Spin splitting of relativistic particles in three dimensions

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Abstract

The behaviour of relativistic particles in an electric and/or magnetic field is considered in the general case when the direction of propagation may differ from the direction of the field. Special attention is paid to the spin splitting and the ensuing Larmor precession frequency of both neutral and charged particles with spin-1/2. For both neutral and charged particles, the Larmor frequency shows a longitudinal motional redshift. For a neutral particle, there is a dynamical upper bound, which depends on both the mass and the transverse momentum of the particle; moreover, the transverse motion leads to a blueshift of the Larmor frequency. For a charged particle, the longitudinal motional decrease of the spin splitting is determined by the formation of Landau levels and it has no upper limit. Unlike the nonrelativistic limit, the relativistic spin splitting depends on the Landau levels and decreases for higher Landau levels, thereby signalling the presence of a Landau ladder redshift effect.

(Some figures may appear in colour only in the online journal)

1. Introduction

Larmor precession is a well-known effect in nonrelativistic quantum mechanics. Within the nonrelativistic theory, the Larmor precession frequency does not depend on the particle velocity but only on the size of the magnetic dipole moment (MDM) and the strength of the applied field. Furthermore, the effect is linear in both the size of the MDM and the strength of the field. There is no upper bound on the Larmor precession frequency: in principle, it increases infinitely as the applied field increases.

Recently [1], we have examined the properties of spin splitting of neutral relativistic particles of spin-1/2 in one dimension and showed a deviation of the behaviour of the Larmor precession frequency from the nonrelativistic result. We have shown two notable effects. First, an upper limit \( \varepsilon = \frac{2mc^2}{\mu} \) of the spin splitting in one dimension exists, which is independent of the size of the EDM or the strength of the applied field. Second, the spin splitting depends on the particle momentum and velocity, which we have referred to as the relativistic motional decrease effect.

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Here, we explore the properties of spin splitting in three dimensions for both neutral and charged relativistic particles of spin-1/2. We show that the relativistic motional decrease effect is present, in a modified form, for both neutral and charged particles in three dimensions. However, in three dimensions there is no upper limit for spin splitting for neither neutral nor charged particles. The reason for this is different in the two cases.

2. A neutral particle in three dimensions

The Hamiltonian of a neutral relativistic particle with the anomalous MDM and eventually electric dipole moment (EDM) propagating in a static magnetic and/or electric field reads

\[
\hat{H} = c\hat{\alpha} \cdot \hat{p} + \hat{\beta} mc^2 + 2\hat{\beta} \hat{S}_\parallel (\hat{d} E - \mu B),
\]

where \( \hat{S}_\parallel \) is the longitudinal component (along the direction of the applied field) of the spin vector operator in relativistic theory, \( E \) is the electric field, \( B \) is the magnetic field, \( d \) is the size of the EDM and \( \mu \) is the size of the MDM of the relativistic particle. We assume that the magnetic and electric fields, if both are present, are applied along the same direction. In order to find the eigenvalues of the Hamiltonian (1), we suppose that their eigenstates are of the form of plane waves. This results in the formation of a \( 4 \times 4 \) determinant, the solutions of which are the four distinct eigenvalues

\[
E_{\pm}^\dagger = \pm \sqrt{p_{\parallel}^2 c^2 + (c\sqrt{m^2 c^2 + p_{\perp}^2} + \delta)^2},
\]

where

\[
\delta = \frac{2mc^2}{\mu}.
\]
that both

\[ E_\perp^\pm = \pm \sqrt{p_\parallel^2 c^2 + \left( c^2 m^2 c^2 + p_\perp^2 - \delta \right)^2}, \]

(2b)

where \( \delta = dE - \mu B \) and \( p_\perp \) is the transverse component of the particle momentum. If \( p_\perp = 0 \), the eigenvalues (2) reduce to the 1D case considered previously [1].

The spin splitting \( \varepsilon = E_\perp^+ - E_\perp^- = E_\parallel^+ - E_\parallel^- \) reads

\[ \varepsilon = \sqrt{p_\parallel^2 c^2 + \left( c^2 m^2 c^2 + p_\perp^2 + \delta \right)^2} - \sqrt{p_\parallel^2 c^2 + \left( c^2 m^2 c^2 + p_\perp^2 - \delta \right)^2}. \]

(3)

It depends not only on the longitudinal momentum \( p_\parallel \), along the direction of the applied electric field, but also on the transverse momentum \( p_\perp \). However, these momentum components enter in the eigenvalues in a different manner and lead to qualitatively different behaviours. Because both the longitudinal momentum \( p_\parallel \) and the transverse momentum \( p_\perp \) enter quadratically in the spin splitting, the latter does not depend on their signs; hence, we shall assume for simplicity that both \( p_\parallel \geq 0 \) and \( p_\perp \geq 0 \). Furthermore, because when \( \delta \) is replaced by \( -\delta \) the spin splitting in equation (3) changes its sign too, we shall assume without loss of generality that \( \delta \geq 0 \); then we find from equation (3) that \( \varepsilon > 0 \).

