AUTOMORPHISMS OF CENTRAL EXTENSIONS OF TYPE I VON NEUMANN ALGEBRAS

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Abstract. Given a von Neumann algebra $M$ we consider the central extension $E(M)$ of $M$. For type I von Neumann algebras $E(M)$ coincides with the algebra $LS(M)$ of all locally measurable operators affiliated with $M$. In this case we show that an arbitrary automorphism $T$ of $E(M)$ can be decomposed as $T = T_a \circ T_\phi$, where $T_a(x) = axa^{-1}$ is an inner automorphism implemented by an element $a \in E(M)$, and $T_\phi$ is a special automorphism generated by an automorphism $\phi$ of the center of $E(M)$. In particular if $M$ is of type $I_\infty$ then every band preserving automorphism of $E(M)$ is inner.

1. Introduction

In the series of paper [1]-[3] we have considered derivations on the algebra $LS(M)$ of locally measurable operators affiliated with a von Neumann algebra $M$, and on various subalgebras of $LS(M)$. A complete description of derivations has been obtained in the case of von Neumann algebras of type I and III.

A comprehensive survey of recent results concerning derivations on various algebras of unbounded operators affiliated with von Neumann algebras is presented in [4].

It is well-known that properties of derivations on algebras are strongly correlated with properties of automorphisms of underlying algebras (see e.g. [8]). Algebraic automorphisms of $C^*$-algebras and von Neumann algebras were considered in the paper of R. Kadison and J. Ringrose [9], which is devoted to automatic continuity and innerness of automorphisms. By this paper we initiate a study of automorphisms of the algebra $LS(M)$ and its various subalgebras. In the commutative case a similar problem has been considered by A.G. Kusraev [12] who proved by means of Boolean-valued analysis the existence of non trivial band preserving automorphism on algebras of the form $L^0(\Omega, \Sigma, \mu)$. The algebra $LS(M)$ and its subalgebras present a non commutative counterparts of the algebra $L^0(\Omega, \Sigma, \mu)$. In the present paper we establish a general form of automorphisms of the algebra $LS(M)$ for type I von Neumann algebras $M$.

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Let $\mathcal{A}$ be an algebra. A one-to-one linear operator $T : \mathcal{A} \rightarrow \mathcal{A}$ is called an automorphism if $T(xy) = T(x)T(y)$ for all $x, y \in \mathcal{A}$. Given an invertible element $a \in \mathcal{A}$ one can define an automorphism $T_a$ of $\mathcal{A}$ by $T_a(x) = axa^{-1}$, $x \in \mathcal{A}$. Such automorphisms are called inner automorphisms of $\mathcal{A}$. It is clear that for commutative (abelian) algebra $\mathcal{A}$ all inner automorphisms are trivial, i.e. acts as unit operator. In the general case inner automorphisms are identical on the center of $\mathcal{A}$. Essentially different classes of automorphisms are those which are generated by automorphisms of the center $Z(\mathcal{A})$ of $\mathcal{A}$. In some cases such automorphisms $\phi$ on $Z(\mathcal{A})$ can be extended to automorphisms $T_\phi$ of the whole algebra $\mathcal{A}$ (see e.g. Kaplansky [10, Theorem 1]). The main result of the present paper shows that for a type I von Neumann algebra $M$ every automorphism $T$ of the algebra $LS(M)$ can be uniquely decomposed as a composition $T = T_a \circ T_\phi$ of an inner automorphism $T_a$ and an automorphism $T_\phi$ generated by an automorphism $\phi$ of the center of $LS(M)$.

In section 2 we recall the notions of the algebras $S(M)$ of measurable operators and $LS(M)$ of locally measurable operators affiliated with a von Neumann algebra $M$. We also introduce the so-called central extension $E(M)$ of the von Neumann algebra $M$. In the general case $E(M)$ is a *-subalgebra of $LS(M)$, which coincides with $LS(M)$ if and only if $M$ does not have direct summands of type II. We also introduce two generalizations of the topology of convergence locally in measure on $LS(M)$ and prove that for the type I case they coincide.

In section 3 we consider automorphisms of the algebra $E(M)$ – the central extension of a von Neumann algebra $M$. We prove (Theorem 3.10) that if $M$ is of the type I then each automorphism $T$ of $E(M)$ which acts identically on the center $Z(E(M))$ of $E(M)$, is inner. We also show that for homogeneous type I von Neumann algebras $M$ every automorphism $\phi$ of the center $Z(E(M))$ of $E(M)$ can be extended to an automorphism $T_\phi$ of the whole $E(M)$. Finally we prove the main result of the present paper which shows that each automorphism $T$ of $E(M)$ for a type I von Neumann algebra $M$ can be uniquely represented as $T = T_a \circ T_\phi$, where $T_a$ is an inner automorphism implemented by an element $a \in E(M)$, and $T_\phi$ is an automorphism generated by an automorphism $\phi$ of the center of $E(M)$. In particular we obtain that each bundle preserving automorphism of $E(M)$ is inner if $M$ is of type $I_\infty$.

2. CENTRAL EXTENSIONS OF VON NEUMANN ALGEBRAS

In this section we give some necessary definitions and a preliminary information concerning algebras of measurable and locally measurable operators affiliated with a von Neumann algebra. We also introduce the notion of the central extension of a von Neumann algebra.
Let $H$ be a complex Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on $H$. Consider a von Neumann algebra $M$ in $B(H)$ with the operator norm $\| \cdot \|_M$. Denote by $P(M)$ the lattice of projections in $M$.

A linear subspace $\mathcal{D}$ in $H$ is said to be affiliated with $M$ (denoted as $x\eta M$), if $u(\mathcal{D}) \subset \mathcal{D}$ for every unitary $u$ from the commutant

$$M' = \{ y \in B(H) : xy = yx, \forall x \in M \}$$

of the von Neumann algebra $M$.

