Waves in a Bose-Einstein condensate of atoms with a dipole moment

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Based on the modified Gross-Pitaevskii equation for atoms with intrinsic dipole moments which accounts for the relaxation of a condensate, the dipole-dipole interaction and the interaction of atoms with the electromagnetic field, the propagation of the sound and electromagnetic waves in a Bose-Einstein condensate is studied. Owing to hybridization of the electromagnetic and sound waves near the resonance frequency of an atom, there arise the two branches of excitations in which the electromagnetic oscillations transform into the sound oscillations and vice versa. It is shown that under hybridization the crucial role is played by the dipole-dipole interaction, which leads to a substantial increase of the repulsion between the branches of the spectrum. The influence of the dissipative effects connected with the relaxation of the macroscopic wave function of the condensate and the imaginary part of the polarizability of an atom on the form of the dispersion curves is investigated.

**Key words:** dipole moment, electromagnetic and sound waves, dipole-dipole interaction, permittivity, dispersion law of excitations, hybridization

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**I. INTRODUCTION**

Recently, considerable attention is paid to investigation of the behavior of the superfluid systems of Bose particles in the electric and magnetic fields. The interest in these problems is stimulated by the experimental investigations of Bose-Einstein condensates (BEC) in the traps created by both magnetic fields and the field of the laser radiation [1,2]. A special character of the interaction at low temperatures of the electromagnetic field with many-particle systems of electrically neutral atoms obeying the Bose-Einstein statistics is indicated by the investigations of the propagation of light in atomic gases [3,4]. An unexpectedly high electrical activity of superfluid helium, manifested in different conditions, was reported in a series of experimental works [5-11]. Understanding of the effects observed in such and similar experiments requires a detailed investigation of the interaction of many-particle systems of Bose atoms, being in the coherent state, with the electromagnetic field.

The interaction of the electromagnetic field with a system of electric charges is realized through the multipole moments of a system. If the system is electrically neutral, then the next in importance characteristic that describes the interaction with the electric field is its dipole moment. There are arguments in favor of that a helium atom, which does not have an intrinsic dipole moment in the free state, can spontaneously acquire an intrinsic dipole moment in the liquid helium due to the interaction with the surrounding atoms [12,13]. In this respect, it appears important a detailed theoretical study of the electromagnetic properties of the superfluid system of atoms possessing an intrinsic dipole moment. This will allow to compare theoretical predictions with the phenomena observed in experiments. The propagation of the electromagnetic field in BEC with account of the inner structure of atoms within the model of an ideal gas was studied in [14-16], and the account for the structure of atoms within the modified Gross-Pitaevskii approach (GP) was suggested in [12].

In this work, basing on the GP equation that accounts for the relaxation in BEC, the interaction with the electromagnetic field and the dipole-dipole interaction, the propagation of the electromagnetic and sound waves in the anisotropic superfluid system of atoms with intrinsic dipole moments is theoretically studied. Both an intrinsic dipole moment and a dipole moment induced by the electric field is taken into account. The permittivity of the condensate of atoms is calculated. It is shown that a BEC of atoms with dipole moments is a medium with the frequency and spatial dispersion. Since particles possess intrinsic dipole moments, there arises a relationship between the electromagnetic
and sound waves in such a medium, which is particularly enhanced at a frequency close to the resonance frequency of an atom. The propagation of the sound waves in the condensate is accompanied by the oscillations of the intensity of the electric field, and while the electromagnetic wave is propagating there arise the oscillations of the condensate. It proves out that taking into account of the dipole-dipole interaction is essential for the description of the hybridization effect. The influence of the dissipative processes on the propagation of waves in a BEC of particles with a dipole moment is considered.

II. GROSS-PITAEVSKII EQUATION FOR ATOMS WITH A DIPOLE MOMENT IN THE ELECTROMAGNETIC FIELD WITH ACCOUNT OF THE DISSIPATION AND DIPOLE-DIPOLE INTERACTION

The dynamical GP equation for the macroscopic wave function of the condensate $\tilde{\psi} = \psi(r, t)$, interacting with the alternating external field $U(r, t)$ [1,2], can be obtained by means of the Lagrange formalism [17,18], if the Lagrange function is chosen in the form

$$L = \int dr \left[ \frac{i}{2} \left( \psi^*(r) \dot{\psi}(r) - \dot{\psi}^*(r) \psi(r) \right) - \frac{\hbar^2}{2m} \nabla \psi(r)^2 - U(r, t) |\psi(r)|^2 \right] - \frac{1}{2} \int drd\!r' U(r, r') |\psi(r)|^2 |\psi(r')|^2. \quad (1)$$

For brevity the designation of dependence of the wave function on time is omitted. The potential of the interparticle interaction can be presented in the form

$$U(r, r') = g\delta(r - r') + U_D(r, r'), \quad (2)$$

where the first term accounts for the short-range interaction, and the second term

$$U_D(r, r') = \frac{d(r) \cdot d(r') - 3 \left[ x \cdot d(r) \right] \left[ x \cdot d(r') \right]}{R^3} \quad (3)$$

– the long-range dipole-dipole interaction, and in $R = r' - r$, $x \equiv R/R$. The Lagrange function [11] leads to the equation

$$i\hbar \dot{\psi} = -\frac{\hbar^2}{2m} \Delta \psi + U(r, t) \psi + \psi \int U(r, r') |\psi(r', t)|^2 dr'. \quad (4)$$

The equation (4) is reversible in time, that is invariant with respect to the transformation $t \rightarrow -t, \psi \rightarrow \psi^*$, and describes the dynamics of the condensate without taking into account of possible dissipative processes. In the absence of the alternating external field the stationary solution of this equation has the form $\psi(r, t) = \tilde{\psi}_0(r) \exp \left[ -i \frac{\mu t}{\hbar} \right]$. In the nonstationary case, in the presence of the alternating external field, the function $\tilde{\psi}_0(r, t)$ depends on time and the equation (4) acquires the form

$$i\hbar \dot{\tilde{\psi}} = -\frac{\hbar^2}{2m} \Delta \tilde{\psi} + \left[ U(r, t) - \mu \right] \tilde{\psi} + \tilde{\psi} \int U(r, r') |\tilde{\psi}(r', t)|^2 dr', \quad (5)$$

where now $\tilde{\psi} \equiv \tilde{\psi}_0(r, t)$. In the following the macroscopic wave function is treated as precisely a function which does not depend on time in the stationary state, and the sign of tilde is omitted. The parameter $\mu$ has the meaning of the chemical potential and in the equilibrium state is related to the total number of particles by the equation

$$N = \int |\psi_0(r)|^2 dr. \quad (6)$$

The equation (5) entails the conservation law for the total number of particles in the condensate. Meanwhile, it is evident that under nonstationary processes the particles from the condensate can pass to the overcondensate quasiparticle states, so that the number of particles in the condensate will be no longer conserved. If at some moment of time the system resides in a nonstationary state, in which a part of the particles is in the one-particle condensate and the rest part of the particles forms a gas of quasiparticles, then such state will relax to the equilibrium state, and at zero temperature with time all the particles will pass into the condensate. The damping of oscillations of the atomic condensate in magnetic traps was observed experimentally [19] and turned out to be small. Dissipative
processes in the one-particle condensate can be taken into account phenomenologically within the Lagrange formalism by introducing the dissipative function \([17]\), which is usually chosen to be quadratic in the velocity \([20]\), so that \[
D = \hbar \gamma \dot{\psi}^* \dot{\psi},
\] (7)
and \(\gamma\) is a real phenomenological dimensionless dissipative coefficient. As a result, in view of \([11]\) and \([7]\), we come to the equation which differs from the equation \([5]\) only by the presence of the time derivative with a real coefficient
\[
i \hbar \dot{\psi} = -\frac{\hbar^2}{2m} \Delta \psi + \left[ U(\mathbf{r}, t) - \mu \right] \psi + \psi \int U(\mathbf{r}, \mathbf{r}') \left| \psi(\mathbf{r}', t) \right|^2 d\mathbf{r}' + \hbar \gamma \dot{\psi}.
\] (8)

Formally \([8]\) is obtained from \([5]\) by the substitution \(i \rightarrow i - \gamma\).

