THE RESTRICTED ALGEBRAS ON INVERSE SEMIGROUPS III,
FOURIER ALGEBRA

MASSOUD AMINI, ALIREZA MEDGHALCHI

Abstract. The Fourier and Fourier-Stieltjes algebras $A(G)$ and $B(G)$ of a locally compact group $G$ are introduced and studied in 60's by Pierre Eymard in his PhD thesis. If $G$ is a locally compact abelian group, then $A(G) \cong L^1(\hat{G})$, and $B(G) \cong M(\hat{G})$, via the Fourier and Fourier-Stieltjes transforms, where $\hat{G}$ is the Pontryagin dual of $G$. Recently these algebras are defined on a (topological or measured) groupoid and have shown to share many common features with the group case. This is the last in a series of papers in which we have investigated a “restricted” form of these algebras on a unital inverse semigroup $S$.

1. Introduction.

In [1] and [2] we introduced the concept of restricted representations on an inverse semigroup and studied the restricted versions of positive definite functions, semigroup algebra, and semigroup $C^*$-algebras.

In this paper, our aim is to study the restricted Fourier and Fourier-Stieltjes algebras $A(S)$ and $B(S)$ on an inverse semigroup $S$. In particular, we prove restricted version of the Eymard’s characterization [5] of the Fourier algebra (Theorem 2.1).

The structure of algebras $B(S)$ and $A(S)$ is far from being well understood, even in special cases. From the results of [4], [7], it is known that for a commutative unital discrete $*$-semigroup $S$, $B(S) = M(\hat{S})$ via Bochner theorem [7]. Even in this case, the structure of $A(S)$ seems to be much more complicated than the group case. This is mainly because of the lack of an appropriate analog of the group algebra. If $S$ is a discrete idempotent semigroup with identical involution. Then $\hat{S}$ is a compact topological semigroup with pointwise multiplication. We believe that in this case $A(S) = L(\hat{S})$ where $L(\hat{S})$ is the Baker algebra on $\hat{S}$ (see for instance [8]), however we are not able to prove it at this stage.
All over this paper, \( S \) denotes a unital inverse semigroup with identity 1. \( E \) denotes the set of idempotents of \( S \) consists of elements the form \( ss^* \), \( s \in S \). \( \Sigma = \Sigma(S) \) is the family of all \(*\)-representations \( \pi \) of \( S \) with
\[
\|\pi\| := \sup_{x \in S} \|\pi(x)\| \leq 1.
\]
The associated groupoid of \( S \) is denoted by \( S_a [9] \). If we adjoin a zero element 0 to this groupoid, and put \( 0^* = 0 \), we get an inverse semigroup \( S_r \) which is called the restricted semigroup of \( S \). A restricted representation \( \{\pi, \mathcal{H}_\pi\} \) of \( S \) is a map \( \pi : S \to \mathcal{B}(\mathcal{H}_\pi) \) such that
\[
\pi(x)\pi(y) = \begin{cases} 
\pi(xy) & \text{if } x^*x = yy^* \ (x, y \in S) \\
0 & \text{otherwise}
\end{cases}
\]
\( \Sigma_r = \Sigma_r(S) \) denotes the family of all restricted representations \( \pi \) of \( S \) with \( \|\pi\| = \sup_{x \in S} \|\pi(x)\| \leq 1 \). Two basic examples of restricted representations are the restricted left and right regular representations \( \lambda_r \) and \( \rho_r \) of \( S [1] \).

2. Restricted Fourier and Fourier-Stieltjes algebras

Let \( S \) be a unital inverse semigroup and \( P(S) \) be the set of all bounded positive definite functions on \( S \) (see [6] for the group case and [2] for inverse semigroups). Following the notations of [2], we use the notation \( P(S) \) with indices \( r, e, f \), and 0 to denote the positive definite functions which are restricted, extendible, of finite support, or vanishing at zero, respectively. Let \( B(S) \) be the linear span of \( P(S) \). Then \( B(S) \) is a commutative Banach algebra with respect to the pointwise multiplication and the following norm \([3],[13]\)
\[
\|u\| = \sup \{ \| \sum_{x \in S} u(x)f(x) \| : f \in \ell^1(S), \sup_{\pi \in \Sigma(S)} \| \tilde{\pi}(f) \| \leq 1 \} \ (u \in B(S)).
\]
Also \( B(S) \) coincides with the set of the coefficient functions of elements of \( \Sigma(S) [1] \).
If one wants to get a similar result for the set of coefficient functions of elements of \( \Sigma_r(S) \), one has to apply the above facts to \( S_r \). But \( S_r \) is not unital in general, so one is led to consider a smaller class of bounded positive definite functions on \( S_r \). The results of [10] suggests that these should be the class of extendible positive definite functions on \( S \). Among these, those which vanish at 0 correspond to elements of \( P_{r,e}(S) \).

In this section we show that the linear span \( B_{r,e}(S) \) of \( P_{r,e}(S) \) is a commutative Banach algebra with respect to the pointwise multiplication and an appropriate
modification of the above norm. We call this the restricted Fourier-Stieltjes algebra of $S$ and show that it coincides with the set of all coefficient functions of elements of $\Sigma_r(S)$.

As before, the indices $e$, 0, and $f$ is used to distinguish extendible elements, elements vanishing at 0, and elements of finite support, respectively. We freely use any combination of these indices. Consider the linear span of $P_{r,e,f}(S)$ which is clearly a two-sided ideal of $B_{r,e}(S)$, whose closure $A_{r,e}(S)$ is called the restricted Fourier algebra of $S$. We show that it is a commutative Banach algebra under pointwise multiplication and norm of $B_r(S)$. We also show that it is the predual of the von Neumann algebra.

