Semi-uniform input-to-state stability of infinite-dimensional systems

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Abstract
We introduce the notions of semi-uniform input-to-state stability and its subclass, polynomial input-to-state stability, for infinite-dimensional systems. We establish a characterization of semi-uniform input-to-state stability based on attractiveness properties as in the uniform case. Sufficient conditions for linear systems to be polynomially input-to-state stable are provided, which restrict the range of the input operator depending on the rate of polynomial decay of the product of the semigroup and the resolvent of its generator. We also show that a class of bilinear systems are polynomially integral input-to-state stable under a certain smoothness assumption on nonlinear operators.

Keywords
Infinite-dimensional systems · Input-to-state stability · Polynomial stability · \(C_0\)-semigroups

Mathematics Subject Classification
47D06 · 47N70 · 93C25 · 93D09

1 Introduction
Consider a semi-linear system with state space \(X\) and input space \(U\) (both Banach spaces):

\[
\dot{x}(t) = Ax(t) + F(x(t), u(t)), \quad t \geq 0; \quad x(0) = x_0 \in X,
\]

where \(A\) with domain \(D(A)\) is the generator of a \(C_0\)-semigroup \((T(t))_{t \geq 0}\) on \(X\) and \(F : X \times U \to X\) is a nonlinear operator such that the solution \(x\) of (1.1) exists on \([0, \infty)\) for each essentially bounded input \(u\). We are interested in the cases where
the $C_0$-semigroup $(T(t))_{t \geq 0}$ is semi-uniformly stable, that is, $(T(t))_{t \geq 0}$ is uniformly bounded and $\|T(t)(I-A)^{-1}\| \to 0$ as $t \to \infty$;

- the $C_0$-semigroup $(T(t))_{t \geq 0}$ is polynomially stable with parameter $\alpha > 0$, that is, $(T(t))_{t \geq 0}$ is semi-uniformly stable and $\|T(t)(I-A)^{-1}\| = O(t^{-1/\alpha})$ as $t \to \infty$, which means that there is a constant $M > 0$ such that for all $t > 0$, $\|T(t)(I-A)^{-1}\| \leq \frac{M}{t^{1/\alpha}}$.

In this paper, we introduce and study new notions of input-to-state stability (ISS), which are closely related to semi-uniformly stable semigroups and polynomially stable semigroups.

The concept of ISS has been introduced for ordinary differential equations in [31]. This concept combines asymptotic stability with respect to initial states and robustness against external inputs. Motivated by robust stability analysis of partial differential equations, ISS has been recently studied for infinite-dimensional systems, e.g., in [10, 13–15, 17, 18, 22, 23, 30]; see also the survey [21]. Exponential stability of $C_0$-semigroups plays an important role in the theory of uniform ISS. The concept of ISS related to strong stability of $C_0$-semigroups has been also introduced in [22]. This stability concept is called strong ISS.

Exponential stability of $C_0$-semigroups is a strong property in terms of quantitative asymptotic character and robustness against perturbations. However, we sometimes encounter systems that is strongly stable but not exponentially stable. Strong stability is distinctly qualitative in character unlike exponential stability and has a much weaker asymptotic property than exponential stability. Hence, it does not hold in general that the system (1.1) with generator $A$ of a strongly stable semigroup is uniformly globally stable, that is, there exist functions $\gamma, \mu \in K_\infty$ such that

$$\|x(t)\| \leq \gamma(\|x_0\|) + \mu \left( \text{ess sup}_{0 \leq t < \infty} \|u(t)\|_U \right)$$

for all $x_0 \in X$, essentially bounded functions $u : \mathbb{R} \to U$, and $t \geq 0$, even when $F(\xi, v) = Bv$ ($\xi \in X$, $v \in U$) for some bounded linear operator $B$ from $U$ to $X$, as shown in Theorem 3 of [23]. Here, $K_\infty$ is the set of the classic comparison functions from nonlinear systems theory; see the notation paragraph at the end of this section. The same is true for the sets $KL$ and $K$ we will use below.

Semi-uniform stability and its subclass, polynomial stability, lie between the two notions of semigroup stability, exponential stability and strong stability, in the sense that semi-uniform stability leads to the quantified asymptotic behavior of trajectories with initial states in the domain of the generator. Semi-uniformly stable semigroups have been extensively studied, and it has been shown that various partial differential equations such as weakly damped wave equations are semi-uniformly stable. We refer, for example, to [3–5, 7, 8, 19, 25–29, 34] for the developments of semi-uniform stable semigroups and polynomially stable semigroups. The main motivation of introducing semi-uniform and polynomial versions of ISS is to bridge the gap between uniform ISS and strong ISS as in the semigroup case.
In a manner analogous to semi-uniform stability of semigroups, we define semi-uniform ISS of the system (1.1) as follows. The semi-linear system (1.1) is semi-uniformly ISS if the system is uniformly globally stable and if there exist functions \( \kappa \in KL \) and \( \mu \in K_\infty \) such that

\[
\|x(t)\| \leq \kappa(\|x_0\|_A, t) + \mu(\text{ess sup}_{0 \leq t < \infty} \|u(t)\|_U)
\]

(1.2)

for all \( x_0 \in D(A) \), essentially bounded functions \( u : [0, \infty) \to U \), and \( t \geq 0 \), where \( \| \cdot \|_A \) is the graph norm of \( A \), i.e., \( \|x_0\|_A := \|x_0\| + \|Ax_0\| \) for \( x_0 \in D(A) \).

We provide a characterization of semi-uniform ISS based on attractivity properties called the limit property and the asymptotic gain property. These properties have been introduced in [33] in order to characterize ISS of ordinary differential equations. For infinite-dimensional systems, such attractivity-based characterizations have been established for uniform ISS [22, Theorem 5], strong ISS [22, Theorem 12], and weak ISS [30, Theorem 3.1]. Using the attractivity properties, we show that semi-uniform ISS implies strong ISS for linear systems and bilinear systems.

The semi-linear system (1.1) is called polynomially ISS with parameter \( \alpha > 0 \) if the system is semi-uniformly ISS and if a \( KL \) function \( \kappa \) in (1.2) satisfies \( \kappa(r, t) = O(t^{-1/\alpha}) \) as \( t \to \infty \) for all \( r > 0 \). We study polynomial ISS of the linear system

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0; \quad x(0) = x_0 \in X,
\]

(1.3)

where \( A \) is the generator of a polynomially stable semigroup \( (T(t))_{t \geq 0} \) with parameter \( \alpha > 0 \) on \( X \) and the input operator \( B \) is a bounded linear operator from \( U \) to \( X \).

It is readily verified that polynomial ISS is equivalent to infinite-time admissibility of input operators (together with polynomial stability of \( C_0 \)-semigroups); see, e.g., [36] and [35, Chapter 4] for admissibility. Using this equivalence, we show that the linear system (1.3) is polynomial ISS if \( B \) maps into the domain of the fractional power of \( -A \) with exponent \( \beta > \alpha \). Similar conditions are placed for the range of a perturbation operator in the robustness analysis of polynomial stability [25–27].

This sufficient condition is refined in the case where \( A \) is a diagonalizable operator (see Definition 4.5) on a Hilbert space and \( B \) is of finite rank. Moreover, when the eigenvalues \( (\lambda_n)_{n \in \mathbb{N}} \) of the diagonalizable operator \( A \) satisfy \( |\text{Im}\lambda_n - \text{Im}\lambda_m| \geq d \) for some \( d > 0 \) and for all distinct \( n, m \in \mathbb{N} \) with \( \lambda_n, \lambda_m \) near the imaginary axis, we give a necessary and sufficient condition for polynomial ISS. To this end, we utilize the relation between Laplace–Carleson embeddings and infinite-time admissibility established in Theorem 2.5 of [16].

Even in the linear case, uniform global stability and hence polynomial ISS impose a strict condition on boundedness of input operators when \( C_0 \)-semigroups are polynomially stable but not exponentially stable. This observation motivates us to study a variant of ISS called integral ISS [32]. We introduce a notion of polynomial integral ISS with parameter \( \alpha > 0 \), which means that there exist functions \( \kappa \in KL, \gamma, \theta \in K_\infty \), and \( \mu \in K \) such that the following three conditions hold: (i) \( \|x(t)\| \leq \gamma(\|x_0\|) \) for all...
$x_0 \in X$ and $t \geq 0$ in the zero-input case $u(t) \equiv 0$; (ii) it holds that

$$
\|x(t)\| \leq \kappa(\|x_0\|_A, t) + \theta \left( \int_0^t \|u(s)\|_U ds \right)
$$

for all $x_0 \in D(A)$, essentially bounded functions $u : [0, \infty) \to U$, and $t \geq 0$; (iii) $\kappa(r, t) = O(t^{-1/\alpha})$ as $t \to \infty$ for all $r > 0$. By definition, we immediately see that the linear system (1.3) is polynomially integral ISS for all generators of polynomially stable semigroups and all bounded input operators. Moreover, we prove that bilinear systems are also polynomially integral ISS provided that the product of the $C_0$-semigroup and the nonlinear operator has the same polynomial decay rate as $\|T(t)(I - A)^{-1}\|$. This result is a polynomial analogue of Theorem 4.2 in [20] on uniform integral ISS.

This paper is organized as follows. In Sect. 2, we review some basic facts on semi-uniform stability and polynomial stability of $C_0$-semigroups. In Sect. 3, we provide a characterization of semi-uniform ISS and investigate the relation between semi-uniform ISS and strong ISS. Polynomial ISS of linear systems and polynomial integral ISS of bilinear systems are studied in Sects. 4 and 5, respectively.

**Notation:** Let $\mathbb{N}_0$ and $\mathbb{R}_+$ denote the set of nonnegative integers and the set of nonnegative real numbers, respectively. Define $t \mathbb{R} := \{is : s \in \mathbb{R}\}$. For real-valued functions $f, g$ on $\mathbb{R}$, we write

$$
f(t) = O(g(t)) \quad \text{as } t \to \infty
$$

if there exist $M > 0$ and $t_0 \in \mathbb{R}$ such that $f(t) \leq Mg(t)$ for all $t \geq t_0$. Let $X$ and $Y$ be Banach spaces. The space of all bounded linear operators from $X$ to $Y$ is denoted by $\mathcal{L}(X, Y)$. We write $\mathcal{L}(X) := \mathcal{L}(X, X)$. The domain and the range of a linear operator $A : X \to Y$ are denoted by $D(A)$ and $\text{ran}(A)$, respectively. We denote by $\sigma(A)$ and $\varrho(A)$ the spectrum and the resolvent set of a linear operator $A : D(A) \subset X \to X$, respectively. We write $R(\lambda, A) := (\lambda I - A)^{-1}$ for $\lambda \in \varrho(A)$. The graph norm $\| \cdot \|_A$ of a linear operator $A : D(A) \subset X \to X$ is defined by $\|x\|_A := \|x\| + \|Ax\|$ for $x \in D(A)$. We denote by $L^\infty(\mathbb{R}_+, X)$ the space of all measurable functions $f : \mathbb{R}_+ \to X$ such that $\|f\|_\infty := \text{ess sup}_{t \in \mathbb{R}_+} \|f(t)\| < \infty$. We denote by $C(\Omega, X)$ the space of all continuous functions from a topological space $\Omega$ to $X$. Let $Z$ and $W$ be Hilbert spaces. The Hilbert space adjoint of $T \in \mathcal{L}(Z, W)$ is denoted by $T^*$.

