Efficient Private SCO for Heavy-Tailed Data via Clipping

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Abstract

We consider stochastic convex optimization for heavy-tailed data with the guarantee of being differentially private (DP). Prior work on this problem is restricted to the gradient descent (GD) method, which is inefficient for large-scale problems. In this paper, we resolve this issue and derive the first high-probability bounds for private stochastic method with clipping. For general convex problems, we derive excess population risks $\hat{O}\left(\frac{d^{1/7} \ln \frac{(ne)^2}{\beta d}}{(ne)^{2/7}}\right)$ and $\hat{O}\left(\frac{d^{1/7} \ln \frac{(ne)^2}{\beta d}}{(ne)^{2/7}}\right)$ under bounded or unbounded domain assumption, respectively (here $n$ is the sample size, $d$ is the dimension of the data, $\beta$ is the confidence level and $\epsilon$ is the private level). Then, we extend our analysis to the strongly convex case and non-smooth case (which works for generalized smooth objectives with H"older-continuous gradients). We establish new excess risk bounds without bounded domain assumption. The results above achieve lower excess risks and gradient complexities than existing methods in their corresponding cases. Numerical experiments are conducted to justify the theoretical improvement.

1 Introduction

Stochastic Convex Optimization (SCO) (1) and its empirical form, Empirical Risk Minimization (ERM) (1), have been widely used in areas such as medicine, finance, genomics and social science. Today, machine learning tasks often involve sensitive data, which leads to privacy-preserving concerns. This means that a machine learning algorithm not only needs to learn effectively from data, but also provides a certain level of privacy-preserving guarantee. As a widely-accepted concept for privacy preservation, differential privacy (DP) (2) provides the provable guarantee that an algorithm

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learns statistical characteristics of the population, but nothing about individuals. Differentially private algorithms have been widely studied and recently deployed in industry (3, 4).

In this work, we focus on Differentially Private Stochastic Convex Optimization (DP-SCO), which is started by Bassily et al. (5). The problem of DP-SCO aims to find a $x^{priv} \in \mathbb{R}^d$ that minimizes the population risk, i.e.,

$$\min_{x \in \mathbb{R}^d} f(x), \ f(x) = \mathbb{E}_\xi[f(x, \xi)],$$

with the guarantee of being differentially private. Here, $\xi$ is a random variable on the probability space $\Omega$ with some unknown distribution $\mathcal{P}$. Function $f(x)$ is a smooth convex loss function and has an expected form on $\xi$. The utility of an algorithm (utility bound) is measured by the excess risk, that is $f(x^{priv}) - \min_{x \in \mathbb{R}^d} f(x)$. Besides, we also introduce the differentially private empirical risk minimization (DP-ERM) over a fixed dataset $D = \{\xi_i\}_{i=1}^n$, i.e., $\min_x \widehat{f}(x, D), \ \widehat{f}(x, D) = \frac{1}{n} \sum_{i=1}^n f(x, \xi_i)$. DP-ERM appears frequently in the literature (5, 6, 7, 8, 9, 10). Besides DP-ERM, Bassily et al. (5) studied DP-SCO and achieved a sub-optimal rate. Later, Bassily et al. (11) established a tight analysis on excess risk. After their work, some works focus on reducing the gradient complexity and running time (12) and different geometries (13, 14).

Almost all the previous results tackle the Problem (1) or its empirical form by estimating the expected mean of a random variable (such as gradient) based on the empirical mean estimator or its variants, and then adding random noise to achieve $(\epsilon, \delta)$-differential privacy. They all assume that either the loss function is $O(1)$-Lipschitz (15) or each data sample has bounded $\ell_2$ or $\ell_\infty$ norm explicitly, which restricts the sensitivity (see Definition 5) in order to establish the DP guarantee of their algorithms. However, the quality of the empirical mean is sensitive to data outliers, especially when the data are heavy-tailed\footnote{Heavy-tailed data, in which the data outliers would be sampled with higher probability than data called “light-tailed” (16). We do not assume the data to be “light-tails”, i.e., sub-Gaussian distribution meaning that for random vector $\eta$ and $b \geq 0$, if there exists $\mathbb{E}[\eta]$ and variance $\sigma$, $\mathbb{P}\{\|\eta - \mathbb{E}[\eta]\|_2 \geq b\} \leq 2 \exp(-b^2/(2\sigma^2))$.} (17). This situation widely exists in real world such as in biomedical engineering and finance (18). Heavy-tailed data could lead to a very large empirical mean and thus ruin the above assumptions. This could cause big oscillations in the convergence trajectories, which may even lead to divergence.

Recently, to tackle the above problems, several works have been proposed to privately estimate the mean of a heavy-tailed distribution via scaling or truncation. Bun and Steinke (19) proposed a private mean estimator with a trimming framework. Wang et al. (20) adopted it in the private gradient descent (GD) under some strong assumptions. Later, Kamath et al. (21) studied private mean estimation under a general setting that the mean of the distribution has bounded $k$-th moments. They proposed a one-round mean estimator called CDPHDME and established its $\ell_2$-error guarantees, which hold with constant probability. Note that the above algorithms of DP-SCO all have guarantees that hold in expectation. It is more desirable to establish excess risk bounds that hold with high probability as the outliers may cause large variances in the gradient-based algorithms, which cannot be captured by guarantees that hold in expectation (16, 20, 22). A recent work Kamath et al. (23) extended the
CDPHDME to a multiple-step framework (DP-GD) and provided improved statistical guarantees on excess risk bounds with a novel analysis. Their utility guarantees also hold with high probability as mentioned by the authors. By adopting the scaling and truncation based mean estimator in (24), Wang et al. (20) also established high probability guarantees of their proposed DP-GD algorithms in general convex and strongly convex cases. Using this estimator, some works focus on gradient expectation maximization (25) and high dimensional space (26). However, due to the usage of this mean estimator, their algorithm needs to solve an additional correction function in each iteration, which leads to a high computational cost. Moreover, in view of the prevalence of large-scale datasets in the industry, their GD-based method does not satisfy the increasing computing needs in the industry. Then, it is natural to ask: is there any private SGD-based algorithm that can deal with the heavy-tailed data? Can we establish its statistical guarantee with high probability?

To answer these questions, we propose new analysis for SGD-based method that is capable of handling heavy-tailed data with private guarantees. We summarize our main contributions as follows.

- **(Theorem 2)** Our first result focuses on convex and \( L \)-smooth objectives with a bounded domain, which is also studied in (20). We present the privacy guarantee for SGD-based algorithm with clipping (clipped-dpSGD). We establish its excess risk bound
  \[
  \tilde{O}\left(\frac{d^{1/7}}{(n\epsilon)^{2/7}}\right)
  \]
  which holds with high probability, and is faster than the known bound
  \[
  \tilde{O}\left(\frac{d^{2/3}}{(n\epsilon)^{1/3}}\right)
  \]
  in (20) if \( n \leq \tilde{\Omega}\left(\frac{d^{11}}{\epsilon^8}\right) \). It also enjoys a better gradient complexity than the one in (20) (see Remark 1). Moreover, our method is less restrictive on the constraint of \( n \), which is \( n \geq \tilde{\Omega}\left(\sqrt{d} \epsilon\right) \) instead of \( n \geq \tilde{\Omega}\left(\frac{d^2}{\epsilon^2}\right) \).

- **(Theorem 5)** Using a restarting technique, we extend clipped-dpSGD to the \( \mu \)-strongly convex setting. For this extension, we prove an excess risk bound
  \[
  \tilde{O}\left(\frac{\sqrt{d^2L^2}}{n\epsilon^3\mu^2}\right)
  \]
  which holds with high probability and is faster than existing one
  \[
  \tilde{O}\left(\frac{d^{11}}{(n\epsilon)^{2/7}}\right)
  \]  
  (20). It also enjoys a better gradient complexity than the one in (20) (see Remark 3). Unlike in (20), our method does not require solving an auxiliary correction function in each iteration, which saves a significant amount of computation and we do not need a bounded domain assumption. We also discuss why restarting is a sensible choice for the strongly convex case (see section 3.1 for details).

- **(Theorem 6)** We refine the non-private convergence analysis provided by Gorbunov et al. (16) and propose a private mean estimator, whose error bound holds with high probability. This estimator reveals a trade-off between data utility and privacy, which also suggests the necessity of using the restarting technique in the strongly convex case.

\[\text{2Here the utility bound is measured in terms of } n \text{ and } d, \text{ which is the main concern in learning theory (9).}\]

\[\text{3The original rate in (20) shall be } \tilde{O}\left(\frac{d^{1/3}}{(n\epsilon)^{2/3}}\right) \text{ and } \tilde{O}\left(\frac{d^{1/3}}{(n\epsilon)^{2/3}}\right) \text{ for the convex and strongly convex cases, as confirmed in (23). The quoted bounds are better because of the distinction in the definition of bounded variance. Letting } \mathcal{P} \text{ be the distribution of interests, } \xi \sim \mathcal{P} \text{ and } E[\nabla f(x, \xi)] = \mu, \text{ Wang et al. (20) considered } E[\|\nabla f(x, \xi) - \mu\|_2^2] \text{ is bounded by constant } v \text{ for each coordinate } j \in [d], \text{ while we consider the } E[\|\nabla f(x, \xi) - \mu\|_2^2] \text{ is bounded by } \sigma^2. \text{ The quoted bounds are obtained with } v = \sigma^2/d, \text{ and the total gradient complexity also changes accordingly.}\]
We propose a novel stepsize rule for clipped-dpSGD to handle the problems with Hölder continuous gradients and derive the first high-probability excess risk bound for stochastic convex optimization problems with heavy-tailed data. Moreover, Hölder continuity implies the problem can be non-smooth and our analysis does not require it to hold on $\mathbb{R}^n$.

## 2 Preliminaries

In this section, we provide necessary background for our analyses, including differential privacy and basic assumptions in convex optimization.

### 2.1 Setting

**Definition 1. (L-smoothness (15)).** A differentiable function $f$ is called $L$-smoothness on $\mathbb{R}^d$ if for all $x, y \in \mathbb{R}^d$, it holds that, $f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2} \|y - x\|^2$.

Additionally, if $f$ is convex, then for all $x, y \in \mathbb{R}^d$, the following holds (15).

$$\|\nabla f(y) - \nabla f(x)\|^2 \leq 2L(f(x) - f(y) - \langle \nabla f(y), x - y \rangle). \tag{2}$$

**Definition 2. (µ-strongly convex (15)).** A differentiable function $f$ is called $µ$-strongly convex on $\mathbb{R}^d$ if for all $x, y \in \mathbb{R}^d$, it holds that, $f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{µ}{2} \|y - x\|^2$.

**Definition 3.** The Projection operator on a convex set $\mathcal{X}$ is defined as $\text{Proj}_{\mathcal{X}}(\theta) = \arg \min_{x \in \mathcal{X}} \|\theta - x\|^2$.

**Gradient Clipping**

It is natural to use clipping to avoid the large norm of stochastic gradient. The framework is given as follows (16):

$$\text{clip}(\nabla f(x, \xi), \lambda) : \hat{\nabla} f(x, \xi) = \begin{cases} \nabla f(x, \xi), & \text{if } \|\nabla f(x, \xi)\|_2 \leq \lambda, \\ \frac{\lambda}{\|\nabla f(x, \xi)\|_2} \nabla f(x, \xi), & \text{otherwise}, \end{cases} \tag{3}$$

where $\nabla f(x, \xi)$ is a mini-batched version of $\nabla f(x)$. That is, in order to compute $\text{clip}(\nabla f(x, \xi), \lambda)$, one needs to get $m$ i.i.d. samples $\nabla f(x, \xi_1), \ldots, \nabla f(x, \xi_m)$ and compute its average $\nabla f(x, \xi) = \frac{1}{m} \sum_{i=1}^m \nabla f(x, \xi_i)$.

**Assumption 1.** For the loss function and the population risk, we assume the following:

1. The loss function $f(x, \xi)$ is non-negative, differentiable and convex.
2. The population risk $f(x)$ is $L$-smooth, and satisfies $\nabla f(x^*) = 0$ at the optimal solution $x^*$.
3. Throughout the paper, we assume that for all $x \in \mathbb{R}^d$, the stochastic gradient $\nabla f(x, \xi)$ of function $f(x, \xi)$ satisfies:
   $$\mathbb{E}_\xi[\nabla f(x, \xi)] = \nabla f(x), \quad \mathbb{E}_\xi[\|\nabla f(x, \xi) - \nabla f(x)\|^2] \leq \sigma^2,$$

where $\sigma$ is some non-negative known number.
These assumptions on the stochastic gradient are standard in (private) SCO or ERM (20, 16, 27, 28).

2.2 Differential Privacy

**Definition 4.** (Differential Privacy (2)). We say that two datasets \( D \) and \( D' \) are neighbors if they differ by only one entry, denoted as \( D \sim D' \). An algorithm \( A \) is \((\epsilon, \delta)\)-differentially private if and only if for all neighboring datasets \( D, D' \) and for all events \( S \) in the output space of \( A \), we have \( P(A(D) \in S) \leq e^{\epsilon} P(A(D') \in S) + \delta \), where \( \delta = 0 \) and \( A \) is \( \epsilon \)-differentially private.

The additive term \( \delta \) (preferably smaller than \( 1/|D| \)) is called the broken probability of \( \epsilon \)-differential privacy (2). The core concept and fundamental tools used in DP are sensitivity and Gaussian mechanism, introduced as follows:

**Definition 5.** (\( \ell_2 \)-sensitivity (29)). The \( \ell_2 \)-sensitivity of a deterministic query \( q(\cdot) \) is defined as \( \Delta_2(q) = \sup_{D \sim D'} \| q(D) - q(D') \|_2 \).

**Definition 6.** (Gaussian Mechanism (29)). Given any function \( q : \Omega \to \mathbb{R}^d \), the Gaussian Mechanism is defined as \( M_G(D, q, \epsilon) = q(D) + Y \), where \( Y \) is drawn from Gaussian Distribution \( N(0, \sigma'^2 I_p) \) with \( \sigma' \geq \sqrt{2 \ln(1.25/\delta) \Delta_2(q)} / \epsilon \).