Let us make a remark as to accounting for the dissipative phenomena in this approach. The standard GP equation is applied for the description of the condensate at zero temperature. Dissipative processes are accompanied with the production of entropy and the heat release and, consequently, with the increase in temperature. In the approach under consideration these temperature effects are disregarded and it is supposed that, as in the standard case, the system is considered at zero temperature. Actually it means that the released heat is being removed from the system so fast that its temperature does not have time to change noticeably. Such approximate account of the dissipation is entirely equivalent to the account of friction in mechanics, where the transformation of the mechanical energy into heat is described by means of the dissipative function, but the notions of temperature and entropy are not used in this case.

It can be shown \([21]\) that the introduced dimensionless dissipative coefficient \(\gamma\) is connected to the third viscosity coefficient \(\zeta_3\) and the time of uniform relaxation of the particle number density in the condensate \(\tau_0\) by the relations
\[
\zeta_3 = \frac{\hbar \gamma}{2m^2 n_0}, \quad \tau_0 = \frac{\hbar}{2g \gamma n_0}.
\] (9)

Let us estimate the value of the coefficient \(\gamma\) under the assumption that relations \([9]\) are valid as well for the superfluid \(^4\text{He}\). For the helium third viscosity coefficient there is an estimation \([22]\)
\[
2 \cdot 10^{-3} \text{ cm}^5/\text{g}\cdot\text{s} < \zeta_3 < 1.6 \cdot 10^{-2} \text{ cm}^5/\text{g}\cdot\text{s}.
\] (10)

From here and \([9]\) it follows that
\[
4 < \gamma < 30.
\] (11)

In calculations for atomic condensates one often uses a less value \(\gamma \approx 0.01\) \([23]\).

The potential energy of interaction of the condensate atom with the electric field in the dipole approximation has the form
\[
U(\mathbf{r}, t) = -\mathbf{d} \cdot \mathbf{E},
\] (12)
where \(\mathbf{d}\) is the dipole moment of an atom, \(\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 + \mathbf{\delta E}(\mathbf{r}, t)\) is the electric field acting on an atom, which is a sum of the constant and alternating fields. As is known \([24]\), the local electric field acting on an atom in the dielectric differs from the external field, but for the diluted systems such as atomic condensates this difference is very small, it is small as well for the superfluid helium. Therefore we consider \(\mathbf{E}_0 = \mathbf{E}_0 e\) as the constant external field directed along the unit vector \(\mathbf{e}\). The alternating part of the field arises during the propagation of the electromagnetic wave.

The dipole moment of an atom is a sum of two terms:
\[
\mathbf{d} = \mathbf{d}_0 + \mathbf{d}_p.
\] (13)
Here \(\mathbf{d}_0\) is the intrinsic dipole moment of an atom which is directed along the external field in the equilibrium state. Its value \(d_0\) is considered constant. The second term is the dipole moment induced by the field
\[
\mathbf{d}_p \equiv \mathbf{d}_p(t) = \int_t^\infty \alpha(t-t') \mathbf{E}(t') dt' = \int_0^\infty \alpha(\tau) \mathbf{E}(t-\tau) d\tau,
\] (14)
where \(\alpha\) is the polarizability of a single atom. The intrinsic dipole moment, generally speaking, can change its direction, so that in general case \(\mathbf{d}_0(t) = d_0 e + \mathbf{\delta d}_0(t)\). Here the first term is the equilibrium dipole moment and the second term is the part depending on time. Since the value of the intrinsic dipole moment is constant, then at small deviations from equilibrium \(\mathbf{\delta d}_0(t) e = 0\). As can be seen below, by virtue of the last condition the fluctuations of the
intrinsic dipole moment fall out of the equations for the macroscopic wave function of the condensate. Hence in this approximation at small oscillations the intrinsic dipole moment can be considered constant both in magnitude and direction, so that in the following we assume \( d_0 = d_0 \mathbf{e} \).

The polarization part of the dipole moment equals \( \mathbf{d}_p(t) = \alpha_0 E_0 \mathbf{e} + \delta \mathbf{d}_p(t) \), where \( \alpha_0 \equiv \alpha(0) \) is the static polarization of an atom. Here the first term is the constant dipole moment induced by the constant field and the second term is the dipole moment induced by the alternating field

\[
\delta \mathbf{d}_p(t) = \int_0^\infty \alpha(\tau) \delta \mathbf{E}(t - \tau) d\tau.
\] (15)

Thus, the total dipole moment of an atom can be presented as a sum of the equilibrium and alternating parts \( \mathbf{d}(t) = d_s \mathbf{e} + \delta \mathbf{d}(t) \), where \( d_s = d_0 + \alpha_0 E_0 \). The index of the nonstationary dipole moment in (15) will be omitted in the following, so that \( \delta \mathbf{d}(t) \equiv \delta \mathbf{d}_p(t) \).

In the stationary equilibrium state the phase of the order parameter \( \psi_0 \) can be chosen so that it would be real, and from (15) it follows

\[
\psi_0^2 \equiv n_0 = \frac{\mu + d_s E_0}{g^*},
\] (16)

where \( g^* \equiv \int U(r, r') dr' \). In the spatially uniform equilibrium state the dipole-dipole interaction does not influence on the value of the equilibrium density, because its contribution becomes zero under integration over angles, so that \( g^* = g \). In view of (10), GP equation which accounts for the relaxation, the interaction with the electric field in the dipole approximation and the dipole-dipole interaction takes the form

\[
h(i - \gamma)\psi = -\frac{\hbar^2}{2m} \Delta \psi + (d_s E_0 - \mathbf{E} \cdot \mathbf{d}) \psi + g|\psi|^2 \psi + \psi \int U_D(r, r')|\psi(r', t)|^2 dr' .
\] (17)

The polarization vector

\[
\mathbf{P} = \mathbf{d} |\psi|^2
\] (18)

is also a sum of constant and alternating terms \( \mathbf{P} = \mathbf{P}_s + \delta \mathbf{P}(t) \). The constant polarization is a sum of the spontaneous polarization of the condensate of atoms with intrinsic dipole moments and the polarization under the action of the constant external field:

\[
\mathbf{P}_s = d_0 n_0 \mathbf{e} + \alpha_0 n_0 E_0 \mathbf{e} = d_s n_0 \mathbf{e},
\] (19)

and the nonstationary contribution into the total vector of polarization has the form

\[
\delta \mathbf{P} = \psi_0 d_s (\delta \psi + \delta \psi^*) \mathbf{e} + n_0 \int_0^\infty \alpha(\tau) \delta \mathbf{E}(t - \tau) d\tau .
\] (20)

Thus, the total electric displacement \( \mathbf{D} = \mathbf{E} + 4\pi \mathbf{P} \) is a sum of the constant displacement \( \mathbf{D}_s = 4\pi \mathbf{P}_s + \mathbf{E}_0 = \varepsilon_0 \mathbf{E}_0 + 4\pi d_0 n_0 \mathbf{e} \) and the displacement induced by the alternating field

\[
\mathbf{D} = \mathbf{D}_s + \delta \mathbf{D},
\] (21)

where \( \delta \mathbf{D} = \delta \mathbf{E} + 4\pi \delta \mathbf{P} \) is the contribution into the displacement that depends on time, \( \varepsilon_0 = 1 + 4\pi \alpha_0 n_0 \) is the static permittivity.

