Abstract

Building on foundations introduced in a previous paper, we give several $p$-adic analytic descriptions of the categories of étale $\mathbb{Z}_p$-local systems and étale $\mathbb{Q}_p$-local systems on an affinoid algebra over a finite extension of $\mathbb{Q}_p$ (or more generally, over the fraction field of the Witt vectors of a perfect field of characteristic $p$). These include generalizations of Fontaine’s theory of $(\phi, \Gamma)$-modules, the refinement of Fontaine’s construction introduced by Cherbonnier and Colmez, and a recent description by Fargues and Fontaine in terms of semistable vector bundles on a certain scheme. Our descriptions depend on the embedding of the associated affinoid space into an affine toric variety, but there are natural functoriality maps relating the constructions for different choices of the embedding; these may be used to give analogous descriptions over more general rigid or Berkovich analytic spaces.

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After its formalization by Deligne [23], the subject of Hodge theory may be viewed as the study of the interrelationship among different cohomology theories associated to algebraic varieties over \( \mathbb{C} \), most notably singular (Betti) cohomology and the cohomology of differential forms (de Rham cohomology). From work of Fontaine and others, there emerged a parallel subject of \( p \)-adic Hodge theory concerning the interrelationship among different cohomology theories associated to algebraic varieties over finite extensions of \( \mathbb{Q}_p \), most notably étale cohomology with coefficients in \( \mathbb{Q}_p \) and algebraic de Rham cohomology.

In ordinary Hodge theory, the relationship between Betti and de Rham cohomologies is forged using the Riemann-Hilbert correspondence, which relates topological data (local systems) to analytic data (integrable connections). In \( p \)-adic Hodge theory, one needs a similar correspondence relating de Rham data to étale \( \mathbb{Q}_p \)-local systems, which arise from the étale cohomology functor on schemes of finite type over \( K \). However, in this case the local systems turn out to be far more plentiful, so it is helpful to first build a correspondence relating them to some sort of intermediate algebraic objects. This is achieved by the theory
of \((\varphi, \Gamma)\text{-modules}\), which gives some Morita-type dualities relating étale \(\mathbb{Q}_p\)-local systems over \(K\) (i.e., continuous representations of the absolute Galois group \(G_K\) on finite-dimensional \(\mathbb{Q}_p\)-vector spaces) with modules over certain mildly noncommutative topological algebras. The latter appear as topological monoid algebras over commutative period rings for some continuous operators (the eponymous \(\varphi\) and \(\Gamma\)).

One of the key features of Hodge theory is that it provides information not just about individual varieties, but also about families of varieties through the mechanism of variations of Hodge structures. This paper is the second in a series beginning with [50], in which we develop some mechanisms for carrying out relative \(p\)-adic Hodge theory; in this paper in particular, we exhibit a generalization of the theory of \((\varphi, \Gamma)\)-modules which describes the étale \(\mathbb{Q}_p\)-local systems on arbitrary nonarchimedean analytic spaces (in the sense of either Tate or Berkovich). In the remainder of this introduction, we recall the results of [50], then describe in detail what is accomplished in this paper and how the results of [50] are used. (We refer back to the introduction to [50] for additional background, including discussion of and contrast between two different possible relativizations of \(p\)-adic Hodge theory.)

### 0.1 \(\varphi\)-modules and local systems

For \(K\) a perfect field of characteristic \(p\), the discrete representations of the absolute Galois group \(G_K\) of \(K\) on finite dimensional \(\mathbb{F}_p\)-vector spaces form a category equivalent to the category of \(\varphi\)-modules over \(K\), i.e., finite-dimensional \(K\)-vector spaces equipped with isomorphisms with their \(\varphi\)-pullbacks. This amounts to a nonabelian generalization of the Artin-Schreier description of \((\mathbb{Z}/p\mathbb{Z})\)-extensions of fields of characteristic \(p\) [24, Exposé XXII, Proposition 1.1]. Katz [42, Proposition 4.1.1] generalized this result in two directions, by exhibiting an equivalence of categories between \(\varphi\)-modules over the length \(n\) Witt vectors \(W_n(R)\) over a perfect \(\mathbb{F}_p\)-algebra \(R\) and the locally constant sheaves in finite free \(\mathbb{Z}/p^n\mathbb{Z}\)-modules on the étale site of \(R\). By taking inverse limits, one gets an equivalence of categories between \(\varphi\)-modules over the full \(p\)-typical Witt ring \(W(R)\) over a perfect \(\mathbb{F}_p\)-algebra \(R\) and étale \(\mathbb{Z}_p\)-local systems on the small étale site of \(\text{Spec}(R)\). Similarly, one has an equivalence of categories between étale \(\varphi\)-modules over \(W(R)[p^{-1}]\) and étale \(\mathbb{Q}_p\)-local systems on the small étale site of \(\text{Spec}(R)\).

The key result from [50] behind the existence of \((\varphi, \Gamma)\)-modules and their generalizations is an equivalence of categories between the finite étale algebras over certain perfect \(\mathbb{F}_p\)-algebras and certain \(\mathbb{Q}_p\)-algebras (extending Faltings’s almost purity theorem). For the purposes of this paper, we may state this result as follows. Let \(R\) be a perfect uniform Banach algebra over \(\mathbb{F}_p((\pi))\). Let \(\mathfrak{o}_R\) be the subring of \(R\) consisting of elements of norm at most 1. Define the element \(z = \sum_{i=0}^{p-1} [(\pi + 1)^i/p] \in W(\mathfrak{o}_R)\). One then obtains an equivalence of categories between the finite étale algebras over \(R\) and over \(W(\mathfrak{o}_R)[[\pi^{-1}]]/(z)\) [50, Theorem 3.6.20].

Any rank-preserving equivalence of categories between the finite étale algebras over two different rings immediately gives corresponding equivalences of categories for \(\mathbb{Z}_p\)-local systems and \(\mathbb{Q}_p\)-local systems. One thus obtains an equivalence of categories between \(\varphi\)-modules over \(W(R)[p^{-1}]\) and étale \(\mathbb{Q}_p\)-local systems on the small étale site of \(\text{Spec}(W(\mathfrak{o}_R)[[\pi^{-1}]]/(z))\). One has some similar equivalences with \(W(R)[p^{-1}]\) replaced with a certain subring of overcon-
vergent elements, or a corresponding extended Robba ring giving by taking a certain Fréchet completion of the overconvergent subring. The last of these rings has the key property that it can be used to describe analytic étale \( \mathbb{Q}_p \)-local systems over \( W(\mathfrak{o}_R)[[\pi^{-1}]]/(z) \), which are obtained by glueing algebraic \( \mathbb{Z}_p \)-local systems up to isogeny not over a Zariski open covering, but rather over an open covering of the Gel’fand spectrum. One can also describe étale cohomology of the aforementioned local systems in terms of the corresponding \( \varphi \)-modules. See [50, §8] for all of these results.

### 0.2 \((\varphi, \Gamma)\)-modules

Using the aforementioned results, one can describe étale \( \mathbb{Q}_p \)-local systems on some nonarchimedean analytic spaces over fields of mixed characteristic. Before indicating how this happens more generally, we first recall how one extracts the original theory of \((\varphi, \Gamma)\)-modules for Galois representations over \( \mathbb{Q} \). (A similar theory is available for finite extensions of \( \mathbb{Q}_p \), or more generally finite extensions of an absolutely unramified \( p \)-adic field with perfect residue field.)

In the previous discussion, let us take \( A = K = \mathbb{F}_p((\pi)) \). In this case, the ring \( F = W(\mathfrak{o}_R)[[\pi^{-1}]]/(z) \) is none other than the \( p \)-adic completion of \( \mathbb{Q}_p(\mu_p) \) for the \( p \)-adic norm, with \([1+\pi]^p^n\) mapping to a primitive \( p^n \)-th root of unity for each nonnegative integer \( n \). One can thus describe \( \mathbb{Q}_p \)-local systems over \( \mathbb{Q}_p(\mu_p) \) using \( \varphi \)-modules over \( W(K)[p^{-1}] \), or over the overconvergent subring of same. To describe local systems over \( \mathbb{Q}_p \), we must add some descent data for the action of the group Gal(\( \mathbb{Q}_p(\mu_p)/\mathbb{Q}_p \)) \( \cong \mathbb{Z}_p^\times \). This is done by lifting the group action to \( W(\mathfrak{o}_R) \) so that \( \gamma \in \mathbb{Z}_p^\times \) takes \([1+\pi]\) to \([(1+\pi)^\gamma]\) (defined using the binomial series), and considering \( \varphi \)-modules equipped with isomorphisms with their \( \gamma \)-pullbacks, or \((\varphi, \Gamma)\)-modules. We are particularly interested in the étale \((\varphi, \Gamma)\)-modules; these satisfy a condition on the \( \varphi \)-action reminiscent of the semistability condition for vector bundles in geometric invariant theory. (We will return to this analogy later in this introduction.)

In the previous construction, it is relatively easy to pass from the ring \( W(\mathfrak{o}_R)[p^{-1}] \) to the two-dimensional local field \( E \) defined as the completion of \( \mathbb{Z}_p((\pi))[p^{-1}] \) for the \( p \)-adic norm. Since \( E \) has imperfect residue field, to make things unambiguous we must pick out a Frobenius lift on \( E \); we use the one sending \( 1+\pi \) to its \( p \)-th power. We then obtain a \( \varphi \)-equivariant embedding \( E \) into \( W(\mathfrak{o}_R)[p^{-1}] \) taking \( 1+\pi \) to \([1+\pi]\). The equivalence of étale \((\varphi, \Gamma)\)-modules over \( E \) and over \( W(\mathfrak{o}_R)[p^{-1}] \), as introduced by Fontaine [28], then follows from the fact that the field \( K \) and its completed perfection have equivalent categories of finite étale algebras.

A further refinement was achieved by Cherbonnier and Colmez [20], who showed that just as one can descend étale \( \varphi \)-modules over \( W(\mathfrak{o}_R)[p^{-1}] \) to its overconvergent subring, one can descend étale \((\varphi, \Gamma)\)-modules over \( E \) to its overconvergent subring \( E^\dagger \). This is not a purely formal observation, as it fails to hold for \( \varphi \)-modules. Rather, this is a subtler phenomenon similar in substance to Sen-Tate decompletion in Galois cohomology of local fields [72]. This refinement turns out to be extremely fruitful, as it allows for numerous constructions in \( p \)-adic Hodge theory to be made at the level of étale \((\varphi, \Gamma)\)-modules. Many of these constructions pass through a further construction of Berger, extending scalars from
to the Robba ring $R$ of germs of analytic functions on open annuli of outer radius 1. (One formally defines the étale $(\varphi, \Gamma)$-modules over $R$ to be the ones arising from base extension from $E^\dagger$; it is easy to check that the base change functor from $E^\dagger$ to $R$ is an equivalence of categories.) For instance, Berger used this approach to give the first proof of Fontaine’s conjecture that de Rham representations are potentially semistable [9], and to give a simple proof of the theorem of Colmez and Fontaine that weakly admissible filtered $(\varphi, N)$-modules are admissible [10]. (See the introduction to [50] for discussion of some other examples.)

0.3 Relative period rings

In this paper, we describe a generalization of the theory of $(\varphi, \Gamma)$-modules in which the field $\mathbb{Q}_p$ can be replaced by an arbitrary reduced affinoid algebra over $\mathbb{Q}_p$, or more generally, over $K = \text{Frac} W(k)$ for $k$ a perfect field of characteristic $p$. Note that this also includes affinoid algebras over finite extensions of $K$, by restriction of scalars. Before discussing the theory, we must indicate the base rings over which the relevant objects are defined.

As in the traditional theory, we cannot hope to directly realize local systems over our original affinoid algebra, but instead must pass to a highly ramified extension and then get back using descent. We do this by working not with a bare affinoid algebra, but rather with an affinoid algebra equipped with a certain choice of local coordinates. More precisely, we fix an unramified morphism from the corresponding reduced affinoid space to an affine space, or more generally to an affine toric variety. This time, in addition to adjoining the $p$-power roots of unity, we also adjoin the $p$-power roots of the coordinate functions. The completion of the result has the form $W(\mathfrak{a}_R)[[\pi^{-1}]]/((1 + \pi - 1)/(1 + \pi^{1/p} - 1))$ for a certain natural Banach algebra $R$ over $k((\pi))$.

From this construction, we make a suite of period rings equipped with an action of a Frobenius lift $\varphi$ and an auxiliary group $\Gamma_N$ of the form $\mathbb{Z}_p^\times \ltimes \mathbb{Z}_p^n$, where $n$ is the number of local coordinates. These are analogues of the rings of type $A$ and $B$ in the notation of Cherbonnier and Colmez [20], and of the Robba ring used in the work of Berger [9, 10]. (In the notation of this paper, the Robba ring is considered to be of a new type $C$.) The setup is modeled closely on the original work of Faltings [26], but with the big difference that since we are not currently trying to study properties sensitive to integral models (like the crystalline condition), we are able to simplify and generalize by avoiding any use of integral models. (This avoidance of integral models is also a feature of the approaches to comparison isomorphisms described recently by Beilinson [8] and Scholze [71].)

In what follows, it will be important to distinguish between perfect and imperfect period rings; the distinction is approximately that $\varphi$ is bijective on perfect period rings and not on imperfect ones. For instance, Fontaine’s ring $E$ from earlier is an example of an imperfect period ring; taking its weakly completed perfection with respect to the specified Frobenius lift $\varphi$ gives a perfect period ring. One key distinction is that imperfect period rings are only defined when the map from the affinoid space to the affine toric variety is not just unramified but étale; in particular, this can only occur when the affinoid space admits at worst toroidal singularities. By contrast, perfect period rings are defined for unramified morphisms to toric varieties, which includes closed immersions; this is critical for discussion of functoriality, as
0.4 Relative \((\varphi, \Gamma)\)-modules

For the moment, let us continue to fix attention on a single affinoid space with a suitable choice of local coordinates. Using the results of [50], we obtain a correspondence between étale local systems (in \(\mathbb{Z}_p\)-modules or \(\mathbb{Q}_p\)-vector spaces) on the original affinoid algebra and certain modules over the period rings equipped with suitable actions of \(\varphi\) and \(\Gamma_N\), called étale \((\varphi, \Gamma_N)\)-modules. This includes an analogue of the decompletion argument of Cherbonnier and Colmez, which when suitably articulated carries through with little change. (The form of the argument we generalize is that given in [49], where the Tate-Sen formalism is avoided.)

Note that by working with type \(C\), we are able to describe analytic local systems on the original affinoid algebra, which in the case of a connected base ring correspond to representations of de Jong’s étale fundamental group of the affinoid space [22]. These representations need not have compact image; for instance, Tate curves give rise to natural one-dimensional representations over \(\mathbb{Q}_p\) with image \(\mathbb{Z}_p^\times\). (Further interesting examples arise on Rapoport-Zink period spaces; see below.) We also obtain descriptions of étale cohomology of local systems, generalizing results of Herr [36, 37] and the second author [59].

In addition, we give a relative analogue of a new variant of the theory of \((\varphi, \Gamma)\)-modules recently introduced by Fargues and Fontaine [27]. In their work, one can replace the \((\varphi, \Gamma)\)-modules over the Robba ring (including those not satisfying the étale condition) by an equivalent category of vector bundles over a certain scheme which are equivariant for an action of \(\Gamma\) on the base scheme. Note that \(\varphi\) is no longer present. The étaleness condition on \((\varphi, \Gamma)\)-modules then becomes equivalent to semistability of vector bundles for a natural degree function.

0.5 Functoriality and globalization

As noted earlier, our description of relative \((\varphi, \Gamma)\)-modules depends on a choice of local coordinates on an affinoid space. Since the category of étale \(\mathbb{Q}_p\)-local systems on that space does not distinguish any such choice, it is imperative to describe how changes of the choice of coordinates are reflected in the structure of relative \((\varphi, \Gamma)\)-modules.

We handle this by considering the category of torically framed affinoid algebras, consisting of affinoid algebras equipped with unramified morphisms from the corresponding affinoid spaces to affine toric varieties, with morphisms induced by toric morphisms of the toric varieties. One recovers the usual category of affinoid algebras by formally inverting those morphisms which are split inclusions of toric varieties which act as the identity on the underlying affinoid spaces; we call these the toric refinements. We show that any toric refinement induces an equivalence of categories on \((\varphi, \Gamma)\)-modules over the corresponding period rings of type \(\check{C}\). Note that we are limited to perfect period rings here because one cannot discuss toric refinements without using closed immersions, whereas imperfect period rings only behave well for étale maps to toric varieties.
This gives rise to a category of not necessarily étale \((\varphi, \Gamma)\)-modules on a fairly general class of analytic spaces, by making a site of framed affinoid subdomains and considering crystals of \((\varphi, \Gamma)\)-modules on these; among these, the étale objects again describe étale \(\mathbb{Q}_p\)-local systems. An alternate approach has been introduced by Scholze \cite{Scholze} in the form of the pro-étale site of an analytic space; we will incorporate Scholze’s point of view in a subsequent paper in this series. (Yet another approach would be to take the completed direct limit over all toric framings; this suggests an interpretation of \(p\)-adic Hodge theory in the language of \(\Lambda\)-rings.)

It is worth mentioning here that Kappen \cite{Kappen} has introduced a notion of uniformly rigid spaces over \(\mathbb{Q}_p\), whose relationship to usual rigid analytic spaces is somewhat akin to the relationship between manifolds with and without boundary. The basic objects in this theory are built not out of affinoid algebras but semiaffinoid algebras, which arise from formal schemes over \(\mathbb{Z}_p\) of formally finite type. Although the glueing of uniformly rigid spaces is somewhat incompatible with that of Berkovich spaces (see §1.2), one can at least use our methods to describe a theory of relative \((\varphi, \Gamma)\)-modules over a semiaffinoid space.

### 0.6 Further remarks

A theory of relative \((\varphi, \Gamma)\)-modules similar to our construction has been introduced by Andreatta and Brinon \cite{Andreatta-Brinon}. Their construction is somewhat different, relying on careful use of integral models. Another significant difference is that Andreatta and Brinon consider only cases for which they can construct imperfect period rings. We instead make a thorough study of the perfect period rings first, which allows both for direct work on nonsmooth analytic spaces (without having to pass to a resolution of singularities) and for considerations of functoriality. At the level of imperfect period rings, neither construction subsumes the other, but either construction alone is sufficient for handling relative \((\varphi, \Gamma)\)-modules locally on smooth analytic space. (A more precise description of the relationship between our constructions and various results in the literature can be found in §4.11.)

In this paper, the cyclotomic tower over \(\mathbb{Q}_p\) plays a crucial role as a sufficiently ramified extension of \(\mathbb{Q}_p\). (The key property is that the completion of the union over the tower is perfectoid in the sense of \cite[Definition 3.5.1]{Scholze}.) However, the setup of \cite{Scholze} leaves open the possibility of using other towers instead. For instance, based on Kisin’s work on the classification of crystalline representations using Breuil modules \cite{Kisin}, Caruso \cite{Caruso} has described (for \(p\) odd) an analogue of the theory of \((\varphi, \Gamma)\)-modules where the role of the cyclotomic tower is played instead by the tower obtained by adjoining the \(p\)-power roots of \(p\) (or more generally of some uniformizer of a base field \(K\)). It seems likely that the optimal way to relate these constructions to ours and to extend them to the relative setting is to compare them to Scholze’s canonically defined objects \cite{Scholze1, Scholze2}.

One can employ relative \((\varphi, \Gamma)\)-modules to construct some natural étale local systems arising in arithmetic geometry. In a subsequent paper, we plan to illustrate this by constructing natural local systems on Rapoport-Zink period domains of filtered isocrystals. This will realize the construction sketched in \cite{Scholze3}.

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Acknowledgments

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1 Analytic spaces

We continue the discussion of analytic geometry initiated in [50, §2]. The emphasis here is on global constructions, including Berkovich’s analytic spaces and Kappен’s uniformly rigid spaces. We retain all notation and terminology from [50, §2].

Hypothesis 1.0.1. Throughout this paper, fix a prime number \( p \) and an analytic field \( K \). Until otherwise specified (see Hypothesis 3.0.1), \( K \) is permitted to carry the trivial norm, and its residue characteristic need not equal \( p \).

1.1 Berkovich’s nonarchimedean analytic spaces

We introduce Berkovich’s nonarchimedean analytic spaces following [14]. The definition in [13] is somewhat more restrictive; see Remark 1.1.5. One should ideally view Berkovich spaces in context together with Huber’s adic spaces [39]; however, we will defer this comparison to a subsequent paper, where it will be needed in order to pass between our framework and that of Scholze [70, 71].

Definition 1.1.1. Let \( X \) be a topological space. Equip each subset of \( X \) with the induced topology. For \( \tau \) a family of subsets of \( X \) and \( Y \subseteq X \), put \( \tau|_Y = \{ V \in \tau : V \subseteq Y \} \). We say \( \tau \) is a patchwork on \( X \) if it satisfies the following conditions.

(a) For each \( V \in \tau \), \( \tau|_V \) contains a fundamental system of neighborhoods in \( V \) around each point of \( V \). (Recall that a neighborhood of \( x \) in \( V \) contains an open set in \( V \) containing \( x \) but need not itself be open in \( V \).)

(b) For each \( x \in X \), there exists a finite subfamily \( \{ V_1, \ldots, V_n \} \subseteq \tau \) such that \( x \in V_1 \cap \cdots \cap V_n \) and \( V_1 \cup \cdots \cup V_n \) is a neighborhood of \( x \) in \( X \).

(c) For each \( U_1, U_2 \in \tau \) and each \( x \in U_1 \cap U_2 \), there exists a finite subfamily \( \{ V_1, \ldots, V_n \} \subseteq \tau|_{U_1 \cap U_2} \) such that \( x \in V_1 \cap \cdots \cap V_n \) and \( V_1 \cup \cdots \cup V_n \) is a neighborhood of \( x \) in \( U_1 \cap U_2 \).

Example 1.1.2. An example of a patchwork on \( \mathbb{R} \) is given by the collection of intersections \( I \cap [m, n] \) for \( I \) an open interval and \( m, n \in \mathbb{Z} \). More generally, let \( X \) be the total space of a locally finite simplicial complex; then the collection of all open subsets of all simplices in the complex forms a patchwork on \( X \).
Remark 1.1.3. In [14], a family is called dense if it satisfies condition (a) of Definition 1.1.1, a quasinet if it satisfies conditions (a) and (b), and a net if it satisfies all three conditions. However, the term net has a far more established meaning in point-set topology, as a sequence indexed by a general directed set. (This concept is used to deal with spaces which are not first-countable; see for instance [52]. Such spaces can also be handled using filters, as in [10].) We have thus opted for a different name, which is meant to suggest a family of subsets which cover \( X \), but only provides a neighborhood basis after the covering subsets are appropriately stitched together.

Definition 1.1.4. Let \( X \) be a locally Hausdorff topological space. A (strictly) \( K \)-affinoid atlas on \( X \) consists of the following data.

(i) A patchwork \( \tau \) on \( X \).

(ii) For each \( U \in \tau \), a (strictly) \( K \)-affinoid algebra \( A_U \) and a homeomorphism \( \mathcal{M}(A_U) \cong U \).

(iii) For each \( U, V \in \tau \) with \( V \subseteq U \), a bounded homeomorphism \( \text{Res}_{U,V} : A_U \to A_V \) which identifies \( V \) with a (strictly) affinoid subdomain of \( U \), satisfying the compatibility condition \( \text{Res}_{U,W} = \text{Res}_{V,W} \circ \text{Res}_{U,V} \).

Any (strictly) \( K \)-affinoid atlas can be uniquely extended to a maximal such atlas [14, Proposition 1.2.13]. We refer to a locally Hausdorff topological space equipped with a (strictly) \( K \)-affinoid atlas as a (strictly) \( K \)-analytic space. We form a category of (strictly) \( K \)-analytic spaces by starting with strong morphisms (continuous maps of spaces equipped with compatible morphisms of affinoids), then formally inverting those morphisms whose underlying maps are homeomorphisms; see [14, §1.2].

We will identify (strictly) \( K \)-affinoid spaces with certain (strictly) \( K \)-analytic spaces via the natural fully faithful (by Tate’s theorem) functor. A (strictly) affinoid subdomain of a \( K \)-analytic space is an element of the maximal (strictly) \( K \)-affinoid atlas; this agrees with the previous definition for (strictly) \( K \)-affinoid spaces.

Remark 1.1.5. Note that the definition of a (strictly) \( K \)-analytic space does not guarantee that every point has an affinoid neighborhood (or equivalently a neighborhood basis consisting of affinoid spaces). Spaces with the latter property are said to be good; they may be viewed as locally ringed spaces for the usual topology on \( \mathcal{M}(A) \) with local rings as described in [50, Definition 2.4.11]. Consequently, they coincide with the analytic spaces introduced in [13].

Remark 1.1.6. In Berkovich’s original development, it was not checked that the natural functor from strictly \( K \)-analytic spaces to \( K \)-analytic spaces is fully faithful [14, Remark 1.5.6]. However, this was subsequently verified by Temkin [73].

Definition 1.1.7. Let \( A \) be a (strictly) \( K \)-affinoid algebra, and put \( X = \mathcal{M}(A) \). The weak G-topology on \( X \) is the G-topology (set-theoretic Grothendieck topology) in which the admissible open sets are the (strictly) affinoid subdomains of \( X \), and the admissible
coverings are the finite coverings by (strictly) affinoid subdomains. For this G-topology, by
the theorems of Tate and Kiehl \[50, Theorem 2.5.13\], the structure presheaf \( \mathcal{O}_X : V \mapsto A_V \)
is a sheaf of rings, over which any coherent module is represented by a finite \( A \)-module. By \[50, Corollary 2.5.12\], local freeness of a finite \( A \)-module can be checked locally in the
G-topology.

The weak G-topology can be enriched in a natural way to provide a G-topology on any
(strictly) \( K \)-analytic space \( X \), in which the admissible open sets are the (strictly) analytic
subdomains of \( X \) (those subsets on which the restriction of the maximal atlas forms a patch-
work), and any admissible covering of a (strictly) affinoid subdomain can be refined to a
finite covering by (strictly) affinoid subdomains; see [14, §1.3]. We use this G-topology to
view \( X \) as a locally G-ringed space.

The relationship with Tate’s theory of rigid analytic spaces is given by the following
theorem.

Theorem 1.1.8. The category of paracompact strictly \( K \)-analytic spaces is equivalent to
the category of quasiseparated rigid \( K \)-analytic spaces which admit an admissible affinoid
covering in which each set in the covering meets only finitely many others. In particular, the
category of compact strictly \( K \)-analytic spaces is equivalent to the category of quasicompact
quasiseparated rigid \( K \)-analytic spaces.

Proof. See [14, Theorem 1.6.1].

Definition 1.1.9. Let \( A \) be an affinoid algebra over \( K \), and let \( X \) be a scheme locally of
finite type over Spec(\( A \)). Let \( C_A \) be the category of good \( K \)-analytic spaces \( Y \) equipped with
morphisms to Spec(\( A \)) in the category of locally ringed spaces. Let \( F : C_A \to \text{Set} \) be the
functor taking \( Y \in C_A \) to the set of morphisms \( Y \to X \) over Spec(\( A \)) in the category of locally
ringed spaces. Then the functor \( F \) is representable by an object \( X^{an} \) [14, Proposition 2.6.1],
called the analytification of \( X \) over \( A \). (The case \( A = K \) is treated in [13, §§3.4–3.5].)

Definition 1.1.10. Let \( \psi : Y \to X \) be a morphism of analytic spaces over \( K \). For \( y \in Y \) an
unspecified point, put \( x = \psi(y) \). Let \( \mathcal{O}_{X,x}, \mathcal{O}_{Y,y} \) denote the local rings of \( X, Y \) at \( x, y \), and
let \( m_x \) be the maximal ideal of \( \mathcal{O}_{X,x} \).

We say \( \psi \) is finite if for every affinoid subdomain \( U \) of \( X \), \( \psi^{-1}(U) \to U \) is induced by a
finite morphism of affinoid algebras; this property is local on the target [14, Lemma 1.3.7].
We say \( \psi \) is finite at \( y \) if there exist open neighborhoods \( U, V \) of \( x, y \) such that the induced
morphism \( V \to U \) is finite; one can then make these neighborhoods arbitrarily small [14,
Lemma 3.1.2]. We say \( \psi \) is quasifinite if it is finite at each point of \( Y \).

Assume for the rest of this definition that \( \psi \) is quasifinite. We say \( \psi \) is flat at \( y \) if there exist open neighborhoods \( U, V \) of \( x, y \) such that for any affinoid subdomain \( W \) of \( U \),
the induced map \( \psi^{-1}(W) \to W \) is induced by a flat morphism of affinoid algebras. We say \( \psi \) is
flat if it is flat at each point of \( Y \).

We say \( \psi \) is unramified at \( y \) if the ring \( \mathcal{O}_{Y,y}/m_x \mathcal{O}_{Y,y} \) is a finite separable field extension
of \( \mathcal{O}_{X,x}/m_x \). It is equivalent to require that the sheaf of relative differentials \( \Omega_{Y/X} \) vanishes
[14, Corollary 3.3.6]. We say \( \psi \) is unramified if it is unramified at each point of \( Y \). We say
\( \psi \) is \( \acute{e} \text{tale} \) (at \( y \) or everywhere) if it is flat and unramified (at \( y \) or everywhere).
Remark 1.1.11. For good analytic spaces, any unramified morphism factors locally for the Berkovich topology on the source as a closed immersion followed by an étale morphism. Namely, by [14, Proposition 3.3.11] this reduces to the fact that an unramified (formally unramified and locally of finite type) morphism of schemes factors locally on the source as a closed immersion followed by an étale morphism [33, Corollaire 18.4.7]. For general analytic spaces, one gets a similar assertion working locally for the G-topology; it is unclear whether the same assertion holds for the Berkovich topology. One issue is the lack of canonicality of the factorization at the level of schemes, which can only be resolved at the level of algebraic spaces [68].

Similarly, using the Berkovich topology for good analytic spaces and the G-topology otherwise, any étale morphism factors locally on the source as an open immersion followed by a finite étale cover. Using the fact that Gel’fand spectra are Hausdorff, any étale morphism can also be factored locally on the source as a finite étale cover followed by an open immersion.

Remark 1.1.12. As for schemes (see [14, §4.1] for more details), we naturally associate to each \( K \)-analytic space an étale site and étale topos. These can be used to define étale fundamental groups of analytic spaces, as in [22]. For strictly \( K \)-analytic spaces, one can also obtain these objects in the framework of adic spaces [39].

1.2 Semiaffinoid algebras and uniformly rigid spaces

We mention in passing an alternate category of analytic spaces due to Kappen [40]. These spaces play a role analogous to that played by manifolds with boundary in ordinary topology.

Hypothesis 1.2.1. Throughout §1.2 assume that \( K \) is discretely valued (of arbitrary characteristic).

Definition 1.2.2. For \( m \) and \( n \) nonnegative integers, let \( R_{m,n} \) denote the completion of \( \mathfrak{o}_K[x_1, \ldots, x_m, y_1, \ldots, y_n] \) with respect to the ideal \( (m_K, x_1, \ldots, x_m) \). Equip this ring with the Gauss norm. A semiaffinoid algebra over \( K \) is a \( K \)-Banach algebra admitting a strict surjection from \( R_{m,n} \otimes_{\mathfrak{o}_K} K \) for some \( m, n \). Note that a strictly \( K \)-affinoid algebra is semiaffinoid, but a general \( K \)-affinoid algebra need not be.

Remark 1.2.3. One can similarly define a semiaffinoid homomorphism between \( K \)-Banach algebras, but the composition of two such homomorphisms need not again be semiaffinoid. For example, \( A = \mathbb{Z}_p[[x]]_{(p)} \) is a \( K \)-semiaffinoid algebra but \( A\{y\} \) is not. Consequently, a rational localization of a semiaffinoid algebra need not again be semiaffinoid.

Since Kappen’s interest in semiaffinoid algebras is their relationship with formal schemes, he opts to introduce a different notion of a semiaffinoid subdomain embedding [40, Definition 2.22]. He then introduces an appropriate G-topology on the set of maximal ideals of a semiaffinoid algebra, called the uniformly rigid G-topology [40, Definition 2.35], for which one has an analogue of the acyclicity theorem of Tate [40, Theorem 2.41]. (Curiously, the analogue of Kiehl’s theorem is not known and is even predicted to fail [40, Conjecture 3.7].)

Using the uniformly rigid G-topology introduced above, Kappen defines a notion of uniformly rigid spaces glued out of spaces associated to semiaffinoid algebras [40, Definition 2.46].
Remark 1.2.4. As one might expect from the incompatibility noted at the start of Remark 1.2.3, the glueing of uniformly rigid spaces is not compatible with the formaton of the Gel'fand spectrum of a semiaffinoid algebra [40, §4]. We will thus be unable to work directly with uniformly rigid spaces in our framework. What we can hope to do is work with a single semiaffinoid algebra by working with its associated Gel'fand spectrum; for this, we must work with a larger category in which we can form rational localizations.

Definition 1.2.5. A mixed affinoid algebra over $K$ is a $K$-Banach algebra which is strictly affinoid over a semiaffinoid $K$-algebra. Any such ring is noetherian because it can be constructed from the noetherian ring $\mathfrak{o}_K$ by a sequence of operations which preserve the noetherian property (polynomial extensions, adic completions, and localizations). One can consider also further compositions of affinoid and semiaffinoid homomorphisms, but we will not do so here.

Proposition 1.2.6. Any mixed affinoid algebra satisfies the Tate sheaf property and the Kiehl glueing property. (As per [50, Definition 2.7.6], these only involve finite projective modules over covering families of rational localizations.)

Proof. Since mixed affinoid algebras are noetherian, the usual proof of Tate’s theorem [18, Theorem 8.2.1/1] carries over. For the Kiehl glueing property, by [50, Proposition 2.4.15], it suffices to check the case of a simple Laurent covering. In this case, using the Tate sheaf property, it is easy to check that one gets a glueing square in the sense of [50, Definition 2.7.3], so we may use [50, Proposition 2.7.5] to deduce the claim. □

1.3 Quasi-Stein spaces

Definition 1.3.1. An analytic space $X$ over $K$ is quasi-Stein if $X$ can be written as an ascending sequence $U_0 \subseteq U_1 \subseteq \cdots$ of affinoid subdomains corresponding to a sequence $\cdots \to A_1 \to A_0$ of homomorphisms each with dense image. For example, any open disc is seen to be quasi-Stein by writing it as a union of closed discs with the same center. Similarly, any open or half-open annulus is quasi-Stein. (Note that by compactness, every affinoid subdomain of $X$ is contained in some $U_i$.)

The following is a result of Kiehl [53, Satz 2.4] in the strictly analytic case, but the proof applies to more general Berkovich analytic spaces without change.

