SIGN-CHANGING SOLUTIONS FOR A PARAMETER-DEPENDENT QUASILINEAR EQUATION

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Abstract. We consider quasilinear elliptic equations, including the following Modified Nonlinear Schrödinger Equation as a special example:

\[
\begin{cases}
\Delta u + \frac{1}{2} u \Delta u^2 + \lambda |u|^{r-2}u = 0, & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \subset \mathbb{R}^N (N \geq 3) \) is a bounded domain with smooth boundary, \( \lambda > 0, r \in (2, 4) \). We prove as \( \lambda \) becomes large the existence of more and more sign-changing solutions of both positive and negative energies.

1. Introduction. In this paper, we consider the following quasilinear elliptic equation

\[
\begin{cases}
\sum_{i,j=1}^{N} D_j(b_{ij}(x,u)D_i u) - \frac{1}{2} \sum_{i,j=1}^{N} D_z b_{ij}(x,u)D_i uD_j u + \lambda f(x,u) = 0, & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \subset \mathbb{R}^N (N \geq 3) \) is a bounded domain with smooth boundary, and we use the notations \( D_i u = \frac{\partial u}{\partial x_i} \), \( D_z b_{ij}(x,z) = \frac{\partial}{\partial z} b_{ij}(x,z) \).

A function \( u \in H^1_0(\Omega) \cap L^\infty(\Omega) \) is called a weak solution of (1.1) if the following equation holds for all \( \varphi \in C_0^\infty(\Omega) \)

\[
\int_{\Omega} \sum_{i,j=1}^{N} b_{ij}(x,u)D_i uD_j \varphi dx + \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{N} D_z b_{ij}(x,u)D_i uD_j u \varphi dx = \lambda \int_{\Omega} f(x,u)\varphi dx.
\]

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Formally the problem has a variational structure, given by the functional
\[
I(u) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{N} b_{ij}(x,u) D_i u D_j u \, dx - \lambda \int_{\Omega} F(x,u) \, dx, \quad u \in H^1_0(\Omega)
\]
where \(F(x,z) = \int_0^z f(x,\tau) \, d\tau\).

We assume
\begin{align*}
(b_1) & \quad b_{ij} \in C^\alpha(\overline{\Omega} \times \mathbb{R}, \mathbb{R}), \alpha \in (0, 1), D_2 b_{ij} \in C^{1,0}(\overline{\Omega} \times \mathbb{R}, \mathbb{R}), b_{ij} = b_{ji}, i,j = 1, \ldots, N. \text{ There exists a constant } c_1 > 0 \text{ such that for } x \in \overline{\Omega}, z_1, z_2 \in \mathbb{R}, \quad |D_z b_{ij}(x,z_1) - D_z b_{ij}(x,z_2)| \leq c_1 |z_1 - z_2|. \\
(b_2) & \quad \text{There exist constants } c_0, c_1 > 0 \text{ such that for } x \in \overline{\Omega}, z \in \mathbb{R}, \xi = (\xi_i) \in \mathbb{R}^N, \quad c_0 (1 + z^2)|\xi|^2 \leq \sum_{i,j=1}^{N} b_{ij}(x,z) \xi_i \xi_j \leq c_1 (1 + z^2)|\xi|^2. \\
(b_3) & \quad \text{There exist constants } c_0, c_1 > 0 \text{ such that for } x \in \overline{\Omega}, z \in \mathbb{R}, \xi = (\xi_i) \in \mathbb{R}^N, \quad c_0 (1 + z^2)|\xi|^2 \leq \sum_{i,j=1}^{N} (b_{ij}(x,z) + \frac{1}{2} z D_z b_{ij}(x,z)) \xi_i \xi_j \leq c_1 (1 + z^2)|\xi|^2. \\
(b_4) & \quad b_{ij}(x,z) \text{ is even in } z. \\
f_1 & \quad f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R}). \\
f_2 & \quad \lim_{z \to 0} \frac{f(x,z)}{z} = 0 \text{ uniformly in } x \in \overline{\Omega}. \\
f_3 & \quad \text{There exist } c_1 > 0 \text{ and } r \in (2, 4) \text{ such that for } x \in \overline{\Omega}, z \in \mathbb{R}, \quad |f(x,z)| \leq c_1 (1 + |z|^{r-1}). \\
(f_4) & \quad f(x,z) \text{ is odd in } z.
\end{align*}

We have the following multiplicity result of sign-changing solutions.

**Theorem 1.1.** Assume (\(b_1\))-(\(b_4\)) and (\(f_1\))-(\(f_4\)). Then for any positive integer \(k\) there exists \(\Lambda_k > 0\) such that if \(\lambda \geq \Lambda_k\), the problem (1.1) has \(2k\) pairs of solutions \(\pm u_1, \ldots, \pm u_k; \pm v_1, \ldots, \pm v_k\) with the property that \(I(u_1) \leq \cdots \leq I(u_k) < 0 < I(v_1) \leq I(v_2) \leq \cdots \leq I(v_k)\), \(u_1, v_1\) are positive and \(u_2, \ldots, u_k, v_2, \ldots, v_k\) are sign-changing.

An important example of the equation (1.1) is the so-called Modified Nonlinear Schrödinger Equation (MNSE)
\[
\begin{cases}
\Delta u + \frac{1}{r} u \Delta u^2 + \lambda |u|^{r-2} u = 0, & \text{in } \Omega, \\
u = 0, & \text{on } \partial\Omega,
\end{cases}
\]
which corresponds to the case of \(b_{ij}(x,z) = (1 + z^2) \delta_{ij}, f(x,z) = |z|^{r-2} z\) with \(2 < r < 4\). This equation and its extensions have been involved in many models of mathematical physics and received considerable attention during the last years, we refer \([5, 9, 13, 14, 18, 20, 31, 32]\) and the references therein.

When \(4 \leq r < \frac{4N}{N-2}\), existence theory and multiple solutions for (MNSE) and the general equation (1.1) have been well developed in recent years, see \([8, 11, 19, 27, 28, 26, 22, 23]\). Multiple sign-changing solutions of positive energies have been obtained for the quasilinear equation (1.1), c.f.,[19]. There are few results for the
case $2 < r < 4$. The authors of [10] obtained a multiplicity result without much sign information and under more restricted conditions they obtained multiple sign-changing solutions. Our paper here provides a different approach for constructing sign-changing solutions for the case $2 < r < 4$. Next we outline our idea and approach.

Notice that the functional $I$ is not differentiable in $H^1_0(\Omega)$. In [22, 19] the authors of the present paper introduced a $p$-Laplacian perturbation method to deal with the quasilinear equation (1.1) with a superlinear function $f(x, z)$. In [22] they considered the functional

$$L_\mu(u) = \frac{1}{4} \mu \int_\Omega |\nabla u|^4 \, dx + \frac{1}{2} \mu \sum_{i,j=1}^N b_{ij}(x, u)D_i u D_j u \, dx - \lambda \int_\Omega F(x, u) \, dx, \quad u \in W^{1,4}_0(\Omega).$$

In order to obtain sign-changing solutions, in [19] they considered the functional for $u \in W^{1,q}_0(\Omega)$

$$L_\mu(u) = \frac{1}{q} \mu \int_\Omega |\nabla u|^q \, dx + \frac{1}{2} \mu \sum_{i,j=1}^N b_{ij}(x, u)D_i u D_j u \, dx$$

$$+ \frac{1}{q} \mu \int_\Omega |\nabla u|^{q-2} u^2 \, dx - \lambda \int_\Omega F(x, u) \, dx,$$

with $q > 4$. In both cases the critical points of the perturbed functionals $L_\mu$ are used as approximate solutions of the original problem. Assume $u_\mu$ is a critical point of $L_\mu$ and the critical value $L_\mu(u_\mu)$ is bounded uniformly in $\mu \in (0, 1]$. We expect $u_\mu$ to converge to a solution of the original problem by the limit process $\mu \to 0$. If we want to obtain multiple solutions of the original equation, the difficulty is that different solutions of the approximate equations converge to the same solution of the original equation. It is a hard work to distinguish these limit functions. For the superlinear case $4 < r < \frac{4N}{N-2}$, we can distinguish the limit functions by showing that the corresponding critical values tend to infinity. For the case $2 < r < 4$, the situation is quite different.

Following the idea of [29, 34], we expect that the original problem shares some solutions with the approximate problem so that the limit process is not needed. The approach was used recently in [24] to treat a semi-classical setting of the quasilinear equation (1.1) with $4 < r < 22^*$. Obviously the functionals $I$ and the perturbed functional can not share any nontrivial solutions. We need to choose the perturbation terms more carefully. We define the perturbed functional $I_\mu$ by the following for $\mu \in (0, 1]$

$$I_\mu(u) = \frac{1}{2} \sigma \int_\Omega \left( \frac{|\nabla u|}{m_\mu(|\nabla u|)} \right)^{q-2} |\nabla u|^2 \, dx + \frac{1}{2} \sigma \int_\Omega \left( \frac{|\nabla u|}{m_\mu(|\nabla u|)} \right)^{q-4} u^2 |\nabla u|^2 \, dx$$

$$+ \frac{1}{2} \sum_{i,j=1}^N \beta_{ij}(x, u)D_i u D_j u \, dx - \lambda \int_\Omega F(x, u) \, dx$$

for $u \in W^{1,q}_0(\Omega)$, where $q > 4$, $\beta_{ij}(x, z) = b_{ij}(x, z) - \sigma(1 + z^2)\delta_{ij}$, $\sigma$ is a small positive number so that $\beta_{ij}, i, j = 1, \cdots, N$ satisfy the assumptions (b1)-(b4) as $b_{ij}$ do (with a smaller positive constant $c_0$). The auxiliary function $m_\mu$ (see Section 2 for the definition of $m_\mu$) has the property that

$$m_\mu(t) = t, \quad \text{for } |t| \leq \frac{1}{\mu},$$

(1.5)
This property allows the approximate problem and the original problem to share solutions. In fact we have the following theorem.

