in §5.2.4 a morphism \(\tilde{\rho} : (U(f_2), \cdot e_1) \to U(f_2)^{\otimes 2}\), which we compute in Lemma 5.7.

In §5.3, we show the commutativity of a diagram relating \(\Delta_*^{lr}\) and \(\tilde{\rho}\) (Lemma 5.10), and derive from there the commutativity of a diagram relating \(\Delta_*^{lr}\) and \(\varpi, \ell, \text{pr}_{12}\) and \(\text{row} \cdot (-) \cdot \text{col}\). (Proposition 5.11 (6.3.3)).

5.1. Material on infinitesimal braid Lie algebras.

5.1.1. The Lie algebras \(t_n, p_n\). For \(n \geq 2\), let \(t_n\) be the graded Lie \(k\)-algebra with generators \(t_{ij}, i \neq j \in [1, n]\) of degree 1, and relations \(t_{ij} = t_{ij}, [t_{ij}, t_{ik} + t_{jk}] = 0\) for \(i, j, k\) all different in \([1, n]\), and \([t_{ij}, t_{kl}] = 0\) for \(i, j, k, l\) all different in \([1, n]\). The Lie algebra \(t_n\) is called the Drinfeld-Kohno, or infinitesimal braid, Lie algebra.

For \(n \geq 4\), we denote by \(p_n\) the graded Lie \(k\)-algebra with generators \(e_{ij}\) of degree 1, where \(i \neq j \in [1, n]\) and relations \(e_{ij} = e_{ij}, \sum_{j \in [1, n]} - e_{ij} = 0, [e_{ij}, e_{kl}] = 0\) for \(i, j, k, l\) all distinct in \([1, n]\). The Lie algebra \(p_n\) is called the sphere infinitesimal braid Lie algebra.

For \(n \geq 3\), there are surjective morphisms of graded Lie algebras \(t_n \to p_{n+1}\) given by \(t_{ij} \mapsto e_{ij}\) for \(i \neq j \in [1, n]\), \(t_{n+1} \to p_{n+1}\) given by \(t_{ij} \mapsto e_{ij}\) for \(i \neq j \in [1, n + 1]\), and an
injective morphism $t_n \to t_{n+1}$ given by $t_{ij} \mapsto t_{ij}$ for $i \neq j \in [1, n]$; they fit in a commutative diagram

$$
\begin{array}{c}
\text{t}_n \\
\downarrow \\
\text{t}_{n+1}
\end{array}
\quad \begin{array}{c}
\rightarrow \\
\text{p}_{n+1}
\end{array}
$$

Moreover, the morphism $t_n \to p_{n+1}$ factorizes as $t_n \to t_n/Z(t_n) \simeq p_{n+1}$, where $Z(t_n)$ is the 1-dimensional center of $t_n$ (it is concentrated in degree 1 and spanned by $\sum_{i<j}[1, n] t_{ij}$).

**Remark 5.1.** Let $P_n$ (resp. $K_n$) be the pure sphere (resp. Artin pure) braid group with $n$ strands (see §6.1.1 and [Bir]). Its lower central series defines a descending group filtration. The associated graded $\mathbb{Z}$-module is a $\mathbb{Z}$-Lie algebra. Then $\text{gr}(P_n) \otimes \mathbb{k} \simeq p_n$, and $\text{gr}(K_n) \otimes \mathbb{k} \simeq t_n$.

### 5.1.2. The morphisms $\ell, \text{pr}_i$ and $\text{pr}_{12}$ between infinitesimal braid Lie algebras.

There is a graded Lie algebra isomorphism $p_4 \simeq f_2$, where $e_0 = e_{14} = e_{23}$, $e_1 = e_{12} = e_{34}$. One also sets $e_\infty := -e_0 - e_1$, so $e_\infty = e_{13} = e_{24}$.

One checks that there are Lie algebra morphisms $\text{pr}_i : p_5 \to p_4$ for $i = 1, 2, 5$, given by

| elt $x \in p_5$ | $e_{12}$ | $e_{13}$ | $e_{14}$ | $e_{15}$ | $e_{23}$ | $e_{24}$ | $e_{25}$ | $e_{34}$ | $e_{35}$ | $e_{45}$ |
|----------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| $\text{pr}_1(x)$ | 0       | 0       | 0       | 0       | $e_0$   | $e_\infty$ | $e_1$   | $e_\infty$ | $e_0$   | $e_0$   |
| $\text{pr}_2(x)$ | 0       | $e_\infty$ | $e_0$   | $e_1$   | 0       | 0       | 0       | $e_1$   | $e_0$   | $e_\infty$ |
| $\text{pr}_5(x)$ | $e_1$   | $e_\infty$ | $e_0$   | 0       | $e_0$   | $e_\infty$ | 0       | $e_1$   | 0       | 0       |

These morphisms give rise to the Lie algebra morphism $\text{pr}_{12} : p_5 \to p_4^{\otimes 2}$ defined by $\text{pr}_{12}(x) := (\text{pr}_1(x), \text{pr}_2(x))$.

There is a Lie algebra morphism $\ell : p_4 \to p_5$, given by $e_0 \mapsto e_{23}$, $e_1 \mapsto e_{12}$. It is such that $\text{pr}_5 \circ \ell$ is the identity of $p_4$.

We again denote by $\text{pr}_{12}$, $\text{pr}_i$ and $\ell$ the morphisms between universal enveloping algebras $U(p_5) \to U(p_4)^{\otimes 2}$, $U(p_5) \to U(p_4)$ and $U(p_4) \to U(p_5)$ induced by $\text{pr}_{12}$, $\text{pr}_i$ and $\ell$.

### 5.2. Algebraic constructions related to an ideal of $U(p_5)$.

#### 5.2.1. The structure of $J(\text{pr}_5)$.

**Definition 5.2.** We denote by $J(\text{pr}_5)$ the kernel $\text{Ker}(U(p_5) \overset{\text{pr}_5}{\longrightarrow} U(p_4))$. This is a two-sided ideal of $U(p_5)$.

In order to study the structure of $J(\text{pr}_5)$, we prove the following Lemma 5.3.

**Lemma 5.3.** 1) The Lie subalgebra of $p_5$ generated by the $e_{ij}$, $i \in [1, 4]$ is freely generated by the $e_{ij}$, $i \in [1, 3]$, and coincides with the ideal $\text{Ker}(\text{pr}_5)$ of $p_5$; it will be denoted as $f_3$.

2) There exists a unique Lie algebra morphism $\text{inj} : p_4 \to p_5$, given by $e_{ij} \mapsto e_{ij}$ if $i, j \in [1, 3]$ and $e_{ii} \mapsto -\sum_{j \in [1, 3]} e_{ij}$. This morphism is injective.

3) There is a direct sum decomposition $p_5 = \text{Im}(\text{inj} : p_4 \to p_5) \oplus f_3$. 
Proof. 1) follows from [1H2], §1.1 with \( n = i = 5 \) (the notation for \( f_3 \) in loc. cit. is \( N_n \)). The existence and uniqueness of \( \text{inj} \) follows from a direct check based on the presentation of \( p_4 \). One also checks that \( \text{pr}_5 \circ \text{inj} = \text{id}_{p_4} \). This proves the injectivity of \( \text{inj} \). All this proves 2). Finally, 3) follows directly from the facts that \( f_3 = \ker(\text{pr}_5) \) and that \( \text{pr}_5 \circ \text{inj} = \text{id}_{p_4} \).

\[ \Box \]

Remark 5.4. Using [1H2], one can prove that the centralizer of \( e_{45} \) in \( p_5 \) decomposes as a direct sum \( ke_{45} \oplus \text{im}(\text{inj}) \).

Lemma 5.5. The map \( U(p_5) \otimes U(f_3) \rightarrow J(\text{pr}_5) \), \((p_i)_{i \in [1,3]} \mapsto \sum_{i \in [1,3]} p_i \cdot e_{i5}\) is an isomorphism of left \( U(p_5) \)-modules.

Proof. Set

\[ \tilde{p}_4 := \text{im}(\text{inj} : p_4 \rightarrow p_5). \]

Then \( p_5 = \tilde{p}_4 \oplus f_3 \) is a decomposition of the Lie algebra \( p_5 \) as a direct sum of two Lie subalgebras. The tensor product of \( \text{inj} : U(p_4) \rightarrow U(p_5) \) with the injection \( U(f_3) \rightarrow U(p_5) \), followed by the product in \( U(p_5) \), induces a linear map

\[ \text{codec} : U(p_4) \otimes U(f_3) \rightarrow U(p_5). \]

This map is compatible with the PBW filtrations on both sides, and its associated graded map is the linear map \( S(p_4) \otimes S(f_3) \rightarrow S(p_5) \), which is an isomorphism of graded vector spaces, so that codec is an isomorphism of filtered vector spaces.

The following diagram

\[ U(p_4) \otimes U(f_3) \xrightarrow{\text{codec}} U(p_5) \]

\[ \xrightarrow{\text{id} \otimes \varepsilon} U(p_4) \]

\[ \xrightarrow{\text{pr}_5} U(p_4) \]

is commutative, where \( \varepsilon \) is the counit of \( U(f_3) \). Indeed, for \( p \in U(p_4) \) and \( f \in U(f_3) \),

\[ \text{pr}_5 \circ \text{codec}(p \otimes f) = \text{pr}_5 \left( \text{inj}(p)f \right) = \left( \text{pr}_5(\text{inj}(p)) \right) \cdot \text{pr}_5(f) = c(p) \cdot \varepsilon(f), \]

where the first equality follows from the definition of dec, the second equality follows from the algebra morphism property of \( \text{pr}_5 \), and the third equality follows from the facts that \( \text{pr}_5 \circ \text{inj} = \text{id}_{p_4} \) and that \( f_3 = \ker(\text{pr}_5 : p_5 \rightarrow p_4) \).

This diagram implies that \( J(\text{pr}_5) \) is equal to the isomorphic image by codec of the subspace \( U(p_4) \otimes U(f_3)_+ \) of \( U(p_4) \otimes U(f_3) \), where \( U(f_3)_+ := \ker(\varepsilon : U(f_3) \rightarrow k) \) is the augmentation ideal of \( U(f_3) \), that is

\[ J(\text{pr}_5) = \text{im}(U(p_4) \otimes U(f_3)_+ \xrightarrow{\text{codec}} U(p_5)). \]
Since the $e_{i5}$, $i \in [1,3]$, belong to $f_3$, the following diagram commutes
\[ (5.2.3) \quad U(p_4) \otimes U(f_3)^{\otimes 3} \xrightarrow{\text{codec}^{\otimes 3}} U(p_4) \otimes U(f_3) \xrightarrow{\text{codec}} U(p_5)^{\otimes 3} \xrightarrow{\text{codec}} U(p_5) \]
where the lower horizontal map is given by $(p_i)_{i \in [1,3]} \mapsto \sum_{i \in [1,3]} p_i e_{i5}$, and the upper horizontal map is given by the tensor product the the identity in $U(p_4)$ with the map $U(f_3)^{\otimes 3} \to U(f_3)$, $(\varphi_i)_{i \in [1,3]} \mapsto \sum_{i \in [1,3]} \varphi_i e_{i5}$.

Since $U(f_3)$ is freely generated, as an associative algebra, by the $e_{i5}$, $i \in [1,3]$, the latter map corestricts to a $k$-module isomorphism $U(f_3)^{\otimes 3} \to U(f_3)_+$. It follows that the upper horizontal map of $(5.2.3)$ corestricts to an isomorphism $U(p_4) \otimes U(f_3)^{\otimes 3} \to U(p_4) \otimes U(f_3)_+$.

Since the vertical maps of $(5.2.3)$ are isomorphisms, this implies that the lower horizontal map of $(5.2.3)$ corestricts to an isomorphism from $U(p_5)^{\otimes 3}$ to the image by codec of $U(p_4) \otimes U(f_3)_+$, which is $J(pr_3)$ according to $(5.2.2)$. \hfill \Box

5.2.2. A morphism $\varpi : U(p_5) \to M_3(U(p_5))$. Lemma 5.3 says that the hypothesis of Lemma 4.1 is satisfied in the following situation: $R = U(p_5)$, $J = J(pr_3)$, $d = 3$, $(j_a)_{a \in [1,d]} = (e_{i5})_{i \in [1,3]}$. We denote by
\[ \varpi : U(p_5) \to M_3(U(p_5)) \]
the algebra morphism given in this situation by Lemma 4.1. Then for $p \in U(p_5)$, $\varpi(p) = (a_{ij}(p))_{i,j \in [1,3]}$, and
\[ \forall i \in [1,3], \quad e_{i5}p = \sum_{j \in [1,3]} a_{ij}(p)e_{j5} \]
(equalities in $U(p_5)$).

5.2.3. Construction and properties of a morphism $\rho : U(f_2) \to M_3(U(f_2)^{\otimes 2})$. Define the algebra morphism
\[ (5.2.4) \quad \rho : U(f_2) \to M_3(U(f_2)^{\otimes 2}) \]
to be the composition
\[ U(f_2) \xrightarrow{\ell} U(p_5) \xrightarrow{\varpi} M_3(U(p_5)) \xrightarrow{M_3(pr_{12})} M_3(U(f_2)^{\otimes 2}), \]
where $\ell$ is as in $(5.1.2)$ $\varpi$ is as in $(5.2.2)$ and $M_3(pr_{12})$ is the morphism induced by $pr_{12}$, i.e., taking $(p_{ij})_{i,j \in [1,3]}$ to $(pr_{12}(p_{ij}))_{i,j \in [1,3]}$.

Lemma 5.6. Set
\[ (5.2.5) \quad \text{row} := (e_1, -f_1, 0) \in M_{1 \times 3}(U(f_2)^{\otimes 2}), \quad \text{col} := \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \in M_{3 \times 1}(U(f_2)^{\otimes 2}) \]
(recall that $e_1, f_1 \in U(f_2)^{\otimes 2}$ are $e_1 \otimes 1, 1 \otimes e_1$), then

$$\rho(e_1) = \text{col} \cdot \text{row}$$

(equality in $M_3(U(f_2)^{\otimes 2})$).

**Proof.** One has $\ell(e_1) = e_{12}$. Let us compute $\varpi(e_{12})$. One has

$$e_{15}e_{12} = e_{12}e_{15} + [e_{15}, e_{12}] = e_{12}e_{15} + [e_{25}, e_{15}] = (e_{12} + e_{25})e_{15} - e_{15}e_{25},$$

$$e_{25}e_{12} = -e_{25}e_{15} + (e_{12} + e_{15})e_{25}$$

(applying the permutation of indices 1 and 2 to the previous equality),

$$e_{35}e_{12} = e_{12}e_{35},$$

which implies that

$$\varpi(e_{12}) = \begin{pmatrix} e_{12} + e_{25} & -e_{15} & 0 \\ -e_{25} & e_{12} + e_{15} & 0 \\ 0 & 0 & e_{12} \end{pmatrix} \in M_3(U(p)).$$

The image of this matrix in $M_3(U(f_2)^{\otimes 2})$ is

$$\rho(e_1) = \begin{pmatrix} e_1 & -f_1 & 0 \\ -e_1 & f_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{col} \cdot \text{row}.$$

Therefore $\rho(e_1) = \text{col} \cdot \text{row}$. \qed

5.2.4. **Construction and properties of a morphism $\tilde{\rho}: (U(f_2), \cdot e_1) \to U(f_2)^{\otimes 2}.$** Lemma 5.6 shows that the hypothesis of Lemma 4.3 is satisfied in the following situation: $R = U(f_2), S = U(f_2)^{\otimes 2}, e = e_1, n = 3, f = \rho, \text{row} \text{ and } \text{col}$ are as in Lemma 5.6. We denote by

$$\tilde{\rho}: (U(f_2), \cdot e_1) \to U(f_2)^{\otimes 2}$$

the algebra morphism given in this situation by Lemma 4.3.

Then for any $f \in U(f_2)$, one has

\begin{equation}
(5.2.6) \quad \tilde{\rho}(f) = \text{row} \cdot \rho(f) \cdot \text{col} = \text{row} \cdot \{M_3(pr_{12}) \circ \varpi \circ \ell(f)\} \cdot \text{col} \in U(f_2)^{\otimes 2}.
\end{equation}

**Lemma 5.7.** For any $n \geq 0$,

$$\tilde{\rho}(e_0^n) = e_1e_0^n + f_1f_0^n - \sum_{i=0}^{n-1} (e_1e_0^i) \cdot (f_1f_0^{n-1-i})$$

(equality in $U(f)^{\otimes 2}$).

**Proof.** According to (5.2.6), $\tilde{\rho}(e_0^n) = \text{row} \cdot \rho(e_0^n) \cdot \text{col}$. As $\rho$ is an algebra morphism, $\rho(e_0^n) = \rho(e_0)^n$. Then $\rho(e_0) = M_3(pr_{12}) \circ \varpi \circ \ell(e_0) = M_3(pr_{12})(\varpi(e_{23})).$
Let us compute $\varpi(e_{23})$. One has
\begin{align*}
e_{15}e_{23} &= e_{23}e_{15}, \\
e_{25}e_{23} &= e_{23}e_{25} + [e_{25}, e_{23}] = e_{23}e_{25} + [e_{35}, e_{25}] = (e_{23} + e_{35})e_{25} - e_{25}e_{35}, \\
e_{35}e_{23} &= -e_{35}e_{25} + (e_{23} + e_{25})e_{35}
\end{align*}
(applying the permutation of indices 2 and 3 to the previous equality),

which implies that

\[
\varpi(e_{23}) = \begin{pmatrix} e_{23} & 0 & 0 \\
0 & e_{23} + e_{35} & -e_{25} \\
0 & -e_{35} & e_{23} + e_{25} \end{pmatrix} \in M_3(U(p_5)).
\]

Then

\[
\rho(e_0) = M_3(pr_{12})(\varpi(e_{23})) = \begin{pmatrix} e_0 & 0 & 0 \\
0 & -e_1 + f_0 & -e_1 \\
0 & e_0 + e_1 - f_0 & e_0 + e_1 \end{pmatrix} \in M_3(U(f_2) \otimes 2).
\]

Set $T := (-e_1 + f_0 \ e_0 + e_1, -e_1 \ e_0 + e_1) \in M_2(U(f_2) \otimes 2)$, then $\rho(e_0^n) = \begin{pmatrix} e_0^n & 0 \\
0 & T^n \end{pmatrix}$, therefore

\[
\tilde{\rho}(e_0^n) = \text{row} \cdot \rho(e_0^n) \cdot \text{col} = e_1 e_0^n + (-f_1 \ 0) T^n \begin{pmatrix} -1 \\
0 \end{pmatrix},
\]

where the last equality follows from the form of row and col.

One checks that $T = \begin{pmatrix} 1 & 0 \\
-1 & 1 \end{pmatrix} \begin{pmatrix} f_0 & -e_1 \\
0 & e_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\
-1 & 1 \end{pmatrix}^{-1}$, therefore

\[
T^n = \begin{pmatrix} 1 & 0 \\
-1 & 1 \end{pmatrix} \begin{pmatrix} f_0 & -e_1 \\
0 & e_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\
1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\
-1 & 1 \end{pmatrix} \begin{pmatrix} f_0^n & -\sum_{i=0}^{n-1} f_0^i e_0^{n-1-i} \\
0 & e_0^n \end{pmatrix} \begin{pmatrix} 1 & 0 \\
1 & 1 \end{pmatrix},
\]

so

\[
\tilde{\rho}(e_0^n) = e_1 e_0^n + (-f_1 \ 0) \left( f_0^n - \sum_{i=0}^{n-1} f_0^i e_0^{n-1-i} \right) \begin{pmatrix} -1 \\
-1 \end{pmatrix} = e_1 e_0^n + f_1 f_0^n - \sum_{i=0}^{n-1} f_1 f_0^i e_0^{n-1-i};
\]

the result then follows from the commutativity of $e_s$ with $f_t$ for $s, t \in \{0, 1\}$.  

\begin{remark}
If $A$ is a unital associative algebra, then the following identity holds in $M_2(A)$
\[
(5.2.7) \quad \text{Ad}(\begin{pmatrix} 1 & 0 \\
a & 1 \end{pmatrix})\begin{pmatrix} u & v \\
0 & w \end{pmatrix} = \text{Ad}(\begin{pmatrix} 1 & a^{-1} \\
0 & 1 \end{pmatrix})\begin{pmatrix} a^{-1}w & 0 \\
au - wa - ava & aua^{-1} \end{pmatrix}
\]

provided $u, v, w \in A$ and $a \in A^\times$. This implies the identity

\[
T = \begin{pmatrix} 1 & -1 \\
0 & 1 \end{pmatrix} \begin{pmatrix} e_0 & 0 \\
e_0 + e_1 - f_0 & f_0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\
0 & 1 \end{pmatrix}^{-1},
\]

allowing for an alternative computation of $T^n$.
\end{remark}

5.3. Relationship between infinitesimal braid Lie algebras and $\Delta_{i,r}^l$. 


5.3.1. **Relationship between \( \tilde{\rho} \) and \( \Delta_{\ast}^{L,L} \).** Denote by \( k[e_0] \) the linear span in \( U(f_2) \) of the elements \( e_0^n, n \geq 0 \).

**Lemma 5.9.** \((U(f_2), \cdot_{e_1})\) is generated, as an associative (non-unital) algebra, by \( k[e_0] \).

**Proof.** For \( k \geq 1 \), the \( k \)-th power of the canonical injection, followed by the \( k \)-th fold product \( \cdot_{e_1} \), sets up a linear map \( k[e_0]^{\otimes k} \to U(f_2) \). One checks that this map is injective and that its image coincides with the part of \( U(f_2) \) of \( e_1 \)-degree equal to \( k - 1 \). So the composition \( \oplus_{k \geq 1} k[e_0]^{\otimes k} \to \oplus_{k \geq 1} U(f_2)^{\otimes k} \to U(f_2) \), where the first map is the canonical injection and the second map is the iteration of the product \( \cdot_{e_1} \), maps \( \oplus_{k \geq 1} k[e_0]^{\otimes k} \) injectively (in fact, bijectively) to \( U(f_2) \).

Recall that \( \mathcal{W}_{i,DR} \) is the subalgebra of \( U(f_2) \) equal to \( k \oplus U(f_2)e_1 \) (see §3.3). We set

\[
(\mathcal{W}^{DR}_{i})_+ := U(f_2)e_1.
\]

This is a (non-unital) subalgebra of \( \mathcal{W}^{DR}_{i} \).

Since the right multiplication by \( e_1 \) is injective in \( U(f_2) \), the algebra morphism

\[
\text{mor}^{U(f_2),e_1}_{U(f_2),e_1} : (U(f_2), \cdot_{e_1}) \to (\mathcal{W}^{DR}_{i})_+
\]

(see §1) is an algebra isomorphism.

**Lemma 5.10.** The following diagram is commutative

\[
\begin{array}{ccc}
(U(f_2), \cdot_{e_1}) & \xrightarrow{\tilde{\rho}} & U(f_2)^{\otimes 2} \\
\text{mor}^{U(f_2),e_1}_{U(f_2),e_1} \downarrow & & \downarrow \Delta_{\ast}^{L,L} \\
(\mathcal{W}^{DR}_{i})_+ & \xrightarrow{(\mathcal{W}^{DR}_{i})_+^{\otimes 2}} (\mathcal{W}^{DR}_{i})_+
\end{array}
\]

where \( \Delta_{\ast}^{L,L} \) is as in §3.1.6 and the inclusion \( \mathcal{W}^{DR}_{r} \hookrightarrow U(f_2) \) is as in §3.1.1.

**Proof.** For \( n \geq 1 \),

\[
\Delta_{\ast}^{L,L} \circ \text{mor}^{U(f_2),e_1}_{U(f_2),e_1}(e_0^n) = \Delta_{\ast}^{L,L}(e_0^n e_1) = \text{Ad}(e_1)^{\otimes 2} \circ \Delta_{\ast}(-y_{n+1})
\]

\[
= \text{Ad}(e_1)^{\otimes 2}(-1 \otimes y_{n+1} - \sum_{i=1}^{n} y_i \otimes y_{n+1-i})
\]

\[
= e_1 e_0^n \otimes 1 + 1 \otimes e_1 e_0^n - \sum_{i=1}^{n} e_1 e_0^{n-1} \otimes e_1 e_0^{n-i} = \tilde{\rho}(e_0^n),
\]

where the second equality follows from Def. 3.10 the third equality follows from (1.1.3), the fourth equality follows from Lemma 5.4 and the last equality follows from Lemma 5.7.

It follows that the two maps of the above diagram agree on \( e_0^n, n \geq 1 \). Since these maps are algebra morphisms, and since the family \( e_0^n, n \geq 0 \) generates \( (U(f_2), \cdot_{e_1}) \) (see Lemma 5.9), this diagram commutes. \( \square \)
Proposition 5.11. The following diagram commutes

$$\begin{array}{ccccccc}
U(f_2) & \xrightarrow{\ell} & U(p_5) & \xrightarrow{\varpi} & M_3(U(p_5)) & \xrightarrow{M_3(pr_{12})} & M_3(U(f_2)^{\otimes 2}) & \xrightarrow{\text{row}(-)\cdot \text{col}} & U(f_2)^{\otimes 2} \\
\simeq & & \circ & & & & & & \\
(U(f_2), \cdot e_1) & \simeq & & & & & & & \Delta_{l,r}^{l,r} \\
\text{mor}_{U(f_2), e_1}^l & & & & & & & & (W_{l}^{\text{DR}})^{\otimes 2}
\end{array}$$

where $\ell, \text{pr}_{12}$ are as in §5.1.2, $\varpi$ is as in §5.2.2, $\Delta_{l,r}^{l,r}$ is as in §3.1.6; in this diagram, all the maps are algebra morphisms, except for the maps marked with $\circ$, which are only $k$-module morphisms.

Proof. This follows from the combination the commutative diagram from Lemma 5.10 with the specialization of the commutative diagram from Lemma 4.3. \qed

5.3.2. Completion of the diagram (5.3.3). Recall that $U(f_2)$ and $U(p_5)$ are graded $k$-algebras, and that $W_{l}^{\text{DR}}, W_{r}^{\text{DR}}$ are graded subalgebras of $U(f_2)$. These gradings equip all the $k$-algebras in the commutative diagram of Proposition 5.9 with natural gradings; in the case of $(U(f_2), \cdot e_1)$, we define the grading by $(U(f_2), \cdot e_1)_n := U(f_2)_{n-1}$.

Lemma 5.12. The commutative diagram (5.3.3) gives rise to a commutative diagram between the degree completions of its constituents.

Proof. All the maps in this commutative diagram have degree 0, except for the maps row$(-)\cdot$col and $\text{mor}_{U(f_2), e_1}^l$ (marked with $\circ$), which have degree 1. These maps therefore induce maps between the degree completions of these algebras, and the completed diagram then commutes. \qed

6. RELATIONSHIP BETWEEN $\Delta_{p}$ AND PURE SPHERE BRAID GROUPS

The purpose of this section is to construct a commutative diagram relating $\Delta_{l,r}^{l,r}$ with braid groups (diagram (6.3.3)); this construction is inspired by that of §5.

In §6.1, we recall the definition of various families of braid groups (the Artin braid group, the sphere braid group and the modular group of the sphere with marked points), of their pure subgroups, and of a diagram of morphisms relating them (see (6.1.1)). We recall the presentation of these groups and relate various generators by morphisms (Lemma 6.3). We then give a presentation of the modular group $P_5^*$ which exhibits an order 5 cyclic symmetry, and may be viewed as an analogue of the presentation of $p_5$ in [1102], Proposition 4 (this presentation is not used is the sequel of the paper). In §6.1.2 we define morphisms $\ell : F_2 \to P_5^*$, $\text{pr}_{12} : P_5^* \to F_2$
(i ∈ [1, 5]), pr_{12}: P^*_5 → (F_2)^2 relating P^*_5 with the free group with two generators F_2 or its square.

In [0.2] we introduce an ideal J(pr_5) of the group algebra kP^*_5 arising from pr_5, and show its freeness as a left kP^*_5-module (Lemma 6.11). Lemma 4.1 then gives rise to an algebra morphism \( \varphi : kP^*_5 → M_3(kP^*_5) \). By composing \( \varphi \) with the morphisms \( l \) and \( pr_{12} \), we construct a morphism \( \rho : kF_2 → M_3((kF_2)^{\otimes 2}) \) (see (0.2.1)). In Lemma 6.12 by introducing suitable elements row \( ∈ M_{1×3}((kF_2)^{\otimes 2}) \) and col \( ∈ M_{3×1}((kF_2)^{\otimes 2}) \), we show that the morphism \( \rho \) satisfies the hypothesis of Lemma 4.3. Applying this lemma, we obtain in [6.2.4] a morphism \( \tilde{\rho} : (kF_2, x_1, -1) → (kF_2)^{\otimes 2} \), which we compute in Lemma 6.13.

In [6.3] we show the commutativity of a diagram relating \( ∆^{1,r} \) and \( \tilde{\rho} \) (Lemma 6.16), and derive from there the commutativity of a diagram relating \( ∆^{1,r} \) and \( \varphi, \rho, pr_{12} \) and row \( \cdot (−) \cdot \text{col} : M_{3×1}((kF_2)^{\otimes 2}) → (kF_2)^{\otimes 2} \) (Proposition 6.17 (6.3.3)). In the end of the subsection, we show that the constituents of this diagram are compatible with the filtrations, which implies the commutativity of the diagram between the completions of these maps.

6.1. Material on braid groups.

6.1.1. Braid groups. For X a topological space, let \( C_n(X) \) denote its configuration space of n distinct points. Let also \( \mathfrak{M}_{0,n+1} \) be the moduli space of smooth complex projective curves of genus zero with \( n + 1 \) marked points. Below, we give a list of topological spaces and simply-connected subspaces, together with standard names and notation for the corresponding fundamental groups (see [Br, Ih1, LoS]).

| space X | \( C_n(\mathbb{C}) \) | \( C_n(\mathbb{P}^1_{\overline{\mathbb{C}}}) \) | \( C_n(\mathbb{P}^1_{\overline{\mathbb{C}}})/\text{PGL}_2(\mathbb{C}) \simeq \mathfrak{M}_{0,n} \) |
|---------|-----------------|-----------------|-----------------|
| subspace b | \( U_n \) | \( U_n \) | \( B^*_n \) |
| notation for \( π_1(X,b) \) | \( K_n \) | \( P_n \) | \( P_n \) |
| name of \( π_1(X,b) \) | pure Artin braid group | pure sphere (Hurwitz) braid group | pure modular group of the sphere with \( n \) marked points |

| space X | \( C_n(\mathbb{C})/S_n \) | \( C_n(\mathbb{P}^1_{\overline{\mathbb{C}}})/S_n \) | \( C_n(\mathbb{P}^1_{\overline{\mathbb{C}}})/(S_n × \text{PGL}_2(\mathbb{C})) \simeq \mathfrak{M}_{0,n}/S_n \) |
|---------|-----------------|-----------------|-----------------|
| subspace b | \( S_n ∦ U_n \) | \( S_n ∞ U_n \) | \( S_n ∞ B^*_n \) |
| notation for \( π_1(X,b) \) | \( B_n \) | \( H_n \) | \( B^*_n \) |
| name of \( π_1(X,b) \) | Artin braid group | sphere (Hurwitz) braid group | modular group of the sphere with \( n \) marked points |

Here we set \( U_n := \{ (x_1, \ldots, x_n) ∈ \mathbb{R}^n| x_1 < \cdots < x_n \} ⊂ C_n(\mathbb{C}) \); we define \( \hat{B}_n ⊂ C_n(\mathbb{P}^1_{\overline{\mathbb{C}}}) \) as the set of \( n \)-tuples \((x_1, \ldots, x_n) \) in \( \mathbb{R}^n \), cyclically ordered when viewed as a \( n \)-tuple of \( \mathbb{P}^1_{\overline{\mathbb{C}}} \); it is stable under the action of \( \text{PGL}_2(\mathbb{C}) \); we define \( B_n \) as the quotient \( \hat{B}_n/\text{PGL}_2(\mathbb{C}) \); it is a simply-connected subspace of \( C_n(\mathbb{P}^1_{\overline{\mathbb{C}}})/\text{PGL}_2(\mathbb{C}) \).
By homotopy exact sequence, the pure groups appear as the kernels of the natural morphisms of their non-pure counterparts to $S_n$, so

$$K_n = \text{Ker}(B_n \to S_n), \quad P_n = \text{Ker}(H_n \to S_n), \quad P_n^* = \text{Ker}(B_n^* \to S_n).$$

According to \[LoS\], Appendix, $B_n^*$ is isomorphic to the quotient of the Artin braid group $B_n$ by the normal subgroup generated by $\eta_n$, which corresponds to the winding of $x_n$ around $(x_1, \ldots, x_{n-1})$ (see (6.1.1) for an expression in terms of standard generators), and by its center $Z(B_n)$, which is isomorphic to $\mathbb{Z}$. One has $Z(B_n) \subset K_n$, and $P_{n+1}^* \simeq K_n/Z(B_n)$. One also has $P_n \simeq P_n^* \times C_2$, where $C_2$ is the cyclic group of order 2 (see \[LoS\], Proposition A4 iii) and also \[In\], Corollary 2.1.2).

