Quantum mechanics without spacetime V
- Why a quantum theory of gravity should be non-linear -

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Abstract

If there exists a formulation of quantum mechanics which does not refer to a background classical spacetime manifold, it then follows as a consequence, (upon making one plausible assumption), that a quantum description of gravity should be necessarily non-linear. This is true independent of the mathematical structure used for describing such a formulation of quantum mechanics. A specific model which exhibits this non-linearity is constructed, using the language of noncommutative geometry. We derive a non-linear Schrodinger equation for the quantum dynamics of a particle; this equation reduces to the standard linear Schrodinger equation when the mass of the particle is much smaller than Planck mass. It turns out that the non-linear equation found by us is very similar to a non-linear Schrodinger equation found by Doebner and Goldin in 1992 from considerations of unitary representations of the infinite-dimensional group of diffeomorphisms in three spatial dimensions. Our analysis suggests that the diffusion constant introduced by Doebner and Goldin depends on the mass of the particle, and that this constant tends to zero in the limit in which the particle mass is much smaller than Planck mass, so that in this limit the non-linear theory reduces to standard linear quantum mechanics. A similar effective non-linear Schrodinger equation was also found for the quantum dynamics of a system of D0-branes, by Mavromatos and Szabo.

1 Introduction

In a Universe in which there are no \textit{classical} matter fields, it is desirable to have a formulation of quantum mechanics which makes no reference to a classical spacetime manifold. This is because the spacetime metric generated by the existing quantum matter fields will exhibit quantum fluctuations, and in accordance with the Einstein hole argument [1], such metric fluctuations will destroy the underlying classical spacetime manifold structure. (For a related discussion see also [2]).

Since one could in principle have a Universe in which there are no classical matter fields (in particular such a situation is likely to have arisen immediately after the Big Bang), one should hence look for a way of describing quantum mechanics without a spacetime manifold. Furthermore, such a description should reduce to standard quantum mechanics as and when

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classical matter fields and hence a classical spacetime manifold are present. However, in the absence of a classical spacetime manifold, one might expect interesting new physics, as the conventional quantum mechanical description (which is true in the presence of a classical spacetime manifold) might not be valid. In this paper we show that if such a formulation exists, it implies that a quantum description of gravity should be non-linear. Also, it will be argued that such a formulation of quantum mechanics is a limiting case of a non-linear quantum mechanics. This argument relies on one plausible assumption that, like in classical general relativity, in a quantum theory of gravity also, gravity is a source for itself.

The above argument, which is straightforward and brief, will be described in Section 2. In Section 3, we will present the outline of a mathematical model for such a non-linear quantum mechanics, building on previous preliminary work which uses the language of non-commutative geometry to construct a description of quantum mechanics without a spacetime manifold. We derive a non-linear Schrodinger equation for a particle of mass $m$, which reduces to the standard linear Schrodinger equation when the mass of the particle is much smaller than Planck mass. It turns out that the non-linear equation found by us is very similar to the Doebner-Goldin equation [3].

2 Why a quantum theory of gravity should be non-linear?

Consider a `box' of elementary particles of masses $m_i$, and let each of the masses be much smaller than the Planck mass $m_{Pl} \equiv (\hbar c/G)^{1/2} \sim 10^{-5} \text{gms}$ (this could be a collection of electrons for instance). In this case, the dynamics of each of the particles will follow the rules of quantum mechanics. Let this box be the whole Universe, so that there will be, in the box, no classical spacetime manifold, nor a classical spacetime metric. Also, since $m_i \ll m_{Pl}$, its the same situation as sending $m_{Pl}$ to infinity, or letting $G \to 0$, which means we can neglect the gravity inside the box.

The quantum dynamics of the above system will have to be described without reference to a classical spacetime manifold. Since gravity is negligible in this situation, whatever replaces the concept of the classical Minkowski manifold here can be called the `quantum version of Minkowski spacetime'. This quantum description of the particles in the box should become equivalent to standard quantum mechanics as and when a classical spacetime manifold becomes externally available. A classical spacetime would become externally available if outside the box there are classical matter fields which dominate the Universe.

