Effect of resonance for $CP$ asymmetry of the decay process $\bar{B}_s \rightarrow P\pi^+\pi^-$ in perturbative QCD

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In the framework of Perturbative QCD (PQCD) approach we study the direct $CP$ asymmetry for the decay channel $\bar{B}_s \rightarrow P\pi^+\pi^-$ around the resonance range via the $\rho - \omega$ mixing mechanism (where $P$ refer to pseudoscalar meson). We find that the $CP$ asymmetry can be enhanced by $\rho - \omega$ mixing when the masses of the $\pi^+\pi^-$ pairs are at the area of $\rho - \omega$ resonance, and the maximum $CP$ asymmetry can reach 59% for the relevant decay channels.

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I. INTRODUCTION

The rich data from $B$ meson factories make the study of $B$ physics a very hot topic. A lot of research has been made, especially for $CP$ asymmetry. $CP$ asymmetry is an important area in test of the Standard Model (SM) and searching new physics signals. The detection of Cabibbo-Kobayashi-Maskawa (CKM) matrix elements play an important role in understanding of $CP$ asymmetry. The nonleptonic decay of $B$ meson is expected to be ideal decay process in searching $CP$ asymmetry. Direct $CP$ asymmetry in $B$ meson decay channel arises from weak phase and strong phase differences. In SM, the weak phase is responsible for the $CP$ asymmetry by CKM matrix $^[1, 2]$. Meanwhile, the large strong phase is needed for producing $CP$ asymmetry which comes from QCD correction. Recently, the large $CP$ asymmetry was found by the LHCb Collaboration in the three-body decay channels of $B^\pm \rightarrow \pi^\mp\pi^+\pi^-$ and $B^\pm \rightarrow K^\mp\pi^+\pi^-$. Hence, more attention about $CP$ asymmetry has been focused on the three body decay channels of $B$ meson.

Direct $CP$ asymmetry arises from the weak phase difference and the strong phase difference. The weak phase difference is determined by the CKM matrix elements, while the strong phase can be produced by the hadronic matrix and interference between intermediate states. The vacuum polarisation of photon are described by coupling the vector meson in the vector meson dominance (VMD) model. The strength of coupling of the $\omega$ meson to the photon is weak comparing with the $\rho$ meson $^[4]$. However, the strong interaction enhances the $\pi^+\pi^-$ pair production amplitudes in the $\rho$ and $\omega$ resonance region. $\rho - \omega$ interference presents the large contribution for the process of
$e^+e^- \rightarrow \pi^+\pi^-$ due to the isospin-breaking effects. Since the strong phase exist, the $\rho$ and $\omega$ interference can affect the direct CP asymmetry and present the sizeable contribution.

The direct CP asymmetry is discussed via $\rho-\omega$ interference in $B$ decays by the the naive factorization approach. But the method bases on the assumption of no strong rescattering, and can not predict direct CP asymmetry effectively. Recently, the CP asymmetry of charmless three-body B-decay is presented in the leading term of QCD factorization by model dependent approach, where focus on the local CP asymmetry. The direct CP asymmetry of the quasi-two-body decay of $B \rightarrow P\rho \rightarrow P\pi\pi$ is calculated in perturbative QCD approach, where does not taking into account the resonance effects. In our opinion, $B \rightarrow P\pi\pi$ have effectively three contributions around the $\rho$ resonance: (a) $B \rightarrow P\rho \rightarrow P\pi\pi$, (b)$B \rightarrow P\omega \rightarrow P\rho \rightarrow P\pi\pi$, and (c)$B \rightarrow P\omega \rightarrow P\pi\pi$. Roughly speaking, the amplitudes of their contributions: $a > b > c$. We have absorbed (c) into (b) effectively, which is just the (effective) $\rho-\omega$ mixing parameter: $\tilde{\Pi}_{\rho\omega}$.

The hadronic matrix elements can be calculated by the factorization approach introducing the strong phase. Adding the QCD corrections, the different dynamic methods are given based on the leading power of $1/m_b$ ($m_b$ is b quark mass). The non-leptonic weak decay amplitudes of $B$ mesons can be calculated by the perturbative QCD (PQCD) approach taking into account transverse momenta. In the PQCD approach, the hard interaction consisting of six quark operator dominates the decay amplitude from short distance. The nonperturbative dynamics are included in the meson wave function which can be extracted from experiment. Finally, we obtain new large strong phases by the phenomenological mechanism of $\rho-\omega$ mixing and the dynamics of the PQCD approach. The large $CP$ asymmetry may be obtained by the resonant region due to the strong phase.

The remainder of this paper is organized as follows. In Sec. II we present the form of the effective Hamiltonian. In Sec. III we give the calculating formalism of $CP$ asymmetry from $\rho-\omega$ mixing in $\bar{B}_s \rightarrow P\pi^+\pi^-$. Input parameters are presented in Sec. IV. We present the numerical results in Sec. V. Summary and discussion are included in Sec. VII. The related function defined in the text are given in the Appendix.

II. THE EFFECTIVE HAMILTONIAN

Based on the expansion of the operator product, the effective weak Hamiltonian can be written as

$$\mathcal{H}_{\Delta B=1} = \frac{G_F}{\sqrt{2}} \left[ V_{ub}V_{ud}^* (c_1 O_1^u + c_2 O_2^u) - V_{tb}V_{td}^* \sum_{i=3}^{10} c_i O_i \right] + H.C. ,$$

(1)

where $G_F$ represents Fermi constant, $c_i$ ($i=1,...,10$) are the Wilson coefficients, $V_{ub}$, $V_{ud}$, $V_{tb}$ and $V_{td}$ are the CKM matrix elements. The operators $O_i$ have the following forms:

$$O_1^u = \bar{d}_\alpha \gamma_\mu (1 - \gamma_5) u_\beta \bar{u}_\beta \gamma_\mu (1 - \gamma_5) b_\alpha,$$

$$O_2^u = \bar{d}_\gamma (1 - \gamma_5) u_\mu \gamma_\mu (1 - \gamma_5) b,$$
\[ O_3 = \bar{d}\gamma(1-\gamma_5)b\sum_{q'} q'^\mu(1-\gamma_5)q', \]
\[ O_4 = \bar{d}_\alpha\gamma(1-\gamma_5)b\bar{q}_\beta\gamma^\mu(1-\gamma_5)q', \]
\[ O_5 = \bar{d}\gamma(1-\gamma_5)b\sum_{q'} q'^\mu(1+\gamma_5)q', \]
\[ O_6 = \bar{d}_\alpha\gamma(1-\gamma_5)b\bar{q}_\beta\gamma^\mu(1+\gamma_5)q', \]
\[ O_7 = \frac{3}{2}\bar{d}\gamma(1-\gamma_5)b\sum_{q'} e_q\bar{q}'^\mu(1+\gamma_5)q', \]
\[ O_8 = \frac{3}{2}\bar{d}_\alpha\gamma(1-\gamma_5)b\bar{q}_\beta\gamma^\mu(1+\gamma_5)q', \]
\[ O_9 = \frac{3}{2}\bar{d}\gamma(1-\gamma_5)b\sum_{q'} e_q\bar{q}'^\mu(1-\gamma_5)q', \]
\[ O_{10} = \frac{3}{2}\bar{d}_\alpha\gamma(1-\gamma_5)b\bar{q}_\beta\gamma^\mu(1-\gamma_5)q', \]
\[
(2)
\]
where \(\alpha\) and \(\beta\) are color indices, and \(q' = u, d\) or \(s\) quarks. In Eq. (2) \(O^u_1\) and \(O^u_2\) are tree operators, \(O_3-6\) are QCD penguin operators, and \(O_7-10\) are the operators associated with electroweak penguin diagrams.

