Ultrarelativistic $N$-boson systems

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Abstract. General analytic energy bounds are derived for $N$-boson systems governed by ultrarelativistic Hamiltonians of the form

$$H = \sum_{i=1}^{N} \|p_i\| + \sum_{1 \leq i < j}^{N} V(r_{ij}),$$

where $V(r)$ is a static attractive pair potential. It is proved that a translation-invariant model Hamiltonian $H_c$ provides a lower bound to $H$ for all $N \geq 2$. This result was conjectured in an earlier paper but proved only for $N = 2, 3, 4$. As an example, the energy in the case of the linear potential $V(r) = r$ is determined with error less than 0.55\% for all $N \geq 2$.

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1. Introduction

We consider first the semirelativistic $N$-body Hamiltonian $H$ given by

$$H = \sum_{i=1}^{N} \sqrt{\|\mathbf{p}_i\|^2 + m^2} + \sum_{1=i<j}^{N} V(r_{ij}), \quad (1)$$

and the following model Hamiltonian $H_c$

$$H_c = \sum_{1=i<j}^{N} \left[ \gamma^{-1} \sqrt{\gamma \|\mathbf{p}_i - \mathbf{p}_j\|^2 + (mN)^2 + V(r_{ij})} \right], \quad (2)$$

where $\gamma = \left( \frac{N}{2} \right) = \frac{1}{2}N(N-1)$. If $\Psi(\rho_2, \rho_3, \ldots, \rho_N)$ is the lowest boson eigenstate of $H$ expressed in terms of Jacobi relative coordinates, then it was proved in Ref. [1] that the model facilitates a ‘reduction’ $\langle H_c \rangle = \langle H \rangle$ to the expectation of a one-body Hamiltonian $\mathcal{H}$ given by

$$\mathcal{H} = N\sqrt{\lambda p^2 + m^2} + \gamma V(r), \quad \lambda = \frac{2(N-1)}{N}. \quad (3)$$

The question remains as to the relation between $H$ and the model $H_c$. It is known from earlier work (discussed in [1]) that the lower bound conjecture

$$\langle H \rangle \geq \langle H_c \rangle \quad (4)$$

is true for the following cases: for the harmonic oscillator $V(r) = vr^2$, for all attractive $V(r)$ in the nonrelativistic large-$m$ limit, and for static gravity $V(r) = -v/r$. This list was augmented in Ref. [1] by the following cases: in general for $N = 3$, and, if $m = 0$, for $N = 4$. The purpose of the present article is to extend this list to include the ultrarelativistic cases $m = 0$ for all $N \geq 2$ and arbitrary attractive $V(r)$.

2. The general lower bound for $m = 0$.

It was shown in Ref. [1] that the non-negativity of the expectation $\langle \delta(m, N) \rangle$ is sufficient to establish the validity of the conjecture (4), where

$$\delta(m, N) = \sum_{i=1}^{N} \sqrt{\|\mathbf{p}_i\|^2 + m^2} - \frac{2}{N-1} \sum_{1=i<j}^{N} \sqrt{\frac{N-1}{2N} \|\mathbf{p}_i - \mathbf{p}_j\|^2 + m^2}. \quad (5)$$

Thus for the new cases we are now able to treat we must consider $\langle \delta(0, N) \rangle$. By using the necessary boson permutation symmetry of $\Psi$, the expectation value we need to study is reduced to

$$\langle \delta(0, N) \rangle = N \left( \|\mathbf{p}_1\| - \sqrt{\frac{N-1}{2N} \|\mathbf{p}_1 - \mathbf{p}_2\|} \right). \quad (6)$$
The principal result of this paper, the lower bound for \( m = 0 \) and all \( N \geq 2 \), is an immediate consequence of the following:

**Theorem 1** \( \langle \delta(0, N) \rangle = 0 \).

**Proof of Theorem 1**

Without loss of generality we adopt in momentum space a coordinate origin such that \( \sum_{i=1}^{N} p_i := p = 0 \). We define the mean lengths
\[
\langle \|p_1\| \rangle := k \quad \text{and} \quad \langle \|p_1 - p_2\| \rangle := d. \tag{7}
\]