A linear operator $x$ on $H$ with the domain $\mathcal{D}(x)$ is said to be affiliated with $M$ (denoted as $x\eta M$) if $\mathcal{D}(x)\eta M$ and $u(x(\xi)) = x(u(\xi))$ for all $\xi \in \mathcal{D}(x)$.

A linear subspace $\mathcal{D}$ in $H$ is said to be strongly dense in $H$ with respect to the von Neumann algebra $M$, if

1) $\mathcal{D}\eta M$;
2) there exists a sequence of projections $\{p_n\}_{n=1}^\infty$ in $P(M)$ such that $p_n \uparrow 1$, $p_n(H) \subset \mathcal{D}$ and $p_n^\perp = 1 - p_n$ is finite in $M$ for all $n \in \mathbb{N}$, where $1$ is the identity in $M$.

A closed linear operator $x$ acting in the Hilbert space $H$ is said to be measurable with respect to the von Neumann algebra $M$, if $x\eta M$ and $\mathcal{D}(x)$ is strongly dense in $H$. Denote by $S(M)$ the set of all measurable operators with respect to $M$ (see [14]).

A closed linear operator $x$ in $H$ is said to be locally measurable with respect to the von Neumann algebra $M$, if $x\eta M$ and there exists a sequence $\{z_n\}_{n=1}^\infty$ of central projections in $M$ such that $z_n \uparrow 1$ and $z_n x \in S(M)$ for all $n \in \mathbb{N}$ (see [15]).

It is well-known [5], [15] that the set $LS(M)$ of all locally measurable operators with respect to $M$ is a unital *-algebra when equipped with the algebraic operations of strong addition and multiplication and taking the adjoint of an operator, and contains $S(M)$ as a solid *-subalgebra.

Let $(\Omega, \Sigma, \mu)$ be a measure space and from now on suppose that the measure $\mu$ has the direct sum property, i. e. there is a family $\{\Omega_i\}_{i \in J} \subset \Sigma$, $0 < \mu(\Omega_i) < \infty$, $i \in J$, such that for any $A \in \Sigma$, $\mu(A) < \infty$, there exist a countable subset $J_0 \subset J$ and a set $B$ with zero measure such that $A = \bigcup_{i \in J_0} (A \cap \Omega_i) \cup B$.

We denote by $L^0(\Omega, \Sigma, \mu)$ the algebra of all (equivalence classes of) complex measurable functions on $(\Omega, \Sigma, \mu)$ equipped with the topology of convergence in measure.

Consider the algebra $S(Z(M))$ of operators which are measurable with respect to the center $Z(M)$ of the von Neumann algebra $M$. Since $Z(M)$ is an abelian von Neumann algebra it is *-isomorphic to $L^\infty(\Omega, \Sigma, \mu)$ for an appropriate measure space $(\Omega, \Sigma, \mu)$. Therefore the algebra $S(Z(M))$ coincides with $Z(LS(M))$ and can be identified with the algebra $L^0(\Omega, \Sigma, \mu)$ of all measurable functions on $(\Omega, \Sigma, \mu)$.
The basis of neighborhoods of zero in the topology of convergence locally in measure on $L^0(Ω, Σ, µ)$ consists of the sets

$$W(A, ε, δ) = \{ f ∈ L^0(Ω, Σ, µ) : \exists B ∈ Σ, B ⊆ A, µ(A \setminus B) ≤ δ, f · χ_B ∈ L^∞(Ω, Σ, µ), ∥f · χ_B∥_{L^∞(Ω, Σ, µ)} ≤ ε \},$$

where $ε, δ > 0$, $A ∈ Σ$, $µ(A) < +∞$, and $χ_B$ is the characteristic function of the set $B ∈ Σ$.

Recall the definition of the dimension functions on the lattice $P(M)$ of projection from $M$ (see [5], [14]).

By $L_+$ we denote the set of all measurable functions $f : (Ω, Σ, µ) → [0, ∞]$ (modulo functions equal to zero $µ$-almost everywhere).

Let $M$ be an arbitrary von Neumann algebra with the center $Z = L^∞(Ω, Σ, µ)$. Then there exists a map $D : P(M) → L_+$ with the following properties:

(i) $d(e)$ is a finite function if only if the projection $e$ is finite;
(ii) $d(e + q) = d(e) + d(q)$ for $p, q ∈ P(M)$, $eq = 0$;
(iii) $d(uu^*) = d(u^*u)$ for every partial isometry $u ∈ M$;
(iv) $d(ze) = zd(e)$ for all $z ∈ P(Z(M))$, $e ∈ P(M)$;
(v) if $\{ e_α \}_{α ∈ J}$, $e ∈ P(M)$ and $e_α ↑ e$, then

$$d(e) = \sup_{α ∈ J} d(e_α).$$

This map $d : P(M) → L_+$, is a called the dimension functions on $P(M)$.

**Remark 2.1.** Recall that for an element $x ∈ M$ the projection defined as

$$c(x) = \inf \{ z ∈ P(Z(M)) : zx = x \}
$$

is called the central cover of $x$.

Let $M$ be a type I von Neumann algebra. If $p, q ∈ P(M)$ are abelian projections with $c(p) = c(q) = 1$, then the property (iii) implies that $0 < d(p)(ω) = d(q)(ω) < ∞$ for $µ$-almost every $ω ∈ Ω$. Therefore replacing $d$ by $d(p)^{-1}d$ we can assume that $d(p) = c(p)$ for every abelian projection $p ∈ P(M)$. Thus for all $e ∈ P(M)$ we have that $d(e) ≥ c(e)$.

The basis of neighborhoods of zero in the topology $t(M)$ of convergence locally in measure on $LS(M)$ consists (in the above notations) of the following sets

$$V(A, ε, δ) = \{ x ∈ LS(M) : ∃ p ∈ P(M), ∃ z ∈ P(Z(M)), xp ∈ M, ∥xp∥_M ≤ ε, z^⊥ ∈ W(A, ε, δ), d(zp^⊥) ≤ εz \},$$

where $ε, δ > 0$, $A ∈ Σ$, $µ(A) < +∞$.