Gaussian Mechanism preserves \((\epsilon, \delta)\)-differential privacy and is popular in the study of DP-SCO and DP-ERM (5, 11, 20, 30).

3 Main Algorithm

In this section, we introduce the clipped-dpSGD algorithm in Algorithm 1 and establish its utility and complexity bounds. Before presenting our results, we discuss the main difficulties as follows.

3.1 Difficulties

Abadi et al. (31) proposed a DP-SGD algorithm, which does a per-sample clipping operation that can handle heavy-tailed data. However, they did not analyze the choice of clipping parameter \( \lambda \). In fact, \( \lambda \) heavily affects the convergence rate and selecting the best \( \lambda \) is quite difficult (25). There is no known statistical guarantee of such clipping method. Recently, Gorbunov et al. (16) conducted a tight convergence analysis for a similar clipping framework (3). Such framework clips the sampled gradient only once per-iteration and thus saves a large amount of comparison operations when using large batch size compared to the one in (23, 31). Inspired by their results, we conduct a thorough study for this clipping technique in the private setting. Clearly, it is insufficient to directly use Gorbunov et al.’s analysis (16) in the private setting. We summarize the main difficulties as follows.
Algorithm 1 Clipped Stochastic Gradient Descent (clipped-dpSGD)

**Input**: data \( \{\xi_i\}_{i=1}^n \), starting point \( x^0 \), number of iterations \( N \), batchsize \( m \), stepsize \( \gamma > 0 \), clipping level \( \lambda > 0 \)

1: for \( k = 0 \) to \( N - 1 \) do  
2: Draw i.i.d samples \( \xi^k_1, \ldots, \xi^k_m \) and compute \( \nabla f(x^k, \xi^k) \) according to \( \nabla f(x, \xi) = \frac{1}{m} \sum_{i=1}^m \nabla f(x, \xi_i) \).
3: Compute \( \tilde{\nabla} f(x^k, \xi^k) = \text{clip}(\nabla f(x^k, \xi^k), \lambda) + z^k \), where \( z^k \sim \mathcal{N}(0, \hat{\sigma}^2 I_d) \).
4: **OPTION I**: \( x^{k+1} = x^k - \gamma \tilde{\nabla} f(x^k, \xi^k) \).
   **OPTION II**: \( x^{k+1} = \text{Proj}_\mathcal{X}(x^k - \gamma \tilde{\nabla} f(x^k, \xi^k)) \).
5: end for  
6: return \( \bar{x}^N = \frac{1}{N} \sum_{k=0}^{N-1} x^k \)

First, although their analysis ensures that the norm of the added random noise and its inner product with the parameter \( x_k \) have sub-Gaussian tail bounds, if we naively extend their result into the private setting by summing up these two terms, we obtain a loose bound on the excess risk.

Second, although Gorbunov et al. (16) also used a restarting technique in the strongly convex case, their analysis fails to work in the private setting. In particular, if we naively follow their analysis, the privacy term in the excess risk, i.e., \( O\left(\frac{N^{5/2}d}{\lambda^2 n^2} \right) \) where \( \lambda = 4L\|x^0 - x^*\|^2 \), \( \gamma = \frac{1}{70L \ln \frac{2N}{\beta}} \) required by their analysis, cannot be upper bounded by \( \frac{2}{n}(f(x^0) - f(x^*)) \), which ruins the condition of using the restarting technique.

Note that it does not seem possible to use the techniques in (8, 20) in the strongly convex case. Unlike GD, the mean estimation error of the gradient clipping (3) does not converge to zero.

To tackle the above difficulties, we conduct a novel analysis of the clipping technique in the private setting, which leads to much tighter excess risks.

In the following sections, we present our results for general convex, strongly convex and non-smooth objectives.

### 3.2 Convex Case

We begin our analysis under the cases that \( f(x) \) is convex and \( L \)-smooth. Algorithm 1 is a type of gradient perturbation mechanism similar to the one in (31), but we do not assume that the \( \ell_2 \) norm of \( \nabla f(x, \xi) \) is bounded by a hyperparameter \( \lambda \), and instead we provide a method to select it. We defer all the proofs in this paper to Appendix.

**Theorem 1.** (Privacy guarantee). For \( \epsilon \leq c_1 \frac{Nm^2}{n^4} \) with some constant \( c_1 \), Algorithm 1 is \((\epsilon, \delta)\)-differentially private for any \( \delta > 0 \) if

\[
\hat{\sigma} = c \frac{\lambda m \sqrt{T \ln(1/\delta)}}{n \epsilon},
\]

for some constant \( c \).
The constraint on \( \epsilon \) can be removed by introducing an additional factor \( \sqrt{\log T/\delta} \) in \( \tilde{\sigma} \). However, there will be a factor of \( \sqrt{\log T/\delta} \) in the utility bounds in Theorems 2, 3 and 5 accordingly (30). In this case, we reserve the constraint as Wang et al. (8) did. To compare with the existing method in (20), we first consider the case where the parameter domain is bounded, which is defined as follows.

**Assumption 2.** *(Bounded domain).* The convex set \( \mathcal{X} \) is defined as the following \( l_2 \)-ball centered at optimal solution \( x^* \) of (1): \( \{\|x - x^*\|_2 \leq \|x^0 - x^*\|_2\} \).

This type of assumption is widely required in DP-SCO (20, 23, 25). Under this assumption, we establish the utility bound of Algorithm 1 in the following theorem.

**Theorem 2.** *(Utility bound).* Under Assumption 1 and 2, if we take \( N = \tilde{O}\left(\frac{R_0^2n\epsilon}{\lambda} \sqrt{d \ln \frac{1}{\delta}}\right)^2 \) and **OPTION I** in Algorithm 1, then for all \( \beta \in (0,1) \), we have that after \( N \) iterations of clipped-dpSGD with

\[
\lambda = 2LR_0, \quad m = \max \left\{ 1, \frac{81N^2\sigma^2}{2LR^2_0 (\ln \frac{1}{\beta})^2} \right\},
\]

where \( R_0 = \|x^0 - x^*\|_2 \), and stepsize \( \gamma = \frac{1}{2L \ln \frac{1}{\beta}} \), with probability at least \( 1 - \beta \), the following holds

\[
f(x^N) - f(x^*) \leq \tilde{O}\left(\frac{(d \ln \frac{1}{\beta})^{\frac{1}{4}} \sqrt{\ln (n\epsilon)^2}}{(n\epsilon)^{\frac{1}{4}} \beta^d}\right),
\]

when \( n \geq \tilde{\Omega}\left(\frac{\sqrt{d}}{\epsilon}\right) \). The total gradient complexity is \( \tilde{O}\left(\max\left\{n^{\frac{7}{2}}d^{\frac{3}{2}}, n^{\frac{5}{2}}d^{\frac{1}{2}}\right\}\right) \), and the Big-\( \tilde{O} \) notation here omits other logarithmic factors and the terms \( \sigma, \ln \frac{1}{\beta}, L \).

**Remark 1.** To the best of our knowledge, this is the first high probability utility bound for clipped-dpSGD for our considered problem. Compared with naively extending the result of (16) into the private setting, we save a factor \( \tilde{O}\left(\frac{N^{3/2}}{d^{\frac{1}{2}}\epsilon^{\frac{3}{2}}}ight) \) on the excess risk by our novel analysis. Below we compare our results with the typical GD based method in (20), which requires similar assumptions. Typically, GD based method is easy to have a faster rate than SGD based one. However, we show that our SGD based method achieves an order \( \tilde{O}\left(\frac{d^{\frac{1}{2}}}{(n\epsilon)^{\frac{1}{2}}}\right) \), which improves upon the existing GD based one \( \tilde{O}\left(\frac{d^{\frac{1}{2}}}{(n\epsilon)^{\frac{1}{2}}}\right) \) in (20) for the case when \( n \leq \tilde{O}\left(\frac{d^{\frac{1}{2}}}{\epsilon}\right) \). In addition, compared with the result in (20), our method needs less total gradient complexity, that is \( \tilde{O}\left(\max\left\{n^{\frac{7}{2}}d^{\frac{3}{2}}, n^{\frac{5}{2}}d^{\frac{1}{2}}\right\}\right) \) instead of \( \tilde{O}(n^{\frac{7}{2}}d^{\frac{3}{2}}) \) (compared under the \( n \geq \tilde{\Omega}\left(\frac{d^2}{\epsilon}\right) \) regime required in (20)). Moreover, we do not need to solve an auxiliary correction function in each iteration and only require \( n \geq \tilde{\Omega}\left(\frac{\sqrt{d}}{\epsilon}\right) \), which is much more practical than \( n \geq \tilde{\Omega}\left(\frac{d^2}{\epsilon}\right) \).

As mentioned before, we will extend our method to the strongly convex case using the restarting technique. However, Assumption 2 hinders this technique, as it is difficult to guarantee that the sequence \( \{\|x^k - x^*\|_2\}_{k=0}^N \) is decreasing with a bounded domain. Therefore, we need to consider a
new analysis in convex case without a bounded domain. The following theorem establishes the utility guarantee under unbounded domain.

**Theorem 3.** *(Utility bound).* Under Assumption 1 and 2, if we take $N = \tilde{O} \left( \frac{R_0^2 \sigma n \epsilon}{\sqrt{d \ln \frac{1}{\beta}}} \right)^{\frac{2}{7}}$ and OPTION II in Algorithm 1, then for all $\beta \in (0, 1)$, we have that after $N$ iterations of clipped-dpSGD with

$$\lambda = 4LR_0 + 2\sqrt{D}, \quad m = \max \left\{ 1, \frac{162N^2\sigma^2}{\lambda^2(\ln \frac{4N}{\beta})^2} \right\},$$

where $R_0 = \|x^0 - x^*\|_2$, $D = \frac{648\gamma LN^3\sigma^2}{n\epsilon \ln^2 \frac{4N}{\beta} \ln \frac{1}{\beta}}$, and stepsize $\gamma = \frac{1}{24L\ln \frac{4N}{\beta}}$, with probability at least $1 - \beta$, the following holds

$$f(\bar{x}^N) - f(x^*) \leq \tilde{O} \left( \frac{(d \ln \frac{1}{\delta})^{\frac{2}{7}} \ln (\frac{\epsilon}{\beta d})}{(n\epsilon)^{\frac{2}{7}}} \right),$$

when $n \geq \tilde{\Omega}(\sqrt{d})$. The total gradient complexity is $\tilde{O} \left( \max \left\{ n^2 \frac{\sigma^2}{d}, n^2 \frac{\sigma^2}{d^2} \right\} \right)$, and the Big-$\tilde{O}$ notation here omits other logarithmic factors and the terms $\sigma, \ln \frac{1}{\beta}, L$.

**Remark 2.** The utility analysis is more complicated without bounded domain assumption. But the result is only slightly worse than Theorem 2, up to square root of logarithmic factor $\sqrt{\ln (\cdot)}$. This is because we are able to bound $\|x^k - x^*\|_2, \forall k \in 0, \cdots, N$ with high probability. Note that this theorem also enjoys the similar advantages of Theorem 2 over the existing GD based method in (20), in terms of excess risk and total gradient complexity.

### 3.3 Strongly Convex Case

Based on the analysis of Theorem 3, we can consider, in this subsection, the restarting technique under the case that $f(x)$ is additionally $\mu$-strongly convex regardless of Assumption 2. For this case, we modify Algorithm 1 to a restarted version called Restarted Clipped Stochastic Gradient Descent (R-clipped-dpSGD), as shown in Algorithm 2. In each iteration, R-clipped-dpSGD runs clipped-dpSGD for $N_0$ iterations from the current point $\tilde{x}^t$ and use its output $\tilde{x}^{t+1}$ as the starting point for the next iteration. This strategy is known as the restarting technique (16, 32, 33, 34).

**Theorem 4.** *(Privacy guarantee).* Algorithm 2 is overall $(\epsilon, \delta)$-differentially private after $\tau$ runs, where each run of clipped-dpSGD is $(\tilde{\epsilon}, \tilde{\delta})$-differentially private with $\tilde{\epsilon} = \frac{\epsilon}{2\sqrt{2\tau \ln \frac{1}{\beta}}}, \tilde{\delta} = \frac{\delta}{2\tau}$.

Finally, we establish the utility guarantee of Algorithm 2 in the following theorem.

**Theorem 5.** *(Utility bound).* Assume that the function $f$ is $\mu$-strongly convex and $L$-smooth. If we choose $\beta \in (0, 1)$, $\tau$ and $N_0 \geq 1$ such that $\frac{N_0}{\ln \frac{N_0}{\beta}} \geq \frac{768L}{\mu}$, together with

$$\lambda_t = 4LR_0 + 2\sqrt{D}, \quad m_t = \max \left\{ 1, \frac{162N_0^2\sigma^2}{\lambda_t^2(\ln \frac{4N_0}{\beta})^2} \right\},$$
Algorithm 2 Restarted Clipped Stochastic Gradient Descent (R-clipped-dpSGD)

**Input**: data \( \{ \xi_i \}_{i=1}^n \), starting point \( x^0 \), number of iterations \( N_0 \) of clipped-dpSGD, number \( \tau \) of clipped-dpSGD runs, batchsize \( \{ m_t \}_{t=0}^{\tau-1} \).