**Remark 6.1.** The isomorphism $P_{n+1}^* \simeq K_n/Z(B_n)$ can be interpreted as follows. There is an isomorphism $\mathfrak{m}_{0,n+1} \simeq C_n(\mathbb{C})/\text{Aff}$, with $\text{Aff} = \{x \mapsto ax + b \mid a \in \mathbb{C}^\times, b \in \mathbb{C}\}$; it gives rise to a homotopy exact sequence $\pi_2(\mathfrak{m}_{0,n+1}) \to \pi_1(\text{Aff}) \to \pi_1(C_n(\mathbb{C})) \to \pi_1(\mathfrak{m}_{0,n+1}) \to 1$. The spaces $\mathfrak{m}_{0,n}$ are $K(\pi,1)$-spaces, as can be seen inductively from the homotopy exact sequences of the fibrations $\mathfrak{m}_{0,n+1} \to \mathfrak{m}_{0,n}$, therefore $\pi_2(\mathfrak{m}_{0,n+1}) = 1$. One has $K_n = \pi_1(C_n(\mathbb{C}))$, $P_{n+1}^* = \pi_1(\mathfrak{m}_{0,n+1})$ and $\pi_1(\text{Aff}) = \mathbb{Z}$, so the above exact sequence implies $P_{n+1}^* \simeq K_n/\mathbb{Z}$.

**Remark 6.2.** The isomorphism $P_n \simeq P_n^* \times C_2$ implies, in the notation of Remark 5.1, the isomorphism $\text{gr}(P_n) \simeq \text{gr}(P_n^*) \times C_2$, therefore $\text{gr}(P_n^*) \otimes \mathfrak{k} \simeq \text{gr}(P_n) \otimes \mathfrak{k} \simeq \mathfrak{p}_n$ as $\mathbb{Q} \subset \mathfrak{k}$.

A diagram of pure braid groups. The canonical projection $C_{n+1}(\mathbb{C}) \to C_{n+1}(\mathbb{P}_1(\mathbb{C}))/\text{PGL}_2(\mathbb{C})$ defines a morphism of topological spaces; this map takes $U_{n+1}$ to $B_{n+1}$, therefore induces a group morphism $K_{n+1} \to P_{n+1}^*$.

Define a morphism $C_n(\mathbb{C}) \to C_{n+1}(\mathbb{P}_1(\mathbb{C}))/\text{PGL}_2(\mathbb{C})$ to be the map taking $(x_1, \ldots, x_n)$ to the class of $(x_1, \ldots, x_n, \infty)$. This map takes $U_n$ to $B_{n+1}$, therefore induces a group morphism $K_n \to P_{n+1}^*$.

Let $D$ be the open unit disc in $\mathbb{C}$, let $C_n^D(\mathbb{C})$ (resp. $U_n^D$) be the intersection of $C_n(\mathbb{C})$ (resp. $U_n$) with $D^n$. Then $(C_n^D(\mathbb{C}), U_n^D)$ is a deformation retract of $(C_n(\mathbb{C}), U_n)$, therefore $K_n \simeq \pi_1(C_n^D(\mathbb{C}), U_n^D)$. The morphism $C_n^D(\mathbb{C}) \to C_{n+1}(\mathbb{C})$ given by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 2)$ takes $U_n^D$ to $U_{n+1}$, and therefore induces a morphism $K_n \to K_{n+1}$.

Then the diagram

$$(6.1.1)$$

$\begin{array}{ccc}
K_n & \xrightarrow{P_n^*} & P_{n+1} \\
\downarrow & & \downarrow \\
K_{n+1} & \end{array}$$

commutes.

Presentations of braid groups. According to \[Ar\], the group $B_n$ is presented by generators $\sigma_1, \ldots, \sigma_{n-1}$, subject to relations $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| > 1$, and $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $i \in [1, n-1]$. The morphism $B_n \to S_n$ is then given by $\sigma_i \mapsto (i, i+1)$. 


For \( i < j \in [1, n] \), set
\[
\hat{x}_{ij} := (\sigma_{j-2} \cdots \sigma_1)^{-1} \sigma_{j-1}^2 (\sigma_{j-2} \cdots \sigma_i) = (\sigma_{j-1} \cdots \sigma_{i+1}) \sigma_i^2 (\sigma_{j-1} \cdots \sigma_{i+1})^{-1} \in B_n.
\]
Then one checks that \( \hat{x}_{ij} \in K_n \). According to [Ar], a presentation of \( K_n \) is given by generators \( \hat{x}_{ij}, i < j \in [1, n] \), subject to relations
\[
(a_{ijk}, \hat{x}_{ij}) = (a_{ijk}, \hat{x}_{ik}) = (a_{ijk}, \hat{x}_{jk}) = 1
\]
for \( i < j < k \) and \( i, j, k \in [1, n] \), where \( a_{ijk} = \hat{x}_{ij} \hat{x}_{ik} \hat{x}_{jk} \), together with
\[
(\hat{x}_{ij}, \hat{x}_{kl}) = (\hat{x}_{ik}, \hat{x}_{ij}^{-1} \hat{x}_{jl} \hat{x}_{ij}) = (\hat{x}_{il}, \hat{x}_{jk}) = 1.
\]
for \( i < j < k < l \) and \( i, j, k, l \in [1, n] \).

Set
\[
\omega_n := \hat{x}_{12} \cdot (\hat{x}_{13} \hat{x}_{23}) \cdots (\hat{x}_{1n} \cdots \hat{x}_{n-1,n}) \in K_n.
\]
Then \( \omega_n \) is a generator of \( Z(B_n) \), and \( P_{n+1}^* = K_n/\langle \omega_n \rangle \). For \( i < j \in [1, n] \), we denote by \( x_{ij} \in P_{n+1}^* \) the image of \( \hat{x}_{ij} \in K_n \) by the natural projection.

For \( i \in [1, n] \), define \( x_{i,i+1} \in P_{n+1}^* \) by
\[
x_{i,i+1} := (x_{ii+1})^{-1}.
\]

Lemma 6.3. If \( i < j \in [1, n+1] \), then \( x_{ij} \in P_{n+1}^* \) is the image if \( \hat{x}_{ij} \in K_{n+1} \) under the morphism \( K_{n+1} \to P_{n+1}^* \).

Proof. If \( j \leq n \), then the image of \( \hat{x}_{ij} \in K_n \) under \( K_n \to P_{n+1}^* \) is \( x_{ij} \in P_{n+1}^* \), and the image of the same element under \( K_{n+1} \to K_{n+1} \) is \( \hat{x}_{ij} \in K_{n+1} \), so the result follows from the commutativity of (6.1.1).

Assume now that \( j = n+1 \). According to [LoS], three lines after (A1), one has
\[
(\sigma_{i} \cdots \sigma_{n-1} \cdot \eta_{n+1}^{-1} \cdot (\sigma_{i} \cdots \sigma_{n})^{-1} = \hat{x}_{1i} \cdots \hat{x}_{i-1,i} \hat{x}_{i,i+1} \cdots \hat{x}_{i,n+1}
\]
(equality in \( B_{n+1} \)). Moreover, one checks that for \( i \in [1, n] \),
\[
(\sigma_{i} \cdots \sigma_{n} \cdot \eta_{n+1}^{-1} \cdot (\sigma_{i} \cdots \sigma_{n})^{-1} = \hat{x}_{1i} \cdots \hat{x}_{i-1,i} \hat{x}_{i,i+1} \cdots \hat{x}_{i,n+1}
\]
(equality in \( B_{n+1} \)). The left-hand side belongs to the kernel of the morphism \( B_{n+1} \to B_{n+1}^* \), and the right-hand side belongs to the subgroup \( K_{n+1} \subset B_{n+1} \), therefore the latter side belongs to the kernel of the composed morphism \( K_{n+1} \to B_{n+1}^* \). As this morphism factors as \( K_{n+1} \to P_{n+1}^* \to B_{n+1}^* \), the right-hand side belongs to \( \text{Ker}(K_{n+1} \to P_{n+1}^*) \).

The following result may be viewed as an analogue of Proposition 4 in [Il2].

Proposition 6.4. For \( i \in C_5 \simeq [1, 5] \), define \( g_i \in P_5^* \) by \( g_i := x_{i,i+1} \) (with the convention \( x_{0,1} := x_{1,5} \)). The group \( P_5^* \) is presented by generators \( g_i \) (\( i \in C_5 \)), subject to the relations
\[
(g_i, g_j) = 1 \text{ if } i, j \in C_5 \text{ and } i - j \neq \pm 1, \quad (g_0, g_1)(g_1, g_2)(g_2, g_3)(g_3, g_4)(g_4, g_5) = 1.
\]
Proof. Relation $\omega_5 = 1$, namely

(6.1.6) \[ x_{12}x_{13}x_{23}x_{14}x_{24}x_{34} = 1, \]

together with relation (6.1.4) for $(i, n) = (4, 4)$, implies $g_T = x_{12}x_{13}x_{23}$, therefore

(6.1.7) \[ x_{13} = g_T^{-1}g_T^{-1}. \]

Relation (6.1.6) together with $(x_{14}, x_{23}) = 1$ yields $x_{12}x_{13}x_{14}x_{23}x_{24}x_{34} = 1$, which together with (6.1.4) for $(i, n) = (1, 4)$, yields $x_{15} = x_{23}x_{24}x_{34}$. This relation yields

(6.1.8) \[ x_{24} = g_T^{-1}g_T^{-1}. \]

By the commutation of $x_{34}$ with $x_{23}x_{24}x_{34}$, this relation also yields $g_T = x_{15} = x_{24}x_{34}x_{23} = x_{24}x_{34}g_T$, therefore $x_{34}^{-1}x_{24}^{-1} = g_T^{-1}$. (6.1.6) implies $x_{14} = x_{23}^{-1}x_{12}^{-1}x_{34}^{-1}x_{24}$, which after combination with the previous equality yields $x_{14} = x_{23}^{-1}x_{13}^{-1}x_{12}g_T^{-1} = g_T^{-1}x_{13}^{-1}g_T^{-1}g_T^{-1}$. Combining with (6.1.7), we obtain

(6.1.9) \[ x_{14} = g_T^{-1}g_T^{-1}. \]

As $P_5^c$ is a quotient of $K_4$ it is generated by $\{ x_{ij} \mid i < j \in [1, 4] \}$, therefore also by the union of this set with $g_T$, which is equal to $\{ g_i \mid i \in C_5 \} \cup \{ x_{13}, x_{24}, x_{14} \}$. Relations (6.1.7), (6.1.8) and (6.1.9) then imply that a generating set is $\{ g_i \mid i \in C_5 \}$.

The group $P_5^c$ may be viewed as generated by $\{ g_i \mid i \in C_5 \} \cup \{ x_{13}, x_{24}, x_{14} \}$, subject to relations (6.1.2), (6.1.3), $\omega_5 = 1$, and (6.1.4) for $(i, n) = (1, 4), (4, 4)$. These relations imply expressions (6.1.7), (6.1.8) and (6.1.9) of $x_{13}, x_{24}, x_{14}$ in terms of $\{ g_i \mid i \in C_5 \}$. Substituting these expressions in relations (6.1.2), (6.1.3), $\omega_5 = 1$, and (6.1.4) for $(i, n) = (1, 4), (4, 4)$, we obtain a presentation of $P_5^c$ in terms of the generators $\{ g_i \mid i \in C_5 \}$. The relations obtained in this way are the following. Relation $\omega_5 = 1$ yields the commutation of $g_T$ with $g_T$. Relation (6.1.2) for $(i, j, k) = (1, 2, 3)$ yields the commutation of $g_T$ with $g_T$ and $g_T$. Relation (6.1.2) for $(i, j, k) = (2, 3, 4)$ yields the commutation of $g_T$ with $g_T$. The first part of relation (6.1.3), namely $(x_{12}, x_{34}) = 1$, yields the commutation of $g_T$ with $g_T$. The last part of relation (6.1.3), namely $(x_{14}, x_{23}) = 1$, yields a consequence of the already obtained commutations of $g_T$ with $g_T$ and $g_T$. The middle part of relation (6.1.3), namely $(x_{13}, x_{24}^{-1}x_{24}x_{12}) = 1$, together with the commutations $(g_T, g_T) = (g_T, g_T) = 1$, yields the relation $g_T^{-1}g_T^{-1}g_T^{-1}g_T^{-1}g_T^{-1}g_T^{-1}g_T^{-1} = 1$, which by again using the commutation relations yields $g_T^{-1}g_T^{-1}g_T^{-1}g_T^{-1}g_T^{-1}g_T^{-1}g_T^{-1} = 1$, which is equivalent to the cyclic relation $(g_T, g_T)(g_T, g_T)(g_T, g_T)(g_T, g_T)(g_T, g_T) = 1$.

Relation (6.1.2) for $(i, j, k) = (1, 2, 4)$ splits as the conjunction of $x_{12}x_{14}x_{24} = x_{14}x_{24}x_{12}$ and $x_{14}x_{24}x_{12} = x_{24}x_{12}x_{14}$. The first relation yields a consequence of already obtained relations, namely $(g_T, g_T) = (g_T, g_T) = 1$. After using $(g_T, g_T) = (g_T, g_T) = 1$, the second relation yields $g_T^{-1}g_T^{-1}g_T^{-1}g_T^{-1}g_T^{-1}g_T^{-1} = 1$ which as above is equivalent to the already obtained cyclic relation.
Relation (6.1.2) for \((i, j, k) = (1, 3, 4)\) splits as the conjunction of \(x_{13}x_{14}x_{34} = x_{34}x_{13}x_{14}\) and \(x_{34}x_{13}x_{14} = x_{14}x_{34}x_{13}\). The first relation yields a consequence of the already obtained relations \((g_{j}, g_{i}) = (g_{j}, g_{i}) = (g_{j}, g_{i}) = 1\). After using \((g_{j}, g_{i}) = 1\) and the commutation of \(g_{i}\) with \(g_{j}^{-1}\), the second relation yields the already obtained relation \(g_{j}^{-1}g_{i}^{-1}g_{j}^{-1}g_{i}^{-1}g_{j}^{-1}g_{i}^{-1} = 1\).

**Remark 6.5.** Under the commutation relations, the cyclic relation \(\prod_{i \in C_{5}} (g_{i}, g_{i+1}) = 1\) (using the notation \(\prod_{i \in C_{5}} a_{i} := a_{i}a_{i+1} \cdots a_{j}a_{j+1}\)) is equivalent to any of the relations \(\prod_{i \in C_{5}} g_{i}g_{i+1}a_{i}^{-1} = 1\), where \(j \in C_{5} \setminus \{0, 5\}\). For \(j = 5\), this relation expresses as \(\prod_{i \in C_{5}} (g_{i}^{-1}, g_{i+1}) = 1\). The cyclic relation is also equivalent to the relation \(\prod_{i \in C_{5}} (g_{i-1}, g_{i-1}) = 1\). All this proves that the group \(D_{5} \times C_{2}\) acts by automorphisms of \(P_{5}^{2}\) as follows: the dihedral group \(D_{5}\) acts by permutation of indices of the generators \((g_{i})_{i \in C_{5}}\) and the cyclic group \(C_{2}\) acts by \(g_{i} \mapsto g_{i}^{-1}\).

**Diagrammatic representation.** The generators of \(B_{n}\) are depicted as follows when \(n = 4\),

\[
\sigma_1 = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}, \quad \sigma_2 = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}, \quad \sigma_3 = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\]

and the convention for the product is \(\sigma_2 \sigma_1 = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}\). The element \(\hat{x}_{ij} \in K_{n}\) is then depicted as follows

\[
\hat{x}_{ij} = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\]

6.1.2. The morphisms \(\ell, \text{pr}_{i}, \) and \(\text{pr}_{12}\) between braid groups. Let \(F_{2}\) be the free group with generators \(X_{0}, X_{1}\). Similarly to (1.7.3), we will abuse notation by setting for \(i = 0, 1\)

(6.1.10) \(X_{i} := (X_{i}, 1) \in (F_{2})^{2}, \quad Y_{i} := (1, X_{i}) \in (F_{2})^{2}\).

We will denote by \(X_{0}, X_{1}\) the elements \((X_{0}, 1)\) and \((X_{1}, 1)\), and by \(Y_{0}, Y_{1}\) its elements \((1, X_{0})\) and \((1, X_{1})\).

**Lemma 6.6.** 1) There are group morphisms \(\text{pr}_{i} : P_{5}^{2} \to F_{2}\) for \(i = 1, 2, 5\), given by

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
x \in P_{5}^{2} & x_{12} & x_{13} & x_{14} & x_{15} & x_{24} & x_{25} & x_{34} & x_{35} & x_{45} \\
\hline
\text{pr}_{1}(x) & 1 & 1 & 1 & 1 & X_{0} & (X_{0}X_{1})^{-1} & X_{1} & X_{1} & X_{0} \\
\hline
\text{pr}_{2}(x) & 1 & (X_{0}X_{1})^{-1} & X_{0} & X_{1} & 1 & 1 & X_{1} & (X_{0}X_{1})^{-1} & (X_{0}X_{1})^{-1} \\
\hline
\text{pr}_{5}(x) & X_{1} & (X_{0}X_{1})^{-1} & X_{0} & 1 & X_{0} & (X_{1}X_{0})^{-1} & 1 & X_{1} & 1 \\
\hline
\end{array}
\]

2) There is a group morphism \(\ell : F_{2} \to P_{5}^{2}\), given by \(X_{0} \mapsto x_{23}, X_{1} \mapsto x_{12}\). We have \(\text{pr}_{5} \circ \ell = \text{id}_{F_{2}}\).

**Proof.** Direct computation. \(\Box\)
Define \( \text{pr}_{12} : P^*_5 \to F^*_2 \) as the morphism \( p \mapsto (\text{pr}_1(p), \text{pr}_2(p)) \). We will denote by \( \underline{F}, \underline{pr} \) and \( \underline{pr}_{12} \) the Hopf algebra morphisms relating the group algebras \( kF_2, (kF_2)^{\oplus 2} \) and \( kP^*_5 \) induced by \( \underline{F}, \underline{pr} \) and \( \underline{pr}_{12} \).

**Remark 6.7.** The pure modular group of the sphere with 4 marked points \( P^*_4 \) is freely generated by \( x_{12}, x_{23} \), and therefore isomorphic to \( F_2 \). Composing with this isomorphism the morphisms \( \underline{pr} \) (resp. \( \underline{F} \)), one obtains the morphisms \( P^*_5 \to P^*_4 \) (resp. \( P^*_4 \to P^*_5 \)) induced by the morphisms between moduli spaces consisting in forgetting the \( i \)-th marked point (resp. doubling the fourth marked point).

### 6.2. Algebraic constructions related to an ideal of \( kP^*_5 \)

#### 6.2.1. The structure of \( J(\underline{pr}) \)

**Definition 6.8.** We denote by \( J(\underline{pr}) \) the kernel \( \text{Ker}(kP^*_5 \xrightarrow{\underline{pr}} (kF_2)^{\oplus 2}) \). This is a two-sided ideal of \( kP^*_5 \).

Let \( F_3 \) be the free group with generators \( a_i, i \in \{1, 3\} \); there is a unique group morphism \( F_3 \to P^*_5 \), given by \( a_i \mapsto x_{i5} \) for \( i \in \{1, 3\} \).

**Lemma 6.9.** 1) The morphisms \( \underline{pr}_5 \) and \( F_3 \to P^*_5 \) fit in an exact sequence \( 1 \to F_3 \to P^*_5 \to F_2 \to 1 \). As \( F_3 \to P^*_5 \) is injective, we will identify \( F_3 \) with its image in \( P^*_5 \).

2) The map \( F_2 \times F_3 \to P^*_5, (f, f') \mapsto \underline{f} f' \) is a bijection.

**Proof.** 1) follows from [Ih1], Proposition 2.1.3, based on [FvB]. 2) then follows from the fact that \( \underline{f} \) is a section of \( \underline{pr}_5 \). \( \square \)

**Lemma 6.10.** Let \( F_n \) be the free group with generators \( (a_i)_{i \in [1, n]} \) and let \( (kF_n)_+ \) be its augmentation ideal. The map \( (kF_n)^{\oplus n} \to (kF_n)_+, (f_i)_{i \in [1, n]} \mapsto \sum_{i \in [1, n]} f_i \cdot (a_i - 1) \) is an isomorphism of left \( kF_n \)-modules.

**Proof.** Let us prove surjectivity. The space \( (kF_n)_+ \) is \( k \)-linearly generated by the differences \( w - 1 \), where \( w \) in a word in \( (a_i^\epsilon)_{i \in [1, n], \epsilon \in \{\pm 1\}} \). Such a word \( w \) is a product \( a_{f(1)}^{\epsilon(1)} \cdots a_{f(k)}^{\epsilon(k)} \), for some \( k \geq 0 \) and some functions \( f : [1, k] \to [1, n] \) and \( \epsilon : [1, k] \to \{\pm 1\} \). Since

\[
a_{f(1)}^{\epsilon(1)} \cdots a_{f(k)}^{\epsilon(k)} - 1 = \sum_{i=1}^{k} \epsilon(i) a_{f(1)}^{\epsilon(1)} \cdots a_{f(i)}^{\epsilon(i)} a_{f(i)}^{\epsilon(i-1)} a_{f(i)}^{\epsilon(i)-1}/2 (a_{f(i)} - 1),
\]

\( w - 1 \) belongs to the image of \( (kF_n)^{\oplus n} \to (kF_n)_+ \).

Let us prove the injectivity of this map. Set \( I := (kF_n)_+ \). Then each side of this map is equipped with a separated descending filtration, given by \( \text{Fil}^k((kF_n)^{\oplus n}) := (I^k)^{\oplus n} \) for \( k \geq 0 \) for the left side, and by \( \text{Fil}^k((kF_n)_+) := I^{k+1} \). The map \( (kF_n)^{\oplus n} \to (kF_n)_+ \) is compatible with the filtrations of both sides. The associated graded spaces of degree \( k \) are \( k(A)^{\oplus n}_k \) for the left-hand side and \( k(A)_{k+1} \) for the right-hand side, where \( k(A) \) is the free algebra.
over generators $\alpha_1, \ldots, \alpha_n$, and the associated graded map $k(A)_k^\oplus \to k(A)_{k+1}$ is given by $(\varphi_i)_{i \in [1, n]} \mapsto \sum_{i \in [1, n]} \varphi_i \cdot \alpha_i$. As this map is injective, so is the map $(kF_n)^\oplus \to (kF_n)_+$. □

**Lemma 6.11.** The map $(kP_5^*)^{\oplus 3} \to J(\underline{pr}_a), (\underline{p}_i)_{i \in [1, 3]} \mapsto \sum_{i \in [1, 3]} p_i \cdot (x_{i5} - 1)$ is an isomorphism of left $kP_5^*$-modules.

**Proof.** The bijection from Lemma [6.9] 2) induces a linear isomorphism $kF_2 \otimes kF_3 \xrightarrow{\sim} kP_5^*$. For $(f, f') \in F_2 \times F_3$, $\underline{pr}_a((f) \cdot f') = f$ as $\underline{pr}_a \circ \underline{f} = \text{id}_{F_2}$ and $F_3 = \text{Ker}(\underline{pr}_a)$. It follows that the diagram

$$
\begin{array}{ccc}
\text{kF}_2 \otimes \text{kF}_3 & \xrightarrow{\sim} & \text{kP}_5^* \\
\text{id} \otimes \epsilon & \downarrow & \text{pr}_5 \\
\text{kF}_2 & \text{pr}_5 & \\
\end{array}
$$

It follows that $J(\underline{pr}_a)$ is the image of $kF_2 \otimes (kF_3)_+$ under the linear isomorphism $kF_2 \otimes kF_3 \xrightarrow{\sim} kP_5^*$.

Since $x_{i5} \in F_3$ for $i \in [1, 3]$, the diagram

$$
\begin{array}{ccc}
(kF_2 \otimes kF_3)^{\oplus 3} & \xrightarrow{\sim} & (kP_5^*)^{\oplus 3} \\
\downarrow & & \downarrow \\
(kF_2 \otimes kF_3) & \xrightarrow{\sim} & kP_5^* \\
\end{array}
$$

commutes, where the vertical maps are, on the left-hand side, the tensor product of the id$_{kF_2}$ with $kF_3^{\oplus 3} \to kF_3, (\underline{f})_{i \in [1, 3]} \mapsto \sum_{i \in [1, 3]} f_i \cdot (x_{i5} - 1)$, and on the right-hand side, the map given by the same formula.

It follows from Lemma [6.11] that $kF_2 \otimes (kF_3)_+$ is the isomorphic image of the left vertical map, therefore $J(\underline{pr}_a)$ is the isomorphic image of $(kP_5^*)^{\oplus 3}$ by the right vertical map, which proves the claimed statement. □

6.2.2. A morphism $\varpi : kP_5^* \to M_3(kP_5^*)$. Lemma [6.11] says that the hypothesis of Lemma 4.1 is satisfied in the following situation: $R = kP_5^*, J = J(\underline{pr}_a), d = 3, (j_a)_{a \in [1, d]} = (x_{i5} - 1)_{i \in [1, 3]}$. We denote by

$$
\varpi : kP_5^* \to M_3(kP_5^*)
$$

the algebra morphism given in this situation by Lemma 4.1. Then for $\underline{p} \in kP_5^*, \varpi(\underline{p}) = (a_{ij}(\underline{p}))_{i,j \in [1, 3]}$, and

$$
\forall i \in [1, 3], \quad (x_{i5} - 1)p = \sum_{j \in [1, 3]} a_{ij}(\underline{p})(x_{j5} - 1)
$$

(equalities in $kP_5^*$).
6.2.3. Construction and properties of a morphism \((kF_2, \cdot, X, -1) \to (kF_2)^{\otimes 2}\). Define the algebra morphism

\[ \rho : kF_2 \to M_3((kF_2)^{\otimes 2}) \]

to be the composition

\[ kF_2 \xrightarrow{\ell} U(p_3) \xrightarrow{\varpi} M_3(kP_3^2) \xrightarrow{M_3 (pr_1^2)} M_3((kF_2)^{\otimes 2}), \]

where \(\ell\) is as in (5.1.2), \(\varpi\) is as in (5.2.2), and \(M_3(pr_{12})\) is the morphism induced by \(pr_{12}\), i.e., taking \((p_{ij})_{i,j \in [1,3]}\) to \((pr_{12}(p_{ij}))_{i,j \in [1,3]}\).

**Lemma 6.12.** Set

\[ \text{row} := (X_1 - 1 \ 1 - Y_1 \ 0) \in M_{1 \times 3}((kF_2)^{\otimes 2}), \quad \text{col} := \begin{pmatrix} Y_1 \\ -1 \\ 0 \end{pmatrix} \in M_{3 \times 1}((kF_2)^{\otimes 2}) \]

(where \(X_1, Y_1 \in F_2^2 \subset (kF_2)^{\otimes 2}\) are defined by (6.1.10), then

\[ \rho(X_1 - 1) = \text{col} \cdot \text{row} \]

(equality in \(M_3((kF_2)^{\otimes 2})\)).

**Proof.** One has \(\ell(X_1) = x_{12}\). Let us compute \(\varpi(x_{12})\). One has

\[(x_{15} - 1)x_{12} = (x_{15} - 1)x_{12}x_{15}x_{25}x_{25}^{-1}x_{15}^{-1} = x_{12}x_{15}x_{25}(x_{15} - 1)x_{25}^{-1}x_{15}^{-1} = x_{12}x_{15}x_{25}(-x_{15}x_{25}^{-1}x_{15}^{-1} + x_{25}^{-1}x_{15}^{-1})(x_{15} - 1) + x_{12}x_{15}x_{25}(1 - x_{15})x_{25}^{-1}(x_{25} - 1), \]

\[(x_{25} - 1)x_{12} = (x_{25} - 1)x_{12}x_{15}x_{25}x_{25}^{-1}x_{15}^{-1} = x_{12}x_{15}x_{25}(x_{25} - 1)x_{25}^{-1}x_{15}^{-1} = x_{12}x_{15}x_{25}(x_{25}^{-1} - 1)x_{15}^{-1}(x_{15} - 1) + x_{12}x_{15}(x_{25} - 1), \]

\[(x_{35} - 1)x_{12} = x_{12}(x_{35} - 1), \]

which implies that

\[ \varpi(x_{12}) = \begin{pmatrix} x_{12}x_{15}x_{25}(x_{25}^{-1} - 1)x_{15}^{-1} & 0 & x_{12}x_{15}(1 - x_{15})x_{25}^{-1} \\ x_{12}x_{15}x_{25}(-x_{15}x_{25}^{-1}x_{15}^{-1} + x_{25}^{-1}x_{15}^{-1}) & x_{12}x_{15}x_{25}(1 - x_{15})x_{25}^{-1} & 0 \\ 0 & x_{12}x_{15} & 0 \end{pmatrix} \in M_3(kP_3^2). \]

The image of \(\varpi(x_{12} - 1)\) in \(M_3((kF_2)^{\otimes 2})\) by \(M_3(pr_{12})\) is therefore

\[ \begin{pmatrix} (X_1 - 1)Y_1 & Y_1(1 - Y_1) & 0 \\ 1 - X_1 & Y_1 - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (X_1 - 1 \ 1 - Y_1 \ 0) \begin{pmatrix} Y_1 \\ -1 \\ 0 \end{pmatrix} = \text{col} \cdot \text{row}. \]

Therefore \(\rho(X_1 - 1) = \text{col} \cdot \text{row}. \)
6.2.4. Construction and properties of a morphism $\tilde{\rho} : (kF_2, \cdot, X_{1-1}) \to (kF_2)^{\otimes 2}$. Lemma 6.12 shows that the hypothesis of Lemma 6.13 is satisfied in the following situation: $R = kF_2$, $S = (kF_2)^{\otimes 2}$, $e = X_1 - 1$, $n = 3$, $f = \rho$, row and col are row and col from Lemma 6.12. We denote by

$$\tilde{\rho} : (kF_2, \cdot, X_{1-1}) \to (kF_2)^{\otimes 2}$$

the algebra morphism given in this situation by Lemma 6.13.

