Geodesic and Path Motion in the Nonsymmetric Gravitational Theory

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We study the problem of test-particle motion in the Nonsymmetric Gravitational Theory (NGT) assuming the four-velocity of the particle is parallel-transported along the trajectory. The predicted motion is studied on a static, spherically symmetric background field, with particular attention paid to radial and circular motions. Interestingly, it is found that the proper time taken to travel between any two non-zero radial positions is finite. It is also found that circular orbits can be supported at lower radii than in General Relativity for certain forms of motion.

We present three interactions which could be used as alternate methods for coupling a test-particle to the antisymmetric components of the NGT field. One of these takes the form of a Yukawa force in the weak-field limit of a static, spherically symmetric field, which could lead to interesting phenomenology.

I. INTRODUCTION

The problem of particle motion in gravitational theory is a long-standing and complicated one. Yet, no problem could be more important in verifying the physical validity of a theory, since we measure a gravitational field by observing its effect on particle motion.

Several studies have been written on the problem of motion in General Relativity (GR), both for test-particles and for massive bodies. Einstein originally postulated that the four-velocity of a test-particle was parallel-transported along itself during the course of the particle’s motion. He also showed that such a motion described a path of extremal length. In 1927, Einstein and Grommer showed that the field equations of GR did not allow for arbitrary motions of mass concentrations, rendering Einstein’s original postulate of geodesic motion for test-particles spurious. Beginning in 1938, several workers sought to derive the equation of particle motion from the field equations, either by considering how the field equations governed the motion of singularities in the gravitational field, or by investigating the matter-response equation, \( \nabla \cdot T = 0 \) where \( T_{\mu\nu} \) is the stress-energy tensor, and its effect on the motion of mass concentrations. In 1927, Wheeler discusses the potential problems associated with attempting to derive the equation of motion from the gravitational field equations. In particular, he points out that describing concentrations of mass-energy by singularities in the metric can render a theory incomplete. Meanwhile, approaching the problem from the point of view of the matter-response equation is troublesome, as different stress tensors yield fundamentally different equations of motion. As Wheeler points out, however, both of these methods are valid in the weak-field régime.

The Nonsymmetric Gravitational Theory (NGT) involves a nonsymmetric, second-rank tensor field \( g_{\mu\nu} \), interpreted as the gravitational potential. This tensor is referred to as the fundamental tensor. Although it is tempting to proceed in analogy with GR and associate with NGT a metric space, it is not clear \textit{a priori} how this structure is to be identified. In old NGT (mathematically equivalent to Einstein-Straus Unified Field Theory, but with a different interpretation), it was found by means of a Cauchy analysis of the field equations (see [7] and [8,9]; note that this work was performed in the framework of Einstein-Straus Unified Field Theory) that the theory contained three possibly different sets of characteristic surfaces, corresponding to three different metric structures (see (30.1)–(30.3) in [6]). It is not clear which should be chosen.

In a recent version of NGT, also known as massive or finite-ranged NGT, the field equations have been modified so as to correct asymptotic consistency problems (see [10,11,12]). In the linearized theory, this corresponds to endowing the antisymmetric sector of the gravitational field with a mass. Certain modifications to the compatibility condition have also been made (see, for instance, (10) of [12]). These changes are sufficiently significant to cast doubt on the direct applicability of the precise results of [6] and [12]. It is evident, however, that the metric structure of a generic NGT spacetime still would not be unique. Current research is exploring the possibility of generalizing the work described in [6] and [12].

Ignoring for the time being these unanswered questions, we will assume the metric structure is given by \( s_{\mu\nu} \equiv g_{(\mu\nu)} \); the generalization of most of our results to some other metric structure amounts essentially to redefining certain metric functions. We will define \( g_{\mu\nu} \) by \( g_{\mu\nu} g_{\nu\sigma} = \delta_{\mu}^{\sigma} \), and \( s_{\mu\nu} \) by \( s_{\mu\nu} s_{\nu\sigma} = \delta_{\mu}^{\sigma} \). Note that in the first of these, the order of the indices is important. We will also define \( a_{\mu\nu} \equiv g_{[\mu\nu]} \). A boldface character will be used to denote a tensor density: \( g_{\mu\nu} = \sqrt{-g} g_{\mu\nu} \).
The problem of particle motion is no simpler in NGT than in GR. Despite its considerable historical value, an analysis of the type performed by Einstein et al. in [3] is not desirable due to the interpretational difficulties associated with introducing singularities into a field theory. On the other hand, the method of Papapetrou [14] demands a priori knowledge of the stress tensor $T^\mu\nu$, or at the very least a physical interpretation of its components. In GR, the stress tensor is a symmetric tensor; this allows a simple physical picture of its components as being energy and momentum densities. In NGT, however, the stress tensor need not be symmetric. Although its symmetric components can be interpreted as in GR, it is unclear how the antisymmetric components should be interpreted. Furthermore, a discussion analogous to the debate on the uniqueness of the metric structure outlined above also applies to the stress tensor: it is not clear which form of the stress tensor ($T^\mu\nu, T_\mu\nu$ or some other combination) should be interpreted as representing the energy and momentum densities associated with the gravitational field. It will nevertheless be informative to investigate the predictions of the matter-response equation for the unambiguous case of a structureless test-particle; this will be done at the end of Section II. The methods of both Einstein [3] and Papapetrou [14] have been, to one extent or another, generalized to the case of NGT [14].

Notwithstanding the criticisms of Wheeler, we will follow Einstein and simply postulate that the four-velocity of a test-particle is parallel-transported along itself during its motion. Section II discusses some generalities about particle motion and parallel-transport; we also introduce the notation used in the remainder of this paper. In Section III, we derive the first integrals arising from the equations of motion due to symmetries in the background spacetime. In Section IV applies the results of the previous sections to the so-called geodesic and path equations for the special test-particle; this will be done at the end of Section III. The methods of both Einstein [3] and Papapetrou [4,5] have been, to one extent or another, generalized to the case of NGT [14].

II. GEODESIC AND PATH EQUATIONS

We define a connection $D$ by its effect on a set of basis vectors $e_\mu$: $D_X e_\mu = C^\beta_\mu X^\nu e_\beta$, where $C^\beta_\mu$ are the coefficients of the connection, which need not be symmetric on $\mu\nu$. We add to this the requirement that $D_\mu$ obey the Leibnitz rule, as well as commuting with contractions. By decomposing vectors onto this basis, we deduce that the effect of the connection $D$ on vectors is given in component form by: $D_\mu v^\lambda = \partial_\mu v^\lambda + C^\lambda_\rho v^\rho$. This establishes the operational use of $D$ on vectors; the generalization to covectors and tensors of higher-rank is immediate.

A vector $u$ is said to be parallel-transported with respect to a given curve $x(\tau)$ if its components at a point $P'$ on the curve are proportional to its components at another point $P$ on the curve, an infinitesimal distance away (see [5], p. 13): $D_x u = \phi(\tau) u$, where $x = dx/d\tau$ and $\phi(\tau)$ is an arbitrary function of $\tau$. Here, $\tau$ is some parameter for the curve $x(\tau)$. In particular, if we take $u = x$, we may write in component form: $u^\nu D_\nu u^\beta = \phi(\tau) u^\beta$. The term on the right-hand side can be eliminated by affinely reparametrizing the curve, leading to the generalized path equation (see also [6]):

$$u^\nu D_\nu u^\beta = \frac{du^\beta}{d\tau} + C^\beta_\mu u^\mu u^\nu = 0$$

(1)

for some affine parameter $\tau$. Being nothing more than the definition of parallel-transport of a vector with respect to a given connection $D$, the generalized path equation is fundamental in itself; it is not clear that it could be derived from any more fundamental principle, say, a variational principle. The relation of the generalized path equation to paths of extremal length will be discussed in more detail below.

We will take $u^\beta$ to represent the components of the particle four-velocity, and we will call (1) the equation of particle motion.

Our definition of the generalized path equation is essentially that of Eisenhart (see [13], p. 12). However, alternate but equivalent definitions of the generalized path equation exist in the literature. For instance, Lichnerowicz defines a path $x(\tau)$ of the connection with coefficients $C^\beta_\mu$, by:

$$\left(\frac{du^\beta}{d\tau} + C^\beta_\mu u^\mu u^\nu\right) u^\alpha - \left(\frac{du^\alpha}{d\tau} + C^\alpha_\mu u^\mu u^\nu\right) u^\beta = 0,$$

(2)

(see [7], pp. 248–249; there, the generalized path equation is referred to as a geodesic equation, regardless of the coefficients $C^\beta_\mu$,), where $u = dx/d\tau$ is the tangent vector to the path. We can recover this definition from the
generalized path equation by multiplying (1) by \( u^\alpha \) and antisymmetrizing on \( \alpha \beta \). Tonnelat (see [17], pp. 145–148) and Hlavatý (see [18], pp. 64–66) propose similar definitions. Although these are equivalent, (2) has the advantage of being independent of the parametrization of the curve \( x(\tau) \), and holds even for a non-affine parameter. For our purposes, the added benefit of being able to use a general parameter is not worth the increase in complexity associated with using a tensor equation of motion over a vector equation of motion. We will therefore use (1) instead of (2).