Classes of comparison functions for ISS are defined as follows:

$$
\mathcal{K} := \{\mu \in C(\mathbb{R}_+, \mathbb{R}_+) : \mu \text{ is strictly increasing and } \mu(0) = 0\}
$$

$$
\mathcal{K}_\infty := \{\mu \in \mathcal{K} : \mu \text{ is unbounded}\}
$$

$$
\mathcal{L} := \left\{\mu \in C(\mathbb{R}_+, \mathbb{R}_+) : \mu \text{ is strictly decreasing with } \lim_{t \to \infty} \mu(t) = 0\right\}
$$

$$
\mathcal{KL} := \{\kappa \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+) : \kappa(\cdot, t) \in \mathcal{K} \forall t \geq 0 \text{ and } \kappa(r, \cdot) \in \mathcal{L} \forall r > 0\}.
$$
2 Basic facts on semi-uniform stability and polynomial stability of semigroups

We start by recalling the notion of semi-uniform stability of $C_0$-semigroups introduced in Definition 1.2 of [4].

**Definition 2.1** A $C_0$-semigroup $(T(t))_{t \geq 0}$ on a Banach space with generator $A$ is semi-uniformly stable if $(T(t))_{t \geq 0}$ is uniformly bounded and satisfies

$$\|T(t)R(1, A)\| \to 0 \quad \text{as } t \to \infty. \quad (2.1)$$

The following characterization of (2.1) in terms of the intersection of $\sigma(A)$ with $i \mathbb{R}$ has been established in Theorem 1.1 of [4].

**Theorem 2.2** Let $A$ be the generator of a uniformly bounded semigroup $(T(t))_{t \geq 0}$ on a Banach space. Then, (2.1) holds if and only if $\sigma(A) \cap i \mathbb{R}$ is empty.

Quantitative statements on the decay rate, as $t \to \infty$, of $\|T(t)R(1, A)\|$ and the blow-up rate, as $s \to \infty$, of $R(is, A)$ have been also given, e.g., in [3–5, 7, 19, 29]. In particular, we are interested in semi-uniform stability with polynomial decay rates studied in [3, 5, 19]. Note that $\|T(t)R(1, A)\|$ and $\|T(t)A^{-1}\|$ are asymptotically of the same order, which easily follows from the resolvent equation.

**Definition 2.3** A $C_0$-semigroup $(T(t))_{t \geq 0}$ on a Banach space with generator $A$ is polynomially stable with parameter $\alpha > 0$ if $(T(t))_{t \geq 0}$ is semi-uniformly stable and satisfies

$$\|T(t)A^{-1}\| = O \left( \frac{1}{t^{1/\alpha}} \right) \quad \text{as } t \to \infty. \quad (2.2)$$

For the generator $A$ of a uniformly bounded semigroup, $-A$ is sectorial in the sense of [12, Chapter 2]. Therefore, if $A$ is injective, then the fractional power $(-A)^{\beta}$ is well defined for $\beta \in \mathbb{R}$. The following result gives the rate of polynomial decay of $\|T(t)(-A)^{-\beta}\|$ for $\beta > 0$; see Proposition 3.1 of [3] for the proof.

**Proposition 2.4** Let $(T(t))_{t \geq 0}$ be a uniformly bounded semigroup on a Banach space with generator $A$ such that $0 \in \varrho(A)$. For fixed $\alpha, \beta > 0$,

$$\|T(t)A^{-1}\| = O \left( \frac{1}{t^{1/\alpha}} \right) \quad \text{as } t \to \infty$$

if and only if

$$\|T(t)(-A)^{-\beta}\| = O \left( \frac{1}{t^{\beta/\alpha}} \right) \quad \text{as } t \to \infty.$$

For $C_0$-semigroups generated by normal operators on Hilbert spaces, a spectral condition equivalent to polynomial decay is known. The proof can be found in Proposition 4.1 of [3].
Proposition 2.5 Let \((T(t))_{t \geq 0}\) be the \(C_0\)-semigroup on a Hilbert space generated by a normal operator \(A\) whose spectrum \(\sigma(A)\) is contained in the open left half-plane \(\{\lambda \in \mathbb{C} : \text{Re}\, \lambda < 0\}\). For a fixed \(\alpha > 0\),
\[
\|T(t)A^{-1}\| = O\left(\frac{1}{t^{1/\alpha}}\right) \quad \text{as } t \to \infty
\]
if and only if there exist \(C, p > 0\) such that
\[
|\text{Im}\, \lambda| \geq \frac{C}{|\text{Re}\, \lambda|^{1/\alpha}}
\]
for all \(\lambda \in \sigma(A)\) with \(\text{Re}\, \lambda > -p\).

3 Characterization of semi-uniform input-to-state stability

In this section, we first present the nonlinear system we consider and introduce the notion of semi-uniform input-to-state stability. Next, we develop a characterization of this stability. Finally, the relation between semi-uniform input-to-state stability and strong input-to-state stability is investigated.

3.1 System class

Let \(X\) and \(U\) be Banach spaces with norm \(\|\cdot\|\) and \(\|\cdot\|_U\), respectively. Let \(U\) be a normed vector space contained in the space \(L^1_{\text{loc}}(\mathbb{R}_+, U)\) of all locally integrable functions from \(\mathbb{R}_+\) to \(U\). We denote by \(\|\cdot\|_U\) the norm on \(U\). Assume that \(u(\cdot + \tau) \in U\) and \(\|u\|_U \geq \|u(\cdot + \tau)\|_U\) for all \(u \in U\) and \(\tau \geq 0\). We are interested in the case \(U = L^\infty(\mathbb{R}_+, U)\), but a general space \(U\) is used for the definition of semi-uniform input-to-state stability and its characterization.

Consider a semi-linear system with state space \(X\) and input space \(U\):
\[
\Sigma(A, F) \quad \begin{cases} 
\dot{x}(t) = Ax(t) + F(x(t), u(t)), & t \geq 0 \\
x(0) = x_0,
\end{cases}
\]
where \(A\) is the generator of a \(C_0\)-semigroup \((T(t))_{t \geq 0}\) on \(X\), \(F : X \times U \to X\) is a nonlinear operator, \(x_0 \in X\) is an initial state, and \(u \in U\) is an input.

Definition 3.1 Suppose that for every \(\tau > 0\), \(f \in C([0, \tau], X)\), and \(g \in U\), the map \(t \mapsto F(f(t), g(t))\) is integrable on \([0, \tau]\). For \(\tau > 0\), a function \(x \in C([0, \tau], X)\) is called a mild solution of \(\Sigma(A, F)\) on \([0, \tau]\) if \(x\) satisfies the integral equation
\[
x(t) = T(t)x_0 + \int_0^t T(t-s)F(x(s), u(s))\,ds \quad \forall t \in [0, \tau].
\]
Moreover, we say that \(x \in C(\mathbb{R}_+, X)\) is a mild solution of \(\Sigma(A, F)\) on \(\mathbb{R}_+\) if \(x\) is a mild solution of \(\Sigma(A, F)\) on \([0, \tau]\) for all \(\tau > 0\).
By Proposition 1.3.4 of [2], the integrability of the map $t \mapsto F(x(t), u(t))$ guarantees that the integral in (3.1) exists as a Bochner integral and is continuous with respect to $t$. If the nonlinear operator $F$ satisfies certain Lipschitz conditions, then $\Sigma(A, F)$ has a unique mild solution on $[0, \tau]$ for some $\tau > 0$. We refer, e.g., to [14, 22] for the mild solution and uniform ISS properties of $\Sigma(A, F)$ in the case where $U$ is the space of piecewise continuous functions that are bounded and right-continuous. In this paper, we sometimes consider a class of bilinear systems whose nonlinear operators $F$ satisfy the next assumption.

Assumption 3.2 The nonlinear operator $F : X \times U \to X$ of $\Sigma(A, F)$ is decomposed into $F(\xi, v) = B\xi + G(\xi, v)$ for all $\xi \in X$ and $v \in U$, where $B \in L(X, U)$ and $G : X \times U \to X$ is a nonlinear operator satisfying the following conditions:

1. $G(0, v) = 0$ for all $v \in U$.
2. For all $r > 0$, there exist $K_r > 0$ and $\chi_r \in K$ such that for all $\xi, \zeta \in X$ with $\|\xi\|, \|\zeta\| \leq r$ and all $v \in U$,
   $$\|G(\xi, v) - G(\zeta, v)\| \leq K_r \|\xi - \zeta\| \chi_r(\|v\|_U).$$
3. For all $\tau > 0$ and all functions $f \in C([0, \tau], X)$, $g \in L^\infty([0, \tau], U)$, the map $t \mapsto G(f(t), g(t))$ is measurable on $[0, \tau]$.

Note that the nonlinear operator $G$ has the following property under Assumption 3.2: For all $r > 0$, there exist $K_r > 0$ and $\chi_r \in K$ such that for all $\xi \in X$ with $\|\xi\| \leq r$ and all $v \in U$,

$$\|G(\xi, v)\| \leq K_r \|\xi\| \chi_r(\|v\|_U).$$

A standard argument using Gronwall’s inequality and Banach’s fixed point theorem shows that there exists a unique mild solution of $\Sigma(A, F)$ for $U = L^\infty(\mathbb{R}_+, U)$ under Assumption 3.2; see, e.g., Section 4.3.1 of [6] or Lemma 2.8 of [13]. More precisely, if Assumption 3.2 holds, then one of the following statements is true for all initial states $x_0 \in X$ and all inputs $u \in L^\infty(\mathbb{R}_+, U)$:

1. There exists a unique mild solution of $\Sigma(A, F)$ on $\mathbb{R}_+$.
2. There exist $t_{\max} \in (0, \infty)$ and $x \in C([0, t_{\max}], X)$ such that $x|_{[0, \tau]}$ is a unique mild solution of $\Sigma(A, F)$ on $[0, \tau]$ for all $\tau \in (0, t_{\max})$ and
   $$\lim_{t \uparrow t_{\max}} \|x(t)\| = \infty.$$

Throughout this section, we consider only forward complete systems; see also [33, p. 1284] and [1] for forward completeness.

