Twin peaks kHz QPOs: mathematics of the 3 : 2 orbital resonance

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Abstract

Using the method of multiple scales, one can derive an analytic solution that describes the behaviour of weakly coupled, non-linear oscillations in nearly Keplerian discs around neutron stars or black holes close to the 3:2 orbital epicyclic resonance. The solution obtained agrees with the previous numerical simulation. Such result may be relevant to the kilohertz quasi-periodic variability in X-ray fluxes observed in many Galactic black hole and neutron star sources. With a particular choice of tunable parameters, the solution fits accurately the observational data for Sco X-1.

Key words: General relativity – Accretion – X-rays: individual (Sco X-1) – QPOs

1. Introduction

Many Galactic black hole and neutron star sources in low mass X-ray binaries show both chaotic and quasi-periodic variability in their observed X-ray fluxes. Some of the quasi-periodic oscillations (QPOs) are in the kHz range and often come in pairs (of the quasi-periodic oscillations (QPOs) are in the kHz range and often come in pairs (e.g., van der Klis, 2000). In all four black hole sources with twin kHz QPOs pairs, $\nu_{\text{upper}}/\nu_{\text{down}} = 3/2$ (McClintock & Remillard, 2003). For neutron stars the ratio of twin-peak frequencies is mostly close to 3/2 too (Abramowicz, Bulik, Bursa, Kluźniak, 2003). Based on these and other properties of QPOs Kluźniak & Abramowicz (2000) concluded that twin peak kHz QPOs are due to a resonance in the accretion disc oscillation modes and they noticed that the specific 3:2 ratio could be a consequence of strong gravity.

According to the standard Shakura-Sunyaev accretion disc model, matter spirals down into the central black hole along stream lines that are located almost on the equatorial plane $\theta = \theta_0 = \pi/2$, and that locally differ only slightly from a family of concentric circles $r = r_0 = \text{const}$. The small deviations, $\delta r = r - r_0$, $\delta \theta = \theta - \theta_0$ are governed, with accuracy to linear terms, by

$$\delta \dot{r} + \omega_r^2 \delta r = \delta a_r, \quad \delta \ddot{r} + \omega_r^2 \delta \theta = \delta a_\theta. \quad (1)$$

Here the dot denotes the time derivative. For purely Keplerian (free) motion $\delta a_r = 0$, $\delta a_\theta = 0$ and the above equations describe two uncoupled harmonic oscillators with the eigenfrequencies $\omega_r$, $\omega_r$.

We shall start with an argument which appeals to physical intuition and which shows that the resonance now discussed is a very natural, indeed necessary, consequence of strong gravity. Assume that in thin discs, random fluctuations have $\delta r \gg \delta \theta$.\footnote{In actual calculations this additional assumption is not made.} Thus, $\delta r \delta \theta$ is a first order term in $\delta \theta$ and should be included in the first order equation for vertical oscillations (1). The equation now takes the form,

$$\delta \ddot{\theta} + \omega_0^2 [1 + h \delta r] \delta \theta = \delta a_\theta, \quad (2)$$

where $h$ is a known constant. The first order equation for $\delta r$ has the solution $\delta r = A_0 \cos(\omega_r t)$. Inserting this in (2) together with $\delta a_\theta = 0$, one arrives to the Mathieu equation ($A_0$ is absorbed in $h$),

$$\delta \ddot{\theta} + \omega_0^2 [1 + h \cos(\omega_r t)] \delta \theta = 0, \quad (3)$$

that describes the parametric resonance. From the theory of the Mathieu equation one knows that when

$$\frac{\omega_r}{\omega_0} = \frac{\nu_r}{\nu_0} = \frac{2}{n}, \quad n = 1, 2, 3, \ldots, \quad (4)$$

the parametric resonance is excited (Landau & Lifshitz, 1976). The resonance is strongest for the smallest possible value of $n$. Because near black holes and neutron stars $\nu_r < \nu_0$ (see Figure 1), the smallest possible value for resonance is $n = 3$, which means that $2 \nu_0 = 3 \nu_r$. This is an example of a situation in which parametric resonance works in a thin accretion disc and it intuitively explains the observed 3:2 ratio (Kluźniak and Abramowicz, 2002), because, obviously,

$$\nu_{\text{upper}} = \nu_r, \quad \nu_{\text{down}} = \nu_r. \quad (5)$$

Of course, in real discs neither $\delta r = A_0 \cos(\omega_r t)$, nor $\delta a_\theta = 0$ exactly; moreover in realistic situations, for purely geodesic motion ($\delta a_\theta = \delta a_r = 0$) the system does not show increasing amplitudes (the higher terms prevent this from occurring).