we can obtain numerical values of \(c_i\). When \(c_i(m_b)\) [11],
\[ c_1 = -0.2703, \quad c_2 = 1.1188, \]
\[ c_3 = 0.0126, \quad c_4 = -0.0270, \]
\[ c_5 = 0.0085, \quad c_6 = -0.0326, \]
\[ c_7 = 0.0011, \quad c_8 = 0.0004, \]
\[ c_9 = -0.0090, \quad c_{10} = 0.0022. \]
\[
(3)
\]
One can obtain numerical values of \(a_i\) including Wilson coefficients and the color index \(N_c\) [9]:
\[ a_1 = C_2 + C_1/N_c, \quad a_2 = C_1 + C_2/N_c, \]
\[ a_3 = C_3 + C_4/N_c, \quad a_4 = C_4 + C_3/N_c, \]
\[ a_5 = C_5 + C_6/N_c, \quad a_6 = C_6 + C_5/N_c, \]
\[ a_7 = C_7 + C_8/N_c, \quad a_8 = C_8 + C_7/N_c, \]
\[ a_9 = C_9 + C_{10}/N_c, \quad a_{10} = C_{10} + C_9/N_c. \]
\[
(4)
\]
III. CP ASYMMETRY IN $B_s^0 \rightarrow \rho^0(\omega)P \rightarrow \pi^+\pi^-P$

A. Formalism

In the vector meson dominance model (VMD), photons are dressed by coupling to the vector mesons. Based on the same mechanism, $\rho - \omega$ mixing was proposed and later gradually applied to B meson physics [13, 20]. Due to the effective Hamiltonian, the amplitude $A$ for the decay process of $\bar{B}_s^0 \rightarrow \pi^+\pi^-P$ ($B_s^0 \rightarrow \pi^+\pi^-\bar{P}$) can be written as [13]:

$$A = \langle \pi^+\pi^-P|H^T|\bar{B}_s^0 \rangle + \langle \pi^+\pi^-P|H^P|\bar{B}_s^0 \rangle,$$

$$\bar{A} = \langle \pi^+\pi^-P|H^T|\bar{B}_s^0 \rangle + \langle \pi^+\pi^-P|H^P|B_s^0 \rangle,$$

with $H^T$ and $H^P$ are the Hamiltonian of the tree and penguin operators, respectively.

The relative amplitudes and phases of $H^T$ and $H^P$ can be expressed as follows [13]:

$$A = \langle \pi^+\pi^-P|H^T|\bar{B}_s^0 \rangle[1 + re^{i(\delta + \phi)}],$$

$$\bar{A} = \langle \pi^+\pi^-P|H^T|\bar{B}_s^0 \rangle[1 + re^{i(\delta - \phi)}],$$

with $\delta$ and $\phi$ are strong and weak phases, respectively. $\phi$ is the weak phase in the CKM matrix that causes the CP asymmetry, which is arg[$V_{ub}\bar{V}_{ub}^\ast/(V_{ub}\bar{V}_{ub}^\ast)](q = d, s)$. The parameter $r$ represents the absolute value of the ratio of penguin and tree amplitudes:

$$r \equiv \left| \frac{\langle \pi^+\pi^-P|H^P|\bar{B}_s^0 \rangle}{\langle \pi^+\pi^-P|H^T|\bar{B}_s^0 \rangle} \right|.$$  

The $CP$ violating asymmetry, $A_{CP}$, can be written as

$$A_{CP} = \frac{|A|^2 - |\bar{A}|^2}{|A|^2 + |\bar{A}|^2} = \frac{-2rsin\delta sin\phi}{1 + 2rcos\delta cos\phi + r^2}.$$  

From Equation (10), it can be seen that the $CP$ asymmetry depends on the weak phase difference and the strong phase difference. The weak phase is determined for a particular decay process. Hence, in order to obtain a large $CP$ asymmetry, we need some mechanism to increase $\delta$. It has been found that $\rho - \omega$ mixing can lead to a large strong phase difference [4, 14–20]. Based on $\rho - \omega$ mixing and working to the first order of isospin violation, we have the following results [13]:

$$\langle \pi^+\pi^-P|H^T|\bar{B}_s^0 \rangle = \frac{g_\rho}{s_\rho s_\omega} \bar{t}_\rho t_\omega + \frac{g_\rho}{s_\rho} t_\rho,$$

$$\langle \pi^+\pi^-P|H^P|\bar{B}_s^0 \rangle = \frac{g_\rho}{s_\rho s_\omega} \bar{t}_\omega p_\omega + \frac{g_\rho}{s_\rho} p_\rho,$$  

where $t_\rho(p_\rho)$ and $t_\omega(p_\omega)$ are the tree (penguin) amplitudes for $\bar{B}_s^0 \rightarrow \rho^0P$ and $\bar{B}_s^0 \rightarrow \omega P$, respectively; $g_\rho$ is the
coupling constant of $\rho^0 \to \pi^+\pi^-$ decay process; $\tilde{\Pi}_{\rho\omega}$ is the effective $\rho - \omega$ mixing amplitude which also effectively absorbed into the direct coupling $\omega \to \pi^+\pi^-$. $s_V$, $m_V$ and $\Gamma_V (V=\rho$ or $\omega$) represent the inverse propagator, mass and decay rate of the vector meson $V$, respectively.

$$s_V = s - m_V^2 + im_V \Gamma_V,$$  \hspace{1cm} (13)

where $\sqrt{s}$ denotes the invariant mass of the $\pi^+\pi^-$ pairs \[13\].