1: for \( t = 0 \) to \( \tau - 1 \) do
2: Run clipped-dpSGD (Algorithm 1) for \( N_0 \) iterations with OPTION I, constant batchsize \( m^t \), stepsize \( \gamma \), Gaussian random noise \( z^t \sim N(0, \hat{\sigma}^2_t I_d) \), \( \hat{\sigma}_t = \frac{\lambda_t m_t \sqrt{N_0 \ln[1/\delta]}}{n\epsilon} \) and starting point \( \hat{x}^t \).
3: Define the output of clipped-dpSGD by \( \hat{x}^{t+1} \).
4: end for
5: return \( \hat{x}^\tau \).

where \( R_0^t = \| \hat{x}^t - x^* \|_2 \), \( D = \frac{648\gamma L N_0^3 \sigma^2}{n\epsilon \ln \frac{4N_0^2}{\beta} \ln \frac{1}{\delta}} \), and stepsize \( \gamma = \frac{1}{248L \ln \frac{4N_0^2}{\beta}} \), we have that with probability at least \( 1 - \tau \beta \),

\[
  f(\hat{x}^\tau) - f(x^*) \leq \frac{\mu R^2}{2^{\tau+1}} + O \left( \frac{N_0^3 \sigma^2}{n\epsilon L \ln^2 \frac{4N_0^2}{\beta} \ln \frac{1}{\delta}} \right),
\]

where \( R = \sqrt{2(f(x^0) - f(x^*))/\mu} \). Moreover, if taking \( \tau = O \left( \frac{L}{\mu} \log n \right) \), and \( N_0 = O \left( \frac{L}{\mu} \ln \frac{L^2}{(\mu \beta)^2} \right) \), with probability at least \( 1 - \tau \beta \), the output \( \hat{x}^\tau \) satisfies

\[
  f(\hat{x}^\tau) - f(x^*) \leq \tilde{O} \left( \frac{d^2 L^2 \sqrt{\log n} \ln \frac{L}{\mu \beta} \ln \frac{L}{\delta \mu}}{\mu^3 n \epsilon} \right). \tag{7}
\]

The total gradient complexity is \( \tilde{O} \left( \max \left\{ \frac{d^2 L^2}{\mu^2}, nd^2 \right\} \right) \), and the Big-\( \tilde{O} \) notation here omits other logarithmic factors and the terms \( \sigma, \ln \frac{1}{\beta} \).

**Remark 3.** R-clipped-dpSGD yields an \( \tilde{O} \left( \frac{1}{\mu^3} \right) \) rate on the excess risk, which is much faster than the existing one \( \tilde{O} \left( \frac{d^2 L^4}{\mu^2} \right) \) in (20). Moreover, R-clipped-dpSGD does not require that parameter domain \( X \) is bounded. The total gradient complexity \( \tilde{O} \left( \max \left\{ \frac{d^2 L^2}{\mu^2}, nd^2 \right\} \right) \) is better than GD based method \( \tilde{O} \left( nd \frac{L^2}{\mu} \right) \) from Theorem 7 in (20). (Note that the above gradient complexity all require \( n \geq \frac{L}{\mu} \).)

### 3.4 Technical Details

In this part, we summarize the techniques we used to deduce the utility bounds in the previous sections. One usual technique to obtain utility bound is through private mean estimator. As we mention in the Introduction, many works have been done in this way. Among them, only Wang et al. (20) provided analysis that guarantees excess risk with high probability. But their method has a high per-iteration cost because it is a GD based method and has to solve an correction function iteratively. In order to
gain faster rate, we consider a simple clipping framework (3). Thus, our mean estimator is given as follows:

$$\tilde{\nabla} f(x, \xi) = \hat{\nabla} f(x, \xi) + z,$$

where $\hat{\nabla} f(x, \xi)$ is the clipped gradient (3) and $z$ is the injected Gaussian random noise. A key observation is that the clipping technique also makes the $\ell_2$-sensitivity to be bounded by $\lambda$. Thus, the estimator will be $(\epsilon, \delta)$-differential privacy if we set $z \sim \mathcal{N}(0, \sigma'^2 I_d)$, $\sigma' = \mathcal{O}\left(\frac{\lambda \sqrt{\ln(1/\delta)}}{\epsilon}\right)$, and by the composition theorem (2), the whole algorithm is still differentially private.

Such estimator has lower per-iteration cost compared with GD based methods but its estimation error under high probability is hard to analyze, as clipping introduces additional bias even in statistical estimation. This difficulty was not solved until recently. Gorbunov et al. (16) implicitly performed a mean error analysis of the clipping method (3) in a non-private way. Although we can privately use their results via adding Gaussian random noise to achieve high probability excess risk for DP-SCO, such naive extension is insufficient to achieve the utility bounds in Theorem 2,3 and cannot extend to strongly convex setting (see section Difficulties for details). This is where the intersection of privacy and heavy-tailed data gives rise to a new technical challenge: no unbiased mean estimation oracle for this setting is known to exist (23). We consider decoupling the bias and noise for application and explicitly derive a mean estimation error in the following theorem.

**Theorem 6.** If $\|\nabla f(x^k)\|_2 \leq \frac{1}{2}$ holds for all iteration $k \geq 0$, and for all $\beta \in (0, 1)$, $N \geq 1$, and the injected Gaussian random noise variance $\tilde{\sigma}^2 I_d$, we have for all $0 \leq k \leq N - 1$, with probability at least $1 - \beta$, if batchsize $m$ is set to be $\max\left\{1, \frac{162 N^2 \tilde{\sigma}^2}{\lambda^4 [\ln \frac{\beta}{3}]^2}\right\}$, then

$$\sum_{t=0}^{k} \|\tilde{\nabla} f(x^t, \xi^t) - \nabla f(x^t)\|_2 \leq \lambda \left(4 \ln \frac{4}{\beta} + \frac{\ln \frac{4}{\beta}}{3} + \frac{2 \ln^2 \frac{4}{\beta}}{81 N} + \frac{k \tilde{\sigma} \sqrt{16d \ln \frac{4k}{\beta}}}{\lambda}\right).$$

The constraint on $\nabla f(x^{k+1})$ in Theorem 6 comes from Lemma 2.4 in (16). It can be automatically satisfied when embedding such mean estimation error in the analyses of Theorems 2 and 3. However, Bernstein’s inequality also makes the above mean estimation error unable to converge to zero. This is the reason why we consider the restarting technique for strongly convex case. The corresponding privacy is guaranteed by advanced composition theorem (29). Unfortunately, advanced composition theorem is not a tighter estimation of privacy loss compared with moments accountant (31). However, our restarting method does not overall fit the way of moments accountant. We leave how to tightly estimate the privacy loss for the restarting technique as an open problem.
3.5 Non-smooth Case

In this section, we extend our analysis of clipped-dpSGD to non-smooth case. Typically, we consider the gradient of function $f$ satisfies Hölder continuity and establish a new excess risk bound. We first introduce the Hölder continuity.

**Level of smoothness.** We assume that the function $f$ has $(\nu, M_\nu)$-Hölder continuous gradients on a compact set $Q \in \mathbb{R}^n$ for some $\nu \in [0, 1]$, $M_\nu > 0$ meaning that

$$
||\nabla f(x) - \nabla f(y)||_2 \leq M_\nu ||x - y||^\nu_{2}, \forall x, y \in Q.
$$

(10)

Hölder continuous covers the $L$-smoothness of $f$. When $\nu = 1$, inequality (10) is equivalent to $M_1$-smoothness of $f$, and when $\nu = 0$, $f$ is non-smooth and has a uniformly boundedness of $\nabla f(x)$. Although the boundedness of $\nabla f(x)$ is not preferred with heavy-tailed data, one can assume the boundedness of subgradients of $f$ for this special case (35). Example with $\nu \in (0, 1)$ can be founded in (36). Moreover, the inequality (10) only need to hold in a compact set $Q$. Explicitly, as we show in the following result, $Q$ should contain a ball that centers at $x^*$ with radius $7R_0 \geq ||x^0 - x^*||_2$.

**Theorem 7.** (Utility bound). Assuming the function $f$ is convex and achieves its minimum at a point $x^*$, and its gradients satisfy (10) with $\nu \in [0, 1]$, $M_\nu$ on $Q = B_{7R_0} = \{x \in \mathbb{R}^n||x - x^*||_2 \leq 7R_0\}$, where $R_0 \geq ||x^0 - x^*||_2$, then for all $\beta \in (0, 1), \alpha > 0$, $N = \tilde{O}\left(\frac{ne}{\sqrt{d\ln \frac{\lambda}{\beta}}}\right)$, we have that after $N$ iterations of Algorithm 1 with OPTION II,

$$
\lambda = 2M_\nu C^\nu R_0^\nu, \ m = \max \left\{1, \frac{27N\sigma^2}{\lambda^2 \ln \frac{\lambda}{\beta}}\right\},
$$

and stepsize

$$
\gamma = \min \left\{\frac{\alpha^{1+\nu}}{8M_\nu^{1+\nu}}, \frac{R_0}{\sqrt{2N\alpha^{1+\nu} M_\nu^{1+\nu}}}, \frac{R_0}{2\lambda \ln \frac{\lambda}{\beta}}, \frac{\lambda R_0}{2DN}\right\},
$$

where $D = \frac{108N^{1.5}\sigma^2}{ne \ln \frac{\lambda}{\beta}}$, $C = 9$, with probability at least $1 - N\beta$, the following holds

$$
f(\bar{x}^N) - f(x^*) \leq \tilde{O}\left(\max \left\{\frac{M_\nu^{1+\nu} (d\ln \frac{1}{\beta})^{\frac{1}{2}}}{\alpha^{1+\nu} (ne)^{\frac{1}{2}}}, \frac{M_\nu (d\ln \frac{1}{\beta})^{\frac{1}{2}}}{(ne)^{\frac{1}{2} M_\nu}}\right\}, \frac{M_\nu^{1+\nu} (d\ln \frac{1}{\beta})^{\frac{1}{2}}}{(ne)^{\frac{1}{2}}} \right) \left(\frac{d\ln \frac{1}{\beta}}{(ne)^{\frac{1}{2}}} \sqrt{\ln \left(\frac{ne^2}{\beta d}\right)} \right)
$$

(11)

when $n \geq \tilde{O}(\sqrt{\frac{\lambda}{\epsilon}})$. The total gradient complexity is $\tilde{O}\left(\max \left\{n^2d^2, n^2d^2\right\}\right)$, and the Big-$\tilde{O}$ notation here omits other logarithmic factors and the terms $\sigma, \alpha, R_0$.

**Remark 4.** This result has an important feature that Hölder continuity is required only on the ball $B_{7R_0}$ centered at the solution. To the best of our knowledge, this is the first theoretical result about
differentially private non-smooth optimization problem for heavy-tailed data. Moreover, if the last term in (11) induced by injected noise is not considered, our result keeps up with the tightest known bound for non-private and non-smooth SGD type method (35).

4 Experiments

We tested our proposed clipped-dpSGD algorithm and Algorithm 4 from (20) which holds with high probability, on ridge regression and logistic regression tasks. We noticed that their method contains practical issues when dealing with sparse data. We handled it by adding a small perturbation into gradient when the data is sparse. Moreover, although they used an SGD version of their method in their experiments, no theoretical guarantee is provided for the variant. To be consistent with the theory, we used their analyzed algorithms in our experiments. We also used the non-private heavy-tailed clipped SCO method in (16), as our baseline method.

The loss function for ridge regression is $f(x, \xi) = (\langle x, \xi \rangle - y)^2$. And for logistic regression, it is $f(x, \xi) = \log(1 + \exp(1 + y\langle x, \xi \rangle))$. To show the efficiency of our algorithm in dealing with real world data, we used the Adult dataset from the LIBSVM library (37) which was been used by Wang et al. (20). More specifically, the task is to predict whether the annual income of an individual is above 50,000. This dataset contains 32,561 samples with 123 dimensions. We selected 28,000 samples as the training set and the rest were used for test.

We evaluated the excess risk $f(x_{\text{priv}}) - \min_{x \in \mathbb{R}^d} f(x)$ and running time of these private algorithms. Explicitly, we tested the performances of our Theorem 2, Theorem 3, Algorithm 4 from (20), the non-private Algorithm 3 from (16), which are denoted by T2, T3, DP-GD and CSGD, respectively. For the clipped based methods, we use constant batchsize $m$. Empirically, the best batchsize is roughly $\sqrt{n}$ where $n$ is the number of training examples. Therefore, we selected $m$ to be equal to 200 and tuned stepsizes and clipping parameters according to the theories in both papers. For the privacy parameters, we chose $\epsilon = \{0.5, 0.75, 1.0, 2.0\}$ and $\delta = O(1/n)$. The failure probability was set to be 0.01. See Appendix for the selections of other parameters.

![Figure 1: Trajectories of DP-GD, clipped-dpSGD of Theorem 2, Theorem 3 and non-private CSGD on Adult dataset for ridge regression.](image)

We can see from the numerical results that our proposed methods outperform the method in (20)
and are comparable with the non-private one. For the two methods we proposed in this paper, T2 has a better estimation error than T3 in almost all cases. These results justify our previous theoretical analysis (this dataset satisfies $n \leq \tilde{\Omega}(d\epsilon^8)$). Table 1 illustrates the corresponding estimation errors (divided by initial error) and running times after being averaged over 100 independent runs. From the results we can see that for all the private methods, i.e., T2, T3, and DP-GD, the estimation errors decrease as $\epsilon$ becomes larger. Moreover, the results in Table 1 validate that our proposed methods have a significantly better performance in terms of running time.

5 Conclusions

In this paper, we conducted a comprehensive study of DP-SCO for heavy-tailed data for the (strongly) convex and (non) smooth objectives. We established the first high probability excess risks and complexities for clipped-dpSGD algorithm under bounded or unbounded domain assumption. Our analysis is inspired by a recently proposed non-private SCO method (16) and overcomes the difficulties when extending to private setting. We showed that for (strongly) convex and (non) smooth objectives, our method achieves better utility bounds for some cases and runs much faster than the existing high
probability DP-GD method (20). Moreover, as the general clipping method we adopt is widely used in private area (38, 39), our theoretical analysis of convergence rate for this method is of great help to related studies. We also conducted a numerical study of the considered methods. The results confirm a better performance of our method.

Admittedly, our approach has some limitations. For example, we do not consider regularized problems. It would also be interesting to generalize our approach to general non-smooth or non-convex problems. In addition, the advanced theorem (29) we use for strongly convex objective may not be the best choice. It is another interesting direction to propose a tighter estimate of the privacy loss for the restarting technique.

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Appendix

6 Omitted Proofs.

6.1 Proof of Theorem 1

Suppose $D$ and $D'$ be the neighbouring datasets drawn from a distribution $\mathcal{P}$. Let $M$ be a sample from $[n]$ and each $i \in [n]$ is chosen independently with probability $\frac{m}{n}$. Then the mechanism $\mathcal{M}(D) = \hat{\nabla} f(x, D) + \mathcal{N}(0, \sigma^2 I^d)$, where $\hat{\nabla} f(x, D) = \frac{1}{m} \sum_{i \in M} \nabla f(x, D_i) / \max \left( 1, \frac{\| \sum_{i \in M} \nabla f(x, D_i) \|_2}{\lambda} \right)$. 

