Consensus with preserved privacy against neighbor collusion

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Abstract
This paper proposes a privacy-preserving algorithm to solve the average-consensus problem based on Shamir’s secret sharing scheme, in which a network of agents reach an agreement on their states without exposing their individual states until an agreement is reached. Unlike other methods, the proposed algorithm renders the network resistant to the collusion of any given number of neighbors (even with all neighbors’ colluding). Another virtue of this work is that such a method can protect the network consensus procedure from eavesdropping.

Keywords Privacy-preserving consensus · Cyber security · Network control · Secret sharing scheme

1 Introduction
As a successful conceptual abstraction of many emerging phenomena in nature, the multi-agent model (MAM) has been extensively studied in the last decades. These studies initiated from the fundamental but also inspiring problem, namely multi-agent consensus, which aims at driving the states of agents to a global agreement. The formal study of the consensus problem was first introduced by Degroot [1] in the 1970s, which sparked many further extensions and developments (see, e.g., [2–4] and the references therein). The methods and results, which were conceived in the study of the consensus problem, were later adapted to network coordination problems in control theory, such as formation control [5–9], distributed optimization [10–13], and also network games [14, 15]. In the control setting, the situation is further complicated, as the agents are governed by nonlinear dynamics [16, 17].

In distributed algorithms, each agent typically requires access to the states of its neighbors to compute a local update. This may leave the agents in the network vulnerable, as some of them may not wish to disclose this local information to their neighbors, especially if some of it is highly private or sensitive. For many applications, it is, therefore, essential to achieve network consensus, while preserving the privacy of each agent. For example, in opinion dynamics [18], opinions may relate to a sensitive topic, and the participating individuals expect these to remain secret, especially from their acquaintances, i.e., each of their neighbors.

The fundamental idea of many existing approaches to privacy-preserving consensus algorithms is to conceal the individual state by adding a deterministic or stochastic disturbance to the real state before communicating it. This idea was first mentioned by Kefayati et al. [19], where a zero-mean normal noise was added to the agents’ states. Based on this work, [20] proposed a synchronization algorithm that blends the true state with a random noise drawn from a Laplace distribution with a time-decaying magnitude. These methods guarantee that the agents reach consensus, but the value agreed upon may not necessarily be the average of the initial states. To achieve average consensus, Mo and Murray [21] shifted the real state by a particular linear combination of Gaussian processes, tailored specifically to ensure that at any time the sum of all the noise injected previously vanishes exactly (also cf. [22–24]). The true states can also be masked using a deterministic state mapping [25], or by simply adding a deterministic disturbance, such as an offset.
Another approach to privacy-preserving consensus takes advantage of homomorphic encryption schemes, which allows algebraic computations to be performed directly on the encrypted data without the need of deciphering it first [29]. Indeed, many prevalent encryption methods are naturally partially homomorphic, meaning that one, but not the other, of the addition and multiplication operations can be performed directly on the ciphertexts. For example, the RSA and ElGamal cryptosystems allow multiplications on the encrypted data, and the Benaloh and Paillier systems allow additions without deciphering [30]. In the classical consensus algorithm, only addition is involved, which is why the Paillier cryptosystems were used in [31] and [32] to achieve privacy-preserving consensus.

In cryptography, the secret sharing scheme is a multi-party encryption method to share a confidential message with multiple parties, which ensures that even with the collusion of a certain number of parties, it is still not possible to uncover the secret message. Shamir [33] presented the first algorithm to solve the secret sharing problem, and the good survey paper [34] provides more insights on the historical evolution of this problem.

In this work, we employ a secret sharing scheme for the communication between agents, to address the problem of privacy-preserving consensus on undirected graphs. As in network coordination, each agent is required to send its state to all its neighbors, which indeed injects into the network the duplicates of the same piece of information. To preserve its privacy, each agent can partition its local information into several so-called secret shares. Then, rather than sending the full information to all of its neighbors, each agent only sends one share to each neighbor. As the information can be reconstructed entirely from a certain specified number of shares, the information injected into the network for each agent is still intact but in a confidential way. In addition, the secret sharing scheme makes the communication naturally resistant to eavesdropping.

Unlike methods based on differential privacy techniques [20–24], the proposed method can reach average consensus with no errors, and also protect the network from eavesdropping. In addition, the privacy security adopted in this paper renders the network immune to the collusion of any given number of neighbors. This is in contrast to [21, 26], where at least one neighbor of each agent must be honest.

In the rest of the paper, we use \( \mathbb{Z}^+ \) to denote the set of all positive integers, and for any \( N \in \mathbb{Z}^+ \), we define the set \( [N] = \{1, 2, \ldots, N\} \). In addition, we denote by \( |S| \) the cardinality of a given set \( S \). For any event \( A \), the indicator function \( \mathbb{1}(A) = 1 \) when \( A \) happens, and otherwise \( \mathbb{1}(A) = 0 \). Moreover, we denote \( \mathbf{1} \in \mathbb{R}^n \) is the vector consisting of all one entries.