**Theorem 1.2.** Assume (b₁)-(b₄) and (f₁)-(f₄). Then

1. For any positive integer \( k \) there exists \( \Lambda_k > 0 \), independent of \( \mu \) such that for \( \lambda \geq \Lambda_k \) the functional \( I_\mu \) has \( 2k \) pairs of critical points \( \pm u_1(\mu), \pm u_2(\mu), \pm v_1(\mu), \cdots, \pm u_k(\mu), \pm v_k(\mu) \) with the property that \( I_\mu(u_1(\mu)) \leq \cdots \leq I_\mu(u_k(\mu)) \leq I_\mu(v_1(\mu)) \leq \cdots \leq I_\mu(v_k(\mu)) \), \( u_1(\mu) \) and \( v_1(\mu) \) are positive, and \( u_j(\mu), v_j(\mu), j = 2, \cdots, k \) are sign-changing.

2. There exist \( \beta \in (0, 1) \), \( M > 0 \), independent of \( \mu \), such that

\[
\|u_j(\mu)\|_{C^1,\lambda(T)} \leq M, \quad \|v_j(\mu)\|_{C^1,\lambda(T)} \leq M, \quad j = 1, \cdots, k.
\]

Consequently for \( \mu \leq \frac{1}{2\beta}M \), \( u_j(\mu), v_j(\mu), j = 1, \cdots, k \) are solutions of the problem (1.1).

Obviously Theorem 1.1 follows from Theorem 1.2.

In this paper, we use \( c, c_0, c_1, \cdots \) to denote variant constants, and \( c(\mu) \), if necessary, to denote constants depending on \( \mu \), \( \| \cdot \| \) to denote the norm \( \|u\| = (\int_\Omega |\nabla u|^q \, dx)^{\frac{1}{q}} \). The paper is organized as follows. In Section 2 we prove the deformation lemma for the functional \( I_\mu \). In Section 3 we construct critical values of \( I_\mu \) by the method of invariant sets with respect to the descending flow. In Section 4 we prove the uniform bound for the gradient of the approximate sign-changing solutions obtained in Section 3 and complete the proof of Theorem 1.2 and Theorem 1.1.

2. **The deformation lemma.** We define the auxiliary function \( m_\mu \). Let \( b \in C_0^\infty([0, \infty), [0, 1]) \) such that \( b(t) = 1 \) for \( 0 \leq t \leq 1 \); \( b(t) = 0 \) for \( t \geq 2 \) and \( b'(t) \leq 0 \). Let \( m(t) = \int_0^t b(\tau) \, d\tau \). For \( \mu \in (0, 1) \) define

\[
b_\mu(t) = b(\mu t), \quad m_\mu(t) = \int_0^t b_\mu(\tau) \, d\tau = \frac{1}{\mu} m(\mu t), \quad t \in [0, \infty).
\]

We have

\[
0 \leq \frac{tb_\mu(t)}{m_\mu(t)} \leq 1, \quad 0 \leq \frac{-t^2b_\mu'(t)}{m_\mu(t)} \leq c, \quad m_\mu(t) \leq \min\{t, \frac{c}{\mu}\}. \tag{2.1}
\]

**Lemma 2.1.** The functional \( I_\mu \) is of \( C^1 \)-class, coercive and bounded from below on \( W_0^{1,q}(\Omega) \).

**Proof.** By the assumptions (b₂) and (f₃), we have

\[
I_\mu(u) = \frac{1}{2} \int_\Omega \left( \frac{|\nabla u|}{m_\mu(|\nabla u|)} \right)^{q-2} |\nabla u|^2 \, dx + \frac{1}{2} \int_\Omega \left( \frac{|\nabla u|}{m_\mu(|\nabla u|)} \right)^{q-4} u^2 |\nabla u|^2 \, dx
\]

\[
+ \frac{1}{2} \int_\Omega \sum_{i,j=1}^N \beta_{ij}(x,u)D_iuD_ju \, dx - \lambda \int_\Omega F(x,u) \, dx
\]

\[
\geq c_\mu^{q-2} \int_\Omega |\nabla u|^q \, dx + c \int_\Omega u^2 |\nabla u|^2 \, dx - c \left( 1 + \int_\Omega |u|^r \, dx \right)
\]

\[
\geq c_\mu^{q-2} \int_\Omega |\nabla u|^q \, dx + c \int_\Omega u^2 |\nabla u|^2 \, dx - c,
\]

where \( c > 0 \) is independent of \( \mu \).
solution of the following equation

Obviously Lemma 2.2.

to the class of quasilinear elliptic problems.

some early work of this approach, and [19, 21] for more development more relevant

presence of invariant sets with respect to the descent flow, see [4, 2, 3, 16, 25] for

Definition 2.1.

Given

since

r < 4 < \frac{4N}{N-2}, I_\mu is bounded from below uniformly in \mu. We have

\langle DI_\mu(u), \varphi \rangle = \sigma \int_\Omega \left( \frac{\nabla u}{m_\mu(\nabla u)} \right)^{q-2} \left( 1 + \frac{q - 2}{2} \left( 1 - \frac{m_\mu(\nabla u)}{m_\mu(\nabla u)} \right) \right) \nabla u \nabla \varphi \, dx

+ \sigma \int_\Omega \left( \frac{\nabla u}{m_\mu(\nabla u)} \right)^{q-4} u^2 \left( 1 + \frac{q - 4}{2} \left( 1 - \frac{m_\mu(\nabla u)}{m_\mu(\nabla u)} \right) \right) \nabla u \nabla \varphi \, dx

+ \sigma \int_\Omega \left( \frac{\nabla u}{m_\mu(\nabla u)} \right)^{q-4} |\nabla u|^2 \varphi \, dx

(2.2)

\begin{align*}
&+ \int_\Omega \sum_{i,j=1}^N \beta_{ij}(x,u) D_i u D_j \varphi \, dx + \frac{1}{2} \int_\Omega \sum_{i,j=1}^N D_i \beta_{ij}(x,u) D_i u D_j u \varphi \, dx \\
&- \lambda \int_\Omega f(x,u) \varphi \, dx .
\end{align*}

\Box

In order to obtain sign-changing solutions we shall use minimax method in the

presence of invariant sets with respect to the descent flow, see [4, 2, 3, 16, 25] for

some early work of this approach, and [19, 21] for more development more relevant
to the class of quasilinear elliptic problems.

First we construct an operator \( A : W_0^{1,q}(\Omega) \to W_0^{1,q}(\Omega) \). The vector field \( u - Au \)

behaves like a pseudo gradient vector field of the functional \( I_\mu \). We define a family

of convex functions for \( \mu \in (0, 1]\)

\begin{align*}
J_\mu(u) &= \frac{1}{2} \sigma \int_\Omega \left( \frac{\nabla u}{m_\mu(\nabla u)} \right)^{q-2} |\nabla u|^2 \, dx + \frac{1}{2} \sigma \int_\Omega \left( \frac{\nabla u}{m_\mu(\nabla u)} \right)^{q-4} u^2 |\nabla u|^2 \, dx \\
&+ \frac{1}{2} \int_\Omega \sum_{i,j=1}^N \beta_{ij}(x,u) D_i u D_j u \, dx + \frac{1}{q} c_1 \int_\Omega |u|^q \, dx + \frac{1}{2} c_2 \int_\Omega u^2 \, dx , \quad u \in W_0^{1,q}(\Omega)
\end{align*}

(2.3)

where \( c_1, c_2 \) are positive constants to be chosen. Also we define

\begin{align*}
R(u) &= \lambda \int_\Omega F(x,u) \, dx + \frac{1}{q} c_1 \int_\Omega |u|^q \, dx + \frac{1}{2} c_2 \int_\Omega u^2 \, dx .
\end{align*}

(4.4)

Obviously

\begin{align*}
I_\mu(u) &= J_\mu(u) - R(u).
\end{align*}

Definition 2.1. Given \( u \in W_0^{1,q}(\Omega) \), define \( v = Au \in W_0^{1,q}(\Omega) \) as the unique solution of the following equation

\begin{align*}
\langle DJ_\mu(v), \varphi \rangle = \langle DR(u), \varphi \rangle \quad \text{for all } \varphi \in W_0^{1,q}(\Omega).
\end{align*}

(2.5)

The definition is well-posed, as shown in Proposition 2.1.