Consider next the case in which, to a first order approximation, the masses $m_i$ in the box are no longer negligible, compared to Planck mass. We need to now take into account the `quantum gravitational field' produced by these masses, and denoted by a set of variables, say $\eta$. Let us assume that we can associate with the system a physical state $\Psi(\eta, m_i)$ and operators which act on the physical state. It is plausible that to this order of approximation the physical state is determined by a linear equation

$$\hat{O}\Psi(\eta, m_i) = 0 \quad (1)$$

where the operator $\hat{O}$, defined on the background quantum Minkowski spacetime, depends
on the gravitational field variables $\eta$ only via the linearized departure of $\eta$ from its ‘quantum Minkowski limit’ and furthermore does not have any dependence on the physical state $\Psi$.

However, when the masses start becoming comparable to Planck mass, the quantum gravitational field described by the state $\Psi(\eta, m_i)$ will contribute to its own dynamics, and the operator $\hat{O}$ will pick up non-linear corrections which depend on $\Psi(\eta, m_i)$. In this sense, the quantum description of the gravitational field should be via a non-linear theory.

One should contrast this situation with that for a quantized non-abelian gauge theory. In the case of gravity, the operator $\hat{O}$ captures information about the ‘evolution’ of the quantized gravitational field, and hence should depend on $\Psi(\eta, m_i)$, because the gravitational field also plays the role of describing spacetime structure. In the case of a non-abelian gauge theory, the analog of the operator $\hat{O}$ again describes evolution of the quantized gauge field, but the wave-functional $\Psi_A(A_i)$ describing the gauge field will not contribute to $\hat{O}$, because the gauge-field does not describe spacetime structure. $\Psi_A(A_i)$ can, on the other hand, be thought of as contributing non-linearly to a description of the quantized geometry of the internal space on which the gauge field lives.

In approaches to quantum gravity wherein one quantizes a classical theory of gravitation, using the standard rules of quantum theory, linearity is inherent, by construction. Such a treatment could by itself yield a self-consistent theory of quantum gravity. However, by requiring that there be a formulation of quantum mechanics which does not refer to a spacetime manifold, one is led to conclude that quantum gravity should be a non-linear theory. This happens because we have introduced the notion of a quantum Minkowski spacetime (unavoidable when there is no classical spacetime manifold available); iterative corrections to this quantum Minkowski spacetime because of gravity bring about the non-linear dependence on the physical quantum gravitational state.

Consider now the dynamical equation which describes the motion of a particle $m_1$, in the absence of a classical spacetime manifold. We assume that there are also present other particles, and in general $m_1$ and all the other particles together determine the ‘quantum gravitational field’ of this ‘quantum spacetime’.

The simplest situation is the one in which the masses of all the particles are much smaller than Planck mass, so that the spacetime is a quantum version of Minkowski spacetime, and gravity is negligible. In this case, the equation of motion of $m_1$ will be the background independent analog of the flat spacetime Klein-Gordon equation.

Keeping the mass $m_1$ small, increase the mass of the other particles so that $m_1$ becomes a test particle, while the ‘quantum gravitational field’ of the other particles obeys the linear equation (1). Here again, the equation of motion of $m_1$ will be the background independent analog of the curved spacetime Klein-Gordon equation, where the ‘quantum gravitational field’ depends linearly on the source.

Still keeping the mass $m_1$ a test particle, increase the mass of the other particles further, to the Planck mass range, so that the ‘quantum gravitational field’ of the other particles now obeys a non-linear equation, where the quantum gravitational field depends non-linearly on the quantum state of these particles. In this case, the equation of motion of $m_1$ will be similar to the previous case but the ‘quantum gravitational field’ now depends non-linearly on the quantum state of the source, the source being all the particles other than $m_1$.