Theorem 1.3.2 (Kiehl). Let $X$ be a quasi-Stein space over $K$ and let $\mathcal{F}$ be a coherent sheaf on $X$.

(a) For every affinoid subdomain $U$ of $X$, the map $\mathcal{F}(X) \to \mathcal{F}(U)$ has dense image.

(b) The sheaf $\mathcal{F}$ is acyclic for Čech cohomology.

(c) For every $\alpha \in X$, $\mathcal{F}(X)$ has dense image in the stalk $\mathcal{F}_{\alpha}$. 

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Corollary 1.3.3. Let $X$ be a quasi-Stein space over $K$ and let $F$ be a coherent sheaf on $X$. Let $s_1, \ldots, s_n$ be global sections of $F$ which generate $F_\alpha$ for each $\alpha \in X$. Then $s_1, \ldots, s_n$ generate $F(X)$ as a module over $\mathcal{O}(X)$.

Proof. Use $s_1, \ldots, s_n$ to define a surjection $\mathcal{O}^n \to F$, then apply Theorem 1.3.2 to the kernel. 

Remark 1.3.4. For $f : X \to Y$ a finite surjective morphism of compact separated $K$-analytic spaces, the fact that $X$ is affinoid does not imply that $Y$ is affinoid, in contrast with the situation for schemes. This is related to the fact that there are compact separated $K$-analytic spaces which are not affinoid, but for which every coherent sheaf is acyclic. For instance, Qing Liu has exhibited examples which occur as immersed subspaces of a two-dimensional unit polydisc [57, 58].

2 Toric geometry

The methods of [50] can be used to describe étale local systems only on certain deeply ramified Banach rings (as in [31 §6] and [70]). To apply these results to affinoid algebras over a discretely valued field, it is helpful to have a systematic way to make deeply ramified covers. (An alternate approach would be to consider all deeply ramified covers; this is the point of view of Scholze. See Remark 3.8.7.)

Over the field itself, we may simply adjoin $p$-power roots of unity; however, for higher-dimensional spaces, one must also kill the differentials over the base field. Our approach to this issue is to consider not bare affinoid spaces, but affinoid spaces equipped with unramified maps to affine toric varieties; one may then use toric morphisms to add ramification in a natural way. This bears a strong resemblance to Payne’s description of Berkovich analytification in terms of tropical geometry [65].

The reader unfamiliar with toric varieties may safely pretend on first reading that all affine toric varieties are affine spaces with marked coordinate axes. For additional background, see the survey article of Danilov [21], the introductory notes of Fulton [30], and the book of Oda [63].

Remark 2.0.1. Multiple conventions exist in the literature regarding the definition of the class of toric varieties. Under a definition in which toric varieties are taken to be partial torus compactifications, we will only be considering normal toric varieties.

2.1 Cones and fans

We begin with the underlying combinatorics behind toric varieties, and how it corresponds to geometry. (The reader unfamiliar with toric varieties is advised to read Example 2.1.5 first.)

Definition 2.1.1. A commutative monoid $M$ (written multiplicatively) is integral if the cancellation law holds, i.e., if $ac = bc$ implies $a = b$. An equivalent condition is that $M$
injects into its group completion $M^{gp}$. An integral monoid $M$ is saturated if the image of $M$ in $M^{gp}$ is saturated, i.e., for all $x \in M^{gp}$ such that $x^n \in M$ for some positive integer $n$, we have $x \in M$. We say $M$ is fine if it is finitely generated and integral. We say $M$ is toric if $M$ is fine and saturated and $M^{gp}$ is torsion-free.

**Definition 2.1.2.** Let $N$ be a finite free $\mathbb{Z}$-module. A convex polyhedral cone in $N_\mathbb{R} = N \otimes_{\mathbb{Z}} \mathbb{R}$ is a subset $\sigma$ of the form $\{r_1 v_1 + \cdots + r_m v_m : r_1, \ldots, r_m \geq 0\}$ for some $v_1, \ldots, v_m \in N_\mathbb{R}$; $\sigma$ is strongly convex if $\sigma \cap (-\sigma) = \{0\}$. The dual cone $\sigma^\vee$ consists of those functionals in $N_\mathbb{R}^\vee = N^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ which are nonnegative on $\sigma$; it is a convex polyhedral cone in $N_\mathbb{R}^\vee$ but is only strongly convex if $\sigma$ has full dimension. (For instance, $\sigma = \{0\}$ is a strongly convex polyhedral cone with dual $\sigma^\vee = N_\mathbb{R}^\vee$.) We say $\sigma$ is rational if $v_1, \ldots, v_m$ can be taken in $N$, in which case $M_\sigma = \sigma^\vee \cap N^\vee$ is a toric monoid (Gordan’s lemma; see [63, Proposition 1.1]). Conversely, for any toric monoid $M$, the $\mathbb{R}_{\geq 0}$-span of $M$ is the dual cone of a strongly convex polyhedral cone in $N_\mathbb{R}$ for $N = (M^{gp})^\vee$.

A face of a cone $\sigma$ is the intersection of $\sigma$ with a supporting hyperplane. Note that if $\sigma$ is a strongly convex rational polyhedral cone, then any face spans a rational subspace of $N_\mathbb{R}$, inside of which the face is itself a strongly convex rational polyhedral cone.

**Definition 2.1.3.** Let $N$ be a finite free $\mathbb{Z}$-module. A fan in $N$ is a finite set $\Delta$ of strongly convex rational polyhedral cones in $N_\mathbb{R}$ satisfying the following conditions.

(a) Each face of a cone in $\Delta$ is itself a cone in $\Delta$.

(b) The intersection of two cones in $\Delta$ is a face of each.

**Definition 2.1.4.** Let $R$ be any ring. An affine toric variety over $R$ is a scheme of the form $\text{Spec}(R[M_\sigma])$ for some strongly convex rational polyhedral cone $\sigma$. (Here $R[M_\sigma]$ denotes the monoid ring over $R$.)

Let $\Delta$ be a fan in some finite free $\mathbb{Z}$-module $N$. For $\sigma, \tau \in \Delta$, the fan condition lets us identify $\text{Spec}(R[M_{\sigma \cap \tau}])$ with an open affine subscheme of each of $\text{Spec}(R[M_\sigma])$ and $\text{Spec}(R[M_\tau])$. Using these identifications to glue the schemes $\text{Spec}(R[M_\sigma])$ yields a scheme $X(\Delta)$, called the toric variety over $R$ associated to $\Delta$. Over a field, any toric variety is reduced, separated, and normal [30, §2.1] (but see Remark 2.0.4).

**Example 2.1.5.** Here are some basic examples of the previous construction, over an arbitrary ring $R$.

- For $\sigma = \mathbb{R}_{\geq 0}^n$, $R[M_\sigma] = R[T_1, \ldots, T_n]$ is the ordinary polynomial ring, and $\text{Spec}(R[M_\sigma])$ is the affine $n$-space over $R$.

- For $\sigma = \{0\} \subseteq \mathbb{R}^n$, $R[M_\sigma] = R[T_1^\pm, \ldots, T_n^\pm]$ is the Laurent polynomial ring in $n$ generators, and $\text{Spec}(R[M_\sigma])$ is the $n$-dimensional (split) algebraic torus over $R$. Consequently, any toric variety may be viewed as a partial compactification of a torus.

- For $n$ a positive integer, let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$, and put $\sigma_0 = \mathbb{R}_{\geq 0}^n$. For $i \in \{1, \ldots, n\}$, let $\sigma_i$ be the cone generated by $e_j$ for $j \neq i$, together with $-e_i - \cdots - e_n$. For $\Delta$ the fan $\{\sigma_0, \ldots, \sigma_n\}$, the associated toric variety over $R$ is the projective $n$-space over $R$. 

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Definition 2.1.6. Let $\Delta, \Delta'$ be fans with ambient $\mathbb{Z}$-modules $N, N'$. A morphism $\Delta \to \Delta'$ of fans is a $\mathbb{Z}$-linear homomorphism $\varphi : N \to N'$ with the property that for $\varphi_\mathbb{R} : N_\mathbb{R} \to N'_\mathbb{R}$ the $\mathbb{R}$-linear extension of $\varphi$, for each $\sigma \in \Delta$ there exists $\sigma' \in \Delta'$ for which $\varphi_\mathbb{R}(\sigma) \subseteq \sigma'$. Any such morphism induces a morphism of schemes $X(\Delta) \to X(\Delta')$ for any base ring; such a morphism of schemes is said to be toric. In particular, a morphism $\text{Spec}(R[\sigma]) \to \text{Spec}(R[\sigma'])$ of affine toric varieties is toric if and only if it is induced by a morphism of ambient $\mathbb{Z}$-modules whose $\mathbb{R}$-linear extension carries $\sigma$ into $\sigma'$.

Since we will be studying analytic spaces by locally embedding them into toric varieties, it is natural to mention the following result about embedding schemes into toric varieties. (We do not know whether the same result holds over an arbitrary field.)

Theorem 2.1.7 (Włodarczyk). Let $X$ be a normal reduced separated scheme of finite type over an algebraically closed field $F$. Then $X$ admits a closed immersion into a toric variety over $F$ if and only if any two closed points of $X$ can be found inside some open affine subscheme of $X$. In this case, if $X$ is also smooth, then it admits a closed immersion into a smooth toric variety.

Proof. See [78, Theorem A].

It is also natural to point out the relationship with logarithmic structures. For more on these, see [41] or [64].

Definition 2.1.8. Let $X$ be a scheme or a $K$-analytic space. A prelogarithmic structure (or prelog structure) on $X$ is a sheaf of monoids $M$ on the étale site of $X$ equipped with a homomorphism $\alpha : M \to \mathcal{O}_X^\times$ of sheaves of monoids, where $\mathcal{O}_X^\times$ refers to the underlying multiplicative monoid of the structure sheaf. A prelogarithmic structure $\alpha$ is a logarithmic structure (or log structure) if $\alpha$ induces an isomorphism $\alpha^{-1}(\mathcal{O}_X^\times) \cong \mathcal{O}_X^\times$, where $\mathcal{O}_X^\times$ refers to the submonoid of $\mathcal{O}_X^\times$ consisting of invertible sections. The forgetful functor from logarithmic to prelogarithmic structures has a left adjoint [41, §1.3].

A chart of a logarithmic structure $\alpha$ on $X$ is a prelogarithmic structure of the form $\beta : P_X \to \mathcal{O}_X$ for $P_X$ the constant sheaf defined by a fine monoid $P$ together with an isomorphism of the associated logarithmic structure of $\beta$ with $\alpha$. A chart is toric if its underlying monoid $P$ is toric. A logarithmic structure is fine (resp. toric) if it étale locally admits charts (resp. toric charts) everywhere.

Example 2.1.9. Any affine toric variety $\text{Spec}(R[\sigma])$ admits a natural toric chart defined by the multiplicative map $M_\sigma \to R[\sigma]$. Consequently, any toric variety admits a natural logarithmic structure, and toric morphisms between toric varieties are precisely those that define morphisms of logarithmic structures.

2.2 Toric frames

We now use affine toric varieties to provide local coordinates on analytic spaces.
Definition 2.2.1. For σ a strongly convex rational polyhedral cone, write $K(\sigma)$ for the analytification of $\text{Spec}(K[M_\sigma])$. This space may be viewed as the union of the affinoid spaces $\mathcal{M}(K\{M_\sigma\}_\lambda)$ over all $\lambda \in N_\mathbb{R}$ (or all $\lambda \in N_\mathbb{Q}$), where $K\{M_\sigma\}_\lambda$ denotes the completion of $K[M_\sigma]$ for the weighted Gauss norm

$$\left| \sum_{s \in M_\sigma} c_s s \right|_\lambda = \max_{s \in M_\sigma} \{|c_s| p^{\lambda(s)}\}.$$  

To lighten notation, we identify elements of $M_\sigma$ with functions on $K(\sigma)$ without putting brackets or other enclosing symbols around them.

By a (strictly) rational subdomain of $K(\sigma)$, we will mean a (strictly) rational subdomain $U$ of $\mathcal{M}(K\{M_\sigma\}_\lambda)$ for some $\lambda \in N_\mathbb{Q}$. Note that $U$ is then a (strictly) rational subdomain of $\mathcal{M}(K\{M_\sigma\}_\lambda)$ for any $\lambda \in N_\mathbb{Q}$ for which $U \subseteq \mathcal{M}(K\{M_\sigma\}_\lambda)$.

Definition 2.2.2. Let $A$ be either a reduced affinoid algebra over $K$ or, in case $K$ is discretely valued, a reduced mixed affinoid algebra over $K$. A toric frame for $A$ is an unramified morphism $\psi : \mathcal{M}(A) \to K(\sigma)$. In the affinoid case, the unramified condition may interpreted using Definition 1.1.10; a consistent interpretation which also covers the mixed affinoid case is that $\psi$ factors through a morphism $\mathcal{M}(A) \to \mathcal{M}(K\{M_\sigma\}_\lambda)$ such that the relative module of continuous differentials vanishes), then the composition $\psi' : \mathcal{M}(B) \to \mathcal{M}(A) \to K(\sigma)$ is again a toric frame. In case $\mathcal{M}(B)$ is a (strictly) affinoid, Weierstrass, Laurent, or rational subdomain of $\mathcal{M}(A)$, we say that $\psi'$ is a (strictly) affinoid, Weierstrass, Laurent, or rational subframe of $\psi$. A finite collection of subframes $\psi_i : \mathcal{M}(B_i) \to K(\sigma)$ of $\psi$ is a covering family (resp. a strong covering family) if $\mathcal{M}(A)$ is covered by the $\mathcal{M}(B_i)$ (resp. by the relative interiors of the $\mathcal{M}(B_i)$ in $\mathcal{M}(A)$).

Remark 2.2.3. Any affinoid algebra over $K$ admits at least one toric frame, because any strict surjection $K\{M_\sigma\}_\lambda \to A$ defines such a frame. A similar construction shows that any mixed affinoid algebra over $K$ admits at least one toric frame.

Remark 2.2.4. A toric frame $\psi : \mathcal{M}(A) \to K(\sigma)$ gives rise naturally to logarithmic structures on $\text{Spec}(A)$ and $\mathcal{M}(A)$, by pulling back the natural logarithmic structure on $K(\sigma)$.

Remark 2.2.5. If $\psi_1 : \mathcal{M}(A_1) \to K(\sigma)$ and $\psi_2 : \mathcal{M}(A_2) \to K(\sigma)$ are two toric frames, then the fibred product $\psi : \mathcal{M}(A_3) \to K(\sigma)$ is also a toric frame.
2.3 Maps between frames

So far, we have been working within a single toric variety. We next introduce some formalism for moving between different toric varieties.

**Definition 2.3.1.** For $\psi : \mathcal{M}(A) \to K(\sigma), \psi' : \mathcal{M}(A') \to K(\sigma')$ two toric frames, a *toric morphism* of frames from $\psi'$ to $\psi$ is a diagram

$$
\begin{array}{ccc}
\mathcal{M}(A') & \xrightarrow{\psi'} & K(\sigma') \\
\downarrow & & \downarrow \\
\mathcal{M}(A) & \xrightarrow{\psi} & K(\sigma)
\end{array}
$$

(2.3.1.1)

of morphisms of analytic spaces in which the right vertical arrow is toric. Such a morphism is a *toric refinement* if the map $\mathcal{M}(A') \to \mathcal{M}(A)$ is an isomorphism and the toric map $K(\sigma') \to K(\sigma)$ is induced by a splitting $M_{\sigma'} \cong M_\sigma \oplus T$ for some toric monoid $T$. A toric refinement is *boundary-free* if the image of $T$ in $A$ consists of invertible elements.

**Definition 2.3.2.** Let $\psi_1 : \mathcal{M}(A_1) \to K(\sigma_1)$ and $\psi_2 : \mathcal{M}(A_2) \to K(\sigma_2)$ be two toric frames, and let $\tau : \mathcal{M}(A_1) \to \mathcal{M}(A_2)$ be a morphism of spaces (necessarily defined by a bounded $K$-linear homomorphism $A_2 \to A_1$). We define the *framed graph* of $\tau$ to be the toric frame $\psi_3 : \mathcal{M}(A_3) \to K(\sigma_3)$ in which $A_3 = A_1$, $\sigma_3 = \sigma_1 \oplus \sigma_2$, and $\psi_3$ is obtained by identifying $K(\sigma_3)$ with $K(\sigma_1) \times_K K(\sigma_2)$ and taking the product of the morphisms $\psi_1$ and $\psi_2 \circ \tau$. From the projections out of the fibred product, we obtain toric morphisms $\psi_3 \to \psi_1$, $\psi_3 \to \psi_2$ whose underlying morphisms on spaces are the identification $\mathcal{M}(A_3) \cong \mathcal{M}(A_1)$ and the composition $\tau : \mathcal{M}(A_3) \cong \mathcal{M}(A_1) \to \mathcal{M}(A_2)$.

**Example 2.3.3.** In Definition 2.3.2 for $\tau$ an isomorphism, the framed graph is a toric refinement of both $\psi_1$ and $\psi_2$.

**Remark 2.3.4.** Using Remark 2.2.3 and Definition 2.3.2 it follows that if we start with the category of toric frames of affinoid (resp. semiaffinoid) spaces, then localize by formally inverting the toric refinements, we recover the category of affinoid (resp. semiaffinoid) spaces over $K$. This will imply that when using toric frames, most questions of functoriality will amount to checking compatibility of constructions of interest with the formation of toric refinements.

**Remark 2.3.5.** The analogue of Remark 2.3.4 would hold even if we used just locally closed immersions into toric varieties, rather than unramified morphisms. However, we will want to work with the étale topology, and it is easier to do so without having to change frames.

**Remark 2.3.6.** One can use toric frames to give a polyhedral interpretation of the geometry of nonarchimedean analytic spaces following Payne [65] or Hrushovski-Loeser [38]. Namely, for each strictly convex rational polyhedral cone $\sigma$, let $\bar{\sigma}$ be the closure of $\sigma$ in the space
of linear maps $\sigma^\vee : [0, +\infty] \to K(\sigma) \to \tilde{\sigma}$ characterized by

$$\text{Trop}(\sigma)(\alpha)(s) = -\log(\alpha(s)) \quad (\alpha \in K(\sigma), s \in M_\sigma).$$

For each frame $\psi : \mathcal{M}(A) \to K(\sigma)$, the composition $\text{Trop}(\psi) : \mathcal{M}(A) \to K(\sigma) \to \tilde{\sigma}$ (which one might call the *tropicalization* of the frame $\psi$) has polyhedral image. The natural map from $\mathcal{M}(A)$ to the inverse limit of the images of the $\text{Trop}(\psi)$ over all toric frames $\psi$ is a homeomorphism (and likewise for affine frames). Moreover, a subset of $\mathcal{M}(A)$ is a rational subdomain if and only if it appears as the inverse image of a rational polyhedral set under some $\text{Trop}(\psi)$.

**Remark 2.3.7.** Our use of framing data to study local systems admits parallels to several other constructions in algebraic geometry. Here are a few examples.

One can similarly work with framed affine schemes of finite type over a field, which come equipped with unramified morphisms to affine spaces. This gives an approach to the construction of algebraic de Rham cohomology (as in [35]): first define the cohomology of a framed affine scheme as the cohomology of the de Rham complex of its formal neighborhood in the ambient affine space, then show that toric refinements induce homotopy equivalences.

A similar setting is the construction of rigid cohomology introduced by Berthelot. Given a framed affine scheme in characteristic $p$, one works with a certain “$p$-adic tubular neighborhood” of the scheme in the generic fibre of the formal affine space. See [56] for details. (In both cases, it is sometimes convenient to use more general smooth affine schemes as the ambient spaces, but affine spaces are sufficient to get the theory started.)

A different but loosely analogous construction is the *thickening space* introduced by Xiao [79, 80] in order to relate the Abbes-Saito ramification filtration on the Galois group of a local field with imperfect residue field [1, 2] to $p$-adic differential equations. It would be interesting to understand how these spaces relate to ours.

### 2.4 Toric frames and perfectoid algebras

We now come to the fundamental role played by toric frames in $p$-adic Hodge theory: they provide a mechanism for converting ordinary affinoid algebras over a $p$-adic field into members of the special class of *perfectoid* Banach algebras considered in [50]. This is analogous to the role played by the arithmetically profinite (and hence deeply ramified) extension $\mathbb{Q}_p(\epsilon)$ in the usual theory of $(\varphi, \Gamma)$-modules. A similar construction has been introduced by Scholze in order to reduce certain cases of the weight-monodromy conjecture in étale cohomology from characteristic 0 to characteristic $p$ [70], and to construct comparison morphisms in $p$-adic Hodge theory [71].

**Hypothesis 2.4.1.** Throughout §2.4 assume that $K = \text{Frac}(W(k))$ for $k$ a perfect field of characteristic $p$. Put

$$K(\epsilon) = K[\mathbb{Q}_p/\mathbb{Z}_p]/\left(\sum_{i=0}^{p-1}[ip^{-1}]\right).$$
For $n \geq 0$, let $\epsilon_n$ be the class of $[p^{-n}]$ in $K(\epsilon)$; it is a primitive $p^n$-th root of unity. Let $A$ be a reduced affinoid or mixed affinoid algebra over $K$, and let $\psi : \mathcal{M}(A) \to K(\sigma)$ be a toric frame. Let $\alpha$ denote the spectral norm on $A$. Let $N$ denote the ambient $\mathbb{Z}$-module of $\sigma$, put $M_\sigma = N^\vee \cap \sigma^\vee$, and put $N_p = N \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

**Definition 2.4.2.** For $n = 0, 1, \ldots$, let $\varphi_n : K(\epsilon_{n+1})(\sigma) \to K(\epsilon_n)(\sigma)$ be the morphism acting as the Witt vector Frobenius on $K$, embedding $\epsilon_n$ into $K(\epsilon_{n+1})$ within $K(\epsilon)$, and taking $s \in M_\sigma$ to $s^p$. Put $\Phi_{n,\sharp} = \varphi_0 \circ \cdots \circ \varphi_{n-1}$. Let $A_{\psi,n}$ be the reduced quotient of $A \otimes_{K[\mathcal{M}_\sigma],\Phi_{n,\sharp}} K(\epsilon_n)[M_\sigma]$. From this description, $A_{\psi,n+1}$ is a finite $A_{\psi,n}$-module, and (since $A_{\psi,n}$ is reduced) the induced map $\varphi_{n,\sharp} : A_{\psi,n} \to A_{\psi,n+1}$ is injective. Equip $A_{\psi,n}$ with its spectral norm $\alpha_{\psi,n}$. By interpreting $\alpha_{\psi,n}$ as a supremum norm [50 Theorem 2.3.10], we deduce that $\varphi_{n,\sharp}$ is isometric. Let $A_{\psi,\infty}$ be the completed direct limit of the $A_{\psi,n}$, and let $\alpha_{\psi,\infty}$ be the induced power-multiplicative norm on $A_{\psi,\infty}$.

**Remark 2.4.3.** For $n$ running over all nonnegative integers, the base change functor

$$\text{F\acute{e}t}(\lim_{\rightarrow} A_{\psi,n}) \to \text{F\acute{e}t}(A_{\psi,\infty})$$

is a tensor equivalence by [50 Proposition 2.6.9].

**Remark 2.4.4.** If $\psi$ is boundary-free, then the morphisms $A_{\psi,n} \to A_{\psi,n+1}$ are étale. Otherwise, they are only log-étale for the toric logarithmic structures indicated in Remark 2.2.4. This will force us to keep track of certain extra conditions when trying to perform Galois descent; see for example Definition 3.3.5.

**Lemma 2.4.5.** If $R$ is a reduced $\mathbb{Z}[p^{-1}]$-algebra and $r \in R$ is not a zero-divisor, then $R[T]/(T^p - r)$ is also reduced.

**Proof.** By quotienting by each minimal prime ideal of $R$ in turn, we reduce to the case where $R$ is an integral domain and $r$ is nonzero. In this case, put $F = \text{Frac}(R)$; then $T^p - r$ is a separable polynomial over $F$, so $F[T]/(T^p - r)$ is a product of fields into which $R[T]/(T^p - r)$ embeds. This proves the claim. \hfill $\square$

**Remark 2.4.6.** Suppose that the image of each irreducible component of $\mathcal{M}(A)$ under $\psi$ meets the torus inside $K(\sigma)$. Then the elements of $M_\sigma$ do not map to zero divisors in $A$, so by Lemma 2.4.5, the tensor product $A \otimes_{K[\mathcal{M}_\sigma],\Phi_{n,\sharp}} K(\epsilon_n)[M_\sigma]$ is reduced. In particular, this holds when $\psi$ is boundary-free, but in this case it follows more directly from the fact that a finite étale algebra over a reduced ring is reduced.

The utility of the above construction depends upon progress towards the following conjecture.

**Conjecture 2.4.7.** The Frobenius map $\varphi$ on $\mathfrak{o}_{A_{\psi,\infty}}/(p)$ is surjective.

**Example 2.4.8.** Conjecture 2.4.7 holds when $A = K$ and $\psi : \mathcal{M}(A) \to \mathcal{M}(K)$ is the identity map, as in this case $\mathfrak{o}_{A_{\psi,\infty}}$ is the $p$-adic completion of $W(k)[\epsilon_1, \epsilon_2, \ldots]$. More generally, Conjecture 2.4.7 also holds whenever $A = K\{M_\sigma\}_\lambda$ for some $\sigma$ and $\lambda$ and $\psi : \mathcal{M}(A) \to K(\sigma)$ is the natural embedding. This follows from the previous observation plus the existence of the $K(\epsilon)$-linear endomorphism of $A_{\psi,\infty}$ induced by multiplication by $p$ on $M_\sigma$.  

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Remark 2.4.9. If Conjecture \([2.4.7]\) holds for a given frame \(\psi\), then \(A_{\psi,\infty}\) is a perfectoid uniform Banach \(\mathbb{Q}_p\)-algebra in the sense of [50, Definition 3.6.1]. This follows from the first part of Example \([2.4.8]\) which implies that in all cases \(A_{\psi,\infty}\) is a Banach algebra over a perfectoid field.

Since we do not have a proof of Conjecture \([2.4.7]\) we will instead establish that the class of rings for which it holds is stable under various natural operations.

Definition 2.4.10. We say that the toric frame \(\psi\) is eligible if the conclusion of Conjecture \([2.4.7]\) holds.

Remark 2.4.11. The fibred product of two eligible frames with the same target is again eligible, by [50, Corollary 3.6.17].

Theorem 2.4.12. Assume that \(\psi\) is eligible. Let \(A \to B\) be a \(K\)-linear bounded unramified homomorphism of affinoid or mixed affinoid algebras over \(K\), and let \(\psi' : \mathcal{M}(B) \to \mathcal{M}(A) \to K(\sigma)\) be the composite frame.

(a) If \(A \to B\) is a rational localization, then so is \(A_{\psi,\infty} \to B_{\psi',\infty}\), and \(\psi'\) is eligible.

(b) If \(A \to B\) has dense image (e.g., if \(\mathcal{M}(B) \to \mathcal{M}(A)\) is a closed immersion), then so does \(A_{\psi,\infty} \to B_{\psi',\infty}\), and \(\psi'\) is eligible.

(c) If \(A \to B\) is finite étale, then so is \(A_{\psi,\infty} \to B_{\psi',\infty}\), and \(\psi'\) is eligible.

(d) If \(B = R[p^{-1}]\) for \(R\) the completion of \(\mathfrak{o}_A\) with respect to a finitely generated ideal containing \(p\), then \(\psi'\) is eligible. (Note that finite generation is automatic if \(A\) is a mixed affinoid algebra over \(K\), as then \(\mathfrak{o}_A\) is noetherian.)

Proof. We may deduce (a), (b), (c) from [50, Theorems 3.6.14, 3.6.16, 3.6.20], respectively. We may deduce (d) from [50, Proposition 3.6.18].

Remark 2.4.13. Beware that surjectivity of \(A \to B\) does not imply that \(A_{\psi,\infty} \to B_{\psi',\infty}\) is surjective, only that it has dense image. See Example [4.1.11].

Theorem \([2.4.12]\) implies a local version of Conjecture \([2.4.7]\) which will be sufficient for global study of relative \((\varphi, \Gamma)\)-modules on Berkovich analytic spaces over \(K\).

Corollary 2.4.14. Any toric frame \(\psi : \mathcal{M}(A) \to K(\sigma)\) admits a strong covering family by eligible rational subframes.

Proof. By Theorem \([2.4.12]\) any subframe of \(\psi\) which is the composition of a closed immersions, a finite étale cover, and a rational subdomain embedding is eligible. In the affinoid case, the claim thus follows from the fact that any unramified morphism locally admits a factorization of this form (Remark [1.1.11]). In the mixed affinoid case, we may make a similar argument once we allow the factorization to include a completion as in Theorem \([2.4.12]\)(d).
For compatibility with the work of Andreatta and Brinon [5], we mention another method for generating eligible frames.

**Theorem 2.4.15.** Let $L$ be a finite extension of $K$. Let $d$ be a nonnegative integer. Let $R$ be a $p$-adically complete and separated, noetherian $\mathfrak{O}_L\{T_1^\pm,\ldots,T_d^\pm\}$-algebra which is flat over $\mathfrak{O}_L$ and satisfies the following conditions.

(a) The set $\{T_1^{e_1},\ldots,T_d^{e_d} : e_1,\ldots,e_d = 0,\ldots,p-1\}$ forms a basis of $R \otimes_{\mathfrak{O}_L} \kappa_L$ as a module over its image under Frobenius.

(b) The ring $R \otimes_{\mathfrak{O}_L} \kappa_L^{\text{alg}}$ is integral.

Put $A = R \otimes_{\mathfrak{O}_L} L$. Then the map $\mathfrak{O}_K\{T_1^\pm,\ldots,T_d^\pm\} \to R$ induces an eligible frame $\psi : \mathcal{M}(A) \to K(\sigma)$.

**Proof.** Condition (a) implies that $\mathfrak{O}_K\{T_1^\pm,\ldots,T_d^\pm\} \to R$ is formally unramified, so $\psi$ is indeed a frame. By [4, Theorem 3.7], there exists an element $\lambda \in \mathfrak{m}_K(\xi)$ such that Frobenius is surjective on $\mathfrak{O}_{A_{\psi,\infty}}(\lambda)$. By [50, Proposition 3.6.2(c)], this implies that $A_{\psi,\infty}$ is perfectoid.

**Remark 2.4.16.** If $\psi$ is eligible, then for any $B \in \mathcal{F}\mathbf{Et}(A)$, $\bigcup_n \mathfrak{O}_{A_{\psi,n}} \otimes_A B$ is almost finite projective and almost finite étale over $\bigcup_n \mathfrak{O}_{A_{\psi,n}}$ in the sense of Faltings (see [50, §5.5] for brief definitions and [31] for fuller context). Namely, this follows from the perfectoid property of $A_{\psi,\infty}$ (Remark 2.4.9) plus a very strong form of Faltings’s almost purity theorem; see either [50, Theorem 5.5.9] or the analogous result of Scholze [70]. By Theorem 2.4.15, this in turn implies Faltings’s original almost purity theorem [25, 26]. We will not use almost purity explicitly in what follows, but it is closely related to the arguments of [50] which feed into our constructions.

**Remark 2.4.17.** At present, we do not know how to prove Conjecture 2.4.7 even in the simple case where $\psi$ is an affinoid subdomain embedding. One plausible approach would be to deduce this from Theorem 2.4.12 using a simplicial covering as in Remark 4.1.6; however, this is likely to run into difficulties of the sort suggested by Remark 1.3.4. Another plausible approach would be to first reduce to the case where $\psi$ is étale using Theorem 2.4.12(b), then go back to the ramification-theoretic approach to the field of norms equivalence used for instance in [1]. To make this work, it may become necessary to incorporate the more sophisticated ramification filtrations introduced by Abbes and Saito [1, 2].

## 3 Perfect period rings

We now make perfect period rings associated to framed affinoid spaces over $p$-adic fields. This involves using toric coordinates as in §2.4 to pass from an affinoid space to a suitably ramified cover. We obtain a notion of relative $(\varphi,\Gamma)$-modules and a relationship between such objects and local systems. We also give an alternate description of $(\varphi,\Gamma)$-modules in
terms of vector bundles on certain schemes, in the style of the work of Fargues and Fontaine [27]. Finally, we globalize the construction by performing descent along toric refinements.

Our results extend work of many authors. To streamline the exposition, we have reserved most comparisons to existing literature to a separate section at the end of the paper (§4.11).

**Hypothesis 3.0.1.** For the remainder of the paper, set notation as in Hypothesis 2.4.1 and assume further that the toric frame \( \psi \) is eligible.

**Notation 3.0.2.** For a positive integer, write \( \mathbb{Z}_{p^d} \) for the finite étale \( \mathbb{Z}_p \)-algebra with residue ring \( \mathbb{F}_{p^d} \), and write \( \mathbb{Q}_{p^d} \) for \( \mathbb{Z}_{p^d}[p^{-1}] \).

### 3.1 Reduction to positive characteristic

In §2.4, it was explained how to associate to the frame \( \psi \) a Banach algebra \( A_{\psi, \infty} \) with the useful property of being perfectoid. This implies a certain strong relationship with analytic geometry in positive characteristic; we now trace back through this relationship, as developed in [50], in order to set some key notations for what follows.

**Definition 3.1.1.** Let \( \mathfrak{o}_{\mathcal{T}_\psi} \) be the inverse perfection of \( \mathfrak{o}_{A_{\psi, \infty}} \) [50, Definition 3.4.1]. For \( \theta : W(\mathfrak{o}_{\mathcal{T}_\psi}) \to \mathfrak{o}_{A_{\psi, \infty}} \) the map defined as in [50, Definition 3.4.3], we may equip \( \mathfrak{o}_{\mathcal{T}_\psi} \) with the power-multiplicative seminorm \( \pi = \mu(\theta^*(\alpha_{\psi, \infty})) \). By [50, Lemma 3.4.5], \( \pi \) is a norm under which \( \mathfrak{o}_{\mathcal{T}_\psi} \) is complete. Put \( \omega = p^{-\nu}/(p-1) \), \( \overline{\mathcal{T}_\psi} = (\ldots, \epsilon_1 - 1, \epsilon_0 - 1) \in \mathfrak{o}_{\mathcal{T}_\psi} \), and \( \pi = [1 + \overline{\mathcal{T}_\psi}]^{-1} \in W(\mathfrak{o}_{\mathcal{T}_\psi}) \). Note that \( \pi^\psi(x) = \omega \pi^\psi(x) \quad (x \in \mathfrak{o}_{\mathcal{T}_\psi}). \) (3.1.1.1)

We extend \( \pi^\psi \) multiplicatively to \( \overline{A}_\psi = \mathfrak{o}_{\mathcal{T}_\psi}[[\pi^{-1}]] \) so that (3.1.1.1) holds also for \( x \in \overline{A}_\psi \). The map \( \theta \) extends to a map \( \theta : \overline{R}_{\mathcal{T}_\psi}^{\text{int}, 1} \to A_{\psi, \infty} \); we will use the symbol \( \theta \) to refer to the extended map unless otherwise specified.

