THE MODULI SPACE OF 4-DIMENSIONAL NILPOTENT COMPLEX ASSOCIATIVE ALGEBRAS

ALICE FIALOWSKI AND MICHAEL PENKAVA

ABSTRACT. In this paper, we study 4-dimensional nilpotent complex associative algebras. This is a continuation of the study of the moduli space of 4-dimensional algebras. The non-nilpotent algebras were analyzed in an earlier paper. Even though there are only 15 families of nilpotent 4-dimensional algebras, the complexity of their behavior warranted a separate study, which we give here.

1. Construction of the algebras by extensions

The authors and collaborators have been carrying out a construction of moduli spaces of low dimensional complex and real Lie and associative algebras in a series of papers. Our method of constructing the moduli spaces of such algebras is based on the principal that algebras are either simple or can be constructed as extensions of lower dimensional algebras. There is a classical theory of extensions, which was developed by many contributors going back as early as the 1930s. In [2], we gave a description of the theory of extensions of an algebra $W$ by an algebra $M$. Consider the diagram

$$0 \rightarrow M \rightarrow V \rightarrow W \rightarrow 0$$

of associative $K$-algebras, so that $V = M \oplus W$ as a $K$-vector space, $M$ is an ideal in the algebra $V$, and $W = V/M$ is the quotient algebra. Suppose that $\delta \in C^2(W) = \text{Hom}(T^2(W), W)$ and $\mu \in C^2(M)$ represent the algebra structures on $W$ and $M$ respectively. We can view $\mu$ and $\delta$ as elements of $C^2(V)$. Let $T^{k,l}$ be the subspace of $T^{k+l}(V)$ given recursively by

$$T^{0,0} = K$$

$$T^{k,l} = M \otimes T^{k-1,l} \oplus V \otimes T^{k,l-1}.$$ 

Let $C^{k,l} = \text{Hom}(T^{k,l}, M) \subseteq C^{k+l}(V)$. If we denote the algebra structure on $V$ by $d$, we have

$$d = \delta + \mu + \lambda + \psi,$$

where $\lambda \in C^{1,1}$ and $\psi \in C^{0,2}$. Note that in this notation, $\mu \in C^{2,0}$. Then the condition that $d$ is associative: $[d,d] = 0$ gives the following relations:

$$[\delta, \lambda] + \frac{1}{2}[\lambda, \lambda] + [\mu, \psi] = 0,$$  

The Maurer-Cartan equation

$$[\mu, \lambda] = 0,$$  

The compatibility condition

$$[\delta + \lambda, \psi] = 0,$$  

The cocycle condition

Date: May 22, 2014.

Research of the first author was partially supported by OTKA grant K77757 and the second author by grants from the University of Wisconsin-Eau Claire.
Since $\mu$ is an algebra structure, $[\mu, \mu] = 0$, so if we define $D_\mu$ by $D_\mu(\varphi) = [\mu, \varphi]$, then $D_\mu^2 = 0$. Thus $D_\mu$ is a differential on $C(V)$. Moreover $D_\mu : C^k,l \to C^{k+1,l}$.

Let
\[
Z_{\mu}^{k,l} = \ker(D_\mu : C^{k,l} \to C^{k+1,l}), \quad \text{the (k,l)-cocycles}
\]
\[
B_{\mu}^{k,l} = \text{Im}(D_\mu : C^{k-1,l} \to C^{k,l}), \quad \text{the (k,l)-coboundaries}
\]
\[
H_{\mu}^{k,l} = Z_{\mu}^{k,l}/B_{\mu}^{k,l}, \quad \text{the } D_\mu\text{-cohomology}
\]

Then the compatibility condition means that $\lambda \in Z^{1,1}$. If we define $D_{\delta+\lambda}(\varphi) = [\delta + \lambda, \varphi]$, then it is not true that $D_{\delta+\lambda}^2 = 0$, but $D_{\delta+\lambda}D_\mu = -D_\mu D_{\delta+\lambda}$, so that $D_{\delta+\lambda}$ descends to a map $D_{\delta+\lambda} : H_{\mu}^{k,l} \to H_{\mu}^{k,l+1}$, whose square is zero, giving rise to the $D_{\delta+\lambda}$-cohomology $H_{\mu,\delta+\lambda}^{k,l}$. If the pair $(\lambda, \psi)$ gives rise to an algebra $d$, and $(\lambda, \psi')$ give rise to another algebra $d'$, then if we express $\psi' = \psi + \tau$, it is easy to see that $[\mu, \tau] = 0$, and $[\delta + \lambda, \tau] = 0$, so that the image $\bar{\tau}$ of $\tau$ in $H_{\mu,2}^{0,1}$ is a $D_{\delta+\lambda}$-cocycle, and thus $\tau$ determines an element $\{\bar{\tau}\} \in H_{\mu,\delta+\lambda}^{0,2}$.

If $\beta \in C^{0,1}$, then $g = \exp(\beta) : T(V) \to T(V)$ is given by $g(m,w) = (m + \beta(w), w)$. Furthermore, $g^* = \exp(-\text{ad}_{\beta}) : C(V) \to C(V)$ satisfies $g^*(d) = d'$, where $d' = \delta + \mu + \lambda' + \psi'$ with $\lambda' = \lambda + [\mu, \beta]$ and $\psi' = \psi + [\delta + \lambda, \frac{1}{2}[\mu, \beta], \beta]$. In this case, we say that $d$ and $d'$ are equivalent extensions in the restricted sense. Such equivalent extensions are also equivalent as algebras on $T(V)$.

Let $G_{\mu} = \text{GL}(M) \times \text{GL}(W) \subseteq \text{GL}(V)$. If $g \in G_{\mu}$, then $g^* : C^{k,l}(W) \to C^{k,l}(W)$, and $g^* : C^k(W) \to C^k(W)$, so $\delta' = g^*(\delta)$ and $\mu' = g^*(\mu)$ are algebra structures on $T(M)$ and $T(W)$ respectively. The group $G_{\delta,\mu}$ is the subgroup of $G_{\mu}$ consisting of those elements $g$ such that $g^*(\delta) = \delta$ and $g^*(\mu) = \mu$. Then $G_{\delta,\mu}$ acts on the restricted equivalence classes of extensions, giving the equivalence classes of general extensions. Also $G_{\delta,\mu}$ acts on $H_{\mu}^{k,l}$, and induces an action on the classes $\lambda$ of $\lambda$ giving a solution to the MC equation.

Next, consider the group $G_{\delta,\mu,\lambda}$ consisting of the automorphisms $h$ of $V$ of the form $h = g \exp(\beta)$, where $g \in G_{\delta,\mu}$, $\beta \in C^{0,1}$ and $\lambda = g^*(\lambda) + [\mu, \beta]$. If $d = \delta + \mu + \lambda + \psi + \tau$, then $h^*(d) = \delta + \mu + \lambda + \psi + \tau'$ where
\[
\tau' = g^*(\psi) - \psi + [\delta + \lambda, \frac{1}{2}[\mu, \beta], \beta] + g^*(\tau).
\]

Thus the group $G_{\delta,\mu,\lambda}$ induces an action on $H_{\mu,\delta+\lambda}^{0,2}$ given by $\{\bar{\tau}\} \to \{\bar{\tau}'\}$.

The general group of equivalences of extensions of the algebra structure $\delta$ on $W$ by the algebra structure $\mu$ on $M$ is given by the group of automorphisms of $V$ of the form $h = \exp(\beta)g$, where $\beta \in C^{0,1}$ and $g \in G_{\delta,\mu}$. Then we have the following classification of such extensions up to equivalence.

**Theorem 1.1** ([2]). The equivalence classes of extensions of $\delta$ on $W$ by $\mu$ on $M$ is classified by the following:

1. Equivalence classes of $\bar{\lambda} \in H_{\mu}^{1,1}$ which satisfy the MC equation
\[
[\delta, \lambda] + \frac{1}{2}[[\lambda, \lambda] + [\mu, \psi]] = 0
\]

for some $\psi \in C^{0,2}$, under the action of the group $G_{\delta,\mu}$. 
(2) Equivalence classes of \( \{ \vec{\tau} \} \in H^{0,2}_{\mu,\lambda} \) under the action of the group \( G_{\delta,\mu,\lambda} \).