Without loss of generality, we assume $\lambda = 1$, i.e., $\|\hat{\nabla} f(x, \cdot)\|_2 \leq 1$. Now, consider the $k$-th query, $\mathcal{M}_k = \hat{\nabla} f(x^k, D^k) + \mathcal{N}(0, \sigma^2 I^d)$.

By Theorem 2.1 in (31), we have $\alpha_{\mathcal{M}}(\omega) \leq \sum_{k=0}^{N-1} \alpha_{\mathcal{M}_k}(\omega)$.

Now we bound $\alpha_{\mathcal{M}_k}(\omega)$. Note that the $\ell_2$-sensitivity of $\hat{\nabla} f(x, \cdot)$ is $\|\hat{\nabla} f(x, D) - \hat{\nabla} f(x, D')\|_2 \leq 1$. Thus we can follow the way of Lemma 3 in (31) with $q = \frac{m}{n}$, and $\nabla f(x, D_n) = me_1$ and $\sum_{i \in M \setminus \{n\}} f(x, D_i) = 0$, by fixing $D'$ and letting $D = D' \cup \{D_n\}$. For some constant $c_1$ and any integer $\omega \leq \tilde{\sigma}^2 \ln m/(n\tilde{\sigma})$, we have 

$$\alpha_{\mathcal{M}_k}(\omega) \leq c_1 \frac{m^2 \omega^2}{n^2 \tilde{\sigma}^2} + O \left( \frac{m^3 \omega^3}{n^3 \tilde{\sigma}^3} \right).$$

After $N$ iterations, we have that for some $c_1$,

$$\alpha_{\mathcal{M}}(\omega) \leq \sum_{k=0}^{N-1} \alpha_{\mathcal{M}_k}(\omega) \leq c_1 \frac{N m^2 \omega^2}{n^2 \tilde{\sigma}^2}.$$

To be $(\epsilon, \delta)$-differentially private, using Theorem 2.2 in (31), it suffices that 

$$c_1 \frac{N m^2 \omega^2}{n^2 \tilde{\sigma}^2} \leq \frac{\omega \epsilon}{2}$$

and

$$\exp \left( -\frac{\omega \epsilon}{2} \right) \leq \delta.$$

In addition, we need

$$\omega \leq \tilde{\sigma}^2 \ln m/(n\tilde{\sigma}).$$

It can be verified that when $\epsilon \leq c_2 \frac{N m^2}{n^2}$ for some $c_2$, we can set

$$\tilde{\sigma} = c \frac{m \sqrt{N \ln(1/\delta)}}{n \epsilon}$$

to satisfy all the conditions for some $c$. Therefore, $N$-fold queries

$$\mathcal{M}_k = \hat{\nabla} f(x^k, D^k) + \mathcal{N}(0, \sigma^2 I^d).$$

will guarantee $(\epsilon, \delta)$-differentially private for $\epsilon \leq c_2 \frac{N m^2}{n^2}$. 

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6.2 Proof of Theorem 2

Proof. For any $k$-th iteration, let $\hat{x}^k := x^k - \gamma \nabla f(x^k, \xi^k)$, we have $\|x^{k+1} - x^*\|_2 \leq \|\hat{x}^k - x^*\|_2$ by the property of projection. Then with the convexity and $L$-smoothness of $f(x)$, we have the following inequality:

$$
\|x^{k+1} - x^*\|_2 \leq \|\hat{x}^k - x^*\|_2
= \|x^k - \gamma \nabla f(x^k, \xi^k) - x^*\|_2
\leq \|x^k - \gamma \left(\nabla f(x^k, \xi^k) - \nabla f(x^k) + \nabla f(x^k)\right) - x^*\|_2
\leq \|x^k - \gamma \nabla f(x^k) - x^*\|_2 + \gamma \|\nabla f(x^k, \xi^k) - \nabla f(x^k)\|_2.
$$

For the first term, we can expand it as

$$
\|x^k - \gamma \nabla f(x^k) - x^*\|_2^2 = \|x^k - x^*\|_2^2 + \gamma^2 \|\nabla f(x^k)\|_2^2 - 2\gamma \langle x^k - x^*, \nabla f(x^k) \rangle
\leq \|x^k - x^*\|_2^2 + \gamma^2 \|\nabla f(x^k)\|_2^2 - 2\gamma (f(x^k) - f(x^*))
\leq \|x^k - x^*\|_2^2 + 2\gamma^2 L (f(x^k) - f(x^*)) - 2\gamma (f(x^k) - f(x^*))
\leq \|x^k - x^*\|_2^2 + 2\gamma (\gamma L - 1) (f(x^k) - f(x^*))
\leq \|x^k - x^*\|_2^2 \left(1 - \frac{2\gamma(\gamma L - 1)}{\|x^k - x^*\|_2^2} (f(x^k) - f(x^*))\right).
$$

Choosing $\gamma$ such that $1 - \gamma L > 0$ and using the inequality $\sqrt{1 - \alpha} \leq 1 - \frac{\alpha}{2}$, we obtain

$$
\|x^k - \gamma \nabla f(x^k) - x^*\|_2 = \sqrt{\|x^k - \gamma \nabla f(x^k) - x^*\|_2^2}
\leq \|x^k - x^*\|_2 \left(1 - \frac{\gamma(\gamma L - 1)}{\|x^k - x^*\|_2^2} (f(x^k) - f(x^*))\right)
\leq \|x^k - x^*\|_2 + \gamma (\gamma L - 1) (f(x^k) - f(x^*)).
$$

Thus, the following holds

$$
\|x^{k+1} - x^*\|_2 \leq \|x^k - x^*\|_2 + \frac{\gamma (\gamma L - 1)}{\|x^k - x^*\|_2^2} (f(x^k) - f(x^*)) + \gamma \|\nabla f(x^k, \xi^k) - \nabla f(x^k)\|_2. \quad (12)
$$

Using the notation $R_k \overset{\text{def}}{=} \|x^k - x^*\|_2$, and summing up (12) for $k = 0, 1, \ldots, N - 1$, we derive that

$$
\sum_{k=0}^{N-1} \frac{\gamma(1 - \gamma L)}{R_k} (f(x^k) - f(x^*)) \leq R_0 - R_N + \gamma \sum_{k=0}^{N-1} \|\nabla f(x^k, \xi^k) - \nabla f(x^k)\|_2.
$$

Since $R_k \geq 0$, it holds that

$$
\gamma(1 - \gamma L) \sum_{k=0}^{N-1} \frac{(f(x^k) - f(x^*))}{R_k} \leq R_0 + \gamma \sum_{k=0}^{N-1} \|\nabla f(x^k, \xi^k) - \nabla f(x^k)\|_2.
$$
By Assumption 2, we have $R_k \leq R_0$ for all $k = 0, \ldots, N - 1$. Let us define $A = \frac{\gamma(1 - \gamma L)}{R_0} > 0$, then we obtain
\[
A \sum_{k=0}^{N-1} (f(x^k) - f(x^*)) \leq \gamma(1 - \gamma L) \sum_{k=0}^{N-1} \frac{(f(x^k) - f(x^*))}{R_k}
\]
\[
\leq R_0 + \gamma \sum_{k=0}^{N-1} \|\nabla f(x^k, \xi^k) - \nabla f(x^k)\|_2.
\]

Noting that $\bar{x}^N = \frac{1}{N} \sum_{k=0}^{N-1} x^k$, then with the Jensen’s inequality $f(\bar{x}^N) = f\left(\frac{1}{N} \sum_{k=0}^{N-1} x^k\right) \leq \frac{1}{N} \sum_{k=0}^{N-1} f(x^k)$, we have
\[
AN(f(\bar{x}^N) - f(x^*)) \leq R_0 + \gamma \sum_{k=0}^{N-1} \|\nabla f(x^k, \xi^k) - \nabla f(x^k)\|_2.
\]

(13)

Since $f$ is $L$-smooth, this implies
\[
\|\nabla f(x^k)\|_2 \leq L\|x^k - x^*\|_2 \leq LR_0 = \frac{\lambda}{2},
\]
for $k = 0, 1, \ldots, N - 1$. Then the clipping level $\lambda$ can be chosen as
\[
\lambda = 2LR_0,
\]
which implies that Theorem 2 holds for all $k$. Thus, for $k = N$ and $\tilde{\sigma} = \frac{\lambda m \sqrt{N \ln(1/\delta)}}{\eta \epsilon}$, using Theorem 6 we have that with probability at least $1 - \beta$,
\[
AN(f(\bar{x}^N) - f(x^*)) \leq R_0 + 4\gamma \lambda \ln \frac{4}{\beta} + \gamma \lambda \ln \frac{4}{\beta} + \gamma \lambda \ln \frac{4}{\beta} + \frac{2\ln^2 \frac{4}{\beta}}{81N} + \frac{2\ln^2 \frac{4}{\beta}}{81N} + \frac{\gamma N \sigma^2 \ln 4 \ln N}{\beta} \leq R_0 + 12\gamma LR_0 \ln \frac{4}{\beta} + \frac{2\gamma LR_0 N \sigma^2 \ln 4 \ln N}{\beta}.
\]

(14)

Since $A = \frac{\gamma(1 - \gamma L)}{R_0}$ and $1 - \gamma L \geq \frac{1}{2}$, we get that with probability at least $1 - \beta$,
\[
f(\bar{x}^N) - f(x^*) \leq \frac{28LR_0^2 \ln \frac{4}{\beta}}{N} + \frac{4LR_0^2 \sigma^2 \ln 4 \ln N}{\beta} \leq O \left(\frac{LR_0^2 \ln \frac{4}{\beta}}{\sqrt{N}} + \frac{N \sigma^2 \ln 4 \ln N}{\beta} \ln \frac{1}{\delta}\right) \leq O \left(\frac{LR_0^2 \ln \frac{4}{\beta}}{\sqrt{N}} + \frac{N \sigma^2 \ln 4 \ln N}{\beta} \ln \frac{1}{\delta}\right).
\]

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Thus if we take $N$ such that $O\left(\frac{LR^2 \ln \frac{1}{\delta}}{N}\right) \leq \left(\frac{N^{\frac{2}{3}} \sigma^2 \sqrt{\ln \frac{1}{\delta}}}{nL}\right)$ we have with probability at least $1 - \beta$, the following holds for $n \geq \Omega\left(\sqrt{\frac{\gamma}{\sigma^2}}\right)$,

$$ f(\bar{x}^N) - f(x^*) \leq O\left(\frac{(d \ln \frac{1}{\delta})^{\frac{1}{2}} \sqrt{\ln (\frac{nL^2}{\delta})}}{(ne)^{\frac{1}{2}} (\ln \frac{1}{\delta})^{\frac{1}{2}}}\right) $$

where $N = O\left(\frac{\|x^0 - x^*\|_2^2 \eta L^2}{\sqrt{d}}\right)^{\frac{2}{3}}$. The total gradient complexity is $O\left(\max\left\{n^2 d^2, n^\gamma d^4\right\}\right)$. 

### 6.3 Proof of Theorem 3

*Proof.* Since $f(x)$ is convex and $L$-smooth, we have the following inequality:

$$ \|x^{k+1} - x^*\|_2 = \|x^k - \gamma \nabla f(x^k, \xi^k) - x^*\|_2 \leq \|x^k - \gamma (\nabla f(x^k, \xi^k) - \nabla f(x^k)) - x^*\|_2 \leq \|x^k - \gamma \nabla f(x^k) - x^*\|_2 + \gamma \|\nabla f(x^k, \xi^k) - \nabla f(x^k)\|_2. $$

For the first term, we can expand it as

$$ \|x^k - \gamma \nabla f(x^k) - x^*\|_2^2 = \|x^k - x^*\|_2^2 + \gamma^2 \|\nabla f(x^k)\|_2^2 - 2\gamma \langle x^k - x^*, \nabla f(x^k) \rangle \leq \|x^k - x^*\|_2^2 + \gamma^2 \|\nabla f(x^k)\|_2^2 - 2\gamma \langle f(x^k) - f(x^*), \nabla f(x^k) \rangle \leq \|x^k - x^*\|_2^2 + 2\gamma (|x^k - x^*|_2^2, \gamma L - 1) (f(x^k) - f(x^*)) \leq \|x^k - x^*\|_2^2 \left( 1 - \frac{-2\gamma (\gamma L - 1)}{|x^k - x^*|_2^2} (f(x^k) - f(x^*)) \right). $$

Choosing $\gamma$ such that $1 - \gamma L > 0$ and using the inequality $\sqrt{1 - x} \leq 1 - \frac{x}{2}$, we obtain

$$ \|x^k - \gamma \nabla f(x^k) - x^*\|_2 = \sqrt{\|x^k - \gamma \nabla f(x^k) - x^*\|_2^2} \leq \|x^k - x^*\|_2 \left( 1 - \frac{-\gamma (\gamma L - 1)}{|x^k - x^*|_2^2} (f(x^k) - f(x^*)) \right) \leq \|x^k - x^*\|_2 + \gamma (\gamma L - 1) (f(x^k) - f(x^*)). $$

Thus, the following holds

$$ \|x^{k+1} - x^*\|_2 \leq \|x^k - x^*\|_2 + \frac{\gamma (\gamma L - 1)}{|x^k - x^*|_2} (f(x^k) - f(x^*)) + \gamma \|\nabla f(x^k, \xi^k) - \nabla f(x^k)\|_2. \quad (16) $$

Using the notation $R_k \overset{\text{def}}{=} \|x^k - x^*\|_2$, and summing up (16) for $k = 0, 1, \ldots, N - 1$, we derive that

$$ \sum_{k=0}^{N-1} \frac{\gamma (1 - \gamma L)}{R_k} (f(x^k) - f(x^*)) \leq R_0 - R_N + \gamma \sum_{k=0}^{N-1} \|\nabla f(x^k, \xi^k) - \nabla f(x^k)\|_2. $$

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Let us define $A = \gamma (1 - \gamma L) > 0$, then

$$A \sum_{k=0}^{N-1} \frac{(f(x^k) - f(x^*))}{R_k} \leq R_0 - R_N + \gamma \sum_{k=0}^{N-1} \| \tilde{\nabla} f(x^k, \xi^k) - \nabla f(x^k) \|_2. \quad (17)$$

Taking into account that

$$A \sum_{k=0}^{N-1} \frac{(f(x^k) - f(x^*))}{R_k} \geq 0,$$

and changing the indices in (29), we get that for all $T \geq 0$,

$$R_T \leq R_0 + \gamma \sum_{k=0}^{T-1} \| \tilde{\nabla} f(x^k, \xi^k) - \nabla f(x^k) \|_2. \quad (18)$$

The remaining part of the proof is based on the analysis of inequality (30). In particular, via induction we prove that for all $T = 0, \ldots, N$ with probability at least $1 - T\beta$, the following statement holds

$$R_t \leq R_0 + \gamma \sum_{k=0}^{t-1} \| \tilde{\nabla} f(x^k, \xi^k) - \nabla f(x^k) \|_2$$

$$\leq R_0 + \gamma \lambda 6 \ln \frac{4}{\beta} + \frac{648 \gamma N^3 \sigma^2 \sqrt{dN \ln \frac{4N}{\beta} \ln \frac{1}{\delta}}}{\lambda \epsilon \ln^2 \frac{4}{\beta}}, \quad (19)$$

for all $t = 0, \ldots, T$ simultaneously where $\gamma$, $\lambda$ will be defined further. Let us define $D := \frac{648 \gamma \lambda N^3 \sigma^2 \sqrt{dN \ln \frac{4N}{\beta} \ln \frac{1}{\delta}}}{\epsilon \ln^2 \frac{4}{\beta}}$, and the probability event when statement (31) holds as $E_T$. Then, our goal is to show that $\mathbb{P}(E_T) \geq 1 - T\beta$ for all $T = 0, \ldots, N$. Clearly, when $T = 0$, inequality (31) holds with probability 1. Next, assuming that for $T \leq N - 1$, we have $\mathbb{P}(E_T) \geq 1 - T\beta$. Let us prove that $\mathbb{P}(E_{T+1}) \geq 1 - (T + 1)\beta$.