**Example 3.1.2.** Suppose that \( A = K \). By [50, Example 3.3.8], \( \overline{A}_\psi \) is the completed perfection of \( k((\overline{\mathcal{T}_\psi})) \).

By the same token, suppose that \( A = K \{M_\nu \}_\lambda \) for some \( \lambda \in N_{\mathbb{Q}} \). In this case, we may identify \( \overline{A}_\psi \) with the completed direct perfection \( R_\lambda \) of \( k((\overline{\mathcal{T}_\psi})) \{M_\nu \}_\lambda \).

**Theorem 3.1.3.** (a) The maps \( \theta : \overline{R}_{\mathcal{T}_\psi}^{\text{int}, 1} \to A_{\psi, \infty} \) and \( \varphi : \mathfrak{o}_{A_{\psi, \infty}}/(p) \to \mathfrak{o}_{A_{\psi, \infty}}/(p) \) are surjective.

(b) For every \( x \in A_{\psi, \infty} \), there exists \( y = \sum_{i=0}^\infty [y_i] p^i \in W(\mathfrak{o}_{\mathcal{T}_\psi})[[\pi^{-1}]] \) with \( \theta(y) = x \) and \( \pi^\psi(y_0) \geq \pi^\psi(y_i) \) for all \( i > 0 \). In particular, \( \lambda(\pi^\psi)(y) = \alpha_{\psi, \infty}(x) \), so \( \theta \) is optimal (and hence strict).

(c) The kernel of \( \theta \) is generated by \( z = \sum_{i=0}^{p-1} [\overline{\mathcal{T}_\psi} + 1]^i/p \).
(d) There is a natural homeomorphism $\mathcal{M}(A_{\psi,\infty}) \cong \mathcal{M}(\bar{A}_\psi)$ under which (strictly) rational subdomains on both sides correspond. Moreover, if $\bar{B}$ represents a rational subdomain of $\mathcal{M}(\bar{A}_\psi)$, then $W(\mathcal{O}_\mathbb{P})[[\mathcal{O}]]/(z)$ represents the corresponding rational subdomain of $\mathcal{M}(A_{\psi,\infty})$.

(e) There is an equivalence of tensor categories between $\mathbf{F\acute{e}t}(A_{\psi,\infty})$ and $\mathbf{F\acute{e}t}(\bar{A}_\psi)$ under which $B \in \mathbf{F\acute{e}t}(A_{\psi,\infty})$ corresponds to $W(o_{\bar{B}})(\mathcal{O})/\mathcal{O}$. Moreover, the objects of both categories may be naturally viewed as uniform Banach algebras which are finite Banach algebras over their respective base rings.

Proof. By [50, Example 3.3.5], the element $z$ described in (c) is primitive of degree 1 in the sense of [50, Definition 3.3.4]. Part (a) holds by Theorem 2.4.12. Given (a), we may deduce (b) and (c) from [50, Lemma 5.5.5], (d) from [50, Theorem 3.3.7, Theorem 3.6.14], and (e) from [50, Theorem 3.6.20].

### 3.2 Action of $\Gamma$

The passage from $A$ to $A_{\psi,\infty}$ gives rise to a Galois group which we must keep track of as we go along.

**Definition 3.2.1.** Let $\Gamma_N$ be the semidirect product $\mathbb{Z}_p^\times \rtimes N_p$. Define an action of $\Gamma_N$ on $A_{\psi,n}$ via isometric automorphisms so that $\gamma \in \mathbb{Z}_p^\times$ acts via the cyclotomic character on $K(\epsilon_n)$ (i.e., carrying $\epsilon_n$ to $\epsilon_n^\gamma$) and fixes $M_{\sigma}$, while $\nu \in N_p$ acts via the formula

$$\nu \left( \sum_{s \in M_{\sigma}} c_s s \right) = \sum_{s \in M_{\sigma}} \epsilon_n^{(\nu,s)} c_s s.$$

A more compact way to express this action is to express any $\gamma \in \Gamma_N$ as $\nu \cdot \gamma_0$ with $\nu \in N_p, \gamma_0 \in \mathbb{Z}_p^\times$ and set $\langle \gamma, s \rangle = \langle \nu, s \rangle$, so that $\gamma(s) = \epsilon_n^{(\gamma, s)} s$ for all $\gamma \in \Gamma_N$. The maps $\varphi_n$ are equivariant for these actions, so we get an action on $A_{\psi,\infty}$; this action is continuous because $\Gamma_N$ acts by isometries and each element of the dense subring $\bigcup_n A_{\psi,n}$ of $A_{\psi,\infty}$ is fixed by some open subgroup of $\Gamma_N$. This action transfers to a continuous action on $\bar{A}_\psi$ with the property that

$$\gamma(1 + \mathfrak{p}) = (1 + \mathfrak{p})^{\gamma_0}, \quad \gamma(s) = (1 + \mathfrak{p})^{(\nu,s)} s \quad (\gamma = \nu \cdot \gamma_0, \nu \in N_p, \gamma_0 \in \mathbb{Z}_p^\times).$$

**Lemma 3.2.2.** For each $n$, the projection $\mathcal{M}(A_{\psi,n}) \to \mathcal{M}(A)$ is surjective and identifies $\mathcal{M}(A)$ with the quotient of $\mathcal{M}(A_{\psi,n})$ by the action of $\Gamma_N$.

Proof. Surjectivity follows from the fact that the morphism $A \to A \otimes_K[M_{\sigma}, \varphi_n] K(\epsilon_n)[M_{\sigma}]$ of $K$-Banach algebras is faithfully finite flat and so induces a surjective map on spectra. Since any continuous surjection between compact spaces is a quotient map [50, Remark 2.3.15(b)], it remains to check that $\Gamma_N$ acts transitively on fibres.

To lighten notation, put $B = A_{\psi,n}$. Choose any $\alpha \in \mathcal{M}(A)$, and put $L = \mathcal{H}(\alpha)$, $A_L = A \hat{\otimes}_K L$, and $B_L = B \hat{\otimes}_K L$. We may view $A$ as the intersection of the kernels of
the maps $g - 1 : B \to B$ for $g \in \Gamma_N$; these maps, being $A$-linear endomorphisms of the finite Banach $A$-module $B$, are necessarily strict because $A$ is noetherian (compare \cite[Lemma 2.5.2]{50}). By \cite[Lemma 2.2.10(b)]{50}, $B^\Gamma_L = A_L$.

From the natural map $A \to \mathcal{H}(\alpha) = L$, we obtain a map $A_L = \widehat{A \otimes_K L} \to \widehat{L \otimes_K L}$; composing with the multiplication map $L \otimes_K L \to L$ gives a surjective map $A_L \to L$. Let $p$ be the kernel of this map; it is a maximal ideal of $A_L$. Note that $G$ permutes the fibres of $\text{Spec}(B_L) \to \text{Spec}(A_L)$ transitively (see for instance \cite[Exercise 5.13]{7}), so $G$ acts transitively on the set of primes of $B_L$ above $p$.

Similarly, for each $\beta \in \mathcal{M}(B)$ extending $\alpha$, we get a map $B_L = B \widehat{\otimes_K L} \to \mathcal{H}(\beta) \widehat{\otimes_K L} \to \mathcal{H}(\beta)$ in which the last arrow is the multiplication map. The kernel of this map is a prime ideal $q$ lying above $p$, from which we may reconstruct $\beta$ by extending the norm on the field $A_L/p \cong L$ to its finite extension $B_L/q$, then restricting along $B \to B_L \to B_L/q$. Consequently, $G$ acts transitively on the fibre of $\mathcal{M}(B) \to \mathcal{M}(A)$ above $\alpha$, as desired. \qed

**Remark 3.2.3.** We may identify $\mathcal{M}(\overline{A}_\psi) \cong \mathcal{M}(A_{\psi, \infty})$ (see Theorem 3.1.3) with the inverse limit of the $\mathcal{M}(A_{\psi, n})$. By Lemma 3.2.2 the projection $\pi : \mathcal{M}(A_{\psi, \infty}) \to \mathcal{M}(A)$ identifies $\mathcal{M}(A)$ with the quotient of $\mathcal{M}(A_{\psi, \infty}) \cong \mathcal{M}(\overline{A}_\psi)$ by the action of $\Gamma_N$. It follows that $\pi$ is not only a closed map, but also an open map: if $U \subseteq \mathcal{M}(A_{\psi, \infty})$ is open, then $\pi^{-1}(\pi(U)) = \bigcup_{\gamma \in \Gamma_N} \gamma(U)$ is also open in $\mathcal{M}(A_{\psi, \infty})$, so $\pi(U)$ is open in $\mathcal{M}(A)$.

**Remark 3.2.4.** Note that if $\epsilon < 1$ and $x \neq g(x)$ for some $x \in \overline{A}_\psi$ and some $g \in U$, then there exists a nonnegative integer $n$ for which

$$\overline{\alpha}_\psi(x^{p^{-n}} - g(x^{p^{-n}})) > \epsilon \overline{\alpha}_\psi(x^{p^{-n}}).$$

It follows that while the actions of $\Gamma_N$ on $\overline{A}_\psi$ and $A_{\psi, \infty}$ are continuous, they are not in general analytic; that is, one does not obtain actions of the Lie algebra of $\Gamma_N$. This can only be remedied by restricting $\psi$ suitably and then forming imperfect period rings; see \cite[4.4]{34}.

**Lemma 3.2.5.** For $R \in \text{FÉt}(\overline{A}_\psi)$ and $B \in \text{FÉt}(A_{\psi, \infty})$ corresponding via Theorem 3.1.3(e) and admitting compatible actions of $\Gamma_N$, the following conditions are equivalent.

(a) The action of $\Gamma_N$ on $R$ is continuous.

(b) The action of $\Gamma_N$ on $B$ is continuous.

(c) There exist a positive integer $n$, an object $B_n \in \text{FÉt}(A_{\psi, n})$, an action of $\Gamma_N$ on $B_n$ which is trivial on some open subgroup of $\Gamma_N$, and a $\Gamma_N$-equivariant isomorphism $B_n \otimes_{A_{\psi, n}} A_{\psi, \infty} \cong B$.

**Proof.** The equivalence of (a) and (b) follows by writing

$$B = W(\mathfrak{o}_R)[[\pi^{-1}]]/(z), \quad R = \mathfrak{o}_B^\text{frep}[[\pi^{-1}]].$$

It is clear that (c) implies (b), so we need only check that (b) implies (c); moreover, by \cite[Theorem 2.6.10]{50}, we may work locally around some $\beta \in \mathcal{M}(A)$. 

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Suppose that the action of $\Gamma_N$ on $B$ is continuous. From Remark 2.4.3 we obtain a positive integer $m$, an element $B_m \in \mathbf{F\acute{E}t}(A_{\psi,m})$, and an isomorphism $B_m \otimes_{A_{\psi,m}} A_{\psi,\infty} \cong B$. Pick any $\beta_m \in \mathcal{M}(A_{\psi,m})$ lifting $\beta$. By passing from $A$ to a suitable rational localization encircling $\beta$, we may ensure that $B_m = A_{\psi,m}[T]/(P(T))$ for some $P(T) \in A_{\psi,m}[T]$ such that for $x \in B_m$ the image of $T$, $P'(x)$ vanishes nowhere on $\mathcal{M}(B_m)$. Note that $P'(x)$ is therefore a unit in $B_m$ by [50] Corollary 2.3.7.

Write $Q(T) = P(T + x) = \sum_{i > 0} Q_i T^i$ with $Q_i \in B_m$ and $Q_1 \in B_m$. For $\gamma$ in any open subgroup of $\Gamma_N$ fixing $A_{\psi,m}$, we have $\gamma(P) = P$, so $0 = \gamma(P(x)) = \gamma(P)(\gamma(x)) = P(\gamma(x)) = Q(\gamma(x) - x)$. However, because $\Gamma_N$ acts continuously on $B$, for $\gamma$ in a sufficiently small open subgroup of $\Gamma_N$, we have

$$\alpha_{\psi,\infty}(Q(\gamma(x) - x)) = \alpha_{\psi,\infty}(Q_1(\gamma(x) - x)) \geq \alpha_{\psi,\infty}(Q_1^{-1})^{-1}\alpha_{\psi,\infty}(\gamma(x) - x).$$

It follows that $\gamma$ fixes $x$, from which the desired result follows.

Remark 3.2.6. When $A$ is an affinoid algebra and $\psi$ is étale, the equivalent conditions of Lemma 3.2.5 are always satisfied. See Proposition A.1.7 for an even stronger statement.

Remark 3.2.7. Beware that the conditions of Lemma 3.2.5 do not imply that $B^{\Gamma_N} = B_n$; in particular, it does not follow that $A^{\Gamma_N}_{\psi,\infty} = A$. This last statement is indeed true, but a far more intricate argument is needed; see Theorem 4.10.5.

Remark 3.2.8. Let $\psi': \mathcal{M}(A') \to K(\sigma')$ be a second toric frame, and specify a morphism of toric frames as in Definition 2.3.1. We then obtain functoriality maps between each period ring $*\psi$ and the corresponding period ring $*\psi'$; these maps are continuous for all available topologies and equivariant with respect to $\varphi$. They are also equivariant for the $\Gamma$-actions in the following sense: the given toric morphism induces a $\mathbb{Z}_p$-linear homomorphism $N_p' \to N_p$ and hence a group homomorphism $\Gamma_{N'} \to \Gamma_N$, and the functoriality maps are equivariant for the action of $\Gamma_{N'}$ on $*\psi$ via $\Gamma_{N'} \to \Gamma_N$ and the usual action of $\Gamma_{N'}$ on $*\psi$.

3.3 Local systems

Using $\Gamma$, we can relate local systems over $A$, $A_{\psi,\infty}$, and $\overline{A}_\psi$ as follows. For convenience, we begin by recopying some relevant facts about faithfully flat descent from [50] Theorems 1.3.4 and 1.3.5.

Theorem 3.3.1 (Faithfully flat descent). Let $f : R \to S$ be a faithfully flat morphism of rings.

(a) The morphism $f$ is an effective descent morphism for the category of modules over rings.

(b) An $R$-module $U$ is finite (resp. finite projective) if and only if $f^*U = U \otimes_R S$ is a finite (resp. finite projective) $S$-module.

(c) An $R$-algebra $U$ is finite étale if and only if $f^*U$ is a finite étale $S$-algebra.

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Definition 3.3.2. Let $G$ be a group acting on a ring $R$. Let $c, d$ be integers with $d > 0$. Let $E$ be an étale sheaf, étale $\mathbb{Z}_{p^d}$-local system, étale $\mathbb{Q}_{p^d}$-local system, or étale $(c, d)$-$\mathbb{Q}_{p^d}$-local system on $\text{Spec}(R)$. An action of $G$ on $E$ is given by specifying for each $g \in G$ an isomorphism $\iota(g) : E \cong g^*E$, subject to the restriction that $\iota(g_1 g_2) = g_2^* (\iota(g_1)) \circ \iota(g_2)$. For example, if $G$ acts trivially on $R$, there is a trivial action on $E$ using the natural identifications $E \cong g^*E$ for all $g \in G$.

Definition 3.3.3. Let $G$ be a profinite topological group acting continuously on a Banach ring $R$. Suppose first that $E$ is a locally constant $\mathbb{Z}/p^m \mathbb{Z}$-sheaf on the étale site of $\text{Spec}(R)$ on which $G$ acts. The sheaf $E$ is then represented by a finite étale $R$-algebra $R_m$ on which $G$ also acts. Equip $R_m$ with a topology by viewing it as a finite projective $R$-algebra and arguing as in [50, Lemma 2.2.12]. We say that the action of $G$ on $E$ is continuous if the action of $G$ on $R_m$ is continuous.

Suppose next that $E = \{ \cdots \to E_2 \to E_1 \}$ is an étale $\mathbb{Z}_{p^d}$-local system on $\text{Spec}(R)$. We say that an action of $G$ on $E$ is continuous if the action of $G$ on $E_m$ is continuous for each $m$.

Suppose next that $E$ is an étale $\mathbb{Q}_{p^d}$-local system on $\text{Spec}(R)$. Write $E = F \otimes_{\mathbb{Z}_{p^d}} \mathbb{Q}_{p^d}$ for some $\mathbb{Z}_{p^d}$-local system $F$ on $\text{Spec}(R)$, on which $G$ need not act. We say that an action of $G$ on $E$ is continuous if for some (and hence any) choice of $F$, there is an open subgroup $H$ of $G$ which acts continuously on $F$.

Suppose next that $E$ is an étale $(c, d)$-$\mathbb{Q}_p$-local system on $\text{Spec}(R)$. We say that an action of $G$ on $E$ is continuous if the action on the underlying $\mathbb{Q}_{p^d}$-local system is continuous.

Suppose finally that $E$ is an étale $(c, d)$-$\mathbb{Q}_p$-local system on $\mathcal{M}(R)$. We say that an action of $G$ on $E$ is continuous if there exist an open subgroup $H$ of $G$ and a covering family $R \to R_1, \ldots, R \to R_n$ of rational localizations stable under $H$, such that $E$ restricts to an étale $(c, d)$-$\mathbb{Q}_{p^d}$-local system on $\text{Spec}(R_1 \oplus \cdots \oplus R_n)$ on which $H$ acts continuously.

Remark 3.3.4. In the case $R \in \mathbf{F\acute{e}t}(A_{\psi, \infty})$ and $G = \Gamma_N$, Lemma 3.2.5 implies that an action of $G$ on an étale $\mathbb{Z}_{p^d}$-local system $E = \{ \cdots \to E_2 \to E_1 \}$ is continuous if and only if for each positive integer $m$, there exist a nonnegative integer $n$ and an open subgroup $H$ of $G$ fixing $A_{\psi,n}$ such that the action of $H$ on $E_m$ arises from the trivial action on some locally constant $\mathbb{Z}/p^m \mathbb{Z}$-sheaf on $\text{Spec}(A_{\psi,n})$. Using this interpretation, the continuity condition may be seen to be local for the étale topology.

In the case at hand, we need an additional condition besides continuity.

Definition 3.3.5. Let $E$ be an étale sheaf, étale $\mathbb{Z}_{p^d}$-local system, étale $\mathbb{Q}_{p^d}$-local system, or étale $(c, d)$-$\mathbb{Q}_{p^d}$-local system on $\text{Spec}(A_{\psi, \infty})$, equipped with an action of an open subgroup $H$ of $\Gamma_N$. We say that this action is effective if for each finitely generated submonoid $T$ of $M_\sigma$, the subgroup of $N_p \cap H$ acting trivially on $A_{\psi, \infty}/(s^{1/p^n} : s \in T, n \geq 0)$ also acts trivially on the pullback of $E$ to $\text{Spec}(A_{\psi, \infty}/(s^{1/p^n} : s \in T, n \geq 0))$. We make similar definitions over $\mathcal{M}(A_{\psi, \infty})$. This condition is automatic when $\psi$ is boundary-free.

Remark 3.3.6. A more geometric interpretation of the effectivity condition is that the restriction of $E$ to the formal completion along the toric boundary descends to the boundary itself. This condition appears naturally in the proof of Theorem 3.3.7 below.

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Theorem 3.3.7. For any open subgroup $H$ of $\Gamma_N$, for any $c,d,m \in \mathbb{Z}$ with $d,m > 0$, the categories of locally constant étale $\mathbb{Z}/p^m\mathbb{Z}$-sheaves, étale $\mathbb{Z}/p^m\mathbb{Z}$-local systems, étale $\mathbb{Q}_p^m$-local systems, and étale $(c,d)$-$\mathbb{Q}_p$-local systems over $\text{Spec}(\bigcup_n A_{\psi,n}^H)$ are respectively equivalent to the categories of locally constant étale $\mathbb{Z}/p^m\mathbb{Z}$-sheaves, étale $\mathbb{Z}/p^m\mathbb{Z}$-local systems, étale $\mathbb{Q}_p^m$-local systems, and étale $(c,d)$-$\mathbb{Q}_p$-local systems over $\text{Spec}(A_{\psi,\infty})$ carrying continuous effective actions of $H$. The same is also true with $\text{Spec}(\bigcup_n A_{\psi,n}^H), \text{Spec}(A_{\psi,\infty})$ replaced by $\mathcal{M}(\bigcup_n A_{\psi,n}^H), \mathcal{M}(A_{\psi,\infty})$, respectively.

We are forced to write $\bigcup_n A_{\psi,n}^H$ instead of $A_{\psi,\infty}^H$ because although these two rings do ultimately coincide, we do not yet have this statement in hand (see Remark 3.2.7).

Proof. It is enough to check the claim for locally constant étale $\mathbb{Z}/p^m\mathbb{Z}$-sheaves. Keep in mind that the reduction to this case requires arbitrary $H$ even if we start with $H = \Gamma_N$.

The functor from objects over $\text{Spec}(\bigcup_n A_{\psi,n}^H)$ to objects over $\text{Spec}(A_{\psi,\infty})$ equipped with continuous effective $H$-actions is just base extension, which we need to check is essentially surjective. We do this by induction primarily on rank($N$) and secondarily on the dimension of the $\mathbb{R}$-span of $\sigma$. Note that by Remark 3.3.3, continuity of the action gives a descent to a sheaf over $\text{Spec}(\bigcup_n A_{\psi,n}^H)$ for some open subgroup $H'$ of $H$.

If $\sigma = \{0\}$, then the effectivity condition is empty. The residual action of $H/H'$ defines a descent datum; since $\bigcup_n A_{\psi,n}^{H'}$ is faithfully finite étale over $\bigcup_n A_{\psi,n}^H$, we may conclude by Theorem 3.3.1.

If $\sigma \neq \{0\}$, we can find some $s \in M_\sigma$ which is not invertible. Let $I$ be the ideal of $\bigcup_n A_{\psi,n}^{H'}$ generated by $s^{1/p^m}$ for all nonnegative integers $m$ for which $s^{1/p^m} \in \bigcup_n A_{\psi,n}^{H'}$; there are only finitely many such $m$, so the $I$-adic and $s$-adic topologies on $\bigcup_n A_{\psi,n}^{H'}$ coincide. Put $B_1 = \bigcup_n A_{\psi,n}^{H'}[s^{-1}], B_1' = \bigcup_n A_{\psi,n}^{H'}[s^{-1}]$, and let $B_2, B_2'$ be the $s$-adic completions of $\bigcup_n A_{\psi,n}^{H'}, \bigcup_n A_{\psi,n}^{H'}$, respectively. The restriction of the given sheaf to $\text{Spec}(B_1')$ descends to $\text{Spec}(B_1)$ by the induction hypothesis (since we are considering the same $N$ but smaller $\sigma$). The restriction to $\text{Spec}(B_2')$ descends uniquely to $\text{Spec}(B_2'/I)$ by the effectivity condition, and then descends to $\text{Spec}(B_2/(I \cap B_2))$ by the induction hypothesis again (this corresponds to finitely many cases with smaller $N$, plus some glueing on overlaps), and then extends to $\text{Spec}(B_2)$ using the equivalence $\text{FÉt}(B_2) \to \text{FÉt}(B_2/(I \cap B_2))$ from [50, Theorem 1.2.8]. Since $\bigcup_n A_{\psi,n}^{H'}$ is an affinoid algebra, it is noetherian, so the map $\bigcup_n A_{\psi,n}^{H'} \to B_2$ is flat; consequently, $\text{Spec}(B_1 \oplus B_2)$ is an fpqc cover of $\text{Spec}(\bigcup_n A_{\psi,n}^{H'})$ on which we have now exhibited a descent datum. By Theorem 3.3.1 we obtain the desired result.  

3.4 Perfect period rings and $(\varphi, \Gamma_N)$-modules

In preparation for a description of étale local systems, we introduce a suite of period rings associated to the frame $\psi$ and a notion of relative $(\varphi, \Gamma)$-modules. We characterize these period rings as perfect because the Frobenius map $\varphi$ acts on them via isomorphisms; we will introduce a contrasting suite of imperfect period rings somewhat later.
Definition 3.4.1. For $0 < s \leq r$, put
\[
\begin{align*}
\tilde{A}_\psi &= W(\mathbf{A}_\psi), & \tilde{A}_\psi^r &= \tilde{R}_{\mathbf{A}_\psi}^\text{int}, & \tilde{A}_\psi^1 &= \tilde{R}_{\mathbf{A}_\psi}^\text{int}, \\
\tilde{B}_\psi &= A_\psi[p^{-1}], & \tilde{B}_\psi^r &= A_\psi^r[p^{-1}], & \tilde{B}_\psi^1 &= A_\psi^1[p^{-1}], \\
\tilde{C}^{[s,r]}_\psi &= \tilde{R}^{[s,r]}_\psi, & \tilde{C}^r_\psi &= \tilde{R}^r_\psi, & \tilde{C}_\psi &= \tilde{R}_{\mathbf{A}_\psi},
\end{align*}
\]

in the sense of [50, Definition 5.1.1]. We will refer collectively to these rings (including $\mathbf{A}_\psi$) as the \textit{perfect period rings} associated to $\psi$. All of these rings inherit bijective actions of $\varphi$ and $\Gamma_N$ except that $\varphi$ maps $\tilde{A}_\psi^1[p^{-1}]$ to $\tilde{A}_\psi^1/p$ and so forth.

Remark 3.4.2. Recall that the rings introduced in Definition 3.4.1 carry certain topologies described in [50, Definition 5.1.3]: the rings contained in $\tilde{B}_\psi$ carry a $p$-adic topology and a weak topology, the rings contained in $\tilde{C}^{[s,r]}_\psi$ or $\tilde{C}^r_\psi$ carry a Fréchet topology, and the rings contained in $\tilde{C}_\psi$ carry an LF (limit-of-Fréchet) topology. The actions of $\varphi$ and $\Gamma_N$ are continuous for all of these topologies except that $\Gamma_N$ does not act continuously for the $p$-adic topology. (This would require the action of $\Gamma_N$ on $\mathbf{A}_\psi$ to be trivial on some open subgroup, which it isn’t.)

Lemma 3.4.3. If $A$ is connected, then $(\tilde{B}_\psi)^{\varphi, \Gamma_N} = (\tilde{C}_\psi)^{\varphi, \Gamma_N} = \mathbb{Q}_p$.

Proof. By [50, Corollary 5.2.4],
\[
(\tilde{B}_\psi)^{\varphi, \Gamma_N} = (\tilde{C}_\psi)^{\varphi, \Gamma_N} = W(\mathbf{A}_\psi)^{\varphi, \Gamma_N}[p^{-1}].
\]

Given $e \in \mathbf{A}_\psi^{\varphi, \Gamma_N}$, decompose $e$ as $\sum_{i \in \mathbb{F}_p} i e_i$ as in [50, Lemma 3.1.2]; then each $e_i$ is a $\Gamma_N$-invariant idempotent in $\mathbf{A}_\psi$. However, since $A$ is connected, so is $\mathcal{M}(A)$ by Kiehl’s theorem [50, Theorem 2.5.13(b)]. Hence $\mathcal{M}(\mathbf{A}_\psi) \cong \mathcal{M}(A_{\psi, \infty})$ admits no proper nonempty $\Gamma_N$-invariant closed and open subsets, so $\mathbf{A}_\psi$ has no $\Gamma_N$-invariant idempotents other than 0 or 1. We conclude that $e \in \mathbb{F}_p$, proving the claim. \hfill \Box

Definition 3.4.4. For $d$ a positive integer, let $M$ be a $\varphi^d$-module or local $\varphi^d$-module over a perfect period ring $R$ equipped with a semilinear action of $\Gamma_N$ on $M$ commuting with $\varphi^d$. The $\Gamma_N$-action is \textit{continuous} if the action map $\Gamma_N \times M \to M$ is continuous for all of the available topologies on $R$. (We omit the $p$-adic topology because the action is not even continuous on the ring $R$.) The action is \textit{effective} if for each finitely generated submonoid $T$ of $M_\eta$, the subgroup of $N_\eta$ fixing $R/(s^{1/n}) : s \in T, n \geq 0$ also acts trivially on $M/(s^{1/n}) : s \in T, n \geq 0)M$. If both conditions are satisfied, we describe $M$ as a $(\varphi^d, \Gamma_N)$-module or local $(\varphi^d, \Gamma_N)$-module; we say $M$ is pure, étale, globally pure, or globally étale as a (local) $(\varphi^d, \Gamma_N)$-module if it is pure, étale, globally pure, or globally étale as a (local) $\varphi^d$-module (in the sense of [50 §7.3]). Note that we do not insist on any compatibility between $\Gamma_N$ and pure models.
Example 3.4.5. The element
\[ t = \log(1 + \pi) = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{\pi^i}{i} \in \tilde{C}_\psi \]
satisfies \( \varphi(t) = pt \) and \( \gamma(t) = \gamma \cdot t \) for \( \gamma \in \mathbb{Z}_p^\times \) and is fixed by \( N_p \). We may thus view \( t^{-1}\tilde{C}_\psi \) as a \((\varphi, \Gamma_N)\)-module over \( \tilde{C}_\psi \) which is pure of slope 1. More generally, for \( M \) a \((\varphi, \Gamma_N)\)-module over \( \tilde{C}_\psi \) and \( m \in \mathbb{Z} \), we may view \( t^{-m}M \) as a \((\varphi, \Gamma_N)\)-module over \( \tilde{C}_\psi \); we also denote this \((\varphi, \Gamma_N)\)-module by \( M(m) \) and describe it as a twist of \( M \).

We may formally factor
\[ t = u\pi \prod_{i=1}^{\infty} \frac{(1 + \pi)^{p^i} - 1}{(1 + \pi)^{p^i-1} - 1} \]
for some unit \( u \in \tilde{C}_\psi \). For each positive integer \( d \), put
\[ t_d = \prod_{i=1}^{\infty} \frac{(1 + \pi)^{p^di} - 1}{(1 + \pi)^{p^di-1} - 1} ; \]
then \( t_d^{-1}\tilde{C}_\psi \) is a \((\varphi^d, \Gamma_N)\)-module over \( \tilde{C}_\psi \) which is pure of slope \( 1/d \).

Remark 3.4.6. Let \( M \) be a \((\varphi, \Gamma_N)\)-module over \( \tilde{C}_\psi \). As a function on \( \mathcal{M}(\tilde{A}_\psi) \cong \mathcal{M}(A_{\psi, \infty}) \), the slope polygon of \( M \) is \( \Gamma_N \)-invariant; it thus descends to a map on \( \mathcal{M}(A) \). Since the map \( \mathcal{M}(A_{\psi, \infty}) \to \mathcal{M}(A) \) is a quotient map (see Remark 3.2.3), we may descend various assertions about the slope polygon from \( \mathcal{M}(A_{\psi, \infty}) \) to \( \mathcal{M}(A) \), such as the following.

- The slope polygon function is bounded and lower semicontinuous on \( \mathcal{M}(A) \) \[50\], Theorem 7.4.5, Proposition 7.4.6]. Consequently, the pure locus and the étale locus are open.
- There is an open dense subset of \( \mathcal{M}(A) \) on which the slope polygon is locally constant \[50\], Corollary 7.4.7].
- If the slope polygon function is locally constant, then \( M \) admits a global slope filtration \[50\], Theorem 7.4.8].

Definition 3.4.7. Let \( d \) be a positive integer. For \( M \) a \((\varphi^d, \Gamma_N)\)-module over any perfect period ring \( R \), start with the complex
\[ 0 \to M \xrightarrow{\varphi^d - 1} M \to 0, \quad (3.4.7.1) \]
then replace \( M \) in each position by the complex of continuous effective \( \Gamma_N \)-cochains with values in \( M \). Here we mean that a cochain, viewed as a function on \( \Gamma_N^i \) for some nonnegative integer \( i \), is effective if for each finitely generated submonoid \( T \) of \( M_\sigma \), for \( H \) the subgroup of \( N_p \) fixing \( R/(s^{1/p^n} : s \in T, n \geq 0) \), elements of \( \Gamma_N^i \) which are termwise congruent modulo \( H \) map to elements of \( M \) which are congruent modulo \( (s^{1/p^n} : s \in T, n \geq 0) \). Let \( H^i_{\varphi^d, T}(M) \) denote the total cohomology of the resulting double complex.
Remark 3.4.8. If $ψ$ is a boundary-free frame, then $H^i_{ψ, Γ}(M)$ is the $i$-th hypercohomology of the complex (3.4.7.1) induced by continuous $Γ_N$-cohomology (i.e., the functor of invariants on the category of topological abelian groups equipped with continuous actions of $Γ_N$). This suggests a reinterpretation in terms of Scholze’s pro-étale topology, which we will introduce in a subsequent paper.

Remark 3.4.9. With definition as in Remark 3.2.8 we obtain base extension functors from (local) $(ϕ, Γ_N)$-modules on $ψ$ to (local) $(ϕ, Γ_N')$-modules on $ψ'$; these preserve the globally pure/étale and pure/étale conditions.

3.5 $(ϕ, Γ_N)$-modules and local systems

Using Theorem 3.3.7 (to pass to a deeply ramified cover) and the results of [50] (to relate characteristic 0 to characteristic $p$), we can now use perfect period rings to describe étale local systems and their étale cohomology.

Theorem 3.5.1. For $d$ a positive integer such that $Q_{pd} ⊆ K$, the following tensor categories are equivalent.

(a) The category of étale $Z_{pd}$-local systems over Spec$(A)$.

(b) The category of $(ϕ^d, Γ_N)$-modules over $\hat{A}_ψ$.

(c) The category of $(ϕ^d, Γ_N)$-modules over $\hat{A}_ψ^\dagger$.

More precisely, the functor from (c) to (b) is base extension.

Proof. The equivalence of (a) and (b) follows from Theorem 3.3.7 and [50] Theorem 8.1.2], as does full faithfulness of base extension from (c) to (b). To check essential surjectivity of this base extension, it suffices to check that for any $ϕ^d$-module $M^\dagger$ over $\hat{A}_ψ^\dagger$, if $M = M^\dagger ⊗_{\hat{A}_ψ^\dagger} A_ψ$ admits a continuous effective $Γ_N$-action, then this action induces a continuous effective action on $M^\dagger$. We obtain an effective action on $M^\dagger$ by [50] Theorem 8.1.2], but it remains to check continuity since there is no LF topology on $\hat{A}_ψ$.