Equivalent extensions will give rise to equivalent algebras on \( V \), but it may happen that two algebraic structures arising from nonequivalent extensions are equivalent. This is because the group of equivalences of extensions is the group of invertible block upper triangular matrices on the space \( V = M \oplus W \), whereas the equivalence classes of algebraic structures on \( V \) are given by the group of all invertible matrices, which is larger.

The fundamental theorem of finite dimensional algebras allows us to restrict our consideration of extensions to two cases. First, we can consider those extensions where \( \delta \) is a semisimple algebra structure on \( W \), and \( \mu \) is a nilpotent algebra structure on \( M \). In this paper let \( K = \mathbb{C} \). Then we can also assume that \( \psi = \tau = 0 \). Thus the classification of the extension reduces to considering equivalence classes of \( \lambda \).

Secondly, we can consider extensions of the trivial algebra structure \( \delta = 0 \) on a 1-dimensional space \( W \) by a nilpotent algebra \( \mu \). This is because a nilpotent algebra has a codimension 1 ideal \( M \), and the restriction of the algebra structure to \( M \) is nilpotent. However, in this case, we cannot assume that \( \psi \) or \( \tau \) vanish, so we need to use the classification theorem above to determine the equivalence classes of extensions. In many cases, in solving the MC equation for a particular \( \lambda \), if there is any \( \psi \) yielding a solution, then \( \psi = 0 \) also gives a solution. So the action of \( G_{\delta,\mu,\lambda} \) on \( H^{0,2} \) takes on a simpler form than the general action we described above.

In addition to the complexity which arises because we cannot take the cocycle term \( \psi \) in the extension to be zero, there is another issue that complicates the construction of the extensions. If an algebra is not nilpotent, then it has a maximal nilpotent ideal which is unique, and it can be constructed as an extension of a semisimple algebra by this unique ideal. Both the semisimple and nilpotent parts in this construction are completely determined by the algebra. Therefore, a classification of extensions up to equivalence of extensions will be sufficient to classify the algebras. This means that the equivalence classes of the module structure \( \lambda \) determine the algebras up to isomorphism.

For nilpotent algebras, we don’t have this assurance. The same algebra structure may arise by extensions of the trivial algebra structure on a 1-dimensional space by two different nilpotent algebra structures on the same \( n - 1 \)-dimensional space.

In addition, the deformation theory of nilpotent algebras is far more involved than the deformation theory of the non nilpotent algebras.

In this paper, we study the complex 4-dimensional nilpotent algebras. In [1], the following idea for construction of nilpotent algebras was discussed, and since it is simpler than the general construction, we outline the idea here.

First, if a nonzero algebra is nilpotent, then it has a nontrivial ideal \( M \) which has the property that the product of any element in \( M \) with an arbitrary element is zero. We call such an ideal completely trivial. There is a unique ideal which is maximal in the set of completely trivial ideals, which we call the kernel of the algebra. If we call the quotient of the algebra by its kernel the core of the algebra, then we see that any nilpotent algebra determines a unique kernel and core, and the algebra is given by an extension of its core by the kernel in a particularly simple manner. In the language we introduced above, we have \( \lambda = 0 \) and \( \mu = 0 \), so that the compatibility relation and the Maurer Cartan equation are satisfied trivially. In terms of the coboundary operator \( D_\delta \) given by \( D_\delta(\varphi) = 0 \), we get that the
cocycle $\psi$ is actually a $D_5$-cocycle, and equivalent cocycles determine equivalent algebras, so that the algebras are classified by the action of the group $G_{\delta,\mu,\lambda}$ on the $D_5$-cohomology classes.

All of this holds for any completely trivial ideal, and it is more convenient to study the case when we don’t assume $M$ is the kernel, even though this means that some of the algebras will be constructed in a non unique manner. In particular, let us consider the case where $d$ vanishes as well as $\mu$ and $\lambda$. Then the group $G_{\mu,\delta,\lambda}$ coincides with the group $G_{M,W}$. If we express $M = \langle e_1 \rangle$, $W = \langle e_2, \ldots, e_n \rangle$, then we can express $\psi = \psi^{1+1,j+1}_{i,j}$, and an element $g$ in $G_{M,W}$ is given in block form by a matrix $G = \begin{bmatrix} g_{1,1} & 0 \\ 0 & g_{2,2} \end{bmatrix}$. If $C = [c_{i,j}]$ is the $n \times n$ matrix determined by the coefficients of $\psi$, and $\psi' = G^*(\psi)$ is the cocycle determined by the action of $G$ on $\psi$, and $C' = [c'_{i,j}]$ is the corresponding matrix of coefficients of $\psi'$, then $C' = g_{1,1}^{-1}G^T CG$, so that $C'$ and $C$ are cogredient matrices. Therefore, the classification of cogredient matrices, or equivalently, complex bilinear forms, is a useful component of the construction.

In the next section, we classify complex bilinear forms on a 2-dimensional complex vector space and then in the following section, we use this classification to give a construction of the nilpotent 3-dimensional complex algebras, which agrees with the classification in [5]. Then in the succeeding sections we use the same idea to construct the nilpotent 4-dimensional complex algebras.

2. Classification of bilinear forms on a 2-dimensional complex vector space.

A complete classification of complex bilinear forms on a finite dimensional space was given e.g. in [5]. However, this classification does not involve a stratification of the moduli space by complex projective orbifolds, which we believe is the right way to understand this classification. Certainly, it is the correct point of view for the purpose for which we will apply it. In this paper, we will give a stratification of the moduli spaces of 2 and 3 dimensional complex bilinear forms by projective orbifolds.

Let $\beta$ be a bilinear form on $\mathbb{C}^2$ which is given by the matrix $B = (b_{ij})$ where $b_{ij} = \beta(e_i, e_j)$ in terms of some basis $\langle e_1, e_2 \rangle$. We say that two matrices $B$ and $C$ are cogredient if there is a nonsingular matrix $P$ such that $C = P^T BP$. This is precisely the condition that $B$ and $C$ represent $\beta$ with respect to different bases.

Then every nontrivial bilinear form is given up to equivalence by either a matrix of the form $B(p : q) = \begin{bmatrix} 1 & p \\ q & 0 \end{bmatrix}$ or $C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Moreover, the matrices of the form $B(p : q)$ form a projective family parameterized by the orbifold $\mathbb{P}^1/\Sigma_2$, where the action of the symmetric group $\Sigma_2$ on $\mathbb{P}^1$ is given by interchanging the projective coordinates $(p : q)$, so that $B(p : q) \sim B(q : p)$ where $\sim$ stands for the equivalence of cogredient matrices.

To see why this is true, we note first that if $\beta$ is nontrivial and $\beta(u, u) = 0$ for all $u \in \mathbb{C}^2$, then $\beta(v, u) = -\beta(u, v)$ for all $u, v \in \mathbb{C}^2$, since $0 = \beta(u + v, u + v) = \beta(u, v) + \beta(v, u)$. Since $\beta$ is nontrivial, there must be some $u$ and $v$ such that $\beta(v, u) = 1$, and the matrix of $\beta$ with respect to this basis is $C$.

Thus we may assume there is some $u$ in $\mathbb{C}^2$ such that $\beta(u, u) = 1$. We claim that there is a nonzero vector $v$ such that $\beta(v, v) = 0$. For suppose that we choose
any vector \( v \) which is linearly independent of \( u \). If \( \beta(u, v) \neq 0 \), let \( w = u + xv \), and then we compute
\[
\beta(u, w) = 1 + (\beta(u, v) + \beta(v, u))x + \beta(v, v)x^2.
\]
This is a quadratic equation in \( x \) which has a nontrivial solution \( x \neq 0 \). In terms of the basis \( u, w, x \), the matrix of \( \beta \) has the form \( B(p : q) \) for some \( (p : q) \). Moreover, it is easy to check that \( B(p : q) \sim B(x : y) \) precisely when \( (p : q) = (x : y) \) or \( (p : q) = (y : x) \).

