Abstract. We construct a new finite difference method for the flow of ideal viscous isentropic gas in one spatial dimension. For the continuity equation, the method is a standard upwind discretization. For the momentum equation, the method is an uncommon upwind discretization, where the moment and the velocity are solved on dual grids. Our main result is convergence of the method as discretization parameters go to zero. Convergence is proved by adapting the mathematical existence theory of Lions and Feireisl to the numerical setting.

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1. Introduction

In this paper, we will develop a convergent finite difference method for the flow of an ideal viscous isentropic gas in one spatial dimension. We will assume that the flow may be modeled by the Navier-Stokes system (cf. [16]):

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \quad \text{in } \mathbb{R}^+ \times (0, L), \quad (1.1) \\
(\rho u)_t + (\rho u^2)_x &= \mu u_{xx} - p(\rho)_x, \quad \text{in } \mathbb{R}^+ \times (0, L). \quad (1.2)
\end{align*}
\]

The unknowns in this system are the fluid density \( \rho = \rho(t, x) \) and the fluid velocity \( u = u(t, x) \). For an isentropic flow, the ideal pressure law takes the form:

\[ p(\rho) = \frac{\gamma - 1}{\gamma} \rho^\gamma, \quad \text{for } \gamma > 1. \]

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form

\[ p = a \rho^\gamma, \quad a > 0, \]

where the value of \( \gamma \) is determined by the specific gas in question. In this paper, we will be forced require that

\[ \frac{3}{2} < \gamma < 2, \]

to prove convergence of the method. Note that this significantly limits the physical applicability of the convergence result. While monoatomic gases (\( \gamma \sim \frac{5}{3} \)), such as helium, are included, diatomic gases (\( \gamma \sim \frac{7}{5} \)), such as air, are not. The condition is necessary for convergence, but not stability, of the method.

At the boundary, (1.1) - (1.2) is augmented with no-slip conditions,

\[ u(t, 0) = u(t, L) = 0. \]

and the initial conditions,

\[ 0 < \rho(0, x) = \rho_0(x) \in L^{\gamma}(0, L), \quad u(0, x) = u_0(x) \in L^{\infty}(0, L). \quad (1.3) \]

Existence of global classical solutions for the system (1.1)-(1.2) is well-known and was first established by Kanel’ [9] for smooth data (see also [14]). For non-smooth initial data, corresponding results have been obtained by David Hoff [1, 7]. These works are all based on the same approach: find pointwise upper and lower bounds on the density, then use these bounds to estimate the second derivative of the velocity. With smooth initial density, the density remain \( H^1 \), while with discontinuous initial density, the density is at most \( BV \) as initial discontinuities persists for all time (see [7]). The pointwise bounds on the density are obtained by tracking certain quantities along streamlines rendering the entire existence theory essentially Lagrangian.

In the literature, one can find a huge variety of numerical methods appropriate for (1.1)-(1.2). However, very few of these methods have been proven to converge. This lack of rigorous results are most likely a consequence of the Lagrangian nature of the existence theory. In fact, prior to this paper, all convergent methods have been discretized in Lagrangian coordinates and are due to David Hoff and collaborators [18, 19, 20]. That being said, there are also several existence results that utilize discrete Lagrangian approximation schemes (e.g. [1, 7, 8]) to construct solutions. For practical applications, results in Eulerian coordinates is often desirable and a mapping from discrete Lagrange to discrete Euler is costly. For this reason, most practitioners would employ an Eulerian discretization.

In this paper, we will discretize the equation in Eulerian coordinates. As a consequence, obtaining pointwise bounds on the density becomes highly involved and will not be pursued in this paper. Instead, we will develop a convergence theory in the spirit of the continuous existence theory [4, 15, 17] as sparked by P. L. Lions. In particular, we will not obtain any form of continuity of the discrete density and instead prove strong convergence of the density using renormalization and what is known as the effective viscous flux (cf. [17]).
In more than one spatial dimension, the literature is almost void of convergent numerical methods. For the full system, the only result is the paper [13], by the author, in which a convergent finite element discretization of (1.1)-(1.2) is developed. Over the last years, there have also been developed a series of convergent methods [2, 3, 5, 10, 11, 12] for the Stokes version of (1.1)-(1.2). The method we shall develop and analyze in this paper can be considered as an archetype for all the methods mentioned above. That is, in one spatial dimension, all of these methods are in a sense equivalent and of the form we shall consider here. In addition, the method we present here is the first finite difference method for which convergence is proved.

The paper is organized as follows: In the next section, we will define the numerical method and state the main convergence result. Then, we will derive stability of the method and some other basic properties. In Section 4, we will derive an equation for the effective viscous flux and also provide a higher integrability estimate for the density. Section 5 concerns passing to the limit in the method. In particular, we prove that the limit almost satisfies (1.1)-(1.2). The remaining ingredient is to prove strong convergence of the density which is proved in the final section.

2. The numerical method and main result

Our method will be posed on a uniform grid in both space and time. In time, we shall approximate at discrete points \( t^k = k \Delta t \), where \( \Delta t \) is assumed to be of the order \( \Delta x \). In space, we will divide the domain \([0, L]\) into \( N \) intervals of length \( \Delta x = L/N \). It will be convenient to write this as

\[
[0, L] = \bigcup_i [x_{i-1/2}, x_{i+1/2}], \quad i = 0, \ldots, N - 1,
\]

where we have introduced the slightly confusing notation

\[
x_{i-1/2} = i \Delta x, \quad i = 0, \ldots, N.
\]

We shall also need the dual grid given by the midpoint nodes

\[
x_i = \left(i + \frac{1}{2}\right) \Delta x, \quad i = 0, \ldots, N - 1.
\]

The numerical method will approximate the density on the dual grid and the velocity on the standard grid:

\[
\varrho(k \Delta t, x_i) \approx \varrho^k_i, \quad i = 0, \ldots, N - 1,
\]

\[
u(k \Delta t, x_{i-1/2}) \approx u_{i-1/2}^k, \quad i = 0, \ldots, N.
\]

Staggered grid of this kind is necessary for incompressible flows, and as a consequence also widely used to develop all speed compressible flow methods.

2.1. Discrete operators. To discretize the convective terms, we shall utilize an upwind method. To facilitate this, we introduce the notation

\[
u^+ = \max\{u, 0\}, \quad u^- = \min\{u, 0\}.
\]

We shall also need the average velocity over an interval

\[
\hat{u}_i = \frac{u_{i-1/2} + u_{i+1/2}}{2}.
\]
For the continuity equation (1.1), we shall use the following upwind flux
\[ \text{Up}(\varrho u)_{i+1/2} = \varrho_i u_{i+1}^+ + \varrho_{i+1} u_{i+1/2}^- . \]
This upwind flux will also be used for the momentum equation, where we upwind the (averaged) momentum
\[ \text{Up}(\varrho \hat{u} u)_{i+1/2} = (\varrho_i \hat{u}_i) u_{i+1}^+ + (\varrho_{i+1} \hat{u}_{i+1}) u_{i+1/2}^- . \]

The remaining derivatives will be discretized using the operators
\[ \partial_{i+1/2} f = \frac{f_{i+1} - f_i}{\Delta x}, \quad \partial_i v = \frac{v_{i+1/2} - v_{i-1/2}}{\Delta x} , \]
which defines the following (standard) Laplace operator
\[ \Delta_{i+1/2} u_h = \partial_{i+1/2} \partial_i u_h = \frac{u_{i-1} - 2u_{i+1} + u_{i+3}}{\Delta x^2} . \]

For time discretization, we will use implicit time-stepping
\[ \partial_t f_i = \frac{f_i^k - f_i^{k-1}}{\Delta t} , \]

2.2. Numerical method. The numerical method is defined as follows:

**Definition 2.1.** Given \( \Delta t, \Delta x < 1 \), and initial data (1.3), define the numerical initial data
\[ \varrho_i^0 = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \varrho_0(y) \, dy, \quad u_i^{0 \leftarrow 1/2} = u_0(x_{i-1/2}), \quad i = 0, \ldots, N - 1 . \]
Determine sequentially the numbers
\[ (\varrho_i^k, u_i^k), \quad i = 0, \ldots, N - 1, \quad k = 0, \ldots, M , \]
solving the nonlinear system
\[ \begin{align*}
\partial_t \varrho_i + \partial_i \text{Up}(\varrho^k u^k) &= 0 \quad \text{(2.1)} \\
\partial_t \left( \varrho_i \hat{u}_i + \varrho_{i+1} \hat{u}_{i+1} \right) &+ \frac{\left( \text{Up}(\varrho^k \hat{u}^k u^k)_{i+3/2} - \text{Up}(\varrho^k \hat{u}^k u^k)_{i-1/2} \right)}{2\Delta x} \\
&= \mu \Delta_{i+1/2} u^k_h - \partial_{i+1/2} p_h^k, \quad \text{(2.2)} \\
u_{i-1/2}^k &= u_{N-1/2}^k = 0 .
\end{align*} \]

It is not completely trivial that our numerical method is well-defined. Since the system of equations (2.1)-(2.2) are both nonlinear and implicit, existence of a solutions needs to be established.

**Lemma 2.2.** Let \( 0 < \Delta t, \Delta x < 1 \) be fixed. The nonlinear implicit system of equations (2.1)-(2.2) admits at least one solution.

**Proof.** The existence can be proved using a topological degree argument. Since the proof is very similar to the corresponding result in [6, 11, 13], we do not give the details here and instead refer the reader to any of the mentioned papers.

From [6] and [11], we also have that the method preserves strict positivity of the density:
Lemma 2.3. Let \( \{ (\rho^k_i, u^k_i) \}_{i} \) satisfy the continuity scheme (2.1). Then,

\[
\min_i \rho^k_i \geq \min_i \rho^k_{i-1/2} \left( 1 + \frac{1}{\Delta t \max_i |u^k_{i+1/2}|} \right) .
\]

2.3. Extension. To analyze the finite difference method, it will be of great convenience to extend the numerical solution to all of \([0, T) \times (0, L)\). Since the density appears nonlinearly in the pressure, we will use piecewise constants to extend it:

\[
\rho_h(t, x) = \rho^k_i, \quad \forall (t, x) \in \left[ t^k, t^{k+1} \right) \times \left[ x_i - 1/2, x_i + 1/2 \right).
\]

(2.3)

For the velocity, we shall utilize piecewise continuous lines in space and piecewise constants in time.

\[
u_h(t, x) = u^k_{i-1/2} + \left( \frac{x - x_i - 1/2}{\Delta x} \right) \left( u^k_{i+1/2} - u^k_{i-1/2} \right), \quad \forall (t, x) \in \left[ t^k, t^{k+1} \right) \times \left[ x_i - 1/2, x_i + 1/2 \right).
\]

(2.4)

for all \((t, x) \in \left[ t^k, t^{k+1} \right) \times \left[ x_i - 1/2, x_i + 1/2 \right)\). Note that an extension of the average velocity \( \hat{u}_h^k \) is now given by the \( L^2 \) projection onto piecewise constant:

\[
\hat{u}_h(t, x) = \Pi^Q_h[u_h] = \frac{1}{\Delta x} \int_{x_i-1/2}^{x_i+1/2} u_h \, dx, \quad x \in (x_i-1/2, x_i+1/2).
\]

2.4. Main result. The following theorem is our main result in this paper.

Theorem 2.4. Assume we are given initial data \((\rho_0, u_0)\) satisfying (1.3) and a finite final time \(T > 0\). Let \( \{ (\rho_h, u_h) \}_{h>0} \) be a family of numerical solutions, constructed through Definition 2.1 and (2.3)-(2.4), with

\[
\Delta t = \Delta x = h.
\]

Then, as \(h \to 0\), \( u_h \to u \) in \( L^2(0, T; W^{1,2}_0(0, L)) \), \( \rho_h \to \rho \) a.e. on \((0, T) \times (0, L)\), where \((\rho, u)\) is a weak solution of (1.1)-(1.2):

\[
\int_0^T \int_0^L \rho(\phi_t + u \cdot \phi_x) \, dx \, dt = \int_0^L \rho_0 \phi(0, \cdot) \, dx,
\]

\[
\int_0^T \int_0^L (\rho u) v_t + \rho u^2 v_x + \mu u_x v_x - p(\rho) v_x \, dx \, dt = \int_0^L \rho_0 u_0 \phi(0, \cdot) \, dx,
\]

for all \((\phi, v) \in C_0^\infty(\mathbb{R}) \times (0, T) \times (0, L)\).

Theorem 2.4 will follow as a consequence of the various results stated and proved in the upcoming sections. The proof will be completed in Section 6.

3. Stability and energy estimates

In this section, we will derive a numerical analog of the continuous energy estimate. This estimate yields in particular stability of the method, but it will also provides us with necessary \( L^p \) bounds uniform in the discretization parameters.
3.1. Renormalized continuity scheme. We shall need the following renormalized continuity scheme at several occasions. The term renormalized is motivated by the corresponding continuous equation and its role in the existence theory (cf. [15]).