By [50] Theorem 8.1.2] again, the underlying $ϕ^d$-module of $M^\dagger$ corresponds to a $Z_{pd}$-local system $T = \{ \cdots → T_2 → T_1 \}$ on Spec$(\overline{A}_ψ)$, which we may safely assume is of constant rank $m > 0$. For each positive integer $n$, let $T_n$ be represented by $\overline{U}_n ∈ \text{FÉt}(\overline{A}_ψ)$. Let $U$ be the completed direct limit of the $\overline{U}_n$, and put $M_U = M ⊗_{\hat{A}_ψ} W(U)$. We then have

$$M^\dagger = (M^\dagger_U ⊗_{W(U)} \overline{R}_{\text{int}})^{GLm(Z_{pd})}, \quad M = (M_U^\dagger ⊗_{W(U)} W(U))^{GLm(Z_{pd})}. $$

The action of $Γ_N$ on $M^\dagger_U$ is continuous for the weak topology, but on this set the $p$-adic, weak, and LF topologies coincide [50] Remark 5.1.5]. We thus deduce that the action of $Γ_N$ on $M^\dagger$ is continuous for all topologies, as desired.

Theorem 3.5.2. For $d$ a positive integer such that $Q_{pd} ⊆ K$, the following tensor categories are equivalent.
(a) The category of étale $(c,d)$-$\mathbb{Q}_p^d$-local systems over $\text{Spec}(A)$.

(b) The category of globally $(c,d)$-pure $(\varphi^d,\Gamma_N)$-modules over $\tilde{B}_\psi$.

(c) The category of globally $(c,d)$-pure $(\varphi^d,\Gamma_N)$-modules over $\tilde{B}_\psi^\dagger$.

(d) The category of globally $(c,d)$-pure $(\varphi^d,\Gamma_N)$-modules over $\tilde{C}_\psi$.

More precisely, the functors from (c) to (b) and (d) are base extensions.

Proof. This follows from Theorem 3.3.7 and [50, Theorem 8.1.4] by arguing as in Theorem 3.5.1 to match up topologies.

Theorem 3.5.3. For $d$ a positive integer such that $\mathbb{Q}_p^d \subseteq K$, the following tensor categories are equivalent.

(a) The category of étale $(c,d)$-$\mathbb{Q}_p^d$-local systems over $\mathcal{M}(A)$.

(b) The category of $(c,d)$-pure local $(\varphi^d,\Gamma_N)$-modules over $\tilde{B}_\psi$.

(c) The category of $(c,d)$-pure local $(\varphi^d,\Gamma_N)$-modules over $\tilde{B}_\psi^\dagger$.

(d) The category of $(c,d)$-pure $(\varphi^d,\Gamma_N)$-modules over $\tilde{C}_\psi$.

More precisely, the functors from (c) to (b) and (d) are base extensions.

Proof. This follows from Theorem 3.5.2 and the fact that $\varphi^d$-modules over $\tilde{C}_\psi$ glue over covering families of rational localizations [50, Remark 6.4.2].

Theorem 3.5.4. For $d$ a positive integer such that $\mathbb{Q}_p^d \subseteq K$, let $T$ be an étale $\mathbb{Z}_p^d$-local system on $\text{Spec}(A)$. Let $M$ be the $(\varphi^d,\Gamma_N)$-module over one of $\tilde{A}_\psi$ or $\tilde{A}_\psi^\dagger$ corresponding to $T$ via Theorem 3.5.1. Then for $i \geq 0$, there is a natural (in $T$ and $A$) bijection $H^i_{\text{ét}}(\text{Spec}(A), T) \cong H^i_{\varphi^d,\Gamma}(M)$.

Proof. The claim follows immediately from [50, Theorem 8.2.1] after noting that one can compute the étale cohomology of $T/(p^n)$ in the fpqc topology [60, Proposition 3.7].

Theorem 3.5.5. For $d$ a positive integer such that $\mathbb{Q}_p^d \subseteq K$, let $E$ be an étale $\mathbb{Q}_p^d$-local system on $\text{Spec}(A)$. Let $M$ be the globally étale $(\varphi^d,\Gamma_N)$-module over one of $\tilde{B}_\psi, \tilde{B}_\psi^\dagger, \tilde{C}_\psi$ corresponding to $E$ via Theorem 3.5.2. Then for $i \geq 0$, there is a natural (in $E$ and $A$) bijection $H^i_{\text{ét}}(\text{Spec}(A), E) \cong H^i_{\varphi^d,\Gamma}(M)$.

Proof. The claim over $\tilde{B}_\psi$ and $\tilde{B}_\psi^\dagger$ follows from Theorem 3.5.4 (as in the proof of [50, Theorem 8.2.3]). We may then deduce the claim over $\tilde{C}_\psi$ from the claim over $\tilde{B}_\psi^\dagger$ by comparing $\varphi$-cohomology using [50, Theorem 8.2.3].
Theorem 3.5.6. For $d$ a positive integer such that $\mathbb{Q}_{p^d} \subseteq K$, let $E$ be an étale $\mathbb{Q}_{p^d}$-local system on $\mathcal{M}(A)$. Let $M$ be the étale $(\varphi^d, \Gamma_N)$-module over $\mathcal{C}_\psi$ corresponding to $E$ via Theorem 3.5.3. Then for $i \geq 0$, there is a natural (in $E$ and $A$) bijection $H^i_{\text{ét}}(\mathcal{M}(A), E) \cong H^i_{\varphi^d, \Gamma}(M)$.

Proof. By Theorem 3.5.4, we may compute the étale cohomology of $E$ using a complex with $\varphi^d$, $\Gamma$, and Čech differentials. On each $\Gamma$-level, we may apply [50, Theorem 8.3.6] to eliminate the higher Čech terms; this leaves behind exactly the complex computing $(\varphi^d, \Gamma)$-cohomology.

3.6 $(\varphi, \Gamma)$-modules, vector bundles, and $B$-pairs

We can interpret $(\varphi, \Gamma_N)$-modules over $\mathcal{C}_\psi$ in the language of vector bundles on certain schemes; this generalizes a result of Fargues-Fontaine for $A = K$ [27]. We also obtain a similar description in the language of $B$-pairs as introduced by Berger [11].

Definition 3.6.1. Write $P_\psi$ for the graded ring $P_{\mathcal{C}_\psi}$ of [50, Definition 6.3.1]. By a $\Gamma_N$-vector bundle on $\text{Proj}(P_\psi)$, we mean a quasicoherent finite locally free sheaf on $\text{Proj}(P_\psi)$ equipped with a continuous (as in [50, Definition 6.3.17]) effective action on $\Gamma_N$.

Definition 3.6.2. Put $B_{e, \psi} = P_\psi[t^{-1}]_0$. Let $B_{dR, \psi}^{\nabla^+}$ be the $\pi$-adic completion of $\hat{B}_{e, \psi}^{\nabla^+}$; note that $\text{Spec}(B_{dR, \psi}^{\nabla^+})$ may be naturally identified with the $t$-adic completion of $\text{Proj}(P_\psi)$. Put $B_{dR, \psi}^{\nabla^+} = B_{dR, \psi}^{\nabla^+}[t^{-1}]$; this ring receives natural maps from both $B_{e, \psi}$ and $B_{dR, \psi}^{\nabla^+}$. All of these rings inherit actions of $\varphi$ and $\Gamma_N$.

By a $B$-pair over $\psi$, we will mean a triple $M = (M_e, M_{dR}^{\nabla^+}, i)$ in which $M_e$ is a finite projective module over $B_{e, \psi}$ equipped with a continuous effective action of $\Gamma_N$, $M_{dR}^{\nabla^+}$ is a finite projective module over $B_{dR, \psi}^{\nabla^+}$ equipped with a continuous effective action of $\Gamma_N$, and

$$i : M_e \otimes_{B_{e, \psi}} B_{dR, \psi}^{\nabla^+} \cong M_{dR}^{\nabla^+} \otimes_{B_{dR, \psi}^{\nabla^+}} B_{dR, \psi}^{\nabla^+}$$

is a $\Gamma_N$-equivariant isomorphism. We denote the object defined by this isomorphism by $M_{dR}^{\nabla^+}$.

Theorem 3.6.3. The following categories are naturally (in $\psi$) equivalent.

(a) The category of $(\varphi, \Gamma_N)$-modules over $\mathcal{C}_\psi$.

(b) The category of $\Gamma_N$-vector bundles on $\text{Proj}(P_\psi)$.

(c) The category of $B$-pairs over $\psi$.

Proof. The equivalence between (a) and (b) follows from [50, Theorem 6.3.12]. The equivalence between (b) and (c) follows from [50, Proposition 8.3.10].
Remark 3.6.4. The definition of $B$-pairs amounts to using a covering of $\operatorname{Proj}(P_\psi)$ in the flat topology rather than the Zariski topology (modulo the fact that the map $\text{Spec}(B^\psi_{\text{dr},\psi}) \to \operatorname{Proj}(P_\psi)$ is not in general known to be flat). This covering has the convenient feature of being $\Gamma_N$-stable; we use this feature next to interpret cohomology of $(\varphi, \Gamma)$-modules in terms of vector bundles. This is necessary to work around the fact that it is tricky to make sense of equivariant coherent cohomology of sheaves on $\operatorname{Proj}(P_\psi)$ with continuous action of $\Gamma_N$, because $\Gamma_N$ does not act continuously on $\operatorname{Proj}(P_\psi)$.

Definition 3.6.5. Let $V$ be a $\Gamma_N$-vector bundle on $\operatorname{Proj}(P_\psi)$. For $j \geq 0$, let $C_{0,j}^i$ (resp. $C_{1,j}^i$) be the set of continuous effective $j$-cochains for $\Gamma_N$ with values in $\Gamma(\text{Spec}(B_{\psi,\psi}) \cup \text{Spec}(B^\psi_{\text{dr},\psi}), V)$ (resp. $\Gamma(\text{Spec}(B^\psi_{\text{dr},\psi}), V)$). The $C_{1,j}^i$ form a double complex (using Čech differentials along $i$ and cochain differentials along $j$), whose total cohomology we call $H^i(\operatorname{Proj}(P_\psi), \Gamma_N; V)$.

Theorem 3.6.6. Let $V$ be the $\Gamma_N$-vector bundle on $\operatorname{Proj}(P_\psi)$ corresponding to the $(\varphi, \Gamma_N)$-module $M$ over $\mathcal{C}_\psi$ via Theorem 3.6.3. Then for $i \geq 0$, there is a natural (in $V$ and $\psi$) bijection $H^i(\operatorname{Proj}(P_\psi), \Gamma_N; V) \cong H^i_{\text{dR}}(\mathcal{M}(A), E) \cong H^i(\operatorname{Proj}(P_\psi), \Gamma_N; V)$.

Proof. This follows at once from [50, Theorem 8.3.3] and the fact that one may compute cohomology of quasicoherent sheaves on $\operatorname{Proj}(P_\psi)$ using the Čech complex for the covering by $\text{Spec}(B_{\psi,\psi})$ and $\text{Spec}(B^\psi_{\text{dr},\psi})$ [50, Proposition 8.3.12].

Remark 3.6.7. As in [50, Remark 8.3.7], we may reinterpret Theorem 3.5.6 as follows. Let $E$ be an étale $\mathbb{Q}_\varphi$-local system on $\mathcal{M}(A)$. Associate to $E$ an étale $(\varphi, \Gamma_N)$-module $M$ over $\mathcal{C}_\psi$ via Theorem 4.9.4, then associate to $M$ a $\Gamma_N$-vector bundle $V$ on $\operatorname{Proj}(P_\psi)$ via Theorem 3.6.3. By Theorem 3.5.6 and Theorem 3.6.6, for $i \geq 0$, we have natural (in $E$ and $A$) bijections $H^i_{\text{dR}}(\mathcal{M}(A), E) \cong H^i(\operatorname{Proj}(P_\psi), \Gamma_N; V)$.

Remark 3.6.8. The symbol $\nabla$ in the notation for the rings $B^\psi_{\text{dr},\psi}, B^\psi_{\text{dr},\psi}$ indicates that these should be considered as the kernels of integrable connections on certain larger rings. Such larger rings appear in the work of Scholze [71]; we will encounter them later in this series.

3.7 Compatibility with toric refinements

In preparation for globalizing the definition of relative $(\varphi, \Gamma)$-modules, we consider their interaction with base extension along toric refinements; for technical reasons, we only get results for boundary-free refinements. It will be useful to set up a formalism akin to that used in the study of faithfully flat descent in algebraic geometry.

Hypothesis 3.7.1. Throughout §3.7 consider a boundary-free toric refinement of toric frames with notation as in §2.3.1.1. Let $d$ be any positive integer.

Let $\psi'' : \mathcal{M}(A) \to K(\sigma'')$ be a second copy of $\psi'$. Write $M_{\sigma'} = M_\sigma \oplus \overline{M}$ and $M_{\sigma''} = M_\sigma \oplus \overline{M}$ with $\overline{M} \cong M$. Let $\psi''' : \mathcal{M}(A) \to K(\sigma''')$ be the diagonal map from $\mathcal{M}(A)$ to $K(\sigma') \times_{K(\sigma)} K(\sigma'')$ and identify $M_{\sigma'''}$ with $M_\sigma \oplus \overline{M} \oplus \overline{M}$.
Lemma 3.7.2. Let \( L \) be an analytic field of characteristic 0. For any \( c \in [p^{-n/(p-1)}, 1] \), if \( x, y \in L^x \) satisfy \( |x^p - y^p| \leq c|x|^p \), then \( |x - y| \leq c^{1/p}|x| \).

Proof. See [17, Lemma 10.2.2].

Lemma 3.7.3. Let \( L \) be an analytic field of characteristic 0, and suppose \( x, y \in L^x \) satisfy \( |x^p - y| \leq c|y| \) for some \( c \in [p^{-n/(p-1)}, 1] \). Then for any \( P(T) \in L[T] \) of degree less than \( p \), the \( p^{-1/(p-1)}|y|^{1/p} \)-Gauss norm of \( P(T) \) is at most \( pc^{(p-1)/p} \) times the spectral seminorm of \( P(T - x) \) in \( L[T]/(T^p - y) \).

Proof. We may enlarge \( L \) to include a primitive \( p \)-th root of unity \( \zeta_p \) and a \( p \)-th root \( w \) of \( y \); by Lemma 3.7.2, we have \( |w - x| \leq c^{1/p}|w| \). The spectral seminorm of \( P(T - w) \) may be computed as \( \max\{P(\zeta_p^i - 1)w) : i = 0, \ldots, p - 1 \} \). This maximum is bounded above by the \( p^{-1/(p-1)}|w| \)-Gauss norm of \( P \), but for \( \deg(P) < p \) the reverse inequality also holds because the ratios \( (\zeta_p^i - 1)w/(\zeta_p - 1)w \) for \( i = 0, \ldots, p - 1 \) have distinct images in \( \kappa_L \). It is thus sufficient to observe that since \( \deg(P) \leq p - 1 \), the \( p^{-1/(p-1)}|y|^{1/p} \)-Gauss norm of \( P(T) \) is at most \( (|w - x|/(p^{-1/(p-1)}|y|^{1/p}))^{p-1} \) times the \( p^{-1/(p-1)}|y|^{1/p} \)-Gauss norm of \( P(T) \) in the generator \( T + x - w \).

Lemma 3.7.4. Let \( R \) be a perfectoid uniform Banach \( \mathbb{Q}_p \)-algebra. For each nonnegative integer \( n \), view \( R_n = R[T_n]/(T_n^p - r) \) as a perfectoid uniform Banach algebra with norm \( \alpha_n \), using the compatibility of the perfectoid correspondence with finite étale algebras [50, Theorem 3.6.20]. For \( n > 0 \), view \( R_n \) as an \( R_{n-1} \)-algebra by identifying \( T_{n-1} \) with \( T_n \). Then for any \( n > 0 \) and any \( \epsilon > 0 \), there exists an \( R_{n-1} \)-linear splitting of the map \( R_{n-1} \rightarrow R_n \) which is bounded of norm at most \( 1 + \epsilon \).

Proof. Choose \( c \in (p^{-n/(p-1)}, 1] \) so that \( pc^{(p-1)/p} \leq 1 + \epsilon \). Choose a nonnegative integer \( m \) for which \( p^{-1-p^{-1} \cdots - p^{-m}} \leq c \). Since \( R_{n-1} \) is perfectoid, we can apply [50, Corollary 3.6.7] to find \( t_n \in R_{n-1} \) for which for each \( \beta \in \mathcal{M}(R_{n-1}) \),

\[ \beta(T_{n-1} - t_n^p) \leq p^{-1-p^{-1} \cdots - p^{-m}} \max\{\beta(T_{n-1}), \alpha_{n-1}(T_{n-1}^{-1})^{-1}\} \leq c\beta(T_{n-1}). \]

We define a splitting by lifting each element of \( R_n \) to an element of \( R_{n-1}[T] \) of degree less than \( p \), then evaluating at \( T = t_n \). This has the desired effect by Lemma 3.7.3.

Proposition 3.7.5. The homomorphism \( A_{\psi, \infty} \rightarrow A_{\psi', \infty} \) of Banach modules over \( A_{\psi, \infty} \) splits.

Proof. By induction, the claim reduces to the case where \( \mathbb{M} \) is the free monoid on a single generator \( T \). Define \( R_n \) as in Lemma 3.7.4 for \( R = A_{\psi, \infty} \) and \( r = \psi'_n(T) \). By that lemma, for each positive integer \( n \), we can find an \( R_{n-1} \)-linear splitting of the map \( R_n \rightarrow R_{n-1} \) which is bounded of norm at most \( p^{p^{-n}} \). We then get a splitting of \( A_{\psi, \infty} \rightarrow \bigcup R_n \) which is bounded of norm at most \( p^{p^{-1}+p^{-2}+\cdots} < p^{1/(p-1)} \). It thus extends to the completion \( A_{\psi', \infty} \) of \( \bigcup R_n \), as desired.
Definition 3.7.6. Put $L = \text{Hom}(\overline{M}, \mathbb{Z}_p)$. By identifying $\text{Hom}(\overline{M}', \mathbb{Z}_p)$ also with $L$ via the identification $\overline{M}' \cong \overline{M}'$, we may define a composition of homomorphisms

$$N_p''' \to \text{Hom}(\overline{M}', \mathbb{Z}_p) \oplus \text{Hom}(\overline{M}', \mathbb{Z}_p) \to L$$

in which the first map is the natural projection and the second map acts by $(a, b) \mapsto b - a$.

Define the action of $N_p'''$ on $L$ by translation via this homomorphism; this extends to an action of $\Gamma_{N'''}$ by taking $\mathbb{Z}_p^\times$ to act on $L$ by scalar multiplication.

Let $\text{Cont}(L, *_{\psi''})$ denote the ring of continuous maps from $L$ to $\ast_{\psi''}$. We obtain an action of $\Gamma_{N'''}$ on this ring by taking the inverse transpose of the action on $L$ and the action on $\ast_{\psi''}$ via $\Gamma_{N'''}$: for $T \in \text{Cont}(L, *_{\psi''})$ and $\nu \in L$, $\gamma(T)(\nu) = \gamma(T(\gamma^{-1}(\nu)))$.

Lemma 3.7.7. For $0 < s \leq r$, for

$$* = A_{\ast_\infty}, \overline{A}, \overline{A}, \overline{A}, \overline{A}, \overline{B}, \overline{B}, \overline{B}, \overline{C}, \overline{C}, \overline{C}, \overline{C},$$

there is a canonical $\Gamma_{N'''}$-equivariant isomorphism $*_{\psi''} \to \text{Cont}(L, *_{\psi''})$. (In case the refinement is not boundary-free, we still obtain an $\Gamma_{N'''}$-equivariant isometric homomorphism.)

Proof. For each $\nu \in L$, there is a homomorphism $\pi_\nu : *_{\psi''} \to *_{\psi''}$ such that for each $s' \in \overline{M}'$, corresponding to $s'' \in \overline{M}'$, we have

$$\pi_\nu(s') = (1 + \overline{\gamma})^{(\nu, s')} s''.$$

Note that for each $\beta \in M(\overline{A}_{\psi''})$, there is a unique choice of $\nu$ for which $\beta$ belongs to the image of $\pi_\nu$. We may thus combine the maps $\pi_\nu$ to obtain the desired isometric homomorphism $*_{\psi''} \to \text{Cont}(L, *_{\psi''})$.

We next check that this homomorphism is $\Gamma_{N'''}$-equivariant. This means that for all $x \in *_{\psi''}$ corresponding to $T_x \in \text{Cont}(L, *_{\psi''})$, all $\nu \in L$, and all $\gamma \in \Gamma_{N'''}$,

$$\gamma(T_x(\gamma^{-1}(\nu))) = T_{\gamma(x)}(\nu).$$

For starters, $\text{(3.7.7.1)}$ is clear for $x \in *_{\psi''}$, as then $T_x$ is the constant function on $L$ with value $x$. In light of this observation, it suffices to check $\text{(3.7.7.1)}$ when $x = s' \in \overline{M}'$. For $\gamma \in \mathbb{Z}_p^\times$, we have $\gamma(s') = s'$ and

$$\gamma(T_{s'}(\gamma^{-1}(\nu))) = \gamma((1 + \overline{\gamma})^{(\nu, s')} s'') = (1 + \overline{\gamma})^{(\nu, s')} s'' = T_{s'}(\nu) = T_{\gamma(s')}(\nu),$$

so we need only check $\text{(3.7.7.1)}$ for $\gamma \in N'''$. Let $a, b$ be the projections of $\gamma$ to $\text{Hom}(\overline{M}', \mathbb{Z}_p) \cong L$, $\text{Hom}(\overline{M}', \mathbb{Z}_p) \cong L$; then

$$\gamma(T_{s'}(\gamma^{-1}(\nu))) = \gamma((1 + \overline{\gamma})^{(\gamma^{-1}(\nu), s')} s'') = (1 + \overline{\gamma})^{(\nu + a - b, s')} s''$$

$$= (1 + \overline{\gamma})^{(\nu, s')} (1 + \overline{\gamma})^{(a, s')} s''$$

$$= T_{\gamma(s')}(\nu).$$

We thus obtain $\text{(3.7.7.1)}$, establishing the desired equivariance.
We finally establish that the homomorphism is bijective. We need only consider the case \( * = A_{\psi, \infty} \); we may further reduce to the case where \( \overline{M} \) is freely generated by a single element \( T' \). Let \( T'' \) be the corresponding element of \( \overline{M}'' \). Since \( \psi' \) is a boundary-free toric refinement of \( \psi \), \( (T'/T'')^{p^n} \) is an element of \( A_{\psi'', \infty} \), and the corresponding function on \( L \) is a primitive character of order \( p^n \). By discrete Fourier analysis, the image of \( A_{\psi'', \infty} \to \text{Cont}(L, A_{\psi', \infty}) \) includes all locally constant functions; since such functions are dense in \( \text{Cont}(L, A_{\psi', \infty}) \), this proves the claim.

**Corollary 3.7.8.** With notation as in Lemma \[3.7.7\], the fixed subrings of \( *_{\psi''} \) under \( \ker(\Gamma_{N''} 
rightarrow \Gamma_{N'}) \) and \( \ker(\Gamma_{N''} \to \Gamma_{N'''}) \) equal \( *_{\psi'} \) and \( *_{\psi''} \), respectively. (This does not require the refinement to be boundary-free.)

**Proposition 3.7.9.** With notation as in Lemma \[3.7.7\], the functoriality homomorphism \( *_{\psi} \to *_{\psi'} \) induces an isomorphism \( *_{\psi} \to *^H_{\psi'} \).

**Proof.** We first check the case \( * = A_{\psi, \infty} \). In this case, Proposition \[3.7.3\] defines an \( A_{\psi', \infty} \)-linear projection \( \pi : A_{\psi', \infty} \to A_{\psi, \infty} \); taking the completed tensor product with \( A_{\psi'', \infty} \) yields an \( A_{\psi'', \infty} \)-linear projection \( \pi'' : A_{\psi'', \infty} \to A_{\psi', \infty} \). We can now form a commutative diagram

\[
\begin{array}{cccc}
A^H_{\psi', \infty} & \rightarrow & A_{\psi', \infty} & \rightarrow & A^H_{\psi', \infty} \\
\downarrow & & \downarrow \pi & & \downarrow \\
A^H_{\psi'', \infty} & \rightarrow & A_{\psi'', \infty} & \rightarrow & A^H_{\psi'', \infty}
\end{array}
\]

in which all of the vertical arrows are injective. Since \( A^H_{\psi', \infty} = A_{\psi'', \infty} \), the composition along the bottom row is the identity map; the same is then true along the top row. It follows that the obviously injective map \( A_{\psi, \infty} \to A^H_{\psi', \infty} \) is also surjective, and hence a bijection.

We next check the claim for \( * = A \). Note first that \( (W(o_{\overline{\tau}_{\psi}})/(z))^H = o^H_{A_{\psi, \infty}} = o_{A_{\psi, \infty}} = W(o_{\overline{\tau}_{\psi}})/(z) \), so every \( H \)-invariant element of \( W(o_{\overline{\tau}_{\psi}}) \) is congruent modulo \( z \) to an element of \( W(o_{\overline{\tau}_{\psi}}) \). Since \( z \) is also \( H \)-invariant, we may divide the difference by \( z \) and repeat: this implies that for every positive integer \( n \), every \( H \)-invariant element of \( W(o_{\overline{\tau}_{\psi}}) \) is congruent modulo \( z^n \) to an element of \( W(o_{\overline{\tau}_{\psi}}) \). Since \( W(o_{\overline{\tau}_{\psi}}) \) is \( z \)-adically complete, it follows that \( W(o_{\overline{\tau}_{\psi}})^H = o_{\overline{\tau}_{\psi}} \), so \( o^H_{A_{\psi, \infty}} = o_{\overline{\tau}_{\psi}} \) and \( \overline{A}_{\psi}^H = \overline{A}_{\psi} \) as desired.

To treat the other cases, note that by Corollary \[3.7.8\], we may write

\[ *_{\psi'} = *_{\ker(\Gamma_{N''} \to \Gamma_N)} = *_{\psi} \cap *_{\psi''}. \]

By the previous paragraph, \( \overline{A}_{\psi} \cap \overline{A}_{\psi'} = \overline{A}_{\psi} \); we may thus deduce the desired equality from [50] Remark 5.2.12.

**Definition 3.7.10.** Define a \((\varphi^d, H)\)-module over a perfect period ring \( *_{\psi} \) to be a \( \varphi^d \)-module equipped with a compatible continuous effective action of \( H \). In particular, any \((\varphi^d, \Gamma_N)\)-module may be viewed as a \((\varphi^d, H)\)-module. By analogy with the cohomology of \((\varphi^d, \Gamma_N)\)-modules (see Definition \[3.4.7\]), we associate to a \((\varphi^d, H)\)-module \( M \) the cohomology groups \( H^i_{\varphi^d, H}(M) \) for \( i \geq 0 \); these vanish for \( i > 1 + \text{rank}(H) \).
**Proposition 3.7.11.** For $* = \tilde{A}, \tilde{A'}, \tilde{B}, \tilde{C}$, the following results hold.

(a) Base extension of $\varphi^d$-modules over $*_{\psi}$ to $(\varphi^d, H)$-modules over $*_{\psi'}$ is fully faithful.

(b) Base extension of $(\varphi^d, \Gamma_N)$-modules over $*_{\psi}$ to $(\varphi^d, \Gamma_{N'})$-modules over $*_{\psi'}$ is fully faithful.

**Proof.** This is immediate from Proposition 3.7.9.

To establish essential surjectivity of the base extension, we need some more intricate arguments specific to the case of type $\tilde{C}$.

**Definition 3.7.12.** For $\delta \in \mathcal{M}(\tilde{A}_{\psi})$, put

$$\mathcal{T}_{\psi, \delta} = \mathcal{H}(\delta), \quad \mathcal{C}_{\psi, \delta} = \mathcal{R}_{\mathcal{T}_{\psi, \delta}}, \quad A_{\psi, \delta, \infty} = W(\mathcal{T}_{\psi, \delta})[[\pi]^{-1}]/(z).$$

Then form the product $\prod_{\beta} \mathcal{H}(\beta)$ over all $\beta \in \mathcal{M}(\tilde{A}_{\psi, \infty})$ lifting $\delta$, let $\tilde{A}_{\psi', \delta}$ be the subring of $\mathcal{C}_{\psi', \delta}$ on which the supremum norm is finite, and put $\mathcal{C}_{\psi', \delta} = \mathcal{R}_{\tilde{A}_{\psi', \delta}}$ and $\tilde{A}_{\psi', \delta, \infty} = W(\tilde{A}_{\psi', \delta})[[\pi]^{-1}]/(z)$. We may then consider $\varphi^d$-modules over $\mathcal{C}_{\psi, \delta}$ and $(\varphi^d, H)$-modules over $\mathcal{C}_{\psi', \delta}$.

**Lemma 3.7.13.** Let $M$ be a $(\varphi^d, H)$-module over $\mathcal{C}_{\psi'}$.

(a) The natural map $H^0_{\varphi^d, H}(M/t_d M) \otimes_{A_{\psi', \infty}} \mathcal{C}_{\psi'} \to M$ is an isomorphism.

(b) We have $H^i_{\varphi^d, H}(M/t_d M) = 0$ for $i > 0$.

**Proof.** Using Proposition 3.7.5 we may view the underlying module of $M/t_d M$ as the base extension of a finite projective module over $A_{\psi', \infty}$. We may then deduce both claims from Lemma 3.7.7.

**Lemma 3.7.14.** Let $M$ be a $(\varphi^d, H)$-module over $\mathcal{C}_{\psi'}$.

(a) The natural map $H^0_{\varphi^d, H}(M/t_d M) \otimes_{A_{\psi, \infty}} \mathcal{C}_{\psi} \to M$ is an isomorphism.

(b) We have $H^i_{\varphi^d, H}(M/t_d M) = 0$ for $i > 0$.

**Proof.** Put $M_d = (M/t_d M)^{\varphi^d}$, which is a finite projective $A_{\psi', \infty}$-module equipped with an action of $H$. Then equip $N_d = M_d \otimes_{A_{\psi', \infty}} A_{\psi'', \infty}$ with the induced action of $\ker(\Gamma_{N''} \to \Gamma_N)$. Using Lemma 3.7.13(a), we may descend $N_d$ to a finite projective module $P_d$ over $A_{\psi', \infty}$ equipped with an action of $\ker(\Gamma_{N''} \to \Gamma_N)$. By tensoring along the projections obtained from Lemma 3.7.7 by restricting to $0 \in L$, we may identify $P_d$ with the base extension of $M_d$ along the identification $A_{\psi', \infty} \cong A_{\psi'', \infty}$. We thus obtain the isomorphism in (a) by tensoring the isomorphism $M_d \otimes_{A_{\psi', \infty}} A_{\psi'', \infty} \cong N_d \cong P_d \otimes_{A_{\psi', \infty}} A_{\psi'', \infty}$ over $A_{\psi, \infty}$ with a projection $A_{\psi', \infty} \to A_{\psi, \infty}$ provided by Proposition 3.7.3. Using the same projection, we deduce (b) from Lemma 3.7.14(b).
We next introduce a dévissage argument inspired by the arguments of [9] and [45, Appendix].

**Lemma 3.7.15.** Let $M$ be a $(\varphi^d, H)$-module over $\tilde{C}_{\varphi'}$, and choose $\delta \in \mathcal{M}(\tilde{A}_{\psi})$. Suppose that the slopes of $M_\delta = M \otimes \tilde{C}_{\varphi'} \tilde{C}_{\varphi',\delta}$ are all nonpositive and not all equal to $0$. Then there exists a short exact sequence $0 \to M \to N \to P \to 0$ of $(\varphi^d, H)$-modules in which $P$ is of rank 1 and degree $1/d$ and the slopes of $N_\delta = N \otimes \tilde{C}_{\varphi'} \tilde{C}_{\varphi',\delta}$ are all nonpositive.

**Proof.** Let $F_\delta$ be the maximal $\varphi^d$-submodule of $M_\delta$ of slope 0; then $F_\delta$ is also stable under $H$. Moreover, $H^0_{\varphi^d,H}(M_\delta/F_\delta) = 0$ because $M_\delta/F_\delta$ has all negative slopes, so

$$H^0_{\varphi^d,H}(F_\delta) = H^0_{\varphi^d,H}(F_\delta)^H = H^0_{\varphi^d,H}(M_\delta)^H = H^0_{\varphi^d,H}(M_\delta).$$

By Lemma 3.7.14(a) and the fact that $F_\delta \neq M_\delta$, we can find an element $x \in H^0_{\varphi^d,H}(M/t_dM)$ whose image $x_\delta$ in $H^0_{\varphi^d,H}(M_\delta/t_dM_\delta)$ does not belong to the image of $H^0_{\varphi^d,H}(F_\delta/t_dF_\delta)$. Let $y$ be the image of $x$ under the connecting homomorphism $H^0_{\varphi^d,H}(M/t_dM) \to H^1_{\varphi^d,H}(t_dM)$, and let $y_\delta$ be the image of $y$ in $H^1_{\varphi^d,H}(t_dM_\delta)$.

Note that $H^0_{\varphi^d}(t_dF_\delta) = H^0_{\varphi^d}(t_dM_\delta) = 0$ because $t_dF_\delta$ and $t_dM_\delta$ have negative slopes. We thus obtain a commuting diagram

$$
\begin{array}{ccccccc}
0 & \to & H^0_{\varphi^d,H}(F_\delta) & \to & H^0_{\varphi^d,H}(F_\delta/t_dF_\delta) & \to & H^1_{\varphi^d,H}(t_dF_\delta) & \to & H^1_{\varphi^d,H}(F_\delta) \\
\end{array}
\begin{array}{ccccccc}
0 & \to & H^0_{\varphi^d,H}(M_\delta) & \to & H^0_{\varphi^d,H}(M_\delta/t_dM_\delta) & \to & H^1_{\varphi^d,H}(t_dM_\delta) & \to & H^1_{\varphi^d,H}(M_\delta) \\
\end{array}
$$

with exact rows. Using the exactness of the sequence $0 = H^0_{\varphi^d,H}(M_\delta/F_\delta) \to H^1_{\varphi^d,H}(F_\delta) \to H^1_{\varphi^d,H}(M_\delta)$, a diagram chase shows that the map

$$\text{coker}(H^0_{\varphi^d,H}(F_\delta/t_dF_\delta) \to H^0_{\varphi^d,H}(M_\delta/t_dM_\delta)) \to \text{coker}(H^1_{\varphi^d,H}(t_dF_\delta) \to H^1_{\varphi^d,H}(t_dM_\delta))$$

is injective. Since the class of $x_\delta$ in the source of this map is nonzero and maps to the class of $y_\delta$, the latter is also nonzero.