If one studies the miniversal deformation of the moduli space of bilinear forms, one discovers that elements of the form \( B(p : q) \) have smooth deformations in a neighborhood of the element \( B(p : q) \) and do not have jump deformations, while the element \( C \) has a jump deformation to \( B(1 : -1) \) and smooth deformations in a neighborhood of \( B(1 : -1) \). We shall see that the deformations of the moduli space of bilinear forms are reproduced in the deformations of the moduli space of three dimensional complex algebras which are determined by these bilinear forms. For basic notions of deformations see [3].

3. Nilpotent 3-dimensional complex associative algebras

The moduli space of complex 3-dimensional associative algebras was constructed in [5]. Here we wish to construct the nilpotent algebras using the following observations. In [1], the author discusses what he calls central extensions of an associative algebra. These are extensions such that the ideal in the extended algebra has trivial multiplication with the entire algebra. This language derives from the similar construction of central extensions of Lie algebras. In this section, we show how to use the classification of bilinear forms as a tool in constructing the 3-dimensional nilpotent algebras.

There is only one nontrivial nilpotent 2-dimensional complex algebra, represented by \( \delta = \psi^3_2 \), on the space \( W = \langle e_2, e_3 \rangle \). If we extend \( \delta \) by a completely trivial ideal \( M = \langle e_1 \rangle \), then this extension is given by a cocycle \( \psi = \psi^2_1 c_1 + \psi^3_1 c_2 + \psi^2_3 c_3 + \psi^3_1 c_4 \). The cocycle condition \( [\delta, \psi] = 0 \) gives \( c_1 = 0 \) and \( c_2 = c_3 \). Moreover, if \( \beta = \psi^2_1 b_1 + \psi^3_1 b_2 \in C^{1,0} \), then \( [\delta, \beta] = -\psi^3 c_1 b_1 \), which means that up to a coboundary term, we can assume \( \psi = (\psi^{2,3}_1 + \psi^{3,3}_1) c_3 \). Applying an element of the group \( G_{\delta, \mu, \lambda} \) to \( \psi \) replaces \( \psi \) by an arbitrary nonzero multiple, so this means we only have two cases to study, when \( c_3 = 1 \) or \( c_3 = 0 \). The first case gives the algebra \( d = \psi^3_2 + \psi^{23}_1 + \psi^{32}_1 \), which is equivalent to the algebra \( d_{19} \) on our list of 3-dimensional algebras. The second case gives \( d = \psi^3_2 \), which is equivalent to \( d_{20}(0 : 0) \) on our list, but note that this algebra has a kernel which is 2-dimensional, so this algebra would arise in a different fashion as well.

Next, consider the trivial nilpotent 2-dimensional complex algebra given by \( \delta = 0 \). In this case the cocycle condition on \( \psi = \psi^{22}_1 c_1 + \psi^{23}_1 c_2 + \psi^{32}_1 c_3 + \psi^{33}_1 c_4 \) is trivial. Let \( C = [c_1^3 \ c_2^3] \). An element \( g \) of the group \( G_{\delta, \mu, \lambda} \) is given by an arbitrary invertible matrix \( G \) of the block diagonal form \( G = \text{diag}(g_1, G_2) \) where \( g_1 \neq 0 \) and \( G_2 \) is an invertible \( 2 \times 2 \) matrix. The action of \( g \) on \( \psi \) transforms \( \psi \) into the element \( \psi' \) whose matrix is \( g_1^{-1} G_2^T CG_2 \), which means that if the matrices representing \( \psi \) and \( \psi' \) are cogredient then they determine equivalent algebras. Using our classification of cogredient matrices, we find that we obtain a family \( d_{20}(p : q) \) given by the matrix \( B(p : q) \) in our classification of bilinear forms on a 2-dimensional complex vector space, and the algebra \( d_{21} \), given by the matrix \( C \) in this classification.
This gives all of the nontrivial 3-dimensional nilpotent complex associative algebras in a very simple fashion.

4. Classification of bilinear forms on a 3-dimensional complex vector space

If $\beta$ is a bilinear form on an $n$-dimensional space $U$, then we say that $\beta$ is decomposable if $U = V \oplus W$, where $V$ and $W$ are nontrivial subspaces of $U$ satisfying $\beta(V, W) = \beta(W, V) = 0$. Using the classification of 2-dimensional complex bilinear forms, it is easy to see that every matrix $B$ representing a nontrivial bilinear form on a complex 3-dimensional space is conjugate to a matrix of one of the six types given below.

$$B_1(p : q) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & p \\ 0 & q & 0 \end{bmatrix}, \quad B_2(p : q) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & p \\ 0 & q & 0 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$B_5 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad B_6 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$ 

The first four matrices correspond to the decomposable bilinear forms, and it follows from our classification of bilinear forms on $\mathbb{C}^2$ that the matrix of every decomposable bilinear form is conjugate to one of these six matrix types. It is not as easy to see that the matrix of any indecomposable bilinear form is conjugate to one of the last two matrices. In fact, from the results of [8], it follows that when $n$ is odd, there are exactly two nonequivalent $n \times n$ indecomposable matrices.

The matrix $B_1(0 : 0)$ is conjugate to the matrix $B_3(1 : 1)$. Moreover, the families $B_1(p : q)$ and $B_3(p : q)$ are parametrized by $\mathbb{P}^1/\Sigma_2$, so they determine projective orbifolds. Other than these identifications, all of the matrices represent distinct equivalence classes. As a consequence, we see that the moduli space of complex bilinear forms on a 3-dimensional vector space is stratified by projective orbifolds. It is also true that the deformations of the elements in the moduli space are either given by jumps between the strata, by smooth deformations which factor through a jump deformation, or by smooth deformations along a stratum. This pattern is consistent with the patterns that we have observed in moduli spaces of algebras.

The decomposition of the moduli space of $3 \times 3$ matrices we have given above has the advantages that it gives a stratification by projective orbifolds and moreover, it is easy to see which matrices are indecomposable. Nevertheless, even this decomposition is not the correct one from the point of view of deformation theory.
Consider the following 6 matrices:

\[
C_1(p : q) = \begin{bmatrix} 0 & 0 & q \\ p & 0 & 1 \end{bmatrix}, \quad C_2(p : q) = \begin{bmatrix} 0 & 0 \\ p & 0 & 1 \end{bmatrix}
\]

\[
C_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\]

\[
C_5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_6 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\]

The matrices above correspond to the algebras \(d_{78}(p : q)\), \(d_{86}(p : q)\), \(d_{81}\), \(d_{84}\), \(d_{85}\) and \(d_{87}\), which we will describe below. In most cases, there is a one to one correspondence between the \(B\) matrices above and the \(C\) matrices. However, there are a couple of very important exceptions. The matrix \(B_1(p : q)\) is cogredient to \(C_1(p : q)\) except for \((1 : 1)\) and \((0 : 0)\). The matrix \(B_1(1 : 1)\) is cogredient to \(C_5\), while the matrix \(B_1(0 : 0)\) is cogredient to \(C_2(1 : 1)\). The matrix \(B_2(p : q)\) is always cogredient to \(C_2(p : q)\), \(B_3\) is cogredient to \(C_3\), \(B_4\) is cogredient to \(C_6\), \(B_5\) is cogredient to \(C_5\), and \(B_6\) is cogredient to \(C_1(1 : 1)\).

The interchange of \(B_1(1 : 1)\) with \(B_6\) seems a bit strange because \(B_6\) is indecomposable, so it would seem that the family of decomposables \(B_1(p : q)\) is more natural than the latter stratification. However, it turns out that decomposability/indecomposability is not preserved under deformations. The stratification given by the \(C\) matrices is consistent with deformation theory in that deformations occur along a stratum, jump deformations to elements of different strata, and smooth deformations along another stratum which “factor through a jump deformation”.

5. Stratification of the nilpotent algebras

The nilpotent algebras are divided into 15 different strata. There are 4 projective 1-parameter families of algebras: \(d_{75}(p : q)\), \(d_{78}(p : q)\), \(d_{83}(p : q)\) and \(d_{86}(p : q)\), where \((p : q)\) is a projective coordinate. For some of these families, there is also an action of the symmetric group \(\Sigma_2\), given by permutation of the coordinates, so that the algebra associated to the parameter \((p : q)\) is isomorphic to the algebra given by the parameter \((q : p)\). In this case, the family is parameterized by \(\mathbb{P}^1/\Sigma_2\). Otherwise the family is parameterized by \(\mathbb{P}^1\). The 11 algebras \(d_{73}\), \(d_{74}\), \(d_{76}\), \(d_{77}\), \(d_{80}\), \(d_{81}\), \(d_{82}\), \(d_{79}\), \(d_{84}\), \(d_{85}\) and \(d_{87}\) each determine a stratum consisting of a single algebra. We will give a description of each of the strata below.