Lemma 3.1. Let $B \in C^1(0,R^+)$ and define $b(z) = zB'(z) - B(z)$. If \{$(q^k_i, u^k_{i+1/2})_{i,k}$ satisfies the continuity scheme (2.1), then the following identity holds

$$\partial^k_t B(q_i) + \partial_i \text{Up} \left(B(q^k_i)u^k\right) + b(q^k_i)\partial_i u^k + \mathcal{P} \left[B(q^k_i), u^k\right] = 0,$$

(3.1)

where $\mathcal{P}$ is given by

$$\mathcal{P} \left[B(q^k_i), u\right] = \Delta tB''(q^*_i)\partial^k_t q^*_i|^2 - B''(q^k_i)[q_{i+1} - q_i]^2u^i_{i+1/2} + B''(q^k_i)[q_{i-1} - q_i]u^i_{i-1/2}.$$ Here, $q^*_i$, $q^k_i$, and $q^k_i$, are some numbers in the range $[q^k_{i-1}, q^k_i], [q^k_i, q^k_{i+1}]$, and $[q^k_{i-1}, q^k_{i+1}]$, respectively.

Proof. We begin by multiplying (2.1) with $B'(q)$ to obtain

$$B'(q_i)\partial^k_t q_i = -B'(q_i)\partial_i \text{Up}(qu).$$

By applying Taylor expansion, we find that

$$\partial^k_t B(q) + \Delta tB''(q^*_i)\partial^k_t q^*_i|^2 = -B'(q_i)\partial_i \text{Up}(qu),$$

(3.2)

where $q^*_i$ is some number in $[q^k_{i-1}, q^k_i]$.

By adding and subtracting and applying another Taylor expansion,

$$(\Delta x)B'(q_i)\partial_i \text{Up}(qu)$$

$$= B'(q_i) \left[q_i u^i_{i+1/2} + q_{i+1}u^i_{i+1/2} - q_{i-1}u^i_{i-1/2} - q_iu^i_{i-1/2}\right]$$

$$= B'(q_i)q_i(u^i_{i+1/2} - u^i_{i-1/2}) + B'(q_i)[q_{i+1} - q_i]u^i_{i+1/2}$$

$$+ B'(q_i)[q_{i-1} - q_i]u^i_{i-1/2}$$

$$= B'(q_i)q_i(u^i_{i+1/2} - u^i_{i-1/2}) + [B(q_{i+1}) - B(q_i)]u^i_{i+1/2}$$

$$+ [B(q_i) - B(q_{i-1})]u^i_{i-1/2} + B''(q^k_i)[q_{i+1} - q_i]u^i_{i+1/2}$$

$$- B''(q^k_i)[q_{i-1} - q_i]u^i_{i-1/2}$$

$$= (\Delta x) \left[B'(q_i)q_i\partial_i u + \partial_{i+1/2}B(q)u^{-i+1/2} + \partial_{i-1/2}B(q)u^i_{i-1/2}\right]$$

$$+ B''(q^k_i)[q_{i+1} - q_i]u^i_{i+1/2} - B''(q^k_i)[q_{i-1} - q_i]u^i_{i-1/2}$$

(3.3)

Here, $q^k_i$ and $q^k_i$ are some numbers in $[q_i, q_{i+1}]$ and $[q_i, q_{i-1}]$, respectively. Next, we calculate

$$B'(q_i)q_i\partial_i u + \partial_{i+1/2}B(q)u^{-i+1/2} + \partial_{i-1/2}B(q)u^i_{i-1/2}$$

$$= b(q_i)\partial_i u + B(q_i)e\partial_i u + \partial_{i+1/2}B(q)u^{-i+1/2} + \partial_{i-1/2}B(q)u^i_{i-1/2}$$

(3.4)

We conclude the proof by combining (3.4) in (3.3) and (3.2). □
3.2. The convection operator. To prove stability of the method, we shall multiply the momentum scheme (2.2) by $u^k_{i+1/2}$ and sum over all $i$. For this purpose, we shall need the following identity for the convective discretization.

Lemma 3.2. The following identity holds

$$
\begin{align*}
\Delta x \sum_{i=0}^{N-1} \left( \frac{U_p(g^k u^k)_{i+3/2} - U_p(g^k \tilde{u}^k u^k)_{i-1/2}}{2\Delta x} \right) u^k_{i+1/2} \\
&= -\Delta x \sum_{i=0}^{N-1} U_p(g^k u^k)_{i+1/2} \frac{1}{2} \left( \frac{\tilde{u}^k_{i+1/2}^2 - \tilde{u}^k_{i-1/2}^2}{2} \right) + N_2 \\
&= -\Delta x \sum_i \partial_i^k \theta_t \left( \frac{\tilde{u}^k_{i+1/2}^2}{2} \right) + N_2.
\end{align*}
$$

where the numerical diffusion term $N_1$ is given by

$$
N_2 = \frac{(\Delta x)^2}{2} \sum_{i=0}^{N-1} \left| U_p(g^k u^k)_{i+1/2} \right| \left| \partial_{i+1/2} \tilde{u}^k \right|^2.
$$

Proof. By applying summation by parts, we see that

$$
\begin{align*}
\Delta x \sum_{i=0}^{N-1} \left( \frac{U_p(g^k u^k)_{i+3/2} - U_p(g^k \tilde{u}^k u^k)_{i-1/2}}{2\Delta x} \right) u^k_{i+1/2} \\
&= -\Delta x \sum_{i=0}^{N-1} U_p(g^k \tilde{u}^k u^k)_{i+1/2} \left( \frac{u^k_{i+3/2} - u^k_{i-1/2}}{2\Delta x} \right) \\
&= -\Delta x \sum_i U_p(g^k \tilde{u}^k u^k)_{i+1/2} \partial_{i+1/2} \tilde{u}^k.
\end{align*}
$$

Next, we apply the definition of $U_p(\cdot)$ and add and subtract to deduce

$$
U_p(g^k \tilde{u}^k u^k)_{i+1/2} \partial_{i+1/2} \tilde{u}^k \\
= \left[ \left( \theta_t^k u^k \right)_{i+1/2}^{k+} + \left( \theta_t^{k+} \tilde{u}^k \right)_{i+1/2}^{k-} \right] \partial_{i+1/2} \tilde{u}^k \\
= \frac{1}{2} \left[ \theta_t^k u^k_{i+1/2}^{k+} + \theta_t^{k+} \tilde{u}^k_{i+1/2}^{k-} \right] \partial_{i+1/2} \tilde{u}^k \\
+ \frac{1}{2} \left[ \theta_t^k u^k_{i+1/2}^{k-} + \theta_t^{k-} \tilde{u}^k_{i+1/2}^{k+} \right] \partial_{i+1/2} \tilde{u}^k \\
= U_p(g^k u^k)_{i+1/2} \partial_{i+1/2} \left( \frac{\tilde{u}^k_{i+1/2}^2}{2} \right) \\
- \frac{\Delta x}{2} \left| U_p(g^k u^k)_{i+1/2} \right| \left| \partial_{i+1/2} \tilde{u}^k \right|^2.
$$

We then conclude the proof by setting this identity in (3.5). □

3.3. Energy estimate. We are now ready to prove stability of the method.
Proposition 3.3. Let \( \{(q^k, u^k)\}_{i,k} \) be the numerical solution obtained through Definition 2.1. The following stability estimate holds,

\[
\max_m \left( \Delta x \sum_i \theta_i^m |{\tilde u_i^m}|^2 + \frac{1}{\gamma - 1} p(\theta_i^m) \right) \\
+ \Delta t \Delta x \sum_{k=1}^M \sum_i \left| \partial_{i+1/2} u^k_h \right|^2 + \sum_{i=1}^N N_i \right) \\
= \Delta x \sum_i \theta_i^0 |{\tilde u_i^0}|^2 + \frac{1}{\gamma - 1} p(\theta_i^0),
\]

where the numerical diffusion terms are

\[
N_1 = (\Delta t)^2 \Delta x \sum_{k=1}^M \sum_i p'' \left( \theta_i^k \right) |\partial_i^k \theta_i|^2 \\
N_2 = \Delta t (\Delta x)^2 \sum_{k=1}^M \sum_i p'' \left( \theta_i^k \right) |\partial_{i+1/2} \theta_i^k| \left| u^k_{i+1/2} \right| \\
N_3 = (\Delta t)^2 \Delta x \sum_{k=1}^M \sum_i \frac{\theta_i^{k-1}}{2} |\partial_i^k \tilde u_i|^2 \\
N_4 = \frac{\Delta t (\Delta x)^2}{2} \sum_{k=1}^M \sum_{i=0}^{N-1} \left| \partial_{i+1/2} \theta_i^k \right| \left| \partial_{i+1/2} \tilde u_i^k \right|^2.
\]

Proof. We begin by multiplying (2.2) by \( u^k_{i+1/2} \Delta x \) and sum over all \( i \) to obtain

\[
\Delta x \sum_i \theta_i^k \left( \frac{q_i \hat u_i + q_{i+1} \hat u_{i+1}}{2} \right) u^k_{i+1/2} \\
= -\Delta x \sum_i \left( \frac{\text{Up}(q^k \hat u^k u^k)_{i+3/2} - \text{Up}(q^k \hat u^k u^k)_{i-1/2}}{2\Delta x} \right) u^k_{i+1/2} \\
- \Delta x \sum_i \left| \partial_{i+1/2} u^k_h \right|^2 - p(\theta_i^k) \partial_i u^k_h,
\]

where we have also applied summation by parts together with the boundary condition \( u^k_{-1/2} = u^k_{N-1/2} = 0 \). Next, we apply Lemma 3.2 and Lemma 3.1 (with \( B(z) = \frac{1}{\gamma - 1} P(z) \)) to deduce

\[
\Delta x \sum_i \theta_i^k \left( \frac{q_i \hat u_i + q_{i+1} \hat u_{i+1}}{2} \right) u^k_{i+1/2} - \partial_i^k \theta_i \left( \frac{|\hat u_i|^2}{2} \right) \\
= -N_2 - \Delta x \sum_i \left| \partial_{i+1/2} u^k_h \right|^2 - \frac{1}{\gamma - 1} \partial_i^k p(\theta_i) - P[p(\theta), u].
\]

To proceed, we observe the following identities

\[
\Delta x \sum_i \theta_i^k \left( \frac{q_i \hat u_i + q_{i+1} \hat u_{i+1}}{2} \right) u^k_{i+1/2} = \Delta x \sum_i \theta_i^k (q_i \hat u_i) \hat u_i^k, \quad (3.8)
\]
By applying (3.8)-(3.9) in (3.7), we deduce
\[
\Delta t \sum_i \frac{|\hat{u}_i|^2}{2} + \frac{\Delta t |\hat{u}_i - \hat{u}_i^{k-1}|^2}{2} = \Delta t \left( \frac{|\hat{u}_i|^2}{2} \right) + \frac{\Delta t |\hat{u}_i - \hat{u}_i^{k-1}|^2}{2}.
\]

By applying (3.8)-(3.9) in (3.7), we deduce
\[
\Delta x \sum_i \left( \frac{|\hat{u}_i|^2}{2} + \frac{\Delta t |\hat{u}_i - \hat{u}_i^{k-1}|^2}{2} + \frac{1}{\gamma - 1} \partial_t p(\hat{u}_i) \right)
+ \Delta x \sum_i \left| \partial_{t+1/2} u_h^k \right|^2 + N_2 + \mathcal{P}[p(\hat{u}), u] = 0.
\]

We conclude by multiplying with \(\Delta t\), summing over \(k = 1, \ldots, M\), and recalling the definition of \(\mathcal{P}\). \(\square\)

The previous stability estimate also provides some uniform integrability estimates on various quantities. To state these, let us introduce the notation
\[
f_h \in L^p(Q),
\]
to denote the case when \(f_h\) is bounded in \(L^p\) uniformly with respect to both \(\Delta t\) and \(\Delta x\).

**Corollary 3.4.** Let \((\hat{u}_h, u_h)\) be the numerical solution constructed through Definition 2.1 and (2.3)-(2.4). Then,
\[
\begin{align*}
\hat{u}_h &\in L^\infty(0, T; L^\gamma(0, L)), & p(\hat{u}_h) &\in L^\infty(0, T; L^1(0, L)), \\
u_h &\in L^2(0, T; W_0^{1,2}(0, L)) \cap L^\infty(0, T; L^\gamma(0, L)).
\end{align*}
\]

As a consequence,
\[
\begin{align*}
\partial_t \hat{u}_h &\in L^\infty(0, T; L^{2\gamma \gamma/(2\gamma - 2)}(0, L)), & \partial_t |\hat{u}_h|^2 &\in L^\infty(0, T; L^1(0, L)), \\
\partial_t u_h &\in L^2(0, T; L^\gamma(0, L)), & \partial_t |u_h|^2 &\in L^2(0, T; L^{2\gamma \gamma/(2\gamma - 2)}(0, L)).
\end{align*}
\]

**Proof.** From Proposition 3.3, we have that \(\partial_t \hat{u}_h \in L^\infty(0, T; L^\gamma(0, L)), u_h \in L^2(0, T; W_0^{1,2}(0, L)), \) and \(\partial_t \hat{u}_h^2 \in L^\infty(0, T; L^1(0, L)).\) The remaining bounds follows directly from these using Sobolev embedding and interpolation estimates. \(\square\)

### 3.4. Control in time

We end this section by establishing some weak control on the time-derivatives. For this purpose, let
\[
\partial_t^k f_h = \frac{f_h(\cdot) - f_h(\cdot - \Delta t)}{\Delta t} = \frac{f^k - f^{k-1}}{\Delta t}, \quad t \in [k \Delta t, (k + 1) \Delta t),
\]
for all \(k = 1, \ldots, M\).