## 2 Preliminaries

In this section, we will introduce the notions used in the paper, and revisit some fundamental results on graph theory and secret sharing schemes.

### 2.1 Network graph and consensus algorithm with switching topology

For a networked system, the topology of inter-agent connectivity can be modeled by a graph \( G = (\mathcal{V}, \mathcal{E}) \), where the set of nodes is \( \mathcal{V} = \{1, \ldots, N\} \), and \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) is the edge set. A graph \( G \) is undirected if \((i,j) \in \mathcal{E}\), for any \((j,i) \in \mathcal{E}\). In this paper, without further indication, we assume that all the graphs are undirected. We also define the neighbor set of a node \( i \) as \( N_i = \{j : (j,i) \in \mathcal{E}\} \), and we say that \( j \) is a neighbor of \( i \) if \( j \in N_i \). Moreover, we say that two edges are adjacent if they are incident to a same endpoint.

The union of any two graphs \( G_1 \) and \( G_2 \) is defined by \( G_1 \cup G_2 = (\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E}_1 \cup \mathcal{E}_2) \). Given a finite set of graphs \( \mathcal{G} = \{G_1, G_2, \ldots, G_M\} \) all having the same set of nodes \( \mathcal{V} \), we say that a function \( \mathcal{H} : \mathbb{Z}^+ \to \mathcal{G} \) is a dynamical graph. Moreover, a dynamical graph \( \mathcal{H}(t) \) is jointly connected across any time interval \( t \in \mathbb{Z}^+ \) if the graph \( \bigcup_{\tau \in I} \mathcal{H}(\tau) \) is connected.

Without loss of generality, we denote by \( x_i(t) \in \mathbb{R} \) the state of node \( i \) at time \( t \in \mathbb{Z}^+ \). We say that the agents in such a network have reached consensus, if, for any initial condition \( x_i(0), i \in \mathcal{V} \), it holds that at some time \( t \in \mathbb{Z}^+ \), \( x_i(t) = x_j(t) \) for all \( i, j \in \mathcal{V} \). To tackle this problem, one can cast to the discrete-time consensus algorithm,

\[
x_i(t+1) = x_i(t) + \alpha_i(t) \sum_{j \in N_i} (x_j(t) - x_i(t)),
\]

for \( i \in [N] \), where \( \alpha_i(t) \in \mathbb{R} \) is a step size. It has been shown that if a communication graph \( \mathcal{G} \) is connected, and the step size \( \alpha_i \) in (1) satisfies \( \alpha_i < \frac{1}{|N_i|} \), then the iterative algorithm given in (1) ensures that the states of all agents converge to the average value \( \bar{x}_0 \), where \( \bar{x}_0 = \frac{1}{N} \sum_{i=1}^{N} x_i(0) \) (see [2, 4, 35] and the references therein).

Moreover, when the communication topology is a dynamical graph \( \mathcal{H}(t), t \in \mathbb{Z}^+ \), the following lemma gives a sufficient condition for asymptotic convergence of the consensus algorithm in (1).

**Lemma 1** (Proposition 2 [36]) Given a dynamical undirected graph \( \mathcal{H}(t) \), let \( \alpha_i(t) = \frac{1}{1 + |N_i(t)|} \) for each time \( t \). If for any time \( T \) the graph \( \mathcal{H}_T = \bigcup_{\tau = T}^{\infty} \mathcal{H}(\tau) \) is connected, then consensus...
is globally asymptotically reached using the iterative algorithm in (1).

Note that Lemma 1 only guarantees consensus but not average consensus for the agents’ states.

### 2.2 Secret sharing schemes

The secret sharing scheme is an encryption method for sharing a confidential message with multiple parties, such that even with the collusion of a certain number of parties, the message should still not be disclosed.

Specifically, an \((n, p)\) secret sharing scheme consists of two algorithms \((\text{Share}, \text{Reconstruct})\) with the forms that:

- **Share** takes as input a secret \(M\) and outputs \(n\) shares \((M_1, \ldots, M_n)\).
- **Reconstruct** takes as input \(p\) different shares \((M_i)_{i \in I}\) for any index set \(I \subset [n]\) with \(|I| = p\), and outputs \(M\).

The generated secret share \(M_i\) is then distributed to the party \(i\) for each \(i \in [n]\). The security of such a scheme requires that any collusion of less than \(p\) parties should reveal no information about the message \(M\). More precisely, this means that, for any index set \(I \subset [n]\) with \(|I| < p\), the distribution of \((M_i)_{i \in I}\) should be independent of the true message \(M\). Secret sharing schemes are used in many applications, e.g., encryption keys, distributed storage, missile launch codes, and numbered bank accounts. In these applications, each of the generated pieces of information must keep the original message confidential, as their exposure is undesirable; however, it is also critical that the message should not be lost.

