OPTIMAL FEEDBACK CONTROL, LINEAR FIRST-ORDER PDE SYSTEMS, AND OBSTACLE PROBLEMS

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ABSTRACT. We introduce an alternative approach for the analysis and numerical approximation of the optimal feedback control mapping. It consists in looking at a typical optimal control problem in such a way that feasible controls are mappings depending both in time and space. In this way, the feedback form of the problem is built-in from the very beginning. Optimality conditions are derived for one such optimal mapping, which by construction is the optimal feedback mapping of the problem. In formulating optimality conditions, costates in feedback form are solutions of linear, first-order transport systems, while optimal descent directions are solutions of appropriate obstacle problems. We treat situations with no constraint-sets for control and state, as well as the more general case where both constraint-sets are considered for state and control.

1. Introduction

Consider the optimal control problem

\[
\begin{align*}
\text{Minimize in } u(s) \in K : & \quad I(u) = \int_0^T F(x(s), u(s)) \, ds + g(x(T)) \\
\text{subject to } & \quad x'(s) = f(x(s), u(s)) \text{ in } (0, T), \\
& \quad x(0) = x_0, x(s) \in \Omega
\end{align*}
\]

where:

- \( T > 0 \) is the time horizon considered;
- \( \Omega \subset \mathbb{R}^N \) is the feasible set for the state variable \( x : (0, T) \to \Omega \);
- \( K \subset \mathbb{R}^m \) is the feasible set for the control variable \( u : (0, T) \to K \);
- \( x_0 \in \mathbb{R}^N \) is the vector determining the state of the system when we start to care about the control problem;
- \( F : \Omega \times K \to \mathbb{R} \) is the density for the cost functional, while \( g : \Omega \to \mathbb{R} \) is the contribution depending on the final state;
- \( f : \Omega \times K \to \mathbb{R}^N \) is the map providing the state equation that governs the dynamics of the system.

With all of these ingredients given to us, we care about the following map

\[
U(t, y) : [0, T] \times \Omega \to K
\]

defined as follows. For \( y \in \Omega \), and \( t \in [0, T] \), consider the problem

\[
\begin{align*}
\text{Minimize in } u(t) \in K : & \quad I(u) = \int_t^T F(x(s), u(s)) \, ds + g(x(T)) \\
\text{subject to } & \quad x'(s) = f(x(s), u(s)) \text{ in } (t, T), \\
& \quad x(t) = y, x(s) \in \Omega.
\end{align*}
\]
Let us assume, to let the discussion move ahead, that there is a unique optimal solution for this optimal control problem for every $t \in [0, T]$, and $y \in \Omega$. Suppose $u(s; t, y)$, for $s \in [t, T]$, is such optimal solution. Then we take

$$U(t, y) = u(t; t, y)$$

for a.e. $t \in [0, T]$, $y \in \Omega$.

**Definition 1.1.** This map $U(t, y)$ is called the optimal, feedback control of the problem.

The relevance of this map is recorded in the following statement, which is hardly in need of further justification. It establishes that all optimal pairs for the optimal control problem are always related through $U$.

**Proposition 1.1.** Let $(x(t), u(t))$ be an optimal pair for the control problem. Then $u(t) = U(t, x(t))$.

The whole point of feedback control is to be able to compute (approximate) this mapping $U$ beforehand, so that when we come to finding the optimal solution of the original problem we are ready to adjust to disturbances that may occur during real processes by measuring (part of) the state of the system, and adjusting the optimal control through the optimal feedback mapping $U$.

The classical way of trying to calculate $U(t, y)$ is by considering the Hamilton-Jacobi-Bellman equation for the value function $v(t, x)$

$$v_t(t, x) + H(\nabla v(t, x), x) = 0 \text{ in } (0, T) \times \mathbb{R}^N \tag{1.1}$$

together with the terminal time condition $v(T, x) = g(x)$ for all $x \in \mathbb{R}^N$. Here we are taking $\Omega = \mathbb{R}^N$. The value function $v(t, x)$, and the hamiltonian $H(p, x)$ are defined as usual

$$H(p, x) = \min_{u \in K} \{F(x, u) + p \cdot f(x, u)\}, \tag{1.2}$$

and $v(t, x)$ is the optimal value of the above problem determining $U(t, x)$; that is, if $v(t, x)$ is known, then the optimal, feedback map $U(t, x)$ is precisely the vector $u = U(t, x)$ where the minimum

$$\min_{u \in K} \{F(x, u) + \nabla v(t, x) \cdot f(x, u)\}$$

is realized. This beautiful theory is by now well-established through viscosity solutions of (1.1). See [4], [5], for instance.

Alternatively, one can focus directly on the field $v(t, x) = \nabla v(t, x)$ instead of on $v(t, x)$. It is elementary to argue that, by differentiating the Hamilton-Jacobi-Bellman equation above with respect to $x$,

$$v_t(t, x) + H_p(v(t, x), x) \nabla v(t, x) + H_x(v(t, x), x) = 0 \text{ in } (0, T) \times \mathbb{R}^N \tag{1.3}$$

together with the terminal time condition

$$v(T, x) = \nabla g(x).$$

If $v(t, x)$ is known, then, as before, the optimal, feedback map is obtained through the optimal solution of the problem

$$\min_{u \in K} \{F(x, u) + v(t, x) \cdot f(x, u)\}.$$

Both approaches have advantages and disadvantages. This second perspective may seem more appealing for two reasons. The first one is that the field we need in order to compute $U(t, y)$ is $v(t, y)$. Bearing in mind that what we compute or approximate with the Hamilton-Jacobi-Bellman equation is $v(t, x)$, and then we need to approximate its spatial gradient $\nabla v(t, x)$, it may look reasonable to deal with a problem which directly furnishes this field $v(t, x) = \nabla v(t, x)$. The

\[\nabla\] designates throughout the gradient only with respect to the spatial variable $x$.\[\]
second reason is more important. Both problems (1.1), and (1.3), can be treated with the method of characteristics. However, this somehow leads us back in both cases to solving the underlying Hamilton ODE system. Approximating either $v(t, x)$ or $v(t, x)$ through the characteristic scheme is like computing $U(t, y)$ directly. Therefore, from a practical point of view, one would decide that problem (1.1) or (1.3) for which numerical methods are better developed, or better known. Put it in this way, the second possibility may seem more attractive because of the semi-linear nature of (1.3) versus the fully-nonlinear equation (1.1). However, (1.3) is a system while (1.1) is a single equation.

The truth is that from a practical viewpoint, either of the two procedures is not easy to implement, even for low dimension $N$, as hyperbolic first-order PDE or systems are delicate, and even more so is its numerical implementation (check [1]). In addition, setting up (1.1) or (1.3), require to have an explicit form of the hamiltonian $H(p, x)$ which involves to go through the minimization calculation with respect to the control variable $u$. Even in simple, academic examples, when a restriction set $K$ should be respected, the hamiltonian may be discontinuous so that the mathematical analysis of problems (1.1) and (1.3) is far from straightforward. These practical difficulties has stirred certain interest in finding other ways to treat and approximate optimal feedback control. See [2], [3], [10].

The objective of this contribution is two-fold.

1. If we are willing to deal with the optimization problem (1.2) determining the hamiltonian $H(p, x)$, then, essentially through the same effort, we can compute the density

$$
\phi(x, \xi) = \min_{u \in \mathbb{R}} \{ F(x, u) : \xi = f(x, u) \in \Lambda(x) \},
$$

where $\Lambda(x)$ is the admissible cone of directions at $x \in \Omega$, defined by taking the closure of the set

$$\{ \xi \in \mathbb{R}^N : x + \epsilon \xi \in \Omega \text{ for some } \epsilon > 0 \},$$

and examine what sometimes is referred to as the variational reformulation of the optimal control problem (see [9])

$$\text{Minimize in } x(s) : \int_t^T \phi(x(s), x'(s)) \, ds$$

subject to $x(t) = y$. If $X(s) \equiv X(s; t, y)$ is the optimal solution for this variational problem, then the optimal $u = U(t, y)$ in the definition of $\phi(X(t), X'(t))$ is precisely the optimal, feedback mapping. Assuming that this density $\phi$ is smooth, we would like to write the differential equation or system whose solution may furnish $U(t, y)$. We will treat the classical linear-quadratic regulator to find the Ricatti vector equation, although in a slightly different form as it is typically done. We will also explore a new scalar, non-linear problem to emphasize the significance of non-linear, first-order PDEs in this framework.