Definition 3.3 The semi-linear system $\Sigma(A, F)$ is forward complete if there exists a unique mild solution of $\Sigma(A, F)$ on $\mathbb{R}_+$ for all $x_0 \in X$ and $u \in U.$
We denote by \( \phi(t, x_0, u) \) the unique mild solution of the forward complete semi-linear system \( \Sigma(A, F) \) with initial state \( x_0 \in X \) and input \( u \in \mathcal{U} \), i.e.,

\[
\phi(t, x_0, u) = T(t)x_0 + \int_0^t T(t-s)F(\phi(s, x_0, u), u(s))\,ds \quad \forall t \geq 0.
\]

The mild solution satisfies the cocycle property

\[
\phi(t + \tau, x_0, u) = \phi(t, \phi(\tau, x_0, u), u(\cdot + \tau))
\]

for all \( x_0 \in X, u \in \mathcal{U}, \) and \( t, \tau \geq 0 \).

### 3.2 Definition of semi-uniform input-to-state stability

For the forward complete semi-linear system \( \Sigma(A, F) \), we introduce the notion of semi-uniform input-to-state stability. Before doing so, we recall the definition of uniform global stability; see [33, p. 1285] and [22, Definition 6].

**Definition 3.4** The semi-linear system \( \Sigma(A, F) \) is called **uniformly globally stable (UGS)** if the following two conditions hold:

1. \( \Sigma(A, F) \) is forward complete.
2. There exist \( \gamma, \mu \in \mathcal{K}_\infty \) such that

\[
\|\phi(t, x_0, u)\| \leq \gamma(\|x_0\|) + \mu(\|u\|_{\mathcal{U}})
\]

for all \( x_0 \in X, u \in \mathcal{U}, \) and \( t \geq 0 \).

**Definition 3.5** The semi-linear system \( \Sigma(A, F) \) is called **semi-uniformly input-to-state stable (semi-uniformly ISS)** if the following two conditions hold:

1. \( \Sigma(A, F) \) is UGS.
2. There exist \( \kappa \in \mathcal{KL} \) and \( \mu \in \mathcal{K}_\infty \) such that

\[
\|\phi(t, x_0, u)\| \leq \kappa(\|x_0\|_A, t) + \mu(\|u\|_{\mathcal{U}})
\]

for all \( x_0 \in D(A), u \in \mathcal{U}, \) and \( t \geq 0 \).

In particular, if there exists \( \alpha > 0 \) such that for all \( r > 0, \kappa(r, t) = O(t^{-1/\alpha}) \) as \( t \to \infty \), then \( \Sigma(A, F) \) is called **polynomially input-to-state stable (polynomially ISS)** with parameter \( \alpha > 0 \).

Assume that the nonlinear operator \( F : X \times U \to X \) satisfies \( F(\xi, 0) = 0 \) for all \( \xi \in X \). Then, one can easily see that if the semi-linear system \( \Sigma(A, F) \) is semi-uniformly (resp. polynomially) ISS, then \( A \) generates a semi-uniformly (resp. polynomially) stable semigroup \( (T(t))_{t \geq 0} \) on \( X \). In fact, \( \phi(t, x_0, 0) = T(t)x_0 \) for all \( x_0 \in X \) and \( t \geq 0 \) by assumption. Therefore, \( (T(t))_{t \geq 0} \) is uniformly bounded by UGS with \( u(\cdot) \equiv 0 \). Take \( \xi \in X \) with \( \|\xi\| = 1 \). Then,

\[
\|AR(1, A)\xi\| \leq 1 + \|R(1, A)\|.
\]
Since the inequality (3.4) with $u(t) ≡ 0$ yields
\[
\|T(t)R(1, A)ξ\| \leq \kappa (\|R(1, A)ξ\| + \|AR(1, A)ξ\|, t)
\]
\[
\leq \kappa (1 + 2\|R(1, A)\|, t)
\]
for all $t ≥ 0$, it follows that
\[
\|T(t)R(1, A)\| \leq \kappa (1 + 2\|R(1, A)\|, t) \to 0 \quad \text{as } t \to \infty.
\]

Hence, $(T(t))_{t ≥ 0}$ is semi-uniformly stable. Note that semi-uniform stability of $(T(t))_{t ≥ 0}$ generated by $A$ is equivalent to $i\mathbb{R} ⊂ \varrho(A)$ by Theorem 2.2. A similar calculation shows that
\[
\|T(t)A^{-1}\| ≤ \kappa (1 + \|A^{-1}\|, t)
\]
for all $t ≥ 0$. Thus, polynomial ISS of $Σ(A, F)$ implies polynomial stability of $(T(t))_{t ≥ 0}$.

We conclude this subsection by giving an example of polynomially ISS (but not necessarily uniform ISS) nonlinear systems.

**Example 3.6** Let $X$ and $U$ be Banach spaces. Let $A$ be the generator of a polynomially stable semigroup $(T(t))_{t ≥ 0}$ with parameter $α > 0$ on $X$ and let $H ∈ \mathcal{L}(U, X)$ satisfy $\text{ran}(H) ⊂ D((-A)^{β})$ for some $β > α$. Assume that $q : \mathbb{R}_+ → \mathbb{R}$ satisfies the following conditions:
1. $q(0) = 0$.
2. For all $r > 0$, there exists $K_r > 0$ such that
   \[
   |q(z) − q(w)| ≤ K_r|z − w| \quad \forall z, w ∈ [0, r].
   \]
3. $\sup_{z ≥ 0} |q(z)| < \infty$.

Define a nonlinear operator $F : X × U → X$ by
\[
F(ξ, v) := q(\|ξ\|)Hv, \quad ξ ∈ X, \; v ∈ U.
\]

A routine calculation shows that Assumption 3.2 holds for the nonlinear operator $F$.

We show that $Σ(A, F)$ is polynomially ISS with parameter $α$ for $U = L^∞(\mathbb{R}_+, U)$. By Proposition 2.4, there is a constant $M > 0$ such that
\[
\|T(t)(−A)^{−β}\| ≤ \frac{M}{(t + 1)β/α} \quad \forall t ≥ 0.
\]
Since $(-A)^β$ is closed and since $\text{ran}(H) ⊂ D((-A)^{β})$, it follows that $(-A)^β H ∈ \mathcal{L}(U, X)$. Let $c := \sup_{z ≥ 0} |q(z)| < \infty$ and $t > 0$. We obtain
\[
\left\| \int_0^t T(t − s)F(x(s), u(s))ds \right\| = \left\| \int_0^t T(t − s)(−A)^{−β}(-A)^β F(x(s), u(s))ds \right\|
\]
\[
\leq \int_0^t cM\|(-A)^{β} H\| \frac{1}{(t − s + 1)^β/α} \|u\|_∞ ds
\]
for all \( x \in C([0, t], X) \) and \( u \in L^\infty(\mathbb{R}^+, U) \). From this estimate, we see that \( \Sigma(A, F) \) is polynomially ISS with parameter \( \alpha \) for \( U = L^\infty(\mathbb{R}^+, U) \).

### 3.3 Characterization of semi-uniform input-to-state stability

We define a semi-uniform version of the properties of uniform attractivity and strong attractivity studied in [22]. The attractivity properties has been originally introduced in [33, pp. 1284–1285] in order to characterize ISS of ordinary differential equations.

**Definition 3.7** The forward complete semi-linear system \( \Sigma(A, F) \) has the \( \text{semi-uniform limit property} \) if there exists \( \mu \in K_{\infty} \) such that the following statement holds: For all \( \varepsilon, r > 0 \), there is \( \tau = \tau(\varepsilon, r) < \infty \) such that for all \( x_0 \in D(A) \),

\[
\|x_0\|_A \leq r \quad \& \quad u \in U \quad \Rightarrow \quad \exists t \leq \tau : \|\phi(t, x_0, u)\| \leq \varepsilon + \mu(\|u\|_U).
\]

**Definition 3.8** The forward complete semi-linear system \( \Sigma(A, F) \) has the \( \text{semi-uniform asymptotic gain property} \) if there exists \( \mu \in K_{\infty} \) such that the following statement holds: For all \( \varepsilon, r > 0 \), there is \( \tau = \tau(\varepsilon, r) < \infty \) such that for all \( x_0 \in D(A) \) with \( \|x_0\|_A \leq r \) and all \( u \in U \),

\[
t \geq \tau \quad \Rightarrow \quad \|\phi(t, x_0, u)\| \leq \varepsilon + \mu(\|u\|_U).
\]

By definition, the asymptotic gain property is stronger than the limit property. We will see that both properties are equivalent if the system is UGS. Moreover, based on these attractivity properties, a characterization of semi-uniform ISS is established. The attractivity-based characterization of ISS is useful when the construction of a \( KL \) function \( \kappa \) is involved. The proof for the semi-uniform case is obtained by a slight modification of the proof of Theorem 5 in [22] for the uniform case. We sketch it for the sake of completeness.

**Theorem 3.9** The following statements on the semi-linear system \( \Sigma(A, F) \) are equivalent:

1. \( \Sigma(A, F) \) is semi-uniformly ISS.
2. \( \Sigma(A, F) \) is UGS and has the semi-uniform limit property.
3. \( \Sigma(A, F) \) is UGS and has the semi-uniform asymptotic gain property.

**Proof** [1. \( \Rightarrow \) 2.] Suppose that \( \Sigma(A, F) \) is semi-uniformly ISS. By definition, \( \Sigma(A, F) \) is UGS. There exist \( \kappa \in KL \) and \( \mu \in K_{\infty} \) such that

\[
\|\phi(t, x_0, t)\| \leq \kappa(\|x_0\|_A, t) + \mu(\|u\|_U)
\]

for all \( x_0 \in D(A), u \in U \), and \( t \geq 0 \). Take \( \varepsilon, r > 0 \). We obtain \( \kappa(r, \tau) \leq \varepsilon \) for some \( \tau = \tau(\varepsilon, r) < \infty \). Therefore, if \( x_0 \in D(A) \) satisfies \( \|x_0\|_A \leq r \), then

\[
\|\phi(\tau, x_0, t)\| \leq \varepsilon + \mu(\|u\|_U) \quad \forall u \in U.
\]
Thus, $\Sigma(A, F)$ has the semi-uniform limit property.