Put $P = t_d^{-1} \tilde{C}_{\varphi'}$, viewed as a $(\varphi^d, H)$-module of rank 1 and degree $1/d$. Let $0 \to M \to N \to P \to 0$ be the exact sequence defined by $y$; we now show that this sequence has the property that the slopes of $N_\delta$ are all nonpositive, as desired. Suppose the contrary. Let $E_\delta$ be the largest nonzero $\varphi^d$-submodule of $N_\delta$ of maximum slope; it is also stable under $H$. Note that $E_\delta$ cannot be contained in $M_\delta$, so we must have an exact sequence of the form $0 \to D_\delta \to E_\delta \to P_\delta \to 0$ for some $(\varphi^d, H)$-submodule $D_\delta$ of $M_\delta$. Note that $D_\delta$ has nonpositive slopes (by virtue of being contained in $M_\delta$), $P_\delta$ has degree $1/d$, and $E_\delta$ has positive degree. Since all degrees involved are integer multiples of $1/d$, $D_\delta$ must be étale (as otherwise the degree of $E_\delta$ would be at most $1/d - 1/d \leq 0$) and hence contained in $F_\delta$. It now follows that $y_\delta$ belongs to the image of $H^1_{\varphi^d,H}(t_dF_\delta) \to H^1_{\varphi^d,H}(t_dM_\delta)$; this contradiction yields the claim. 

\qed
Corollary 3.7.16. Let $M$ be a $(\varphi^d, H)$-module over $\tilde{\mathcal{C}}$, and choose $\delta \in \mathcal{M}(\overline{A}_\psi)$. Suppose that the slopes of $M_\delta = M \otimes_{\overline{C}} \overline{\mathcal{C}}$, $\overline{C}_\psi, \delta$ are all nonpositive. Then there exists another $(\varphi^d, H)$-module $N$ over $\tilde{\mathcal{C}}$ which is a successive extension of $M$ by $(\varphi^d, H)$-modules of rank 1, such that $N_\delta = N \otimes_{\overline{C}} \overline{\mathcal{C}}, \delta$ is étale.

Proof. This follows by repeated application of Lemma 3.7.15.

We can now establish essential surjectivity of base extension.

Theorem 3.7.17. Suppose (as throughout 3.7) that $\psi'$ is a boundary-free toric refinement of $\psi$.

(a) Base extension from $\varphi^d$-modules from $\tilde{\mathcal{C}}$ to $(\varphi^d, H)$-modules over $\tilde{\mathcal{C}}$ is an equivalence of categories.

(b) Base extension from $(\varphi^d, \Gamma_N)$-modules from $\tilde{\mathcal{C}}$ to $(\varphi^d, \Gamma_{N'})$-modules over $\tilde{\mathcal{C}}'$ is an equivalence of categories.

Proof. In both (a) and (b), full faithfulness follows from Proposition 3.7.11. To check essential surjectivity, it suffices to treat (a); using full faithfulness, we may also work locally around a single $\delta \in \mathcal{M}(\overline{A}_\psi)$. Let $M'$ be a $(\varphi^d, H)$-module over $\tilde{\mathcal{C}}$. By twisting, we may reduce to the case where the slopes of $M'_\delta$ are all nonpositive. By Corollary 3.7.16, there exists a short exact sequence

$$0 \to M' \to N' \to P' \to 0$$

of $(\varphi^d, H)$-modules in which $N'_\delta$ is étale and $P'$ admits a filtration $0 = P'_0 \subset \cdots \subset P'_i = P'$ in which each successive quotient $Q'_i = P'_i/P'_{i-1}$ is of rank 1. Let $M'_i$ be the preimage of $P'_i$ in $N'$, so that $M'_0 = M'$ and $M'_i = \varphi M'_i$.

Since the étale locus of $N'$ on $\mathcal{M}(\overline{A}_\psi)$ is open (as in Remark 3.4.6) and contains $\delta$, it also contains a rational subspace corresponding to a rational localization $\overline{A}_\psi \to \overline{B}$ encircling $\delta$. Put $B' = \overline{A}_\psi \hat{\otimes}_{\overline{C}} \overline{B}$ and let $M'_{i,B}, Q'_{i,B}$ denote the base extensions of $M'_i, Q'_i$ to $\overline{B}$.

Since $M'_{i,B}$ and the $Q'_{i,B}$ are pure, they descend to $\varphi^d$-modules $M'_{i,B}, Q'_{i,B}$ over $\overline{\mathcal{B}}$ by [50] Theorem 8.1.9 and Theorem 3.3.7. We now prove that $M'_{i,B}$ descends to a $\varphi^d$-module $M_{i,B}$ over $\overline{\mathcal{B}}$ for $i = l, l-1, \ldots, 0$, taking $i = l$ as the base case. Given the claim for some $i > 0$, note that base extension of $\varphi^d$-modules over $\overline{\mathcal{B}}$ to $(\varphi^d, H)$-modules over $\overline{\mathcal{B}}'$ is fully faithful (as in Proposition 3.7.11). We thus achieve the induction by writing $M'_{i-1,B}$ as the kernel of the map $M'_{i,B} \to Q'_{i,B}$ between two descendable $(\varphi^d, H)$-modules.

Taking $i = 0$ in the previous paragraph, we find that $M'_{0,B}$ descends to a $\varphi^d$-module over $\overline{\mathcal{B}}$. Again by full faithfulness, we may glue to obtain a $\varphi^d$-module over $\tilde{\mathcal{C}}$ whose base extension is $M'$.

We use this to compare cohomology before and after base extension.

Theorem 3.7.18. Suppose (as throughout 3.7) that $\psi'$ is a boundary-free toric refinement of $\psi$. 

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(a) Let $M$ be a $\varphi^d$-module over $\tilde{\mathcal{C}}_\psi$. Then the natural maps $H^i_{\varphi^d}(M) \to H^i_{\varphi^d,\mathcal{H}}(M \otimes \tilde{\mathcal{C}}_\psi \tilde{\mathcal{C}}_\psi')$ are bijections for all $i \geq 0$.

(b) Let $M$ be a $(\varphi^d, \Gamma_N)$-module over $\tilde{\mathcal{C}}_\psi$. Then the natural maps $H^i_{\varphi^d,\Gamma}(M) \to H^i_{\varphi^d,\Gamma}(M \otimes \tilde{\mathcal{C}}_\psi \tilde{\mathcal{C}}_\psi')$ are bijections for all $i \geq 0$.

Proof. It suffices to prove (a). By induction, we may reduce to the case where $M'$ is the free monoid generated by a single element $T$. The claim is clear for $i = 0$ by Proposition 3.7.11 and for $i \geq 3$ because both sides of the map in question are zero.

For $i = 1$, by interpreting the cohomology groups as Yoneda extension groups, we may deduce injectivity from Proposition 3.7.11 again and surjectivity from Theorem 3.7.17(a).

For $i = 2$, the claim is that $H^2_{\varphi^d,\mathcal{H}}(M \otimes \tilde{\mathcal{C}}_\psi \tilde{\mathcal{C}}_\psi') = 0$. This is obvious if $H^1_{\varphi^d}(M) = 0$; also, the claim for $t_d^{-1}M$ implies the claim for $M$ because $H^1_{\varphi^d,\mathcal{H}}((t_d^{-1}M/M) \otimes \tilde{\mathcal{C}}_\psi \tilde{\mathcal{C}}_\psi') = H^2_{\varphi^d,\mathcal{H}}((t_d^{-1}M/M) \otimes \tilde{\mathcal{C}}_\psi \tilde{\mathcal{C}}_\psi') = 0$ by Lemma 3.7.14(b). However, by [50, Proposition 6.2.2], any sufficiently large integer $m$ satisfies $H^1_{\varphi^d}(t_d^{-m}M) = 0$, so we may deduce the claim in general.

Remark 3.7.19. Suppose now that $\psi'$ is a general refinement of $\psi$, not necessarily boundary-free. We would expect the analogues of Theorem 3.7.17 and Theorem 3.7.18 to hold, but our arguments break down completely. For instance, Lemma 3.7.7 fails outright in this generality; for another example, the proof of Proposition 3.7.5 breaks down because the analogues of the rings $R_n$ are no longer perfectoid. One special case which can be salvaged is that of étale $(\varphi, \Gamma)$-modules, thanks to the results of §3.5.

3.8 Globalization

We conclude this discussion by describing relative $(\varphi, \Gamma)$-modules (of type $\tilde{\mathcal{C}}$) over an arbitrary $K$-analytic space. (A similar construction is possible for uniformly rigid spaces, but this requires a better understanding of the G-topology of such spaces than we are prepared to demonstrate here. See the discussion in §1.2.) We globalize using the following construction.

Definition 3.8.1. Let $X$ be a reduced $K$-analytic space. Define the (boundary-free) framed affinoid site of $X$ to be the site consisting of the following data.

- The objects of the underlying category are pairs $(j, \psi)$, where $j : \mathcal{M}(A) \to X$ is an affinoid subdomain and $\psi : \mathcal{M}(A) \to K(\sigma)$ is an eligible (boundary-free) toric frame. Note that the spaces $\mathcal{M}(A)$ cover $X$ thanks to Corollary 2.4.14.

- Morphisms between two pairs $(j_1 : \mathcal{M}(A_1) \to X, \psi_1 : \mathcal{M}(A_1) \to K(\sigma_1))$ and $(j_2 : \mathcal{M}(A_2) \to X, \psi_2 : \mathcal{M}(A_2) \to K(\sigma_2))$ exist only when $j_1$ factors through $j_2$. In this case, any factorization corresponds to a morphism $\mathcal{M}(A_1) \to \mathcal{M}(A_2)$ of $K$-affinoid spaces, and the morphisms in the category consist of toric morphisms from $\psi_1$ to $\psi_2$ lifting such a factorization.
A family of morphisms to a common target is a covering family if the underlying morphisms of affinoid spaces form a covering family of affinoid subdomains.

**Remark 3.8.2.** The sites constructed in Definition 3.8.1 are analogous to the $G$-topology introduced in Definition 1.1.7. One can make analogous definitions corresponding to the étale topology, but we will not need these here.

**Definition 3.8.3.** Let $X$ be a $K$-analytic space. We define a relative $(\varphi, \Gamma)$-module over $X$ to be an assignment to each pair $(j, \psi)$ in the framed affinoid site of $X$ a $(\varphi, \Gamma_N)$-module $M_\psi$ over $\tilde{C}_\psi$ and to each morphism $(j_1, \psi_1) \to (j_2, \psi_2)$ an isomorphism $M_{\psi_2} \otimes_{\tilde{C}_{\psi_2}} \tilde{C}_{\psi_1} \cong M_{\psi_1}$ of $(\varphi, \Gamma_{N_1})$-modules, satisfying the cocycle condition and the sheaf property for covering families of rational localizations. (In other words, we are specifying a crystal of $(\varphi, \Gamma_N)$-modules.) Note that it is equivalent to use the boundary-free framed affinoid site in this construction, by Theorem 3.7.17. For $f : Y \to X$ a morphism of $K$-analytic spaces, we obtain a pullback functor $f^*$ from relative $(\varphi, \Gamma)$-modules over $X$ to relative $(\varphi, \Gamma)$-modules over $Y$.

**Theorem 3.8.4.** (a) For any eligible boundary-free toric frame $\psi : \mathcal{M}(A) \to K(\sigma)$, restriction to $(\text{id}, \psi)$ defines a fully faithful functor from relative $(\varphi, \Gamma)$-modules over $\mathcal{M}(A)$ to $(\varphi, \Gamma_N)$-modules on $\tilde{C}_\psi$.

(b) Any admissible covering of a $K$-analytic space gives rise to an effective descent morphism for relative $(\varphi, \Gamma)$-modules.

**Proof.** Part (a) follows from Theorem 3.7.17. Given (a), (b) reduces using the Gerritzen-Grauert theorem [50, Theorem 2.5.4] to the case of a covering of a single frame by rational subframes, for which we may apply [50, Remark 6.4.2].

**Definition 3.8.5.** Let $X$ be a $K$-analytic space. Let $M$ be a relative $(\varphi, \Gamma)$-module on $X$. We define the cohomology $H^i_{\varphi, \Gamma}(M)$ of $M$ as follows. Choose an admissible covering $\{(j_i, \psi_i)\}_{i \in I}$ of $X$ in the framed affinoid site. For each finite fibre product of terms in the covering, form the complex of continuous effective $\Gamma$-cochains, then assemble these into a double complex using Čech differentials. The cohomology of the resulting site does not depend on the choice of the covering: this reduces to the case of a single frame being covered by finitely many rational subframes, in which case [50, Theorem 5.3.4, Remark 6.4.2] implies the claim.

**Remark 3.8.6.** For $X$ a $K$-analytic space, the existence of a resolution of singularities $f : Y \to X$ follows from Temkin’s functorial desingularization of quasixcellent $\mathbb{Q}$-schemes [76, 75]. We expect that the category of relative $(\varphi, \Gamma)$-modules over $X$ is equivalent to the category of descent data in the category of relative $(\varphi, \Gamma)$-modules over $Y$ (i.e., objects over $Y$ equipped with isomorphisms of their two pullbacks to $Y \times_X Y$ satisfying the cocycle condition). If so, then one can use simplicial hypercoverings by smooth $K$-analytic spaces to study relative $(\varphi, \Gamma)$-modules over arbitrary $K$-analytic spaces.
Remark 3.8.7. An alternate approach to globalization can be obtained using Scholze’s definition of the pro-étale site associated to an analytic space [71]. This approach has several technical advantages: it absorbs various continuity conditions in such a way that one can typically work with true sheaves rather than inverse systems, and it allows for the perfect period rings to be viewed as sheaves in a natural way. We will incorporate Scholze’s viewpoint at a later stage in this series of papers.

4 Imperfect period rings

While the perfect period rings are sufficient for many purposes (e.g., the construction of universal local systems on Rapoport-Zink period domains, to be discussed in a subsequent paper), they suffer from some defects that cause problems for some applications; notably, one cannot differentiate the $\Gamma_N$-action on them (Remark 3.2.4). We now tackle the subtle task of defining a suite of imperfect period rings, on which $\varphi$ does not act bijectively. We can only define these under some technical hypotheses (see Hypothesis [4.1.1]; notably, we must assume $A$ is a mixed affinoid algebra. This includes the case of a strictly affinoid algebra; in this case, we only cover cases in which $\psi$ is étale, not just unramified. (A similar restriction holds in the mixed affinoid case, modulo defining what an étale morphism is in this setting.)

As a result, imperfect period rings seem to be most relevant for studying $(\varphi, \Gamma)$-modules over smooth strictly $K$-analytic spaces (or the corresponding subcategory among the uniformly rigid spaces). Moreover, we cannot discuss functoriality directly in terms of imperfect period rings, because the étaleness of $\psi$ is not preserved by toric refinements; fortunately, when the imperfect period rings occur, they give rise to the same $(\varphi, \Gamma)$-modules as over the perfect period rings (see §4.10), so we may use the functoriality formalism from the perfect case instead.

Given that the perfect period rings are easier to construct than the imperfect ones, one may wonder whether the imperfect period rings are necessary. We expect that there will ultimately be numerous constructions in relative $p$-adic Hodge theory which can only be made using imperfect period rings; we provide one example at the end of this section, a relative version of the Ax-Sen-Tate theorem (Theorem 4.10.5).

As in §3, the constructions in this section make contact with a fair bit of existing work; comparisons between our work and existing literature are again reserved to §4.11.

4.1 The basic hypothesis

Recall that Hypothesis [3.0.1] is still in effect. We now introduce the additional running hypothesis under which imperfect period rings will be studied, and verifying the hypothesis in some cases of interest.

Hypothesis 4.1.1. Throughout [4] (unless otherwise specified), continue to assume Hypothesis [3.0.1] but also assume that $A$ is a mixed affinoid algebra over $K$ satisfying the following conditions.
(a) There exists a unique mixed affinoid algebra $R$ over $k(\langle \pi \rangle)$ with completed perfect closure $\bar{A}_\psi$ such that $\Omega_{R/k}$ is freely generated by $d\pi, d\bar{T}_1, \ldots, d\bar{T}_n$ for some (and hence any) elements $\bar{T}_1, \ldots, \bar{T}_n$ of $M_{\sigma}$ which form a $\mathbb{Z}$-basis of $N^\vee$. (Equivalently, $\pi, \bar{T}_1, \ldots, \bar{T}_n$ form an absolute $p$-basis of $R$.)

(b) There exists a unique weakly complete $\varphi$-stable $W(k)$-flat subring $A_\psi$ of $\tilde{A}_\psi$ such that $A_\psi/(p) = R$ and for some (and hence any) elements $T_1, \ldots, T_n$ of $M_{\sigma}$ which form a $\mathbb{Z}$-basis of $N^\vee$, the module of continuous differentials $\Omega_{A_\psi/W(k)}$ admits the basis $d\pi, dT_1, \ldots, dT_n$.

(c) The map $A_\psi \cap \tilde{A}_\psi \rightarrow R$ is surjective.

**Example 4.1.2.** Hypothesis 4.1.1 holds in case $A = K\{M_{\sigma}\}_\lambda$ for some $\lambda \in N_{\mathbb{Q}}$.

As in the case of Conjecture 2.4.7, we may verify Hypothesis 4.1.1 in a number of useful cases by establishing stability under some basic operations on frames.

**Theorem 4.1.3.** Let $\psi : \mathcal{M}(A) \rightarrow K(\sigma)$ be a frame satisfying Hypothesis 4.1.1. Let $A \rightarrow B$ be a $K$-linear bounded homomorphism of mixed affinoid algebras over $K$, and let $\psi' : \mathcal{M}(B) \rightarrow \mathcal{M}(A) \rightarrow K(\sigma)$ be the composite frame.

(a) If $A \rightarrow B$ is a strictly rational localization, then $\psi'$ also satisfies Hypothesis 4.1.1.

(b) If $A \rightarrow B$ is finite étale, then $\psi'$ also satisfies Hypothesis 4.1.1.

**Proof.** In both cases, $\psi'$ satisfies Hypothesis 3.0.1 by Theorem 2.4.12, so we focus on the other conditions of Hypothesis 4.1.1. We first check (a). By Theorem 3.1.3, we obtain elements $\tilde{f}_1, \ldots, \tilde{f}_m, \tilde{g} \in \bar{A}_\psi$ defining a strictly rational localization $\bar{A}_\psi \rightarrow B_{\psi'}$. By [50, Remark 2.4.8], we can also take $\tilde{f}_1, \ldots, \tilde{f}_m, \tilde{g}$ in the dense subring $\cup_{i=0}^{\infty} \varphi^{-i}(A_\psi/(p))$ of $A_\psi$. By applying $\varphi$ repeatedly, we obtain $\tilde{f}_1, \ldots, \tilde{f}_m, \tilde{g} \in A_\psi/(p)$ defining a strictly rational localization $A_\psi/(p) \rightarrow R$. By lifting $\tilde{f}_1, \ldots, \tilde{f}_m, \tilde{g}$ to $f_1, \ldots, f_m, g \in A^\dagger_\psi$, we establish that $\psi'$ satisfies Hypothesis 4.1.1 for the ring $R$ just constructed.

We next check (b). For this, it suffices to note that $(A^\dagger_\psi/(p))$ is henselian (as in the proof of [50, Proposition 5.5.3(a)]), so that $\mathbf{F}\acute{e}t(A^\dagger_\psi/(p)) \cong \mathbf{F}\acute{e}t(A_\psi/(p))$ by [50, Theorem 1.2.8], and that $\mathbf{F}\acute{e}t(A_\psi/(p)) \cong \mathbf{F}\acute{e}t(\bar{A}_\psi)$ by [50, Theorem 3.1.15].

As in the case of Corollary 2.4.14, this implies a covering result for somewhat more general frames; however, because Theorem 4.1.3 does not apply to closed immersions (see Example 4.1.11), we only cover frames of a special type.

**Definition 4.1.4.** We say a toric frame $\psi : \mathcal{M}(A) \rightarrow K(\sigma)$ is étale if $A$ is an affinoid algebra over $K$ and the morphism $\psi$ is étale; if $A$ is strictly $K$-affinoid, we say that the frame $\psi$ is strictly étale. We say that $\psi$ is of (strictly) rational type if $\psi$ factors as a finite étale cover of a (strictly) rational subdomain.
**Corollary 4.1.5.** Let \( \psi : \mathcal{M}(A) \to K(\sigma) \) be an étale toric frame. (We need not assume that \( \psi \) is eligible or that it satisfies Hypothesis [4.1.1].) Then any strictly rational subframe of \( \psi \) which is of strictly rational type satisfies Hypothesis [4.1.1]; consequently, \( \psi \) admits a strong covering family by strictly rational subframes satisfying Hypothesis [4.1.1].

**Proof.** The first assertion follows from Example 4.1.2 and Theorem 4.1.3. The second assertion follows from Remark 1.1.11 plus the first assertion.

**Remark 4.1.6.** By Remark 1.1.11 plus the Gerritzen-Grauert theorem [50, Theorem 2.5.4], any strictly étale frame \( \psi : \mathcal{M}(A) \to K(\sigma) \) admits a covering family of rational subframes of strictly rational type; however, the overlaps are not themselves guaranteed to be of strictly rational type. The standard way to work around this is to construct a simplicial covering

\[
\cdots \to \coprod_j \mathcal{M}(A_{2j}) \to \coprod_i \mathcal{M}(A_{1j}) \to \mathcal{M}(A)
\]

of \( \mathcal{M}(A) \) by rational subdomains in which each frame \( \psi_{ij} : \mathcal{M}(A_{ij}) \to K(\sigma) \) is of (strictly) rational type.

**Remark 4.1.7.** If \( \psi \) is a strictly étale frame, then in Hypothesis 4.1.1(a) it is sufficient to check existence of the ring \( R \), as uniqueness will then be implied. The argument is as follows. (A similar argument applies to part (b).)

As in Remark 4.1.6, form a simplicial hypercovering by rational subframes, each of strictly rational type. By Corollary 4.1.5, this hypercovering uniquely determines a finite collection of affinoid spaces over \( k((\pi)) \) together with some glueing data; the ring \( R \) can be recovered as the ring of global sections of the resulting analytic space.

**Remark 4.1.8.** It is tempting to conjecture that Hypothesis 4.1.1 is satisfied whenever \( \psi \) is a strictly étale frame, as in this case Remark 4.1.7 reduces the problem to showing that a certain compact analytic space is affinoid. However, this latter problem can be rather subtle (see Remark 1.3.4), so we refrain from making the conjecture. In any case, we get some information by working locally using Corollary 4.1.5; see for instance Theorem 4.9.4.

**Remark 4.1.9.** It is unclear whether one can obtain useful analogues of Hypothesis 4.1.1 and Corollary 4.1.5 in any cases where \( A \) is affinoid but not strictly affinoid over \( K \). If one removes the adjective *strictly* from condition (a) of Hypothesis 4.1.1, then it holds when \( \psi \) is of rational type by the same proof as in Theorem 4.1.3. However, it is unclear whether this is ultimately helpful for the theory of imperfect period rings; see Remark 4.2.7.

**Remark 4.1.10.** The condition in Corollary 4.1.5 that \( \psi \) must be étale arises from the fact that Theorem 4.1.3 does not apply to closed immersions. In fact, it is easy to see that Theorem 4.1.3 cannot be extended to closed immersions; see for instance Example 4.1.11.

Here is an archetypal example in which \( A \) is a strictly affinoid algebra over \( K \) but Hypothesis 4.1.1 is not satisfied.
Example 4.1.11. Take $K(\sigma)$ to be the one-dimensional affine space over $K$ with coordinate $T$ and $A = K\{T\}/(T - 1)$. In this case, the ring $\widetilde{A}_\psi$ can have infinitely many idempotent elements; it therefore cannot be the completed perfect closure of an affinoid algebra over $k((\pi))$.

Remark 4.1.12. It is unclear whether one can formulate a meaningful form of Hypothesis 4.1.1 which allows $A$ to be an arbitrary $K$-affinoid algebra. See Remark 4.2.7 for more discussion.

For compatibility with the work of Andreatta and Brinon [5], we mention another method for generating frames satisfying Hypothesis 4.1.1, in which one encounters some mixed affinoid algebras which are not affinoid.

Proposition 4.1.13. Let $L$ be a finite extension of $K$. Let $d$ be a nonnegative integer. Let $R$ be a ring obtained from $o_L\{T_1^{\pm}, \ldots, T_d^{\pm}\}$ by a finite sequence of operations, each of one of the following types.

(a) Form the $p$-adic completion of an étale extension.

(b) Form the $p$-adic completion of the localization with respect to a multiplicative system.

(c) Form the completion with respect to an ideal containing $p$.

Put $A = R \otimes_{o_L} L$. Then the map $o_K\{T_1^{\pm}, \ldots, T_d^{\pm}\} \to R$ induces a frame $\psi : M(A) \to K(\sigma)$ which is eligible and satisfies Hypothesis 4.1.1.

Proof. For conditions (a) and (b) of Hypothesis 4.1.1, see [4, Theorem 7.6]. For condition (c) of Hypothesis 4.1.1, see [5, §4.9].

4.2 Imperfect period rings and reality checks

Under Hypothesis 4.1.1, we now construct the imperfect period rings.

Definition 4.2.1. Put $A^{[s,r]} = A_\psi \cap \tilde{A}_\psi^{[s,r]}$ within $\tilde{A}_\psi$, and put $A^{[s,r]} = \cup_{r>0} A^{[s,r]} = A_\psi \cap \tilde{A}_\psi^{[s,r]}$. The pair $(A_\psi^{[s,r]}, (p))$ is obviously henselian; one checks that $(A^{[s,r]}, (p))$ is also henselian by imitating the proof of [50, Proposition 5.5.3(a)].

Define $B^{[s,r]}_\psi, B^{[s,r]}_\psi, B^{[s,r]}_\psi$ by inverting $p$ in the corresponding rings of type $A$. Note that $B^{[s,r]}_\psi = B_\psi \cap \tilde{B}_\psi^{[s,r]}$ and $B^{[s,r]}_\psi = B_\psi \cap \tilde{B}_\psi^{[s,r]}$ within $\tilde{B}_\psi$.

Let $C^{[s,r]}_\psi$ be the Fréchet completion of $B^{[s,r]}_\psi$ with respect to $\lambda(\overline{a}_\psi^t)$ for $t \in [s, r]$; it embeds naturally into $\tilde{C}^{[s,r]}_\psi$. Put $C^{[s,r]}_\psi = \cap_{0<s<r} C^{[s,r]}_\psi$ and $C^{[s,r]}_\psi = \cup_{r>0} C^{[s,r]}_\psi$.

We refer collectively to the rings

$A_\psi, A_\psi/(p), A^{[s,r]}_\psi, A^{[s,r]}_\psi, B^{[s,r]}_\psi, B^{[s,r]}_\psi, B^{[s,r]}_\psi, C^{[s,r]}_\psi, C^{[s,r]}_\psi, C_\psi$

as the imperfect period rings associated to $\psi$. (We need not include $A^{[s,r]}_\psi/(p)$ separately, because it coincides with $A_\psi/(p)$.) All of these rings inherit actions of $\varphi$ and $\Gamma_N$ (with the same proviso about $\varphi$ and $r$ as in Definition 3.4.1).
Remark 4.2.2. Each imperfect ring embeds into its perfect counterpart, and thus inherits some topologies (see Remark 3.4.2). Since the actions of ϕ and Γ_N are continuous on the perfect period rings (excluding the action of Γ_N for the p-adic topology), the same is true of the imperfect period rings. In fact, somewhat more is true; see Proposition 4.4.7.

We need some reality checks in the style of [50, §5.2] but more in the spirit of [43, §2], where the condition of having enough units plays a key role.

Lemma 4.2.3. There exists r_0 > 0 such that for r ∈ (0, r_0], every x ∈ A_ψ/(p) admits a lift \overline{x} ∈ A_ψ for which

\[ \lambda(\overline{x})(x - [\overline{x}]) \leq p^{-1/2} \lambda(\overline{x})(x) \]

In particular, \( \lambda(\overline{x})(x) = \overline{\psi}(\overline{x})^r \).

The key point here is that r_0 may be chosen uniformly in \( \overline{x} \).

Proof. We first show that for some \( r_1 > 0 \), there exists \( c > 0 \) such that every \( \overline{x} \in A_ψ/(p) \) admits a lift \( x \in A_ψ^{1,r} \) for which \( \lambda(\overline{x})(x) \leq c \overline{x}(\overline{x})^r \) for all \( r \in (0, r_1] \). When \( A = K\{M_\lambda \} \) for some \( \lambda \in N_Q \), this can be trivially done for any \( r_1 \) with \( c = 1 \). In case \( A \) is affinoid, the homomorphism \( k((\overline{\psi}))\{M_\lambda \} \rightarrow A_ψ/(p) \) is strictly affinoid, so there exists a strict surjection \( k((\overline{\psi}))\{M_\lambda \} \{T_1, \ldots, T_m\} \rightarrow A_ψ/(p) \) for some nonnegative integer m. We thus obtain the desired lifts by first lifting to \( k((\overline{\psi}))\{M_\lambda \} \{T_1, \ldots, T_m\} \), then lifting each \( T_i \) to \( T_i \) (and lifting elements of \( k((\overline{\psi}))\{M_\lambda \} \) as before). A similar argument applies in the semiaffinoid case with the Tate algebra replaced by a mixed power series ring.

For any sufficiently small \( r_0 > 0 \), we have \( c^{r_0/r_1}p^{r_0/r_1-1} \leq p^{-1/2} \). Choose some such \( r_0 \); by [50, Lemma 5.2.1], for \( r \in (0, r_0] \),

\[ \lambda(\overline{x})(x - [\overline{x}]) \leq p^{r_0/r_1-1}\lambda(\overline{x}^{1/r_0})(x - [\overline{x}])^r \]

\[ \leq p^{r_0/r_1-1}(x - [\overline{x}]) \]

\[ \leq p^{-1/2}\overline{x}(\overline{x})^r \]

This yields the claim.

Corollary 4.2.4. For \( r_0 \) as in Lemma 4.2.3 for \( r \in (0, r_0] \), any \( x \in B_ψ^{1,r} \) can be written as a \( \lambda(\overline{x})^r \)-convergent sum \( \sum_{n=m}^\infty p^n y_n \) with \( y_n \in A_ψ^{1,r_0} \) such that, if we write \( \overline{y}_n \) for the image of \( y_n \) in \( A_ψ/(p) \), then \( \lambda(\overline{x})^r(y_n - [\overline{y}_n]) \leq p^{-1/2}\lambda(\overline{x}^r)(y_n) \) for all \( n \geq 0 \). Moreover, for any such representation, for all \( s \in (0, r] \),

\[ \lambda(\overline{x})(x) = \max_{n\geq0}\{p^{-s}\overline{x}(\overline{y}_n)\} \] (4.2.4.1)

Proof. We may assume \( x \in A_ψ^{1,r} \). The existence of such a representation within \( A_ψ \) is immediate from Lemma 4.2.3. To prove convergence with respect to \( \lambda(\overline{x})^r \), write \( x = \sum_{n=0}^\infty p^n [\overline{x}_n] \) and note that \( \lambda(\overline{x}^r)(p^n y_n) \) is bounded above by the greater of \( \lambda(\overline{x}^r)(p^n y_n) \)
and the maximum of $p^{-1/2}(\bar{\alpha}_\psi^t)(p^t y_i)$ over $i < n$. By an easy induction argument, it follows that there exists $n$ such that

$$
\lambda(\bar{\alpha}_\psi^t) \left( x - \sum_{i=0}^{n-1} p^t y_i \right) \leq p^{-1/2}(\bar{\alpha}_\psi^t)(x).
$$

Applying this argument repeatedly yields the desired convergence, which in turn immediately implies (4.2.4.1). \qed

**Lemma 4.2.5.** There exists $r_0 > 0$ such that for $0 < s \leq r \leq r_0$, $C^{[s,r]}_\psi$ is a mixed affinoid algebra over $K$.

**Proof.** As in the proof of Lemma 4.2.3. \qed

**Corollary 4.2.6.** For $r_0 > 0$ as in Lemma 4.2.5, for $0 < s \leq s' \leq r' \leq r_0$, inside $C^{[s',r']}_\psi$ we have

$$
C^{[s,r]}_\psi \cap C^{[s',r']}_\psi = C^{[s',r']}_\psi.
$$

**Proof.** By Lemma 4.2.5, all of the rings in question are mixed affinoid algebras over $K$. The claim then follows from the Tate acyclicity property for mixed affinoid algebras (Proposition 1.2.6). \qed

**Remark 4.2.7.** There is no analogue of Lemma 4.2.5 when $\psi$ is rational but not strictly rational. For example, take $\psi$ to be defined by the homomorphism $K\{T\} \to K\{T/q\}$ for some $q \in (0,1) \setminus p\mathbb{Q}$. In this case,

$$
\{ \beta \in K(\sigma) \times_K \mathcal{M}(K\{\pi\}) : \beta(\pi) \in [\omega^s, \omega^r], \beta(T) \leq \beta(\pi)^{\log_\omega q} \}
$$

is not an affinoid subspace of $K(\sigma) \times_K \mathcal{M}(K\{\pi\})$.

As a result, we are forced to assume that the ring $R$ in Hypothesis 4.1.1(a) is strictly affinoid (or more generally mixed affinoid) over $k((\pi))$ in order to use the theorems of Tate and Kiehl. It may be possible to extend these to the cases that occur when $R$ is not strictly affinoid over $k((\pi))$, but we did not look into this. The main difficulty is the Tate sheaf property, as then the Kiehl glueing property would follow from [50, Proposition 2.7.5].

Using the lifts from Lemma 4.2.3 in place of Teichmüller lifts, we obtain analogues of [50, Lemmas 5.2.5 and 5.2.7].

**Lemma 4.2.8.** For $r_0$ as in Lemma 4.2.3, for $0 < s \leq r \leq r_0$, within $C^{[s,s]}_\psi$ we have $A^{[s,s]}_\psi \cap C^{[s,r]}_\psi = A^{[s,r]}_\psi$.

**Proof.** Take $x$ in the intersection, and write $x$ as the limit in $C^{[s,r]}_\psi$ of a sequence $x_0, x_1, \ldots$ with $x_i \in B^{[s,r]}_\psi$. For each positive integer $j$, we can find $N_j > 0$ such that

$$
\lambda(\bar{\alpha}_\psi^t)(x_i - x) \leq p^{-j} \quad (i \geq N_j, t \in [s, r]).
$$
Write \( x_i = \sum_{l=m}^{\infty} p^l x_{il} \) as in Corollary 4.2.4. Put \( y_i = \sum_{l=0}^{\infty} p^l x_{il} \in A_{\psi}^{r,t} \). For \( i \geq N_j \), having \( x \in \hat{A}_{\psi}^{t,r} \) and \( \lambda(\overline{\alpha}_{\psi})(x_i - x) \leq p^{-j} \) implies that \( \lambda(\overline{\alpha}_{\psi})(p^l x_{il}) \leq p^{-j} \) for \( l < 0 \) by 4.2.4.1. That is,
\[
\overline{\alpha}_{\psi}(x_{il}) \leq p^{(l-j)/s} \quad (i \geq N_j, l < 0).
\]
Since \( p^{-l} p^{(l-j)r/s} \leq p^{1+(1-j)r/s} \) for \( l \leq -1 \), we deduce that \( \lambda(\overline{\alpha}_{\psi})(x_i - y_i) \leq p^{1+(1-j)r/s} \) for \( i \geq N_j \). Consequently, the sequence \( y_0, y_1, \ldots \) converges to \( x \) under \( \lambda(\overline{\alpha}_{\psi}) \); it follows that \( x \in A_{\psi}^{t,r} \) as desired.