One celebrated secret sharing scheme is the Shamir’s scheme proposed by Adi Shamir [33]. In this scheme, the Share algorithm samples the values of a secret \((p - 1)\)-order polynomial at \(n\) different points, and the Reconstruct algorithm can recover the secret polynomial from any \(p\) of these samples. In addition, as the \((p - 1)\)-order polynomial contains the true message \(M\) as its constant term, \(M\) can be reconstructed (see the appendix for more details).

### 3 Problem formulation

In this paper, we want to achieve network consensus subject to communication safety, and also anti-collusion of neighbors.

Define the security degree by an integer tuple:

\[
p = (p_1, p_2, \ldots, p_N) \in \mathbb{Z}^+^N
\]

with \((p_i - 1)\) indicating the maximum number of neighbors of agent \(i\) that are allowed to collude without a privacy leak. Next, we give the detailed definition of the security adopted in the paper.

**Definition 1** \((p\)-degree security\) We say that an algorithm in a network is of \(p\)-degree security if for each agent \(i \in [N]\) at any time \(t \in \mathbb{Z}^+\):

1. the state \(x_i(t)\) is safe even with the collusion of less than \(p_i\) neighbors in \(\mathcal{N}_i\), and
2. it is not possible to disclose the state \(x_i(t)\) by eavesdropping the communication on less than \(p_i\) edges in \(\{(i, j) \in \mathcal{E} : j \in \mathcal{N}_i\}\).

In this definition, we set the security degree \(p_i\) for each agent \(i\) which is to assure that even a certain number of \(i\)'s neighbors betray or get attacked, the state \(x_i(k)\) is still not leaked.

**Remark 1** For an algorithm of \(p\)-degree security:

(i) if \(p_i \geq 2\), the state \(x_i(t)\) is kept secret from every neighbor \(j \in \mathcal{N}_i\);
(ii) if \(p_i = |\mathcal{N}_i|\), then \(x_i(t)\) is disclosed only when all neighbors of a node are colluding;
(iii) if \(p_i > |\mathcal{N}_i|\), then the state \(x_i(t)\) is completely confidential in the network.

Now, we are ready to state the problem of privacy-preserving consensus that will be solved in this paper.

**Problem 1** In a network consisting of \(N\) agents, the problem of privacy-preserving consensus with security degree \(p\) is to achieve for every agent \(i\) that

(a) average consensus is reached, that is

\[
\lim_{t \to \infty} x_i(t) = \frac{1}{N} \sum_{j \in [N]} x_j(0), \quad \forall i \in [N],
\]

(b) the consensus algorithm is of \(p\)-degree security.

We note that the privacy-preserving requirement of (2) in Problem 1 is only valid before consensus has been reached. After that, although the information transmitted between agents is still encrypted, or more precisely the communication satisfies \(p\)-degree security, the agent state \(x_i(k)\) is nevertheless already known to all the other agents, due to the state consensus.
4 Privacy-preserving consensus based on secret sharing

Two algorithms are proposed in this section. The first one solves the privacy-preserving consensus problem using a secret sharing scheme in communication. The second one is a key distribution algorithm which synchronizes a secret key across the network within finite steps.

4.1 Privacy-preserving consensus algorithm

In this subsection, we propose an algorithm inspired by the Shamir’s secret sharing scheme to solve Problem 1.

As a private key, each agent \(i\) will randomly initialize\(^{1}\) a coefficient vector \(a^{(i)} = (a^{(i)}_1, \ldots, a^{(i)}_{p_i-1}) \in [−1, 1]^{p_i−1}\), and keep this vector \(a^{(i)}\) secret from all other agents. Then, for each \(i \in [N]\), we define an encryption polynomial of order \((p_i−1)\) by

\[
f_i(\theta, t) = \sum_{j=1}^{p_i-1} a^{(i)}_j \theta^j + x_i(t)
\]

for \(\theta \in \mathbb{R}, t \in \mathbb{Z}^+\). This polynomial is only known to agent \(i\), since the coefficients \(a^{(i)}\) and state \(x_i(t)\) are both hidden from the others. Note that the state \(x_i(t)\) is equal to \(f_i(0, t)\).

Let a security degree \(p \in \mathbb{Z}^+\), and we say that \(p\) has maximal order \(\bar{p}\) if it holds that

\[
p = \max_{i \in [N]} p_i.
\]

Note that such an upper bound \(\bar{p}\) of secret degrees indicates the maximal capacity of the tolerance for attacks on the neighbors, and can be preassigned for a privacy-preserving algorithm.

Then, we define the key sequence for \(\bar{p}\) communication channels by an integer vector:

\[
S = (S_1, \ldots, S_{\bar{p}}) \in [1, \kappa]^{\bar{p}} \subset \mathbb{Z}^\bar{p},
\]

where \(\kappa \gg p\) is the maximal possible key and in general \(\kappa \gg p\).

