A Fractional $p$-Laplacian Problem with Multiple Critical Hardy–Sobolev Nonlinearities

Ronaldo B. Assunção, Jeferson C. Silva and Olímpio H. Miyagaki

Abstract. In this work, we study the existence of a weak solution to the following quasi linear elliptic problem involving the fractional $p$-Laplacian operator, a Hardy potential, and multiple critical Sobolev nonlinearities with singularities,

\begin{align*}
(-\Delta_p)^s u - \mu \frac{|u|^{p-2}u}{|x|^s} &= \left| u \right|^{p_s(\beta)-2}u \frac{1}{|x|^\beta} + \left| u \right|^{p_s(\alpha)-2}u \frac{1}{|x|^{\alpha}},
\end{align*}

where $x \in \mathbb{R}^N$, $u \in D^{s,p}(\mathbb{R}^N)$, $0 < s < 1$, $1 < p < +\infty$, $N > sp$, $0 < \alpha < sp$, $0 < \beta < sp$, $\beta \neq \alpha$, $\mu < \mu_H := \inf_{u \in D^{s,p}(\mathbb{R}^N) \setminus \{0\}} \{ \left| u \right|_{s,p}^p / \| u \|_{s,p}^p \} > 0$. To prove the existence of solution to the problem, we have to formulate a refined version of the concentration-compactness principle and, as an independent result, we have to show that the extremals for the Sobolev inequality are attained.

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1. Introduction and main result

The fractional $p$-Laplacian operator is a non-linear and non-local operator defined for differentiable functions $u : \mathbb{R}^N \to \mathbb{R}$ by

\begin{align*}
(-\Delta_p)^s u(x) := 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N + sp}} \, dy,
\end{align*}

where $x \in \mathbb{R}^N$, $u \in D^{s,p}(\mathbb{R}^N)$, $0 < s < 1$, $1 < p < +\infty$, $N > sp$, and $\mu_H := \inf_{u \in D^{s,p}(\mathbb{R}^N) \setminus \{0\}} \{ \left| u \right|_{s,p}^p / \| u \|_{s,p}^p \} > 0$. To prove the existence of solution to the problem, we have to formulate a refined version of the concentration-compactness principle and, as an independent result, we have to show that the extremals for the Sobolev inequality are attained.

Corresponding author: Jeferson C. Silva.

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where \( x \in \mathbb{R}^N, p \in (1, +\infty), s \in (0, 1) \) and \( N > sp \). The definition (1.1) is consistent, up to a normalization constant dependent only on \( N \) and on \( s \), with the usual definiton of the linear fractional Laplacian operator when \( p = 2 \). In this special case it is simply denoted by \((-\Delta)^s\) and is defined by

\[
(-\Delta)^s u(x) := C(N, s) \text{ p.v. } \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N + 2s}} dy,
\]

where p.v. stands for the Cauchy’s principal value and the normalization constant is given by

\[
C(N, s) := 2^{2s-1} \pi^\frac{n}{2} \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(-s)}.
\]

In this way the identity

\[
(-\Delta)^s u = \mathcal{F}^{-1} (|\xi|^{2s} (\mathcal{F} u))
\]

holds, where \( \xi \in \mathbb{R}^N, u \in \mathcal{S}(\mathbb{R}^N) \), the class of Schwartz differentiable functions with rapid decay, and

\[
\mathcal{F} u(\xi) = \int_{\mathbb{R}^N} \exp(-2\pi i x \cdot \xi) u(x) dx
\]

denotes the Fourier transform of \( u \).

For more basic information about the fractional \( p \)-Laplacian operator we cite the article by Di Nezza, Palatucci and Valdinoci [14] and the book by Molica Bisci, Rădulescu, Servadei [29], as well as the references therein; for some motivations in physics, chemistry, and economy that lead to the study of this kind of operator we mention the article by Caffarelli [9]. We would like to cite a recent survey book due to Bucur and Valdinoci [8], where some of the applications above cited are more detailed, for instance, in probabilistic theory: the random walk with long jumps and payoff model; in physics: water waves model, crystal dislocation, geocentric analysis, etc.

Non-local problems involving the fractional \( p \)-Laplacian operator \((-\Delta_p)^s\) have received the attention of several authors in the last decade, mainly in the case \( p = 2 \) and in the cases where the nonlinearities have pure polynomial growth involving subcritical exponents (in the sense of the Sobolev embeddings). For example, this operator leads naturally to the class of quasi-linear problems

\[
\begin{aligned}
(-\Delta_p)^s u &= f(x, u), \quad x \in \Omega \subset \mathbb{R}^N \\
u &= 0, \quad x \in \mathbb{R}^N \setminus \Omega,
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^N \) is a domain. Nowadays there exists an extensive and ever growing literature about the class of quasi-linear problems (1.3) in the case where \( \Omega \) is bounded and with Lipschitz boundary. In particular, we cite Franzina and Palatucci [20] and Lindgren and Lindqvist [27] for problems involving \( p \)-eigenvalues; Di Castro, Kuusi and Palatucci [13] and Iannizzotto, Mosconi and Squassina [24, 25] for regularity theory; Iannizzotto, Liu, Perera, and Squassina [23], Molica Bisci, Rădulescu, and Servadei [29] and Servadei and Valdinoci [31] for the theory of existence of solutions
in the case of nonlinearities with pure polynomial growth involving subcritical exponents; Alves and Miyagaki [2], Fiscella, Molica Bisci and Servadei [18], Servadei and Valdinoci [32] for the theory of existence of solutions in the case of nonlinearities with pure polynomial growth involving critical exponents. Moreover, great attention has been given to the study of existence of solutions to nonlocal problems with Hardy potential and also with other types of nonlinearities; for these cases, we cite Abdellaoui, Peral and Primo [1], Barrios, Medina and Peral [4], Cotioli and Tavoularis [12] and Yang and Wu [34], as well as the references therein.

In what follows, we mention an interesting class of quasi-linear elliptic problems in the general class of problems (1.3); more precisely, we consider problems with multiple critical nonlinearities in the sense of the Sobolev embeddings and also a nonlinearity of Hardy type, which consistently appears on the side of the nonlocal operator. Fillippucci, Pucci and Robert [17] considered the quasi-linear elliptic problem

$$-\Delta_p u - \mu \frac{|u|^{p-1}}{|x|^p} = u^{p^* - 1} + \frac{p^* - 1}{|x|^\alpha} \left( x \in \mathbb{R}^N \right), \quad (1.4)$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ is the $p$-Laplacian operator, $N \geq 2$ is an integer, $p \in (1, N)$, $\alpha \in (0, p)$, $p^*(\alpha) = [p(N - \alpha)]/(N - p)$, $p^* = p^*(0) = Np/(N - p)$, and $\mu$ is a real parameter. The combination of two nonlinearities leads to some serious difficulties and subtleties to problem (1.4). When only one nonlinearity appears with critical exponent, several results about the existence of weak solutions are already known. In general, these weak solutions are radially symmetric with respect to some point. The common strategy to obtain a solution to problem (1.4) consists in the construction of solutions as critical points of the energy functional naturally associated to this class of problems, since they have variational structure. To do this, the authors used a version of the mountain pass theorem due to Ambrosetti and Rabinowitz. However, since the problem is invariant under the action of the group of conformal transformations $u \mapsto u_r(x) := r^{(N-p)/p} u(rx)$, the mountain pass theorem yields only Palais–Smale sequences and not necessarily critical points for the energy functional. So, an important step in the proof of their existence result consists in showing a refined version of the concentration-compactness principle in order to better understand the behavior of the Palais–Smale sequences. The main difficulty is that there is an asymptotic competition between the energy carried by the two critical nonlinearities. If one of them dominates the other, then there is the annihilation of the weaker one; in this case, the limit of the Palais–Smale sequence is a weak solution of a problem involving only one critical nonlinearity. Of course, in this case we do not obtain a weak solution to problem (1.4). Therefore, the crucial point consists in avoiding the domination of one nonlinearity over the other.

Afterwards, Ghoussoub and Shakerian [21] considered the quasi-linear nonlocal elliptic problem

$$(-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = |u|^{2^*(\beta) - 2} u + \frac{|u|^{2^*(\beta) - 2} u}{|x|^{\beta}} \quad (x \in \mathbb{R}^N). \quad (1.5)$$
This problems generalizes the one studied by Filippucci, Pucci and Robert to the case of nonlocal operators; more specifically, to the fractional Laplacian operator with \( p = 2 \). Besides the above-mentioned difficulties caused by the presence of multiple critical nonlinearities, in the case of problem (1.5) there exist additional difficulties. To show the existence of a weak solution, the authors considered an idea proposed by Caffarelli and Silvestre [9] that uses the harmonic extension of the fractional Laplacian to the upper half-space \( \mathbb{R}^{n+1}_+ \), changing the given nonlocal problem to a local problem with Neumann boundary condition.

Recently, Chen [10] considered the quasi-linear nonlocal elliptic problem

\[
(-\Delta)^s u - \mu \frac{|u|^{p^*_s(\alpha)-2} u}{|x|^\alpha} + \frac{|u|^{p^*_s(\beta)-2} u}{|x|^\beta} \quad (x \in \mathbb{R}^N).
\]

This problem generalizes the problem studied by Ghoussoub and Shakerian, still in the case \( p = 2 \) but for the case where both nonlinearities have singularities at the origin. Again, the basic strategy used by the author to show the existence of a weak, positive solution to problem (1.6) was the use of the harmonic extension of the fractional Laplacian proposed by Caffarelli and Silvestre [9] as well as the mountain pass theorem and the concentration-compactness principle.

We would like to mention that the fractional-type concentration-compactness principle was also introduced by Palatucci and Pisante [30], and in a recent survey book by Dipierro, Medina and Valdinoci [15]. Motivated by the above-mentioned results, in this work we consider the quasi-linear elliptic problem involving the fractional \( p \)-Laplacian problem with multiple critical nonlinearities with singularities at the origin and a Hardy term,

\[
(-\Delta_p)^s u - \mu \frac{|u|^{p-2} u}{|x|^p} = \frac{|u|^{p^*_s(\beta)-2} u}{|x|^\beta} + \frac{|u|^{p^*_s(\alpha)-2} u}{|x|^\alpha} \quad (x \in \mathbb{R}^N),
\]

where \( 0 < s < 1, 1 < p < +\infty, N > sp \), \( 0 < \alpha < sp \), \( 0 < \beta < sp \), \( \beta \neq \alpha \), \( \mu < \mu_H \) (the constant \( \mu_H \) is defined below), and \( p^*_s(\alpha) = [p(N - \alpha)]/(N - ps) \); in particular, if \( \alpha = 0 \), then \( p^*_s(0) = p^*_s = Np/(N - p) \).

The choice of the function space where we look for the solutions to problems with variational structure such as problem (1.7) is an important step in its study. Let \( \Omega \subset \mathbb{R}^N \) be an open, bounded subset with differentiable boundary. We consider tacitly that all the functions are Lebesgue integrable and we introduce the fractional Sobolev space

\[
W^{s,p}_0(\Omega) := \{ u \in L^1_{loc}(\mathbb{R}^N) : [u]_{s,p} < +\infty; \ u \equiv 0 \text{ a.e. } \mathbb{R}^N \setminus \Omega \}
\]

and the fractional homogeneous Sobolev space

\[
D^{s,p}(\mathbb{R}^N) := \{ u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty \} \supset W^{s,p}_0(\Omega).
\]

In these definitions, the symbol \([u]_{s,p}\) stands for the Gagliardo seminorm, defined by

\[
u \mapsto [u]_{s,p} = \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \right)^{1/p} \quad (u \in C_0^\infty(\mathbb{R}^N)).
\]
For $1 < p < +\infty$, the function spaces $W_0^{s,p}(\Omega)$ and $D^{s,p}(\mathbb{R}^N)$ are separable, reflexive Banach spaces with respect to the Gagliardo seminorm $[\cdot]_{s,p}$. These spaces can also be understood as the respective completions of the spaces of differentiable functions with compact support $C_0^\infty(\Omega)$ and $C_0^\infty(\mathbb{R}^N)$ with respect to $[\cdot]_{s,p}$; see, for example, Brasco, Mosconi and Squassina [5]. The topological dual of the space $W_0^{s,p}(\Omega)$ is denoted by $W^{-s,p'}(\Omega)$ where $1/p + 1/p' = 1$ or by $(W_0^{s,p}(\Omega))'$, with the corresponding duality product $\langle \cdot, \cdot \rangle: W^{-s,p'}(\Omega) \times W_0^{s,p}(\Omega) \to \mathbb{R}$. Due to the reflexivity of the space, the weak convergence and the weak* convergence in $W^{-s,p'}(\Omega)$ coincide. Moreover, in the Sobolev space $D^{s,p}(\mathbb{R}^N)$, the space where we look for solutions to problem (1.7), the Gagliardo seminorm $[\cdot]_{s,p}$ is in fact a norm and $(D^{s,p}(\mathbb{R}^N); [\cdot]_{s,p})$ is a uniformly convex Banach space.

The variational structure of problem (1.7) can be established with the help of the following version of the Hardy–Sobolev inequality, which can be found in the paper by Chen, Mosconi and Squassina [11].