The topology $t(M)$ is metrizable if and only if the center $Z(M)$ is $σ$-finite (see [5]).

Given an arbitrary family $\{ z_i \}_{i ∈ I}$ of mutually orthogonal central projections in $M$ with $\bigvee_{i ∈ I} z_i = 1$ and a family of elements $\{ x_i \}_{i ∈ I}$ in $LS(M)$ there exists a
unique element \( x \in LS(M) \) such that \( z_i x = z_i x_i \) for all \( i \in I \). This element is denoted by \( x = \sum_{i \in I} z_i x_i \).

We denote by \( E(M) \) the set of all elements \( x \) from \( LS(M) \) for which there exists a sequence of mutually orthogonal central projections \( \{ z_i \}_{i \in I} \) in \( M \) with \( \bigvee_{i \in I} z_i = 1 \), such that \( z_i x \in M \) for all \( i \in I \), i.e.

\[
E(M) = \{ x \in LS(M) : \exists z_i \in P(Z(M)), z_i z_j = 0, i \neq j, \bigvee_{i \in I} z_i = 1, z_i x \in M, i \in I \},
\]

where \( Z(M) \) is the center of \( M \).

It is known [3] that \( E(M) \) is *-subalgebras in \( LS(M) \) with the center \( S(Z(M)) \), where \( S(Z(M)) \) is the algebra of all measurable operators with respect to \( Z(M) \), moreover, \( LS(M) = E(M) \) if and only if \( M \) does not have direct summands of type II.

A similar notion (i.e. the algebra \( E(A) \)) for arbitrary *-subalgebras \( A \subset LS(M) \) was independently introduced recently by M.A. Muratov and V.I. Chilin [6]. The algebra \( E(M) \) is called the central extension of \( M \).

It is known ([3], [6]) that an element \( x \in LS(M) \) belongs to \( E(M) \) if and only if there exists \( f \in S(Z(M)) \) such that \( |x| \leq f \). Therefore for each \( x \in E(M) \) one can define the following vector-valued norm

\[
||x|| = \inf\{ f \in S(Z(M)) : |x| \leq f \}
\]

and this norm satisfies the following conditions:

1) \( ||x|| \geq 0; ||x|| = 0 \iff x = 0; \)
2) \( ||fx|| = |f||x||; \)
3) \( ||x + y|| \leq ||x|| + ||y||; \)
4) \( |xy| \leq ||x|| ||y||; \)
5) \( ||xx^*|| = ||x||^2 \)

for all \( x, y \in E(M), f \in S(Z(M)) \).

Let us equip \( E(M) \) with the topology which is defined by the following system of zero neighborhoods:

\[
O(A, \varepsilon, \delta) = \{ x \in E(M) : ||x|| \in W(A, \varepsilon, \delta) \},
\]

where \( \varepsilon, \delta > 0, A \in \Sigma, \mu(A) < +\infty. \)

Denote the above topology by \( t_c(M) \).

**Proposition 2.2.** The topology \( t_c(M) \) is stronger that the topology \( t(M) \) of convergence locally in measure.

**Proof.** It is sufficient to show that

\[
O(A, \varepsilon, \delta) \subset V(A, \varepsilon, \delta).
\]

Let \( x \in O(A, \varepsilon, \delta) \), i.e. \( ||x|| \in W(A, \varepsilon, \delta) \). Then there exists \( B \in \Sigma \) such that \( B \subseteq A, \mu(A \setminus B) \leq \delta \).
and

$$||x||\chi_B \in L^\infty(\Omega, \Sigma, \mu), \ ||||x||\chi_B||_M \leq \varepsilon.$$ 

Put $z = p = \chi_B$. Then $||xp|| = ||x\chi_B|| = ||x||\chi_B \in L^\infty(\Omega, \Sigma, \mu)$, i.e. $xp \in M$ and moreover $||xp||_M \leq \varepsilon$. Since $\mu(A \setminus B) \leq \delta$ and $\chi^\perp_B \chi_B = \chi^\perp_B \chi_B = 0$, one has $z^\perp \in W(A, \varepsilon, \delta)$. Therefore

$$||xp||_M \leq \varepsilon, \ z^\perp \in W(A, \varepsilon, \delta), \ zp^\perp = \chi_B \chi_B^\perp = 0$$

and hence $x \in V(A, \varepsilon, \delta)$. □

**Proposition 2.3.** If $M$ is a type I von Neumann algebra and $0 < \varepsilon < 1$, then

$$O(A, \varepsilon, \delta) = V(A, \varepsilon, \delta).$$

**Proof.** From above (2.2) we have that $O(A, \varepsilon, \delta) \subset V(A, \varepsilon, \delta)$. Therefore it is sufficient to show that $V(A, \varepsilon, \delta) \subset O(A, \varepsilon, \delta)$.