Lemma 2.2. \( DJ_\mu \) is locally Lipschitz continuous, there exists a constant \( c_0 = c_0(\mu) \)
such that

\begin{align*}
\| DJ_\mu(u) - DJ_\mu(v) \| \leq c_0(1 + \|u\|^{q-2} + \|v\|^{q-2})\|u-v\| , \quad \text{for } u, v \in W_0^{1,q}(\Omega).
\end{align*}

(2.6)
Proof. Denote $w_t = tu + (1-t)v$, $t \in [0, 1]$. By (2.3), for $u, v, \varphi \in W_0^{1,q}(\Omega)$, we have
\[
\langle D\mu(u) - D\mu(v), \varphi \rangle = \int_0^1 \frac{d}{dt} \langle D\mu(w_t), \varphi \rangle dt \tag{2.7}
\]
and
\[
\langle D\mu(w_t), \varphi \rangle = \left(1 + \frac{q - 2}{2} \frac{1}{m_\mu(\nabla w_t)} \right) \nabla w_t \nabla \varphi dx
\]
\[
+ \frac{q - 1}{2} \frac{1}{m_\mu(\nabla w_t)} \nabla w_t \nabla \varphi dx
\]
\[
+ \sum_{i,j=1}^N \beta_{ij}(x, w_t) D_i w_t D_j \varphi dx + \frac{1}{2} \left( \sum_{i,j=1}^N D_i \beta_{ij}(x, w_t) D_i w_t D_j \varphi dx \right)
\]
\[
+ c_1 \int_\Omega |w_t|^q w_t \varphi dx + c_2 \int_\Omega w_t \varphi dx . \tag{2.8}
\]
Also
\[
\frac{d}{dt} \langle D\mu(w_t), \varphi \rangle = \left(1 + \frac{q - 2}{2} \frac{1}{m_\mu(\nabla w_t)} \right) \nabla (u - v) \nabla \varphi dx
\]
\[
+ \frac{q - 1}{2} \frac{1}{m_\mu(\nabla w_t)} \nabla (u - v) \nabla \varphi dx
\]
\[
+ \sum_{i,j=1}^N \beta_{ij}(x, w_t) D_i (u - v) D_j \varphi dx + \frac{1}{2} \left( \sum_{i,j=1}^N D_i \beta_{ij}(x, w_t) D_i (u - v) D_j \varphi dx \right)
\]
\[
+ c_1 \int_\Omega |w_t|^q (u - v) \varphi dx + c_2 \int_\Omega (u - v) \varphi dx . \tag{2.9}
\]
+ \int_{\Omega} \sum_{i,j=1}^{N} D_{ij} \beta_{ij}(x, w_t) D_i w_t D_j (u - v) \varphi \, dx + \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{N} \left( \frac{d}{dt} D_{ij} \beta_{ij}(x, w_t) \right) D_i w_t D_j w_t \varphi \, dx \\
+ c_1 (q - 1) \int_{\Omega} |w_t|^{q - 2} (u - v) \varphi \, dx + c_2 \int_{\Omega} (u - v) \varphi \, dx.

We can check the right hand side of (2.8) term by term and show that all terms can be controlled by \(c(1 + ||w_t||^{q - 2})||u - v|| \cdot ||\varphi||\) and obtain (2.6). As an example we consider two terms and omit the rest. We have

\[
\left| \frac{1}{m_\mu(|\nabla w_t|)} \left( 1 + \frac{q - 2}{2} \left( 1 - \frac{|\nabla w_t| b_\mu(|\nabla w_t|)}{m_\mu(|\nabla w_t|)} \right) \right) \nabla(u - v) \nabla \varphi \, dx \right| \\
\leq c \int_{\Omega} (1 + \mu^{q - 2} |\nabla w_t|^{q - 2}) |\nabla(u - v)||\nabla \varphi| \, dx \\
\leq c(1 + ||w_t||^{q - 2})||u - v|| \cdot ||\varphi||,
\]

and

\[
\left| \int_{\Omega} \sum_{i,j=1}^{N} \left( \frac{d}{dt} D_{ij}(t, w_t) \right) D_i w_t D_j w_t \varphi \, dx \right| \\
\leq c \int_{\Omega} |u - v||\nabla w_t|^2 |\varphi| \, dx \\
\leq c \int_{\Omega} (1 + |\nabla w_t|^{q - 2}) |u - v||\varphi| \, dx \\
\leq c(1 + ||w_t||^{q - 2})||u - v|| \cdot ||\varphi||.
\]

In the above we have used the fact \(D_t b_{ij}\), as well \(D_z \beta_{ij}\), satisfies the Lipschitz condition \((b_1)\), that is,

\[|D_z \beta_{ij}(x, z_1) - D_z \beta_{ij}(x, z_2)| \leq c|z_1 - z_2| \quad \text{for} \quad z_1, z_2 \in \mathbb{R}, x \in \overline{\Omega}\]

hence

\[
\left| \frac{d}{dt} D_z \beta_{ij}(x, w_t) \right| \leq c|u - v|.
\]

Now

\[
\left| \frac{d}{dt} \langle DJ_\mu(w_t), \varphi \rangle \right| \leq c(1 + ||w_t||^{q - 2})||u - v|| \cdot ||\varphi|| \quad (2.10)
\]

and

\[
|\langle DJ_\mu(u) - DJ_\mu(v), \varphi \rangle| = \left| \int_0^1 \frac{d}{dt} \langle DJ_\mu(w_t), \varphi \rangle \, dt \right| \\
\leq c \int_0^1 (1 + ||w_t||^{q - 2}) ||u - v|| \cdot ||\varphi|| \, dt \leq c(1 + ||u||^{q - 2} + ||v||^{q - 2}) ||u - v|| \cdot ||\varphi||.
\]

\[
\square
\]

**Lemma 2.3.** Given the proper constants \(c_1, c_2\), the operator \(DJ_\mu : X = W_0^{1,q}(\Omega) \rightarrow X^*\) is strongly monotone, that is, there exists \(c = c(\mu) > 0\) such that

\[
\langle DJ_\mu(u) - DJ_\mu(v), u - v \rangle \geq c ||u - v||^q, \quad \text{for} \quad u, v \in W_0^{1,q}(\Omega).
\]
Proof. Take $\varphi = u - v$ in (2.9), we have

\[
\frac{d}{dt} (DJ_\mu(w_t), u - v) \\
= \int_{\Omega} \left( \frac{|\nabla w_t|}{m_\mu(|\nabla w_t|)} \right)^{q-2} \left( 1 + \frac{q - 2}{2} \left( 1 - \frac{|\nabla w_t|b_\mu(|\nabla w_t|)}{m_\mu(|\nabla w_t|)} \right) \right) |
\nabla (u - v)|^2 \, dx \\
+ (q - 2) \sigma \int_{\Omega} \left( \frac{|\nabla w_t|}{m_\mu(|\nabla w_t|)} \right)^{q-2} \left( 1 - \frac{|\nabla w_t|b_\mu(|\nabla w_t|)}{m_\mu(|\nabla w_t|)} \right) \frac{1}{2} + \frac{q - 1}{2} \left( 1 - \frac{|\nabla w_t|b_\mu(|\nabla w_t|)}{m_\mu(|\nabla w_t|)} \right) \\
\cdot \left( \frac{\nabla w_t}{|\nabla w_t|} \nabla (u - v) \right)^2 \, dx \\
- \frac{q - 2}{2} \sigma \int_{\Omega} \left( \frac{|\nabla w_t|}{m_\mu(|\nabla w_t|)} \right)^{q-4} w_t^2 \left( 1 + \frac{q - 2}{2} \left( 1 - \frac{|\nabla w_t|b_\mu(|\nabla w_t|)}{m_\mu(|\nabla w_t|)} \right) \right) |
\nabla (u - v)|^2 \, dx \\
+ \frac{1}{2} \sum_{i,j=1}^N \int_{\Omega} \frac{\partial \beta_{ij}}{\partial \theta} \cdot \nabla \beta_{ij} \nabla w_t \nabla (u - v) \, dx \\
+ \frac{1}{2} \sum_{i,j=1}^N \int_{\Omega} \frac{\partial D_{ij}}{\partial \theta} \cdot \nabla D_{ij} \nabla w_t \nabla (u - v) \, dx \\
+ \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^N (\frac{\partial D_{ij}}{\partial \theta} D_{ij}) \nabla w_t \nabla w_t \nabla (u - v) \, dx \\
+ \frac{1}{2} \int_{\Omega} (\frac{\partial D_{ij}}{\partial \theta} D_{ij}) \nabla w_t \nabla w_t \nabla (u - v) \, dx \\
+ c_1 \int_{\Omega} |w_t|^{q-2}(u - v)^2 \, dx + c_2 \int_{\Omega} (u - v)^2 \, dx.
\]

