Spectral Expansion for the Asymptotically Spectral Periodic Differential Operators

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Abstract
In this paper we investigate the spectral expansion for the asymptotically spectral differential operators generated in \( L^2_{m} (-\infty, \infty) \) by ordinary differential expression of arbitrary order with periodic matrix-valued coefficients.

Key Words: Periodic nonself-adjoint differential operator, Spectral singularities, Spectral expansion.

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1 Introduction and preliminary Facts
Let \( T^{(m)}(p_1, P_2, P_3, ..., P_n) =: T^{(m)} \) be the differential operator generated in the space \( L^2_{m} (-\infty, \infty) \) by the differential expression
\[
l^{(m)}(y) = y^{(n)} + p_1 I_m y^{(n-1)} + P_2 y^{(n-2)} + P_3 y^{(n-3)} + ... + P_n y
\]
and \( T^{(m)}_t(p_1, P_2, P_3, ..., P_n) =: T^{(m)}_t \) for \( t \in \mathbb{C} \) be the differential operator generated in \( L^2_{m} (0, 1) \) by the same differential expression and the boundary conditions
\[
U_{\nu, t}(y) =: y^{(\nu)} (1) - e^{it} y^{(\nu)} (0) = 0, \quad \nu = 0, 1, ..., (n - 1),
\]
where \( n \geq 2 \), \( p_1 \) is \( (n - 1) \) times continuously differentiable scalar function, \( p_1 (x + 1) = p_1 (x) \), \( I_m \) is \( m \times m \) unit matrix, \( P_v \) for \( v = 2, 3, ..., n \) are the \( m \times m \) matrix with the complex-valued summable on \([0, 1]\) entries, \( P_v (x + 1) = P_v (x) \) and \( y = (y_1, y_2, ..., y_m) \) is a vector-valued function. Here \( L^2_{m} (a, b) \) is the space of the vector-valued functions \( f = (f_1, f_2, ..., f_m) \) with the norm \( ||f||_{(a,b)} \) and inner product \( \langle \cdot, \cdot \rangle_{(a,b)} \) defined by
\[
||f||_{(a,b)}^2 = \int_a^b |f (x)|^2 \, dx, \quad \langle f, g \rangle_{(a,b)} = \int_a^b \langle f (x), g (x) \rangle \, dx,
\]
where \(|\cdot|\) and \( \langle \cdot, \cdot \rangle \) are the norm and inner product in \( \mathbb{C}^m \).

In this paper we consider the spectral expansion for the operator \( T^{(m)} \). The spectral expansion for the self-adjoint differential operators with periodic coefficients was constructed by Gelfand [1], Titchmarsh [8] and Tkachenko [9]. The existence of the spectral singularities and the absence of the Parseval’s equality for the nonself-adjoint operators \( T^{(m)}_t \) do not allow us to apply the elegant method of Gelfand (see [1]) for the construction of the spectral expansion for the nonself-adjoint periodic operators. These situation essentially complicate the construction of the spectral expansion for the nonself-adjoint case.
Note that the spectral singularity of $T^{(m)}$ is a point of its spectrum $\sigma(T^{(m)})$ in neighborhood on which the projections of $T^{(m)}$ are not uniformly bounded or equivalently a point $\lambda \in \sigma(T^{(m)})$ is called a spectral singularity of $T^{(m)}$ if the spectral projections of the operators $T^{(m)}_t$ for $t \in (-\pi, \pi]$ corresponding to the eigenvalues lying in the small neighborhood of $\lambda$ are not uniformly bounded (see [10, 11, 2]). Thus here we use the following definition of the spectral singularity.

**Definition 1** Let $e(t, \gamma)$ be the projection of $T^{(m)}_t$ defined by contour integration of the resolvent of $T^{(m)}_t$ over the closed curve $\gamma$. We say that $\lambda \in \sigma(T^{(m)})$ is a spectral singularity of $T^{(m)}$ if for all $\varepsilon > 0$, there exists a sequence of closed curves $\gamma_n \subset \{ z \in \mathbb{C} : |z - \lambda| < \varepsilon \}$ such that

$$\lim_{n \to \infty} \sup_{t} \| e(t, \gamma_n) \| = \infty,$$

where $\sup$ is taken over all $t$ for which $\gamma_n$ lies in the resolvent set of $T^{(m)}_t$ and $T^{(m)}_t$ has a unique simple eigenvalue inside $\gamma_n$.

This paper can be considered as continuation of the paper [11]. To describe the scheme of this paper let us introduce some well-known facts and some results of [11] about eigenvalues (Bloch eigenvalues) and eigenfunction (Bloch functions) of $T^{(m)}_t$ and the problems of the spectral expansion of $T^{(m)}$ which are used essentially.

(a) **On the Bloch eigenvalues and Bloch functions.** It is well-known that (see [7, 4]) the spectrum $\sigma(T^{(m)})$ of $T^{(m)}$ is the union of the spectra $\sigma(T^{(m)}_t)$ of $T^{(m)}_t$ for $t \in (-\pi, \pi]$. Denote by $T^{(m)}_t(0)$ and $T^{(m)}_t(C)$ respectively the operator $T^{(m)}_t(0, P_2, 0_m, 0_m, ..., 0_m)$ if $P_2(x) = 0_m$ and $P_2(x) = C$, where $0_m$ is $m \times m$ zero matrix and

$$C = \int_{0}^{1} P_2(x) \, dx.$$

It is clear that

$$\varphi_{k,j,t}(x) = e(t)e_j e^{i(2\pi k + t)x}$$

for $k \in \mathbb{Z}$, $j = 1, 2, ..., m$, where $(e(t))^{-2} = \int_{0}^{1} | e^{itx} |^2 \, dx$ and $e_1 = (1, 0, 0, ..., 0)$, $e_2 = (0, 1, 0, ..., 0)$, $..., e_m = (0, 0, ..., 0, 1)$, are the normalized eigenfunctions of the operator $T^{(m)}_t(0)$ corresponding to the eigenvalue $(2\pi ki + ti)^n$. One can easily verify that the eigenvalues and normalized eigenfunctions of $T^{(m)}_t(C)$ are

$$\mu_{k,j}(t) = (2\pi ki + ti)^n + \mu_j (2\pi ki + ti)^{n-2}, \quad \Phi_{k,j,t}(x) = e(t)v_j e^{i(2\pi k + t)x}$$

for $k \in \mathbb{Z}$, $j = 1, 2, ..., m$, where $v_1, v_2, ..., v_m$ are the normalized eigenvectors of the matrix $C$ corresponding to the eigenvalues $\mu_1, \mu_2, ..., \mu_m$, if the eigenvalues of the matrix $C$ are simple.