2. As already pointed out, computing $\phi$ or dealing with the hamiltonian $H$ may be unfeasible in practice; or they may lack the appropriate regularity so that the underlying equation or system one may write with them may make no sense, or may require a highly non-trivial analytical study. Therefore, since according to Proposition [11] the optimal control $u(t)$ for the initial problem can be represented in the form $u(t) = U(t, x(t))$, it may be worthwhile making an attempt to look at the initial optimal control problem in the following form:

Find the optimal map $u(s, x) \in K$ that minimizes the integral

$$\int_t^T F(x(s), u(s, x(s))) \, ds + g(x(T))$$
subject to
\[ x'(s) = f(x(s), u(s, x(s))) \text{ in } (t, T), \quad x(t) = y, x(s) \in \Omega, \]
for fixed, but otherwise arbitrary \((t, y) \in [0, T] \times \Omega\). In this format, feasible controls are Lipschitz mappings \(u(t, x) : [0, T] \times \Omega \to K\) so that integral curves of the ODE system
\[ x'(s) = f(x(s), u(s, x(s))) \text{ in } (t, T), \quad x(t) = y, \]
do not exit \(\Omega\) provided \(y \in \Omega\). If \(\Omega\) is a Lipschitz domain, this requires that \(f(x, u(s, x)) \cdot \nu(x) \leq 0\) whenever \(x \in \partial \Omega\), if \(\nu(x)\) is the unit outer normal at \(x\). More specifically, we will take
\[ \mathcal{A} = \{ u \in L^\infty(0, T; H^1(\Omega, K)) : f(x, u(s, x)) \cdot \nu(x) \leq 0 \text{ for } x \in \partial \Omega, s \in [0, T] \}. \]

Our main point is to explore to what extent an optimal such map can be determined by looking directly at the optimal control problem in this form, and, in particular, if one can derive optimality conditions that could help in finding or approximating the optimal feedback mapping in a reasonable way in practice. This point of view also requires an analytical investigation that will be addressed in the final sections. Difficulties involved in such an endeavor are well-known (see for instance \[5\]). Our goal in the first few sections is to formally introduce this perspective, and assess to what extent it might provide a reasonable way to approximate optimal, feedback mappings in real-world problems at least in some simplified situations. For obvious reasons, we will identify the optimal control problem in this form as the feedback control form.

Our main result states optimality conditions for an optimal map for the control problem in feedback form. The statement of this result actually ties together the three topics occurring in the title of the paper: optimal feedback control, linear first-order PDE systems, and obstacle problems. At the same time, a descent direction is provided which may be the basis for a practical, iterative approximation scheme. We take here \(\Omega = \mathbb{R}^N\), leaving a proper \(\Omega\) for the last part of the paper. Recall that
\[ I(u) = \int_0^T F(x(t), u(t, x(t))) \, dt, \quad x'(t) = f(x(t), u(t, x(t))) \text{ in } (0, T), x(0) = x_0. \]

Let the costate \(p(t, x)\), associated with the control \(u(t, x)\), be the solution of the problem
\[ p_t(t, x) + \nabla p(t, x) f(x, u(t, x)) + p(t, x) \nabla[f(x, u(t, x))] = \nabla[F(x, u(t, x))] \text{ in } [0, T] \times \mathbb{R}^N, \]
under the terminal time condition \(p(T, x) = \nabla g(x)\). Set
\[ \nabla I(u)(t, x) = F_u(u, x) + p(t, x) f_u(u, x), \]
The reason for this special notation comes from the fact (Section 7) that
\[ \frac{d}{dt} I(u + \epsilon U) \big|_{\epsilon=0} = \int_0^T \nabla I(u)(t, x) U(t, x) \, dt. \]

**Theorem 1.2.** Let \(u(t, x)\) be a feasible map for the optimal control problem under a constraint set \(K\), which is assumed to be compact and convex, and let \(p(t, x)\) be its associated costate as just indicated. Then the solution of the obstacle problem, for each fixed time \(t \in [0, T]\),
\[ \text{Minimize in } U(t, x) \in K : \int_{\mathbb{R}^N} \left( \frac{1}{2} \nabla U(t, x) - \nabla u(t, x))^2 + \nabla I(u)(t, x)(U(t, x) - u(t, x)) \right) \, dx \]
is a feasible, descent direction for the optimal control at \(u(t, x)\), in an average sense
\[ \int_{\mathbb{R}^N} \nabla I(u)(t, x)(U(t, x) - u(t, x)) \, dx \leq 0, \]
for all $t \in [0, T]$. If such $u(t, x)$ is indeed optimal for the control problem, then the solution of
the obstacle problem is $u(t, x)$ itself for all $t \in [0, T]$.

The initial condition $x_0$ is irrelevant here. Said differently, we are finding the optimal feedback mapping $u(t, x)$ for all arbitrary initial conditions. On the other hand, the structure of the first-order system for the costate $p$ is such that it always admit a unique solution through the method of characteristics. Indeed, that system arises precisely through the characteristics, as we will clearly see later.

One can therefore establish an iterative approximation procedure based on this descent direction:

1. Initialization. Take any initial $u_0(t, x) \in K$.
2. Iterative scheme until convergence: if $u_j(t, x)$ is known, then
   a. Compute the costate $p_j(t, x)$ by solving the corresponding linear, first-order PDE system for $u = u_j$.
   b. Set $\nabla I(u_j)(t, x) = F_u(x, u_j(t, x)) - p_j(t, x)f_u(x, u_j(t, x))$.
   c. Solve the obstacle problem to determine $U_j(t, x)$.
   d. Update $u_j$ to $u_j + \epsilon U_j$ for some small $\epsilon$.

In practice, solving the obstacle problem Step 2.c. may be avoided by simply taking $U_j(t, x)$ as the solution of the mathematical programming problem

$$\nabla I(U_j)(t, x) = \min_{v \in K} (F_u(x, v) - p_j(t, x)f_u(x, v)).$$

Formally, however, this $U_j(t, x)$ might not be feasible since the spatial regularity required on $U_j$ may not be valid as it also depends on the $x$-regularity of the costate $p_j$ solution of (1.4). This is the main reason to consider the obstacle problem in order to ensure the required $x$-regularity.

Another important fact refers to existence of an optimal feedback map. For our main existence result, we are going to use, as a main tool, the variational reformulation as described above. We will take

\begin{equation}
\phi(x, \xi) = \min_{u \in K} \{F(x, u) : \xi = f(x, u) \in \Lambda(x)\},
\end{equation}

with

$$\Lambda(x) = \text{closure}\{\xi \in \mathbb{R}^N : x + \epsilon \xi \in \Omega, \text{ for some } \epsilon > 0\}.$$ 

Note that here we are considering a bounded, regular domain $\Omega \subset \mathbb{R}^N$.

**Theorem 1.3.** Suppose the ingredients ($F$, $f$, $K$, and $\Omega$) of the original optimal control problem are such that the integrand $\phi$ in (1.2) is convex in $\xi$, and coercive in $\xi$ uniformly in $x$, i.e.

$$\lim_{|\xi| \to \infty} \frac{\phi(x, \xi)}{|\xi|} = +\infty$$

uniformly in $x$. Then there is an optimal feedback control map for the control problem.

