A STOCHASTIC MODEL FOR COMPUTER VIRUS PROPAGATION

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(Communicated by Georges Zaccour)

ABSTRACT. A three-dimensional continuous-time stochastic model based on the classic Kermack-McKendrick model for the spread of epidemics is proposed for the propagation of a computer virus. Moreover, control variables are introduced into the model. We look for the controls that either minimize or maximize the expected time it takes to clean the infected computers, or to protect them from the virus. Using dynamic programming, the equations satisfied by the value functions are derived. Particular problems are solved explicitly.

1. Introduction. In recent years, numerous papers have been published on the mathematical modelling of computer virus propagation. A special issue of the journal Security and Communication Networks devoted to mathematical models for malware propagation was published in 2019. Earlier papers on this subject include the ones by Mishra and Pandey [6], Mishra and Saini [7], Gan et al. [1], Peng et al. [8], Song et al. [11], Qin [9] and Xu and Ren [15].

Many of the papers mentioned above were based on the classic Kermack-McKendrick model for the dynamics of an infectious disease. Let \(X(t)\) be the number of individuals, at time \(t\), in a certain population who are susceptible to a given disease, \(Y(t)\) be the number of those who are infected and \(Z(t)\) be the number of those who are removed from the population because they are either recovered and immune, or quarantined, or dead. Instead of considering the number of individuals, the variables \(X(t)\), \(Y(t)\) and \(Z(t)\) could represent the percentage of individuals in each category.

Kermack and McKendrick [3] proposed the following model for the spread of epidemics:

\[
\begin{align*}
\frac{dX(t)}{dt} &= -k_1 X(t) Y(t) dt, \\
\frac{dY(t)}{dt} &= k_1 X(t) Y(t) dt - k_2 Y(t) dt, \\
\frac{dZ(t)}{dt} &= k_2 Y(t) dt, \\
\end{align*}
\]

(1)

where \(k_1\) and \(k_2\) are positive constants. It is therefore assumed that the rate at which the susceptible individuals become infected is proportional to the product \(X(t)Y(t)\).

The model defined in (1) is a particular SIR model (for Susceptible, Infected, Removed). SIR models have been used by many authors to explain the spread of various diseases; see, for instance, Rachah and Torres [10] where such a model was
proposed to explain the 2014 outbreak of the Ebola virus in West Africa. Ionescu et al. [2] and Lefebvre [4] solved optimal control problems for stochastic versions of the classic Kermack-McKendrick model. Tong et al. [12] also considered an optimal control problem based on this virus propagation model.

Here, the Kermack-McKendrick model is first generalized by defining

\[
\begin{align*}
\frac{dX(t)}{dt} &= -k_0 X(t) \, dt - k_1 X(t) Y(t) \, dt, \\
\frac{dY(t)}{dt} &= k_1 X(t) Y(t) \, dt - k_2 Y(t) \, dt, \\
\frac{dZ(t)}{dt} &= k_0 X(t) \, dt + k_2 Y(t) \, dt,
\end{align*}
\]

where \( k_0 \geq 0 \) is a constant. Moreover, rather than the spread of an epidemic, we are interested in the propagation of a computer virus. Therefore, \( X(t) \) is now the number of computers that are susceptible to the virus at time \( t \), \( Y(t) \) is the number of those that are infected, and \( Z(t) \) is the number of those that are cured from or protected against the virus. Again, we could consider percentages instead of numbers.

The total number of computers is equal to \( N \). It follows that

\[
\dot{X}(t) + \dot{Y}(t) + \dot{Z}(t) = 0
\]

in the system defined in (2).