Now increase the mass $m_1$ also, to the Planck mass range, so that its no longer a test particle. The equation of motion of $m_1$ thus now depends non-linearly on the quantum
gravitational state of the system, where system now includes the mass $m_1$ as well. A special case would be one where we remove all particles, except $m_1$, from the system. The equation of motion for $m_1$ would then depend non-linearly on its own quantum state. This is what gives rise to non-linear quantum mechanics, which when seen from a classical spacetime manifold (as and when the latter becomes available) would be a non-linear Schrödinger equation, in the non-relativistic limit of the Klein-Gordon equation. In the next section we construct a model for such a non-linear equation, based on the language of non-commutative geometry.

3 A model based on noncommutative geometry

In order to build a model of quantum mechanics which does not refer to a classical spacetime manifold, we propose to use the language of noncommutative differential geometry. While one could not give a foolproof reason that using noncommutative geometry is the correct approach, there are plausible arguments [2], [4], [5] which, in our opinion, make this approach an attractive one. The most attractive reason is that in a noncommutative geometry there is a natural generalisation of diffeomorphism invariance.

Nonetheless, it is fair to say that, from the viewpoint of physics, noncommutative differential geometry is a subject still under development, especially with regard to Lorentzian (as opposed to Euclidean) spaces. Thus we have here introduced an asymmetric metric - the relation of this metric with the more common treatment of the distance function (see for instance [6]) in noncommutative geometry remains to be understood. Our basic outlook here is that, as a consequence of the assumptions we make, we derive a mass-dependent non-linear Schrödinger equation, which is in principle falsifiable through laboratory tests of quantum mechanics for mesoscopic systems; tests which could possibly be performed in the foreseeable future.

We shall assume that in nature, at the fundamental level, spacetime coordinates are non-commuting, and that one can choose to work with one of many possible equivalent coordinate systems. A coarse grained description of these noncommuting coordinates yields the familiar commuting coordinates of the classical spacetime manifold.

In the language of noncommutative geometry, diffeomorphisms of ordinary coordinates are replaced by automorphisms of noncommuting coordinates, when one goes from a commutative geometry to a noncommutative one. Diffeomorphism invariance could then in principle be replaced by automorphism invariance if one is trying to construct a theory of gravitation using noncommuting coordinates. Our proposal is that such a noncommutative theory of gravity also describes quantum mechanics [5].

A possible description of quantum Minkowski spacetime

A special case, which is much simpler and more tractable, as compared to the general non-linear case, is a description of the ‘quantum Minkowski spacetime’ (defined in the previous section) using noncommuting coordinates. We recall this construction here [5], before going on to use this construction to propose a model of non-linear quantum mechanics. As mentioned above, a quantum Minkowski spacetime should be invoked if the masses of all the particles in the box are small compared to Planck mass. One will have a classical spacetime when the masses involved become large compared to Planck mass. For simplicity we here
assume that the box has only one particle. It is not difficult to generalize from here to the case of many particles.

Let us start by recalling the Klein-Gordon equation (which we here think of as a relativistic Schrodinger equation) \[7\] for the quantum mechanics of a particle in 2-d spacetime

\[-\hbar^2 \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \psi = m^2 \psi.\] \(2\)

This equation, after the substitution \(\psi = e^{iS/\hbar}\), becomes

\[\left( \frac{\partial S}{\partial t} \right)^2 - \left( \frac{\partial S}{\partial x} \right)^2 - i \hbar \left( \frac{\partial^2 S}{\partial t^2} - \frac{\partial^2 S}{\partial x^2} \right) = m^2 \] \(3\)

and can further be written as

\[p^\mu p_\mu + \hbar \frac{\partial S}{\partial x^\mu} = m^2 \] \(4\)

where we have defined

\[p^t = -\frac{\partial S}{\partial t}, \quad p^x = \frac{\partial S}{\partial x} \] \(5\)

and the index \(\mu\) takes the values 1 and 2. We assume that \(p\) gives a definition of a ‘generalised momentum’, in terms of the complex action \(S\), and carrying the same information as the standard momentum operator.