**Lemma 3.5.** Let \((\hat{u}_h, u_h)\) be the numerical solution constructed through Definition 2.1 and (2.3)-(2.4). Then,
\[
\begin{align*}
\partial_t^k \hat{u}_h &\in L^2(0, T; W^{-1,\gamma}(0, L)), & \partial_t^k \hat{u}_h &\in L^1(0, T; W^{-1,\gamma}(0, L)),
\end{align*}
\]
Proof. 1. We first prove (3.10). Let \( \phi \in C^\infty(0, L) \) be arbitrary and define
\[
\phi_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \phi(y) \, dy, \quad i = 0, \ldots, N - 1.
\]

Now, multiply the continuity scheme (2.1) with \( \Delta x \phi_i \), sum over all \( i \), and apply summation by parts, to deduce
\[
\Delta x \sum_i \left( \frac{\partial_t \phi_i^k}{\Delta t} \right) \phi_i = \Delta x \sum_i \uparrow \phi^{(k)}(x_{i+1/2}) - \Delta x \sum_i \left( \partial_t \phi_i^k \phi_i \phi^{(k)}(x_{i+1/2}) \right)
\leq C \| \varphi_h \|_{L^\infty(0, L)} \| \varphi_{h_i} \|_{L^\infty(0, L)} \| \phi_x \|_{L^{\frac{2}{1-\gamma}}(0, L)}.
\]

The last inequality is an application of the Hölder inequality. Now, since (3.12) holds for all \( \phi \), we can conclude that it continues to hold for all \( \phi \in W^{1, \frac{2}{1-\gamma}}(0, L) \) By multiplying with \( \Delta t \) and summing over all \( k \), we obtain
\[
\left\| \partial_t \varphi_h \right\|_{L^2(0, T; W^{1, \frac{2}{1-\gamma}}(0, L))} \leq \| \varphi_h \|_{L^\infty(0, T; L^\gamma(0, L))} \| \varphi_{h_i} \|_{L^2(0, T; L^\infty(0, L))},
\]
which is (3.10).

2. Let \( \phi \in W^{1, \infty}_0(0, L) \) be arbitrary and define
\[
v_{i-1/2} = v(x_{i-1/2}), \quad i = 0, \ldots, N.
\]

By multiplying the momentum scheme (2.2) with \( \Delta x v_{i+1/2} \), summing over \( i \), and applying summation by parts, we obtain
\[
\Delta x \sum_i \partial_t \left( \phi_i \tilde{u}^k_i + \phi_{i+1} \tilde{u}^k_{i+1} \right) v_{i+1/2}
= \Delta x \sum_i \uparrow \phi^{(k)}(x_{i+1/2}) - \Delta x \sum_i \left( \partial_t \phi_i^k \phi_i \phi^{(k)}(x_{i+1/2}) \right)
\leq C \| \varphi_h \|_{L^\infty(0, L)} \| \varphi_{h_i} \|_{L^\infty(0, L)} \| \phi_x \|_{L^{\frac{2}{1-\gamma}}(0, L)}
\]

Next, we multiply with \( \Delta t \) and sum over all \( k \) to conclude
\[
\Delta t \sum_k \left| \Delta x \sum_i \partial_t \left( \phi_i \tilde{u}^k_i + \phi_{i+1} \tilde{u}^k_{i+1} \right) v_{i+1/2} \right|
\leq C \| v_x \|_{L^\infty(0, L)} \left( \| \varphi_h \|_{L^2(0, T; L^\infty(0, L))} \| \varphi_{h_i} \|_{L^\infty(0, T; L^\gamma(0, L))} \| \phi_x \|_{L^\infty(0, T; L^\infty(0, L))} \| \varphi_{h_{i+1}} \|_{L^\infty(0, T; L^\infty(0, L))} \right)
\leq C \| v_x \|_{L^\infty(0, L)}
\]
where the last inequality is Corollary 3.4. Finally, we calculate
\[
\int_0^T \int_0^L \partial_t \left( \phi_i \tilde{u}^k_i \right) v \, dx \, dt
\leq \Delta t \Delta x \sum_k \left| \sum_i \partial_t \left( \phi_i \tilde{u}^k_i \right) \left( \frac{v_{i+1/2} + v_{i-1/2}}{2} \right) \right|
\]
\[ + \Delta t \sum_k \sum_i \partial_i^k (\rho_i \hat{u}_i) \int_{x_i-1/2}^{x_i+1/2} \left( \frac{v_i + v_{i-1}}{2} - v(x) \right) \, dx \]

\[ = \Delta t \Delta x \sum_k \sum_i \partial_i^k (\rho_i \hat{u}_i + \rho_{i+1} \hat{u}_{i+1}) v_{i+1/2} \]

\[ + \Delta t \sum_k \sum_i \partial_i^k (\rho_i \hat{u}_i) \int_{x_i-1/2}^{x_i+1/2} \left( \frac{v_i + v_{i-1}}{2} - v(x) \right) \, dx \]

\[ \leq C \left( 1 + \| \rho_h \hat{u}_h \|_{L^1(0,T;L^1(0,L))} \right) \| v_x \|_{L^\infty(0,L)}, \]

form which (3.11) follows.

\[ \square \]

4. The effective viscous flux

The purpose of this section is to derive an equation for the quantity

\[ F_h = \mu(u_h)_x - p(\rho_h). \]

This quantity is often termed the effective viscous flux and stands at the center of both the available existence results (in more than 1D) [15, 4] and the available numerical results (cf. [13]). Specifically, both higher (than \( L^1 \)) integrability on the pressure and strong convergence of the density is proved using the upcoming equation (see Proposition 4.2 for details).

To derive the desired equation, we shall need the following discrete Neumann Laplace operator

\[ -\Delta_i q_h = -\partial_i \partial_{i+1/2} q_h = f_i, \]

\[ q_N = q_{N-1}, \]

\[ q_{-1} = q_1. \]

and we observe that \(-\Delta_i\) is nothing but the standard 3-point Laplacian on the density grid. Moreover, the Neumann condition is realized by adding a shadow cell on both sides of the domain \((0, L)\). Due to the Neumann condition, \(\Delta_i\) is only well-defined for sources \(f_h\) of zero mean.

In the upcoming analysis, we shall not need \(\Delta_i\) directly, but its discrete derivative:

\[ \partial_i^{i+1/2} \Delta_i^{-1} \left[ f^k_h \right] = -\partial_i^{i+1/2} g^k_h, \]

where we observe that the Neumann condition renders \(\partial_{N-1/2} \Delta_i^{-1} \left[ f^k_h \right] = 0\) and \(\partial_{-1/2} \Delta_i^{-1} \left[ f^k_h \right] = 0\). We shall also need the inverse of the \(\Delta_i^{i+1/2}\) operator occurring in the momentum scheme (2.2):

\[ -\Delta_i^{-1} \left[ v_h \right] = w_{i+1/2}, \]

where \(w_h\) solves the linear Dirichlet system

\[ -\Delta_i^{i+1/2} w_h = -\partial_i^{i+1/2} \partial_i w_h = v_{i+1/2}, \quad w_{-1/2} = w_{N-1/2} = 0. \]

We will mostly be interested in the discrete derivative

\[ \partial_i \Delta_i^{-1} \left[ v_h \right] = -\partial_i w_h, \]

The following result follows from standard summation by parts.
Lemma 4.1. The following duality holds,
\[ \Delta x \sum_i v_{i+1/2} \partial_{i+1/2} \Delta^i \left[ f_h \right] = -\Delta x \sum_i \partial_i \Delta_{i+1/2}^{-1} \left[ v_h \right] f_i. \]

Proof. By direct calculation, using the definition of \( \Delta_i \) and \( \Delta_{i+1/2}^{-1} \), and the respective boundary conditions, we deduce
\[
\Delta x \sum_i v_{i+1/2} \partial_{i+1/2} \Delta^i \left[ f_h \right] \\
= \Delta x \sum_i \left( \partial_{i+1/2} \partial_i \Delta_{i+1/2}^{-1} \left[ v_h \right] \right) \left( \partial_i \Delta_{i+1/2}^{-1} \left[ f_h \right] \right) \\
= -\Delta x \sum_i \left( \partial_i \Delta_{i+1/2}^{-1} \left[ v_h \right] \right) \left( \partial_i \Delta_{i+1/2}^{-1} \left[ f_h \right] \right) \\
= -\Delta x \sum_i \left( \partial_i \Delta_{i+1/2}^{-1} \left[ v_h \right] \right) f_i,
\]
which concludes the proof. \( \square \)

The following proposition gives the effective viscous flux equation we shall need in the convergence analysis. The two error terms appearing in (4.1) will be bounded below.

Proposition 4.2. Let \((\rho_h, u_h)\) be the numerical solution constructed through Definition 2.1 and (2.3)-(2.4). For any \(m = 1, \ldots, M\), we have that
\[
- \Delta t \Delta x \sum_{k=0}^m \sum_i \left( \mu \partial_i u^k_i - p \left( \frac{\rho^k_i}{\rho^i} \right) \right) \left( \frac{\rho^{M}_i}{L} \right) \\
= \Delta t \Delta x \sum_{k=0}^m \sum_i U_p \left( \rho^k \hat{u}^k \right)_{i+1/2} \left( \frac{\rho^M}{L} \right) \\
- \Delta x \sum_i \partial_i \Delta_{i+1/2}^{-1} \left[ \frac{\rho^m_i \hat{u}_i^m + \rho_{i+1}^m \hat{u}_{i+1}^m}{2} \right] \rho^M_i \\
+ \Delta x \sum_i \partial_i \Delta_{i+1/2}^{-1} \left[ \frac{\rho^0_i \hat{u}_i^0 + \rho_{i+1}^0 \hat{u}_{i+1}^0}{2} \right] \rho^1_i + E^m_1 + E^m_2,
\]
where the numerical error terms \(E^m_1\) and \(E^m_2\) are given by (4.6) and (4.8), respectively.

Proof. Let \(m = 1, \ldots, M\) be arbitrary. By multiplying the momentum scheme (2.2) with \(v_{i+1/2}(\Delta t \Delta x)\), where
\[
v^k_{i+1/2} = \partial_{i+1/2} \Delta_{i+1/2}^{-1} \left[ \rho^k_i - \frac{\rho^M_i}{L} \right], \quad \mathcal{M} = \int_0^L \rho^0 dx,
\]
and summing over all \( i \) and \( k = 0, \ldots, m \), we obtain the starting point

\[
\Delta t \Delta x \sum_{k=0}^{m} \sum_{i} \left( \mu \Delta_{i+1/2} u^k_h - \partial_{i+1/2} p(\hat{\theta}^k_h) \right) v^k_{i+1/2}
\]

\[
= \Delta t \Delta x \sum_{k=0}^{m} \sum_{i} \partial^k_t \left( \frac{\theta^k_i \hat{u}_i + \theta^k_{i+1} \hat{u}_{i+1}}{2} \right) v^k_{i+1/2}
\]

\[
+ \Delta t \Delta x \sum_{k=0}^{m} \sum_{i} \left( \frac{U_p(\theta^k u^k)_{i+3/2} - U_p(\theta^k u^k)_{i-1/2}}{2 \Delta x} \right) v^k_{i+1/2}
\]

\[
= S_1 + S_2.
\]

(4.2)

Here, we have introduced the quantities

\[
S_1 = \Delta t \Delta x \sum_{k=0}^{m} \sum_{i} \partial^k_t \left( \frac{\theta^k_i \hat{u}_i + \theta^k_{i+1} \hat{u}_{i+1}}{2} \right) v^k_{i+1/2}
\]

\[
S_2 = \Delta t \Delta x \sum_{k=0}^{m} \sum_{i} \left( \frac{U_p(\theta^k u^k)_{i+3/2} - U_p(\theta^k u^k)_{i-1/2}}{2 \Delta x} \right) v^k_{i+1/2}
\]