However, one may expect that because these equations are approximately obeyed in thin discs, the parametric resonance will indeed be excited. Such a resonance was found in numerical simulations of oscillations in a nearly Keplerian accretion disc by Abramowicz et al. (2003).

2. Nearly-geodesic motion

The equations roughly outlined in the previous Section will now be derived.
The equations of the geodesic motion of a unit mass test-particle in spherical coordinates can be written in the form ($\tilde{t} = t/r_G$):

$$\dot{r}(\tau) + \frac{E^2}{2g_{\theta\theta}} \frac{\partial U_{\text{eff}}(r, \theta)}{\partial \theta} + 2\Gamma_{\theta\rho}^\rho \dot{\theta}(\tau) = 0$$

$$\dot{\theta}(\tau) = 0$$

$$\dot{r}(\tau) + \frac{E^2}{2g_{rr}} \frac{\partial U_{\text{eff}}(r, \theta)}{\partial r} + \Gamma_{\theta\rho}^\rho \dot{\theta}(\tau)^2 + \Gamma_{rr}^r \dot{r}(\tau)^2 = 0$$

where $U_{\text{eff}} = g^{tt} + l^2 g^{\phi\phi}$ is the effective potential and $E = -u_t$ is the energy.

Consider a geodesic on the equatorial plane ($r_0, \theta_0 = \pi/2, \phi = \Omega \tilde{t}$) and perturb it slightly, keeping constant the angular momentum:

$$r(\tilde{t}) = r_0 + \rho(\tilde{t}) \quad \rho << r_0$$

$$\theta(\tilde{t}) = \frac{\pi}{2} + z(\tilde{t}) \quad z << \pi/2$$

Here $\rho(\tilde{t})$ and $z(\tilde{t})$ denote small deviations from the circular orbit, and to first order they describe two uncoupled harmonic oscillators with epicyclic eigenfrequencies:

$$\omega^2_\rho = \left( \frac{1}{g_{\theta\theta}} \frac{\partial^2 U_{\text{eff}}}{\partial \theta^2} \right)_{t, r_0, \pi/2}$$

$$\omega^2_z = \left( \frac{1}{g_{rr}} \frac{\partial^2 U_{\text{eff}}}{\partial r^2} \right)_{t, r_0, \pi/2}$$

The ratio of the two frequencies is

$$\frac{\omega^2_z}{\omega^2_\rho} = \frac{r_0}{r_0 - 3} > 1$$

This does not happen in the Newtonian case where the radial and vertical frequencies are always equal to the orbital frequency ($\Omega = GM/r_G^3$). When continuing to higher orders in the Taylor expansion, the two motions become coupled, and this is why the so-called parametric resonance can arise. The perturbed equations are nonlinear differential equations with constant coefficients which are

the values of the different derivatives of the effective potential $U_{\text{eff}}$ and the metric at equilibrium (see Appendix 2):

$$\delta a_\theta = \ddot{z}(\tilde{t}) + \omega^2_\rho z(\tilde{t}) - f(\rho(\tilde{t}), z(\tilde{t}), r_0, \theta_0)$$

$$\delta a_r = \ddot{\rho}(\tilde{t}) + \omega^2_\rho \dot{\rho}(\tilde{t}) - g(\rho(\tilde{t}), z(\tilde{t}), r_0, \theta_0)$$

In the case of Schwarzschild’s metric,

$$f(\rho, z, r_0, \theta_0) = c_{11} \rho^2 + c_{12} \rho z + c_{13} z^2$$

$$g(\rho, z, r_0, \theta_0) = c_{21} \rho^2 + c_{22} \rho z + c_{23} z^2 + c_{33} z^3$$

The third order terms in the Taylor expansion provide the damping which saturates the increasing amplitudes: this motivates stopping the expansion at this order of the perturbation.