### III. PERMITTIVITY OF BEC ATOMS WITH A DIPOLE MOMENT

The relation (20) and equation (17) allow to obtain the permittivity of BEC atoms with a dipole moment as a function of the frequency and the wave vector. It should be noted that the de Broglie wavelength of the condensate atoms at low temperatures is large and greatly exceeds the mean interatomic distance, so that all atoms are collectivized and the many-particle system actually loses its discrete structure. For this reason BEC can be considered in a good approximation as a continuous medium, described by the macroscopic wave function even at high frequencies, when the wavelength of propagating oscillations is of the order or less than the mean distance between particles. Therefore we will use the equation (17) also for the description of the short sound waves at high frequencies comparable with the internal frequencies of an atom.

\[
\delta \varepsilon = 1 + 4 \pi \alpha_0 n_0 = 1 + 4 \pi \frac{\alpha_0}{\alpha(0)} n_0.
\]
Let us linearize the equation (17), taking into account the terms linear in the alternating field and the fluctuation of the dipole moment and assuming \( \delta \psi = \psi - \psi_0 \). Writing the dipole moment of an atom in the form \( d(r) = d_0 e + \delta d(r) \), we find accurate to the linear terms in the fluctuation of the dipole moment the expression for the dipole-dipole potential \( U_D(r, r') \approx U_D^{(0)} + U_D^{(1)}(r, r') \), where the contributions of the zero and first approximations have the form

\[
U_D^{(0)} = \frac{d_0^2}{R^3} \left[ 1 - 3(x \cdot e)^2 \right],
\]

\[
U_D^{(1)} = \frac{d_0^2}{R^3} \left[ (e \cdot \delta d(r)) \right] - \frac{3d_0^2}{R^3} (e \cdot x) \left[ x \cdot \delta d(r) + x \cdot \delta d(r') \right].
\]

In the linearized equation it is convenient to pass to the real quantities

\[
\delta \Psi = \tilde{\delta} \psi + \delta \psi^*, \quad \delta \Phi = \delta \psi - \delta \psi^*.
\]

If in the macroscopic wave function one singles out the modulus and the phase \( \psi = \rho e^{i\chi} \), assuming in the equilibrium \( \rho_0 = \psi_0 \) and \( \chi_0 = 0 \), then it turns out that the functions introduced in (23) are connected with the fluctuations of the modulus \( \delta \rho \) and the phase \( \delta \chi \) by the relations \( \delta \Psi = 2 \delta \rho, \delta \Phi = 2 \psi_0 \delta \chi \). The linearized system of equations for the real functions (23) takes the form

\[
\hbar \delta \Phi - \hbar \gamma \delta \Psi = -\frac{\hbar^2}{2m} \Delta \delta \Psi + 2g_0 \delta \Psi - 2 \psi_0 \left( \frac{d_0 e \cdot \delta E + E_0 e \cdot \delta d}{R^3} \right) + 2 \delta J_D,
\]

\[
\hbar \delta \Psi + \hbar \gamma \delta \Phi = \frac{\hbar^2}{2m} \Delta \delta \Phi,
\]

where

\[
\delta J_D = n_0 \psi_0 d_s \left\{ \frac{1}{R^3} \int \left[ \left( e \cdot \delta d(r) + e \cdot \delta d(r') \right) \right] dr' - 3 \int \left( e \cdot x \right) \left[ x \cdot \delta d(r) + x \cdot \delta d(r') \right] dr' \right\} +
\]

\[
+ n_0 d_s^2 \int \frac{1 - 3(x \cdot e)^2}{R^3} \delta \Psi(r') dr'.
\]

This system of equations describes in the linear approximation the dynamics of the superfluid system in the weak electric field. Let us assume that the alternating electric field varies according to the harmonic law \( \delta E = e^{iQ(r,t)} \), where \( Q(r,t) = kr - \omega t \), \( k \) is the wave vector, \( \omega \) is the frequency, \( b \) is the constant complex vector. The solutions of the system of equations (24) also have the form \( \delta \Psi(r,t) = \Psi_0 e^{iQ(r,t)}, \delta \Phi(r,t) = \Phi_0 e^{iQ(r,t)} \). Since the fluctuations of the oscillating quantities are chosen in the form of plane waves, then \( \delta d(r) = \delta d(r) e^{-ikr}, \delta \Psi(r) = \delta \Psi(r) e^{-ikr} \). In view of this we find

\[
\delta J_D = -\frac{4\pi d_s n_0}{3} \left( \delta_{ij} - 3s_i s_j \right) \left[ \psi_0 e_i \delta d_j(r) + d_s e_i e_j \delta \Psi(r) \right].
\]

As a result, we come to the system of linear equations

\[
- \hbar \omega \Phi_0 + (i \hbar \omega - \xi_k - 2g_0 + u) \Psi_0 = -2 \psi_0 \left[ d_s + \alpha(\omega) E_0 \right] e \cdot b + L \cdot b,
\]

\[
(i \hbar \omega - \xi_k) \Phi_0 + i \hbar \omega \Psi_0 = 0,
\]

where the quantities

\[
u = \frac{8\pi d_s^2 n_0}{3} \left[ 1 - 3(e \cdot s)^2 \right],
\]

\[
L = -2 \psi_0 \left[ \frac{4\pi d_s n_0}{3} \alpha(\omega) \right] e + 8 \pi \psi_0 n_0 d_s \alpha(\omega) (e \cdot s) s
\]

appeared due to taking into account of the dipole-dipole interaction, and \( \xi_k = \hbar^2 k^2/2m \). The solutions of the system of equations (27) have the form

\[
\Psi_0 = \frac{\xi_k - i \gamma \hbar \omega}{\nu D} \left\{ -2 \psi_0 \left[ d_s + \alpha(\omega) E_0 \right] b \cdot e + b \cdot L \right\},
\]

\[
\Phi_0 = \frac{i \hbar \omega}{D} \left\{ -2 \psi_0 \left[ d_s + \alpha(\omega) E_0 \right] b \cdot e + b \cdot L \right\},
\]
and here

\[ D \equiv D(\omega, k) \equiv (\hbar \omega)^2 - (i\hbar \omega \gamma - \xi_k)(i\hbar \omega \gamma - \xi_k - 2\gamma n_0 + u). \]  

(30)

Taking into account the relations (29), and that the amplitude of oscillations of the polarization vector (20) \( p = \psi_0 d_s \Psi_0 e_n n_0 \alpha(\omega) b \), we find the form of the polarization tensor of BEC with allowance for the dipole-dipole interaction

\[ \kappa_{ij}(\omega, k) = \kappa_0(\omega) \delta_{ij} + \kappa_{\perp}(\omega, k) e_i e_j + \kappa_d(\omega, k)(e \cdot s) e_i s_j, \]

(31)

where

\[ \kappa_0(\omega) = n_0 \alpha(\omega), \]

\[ \kappa_{\perp}(\omega, k) = -2n_0 d_s (\xi_k - i\hbar \omega \gamma) D^{-1} \left[ d_s + \alpha(\omega) E_0 + \frac{4\pi d_s n_0}{3} \alpha(\omega) \right], \]

\[ \kappa_d(\omega, k) = 8\pi d_s^2 n_0^2 \alpha(\omega)(\xi_k - i\hbar \omega \gamma) D^{-1}. \]

In neglect of the dipole-dipole interaction in the formula (31) we have to put \( \kappa_{\perp}(\omega, k) = -2n_0 d_s (\xi_k - i\hbar \omega \gamma) D^{-1} [d_s + \alpha(\omega) E_0] \) and \( \kappa_d(\omega, k) = 0 \). Thus, in this case the tensor of permittivity \( \varepsilon_{ij}(\omega, k) = \delta_{ij} + 4\pi \kappa_{ij}(\omega, k) \) can be presented in the form

\[ \varepsilon_{ij}(\omega, k) = \varepsilon_{ij}^s(\omega, k) + \varepsilon_{ij}^n(\omega, k), \]