In order to study properties of $B_{r,e}(S)$, we are led by Proposition 5.1 to consider $B_{0,e}(S)$. More generally we calculate this algebra for any inverse 0-semigroup $T$. Let $B_e(T)$ be the linear span of $P_e(T)$ with pointwise multiplication and the norm

$$\|u\| = \sup\{ |\sum_{x \in T} f(x)u(x)| : f \in \ell^1(T), \|f\|_{\Sigma(T)} \leq 1 \} \quad (u \in B_e(T)),$$

and $B_{0,e}(T)$ be the closed ideal of $B_e(T)$ consisting of elements vanishing at 0. First let us show that $B_e(T)$ is complete in this norm. The next lemma is quite well known and follows directly from the definition of the functional norm.

**Lemma 2.1.** If $X$ is a Banach space, $D \subseteq X$ is dense, and $f \in X^*$, then

$$\|f\| = \sup\{|f(x)| : x \in D, \|x\| \leq 1\}.$$

**Lemma 2.2.** If $T$ is an inverse 0-semigroup (not necessarily unital), then we have the following isometric isomorphism of Banach spaces:

(i) $B_e(T) \simeq C^*(T)^*$,

(ii) $B_{0,e}(T) \simeq (C^*(T)/\mathbb{C}\delta_0)^*$.

In particular $B_e(T)$ and $B_{0,e}(T)$ are Banach spaces.

**Proof** (ii) clearly follows from (i). To prove (i), first recall that $P_e(S)$ is affinely isomorphic to $\ell^1(S)_+^*$ [10, 1.1] via

$$<u, f> = \sum_{x \in S} f(x)u(x) \quad (f \in \ell^1(S), u \in P_e(S)).$$

This defines an isometric isomorphism $\tau_0$ from $B_e(T)$ into $\ell^1(T)^*$ (with the dual norm). By above lemma, one can lift $\tau_0$ to an isometric isomorphism $\tau$ from $B_e(T)$ into $C^*(T)^*$. We only need to check that $\tau$ is surjective. Take any $v \in C^*(T)$, and
let $w$ be the restriction of $v$ to $\ell^1(T)$. Since $\|f\|_{\Sigma(T)} \leq \|f\|_1$, for each $f \in \ell^1(T)$, The norm of $w$ as a linear functional on $\ell^1(T)$ is not bigger than of the norm of $v$ as a functional on $C^*(T)$. In particular, $w \in \ell^1(T)^*$ and so there is a $u \in B_\tau(T)$ with $\tau_0(u) = w$. Then $\tau(u) = v$, as required. 

According to Notation 2.1 of [2], we know that the restriction map $\tau : B_{0,e}(S_r) \to B_{r,e}(S)$ is a surjective linear isomorphism. Also $\tau$ is clearly an algebra homomorphism ($B_{0,e}(S_r)$ is an algebra under pointwise multiplication [10, 3.4], and the surjectivity of $\tau$ implies that the same fact holds for $B_{r,e}(S)$). Now we put the following norm on $B_r(S)$,

$$\|u\|_r = \sup\{ \left| \sum_{x \in S} f(x)u(x) \right| : f \in \ell^1_r(S), \|f\|_{\Sigma_r(S)} \} \quad (u \in B_r(S)),$$

then using the fact that $B_{0,e}(S_r)$ is a Banach algebra (it is a closed subalgebra of $B(S_r)$ which is a Banach algebra [3, Theorem 3.4]) we have

**Lemma 2.3.** The restriction map $\tau : B_{0,e}(S_r) \to B_{r,e}(S)$ is an isometric isomorphism of normed algebras. In particular, $B_{r,e}(S)$ is a commutative Banach algebra under pointwise multiplication and above norm.

**Proof** The second assertion follows from the first and Lemma 2.2 applied to $T = S_r$. For the first assertion, we only need to check that $\tau$ is an isometry. But this follows directly from [1, Proposition 3.3] and the fact that $\Sigma_r(S) = \Sigma_0(S_r)$. 

**Corollary 2.1.** $B_{r,e}(S)$ is the set of coefficient functions of elements of $\Sigma_r(S)$.

**Proof** Given $u \in P_{r,e}(S)$, let $v$ be the extension by zero of $u$ to a function on $S_r$, then $v \in P_{0,e}(S_r)$, so there is a cyclic representation $\pi \in \Sigma(S_r)$, say with cyclic vector $\xi \in \mathcal{H}_\pi$, such that $v = < \pi(\cdot)\xi, \xi >$ (see the proof of [10, 3.2]). But

$$0 = v(0) = < \pi(0)\xi, \xi >= < \pi(0^*0)\xi, \xi >= \|\pi(0)\xi\|,$$

that is $\pi(0)\xi = 0$. But $\xi$ is the cyclic vector of $\pi$, which means that for each $\eta \in \mathcal{H}_\pi$, there is a net of elements of the form $\sum_{i=1}^n c_i \pi(x_i)\xi$, converging to $\eta$ in the norm topology of $\mathcal{H}_\pi$, and

$$\pi(0)(\sum_{i=1}^n c_i \pi(x_i)\xi) = \sum_{i=1}^n c_i \pi(0)\xi = 0,$$

so $\pi(0)\eta = 0$, and so $\pi(0) = 0$. This means that $\pi \in \Sigma_0(S_r) = \Sigma_r(S)$. Now a standard argument, based on the fact that $\Sigma_r(S) = \Sigma_0(S_r)$ is closed under direct sum, shows that each $u \in B_{r,e}(S)$ is a coefficient function of some element of $\Sigma_r(S)$. The converse follows from [2, Lemma 2.6].
Corollary 2.2. We have the isometric isomorphism of Banach spaces $B_{r,e}(S) \simeq C_r(S)^*$.