Equation (3) could be thought of as a generalisation of the classical Hamilton-Jacobi equation to the quantum mechanical case [also for reasons which will become apparent as we proceed], where the ‘action’ function \(S(x, t)\) is now complex. Evidently, in (4) the \(\hbar\) dependent terms appear as corrections to the classical term \(p^\mu p_\mu\). We chose to consider the relativistic case, as opposed to the non-relativistic one, because the available spacetime symmetry makes the analysis more transparent. Subsequently, we will consider the non-relativistic limit.

Taking clue from the form of Eqn. (4) we now suggest a model for the dynamics of the ‘quantum Minkowski spacetime’, in the language of two noncommuting coordinates \(\hat{x}\) and \(\hat{t}\). We ascribe to the particle a complex ‘generalised momentum’ \(\hat{p}\), having two components \(\hat{p}^t\) and \(\hat{p}^x\), which do not commute with each other. The noncommutativity of these momentum components is assumed to be a consequence of the noncommutativity of the coordinates, as the momenta are defined to be the partial derivatives of the complex action \(S(\hat{x}, \hat{t})\), with respect to the corresponding noncommuting coordinates.

We further assume that the coordinates \(\hat{x}\) and \(\hat{t}\) describe the non-commutative version of Minkowski space, and that the noncommutative flat metric is

\[\hat{\eta}_{\mu\nu} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \] \(6\)

We now propose that the dynamics in the quantum Minkowski spacetime is given by the equation

\[\hat{p}^\mu \hat{p}_\mu = m^2 \] \(7\)
where $\hat{p}^\mu$ as mentioned above are non commuting. Here, $\hat{\eta}_{\mu\nu}\hat{p}^\nu$ is well-defined. Written explicitly, this equation becomes

$$(\hat{p}^t)^2 - (\hat{p}^x)^2 + \hat{p}^t \hat{p}^x - \hat{p}^x \hat{p}^t = m^2$$

Eqn. (7) appears an interesting and plausible proposal for the dynamics, because it generalizes the corresponding special relativistic equation to the noncommutative case. The noncommutative Hamilton-Jacobi equation is constructed from (8) by defining the momentum as gradient of the complex action function. Arguments in favour of introducing such particle dynamics were given in [5] (Section 3 therein), motivated by considerations of a possible spacetime symmetry that the quantum Minkowski spacetime could have.

Since we know from the above arguments that the quantum Minkowski metric can be replaced by the usual Minkowski metric as and when a classical spacetime becomes available, we propose the following rule for the transformation of the expression on the left of (8), whenever a classical spacetime is available:

$$(\hat{p}^t)^2 - (\hat{p}^x)^2 + \hat{p}^t \hat{p}^x - \hat{p}^x \hat{p}^t = (p^t)^2 - (p^x)^2 + i\hbar \frac{\partial p^\mu}{\partial x^\mu}$$

Here, $p$ is the ‘generalised momentum’ of the particle as seen from the commuting coordinate system, and is related to the complex action by Eqn. (5). This equality should be seen as two ways of describing the same physics - one written in the noncommuting coordinate system, and the other written in the standard commuting coordinate system.

The idea here is that by using the Minkowski metric of ordinary spacetime one does not correctly measure the length of the ‘momentum’ vector, because the noncommuting off-diagonal part is missed out. The last, $\hbar$ dependent term in (9) provides the correction. - the origin of this term’s relation to the commutator $\hat{p}^t \hat{p}^x - \hat{p}^x \hat{p}^t$ can be understood as follows.

Let us write this commutator by scaling all momenta with respect to Planck momenta: define $\hat{P}^\mu = \hat{p}^\mu / P_{pl}$. Also, all lengths are scaled with respect to Planck length: $\hat{X}^\mu = \hat{x}^\mu / L_{pl}$. Let the components of $X^\mu$ be denoted as $(\hat{T}, \hat{X})$. The commutator $\hat{P}^T \hat{P}^X - \hat{P}^X \hat{P}^T$ represents the ‘non-closing’ of the basic quadrilateral when one compares (i) the operator obtained by moving first along $\hat{X}$ and then along $\hat{T}$, with (ii) the operator obtained by moving first along $\hat{T}$ and then along $\hat{X}$. When seen from a commuting coordinate system, this deficit (i.e. non-closing) can be interpreted as a result of moving to a neighbouring point, and in the infinitesimal limit the deficit will be the sum of the momentum gradients in the various directions. The deficit is thus given by $i\hbar \partial p^\mu / \partial x^\mu = i(L_{pl}/P_{pl})\partial p^\mu / \partial x^\mu$. This gives that $\hat{p}^t \hat{p}^x - \hat{p}^x \hat{p}^t = i\hbar \partial p^\mu / \partial x^\mu$.