(33)

where the symmetric and antisymmetric parts of the tensor are singled out:

\[ \varepsilon_{ij}^s(\omega, k) = \varepsilon_{ji}^s(\omega, k) = [1 + 4\pi \kappa_0(\omega)] \delta_{ij} + 4\pi \kappa_{\perp}(\omega, k) e_i e_j + 2\pi \kappa_d(\omega, k)(e \cdot s)(e_i s_j + e_j s_i), \]

\[ \varepsilon_{ij}^n(\omega, k) = -\varepsilon_{ji}^n(\omega, k) = 2\pi \kappa_d(\omega, k)(e \cdot s)(e_i s_j - e_j s_i). \]

(34)

As it is seen, the tensor of permittivity is a function of both the frequency and the wave vector, so that the medium possesses both the frequency and the spatial dispersion. The account of the dipole-dipole interaction results in that the tensor of permittivity ceases to be symmetric. Note that in the most general case the tensor of permittivity is not required to be either hermitian or symmetric [25]. The proof of its symmetry, based on the principle of symmetry of the kinetic coefficients [26], is not applicable in this case. In the presence of the antisymmetric additional term to the hermitian tensor of permittivity, for example in the magnetic field, the medium becomes optically active or gyrotropic [26]. The phase velocities of waves with the right and left circular polarization differ in the gyrotropic medium, which leads to the rotation of the plane of polarization of the linearly polarized wave. In our case in the absence of dissipation the tensor of permittivity proves to be real and unsymmetric. However, as will be seen, this does not lead to the effects of rotation of the plane of polarization.

A Fourier component of the polarizability of an atom \( \alpha(\omega) \equiv \int_0^\infty \alpha(\tau)e^{i\omega \tau} d\tau \) is often chosen in the model of the damped oscillator in the form

\[ \alpha(\omega) = \frac{N_\alpha e^2}{m_0 \omega_0^2 - \omega^2 - i\nu \omega}, \]

(35)

where \( N_\alpha \) is a number of electrons in an atom which give a contribution to the polarization, \( \nu \) is a phenomenological coefficient of damping, \( m_0 \) is the mass of an electron, \( \omega_0 \) is the characteristic resonance frequency of oscillations of an electron in an atom. In this model the static polarizability of an atom \( \alpha_0 \equiv \alpha(0) = N_\alpha e^2/m_0 \omega_0^2 \). The real and imaginary parts of the polarizability (35) \( \alpha(\omega) = \alpha'(\omega) + i\alpha''(\omega) \):

\[ \alpha'(\omega) = \alpha_0 \frac{\omega_0^2 (\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \nu^2 \omega^2}, \quad \alpha''(\omega) = \alpha_0 \frac{\nu \omega \omega_0^2}{(\omega_0^2 - \omega^2)^2 + \nu^2 \omega^2}. \]

(36)

Far from the resonant frequency \( \omega_0 \) the mutual influence of the sound waves in the condensate and the electromagnetic waves turns out to be small, so we pay the main attention to the propagation of waves near the resonant frequency. We are going to consider the propagation of electromagnetic waves of a certain frequency, assuming \( \omega \) to be real. Owing to the presence of damping the wave vector of the propagating wave is complex \( k = k' + ik'' \). Generally speaking, the real and imaginary parts of this vector can be directed differently. In the following we confine ourselves to considering the uniform wave, for which \( k' = (k' + ik'')s \), where \( s \) is the unit vector determining the direction of propagation of the wave.
IV. DISPERSION EQUATION FOR WAVES IN BEC

In neglect of the magnetic properties of the medium, from the Maxwell system of equations

\[
\text{div } \delta \mathbf{D} = 0, \quad \text{div } \delta \mathbf{B} = 0,
\]
\[
\text{rot } \delta \mathbf{B} = \frac{1}{c} \frac{\partial \delta \mathbf{D}}{\partial t}, \quad \text{rot } \delta \mathbf{E} = -\frac{1}{c} \frac{\partial \delta \mathbf{B}}{\partial t},
\]
there follows the system of linear equations describing the propagation of the electromagnetic waves in BEC

\[
\left[ k^2 (\delta_{ij} - s_is_j) - \frac{\omega^2}{c^2} \epsilon_{ij} \right] b_j = 0.
\]

For the dipole moment oriented along the vector \( \mathbf{e} = (0, 0, 1) \) and the direction of propagation of the wave \( \mathbf{s} = (\sin \theta, 0, \cos \theta) \), the tensor of permittivity has the following components

\[
\epsilon_{xx} = \epsilon_{yy} = \epsilon, \quad \epsilon_{zz} = \epsilon + 4\pi (\kappa_{\perp} + \kappa_d \cos^2 \theta),
\]
\[
\epsilon_{xy} = \epsilon_{yx} = \epsilon_{yz} = \epsilon_{zy} = 0,
\]
\[
\epsilon_{xx} = 2\pi \kappa_d \sin \theta \cos \theta, \quad \epsilon_{zz} = 0.
\]

Here \( \epsilon \equiv \epsilon(\omega) = 1 + 4\pi \kappa_0(\omega), \kappa_{\perp} \equiv \kappa_{\perp}(\omega, k), \kappa_d \equiv \kappa_d(\omega, k) \). As we see, \( \omega = 0 \) and \( \theta = \pi/2 \) the antisymmetric component of the tensor of permittivity becomes zero.

It proves more convenient to pass from the oscillations of the electric field intensity to the oscillations of the electric displacement \( \delta \mathbf{D}(r, t) = \mathbf{f} e^{i \omega t} \), so that \( f_i = \epsilon_{ij} b_j \). The reciprocal transformation is \( b_i = \eta_{ij} f_j \), where \( \eta_{ij} \equiv \epsilon_{ij}^{-1} \) is the inverse tensor of permittivity: \( \eta_{ik} \epsilon_{kj} = \delta_{ij} \). This tensor has the form

\[
\eta_{ij} \equiv \eta_{ij}(\omega, k) = \frac{1}{\epsilon} \left[ \delta_{ij} - 4\pi \frac{\kappa_{\perp} \epsilon_i \epsilon_j + \kappa_d (\mathbf{e} \cdot \mathbf{s}) \epsilon_i s_j}{\epsilon + 4\pi \kappa_{\perp} + 4\pi \kappa_d (\mathbf{e} \cdot \mathbf{s})} \right].
\]

The inverse tensor \( [40] \) is also unsymmetric and has the following components:

\[
\eta_{xx} = \eta_{yy} = \frac{1}{\epsilon}, \quad \eta_{zz} = \frac{1}{\epsilon + 4\pi \kappa_{\perp} + 4\pi \kappa_d \cos^2 \theta}, \quad \eta_{xy} = \eta_{yx} = \eta_{yz} = \eta_{zy} = 0,
\]
\[
\eta_{xx} = 0, \quad \eta_{zz} = -\frac{4\pi \kappa_d \sin \theta \cos \theta}{\epsilon (\epsilon + 4\pi \kappa_{\perp} + 4\pi \kappa_d \cos^2 \theta)}.
\]

The amplitudes of oscillations of the displacement satisfy the system of linear equations following from \( [38] \)

\[
\left( \omega^2 \theta_{ik} - k^2 \theta_{ik} \right) f_k = 0,
\]

where \( \theta_{ik} = (\delta_{ij} - s_is_j) \eta_{jk} \). Since \( s_i \theta_{ik} = 0 \), then from \( [42] \) it is obvious that \( \mathbf{s} \cdot \mathbf{f} = 0 \) and therefore the vector of the displacement lies in the plane perpendicular the direction of propagation of the wave. The components of the tensor \( \theta_{ik} \) have the form

\[
\theta_{xx} = \frac{\cos^2 \theta (\epsilon + 4\pi \kappa_{\perp} + 4\pi \kappa_d)}{\epsilon (\epsilon + 4\pi \kappa_{\perp} + 4\pi \kappa_d \cos^2 \theta)}, \quad \theta_{yy} = \frac{1}{\epsilon}, \quad \theta_{zz} = \frac{\sin^2 \theta}{(\epsilon + 4\pi \kappa_{\perp} + 4\pi \kappa_d \cos^2 \theta)},
\]
\[
\theta_{xy} = \theta_{yx} = \theta_{yz} = \theta_{zy} = 0,
\]
\[
\theta_{xx} = -\frac{\sin \theta \cos \theta}{(\epsilon + 4\pi \kappa_{\perp} + 4\pi \kappa_d \cos^2 \theta)}, \quad \theta_{zz} = -\frac{\sin \theta \cos \theta (\epsilon + 4\pi \kappa_{\perp} + 4\pi \kappa_d)}{\epsilon (\epsilon + 4\pi \kappa_{\perp} + 4\pi \kappa_d \cos^2 \theta)}.
\]