We will now rewrite the sums \( S_1 \) and \( S_2 \) using the definition of \( v^k_{i+1/2} \) and the continuity scheme. We will treat the least involved term, namely \( S_2 \), first. For this purpose, we apply summation by parts and the definition of \( v^k_{i+1/2} \) to calculate

\[
S_2 = -\Delta t \Delta x \sum_{k=0}^{m} \sum_{i} U_p(\theta^k u^k)_{i+1/2} \left( \frac{v^k_{i+3/2} - v^k_{i-1/2}}{2 \Delta x} \right)
\]

\[
= -\Delta t \Delta x \sum_{k=0}^{m} \sum_{i} U_p(\theta^k u^k)_{i+1/2} \left( \frac{\hat{\theta}^k_{i+1} + \hat{\theta}^k_i}{2} \right) \tag{4.3}
\]

\[
+ \Delta t \Delta x \sum_{k=0}^{m} \sum_{i} U_p(\theta^k u^k)_{i+1/2} \left( \frac{M}{L} \right).
\]

For the \( S_1 \) term, we first apply summation by parts in time followed by an application of Lemma 4.1 to deduce

\[
S_1 = \Delta t \Delta x \sum_{k=0}^{m} \sum_{i} \partial^k_t \left( \frac{\theta^k_i \hat{u}_i + \theta^k_{i+1} \hat{u}_{i+1}}{2} \right) \left( \partial_{i+1/2} \Delta^{-1} \left[ \theta^k - \frac{M}{L} \right] \right)
\]

\[
= -\Delta t \Delta x \sum_{k=0}^{m} \sum_{i} \left( \frac{\theta^k_{i-1} \hat{u}_i - \theta^k_{i+1} \hat{u}_{i+1}}{2} \right) \left( \partial_{i+1/2} \Delta^{-1} \left[ \theta^k - \frac{M}{L} \right] \right)
\]

\[
= \Delta t \Delta x \sum_{k=0}^{m} \sum_{i} \partial^k_t \Delta^{-1} \left[ \frac{\theta^k_{i-1} \hat{u}_i - \theta^k_{i+1} \hat{u}_{i+1}}{2} \right] \partial^k_t \theta^k \tag{4.4}
\]

\[
- \Delta x \sum_{i} \partial^k_t \Delta^{-1} \left[ \frac{\theta^m \hat{u}_i - \theta^m_{i+1} \hat{u}_{i+1}}{2} \right] \theta^k \tag{4.4}
\]

\[
+ \Delta x \sum_{i} \partial^k_t \Delta^{-1} \left[ \frac{\theta^0 \hat{u}_i - \theta^0_{i+1} \hat{u}_{i+1}}{2} \right] \theta^k.
\]
Next, we multiply the continuity scheme (2.1) by $\Delta t \Delta x q^k_i$, where

$$q^k_i = \partial_i \Delta^{i+1/2} \left[ \frac{g_{i}^{k-1} u_{i}^{k-1} + g_{i+1}^{k-1} \hat{u}_{i+1}^{k-1}}{2} \right],$$

and sum over all $i$ and $k = 0, \ldots, m$ to obtain

$$\Delta t \Delta x \sum_{k=0}^{m} \sum_{i} \partial_i \Delta^{i+1/2} \left[ \frac{g_{i}^{k-1} u_{i}^{k-1} + g_{i+1}^{k-1} \hat{u}_{i+1}^{k-1}}{2} \right] \partial_i q_i = -\Delta t \Delta x \sum_{k=0}^{m} \sum_{i} \partial_i \text{Up}(q^k u^k) \left( \partial_i \Delta^{-1}_{i+1/2} \left[ \frac{g_{i}^{k-1} u_{i}^{k-1} + g_{i+1}^{k-1} \hat{u}_{i+1}^{k-1}}{2} \right] \right)$$

$$= \Delta t \Delta x \sum_{k=0}^{m} \sum_{i} \text{Up}(q^k u^k)_{i+1/2} \left( \frac{g_{i}^{k-1} \hat{u}_{i}^{k-1} + g_{i+1}^{k-1} \hat{u}_{i+1}^{k-1}}{2} \right)$$

$$= \Delta t \Delta x \sum_{k=0}^{m} \sum_{i} \text{Up}(q^k u^k)_{i+1/2} \left( \frac{g_{i}^{k} \hat{u}_{i}^{k} + g_{i+1}^{k} \hat{u}_{i+1}^{k}}{2} \right) + E^m_1, \quad (4.5)$$

where the last identity follows from summation by parts, the definition of $\Delta^{-1}_{i+1/2}$, and where we have introduced the error term

$$E^m_1 = -\Delta t \Delta x \sum_{k=0}^{m} \sum_{i} \text{Up}(q^k u^k)_{i+1/2} \left( \frac{g_{i}^{k} \hat{u}_{i}^{k} + g_{i+1}^{k} \hat{u}_{i+1}^{k}}{2} \right) - \frac{g_{i}^{k-1} \hat{u}_{i}^{k-1} + g_{i+1}^{k-1} \hat{u}_{i+1}^{k-1}}{2}. \quad (4.6)$$

To conclude the proof, we shall need to further rewrite (4.5). By adding and subtracting, we obtain the identity

$$\text{Up}(q^k u^k)_{i+1/2} \left( \frac{g_{i}^{k} \hat{u}_{i}^{k} + g_{i+1}^{k} \hat{u}_{i+1}^{k}}{2} \right)$$

$$= \left( g_{i}^{k} u_{i+1/2}^k + g_{i+1}^{k} |u_{i+1/2}^k| \right) \left( \frac{g_{i}^{k} \hat{u}_{i}^{k} + g_{i+1}^{k} \hat{u}_{i+1}^{k}}{2} \right)$$

$$= g_{i}^{k} u_{i+1/2}^k \hat{u}_{i}^{k} \left( \frac{g_{i}^{k} + g_{i+1}^{k}}{2} \right) + g_{i+1}^{k} u_{i+1/2}^k \hat{u}_{i+1}^{k} \left( \frac{g_{i}^{k} + g_{i+1}^{k}}{2} \right)$$

$$+ g_{i+1}^{k} |u_{i+1/2}^k| \hat{u}_{i+1}^{k} \left( \frac{g_{i}^{k} + g_{i+1}^{k}}{2} \right)$$

$$= \text{Up}(g^k \hat{u}^k u^k) \left( \frac{g_{i}^{k} + g_{i+1}^{k}}{2} \right) + g_{i}^{k} g_{i+1}^{k} |u_{i+1/2}^k| \left( \frac{\hat{u}_{i+1}^{k} - \hat{u}_{i}^{k}}{2} \right).$$
Consequently, by combining this identity (4.5) and (4.4), we conclude

\[ S_1 = \Delta t \Delta x \sum_{k=0}^{m} \sum_i U_p(q^k \hat{u}_i^k u^k) \left( \frac{\hat{q}_i^k + q_{i+1}^k}{2} \right) \]

\[- \Delta x \sum_i \partial_i \Delta i+1/2 \left[ \frac{\hat{q}_i^m \hat{q}_i^m + \hat{q}_{i+1}^m \hat{q}_{i+1}^m}{2} \right] \theta_i^M \]

\[ + \Delta x \sum_i \partial_i \Delta i+1/2 \left[ \frac{\hat{q}_i^0 \hat{q}_i^0 + \hat{q}_{i+1}^0 \hat{q}_{i+1}^0}{2} \right] \theta_i^1 + E_1^m + E_2^m, \]

where

\[ E_2^m = \Delta t \Delta x \sum_{k=0}^{m} \sum_i \theta_i^k \hat{u}_i^k \left| u_{i+1/2}^k \right| \left( \frac{\hat{u}_{i+1}^k - \hat{u}_i^k}{2} \right). \]

Finally, we apply (4.3) and (4.7) to (4.2) to discover

\[ \Delta t \Delta x \sum_{k=0}^{m} \sum_i \left( \mu \Delta i+1/2 u_{i}^k - \partial_i u_{i+1/2} p(\hat{q}_i^k) \right) v_{i+1/2}^k \]

\[ = \Delta t \Delta x \sum_{k=0}^{m} \sum_i U_p(q^k \hat{u}_i^k u^k) \left( \frac{M}{L} \right) \]

\[- \Delta x \sum_i \partial_i \Delta i+1/2 \left[ \frac{\hat{q}_i^m \hat{q}_i^m + \hat{q}_{i+1}^m \hat{q}_{i+1}^m}{2} \right] \theta_i^m \]

\[ + \Delta x \sum_i \partial_i \Delta i+1/2 \left[ \frac{\hat{q}_i^0 \hat{q}_i^0 + \hat{q}_{i+1}^0 \hat{q}_{i+1}^0}{2} \right] \theta_i^1 + E_1^m + E_2^m. \]

This and a final summation by parts,

\[ \Delta t \Delta x \sum_{k=0}^{m} \sum_i \left( \mu \Delta i+1/2 u_{i}^k - \partial_i u_{i+1/2} p(\hat{q}_i^k) \right) v_{i+1/2}^k \]

\[ = -\Delta t \Delta x \sum_{k=0}^{m} \sum_i \left( \mu \partial_i u_{i}^k - p(\hat{q}_i^k) \right) \left( \hat{q}_i^k - \frac{M}{L} \right), \]

concludes the proof. \[\square\]

4.1. **Bound on the error terms in (4.1).** In order for the previous proposition to be useful, we will need to prove suitable bounds on the error terms \(E_1^h\) and \(E_2^h\). It is at this stage we will need to impose some requirements on the value of \(\gamma\). No other results in the paper imposes any unphysical restrictions on \(\gamma\).

We begin by deriving a bound on \(E_2^h\).

**Lemma 4.3.** Assume that the adiabatic exponent satisfies

\[ \gamma > \frac{4}{3}, \]

and let \(E_2^h\) be given by (4.8). There is a constant \(C > 0\), independent of \(\Delta t\) and \(\Delta x\), such that for any \(m = 1, \ldots, M\),

\[ |E_2^m| \leq (\Delta x)^{3\gamma-4}, \]
Proof. By two applications of the Cauchy-Schwartz inequality and a standard inverse estimate, we deduce

\[ |E_m^2| \leq |E_m^{3j}| \leq \Delta t \Delta x \sum_k \sum_i \varrho_i^k \varrho_i^{k+1} \left| u_i^{k+1} \right| \frac{\hat{u}_{i+1}^k - \hat{u}_i^k}{2} \]

\[ \leq (\Delta x)C \left( \Delta t \Delta x \sum_k \sum_i \left| \partial_t u_i^{k+1/2} \right|^2 \right)^{1/2} \]

\[ \times \left\| u_h \right\|_{L^2(0,T;L^\infty(0,L))} \left\| \theta_h \right\|_{L^\infty(0,T;L^4(0,L))} \]

\[ \leq (\Delta x)^1 (\Delta x)^2 \left( \frac{1}{4} - \frac{1}{4} \right) C \leq (\Delta x)^\frac{2\gamma-4}{2\gamma} C, \]

where the norms are bounded due to Corollary 3.4. \square

We are now ready to bound the other error term in (4.1). The following lemma is the sole reason for the requirement \( \gamma > \frac{3}{2} \).

Lemma 4.4. Assume that the adiabatic coefficient satisfies

\[ \gamma > \frac{3}{2}, \]

and let \( E_h^1 \) be given by (4.6). There exists a constant \( C > 0 \), independent of discretization parameters, such that for any \( m = 0, \ldots, M \),

\[ |E_m^1| \leq h^{\frac{2\gamma-3}{2\gamma}} C. \]

Proof. From (4.6), we have that

\[ |E_m^1| = \left| (\Delta t)^2 \Delta x \sum_{k=0}^m \sum_i U_p(\varrho^k u_i^k)_{i+1/2} \left( \frac{\partial_t^k \theta_i \hat{u}_i}{2} + \partial_t^k (\theta_{i+1} \hat{u}_{i+1}) \right) \right| \] (4.9)

Let us now examine one of the terms in \( E_h^1 \). By adding and subtracting, we write

\[ \left| (\Delta t)^2 \Delta x \sum_{k=0}^m \sum_i U_p(\varrho^k u_i^k)_{i+1/2} \hat{\varrho}_i \right| \]

\[ \leq (\Delta t)^2 \Delta x \sum_{k=0}^m \sum_i U_p(\varrho^k u_i^k)_{i+1/2} \left| \hat{\varrho}_i \right| \left| \partial_t^k \varrho_i \right| \] (4.10)

\[ + (\Delta t)^2 \Delta x \sum_{k=0}^m \sum_i U_p(\varrho^k u_i^k)_{i+1/2} \left| \varrho_i^{k-1} \right| \left| \partial_t^k \hat{u}_i \right| =: S_1 + S_2. \]
To bound the $S_1$ term, we multiply and divide by $\sqrt{\mathcal{F}''(\varrho^k_i)}$ and apply the Cauchy-Schwartz inequality

\[ S_1 = (\Delta t)^2 \Delta x \sum_{k=0}^{m} \sum_{i} \left| \nabla (\varrho^k u^k)_{i+1/2} \right| \left| \partial_t \varrho^k_i \right| \]

\[ \leq (\Delta t)^{3/2} C \left( \Delta x \Delta t \sum_{k=0}^{m} \sum_{i} \left| \nabla (\varrho^k u^k)_{i+1/2} \right|^2 (\varrho^k_i)^{2-\gamma} \right)^{1/2} \]

\[ \times \left( (\Delta t)^2 \Delta x \sum_{k=0}^{m} \sum_{i} p''(\varrho^k_i) \left| \partial_t \varrho^k_i \right| \right)^{1/2}. \]  

(4.11)

Next, we make several applications of the Hölder inequality together with standard inverse estimates, to deduce

\[ \Delta x \Delta t \sum_{k=0}^{m} \sum_{i} \left| \nabla (\varrho^k u^k)_{i+1/2} \right|^2 (\varrho^k_i)^{2-\gamma} \]

\[ \leq \| u_h \|_{L^\infty(0,T;L^2(0,L))} \| \varrho_h \|_{L^\infty(0,T;L^2(0,L))} \| \varrho_h u_h \|_{L^\infty(0,T;L^2(0,L))} \]

\[ \leq C(\Delta x)^{-\frac{2-\gamma}{\gamma}} \| \varrho_h \|_{L^\infty(0,T;L^\gamma(0,L))} \] \[ \times \left( \Delta x \right)^2 (\frac{1}{2} - \frac{\gamma + 1}{\gamma}) \| \varrho_h u_h \|_{L^\infty(0,T;L^{2\gamma}(0,L))} \]

\[ \leq (\Delta x)^{-1 + \frac{2\gamma - 3}{\gamma}} C, \]  

(4.12)

where we have used Corollary 3.4 to conclude the last inequality.