6. The fifteen families of algebras

6.1. Type 73. The algebra \(d_{73} = \psi_2^{31} + \psi_2^{13} + \psi_3^{33} + \psi_2^{34} + \psi_2^{34}\) has a 1-dimensional kernel \(M = \langle c_2 \rangle\). Its core is the 3-dimensional nilpotent algebra \(d_{20}(0 : 0) = \psi_2^{33}\). The algebra \(d_{73}\) is isomorphic to its opposite algebra. We compute

\[
H^2(d_{73}) = \langle \delta^1, \delta^2, \delta^3 \rangle,
\]
\[ d_{73} = \psi_{23}^3 + \psi_{2}^{13} + \psi_{2}^{33} + \psi_{2}^{34} + \psi_{2}^{44} \]
\[ d_{74} = \psi_{23}^{21} + \psi_{23}^{41} + \psi_{2}^{23} + \psi_{2}^{43} + \psi_{2}^{44} \]
\[ d_{75}(p : q) = \psi_{33}^{pq} + p\psi_{31}^{44} + q\psi_{34}^{44} \]
\[ d_{75}(1 : 1) = \psi_{23}^{12} + \psi_{2}^{13} + \psi_{2}^{14} \]
\[ d_{75}(1 : -1) = \psi_{23}^{22} + \psi_{2}^{33} + \psi_{2}^{44} \]
\[ d_{75}(1 : 0) = \psi_{23}^{33} + \psi_{2}^{34} + \psi_{2}^{44} \]
\[ d_{75}(0 : 0) = \psi_{23}^{43} + \psi_{2}^{44} \]
\[ d_{76} = \psi_{23}^{31} + \psi_{23}^{32} + \psi_{2}^{33} + \psi_{2}^{34} \]
\[ d_{77} = \psi_{23}^{41} + \psi_{23}^{42} + \psi_{2}^{43} + \psi_{2}^{44} \]
\[ d_{78}(p : q) = \psi_{33}^{pq} + p\psi_{31}^{44} + q\psi_{34}^{44} \]
\[ d_{78}(1 : 1) = \psi_{23}^{12} + \psi_{23}^{32} + \psi_{2}^{23} + \psi_{2}^{24} + \psi_{2}^{24} \]
\[ d_{78}(1 : -1) = \psi_{23}^{22} + \psi_{2}^{33} + \psi_{2}^{34} + \psi_{2}^{44} \]
\[ d_{78}(1 : 0) = \psi_{23}^{32} + \psi_{2}^{33} + \psi_{2}^{34} + \psi_{2}^{44} \]
\[ d_{79} = \psi_{23}^{21} + \psi_{23}^{22} + \psi_{23}^{13} + \psi_{23}^{14} \]
\[ d_{80} = \psi_{23}^{13} + \psi_{23}^{14} + \psi_{2}^{13} + \psi_{2}^{14} \]
\[ d_{81} = \psi_{23}^{31} + \psi_{23}^{32} + \psi_{2}^{33} + \psi_{2}^{34} \]
\[ d_{82} = \psi_{23}^{41} + \psi_{23}^{42} + \psi_{2}^{43} + \psi_{2}^{44} \]
\[ d_{83}(p : q) = p\psi_{1}^{42} + q\psi_{1}^{24} + \psi_{1}^{44} + \psi_{1}^{33} \]
\[ d_{83}(1 : 1) = \psi_{1}^{24} + \psi_{1}^{42} + \psi_{1}^{44} + \psi_{1}^{33} \]
\[ d_{83}(1 : -1) = \psi_{1}^{24} + \psi_{1}^{44} + \psi_{1}^{34} + \psi_{1}^{33} \]
\[ d_{83}(1 : 0) = \psi_{1}^{34} + \psi_{1}^{44} + \psi_{1}^{33} \]
\[ d_{84} = \psi_{2}^{31} + \psi_{2}^{32} + \psi_{2}^{13} + \psi_{2}^{14} \]
\[ d_{85} = \psi_{2}^{31} + \psi_{2}^{32} + \psi_{2}^{13} + \psi_{2}^{14} \]
\[ d_{86}(p : q) = p\psi_{1}^{42} + q\psi_{1}^{24} + \psi_{1}^{44} \]
\[ d_{86}(1 : 1) = \psi_{1}^{24} + \psi_{1}^{42} + \psi_{1}^{44} \]
\[ d_{86}(1 : -1) = \psi_{1}^{24} + \psi_{1}^{44} + \psi_{1}^{34} + \psi_{1}^{33} \]
\[ d_{86}(1 : 0) = \psi_{1}^{34} + \psi_{1}^{44} \]
\[ d_{87} = \psi_{2}^{31} + \psi_{2}^{32} \]

| Algebra      | $H^0$ | $H^1$ | $H^2$ | $H^3$ |
|--------------|-------|-------|-------|-------|
| $d_{73}$     | 0 2   | 0 2   | 0 3   | 4 0   |
| $d_{74}$     | 0 4   | 0 4   | 0 4   | 4 0   |
| $d_{75}(p : q)$ | 0 2 3 | 0 5   | 8 0   |       |
| $d_{75}(1 : 1)$ | 0 2 3 | 0 6   | 10 0  |       |
| $d_{75}(1 : -1)$ | 0 2 3 | 0 6   | 11 0  |       |
| $d_{75}(1 : 0)$ | 0 2 3 | 0 5   | 8 0   |       |
| $d_{75}(0 : 0)$ | 0 2 3 | 0 6   | 10 0  |       |
| $d_{76}$     | 0 4   | 0 4   | 0 4   | 4 0   |
| $d_{77}$     | 0 4   | 0 4   | 0 4   | 4 0   |
| $d_{78}(p : q)$ | 0 2 3 | 0 5   | 8 0   |       |
| $d_{78}(1 : 1)$ | 0 2 3 | 0 6   | 12 0  |       |
| $d_{78}(1 : -1)$ | 0 2 3 | 0 6   | 12 0  |       |
| $d_{78}(1 : 0)$ | 0 2 3 | 0 6   | 12 0  |       |
| $d_{79}$     | 0 4   | 0 4   | 0 4   | 4 0   |
| $d_{80}$     | 0 4   | 0 4   | 0 4   | 4 0   |
| $d_{81}$     | 0 4   | 0 4   | 0 4   | 4 0   |
| $d_{82}$     | 0 4   | 0 4   | 0 4   | 4 0   |
| $d_{83}(p : q)$ | 0 2 3 | 0 5   | 9 0   |       |
| $d_{83}(1 : 1)$ | 0 4   | 0 4   | 0 4   | 4 0   |
| $d_{83}(1 : -1)$ | 0 4   | 0 4   | 0 4   | 4 0   |
| $d_{83}(1 : 0)$ | 0 4   | 0 4   | 0 4   | 4 0   |
| $d_{84}$     | 0 4   | 0 4   | 0 4   | 4 0   |
| $d_{85}$     | 0 4   | 0 4   | 0 4   | 4 0   |
| $d_{86}(p : q)$ | 0 2 3 | 0 5   | 9 0   |       |
| $d_{86}(1 : 1)$ | 0 4   | 0 4   | 0 4   | 4 0   |
| $d_{86}(1 : -1)$ | 0 4   | 0 4   | 0 4   | 4 0   |
| $d_{86}(1 : 0)$ | 0 4   | 0 4   | 0 4   | 4 0   |
| $d_{87}$     | 0 4   | 0 4   | 0 4   | 4 0   |

Table 1. The cohomology of the algebras $d_{73} \ldots d_{87}$

where

\[ \delta^1 = -\psi_{23}^{1,1} - \psi_{2}^{3,2} - \psi_{2}^{3,4} + \psi_{2}^{4,4} \]
\[ \delta^2 = -\psi_{2}^{1,1} + \psi_{23}^{2,3} + \psi_{2}^{3,2} + 2\psi_{2}^{3,3} + \psi_{2}^{3,4} + \psi_{2}^{4,3} \]
\[ \delta^3 = -2\psi_{2}^{3,1} - 2\psi_{2}^{3,2} + \psi_{2}^{3,4} + 3\psi_{2}^{3,4} + \psi_{2}^{3,4} - 2\psi_{2}^{3,4} + \psi_{2}^{4,1} + 3\psi_{2}^{4,2} + \psi_{2}^{4,3} \].