From the condition

\[
\det \left[ k^2 \theta_{ik} - \omega^2 \frac{\theta_{ik}}{c^2} \right] = 0
\]

we find the equation determining the dispersion laws for the propagating waves:

\[
\left( k^2 \theta_{yy} - \frac{\omega^2}{c^2} \right) \left( k^2 \theta_{xx} - \frac{\omega^2}{c^2} \right) \left( k^2 \theta_{zz} - \frac{\omega^2}{c^2} \right) - k^4 \theta_{xx} \theta_{zz} = 0.
\]
It follows from here that the wave, in which oscillates the component $f_y$ perpendicular to the plane containing the vectors $e$ and $s$, propagates with the dispersion law

$$\frac{\omega^2}{c^2} \varepsilon = k^2,$$

(46)

and oscillations of the condensate are absent in this wave. For the wave in which the displacement vector oscillates in the plane $(x, z)$, where the vectors $e$ and $s$ lie, the dispersion law, according to [35], is determined by the equation

$$\left(\frac{\omega^2}{c^2k^2} - \vartheta_{xx}\right) \left(\frac{\omega^2}{c^2k^2} - \vartheta_{zz}\right) - \vartheta_{xx}\vartheta_{zz} = 0. $$

(47)

Since, as can be seen from formulas [39], $\vartheta_{xx}\vartheta_{zz} = \vartheta_{xz}\vartheta_{zx}$, then from (47) apart from the trivial solution $\omega^2 = 0$ there follows the equation determining the dispersion law

$$\frac{\omega^2}{c^2k^2} \varepsilon = 1 - \frac{4\pi k_\perp \sin^2 \theta}{\varepsilon + 4\pi k_\perp + 4\pi k_d \cos^2 \theta}.$$

(48)

As seen, the nontrivial effects under propagation of waves arise if the wave propagates at the angle to the direction of the dipole moments, so in the following we consider the cases when $\theta \neq 0, \pi$. The relation between the components of the displacement vector in the plane $(x, z)$ is determined by the system of linear equations

$$\left(\frac{\omega^2}{c^2} - k^2\vartheta_{xx}\right)f_x - k^2\vartheta_{xz}f_z = 0,$$

$$k^2\vartheta_{xx}f_x - \left(\frac{\omega^2}{c^2} - k^2\vartheta_{zz}\right)f_z = 0.$$

(49)

Taking into account formulas [32], the equation (48), which determines the dispersion laws for the sound waves in BEC and the electromagnetic waves with account of their damping, can be written in the form

$$\left[k^2 - \frac{\omega^2}{c^2} \varepsilon(\omega)\right]\left\{(i\hbar\omega^2 - (i\hbar\omega + \xi_k)(i\hbar\omega - \xi_k - 2gn_0 + u))\right\} =$$

$$= (i\hbar\omega - \xi_k)\left\{8\pi\eta_0d_s [d_s + \alpha(\omega)E_0] \left(\frac{\omega^2}{c^2} - \cos^2 \theta \varepsilon(\omega)\right) +

+ 32\pi^2 d_s^2 n_0^2 \alpha(\omega) \left[\frac{\omega^2}{c^2} \left(\frac{1}{3} - \cos^2 \theta\right) + \frac{2}{3} \cos^2 \theta \varepsilon(\omega)\right]\right\}$$

(50)

where, as in [28], it is designated $u \equiv 8\pi d_s^2 n_0 (1/3 - \cos^2 \theta)$. In the following we consider the external electric field weak, assuming $E_0 \to 0$, so that $d_s = d_0$. Actually, in the problem under consideration the external field is needed only to single out a direction along which the dipoles are oriented. The account for the influence of the external static field on the propagation of waves in the condensate is also of interest and needs a separate research. Neglecting the quantity $u$ in the left side of equation (50) and the second term in the curly brackets in the right side of this equation, we obtain the equation which determines the dispersion laws of waves without account of the dipole-dipole interaction.

It is convenient to represent the obtained dispersion equation in the dimensionless form. It is natural to define the two characteristic wave numbers and the dimensionless frequency:

$$k_1 \equiv \frac{\omega_0}{c}, \quad k_2 \equiv \sqrt{\frac{2m_0 \omega_0}{\hbar}}, \quad x \equiv \frac{\omega}{\omega_0}.$$

(51)

Then in the zero constant electric field $E_0 = 0$ the equation (50) takes the form

$$\left[\left(\frac{k}{k_1}\right)^2 - x^2 \varepsilon'(x) - ix^2 \varepsilon''(x)\right]\left[\left(\frac{k}{k_2}\right)^2 - \beta_-(x) - i\gamma x\right]\left[\left(\frac{k}{k_2}\right)^2 + \beta_+(x) - i\gamma x\right] = -\eta_B \Omega(k^2, x),$$

(52)

where

$$\Omega(k^2, x) \equiv \left[ix\gamma - \left(\frac{k}{k_2}\right)^2\right]\left[x^2 \left[1 + (\varepsilon(x) - 1)f_D(\theta)\right] + \frac{5}{3} \cos^2 \theta \left(\frac{k}{k_1}\right)^2 \left[\frac{2}{5} - \frac{1}{\varepsilon(x)}\right]\right].$$

(53)
Here the designation for the function $f_D(\theta) \equiv 1/3 - \cos^2 \theta$ is introduced, being typical for the systems with the dipole-dipole interaction, and also the designation for the function of the dimensionless frequency

$$\beta_{\pm}(x) \equiv \sqrt{\frac{\phi^2}{4} + x^2 \pm \frac{\phi}{2}},$$

(54)

where $\phi \equiv \phi(\theta) = 2v - \eta_B f_D(\theta)$, the dimensionless parameter $v \equiv gn_0/\hbar \omega_0$ is determined by the constant $g$ of the short-range interaction of atoms of the condensate. Also (52) contains the dimensionless constant which characterizes the strength of the interaction of the condensate of particles with a dipole moment with the electric field:

$$\eta_B \equiv \frac{8\pi d_B^3 n_0}{\hbar \omega_0}.$$  

(55)

In (52) the real and imaginary parts of the permittivity, depending on the dimensionless frequency, are given by the formulas

$$\varepsilon'(x) = 1 + (\varepsilon_0 - 1) \frac{(1 - x^2)}{(1 - x^2)^2 + \nu^2 x^2}, \quad \varepsilon''(x) = (\varepsilon_0 - 1) \frac{\nu x}{(1 - x^2)^2 + \nu^2 x^2},$$

(56)

where $\nu \equiv \nu/\omega_0$. In the presence of damping the wave number is complex $k = k' + ik''$.