To ensure that the agents communicate using the same key sequence, they first need to agree on a common key sequence \(S\). To this end, one method is to preassign a random, or a default key sequence for all the agents. For example, we can set a default key \(S = (1, 2, 3, \ldots, p)\), which is actually widely used in many secret sharing applications. An alternative method to establishing a common key sequence \(S\) is to use a consensus algorithm, which can synchronize the key sequences of all the agents starting from any random initial keys. The latter will be further detailed in Sect. 4.2.

Next, we present Algorithm 1 to reach asymptotic consensus with \(p\)-degree security for a group of agents \(\mathcal{V}\). To store the information transmitted through each channel, every node \(i \in \mathcal{V}\) sets a local buffer \(r_i(k) \in \mathbb{R}\), for every \(k \in S\).\(^{2}\)

\[
\text{Algorithm 1 Privacy-preserving consensus with security degree } p.
\]

\[
\text{Input: Security degree } p = (p_1, p_2, \ldots, p_N); \text{ initial } x_i(0), \quad \forall i \in [N]; \text{ graph } \mathcal{G} = (\mathcal{V}, \mathcal{E}); \text{ secret key } S \in [1, \kappa]^p, \text{ which has been successfully distributed to all nodes, and step size } \alpha.
\]

1. **Initialization:** For each node \(i\), randomly choose \(^4\) a coefficient vector \(a^{(i)} \in [−1, 1]^{p_i−1}\), and Handshake Pairs \((b^{(i)}_j, c^{(i)}_j)\), \(\forall j \in \mathcal{N}_i\), where \(b^{(i)}_j \in [-1, 1]\) and \(c^{(i)}_j \in S\). Initialize the buffers as \(r_i(k) = f_i(k, 0), \forall i \in [N], \text{ and } k \in S\).

2. for \(t = 1, 2, \ldots, \) do
3. for \(i = 1 \text{ to } N\) do
4. **Step 1 (Handshake):** decide a channel \(c_{ij}\) for each edge \((i, j) \in \mathcal{E}\).
5. for \(j \in \mathcal{N}_i\) do
6. if \(b^{(i)}_j \geq b^{(j)}_i\) then
7. \(c_{ij} \leftarrow c^{(i)}_j\);
8. else
9. \(c_{ij} \leftarrow c^{(j)}_i\);
10. end if
11. end for
12. **Step 2 (Channel-wise consensus):** update \(i\)'s buffer using the information over channel \(c_{ij}\) for each neighbor \(j\). Note that \(r_j(c_{ij})\) is the only information needed by \(i\).
13. for \(k \in S\) do
14. \(r_i(k) \leftarrow r_i(k) + \alpha \sum_{j \in \mathcal{N}_i} (c_{ij} = k)[r_j(k) - r_i(k)];
\]
15. end for
16. **Step 3:** Reconstruct the state \(x_i(t)\) from buffers.
17. Update \(x_i(t)\):

\[
x_i(t) = \prod_{k \in S} \frac{r_i(k) \prod_{\ell \in S} (-\ell)}{\prod_{k \in S} (k - \ell)};
\]
18. end for
19. Reset Handshake pairs.
20. For each node \(i\), randomly choose Handshake Pairs \((b^{(i)}_j, c^{(i)}_j)\), \(\forall j \in \mathcal{N}_i\).
21. end for

Algorithm 1 starts with a Handshake procedure (Step 1 in Algorithm 1), in which a communication channel \(c_{ij} \in S\)

\[\text{\footnotesize \(^{1}\) In this paper, any random variables are drawn from the uniform distribution on their supported sets.}\]

\[\text{\footnotesize \(^{2}\) The random selection is drawn from the uniform distribution, in particular } a^{(i)} \sim U([-1, 1]^{p_i-1}), \quad b^{(i)} \sim U([-1, 1]), \quad \text{and } c^{(i)} \sim U(S), \text{ where } U(\cdot) \text{ is the uniform distribution.}\]
is assigned for each edge \((i, j)\). Particularly, at the begin-
ing of each iteration, the agent \(i, i \in V\), randomly selects a Handshake Pair \((b_{ij}^{(0)}, c_{ij}^{(0)})\) for each edge \((i, j) \in E\), with \(b_{ij}^{(0)} \in [-1, 1]\) being a random bid value, and \(c_{ij}^{(0)} \in S\) being the preferred channel for edge \((i, j)\) according to agent \(i\). Then, the edge \((i, j)\) will be assigned the preferred channel with the larger bid value from the nodes \(i\) and \(j\).

After each edge has been assigned a channel, a channel-wise consensus iteration (Step 2 in Algorithm 1) is performed among the agents. In particular, for a given channel \(k \in S\), each agent \(i\) updates the buffer \(r_j(k)\) using the information only from the neighbors in \(\{j \in N_i' : c_{ij} = k\}\), i.e., only performs a consensus iteration on the edges whose communication channels have been selected to be \(k\).