There are only three indefinite terms in the right hand side of (2.11) and they can be controlled by other definite terms. In fact

\[
\frac{d}{dt} (DJ_\mu(w_t), u - v) \\
\geq c \int_{\Omega} \left( \frac{|\nabla w_t|}{m_\mu(|\nabla w_t|)} \right)^{q-2} |
\nabla (u - v)|^2 \, dx \\
+ \frac{1}{2} \sum_{i,j=1}^N \int_{\Omega} \frac{\partial \beta_{ij}}{\partial \theta} \cdot \nabla \beta_{ij} \nabla w_t \nabla (u - v) \, dx \\
+ \frac{1}{2} \sum_{i,j=1}^N \int_{\Omega} \frac{\partial D_{ij}}{\partial \theta} \cdot \nabla D_{ij} \nabla w_t \nabla (u - v) \, dx \\
+ \frac{1}{2} \sum_{i,j=1}^N (\frac{\partial D_{ij}}{\partial \theta} D_{ij}) \nabla w_t \nabla w_t \nabla (u - v) \, dx \\
+ \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^N (\frac{\partial D_{ij}}{\partial \theta} D_{ij}) \nabla w_t \nabla w_t \nabla (u - v) \, dx.
\]
We estimate the last three terms in the right hand side of (2.12),
\[
\left| \int_\Omega \left( \frac{\nabla w_1}{m_\mu(\nabla w_1)} \right)^{q-4} \left( 1 + \frac{q-4}{2} \left( 1 - \frac{\nabla w_1 |_{\partial \Omega}(\nabla w_1)}{m_\mu(\nabla w_1)} \right) \right) w_1(u-v) \nabla w_2 \nabla (u-v) \, dx \right|
\leq \varepsilon \int_\Omega \left( \frac{\nabla w_1}{m_\mu(\nabla w_1)} \right)^{q-2} |\nabla(u-v)|^2 \, dx + c_\varepsilon \int_\Omega \left( \frac{\nabla w_1}{m_\mu(\nabla w_1)} \right)^{q-4} m_\mu^2(|\nabla w_1|) w_1^2(u-v)^2 \, dx
\]
\[
\leq \varepsilon \int_\Omega \left( \frac{\nabla w_1}{m_\mu(\nabla w_1)} \right)^{q-2} |\nabla(u-v)|^2 \, dx + \varepsilon \int_\Omega \left( \frac{\nabla w_1}{m_\mu(\nabla w_1)} \right)^{q-4} m_\mu^2(|\nabla w_1|)(u-v)^2 \, dx
\]
\[
+ c_\varepsilon \mu^{-2} \int_\Omega |w_1|^{q-2}(u-v)^2 \, dx
\]
(2.13)

and
\[
\left| \int_\Omega \sum_{i,j=1}^N D_i \beta_{ij}(x, w_1) D_i w_1 D_j(u-v) \, dx \right|
\leq \varepsilon \int_\Omega \left( \frac{\nabla w_1}{m_\mu(\nabla w_1)} \right)^2 |\nabla(u-v)|^2 \, dx + c_\varepsilon \int_\Omega \sum_{i,j=1}^N (D_i \beta_{ij}(x, w_1))^2 (u-v)^2 m_\mu^2(|\nabla w_1|) \, dx
\]
\[
\leq \varepsilon \int_\Omega \left( 1 + \left( \frac{\nabla w_1}{m_\mu(\nabla w_1)} \right)^{-q^{-2}} \right) |\nabla(u-v)|^2 \, dx + c_\varepsilon \mu^{-2} \int_\Omega (1 + |w_1|^{q-2})(u-v)^2 \, dx .
\]
(2.14)

Also
\[
\left| \int_\Omega \left( \frac{d}{dt} D_{ij}(x, w_1) \right) D_i w_1 D_j(u-v) \, dx \right|
\leq c \int_\Omega |\nabla w_1|^2 (u-v)^2 \, dx
\leq \varepsilon \int_\Omega \left( \frac{\nabla w_1}{m_\mu(\nabla w_1)} \right)^{q-2} m_\mu^2(|\nabla w_1|) (u-v)^2 \, dx + c_\varepsilon \int_\Omega m_\mu^2(|\nabla w_1|)(u-v)^2 \, dx
\leq \varepsilon \int_\Omega \left( \frac{\nabla w_1}{m_\mu(\nabla w_1)} \right)^{q-4} |\nabla w_1|^2 (u-v)^2 \, dx + c_\varepsilon \mu^{-2} \int_\Omega (u-v)^2 \, dx .
\]
(2.15)

By taking \( \varepsilon \) small enough first, then \( c_1, c_2 \) large enough, by (2.12)-(2.15) we obtain
\[
\frac{d}{dt} (D\mu(w_1), u-v) \geq c \int_\Omega \left( \frac{\nabla w_1}{m_\mu(\nabla w_1)} \right)^{q-2} |\nabla(u-v)|^2 \, dx
\geq c_\mu^{q-2} \int_\Omega |\nabla w_1|^{q-2} |\nabla(u-v)|^2 \, dx
\]
\[
\langle DJ_\mu(u) - DJ_\mu(v), u-v \rangle = \int_0^1 \frac{d}{dt} (DJ_\mu(w_1), u-v) \, dt
\geq c_\mu^{q-2} \int_\Omega \left( \int_0^1 |\nabla w_1|^{q-2} \, dt \right) \cdot |\nabla(u-v)|^2 \, dx
\]
\[
\begin{align*}
&\geq c\mu^{q-2} \int_{\Omega} (|\nabla u|^q + |\nabla v|^q - |\nabla(u - v)|^2) \, dx \\
&\geq c\mu^{q-2} \int_{\Omega} |\nabla (u - v)|^q \, dx = c\mu^{q-2} ||u - v||^q.
\end{align*}
\]

Lemma 2.4. The functional \( I_\mu \) satisfies the Palais-Smale condition.

Proof. We have
\[
I_\mu(u) = J_\mu(u) - R(u), \quad DI_\mu(u) = DJ_\mu(u) - DR(u), \quad u \in W_0^{1,q}(\Omega). \quad (2.16)
\]

The operator \( DR : X = W_0^{1,q}(\Omega) \to X^* \) is weakly continuous, that is, if \( u_n \to u \) in \( W_0^{1,q}(\Omega) \), then \( ||DR(u_n) - DR(u)||_{X^*} \to 0 \) as \( n \to \infty \).

Let \( \{u_n\} \subset W_0^{1,q}(\Omega) \) be a Palais-Smale sequence of \( I_\mu \). By Lemma 2.1, \( \{u_n\} \) is bounded in \( W_0^{1,q}(\Omega) \). Assume \( u_n \to u \) in \( W_0^{1,q}(\Omega) \), then by (2.16) and Lemma 2.3, we have
\[
o(1) = (DI_\mu(u_n) - DI_\mu(u), u_n - u)
= (DJ_\mu(u_n) - DJ_\mu(u), u_n - u) - (DR(u_n) - DR(u), u_n - u)
\geq c\|u_n - u\|^q - ||DR(u_n) - DR(u)||_{X^*} \|u_n - u\|
= c\|u_n - u\|^q + o(1),
\]

therefore \( u_n \to u \) in \( W_0^{1,q}(\Omega) \). \( \square \)

Proposition 2.1. (1) The operator \( A : u \in W_0^{1,q}(\Omega) \to v = Au \in W_0^{1,q}(\Omega) \) is well-defined and continuous. Moreover, there exist constants \( a_1, a_2 > 0 \) such that
\[
(2) \quad \langle DI_\mu(u), u - Au \rangle \geq a_1 \|u - Au\|^q.
(3) \quad ||DI_\mu(u)|| \leq a_2 (1 + ||I_\mu(u)|| + \|u - Au\|^{q-2})\|u - Au\|.
\]

Proof. (1) It follows from Lemma 2.3 that the operator \( A \) is strongly monotone and coercive, by Minty-Browder’s Theorem (e.g., [6] Theorem 5.16) the operator \( A \) is well-defined. Let \( u, \bar{u} \in W_0^{1,q}(\Omega) \), \( v = Au, \bar{v} = A\bar{u} \). Then by Lemma 2.3, we have
\[
c\|v - \bar{v}\|^q \leq \langle DJ_\mu(v) - DJ_\mu(\bar{v}), v - \bar{v}\rangle
= \langle DR_\mu(u) - DR_\mu(\bar{u}), v - \bar{v}\rangle
\leq ||DR(u) - DR(\bar{u})||_{X^*} \|v - \bar{v}\|.
\]

So \( \|v - \bar{v}\| \leq c||DR(u) - DR(\bar{u})||^{1\over q} \). Since \( DR : X = W_0^{1,q}(\Omega) \to X^* \) is weakly continuous, \( A \) is continuous.

(2) Given \( u \in W_0^{1,q}(\Omega) \), \( v = Au \), we have
\[
DI_\mu(u) = DJ_\mu(u) - DR(u) = DJ_\mu(u) - DJ_\mu(v).
(2.17)
\]

By Lemma 2.3, we have
\[
\langle DI_\mu(u), u - v \rangle = \langle DJ_\mu(u) - DJ_\mu(v), u - v \rangle \geq a_1 \|u - v\|^q.
(3) \quad \text{By Lemma 2.2, we have}
\]
\[
||DI_\mu(u)|| = ||DJ_\mu(u) - DJ_\mu(v)||
\leq c(1 + \|u\|^{q-2} + \|v\|^{q-2})\|u - v\|
(2.18)
\leq c(1 + \|u\|^{q-2} + \|u - v\|^{q-2})\|u - v\|.
\]
By Lemma 2.1, \( I_\mu \) is bounded from below

\[
I_\mu(u) \geq c \int_\Omega |\nabla u|^q \, dx - c. \tag{2.19}
\]

Substituting (2.19) into (2.18), we obtain

\[
\|D I_\mu(u)\| \leq c(1 + \|u\|^{q-2} + \|u - v\|^{q-2})\|u - v\|
\begin{align*}
&\leq c(1 + |I_\mu(u)|^{\frac{q-2}{q}} + \|u - v\|^{q-2})\|u - v\|
&\leq a_2(1 + |I_\mu(u)| + \|u - v\|^{q-2})\|u - v\|. 
\end{align*}
\]

\[
\square
\]

For \( \varepsilon > 0 \), let \( P_\varepsilon \) be an open convex neighborhood of the positive cone \( P \) of \( W^{1,q}_0(\Omega) \):

\[
P = \{ u \in W^{1,q}_0(\Omega) | u \geq 0, \text{ a.e. } x \in \Omega \}
\]

\[
P_\varepsilon = \{ u \in W^{1,q}_0(\Omega) | \left(\frac{3}{4}c_0\lambda_1 + \frac{1}{2}c_2\right) \int_\Omega u^2 \, dx + \frac{3}{16}c_0S\left( \int_\Omega u^{\frac{4N}{N-4}} \, dx \right)^\frac{N-2}{N} + \frac{q-1}{q}c_1 \int_\Omega u^\eta \, dx < \varepsilon \}
\]

and \( Q_\varepsilon = -P_\varepsilon \), where \( \lambda_1 \) is the first eigenvalue of the Laplacian \(-\Delta\) with Dirichlet boundary condition, \( S \) is the best constant of the Sobolev imbedding \( H^1_0(\Omega) \hookrightarrow L^{\frac{4N}{N-4}}(\Omega) \), \( c_0 \) is the constant given in the assumption \( b_3 \), \( c_1 \) and \( c_2 \) are chosen in Lemma 2.3. \( P_\varepsilon \) and \( Q_\varepsilon \) are convex sets.