In first approximation we used Paczyński & Wiita (1980) pseudo-Newtonian model. The perturbed equations in the model and in general relativity look the same, apart from those terms in $g(\rho, z, r_0, \theta_0)$ which involve $\rho^2$ that here are absent. On the geodesics $\delta a_\theta = \delta a_r = 0$; however, in order to explore the features of the resonance, a small additional isotropic force, parametrized by $|\alpha| \in [0, 1]$, can be added (Abramowicz et al. 2003): this force at the moment has no physical meaning (it may be a coupling due for example to pressure or viscosity), it is just a mathematical tool. Then

$$\delta a_\theta = -\alpha(c_{21} \rho^2 + c_{23} z^2) z$$

$$\delta a_r = -\alpha(c_{22} + c_{12} \rho^2) z^2$$

When $|\alpha| = 0$ the previous equations are just the approximation of the geodesics, while when $|\alpha|$ increases, they describe slight deviations from the free motion.

3. The method of multiple scales

This method is a variation of the straightforward expansion which leads to a uniformly valid approximate solution of systems of weakly nonlinear differential equations; the underlying idea is to consider the expansion which represents the approximate solution to be a function of multiple independent variables, or scales, instead of a single variable. The new “time-like” independent variables are defined: $T_k = e^k t$ for $k = 0, 1, 2, \ldots$. Expressing the solution as a function of more variables, treated as independent, is an artifice to remove the secular terms, which would make the solution to explode unphysically.

By writing $T_k$ one makes a formal assumption of physical slow variation explicit. Indeed $T_k$ define progressively longer time scales, which are not negligible when $t$ is of the order of $1/e^k$ or longer. The characteristic time scale of the orbital motion near a neutron star or a stellar mass...
black hole is short \( T_0 \sim 1\) ms): hence for \( \epsilon \) of the order of \( 10^{-1} \) the dynamical effects on the scale \( T_3 \) become important in the interval of 1s.

The conditions to eliminate the secular terms give a simpler system of differential equations: for a set of initial conditions the solution to such a system is a sum of trigonometric functions. The same solution can be found by assuming a priori that the asymptotic solution is the sum of trigonometric functions in which the dominant terms have frequencies not far from the original eigenfrequencies of the uncoupled harmonic oscillators (method of Lindstedt-Poincaré). The corrections to the eigenfrequencies depend on the amplitudes and on the constant coefficients hence in a Fourier transform of the geodesics perturbed in the vicinity of \( r_{3:2} = 27/5 \), two peaks centered near \( 3:2 \) would be found, the position of which and the ratio depend on the perturbation \( (\alpha) \) and on the initial amplitudes.

This method was used to find the approximate solution to equations (11) (both when \( \alpha = 0 \), hence \( \delta \alpha_\theta = \delta \alpha_r = 0 \) and not):

\[
\begin{align*}
\ddot{r}(t) + \omega_0^2 z(t) &= \sum_{i,j,k,l} a_{ijkl} z(t)^i \rho(t)^j \dot{z}(t)^k \dot{\rho}(t)^l \quad (15) \\
\ddot{r}(t) &+ \omega_0^2 r(t) = \sum_{q,r,o,p} b_{qrop} z(t)^q \rho(t)^r \dot{z}(t)^o \dot{\rho}(t)^p .
\end{align*}
\]

The coefficients of the trigonometric functions \( (a,g,c,..) \) are constants which depend on \( r_0 \) and the initial conditions. For simplicity of notation here the initial phases are equal to zero.