The $\rho - \omega$ mixing parameters were recently determined precisely by Wolfe and Maltman \[21, 22\],

$$\Re \tilde{\Pi}_{\rho\omega}(m^2_\rho) = -4470 \pm 250_{\text{model}} \pm 160_{\text{data}} \text{MeV}^2,$$

$$\Im \tilde{\Pi}_{\rho\omega}(m^2_\rho) = -5800 \pm 2000_{\text{model}} \pm 1100_{\text{data}} \text{MeV}^2$$  \hspace{1cm} (14)

One can find that the mixing parameter is the momentum dependence including the non-resonant contribution that absorbs the direct decay $\omega \to \pi^+\pi^-$. We introduce the momentum dependence of the mixing parameter $\tilde{\Pi}_{\rho\omega}(s)$ for $\rho - \omega$ mixing, which leads to the explicit $s$ dependence. It is reasonable to devote one's energies to search the mixing contribution at the region of $\omega$ mass where the two pions can be produced. We write $\tilde{\Pi}_{\rho\omega}(s) = \Re \tilde{\Pi}_{\rho\omega}(m^2_\omega) + \Im \tilde{\Pi}_{\rho\omega}(m^2_\omega)$, and update the values as follows \[23\]:

$$\Re \tilde{\Pi}_{\rho\omega}(m^2_\omega) = -4760 \pm 440 \text{MeV}^2,$$

$$\Im \tilde{\Pi}_{\rho\omega}(m^2_\omega) = -6180 \pm 3300 \text{MeV}^2.$$  \hspace{1cm} (15)

In fact, the contribution of the $s$ dependence of $\tilde{\Pi}_{\rho\omega}$ is negligible. We can make the expansion $\tilde{\Pi}_{\rho\omega}(s) = \tilde{\Pi}_{\rho\omega}(m^2_\omega) + (s - m_\omega)\tilde{\Pi}_{\rho\omega}'(m^2_\omega)$. From Eqs. \[13\] \[11\] \[11\] \[12\] one has

$$r e^{i\delta} e^{i\phi} = \frac{\tilde{\Pi}_{\rho\omega} p_\omega + s_\omega p_\rho}{\tilde{\Pi}_{\rho\omega} t_\omega + s_\omega t_\rho}.$$  \hspace{1cm} (16)

Defining

$$\frac{p_\omega}{t_\rho} = r' e^{i(\delta + \phi)} , \quad \frac{t_\omega}{t_\rho} = \alpha e^{i\delta_\alpha} , \quad \frac{p_\rho}{p_\omega} = \beta e^{i\delta_\beta} ,$$  \hspace{1cm} (17)

with $\delta_\alpha$, $\delta_\beta$ and $\delta_q$ are strong phases. It is available from Eqs. \[13\] \[17\]:

$$r e^{i\delta} = r' e^{i\delta} \frac{\tilde{\Pi}_{\rho\omega} + \beta e^{i\delta_\beta} s_\omega}{\tilde{\Pi}_{\rho\omega} \alpha e^{i\delta_\alpha} + s_\omega}.$$  \hspace{1cm} (18)

In order to obtain the $CP$ violating asymmetry in Eq. \[13\], $\sin \phi$ and $\cos \phi$ are necessary. The weak phase $\phi$ is fixed
by the CKM matrix elements. In the Wolfenstein parametrization \[24\], one has

\[
\begin{align*}
\sin \phi & = \frac{\eta}{\sqrt{\rho(1-\rho) - \eta^2}^2 + \eta^2}, \\
\cos \phi & = \frac{\rho(1-\rho) - \eta^2}{\sqrt{\rho(1-\rho) - \eta^2}^2 + \eta^2}.
\end{align*}
\] (19)

where the same result has been found for $b \to d$ transition from $\Lambda_b$ decay process \[14\].

IV. CALCULATION

For the simplification, we take the decay process of $\bar{B}_s^0 \to \rho^0(\omega)K^0 \to \pi^+\pi^-K^0$ as example for the study of the $\rho - \omega$ interference. The other decay channels can be obtained similarly. According to the Hamiltonian(1), based on CKM matrix elements of $V_{ub}V_{ud}^*, V_{tb}V_{td}^*$, the decay amplitude of $\bar{B}_s^0 \to \rho^0K^0$ in perturbation QCD approach can be written as

\[
\sqrt{2}M(\bar{B}_s^0 \to \rho^0K^0) = V_{ub}V_{ud}^* t_\rho - V_{tb}V_{td}^* p_\rho
\] (20)

where $t_\rho$ and $p_\rho$ refer to the tree and penguin contributions respectively. We write:

\[
t_\rho = f_\rho F_{B_s \to K}^{LL} [a_2] + M_{B_s \to K}^{LL} [C_2]
\] (21)

and

\[
p_\rho = f_\rho F_{B_s \to K}^{LL} \left[ -a_4 + \frac{3}{2} a_7 + \frac{1}{2} a_{10} + \frac{3}{2} a_9 \right] + M_{B_s \to K}^{LR} \left[ -C_5 + \frac{1}{2} C_7 \right] \\
+ M_{B_s \to K}^{LL} \left[ -C_3 + \frac{1}{2} C_9 + \frac{3}{2} C_{10} \right] - M_{B_s \to K}^{SP} \left[ \frac{3}{2} C_8 \right] + f_{B_s} F_{ann}^{LL} \left[ -a_4 + \frac{1}{2} a_{10} \right] \\
+ f_{B_s} F_{ann}^{SP} \left[ -a_6 + \frac{1}{2} a_8 \right] + M_{ann}^{LL} \left[ -C_3 + \frac{1}{2} C_9 \right] + M_{ann}^{LR} \left[ -C_5 + \frac{1}{2} C_7 \right]
\] (22)

The decay amplitude for $\bar{B}_s^0 \to \omega K^0$ can be written as

\[
\sqrt{2}M(\bar{B}_s^0 \to \omega K^0) = V_{ub}V_{ud}^* t_\omega - V_{tb}V_{td}^* p_\omega,
\] (23)

One can also present the contributions of $t_\omega$ and $p_\omega$ as well.

\[
t_\omega = f_\omega F_{B_s \to K}^{LL} [a_2] + M_{B_s \to K}^{LL} [C_2]
\] (24)
\[ p_\omega = f_\omega F_{B_s \rightarrow K}^{LL} \left[ 2a_3 + a_4 + 2a_5 + \frac{1}{2}a_7 + \frac{1}{2}a_9 - \frac{1}{2}a_{10} \right] \]
\[ + M_{B_s \rightarrow K}^{LL} \left[ C_3 + 2C_4 - \frac{1}{2}C_9 + \frac{1}{2}C_{10} + M_{B_s \rightarrow K}^{LR} \left[ C_5 - \frac{1}{2}C_7 \right] \right] \]
\[ - M_{B_s \rightarrow K}^{SP} \left[ 2C_6 + \frac{1}{2}C_8 + f_{B_s} F_{ann}^{LL} a_4 - \frac{1}{2}a_{10} \right] + f_{B_s} F_{ann}^{SP} a_6 - \frac{1}{2}a_8 \]
\[ + M_{ann}^{LL} \left[ C_3 - \frac{1}{2}C_9 + M_{ann}^{LR} \left[ C_5 - \frac{1}{2}C_7 \right] \right] \]  

(25)

The function \( F \) and \( M \) are given in Sec.IX. The index \( LL, LR \) and \( SP \) arise from the \((V - A)(V - A), (V - A)(V + A)\) and \((S - P)(S + P)\) operators, respectively.

\[ \alpha e^{i\delta_\alpha} = \frac{t_\omega}{t_p}, \]  
\[ \beta e^{i\delta_\beta} = \frac{p_\rho}{p_\omega}, \]  
\[ \gamma' e^{i\delta_q} = \frac{p_\omega}{t_p} \times \frac{V_{tb}V_{td}^*}{|V_{ub}V_{ud}|}, \]  

where

\[ \frac{|V_{tb}V_{td}^*|}{|V_{ub}V_{ud}|} = \sqrt{\frac{\rho(1 - \rho) - \eta^2/2 + \eta^2}{(1 - \lambda^2/2)(\rho^2 + \eta^2)}} \]  

(29)

From above equations, the new strong phases \( \delta_\alpha, \delta_\beta \) and \( \delta_q \) are introduced by the interference of \( \rho - \omega \) mesons. The strong phase \( \delta \) are obtained by the equations 17 and 18 in the framework of PQCD.