Lemma 4.2.9. For \( r_0 \) as in Lemma 4.2.3, for \( 0 < s \leq r \leq r_0 \), each \( x \in C_{\psi}^{[s,s]} \) can be decomposed as \( y + z \) with \( y \in A_{\psi}^{t,s} \), \( z \in C_{\psi}^{[s,r]} \), and
\[
\lambda(\overline{\alpha}_{\psi})(z) \leq p^{1-t/s} \lambda(\overline{\alpha}_{\psi})(x)^{t/s} \quad (t \in [s,r]).
\]

Proof. As in the proof of \([50]\) Lemma 5.2.7, we may reduce to the case \( x \in B_{\psi}^{t,s} \). Write \( x = \sum_{n=m}^{\infty} p^n y_n \) as in Corollary 4.2.4 and put \( y = \sum_{n=0}^{\infty} p^n y_n \) and \( z = x - y \). For \( n < 0 \), we then have \( \lambda(\overline{\alpha}_{\psi})(p^n x_n) \leq \lambda(\overline{\alpha}_{\psi})(x) \) and so
\[
\lambda(\overline{\alpha}_{\psi})(p^n x_n) = \lambda(\overline{\alpha}_{\psi})(p^n[\overline{\alpha}_n])
= p^{-n(1-t/s)} \lambda(\overline{\alpha}_{\psi})(p^n x_n)^{t/s}
\leq p^{1-t/s} \lambda(\overline{\alpha}_{\psi})(x)^{t/s}.
\]

This proves the claim.

4.3 Period rings and étale covers

One of the key results from \([50]\) making it possible to relate \((\varphi, \Gamma_N)\)-modules over perfect period rings to étale local systems is the matching of certain finite étale algebras in characteristic 0 and characteristic \( p \) \([50]\) Theorem 3.6.20, Corollary 5.5.6]. For imperfect period rings, we need similar results, but fortunately these are easily deduced from the perfect case.

Definition 4.3.1. In this definition, all unions are taken as \( n \) runs over all nonnegative integers. Let \( \hat{\varphi}^{-n}(A_{\psi}) \) denote the \( p \) adic completion of \( \varphi^{-n}(A_{\psi}) \) within \( \hat{A}_{\psi}^{t} \). For \( r > 0 \), let \( \hat{\varphi}^{-n}(A_{\psi}^{t,r}) \) denote the completion of \( \varphi^{-n}(A_{\psi}^{t,r}) \) for the supremum of \( \lambda(\overline{\alpha}_{\psi}) \) and the \( p \) adic norm. Let \( \hat{\varphi}^{-n}(A_{\psi}^{r}) \) denote the union of \( \hat{\varphi}^{-n}(A_{\psi}^{t,r}) \) over all \( r > 0 \).
Theorem 4.3.2. Applying $\text{F}^\text{Et}$ to any arrow in the diagram

\[
\begin{array}{c}
\bigcup \varphi^{-n}(A_\psi^{\dagger,p^{-n}}) \\
\downarrow \varphi^{-n}(A_\psi^{\dagger,p^{-n}}) \\
A_\psi^{\dagger} \rightarrow \bigcup \varphi^{-n}(A_\psi) \\
\downarrow \bigcup \varphi^{-n}(A_\psi) \\
A_\psi/\langle p \rangle \rightarrow \bigcup \varphi^{-n}(A_\psi/\langle p \rangle)
\end{array}
\Rightarrow
\begin{array}{c}
\bigcup A_{\psi,n} \\
\downarrow \hat{A}^{\dagger,1}_\psi \\
A_{\psi,\infty}
\end{array}
\]

produces a tensor equivalence. (All unions are taken as $n$ runs over all nonnegative integers.)

Proof. The arrows in the bottom row of the diagram induce equivalences by [50, Theorem 3.1.15]. It is thus sufficient to link each entry of the diagram with the bottom row.

Consider first the bottom three rows of the diagram. Each of the rings

\[A_\psi^{\dagger}, \hat{A}_\psi^{\dagger}, A_\psi, \bigcup \varphi^{-n}(A_\psi), \hat{A}_\psi\]

is henselian with respect to $(p)$; consequently, the vertical arrows from these rings induce equivalences by [50, Theorem 1.2.8]. The arrows $\bigcup \varphi^{-n}(A_\psi^{\dagger}) \rightarrow \bigcup \varphi^{-n}(A_\psi)$ and $\bigcup \varphi^{-n}(A_\psi) \rightarrow \hat{A}_\psi$ thus induce functors which are essentially surjective, but also fully faithful by [50, Lemma 2.2.4(a)], and thus equivalences.

It remains to link the top two rows to the rest of the diagram. To begin with, the arrows out of $\hat{A}^{\dagger,1}_\psi$ induce equivalences by [50, Corollary 5.5.6], while the arrow $\bigcup A_{\psi,n} \rightarrow A_{\psi,\infty}$ induces an equivalence by [50, Proposition 2.6.9].

We next link $\bigcup \varphi^{-n}(A_\psi^{\dagger,p^{-n}})$ to the bottom of the diagram. Any finite étale algebra $U$ over $A_\psi^{\dagger}$ may be defined over $A_\psi^{\dagger,p^{-n}}$ for some $n$. Since any finite étale algebra over $\hat{A}_\psi$ is isomorphic to its $\varphi$-pullback, we deduce (using the equivalences already established) that any finite étale algebra over $\hat{A}_\psi$ lifts to a finite étale algebra over $\varphi^{-n}(A_\psi^{\dagger,p^{-n}})$ for some $n$. That is, the arrow $\cup \varphi^{-n}(A_\psi^{\dagger,p^{-n}}) \rightarrow \cup \varphi^{-n}(A_\psi^{\dagger})$ induces a functor which is essentially surjective. However, this functor is also fully faithful by the following argument. Given $U, V \in \text{F}^\text{Et}(\cup \varphi^{-n}(A_\psi^{\dagger,p^{-n}}))$, any morphism between $U$ and $V$ in $\text{F}^\text{Et}(\cup \varphi^{-n}(A_\psi^{\dagger}))$ gives rise (using the equivalences already established) to a morphism in $\text{F}^\text{Et}(\hat{A}^{\dagger}_\psi)$ and hence a morphism in $\text{F}^\text{Et}(\hat{A}^{\dagger,1}_\psi)$. Since $U, V$ are finite projective modules over $\cup \varphi^{-n}(A_\psi^{\dagger,p^{-n}})$ (see [50, Definition 1.2.1]), we also get a morphism between $U$ and $V$ in $\text{F}^\text{Et}(R)$ for $R$ equal to the intersection $(\cup \varphi^{-n}(A_\psi^{\dagger})) \cap \hat{A}^{\dagger,1}_\psi$ within $\hat{A}^{\dagger}_\psi$. This intersection equals $\cup \varphi^{-n}(A_\psi^{\dagger}) \times \hat{A}^{\dagger,1}_\psi$, and hence $U$ and $V$ become isomorphic in $\text{F}^\text{Et}(R)$. Therefore, $\text{F}^\text{Et}(\cup \varphi^{-n}(A_\psi^{\dagger,p^{-n}}))$ is fully faithful.
We finally link $\hat{\cup}\varphi^{-n}(A^\dagger_p,\psi^{-n})$ to the bottom of the diagram. The functor induced by the arrow from this ring to $\hat{\cup}\varphi^{-n}(A^\dagger_p)$ is essentially surjective, because we have an equivalence that factors through it. We may check full faithfulness by imitating the argument given for $\cup\varphi^{-n}(A^\dagger,\psi^{-n})$.

**Remark 4.3.3.** If $\psi'$ is another frame obtained by making a finite étale extension of $A$, then we deduce from Theorem 4.3.2 (as in the proof of [50, Proposition 5.5.4]) that $C_{\psi'}$ is finite étale over $C_\psi$.

### 4.4 Analyticity of the $\Gamma_N$-action

As noted earlier, one key feature of the imperfect period rings which distinguishes them from their perfect counterparts is analyticity of the action of $\Gamma_N$.

**Notation 4.4.1.** Throughout §4.4, for $n$ a positive integer, let $U_n$ denote the open subgroup $(1 + p^n\mathbb{Z}_p) \ltimes p^n\mathbb{N}_p$ of $\Gamma_N$.

**Remark 4.4.2.** We will make repeated use of the fact that for $\gamma$ an endomorphism of a ring $R$, the operator $\gamma - 1$ satisfies the following analogue of the Leibniz rule: for all $x, y \in R$,

$$
(\gamma - 1)(xy) = (\gamma - 1)(x)y + \gamma(x)(\gamma - 1)(y). \tag{4.4.2.1}
$$

For $M$ an $R$-module equipped with a semilinear $\gamma$-action, [4.4.2.1] also holds for $x \in R, y \in M$. This observation can be used to fold differential and difference algebra into a common framework; see [3].

**Lemma 4.4.3.** Let $C$ be a reduced affinoid or mixed affinoid algebra over a nontrivially normed analytic field, and let $\beta$ be the spectral norm on $C$. Let $U$ be an open subgroup of $\Gamma_N$ acting on $C$, and assume that $\mathcal{M}(C)$ admits a neighborhood basis each element of which is a rational subdomain stable under an open subgroup of $U$. Suppose also that there exists $c > 0$ such that for any positive integer $n$ for which $U_n \subseteq U$, we have

$$
\beta(\gamma(x) - x) \leq cp^{-p^n+1/(p-1)}\beta(x) \quad (x \in C, \gamma \in U_n). \tag{4.4.3.1}
$$

Let $D$ be a finite étale $C$-algebra (which is again a reduced affinoid or mixed affinoid algebra) and extend $\beta$ to a power-multiplicative norm on $D$. Assume that $D$ admits a continuous extension of the action of $U$. Then there exists $d > 0$ such that for any positive integer $n$ for which $U_n \subseteq U$, we have

$$
\beta(\gamma(y) - y) \leq dp^{-p^n+1/(p-1)}\beta(y) \quad (y \in D, \gamma \in U_n). \tag{4.4.3.2}
$$

**Proof.** Using [4.4.2.1], it is enough to check [4.4.3.2] for some generators of $D$ as a finite Banach module over $C$ (which exist by [50, Lemma 2.5.2 and Remark 2.5.3]). From this, it follows that it is enough to check [4.4.3.2] after replacing $U$ by an open subgroup; by our hypothesis on the action of $U$ on $\mathcal{M}(C)$, we may work locally around an arbitrary $\delta \in \mathcal{M}(C)$. 50
(We are using here the fact that (4.4.3.1) continues to hold when $C$ is replaced by a rational localization on which $U_n$ acts, though possibly with a different constant $c$. This can be seen by writing the localization as an affinoid homomorphism.)

Since the local ring $C_\delta$ is henselian \[50\] Lemma 2.4.12, we may reduce to the case where the fibre of $M(D) \to M(C)$ above $\delta$ consists of a single point $\tilde{\delta}$. In this case, we may choose $y \in D$ whose image in $H(\delta)$ is a generator of this field over $H(\delta)$. By shrinking $C$ further, we can ensure that $y$ is a root of a polynomial $P(T) \in C[T]$ of degree $d = [H(\delta) : H(\delta)]$. The discriminant $\Delta$ of this polynomial has nonzero image in $H(\delta)$; by shrinking $C$ further, we can ensure that $\Delta$ is a unit in $C$.

Write $P(T) = \sum_i P_i T^i$. Since $P(y) = \gamma(P)(\gamma(y)) = 0$, we have on one hand
\[
P(\gamma(y)) = (P - \gamma(P))(\gamma(y)) = \sum_i (1 - \gamma)(P_i)\gamma(y).
\]
From (4.4.3.1), we can find a constant $c > 0$ such that $\beta(\gamma(P_i) - P_i) \leq cp^{-p^{n+1}/(p-1)}$ for all positive integers $n$ such that $U_n \subseteq U$ and all $\gamma \in U_n$.

On the other hand, we may write
\[
P(\gamma(y)) = P'(y)(\gamma(y) - y) + Q(\gamma(y), y)(\gamma(y) - y)^2
\]
for some polynomial $Q \in C[T_1, T_2]$. Since $\Delta$ is invertible in $C$, $P'(y)$ is invertible in $D$. Since the extension of the action of $\gamma$ to $D$ is continuous, for $\gamma \in U_n$ with $n$ sufficiently large, we have $\beta(\gamma(y) - y) = \beta(P(\gamma(y))/P'(y)) \leq \beta(P(\gamma(y)))/\beta(1/P'(y))$. We thus obtain (4.4.3.2) for this particular choice of $y$, and (using (4.4.2.1)) also for each power of $y$. Since $D$ is generated over $C$ by finitely many powers of $y$, we may deduce (4.4.3.2) in full generality. \hfill $\square$

**Lemma 4.4.4.** There exists $c > 0$ such that for every positive integer $n$,
\[
\overline{\alpha}_\psi(\gamma(\overline{x}) - \overline{x}) \leq cp^{-p^{n+1}/(p-1)}\overline{\alpha}_\psi(\overline{x}) \quad (\gamma \in U_n, \overline{x} \in A_\psi/(p)). \tag{4.4.4.1}
\]

**Proof.** For $A = K\{M_\lambda\}$ for some $\lambda \in N_Q$, using the observation that $\overline{\alpha}_\psi(\overline{p}^\alpha) = p^{-p^{n+1}/(p-1)}$, it is easy to check (4.4.4.1) for $\overline{x} = 1 + \overline{\pi}$ and for $\overline{x} \in S$. The same then follows for all $\overline{x}$ thanks to (4.4.2.1).

Suppose next that $\psi$ is a strictly rational subdomain. Define $\overline{f}_1, \ldots, \overline{f}_n, g$ as in the proof of Theorem 4.1.3 with $A = K\{M_\lambda\}$ for some $\lambda \in N_Q$. By writing
\[
(\gamma - 1) \left( \frac{1}{g} \right) = \frac{1 - \gamma(\overline{g})}{\overline{g}(\gamma(\overline{g}))},
\]
we deduce (4.4.4.1) for $\overline{x} = 1/\overline{g}$. We may then use (4.4.2.1) to check (4.4.4.1) for $\overline{x} = \overline{f}_1/\overline{g}, \ldots, \overline{f}_m/\overline{g}$, and then for all $\overline{x} \in A_\psi/(p)$.

Suppose next that $A$ is a strictly affinoid algebra over $K$. Since $M(A_\psi/(p)) \cong M(A_\psi) \cong M(A_{\psi, \infty})$ is the inverse limit of the spaces $M(A_{\psi, n})$, it admits a neighborhood basis each element of which is a rational subdomain stable under an open subgroup of $\Gamma_N$. We may thus apply Lemma 4.4.3 to deduce the desired result. The mixed affinoid case is similar. \hfill $\square$
**Proposition 4.4.5.** There exists \( c > 0 \) such that for every positive integer \( n \), there exists \( r_n > 0 \) such that

\[
\lambda(\overline{\alpha}_\psi^n)(\gamma(x) - x) \leq c^r p^{-p^{n+1}r/(p-1)} \lambda(\overline{\alpha}_\psi^n)(x) \quad (\gamma \in U_n, r \in (0, r_n], x \in C^r_\psi).
\]

**Proof.** Choose \( r_0 > 0 \) as in Lemma 4.2.3. Given \( \bar{x} \in A_\psi/(p) \), lift \( \bar{x} \) to \( x \in A_\psi^{r_0} \) as in Lemma 4.2.3 so that \( \lambda(\overline{\alpha}_\psi^n)(x - [\bar{x}]) \leq p^{-1/2} \lambda(\overline{\alpha}_\psi^n)(x) \). Write \( x = \sum_{i=0}^\infty p^i [\gamma_i] \) with \( \gamma_i, \gamma_i \in \overline{A}_\psi \). Since \( \overline{x}_0 = \overline{x} \in A_\psi/(p) \), for \( c > 0 \) as in Lemma 4.4.3, we have \( \overline{\alpha}_\psi^n(\gamma_i) \leq c r_0 p^{-p^{n+1}r_0/(p-1)} \overline{\alpha}_\psi^n(\overline{x}_0) \). On the other hand, since \( \gamma \) is an isometry, we have \( \lambda(\overline{\alpha}_\psi^n)(\gamma(x) - x) \leq \lambda(\overline{\alpha}_\psi^n)(x) \), so \( \lambda(\overline{\alpha}_\psi^n(\gamma_i)) \leq \lambda(\overline{\alpha}_\psi^n(\overline{x})) \) for \( i > 0 \). Now choose \( r_n \) so that

\[
p^{-1/2} \leq c r_n p^{-p^{n+1}r_n/(p-1)},
\]
then \( \lambda(\overline{\alpha}_\psi^n)(\gamma(x) - x) \leq c^r n p^{-p^{n+1}r_n/(p-1)} \lambda(\overline{\alpha}_\psi^n)(x) \). From this calculation, we may deduce the general case using Corollary 4.2.4.\( \square \)

**Corollary 4.4.6.** There exist \( r_0 > 0, c < 1 \), and a positive integer \( m \) such that

\[
\lambda(\overline{\alpha}_\psi^n)(\gamma(x) - x) \leq c^r m \lambda(\overline{\alpha}_\psi^n)(x) \quad (\gamma \in U_m, r \in (0, r_0], x \in C^r_\psi).
\]

Corollary 4.4.6 has the following essentially formal consequence.

**Proposition 4.4.7.** For \( r_0 \) as in Corollary 4.4.6, for all \( 0 < s \leq r \leq r_0 \), \( \gamma \in \Gamma_N \), and \( x \in C_\psi^{[s,r]} \), the limit

\[
(d\gamma)(x) = \lim_{n \to \infty} \frac{\gamma^{p^m(p-1) - 1}(x)}{p^n(p-1)}
\]
exists and defines a bounded derivation \( d\gamma \) on \( C_\psi^{[s,r]} \). This derivation may also be written as \( \log \gamma; \) that is,

\[
(d\gamma)(x) = \sum_{i=1}^\infty (-1)^{i-1} (\gamma - 1)^i(x).
\]

(The same then holds for \( x \in C_\psi^r \) or \( C_\psi^s \).

**Proof.** Take \( m \) as in Corollary 4.4.6. For \( n \geq m \), write

\[
\gamma^{p^m(p-1) - 1} = \sum_{i=1}^{p^n-m} \binom{p^n-m}{i} (\gamma^{p^m(p-1) - 1})^i,
\]

since \( \gamma^{p^m(p-1)} \in U_m \), we deduce that for \( n \) large,

\[
\lambda(\overline{\alpha}_\psi^n)((\gamma^{p^m(p-1) - 1})(x)) \leq p^{-(n-m)} \lambda(\overline{\alpha}_\psi)(x) \quad (t \in [s,r], x \in C_\psi^{[s,r]}).
\]

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By writing
\[
\frac{\gamma^{p^n+1(p-1)} - 1}{p^{n+1}(p-1)} - \frac{\gamma^{p^n(p-1)} - 1}{p^n(p-1)} = \frac{\gamma^{p^n+1(p-1)} - 1 - p(\gamma^{p^n(p-1)} - 1)}{p^{n+1}(p-1)}
\]
\[
= \left(\frac{\gamma^{p^n(p-1)} - 1}{p^{n+1}(p-1)}\right)^2 \sum_{i=0}^{p-2} \binom{p}{i+2} (\gamma^{p^n(p-1)} - 1)^i
\]
and using the fact that \(\gamma\) acts as an isometry, we obtain the first assertion. The second assertion follows from the observation that the binomial expansion of \(\left(\frac{\gamma^{p^n(p-1)} - 1}{p^{n+1}(p-1)}\right)^2\) as a power series in \(\gamma - 1\) converges term-by-term to the series expansion of \(\log(\gamma)\).

**Remark 4.4.8.** To make Proposition 4.4.7 more explicit, choose a basis \(T_1, \ldots, T_n\) of \(N^\vee\), and let \(\frac{\partial}{\partial \pi}, \frac{\partial}{\partial T_1}, \ldots, \frac{\partial}{\partial T_n}\) be the partial derivations on \(R = (\cap_{q \in (0,1)} K \{\pi/q\})[M_\sigma]\) with respect to \(\pi, T_1, \ldots, T_n\). For \(\gamma \in \mathbb{Z}^\times\), the action of \(d\gamma\) on \(R\) is given by
\[
d\gamma = (1 + \pi) \log(\gamma) \frac{\partial}{\partial \pi}.
\]
For \(\gamma \in \mathbb{Z}_p\), the action of \(d\gamma\) on \(R\) is given by
\[
d\gamma = \sum_{i=1}^n \langle \gamma, T_i \rangle T_i \log(1 + \pi) \frac{\partial}{\partial T_i}.
\]
It follows that \(\gamma \to d\gamma / \log(1 + \pi)\) defines an action of the Lie algebra \(\text{Lie}(\Gamma_N)\) on \(C^{[s,r]}\).

### 4.5 Decomposition of perfect period rings

Recall that every element of \(\mathbb{F}_p((\pi))^{\text{perf}}\) can be written uniquely as a sum
\[
\sum_{i \in [0,1) \cap \mathbb{Z}[p^{-1}]} c_i \pi^i \quad (c_i \in \mathbb{F}_p((\pi)))
\]
which converges in the sense that for any \(\epsilon > 0\), there are only finitely many indices \(i\) for which \(|c_i| > \epsilon\). We now obtain some similar decompositions of perfect period rings in terms of their imperfect counterparts.

**Definition 4.5.1.** For \(\mu \in (\mathbb{Z} \times N^\vee) \otimes_{\mathbb{Z}} (\mathbb{Z}[p^{-1}]/\mathbb{Z})\), let \(A^{[s,r]}[\mu]\) denote the \(A^{[s,r]}\)-submodule of \(\tilde{A}^{[s,r]}\) generated by \((1 + \pi)^e s\) for all \((e, s)\) in the inverse image of \(\mu\) in \((\mathbb{Z} \times N^\vee)[p^{-1}]\). Define \(A_\psi[\mu], B^{[s,r]}[\mu], C^{[s,r]}[\mu], C^r[\mu]\) analogously.

**Remark 4.5.2.** If \(\psi\) is boundary-free, then \(A^{[s,r]}[\mu]\) is always nonzero and is freely generated by any single element of the form \((1 + \pi)^e s\). In general, one needs a finite number (possibly zero) of such generators. One can often reduce to the boundary-free case by replacing \(\psi\) with each of a sequence of rational subframes whose union is dense (using the fact that subframes of strictly rational type form a neighborhood basis in \(\mathcal{M}(A)\), as in Remark 4.1.6). However, a bit of care is required: for instance, if one wishes to use this method to show that a certain map is bounded, one must check that the bound is uniform over the sequence of subframes.
Lemma 4.5.3. The natural map

\[
\bigoplus_{\mu} A_\psi[\mu]/(p) \to \tilde{A}_\psi/(p)
\]

is an isomorphism of Banach modules over \( A_\psi/(p) \) for the norm \( \overline{\psi} \) on \( \tilde{A}_\psi/(p) \) and the supremum norm on \( \bigoplus_{\mu} A_\psi[\mu]/(p) \) for the restriction of \( \overline{\psi} \) to each summand.

Proof. It suffices to check that the map \( \bigoplus_{\mu} A_\psi[\mu]/(p) \to \tilde{A}_\psi/(p) \) is strict. We have an isomorphism

\[
\overline{\psi}^{-1}(A_\psi/(p)) \cong \bigoplus_{\nu} A_\psi[\nu]/(p)
\]

of finite \( A_\psi/(p) \)-modules in which \( \nu \) runs over \((Z \times N^\psi) \otimes_Z (p^{-1}Z/\mathbb{Z})\). Since \( A_\psi/(p) \) is a mixed affineoid algebra over a complete discretely valued field, this is also an isomorphism of finite Banach modules \([50] \text{Lemma 2.5.2}\). Consequently, there exists \( c \geq 1 \) such that for any \( x \in \overline{\psi}^{-1}(A_\psi/(p)) \), the unique decomposition \( x = \sum_{\nu} x_\nu \) with \( x_\nu \in A_\psi[\nu]/(p) \) satisfies

\[
\max_{\nu}\{\overline{\psi}(x_\nu)\} \leq c\overline{\psi}(x).
\] (4.5.3.1)

By multiplying through by some \((1 + \pi)^es\), we see that for each \( \mu \in (Z \times N^\psi) \otimes_Z (Z[p^{-1}]/\mathbb{Z}) \), every \( x \in \overline{\psi}^{-1}(A_\psi[\mu]/(p)) \) has a unique representation as a sum \( \sum_{\nu} x_\nu \) in which \( \nu \) runs over elements of \((Z \times N^\psi) \otimes_Z (Z[p^{-1}]/\mathbb{Z})\) with \( p\nu = \mu \). Moreover, this decomposition again satisfies (4.5.3.1) for the same constant \( c \). (This requires a reduction to the boundary-free case as in Remark 4.5.2, which is valid because the constant \( c \) can be used uniformly.)

By applying this repeatedly, we obtain for each positive integer \( n \) an isomorphism

\[
\overline{\psi}^{-n}(A_\psi/(p)) \cong \bigoplus_{\nu} A_\psi[\nu]/(p)
\]

of finite Banach \( A_\psi/(p) \)-modules in which \( \nu \) runs over \((Z \times N^\psi) \otimes_Z (Z[p^{-n}]/\mathbb{Z})\). Moreover, for any \( x \in \overline{\psi}^{-n}(A_\psi/(p)) \), the unique decomposition \( x = \sum_{\nu} x_\nu \) with \( x_\nu \in A_\psi[\nu]/(p) \) satisfies

\[
\max_{\nu}\{\overline{\psi}(x_\nu)\} \leq c^{1+p^{-1}+\ldots+p^{-n-1}}\overline{\psi}(x) \leq c^{p/(p-1)}\overline{\psi}(x).
\] (4.5.3.2)

Since the final constant in (4.5.3.2) is independent of \( n \), we may take the union over \( n \) and then complete to obtain the desired isomorphism.

\[ \square \]

Corollary 4.5.4. For \( r_0 > 0 \) as in Lemma 4.2.3, there exists \( c > 0 \) such that for \( 0 < s \leq r \leq r_0 \), for \( x = \sum_{\mu} x_\mu \in \tilde{C}_\psi^{[s,r]} \), we have

\[
\sup_{\mu}\{\lambda(\overline{\psi})(x_\mu)\} \leq c\lambda(\overline{\psi})(x) \quad (t \in [s,r]).
\]

Consequently, the natural maps

\[
\bigoplus_{\mu} A_\psi^r[\mu] \to \tilde{A}_\psi^r, \quad \bigoplus_{\mu} B_\psi^r[\mu] \to \tilde{B}_\psi^r, \quad \bigoplus_{\mu} C_\psi^{[s,r]}[\mu] \to \tilde{C}_\psi^{[s,r]}
\]

are isomorphisms of Banach modules over \( A_\psi^r, B_\psi^r, C_\psi^{[s,r]} \), respectively.
Definition 4.5.5. For \( r_0 > 0 \) as in Lemma 4.2.3, for \( 0 < s \leq r \leq r_0 \), we obtain from 4.5.4 a \( \mathcal{C}^{[s,r]}_{\psi} \)-linear projection \( \Pi_{\mu} : \mathcal{C}^{[s,r]}_{\psi} \to \mathcal{C}^{[s,r]}_{\psi}[\mu] \) by picking out one term in the decomposition \( \mathcal{C}^{[s,r]}_{\psi} \cong \bigoplus_{\mu} \mathcal{C}^{[s,r]}_{\psi}[\mu] \). In case \( \psi \) is boundary-free, we can compose \( \Pi_{\mu} \) with an isomorphism \( \mathcal{C}^{[s,r]}_{\psi}[\mu] \cong \mathcal{C}^{[s,r]}_{\psi} \) (see Remark 4.5.2) to obtain a projection \( \mathcal{C}^{[s,r]}_{\psi} \to \mathcal{C}^{[s,r]}_{\psi} \). See [44, §3.5] for a similar construction.

4.6 \( \varphi \)-modules over imperfect period rings

We now introduce imperfect period rings into the study of local systems, starting with a study of \( \varphi \)-modules parallel to the study made for perfect period rings in [50, §6–7]. We do not complete the analogy, however, because for \( (\varphi, \Gamma_N) \)-modules we will be able to descend results from the perfect case more easily (see Theorem 4.10.1).

Hypothesis 4.6.1. Throughout §4.6, take \( r_0 > 0 \) as in Lemma 4.2.3 and Lemma 4.2.5, let \( a \) be a positive integer, and put \( q = p^a \).

We start with the analogue of [50, Definition 6.1.1].

Definition 4.6.2. A \( \varphi^a \)-module over an imperfect period ring is a finite projective module equipped with a semilinear \( \varphi^a \)-action (i.e., an isomorphism with its pullback by \( \varphi^a \)). These form an exact (but not abelian) category in which the morphisms are \( \varphi^a \)-equivariant morphisms of underlying modules. As in [50, Definition 1.5.3], for \( M \) a \( \varphi^a \)-module, we write

\[
H^0_{\varphi^a}(M) = \ker(\varphi^a - 1, M), \quad H^1_{\varphi^a}(M) = \text{coker}(\varphi^a - 1, M),
\]

and \( H^i_{\varphi^a}(M) = 0 \) for \( i \geq 2 \).

For \( * \in \{ A^*, B^*, C \} \) and \( M \) a \( \varphi^a \)-module over \( *_{\psi} \), for \( r > 0 \) sufficiently small, \( M \) can be realized as the base extension of a finite projective module \( M_r \) over \( *_{\hat{\psi}} \) equipped with an isomorphism \( \varphi^a M_r \cong M_r \otimes_{*_{\psi}} *_{\psi}^{r/q} \). We call \( M_r \) a model of \( M \) over \( *_{\hat{\psi}} \).

Lemma 4.6.3. Take \( R \) to be one of \( A_{\psi}, \hat{\cup}_{\varphi^{-n}}(A_{\psi}), \hat{A}_{\psi} \). Let \( M \) be a finite projective module over \( R/(p^m) \) for some positive integer \( m \) equipped with a semilinear \( \varphi^a \)-action. Then there exists a faithfully finite étale \( R \)-algebra \( U \) such that \( M \otimes_R U/(p^m) \) admits a basis fixed by \( \varphi^a \). More precisely, if \( l < m \) is another positive integer and \( M \otimes_R U/(p^l) \) admits a \( \varphi^a \)-fixed basis, then \( U \) can be chosen so that this basis lifts to a \( \varphi^a \)-fixed basis of \( M \otimes_R U/(p^m) \).

Proof. As in [50, Lemma 3.2.6].

Corollary 4.6.4. The base change functors among the categories of \( \varphi^a \)-modules over the rings

\[
A_{\psi} \to \bigcup \varphi^{-n}(A_{\psi}) \to \bigcup \varphi^{-n}(A_{\psi}) \to \hat{A}_{\psi}
\]

are tensor equivalences.
Proof. It is enough to check that for each ring $R$ in the diagram, the base change functor from $R$ to $\hat{A}_\psi$ is an equivalence of categories. To prove full faithfulness, it suffices to prove that for any $\varphi^a$-module $M$ over $R$, any $v \in M \otimes_R \hat{A}_\psi$ fixed by $\varphi^a$ must belong to $M$. For this, we may omit the case $R = \cup \varphi^{-n}(\hat{A}_\psi)$ by reducing it to the case $R = A_\psi$; we may thus assume that $R$ is $p$-adically complete. Apply Lemma 4.3.3 to construct a $R$-algebra $U$ which is the $p$-adic completed direct limit of faithfully finite étale $R$-subalgebras, such that $M \otimes_R U$ admits a $\varphi^a$-fixed basis. Put $\hat{U} = U \otimes_R \hat{A}_\psi$, so that we may view $v$ as an element of $M \otimes_R \hat{U}$. By writing $v$ in terms of a $\varphi^a$-fixed basis of $M \otimes_R U$ and noting that $U^\varphi = \hat{U}^\varphi$, we deduce that $v \in M \otimes_R U$. Since within $\hat{U}$ we have $\hat{A}_\psi \cap U = R$, we deduce that $v \in M$ as desired.

To prove essential surjectivity, it is enough to factor the functor from $\mathbb{Z}_p$-$\text{Loc}(\hat{A}_\psi)$ to $\varphi^a$-modules over $\hat{A}_\psi$ through the category of $\varphi^a$-modules over $A_\psi$. To do this, use Lemma 4.6.3 to imitate the construction in the proof of [50, Theorem 8.1.2].

As in [50], it is occasionally useful to consider a variant defined in more geometric terms.

Definition 4.6.5. For $r \in (0, r_0]$, let $X_{\psi, r}$ be the union of the spaces $M(C^{[s, r]}_{\psi})$ over all $s > 0$. In case $A$ is a strictly affinoid algebra over $K$, each $C^{[s, r]}_{\psi}$ is an affinoid algebra over $K$ by Lemma 4.2.2, so $X_{\psi, r}$ is a quasi-Stein space (Definition 1.3.1). Note that $\varphi^a$ induces a map $X_{\psi, r/q} \to X_{\psi}$; a $\varphi^a$-bundle over $C_{\psi}$ is a vector bundle $E$ over $X_{\psi, r}$ equipped with an isomorphism $(\varphi^a)^*E \cong E$ of vector bundles over $X_{\psi, r/q}$. By taking the categorical direct limit [50, Remark 1.2.9] over all $r \in (0, r_0]$, we obtain the category of $\varphi^a$-bundles over $C_{\psi}$.

For $s \in (0, r/q]$, a $\varphi^a$-module over $C^{[s, r]}_{\psi}$ is a finite locally free module $M$ over $C^{[s, r]}_{\psi}$ equipped with an isomorphism $(\varphi^a)^*M \otimes_{C^{[s, r]/q}_{\psi}} C^{[s, r/q]}_{\psi} \cong M \otimes_{C^{[s, r]}_{\psi}} C^{[s, r]}_{\psi}$ of modules over $C^{[s, r/q]}_{\psi}$. There is a functor from $\varphi^a$-bundles over $C_{\psi}$ to $\varphi^a$-modules over $C^{[s, r]}_{\psi}$ obtained by taking sections of the bundle over the semiaffinoid space $M(C^{[s, r]}_{\psi})$; as in [50, Lemma 6.1.3], this functor is a tensor equivalence thanks to the Kiehl gluing property for mixed affinoid algebras (Proposition 1.2.6).

Lemma 4.6.6. Let $E$ be a $\varphi^a$-bundle over $C_{\psi}$ for some $r \in (0, r_0]$. Let $M_r$ be the module of global sections of $E$. Then $M_r$ is a finite projective module over $C_{\psi}$.

Proof. Imitate the proof of [51, Proposition 2.2.7].