Let $0 < s < 1, 1 < p < +\infty$ and $0 \leq \alpha < sp < N$. Then there exists a positive constant $C \in \mathbb{R}_+$ such that

$$\left( \int_{\Omega} \frac{|u|^p}{|x|^\alpha} \, dx \right)^{1/p^*_\alpha} \leq C \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N + ps}} \, dx \, dy \right)^{1/p} \tag{1.8}$$

for every $u \in W_0^{s,p}(\Omega)$. The parameter $p^*_\alpha(\alpha)$ is the critical fractional exponent of the Hardy–Sobolev embeddings $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N; |x|^{-sp})$ where the Lebesgue space $L^p(\mathbb{R}^N; |x|^{-sp})$ is equipped with the norm

$$||u||_{L^p(\mathbb{R}^N; |x|^{-sp})} := \left( \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{sp}} \, dx \right)^{1/p}.$$

Indeed, the embeddings $W_0^{s,p}(\Omega) \hookrightarrow L^q(\Omega; |x|^\alpha)$ are continuous for $0 \leq \alpha \leq ps$ and for $1 \leq q \leq p^*_\alpha(\alpha)$; and these embeddings are compact for $1 \leq q < p^*_\alpha(\alpha)$. Moreover, the best constants of these embeddings are positive numbers, and we define

$$\mu_H := \inf_{u \in D^{s,p}(\mathbb{R}^N) \setminus \{0\}} \frac{[u]_{s,p}^p}{||u||_{L^p(\mathbb{R}^N; |x|^{-sp})}^p}. \tag{1.9}$$

The functional $u \mapsto (1/p)[u]_{s,p}^p$ is convex and it belongs to the class $C^1(D^{s,p}(\mathbb{R}_+); \mathbb{R})$, so that for every function $u \in W_0^{s,p}(\Omega)$, its subdifferential is exactly $(-\Delta_p)^s u$, that is, the unique element of the topological dual space $W^{-s,p'}(\Omega)$ such that

$$\langle (-\Delta_p)^s u, \phi \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{J_p(u(x) - u(y)) (\phi(x) - \phi(y))}{|x - y|^{N + ps}} \, dx \, dy; \quad \forall \phi \in W_0^{s,p}(\Omega).$$

In the previous formula, by way of simplicity we introduced the notation: given $1 < m < +\infty$, we define the function $J_m: \mathbb{R} \to \mathbb{R}$ by $J_m(t) = |t|^{m-2}t$.

Now we can define precisely the notion of weak solution to problem (1.7). We say that the function $u \in D^{s,p}(\mathbb{R}^N)$ is a weak solution to problem (1.7) if

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{J_p(u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N + sp}} \, dx \, dy - \mu \int_{\mathbb{R}^N} \frac{J_p(u) \varphi(x)}{|x|^{ps}} \, dx$$
In other terms, \( u \) by Chen [10]. However, since we consider the case 1 by Filippucci, Pucci, and Robert [17], by Ghoussoub and Shakerian [21], and also difficulty, we used directly the definition of the Silvestre [9] because this idea is valid only in the case the harmonic extension of the fractional Laplacian as described by Caffarelli and D posed several difficulties to prove that bounded Palais–Smale in the reflexive Banach space \( D^{s,p}(\mathbb{R}^N) \) has at least a subsequence that converges strongly to a nontrivial named energy functional. In fact, for the parameters in the intervals already specified, we have \( \Phi \in C^1(D^{s,p}(\mathbb{R}^N); \mathbb{R}) \):

\[
\langle \Phi'(u), \varphi \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{J_p(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} \, dx \, dy - \mu \int_{\mathbb{R}^N} \frac{J_p(u(x)) \varphi(x)}{|x|^{ps}} \, dx - \int_{\mathbb{R}^N} \frac{J_p(\beta)(u) \varphi(x)}{|x|^{\beta}} \, dx - \int_{\mathbb{R}^N} \frac{J_p(\alpha)(u) \varphi(x)}{|x|^{\alpha}} \, dx, \quad \forall \varphi \in D^{s,p}(\mathbb{R}^N).
\]

In other terms, \( u \in D^{s,p}(\mathbb{R}^N) \) is a weak solution to problem (1.7) if, and only if, \( \Phi'(u) = 0 \).

Now we can state our result.

**Theorem 1.1.** Let \( 0 < s < 1, \ 1 < p < +\infty, \ N > sp, \ 0 < \alpha < sp, \ 0 < \beta < sp, \ \beta \neq \alpha, \ \mu < \mu_H \). Then there exists a weak solution \( u \in D^{s,p}(\mathbb{R}^N) \) to problem (1.7).

The proof of Theorem 1.1 follows several ideas that have appeared in the papers by Filippucci, Pucci, and Robert [17], by Ghoussoub and Shakerian [21], and also by Chen [10]. However, since we consider the case \( 1 < p < +\infty \), we cannot apply the harmonic extension of the fractional Laplacian as described by Caffarelli and Silvestre [9] because this idea is valid only in the case \( p = 2 \). To deal with this difficulty, we used directly the definition of the \( p \)-fractional operator to prove our result. Moreover, since we consider the whole space \( \mathbb{R}^N \) and since problem (1.7) contains critical nonlinearities in the sense of the Hardy–Sobolev embeddings, it follows that the Hardy–Sobolev embedding \( D^{s,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N; |x|^{-sp}) \) is noncompact. This poses several difficulties to prove that bounded Palais–Smale in the reflexive Banach space \( D^{s,p}(\mathbb{R}^N) \) have at least a subsequence that converges strongly to a nontrivial function in this space. Clear enough, the presence of multiple Sobolev critical nonlinearities also contributes to the difficulties in the proof of the theorem. Moreover, due to the presence of a Hardy potential, with the parameters in the already specified intervals, the functional \( u \mapsto ([u]_{s,p}^p - \mu \int_{\mathbb{R}^N} |u|^p/|x|^{sp} \, dx)^{1/p} \) does not define a norm in the Sobolev space \( D^{s,p}(\mathbb{R}^N) \), although it can be compared to a suitable norm (see Goyal [22] and Filippucci, Pucci and Robert [17]); as a consequence, the
energy functional $\Phi$ is not lower semicontinuous. Based on some estimates proved by Brasco, Mosconi, and Squassina [5], by Xiang, B. Zhang, and X. Zhang [33], and by Brasco, Squassina, and Yang [6], we managed to overcome these difficulties and prove a refined version of the concentration-compactness principle.

Finally, we should mention that Theorem 1.2 below is crucial in the proof of Theorem 1.1.

**Theorem 1.2.** The best Hardy constant, defined by

$$
\frac{1}{K(\mu, \alpha)} = \inf_{u \in D^{s,p}(\mathbb{R}^N), u \neq 0} \frac{[u]_{s,p}^p - \mu \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{ps}} \, dx}{\left( \int_{\mathbb{R}^N} \frac{|u|^{p^*_s(\alpha)}}{|x|^{\alpha}} \, dx \right)^{\frac{p}{p^*_s(\alpha)}}},
$$

(1.11)

is attained by a nontrivial function $u \in D^{s,p}(\mathbb{R}^N)$.

The paper is divided in several sections. In Section 2, we use the mountain pass theorem to show the existence of suitable Palais–Smale sequences; in Section 3, we study the behavior of these Palais–Smale sequences; in Section 4, we prove Theorem 1.1; and in Section 5, we prove Theorem 1.2.

**2. Preliminary results**

In this section we present some preliminary results that will be useful in the proof of Theorem 1.1. By the definition of $\mu_H$, the following inequality is valid,

$$
\mu_H \leq \frac{[u]_{s,p}^p}{\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{ps}} \, dx}
$$

for all $u \in D^{s,p}(\mathbb{R}^N) \setminus \{0\}$. (2.1)

It is well known that the sharp constant $\mu_H$ is attained; the proof of this claim can be found in a paper by Frank and Seiringer [19]. For $0 < \mu < \mu_H$, it follows from inequality (2.1) that

$$
\|u\| := \left( [u]_{s,p}^p - \mu \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{ps}} \, dx \right)^{\frac{1}{p}}
$$

(2.2)

is well defined in the space $D^{s,p}(\mathbb{R}^N)$. Note that $\| \cdot \|$ is comparable to the Gagliardo norm $[\cdot]_{s,p}$; to see this, it is sufficient to use the pair of inequalities

$$
\left( 1 - \frac{\mu_+}{\mu_1} \right) [u]_{s,p}^p \leq \|u\|^p \leq \left( 1 + \frac{\mu_-}{\mu_1} \right) [u]_{s,p}^p,
$$

(2.3)

valid for all $u \in D^{s,p}(\mathbb{R}^N)$ where $\mu_+ := \max\{\mu, 0\}$ and $\mu_- := \max\{-\mu, 0\}$. By combining the Hardy inequality proved in [19] together with the Sobolev inequality proved in Brasco, Mosconi and Squassina [5], we obtain an inequality of Hardy–Sobolev type. Indeed, for $0 < \alpha < ps$, by the Hölder inequality and by the fractional
versions of the Hardy and the Sobolev inequalities, the embedding \( D^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p_\ast(\alpha)}(\mathbb{R}^N, |x|^{-\alpha}) \) is continuous. Using the sharp constant of this embedding, we defined the constant \( K(\mu, \alpha) \) in (1.11) and in Section 5 we show the Theorem 1.2.

Now we recall the definition of the energy functional naturally associated to the variational problem (1.7) and rewrite it in the form

\[
\Phi(u) = \frac{\|u\|^p}{p} - \frac{1}{p_\ast(\beta)} \int_{\mathbb{R}^N} \frac{|u|^{p_\ast(\beta)}}{|x|^\beta} dx - \frac{1}{p_\ast(\alpha)} \int_{\mathbb{R}^N} |u|^{p_\ast(\alpha)} dx.
\]

Using the fractional versions of the Hardy and the Hardy–Sobolev inequalities above-mentioned, it is easy to show that the functional \( \Phi \) is well defined in the space \( D^{s,p}(\mathbb{R}^N) \).

Recall that the sequence \( \{u_n\} \subset D^{s,p}(\mathbb{R}^N) \) is a Palais–Smale sequence for the energy functional \( \Phi \) at the level \( c \in \mathbb{R} \), in short \( (PS)_c \), if \( \Phi(u_n) \to c \) and \( \langle \Phi'(u_n), \phi \rangle \to 0 \) for every \( \phi \in D^{s,p}(\mathbb{R}^N) \). Our first preliminary result concerns the existence of the Palais–Smale sequence for the energy functional \( \Phi \).

**Proposition 2.1.** Let \( \mu \in [0, \mu_H) \) and \( \alpha \in [0, sp) \). Then there exists a sequence \( \{u_n\}_n \subset D^{s,p}(\mathbb{R}^N) \) such that

\[
\lim_{n \to +\infty} \Phi(u_n) = c \quad \text{and} \quad \lim_{n \to +\infty} \Phi'(u_n) = 0 \quad \text{strongly in} \quad (D^{s,p}(\mathbb{R}^N))',
\]

where \( 0 < c < c^* \) and \( c^* \) is defined as being

\[
\min \left\{ \left( \frac{p_\ast(\beta)}{pp_\ast(\beta)} \right) K(\mu, \beta)^{\frac{p_\ast(\beta)}{pp_\ast(\beta)}}, \left( \frac{p_\ast(\alpha)}{pp_\ast(\alpha)} \right) K(\mu, \alpha)^{\frac{p_\ast(\alpha)}{pp_\ast(\alpha)}} \right\}.
\]

To prove Proposition 2.1 we need the following version of the mountain-pass theorem by Ambrosetti and Rabinowitz [3].

**Proposition 2.2.** Let \( (V, \| \cdot \|) \) be a Banach space and consider a function \( F \in C^1(V; \mathbb{R}) \). Suppose that

1. \( F(0) = 0 \).
2. There exist \( \lambda > 0 \) and \( R > 0 \) such that \( F(u) \geq \lambda \) for all \( u \in V \), with \( \|u\| = R \).
3. There exists \( v_0 \in V \) such that \( \liminf_{t \to +\infty} F(tv_0) < 0 \).

Let \( t_0 > 0 \) be a positive real number such that \( \|tv_0\| > R \) and \( F(t_0v_0) < 0 \). Define

\[
c := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} F(\gamma(t))
\]

where \( \Gamma := \{ \gamma \in C([0,1], V) : \gamma(0) = 0 \quad \text{and} \quad \gamma(1) = t_0v_0 \} \). Then there exists a Palais–Smale \( (PS)_c \) sequence at the level \( c \in \mathbb{R} \) for the functional \( F \).

The proof of Proposition 2.1 follows from the next lemma.

**Lemma 2.3.** The energy functional \( \Phi \) verifies the hypotheses of Proposition 2.2 for every function \( u \in D^{s,p}(\mathbb{R}^N) \setminus \{0\} \).
Proof. Clearly, $\Phi \in C^1(D^{s,p}(\mathbb{R}^N))$ and $\Phi(0) = 0$. By definition (1.11) of the best constant $1/K(\mu, \alpha)$, we have

$$\Phi(u) \geq \frac{\|u\|^p}{p} - \frac{K(\mu, \beta)}{p_s(\beta)} \frac{\|u\|^{p_s(\beta)}}{p_s(\beta)} \int_{\mathbb{R}^N} \frac{|u|^{p_s(\beta)}}{|x|^\beta} \, dx - \frac{K(\mu, \alpha)}{p_s(\alpha)} \frac{\|u\|^{p_s(\alpha)}}{p_s(\alpha)} \int_{\mathbb{R}^N} \frac{|u|^{p_s(\alpha)}}{|x|^\alpha} \, dx.$$

Since $\alpha, \beta \in (0, sp)$, it follows that $p_s(\alpha) > p$ and $p_s(\beta) > p$. Therefore, from the pair of inequalities (2.3) we deduce, for $\lambda > 0$ such that $\Phi(u) > \Phi(0)$, we have

$$\lim_{t \to +\infty} \Phi(tu) = -\infty \quad \text{as} \quad t \to +\infty.$$

So, we consider $t_u > 0$ such that $\Phi(t_u) < 0$ for all $t \geq t_u$ and $[t_u u]_{s,p} > R$. Now we can define

$$\Gamma_u := \left\{ \gamma \in C([0,1], D^{s,p}(\mathbb{R}^N)) : \gamma(0) = 0 \right\}.$$

Using Lemma 2.3 as well as Proposition 2.2, we deduce the existence of a Palais–Smale sequence $\{u_n\}_n \subset D^{s,p}(\mathbb{R}^N)$ for the energy functional $\Phi$ such that

$$\lim_{n \to +\infty} \Phi(u_n) = c_u \quad \text{and} \quad \lim_{n \to +\infty} \Phi'(u_n) = 0 \quad \text{strongly in} \quad (D^{s,p}(\mathbb{R}^N))'.$$

Moreover, in the definition of $c_u$ we deduce also that $c_u \geq \lambda > 0$; so, $c_u > 0$ for all $u \in D^{s,p}(\mathbb{R}^N) \setminus \{0\}$.