Combining (4.12)-(4.11) and recalling that $\Delta t = \Delta x$, yields

\[ S_1 \leq (\Delta x)^{\frac{2\gamma - 3}{\gamma}}. \]  

(4.13)

Next, we turn to the $S_2$ term in (4.10). An application of the Cauchy-Schwartz inequality yields

\[ S_2 = (\Delta t)^2 \Delta x \sum_{k=0}^{m} \sum_{i} \left| \nabla (\varrho^k u^k)_{i+1/2} \right| \left| \partial_t \varrho^k_i \right| \]

\[ \leq \left( (\Delta t)^2 \Delta x \sum_{k=0}^{m} \sum_{i} \varrho^k_i \left| \partial_t \varrho^k_i \right| \right)^{1/2} \]

\[ \times \left( (\Delta t)^2 \Delta x \sum_{k=0}^{m} \sum_{i} \left| \nabla (\varrho^k u^k)_{i+1/2} \right|^2 \varrho^k_i^{-1} \right)^{1/2} \]

\[ \leq (\Delta t)^{1/2} C \| \varrho_h u_h \|_{L^2(0,T;L^2(0,L))} \| \varrho_h \|_{L^\infty(0,T;L^\infty(0,L))} \]  

(4.14)
Next, we proceed as in (4.12) to discover

\[ \Delta t \Delta x \sum_{k=0}^{m} \sum_{i} \left| U_p(\varrho^k u^k)_{i+1/2} \right|^2 \theta^k_i \]

\[
\leq \| \varrho h u h \|_{L^2(0,T;L^2(0,L))}^2 \| \varrho h \|_{L^\infty(0,T;L^\infty(0,L))}^2 \\
\leq C(\Delta x)^{2\left(\frac{3}{2} - \frac{3}{\gamma}\right)} \| \varrho h u h \|_{L^2(0,T;L^\gamma(0,L))}^2 (\Delta x)^{-\frac{1}{\gamma}} \| \varrho h \|_{L^\infty(0,T;L^\gamma(0,L))} \\
\leq (\Delta x)^{-1 + \frac{3 - \gamma}{\gamma}} C, \tag{4.15}
\]

where we have used Corollary 3.4 in the last inequality.

By combining (4.14) and (4.15), and recalling that \( \Delta t = \Delta x \), we conclude

\[ S_2 \leq (\Delta x)^{\frac{3 - \gamma}{\gamma}} C. \tag{4.16} \]

By setting (4.13) and (4.16) in (4.10) we obtain

\[ S_1 + S_2 \leq (\Delta x)^{\frac{3 - \gamma}{\gamma}} C, \]

and hence we have the desired bound for the first term in (4.9). The second term in (4.9) can be bounded by the exact same arguments. \( \square \)

4.2. Higher integrability on the density. From Corollary 3.4, we only know that \( p(\varrho_h) \) is uniformly (in \( \Delta t \) and \( \Delta x \)) bounded in \( L^\infty(0,T;L^1(0,L)) \). Hence, it is unclear whether \( p(\varrho_h) \) actually converges to an integrable function. In the following lemma, we prove that the pressure has more integrability than provided by the energy estimate.

**Lemma 4.5.** Let \((\varrho_h, u_h)\) be the numerical solution constructed using Definition 2.1 and (2.3)-(2.4), with \( \gamma > \frac{3}{2} \).

There is a constant \( C > 0 \), independent of discretization parameters, such that

\[ a \int_0^T \int_0^L \varrho_h^{\gamma + 1} \, dx \, dt \leq C. \]

In other words, \( p(\varrho_h) \in L^\infty(0,T;L^{\frac{\gamma + 1}{\gamma}}(0,L)) \).
Proof. First, we rewrite the equation (4.1) in the form (4.1)
\[ \int_0^T \int_0^L p(\theta_h) \rho_h \, dx \, dt = \int_0^T \int_0^L p(\theta_h) \frac{\mathcal{M}}{L} \, dx + \mu \int_0^T \int_0^L (u_h)_x \theta_h \, dx \, dt + \Delta t \Delta x \sum_{k=0}^m \sum_{i} \text{Up}(\theta_h) \left( \frac{\mathcal{M}}{L} \right) \]
\[ - \Delta x \sum_i \partial_i \Delta_i^{-1} \left[ \frac{\theta_i^M \hat{u}^M + \theta_{i+1}^M \hat{u}_{i+1}^M}{2} \right] \theta_i^M \]
\[ + \Delta x \sum_i \partial_i \Delta_i^{-1} \left[ \frac{\theta_i^0 \hat{u}^0 + \theta_{i+1}^0 \hat{u}_{i+1}^0}{2} \right] \theta_i^1 + E_1^M + E_2^M. \]

To bound the terms involving the discrete inverse Laplacian, we shall use the elementary bound
\[ \left\| \partial_i \Delta_i^{-1/2} \left[ \theta_i^k \hat{u}_i^k \right] \right\|_{L^\infty(0,T;L^\infty(0,L))} \leq C \| \theta_h u_h \|_{L^\infty(0,T;L^1(0,L))}. \]
Together with the Hölder inequality and Lemmas 4.4 and 4.3, this readily provides the bound
\[ \int_0^T \int_0^L p(\theta_h) \rho_h \, dx \, dt \]
\[ \leq \frac{\mathcal{M}}{L} \| \theta_h \|_{L^\infty(0,T;L^\gamma(0,L))} + \frac{\mu}{\epsilon} \left\| (u_h)_x \right\|_{L^2(0,T;L^2(0,L))}^2 \]
\[ + \epsilon \| \theta_h \|_{L^{\gamma+1}(0,T;L^{\gamma+1}(0,L))} + \frac{\mathcal{M} C}{L} \| \theta_h u_h^2 \|_{L^1(0,T;L^1(0,L))} \]
\[ + \mathcal{M} C \| \theta_h \hat{u}_h \|_{L^\infty(0,T;L^1(0,L))} + (\Delta x)^{2-\gamma} C. \]
The proof is completed by fixing $\epsilon$ sufficiently small.

\[ \square \]

5. Weak convergence

In this section, we will pass to the limit in the numerical method and prove that the limit is almost a weak solution to the compressible Navier-Stokes equations. Our starting point is that Corollary 3.4 allow us to assert the existence of functions
\begin{align*}
  u &\in L^2(0, T; W^{1,2}_0(0, L)), \\
  \rho &\in L^\infty(0, T; L^7(0, L)), \\
  \frac{p(\rho)}{\rho} &\in L^{\gamma+1}(0, T; L^{\gamma+1}(0, L)) \tag{5.1}
\end{align*}
and a subsequence $h_j \to 0$, such that
\begin{align*}
  \theta_h &\rightharpoonup \theta \text{ in } L^\infty(0, T; L^\gamma(0, L)), \\
  u_h &\to u \text{ in } L^2(0, T; W^{1,2}_0(0, L)), \\
  p(\theta_h) &\to \frac{p(\theta)}{\theta} \text{ in } L^{\gamma+1}(0, T; L^{\gamma+1}(0, L)). \tag{5.2}
\end{align*}
Note that we cannot make the identification \( \overline{\rho(\varrho)} = \rho \) as this would require the density to converge strongly. We will prove that this is indeed true in the next section.

To conclude convergence of the product terms appearing in the method, we shall need the following lemma from [11]:

**Lemma 5.1.** Given \( T > 0 \) and a small number \( h > 0 \), write \( [0, T) = \bigcup_{k=1}^{M} [t_{k-1}, t_{k}] \) with \( t_{k} = hk \) and \( Mh = T \). Let \( \{f_{h}\}_{h>0}^{\infty}, \{g_{h}\}_{h>0}^{\infty} \) be two sequences such that the mappings \( t \mapsto f_{h}(t, x) \) and \( t \mapsto g_{h}(t, x) \) are constant on each interval \( (t_{k-1}, t_{k}] \) and assume that \( \{f_{h}\}_{h>0}^{\infty}, \{g_{h}\}_{h>0}^{\infty} \) converges weakly to \( f \) and \( g \) in \( L^{p_{1}}(0, T; L^{q_{1}}(0, L)) \) and \( L^{p_{2}}(0, T; L^{q_{2}}(0, L)) \), respectively, where \( 1 < p_{1}, q_{1} < \infty \) and \( \frac{1}{p_{1}} + \frac{1}{p_{2}} = \frac{1}{q_{1}} + \frac{1}{q_{2}} = 1 \). If \( \partial_{t}^{h} g_{h} \in_{b} L^{1}(0, T; W^{-1,1}(0, L)) \) and \( h^{\alpha}f(\cdot, x + h) - f(\cdot, x) \in_{b} L^{p_{3}}(0, T; L^{q_{3}}(0, L)) \), for some \( \alpha < 1 \), then \( g_{h}f_{h} \rightharpoonup gf \) in the sense of distributions on \( (0, T) \times \Omega \).

**Lemma 5.2.** Given the convergences (5.2),
\[
\begin{align*}
g_{h}u_{h} & \rightharpoonup gu \text{ in } L^{2}(0, T; L^{7}(0, L)), \\
g_{h} \hat{u}_{h} & \rightharpoonup gu \text{ in } L^{\infty}(0, T; L^{\frac{2}{7\gamma}}(0, L)),
\end{align*}
\]
and (5.3),
\[
\begin{align*}
g_{h}u_{h}, \ g_{h}|u_{h}|^{2}, \ g_{h}|u_{h}|^{2} & \rightharpoonup gu^{2} \text{ in } L^{1}(0, T; L^{\frac{2}{7\gamma}}(0, L)).
\end{align*}
\]

**Proof.** From Corollary 3.4 and Lemma 3.5, we have that \( u_{h} \in_{b} L^{2}(0, T; W_{0}^{1,2}(0, L)) \) and \( \partial_{t}^{h} g_{h} \in_{b} L^{2}(0, T; W^{-1,\gamma}(0, L)) \). We can then apply Lemma 5.1, with \( g_{h} = g_{h} \) and \( f_{h} = u_{h} \), to conclude
\[
g_{h}u_{h} \rightharpoonup gu \text{ in } L^{2}(0, T; L^{7}(0, L)).
\]

Next, we notice that
\[
\begin{align*}
\|\hat{u}_{h} - u_{h}\|_{L^{2}(0, L)}^{2} & = \sum_{i} \int_{x_{i-1}/2}^{x_{i+1}/2} \left| \frac{1}{\Delta x} \int_{x_{i-1}/2}^{x_{i+1}/2} u_{h}(y) \, dy - u_{h}(x) \right|^{2} \, dx \\
& \leq (\Delta x)^{2} \| (u_{h})_{x} \|_{L^{2}(0, L)},
\end{align*}
\]
where the last inequality is the Poincaré inequality. Hence, by writing
\[
g_{h} \hat{u}_{h} = g_{h}u_{h} + (g_{h} \hat{u}_{h} - u_{h}),
\]
and passing to the limit, we conclude the second convergence of (5.3).