The paper is organized as follows. We start by considering a classical problem in the Calculus of Variations, and regard its Euler-Lagrange system as a condition for a field $v(t, x)$ so that solutions of $v'(t) = v(t, x(t))$ are minimizers. This idea is reminiscent of the concept of exact fields in the Calculus of Variations (see [2]). By considering the variational reformulation of an optimal control problem, and applying these ideas (whenever possible) to the underlying equivalent variational problem, one can write down differential problems that feasible fields $v$ must solve. The classical linear-quadratic regulator is the typical first example in which these ideas can be implemented (Section 4). We also treat a non-standard academic example to stress how a backwards Burguer’s equation occurs in this setting (Section 5). A variational problem directly in feedback form, as a preliminary step for a more general optimal control problem is
considered next. In Section 7 we explore, in several steps, optimality conditions for optimal controls in feedback form, providing explicitly descent directions that may be used iteratively to find approximations for the optimal feedback mapping $U(t, y)$. This is for $\Omega$ all of space, while the treatment for a bounded $\Omega$ is reserved to the final section (Section 9).

The immediate future asks for testing this viewpoint in concrete examples starting with simple academic situations and proceeding with more and more elaborate problems. We are already working on that (R).

2. A situation coming from the Calculus of Variations

Consider the following variational problem

$$\text{Minimize in } x(t) : \quad I(x) = \int_0^T \phi(x(t), x'(t)) \, dt$$

subject to $x(0) = x_0$, where feasible paths $x : [0, T] \to \mathbb{R}^N$ are taken to be absolutely continuous. Suppose that the integrand $\phi(x, \xi) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is smooth, strictly convex, and coercive in the sense

$$\lim_{|\lambda| \to \infty} \frac{\phi(x, \lambda)}{|\lambda|} = +\infty,$$

uniformly in $x$. Then, it is standard to show that optimal solutions will be found as solutions of the corresponding Euler-Lagrange system completed with the appropriate natural (or transversality) boundary condition at the right-endpoint $T$

$$- [\phi_x(x(t), x'(t))]' + \phi_x(x(t), x'(t)) = 0 \text{ in } (0, T), \quad x(0) = x_0, \phi_x(x(T), x'(T)) = 0.$$ 

We would like to be able to recover these optimal paths as solutions of a first-order system, instead of as solutions of a second-order system, with some condition at the final time. This is the whole point of our analysis here: first-order systems like $x'(t) = v(t, x(t))$ furnish directly the value $x'$ at any time where $x$ is known. This is not so with boundary-value second-order systems. But since in the previous variational problem, the condition on the final time is the natural condition, we cherish the possibility of being able to find a suitable map $v : (0, T) \times \mathbb{R}^N \to \mathbb{R}^N$ so that solutions of the first-order system

$$x'(s) = v(s, x(s)) \text{ in } (t, T), \quad x(t) = y$$

will automatically be the optimal solutions of

$$- [\phi_x(x(s), x'(s))]' + \phi_x(x(s), x'(s)) = 0 \text{ in } (t, T), \quad x(t) = y, \phi_x(x(T), x'(T)) = 0.$$ 

Put therefore $x'(s) = v(s, x(s))$ and substitute in the Euler-Lagrange system above to get, after differentiation and some algebra,

$$\nabla v \cdot v + v_s + \phi_{\xi x}(x, v)^{-1} (\phi_{\xi x}(x, v) v - \phi_x(x, v)) = 0$$

where we have just put $x$ to mean $x(s)$, and $v$ to actually mean $v(s, x(s))$. We can regard (2.1) as a first-order, non-linear, PDE system for the vector unknown field $v : (0, T) \times \mathbb{R}^N \to \mathbb{R}^N$ if we just simply regard $x(s)$ as the spatial variable $x$. The natural condition for $s = T$ provides the condition

$$\phi_x(x, v(T, x)) = 0.$$ 

These computations show that if we put

$$C(x, v) = \phi_{\xi x}(x, v)^{-1} (\phi_{\xi x}(x, v) v - \phi_x(x, v)), \quad D(x, v) = \phi_x(x, v),$$

then $v(t, x)$ must be a solution of the first-order system

$$v_t + \nabla v \cdot v + C(x, v) = 0$$

together with the final condition $D(x, v(T, x)) = 0$. 

Proposition 2.1. Suppose the density $\phi(x, \xi) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is smooth, strictly convex, and coercive as indicated earlier. Let the field $v(t, x)$ be a regular (Lipschitz) solution of the problem
\[ v_t + \nabla v \cdot v + C(x, v) = 0 \text{ in } (0, T) \times \mathbb{R}^N \]
under the condition $D(x, v(T, x)) = 0$. Then each integral curve $x(s)$ of the dynamical system
\[ x'(s) = v(s, x(s)) \text{ in } (t, T), \quad x(t) = y \]
is the optimal solution of the variational problem
\[ \text{Minimize in } x(s) : \int_t^T \phi(x(s), x'(s)) \, ds \]
subject to $x(t) = y$.

3. A Varitational Reformulation of Optimal Control Problems

There is a standard transformation of an optimal control problem into a true variational problem like the one in the previous section, so that we can take advantage of the ideas discussed above to find the optimal feedback map $U(t, y)$.

Recall the optimal control problem in which we are interested
\[ \text{Minimize in } u(s) \in K : \quad I(u) = \int_t^T F(x(s), u(s)) \, ds \]
subject to
\[ x'(s) = f(x(s), u(s)) \text{ in } (t, T), \quad x(t) = y, x(s) \in \Omega. \]
It is well-known that the contribution $g(x(T))$ can be incorporated into the integrand $F$. Define the following density
\[ (3.1) \quad \phi(x, \xi) = \min_{u \in K} \{ F(x, u) : \xi = f(x, u) \in \Lambda(x) \}, \]
where $\Lambda(x)$ is the admissible cone of directions at $x \in \Omega$, defined by taking the closure of the set
\[ \{ \xi \in \mathbb{R}^N : x + \epsilon \xi \in \Omega \text{ for some } \epsilon > 0 \}. \]
Consider the variational problem
\[ \text{Minimize in } x(s) : \int_t^T \phi(x(s), x'(s)) \, ds \]
subject to $x(t) = y$.

Proposition 3.1. These two optimization problems, the initial optimal control problem and the variational problem with integrand $\phi$, are equivalent in the following sense: if the pair $(x, u)$ is optimal for the first, then $x$ is optimal for the second; if $x$ is optimal for the second, and we take $u(s)$ such that $\phi(x(s), x'(s)) = F(x(s), u(s))$ then $(x, u)$ is optimal for the control problem.

The proof of this proposition is straightforward, and does not require any special idea (see for instance [9]).

Whenever the variational reformulation of a certain optimal control problem leads to a regular variational problem which we can apply Proposition 2.1 to, we can find the optimal feedback mapping $U(t, y)$ through a solution of the corresponding first-order system.

Proposition 3.2. Suppose the density $\phi$ in (3.1) complies with all of the regularity assumptions in Proposition 2.1, and let $C, D$ be the corresponding fields. Let $u(t, y)$ be a regular solution of the problem
\[ u_t + \nabla v \cdot v + C(x, u) = 0 \text{ in } (0, T) \times \mathbb{R}^N, \quad D(x, u(T, x)) = 0. \]
Then the optimal, feedback mapping $U(t, y)$ can be obtained from the identity
\[
\phi(y, u(t, y)) = F(y, U(t, y)).
\]
This proposition is a direct consequence of both Propositions 3.1 and 2.1.