**Lemma 2.4.** Suppose that $\mu \in [0, \mu_H]$ and that $\alpha \in [0, sp)$. Then there exists $u \in D^{s,p}(\mathbb{R}^N) \setminus \{0\}$ such that $0 < c_u < c^*$, where $c^*$ is defined in (2.4).

**Proof.** By hypothesis on the parameters $\mu$ and $\alpha$, we can consider a function $u \in D^{s,p}(\mathbb{R}^N) \setminus \{0\}$ for which $1/K(\mu, \alpha)$ is attained; see Theorem 1.2 and its independent proof in Section 5. By definition of $t_u$ and by the fact that $c_u > 0$, we get

$$0 < c_u \leq \sup_{t \geq 0} \Phi(tu) \leq \sup_{t \geq 0} f(t),$$

where $f: \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$f(t) := \frac{t^p \|u\|^p}{p} - \frac{t^{p_s(\alpha)} \|u\|^{p_s(\beta)}}{p_s(\alpha)} \int_{\mathbb{R}^N} \frac{|u|^{p_s(\beta)}}{|x|^\alpha} \, dx.$$
Note that the supremum of the function $f$ is attained in $t_u \in \mathbb{R}_+$ such that
\[ f(t_u) = \frac{sp - \alpha}{p(N - \alpha)} K(\mu, \alpha) \frac{-(N-\alpha)}{sp-\beta} . \]
We deduce that $0 < c_u \leq \left( \frac{1}{p} - \frac{1}{p^*_\alpha(\alpha)} \right) K(\mu, \alpha) \frac{-(N-\alpha)}{sp-\beta}$, since $u$ attains the constant $1/K(\mu, \alpha)$.

Note that the inequality is strict, that is, the equality does not occur. Indeed, suppose that it is valid the equality; therefore, we have
\[ 0 < c_u = \sup_{t \geq 0} \Phi(tu) = \sup_{t \geq 0} f(t). \]
Consider two positive real numbers $t_1, t_2 \in \mathbb{R}_+$ where two extrema are attained. Then
\[ f(t_1) - \frac{p^{*\beta}(\beta)}{p^*_\beta(\beta)} \int_{\mathbb{R}^N} \frac{|u|^{p^*_\beta(\beta)}}{|x|^\beta} dx = f(t_2). \]
This implies that $f(t_1) > f(t_2)$ since $u \in D^{s,p}(\mathbb{R}^N) \setminus \{0\}$ and $t > 0$. In this way we get a contradiction, for $\sup_{t \geq 0} f(t)$ is attained at $t_2 > 0$.

Similarly, we get $c_u < \left( \frac{1}{p} - \frac{1}{p^*_\alpha(\beta)} \right) K(\mu, \beta) \frac{-(N-\beta)}{sp-\beta}$.

The lemma is proved. \hfill \Box

3. The structure of the Palais–Smale sequences

In this section we consider the parameter $\alpha \in (0, sp)$.

**Proposition 3.1.** Let $\{u_n\}_n \subset D^{s,p}(\mathbb{R}^N)$ be a Palais–Smale sequence $(PS)_c$ for the energy functional $\Phi$ at the level $c \in (0, c^*)$, as defined in Proposition 2.1. Suppose that $u_n \rightharpoonup 0$ weakly in $D^{s,p}(\mathbb{R}^N)$ as $n \to +\infty$. Then there exists a positive constant $\varepsilon_0 = \varepsilon_0(N, p, \mu, \alpha, s, c)$ such that for every $\delta > 0$ one of the following limits is valid,
\[
\lim_{n \to +\infty} \int_{B_\delta(0)} \frac{|u_n|^{p^*_\alpha(\alpha)}}{|x|^\alpha} dx = 0 \quad \text{or} \quad \lim_{n \to +\infty} \int_{B_\delta(0)} \frac{|u_n|^{p^*_\alpha(\alpha)}}{|x|^\alpha} dx \geq \varepsilon_0. \quad (3.1)
\]

The proof of Proposition 3.1 uses three lemmas and one remark.

**Lemma 3.2.** Let $\{u_n\}_n \subset D^{s,p}(\mathbb{R}^N)$ be a Palais–Smale sequence $(PS)_c$ for the energy functional $\Phi$ as defined in Proposition 3.1. If $u_n \rightharpoonup 0$ weakly in $D^{s,p}(\mathbb{R}^N)$ as $n \to +\infty$, then for every compact subset $\omega \subset \mathbb{R}^N \setminus \{0\}$, up to a subsequence, the following limits are valid,
\[
\lim_{n \to +\infty} \int_\omega \frac{|u_n|^p}{|x|^{ps}} dx = \lim_{n \to +\infty} \int_\omega \frac{|u_n|^{p^*_\alpha(\alpha)}}{|x|^\alpha} dx = 0, \quad (3.2)
\]
\[
\lim_{n \to +\infty} \int_\omega \frac{|u_n|^{p^*_\beta(\beta)}}{|x|^\beta} dx = \lim_{n \to +\infty} \int_\omega \int_\omega \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+sp}} dx 
\]
Proof. We consider a fixed compact subset $\omega \subseteq \mathbb{R}^N \setminus \{0\}$. It is well known that the embedding $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^q(\omega)$ is compact for the parameter $q$ in the interval $1 \leq q < p_s^*$, where $p_s^* = \frac{Np}{N-sp}$ is the critical Sobolev exponent for the embedding; see Di Nezza, Palatucci and Valdinoci [14]. Note that both $|x|^{-sp}$ and $|x|^{-\alpha}$ are bounded in the subset $\omega$. Therefore, the limits in (3.2) follow from the compact embedding, since by hypothesis $u_n \rightharpoonup 0$ weakly in $D^{s,p}(\mathbb{R}^N)$ as $n \to +\infty$ and we also have $p < p_s^*(\alpha) < p_s^*$ due to the condition $\alpha \in (0, sp)$.

Now we consider the limits in (3.3). Let $\eta \in C_0^\infty(\mathbb{R}^N; \mathbb{R})$ be a cut-off function such that $\text{supp}(\eta) \subseteq \mathbb{R}^N \setminus \{0\}$ with $0 \leq \eta \leq 1$ and $\eta|_{\omega} \equiv 1$. Using a result in Brasco, Squassina and Yang [6, Lemma A.1] we deduce that $\eta^p u_n \in D^{s,p}(\mathbb{R}^N)$ $n \in \mathbb{N}$. So, since the sequence $\{u_n\}_n \in D^{s,p}(\mathbb{R}^N)$ is bounded by the fact that $\{u_n\}_n$ converges weakly to zero in $D^{s,p}(\mathbb{R}^N)$. In this way, from the estimates (3.4) it follows that

$$o_n(1) = \langle \Phi'(u_n), \eta^p u_n \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J_p(u_n(x) - u_n(y))(\eta^p(x)u_n(x) - \eta^p(y)u_n(y)) \frac{dx dy}{|x-y|^{N+sp}} - \int_{\mathbb{R}^N} J_{p^*_s(\alpha)} \eta^p u_n dx - \int_{\mathbb{R}^N} J_{p^*_s(\beta)} \eta^p u_n \frac{dx}{|x|^\beta}.$$  

(3.5)

Note that both $|x|^{-\alpha}$ and $|x|^{-\beta}$ are bounded in $\text{supp}(\eta) \subseteq \mathbb{R}^N \setminus \{0\}$. Since $u_n \rightharpoonup 0$ weakly in $D^{s,p}(\mathbb{R}^N)$ as $n \to +\infty$ and since $D^{s,p}(\mathbb{R}^N)$ is compactly embedded in both $L_{loc}^{p_s^*(\alpha)}(\mathbb{R}^N)$ and $L_{loc}^{p_s^*(\beta)}(\mathbb{R}^N)$, we get

$$\int_{\mathbb{R}^N} J_{p_s^*(\alpha)}(u_n) \frac{dx}{|x|^\alpha} \to 0 \quad \text{as} \quad n \to +\infty.$$ 

and

$$\int_{\mathbb{R}^N} J_{p_s^*(\beta)}(u_n) \frac{dx}{|x|^\beta} \to 0 \quad \text{as} \quad n \to +\infty.$$ 

Therefore, from the estimate (3.5) we obtain

$$o_n(1) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J_p(u_n(x) - u_n(y))(\eta^p(x)u_n(x) - \eta^p(y)u_n(y)) \frac{dx dy}{|x-y|^{N+sp}} - \int_{\mathbb{R}^N} J_{p_s^*(\alpha)} \eta^p u_n dx - \int_{\mathbb{R}^N} J_{p_s^*(\beta)} \eta^p u_n \frac{dx}{|x|^\beta}.$$  

(3.6)

To proceed further, we define the functions

$$D^s f(y) := \int_{\mathbb{R}^N} \frac{|f(x) - f(y)|^p}{|x-y|^{N+sp}} dx$$

and also

$$I := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J_p(u_n(x) - u_n(y))(\eta^p(x) - \eta^p(y))u_n(y) \frac{dx dy}{|x-y|^{N+sp}}.$$
Our goal now is to show that $I \to 0$ as $n \to +\infty$. To do this, we use the Hölder inequality and deduce that

$$I \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p-1}|u_n(y)||\eta^p(x) - \eta^p(y)|}{|x - y|^{N+sp}} \, dx \, dy$$

$$\leq [u_n]_{s,p} \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(y)|^p|\eta^p(x) - \eta^p(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \right)^{\frac{1}{p}}$$

$$\leq C \left( \int_{\mathbb{R}^N} |u_n(y)|^p |D^s\eta^p(y)|^p \, dy \right)^{\frac{1}{p}}, \quad (3.7)$$

where to get the last inequality we used the fact that $u_n \to 0$ weakly in $D^{s,p}(\mathbb{R}^N)$ as $n \to +\infty$. Therefore, the sequence $\{u_n\} \subset D^{s,p}(\mathbb{R}^N)$ is bounded.

Now we show that $D^s\eta^p(y) \in L^\infty(\mathbb{R}^N)$. Indeed, for $y \in \mathbb{R}^N$ we have

$$D^s\eta^p(y) = \int_{\mathbb{R}^N} \frac{|\eta^p(x) - \eta^p(y)|^p}{|x - y|^{N+sp}} \, dx$$

$$= \int_{\mathbb{R}^N \setminus B_r(y)} \frac{|\eta^p(x) - \eta^p(y)|^p}{|x - y|^{N+sp}} \, dx + \int_{B_r(y)} \frac{|\eta^p(x) - \eta^p(y)|^p}{|x - y|^{N+sp}} \, dx,$$

for every positive real number $r > 0$.

To estimate the first term we note that since $\eta \in C^\infty_0(\mathbb{R}^N)$, it follows that $\eta$ is a bounded function; thus, there exists a positive constant $C_1 > 0$ such that

$$\int_{\mathbb{R}^N \setminus B_r(y)} \frac{|\eta^p(x) - \eta^p(y)|^p}{|x - y|^{N+sp}} \, dx \leq C_1 \int_{\mathbb{R}^N \setminus B_r(y)} \frac{1}{|x - y|^{N+sp}} \, dx$$

$$\leq C_1 \int_r^\infty t^{-1-sp} \, dt < +\infty.$$ 

To estimate the second term, we also use the fact that $\eta^p \in C^\infty_0(\mathbb{R}^N)$, and the mean value inequality to deduce that there is $t \in [0, 1]$ such that

$$\int_{B_r(y)} \frac{|\eta^p(x) - \eta^p(y)|^p}{|x - y|^{N+sp}} \, dx \leq \int_{B_r(y)} |\nabla \eta^p(ty + (1-t)x) \cdot (x - y)|^p \, dx$$

$$\leq C \int_{B_r(y)} |x - y|^{-(N+sp)} \, dx < +\infty,$$

where $C$ is an estimate for the growth of the gradient of $\eta^p$. Thus we can deduce from both estimates that $D^s\eta^p \in L^\infty(\mathbb{R}^N)$. Returning to the analysis of inequality (3.7), we also have

$$\int_{\mathbb{R}^N} |u_n(y)|^p |D^s\eta^p(y)| \, dy = \int_{B_R(0)} |u_n(y)|^p |D^s\eta^p(y)| \, dy$$

$$+ \int_{\mathbb{R}^N \setminus B_R(0)} |u_n(y)|^p |D^s\eta^p(y)| \, dy, \quad (3.8)$$

for every fixed positive radius $R \in \mathbb{R}_+$ which will be defined below.