Hence, since the relation (9) holds, there is equivalence between the background independent description (7) and standard quantum dynamics given by (4).

The proposal proceeds along analogous lines for four-dimensional spacetime. The metric $\hat{\eta}_{\mu\nu}$ is defined by adding an antisymmetric part (all entries of which are 1 and $-1$) to the Minkowski metric, and the off-diagonal contribution on the left-hand side of (9) is to be set equal to $i\hbar \partial p^\mu / \partial x^\mu$ on the right hand side.

Our proposal for the dynamics of a ‘quantum Minkowski spacetime’, described via equations (6)-(9), introduces new and unusual features, and one should ask for a justification for doing so, apart from the need to have for a description of quantum mechanics which does
not refer to a Minkowski spacetime. Partly, the justification comes from considerations of possible symmetries that a noncommutative Minkowski spacetime could have [5]. Equally importantly, the validity of the features introduced above can be experimentally tested by verifying or ruling out the non-linear generalisation of the quantum Minkowski spacetime constructed below.

A non-linear Schrödinger equation

In Section 1 we argued that when the particle masses become comparable to Planck mass, the equation of motion for the particles should become non-linear. We now present an approximate construction for such a non-linear equation, based on the dynamics for the ‘quantum Minkowski spacetime’ proposed above.

The starting point for our discussion will be Eqn. (7) above - we will assume that a natural generalisation of this equation describes quantum dynamics when the particle mass $m$ is comparable to the Planck mass $m_{Pl}$. In this case, we no longer expect the metric $\eta$ to have the ‘flat’ form given in (6) above. The symmetric (i.e. diagonal) components of the metric are of course expected to start depending on $m$ (this is usual gravity, analogous to the Schwarzschild geometry) and in general the antisymmetric (i.e. off-diagonal) components are also expected to depend on $m$. So long as the antisymmetric components are non-zero, we can say that quantum effects are present. As $m$ goes to infinity, the antisymmetric component should go to zero - since in that limit we should recover classical mechanics. In fact the antisymmetric, off-diagonal part (let us call it $\theta_{\mu\nu}$) should already start becoming ignorable when $m$ exceeds $m_{Pl}$. It is interesting to note that the symmetric part should grow with $m$, while the antisymmetric part should fall with increasing $m$. There probably is a deep reason why this is so. So, in this case, instead of eqn. (9), we will have the equation

$$\hat{g}_{tt}(\hat{p}^t)^2 - \hat{g}_{xx}(\hat{p}^x)^2 + \hat{\theta}(\hat{p}^t \hat{p}^x - \hat{p}^x \hat{p}^t) = g_{tt}(p^t)^2 - g_{xx}(p^x)^2 + i\hbar \theta \frac{\partial p^\mu}{\partial x^\mu} = m^2$$

(10)

Here, $g$ is the symmetric part of the metric, and $\theta$ is the value of a component of $\theta_{\mu\nu}$.