[2. $\Rightarrow$ 3.] Suppose that $\Sigma(A, F)$ is UGS and has the semi-uniform limit property. By assumption, there exist $\gamma, \mu \in K_\infty$ such that the following statement holds: For every $\varepsilon, r > 0$, there is $\tau = \tau(\varepsilon, r) < \infty$ such that for all $x_0 \in D(A)$,

$$\|x_0\|_A \leq r \land u \in U \Rightarrow \exists t_1 \leq \tau : \|\phi(t_1, x_0, u)\| \leq \varepsilon + \mu(\|u\|_U),$$

and for all $s \geq 0$,

$$\|\phi(s, \phi(t_1, x_0, u), u(\cdot + t_1))\| \leq \gamma(\|\phi(t_1, x_0, u)\|) + \mu(\|u(\cdot + t_1)\|_U).$$

Using the cocycle property (3.2), we obtain

$$\|\phi(t_1 + s, x_0, u)\| = \|\phi(s, \phi(t_1, x_0, u), u(\cdot + t_1))\| \leq \gamma(\|\phi(t_1, x_0, u)\|) + \mu(\|u(\cdot + t_1)\|_U).$$

Since $\gamma(a + b) \leq \gamma(2a) + \gamma(2b)$ for all $a, b \geq 0$, it follows that

$$\|\phi(t_1 + s, x_0, u)\| \leq \gamma(2\varepsilon) + \gamma(2\mu(\|u\|_U)) + \mu(\|u\|_U) \leq \gamma(2\varepsilon) + \tilde{\mu}(\|u\|_U),$$

where $\tilde{\mu} := \gamma \circ (2\mu) + \mu \in K_\infty$.

Choose $\tilde{\varepsilon}, \tilde{r} > 0$ arbitrarily and set

$$\varepsilon := \frac{\gamma^{-1}(\tilde{\varepsilon})}{2}, \quad r := \tilde{r}.$$

We have shown that there is

$$\tau = \tau(\varepsilon, r) = \tau\left(\frac{\gamma^{-1}(\tilde{\varepsilon})}{2}, \tilde{r}\right) < \infty$$

such that for all $x_0 \in D(A)$ with $\|x_0\|_A \leq \tilde{r}$ and all $u \in U$,

$$t \geq \tau \Rightarrow \|\phi(t, x_0, u)\| \leq \tilde{\varepsilon} + \tilde{\mu}(\|u\|_U).$$

Hence, $\Sigma(A, F)$ has the semi-uniform asymptotic gain property.

[3. $\Rightarrow$ 1.] Suppose that $\Sigma(A, F)$ is UGS and has the semi-uniform asymptotic gain property. There exist $\gamma, \mu \in K_\infty$ such that the following two properties hold:

(a) For all $x_0 \in X, u \in U$, and $t \geq 0$,

$$\|\phi(t, x_0, u)\| \leq \gamma(\|x_0\|) + \mu(\|u\|_U).$$

(3.5)
(b) For all $\varepsilon, r > 0$, there is $\tau = \tau(\varepsilon, r) < \infty$ such that for all $x_0 \in D(A)$ with $\|x_0\|_A \leq r$ and all $u \in U$,
\[
t \geq \tau \quad \Rightarrow \quad \|\phi(t, x_0, u)\| \leq \varepsilon + \mu(\|u\|_U).
\]  
\tag{3.6}

Let $r > 0$. Set $\varepsilon_n := 2^{-n} \gamma(r)$ for $n \in \mathbb{N}_0$ and $\tau_0 := 0$. By the property (3.6), there exist $\tau_n = \tau_n(\varepsilon_n, r), n \in \mathbb{N}$, such that for all $x_0 \in D(A)$ with $\|x_0\|_A \leq r$ and all $u \in U$,
\[
t \geq \tau_n \quad \Rightarrow \quad \|\phi(t, x_0, u)\| \leq \varepsilon_n + \mu(\|u\|_U).
\]  
\tag{3.7}

For $n = 0$, we also obtain (3.7) by the property (3.5) and the inequality
\[
\gamma(\|x_0\|) \leq \gamma(\|x_0\|_A) \leq \gamma(r) = \varepsilon_0.
\]

We may assume without loss of generality that $\inf_{n \in \mathbb{N}} (\tau_n - \tau_{n-1}) > 0$. For these sequences $(\varepsilon_n)_{n \in \mathbb{N}_0}$ and $(\tau_n)_{n \in \mathbb{N}_0}$, one can construct a function $\kappa \in \mathcal{K}$ satisfying
\[
\varepsilon_n \leq \kappa(r, t) \quad \forall t \in [\tau_n, \tau_{n+1}), \forall n \in \mathbb{N}_0;
\]  
\tag{3.8}

see the proof of Lemma 7 of [22] for the detailed construction. From (3.7) and (3.8), we have that for all $x_0 \in D(A)$ with $\|x_0\|_A \leq r$ and all $u \in U$,
\[
\|\phi(t, x_0, u)\| \leq \kappa(\|x_0\|_A, t) + \mu(\|u\|_U) \quad \forall t \geq 0.
\]  
\tag{3.9}

Take $x_0 \in D(A)$ and $u \in U$ arbitrarily. If $\|x_0\|_A = 0$, then the property (3.5) yields
\[
\|\phi(t, x_0, u)\| \leq \mu(\|u\|_U) = \kappa(\|x_0\|_A, t) + \mu(\|u\|_U) \quad \forall t \geq 0.
\]

If $\|x_0\|_A > 0$, then it follows from (3.9) with $r := \|x_0\|_A$ that
\[
\|\phi(t, x_0, u)\| \leq \kappa(\|x_0\|_A, t) + \mu(\|u\|_U) \quad \forall t \geq 0.
\]

Thus, $\Sigma(A, F)$ is semi-uniformly ISS.

\[\square\]

### 3.4 Relation between semi-uniform input-to-state stability and strong input-to-stability

After recalling the notion of strong input-to-state stability introduced in [22, Definition 13], we study its relation to semi-uniform ISS with the help of the characterization in Theorem 3.9.

**Definition 3.10** The semi-linear system $\Sigma(A, F)$ is strongly input-to-state stable (strongly ISS) if $\Sigma(A, F)$ is forward complete and if there exist $\gamma, \mu \in \mathcal{K}_\infty$ and $\kappa : X \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that the following three conditions hold:
1. \( \kappa(x_0, \cdot) \in \mathcal{L} \) for all \( x_0 \in X \) with \( x_0 \neq 0 \).
2. \( \kappa(x_0, t) \leq \gamma(\|x_0\|) \) for all \( x_0 \in X \) and \( t \geq 0 \).
3. \( \|\phi(t, x_0, u)\| \leq \kappa(x_0, t) + \mu(\|u\|_U) \) for all \( x_0 \in X, u \in U, \) and \( t \geq 0 \).

For some special classes of semi-linear systems, semi-uniform ISS implies strong ISS.

**Theorem 3.11** Assume that the operator \( F \) of \( \Sigma(A, F) \) satisfies one of the following conditions:
1. There exists \( B \in \mathcal{L}(U, X) \) such that \( F(\xi, v) = Bv \) for all \( \xi \in X \) and \( v \in U \).
2. \( F(\xi - \zeta, v) = F(\xi, v) - F(\zeta, v) \) for all \( \xi, \zeta \in X \) and \( v \in U \).

Then, semi-uniform ISS implies strong ISS for \( \Sigma(A, F) \).

**Proof** By Theorem 12 of [22], the semi-linear system \( \Sigma(A, F) \) is strongly ISS if and only if \( \Sigma(A, F) \) is UGS and has the strong asymptotic gain property, which means that there exists \( \mu \in \mathcal{K}_\infty \) such that the following statement holds: For all \( \varepsilon > 0 \) and \( x_0 \in X \), there exists \( \tau = \tau(\varepsilon, x_0) < \infty \) such that for all \( u \in U, \)

\[
t \geq \tau \implies \|\phi(t, x_0, u)\| \leq \varepsilon + \mu(\|u\|_U). \tag{3.10}
\]

It suffices to show that semi-uniform ISS implies the strong asymptotic gain property.

1. Assume that there exists \( B \in \mathcal{L}(U, X) \) such that \( F(\xi, v) = Bv \) for all \( \xi \in X \) and \( v \in U \). By linearity, \( \phi(t, x_0, u) = \phi(t, x_0, 0) + \phi(t, 0, 0) \) for all \( x_0 \in X, u \in U, \) and \( t \geq 0 \). Since \( \Sigma(A, F) \) is semi-uniformly ISS, \( (T(t))_{t \geq 0} \) is uniformly bounded and \( \lim_{t \to \infty} T(t)x_0 = 0 \) as \( t \to \infty \) for all \( x_0 \in D(A) \). Hence, strong stability of \( (T(t))_{t \geq 0} \) follows by the density of \( D(A) \); see also Proposition A.3 of [11]. For all \( \varepsilon > 0 \) and \( x_0 \in X \), there exists \( \tau = \tau(\varepsilon, x_0) < \infty \) such that

\[
\|\phi(t, x_0, 0)\| = \|T(t)x_0\| \leq \varepsilon \quad \forall t \geq \tau.
\]

Since \( \Sigma(A, F) \) is UGS, there exists \( \mu \in \mathcal{K}_\infty \) such that

\[
\|\phi(t, 0, u)\| \leq \mu(\|u\|_U) \quad \forall u \in U, \forall t \geq 0.
\]

Thus, \( \Sigma(A, F) \) has the strong asymptotic gain property.

2. Assume that \( F(\xi - \zeta, v) = F(\xi, v) - F(\zeta, v) \) for all \( \xi, \zeta \in X \) and \( v \in U \). Since \( \Sigma(A, F) \) is UGS and has the semi-uniform asymptotic gain property by Theorem 3.9, there exist \( \gamma, \mu \in \mathcal{K}_\infty \) such that the following two properties hold:

(a) For all \( x_0 \in X, u \in U, \) and \( t \geq 0, \)

\[
\|\phi(t, x_0, u)\| \leq \gamma(\|x_0\|) + \mu(\|u\|_U). \tag{3.11}
\]

(b) For all \( \varepsilon > 0 \), there exists \( \tau = \tau(\varepsilon, r) < \infty \) such that for all \( x_0 \in D(A) \) with \( \|x_0\|_A \leq r \) and all \( u \in U, \)

\[
t \geq \tau \implies \|\phi(t, x_0, u)\| \leq \varepsilon + \mu(\|u\|_U). \tag{3.12}
\]
Take $\varepsilon > 0$ and $x_0 \in X$. There exists $y_0 \in D(A)$ such that $\|x_0 - y_0\| \leq \gamma^{-1}(\varepsilon/2)$. By assumption, for all $u \in \mathcal{U}$, $\phi(t, x_0, u) - \phi(t, y_0, u)$ is the mild solution of $\Sigma(A, F)$ with initial state $x_0 - y_0$ and input $u$. Therefore, the property (3.11) implies that

$$\|\phi(t, x_0, u) - \phi(t, y_0, u)\| = \|\phi(t, x_0 - y_0, u)\| \leq \gamma(\|x_0 - y_0\|) + \mu(\|u\|_\mathcal{U})$$

$$= \frac{\varepsilon}{2} + \mu(\|u\|_\mathcal{U})$$

(3.13)

for all $u \in \mathcal{U}$ and $t \geq 0$.

