Thermodynamics of black branes
in asymptotically Lifshitz spacetimes

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Abstract

Recently, a class of gravitational backgrounds in 3 + 1 dimensions have been proposed as holographic duals to a Lifshitz theory describing critical phenomena in 2 + 1 dimensions with critical exponent \( z \geq 1 \). We continue our earlier work [17], exploring the thermodynamic properties of the “black brane” solutions with horizon topology \( \mathbb{R}^2 \). We find that the black branes satisfy the relation \( \mathcal{E} = \frac{2}{2+z} Ts \) where \( \mathcal{E} \) is the energy density, \( T \) is the temperature, and \( s \) is the entropy density. This matches the expected behavior for a 2 + 1 dimensional theory with a scaling symmetry \( (x_1, x_2) \to \lambda(x_1, x_2), t \to \lambda^z t \).
1 Introduction

Since the Maldacena conjecture [1], holography has offered an interesting new tool to explore strongly coupled field theories (for a review, see [2]). In this framework, black hole backgrounds are dual to strongly coupled plasmas, and using these backgrounds, one can extract hydrodynamic and thermodynamic properties of the plasma [3–5].

Recently, much effort has gone into describing quantum critical behavior in condensed matter systems using holographic techniques (for a review, see [6]). Quantum critical systems exhibit a scaling symmetry

$$t \to \lambda^z t, \quad x_i \to \lambda x_i$$

(1)
similar to the scaling invariance of pure AdS ($z = 1$) in the Poincaré patch. From a holographic standpoint, this suggests the form of the spacetime metric

$$ds^2 = L^2 \left( r^{2z} dt^2 + r^2 dx^i dx^j \delta_{ij} + \frac{dr^2}{r^2} \right),$$

(2)

where the above scaling is realized as an isometry of the metric along with $r \to \lambda^{-1} r$ (for our purposes, $i = 1, 2$). Other metrics exist with the above scaling symmetry, but also with an added Galilean boost symmetry [7–9] which we will not consider here (black brane solutions in these backgrounds were discussed in [10]). There has also been some success at embedding a related metric into string theory [11] with anisotropic (in space) scale invariance. This may serve as a template for embedding metrics of the form (2) into string theory. Here, however, we will continue our study of the model in [12] (some analysis of generalizations of this model appear in [13–15]). The authors of [12] constructed a 4D action that admits the metric (2) as a solution, which is equivalent to the action [13]

$$S = \frac{1}{16\pi G_4} \int d^4 x \sqrt{-g} \left( R - 2\Lambda - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{c^2}{2} A_\mu A^\mu \right)$$

(3)

($F = dA$) with terms in the action parameterized by

$$c = \frac{\sqrt{2Z}}{L}, \quad \Lambda = -\frac{1}{2} \frac{Z^2 + Z + 4}{L^2}. $$

(4)

The solution discussed in [12] has the metric (2) and

$$A = L^2 r^z \frac{2 z (z - 1)}{L^2} dt$$

(5)

1This background was earlier studied in [16], however, without an action principle that admits the above metric as a solution.
with the identification $z = Z$ and $L = \hat{L}$ or the identification $z = 4/Z, L = \frac{2}{Z}\hat{L}$ and is defined for $z \geq 1$ (with $z = 1$ giving AdS$_4$).

In our current work, we will analyze the thermodynamics of black brane solutions which asymptote to (2). These black brane solutions have been studied numerically in [17–19]. However, we will show that their energy density $E$, entropy density $s$ and temperature $T$ are related by

$$E = \frac{2}{2 + z} Ts$$

(6)

using purely analytic methods.

In the following section we will introduce the ingredients needed to prove (6). Particularly important is the existence of a conserved quantity, used to relate horizon data to boundary data. In the final section, we combine these ingredients into the result (6). We also show that this relation is expected in the dual field theory as a result of the scaling symmetry $(x_1, x_2) \rightarrow \lambda (x_1, x_2), t \rightarrow \lambda^2 t$.