Theorem 4.6.7. The natural functor from $\varphi^a$-modules over $C_{\psi}$ to $\varphi^a$-bundles over $C_{\psi}$ is a tensor equivalence.

Proof. Full faithfulness follows from Corollary 4.2.6 while essential surjectivity follows from Lemma 4.6.6.

Corollary 4.6.8. The category of $\varphi^a$-modules over $C_{\psi}$ admits gluing for covering families of rational subframes.

Proof. This follows from Theorem 4.6.7 together with the Kiehl gluing property for semi-affinoid algebras (Proposition 1.2.6).

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Remark 4.6.9. When \( A \) is a finite extension of \( K \) and \( \psi \) is the structure morphism, one can obtain Theorem 4.6.7 without using the \( \varphi \)-action, using Lazard’s theorem that any vector bundle on an annulus over a spherically complete field is freely generated by global sections [55]. It is unclear whether such an argument can be made more generally.

We have the following analogue of [50, Proposition 6.4.5].

Proposition 4.6.10. Let \( M \) be a \( \varphi^a \)-module over \( C_\psi \), and let \( M_r \) be a model of \( M \) over \( C_\psi^r \) for some \( r \in (0, r_0] \). For \( s \in (0, r/q] \), put \( M_{[s,r]} = M_r \otimes_{C_\psi} C_\psi^{[s,r]} \). Then the vertical arrows in the diagram

\[
0 \to M_r \xrightarrow{\varphi^a - 1} M_{r/q} \to 0 \\
0 \to M_{[s,r]} \xrightarrow{\varphi^a - 1} M_{[s,r/q]} \to 0
\]

induce an isomorphism on the cohomology of the horizontal complexes.

Proof. Identify the kernel and cokernel with extension groups and apply Lemma 4.6.6.

Definition 4.6.11. We define rank, degree, and slope for \( \varphi^a \)-modules over an imperfect period ring by analogy with the perfect case (see [50, Definition 7.2.1]). In fact, these invariants can all be computed by first extending scalars to the corresponding perfect ring.

Define the pure locus (resp. the \( \acute{e}tale \) locus) of \( M \) by base extension to the corresponding perfect period ring. These are open subsets of \( \mathcal{M}(A) \) by Remark 3.4.6.

We define the pure and \( \acute{e}tale \) conditions over imperfect period rings by following [50, §7.3].

Definition 4.6.12. Choose \( R \) from among \( B_\psi, B_\psi^\dagger, C_\psi \), then define \( R_0 \) to be \( A_\psi, A_\psi^\dagger, A_\psi^\dagger \), respectively. Fix integers \( c, d \) with \( d \) a positive multiple of \( a \). Let \( M \) be a \( \varphi^a \)-module over \( R \). A \((c,d)\)-pure model for \( M \) is a finite \( R_0 \)-submodule \( M_0 \) of \( M \) for which the map \( M_0 \otimes_{R_0} R \to M \) is an isomorphism, and the action of \( \varphi^a \) on \( M \) induces an isomorphism \((p^c \varphi^d)^* M_0 \cong M_0 \). We say the pure model is free or locally free if the underlying \( R_0 \)-module \( M_0 \) has that property.

For \( \gamma \in \mathcal{M}(A) \), a (free, locally free) local \((c,d)\)-pure model for \( M \) at \( \gamma \) consists of a rational subframe \( \psi' \) encircling \( \gamma \) and a (free, locally free) \((c,d)\)-pure model of \( M \otimes_R R' \) for \( R' \) the analogue of \( R \) with \( \psi \) replaced by \( \psi' \). We write \( \acute{e}tale \) model as shorthand for \((0,1)\)-pure model, and likewise in the presence of modifiers. (Note that we must assume that \( \psi' \) again satisfies Hypothesis 4.1.1, but this is harmless in practice because such subframes form a neighborhood basis of \( \gamma \) in \( \mathcal{M}(A) \), as in Remark 4.1.6.)

It can be shown (by arguing as in [50, Definition 8.1.5]) that the existence of a \((c,d)\)-pure model depends only on the ratio \( c/d \). If \( M \) admits a local \((c,d)\)-pure model at \( \gamma \), we say \( M \) is pure or pure of slope \( c/d \) at \( \gamma \). If \( M \) is pure (resp. pure of slope \( s \)) at all \( \gamma \in \mathcal{M}(A) \), we simply say that \( M \) is pure (resp. pure of slope \( s \)). In these definitions, we write \( \acute{e}tale \) as shorthand for pure of slope \( 0 \).
Lemma 4.6.13. Keep notation as in Definition 4.6.12.

(a) The $\varphi^a$-module $M$ admits a locally free local $(c,d)$-pure model at $\beta$ if and only if it admits a free local $(c,d)$-pure model at $\beta$.

(b) If $M$ is a $\varphi^a$-module over $B^\dagger_\psi$, then $M$ admits a free local $(c,d)$-pure model at $\beta$ if and only if $M \otimes_{B^\dagger_\psi} B_\psi$ does.

Proof. As in [50, Lemma 7.3.3].

Proposition 4.6.14. (a) The base change functor from $\varphi^a$-modules over $A^\dagger_\psi$ to $\varphi^a$-modules over $A_\psi$ is fully faithful.

(b) The base change functor from globally étale $\varphi^a$-modules over $B^\dagger_\psi$ to $\varphi^a$-modules over $C_\psi$ is fully faithful.

Proof. As in [50, Remark 4.3.4], part (a) reduces to checking that any $\varphi^a$-invariant element $v$ of $M \otimes_{A^\dagger_\psi} A_\psi$ belongs to $M$ itself. By extending scalars to $\hat{A}_\psi$ and applying [50, Theorem 8.1.2], we deduce that $v$ belongs to $M \otimes_{A^\dagger_\psi} \hat{A}_\psi$. Since $M$ is projective and we have $A_\psi \cap \hat{A}^\dagger_\psi = \hat{A}^\dagger_\psi$ within $\hat{A}_\psi$, we deduce that $v \in M$ as desired.

Similarly, part (b) reduces to checking that any $\varphi^a$-invariant element $v$ of $M \otimes_{B^\dagger_\psi} C_\psi$ belongs to $M$ itself. By extending scalars to $\hat{B}^\dagger_\psi$ and applying [50, Theorem 8.1.4], we deduce that $v$ belongs to $M \otimes_{B^\dagger_\psi} \hat{B}^\dagger_\psi$. Since $B_\psi \cap \hat{B}^\dagger_\psi = \hat{B}^\dagger_\psi$ within $\hat{B}_\psi$, we deduce that $v \in M$ as desired.

4.7 $(\varphi, \Gamma_N)$-modules

We now introduce $(\varphi, \Gamma_N)$-modules over imperfect period rings.

Definition 4.7.1. For $d$ a positive integer, a $(\varphi^d, \Gamma_N)$-module over an imperfect period ring is (as expected) a $\varphi^d$-module equipped with a compatible action of $\Gamma_N$ which is effective and continuous for all available topologies. (Effectivity here means that for each $s \in M_{\sigma}$, the subgroup of $N_p$ fixing $s$ acts trivially on the module modulo $s$.) We say such an object is (globally) pure/étale if the same is true of the underlying $\varphi^d$-module, with no regard for the action of $\Gamma_N$. Define the cohomology groups $H^i_{\varphi^d, \Gamma_N}(M)$ as in Definition 3.4.7.

Remark 4.7.2. Because $\psi$ is now forced to be étale by hypothesis, Remark 3.2.6 implies that one may omit the hypothesis of continuity of the $\Gamma_N$-action in the definition of a $(\varphi^d, \Gamma_N)$-module over $A_\psi$ or $A^\dagger_\psi$. Similarly, by Proposition A.1.7, one may replace continuity with boundedness in the definition of a $(\varphi^d, \Gamma_N)$-module over $C_\psi$.

Remark 4.7.3. Over an imperfect period ring, continuity of the action of $\Gamma_N$ on a $(\varphi^d, \Gamma_N)$-module implies that the operator norm of $\gamma - 1$ tends to 0 as $\gamma \in \Gamma_N$ tends to 1. Namely,
this holds for the trivial \((\varphi^d, \Gamma_N)\)-module by Corollary \([4.4.6]\) and extends to the general case by \([4.4.2.1]\).

Once \(\gamma - 1\) acts with sufficiently small operator norm (e.g., less than \(p^{-1}\)), the identity

\[
\gamma^p - 1 = (1 + (\gamma - 1))^p - 1 = (\gamma - 1)^p + p \sum_{i=1}^{p-1} \frac{(p-1)^{i}}{i}
\]

may be used repeatedly to deduce that the action of \(\Gamma_N\) is analytic.

**Remark 4.7.4.** Continuity of the action of \(\Gamma_N\) on a \((\varphi^d, \Gamma_N)\)-module implies that the action of the group algebra \(\mathfrak{o}_K[\Gamma_N]\) extends to an action of the completion \(\mathfrak{o}_K[\Gamma_N]\) with respect to the augmentation ideal \(I = \ker(\mathfrak{o}_K[\Gamma_N] \to \mathfrak{o}_K)\). Over an imperfect period ring, Remark \([4.7.3]\) implies somewhat more: for \(n\) a sufficiently large positive integer (depending on the module in question), the action of \(\mathfrak{o}_K[\Gamma_n]\) extends to the \(p\)-adic completion \(\mathfrak{o}_K[\Gamma_n][I^n/p]\). It thus follows that for any \((\varphi^d, \Gamma_N)\)-module, the action of \(\mathfrak{o}_K[\Gamma_n]\) extends to an action of the ring \((\lim_{\to n} \mathfrak{o}_K[\Gamma_n])[p^{-1}]\), which is a disjoint union of copies of the ring of rigid analytic functions on an open unit polydisc over \(K\). This ring should play a role in relative \(p\)-adic Hodge theory extending the work of Pottharst relating Iwasawa cohomology to \((\varphi, \Gamma)\)-modules \([66]\); for instance, this role is apparent in \([51]\).

### 4.8 Deperfection

We make some calculations with the aim of descending \((\varphi, \Gamma_N)\)-modules from \(\tilde{C}_\psi\) to \(C_\psi\). These are in the spirit of the work of Sen \([72]\) as reinterpreted by Cherbonnier and Colmez \([20]\).

**Hypothesis 4.8.1.** Throughout \([4.8]\) choose \(r_0 > 0\) as in Lemma \([4.2.3]\) Fix an identification \(N \cong \mathbb{Z}_\kappa\), let \(H\) be the subgroup \((1 + p^2\mathbb{Z}_p) \ltimes p^2\mathbb{Z}_p^n\) of \(\Gamma_N\), put \(\gamma_0 = 1 + p^2 \in 1 + p^2\mathbb{Z}_p\), and let \(\gamma_1, \ldots, \gamma_n\) be the standard generators of \(p^2N_p \cong p^2\mathbb{Z}_p^n\). We may view \(\gamma_0, \ldots, \gamma_n\) as topological generators of \(H\), even when \(p = 2\).

We construct some subrings intermediate between the imperfect and perfect period rings.

**Definition 4.8.2.** Take \(r \in (0, r_0]\) and \(s \in (0, r]\). From Corollary \([4.5.4]\) we have isomorphisms

\[
\overline{A}_\psi = \hat{A}_\psi/(p) \cong \bigoplus_\mu A_\psi[\mu]/(p), \quad \tilde{C}_\psi^{[s,r]} \cong \bigoplus_\mu C_\psi^{[s,r]}[\mu]
\]

of Banach modules over \(A_\psi, C_\psi^{[s,r]}\), respectively. Regroup terms to obtain

\[
\overline{A}_\psi \cong A_\psi/(p) \oplus N_{0,\psi} \oplus \cdots \oplus N_{n,\psi}, \quad \tilde{C}_\psi^{[s,r]} \cong C_\psi^{[s,r]} \oplus N_{[s,r],0,\psi} \oplus \cdots \oplus N_{[s,r],n,\psi}
\]

with \(N_{0,\psi}, N_{[s,r],0,\psi}\) accounting for those \(\mu\) with nonzero image in \(\mathbb{Z}_p[\mathbb{Z}_p]/\mathbb{Z}\) and \(N_{i,\psi}, N_{[s,r],i,\psi}\) for \(i > 0\) accounting for those \(\mu\) for which \(\mu \in N^\vee \otimes \mathbb{Z}[\mathbb{Z}_p]/\mathbb{Z}\), \(\langle \gamma_i/p^2, \mu \rangle \neq 0\), and \(\langle \gamma_j/p^2, \mu \rangle = 0\) for \(j = i + 1, \ldots, n\). Then put

\[
\overline{A}_\psi^i = A_\psi/(p) \oplus N_{0,\psi} \oplus \cdots \oplus N_{i,\psi}; \quad C_\psi^{[s,r]}[i] = C_\psi^{[s,r]} \oplus N_{[s,r],0,\psi} \oplus \cdots \oplus N_{[s,r],i,\psi};
\]
these are complete $\Gamma_N$-stable subrings of $\bar{A}_\psi, \bar{C}_\psi^{[s,r]}$. Note that $\bar{A}_\psi = A_\psi, C_\psi^{[s,r]};$ by convention, we also put $\bar{A}_\psi^{-1} = A_\psi/(p), C_\psi^{[s,r]-1} = C_\psi^{[s,r]}$.

Using these intermediate rings as stepping stones, we descend $(\phi, \Gamma_N)$-modules from $\bar{C}_\psi$ to $C_\psi$ as follows.

**Lemma 4.8.3.** For $i = 0, \ldots, n$, for any nonnegative integer $h$, $\gamma_i^{p^h} - 1$ is bijective on $\bar{N}_{i,\psi}$ with bounded inverse.

**Proof.** Using the identity

$$\gamma_i^{p^h} - 1 = (\gamma_i - 1)(1 + \gamma_i^{p^h} + \cdots + \gamma_i^{p^h(p^{-1}-1)}),$$

it suffices to check the claim for $h$ sufficiently large. To be precise, we choose $c > 0$ as in Lemma 4.4.4 and prove the claim for all $h$ for which $c < p^{h+2}$.

By Lemma 4.5.3, it is sufficient to check that for each $\mu$ which contributes to $\bar{N}_{i,\psi}$, $\gamma_i^{p^h} - 1$ is bijective on $A_\psi[\mu]/(p)$ and its inverse is bounded uniformly in $\mu$. Suppose first that $\psi$ is boundary-free; in this case, $A_\psi[\mu]/(p) = (A_\psi/(p))\bar{y}$ for $\bar{y} = (1 + \pi)^e s$ for any $(e, s) \in (\mathbb{Z} \times N^\vee)[\frac{1}{p}]$ lifting $\mu$. Note that $\gamma_i^{p^h}(\bar{y}) - \bar{y} = ((1 + \pi)^t - 1)\bar{y}$ for some $t \in \mathbb{Z}[p^{-1}]$ of $p$-adic valuation at most $h + 1$, so

$$\overline{\alpha}_\psi(\gamma_i^{p^h}(\bar{y}) - \bar{y}) \geq p^{-h+2/(p-1)}\overline{\alpha}_\psi(\bar{y}). \quad (4.8.3.1)$$

On the other hand, for any $\bar{x} \in A_\psi/(p)$, as in (4.4.2.1) we may write

$$(\gamma_i^{p^h} - 1)(\bar{x}) = (\gamma_i^{p^h} - 1)(\bar{y})\bar{x} + \gamma_i^{p^h}(\bar{y})(\gamma_i^{p^h} - 1)(\bar{x})$$

$$= ((1 + \pi)^t - 1)\bar{x}\bar{y} + \gamma_i^{p^h}(\bar{y})(\gamma_i^{p^h} - 1)(\bar{x}),$$

and by (4.4.1.1) and our choice of $c$, we have

$$\overline{\alpha}_\psi(\gamma_i^{p^h}(\bar{x}) - \bar{x}) \leq cp^{-h+3/(p-1)}\overline{\alpha}_\psi(\bar{x}) < p^{-h+2/(p-1)}\overline{\alpha}_\psi(\bar{x}) \quad (\bar{x} \in A_\psi/(p) \setminus \{0\}). \quad (4.8.3.2)$$

By (4.8.3.1) and (4.8.3.2), the operator on $A_\psi/(p)$ defined by

$$\bar{x} \mapsto ((1 + \pi)^t - 1)^{-1}\bar{y}^{-1}(\gamma_i^{p^h} - 1)(\bar{x}) = \bar{x} + (1 + \pi)^{-1}\bar{y}^{-1}\gamma_i^{p^h}(\bar{y})(\gamma_i^{p^h} - 1)(\bar{x})$$

is the sum of the identity map with an operator of norm strictly less than 1, and hence itself is invertible with norm 1. It follows that $\gamma_i^{p^h} - 1$ is invertible on $A_\psi[\mu]/(p)$ and

$$\overline{\alpha}_\psi(\gamma_i^{p^h}(\bar{x}) - \bar{x}) = ((1 + \pi)^t - 1)\overline{\alpha}_\psi(\bar{x}) \quad (\bar{x} \in A_\psi[\mu]/(p)) \quad (4.8.3.3)$$

at least if $\psi$ is boundary-free. If $\psi$ is not boundary-free, we may approximate $\psi$ by a sequence of boundary-free frames as in Remark 4.5.2, choose $c$ uniformly over the sequence (which is possible by inspection of the proof of Lemma 4.4.4), then deduce the claim for $\psi$. \hfill \Box
Corollary 4.8.4. For $i = 0, \ldots, n$ and $h$ a nonnegative integer, the map
\[ \overline{A}^i \psi \cong A^{-1} \psi \oplus N_{i,\psi} \xrightarrow{1 \otimes (\gamma^h_i - 1)} A^{-1} \psi \oplus N_{i,\psi} \cong \overline{A}^i \psi \]
is bijective with bounded inverse.

Proof. Combine Lemma [4.5.3] and Lemma [4.8.3] \qed

Corollary 4.8.5. For $i = 0, \ldots, n$ and $h$ a nonnegative integer, there exist $r_1 > 0$ and $c \geq 1$ such that for $0 < s \leq r \leq r_1$, every $x \in C_{\psi}^{[s,r],i}$ has a unique representation as $y + (\gamma^h_i - 1)z$ with $y \in C_{\psi}^{[s,r],i-1}$, $z \in N_{[s,r],i,\psi}$, and this representation satisfies
\[ \lambda(\alpha_{\psi}^{-1})(y), \lambda(\overline{x})^{-1}(z) \leq c^{t} \lambda(\alpha_{\psi}^{-1})(x) \quad (t \in [s, r]). \]

Proof. Uniqueness follows from the direct sum decomposition (4.8.2.1) and the bijectivity of $\gamma^h_i - 1$ on $N_{[s,r],i,\psi}$ (Lemma [4.8.3]), so it suffices to prove existence.

We first check the claim for $x \in \tilde{A}^{i, \psi 0} \cap C_{\psi}^{[s,r],i}$. Write $x = \sum_{n=0}^{\infty} p^n \overline{\tau}_{0,n}$ with $\overline{\tau}_{0,n} \in \overline{A}_{\psi}$. By Corollary [4.8.4], we can choose a constant $c_0 > 0$ such that for any $x$, we can write $\overline{x} = \overline{y} + (\gamma^h_i - 1)\overline{z}$ with $\overline{y} \in \overline{A}_{\psi}^{-1}$, $\overline{z} \in N_{i,\psi}$, and $\overline{\alpha}_{\psi}(\overline{y}), \overline{\alpha}_{\psi}(\overline{z}) \leq c_0 \overline{\alpha}_{\psi}(x)$. By Corollary [4.5.4] plus Corollary [4.2.1], we can choose another constant $c_1 > 0$ such that for any $x$, we can lift $\overline{y}, \overline{z}$ to elements $y_0 \in \tilde{A}_{\psi}^{i, ro}$, $z_0 \in \tilde{A}_{\psi}^{i, r0} \cap N_{[s,r],i,\psi}$ with $\lambda(\overline{\alpha}_{\psi}^{-1})(y_0), \lambda(\overline{\alpha}_{\psi}^{-1})(z_0) \leq c_1 \overline{\alpha}_{\psi}^{-1}(x)$. By subtracting these elements off, dividing by $p$, and repeating, we produce sequences $y_0, y_1, \ldots \in \tilde{A}_{\psi}^{i, ro}$ and $z_0, z_1, \ldots \in \tilde{A}_{\psi}^{i, r0} \cap N_{[s,r],i,\psi}$ such that for $m = 0, 1, \ldots$, $x - \sum_{n=0}^{m} p^n \lambda(\overline{\alpha}_{\psi}^{-1})(y_n + (\gamma^h_i - 1)(z_n)) \in p^{m+1} \tilde{A}_{\psi}^{i, ro}$ and
\[ \lambda(\overline{\alpha}_{\psi}^{-1})(p^m y_m), \lambda(\overline{\alpha}_{\psi}^{-1})(p^m z_m) \leq \max\{ c_1^{m-n} p^{-n} \lambda(\overline{\alpha}_{\psi}^{-1})(x_n) : n = 0, \ldots, m \}. \]

For any $r_1 \in (0, r_0]$, we have $p^{r_1/r_0 - 1} c_1^{r_1/r_0} \leq 1$. Choose one such value once and for all (independently of $x$). Then for $0 < s \leq r \leq r_1$, for $y = \sum_{n=0}^{\infty} y_n p^n$ and $z = \sum_{n=0}^{\infty} z_n p^n$,
\[ \lambda(\overline{\alpha}_{\psi}^{-1})(y), \lambda(\overline{\alpha}_{\psi}^{-1})(z) \leq c_1^{t/r_0} \lambda(\overline{\alpha}_{\psi}^{-1})(x) \quad (t \in [s, r]). \]

This proves the claim for $x \in \tilde{A}_{\psi}^{i, ro} \cap C_{\psi}^{[s,r],i}$; the general case follows by the density of $\tilde{A}_{\psi}^{i, ro}[p^{-1}] \cap C_{\psi}^{[s,r],i}$ in $C_{\psi}^{[s,r],i}$. \qed

Lemma 4.8.6. Choose $r_1 > 0$ as in Corollary [4.8.5]. Choose $i \in \{0, \ldots, n\}$, $h$ a nonnegative integer, $r \in (0, r_1]$, and $s \in (0, r]$. Let $G_i$ be a square matrix over $C_{\psi}^{[s,r],i}$ such that $\lambda(\overline{\alpha}_{\psi}^{-1})(G_i - 1) < 1$ for all $t \in [s, r]$. Then for any sufficiently large nonnegative integer $a$, there exists a square matrix $U$ over $\varphi^{-a}(C_{\psi}^{[sp^a, rp^a],i})$ of the same size such that $\lambda(\overline{\alpha}_{\psi}^{-1})(U - 1) < 1$ for all $t \in [s, r]$ and $U^{-1}G_i \gamma^h_i(U)$ has entries in $\varphi^{-a}(C_{\psi}^{[sp^a, rp^a],i-1})$. 61
Proof. Take $c \geq 1$ as in Corollary 4.8.5 and put
\[ \epsilon = \sup \{ \lambda(\overline{\alpha}_i)(G_i - 1)^{1/3} : t \in [s, r] \} . \]

We prove the claim for any $a$ for which $ce^{c^p-a} < 1$, by defining a sequence of invertible matrices $U_0, U_1, \ldots$ over $\varphi^{-a}(C_{[s, a, r, p-a], i})$ for which that the representation of $G_i, t = U_i^{-1}G_i' [Y_i] (U_i)$ as $1 + X_i + (\gamma_i^b - 1)(Y_i)$ with $X_i$ having entries in $\varphi^{-a}(C_{[s, a, r, p-a], i, \psi})$ and $Y_i$ having entries in $\varphi^{-a}(N_{[s, p-a, r, p-a], i, \psi})$ satisfies
\[ \lambda(\overline{\alpha}_i)(X_i) \leq \epsilon^2, \quad \lambda(\overline{\alpha}_i)(Y_i) \leq \epsilon^{l+2} \quad (t \in [s, r]) . \]

The product $U_0U_1 \cdots$ will then converge under $\lambda(\overline{\alpha}_i)$ for $t \in [s, r]$ to a matrix $U$ of the desired form.

To begin with, put $U_0 = 1$ and apply Corollary 4.8.5 to construct $X_0$ and $Y_0$ of the desired form with $\lambda(\overline{\alpha}_i)(X_0), \lambda(\overline{\alpha}_i)(Y_0) \leq c^{p-a} \lambda(\overline{\alpha}_i)(G_i - 1) \leq \epsilon^2$ for $t \in [s, r]$. Given $U_i, X_i, Y_i$, set $U_{i+1} = U_i(1 - Y_i)$, so that
\[ G_i, t+1 = (1 - Y_i)^{-1}(1 + X_i + (\gamma_i^b - 1)(Y_i))(1 - \gamma_i^b (Y_i)) = 1 + X_i + Y_i X_i - X_i \gamma_i^b (Y_i) + E_i \]
for some matrix $E_i$ with $\lambda(\overline{\alpha}_i)(G_i, t+1 - E_i) \leq \epsilon^{2(l+2)}$ for $t \in [s, r]$. Note that $Y_i X_i - X_i \gamma_i^b (Y_i)$ has entries in $\varphi^{-a}(N_{[s, p-a, r, p-a], i, \psi})$ and $\lambda(\overline{\alpha}_i)(Y_i X_i - X_i \gamma_i^b (Y_i)) \leq \epsilon^{l+4}$ for $t \in [s, r]$. By Corollary 4.8.5 we can split $E_i = A_i + (\gamma_i^b - 1)(B_i)$ with $A_i$ having entries in $\varphi^{-a}(C_{[s, a, r, p-a], i, \psi})$, $B_i$ having entries in $\varphi^{-a}(N_{[s, p-a, r, p-a], i, \psi})$, and $\lambda(\overline{\alpha}_i)(A_i), \lambda(\overline{\alpha}_i)(B_i) \leq 2^{p-a} \lambda(\overline{\alpha}_i)(E_i) \leq c^{p-a} \epsilon^{l+4} \leq \epsilon^{l+3}$. We may then take $X_{i+1} = X_i + A_i$ and $Y_{i+1} = B_i$. \(\square\)

Lemma 4.8.7. Choose $r_1 > 0$ as in Corollary 4.8.5. Let $d$ be a positive integer and put $q = p^d$. Let $M$ be a $(\varphi^d, \Gamma_N)$-module over $\mathcal{C}_{[r, q]}$. Let $M_r$ be a model of $M$ over $\mathcal{C}_{[r, q]}$ for some $r \in (0, r_1]$. Suppose that $M_r \otimes \mathcal{C}_{[r, q]} \mathcal{C}_{[r, q]}$ admits a basis $e_1, \ldots, e_m$, and let $F, G_0, \ldots, G_n$ be the matrices via which $\varphi, \gamma_0, \ldots, \gamma_n$ act on the given basis. Suppose further that for some $i \in \{0, \ldots, n\}$ and some nonnegative integer $h$, $F$ has entries in $\mathcal{C}_{[r, q]}$, $G_0, \ldots, G_n$ have entries in $\mathcal{C}_{[r, q]}$, and $H_h = \gamma_i^h (G_i) \cdots \gamma_i^{h-1} (G_i)$ has entries in $\mathcal{C}_{[r, q]}$. Then $F$ has entries in $\mathcal{C}_{[r, q]}$ and $G_0, \ldots, G_n$ all have entries in $\mathcal{C}_{[r, q]}$.

Proof. Since $\Gamma_N$ acts continuously on $M$, we may increase $h$ to ensure that $\lambda(\overline{\alpha}_i)(H_h - 1) < p^{-1}$ for $s \in [r, q]$. There is no harm in applying $\varphi^d$ to everything in order to replace $r$ by $r/q$; by so doing, thanks to Corollary 4.8.5 we can ensure that
\[ \lambda(\overline{\alpha}_i)(x) \leq p \lambda(\overline{\alpha}_i)(\gamma_i^b (x) - x) \quad (x \in N_{[r, q, r, \alpha]}, s \in [r, q]) . \]

Since $\gamma_i$ commutes with $\varphi$ and $\gamma_j$, we have
\[ H_h^{-1} F \varphi (H_h) = \gamma_i^b (F), \quad H_h^{-1} G_j \gamma_j (H_h) = \gamma_i^b (G_j) . \]
Write \( F = F_1 + F_2, G_j = G_{j,1} + G_{j,2} \) with \( F_1 \) having entries in \( C^{[r/q,r/q],i-1}_\psi \), \( F_2 \) having entries in \( N_{[r/q,r/q],i,\psi} \), \( G_{j,1} \) having entries in \( C^{[r/r,q],i-1}_\psi \), and \( G_{j,2} \) having entries in \( N_{[r/q,r],i,\psi} \).

Then

\[
H^{-1}_h F_2 \varphi(H_h) - F_2 = (\gamma_i^{p_h} - 1)(F_2), \quad H^{-1}_h G_{j,2} \gamma_j(H_h) - G_{j,2} = (\gamma_i^{p_h} - 1)(G_{j,2}).
\]

Suppose that \( F_2 \neq 0 \). Since \( \lambda(\pi^{r/q}_\psi)(H_h^{-1}) < p^{-1} \) for \( s \in [r/q,r] \), we have \( \lambda(\pi^{r/q}_\psi)(H_h^{-1} F_2 \varphi(H_h) - F_2) < p^{-1} \lambda(\pi^{r/q}_\psi)(F_2) \), a contradiction. We deduce that \( F_2 = 0; \) a similar argument shows that \( G_{j,2} = 0 \). This proves the claim. \( \square \)

**Proposition 4.8.8.** Choose \( r_1 > 0 \) as in Corollary \([4.8.3]\). Let \( d \) be a positive integer and put \( q = p^d \). Let \( M \) be a \((\varphi^d, \Gamma_N)\)-module over \( C_\psi \). Let \( M_r \) be a model of \( M \) over \( \bar{C}_\psi \) for some \( r \in (0, r_1) \). Suppose that \( M_{[r/q,r]} = M_r \otimes C_\psi \bar{C}_{[r/r,q],i} \) is a free module. Then for a suitable choice of \( r \), \( M_{[r/q,r]} \) admits a basis on which \( \varphi \) acts via a matrix over \( C_{[r/q,r/q],i} \) and \( \gamma_0, \ldots, \gamma_n \) act via matrices over \( C_{[r/r,q],i} \).

**Proof.** We check that for \( i = n, \ldots, -1 \), we can find a basis of \( M \) on which \( \varphi \) acts via a matrix over \( C_{[r/q,r/q],i} \) and \( \gamma_0, \ldots, \gamma_n \) act via matrices over \( C_{[r/r,q],i} \). This holds for \( i = n \) because \( C_{[r/q,r/q],n} = C_{[r/r,q]} \). Given the claim for some \( i \), by Lemma \([4.8.6]\) and continuity of the action of \( \gamma \), we can change basis so that for some nonnegative integer \( h \), \( \gamma_i^{p_h} \) acts via a matrix over \( C_{[r/q,r/q],i-1} \) for some \( r > 0 \). (More precisely, after applying Lemma \([4.8.6]\) apply \( \varphi^a \) and replace \( r \) with \( rq^{-a} \).) On this new basis, \( \varphi \) acts via a matrix over \( C_{[r/q,r/q],i-1} \) and \( \gamma_0, \ldots, \gamma_n \) act via matrices over \( C_{[r/q,r/q],i-1} \) byLemma \([4.8.7]\). This completes the induction; the case \( i = -1 \) yields the desired result. \( \square \)

Similar but simpler arguments allow us also to compare cohomology over \( C_\psi \) and \( \bar{C}_\psi \).

**Lemma 4.8.9.** Let \( d \) be a positive integer and put \( q = p^d \). Let \( M \) be a \((\varphi^d, \Gamma_N)\)-module over \( C_\psi \). Let \( M_r \) be a model of \( M \) over \( \bar{C}_\psi \) for some \( r \in (0, r_0) \). Then for \( i = 0, \ldots, n \), there exists a positive integer \( a \) such that \( \gamma_i - 1 \) acts bijectively on \( M_r \otimes C_\psi \varphi^{-ad}(N_{[r/q,r+1]/[r/q],i,\psi}) \).

**Proof.** Choose a finite Banach norm on \( M_{[r/q,r]} = M_r \otimes C_{[r/q,r]} \). By continuity of the \( \Gamma_N \)-action, for any \( \epsilon > 0 \), we can find a positive integer \( h \) such that \( \gamma_i^{p_h} - 1 \) acts on \( M_{[r/q,r]} \) with operator norm less than 1. On the other hand, for any \( \mu \) which contributes a nonzero summand to \( N_{[r/q,r],i,\psi} \) and any \( (e,s) \in (\mathbb{Z} \times N^\vee)^{[1]}_p \) lifting \( \mu \), for \( x = \varphi^{-ad}((1 + \pi)^e s) \), we have \( (\gamma_i^{p_h} - 1)(x) = (1 + \pi)^t x \) for some \( t \in \mathbb{Z}[p^{-1}] \) of \( p \)-adic valuation at most \( h + 1 - ad \). By taking \( t \) large enough, we can ensure that for \( v \in M_r \), the expression

\[
(\gamma_i^{p_h} - 1)(x v) = (\gamma_i^{p_h} - 1)(x) v + \gamma_i^{p_h}(x)(\gamma_i^{p_h} - 1)(v)
\]

from \([4.4.2.1]\) is dominated by the term \( (\gamma_i^{p_h} - 1)(x) v \). This implies (by reducing to the boundary-free case as in Remark \([4.5.2]\)) that for a sufficiently large, \( \gamma_i - 1 \) acts bijectively on
For \( d \) a positive integer, let \( M \) be a \((\varphi^d, \Gamma_N)\)-module over \( \mathbb{C}_\psi \). Then the continuous effective \((\varphi^d, \Gamma_N)\)-cohomology groups of \( M \otimes_{\mathbb{C}_\psi} (\mathbb{C}_\psi/\mathbb{C}_\psi) \) are all trivial.

**Proof.** For the purposes of computing cohomology, [50] Proposition 6.4.5 allows to pass freely between \( \tilde{\mathbb{C}}_\psi \) and \( \mathbb{C}_\psi^{[r/q,r]} \), while Proposition 4.6.10 does the same between \( \mathbb{C}_\psi \) and \( \mathbb{C}_\psi^{[r/q,r]} \). Consequently, by Lemma 4.8.9, the continuous effective \( \Gamma_N \)-cohomology groups of \( M \otimes_{\mathbb{C}_\psi} (\mathbb{C}_\psi/\varphi^{-ad}(\mathbb{C}_\psi)) \) are trivial. On the other hand, it is clear that the \( \varphi^d \)-cohomology groups of \( M \otimes_{\mathbb{C}_\psi} (\mathbb{C}_\psi/\varphi^{-ad}(\mathbb{C}_\psi)/\mathbb{C}_\psi) \) are trivial.

Finally, we note that the above results can also be established with \( \mathbb{C}_\psi, \tilde{\mathbb{C}}_\psi \) replaced by \( \mathbb{A}_\psi^\dagger, \tilde{\mathbb{A}}_\psi^\dagger \). The proofs are somewhat simpler in this case, so we can safely omit most details.