Before proceeding any further, we must specify the coefficients \( C^\beta_{\mu\nu} \). Although a multitude of connections can be formed in NGT, there are three that arise more commonly. The first is the symmetric Levi-Civita connection, whose coefficients are the usual Christoffel symbols:

\[
\left( \{ \frac{\partial}{\partial \tau} \}^\beta_{\mu\nu} \right) = \frac{1}{2} \delta^{\beta\lambda} (\partial_\nu s_{\mu\lambda} + \partial_\mu s_{\lambda\nu} - \partial_\lambda s_{\mu\nu}).
\]  

(3)

Being compatible with \( s_{\mu\nu} \), it is an appealing candidate for describing test-particle motion, as it leaves the magnitude of vectors, and hence the four-velocity, unchanged under parallel transport. The second is the unconstrained W-connection whose torsion vector \( W_\mu \) appears in the NGT action [11–13]. The third is the \( \Gamma \)-connection, whose torsion vector is constrained to vanish. These last two connections are not independent, but are related by

\[
\Gamma^\beta_{\mu\nu} = W^\beta_{\mu\nu} - \frac{2}{3} \delta^\beta_\mu W_\nu,
\]

(4)

which ensures that the torsion vector \( \Gamma_\nu = 0 \). It can be shown (see [15], pp. 12–13 and pp. 30–31) that the relation (4) is sufficient to guarantee that (1) will predict the same motion, regardless of which of these two connections is used. This can be seen immediately by introducing (4) into (2). The problem is therefore reduced to studying motion predicted by a Levi-Civita connection and that predicted by a \( \Gamma \)-connection.

Note that for a non-Levi-Civita connection, the solutions of (1) are not paths of extremal length \([19]\). By contracting \( \Gamma^\beta_{\mu\nu} \) and \( \beta^\nu \) and collecting terms, we arrive at

\[
\frac{1}{2} \left( \frac{d}{d\tau} \frac{\partial L}{\partial u^\alpha} - \frac{\partial L}{\partial x^\alpha} \right) = s_{\alpha\beta} u^\mu u^\nu \left( \{ \frac{\partial}{\partial \tau} \}^\beta_{\mu\nu} \right) - C^\beta_{\mu\nu},
\]

(5)

where we have introduced \( L = s_{\mu\nu} u^\mu u^\nu \) as the magnitude of the vector \( u \). We recognize the left-hand side of (5) as the Euler-Lagrange equation for the Lagrangian \( L \), the very Lagrangian used when extremizing the length of a curve whose tangent vector is \( u \). We conclude that if the right-hand side of (1) vanishes, which is by no means assured, then the length of the path described by (1) will satisfy the Euler-Lagrange equation and the path will have extremal length. This occurs trivially when the connection appearing in (1) is the Levi-Civita connection. Other connections must be examined on a case-by-case basis.

To be specific, we will refer to the case of (1) with the Levi-Civita connection as the geodesic equation; when using the \( \Gamma \)-connection, we will refer to this as the path equation.

### III. Conservation Laws

We will now show that the generalized path equation contains first integrals of the motion that reflect a symmetry in the background geometry, allowing us to simplify the description of the motion. Interestingly, we will see that the first integrals are not always explicitly guaranteed to exist, leading us to question whether a conserved quantity is necessarily associated with every symmetry in the background geometry.

Consider \( X_\beta u^\beta \), where \( X_\beta = s_{\beta\alpha} X^\alpha \) and where \( u^\beta \) satisfies (1); \( X_\beta \) are the components of an arbitrary vector. Take

\[
\frac{d(X_\beta u^\beta)}{d\tau} = u^\beta u^\nu D_\nu X_\beta + X_\beta u^\nu D_\nu u^\beta = u^\beta u^\nu D_\beta X_\beta;
\]

by direct calculation,

\[
D_\beta X_\mu = \frac{1}{2} \mathcal{L}_X [s]_{\mu\nu} + X_\beta \left( \{ \frac{\partial}{\partial \tau} \}^\beta_{\mu\nu} \right) - C^\beta_{\mu\nu},
\]

where \( \mathcal{L}_X [s]_{\mu\nu} = X^\alpha \partial_\alpha s_{\mu\nu} + s_{\mu\alpha} \partial_\nu X^\alpha + s_{\nu\alpha} \partial_\mu X^\alpha \) is the \( \mu\nu \)-component of the Lie derivative of the symmetric part of the fundamental tensor along a flow generated by \( X \). Therefore,

\[
\frac{d(X_\beta u^\beta)}{d\tau} = \frac{1}{2} \mathcal{L}_X [s]_{\mu\nu} u^\mu u^\nu + X_\beta u^\mu u^\nu \left( \{ \frac{\partial}{\partial \tau} \}^\beta_{\mu\nu} \right) - C^\beta_{\mu\nu}.
\]

(6)
If $X$ is a Killing vector, then $\mathcal{L}_X [s] = 0$ and (3) reduces to
\[
\frac{d(X_\beta u^\beta)}{dt} = X_\beta u^\mu u^\nu \left( \gamma_{\mu\nu} - C_{\mu\nu}^\beta \right).
\] (7)

If the connection coefficients $C_{\mu\nu}^\beta$ are taken to be Christoffel symbols, we see that (7) is sufficient to guarantee the existence of a conserved quantity. Two notable examples are $X = \partial/\partial t$ and $X = \partial/\partial \phi$, both Killing vectors of a static, spherically symmetric system. Writing $s_{tt} = \gamma$ and $s_{t\phi} = \beta \sin^2 \theta$, we conclude from (7) that $s_{t\beta} u^\beta = \gamma t \equiv E$ and $s_{\phi\beta} u^\beta = \beta \phi \sin^2 \theta \equiv J$ are conserved quantities, corresponding physically to the conservation of energy per unit mass and angular momentum per unit mass.

The situation is radically different when the $C_{\mu\nu}^\beta$ are taken to be some connection coefficients other than the Christoffel symbols. In particular, take $C_{\mu\nu}^\beta = \Gamma_{\mu\nu}^\beta$. In this case, we see from (7) that the fact that $X$ may be a Killing vector is not sufficient to guarantee the existence of a conserved momentum; we must further require that the relevant connection coefficients be equal to the corresponding Christoffel symbols. A case in point is the Killing vector $X = \partial/\partial t$ of a static, spherically symmetric system. From Appendix A, we see that the only relevant connection coefficient, $\Gamma_{(tt)}^\beta$, is indeed equal to the corresponding Christoffel symbol. It therefore follows that $s_{t\beta} u^\beta = \gamma t \equiv E$ is a conserved quantity, again corresponding physically to the conservation of energy per unit mass.

In contrast, consider the Killing vector $X = \partial/\partial \phi$ of this same system. An inspection of the connection coefficients listed in Appendix A will show that these are not equal to the corresponding Christoffel symbols. It follows that $s_{\phi\beta} u^\beta = r^2 \phi \sin^2 \theta$ is not conserved, despite the postulated spherical symmetry of the situation. This might lead one to believe that angular momentum would therefore not be conserved; in fact, it will be shown in the next section that there exists a conserved quantity whose asymptotic value is the conventional angular momentum.

Let $X^\beta = u^\beta$ in (3), giving
\[
\frac{d(s_{\alpha\beta} u^\alpha u^\beta)}{dt} = 2s_{\alpha\beta} u^\mu u^\nu u^\sigma \left( \gamma_{\mu\nu} - C_{\mu\nu}^\beta \right) = -u^\mu u^\nu u^\sigma D_\alpha s_{\mu\nu}.
\] (8)

In cases where the right-hand side of (8) vanishes, this yields another constant of the motion: the magnitude of the velocity. This will obviously be true for the geodesic equation, as the Levi-Civita connection is by definition compatible with the symmetric part of the fundamental tensor. In other situations, we are faced with the possibility that the magnitude of the velocity of a particle will not be conserved. Certain special cases arise when the chosen connection is not that of Levi-Civita, yet the right-hand side of (8) still vanishes. This is the case when considering a static, spherically symmetric background field. As the asymptotic properties of such a field in NGT force $W_\mu = 0$ [13,20], the right-hand side of (8) is seen to vanish, allowing the magnitude of the velocity to be conserved. However, it must be emphasized that this result is true only for a static, spherically symmetric field; non-Levi-Civita motion in a more generic NGT field would most likely leave the magnitude of the velocity unconserved.