In [11] to obtain the asymptotic formulas for $T^{(m)}_t$ we took the operator $T^{(m)}_t(C)$, for an unperturbed operator and $T^{(m)}_t - T^{(m)}_t(C)$ for a perturbation and proved the following.

**Theorem 1.1** of [11] Suppose $p_1 = 0$ and the eigenvalues of $C$ are simple.

(a) The eigenvalues of $T^{(m)}_t$ consist of $m$ sequences $\{\lambda_{k,j}(t) : k \in \mathbb{Z} \}$ for $j = 1, 2, ..., m$ satisfying the following, uniform with respect to $t$ in $Q_\varepsilon(n)$, formula

$$\lambda_{k,j}(t) = (2\pi ki + ti)^n + \mu_j (2\pi ki + ti)^{n-2} + O(k^{-3} \ln |k|)$$

as $k \to \infty$, where

$$Q_\varepsilon(2\mu) = \{ t \in Q : |t - \pi k| > \varepsilon, \forall k \in \mathbb{Z} \}, \quad Q_{\varepsilon}(2\mu + 1) = Q, \quad \varepsilon \in (0, \frac{\pi}{4})$$
and \( Q \) is a compact subset of \( \mathbb{C} \) containing a neighborhood of the interval \([−\pi, \pi]\). There exists a constant \( N(ε) \) such that if \( |k| \geq N(ε) \) and \( t \in Q_{c}(n) \), then \( \lambda_{k,j}(t) \) is a simple eigenvalue of \( T_{t}^{(m)} \) and the corresponding normalized eigenfunction \( ψ_{k,j,t}(x) \) satisfies

\[
ψ_{k,j,t}(x) = e(t)v_{j}e^{i(2πk + t)x} + O(k^{-1} \ln |k|)
\]

as \( k \to \infty \). This formula is uniform with respect to \( t \) and \( x \) in \( Q_{c}(n) \) and in \([0,1]\).

\( b \) If \( t \in C(n) \) then the root functions of \( T_{t}^{(m)} \) form a Riesz basis in \( L^{2}(0,1) \), where \( C(2μ) = \mathbb{C} \{ \pi k : k \in \mathbb{Z} \} \), \( C(2μ + 1) = \mathbb{C} \).

\( c \) Let \( \left( T_{t}^{(m)} \right)^{*} \) be the adjoint operator of \( T_{t}^{(m)} \) and \( X_{k,j,t} \) be the eigenfunction of

\[
\left( T_{t}^{(m)} \right)^{*}
\]

corresponding to the eigenvalue \( \lambda_{k,j}(t) \) and satisfying \( (X_{k,j,t}, Ψ_{k,j,t}) = 1 \), where \( |k| \geq N(ε) \) and \( t \in Q_{c} \). Then \( X_{k,j,t}(x) \) satisfies the following, uniform with respect to \( t \) and \( x \) in \( Q_{c}(n) \) and in \([0,1]\) formula

\[
X_{k,j,t}(x) = u_{j}(e(t))^{-1}e^{i(2kπt + x)} + O(k^{-1} \ln |k|)
\]

as \( k \to \infty \), where \( u_{j} \) is the eigenvector of \( C^{*} \) corresponding to \( \overline{π_{j}} \) and satisfying \( (u_{j}, v_{j}) = 1 \).

Note that the formula \( f(k,t) = O(h(k)) \) as \( k \to \infty \) is said to be uniform with respect to \( t \) in a set \( E \) if there exist positive constants \( N \) and \( c \), independent of \( t \), such that \( |f(k,t)| < c |h(k)| \) for all \( t \in E \) and \( |k| \geq N \).

**Remark 1** It is well-known that [6] the substitution

\[
Y(x) = \exp \left(-\frac{1}{n} \int_{0}^{x} p_{1}(t)dt \right) \overline{Y(x)}
\]

reduce the matrix equation

\[
Y^{(n)}(x) + p_{1}(x)Y^{(n-1)}(x) + \cdots + p_{n}(x)Y = \lambda Y(x)
\]

to the equation of the form

\[
\overline{Y}^{(n)}(x) + \overline{P}_{2}(x)\overline{Y}^{(n-2)}(x) + \cdots + \overline{P}_{n}(x)\overline{Y} = \lambda \overline{Y}(x).
\]

One can easily verify that

\[
\overline{P}_{2}(x) = q(x)I_{m} + P_{2}(x),
\]

the eigenvalues \( \lambda_{k,j}(t) \) and eigenfunctions \( ψ_{k,j,t}(t) \) of the operator \( T_{t}^{(m)} \) corresponding to (12) satisfies the formula

\[
\lambda_{k,j}(t + ir) = \lambda_{k,j}(t), \quad ψ_{k,j,t+ir}(x) = ψ_{k,j,t}(t),
\]

where \( q \) is a scalar function \( r = \frac{1}{2} \int_{0}^{1} p_{1}(t)dt \). It follows from (13) and (4) that the eigenvalues of \( \int_{0}^{1} \overline{P}_{2}(x)dx \) are simple whenever the eigenvalues of \( C \) are simple. Therefore the results obtained in Theorem 1.1 of [11] for \( T_{t}^{(m)} \) continues to hold for \( T_{t}^{(m)} \). Moreover, Theorem 1.1 of [11] with (14) immediately implies that there exists \( N_{0} \) such that the eigenvalues \( \lambda_{k,j}(t) \) of \( T_{t}^{(m)} \) for \( |k| \geq N_{0} \) and \( t \in (-π, π] \) are simple, the corresponding functions \( Ψ_{k,j,t}(x) \) and \( X_{k,j,t}(x) \) satisfy the uniform with respect to \( t \) and \( x \) in \((-π, π] \) and in \([0,1]\) formulas

\[
Ψ_{k,j,t}(x) = e(t + ir)v_{j}e^{i(2πk + t)x} + O(k^{-1} \ln |k|)
\]

and

\[
X_{k,j,t}(x) = (e(t + ir))^{-1}u_{j}e^{i(2πk + t)x} + O(k^{-1} \ln |k|)
\]

if one of the following conditions hold
Condition 1 \( n \) is an odd number and the eigenvalues of \( C \) are simple.