To estimate the first integral in (3.8) we recall that $u_n \to 0$ weakly in $D^{s,p}(\mathbb{R}^N)$; it follows that $u_n \to 0$ strongly in $L^q_{loc}(\mathbb{R}^N)$ for every $q \in [1, p^*_s)$, as $n \to +\infty$. As
we have already seen, $D^s \eta^p \in L^\infty(\mathbb{R}^N)$; moreover, $u_n \to 0$ in $L^p_{\text{loc}}(\mathbb{R}^N)$ as $n \to +\infty$. Thus, it follows that for every positive real number $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ the following inequalities hold:

$$
\int_{B_R(0)} |u_n(y)|^p |D^s \eta^p(y)| \, dy \leq \|D^s \eta^p\|_\infty \int_{B_R(0)} |u_n(y)|^p \, dy < \frac{\varepsilon}{2}.
$$

(3.9)

Now we are going to estimate the second integral on the right-hand side of equality (3.8). By using the Hölder inequality with exponents $p^*_s/p$ and $N/sp$ and since $\{u_n\}$ is bounded in $L^{p^*_s}$($\mathbb{R}^N$), we have

$$
\int_{\mathbb{R}^N \setminus B_R(0)} |u_n(y)|^p |D^s \eta^p(y)| \, dy \leq \|u_n\|_{L^{p^*_s}(\mathbb{R}^N)} \|D^s \eta^p\|_{L^{N/sp}(\mathbb{R}^N \setminus B_R(0))}
$$

$$
\leq C \|D^s \eta^p\|_{L^{N/sp}(\mathbb{R}^N \setminus B_R(0))}.
$$

By the definition of $D^s \eta^p$, we have

$$
\int_{\mathbb{R}^N \setminus B_R(0)} |D^s \eta^p(y)|^\frac{N}{sp} \, dy = \int_{\mathbb{R}^N \setminus B_R(0)} \left( \int_{\mathbb{R}^N} \frac{|\eta^p(x) - \eta^p(y)|^p}{|x - y|^{N+sp}} \, dx \right)^\frac{N}{sp} \, dy.
$$

By the fact that $\eta \in C^\infty_0(\mathbb{R}^N)$, there exists a positive real number $r > 0$ such that $\text{supp}(\eta) \subset B_r(0)$. So, we choose $R > r > 0$ as a first condition on the constant $R$. In this way, we get

$$
\int_{\mathbb{R}^N \setminus B_R(0)} \left( \int_{\mathbb{R}^N} \frac{|\eta^p(x) - \eta^p(y)|^p}{|x - y|^{N+sp}} \, dx \right)^\frac{N}{sp} \, dy
$$

$$
= \int_{\mathbb{R}^N \setminus B_R(0)} \left( \int_{B_r(0)} \frac{|\eta^p(x)|^p}{|x|^{N+sp}} \, dx \right)^\frac{N}{sp} \, dy
$$

$$
= \int_{\mathbb{R}^N \setminus B_R(0)} \left( \int_{B_r(0)} \frac{|\eta^p(x)|^p}{|x|^{N+sp}} \, dx \right)^\frac{N}{sp} \, dy
$$

$$
\leq \int_{\mathbb{R}^N \setminus B_R(0)} \left( \int_{B_r(0)} \frac{|\eta^p(x)|^p}{(|y| - r)^{N+sp}} \, dx \right)^\frac{N}{sp} \, dy
$$

$$
\leq \left( \int_{B_r(0)} |\eta^p(x)|^p \, dx \right)^\frac{N}{sp} \, \int_{\mathbb{R}^N \setminus B_R(0)} \frac{1}{(|y| - r)^{(N+sp)N}} \, dy
$$

$$
= \|\eta^p\|^\frac{N}{p^*_s} \int_{B_R(0)} \frac{1}{|y|^p} \, dy
$$

$$
\leq C \int_{\partial B_r(0)} \frac{1}{(|y| - r)^{(N+sp)N}} \, dt
$$

$$
\leq C \int_0^\infty \frac{\rho^{N-1} + r^{N-1}}{\rho - r} \, d\rho.
$$

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Thus, by using inequality (3.7) and the limit (3.11) it follows that
\[ \int_{\mathbb{R}^N \setminus B_R(0)} |D^s\eta^p(y)|^\frac{N}{sp} dy \leq C \left( \frac{1}{(R-r)^{\frac{N}{sp}}} + \frac{1}{(R-r)^{\frac{(N+sp)N}{sp}}} \right). \]

Passing to the limit as $R \to +\infty$ we get
\[ \int_{\mathbb{R}^N \setminus B_R(0)} |D^s\eta^p(y)|^\frac{N}{sp} dy \to 0. \]

Moreover, for every positive real number $\varepsilon > 0$ there exists $R(\varepsilon) > 0$ big enough such that
\[ \int_{\mathbb{R}^N \setminus B_R(0)} |D^s\eta^p(y)|^\frac{N}{sp} dy < \frac{\varepsilon}{2}. \]  
(3.10)

Substituting inequalities (3.9) and (3.10) into equality (3.8), for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ and $R = R(\varepsilon) > 0$ such that if $n > n_0$, then
\[ \int_{\mathbb{R}^N} |u_n(y)|^p |D^s\eta^p(y)| dy < \varepsilon, \]
that is,
\[ \int_{\mathbb{R}^N} |u_n(y)|^p |D^s\eta^p(y)| dy \to 0 \quad \text{as } n \to +\infty. \]  
(3.11)

Thus, by using inequality (3.7) and the limit (3.11) it follows that
\[ I := \int_{\mathbb{R}^N} J_p(u_n(x) - u_n(y)) \frac{\eta^p(x) - \eta^p(y)}{|x-y|^{N+sp}} u_n(y) \, dx \, dy = o_n(1). \]  
(3.12)

Combining the estimates (3.12) and (3.6), we get
\[ o_n(1) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \eta^p(x) |u_n(x) - u_n(y)|^p \, dx \, dy. \]  
(3.13)

Our goal now is to show that
\[ [\eta u_n]_{s,p}^p = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \eta^p(x) |u_n(x) - u_n(y)|^p \, dx \, dy + o_n(1). \]  
(3.14)

To accomplish this, we use the elementary inequality
\[ ||X + Y|^p - |X|^p| \leq C_p (|X|^{p-1} + |Y|^{p-1}) |Y| \quad \text{for all } X, Y \in \mathbb{R}^N, \]
valid for $p > 1$; here, $C_p > 0$ is a constant that depends only on the exponent $p$. We use this inequality with the choices
\[ X = \eta(x)(u_n(x) - u_n(y)) \quad \text{and} \quad Y = u_n(y)(\eta(x) - \eta(y)). \]

Thus,
\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\eta(x)(u_n(x) - u_n(y))|^p - |\eta(x)(u_n(x) - u_n(y))|^p}{|x-y|^{N+sp}} \, dx \, dy \leq C_p \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\eta(x)(u_n(x) - u_n(y))|^{p-1} |u_n(y)(\eta(x) - \eta(y))|}{|x-y|^{N+sp}} \, dx \, dy \]
Using the H"older inequality, the definition of $D^s\eta$ and the fact that $\eta \in C_0^\infty(\mathbb{R}^N)$, we obtain

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\eta(x) u_n(x) - \eta(y) u_n(y)|^p - |\eta(x) (u_n(x) - u_n(y))|^p}{|x - y|^{N+sp}} \, dx \, dy \leq C[u_n]^{p-1}_{s,p} \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(y)(\eta(x) - \eta(y))|^p}{|x - y|^{N+sp}} \, dx \, dy \right)^{1/p} + C_p \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(y)(\eta(x) - \eta(y))|^p}{|x - y|^{N+sp}} \, dx \, dy.
\]  

(3.15)

We remark that by the definition of $D^s\eta$,

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(y)(\eta(x) - \eta(y))|^p}{|x - y|^{N+sp}} \, dx \, dy = \int_{\mathbb{R}^N} |u_n(y)|^p \, |D^s\eta(y)| \, dy.
\]

Using the limit (3.11) in the inequality (3.15), we get

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\eta(x) u_n(x) - \eta(y) u_n(y)|^p - |\eta(x) (u_n(x) - u_n(y))|^p}{|x - y|^{N+sp}} \, dx \, dy = o_n(1).
\]

This implies that the estimate (3.14) is valid. Following up, using inequalities (2.3) and the estimates (3.12) and (3.14) in the estimate (3.6), we deduce that

\[
o_n(1) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\eta^p(x)|u_n(x) - u_n(y)|^p}{|x - y|^{N+sp}} \, dx \, dy = [\eta u_n]^{p}_{s,p} + o_n(1) \geq ||\eta u_n||^p + o_n(1).
\]

Thus, we get

\[||\eta u_n||^p \rightarrow 0 \text{ as } n \rightarrow +\infty.\]

Since $\eta \equiv 1$ in $\omega \subset \mathbb{R}^N \setminus \{0\}$, it follows that

\[\int_{\omega} \int_{\omega} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \rightarrow 0 \text{ as } n \rightarrow +\infty.\]

This establishes the limits (3.3). The lemma is proved.

**Remark 3.3.** In the proof of Lemma 3.2 we showed that for every cut-off function $\eta \in C_0^\infty(\mathbb{R}^N)$ and for every sequence $\{u_n\}_n \subset D^{s,p}(\mathbb{R}^N)$ such that $u_n \rightharpoonup 0$ weakly in $D^{s,p}(\mathbb{R}^N)$, it holds that

\[\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\eta^p(x)|u_n(x) - u_n(y)|^p}{|x - y|^{N+sp}} \, dx \, dy = o_n(1).
\]

Moreover, if we consider the cut-off function $\eta \in C_0^\infty(\mathbb{R}^N)$ as in the proof of Lemma 3.2 and the sequence $\{u_n\}_n \subset D^{s,p}(\mathbb{R}^N)$ is a Palais-Smale sequence $(PS)_c$ at the level $c \in (0, c^*)$ as in Proposition 3.1, then we get

\[\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\eta^p(x)|u_n(x) - u_n(y)|^p}{|x - y|^{N+sp}} \, dx \, dy = o_n(1).
\]
In this way, we deduce that
\[
\lim_{n \to +\infty} \int_\omega \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+sp}} \, dx \, dy = 0, 
\]
\[
\lim_{n \to +\infty} \int_\omega \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+sp}} \, dx \, dy = 0, 
\]
for every subset \( \omega \subset \mathbb{R}^N \setminus \{0\} \).

Let \( \delta \in \mathbb{R}_+ \) be fixed; we define the quantities
\[
\gamma := \lim_{n \to +\infty} \int_{B_\delta(0)} \int_{B_\delta(0)} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+sp}} \, dx \, dy - \mu \int_{B_\delta(0)} \frac{|u_n|^p}{|x|^\alpha} \, dx, 
\]
\[
\lambda := \lim_{n \to +\infty} \int_{B_\delta(0)} \frac{|u_n|^{p_\gamma(\alpha)}}{|x|^\alpha} \, dx, 
\]
\[
\xi := \lim_{n \to +\infty} \int_{B_\delta(0)} \frac{|u_n|^{p_\gamma(\beta)}}{|x|^\beta} \, dx. 
\]

From Remark 3.3 and from Lemma 3.2 we deduce that the above quantities are well defined and are independent of the particular choice of \( \delta > 0 \). Indeed, consider a cut-off function \( \eta \in C_0^\infty(\mathbb{R}^N) \) such that \( \eta|_{B_\delta(0)} \equiv 1 \). It follows that there exist positive real numbers \( R > r > 0 \) such that \( \text{supp}(\eta) \subset B_r(0) \subset B_R(0) \). Note that
\[
[\eta u_n]_{s,p}^p = \int_{\mathbb{R}^N} \int_{B_R(0)} \left| \frac{\eta(x) u_n(x) - \eta(y) u_n(y)}{|x-y|^{N+sp}} \right|^p \, dx \, dy 
\]
\[
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N \setminus B_R(0)} \frac{|\eta(x) u_n(x) - \eta(y) u_n(y)|^p}{|x-y|^{N+sp}} \, dx \, dy 
\]
\[
+ \int_{\mathbb{R}^N} \int_{B_R(0)} \left| \frac{|\eta(x) u_n(x) - \eta(y) u_n(y)|^p}{|x-y|^{N+sp}} \right| \, dx \, dy 
\]
\[
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N \setminus B_R(0)} \left| \frac{|\eta(y) u_n(y)|^p}{|x-y|^{N+sp}} \right| \, dx \, dy 
\]
\[
+ \int_{\mathbb{R}^N} \int_{B_R(0)} \left| \frac{|\eta(x) u_n(x) - \eta(y) u_n(y)|^p}{|x-y|^{N+sp}} \right| \, dx \, dy. 
\]

Now we estimate both integrals on the right-hand side of equality (3.21). Considering the first integral, we have
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N \setminus B_R(0)} \frac{|\eta(y) u_n(y)|^p}{|x-y|^{N+sp}} \, dx \, dy 
\]
\[
= \int_{\mathbb{R}^N} |\eta(y) u_n(y)|^p \left( \int_{\mathbb{R}^N \setminus B_R(0)} \frac{1}{|x-y|^{N+sp}} \, dx \right) \, dy 
\]
\[
= \int_{B_r(0)} |\eta(y) u_n(y)|^p \left( \int_{\mathbb{R}^N \setminus B_R(0)} \frac{1}{|x-y|^{N+sp}} \, dx \right) \, dy. 
\]
From these results we deduce that

\[ u_n \to 0 \quad \text{weakly in} \quad D^{s,p}(\mathbb{R}^N) \quad \text{as} \quad n \to +\infty \]

and that the embedding

\[ D^{s,p}(\mathbb{R}^N) \hookrightarrow L^q_{\text{loc}}(\mathbb{R}^N) \]

is compact for \( q \in [1, p^*_s) \); so,

\[ \int_{B_r(0)} |\eta(y)u_n(y)|^p \, dy \to 0 \quad \text{as} \quad n \to \infty. \]

From these results we deduce that

\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N \setminus B_R(0)} \frac{|\eta(y)u_n(y)|^p}{|x-y|^{N+sp}} \, dx \, dy \to 0 \quad \text{as} \quad n \to \infty. \]

Thus, from equality (3.21) we obtain

\[ [\eta u_n]_{s,p} = \int_{\mathbb{R}^N} \int_{B_R(0)} \frac{|\eta(x)u_n(x) - \eta(y)u_n(y)|^p}{|x-y|^{N+sp}} \, dx \, dy + o_n(1). \quad (3.22) \]

Considering the second integral in (3.21), we also have

\[ \int_{\mathbb{R}^N} \int_{B_R(0)} \frac{|\eta(x)u_n(x) - \eta(y)u_n(y)|^p}{|x-y|^{N+sp}} \, dx \, dy \]

\[ = \int_{\mathbb{R}^N \setminus B_R(0)} \int_{B_R(0)} \frac{|\eta(x)u_n(x) - \eta(y)u_n(y)|^p}{|x-y|^{N+sp}} \, dx \, dy \]

\[ + \int_{B_R(0)} \int_{B_R(0)} \frac{|\eta(x)u_n(x) - \eta(y)u_n(y)|^p}{|x-y|^{N+sp}} \, dx \, dy. \quad (3.23) \]