**Dynamical upper limit.** Because \( \partial \varepsilon / \partial p_\parallel \varepsilon = -p_\parallel / (E_\parallel^0 E_\parallel^\perp) < 0 \), we conclude that \( \varepsilon (p_\parallel) \) is a monotonically decreasing function of \( p_\parallel \), with the maximum value of \( \varepsilon (p_\parallel) \) achieved for \( p_\parallel = 0 \). This behaviour is the same as in the 1D case [1].

The expression for \( \varepsilon_0 = \varepsilon (p_\parallel = 0) \) reads

\[ \varepsilon_0 = |c\sqrt{m^2 c^2 + p_\parallel^2} + \delta| - |c\sqrt{m^2 c^2 + p_\parallel^2} - \delta|. \]

(4)

There are two distinct regimes depending on the size of the interaction energy \( \delta \):

\[ \delta < c\sqrt{m^2 c^2 + p_\parallel^2} \Rightarrow \varepsilon_0 = 2\delta, \]

(5a)

\[ \delta > c\sqrt{m^2 c^2 + p_\parallel^2} \Rightarrow \varepsilon_0 = c\sqrt{m^2 c^2 + p_\parallel^2}. \]

(5b)

Similar to the 1D model [1], \( \varepsilon_0 \) grows linearly with \( \delta \) until a threshold value is reached, after which \( \varepsilon_0 \) stays constant. However, unlike the 1D model [1], the threshold value \( c\sqrt{m^2 c^2 + p_\parallel^2} \) and the saturation values depend not only on the rest mass energy \( mc^2 \), but also on the value of the transverse momentum \( p_\perp \). Consequently, in the 3D model a global upper limit for the spin splitting does not exist, since \( p_\perp \) may be arbitrarily large. However, for any fixed \( p_\perp \), the value \( c\sqrt{m^2 c^2 + p_\parallel^2} \) is the maximum value that the spin splitting \( \varepsilon_0 \) can have. We refer to it as the **dynamical upper limit for a given transverse momentum** \( p_\perp \). It is demonstrated in figure 1.

**Transverse motional blueshift.** For small values of \( p_\perp \), the spin splitting (3) can be expanded in a Taylor series versus \( p_\perp \),

\[ \varepsilon \approx \varepsilon_0 - \frac{p_\perp^2 c^2}{2m^2 c^4 + p_\perp^2 c^2 - \delta^2} \varepsilon_0, \]

(6)

When \( p_\perp = 0 \), equation (6) reduces to the 1D expression [1]. As \( p_\perp \) increases, the size of the energy shift decreases—a dependence, which is opposite to the one on the longitudinal momentum \( p_\parallel \).

![Figure 1](https://example.com/figure1.png)

**Figure 1.** Spin splitting of a neutral particle versus interaction energy \( \delta \) for different values of the transverse momentum \( p_\perp \). The dynamical upper limit \( c\sqrt{m^2 c^2 + p_\parallel^2} \) increases with the transverse momentum \( p_\perp \). The longitudinal momentum is \( p_\parallel = 0.3 mc \).

For large values of \( p_\perp \), we find from equation (3) the high-speed approximation

\[ \varepsilon \approx \frac{2\delta \sqrt{m^2 c^2 + p_\parallel^2}}{p_\parallel}. \]

(7)

For \( p_\parallel = 0 \), equation (7) reduces to the high-speed limit in the 1D case [1]. Obviously, the spin splitting in this limit increases with the transverse momentum \( p_\perp \).

Because \( \partial \varepsilon / \partial p_\perp \varepsilon = \frac{p_\parallel \left( E_\parallel^0 E_\parallel^\perp - \delta \right)}{(E_\parallel^0 E_\parallel^\perp)} > 0 \), the spin splitting \( \varepsilon \) is a monotonically increasing function of \( p_\perp \). This is also evident in equations (6) and (7), and figures 2 and 3. We refer to this behaviour as **transverse motional increase** for the spin splitting \( \varepsilon \), and **transverse motional blueshift** for the associated Larmor precession frequency \( \omega_\perp = \varepsilon / \hbar \).

This effect is illustrated in figure 2 for a fixed interaction energy \( \delta = 0.3 mc^2 \) < \( c\sqrt{m^2 c^2 + p_\parallel^2} \). The line for \( p_\perp = 0 \) shows no transverse motional increase, as expected: then the spin splitting is given by equation (5a). If \( \delta > c\sqrt{m^2 c^2 + p_\parallel^2} \), the spin splitting would not be constant with \( p_\perp \), but would manifest the transverse motional blueshift effect described by equation (5b). Figure 2 shows that the transverse blueshift is more pronounced for larger values of the longitudinal momentum \( p_\parallel \). Furthermore, the transverse blueshift effect is larger for larger values of the interaction energy \( \delta \), as shown in figure 3.