4. Symmetry and regions of resonance

The Taylor expansion at constant angular momentum to third order near the stationary point of the effective potential leads to two nonlinear differential equations with constant coefficients:

\[
\begin{align*}
\dot{\theta}(t) &= \frac{\pi}{2} + \epsilon g \cos(\omega_0^* t) t + \\
\dot{\rho}(t) &= \epsilon d \cos(\omega_0^* t) t + \\
\dot{\phi}(t) &= \epsilon c \cos(\omega_0^* t) t + \\
\dot{\psi}(t) &= \epsilon b \cos(\omega_0^* t) t + \\
\dot{\eta}(t) &= \epsilon a \cos(\omega_0^* t) t + \\
\end{align*}
\]

The only possible resonances to this degree of approximation are then:

\[\frac{1}{2}, \frac{3}{1}, \frac{4}{1}, \frac{1}{4}, \frac{1}{3}, \frac{2}{3}\]

Since in General Relativity \( \omega_\rho < \omega_\theta \) (this is due to the curvature of the space), then the only possible resonance is the \( 3:2 \), which is indeed observed most of the times in black holes candidates. In the next approximation a new resonance appears, \( 4:2 \), and one may find many other possible \( m:n \). However a property of parametric resonance is that the region of instability becomes thinner when \( m+n \) increases: the “higher” resonances are less probable (Bell 1957). This is why we do not observe them.

Landau & Lifshitz (“Mechanics” 1976), when studying systems similar to ours (chapter 5 par.28) with the
non-geodesic terms $\alpha = 0$, use the method of Lindstedt-Poincaré and they conclude that these are the solution we are interested in, because in a closed system, without any source of energy, there cannot be a spontaneous increment in the intensity of oscillations. In this sense we asserted that “no parametric resonance occurs for strictly geodesic motion” (Abramowicz et al. 2003); the initial conditions of the perturbation of the geodesics are related to the energy; even if the initially small two perturbations couple and exchange energy, their amplitudes cannot become observable.

5. 3:2 resonance near non-rotating compact objects

An approximate solution for Schwarzschild’s perturbed geodesics was obtained and the validity of the approximation was verified a posteriori for initial conditions of the order of $10^{-1}$ (which were used in the numerical integration).

In this solution two modes remain locked, with the two frequencies near to the 3:2 ratio: the value of these frequencies is one we would see the twin peaks in the Fourier power spectra of the observed signal. The correction to the eigenfrequencies can be written in the following form (higher order corrections can be neglected):

$$\omega_\omega \simeq \omega + \omega_\omega \left(r_0, \alpha, \alpha_0(0), \alpha_\alpha(0)\right)$$
$$\omega_\alpha \simeq \omega + \omega_\alpha \left(r_0, \alpha, \alpha_0(0), \alpha_\alpha(0)\right)$$

$$\omega_\alpha \left(r_0, \alpha, \alpha_0(0), \alpha_\alpha(0)\right) = \frac{1}{8 \omega_{\omega} \omega_{\alpha} \left(\omega - \omega\right)} \left\{ 2 e^{2} \alpha^{2} 1 - \hat{c}_{21} \omega + \frac{1}{4} \hat{c}_{22} \omega \omega_{\alpha}^{2} + \frac{1}{2} \hat{c}_{21} \omega + \frac{1}{2} \hat{c}_{22} \omega \omega_{\alpha} + c_{11} \left(\omega_{\omega} + e_{20} \omega_{\alpha} + 2 c_{20} \omega_{\alpha} \right) \right\}
+ \omega_{\alpha} \left(3 e_{11} \omega_{\alpha} + 2 \hat{c}_{20} \omega_{\alpha} \right)
+ e_{20} \left(-2 c_{20} \omega_{\alpha} + 8 c_{21} \omega + \left(3 e_{20} \omega_{\alpha} + 2 c_{21} \omega \right) \right)\alpha_{\alpha}(0) \right\},$$

The analytic approximation in figure (2) was obtained by fixing $|\alpha| = 0.99$ (far from geodesics) and $r_0 = r_{2:1}$: each point on the segment corresponds to the state reached by the system starting from different initial conditions $(r(0) \in [0.23, 0.21])$ and $z(0) \in [-0.26, 0.19])$. The same slope was matched numerically (in the pseudo-Newtonian case) by fixing the initial conditions and by varying $\alpha$.

The fact that the slope is smaller than 3:2 depends on the choice of the initial conditions. This analytic result is important in explaining the behaviour of the solution more than the numerical values: the fact that the observed ratios are not exactly in 3:2 ratio is a feature of parametric resonance and the position of the centroid frequencies depends on the strength and features of the non-linear perturbation.