In the next section, the deterministic model will be solved explicitly. Next, a stochastic version of this model will be considered in Section 3. Moreover, control variables will be introduced into the model:

\[
\begin{align*}
\frac{dX(t)}{dt} &= -k_0 u_0(t) X(t) \, dt - k_1 u_1(t) X(t) Y(t) \, dt - v_1^{1/2} \, dB(t), \\
\frac{dY(t)}{dt} &= k_1 u_1(t) X(t) Y(t) \, dt - k_2 u_2(t) Y(t) \, dt + v_1^{1/2} \, dB(t), \\
\frac{dZ(t)}{dt} &= k_0 u_0(t) X(t) \, dt + k_2 u_2(t) Y(t) \, dt,
\end{align*}
\]

where \( u_i(t), i = 0, 1, 2, \) is a control variable, \( v \) is a positive function and \( \{B(t), t \geq 0\} \) is a standard Brownian motion starting at \( B(0) = 0 \). In Lefebvre [5], the author considered a two-dimensional modification of the above model, with \( k_0 = 0 \):

\[
\begin{align*}
\frac{dX(t)}{dt} &= -k_1 X(t) Y(t) u_1(t) \, dt, \\
\frac{dY(t)}{dt} &= k_1 X(t) Y(t) u_1(t) \, dt - k_2 Y(t) u_2(t) \, dt + \{v[Y(t)]\}^{1/2} \, dB(t).
\end{align*}
\]

Notice that the sum \( X(t) + Y(t) \) is not fixed in this two-dimensional system.

We will look for the controls \( u_0(t) \) and \( u_2(t) \) (respectively \( u_1(t) \)) that minimize (respectively maximize) the expected cost it takes to clean the infected computers, or to protect them from the virus, while taking the quadratic control costs into account. The problem is thus expressed as a stochastic differential game. There are three optimizers: one (using \( u_0(t) \)) who works to protect the computers, one (using \( u_2(t) \)) who cleans the infected ones, and one (the hacker, using \( u_1(t) \)) whose aim is to infect the computers. Notice that the above model could also be used in the case of a biological war between two countries.

Making use of dynamic programming, we will derive in Section 3 the equation satisfied by the value function, which is the optimal value of the expected cost, in various cases. Two particular problems will be solved explicitly in Section 4. Finally, we will end this paper with a few concluding remarks in Section 5.

2. Deterministic model. In this section, we consider the system defined in (2).

We can obtain an explicit solution for any choice of the constants \( k_i \), for \( i = 1, 2, 3 \). Let us choose \( k_1 = k_3 = 0.1 \) and \( k_2 = 0.001 \). Moreover, assume that the initial conditions are

\[
X(0) = 1000, \quad Y(0) = 1 \quad \text{and} \quad Z(0) = 0.
\]
Using the mathematical software Maple, we find that

\[
X(t) = \frac{1001000 \exp\left\{\frac{1001}{100} \left[e^{-t/10} - 1\right] - \frac{t}{10}\right\}}{1000 \exp\left\{\frac{1001}{100} \left[e^{-t/10} - 1\right]\right\} + 1},
\]

\[
Y(t) = \frac{1001 e^{-t/10}}{1000 \exp\left\{\frac{1001}{100} \left[e^{-t/10} - 1\right]\right\} + 1},
\]

\[
Z(t) = -1001 e^{-t/10} + 1001.
\]

The functions \(X(t), Y(t)\) and \(Z(t)\) are presented in Figures 1 to 3, respectively. We are particularly interested in the number \(Y(t)\) of infected computers at time \(t\). We see that, in the absence of control variables, this number, in the example considered here, will increase rapidly from the initial value \(Y(0) = 1\) to a maximum of about 162 computers at time 13.4, and then it will decrease to zero. Hence, more than 16% of the computers will be infected by the virus. The number \(X(t)\) of computers that are susceptible to the virus decreases steadily, while the number \(Z(t)\) of those that are cured from or protected against the virus increases rapidly. These results are obtained in the case when the constant \(k_2\) is much smaller than \(k_1\) and \(k_3\). When \(k_i = 0.1\) for \(i = 1, 2, 3\), \(Y(t)\) increases very rapidly: the maximum of more than 986 infected computers is attained at \(t \simeq 0.14\). See Figure 4.

![Figure 1. Function X(t) in the interval [0, 80] when k_1 = k_3 = 0.1 and k_2 = 0.001.](image)

Of course, contrary to humans who can in some cases recover from a disease without any special treatment, in general infected computers cannot cure themselves. Therefore, the control variables \(u_0(t)\) and \(u_2(t)\) in the system can be interpreted as additional or outside help to stop the virus propagation. With the above numerical examples, we see that without extra effort, the vast majority of the computers might get infected.