**Proposition 4.8.11.** Let \( d \) be a positive integer and put \( q = p^d \). Let \( M \) be a \((\varphi^d, \Gamma_N)\)-module over \( \mathbb{A}_\psi^\dagger \) whose underlying module is free. Then there exists a basis of \( M \) on which \( \varphi, \gamma_0, \ldots, \gamma_n \) act via matrices over \( \mathbb{A}_\psi^\dagger \).

**Proof.** Note that if we trace through the proof of Proposition 4.8.8 applied to \( M \otimes_{\mathbb{A}_\psi^\dagger} \tilde{\mathbb{C}}_\psi \) using a basis of \( M \), all of the coordinate changes produced by Lemma 4.8.6 are defined over \( \mathbb{A}_\psi^\dagger \).

**Lemma 4.8.12.** For \( d \) a positive integer, let \( M \) be a \((\varphi^d, \Gamma_N)\)-module over \( \mathbb{A}_\psi^\dagger \). Then the continuous effective \((\varphi^d, \Gamma_N)\)-cohomology groups of

\[
\begin{align*}
M &\otimes_{\mathbb{A}_\psi^\dagger} (\mathbb{A}_\psi^\dagger/\mathbb{A}_\psi^\dagger), \\
M &\otimes_{\mathbb{A}_\psi^\dagger} (\mathbb{B}_\psi^\dagger/\mathbb{B}_\psi^\dagger), \\
M &\otimes_{\mathbb{A}_\psi^\dagger} (\tilde{\mathbb{C}}_\psi/\mathbb{C}_\psi)
\end{align*}
\]

are all trivial.

**Proof.** As in Lemma 4.8.10.

### 4.9 Local systems and cohomology

We are now ready to relate imperfect period rings to étale local systems and their cohomology.

**Theorem 4.9.1.** For \( d \) a positive integer such that \( \mathbb{Q}_{p^d} \subseteq K \), the categories of \((\varphi^d, \Gamma_N)\)-modules over the rings in the diagram

\[
\begin{align*}
\mathbb{A}_\psi^\dagger &\longrightarrow \bigcup \varphi^{-n}(\mathbb{A}_\psi^\dagger) & &\longrightarrow & &\bigcup \varphi^{-n}(\mathbb{A}_\psi^\dagger) & &\longrightarrow & &\tilde{\mathbb{A}}_\psi^\dagger \\
\downarrow & &\downarrow & &\downarrow & &\downarrow & &\downarrow \\
\mathbb{A}_\psi &\longrightarrow \bigcup \varphi^{-n}(\mathbb{A}_\psi) & &\longrightarrow & &\bigcup \varphi^{-n}(\mathbb{A}_\psi) & &\longrightarrow & &\tilde{\mathbb{A}}_\psi
\end{align*}
\]

are equivalent via the apparent base change functors, and are all equivalent to \( \mathbb{Z}_{p^d}\text{-Loc}(A) \) (and hence to \( \mathbb{Z}_{p^d}\text{-Loc}(\mathcal{M}(A)) \) as in [50, Remark 2.8.2]).

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Proof. The categories of \((\varphi^d, \Gamma_N)\)-modules over \(\tilde{A}^\dagger_\psi\) and \(\tilde{A}_\psi\) are equivalent to each other and to \(\mathbb{Z}_{p^d}\text{-Loc}(A)\) by Theorem 3.5.1. It thus remains to produce enough equivalences of categories to link the rings in the right column to all of the other rings in the diagram. Among the rings in the bottom row of the diagram (which all carry the weak topologies), all of the claimed equivalences follow from Corollary 4.6.4.

We next check that the base change functors on \((\varphi^d, \Gamma_N)\)-modules from all of the rings in the top row of the diagram down to \(\tilde{A}_\psi\) are essentially surjective. To check that a given \((\varphi^d, \Gamma_N)\) is equivalent to \((\varphi, \Gamma_N)\) and are all equivalent to \(\mathbb{Z}_{p^d}\text{-Loc}(A)\). Let \(\tilde{M}\) be the \((\varphi^d, \Gamma_N)\)-module corresponding to \(M \otimes_{A_\psi} \tilde{A}_\psi\). By our assumption about \(T, \tilde{M}\) admits a basis fixed modulo \(p\) by \(\varphi^d\) and \(\Gamma_N\). Let \(H\) be the subgroup of \(\hat{G}\) indicated in Hypothesis 4.8.1. By Proposition 4.8.11 there exists a basis of \(\tilde{M}\) on which \(\varphi^d\) acts via matrices over \(A^\dagger_\psi\). In particular, the \(A^\dagger_\psi\)-span \(M_0\) of this basis is a \(\varphi^d\)-module over \(A^\dagger_\psi\). By Proposition 4.6.14(a), \(M_0\) inherits an action of \(\Gamma_N\); this action is effective and continuous because the same is true of the action on \(M_0 \otimes_{A^\dagger_\psi} \tilde{A}_\psi^\dagger \cong \tilde{M}\). From the isomorphism \(M_0 \otimes_{A^\dagger_\psi} \tilde{A}_\psi \cong M \otimes_{A_\psi} \tilde{A}_\psi\), we obtain an isomorphism \(M_0 \otimes_{A^\dagger_\psi} A_\psi \cong M\) by applying Proposition 4.6.14(a) again. This yields the desired essential surjectivity.

To consider \(\mathbb{Q}_p\text{-local systems}, we must as usual distinguish between the scheme \(\text{Spec}(A)\) and the space \(\mathcal{M}(A)\). We consider the former first.

**Theorem 4.9.2.** For \(d\) a positive integer such that \(\mathbb{Q}_{p^d} \subseteq K\), the categories of globally étale \((\varphi^d, \Gamma_N)\)-modules over the rings in the diagram

\[
\begin{array}{ccc}
C_{\psi} & \longrightarrow & C_{\psi} \\
\mid & & \mid \\
B^\dagger_\psi \cup \varphi^{-n}(B^\dagger_\psi) & \longrightarrow & \hat{B}^\dagger_\psi \\
\mid & & \mid \\
B_\psi \cup \varphi^{-n}(B_\psi) & \longrightarrow & \hat{B}_\psi \\
\end{array}
\]

are equivalent via the apparent base change functors, and are all equivalent to \(\mathbb{Q}_{p^d}\text{-Loc}(A)\). Consequently, for any \(c \in \mathbb{Z}\), the categories of globally \((c, d)\)-pure \((\varphi, \Gamma_N)\)-modules over the rings in the diagram are equivalent via the apparent base change functors, and are all equivalent to the category of étale \((c, d)\)-\(\mathbb{Q}_p\)-local systems on \(\text{Spec}(A)\).
Proof. If we omit $C_{\psi}$ and $\tilde{C}_{\psi}$ from the diagram, then the claim follows from Theorem 4.9.1. We may join $\tilde{C}_{\psi}$ to the rest of the diagram using Theorem 3.5.2. To connect $C_{\psi}$ to the rest of the diagram, note that base extension of globally étale $\varphi^d$-modules from $B^\dagger_{\psi}$ to $C_{\psi}$ is an equivalence by Proposition 4.6.14(b). This gives full faithfulness of base extension of globally étale $(\varphi^d, \Gamma_N)$-modules from $B^\dagger_{\psi}$ to $C_{\psi}$. Also, any globally étale $(\varphi^d, \Gamma_N)$-module over $C_{\psi}$ descends uniquely to a globally étale $\varphi^d$-module $M$ over $B^\dagger_{\psi}$ equipped with an action of $\Gamma_N$ which commutes with $\varphi^d$, is effective, and is continuous for the LF topology. We may check continuity for the weak topology by extending scalars to $\tilde{C}_{\psi}$ and invoking Theorem 3.5.2 again. This completes the proof.

To handle the space $\mathcal{M}(A)$, we must sheafify the previous construction; in so doing, we get to globalize to the case where $\psi$ is strictly étale (so in particular $A$ is strictly affinoid) but need not satisfy Hypothesis 4.1.1.

Definition 4.9.3. Throughout this definition, suppose that the frame $\psi$ is strictly étale but does not necessarily satisfy Hypothesis 4.1.1, and let $d$ be a positive integer. Let $\psi', \psi''$ be frames satisfying Hypothesis 4.1.1 which factor through $\psi$. For $\varphi^d$-modules $M_1, M_2$ over the period rings $\mathfrak{m}_{\psi'}, \mathfrak{m}_{\psi''}$, we define a local morphism $M_1 \otimes_{\varphi_{\psi'}} \varphi_{\psi} \to M_2 \otimes_{\varphi_{\psi''}} \varphi_{\psi}$ of $\varphi^d$-modules to be a collection of morphisms $M_1 \otimes_{\varphi_{\psi'}} \varphi_{\psi_i} \to M_2 \otimes_{\varphi_{\psi''}} \varphi_{\psi_i}$ for some covering family $\psi_1, \ldots, \psi_n$ of strictly rational subframes of $\psi$ satisfying Hypothesis 4.1.1, subject to the condition of compatibility on overlaps. This last condition means that the morphisms defined on $\psi_i$ and $\psi_j$ agree on any rational subframe of both $\psi_i$ and $\psi_j$ which satisfies Hypothesis 4.1.1 (Such covering families exist thanks to Corollary 4.1.5). We consider a local morphism to be the same as the local morphism obtained by replacing the covering family with a refinement thereof. When $\psi$ itself satisfies Hypothesis 4.1.1 and $\mathfrak{m} = C$, local morphisms are just morphisms in the usual sense by Corollary 4.6.8.

A local $\varphi^d$-module over $\mathfrak{m}_{\psi}$ is a glueing datum for $\varphi^d$-modules with respect to a covering family of rational subframes satisfying Hypothesis 4.1.1 (with the glueing maps given by local morphisms as in the previous paragraph). The natural functor from $\varphi^d$-modules to local $\varphi^d$-modules over any given period ring is fully faithful; it is an equivalence over $C_{\psi}$ when $\psi$ satisfies Hypothesis 4.1.1 by Corollary 4.6.8 again. We define a local $(\varphi^d, \Gamma_N)$-module to be a local $\varphi^d$-module equipped with a compatible action of $\Gamma_N$ which is effective and continuous for all available topologies, and pass along local properties of the underlying local $\varphi^d$-module such as being étale or pure.

Theorem 4.9.4. Assume that $\psi$ is strictly étale but does not necessarily satisfy Hypothesis 4.1.1. (It is not even necessary to assume that $\psi$ is eligible.) For $d$ a positive integer such that $\mathbb{Q}_p^d \subseteq K$, the categories of étale local $(\varphi^d, \Gamma_N)$-modules over the rings in (4.9.2.1) are equivalent via the apparent base change functors, and are all equivalent to $\mathbb{Q}_p^d \text{-Loc}(\mathcal{M}(A))$. Consequently, for any $c \in \mathbb{Z}$, the categories of $(c, d)$-pure local $(\varphi, \Gamma_N)$-modules over the rings in the diagram are equivalent via the apparent base change functors, and are all equivalent to the category of étale $(c, d)$-$\mathbb{Q}_p$-local systems on $\mathcal{M}(A)$.

Proof. This follows at once from Theorem 4.9.2. □
We next turn to étale cohomology.

**Theorem 4.9.5.** For \( d \) a positive integer such that \( \mathbb{Q}_{p^d} \subseteq K \), let \( T \) be an étale \( \mathbb{Z}_{p^d} \)-local system on \( \text{Spec}(A) \). Let \( M \) be the \((\varphi,d,\Gamma_N)\)-module over one of \( \tilde{A}_{\psi}, \tilde{A}_{\psi}^\dagger, A_{\psi}, A_{\psi}^\dagger \) corresponding to \( T \) via Theorem 4.9.1. Then for \( i \geq 0 \), there is a natural (in \( T \) and \( A \)) bijection \( H^i_{\text{ét}}(\text{Spec}(A), T) \cong H^i_{\varphi^d,\Gamma}(M) \).

**Proof.** This follows from Theorem 3.5.4 using Lemma 4.8.12.

**Theorem 4.9.6.** For \( d \) a positive integer such that \( \mathbb{Q}_{p^d} \subseteq K \), let \( E \) be an étale \( \mathbb{Q}_{p^d} \)-local system on \( \text{Spec}(A) \). Let \( M \) be the globally étale \((\varphi,d,\Gamma_N)\)-module over one of \( \tilde{B}_{\psi}, \tilde{B}_{\psi}^\dagger, B_{\psi}, B_{\psi}^\dagger, C_{\psi}, \tilde{C}_{\psi} \) corresponding to \( E \) via Theorem 4.9.2. Then for \( i \geq 0 \), there is a natural (in \( E \) and \( A \)) bijection \( H^i_{\text{ét}}(\text{Spec}(A), E) \cong H^i_{\varphi^d,\Gamma}(M) \).

**Proof.** This follows from Theorem 3.5.5 plus Lemma 4.8.12.

**Theorem 4.9.7.** For \( d \) a positive integer such that \( \mathbb{Q}_{p^d} \subseteq K \), let \( E \) be an étale \( \mathbb{Q}_{p^d} \)-local system on \( \mathcal{M}(A) \). Let \( M \) be the étale \((\varphi,d,\Gamma_N)\)-module over one of \( C_{\psi}, \tilde{C}_{\psi} \) corresponding to \( E \) via Theorem 3.5.3. Then for \( i \geq 0 \), there is a natural (in \( E \) and \( A \)) bijection \( H^i_{\text{ét}}(\mathcal{M}(A), E) \cong H^i_{\varphi^d,\Gamma}(M) \).

**Proof.** This follows from Theorem 3.5.6 plus Theorem 4.9.6.

### 4.10 Some descent results

We have a descent result for \((\varphi,\Gamma_N)\)-modules from perfect to imperfect period rings, which allows us to derive some facts about \((\varphi,\Gamma_N)\)-modules over imperfect period rings by reduction to the perfect case.

**Theorem 4.10.1.** Let \( d \) be a positive integer.

(a) For \( M \) a \((\varphi,d,\Gamma_N)\)-module over \( C_{\psi} \) and \( i \geq 0 \), the natural maps \( H^i_{\varphi^d,\Gamma}(M) \rightarrow H^i_{\varphi^d,\Gamma}(M \otimes_{C_{\psi}} \tilde{C}_{\psi}) \) are bijections.

(b) Base extension of \((\varphi,d,\Gamma_N)\)-modules from \( C_{\psi} \) to \( \tilde{C}_{\psi} \) is an equivalence of categories.

**Proof.** Part (a) follows by Lemma 4.8.10. For part (b), full faithfulness follows from the case \( i = 0 \) of (a). Thanks to full faithfulness and the glueing property for \( \varphi \)-modules over \( C_{\psi} \) (Corollary 4.6.8), we may deduce essential surjectivity locally, so we may fix \( \gamma \in \mathcal{M}(A) \) and prove the claim after replacing \( \psi \) with a strictly rational subframe encircling \( \gamma \). Using the fact that \( \tilde{R}_L^{[s,r]} \) is a principal ideal domain for any analytic field \( L \) [13, Proposition 2.6.8], we may reduce to the case of a \((\varphi,d,\Gamma_N)\)-module over \( \tilde{C}_{\psi} \) represented by a finite free module over \( \tilde{C}_{\psi}^{[r/p^d,r]} \) for some \( r > 0 \) equipped with appropriate actions of \( \varphi \) and \( \Gamma_N \). We may then invoke Proposition 4.8.8 to descend to a \((\varphi,d,\Gamma_N)\)-module over \( C_{\psi} \).
Corollary 4.10.2. For a positive integer, let $M$ be a $(\varphi^d, \Gamma_N)$-module over $C_\psi$. Then $M$ is (globally) pure/étale if and only if $\tilde{M} = M \otimes_{\psi} \tilde{\psi}$ is (globally) pure/étale.

Proof. Suppose first that $\ast = C$. If $\tilde{M}$ is globally pure, then by Theorem 4.9.2 it descends uniquely to a globally pure $(\varphi^d, \Gamma_N)$-module $N$ over $C_\psi$. By Theorem 4.10.1(a), we obtain an isomorphism $M \cong N$, so $M$ is globally pure. The other cases are similar.

Corollary 4.10.3. For a positive integer, let $M$ be a $(\varphi^d, \Gamma_N)$-module over $C_\psi$ such that the slope polygon of $\tilde{M} = M \otimes_{\psi} \tilde{\psi}$ is constant over $M(A_\psi)$. Then there exists a filtration $0 = M_0 \subset \cdots \subset M_l = M$ of $M$ by $(\varphi^d, \Gamma_N)$-submodules such that $M_i/M_{i-1}$ are vector bundles (and hence $(\varphi^d, \Gamma_N)$-modules) which are pure of constant slope, and $\mu(M_l/M_0) > \cdots > \mu(M_l/M_{l-1})$. In other words, the filtration of $\tilde{M}$ given in [50, Theorem 7.4.8] descends to $M$.

Proof. Apply [50, Theorem 7.4.8] to $\tilde{M}$, then apply Theorem 4.10.1(b) and Corollary 4.10.2 to descend each step of the filtration.

We have the following analogue of [50, Remark 7.3.5].

Remark 4.10.4. Consider the following conditions on a $(\varphi^d, \Gamma_N)$-module $M$.

(a) The $(\varphi^d, \Gamma_N)$-module $M$ is globally étale (i.e., admits a locally free étale model).

(b) The $(\varphi^d, \Gamma_N)$-module $M$ admits an étale model.

(c) The $(\varphi^d, \Gamma_N)$-module $M$ is étale (i.e., admits locally free local étale models).

(d) The $(\varphi^d, \Gamma_N)$-module $M$ admits local étale models.

In all cases, (a) implies (b) and (c), which in turn each imply (d).

Over $B_\psi$ or $B_\pi$, (d) implies (b). If $A$ is normal, then (a) and (b) are equivalent as in [50, Proposition 8.1.13] (i.e., using Theorem 4.9.3), which implies that all four conditions are equivalent.

Over $C_\psi$, (c) and (d) are equivalent by Corollary 4.10.2 plus [50, Theorem 7.3.7]. If $A$ is normal, then again (a) and (b) are equivalent. However, the two pairs of conditions are not equivalent to each other, as in [50, Example 8.1.14].

As a sample application of the combination of perfect and imperfect methods, we offer a result in the spirit of the Ax-Sen-Tate theorem; this globalizes a theorem of Brinon [17] which applies to sufficiently small affinoid spaces. A similar globalization appears in [71]. (A similar argument should apply to uniformly rigid spaces once one sets up the notion of relative $(\varphi, \Gamma)$-modules for them.)

Theorem 4.10.5. Let $X$ be any Berkovich strictly analytic space or rigid analytic space. Define a relative $(\varphi, \Gamma)$-module $M$ on $X$ by assigning to each pair $(j, \psi)$ in the framed affinoid site the object $C_\psi$ with its usual action of $\varphi$ and $\Gamma$. Then there is a natural isomorphism $H^0(X, \mathcal{O}_X) \to H^0_{\varphi, \Gamma}(M/tM)$. 

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Proof. The existence, naturality, and injectivity of the map comes from the natural identification \( H^0_\psi(\hat{C}_\psi) \cong A_{\psi,\infty} \). To prove surjectivity, using resolution of singularities (Remark 3.8.6), we may reduce to the case of \( X \) smooth; we may also work locally. We may thus assume that \( X = M(A) \) is strictly affine over \( K \) and that there exists a boundary-free frame \( \psi : M(A) \to K(\sigma) \) of strictly rational type. By Theorem 3.8.4 (plus the snake lemma), we have \( H^0_{\psi,\Gamma}(M/tM) = H^0_{\psi,\Gamma}(C_\psi/tC_\psi) \). By Theorem 4.10.1(a), we also have \( H^0_{\psi,\Gamma}(M/tM) = H^0_{\psi,\Gamma}(C_\psi/tC_\psi) \). However, \( H^0_{\psi}(C_\psi/tC_\psi) = \bigcup_n A_{\psi,n}, \) so \( H^0_{\psi,\Gamma}(M/tM) = (\bigcup_n A_{\psi,n})^\Gamma_N = A \) as desired.

Corollary 4.10.6. Let \( A \) be a strictly affinoid algebra over \( K \), and let \( \psi : M(A) \to K(\sigma) \) be any toric frame (not necessarily eligible). Then \( A_{\psi,\infty}^{\Gamma_N} = A \).

Proof. If \( \psi \) is eligible, this follows from Theorem 4.10.5 and Theorem 3.8.4. We may reduce the general case to this one using Corollary 2.4.14.

4.11 Prior art

The results described here include a number of prior results, some of whose proofs may not immediately resemble the ones given here. We include some discussion of these results here.

Remark 4.11.1. In Theorem 4.9.4 if we take the frame \( \psi : M(K) \to M(K) \) with \( \sigma \) the zero cone in the zero-dimensional vector space over \( \mathbb{R} \), then the rings \( B_\psi, B^\dagger_\psi, C_\psi \) may be identified with \( E_K, R^bd_K, R_K \).

Next, let \( L \) be a finite extension of \( K \) and consider the frame \( \psi : M(L) \to M(K) \) with \( \sigma \) as in the previous case. The rings \( B^\dagger_{\psi',}(B^\dagger_\psi)^H, C^H_{\psi'} \) each have finitely many connected components, acted upon transitively by \( \Gamma = \Gamma_N = \mathbb{Z}_p^\times \). Choose connected components \( B_L, B^\dagger_L, C_L \), respectively; each of them is stabilized by the subgroup \( \Gamma_L \) of \( \Gamma \) corresponding to \( \text{Gal}(L(\mu_{p^\infty})/L) \) via the cyclotomic character. The categories of étale \((\varphi, \Gamma)-\text{modules over } B_{\psi,L}, (B^\dagger_{\psi,L})^H, C^H_{\psi,L} \) may then be canonically identified with the categories of étale \((\varphi, \Gamma_L)-\text{modules over } B_L, B^\dagger_L, C_L \). (The ring \( C_L \) is often denoted \( B^\dagger_{\text{rig},L}, \) e.g., in the work of Berger [2][10].)

With these observations, the equivalence between \( Z_{\varphi,L}\text{-Loc}(A) \) and \((\varphi, \Gamma_N)-\text{modules over } A_\psi \) in Theorem 4.9.1 becomes Fontaine’s original theorem on \((\varphi, \Gamma)-\text{modules [28 Théorème 3.4.3]}.\) (Fontaine uses a slightly different method to pass from \( \mathcal{F}\text{Et}(A_{\psi,\infty}) \) to \( \mathcal{F}\text{Et}(A_\psi) \), based on the field of norms construction [29][77]. This agrees with our construction thanks to [29][§3.4, Proposition].)

Remark 4.11.2. Generalizing the work of Fontaine, the equivalence between \( Z_{\varphi,L}\text{-Loc}(A) \) and \((\varphi, \Gamma_N)-\text{modules over } A_\psi \) in Theorem 4.9.1 was established in some additional cases by Andreatta [4][Theorem 7.11]; the cases treated are those enumerated in Proposition 4.1.13. Similar results can be found in the work of Scholl [69].

The limitation of the applicability of Andreatta’s method arises from a corresponding limitation in Faltings’s almost purity theorem [25][26]. Faltings’s approach to the almost purity theorem is based on formal schemes, which makes it necessary to impose hypotheses.
in order to have reasonable choices of formal models of the affinoid spaces in question. However, the generalization of the field of norms correspondence introduced in [50, §3.6] (and independently by Scholze [70, 71]) eliminates any such reliance on integral models.

We suspect that one can use the generalization of Faltings’s theorem given by Gabber and Ramero [31, 32] to obtain results of the same strength, but we did not check this carefully.

**Remark 4.11.3.** Keep notation as in Remark 4.11.1. In Theorem 4.9.1, the equivalence between \((\varphi, \Gamma_N)\)-modules over \(A^\dagger_\psi\) and \(A_\psi\) was originally established by Cherbonnier and Colmez [20, Corollaire III.5.2], using the Sen-Tate method of decomposition in continuous Galois cohomology. A more streamlined argument may be inferred from the generalization given by Berger and Colmez [12, Théorème 4.2.9]. Our method is similar in spirit but somewhat different in technical details, and leads to a simpler argument overall; see [49] for a presentation of the resulting proof of the Cherbonnier-Colmez theorem.

**Remark 4.11.4.** In those cases considered in Remark 4.11.2, the equivalence between \((\varphi, \Gamma_N)\)-modules over \(A^\dagger_\psi\) and \(A_\psi\) in Theorem 4.9.1 was established by Andreatta and Brinon [5, Théorème 4.35]. As in Remark 4.11.3, the method is adapted from the Sen-Tate method, which we have short-circuited here.

**Remark 4.11.5.** Keep notation as in Remark 4.11.1. In Theorem 4.9.4, the equivalence between étale \((\varphi, \Gamma_N)\)-modules over \(B^\dagger_\psi\) and \(C_\psi\) was originally observed by Berger [10, Proposition IV.2.2] but without discussion of the mismatch of topologies.

**Remark 4.11.6.** Keep notation as in Remark 4.11.1. In this case, the case of Theorem 4.9.5 over \(A_\psi\) was originally established by Herr [36]. The case over \(A^\dagger_\psi\), as well as Theorem 4.9.6, are due to the second author [59]. It would be interesting to use \((\varphi, \Gamma_N)\)-modules to study cohomology of étale local systems on more general analytic spaces, e.g., to derive finiteness results. Some work exists in this direction based on the construction of Andreatta and Brinon; see for instance the work of Andreatta and Iovita [6].

**Remark 4.11.7.** To conclude, let us point out one more time the primary features of the present work which we feel constitutes a significant departure from the preceding literature: the ability to treat the product of two different toric embeddings of the same space. This makes it possible to make functoriality considerations which are not accessible in the framework of Faltings’s almost purity theorem (in which the work of Andreatta and Brinon takes place).

This feature is shared by the recent work of Scholze [71] using the *pro-étale site* associated to an analytic space. One feature of our development not present in [71] is the crystalline nature of relative \((\varphi, \Gamma)\)-modules, i.e., the fact that one can construct relative \((\varphi, \Gamma)\)-modules on affinoid spaces and compute their cohomology using a single frame (Theorem 3.7.17).
A Appendix

A.1 Boundedness and continuity of $\Gamma$-actions

In case $A$ is an affinoid algebra and $\psi$ is an étale morphism, the continuity of the action of $\Gamma_N$ in the definition of a $(\varphi, \Gamma_N)$-module can be replaced by a formally weaker condition, as follows. (It should be possible to adapt the argument also to the mixed affinoid case, but we did not attempt this.)

**Lemma A.1.1.** Let $G$ be a profinite group each of whose subgroups of finite index is open, and which admits a continuous isometric action on an analytic field $F$. Let $E$ be a finite extension of $F$ to which the action of $G$ extends. Then the extended action is also continuous.

**Proof.** Since the claim is evident for $E$ contained in the direct perfection of $F$, we may reduce to the case where $E/F$ is Galois. Given $x \in E$, let $P(T) = \prod_{z \in S}(T - z) \in F[T]$ be the minimal polynomial of $x$ over $F$ and put $n = \deg(P)$. For $\gamma \in G$, we then have

$$0 = \gamma(P(x)) = \gamma(P)(\gamma(x)) = (\gamma(P) - P)(\gamma(x)) + \prod_{z \in S}((\gamma(x) - z)).$$

Choose $\epsilon > 0$ so that $|y - z| > \epsilon$ whenever $y$ and $z$ are distinct elements of $S$. We can then find an open subgroup $G_1$ of $G$ such that for all $\gamma \in G_1$ and all $y \in S$, $|\gamma(P) - P|(\gamma(y))| < \epsilon^n$. This means that $\prod_{z \in S}|\gamma(y) - z| < \epsilon^n$, so there exists $z = z(\gamma, y) \in S$ such that $|\gamma(y) - z| < \epsilon$; moreover, $z$ is uniquely determined by $\gamma$ and $y$. For $\gamma' \in G_1$, we have both $|\gamma'(y) - z(\gamma', y)| < \epsilon$ and $|\gamma'(y) - \gamma'(z(\gamma', y))| < \epsilon$, forcing $z(\gamma', y) = z(\gamma', z(\gamma, y))$. That is, the map $z$ defines a group action of $G_1$ on $S$. The stabilizer of $x$ under this group action is a subgroup $G_2$ of $G_1$ of finite index.

Since $G_2$ has finite index in $G$, it is open by hypothesis. For $\gamma \in G_2$, we have $|\gamma(x) - x| < \epsilon$ and so $|x - \gamma(w)| < \epsilon$ whenever $|x - w| < \epsilon$. This proves that the action on $E$ is continuous. □

**Remark A.1.2.** We will only apply Lemma A.1.1 in cases where it is obvious that the subgroups of the profinite group $G$ of finite index are all open. Nonetheless, it is worth pointing out that this is true whenever $G$ is topologically finitely generated, by a theorem of Nikolov and Segal [61, 62].

**Lemma A.1.3.** Let $G$ be a profinite group each of whose subgroups of finite index is open. Then for any positive integers $n$ and $d$, any homomorphism $\tau : G \to GL_n(\mathbb{Q}_p^d)$ with bounded image is continuous.

**Proof.** The closure $H$ of the image of $\tau$ is a subgroup of the locally compact group $GL_n(\mathbb{Q}_p^d)$ which is closed and bounded, and hence compact. For each open subgroup $J$ of $H$, $\tau^{-1}(J)$ is a subgroup of $G$ of finite index, which is thus open. Hence $\tau$ is continuous. □

**Remark A.1.4.** Note that the boundedness condition in Lemma A.1.3 is needed to avoid pathologies such as the following. Choose a basis of $\mathbb{Q}_p$ as a $\mathbb{Q}$-vector space, then use...
this basis to define a $\mathbb{Q}$-linear projection $\tau : \mathbb{Q}_p \to \mathbb{Q}$. We then obtain a homomorphism $\mathbb{Z}_p \to \text{GL}_2(\mathbb{Q}_p)$ which is not continuous by taking

$$t \mapsto \begin{pmatrix} 1 & \tau(t) \\ 0 & 1 \end{pmatrix}.$$ 

One obtains similar pathologies if one omits the boundedness condition in Proposition A.1.7.

**Definition A.1.5.** For $M$ an abelian group equipped with a norm $|\cdot|$, an action of a group $G$ on $M$ is **bounded** if there exists $c > 0$ such that $|g(x)| \leq c|x|$ for all $g \in G$, $x \in M$. This condition evidently depends only on the equivalence class of the norm on $M$.

For $R$ a perfect uniform Banach $\mathbb{Q}_p$-algebra with norm $\alpha$ and $M$ a $\varphi^d$-module over $\hat{R}_R$, we say that the action of $G$ on $M$ is **bounded** if for each $r, s$ with $0 < s \leq r/p^d$, for some (and hence any) norm on the model $M_{[s,r]}$ of $M$ over $\hat{R}_R^{[s,r]}$ corresponding to the norm $\max\{\lambda(\alpha^s), \lambda(\alpha^r)\}$ on $\hat{R}_R^{[s,r]}$ as in [50] Lemma 2.2.12, the action of $G$ on $M_{[s,r]}$ is bounded as in the previous paragraph.

**Lemma A.1.6.** Let $L$ be a perfect nontrivially normed analytic field of characteristic $p$. Let $G$ be a profinite group each of whose subgroups of finite index is open, and suppose $G$ acts continuously on $L$. For $d$ a positive integer, let $M$ be a $\varphi^d$-module over $\hat{R}_L$ admitting a semilinear action of $G$ commuting with $\varphi$. Then the action of $G$ on $M$ is continuous if and only if it is bounded.

**Proof.** Since $G$ is compact, continuity evidently implies boundedness, so we need only check the converse. Suppose first that $M$ is trivial. In this case, on any basis fixed by $\varphi^d$, $G$ acts via matrices over $\hat{R}_L^{\varphi^d}$, which equals $\mathbb{Q}_p$ by [50] Corollary 5.2.4. The resulting homomorphism $G \to \text{GL}_n(\mathbb{Q}_p)$ for $n = \text{rank}(M)$ is continuous by Lemma A.1.3 so $G$ acts continuously on $M$ as desired.

Suppose next that $M$ is pure. There is no harm in enlarging $d$ so that $d\mu(M) \in \mathbb{Z}$; then $M$ corresponds via [50] Theorem 8.1.6 to an étale $\mathbb{Q}_p$-local system over $\text{Spec}(L)$. It follows that we can exhibit a perfect analytic field $L'$ which is the completion of a possibly infinite Galois extension of $L$, such that $G$ extends to $L'$ (necessarily continuously by Lemma A.1.1) and $M \otimes_{\hat{R}_L} \hat{R}_{L'}$ admits a basis over $\hat{R}_{L'}$ on which $\varphi^d$ acts by multiplication by $p^{c_i}$ for some $c_i \in \mathbb{Z}$. By passing from $L$ to $L'$, we may thus reduce to the previous case.

To handle the general case, let $0 = M_0 \subset \cdots \subset M_t = M$ be the slope filtration of $M$ provided by [50] Theorem 4.2.12. Since this filtration is unique, it admits an action of $G$. By the previous paragraph, the induced action of $G$ on each $M_i/M_{i-1}$ is continuous. It follows easily that the action on $M$ is also continuous. 

**Proposition A.1.7.** Suppose that $A$ is an affinoid algebra and that $\psi$ is étale. For $d$ a positive integer, let $M$ be a $\varphi^d$-module over $\mathbb{C}_\psi$ equipped with a semilinear action of $\Gamma_N$ commuting with $\varphi^d$. Then this action is continuous if and only if it is bounded. (In particular, if the action is bounded and effective, then $M$ is a $(\varphi^d, \Gamma_N)$-module.)
Proof. The claim is local on \(\mathcal{M}(A)\), so we may assume \(\psi\) is of rational type. In this case, as in the proof of Theorem 4.1.3(a), \(\overline{A}_\psi\) is the completed perfection of an affinoid algebra over \(k((\pi))\). Let \(\Delta\) be the Shilov boundary of \(\overline{A}_\psi\), the unique minimal finite subset of \(\mathcal{M}(\overline{A}_\psi)\) whose supremum computes the spectral norm on \(\overline{A}_\psi\) (see [13, Corollary 2.4.5]). Since \(\Delta\) is finite, its elements are all fixed by some subgroup \(G\) of \(\Gamma_N\) of finite index. By making \(G\) a bit smaller, we can ensure that the points of \(\mathcal{M}(R)\) above points of \(\Delta\) are all fixed by \(G\) also. Since every subgroup of \(\Gamma_N\) of finite index is open, by Lemma A.1.6, the action of \(G\) on \(M \otimes_{\overline{C}_\psi} \hat{\mathcal{R}}_{H(\delta)}\) is continuous for each \(\delta \in \Delta\). From the definition of the Shilov boundary, it follows that \(G\) acts continuously on \(M\), as then does \(\Gamma_N\).

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