Since $y_0 \in D(A)$, it follows from the property (3.12) that in the case $y_0 \neq 0$, there exists $\tau = \tau(\varepsilon, \|y_0\|_A) < \infty$ such that for all $u \in \mathcal{U}$,

$$t \geq \tau \implies \|\phi(t, y_0, u)\| \leq \frac{\varepsilon}{2} + \mu(\|u\|_\mathcal{U}).$$

(3.14)

In the case $y_0 = 0$, the property (3.11) yields that (3.14) holds with $\tau = 0$. Combining the estimates (3.13) and (3.14), we obtain

$$t \geq \tau \implies \|\phi(t, x_0, u)\| \leq \varepsilon + \tilde{\mu}(\|u\|_\mathcal{U})$$

for all $u \in \mathcal{U}$, where $\tilde{\mu} := 2\mu \in K_\infty$. Since $y_0$ depends only on $\varepsilon$ and $x_0$, it follows that $\varepsilon$ and $x_0$ determine $\tau = \tau(\varepsilon, \|y_0\|_A)$. Thus, $\Sigma(A, F)$ has the strong asymptotic gain property. \hfill \Box

Suppose that $\Sigma(A, F)$ is strong ISS. If the input $u \in \mathcal{U}$ satisfies

$$\lim_{\tau \to \infty} \|u(\cdot + \tau)\|_\mathcal{U} \to 0,$$

then $\|\phi(t, x_0, u)\| \to 0$ as $t \to \infty$ for all $x_0 \in X$; see Lemma 2.5 of [30], where this convergence result has been proved under a weaker assumption. We obtain a convergence property of semi-uniform ISS as a corollary of Theorem 3.11.

**Corollary 3.12** Under the same assumption on the operator $F$ as in Theorem 3.11, if $\Sigma(A, F)$ is semi-uniformly ISS, then $\|\phi(t, x_0, u)\| \to 0$ as $t \to \infty$ for all $x_0 \in X$ and all $u \in \mathcal{U}$ satisfying $\|u(\cdot + \tau)\|_\mathcal{U} \to 0$ as $\tau \to \infty$.

### 4 Polynomial input-to-state stability of linear systems

In this section, we focus on polynomial ISS of linear systems for $\mathcal{U} = L_\infty(\mathbb{R}_+, \mathcal{U})$. First, we give a sufficient condition for general linear systems to be polynomially ISS. Next, we consider linear systems with diagonalizable generators and finite-rank input operators and refine the sufficient condition. Finally, a necessary and sufficient condition for polynomial ISS is presented in the case where the eigenvalues of the diagonalizable generator near the imaginary axis have uniformly separated imaginary parts.
4.1 Polynomial input-to-state stability for general linear systems

Let $X$ and $U$ be Banach spaces. Consider a linear system with state space $X$ and input space $U$:

$$
\Sigma_{\text{lin}}(A, B) \quad \left\{ \begin{array}{ll}
\dot{x}(t) = Ax(t) + Bu(t), & t \geq 0 \\
x(0) = x_0,
\end{array} \right.
$$

where $A$ is the generator of a $C_0$-semigroup $(T(t))_{t \geq 0}$ on $X$, $B \in \mathcal{L}(U, X)$ is an input operator, $x_0 \in X$ is an initial state, and $u \in L^\infty(\mathbb{R}_+, U)$ is an input.

To study ISS of linear systems, we employ the notion of admissibility studied in the seminal work \[36\].

**Definition 4.1** We call the operator $B \in \mathcal{L}(U, X)$ infinite-time $L^\infty$-admissible for a $C_0$-semigroup $(T(t))_{t \geq 0}$ on $X$ if there exists a constant $c > 0$ such that

$$
\left\| \int_0^t T(s) Bu(s) \, ds \right\| \leq c \|u\|_\infty
$$

for all $u \in L^\infty(\mathbb{R}_+, U)$ and $t \geq 0$.

Let $(T(t))_{t \geq 0}$ be a $C_0$-semigroup on $X$ and let $B \in \mathcal{L}(U, X)$. If there exists $\mu \in \mathcal{K}_\infty$ such that

$$
\left\| \int_0^t T(t-s) Bu(s) \, ds \right\| \leq \mu(\|u\|_\infty) \quad \forall u \in L^\infty(\mathbb{R}_+, U), \forall t \geq 0,
$$

then

$$
\left\| \int_0^t T(s) Bu(s) \, ds \right\| = \left\| \int_0^t T(s) B \frac{u(s)}{\|u\|_\infty} \, ds \right\| \|u\|_\infty \leq \mu(1) \|u\|_\infty
$$

for all $u \in L^\infty(\mathbb{R}_+, U) \setminus \{0\}$ and $t \geq 0$. Hence, $B$ is infinite-time $L^\infty$-admissible for $(T(t))_{t \geq 0}$.

As in the case of strong ISS \[23, Proposition 1\], polynomial ISS for $\mathcal{U} = L^\infty(\mathbb{R}_+, U)$ is equivalent to the combination of polynomial stability of $C_0$-semigroups and infinite-time $L^\infty$-admissibility of input operators.

**Lemma 4.2** Let $X$ and $U$ be Banach spaces. The linear system $\Sigma_{\text{lin}}(A, B)$ is polynomially ISS with parameter $\alpha > 0$ for $\mathcal{U} = L^\infty(\mathbb{R}_+, U)$ if and only if the $C_0$-semigroup $(T(t))_{t \geq 0}$ on $X$ generated by $A$ is polynomially stable with parameter $\alpha$ and the input operator $B \in \mathcal{L}(U, X)$ is infinite-time $L^\infty$-admissible for $(T(t))_{t \geq 0}$.

**Proof** By the remarks following Definitions 3.5 and 4.1, polynomial ISS of $\Sigma_{\text{lin}}(A, B)$ for $\mathcal{U} = L^\infty(\mathbb{R}_+, U)$ implies polynomial stability of $(T(t))_{t \geq 0}$ and infinite-time $L^\infty$-admissibility of $B$. The converse implication immediately follows, since there exist...
constants $M, c > 0$ such that

$$\|\phi(t, x_0, u)\| \leq \frac{M\|x_0\|_A}{(t + 1)^{1/\alpha}} + c\|u\|_\infty$$

for all $x_0 \in D(A), u \in L^\infty(\mathbb{R}^+, U)$, and $t \geq 0$. □

We provide a simple sufficient condition for $\Sigma_{\text{lin}}(A, B)$ to be polynomially ISS, by restricting the range of the input operator $B$.

**Proposition 4.3** Let $X$ and $U$ be Banach spaces. Suppose that $A$ is the generator of a polynomially stable semigroup with parameter $\alpha > 0$ on $X$. If $B \in \mathcal{L}(U, X)$ satisfies $\text{ran}(B) \subset D((-A)^\beta)$ for some $\beta > \alpha$, then $\Sigma_{\text{lin}}(A, B)$ is polynomially ISS with parameter $\alpha$ for $U = L^\infty(\mathbb{R}^+, U)$.

**Proof** Let $(T(t))_{t \geq 0}$ be the polynomially stable semigroup on $X$ generated by $A$. By Proposition 2.4, there exists $M > 0$ such that

$$\|T(t)(-A)^{-\beta}\| \leq \frac{M}{(t + 1)^{\beta/\alpha}} \quad \forall t \geq 0.$$

Since $(-A)^\beta$ is closed, we have that $(-A)^\beta B \in \mathcal{L}(U, X)$ by assumption. For all $u \in L^\infty(\mathbb{R}^+, U)$ and $t \geq 0$, we obtain

$$\left\| \int_0^t T(s)Bu(s)ds \right\| = \left\| \int_0^t T(s)(-A)^{-\beta}(-A)^\beta Bu(s)ds \right\|$$

$$\leq \int_0^t M\|(-A)^\beta B\| \|u\|_\infty ds$$

$$\leq \frac{\alpha M\|(-A)^\beta B\|}{\beta - \alpha} \|u\|_\infty.$$

Hence, $B$ is infinite-time $L^\infty$-admissible for $(T(t))_{t \geq 0}$. Thus, $\Sigma_{\text{lin}}(A, B)$ is polynomially ISS by Lemma 4.2. □

From an argument similar to that in Example 18 of [28], we see that if $\beta < \alpha$, then the condition $\text{ran}(B) \subset D((-A)^\beta)$ may not lead to UGS.

**Example 4.4** Let $A$ be the generator of a polynomially stable semigroup $(T(t))_{t \geq 0}$ with parameter $\alpha > 0$ on a Banach space $X$. Set $U := X$ and $B := (-A)^{-\beta}$ with $0 < \beta < \alpha$. Taking the input $u(t) := T(t)y_0$ with $y_0 \in X$, we obtain

$$\left\| \int_0^t T(t - s)Bu(s)ds \right\| = \|tT(t)(-A)^{-\beta}y_0\|$$

for all $t \geq 0$. If the linear system $\Sigma_{\text{lin}}(A, B)$ is UGS for $U = L^\infty(\mathbb{R}^+, U)$, then the uniform boundedness principle implies that

$$\sup_{t \geq 0} \|tT(t)(-A)^{-\beta}\| < \infty. \quad (4.1)$$
However, one can easily find polynomially stable semigroups with parameter $\alpha$ for which the condition (4.1) does not hold. Hence, the condition $\text{ran}(B) \subset D((-A)^{\beta})$ with $\beta < \alpha$ does not imply UGS in general. The case $\alpha = \beta$ remains open except in the diagonalizable case studied in the next subsection.

### 4.2 Polynomial input-to-state stability for diagonalizable linear systems

In this subsection, we consider linear systems with diagonalizable generators and finite-rank input operators. We aim to refine the condition on the range of the input operator obtained in Proposition 4.3. To this end, we first review the definition and basic properties of diagonalizable operators; see Section 2.6 of [35] for details.

**Definition 4.5** Let $X$ be a Hilbert space. The linear operator $A : D(A) \subset X \rightarrow X$ is diagonalizable if $\rho(A) \neq \emptyset$ and there exists a Riesz basis $(\varphi_n)_{n \in \mathbb{N}}$ in $X$ consisting of eigenvectors of $A$.

Throughout this subsection, we place the following assumption.

**Assumption 4.6** Let $X$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. The operator $A : D(A) \subset X \rightarrow X$ is diagonalizable, and $(\varphi_n)_{n \in \mathbb{N}}$ is a Riesz basis in $X$ consisting of eigenvectors of $A$. The biorthogonal sequence for $(\varphi_n)_{n \in \mathbb{N}}$ and the eigenvalue corresponding to the eigenvector $\varphi_n$ are given by $(\psi_n)_{n \in \mathbb{N}}$ and $\lambda_n$, respectively.