Moreover, the Handshake procedure selects a channel for each edge randomly and repeatedly at each time \(t \in \mathbb{Z}^+\). This guarantees that, over any long enough time period, the dynamical graph corresponding to each channel will be jointly connected. As a result, the buffers on each channel will reach consensus asymptotically, and then so does the reconstructed state \(x_i(t)\). Proposition 1 shows that Algorithm 1 achieves average consensus asymptotically for all the agents.

**Proposition 1** If the communication graph \(G\) is connected, and the step size \(\alpha\) satisfies

\[
\alpha < \frac{1}{\max_{i \in V} |N_i'|},
\]

then Algorithm 1 solves Problem 1.

**Proof** For any time \(t \in \mathbb{Z}^+\), through the Handshake procedure, we denote by \(c_i(t) \in S\) the assigned communication channel on the edge \((i, j)\).

Then, given a key sequence \(S \in \mathbb{Z}^p\), for each channel \(k \in S\), we define a channel vector:

\[
\xi_k(t) = (r_1(k), r_2(k), \ldots, r_N(k)), \quad \xi_k(t) \in \mathbb{R}^N,
\]

where \(r_j(k)\) is agent \(i\)'s buffer corresponding to the channel \(k\) at time \(t\). Then, this channel vector satisfies

\[
\xi_k(t+1) = \xi_k(t) - aL_k(t)\xi_k(t),
\]

where the matrix \(L_k(t)\) is the Laplacian matrix of channel \(k\) at time \(t\). In particular, the matrix \(L_k(t) = [l_{ij}(k)]_{ij} \in \mathbb{R}^{N \times N}\) satisfies

\[
l_{ij}(k) = \begin{cases} 
-1, & \text{if } i \neq j, \text{ and } c_{ij}(t) = k, \\
\sum_{r \in N_i'} (c_{ir}(t) = k), & \text{if } i = j, \\
0, & \text{otherwise}.
\end{cases}
\]

Denote the set of all the subgraphs of \(G\) by \(\tilde{G}\). Then, for each channel \(k\), define the dynamical graph \(H_k(t) : \mathbb{Z}^+ \rightarrow \tilde{G}\), such that the Laplacian matrix of \(H_k(t)\) is \(L_k(t)\). Then, for all \(k \in S\), dynamical graph \(H_k(t)\) satisfies that, for any time \(t_0\), there exists an infinite subset of time instants, denoted by \(T_{k_0} \subset \{t \in \mathbb{Z}^+ : t \geq t_0\}\), such that \(H_k(t)\) is connected for any \(t \in T_{k_0}\). The existence of \(T_{k_0}\) is because, in the Handshake procedure, the channels \(c_k(t)\) are randomly selected at each time \(t\).

Then, for each channel \(k\), we define the average buffer vector \(\bar{\xi}_k(t) = 1/N^T\xi_k(t)\), which can be shown to be invariant along the trajectory of system (3). This is due to the fact that \(1^T(I - aL_k(t)) = 1^T\), for any \(t \in \mathbb{Z}^+\). This implies that

\[
\bar{\xi}_k(t) = 1/N^T\bar{\xi}_k(0), \quad \forall t.
\]

Then, the consensus error is given by \(\delta_k(t) = \bar{\xi}_k(t) - \mathbf{1}\bar{\xi}_k(0)\). Note that \(\overline{\delta}_k^2(t) = 0\) for any \(t\). Furthermore, for any \(t_p\), such that \(L_k(t_p)\) corresponds to a connected graph, we know that the matrix \(L_k(t_p)\) has a simple smallest eigenvalue \(0\) with the eigenvector \(\mathbf{1}\). This implies that

\[
\max_{x \in \mathbb{R}^1 t=0} \frac{x^T(I - aL_k(t_p))x}{x^Tx} = 1 - \alpha \min_{x \in \mathbb{R}^1 t=0} \frac{x^TL_k(t_p)x}{x^Tx} = 1 - \alpha \lambda_2^k(t_p),
\]

where \(\lambda_2^k(t_p)\) is the second smallest eigenvalue of the Laplacian \(L_k(t_p)\).

Then, we consider a discrete Lyapunov function \(\Phi_k(t) = \overline{\delta}_k^2(t)\overline{\delta}_k(t)\), for each \(k \in S\). Let \(t_0 = 1\). Then, for any time instant \(t_p \in T_{k_0}\), we have

\[
\Phi_k(t_p + 1) = ||(I - aL_k(t_p))\overline{\delta}_k(t_p) - \mathbf{1}\overline{\delta}_k(0)||_2^2 = ||(I - aL_k(t_p))\overline{\delta}_k(t_p)||_2^2 \leq (1 - \alpha \lambda_2^k(t_p))||\overline{\delta}_k(t_p)||_2^2,
\]

where we use \(L_k(t_p)\mathbf{1} = 0\) in the second equality, and equation (6) in the last inequality. By Gershgorin theorem, all eigenvalues of \(L_k(t_p)\) are located in the interval \([0, 2 \max_i |d_i^k(t_p)|]\), where \(d_i^k(t_p)\) is the \(i\)th diagonal element of Laplacian \(L_k(t_p)\). Thus, if \(\alpha < \frac{1}{\max_{i \in V} |N_i'|} < \frac{1}{\max_{i \in V} |d_i^k(t_p)|}\), then

\[|1 - \alpha \lambda_2^k(t_p)| < 1\]. Therefore, the Lyapunov function \(\Phi_k(t)\) is strictly decreasing at any time \(t \in T_{k_0}\). Combine the argument (4), average consensus follows for all channels \(k \in S\).

