**Diskoseismology**

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**Abstract.** The normal mode oscillations of (geometrically thin) accretion disks around black holes and other compact objects will be analyzed and contrasted with those in stars. For black holes, the most robust modes are gravitationally trapped near the radius at which the radial epicyclic frequency is maximum. Their eigenfrequencies depend mainly on the mass and angular momentum of the black hole. The fundamental g-mode has recently been seen in numerical simulations of black hole accretion disks. For stars such as white dwarfs, the modes are trapped near the inner boundary (magnetospheric or stellar) of the accretion disk. Their eigenfrequencies are approximately multiples of the (Keplerian) angular velocity of the inner edge of the disk. The relevance of these modes to the high frequency quasi-periodic oscillations observed in the power spectra of accreting binaries will be discussed. In contrast to most stellar oscillations, most of these modes are unstable in the presence of viscosity (if the turbulent viscosity induced by the magnetorotational instability acts hydrodynamically).

1. Introduction; comparison with helio- and asteroseismology  
We shall briefly review the application of normal mode analysis, so successfully employed in stars (as this conference illustrates), to the ubiquitous accretion disks that surround a wide variety of objects (black holes, neutron stars, white dwarfs, and protostars). In the first three cases (except for the supermassive black holes observed at the center of galaxies), they are usually maintained by gas tidally drawn through the inner Lagrange point from their companion star in a binary. The relativistic formulation of diskoseismology was initiated by S. Kato and J. Fukue in 1980 [1], and reviews can be found in [2, 3, 4].

Let us briefly compare the adiabatic hydrodynamic perturbations of stars and accretion disks in the Newtonian limit. We shall assume that the gravitational field of the disk (including that of the perturbations) is negligible, which is an excellent approximation for most accretion disks since their mass is much less than that of the central object. To facilitate the comparison, we shall also make the same (Cowling) approximation for the perturbations of the star. We shall consider the usual case of geometrically thin (and optically thick) accretion disks: thickness $h(r) \sim c_s/\Omega$ much less than the radial distance $r$, where $c_s$ (then much less than $v_\phi = r\Omega$) is the sound speed and $\Omega(r)$ is the angular velocity of the disk. The pressure gradient is then much greater vertically than radially; and the motion of the unperturbed gas is close to that of free–particle orbits, with $v_z \ll v_r \ll v_\phi$. The unperturbed disk is axially symmetric, with $z$ the distance from its midplane (about which it is reflection symmetric). Throughout, we also make a WKB approximation (usually valid) that the perturbation radial wavelength $\lambda_r \ll r$.

The governing equations of motion of perturbations of a (slowly rotating) star (left) and an
accretion disk (right) are then

\[
(\omega^2 - N_z^2)\xi_\phi = (\nabla_r + \rho A_\phi)\delta U \\
(\omega^2 - \kappa^2)\xi_\phi = (\nabla_r + \rho A_\phi)\delta U \\
\omega^2\xi_\phi = \nabla_\phi \delta U \\
\omega^2\xi_\phi = \nabla_\phi \delta U , \\
i\omega\xi_\phi = 2(\Omega + rd\Omega/dr)\xi_\phi .
\]

The perturbation of the specific enthalpy \( \delta U = \delta p/\rho \) and the displacement \( \zeta \) are of the form

\[ f(r, z) \exp[i(m\phi + \sigma t)]. \]

The corotating frequency is \( \omega(r) = \sigma + m\Omega(r) \), the radial epicyclic frequency is \( \kappa(r) \), and \( \hat{N}(r, z) \) is the buoyancy (Brunt-Väisälä) frequency.

Note first that the vertical equation of motion in the disk is of the same form as the radial equation for a star. The key difference is in the radial equation for the disk, with the (negligible) radial buoyancy frequency replaced by the radial epicyclic frequency. This reflects the dominance of centrifugal support. The third equation of motion of disk perturbations simply reflects conservation of angular momentum.

2. Relativistic diskoseismology

We shall employ the same assumptions as above, but in the context of relativistic hydrodynamics in the (Kerr) metric of a black hole. We take \( c = 1 \), and express all distances in units of \( GM/c^2 \) and all frequencies in units of \( c^3/GM \) (where \( M \) is the mass of the black hole) unless otherwise indicated.