More precisely, the cohomology classes of these algebras give a basis of $H^2$, but it is convenient to identify the cohomology classes with representative cocycles, which give a pre-basis of the cohomology.

The third order deformation is versal, with four relations

\[ t_2^2 = 0, \quad t_2 t_3 = 0, \quad t_2 t_3 = 0, \quad t_2^2 (t_1 + t_3) = 0. \]
Note that the number of relations is precisely the dimension of $H^3$, and that the third relation is redundant. From the first relation, we see that in any deformation, $t_3 = 0$. Since $t_3$ vanishes by the first relation, the second and third relations give no additional constraints. Finally, the fourth relation gives that either $t_1 = 0$ or $t_2 = 0$. When $t_1 = 0$, we obtain a 1-parameter deformation
\[
d_{t_2} = \psi_2^{11} + \psi_1^{13} + \psi_1^{33} + \psi_2^{44} - t_2 \psi_1^{11} + t_2 \psi_1^{23} + t_2 \psi_2^{32} + 2 t_2 \psi_3^{33} + t_2 \psi_4^{34}
\]
+ $t_2 \psi_4^{43} - t_2^2 \psi_1^{11} - t_2^2 \psi_1^{12} - t_2^2 \psi_1^{21} - t_2^2 \psi_3^{13} - t_2^2 \psi_3^{31} - t_2^2 \psi_4^{14} - t_2^2 \psi_4^{41}$

which is a deformation to a neighborhood of the algebra $d_{75}(1 : 1)$. In fact, one can show that $d_{t_2} \sim d_{75}(x : y)$ where $t_2 = -\frac{(x-y)^2}{2y}$, whenever $t_2 \neq 0$. From this expression, we see that as $t_2 \to 0$, the algebra $d_{t_2}(x : y) \to d_{75}(1 : 1)$. However, when $t_2 = 0$, we obtain the original algebra $d_{75}$. What happens is that the matrix which expresses the isomorphism between the algebras $d_{t_2}$ and $d_{75}(x : y)$ becomes singular when $x = y$.

The second solution of the relations, given by setting $t_2 = 0$, gives a 1-parameter deformation in a neighborhood of $d_{36}(1 : 1)$. As in the case of the first solution, we do not obtain a deformation to the algebra $d_{36}(1 : 1)$.

There are no jump deformations to either $d_{75}(1 : 1)$ or $d_{36}(1 : 1)$, which we found surprising, since in our previous constructions, we have always observed that when a member of one stratum deforms smoothly to a neighborhood of a point in a different stratum, then the smooth deformation factored through a jump deformation to that point, which just means that there is a jump deformation to the point.

6.2. Type 74. The algebra $d_{74} = \psi_2^{33} + \psi_1^{41} + \psi_1^{43} + \psi_1^{34} + \psi_1^{14} + \psi_1^{44}$ has a 1-dimensional kernel $M = \langle e_2 \rangle$. Its core is the 3-dimensional nilpotent algebra $d_{19} = \psi_1^{43} + \psi_1^{34} + \psi_3^{44}$. This algebra is commutative.

When we compute the miniversal deformation, we obtain that the fourth order deformation is versal, and all the relations vanish. The miniversal deformation gives jump deformations to $d_2$, $d_6$, $d_8$, $d_9$, $d_{31}$, $d_{32}$, $d_{35}$, $d_{36}$, $d_{49}$, and $d_{50}$.

It is interesting to note that all the algebras to which $d_{74}$ deforms are also commutative. It is well-known that in order for an algebra to deform to a commutative algebra, it must be commutative, and, however, it is possible, and even common, for commutative algebras to deform to noncommutative algebras.

6.3. Type 75($p : q$). The family of algebras $d_{75}(p : q)$ are generically given by the algebras $\psi_2^{33} + p \psi_1^{43} + q \psi_1^{34} + \psi_1^{44}$. However, for certain special values of the parameters, the algebra is given by a different algebraic structure. We have $d_{75}(1 : -1) = \psi_1^{43} + \psi_2^{41} + \psi_1^{44}$ and $d_{75}(1 : 1) = \psi_2^{33} + \psi_3^{34} + \psi_4^{14}$. The family is parameterized by $\mathbb{P}^1/\Sigma_2$, so that $d_{75}(p : q) \sim d_{75}(q : p)$. It might be possible to find a family $d(p : q)$ of algebras which give a parametrization of the entire family, but we did not discover such a representation. In all cases, the algebra $d_{75}(p : q)$ has a 2-dimensional kernel $M = \langle e_1, e_2 \rangle$ and its core is the trivial nilpotent 2-dimensional algebra.

For projective families of algebras, generically, the dimension of $H^n$ does not vary, and there is a generic pattern for the deformations. There are a few values of $(p : q)$ for which the pattern varies from the generic case, in that the dimension of $H^n$ may be larger, and there may be extra deformations. Usually, these special values are $(1 : 1)$, $(1 : -1)$, $(1 : 0)$ and $(0 : 0)$. In fact, $(0 : 0)$ is always special, and...
there are always jump deformations from the algebra corresponding to \((0 : 0)\), to every member in the family. When there is no action of \(\Sigma_2\) on the family, so the algebra corresponding to \((p : q)\) is not generically isomorphic to that for \((q : p)\), the value \((0 : 1)\) is also special. Occasionally, there are additional special values.

6.3.1. **Generic Case.** In the case of \(d_{75}(p : q)\), generically, there are jump deformations to \(d_3\), \(d_4\) and \(d_5\), as well as smooth deformations along the family in a neighborhood of \(d_{75}(p : q)\).

6.3.2. \((p : q) = (1 : 0)\). The algebra \(d_{75}(1 : 0)\) has additional jump deformations to the algebras \(d_{22}, \ldots, d_{40}\) and the algebras \(d_{45}, \ldots, d_{48}\).

6.3.3. \((p : q) = (1 : -1)\). The algebra \(d_{75}(1 : -1)\) has additional jump deformations to \(d_{14}, d_{15}, d_{16}, d_{22}, d_{23}, d_{27}, d_{28}, d_{29}, d_{30}, d_{58},\) and \(d_{59}\).

6.3.4. \((p : q) = (1 : 1)\). The algebra \(d_{75}(1 : 1)\) has additional jump deformations to \(d_{22}, d_{23}, d_{27}, d_{28}, d_{29}, d_{30}, d_{37}(x : y)\) for all \((x : y)\) except \((1 : 1)\) and \((0 : 0)\), \(d_{38}(x : y)\) for all \((x : y)\) except \((1 : 1)\) and \((0 : 0)\), and \(d_{73}\).

6.4. \((p : q) = (0 : 0)\). The algebra \(d_{75}(0 : 0)\) representing the "generic element" in \(P^1\), will automatically have jump deformations to every element in the family \(d_{75}(p : q)\) except itself. In fact, it is not uncommon for the generic element to coincide with an element in some other family, and this happens here, as \(d_{75}(p : q) \sim d_{86}(1 : 1)\). As a consequence, we also know that \(d_{75}(0 : 0)\) will deform in a neighborhood of \(d_{86}(1 : 1)\). Our general method of listing the algebras is organized around the principal that the order should be preserved by deformations in the sense that elements only deform to elements whose numbers are lower. However, the generic element in a family cannot be expected to preserve that principle, as in some sense, it is on a higher level than members of its family.