6. Conclusions

The Taylor expansion of the relativistic geodesics to the third order leads to two coupled harmonic oscillators: the purely geodesics motion is stable, while when it is perturbed there are cases for which parametric resonance may occur.

Using the method of multiple scales one can highlight that, owing to the curvature (through the way it determines the effective potential), the first allowed resonance between the radial and the vertical epicyclic frequencies is the 3:2, in agreement with the numerical analysis.

Moreover the deviation of the slope from 3:2 is easily explained as a property of such a non-linear resonance.

The analysis of this toy-model reinforces the theory that indeed the observed pairs of QPOs may be due to parametric resonance, and finally to the strong gravitational field alone.

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$^2$\footnote{The perturbation $\alpha$ is included in the coefficients with tilde in the following way: $c_{\alpha} = c_{ij}(1 - \alpha)$. The initial conditions $\alpha_0(0)$ and $\alpha_\alpha(0)$ (g and $\alpha$ in equations (14)) are the initial values of the amplitudes of the zeroth order approximation (the harmonic oscillators).}
Appendix 1. Errata corrigé

The equations studied in (Abramowicz et al. 2003) for the perturbed geodesics in the Pseudo-Newtonian potential were incorrect. The correct ones are:

\begin{equation}
\begin{aligned}
f(\rho, z, r_0, \theta_0) & = c_{11} \rho \dot{r} + c_{03} \dot{z} + c_{21} \rho^2 \ddot{r} + c_{10} \rho \dot{z} \dot{\theta} + c_{01} \rho z \dot{\theta} + c_{02} \rho^2 z \dot{\theta} + c_{30} \rho^3 + c_{12} \rho^2 \dot{z}^2 + c_{11} \rho \dot{z}^2 \ .
\end{aligned}
\end{equation}

This does not affect the numerical results: indeed the symmetry (then the possible resonances) is the same and the only difference is in the values of the initial conditions which lead to the resonance.

Appendix 2. Coefficients

In equations (11) I named the constant coefficients in a way practical to remember their origin: for example \( c_{21} \) is a coefficient in the equation \( z \) (it has the letter \( c \)) and it is the coefficient of \( \rho \) at the 2nd power and \( z \) at the first power (numbers (21) ). Every coefficient contains the derivatives of the effective potential and of the metric at equilibrium. The explicit form of each coefficient is:

\begin{align}
c_{11} & = E^2 \left( \frac{1}{2} \frac{\partial^2 U}{\partial r^2} - \frac{1}{r_0^2} \frac{\partial^2 U}{\partial \theta^2} \right) , \\
c_b & = \frac{2}{r_0} , \\
c_{21} & = E^2 \left( \frac{3}{2r_0^4} \frac{\partial^2 U}{\partial \theta^2} + \frac{1}{4r_0^2} \frac{\partial U}{\partial r^2} \right) , \\
c_{10} & = \frac{2}{r_0} , \\
c_{03} & = E^2 \frac{\partial^4 U}{\partial \theta^4} , \\
e_{02} & = E^2 \left[ \frac{1}{4} \left( 1 - \frac{1}{r_0} \right) \frac{\partial^2 U}{\partial r^2 \partial \theta^2} \right] , \\
e_{20} & = E^2 \left[ \frac{1}{2} \frac{\partial^2 U}{\partial r^2 \partial \theta^2} + \frac{1}{4} \left( 1 - \frac{1}{r_0} \right) \frac{\partial^4 U}{\partial r^4} \right] , \\
e_{22} & = \frac{r_0 - 1}{r_0} , \\
e_{30} & = E^2 \left[ \frac{1}{4r_0^4} \frac{\partial^2 U}{\partial r^4} - \frac{1}{2r_0^2} \frac{\partial^2 U}{\partial r^2 \partial \theta^2} + \frac{1}{12} \left( 1 - \frac{1}{r_0} \right) \frac{\partial^4 U}{\partial \theta^4} \right] , \\
e_{12r2} & = 1 , \end{align}

where the derivatives of the effective potential are evaluated in the equilibrium and \( E = -u_t \) is the energy.

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