Condition 2 \( n \) is an even number, the eigenvalues of \( C \) are simple and

\[
\text{Re} \int_0^1 p_t(x) \, dx = \text{Re} \, nr \neq 0.
\]

Let \( Y_1(x, \lambda), Y_2(x, \lambda), \ldots, Y_n(x, \lambda) \) be the solutions of the matrix equation (11) satisfying \( Y_k^{(j)}(0, \lambda) = 0_m \) for \( j \neq k - 1 \) and \( Y_k^{(k-1)}(0, \lambda) = I_m \). The eigenvalues of the operator \( T_{t}^{(m)} \) are the roots of the characteristic determinant

\[
\Delta(\lambda, t) = \det(Y_j^{(\nu - 1)}(1, \lambda) - e^{it}Y_j^{(\nu - 1)}(0, \lambda))_{\nu = 1}^n = e^{inmt} + f_1(\lambda)e^{i(nm - 1)t} + f_2(\lambda)e^{i(nm - 2)t} + \cdots + f_{nm - 1}(\lambda)e^{it} + 1
\]

which is a polynomial of \( e^{it} \) with entire coefficients \( f_1(\lambda), f_2(\lambda), \ldots \). Therefore the multiple eigenvalues of the operators \( T_{t}^{(m)} \) are the zeros of the resultant \( R(\lambda) \equiv R(\Delta, \Delta') \) of the polynomials \( \Delta(\lambda, t) \) and \( \frac{d}{dt}\Delta(\lambda, t) \). Since \( R(\lambda) \) is entire function and the large eigenvalues of \( T_{t}^{(m)} \) for \( t \in (-\pi, \pi) \) are simple if Condition 1 or Condition 2 holds (see the end of Remark 1), there exist at most finite number of multiple eigenvalues lying in the spectrum \( \sigma(T^{(m)}) \) of \( T^{(m)} \). Denote they by \( \alpha_1, \alpha_2, \ldots, \alpha_p \). For each \( \alpha_k \) there are at most \( nm \) values of \( t \in (-\pi, \pi) \) satisfying \( \Delta(\alpha_k, t) = 0 \). Hence the sets

\[
A_k = \{ t \in (-\pi, \pi) : \Delta(\alpha_k, t) = 0 \} \quad & \quad A = \bigcup_{k=1}^{p} A_k
\]

are finite and for \( t \notin A \) all eigenvalues of \( T_{t}^{(m)} \) are simple.

In [11] we proved the following lemma for \( \tilde{T}_{t}^{(m)} \) that continues to hold for \( T_{t}^{(m)} \).

**Lemma 3.1 of [11]** The eigenvalues \( \lambda_{k, j}(t) \) of \( T_{t}^{(m)} \) can be numbered as \( \lambda_1(t), \lambda_2(t), \ldots \), such that for each \( p =: p(k, j) \) the function \( \lambda_p(t) \) is continuous in \((-\pi, \pi)\). Moreover, if \( |k| \geq N_0, t \in (-\pi, \pi) \) then

\[
\lambda_{p(k, j)}(t) = \lambda_{k, j}(t),
\]

\[
p(k, j) = 2|k|m + j, \quad \forall k > 0,
\]

\[
p(k, j) = (2|k| - 1)m + j, \quad \forall k < 0,
\]

where \( p > N_1, N_1 = (2N_0 - 1)m \) and \( N_0 \) is defined in Remark 1.

Thus if Condition 1 or Condition 2 holds, then for \( t \in ((-\pi, \pi) \setminus A) \) and \( k = 1, 2, \ldots \) the eigenvalues \( \lambda_k(t) \) are simple. The corresponding eigenfunctions of \( T_{t}^{(m)} \) and \( \left(T_{t}^{(m)}\right)^{*} \) are denoted by \( \Psi_{k, t} \) and \( X_{k, t} \) respectively. Apart from the eigenvalues \( \lambda_{k, j}(t) \), where \( |k| \geq N_0 \), there exist \( N_1 \) eigenvalues of the operator \( T_{t}^{(m)} \) denoted by \( \lambda_1(t), \lambda_2(t), \ldots, \lambda_{N_1}(t) \) (see p. 12 of [11]). We define \( \lambda_p(t) \) for \( p > N_1 \) and \( t \in ((-\pi, \pi) \setminus A) \) by (20). In this paper both the notations of Theorem 1.1 of [11] and the notation of Lemma 3.1 of [11] are used.

(b) On the problems of the spectral expansion of \( T^{(m)} \). By Gelfand’s Lemma (see [1]) for every \( f \in L_{2}^{(m)}(-\infty, \infty) \) there exists \( f_t(x) \) such that

\[
f(x) = \frac{1}{2\pi} \int_{0}^{2\pi} f_t(x) dt, \quad \int_{-\infty}^{\infty} |f(x)|^2 \, dx = \frac{1}{2\pi} \int_{0}^{1} \int_{0}^{1} |f_t(x)|^2 \, dx \, dt
\]

and

\[
f_t(x + 1) = e^{it}f_t(x), \quad \int_{0}^{1} f_t(x)\overline{X_{k, t}(x)} \, dx = \int_{-\infty}^{\infty} f(x)\overline{X_{k, t}(x)} \, dx,
\]
where $\Psi_{k,t}$ and $X_{k,t}$ are extended to $(-\infty, \infty)$ by
\[
\Psi_{k,t}(x+1) = e^{it} \Psi_{k,t}(x) \quad \& \quad X_{k,t}(x+1) = e^{it} X_{k,t}(x).
\] (23)