4. The linear quadratic regulator

All of the preceding formalism can be made quite explicit for the classical situation of the linear quadratic regulator (LQR). We believe it is worthwhile to revise this important situation from this perspective, as it is a very good way to illustrate the ideas discussed so far. In this case, we have the following ingredients:

1. $K$ and $\Omega$ are all of space, so that we do not have restrictions on state or control.
2. $F(x, u)$ is a quadratic integrand separately in both sets of variables
   \[
   F(x, u) = \frac{1}{2} x^* Q x + \frac{1}{2} u^* R u
   \]
   where $Q$ and $R$ are constant, symmetric, positive definite (R strictly) matrices of the appropriate dimensions.
3. $f(x, u)$ is linear in both sets of variables
   \[
   f(x, u) = Ax + Bu
   \]
   for constant matrices $A$ and $B$ of the appropriate dimensions.
4. The cost functional typically incorporates a contribution involving the final state $x(T)$ in the form
   \[
   \frac{1}{2} x(T)^* H x(T)
   \]
   with $H$, again, a symmetric, positive definite (not necessarily strictly) matrix.

It is customary to include this last contribution into the cost in an integral form. It will also be advantageous to do so for us here. Note that
\[
\frac{1}{2} x(T)^* H x(T) - \frac{1}{2} x_0^* H x_0 = \int_0^T \frac{d}{dt} \left[ \frac{1}{2} x(t)^* H x(t) \right] dt.
\]
Therefore
\[
\frac{1}{2} x(T)^* H x(T) = \frac{1}{2} x_0^* H x_0 + \int_0^T x'(t)^* H x(t) dt = \frac{1}{2} x_0^* H x_0 + \int_0^T (Ax(t) + Bu(t))^* H x(t) dt.
\]
Since the constant term involving $x_0$ does not interfere with the optimization process, we can transform our $F(x, u)$ to incorporate this new contribution, and so take
\[
F(x, u) = x^* A^* H x + u^* B^* H x + \frac{1}{2} x^* Q x + \frac{1}{2} u^* R u.
\]
It is to the problem in this final form that we would like to apply the variational reformulation. According to what was written in the previous section, the integrand for the equivalent variational problem is determined through the minimization problem
\[
\phi(x, \xi) = \min_u \{ x^* A^* H x + u^* B^* H x + \frac{1}{2} x^* Q x + \frac{1}{2} u^* R u : \xi = Ax + Bu \}.
\]
The sum of the first two terms amounts to the quantity $\xi^* H x$ actually coming from the computations performed above concerning the contribution to the cost at the final time. Therefore
\[
\phi(x, \xi) = \xi^* H x + \frac{1}{2} x^* Q x + \min_u \{ \frac{1}{2} u^* R u : \xi = Ax + Bu \}.
\]
This optimization problem is a non-linear mathematical programming problem that can be solved in a standard way, though the explicit computations are a bit tedious. Since \( \xi = Ax + Bu \), we focus on computing

Minimize in \( u : \frac{1}{2}u^*Ru \) subject to \( \xi = Ax + Bu \).

Since in this problem \( x \) and \( \xi \) are regarded as parameters, if we put \( z = \xi - Ax \), we will have to find the solution of the problem

Minimize in \( u : \frac{1}{2}u^*Ru \) subject to \( z = Bu \).

After some elementary computations performed with some care, we can write the final outcome

\[
\begin{align*}
\mathbf{u} &= \mathbf{R}^{-1}B^*(\mathbf{BR}^{-1}B^*)^{-1}\mathbf{z}, & \frac{1}{2}u^*Ru &= \frac{1}{2}z^*(\mathbf{BR}^{-1}B^*)^{-1}z.
\end{align*}
\]

Putting back \( z = \xi - Ax \), we find the law furnishing the optimal control \( \mathbf{u} \) if we know both \( x \) and its derivative \( \dot{x} \), namely

\[
(4.1) \quad \mathbf{u}(t) = \mathbf{R}^{-1}B^*(\mathbf{BR}^{-1}B^*)^{-1}(\dot{x}(t) - Ax(t)).
\]

If we take back this information to the computation of the density \( \phi \), and exercise some care with the calculation (recall \( \mathbf{R} \) and \( \mathbf{R}^{-1} \) are symmetric), we find that

\[
\phi(x, \xi) = \xi^*Hx + \frac{1}{2}x'^*Qx + \frac{1}{2}\xi'^*C\xi - \xi'^*CAx + \frac{1}{2}x'^*A^*CAx
\]

where \( \mathbf{C} = (\mathbf{BR}^{-1}B^*)^{-1} \) is a symmetric matrix. We can put some of the terms together and set

\[
\phi(x, \xi) = \frac{1}{2}\xi'^*C\xi + \xi'^*Dx + \frac{1}{2}x'^*Ex, \quad \mathbf{D} = \mathbf{H} - \mathbf{CA}, \mathbf{E} = \mathbf{Q} + \mathbf{A}^*\mathbf{CA}.
\]

In this way we come to get interested in the variational problem

Minimize in \( x(s) : \int_t^T \left( \frac{1}{2}x'(s)^*C\dot{x}'(s) + x'(s)^*Dx(s) + \frac{1}{2}x(s)^*Ex(s) \right) ds \)

subject to \( x(t) = y \). This is a regular variational problem whose unique optimal solution must be necessarily the solution of the second-order linear system with constant coefficients

\[
(4.2) \quad -C x''(s) + Ex(s) = 0 \text{ in } (0, T), \quad x(t) = y, Cx'(T) + Dx(T) = 0.
\]

We can therefore apply Proposition 2.1 and find the first-order system whose solution provides the optimal, feedback mapping \( \mathbf{U}(t, y) \).

To this aim, put \( \dot{x}'(t) = \mathbf{u}(t, x(t)) \), and substitute into system \( (4.2) \), to get

\[
-C (u_t(t, x(t)) + \nabla u(t, x(t))u(t, x(t))) + Ex(t) = 0 \text{ in } (0, T), \quad Cu(T, x(T)) + Dx(T) = 0.
\]

If we regard \( x(t) \) as the spatial variable, we have that \( \mathbf{u}(t, x) \) must be a solution of the first-order system

\[
(4.3) \quad u_t(t, x) + \nabla u(t, x)u(t, x) = C^{-1}Ex \text{ in } (0, T) \times \mathbb{R}^N
\]

together with the final condition

\[
(4.4) \quad u(T, x) = -C^{-1}Dx.
\]

Suppose that \( \mathbf{u}(t, x) \) is indeed a smooth solution of this Cauchy problem for the first-order system, then we can exploit the relationship \( (4.1) \) to find

\[
\mathbf{U}(t, y) = \mathbf{R}^{-1}B^*(\mathbf{BR}^{-1}B^*)^{-1}(\mathbf{u}(t, y) - Ay).
\]
In this very particular situation, the solution of the system (1.3) with final condition (1.4) can be obtained in the form

\[ u(t, x) = F(t)x. \]

In fact, by substituting this \( u \) into the system and final condition, we see that the matrix-valued function \( F(t) \) ought to be a solution of the Ricatti-type system

\[ F'(t) + F^2(t) = C^{-1}E \text{ in } (0, T), \quad F(T) = -C^{-1}D, \]

and then

\[ U(t, y) = R^{-1}B^*(BR^{-1}B^*)^{-1}(F(t)y - Ay). \]

5. An academic example

To briefly illustrate the significance of the problem that we need to solve in order to find the optimal feedback mapping, let us consider the following academic situation in dimension one

Minimize in \( u \) : \[ \int_0^T \left( \frac{1}{2} u(t)^2 - \frac{1}{2} f(x(t))^2 \right) \, dt \]

subject to

\[ x' = f(x) + u \text{ in } (0, T), \quad x(0) = x_0. \]

In such a situation, it is straightforward to find the equivalent variational problem, namely substitute \( u = x' - f(x) \) into the cost functional to find the variational problem

Minimize in \( x(t) \) : \[ \int_0^T \left( \frac{1}{2} (x'(t) - f(x(t)))^2 - \frac{1}{2} f(x(t))^2 \right) \, dt \]

under \( x(0) = x_0 \). By performing some computations, we also find that the functional can be written

\[ \int_0^T \left[ \frac{1}{2} x'(t)^2 - x'(t)f(x(t)) \right] \, dt. \]