Then for any $f \in kF_2$, one has

$$\tilde{\rho}(f) = \text{row} \cdot \rho(f) \cdot \text{col} = \text{row} \cdot \{M_3(\text{pr}_{12}) \circ \varpi \circ \xi(f)\} \cdot \text{col} \in (kF_2)^{\otimes 2}.$$  

**Lemma 6.13.** For any $k \in \mathbb{Z}$,

$$\tilde{\rho}(X_0^k) = (X_1 - 1)X_0^k \otimes 1 + 1 \otimes (1 - X_1^{-1})X_0^kX_1 - \sum_{i=1}^{k-1} (X_1 - 1)X_0^i \otimes (1 - X_1^{-1})X_0^{k-i}X_1,$$

and

$$\tilde{\rho}(X_0^kX_1^{-1}) = \tilde{\rho}(X_0^k)(X_1^{-1} \otimes X_1^{-1})$$

(= \text{equalities in } (kF_2)^{\otimes 2}).

**Proof.** According to (6.2.3), $\tilde{\rho}(X_0^k) = \text{row} \cdot \rho(X_0^k) \cdot \text{col}$. As $\rho$ is an algebra morphism, $\rho(X_0^k) = \rho(X_0)^k$. Then $\rho(X_0) = M_3(\text{pr}_{12}) \circ \varpi \circ \xi(X_0) = M_3(\text{pr}_{12})(\varphi(x_{23})).$

Let us compute $\varphi(x_{23})$.

As $x_{15}$ commutes with $x_{23}$, one has

$$(x_{15} - 1) \cdot x_{23} = x_{23} \cdot (x_{15} - 1).$$

Since $x_{25}$ commutes with $x_{23}x_{25}x_{35}$, one has

$$x_{23}^{-1}x_{25}x_{23} = x_{23}^{-1}x_{25}x_{25}^{-1}x_{35}^{-1}x_{25}$$

which implies the first equality in the following chain of equalities

$$x_{23}^{-1}(x_{25} - 1)x_{23} = x_{25}x_{35}x_{25}x_{35}^{-1}x_{25}^{-1} - 1 = x_{25}(x_{35}x_{25}x_{35}^{-1}x_{25}^{-1} - 1) + x_{25} - 1$$

$$= x_{25}\{x_{35}(x_{25}x_{35}^{-1}x_{25}^{-1} - 1) + x_{35} - 1\} + x_{25} - 1$$

$$= x_{25}[x_{35}x_{25}x_{35}^{-1}x_{25}^{-1} - 1 + x_{35} - 1] + x_{35} - 1 + x_{25} - 1$$

$$= x_{25}(x_{35}[x_{35}x_{25}x_{35}^{-1}x_{25}^{-1} - 1 + x_{35} - 1] + x_{25} - 1) + x_{35} - 1 + x_{25} - 1$$

$$= x_{25}x_{25}x_{35}x_{25}^{-1}x_{25}^{-1} - 1 + x_{25}x_{35}x_{25}(x_{35}^{-1} - 1) + x_{25}x_{35}(x_{25} - 1) + x_{25}(x_{35} - 1) + x_{25} - 1$$

$$= (-x_{25}x_{35}x_{25}^{-1}x_{25}^{-1} + x_{25}x_{35} + 1)(x_{25} - 1) + (-x_{25}x_{35}x_{25}x_{35}^{-1} + x_{25})(x_{35} - 1),$$

the following equalities being immediate. Therefore

$$x_{25}^{-1}x_{23} = x_{25}(-x_{25}x_{35}x_{25}x_{35}^{-1}x_{25}^{-1} + x_{25}x_{35} + 1)(x_{25} - 1) + x_{23}(-x_{25}x_{35}x_{25}x_{35}^{-1} + x_{25})(x_{35} - 1).$$
Since $x_{23}x_{25}x_{35} = x_{35}x_{23}x_{25}$, one has $x_{23}^{-1}x_{35}x_{23} = x_{25}x_{35}x_{25}^{-1}$, which implies the first equality in the following chain of equalities

\begin{align*}
x_{23}^{-1}(x_{35} - 1)x_{23} &= x_{25}x_{35}x_{25}^{-1} - 1 = x_{25}(x_{35}x_{25}^{-1} - 1) + x_{25} - 1 \\
x_{25}(x_{35}x_{25}^{-1} - 1 + x_{35} - 1) + x_{25} - 1 &= x_{25}x_{35}(x_{25}^{-1} - 1) + x_{25} - 1 + x_{25}(x_{35} - 1) \\
&= (-x_{25}x_{35}x_{25}^{-1} + 1)(x_{25} - 1) + x_{25}(x_{35} - 1),
\end{align*}

the following equalities being immediate. Therefore

\begin{equation}
(x_{35} - 1) \cdot x_{23} = x_{23}(-x_{25}x_{35}x_{25}^{-1} + 1) \cdot (x_{25} - 1) + x_{23}x_{25} \cdot (x_{35} - 1).
\end{equation}

Equalities (6.2.4), (6.2.5) and (6.2.6) imply that

\[
\varpi(x_{23}) = \begin{pmatrix}
  x_{23} & 0 & 0 \\
  0 & x_{23}(-x_{25}x_{35}x_{25}^{-1} + x_{25}x_{35} + 1) & x_{23}(-x_{25}x_{35}x_{25}^{-1} + x_{25}) \\
  0 & x_{23}(-x_{25}x_{35}x_{25}^{-1} + 1) & x_{23}x_{25}
\end{pmatrix} \in M_3(kP_5^*).
\]

Then

\[
\rho(X_0) = M_3(\text{pr}_{12})(\varpi(x_{23})) = \begin{pmatrix}
  X_0 & 0 & 0 \\
  0 & (1 - X_1)X_0 + Y_1^{-1}Y_0Y_1 & (1 - X_1)X_0X_1 \\
  0 & X_0 - Y_1^{-1}Y_0Y_1X_1^{-1} & X_0X_1
\end{pmatrix} \in M_3((kF_2)^{\otimes 2}).
\]

Set $T := \begin{pmatrix}
  (1 - X_1)X_0 + Y_1^{-1}Y_0Y_1 \\
  X_0 - Y_1^{-1}Y_0Y_1X_1^{-1}
\end{pmatrix} \in M_2((kF_2)^{\otimes 2})$, then \[\tilde{\rho}(X_0^k) = \begin{pmatrix}
  X_0^k & 0 \\
  0 & T^k
\end{pmatrix},\]

then

\[
\tilde{\rho}(X_0^k) = \text{row}\cdot \rho(X_0^k) \cdot \text{col} = (X_1 - 1)X_0^k + (1 - Y_1) \begin{pmatrix}
  -1 \\
  0
\end{pmatrix} T^k \begin{pmatrix}
  -1 \\
  0
\end{pmatrix} = (X_1 - 1)X_0^kY_1 - (1 - Y_1)(T^k)_{11},
\]

where the second equality follows from the form of \text{row} and \text{col}, and where $(T^k)_{11}$ means the $(1,1)$-entry of $T^k$.

One checks that

\[
T = \begin{pmatrix}
  1 & 0 \\
  -X_1^{-1} & 1
\end{pmatrix} \begin{pmatrix}
  a & b \\
  0 & c
\end{pmatrix} \begin{pmatrix}
  1 & 0 \\
  -X_1^{-1} & 1
\end{pmatrix}^{-1},
\]

where

\[
\begin{pmatrix}
  1 & 0 \\
  -X_1^{-1} & 1
\end{pmatrix}^{-1} = \begin{pmatrix}
  1 & 0 \\
  X_1^{-1} & 1
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
  a & b \\
  0 & c
\end{pmatrix} = \begin{pmatrix}
  Y_1^{-1}Y_0Y_1 & (1 - X_1)X_0X_1 \\
  0 & X_1^{-1}X_0X_1
\end{pmatrix}.
\]

For $k \in \mathbb{Z}$, one has

\[
\begin{pmatrix}
  a & b \\
  0 & c
\end{pmatrix}^k = \begin{pmatrix}
  a^k \sum_{i=0}^{k-1} a^i b c^{k-i-1} \\
  0 & c^k
\end{pmatrix},
\]

using the notation (2.2.1) so that

\[
T^k = \begin{pmatrix}
  1 & 0 \\
  -X_1^{-1} & 1
\end{pmatrix} \begin{pmatrix}
  a^k \sum_{i=0}^{k-1} a^i b c^{k-i-1} \\
  0 & c^k
\end{pmatrix} \begin{pmatrix}
  1 & 0 \\
  X_1^{-1} & 1
\end{pmatrix} = \begin{pmatrix}
  a^k + \sum_{i=0}^{k-1} a^i b c^{k-i-1} \cdot X_1^{-1} * \\
  0 & c^k
\end{pmatrix}.
\]
therefore

\[(T^k)_{11} = a^k + \sum_{i=0}^{k-1} a^i b c^{k-i-1} \cdot X_1^{-1}\]

\[= Y_1^{-1} Y_0^k Y_1 + \sum_{i=0}^{k-1} Y_1^{-1} Y_0^i Y_1 \cdot (1 - X_1) X_0 X_1 \cdot X_1^{-1} X_0^{-1} X_1^{-1} X_1^{-1} X_1^{-1} X_1^{-1}
\]

\[= Y_1^{-1} Y_0^k Y_1 + \sum_{i=0}^{k-1} Y_1^{-1} Y_0^i Y_1 \cdot (1 - X_1) X_0^{k-i}
\]

\[= Y_1^{-1} Y_0^k Y_1 + (1 - X_1) X_0^k + \sum_{i=1}^{k-1} Y_1^{-1} Y_0^i Y_1 \cdot (1 - X_1) X_0^{k-i}.
\]

It follows that

\[\tilde{\rho}(X_0^k) = (X_1 - 1) Y_1 X_0^k - (1 - Y_1) (T^k)_{11}
\]

\[= (X_1 - 1) Y_1 X_0^k - (1 - Y_1) \{(1 - X_1) X_0^k + Y_1^{-1} Y_0^k Y_1 + \sum_{i=0}^{k-1} Y_1^{-1} Y_0^i Y_1 \cdot (1 - X_1) X_0^{k-i}\}
\]

\[= (X_1 - 1) Y_1 X_0^k - (1 - Y_1) (1 - X_1) X_0^k - (1 - Y_1) Y_1^{-1} Y_0^k Y_1 - (1 - Y_1) \sum_{i=1}^{k-1} Y_1^{-1} Y_0^i Y_1 \cdot (1 - X_1) X_0^{k-i}
\]

\[= (X_1 - 1) X_0^k + (Y_1 - 1) Y_1^{-1} Y_0^k Y_1 - \sum_{i=1}^{k-1} (1 - X_1) X_0^{k-i} \cdot Y_1^{-1} (1 - Y_1) Y_0^i Y_1,
\]

which implies the first identity.

By Lemma 6.12, one has \(\rho(X_1) = 1 + \text{col} \cdot \text{row}\). As \(1 + \text{row} \cdot \text{col}\) is equal to \(X_1 Y_1\) and is therefore invertible, one checks that the inverse of \(1 + \text{col} \cdot \text{row}\) is \(1 - \text{col} \cdot (1 + \text{row} \cdot \text{col})^{-1} \cdot \text{row}\), therefore

\[\rho(X_1^{-1}) = 1 - \text{col} \cdot (1 + \text{row} \cdot \text{col})^{-1} \cdot \text{row} = 1 - \text{col} \cdot (X_1 Y_1)^{-1} \cdot \text{row}.
\]

Then

\[\rho(X_0^k X_1^{-1}) = \text{row} \cdot \rho(X_0^k) \cdot (1 - \text{col} \cdot (X_1 Y_1)^{-1} \cdot \text{row}) \cdot \text{col}
\]

\[= \text{row} \cdot \rho(X_0^k) \cdot \text{col} \cdot (1 - (X_1 Y_1)^{-1} \cdot \text{row} \cdot \text{col}) = \rho(X_0^k) \cdot (X_1 Y_1)^{-1},
\]

which proves the second statement.

\[\square\]

**Remark 6.14.** Identity (5.2.7) from Remark 5.8 implies another decomposition of \(T\), namely

\[T = \begin{pmatrix} 1 & -X_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X_0 & 0 \\ X_0 - Y_1^{-1} Y_0 Y_1 X_1^{-1} & Y_1^{-1} Y_0 Y_1 \end{pmatrix} \begin{pmatrix} 1 & -X_1 \\ 0 & 1 \end{pmatrix}^{-1},
\]

which as in this remark allows for an alternative computation of \(T^k\).

6.3. Relationship between braid groups and \(\Delta^l_r\).
6.3.1. **Relationship between \( \rho \) and \( \Delta^l_r \).** Denote by \( k[X_0^{\pm 1}] \) the linear span in \( kF_2 \) of the elements \( X_0^k, k \in \mathbb{Z} \), by \( k[X_0^{\pm 1}]X_1^{-1} \) the linear span of the elements \( X_0^kX_1^{-1}, k \in \mathbb{Z} \). The sum of these submodules of \( kF_2 \) is direct.

**Lemma 6.15.** \((kF_2, \cdot, X_1, -1)\) is generated, as an associative (non-unital) algebra, by \( k[X_0^{\pm 1}] \oplus k[X_0^{\pm 1}]X_1^{-1} \).

**Proof.** For \( s \geq 0 \) and for \((k_0, \ldots, k_s) \in \mathbb{Z}^{s+1}, (\epsilon_1, \ldots, \epsilon_s) \in \{\pm 1\}^s\), set

\[
(6.3.1) \quad w(k_0, \ldots, k_s|\epsilon_1, \ldots, \epsilon_s) := X_0^{k_0}X_1^{\epsilon_1}X_0^{k_1} \cdots X_0^{k_s-1}X_1^{\epsilon_s}X_0^{k_s} \in F_2.
\]

If \( s \geq 0 \), let \((F_2)_s\) be the subset of \( F_2 \) of all the elements (6.3.1), where \((k_0, \ldots, k_s) \in \mathbb{Z}^{s+1}, (\epsilon_1, \ldots, \epsilon_s) \in \{\pm 1\}^s\); so when \( s = 0 \), \((F_2)_0\) is the set of all \( X_0^k, k \in \mathbb{Z} \).

One has for \( s \geq 0, (k_0, \ldots, k_{s+1}) \in \mathbb{Z}^{s+1}, (\epsilon_2, \ldots, \epsilon_{s+1}) \in \{\pm 1\}^s \),

\[
w(k_0, \ldots, k_{s+1}|1, \epsilon_2, \ldots, \epsilon_{s+1}) = X_0^{k_0}X_1^{-1}w(k_1, \ldots, k_{s+1}|\epsilon_2, \ldots, \epsilon_{s+1}) + w(k_0 + k_1, k_2, \ldots, k_{s+1}|\epsilon_2, \ldots, \epsilon_{s+1}),
\]

\[
w(k_0, \ldots, k_{s+1} - 1, \epsilon_2, \ldots, \epsilon_{s+1}) = -X_0^{k_0}X_1^{-1}w(k_1, \ldots, k_{s+1}|\epsilon_2, \ldots, \epsilon_{s+1}) + w(k_0 + k_1, k_2, \ldots, k_{s+1}|\epsilon_2, \ldots, \epsilon_{s+1}).
\]

These identities enable one to prove by induction on \( s \geq 0 \) that \((F_2)_s\) is contained in the associative, non-unital subalgebra of \((kF_2, \cdot, X_1, -1)\) generated by \( k[X_0^{\pm 1}] \oplus k[X_0^{\pm 1}]X_1^{-1} \). The statement then follows from the fact that the union for \( s \geq 0 \) of all \((F_2)_s\) is equal to \( F_2 \). \(\Box\)

Recall that \( W^B_l \) is the subalgebra of \( kF_2 \) equal to \( k \oplus kF_2(X_1 - 1) \) (see \( \S 1.3 \)). We set

\[
(6.3.2) \quad (W^B_l)_+ := kF_2(X_1 - 1).
\]

This is a (non-unital) subalgebra of \( W^B_l \).

Since the right multiplication by \( X_1 - 1 \) is injective in \( kF_2 \), the algebra morphism

\[
\text{mor}_{kF_2, X_1-1}^l : (kF_2, \cdot, X_1, -1) \to (W^B_l)_+
\]

(see \( \S 1 \)) is an algebra isomorphism.

According to Proposition \( 2.2 \), \( X_1 \) is an invertible element of \( W^B_l \); \( Y_1 = (1, X_1) \) is therefore an invertible element of \( (W^B_l)^{\otimes 2} \). We denote by \( \text{Ad}(Y_1) \) the inner automorphism of this algebra given by conjugation by \( Y_1 \).

**Lemma 6.16.** The following diagram is commutative

\[
\begin{array}{ccc}
(kF_2, \cdot, X_1, -1) & \xrightarrow{\Delta^l_r} & (kF_2)^{\otimes 2} \\
\text{mor}_{kF_2, X_1-1} & \simeq & \text{Ad}(Y_1)^{-1}\circ \Delta^l_r \\
(W^B_l)_+ & \xrightarrow{\rho} & (W^B_l)^{\otimes 2}
\end{array}
\]

where \( \Delta^l_r \) is as in \( \S 3.1.6 \) and the inclusion \( W^B_r \hookrightarrow kF_2 \) is as in \( \S 3.1.1 \).
Proof. For \( k \in \mathbb{Z} \),
\[
\Delta^{l,r}_{\mathbb{A}} \circ \text{mor}^{I}_{kF_{2}, X_{1}-1}(X_{0}^{k}X_{1}^{-1}) = \Delta^{l,r}_{\mathbb{A}}(X_{0}^{k}(1 - X_{1})) = \text{Ad}(X_{1} - 1)^{\otimes 2} \circ \Delta_{\mathbb{A}}(-X_{0}^{k}(1 - X_{1}))
\]
\[= \text{Ad}(X_{1} - 1)^{\otimes 2}(-X_{0}^{k}(1 - X_{1}) \otimes 1 - 1 \otimes X_{0}^{k}(1 - X_{1}) - \sum_{i=1}^{k-1} X_{0}^{i}(1 - X_{1}) \otimes X_{0}^{k-i}(1 - X_{1}))
\]
\[= (X_{1} - 1)X_{0}^{k} \otimes 1 + 1 \otimes (X_{1} - 1)X_{0}^{k} - \sum_{i=1}^{k-1} (X_{1} - 1)X_{0}^{i}X_{0}^{k-i}(1 - X_{1})
\]
\[= (id \otimes \text{Ad}(X_{1}))(\tilde{\rho}(X_{0}^{k})) = \text{Ad}(Y_{1})(\tilde{\rho}(X_{0}^{k}))
\]
and similarly
\[
\Delta^{l,r}_{\mathbb{A}} \circ \text{mor}^{I}_{kF_{2}, X_{1}-1}(X_{0}^{k}X_{1}^{-1}) = \Delta^{l,r}_{\mathbb{A}}(X_{0}^{k}(1 - X_{1}^{-1})) = \text{Ad}(X_{1} - 1)^{\otimes 2} \circ \Delta_{\mathbb{A}}(X_{0}^{k}(1 - X_{1}^{-1}))
\]
\[= \text{Ad}(X_{1} - 1)^{\otimes 2}(X_{0}^{k}(1 - X_{1}^{-1}) \otimes 1 + 1 \otimes X_{0}^{k}(1 - X_{1}^{-1}) - \sum_{i=0}^{k} X_{0}^{i}(1 - X_{1}^{-1}) \otimes X_{0}^{k-i}(1 - X_{1}^{-1}))
\]
\[= (X_{1} - 1)X_{0}^{k}X_{1}^{-1} \otimes 1 + 1 \otimes (X_{1} - 1)X_{0}^{k}X_{1}^{-1} - \sum_{i=0}^{k-1} (X_{1} - 1)X_{0}^{i}X_{0}^{k-i}(1 - X_{1}^{-1})
\]
\[= (id \otimes \text{Ad}(X_{1}))(\tilde{\rho}(X_{0}^{k}))(X_{1}^{-1} \otimes X_{1}^{-1}) = (id \otimes \text{Ad}(X_{1}))(\tilde{\rho}(X_{0}^{k}X_{1}^{-1})) = \text{Ad}(Y_{1})(\tilde{\rho}(X_{0}^{k}X_{1}^{-1}))
\]
in each of these sequence of equalities, the second equality follows from Def. 3.11, the third equality follows from Lemma 2.5, the fourth equality follows from Lemma 3.5, the fifth equality follows from Lemma 6.13, and the last equality follows from \( Y_{1} = 1 \otimes X_{1} \).

It follows that the two maps of the announced diagram agree on the family of elements \( X_{0}^{k}, X_{0}^{k}X_{1}^{-1}, k \in \mathbb{Z} \). Since these maps are algebra morphisms, and since this family generates \((kF_{2}, X_{1}-1)\) (see Lemma 6.15), this diagram commutes. \( \square \)

**Proposition 6.17.** The following diagram commutes
\[
(6.3.3)
\]
\[
\xymatrix{ kF_{2} \ar[r]^{\ell} \ar[d]^{\cong} & kP_{0} \ar[r]^{\tilde{\mathcal{M}}_{\mathbb{A}}(kP_{0}^{*})} & \text{M}_{3}(kP_{0}^{*}) \ar[r]^{M_{3}((kF_{2})^{\otimes 2})} & \text{M}_{3}((kF_{2})^{\otimes 2}) \ar[r]^{\text{row}(-) \circ \text{col}} & (kF_{2})^{\otimes 2} \ar@{^{(}->}[u] \ar[d]^{(W_{i}^{B})^{+}} \ar[r]^{\oplus \text{Ad}(Y_{1}^{-1}) \circ \Delta_{\mathbb{A}}^{l,r}} & (W_{i}^{B})^{+} \otimes 2 \ar[u]}
\]
where \( \ell, \text{pr}_{12} \) are as in (6.1.2), \( \cong \) is as in (6.2.2), and \( \Delta_{\mathbb{A}}^{l,r} \) is as in (6.1.4); in this diagram, all the maps are algebra morphisms, except for the maps marked with \( \circ \), which are only \( k \)-module morphisms.
Proof. This follows from the combination the commutative diagram from Lemma 6.10 with the specialization of the commutative diagram from Lemma 4.3.

Remark 6.18. Set \( \text{row}' := (Y_1(X_1 - 1) \ Y_1(1 - Y_1) \ 0) \) and \( \text{col}' := \begin{pmatrix} 1 \\ -Y_1^{-1} \\ 0 \end{pmatrix} \). The diagram obtained from (6.3.3) by replacing \( \text{row} \cdot (\cdot) \cdot \text{col} \) by \( \text{row}' \cdot (\cdot) \cdot \text{col}' \) and \( \text{Ad}(Y_1^{-1}) \circ \Delta_{i,r}^{l,r} \) by \( \Delta_{i,n}^{l,r} \) also commutes.

6.3.2. Completion of the diagram (6.3.3). Assume that \( \mathcal{A} \) is a filtered \( \mathbb{k} \)-algebra equipped with a descending filtration \( (F^n \mathcal{A})_{n \geq 0} \). If \( k \) is an integer \( \geq 1 \), then \( F^n(M_k(\mathcal{A})) := M_k(F^n \mathcal{A}) \) for \( n \geq 0 \) defines a descending filtration on the algebra \( M_k(\mathcal{A}) \). The associated graded algebra is \( \text{gr}(M_k(\mathcal{A})) \simeq M_k(\text{gr}(\mathcal{A})) \) and the associated graded algebra is \( M_k(\mathcal{A})^\wedge \simeq M_k(\mathcal{A}) \).

Definition 6.19. If \( V, W \) are \( \mathbb{k} \)-modules equipped with descending filtrations \( (F^n V)_{n \geq 0} \), \( (F^n W)_{n \geq 0} \), then we say that the \( \mathbb{k} \)-module map \( f : V \to W \) is weakly compatible with the filtrations if for some \( d \in \mathbb{Z} \), one has \( f(F^n V) \subset F^{n+d} W \) for any \( n \geq 0 \); one then says that \( f \) has filtration shift \( \leq d \).

If \( f \) has filtration shift \( \leq d \), then it induces a degree \( d \) map \( \text{gr}(f; d) : \text{gr}(V) \to \text{gr}(W) \) and a morphism of topological \( \mathbb{k} \)-modules \( \hat{f} : \hat{V} \to \hat{W} \). (Although we will not need the notion, one can consistently define the filtration shift of \( f \) to be the largest \( d \) such that \( f(F^n V) \subset F^{n+d} W \) for any \( n \geq 0 \) if it exists, and \( +\infty \) otherwise.)

Filtrations on the constituents of (6.3.3). The spaces in diagram (6.3.3) are all \( \mathbb{k} \)-algebras. They are equipped with descending filtrations in the following way:

(a) \( \mathbb{k}F_2, \mathbb{k}P_n^* \) are group algebras and are therefore equipped with the adic filtration of their augmentation ideals;

(b) \( M_3(\mathbb{k}P_n^*) \) is a matrix algebra over a filtered algebra and is therefore filtered by the above;

(c) \( (\mathbb{k}F_2)^{\otimes 2} \) is isomorphic to the group algebra of \( (F_2)^2 \) and is therefore filtered. Then \( M_3((\mathbb{k}F_2)^{\otimes 2}) \) is a matrix algebra over this algebra and is therefore filtered;

(d) there is a sequence of inclusions \( (\mathcal{W}_i^B)_+ \subset \mathcal{W}_i^B \subset \mathbb{k}F_2 \), which induce filtrations on \( (\mathcal{W}_i^B)_+ \) and on \( \mathcal{W}_i^B \), and therefore on \( (\mathcal{W}_i^B)^{\otimes 2} \) (see (1.3.3));

(e) the non-unital algebra \( (\mathbb{k}F_2, \cdot X_i-1) \) is filtered by \( F^0(\mathbb{k}F_2) := \mathbb{k}F_2, F^n(\mathbb{k}F_2) := I_{n-1} \) for \( n \geq 1 \).

Compatibilities of the morphisms of (6.3.3) with filtrations.

Lemma 6.20. Recall that \( F_3 \) may be viewed as a normal subgroup of \( P_3^* \) (see Lemma 6.9). For any \( g \in P_3^* \), for any \( i \in [1, 3] \), there exists an element \( w(g, i) \in F_3 \), such that

\[
(6.3.4) \quad g^{-1}x_{i5}g = w(g, i)x_{i5}w(g, i)^{-1}.
\]
Proof. As $F_3$ is normal in $P_5^*$, for any $g \in P_5^*$, the inner automorphism of $P_5^*$ given by conjugation by $g^{-1}$ restricts to an automorphism $\theta_g$ of $F_3$. Then $g \mapsto \theta_g$ defines a group anti-homomorphism $\theta : P_5^* \to \text{Aut}(F_3)$. Let $\text{Aut}^*(F_3)$ be the subgroup of $\text{Aut}(F_3)$, consisting in all the automorphisms taking each $x_{i,j}, i \in [1,3]$, to a conjugate of this element. When $g \in F_3$, $\theta_g$ is an inner automorphism of $F_3$, therefore belongs to $\text{Aut}^*(F_3)$. One computes

$$
\theta_{12} : x_{15} \mapsto \text{Ad}(x_{15}x_{25})(x_{15}), \ x_{25} \mapsto \text{Ad}(x_{15})(x_{25}), \ x_{35} \mapsto x_{35},
$$

$$
\theta_{23} : x_{15} \mapsto x_{15}, \ x_{25} \mapsto \text{Ad}(x_{25}x_{35})(x_{25}), \ x_{35} \mapsto \text{Ad}(x_{25})(x_{35}),
$$

which implies that the images by $\theta$ of $x_{12}$ and $x_{23}$, and therefore also of $F_2$, lie in $\text{Aut}^*(F_3)$. Together with Lemma 6.9, this implies that the image of $P_5^*$ is contained in $\text{Aut}^*(F_3)$, and therefore the announced statement. $\square$

Lemma 6.21. The algebra morphism $\varpi : kP_5^* \to M_3(kP_5^*)$ has filtration shift $\leq 0$.

Proof. Let $J := (kP_5^*)_+$. The announced statement means that for any $n \geq 0$, $\varpi(J^n) \subset M_3(J^n)$. This is obvious for $n = 0$, let us prove it for $n = 1$.

As $\varpi$ is linear, and as $J$ is spanned by the $g - 1$, where $g \in P_5^*$, it suffices to show that $\varpi(g) - id \in M_3(J)$ for any $g \in P_5^*$, where id is the unit matrix in $M_3(kP_5^*)$. Let $i \in [1,3]$; then

$$(x_{i5} - 1)g = g \cdot w(g,i)(x_{i5} - 1)w(g,i)^{-1} = g \cdot w(g,i)(x_{i5} - 1) + g \cdot w(g,i)(x_{i5} - 1)(w(g,i)^{-1} - 1)$$

where the first equality follows from (6.3.4).

The map $kF_3^{\geq 3} \to (kF_3)_+(f_{i})_{i \in [1,3]} \mapsto \sum_{i \in [1,3]} f_{i} \cdot (x_{i5} - 1)$ is bijective. For $w \in F_3$, we denote by $(f_{i}(w))_{i \in [1,3]}$ the preimage of $w - 1$ in $kF_3^{\geq 3}$. Then $\sum_{i \in [1,3]} f_{i} \cdot (w) \cdot (x_{i5} - 1) = w - 1$. Substituting $w$ by $(w, g, i)^{-1}$ in this identity, one obtains

$$(x_{i5} - 1)g = g \cdot w(g,i)(x_{i5} - 1) + \sum_{j \in [1,3]} g \cdot w(g,i)(x_{j5} - 1)f_{i}(w,g,i)^{-1} \cdot (x_{j5} - 1).$$

It follows that

$$\varpi(g)_{ij} = g \cdot w(g,i)\delta_{ij} + g \cdot w(g,i)(x_{i5} - 1)f_{i}(w,g,i)^{-1}.$$ 

As $g \cdot w(g,i) \in P_5^*$, $g \cdot w(g,i) \equiv 1 \mod J$. Moreover, as $J$ is a two-sided ideal, $g \cdot w(g,i)(x_{i5} - 1)f_{i}(w,g,i)^{-1} \in J$. It follows that $\varpi(g)_{ij} \equiv \delta_{ij} \mod J$, and therefore that $\varpi(g) - id \in M_3(J)$. Therefore $\varpi(J) \subset M_3(J)$.

Then for $n \geq 1$, $\varpi(J^n) \subset \varpi(J) \subset M_3(J) \subset M_3(J^n)$. $\square$

Proposition 6.22. The morphisms in diagram (6.3.3) are weakly compatible with the filtrations in the sense of Definition 6.19: they all have filtration shift $\leq 0$, except for $kF_2 \simeq (kF_2, \cdot x_{1,-1})$ and $\text{row} \cdot (-) \cdot \text{col}$ (marked with a $\circ$), which have filtration shift $\leq 1$.

Proof. The morphisms $\underline{\ell}$ and $\underline{pr}_{12}$ have filtration shift $\leq 0$ as they arise from a group morphism, so that $M_3(\underline{pr}_{12})$ also has filtration shift $\leq 0$. The map $kF_2 \simeq (kF_2, \cdot x_{1,-1})$ has filtration shift $\leq 1$ by construction. The map $\text{row} \cdot (-) \cdot \text{col}$ has filtration shift $\leq 1$ as $X_1 - 1, Y_1 - 1$ both belong
to $F^1(kF_2)^{\otimes 2}$. The morphism $\text{mor}_{kF_2, X_1 - 1}$ has filtration shift $\leq 0$ as $I^n(X_1 - 1) \subset I^{n+1}$. The morphism $\text{Ad}(Y_1 - 1) \circ \Delta^\ell_r$ has filtration shift $\leq 0$ for the following reasons: $\Delta^\ell_r$ has filtration shift $\leq 0$ (Proposition 2.14), $\text{Ad}(X_1 - 1) : W^B_l \to W^B_r$ has filtration shift $\leq 0$, and the automorphism $\text{Ad}(X_1 - 1)$ of $W^B_l$ has filtration shift $\leq 0$. The algebra morphism $(W^B_l)^{\otimes 2} \hookrightarrow (kF_2)^{\otimes 2}$ has filtration shift $\leq 0$ by construction. Finally, the fact that the algebra morphism $\Delta$ has filtration shift $\leq 0$ follows from Lemma 6.21. □

Consequences of compatibility with filtration.