In a similar way, we can get the \( t_\rho, t_\omega, p_\rho, \) and \( p_\omega \) for the processes of \( B_s^0 \rightarrow \rho^0(\omega)\eta \) and \( B_s^0 \rightarrow \rho^0(\omega)\eta' \), respectively. The relevant \( CP \) asymmetry can also be produced in similar approach. In the calculation, \( \eta \) and \( \eta' \) mesons are introduced. The \( \eta \) and \( \eta' \) mixing depend on the quark flavor basis 23. The mesons are consisted of \( \bar{n}n = (\bar{u}d + \bar{d}u)/\sqrt{2} \) and \( \bar{s}s \):

\[ \begin{pmatrix} |\eta\rangle \\ |\eta'\rangle \end{pmatrix} = U(\phi) \begin{pmatrix} |\eta_n\rangle \\ |\eta_s\rangle \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} |\eta_n\rangle \\ |\eta_s\rangle \end{pmatrix} \]  

(30)

where the mixing angle \( \phi = 39.3^\circ \pm 1.0^\circ \). Explicitly, only two decay constants are needed is the advantage here:

\[ \langle 0|\bar{n}\gamma^\mu\gamma_5 n|\eta_n, P^\mu\rangle = \frac{i}{\sqrt{2}} f_n P^\mu, \]  
\[ \langle 0|\bar{s}\gamma^\mu\gamma_5 s|\eta_s, P^\mu\rangle = i f_s P^\mu. \]  

(31)  

(32)

We use 26

\[ f_n = 139.1 \pm 2.6 \text{MeV}, \quad f_s = 174.2 \pm 7.8 \text{MeV}. \]  

(33)

For the pure annihilation type decay process, one can also divides the amplitudes into \( t_\rho, t_\omega, p_\rho, \) and \( p_\omega \) depending on \( V_{ub}V_{us}^* \) and \( V_{tb}V_{ts}^* \). The amplitudes can be given as following for the channel \( B_s^0 \rightarrow \pi^0\rho^0(\omega) \): \( M(B_s^0 \rightarrow \rho^0\pi^0) = \)
\[ V_{ub}V_{us}^* t_\rho - V_{tb}V_{ts}^* p_\rho \text{ and } M(\bar{B}_s^0 \rightarrow \omega \pi^0) = V_{ub}V_{us}^* t_\omega - V_{tb}V_{ts}^* p_\omega. \]

V. INPUT PARAMETERS

The CKM matrix, which elements are determined from experiments, can be expressed in terms of the Wolfenstein parameters \(A, \rho, \lambda\) and \(\eta\) \[24\]:

\[
\begin{pmatrix}
1 - \frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\
-\lambda & 1 - \frac{1}{2}\lambda^2 & A\lambda^2 \\
A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1
\end{pmatrix},
\]

where \(O(\lambda^4)\) corrections are neglected. The latest values for the parameters in the CKM matrix are \[27\]:

\[
\lambda = 0.22506 \pm 0.00050, \quad A = 0.811 \pm 0.026, \\
\bar{\rho} = 0.124^{+0.019}_{-0.018}, \quad \bar{\eta} = 0.356 \pm 0.011.
\]

where

\[
\bar{\rho} = \rho(1 - \frac{\lambda^2}{2}), \quad \bar{\eta} = \eta(1 - \frac{\lambda^2}{2}).
\]

From Eqs. (35) (36) we have

\[
0.109 < \rho < 0.147, \quad 0.354 < \eta < 0.377.
\]

The other parameters are given as following \[24, 28, 29\]:

\[
\begin{align*}
f_\pi &= 0.131\text{GeV}, & f_K &= 0.160\text{GeV}, \\
m_{B_s^0} &= 5.36677\text{GeV}, & \tau_{B_s^0} &= 1.512 \times 10^{-12}\text{s} \\
m_{\rho(770)} &= 0.77526\text{GeV}, & \Gamma_{\rho(770)} &= 0.1491\text{GeV}, \\
m_{\omega(782)} &= 0.78265\text{GeV}, & \Gamma_{\omega(782)} &= 8.49 \times 10^{-3}\text{GeV}, \\
m_\pi &= 0.13957\text{GeV}, & m_W &= 80.385\text{GeV}, \\
f_\rho &= 209 \pm 2\text{MeV}, & f_\rho^T &= 165 \pm 9\text{MeV}, \\
f_\omega &= 195.1 \pm 3\text{MeV}, & f_\omega^T &= 145 \pm 10\text{MeV}.
\end{align*}
\]

VI. NUMERICAL RESULTS

In the numerical results, we find the \(CP\) asymmetry can be enhanced when the masses of the \(\pi^+\pi^-\) pairs are in the area around the \(\rho - \omega\) resonance, and the maximum \(CP\) asymmetry for our considering the decay channels can reach 59%. We also discuss the numerical results from the case of tree and penguin dominated type decay and the
FIG. 1: Plot of $A_{CP}$ as a function of $\sqrt{s}$ corresponding to central parameter values of CKM matrix elements. The dashed line, dash-dotted, solid line refer to the decay channels of $\bar{B}_s^0 \to \rho^0(\omega)K^0 \to \pi^+\pi^-K^0$, $\bar{B}_s^0 \to \rho^0(\omega)\eta' \to \pi^+\pi^-\eta'$ and $\bar{B}_s^0 \to \rho^0(\omega)\eta \to \pi^+\pi^-\eta$, respectively.

FIG. 2: Plot of $\sin\delta$ as a function of $\sqrt{s}$ corresponding to central parameter values of CKM matrix elements. The dashed line, dash-dotted, solid line refer to the decay channels of $\bar{B}_s^0 \to \rho^0(\omega)K^0 \to \pi^+\pi^-K^0$, $\bar{B}_s^0 \to \rho^0(\omega)\eta' \to \pi^+\pi^-\eta'$ and $\bar{B}_s^0 \to \rho^0(\omega)\eta \to \pi^+\pi^-\eta$, respectively.)

case of pure annihilation type decay in the framework of Perturbative QCD. The CP violation is associated with the CKM matrix elements and $\sqrt{s}$. In our numerical calculations, we find that the CP asymmetry depend weakly on the variation of the CKM matrix elements. Hence, we let $(\rho, \eta)$ vary between the central values $(\rho_{central}, \eta_{central})$.