Proof We have the following of isometric linear isomorphisms: first $B_{r,e}(S) \simeq B_{0,e}(S_r)$ (Lemma 2.3), then $B_{0,e}(S_r) \simeq (C^*(S_r)/\mathbb{C}\delta_0)^*$ (Lemma 2.2, applied to $T = S_r$), and finally $C_r^*(S) \simeq C^*(S_r)/\mathbb{C}\delta_0$ [1, Proposition 4.2].

Next, as in [7], we give an alternative description of the norm of the Banach algebra $B_{r,e}(S)$. For this we need to know more about the universal representation of $S_r$. Applying the discussion before Example 2.1 in [2] to $T = S_r$, we know that the universal representation $\omega$ of $S_r$ is the direct sum of all cyclic representations corresponding to elements of $P_r(S_r)$. To be more precise, this means that given any $u \in P_r(S_r)$ we consider $u$ as a positive linear functional on $\ell^1(S_r)$, then by [10, 21.24], there is a cyclic representation $\{\tilde{\pi}_u, H_u, \xi_u\}$ of $\ell^1(S_r)$, with $\pi_u \in \Sigma(S_r)$, such that

$$< u, f > = \langle \tilde{\pi}_u(f) \xi_u, \xi_u \rangle \quad (f \in \ell^1(S_r)).$$

Therefore $\pi_u$ is a cyclic representation of $S_r$ and $u = < \pi_u(.) \xi_u, \xi_u >$ on $S_r$. Now $\omega$ is the direct sum of all $\pi_u$’s, where $u$ ranges over $P_r(S_r)$. There is an alternative construction in which one can take the direct sum of $\pi_u$’s with $u$ ranging over $P_{0,e}(S_r)$ to get a subrepresentation $\omega_0$ of $\omega$. Clearly $\omega \in \Sigma(S_r)$ and $\omega_0 \in \Sigma_0(S_r)$. It follows from [10,3.2] that the set of coefficient functions of $\omega$ and $\omega_0$ are $B_e(S_r)$ and $B_{0,e}(S_r) = B_{r,e}(S_r)$, respectively (c.f. Notation 2.1 in [2]). As far as the original semigroup $S$ is concerned, we prefer to work with $\omega_0$, since it could be considered as an element of $\Sigma_r(S)$. Now $\tilde{\omega}_0$ is a non degenerate $*$-representation of $\ell^1_r(S)$ which uniquely extends to a non degenerate representation of the restricted full $C^*$-algebra $C^*_r(S)$, which we still denote by $\tilde{\omega}_0$. We gather some of the elementary facts about $\tilde{\omega}_0$ in the next lemma.

Lemma 2.4. With the above notation, we have the following:

(i) $\tilde{\omega}_0$ is the direct sum of all non degenerate representations $\pi_u$ of $C^*_r(S)$ associated with elements $u \in C^*_r(S)^*_+$ via the GNS-construction, namely $\tilde{\omega}_0$ is the universal representation of $C^*_r(S)$. In particular, $C^*_r(S)$ is faithfully represented in $H_{\omega_0}$.

(ii) The von Neumann algebras $C^*_r(S)^{**}$ and the double commutant of $C^*_r(S)$ in $B(H_{\omega_0})$ are isomorphic. They are generated by elements $\tilde{\omega}_0(f)$, with $f \in \ell^1_r(S)$, as well as by elements $\omega_0(x)$, with $x \in S$.

(iii) Each representation $\pi$ of $C^*_r(S)$ uniquely decomposes as $\pi = \pi^{**} \circ \omega_0$. 
(iv) For each \( \pi \in \Sigma_r(S) \) and \( \xi, \eta \in \mathcal{H}_\pi \), let \( u = \langle \pi(\cdot)\xi, \eta \rangle \), then \( u \in C_r(S)^* \) and
\[
< T, u > = \langle \hat{T}\circ \pi(\cdot)\xi, \eta \rangle \quad (T \in C_r(S)^*).
\]

**Proof** Statement (i) follows by an standard argument. Statement (iii) and the first part of (ii) follow from (i) and the second part of (ii) follows from the fact that both set of elements described in (ii) have clearly the same commutant in \( \mathcal{B}(\mathcal{H}_\omega_0) \) as the set of elements \( \tilde{\omega}_0(u) \), with \( u \in C_r(S)^* \) which generate \( C_r(S)^{''} \). The first statement of (iv) follows from [2, Lemma 2.6] and Corollary 2.2. As for the second statement, first note that for each \( f \in \ell^1_r(S) \), \( \tilde{\omega}_0(f) \) is the image of \( f \) under the canonical embedding of \( C_r(S)^* \) in \( C_r(S)^{**} \). Therefore, by (iii),
\[
< \tilde{\omega}_0(f), u > = \langle u, f \rangle = \sum_{x \in S} f(x)u(x)
= \langle \hat{\pi}(f)\xi, \eta \rangle = \langle \hat{\pi}\circ \tilde{\omega}_0(f)\xi, \eta \rangle.
\]
Taking limit in \( \|f\|_{\Sigma_r} \) we get the same relation for any \( f \in C_r(S)^* \), and then, using (ii), by taking limit in the ultraweak topology of \( C_r(S)^{**} \), we get the desired relation. \( \square \)

**Lemma 2.5.** Let \( 1 \) be the identity of \( S \), then for each \( u \in P_{r,e}(S) \) we have \( \|u\|_r = u(1) \).