Let $x \in V(A, \varepsilon, \delta)$. Then there exist $p \in P(M)$ and $z \in P(Z(M))$ such that

$$xp \in M, \ ||xp||_M \leq \varepsilon, \ z^\perp \in W(A, \varepsilon, \delta), \ d(zp^\perp) \leq \varepsilon z.$$ 

Since $M$ is of type I Remark 2.1 implies that $d(zp^\perp) \geq c(zp^\perp)$. Now from $d(zp^\perp) \leq \varepsilon z$ it follows that $c(zp^\perp) \leq \varepsilon z$. From $0 < \varepsilon < 1$ we obtain that $zp^\perp = 0$. Therefore $z \leq c(p)$, where $c(p)$ is the central cover of $p$. Thus $z = zp$. Put $z = \chi_E$ for an appropriate $E \in \Sigma$. Since $z^\perp \in W(A, \varepsilon, \delta)$ one has that $\chi_{\Omega \setminus E} \in W(A, \varepsilon, \delta)$. Thus there exists $B \in \Sigma$ such that $B \subseteq A, \ \mu(A \setminus B) \leq \delta, \ |\chi_{\Omega \setminus E} \chi_B| \leq \varepsilon < 1$. Hence $\chi_B \leq \chi_E$. So we obtain

$$||x||\chi_B \leq ||x||\chi_E = ||x||z = ||xz|| = ||xzp|| = ||xp|| \leq \varepsilon.$$ 

This means that $x \in O(A, \varepsilon, \delta)$. □

**Corollary 2.4.** If $M$ is a type I von Neumann algebra then the topologies $t(M)$ and $t_c(M)$ coincide.

**Proposition 2.5.** Let $M$ be a type I von Neumann algebra and $x \in LS(M)$, $x \geq 0$. If $pxp = 0$ for all abelian projections $p \in M$ then $x = 0$.

**Proof.** Since $x \geq 0$ we have that $x = yy^*$ for an appropriate $y \in LS(M)$. Then

$$0 = pxp = pyy^*p = py(py)^*$$

and hence $py = 0$. Therefore $y^*py = 0$ for all abelian projections $p \in M$. But since $M$ has the type I there exists a family $\{p_i\}_{i \in J}$ of mutually orthogonal abelian projections such that $\sum_{i \in J} p_i = 1$. For any finite subset $F \subseteq J$ put $p_F = \sum_{i \in F} p_i$. Since $p_F \uparrow 1$ from $yp_Fy^* = 0$ we have that $yy^* = 0$, i.e. $x = yy^* = 0$. □
3. AUTOMORPHISMS OF CENTRAL EXTENSIONS FOR TYPE I VON NEUMANN ALGEBRAS

Let \( \mathcal{A} \) be an arbitrary algebra with the center \( Z(\mathcal{A}) \) and let \( T : \mathcal{A} \to \mathcal{A} \) be an automorphism. It is clear that \( T \) maps \( Z(\mathcal{A}) \) onto itself. Indeed for all \( a \in Z(\mathcal{A}) \) and \( x \in \mathcal{A} \) one has

\[
T(a)T(x) = T(ax) = T(xa) = T(x)T(a)
\]

which means that \( T(a) \in Z(\mathcal{A}) \).

An operator \( T : \mathcal{A} \to \mathcal{A} \) is said to be \( Z(\mathcal{A}) \)-linear if \( T(ax) = aT(x) \) for all \( a \in Z(\mathcal{A}) \) and \( x \in \mathcal{A} \). It is easy to see that an automorphism \( T : \mathcal{A} \to \mathcal{A} \) of a unital algebra \( \mathcal{A} \) is \( Z(\mathcal{A}) \)-linear if and only if it is identical on the center \( Z(\mathcal{A}) \).

**Theorem 3.1.** Let \( M \) be a von Neumann algebra of type I and let \( E(M) \) be its central extension. Then each \( Z(E(M)) \)-linear automorphism \( T \) of the algebra \( E(M) \) is inner.

**Proof.** Let us show that \( T \) is \( t(M) \)-continuous. First suppose that the center \( Z(M) \) of the von Neumann algebra \( M \) is \( \sigma \)-finite. Then the topology \( t(M) \) is metrizable and hence it is sufficient to prove that the operator \( T \) is \( t(M) \)-closed.

Consider a sequence \( \{x_n\} \subset E(M) \) such that \( x_n \xrightarrow{t(M)} 0, \; T(x_n) \xrightarrow{t(M)} y \). Take \( x \in E(M) \) such that \( T(x) = y \) and let us show that \( x = 0 \). Since

\[
x^*x_n \xrightarrow{t(M)} 0
\]

and

\[
T(x^*x_n) = T(x^*)T(x_n) \xrightarrow{t(M)} T(x^*)y = T(x^*)T(x) = T(x^*x),
\]

we may suppose (by replacing the sequence \( \{x_n\} \) by the sequence \( \{x^*x_n\} \)) that \( x \geq 0 \).

Let \( p \in M \) be an arbitrary abelian projection with \( c(p) = 1 \). Then \( px_n p = a_n p \) for an appropriate \( a_n \in S(Z(M)) \), \( n \in \mathbb{N} \). Since \( x_n \xrightarrow{t(M)} 0 \) and \( c(p) = 1 \) it follows that \( a_n \xrightarrow{t(M)} 0 \). Therefore

\[
T(p)T(x_n)T(p) = T(px_n p) = T(a_n p) = a_n T(p) \xrightarrow{t(M)} 0.
\]

On the other hand

\[
T(p)T(x_n)T(p) \xrightarrow{t(M)} T(p)yT(p),
\]

thus \( T(p)yT(p) = 0 \) and hence

\[
pxp = T^{-1}(T(p)yT(p)) = T(0) = 0,
\]

i.e. \( pxp = 0 \) for all abelian projections with \( c(p) = 1 \). Therefore Proposition 2.5 implies that \( x = 0 \), i.e. \( T \) is \( t(M) \)-continuous.

Now consider the general case, i.e. when the center \( Z(M) \) is arbitrary. Take a family \( \{z_i\}_{i \in I} \) of mutually orthogonal central projections in \( M \) with \( \bigvee_i z_i = 1 \)
such that $z_i Z(M)$ is $\sigma$-finite for all $i \in I$. From the above we have that $z_i T$ is $t(z_i M)$ continuous on $z_i E(M)$ for all $i \in I$, where $(z_i T)(x) = T(z_i x) = z_i T(x)$ is the restriction of $T$ onto $z_i E(M)$ which is well-defined in view of the $Z(E(M))$-linearity of $T$. Therefore $T$ is $(M)$-continuous of whole $E(M) = \bigoplus_{i \in I} z_i E(M)$.