All perturbations can be obtained from the function \( \delta V = V(r, z) \exp[i(m\phi + \sigma t)] = \delta p/\rho \beta \omega \), which satisfies the master equation

\[
\frac{1}{r} \frac{\partial}{\partial r} \left[ r^2 Y^3 \rho \left( \frac{\omega^2}{\omega^2 - \kappa^2} \right) \frac{\partial V}{\partial r} \right] + \frac{\partial}{\partial z} \left[ r Y^2 \rho \left( \frac{\omega^2}{\omega^2 - N_z^2} \right) \frac{\partial V}{\partial z} \right] + QV = 0 .
\]

The relativistic functions \( \beta \) and \( Y \) are approximately 1, and the function \( Q \) depends upon \( \omega \), \( m \), and various properties of the unperturbed accretion disk.

We now make the additional assumption that the equation of state is approximately barotropic \([p(\rho)]\), so the buoyancy frequency \( N_z \ll |\omega| \). Because the induced turbulence should limit the \( z \) dependence of the specific entropy, this should be a reasonable approximation. We look for a separated solution \( V = V_r(r)V_z(z) \) to the master equation, assuming that \( V_z \) and all other properties of the equilibrium disk vary radially much less than \( V_r \), \( \omega^{-1} \), and \( (\omega^2 - \kappa^2)^{-1} \). We then obtain the separated equations

\[
(1 - y^2) \frac{\partial^2 V_r}{\partial y^2} - 2gy \frac{\partial V_r}{\partial y} + 2g[y^2 + \Psi(1 - y^2)]V_y = 0 ,
\]

\[
\frac{d}{dr} \left[ \left( \frac{\omega^2}{\omega^2 - \kappa^2} \right) \frac{dV_r}{dr} \right] + \left( G - \frac{\Psi - \omega^2 \Gamma}{\Gamma h^2} \right) \frac{V_r}{Y^2} = 0 .
\]

We have introduced \( \omega_\perp(r) \equiv \omega/\Omega_\perp \), where \( \Omega_\perp \) is the vertical epicyclic frequency, and the dimensionless vertical coordinate \( y = fz/h(r) \). Both \( f \) and \( g \) are functions of only the adiabatic index \( \Gamma \), while the function \( G(r) \) is usually negligible.

Then the only properties of the accretion disk which govern the modes are \( \Gamma \), the speed of sound (through \( \rho \)), and the orbital frequencies (whose radial dependences are shown in Figure 1). The speed of sound is a weak function of the luminosity of the disk, its viscosity, and the black hole mass. The separation function \( \Psi(r) \) is slowly varying. Together with the usual boundary conditions, these two equations specify vertical eigenvalues \( \Psi = \Psi(\sigma, r) \) and eigenfrequencies \( \sigma \), as well as the corresponding vertical \( (V_y) \) and radial \( (V_r) \) eigenfunctions for modes of all types.
Figure 1. Radial dependence of the key free–particle orbital frequencies, for three values of the black hole angular momentum $J = aGM^2/c$. The parameter is bounded by $a = -1$ (maximally counter–rotating) to $a = 1$ (maximally co–rotating with the disk). The event horizon moves from $r = 2$ for $a = 0$ to $r = 1$ for $a = 1$. The inner edge of the disk ($r_i$) is defined to be where the orbits become unstable ($\kappa = 0$).
The mathematical classification of our modes is similar to that for stars,

- **g–modes**: $\Psi > \omega^2 \Rightarrow \omega^2 < \kappa^2$
- **p–modes**: $\Psi < \omega^2 \Rightarrow \omega^2 > \kappa^2$
- **c–modes**: $\Psi \approx \omega^2$, 

but the physical meaning is much different. The g–modes are inertial, with most of the restoring force being gravitational–centrifugal. So are the p–modes, but the small pressure force acts in the same way as in acoustic modes. These properties can be seen by employing the extreme (local) WKB limit, in which case one obtains the dispersion relation $\omega^2 - \kappa^2 \sim \pm C k^2 c_s^2$. The plus (minus) sign refers to the p(g)–modes, and $C \sim 1$.