From Corollary 3.4 and Lemma 3.5, we have that
\[
g_{h}|u_{h}|^{2} \in_{b} L^{1}(0, T; L^{\frac{2}{7\gamma}}(0, L)) \quad \text{and} \quad \partial_{t}(g_{h} \hat{u}_{h}) \in_{b} L^{1}(0, T; W^{-1,1}(0, L)).
\]
Lemma 5.1 is then applicable, with \( g_{h} = g_{h} \hat{u}_{h} \) and \( f_{h} = u_{h} \), yielding
\[
g_{h} \hat{u}_{h}u_{h} \rightharpoonup gu^{2} \text{ in } L^{1}(0, T; L^{\frac{2}{7\gamma}}(0, L)).
\]

The remaining convergences can be obtained similarly (i.e (5.4)). \( \square \)

5.1. **Convergence of the density scheme.** We now prove that the limit \((\varrho, u)\) is a weak solution of the continuity equation.

**Lemma 5.3.** Let \((\varrho_{h}, u_{h})\) be the numerical approximation constructed using Definition 2.1 and (2.3)-(2.4). The limit \((\varrho, u)\) is a weak solution of the continuity equation. That is,
\[
\varrho_{t} + (\varrho u)_{x} = 0,
\]
in the sense of distributions on \([0, T) \times [0, L] \).
We then apply Green’s theorem to obtain \( \phi \) for all \( \phi \in C_0^\infty([0,T] \times [0,L]) \), where \( P_1(\phi) = \Delta t \sum_k P_k^t(\phi) \) is given by (5.10).

To prove the claim (5.5), let \( \phi \in C_0^\infty([0,T] \times [0,L]) \) be arbitrary and define

\[
\phi_i^k = \frac{1}{\Delta t \Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{t_{k-1}}^{t_k} \phi(t,x) \, dt \, dx, \quad i = 0, \ldots, N-1.
\]

Now, multiply (2.1) with \( \phi_i^k \Delta x \), and sum over all \( i \), to discover

\[
\Delta x \sum_i (\partial_t \phi_i^k)\phi_i^k = \Delta x \sum_i \left( \partial_t \text{Up}(q^k u^k) \right) \phi_i^k = \sum_i \text{Up}(q^k u^k)_{i+1/2} \left( \phi_i^k - \phi(\Delta tk, x_{i+1/2}) \right) - \text{Up}(q^k u^k)_{i-1/2} \left( \phi_i^k - \phi(\Delta tk, x_{i-1/2}) \right). \tag{5.6}
\]

To proceed, we add and subtract to derive the identities

\[
\text{Up}(q^k u^k)_{i+1/2} \left( \phi_i^k - \phi(\Delta tk, x_{i+1/2}) \right) = \phi_i^k u_{i+1/2} \left( \phi_i^k - \phi(\Delta tk, x_{i+1/2}) \right) + (\phi_i^{k+1} - \phi_i^k) u_{i+1/2} \left( \phi_i^k - \phi(\Delta tk, x_{i+1/2}) \right), \tag{5.7}
\]

and

\[
\text{Up}(q^k u^k)_{i-1/2} \left( \phi_i^k - \phi(\Delta tk, x_{i-1/2}) \right) = \phi_i^k u_{i-1/2} \left( \phi_i^k - \phi(\Delta tk, x_{i-1/2}) \right) - (\phi_i^k - \phi_{i-1}^k) u_{i-1/2} \left( \phi_i^k - \phi(\Delta tk, x_{i-1/2}) \right). \tag{5.8}
\]

By applying (5.7)-(5.8) in (5.6), we obtain

\[
\Delta x \sum_i (\partial_t \phi_i^k)\phi_i^k = \sum_i \phi_i^k u_{i+1/2} \left( \phi_i^k - \phi(\Delta tk, x_{i+1/2}) \right) - \phi_i^k u_{i-1/2} \left( \phi_i^k - \phi(\Delta tk, x_{i-1/2}) \right) + (\phi_i^{k+1} - \phi_i^k) u_{i+1/2} \left( \phi_i^k - \phi(\Delta tk, x_{i+1/2}) \right) + (\phi_i^k - \phi_{i-1}^k) u_{i-1/2} \left( \phi_i^k - \phi(\Delta tk, x_{i-1/2}) \right).
\]

We then apply Green’s theorem to obtain

\[
\Delta x \sum_i (\partial_t \phi_i^k)\phi_i^k = \sum_i \int_{x_{i-1/2}}^{x_{i+1/2}} \phi_i^k \frac{d}{dx} \left( u_i^k (\phi_i^k - \phi(\Delta tk, x)) \right) \, dx + P_k^t(\phi), \tag{5.9}
\]
where we have defined
\[ P^k_t(\phi) = \Delta x \sum_i \left[ \left( \partial_{i-1/2} \varrho^k \right) u^+_{i-1/2} + \left( \partial_{i+1/2} \varrho^k \right) u^-_{i+1/2} \right] \times \left( \phi^k_i - \phi(\Delta tk, x_{i-1/2}) \right). \] (5.10)

To proceed, we observe the two identities
\[ \Delta t \Delta x \sum_i (\partial^k \varrho) \phi^k_i = \int_0^T \int_0^L (\partial^k \varrho_h) \phi \, dx \, dt \] (5.11)
and
\[ \Delta t \sum_i \int_{x_{i-1/2}}^{x_{i+1/2}} \varrho^k_i \frac{d}{dx} \left( u^k_h(\phi^k_i - \phi(\Delta tk, x)) \right) \, dx \]
\[ = \int_0^T \int_0^L \varrho_h(u_h(\phi_i - \phi(x)) - \varrho_h u_h \phi \, dx = - \int_0^T \int_0^L \varrho_h u_h \phi \, dx, \] (5.12)
where the last equality follows by definition of \( \phi^k_i \).

By combining (5.9)-(5.12), we obtain (5.5), which was our claim.

2. Next, we prove that the \( P_1(\phi) \) converges to zero as \( h \to 0 \). More precisely, we claim that
\[ |P_1(\phi)| \leq h^2 C. \] (5.13)
To prove this, we apply the Cauchy-Schwartz inequality to obtain
\[ P_1(\phi) = \Delta x \Delta t \sum_k \sum_i \left[ \left( \partial_{i-1/2} \varrho^k \right) u^+_{i-1/2} + \left( \partial_{i+1/2} \varrho^k \right) u^-_{i+1/2} \right] \times \left( \phi^k_i - \phi(\Delta tk, x_{i-1/2}) \right) \]
\[ \leq C \left( \Delta t(\Delta x)^2 \sum_k \sum_i p''(\varrho^k_i) \left| \partial_{i+1/2} \varrho^k_h \right|^2 |u^-_{i+1/2}| \right)^{1/2} \]
\[ \times (\Delta x)^{-1/2} \left( \Delta t \Delta x \sum_k \sum_i |u^-_{i+1/2}|(\varrho^k_i)^{2-\gamma} \right)^{1/2} \times \left| \phi^k_i - \phi(\Delta tk, x_{i-1/2}) \right|^{1/2} \] (5.14)
Note that the first term after the inequality is bounded by the energy estimate (3.6). Now, the Hölder and Poincaré inequalities provides the bound
\[ (\Delta x)^{-1/2} \left( \Delta t \Delta x \sum_k \sum_i |u^-_{i+1/2}|(\varrho^k_i)^{2-\gamma} \left| \phi^k_i - \phi(\Delta tk, x_{i-1/2}) \right|^{1/2} \right)^{1/2} \]
\[ \leq (\Delta x)^{1/2} C \left\| \nabla \phi \right\|_{L^\infty(0,T;L^\infty(0,L))} \left\| u_h \right\|_{L^2(0,T;L^\infty(0,L))} \left\| \varrho_h \right\|_{L^\infty(0,T;L^\gamma(0,L))} \]
\[ \leq (\Delta x)^{1/2} C \left\| \nabla \phi \right\|_{L^\infty(0,T;L^\infty(0,L))}, \]
where we have used Corollary 3.4 to conclude the last bound. Together with (5.14), this proves our claim (5.13).
3. Let us now send \( h \to 0 \) in (5.5) and thereby conclude the proof. For this purpose, we shall need the following elementary identity

\[
\int_0^T \int_0^L \partial_t^k (g_h) \phi \, dx \, dt = - \int_0^T \int_0^L g_h(t - \Delta t) \partial_t^k \phi \, dx \, dt - \int_0^T \frac{\partial \phi}{\partial t}(\Delta t) \, dx,
\]

where we have also used that \( \phi(T, \cdot) = 0 \). With this identity, (5.5) tell us that

\[
- \int_0^T \int_0^L g_h(t - \Delta t) \partial_t^k \phi \, dx \, dt = \int_0^T \int_0^L \rho u \phi_x \, dx \, dt
\]

in the sense of distributions. This concludes the proof.

5.2. Weak limit of the momentum scheme. In this subsection, we pass to the limit in the momentum scheme to conclude that \((\rho, u)\) is almost a weak solution of the momentum equation. We begin by deriving an integral formulation of the momentum scheme (2.2).

Lemma 5.4. Let \((\rho_h, u_h)\) be the numerical solution constructed through Definition 2.1 and (2.3)-(2.4). Then, for all sufficiently smooth \( v \),

\[
\int_0^T \int_0^L \partial_t^k (\rho_h \hat{u}_h) v - \rho_h (\hat{u}_h)^2 v_x \, dx \, dt = \int_0^T \int_0^L (p(\rho_h) - \mu(u_h)_x)v_x \, dx \, dt + P_2(v),
\]

where the numerical error term is given by

\[
P_2(v) = -\Delta t \sum_k \sum_i \int_{x_{i-1/2}}^{x_{i+1/2}} \partial_t^k (\rho_i \hat{u}_i) \left( \frac{v_{i-1/2}^k + v_{i+1/2}^k}{2} - v(x) \right) dx
\]

\[
+ \frac{\Delta t}{2} \sum_k \sum_i (\rho_{i+1} \hat{u}_{i+1}^k - \rho_i \hat{u}_i^k) \left[ u_{i+1/2}^k (v_{i+3/2} - v_{i+1/2}) - u_{i+1/2}^- (v_{i+1/2} - v_{i-1/2}) \right].
\]

Proof. Let \( v \in C_0^\infty([0,T) \times 0, L) \) be arbitrary and introduce the notation

\[
v_{i-1/2}^k = v(k \Delta t, x_{i-1/2}), \quad i = 0, \ldots, N, \quad k = 1, \ldots, M.
\]
Now, multiply (2.2) with $v_{i+1/2}^k \Delta x$, and sum over all $i$ to obtain

$$\frac{\Delta x}{2} \sum_i \partial^k_i (\varrho i + \varrho_{i+1/2} \hat{u}_{i+1/2}) v_{i+1/2}^k$$

$$+ \frac{1}{2} \sum_i \left( U_p \left( \varrho_{i+1/2}^k \hat{u}_{i+1/2}^k \right)_{i+3/2} - U_p \left( \varrho_{i-1/2}^k \hat{u}_{i-1/2}^k \right)_{i-1/2} \right) v_{i+1/2}^k$$

$$= \Delta x \sum_i \left( \mu \Delta_{i+1/2} u^k - \partial_{i+1/2} p(\varrho^k_h) \right) v_{i+1/2}^k. \quad (5.16)$$

We will now write each integral term in (5.16) in the form (5.15). Let us begin with the right-hand side.

1. Using summation by parts, we calculate

$$\Delta x \sum_i \left( \mu \Delta_{i+1/2} u^k - \partial_{i+1/2} p(\varrho^k_h) \right) v_{i+1/2}^k$$

$$= \sum_i \left( p(\varrho^k_h) - \mu \partial_{i+1/2} u^k \right) \left( v_{i+1/2}^k - v_{i-1/2}^k \right)$$

$$= \sum_i \left( p(\varrho^k_h) - \mu \partial_{i+1/2} u^k \right) \int_{x_{i-1/2}}^{x_{i+1/2}} v_x(k \Delta t, x) \, dx$$

$$= \int_0^L \left( p(\varrho^k_h) - \mu (u^k_h)_x \right) v_x(k \Delta t, x) \, dx, \quad (5.17)$$

which is of the form we wanted.