Lemma 6.23. (1) The associated graded algebras of the diagram (6.3.3) can be canonically identified with the algebras of diagram (5.3.3);
(2) The associated graded maps $\text{gr}(f, d)$ of the diagram (6.3.3) (where $d = 0$ except for the maps marked $\diamond$, where $d = 1$) can be identified with the corresponding maps of diagram (5.3.3).

Proof. This can be checked directly. □

Corollary 6.24. The maps from the diagram (5.3.3) induce continuous maps between the completions of the algebras of this diagram. These completions will be denoted with a hat ($\hat{l}$, $\hat{c}$, etc.)

Proof. This follows from the fact that these maps are all weakly compatible with the filtrations, see Proposition 6.22. □

7. Associators

The purpose of this section is to recall some facts on associators, and in particular how these objects relate braid groups to infinitesimal braid Lie algebras.

In §7.1 we recall the definition of the set of associators over $k$ with a given parameter $\mu \in k^\times$ (Definition 7.3), as well as the basic nonemptiness result (Theorem 7.3 see [Dr]). In §7.2 we recall the construction and properties of $\Gamma$-functions of associators (Lemma 7.3). In §7.3 we recall from [BN] that a choice of an associator, together with combinatorial data (family of parenthesizations), gives rise to a family of isomorphisms between algebras associated with braid groups and the ones with infinitesimal braid Lie algebras; we also describe the behavior of these isomorphisms with respect to a change of parenthesization and with respect to cabling. In §7.4 we apply these results to explicitly compute images of particular elements of the modular group $P_5^t$ in the group of invertible elements of the completion $U(p_5)^\wedge$ of the enveloping algebra of the infinitesimal braid Lie algebra $p_5$ (Lemmas 7.7 and 7.8 leading to Proposition 7.9).

7.1. The set $M_\mu(k)$ of associators.
7.1.1. The notation \( \Phi(a_0, a_1) \). Let \( \mathcal{A} = F^0 \mathcal{A} \supset F^1 \mathcal{A} \supset \cdots \) be a filtered algebra over \( k \), complete with respect to its filtration. Let \( a_0, a_1 \in F^1 \mathcal{A} \). Then there is a morphism \( \text{ev}_{a_0, a_1} : U(f_2)^\wedge \to \mathcal{A} \), uniquely determined by the conditions \( e_i \mapsto a_i \) for \( i = 0, 1 \).

**Definition 7.1.** For \( \Phi \in U(f_2)^\wedge \), we set \( \Phi(a_0, a_1) := \text{ev}_{a_0, a_1}(\Phi) \) (this is an element of \( \mathcal{A} \)).

**Remark 7.2.**

1. Assume that \( \mathcal{A} = U(f_2)^\wedge \), \( \Phi = \Phi(e_0, e_1) \), then \( \text{ev}_{a_0, a_1} = \text{id}_{U(f_2)^\wedge} \), therefore \( \Phi = \Phi(e_0, e_1) \).

2. Assume that \( \mathcal{A} = k\langle \langle X \rangle \rangle \), \( \Phi = \Phi(e_0, e_1) \), then \( \text{ev}_{a_0, a_1} : U(f_2)^\wedge \to k\langle \langle X \rangle \rangle \) is the isomorphism \( \Phi \), and \( \Phi(x_0, -x_1) \) is the image in \( k\langle \langle X \rangle \rangle \) of \( \Phi \in U(f_2)^\wedge \) under this isomorphism.

7.1.2. **Definition of \( M_\mu(k) \).** Recall the graded Lie algebra \( t_4 \) from \( \|$5.1.1$\$.

**Definition 7.3.** \( \Phi(t_{12}, t_{23} + t_{24}) \cdot \Phi(t_{13} + t_{23}, t_{34}) = \Phi(t_{23}, t_{34}) \cdot \Phi(t_{12} + t_{13}, t_{24} + t_{34}) \cdot \Phi(t_{12}, t_{23}) \) (equality in \( U(t_4)^\wedge \), called the pentagon condition).

In \( \|$5.1.1$\$, it was proved that \( M_\mu(k) \) that the first two equalities are consequences of the latter one, where \( \mu \) is obtained from the expansion \( \Phi = 1 + \sum [e_0, e_1] + \text{terms of degree} \geq 3 \).

7.1.3. **Nonemptiness of \( M_\mu(k) \).**

**Theorem 7.4** \( \|$5.1.1$\$. (1) Let \( k := \mathbb{C} \). Identify the series \( \varphi_{KZ} \in \mathbb{C} \langle \langle A, B \rangle \rangle \) from \( \|$2$\$ in \( \|$5.1.1$\$ with an element of \( U(f_2)^\wedge \) via the isomorphism given by \( A \mapsto e_0, B \mapsto e_1 \). Then \( \varphi_{KZ} \in M_1(\mathbb{C}) \) (see \( \|$5.1.1$\$, \( \|$2$\$).

2. If \( k \) is a \( \mathbb{Q} \)-algebra and \( \mu \in k^\times \), then the set \( M_\mu(k) \) is nonempty (see Proposition \( \|$5.3$\$ in \( \|$5.1.1$\$).

Proposition \( \|$5.3$\$ from \( \|$5.1.1$\$ contains the statement that \( M_1(k) \neq \emptyset \) for \( k \) a field containing \( \mathbb{Q} \) and \( \mu \in k^\times \). In particular, \( M_1(\mathbb{Q}) \neq \emptyset \). If \( k \) is a \( \mathbb{Q} \)-algebra, then the ring morphism \( \mathbb{Q} \to k \) gives rise to a map \( M_1(\mathbb{Q}) \to M_1(k) \), proving the nonemptiness of \( M_1(k) \). The automorphism of \( U(f_2)^\wedge \) given by \( e_i \mapsto \mu e_i \) (\( i = 0, 1 \)) then gives rise to a bijection \( M_1(k) \to M_\mu(k) \), thus proving (2) in Theorem \( \|$7.4$\$.
7.2. Γ-functions of associators. Let \( \Phi \in M_\mu(k) \). Let \( \varphi_0, \varphi_1 \) be the elements of \( U(f_2)^\wedge \) defined by the equality \( \Phi = 1 + \varphi_0 e_0 + \varphi_1 e_1 \) (equality in \( U(f_2)^\wedge \)). Let \( \tau_0, \tau_1 \) be free commutative formal variables; there is a unique continuous \( k \)-algebra morphism \( U(f_2)^\wedge \to k[[\tau_0, \tau_1]] \), denoted \( f \mapsto f^{ab} \), such that \( e_i \mapsto \tau_i \) for \( i = 0, 1 \).

**Lemma 7.5.** Let \( \mu \in k \) and \( \Phi \in M_\mu(k) \). Recall that \( \Phi \) may be viewed as an element of \( \exp(\mathfrak{Lib}_k(X)) \) through the isomorphism \([E]\). One then defines \( \Gamma_\Phi(t) \in k[t]^\times \) according to (1.1.2).

(1) One has the identity

\[
(1 + \varphi_1 e_1)^{ab} = \frac{\Gamma_\Phi(-\tau_0)\Gamma_\Phi(-\tau_1)}{\Gamma_\Phi(-\tau_0 - \tau_1)}
\]
in \( k[[\tau_0, \tau_1]] \).

(2) \( \Gamma_\Phi \) satisfies the identity

\[
\Gamma_\Phi(t)\Gamma_\Phi(-t) = \frac{\mu t}{e^{\mu t/2} - e^{-\mu t/2}}
\]
in \( 1 + t^2 k[[t]] \).

**Proof.** In \([E]\), one attaches to \( \Phi \) a collection \( (\zeta_\Phi(n))_{n \geq 2} \) of elements of \( k \) with the following properties: (a) for \( n \) even \( \geq 2 \), one has \( \zeta_\Phi(n) = \mu^n \cdot \zeta(n)/(2\pi)^n \); (b) the series \( \tilde{\Gamma}_\Phi(t) \) (denoted \( \Gamma_\Phi(t) \) in \([E]\)) defined by \( \tilde{\Gamma}_\Phi(t) := \exp(-\sum_{n \geq 2} \zeta_\Phi(n)t^n/n) \) is such that

\[
(1 + \varphi_1 e_1)^{ab} = \frac{\tilde{\Gamma}_\Phi(\tau_0 + \tau_1)}{\Gamma_\Phi(\tau_0)\Gamma_\Phi(\tau_1)}
\]
(identity in \( k[[\tau_0, \tau_1]] \)). Then

\[
(\text{r.h.s. of (7.2.2)}) = \exp(-\sum_{n \geq 2} \zeta_\Phi(n)\left\{ (\tau_0 + \tau_1)^n - \tau_0^n - \tau_1^n \right\})
\]

\[
= \exp(-\sum_{n \geq 2} \frac{\zeta_\Phi(n)}{n}\left\{ n\tau_0^{n-1}\tau_1 + O(\tau_1^2) \right\}) = 1 - \sum_{n \geq 2} \zeta_\Phi(n)\tau_0^{n-1}\tau_1 + O(\tau_1^2),
\]

and

\[
(\text{l.h.s. of (7.2.2)}) = 1 + \sum_{n \geq 2} (\Phi|e_0^{n-1}e_1)\tau_0^{n-1}\tau_1 + O(\tau_1^2).
\]

Then \((7.2.2)\) implies \( \zeta_\Phi(n) = -(\Phi|e_0^{n-1}e_1) \) for \( n \geq 2 \). It follows that \( \Gamma_\Phi(t) = 1/\tilde{\Gamma}_\Phi(-t) \), where \( \Gamma_\Phi \) is as in (1.1.2). Plugging this equality in \((7.2.2)\), we obtain \((7.2.1)\). This proves 1.

Let us prove 2). Since \( \log \Gamma(1-t) = \gamma t + \sum_{n \geq 2} \zeta(n)t^n/n \), the series \( \exp(2\sum_{n \text{ even}, n \geq 2} \zeta_\Phi(n)t^n/n) \) is equal to \( \Gamma(1+t)\Gamma(1-t) \). Using the computation of this series in the proof of Lemma 1.2 one obtains

\[
\exp(2\sum_{n \text{ even}, n \geq 2} \frac{\zeta(n)t^n}{n}) = \frac{2\pi it}{e^{(2\pi it)/2} - e^{-(2\pi it)/2}}.
\]
Then

\[ \Gamma_\Phi(t)\Gamma_\Phi(-t) = (\tilde{\Gamma}_\Phi(t)\tilde{\Gamma}_\Phi(-t))^{-1} = \exp(2 \sum_{n \text{ even}, n \geq 2} \frac{\zeta(n)t^n}{n}) = \exp(2 \sum_{n \text{ even}, n \geq 2} \frac{\zeta(n)(\mu t/(2\pi i))^n}{n}) \]

\[ = \frac{\mu t}{e^{\mu t/2} - e^{-\mu t/2}}, \]

which proves (2). □

Remark 7.6. Lemma 7.5 can also be derived by combining two results from [Fu2], namely the inclusion result \( M_\mu(k) \subset \text{DMR}_\mu(k) \) and the result on \( \Gamma \)-functions for elements of \( \text{DMR}_\mu(k) \).

7.3. Isomorphisms arising from associators.

7.3.1. Comparison isomorphisms. The permutation group \( S_n \) acts by automorphisms of \( t_n \) by permutation of indices via \( \sigma \cdot t_{ij} = t_{\sigma^{-1}(i)\sigma^{-1}(j)} \). This action gives rise to an algebra structure on the tensor product \( U(t_n)^\wedge \otimes kS_n \), and to a topological Hopf algebra on it, defined by the conditions that the elements of \( S_n \) be group-like and the elements of \( t_n \) be primitive; the resulting topological Hopf algebra structure is denoted by \( U(t_n)^\wedge \rtimes kS_n \).

According to [BN], each integer \( n \geq 2 \) and parenthesization \( P \) of a word with \( n \) identical letters gives rise to a morphism of Hopf algebras

\[ (7.3.1) \tilde{b}^P_{(\mu, \Phi)} : kB_n \to U(t_n)^\wedge \rtimes kS_n. \]

If \( P \) is the leftmost parenthesization \((\bullet \cdots \bullet)\), then

\[ \tilde{b}^P_{(\mu, \Phi)} : kB_n \to U(t_n)^\wedge \rtimes kS_n \]

is given by \( \sigma_1 \mapsto (12)e^{(\mu/2)t_{12}} \) and

\[ (7.3.2) \sigma_i \mapsto \Phi(t_{i1} + \cdots + t_{i-1,i} + t_{i,i+1})^{-1}(i,i+1)e^{(\mu/2)t_{i,i+1}}\Phi(t_{i1} + \cdots + t_{i-1,i} + t_{i,i+1}) \]

for \( i \in [2, n-1] \).

For arbitrary \( (n, P) \), the morphism \( \tilde{b}^P_{(\mu, \Phi)} \) fits in a commutative diagram

\[
\begin{array}{ccc}
\mathbf{k}B_n & \longrightarrow & U(t_n)^\wedge \rtimes \mathbf{k}S_n \\
\downarrow & & \downarrow \\
\mathbf{k}S_n & \rightarrow & \\
\end{array}
\]

therefore it restricts to a morphism \( kK_n \to U(t_n)^\wedge \), which induces an isomorphism of topological Hopf algebras

\[ b^P_{(\mu, \Phi)} : (kK_n)^\wedge \to U(t_n)^\wedge. \]

One checks that \( b^P_{(\mu, \Phi)}(\omega_n) = e^{\mu \sum_{i<j \in [1,n]} t_{ij}} \). As the group \( P_{n+1}^* \) (resp. the Lie algebra \( p_{n+1} \)) is the quotient of \( K_n \) (resp. of \( t_n \)) by \( \omega_n \) (resp. by \( \sum_{i<j \in [1,n]} t_{ij} \)), the morphism \( b^P_{(\mu, \Phi)} \) induces an isomorphism of topological Hopf algebras

\[ \Phi^P_{(\mu, \Phi)} : (kP_{n+1}^*)^\wedge \to U(p_{n+1})^\wedge. \]
We will set

\[(7.3.3) \quad e_{(\mu, \varphi)}^{(5)} := e_{(\mu, \varphi)}^{(\bullet \bullet \bullet \bullet \bullet)}; \]

it is an isomorphism \((kP_n^+)\wedge \to U(p_n)^\wedge.\)

7.3.2. Change of parenthesization. Let \(\text{Par}_n\) be the set of parenthesizations of a word with \(n\) identical letters, and let \(G(U(t_n)^\wedge)\) be the group of group-like elements of \(U(t_n)^\wedge.\) There is a unique map

\[(\text{Par}_n)^2 \to G(U(t_n)^\wedge), \quad (P, P') \mapsto \Phi_{P'}^{P'},\]

satisfying the identity \(\Phi_{P'}^{P'} \Phi_{P'}^{P'} = \Phi_{P'}^{P'}\) (equality in \(G(U(t_n)^\wedge)\)), and such that:

- if \(a, b, c\) are integers \(\geq 1\) with \(a + b + c = n\), and if \(A \in \text{Par}_a, B \in \text{Par}_b, C \in \text{Par}_c,\) then
  \[\Phi_{(AB)C} = \Phi(\sum_{i \in [1, a], j \in a+1, b} t_{ij} \sum_{j \in a+1, b, k \in a+b+c, j} t_{jk}) \in G(U(t_n)^\wedge);\]

- if \(a, b\) are integers \(\geq 1\) with \(a + b = n\), and if \(A, A' \in \text{Par}_a\) and \(B, B' \in \text{Par}_b,\) then
  \[\Phi_{A'B'} = \text{mor}_{a,b}(\Phi_A^{A'} \otimes \Phi_B^{B'}) \in G(U(t_n)^\wedge),\]

where \(\text{mor}_{a,b} : U(t_n)^\wedge \otimes U(t_n)^\wedge \to U(t_n)^\wedge\) is the algebra morphism induced by the Lie algebra morphism \(t_a \oplus t_b \to t_n\) induced by \((t_{ij}, 0) \mapsto t_{ij}\) for \(i \neq j \in [1, a]\) and \((0, t_{ij}) \mapsto t_{a+i, a+j}\)

Then if \(P, P'\) belongs to \(\text{Par}_n,\) one has the identity

\[\tilde{b}_{(\mu, \varphi)}^{P', P} = \text{Ad}(\Phi_{P'}^{P}) \circ \tilde{b}_{(\mu, \varphi)}^{P'}\]

(equality of morphisms \(kB_n \to U(t_n)^\wedge \times kS_n,\) which implies the identities

\[(7.3.4) \quad \tilde{b}_{(\mu, \varphi)}^{P'} = \text{Ad}(\Phi_{P'}^{P}) \circ \tilde{b}_{(\mu, \varphi)}^{P'}\]

(equality of isomorphisms \((kK_n)^\wedge \rightarrow U(t_n)^\wedge)\) and

\[\tilde{s}_{(\mu, \varphi)}^{P'} = \text{Ad}(\Phi_{P'}^{P}) \circ \tilde{s}_{(\mu, \varphi)}^{P'}\]

(equality of morphisms \((kP_{n+1}^+)\wedge \rightarrow U(p_{n+1})^\wedge).\)

7.3.3. Behavior with respect to cabling morphisms. To the data of an integer \(n \geq 1\) and of a sequence \(m = (m_1, \ldots, m_n)\) of integers \(\geq 1,\) one associates the cabling morphisms

\[\text{cab}^B_m : K_n \rightarrow K_m, \quad \text{cab}^D_m : t_n \rightarrow t_m,\]

where \(m = m_1 + \cdots + m_n,\) which are defined as follows: the effect of \(\text{cab}^B_m\) consists in successively replacing, for \(i \in [1, n],\) the \(i\)-th strand in a given pure braid by \(m_i\) parallel strands; \(\text{cab}^D_m\) takes \(t_{ij},\) for \(i \neq j \in [1, n]\) to \(s_{\sum b_{m_1+\cdots+m_{j-1}+1, m_j}} \sum_{s_{m_1+\cdots+m_{j-1}+1, m_j}} t_{ab}.\)

Let \(n \geq 1,\) let \(P \in \text{Par}_n\) and let \(m = (m_1, \ldots, m_n)\) be a sequence of integers \(\geq 1.\) For each \(i \in [1, n],\) let \(p_i \in \text{Par}_{m_i}.\) Let \(m := m_1 + \cdots + m_n\) and \(P_{m, (p_i)} \in \text{Par}_m\) be the parenthesization obtained from \(P\) by replacing the \(i\)-th letter by the word of length \(m_i\) with parenthesization \(p_i.\)
As \( P_{m,(r)} \) can be obtained from \( P \) by a repeated application of the operation of doubling of strands (called cabling operation in \[BN\], §2.1), and as the assignment \((n, P) \mapsto b_{(\mu, \Phi)}^n \) is compatible with this operation (see \[BN\], §3), the cabling morphisms exhibit the following compatibility properties with the comparison isomorphisms

\[
(7.3.5) \quad \text{cab}^\text{DR}_{m,n} \circ b_{(\mu, \Phi)}^n = b_{(\mu, \Phi)}^m \circ \text{cab}^\text{B}_m
\]

(equality of morphisms \((kK_n)^\wedge \to U(t_m)^\wedge\)).

### 7.4. Computation of images of elements.

For \( i \in [1, 5] \) and \( I \) any collection of indices in \([1, 5] \setminus \{i\} \), we use the notation \( e_{I,j} := \sum_{i \in I} e_{ij} \).

**Lemma 7.7.** One has

\[
(7.4.1) \quad U_{\mu, \Phi}(x_{12}) = e^{\mu e_{12}},
\]

\[
(7.4.2) \quad U_{\mu, \Phi}(x_{23}) = \Phi(e_{12}, e_{23})^{-1} e^{\mu e_{23}} \Phi(e_{12}, e_{23}),
\]

\[
(7.4.3) \quad U_{\mu, \Phi}(x_{34}) = \Phi(e_{12,3}, e_{34})^{-1} e^{\mu e_{34}} \Phi(e_{12,3}, e_{34}),
\]

(equalities in \( U(p_5)^\wedge \)) where \( U_{\mu, \Phi}(x) \) is as in 7.3.3).

**Proof.** Let \( i \in [1, 3] \). Specializing (7.3.2) to \( n = 4 \) yields \( b_{(\mu, \Phi)}^{((**))} (\sigma_i) \). Taking squares, one obtains \( b_{(\mu, \Phi)}^{((**))} (x_{i,i+1}) \). Projecting in \( U(p_5)^\wedge \), one obtains \( U_{\mu, \Phi}^{((**))} (x_{i,i+1}) \), which is \( U_{\mu, \Phi}(x_{i,i+1}) \). \( \square \)

**Lemma 7.8.** One has the following equalities in \( U(p_5)^\wedge \):

\[
(7.4.4) \quad U_{\mu, \Phi}(x_{15}) = \Phi(e_{12,3}, e_{34})^{-1} \Phi(e_{12}, e_{2,34})^{-1} e^{\mu e_{15}} \cdot (\Phi(e_{12,3}, e_{34})^{-1} \Phi(e_{12}, e_{2,34})^{-1})^{-1},
\]

\[
(7.4.5) \quad U_{\mu, \Phi}(x_{25}) = \Phi(e_{12,3}, e_{34})^{-1} e^{\mu e_{25}} \cdot (\Phi(e_{12,3}, e_{34})^{-1} e^{\mu e_{25}} \Phi(e_{12}, e_{1,34})^{-1})^{-1},
\]

\[
(7.4.6) \quad U_{\mu, \Phi}(x_{35}) = e^{(\mu/2) e_{12,3}} \Phi(e_{4,12}, e_{1,3}) \cdot e^{\mu e_{35}} \cdot (e^{(\mu/2) e_{12,3}} \Phi(e_{4,12}, e_{1,3}))^{-1},
\]

\[
(7.4.7) \quad U_{\mu, \Phi}(x_{45}) = e^{\mu e_{45}}.
\]

**Proof.** The elements \( \tilde{x}_{15} \in K_4 \) and the cabling morphisms. For \( i \in [1, 4] \), let \( \tilde{x}_{i5} \in K_4 \) be given by the following diagrams

\[
\tilde{x}_{15} = \quad \quad \tilde{x}_{25} = \quad \quad \tilde{x}_{35} = \quad \quad \tilde{x}_{45} =
\]

under the convention explained after Remark 6.5.

Then each \( \tilde{x}_{i5} \) is a lift of \( x_{i5} \in P_5^* \) under the projection \( K_4 \to P_5^* \).
By the hexagon equality, one has

\[ g := e^{(\mu/2)t_{12}} \Phi(t_{13}, t_{12}) \in U(t_3)^. \]

Then

\begin{align*}
  b^{(\bullet \bullet \bullet)}_{(\mu, \Phi)}(x^{(2)}_{13}) &= e^{-\mu t_{12}}, \\
  b^{(\bullet \bullet \bullet)}_{(\mu, \Phi)}(x^{(3)}_{24}) &= g \cdot e^{-\mu t_{23}} \cdot g^{-1}.
\end{align*}

Let us prove identity (7.4.9). We have \( \tilde{\Phi} = \Phi(t_{12}, t_{23})^{-1} \Phi(t_{12}, t_{12})^{-1} = \Phi(t_{12}, t_{23})^{-1} e^{-\mu t_{23}} \Phi(t_{12}, t_{23}) e^{-\mu t_{12}}. \)

By the hexagon equality, one has

\[ e^{(\mu/2)t_{23}} \Phi(t_{12}, t_{23}) e^{(\mu/2)t_{12}} = \Phi(t_{13}, t_{23}) e^{(\mu/2)t_{23}, 13} \Phi(t_{12}, t_{13}) \]

and by the exchange of \( t_{12} \) and \( t_{23} \), also

\[ e^{(\mu/2)t_{12}} \Phi(t_{23}, t_{12}) e^{(\mu/2)t_{23}} = \Phi(t_{13}, t_{12}) e^{(\mu/2)t_{23}, 13} \Phi(t_{23}, t_{13}). \]

Taking the product of the last equality with the previous one and using the 2-cycle relation, we get

\[ e^{(\mu/2)t_{12}} \Phi(t_{23}, t_{12}) e^{(\mu/2)t_{23}} e^{(\mu/2)t_{12}} = \Phi(t_{13}, t_{12}) e^{(\mu/2)t_{23}, 13} \Phi(t_{12}, t_{13}) \]

Inverting, using the duality identity and conjugating by \( e^{(\mu/2)t_{12}} \), we get

\begin{align*}
  \Phi(t_{12}, t_{23})^{-1} e^{-\mu t_{23}} \Phi(t_{23}, t_{12})^{-1} e^{-\mu t_{12}} &= e^{(\mu/2)t_{12}} \Phi(t_{13}, t_{12}) \cdot e^{-\mu t_{23}} \cdot e^{(\mu/2)t_{12}} \Phi(t_{13}, t_{12})^{-1}.
\end{align*}

This proves identity (7.4.9).

Computation of the images of \( x_{i5} \in P^*_5 \) by the comparison isomorphisms. Consider the following table

| \( i \) | \( n(i) \) | \( P(i) \) | \( Q(i) \) | \( m(i) \) | \( p(i) = (p_{1(i)}, \ldots, p_{n(i)}) \)
|---|---|---|---|---|---|
| 1 | 2 | \( \bullet \bullet \bullet \) | \( \bullet \bullet \bullet \) | \( (1, 3) \) | \( (p_1, p_2) = (\bullet, \bullet \bullet) \)
| 2 | 3 | \( \bullet \bullet \bullet \) | \( \bullet \bullet \bullet \) | \( (1, 1, 2) \) | \( (p_1, p_2, p_3) = (\bullet, \bullet, \bullet) \)
| 3 | 3 | \( \bullet \bullet \bullet \) | \( \bullet \bullet \bullet \) | \( (2, 1, 1) \) | \( (p_1, p_2, p_3) = (\bullet, \bullet, \bullet) \)
| 4 | 2 | \( \bullet \bullet \) | \( \bullet \bullet \) | \( (1, 3) \) | \( (p_1, p_2) = (\bullet \bullet, \bullet) \)
For any $i \in [1, 4]$, $Q(i) = P_{m(i), p(i)}$, which together with (7.3.9) implies the commutativity of the left square of the following diagram

\[
\begin{array}{c}
(kK_{n(i)})^\wedge & \xrightarrow{\text{cabl}^B_{m(i)}} & (kK_4)^\wedge \\
\downarrow & & \downarrow \\
U(t_{n(i)})^\wedge & \xrightarrow{\text{cabl}^B_{m(i)}} & U(t_4)^\wedge
\end{array}
\]

where the upper and lower maps can in the right square are the canonical projections; the commutativity of the middle square follows from (7.3.4).

Consider the following table

| $i$ | $k(i) \in K_{n(i)}$ | $h(i) \in U(t_{n(i)})^\wedge$ | $g(i) \in U(t_4)^\wedge$ |
|-----|---------------------|---------------------|---------------------|
| 1   | $\frac{3}{2}x_{13}$ | $e^{-\mu_1t_2}$     | 1                   |
| 2   | $\frac{3}{2}x_{24}$ | $\text{Ad}(g(e^{-\mu_2t_{13}}))$ | $\text{cabl}^B_{1,1,2}(g)$ |
| 3   | $\frac{3}{2}x_{24}$ | $\text{Ad}(g(e^{-\mu_2t_{13}}))$ | $\text{cabl}^B_{1,1,1}(g)$ |
| 4   | $\frac{3}{2}x_{13}$ | $e^{-\mu_1t_2}$     | 1                   |

For any $i \in [1, 4]$, $\text{cabl}^B_{m(i)}(k(i)) = \tilde{x}_{i5}$ and $\text{can}(\tilde{x}_{i5}) = x_{i5} \in P_5^\wedge$. On the other hand, $b_{(\mu, \Phi)}^{P(i)}(k(i)) = h(i) \in U(t_{n(i)})^\wedge$. Since $\text{cabl}^B_{m(i)}(g) = g(i)$ for $i \in \{2, 3\}$, we have $\text{cabl}^B_{m(i)}(h(i)) = \text{Ad}(g(i))(e^{-\mu_1t_{12} \cdots i \cdots 4})$ for any $i \in [1, 4]$. The image of this element by $\text{can} \circ \text{Ad}(\Phi_{Q(i)}^{(\mu)}(\bullet \bullet \bullet \bullet \bullet))$ is then

\[
[\text{Ad}(\Phi_{Q(i)}^{(\mu)}(\bullet \bullet \bullet \bullet \bullet)) \circ \text{Ad}(g(i))(e^{-\mu_1t_{12} \cdots i \cdots 4})] = \text{Ad}([\Phi_{Q(i)}^{(\mu)}(\bullet \bullet \bullet \bullet \bullet) \cdot g(i)])(e^{\mu_{i5}}),
\]

where we use the notation $[g] := \text{can}(g)$ for $g \in U(p_5)^\wedge$.

By the commutativity of the diagram, one has therefore

(7.4.11) $\Phi_{Q(i)}^{(\mu)}(x_{i5}) = \text{Ad}([\Phi_{Q(i)}^{(\mu)}(\bullet \bullet \bullet \bullet \bullet) \cdot g(i)])(e^{\mu_{i5}})$.