**Proof** As \( \|\delta_e\|_{\Sigma_r} = 1 \) and \( u(1) = \lambda_r(1)u(1) \geq 0 \), we have \( \|u\|_r \geq |u(1)| = u(1) \). Conversely, by the proof of Corollary 2.1, there is \( \pi \in \Sigma_r(S) \) and \( \xi \in \mathcal{H}_\pi \) such that \( u = \langle \pi(\cdot)\xi, \xi \rangle \). Hence \( u(1) = \langle \pi(1)\xi, \xi \rangle = \|\xi\|^2 \geq \|u\|_r \). \( \square \)

**Lemma 2.6.** For each \( \pi \in \Sigma_r(S) \) and \( \xi, \eta \in \mathcal{H}_\pi \), consider \( u = \langle \pi(\cdot)\xi, \eta \rangle \in B_{r,e}(S) \), then \( \|u\|_r \leq \|\xi\|\|\eta\| \). Conversely each \( u \in B_{r,e}(S) \) is of this form and we may always choose \( \xi, \eta \) so that \( \|u\|_r = \|\xi\|\|\eta\| \).

**Proof** The first assertion follows directly from the definition of \( \|u\|_r \) (see the paragraph after Lemma 2.2). The first part of the second assertion is the content of Corollary 2.1. As for the second part, basically the proof goes as in [5]. Consider \( u \) as an element of \( C_r(S)^* \) and let \( u = v, |u| \) be the polar decomposition of \( u \), with \( v \in C_r(S)^{**} \) and \( |u| \in C_r(S)^*_+ = P_{r,e}(S) \), and the dot product is the module action of \( C_r(S)^* \) on \( C_r(S)^* \) [5]. Again, by the proof of Corollary 2.1, there is a cyclic representation \( \pi \in \Sigma_r(S) \), say with cyclic vector \( \eta \), such that \( |u| = \langle \pi(\cdot)\eta, \eta \rangle \).
Put \( \xi = \tilde{\pi}^{**}(v)\eta \), then \( \| \xi \| \leq \| \eta \| \) and by Lemma 2.4 (iv) applied to \( |u| \),
\[
u(x) = < \omega_0(x), u > = < \omega_0(x)v, |u| > = < \tilde{\pi}^{**} \circ \omega_0(x)v \eta, \eta > = < \pi(x)\xi, \eta >,
\]
and, by Corollary 2.2 and Lemma 2.5,
\[
\| u \|_r = \| |u| \| = |u|(1) = \| \eta \|_2 \geq \| \xi \|.
\]
□

Note that the above lemma provides an alternative (direct) way of proving the second statement of Lemma 2.3 (just take any two elements \( u, v \) in \( B_{r,e}(S) \) and represent them as coefficient functions of two representations such that the equality hold for the norms of both \( u \) and \( v \), then use the tensor product of those representations to represent \( uv \) and apply the first part of the lemma to \( uv \).) Also it gives the alternative description of the norm on \( B_{r,e}(S) \) as follows:

**Corollary 2.3.** For each \( u \in B_{r,e}(S) \),
\[
\| u \|_r = \inf \{ \| \xi \|_2 \| \eta \| : \xi, \eta \in H_\pi, \pi \in \Sigma_r(S), u = < \pi(.), \xi > \}.
\]
□

**Corollary 2.4.** For each \( u \in B_{r,e}(S) \),
\[
\| u \|_r = \sup \{ \| \sum_{n=1}^\infty c_n u(x_n) \| : c_n \in \mathbb{C}, x_n \in S (n \geq 1), \| \sum_n c_n \delta_{x_n} \|_{\Sigma_r} \leq 1 \}.
\]

**Proof** Just apply Kaplansky’s density theorem to the unit ball of \( C_r^*(S)^{**} \). □

**Corollary 2.5.** The unit ball of \( B_{r,e}(S) \) is closed in the topology of pointwise convergence.

**Proof** If \( u \in B_{r,e}(S) \) with \( \| u \|_r \leq 1 \), then for each \( n \geq 1 \), each \( c_1, \ldots, c_n \in \mathbb{C} \), and each \( x_1, \ldots, x_n \in S \),
\[
\| \sum_{k=1}^n c_k u(x_k) \| \leq \| \sum_{k=1}^n c_k \delta_{x_k} \|_{\Sigma_r}.
\]
If \( u_\alpha \to u \), pointwise on \( S \) with \( u_\alpha \in B_{r,e}(S) \), \( \| u_\alpha \|_r \leq 1 \), for each \( \alpha \), then all \( u_\alpha \)'s satisfy above inequality, and so does \( u \). Hence, by above corollary, \( u \in B_{r,e}(S) \) and \( \| u \|_r \leq 1 \). □

**Lemma 2.7.** For each \( f, g \in \ell^2(S) \), \( f \cdot \tilde{g} \in B_{r,e}(S) \) and \( if \cdot \|_r \) is the norm of \( B_{r,e}(S) \), \( \| f \cdot \tilde{g} \|_r \leq \| f \|_2 \cdot \| g \|_2 \).
Theorem 2.1. Consider the following sets:

\[ E_1 = \langle f \cdot \tilde{g} : f, g \in \ell_f^2(S) \rangle, \]
\[ E_2 = \langle h \cdot \tilde{h} : h \in \ell_f^1(S) \rangle, \]
\[ E_3 = \langle P_{r,e,f}(S) \rangle, \]
\[ E_4 = \langle P(S) \cap \ell^2(S) \rangle, \]
\[ E_5 = \langle h \cdot \tilde{h} : h \in \ell^2(S) \rangle, \]
\[ E_6 = \langle f \cdot \tilde{g} : f, g \in \ell^2(S) \rangle. \]

Then \( E_1 \subseteq E_2 \subseteq E_3 \subseteq E_4 \subseteq E_5 \subseteq E_6 \subseteq B_{r,e}(S) \) and the closures of all of these sets in \( B_{r,e}(S) \) are equal to \( A_{r,e}(S) \).