Using the facts that \( \text{supp}(\eta) \subset B_r(0) \subset B_R(0) \) and \( R > r > 0 \), we deduce that

\[ \int_{\mathbb{R}^N \setminus B_R(0)} \int_{B_R(0)} \frac{|\eta(x)u_n(x) - \eta(y)u_n(y)|^p}{|x-y|^{N+sp}} \, dx \, dy \]

\[ = \int_{\mathbb{R}^N \setminus B_R(0)} \frac{|\eta(x)u_n(x)|^p}{|x-y|^{N+sp}} \, dx \, dy \]

\[ = \int_{\mathbb{R}^N \setminus B_R(0)} \frac{|\eta(x)u_n(x)|^p}{|x-y|^{N+sp}} \, dx \, dy \]

\[ \leq \left( \int_{\mathbb{R}^N \setminus B_R(0)} \frac{1}{|y-r|^{N+sp}} \, dy \right) \left( \int_{B_r(0)} |\eta(x)u_n(x)|^p \, dx \right). \]
Under the conditions on the parameters \( N \geq p < N \), we know that
\[
\int_{\mathbb{R}^N \setminus B_R(0)} \frac{1}{(|y| - r)^{N + sp}} \, dy < +\infty.
\]

Using again the facts that \( u_n \to 0 \) weakly in \( D^{s,p}(\mathbb{R}^N) \) and that the embedding \( D^{s,p}(\mathbb{R}^N) \hookrightarrow L^q_{loc}(\mathbb{R}^N) \) is compact for \( q \in [1, p^*_s) \), it follows that
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N \setminus B_R(0)} \int_{B_R(0)} \frac{\eta(x)u_n(x) - \eta(y)u_n(y)}{|x - y|^{N + sp}} \, dx \, dy = 0.
\]

(3.24)

Using the estimates (3.22), (3.23) and (3.24), it follows that
\[
[\eta u_n]_{s,p}^p = \int_{B_R(0)} \int_{B_R(0)} \frac{\eta(x)u_n(x) - \eta(y)u_n(y)}{|x - y|^{N + sp}} \, dx \, dy + o_n(1).
\]

(3.25)

To proceed further, we must estimate the previous integral. To do this, we consider positive real numbers \( R > \delta > 0 \) and we write
\[
B_R(0) \times B_R(0) = ((B_R(0) \setminus B_\delta(0)) \cup B_\delta(0)) \times ((B_R(0) \setminus B_\delta(0)) \cup B_\delta(0)).
\]

In the case of the domain of integration \( (B_R(0) \setminus B_\delta(0)) \times B_R(0) \) we have
\[
\int_{B_R(0) \setminus B_\delta(0)} \int_{B_R(0)} \frac{\eta(x)u_n(x) - \eta(y)u_n(y)}{|x - y|^{N + sp}} \, dx \, dy
\leq 2^p \int_{B_R(0) \setminus B_\delta(0)} \int_{B_R(0)} \frac{\eta(x)(u_n(x) - u_n(y))}{|x - y|^{N + sp}} \, dx \, dy
+ 2^p \int_{B_R(0) \setminus B_\delta(0)} \int_{B_R(0)} \frac{|u_n(y)(\eta(x) - \eta(y))|}{|x - y|^{N + sp}} \, dx \, dy.
\]

(3.26)

To estimate the first integral on the right-hand side of inequality (3.26) we use limit (3.16) with \( \omega = B_R(0) \setminus B_\delta(0) \in \mathbb{R}^N \setminus \{0\} \) to deduce that
\[
\lim_{n \to +\infty} \int_{B_R(0) \setminus B_\delta(0)} \int_{B_R(0)} \frac{\eta(x)(u_n(x) - u_n(y))}{|x - y|^{N + sp}} \, dx \, dy = 0.
\]

To estimate the second integral on the right-hand side of inequality (3.26), we use the same subset \( \omega = B_R(0) \setminus B_\delta(0) \in \mathbb{R}^N \setminus \{0\} \) together with limit (3.11) to obtain
\[
\lim_{n \to +\infty} \int_{B_R(0) \setminus B_\delta(0)} \int_{B_R(0)} \frac{|u_n(y)(\eta(x) - \eta(y))|}{|x - y|^{N + sp}} \, dx \, dy = 0.
\]

Therefore, from the two previous limits and from inequality (3.26) we obtain
\[
\lim_{n \to +\infty} \int_{B_R(0) \setminus B_\delta(0)} \int_{B_R(0)} \frac{\eta(x)u_n(x) - \eta(y)u_n(y)}{|x - y|^{N + sp}} \, dx \, dy = 0.
\]

(3.27)

In the case of the domain of integration \( B_\delta(0) \times (B_R(0) \setminus B_\delta(0)) \) we proceed as in the previous case. We write an inequality analogous to inequality (3.26); afterwards, we use inequality (3.17) together with limit (3.11) to obtain
\[
\lim_{n \to +\infty} \int_{B_\delta(0)} \int_{B_R(0) \setminus B_\delta(0)} \frac{\eta(x)u_n(x) - \eta(y)u_n(y)}{|x - y|^{N + sp}} \, dx \, dy = 0.
\]

(3.28)
Therefore, combining limits (3.27) and (3.28) with estimate (3.25), we deduce that

\[
[\eta u_n]_{s,p}^p = \int_{B_R(0)} \int_{B_R(0)} \frac{|\eta(x)u_n(x) - \eta(y)u_n(y)|^p}{|x-y|^{N+sp}} \, dx \, dy + o_n(1)
\]

\[
= \int_{B_\delta(0)} \int_{B_\delta(0)} \frac{|\eta(x)u_n(x) - \eta(y)u_n(y)|^p}{|x-y|^{N+sp}} \, dx \, dy + o_n(1).
\]  

(3.29)

Now we can state the following result.

**Lemma 3.4.** Let \( \{u_n\}_n \subset D^{s,p}(\mathbb{R}^N) \) be a Palais–Smale sequence \((PS)\_c\) for the functional \( \Phi \) at the level \( c \in (0,c^*) \) and let \( \lambda, \xi, \) and \( \gamma \) be defined in (3.18), (3.19) and (3.20). If \( u_n \rightharpoonup 0 \) weakly in \( D^{s,p}(\mathbb{R}^N) \) as \( n \to +\infty \), then

\[
\lambda \frac{p}{p_s} \leq K(\mu, \alpha) \gamma \quad \text{and} \quad \xi \frac{p}{p_s(\beta)} \leq K(\mu, \beta) \gamma.
\]  

(3.30)

**Proof.** Let \( \eta \in C_0^\infty(\mathbb{R}^N) \) be a cut-off function such that \( \eta|_{B_\delta(0)} = 1 \), with \( \delta > 0 \).

Using the definition (1.11) of the constant \( K(\mu, \alpha) \) we get

\[
\left( \int_{B_\delta(0)} \frac{|u_n|^{p_s(\alpha)}}{|x|^\alpha} \, dx \right)^{\frac{p}{p_s(\alpha)}} \leq \left( \int_{\mathbb{R}^N} \frac{|\eta u_n|^{p_s(\alpha)}}{|x|^\alpha} \, dx \right)^{\frac{p}{p_s(\alpha)}} \leq K(\mu, \alpha) \|\eta u_n\|^p.
\]

Using the estimate (3.29) and that \( u_n \rightharpoonup 0 \) in \( D^{s,p}(\mathbb{R}^N) \), we conclude by taking the limit as \( n \to +\infty \) that \( \lambda \frac{p}{p_s} \leq K(\mu, \alpha) \gamma \). The other inequality in (3.30) can be obtained in a similar way. This concludes the proof of the lemma. \( \square \)

Now we state another lemma that will be useful in the proof of Proposition 3.1.

**Lemma 3.5.** Let \( \{u_n\}_n \subset D^{s,p}(\mathbb{R}^N) \) be a Palais–Smale sequence \((PS)\_c\) for the functional \( \Phi \) at the level \( c \in (0,c^*) \) and let \( \gamma, \lambda, \) and \( \xi \) be defined in (3.18), (3.19), and (3.20). If \( u_n \rightharpoonup 0 \) weakly in \( D^{s,p}(\mathbb{R}^N) \) as \( n \to +\infty \), then \( \gamma \leq \lambda + \xi \).

**Proof.** Let \( \eta \in C_0^\infty(\mathbb{R}^N) \) be a cut-off function such that \( \eta|_{B_\delta(0)} = 1 \), with \( \delta > 0 \).

Using Brasco, Squassina and Yang [6, Lemma A.1], it follows that \( \eta^p u_n \in D^{s,p}(\mathbb{R}^N) \); hence,

\[
\lim_{n \to +\infty} \langle \Phi'(u_n), \eta^p u_n \rangle = 0,
\]

that is,

\[
o_n(1) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J_p(\eta u_n(x) - \eta u_n(y)) (\eta^p(x)u_n(x) - \eta^p(y)u_n(y)) \frac{dx \, dy}{|x-y|^{N+sp}}
\]

\[
- \mu \int_{\mathbb{R}^N} |u_n|^p \, dx - \int_{\mathbb{R}^N} \frac{|u_n|^{p_s(\beta)} \eta^p}{|x|^\beta} \, dx - \int_{\mathbb{R}^N} \frac{|u_n|^{p_s(\alpha)} \eta^p}{|x|^\alpha} \, dx.
\]

We know that

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J_p(\eta u_n(x) - \eta u_n(y)) (\eta^p(x)u_n(x) - \eta^p(y)u_n(y)) \frac{dx \, dy}{|x-y|^{N+sp}} = [\eta u_n]_{s,p} + o_n(1) = \int_{B_{\delta}(0)} \int_{B_{\delta}(0)} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+sp}} \, dx \, dy + o_n(1).
\]
Therefore, by the lim sup property and Lemma 3.2 with \( \omega = \text{supp}(\eta) \setminus B_\delta(0) \in \mathbb{R}^N \setminus \{0\} \) we have \( \gamma \leq \lambda + \xi \). This concludes the proof of the lemma. \( \square \)

Finally, we can prove Proposition 3.1, which states that every Palais–Smale sequence \( \{u_n\}_n \subset D^{s,p}(\mathbb{R}^N) \) for the functional \( \Phi \) at the level \( c \in (0, c^*) \) such that \( u_n \rightharpoonup 0 \) weakly in \( D^{s,p}(\mathbb{R}^N) \) as \( n \to +\infty \) verifies one of the limits \( \lim_{n \to +\infty} \int_{B_\delta(0)} \frac{|u_n|^p c^*_p(\alpha)}{|x|\alpha} \, dx = 0 \) or \( \lim_{n \to +\infty} \int_{B_\delta(0)} \frac{|u_n|^p c^*_p(\alpha)}{|x|\alpha} \, dx \geq \varepsilon_0 \) with arbitrary \( \delta > 0 \) and a positive constant \( \varepsilon_0 = \varepsilon_0(N, p, \mu, \alpha, s, \beta, c) \) independent of \( \delta \).

**Proof of Proposition 3.1.** Let \( \{u_n\}_n \subset D^{s,p}(\mathbb{R}^N) \) be a Palais–Smale sequence for the functional \( \Phi \) at the level \( c \in (0, c^*) \) as in Proposition 2.1 and consider \( \alpha \neq 0 \). From Lemmas 3.4 and 3.5, we infer that

\[
\lambda^\mu_{\rho^s} \leq K(\mu, \alpha) \gamma \leq K(\mu, \alpha) \lambda + K(\mu, \alpha) \xi.
\]

So,

\[
\lambda^\mu_{\rho^s(\alpha)} \left( 1 - K(\mu, \alpha) \lambda^{1 - \frac{p}{\rho^s(\alpha)}} \right) \leq K(\mu, \alpha) \xi. \tag{3.31}
\]

Since

\[
\Phi(u_n) - \left\langle \frac{1}{p^*_s(\alpha)} \Phi'(u_n), u_n \right\rangle = c + o_n(1),
\]

it follows from Lemma 3.2 that

\[
\lambda \leq \frac{cp(N - \alpha)}{sp - \alpha}. \tag{3.32}
\]

Combining inequalities (3.31) and (3.32), we deduce that

\[
\left( 1 - \left( \frac{cp(N - \alpha)}{sp - \alpha} \right)^\frac{sp - \alpha}{p(N - \alpha)} K(\mu, \alpha) \right) \lambda^\mu_{\rho^s(\alpha)} \leq K(\mu, \alpha) \xi.
\]

By the definition (2.4) of \( c^* \), it follows that \( 1 - \left( \frac{cp(N - \alpha)}{sp - \alpha} \right)^\frac{sp - \alpha}{p(N - \alpha)} K(\mu, \alpha) > 0 \). So, there exists a positive real number \( \delta_1 = \delta_1(N, p, \mu, s, \alpha, c) > 0 \) such that \( \lambda^\mu_{\rho^s} \leq \delta_1 \xi \).

Similarly, we deduce the existence of another positive real number \( \delta_2 = \delta_2(N, p, \mu, s, \beta, c) > 0 \) such that \( \xi^\mu_{\rho^s(\alpha)} \leq \delta_2 \lambda \).

In this way, we infer that \( \lambda = 0 \) if, and only if, \( \xi = 0 \) and \( \lambda > 0 \) if, and only if, \( \xi > 0 \). Thus, by definition 3.19, we can guarantee the existence of a real positive number \( \varepsilon_0 = \varepsilon_0(N, p, \mu, \alpha, s, c) > 0 \) such that for all \( \delta > 0 \) one of the limits is valid,

\[
\lim_{n \to +\infty} \int_{B_\delta(0)} \frac{|u_n|^p c^*_p(\alpha)}{|x|\alpha} \, dx = 0 \quad \text{or} \quad \lim_{n \to +\infty} \int_{B_\delta(0)} \frac{|u_n|^p c^*_p(\alpha)}{|x|\alpha} \, dx \geq \varepsilon_0.
\]

The proposition is proved. \( \square \)
4. Proof of Theorem 1.1

The proof of Theorem 1.1 uses three auxiliary lemmas.