Since the system $\{\Psi_{k,t}(x) : k \in \mathbb{N}\}$ for $t \in ((-\pi, \pi) \setminus A)$ form a Riesz basis, we have
\[
f_t(x) = \sum_{k \in \mathbb{N}} a_k(t) \Psi_{k,t}(x),
\] (24)
where $\mathbb{N} = \{1, 2, \ldots\}$,
\[
a_k(t) = \int_0^1 f_t(x) X_{k,t}(x) \, dx,
\] (25)
\[
X_{k,t}(x) = \frac{1}{\alpha_k(t)} \Psi_{k,t}^*, \quad \alpha_k(t) = (\Psi_{k,t}, \Psi_{k,t}^*),
\] (26)
and $\Psi_{k,t}^*$ is the normalized eigenfunctions of $\left(T_t^{(m)}\right)^*$. Using (24) in (21), we get
\[
f(x) = \frac{1}{2\pi} \int_{(-\pi, \pi)} \sum_{k \in \mathbb{N}} a_k(t) \Psi_{k,t}(x) \, dt.
\] (27)

Thus, to construct the spectral expansion we need to consider the following.

(i) The existence of the integral over $(-\pi, \pi)$ of the expression $a_k(t) \Psi_{k,t}$
(ii) The investigation of the term by term integration in (27).

Now we are ready to describe the scheme of this paper. In [5] the spectrality at $\infty$ of the operator $T^{(m)}$ for the case $m = 1$ was investigated in detail. From the Theorem 1.1 of [11] we immediately obtain that if one of the Condition 1 and Condition 2 hold, then the operator $T^{(m)}$ for general $m$ is an asymptotically spectral operator (see Theorem 1). According to the definition of the spectrality at $\infty$ given in [5] for the case $m = 1$ (see Definition 3.2 of [5]), and taking into account that the eigenfunctions of $T_t^{(m)}$ for almost all $t$ form a Riesz basis in $L^2_\mathbb{R}(0, 1)$ we give the following definition of the asymptotic spectrality.

Definition 2 The operator $T^{(m)}$ is said to be an asymptotically spectral operator if there exists a positive constant $M$ such that
\[
\sup_{\gamma \in \text{R}(M)} \sup_{t \in (-\pi, \pi)} \|e(t, \gamma)\| < \infty,
\] (28)
where $\text{R}(M)$ is the ring consisting of all sets which are the finite union of the half closed rectangles lying in $\{\lambda \in \mathbb{C} : |\lambda| > M\}$.

The main result of this paper is the construction of the spectral expansion for each $f \in L^m_\mathbb{R}(-\infty, \infty)$ if one of the Condition 1 and Condition 2 holds. For this we introduce new concepts as singular quasimomentum and essential spectral singularities (ESS) (see Definition 3 in the next section) and consider the effect of these concepts to the spectral expansion. In particular, for $m = 1$ we obtain the spectral expansion for the operator generated by
\[
y^{(n)} + p_1 y^{(n-1)} + p_2 y^{(n-2)} + p_3 y^{(n-3)} + \ldots + p_n y,
\]
if $n = 2\mu + 1$ or if $n = 2\mu$ and (17) holds. If $n = 2\mu$ and $r = 0$ then in general the operator $T^{(1)}$ is not asymptotically spectral operator (see [12]) and this case will not be considered in this paper. Thus in this paper we investigate the spectral expansion for asymptotically spectral differential operators generated by the system of differential expressions with periodic complex-valued coefficients by using the singular quasimomentum and ESS.
2 Main Results

To consider the asymptotic spectrality and spectral expansion we need to study the series

$$
\sum_{k \in \mathbb{N}} (f, X_{k,t}) \Psi_{k,t}(x).
$$

(29)

For this let us estimate the remainder

$$
R_l(x,t) = \sum_{k > l} (f, X_{k,t}) \Psi_{k,t}(x),
$$

(30)

where \( l \geq N_1 \) and \( N_1 \) is defined in (20), of the series (29) by using the notations and results of Theorem 1.1 of [11].

Lemma 1 Suppose one of the Condition 1 and Condition 2 holds. Let \( J \) be a subset of the set \( \mathbb{Z}(s) \), where

$$
\mathbb{Z}(s) = \{(k, j) : k \in \mathbb{N}, |k| \geq s, j = 1, 2, \ldots, m\},
$$

(31)

\( s \geq N_0 \) and \( N_0 \) is defined in Remark 1. There exist a positive constant \( c \), independent of \( t, J \) and \( f \) such that

$$
\| \sum_{(k,j) \in J} (f, X_{k,j,t}) \Psi_{k,j,t} \|^2 \leq c \left( \sum_{(k,j) \in J} |(f, e_j e^{i(2\pi k + t) x})|^2 + \|f\|^2 \right)
$$

(32)

and

$$
\| \sum_{(k,j) \in J} (f, X_{k,j,t}) \Psi_{k,j,t}(x) \|^2 \leq c \|f\|^2
$$

(33)

for \( f \in L_2^n(0, 1) \) and \( t \in (-\pi, \pi] \), where \( \{e_i : i = 1, 2, \ldots, m\} \) is a standard basis of \( \mathbb{C}^m \) defined in (5), \( \| \cdot \| \) and \( (\cdot, \cdot) \) denotes the norm and inner product of \( L_2^n(0, 1) \).