Let us give numerical estimates of the introduced quantities. Assuming $m \approx 10^{-23}$ g, $\omega_0 \approx 10^{15} \text{Hz}$, we have $k_1 \sim 10^4 \pm 10^5 \text{cm}^{-1}$ and $k_2 \sim 10^6 \pm 10^7 \text{cm}^{-1}$, so that $k_2/k_1 \sim 10^5$. Since the Bogolyubov speed of sound in the condensate $c_B$ is determined by the formula $c_B^2 = g n_0/\hbar$, then $\nu = m c_B^2/\hbar \omega_0$. In the atomic condensates $c_B \approx 1 \text{sm/s}$ [27,28], so that $\nu \sim 10^{-11}$. If we take $c_B \approx 10^4 \text{sm/s}$ – the order of magnitude of the first speed of sound in helium, then $\nu \sim 10^{-3}$. Let us give an estimate for the value of the parameter (55). The dipole moment of an atom can be written in the form $d_0 = ad_D$, where $d_D = 1D = 10^{-18} \text{cm}^{5/2}\text{g}^{1/2} \text{s}^{-1}$ is a typical value of the dipole moment for the polar molecules, $a$ is a dimensionless constant. In the atomic condensates the characteristic density $n_0 \approx 10^{14} \text{cm}^{-3}$, so that $\eta_B \sim 10^{-5}a^2$. In this case this parameter is small, even if the dipole moment of an atom is of the order of the dipole moment of the polar molecule, when $a \sim 1$. For the particle number density close to the density in the liquid helium $n_0 \approx 10^{22} \text{cm}^{-3}$ we obtain $\eta_B \sim 10^{-1}a^2$. In works [12,13] there were given arguments in favor of that at some temperature somewhat higher than the temperature of the superfluid transition, the liquid helium transforms into the polarized state. In this case atoms of the helium in the medium acquire the intrinsic dipole moment. A rough estimate in [12,13] gave the value of this moment which proved to be by 5 – 6 orders of magnitude less than the characteristic dipole moment of the polar molecules, so that the parameter $a \sim 10^{-6} \pm 10^{-5}$, which gives $\eta_B \sim 10^{-13} \pm 10^{-11}$. Thus, we can always consider $\eta_B \ll 1$.

V. SOLUTION OF DISPERSION EQUATION FOR WAVES IN BEC

In this section we obtain the solution of the dispersion equation (52), taking into account a small value of the parameter $\eta_B$. In view of the fact that in general case the wave number is complex $k = k' + ik''$, let us introduce the following notations:

$$z^2 = \frac{(k'^2 - k''^2)}{k_1 k_2}, \quad y^2 = \frac{k' k''}{k_1 k_2}, \quad \kappa = \frac{k_1}{k_2},$$

(57)

$$\left(\frac{k}{k_1}\right)^2 = \frac{1}{\kappa} \left(\frac{\varepsilon'}{\nu} \pm 2i y^2\right), \quad \left(\frac{k}{k_2}\right)^2 = \kappa \left(z^2 + 2i y^2\right).$$

The real and imaginary parts of the wave number are expressed through the parameters $z$ and $y$ by the relations:

$$k'^2/k_1 k_2 = \frac{1}{2} \left[\sqrt{z^4 + 4y^4 + z^2}\right], \quad k''^2/k_1 k_2 = \frac{1}{2} \left[\sqrt{z^4 + 4y^4 - z^2}\right].$$

(58)

In terms of variables (54) the original equation (52) takes the form

$$\left[\left(\frac{z^2}{\kappa} - x^2 \varepsilon'(x) + i \left(\frac{2y^2}{\kappa} - x^2 \varepsilon''(x)\right)\right)\left(\kappa z^2 - \beta_-(x)\right) + i(2\kappa y^2 - \gamma x)\right] \times$$

$$\times \left[\left(\kappa z^2 + \beta_+(x)\right) + i(2\kappa y^2 - \gamma x)\right] = -\eta_B \Omega(z^2, y^2, x),$$

(59)
where now

\[
\Omega(z^2, y^2, x) \equiv \left[ i\omega \gamma - \kappa (z^2 + 2iy^2) \right] \left\{ x^2 \left[ 1 + (\varepsilon(x) - 1) f_D(\theta) \right] + \frac{5}{3\kappa} \cosh^2 \theta (z^2 + 2iy^2) \left[ \frac{2}{5} - \frac{1}{\varepsilon(x)} \right] \right\}. \tag{60}
\]

Since the damping is assumed to be sufficiently weak and the parameter \(\eta_B\) is small, then the dissipative terms in function (60) are neglected, so that it takes the form

\[
\Omega(z^2, x) = -\kappa z^2 \left\{ x^2 \left[ 1 + (\varepsilon' - 1) f_D(\theta) \right] + \frac{5}{3\kappa} z^2 \cosh^2 \theta \left[ \frac{2}{5} - \frac{1}{\varepsilon'(x)} \right] \right\}. \tag{61}
\]

In the left part of equation (61) the terms quadratic in the dissipative coefficients in the real part are neglected, and the imaginary part, due to the fact that the right part is real, is equated to zero. As a result we come to the equation

\[
\left( \frac{z^2}{\kappa} - x^2 \varepsilon'(x) \right) \left( \kappa z^2 - \beta_-(x) \right) \left( \kappa z^2 + \beta_+(x) \right) = -\eta_B \Omega(z^2, x), \tag{62}
\]

and the function describing the damping of waves is expressed through the solutions of equation (62):

\[
y^2(x) = \frac{\kappa x^2 \varepsilon'(x) \left( \kappa z^2 - \beta_-(x) \right) \left( \kappa z^2 + \beta_+(x) \right)}{2 \left( \kappa z^2 - \beta_-(x) \right) \left( \kappa z^2 + \beta_+(x) \right) + \gamma x \left( \kappa z^2 - \kappa x^2 \varepsilon'(x) \right) \left( 2\kappa z^2 + \phi \right)}. \tag{63}
\]

In neglect of damping and the right side equation (62) has the three solutions:

\[
z_{em}^2 = \kappa x^2 \varepsilon'(x), \quad z_s^2 = \frac{\beta_-(x)}{\kappa}, \quad z_0^2 = \frac{\beta_+(x)}{\kappa}. \tag{64}
\]

The first two solutions \(z_{em}^2, z_s^2\) describe, respectively, the propagating electromagnetic and sound waves. The third solution \(z_0^2\) does not correspond to the propagating waves and will not be considered.

Equation (62) allows to obtain the corrections to the solutions \(z_{em}^2, z_s^2\), caused by the mutual influence of the sound and electromagnetic waves. The solutions of (62) are sought in the form \(z^2 = z_{em}^2 + \xi_{em}\) or \(z^2 = z_s^2 + \xi_s\), where \(\xi_{em} \ll z_{em}^2\) and \(\xi_s \ll z_s^2\). Then the corrections to the spectrum of the electromagnetic and sound waves will be determined by equations

\[
\xi_{em}^2 \left( \frac{\zeta_+ + \zeta_-}{\zeta_+ - \zeta_-} \right) \xi_{em} - \eta_B \frac{\Theta_{em}(x)}{\zeta_+ - \zeta_-} = 0, \tag{65}
\]

\[
\xi_s^2 \left( \frac{\zeta_+ - \zeta_-}{\zeta_+ - 2\zeta_-} \right) \xi_s - \eta_B \frac{\Theta_s(x)}{\zeta_+ - 2\zeta_-} = 0, \tag{66}
\]

where the designations are introduced

\[
\zeta_- \equiv \zeta_-(x) \equiv \kappa x^2 \varepsilon'(x) - \frac{\beta_-(x)}{\kappa}, \quad \zeta_+ \equiv \zeta_+(x) \equiv \kappa x^2 \varepsilon'(x) + \frac{\beta_+(x)}{\kappa}, \tag{67}
\]

and also

\[
\Theta_{em}(x) \equiv -\kappa^{-1} \Omega(z_{em}^2, x) = \frac{1}{3} \kappa x^4 \varepsilon'(x) \left( \varepsilon'(x) + 2 \right) \sin^2 \theta, \tag{68}
\]

\[
\Theta_s(x) \equiv -\kappa^{-1} \Omega(z_s^2, x) = \frac{\beta_-(x)}{\kappa} \left\{ x^2 \left( \frac{\varepsilon'(x) + 2}{3} + \cosh^2 \theta \left( \frac{\beta_-(x)}{\kappa^2} \left[ \frac{2}{3} \frac{5}{3} \varepsilon'(x) - \frac{1}{\varepsilon'(x)} \right] - x^2 \left( \varepsilon'(x) - 1 \right) \right) \right\}. \tag{69}
\]