Further by Corollary 2.4 the topologies $t(M)$ and $t_c(M)$ coincide and hence $T$ is also $t_c(M)$-continuous and according to [16, Theorem 2] there exists $c \in S(Z(M))$ such that $\|T(x)\| \leq c\|x\|$ for all $x \in E(M)$.

Take a sequence $\{z_n\}_{n \in \mathbb{N}}$ of mutually orthogonal central projections in $M$ with $\bigvee z_n = 1$ such that $z_n c \in Z(M)$ for all $n \in \mathbb{N}$. This means that the automorphism $z_n T$ maps bounded elements from $z_n E(M)$ to bounded elements, i.e. $z_n T(z_n M) \subseteq z_n M$. Then given any $n \in \mathbb{N}$ the automorphism $z_n T|_{z_n M}$ is identical on the center of $z_n M$. By theorem of Kaplansky [11, Theorem 10] there exist elements $a_n \in z_n M$ which are invertible in $z_n M$, such that $z_n T(x) = a_n xa_n^{-1}$ for all $x \in z_n M$. Put $a = \sum_{n \geq 1} z_n a_n$. It is clear that $a \in E(M)$ and

$$T(x) = \sum_{n \geq 1} z_n T(x) = \sum_{n \geq 1} z_n T(z_n x) = \sum_{n \geq 1} a_n(z_n x)a_n = axa^{-1}$$

for all $x \in E(M)$.

Let $M$ be a von Neumann algebra of type $I_n$, $n \in \mathbb{N}$, with the center $Z(M)$. Then $M$ is $^*$-isomorphic to the algebra $M_n(Z(M))$ of all $n \times n$ matrices over $Z(M)$ (cf. [13, Theorem 2.3.3]). Moreover the algebra $S(M) = E(M)$ is $^*$-isomorphic to the algebra $M_n(Z(S(M)))$, where $Z(S(M)) = Z(M)$ is the center of $S(M)$ (see [2, Proposition 1.5]). If $e_{ij}$, $i, j = 1, \ldots, n$ are matrix units in $M_n(S(Z(M)))$ then each element $x \in M_n(S(Z(M)))$ has the form

$$x = \sum_{i,j=1}^n a_{ij} e_{ij}, a_{ij} \in S(Z(M)), \ i, j = 1, \ldots, n.$$ 

Let $\phi : S(Z(M)) \to S(Z(M))$ be an automorphism. Setting

$$T_\phi \left( \sum_{i,j=1}^n a_{ij} e_{ij} \right) = \sum_{i,j=1}^n \phi(a_{ij}) e_{ij} \quad (3.1)$$

we obtain a linear operator $T_\phi$ on $M_n(S(Z(M)))$, which is in fact an automorphism of $M_n(S(Z(M)))$. Indeed, for

$$x = \sum_{i,j=1}^n a_{ij} e_{ij}, y = \sum_{i,j=1}^n b_{ij} e_{ij}, a_{ij}, b_{ij} \in S(Z(M)), \ i, j = 1, \ldots, n$$

we have

$$T_\phi(xy) = T_\phi \left( \sum_{i,j=1}^n a_{ij} e_{ij} \sum_{k,s=1}^n b_{ks} e_{ks} \right) = T_\phi \left( \sum_{i,j,s=1}^n a_{ij}b_{js} e_{is} \right) =$$
\[
= \sum_{i,j,s=1}^{n} \phi(a_{ij}b_{js})e_{is} = \sum_{i,j,s=1}^{n} \phi(a_{ij})\phi(b_{js})e_{is} = \\
= \sum_{i,j=1}^{n} \phi(a_{ij})e_{ij} \sum_{k,s=1}^{n} \phi(b_{ks})e_{ks} = T_{\phi}(x)T_{\phi}(y),
\]

i.e. \( T_{\phi}(xy) = T_{\phi}(x)T_{\phi}(y) \).

The following property immediately follows from the definition of \( T_{\phi} \):

if \( \varphi \) and \( \phi \) are two automorphisms of \( S(Z(M)) \) then \( T_{\phi} \circ T_{\varphi} = T_{\phi_{\varphi}} \), in particular \( T_{\phi^{-1}} = T_{\phi^{-1}} \).

**Remark 3.2.** (i) If the automorphism \( \phi \) on \( S(Z(M)) \) is non trivial (i.e. not identical) then it is clear that \( T_{\phi} \) can not be an inner automorphism on \( M_n(S(Z(M))) \).

(ii) It is known [9, Lemma 1] that every (algebraic) automorphism of \( C^* \)-algebra is automatically norm continuous. But in our case this is not true in general. Suppose that the abelian algebra \( S(Z(M)) \) is represented as \( L^0(\Omega, \Sigma, \mu) \), with a continuous Boolean algebra \( \Sigma \). Then A.G. Kusraev [12, Theorem 3.4] has proved that \( S(Z(M)) \) admits a non trivial band preserving automorphism which is, in particular \( t(M) \)-discontinuous. Therefore \( T_{\phi} \) gives an example of a \( t(M) \)-discontinuous automorphism of \( E(M) \). In particular, \( T_{\phi} \) is not inner.

**Proposition 3.3.** If \( M \) is a von Neumann algebra of type \( I_\alpha \), then each automorphism \( T \) of \( E(M) \) can be uniquely represented in the form

\[
T = T_a \circ T_{\phi},
\]

where \( T_a \) is an inner automorphism implemented by an element \( a \in E(M) \), and \( T_{\phi} \) is the automorphism of the form (3.1) generated by an automorphism \( \phi \) of the center \( S(Z(M)) \).