A. The case of tree and penguin dominated type decay

We refer to the decay processes of $\bar{B}_s^0 \to \rho^0(\omega)K^0 \to \pi^+\pi^-K^0$, $\bar{B}_s^0 \to \rho^0(\omega)\eta \to \pi^+\pi^-\eta$ and $\bar{B}_s^0 \to \rho^0(\omega)\eta' \to \pi^+\pi^-\eta'$ as the case of tree and penguin dominated type decay. In Fig.1, we show the plot of CP asymmetry as a function of $\sqrt{s}$. One can find the CP asymmetry varies sharply when the masses of the $\pi^+\pi^-$ pairs are in the area around the $\rho-\omega$ resonance range. For the decay process of $\bar{B}_s^0 \to \rho^0(\omega)K^0 \to \pi^+\pi^-K^0$, the maximum CP asymmetry can reach 40%. For the decay channels of $\bar{B}_s^0 \to \rho^0(\omega)\eta' \to \pi^+\pi^-\eta'$ and $\bar{B}_s^0 \to \rho^0(\omega)\eta \to \pi^+\pi^-\eta$, we obtain the maximum CP asymmetry is 59% and 21%, respectively. From Equation (10), one can find the CP asymmetry is
FIG. 3: Plot of $r$ as a function of $\sqrt{s}$ corresponding to central parameter values of CKM matrix elements. The dashed line, dash-dotted, solid line refer to the decay channels of $\bar{B}_s^0 \to \rho^0(\omega)K^0 \to \pi^+\pi^-K^0$, $\bar{B}_s^0 \to \rho^0(\omega)\eta' \to \pi^+\pi^-\eta'$ and $\bar{B}_s^0 \to \rho^0(\omega)\eta \to \pi^+\pi^-\eta$, respectively.

FIG. 4: Plot of $A_{CP}$ as a function of $\sqrt{s}$ corresponding to central parameter values of CKM matrix elements for $\bar{B}_s^0 \to \rho^0(\omega)\pi^0 \to \pi^+\pi^-\pi^0$.

affected by the weak phase difference, the strong phase difference and $r$. The weak phase depends on the CKM matrix elements. Hence, the change of $CP$ asymmetry is derived from the variation of strong phase $\delta$ and $r$ except the CKM matrix. We take the central values from the parameters of $(\rho_{central}, \eta_{central})$. Taking into account of $\rho - \omega$ mixing, we can see that $\sin \delta$ oscillate considerably at the area of $\rho - \omega$ resonance from Fig.2 for the considering decay processes. The plot of $r$ as a function of $\sqrt{s}$ is presented in Fig.3. One can see that the $r$ change sharply for the process of $\bar{B}_s^0 \to \rho^0(\omega)\eta' \to \pi^+\pi^-\eta'$ and $\bar{B}_s^0 \to \rho^0(\omega)\eta \to \pi^+\pi^-\eta$.

B. The case of pure annihilation decay type

In Fig.4, we present the plot of CP asymmetry parameter as a function $\sqrt{s}$ corresponding to central parameter values of CKM matrix elements for the pure annihilation decay type of $\bar{B}_s^0 \to \rho^0(\omega)\pi^0 \to \pi^+\pi^-\pi^0$. One can find the maximum CP asymmetry reach 28% when the masses of the $\pi^+\pi^-$ pairs are in the area around the $\rho - \omega$ resonance range. The plots of $\sin \delta$ and $r$ as a function of $\sqrt{s}$ are given in Fig.5 and Fig.6, respectively. We can see that $\sin \delta$
FIG. 5: Plot of $\sin \delta$ as a function of $\sqrt{s}$ corresponding to central parameter values of CKM matrix elements for $\bar{B}_s^0 \rightarrow \rho^0(\omega) \pi^0 \rightarrow \pi^+ \pi^- \pi^0$.

FIG. 6: Plot of $r$ as a function of $\sqrt{s}$ corresponding to central parameter values of CKM matrix elements for $\bar{B}_s^0 \rightarrow \rho^0(\omega) \pi^0 \rightarrow \pi^+ \pi^- \pi^0$.

and $r$ oscillate sharply taking into account $\rho - \omega$ resonance. Generally, the $CP$ asymmetry is tiny in the case of pure annihilation decay. However, the maximum $CP$ asymmetry can reach 28% at the area of $\rho - \omega$ resonance, which give us a chance to search $CP$ asymmetry from the pure annihilation decay type.

VII. SUMMARY AND CONCLUSION

In this paper, we study the $CP$ asymmetry for the decay process of $\bar{B}_s \rightarrow P \pi^+ \pi^-$ in Perturbative QCD. It has been found the $CP$ asymmetry can be enhanced greatly at the area of $\rho - \omega$ resonance. The maximum $CP$ asymmetry can reach 40% for the process of $\bar{B}_s^0 \rightarrow \rho^0(\omega) K^0 \rightarrow \pi^+ \pi^- K^0$. However, the paper has also discussed the $CP$ asymmetry of the decay process of $\bar{B}_s \rightarrow \rho^0(\omega) K^0 \rightarrow \pi^+ \pi^- K^0$ from $b \rightarrow d$ transition in QCD factorization. The maximum $CP$ asymmetry reach 46% when the invariant mass of the $\pi^+ \pi^-$ pair is in the vicinity of the $\omega$ resonance from QCD factorization [19]. The difference of $CP$ asymmetry mainly comes from the strong phase difference between QCD factorization and Perturbative QCD. The hadronic matrix elements can be calculated from first principles in the decays
of B-meson. Due to the power expansion of $1/m_b$ ($m_b$ is b quark mass), all of the theories of factorization are shown to deal with the hadronic matrix elements in the leading power of $1/m_b$. But these methods are different significantly due to the collinear degree or transverse momenta. The power counting is different from the hard kernels between QCDF and PQCD. It is important to extract the strong phase difference for CP violation. The more different feature of QCDF and PQCD is the strong interaction scale at which of PQCD is low, typically of order $1\sim 2$ GeV, the case of QCDF is order $O(m_b)$ for the Wilson coefficients.

Meanwhile, we find that the CP asymmetry associated with the case of pure annihilation type decay process of $\bar{B}_s^0 \rightarrow \rho^0(\omega)\pi^0 \rightarrow \pi^+\pi^-\pi^0$ can be enhanced and the maximum value reach 28%. Hence, one can search for the large CP asymmetry at the area of $\rho - \omega$ resonance from pure annihilation type decay process of $\bar{B}_s^0 \rightarrow \rho^0(\omega)\pi^0 \rightarrow \pi^+\pi^-\pi^0$.

In this work, we have take the Perturbative QCD approximation which add the QCD correction to the naive factorization which is based on the power expansion of $1/m_b$. The final state interaction is also neglected in this approximation which may give some uncertainties. There are some uncertainties from the input parameters, the hard scattering scale and CKM matrix elements. The theoretical results can be improved by high order correction from $\alpha_s$ and $1/m_b$.