No discussion of conservation laws for particle motion would be complete without mentioning the matter-response equation, itself a conservation law. In NGT, the matter-response equation can be written [12]
\[
\partial_\mu t^\mu_\lambda - \frac{1}{2} \partial_\lambda g_{\mu\nu} T^{\mu\nu} = 0,
\] (9)
where
\[
t^\mu_\lambda = \frac{1}{2} (g_{\mu\lambda} T^{\rho\rho} + g_{\lambda\rho} T^{\rho\mu}).
\] (10)

Given the nonsymmetric nature of both $g_{\mu\nu}$ and $T^{\mu\nu}$, it is not obvious how (10) can be inverted to give $T^{\mu\nu}$ in terms of $t^\mu_\lambda$. It will be sufficient for our needs to note that the relationship between $t^\mu_\lambda$ and $T^{\mu\nu}$ is independent of the properties of $T^{\mu\nu}$, and thus does not involve any acceleration terms.

If we assume the simplest case of a monopole test-particle, we have
\[
\int T^{\mu\nu} d^3x \neq 0
\] (11a)
\[
\int (x^\alpha - X^\alpha) T^{\mu\nu} d^3x = 0,
\] (11b)
and similarly for higher-order moments, the integration being carried out over a hypersurface of constant $t$, as per the definition of Papapetrou (see [3], p. 154). Here, $X^\alpha$ is the position of the monopole, while $x^\alpha$ is the position of the
observer. We take \( x^0 = X^0 = t \) and parametrize the motion using \( t \). It is a simple matter to show that \([11]\) imply similar multipole relationships for \( t_\nu^\mu \).

Using a method similar in spirit to that of Papapetrou (see \([3]\), pp. 152–157), we can show that \([11]\) leads to

\[
\frac{d}{dt} \left( \frac{dX^\beta}{dt} \int T^{00} d^3x \right) + \{ \beta \}_{\mu\nu} \frac{dX^\mu}{dt} \frac{dX^\nu}{dt} = \frac{1}{2} s^{\beta\lambda} \left( \partial_\lambda g_{\mu\nu} \int T^{\mu\nu} d^3x - \partial_\lambda s_{\mu\nu} \frac{dX^\mu}{dt} \frac{dX^\nu}{dt} \right) \int T^{00} d^3x, \tag{12}
\]

where we define \( \tau^{\mu\nu} \) by \( s^{\lambda\nu} t_\nu^\lambda \). The first term on the right-hand side of \([12]\) still contains \( T^{\mu\nu} \); however, given the relationship between \( T^{\mu\nu} \) and \( \tau^{\mu\nu} \), we can assert that this term does not contain the acceleration of the monopole, and that this acceleration is contained only in the first term on the left-hand side.

If \( T^{\mu\nu} \) is taken to be symmetric, we can show that \( \tau^{\mu\nu} = T^{\mu\nu} \); this is the motivation for the definition of \( \tau^{\mu\nu} \). It then follows that

\[
\int T^{\mu\nu} d^3x = \frac{dX^\mu}{dt} \frac{dX^\nu}{dt} \int T^{00} d^3x,
\]

which causes the right-hand side of \([12]\) to vanish. In such a scenario, we find that \([12]\) reduces to

\[
\frac{d}{ds} \left( m \frac{dX^\beta}{ds} \right) + m \{ \beta \}_{\mu\nu} \frac{dX^\mu}{ds} \frac{dX^\nu}{ds} = 0,
\]

where we have suggestively introduced

\[
m = \frac{ds}{dt} \int T^{00} d^3x.
\]

Thus \([12]\) reduces to the geodesic equation when \( T^{[\mu\nu]} \) vanishes. However, note that we have not required that \( g_{[\mu\nu]} \) vanish: we are still working within the realm of NGT.

Of course, \( T^{[\mu\nu]} = 0 \) is a special case; indeed, \([12]\) would seem to indicate that even a simple monopole particle couples directly to both \( T^{[\mu\nu]} \) and \( g_{[\mu\nu]} \), and in NGT there is certainly no need to force such a contribution to vanish.

The preceding arguments would certainly seem to lend credence to the choice of the geodesic equation over the path equation as the correct description of motion in NGT; at the very least, \([12]\) is a strong argument for considering a geodesic equation with a supplementary coupling term on its right-hand side. However, the path equation has appeared sufficiently often in the NGT literature (see, for example, \([10]\) and \([19]\), and the references cited therein) to warrant the investigation of its physical predictions. In the remainder of this work, we will describe in parallel some of the properties of the geodesic and path equations that we have uncovered. In Section \([\text{VII}]\) below, we will return to the question outlined above: What happens if \( T^{[\mu\nu]} \neq 0? \)

### IV. MOTION IN A STATIC, SPHERICALLY SYMMETRIC FIELD

This section presents the equations and first integrals of \([11]\) in the background geometry of the static, spherically symmetric solution described in Appendix \([4]\). These results will be used in the forthcoming sections to study radial and circular motions. We consider first the Levi-Civita connection, followed by the \( \Gamma \)-connection.

We have already found in Section \([\text{II}]\) that geodesic motion in a static, spherically symmetric field yields as constants of the motion the energy per unit mass \( E = \gamma t \), and the angular momentum per unit mass \( J = \beta \phi \sin^2 \theta \), as well as leaving \( \kappa^2 = s_{\alpha\beta} u^\alpha u^\beta \) unchanged. The \( \theta \)-component of the geodesic equation,

\[
\frac{d^2 \theta}{d\tau^2} + \frac{\beta'}{\beta} \frac{d\theta}{d\tau} - \sin \theta \cos \theta \left( \frac{d\phi}{d\tau} \right)^2 = 0,
\]

can be satisfied identically by choosing \( \theta(\tau_0) = \pi/2 \) and \( \dot{\theta}(\tau_0) = 0 \) for some initial proper time \( \tau_0 \), corresponding to planar orbits. Using this, the \( r \)-component of the geodesic equation becomes

\[
\frac{d^2 r}{d\tau^2} + \frac{\alpha'}{2\alpha} \left( \frac{dr}{d\tau} \right)^2 - \frac{\beta' j^2}{2\beta^2 \alpha} + \frac{\gamma' E^2}{2\alpha \gamma^2} = 0. \tag{13}
\]

Although this equation is identical in form to the corresponding equation in GR, the predicted motion is different owing to the different functional content of the metric functions.
Given these constants, the conservation of $\kappa^2$ may be written

$$-\kappa^2 = -s_{\alpha\beta}u^\alpha u^\beta = \alpha \left( \frac{dr}{d\tau} \right)^2 - \frac{E^2}{\gamma} + \frac{J^2}{\beta^2 + f^2}. \tag{14}$$

Results similar to those listed above can also be derived for the path equation, the only difficulty arising from the slightly more complex nature of the connection coefficients.

The $t$-component of the path equation was integrated in Section II, and found to lead to a constant of the motion: $E = \gamma \dot{t}$. This constant can be interpreted physically by examining its value asymptotically in the flat-space region, where it is found to correspond to the particle energy per unit mass.

We have also discussed in Section III how, barring unusual circumstances, the path equation will in general not leave the magnitude of the velocity, $\kappa^2$, unchanged. However, in certain special cases, of which the static, spherically symmetric field is one, a conserved $\kappa^2$ can be extracted from the path equation. This is by virtue of the form of (8), whose rightmost member precludes any contribution from the antisymmetric piece of $D_\alpha s_{\mu\nu}$. It can be shown that, in the static, spherically symmetric solution to the NGT field equations, the only contribution coming from this covariant derivative is indeed antisymmetric on its indices. We therefore have a conserved $\kappa^2$: $\kappa^2 = s_{\alpha\beta}u^\alpha u^\beta$.

As was the case for the geodesic equation, the $\theta$-component of the path equation,

$$\frac{d^2 \theta}{d\tau^2} + \frac{\beta \beta' + f f' dr}{\beta^2 + f^2} \frac{d\theta}{d\tau} - \sin \theta \cos \theta \left( \frac{d\phi}{d\tau} \right)^2 = 0,$$

can be satisfied identically by choosing $\theta(\tau_0) = \pi/2$ and $\dot{\theta}(\tau_0) = 0$, so that orbits will lie in a plane. Here, $f$ is defined by $f \sin \theta = g_{\theta\phi}$. The $\phi$-component of the path equation then leads to

$$\frac{d^2 \phi}{d\tau^2} + \frac{\beta \beta' + f f' dr}{(\beta^2 + f^2)} \frac{d\phi}{d\tau} = \frac{d\ln(\sqrt{\beta^2 + f^2} \phi/J)}{d\tau} = 0,$$

where $J$ is a constant having dimensions of a length. Solving for $J$, we have

$$J = \sqrt{\beta^2 + f^2} \frac{d\phi}{d\tau}.$$

The physical significance of $J$ is found by taking the limit $r \rightarrow \infty$. From (A.3), we see that $f \rightarrow 0$ exponentially, so that $J$ behaves asymptotically as $r \beta \dot{\phi}$, which we identify with the conventional angular momentum (per unit mass). We therefore conclude that it is $J = \sqrt{\beta^2 + f^2} \beta \dot{\phi}$ and not $\beta \dot{\phi}$ which represents the axial component of the angular momentum of the path equation. Near the origin, where $f$ is not small with respect to $\beta$, the value of $J$ for a given $\phi$ will be very different from the conventional angular momentum.