**Proposition 4.7** Suppose that Assumption 4.6 is satisfied. Then, the following statements hold:

1. The operator $A$ may be written as
   \[ Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, \psi_n \rangle \varphi_n \quad \forall x \in D(A) \]
   and
   \[ D(A) = \left\{ x \in X : \sum_{n=1}^{\infty} |\lambda_n|^2 |\langle x, \psi_n \rangle|^2 < \infty \right\}. \]

2. The operator $A$ is the generator of a $C_0$-semigroup $(T(t))_{t \geq 0}$ on $X$ if and only if
   \[ \sup_{n \in \mathbb{N}} \text{Re} \lambda_n < \infty. \]
   In this case, the exponential growth bound of $(T(t))_{t \geq 0}$ is given by $\sup_{n \in \mathbb{N}} \text{Re} \lambda_n$, and for all $x \in X$ and $t \geq 0$,
   \[ T(t)x = \sum_{n=1}^{\infty} e^{t\lambda_n} \langle x, \psi_n \rangle \varphi_n. \]
Suppose that the eigenvalues \((\lambda_n)_{n \in \mathbb{N}}\) of a diagonalizable operator \(A\) satisfy \(\text{Re} \lambda_n \leq 0\) for all \(n \in \mathbb{N}\). Then, \(A\) generates a uniformly bounded semigroup. Moreover, \(-A\) is sectorial in the sense of [12, Chapter 2], and hence, the fractional power \((-A)^\alpha\) is well defined for every \(\alpha > 0\). The domain of the fractional power \((-A)^\alpha\) is given by

\[
D((-A)^\alpha) = \left\{ x \in X : \sum_{n=1}^{\infty} |\lambda_n|^{2\alpha} |\langle x, \psi_n \rangle|^2 < \infty \right\}
\]

for all \(\alpha > 0\), where \((\psi_n)_{n \in \mathbb{N}}\) is as in Assumption 4.6.

A diagonalizable operator is similar to a normal operator. Hence, by Proposition 2.5, a diagonalizable operator with eigenvalues \((\lambda_n)_{n \in \mathbb{N}}\) generates a polynomially stable semigroup with parameter \(\alpha > 0\) if and only if \(\text{Re} \lambda_n < 0\) for all \(n \in \mathbb{N}\) and there exist \(C, p > 0\) such that

\[
|\text{Im} \lambda_n| \geq \frac{C}{|\text{Re} \lambda_n|^{1/\alpha}} \quad \text{if} \ \text{Re} \lambda_n > -p. \tag{4.2}
\]

We obtain a refined sufficient condition for linear systems with diagonalizable generators and finite-rank input operators to be polynomially ISS.

**Theorem 4.8** Let Assumption 4.6 be satisfied, and let \(U\) be a Banach space. Suppose that the diagonalizable operator \(A\) generates a polynomially stable semigroup with parameter \(\alpha > 0\) on \(X\). If \(B \in \mathcal{L}(U, X)\) is a finite-rank operator and satisfies \(\text{ran}(B) \subset D((-A)^\alpha)\), then \(\Sigma_{\text{lin}}(A, B)\) is polynomially ISS with parameter \(\alpha\) for \(U = L^\infty(\mathbb{R}_+, U)\).

**Proof** By Lemma 4.2, it suffices to show that \(B\) is infinite-time \(L^\infty\)-admissible for \((T(t))_{t \geq 0}\) in the case \(B \neq 0\).

By a property of a Riesz basis (see, e.g., Proposition 2.5.2 of [35]), there exists a constant \(M_1 > 0\) such that

\[
\left\| \int_0^t T(s)Bu(s)ds \right\|^2 \leq M_1 \sum_{n=1}^{\infty} \left| \left( \int_0^t T(s)Bu(s)ds, \psi_n \right) \right|^2 \tag{4.3}
\]

for all \(u \in L^\infty(\mathbb{R}_+, U)\) and \(t \geq 0\). Since \(B\) is a finite-rank operator, there is an orthonormal basis \((\xi_k)_{k=1}^m\) of the finite-dimensional space \(\text{ran}(B)\), where \(m \in \mathbb{N}\) is the dimension of \(\text{ran}(B)\). Then, we obtain

\[
Bu = \sum_{k=1}^{m} \langle Bu, \xi_k \rangle \xi_k \quad \forall v \in U.
\]

Since

\[
\langle T(s)\xi_k, \psi_n \rangle = \langle \xi_k, T(s)^*\psi_n \rangle = e^{s\lambda_n} \langle \xi_k, \psi_n \rangle
\]
for all $s \geq 0$, it follows that
\[
\left\langle \int_0^t T(s)Bu(s)ds, \psi_n \right\rangle = \left\langle \int_0^t \sum_{k=1}^m \langle Bu(s), \xi_k \rangle T(s) \xi_k ds, \psi_n \right\rangle = \sum_{k=1}^m \int_0^t e^{s\lambda_n} \langle Bu(s), \xi_k \rangle ds \langle \xi_k, \psi_n \rangle
\]
for all $u \in L^\infty(\mathbb{R}_+, U)$ and $t \geq 0$. Therefore,
\[
\left| \left\langle \int_0^t T(s)Bu(s)ds, \psi_n \right\rangle \right| \leq \sum_{k=1}^m \int_0^t e^{s\operatorname{Re} \lambda_n} |\langle Bu(s), \xi_k \rangle| ds |\langle \xi_k, \psi_n \rangle| \leq \frac{\|B\|\|u\|_\infty}{|\operatorname{Re} \lambda_n|} \sum_{k=1}^m |\langle \xi_k, \psi_n \rangle|.
\]

Combining $\xi_k \in D((-A)\alpha)$ with the geometric condition (4.2) on $(\lambda_n)_{n \in \mathbb{N}}$, we obtain
\[
\sum_{n=1}^\infty \frac{|\langle \xi_k, \psi_n \rangle|^2}{|\operatorname{Re} \lambda_n|^2} \leq \frac{1}{p^2} \sum_{n=1}^\infty |\langle \xi_k, \psi_n \rangle|^2 + \frac{1}{C^{2\alpha}} \sum_{n=1}^\infty |\lambda_n|^{2\alpha} |\langle \xi_k, \psi_n \rangle|^2 =: c_k < \infty
\]
for every $k = 1, \ldots, m$. Therefore,
\[
\sum_{n=1}^\infty \left| \left\langle \int_0^t T(s)Bu(s)ds, \psi_n \right\rangle \right|^2 \leq m (\|B\|\|u\|_\infty)^2 \sum_{n=1}^\infty \frac{|\langle \xi_k, \psi_n \rangle|^2}{|\operatorname{Re} \lambda_n|^2} \leq m (\|B\|\|u\|_\infty)^2 \sum_{k=1}^m c_k
\]
for all $u \in L^\infty(\mathbb{R}_+, U)$ and $t \geq 0$. From the estimates (4.3) and (4.4), we obtain
\[
\left\| \int_0^t T(s)Bu(s)ds \right\| \leq \left( \|B\| \sqrt{m M_1 \sum_{k=1}^m c_k} \right) \|u\|_\infty
\]
for all $u \in L^\infty(\mathbb{R}_+, U)$ and $t \geq 0$. Thus, $B$ is infinite-time $L^\infty$-admissible for $(T(t))_{t \geq 0}$. \square

We apply Theorem 4.8 to an Euler–Bernoulli beam with weak damping.
Example 4.9 Consider a simply supported Euler–Bernoulli beam with weak damping, which is described by the following partial differential equation on \((0, 1)\):

\[
\begin{cases}
\frac{\partial^2 z}{\partial t^2}(\zeta, t) + \frac{\partial^4 z}{\partial \zeta^4}(\zeta, t) + h(\zeta) \int_0^1 h(r) \frac{\partial z}{\partial t}(r, t) \, dr + b(\zeta) u(t) = 0, \\
0 < \zeta < 1, \ t \geq 0 \\
z(0, t) = z(1, t), \ \frac{\partial^2 z}{\partial \zeta^2}(0, t) = 0 = \frac{\partial^2 z}{\partial \zeta^2}(1, t), \ t \geq 0 \\
z(\zeta, 0) = z_0(\zeta), \ \frac{\partial z}{\partial t}(\zeta, 0) = z_1(\zeta), \ 0 < \zeta < 1,
\end{cases}
\]

(4.5)

where \(b \in L^2(0, 1)\) is a “shaping function” for the external input \(u\) and \(h\) is the damping coefficient. Here, we set \(h(\zeta) := 1 - \zeta\) for \(\zeta \in (0, 1)\).

It is well known that the partial differential Equ. (4.5) can be written as a first-order linear system in the following way; see, e.g., Exercise 3.18 of [9]. Define \(X_0 := L^2(0, 1)\) and

\[
A_0 f := \frac{d^4 f}{d \zeta^4}
\]

with domain

\[
D(A_0) := \left\{ f \in W^{4,2}(0, 1) : f(0) = 0 = f(1) \text{ and } \frac{d^2 f}{d \zeta^2}(0) = 0 = \frac{d^2 f}{d \zeta^2}(1) \right\}.
\]

The operator \(A_0\) has a positive self-adjoint square root \(A_0^{1/2} = -\frac{d^2}{d \zeta^2}\) with domain

\[
D\left(A_0^{1/2}\right) = \{ f \in W^{2,2}(0, 1) : f(0) = 0 = f(1) \}.
\]

The space \(X := D(A_0^{1/2}) \times L^2(0, 1)\) equipped with an inner product

\[
\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle := \left\langle A_0^{1/2} x_1, A_0^{1/2} y_1 \right\rangle_{L^2} + \langle x_2, y_2 \rangle_{L^2}
\]

is a Hilbert space. Define the operators \(A_1 : D(A_1) \subset X \to X\) and \(B, H \in \mathcal{L}(\mathbb{C}, X)\) by

\[
A_1 := \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}
\]

with domain \(D(A_1) = D(A_0) \times D(A_0^{1/2})\) and

\[
Bv := \begin{bmatrix} 0 \\ -bv \end{bmatrix}, \quad Hv := \begin{bmatrix} 0 \\ hv \end{bmatrix}, \quad v \in \mathbb{C}.
\]
For

\[ x := \begin{bmatrix} z \\ \frac{\partial z}{\partial t} \end{bmatrix}, \quad x_0 := \begin{bmatrix} z_0 \\ z_1 \end{bmatrix}, \]

the partial differential Equ. (4.5) can be written as

\[ \dot{x}(t) = (A_1 - HH^*)x(t) + Bu(t), \quad t \geq 0; \quad x(0) = x_0. \]

The operator \( A_1 \) is diagonalizable with simple eigenvalues

\[ \lambda_n := in^2\pi^2, \quad \lambda_{-n} := -in^2\pi^2, \quad n \in \mathbb{N}. \]

Since \( A_1 \) has compact resolvents by Lemma 3.2.12 of [9], it follows from Theorem 1 of [37] that \( A := A_1 - HH^* \) is also diagonalizable. Moreover, \( A \) generates a polynomially stable semigroup with parameter \( \alpha = 1 \) by Corollary 6.6 of [7]. Thus, Theorem 4.8 shows that \( \Sigma_{\text{lin}}(A, B) \) is polynomially ISS with parameter \( \alpha = 1 \) if \( \text{ran}(B) \subset D(A) = D(A_1) \), i.e., \( b \in D(A_0^{1/2}) \).