2 Summary of earlier results

In our earlier work [17], the action (3) was reduced on the Ansatz

$$ds^2 = -e^{2A(r)} dt^2 + e^{2B(r)} ((dx_1)^2 + (dx_2)^2) + e^{2C(r)} dr^2$$

$$A = e^{G(r)} dt$$

(7)

to give a one dimensional Lagrangian

$$L_{1D} = 4e^{(2B+A-C)} \partial B \partial A + 2e^{(2B+A-C)} (\partial B)^2 + \frac{1}{2} e^{(-A+2B-C+2G)} (\partial G)^2$$

$$-2\Lambda e^{(A+2B+C)} + \frac{1}{2} e^{(-A+2B+C+2G)}.$$ 

(8)

The equations of motion following from this action have solutions given by (2), (5). Further, there are the known black brane solutions that asymptote to AdS$_4$,

$$ds^2 = \left(\frac{-3}{\Lambda}\right) \left(-r^2 f(r) dt^2 + r^2 (dx_1^2 + dx_2^2) + \frac{dr^2}{r^2 f(r)}\right),$$

$$f(r) = 1 - \frac{r^3}{r_0^3}$$

$$A = 0.$$ 

(9)

\footnote{We constructed numeric solutions for this system in [17] (seen as the large $r_0$ limit of this earlier work), however, our current work does not depend on any numeric analysis.}
For the remainder of the paper, we will be concerned with black branes which asymptote to the Lifshitz background \cite{2,5} for \( z \neq 1 \).

In \cite{17}, a Noether charge was found which is associated with the shift

\[
\begin{pmatrix}
A(r) \\
B(r) \\
C(r) \\
G(r)
\end{pmatrix} \rightarrow \begin{pmatrix}
A(r) + \delta \\
B(r) - \delta \\
C(r) + 0 \\
G(r) + \delta
\end{pmatrix}
\]

with \( \delta \) a constant. The above represents a diffeomorphism which preserves the volume element \( dt dx_1 dx_2 \). This is why it is inherited as a Noether symmetry in the reduced Lagrangian. The associated conserved quantity is

\[
(2e^{(A+2B-C)} \partial A - 2e^{(A+2B-C)} \partial B - e^{(-A+2B-C+2G)} \partial G) \equiv D_0.
\]

2.1 The perturbed solution near the horizon

We begin by first reviewing the results found in \cite{17} for the expansion near the horizon. We require that \( e^{2A} \) goes to zero linearly, \( e^{2C} \) has a simple pole, and \( e^G \) goes to zero linearly to make the flux \( dA \) go to a constant (in a local frame or not). Further, we take the gauge \( B(r) = \ln(Lr) \) for this section. We expand

\[
A(r) = \ln \left( r^2 L \left( a_0 (r - r_0)^{\frac{1}{2}} + a_0 a_1 (r - r_0)^{\frac{3}{2}} + \cdots \right) \right), \quad B(r) = \ln(rL)
\]

\[
C(r) = \ln \left( \frac{L}{r} \left( c_0 (r - r_0)^{-\frac{1}{2}} + c_1 (r - r_0)^{\frac{1}{2}} + \cdots \right) \right),
\]

\[
G(r) = \ln \left( \frac{L^2 r^z}{z} \sqrt{\frac{2(z-1)}{L^2}} \left( a_0 g_0 (r - r_0) + a_0 g_1 (r - r_0)^2 + \cdots \right) \right).
\]

Note that by scaling time we can adjust the constant \( a_0 \) by an overall multiplicative factor (note the use of \( a_0 \) in the expansion of \( G(r) \) as well, as \( e^G \) multiplies \( dt \) for the one-form \( A \)). We will need to use this to fix the asymptotic value of \( A(r) \) to be exactly \( \ln(r^z L) \) with no multiplicative factor inside the log.

We plug this expansion into the equations of motion arising from \cite{8}, and solve for the various coefficients. We find a constraint on the 0th order constants: as expected not all boundary conditions are allowed. We solve for \( c_0 \) in terms of the other \( g_0 \) and \( r_0 \), and find

\[
c_0 = \frac{\sqrt{(2z + g_0^2 r_0 (z-1)) r_0^2}}{\sqrt{z} \sqrt{(z^2 + z + 4)r_0^2}}.
\]
All further coefficients (e.g. $g_1$) are determined from the two constants $r_0$ and $g_0$. We evaluate (11) at $r = r_0$ using the above expansion to find

$$D_0 = \frac{r_0^{z+3}L^2a_0}{c_0}. \quad (14)$$

This must be preserved along the flow in $r$. We will use this to relate constants at the horizon to coefficients that appear in the expansion at $r = \infty$.