Despite the assumed small value of the corrections \(\xi_{em}\) and \(\xi_s\), as will be seen, it is not sufficient to restrict ourselves to the linear in these quantities terms in equations (65) and (66). First we consider separately the case of propagation of waves without damping, and then take into account the influence of the dissipative effects.
VI. ELECTROMAGNETIC AND SOUND WAVES IN BEC WITHOUT ACCOUNT OF DAMPING

The dispersion dependencies of the electromagnetic \( z_{em}^2(x) \) (curve \( A_1OC_2 \)) and sound \( z_s^2(x) \) (curve \( A_2OC_1 \)) waves without account of their hybridization are presented graphically in Fig. 1. These curves always intersect at some frequency \( x_* < 1 \). At this frequency the function \( \zeta^2(x_*) = \kappa x^2 \varepsilon'(x_*) - \kappa^{-1} \beta_-(x_*) = 0 \), and thus near \( x_* \) in the linear approximation the required small corrections tend to infinity, so that it becomes necessary to take into account the quadratic terms in (65) and (66). Neglecting the damping, when \( \gamma = \nu = 0 \), the corrections to the dispersion laws of waves \( z_{em}^2(x_*) \), \( z_s^2(x_*) \) are determined by the equations (65) and (66). The permittivity is real in this case

\[
\varepsilon'(x) \equiv \varepsilon(x) = 1 + \frac{\varepsilon_0 - 1}{1 - x^2},
\]

and tends to infinity at approaching from the side of low frequencies to the resonance frequency \( \omega_0 \). In the frequency range \( \omega_0 < \omega < \sqrt{\varepsilon_0\omega_0} \) the permittivity is negative, and electromagnetic waves cannot propagate in such medium. At \( \omega > \sqrt{\varepsilon_0\omega_0} \) a possibility of propagating of the electromagnetic waves appears again, but we are not going to consider them in this high-frequency region. The coordinates of the point of intersection of the dispersion curves \( x_*, k_* \) in neglect of hybridization (point \( O \) in Fig. 1) are determined by the equations

\[
\frac{\beta_-(x_*)}{x^2 \varepsilon(x_*)} = \kappa^2 = \left( \frac{k_1}{k_2} \right)^2, \quad k^2_* = k_1^2 x^2 \varepsilon(x_*) = k_2^2 \beta_-(x_*).
\]

In the point of intersection

\[
\Theta_{em}(x_*) = \Theta_s(x_*) \equiv \Theta(x_*) = \frac{\kappa^2 x^4 \varepsilon(x_*) \left( \varepsilon(x_*) + 2 \right)}{3 \left( \beta_-(x_*) + \beta_+(x_*) \right)} \sin^2 \theta.
\]

The account of the relation between the electromagnetic and sound waves results in that at the frequency \( x_* = \omega_1/\omega_0 \) there appear the jumps in the both branches determined by the equations (65) and (66). The magnitudes of these jumps are equal for both branches, but have the different signs. For the electromagnetic wave:

\[
k^2_{em}(x_* + 0) = k_1 k_2 \sqrt{\eta_B \Theta(x_*)}, \quad k^2_{em}(x_* - 0) = -k_1 k_2 \sqrt{\eta_B \Theta(x_*)},
\]

and for the sound wave:

\[
k^2_s(x_* + 0) = -k_1 k_2 \sqrt{\eta_B \Theta(x_*)}, \quad k^2_s(x_* - 0) = k_1 k_2 \sqrt{\eta_B \Theta(x_*)}.
\]

Due to hybridization the two continuous dispersion curves appear \( (A_1B_1C_1) \) and \( (A_2B_2C_2) \) in Fig. 1).

In the waves, propagation of which is described by these curves, generally speaking, both the density of the condensate and the electromagnetic field oscillate, but the amplitudes of the oscillations prove to be comparable only near the frequency \( x_* = \omega_1/\omega_0 \). The region \( A_1B_1 \) of the first curve mainly corresponds to the electromagnetic waves, which in the region \( B_1C_1 \) transform into the sound waves. The region \( A_2B_2 \) of the second curve mainly corresponds to the sound waves, which in the region \( B_2C_2 \) transform into the electromagnetic waves. Note that Fig. 1 and other figures in the paper are given with an illustrative purpose, so the parameters at which the calculations are done are chosen from the illustrative considerations and can essentially differ from the parameters of real systems.

It should be pointed out the important role of the dipole-dipole interaction near the point of intersection of curves. Without account of the dipole-dipole interaction the function (61) has the form

\[
\hat{\Omega}(z^2, x) = -\kappa^2 \left\{ x^2 - \frac{x^2}{\varepsilon(x)} \cos^2 \theta \right\}.
\]

In contrast to the function (61), which accounts for the dipole-dipole interaction, the function (75) contains only the inverse permittivity giving a small contribution but does not contain the permittivity in the first power which is large near the resonant frequency. In the point of intersection of spectrums \( x_* \) the function (75) takes the value \( \hat{\Omega}(z_s^2, x_*) = -\kappa^2 x^4 \varepsilon(x_*) \sin^2 \theta \), and the function (61), accounting for the dipole-dipole interaction, is \( \Omega(z_s^2, x_*) = (-1/3)\kappa^2 x^4 \varepsilon(x_*) \left( \varepsilon(x_*) + 2 \right) \sin^2 \theta \). Their ratio is

\[
\frac{\hat{\Omega}(z_s^2, x_*)}{\Omega(z_s^2, x_*)} = \frac{3}{\varepsilon(x_*) + 2}
\]

small, since \( \varepsilon(x_*) \gg 1 \). Thus, taking into account of the dipole-dipole interaction leads to a substantial increase of the “repulsion” of the branches of the spectrum.
Figure 1: Dispersion curves without account of damping. The curve $A_1OC_2$ is the electromagnetic wave $k'^2/k_1^2 = x^2\varepsilon'(x)$, the curve $A_2OC_1$ is the sound wave $k'^2/k_1^2 = \beta_-(x)/k_1^2$. The dispersion curves $A_1B_1C_1$ and $A_2B_2C_2$ arise as a result of hybridization. The calculation is carried out with an illustrative purpose with the parameters: $\kappa \equiv k_1/k_2 = \sqrt{0.1}, \phi = 10^{-3}, \varepsilon_0 = 2, \theta = \pi/2, \eta_B = 0.1, \gamma = 0.1, \tilde{\nu} = 0$.

The relation between the amplitudes of oscillations of the BEC density $\delta n = \psi_0 \delta \psi = \tilde{n} e^{iQ(r,t)}$ ($\tilde{n} \equiv \psi_0 \Psi_0$ is the amplitude of oscillations of the density) and the electromagnetic field for the curves which arose due to hybridization can be obtained from the equations \text{\cite{27}}. The dependencies of the ratio $R \equiv |(\tilde{n}/n_0)/(|\tilde{b} \cdot L_1|)$ (where $\tilde{b}$ is the dimensionless amplitude of oscillations of the electric field, and $L_1 \equiv -3/(\varepsilon(\omega) + 2) e + (\varepsilon(\omega) - 1)(e \cdot s) s$ for the branches $A_1B_1C_1$ and $A_2B_2C_2$ are shown in Fig. 2. As seen, the most strong coupling between the electromagnetic and sound oscillations arises near the frequency $x^*$, where the sound wave is transformed into the electromagnetic wave and vice versa.