Proof The inclusion \( E_1 \subseteq E_2 \) follows from [2, Lemma 2.3], and the inclusions \( E_2 \subseteq E_3 \) and \( E_4 \subseteq E_5 \) follow from [2, Theorem 2.1]. The inclusions \( E_3 \subseteq E_4 \) and \( E_5 \subseteq E_6 \) are trivial. Now \( E_1 \) is dense in \( E_6 \) by lemma 2.7 and the fact that \( \ell_f^2(S) \) is dense in \( \ell^2(S) \). Finally \( \tilde{E}_3 = A_{r,e}(S) \), by definition, and \( E_3 \subseteq E_2 \subseteq E_1 \) by [2, Theorem 2.1], hence \( \tilde{E}_i = A_{r,e}(S) \), for each \( 1 \leq i \leq 6 \).

Lemma 2.8. \( P_{r,e}(S) \) separates the points of \( S \).

Proof We know that \( S_r \) has a faithful representation (namely the left regular representation \( \Lambda \)), so \( P_r(S_r) \) separates the points of \( S_r \) [10, 3.3]. Hence \( P_{0,e}(S_r) = P_{r,e}(S) \) separates the points of \( S_r \backslash \{0\} = S \).
Proposition 2.1. For each \( x \in S \) there is \( u \in A_{r,e}(S) \) with \( u(x) = 1 \). Also \( A_{r,e}(S) \) separates the points of \( S \).

**Proof** Given \( x \in S \), let \( u = \delta_{(x^*x) \in E_1} \subseteq A_{r,e}(S) \), then \( u(x) = 1 \). Also given \( y \neq x \) and \( u \) as above, if \( u(y) \neq 1 \), then \( u \) separates \( x \) and \( y \). If \( u(y) = 1 \), then use above lemma to get some \( v \in B_{r,e}(S) \) which separates \( x \) and \( y \). Then \( u(x) = u(y) = 1 \), so \( (uv)(x) = v(x) \neq u(y) = (uv)(y) \), i.e. \( uv \in A_{r,e}(S) \) separates \( x \) and \( y \). \( \square \)

Proposition 2.2. For each finite subset \( K \subseteq S \), there is \( u \in P_{r,e,f}(S) \) such that \( u|_K \equiv 1 \).

**Proof** For \( F \subseteq S \), let \( F_e = \{ x^*x : x \in F \} \) and note that \( F \subseteq F \cdot F_e \) (since \( x = x(x^*x) \), for each \( x \in F \)). Now given a finite set \( K \subseteq S \), put \( F = K \cup K^* \cup K_e \), then since \( K_e = K_e^* \) we have \( F = F^* \), and since \( K_e = (K^*)_e \) and \( (K_e)_e = K_e \) we have \( F_e \subseteq F \). Hence \( K \subseteq F \subseteq F \cdot F_e \). Now \( F \cdot F_e \) is a finite set and if \( f = \chi_F \), then \( u = f \cdot \hat{f} = \chi_F \cdot \hat{\chi}_F = \chi_{F \cdot F_e} = \chi_{F_e} \in P_{r,e,f}(S) \) and \( u|_K \equiv 1 \). \( \square \)

Corollary 2.6. \( B_{r,e,f}(S) = \langle P_{r,e,f}(S) \rangle \) and \( B_{r,f}(S) = A_{r,e}(S) \).

**Proof** Clearly \( \langle P_{r,e,f}(S) \rangle \subseteq B_{r,e,f}(S) \). Now if \( v \in B_{r,e,f}(S) \), then \( v = \sum_{i=1}^{4} \alpha_i v_i \), for some \( \alpha_i \in \mathbb{C} \) and \( v_i \in P_{r,e,f}(S) \) (\( 1 \leq i \leq 4 \)). Let \( K = \text{supp}(v) \subseteq S \) and \( u \in P_{r,e,f}(S) \) be as in the above proposition, then \( u|_K \equiv 1 \) so \( v = uv = \sum_{i=1}^{4} \alpha_i (uv_i) \) is in the linear span of \( P_{r,e,f}(S) \). \( \square \)

3. **Fourier and Fourier-Stieltjes Algebras of Associated Groupoids**

We observed in section 1 that one can naturally associate a (discrete) groupoid \( S_a \) to any inverse semigroup \( S \). The Fourier and Fourier-Stieltjes algebras of (topological and measured) groupoids are studied in [11], [12], [13], and [14]. It is natural to ask if the results of these papers, applied to the associated groupoid \( S_a \) of \( S \), could give us some information about the associated algebras on \( S \). In this section we explore the relation between \( S \) and its associated groupoid \( S_a \), and resolve some technical difficulties which could arise when one tries to relate the corresponding function algebras.

Let us remind some general terminology and facts about groupoids. There are two parallel approaches to the theory of groupoids, theory of measured groupoids versus theory of locally compact groupoids (compare [13] with [14]). Here we deal with discrete groupoids (like \( S_a \)) and so basically it doesn’t matter which approach
we take, but the topological approach is more suitable here. Even if one wants to look at the topological approach, there are two different interpretation about what we mean by a "representation" (compare [12] with [13]). The basic difference is that whether we want representations to preserve multiplications everywhere or just almost everywhere (with respect to a Borel measure on the unit space of our groupoid which changes with each representation). Again the "everywhere approach" is more suitable for our setting. This approach, mainly taken by [11] and [12], is the best fit for the representation theory of inverse semigroups (when one wants to compare representation theories of $S$ and $S_a$). Even then, there are some basic differences which one needs to deal with them carefully.