Proof. During the proof of the lemma we denote by \( c_1, c_2, \ldots \) the positive constants that do not depend on \( t, J \) and \( f \). They will be used in the sense that there exists \( c_i \) such that the inequality holds. We prove the lemma when Condition 1 holds. The prove of the case when Condition 2 holds is the same. Moreover, formulas (15) and (16) show that without loss of generality and for simplicity the notations it can be assumed that \( r = 0 \). Then \( e(t + ir) = 1 \) for \( t \in (-\pi, \pi] \). Now we prove (32) by showing that the inequalities

$$
\sum_{(k,j) \in J} |(f, X_{k,j,t})|^2 \leq c_1 \left( \sum_{(k,j) \in J} |(f, e_j e^{i(2\pi k + t) x})|^2 + \frac{1}{\sqrt{s}} \|f\|^2 \right),
$$

(34)

$$
\| \sum_{(k,j) \in J} (f, X_{k,j,t}) \Psi_{k,j,t}(x) \|^2 \leq c_2 \sum_{(k,j) \in J} |(f, X_{k,t})|^2
$$

(35)

hold. Under the above assumption, it follows from (16) that

$$
|(f, X_{k,j,t})|^2 \leq c_3 \left( |(f, u_j e^{i(2\pi k + t) x})|^2 + \|f\|^2 \|k^{-1} \ln |k|\|^2 \right)
$$

(36)

for \( |k| \geq N_0 \). Since \( \{u_1, u_2, \ldots, u_m\} \) is a basis of \( \mathbb{C}^m \) and \( \{e_i e^{i(2\pi k + t) x} : k \in \mathbb{Z}, i = 1, 2, \ldots, m\} \) is an orthonormal basis in \( L_2^n(0, 1) \) we have

$$
\sum_{(k,j) \in J} |(f, u_j e^{i(2\pi k + t) x})|^2 \leq c_4 \sum_{(k,j) \in J} |(f, e_j e^{i(2\pi k + t) x})|^2 \leq c_4 \|f\|^2.
$$

(37)
Thus (34) follows from (36) and the first inequality of (37).

Now we prove (35). For this we use the relations

\[ \Psi_{k,j,t}(x) = v_j e^{i(2\pi k + t)x} + h_{k,j,t}(x), \quad \|h_{k,j,t}\| = O(k^{-1} \ln |k|), \]  

\[ \|(f, X_{k,j,t})h_{k,j,t}(x)\| \leq c_5 \left( |(f, X_{k,j,t})|^2 + |k^{-1} \ln |k||^2 \right) \]  

obtained from (15) for \(|k| \geq N_0\) under the above assumption. By (36) and (37)

\[ \sum_{(k,j) \in J} |(f, X_{k,j,t})|^2 \leq c_6 \|f\|^2. \]  

This and (39) imply that the series

\[ \sum_{(k,j) \in J} (f, X_{k,j,t})v_j e^{i(2\pi k + t)x} \quad \& \quad \sum_{(k,j) \in J} (f, X_{k,j,t})h_{k,j,t}(x) \]

converge in the norm of \(L^2_w(0, 1)\) and by (38) we have

\[ \| \sum_{(k,j) \in J} (f, X_{k,j,t})\Psi_{k,j,t}(x) \|^2 \leq 2S_1 + 2S_2^2, \]  

where

\[ S_2 = \| \sum_{(k,j) \in J} (f, X_{k,j,t})h_{k,j,t} \|, \]

\[ S_1 = \| \sum_{(k,j) \in J} (f, X_{k,j,t})v_j e^{i(2\pi k + t)x} \|^2 \leq c_7 \sum_{(k,j) \in J} |(f, X_{k,j,t})|^2. \]  

Now let us estimate \(S_2\). It follows from the second equality of (38) that

\[ S_2 \leq c_8 \sum_{(k,j) \in J} |(f, X_{k,j,t})| \frac{\ln |k|}{|k|}. \]

Now using the Schwarz inequality for \(L_2\) we obtain

\[ S_2^2 = ( \sum_{(k,j) \in J} |(f, X_{k,j,t})|^2 ) O(s^{-\frac{3}{2}}). \]

Therefore (35) follows from (41)-(44). Thus (34) and (35) and hence (32) is proved. It with the second inequality of (37) yields (33).  

Theorem 1 If one of the Condition 1 and Condition 2 holds then \(T^{(m)}\) is an asymptotically spectral operator.

Proof. Let \(M\) be a positive constant such that if \(\lambda_{k,j}(t) \in \{ \lambda \in \mathbb{C} : |\lambda| > M \}\), then \((k, j) \in \mathbb{Z}(N_0)\) for all \(t \in (-\pi, \pi]\), where \(\mathbb{Z}(s)\) is defined in (31). If \(\gamma \in R(M)\), where \(R(M)\) is defined in Definition 2, then \(\gamma\) encloses finite number of the eigenvalues of \(T^{(m)}_l\). Thus, there exists a finite subset \(J\) of \(\mathbb{Z}(N_0)\) such that the eigenvalue \(\lambda_{k,j}(t)\) lies inside \(\gamma\) if and only if \((k, j) \in J\). Moreover, by Remark 1, \(\lambda_{k,j}(t)\) for \((k, j) \in J\) is a simple eigenvalue. It is well-known that the simple eigenvalues are the simple poles of the Green function of \(T^{(m)}_l\) and the projection \(e(t, \gamma)\) has the form

\[ e(t, \gamma)f = \sum_{(k,j) \in J} (f, X_{k,j,t})\Psi_{k,j,t}. \]  

Therefore the proof of the theorem follows from (33) and Definition 2.
Now we are ready to consider the Spectral Expansion for \( T^{(m)} \), when Condition 1 or Condition 2 holds. As we noted in the introduction (see (i) and (iii)) for the spectral expansion we need to consider the integrals of the expression \( a_k(t) \Psi_{k,t}(x) \) over \((-\pi, \pi]\) for almost all \( x \) and the term by term integration of the series in \((27)\). The functions \( \Psi_{k,t}(x) \) and \( X_{k,t}(x) \) for each \( x \in [0, 1] \) are defined in \((-\pi, \pi]\) and in \((-\pi, \pi] \) for \( k \leq N_1 \) and \( k > N_1 \) respectively, because the corresponding eigenvalue \( \lambda_k(t) \) is simple (see Remark 1 and the definition of \( N_1 \) in \((20)\)). Since \( A \) is a finite set the integrals over \((-\pi, \pi]\) and \((-\pi, \pi]\) are the same. Now using Lemma 1 we prove the following

**Theorem 2** If one of the Condition 1 and Condition 2 are satisfied, then for each \( f \in L_2^m(\mathbb{R}), (-\infty, \infty) \) the following equality holds

\[
f(x) = \frac{1}{2\pi} \int_{(-\pi, \pi]} \sum_{k \leq N_1} a_k(t) \Psi_{k,t}(x) dt + \frac{1}{2\pi} \sum_{k > N_1, t} a_k(t) \Psi_{k,t}(x) dt. \quad (46)
\]

The series in \((46)\) converges in the norm of \( L_2^m(a, b) \) for every \( a, b \in \mathbb{R} \).