In addition, we also expect the generic element to have jump deformations to all algebras to which any other member of its family jumps. In fact, the algebra \(d_{75}(0 : 0)\) has jump deformations to \(d_1, \ldots, d_9, d_{14}, d_{15}, d_{16}, d_{20}, \ldots, d_{30}, d_{37}(x : y), d_{38}(x : y), d_{39}, d_{40}, d_{45}, \ldots, d_{50}, d_{57}, \ldots, d_{60}, d_{65}, d_{68}, d_{73}, d_{74}, d_{75}(x : y), d_{76}, d_{77}, d_{78}(x : y)\) except \((0 : 0)\), \(d_{83}(1 : 1)\), \(d_{80}, d_{82}, \ldots, d_{85}\), as well as deforming in a neighborhood of \(d_{83}(1 : 1)\).

6.5. **Type 76.** The algebra \(d_{76} = \psi_2^{31} + \psi_2^{13} + \psi_1^{33} + \psi_2^{44}\) has a 1-dimensional kernel \(M = \langle e_2 \rangle\). Its core is the 3-dimensional nilpotent algebra \(d_{20}(0 : 0) = \psi_1^{33}\). It is commutative. It has jump deformations to \(d_2, d_6, d_7, d_8, d_9, d_{31}, d_{32}, d_{35}, d_{36}, d_{37}(1 : 1), d_{38}(1 : 1), d_{49}, d_{50}, d_{73}\), and \(d_{74}\), as well as deformations smoothly in a neighborhood of \(d_{37}(1 : 1)\) and \(d_{38}(1 : 1)\). With the exception of \(d_{73}\), the algebra jumps to are all commutative, but the algebras \(d_{37}(x : y)\) and \(d_{38}(x : y)\) to which it deforms smoothly are not in general commutative.

6.6. **Type 77.** The algebra \(d_{77} = \psi_2^{31} + \psi_2^{13} + \psi_1^{33} + \psi_2^{34}\) has a 1-dimensional kernel \(M = \langle e_2 \rangle\). Its core is the 3-dimensional nilpotent algebra \(d_{20}(0 : 0) = \psi_1^{33}\). It is isomorphic to its opposite algebra. It has jump deformations to \(d_3, d_4, d_5, d_{22}, \ldots, d_{30}, d_{37}(x : y)\) except \((1 : 1)\) and \((0 : 0)\), \(d_{38}(x : y)\) except \((1 : 1)\) and \((0 : 0)\), \(d_{45}, \ldots, d_{48}, d_{58}, d_{59},\) and \(d_{73}\).
6.7. **Type 78** ($p : q$). The family of algebras $d_{78}(p : q)$ are given by $\psi_{1}^{3} + p\psi_{1}^{31} + q\psi_{2}^{3} + \psi_{2}^{34} + \psi_{2}^{44}$. The family is parameterized by $\mathbb{P}^{1}/2$, but the linear automorphism which transforms $d_{78}(p : q)$ to $d_{78}(q : p)$ cannot be given in a completely generic form. When $p = 1$ and $q = 0$ or $p = 1$ and $q = 2$, the matrix which gives the equivalence does not fall into the same pattern as will work generically. None of the algebras in this family are commutative, owing to the $\psi_{2}^{34}$ term in the algebra.

Generically, the kernel of this algebra is the 1-dimensional ideal $M = \langle e_{2} \rangle$, and its core is the trivial 3-dimensional algebra. For the element $d_{78}(0 : 0)$, the dimension of the kernel is 2, and we also have that $d_{78}(0 : 0) \sim d_{86}(1 : \gamma)$, where $\gamma$ is a primitive 6-th root of unity.

The special cases of the parameters $(p : q)$ include not only the standard values $(1 : 1), (1 : -1), (1 : 0)$ and $(0 : 0)$, but also the values $(1 : 2)$ and $(1 : \gamma)$, where $\gamma$ is a primitive 6th root of unity.

6.7.1. **Generic Case.** Generically, an algebra in the family $d_{78}(p : q)$ has jump deformations to $d_{1}, d_{37}(p : q)$ and $d_{38}(p : q)$, as well as smooth deformations in a neighborhood of $d_{37}(p : q), d_{38}(p : q)$ and $d_{78}(p : q)$.

6.7.2. $(p : q) = (1 : 2)$. We mentioned something unusual in this special case already, namely that the matrices which give rise to the isomorphisms between $d_{78}(1 : 2)$ and $d_{78}(2 : 1)$ do not fit the generic pattern. Other than this, the deformation pattern is entirely generic.

6.7.3. $(p : q) = (1 : 0)$. The algebra $d_{78}(1 : 0)$, in addition to the generic deformations has jump deformations to $d_{1}, d_{3}, d_{4}, d_{5}, d_{22}, \ldots, d_{30}, d_{39}, d_{40},$ and $d_{45}, \ldots, d_{48}$.

6.7.4. $(p : q) = (1 : -1)$. The algebra $d_{78}(1 : -1)$ has jump deformations to $d_{20}$ and $d_{21}$ in addition to the generic deformations.

6.7.5. $(p : q) = (1 : 1)$. The algebra $d_{78}(1 : 1)$ does not quite follow the generic pattern in that while it does deform in a neighborhood of $d_{37}(1 : 1)$ and $d_{38}(1 : 1)$, it does not jump to either one of them. In fact, since both $d_{37}(1 : 1)$ and $d_{38}(1 : 1)$ are commutative, while $d_{78}(1 : 1)$ is not commutative, it has no possibility of having a jump deformation to either of them. It does have a jump deformation to $d_{1}$, which fits the generic pattern.

6.8. **Type 79**. The algebra $d_{79} = \psi_{2}^{31} + \psi_{1}^{33} + \psi_{1}^{33}$ has a 2-dimensional kernel $M = \langle e_{2}, e_{4} \rangle$. Its core is the nontrivial 2-dimensional nilpotent algebra $d_{6}$. It is commutative. It has jump deformations to $d_{2}, d_{9}, d_{22}, d_{36}, d_{37}(x : y), d_{38}(x : y)$, both for all $(x : y), d_{45}, d_{50}, d_{58}, d_{59}, d_{65}, d_{68}, d_{73}, d_{74}, d_{75},$ and $d_{77}$.

6.9. **Type 80**. The algebra $d_{80} = \psi_{2}^{33} + \psi_{1}^{43} - \psi_{1}^{43} + \psi_{4}^{44}$ has a 2-dimensional kernel $M = \langle e_{1}, e_{2} \rangle$ and its core is the trivial nilpotent 2-dimensional algebra. In fact, this element is the $(1 : -1)$ case of the family of algebras $\psi_{2}^{33} + p\psi_{1}^{31} + q\psi_{1}^{44} + \psi_{2}^{44}$, which correspond to $d_{75}(p : q)$ in most cases.

The reason this element is relegated to a different position than one would expect is that there are two algebras which deform in a neighborhood of $d_{75}(1 : -1)$. To decide which one actually belongs in the family, we consider the following data. First, the one in the family has the smaller cohomology $H^{2}$. Secondly, this element deforms to the other one, and we would expect that the one which does not deform to the other actually belongs in the family.
This algebra has jump deformations to \(d_1, d_3, d_4, d_5, d_{14}, d_{15}, d_{16}, d_{20}, \cdots, d_{23}, d_{27}, \cdots, d_{30}, d_{39}, d_{40}, d_{57}, \cdots, d_{60}, \) and \(d_{75}(1 : -1).\)

6.10. **Type 81.** The algebra \(d_{81} = \psi_2^{13} - \psi_2^{31} + \psi_2^{44}\) has a 1-dimensional kernel \(M = \langle e_2 \rangle.\) Its core is the 3-dimensional trivial nilpotent algebra. It is isomorphic to its opposite algebra. It has jump deformations to \(d_1, d_{20}, d_{21}, d_{37}(1 : -1), d_{38}(1 : -1), d_{40}, d_{51}, d_{52}, d_{78}(1 : -1)\) and deforms in neighborhoods of the algebras \(d_{37}(1 : -1), d_{38}(1 : -1)\) and \(d_{78}(1 : -1).\)

6.11. **Type 82.** The algebra \(d_{81} = \psi_2^{33} + \psi_1^{34} + \psi_1^{44}\) has a 2-dimensional kernel \(M = \langle e_1, e_2 \rangle.\) Its core is the 2-dimensional trivial nilpotent algebra. It is isomorphic to its opposite algebra. It has jump deformations to \(d_3, d_4, d_5, d_{22}, d_{23}, d_{27}, \cdots, d_{30}, d_{37}(x : y)\) except \((1 : 1)\) and \((0 : 0), d_{38}(x : y)\) except \((1 : 1)\) and \((0 : 0), d_{73}, d_{75}(1 : 1)\) and deforms in a neighborhood of \(d_{75}(1 : 1).\)