The statement then follows from

\[
\Phi^{(\bullet \bullet \bullet \bullet \bullet)} = \Phi(t_{123}, t_{34})^{-1}, \quad \Phi^{(\bullet \bullet \bullet \bullet \bullet \bullet)} = \Phi(t_{123}, t_{34})^{-1} \Phi(t_{12}, t_{23, 34})^{-1},
\]

\[
\text{cabl}^{\text{DR}}_{1,1,2}(g) = e^{(\mu/2)t_{12}} \Phi(t_{1,34}, t_{12}), \quad \text{cabl}^{\text{DR}}_{2,1,1}(g) = e^{(\mu/2)t_{123}} \Phi(t_{12,4}, t_{12,3}).
\]

Recall that $p_5$ contains a graded Lie subalgebra $f_5$, freely generated by the elements $e_{i5}$, $i \in [1, 3]$ of degree 1. Its degree completion $U(f_3)^\wedge$ is then a closed subalgebra of $U(p_5)^\wedge$. \qed
Proposition 7.9. We have

\[(7.4.12)\] \(\varphi^{(5)}_{(\mu, \phi)}(x_{15}) = \Phi(-e_{123,5}, e_{12,5})^{-1} \Phi(-e_{12,5}, e_{15})^{-1} \cdot e^{\mu e_{15}} \cdot \Phi(-e_{12,5}, e_{15}) \Phi(-e_{123,5}, e_{12,5}).\]

\[(7.4.13)\] \(\varphi^{(5)}_{(\mu, \phi)}(x_{25}) = \Phi(-e_{123,5}, e_{12,5})^{-1} e^{(-\mu/2)e_{12,5}} \Phi(-e_{12,5}, e_{25})^{-1} \cdot e^{\mu e_{25}} \cdot (\Phi(-e_{123,5}, e_{12,5})^{-1} e^{(-\mu/2)e_{12,5}} \Phi(-e_{12,5}, e_{25})^{-1})^{-1}.\]

\[(7.4.14)\] \(\varphi^{(5)}_{(\mu, \phi)}(x_{35}) = e^{(-\mu/2)e_{123,5}} \Phi(e_{35}, -e_{123,5}) \cdot e^{\mu e_{35}} \cdot (e^{(-\mu/2)e_{123,5}} \Phi(e_{35}, -e_{123,5}))^{-1}.\)

(\textit{equalities in } U(f_3)^\wedge). In particular, for \(i \in [1, 4],\) the elements \(\varphi^{(5)}_{(\mu, \phi)}(x_{15})\) of \(U(p_5)^\wedge\) in fact belong to the subalgebra \(U(f_3)^\wedge.\)

Proof. One has the identity \(e_{ij} = e_{kl} + e_{km} + e_{lm}\) for \(i, j, k, l, m\) are five distinct elements of \([1, 5].\) It implies the equality

\[e_{34} = e_{12,5} + e_{12}.\]

The same identity implies \(e_{45} = e_{12} + e_{13} + e_{23} = e_{12,3} + e_{12},\) therefore, using \(e_{14} = -e_{123,5},\) the equality

\[e_{12,3} = -e_{123,5} - e_{12}.\]

The two equalities imply the equality

\[(7.4.15)\] \(\Phi(e_{12,3}, e_{34}) = \Phi(-e_{123,5} - e_{12}, e_{12,5} + e_{12}).\)

As the series \(\Phi\) is the exponential of a Lie series without linear term, it satisfies the identity

\[(7.4.16)\] \(\Phi(a, b) = \Phi(a + z, b + z'),\)

where \(z, z'\) is any pair of elements which commute together and with both \(a\) and \(b.\) By applying this with \(a = -e_{123,5} - e_{12}, b = e_{12,5} + e_{12}, z = e_{12}, z' = -e_{12},\) this gives

\[(7.4.17)\] \(\Phi(-e_{123,5} - e_{12}, e_{12,5} + e_{12}) = \Phi(-e_{123,5}, e_{12,5}).\)

The equations (7.4.15) and (7.4.17) then imply

\[(7.4.18)\] \(\Phi(e_{12,3}, e_{34}) = \Phi(-e_{123,5}, e_{12,5}).\)

The relation \(\sum_{i \in [1, 5] \setminus \{2\}} e_{12} = 0\) implies the equality

\[e_{2,34} = -e_{12} - e_{25},\]

which implies

\[(7.4.19)\] \(\Phi(e_{12}, e_{2,34}) = \Phi(e_{12}, -e_{12} - e_{25}).\)

Equation \((7.4.16)\) with \(a = e_{12}, b = -e_{12} - e_{25}, z = -e_{12} - e_{12,5}, z' = -z,\) gives

\[(7.4.20)\] \(\Phi(e_{12}, -e_{12} - e_{25}) = \Phi(-e_{12,5}, e_{15}).\)
The equations (7.4.19) and (7.4.20) then imply
\[ (7.4.21) \quad \Phi(e_{12}, e_{2,34}) = \Phi(-e_{12,5}, e_{15}). \]
By exchange of indices 1 and 2, the same argument yields the equality
\[ (7.4.22) \quad \Phi(e_{12}, e_{1,34}) = \Phi(-e_{12,5}, e_{25}). \]

Proof of (7.4.12). Combining (7.4.18) and (7.4.21), we get the equality
\[ \Phi(e_{12}, e_{34})^{-1}\Phi(e_{12}, e_{2,34})^{-1} = \Phi(-e_{123,5}, e_{125})^{-1}\Phi(-e_{12,5}, e_{15})^{-1}. \]
Plugging this in equation (7.4.4), we get the claimed equality.

Proof of (7.4.13). Since \( e_{12} \) commutes with both \( e_{4,12} \) and \( e_{12,3} \), and since \( \Phi(e_{4,12}, e_{12,3}) \) is the exponential of a Lie polynomial in \( e_{4,12}, e_{12,3} \), one has \( \Phi(e_{4,12}, e_{12,3}) = \Phi(e_{4,12} + e_{12}, e_{12,3} + e_{12}) \).

8. Commutative diagrams relating de Rham and Betti morphisms

In \[ \ref{8.1} \] we recalled the properties of associators of relating braid groups with infinitesimal braid Lie algebras. In this section, we fix a pair \( (\mu, \Phi) \), where \( \mu \in \ker \Phi \) and \( \Phi \) is in the set \( \Phi_{\mu}(k) \) of associators with the parameter \( \mu \), and we use these properties for proving the commutativity of various diagrams relating \( (\mu, \Phi) \) and constituents of the completed versions of \( \ref{5.3.3} \) and \( \ref{6.3.3} \); recall that these constituents include, on the de Rham side (diagram \( \ref{5.3.3} \)), morphisms \( \hat{\ell}, \hat{M}_3(\text{pr}_{12}^\kappa), \hat{\omega}, \) and the map \( \text{row}(-) \cdot \text{col} \), and on the Betti side (diagram \( \ref{6.3.3} \)), the underlined versions of these maps. Using these results, we then prove the commutativity of \( \ref{8.1} \) for any \( (\mu, \Phi) \).

More precisely, in \[ \ref{8.1} \] we construct a commutative diagram relating \( \hat{\ell} \) to \( \hat{\ell} \) by \( (\mu, \Phi) \). In \[ \ref{8.2} \] we similarly relate \( \hat{M}_3(\text{pr}_{12}^\kappa) \) to \( \hat{M}_3(\text{pr}_{12}^\kappa) \) by \( (\mu, \Phi) \). In \[ \ref{8.3} \] we relate \( \hat{\omega} \) to \( \hat{\omega} \) by \( (\mu, \Phi) \). As the morphism \( \hat{\omega} \) (resp. \( \hat{\omega} \)) is related to a specific basis of the ideal \( J(\text{pr}_3) \) (resp. \( J(\text{pr}_3) \)), this relation involves a matrix \( P \in \text{GL}_3(U(\mathfrak{p}_3)^\kappa) \) which accounts for the discrepancy of these choices of bases. The results of \[ \ref{8.1}-\ref{8.3} \] are used in \[ \ref{8.4} \] to relate \( \hat{\rho} \) to \( \hat{\rho} \) by \( (\mu, \Phi) \), where we recall that
\(\hat{\rho}\) and \(\hat{\sigma}\) are compositions of constituents of \([5.3.3]\) and \([6.3.3]\). In §8.3, using the computations from \([5.3.3]\), we compute some linear combinations of entries of the matrix \(\mathcal{P} \in \text{GL}_3((U(f_2)^{\otimes 2})^\wedge)\), which is the image of \(P\) under the morphism \(\text{pr}_{12} : U(p_5) \to U(f_2)^{\otimes 2}\) (Lemma 8.8). In the same subsection, we then use this information to relate col to col \(\otimes\) row by \((\mu, \Phi)\) (Lemma 8.9). In §8.6 we relate the matrix col \(\otimes\) row to the matrix col \(\otimes\) row by \((\mu, \Phi)\) (Lemma 8.11); together with Lemma 8.9 this result enables us to relate row \(\otimes\) col to row \(\otimes\) col by \((\mu, \Phi)\). Finally, in §8.8 (Proposition 8.17), the results of §§8.5 and 8.6 are combined in §8.7 to relate row \(\otimes\) row \(\otimes\) col \(\otimes\) col to row \(\otimes\) row \(\otimes\) col \(\otimes\) col by \((\mu, \Phi)\). Finally, in §8.8 (Proposition 8.17), the results of §§8.5 and 8.6 are combined in §8.7 to relate row \(\otimes\) row \(\otimes\) col \(\otimes\) col to row \(\otimes\) row \(\otimes\) col \(\otimes\) col by \((\mu, \Phi)\). Finally, in §8.8 (Proposition 8.17), the results of §§8.5 and 8.6 are combined in §8.7 to relate row \(\otimes\) row \(\otimes\) col \(\otimes\) col to row \(\otimes\) row \(\otimes\) col \(\otimes\) col by \((\mu, \Phi)\).

Throughout this section, we fix \(\mu \in k^\times\) and \(\Phi \in M_\mu(k)\).

8.1. Commutative diagram relating \(\hat{\ell}\) and \(\hat{\ell}\). One checks that the morphism
\[
(8.1.1) \quad \mathcal{g}_{(\mu, \Phi)} : (kF_2)^\wedge \to U(f_2)^\wedge
\]
defined in §17 is an isomorphism of topological Hopf algebras and is given by
\[
(8.1.2) \quad X_0 \mapsto \Phi(e_0, e_1)e^{\mu e_0}\Phi(e_0, e_1)^{-1}, \quad X_1 \mapsto e^{\mu e_1}.
\]
Then:

**Lemma 8.1.** The following diagram of topological Hopf algebras
\[
\begin{array}{ccc}
(kF_2)^\wedge & \xrightarrow{\hat{\ell}} & (kP_5)^\wedge \\
\mathcal{g}_{(\mu, \Phi)} \downarrow & & \downarrow \mathcal{g}_{(\mu, \Phi)}^{(5)} \\
U(f_2)^\wedge & \xrightarrow{\hat{\ell}} & U(p_5)^\wedge
\end{array}
\]
is commutative, where \(\mathcal{g}_{\Phi}\) is as in (8.1.1), \(\hat{\ell}\) is the completed version of \(\ell\) from (6.1), \(\mathcal{g}_{(\mu, \Phi)}^{(5)}\) is as in (5.3.3), and \(\hat{\ell}\) is the completed version of \(\ell\) from (6.1.2).

**Proof.** Using \(x_{12} = \sigma_1^2\), one computes \(\mathcal{g}_{(\mu, \Phi)}^{(5)} \circ \hat{\ell}(X_1) = e^{\mu e_{12}}\), which is computed to be equal to \(\hat{\ell} \circ \mathcal{g}_{(\mu, \Phi)}(X_1)\). Using \(x_{23} = \sigma_2^2\), one computes \(\mathcal{g}_{(\mu, \Phi)}^{(5)} \circ \hat{\ell}(X_0) = \Phi(e_{23}, e_{12})e^{\mu e_{23}}\Phi(e_{23}, e_{12})^{-1}\), whereas \(\hat{\ell} \circ \mathcal{g}_{(\mu, \Phi)}(X_1) = \Phi(e_{12}, e_{23})^{-1}e^{\mu e_{23}}\Phi(e_{12}, e_{23})\). The relation \(\Phi(e_{12}, e_{23})\Phi(e_{23}, e_{12}) = 1\) then implies that these images are equal. \(\square\)

8.2. Commutative diagram relating \(M_3(pr_{12}^*)\) and \(M_3(pr_{12}^*)\).

**Lemma 8.2.** The following
\[
\begin{array}{ccc}
(kP_5)^\wedge & \xrightarrow{pr_1^\wedge} & (kF_2)^\wedge \\
\mathcal{g}_{(\mu, \Phi)} \downarrow & & \downarrow \mathcal{g}_{(\mu, \Phi)}^{(5)} \\
U(p_5)^\wedge & \xrightarrow{pr_1^\wedge} & U(f_2)^\wedge \xrightarrow{\text{Ad}(\Phi(e_{02}, e_{12})^{-1})} U(f_2)^\wedge \\
\mathcal{g}_{(\mu, \Phi)} \downarrow & & \downarrow \mathcal{g}_{(\mu, \Phi)}^{(5)} \\
U(p_5)^\wedge & \xrightarrow{pr_2^\wedge} & U(f_2)^\wedge \xrightarrow{\text{Ad}(\Phi(e_{02}, e_{12})^{-1}e_{02}^{-1}e_{12})^{-1}} U(f_2)^\wedge
\end{array}
\]
are commutative diagrams of topological Hopf algebras.
Proof. If follows from Proposition [5.4] that the elements \(x_{i,i+1}, i \in C_5\) generate \(P_5^*\). So it suffices to check the commutativity of the diagrams on these elements. The generators can be shown to have the same images under the two maps of the first diagram, these images being given by the following table.

| generator \(x\) of \(P_5^*\) | \(x_{12}\) | \(x_{23}\) | \(x_{34}\) | \(x_{15}\) | \(x_{15}\) |
|---------------------------|--------|--------|--------|--------|--------|
| \(\text{Ad}(\Phi(e_0, e_1)^{-1}) \circ \varrho_{(\mu, \Phi)} \circ \text{pr}^\lambda_1(x)\) | \(e^{\mu e_0}\) | \(\Phi(e_0, e_1)^{-1} e^{\mu e_1} \Phi(e_0, e_1)\) | \(e^{\mu e_0}\) | 1 |

This implies that the first diagram commutes.

The situation in the case of the second diagram is given by the following table.

| generator \(x\) of \(P_5^*\) | \(x_{12}\) | \(x_{23}\) | \(x_{34}\) | \(x_{15}\) | \(x_{15}\) |
|---------------------------|--------|--------|--------|--------|--------|
| \(\varrho_{(\mu, \Phi)} \circ \text{pr}^\lambda_1(x)\) | \(1\) | \(\Phi(e_0, e_1)^{-1} e^{\mu e_1}, \cdot \Phi(e_\infty, e_1)\) | \(\Phi(e_0, e_1)^{-1} e^{\mu e_1}, \cdot \Phi(e_\infty, e_1)\) | \(\Phi(e_0, e_1)^{-1} e^{\mu e_1}, \cdot \Phi(e_\infty, e_1)\) | 1 |

where \(* = \Phi(e_\infty, e_1)^{-1} e^{-(\mu/2)} e_1 \Phi(e_0, e_1)^{-1} e^{-\mu e_0} \Phi(e_0, e_1)^{-1} e^{-(\mu/2)} e_1 \Phi(e_\infty, e_1)\). Conjugating \(\mu_{[7.4.10]}\) by \(\Phi(t_{13}, t_{12})^{-1} e^{-(\mu/2)} t_{12}\), taking the image of the resulting identity by the morphism \(U(t_3)^\wedge \to U(f_2)^\wedge\) given by \(t_{12} \mapsto e_1, t_{23} \mapsto e_0, t_{13} \mapsto e_\infty\) and using the identity \(\Phi(e_1, e_0)^{-1} = \Phi(e_0, e_1)\), one obtains the equality \(* = e^{\mu e_\infty}\). It follows that the second diagram commutes. 

\[\square\]

Lemma 8.3. The following diagram commutes

\[
\begin{array}{cccc}
M_3((kP_5^*)^\wedge) & M_3((kF_2)^{\otimes 2})^\wedge \\
\downarrow \text{M}_3(\varrho_{(\mu, \Phi)}^{(5)}) & \downarrow \text{M}_3(\varrho_{(\mu, \Phi)}^{(5)} \circ \text{pr}^\lambda_2) \\
M_3(U(p_5)^\wedge) & M_3((U(f_2)^{\otimes 2})^\wedge) \\
\end{array}
\]

where

\[
(8.2.1) \quad \kappa := e^{-(\mu/2)} f_1 \Phi(e_0, e_1) \Phi(f_\infty, f_1) \in ((U(f_2)^{\otimes 2})^\wedge)^\times,
\]

and \(\text{Ad}(\kappa)\) denotes the automorphism taking each entry of a matrix to its image by the automorphism \(\text{Ad}(\kappa)\) of \((U(f_2)^{\otimes 2})^\wedge\).

Proof. Combining the tensor product of the two diagrams from Lemma 8.2 with the diagram expressing the compatibility of \(\varrho_{(\mu, \Phi)}^{(5)}\) with the coproducts of its source and target, one obtains the following commutative diagram

\[
\begin{array}{cccc}
(kP_5^*)^\wedge & \varrho_{(\mu, \Phi)}^{(5)} \\
\downarrow \text{pr}^\lambda_{12} & \downarrow \varrho_{(\mu, \Phi)}^{(5)} \circ \text{pr}^\lambda_{12} \\
U(p_5)^\wedge & M_3((U(f_2)^{\otimes 2})^\wedge) \\
\end{array}
\]
from where we derive the announced commutative diagram.

\[ \square \]

8.3. Commutative diagram relating \( \hat{\omega} \) and \( \hat{\varpi} \).

**Lemma 8.4.** There exists a unique element \( P \in M_3(U(p_5)^{\wedge}) \), such that

\[
\begin{pmatrix}
\mathfrak{u}_{(\mu, \Phi)}^{(5)}(x_{15} - 1) \\
\mathfrak{u}_{(\mu, \Phi)}^{(5)}(x_{25} - 1) \\
\mathfrak{u}_{(\mu, \Phi)}^{(5)}(x_{35} - 1)
\end{pmatrix}
= P
\begin{pmatrix}
e_{15} \\
e_{25} \\
e_{35}
\end{pmatrix}
\]

(equality in \( M_{3 \times 1}(U(p_5)^{\wedge}) \)). Then \( P \in \text{GL}_3(U(p_5)^{\wedge}) \) and the following diagram is commutative

\[
\begin{array}{ccc}
(kP_5^\wedge) & \xrightarrow{\hat{\omega}} & M_3((kP_5^\wedge)^\wedge) \\
\downarrow \mathfrak{u}_{(\mu, \Phi)}^{(5)} & & \downarrow M_3(\mathfrak{u}_{(\mu, \Phi)}^{(5)}) \\
U(p_5)^\wedge & \xrightarrow{\varpi} & M_3(U(p_5)^{\wedge}) \xrightarrow{\text{Ad}(P)} M_3(U(p_5)^{\wedge})
\end{array}
\]

where \( \hat{\omega}, \hat{\varpi} \) are the completions of \( \omega, \varpi \).

**Proof.** One checks that the following diagram commutes

\[
\begin{array}{ccc}
(kP_5^\wedge) & \xrightarrow{pr_5^\wedge} & (kF_2)^\wedge \\
\downarrow \mathfrak{u}_{(\mu, \Phi)}^{(5)} & & \downarrow \mathfrak{u}_{(\mu, \Phi)}^{(5)} \\
U(p_5)^\wedge & \xrightarrow{pr_5^\wedge} & U(f_2)^\wedge
\end{array}
\]

Since its vertical arrows are isomorphisms, it follows that \( \mathfrak{u}_{(\mu, \Phi)}^{(5)} \) induces an isomorphism between the kernels of the horizontal maps, which are the completions \( J(pr_5)^{\wedge} \) and \( J(pr_5)^{\wedge} \) of the ideals \( J(pr_5) \) and \( J(pr_5) \). The existence and uniqueness of \( P \in M_3(U(p_5)^{\wedge}) \) with the indicated properties follows from the facts that \( (e_{15})_{i \in [1, 3]} \) is a basis of \( J(pr_5) \) and that the \( \mathfrak{u}_{(\mu, \Phi)}^{(5)}(x_{j5} - 1) \), \( j \in [1, 3] \) belong to this space. The fact that \( P \in \text{GL}_3(U(p_5)^{\wedge}) \) follows from the fact that \( \mathfrak{u}_{(\mu, \Phi)}^{(5)}(x_{j5} - 1) \), \( j \in [1, 3] \) is also a basis of this module. It remains to apply Lemma 8.2 with \( R = (kP_5^\wedge)^\wedge, J = J(pr_5)^{\wedge}, (j_a)_{a \in [1, d]} = (x_{i5} - 1)_{i \in [1, 3]}, R' = U(p_5)^{\wedge}, J' = J(pr_5)^{\wedge}, (j'_a)_{a \in [1, d]} = (e_{i5})_{i \in [1, 3]}, f = \mathfrak{u}_{(\mu, \Phi)}^{(5)}. \]

Recall the element \( P \in \text{GL}_3(U(p_5)^{\wedge}) \) from Lemma 8.4. Define \( \mathcal{P} \in \text{GL}_3((U(f_2)^{\otimes 2})^{\wedge}) \) by

\[
\mathcal{P} := M_3(pr_{12}^\wedge)(P).
\]

Let \( \kappa \) be as in 8.2.1 and let \( \kappa \cdot \mathcal{P} \) be the matrix obtained by the left multiplication by the scalar \( \kappa \) of the entries of the matrix \( \mathcal{P} \). As \( \kappa \) is invertible, one has

\[
\kappa \cdot \mathcal{P} \in \text{GL}_3((U(f_2)^{\otimes 2})^{\wedge}).
\]
Then one checks that the following diagram commutes

(8.3.4) \[
\begin{array}{ccc}
\mathcal{M}_3(U(p_3)^\wedge) & \xrightarrow{\operatorname{Ad}(P)} & \mathcal{M}_3(U(p_5)^\wedge) \\
\rho & \downarrow & \rho \\
\mathcal{M}_3(U(f_2)^\wedge) & \xrightarrow{\operatorname{Ad}(\kappa)} & \mathcal{M}_3(U(f_2)^\wedge)
\end{array}
\]

8.4. **Commutative diagram relating \(\hat{\rho}\) and \(\hat{\rho}\).** As the morphism \(\rho : U(f_2) \to \mathcal{M}_3(U(f_2)^\wedge)\) given by (8.2.4) is graded, it gives rise to a continuous algebra morphism

(8.4.1) \[
\hat{\rho} : U(f_2)^\wedge \to \mathcal{M}_3((U(f_2)^\wedge)^\wedge).
\]

On the other hand, the morphism \(\hat{\rho} : kF_2 \to \mathcal{M}_3((kF_2)^\wedge)\) is the composition of the continuous morphisms \(\hat{\rho} : kF_2 \to \mathcal{M}_3(p_{12})\), therefore it gives rise to a continuous algebra morphism

(8.4.2) \[
\hat{\rho} : (kF_2)^\wedge \to \mathcal{M}_3(((kF_2)^\wedge)^\wedge).
\]

**Lemma 8.5.** The following diagram commutes

(8.4.3) \[
\begin{array}{ccc}
(kF_2)^\wedge & \xrightarrow{\hat{\rho}} & \mathcal{M}_3((kF_2)^\wedge) \\
\mathcal{M}_3(M_{\mu,\Phi}) & \xrightarrow{\mathcal{M}_3(\mu_{\Phi})} & \mathcal{M}_3((kF_2)^\wedge) \\
U(f_2)^\wedge & \xrightarrow{\hat{\rho}} & \mathcal{M}_3(U(f_2)^\wedge) \\
\mathcal{M}_3(U(f_2)^\wedge) & \xrightarrow{\mathcal{M}_3(\mu_{\Phi})} & \mathcal{M}_3(U(f_2)^\wedge)
\end{array}
\]

**Proof.** This follows from the juxtaposition of the diagram (8.3.4) and of the diagrams from Lemmas 8.1, 8.3 and 8.4.

8.5. **Relationship between \(\text{col}f\) and \(\text{col}\).**

**Lemma 8.6.** The matrix \(P \in \text{GL}_3(U(p_3)^\wedge)\) from Lemma 8.4 in fact belongs to \(\text{GL}_3(U(f_3)^\wedge)\).

**Proof.** It follows from Proposition 7.9 that for \(i \in [1,3]\), \(\frac{\mu}{(5)(\mu,\Phi)}(x_{i5} - 1)\) belongs to \(U(f_3)^\wedge\). Since it also belongs to the augmentation ideal of \(U(p_3)^\wedge\), it belongs to the augmentation ideal \(U(f_3)^\wedge\) of \(U(f_3)^\wedge\). As \(U(f_3)^\wedge\) is a free topological algebra over generators \((e_{ij})_{i \in [1,3]}\), there exists a unique family \((\tilde{p}_{ij})_{i,j \in [1,3]}\) of elements of \(U(f_3)^\wedge\), such that for any \(i \in [1,3]\), one has

\[
\frac{\mu}{(5)(\mu,\Phi)}(x_{i5} - 1) = \sum_{j \in [1,3]} \tilde{p}_{ij} \cdot e_j.
\]

Then \(\tilde{P} := (\tilde{p}_{ij})_{i,j \in [1,3]}\) satisfies the equality

\[
\begin{pmatrix}
\frac{\mu}{(5)(\mu,\Phi)}(x_{15} - 1) \\
\frac{\mu}{(5)(\mu,\Phi)}(x_{25} - 1) \\
\frac{\mu}{(5)(\mu,\Phi)}(x_{35} - 1)
\end{pmatrix} = \hat{P} \begin{pmatrix} e_{15} \\
e_{25} \\
e_{35}
\end{pmatrix}
\]

The uniqueness of a matrix satisfying this equality (see Lemma 8.4) then implies that \(P = \tilde{P}\), therefore that \(P \in \mathcal{M}_3(U(f_3)^\wedge)\). For each \(i \in [1,3]\), one has \(\frac{\mu}{(5)(\mu,\Phi)}(x_{i5} - 1) = \mu e_{i5} + \text{(element of } U(f_3)^\wedge \text{ of degree } \geq 2)\). This implies that \(P \in \mu \text{id} + \mathcal{M}_3(U(f_3)^\wedge)\), therefore that \(P^{-1}\) has coefficients in \(U(f_3)^\wedge\). Therefore \(P \in \text{GL}_3(U(f_3)^\wedge)\).
Lemma 8.8. Set \( a \in \mathbb{K} \langle \langle e_1, \ldots, e_n \rangle \rangle \) to be the elements of \( U(f_3)^\wedge \) such that

\[
(8.5.1) \quad P = \begin{pmatrix}
p_{11} & p_{12} & p_{13} \\
p_{21} & p_{22} & p_{23} \\
p_{31} & p_{32} & p_{33}
\end{pmatrix},
\]

Lemma 8.7. Let \( \mathbb{K} \langle \langle e_1, \ldots, e_n \rangle \rangle \) be the noncommutative formal power series algebra over generators \( e_1, \ldots, e_n \). For \( i \in [1, n] \), let \( \Pi_i \) be the endomorphism of \( \mathbb{K} \langle \langle e_1, \ldots, e_n \rangle \rangle \) defined by the identity

\[
\forall a \in \mathbb{K} \langle \langle e_1, \ldots, e_n \rangle \rangle, \quad a = \epsilon(a)1 + \sum_{i=1}^{n} \Pi_i(a)e_i,
\]

where \( \epsilon \) is the augmentation of \( \mathbb{K} \langle \langle e_1, \ldots, e_n \rangle \rangle \). Then one has for any \( k \geq 1 \) and \( a_1, \ldots, a_k \in \mathbb{K} \langle \langle e_1, \ldots, e_n \rangle \rangle \),

\[
\Pi_i(a_1 \cdots a_k) = \sum_{s=1}^{k} a_1 \cdots a_{s-1} \Pi_i(a_s) \epsilon(a_{s+1} \cdots a_k).
\]

Proof. The statement is trivial for \( k = 1 \). When \( k = 2 \), it is proved by using the decompositions of \( a_1 \) and \( a_2 \). The general case can be proved by induction, using the statement for \( k = 2 \). \( \square \)

Lemma 8.8. Set

\[
(8.5.2) \quad v := \frac{1}{\mu} e^{\mu f_1} \frac{\Gamma_\phi(e_1) \Gamma_\phi(f_1)}{\Gamma_\phi(e_1 + f_1)} \in ((U(f_2)^{\otimes 2})^\wedge)^\times.
\]

Then

\[
(8.5.3) \quad \kappa \cdot (\overline{p}_{11} - \overline{p}_{12}) \cdot v = e^{\mu f_1},
\]

\[
(8.5.4) \quad \kappa \cdot (\overline{p}_{21} - \overline{p}_{22}) \cdot v = -1,
\]

\[
(8.5.5) \quad \overline{p}_{31} - \overline{p}_{32} = 0
\]

(\textit{equalities in } \( U(f_2)^{\otimes 2)^\wedge} \), \( \kappa \) is as in \( 8.3.1 \) and \( p \mapsto \overline{p} \) is the map \( p_{ij}^\wedge \), so that \( \overline{P} = (\overline{p}_{ij})_{i,j \in [1,3]} \) and \( (p_{ij})_{i,j \in [1,3]} \) is as in \( 8.3.1 \)).

Proof. Let \( i \in [1, 3] \). Recall that \( (p_{ij})_{i,j \in [1,3]} \) is the triple of elements of \( U(f_3)^\wedge \) such that \( a_{(5, \phi)}(x_{i5} - 1) = \sum_{j \in [1,3]} p_{ij} e_{j5} \). In the notation of Lemma 8.7 with \( n = 3 \), \( (e_1, e_2, e_3) := (e_{15}, e_{25}, e_{35}) \), this means that \( p_{ij} = \Pi_j(\mathfrak{u}_{(5, \phi)}(x_{i5} - 1)) \) for any \( i, j \in [1, 3] \). Therefore

\[
(8.5.6) \quad p_{11} - p_{12} = (\Pi_1 - \Pi_2)(\mathfrak{u}_{(5, \phi)}(x_{15} - 1)).
\]

\textbf{Proof of (8.5.3).} Equation \( (8.5.6) \) for \( i = 1 \) and equation \( (8.4.12) \) yield

\[
p_{11} - p_{12} = (\Pi_1 - \Pi_2) \left( \Phi(-e_{123,5}, e_{125})^{-1} \Phi(-e_{12,5}, e_{15})^{-1} \cdot (e^{\mu e_{15}} - 1) \cdot \Phi(-e_{12,5}, e_{15}) \Phi(-e_{123,5}, e_{12,5}) \right).
\]

We now apply Lemma 8.7 for \( k = 5 \) and \( a_1 = \Phi^{-1}(-e_{35}, e_{12,5}), \ldots, a_5 := \Phi(-e_{35}, e_{12,5}). \) Since \( \epsilon(a_4) = \epsilon(e^{\mu e_{15}} - 1) = 0 \), we obtain

\[
p_{11} - p_{12} = a_1 a_2 a_3 a_4 (\Pi_1 - \Pi_2)(a_5) + a_1 a_2 a_3 (\Pi_1 - \Pi_2)(a_4)(a_5) + a_1 a_2 (\Pi_1 - \Pi_2)(a_3)(a_4)(a_5).
\]
One computes $\epsilon(a_4) = \epsilon(a_5) = 1$, and $(\Pi_1 - \Pi_2)(a_3) = \frac{e^{\mu_{e_{15}} - 1}}{e_{15}}$. Lemma 8.7 implies that the restriction of $\Pi_1 - \Pi_2$ to the subalgebra of $U(f_1)^{\wedge}$ generated by $e_{12,5}$ and $e_{35}$ is zero. It follows that $(\Pi_1 - \Pi_2)(a_5) = 0$. All this implies that

$$p_{11} - p_{12} = a_1a_2a_3(\Pi_1 - \Pi_2)(a_4) + a_1a_2\frac{e^{\mu_{e_{15}} - 1}}{e_{15}}.$$ 

Decompose $\Phi(a, b) = 1 + \varphi_0(a, b)a + \varphi_1(a, b)b$, where $\varphi_0$, $\varphi_1 \in k\langle a, b \rangle$. Then $a_4 = \Phi(-e_{12,5}, e_{15}) = 1 + \varphi_0(-e_{12,5}, e_{15}) + \varphi_1(-e_{12,5}, e_{15})e_{15}$. So $\Pi_1(a_4) = -\varphi_0(-e_{12,5}, e_{15}) + \varphi_1(-e_{12,5}, e_{15})$, $\Pi_2(a_4) = -\varphi_0(-e_{12,5}, e_{15})$, so $(\Pi_1 - \Pi_2)(a_4) = \varphi_1(-e_{12,5}, e_{15})$. Finally

$$p_{11} - p_{12} = a_1a_2a_3\varphi_1(-e_{12,5}, e_{15}) + a_1a_2\frac{e^{\mu_{e_{15}} - 1}}{e_{15}}.$$ 

(equality in $U(f_1)^{\wedge}$).