Then, we have for any \(k \in S\) and \(i \in V\),
\[ \lim_{t \to \infty} r_i(k)|_t = \frac{1}{N} \sum_{i \in V} r_i(k)|_{t=0} = \frac{1}{N} \sum_{i \in V} f_i(k, 0) \]
\[ = \frac{1}{N} \sum_{i \in V} \left( \sum_{j=1}^{\bar{p}-1} (a_{ij}^{(0)})^k \right) \frac{1}{N} \sum_{i \in V} x_i(0) \]
\[ = \frac{\bar{p}-1}{N} \left( \sum_{i \in V} a_{ij}^{(0)} \right)^k \frac{1}{N} \sum_{i \in V} x_i(0), \]

in which we use \( \bar{p} \geq \max p_i \), and we set \( a_{ij}^{(0)} = 0 \) for any \( j \neq j_i \). Then, we see that after consensus is reached, i.e., when \( t \) is large enough, every node obtains \( \delta \) observations \( \{(k, F(k)) : k \in S\} \) of the average polynomial \( F(\theta) \), where \( F(\theta) \) is
\[ F(\theta) = \frac{\bar{p}-1}{N} \left( \sum_{i \in V} a_{ij}^{(0)} \right) \bar{\theta}^k + \frac{1}{N} \sum_{i \in V} x_i(0). \]

According to the Shamir’s algorithm, the constant term \( \frac{1}{N} \sum_{i \in V} x_i(0) \) can be reconstructed by each agent \( i \) as
\[ \frac{1}{N} \sum_{i \in V} x_i(0) = \sum_{k \in S} r_i(k)|_{t=\infty} \prod_{\varepsilon \in \varepsilon(k)} (-\varepsilon) \prod_{\varepsilon \in \varepsilon(k)} (k-\varepsilon) = x_i(\infty). \]

Thus, the assertion holds.

The \( \bar{p} \)-degree security of Algorithm 1 follows that for any agent \( i \in \mathcal{V} \). Moreover, any collusion of less than its \( p_i \) neighbors cannot reconstruct \( x_i(t) \).

For any adversary, since the key sequence \( S \) is unknown, the interception on the communication link does not give any useful information to reconstruct \( x_i(t) \).

Note that, in Algorithm 1, we can use an alternative update law for each \( r_i(k) \) in (2), which is
\[ r_i(k) \leftarrow \alpha_i r_i(k) + \alpha \sum_{j \in N_i} 1(c_{ij} = k) r_j(k), \]

where \( \alpha = \frac{1}{1 + \sum_{j \in N_i} 1(c_{ij} = k)} \). However, by Lemma 1, this update law can only achieve consensus, but the reached state does not necessarily equal to \( \frac{1}{N} \sum_{i \in V} x_i(0) \).

### 4.2 Key distribution

To protect the privacy-preserving algorithm from eavesdropping, the secret key sequence \( S \) must first be distributed to each individual in the network. Using such a key sequence, the information exchanged between the agents afterwards becomes resistant to the wiretapping on communication links.

Indeed, the key sequence can be set to some default value, e.g., \( S = (1, 2, \ldots, \bar{p}) \). Even with such kinds of default keys being public, any eavesdropper would still need at least \( p_i \) pieces of information transmitted to/from agent \( i \) to reconstruct the state \( x_i(t) \). However, we still provide here a key distribution algorithm for the scenario in which the key \( S \) must be kept secret.

We note that consensus algorithms typically only provide asymptotic convergence to a synchronized state. This means that, within finite time, one agent’s key updated by the algorithm can only approach, but not equal, the keys of the others. To reach an exact synchronization for the key sequence, finite-time consensus algorithms are needed, such as, e.g., [37–40]. These finite-time consensus algorithms, however, often depend on particular communication graphs [41], or require first computing a matrix factorization [37] or minimal polynomial [38] of the weight matrix. For our purpose, with the elements of the key sequence being integers, we can adopt the following finite-step consensus algorithm that can be computed much more easily.

Define the communication weight matrix \( W = [w_{ij}] \in \mathbb{R}^{n \times n} \), satisfying the following conditions:

(a) \( w_{ij} > 0 \) if \((i, j) \in \mathcal{E}\), and \( w_{ij} = 0 \) if \((i, j) \notin \mathcal{E}\). Moreover, \( w_{ii} > 0 \) for \( i \in \mathcal{N} \).