### 2.2 The perturbed solution near $r = \infty$

We now turn to the question of the deformation space around the solution given in (5) and (4). We take the expansion of the functions

$$A(r) = \ln(r^z L) + \epsilon A_1(r), \quad B(r) = \ln(rL) + \epsilon B_1(r)$$

$$C(r) = \ln \left( \frac{L}{r} \right) + \epsilon C_1(r), \quad G(r) = \ln \left( \frac{L^2 r^z}{z} \sqrt{\frac{2z(z-1)}{L^2}} \right) + \epsilon G_1(r). \quad (15)$$

Using straightforward perturbation theory, we may find the solutions in the $B_1 = 0$ gauge

$$A_1(r) = C_0 \frac{(z-1)(z-2)}{(z+2)} r^{-z-2} + C_2 \left( z^2 + 3z + 2 - (z+1)\gamma \right) r^{-\frac{z-1}{2} - \frac{z}{2}} \quad (16)$$

$$B_1(r) = 0 \quad (17)$$

$$C_1(r) = -C_0 (z-1) r^{-z-2} + C_2 \left( z^2 - 7z + 6 - (z-1)\gamma \right) r^{-\frac{z-1}{2} - \frac{z}{2}} \quad (18)$$

$$G_1(r) = C_0 \frac{2(z^2 + 2)}{z + 2} r^{-z-2} + C_2 4z(z+1) r^{-\frac{z-1}{2} - \frac{z}{2}} \quad (19)$$

where we have defined the useful constant

$$\gamma = \sqrt{9z^2 - 20z + 20}. \quad (20)$$

In the above, we have dropped certain terms in the expansion from [17]. We have dropped them so that we meet the criterion of [20] to be sufficiently close to the Lifshitz background [2], [3]. We may evaluate the conserved quantity, and we find

$$D_0 = -\frac{2(z-1)(z-2)(z+2)L^2}{z} c_0. \quad (21)$$

One may worry that nonlinearities may contribute to the value of this constant. However, one may examine the powers of $r$ available in [17], and quickly be convinced that the higher nonlinear contributions will be zero once we meet the criterion of [20] (i.e. dropping the “bad” modes). Hence, the above constant is the value of $D_0$ throughout the flow in $r$. 

4
2.3 Gauge invariance

In the previous sections, we have gauge fixed by taking $B_1(r) = 0$. Here, we write down the linearized gauge transformations that will allow us to switch to other gauges in perturbation theory (used near $r = \infty$). The transformation

\begin{align}
A_1(r) &\to A_1(r) + \frac{z}{r}\delta(r), \quad B_1(r) \to B_1(r) + \frac{1}{r}\delta(r) \\
C_1(r) &\to C_1(r) + r\partial_r \left( \frac{\delta(r)}{r} \right), \quad G_1(r) \to G_1(r) + \frac{z}{r}\delta(r)
\end{align}

(22)

corresponds to infinitesimal coordinate transformations $r \to r + \epsilon \delta(r)$. One can see that such a shift leaves the equations invariant to leading order in $\epsilon$ when expanding about the solution [2], [5].

3 The black brane thermodynamics

In the above, we have calculated the conserved quantity $D_0$ in two regions: near the horizon $r = r_0$ and in the asymptotic region $r = \infty$. We may use this to solve for $C_0$ in terms of the horizon data

\[ C_0 = -\frac{1}{2} \frac{r_0^{z+3}a_0}{c_0(z-1)(z-2)(z+2)}. \]

(23)

We will see that $C_0$ is proportional to the energy density $\mathcal{E}$ of the background.

Indeed, the authors of [20] identified the energy density of the background in terms of the coefficient of the $r^{-z-2}$ term in the expansion at infinity for any $z$. This mode was also identified in [17] as the mass mode using background subtraction. However, for $1 \leq z \leq 2$ there were additional divergences that were not canceled. These were cured using the local counter terms of [20].