2. For the time derivative term, we perform summation by parts to deduce

$$\frac{\Delta x}{2} \sum_i \partial^k_i (\varrho i + \varrho_{i+1/2} \hat{u}_{i+1/2}) v_{i+1/2}^k$$

$$= \int_0^L \partial^k_i (\varrho \hat{u}_h) v \, dx$$

$$+ \sum_i \int_{x_{i-1/2}}^{x_{i+1/2}} \partial^k_i (\varrho \hat{u}_h) \left( \frac{v_{i-1/2}^k + v_{i+1/2}^k}{2} - v(x) \right) \, dx$$

$$=: \int_0^L \partial^k_i (\varrho \hat{u}_h) v \, dx + H^k_i \quad (5.18)$$
3. For the last term, we begin applying summation by parts

\[
\frac{1}{2} \sum_i \left( \text{Up} \left( \varrho^k \hat{u}^k u^k \right)_{i+3/2} - \text{Up} \left( \varrho^k \hat{u}^k u^k \right)_{i-1/2} \right) v^k_{i+1/2}
\]

\[
= -\frac{1}{2} \sum_i \text{Up}(\varrho^k \hat{u}^k u^k)(v_{i+3/2} - v_{i-1/2})
\]

\[
= -\frac{1}{2} \sum_i \text{Up}(\varrho^k \hat{u}^k u^k)(v_{i+3/2} - v_{i+1/2})
\]

\[
= -\frac{1}{2} \sum_i \text{Up}(\varrho^k \hat{u}^k u^k)(v_{i+1/2} - v_{i-1/2}).
\]

Now, by adding and subtracting, we develop the identity

\[
\text{Up}(\varrho^k \hat{u}^k u^k)(v_{i+3/2} - v_{i+1/2})
\]

\[
= \varrho_{i+1}^k \hat{u}_{i+1}^k u_{i+1/2}(v_{i+3/2} - v_{i+1/2})
\]

\[
- \left( \varrho_i^k \hat{u}_i^k - \varrho_i^k \hat{u}_i^k \right) u_{i+1/2}^+(v_{i+3/2} - v_{i+1/2}).
\]

Similarly, we derive the identity

\[
\text{Up}(\varrho^k \hat{u}^k u^k)(v_{i+1/2} - v_{i-1/2})
\]

\[
= \varrho_i^k \hat{u}_i^k u_{i+1/2}(v_{i+1/2} - v_{i-1/2})
\]

\[
+ \left( \varrho_i^k \hat{u}_i^k - \varrho_i^k \hat{u}_i^k \right) u_{i+1/2}^-(v_{i+1/2} - v_{i-1/2}).
\]

By applying (5.20) and (5.21) in (5.19), we obtain

\[
\frac{1}{2} \sum_i \left( \text{Up} \left( \varrho^k \hat{u}^k u^k \right)_{i+3/2} - \text{Up} \left( \varrho^k \hat{u}^k u^k \right)_{i-1/2} \right) v^k_{i+1/2}
\]

\[
= -\sum_i \varrho_i^k |\hat{u}_i^k|^2 \int_{x_{i-1/2}}^{x_{i+1/2}} v_x \, dx
\]

\[
+ \frac{1}{2} \sum_i \left( \varrho_{i+1}^k \hat{u}_{i+1}^k - \varrho_i^k \hat{u}_i^k \right) \left[ u_{i+1/2}^+(v_{i+3/2} - v_{i+1/2})
\]

\[
- u_{i+1/2}^-(v_{i+1/2} - v_{i-1/2}) \right]
\]

\[
= -\int_0^L \varrho_h \left( \hat{u}_h^k \right)^2 v_x \, dx + H_2^k.
\]

4. By applying (5.17), (5.18), and (5.22), to (5.16), multiplying with \( \Delta t \), and summing over all \( k \), we discover

\[
\int_0^T \int_0^L \partial_t (\varrho_h \hat{u}_h)v - p(\varrho_h) v_x \, dx \, dt + \Delta t \sum_k \left( E_1^k + E_2^k \right)
\]

\[
= \int_0^T \int_0^L \left( p(\varrho_h) - \mu(u_h)_x \right) v_x \, dx \, dt,
\]

which is (5.15) with

\[
P_2(v) = -\Delta t \sum_k \left( H_1^k + H_2^k \right).
\]
This concludes the proof. □

Next, we prove that the error term in (5.15) converges to zero as $h \to 0$.

**Lemma 5.5.** Let $P_2(v)$, be as in the previous lemma. There is a constant $C > 0$, independent of $h$, such that

$$|P_2(v)| \leq h^\frac{1}{2} C \|v_x\|_{L^\infty(0,T;L^{2}(0,L))},$$

for all $v \in L^\infty(0,T;W_0^{1,\infty}(0,L))$.

**Proof.** By definition, we have that

$$P_2(v) = -\Delta t \sum_k \sum_i \int_{x_{i-1/2}}^{x_{i+1/2}} \partial_t^k \left( \partial_t \hat{u}_i \right) \left( \frac{v_{i-1/2}^k + v_{i+1/2}^k}{2} - v(x) \right) dx$$

$$+ \frac{\Delta t}{\hat{H}} \sum_k \sum_i \left( \hat{\theta}_{i+1}^k \hat{\theta}_{i+1}^k - \hat{\theta}_i^k \hat{\theta}_i^k \right) \left[ u_{i+1/2}^+(v_{i+3/2}^k - v_{i+1/2}^k) - u_{i+1/2}^-(v_{i+1/2}^k - v_{i-1/2}^k) \right]$$

$$=: S_1 + S_2.$$

Let us now bound the $S_1$ and $S_2$ term separately.

1. **Bound on $S_1$:** By adding and subtracting, we write

$$S_1 = -\Delta t \sum_k \sum_i \int_{x_{i-1/2}}^{x_{i+1/2}} \partial_t^k \left( \partial_t \hat{u}_i \right) \left( \frac{v_{i-1/2}^k + v_{i+1/2}^k}{2} - v(x) \right) dx$$

$$= -\Delta t \sum_k \sum_i \int_{x_{i-1/2}}^{x_{i+1/2}} \partial_t^{k-1} \hat{\theta}_i^k \left( \frac{v_{i-1/2}^k + v_{i+1/2}^k}{2} - v(x) \right) dx$$

$$- \Delta t \sum_k \sum_i \int_{x_{i-1/2}}^{x_{i+1/2}} \hat{\theta}_i^k \partial_t^{k-1} \left( \partial_t \hat{u}_i \right) \left( \frac{v_{i-1/2}^k + v_{i+1/2}^k}{2} - v(x) \right) dx$$

$$=: K_1 + K_2.$$

To bound the $K_1$ term, we apply the Cauchy-Schwartz and Poincaré inequalities to obtain

$$|K_1| = \left| \Delta t \sum_k \sum_i \int_{x_{i-1/2}}^{x_{i+1/2}} \partial_t^{k-1} \hat{\theta}_i^k \left( \frac{v_{i-1/2}^k + v_{i+1/2}^k}{2} - v(x) \right) dx \right|$$

$$\leq (\Delta t)^{-\frac{1}{2}} \left( (\Delta t)^2 \Delta x \sum_k \sum_i \hat{\theta}_i^{k-1} \left| \partial_t^{k-1} \hat{u}_i^k \right|^2 \right)^{\frac{1}{2}}$$

$$\times \left( \Delta t \Delta x \sum_k \sum_i \hat{\theta}_i^{k-1} \left( \frac{v_{i-1/2}^k + v_{i+1/2}^k}{2} - v(x) \right) dx \right)^{\frac{1}{2}}$$

$$\leq (\Delta t)^{-\frac{1}{2}} C M_{\hat{H}}^\frac{1}{2} \left( \Delta x \right) \|v_x\|_{L^\infty(0,T;L^{2}(0,L))} \leq h^\frac{1}{2} C \|v_x\|_{L^\infty(0,T;L^{\infty}(0,L))},$$

where we have used that the term involving $\partial_t^{k} \hat{u}_h$ is bounded by the energy estimate (3.6).
To bound the $K_2$ term, we apply yet another application of the Cauchy-Schwartz and Poincaré inequalities to discover

$$|K_2| = \left| \Delta t \sum_k \sum_i \int_{x_{i-1/2}}^{x_{i+1/2}} \Delta_t \partial_i^k \hat{u}_i^k \left( \frac{\hat{v}_i^{k-1/2} + v_i^{k+1/2}}{2} - v(x) \right) \, dx \right|$$

$$\leq (\Delta t)^{-\frac{1}{2}} \left( (\Delta t)^2 \Delta x \sum_k \sum_i \partial_i^{\prime \prime} \left( \hat{u}_i^k \right) \left( \partial_i^k \hat{u}_i^k \right)^2 \right)^{\frac{1}{2}}$$

$$\times \left( \Delta t \Delta x \sum_k \sum_i \left( \partial_i^k \hat{u}_i^k \right)^{2-\gamma} \left| \partial_i^k \hat{u}_i^k \right|^2 \right) \left( \frac{\hat{v}_i^{k-1/2} + v_i^{k+1/2}}{2} - v(x) \right)^2 \right)^{\frac{1}{2}}$$

(5.26)

$$\leq (\Delta t)^{-\frac{1}{2}} C \| \hat{u}_h \|_{L^2(0, T; L^\infty(0, L))} \mathcal{M}^{\frac{1}{2}}(\Delta x) \| v_x \|_{L^\infty(0, T; L^\infty(0, L))}.$$ 

Setting (5.25)-(5.26) in (5.24) yields

$$S_1 \leq h^2 C \| v_x \|_{L^\infty(0, T; L^2(0, L))}.$$ 

(5.27)

2. **Bound on $S_2$:** By adding and subtracting, we calculate

$$|S_2| = \left| \frac{\Delta t}{2} \sum_k \sum_i \left( \hat{u}_i^k \hat{u}_i^k - \hat{u}_i^k \hat{u}_i^k \right) \right|$$

$$\leq (\Delta x) \| v_x \|_{L^\infty(0, T; L^\infty)} \left( \Delta x \Delta t \sum_k \sum_i \left| \partial_i^{1/2} \rho \right| \left| u_i^{k+1/2} \right| \left| \hat{u}_i^k \right| \right)$$

(5.28)

$$\leq (\Delta x) \| v_x \|_{L^\infty(0, T; L^\infty)} \left( K_3 + K_4 \right).$$ 

To bound the $K_3$ term, we apply the Cauchy-Schwartz inequality, followed by the Hölder inequality, and obtain

$$K_3 = \Delta t \Delta x \sum_k \sum_i \left| \partial_i^{1/2} \rho \right| \left| u_i^{k+1/2} \right| \left| \hat{u}_i^k \right|$$

$$\leq (\Delta t)^{-\frac{1}{2}} \left( (\Delta t)^2 \Delta x \sum_k \sum_i \rho \left( \partial_i^{1/2} \rho \right) \left| u_i^{k+1/2} \right| \right)^{\frac{1}{2}}$$

(5.29)

$$\times \left( \Delta t \Delta x \sum_k \sum_i \left| u_i^{k+1/2} \right| \left| \hat{u}_i^k \right|^2 \left( \hat{u}_i^k \right)^{2-\gamma} \right)^{\frac{1}{2}}$$

$$\leq (\Delta t)^{-\frac{1}{2}} C \mathcal{M}^{\frac{1}{2}} \| \hat{u}_h \|_{L^3(0, T; L^\infty(0, L))}$$

$$\leq (\Delta t)^{-\frac{1}{2}} C \left( (\frac{3}{2} - \frac{1}{2}) \right) \| \hat{u}_h \|_{L^3(0, T; L^\infty(0, L))} \leq (\Delta t)^{-\frac{1}{2}} C,$$

where we have also utilized a standard inverse estimate in time and the energy estimate (3.6).
Next, we apply the Hölder inequality to deduce
\[
K_4 = \Delta t \Delta x \sum_k \sum_i |\partial_t^{1/2} u_h^k| u_{i+1/2}^k \|u_{i+1/2}^k\|_{\dot{L}^1} \leq \|(u_h)_x\|_{L^2(0,T;L^2(0,L))} \|u_h\|_{L^2(0,T;L^\infty(0,L))} \|\dot{\varrho}_h\|_{L^\infty(0,T;L^2(0,L))} \leq (\Delta x)^{1/2} \|\dot{\varrho}_h\|_{L^\infty(0,T;L^\gamma(0,L))} C,
\]
where we have utilized an inverse estimate and the energy estimate (3.6).

By setting (5.29) and (5.30) in (5.28), we conclude
\[
|S_2| \leq h^4 \|v_x\|_{L^\infty(0,T;L^\infty(0,L))}.
\]

Finally, we apply (5.27) and (5.31) in (5.23) to conclude the proof.

\[\square\]

**Lemma 5.6.** Let \((\varrho_h, u_h)\) be the numerical approximation constructed using Definition 2.1 and (2.3)-(2.4). The limit \((\varrho, u, \overline{p(\varrho)})\) satisfies
\[
(\varrho u)_t + (\varrho u^2)_x = \mu u_{xx} - \overline{p(\varrho)}_x,
\]
in the sense of distributions on \([0,T] \times 0, L\).

**Proof.** We begin by writing (5.15) in the form
\[
-\int_0^T \int_0^L (\varrho(\dot{\varrho}_h)(-\Delta t, \cdot) \partial_t^2 v + \varrho(\dot{\varrho}_h)\varrho v_x) \ dx \ dx dt = -\int_0^T \int_0^L (\varrho(\dot{\varrho}_h) - \mu(u_h)_x) v_x \ dx \ dx dt + \int_0^T \int_0^L (\varrho(\dot{\varrho}_h)\varrho v_x) \ dx \ dx dt + P_2(v).
\]

The convergence (5.2) provides \(p(\varrho_h) \to \overline{p(\varrho)}\) and from Lemma 5.5, we have that \(P_2(v) \to 0\) as \(h \to 0\). Moreover, from Lemma 5.2, we know that \(\varrho_h \dot{\varrho}_h \to \varrho u\) in \(L^\infty(0,T;L^{2\gamma}(0,L))\) and \(\varrho_h \dot{\varrho}_h \to \varrho u\) in \(L^1(0,T;L^{2\gamma/(2\gamma-1)}(0,L))\) for each fixed \(t \in [k\Delta t, (k+1)\Delta t)\) and \(i \in [x_{i-1/2}, x_{i+1/2})\).