The element $\overline{p}_{11} - \overline{p}_{12}$ in $(U(f_2)^{\wedge})^{\wedge}$ is the image of $p_{11} - p_{12}$ by the morphism $pr_{12}$, given by $e_{15} \mapsto f_1$, $e_{25} \mapsto e_1$, $e_{35} \mapsto e_\infty + f_0$, which is such that $e_{123,5} \mapsto (e_1 + e_\infty) + (f_1 + f_0) = -e_0 - f_\infty$. Therefore

$$\overline{p}_{11} - \overline{p}_{12} = \Phi(e_0 + f_\infty, e_1 + f_1)^{-1}\Phi(-e_1 - f_1, f_1)^{-1}\left((e^{\mu_f_1} - 1)\varphi_1(-e_1 - f_1, f_1) + \frac{e^{\mu_f_1} - 1}{f_1}\right).$$

Since $\Phi$ is the exponential of a Lie series without linear terms, and since $e_1$ and $f_1$ commute, we have $\Phi(-e_1 - f_1, f_1) = 1$. So,

$$\overline{p}_{11} - \overline{p}_{12} = \Phi(e_0 + f_\infty, e_1 + f_1)^{-1}\left((e^{\mu_f_1} - 1)\varphi_1(-e_1 - f_1, f_1) + \frac{e^{\mu_f_1} - 1}{f_1}\right).$$

By (8.2.1), and by the commutation of $e_i$ with $f_j$ ($i, j \in \{0, 1, \infty\}$), we have $\Phi(e_0 + f_\infty, e_1 + f_1)^{-1} = \kappa^{-1}e^{-(\mu/2)f_1}$. Therefore

$$\overline{p}_{11} - \overline{p}_{12} = \kappa^{-1}\frac{e^{(\mu/2)f_1} - e^{-(\mu/2)f_1}}{f_1}\left(1 + f_1\varphi_1(-e_1 - f_1, f_1)\right).$$

Equality (7.2.1), together with the fact that the variables $-e_1 - f_1$ and $f_1$ commute, implies

$$1 + f_1\varphi_1(-e_1 - f_1, f_1) = \frac{\Gamma_\Phi(e_1 + f_1)\Gamma_\Phi(-f_1)}{\Gamma_\Phi(e_1)},$$

therefore

$$\overline{p}_{11} - \overline{p}_{12} = \kappa^{-1}\frac{e^{(\mu/2)f_1} - e^{-(\mu/2)f_1}}{f_1}\frac{\Gamma_\Phi(e_1 + f_1)\Gamma_\Phi(-f_1)}{\Gamma_\Phi(e_1)}.$$ 

The identity from Lemma 7.3 (2), together with (8.5.2), then implies $\overline{p}_{11} - \overline{p}_{12} = \kappa^{-1}e^{\mu_{f_1} f_1 - 1}$, as claimed.

Proof of (8.5.4). Equation (8.5.6) for $i = 2$ and equation (7.4.13) yield

$$p_{21} - p_{22} = (\Pi_1 - \Pi_2)\left(\Phi(-e_{123,5}, e_{12,5})^{-1}e^{-(\mu/2)e_{12,5}}\Phi(-e_{12,5}, e_{25})^{-1}\right)\left((e^{\mu_{e_{25}} - 1}).

\Phi(-e_{12,5}, e_{25})e^{(\mu/2)e_{12,5}}\Phi(e_{123,5}, e_{12,5})\right).$$
We now apply Lemma 8.7 with \( a_1 := \Phi(-e_{123,5}, e_{123,5})^{-1}, \ldots, a_7 := \Phi(-e_{123,5}, e_{123,5}) \). Since \( \epsilon(a_4) = \epsilon(e^{\mu e_{25}} - 1) = 0 \), we get

\[
p_{21} - p_{22} = a_1 a_2 a_3 a_4 a_5 (\Pi_1 - \Pi_2)(a_7) + a_1 a_2 a_3 a_4 a_5 (\Pi_1 - \Pi_2)(a_6) e(a_7)
+ a_1 a_2 a_3 a_4 (\Pi_1 - \Pi_2)(a_5) e(a_7) + a_1 a_2 a_3 (\Pi_1 - \Pi_2)(a_4) e(a_7).
\]

The elements \( a_6 \) and \( a_7 \) belong to the subalgebra of \( U(f_3)^\wedge \) generated by \( e_{123,5} \) and \( e_{35} \). Lemma 8.7 implies that the restriction of \( \Pi_1 - \Pi_2 \) to this subalgebra is zero. This implies that \( (\Pi_1 - \Pi_2)(a_6) = (\Pi_1 - \Pi_2)(a_7) = 0 \). One also has \( \epsilon(a_5) = \epsilon(a_6) = \epsilon(a_7) = 1 \), and \( (\Pi_1 - \Pi_2)(a_4) = \frac{1 - e^{\mu e_{25}}}{e_{35}} \). All this implies

\[
p_{21} - p_{22} = a_1 a_2 a_3 \left( a_4 (\Pi_1 - \Pi_2)(a_5) + \frac{1 - e^{\mu e_{25}}}{e_{25}} \right).
\]

Recall the decomposition \( \Phi(a, b) = 1 + \varphi_0(a, b)a + \varphi_1(a, b)b \), where \( \varphi_0, \varphi_1 \in k(\langle a, b \rangle) \). Then \( a_5 = \Phi(-e_{123,5}, e_{25}) = 1 + \varphi_0(-e_{123,5}, e_{25})(-e_{123,5}) + \varphi_1(-e_{123,5}, e_{25})e_{25} \). So \( \Pi_1(a_5) = -\varphi_0(-e_{123,5}, e_{15}), \Pi_2(a_5) = -\varphi_0(-e_{123,5}, e_{15}) + \varphi_1(-e_{123,5}, e_{25}), \) so \( (\Pi_1 - \Pi_2)(a_5) = -\varphi_1(-e_{123,5}, e_{25}) \). So

\[
p_{21} - p_{22} = a_1 a_2 a_3 \left( - (e^{\mu e_{25}} - 1) \varphi_1(-e_{123,5}, e_{25}) + \frac{1 - e^{\mu e_{25}}}{e_{25}} \right)
= \Phi(-e_{123,5}, e_{123,5})^{-1} e^{-(\mu/2)}e_{123,5} \Phi(-e_{123,5}, e_{35})^{-1} \left( 1 + e_{25} \varphi_1(-e_{123,5}, e_{25}) \right) \frac{1 - e^{\mu e_{25}}}{e_{25}}
\]
(equality in \( U(f_3)^\wedge \)).

The morphism \( \text{pr}_{12} : U(p_5) \to U(f_2)^{\otimes 2} \) is such that \( e_{25} \mapsto e_1, e_{123,5} \mapsto e_1 + f_1, e_{123,5} \mapsto -e_0 - f_\infty \). The image of the last equality under \( \text{pr}_{12} \) is therefore

\[
\overline{p}_{21} - \overline{p}_{22} = \Phi(e_0 + f_\infty, e_1 + f_1)^{-1} e^{-(\mu/2)}(e_1 + f_1) \Phi(-e_1 - f_1, e_1)^{-1} \left( 1 + e_1 \varphi_1(-e_1 - f_1, e_1) \right) \frac{1 - e^{\mu e_1}}{e_1}.
\]

Since \( \Phi \) is the exponential of a Lie series without linear terms, and since \( e_1 \) and \( f_1 \) commute, we have \( \Phi(-e_1 - f_1, e_1) = 1 \). Using again \( \Phi(e_0 + f_\infty, e_1 + f_1)^{-1} = \kappa^{-1} e^{-(\mu/2)f_1} \), and the commutation of \( e_1 \) and \( f_1 \), we obtain

\[
\overline{p}_{21} - \overline{p}_{22} = \kappa^{-1} e^{-(\mu/2)e_1 - \mu f_1} \left( 1 + e_1 \varphi_1(-e_1 - f_1, e_1) \right) \frac{1 - e^{\mu e_1}}{e_1}.
\]

Equality (7.2.1), together with the fact that the variables \( -e_1 - f_1 \) and \( e_1 \) commute, implies

\[
1 + e_1 \varphi_1(-e_1 - f_1, e_1) = \frac{\Gamma_\Phi(e_1 + f_1) \Gamma_\Phi(-e_1)}{\Gamma_\Phi(f_1)},
\]

therefore

\[
\overline{p}_{21} - \overline{p}_{22} = \kappa^{-1} e^{-(\mu/2)e_1 - \mu f_1} \cdot \frac{\Gamma_\Phi(e_1 + f_1) \Gamma_\Phi(-e_1)}{\Gamma_\Phi(f_1)} \cdot \frac{1 - e^{\mu e_1}}{e_1}.
\]

The identity from Lemma 8.6 (2), together with \( 8.5.2 \), then implies \( \overline{p}_{21} - \overline{p}_{22} = -\kappa^{-1} v^{-1} \), as claimed.

Proof of 8.5.5. Equation 8.5.6 for \( i = 3 \) and equation 7.4.14 yield

\[
p_{31} - p_{32} = (\Pi_1 - \Pi_2) \left( e^{-(\mu/2)e_{123,5}} \Phi(e_{35}, -e_{123,5}) \cdot (e^{\mu e_{35}} - 1) \cdot (e^{-(\mu/2)e_{123,5}} \Phi(e_{35}, -e_{123,5}))^{-1} \right).
\]

As we have seen, Lemma 8.7 implies that the restriction of \( \Pi_1 - \Pi_2 \) to the subalgebra of \( U(f_3)^\wedge \) generated by \( e_{123,5} \) and \( e_{35} \) is zero. Since the argument of \( \Pi_1 - \Pi_2 \) in the above equality belongs to this subalgebra, we have \( p_{31} - p_{32} = 0 \). It follows that \( \overline{p}_{31} - \overline{p}_{32} = 0 \), as claimed. \( \square \)
Lemma 8.9. We have

\[(8.5.7)\]
\[M_{3\times 1}(a_{(\mu, \phi)}^{\otimes 2})(\text{col}) = \kappa \cdot \mathcal{P} \cdot \text{col} \cdot v\]

\[(equality \ in \ M_{3\times 1}(\mathbb{U}(f_2)^{\otimes 2})^\wedge)\], \ where \ v \ is \ given \ by \ \[(8.5.2)\].

\[\text{Proof.} \ By \ (8.1.2), \ the \ left-hand \ side \ is \ equal \ to \ \begin{pmatrix} e^{\mu f_1} - 1 \\ 0 \end{pmatrix}. \ By \ the \ definition \ of \ \text{col} \ in \ Lemma \ 5.6 \ and \ by \ (8.5.1), \ the \ right-hand \ side \ is \ equal \ to \ \kappa \cdot \begin{pmatrix} \mathcal{P}_{11} - \mathcal{P}_{12} \\ \mathcal{P}_{21} - \mathcal{P}_{22} \\ \mathcal{P}_{31} - \mathcal{P}_{32} \end{pmatrix} \cdot v. \ The \ result \ then \ follows \ from \ Lemma \ 8.8. \Box\]

8.6. Commutative diagram relating row and row.

Lemma 8.10. The morphism \(\hat{\rho}\) from \[(8.4)\] is such that

\[
\hat{\rho}(e^{\mu e_1} - 1) = \frac{e^{\mu(e_1 + f_1)} - 1}{e_1 + f_1} \cdot \text{col} \cdot \text{row}
\]

\[(equality \ in \ M_3((\mathbb{U}(f_2)^{\otimes 2})^\wedge)).\]

\[\text{Proof.} \ By \ Lemma \ 5.6, \ one \ has \ \rho(e_1) = \text{col} \cdot \text{row}. \ If \ k \geq 1, \ then \ \rho(e_1^k) = (\text{col} \cdot \text{row})^k = \text{col} \cdot (\text{row} \cdot \text{col})^{k-1} \cdot \text{row}. \ Since \ \text{row} \cdot \text{col} = e_1 + f_1, \ and \ since \ this \ element \ commutes \ with \ the \ entries \ of \ \text{col}, \ one \ has \ \rho(e_1^k) = (e_1 + f_1)^{k-1} \cdot \text{col} \cdot \text{row}.

\[
\hat{\rho}(e^{\mu e_1} - 1) = \sum_{k \geq 1} \frac{\mu^k}{k!} \rho(e_1^k) = \sum_{k \geq 1} \frac{\mu^k}{k!} (e_1 + f_1)^{k-1} \cdot \text{col} \cdot \text{row} = \frac{e^{\mu(e_1 + f_1)} - 1}{e_1 + f_1} \cdot \text{col} \cdot \text{row}.
\]

\[\Box\]

In Lemma 6.12, we defined \(\text{row} \in M_{1\times 3}((kF_2)^{\otimes 2})\) and \(\text{col} \in M_{3\times 1}((kF_2)^{\otimes 2})\).

Lemma 8.11. The elements \(\text{col} \cdot \text{row} \in M_3((kF_2)^{\otimes 2})\) and \(\text{col} \cdot \text{row} \in M_3((\mathbb{U}(f_2)^{\otimes 2})^\wedge)\) are related by

\[(8.6.1) \quad M_3((a_{(\mu, \phi)}^{\otimes 2})(\text{col} \cdot \text{row}) = \kappa \cdot \mathcal{P} \cdot \frac{e^{\mu(e_1 + f_1)} - 1}{e_1 + f_1} \cdot \text{col} \cdot \text{row} \cdot (\kappa \cdot \mathcal{P})^{-1}\]

\[\text{where } \kappa \text{ is as in } (8.2.1) \ (equality \ in \ M_3((\mathbb{U}(f_2)^{\otimes 2})^\wedge)).\]

\[\text{Proof.} \ According \ to \ Lemma \ 8.3, \ there \ is \ an \ equality\]

\[(8.6.2) \quad M_3((a_{(\mu, \phi)}^{\otimes 2}) \circ \hat{\rho} = \text{Ad}((\kappa \cdot \mathcal{P}) \circ \hat{\rho} \circ a_{(\mu, \phi)})\]

\[\text{of morphisms } (kF_2)^\wedge \to M_3((\mathbb{U}(f_2)^{\otimes 2})^\wedge). \ Apply \ this \ equality \ to \ X_1 - 1 \in kF_2.\]

By Lemma 6.12 \(\hat{\rho}(X_1 - 1) = \text{col} \cdot \text{row}\) so the image of \(X_1 - 1\) by the left-hand side of \(8.6.2\) is the left-hand side of \(8.6.1\).
By (8.1.2), the image of $X_1 - 1$ by $a_{(\mu, \phi)}$ is $e^{\mu f_1} - 1$. By Lemma 8.10 the image of this element under $\hat{\rho}$ is $\frac{e^{\mu (e_1 + f_1)} - 1}{e_1 + f_1}$ col · row, and the image of the latter element by $\text{Ad}(\Xi \cdot \mathcal{P})$ is the right-hand side of (8.6.2). This proves (8.6.2). □

**Lemma 8.12.** Set

\[(8.6.3)\quad u := \mu e^{-\mu f_1} \frac{\Gamma_\phi(e_1 + f_1)}{\Gamma_\phi(e_1)\Gamma_\phi(f_1)} \frac{e^{\mu (e_1 + f_1)} - 1}{e_1 + f_1} \in \left( (U(f_2)^{\otimes 2})^\wedge \right)^{\times},\]

where $v$ is as in (8.5.2).

Then

\[(8.6.4)\quad M_{1 \times 3}(a_{(\mu, \phi)})^{\otimes 2} (\text{row}) = u \cdot \text{row} \cdot (\kappa \cdot \mathcal{P})^{-1}\]

(equality in $M_{3 \times 1}(U(f_2)^{\otimes 2})^\wedge$).

**Proof.** Since $\frac{e^{\mu (e_1 + f_1)} - 1}{e_1 + f_1}$ commutes with the entries of col, and since

\[(8.6.5)\quad \frac{e^{\mu (e_1 + f_1)} - 1}{e_1 + f_1} = v \cdot u,\]

the equation (8.6.1) may be rewritten as

\[M_{3 \times 1}(a_{(\mu, \phi)})^{\otimes 2} (\text{col}) \cdot M_{1 \times 3}(a_{(\mu, \phi)})^{\otimes 2} (\text{row}) = \kappa \cdot \mathcal{P} \cdot \text{col} \cdot v \cdot u \cdot \text{row} \cdot (\kappa \cdot \mathcal{P})^{-1}.\]

The equation (8.5.7) then enables one to rewrite this equality as follows

\[\kappa \cdot \mathcal{P} \cdot \text{col} \cdot v \cdot \left( M_{1 \times 3}(a_{(\mu, \phi)})^{\otimes 2} (\text{row}) - u \cdot \text{row} \cdot (\kappa \cdot \mathcal{P})^{-1} \right) = 0.\]

Since $v$ (resp. $\kappa \cdot \mathcal{P}$) is an invertible scalar (resp. matrix), and since the map $M_{1 \times 3}(U(f_2)^{\otimes 2})^\wedge \to M_3(U(f_2)^{\otimes 2})^\wedge$, $X \mapsto \text{col} \cdot X$ is injective, the map with the same source and target given by $X \mapsto \kappa \cdot \mathcal{P} \cdot \text{col} \cdot v \cdot X$ is also injective. This implies the statement. □

8.7. Commutative diagram relating $\text{row} \cdot (\ -) \cdot \text{col}$ and $\text{row} \cdot (\ -) \cdot \text{col}$.

**Lemma 8.13.** The following diagram of $k$-module morphisms is commutative

\[
\begin{array}{ccc}
M_3((kF_2)^{\otimes 2})^\wedge & \xrightarrow{\text{row} \cdot (\ -) \cdot \text{col}} & ((kF_2)^{\otimes 2})^\wedge \\
M_3((a_{(\mu, \phi)})^{\otimes 2}) & \simeq & M_3((U(f_2)^{\otimes 2})^\wedge) \\
M_3(U(f_2)^{\otimes 2})^\wedge & \simeq & \text{Ad}(\kappa \cdot \mathcal{P}) \\
\end{array}
\]

where $u \cdot (\ -) \cdot v$ is the linear map $x \mapsto u \cdot x \cdot v$. 

Proof. This follows from the juxtaposition of the obviously commutative diagram

\[ M_3((kF_2)^{\otimes 2})^\wedge \xrightarrow{\text{row}(-)\col} ((kF_2)^{\otimes 2})^\wedge \]

\[ M_3((\mathcal{A}_{\mu,\phi})^{\otimes 2}) \cong \]

\[ M_3((U(f_2)^{\otimes 2})^\wedge) \xrightarrow{\text{row}(-)\col} (U(f_2)^{\otimes 2})^\wedge \]

with the following diagram

\[ M_3((U(f_2)^{\otimes 2})^\wedge) \xrightarrow{\text{Ad}(\kappa,\ell)} (U(f_2)^{\otimes 2})^\wedge \]

\[ M_3((U(f_2)^{\otimes 2})^\wedge) \xrightarrow{\text{row}(-)\col} (U(f_2)^{\otimes 2})^\wedge \]

whose commutativity follows from (8.5.7) and (8.6.4).

\[ \boxdot \]

8.8. Commutativity of the diagram (8.2.1). Recall the elements \( u, v \in (U(f_2)^{\otimes 2})^\wedge \) (8.6.3, 8.5.2).

Lemma 8.14. One has

\[ e^{\mu(f_1)}u = \mu \frac{e^{\mu(e_1+f_1)} - 1}{e_1 + f_1} \frac{\Phi(e_1 + f_1)}{\Phi(e_1)\Phi(f_1)} \]

\[ e_1 + f_1 \frac{\Gamma(e_1 + f_1)}{\Gamma(e_1)\Gamma(f_1)} \]

\[ e^{-\mu(f_1)} = \left( e_1 + f_1 \frac{\Gamma(e_1 + f_1)}{\Gamma(e_1)\Gamma(f_1)} \right)^{-1} \]

(identities in \((U(f_2)^{\otimes 2})^\wedge\)).

Proof. The first claimed equality follows directly from (8.6.3). Then

\[ \frac{e_1 + f_1}{e^{\mu(e_1+f_1)} - 1} e^{-\mu(f_1)} = \left( \mu \frac{e^{\mu(e_1+f_1)} - 1}{e_1 + f_1} \frac{\Phi(e_1 + f_1)}{\Phi(e_1)\Phi(f_1)} \right)^{-1} , \]

where the first equality follows from (8.6.5) and the second equality follows form the first claimed equality. This proves the second claimed equality.

\[ \boxdot \]

Lemma 8.15. The following diagram is commutative

\[ \xymatrix{ (kF_2)^\wedge \ar[r]^{\mathcal{A}_{\mu,\phi}} \ar[d]_{\text{mor}_{kF_2,x_1,-1}} & U(f_2)^\wedge \ar[r]^{\text{mor}_{U(f_2)\cdot e_1}} & (\hat{\mathcal{V}}_i^{\text{DR}})^+ \ar[d]_{(-)\otimes e_1^{-1}} \\
(\hat{\mathcal{V}}_i^{\text{B}})^+ \ar[r]_{\mathcal{A}_{\mu,\phi}} & (\hat{\mathcal{V}}_i^{\text{DR}})^+ } \]
Proof. Consider the diagram

The commutativity of the upper left trapezoid follows from (1.7.2). The commutativity of the upper right trapezoid follows from the fact that $(\mathcal{W}^\text{DR}_l)_+$ is a subalgebra of $U(f_2)^\wedge$. The commutativity of the left and right triangles follows from the definition of $\text{mor}^l_{R,e}$ (see (1.2). The commutativity of the lower trapezoid follows from the fact that $\omega_{l,e} : (kF_2)^\wedge \to U(f_2)^\wedge$ is an algebra morphism, such that $X_1-1 \to e^{\mu e_1-1}$. The fact that the middle triangle is commutative follows from the fact that $U(f_2)^\wedge$ is an associative algebra and from $(e^{\mu e_1-1})_1 \cdot x = e_1$.}

Lemma 8.16. The following diagram is commutative

where $(-) \cdot a$ is the endomorphism of right multiplication by $a$.

Proof. It follows from its definition (3.1.2) that $\hat{\Delta}^l_{r}$ is an algebra morphism. It therefore intertwines the maps $(-) \frac{e_1}{e^{\mu e_1} - 1} : \mathcal{W}^\text{DR}_l \to \mathcal{W}^\text{DR}_l$ and $(-) \hat{\Delta}^l_{r} \left( \frac{e_1}{e^{\mu e_1} - 1} \right) : (\mathcal{W}^\text{DR}_l)^\otimes 2 \to (\mathcal{W}^\text{DR}_l)^\otimes 2$.

One computes

where the first equality follows from the definition of $\hat{\Delta}^l_{r}$, the second equality follows from the fact that $\hat{\Delta}^l_{r}$ is an algebra morphism, and the last equality follows the commutativity of the diagram

in which the inclusions $k[e_1] \hookrightarrow \mathcal{W}^\text{DR}_x$, $x \in \{l, r\}$ are defined by the condition that the composed maps $k[e_1] \hookrightarrow \mathcal{W}^\text{DR}_x \hookrightarrow U(f_2)^\wedge$ coincide with the canonical inclusion $k[e_1] \hookrightarrow U(f_2)^\wedge$. 

\[ \text{Ad}(e_1) \]
This implies that $\Delta^1_a$ intertwines $(-) \cdot \frac{e_1}{e_1+1}$ and $(-) \cdot \frac{e_1+f_1}{e_1(f_1+1)+1}$, proving the claimed commutativity.

\[ \square \]

**Proposition 8.17.** If $\mu \in k^\times$ and $\Phi \in M_\mu(k)$, then the diagram (8.3.1) commutes.

**Proof.** Consider the following diagram. It is divided into the subdiagrams S1, ..., S11.

\[ (8.8.1) \]

Let us explain the notation in this diagram. If $A$ is an algebra and $a, b$ are elements in $A$, we denote by $a \cdot (-) \cdot b$ the linear endomorphism of $A$ given by $x \mapsto axb$, so if $a \in A$ is invertible, the endomorphism $\text{Ad}(a)$ is then equal to $a \cdot (-) \cdot a^{-1}$. We also denote by $(-) \cdot a$ the linear endomorphism of $A$ given by $x \mapsto xa$.

According to Proposition 2.2, $X_1$ is an invertible element in $\hat{W}_r^B$, therefore $Y_1 = 1 \otimes X_1$ is an invertible element in $(\hat{W}_r^B)^{\otimes 2}$, so that the automorphism $\text{Ad}(Y_1)$ of $(\hat{W}_r^B)^{\otimes 2}$ (top map of S6) is well-defined. According to (8.5.2) and (8.6.3), $u$ and $v$ are invertible elements of $(U(f_2)^{\otimes 2})^\wedge$ which belong to the topological subalgebra of this algebra generated by $e_1$ and $f_1$, therefore they are invertible elements in $(\hat{W}_r^B)^{\otimes 2}$. Moreover $e_1^{\mu f_1}$, $\frac{e_1+f_1}{e_1(f_1+1)+1}$ and $\frac{f_1(e_1+f_1)}{e_1(f_1+1)+1}$ are also obviously elements with the same properties. All this implies the well-definedness of all the maps of the subdiagrams S10 and S11. As $\frac{e_1}{e_1+1}$ belongs to the topological subalgebra of $U(f_2)^\wedge$ generated by $e_1$, this element belongs to its subalgebra $\hat{W}_r^B$, so that there is a well-defined endomorphism $(-) \cdot \frac{e_1}{e_1+1}$ of $\hat{W}_r^B$. It follows from (8.3.1) that $(\hat{W}_r^B)^+ \to k$ obtained by the restriction of the counit map
$U(f_2)^\wedge \to k$ to $\hat{W}^{\text{DR}}_l$. As $e(\frac{e_1}{e_1}) = \mu^{-1}$, we have $e \circ ((-) \cdot \frac{e_1}{e_1}) = \mu^{-1} \cdot e$ (equality of $k$-module morphisms $\hat{W}^{\text{DR}}_l \to k$), therefore $(-) \cdot \frac{e_1}{e_1}$ restricts to a $k$-module endomorphism of $(\hat{W}^{\text{DR}}_l)^+$, which is the common map of the pair of subdiagrams (S2, S9).

The other maps of the diagram (8.8.1) are introduced in the parts of this text indicated by the following table, where by (S,S') we understand the map(s) common to subdiagrams S and S'.

| map    | (S1,S2) | (S1,S3) | (S1,S4) | (S2,S3), (S3,S4) top, (S4, S5) | (S2,S7) | (S3,S4) bottom | (S3,S7) |
|--------|---------|---------|---------|---------------------------------|---------|----------------|---------|
| ref.   | (6.3.2) | (8.4.2) |         | (6.2.2), (8.2.2)                | (8.1.1) | (5.3.2)        | (8.3.3) |
| ref.   | (8.2.2) | (8.5.2) | (6.3.3) | (8.1.1)                         | (3.1.1) | Def. 3.2       | (3.1.3) |
| ref.   | (8.4.1) |         |         |                                 |         | (8.4.1)        |         |

Let us prove the commutativity of the various subdiagrams of (8.8.1):

- The commutativity of S1 follows from Proposition 6.17, Corollary 6.24, (6.2.1) and (8.1.2).
- The commutativity of S2 (resp. S3, S4, S5) follows from Lemma 8.16 (resp. Lemma 8.5, Lemma 8.13, Lemma 8.3).
- As $(\varphi_{\mu, \Phi})^\otimes$ is an algebra morphism, it intertwines the automorphisms $\text{Ad}(Y_1)$ of $(\hat{W}^{\text{DR}}_r)^\otimes$ and $\text{Ad}((\varphi_{\mu, \Phi})^\otimes)(Y_1)$ of $(\hat{W}^{\text{DR}}_r)^\otimes$. According to (8.1.2), one has $\varphi_{\mu, \Phi}(X_1) = e^{c_1}$. Lemma 8.3 then implies that $\varphi_{\mu, \Phi}(X_1) = e^{c_1}$. Therefore $(\varphi_{\mu, \Phi})^\otimes(Y_1) = e^{c_1}$. This implies the commutativity of S6.
- The commutativity of S7 follows from Proposition 5.11, Lemma 5.12, (5.2.4) and (8.4.1).
- The commutativity of S8 follows from the fact that $\hat{W}^{\text{DR}}_r$ is a subalgebra of $U(f_2)^\wedge$.
- The commutativity of S9 follows from Lemma 8.16 combined to the fact that the endomorphism $(-) \cdot \frac{e_1}{e_1}$ restricts to an endomorphism of $(\hat{W}^{\text{DR}}_r)^+$. The commutativity of S10 follows from the associativity of the product in $(\hat{W}^{\text{DR}}_r)^\otimes$. The commutativity of S11 follows from associativity of the product in $(\hat{W}^{\text{DR}}_r)^\otimes$ combined with Lemma 8.13.

This implies that the external diagram of (8.8.1) commutes, therefore that so does the restriction of (8.2.1) to the submodule $(\hat{W}^{\text{B}}_l)^+_l$ of $\hat{W}^{\text{B}}_l$. On the other hand, the restriction of (8.2.1) to the submodule $k1$ of $\hat{W}^{\text{B}}_l$ also commutes, as it takes $1 \in \hat{W}^{\text{B}}_l$ to the unity element $1^\otimes$ of $(\hat{W}^{\text{DR}}_l)^\otimes$ by the two composed morphisms of this diagram. As $\hat{W}^{\text{B}}_l = k1 \oplus (\hat{W}^{\text{DR}}_l)^+$, it follows that (8.2.1) commutes.

□
9. Proof of Theorem 3.1

Recall that by Proposition 3.13, the proof of the Theorem 3.1 can be reduced to the commutativity of the diagram (3.2.1) relating $\hat{\Delta}_{l,r}^\star$ and $\hat{\Delta}_{l,r}^\#$ for some pair $(\mu, \Phi)$ where $\mu \in k^\times$ and $\Phi \in \text{DMR}_\mu(k)$. On the other hand, Proposition 8.17 establishes the commutativity of the same diagram for any pair $(\mu, \Phi)$ with $\mu \in k^\times$ and $\Phi \in M^\mu_1(k)$. In this section, we give two proofs of Theorem 3.1:

1) the first proof (§9.1) is based on the nonemptiness of $M^\mu_1(k)$ for $\mu \in k^\times$ (Proposition 5.3 from [Dr], see (2) in Theorem 7.4), on the inclusion result $M^\mu_1(k) \subset \text{DMR}_\mu(k)$ from [Fu2], and on Propositions 3.13 and 8.17;

2) the second proof (§9.2) is based on the existence of an explicit complex associator with $\mu = 1$ (§2 from [Dr], see (1) from Theorem 7.4), on the fact that this element belongs to $\text{DMR}_1(C)$, on the nonemptiness of $\text{DMR}_\mu(k)$ for $\mu \in k^\times$ (recall that this nonemptiness, which follows from §3.2.3 from [R], see (3) in Theorem 1.1, is necessary to the definition of one of the ingredients of Theorem 3.1, namely $\hat{\Delta}_\star$), and again on Propositions 3.13 and 8.17.

As the diagram (3.2.1), which depends on a pair $(\mu, \Phi)$ with $\mu \in k^\times$ and $\Phi \in (U(f_2)^\wedge)^\times$, plays a crucial role in both proofs, we will denote it by $\text{diagr}(\mu, \Phi)$.

9.1. First proof of Theorem 3.1. Recall that one may compose the inclusion $\text{DMR}_\mu(k) \subset k\langle\langle X\rangle\rangle^\times$ with the isomorphism $k\langle\langle X\rangle\rangle^\times \cong (U(f_2)^\wedge)^\times$ given in (1.3.1), and view $\text{DMR}_\mu(k)$ as a subset of $(U(f_2)^\wedge)^\times$.

We recall:

**Theorem 9.1** (Theorem 0.2 in [Fu2]). One has the inclusion $M^\mu_1(k) \subset \text{DMR}_\mu(k)$ of subsets of $(U(f_2)^\wedge)^\times$.

Let us now prove Theorem 3.1. Let $k$ be a $\mathbb{Q}$-algebra. According to Proposition 5.3 from [Dr] (see (2) in Theorem 7.4), there exists an element $\Phi_k \in M_1(k)$. Proposition 8.17 then implies that $\text{diagr}(1, \Phi_k)$ commutes. Since $\Phi_k \in M_1(k)$, Theorem 9.1 implies $\Phi_k \in \text{DMR}_\mu(k)$. Combining this fact, the commutativity of $\text{diagr}(1, \Phi_k)$ and Proposition 3.13 one obtains $\hat{\Delta}_\star = \hat{\Delta}_\star$.