**Lemma 4.1.** Let \( \{u_n\}_n \subset D^{s,p}(\mathbb{R}^N) \) be a Palais–Smale sequence \((PS)_c\) for the functional \( \Phi \) at the level \( c \in (0,c^*) \) as in Proposition 3.1. Therefore,

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} \frac{|u_n|^{p_*(\alpha)}}{|x|^\alpha} \, dx > 0.
\]

**Proof.** We argue by contradiction. Suppose that

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} \frac{|u_n|^{p_*(\alpha)}}{|x|^\alpha} \, dx = 0. 
\quad (4.1)
\]

Using this hypothesis and the fact that \( \langle \Phi'(u_n), u_n \rangle = o_n(1) \), it follows that

\[
\|u_n\|^p = \|u_n\|_{L^{p_*(\beta)}(\mathbb{R}^N;|x|^{-\beta})}^{p_*(\beta)} + o_n(1).
\]

From this estimate and by the definition (1.11) of the constant \( K(\mu, \beta) \), it follows that

\[
\|u_n\|_{L^{p_*(\beta)}(\mathbb{R}^N;|x|^{-\beta})}^{p_*(\beta)} \leq K(\mu, \beta) \|u_n\|^p
\]

\[
= K(\mu, \beta) \left( \|u_n\|_{L^{p_*(\beta)}(\mathbb{R}^N;|x|^{-\beta})}^{p_*(\beta)} + o_n(1) \right),
\]

that is,

\[
\|u_n\|_{L^{p_*(\beta)}(\mathbb{R}^N;|x|^{-\beta})}^{p_*(\beta)} \left( 1 - K(\mu, \beta) \|u_n\|_{L^{p_*(\beta)}(\mathbb{R}^N;|x|^{-\beta})}^{p_*(\beta) - p} \right) \leq o_n(1).
\]

We already know that

\[
\Phi(u_n) - \frac{1}{p} \langle \Phi'(u_n), u_n \rangle = c + o_n(1);
\]

thus, using the hypothesis (4.1) again, we get

\[
\|u_n\|_{L^{p_*(\beta)}(\mathbb{R}^N;|x|^{-\beta})}^{p_*(\beta)} = \int_{\mathbb{R}^N} \frac{|u_n|^{p_*(\beta)}}{|x|^\beta} \, dx = \frac{cp(N - \beta)}{(ps - \beta)} + o_n(1) \to 0 \quad \text{as} \quad n \to +\infty. 
\quad (4.2)
\]

These estimates mean that

\[
\|u_n\|_{L^{p_*(\beta)}(\mathbb{R}^N;|x|^{-\beta})}^{p_*(\beta)} \left( 1 - K(\mu, \beta) \left( \frac{cp(N - \beta)}{(ps - \beta)} \right)^{\frac{p_*(\beta) - p}{p_*(\beta)}} \right) \leq o_n(1).
\]

By the definition (2.4) of the constant \( c^* \) and the hypothesis \( c \in (0,c^*) \), it follows that

\[
\lim_{n \to +\infty} \|u_n\|_{L^{p_*(\beta)}(\mathbb{R}^N;|x|^{-\beta})}^{p_*(\beta)} = 0.
\]

But this is a contradiction with inequality (4.2). The lemma is proved. \( \square \)
Lemma 4.2. Let \( \{u_n\}_n \subset D^{s,p}(\mathbb{R}^N) \) be a Palais–Smale sequence \((PS)_c\) for the functional \( \Phi \) at the level \( c \in (0, c^*) \). Then there exists a positive real number \( \varepsilon_1 \in \left(0, \frac{\varepsilon_0}{2}\right] \), with \( \varepsilon_0 \) given in the limit (3.1), such that for every positive real number \( \varepsilon \in (0, \varepsilon_1) \), there exists a sequence of positive real numbers \( \{r_n\}_n \subset \mathbb{R}_+ \) with the property that the sequence \( \{\tilde{u}_n\}_n \subset D^{s,p}(\mathbb{R}^N) \), defined by the conformal transformation
\[
\tilde{u}_n(x) := r_n \frac{u_n(r_n x)}{u_n(r_n x)} \quad \text{for all } x \in \mathbb{R}^N,
\]
is another \((PS)_c\) sequence which verifies
\[
\int_{B_1(0)} \frac{u_n \left| \frac{u_n^*}{|x|^\alpha} \right|}{|x|^\alpha} dx = \varepsilon \quad \text{for all } n \in \mathbb{N}.
\]

Proof. We begin by setting
\[
\sigma := \lim_{n \to +\infty} \int_{\mathbb{R}^N} \frac{u_n \left| \frac{u_n^*}{|x|^\alpha} \right|}{|x|^\alpha} dx.
\]

From Lemma 4.1 it follows that \( \sigma > 0 \). Let \( \varepsilon_1 := \min\{\frac{\varepsilon_0}{2}, \sigma\} \), with \( \varepsilon_0 > 0 \) given in the limit (3.1) and let us fix \( \varepsilon \in (0, \varepsilon_1) \). Passing to a subsequence if necessary, still denoted in the same way, for every natural number \( n \in \mathbb{N} \) there exists a positive real number \( r_n > 0 \) such that
\[
\int_{B_{r_n}(0)} \frac{|u_n| |u_n^*|}{|x|^\alpha} dx = \varepsilon.
\]
The change of variables \( x = r_n y \) yields
\[
\varepsilon = \int_{B_{r_n}(0)} \frac{|u_n(x)| |u_n^*(x)|}{|x|^\alpha} dx = \int_{B_1(0)} \frac{|u_n(r_n y)| |u_n^*(r_n y)| r_n^N}{|y|^\alpha r_n^\alpha} dy
\]
\[
= \int_{B_1(0)} \frac{|u_n^*(r_n y)|}{|y|^\alpha} dy
\]
A similar change of variables also yields
\[
[\tilde{u}_n]_{s,p}^p = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|r_n \frac{u_n(r_n x)}{u_n(r_n x)} - r_n \frac{u_n(r_n y)}{u_n(r_n y)}|^p}{|x - y|^{N+sp}} dx dy
\]
\[
= r_n^{-N+sp} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(z) - u_n(t)|^p}{|z - t|^{N+sp}} r_n^{N-2N} dz dt
\]
\[
= [u_n]_{s,p}^p.
\]
Moreover,
\[
\int_{\mathbb{R}^N} \frac{|\tilde{u}_n|^p}{|x|^{ps}} dx = r_n^{-N+sp} \int_{\mathbb{R}^N} \frac{|u_n(y)|^p}{r_n^{-ps}|y|^{ps}} r_n^{-N} dy = \int_{\mathbb{R}^N} \frac{|u_n|^p}{|y|^{ps}} dy,
\]
and also
\[ \int_{\mathbb{R}^N} \frac{|\tilde{u}_n|^p_s(\beta)}{|x|^\beta} \, dx = r_n^{N-\beta} \int_{\mathbb{R}^N} \frac{|u_n(y)|^p_s(\beta)}{r_n^{N-\beta}|y|^\beta} \, dy = \int_{\mathbb{R}^N} \frac{|u_n|^p_s(\beta)}{|y|^\beta} \, dy. \]

Therefore, the sequence \( \{\tilde{u}_n\}_n \subset D^{s,p}(\mathbb{R}^N) \) is also a \((PS)_c\) sequence for the functional \( \Phi \) and verifies equality (4.4). The lemma is proved. \( \square \)

Finally, we can present the proof of our theorem.

**Proof of Theorem 1.1.** Let \( \{u_n\}_n \subset D^{s,p}(\mathbb{R}^N) \) be a Palais–Smale sequence \((PS)_c\) for the functional \( \Phi \) at the level \( c \in (0, c^*) \). We claim that this sequence is bounded in the function space \( D^{s,p}(\mathbb{R}^N) \); after a passage to a subsequence if necessary, still denoted in the same way, \( u_n \rightharpoonup \tilde{u} \) weakly in \( D^{s,p}(\mathbb{R}^N) \) as \( n \to +\infty \) and \( \tilde{u} \) is a weak solution to problem (1.7).

Indeed, first we have to show that the sequence \( \{\tilde{u}_n\}_n \subset D^{s,p}(\mathbb{R}^N) \) is bounded in the function space \( D^{s,p}(\mathbb{R}^N) \). To accomplish this, we use the pair of inequalities (2.3) and the fact that \( \{\tilde{u}_n\}_n \) is a \((PS)_c\) sequence. It follows that there exist positive real numbers \( C_1 \) and \( C_2 \) such that
\[ \Phi(\tilde{u}_n) - \frac{1}{p_s(\alpha)} \langle \Phi'(\tilde{u}_n), \tilde{u}_n \rangle = \left( \frac{1}{p} - \frac{1}{p_s(\alpha)} \right) \|\tilde{u}_n\|^p 
+ \left( \frac{1}{p_s(\alpha)} - \frac{1}{p_s(\beta)} \right) \int_{\mathbb{R}^N} \frac{|\tilde{u}_n|^p_s(\beta)}{|x|^\beta} \, dx \]
\[ \leq C_1 + C_2 \|\tilde{u}_n\| + o_n(1); \]

thus,
\[ \left( \frac{1}{p} - \frac{1}{p_s(\alpha)} \right) \|\tilde{u}_n\|^p \leq C_1 + C_2 \|\tilde{u}_n\|. \]

This inequality assures us that \( \{\|\tilde{u}_n\|\}_n \) is bounded. Using once more the pair of inequalities (2.3), we deduce that the sequence \( \{\tilde{u}_n\}_n \subset D^{s,p}(\mathbb{R}^N) \) is bounded. So, after a passage to a subsequence if necessary, still denoted in the same way, there exists \( \tilde{u} \in D^{s,p}(\mathbb{R}^N) \) such that \( \tilde{u}_n \rightharpoonup \tilde{u} \) weakly in \( D^{s,p}(\mathbb{R}^N) \) as \( n \to +\infty \).

Notice that if \( \tilde{u} \equiv 0 \), then Proposition 3.1 guarantees that
\[ \lim_{n \to +\infty} \int_{B_1(0)} \frac{|\tilde{u}_n|^{p_s(\alpha)}}{|x|^{\alpha}} \, dx = 0 \quad \text{or} \quad \lim_{n \to +\infty} \int_{B_1(0)} \frac{|\tilde{u}_n|^{p_s(\alpha)}}{|x|^{\alpha}} \, dx \geq \varepsilon_0. \]

Since \( 0 < \varepsilon < \frac{\varepsilon_0}{2} \), we get a contradiction in both cases in view of equality (4.4). We deduce that \( \tilde{u} \neq 0 \).

Notice that from the weak convergence \( \tilde{u}_n \rightharpoonup \tilde{u} \) in \( D^{s,p}(\mathbb{R}^N) \) we get
\[ \tilde{u}_n \to \tilde{u} \quad \text{a.e.} \ \mathbb{R}^N. \]

We also have, from inequalities (2.1) and (2.3) and from definition (1.11), that the sequence \( \{\tilde{u}_n\}_n \subset D^{s,p}(\mathbb{R}^N) \) is bounded in \( L^{p_s(\beta)}(\mathbb{R}^N, |x|^{-\beta}) \) and in \( L^{p_s(\alpha)}(\mathbb{R}^N, |x|^{-\alpha}) \). Thus, using a result in Kavian [26, Lemme 4.8], we deduce that
\[ \tilde{u}_n \rightharpoonup \tilde{u} \ \text{weakly in} \ L^{p_s(\beta)}(\mathbb{R}^N, |x|^{-\beta}) \ \text{as} \ n \to +\infty, \]
\[ \tilde{u}_n \to \tilde{u} \quad \text{weakly in } L^{p^*_s(\alpha)}(\mathbb{R}^N, |x|^{-\alpha}) \quad \text{as } n \to +\infty. \]

From these convergences, for an arbitrary test function \( \varphi \in D^{s,p}(\mathbb{R}^N) \) we have

\[
\langle \Phi'(\tilde{u}_n), \varphi \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{J_p(\tilde{u}_n(x) - \tilde{u}_n(y))}{|x-y|^{N+sp}} \varphi(x-y) \, dx \, dy \\
- \mu \int_{\mathbb{R}^N} \frac{J_p(\tilde{u}_n)}{|x|^{ps}} \, dx - \int_{\mathbb{R}^N} \frac{J_{p^*_s(\beta)}(\tilde{u}_n) \varphi}{|x|^\beta} \, dx - \int_{\mathbb{R}^N} \frac{J_{p^*_s(\alpha)}(\tilde{u}_n) \varphi}{|x|^\alpha} \, dx \\
= o_n(1) + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{J_p(\tilde{u}(x) - \tilde{u}(y))}{|x-y|^{N+sp}} \varphi(x-y) \, dx \, dy \\
- \mu \int_{\mathbb{R}^N} \frac{J_p(\tilde{u})}{|x|^{ps}} \, dx - \int_{\mathbb{R}^N} \frac{J_{p^*_s(\beta)}(\tilde{u}) \varphi}{|x|^\beta} \, dx - \int_{\mathbb{R}^N} \frac{J_{p^*_s(\alpha)}(\tilde{u}) \varphi}{|x|^\alpha} \, dx \\
= \langle \Phi'(\tilde{u}), \varphi \rangle + o_n(1). \]

Since \( \{\tilde{u}_n\}_n \subset D^{s,p}(\mathbb{R}^N) \) is a \((PS)_c\) sequence, it follows that \( \langle \Phi'(\tilde{u}_n), \varphi \rangle \to 0 \) as \( n \to +\infty \); and since the test function \( \varphi \in D^{s,p}(\mathbb{R}^N) \) is arbitrary, it follows that

\[ \langle \Phi'(\tilde{u}), \varphi \rangle = 0 \quad \text{for all } \varphi \in D^{s,p}(\mathbb{R}^N). \]

We conclude that \( \tilde{u} \in D^{s,p}(\mathbb{R}^N) \) is a nontrivial weak solution to problem (1.7). \( \square \)

### 5. Extremals for the Sobolev inequality

In this section we show Theorem 1.2.

**Proposition 5.1.** The constant \( K(\mu, \alpha) \) defined in (1.11) is attained by a nontrivial function \( \tilde{u} \in D^{s,p}(\mathbb{R}^N) \).