Hence, there is no problems with sending \(h \to 0\) in (5.33) to conclude the proof.

\[\square\]

### 5.3. The effective viscous flux limit.

We are going to end this section by passing to the limit in the effective viscous flux equation (4.1). To achieve this, we shall need the following result.

**Lemma 5.7.** Let \(\{f_h\}_{h>0}, \{g_h\}_{h>0}\) be two sequences satisfying

- For each fixed \(h > 0\), \(f_h\) and \(g_h\) are piecewise linear and piecewise constant, respectively, with respect to our grid with \(\Delta t = \Delta x = h\):

  \[
g_h(t, x) = g_h^k, \quad v_h(t, x) = v_h^k = \frac{x - x_{i-1/2}}{\Delta x} (v_{i+1/2} - v_{i-1/2}),
\]

  for \((t, x) \in [k\Delta t, (k+1)\Delta t) \times (x_{i-1/2}, x_{i+1/2})\), \forall k, i.

- As \(h \to 0\), \(f_h \to f\) and \(g_h \to g\) in \(L^\infty(0,T;L^p(0,L))\) and \(L^\infty(0,T;L^{p/(1+p)}(0,L))\), respectively, where \(1/p + 1/q < 1\).

- The discrete time derivative of \(f_h\) satisfies

  \[
\partial_t^k v_h \in L^1(0,T;W^{-1,1}(0,L)).
\]
Then, for any \( t \in (0, T) \),

\[
\lim_{h \to 0} \left( \Delta x \sum_i \partial_i \Delta_{i+1/2}^{-1} \left[ v_i^k \right] g_i^k \right) = \int_0^L \partial_x \Delta_D^{-1} [f(t)] g(t) \, dx,
\]

(5.34)

where \( k \) is given by \( k = \lfloor t/h \rfloor \) and \( \Delta_D \) denotes the Laplace operator with homogenous Dirichlet conditions.

**Proof.** Let \( q_h \) be the extension of \( \partial_i \Delta_{i+1/2}^{-1} \left[ v_i^k \right] \) to all of \( [0, T) \times (0, L) \);

\[
q_h(t, x) = \partial_i \Delta_{i+1/2}^{-1} \left[ v_i^k \right],
\]

\( \forall (t, x) \in [k \Delta t, (k + 1) \Delta t) \times (x_{i-1/2}, x_{i+1/2}) \) and relevant \( i, k \). From the requirements on \( f_h \), it is clear that

\[
\partial_i \Delta_{i+1/2} q_h, \quad \partial^k t q_h \in L^1(0, T; L^1(0, L)),
\]

Then, we can apply Lemma 5.1, with \( f_h = g_h = q_h \) to conclude that \( q_h^2 \rightharpoonup q^2 \) in the sense of distributions. Thus, \( q_h \to q \) a.e as \( h \to 0 \), where the convergence may take place along a subsequence. Observe that the limit must satisfy

\[
q = \partial_x \Delta_D^{-1} [v],
\]

where \( \Delta_D \) is the Laplacian with Dirichlet conditions.

Next, let \( q_h^L \) be the linear-in-time interpolation of \( q_h \):

\[
q_h^L(t, x) = q_h^L(x) + \frac{t - k \Delta t}{\Delta t} \left( q_h^{k+1}(x) - q_h^k(x) \right), \quad t \in [k \Delta t, (k + 1) \Delta t),
\]

for \( k = 0, \ldots, M - 1 \). Since we have that

\[
\frac{\partial q_h^L(t, x)}{\partial t} = \partial^k t q_h \in L^1(0, T; L^1(0, L)),
\]

we have that \( q_h^L \to q \) in \( C(0, T; L^p(0, L)) \) and \( \sup_t \| q_h^L - q_h \|_{L^p(0, L)} \to 0 \), for any \( p < \infty \). Finally, we write

\[
\Delta x \sum_i \partial_i \Delta_{i+1/2}^{-1} \left[ v_i^k \right] g_i^k = \int_0^L q_h^L g_h \, dx + \int_0^L (q_h^L - q_h) g_h \, dx,
\]

and send \( h \to 0 \) to discover (5.34). \( \square \)

Equipped with the previous lemma, we now pass to the limit in the effective viscous flux equation (4.1).

**Lemma 5.8.** Assume that \( \gamma > \frac{3}{2} \) and that the convergences (5.2) and (5.3) holds. Then, for any \( t \in (0, T) \),

\[
- \int_0^t \int_0^L \left( \mu u_x - p(\varrho) \right) \left( \varrho - \frac{M}{L} \right) \, dx dt
= \int_0^t \int_0^L gu^2 \, dx dt - \int_0^L \partial_x \Delta_D^{-1} [gu] \varrho \, dx dt \bigg|_{s=0}^t,
\]

(5.35)

where the overline denotes the weak \( L^1 \) limit.
Proof. Let $m = \lfloor t/\Delta t \rfloor$, then (4.1) provides the identity
\[
- \int_0^{m\Delta t} \int_0^L (\mu(u_h)_x - p(u_h)) \left( \frac{\varrho_h - M}{L} \right) \, dx \, dt \\
= -\Delta x \sum_i \partial_i \Delta^{-1} \left[ \frac{\varrho_i^m \varrho_i^m + \varrho_i^m \varrho_i^m}{2} \right] \varrho_i^M \\
+ \Delta t \Delta x \sum_{k=0}^m \sum_i \Up(\varrho^k \hat{u}^k u^k)_{t+1/2} \left( \frac{M}{L} \right) \\
+ \Delta x \sum_i \partial_i \Delta^{-1} \left[ \frac{\varrho_i^0 \varrho_i^0 + \varrho_i^0 \varrho_i^0}{2} \right] \varrho_i^1 + E_1^m + E_2^m.
\]
6. Strong convergence (proof of Theorem 2.4)

In the previous section, we proved that the method almost converges to a weak solution of the compressible Navier-Stokes equations. To conclude convergence of the method, and thereby conclude the main theorem, it only remains to prove that

\[ p(\varrho) = \varrho \quad \text{a.e.} \]

This is the topic of this section. To make this identification, we shall prove that \( \varrho_h \) converges strongly a.e. Our strategy will be to adopt the continuous arguments of Lions [15] to the numerical setting. For this purpose, we shall need the following well-known lemma. The reader may consult [4] for a proof.

**Lemma 6.1.** Let \( (\varrho, u) \) satisfy the continuity equation (1.1) in the sense of distributions. If \( \varrho \in L^2(0,T; L^2(0,L)) \) and \( u \in L^2(0,T; W^{1,2}_0(0,L)) \), then for all \( B \in C^1(\mathbb{R}^+) \) such that \( B(\varrho) \in L^1(0,T; L^1(0,L)) \), \( B(\varrho)_t + (B(\varrho)u)_x + b(\varrho)u_x = 0 \), \( b(\varrho) = \varrho B'(\varrho) - B(\varrho) \), in the sense of distributions on \([0,T] \times [0,L] \).

From Lemma 5.3, we know that the limit \((\varrho, u)\) is a weak solution of the continuity equation and moreover that \( \varrho \in L^2(0,T; L^2(0,L)) \) and \( u \in L^2(0,T; W^{1,2}_0(0,L)) \). The previous lemma is then applicable. By setting \( B(z) = z \log z \) and integrating over the domain, we obtain

\[ \int_0^L \varrho \log \varrho \, dx(t) = \int_0^L \varrho_0 \log \varrho_0 \, dx - \int_0^t \int_0^L \varrho u_x \, dx, \tag{6.1} \]

for any \( t \in (0,T) \).

Next, we set \( B(z) = z \log z \) in the renormalized continuity scheme (3.1), multiply with \( \Delta x \), and sum over \( k = 1, \ldots, m \), to obtain, for any \( m = 1, \ldots, M \),

\[ \int_0^L \varrho_h^m \log \varrho_h^m \, dx \leq \int_0^L \varrho_h^0 \log \varrho_h^0 \, dx - \int_0^{\Delta t} \int_0^L \varrho_h(u_h)_x \, dx dt, \tag{6.2} \]

where we have also used convexity of the map \( z \mapsto B(z) \) to conclude a sign on the error terms.

By subtracting (6.1) from (6.2) and passing to the limit \( h \to 0 \), we deduce

\[ 0 \leq \left( \int_0^L \bar{\varrho} \log \varrho - \varrho \log \varrho \, dx \right)(t) \leq \int_0^t \int_0^L \varrho u_x - \bar{\varrho} u_x \, dx dt, \tag{6.3} \]

for any \( t \in (0,T) \). Here, the overbar denotes weak \( L^1 \)-limit and the first inequality is a consequence of convexity.

The remaining ingredient to prove strong convergence of the density is to prove that the right-hand side in (6.3) is negative:

**Lemma 6.2.** Let \( (\varrho_h, u_h) \) be the numerical solution constructed using Definition 2.1 and (2.3)-(2.4). Then,

\[ \int_0^t \int_0^L \varrho u_x - \bar{\varrho}u_x \, dx dt \leq 0. \tag{6.4} \]

As a consequence, \( \varrho_h \to \varrho \) a.e as \( h \to 0 \).
Proof. Let \( \Delta_N \) denote the Laplace operator with homogenous Neumann conditions on \((0, L)\). From (5.1) and Lemma 3.5, we have that \( \varrho \in L^\infty(0, T; L^\gamma(0, L)) \) and \( \varrho_t \in L^2(0, T; W^{-1, \gamma}(0, L)) \), respectively. As a consequence, \( v = \partial_x \Delta_N^{-1} \left[ \varrho - \frac{M}{L} \right] = \int_0^x \varrho - \frac{M}{L} \, dx \), satisfies \( v \in L^2(0, T; L^\gamma(0, L)) \). Hence, we can set \( v \) as test-function in (5.32) to obtain

\[
\int_0^t \int_0^L \left( \mu u_x - \tilde{p}(\varrho) \right) \left( \varrho - \frac{M}{L} \right) \, dx \, dt = - \int_0^t \int_0^L \left( \varrho u \partial_x \Delta_N^{-1}[\varrho] \right) \left( \varrho - \frac{M}{L} \right) \, dx \, dt
\]

where the last equality is integration by parts and the duality of the Neumann and Dirichlet Laplacians \( \Delta_N \) and \( \Delta_D \), respectively.

To proceed, we note that \( \varphi = \partial_x \Delta_D^{-1}[\varrho u] \) is a valid test-function for the continuity equation (1.1). Since \((\varrho, u)\) is a weak solution of the continuity equation (Lemma 5.3), we deduce the identity

\[
\int_0^t \int_0^L \partial_t \left( \partial_x \Delta_D^{-1}[\varrho u] \right) \varrho \, dx \, dt = - \int_0^t \int_0^L \varrho u \partial_x \Delta_D^{-1}[\varrho u] \varrho \, dx \, dt - \int_0^t \partial_x \Delta_D^{-1}[\varrho_0 u_0] \varrho_0 \, dx \Big|_{t=0} \]  

(6.6)

By setting (6.6) in (6.5), we discover the identity

\[
\int_0^t \int_0^L \left( \mu u_x - \tilde{p}(\varrho) \right) \left( \varrho - \frac{M}{L} \right) \, dx \, dt = \left( \frac{M}{L} \right) \int_0^t \int_0^L \varrho u^2 \, dx \, dt + \int_0^t \partial_x \Delta_D^{-1}[\varrho_0 u_0] \varrho_0 \, dx \Big|_{t=0} \]  

(6.7)

where the last equality is (5.35).

Finally, we rewrite (6.7) in the form

\[
\int_0^t \int_0^L \left( \mu u_x \varrho - \mu \varrho u_x \varrho \right) \, dx \, dt = \int_0^t \int_0^L \tilde{p}(\varrho) \varrho - \tilde{p}(\varrho) \varrho \, dx \, dt \leq 0, \]  

(6.8)

where the last inequality follows from the convexity of \( z \mapsto p(z) \). This concludes the proof of (6.4).
By combining (6.8) and (6.3), we conclude that

$$\rho \log \rho = \rho \log \rho \text{ a.e on } (0, T) \times (0, L),$$

and hence $$\varrho_h \to \varrho \text{ a.e.}$$ □

**Proof of Theorem 2.4:** From Lemma 5.6, we have that $$(\varrho, u)$$ solves

$$(\varrho u)_t + (\varrho u^2)_x = \mu u_{xx} - p(\varrho)_x,$$

in the sense of distributions. The previous lemma tell us that

$$p(\varrho) = p(\varrho) \text{ a.e in } (0, T) \times (0, L).$$

Hence, $$(\varrho, u)$$ is a weak solution of the compressible Navier-Stokes system (1.1)-(1.2). This concludes the proof of our main theorem (Theorem 2.4). □

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