**Lemma 2.5.** For sufficiently small \( \varepsilon \), we have

\[
A(\partial P_\varepsilon) \subset P_\varepsilon, \quad A(\partial Q_\varepsilon) \subset Q_\varepsilon.
\]

**Proof.** Let \( u \in W^{1,q}_0(\Omega), v = A u \in W^{1,q}_0(\Omega) \). By the definition of the operator \( A \), we have

\[
\sigma \int_\Omega \left( \frac{|\nabla v|}{m_\mu(|\nabla v|)} \right)^{q-2} \left( 1 + \frac{q-2}{2} \left( 1 - \frac{|\nabla v|b_\mu(|\nabla v|)}{m_\mu(|\nabla v|)} \right) \right) \nabla v \nabla \varphi \, dx
\]

\[
+ \sigma \int_\Omega \left( \frac{|\nabla v|}{m_\mu(|\nabla v|)} \right)^{q-4} \left( 1 + \frac{q-4}{2} \left( 1 - \frac{|\nabla v|b_\mu(|\nabla v|)}{m_\mu(|\nabla v|)} \right) \right) v^2 \nabla v \nabla \varphi \, dx
\]

\[
+ \sigma \int_\Omega \left( \frac{|\nabla v|}{m_\mu(|\nabla v|)} \right)^{q-4} |\nabla v|^2 v \varphi \, dx
\]

\[
+ \frac{1}{2} \int_\Omega \sum_{i,j=1}^N \beta_{ij}(x,v) D_i v D_j \varphi \, dx
\]

\[
+ \frac{1}{2} \int_\Omega \sum_{i,j=1}^N D_z \beta_{ij}(x,v) D_i v D_j \varphi \, dx
\]

\[
+ c_1 \int_\Omega |v|^{q-2} v \varphi \, dx + c_2 \int_\Omega v \varphi \, dx
\]

\[
= \int_\Omega (\lambda f(x,u) + c_1 |u|^{q-2} u + c_2 u) v \, dx.
\]

\[
\square
\]
Take $\varphi = v_+$ as test function in (2.20). We estimate the left hand side of (2.20)

\[
\text{LHS} \geq \sigma \int_\Omega |\nabla v_+|^2 \, dx + 2\sigma \int_\Omega v_+^2 |\nabla v_+|^2 \, dx \\
+ \int_\Omega \sum_{i,j=1}^N (\beta_{ij}(x, v_+) + \frac{1}{2} v_+ D_2 \beta_{ij}(x, v_+)) D_i v_+ D_j v_+ \, dx \\
+ c_1 \int_\Omega v_+^q \, dx + c_2 \int_\Omega v_+^2 \, dx \\
\geq \int_\Omega \sum_{i,j=1}^N (b_{ij}(x, v_+) + \frac{1}{2} v_+ D_2 b_{ij}(x, v_+)) D_i v_+ D_j v_+ \, dx + c_1 \int_\Omega v_+^q \, dx \\
+ c_2 \int_\Omega v_+^2 \, dx \\
\geq c_0 \int_\Omega (1 + v_+^2) |\nabla v_+|^2 \, dx + c_1 \int_\Omega v_+^q \, dx + c_2 \int_\Omega v_+^2 \, dx \\
\geq c_0 \lambda_1 \int_\Omega v_+^2 \, dx + \frac{1}{4} c_0 S \left( \int_\Omega \frac{4N}{\alpha} \, dx \right)^\frac{N-2}{N} + c_1 \int_\Omega v_+^q \, dx + c_2 \int_\Omega v_+^2 \, dx.
\]  
(2.21)

By the assumptions (f_2), (f_4), without loss of generality, we assume

\[
(\lambda f(x, z) + c_1 |z|^{q-2} z + c_2 z) z \geq 0, \quad \text{for } z \in \mathbb{R}.
\]  
(2.22)

The right hand side of (2.20),

\[
\text{RHS} = \int_\Omega \left( (\lambda f(x, u) + c_1 |u|^{q-2} u + c_2 u) v_+ \right) \, dx \\
\leq \int_\Omega \left( \lambda f(x, u_+) + c_1 u_+^{q-1} + c_2 u_+ \right) v_+ \, dx \\
\leq \int_\Omega \left( \frac{1}{2} c_0 \lambda_1 u_+ + c_\lambda \frac{4N}{\alpha} u_+^{q-1} + c_1 u_+^{q-1} + c_2 u_+ \right) v_+ \, dx \\
\leq \left( \frac{1}{2} c_0 \lambda_1 + c_2 \right) \int_\Omega u_+ v_+ \, dx + c_1 \int_\Omega u_+^{q-1} v_+ \, dx \\
+ c_\lambda \left( \int_\Omega \frac{4N}{\alpha} \, dx \right)^\frac{q}{2} \left( \int_\Omega \frac{4N}{\alpha} \, dx \right)^\frac{N-2}{2} \left( \int_\Omega \frac{4N}{\alpha} \, dx \right)^\frac{N-2}{N} \\
\leq \left( \frac{1}{2} c_0 \lambda_1 + c_2 \right) \int_\Omega u_+ v_+ \, dx + c_1 \int_\Omega u_+^{q-1} v_+ \, dx \\
+ \frac{1}{4} c_0 S \left( \int_\Omega \frac{4N}{\alpha} \, dx \right)^\frac{N-2}{N} \left( \int_\Omega \frac{4N}{\alpha} \, dx \right)^\frac{N-2}{N}
\]

provided

\[
c_\lambda \left( \int_\Omega \frac{4N}{\alpha} \, dx \right)^\frac{q}{2} \leq \frac{1}{4} c_0 S.
\]  
(2.24)

It follows from (2.21) and (2.23) that

\[
(c_0 \lambda_1 + c_2) \int_\Omega v_+^2 \, dx + \frac{1}{4} c_0 S \left( \int_\Omega \frac{4N}{\alpha} \, dx \right)^\frac{N-2}{N} + c_1 \int_\Omega v_+^q \, dx
\]
that is, \( v \) Then there exists \( \varepsilon_0 \) two sequences of critical values of the approximate functional \( I \) Critical values of the approximate functionals. \[ \text{Proposition 2.2.} \]

For sufficiently small \( \varepsilon \), \( u \) Let \( \text{Lemma 3.1.} \)

Now let \( u \in \partial \Omega \), then

\[
\left( \frac{3}{4} c_0 \lambda_1 + \frac{1}{2} c_2 \right) \int_{\Omega} u_+^2 \, dx + \frac{3}{16} c_0 \mathcal{S} \left( \int_{\Omega} u_+^{4N} \, dx \right)^{\frac{N-2}{N}} + \frac{q-1}{q} c_1 \int_{\Omega} v_+^q \, dx
\]

and

\[
\left( \frac{3}{4} c_0 \lambda_1 + \frac{1}{2} c_2 \right) \int_{\Omega} v_+^2 \, dx + \frac{3}{16} c_0 \mathcal{S} \left( \int_{\Omega} v_+^{4N} \, dx \right)^{\frac{N-2}{N}} + \frac{q-1}{q} c_1 \int_{\Omega} v_+^q \, dx
\]

(2.25)

Now let \( u \in \partial \Omega \), then

\[
\left( \frac{3}{4} c_0 \lambda_1 + \frac{1}{2} c_2 \right) \int_{\Omega} u_+^2 \, dx + \frac{3}{16} c_0 \mathcal{S} \left( \int_{\Omega} u_+^{4N} \, dx \right)^{\frac{N-2}{N}} + \frac{q-1}{q} c_1 \int_{\Omega} u_+^q \, dx = \varepsilon.
\]

For sufficiently small \( \varepsilon \), (2.24) holds. Hence by (2.25), we have

\[
\left( \frac{3}{4} c_0 \lambda_1 + \frac{1}{2} c_2 \right) \int_{\Omega} v_+^2 \, dx + \frac{3}{16} c_0 \mathcal{S} \left( \int_{\Omega} v_+^{4N} \, dx \right)^{\frac{N-2}{N}} + \frac{q-1}{q} c_1 \int_{\Omega} v_+^q \, dx < \varepsilon
\]

that is, \( v = Au \in Q_{\varepsilon} \). We have proved that \( A(\partial \Omega) \subset Q_{\varepsilon} \). Similarly, we have \( A(\partial \Omega) \subset P_{\varepsilon} \). \( \square \)

**Proposition 2.2. (The deformation lemma)** Let \( K_{\varepsilon} = \{ u \in W_0^{1,q}(\Omega) \mid DI_{\mu}(u) = 0, \, I_{\mu}(u) = c \}, \Sigma = P_{\varepsilon} \cup Q_{\varepsilon}, \, K_{\varepsilon} = K_{\varepsilon} \setminus \Sigma \). Let \( N \) be an open neighborhood of \( K_{\varepsilon} \). Then there exists \( \varepsilon_0 \) such that for \( 0 < \varepsilon < \varepsilon_0 \) there exists a map \( \eta : W_0^{1,q}(\Omega) \rightarrow W_0^{1,q}(\Omega) \) satisfying

(i) \( \eta(I_{\mu}^{-\varepsilon}(\Omega \cup \Sigma)) \subset I_{\mu}^{1-\varepsilon} \);

(ii) \( \eta|_{I_{\mu}^{-\varepsilon}} = Id \);

(iii) \( \eta(P_{\varepsilon}) \subset P_{\varepsilon}, \, \eta(Q_{\varepsilon}) \subset Q_{\varepsilon} \);

(iv) \( \eta \) is odd.