We should now make some realistic simplifications, in order to get an equation we can possibly tackle. Here, we are interested in the case $m \sim m_{Pl}$. The symmetric part of the metric - $g$ - should resemble the Schwarzschild metric, and assuming we are not looking at regions close to the Schwarzschild radius (which is certainly true for objects of such masses which we expect to encounter in the laboratory) we can set $g$, approximately, to unity. The key quantity is $\theta = \theta(m)$, and we assume that $\theta$ should be retained - it carries all the new information about any possible non-linear quantum effects. $\theta$ in principle should also depend on the quantum state via the complex action $S$, but we can know about the explicit dependence of $\theta$ on $m$ and $S$ only if we know the dynamics of the quantum gravitational field, which at present we do not. It is like having to know the analog of the Einstein equations for $\theta$. But can we extract some useful conclusions just by retaining $\theta(m)$ and knowing its asymptotic behaviour? Retaining $\theta$, the above dynamical equation can be written in terms of the complex action $S$ as follows:

$$\left(\frac{\partial S}{\partial t}\right)^2 - \left(\frac{\partial S}{\partial x}\right)^2 - i\hbar \theta(m) \left(\frac{\partial^2 S}{\partial t^2} - \frac{\partial^2 S}{\partial x^2}\right) = m^2$$

(11)
This is the equation we would like to work with. We know that $\theta = 1$ is quantum mechanics, and $\theta = 0$ is classical mechanics. We expect $\theta$ to decrease from one to zero, as $m$ is increased. It is probably more natural that $\theta$ continuously decreases from one to zero, as one goes from quantum mechanics to classical mechanics, instead of abruptly going from one to zero. In that case we should expect to find experimental signatures of $\theta$ when it departs from one, and is not too close to zero - we expect this to happen in the vicinity of the Planck mass scale. By substituting the earlier definition $S = -\hbar \ln \psi$ in (11) we get the following non-linear equation for the Klein-Gordon wave-function $\psi$:

$$ -\hbar^2 \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \psi + \frac{\hbar^2}{\psi} \left( 1 - \frac{1}{\theta} \right) \left( \dot{\psi}^2 - \psi'^2 \right) = \frac{m^2}{\theta} \psi $$

(12)

We are interested in working out possible consequences of the non-linearity induced by $\theta$, even though we do not know the explicit form of $\theta$.

Let us go back to the equation (11). As mentioned above, in general $\theta$ will also depend on the state $S$ but for all states $\theta$ should tend to zero for large masses, and if we are looking at large masses we may ignore the dependence on the state, and take $\theta = \theta(m)$. Let us define an effective Planck’s constant $\hbar_{\text{eff}} = \hbar \theta(m)$, i.e. the constant runs with the mass $m$. Next we define an effective wave-function $\psi_{\text{eff}} = e^{iS/\hbar_{\text{eff}}}$. It is then easy to see from (11) that the effective wave function satisfies a linear Klein-Gordon equation

$$ -\hbar_{\text{eff}}^2 \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \psi_{\text{eff}} = m^2 \psi_{\text{eff}} $$

(13)

and is related to the usual wave function $\psi$ through

$$ \psi_{\text{eff}} = \psi^{\frac{1}{\theta(m)}} $$

(14)

For small masses, the effective wave-function approaches the usual wave function, since $\theta$ goes to unity.

We would now like to obtain the non-relativistic limit for this equation. Evidently, this limit should be

$$ i\hbar_{\text{eff}} \frac{\partial \psi_{\text{eff}}}{\partial t} = \frac{\hbar_{\text{eff}}^2 \partial^2 \psi_{\text{eff}}}{2m \partial x^2} $$

(15)

By rewriting $\psi_{\text{eff}}$ in terms of $\psi$ using the above relation we arrive at the following non-linear Schrodinger equation

$$ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + \frac{\hbar^2}{2m} \left( 1 - \theta \right) \left( \frac{\partial^2 \psi}{\partial x^2} - \left[ \ln \psi \right]^2 \right) \psi. $$

(16)

The correction terms can also be combined so that the equation reads

$$ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + \frac{\hbar^2}{2m} \left( 1 - \theta \right) \frac{\partial^2 (\ln \psi)}{\partial x^2} \psi. $$

(17)

It is reasonable to propose that if the particle is not free, a term proportional to the potential, $V(q)\psi$, can be added to the above non-linear equation.
In terms of the complex action function $S$ defined above (3) as $\psi = e^{iS/\hbar}$ this non-linear Schrodinger equation is written as

$$\frac{\partial S}{\partial t} = -\frac{S'^2}{2m} + \frac{i\hbar}{2m}\theta(m)S''.$$  \hspace{1cm} (18)

This equation is to be regarded as the non-relativistic limit of Eqn. (11).