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IX. APPENDIX: RELATED FUNCTIONS DEFINED IN THE TEXT

The functions associated with the tree and penguin contributions are presented for the factorization and non-factorization amplitudes in PQCD approach $^{10,11,30}$. The functions of the case of tree and penguin dominated type decay are written as

- $(V - A)(V - A)$ operators:
  \[
  f_{M_2} F_{B_s \rightarrow M_3}^{LL} (a_i) = \frac{8\pi C_F M_{B_s}^4}{4\pi f_{M_2}} \int_0^1 dx_1 dx_3 \int_0^\infty b_1 db_1 b_3 db_3 \phi_{B_s}(x_1, b_1) \left\{ a_i(t_a) E_e(t_a) 
  \times \left[ (1 + x_3) \phi_3^2(x_3) + r_3(1 - 2x_3)(\phi_3^F(x_3) + \phi_3^T(x_3)) \right] h_e(x_1, x_3, b_1, b_3) 
  + 2r_3 \phi_3^P(x_3) a_i(t'_a) E_e(t'_a) h_e(x_3, x_1, b_3, b_1) \right\},
  \]

- $(V - A)(V + A)$ operators:
  \[
  F_{B_s \rightarrow M_3}^{LR} (a_i) = -F_{B_s \rightarrow M_3}^{LL} (a_i),
  \]
\[ f_{M_2} F_{B_2 \rightarrow M_3}^{S P}(a_i) = 16 \pi r_2 C F M_{B_2}^4 \int_0^1 dx_1 dx_3 \int_0^\infty b_1 db_1 b_2 b_2 b_3 \phi_{B_2}(x_1, b_1) \left\{ a_i(t_b) E_e(t_a) \times \left[ \phi_3^A(x_3) + r_3(2 + x_3) \phi_3^P(x_3) - r_3 x_3 \phi_3^T(x_3) \right] h_e(x_1, x_3, b_3) + 2r_3 \phi_3^P(x_3) a_i(t'_b) E_e(t'_a) h_e(x_3, x_1, b_3, b_1) \right\}, \] (41)

- \( (V - A)(V - A) \) operators:

\[ M_{B_2 \rightarrow M_3}^{L L}(a_i) = 32 \pi C F M_{B_2}^4 / \sqrt{6} \int_0^1 dx_1 dx_2 dx_3 \int_0^\infty b_1 db_1 b_2 dB \phi_{B_2}(x_1, b_1) \left\{ a_i(t_b) E_e(t_b) \times \left[ (1 - x_2) \phi_3^A(x_3) - r_3 x_3 (\phi_3^P(x_3) - \phi_3^T(x_3)) \right] a_i(t_b) E_e(t_b) + h_n(x_1, 1 - x_2, x_3, b_1, b_2) + h_n(x_1, x_2, x_3, b_1, b_2) \times \left[ -(x_2 + x_3) \phi_3^A(x_3) + r_3 x_3 (\phi_3^P(x_3) + \phi_3^T(x_3)) \right] a_i(t_b) E_e(t_b) \right\}, \] (42)

- \( (V - A)(V + A) \) operators:

\[ M_{B_2 \rightarrow M_3}^{L R}(a_i) = 32 \pi C F M_{B_2}^4 / \sqrt{6} \int_0^1 dx_1 dx_2 dx_3 \int_0^\infty b_1 db_1 b_2 b_2 b_2 b_2 \phi_{B_2}(x_1, b_1) \left\{ a_i(t_b) E_e(t_b) \times \left[ h_n(x_1, 1 - x_2, x_3, b_1, b_2) \left[ (1 - x_2) \phi_3^A(x_3) (\phi_3^P(x_2) + \phi_3^T(x_2)) + r_3 x_3 (\phi_3^P(x_3) - \phi_3^T(x_3)) \right] a_i(t_b) E_e(t_b) + (1 - x_2) r_3 (\phi_3^P(x_2) + \phi_3^T(x_2)) (\phi_3^P(x_3) - \phi_3^T(x_3)) a_i(t_b) E_e(t_b) \right] \times \left[ x_2 \phi_3^A(x_3) (\phi_3^P(x_2) - \phi_3^T(x_2)) \right] a_i(t_b) E_e(t_b) + (1 - x_2) r_3 (\phi_3^P(x_2) + \phi_3^T(x_2)) (\phi_3^P(x_3) - \phi_3^T(x_3)) \right] a_i(t_b) E_e(t_b) \times \left[ x_2 \phi_3^A(x_3) (\phi_3^P(x_2) + \phi_3^T(x_3)) \right] a_i(t_b) E_e(t_b) \right\}, \] (43)

- \( (S - P)(S + P) \) operators:

\[ M_{B_2 \rightarrow M_3}^{S P}(a_i) = 32 \pi C F M_{B_2}^4 / \sqrt{6} \int_0^1 dx_1 dx_2 dx_3 \int_0^\infty b_1 db_1 b_2 b_2 b_2 b_2 \phi_{B_2}(x_1, b_1) \phi_3^A(x_2) \left\{ a_i(t_b) E_e(t_b) h_n(x_1, 1 - x_2, x_3, b_1, b_2) + a_i(t_b) E_e(t_b) \times \left[ x_2 \phi_3^A(x_3) + r_3 x_3 (\phi_3^P(x_3) - \phi_3^T(x_3)) \right] h_n(x_1, x_2, x_3, b_1, b_2) \right\}. \] (44)

The functions are associated with the annihilation type process as following:
\( (V - A)(V - A) \) operators:

\[
f_{B, F_{ann}^{LL}}(a_i) = 8\pi C_F M_B^4 f_B \int_0^1 dx_2 dx_3 \int_0^{\infty} b_2 db_2 b_3 db_3 \left\{ a_i(t_c) E_a(t_c) \times \right. \\
\left. \left[ (x_3 - 1)\phi_3^A(x_2)\phi_3^A(x_3) - 4r_2 r_3 \phi_3^P(x_2)\phi_3^P(x_3) + 2r_2 r_3 \phi_3^P(x_2)\phi_3^P(x_3) - \phi_3^T(x_3) \right] h_a(x_3, 1 - x_3, b_2, b_3) + \\
\left[ x_2 \phi_3^A(x_2)\phi_3^A(x_3) + 2r_2 r_3 (\phi_3^P(x_2) - \phi_3^T(x_2))\phi_3^P(x_3) + 2r_2 r_3 (\phi_3^P(x_2) + \phi_3^T(x_2))\phi_3^P(x_3) \right] a_i(t_c') E_a(t_c') h_a(1 - x_3, x_2, b_3, b_2) \right\}. \tag{45} \]

\( (V - A)(V + A) \) operators:

\[
F_{ann}^{LR}(a_i) = F_{ann}^{LL}(a_i). \tag{46} \]