6.12. **Type 83(\(p : q\)).** The family of algebras \(d_{83}(p : q)\) are given by \(p\psi_1^{42} + q\psi_1^{24} + \psi_1^{44} + \psi_2^{44}\). The family is parameterized by \(\mathbb{P}^1,\) and does not have an action of \(\mathbb{Z}_2,\) so that \(d_{83}(p : q)\) is not, in general, isomorphic to \(d_{83}(q : p)\). Generically, the kernel of this algebra is the 2-dimensional ideal \(M = \langle e_1, e_3 \rangle,\) and its core is the trivial 2-dimensional algebra. For the element \(d_{83}(0 : 0),\) the dimension of the kernel is 3, and we also have that \(d_{83}(0 : 0) \sim d_{86}(0 : 0).\)

The special cases of the parameters \((p : q)\) include not only the standard values \((1 : 1), (1 : -1), (1 : 0)\) and \((0 : 0),\) but also \((0 : 1)\) as \(d_{83}(1 : 0)\) is not isomorphic to \(d_{83}(1 : 0),\) so must be treated separately.

6.12.1. **Generic Case.** Generically, an algebra in the family \(d_{83}(p : q)\) has jump deformations to \(d_3, d_4, d_5, d_{14}, d_{15}, d_{16}, d_{22}, \cdots, d_{30}, d_{37}(x : y)\) except \((1 : 1)\) and \((0 : 0), d_{38}(x : y),\) except \((1 : 1)\) and \((0 : 0), d_{45}, \cdots, d_{48}, d_{58}, d_{59}, d_{73}, d_{75}(x : y)\) except \((0 : 0),\) and \(d_{77}\) as well as smooth deformations in a neighborhood of \(d_{83}(p : q).\)

6.12.2. **\((p : q) = (1 : 0)\).** The algebra \(d_{83}(1 : 0),\) in addition to the generic deformations has jump deformations to \(d_{11}, d_{12}, d_{42}, d_{44}, d_{67},\) and \(d_{69}.\)

6.13. **\((p : q) = 0 : 1,\) in addition to the generic deformations has jump deformations to \(d_{10}, d_{13}, d_{41}, d_{43}, d_{66},\) and \(d_{70}.\) Note that these are precisely the opposite algebras to the extra deformations of the algebra \(d_{83}(1 : 0).\)

6.14. **\((p : q) = (1 : -1)\).** The algebra \(d_{83}(1 : -1)\) has jump deformations to \(d_1, d_{20}, d_{21}, d_{39}, d_{40}, d_{51}, d_{52}, d_{60},\) and \(d_{69},\) in addition to the generic deformations.

6.15. **\((p : q) = (1 : -1)\).** In addition to the generic deformations, the algebra \(d_{83}(1 : 1)\) has jump deformations to \(d_2, d_6, d_7, d_8, d_9, d_{31}, \cdots, d_{36}, d_{37}(1 : 1), d_{38}(1 : 1), d_{49}, d_{50}, d_{56}, d_{68}, d_{74},\) and \(d_{82}.\) Note that except for \(d_{65}\) all of these algebras are commutative.

6.16. **Type 84.** The algebra \(d_{84} = \psi_2^{13} - \psi_2^{31} + \psi_2^{44}\) has a 1-dimensional kernel \(M = \langle e_2 \rangle.\) Its core is the trivial 3-dimensional nilpotent algebra. It has jump deformations to \(d_1, d_3, d_4, d_5, d_4, d_{15}, d_{16}, d_{20} - d_{30}, d_{37}(x : y)\) except \((1 : 1)\) and \((0 : 0), d_{38}(x : y)\) except \((1 : 1)\) and \((0 : 0), d_{39}, d_{40}, d_{45} - d_{48}, d_{57} - d_{60}, d_{73}, d_{77}, d_{78}(x : y)\) except \((0 : 0).\)
6.17. **Type 85.** The algebra \( d_{85} = \psi_2^{11} + \psi_2^{13} + \psi_2^{33} + \psi_4^{44} \) has a 1-dimensional kernel \( M = \langle e_2 \rangle \). Its core is the trivial 3-dimensional nilpotent algebra. It has jump deformations to \( d_{1}, \cdots, d_{20}, d_{22}, d_{23}, d_{27}, \cdots, d_{32}, d_{35}, d_{36}, d_{37}(x : y) \) except \((0 : 0), d_{38}(x : y) \) except \((0 : 0), d_{73}, d_{74}, d_{76}, d_{78}(1 : 1)\), and deforms in a neighborhood of \( d_{78}(1 : 1) \).

6.18. **Type 86(p : q).** The family of algebras \( d_{86}(p : q) = \psi_1^{12}p + \psi_1^{14}q + \psi_4^{44} \) is parameterized by \( \mathbb{P}^1/\Sigma_2 \). The algebras are not commutative, except for \( d_{86}(1 : 1) \) and \( d_{86}(0 : 0) \). Generically, the elements in this family have a 2-dimensional kernel \( M = \langle e_1, e_3 \rangle \), and their core is the trivial 2-dimensional algebra, except that \( d_{86}(0 : 0) \) has 3-dimensional kernel and 1-dimensional core.

6.18.1. **Generic Case.** Generically, \( d_{86}(p : q) \) has jump deformations to \( d_{1}, d_{3}, d_{4}, d_{5}, d_{14}, d_{15}, d_{16}, d_{20} - d_{30}, d_{37}(x : y) \) except \((1 : 1) \) and \((0 : 0), d_{38}(x : y) \) except \((1 : 1) \) and \((0 : 0), d_{39}, d_{40}, d_{45} - d_{48}, d_{57} - d_{60}, d_{73}, d_{75}(x : y) \) except \((0 : 0), d_{77}, d_{78}(x : y) \) except \((0 : 0), d_{83}(p : q), d_{80}, d_{84} \), as well as deforming smoothly in a neighborhood of \( d_{83}(p : q) \) and \( d_{86}(p : q) \).

6.18.2. \((p : q) = (1 : 0)\). The algebra \( d_{86}(1 : 0) \) has additional jump deformations to \( d_{10}, \cdots, d_{14}, d_{41}, d_{42}, d_{43}, d_{61}, d_{62}, d_{63}, d_{77} \) and \( d_{84} \).

6.18.3. \((p : q) = (1 : 1)\). The algebra \( d_{86}(1 : 1) \) has additional jump deformations to \( d_{2}, d_{6}, d_{7}, d_{8}, d_{9}, d_{31}, \cdots, d_{36}, d_{49}, d_{50}, d_{65}, d_{68}, d_{74}, d_{76}, d_{82} \) and \( d_{79} \). With the exception of \( d_{85} \), all of these additional jump deformations are to commutative algebras, to which it is not possible for generic elements of the family to deform.

6.18.4. \((p : q) = (1 : -1)\). The algebra \( d_{86}(1 : 1) \) has additional jump deformations to \( d_{51}, d_{52} \) and \( d_{81} \).

6.19. **Type 87.** The algebra \( d_{87} = -\psi_2^{31} + \psi_2^{13} \) has a 2-dimensional kernel \( M = \langle e_2, e_4 \rangle \). Its core is the trivial 2-dimensional nilpotent algebra. It has jump deformations to \( d_{1}, d_{3}, d_{4}, d_{5}, d_{14}, d_{15}, d_{16}, d_{20}, \cdots, d_{30}, d_{37}(x : y) \) except \((1 : 1) \) and \((0 : 0), d_{38}(x : y) \) except \((1 : 1) \) and \((0 : 0), d_{39}, d_{40}, d_{45}, \cdots, d_{48}, d_{51}, \cdots, d_{54}, d_{57}, \cdots, d_{60}, d_{73}, d_{75}(x : y) \) except \((0 : 0), d_{77}, d_{78}(x : y) \) except \((0 : 0), d_{83}(1 : -1), d_{80}, d_{81}, d_{84} \) and \( d_{86}(1 : 1) \) and deforms in a neighborhood of \( d_{83}(1 : -1) \) and \( d_{86}(1 : 1) \).