Having determined the expression for the constant $J$, we can use this to give an expression for $\kappa^2$:

$$-\kappa^2 = -s_{\alpha\beta}u^\alpha u^\beta = \alpha \left( \frac{dr}{d\tau} \right)^2 - \frac{E^2}{\gamma} + \frac{J^2}{\beta^2 + f^2}. \tag{15}$$

Use of these constants of the motion allows us to simplify the $r$-component of the path equation:

$$\frac{d^2 r}{d\tau^2} + \frac{\alpha'}{2\alpha} \left( \frac{dr}{d\tau} \right)^2 + \frac{\gamma' E^2}{2\alpha \gamma^2} - \frac{J^2 \beta' (\beta^2 - f^2) + 2 f f' \beta}{2 \alpha (\beta^2 + f^2)^2} = 0. \tag{16}$$

We can combine path and geodesic motion by introducing the function $\psi(r)$, given by $\psi = \psi_g = 1/\beta$ for the geodesic equation, and $\psi = \psi_p = \beta/(\beta^2 + f^2)$ for the path equation. We then have

$$\frac{d^2 r}{d\tau^2} + \frac{\alpha'}{2\alpha} \left( \frac{dr}{d\tau} \right)^2 + \frac{\gamma' E^2}{2\alpha \gamma^2} - \frac{J^2 \psi'}{2 \alpha} = 0 \tag{17}$$

instead of (13) and (14), and

$$-\kappa^2 = \alpha \left( \frac{dr}{d\tau} \right)^2 - \frac{E^2}{\gamma} + J^2 \psi \tag{18}$$

instead of (14) and (14).
We can get a simple comparison of the geodesic and path equations by first expressing (17) in terms of \( r(t) \):

\[
\frac{d^2 r}{dt^2} + \frac{\alpha'}{2\alpha} \left( \frac{dr}{dt} \right)^2 + \frac{\gamma'}{2\alpha} - \frac{\beta}{2\alpha} \frac{d\ln(\psi)}{dr} \left( \frac{d\phi}{dt} \right)^2 = 0.
\]

The initial conditions for this differential equation are \( t = t_0, r = r(t_0), \dot{r}_0 = \dot{r}(t_0), \) and \( \dot{\phi}_0 = \dot{\phi}(t_0) \), where a dot represents differentiation with respect to \( t \); note that we do not need to specify \( \phi_0 = \phi(t_0) \) as it does not appear in the differential equation. From the constancy of \( J \) and \( E \), we have that

\[
\dot{\phi}^2 = \frac{\psi/\psi_0}{\beta/\beta_0} \dot{\phi}_0^2,
\]

where we have adopted the notation that a subscript “0” on a quantity indicates that the quantity is to be evaluated using the initial data. It follows that the initial acceleration of the particle is

\[
\left. \frac{d^2 r}{dt^2} \right|_0 = -\frac{\alpha_0 \dot{r}_0^2}{2\alpha_0} - \frac{\gamma_0'}{2\alpha_0} + \frac{\beta_0 \dot{\phi}_0^2}{2\alpha_0} \frac{d\ln(\psi)}{dr} \left|_0 \right.
\]

The rightmost term distinguishes the geodesic and path equations. It is always true that \( \ln(\psi_g/\psi_p) \geq 0 \), and for all \( r \geq M \) we have \( \ln(M^2\psi_g) \leq 0 \); therefore \( \ln(M^2\psi_g) \leq \ln(M^2\psi_p) \leq 0 \). In the large \( r \) limit, \( \ln(\psi_p) \approx \ln(\psi_g) \). If we assume that \( f(r) \) does not behave pathologically anywhere between \( r = M \) and \( r \to +\infty \), it then follows that

\[
\frac{d\ln(\psi_g)}{dr} \leq \frac{d\ln(\psi_p)}{dr},
\]

with \( d\ln(\psi_p)/dr \leq 0 \). This allows us to conclude that initially, given identical initial conditions, a particle obeying the path equation would feel a greater inward radial acceleration than a particle obeying the geodesic equation. We could imagine a thought experiment where two particles, one obeying the path equation and the other the geodesic equation, are weakly deflected by some gravitational source. Given our previous analysis, we would expect that the first particle, in obeying the path equation, would be deflected through a larger angle than the second particle. In fact, although such a calculation would take us too far off course, an interesting method to determine experimentally whether an NGT test-particle obeys the path equation or the geodesic equation would be to set up just such a scattering experiment (see, for instance, \cite{2}, pp. 105–119). Given certain values for the parameters, one might be able to determine a bound on the scattering angle with sufficient precision to distinguish between the geodesic and path equations.

We will now use the results of this section to study the properties of radial and circular test-particle motion in a static, spherically-symmetric NGT background.

V. RADIAL MOTION

It is interesting to note that the geodesic and path equations differ only in their angular content: if \( J = 0 \), they predict the same motion. This can be seen in both (17) and (18). It follows that a study of the radial motion of a particle is, in fact, a generic result, independent of which connection was used to study the motion.

A particularly convenient form of the equation of motion is found by introducing reduced variables into (18):

\[
\gamma\alpha \left( \frac{dr}{d\tau} \right)^2 + \gamma(1 + \tilde{J}^2\psi) = \tilde{K} + \tilde{V} = \tilde{E}^2,
\]

where \( \tilde{E} = E/\kappa, \tilde{J} = J/\kappa \) and \( \tilde{\tau} = \kappa \tau \). \( \tilde{K} = \gamma\alpha\left( dr/d\tilde{\tau} \right)^2 \) and \( \tilde{V} = \gamma(1 + \tilde{J}^2\psi) \) are the reduced kinetic and potential energies, respectively.

From (19), we can compute the acceleration of a radially infalling particle (\( \tilde{J} = 0 \)):

\[
\frac{d}{d\tilde{\tau}} \left( \frac{dr}{d\tilde{\tau}} \right)^2 = 2\frac{dr}{d\tilde{\tau}} \frac{d^2 r}{d\tilde{\tau}^2} = \frac{\tilde{E}^2 d(\alpha\gamma)}{\alpha\gamma} \frac{dr}{d\tilde{\tau}} + \frac{1}{\alpha^2} \frac{d\alpha}{dr} \frac{dr}{d\tilde{\tau}}.
\]

Dividing through by \( dr/d\tilde{\tau} \) gives
\[ \frac{d^2 r}{d\tau^2} \bigg|_{\text{in}} = \frac{1}{2\alpha^2} \left( \frac{d\alpha}{dr} - \frac{E^2}{\gamma^2} \frac{d(\alpha\gamma)}{dr} \right). \]

This expression is best evaluated in the \( \nu \)-coordinate system (the \( \nu \)-coordinate system is described in Appendix A). It is found that

\[ \frac{d^2 \nu}{d\tau^2} \bigg|_{\text{in}} = \frac{1}{2\xi(\nu)} \left[ \frac{2(a \sinh(\alpha\nu) + b \sin(b\nu))}{\cosh(\alpha\nu) - \cos(b\nu)} (E^2 - e^\nu) - e^\nu \right]. \tag{20} \]

where

\[ \xi(\nu) = \frac{M^2(1 + s^2)}{(\cosh(\alpha\nu) - \cos(b\nu))^2}. \]

Since \( \nu \leq 0 \), we see that the particle is always attracted toward the larger negative values of \( \nu \).