The g–modes [5] are the most robust, since they are trapped gravitationally (where $\omega^2 < \kappa^2$)

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**Figure 2.** The lower curve is the radius $[r_i(a)]$ of the inner edge of the disk (see Figure 1). The fundamental ($m = 0$) p–mode is trapped between $r_i$ and the curve that passes through $r = 6.5$ for a nonrotating black hole ($a = 0$). The fundamental ($m = 1$) c–mode is trapped between $r_i$ and the curve which becomes infinite for $a = 0$. The fundamental ($m = 0$) g–mode is trapped in the region away from the inner edge of the disk ($7.5 < r < 8.7$ for $a = 0$).
by the radial dependence of $\kappa$. The (inner) $p$–modes are trapped (where $\omega^2 > \kappa^2$) between $\kappa$ and the inner edge of the disk, where the gas begins to leak rapidly onto the black hole and the physical conditions are uncertain. The $c$–modes are almost incompressible (analogous to $f$–modes in stars) and also trapped against the inner edge of the disk, but they are controlled by the radial dependence of $\Omega_{\perp}$. This is illustrated in Figure 2. In principal, the nonaxisymmetric ($m \geq 1$) $g$–modes could be affected by the singularity at the corotation resonance $[\omega(r_c) = 0]$. However, it lies outside the disk ($r_c < r_i$) where the gas is plunging into the black hole.

For all the $g$–modes, the dependence of the eigenfrequencies on the radial and vertical mode numbers is weak, since the speed of sound is much less than the orbital speed. For $m \geq 1$, they obey $|\sigma| \approx m\Omega(r_i)$, where $r_i(a)$ is the inner radius of the disk. The fundamental ($m = 0$) frequency is plotted in Figure 3, along with that of the other modes. It is close to the maximum epicyclic frequency $\kappa$, whose dependence on $a$ is illustrated in Figure 1. The physical frequency of all modes is proportional to $1/M$, reflecting their gravitational trapping.

Introducing a small viscosity ($\eta \sim \alpha \rho h^2 \Omega$, $\alpha \ll 1$), most modes grow at a rate $\sim \alpha \Omega$ [6], so such disks should be unstable.

### 3. Comparison with numerical simulations

Recently, Chris Reynolds and Cole Miller [7] carried out numerical simulations of accretion disks which for the first time have shown one of these modes. The nonrotating black hole is represented by a modified Newtonian gravitational potential, which reproduces fairly accurately the behavior of the radial epicyclic frequency $\kappa(r)$. The accretion disk (of modest radial extent)
Figure 4. Snapshots of an axisymmetric hydrodynamic simulation after 5 (left) and 50 (right) ISCO periods. There are 10 contours per decade (3) of density. The curve at the left is the event horizon of the black hole.

Figure 5. Left: (a) Power in pressure fluctuation on the disk midplane ($z = 0$). In the logarithmic color table, black represents a PSD amplitude $10^5$ times greater than light yellow. Also shown are the radial epicyclic frequency $\kappa$ (solid), orbital frequency $\Omega$ (dashed), and $2\Omega$ (dot–dashed). Right: (b) The PSD within radii $7.75 < r c^2 / GM < 8.25$, where the axisymmetric $g$–mode is predicted to be concentrated. The dashed line indicates the maximum value of $\kappa(r)$.

was evolved for many periods of the innermost stable circular orbit (ISCO, at $r_i = 6GM/c^2$). They performed both 2D and 3D hydrodynamic, and 3D magnetohydrodynamic, simulations.

The density structure of the disk at two epochs is shown in Figure 4. An initial excitation is provided, but the waves damp out due to numerical viscosity and leakage. There is no mass supplied to the disk. However, power spectra of pressure fluctuations at many radii over the extent of the disk show a strong peak whose frequency range and radial extent matches that of the $m = 0$ g–mode, as shown in Figure 5(a). From other runs, it was shown that the difference between the peak frequency and $\kappa_{\text{max}}$ shown in Figure 5(b) is proportional to the speed of sound by the predicted [5] amount. Because of the limited range of the angle $\phi$ and frequencies in these simulations, g–modes with somewhat larger values of the axial mode number $m$ could not have been seen.