**Proof.** The proof follows from Lemma 2.4, Proposition 2.1 and Lemma 2.5. For the detail of the proof see [20]. \( \square \)

3. **Critical values of the approximate functionals.** In this section we define two sequences of critical values of the approximate functional \( I_{\mu} \). Recall that \( \Omega_0 \subset \Omega \) is from condition \( (f_3) \).

**Lemma 3.1.** Let \( X_k \) be a \( k \)-dimensional subspace of \( W_0^{1,q}(\Omega_0) \), \( S_k = \{ u \mid u \in X_k, \| u \| = 1 \} \). Then

\[
\int_{\Omega} F(x, Ru) \, dx \rightarrow \infty \quad \text{as} \, R \rightarrow \infty, \quad \text{uniformly in} \, u \in S_k.
\]

**Proof.** Otherwise there exist \( u_n \in S_k, \, R_n \rightarrow \infty, \, M > 0 \) such that

\[
\int_{\Omega} F(x, R_n u_n) \, dx \leq M, \, n = 1, 2, \ldots
\]

Since \( X_k \) is finite-dimensional, we may assume \( u_n \rightarrow u \neq 0 \) in \( W_0^{1,q}(\Omega_0) \). Then there is \( \alpha > 0 \) such that the set \( A_1 = \{ x \mid u(x) \geq \alpha \} \) has a positive measure. Then by Egorov’s Theorem (e.g., [6, Theorem 4.9]) we find \( A_2 \subset A_1 \) with positive measure
such that \( u_n \to u \) uniformly on \( A_2 \). By \((f_3)\), there is \( a > 0 \) such that \( F(x, u) \geq 0 \) for \( |u| \geq a \) and \( x \in \Omega_0 \). Now by the assumption \((f_3)\) we have
\[
\int_{\Omega} F(x, R_n u_n) \, dx = \int_{A_2} F(x, R_n u_n) \, dx + \int_{\Omega_0 \setminus A_2} F(x, R_n u_n) \, dx
\geq \int_{\Omega_0} F(x, R_n u_n) \, dx - c \to \infty \quad \text{as} \ n \to \infty.
\]
We arrive at a contradiction. \( \square \)

**Lemma 3.2.** For any positive integer \( k \) there exist \( \Lambda_k > 0, R_k > 0 \) and a \( k \)-dimensional subspace \( X_k \) of \( W_0^{1,q}(\Omega) \) such that for \( \lambda \geq \Lambda_k \)
\[
I_\mu(u) < 0 \quad \text{for} \ u \in B_k := \{ u \in X_k ||u|| = R_k \}.
\]

**Proof.** The proof follows from Lemma 3.1. \( \square \)

**Proposition 3.1.** Define
\[
c_j(\mu) = \inf_{B \in I_j} \sup_{u \in B \setminus \Sigma} I_\mu(u) \quad 2 \leq j \leq k \tag{3.1}
\]
where \( \Sigma = P_\varepsilon \cup Q_\varepsilon \) and
\[
\Gamma_j = \{ B \subset W_0^{1,q}(\Omega) | B \text{ is compact, symmetric and } \gamma(B) \geq j \},
\]
\( \gamma(B) \) is the genus of a closed, compact, symmetric set of \( B \). Then for \( \lambda \geq \Lambda_k \),
\[
c_2(\mu), \ldots, c_k(\mu) \text{ are critical values of the functional } I_\mu,
\]
\[-\infty < c_2(\mu) \leq \cdots \leq c_k(\mu) < 0. \tag{3.2}\]
Moreover if \( c = c_j(\mu) = \cdots = c_{j+l-1}(\mu) \), then \( \gamma(K^*_c) \geq l \), where \( c = \inf I_\mu(u) \).

We remark that if \( \Sigma = \emptyset \), this is the classical Clark’s theorem [7, 33]. Clark’s Theorem in the setting of ordered Hilbert spaces was given in [16]. Multiple sign-changing solutions of negative energies in the semilinear cases were obtained in [4, 2, 16]. For reader’s convenience we give the proof for the case \( \Sigma \neq \emptyset \).

**Proof.** By Lemma 2.1 \( I_\mu \) is bounded from below. Using the sets \( B_j \) in Lemma 3.2 we have \( \gamma(B_j) = j \) and for \( j \geq 2 \), \( B_j \setminus \Sigma \neq \emptyset \). Thus formula \((3.2)\) follows from Lemma 2.1 and Lemma 3.2. Assume \( c = c_j(\mu) = \cdots = c_{j+l-1}(\mu) \). Let \( N \) be an open neighborhood of \( K^*_c \) with \( \gamma(I) = \gamma(K^*_c) \). By the deformation lemma there exist \( \varepsilon > 0 \) and an odd map \( \eta : W_0^{1,q}(\Omega) \to W_0^{1,q}(\Omega) \) satisfying \( \eta(\Sigma) \subset \Sigma \) and \( \eta(I_\mu(\frac{1}{2}\varepsilon)(\Omega \cup \Sigma)) \subset I_\mu(\frac{1}{2}\varepsilon) \). By the definition we have a set \( B \) in \( \Gamma_j \setminus \Sigma \) such that \( B \setminus \Sigma \subset I_\mu(\frac{1}{2}\varepsilon) \). Now we have
\[
\eta(B \setminus N) \setminus \Sigma \subset (\eta(B \setminus (N \cup \Sigma)) \cup \eta(\Sigma)) \setminus \Sigma
\leq \eta(I_\mu(\frac{1}{2}\varepsilon) \setminus (N \cup \Sigma)) \subset I_\mu(\frac{1}{2}\varepsilon).
\]
Again by the definition \( \eta(B \setminus N) \notin \Gamma_j \), that is \( j - 1 \geq \gamma(\eta(B \setminus N)) \). We have
\[
j - 1 \geq \gamma(\eta(B \setminus N)) \geq \gamma(B \setminus N) \geq \gamma(B) - \gamma(\Sigma) \geq (j + l - 1) - \gamma(\Sigma)
\]
hence \( \gamma(K^*_c) = \gamma(\Sigma) \geq l \). \( \square \)

**Lemma 3.3.** For sufficiently small \( \varepsilon > 0 \), there exists \( \alpha = \alpha(\varepsilon) > 0 \) such that
\[
I_\mu(u) \geq \alpha > 0 \quad \text{for} \ u \in \partial P_\varepsilon \cap \partial Q_\varepsilon,
\]
\[
I_\mu(u) \geq 0 \quad \text{for} \ u \in M = \overline{P_\varepsilon \cap \overline{Q_\varepsilon}}.
\]
Proof.

\[ I_\mu(u) = \frac{1}{2} \sigma \int_\Omega \left( \frac{1}{m_\mu(|\nabla u|)} \right)^{q-2} |\nabla u|^2 \, dx + \frac{1}{2} \int_\Omega \left( \frac{1}{m_\mu(|\nabla u|)} \right)^{q-4} u^2 |\nabla u|^2 \, dx \\
\quad + \frac{1}{2} \int_\Omega \sum_{i,j=1}^N \beta_{ij}(x,u) D_i u D_j u \, dx - \lambda \int_\Omega F(x,u) \, dx \]

\[ \geq \frac{1}{2} c_0 \int_\Omega (1 + u^2) |\nabla u|^2 \, dx - \int_\Omega \left( \frac{1}{2} c_0 \lambda_1 u^2 + c_\lambda |u|^{\frac{4N}{N-2}} \right) \, dx \]

\[ \geq \frac{1}{8} c_0 S \left( \int_\Omega |u|^{\frac{4N}{N-2}} \, dx \right)^{\frac{N-2}{N}} - c_\lambda \int_\Omega |u|^{\frac{4N}{N-2}} \, dx \]

\[ = \left( \int_\Omega |u|^{\frac{4N}{N-2}} \, dx \right)^{\frac{N-2}{N}} \left( \frac{1}{8} c_0 S - c_\lambda \left( \int_\Omega |u|^{\frac{4N}{N-2}} \, dx \right)^{\frac{N}{N-2}} \right). \]

If \( u \in O = \partial P_\varepsilon \cap \partial Q_\varepsilon \), then

\[ \left( \frac{3}{4} c_0 \lambda_1 + \frac{1}{2} c_2 \right) \int_\Omega u^2 \, dx + \frac{3}{16} c_0 S \int_\Omega |u|^{\frac{4N}{N-2}} \, dx + \frac{q-1}{q} c_1 \int_\Omega |u|^q \, dx = 2 \varepsilon. \]  (3.3)