The acceleration of a static particle is more easily found from (16) or (17). Dividing this through by \( \kappa^2 \) and setting \( dr/d\tau = 0 \) and \( \dot{J} = 0 \) gives

\[ \frac{d^2 r}{d\tau^2} \bigg|_{\text{st}} = -\frac{\gamma' \dot{E}^2}{2\alpha^2 \gamma^2}. \tag{21} \]

Again, the \( \nu \)-coordinate system is more useful for evaluating this expression, giving

\[ \frac{d^2 \nu}{d\tau^2} \bigg|_{\text{st}} = -\frac{\dot{E}^2}{2\xi(\nu)}. \tag{22} \]

To work out the turning points, we consider (13) in the \( \nu \)-coordinate system; we find that there is only one turning point, at

\[ \nu_t = \ln(\dot{E}^2) = \ln(\xi(\nu_t) \dot{\nu}_t^2 + e^{\nu_t}), \tag{23} \]

where \( \nu_t \) is the initial radial position and \( \dot{\nu}_t \) is the initial radial velocity. In order to ensure that the particle velocity is real, we must have \( \nu \leq \nu_t \). However, since there is only one turning point, once the particle begins to head towards \( \nu \to -\infty \), it continues to do so, barring the application of some other force. It is therefore informative to compute the proper time required for the particle to reach some \( \nu_f \) after starting from \( \nu_t \). This is found by inverting (14) in the \( \nu \)-coordinate system:

\[ \tau = (M^2(1 + s^2))^{1/2} \int_{\nu_t}^{\nu_f} \frac{d\nu}{(\cosh(\alpha\nu) - \cos(b\nu))(E^2 - e^\nu)^{1/2}}. \]

We consider the case \( \nu_f < \nu_t \leq 0 \). Bounding \( E^2 - e^\nu \) by \( E^2 - e^\nu = \xi(\nu_t) \dot{\nu}_t^2 \) and \( \cos(b\nu) \) by 1, we have that

\[ \tau \leq \left( \frac{M^2(1 + s^2)}{a^2 \xi(\nu_t) \dot{\nu}_t^2} \right)^{1/2} (\coth(\nu_t/2) - \coth(\nu_f/2)). \tag{24} \]

This is a finite number for all \( \nu_t \) and \( \nu_f \), with the exception \( \nu_t \to 0 \), where \( \tau \to +\infty \). We see that a particle can travel between any two non-zero \( \nu \) values in a finite amount of proper time.

There is a further complication associated with the physical interpretation of the results we have arrived at: it can be seen from (A4) that when \( \nu \) goes from \( \nu > \nu_0 \) to \( \nu < \nu_0 \), the metric undergoes a signature change, with \( \beta(\nu) \) going from being positive to being negative. At the point \( \nu = \nu_0 \), the metric is necessarily degenerate. However we have found that a particle, regardless of whether it obeys the geodesic equation or the path equation, proceeds through the point \( \nu = \nu_0 \) and beyond without hindrance, despite the fact that at \( \nu = \nu_0 \), it would appear that the fundamental assumption of metric-non-degeneracy, so crucial to the postulates of the theory, is no longer valid.

Although these results are reminiscent of the Schwarzschild singularity in GR (see [22], p. 663), it should be kept in mind that we are dealing here with a vacuum solution. It has yet to be established whether the Wyman solution is actually the final state of some generic stellar collapse problem (see [23]).
VI. CIRCULAR MOTION

It is interesting to compare the predictions of the geodesic and path equations in the special case of circular motion as this displays the most striking difference between these two forms of motion: their ability or inability to support circular orbits for certain values of the parameters.

In order to explore the existence of circular orbits, we minimize $\tilde{V}$ with respect to $r$:

$$0 = \frac{d\tilde{V}}{dr} = \frac{d\gamma}{dr} + J^2 \frac{d(\gamma \psi)}{dr}.$$  

Solving for $\tilde{J}$ gives

$$\tilde{J}^2 = -\frac{\frac{d\gamma}{dr}}{\frac{d(\gamma \psi)}{dr}}. \tag{25}$$

The location of the minimum of $\tilde{V}$ is the radius of a circular orbit for a particular value of $\tilde{J}$. Typically in GR, the approach is to solve (24) for $r$ as a function of $\tilde{J}$. Analysis of this equation shows that there is a minimum allowable value of the angular momentum, $\tilde{J}_0$. However, this approach is not practical in NGT, as the resulting equations are seemingly intractable. In turns out to be more profitable to turn the problem around and to view (24) as giving the minimum associated with having $\tilde{J}$ necessary to establish an orbit of a given radius. It is also useful to work in the $\nu$-coordinate system, as this allows us to investigate the region near $r = M$. We can use (A3) to convert $\nu$ values to conventional radial values.

$\tilde{J}$ is a one-to-many function of $\nu$. In particular, in the region $\nu > \nu_0$, $\tilde{J}$ is a one-to-two function of $\nu$. There must therefore be a local extremum, which turns out to be a minimum. This local minimum will correspond to the smallest allowable angular momentum. We find this local minimum by solving

$$\frac{d\tilde{J}^2}{d\nu} = -\frac{d}{d\nu} \left( \frac{d\gamma}{dr} \right) = 0 \tag{26}$$

for $\nu = \nu_m$, and then evaluating $r_m^2 = \beta(\nu_m)$ and $\tilde{J}^2(\nu_m) = \tilde{J}_m^2$. In GR, the resulting values (in conventional spherical coordinates) are $r_m = 6M$ and $\tilde{J}_m^2 = 12M^2$ (see [22], p. 662). Unfortunately, (24) cannot be solved in closed form when $\gamma(\nu)$ and $\psi(\nu)$ take on their NGT form (A2). However, in formulating the problem in this fashion, it becomes straightforward to solve (24) numerically for $\nu_m$. As an example, we take $s = 0.9$ and $\mu M = 0$. Typically, a more reasonable choice might be to take $\mu M \sim 10^{-9}$ or so; however, since we are concerned with values of $\nu$ near the origin, the problems generated in NGT by having $\mu M = 0$ are not relevant here (see [10] for a discussion of the problems associated with having $\mu M = 0$), and taking $\mu M = 0$ simply represents an extreme case. The value $s = 0.9$ is chosen so as to emphasize the observed behaviour. Numerically, it is found that for the geodesic equation, $\nu_m \approx -0.40602$, $r_m \approx 5.9930 M$, and $\tilde{J}_m^2 \approx 11.995 M^2$, while for the path equation, $\nu_m \approx -0.40511$, $r_m \approx 6.0040 M$, and $\tilde{J}_m^2 \approx 12.003 M^2$. Generically, the geodesic equation yields smaller values than in GR, while the path equation yields values larger than the corresponding GR values.

The advantage of using the $\nu$-coordinate system is that the statements of the previous paragraphs are not tied down to some expansion, valid only in a limited region of spacetime; results calculated using the $\nu$-coordinate system can be considered exact and valid for all $\nu$ in the case of infinite-ranged NGT, or approximately true for finite-ranged NGT, in those regions where $\mu r \ll 1$ and when $\mu M$ can be considered small. In particular, this is the case near the origin, where $r \sim M \ll 1/\mu$, and we may conclude that the path equation does not support circular orbits lying below $r = 2M$. On the other hand, circular orbits for the geodesic equation can be made to approach the origin arbitrarily closely by selecting successively larger values of the parameter $s$.

Since the exact Wyman solution is not a solution to the field equations of massive NGT, one may question the wisdom of using it to investigate particle motion in NGT, not to mention the preceding treatment of circular orbits. Indeed, if applicable at all, the Wyman solution can only be used near the origin, where we would expect the range $1/\mu$ of the antisymmetric components of the field to have little effect. The study of circular orbits provides an excellent opportunity to show that using the Wyman solution gives qualitatively similar results to using an asymptotic solution to the field equations. In fact, if we repeat the preceding analysis of circular orbits in the path equation, but use (A3) instead of the Wyman solution, we find that the minimum circular orbit for $s = 0.9$ occurs at $r_m \approx 6.0033 M$, with $\tilde{J}_m^2 \approx 12.003 M^2$. The geodesic equation cannot be similarly verified, as its minimum circular orbits lie below $r = 2M$, where (A3) can no longer be trusted.
VII. BEHAVIOUR IN THE ASYMPTOTIC REGIONS OF SPACETIME

By studying the geodesic and path equations in a static, spherically symmetric field, we were able to see in the previous two sections that there are differences in the resulting motion in the strong-field régime, in particular for the case of circular motion. In this brief section, we will demonstrate explicitly for the case of a static, spherically symmetric background field that the difference between the geodesic and path equations in the weak-field régime is of higher-order, and can safely be neglected.

From (A3), we know that \( f(r) < r^2 \) for \( r > M \). This behaviour is emphasized at larger \( r \), as \( f(r) \) is damped exponentially. Given this behaviour, we will consider the difference between the path equation and the geodesic equation by expanding in powers of \( \varepsilon = f(r)/r^2 \); we will show that this difference must be neglected in order to be consistent with our other approximations.

Take (1) with \( C_{\mu\nu}^\beta = \Gamma^\beta_{\mu\nu} \), and rewrite this as

\[
\frac{dw^\beta}{d\tau} + \{\mu\nu\} u^\mu u^\nu = \left( \{\beta\} - \Gamma^\beta_{\mu\nu} \right) u^\mu u^\nu
\]

by adding and subtracting a Christoffel symbol.