Even further, if \( F(x) \) is a primitive of \( f(x) \), then the previous integral equals

\[ \int_0^T \frac{1}{2} x'(t)^2 \, dt + F(x_0) - F(x(T)). \]

The corresponding Euler-Lagrange problem for optimal solutions is

\[-(x'(t) - f(x(t)))' - x'(t)f'(x(t)) = 0 \text{ in } (0, T), \quad x(0) = x_0, x'(T) - f(x(T)) = 0.\]

We now put \( x'(t) = u(t, x(t)) \), and substitute into this problem to find that \( u \) must be a solution of the problem

\[ u_t(t, x(t)) + u(t, x(t))u_x(t, x(t)) = 0 \text{ in } (0, T) \times \mathbb{R}, \quad u(T, x(T)) = f(x(T)). \]

Replacing \( x(t) \) by \( x \), we arrive at a backwards Burguer’s equation

\[ u_t(t, x) + u(t, x)u_x(t, x) = 0 \text{ in } (0, T) \times \mathbb{R}, \quad u(T, x) = f(x), \]

or equivalently, by reversing time,

\[ u_t(t, x) - u(t, x)u_x(t, x) = 0 \text{ in } (0, T) \times \mathbb{R}, \quad u(0, x) = f(x). \]

We need solutions of this problem to be able to build the optimal, feedback mapping for the initial problem. This is a somewhat non-standard problem in the theory of PDE because we are facing a conservation law for a concave flux function. All kinds of issues raise at this point about the existence of smooth solutions, the multiplicity of solutions, the significance of discontinuous solutions, etc.
6. A Variational Principle in Feedback Form

As a preliminary step for an investigation of an optimal control problem in feedback form as described in the Introduction, let us look at a standard variational problem like

Minimize in $x(t) : \int_0^T \phi(x(t), x'(t)) dt$

under $x(0) = x_0$, for absolutely continuous paths $x(t) \in \mathbb{R}^N$. We assume, to avoid technical issues, that the density $\phi$ is differentiable. Assume we insist in describing the class of admissible paths $x$ for this optimization problem as the solutions of the dynamical system

$$x'(t) = u(t, x(t)) \text{ in } [0, T], \quad x(0) = x_0,$$

so that our optimization variable is no longer the path $x$ but rather the field $u : [0, T] \times \mathbb{R}^N \to \mathbb{R}^N$. In this way, for each such feasible $u$, $x$ is the integral curve starting at $x_0$. We would write

Minimize in $u(t, x) : \int_0^T \phi(x(t), x'(t)) dt$

where $x$ is obtained from $u$ through (6.1).

There are, obviously, several important issues concerning the class of admissible vector fields $u$. To begin with $u$ must be continuous, even Lipschitz, in $x$ so that (6.1) determines $x$ in a unique, unambiguous way. Let us however proceed formally, and suppose that a certain $u$ is optimal. We would like to derive optimality conditions. To this end, we take another feasible $U$, and for small arbitrary $\epsilon$, consider the equation

$$x'(t) + \epsilon X'(t) = u(t, x(t)) + \epsilon X(t) \text{ in } [0, T], \quad x(0) = 0.$$

The path $X$ is the variation produced in $x$, the solution for $u$ in (6.1), by the variation $U$ of $u$. By differentiating with respect to $\epsilon$, and setting $\epsilon = 0$, we immediately obtain

$$(6.2) \quad X'(t) = \nabla u(t, x(t)) X(t) + U(t, x(t)) \text{ in } [0, T], \quad X(0) = 0.$$

Concerning the cost functional, we also differentiate with respect to $\epsilon$ the expression

$$\int_0^T \phi(x(t) + \epsilon X(t), u(t, x(t)) + \epsilon X(t)) dt$$

and then set $\epsilon = 0$. We arrive at the quantity

$$(6.3) \quad \int_0^T (\phi_u X + \phi_x \nabla u X + \phi_t U) dt$$

where, for notational simplicity, we have not written the arguments. If $u$ is a true minimizer for our problem, then (6.3) should vanish for all $U$ when $X$ is determined from $U$ by (6.2).

The usual way of transferring this information on $u$ itself is by means of the costate $p(t, x)$. Suppose this field is a solution of the linear, first-order partial differential system

$$(6.4) \quad p_t(t, x) + \nabla p(t, x)u(t, x) + p(t, x)\nabla u(t, x) = \phi_u(x, u(t, x)) + \phi_x(x, u(t, x)) \nabla u(t, x)$$

in $[0, T] \times \mathbb{R}^N$ under the terminal time condition $p(T, x) = 0$. Then (6.3) can be put in the form

$$\int_0^T [(p(t, x(t))' + p(t, x(t))\nabla u(t, x(t)) X(t) + \phi_x(x, u(t, x)) U(t, x(t))] dt.$$

Note that $p(t, x(t))' = p_t(t, x(t)) + \nabla p(t, x(t))u(t, x(t))$ because of (6.1). An integration by parts leads to (boundary terms drop out)

$$\int_0^T (-pX' + p\nabla u X + \phi_t U) dt.$$
Finally, by taking into account (6.2), we finally arrive at
\[ \int_0^T (\phi_t - p) U \, dt, \]
which is another way of writing (6.3). Since this expression must vanish for arbitrary \( U \) and arbitrary initial condition \( x_0 \), we can conclude that if \( u \) is optimal then \( p = \phi_t \) ought to be the solution of (6.4). Of course, if we take \( \phi_t \) to (6.4), we get back the Euler-Lagrange equation for the minimizer \( x \) which shows the consistency of our computations.

We have therefore learned how to recast optimality conditions for a typical variational problem in a feedback form.

**Proposition 6.1.** Under the regularity assumptions made at the beginning of this section, if the field \( u(t, x) \) is an optimal solution of the variational problem in feedback form, then the solution of the linear transport system (6.4)
\[ p_t(t, x) + \nabla p(t, x) u(t, x) + p(t, x) \nabla u(t, x) = \nabla_x [f(x, u(t, x))] \]
in \([0, T] \times \mathbb{R}^N\) under the terminal time condition \( p(T, x) = 0 \), must be \( p(t, x) \equiv \phi_t(x, u(t, x)) \).

We plan to use this procedure for an optimal control problem in feedback form.

7. **Optimality conditions for an optimal control in feedback form**

With the ingredients indicated at the end of the Introduction, we would like to prove the following optimality criterium. We take \( \Omega = \mathbb{R}^N \) throughout this section, and treat the case of a bounded \( \Omega \) in Section 9. The first statements correspond to having no restriction set \( K \) for the control. Namely, we focus on the optimal feedback control problem yielding the optimal feedback mapping

Minimize in \( u(t, x) : \int_0^T F(x(s), u(s, x(s))) \, ds \)

subject to
\[ x'(s) = f(x(s), u(s, x(s))) \text{ in } (t, T), \quad x(t) = y, \]
for \((t, y) \in [0, T] \times \mathbb{R}^N\) given.

**Theorem 7.1.** Let \( u(t, x) \) be a feasible map for the optimal control problem. Let \( p(t, x) : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) be a solution of the linear transport system
\[ p_t(t, x) + \nabla p(t, x) f(x, u(t, x)) + p(t, x) \nabla_x [f(x, u(t, x))] = \nabla_x [F(x, u(t, x))], \]
with the terminal time condition \( p(T, x) = 0 \). If
\[ F_u(x, u(t, x)) - p(t, x) f_u(x, u(t, x)) \equiv 0, \]
then \( u(t, x) \) is an equilibrium mapping for the feedback optimal control problem.