One may not, however, directly use the results of [20] because of a different choice of gauge: above we use $B_1(r) = 0$ and the authors of [20] use $C_1(r) = 0$. We may easily switch to this gauge by taking

\[ \delta(r) = -r \int \frac{C_1(r)}{r} dr \]

(24)

and transforming the other fields appropriately (near $r = \infty$). In the above integral, we make sure to take the constant of integration so that at large $r$ the correction fields vanish. We find that in the $C_1(r) = 0$ gauge

\[ A_1(r) = -2 \frac{(z-1)c_0r^{-(z-2)}}{(2+z)} + \ldots \]

(25)
where \( \cdots \) are the other terms we are not concerned about. This allows us to compare our results directly to the calculation of [20]. We identify \( 2A_1(r) = f(r) \) where \( f(r) \) appears in equation (5.27) [20]. Therefore, we identify

\[
c_{1,RS} = -(z - 1)c_0
\]  

(26)

where \( c_{1,RS} \) appears in [20] as the coefficient of \( r^{-z-2} \). Therefore, equation (5.31) of [20] becomes

\[
\mathcal{E} = -\frac{4(z - 2)(z - 1)}{z} c_0 = \frac{2r_0^{z+3} a_0}{(2 + z)c_0}.
\]  

(27)

where \( \mathcal{E} \) is the energy density. However, we should note that this is a unitless energy. Restoring units we find

\[
\mathcal{E} = \frac{2r_0^{z+3} a_0}{(2 + z)c_0} \frac{1}{16\pi G_4 L}.
\]  

(28)

From the metric, it is easy to read off the area of the horizon, and therefore the entropy density

\[
s = \frac{4\pi r_0^2}{16\pi G_4}.
\]  

(29)

Further, one can easily read off the temperature from the expansion at the horizon [17]

\[
T = \frac{r_0^{z+1} a_0}{4\pi c_0 L}.
\]  

(30)

From this, we may read an interesting thermodynamic relationship

\[
\mathcal{E} = \frac{2}{2 + z} Ts,
\]  

(31)

which is the main result of our work.

We compare this expression with the known \( z = 1 \) black brane solution in \( AdS_4 \). For this we have \( \mathcal{E} = \frac{2r_0^3}{16\pi G_4 L}, \) \( T = \frac{3c_0}{4\pi L} \) and \( s = \frac{4\pi r_0^2}{16\pi G_4} \). These satisfy the relations \( d\mathcal{E} = T ds, \mathcal{E} = \frac{2}{3} Ts \). The second relation agrees with our expression above for \( z = 1 \).

In general, one may use 2 relations of the form

\[
d\mathcal{E} = T ds
\]  

(32)

\[
\mathcal{E} = KT s
\]  

(33)

(\( K \) a constant, which for our purposes is a function of \( z \)) to find the functional forms \( \mathcal{E}(r_0), \mathcal{E}(T) \) and \( s(r_0), s(T) \). First, one may use (33) to eliminate \( ds \) in the relation
\[ \frac{dE}{dT} \text{ directly relating } \frac{dE}{dT} \text{ and } \frac{dT}{dt}. \text{ One may integrate this to find } E(T). \text{ One may then use this in } \frac{dE}{dT} \text{ to find } s(T) = \frac{4\pi r_0^2}{16\pi G_4}. \text{ Such a procedure will furnish } T(r_0) \text{ and so we can find } E(r_0) \text{ and } s(r_0). \text{ Doing so, we find}

\[
\begin{align*}
\frac{dE}{dT} &= \Theta \left( \frac{K(4\pi r_0^2)}{16\pi G_4} \right)^\frac{1}{K(z)} \\
\frac{dT}{dt} &= \left( \frac{K(4\pi r_0^2)}{16\pi G_4} \right)^\frac{1}{K(z)}
\end{align*}
\]  

(34)

where \( \Theta \) is a constant of integration independent of \( r_0 \). \( \Theta \) has units of length \( \frac{2^{3/2}}{1-K} \) so that the units of \( s \) and \( E \) are canonical. We can rewrite \( \Theta \) in the following way.

\[
\Theta = \frac{L}{16\pi G_4} n(z).
\]

Therefore, we take that \( \Theta = \frac{L}{16\pi G_4} n(z) \). This gives

\[
\begin{align*}
\frac{dE}{dT} &= \frac{4\pi r_0^2}{16\pi G_4} \\
\frac{dT}{dt} &= \frac{L^{K(z)}}{16\pi G_4} n(z) T^{K(z)} \\
\frac{dE}{dT} &= \frac{L^{K(z)}}{16\pi G_4} n(z) T^{K(z)}
\end{align*}
\]  

(35)

where we now explicitly write that \( K \) is a function of \( z \) (\( K(z) = \frac{2}{2+z} \)). To the right of each of these expressions, we have given the relation in terms of only the thermodynamic variable \( T \), rather than referring to the geometric variable \( r_0 \).