**Proof.** Let \( \phi \) be the restriction of \( T \) onto the center \( Z(E(M)) = S(Z(M)) \). As it was mentioned earlier \( \phi \) map \( Z(E(M)) \) onto itself, i.e. \( \varphi \) is an automorphism of \( Z(E(M)) \). Consider the automorphism \( T_{\phi} \) defined by (3.1) and put \( S = T \circ T_{\phi}^{-1} \).

Since \( T \) and \( T_{\phi} \) coincide on \( Z(E(M)) \), one has that \( S \) is identical on the center \( Z(E(M)) \), i.e. \( S \) is a \( Z(E(M)) \)-linear automorphism of \( E(M) \). By Theorem 3.1 there exists an invertible element \( a \in E(M) \) such that \( S = T_a \), i.e. \( S(x) = axa^{-1} \) for all \( x \in E(M) \). Therefore \( T = S \circ T_{\phi} = T_a \circ T_{\phi} \).

Suppose that \( T = T_{a} \circ T_{\phi} = T_{b} \circ T_{\varphi} \) for \( a, b \in E(M) \) and automorphisms \( \phi \) and \( \varphi \) of \( Z(E(M)) \). Then \( T_{b^{-1}} \circ T_{a} = T_{\varphi} \circ T_{\phi}^{-1} \), i.e. \( T_{b^{-1}a} = T_{\varphi_{\phi^{-1}}} \). Since \( T_{b^{-1}a} \) is identical on the center \( Z(E(M)) \) of \( E(M) \), it follows that \( \varphi \circ \phi \) is identical on the center \( Z(E(M)) \), i.e. \( \varphi = \phi \). Therefore \( T_{\varphi} = T_{\phi} \), i.e. \( T_{b^{-1}} \circ T_{a} = Id \) and hence \( T_a = T_b \).

**Proposition 3.4.** Let \( M \) be a von Neumann algebra and let \( T : E(M) \to E(M) \) be an automorphism. If \( x \in E(M) \) and its central cover \( c(x) = 1 \) then \( c(T(x)) = 1 \).
Proposition 3.5. Let \( M \) be an abelian von Neumann algebra and let \( \phi : E(M) \to E(M) \) be a \( t(M) \)-continuous automorphism. Then \( \phi(M) \subseteq M \).

Proof. Let \( x \in M \) be a simple element, i.e.

\[
x = \sum_{i=1}^{n} \lambda_i e_i,
\]

where \( \lambda_i \in \mathbb{C}, e_i \in P(M), e_i e_j = 0, i \neq j, i, j = 1, \ldots, n \). Let us prove that \( \phi(x) \in M \) and \( ||\phi(x)||_M = ||x||_M \). Since \( M \) is abelian and \( \phi(e_i)^2 = \phi(e_i) \), it follows that \( \phi(e_i) \) is a projection for each \( i = 1, \ldots, n \). Therefore from the equality

\[
\phi(x) = \sum_{i=1}^{n} \lambda_i \phi(e_i)
\]

we obtain that \( \phi(x) \in M \) and moreover

\[
||\phi(x)||_M = \max_{1 \leq i \leq n} |\lambda_i| = ||x||_M.
\]

Let now \( x \in M \) be an arbitrary element. Consider a sequence of simple elements \( \{x_n\} \) in \( M \) which \( t(M) \)-converges to \( x \) and \( |x_n| \leq |x| \) for all \( n \in \mathbb{N} \). Then \( \phi(x_n) \xrightarrow{t(M)} \phi(x) \) and \( ||\phi(x_n)||_M = ||x_n||_M \leq ||x||_M \) for all \( n \in \mathbb{N} \). Therefore \( |\phi(x)| \leq ||x||_M 1 \), i.e. \( \phi(x) \in M \).

We are now in a position to consider automorphisms of central extensions for type \( I_\infty \) von Neumann algebras.

Proposition 3.6. Let \( M \) be a von Neumann algebra of type \( I_\infty \), and let \( T : E(M) \to E(M) \) be an automorphism of the central extension \( E(M) \) of \( M \). Then \( T \) is \( t(Z(M)) \)-continuous on \( E(Z(M)) \) and maps \( Z(M) \) onto itself.

Proof. Since \( M \) is of type \( I_\infty \), there exists a sequence of mutually orthogonal abelian projections \( \{p_n\}_{n=1}^{\infty} \) in \( M \) with central covers equal to \( 1 \). For a bounded sequence \( \{a_n\} \) from \( Z(M) \) put

\[
x = \sum_{n=1}^{\infty} a_n p_n.
\]
Then
\[ xp_n = p_nx = a_n p_n \]
for all \( n \in \mathbb{N} \).

Now let \( T \) be an automorphism of \( E(M) \) and denote by \( \phi \) its restriction onto the center of \( E(M) \). If \( q_n = T(p_n) \), \( n \in \mathbb{N} \), then we have
\[ T(xp_n) = T(x)T(p_n) = T(x)q_n \]
and
\[ T(xp_n) = T(a_n p_n) = T(a_n)T(p_n) = \phi(a_n)q_n, \]
therefore
\[ T(x)q_n = \phi(a_n)q_n. \]

For the center-valued norm \( \| \cdot \| \) on \( E(M) \) (see (2.1)) we have
\[ \| q_n \|||T(x)|| \geq \| q_n T(x) \| = \| \phi(a_n)q_n \| = |\phi(a_n)|||q_n||, \]
i.e.
\[ \| q_n \|||T(x)|| \geq |\phi(a_n)|||q_n||. \]
Since \( c(q_n) = c(p_n) = 1 \) (Proposition 3.4) the latter inequality implies that
\[ ||T(x)|| \geq |\phi(a_n)|. \tag{3.3} \]

Let us show that \( \phi \) is \( t(Z(M)) \)-continuous on \( E(Z(M)) \). If we suppose the opposite, then there exists a bounded sequence \( \{a_n\} \) in \( Z(M) \) such that \( \{\phi(a_n)\} \) is not \( t(Z(M)) \)-bounded, which contradicts (3.3). Thus \( \phi \) is \( t(Z(M)) \)-continuous and Proposition 3.5 implies that \( T \) maps \( Z(M) \) onto itself.

\[ \mathbf{□} \]

Remark 3.7. The \( t(Z(M)) \)-continuity of \( T \) on the center \( E(Z(M)) \) easily implies that the restriction of \( T \) on \( E(Z(M)) \) and hence on \( Z(M) \) is a \(*\)-automorphism (cf. [9, Lemma 1]).