We mainly follow the approach and terminology of [12]. As we only deal with discrete groupoids we drop the topological considerations of [12]. This would simplify our short introduction and facilitates our comparison. A (discrete) groupoid is a set $G$ with a distinguished subset $G^2 \subseteq G \times G$ of pairs of multiplicable elements, a multiplication map : $G^2 \to G; (x, y) \mapsto xy$, and an inverse map : $G \to G; x \mapsto x^{-1}$, such that for each $x, y, z \in G$

(i) $(x^{-1})^{-1} = x$,  
(ii) If $(x, y), (y, z)$ are in $G^2$, then so are $(xy, z), (x, yz)$, and $(xy)z = x(yz)$,  
(iii) $(x^{-1}, x)$ is in $G^2$ and if $(x, y)$ is in $G^2$ then $x^{-1}(xy) = y$,  
(iv) If $(y, x)$ is in $G^2$ then $(yx)x^{-1} = y$.

For $x \in G$, $s(x) = x^{-1}x$ and $r(x) = xx^{-1}$ are called the source and range of $x$, respectively. $G^0 = s(G) = r(G)$ is called the unit space of $G$. For each $u, v \in G^0$ we put $G^u = r^{-1}(u), G_v = s^{-1}(v)$, and $G^u_v = G_v \cap G^u$. Note that for each $u \in G^0$, $G^u$ is a (discrete) group, called the isotropy group at $u$. Any (discrete) groupoid $G$ is endowed with left and right Haar systems $\{\lambda_u\}$ and $\{\lambda^u\}$, where $\lambda_u$ and $\lambda^u$ are simply counting measures on $G_u$ and $G^u$, respectively. Consider the algebra $c_{00}(G)$ of finitely supported functions on $G$. We usually make this into a normed algebra using the so-called $I$-norm

$$\|f\|_I = \max\{\sup_{u \in G^0} \sum_{x \in G_u} |f(x)|, \sup_{u \in G^0} \sum_{x \in G^u} |f(x)|\} \quad (f \in c_{00}(G)),$$

where the above supremums are denoted respectively by $\|f\|_{I,s}$ and $\|f\|_{I,r}$. Note that in general $c_{00}(G)$ is not complete in this norm. We show the completion of $c_{00}(G)$ in $\|\cdot\|_I$ by $\ell^1(G)$. There are also natural $C^*$-norms in which one can complete $c_{00}(G)$ and get a $C^*$-algebra. Two well known groupoid $C^*$-algebras obtained in
this way are the **full and reduced groupoid** $C^*$-algebras $C^*_r(G)$ and $C^*_f(G)$. Here we briefly discuss their construction and refer the reader to [] for more details.

A Hilbert bundle $\mathcal{H} = \{\mathcal{H}_u\}$ over $G^0$ is just a field of Hilbert spaces indexed by $G^0$. A **representation** of $G$ is a pair $\{\pi, \mathcal{H}\}$ consisting of a map $\pi$ and a Hilbert bundle $\mathcal{H} = \{\mathcal{H}_u\}$ over $G^0$ such that For each $x, y \in G$,

(i) $\pi(x) : \mathcal{H}_s(x) \to \mathcal{H}_{r(x)}$ is a surjective linear isometry,

(ii) $\pi(x^{-1}) = \pi(x)^*$,

(iii) If $(x, y)$ is in $G^2$, $\pi(xy) = \pi(x)\pi(y)$.

We usually just refer to $\pi$ as the representation and it is always understood that there is a Hilbert bundle involved. We denote the set of all representations of $G$ by $\Sigma(G)$. Note that here a representation corresponds to a (continuous) Hilbert bundle, where as in the usual approach to (locally compact or measured) categories representations are given by measurable Hilbert bundles (see [12] for more details).

A natural example of such a representation is the **left regular representation** $L$ of $G$. The Hilbert bundle of this representation is $L^2(G)$ whose fiber at $u \in G^0$ is $L^2(G^u, \lambda^u)$. In our case that $G$ is discrete, this is simply $\ell^2(G^u)$. Each $f \in c_{00}(G)$ could be regarded as a section of this bundle (which sends $u \in G^0$ to the restriction of $f$ to $G^u$). Also $G$ acts on bounded sections $\xi$ of $L^2(G)$ via

$$L_x\xi(y) = \xi(x^{-1}y) \quad (x \in G, y \in G^{r(x)}).$$

Let $E^2(G)$ be the set of sections of $L^2(G)$ vanishing at infinity. This is a Banach space under the sup-norm and contains $c_{00}(G)$. Furthermore, it is a canonical $c_0(G^0)$-module via

$$b\xi(x) = \xi(x)b(r(x)) \quad (x \in G, \xi \in E^2(G), b \in c_0(G^0)).$$

Now $E^2(G)$ with the $c_0(G)$-valued inner product

$$<\xi, \eta> (u) = \langle L(\cdot)\xi^u \circ s(\cdot), \eta^u \circ r(\cdot) \rangle$$

is a Hilbert $C^*$-module. The action of $c_{00}(G)$ on itself by left convolution extends to a $*$-antirepresentation of $c_{00}(G)$ in $E^2(G)$, which is called the left regular representation of $c_{00}(G)$ [12, Proposition 10]. The map $f \mapsto L_f$ is a norm decreasing homomorphism from $(c_{00}(G), \|\cdot\|_r)$ into $\mathcal{B}(E^2(G))$. Also the former has a left bounded approximate identity $\{e_\alpha\}$ consisting of positive functions such that $\{L_{e_\alpha}\}$ tends to the identity operator in the strong operator topology of the later [12, Proposition 11]. The closure of the image of $c_{00}(G)$ under $L$ is a $C^*$-subalgebra $C^*_r(G)$ of $\mathcal{B}(E^2(G))$ which is called the **reduced $C^*$-algebra** of $G$. We should warn
the reader that $B(E^2(G))$ is merely a $C^*$-algebra and, in contrast with the Hilbert space case, it is not a von Neumann algebra in general. The above construction simply means that we have used the representation $L$ to introduce an auxiliary $C^*$-norm on $c_{00}(G)$ and took the completion of $c_{00}(G)$ with respect to this norm. A similar construction using all non degenerate $\ast$-representations of $c_{00}(G)$ in Hilbert $C^*$-modules yields a $C^*$-completion $C^*(G)$ of $c_{00}(G)$, called the full $C^*$-algebra of $G$.