One may use the above relations to show that the pressure \( P \) is related to the energy density by \( 2P = zE \), which matches the holographic result of [20]. Actually, starting from the equation of state \( 2P = zE \) one can derive the relation (31). The value of \( n(z) \), which is a unitless number, may be determined numerically by plotting \( \log(LT) \) vs \( \log(r_0) \), but we did not attempt to do this here.

The slope of the graph of \( \log(T(r_0)) \) vs \( \log(r_0) \) is

\[
\log(T(r_0)) = \frac{2(1-K(z))}{K(z)} \log(r_0) + \text{constant} = z \log(r_0) + \text{constant},
\]

(36)

which can be quantitatively checked using our earlier data [17].

We may easily compare the thermodynamic relationship (31) with the expected behavior from a system with a scaling symmetry \( (x_1, x_2) \rightarrow \lambda(x_1, x_2), t \rightarrow \lambda^2 t \). This
scaling symmetry (along with $SO(2)$ symmetry rotating the $x_i$) implies that the dispersion relation is

$$\omega^2 = \alpha^2 (k_1^2 + k_2^2) \equiv \alpha^2 k^2 z$$

(37)

where $\alpha$ is some parameter to restore canonical units. We will work with a finite system (a box with sides of length $\ell$), and assume that the occupation number of a given mode (with energy $E_n$) in the box is $Q' \left( e^{-\beta E_n} \right) e^{-\beta E_n}$. This way, the energy of the system is written as

$$E = -\frac{\partial}{\partial \beta} Q, \quad Q = \sum_n Q \left( e^{-\beta E_n} \right).$$

(38)

Inside the box, $k_i = \frac{2\pi}{\ell} n_i$, and we have $d = 2$ spatial dimensions, so there are two $n_i$. We approximate the sum by an integral, and realize that the density of integer $(n_1, n_2)$ lattice points is uniform in the $(n_1, n_2)$ plane to find

$$Q = 2\pi \int n^2 \frac{dn}{n} Q \left( e^{-\beta \alpha \left( \frac{2\pi}{\ell} n \right)^2} \right), \quad n_1^2 + n_2^2 \equiv n^2.$$

(39)

Redefining the integration variable, we find

$$\ell^2 \frac{\alpha^{\frac{d}{2}} \beta^{-\frac{d}{2}}}{2\pi z} \int_0^\infty d\lambda \lambda^{\frac{d}{2} - 1} Q \left( e^{-\lambda} \right) \equiv \frac{2}{\Xi} V \beta^{-\frac{2}{z}},$$

(40)

where the constant $\Xi$ is independent of $\beta$, and we have assumed that the function $Q$ is well behaved at infinity (this is used to construct $\Xi$). This yields all thermodynamic quantities

$$E = \Xi V T^{\frac{d+z}{d}}, \quad S = \int_{d\nu = 0} dT \frac{1}{T} \left( \frac{dE}{dT} \right)_V = \Xi V T^2 \frac{2 + z}{2}$$

(41)

and so indeed relation (31) holds, as well as all subsequent formulae. Using the above argument, we can generalize the result to arbitrary spatial dimension $d$ where the relation (31) becomes

$$\mathcal{E} = Ts \frac{d}{d+z}.$$  

(42)

which generalizes the function $K(z) \to K(z, d) = \frac{d}{d+z}$ (we also promote $n(z) \to n(z, d)$). For $d = 3$ and $z = 1$ (giving $K = \frac{3}{2}$) one can check that (42) and (35) are correct by comparing to the known results for black D3 branes [3] for the functions $s(T)$ and $\mathcal{E}(T)$ up to the normalization $n(z, d)$. The functions of $s(r_0), \mathcal{E}(r_0), T(r_0)$ need to be modified by promoting $r_0^2 \to r_0^d$, so that the “area” is measured appropriately. With this modification, the expressions for $s(r_0), \mathcal{E}(r_0), T(r_0)$ also agree; see for example [21], where similar coordinates are used.
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