The longitudinal motional redshift and transverse motional blueshift effects that we have discussed may be of importance for experiments searching for the EDM of neutral particles [2, 3]. The motional redshift effect will lead to the decrease of the precession frequency of fast moving particles, thus reducing even more the small energy scale and precession...
frequency associated with the EDM of neutral particles. The transverse blueshift effect will have the opposite behaviour—
for particles with large transverse momentum it will enhance
the energy and precession frequency. This can be used in
experimental set-ups for the enhancement of the measured
signal.

3. A charged particle in a magnetic field

The problem for spin precession in relativistic but nonquantum
mechanical framework has been treated previously [4, 5].

Here, we treat the problem of a charged particle without an
anomalous MDM in an external magnetic field within Dirac
theory. The Hamiltonian for a charged particle without the
anomalous MDM in an external magnetic field in the Landau
gauge reads

$$\hat{H} = \hat{\alpha} \cdot \left( \hat{p} - \frac{e}{c} \hat{A} \right) + \beta mc^2,$$

(8)

where $\hat{A}$ is the vector magnetic potential. The eigenvalues of
the Hamiltonian (8) are the relativistic Landau levels [6–9],

$$E_n^\sigma = c \sqrt{m^2 c^2 + p_{\parallel}^2 + (2n + 1 + \sigma) \frac{\hbar e}{c} B},$$

(9)

where $\sigma = \pm 1$ denotes ‘up’ and ‘down’ spin states along the
propagation direction. The spin splitting for the $n$th Landau
level is given by

$$\epsilon_n = c \sqrt{m^2 c^2 + p_{\parallel}^2 + 2(n + 1) \frac{\hbar e}{c} B}$$

$$- c \sqrt{m^2 c^2 + p_{\parallel}^2 + 2n \frac{\hbar e}{c} B}. $$

(10)

The spin splitting for a charged particle in three dimensions is
qualitatively different from expression (3) for a neutral particle.
The difference is due to the coupling of the magnetic field to
the charge of the particle, which modifies the orbital dynamics
by the formation of Landau levels. The relativistic expression
(10) is also different from the nonrelativistic result,

$$\epsilon_n^{\text{nonrel}} = \frac{\hbar e B}{mc}$$

(11)

which is readily obtained from the nonrelativistic expression
for the Landau levels:

$$E_n^{\text{nonrel}} = mc^2 + \frac{\hbar^2 k_n^2}{2m} + \frac{2n + 1 + \sigma \frac{\hbar e}{c} B}{2 \frac{mc}{c}}.$$  

(12)

We note that in the nonrelativistic expression, the spin splitting
is the same for all Landau levels—it does not depend on the
index $n$.

Motional redshift and the Landau ladder redshift effects.
Because it follows from equation (10) that, for neutral particles,
$\partial_n \epsilon_n = -p_i \epsilon_n / (E_{\parallel}^n E_{\perp}^n) < 0$, we conclude that the spin
splitting of a charged particle also exhibits the relativistic
motional redshift effect along the direction of the applied field.
However, this effect shows some quantitative deviations from
the motional decrease effect of neutral particles because of the
difference in the definitions of $E_{\parallel}$ and $E_{\perp}$.

The low-speed and low-field approximation of equation (10) reads (up to second order in $p_\parallel$ and $B$)

$$\epsilon \approx \frac{\hbar e B}{mc} - \frac{2n + 1}{2mc^2} \left( \frac{\hbar e B}{mc} \right)^2 - \frac{1}{2 mc} \left( \frac{p_{\parallel}}{mc} \right)^2.$$  

(13)

The first term is the nonrelativistic result (11); within it the
spin splitting does not depend on the particle longitudinal
momentum $p_{\parallel}$ and the Landau level index $n$, and the splitting
is linear in the magnetic field $B$. The next terms depend on
both $p_{\parallel}$ and $n$, and introduce a quadratic dependence on the
magnetic field. In this low-speed limit, the relativistic motional
decrease effect is embodied in the $p_{\parallel}^2$-term. Obviously, for
heavier particles the spin splitting decrease is smaller.