Next one can define positive definiteness in this context. Let $\pi \in \Sigma(G)$, for bounded sections $\xi, \eta$ of $H^{\pi}$, the function $x \mapsto < \pi(x)\xi(s(x)), \eta(r(x))>$ (where the inner product is taken in the Hilbert space $H^{\pi}_{s(x)}$) is called a coefficient function of $\pi$. A function $\varphi \in \ell^\infty(G)$ is called positive definite if for all $u \in G^0$ and all $f \in c_{00}(G)$

$$\sum_{x,y \in G^u} \varphi(y^{-1}x)f(y)\overline{f}(x) \geq 0,$$

or equivalently, for each $n \geq 1, u \in G^0, x_1, \ldots, x_n \in G^u$, and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$

$$\sum_{i,j=1}^n \overline{\alpha}_i\alpha_j\varphi(x_i^{-1}x_j) \geq 0.$$

We denote the set of all positive definite functions on $G$ by $P(G)$. The linear span $B(G)$ of $P(G)$ is called the Fourier-Stieltjes algebra of $G$. It is equal to the set of all coefficient functions of elements of $\Sigma(G)$ [12, Theorem 1]. It is a unital commutative Banach algebra [12, Theorem 2] under pointwise operations and the norm $\|\varphi\| = \inf\|\xi\|\|\eta\|$, where the infimum is taken over all representations $\varphi = < \pi(.)\xi \circ s(.), \eta \circ r(.) >$. On the other hand each $\varphi \in B(G)$ could be considered as a completely bounded linear operator on $C^*(G)$ via

$$< \varphi, f > = \varphi.f \quad (\varphi \in B(G), f \in c_{00}(G)),$$

such that $\|\varphi\|_\infty \leq \|\varphi\|_{cb} \leq \|\varphi\|$ [12, Theorem 3]. The last two norms are equivalent on $B(G)$ (they are equal in the group case, but it is not known if this is the case for groupoids). Following [12] we denote $B(G)$ endowed with $cb$-norm with $B(G)$. This is known to be a Banach algebra (This is basically [13, Theorem 6.1] adapted to this framework [12, Theorem 3]).

There are four candidates for the Fourier algebra $A(G)$. The first is the closure of the linear span of the coefficients of $E^2(G)$ in $B(G)$ [14], the second is the closure of $B(G) \cap c_{00}(G)$ in $B(G)$ [11], the third is the closure of the of the subalgebra generated by the coefficients of $E^2(G)$ in $B(G)$, and the last one is the completion
of the normed space of the quotient of $E^2(G) \hat{\otimes} E^2(G)$ by the kernel of $\theta$ from $E^2(G) \hat{\otimes} E^2(G)$ into $c_0(G)$ induced by the bilinear map $\theta : c_{00}(G) \times c_{00}(G) \to c_0(G)$ defined by

$$\theta(f, g) = g * \check{f} \quad (f, g \in c_{00}(G)).$$

These four give rise to the same algebra in the group case. We refer the interested reader to [12] for a comparison of these approaches. Here we adapt the third definition. Then $A(G)$ is a Banach subalgebra of $B(G)$ and $A(G) \subseteq c_0(G)$. Moreover if $V N(G) = \{T \in B(E^2(G)) : TRf = RfT \ (f \in c_{00}(G))\}$, where $R$ is the right regular representation of $c_{00}(G)$ in $E^2(G)$, then $V N(G)$ is the strong closure of $C^*_r(G)$ in $B(E^2(G))$. Note that here $V N(G)$ is not necessarily a von Neumann algebra. Also the isometric isomorphism between the linear dual of $A(G)$ and $V N(G)$ may fail to exist, unless we replace $A(G)$ with an appropriate space [12, Theorem 4].

Now we are ready to compare the function algebras on inverse semigroup $S$ and its associated groupoid $S_a$. We would apply the above results to $G = S_a$. First let us look at the representation theory of these objects. As a set, $S_r$ compared to $S_a$ has an extra zero element. Moreover, the product of two non-zero elements of $S_r$ is 0, exactly when it is undefined in $S_a$. Hence it is natural to expect that $\Sigma(S_a)$ is related to $\Sigma_0(S_r) = \Sigma_r(S)$. The major difficulty to make sense of this relation is the fact that representations of $S_a$ are defined through Hilbert bundles, where as restricted representations of $S$ are defined in Hilbert spaces. But a careful interpretation shows that these are two sides of one coin.