In the next section, the stochastic differential game problem will be set up and it will be solved explicitly in two particular cases in Section 4.
3. **Stochastic differential game.** We now turn to the stochastic controlled system defined in (4). Let us denote \((X(0), Y(0), Z(0))\) by \((x, y, z)\), and define

\[
T(x, y, z) := \inf \{ t > 0 : (X(t), Y(t), Z(t)) \in D \in \mathbb{R}^3 \mid (x, y, z) \notin D \}. \tag{8}
\]

Because we have

\[
X(t) + Y(t) + Z(t) = N, \tag{9}
\]

the first-passage time \(T(x, y, z)\) is actually a function of two variables.
Next, we define the cost criterion \( J(x, y, z) \) as follows:

\[
J(x, y, z) = \int_0^T \left\{ \frac{1}{2} \left[ q_0 u_0^2(t) + q_2 u_2^2(t) - q_1 u_1^2(t) \right] + \lambda \right\} dt + K[X(T), Y(T), Z(T)],
\]

where \( q_i \) is a positive function, for \( i = 0, 1, 2 \), \( \lambda \in \mathbb{R} \) and \( K \) is a general termination cost function.

Our aim is to find the values of the control variables \( u_0(t) \) and \( u_2(t) \) (respectively \( u_1(t) \)) that minimize (respectively maximizes) the expected value of \( J(x, y, z) \). If the parameter \( \lambda \) is positive, then the optimizers that are using \( u_0(t) \) and \( u_2(t) \) want the controlled three-dimensional process \((X(t), Y(t), Z(t))\) to leave the continuation region \( C := D^c \) as soon as possible, while the one using \( u_1(t) \) wants this process to remain in \( D \) as long as possible (and vice versa when \( \lambda < 0 \)). The optimizers must of course take the quadratic control costs into account, as well as the termination cost. See Whittle [13].

In the first particular problem that will be considered in Section 4, the random variable \( T(x, y, z) \) will be the first time that the sum \( X(t) + Y(t) \) is equal to zero, so that there are no more computers that are susceptible to the virus. The two optimizers that want to minimize the (expected) time until \( X(t) + Y(t) = 0 \) could be technicians working for the company or organization that is the victim of a cyber-attack; one technician tries to protect the computers against the virus, while the second one tries to remove the virus from the infected computers. The third optimizer is the hacker or attacker. His/her aim is to infect as many computers as possible and to maximize the (expected) duration of the attack. So, the three persons involved in the controlled system play a game (in a broad sense). The winner could be determined by the value of \( T(x, y, z) \) and/or that of \( K[X(T), Y(T), Z(T)] \). We could define \( T(x, y, z) \) to be the first time that \( Y(t) = d_1 \) or \( d_2 \), with \( 0 \leq d_1 < d_2 \leq N \). If \( Y(T) = d_1 \), the technicians were able to stop the propagation of the virus before it became out of control, while \( Y(T) = d_2 \) entails that maximum damage

![Figure 4. Function \( Y(t) \) in the interval \([0, 2]\) when \( k_i = 0.1 \) for \( i = 1, 2, 3 \).](image-url)
to the computer system has been inflicted, so that the hacker won. In the second particular example treated in Section 4, we will choose \( d_1 = 0 \) and \( d_2 = N \).

We assume that the hacker knows how many computers have been infected so far. Moreover, he/she only wants to cause damage to the computer system. We could instead assume that the virus is a ransomware, so that the hacker wants to extort money from the company. We would then choose a termination cost function that takes this objective into account. Similarly, the two technicians could be outside consultants hired by the company, and their reward could depend on the successful removal of the virus.

To solve our problem, we will make use of dynamic programming. We define the value function

\[
F(x, y, z) = \inf_{u_0(t)} \inf_{u_2(t)} \sup_{u_1(t)} E[J(x, y, z)],
\]

in which the supremum and the infinums are over the interval \([0, T]\). We will derive the dynamic programming equation satisfied by \( F(x, y, z) \). However, the form of this equation depends on whether \( T \) is expressed as a function of \((x, y, z)\), \((x, z)\) or \((y, z)\).

With the help of the usual arguments, we can prove the following proposition.