### 4.3 Case where eigenvalues near the imaginary axis have uniformly separated imaginary parts

We investigate how sharp the condition \( \text{ran}(B) \subset D((-A)\alpha) \) is. To this end, we employ the relation between Laplace–Carleson embeddings and infinite-time \( L^\infty \)-admissibility established in [16].

Let \( A : D(A) \subset X \to X \) be diagonalizable and generate a strongly stable semigroup \( (T(t))_{t \geq 0} \) on \( X \). Let \( B \in \mathcal{L}(\mathbb{C}, X) \) be represented as \( Bv = bv \) for some \( b \in X \) and all \( v \in \mathbb{C} \). Define the Borel measure \( \nu \) on the open right half-plane \( \{ \lambda \in \mathbb{C} : \text{Re} \lambda > 0 \} \) by

\[ \nu := \sum_{n \in \mathbb{N}} |\langle b, \psi_n \rangle|^2 \delta_{-\lambda_n}, \]

where \( (\lambda_n)_{n \in \mathbb{N}} \) and \( (\psi_n)_{n \in \mathbb{N}} \) are as in Assumption 4.6 and \( \delta_{-\lambda_n} \) is the Dirac measure at the point \( -\lambda_n \) for \( n \in \mathbb{N} \). Define the Carleson square \( Q_I \) associated with an interval \( I \subset i\mathbb{R} \) and the dyadic stripe \( S_k \) for \( k \in \mathbb{Z} \) by

\[ Q_I := \{ \lambda \in \mathbb{C} : i\text{Im} \lambda \in I, \ 0 < \text{Re} \lambda < |I| \} \quad (4.6) \]

\[ S_k := \{ \lambda \in \mathbb{C} : 2^k \leq \text{Re} \lambda < 2^{k+1} \}. \quad (4.7) \]

Then, Theorem 2.5 of [16] shows that \( B \) is infinite-time \( L^\infty \)-admissible for \( (T(t))_{t \geq 0} \) if and only if

\[ \sum_{k \in \mathbb{Z}} \sup_{I \subset i\mathbb{R}} \frac{\nu(Q_I \cap S_k)}{|I|^2} < \infty. \]

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Using this equivalence of admissibility, we obtain a necessary and sufficient condition for polynomial ISS. We write

\[
\Phi_k := \begin{cases} 
\sup_{-\lambda_n \in S_k} \frac{|\langle b, \psi_n \rangle|^2}{|\operatorname{Re} \lambda_n|^2} & \text{if } \{-\lambda_n : n \in \mathbb{N}\} \cap S_k \neq \emptyset \\
0 & \text{if } \{-\lambda_n : n \in \mathbb{N}\} \cap S_k = \emptyset
\end{cases}
\]  

for \( k \in \mathbb{Z} \).

**Theorem 4.10** Let Assumption 4.6 hold and let \( B \in \mathcal{L}(\mathbb{C}, X) \) be represented as \( Bv = bv \) for some \( b \in X \) and all \( v \in \mathbb{C} \). Assume that there exists \( p > 0 \) such that

\[
\inf \{|\operatorname{Im} \lambda_n - \operatorname{Im} \lambda_m| : n, m \in \mathbb{N}, n \neq m, |\operatorname{Re} \lambda_n|, |\operatorname{Re} \lambda_m| < p\} > 0. \tag{4.10}
\]

Then, \( \Sigma_{\text{lin}}(A, B) \) is polynomially ISS with parameter \( \alpha > 0 \) for \( U = L^\infty(\mathbb{R}_+, U) \) if and only if the diagonalizable operator \( A \) generates a polynomially stable semigroup with parameter \( \alpha \) and

\[
\sum_{k \in \mathbb{Z}} \Phi_k < \infty, \tag{4.11}
\]

where \( \Phi_k \) is defined as in (4.9) for \( k \in \mathbb{Z} \).

**Proof** We first note that a polynomially stable semigroup is strongly stable. By Lemma 4.2, it suffices to show that the conditions (4.8) and (4.11) are equivalent under the assumption (4.10).

Let \( p > 0 \) satisfy (4.10). There exists \( d > 0 \) such that \( |\operatorname{Im} \lambda_n - \operatorname{Im} \lambda_m| \geq d \) for all \( n, m \in \mathbb{N} \) satisfying \( n \neq m \) and \( |\operatorname{Re} \lambda_n|, |\operatorname{Re} \lambda_m| < p \). If an interval \( I \subset i\mathbb{R} \) satisfies \( |I| \geq d \), then for all \( k \in \mathbb{Z} \),

\[
\frac{\nu(Q_I \cap S_k)}{|I|^2} \leq \frac{\nu(Q_I \cap S_k)}{d^2} \leq \frac{\nu(S_k)}{d^2}.
\]

Suppose next that an interval \( I \subset i\mathbb{R} \) satisfies \( |I| < d \). If \( k \in \mathbb{Z} \) satisfies \( p \leq 2^{k+1} \), then \( \nu(Q_I \cap S_k) > 0 \) implies \( |I| \geq p/2 \), and therefore

\[
\frac{\nu(Q_I \cap S_k)}{|I|^2} \leq \frac{4\nu(Q_I \cap S_k)}{p^2} \leq \frac{4\nu(S_k)}{p^2}.
\]

Let \( k \in \mathbb{Z} \) satisfy \( p > 2^{k+1} \). For \(-\lambda_n, -\lambda_m \in S_k \) with \( n \neq m \), we obtain \( |\operatorname{Im} \lambda_n - \operatorname{Im} \lambda_m| \geq d \) by assumption. Recalling that the interval \( I \) is chosen so that \( |I| < d \), we have that \( Q_I \cap S_k \) contains at most one element of \( (-\lambda_n)_{n \in \mathbb{N}} \). Since \( \nu(Q_I \cap S_k) = |\langle b, \psi_n \rangle|^2 \) for some \( n \in \mathbb{N} \) with \(-\lambda_n \in S_k \) or \( \nu(Q_I \cap S_k) = 0 \), it follows that

\[
\frac{\nu(Q_I \cap S_k)}{|I|^2} \leq \Phi_k.
\]
We have shown that for every interval $I \subset i\mathbb{R}$ and $k \in \mathbb{Z}$,
\[
\frac{\nu(Q_I \cap S_k)}{|I|^2} \leq \max \left\{ \frac{\nu(S_k)}{d^2}, \frac{4\nu(S_k)}{p^2}, \Phi_k \right\}.
\]
Hence,
\[
\sum_{k \in \mathbb{Z}} \sup_{I \subset i\mathbb{R}} \frac{\nu(Q_I \cap S_k)}{|I|^2} \leq \left( \frac{1}{d^2} + \frac{4}{p^2} \right) \sum_{k \in \mathbb{Z}} \nu(S_k) + \sum_{k \in \mathbb{Z}} \Phi_k.
\]
Since $b \in X$, it follows that $\sum_{n \in \mathbb{N}} |\langle b, \psi_n \rangle|^2 < \infty$. Therefore, (4.11) implies (4.8).

Conversely, for all $k \in \mathbb{Z}$, if $-\lambda_n \in S_k$, then
\[
\frac{\nu(Q_I \cap S_k)}{|I|^2} \leq \sup_{I \subset i\mathbb{R}} \frac{\nu(Q_I \cap S_k)}{|I|^2},
\]
and hence
\[
\Phi_k \leq \sup_{I \subset i\mathbb{R}} \frac{\nu(Q_I \cap S_k)}{|I|^2}.
\]
This yields
\[
\sum_{k \in \mathbb{Z}} \Phi_k \leq \sum_{k \in \mathbb{Z}} \sup_{I \subset i\mathbb{R}} \frac{\nu(Q_I \cap S_k)}{|I|^2}.
\]
Thus, (4.8) implies (4.11).

For $k \in \mathbb{Z}$, define
\[
\tilde{\Phi}_k := \begin{cases} 
\sup_{-\lambda_n \in S_k} |\lambda_n|^{2\alpha} |\langle b, \psi_n \rangle|^2 & \text{if } \{-\lambda_n : n \in \mathbb{N}\} \cap S_k \neq \emptyset \\
0 & \text{if } \{-\lambda_n : n \in \mathbb{N}\} \cap S_k = \emptyset.
\end{cases}
\]
A routine calculation shows that if $\lim_{n \to \infty} \text{Re} \lambda_n = 0$ and if there exists $C > 0$ such that
\[
|\text{Im} \lambda_n| - \frac{C}{|\text{Re} \lambda_n|^{1/\alpha}} \to 0 \quad \text{as } n \to \infty,
\]
then the condition (4.11) is equivalent to
\[
\sum_{k \in \mathbb{Z}} \tilde{\Phi}_k < \infty.
\]
From this, we observe that the condition \((4.11)\) is milder than \(b \subset D((-A)^{\alpha})\). However, the following example shows that if the assumption \((4.10)\) is not satisfied, then \(b \in D((-A)^{\alpha})\) may be necessary and sufficient for infinite-time \(L^\infty\)-admissibility.

**Example 4.11** Consider a diagonalizable operator \(A\) whose eigenvalues \((\lambda_n)_{n \in \mathbb{N}}\) are given by

\[
\lambda_n := -\frac{1}{2^k} - i2^k, \quad 2^k \leq n \leq 2^{k+1} - 1, \quad k \in \mathbb{N}_0.
\]

Since \((\lambda_n)_{n \in \mathbb{N}}\) satisfies the geometric condition \((4.2)\) with \(\alpha = 1\), it follows that \(A\) generates a polynomially stable semigroups with parameter \(\alpha = 1\). For all \(k \in \mathbb{N}_0\), taking intervals \(I \subset i\mathbb{R}\) with center \(i2^k\), we obtain

\[
\sup_{I \subset i\mathbb{R} \text{ interval}} \frac{\nu(Q_I \cap S_{-k})}{|I|^2} = \frac{\sum_{n=2^k}^{2^{k+1}-1} |\langle b, \psi_n \rangle|^2}{1/2^{2k}} \geq \frac{1}{2} \sum_{n=2^k}^{2^{k+1}-1} |\lambda_n|^2 |\langle b, \psi_n \rangle|^2.
\]

This yields

\[
\sum_{n=1}^{\infty} |\lambda_n|^2 |\langle b, \psi_n \rangle|^2 = \sum_{k \in \mathbb{N}_0} \sum_{n=2^k}^{2^{k+1}-1} |\lambda_n|^2 |\langle b, \psi_n \rangle|^2 \\
\leq 2 \sum_{k \in \mathbb{N}_0} \sup_{I \subset i\mathbb{R} \text{ interval}} \frac{\nu(Q_I \cap S_k)}{|I|^2}.
\]

Thus, infinite-time \(L^\infty\)-admissibility implies \(b \in D(A)\).