\( (S - P)(S + P) \) operators:

\[
f_{B, F_{ann}^{SP}}(a_i) = 16\pi C_F M_B^4 f_B \int_0^1 dx_2 dx_3 \int_0^{\infty} b_2 db_2 b_3 db_3 \left\{ 2r_2 \phi_3^P(x_2)\phi_3^A(x_3) + (1 - x_3) r_3 \phi_3^A(x_2) (\phi_3^P(x_3) + \phi_3^T(x_3)) a_i(t_c) E_a(t_c) h_a(x_3, 1 - x_3, b_2, b_3) + \\
2r_3 \phi_3^A(x_2) \phi_3^A(x_3) + r_2 r_3 (\phi_3^P(x_2) - \phi_3^T(x_2))\phi_3^P(x_3) \right\} a_i(t_c') E_a(t_c') h_a(1 - x_3, x_2, b_3, b_2) \right\}. \tag{47} \]

\( (V - A)(V - A) \) operators:

\[
M_{ann}^{LL}(a_i) = 32\pi C_F M_B^4 / \sqrt{6} \int_0^1 dx_1 dx_2 dx_3 \int_0^{\infty} b_1 db_2 b_2 db_2 \phi_B(x_1, b_1) \left\{ h_{na}(x_1, x_2, x_3, b_1, b_2) \left[ - x_2 \phi_3^P(x_2)\phi_3^A(x_3) - 4r_2 r_3 \phi_3^P(x_2)\phi_3^P(x_3) + r_2 r_3 (1 - x_2) (\phi_3^P(x_2) + \phi_3^T(x_2))(\phi_3^P(x_3) - \phi_3^T(x_3)) + r_2 r_3 x_2 (\phi_3^P(x_2) - \phi_3^T(x_2))(\phi_3^P(x_3) + \phi_3^T(x_3)) \right] a_i(t_d) E_a(t_d) + \\
h_{na}'(x_1, x_2, x_3, b_1, b_2) \left[ (1 - x_3) \phi_3^A(x_2)\phi_3^A(x_3) + (1 - x_3) r_2 r_3 (\phi_3^P(x_2) + \phi_3^T(x_2))(\phi_3^P(x_3) - \phi_3^T(x_3)) + x_2 r_2 r_3 (\phi_3^P(x_2) - \phi_3^T(x_2))(\phi_3^P(x_3) + \phi_3^T(x_3)) \right] a_i(t_d') E_a(t_d') \right\}. \tag{48} \]
\[ M^{LR}_{\text{ann}}(M_2, M_3, a_i) = 32\pi C_F f^4 M_B \sqrt{6} \int_0^1 dx_1 dx_2 dx_3 \int_b^\infty b_1 db_1 b_2 db_2 \phi_B \left( x_1, b_1 \right) \] 
\[ \times \left\{ h_{n_a}(x_1, x_2, x_3, b_1, b_2) \left[ r_2(2 - x_2)(\phi_2^P(x_2) + \phi_2^T(x_2)) \phi_3^A(x_3) \right. \right. \] 
\[ - \left. \left. r_3(1 + x_3)\phi_3^A(x_3)(\phi_3^P(x_3) - \phi_3^T(x_3)) \right\} a_i(t_d) E'_a(t_d) \] 
\[ + h'_{n_a}(x_1, x_2, x_3, b_1, b_2) \left[ r_2 x_2(\phi_2^P(x_2) + \phi_2^T(x_2)) \phi_3^A(x_3) \right. \] 
\[ + \left. r_3(x_3 - 1)\phi_3^A(x_3)(\phi_3^P(x_3) - \phi_3^T(x_3)) \right\} a_i(t'_d) E'_a(t'_d) \} \].

(49)

\[ M^{SP}_{\text{ann}}(a_i) = 32\pi C_F f^4 M_B \sqrt{6} \int_0^1 dx_1 dx_2 dx_3 \int_b^\infty b_1 db_1 b_2 db_2 \phi_B \left( x_1, b_1 \right) \] 
\[ \times \left\{ a_i(t_d) E'_a(t_d) h_{n_a}(x_1, x_2, x_3, b_1, b_2) \left[ (x_3 - 1)\phi_2^A(x_2)\phi_3^A(x_3) \right. \right. \] 
\[ - 4 r_2 r_3 \phi_2^P(x_2)\phi_3^P(x_3) + r_2 r_3 x_3(\phi_2^P(x_2) + \phi_2^T(x_2))(\phi_3^P(x_3) - \phi_3^T(x_3)) \] 
\[ + r_2 r_3(1 - x_2)(\phi_2^P(x_2) - \phi_2^T(x_2))(\phi_3^P(x_3) + \phi_3^T(x_3)) \] 
\[ + a_i(t'_d) E'_a(t'_d) h'_{n_a}(x_1, x_2, x_3, b_1, b_2) \left[ x_2\phi_2^A(x_2)\phi_3^A(x_3) \right. \] 
\[ + x_2 r_2 r_3(\phi_2^P(x_2) + \phi_2^T(x_2))(\phi_3^P(x_3) - \phi_3^T(x_3)) \] 
\[ + r_2 r_3(1 - x_3)(\phi_2^P(x_2) - \phi_2^T(x_2))(\phi_3^P(x_3) + \phi_3^T(x_3)) \right\} \].

(50)

The hard scales are chosen as

\[ t_a = \max\{\sqrt{x_3} M_B, 1/b_1, 1/b_3\}, \]
\[ t'_a = \max\{\sqrt{x_1} M_B, 1/b_1, 1/b_3\}, \]
\[ t_b = \max\{\sqrt{x_1 x_3} M_B, \sqrt{1 - x_1 - x_2 x_3} M_B, 1/b_1, 1/b_2\}, \]
\[ t'_b = \max\{\sqrt{x_1 x_3} M_B, \sqrt{x_1 - x_2 x_3} M_B, 1/b_1, 1/b_2\}, \]
\[ t_c = \max\{\sqrt{1 - x_3 x_2} M_B, 1/b_2, 1/b_3\}, \]
\[ t'_c = \max\{\sqrt{x_2} M_B, 1/b_2, 1/b_3\}, \]
\[ t_d = \max\{\sqrt{x_2 (1 - x_3)} M_B, \sqrt{1 - (1 - x_1 - x_2) x_3 M_B}, 1/b_1, 1/b_2\}, \]
\[ t'_d = \max\{\sqrt{x_2 (1 - x_3)} M_B, \sqrt{x_1 - x_2 (1 - x_3) M_B, 1/b_1, 1/b_2}\}. \]

(51) - (58)