It is clearly seen that the non-linear Schrodinger equation obtained here results from making the Planck constant mass-dependent, in the quantum mechanical Hamilton-Jacobi equation. Equation (16) is in principle falsifiable by laboratory tests of quantum mechanics, and its confirmation, or otherwise, will serve as a test of the various underlying assumptions of the noncommutative model. Properties of this equation, and the possible modifications it implies for quantum mechanics of mesoscopic systems, are at present under investigation.

A comparison with the Doebner-Goldin equation

Remarkably enough, a non-linear Schrodinger equation very similar to (16) was found some years ago by Doebner and Goldin [3]. (For a recent discussion on non-linear quantum mechanics see [8] and also [9]). They inferred their equation from a study of unitary representations of an infinite-dimensional Lie algebra of vector fields $\text{Vect}(\mathbb{R}^3)$ and the group of diffeomorphisms $\text{Diff}(\mathbb{R}^3)$. These representations provide a way to classify physically distinct quantum systems. There is a one-parameter family, labelled by a real constant $D$, of mutually inequivalent one-particle representations of the Lie-algebra of probability and current densities. The usual one-particle Fock representation is the special case $D = 0$. The probability density $\rho$ and the current density $j$ satisfy, not the continuity equation, but a Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = -\nabla j + D \nabla^2 \rho.$$  \hspace{1cm} (19)

A linear Schrodinger equation cannot be consistent with the above Fokker-Planck equation with $D \neq 0$, but Doebner and Goldin found that the following non-linear Schrodinger equation leads to the above Fokker-Planck equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + iD\hbar \left(\nabla^2 \psi + \frac{|\nabla \psi|^2}{|\psi|^2} \psi\right).$$  \hspace{1cm} (20)

The Doebner-Goldin equation should be compared with the equation (16) found by us. Although we have considered a 2-d case, and although there are some important differences, the similarity between the two equations is striking, considering that the two approaches to this non-linear equation are, at least on the face of it, quite different. It remains to be seen as to what is the connection between the representations of $\text{Diff}(\mathbb{R}^3)$, quantum mechanics, and the antisymmetric part $\theta$ of the asymmetric metric introduced by us. (A similar effective non-linear Schrodinger equation was also found for the quantum dynamics of a system of D0-branes, by Mavromatos and Szabo [10]).

The comparison between the two non-linear equations suggests the following relation between the new constants $D$ and $\theta$

$$D \sim \frac{\hbar}{m} (1 - \theta).$$  \hspace{1cm} (21)
There is a significant difference of an $i$ factor in the correction terms in the two equations, and further, the relative sign of the two correction terms is different in the two equations, and in the last term we do not have absolute values in the numerator and denominator, unlike in the Doebner-Goldin equation. The origin of these differences is not clear to us, nonetheless the similarity between the two equations is noteworthy and we believe this aspect should be explored further. It is encouraging that there is a strong parallel between the limits $D \to 0$ and $\theta \to 1$ - both limits correspond to standard linear quantum mechanics. In their paper Doebner and Goldin note that the constant $D$ could be different for different particle species. In the present analysis we clearly see that $\theta(m)$ is labelled by the mass of the particle.

If we substitute $\psi = e^{iS/\hbar}$ in the Doebner-Goldin equation we get the following corrected quantum mechanical Hamilton-Jacobi equation (written for one spatial dimension), after defining $D = -i\hbar (1 - \theta)/2m$

$$\frac{\partial S}{\partial t} = -\frac{S'}{2m} \left[ \theta S' + (1 - \theta)S'' \right] + \frac{i\hbar}{2m} \theta(m) S''.$$  \hspace{1cm} (22)

This equation should be compared with our Eqn. (18).