We wish to make a correspondence between mean lengths such as \( k \) and \( d \) and the sides of triangles that can be constructed with these lengths. We consider the triangle formed by the three vectors \( \{p_1, p_2, p_1 - p_2\} \). We suppose that the corresponding angles in this triangle are \( \{\phi_{12}, \theta_1, \theta_2\} \) (the same notation is used for other similar triples). We now consider projections of one side on a unit vector along an adjacent side and define the mean angles \( \phi \) and \( \theta \) by the relations
\[
\langle \|p_1\| \cos(\phi_{12}) \rangle := \langle \|p_1\| \rangle \cos(\phi)
\]
and
\[
\langle \|p_1 - p_2\| \cos(\theta_1) \rangle := \langle \|p_1 - p_2\| \rangle \cos(\theta).
\]

Thus, on the average, this triangle is isosceles with one angle \( \phi \) and the other two angles \( \theta \). Since \( p = 0 \), we have \( \langle p_1 \cdot p \rangle = 0 \). Hence
\[
\|p_1\|^2 + \sum_{i=2}^{N} \|p_1\||\|p_i\|\cos(\phi_{1i}) = 0.
\]

Thus, by dividing by \( \|p_1\| \) and using boson symmetry, we find
\[
\langle \|p_1\| + (N-1)\|p_2\| \cos(\phi_{12}) \rangle = \langle \|p_1\| (1 + (N-1) \cos(\phi_{12})) \rangle = 0.
\]

We therefore conclude that \( k(1 + (N-1) \cos(\phi)) = 0 \), that is to say
\[
\cos(\phi) = -\frac{1}{N-1}.
\]

We now consider again the triangle formed by the three vectors \( \{p_1, p_2, p_1 - p_2\} \). We have immediately from the dot product \( p_1 \cdot (p_1 - p_2) \)
\[
\|p_1\|\|p_1 - p_2\| \cos(\theta_1) = \|p_1\| (\|p_1\| - \|p_2\| \cos(\phi_{12})).
\]

By dividing by \( \|p_1\| \) and taking means we obtain
\[
d \cos(\theta) = k(1 - \cos(\phi)).
\]

But \( \theta = (\pi/2 - \phi/2) \) and \( \cos(\phi) = -1/(N-1) \). Hence we conclude
\[
\frac{k}{d} = \left( \frac{N-1}{2N} \right)^{\frac{1}{2}}.
\]

This equality establishes Theorem 1. \( \square \)
3. The linear potential

We apply the new bound to the case of the linear potential \( V(r) = r \). The weaker \( N/2 \) lower bound (discussed in Ref. [1]) is always available, but, up to now, we knew no way of obtaining tight bounds for this problem. For a comparison upper bound, we use a Gaussian trial function \( \Phi \) and the original Hamiltonian \( H \) to obtain a scale-optimized variational upper bound \( E \leq E_g = (\Phi, H\Phi) \). As we showed in Ref. [1], for the linear potential \( V(r) = r \) in three spatial dimensions, the conjecture (now proven) implies that the \( N \)-body bounds are given for \( N \geq 2 \) by

\[
N \left( \frac{(N-1)^3}{2N} \right)^{\frac{1}{4}} e = E_c^L \leq E \leq E_g^{LU} = 4N \left( \frac{(N-1)^3}{2N\pi^2} \right)^{\frac{1}{4}},
\]

where \( e \approx 2.2322 \) is the bottom of the spectrum [2] of the one-body problem \( h = \|p\| + r \). From (8) we see that the ratio \( R = E_g/E_c = 4/(\pi^2 e) \approx 1.011 \). The energy of the ultrarelativistic many-body system with linear pair potentials is therefore determined by these inequalities with error less than 0.55% for all \( N \geq 2 \). Earlier we were able to obtain such close bounds for all \( N \) only for the harmonic oscillator [3].

4. Conclusion

We have enlarged the number of semirelativistic problems that satisfy the lower-bound conjecture \( \langle H \rangle \geq \langle H_c \rangle \) to include all problems with \( m = 0 \) and \( N \geq 2 \). An extension of the geometric reasoning used in Ref. [1] from pyramids to more general simplices would perhaps have provided an alternative proof. However, the more algebraic approach adopted here, relying in the end on mean angles in a triangle, seemed to provide a more independent and robust approach.

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