The functions \( h \) in the decay amplitudes consist of two parts: one is the jet function \( S_t(x_i) \) derived by the threshold
re-summation \( h_c(x_1, x_2, b_1, b_2) \) of the other is the propagator of virtual quark and gluon. They are defined by

\[
h_c(x_1, x_3, b_1, b_3) = [\theta(b_1 - b_3)I_0(\sqrt{x_1M_B}b_3)K_0(\sqrt{x_3M_B}b_1)
+\theta(b_3 - b_1)I_0(\sqrt{x_3M_B}b_1)K_0(\sqrt{x_1M_B}b_3)] K_0(\sqrt{x_1x_3M_B}b_1)S_t(x_3),
\]

\[
h_n(x_1, x_2, x_3, b_1, b_2) = [\theta(b_2 - b_1)K_0(\sqrt{x_1x_3M_B}b_2)I_0(\sqrt{x_1x_3M_B}b_1)
+\theta(b_1 - b_2)K_0(\sqrt{x_1x_3M_B}b_1)I_0(\sqrt{x_1x_3M_B}b_2)]
\times \begin{cases}
\frac{i\pi}{2}H_0^{(1)}(\sqrt{(x_2-x_1)x_3M_B}b_2), & x_1 - x_2 < 0 \\
K_0(\sqrt{(x_1-x_2)x_3M_B}b_2), & x_1 - x_2 > 0,
\end{cases}
\]

\[
h_a(x_2, x_3, b_2, b_3) = \left(\frac{i\pi}{2}\right)^2S_t(x_3) \left[\theta(b_2 - b_3)H_0^{(1)}(\sqrt{x_3M_B}b_2)J_0(\sqrt{x_3M_B}b_3)
+\theta(b_3 - b_2)H_0^{(1)}(\sqrt{x_3M_B}b_3)J_0(\sqrt{x_3M_B}b_2)\right]H_0^{(1)}(\sqrt{x_2x_3M_B}b_2),
\]

\[
h_{na}(x_1, x_2, x_3, b_1, b_2) = \left(\frac{i\pi}{2}\right)^2 \left[\theta(b_1 - b_2)H_0^{(1)}(\sqrt{x_2(1-x_3)M_B}b_1)J_0(\sqrt{x_2(1-x_3)M_B}b_2)
+\theta(b_2 - b_1)H_0^{(1)}(\sqrt{x_2(1-x_3)M_B}b_2)J_0(\sqrt{x_2(1-x_3)M_B}b_1)\right]
\times K_0(\sqrt{1-(x_1-x_2)x_3M_B}b_1),
\]

\[
h'_{na}(x_1, x_2, x_3, b_1, b_2) = \left(\frac{i\pi}{2}\right)^2 \left[\theta(b_1 - b_2)H_0^{(1)}(\sqrt{x_2(1-x_3)M_B}b_1)J_0(\sqrt{x_2(1-x_3)M_B}b_2)
+\theta(b_2 - b_1)H_0^{(1)}(\sqrt{x_2(1-x_3)M_B}b_2)J_0(\sqrt{x_2(1-x_3)M_B}b_1)\right]
\times \begin{cases}
\frac{i\pi}{2}H_0^{(1)}(\sqrt{(x_2-x_1)(1-x_3)M_B}b_1), & x_1 - x_2 < 0 \\
K_0(\sqrt{(x_1-x_2)(1-x_3)M_B}b_1), & x_1 - x_2 > 0,
\end{cases}
\]

where \( H_0^{(1)}(z) = J_0(z) + iY_0(z) \).

The \( S_t \) re-sums the threshold logarithms \( \ln^2 x \) appearing in the hard kernels to all orders and it has been parameterized as

\[
S_t(x) = \frac{2^1+2c^{(3/2+c)}}{\sqrt{\pi}^{(1+c)}} [x(1-x)]^c,
\]

with \( c = 0.4 \). In the nonfactorizable contributions, \( S_t(x) \) gives a very small numerical effect to the amplitude \( 32 \).

Therefore, we drop \( S_t(x) \) in \( h_n \) and \( h_{na} \).

The evolution factors \( E^{(t)}_c \) and \( E^{(t)}_a \) entering in the expressions for the matrix elements (see section 3) are given by

\[
E_c(t) = \alpha_s(t)\exp[-SB(t) - S_0(t)], \quad E'_c(t) = \alpha_s(t)\exp[-SB(t) - S_2(t) - S_3(t)]|_{b_1 = b_2},
\]

\[
E_a(t) = \alpha_s(t)\exp[-S_2(t) - S_3(t)], \quad E'_a(t) = \alpha_s(t)\exp[-SB(t) - S_2(t) - S_3(t)]|_{b_2 = b_3},
\]
in which the Sudakov exponents are defined as

\[ S_B(t) = s \left( x_1 \frac{M_{B_1}}{\sqrt{2}}, b_1 \right) + \frac{5}{3} \int_{1/b_1}^t \frac{d\bar{\mu}}{\bar{\mu}} \gamma_q(\alpha_s(\bar{\mu})), \tag{67} \]

\[ S_2(t) = s \left( x_2 \frac{M_{B_2}}{\sqrt{2}}, b_2 \right) + s \left( (1 - x_2) \frac{M_{B_2}}{\sqrt{2}}, b_2 \right) + 2 \int_{1/b_2}^t \frac{d\bar{\mu}}{\bar{\mu}} \gamma_q(\alpha_s(\bar{\mu})), \tag{68} \]

with the quark anomalous dimension \( \gamma_q = -\alpha_s/\pi \). Replacing the kinematic variables of \( M_2 \) to \( M_3 \) in \( S_2 \), we can get the expression for \( S_3 \). The explicit form for the function \( s(Q, b) \) is:

\[ s(Q, b) = \frac{A^{(1)}}{2\beta_1} \hat{q} \ln \left( \frac{\hat{q}}{\bar{b}} \right) - \frac{A^{(2)}}{4\beta_1^2} \left( \hat{b} - 1 \right) - \frac{A^{(1)} - \frac{A^{(2)}}{4\beta_1}}{2} \ln \left( \frac{e^{2\gamma_E} - 1}{2} \right) \ln \left( \frac{\hat{q}}{\bar{b}} \right) \]

\[ + \frac{A^{(1)} \beta_2}{4\beta_1^3} \left[ \ln(2\hat{q}) + 1 - \ln(2\bar{b}) + 1 \right] + \frac{A^{(1)} \beta_2}{8\beta_1} \left[ \ln^2(2\hat{q}) - \ln^2(2\bar{b}) \right], \tag{69} \]

where the variables are defined by

\[ \hat{q} \equiv \ln(\sqrt{2} Q), \quad \bar{b} \equiv \ln[1/(b\Lambda)], \tag{70} \]

and the coefficients \( A^{(i)} \) and \( \beta_i \) are

\[ \beta_1 = \frac{33 - 2n_f}{12}, \quad \beta_2 = \frac{153 - 19n_f}{24}, \]

\[ A^{(1)} = \frac{4}{3}, \quad A^{(2)} = \frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{27}n_f + \frac{8}{3} \beta_1 \ln(1/2\gamma_E), \tag{71} \]

\( n_f \) is the number of the quark flavors and \( \gamma_E \) is the Euler constant. We will use the one-loop running coupling constant, i.e. we pick up the four terms in the first line of the expression for the function \( s(Q, b) \).
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