**Proof.** As in the proof of Lemma 1 we prove \((46)\) when Condition 1 holds and without loss of generality assume that \( r = 0 \). The proof of the case when Condition 2 holds is the same. It follows from \((21)\) that if \( f \in L_2^m(-\infty, \infty) \) then \( f_t \in L_2^m(0, 1) \) for almost all \( t \). Therefore using \((32)\) for \( J = \mathbb{Z}(s) \) and \( f = f_t \) we obtain

\[
\left\| \sum_{(k,j) \in \mathbb{N}(s)} (f_t, X_{k,j,t}) \Psi_{k,j,t}(x) \right\|^2 \leq c \left( \sum_{(k,j) \in \mathbb{N}(s)} \left| (f_t, e_j e^{i(2\pi k + t)x}) \right|^2 + \frac{\|f_t\|^2}{\sqrt{s}} \right). \quad (47)
\]

for almost all \( t \). By \((22)-(24)\),

\[
\sum_{(k,j) \in \mathbb{N}(s)} (f_t, X_{k,j,t}) \Psi_{k,j,t}(x + 1) = e^{it} \sum_{(k,j) \in \mathbb{N}(s)} (f_t, X_{k,j,t}) \Psi_{k,j,t}(x).
\]

and hence by \((47)\) we have

\[
\left\| \sum_{(k,j) \in \mathbb{N}(s)} (f_t, X_{k,j,t}) \Psi_{k,j,t} \right\|^2_{(p, p)} \leq 2p c \left( \sum_{(k,j) \in \mathbb{N}(s)} \left| (f_t, e_j e^{i(2\pi k + t)x}) \right|^2 + \frac{\|f_t\|^2}{\sqrt{s}} \right). \quad (48)
\]

On the other hand by Parseval equality for \( T^{(m)}(0_m) \),

\[
\sum_{(k,j) \in \mathbb{N}(s)} \left( \int_{(-\pi, \pi]} \left| (f_t, e_j e^{i(2\pi k + t)x}) \right|^2 dt \right) = \int_{(-\pi, \pi]} \|f_t\|^2 dt.
\]

Therefore using \((48)\), \((21)\), \((30)\) and the notation of Lemma 3.1 of [11] we obtain

\[
\int_{(-\pi, \pi]} \int_{(-\pi, \pi]} |R_t(x, t)|^2 dxdt \to 0 \quad (49)
\]

as \( l \to \infty \). Thus by Fubini theorem \( R_t(x, t) \) for \( l \geq N_1 \) is integrable with respect to \( t \) for almost all \( x \). Now using the obvious inequality

\[
\left| \int_{(-\pi, \pi]} f(t) dt \right|^2 \leq 2\pi \int_{(-\pi, \pi]} |f(t)|^2 dt,
\]

\((30)\) and \((49)\) we obtain

\[
\left\| \int_{(-\pi, \pi]} \sum_{k \leq l} a_k(t) \Psi_{k,t} dt \right\|^2 \leq 2\pi \int_{(-\pi, \pi]} \int_{(-\pi, \pi]} \left| \sum_{k \leq l} a_k(t) \Psi_{k,t}(x) \right|^2 dtdx \to 0 \quad (50)
\]
as \( l \to \infty \). Therefore
\[
\sum_{k \geq l} a_k(t) \Psi_{k,t}(x) \tag{51}
\]
for \( l \geq N_1 \) and hence \( a_k(t) \Psi_{k,t}(x) \) for \( k > N_1 \) are integrable and we have
\[
\int_{(-\pi,\pi]} \sum_{k \geq N_1} a_k(t) \Psi_{k,t}(x) dt = \sum_{k \geq N_1} \int_{(-\pi,\pi]} a_k(t) \Psi_{k,t}(x) dt, \tag{52}
\]
where the last series converges in the norm of \( L^p_{\omega}(\mathbb{R}^n, (-p, p)) \) for every \( p \in \mathbb{N} \). The existence of the first integral in (46) follows from (24) and the integrability of \( f_t(x) \) and (51). Now using (27) we get the proof of the theorem. \( \blacksquare \)

To obtain the spectral expansion in term of \( t \) we need to consider the existence of
\[
\int_{(-\pi,\pi]} a_k(t) \Psi_{k,t}(x) dt \tag{53}
\]
for \( k \leq N_1 \). To consider the existence of (53) we classify the spectral singularity defined in Definition 1. It is well-known that [6] if \( \lambda_k(t) \) is the simple eigenvalues of \( T_t^{(m)} \) then the projection \( e(t, \gamma) \) defined in Definition 1 has the form
\[
e(t, \gamma)f = (f, X_{k,t})\Psi_{k,t} \tag{54}
\]
where \( \gamma \) contains inside only the eigenvalue \( \lambda_k(t) \) of \( T_t^{(m)} \). One can readily see
\[
||e(t, \gamma)|| = \frac{1}{|\alpha_k(t)|}. \tag{55}
\]
Moreover, if \( \lambda_k(t) \) is a simple eigenvalue then \( |\alpha_k(t)| \neq 0 \) and the function \( \frac{1}{|\alpha_k|} \) is continuous in some neighborhood of \( t \). Therefore, it follows from (55) and Definitions 1 that the set of spectral singularities is the subset of the set of the multiple eigenvalues \( a_1, a_2, ..., a_s \) defined in the introduction (see (19)). Thus there are at most finite number of spectral singularities denoted by \( a_1, a_2, ..., a_s \), where \( s \leq p \), if Condition 1 or Condition 2 holds.