For comparison with Doebner and Goldin we also write down the corrections to the continuity equation which follow as a consequence of the non-linear terms in (16). These can be obtained by first noting, from (15), that the effective wave-function $\psi_{eff}$ obeys the following continuity equation

$$\frac{\partial}{\partial t} (\psi_{eff}^* \psi_{eff}) - \frac{i\hbar_{eff}}{2m} (\psi_{eff}^* \psi_{eff}' - \psi_{eff} \psi_{eff}')' = 0.$$  \hspace{1cm} (23)

By substituting $\psi_{eff} = \psi^{1/\theta(m)}$ and $\hbar_{eff} = \hbar \theta(m)$ in this equation we get the following corrections to the continuity equation for the probability and current density constructed from $\psi(x)$

$$\frac{\partial}{\partial t} (\psi^* \psi) - \frac{i\hbar}{2m} \left( \psi^* \psi' - \psi \psi^* \right)' = \frac{\hbar (1 - \theta)}{m} |\psi|^2 \phi''$$  \hspace{1cm} (24)

where $\phi$ is the phase of the wave-function $\psi$, i.e. $\phi = Re(S)/\hbar$. It is interesting that the phase enters in a significant manner in the correction to the continuity equation. This equation should be contrasted with Eqn. (19). The fact that the evolution is not norm-preserving when the mass becomes comparable to Planck mass perhaps suggests that the appropriate description could be in terms of the effective wave-function $\psi_{eff}$.

4 Discussion

In our model, based on noncommuting coordinates, we have introduced a few new assumptions, a detailed understanding of which should emerge when the relationship between noncommutative differential geometry and quantum gravity is better understood. Thus we have proposed the notion of a generalised momentum, an asymmetric metric, and a generalised energy-momentum equation. It should be said that these ideas do find support in the generalisation from diffeomorphism invariance, to automorphism invariance, when one goes over to noncommuting coordinates, from commuting ones [5]. Nonetheless, we would like to take the stand that such underlying ideas should be put to experimental test, by confirming or
ruling out, in the laboratory, the non-linearities predicted in the vicinity of the Planck mass scale. Such tests could also help decide whether or not quantum gravity is a linear theory.

There are very stringent experimental constraints on possible non-linearities in quantum mechanics, in the atomic domain. However, quantum mechanics has not really been tested in the mesoscopic domain, say for an object having a mass of a billionth of a gram. Such experiments are very difficult to perform with current technology, but given the advances in nanotechnology, perhaps not impossible to carry out in the foreseeable future. While most people do not expect any surprises, we feel that if there are theoretical models predicting non-linearity precisely in the domain which has not been experimentally explored thus far, we should keep an open mind about the possible outcome. At the very least, such experiments can put strong bounds on non-linearities in mesoscopic quantum mechanics.

In the context of the model discussed here, how can we possibly interpret the effective wave function $\psi_{\text{eff}}$? It would appear correct to say that in the present model, the correct physics is described by the effective wave function, and not by the usual wave function, even though the two carry the same information. Now for large masses, $\theta$ goes to zero, this makes $1/\theta$ very large. If we consider a $\psi$ which is a superposition of two eigenstates of an observable which we are measuring, we can see that the exponent $1/\theta$ will lead to extremely rapid oscillations between the two eigenstates. In this case we can possibly set the oscillatory phase to zero - this essentially destroys superpositions between states. This seems to behave like a decohering mechanism which destroys superpositions, replacing sum of amplitudes by sum of classical probabilities. Furthermore, the $1/\theta$ term will lead to a very rapid fall-off of the effective wave function squared with distance, possibly providing an explanation for spatial localization of macroscopic systems.

The presence of the $\theta$ term may thus provide for decoherence. What is not clear yet, though, is the origin of the Born probability rule. A model such as the present one should provide an explanation for the collapse of the wave function according to the Born rule. Also, the $\theta$ term should be able to explain why the quantum system jumps into one of the eigenstates, upon measurement, even though originally its in a superposition of the eigenstates. This issue, as well as the possibility of a connection with the Doebner-Goldin equation, is under investigation. Another aspect under study, and discussed briefly in [5], is the nature of commutation relations for position and momenta, in noncommuting coordinate systems.

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