For \( \varepsilon \) small enough

\[ c_\lambda \left( \int_\Omega |u|^{\frac{4N}{N-2}} \, dx \right)^{\frac{N}{N-2}} \leq \frac{1}{16} c_0 S. \]  (3.4)

Moreover, by (3.3) we have

\[ c \left( \left( \int_\Omega |u|^{\frac{4N}{N-2}} \, dx \right)^{\frac{2-N}{N}} + \left( \int_\Omega |u|^{\frac{4N}{N-2}} \, dx \right)^{\frac{N}{N-2}} + \int_\Omega |u|^{\frac{4N}{N-2}} \, dx \right) \geq 2 \varepsilon \]

hence \( \int_\Omega |u|^{\frac{4N}{N-2}} \, dx \geq c(\varepsilon) > 0 \). Now

\[ I_\mu(u) \geq \left( \int_\Omega |u|^{\frac{4N}{N-2}} \, dx \right)^{\frac{N-2}{N}} \left( \frac{1}{8} c_0 S - c_\lambda \left( \int_\Omega |u|^{\frac{4N}{N-2}} \, dx \right)^{\frac{N}{N-2}} \right) \]

\[ \geq \frac{1}{16} c_0 S \left( \int_\Omega |u|^{\frac{4N}{N-2}} \, dx \right)^{\frac{N-2}{N}} \geq \alpha(\varepsilon) \text{ for } u \in O = \partial P_\varepsilon \cap \partial Q_\varepsilon. \]

Similarly, \( I_\mu(u) \geq 0 \) for \( u \in P_\varepsilon \cap Q_\varepsilon \). \( \square \)

**Proposition 3.2.** Define

\[ d_j(\mu) = \inf_{A \in \Gamma_j} \sup_{u \in A \setminus \Sigma} I_\mu(u), \quad j = 2, \cdots, k \]

where

\[ \Gamma_j = \{ A | A \subset W^{1,q}_0(\Omega), -A = A, A \text{ compact}, \gamma(A \cap \sigma^{-1}(O)) \geq j, \forall \sigma \in \Lambda \} \]

\[ \Lambda = \{ \sigma : W^{1,q}_0(\Omega) \to W^{1,q}_0(\Omega), \text{ continuous, odd}, P_\varepsilon \subset P_\varepsilon, \sigma(Q_\varepsilon) \subset Q_\varepsilon \}
\]

and \( \sigma(u) = u \text{ if } I_\mu(u) \leq 0 \}

Then for \( \lambda \geq \Lambda_k, d_2(\mu), \cdots, d_k(\mu) \) are critical values of the functional \( I_\mu \),

\[ 0 < d_2(\mu) \leq \cdots \leq d_k(\mu) \leq d < +\infty \]  (3.5)

where \( d := d_k(1) \). Moreover, if \( d = d_j(\mu) = \cdots = d_{j+l-1}(\mu) \), then \( \gamma(K_\varepsilon) \geq l \).

We remark that if \( \Sigma = \emptyset \), basically this is the symmetric mountain pass lemma due to Ambrosetti and Rabinowitz [1]. Some earlier versions of this appeared in [16, 19]. We give the proof for completeness.
Proof. First the family $\Gamma_j$ is nonempty, the set $B_j$ belongs to $\Gamma_j$ with $B_j$ from Lemma 3.2. Moreover, for $j \geq 2$ and $A \in \Gamma_j$, $(A \setminus \Sigma) \cap O \neq \emptyset$ (see [22]). By Lemma 3.3,
\[
\sup_{u \in A \setminus \Sigma} I_\mu(u) \geq \inf_{u \in O} I_\mu(u) \geq \alpha > 0,
\]
so we obtain the formula (3.5). Now let $N$ be an open neighborhood of $K^*_\eta$ with $\gamma(\bar{N}) = \gamma(K^*_\eta)$. By the deformation lemma there exist $\varepsilon > 0$ and an odd map 
\[
\eta : W_0^{1,q}(\Omega) \to W_0^{1,q}(\Omega)
\]
satisfying $\eta(\Sigma) \subset \Sigma$, $\eta(I^{s+\frac{1}{2}}_\mu \setminus (N \cup \Sigma)) \subset I^{s-\frac{1}{2}}_\mu$ and $\eta|_{I^{s-\varepsilon}} = \text{Id}$. By the definition we have a set $A$ in $\Gamma_{j+l-1}$ such that $A \setminus \Sigma \subseteq I^{s+\frac{1}{2}}_\mu$. We have
\[
\eta(A \setminus N) \setminus \Sigma \subseteq (\eta(A \setminus (N \cup \Sigma)) \cup \eta(\Sigma)) \setminus \Sigma \subseteq (I^{s+\frac{1}{2}}_\mu \setminus (N \cup \Sigma)) \setminus I^{s-\frac{1}{2}}_\mu.
\]
Again by the definition $\eta(A \setminus N) \notin \Gamma_j$, there exists $\sigma \in \Lambda$ such that
\[
\gamma(\eta(A \setminus N) \cap \sigma^{-1}(O)) \leq j - 1.
\]
Since $\sigma\eta \in \Lambda$, we have
\[
j - 1 \geq \gamma(\eta(A \setminus N) \cap \sigma^{-1}(O)) \\
\geq \gamma(A \setminus N \cap (\sigma\eta)^{-1}(O)) \\
\geq \gamma(A \setminus (\sigma\eta)^{-1}(O)) - \gamma(\bar{N}) \\
\geq (j + l - 1) - \gamma(\bar{N})
\]
hence $\gamma(K^*_\eta) = \gamma(\bar{N}) \geq l$. \hfill \Box

**Proof of Part 1 of Theorem 1.2.** By Proposition 3.1 and Proposition 3.2, we obtain $k - 1$ pairs sign-changing solutions with negative critical values and $k - 1$ pairs of sign-changing solutions with positive critical values, respectively. By minimization of $I_\mu$ we obtain signed solutions with negative critical value, by mountain pass lemma we obtain signed solutions with positive critical value. \hfill \Box

4. Sign-changing solutions of the quasilinear equations. In this section we prove the second part of Theorem 1.2, consequently Theorem 1.1. We have

(1) the functional
\[
I_\mu(u) = \frac{1}{2} \sigma \int_{\Omega} \left( \frac{|\nabla u|}{m_\mu(|\nabla u|)} \right)^{q-2} |\nabla u|^2 \, dx + \frac{1}{2} \sigma \int_{\Omega} \left( \frac{|\nabla u|}{m_\mu(|\nabla u|)} \right)^{q-4} (1 + u^2)|\nabla u|^2 \, dx
\]
\[
+ \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^N \beta_{ij}(x,u)D_i u D_j u \, dx - \lambda \int_{\Omega} F(x,u) \, dx \quad \text{for } u \in W_0^{1,q}(\Omega). \tag{4.1}
\]

(2) The equation in weak form
\[
\langle DI_\mu(u), \phi \rangle = \sigma \int_{\Omega} \left( \frac{|\nabla u|}{m_\mu(|\nabla u|)} \right)^{q-2} \left( \frac{q}{2} - \frac{q-2}{2} \frac{|\nabla u| b_\mu(|\nabla u|)}{m_\mu(|\nabla u|)} \right) \nabla u \nabla \phi \, dx
\]
\[
+ \sigma \int_{\Omega} \left( \frac{|\nabla u|}{m_\mu(|\nabla u|)} \right)^{q-4} \left( \frac{q-2}{2} - \frac{q-4}{2} \frac{|\nabla u| b_\mu(|\nabla u|)}{m_\mu(|\nabla u|)} \right) (1 + u^2)|\nabla u| \nabla \phi \, dx \tag{4.2}
\]
where the structure of elliptic operators in divergence form, see [15, 17]. It holds that for Lemma 4.1.

Proof. Increasing.

\[ pA(x, u, \nabla u) = \sigma \left( \frac{|p|}{\mu(|p|)} \right)^{q-2} \left( q - \frac{2}{\mu(|p|)} \right) |p| \]

\[ + \sigma \left( \frac{|p|}{\mu(|p|)} \right)^{q-4} \left( q - \frac{2}{\mu(|p|)} - \frac{4}{\mu(|p|)} \right) (1 + z^2) |p| \]

\[ + \sum_{i,j=1}^{N} \beta_{ij}(x, z) p_i p_j \]

for \( \varphi \in W^{1,q}_0(\Omega) \).

(3) The equation in divergence form

\[ Q(x, u, \nabla u) = \text{div} A_\mu(x, u, \nabla u) + B_\mu(x, u, \nabla u) = 0, \quad (4.3) \]

where

\[ Q(x, z, p) = \text{div} A_\mu(x, z, p) + B_\mu(x, z, p) \]

for \( (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \), and

\[ A_\mu(x, z, p) = \sigma \left( \frac{|p|}{\mu(|p|)} \right)^{q-2} \left( q - \frac{2}{\mu(|p|)} \right) |p| \]

\[ + \sigma \left( \frac{|p|}{\mu(|p|)} \right)^{q-4} \left( q - \frac{2}{\mu(|p|)} - \frac{4}{\mu(|p|)} \right) (1 + z^2) |p| \]

\[ + \sum_{j=1}^{N} \beta_{ij}(x, z) p_j \]

\[ B_\mu(x, z, p) = -\sigma \left( \frac{|p|}{\mu(|p|)} \right)^{q-4} |p|^2 z - \frac{1}{2} \sum_{i,j=1}^{N} D_z \beta_{ij}(x, z) p_i p_j + \lambda f(x, z). \]

We verify that the coefficients \( A_\mu, B_\mu \) satisfy the general natural conditions for the structure of elliptic operators in divergence form, see [15, 17].