**Proof.** For the case where \( \mu = 0 \) and \( \alpha = 0 \) we refer the reader to the paper by Brasco, Mosconi and Squassina [5]; and for the case where \( \mu = 0 \) and \( \alpha \in (0, sp) \) we refer the reader to the paper by Marano and Mosconi [28]. Here we consider the case where \( 0 < \mu < \mu_H \) and \( \alpha \in (0, sp) \). Let the functional \( I_{\mu, \alpha} : D^{s,p}(\mathbb{R}^N) \to \mathbb{R} \) be given by

\[
I_{\mu, \alpha}(u) := \frac{\|u\|^p}{\left( \int_{\mathbb{R}^N} \frac{|u|^{p^*_s(\alpha)}}{|x|^\alpha} \, dx \right)^{p/p^*_s(\alpha)}}.
\]

Notice that

\[
\int_{\mathbb{R}^N} \frac{|u|^{p^*_s(\alpha)}}{|x|^\alpha} \, dx = \int_{\mathbb{R}^N} \frac{|u|^{p^*_s(\alpha)\varepsilon}|u|^{p^*_s(\alpha)(1-\varepsilon)}}{|x|^\alpha} \, dx \quad \text{for all } \varepsilon \in \mathbb{R}.
\]

Now we choose \( \varepsilon = \frac{p^*_s(p^*_s(\alpha) - p)}{p^*_s(\alpha)(p^*_s(\alpha) - p)} = \frac{N(sp-\alpha)}{s(N-\alpha)} \), thus, \( 1 - \varepsilon = \frac{p(p^*_s-\alpha)}{p^*_s(\alpha)(p^*_s-\alpha)} = \frac{a(N-sp)}{sp(N-\alpha)} > 0 \). Using the Hölder inequality with exponents \( r = \frac{sp}{sp-\alpha} \) and \( r' = \frac{sp}{\alpha} \), it follows that

\[
\int_{\mathbb{R}^N} \frac{|u|^{p^*_s(\alpha)}}{|x|^\alpha} \, dx = \int_{\mathbb{R}^N} \left( |u|^{N(sp-\alpha)} \right)^{1/r} \left( \frac{|u|^{2}}{|x|^\alpha} \right)^{r'} \, dx
\]
such that
\[ I_{\mu,\alpha} \]

functional
\[ I \]

defined in (1.9). These inequalities imply that

where
\[ S \]

is the best constant of the embedding \( D^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p_s}(\mathbb{R}^N) \) and \( \mu_H \) is defined in (1.9). These inequalities imply that

\[
\left( \int_{\mathbb{R}^N} \frac{|u|^{p_s(\alpha)}_{s,p}}{|x|^\alpha} \, dx \right)^{\frac{p}{p_s}} \leq C(N, p, \mu, s, \alpha) \frac{p^{p(N-\alpha)}}{p_s^{p(N-\alpha)}} |u|_{s,p,s}^{p}.
\]

We conclude that there exists a positive constant, still denoted by \( C = C(N, p, \mu, s, \alpha) \), such that

\[
\frac{1}{C} \leq \frac{|u|_{s,p,s}^{p}}{\left( \int_{\mathbb{R}^N} \frac{|u|^{p_s(\alpha)}_{s,p}}{|x|^\alpha} \, dx \right)^{\frac{p}{p_s}}} \quad \text{for all } u \in D^{s,p}(\mathbb{R}^N) \setminus \{0\}.
\]

By the pair of inequalities (2.3) we infer that the functional \( I_{\mu,\alpha} \) is well defined and is bounded from below by a positive constant; so, \( \inf I_{\mu,\alpha} > 0 \). Note that \( I_{\mu,\alpha} \in C^1(D^{s,p}(\mathbb{R}^N), \mathbb{R}) \) and we can apply the Ekeland variational principle [16] to the functional \( I_{\mu,\alpha}(u) \) to guarantee the existence of a minimizing sequence \( \{u_n\}_n \subset D^{s,p}(\mathbb{R}^N) \) with the additional properties

\[
\int_{\mathbb{R}^N} \frac{|u_n|^{p_s(\alpha)}_{s,p}}{|x|^\alpha} \, dx = 1 \quad \text{for all } n \in \mathbb{N}
\]

and also

\[
I_{\mu,\alpha}(u_n) \to \inf_{u \neq 0} I_{\mu,\alpha}(u) := \frac{1}{K(\mu, \alpha)} \quad \text{as } n \to +\infty;
\]

\[
I'_{\mu,\alpha}(u_n) \to 0 \quad \text{in } (D^{s,p}(\mathbb{R}^N))'.
\]

Now we consider two auxiliary functionals \( J, G : D^{s,p}(\mathbb{R}^N) \to \mathbb{R} \) defined by

\[
J(u) := \frac{|u|^p}{p} \quad \text{and} \quad G(u) := \frac{1}{p_s^{s(\alpha)}} \int_{\mathbb{R}^N} \frac{|u|^{p_s(\alpha)}_{s,p}}{|x|^\alpha} \, dx.
\]

Claim 1. The following statements are valid.
1. \( J(u_n) \to \frac{1}{pK(\mu, \alpha)} \) as \( n \to +\infty \).

2. \( J'(u_n) - \frac{1}{K(\mu, \alpha)} G'(u_n) \to 0 \) in \( (D^{s,p}(\mathbb{R}^N))' \) as \( n \to +\infty \).

The proof of the first item is immediate from the definitions involved. To prove the second item we consider \( v \in D^{s,p}(\mathbb{R}^N) \) and note that

\[
J'(u_n)v = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J_p(u_n(x) - u_n(y))(v(x) - v(y)) \frac{dx dy}{|x - y|^{N+sp}} - \mu \int_{\mathbb{R}^N} J_p(u_n)v \frac{dx}{|x|^{ps}},
\]

and

\[
G'(u_n)v = \int_{\mathbb{R}^N} \frac{|u_n|^{p_\star(\alpha)} - 2u_nv}{|x|^{\alpha}} \frac{dx}{|x|^{\alpha}}.
\]

Thus,

\[
J'(u_n)v - \frac{1}{K(\mu, \alpha)} G'(u_n)v = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J_p(u_n(x) - u_n(y))(v(x) - v(y)) \frac{dx dy}{|x - y|^{N+sp}} - \mu \int_{\mathbb{R}^N} J_p(u_n)v \frac{dx}{|x|^{ps}} - \frac{1}{K(\mu, \alpha)} \int_{\mathbb{R}^N} J_{p_\star(\alpha)}(u_n)v \frac{dx}{|x|^{\alpha}}.
\]

We remark that

\[
J_{\mu,\alpha}'(u_n)v = \frac{p(J'(u_n)v)}{\left( \int_{\mathbb{R}^N} \frac{|u_n|^{p_\star(\alpha)}}{|x|^{\alpha}} \frac{dx}{|x|^{\alpha}} \right)^{\frac{p}{p_\star(\alpha)}}} - \frac{p\|u_n\|^{p'}G'(u_n)v}{\left( \int_{\mathbb{R}^N} \frac{|u_n|^{p_\star(\alpha)}}{|x|^{\alpha}} \frac{dx}{|x|^{\alpha}} \right)^{\frac{p}{p_\star(\alpha)} + 1}}.
\]

Since \( \|u_n\|^p \to \frac{1}{K(\mu, \alpha)} \) as \( n \to +\infty \) and \( \int_{\mathbb{R}^N} \frac{|u_n|^{p_\star(\alpha)}}{|x|^{\alpha}} dx = 1 \) for all \( n \in \mathbb{N} \), we obtain

\[
J_{\mu,\alpha}'(u_n)v = p \left( J'(u_n)v - \left( \frac{1}{K(\mu, \alpha)} + o_n(1) \right) G'(u_n)v \right).
\]

And by the fact that \( J_{\mu,\alpha}'(u_n)v \to 0 \) as \( n \to +\infty \) for an arbitrary function \( v \in D^{s,p}(\mathbb{R}^N) \), we deduce that

\[
J'(u_n)v - \frac{1}{K(\mu, \alpha)} G'(u_n)v \to 0 \quad \text{for all } v \in D^{s,p}(\mathbb{R}^N) \text{ as } n \to +\infty.
\]

Therefore,

\[
J'(u_n) - \frac{1}{K(\mu, \alpha)} G'(u_n) \to 0 \quad \text{in } (D^{s,p}(\mathbb{R}^N))' \text{ as } n \to +\infty. \quad (5.1)
\]

This concludes the proof of the claim.

Now we define the Levy concentration function \( Q: \mathbb{R}_+ \to \mathbb{R} \) associated to \( \frac{|u_n|^{p_\star(\alpha)}}{|x|^{\alpha}} \) by

\[
Q(r) := \int_{B_r(0)} \frac{|u_n|^{p_\star(\alpha)}}{|x|^{\alpha}} dx.
\]
Using the continuity of the function $Q$ and since \( \int_{\mathbb{R}^N} \frac{|u_n|^{p^*_s(\alpha)}}{|x|^\alpha} \, dx = 1 \), passing to a subsequence if necessary, still denoted in the same way, for every natural number \( n \in \mathbb{N} \) there exists a positive real number \( r_n > 0 \) such that
\[
Q(r_n) = \int_{B_{r_n}(0)} \frac{|u_n|^{p^*_s(\alpha)}}{|x|^\alpha} \, dx = \frac{1}{2} \quad \text{for all } n \in \mathbb{N}.
\]

As we have already done, we consider the sequence \( \{\tilde{u}_n\} \subset D^{s,p}(\mathbb{R}^N) \) defined in (4.3). We note that \( I_{\mu,\alpha}(\tilde{u}_n) = I_{\mu,\alpha}(u_n) \) because the several integrals present in the functional \( I_{\mu,\alpha} \) are invariant under the conformal transformations defined in (4.3). In this way, \( \{\tilde{u}_n\} \subset D^{s,p}(\mathbb{R}^N) \) is also a minimizing sequence for the functional \( I_{\mu,\alpha} \); moreover,
\[
\int_{B_1(0)} \frac{|\tilde{u}_n|^{p^*_s(\alpha)}}{|x|^\alpha} \, dx = \frac{1}{2} \quad \text{for all } n \in \mathbb{N}. \tag{5.2}
\]

We also note that \( \|\tilde{u}_n\|^p = \frac{1}{K(\mu,\alpha)} + o_n(1) \); so, by inequalities (2.3) it follows that the sequence \( \{\tilde{u}_n\} \subset D^{s,p}(\mathbb{R}^N) \) is bounded. We deduce that, up to a passage to a subsequence, there exists a function \( \tilde{u} \in D^{s,p}(\mathbb{R}^N) \) such that, as \( n \to +\infty \) we have
\[
\tilde{u}_n \rightharpoonup \tilde{u} \quad \text{weakly in } D^{s,p}(\mathbb{R}^N),
\]
\[
\tilde{u}_n \to \tilde{u} \quad \text{strongly in } L^q_{\text{loc}}(\mathbb{R}^N), \quad \text{for all } q \in [1, p^*_s).
\]

Our goals now are to prove that \( \tilde{u} \not\equiv 0 \) and that \( \tilde{u} \) is a minimizer for the functional \( I_{\mu,\alpha} \). To accomplish the first goal we argue by contradiction and suppose that \( \tilde{u} \equiv 0 \). So, we have
\[
\tilde{u}_n \to 0 \quad \text{weakly in } D^{s,p}(\mathbb{R}^N), \tag{5.3}
\]
\[
\tilde{u}_n \to 0 \quad \text{strongly in } L^q_{\text{loc}}(\mathbb{R}^N) \quad \text{for all } q \in [1, p^*_s). \tag{5.4}
\]

Now we set \( 0 < \delta < 1 \) and define \( B_\delta(0) := \{x \in \mathbb{R}^N; |x| \leq \delta < 1\} \). We also consider a cut-off function \( \eta \in C^\infty_0(\mathbb{R}^N) \) such that
\[
\eta \equiv 1 \quad \text{in } B_\delta(0); \quad \eta \equiv 0 \quad \text{in } \mathbb{R}^N \setminus B_1(0); \quad 0 \leq \eta \leq 1 \quad \text{in } \mathbb{R}^N. \tag{5.5}
\]

Using a result in Brasco, Squassina and Yang [6, Lemma A.1], we can also consider \( \eta^p \tilde{u}_n \) as test function in (5.1); so,
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(p)(\tilde{u}_n(x) - \tilde{u}_n(y))(\eta^p \tilde{u}_n(x) - \eta^p \tilde{u}_n(y)) \frac{dx \, dy}{|x - y|^{N + sp}} - \mathcal{K}(\mu, \alpha) \int_{\mathbb{R}^N} \frac{|\eta \tilde{u}_n|^p}{|x|^\alpha} \, dx = \frac{1}{K(\mu, \alpha)} \int_{\mathbb{R}^N} \frac{|\tilde{u}_n|^{p^*_s(\alpha)} \eta^p}{|x|^\alpha} \, dx + o_n(1). \tag{5.6}
\]

As we have already seen, if \( \tilde{u}_n \to 0 \) weakly in \( D^{s,p}(\mathbb{R}^N) \) and \( \eta \in C^\infty_0(\mathbb{R}^N) \), then
\[
\int \int \frac{\eta^p(x) |\tilde{u}_n(x) - \tilde{u}_n(y)|^p}{|x - y|^{N + sp}} \, dx \, dy = [\eta \tilde{u}_n]_{s,p}^p + o_n(1) \quad \text{as } n \to +\infty
\]
and
\[
\int \int \frac{|\tilde{u}_n(y)| |\eta^p(x) - \eta^p(y)|^p}{|x - y|^{N + sp}} \, dx \, dy \to 0 \quad \text{as } n \to +\infty.
\]
Both these estimates together with the estimate (5.6) and Hölder inequality with exponents $p^*_s(\alpha)$ and $p^*_s(\alpha)/(p^*_s(\alpha) - p)$ allow us to deduce that

$$o_n(1) + \|\eta \tilde{u}_n\|^p = \frac{1}{K(\mu, \alpha)} \int_{\mathbb{R}^N} \frac{|\tilde{u}_n|^{p^*_s(\alpha)} \eta^p}{|x|^\alpha} \, dx$$

$$\leq \frac{1}{K(\mu, \alpha)} \int_{B_1(0)} \frac{|\tilde{u}_n|^{p^*_s(\alpha)}}{|x|^\alpha} \, dx$$

$$= \frac{1}{K(\mu, \alpha)} \left( \int_{B_1(0)} \frac{|\tilde{u}_n|^{p^*_s(\alpha)}}{|x|^\alpha} \, dx \right)^{\frac{p^*_s(\alpha)}{p^*_s(\alpha)}} \left( \int_{B_1(0)} \frac{|\tilde{u}_n|^{p^*_s(\alpha)}}{|x|^\alpha} \, dx \right)^{\frac{p^*_s(\alpha) - p}{p^*_s(\alpha)}}$$

$$\leq \frac{1}{K(\mu, \alpha)} \left( \frac{1}{2} \right)^{\frac{p^*_s(\alpha) - p}{p^*_s(\alpha)}} \left( \int_{B_1(0)} \frac{|\tilde{u}_n|^{p^*_s(\alpha)}}{|x|^\alpha} \, dx \right)^{\frac{p}{p^*_s(\alpha)}} + o_n(1), \quad (5.7)$$

where in the last passage we used equality (5.2).