Recall that the \( t- \) and \( \theta- \) components had the same behaviour in both the geodesic and path equations. We need only consider the \( \phi- \) and \( r- \) components. Since the \( \phi- \) component is immediately integrated, its expansion is trivial:

\[
\frac{d\phi}{d\tau} \approx \frac{J^2}{r^4} \left( 1 - \frac{f^2}{r^4} \right).
\]

where \( J \) represents the value of the angular momentum in both the geodesic and path equations. Expanding the right-hand side of the radial component of (27) to lowest order in \( f(r)/r^2 \), we arrive at

\[
\frac{d^2r}{d\tau^2} + \frac{\alpha'}{2\alpha} \left( \frac{dr}{d\tau} \right)^2 + \frac{\alpha' E^2}{2\alpha \gamma^2} - \frac{J^2}{\alpha r^3} = \left( \frac{f f'}{r^6} - \frac{3 f^2}{r^7} \right) \frac{J^2}{\alpha}.
\]

The left-hand side of this is nothing more than the geodesic equation. The right-hand side is of order \( \varepsilon^2 \); approximating \( f(r) \) and \( f'(r) \) by their maximum values of \( f(r) \sim sM^2/3 \) and \( f'(r) \sim -s\mu M^2/3 \), we could conclude that, for large \( r \), the difference between the path equation and the geodesic equation is of order \( 1/r^6 \). Adding to this the exponentially-damped behaviour of \( f(r) \), we could conclude that for any \( r > 2M \) where (A3) can be trusted, the geodesic and path equations will yield identical results. Much stronger than this, however, is the fact that the corrections to \( \gamma(r) \) and \( \alpha(r) \) in (A3) are themselves of order \( \varepsilon^2 \) (see (4.6) in [13] and the discussion preceding it). Thus, if we accept (A3) as valid approximations to some exact solution of the NGT field equations, in order to be consistent with this approximation, we must drop the right-hand side of (28).

This motivates the use of the geodesic equation in weak-field regions of spacetime.

VIII. MODIFYING GEODESIC MOTION

In the previous section, it was shown that path motion converges very rapidly to geodesic motion. However, by its very nature geodesic motion does not couple the particle directly to the antisymmetric components of the NGT field. Moreover, we saw at the end of Section III that even a monopole test-particle must, in the most general case, couple to \( a_{\mu\nu} \). In this section, we present three possible couplings. For simplicity, these are linear in the particle velocity.

Since we are dealing in this section only with the weak-field regions of spacetime, we will consider modifying only the geodesic equation; from the previous section, we know that the path equation will yield similar results in these regions.

The couplings will be derived from a scalar Lagrangian \( L \), which we take to have the form \( L = (1/2) A_{\mu} u^\mu \), where \( A \) is a covector independent of the particle velocity. The Euler-Lagrange equation of particle mechanics allows us to conclude that the contribution of such a Lagrangian to the equation of motion will be of the form

\[
\frac{d}{d\tau} \frac{\partial L}{\partial u^\alpha} - \frac{\partial L}{\partial x^\alpha} = - f_{[\alpha\mu]} u^\mu,
\]

where \( f_{[\alpha\mu]} = \partial_{[\alpha} A_{\mu]} \).

Let \( L = L_1 + L_2 + L_3 \), where
\[ L_1 = \frac{1}{2} \lambda_1 F^\alpha u_\alpha = \frac{\lambda_1}{2} \epsilon^{\mu \nu \lambda} F_{[\mu \nu \lambda]} s^\eta u_\eta \]  
(29a)

\[ L_2 = \frac{1}{2} \lambda_2 g^{[\mu \nu]} F_{[\mu \nu \lambda]} u_\lambda \]  
(29b)

\[ L_3 = \frac{\lambda_3}{2} \frac{\partial_\alpha g^{[\lambda \nu]}}{\sqrt{-g}} s_{\lambda \mu} u^\mu. \]  
(29c)

The components of the field-strength tensor \( F_{[\mu \nu \lambda]} \) are defined by

\[ F_{[\mu \nu \lambda]} = \partial_\lambda g_{\mu \nu} = \frac{1}{3} (\partial_\lambda a_{\mu \nu} + \partial_\mu a_{\nu \lambda} + \partial_\nu a_{\lambda \mu}). \]  
(30)

The constants \( \lambda_i \) \( (i = 1, 2, 3) \) couple the test-particle to the NGT skew field, and have dimensions of a length. The symbol \( \epsilon^{\mu \nu \lambda \eta} \) is the fully antisymmetric Levi-Civita tensor density, defined by

\[ \epsilon^{\mu \nu \lambda \eta} = \begin{cases} +1 & \text{if } \mu \nu \lambda \eta \text{ is an even permutation of 0123,} \\ -1 & \text{if } \mu \nu \lambda \eta \text{ is an odd permutation of 0123,} \\ 0 & \text{otherwise.} \end{cases} \]

It will be found below that only \( L_1 \) will generate any interaction at all in a static, spherically symmetric field. This does not, of course, exclude the possibility that \( L_2 \) and \( L_3 \) could more accurately reflect the desired coupling between the test-particle and the antisymmetric components of the NGT field. It does, however, make their contribution trivial in our particular case of interest. Although for the sake of completeness we will derive the additional terms in the equation of motion for the case when all three interactions are present, we will in the end ignore \( L_2 \) and \( L_3 \), knowing that they will in no way affect our results.

Using the Euler-Lagrange equation, we find that

\[ f_{[\alpha \mu]} = \lambda_1 \partial_\alpha \left( \frac{\epsilon^{[\eta \nu \lambda \eta]} F_{[\sigma \nu \lambda \eta]} s_{\lambda \mu}}{\sqrt{-g}} \right) + \lambda_2 \partial_\alpha \left( g^{[\mu \nu]} F_{[\nu \mu \lambda]} \right) + \lambda_3 \partial_\alpha \left( \frac{\partial_\gamma g^{[\nu \rho]}}{\sqrt{-g}} s_{\nu \mu} \right), \]  
(31)

such that the equation of motion is now written

\[ \frac{d u^\beta}{d \tau} + \{ \beta \}_\mu u^\mu = \kappa^2 s_{\mu \nu} f_{[\alpha \mu]} u^\alpha, \]  
(32)

where \( \kappa^2 = s_{\mu \nu} u^\mu u^\nu \) is the magnitude of the velocity.

As we will only be interested in the behaviour of (32) in a static, spherically symmetric field, it will be of no interest to re-derive the conservation laws discussed in Section III. Suffice it to say that the techniques of that section can easily be generalized, and the first integrals are found to be slightly different. It will be simpler for us to derive the constants of the motion “by hand” below. It is straightforward to show that \( \kappa^2 \) is conserved. The appearance of the magnitude of the velocity in the equation of motion in this fashion has an important implication: for a massless particle (such as a photon), \( \kappa^2 = 0 \) and the right-hand side of (32) is seen to vanish. Therefore, in this scheme, massless particles will not couple to the antisymmetric field.

The case \( \kappa^2 = 0 \) corresponds to pure geodesic motion, which was discussed in previous sections. We will therefore concentrate on the case \( \kappa^2 \neq 0 \). Since \( \kappa^2 \) is conserved, we can scale the proper time \( \tau \) in such a way that \( \kappa^2 = 1 \); henceforth, we assume that this has been done.

In the static, spherically symmetric field, the skew field-strength tensor \( F_{[\mu \nu \lambda]} \) has only one independent, non-zero component,

\[ F_{[\theta \phi]} = \frac{1}{3} \partial_\tau a_{\theta \phi} = \frac{1}{3} f' \sin \theta. \]

On the other hand, because of the chosen form of the fundamental tensor, \( \partial_\phi g^{[\lambda \nu]} \) can be shown to vanish. From (31), it follows that the tensor \( f_{[\alpha \sigma]} \) also has one independent component:

\[ f_{[rt]} = \frac{d}{d r} \left( \frac{\lambda_1 \gamma f'}{\sqrt{\alpha \gamma (r^4 + f^2)}} \right). \]

For convenience, we will write \( \lambda_1 = \lambda \), since the \( \lambda_2 \) and \( \lambda_3 \) terms in (31) make no contribution to the equation of motion.
orbits lie in a plane. Meanwhile, we conclude from (33d) that 
\( J \) corresponds to regions of spacetime where 
\( \frac{\lambda}{r} < \frac{\dot{r}}{r} \), for some initial proper time \( \tau_0 \). It follows that these orbits lie in a plane. Moreover, we find from (33d) that \( J = r^2 \dot{\phi} \) is a constant of the motion. This might have been predicted from the fact that \( s^\alpha_s f_{\alpha\mu} u^\mu \), the right-hand side of (32), does not have a \( \phi \)-component, and as was seen in Section 111, \( J \) is conserved for geodesic motion.