**Proof.** Let \( u(t, x) \) be feasible, and \( x(t) \) be its associated state so that
\[ x'(s) = f(x(s), u(s, x(s))) \text{ in } (t, T), \quad x(t) = y. \]
Let \( U(t, x) \) be a feasible variation of \( u(t, x) \), and write \( X(t) \) for the variation produced on \( x \) by \( U \) on \( u \). Then
\[ x'(s) + \epsilon X'(s) = f(x(s) + \epsilon X(s), u(s, x(s) + \epsilon X(s))) + \epsilon U(s, x(s) + \epsilon X(s)) \text{ in } (t, T), \quad X(t) = 0. \]
By differentiation with respect to \( \epsilon \), and setting \( \epsilon = 0 \) afterwards, we should have
\[ X' = (f_x + f_u \nabla u) X + f_u U \text{ in } (t, T), \quad X(t) = 0, \]
where \( f_x, f_u \) are evaluated at \((x(s), u(s, x(s)))\), and \( \nabla u \) and \( U \) are evaluated at \((s, x(s))\). Going over the same kind of calculations for the cost functional, we arrive at

\[
\int_t^T \left[ (F_x + F_u \nabla u) X + F_u U \right] ds.
\]

Suppose that the field \( u(s, x) \) is such that the conditions on the statement holds for \( p(s, x) \). Then, it is clear that if we put \( p(s) = p(s, x(s)) \) for \( x(s) \) the corresponding state,

\[
p'(s) + p(s)[f_x(x(s), u(s, x(s))) + f_u(x(s), u(s, x(s))) \nabla u(s, x(s))] = F_x(x(s), u(s, x(s))) + F_u(x(s), u(s, x(s))) \nabla u(s, x(s)), \quad s \in [0, T],
\]

with \( p(T) = 0 \). If we take this information back to (7.3), it is straightforward to get

\[
\int_t^T \left[ (p'(s) + p(s)(f_x + f_u \nabla u)) X + F_u U \right] ds.
\]

Integrating by parts in the first term, and bearing in mind that the boundary terms drop out, we obtain

\[
\int_t^T [-pX' + p(f_x + f_u \nabla u)X + F_u U] ds.
\]

Taking into account (7.2), we can also write

\[
\int_t^T (-pf_u U + F_u U) dt,
\]

which vanishes. The arbitrariness of \( U \) finishes the proof.

Notice how the system for the costate in the statement of the theorem comes from (7.3), and so the existence of solution for the costate \( p \) is guaranteed.

This strategy yields, when appropriately interpreted, an iterative approximation procedure for equilibrium mappings based on a typical steepest descent scheme with respect to a norm ensuring differentiability with respect to \( x \). This differentiability issue makes the direction found a descent direction in the average with respect to the spatial variable \( x \). Let \( I(u) \) be the cost functional for the optimal control problem with an initial time \( t \in [0, T] \). The computation in the last proof shows that the field \( \nabla I(u(t, x)) \equiv F_u(x, u(t, x)) - p(t, x)f_u(x, u(t, x)) \) regarded as a mapping of \((t, x) \in [0, T] \times \mathbb{R}^N \) represents the derivative

\[
\frac{d}{dt} I(u + \epsilon U)\bigg|_{\epsilon=0}
\]

in the sense that this derivative is actually the integral

\[
\int_t^T \nabla I(u(s, x(s))) U(s, x(s)) ds
\]

for \( x(s) \) the state associated with \( u \), and initial condition \( y \).

**Corollary 7.2.** Let \( u(t, x) \) be a feasible map for the optimal control problem, and determine the costate \( p(t, x) : [0, T] \times \mathbb{R}^N \to \mathbb{R}^N \) as a solution of the problem

\[
p_t(t, x) + \nabla p(t, x) f(x, u(t, x)) + p(t, x) \nabla_x [f(x, u(t, x))] = \nabla_x [F(x, u(t, x))],
\]

under the terminal time condition \( p(T, x) = 0 \). Then the solution of the problem

\[-\Delta U(t, x) + \nabla I(u)(t, x) = 0 \text{ in } \mathbb{R}^N \]
for every \( t \in [0,T] \), is a descent direction for the optimization problem at \( u(t,x) \) in an average sense

\[
\int_{\mathbb{R}^N} \nabla I(u)(t,x)U(t,x) \, dx = -\int_{\mathbb{R}^N} (|\nabla U(t,x)|^2) \, dx \leq 0
\]

for all \( t \in [0,T] \).

Constraints on the control variable through a set \( K \subset \mathbb{R}^m \) can be easily incorporated. We assume that \( K \) is compact and convex. We need to take into account that variations are now of the form \( u + \epsilon(U - u) \) for arbitrary, feasible \( U \), and so optimality conditions are one-sided conditions. This leads naturally to variational inequalities and obstacle problems.

**Theorem 7.3.** Let \( u(t,x) \) be a feasible map for the optimal control problem for a convex, compact set \( K \). Let \( p(t,x) : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N \) be a solution of the problem

\[
p_t(t,x) + \nabla p(t,x)f(x,u(t,x)) + p(t,x)\nabla_x[f(x,u(t,x))] = \nabla_x[F(x,u(t,x))],
\]

with the terminal time condition \( p(T,x) = 0 \). If \( u(t,x) \) turns out to realize the minimum of the hamiltonian \( F(x,v) - p(t,x)f(x,v) \) in the control variable \( v \in K \)

\[
F(x,u(t,x)) - p(t,x)f(x,u(t,x)) = \min_{v \in K} \{F(x,v) - p(t,x)f(x,v)\},
\]

then \( u(t,x) \) is a local minimum for the feedback optimal control problem.

**Proof.** Note that the condition on the minimum implies that

\[
[F_v(x,u(t,x)) - p(t,x)f_v(x,u(t,x))](v - u(t,x)) \geq 0,
\]

for all \( v \in K \). In particular, by choosing \( v = U(t,x) \), we would have

\[
[F_v(x,u(t,x)) - p(t,x)f_v(x,u(t,x))](U(t,x) - u(t,x)) \geq 0.
\]

This implies that the local change on the cost functional for the variation \( (U(t,x) - u(t,x)) \) which is given, as above, by

\[
\int_0^T [-pf_u(U-u) + Fu(U-u)] \, dt
\]

is non-negative. The arbitrariness of \( U \) in \( K \) yields the result. \( \square \)

This result does translate easily in an iterative, approximation scheme as with the non-restricted situation. Let \( u(t,x) \) be a feasible map for the optimal control problem under a constraint set \( K \). Determine the costate \( p(t,x) : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N \) as a solution of the problem

\[
p_t(t,x) + \nabla p(t,x)f(x,u(t,x)) + p(t,x)\nabla_x[f(x,u(t,x))] = \nabla_x[F(x,u(t,x))],
\]

under the terminal time condition \( p(T,x) = 0 \). As indicated earlier, set

\[
\nabla I(u)(t,x) = F_v(x,u(t,x)) - p(t,x)f_v(x,u(t,x)).
\]

Consider the obstacle problem, for each fixed time \( t \in [0,T] \),

\[(7.5)\]

Minimize in \( U(t,x) \in K : \int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla U(t,x) - \nabla u(t,x)|^2 + \nabla I(u)(t,x)(U(t,x) - u(t,x)) \right) \, dx. \]

It is standard to show that this problem has a unique solution \( U(t,x) \) (see [3]). It is elementary to check through the underlying variational inequality that

\[
\int_{\mathbb{R}^N} \nabla I(u)(t,x)(U(t,x) - u(t,x)) \, dx \leq -\int_{\mathbb{R}^N} |\nabla U(t,x) - \nabla u(t,x)|^2 \leq 0
\]

for all \( t \). This is the result stated in the Introduction.
8. Existence result

8.1. Variational problem. Let $\Omega \subset \mathbb{R}^N$ be a bounded, Lipschitz domain with unit, outward normal $\nu(y)$ for a.e. $y \in \partial\Omega$, and $T > 0$. We would like to focus on the variational problem

$$\text{Minimize in } u \in A: \quad I(u) = \int_0^T \int_{\Omega} \phi(x(s,t,y)), x'(s,t,y)) \, ds \, dt \, dy$$

where

$$A = \{ u \in L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^N)) : u(t,y) \cdot \nu(y) \leq 0 \text{ for a.e. } t \in (0,T), y \in \partial\Omega \},$$

and $x(s,t,y)$ is the flow associated with the feasible vector field $u(t,x)$, i.e.

$$x'(s,t,y) = u(t,x(s,t,y)) \text{ in } (t,T), \quad x(t; t, y) = y.$$  

Indeed, if we put

$$X'(t,y) = u(t,X(t,y)) \text{ in } (0,T), \quad X(0) = y,$$

then $x(s,t,y) = X(s,X^{-1}(t,y))$. Note that the flow cannot exit $\Omega$ under our assumption for feasibility over $\partial\Omega$. The mapping $X^{-1}$ is the flow associated with $-u$.