9.2. Second proof of Theorem 3.1

9.2.1. Change of notation. As this proof uses explicitly the dependence of objects of the paper in the base ring $k$, we make in this subsection the following changes of notation. For $X$ one of the following discrete graded objects of filtered objects from the text: $U(f_2)$, $W_l^{\text{DR}}$, $\Delta_\star$ (graded objects), $W_l^{B}$, $\Delta_\star$ (filtered objects), the notation will be $U(f_2)^k$, $W_l^{\text{DR}, k}$, $\Delta_\star$, $W_l^{B, k}$, $\Delta_\star$. The completion of $X$ was denoted $\hat{X}$ or $X^\wedge$ and will henceforth be denoted $(X^k)^\wedge$, so that the objects $U(f_2)^\wedge$, $W_l^{\text{DR}}$, $\Delta_\star$, $W_l^{B}$, $\Delta_\star$ will be
denoted \((U(f_2)k)^\wedge, (W_i^{\text{DR},k})^\wedge, (\Delta_k^*)^\wedge, (W_i^{B,k})^\wedge, (\underline{\Delta}_k)^\wedge\). The object \((\mathcal{W}_i^{\text{DR}})^{\otimes 2}\) will be denoted \(((W_i^{\text{DR},k})^{\otimes 2})^\wedge\).

The object formerly denoted \(\underline{\Delta}_k\) has been defined not as a completion, but as the common value of the result of a collection of operations on \(\Delta\) (henceforth denoted \((\Delta_k^*)^\wedge\)). As this object also depends on \(k\), its notation will henceforth be \((\Delta_k^*)^\wedge\).

The new notation for the map \(\bar{\mu}_{(\mu, \Phi)}: \mathcal{W}_i^{\text{B}} \rightarrow \mathcal{W}_i^{\text{DR}}\) corresponding to \((\mu, \Phi) \in k^\times \times ((U(f_2)k)^\wedge)^\wedge\) will be \(\bar{\mu}_{(\mu, \Phi)}: (W_i^{B,k})^\wedge \rightarrow (W_i^{\text{DR},k})^\wedge\).

9.2.2. Base change. For \(k\) a \(\mathbb{Q}\)-algebra, denote by \(\text{SCFM}_k\) the category of separated complete filtered \(k\)-modules, i.e., filtered modules such that the intersection of the filtration submodules is 0, and which are complete for the topology defined by the filtration submodules. There is a functor \(-\hat{\otimes}k : \text{SCFM}_{\mathbb{Q}} \rightarrow \text{SCFM}_k\), taking object \(V\) with filtration \((F^nV)_n\) to \(V\hat{\otimes}k := \lim(V/F^nV) \otimes k\).

Then for \(X\) in the list of symbols \(U(f_2), W_i^{\text{DR}}, \Delta, W_i^{B}, \underline{\Delta}_k\), one has \((X^\wedge)^\wedge = (X^{\otimes})^\wedge \hat{\otimes}k\).

Recall that \((\Delta_k^*)^\wedge\) belongs to \(\text{Mor}_{\text{SCFM}_k}(W_i^{B,k})^\wedge, ((W_i^{B,k})^{\otimes 2})^\wedge\) and that \((\Delta_0^*)^\wedge\) belongs to \(\text{Mor}_{\text{SCFM}_0}(W_i^{B,0})^\wedge, ((W_i^{B,0})^{\otimes 2})^\wedge\). Moreover, \(-\hat{\otimes}k\) induces a map
\[(9.2.1) -\hat{\otimes}k : \text{Mor}_{\text{SCFM}_0}(W_i^{B,0})^\wedge, ((W_i^{B,0})^{\otimes 2})^\wedge \rightarrow \text{Mor}_{\text{SCFM}_k}(W_i^{B,k})^\wedge, ((W_i^{B,k})^{\otimes 2})^\wedge\).

Lemma 9.2. One has
\[(\Delta_k^*)^\wedge = (\Delta_0^*)^\wedge \hat{\otimes}k,\]
i.e., \((\Delta_0^*)^\wedge\) is mapped to \((\Delta_k^*)^\wedge\) through the map \((9.2.1)\).

Proof. If follows from §3.2.3 from [R] (see (3) in Theorem 1.1) that \(\text{DMR}_1(k)\) is nonempty for any \(k\) and any \(\mu \in k^\times\). We may therefore choose an element \(\varphi_0\) in \(\text{DMR}_1(\mathbb{Q})\).

According to Proposition 1.19 any \(\phi \in \text{DMR}_1(k)\) gives rise to a commutative diagram
\[
\begin{array}{ccc}
(W_i^{B,k})^\wedge & \rightarrow & (W_i^{B,k})^\wedge \\
\bar{\mu}_{(1,\phi)} & & (\bar{\mu}_{(1,\phi)})^{\otimes 2} \\
(W_i^{\text{DR},k})^\wedge & \rightarrow & ((W_i^{\text{DR},k})^{\otimes 2})^\wedge \\
\end{array}
\]
\[
\begin{array}{ccc}
(W_i^{B,k})^\wedge & \rightarrow & (W_i^{B,k})^\wedge \\
\bar{\mu}_{(1,\varphi_k)} & & (\bar{\mu}_{(1,\varphi_k)})^{\otimes 2} \\
(W_i^{\text{DR},k})^\wedge & \rightarrow & ((W_i^{\text{DR},k})^{\otimes 2})^\wedge \\
\end{array}
\]
Let \(\varphi_k \in \text{DMR}_1(k)\) be the image of \(\varphi_0 \in \text{DMR}_1(\mathbb{Q})\) under the natural map \(\text{DMR}_1(\mathbb{Q}) \rightarrow \text{DMR}_1(k)\). One may set \(\Phi := \varphi_k\) and obtain the following commutative diagram
\[
\begin{array}{ccc}
(W_i^{B,k})^\wedge & \rightarrow & (W_i^{B,k})^\wedge \\
\bar{\mu}_{(1,\phi)} & & (\bar{\mu}_{(1,\phi)})^{\otimes 2} \\
(W_i^{\text{DR},k})^\wedge & \rightarrow & ((W_i^{\text{DR},k})^{\otimes 2})^\wedge \\
\end{array}
\]
\[
\begin{array}{ccc}
(W_i^{B,k})^\wedge & \rightarrow & (W_i^{B,k})^\wedge \\
\bar{\mu}_{(1,\varphi_k)} & & (\bar{\mu}_{(1,\varphi_k)})^{\otimes 2} \\
(W_i^{\text{DR},k})^\wedge & \rightarrow & ((W_i^{\text{DR},k})^{\otimes 2})^\wedge \\
\end{array}
\]
Specializing Proposition 1.19 to $(k, \mu, \Phi) = (Q, 1, \varphi_Q)$, and applying $-\hat{\otimes}k$, one obtains the following diagram

$$
\begin{array}{c}
(W_i^{B,Q})^\otimes \hat{\otimes}k \\
\downarrow \varphi_{(i, x_q)}^Q \hat{\otimes}k \\
(W_i^{DR,Q})^\otimes \hat{\otimes}k
\end{array}
\rightarrow
\begin{array}{c}
(W_i^{B,Q})^\otimes \hat{\otimes}k \\
\downarrow \varphi_{(i, x_q)}^Q \hat{\otimes}k \\
(W_i^{DR,Q})^\otimes \hat{\otimes}k\end{array}
\rightarrow
\begin{array}{c}
((W_i^{DR,Q})^\otimes 2)^\otimes \hat{\otimes}k \\
\downarrow \text{Ad}_{\varphi_{(i, x_q)}^Q \varphi_{(i, x_q)}^Q \varphi_{(i, x_q)}^Q} \hat{\otimes}k \\
((W_i^{DR,Q})^\otimes 2)^\otimes \hat{\otimes}k
\end{array}

One can identify canonically all the spaces and morphisms of the two previous diagrams, except for the top horizontal morphism. Moreover, the vertical morphisms are isomorphisms. This implies the equality of the top horizontal morphisms of the two diagrams, and therefore the announced equality. □

9.2.3. End of the argument. According to §2 from [Dr] (see (1) in Theorem 7.4), there exists an element $\varphi_{KZ} \in M_1(C)$. Proposition 8.17 then implies that $\text{diagr}(1, \varphi_{KZ})$ commutes. According to §3.2.3 from [R] (see (3) in Theorem 1.1), one has $\varphi_{KZ} \in \text{DMR}_1(C)$. Combining this fact, the commutativity of $\text{diagr}(1, \varphi_{KZ})$ and Proposition 3.13, one obtains $\Delta^C_i = (\Delta^C_i)^\wedge$. One has $(\Delta^C_i)^\wedge = (\Delta^Q_i)^\otimes \hat{\otimes}C$, and according to Lemma 9.2 one has $(\Delta^C_i)^\wedge = (\Delta^Q_i)^\otimes \hat{\otimes}C$, therefore $(\Delta^Q_i)^\otimes \hat{\otimes}C = (\Delta^Q_i)^\otimes \hat{\otimes}C$. As the map (9.2.1) is injective for general $k$, it also injective for $k = C$; we then derive $\Delta^Q_i = (\Delta^Q_i)^\wedge$. Applying $-\hat{\otimes}k$ to this equality and using $(\Delta^Q_i)^\wedge = (\Delta^Q_i)^\otimes \hat{\otimes}C$, (again a consequence of Lemma 9.2), one obtains $\Delta^k_i = (\Delta^k_i)^\wedge$, as wanted. □
10. Categorical aspects

In this section, we discuss the categorical aspects of some of the constructions used in part 2. Recall that the main steps of this part are §§5, 6 and 8. The main result of §5 (resp. §6) is diagram (5.3.3) (resp. (6.3.3)) relating $\Delta^{l,r}_\mu$ (resp. $\Delta^{l,r}_\mu$) with maps arising from infinitesimal braid Lie algebras (resp. braid groups). In §7, we constructed, for any pair ($\mu$, $\Phi$), where $\mu \in k^*$ and $\Phi \in M_\mu(k)$, algebra morphisms $a_l^{\mu,\Phi} : \hat{W}_B^l \to \hat{W}^{\text{DR}}_l$ and $a_r^{\mu,\Phi} : \hat{W}_B^r \to \hat{W}^{\text{DR}}_r$. In §8, we constructed additional morphisms depending on ($\mu$, $\Phi$) relating each vertex of (5.3.3) with the corresponding vertex of (6.3.3), and in §8 we proved the commutativity of various diagrams of algebras involving these morphisms, and as a consequence obtained the commutativity of (3.2.1).

The categorical counterpart of (5.3.3) is discussed in §10.2. The main result here is Proposition 10.3, where it is shown that the functor $(\Delta^{l,r}_\mu)^*$ fits in the categorical diagram (10.2.2), which commutes up to natural equivalence (the category of quadruples $\text{Quad}^{\text{DR}}_{\text{fin}}$ in this diagram was inspired by [DeT], §5, and the diagram itself was inspired by the computations of [DeT], §6.3).

In §10.3, we discuss the categorical counterpart of (6.3.3). The main result here is Proposition 10.6, where it is shown that the functor $(\Delta^{l,r}_\mu)^*$ fits in the diagram (10.3.2), which commutes up to natural equivalence and should be viewed as a "Betti" analogue of the "de Rham" diagram (10.2.2).

In §10.4, we discuss the categorical counterpart of §§8. In Lemma 10.7, we obtain the categorical counterpart of the main result of this section, the diagram (3.2.1). We also construct the categorical counterparts of the various steps of §§8. For this, we construct a functor $\text{comp}_{\mu,\Phi} : \text{Quad}^{\text{DR}}_{\text{fin}} \to \text{Quad}^{\text{B}}_{\text{fin}}$ (see Lemma 10.8) which relates two of the vertices of diagrams (10.2.2) and (10.3.2). In Lemmas 10.9, 10.10 and 10.11 using results from §§8 we prove the commutativity up to natural equivalence of three diagrams of categories relating vertices of these two diagrams to one another, and involving the comparison functors $\text{comp}_{\mu,\Phi}(\Delta^{l,r}_\mu)^*$, $(\Delta^{l,r}_\mu)^{\otimes 2}$ and $((\Delta^{l,r}_\mu)^{\otimes 2})^*$.

In §10.5, we introduce the notion of a cube of categories, where vertices are categories, edges are functors, and faces are natural equivalences, and the notion of the flatness of such an object. We show that one may combine the natural equivalences (10.2.2), (10.3.2) and those of Lemmas 10.7, 10.9, 10.10 and 10.11 to construct such a cube of categories. It can be shown that this cube is flat.

10.1. Categorical background. Let $\text{Cat}$ be the 2-category of categories, where arrows are functors and 2-arrows are natural transformations. If $f, g : C \to D$ are two functors between the categories $C$ and $D$, we denote by $\text{Eq}(f, g)$ the set of natural equivalences between them, i.e. the
set of functorial assignments \( \text{Ob}(C) \ni X \mapsto \eta_X \in \text{Iso}_D(f(X), g(X)) \). An element \( \eta \in \text{Eq}(f, g) \) will be depicted as follows:

\[
\begin{array}{c}
\mathcal{C} \\
\downarrow \eta \\
\mathcal{D}
\end{array}
\]

If \( f, g, h : \mathcal{C} \to \mathcal{D} \) are functors, and if \( \eta \in \text{Eq}(f, g), \eta' \in \text{Eq}(g, h) \), we denote by \( \eta' \circ \eta \in \text{Eq}(f, h) \) the composed natural equivalence (vertical composition), and by \( \text{id}_f \in \text{Eq}(f, f) \) the trivial equivalence of \( f \) with itself. If \( f_1, f_2 : \mathcal{C} \to \mathcal{D} \) and \( g_1, g_2 : \mathcal{D} \to \mathcal{E} \) are functors, if \( \eta \in \text{Eq}(f_1, f_2), \eta' \in \text{Eq}(g_1, g_2) \), we denote by \( \eta' \cdot \eta \in \text{Eq}(g_1 \circ f_1, g_2 \circ f_2) \) the composed natural equivalence (horizontal composition).

Let \( \text{GrAlg} \) (resp. \( \text{FilAlg} \)) be the category of \( Z_{\geq 0} \)-graded \( k \)-algebras (resp. of filtered \( k \)-algebras, in the sense of \([13,1]\)).

Objects of \( \text{GrAlg} \) include the graded algebras \( k \) (with only nonzero component in degree 0), \( U(f_2) \otimes^2, U(p_5) \), \( W_l^{\text{DR}} \), \( (W_r^{\text{DR}})^{\otimes 2} \). Objects of \( \text{FilAlg} \) include the filtered algebras \( k \) (with \( F^k k = 0 \) for \( k \geq 1 \), \( (kF_2)^{\otimes 2} \), \( kP_3 \), \( W_l^{\text{B}} \), \( (W_r^{\text{B}})^{\otimes 2} \)).

Let \( \text{GrMod}, \text{GrMod}_{\text{fin}} \) be the contravariant functors \( \text{GrAlg} \to \text{Cat} \) such that for \( A \) a graded algebra, \( \text{GrMod}(A) \) is the category of \( Z_{\geq 0} \)-graded \( A \)-modules and \( \text{GrMod}_{\text{fin}}(A) \) is the full subcategory of objects with finite support.

For \( B \) is a graded \( A \)-bimodule, define an endofunctor \( F_B \) of \( \text{GrMod}(A) \) by \( \text{GrMod}(A) \ni M \mapsto F_B(M) := \text{Hom}_{\text{GrMod}(A)}(B, M) \), in which \( B \) is viewed as a left \( A \)-module, the \( A \)-module structure on \( F_B(M) \) being given by \( (a \varphi)(b) = \varphi(ba) \), for \( a \in A, \varphi \in \text{Hom}_{\text{GrMod}(A)}(B, M) \), and \( b \in B \). Note that the endofunctor \( F_B \) of \( \text{GrMod}(A) \) restricts to an endofunctor, also denoted \( F_B \), of \( \text{GrMod}_{\text{fin}}(A) \).

Let also \( \text{FilMod}, \text{FilMod}_{\text{fin}} : \text{FilAlg} \to \text{Cat} \) be the contravariant functors such that for \( \hat{A} \) a filtered algebra, \( \text{FilMod}(\hat{A}) \) is the category of filtered modules, i.e. modules \( \underline{M} \) equipped with a decreasing filtration \( \underline{M} = F_0 M \supset F_1 M \supset \cdots \), compatible with the filtration of \( \hat{A} \), and \( \text{FilMod}_{\text{fin}}(\hat{A}) \) is the full subcategory of filtered modules \( \underline{M} \) such that such that \( F^k \underline{M} = 0 \) for large enough \( k \). Note that if \( \hat{A} \) is the completion of \( \underline{A} \), then there is a natural equivalence \( \text{FilMod}_{\text{fin}}(\underline{A}) \simeq \text{FilMod}_{\text{fin}}(\hat{A}) \).

For \( \underline{B} \) a filtered (i.e. equipped with a decreasing filtration compatible with the filtration of \( A \)) \( \underline{A} \)-bimodule, define an endofunctor \( F_{\underline{B}} \) of \( \text{FilMod}(\hat{A}) \) by \( \text{FilMod}(\hat{A}) \ni \underline{M} \mapsto F_{\underline{B}}(\underline{M}) := \text{Hom}_{\text{FilMod}(\hat{A})}(\underline{B}, \underline{M}) \), in which \( \underline{B} \) is viewed as a left \( \underline{A} \)-module, the \( \underline{A} \)-module structure on \( F_{\underline{B}}(\underline{M}) \) being defined as above. The endofunctor \( F_{\underline{B}} \) of \( \text{FilMod}(\hat{A}) \) restricts to an endofunctor, also denoted \( F_{\underline{B}} \), of \( \text{FilMod}_{\text{fin}}(\hat{A}) \).

If now \( A \) (resp. \( \underline{A} \)) is a graded (resp. filtered) algebra, any morphism of filtered algebras \( f : \hat{A} \to \hat{A} \) induces a functor \( f^* : \text{GrMod}_{\text{fin}}(A) \to \text{FilMod}_{\text{fin}}(\hat{A}) \) obtained by the composition \( \text{GrMod}_{\text{fin}}(A) \to \text{FilMod}_{\text{fin}}(\hat{A}) \to \text{FilMod}_{\text{fin}}(\hat{A}) \simeq \text{FilMod}_{\text{fin}}(\hat{A}) \).
10.2. Categorical interpretation of the relationship between $\Delta$, and infinitesimal braid Lie algebras \( (35) \).

10.2.1. A category $\text{Quad}^{\text{DR}}$. We define $\text{Quad}^{\text{DR}}$ to be the category where objects are the quadruples $(V, W, \alpha, \beta)$, where $V$ is an object of $\text{GrMod}(U(f_2))$, $W$ is a graded $k$-module (i.e. an object of $\text{GrMod}(k)$), $\alpha : V \to W$ and $\beta : W \to V$ are $k$-module morphisms of respective degrees 1 and 0, such that $(e_1)_V = \beta \circ \alpha$ (equality in $\text{End}_k(V)$), where $U(f_2) \ni x \mapsto x_{|V} \in \text{End}_k(V)$ is the action morphism. A morphism $(V, W, \alpha, \beta) \to (V', W', \alpha', \beta')$ is a pair of a $U(f_2)$-module morphism $V \to V'$ and a $k$-module morphism $W \to W'$, both of degree 0, such that the natural diagrams commute.

We define $\text{Quad}^{\text{DR}}_{\text{fin}}$ to be the full subcategory of $\text{Quad}^{\text{DR}}$ where the objects $(V, W, \alpha, \beta)$ are such that $V$ is an object of $\text{GrMod}_{\text{fin}}(U(f_2))$ and $W$ is an object of $\text{GrMod}_{\text{fin}}(k)$.

10.2.2. A functor $F_{\text{QW}}^{\text{DR}} : \text{Quad}^{\text{DR}} \to \text{GrMod}(\mathcal{W}_1^{\text{DR}})$. If $(V, W, \alpha, \beta)$ is an object of $\text{Quad}^{\text{DR}}$, then $W$ is equipped with the following action of $\mathcal{W}_1^{\text{DR}}$: the element $1 \in \mathcal{W}_1^{\text{DR}}$ acts by id$_W \in \text{End}_k(W)$, and for $x \in U(f_2)$, the element $xe_1 \in \mathcal{W}_1^{\text{DR}}$ acts by $\alpha \circ x_{|V} \circ \beta \in \text{End}_k(W)$.

There is a functor
\[
F_{\text{QW}}^{\text{DR}} : \text{Quad}^{\text{DR}} \to \text{GrMod}(\mathcal{W}_1^{\text{DR}}),
\]
taking the quadruple $(V, W, \alpha, \beta)$ to the $k$-module $W$ equipped with this action. It restricts and corestricts to a functor $F_{\text{QW}}^{\text{DR}} : \text{Quad}^{\text{DR}}_{\text{fin}} \to \text{GrMod}_{\text{fin}}(\mathcal{W}_1^{\text{DR}})$.

10.2.3. A functor $F_{\text{VQ}}^{\text{DR}} : \text{GrMod}(U(f_2)^{\otimes 2}) \to \text{Quad}^{\text{DR}}$. In $[5.1.2]$, we defined graded algebra morphisms $\text{pr}_{12} : U(p_5) \to U(p_4)^{\otimes 2} \simeq U(f_2)^{\otimes 2}$ and $\ell : U(f_2) \simeq U(p_4) \to U(p_5)$. They induce functors $\text{pr}_{12}^* : \text{GrMod}(U(f_2)^{\otimes 2}) \to \text{GrMod}(U(p_5))$ and $\ell^* : \text{GrMod}(U(p_5)) \to \text{GrMod}(U(f_2)^{\otimes 2})$.

As $J(\text{pr}_5)$ is a graded $U(p_5)$-bimodule, one may construct the endofunctor $F_{J(\text{pr}_5)}$ of $\text{GrMod}(U(p_5))$, such that
\[
F_{J(\text{pr}_5)}(N) = \text{Hom}_{\text{GrMod}(U(p_5))}(J(\text{pr}_5), N)
\]
(see $[10.1]$).

Recall that $J(\text{pr}_5)$ is freely generated, as a left $U(p_5)$-module, by $e_{i5}$, $i \in [1, 3]$. This implies that if $N$ is any graded $U(p_5)$-module, the map $\text{ev}_N : F_{J(\text{pr}_5)}(N) = \text{Hom}_{\text{GrMod}(U(p_5))}(J(\text{pr}_5), N) \to N^{\otimes 3}$, $\varphi \mapsto (\varphi(e_{i5}))_{i \in [1, 3]}$ is a graded $k$-module isomorphism.

If $C$ is an object in $\text{GrMod}(U(f_2)^{\otimes 2})$, we set:

- $V(C) := \ell^* \circ F_{J(\text{pr}_5)} \circ \text{pr}_{12}^*(C)$; this is an object in $\text{GrMod}(U(f_2))$;
- $W(C)$ is $C$ viewed as a graded $k$-module;
- $\alpha(C) : V(C) \to W(C)$ takes $\varphi \in V(C)$, viewed as a $U(p_5)$-equivariant map $J(\text{pr}_5) \to \text{pr}_{12}^*C$, to $e_1 \cdot \varphi(e_{i5}) - f_1 \cdot \varphi(e_{25}) \in C \simeq W(C)$;
- $\beta(C) : W(C) \to V(C)$ takes $c \in C \simeq W(C)$ to the unique element $\varphi_c$ of
\[
\text{Hom}_{\text{GrMod}(U(p_5))}(J(\text{pr}_5), \text{pr}_{12}^*C) \simeq V(C),
\]
such that $\varphi_c(e_{15}) = c$, $\varphi_c(e_{25}) = -c$, $\varphi_c(e_{35}) = 0$, i.e. to the preimage of $(c, -c, 0)$ by $\text{ev}_{pr_{12}(C)}$.

**Lemma 10.1.** If $C$ is an object in $\text{GrMod}(U(f_2) \otimes 2)$, then the diagram

\[
\begin{array}{ccc}
V(C) & \xrightarrow{x_{|V(C)}} & V(C) \\
\downarrow{\text{ev}_{pr_{12}(C)}} & & \downarrow{\text{ev}_{pr_{12}(C)}} \\
C^{\oplus 3} & \xrightarrow{\rho(x)} & C^{\oplus 3}
\end{array}
\]

commutes for any $x \in U(f_2)$, where $\rho$ is as in $\text{[5.2.4]}$.

**Proof.** Let $\varphi \in V(C)$. Then $x_{|V(C)}(\varphi)$ is the element $\ell(x)_{|V(C)}(\varphi)$, which is the element $\varphi(- \cdot \ell(x))$ of $V(C) \cong \text{Hom}_{\text{GrMod}(U(p_3))} (J(pr_3), pr_{12}^* C)$. The image of this element by $\text{ev}_{pr_{12}(C)}$ is the triple $(\varphi(e_{i5}\ell(x)))_{i \in [1,3]}$. Since $e_{i5}\ell(x) = \sum_{j \in [1,3]} \varpi(\ell(x))_{ij}e_{j5}$ (see §5.2.2), and since $\varphi$ is $U(p_3)$-equivariant, one has $\varphi(e_{i5}\ell(x)) = \sum_{j \in [1,3]} \text{pr}_{12}(\varpi(\ell(x))_{ij})\varphi(e_{j5})$ which is equal to $\sum_{j \in [1,3]} \rho(x)_{ij}\varphi(e_{j5})$ by the definition of $\rho$. On the other hand, the image of $\varphi$ by $\text{ev}_{pr_{12}(C)}$ is the triple $(\varphi(e_{15}))_{i \in [1,3]}$, whose image by $\rho(x) \cdot -$ is the same element. \par

**Lemma 10.2.** The assignment $C \mapsto (V(C), W(C), \alpha(C), \beta(C))$ is a functor

$F_{VQ}^{DR} : \text{GrMod}(U(f_2) \otimes 2) \to \text{Quad}^{DR}$.

It restricts and corestricts to a functor $F_{VQ}^{DR} : \text{GrMod}_{\text{fin}}(U(f_2) \otimes 2) \to \text{Quad}^{DR}_{\text{fin}}$.

**Proof.** Let $C$ be a $U(f_2) \otimes 2$-module. Consider the diagram

\[
\begin{array}{ccc}
V(C) & \xrightarrow{\text{ev}_{pr_{12}(C)}} & V(C) \\
\downarrow{\beta(C) \circ \alpha(C)} & & \downarrow{\beta(C)} \\
W(C) & \xrightarrow{\cong} & C \\
\downarrow{\text{ev}_{pr_{12}(C)}} & & \downarrow{\text{ev}_{pr_{12}(C)}} \\
V(C) & \xrightarrow{\alpha(C)} & C^{\oplus 3} \\
\downarrow{\text{ev}_{pr_{12}(C)}} & & \downarrow{\text{ev}_{pr_{12}(C)}} \\
C^{\oplus 3} & \xrightarrow{\rho(e_1) \cdot -} & V(C)
\end{array}
\]

In this diagram, the commutativity of the triangles is obvious. The commutativity of the upper and lower left quadrilaterals follows from the definitions of the involved morphisms. The commutativity of the middle (resp. right) square follows from Lemma §5.6 (resp. Lemma 10.1). This implies the equality $\beta(C) \circ \alpha(C) = (e_1)_{|V(C)}$, therefore $(V(C), W(C), \alpha(C), \beta(C))$ is an object of $\text{Quad}^{DR}$. The functoriality of the assignment $C \mapsto (V(C), W(C), \alpha(C), \beta(C))$ is obvious. \par
10.2.4. A natural equivalence $\eta^{DR}$.

**Proposition 10.3.** There is a natural equivalence $\eta^{DR}$

$$\text{GrMod}(U(f_2)^{\otimes 2}) \xrightarrow{(\text{inc}^{\otimes 2})^*} \text{Quad}^{\text{DR}}$$

between the two composite functors $\text{GrMod}(U(f_2)^{\otimes 2}) \to \text{GrMod}(W_r^{\text{DR}})$ of the above diagram, in which the top and right functors are those defined in Lemma 10.2, 10.2.2, and where inc is the inclusion morphism $W_r^{\text{DR}} \hookrightarrow U(f_2)$. It restricts to a natural equivalence

$$\text{GrMod}_{\text{fin}}(U(f_2)^{\otimes 2}) \xrightarrow{(\text{inc}^{\otimes 2})^*} \text{Quad}_{\text{fin}}^{\text{DR}}$$

Proof. Let $C$ be a $U(f_2)^{\otimes 2}$-module. Its image by the composite functor $\text{GrMod}(U(f_2)^{\otimes 2}) \to \text{Quad}^{\text{DR}} \to \text{GrMod}(W_r^{\text{DR}})$ is the $k$-module $W(C)$, equipped with the action of $W_r^{\text{DR}}$ given by $xe_1 \mapsto (xe_1)|_{W(C)}$, in which $(xe_1)|_{W(C)}$ is the endomorphism $W(C) \xrightarrow{\beta(C)} V(C) \xrightarrow{x_{V(C)}} V(C) \xrightarrow{\alpha(C)} W(C)$ of $W(C)$. Its image by the functor $(\Delta_{l,r}^{\text{DR}})^* \circ (\text{inc}^{\otimes 2})^*$ is the $k$-module $C$, equipped with the action of $W_r^{\text{DR}}$ given by $xe_1 \mapsto (xe_1)|_C$, in which $(xe_1)|_C$ is the endomorphism of $C$ induced by $\text{inc}^{\otimes 2} \circ \Delta_{l,r}^{\text{DR}}(xe_1) \in U(f_2)^{\otimes 2}$.

For $x \in U(f_2)$, consider the diagram

$$\begin{array}{ccc}
W(C) & \xrightarrow{\simeq} & C \\
\downarrow{\beta(C)} & & \downarrow{=}
\end{array}$$

$$\begin{array}{ccc}
V(C) & \xrightarrow{\text{ev}_{pr_{12}C}} & C^{\oplus 3} \\
\downarrow{x_{V(C)}} & & \downarrow{\rho(x) \cdot \text{col}}
\end{array}$$

First consider the polygons inside the left square: the left quadrilateral commutes by the definition of $(xe_1)|_{W(C)}$: the commutativity of the upper and lower quadrilaterals have been checked in the proof of Lemma 10.2, the small square commutes by Lemma 10.1 and the commutativity of the right quadrilateral is obvious. The commutativity of the middle rectangle follows from the identity $\text{inc}^{\otimes 2} \circ \Delta_{l,r}^{\text{DR}}(xe_1) = \text{row} \cdot \rho(x) \cdot \text{col}$, which follows from (5.3.3). The commutativity of the right square follows from the definition of $(xe_1)|_C$. All this implies the
com mutativity of the overall diagram, and therefore that the assignment
\[ \eta^{\text{DR}} : C \mapsto (W(C) \cong C) \]

is a natural equivalence. \qed

10.3. Categorical interpretation of the relationship between \( \Delta_c \) and braid groups (§6).

10.3.1. A category \( \text{Quad}^B \). We define \( \text{Quad}^B \) to be the category whose objects are the quadruples \((V, W, \alpha, \beta)\), where \( V \) is an object of \( \text{FilMod}(kF_2) \), \( W \) is a filtered \( k \)-module (i.e., an object of \( \text{FilMod}(k) \)), \( \alpha : V \to W \) and \( \beta : W \to V \) are \( k \)-module morphisms, where \( \alpha \) (resp. \( \beta \)) has filtration shift \( \leq 1 \) (resp. \( \leq 0 \)), such that \((X_1 - 1)_V = \beta \circ \alpha \) (equality in \( \text{End}_k(V) \)), where \( kF_2 \ni x \mapsto x|_V \in \text{End}_k(V) \) is the action morphism. Morphisms are defined as in \([10.2.1]\).

We define \( \text{Quad}^B_{\text{fin}} \) to be the full subcategory of \( \text{Quad}^B \) where the objects \((V, W, \alpha, \beta)\) are such that \( V \) is an object of \( \text{FilMod}_{\text{fin}}(kF_2) \) and \( W \) is an object of \( \text{FilMod}_{\text{fin}}(k) \).