Contrary to the situation for geodesic and path motion, \( s_{i\beta} u^\beta = \gamma_i \) is not conserved. In fact, we see from (33a) that the actual first integral of the motion is

\[
E = \gamma \frac{dt}{d\tau} + \frac{\lambda \gamma f'}{\sqrt{\alpha \gamma (r^4 + f^2)}}. \tag{34}
\]

That this represents the energy per unit mass can be seen by taking the large \( r \) limit and using (A3).

With these results, we can rewrite (33a) as

\[
0 = \frac{d^2r}{dt^2} + \frac{\alpha'}{2\alpha} \left( \frac{dr}{dt} \right)^2 - \frac{j^2}{r^2 \alpha} + \frac{\lambda \gamma f'}{\sqrt{\alpha \gamma (r^4 + f^2)}} \frac{d}{dr} \left( \frac{\gamma f'}{\sqrt{\alpha \gamma (r^4 + f^2)}} \right) + \frac{\gamma'}{2\alpha \gamma^2} \left( E - \frac{\lambda \gamma f'}{\sqrt{\alpha \gamma (r^4 + f^2)}} \right)^2. \tag{35}
\]

There are two calculations to perform. First, we calculate the corrections to the Newtonian gravitational force coming from the supplementary forcing term. Secondly, we will determine the equation for the orbit of a particle acted upon by this extra force. In order to extract a useful result from this, we will make certain simplifying assumptions. In particular, we will assume that \( f(r)/r^2 \ll 1 \) and \( \lambda f'(r)/r^2 \ll 1 \). If \( s \ll 1 \) and \( \mu M \ll 1 \), this corresponds to regions of spacetime where \( M/r \ll 1 \).

Consider first the Newtonian gravitational force. We rewrite (32) in terms of \( r(t) \):

\[
\frac{d^2r}{dt^2} + \frac{\alpha'}{2\alpha} \left( \frac{dr}{dt} \right)^2 - \frac{J_N^2}{r^3 \alpha} + \frac{\gamma'}{2\alpha} = - \frac{\lambda \gamma f'}{\sqrt{\alpha \gamma (r^4 + f^2)}} \frac{d}{dr} \left( \frac{\gamma f'}{\sqrt{\alpha \gamma (r^4 + f^2)}} \right) - \frac{\lambda \gamma f'}{\sqrt{\alpha \gamma (r^4 + f^2)}} \left( E - \frac{\lambda \gamma f'}{\sqrt{\alpha \gamma (r^4 + f^2)}} \right)^2. \tag{36}
\]

Here, \( J_N \equiv r^2 \dot{\phi}/dt \) is the Newtonian value of the angular momentum per unit mass. We recognize the left-hand side as the usual GR contributions to the equation of motion.

To our order of approximation,

\[
\frac{\lambda \gamma}{\alpha} \left( E - \frac{\lambda \gamma f'}{\sqrt{\alpha \gamma (r^4 + f^2)}} \right)^{-1} \frac{d}{dr} \left( \frac{\gamma f'}{\sqrt{\alpha \gamma (r^4 + f^2)}} \right) \approx - \frac{\lambda}{E} d \left( \frac{\gamma f'}{\sqrt{\alpha \gamma (r^4 + f^2)}} \right) \approx \frac{\lambda s M^2 \mu^2}{3E} e^{-\mu r} \frac{1 + \mu r}{r^2}. \tag{37}
\]

Therefore, the radial equation of motion may be written

\[
\frac{d^2r}{dt^2} - \frac{J_N^2}{r^3} = \frac{M}{r^2} - \frac{\lambda s M^2 \mu^2}{3E} e^{-\mu r} \frac{1 + \mu r}{r^2}, \tag{38}
\]

where we have assumed that the particle is moving slowly, so that \( dr/dt \ll 1 \). If \( \lambda < 0 \), this yields a repulsive Yukawa force in (38), while \( \lambda > 0 \) renders this force attractive.

The Yukawa behaviour generated by this modification of geodesic motion has interesting phenomenological implications. Take, for instance, the case of very large \( r \). We find in this case that the right-hand side of (38) reduces to
As to whether unlike the electromagnetic charge, it does not follow from any sort of flux integral. The question immediately arises the strength of the coupling between the test-particle and the NGT field; it could be thought of as a charge, although it would seem more sensible to choose the latter route and take since there is no compelling evidence to suggest that the weak equivalence principle is actually violated in reality [27], the violations of the weak equivalence principle that this material-dependence will cause (see [26], p. 13). However, we have shown that in the weak-field limit, both of these equations lead to virtually identical equations may differentiate between the two. However, despite the differences between these two equations of motion, we would expect that the weak equivalence principle is actually violated in reality [27], it would seem more sensible to choose the latter route and take \( \lambda \) to be some material-independent constant.

Thus far, the treatment has been entirely general. We can specialize this to the case where \( f(r)/r^2 \ll 1 \) and \( \lambda f'(r)/r^2 \ll 1 \). Taking the Schwarzschild forms \( \gamma = 1/\alpha = 1 - 2M/r \) and using \( [A3] \), we arrive at the orbit equation:

\[
\frac{d^2 u}{d\phi^2} + u = \frac{M}{J^2} \left( 1 + \frac{E\lambda s M \mu^2}{12 (1 + \mu r) e^{-\mu r}} \right) + 3Mu^2.
\]

(38)

This equation is similar to the orbit equation from GR (see [22], p. 186), the only difference being the factor multiplying the angular momentum term on the right-hand side.

A comment should be made about the role played by the quantity \( \lambda \) in these equations. In a sense, \( \lambda \) represents the strength of the coupling between the test-particle and the NGT field; it could be thought of as a charge, although unlike the electromagnetic charge, it does not follow from any sort of flux integral. The question immediately arises as to whether \( \lambda \) is a property of the test-particle or some universal constant. If the former, then we must deal with the violations of the weak equivalence principle that this material-dependence will cause (see [22], p. 13). However, since there is no compelling evidence to suggest that the weak equivalence principle is actually violated in reality [27], it would seem more sensible to choose the latter route and take \( \lambda \) to be some material-independent constant.

**IX. CONCLUSIONS**

We have described observational differences between the geodesic and path equations, in the hope that experiments may differentiate between the two. However, despite the differences between these two equations of motion, we have found that under certain circumstances, for example the case of radial motion, they predict identical results. Furthermore, we have shown that in the weak-field limit, both of these equations lead to virtually identical equations of motion, owing to the fact that NGT is constructed so as to recover GR in the weak-field limit.

Many of our results are implicitly linked to the case of a static, spherically symmetric field. This is a question of practicality: as it stands, this is the only available solution to the field equations. However, we would expect that the qualitative effects that we have described would be generic. In particular, as pertains to constants of the motion and the like, the geodesic equation of NGT behaves very much like its counterpart in GR. This is to be expected: the geodesic equation of NGT is constructed so as to couple the particle only indirectly to the antisymmetric field. On the other hand, it was found that the path equation had different constants of the motion. In particular, we found that the conventional angular momentum, \( J = \beta \dot{\phi} \sin^2 \theta \), was not a first integral of the motion. Rather, a different first integral was derived, and was found to be a sort of “generalized” angular momentum. Although we have no proof to support this, the work we have performed thus far would lead us to expect that this generalization of constants of the motion would be a generic feature, owing to the right-hand side of (7): whenever we have come across a case where (7) did not yield an explicit constant of the motion, the resulting equation could nonetheless be integrated to yield a first integral.
Particularly interesting behaviour was found by studying radial motion in a static, spherically symmetric field. Firstly, it was found that an infalling particle need not stop at the origin \( r = 0 \). This was seen using the \( \nu \)-coordinate system, where it was found that a particle can pass through \( \nu_0 \) and proceed toward \( \nu < \nu_0 \). Furthermore, the particle could reach any radius \( \nu_f \) from any non-zero radius \( \nu_i \) in a finite proper time. An upper bound was placed on the proper time taken to travel this distance. It is unclear at this time exactly what is implied by this result: the Wyman solution to the infinite-ranged NGT field equations is a vacuum solution, devoid of any matter contributions. As the gravitational collapse problem in NGT remains unsolved, it is not obvious that the vacuum Wyman solution, as opposed to, say, the Schwarzschild solution, is in fact the final state of a stellar collapse. Regardless, the comparison with the corresponding behaviour in GR is striking, and should be kept in mind.

The major difference between the geodesic and path equations was found by studying circular motion. Here, it was shown that circular orbits for the geodesic equation in NGT extend to lower radii than in GR, while the last circular orbit for the path equation lies outside its GR counterpart. This behaviour was found to be strengthened for larger values of the parameter \( s \) appearing in the static, spherically symmetric field.