The spatial structure of the pressure fluctuations, shown in Figure 6, matches reasonably well that of the predicted g–mode eigenfunction [5].
Figure 6. Maps of the pressure deviation, folded at the period of the peak of the PSD. The phases are 0.0 (left) and 0.5 (right).

Figure 7. Snapshots of a 3D MHD simulation after 10 (left) and 100 (right) ISCO periods. There are 10 contours per decade (3) of density.

Figure 8. Power in azimuthally–averaged MHD radial velocity (left) and pressure (right) fluctuations on the disk midplane. Also shown are \(\kappa\) (solid), \(\Omega\) (dashed), and the \(n = 1, 2, 3\) local vertical pressure oscillation frequencies [9] (dot–dashed).
Table 1. QPOs in binary black hole X-ray sources. Assuming an $m = 0$ g–mode, the value of the dimensionless back hole angular momentum ($a = cJ/GM^2$) follows from Figure 3.

| Black hole binary | $f$ (Hz) | $(M/10M_\odot)f$ | $a$ (g–mode) |
|-------------------|----------|------------------|--------------|
| GRS 1915+105      | 41       | 57 ± 16          | < 0.1        |
| $(M = 14 \pm 4M_\odot)$ | 67       | 94 ± 27          | 0.0 − 0.8    |
|                   | 113      | 158 ± 45         | 0.6 − 1.0    |
|                   | 168      | 235 ± 67         | 0.8 − 1.0    |
| XTE J1550–564     | 92       | 88 ± 11          | 0.1 − 0.5    |
| $(M = 9.6 \pm 1.2M_\odot)$ | 184      | 177 ± 23         | 0.8 − 1.0    |
|                   | 276      | 265 ± 33         | (> 1.0)      |
| GRO J1655–40      | 300      | 189 ± 9          | 0.9 − 1.0    |
| $(M = 6.3 \pm 0.3M_\odot)$ | 450      | 284 ± 14         | (> 1.0)      |

The results of the MHD simulations were quite different, due mainly to the strong turbulence generated by the magnetorotational instability (MRI) [8]. In Figure 7, to be compared with Figure 4, one sees the turbulence that develops from the unperturbed disk after just a few ISCO periods. It persists throughout the remainder of the simulation, as predicted [8]. The initially weak magnetic field is amplified, but the magnetic pressure then remains about 30 times less than the gas pressure.

However, power spectra of these MHD simulations, seen in Figure 8, show no evidence of the g–mode; although they do show local radial epicyclic oscillations (left) and local vertical pressure oscillations (right). The spatial and time dependence of turbulence might not allow such modes to exist. However, if the g–mode reached the same amplitude produced in the hydrodynamical simulation, its fluctuations would be too small to have been seen against those of the turbulence. On the other hand, if the effective viscosity produced by the MHD turbulence acts hydrodynamically, the g–mode should grow at the rate mentioned above.

4. Observations of quasi–periodic oscillations

The Rossi X–ray Timing Explorer satellite (RXTE) has greatly enhanced our understanding of compact objects which are fed by accretion of gas from a companion star. Its timing measurements have produced power spectra of fluctuations in the X–ray flux from such binaries. The high frequency region of such PSDs from three sources are shown in Figure 9 [10]. These sources are chosen because observations of the companion determine the mass of the compact object to be greater than that of a neutron star or white dwarf, so they are presumably black holes. In addition, they have multiple high frequency peaks in the PSD, dubbed quasi–periodic oscillations (QPOs). However, often the QPOs from a particular source are not seen simultaneously. Their frequencies are usually fairly stable, in contrast to QPOs from neutron star binaries.

In Table 1, we present the frequencies and black hole masses in these three sources. If any particular frequency corresponded to that of an axisymmetric g–mode (or any other mode), that would provide a determination of the angular momentum of the black hole, as indicated above (Figure 3). The X-rays would be modulated by the changing temperature and density of the mode, as well as the fluctuating location of the photosphere in the disk. One expects the $m = 0$ g–mode to be the most visible, for three reasons. The higher axial modes are washed out because the full disk is observed. In addition, they cover a much smaller region of the disk and
are located nearer its (leaky) inner edge [5].