First of all, probability event $E_T$ implies that

$$R_t^{(30)} \leq R_0 + \gamma \sum_{k=0}^{t-1} \| \tilde{\nabla} f(x^k, \xi^k) - \nabla f(x^k) \|_2$$

holds for $t = 0, \ldots, T$. Since $f(\cdot)$ is $L$-smooth, this implies

$$\| \nabla f(x^t) \|_2 \leq L \| x^t - x^* \|_2 \leq L \left( R_0 + \gamma \lambda 6 \ln \frac{4}{\beta} + \frac{D}{\lambda} \right) \leq \frac{\lambda}{2},$$

for $t = 0, \ldots, T$. With $\gamma = \frac{1}{24L \ln \frac{4}{\beta}}$, we have

$$\frac{\lambda^2}{4} - LR_0 \lambda - D \geq 0.$$
Hence,
\[
\lambda \geq 2 \left( LR_0 + \sqrt{L^2 R_0^2 + D} \right).
\]

Then the clipping level \( \lambda \) can be chosen as
\[
\lambda = 4LR_0 + 2\sqrt{D}. \tag{20}
\]

Secondly, since event \( E_T \) implies \( \| \nabla f(x^t) \|_2 \leq \lambda \) holds for \( t = 0, \cdots, T \), using Theorem 6 with \( \hat{\sigma} = \frac{\lambda m \sqrt{N \ln(1/\delta)}}{n_t} \), we have the following probability event, defined as \( E_{(1)} \), holds with probability at least \( 1 - \beta \),
\[
\sum_{k=0}^{T} \| \tilde{\nabla} f(x^k, \xi^k) - \nabla f(x^k) \|_2 \leq \lambda \left( 4 \ln \frac{4}{\beta} + \frac{\ln \frac{4}{\beta}}{3} + \frac{2 \ln^2 \frac{4}{\beta}}{81N} + \frac{(T + 1)\hat{\sigma} \sqrt{16d \ln \frac{4(T + 1)}{\beta}}}{\lambda} \right)
\]
\[
\leq \lambda \left( 4 \ln \frac{4}{\beta} + \frac{\ln \frac{4}{\beta}}{3} + \frac{2 \ln^2 \frac{4}{\beta}}{81N} + \frac{N\hat{\sigma} \sqrt{16d \ln \frac{4N}{\beta}}}{\lambda} \right)
\]
\[
\leq \lambda 6 \ln \frac{4}{\beta} + \frac{D}{L\lambda\gamma}.
\]

Finally, event \( E_{T+1} \):
\[
R_{T+1} \leq R_0 + \gamma \sum_{k=0}^{T} \| \tilde{\nabla} f(x^k, \xi^k) - \nabla f(x^k) \|_2 \leq R_0 + \gamma \lambda 6 \ln \frac{4}{\beta} + \frac{D}{L\lambda},
\]
holds with probability \( \mathbb{P}\{E_{T+1}\} = \mathbb{P}\{E_T \cap E_{(1)}\} = 1 - \mathbb{P}\{\bar{E}_T \cup \bar{E}_{(1)}\} \geq 1 - (T + 1)\beta \).

Thus, we have proved that for all \( k = 0, \ldots, N \), we have \( \mathbb{P}\{E_k\} \geq 1 - k\beta \), which implies \( R_k \leq R_0 + \gamma \lambda 6 \ln \frac{4N}{\beta} + \frac{D}{L\lambda} \leq \frac{\lambda}{2L} \) with probability at least \( 1 - k\beta \).

Then, using (17), for \( T = N \) we have that with probability at least \( 1 - N\beta \),
\[
2LA \sum_{k=0}^{N-1} \frac{(f(x^k) - f(x^*))}{\lambda} \leq A \sum_{k=0}^{N-1} \frac{(f(x^k) - f(x^*))}{R_k}
\]
\[
\leq R_0 + \gamma \sum_{k=0}^{N-1} \| \tilde{\nabla} f(x^k, \xi^k) - \nabla f(x^k) \|_2
\]
\[
\leq R_0 + \gamma \lambda 6 \ln \frac{4}{\beta} + \frac{D}{L\lambda} \leq \frac{\lambda}{2L}.
\]

Noting that \( \bar{x}^N = \frac{1}{N} \sum_{k=0}^{N-1} x^k \), then with the Jensen’s inequality, we have
\[
\frac{2ANL}{\lambda} (f(\bar{x}^N) - f(x^*)) \leq \frac{\lambda}{2L}. \tag{21}
\]
Since $A = \gamma(1 - \gamma L)$ and $1 - \gamma L \geq \frac{1}{2}$, we get that with probability at least $1 - \beta$,

$$f(\bar{x}^N) - f(x^*) \leq \lambda^2 \left( \frac{4L^2 \beta^2 N}{N} + \frac{16L^2 R_0^2 + 4D}{2\gamma L^2 N} \right) \leq O \left( \frac{LR_0^2 \ln \frac{4N}{\beta}}{N} + \frac{N^2 \sigma^2 \sqrt{d \ln \frac{4N}{\beta} \ln \frac{1}{\delta}}}{n\epsilon L \ln^2 \frac{4N}{\beta}} \right) \leq O \left( \frac{LR_0^2 \ln \frac{4N}{\beta}}{N} + \frac{N^2 \sigma^2 \ln \frac{4N}{\beta} \sqrt{d \ln \frac{1}{\delta}}}{n\epsilon L} \right)$$

Thus if we take $N$ such that $O \left( \frac{LR_0^2 \ln \frac{4N}{\beta}}{N} \right) \leq O \left( \frac{N^5 \sigma^2 \ln \frac{4N}{\beta} \sqrt{d \ln \frac{1}{\delta}}}{n\epsilon L} \right)$, we have with probability at least $1 - \beta$, the following holds for $n \geq \Omega(\frac{\sqrt{d}}{\epsilon})$,

$$f(\bar{x}^N) - f(x^*) \leq O \left( \frac{\left( \frac{d \ln \frac{1}{\delta}}{\beta \epsilon} \right)^{\frac{1}{2}} \ln \left( \frac{(n\epsilon)^2}{\beta} \right)}{(n\epsilon)^{\frac{1}{2}}} \right),$$

where $N = O \left( \frac{\|x^0 - x^*\|^2_{\infty}}{\sqrt{d \ln \frac{1}{\delta}}} \right)^{\frac{2}{3}}$. The total gradient complexity is $O \left( \max \left\{ \frac{n^2 d^6}{\delta}, \frac{n^7 d^4}{\epsilon} \right\} \right)$.

6.4 Proof of Theorem 4

Let each run of Algorithm 1 to be $(\hat{\epsilon}, \hat{\delta})$-DP with $\hat{\epsilon} = \frac{\epsilon}{2\sqrt{2} \ln \frac{2}{\beta}}, \hat{\delta} = \frac{\delta}{2\epsilon}$. Then apply the following Lemma to guarantee the $(\epsilon, \delta)$-DP of Algorithm 2.

Lemma 1. (Advanced Composition (29)). For all $\epsilon, \delta, \delta' \geq 0$, the class of $(\epsilon, \delta)$-differentially private mechanisms satisfies $(\epsilon', \kappa\delta + \delta')$-differential privacy under $\kappa$-fold adaptive composition for: $\epsilon' = \sqrt{2k \ln \frac{1}{\delta}} \epsilon + k\epsilon (e^\epsilon - 1)$. Typically, it suffices that each mechanism is $(\epsilon, \delta)$-differentially private, where $\epsilon = \frac{\epsilon'}{2\sqrt{2} \ln \frac{2}{\beta}}$.

6.5 Proof of Theorem 5

Proof. Consider the first run of clipped-dpSGD (Algorithm 1). Observed that the proof of Theorem 3 still holds if we substitute $R_0$ everywhere by its upper bound $R$, using $\mu$-strongly convexity of $f(\cdot)$, we have

$$R^2_0 = \|x^0 - x^*\|^2_2 \leq \frac{2}{\mu} (f(x^0) - f(x^*)).$$

It implies that after $N_0$ iterations of clipped-dpSGD, we have

$$f(\bar{x}^{N_0}) - f(x^*) \leq \frac{384L \ln \frac{4N_0}{\beta}}{N_0 \mu} (f(x^0) - f(x^*)) + O \left( \frac{N_0^2 \sigma^2 \sqrt{d \ln \frac{4N_0}{\beta} \ln \frac{1}{\delta}}}{n\epsilon L \ln^2 \frac{4N_0}{\beta}} \right),$$

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with probability at least $1 - \beta$.

Thus, taking $\frac{N_0}{\ln \frac{N_0}{\beta}} \geq \frac{768L}{\mu}$, we have

$$f(\tilde{x}^{N_0}) - f(x^*) \leq \frac{1}{2}(f(x^0) - f(x^*)) + O\left(\frac{N_0^5\sigma^2\sqrt{d\ln \frac{4N_0^2}{\beta} \ln \frac{1}{\delta}}}{n\epsilon L \ln^2 \frac{4N_0}{\beta}}\right).$$

Then, by induction, we can show that for arbitrary $k = 0, 1, \ldots, \tau - 1$, the inequality

$$f(\tilde{x}^{k+1}) - f(x^*) \leq \frac{1}{2}(f(\tilde{x}^k) - f(x^*)) + O\left(\frac{N_0^5\sigma^2\sqrt{d\ln \frac{4N_0^2}{\beta} \ln \frac{1}{\delta}}}{n\epsilon L \ln^2 \frac{4N_0}{\beta}}\right),$$

holds with probability at least $1 - \beta$. Therefore, by induction, we get

$$f(\tilde{x}^\tau) - f(x^*) \leq \frac{1}{2\tau}(f(\tilde{x}^0) - f(x^*)) + O\left(\frac{N_0^5\sigma^2\sqrt{d\ln \frac{4N_0^2}{\beta} \ln \frac{1}{\delta}}}{n\epsilon L \ln^2 \frac{4N_0}{\beta}}\right) \leq \frac{\mu R^2}{2\tau+1} + O\left(\frac{N_0^5\sigma^2\sqrt{d\ln \frac{4N_0^2}{\beta} \ln \frac{1}{\delta}}}{n\epsilon L \ln^2 \frac{4N_0}{\beta}}\right),$$

holds with probability at least $1 - \tau\beta$. Thus, taking $\tau = O\left(\frac{L}{\mu} \log n\right)$ and $N_0 = O\left(\frac{L}{\mu} \ln \frac{L^2}{(\mu \beta)^2}\right)$, we have

$$f(\tilde{x}^\tau) - f(x^*) \leq O\left(\frac{d^2L^2\sqrt{\log n}\ln \frac{L^2}{(\mu \beta)^2}^3\ln \frac{L\log n}{\delta \mu}}{\mu^3n\epsilon \ln \frac{L}{\mu \beta}^2}\right) + \frac{f(x^0) - f(x^*)}{n\epsilon} \leq O\left(\frac{d^2L^2\sqrt{\log n}\ln \frac{L}{\mu \beta} \ln \frac{L}{\delta \mu}}{\mu^3n\epsilon}\right),$$

holds with probability at least $1 - \tau\beta$. The total gradient complexity is $O\left(\max\left\{d\left(\frac{L}{\mu}\right)^2, n\epsilon d^2\right\}\right)$. 

6.6 Proof of Theorem 6

We first recall the following useful lemmas.

**Definition 7.** (Sub-Gaussian vector (40)). A random vector $Z \in \mathbb{R}^d$ is said to be sub-Gaussian with variance $\sigma^2$ if it is centered and for any $u \in \mathbb{R}^d$ such that $|u| = 1$, the real random variable $u^T Z$ is sub-Gaussian with variance $\sigma^2$. We write $Z \sim \text{subG}(\sigma^2)$.