After demonstrating that the geodesic and path equations have similar weak-field limits, we introduced three interactions which could serve as alternate methods for coupling a test-particle to the antisymmetric components of the NGT field. Of these three, only one was found to generate any interaction at all in a static, spherically symmetric field. The correction to the Newtonian gravitational force acting on the particle was worked out in the weak-field limit, the NGT field. The correction to the orbit of a particle was also calculated; the lowest-order NGT correction was shown to be a \( 1/r^3 \) term.

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**APPENDIX A: THE STATIC, SPHERICALLY SYMMETRIC SOLUTION IN NGT**

We list here the salient features and results of the static, spherically symmetric solution to the new NGT field equations. The coordinate system used is \( x^0 = t, \ x^1 = r, \ x^2 = \theta, \) and \( x^3 = \phi \), where \( r, \ \theta, \) and \( \phi \) are the usual spherical coordinates.

The NGT field equations contain a parameter \( 1/\mu \) which is identified with the range of the skew components of the fundamental tensor. The static, spherically symmetric solution introduces two constants of integration: \( M \) and \( s \). \( M \) plays the same role as the Schwarzschild mass in GR, while \( s \) is a dimensionless constant. Beyond being a measure of the strength of the antisymmetric contributions to the gravitational field, the physical meaning of \( s \) is still unclear. To be specific, we will assume that \( M \ll 1/\mu \).

The fundamental tensor is written

\[
g_{\mu \nu} = \begin{bmatrix}
\gamma(r) & w(r) & 0 & 0 \\
-w(r) & -\alpha(r) & 0 & 0 \\
0 & 0 & -\beta(r) & f(r) \sin \theta \\
0 & 0 & -f(r) \sin \theta & -\beta \sin^2 \theta
\end{bmatrix}.
\] (A1)

In [13] and [20], it is shown that the only solution which yields an asymptotically Minkowskian space has \( w(r) = 0 \).

In the case of infinite-ranged NGT, corresponding to setting \( \mu = 0 \) in the field equations, an exact solution to the field equations exists:

\[
\gamma(r) = e^{\nu} \] (A2a)
\[
\alpha(r) = \frac{M^2 e^{-\nu}(1+s^2)}{(\cosh(\alpha \nu) - \cos(\beta \nu))^2} \left( \frac{dv}{dr} \right)^2 \] (A2b)
\[
\beta(r) = r^2 \] (A2c)
\[
f(r) = \frac{2M^2 (\sinh(\alpha \nu) \sin(\beta \nu) + s(1-\cosh(\alpha \nu) \cos(\beta \nu)))}{e^{\nu}(\cosh(\alpha \nu) - \cos(\beta \nu))^2}.\] (A2d)
where

\[ a = \sqrt{\frac{1 + s^2 + 1}{2}} \quad \text{and} \quad b = \sqrt{\frac{1 + s^2 - 1}{2}}. \]

The function \( \nu(r) \) is determined from the relation

\[ e^\nu (\cosh(\nu r) - \cos(\nu r))^2 \frac{r^2}{2M^2} = \cosh(\nu r) \cos(\nu r) - 1 + s \sinh(\nu r) \sin(\nu r). \] (A3)

This exact solution is referred to as the Wyman solution [28,29], and holds for all \( s \).

There are two coordinate systems of interest in the Wyman solution. The first is the conventional set of spherical coordinates used above. However, (A3) can be viewed as defining a transformation from a different coordinate system, the \( \nu \)-coordinates, to conventional spherical coordinates [30]. The \( \nu \)-coordinates are interesting, as (A3) need not be inverted; the coordinate system is \( x^0 = t, x^1 = \nu, x^2 = \theta \) and \( x^3 = \phi \), where

\begin{align*}
\alpha(\nu) &= \frac{M^2 e^{-\nu (1 + s^2)} \left( \cosh(\nu r) - \cos(\nu r) \right)}{2M^2}, \\
\beta(\nu) &= \frac{2M^2 (\cosh(\nu r) \cos(\nu r) - 1 + s \sinh(\nu r) \sin(\nu r))}{e^\nu (\cosh(\nu r) - \cos(\nu r))^2}, \quad (A4)
\end{align*}

and with \( \gamma(\nu) \) and \( f(\nu) \) given as above. Since (A3) gives \( r \) as a many-to-one function of \( \nu \), a particular branch of the solution must be picked. This selection is done by picking the branch that will yield the positive-mass Schwarzschild solution as a limit. This branch begins at \( \nu = 0 \) and extends toward negative \( \nu \). In such a coordinate system, the asymptotic, weak-field region is at \( \nu = 0 \), while the “origin” occurs at \( \nu_0 \) defined by \( \beta(\nu_0) = 0 \). The particular value of \( \nu_0 \) depends on the value of \( s \); for \( s = 1 \), we find numerically that \( \nu_0 \approx -5.1667 \).

When \( f(r)/r^2 \ll 1 \), a solution to the linearized field equations is approximated by [20]

\begin{align*}
\gamma(r) &\approx 1 - 2M \frac{r}{r} \quad \text{(A5a)} \\
\alpha(r) &\approx \left( 1 - \frac{2M}{r} \right)^{-1} \quad \text{(A5b)} \\
f(r) &\approx \frac{sM^2}{3} e^{-\mu r} (1 + \mu r); \quad \text{(A5c)}
\end{align*}

the coordinates are the conventional spherical coordinates. In [13], a similar result is derived. The two forms can be seen to be asymptotically equivalent, notwithstanding a change in notation, by taking \( \mu M \ll 1 \) in (2.12) of [13] and taking \( \mu r \gg 1 \) in the above result.

For both conventional spherical coordinates and the \( \nu \)-coordinates, the Christoffel symbols are given by:

\begin{align*}
\{ t_x \} &= \gamma' \frac{2}{\gamma} \\
\{ x_x \} &= \gamma' \frac{2}{\alpha} \\
\{ x_{xx} \} &= \alpha' \frac{2}{\alpha} \\
\{ x_{\phi\phi} \} &= \sin^2 \theta \{ x_{\theta\theta} \} = -\beta' \frac{\sin^2 \theta}{2\alpha} \\
\{ \theta_{\theta} \} &= \{ \phi_{\phi} \} = \beta' \frac{2}{\beta} \\
\{ \theta_{\phi} \} &= -\sin \theta \cos \theta \\
\{ \phi_{\theta} \} &= \cos \theta \frac{1}{\sin \theta}. \quad \text{(A6h)}
\end{align*}
where \( x \) represents either \( r \) or \( \nu \). A prime denotes differentiation with respect to \( x \).

The \( \Gamma \)-connection coefficients, \( \Gamma^\beta_{\mu\nu} \), are determined by solving the NGT compatibility condition:

\[
\partial_\eta g_{\lambda\xi} - g_{\xi\eta} \Gamma^\rho_{\lambda\eta} - g_{\lambda\rho} \Gamma^\eta_{\eta\rho} = \frac{1}{6} g^{(\mu\rho)} (g_{\rho\xi} g_{\lambda\eta} - g_{\xi\eta} g_{\lambda\rho} - g_{\lambda\xi} g_{\rho\eta}) W_\mu,
\]

(A7)

where \( W_\mu \) is determined from

\[
\partial_\rho g_{[\sigma\rho]} = -\frac{1}{2} g^{(\rho\eta)} W_\eta.
\]

When \( w(r) = 0 \), it can be shown that \( W_\mu = 0 \). The right-hand side of (A7) therefore vanishes. For both conventional spherical coordinates and the \( \nu \)-coordinates, the nonvanishing components of the \( \Gamma \)-connection are:

\[
\begin{align*}
\Gamma^t_{(tx)} &= \gamma' \frac{2}{2\gamma}, \\
\Gamma^x_{(tt)} &= \frac{\gamma'}{2\alpha}, \\
\Gamma^x_{(xx)} &= \alpha' \frac{2}{2\alpha}, \\
\Gamma^x_{(\phi\phi)} &= \sin^2 \theta \frac{2}{2\alpha}, \\
\Gamma^\theta_{(x\theta)} &= \frac{\beta \beta' + ff'}{2(\beta^2 + f^2)}, \\
\Gamma^\phi_{(x\phi)} &= -\sin \theta \cos \theta, \\
\Gamma^\phi_{(\theta\phi)} &= \sin \theta \cos \theta, \\
\Gamma^x_{[\theta\phi]} &= \frac{f' f^2 + 2f \beta' f}{2(\beta^2 + f^2)} \sin \theta, \\
\Gamma^\theta_{[x\phi]} &= -\sin^2 \theta \Gamma^\phi_{[x\theta]} \frac{2(\beta^2 + f^2)}{2(\beta^2 + f^2)}.
\end{align*}
\]

(A8a) (A8b) (A8c) (A8d) (A8e) (A8f) (A8g) (A8h) (A8i)

Again, \( x \) denotes either \( r \) or \( \nu \), and a prime denotes differentiation with respect to \( x \).

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