Figure 2: The ratio of the amplitude of oscillations of the BEC density to the amplitude of oscillations of the electric field. The curves $A_1B_1C_1$ and $A_2B_2C_2$ correspond to the respective branches in Fig. 1. The calculation is carried out with the parameters: $\kappa = \sqrt{0.1}, \phi = 10^{-3}, \varepsilon_0 = 2, \theta = \pi/2, \eta_B = 0.1, \tilde{\nu} = 0$. 
In the long-wave limit one obtains the following dispersion law of the sound waves

$$\omega^2 = \left(1 - \frac{\hbar \omega_0}{2mc_B^2} \eta_B f_D(\theta)\right)c_B^2 k^2,$$

(77)

so that the speed of a sound wave in BEC with account of the dipole-dipole interaction is given by the relation

$$c_D^2 = \left(1 - \frac{\hbar \omega_0}{2mc_B^2} \eta_B f_D(\theta)\right)c_B^2.$$

(78)

At the angle $\theta_0$, determined by formula $\cos^2 \theta_0 = 1/3$, the function $f_D(\theta_0) = 0$. In the angle intervals $0 < \theta < \theta_0$ and $\pi - \theta_0 < \theta < \pi$ the function $f_D(\theta)$ is negative, so that here the dipole-dipole interaction leads to an increase of the speed of sound waves in comparison with the Bogolyubov speed. In the angle interval $\theta_0 < \theta < \pi$ the speed decreases reaching its minimal value $c_{D,min}^2 = c_B^2(1 - \eta_B \hbar \omega_0 / 6mc_B^2)$ at $\theta = \pi/2$. Formula (78) coincides with the formula for the speed of propagation of sound in a system of Bose particles without an intrinsic dipole moment, which is placed in a constant electric field $E_0$, when the dipole moment $d_0 = \alpha_0 E_0$ is induced by this field [29].

VII. INFLUENCE OF DISSIPATION ON PROPAGATION OF WAVES IN BEC

When the dissipation is taken into account, the real part of the permittivity $\varepsilon'(x)$ is determined by formula (56). Its value remains finite near the resonance frequency $x_0$. Thus, there are two possibilities: a) for a small damping rate of the electromagnetic wave, just as in the non-dissipative case, the sound and electromagnetic dispersion curves intersect in neglect of hybridization; b) the dispersion curve of the electromagnetic wave lies below the sound branch, so that the intersection of branches is absent. Let us consider both possibilities.

When one accounts for the damping, the points of the possible intersection of the electromagnetic and sound dispersion curves are determined by the solutions of the equation

$$\varepsilon'(x_\ast) - \frac{\beta_-(x_\ast)}{\kappa^2 x_\ast^2} = 0,$$

(79)

which has the solutions if the condition is satisfied

$$\tilde{\nu} < \frac{(\varepsilon_0 - 1)}{2}\kappa^2, \quad \nu < \frac{(\varepsilon_0 - 1)}{\omega_0} \frac{\hbar \omega_0}{4mc^2}.$$

(80)

\[ k'^2/k_1^2, \quad k''^2/k_1^2 \]

Figure 3: The curve $A_2B_2C_2D_2E_2$ ($k'^2/k_1^2$) corresponds to the branch $A_2B_2C_2$ for the non-dissipation case in Fig. 1. The curve $A'_2B'_2C'_2D'_2E'_2$ ($k''^2/k_1^2$) is the damping on this branch. The calculation is carried out with the parameters: $\kappa = \sqrt{0.1}, \phi = 10^{-3}, \varepsilon_0 = 2, \theta = \pi/2, \eta_B = 0.1, \gamma = 0.1, \nu = 3 \cdot 10^{-2}$. 
The equation \( (79) \) under the condition \( (80) \) has two solutions. The smaller root \( x_{s-} \) corresponds to the intersection of the sound wave with the electromagnetic wave in the region of normal dispersion, and the greater root \( x_{s+} \) in the region of anomalous dispersion. The dependencies of the squares of the real and imaginary parts of the wave numbers on the frequency for two branches, calculated by formulas \( (58) \), \( (63) \), \( (65) \), \( (66) \), are shown in Figs. 3, 4. As seen, accounting for dissipation results in that both dispersion curves become nonmonotonous and in the region of anomalous dispersion the damping of waves substantially increases.

If the condition \( (80) \) is not fulfilled, then the equation \( (79) \) does not have a solution and the sound and electromagnetic branches do not intersect. The dispersion curves and the damping curve for the electromagnetic wave in

\[
\frac{k'^2}{k_1^2}, \frac{k''^2}{k_1^2}
\]

Figure 5: Dispersion curves for the sound and electromagnetic waves in the absence of intersection of spectrums. The curve \( A_1C_1E_1 \left(\frac{k'^2}{k_1^2}\right) \) is the dispersion law of the electromagnetic waves, and the curve \( A'_1C'_1E'_1 \left(\frac{k''^2}{k_1^2}\right) \) is their damping. The curve \( A_2C_2E_2 \left(\frac{k'^2}{k_1^2}\right) \) is the dispersion law of the sound waves, which damping is small for the chosen parameters. The calculation is carried out with the parameters: \( \kappa = \sqrt{0.1}, \phi = 10^{-3}, \varepsilon_0 = 2, \theta = \pi/2, \eta_B = 0.1, \gamma = 0.1, \tilde{\nu} = 7 \cdot 10^{-2} \).
this case are shown in Fig. 5. Here the mutual influence of the sound and electromagnetic oscillations manifests itself in the appearance of maximums in the points $C_1$ and $C_2$ on the dispersion curves $A_1E_1C_1$ and $A_2E_2C_2$. Thus, even in the absence of intersection of the electromagnetic and sound dispersion curves, near the resonance frequency there exists their considerable “interaction”.

VIII. CONCLUSION

In the paper it is theoretically studied the propagation of the electromagnetic waves and sound oscillations in a Bose-Einstein condensate of particles with an intrinsic dipole moment. At that also the electric polarization of atoms under the action of the constant electric field was taken into consideration. Bose-Einstein condensate is described by the modified Gross-Pitaevskii equation which accounts for the relaxation effects by means of introducing a phenomenological dissipative coefficient connected to the third viscosity coefficient and the time of uniform relaxation of the particle number density in the condensate. The interaction of an atom of the condensate with the electric field is accounted for in the dipole approximation. The medium under study is anisotropic, because there is a preferential direction, along which the dipole moments of atoms are oriented. The tensor of permittivity of BEC, in which atoms interact by means of the short-range forces and the dipole-dipole interaction, is obtained. It is shown that such medium possesses both the time and the spatial dispersion.

In the absence of dissipation or at a small dissipation the dispersion curves of the electromagnetic and sound waves in neglect of the interaction intersect at a frequency which is somewhat less than the resonance frequency of an atom. Since atoms possess an intrinsic dipole moment, there arises hybridization of the electromagnetic and sound branches near the frequency of intersection of the dispersion curves. Here the sound wave transforms into the electromagnetic wave and vice versa, and the oscillations of the condensate density are accompanied with the oscillations of the electric field. It is obtained that taking into account of the dipole-dipole interaction leads to a substantial increase of the “repulsion” of the branches of the excitation spectrum. It is shown that the account for the dissipation conditioned by the relaxation of the macroscopic wave function of the condensate and by the imaginary part of the permittivity, leads to the appearance of regions with the anomalous dispersion.

In this work we considered hybridization of the sound and electromagnetic waves at high frequencies close to the resonance frequency of an atom. In the experiment [10] there was found a resonant absorption of the microwave radiation in the superfluid helium at a frequency close to 180 GHz, which in [21] was interpreted as an indication of the existence of the elementary excitations with the energy gap, which exist together with the sound excitations. Note also that the branch of excitations with the energy gap can exist, for example, in the condensate of atoms and their two-atom bound states [30]. Taking into account of the pair correlations in a Bose system also leads to the appearance of a new branch of excitations with the energy gap [31]. In the presence of excitations with a gap the dispersion curve of these excitations always intersects the dispersion curve of the electromagnetic waves and there arises a phenomenon of the strong hybridization of spectrums and the resonant absorption of radiation at more low frequencies [21]. These effects also can be studied by the same method that was used in this work.