The other distinguishing characteristic of these three sources is that the frequencies in the first are in two pairs with 2/3 ratios, those in the second are in a 1/2/3 ratio, and those in the third are in a 2/3 ratio. If one tries to explain these as coming from $m = 1, 2, 3$ g-modes (which are approximately harmonic, as indicated above), the predicted frequencies are too high. The origin of these QPOs remains a mystery, although their frequencies indicate that they are produced near the black hole.

5. Newtonian g-modes
We now consider Newtonian accretion disks, such as those surrounding a white dwarf (or neutron star with a magnetosphere that keeps the disk away from the surface). The white dwarf may or may not have a magnetic field strong enough to provide a magnetosphere. There are two
The horizontal lines represent the scaled frequencies and radial extent of the lowest axial g–modes in Newtonian accretion disks with an inner boundary (at $r_i$). Key differences from the black hole case, assuming the star is spherically symmetric (i.e., slowly rotating). In this weak–field case, $\kappa = \Omega = \Omega_\perp$, so there is no turnover in the radial epicyclic frequency. There is an inner boundary (at radius $r_i$), which we have taken to be impenetrable in this initial investigation [11].

As shown in Figure 10, g–modes are trapped as usual where $\omega^2 < \kappa^2 = \Omega^2$, which gives the eigenfrequency $|\sigma_m| < (m+1)\Omega$. Since again $c_s \ll r\Omega$, the eigenfrequency is close to its maximum possible value:

$$|\sigma_m| \approx (m + 1)\Omega(r_i) = (m + 1)(GM/r_i^3)^{1/2}.$$  \hspace{1cm} (4)

Note that no (inner) p–modes can exist.

Shown in Figure 11 are observations of the frequencies of dwarf nova oscillations (DNOs) (the analog of high frequency QPOs) from the white dwarf binary VW Hyi [12]. Enhanced accretion events produce the outbursts. It is plausible that the position of the boundary ($r_i$) is determined by a balance of the magnetic pressure of the magnetosphere and the ram pressure ($\rho v^2$) of the accreting gas. The disk luminosity is proportional to the mass accretion rate $2\pi r h \rho v_r$. Thus as the disk luminosity decreases during the burst, so should the ram pressure, resulting in an increase in $r_i$. From equation (4), this will result in an increasing DNO period, as observed.
The presence of the 1/2/3 harmonics can be understood in terms of the $m = 0, 1, 2$ g–modes. A few other accreting white dwarf sources exhibit 1/2 harmonics, but most show no harmonics [13].

### 6. Issues for future investigations

A major unresolved issue is the state of the ‘unperturbed’ accretion disk. Since the (MRI induced) turbulence appears to be vigorous, does it allow the existence of ‘normal modes’? Although the vertical extent of the modes is comparable to the vertical (and radial) extent ($h$) of the largest turbulent ‘eddies’, the g–modes have a somewhat greater radial extent [of order $(rh)^{1/2}$]. There is also significant power in the turbulence at the periods of the fundamental modes [7]. If the g–mode does exist, will the turbulence produce the type of viscosity that makes it grow?

A major mystery is the origin of the 2/3 harmonics observed in some black hole sources (Figure 9). Is mode–mode coupling important? Other models include a local resonance between the radial and vertical epicyclic frequencies [14], which produces the relation $\kappa(r_*) = (2/n)\Omega_{\perp}(r_*)$. However, such local oscillations were not seen in the simulations discussed above.

For the Newtonian accretors, the major issue is the nature of the ‘boundary layer’. To what extent will penetration and damping of the g–modes affect the above exploratory results? For
the accreting neutron star sources, the high frequency QPO frequencies \((f_1, f_2)\) are in many cases related to the spin frequency \((f_s)\) of the neutron star by \(f_2 - f_1 = f_s / n\), where the value of \(n = 1\) or \(2\) is such that \(f_2 - f_1 \sim 200 - 300\) Hz \([15]\). This indicates magnetic coupling between the neutron star and the disk.

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