Now we are going to show that similar to the case of type \( I_n \) (\( n \in \mathbb{N} \)) von Neumann algebras, automorphisms of the algebras \( E(M) \) for homogeneous type \( I_\alpha \) von Neumann algebras (\( \alpha \) is an infinite cardinal numbers) also can be represented in the form (3.2).

Suppose that \( \phi : Z(M) \to Z(M) \) is an automorphism. According to [10, Theorem 1] \( \phi \) can be extended to a \(*\)-automorphism of \( M \), which we denote by \( T_\phi \). Since each \(*\)-automorphism is an order isomorphism and each hermitian element of \( E(M) \) is an order limit of hermitian elements from \( M \), we can naturally extend \( T_\phi \) to a \(*\)-automorphism of \( E(M) \).

Theorem 3.8. If \( M \) is a type \( I_\alpha \) von Neumann algebra, where \( \alpha \) is an infinite cardinal number, then each automorphism \( T \) on \( E(M) \) can be uniquely represented as
\[ T = T_\alpha \circ T_\phi, \]
where $T_a$ is an inner automorphism implemented by an element $a \in E(M)$ and $T_\phi$ is an $^*$-automorphism, generated by an automorphism $\phi$ of the center $Z(M)$ as above.

Proof. Let $M$ be an automorphism of $E(M)$ where $M$ is a type $I_\alpha$ von Neumann algebra with the center $Z(M)$. If $\phi$ is the restriction of $T$ onto the center $S(Z(M))$ of $E(M)$, then by Proposition 3.6 $\phi$ maps $Z(M)$ onto itself. By [10, Theorem 1] as above $\phi$ can be extended to a $^*$-automorphism of $E(M)$. Now similar to the Proposition 3.3 there exists an element $a \in E(M)$ such that $T = T_a \circ T_\phi$ and this representation is unique. \hfill \Box

Proposition 3.9. Let $M$ and $N$ be von Neumann algebras of type $I$ and suppose that $M$ is homogeneous of type $I_\alpha$. If there exists an isomorphism (not necessary $^*$-isomorphism) $T$ from $E(M)$ onto $E(N)$ then $N$ is also of type $I_\alpha$.

Proof. Let $z_N$ be a central projection in $N$ such that $z_N N$ is of type $I_\beta$, where $\beta$ is a cardinal number. Take a central projection $z_M$ in $M$ such that $T(z_M) = z_N$. Replacing $M$ and $N$ by $z_M M$ and $z_N N$ respectively we may assume that $z_M = 1_M$, $z_N = 1_N$.

Let $\{p_i\}_{i \in I}$ (respectively $\{e_j\}_{j \in J}$) be a family of mutually equivalent and orthogonal abelian projections in $M$ (respectively in $N$) with $\bigvee_{i \in I} p_i = 1_M$, (respectively $\bigvee_{j \in J} e_j = 1_N$,) where $|I| = \alpha$, $|J| = \beta$. It is clear that $c(p_i) = 1_M$ for all $i \in I$.

Then $q_i = T(p_i)$ is an idempotent ($q_i^2 = q_i$) but not a projection in general. Let $f_i = s_t(q_i)$ be the left projection of the idempotent $q_i$. Since $f_i$ is the projection onto the range of the idempotent $q_i$ we have that $q_i f_i = f_i$, i.e. $f_i q_i f_i = f_i$, and moreover $c(f_i) = 1_N$, because $c(q_i) = 1_N$ (see Proposition 3.4). The equalities

$$q_i E(N) q_i = T(p_i E(M) p_i) = T(Z(E(M)) p_i) = E(Z(N)) q_i,$$

imply that for each $x \in E(N)$ there exists $a_x \in E(Z(N))$ such that $q_i x q_i = a_x q_i$.

Now we show that $f_i$ is an abelian projection. For $x \in E(N)$ and each $f_i$ there exist $a_i \in E(Z(N))$ such that

$$q_i f_i x f_i q_i = a_i q_i.$$

Thus

$$f_i x f_i = (f_i q_i f_i) x (f_i q_i f_i) = f_i (q_i x f_i q_i) f_i = f_i a_i q_i f_i = a_i f_i q_i f_i = a_i f_i,$$

i.e. $f_i E(N) f_i = E(Z(N)) f_i$. This means that $f_i$ is an abelian projection.

Case 1. $\alpha$ and $\beta$ are finite. Let $\Phi$ be a normed center-valued trace on $N$. Then $1_N = \Phi(1_N) = \sum_{i \in I} \Phi(q_i) = \alpha \Phi(q_1) = \alpha \Phi(f_1 q_1) = \alpha \Phi(f_1 q_1 f_1) = \alpha \Phi(f_1)$.

Since $N$ is of type $I_\beta$, we have that

$$1_N = \beta \Phi(f_1).$$
Therefore $\alpha = \beta$.

Case 2. $\alpha$ and $\beta$ are infinite. For a faithful normal semi-finite trace $\tau$ on $N$ put
\[
\tau_i(x) = \tau(f_ix), x \in N.
\]
For each $i \in I$ set
\[
J_i = \{ j \in J : \tau_i(e_j) \neq 0 \}.
\]
Since $\{e_j\}$ is an orthogonal family, one has that $J_i$ is countable for each $i \in I$.