**Lemma 3.1.** $\Sigma_r(S) = \Sigma(S_a)$.

**Proof** First let us show that each $\pi \in \Sigma_r(S)$ could be regarded as an element of $\Sigma(S_a)$. Indeed, for each $x \in S$, $\pi(x) : \mathcal{H}_x \to \mathcal{H}_x$ is a partial isometry, so if we put $\mathcal{H}_u = \pi(u)\mathcal{H}_x \ (u \in E_S)$, then we could regard $\pi(x)$ as an isomorphism from $\mathcal{H}_{x^{-1}x} \to \mathcal{H}_{xx^{-1}}$. Using the fact that the unit space of $S_a$ is $S^0_a = E_S$, it is easy now to check that $\pi \in \Sigma(S_a)$. Conversely suppose that $\pi \in \Sigma(S_a)$, then for each $x \in S_a$, $\pi(x) : \mathcal{H}_{s(x)} \to \mathcal{H}_{r(x)}$ is an isomorphism of Hilbert spaces. Let $\mathcal{H}_x$ be the direct
sum of all Hilbert spaces $H_u$, $u \in E_S$, and define $\pi(x)(\xi_u) = (\eta_v)$, where

$$\eta_v = \begin{cases} 
\pi(x)\xi_x & \text{if } v = xx^* \\
0 & \text{otherwise}
\end{cases}$$

$(x \in S, v \in E_S)$,

then we claim that

$$\pi(x)\pi(y) = \begin{cases} 
\pi(xy) & \text{if } x^*x = yy^* \\
0 & \text{otherwise}
\end{cases}$$

$(x, y \in S)$.

First let’s assume that $x^*x = yy^*$, then $\pi(xy)(\xi_u) = (\theta_v)$, where $\theta_v = \pi(x)\xi_y$ for $v = yy^*$, for which $\pi(x)\xi_y = \pi(xy)\xi_y = \pi(xy)\xi_y$. On the other hand, $\pi(y)(\xi_u) = (\eta_v)$, where $\eta_v = 0$ except for $v = yy^*$, for which $\eta_v = \pi(y)\xi_y$, and $\pi(x)(\eta_v) = (\zeta_u)$, with $\zeta_u = 0$ except for $w = xx^*$, for which $\zeta_u = \pi(x)\eta_v = \pi(x)\eta_v = \pi(x)\pi(y)\xi_y$. Hence $\pi(xy)(\xi_u) = \pi(x)\pi(y)(\xi_u)$, for each $(\xi_u) \in H_x$.

Next assume that $x^*x \neq yy^*$, then the second part of the above calculation clearly shows that $\pi(x)\pi(y)(\xi_u) = 0$. This shows that $\pi$ could be considered as an element of $\Sigma_r(S)$. Finally it is clear that these two embeddings are inverse of each other. □

Next, $S_r = S_a \cup \{0\}$ as sets, and for each bounded map $\varphi : S_r \to \mathbb{C}$ with $\varphi(0) = 0$, it immediately follows from the definition that $\varphi \in P(S_a)$ if and only if $\varphi \in P_0(S_r)$. Hence by above lemma we have

**Proposition 3.1.** The Banach spaces $B_r(S) = B_0(S_r)$ and $B(S_a)$ are isometrically isomorphic. □

This combined with [12, Theorem 2] (applied to $G = S_a$) shows that $B_r(S)$ is indeed a Banach algebra under pointwise multiplication and the above linear isomorphism is also an isomorphism of Banach algebras. By [12, Theorem 1] now we conclude that

**Corollary 3.1.** $B_r(S)$ is the set of coefficient functions of $\Sigma_r(S)$. □

**acknowledgement.** The first author would like to thank hospitality of Professor Mahmood Khoshkam during his stay in University of Saskatchewan, where the main part of the revision was done.

**References**

[1] M. Amini, A. Medghalchi, restricted algebras on inverse semigroups I, representation theory, preprint, Shahid Beheshti University, 2000.
[2] M. Amini, A. Medghalchi, restricted algebras on inverse semigroups II, positive definite functions, preprint, Shahid Beheshti University, 2000.

[3] M. Amini, A. Medghalchi, Fourier algebras on topological foundation *-semigroups, preprint, Shahid Beheshti University, 2000.

[4] C.F. Dunkl, D.E. Rumirez, $L^\infty$-representations of commutative semitopological semigroups, Semigroup Forum 7 (1974) 180-199.

[5] P. Eymard, L’algebra de Fourier d’un groupe localement compact, Bull. Soc. Math. France, 92 (1964) 181-236.

[6] R. Godement, Les fonctions de type positive et la theorie des groupes, Trans. Amer. Math. Soc. 63 (1948) 1-84.

[7] M. Lashkarizadeh Bami, Bochner’s theorem and the Hausdorff moment theorem on foundation topological semigroups, Can. J. Math. 37 (1985) 785-809.

[8] M. Lashkarizadeh Bami, Representations of foundation semigroups and their algebras, Can. J. Math. 37 (1985) 29-47.

[9] Mark V. Lawson, Inverse semigroups, the theory of partial symmetries, World Scientific, Singapore, 1998.

[10] R.J. Lindahl, P.H. Maserick, Positive-definite functions on involution semigroups, Duke Math. J. 38 (1971) 771-782.

[11] K. Oty, Fourier-Stieltjes algebras of r-discrete groupoids, J. Operator Theory, 41 (1999) 175-197.

[12] A.T. Paterson, The Fourier algebra for locally compact groupoids, preprint, 2002.

[13] A. Ramsay, M.E. Walter, Fourier-Stieltjes algebras of locally compact groupoids, J. Functional Analysis, 148 (1997) 314-365.

[14] J. N. Renault, The Fourier algebra of a measured groupoid and its multipliers, J. Functional Analysis, 145 (1997) 455-490.

Department of Mathematics, Tarbiat Modarres University, P.O.Box 14115-175, Tehran, Iran, mamini@modares.ac.ir

Department of Mathematics, Teacher Training University, Tehran, Iran medghalchi@saba.tmu.ac.ir