**Proposition 3.1.** If \( T(x, y, z) = T(x, y) \), then the value function \( F(x, y) \) satisfies the following dynamic programming equation:

\[
0 = \inf_{u_0} \inf_{u_2} \sup_{u_1} \left\{ \frac{1}{2} \left( q_0 u_0^2 + q_2 u_2^2 - q_1 u_1^2 \right) + \lambda - k_0 u_0 x F_x \
- k_1 u_1 xy F_x + k_1 u_1 xy F_y - k_2 u_2 y F_y + \frac{1}{2} v F_{xx} + \frac{1}{2} v F_{yy} \right\},
\]

where all the functions are evaluated at \( t = 0 \). This equation is valid for \((x, y)\) in the continuation region \( C \). Furthermore, we have the boundary condition

\[
F(x, y) = K(x, y) \quad \text{if} \quad (x, y) \in \partial D.
\]

Similarly, if \( T(x, y, z) = T(x, z) \), we have

\[
0 = \inf_{u_0} \inf_{u_2} \sup_{u_1} \left\{ \frac{1}{2} \left( q_0 u_0^2 + q_2 u_2^2 - q_1 u_1^2 \right) + \lambda - k_0 u_0 x F_x \
- k_1 u_1 xy F_x + k_0 u_0 x F_z + k_2 u_2 y F_z + \frac{1}{2} v F_{xx} \right\}
\]

for \((x, z)\) \( \in C \), with \( F(x, z) = K(x, z) \) on the boundary \( \partial D \) of the stopping region \( D \). Finally, when \( F(x, y, z) = F(y, z) \), we can write that

\[
0 = \inf_{u_0} \inf_{u_2} \sup_{u_1} \left\{ \frac{1}{2} \left( q_0 u_0^2 + q_2 u_2^2 - q_1 u_1^2 \right) + \lambda + k_1 u_1 xy F_y \
- k_2 u_2 y F_y + k_0 u_0 x F_z + k_2 u_2 y F_z + \frac{1}{2} v F_{yy} \right\}
\]

in \( C \), together with \( F(y, z) = K(y, z) \) on \( \partial D \).

After having derived the above dynamic programming equations, we can express the optimal controls in terms of the value function \( F \). In the case of Eq. (12),
denoting the expression between the curly brackets by $L$, we have

\[
\frac{\partial L}{\partial u_0} = 0 \implies u_0^* = \frac{k_0 x}{q_0} F_x, \tag{16}
\]
\[
\frac{\partial L}{\partial u_1} = 0 \implies u_1^* = \frac{k_1 x y}{q_1} (F_y - F_x), \tag{17}
\]
\[
\frac{\partial L}{\partial u_2} = 0 \implies u_2^* = \frac{k_2 y}{q_2} F_y. \tag{18}
\]

And similarly for the other dynamic programming equations.

In the next section, we will set up and solve explicitly two particular problems.

4. Particular problems. First, let

\[
T(x, y, z) = T(x, y) := \inf \{ t > 0 : X(t) + Y(t) = 0 \mid X(0) = x, Y(0) = y \}. \tag{19}
\]

Since $X(t) + Y(t) + Z(t) = N$, the random variable $T(x, y)$ is the first time that all the computers are either cured from or protected against the virus. We assume that $x > 0$ and $y > 0$.

The value function $F(x, y)$ satisfies the dynamic programming equation (12), and the optimal controls are given in (16)–(18). Substituting these expressions into (12), we obtain that

\[
0 = \lambda - \frac{1}{2} \frac{k_0^2 x^2}{q_0} F_x^2 - \frac{1}{2} \frac{k_1^2 y^2}{q_1} F^2 + 2 \frac{k_2^2 x^2 y^2}{q_2} (F_y - F_x)^2 + \frac{1}{2} v F_{xx} + \frac{1}{2} v F_{yy}. \tag{20}
\]

This equation is valid in $C_1 := \{ (x, y) \in \mathbb{R}^2 : x + y > 0 \}$. Moreover, we take $K(X(T), Y(T)) \equiv 0$. It follows that we have the boundary condition

\[
F(x, y) = 0 \quad \text{if } x + y = 0. \tag{21}
\]