(b) The matrix \( W \) is doubly stochastic, i.e., \( \sum_{k} w_{ki} = 1 \), and \( \sum_{k} w_{ik} = 1 \) for any \( i \in \mathcal{N} \).

Note that, for any connected graph \( G \), the nonnegative matrix \( W \) is irreducible, and moreover, \( W \) is primitive, i.e., \( W \) has only one eigenvalue with maximum modulus. In particular, we have the largest eigenvalue of \( W \), which is 1, and is simple with eigenvector \( \mathbf{1} \).

Denote the spectral norm\(^3\) of \( W - \frac{1}{n} \mathbf{1} \mathbf{1}^T \) by \( \gamma \). Then, we have \( \gamma \in (0, 1) \). This is because the positive semidefinite matrix \( W^T W \) has a unique largest eigenvalue 1 with eigenvector \( \mathbf{1} \). Consequently, all the eigenvalues of \( (W - \frac{1}{n} \mathbf{1} \mathbf{1}^T)^T(W - \frac{1}{n} \mathbf{1} \mathbf{1}^T) = W^T W - \frac{1}{n} \mathbf{1} \mathbf{1}^T \) belong to \( (0, 1) \).

We employ the classical consensus algorithm:
\[ S_i(t + 1) = \sum_{j=1}^{N} w_{ij} S_j(t) \] (7)
for \( S_i(t) \in \mathbb{R}^p \) and \( i \in [N] \). Then, the following lemma gives the convergence rate for the consensus dynamics (7).

**Lemma 2** Let \( \kappa \in \mathbb{R}^+ \), and let a threshold \( \delta \in \mathbb{R}^+ \) be given. Then, for any initial condition \( S_i(0) \in [0, \kappa]^p \), \( i \in [N] \), the trajectory of the consensus algorithm (7) satisfies
\[ \|S_i(T) - \frac{1}{N} \sum_{j=1}^{N} S_j(0)\|_\infty \leq \delta, \quad \forall i \in [N]. \]

---

\(^3\) Spectral norm of a matrix \( A \in \mathbb{C}^{n \times n} \) is defined by \( \max_{\|x\|_2 \leq 1} \|Ax\|_2 \|x\|_2^{-1} \), which is equal to the square root of the maximum eigenvalue of \( A^* A \).
where $T = \left\lceil \log_\kappa \frac{\delta}{\sqrt{N}} \right\rceil$.

**Proof** See the appendix. \qed

Next, we give the algorithm to distribute an integer key sequence to all the agents. Note that, in the algorithm, we denote by $S_{i,\ell}$ the $\ell$th element of agent $i$’s key sequence $S_i$.

By Lemma 2, Algorithm 2 first guarantees that after a number $\left\lceil \frac{1}{2} \log_\kappa \frac{1}{\sqrt{N}} \right\rceil$ of consensus iterations, the consensus error from the average is smaller than the threshold $\delta = 0.1$. Then, if each element in agent $i$’s state keeps some distance away from the closest integer, and if there are no two identical elements in agent $i$’s key sequence $S_i$, we take the integer part of the key sequence $S_i$ as $i$’s key sequence.

Although, using Algorithm 2, we do not need to compute a matrix factorization, or the minimal polynomial of the weight matrix [37, 38], the algorithm requires some global information, i.e., the spectral norm $\gamma$ of the graph.

To avoid this, we can employ the Metropolis matrix as the weight matrix, which is defined by

$$w_{ij} = \begin{cases} \frac{1}{\max(d_i, d_j)}, & \text{if } (i, j) \in E \text{ for } i \neq j, \\ 1 - \sum_{c \in N_i} \frac{1}{2\max(d_i, d_c)}, & \text{if } i = j, \\ 0, & \text{otherwise}, \end{cases}$$

where $d_i$ is the degree of node $i$, i.e., $d_i = |\mathcal{N}_i|$. Using this weight matrix, each node is only required to know the set of its neighbors. Moreover, in this case, the spectral norm of the matrix $W - \frac{1}{n} \mathbf{1} \mathbf{1}^T$ satisfies $\gamma \leq 1 - \frac{1}{71N^2}$ [42, Lemma 2.2].

Actually, when the number of nodes is large, the convergence rate of the consensus algorithm, or the computation complexity for finite-step consensus algorithms becomes enormous. Therefore, one solution to this is using a default (or even public) key. By the results in Sect. 4.1, we see that even if an adversary is aware of the secret key, the privacy of agent $i$ is not disclosed if the number of $i$’s communication channels eavesdropped by the adversary is less than $\mu_i$.

### 5 Simulation

In this section, we will illustrate the algorithm using a numerical example, in which a network consisting of 5 nodes with a connection topology of cyclic graph achieves average consensus using the proposed privacy-preserving method. As reported in [2], if the step size in the synchronization algorithm is not carefully chosen, the system under a cyclic connection graph will only reach consensus but not average consensus.