10.3.2. A functor \( F^{B}_{QW} : \text{Quad}^B \to \text{FilMod}(W^B) \). If \((V, W, \alpha, \beta)\) is an object of \( \text{Quad}^B \), then \( W \) is equipped with the following action of \( W^B \): the element \( 1 \in W^B \) acts by \( \text{id}_W \in \text{End}_k(W) \), and for \( \varphi \in kF_2 \), the element \( \varphi(X_1 - 1) \in W^B \) acts by \( \alpha \circ \varphi|_V \circ \beta \in \text{End}_k(W) \).

There is a functor
\[ F^{B}_{QW} : \text{Quad}^B \to \text{FilMod}(W^B) , \]

taking the quadruple \((V, W, \alpha, \beta)\) to the \( k \)-module \( W \) equipped with this action. It restricts and corestricts to a functor \( F^{B}_{QW} : \text{Quad}^B_{\text{fin}} \to \text{FilMod}_{\text{fin}}(W^B) \).

10.3.3. A functor \( F^{B}_{Q} : \text{FilMod}((kF_2)^{\otimes 2}) \to \text{Quad}^B \). In \([6.1.2]\) we defined graded algebra morphisms \( \text{pr}_{12}^* : kP_5 \to (kF_2)^{\otimes 2} \) and \( \xi : kF_2 \to kP_5 \). They induce functors \( \text{pr}_{12}^* : \text{FilMod}((kF_2)^{\otimes 2}) \to \text{FilMod}(kP_5) \) and \( \xi^* : \text{FilMod}(kP_5) \to \text{FilMod}(kF_2) \). As \( J(\text{pr}_5) \) is a filtered \( kP_5 \)-bimodule, one may construct the endofunctor \( F^{B}_{J(\text{pr}_5)} \) of \( \text{FilMod}(kP_5) \), such that
\[ F^{B}_{J(\text{pr}_5)}(N) = \text{Hom}_{\text{FilMod}(kP_5)}(J(\text{pr}_5), N) \]
(see \([10.1]\)).

Recall that \( J(\text{pr}_5) \) is freely generated, as a left \( kP_5 \)-module, by \( x_{i5} - 1, i \in [1, 3] \). This implies that if \( N \) is any graded \( kP_5 \)-module, the map \( \psi_{\text{ev}, N} : F_{J(\text{pr}_5)}(N) = \text{Hom}_{\text{FilMod}(kP_5)}(J(\text{pr}_5), N) \to N^{|5, 3|} \), \( \varphi \mapsto (\varphi(x_{i5} - 1))_{i \in [1, 3]} \) is a graded \( k \)-module isomorphism.

If \( C \) is an object in \( \text{FilMod}((kF_2)^{\otimes 2}) \), we set:
- \( V(C) := \xi^* \circ F_{J(\text{pr}_5)} \circ \text{pr}_{12}^*(C) \); this is an object in \( \text{FilMod}(kF_2) \);
- \( W(C) \) is \( C \) viewed as a filtered \( k \)-module;
- \( \alpha(C) : V(C) \to W(C) \) takes \( \varphi \in V(C) \), viewed as a \( kP_5 \)-equivariant map \( J(\text{pr}_5) \to \text{pr}_{12}^* \) to \( (X_1 - 1) \cdot \varphi(x_{15} - 1) - (Y_1 - 1) \cdot \varphi(x_{25} - 1) \in C \approx W(C) \);
\begin{itemize}
\item \( \beta(C) : W(C) \to V(C) \) takes \( c \in C \cong W(C) \) to the unique element \( \varphi_c \) of 
\[ \text{Hom}_{\text{FilMod}(kF_2)}(J(pr_{\ast}), pr_{12}^{\ast}C) \cong V(C), \]

such that \( \varphi_c(x_{15} - 1) = Y_1 \cdot c \), \( \varphi_c(x_{25} - 1) = -c \), \( \varphi_c(x_{35} - 1) = 0 \), i.e. to the preimage of \( (Y_1 \cdot c, -c, 0) \) by \( \text{ev}_{pr_{12}^{\ast}C} \).
\end{itemize}

**Lemma 10.4.** If \( C \) is an object in \( \text{FilMod}((kF_2)^{\otimes 2}) \), then the diagram
\[
\begin{array}{ccc}
V(C) & \xrightarrow{x_{W(C)}} & V(C) \\
\downarrow_{\text{ev}_{pr_{12}^{\ast}C}} & & \downarrow_{\text{ev}_{pr_{12}^{\ast}C}} \\
C^{\otimes 3} & \xrightarrow{\rho(x) -} & C^{\otimes 3}
\end{array}
\]
commutes for any \( x \in kF_2 \), where \( \rho \) is as in (6.2.1).

**Proof.** The argument is similar to the proof of Lemma 10.1 and is based on the definition of \( \rho \) and of \( \varphi_c \) (6.2.2). \( \square \)

**Lemma 10.5.** The assignment \( C \mapsto (V(C), W(C), \alpha(C), \beta(C)) \) is a functor
\[ F_{VQ}^B : \text{FilMod}((kF_2)^{\otimes 2}) \to \text{Quad}^B. \]
It restricts and corestricts to a functor \( F_{VQ}^B : \text{FilMod}_{\text{fin}}((kF_2)^{\otimes 2}) \to \text{Quad}_{\text{fin}}^B. \)

**Proof.** The proof is similar to that of Lemma 10.2 and is based on the definitions of \( \text{row} \) and \( \text{col} \) (see (6.2.2)), Lemma 6.12 and Lemma 10.4 for \( x = X_1 - 1 \). \( \square \)

10.3.4. A natural equivalence \( \eta^B \).

**Proposition 10.6.** There is a natural equivalence \( \eta^B \)
\[
\begin{array}{ccc}
\text{FilMod}((kF_2)^{\otimes 2}) & \xrightarrow{F_{VQ}^B} & \text{Quad}^B \\
\text{FilMod}_{\text{fin}}((kF_2)^{\otimes 2}) & \xrightarrow{F_{QW}^B} & \text{Quad}^B_{\text{fin}} \\
\text{FilMod}((W_l^B)^{\otimes 2}) & \xrightarrow{(\Delta^l)^{\ast}} & \text{FilMod}(W_l^B)
\end{array}
\]

between the two composite functors \( \text{FilMod}((kF_2)^{\otimes 2}) \to \text{FilMod}(W_l^B) \) of the above diagram, where \( \text{inc} \) is the inclusion morphism \( W_l^B \hookrightarrow kF_2 \). It restricts to a natural equivalence
\[
\begin{array}{ccc}
\text{FilMod}_{\text{fin}}((kF_2)^{\otimes 2}) & \xrightarrow{F_{VQ}^B} & \text{Quad}_{\text{fin}}^B \\
\text{FilMod}_{\text{fin}}((W_l^B)^{\otimes 2}) & \xrightarrow{(\Delta^l)^{\ast}} & \text{FilMod}_{\text{fin}}(W_l^B)
\end{array}
\]
Proof. Making use of Lemma 10.4 and of the identity

\[ \text{inc}^{\otimes 2} \circ \text{Ad}(Y_1^{-1}) \circ \Delta_l^r(x(X_1 - 1)) = \text{row} \cdot \rho(x) \cdot \text{col} \]

for \( x \in kF_2 \) (identity in \((kF_2)^{\otimes 2}\)), which follows from (6.3.3), one can repeat the arguments of the proof of Proposition 10.3 to construct a natural equivalence

\[ \text{FilMod}_\text{fin}((kF_2)^{\otimes 2}) \xrightarrow{\text{F}_W^B \quad \text{Quad}^B} \text{FilMod}_\text{fin}(kF_2) \]

which upon composition with the natural equivalence

\[ \text{FilMod}((W^B_r)^{\otimes 2}) \xrightarrow{\text{id}} \text{FilMod}(W^B_r) \]

induced by \( \text{FilMod}((W^B_r)^{\otimes 2}) \ni X \mapsto (Y_1^{-1})_X \in \text{Aut}_{k\text{-mod}}(X) \) yields the natural equivalence (10.3.1). The natural equivalence (10.3.2) is then obtained by restriction as in Proposition 10.3. \( \square \)

10.4. Categorical interpretation of the diagrams relating the de Rham and Betti morphisms (§).

10.4.1. Categorical analogue of Proposition 8.17

Lemma 10.7. There is a natural equivalence \( \eta' \)

(10.4.1)

\[ \begin{array}{ccc}
\text{GrMod}_\text{fin}(W^\text{DR}_r)^{\otimes 2} & \xrightarrow{\Delta_l^r} & \text{FilMod}_\text{fin}(W^B_l) \\
(\Delta_l^r)^* & \xleftarrow{\eta'} & (\Delta_l^r)^* \\
\text{GrMod}_\text{fin}((W^\text{DR}_r)^{\otimes 2}) & \xrightarrow{\text{id}} & \text{FilMod}_\text{fin}((W^B_r)^{\otimes 2})
\end{array} \]

\[ \eta'_M : (\Delta_l^r)^*(\Delta_l^r)^{\otimes 2} \ast (M) \to (\Delta_l^r)^*(\Delta_l^r)^{\ast} (M) \quad \text{in } \text{FilMod}_\text{fin}(W^B_l). \]

Proof. Dualizing the commutative diagram of Proposition 8.17 we see that for any \( M \in \text{GrMod}_\text{fin}((W^\text{DR}_r)^{\otimes 2}) \), the \( k \)-module automorphism of \( M \) induced by the action of

\[ \frac{e^{\mu(e_1+f_1)} - 1}{e_1 + f_1} \frac{\Gamma_\Phi(e_1 + f_1)}{\Gamma_\Phi(e_1) \Gamma_\Phi(f_1)} \in ((W^\text{DR}_r)^{\otimes 2})^\wedge \]

is an isomorphism

\[ \eta'_M : (\Delta_l^r)^*(\Delta_l^r)^{\otimes 2} \ast (M) \to (\Delta_l^r)^*(\Delta_l^r)^{\ast} (M) \]

in \( \text{FilMod}_\text{fin}(W^B_l) \). \( \square \)
10.4.2. A functor \( \text{comp}(\mu, \Phi) : \text{Quad}_{\text{fin}}^{\text{DR}} \rightarrow \text{Quad}_{\text{fin}}^{\text{B}} \). Let \( \mu \in k^\times \) and \( \Phi \in M_{\mu}(k) \). The pair \((\mu, \Phi)\) induces a filtered algebra isomorphism \((\mu, \Phi)^{\wedge} : (kF_2)^{\wedge} \rightarrow U(f_2)^{\wedge} \) (see (10.4.1)) and therefore a functor \((\mu, \Phi)^* : \text{GrMod}_{\text{fin}}(U(f_2)) \rightarrow \text{FilMod}_{\text{fin}}(kF_2) \). The identity of \( k \) also induces a functor \( \text{id}^* : \text{GrMod}_{\text{fin}}(k) \rightarrow \text{FilMod}_{\text{fin}}(k) \).

Let \((V, W, \alpha, \beta)\) be an object in \( \text{Quad}_{\text{fin}}^{\text{DR}} \). Set \( \underline{V} := (\mu, \Phi)^*(V), \underline{W} := \text{id}^*(W), \underline{\alpha} := \alpha \circ (\frac{e^{\mu e_1} - 1}{e_1})_{|V}, \underline{\beta} := \beta \). Then \( \underline{V}, \underline{W} \) are objects of \( \text{FilMod}_{\text{fin}}(kF_2) \), \( \text{FilMod}_{\text{fin}}(k) \), and \( \underline{\alpha}, \underline{\beta} \) are \( k \)-linear maps \( \underline{V} \rightarrow \underline{W}, \underline{W} \rightarrow \underline{V} \).

**Lemma 10.8.** The assignment \((V, W, \alpha, \beta) \rightarrow (\underline{V}, \underline{W}, \underline{\alpha}, \underline{\beta})\) defines a functor \( \text{comp}(\mu, \Phi) : \text{Quad}_{\text{fin}}^{\text{DR}} \rightarrow \text{Quad}_{\text{fin}}^{\text{B}} \).

**Proof.** As \( \alpha \) (resp. \( \beta \)) has degree 1 (resp. 0), it has filtration shift \( \leq 1 \) (resp. \( \leq 0 \)) when viewed as a \( k \)-linear map \( \underline{V} \rightarrow \underline{W} \) (resp. \( \underline{W} \rightarrow \underline{V} \)). Since \( \hat{u}, \hat{v} \) are sums of terms of degree \( \geq 0 \), \( \hat{u}|_{\underline{V}}, \hat{v}|_{\underline{V}} \) have filtration shift 0 when viewed as \( k \)-linear endomorphisms of \( \underline{V} \). It follows that \( \underline{\alpha} \) (resp. \( \underline{\beta} \)) has filtration shift \( \leq 1 \) (resp. \( \leq 0 \)).

One then computes \((X_1 - 1)|_{\underline{V}} = (\mu, \Phi)(X_1) - 1|_{\underline{V}} = (e^{\mu e_1} - 1)|_{\underline{V}} = (\frac{e^{\mu e_1} - 1}{e_1})_{|V} = \beta \circ \alpha \circ (\frac{e^{\mu e_1} - 1}{e_1})_{|V} = \beta \circ \alpha \) (equality in \( \text{End}_k(\underline{V}) \)), where the fifth equality follows from the fact that \((V, W, \alpha, \beta)\) is an object of \( \text{Quad}_{\text{fin}}^{\text{DR}} \). \( \square \)

10.4.3. Three natural equivalences \( \eta_{V,W}, \eta_{W,Q} \) and \( \eta_{Q,V} \).

**Lemma 10.9.** Let \( \text{res}^{\text{DR}} : \text{GrMod}_{\text{fin}}(U(f_2)^{\otimes 2}) \rightarrow \text{GrMod}_{\text{fin}}(\mathcal{W}_r^{\text{DR}})^{\otimes 2} \), \( \text{res}^{\text{B}} : \text{FilMod}_{\text{fin}}((kF_2)^{\otimes 2}) \rightarrow \mathcal{W}_r^{\text{B}})^{\otimes 2} \) be the functors dual to the canonical inclusions \( \text{can}^{\text{DR}} : \mathcal{W}_r^{\text{DR}})^{\otimes 2} \subset (U(f_2)^{\otimes 2}, \text{can}^{\text{B}} : \mathcal{W}_r^{\text{B}})^{\otimes 2} \subset (kF_2)^{\otimes 2} \).

There is a natural equivalence \( \eta_{V,W} \)

\[
\begin{array}{ccc}
\text{GrMod}_{\text{fin}}(U(f_2)^{\otimes 2}) & \xrightarrow{(\mu, \Phi)^{\otimes 2}} & \text{FilMod}_{\text{fin}}((kF_2)^{\otimes 2}) \\
\downarrow \text{res}^{\text{DR}} & & \downarrow \text{res}^{\text{B}} \\
\text{GrMod}_{\text{fin}}(\mathcal{W}_r^{\text{DR}})^{\otimes 2} & \xrightarrow{(\mu, \Phi)^{\otimes 2}} & \text{FilMod}_{\text{fin}}(\mathcal{W}_r^{\text{B}})^{\otimes 2}
\end{array}
\]

**Proof.** Since the composite morphisms \((\mu, \Phi)^{\otimes 2} \circ \text{can}^{\text{B}}\) and \( (\mu, \Phi)^{\otimes 2} \circ \text{can}^{\text{DR}}\) are equal, one obtains the desired natural equivalence, with \( \eta_{V,W} \) acting by the identity. \( \square \)

**Lemma 10.10.** There is a natural equivalence \( \eta_{W,Q} \)

\[
\begin{array}{ccc}
\text{GrMod}_{\text{fin}}(\mathcal{W}_r^{\text{DR}}) & \xrightarrow{(\mu, \Phi)^{\otimes 2}} & \text{GrMod}_{\text{fin}}(\mathcal{W}_r^{\text{B}}) \\
\downarrow \text{F}^{\text{DR}}_{Q,W} & & \downarrow \text{F}^{\text{B}}_{Q,W} \\
\text{Quad}_{\text{fin}}^{\text{DR}} & \xrightarrow{\text{comp}(\mu, \Phi)} & \text{Quad}_{\text{fin}}^{\text{B}}
\end{array}
\]

\( \square \)
Proof. Let \((V, W, \alpha, \beta)\) be an object in \(\text{Quad}^{\text{DR}}_{\text{fin}}\). Its image by \(F^{\text{DR}}_{QW}\) is the \(k\)-module \(W\), equipped with the action of \(W^1\) such that \((x e_1)|_W = \alpha \circ x|_V \circ \beta\); the image of the latter object by \((\underline{\mathfrak{a}}_{(\mu, \Phi)}^l, \Phi)^*\) is the same \(k\)-module, equipped with the action of \(W^l\) such that \((\underline{\mathfrak{a}}(X_1 - 1))|_W = (\underline{\mathfrak{a}}(\mu, \Phi)(x)(e^{\mu e_1} - 1))|_W\) is equal to

\[
\alpha \circ (\underline{\mathfrak{a}}(\mu, \Phi)(x)e^{\mu e_1} - 1)|_V \circ \beta = \alpha \circ (\underline{\mathfrak{a}}(\mu, \Phi)|_V \circ \beta \circ \frac{e^{\mu e_1} - 1}{\alpha \circ \beta}.
\]

On the other hand, the image of \((V, W, \alpha, \beta)\) in \(\text{Quad}^B_{\text{fin}}\) is the quadruple

\[
((\underline{\mathfrak{a}}(\mu, \Phi))^*(V), W, \alpha \circ (e^{\mu e_1} - 1)|_V, \beta).
\]

The image of the latter object by \(F^B_{QW}\) is \(W\), equipped with the action of \(W^l\) such that

\[
(\underline{\mathfrak{a}}(X_1 - 1))|_W = \alpha \circ (e^{\mu e_1} - 1)|_V \circ (\underline{\mathfrak{a}}(\mu, \Phi)|_V \circ \beta) = \alpha \circ (\underline{\mathfrak{a}}(\mu, \Phi)|_V \circ \beta).
\]

One obtains the desired natural equivalence, with \(\eta_{QW}\) acting as follows

\[
\eta_{QW}(V, W, \alpha, \beta) : (\underline{\mathfrak{a}}(\mu, \Phi))^* \circ F^{\text{DR}}_{QW}(V, W, \alpha, \beta) \simeq W \xrightarrow{\frac{\alpha e_1^{\mu e_1} - \beta}{\alpha \circ \beta}} W \simeq F^B_{QW} \circ \text{comp}_{(\mu, \Phi)}(V, W, \alpha, \beta).
\]

Lemma 10.11. There is a natural equivalence \(\eta_{QV}\)

\[
\begin{array}{ccc}
\text{Quad}^{\text{DR}}_{\text{fin}} & \overset{\text{comp}_{(\mu, \Phi)}}{\longrightarrow} & \text{Quad}^B_{\text{fin}} \\
F^B_{QW} & \underset{\eta_{QV}}{\swarrow} & \searrow F^B_{VQ} \\
\text{GrMod}^*_{\text{fin}}(U(f_2)^{\otimes 2}) & \overset{(\underline{\mathfrak{a}})^{\otimes 2}}{\longrightarrow} & \text{FilMod}^*_{\text{fin}}((kF_2)^{\otimes 2})
\end{array}
\]

Proof. Let \(C\) be an object in \(\text{GrMod}^*_{\text{fin}}(U(f_2)^{\otimes 2})\). Then \(\text{comp}_{(\mu, \Phi)} \circ F^{\text{DR}}_{QV}(C)\) is the object

\[
((\underline{\mathfrak{a}}(\mu, \Phi))^*V(C), W(C), \alpha(C) \circ (\ell\frac{e^{\mu e_1} - 1}{e_1})|_{V(C)}, \beta(C))
\]

where

\[
(\underline{\mathfrak{a}}(\mu, \Phi))^*V(C) \simeq (\underline{\mathfrak{a}}(\mu, \Phi))^*\ell^*\text{Hom}_U(p_5)(J(p_5), \text{pr}_1^*12(C)) \simeq \ell^*(\underline{\mathfrak{a}}(\mu, \Phi))^*\text{Hom}_U(p_5)(J(p_5), \text{pr}_1^*12(C))
\]

by Lemma 8.1 while \(F^B_{VQ} \circ (\underline{\mathfrak{a}}^{\otimes 2})^*(C)\) is the object

\[
(\ell^*\text{Hom}_{kF_2^*}(J(p_5), \text{pr}_1^*12(C)), C, \alpha(C), \beta(C))
\]

where \(C := (\underline{\mathfrak{a}}(\mu, \Phi))^*(C)\).

As \(C\) has finite support as a \((U(f_2)^{\otimes 2})\)-module, the same is true of \(\text{pr}_1^*12(C)\) as a \((U(p_5))\)-module. Moreover, \(J(p_5)\) is free as a \((U(p_5))\)-module. It follows that there is an isomorphism

\[
\text{Hom}_{U(p_5)}(J(p_5), \text{pr}_1^*12(C)) \simeq \text{Hom}_{U(p_5)}(\text{pr}_1^*12(C), J(p_5))^*.
\]

The same arguments imply the isomorphism

\[
\text{Hom}_{kF_2^*}(J(p_5), \text{pr}_1^*12(C)) \simeq \text{Hom}_{kF_2^*}(\text{pr}_1^*12(C), J(p_5))^*.
\]
It follows from (8.3.2) that \( \underline{a}^{(5)}_{(\mu, \Phi)} \) induces an isomorphism \( J(\text{pr}_5)^\land \to J(\text{pr}_5)^\land \), moreover Lemma 8.8 implies that there is a well-defined \( kP_5^* \)-module morphism

\[
(\underline{a}^{(5)}_{(\mu, \Phi)})^* \text{Hom}_{U(p_5)}(J(\text{pr}_5)^\land, \text{pr}_{12}^*(C)) \to \text{Hom}_{kP_5^*}(J(\text{pr}_5)^\land, \text{pr}_{12}^*(C)),
\]

(10.4.6)

\[ \varphi \mapsto \kappa \cdot \varphi \circ \underline{a}^{(5)}_{(\mu, \Phi)} \]

where \( \kappa \) is as in (8.2.1). Combining this with the isomorphisms (10.4.4), (10.4.5), we obtain the isomorphism

\[ \eta_{1, QV}(C) : (\underline{a}^{(5)}_{(\mu, \Phi)})^* \text{Hom}_{U(p_5)}(J(\text{pr}_5), \text{pr}_{12}^*(C)) \to \text{Hom}_{kP_5^*}(J(\text{pr}_5), \text{pr}_{12}^*(C)), \]

given by (10.4.6).

In (8.3.2), we defined \( v \in (U(f_2)^{\otimes 2})^* \); then \( v^{-1} \) belongs to the same group. As \( C \) is an object of \( \text{GrMod}_{\text{fin}}(U(f_2)^{\otimes 2}) \), this element acts on \( C \). We define a \( k \)-linear map

\[ \eta_{2, QV}(C) : C \to C, \quad c \mapsto v^{-1} \cdot c. \]

Let us show that the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\beta(C)} & \text{Hom}_{U(p_5)}(J(\text{pr}_5), \text{pr}_{12}^*(C)) \\
\eta_{2, QV}(C) \downarrow & & \downarrow \eta_{1, QV}(C) \\
\text{Hom}_{kP_5^*}(J(\text{pr}_5), \text{pr}_{12}^*(C)) & \xrightarrow{\beta(C)} & C
\end{array}
\]

(10.4.7)

commutes. Let \( c \in C \). Let \( \varphi_c := \beta(C)(c), \quad \xi := v^{-1} \cdot c, \quad \varphi := \beta(C)(c) \). Then the \( k \)-linear map

\[ \eta_{1, QV}(C) \circ \beta(C)(c) \]

is equal to \( \eta_{1, QV}(C)(\varphi_c) \), which is \( \kappa \cdot \varphi_c \circ \underline{a}^{(5)}_{(\mu, \Phi)} \), and is therefore such that

\[ x_{i5} - 1 \mapsto \kappa \cdot \varphi_c(\underline{a}^{(5)}_{(\mu, \Phi)}(x_{i5} - 1)) = \kappa \cdot (\overline{y}_{11} - \overline{y}_{12}) \cdot c = \begin{cases} e^{\mu f_1} \cdot v^{-1} \cdot c & \text{if } i = 1, \\ -v^{-1} \cdot c & \text{if } i = 2, \\ 0 & \text{if } i = 3, \end{cases} \]

using the results and notation of Lemma 8.8.

On the other hand, the \( k \)-linear map

\[ \beta(C) \circ \eta_{2, QV}(C)(c) \]

is equal to \( x_{i5} - 1 \mapsto \begin{cases} Y_1 \cdot \xi & \text{if } i = 1, \\ -\xi & \text{if } i = 2, \\ 0 & \text{if } i = 3. \end{cases} \)

As \( \underline{\xi} = e^{-1} \cdot c \) and \( \underline{a}^{(5)}_{(\mu, \Phi)}(Y_1) = e^{\mu f_1} \), we have \( \beta(C) \circ \eta_{2, QV}(C)(c) = \eta_{1, QV}(C) \circ \beta(C)(c) \), therefore (10.4.7) commutes.

Recall that \( \beta(C) \circ \alpha(C) = \ell(e_1)|_{V(C)} \); it follows that

\[ \beta(C) \circ \alpha(C) \circ \ell\left(\frac{e^{\mu e_1} - 1}{e_1}\right)|_{V(C)} = \ell(e_1)|_{V(C)}. \]
On the other hand,

\[ \beta(C) \circ \alpha(C) = (x_{12} - 1)_{\psi(C)}. \]

For \( \varphi \in \text{Hom}_{U(p)}(J(pr_5), pr_{12}^*(C)) \), one then has

\[ \eta_{\psi(C)}(C) \circ \beta(C) \circ \alpha(C)(\varphi) = \eta_{\psi(C)}(C) \left( J(pr_5) \to C, x \mapsto \varphi(x \cdot (e^{\mu_{12}} - 1)) \right) \]

while

\[ \beta(C) \circ \alpha(C) \circ \eta_{\psi(C)}(C)(\varphi) = \beta(C) \circ \alpha(C) \left( J(pr_5) \to C, x \mapsto \kappa \cdot \varphi(a(\mu, \Phi)(x)) \right) \]

\[ = \left( J(pr_5) \to C, x \mapsto \kappa \cdot \varphi(a(\mu, \Phi)(x)(x_{12} - 1)) \right). \]

The two elements coincide as \( a(\mu, \Phi)(x_{12} - 1) = e^{\mu_{12}} - 1. \)

It follows that the diagram

\[ \begin{array}{ccc}
\text{Hom}_{U(p)}^*(J(pr_5), pr_{12}^*(C)) & \xrightarrow{\beta(C) \circ \alpha(C)} & \text{Hom}_{U(p)}^*(J(pr_5), pr_{12}^*(C)) \\
\eta_{\psi(C)}(C) \downarrow & & \downarrow \eta_{\psi(C)}(C) \\
\text{Hom}_{kP}^*(J(pr_5), pr_{12}^*(C)) & \xrightarrow{\beta(C) \circ \alpha(C)} & \text{Hom}_{kP}^*(J(pr_5), pr_{12}^*(C))
\end{array} \]

commutes. Since \((10.4.7)\) commutes, and by the injectivity of \( \beta(C) : C \to \text{Hom}_{kP}^*(J(pr_5), pr_{12}^*(C)) \), it follows that the diagram

\[ \begin{array}{ccc}
\text{Hom}_{U(p)}^*(J(pr_5), pr_{12}^*(C)) & \xrightarrow{\alpha(C)} & C \\
\eta_{\psi(C)}(C) \downarrow & & \downarrow \eta_{\psi(C)}(C) \\
\text{Hom}_{kP}^*(J(pr_5), pr_{12}^*(C)) & \xrightarrow{\alpha(C)} & C
\end{array} \]

commutes. This and the commutativity of \((10.4.7)\) imply that the pair \( (\eta_{\psi(C)}(C), \eta_{2,\psi(C)}(C)) \) is an isomorphism from the object \((10.4.2)\) to the object \((10.4.3)\), which is obviously functorial with respect to \( C \). We then set

\[ \eta_{\psi(C)} := (C \mapsto (\eta_{1,\psi(C)}(C), \eta_{2,\psi(C)}(C))). \]

\[ \square \]

10.5. A cube of categories. By a cube of categories, we mean the following data:

- a collection of categories \( (C)_{\xi \in \Xi} \), indexed by \( \Xi := \{0, 1\}^3 \subset \oplus_{i=1}^3 \mathbb{Z}^\xi \)
- a collection of functors \( f_{\xi,\xi+i} : C_{\xi} \to C_{\xi+i} \) for any pair \( (\xi, \tilde{i}) \in \Xi \times \{\tilde{1}, \tilde{2}, \tilde{3}\} \) such that \( \xi + \tilde{i} \in \Xi \)
- a collection of natural equivalences \( \eta^{a,i} \in \text{Eq}(f_{a+i, \tilde{i}+\tilde{j}} \circ f_{a+i, \tilde{j}} \circ f_{a+i, \tilde{j}}) \), indexed by \( (a, i) \in \{0, 1\} \times [1, 3] \), where \( (i, j, k) \) is the cyclic permutation of \( (1, 2, 3) \) starting at \( i \).
There are therefore 8 categories, 12 functors, and 6 natural equivalences. The situation is depicted as follows

The element

$$\eta(1) \circ \eta(2) \circ \eta(3),$$

where

$$\eta(i) = \left( \text{id}_{f_{i,j,k}} \cdot \eta^{0,k} \right) \circ \left( \eta^{1,j} \cdot \text{id}_{f_{0,j}} \right)^{-1}$$

(where \((i,j,k)\) is a cyclic permutation of \((1,2,3)\)) is then an element of \(\text{Eq}(f_{1,2,3} \circ f_{1,2} \circ f_{0,1})\).

We say that the cube of categories is flat iff this element is the identity equivalence.

One then shows that the data

\[
\begin{align*}
C_0 &:= \text{GrMod}_{\text{fin}}(U(j_2)^{\otimes 2}), & C_{1} &:= \text{GrMod}_{\text{fin}}((W_{r_{DR}})^{\otimes 2}), & C_{2} &:= \text{FilMod}_{\text{fin}}((kF_2)^{\otimes 2}), & C_{3} &:= \text{Quad}_{\text{fin}}^{DR}, \\
C_{1+2} &:= \text{FilMod}_{\text{fin}}((W_{r_{B}})^{\otimes 2}), & C_{2+3} &:= \text{Quad}_{\text{fin}}^{B}, & C_{3+1} &:= \text{GrMod}_{\text{fin}}(W_{r_{DR}}), & C_{1+2+3} &:= \text{FilMod}_{\text{fin}}(W_{r_{B}}), \\
f_{0,1} &:= \text{res}^{DR}, & f_{0,2} &:= (\mathfrak{a}_{(\mu,\phi)})^*, & f_{0,3} &:= F_{V_{Q}}^{DR}, & f_{1,2} &:= (\mathfrak{a}_{(\mu,\phi)})^{\otimes 2}, & f_{1,3} &:= (\Delta_{l,r}^*)^*, & f_{2,3} &:= \text{res}^{B}, \\
f_{1,2} &:= F_{V_{Q}}, & f_{1,3} &:= F_{V_{Q}}^{DQ}, & f_{2,2} &:= \text{comp}_{(\mu,\phi)}, & f_{2,3} &:= (\Delta_{l,r}^*)^*, & f_{2,3} &:= F_{Q_{W}}, & f_{1,3} &:= (\mathfrak{a}_{(\mu,\phi)})^*, \\
\eta^{0,1} &:= \eta_{QV}, & \eta^{0,2} &:= \eta_{DR}, & \eta^{0,3} &:= \eta_{QW}, & \eta^{1,1} &:= \eta', & \eta^{1,2} &:= \eta^B, & \eta^{1,3} &:= (\eta_{QW})^{-1}.
\end{align*}
\]

give rise to a flat cube of categories in the above sense.
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