To solve the non-linear second-order partial differential equation (20), subject to (21), we will make use of the method of similarity solutions. Based on symmetry and the boundary condition, we try a solution of the form

\[
F(x, y) = H(w), \tag{22}
\]

where $w := x + y$ is the similarity variable. Equation (20) reduces to

\[
0 = \lambda - \frac{1}{2} \frac{k_0^2 x^2}{q_0} [H'(w)]^2 - \frac{1}{2} \frac{k_1^2 y^2}{q_1} [H'(w)]^2 + v H''(w), \tag{23}
\]

and the boundary condition becomes

\[
H(0) = 0. \tag{24}
\]

For the method to apply, we must express all the terms (after simplification) in (23) as functions of $w$. Let us consider the following particular case: $\lambda = 1$, $k_0 = k_2 = 1$, $v \equiv 1$, $q_0 = x^2$ and $q_2 = y^2$. Then, we obtain that

\[
0 = 1 - [H'(w)]^2 + H''(w), \tag{25}
\]

which is a Riccati equation for $G(w) := H'(w)$.

**Remark.** The above particular values for the various parameters in the system were chosen for the sake of simplicity in order to illustrate our results. In practice, these parameters must be estimated by using real-life data or expert intuition.

The general solution of Eq. (25) can be written as follows:

\[
H(w) = w - \ln \left( c_1 e^{2w} + c_2 \right). \tag{26}
\]
The solution that satisfies the boundary conditions $H(a) = H(b) = 0$ is
\[ H(w) = w - \ln \left( \frac{e^{2w+b} - e^{2w+a} + e^{2a+b} - e^{2a}}{e^{2b} - e^{2a}} \right). \] (27)

With $a = 0$, the above expression reduces to
\[ H(w) = w - \ln \left( \frac{e^{2w} + e^b}{1 + e^b} \right). \] (28)

Finally, taking the limit as $b$ tends to infinity, we find that
\[ H(w) = w \implies F(x,y) = x + y. \] (29)

It follows that the optimal controls are given by
\[ u_0^* = \frac{1}{x}, \quad u_1^* = 0, \quad u_2^* = \frac{1}{y}. \] (30)

**Remark.** (i) In this particular case, we see that when the objective is to maximize the (expected) time required for all the computers to become cured from or protected against the virus, the hacker’s optimal strategy consists in using no control at all (after at least one computer gets infected).

(ii) If we substitute the optimal controls into (4), we obtain the following system:
\[
\begin{align*}
\frac{dX^*(t)}{dt} &= -1 + dB(t), \\
\frac{dY^*(t)}{dt} &= -1 + dB(t), \\
\frac{dZ^*(t)}{dt} &= 2dt.
\end{align*}
\] (31)

Hence, $X^*(t) = x - t - B(t)$, $Y^*(t) = y - t + B(t)$ and $Z^*(t) = N - x - y + 2t$. The processes $\{X^*(t), t \geq 0\}$ and $\{Y^*(t), t \geq 0\}$ are Wiener processes with drift equal to $-1$, whereas $\{Z^*(t), t \geq 0\}$ is a deterministic function or a degenerate stochastic process.

Moreover, we have
\[ T^*(x,y) = \inf\{t > 0 : X^*(t) + Y^*(t) = 0 \mid X(0) = x, Y(0) = y\} \]
\[ = \inf\{t > 0 : x + y - 2t = 0\}. \] (32)

That is,
\[ T^*(x,y) = \frac{x + y}{2}. \] (33)

It follows that
\[ F(x,y) = \int_0^{(x+y)/2} \left\{ \frac{1}{2} (1 + 1 - 0) + 1 \right\} dt = x + y. \] (34)

Thus, we retrieve the result obtained above.

(iii) At time $T^* = (x + y)/2$, we have
\[ X^*(T^*) = x - \frac{(x + y)}{2} - B\left(\frac{x + y}{2}\right) \] (35)
and
\[ Y^*(T^*) = y - \frac{(x + y)}{2} + B\left(\frac{x + y}{2}\right), \] (36)
so that $X^*(T^*) + Y^*(T^*) = 0$, as required. Moreover,
\[ E[X^*(T^*)] = \frac{(x - y)}{2} = -E[Y^*(T^*)]. \] (37)
If $x = y$, the probability that $X^*(T^*)$ or $Y^*(T^*)$ will be negative is equal to 1. Since in the application considered both $X(t)$ and $Y(t)$ must be non-negative, we could modify the definition of $T(x, y)$ as follows:

$$T_a(x, y) := \inf \{ t > 0 : X(t) + Y(t) = a \mid X(0) = x, Y(0) = y \},$$  \hspace{1cm} (38)

where $0 \leq a < x + y$. Proceeding as above, we find that

$$H_a(w) = w - a.$$  \hspace{1cm} (39)