We point out that the $B^2$-term in equation (13) also reduces
the spin splitting. This term introduces an $n$-dependence in the
relativistic spin splitting: the larger the Landau level index \( n \), the greater the redshift. We refer to this effect as the Landau ladder redshift. This effect signals yet another difference from the nonrelativistic result, in which the spin splitting is the same for all Landau levels. We note that the Landau level index \( n \) in some sense substitutes the quantum numbers \( p_\parallel \) and \( p_\perp \) of transverse motion (if the \( B \)-field is along \( x \)). We have shown above that for neutral particles the transverse motion leads to a blueshift of the spin splitting, which is quantitatively the opposite of the effect of \( n \) on the spin splitting for charged particles here. This difference is due to the formation of Landau levels. The Landau ladder redshift effect is demonstrated in figure 4 where it is evident that the spin splitting is smaller for higher Landau levels.

In the limit of very high speeds \( (v \sim c) \), the spin splitting \( (10) \) has the asymptotic behaviour

\[
\varepsilon_n(v \sim c) \approx \frac{\hbar e B}{p_\parallel}.
\]

(14)

This expression describes how the spin splitting tends to zero when \( v \sim c \). Notably, it does not depend on the mass of the particle and the Landau levels. It is proportional only to the magnetic field strength and inversely proportional to the longitudinal momentum. The behaviour versus \( p_\parallel \) is very similar to the high-speed limit \( (7) \) of the spin splitting of neutral particles.

With respect to the strength of the magnetic field \( B \), the spin splitting \( (10) \) grows monotonically but nonlinearly, as can also be seen in figure 4. In contrast to neutral particles, there is no upper bound for the spin splitting versus the magnetic field strength. However, for a given value of \( B \), the nonrelativistic value \( (11) \) is always larger than the relativistic one, which is subjected to the motional redshift and the Landau ladder redshift effects.

The predicted Landau ladder redshift effect can be tested in storage rings. It may be demonstrated by the injection of particles with different total energy filling different Landau levels. It will lead to a decrease of the Larmor precession frequency according to \( v \approx \frac{\hbar e B}{m c} \frac{2n+1}{\sqrt{\pi}} \left( \frac{\hbar e B}{m c} \right)^2 \) compared with the nonrelativistic quantum mechanical value. The predicted Landau ladder redshift effect may have a relation with the muon g-2 experiment [10], since it predicts a decreased value of the Larmor precession frequency and may be important for the correct interpretation of the experiments, seeking to measure the MDM. The longitudinal relativistic redshift effect may be detectable in cyclotron-type experimental set-ups in which the particle has a nonzero momentum \( p_\parallel \) along the direction of the applied field which is perpendicular to the plane of the cyclotron orbit.

4. Conclusions

In conclusion, we have investigated the spin splitting of neutral and charged relativistic particles in three dimensions. We have shown that for a neutral particle, the upper bound for the 1D case is modified to a dynamical upper bound in the 3D case. The dynamical upper bound depends on both the mass of the particle and the transverse momentum. We have shown that the transverse motion of the neutral particle leads to a motional increase of the spin splitting and correspondingly to a transverse motional blueshift for the associated Larmor frequency. We have shown that the longitudinal motional decrease of the spin splitting is also present for charged particles. However, there are differences between neutral and charged particles due to the formation of Landau levels in the latter. Furthermore, we have shown that for charged particles, there is no analogue of the transverse motional blueshift. Instead, the spin splitting depends on the Landau levels and decreases for higher Landau levels. We have referred to this effect as the Landau ladder redshift.

References

[1] Tenev T G and Vitanov N V 2012 Relativistic effects for spin splitting of neutral particles: upper bound and motional decrease Phys. Rev. A 86 052114
[2] Dress W B, Miller P D, Pendlebury J M, Perrin P and Ramsey N F 1977 Search for an electric dipole moment of the neutron Phys. Rev. D 15 9
[3] Semertzidis Y K 2011 Review of EDM experiments J. Phys.: Conf. Ser. 335 012012
[4] Selva A D, Magnin J and Maspersi L 1996 Bargmann–Michel–Telegdi equation and one-particle relativistic approach Nuovo Cimento B 111 855
[5] Bargmann V, Michel L and Telegdi V L 1959 Precession of the polarization of particles moving in a homogeneous electromagnetic field Phys. Rev. Lett. 2 435–6
[6] Greiner W 2000 Relativistic Quantum Mechanics (Berlin: Springer)
[7] Bermudez A, Martin-Delgado M A and Solano E 2007 Mesoscopic superposition states in relativistic Landau levels Phys. Rev. Lett. 99 123602
[8] Sakurai J J 1967 Advanced Quantum Mechanics (Reading, MA: Addison-Wesley)
[9] Rusin T M and Zawadzki W 2010 Zitterbewegung of relativistic electrons in a magnetic field and its simulation by trapped ions Phys. Rev. D 82 125031
[10] Bennett G W et al 2009 Improved limit on the muon electric dipole moment Phys. Rev. D 80 052008

Figure 4. Spin splitting of a charged particle versus the magnetic field for different Landau levels \( n \) and for \( p_\parallel = 0.2mc \).