**Proposition 8.1.** Suppose the mapping $\phi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is Lipschitz, whenever finite, and bounded from below. Then the cost functional $I$ is well-defined over $A$.

**Proof.** First of all, notice that, as already pointed out, the flow $x$ cannot exit $\Omega$. On the other hand, by differentiating with respect to $y$, in the ODE system, we arrive at

$$\nabla x'(s,t,y) = \nabla u(t,x(s,t,y)) \nabla x(s,t,y) \text{ in } (t,T), \quad \nabla x(t; t, y) = 1,$$

where $\nabla$ always means differentiation with respect to spatial variables, and 1 is the identity matrix of suitable size. Therefore $\nabla x(s,t,y)$ is the solution of a linear ODE system with a bounded-in-time matrix coefficient, and so the flow $x(s,t,y) \in L^\infty(t,T;W^{1,\infty}(\mathbb{R}^N))$. Hence the composition $\phi(x(s,t,y), u(t,x(s,t,y)))$ is also Lipschitz in $y$, and bounded in $s$, and $t$. The functional $I(u)$ is therefore finite. \qed

**Theorem 8.2.** Assume the following two assumptions hold for $\phi$:

1. it is strictly convex, whenever finite, and coercive;
2. for a.e. $y \in \partial\Omega$, $\phi(y, \xi)$ is finite only when $\xi \cdot \nu(y) \leq 0$.

Then the variational problem above admits a unique optimal map $U(t,y)$.

**Proof.** For fixed $(t,y) \in [0,T] \times \Omega$, consider the standard variational problem

$$\text{Minimize in } x(s) : \quad \int_1^T \phi(x(s), x'(s)) \, ds$$

under the initial condition $x(t) = y$. By our main hypotheses on $\phi$, it is a classical result in the Calculus of Variations that there is a unique minimizer $X(s,t,y)$ which is Lipschitz in $s \in (t,T)$. In particular, we ought to have, for $s \in [t,T]$,

$$X(s,s, X(s,t,y)) = X(s,t,y), \quad X'(s,s, X(s,t,y)) = X'(s,t,y).$$  

Define $U(t,y) = X'(t,t,y)$ for a.e. $t \in [0,T]$, and all $y$. Then, the second property assumed on $\phi$ about the finiteness, implies that $U$ is feasible. Moreover, the flow associated with $U$ is precisely $X(s,t,y)$ for all fixed $t \in [0,T]$ and $y \in \Omega$, because by (8.1),

$$U(s, X(s,t,y)) = X'(s,s, X(s,t,y)) = X'(s,t,y),$$

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and $X(0; 0, y) = y$. As a matter of fact, the flow is $x(s, y) = X(s; 0, y)$. Hence for fixed $t \in [0, T]$ and $y \in \Omega$, we have that $x(s, x^{-1}(t, y)) = X(s; t, y)$, and the inner integral for $I(U)$ would be

$$
\int_t^T \phi(X(s; t, y), X'(s; t, y)) \, ds
$$

which is optimal for all such $t \in [0, T]$ and $y \in \Omega$. $U$ is then the unique minimizer for the variational problem in feedback form. \hfill \Box

If we relax the strict convexity to just convexity, then we might have several optimal maps $U(t, y)$.

Under smoothness assumptions for $\phi$, we can also write down optimality conditions for $U(t, y)$ as has been examined in the preceding section.

**Proposition 8.3.** Suppose, in addition to the previous hypotheses, that the integrand $\phi$ is twice differentiable. Put

$$
C(x, v) = \phi_{xx}(x, v)^{-1} (\phi_x(x, v)v - \phi_x(x, v)), \quad D(x, v) = \phi_x(x, v).
$$

Then the optimal feedback map $U(t, x)$ must be a solution of the problem

$$
U_t(t, x) + \nabla_x U(t, x) U(t, x) + C(x, U(t, x)) = 0 \text{ in } (0, T) \times \Omega,
$$

together with the terminal time condition $D(T, U(T, x)) = 0$, and boundary condition $U(t, x) \cdot \nu(x) = 0$ over $\partial\Omega$.

### 8.2. Existence result for bounded $\Omega$

We go back to our optimal control problem in feedback form

$$
\text{Minimize in } u(t, x) \in K : \int_0^T F(x(s), u(s, x(s))) \, ds + g(x(T))
$$

subject to

$$
x'(s) = f(x(s), u(s, x(s))) \text{ in } (0, T), \quad x(0) = x_0, x(s) \in \Omega.
$$

Recall that:

- $T > 0$ is the time horizon considered;
- $\Omega \subseteq \mathbb{R}^N$ is the feasible set for the state variable $x : (0, T) \to \Omega$;
- $K \subseteq \mathbb{R}^m$ is the feasible set for the control variable $u : (0, T) \to K$;
- $x_0 \in \mathbb{R}^N$ is the vector determining the state of the system when we start to care about the control problem;
- $F : \Omega \times K \to \mathbb{R}$ is the density for the cost functional, while $g : \Omega \to \mathbb{R}$ is the contribution depending on the final state;
- $f : \Omega \times K \to \mathbb{R}^N$ is the map providing the state equation that governs the dynamics of the system.

For our main existence result for optimal feedback controls, we are going to use, as a main tool, the variational reformulation. We will take

$$
(8.2) \quad \phi(x, \xi) = \min_{u \in K} \{ F(x, u) : \xi = f(x, u) \in \Lambda(x) \},
$$

with

$$
\Lambda(x) = \text{closure}\{ \xi \in \mathbb{R}^N : x + \epsilon \xi \in \Omega, \text{ for some } \epsilon > 0 \}.
$$

It is elementary to realize that $\Lambda(x) = \mathbb{R}^N$ if $x \in \Omega$, $\Lambda(x) = \emptyset$ if $x \notin \Omega$, and

$$
\Lambda(x) = \{ z \in \mathbb{R}^N : z \cdot \nu(x) \leq 0 \}
$$

for $x \in \partial\Omega$. Then we know that the variational problem

$$
\text{Minimize in } x : \int_t^T \phi(x(s), x'(s)) \, ds
$$

...
under \( x(0) = y \), is equivalent to the optimal control problem

\[
\text{Minimize in } u \in K : \int_{t}^{T} F(x(s), u(s)) \, ds
\]

subject to

\[
x'(s) = f(x(s), u(s)) \text{ in } (t, T), \quad x(0) = y, x(s) \in \Omega.
\]

This equivalence should be understood in the sense that one can go from optimal solutions of one of the two to optimal solutions for the other as was indicated earlier. In particular, we are especially interested in the passage from an optimal solution \( x \) for the variational version to an optimal solution for the optimal control problem. Let us briefly recall how this is accomplished.