The optimal controls are the same as before. However, at time $T_a(x, y) = (x + y - a)/2$ we have

$$X^*(T_a^*) = x - \left(\frac{x + y - a}{2}\right) - B\left(\frac{x + y - a}{2}\right)$$  \hspace{1cm} (40)

and

$$Y^*(T_a^*) = y - \left(\frac{x + y - a}{2}\right) + B\left(\frac{x + y - a}{2}\right).$$  \hspace{1cm} (41)

If $a$ is large enough, and $x = y$, the probability that both $X^*(T_a^*)$ and $Y^*(T_a^*)$ are positive is also relatively large. For instance, assume that $x = y = 500$ and $a = 100$. Then,

$$E[X^*(T_a^*)] = E[Y^*(T_a^*)] = 50$$  \hspace{1cm} (42)

and

$$P[X^*(T_a^*) \geq 0] = P[Y^*(T_a^*) \geq 0] = P[B(450) < 50] \approx \Phi(0.11) \approx 0.54.$$  \hspace{1cm} (43)

This probability can be much larger if $v$ is small. With $v \equiv 0.01$, it becomes

$$P[X^*(T_a^*) \geq 0] = P[Y^*(T_a^*) \geq 0] = P[B(450) < 500] \approx \Phi(1.11) \approx 0.87.$$  \hspace{1cm} (44)

In the deterministic case when $v$ decreases to zero, $X^*(t)$ and $Y^*(t)$ will be strictly decreasing from $x = y$ to zero and will never become negative.

We could force the optimally controlled process $(X^*(t), Y^*(t))$ to be equal to $(0, 0)$ at time $T(x, y)$ by defining

$$T(x, y) := \inf \{ t > 0 : X^2(t) + Y^2(t) = 0 \mid X(0) = x, Y(0) = y \}.$$  \hspace{1cm} (45)

However, $X^*(t)$ and $Y^*(t)$ still could become negative before the final time.

Suppose now that

$$T(x, y, z) := \inf \{ t > 0 : Y(t) = 0 \text{ or } N \mid X(0) = x, Y(0) = y, Z(0) = z \}$$

$$:= \inf \{ t > 0 : X(t) + Z(t) = N \text{ or } 0 \mid X(0) = x, Z(0) = z \}.$$  \hspace{1cm} (46)

We can denote the random variable $T(x, y, z)$ by $T(x, z)$. We assume that $0 < x + z < N$. Therefore, the stochastic differential game ends the first time either all the computers are infected $(Y(T) = N)$ or there are no more infected computers $(Y(T) = 0)$.

In this second particular case, we take

$$K[X(T), Z(T)] = \begin{cases} 0 & \text{if } X(T) + Z(T) = N, \\ K & \text{if } X(T) + Z(T) = 0, \end{cases}$$  \hspace{1cm} (47)

where $K > 0$. 

The value function $F(x, z)$ satisfies the dynamic programming equation (14). If we denote the expression between the curly brackets by $L$, we can write that

\[
\frac{\partial L}{\partial u_0} = 0 \implies u_0^* = \frac{k_0 x}{q_0} (F_x - F_z),
\]

(48)

\[
\frac{\partial L}{\partial u_1} = 0 \implies u_1^* = -\frac{k_1 y x}{q_1} F_x,
\]

(49)

\[
\frac{\partial L}{\partial u_2} = 0 \implies u_2^* = -\frac{k_2 y z}{q_2} F_z.
\]

(50)

We must solve the non-linear second-order partial differential equation obtained by substituting the above expressions for the optimal controls into Eq. (14), subject to

\[
F(x, z) = \begin{cases} 
0 & \text{if } x + z = N, \\
K & \text{if } x + z = 0.
\end{cases}
\]

(51)