## 5 Polynomial integral input-to-state stability of bilinear systems

In the previous section, we saw that polynomial ISS is restrictive even for linear systems with bounded input operators. This is because infinite-time \(L^\infty\)-admissibility cannot be achieved for all bounded input operators due to the weak asymptotic property of polynomially stable semigroups. This motivates us to study a semi-uniform version of integral input-to-state stability, which provides norm estimates of trajectories with respect to a kind of energy fed into systems.

We recall a stability notion for systems without inputs; see [22, Definition 5].

**Definition 5.1** The semi-linear system \(\Sigma(A, F)\) is called *uniformly globally stable at zero* if the following two conditions hold:
1. \( \Sigma(A, F) \) is forward complete.
2. There exists \( \gamma \in K_\infty \) such that
   \[
   \| \phi(t, x_0, 0) \| \leq \gamma(\|x_0\|)
   \]
   for all \( x_0 \in X \) and \( t \geq 0 \).

We define the concept of semi-uniform integral input-to-state stability.

**Definition 5.2** The semi-linear system \( \Sigma(A, F) \) is called semi-uniformly integral input-to-state stable (semi-uniformly iISS) if the following two conditions hold:

1. \( \Sigma(A, F) \) is uniformly globally stable at zero.
2. There exist \( \kappa \in KL, \theta \in K_\infty, \) and \( \mu \in K \) such that
   \[
   \| \phi(t, x_0, u) \| \leq \kappa(\|x_0\|_A, t) + \theta \left( \int_0^t \mu(\|u(s)\|_U)ds \right)
   \]
   \hspace{1cm} (5.1)
   for all \( x_0 \in D(A), u \in U, \) and \( t \geq 0 \).

In particular, if there exists \( \alpha > 0 \) such that for all \( r > 0, \kappa(r, t) = O(t^{-1/\alpha}) \) as \( t \to \infty \), then \( \Sigma(A, F) \) is called polynomially integral input-to-state stable (polynomially iISS) with parameter \( \alpha > 0 \).

Note that the integral \( \int_0^t \mu(\|u(s)\|_U)ds \) in the right-hand side of the inequality (5.1) may be infinite. In that case, the inequality (5.1) trivially holds.

For every generator \( A \) of a semi-uniformly stable semigroup and every bounded input operator \( B \), the linear system \( \Sigma_{lin}(A, B) \) is semi-uniform iISS. Moreover, if the linear system \( \Sigma_{lin}(A, B) \) is semi-uniform iISS, then \( \Sigma_{lin}(A, B) \) is strong iISS in the sense of Definition 4 in [23]. This can be seen by using the equality \( \phi(t, x_0, u) = \phi(t, x_0, 0) + \phi(t, 0, u) \) as in the case of semi-uniform ISS discussed in Theorem 3.11.

The aim of this section is to give a sufficient condition for bilinear systems satisfying Assumption 3.2 to be polynomially iISS for \( U = L^\infty(\mathbb{R}_+, U) \). We prove that if the nonlinear operator additionally satisfies a certain smoothness assumption, then the bilinear system is polynomially iISS. To this end, we use a non-Lyapunov method devised in Theorem 4.2 of [23] for uniform iISS.

**Theorem 5.3** Let \( A \) be the generator of a polynomially stable semigroup \( (T(t))_{t \geq 0} \) with parameter \( \alpha > 0 \) on a Banach space \( X \). Suppose that the nonlinear operator \( F \) satisfies Assumption 3.2 for another Banach space \( U \) and that there exist \( K > 0 \) and \( \chi \in K \) such that for all \( \xi \in X, v \in U, \) and \( t \geq 0, \)

\[
\| T(t)G(\xi, v) \| \leq \frac{K \| \xi \| \chi(\|v\|_U)}{(t + 1)^{1/\alpha}}.
\]

Then, the bilinear system \( \Sigma(A, F) \) is polynomially iISS with parameter \( \alpha \) for \( U = L^\infty(\mathbb{R}_+, U) \).

**Proof** Since \((T(t))_{t \geq 0}\) is polynomially stable with parameter \( \alpha > 0 \), there exists \( M \geq 1 \) such that
\[ \| T(t) \| \leq M, \quad \| T(t) R(1, A) \| \leq \frac{M}{(t + 1)^{1/\alpha}} \quad \forall t \geq 0. \]

By Gronwall's inequality (see Appendix A of [24] for a simple proof), we have that for all \( x_0 \in X, u \in L^\infty(\mathbb{R}_+, U) \), and \( t \geq 0 \),

\[
\| x(t) \| \leq M \left( \| x_0 \| + \| B \| \int_0^t \| u(s) \| U \, ds \right) + K \int_0^t \| x(s) \| \chi(\| u(s) \| U) \, ds \\
\leq M (\| x_0 \| + t \| B \| \| u \|_\infty) e^{t K \chi(\| u \|_\infty)}
\]

as long as \( x \) is a mild solution of \( \Sigma(A, F) \) on \([0, t]\). Hence, \( \Sigma(A, F) \) is forward complete by the remark following Assumption 3.2. Moreover, \( F(\xi, 0) = 0 \) for all \( \xi \in X \) under Assumption 3.2. Therefore, if \( u(t) \equiv 0 \), then the mild solution \( x \) of \( \Sigma(A, F) \) satisfies

\[
\| x(t) \| = \| T(t) x_0 \| \leq M \| x_0 \|
\]

for all \( x_0 \in X \) and \( t \geq 0 \), which implies that \( \Sigma(A, F) \) is uniformly globally stable at zero.

Take \( x_0 \in D(A) \) and \( u \in L^\infty(\mathbb{R}_+, U) \). The mild solution \( x \) of \( \Sigma(A, F) \) satisfies

\[
\| x(t) \| \leq \frac{M}{(t + 1)^{1/\alpha}} \| x_0 \| A + \int_0^t \left( \frac{M \| B \| \| u(s) \| U}{(t - s + 1)^{1/\alpha}} \right) \| x(s) \| \chi(\| u(s) \| U) \, ds \\
+ \frac{K}{(t - s + 1)^{1/\alpha}} \| x(s) \| \chi(\| u(s) \| U) \, ds
\]

for all \( t \geq 0 \). Define \( z(t) := (t + 1)^{1/\alpha} \| x(t) \| \) for \( t \geq 0 \). Then,

\[
z(t) \leq M \left( \| x_0 \| A + \| B \| (t + 1)^{1/\alpha} \int_0^t \| u(s) \| U \, ds \right) \\
+ K \int_0^t \left( \frac{t + 1}{(t - s + 1)(s + 1)} \right)^{1/\alpha} z(s) \chi(\| u(s) \| U) \, ds
\]

for all \( t \geq 0 \). Since

\[
\max_{0 \leq s \leq t} \left( \frac{t + 1}{(t - s + 1)(s + 1)} \right)^{1/\alpha} = 1,
\]

Gronwall's inequality implies that for all \( t \geq 0 \),

\[
z(t) \leq M \left( \| x_0 \| A + \| B \| (t + 1)^{1/\alpha} \int_0^t \| u(s) \| U \, ds \right) e^{K \int_0^t \chi(\| u(s) \| U) \, ds},
\]

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which is equivalent to
\[
\|x(t)\| \leq M \left( \frac{\|x_0\|_A}{(t+1)^{1/\alpha}} + \|B\| \int_0^t \|u(s)\|_U \, ds \right) e^{K \int_0^t \chi(\|u(s)\|_U) \, ds}.
\]

Using the inequality
\[
\ln(1 + ae^b) \leq \ln(1 + a) + b \quad \forall a, b \geq 0,
\]
we obtain
\[
\ln(1 + \|x(t)\|) \
\leq \ln \left( 1 + M \left( \frac{\|x_0\|_A}{(t+1)^{1/\alpha}} + \|B\| \int_0^t \|u(s)\|_U \, ds \right) \right) + K \int_0^t \chi(\|u(s)\|_U) \, ds
\]
for all \( t \geq 0 \). Since
\[
\ln(1 + a + b) \leq \ln(1 + a) + \ln(1 + b) \quad \forall a, b \geq 0,
\]
it follows that
\[
\ln(1 + \|x(t)\|) \leq \ln \left( 1 + M \frac{\|x_0\|_A}{(t+1)^{1/\alpha}} \right) + \ln \left( 1 + M \|B\| \int_0^t \|u(s)\|_U \, ds \right) + K \int_0^t \chi(\|u(s)\|_U) \, ds
\]
for all \( t \geq 0 \). The inverse function of \( q(r) := \ln(1 + r), r \geq 0, \) is given by \( q^{-1}(r) = e^r - 1 \). Using the inequality
\[
e^{a+b} - 1 \leq (e^{2a} - 1) + (e^{2b} - 1) \quad \forall a, b \geq 0
\]
twice, we obtain
\[
\|x(t)\| \leq \left( 1 + \frac{M \|x_0\|_A}{(t+1)^{1/\alpha}} \right)^2 - 1 + \left( 1 + M \|B\| \int_0^t \|u(s)\|_U \, ds \right)^4 - 1 + e^{4K \int_0^t \chi(\|u(s)\|_U) \, ds} - 1
\]
for all \( t \geq 0 \). Thus, the bilinear system \( \Sigma(A, F) \) is polynomially iISS with parameter \( \alpha \), where \( \kappa \in \mathcal{K}\mathcal{L}, \theta \in \mathcal{K}_\infty \), and \( \mu \in \mathcal{K} \) are given by
\[
\kappa(r, t) := \left( \frac{Mr}{(t+1)^{1/\alpha}} \right)^2 + \frac{2Mr}{(t+1)^{1/\alpha}}
\]
\[
\theta(r) := r^4 + 4r^3 + 6r^2 + 4r + e^r - 1
\]
\[
\mu(r) := \max\{M \|B\| r, 4K \chi(r)\}
\]
for the estimate (5.1). \( \square \)
6 Conclusion

We have introduced the notion of semi-uniform ISS and have established its characterization based on attractivity properties. We have given sufficient conditions for linear systems to be polynomially ISS. In the sufficient conditions, the range of the input operator is restricted, depending on the polynomial decay rate of the product of the $C_0$-semigroup and the resolvent of its generator. We have also shown that a class of bilinear systems are polynomially iISS if the nonlinear operator satisfies a smoothness assumption like the range condition of input operators for polynomial ISS of linear systems. Important directions for future research are to explore the relation between semi-uniform ISS and semi-uniform iISS and to construct Lyapunov functions for polynomial ISS and polynomial iISS.

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