By (55) and Definition 1 for \( j \leq s \) there exist \( t_0 \in A_j, \) where \( A_j \) is defined in (19), \( k \in \mathbb{N} \) and a sequence \( \{t_n\} \) such that \( \lambda_k(t_0) = a_j, t_n \to t_0 \) and \( \frac{1}{|\alpha_k(t_n)|} \to \infty \) as \( n \to \infty \), that is, \( \frac{1}{|\alpha_k|} \) is unbounded in any deleted neighborhood \( U \) of \( t_0 \). If \( \lambda_k(t_0) \) is not a spectral singularity then for some deleted neighborhood \( U \) of \( t_0 \) we have
\[
\sup_{t \in U} \frac{1}{|\alpha_k(t)|} < \infty. \tag{56}
\]
Thus the boundlessness of \( \frac{1}{|\alpha_n|} \) is the characterization of the spectral singularities. The considerations of the spectral singularities play the crucial rule for the investigations of the spectrality of \( T^{(m)} \). However our aim is the construction the spectral expansion and by Theorem 2 the spectral expansion is connected with the existence of (53) for all \( k \). If (56) holds, then arguing as above one can readily see that
\[
\int_U \int_{(0,1)} |a_k(t) \Psi_{k,t}(x)|^2 dx dt \leq \sup_{t \in U} \frac{1}{|\alpha_k(t)|} \int_U \int_{(0,1)} |f_t(x)|^2 dx dt < \infty \tag{57}
\]
Therefore by Fubini theorem the integral
\[
\int_U \frac{1}{\alpha_k(t)} (f, \Psi_{k,t}) \Psi_{k,t}(x) dt \tag{58}
\]
exists for almost all \( x \) if \( \lambda_k(t_0) \) is not a spectral singularity. In general, the converse statement is not true, since \( \frac{1}{\lambda_k} \) may have an integrable boundlessness, and the integral (58) may exist when \( \lambda_k(t_0) \) is a spectral singularity. Hence to construct the spectral expansion for the operator \( T^{(m)} \) we need to introduce a new concept connected with the existence of the integral (58) for \( U \subset (-\pi, \pi) \). Therefore we introduce the following notions for the construction of the spectral expansion. Note that everywhere the integral over \( U \) denotes the integral over \( U \setminus A \).

**Definition 3** Spectral singularity \( \lambda_0 \) is said to be an essential spectral singularity (ESS) of \( T^{(m)} \) if there exist \( k \in \mathbb{N} \), \( t_0 \in (-\pi, \pi] \) and \( f \in L^m_{2\pi}(-\infty, \infty) \) such that \( \lambda_0 = \lambda_k(t_0) \) and for each \( \varepsilon \) the expression

\[
\frac{1}{\lambda_k(t)} \int_{\Gamma_k} f(x,t) \Psi_{k,t}(x) \, dx
\]

for almost all \( x \) is not integrable on \((t_0 - \varepsilon, -t_0 + \varepsilon)\). In this case \( t_0 \) is called a singular quasimomentum.

Let \( S \) be the set of all \( k \in \mathbb{N} \) such that \( \Gamma_k = \{ \lambda_k(t) : t \in (-\pi, \pi] \} \) contains ESS. It follows from Definition 3 that, if \( k \notin S \) then the integral (53) exist. Therefore Theorem 2 can be written in the form.

**Theorem 3** If one of the **Condition 1** and **Condition 2** are satisfied, then for each \( f \in L^m_{2\pi}(-\infty, \infty) \) the equality

\[
\int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{2\pi} \sum_{k \in S} a_k(t) \Psi_{k,t}(x) \, dt + \frac{1}{2\pi} \sum_{k \notin S} a_k(t) \Psi_{k,t}(x) \, dt
\]

holds. The series in (59) converges in the norm of \( L^m_{2\pi}(a,b) \) for every \( a, b \in \mathbb{R} \). In particular, if \( T^{(m)} \) has no ESS, then

\[
\int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{2\pi} \sum_{k = 1}^{\infty} \int_{0}^{2\pi} a_k(t) \Psi_{k,t}(x) \, dt.
\]

Let \( E = \{ t_1, t_2, ..., t_s \} \), where \( -\pi < t_1 < t_2 < ... < t_s \leq \pi \), be the set of all singular quasimomenta of \( T^{(m)} \). Define \( I(\delta) \) by

\[
I(\delta) = (-\pi, \pi) \setminus \bigcup_{j=1}^{s} (t_j - \delta, t_j + \delta),
\]

where \( \delta < \frac{1}{2} \min_j \{ t_1 + \pi, t_j - t_j - 1 \} : j = 2, 3, ..., s \} \), that is, the intervals \((t_j - \delta, t_j + \delta)\) for \( j = 2, 3, ..., s \) are pairwise disjoint.

**Theorem 4** If one of the **Condition 1** and **Condition 2** are satisfied, then for each \( f \in L^m_{2\pi}(-\infty, \infty) \) the following spectral expansion holds

\[
f(x) = \frac{1}{2\pi} \lim_{\delta \to 0} \left( \sum_{k \in S} \int_{I(\delta)} a_k(t) \Psi_{k,t}(x) \, dt \right) + \frac{1}{2\pi} \sum_{k \notin S} \int_{0}^{2\pi} a_k(t) \Psi_{k,t}(x) \, dt,
\]

where the series converges in the norm of \( L^m_{2\pi}(a,b) \) for every \( a, b \in \mathbb{R} \).

**Proof.** Since

\[
\sum_{k \in S} a_k(t) \Psi_{k,t}(x)
\]

is integrable over \((-\pi, \pi] \), we have

\[
\lim_{\delta \to 0} \int_{(t_j - \delta, t_j + \delta)} a_k(t) \Psi_{k,t}(x) \, dt = 0
\]

(64)
Hence the set $S$ in finite dimensional spaces (see Chapter 2 of [3]) one can prove that

$$
\int_{I(\delta)} a_k(t)\Psi_{k,t}(x)dt
$$

exists for all $k \in \mathbb{S}$. Therefore (62) follows from (59).