For \( \mu \in (0, 1] \), define \( g_\mu(t) = \mu^{q-2} t^{q-1} + \mu^{q-4} t^{q-3} + t, t > 0 \). The function \( g_\mu, \mu \in (0, 1] \) satisfies

\[ 1 \leq \frac{t g_\mu(t)}{g_\mu(t)} \leq q - 1. \quad (4.4) \]

Lemma 4.1. It holds that for \( x \in \Omega, |z| \leq M, p \in \mathbb{R}^N \),

\begin{enumerate}
  \item \( p \cdot A_\mu(x, z, p) \geq \lambda(M) g_\mu(|p|)|p| \),
  \item \( |A_\mu(x, z, p)| \leq \Lambda(M) g_\mu(|p|) \),
  \item \( |B_\mu(x, z, p)| \leq \Lambda(M)(1 + g_\mu(|p|)|p|) \),
\end{enumerate}

where \( \lambda, \Lambda \) are two functions from \( \mathbb{R}^+ \) to \( \mathbb{R}^+ \) such that \( \lambda \) is decreasing and \( \Lambda \) is increasing.

Proof.

\begin{enumerate}
  \item \( p A_\mu(x, z, p) = \sigma \left( \frac{|p|}{\mu(|p|)} \right)^{q-2} \left( q - \frac{2}{\mu(|p|)} \right) |p|^2 \]

\[ + \sigma \left( \frac{|p|}{\mu(|p|)} \right)^{q-4} \left( q - \frac{2}{\mu(|p|)} - \frac{4}{\mu(|p|)} \right) (1 + z^2) |p|^2 \]

\[ + \sum_{i,j=1}^{N} \beta_{ij}(x, z) p_i p_j \]

\[ \geq \sigma \left( \frac{|p|}{\mu(|p|)} \right)^{q-2} |p|^2 + \sigma \left( \frac{|p|}{\mu(|p|)} \right)^{q-4} (1 + z^2) |p|^2 + c_0 (1 + z^2) |p|^2 \]

\end{enumerate}
Lemma 4.2. Let $a_{ij} = \frac{\partial A_{ij}}{\partial p_j}$. Then

1. $\sum_{i,j=1}^{N} a_{ij} \xi_i \xi_j \geq \lambda(M) \frac{g_{\mu}(|p|)}{|p|} |\xi|^2$, $\xi \in \mathbb{R}^N$;

2. $|a_{ij}| \leq \Lambda(M) g_{\mu}(|p|)$;

3. $|A_{ij}(x,z,p) - A_{ij}(y,w,p)| \leq \Lambda(M) (|x-y|^\alpha + |z-w|^\alpha) g_{\mu}(|p|)$, $x, y \in \mathbb{R}$, where $\alpha \in (0, 1)$ is the constant in the assumption (b₁);

4. $|B_{ij}(x,z,p)| \leq \Lambda(M) (1 + g_{\mu}(|p|)|p|)$, the same as (3), Lemma 4.1.

Proof. We have

$$a_{ij} = \frac{\partial A_{ij}}{\partial p_j} = \sigma \left( \frac{|p|}{m_{\mu}(|p|)} \right)^{q-2} \left( \frac{q}{2} - \frac{q-2 |p| b_{\mu}(|p|)}{m_{\mu}(|p|)} \right) \delta_{ij}$$

$$+ \frac{q-2}{2} \sigma \left( \frac{|p|}{m_{\mu}(|p|)} \right)^{q-2} \left( 1 - \frac{|p| b_{\mu}(|p|)}{m_{\mu}(|p|)} \right) \left( q - (q-1) \frac{|p| b_{\mu}(|p|)}{m_{\mu}(|p|)} \right) \frac{p_i}{p} \frac{p_j}{p}$$

$$- \frac{q-2}{2} \sigma \left( \frac{|p|}{m_{\mu}(|p|)} \right)^{q-2} |p| b_{\mu}(|p|) \frac{p_i}{m_{\mu}(|p|)} \frac{p_j}{p}$$

$$+ \sigma \left( \frac{|p|}{m_{\mu}(|p|)} \right)^{q-4} \left( q - \frac{q-4 |p| b_{\mu}(|p|)}{2 m_{\mu}(|p|)} \right) (1 + z^2) \delta_{ij}$$

$$+ \frac{q-4}{2} \sigma \left( \frac{|p|}{m_{\mu}(|p|)} \right)^{q-4} \left( 1 - \frac{|p| b_{\mu}(|p|)}{m_{\mu}(|p|)} \right) \left( (q-2) - (q-3) \frac{|p| b_{\mu}(|p|)}{m_{\mu}(|p|)} \right) (1 + z^2) \frac{p_i}{p} \frac{p_j}{p}$$

$$- \frac{q-4}{2} \sigma \left( \frac{|p|}{m_{\mu}(|p|)} \right)^{q-4} |p|^2 b_{\mu}(|p|) \frac{p_i}{m_{\mu}(|p|)} (1 + z^2) \frac{p_i}{p} + \beta_{ij}(x,z).$$

Then

$$\sum_{i,j=1}^{N} a_{ij} \xi_i \xi_j \geq \sigma \left( \frac{|p|}{m_{\mu}(|p|)} \right)^{q-2} \left( \frac{q}{2} - \frac{q-2 |p| b_{\mu}(|p|)}{m_{\mu}(|p|)} \right) |\xi|^2$$

$$+ \sigma \left( \frac{|p|}{m_{\mu}(|p|)} \right)^{q-4} \left( q - \frac{q-4 |p| b_{\mu}(|p|)}{2 m_{\mu}(|p|)} \right) (1 + z^2) |\xi|^2$$

$$+ \sum_{i,j=1}^{N} \beta_{ij}(x,z) \xi_i \xi_j.$$
\[
\geq \lambda(M)(\mu^{q-2}|p|^{q-2} + \mu^{q-4}|p|^{q-4} + 1)|\xi|^2 = \lambda(M)\frac{g_{\mu}|p|}{|p|}|\xi|^2.
\]

(2)

\[
|a_{ij}| \leq c \left( \frac{|p|}{m_{\mu}(|p|)} \right)^{q-2} + \left( \frac{|p|}{m_{\mu}(|p|)} \right)^{q-4}(1 + z^2) + (1 + z^2) \\
\leq \Lambda(M)(\mu^{q-2}|p|^{q-2} + \mu^{q-4}|p|^{q-4} + 1) = \Lambda(M)\frac{g_{\mu}|p|}{|p|}.
\]

(3)

\[
A_{\mu}(x, z, p) - A_{\mu}(y, w, p) \\
= \sigma\left( \frac{|p|}{m_{\mu}(|p|)} \right)^{q-4}\left( \frac{q - 2}{2} - \frac{q - 4 |p|}{2m_{\mu}(|p|)} \right)(z^2 - w^2)p_i \\
+ \sum_{i,j=1}^N (\beta_{ij}(x, z) - \beta_{ij}(y, w))p_j
\]

and

\[
|A_{\mu}(x, z, p) - A_{\mu}(y, w, p)| \\
\leq \Lambda(M)\left( \left( \frac{|p|}{m_{\mu}(|p|)} \right)^{q-4}|z - w|^\alpha|p| + (|x - y|^\alpha + |z - w|^\alpha)|p| \right) \\
\leq \Lambda(|x - y|^\alpha + |z - w|^\alpha)(\mu^{q-4}|p|^{q-3} + |p|) \\
\leq \Lambda(M)(|x - y|^\alpha + |z - w|^\alpha)g_{\mu}(|p|).
\]

\[\square\]

**Proof of Part 2 of Theorem 1.2.** By Corollary 1.5, Theorem 1.7 in [17], any bounded solutions of the perturbed problems satisfies

\[\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leq M\]

where \(\beta = \beta(M_0)\), \(M = M(M_0)\) and \(M_0 = \max_{u \in \overline{\Omega}}|u|\), the constants \(\beta, M\) are independent of \(\mu\). Using the condition \((b_3)\) and the Moser iteration we can show that the solutions \(u_j(\mu), v_j(\mu), j = 1, 2, \cdots, k\) have a uniform bound \(M_0\) independent of \(\mu\), hence are uniformly bounded in \(C^{1,\beta}(\overline{\Omega})\) for some \(\beta \in (0, 1)\).

\[\square\]

**Remark 4.1.** Though our conditions on \(f\) are modeled on the special case \(f(x, z) = |z|^{r-2}z\) with \(2 < r < 4\), our method allows more other cases of the nonlinearity \(f\). For example, the second part of the condition \((f_3)\) is: There is an open subset \(\Omega_0 \subset \Omega\) such that \(F(x, z) \to +\infty\), as \(|z| \to \infty\) uniformly for \(x \in \Omega_0\). This can be replaced by the following: There is an open subset \(\Omega_0 \subset \Omega\) and a constant \(a > 0\) such that \(F(x, z) > 0\) for \(0 < |z| \leq a\) and \(x \in \Omega_0\). Then the same result still holds with slight modification of the proofs.

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