**Lemma 2.** (Lemma 2.4 in (16)) For all iteration $k \geq 0$ the following inequality holds:

$$\left\|\nabla f(x^k, \xi^k) - \mathbb{E}_{\xi^k} \left[\nabla f(x^k, \xi^k)\right]\right\|_2 \leq 2\lambda.$$

(22)
Moreover, if \( \| \nabla f(x^k) \|_2 \leq \frac{1}{2} \) for some \( k \geq 0 \), then for this \( k \) we have:

\[
\left\| \mathbb{E} \xi_k \left[ \nabla f(x^k, \xi^k) \right] - \nabla f(x^k) \right\|_2 \leq \frac{4\sigma^2}{m\lambda},
\]

(23)

\[
\mathbb{E} \xi_k \left\| \nabla f(x^k, \xi^k) - \nabla f(x^k) \right\|_2^2 \leq \frac{18\sigma^2}{m},
\]

(24)

\[
\mathbb{E} \xi_k \left\| \nabla f(x^k, \xi^k) - \mathbb{E} \xi_k \left[ \nabla f(x^k, \xi^k) \right] \right\|_2^2 \leq \frac{18\sigma^2}{m}.
\]

(25)

**Lemma 3. (Lemma 1 in (40))** There exists absolute constant \( \tilde{c} \) so that random vector \( Z \in \mathcal{R}^d \) are norm-sub-Gaussian (\( nSG(\tilde{c} \cdot \tilde{\sigma}) \)), i.e., \( \exists \tilde{c} \),

\[
\mathbb{P}(\| Z - E(Z) \|_2 \geq t) \leq 2e^{-\frac{ct^2}{2\tilde{\sigma}^2}},
\]

(26)

if \( Z \) is \( subG(\tilde{\sigma}^2 / d) \).

**Proof of Theorem 6.** For all iteration \( 0 \leq k \leq N - 1 \),

\[
\sum_{t=0}^{k} \left\| \nabla f(x^t, \xi^t) - \nabla f(x^t) \right\|_2 \leq \sum_{t=0}^{k} \left( \left\| \nabla f(x^t, \xi^t) - \nabla f(x^t) \right\|_2 + \| z^t \|_2 \right)
\]

\[
= \sum_{t=0}^{k} \left\| \nabla f(x^t, \xi^t) - \nabla f(x^t) \right\|_2 + \sum_{t=0}^{k} \| z^t \|_2
\]

\[
= \sum_{t=0}^{k} \left( \| \theta^o_t \|_2 - \mathbb{E} \xi_t[\| \theta^o_t \|_2] \right) + \sum_{t=0}^{k} \mathbb{E} \xi_t[\| \theta^o_t \|_2]
\]

\[
+ \sum_{t=0}^{k} \| \theta^b_t \|_2 + \sum_{t=0}^{k} \| z^t \|_2,
\]

where we introduce new notations:

\[
\theta^o_t \overset{\text{def}}{=} \nabla f(x^t, \xi^t) - \mathbb{E} \xi_t \left[ \nabla f(x^t, \xi^t) \right], \quad \theta^b_t \overset{\text{def}}{=} \mathbb{E} \xi_t \left[ \nabla f(x^t, \xi^t) \right] - \nabla f(x^t).
\]

Upper bound for (1). First of all, we notice that the terms in (1) are conditionally unbiased:

\[
\mathbb{E} \xi_t \left[ \| \theta^o_t \|_2 - \mathbb{E} \xi_t[\| \theta^o_t \|_2] \right] = 0.
\]

Secondly, the terms are bounded with probability 1:

\[
\| \theta^o_t \|_2 - \mathbb{E} \xi_t[\| \theta^o_t \|_2] \leq \| \theta^o_t \|_2 + \mathbb{E} \xi_t[\| \theta^o_t \|_2] \overset{(22)}{=} 4\lambda \overset{\text{def}}{=} c.
\]

Finally, we can bound the conditional variances \( \tilde{\sigma}^2_t = \mathbb{E} \xi_t \left[ \| \theta^o_t \|_2 - \mathbb{E} \xi_t[\| \theta^o_t \|_2] \right]^2 \) as follows:

\[
\tilde{\sigma}^2_t \leq c\mathbb{E} \xi_t \left[ \| \theta^o_t \|_2 - \mathbb{E} \xi_t[\| \theta^o_t \|_2] \right] \leq c\mathbb{E} \xi_t \left[ \| \theta^o_t \|_2 + \mathbb{E} \xi_t[\| \theta^o_t \|_2] \right] = 2c\mathbb{E} \xi_t[\| \theta^o_t \|_2].
\]
Thus, the sequence \( \{ \| \theta_t^a \|_2 - \mathbb{E}_{\xi^t} [\| \theta_t^a \|_2] \}_{t \geq 0} \) is a bounded martingale differences sequence with bounded conditional variances \( \{ \bar{\sigma}_t^2 \}_{t \geq 0} \). Therefore, we can apply Bernstein’s inequality (41, 42) with

\[
X_t = \| \theta_t^a \|_2 - \mathbb{E}_{\xi^t} [\| \theta_t^a \|_2], \quad c = 4\lambda \text{ and } F = \frac{c^2 \ln \frac{4}{\beta}}{6},
\]

and get for all \( b > 0 \), it holds that

\[
\mathbb{P} \left\{ \sum_{t=0}^{k} |X_t| > b \text{ and } \sum_{t=0}^{k} \bar{\sigma}_t^2 \leq F \right\} \leq 2e^{-\frac{b^2}{2F + 2cb/3}}.
\]

Thus, with probability at least \( 1 - 2\exp \left( -\frac{b^2}{2F + 2cb/3} \right) \), we have

either \( \sum_{t=0}^{k} |X_t| \leq b \) or \( \sum_{t=0}^{k} \bar{\sigma}_t^2 > F \).

Here we choose \( b \) in a way such that \( 2\exp \left( -\frac{b^2}{2F + 2cb/3} \right) \leq \frac{\beta}{2} \). This implies that

\[
b^2 - \frac{2c \ln \frac{4}{\beta}}{3} b - 2F \ln \frac{4}{\beta} \geq 0.
\]

Hence,

\[
b \geq \frac{c \ln \frac{4}{\beta}}{3} + \sqrt{\frac{c^2 \ln^2 \frac{4}{\beta}}{9} + 2F \ln \frac{4}{\beta}}.
\]

Next, in order to bound \( \sum_{t=0}^{k} \bar{\sigma}_t^2 \) with probability 1, we have the following inequality for \( F \)

\[
\sum_{t=0}^{k} \bar{\sigma}_t^2 \leq 2c \sum_{t=0}^{k} \mathbb{E}_{\xi^t} [\| \theta_t^a \|_2] \leq 6\sqrt{2c} \sum_{t=0}^{k} \frac{1}{\sqrt{m}} = 6\sqrt{2c} \sigma \frac{k}{\sqrt{m}} \leq \frac{c^2 \ln \frac{4}{\beta}}{6} = F,
\]

where the second step uses the Jensen’s inequality, i.e.,

\[
\left[ \mathbb{E}_{\xi^k} \left[ \| \hat{\nabla} f(x^k, \xi^k) - \mathbb{E}_{\xi^k} [\hat{\nabla} f(x^k, \xi^k)] \|_2 \right] \right]^2 \leq \mathbb{E}_{\xi^k} \left[ \| \hat{\nabla} f(x^k, \xi^k) - \mathbb{E}_{\xi^k} [\hat{\nabla} f(x^k, \xi^k)] \|_2 \right]^2 \leq \frac{18 \sigma^2}{m}.
\]

Therefore, we have shown that with probability at least \( 1 - \frac{\beta}{2} \), \( \sum_{t=0}^{k} |X_t| \leq b \), i.e.,

\[
\sum_{t=0}^{k} |\| \theta_t^a \|_2 - \mathbb{E}_{\xi^t} [\| \theta_t^a \|_2] | \leq b,
\]

where \( b = c \ln \frac{4}{\beta} = 4\lambda \ln \frac{4}{\beta} \) as desired.
Upper bound for (2).

\[ (2) = \sum_{t=0}^{k} \mathbb{E}[\|\theta_t\|_2] \overset{(25)}{\leq} 3\sqrt{2}\sigma \sum_{t=0}^{k} \frac{1}{\sqrt{m}} \]
\[ = 3\sqrt{2}\sigma \frac{k}{\sqrt{m}} \overset{k \leq N}{\leq} 3\sqrt{2}\sigma \frac{N}{\sqrt{m}} \]
\[ = \frac{\lambda \ln \frac{4}{\beta}}{3}. \]

Upper bound for (3).

\[ (3) = \sum_{t=0}^{k} \|\theta_t\|_2 \overset{(23)}{\leq} 4\sigma^2 \sum_{t=0}^{k} \frac{1}{m\lambda} \]
\[ = 4\sigma^2 \frac{k}{m\lambda} \overset{k \leq N}{\leq} 4\sigma^2 \frac{N}{m\lambda} \]
\[ = \frac{2\lambda \ln^2 \frac{4}{\beta}}{81N}. \]

Upper bound for (4). It is easy to verify that the injected Gaussian random noise \( z \in \mathbb{R}^d, z \sim \mathcal{N}(0, \tilde{\sigma}^2 I_d) \) is subG(\( \tilde{\sigma}^2 \)). Together with (26) in the Lemma 3, the following holds

\[ \mathbb{P}(\|z\|_2 \geq \tilde{t}) \leq 2e^{-\frac{\tilde{t}^2}{2\tilde{c}^2 \tilde{\sigma}^2}}, \]

where constant \( \tilde{c} \) is selected as \( 2\sqrt{2} \) as the derivations in (40). We choose \( \tilde{t} \) such that \( 2\exp\left[-\frac{\tilde{t}^2}{2\tilde{c}^2 \tilde{\sigma}^2}\right] = \frac{\beta}{2\pi} \).

That is, we choose \( \tilde{t} = \tilde{\sigma}\sqrt{16d \ln \frac{4k}{\beta}} \) so that with the probability at least \( 1 - \frac{\beta}{2} \),

\[ \sum_{t=0}^{k} \|z_t\|_2 \leq k\tilde{\sigma}\sqrt{16d \ln \frac{4k}{\beta}}. \]
Finally, summarizing all the above bounds we have derived

\[
\sum_{t=0}^{k} \| \tilde{\nabla} f(x^t, \xi^t) - \nabla f(x^t) \|_2 \\
\leq \sum_{t=0}^{k} (||\theta_t^2||_2 - E_{\xi_t} [||\theta_t^2||_2]) + \sum_{t=0}^{k} E_{\xi_t} [||\theta_t^2||_2] \\
+ \sum_{t=0}^{k} ||\theta_t^2||_2 + \sum_{t=0}^{k} ||z^t||_2,
\]

\[E_1 : \mathbb{P}\left( 1 \leq 4\lambda \ln \frac{4}{\beta} \right) \geq 1 - \frac{\beta}{2},\]

\[E_2 : 2 \leq \frac{\lambda \ln \frac{4}{\beta}}{3}, \quad E_3 : 3 \leq \frac{2\lambda \ln^2 \frac{4}{\beta}}{81N},\]

\[E_4 : \mathbb{P}\left\{ 4 \leq k\tilde{\sigma} \sqrt{16d \ln \frac{4k}{\beta}} \right\} \geq 1 - \frac{\beta}{2}.\]

Then, we have for each event \(E_i, i = 1, 2, 3, 4, \bigcap_{i=1}^{4} \mathbb{P}(E_i) = 1 - \bigcup_{i=1}^{4} \mathbb{P}(\bar{E}_i) \geq 1 - \beta\), i.e., with probability at least \(1 - \beta\),

\[
\sum_{t=0}^{k} \| \tilde{\nabla} f(x^t, \xi^t) - \nabla f(x^t) \|_2 \leq \lambda \left( 4 \ln \frac{4}{\beta} + \frac{\ln \frac{4}{\beta}}{3} + \frac{2\ln^2 \frac{4}{\beta}}{81N} + \frac{k\tilde{\sigma} \sqrt{16d \ln \frac{4k}{\beta}}}{\lambda} \right).
\]

(27)
6.7 Proof of Theorem 7

Proof. Since \( f(x) \) is convex and its gradient satisfies inequality (10), we have the following inequality with assuming \( x^k \in B_{\gamma R_0} \):

\[
\|x^{k+1} - x^*\|^2_2 = \|x^k - \gamma \nabla f(x^k, \xi^k) - x^*\|^2_2 \\
\leq \|x^k - x^*\|^2_2 + 2\gamma^2 \|\nabla f(x^k, \xi^k)\|^2_2 - 2\gamma \left\langle x^k - x^*, \nabla f(x^k, \xi^k) \right\rangle \\
\leq \|x^k - x^*\|^2_2 + 2\gamma^2 \|\nabla f(x^k, \xi^k)\|^2_2 + 2\gamma^2 \|z^k\|^2_2 - 2\gamma \left\langle x^k - x^*, \theta_k + \nabla f(x^k) + z^k \right\rangle \\
= \|x^k - x^*\|^2_2 + 2\gamma^2 \|\theta_k + \nabla f(x^k)\|^2_2 + 2\gamma^2 \|\nabla f(x^k)\|^2_2 + 2\gamma^2 \|z^k\|^2_2 \\
- 2\gamma \left\langle x^k - x^*, \theta_k \right\rangle - 2\gamma \left\langle x^k - x^*, \nabla f(x^k) \right\rangle - 2\gamma \left\langle x^k - x^*, z^k \right\rangle \\
\leq \|x^k - x^*\|^2_2 + 2\gamma \left( 4\gamma \left( \frac{1}{\alpha} \right)^{\frac{1}{\alpha+\nu}} M_0^{\frac{2}{\alpha+\nu}} - 1 \right) (f(x^k) - f(x^*)) + 4\gamma^2 \|\theta_k\|^2_2 \\
+ 2\gamma^2 \|z^k\|^2_2 - 2\gamma \left\langle x^k - x^*, \theta_k \right\rangle - 2\gamma \left\langle x^k - x^*, z^k \right\rangle + 4\gamma^2 \alpha^{\frac{2\nu}{\alpha+\nu}} M_0^{\frac{2}{\alpha+\nu}},
\]

where \( \theta_k = \nabla f(x^k, \xi^k) - \nabla f(x^k) \) and the last inequality follows from the convexity of \( f \) and Lemma A.5 from (35).