Specifically, we consider a cyclic graph $\mathcal{G}_c(5)$ consisting of 5 nodes as shown in Fig. 1. Let the upper bound of the security degree $\bar{p} = 4$, and set the security degree $p = (2, 3, 4, 2, 3)$. First, a secret key $S \in \{1, 20\}^4$ is distributed using Algorithm 2, after 28 iterations, all agents share a common secret key $S = (4, 7, 15, 3)$. Then, Algorithm 1 is used to achieve average consensus with the common secret key $S$. In Fig. 2, we show the trajectories of the buffer $r_i(4)$ for all five agents. Figure 3 presents the states $x_i(t)$ reconstructed from each channel’s buffer. We can
6 Conclusions

This paper proposes a privacy-preserving mechanism for the average-consensus problem based on the secret sharing scheme. The proposed algorithm renders the network resistant to the collusion of any given number of neighbors, and protects the consensus procedure from eavesdropping. In future work, we will extend this idea to formation control and distributed optimization, and also exploit it in the relevant applications, such as opinion agreement, sensor network averaging, survey mechanism, and distributed decision making.

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Appendix

Shamir’s scheme The Shamir’s secret sharing scheme [33] allows a user to “share” a secret \( M \) among a group of \( n \) participants, such that (a) any \( p \) or more of the participants can reconstruct \( M \), and (b) any set of less than \( p \) participants learns nothing about \( M \).

For a secret \( M \in \mathcal{F} \), where \( \mathcal{F} \) is a field, the algorithm \( \text{Share}(S) \) is

1. Let \( a_0 = M \).
2. Choose at random \( a_1, \ldots, a_{p-1} \in \mathcal{F} \).
3. Let \( f(x) = \sum_{i=0}^{p-1} a_i x^i \).
4. The secret share \( M_i = (x_i, y_i) \) for any \( i \in [n] \), where \( y_i = f(x_i) \).

To reconstruct \( M \), we need at least \( p \) shares of secret \( \{M_j = (x_j, y_j)\}_{j=1}^p \). Then, one can use the Lagrange interpolation formula:

\[
 f(x) = \sum_{j=1}^p y_j L_j(x),
\]

where the Lagrange basis polynomial \( L_j(x) \) is defined by

\[
 L_j(x) = \prod_{k\in[p]\setminus\{j\}} \frac{x - x_k}{x_j - x_k},
\]

for each \( j \in [p] \). Once the polynomial \( f(x) \) has been reconstructed, the secret message can be recovered as \( M = f(0) \), due to the definition of \( f(x) \). Equivalently, the true message \( M \) can be reconstructed by

\[
 M = f(0) = \sum_{i=1}^M y_i L_i(0) = \sum_{j=1}^M y_j \prod_{k\in[p]\setminus\{j\}} \frac{(-x_k)}{x_j - x_k}.
\]

Proof of Lemma 2 For each \( \ell \in [\overline{p}] \), we denote the \( \ell \)-component vector by

\[
 S^{(\ell)}(t) = (S_{1\ell}(t), S_{2\ell}(t), \ldots, S_{N\ell}(t)) \in \mathbb{R}^N,
\]

where \( S_{j\ell}(t) \) is the \( j \)th element of the key \( S_{\ell}(t) \). Then, for each \( \ell \), the consensus iteration reads \( S^{(\ell)}(t+1) = W S^{(\ell)}(t) \).
Next, the average for element $\ell$ is $\alpha_r(t) = \frac{1}{N} \mathbf{1}^T S^{(r)}(t)$. Moreover, for any $t$, $\alpha_r(t + 1) = \alpha_r(t)$, due to the fact that $\mathbf{1}^T W = \mathbf{1}^T$. Thus, we can denote the invariant average for component $\ell$ by $\alpha_r = \alpha_r(0)$. Then, we have for each $\ell \in [p]$,

$$\|S^{(r)}(t + 1) - \alpha_r \mathbf{1}\|_\infty \leq \|S^{(r)}(t + 1) - \alpha_r \mathbf{1}\|_2 = \|W S^{(r)}(t + 1) - \alpha_r \mathbf{1}\|_2 = \left\|\left(W - \frac{1}{n} \mathbf{1} \mathbf{1}^T\right) S^{(r)}(t) - \alpha_r \mathbf{1}\right\|_2 \leq \lambda \|S^{(r)}(t) - \alpha_r \mathbf{1}\|_2.$$ 

Then, for any time $t$, $\|S^{(r)}(t) - \alpha_r \mathbf{1}\|_\infty \leq \lambda^t \|S^{(r)}(0) - \alpha_r \mathbf{1}\|_2 \leq \lambda^t \sqrt{N} \|S^{(r)}(0) - \alpha_r \mathbf{1}\|_\infty$. According to the initial condition, $\|S^{(r)}(0) - \alpha_r \mathbf{1}\|_\infty \leq \kappa$. Then, the assertion follows. □

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