Suppose that there exists $j \in J$ such that $\tau_i(e_j) = 0$ for all $i \in I$. Since $\tau(f_ie_jf_i) = \tau(f_i(e_j)) = \tau_i(e_j) = 0$, we obtain that $f_ie_jf_i = 0$. But from
\[
0 = f_ie_jf_i = f_i e_jf_i = f_i(e_j)(f_i e_j)^*.
\]
it follows that $f_ie_j = 0$ for all $i \in I$. And since $\bigvee_{i \in I} f_i = 1_N$, this implies that $e_j = 0$ – a contradiction. Therefore given any $j \in J$ there exists $i \in I$ such that $\tau_i(e_j) \neq 0$, i.e. $j \in J_i$. Hence
\[
J = \bigcup_{i \in I} J_i,
\]
i.e.
\[
\beta \leq \alpha N_0,
\]
therefore $\beta \leq \alpha$. Similarly $\alpha = \beta$.

This means that every homogeneous direct summand of the von Neumann algebra $N$ is of type $I_\alpha$, i.e. $N$ itself is homogeneous of type $I_\alpha$. \hfill $\Box$

It is well-known [13] that if $M$ is an arbitrary von Neumann algebra of type I with the center $Z(M)$ then there exists an orthogonal family of central projections $\{z_\alpha\}_{\alpha \in J}$ in $M$ with $\sup_{\alpha \in J} z_\alpha = 1$ such that $M$ is $*$-isomorphic to the $C^*$-product of von Neumann algebras $z_\alpha M$ of type $I_\alpha, \alpha \in J$, i.e.
\[
M \cong \bigoplus_{\alpha \in J} z_\alpha M.
\]
In this case by definition of the central extension we have that
\[
E(M) = \prod_{\alpha \in J} E(z_\alpha M).
\]

Suppose that $T$ is an automorphism of $E(M)$ and $\phi$ is its restriction onto the center $E(Z(M))$. Let us show that $T$ maps each $z_\alpha E(M) \cong E(z_\alpha M)$ onto itself. The automorphism $T$ maps $z_\alpha E(M)$ onto $T(z_\alpha)E(M)$. From Proposition 3.9 it follows that the von Neumann algebra $T(z_\alpha)M$ is of type $I_\alpha$. Thus $T(z_\alpha) \leq z_\alpha$. Suppose that $z_\alpha' = z_\alpha - T(z_\alpha) \neq 0$. By Proposition 3.9 we have that $T^{-1}(z_\alpha')M$ is of type $I_\alpha$, i.e.
\[
0 \neq z_\alpha'' = T^{-1}(z_\alpha') \leq z_\alpha.
\]
On other hand
\[
T(z_\alpha z_\alpha') = T(z_\alpha)T(z_\alpha'') = T(z_\alpha)z_\alpha' = T(z_\alpha)(z_\alpha - T(z_\alpha)) = T(z_\alpha) - T(z_\alpha) = 0,
\]
i.e. \( z_\alpha z'_\alpha = 0 \). Therefore since \( z''_\alpha \leq z_\alpha \) we have that \( z''_\alpha = 0 \), a contradiction with the inequality \( z'_\alpha \neq 0 \). Hence \( z'_\alpha = 0 \), i.e. \( T(z_\alpha) = z_\alpha \).

Therefore \( \phi \) generates an automorphism \( \phi_\alpha \) on each \( z_\alpha S(Z(M)) \cong Z(E(z_\alpha M)) \), for \( \alpha \in J \). Let \( T_{\phi_\alpha} \) be the automorphism of \( z_\alpha E(M) \) generated by \( \phi_\alpha, \alpha \in J \). Put

\[
T_{\phi} (\{x_\alpha\}_{\alpha \in J}) = \{T_{\phi_\alpha}(x_\alpha)\}, \{x_\alpha\}_{\alpha \in J} \in E(M). \tag{3.4}
\]

Then \( T_{\phi} \) is an automorphism of \( E(M) \).

Now we can state the main result of the present paper.

**Theorem 3.10.** If \( M \) is a type I von Neumann algebra, then each automorphism \( T \) of \( E(M) \) can be uniquely represented in the form

\[
T = T_a \circ T_{\phi},
\]

where \( T_a \) is an inner automorphisms implemented by an element \( a \in E(M) \) and \( T_{\phi} \) is an automorphism of the form \( (3.4) \).

**Proof.** Let \( T \) be an automorphism of \( E(M) \) and \( \phi \) be its restriction on \( Z(E(M)) \) – the center of \( E(M) \). Consider the automorphism \( T_{\phi} \) on \( E(M) \) generated by the automorphism \( \phi \) as in \( (3.4) \) above. Similar to the proof of Proposition 3.3 we find an element \( a \in E(M) \) such that \( T = T_a \circ T_{\phi} \) and show that this representation is unique. \( \Box \)

Recall [7], [12] that an operator \( T : E(M) \to E(M) \) is called band preserving if \( T(zx) = zT(x) \) for all \( z \in P(Z(M)), x \in E(M) \).

Proposition 3.6 and Theorem 3.10 imply the following result which is an analogue of [9, Theorem 5, Remark A] giving a sufficient condition for innerness of algebraic automorphisms.

**Corollary 3.11.** If \( M \) is a von Neumann algebra of type I\(_\infty \) then each band preserving automorphism of \( E(M) \) is inner.

**Proof.** Let \( \phi \) be the restriction of \( T \) onto \( E(Z(M)) \). Since \( T \) is band preserving it follows that \( \phi \) acts identically on the simple elements from \( Z(M) \). Proposition 3.6 implies that \( \phi \) is \( t(Z(M)) \)-continuous. Hence \( \phi \) is identical on the whole \( S(Z(M)) = E(Z(M)) \) and therefore by Theorem 3.10 \( T \) is an inner automorphism. \( \Box \)

**Remark 3.12.** It is clear that the conditions of the above Corollary is also necessary for the innerness of automorphisms of \( E(M) \).

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