Letting the constant \( M := \left( \frac{1}{\alpha} \right)^{\frac{1}{\alpha+\nu}} M_0^{\frac{2}{\alpha+\nu}} \) and choosing \( \gamma \) such that \( 1 - 4\gamma M > 0 \), together with the notation \( R_k := \|x^k - x^*\|_2, k > 0 \), we obtain that for all \( k \geq 0 \)

\[
R_{k+1}^2 \leq R_k^2 + 2\gamma (4\gamma M - 1) (f(x^k) - f(x^*)) + 4\gamma^2 \|\theta_k\|^2_2 + 2\gamma^2 \|z^k\|^2_2 \\
- 2\gamma \left\langle x^k - x^*, \theta_k \right\rangle - 2\gamma \left\langle x^k - x^*, z^k \right\rangle + 4\gamma^2 \alpha^{\frac{2\nu}{\alpha+\nu}} M_0^{\frac{2}{\alpha+\nu}} \tag{28}
\]

under the assumption \( x^k \in B_{\gamma R_0} \). Let us define \( A = 2\gamma (1 - 4\gamma M) > 0 \), and summing up (28) for \( k = 0, 1, \ldots, N - 1 \), we derive that under the assumption \( x^k \in B_{\gamma R_0} \) for \( k = 0, \ldots, N - 1 \),

\[
A \sum_{k=0}^{N-1} (f(x^k) - f(x^*)) \leq R_0^2 - R_N^2 + 4\gamma^2 \sum_{k=0}^{N-1} \|\theta_k\|^2_2 + 2\gamma^2 \sum_{k=0}^{N-1} \|z^k\|^2_2 + 4\gamma^2 N \alpha^{\frac{2\nu}{\alpha+\nu}} M_0^{\frac{2}{\alpha+\nu}} \tag{29}
- 2\gamma \sum_{k=0}^{N-1} \left\langle x^k - x^*, \theta_k \right\rangle - 2\gamma \sum_{k=0}^{N-1} \left\langle x^k - x^*, z^k \right\rangle.
\]

Taking into account that

\[
A \sum_{k=0}^{N-1} (f(x^k) - f(x^*)) \geq 0,
\]

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and changing the indices in (29), we get that for all \( T = 0, \cdots, N, \)
\[
R_T^2 \leq R_0^2 + 4\gamma^2 \sum_{t=0}^{T-1} \|\theta_t\|_2^2 + 2\gamma^2 \sum_{t=0}^{T-1} \|z^t\|_2^2 + 4\gamma^2 T \alpha \frac{2\nu}{1+\nu} M_{\nu}^2 \tag{30}
\]
\[
- 2\gamma \sum_{t=0}^{T-1} \langle x^t - x^*, \theta_t \rangle - 2\gamma \sum_{t=0}^{T-1} \langle x^t - x^*, z^t \rangle
\]
under assumption that \( x^t \in B_{\gamma R_0} \) for \( t = 0, \cdots, T - 1 \). The remaining part of the proof is based on the analysis of inequality (30). In particular, via induction we prove that for all \( T = 0, \cdots, N \) with probability at least \( 1 - T\beta \), the following statement holds
\[
R_k^2 \leq R_0^2 + 4\gamma^2 \sum_{t=0}^{k-1} \|\theta_t\|_2^2 + 2\gamma^2 \sum_{t=0}^{k-1} \|z^t\|_2^2 + 4\gamma^2 k \alpha \frac{2\nu}{1+\nu} M_{\nu}^2 \tag{31}
\]
\[
- 2\gamma \sum_{t=0}^{k-1} \langle x^t - x^*, \theta_t \rangle - 2\gamma \sum_{t=0}^{k-1} \langle x^t - x^*, z^t \rangle
\]
\[
\leq C^2 R_0^2
\]
for all \( k = 0, \cdots, T \) simultaneously where \( C \) is defined as 7. Let us define the probability event when statement (31) holds as \( E_T \). Then, our goal is to show that \( \mathbb{P}(E_T) \geq 1 - T\beta \) for all \( T = 0, \cdots, N \). Clearly, when \( T = 0 \), inequality (31) holds with probability 1. Next, assuming that for \( T \leq N - 1 \), we have \( \mathbb{P}(E_T) \geq 1 - T\beta \). Let us prove that \( \mathbb{P}(E_{T+1}) \geq 1 - (T + 1)\beta \).

First of all, probability event \( E_T \) implies that \( x^t \in B_{\gamma R_0} \) for \( t = 0, \cdots, T \). Since \( f(\cdot) \) is \((\nu, M_{\nu})\)-Hölder continuous on \( B_{\gamma R_0}(x^*) \), probability event \( E_T \) implies that
\[
\|\nabla f(x^t)\|_2 \leq M_{\nu} \|x^t - x^*\|_2^\nu \leq M_{\nu} C^\nu R_0^\nu \leq \frac{\lambda}{2},
\]
holds for \( t = 0, \cdots, T \).

Next, we introduce new random variables:
\[
\eta_t = \begin{cases} 
  x^* - x^t, & \text{if } \|x^* - x^t\|_2 \leq CR_0 \\
  0, & \text{otherwise}
\end{cases} \tag{32}
\]
for \( t = 0, \cdots, T \). Note that these random variables are bounded with probability 1, i.e. with probability 1 we have
\[
\|\eta_t\|_2 \leq CR_0.
\]

Using the introduced notation, let \( \gamma \leq \frac{R_0}{2 \sqrt{N} \alpha^{1+\nu} M_{\nu}^{1+\nu}} \), we obtain that event \( E_{T+1} \) implies
\[
R_{T+1}^2 \leq 2R_0^2 + 4\gamma^2 \sum_{t=0}^{T} \|\theta_t\|_2^2 + 2\gamma^2 \sum_{t=0}^{T} \|z^t\|_2^2 + 2\gamma \sum_{t=0}^{T} \langle \eta_t, \theta_t \rangle + 2\gamma \sum_{t=0}^{T-1} \langle \eta_t, z^t \rangle \tag{33}
\]
Finally, we do some preliminaries in order to apply Bernstein’s inequality and rewrite:

\[
R_{T+1}^2 \leq 2R_0^2 + 8\gamma^2 \sum_{t=0}^{T} (\|\theta_t^a\|_2^2 - \mathbb{E}_{\xi_t}[\|\theta_t^a\|_2^2]) + 8\gamma^2 \sum_{t=0}^{T} (\mathbb{E}_{\xi_t}[\|\theta_t^a\|_2^2]) + 8\gamma^2 \sum_{t=0}^{T} \|\theta_t^a\|_2^2 \\
+ 2\gamma \sum_{t=0}^{T} \langle \eta_t, \theta_t^a \rangle + 2\gamma \sum_{t=0}^{T} \langle \eta_t, \theta_t^b \rangle + 2\gamma \sum_{t=0}^{T} \langle \eta_t, z_t^1 \rangle + 2\gamma^2 \sum_{t=0}^{T} \|z_t^1\|_2^2
\]

(34)

where we introduce new notations:

\[
\theta_t^a \overset{\text{def}}{=} \nabla f(x_t^i, \xi_t^j) - \mathbb{E}_{\xi_t}[\nabla f(x_t^i, \xi_t^j)], \quad \theta_t^b \overset{\text{def}}{=} \mathbb{E}_{\xi_t}[\nabla f(x_t^i, \xi_t^j)] - \nabla f(x_t^i).
\]

Upper bound for (1). First of all, we notice that the terms in (1) are conditionally unbiased:

\[
\mathbb{E}_{\xi_t}[\|\theta_t^a\|_2^2 - \mathbb{E}_{\xi_t}[\|\theta_t^a\|_2^2]] = 0.
\]

Secondly, the terms are bounded with probability 1:

\[
\|\theta_t^a\|_2^2 - \mathbb{E}_{\xi_t}[\|\theta_t^a\|_2^2] \leq \|\theta_t^a\|_2^2 + \mathbb{E}_{\xi_t}[\|\theta_t^a\|_2^2] \overset{(22)}{=} 8\lambda^2 \overset{\text{def}}{=} c.
\]

Finally, we can bound the conditional variances \(\tilde{\sigma}_t^2 = \mathbb{E}_{\xi_t}[[\|\theta_t^a\|_2^2 - \mathbb{E}_{\xi_t}[\|\theta_t^a\|_2^2]]]\) as follows:

\[
\tilde{\sigma}_t^2 \leq c\mathbb{E}_{\xi_t}[[\|\theta_t^a\|_2^2 - \mathbb{E}_{\xi_t}[\|\theta_t^a\|_2^2]]] \leq c\mathbb{E}_{\xi_t}[[\|\theta_t^a\|_2^2 + \mathbb{E}_{\xi_t}[\|\theta_t^a\|_2^2]] = 2c\mathbb{E}_{\xi_t}[\|\theta_t^a\|_2^2].
\]

Thus, the sequence \(\{\|\theta_t^a\|_2^2 - \mathbb{E}_{\xi_t}[\|\theta_t^a\|_2^2]\}_{t\geq0}\) is a bounded martingale differences sequence with bounded conditional variances \(\{\tilde{\sigma}_t^2\}_{t\geq0}\). Therefore, we can apply Bernstein’s inequality \((41, 42)\) with \(X_t = \|\theta_t^a\|_2^2 - \mathbb{E}_{\xi_t}[\|\theta_t^a\|_2^2], c = 8\lambda^2\) and \(F = \frac{c^2\ln \frac{3}{\beta}}{6}\) and get for all \(b > 0\), it holds that

\[
\mathbb{P}\left\{\sum_{t=0}^{T} X_t > b \land \sum_{t=0}^{T} \tilde{\sigma}_t^2 \leq F\right\} \leq 2\exp\left(-\frac{b^2}{2F + 2cb/3}\right).
\]

Thus, with probability at least \(1 - 2\exp\left(-\frac{b^2}{2F + 2cb/3}\right)\), we have

either \(\sum_{t=0}^{T} X_t \leq b\) or \(\sum_{t=0}^{T} \tilde{\sigma}_t^2 > F\).

Here we choose \(b\) in a way such that \(2\exp\left(-\frac{b^2}{2F + 2cb/3}\right) \leq \frac{\beta}{4}\). This implies that

\[
b^2 - \frac{2c\ln \frac{3}{\beta}}{3} - b - 2F \ln \frac{8}{\beta} \geq 0.
\]

Hence,

\[
b \geq \frac{c\ln \frac{3}{\beta}}{3} + \sqrt{\frac{c^2\ln^2 \frac{8}{\beta}}{9} + 2F \ln \frac{8}{\beta}}.
\]
Next, in order to bound $\sum_{t=0}^{T} \tilde{\sigma}_t^2$ with probability 1, we have the following inequality for $F$

\[
\sum_{t=0}^{T} \tilde{\sigma}_t^2 \leq 2c \sum_{t=0}^{T} \mathbb{E}_{\xi_t}[\|\theta_t^o\|_2^2] \overset{(25)}{\leq} 36c \sigma^2 \sum_{t=0}^{T} \frac{1}{m} \\
= 36c \sigma^2 \frac{T + 1}{m} \overset{T \leq N - 1}{\leq} \frac{c^2 \ln \frac{8}{\beta}}{6} \\
= F.
\]

Therefore, we have shown that with probability at least $1 - \frac{\beta}{4}$, $\sum_{t=0}^{T} |X_t| \leq b$, i.e.,

\[
\sum_{t=0}^{T} ||\theta_t^o||_2^2 - \mathbb{E}_{\xi_t}[||\theta_t^o||_2^2] | \leq b,
\]

where $b = c \ln \frac{8}{\beta} = 8\lambda^2 \ln \frac{8}{\beta}$ as desired.

**Upper bound for (2).**

\[
(2) = \sum_{t=0}^{T} \mathbb{E}_{\xi_t}[||\theta_t^o||_2^2] \overset{(25)}{\leq} 18\sigma^2 \sum_{t=0}^{T} \frac{1}{m} \\
= 18\sigma^2 \frac{T + 1}{m} \overset{T \leq N - 1}{\leq} 18\sigma^2 \frac{N}{m} \\
= \frac{2\lambda^2 \ln \frac{8}{\beta}}{3}.
\]

**Upper bound for (3).**

\[
(3) = \sum_{t=0}^{T} ||\theta_t^o||_2^2 \overset{(23)}{\leq} 16\sigma^4 \sum_{t=0}^{T} \frac{1}{m^2\lambda^2} \\
= 16\sigma^4 \frac{T + 1}{m^2\lambda^2} \overset{T \leq N - 1}{\leq} 16\sigma^4 \frac{N}{m^2\lambda^2} \\
= \frac{16\lambda^2 \ln^2 \frac{8}{\beta}}{729N}.
\]

**Upper bound for (4).** First of all, we notice that the $\mathbb{E}_{\xi_t}[\theta_t^o] = 0$ summands in (4) are conditionally unbiased:

\[
\mathbb{E}_{\xi_t}[\langle \eta_t, \theta_t^o \rangle] = 0.
\]

Secondly, the terms are bounded with probability 1:

\[
|\langle \eta_t, \theta_t^o \rangle| \leq \|\eta_t\|_2 \|\theta_t^o\|_2 \overset{(22)}{\leq} 2\lambda CR_0.
\]

Finally, we can bound the conditional variances $\tilde{\sigma}_t^2 = \mathbb{E}_{\xi_t}[\langle \eta_t, \theta_t^o \rangle^2]$ as follows:

\[
\tilde{\sigma}_t^2 \leq \mathbb{E}_{\xi_t}[\|\eta_t\|_2^2 \|\theta_t^o\|_2^2] \leq (CR_0)^2 \mathbb{E}_{\xi_t}[\|\theta_t^o\|_2^2].
\]
Thus, the sequence \( \{\langle \eta_t, \theta_t^0 \rangle\}_{t \geq 0} \) is a bounded martingale differences sequence with bounded conditional variances \( \{\tilde{\sigma}_t^2\}_{t \geq 0} \). Therefore, we can apply Bernstein’s inequality (41, 42) with \( X_t = \langle \eta_t, \theta_t^0 \rangle \), \( c_1 = 2\lambda CR_0 \) and \( F = \frac{c_1^2 \ln \frac{4}{\beta}}{6} \) and get for all \( b > 0 \), it holds that

\[
P\left\{ \sum_{t=0}^{T} |X_t| > b \text{ and } \sum_{t=0}^{T} \tilde{\sigma}_t^2 \leq F \right\} \leq 2 \exp \left( -\frac{b^2}{2F + 2c_1 b/3} \right).
\]

Thus, with probability at least \( 1 - 2 \exp \left( -\frac{b^2}{2F + 2c_1 b/3} \right) \), we have

either \( \sum_{t=0}^{T} |X_t| \leq b \) or \( \sum_{t=0}^{T} \tilde{\sigma}_t^2 > F \).