As in the first particular case, we will use the method of similarity solutions to solve our problem. Assume that

\[
F(x, z) = H(w),
\]

(52)

where $w := x + z$. It follows that $F_x(x, z) = F_z(x, z) = H'(w)$, so that $u_0^* = 0$. The partial differential equation reduces to

\[
0 = \lambda - \frac{1}{2} \frac{k_2^2 y^2}{q_2} [H'(w)]^2 + \frac{1}{2} \frac{k_1^2 x^2 y^2}{q_1} [H'(w)]^2 + \frac{1}{2} v H''(w)
\]

(53)

for $0 < w < N$. The boundary conditions are

\[
H(w) = \begin{cases} 
0 & \text{if } w = N, \\
K & \text{if } w = 0.
\end{cases}
\]

(54)

Again, for the sake of simplicity, let us take $k_1 = k_2 = 1$, $\lambda = 1$, $v = 1$, $q_1 = x^2 y^2$ and $q_2 = y^2$. Equation (53) becomes

\[
0 = 1 + \frac{1}{2} H''(w).
\]

(55)

The solution that satisfies (54) is

\[
H(w) = -w^2 + \frac{(N^2 - K)}{N} w + K.
\]

(56)

It follows that

\[
H'(w) = -2 w + \frac{(N^2 - K)}{N},
\]

(57)

so that

\[
F_x = F_z = -2(x + z) + \frac{(N^2 - K)}{N},
\]

(58)

and

\[
u_1^* = -\frac{1}{xy} \left[ -2(x + z) + \frac{(N^2 - K)}{N} \right] = -\frac{1}{xy} \left( 2y - N - \frac{K}{N} \right),
\]

(59)

\[
u_2^* = -\frac{1}{y} \left[ -2(x + z) + \frac{(N^2 - K)}{N} \right] = -\frac{1}{y} \left( 2y - N - \frac{K}{N} \right).
\]

(60)

If we impose the conditions $u_1^* \geq 0$ and $u_2^* \geq 0$, then the above solution is valid if

\[
y \leq \frac{N^2 + K}{2N},
\]

(61)

which is always fulfilled when $K$ is larger than or equal to $N^2$. 

The optimally controlled process satisfies the following stochastic differential equations:

\[
\begin{align*}
\frac{dX^*(t)}{dt} &= \left[2Y^*(t) - N - \frac{K}{N}\right] dt - dB(t), \\
\frac{dY^*(t)}{dt} &= dB(t), \\
\frac{dZ^*(t)}{dt} &= -\left[2Y^*(t) - N - \frac{K}{N}\right] dt.
\end{align*}
\]

Thus, \(\{Y^*(t), t \geq 0\}\) is a standard Brownian motion process.

5. **Conclusion.** In this paper, we considered a stochastic differential game in three dimensions, for which the final time was a first-passage time. A possible application of the model is the propagation of a computer virus. In this game, the total number of computers susceptible to the virus was fixed, so that it was possible to reduce the optimization problems to two-dimensional problems.

If we assume instead that the number of computers is not fixed, there could be two independent Brownian motions:

\[
\begin{align*}
\frac{dX(t)}{dt} &= -k_0 u_0(t) X(t) dt - k_1 u_1(t) X(t) Y(t) dt + v_1^{1/2} dB_1(t), \\
\frac{dY(t)}{dt} &= k_1 u_1(t) X(t) Y(t) dt - k_2 u_2(t) Y(t) dt + v_2^{1/2} dB_2(t), \\
\frac{dZ(t)}{dt} &= k_0 u_0(t) X(t) dt + k_2 u_2(t) Y(t) dt.
\end{align*}
\]

Then, the optimization problems would be in three dimensions.

In Section 4, we found explicit and exact solutions to two particular problems. In both cases, we made use of the method of similarity solutions to solve the partial differential equation satisfied by the value function. When this method does not apply, we could at least try to solve the differential equation, subject to the appropriate boundary conditions, numerically.

Finally, we could have considered a cost criterion that takes the risk-sensitivity of the optimizers into account; see Whittle [14].

**Acknowledgments.** This research was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC). The author also wishes to thank the anonymous reviewer of this paper for the constructive comments.

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Received February 2020; revised March 2020.

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