We note that

$$\|\tilde{u}_n\|_{L^{p^*_s(\alpha)}(B_1(0), |x|^{-\alpha})} = \|\eta \tilde{u}_n + (1 - \eta) \tilde{u}_n\|_{L^{p^*_s(\alpha)}(B_1(0), |x|^{-\alpha})}$$

$$\leq \|\eta \tilde{u}_n\|_{L^{p^*_s(\alpha)}(B_1(0), |x|^{-\alpha})} + \|(1 - \eta) \tilde{u}_n\|_{L^{p^*_s(\alpha)}(B_1(0), |x|^{-\alpha})};$$

and since $\eta \equiv 1$ in $B_{\delta}(0)$, it follows that

$$\|(1 - \eta) \tilde{u}_n\|_{L^{p^*_s(\alpha)}(B_1(0), |x|^{-\alpha})} \leq C \int_{B_1(0) \setminus B_{\delta}(0)} |\tilde{u}_n|^{p^*_s(\alpha)} \, dx.$$

We also have $\tilde{u}_n \to 0$ strongly in $L^q_{\text{loc}}(\mathbb{R}^N)$ for $q \in [1, p^*_s)$; and since $p^*_s(\alpha) < p^*_s$, it follows that

$$\|(1 - \eta) \tilde{u}_n\|_{L^{p^*_s(\alpha)}(B_1(0), |x|^{-\alpha})} = o_n(1).$$

Consequently,

$$\|\tilde{u}_n\|_{L^{p^*_s(\alpha)}(B_1(0), |x|^{-\alpha})} = \left( \int_{B_1(0)} \frac{|\tilde{u}_n|^{p^*_s(\alpha)}}{|x|^\alpha} \, dx \right)^{\frac{p}{p^*_s(\alpha)}}$$

$$\leq \left( \int_{\mathbb{R}^N} \frac{|\eta \tilde{u}_n|^{p^*_s(\alpha)}}{|x|^\alpha} \, dx \right)^{\frac{p}{p^*_s(\alpha)}} + o_n(1). \quad (5.8)$$

Substituting inequality (5.8) into the estimate (5.7) yields

$$\|\eta \tilde{u}_n\|^p \leq \frac{1}{K(\mu, \alpha)} \left( \frac{1}{2} \right) \left( \int_{\mathbb{R}^N} \frac{|\eta \tilde{u}_n|^{p^*_s(\alpha)}}{|x|^\alpha} \, dx \right)^{\frac{p}{p^*_s(\alpha)}} + o_n(1). \quad (5.9)$$

On the other hand, by the definition (1.11) of the constant $K(\mu, \alpha)$, we obtain

$$\frac{1}{K(\mu, \alpha)} \left( \int_{\mathbb{R}^N} \frac{|\eta \tilde{u}_n|^{p^*_s(\alpha)}}{|x|^\alpha} \, dx \right)^{\frac{p}{p^*_s(\alpha)}} \leq \|\eta \tilde{u}_n\|^p. \quad (5.10)$$
Combining inequalities (5.9) and (5.10) we arrive at
\[
\frac{1}{K(\mu, \alpha)} \left( 1 - \left( \frac{1}{2} \right)^{\frac{p^*_\alpha - \alpha}{\alpha}} \right) \int_{\mathbb{R}^N} \frac{\eta \tilde{u}_n |p^*_\alpha(\alpha)}{|x|^{\alpha}} dx \leq o_n(1).
\]

Using the assumptions \( \alpha \in (0, sp) \) and \( N > sp \) it follows from the previous inequality that
\[
\int_{\mathbb{R}^N} \frac{\eta \tilde{u}_n |p^*_\alpha(\alpha)}{|x|^{\alpha}} dx \to 0.
\]
So, this inequality together with estimate (5.8) guarantees that
\[
\int_{B_1(0)} \frac{\tilde{u}_n |p^*_\alpha(\alpha)}{|x|^{\alpha}} dx \to 0 \quad \text{as} \quad n \to +\infty.
\]

But this is a contradiction with equality (5.2). As a result, we deduce that \( \tilde{u} \not\equiv 0 \).

It remains to show that the weak limit \( \tilde{u} \in D^{s,p}(\mathbb{R}^N) \) is in fact a minimizer to \( \frac{1}{K(\mu, \alpha)} \) and that \( \int_{\mathbb{R}^N} \frac{\tilde{u} |p^*_\alpha(\alpha)}{|x|^{\alpha}} dx = 1 \). To do this, for every natural number \( n \in \mathbb{N} \) we define \( \theta_n := \tilde{u}_n - \tilde{u} \). Hence, applying Brezis–Lieb lemma [7, Theorem 1.1] we obtain
\[
1 = \int_{\mathbb{R}^N} \frac{\tilde{u}_n^{p^*_\alpha(\alpha)}}{|x|^{\alpha}} dx = \int_{\mathbb{R}^N} \frac{\tilde{u}^{p^*_\alpha(\alpha)}}{|x|^{\alpha}} dx + \int_{\mathbb{R}^N} \frac{\theta_n^{p^*_\alpha(\alpha)}}{|x|^{\alpha}} dx + o_n(1).
\]

From this estimate we get the inequalities
\[
0 \leq \int_{\mathbb{R}^N} \frac{\tilde{u}^{p^*_\alpha(\alpha)}}{|x|^{\alpha}} dx \leq 1 \quad \text{and} \quad 0 \leq \int_{\mathbb{R}^N} \frac{\theta_n^{p^*_\alpha(\alpha)}}{|x|^{\alpha}} dx \leq 1.
\]

Using the fact that \( \theta_n \rightharpoonup 0 \) weakly in \( D^{s,p}(\mathbb{R}^N) \) as \( n \to +\infty \), applying a result by Brasco, Squassina and Yang [6, Lemma 2.2], we infer that
\[
\|\tilde{u}_n\|^p = \|\tilde{u}\|^p + \|\theta_n\|^p + o_n(1).
\]

In this way, using estimate (5.1) and the definition (1.11) of the constant \( K(\mu, \alpha) \) we get
\[
o_n(1) = \|\tilde{u}_n\|^p - \frac{1}{K(\mu, \alpha)} \int_{\mathbb{R}^N} \frac{\tilde{u}_n |p^*_\alpha(\alpha)}{|x|^{\alpha}} dx
\[
= \left( \|\tilde{u}\|^p - \frac{1}{K(\mu, \alpha)} \int_{\mathbb{R}^N} \frac{\tilde{u} |p^*_\alpha(\alpha)}{|x|^{\alpha}} dx \right)
\[
+ \left( \|\theta_n\|^p - \frac{1}{K(\mu, \alpha)} \int_{\mathbb{R}^N} \frac{\theta_n |p^*_\alpha(\alpha)}{|x|^{\alpha}} dx \right) + o_n(1)
\]
\[
\geq \frac{1}{K(\mu, \alpha)} \left\{ \left( \int_{\mathbb{R}^N} \frac{\tilde{u} |p^*_\alpha(\alpha)}{|x|^{\alpha}} dx \right)^{\frac{p}{p^*_\alpha(\alpha)}} - \int_{\mathbb{R}^N} \frac{\tilde{u} |p^*_\alpha(\alpha)}{|x|^{\alpha}} dx \right\}
\]
\[
+ \frac{1}{K(\mu, \alpha)} \left\{ \left( \int_{\mathbb{R}^N} \frac{\theta_n |p^*_\alpha(\alpha)}{|x|^{\alpha}} dx \right)^{\frac{p}{p^*_\alpha(\alpha)}} - \int_{\mathbb{R}^N} \frac{\theta_n |p^*_\alpha(\alpha)}{|x|^{\alpha}} dx \right\} + o_n(1)
\]
Clearly, \( A + B = o_n(1) \); and using the fact that \( \frac{p}{p^*_s} \in (0,1) \) and that both integrals 
\[
\int_{\mathbb{R}^N} \frac{|\tilde{u}|^{p^*_s(\alpha)}}{|x|^\alpha} \, dx \quad \text{and} \quad \int_{\mathbb{R}^N} \frac{|\theta_n|^{p^*_s(\alpha)}}{|x|^\alpha} \, dx
\]
take their values in the closed interval \([0,1]\), we deduce that \( A, B \geq 0 \) and that \( B = -A + o_n(1) \geq 0 \); this means that \( A = 0 \) and that \( B = o_n(1) \), that is,
\[
\left( \int_{\mathbb{R}^N} \frac{|\tilde{u}|^{p^*_s(\alpha)}}{|x|^\alpha} \, dx \right)^{\frac{p^*_s}{p}} = \int_{\mathbb{R}^N} \frac{|\tilde{u}|^{p^*_s(\alpha)}}{|x|^\alpha} \, dx.
\]

We have already seen that \( \tilde{u} \neq 0 \); so, the previous equality implies that
\[
\int_{\mathbb{R}^N} \frac{|\tilde{u}|^{p^*_s(\alpha)}}{|x|^\alpha} \, dx = 1.
\]

Using again the estimate (5.1) and the fact that \( \tilde{u}_n \rightharpoonup \tilde{u} \) weakly in \( D^{s,p}(\mathbb{R}^N) \) as \( n \to +\infty \), it follows that,
\[
J'(\tilde{u}_n)\tilde{u} - \frac{1}{K(\mu, \alpha)} G'(\tilde{u}_n)\tilde{u} \to J'(\tilde{u})\tilde{u} - \frac{1}{K(\mu, \alpha)} G'(\tilde{u})\tilde{u} = 0
\]
as \( n \to +\infty \).

Finally, we conclude that
\[
\|\tilde{u}\|^p = \frac{1}{K(\mu, \alpha)} \int_{\mathbb{R}^N} \frac{|\tilde{u}|^{p^*_s(\alpha)}}{|x|^\alpha} \, dx = \frac{1}{K(\mu, \alpha)},
\]
that is, the best constant \( \frac{1}{K(\mu, \alpha)} \) is attained by a nontrivial function \( \tilde{u} \in D^{s,p}(\mathbb{R}^N) \). This concludes the proof of the proposition. \( \square \)

Remark 5.2. If \( \mu \leq 0 \), then \( 1/K(\mu, 0) = 1/K(0, 0) \). Therefore there is no extremal for \( 1/K(\mu, \alpha) \) when \( \mu \leq 0 \).

As we have \( \mu \leq 0 \), clearly \( 1/K(\mu, 0) \geq 1/K(0, 0) \). We consider a function \( w \in D^{s,2}(\mathbb{R}^N) \setminus \{0\} \) for which \( 1/K(0, 0) \) is attained. For the existence of such a function we refer Brasco, Mosconi and Squassina [5]. Now for \( \delta \in \mathbb{R} \) and \( \overline{x} \in \mathbb{R}^N \) we define the function \( w_\delta = w(x - \delta \overline{x}) \) for \( x \in \mathbb{R}^N \). Then, by changing variables we get
\[
\frac{1}{K(\mu, 0)} \leq I_\delta = \frac{\|w_\delta\|^p}{\left( \int_{\mathbb{R}^N} |w_\delta|^{p^*_s} \, dx \right)^{\frac{p}{p^*_s}}} = \frac{\|w\|_{D^{s,p}(\mathbb{R}^N)}^p - \mu \int_{\mathbb{R}^N} \frac{|w(x)|^p}{|x + \delta \overline{x}|^{p^*_s}} \, dx}{\left( \int_{\mathbb{R}^N} |w|^{p^*_s} \, dx \right)^{\frac{p}{p^*_s}}};
\]
therefore,
\[
\frac{1}{K(\mu, 0)} \leq \lim_{\delta \to +\infty} I_\delta = \frac{\|w\|_{D^{s,p}(\mathbb{R}^N)}^p}{\left( \int_{\mathbb{R}^N} |w|^{p^*_s} \, dx \right)^{\frac{p}{p^*_s}}} = \frac{1}{K(0, 0)}.
\]
So,
\[
\frac{1}{K(\mu, 0)} = \frac{1}{K(0, 0)}.
\]
We conclude that there is no function that attains the constant $1/K(\mu, 0)$ when $\mu < 0$.

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Ronaldo B. Assunção and Jeferson C. Silva
Departamento de Matemática
Universidade Federal de Minas Gerais (UFMG)
Av. Antônio Carlos, 6627, CEP 30161-970, Belo Horizonte, MG, Brasil
e-mail: ronaldo@mat.ufmg.br
    jefersoncs@ufmg.br

Olímpio H. Miyagaki
Departamento de Matemática
Universidade Federal de São Carlos (UFSCar)
Km 235, Rodovia Washington Luís – Jardim Guanabara, CEP 13565-905
São Carlos, SP, Brasil
e-mail: ohmiyagaki@gmail.com

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