Assume \( X(s; t, y) \) is an optimal solution for the variational problem with derivative \( X'(s; t, y) \). The definition of the integrand \( \phi \) in (8.2) delivers an optimal \( U(t, y) \) so that

\[
(8.3) \quad \phi(X(t; t, y), X'(t; t, y)) = F(X(t; t, y), U(t, y))
\]

and, in addition,

\[
(8.4) \quad U(t, y) \in K, \quad X'(t; t, y) = f(X(t; t, y), U(t, y)).
\]

Since

\[
X(t; t, X(s; 0, y)) = X(s; 0, y), \quad X'(t; t, X(s; 0, y)) = X'(s; 0, y),
\]

for all \( t, s \in [0, T] \), \( y \in \Omega \), we can conclude that by putting \( x(s) = X(s; 0, y) \), and replacing \( y \) by \( X(s; 0, y) \) in (8.3) and in (8.4), then

\[
F(x(s), U(s, x(s))) = \phi(x(s), x'(s)), \quad s \in (0, T),
\]

\[
x'(s) = f(x(s), U(s, x(s))), \quad s \in (0, T).
\]

In this way \( U(t, y) \) is the optimal feedback control map. Notice that this optimal map is such that integral curves of

\[
x'(s) = f(x(s), U(s, x(s))
\]

cannot exit \( \Omega \) by construction of \( \phi \): the integrand would attain an infinite value were that to occur.

Theorem 4.3 is then a direct consequence of Theorem 8.2 applied to the variational reformulation just described.

9. Optimality conditions for bounded \( \Omega \)

The issue of optimality conditions is a bit more involved as the integrand \( \phi \) in (8.2) cannot be smooth. Even if it were, dealing with this integrand is not efficient from a practical point of view as this would require to have it available in close form. To avoid this issue, is, in part, a main reason for this work. To state our new optimality fact under a convex, compact restriction set \( K \), and a feasible domain \( \Omega \) for states, let us set

\[
S_{\nu} = \{ z \in \mathbb{R}^N : \langle z, \nu \rangle \leq 0 \},
\]

and

\[
(9.1) \quad K(y) = \{ u \in K : f(y, u) \in S_{\nu(y)} \} = f^{-1}(y, S_{\nu(y)}) \cap K,
\]

for \( y \in \partial \Omega \), being \( \nu(y) \) the unit, outer normal at \( y \).

**Definition 9.1.** A map \( u(t, x) : [0, T] \times \Omega \rightarrow K \) is feasible for our problem if \( u(t, x) \in K(x) \) for \( x \in \partial \Omega \), while \( u(t, x) \in K \) for \( x \in \Omega \). We will simply write \( u(t, x) \in K(x) \) to mean exactly this: \( K(x) = K \) for \( x \in \Omega \); and \( K(x) \) stands for the set in (9.1) if \( x \in \partial \Omega \).
The motivation for such a definition is clear: if $u$ is a feasible map, then the state path solution of $x'(t) = f(x(t), u(t, x(t)))$ in $[0, T]$ with $x(0) = x_0 \in \Omega$, cannot leave $\Omega$.

Recall that

$$I(u) = \int_0^T F(x(t), u(t, x(t))) dt, \quad x'(t) = f(x(t), u(t, x(t))) \text{ in } (0, T).$$

Let the costate $p(t, x)$, associated with the control $u(t, x)$, be a solution of the problem

$$(9.2) \quad p_t(t, x) + \nabla p(t, x) f(x, u(t, x)) + p(t, x) \nabla_x [F(x, u(t, x))] = \nabla_x [F(x, u(t, x))] \text{ in } [0, T] \times \Omega$$

under the terminal time condition $p(T, x) = 0$ for $x \in \Omega$, but no specific boundary condition around $\partial \Omega$, so there might be several solutions. Set

$$\nabla I(u)(t, x) \equiv F_u(x, u(t, x)) - p(t, x)f_u(x, u(t, x)).$$

**Theorem 9.1.** Assume the sets in (9.1) are all convex for arbitrary $y \in \partial \Omega$. Let $u(t, x)$ be a feasible map for the optimal control problem under a constraint set $K$, which is assumed to be compact and convex, and let $p(t, x)$ be one associated costate as just indicated. Then the solution of the obstacle problem, for each fixed time $t \in [0, T]$,

Minimize in $U(t, x) \in K(x) : \int_{\Omega} \left( \frac{1}{2} \|\nabla U(t, x) - \nabla u(t, x)\|^2 + \nabla I(u)(t, x)(U(t, x) - u(t, x)) \right) dx$

is a descent direction for the optimal control at $u(t, x)$, in an average sense

$$\int_{\Omega} \nabla I(u)(t, x)(U(t, x) - u(t, x)) dx \leq 0,$$

for all $t \in [0, T]$. If such $u(t, x)$ is indeed optimal for the control problem then the solution of the obstacle problem is $u(t, x)$ itself for all $t \in [0, T]$.

**Proof.** Due to the convexity of the sets in (9.1), the perturbation $u + \epsilon(U - u)$ is feasible for small, positive $\epsilon$. Here $u$ is an arbitrary feasible field, and $U$ is the solution of the obstacle problem associated with $u$. We would like to check that the derivative of the cost functional $I(u + \epsilon(U - u))$ with respect to $\epsilon$ at $\epsilon = 0$ is non-positive. This fact will tell us that $U$ is indeed a descent direction to follow from $u$. Though the proof is a bit repetitive with respect to previous situations, for the sake of clarity and completeness, we rewrite the various steps.

Let $X$ be the map defined through the identity

$$x'(s) + \epsilon X'(s) = f(x(s) + \epsilon X(s), u(s, x(s) + \epsilon X(s)) + \epsilon(U(s, x(s) + \epsilon X(s))) - u(s, x(s) + \epsilon X(s))),$$

in $(0, T)$ with $X(0) = 0$. By differentiating with respect to $\epsilon$, and setting $\epsilon = 0$, we find

$$(9.3) \quad X'(s) = \nabla_x f(x, u)X + \nabla u f(x, u)\nabla uX + \nabla x u f(x, u)(U - u) \text{ in } (0, T), \quad X(0) = 0. $$

On the other hand, the same computations performed in the cost functional lead to

$$\int_0^T [\nabla_x F(x(s), u(s, x(s)))X(s) + \nabla u F(x(s), u(s, x(s)))\nabla u(s, x(s)))X(s)$$

$$- \nabla u F(x(s), u(s, x(s)))(U(s, x(s)) - u(s, x(s)))) ds.$$

It is the non-positivity of this last expression, under (9.3), that we would like to check. Or rather, its integral with respect to the spatial variable $x = x_0$, the initial condition for the control problem.

Note that

$$\nabla_x [F(x(s), u(s, x(s)))] = \nabla_x F(x(s), u(s, x(s))) + \nabla u F(x(s), u(s, x(s)))\nabla u(s, x(s)).$$
and so, using the differential system (9.2) for the costate \( p \), the expression we are interested in, can be recast as

\[
\int_0^T \left[ p_t + \nabla_p f + p \nabla_x f \right] X + \nabla_u F(U - u) \, ds.
\]

Because \( f = x' \), we see that \( p_t + \nabla_p f \) is precisely the derivative \( p' \) of the composition \( p(s, x(s)) \), and so the last integral becomes

\[
\int_0^T \left[ p' + p \nabla_x f \right] X + \nabla_u F(U - u) \, ds.
\]

An integration by parts in the first product (boundary terms drop out), and (9.3), permit to write

\[
\int_0^T -p(\nabla_x f X + \nabla_u f \nabla u X + \nabla_u f(U - u)) + p \nabla_x (f) X + \nabla_u F(U - u) \, ds.
\]

But \( \nabla_x (f) = \nabla_x f + \nabla_u f \nabla u \), and hence our integral simplifies to

\[
\int_0^T (\nabla_u F - p \nabla_u f)(U - u) \, ds.
\]

The integrand is the product \( \nabla I(u)(t, x)(U - u) \). If \( U \) is the solution of the obstacle problem indicated in the statement, then for each fixed time \( s \) the integral of \( (9.4) \) with respect to the spatial variable \( x_0 \), the initial condition for the control problem, will be non-positive. This is exactly what was claimed in the statement.

If \( u(t, x) \) is indeed optimal for the problem, then

\[
\int_\Omega \nabla I(u)(t, x)(U(t, x) - u(t, x)) \, dx \geq 0
\]

for all \( t \), and so it is clear that the solution of the obstacle problem is \( U = u \). \( \square \)

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