**Remark 2** By Definition 3 the elements $a_k(t)\Psi_{k,t}$ of the set $\{a_k(t)\Psi_{k,t}(x) : k \in \mathbb{S}\}$ are not integrable on $(-\pi, \pi]$. The sum of elements of this set is integrable (see (59)) due to the cancellations of the singular parts of the nonintegrable elements. Thus, to obtain the spectral expansion (see Theorems 3 and 4) we huddle together the nonintegrable on $(-\pi, \pi]$ elements. The following example shows that, in the general case, for the considerations of the integrals over $(-\pi, \pi]$ the huddling over $\mathbb{S}$ (see (59) and (62)) is necessary and one can not divide the set $\mathbb{S}$ into two disjoint subsets $\mathbb{S}_1$ and $\mathbb{S}_2$ such that the summations

$$
\sum_{k \in \mathbb{S}_1} a_k(t)\Psi_{k,t} \quad \& \quad \sum_{k \in \mathbb{S}_2} a_k(t)\Psi_{k,t}
$$

are integrable over $(-\pi, \pi]$.

**Example 1** For the singular quasimomentum $t_i$ denote by $\mathbb{S}(i)$ the set of all $k$ for which $a_k(t)\Psi_{k,t}$ is nonintegrable over the set $(t_i - \delta, t_i + \delta)$. It is clear that $\mathbb{S} = \bigcup_{i=1,2,\ldots,s} \mathbb{S}(i)$. Let $\mathbb{S}(1) = \{1, 2\}$, $\mathbb{S}(2) = \{2, 3\}$, $\ldots$, $\mathbb{S}(s) = \{s, s + 1\}$. Then $\mathbb{S} = \{1, 2, \ldots, s + 1\}$ and $a_k(t)\Psi_{k,t}$ is nonintegrable over $(-\pi, \pi]$ for $k = 1, 2, \ldots, s + 1$. If $s = 2$ then it is clear that for any proper subset $\mathbb{S}_1$ of $\mathbb{S}$ the summations in (66), where $\mathbb{S}_2 = \mathbb{S}\setminus\mathbb{S}_1$ are not integrable over $(-\pi, \pi]$. This statement for arbitrary $s$ can be proved by induction method.

**Remark 3** If we consider the integral over $(-\pi, \pi]$ as sum of integrals over $I(\delta)$ and

$$(t_j - \delta, t_j + \delta)$$

for $j = 1, 2, \ldots, s$, where $I(\delta)$ is defined in (61), then the first integral in (59) can be written in the form

$$
\int_{(-\pi, \pi]} \sum_{k \in \mathbb{S}} a_k(t)\Psi_{k,t}dt = \sum_{k \in \mathbb{S}} \int_{I(\delta)} a_k(t)\Psi_{k,t}dt + \sum_{i=1}^{s} \int_{(t_i - \delta, t_i + \delta)} \sum_{k \in \mathbb{S}} a_k(t)\Psi_{k,t}dt.
$$

By the definition of $\mathbb{S}(i)$ (see Example 1) we have

$$
\int_{(t_j - \delta, t_j + \delta)} \sum_{k \in \mathbb{S}} a_k(t)\Psi_{k,t}dt = \sum_{k \in \mathbb{S}(i)\setminus(t_j - \delta, t_j + \delta)} a_k(t)\Psi_{k,t}dt + \sum_{k \in \mathbb{S}(i)} a_k(t)\Psi_{k,t}dt.
$$

For each singular quasimomentum $t_i \in \mathbb{E}$ denote by $\Lambda_1(t_i), \Lambda_2(t_i), \ldots, \Lambda_{s_1}(t_i)$ the different ESS of $T^{(m)}$ lying in $\sigma(T^{(m)}_i)$ and put $\mathbb{S}(i,j) := \{k \in \mathbb{S}(i) : \Lambda_k(t_i) = \Lambda_j(t_i)\}$. Thus the set of ESS is $\{\Lambda_j(t_i) : i = 1, 2, \ldots, s; j = 1, 2, \ldots, s_1\}$. It is clear that $\mathbb{S}(i,j) \cap \mathbb{S}(i,v) = \emptyset$ for $j \neq v$. Hence the set $\mathbb{S}(i)$ can be divided into pairwise disjoint subsets $\mathbb{S}(i,j)$ for $j = 1, 2, \ldots, s_1$. On the other hand, using (45) and the well-known argument of the general perturbation theory in finite dimensional spaces (see Chapter 2 of [3]) one can prove that

$$
e(f(t, \gamma))f_t(x) = \sum_{k \in \mathbb{S}(i,j)} (f_t, X_{k,t})\Psi_{k,t}(x), \quad \forall t \in (t_i - \delta, t_i + \delta) \setminus \{t_j\}
$$

and $e(t, \gamma)f_t(x)$ is integrable in $(t_i - \delta, t_i + \delta)$ for almost all $x$, where $e(t, \gamma)$ is the projection defined in Definition 1, $\delta$ is a sufficiently small number and $\gamma$ contains inside only the
eigenvalues $\lambda_k(t)$ for $k = S(i,j)$ of $T_i^{(m)}$ for $t \in (t_i - \delta, t_i + \delta)$. Therefore the summations over $S(i)$ in (68) can be written as the sum of summations over $S(i,j)$ for $j = 1, 2, ..., s_i$:

$$
\int_{(t_j - \delta, t_j + \delta)} \sum_{k \in S(i)} a_k(t) \Psi_{k, t} dt = \sum_{j=1}^{s_i} \left( \int_{(t_j - \delta, t_j + \delta)} \sum_{k \in S(i,j)} a_k(t) \Psi_{k, t} dt \right).
$$

(69)

We say that the set $\{ a_k(t) \Psi_{k, t}(x) : k \in S(i,j) \}$ is a bundle corresponding to the ESS $\Lambda_j(t_i)$. For almost all $x$ the total sum $S(x, t)$ of elements of the bundle is an integrable function in $(t_i - \delta, t_i + \delta)$. However, each element of the bundle is nonintegrable over $(t_i - \delta, t_i + \delta)$ and we huddle together the nonintegrable elements. Hence for the considerations of the integrals over $(t_i - \delta, t_i + \delta)$ the huddling over $S(i,j)$ is necessary. Thus in any case the huddling in the spectral expansion of $T^{(m)}$ is necessary if it has the ESS. Using (67)-(69) one can minimize the number of terms in the huddling in the spectral expansion. In theorems 3 and 4 to avoid the complicated notations, we prefer to use the integrals over $(-\pi, \pi]$ and hence the summations (huddling) over $S$.

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