Here we choose \( b \) in a way such that \( 2 \exp \left( -\frac{b^2}{2F + 2c_1 b/3} \right) \leq \frac{\beta}{4} \). This implies that

\[
b^2 - \frac{2c_1 \ln \frac{8}{\beta}}{3} b - 2F \ln \frac{8}{\beta} \geq 0.
\]

Hence,

\[
b \geq \frac{c_1 \ln \frac{8}{\beta}}{3} + \sqrt{\frac{c_1^2 \ln^2 \frac{8}{\beta}}{9} + 2F \ln \frac{8}{\beta}}.
\]

Next, in order to bound \( \sum_{t=0}^{T} \tilde{\sigma}_t^2 \) with probability 1, we have the following inequality for \( F \)

\[
\sum_{t=0}^{T} \tilde{\sigma}_t^2 \leq (CR_0)^2 \sum_{t=0}^{T} \mathbb{E} \xi_t \|\theta_t^0\|_2^2 \overset{(25)}{\leq} 18(CR_0)^2 \sigma^2 \sum_{t=0}^{T} \frac{1}{m} \leq 18(CR_0)^2 \sigma^2 \frac{T}{m} + 18 \sigma^2 \frac{T}{m} \leq \frac{c_1^2 \ln \frac{8}{\beta}}{6} + 2F \ln \frac{8}{\beta}
\]

where \( c_1 = 2\lambda CR_0 \).

Therefore, we have shown that with probability at least \( 1 - \frac{\beta}{4} \), \( \sum_{t=0}^{T} |X_t| \leq b \), i.e.,

\[
\sum_{t=0}^{T} |\langle \eta_t, \theta_t^0 \rangle| \leq b,
\]

where \( b = c_1 \ln \frac{8}{\beta} = 2\lambda CR_0 \ln \frac{8}{\beta} \) as desired.

Upper bound for (5).

\[
|\langle \eta_t, \theta_t^0 \rangle| \leq \|\eta_t\|_2 \|\theta_t^0\|_2 \overset{(23)}{\leq} \frac{4\sigma^2}{m\lambda} CR_0.
\]

Thus

\[
(5) = \sum_{t=0}^{T} \langle \eta_t, \theta_t^0 \rangle \leq \sum_{t=0}^{T} |\langle \eta_t, \theta_t^0 \rangle| \overset{T \leq N-1}{\leq} \frac{4\sigma^2 CR_0 N}{m\lambda} \leq \frac{4\lambda CR_0 \ln \frac{8}{\beta}}{27}.
\]

Upper bound for (6).

\[
|\langle \eta_t, z^t \rangle| \leq \|\eta_t\|_2 \|z^t\|_2 \leq CR_0 \|z^t\|_2.
\]
Thus

$$\sum_{t=0}^{T} \langle \eta_t, z^t \rangle \leq \sum_{t=0}^{T} |\langle \eta_t, z^t \rangle| \leq CR_0 \sum_{t=0}^{T} \|z^t\|_2$$

Next we bound the $\sum_{t=0}^{T} \|z^t\|_2$. It is easy to verify that a given random Gaussian noise $z \in \mathbb{R}^d, z \sim \mathcal{N}(0, \hat{\sigma}^2 I^d)$ is subG($\hat{\sigma}^2$). Together with (26) in the Lemma 3, the following holds

$$\mathbb{P}(\|z\|_2 \geq \hat{\ell}) \leq 2e^{-\hat{\ell}^2 \hat{c}^2 d / \hat{\sigma}^2},$$

where constant $\hat{c}$ is selected as $2\sqrt{2}$ as the derivations in (40).

To fit in our case, we choose $\hat{\ell}$ such that $2\exp \left[ -\frac{\hat{\ell}^2}{2\hat{c}^2 d \hat{\sigma}^2} \right] = \frac{\beta}{4(T+1)}$. That is, we choose $\hat{\ell} = \hat{\sigma} \sqrt{16d \ln \frac{8(T+1)}{\beta}}$ so that with the probability at least $1 - \frac{\beta}{4}$,

$$\sum_{t=0}^{T} \|z^t\|_2 \leq (T+1)\hat{\sigma} \sqrt{16d \ln \frac{8(T+1)}{\beta}}.$$

Therefore, we bound the 6 with the probability at least $1 - \frac{\beta}{4}$,

$$\sum_{t=0}^{T} \|z^t\|_2 \leq (T+1)\hat{\sigma} \sqrt{16d \ln \frac{8(T+1)}{\beta}} \\ \leq CR_0 \hat{\sigma} \sqrt{16d \ln \frac{8(T+1)}{\beta}} \leq CR_0 \hat{\sigma} \sqrt{16d \ln \frac{8(T+1)}{\beta}} \leq \frac{108CR_0N^{2.5} \sigma^2 \sqrt{d \ln \frac{8N}{\beta} \ln \frac{1}{\beta}}}{n \epsilon \lambda \ln \frac{8}{\beta}} \leq \frac{272^* \sigma^2 \sqrt{d \ln \frac{8N}{\beta} \ln \frac{1}{\beta}}}{n \epsilon \lambda \ln \frac{8}{\beta}}.$$
bounds, we have derived

$$R_{T+1}^2 \leq 2R_0^2 + 8\gamma^2 \sum_{t=0}^{T} (\|\theta_t^o\|_2^2 - \mathbb{E}_{\xi_t}[\|\theta_t^o\|_2^2]) + 8\gamma^2 \sum_{t=0}^{T} (\mathbb{E}_{\xi_t}[\|\theta_t^o\|_2^2]) + 8\gamma^2 \sum_{t=0}^{T} \|\theta_t^e\|_2^2$$

$$+ 2\gamma \sum_{t=0}^{T} \langle \eta_t, \theta_t^o \rangle + 2\gamma \sum_{t=0}^{T} \langle \eta_t, \theta_t^e \rangle + 2\gamma \sum_{t=0}^{T} \langle \eta_t, z_t^e \rangle + 2\gamma^2 \sum_{t=0}^{T} \|z_t^e\|_2^2,$$

(35)

$$E_1 : \mathbb{P}\left\{ \frac{1}{2} \leq 64\gamma^2\lambda^2 \ln \frac{8}{\beta^2} \right\} \geq 1 - \frac{\beta}{4},$$

$$E_2 : \frac{16\gamma^2\lambda^2 \ln \frac{8}{\beta^2}}{3} \leq \frac{128\gamma^2\lambda^2 \ln^2 \frac{8}{\beta^2}}{729N},$$

$$E_4 : \mathbb{P}\left\{ \frac{4}{4} \leq 4\gamma\lambda CR_0 \ln \frac{8}{\beta} \right\} \geq 1 - \frac{\beta}{4},$$

$$E_5 : \frac{8\gamma\lambda CR_0 \ln \frac{8}{\beta}}{27},$$

$$E_6 : \mathbb{P}\left\{ \frac{2\gamma\lambda CR_0 DN}{\lambda} \right\} \geq 1 - \frac{\beta}{4},$$

$$E_7 : \mathbb{P}\left\{ 7 \leq \frac{2\gamma^2 D_2^2 N}{\lambda^2} \right\} \geq 1 - \frac{\beta}{4}.$$

Taken into account these inequalities and our assumptions on $\lambda$ and $\gamma$, we get that probability event Then, we have for each event $E_i \cap E_T$ implies

$$R_{T+1}^2 \leq R_0^2 + 4\gamma^2 \sum_{k=0}^{T} \|\theta_k^o\|_2^2 + 2\gamma^2 \sum_{k=0}^{T} \|z_k^e\|_2^2 + 4\gamma^2 \sum_{k=0}^{T} \|z_k^e\|_2^2 + 4\gamma^2 T^2 \alpha^{\frac{2v}{1+v}} M_\nu^{\frac{2v}{1+v}}$$

$$- 2\gamma \sum_{k=0}^{T} \langle x_k - x^*, \theta_k \rangle - 2\gamma \sum_{k=0}^{T} \langle x_k - x^*, z_k \rangle$$

$$\leq 2R_0^2 + \left( \frac{8}{49} + \frac{2}{147} + \frac{32}{35721} + \frac{2}{7} + \frac{4}{189} + \frac{1}{7} + \frac{1}{98} \right) C^2 R_0^2$$

$$\leq C^2 R_0^2.$$

Moreover, using the union bound we derive

$$\mathbb{P}\left( \bigcap_{i \in \{1,4,6,7\}} E_i \cap E_T \right) = 1 - \mathbb{P}\left( \bigcap_{i \in \{1,4,6,7\}} \bar{E}_i \cap \bar{E}_T \right) \geq 1 - (T + 1)\beta.$$

That is, by definition of $E_{T+1}$ and $E_T$ we have proved that

$$\mathbb{P}(E_{T+1}) \geq \mathbb{P}\left( \bigcap_{i \in \{1,4,6,7\}} E_i \cap E_T \right) \geq 1 - (T + 1)\beta,$$

which implies that for all $T = 0, \cdots, N$ we have $E_T \geq 1 - T\beta$. Then for $T = N$, we have that with
probability at least $1 - N\beta$,

$$AN(f(\bar{x}^N) - f(x^*)) \leq A \sum_{k=0}^{N-1} (f(x^k) - f(x^*))$$

\[(29)\]

\[\leq 2R_0^2 + 4\gamma^2 \sum_{k=0}^{N-1} \|\theta_k\|_2^2 + 2\gamma^2 \sum_{k=0}^{N-1} \|z^k\|_2^2 - 2\gamma \sum_{k=0}^{N-1} \langle x^k - x^*, \theta_k \rangle - 2\gamma \sum_{k=0}^{N-1} \langle x^k - x^*, z^k \rangle\]

\[(37)\]

\[\leq C^2 R_0^2,
\]

where the first inequality follows from Jensen’s inequality that $f(\bar{x}^N) = f(\frac{1}{N} \sum_{k=0}^{N-1} x^k) \leq \frac{1}{N} \sum_{k=0}^{N-1} f(x^k)$. Since $A = 2\gamma(1 - 4\gamma M) > \gamma$, we get that with probability at least $1 - N\beta$,

$$f(\bar{x}^N) - f(x^*) \leq \frac{C^2 R_0^2}{AN} \leq \frac{C^2 R_0^2}{\gamma N}.$$

When

$$\gamma = \min \left\{ \frac{1-\nu}{8M_0^2 \nu^{1+\nu}}, \frac{R_0}{\sqrt{2N} \alpha^{-1+\nu} M_0^{-1+\nu}}, \frac{R_0}{2\lambda \ln \frac{8}{\beta}}, \frac{1}{2DN} \right\}, \quad \lambda = 2M_\nu C^\nu R_0^\nu, \quad m = \max \left\{ 1, \frac{27N\sigma^2}{\lambda^2 \ln \frac{8}{\beta}} \right\},$$

we have that with probability at least $1 - N\beta$,

$$f(\bar{x}^N) - f(x^*) \leq \max \left\{ \frac{8C^2 R_0^2 \nu^{2+\nu}}{\alpha^{-1+\nu} N}, \frac{\sqrt{2C^2 M_\nu^{-1+\nu} R_0^{-1+\nu}}}{\nu}, \frac{4C^{2+\nu} M_\nu R_0^{1+\nu} \ln \frac{8}{\beta}}{N}, \frac{108C^{2-\nu} R_0^{-1+\nu} N^{1.5}\sigma^2 \sqrt{\ln \frac{8N}{\beta} \ln \frac{1}{\delta}}}{nM_\nu \epsilon \ln \frac{8}{\beta}} \right\}.$$

If taking $N = \tilde{O} \left( \frac{ne}{\sqrt{d \ln \frac{1}{\delta}}} \right)^{\frac{1}{\delta}}$, we have that with probability at least $1 - N\beta$,

$$f(\bar{x}^N) - f(x^*) \leq \tilde{O} \left( \max \left\{ \frac{M_\nu^{2+\nu} (d \ln \frac{1}{\delta})^{\frac{1}{2}}}{(ne)^{\frac{1}{2}}}, \frac{M_\nu (d \ln \frac{1}{\delta})^{\frac{1}{2}}}{(ne)^{\frac{1}{2}}}, \frac{M_\nu^{1+\nu} \alpha^{-1+\nu} (d \ln \frac{1}{\delta})^{\frac{1}{1+\nu}}}{(ne)^{\frac{1}{1+\nu}}}, \frac{(d \ln \frac{1}{\delta})^{\frac{2}{2}} \sqrt{\ln \frac{1}{\beta d \nu}}}{(ne)^{\frac{2}{2}} M_\nu} \right\} \right).$$

The total gradient complexity is $\tilde{O} \left( \max \left\{ n^{\frac{1}{2}}, n^{\frac{1}{2}} d^2 \right\} \right)$, and the $\tilde{O}$ notation here omits other logarithmic factors and the terms $\sigma, \alpha, R_0$. 

\[\square\]
7 Detailed Description of Experiments

We run 30 epochs for DP-GD, T2, T3 and non-private CSGD. We fix the batchsize \( m \) to 200 and tune the stepsizes and clipping parameters \( \lambda \). We select the \( \sigma \) to 1. Confidence level is set to 0.01. Correspondingly, stepsizes are set to \( \frac{1}{24L \ln \frac{4}{\delta}} \) for DP-GD, T2 and CSGD and \( \frac{1}{24L \ln \frac{4N}{\delta}} \) for T3. \( \lambda \) is set to 0.54, 0.87 and 0.63 for T2, T3 and non-private CSGD, respectively.

Corresponding to Figure 1 and 2, we give the results which mark the difference between the best and the worst performances as follows.

![Figure 3: Trajectories of DP-GD, clipped-dpSGD of Theorem 2 and Theorem 3 and non-private CSGD on Adult dataset for ridge regression.](image)

(a) \( \epsilon = 0.5 \)  (b) \( \epsilon = 0.75 \)  (c) \( \epsilon = 1.0 \)  (d) \( \epsilon = 2.0 \)

![Figure 4: Trajectories of DP-GD, clipped-dpSGD of Theorem 2 and Theorem 3 and non-private CSGD on Adult dataset for logistic regression.](image)

(a) \( \epsilon = 0.5 \)  (b) \( \epsilon = 0.75 \)  (c) \( \epsilon = 1.0 \)  (d) \( \epsilon = 2.0 \)