6.20. **How we constructed the algebras.** In section 1, we discussed that fact that any nilpotent algebra has a unique kernel, consisting of all elements whose product with any element is zero, and core, consisting of the quotient of the algebra by its kernel, so that, in principle, it should be easier to construct all algebras of a given dimension by considering extensions of algebras by ideals in this manner.

However, our motivation for the constructions is not simply to give a list of all algebras, but to give a decomposition of the moduli space into strata which are dictated by deformation theory. A single stratum may include algebras whose kernels do not all have the same dimension, so this fact alone dictates the need for another method of identifying the strata. Moreover, our constructions have revealed that the strata seem to consist of projective orbifolds, and it is necessary to identify these orbifolds in some natural way. This has led us to the following method of construction.

We first need to know all \((n - 1)\)-dimensional nilpotent algebras, and then we use the fact that there is always a codimension 1 ideal in any nilpotent algebra to
realize all algebras as extensions of the trivial 1-dimensional algebra by an \((n - 1)\)-dimensional nilpotent algebra. In the case of dimension 4 complex nilpotent algebras, there are 3 nontrivial nilpotent 3-dimensional algebras, as well as the trivial nilpotent algebra, which come into play. Here we will consider how our algebras arise by this method of construction.

6.21. Extensions by the nilpotent algebra \(d_{19} = \psi_2^{31} + \psi_2^{13} + \psi_2^{33}\). The algebras \(d_{73}, d_{76}, d_{77}\), and \(d_{79}\) all arise as extensions of the trivial algebra by the 3-dimensional algebra \(d_{19}\). From compatibility condition \([\mu, \lambda] = 0\) and the Maurer Cartan equation, we can reduce \(\lambda\) to the form \(\lambda = \psi_2^{35}x\), where \(x\) can be chosen to be either 1 or 0, \(\psi\) can be taken to be 0, and \(\tau\) is of the form \(\tau = \psi_2^{14}c\), where again, \(c\) is either 0 or 1. This gives 4 distinct possibilities, which are the four algebras listed above.

6.22. Extensions by the nilpotent algebra \(d_{20}(p : q) = \psi_2^{13}p + \psi_2^{31}q + \psi_2^{33}\). The algebras \(d_{73}, d_{74}, d_{75}(x : y)\), \(d_{76}, d_{78}(x : y)\), \(d_{83}(1 : 1)\), \(d_{85}(0 : 0)\), \(d_{80}\), \(d_{81}\), \(d_{82}\), \(d_{84}\), \(d_{85}\), and \(d_{86}(x : y)\) arise as extensions of the trivial algebra by the nilpotent algebra \(d_{20}(p : q)\). In this case, the compatibility condition gives rise to more than one solution, and these solutions depend on the variable \((p : q)\). For the special values \((1 : 0)\) and \((0 : 0)\), there are nongeneric solutions in addition to the generic case.

Let us examine the generic case first. In this case, we can reduce \(\lambda\) to the form \(\lambda = \psi_2^{43}r + \psi_2^{14}s\), where \((r : s)\) is a projective coordinate. However, after further analysis, we obtain that the algebras which arise in this fashion are \(d_{81}\), \(d_{84}\), \(d_{85}\) and \(d_{86}(p : q)\). Thus the projective coordinates \((r : s)\) don’t arise in the final description of the algebra.

When \((p : q) = (1 : 0)\), we obtain the same format for \(\lambda\) and \(\tau\), and the algebras we obtain are \(d_{78}(x : y)\) except \((0 : 0)\), \(d_{84}\) and \(d_{86}(1 : 0)\).

Finally, when \((p : q) = (0 : 0)\), we obtain 4 distinct solutions for the format of \(\lambda\). For example, in one of the patterns, we have \(\lambda = \psi_2^{41}p + \psi_2^{14}q + \psi_2^{34}u\), where \((p : q)\) is a projective coordinate, and the choice of \(u\) can be reduced to \(u = 1\) or \(u = 0\). Thus we obtain a new projective coordinate, and this coordinate is retained in some of the solutions, so we obtain new projective families of algebras which don’t arise from the old projective families. After some analysis, we obtain that algebras arising in this case are \(d_{73}, d_{74}, d_{75}(x : y), d_{76}, d_{78}(x : y), d_{83}(1 : 1), d_{83}(0 : 0), d_{80}, d_{82}\), and \(d_{85}\).

6.23. Extensions by the nilpotent algebra \(d_{21} = \psi_2^{13} - \psi_2^{31}\). The algebras \(d_{78}(1 : -1), d_{81}, d_{84}\), and \(d_{87}\) all arise as extensions of the nilpotent 3-dimensional algebra \(d_{21}\).

6.24. Extensions by the trivial 3-dimensional nilpotent algebra. The algebras \(d_{77}, d_{83}(x : y), d_{90}, d_{79}, d_{84}, d_{86}(x : y)\), and \(d_{87}\) arise as extensions of the trivial 1-dimensional algebra by the trivial 3-dimensional nilpotent algebra.

From the description above, it is clear that many of these algebras arise as extensions of the trivial algebra by different 3-dimensional nilpotent algebras.

7. Commutative Algebras

There are 20 distinct non-nilpotent commutative algebras, of which 9 are unital. Every commutative algebra is a direct sum of algebras which are ideals in quotients
of polynomial algebras. Every finite dimensional unital commutative algebra is a quotient of a polynomial algebra, while every finite dimensional nonunital algebra is an ideal in such an algebra. In [4] we gave a table showing the non-nilpotent commutative algebras, which we will not reproduce here.

Nilpotent commutative algebras were classified by Hazlett [7], and also given in [9]. There are 8 nontrivial nilpotent commutative algebras.

We note that commutative algebras may deform into noncommutative algebras, but noncommutative algebras never deform into a commutative algebras. The fact that commutative algebras have noncommutative deformations plays an important role in physics, and deformation quantization describes a certain type of deformation of a commutative algebra into a noncommutative one.

| Algebra | Structure |
|---------|-----------|
| \( d_{74} \) | \( x\mathbb{C}[x]/(x^5) \) |
| \( d_{75}(1 : 1) \) | \( (x, y) \leq \mathbb{C}[x, y]/(x^2 - y^2, xy) \) |
| \( d_{75}(0 : 0) = d_{86}(1 : 1) \) | \( \mathbb{C}_0 \oplus (x, y) \leq \mathbb{C}_0 \oplus \mathbb{C}[x, y]/(x^2 - y^2, xy) \) |
| \( d_{76} \) | \( (x, y) \leq \mathbb{C}[x, y]/(y^3 - x^2, xy) \) |
| \( d_{79} \) | \( \mathbb{C}_0 \oplus x\mathbb{C}[x]/(x^4) \) |
| \( d_{83}(1 : 1) \) | \( (x, y) \leq \mathbb{C}[x, y]/(y^2, x^2y, x^3) \) |
| \( d_{83}(0 : 0) = d_{86}(0 : 0) \) | \( \mathbb{C}_0^3 \oplus x\mathbb{C}[x]/(x^3) \) |
| \( d_{85} \) | \( (x, y, z) \leq \mathbb{C}[x, y, z]/(x^2 - y^2, y^2 - yz, xz, yz, xz, z^2) \) |

Table 2. Nilpotent 4-dimensional commutative algebras

8. Levels of algebras

We give a table showing the levels of each algebra. For completeness, we include the levels of the non-nilpotent algebras, so that the reader can see how the nilpotent algebras fit into the general picture. To define the level, we say that a rigid algebra has level 1, an algebra which has only jump deformations to an algebra on level one has level two and so on. To be on level \( k + 1 \), an algebra must have a jump deformation to an algebra on level \( k \), but no jump deformations to algebras on a level higher than \( k \). For families, if one algebra in the family has a jump to an element on level \( k \), then we place the entire family on at least level \( k + 1 \). Thus, even though generically, elements of the family \( d_{37}(p : q) \) deform only to members of the same family, there is an element in the family which has a jump to an element on level 4. For the generic element in a family, we consider it to be on a higher level than the other elements, because it has jump deformations to the other elements in its family.

9. Analysis

The reader may note that every element in the family \( d_{86}(p : q) \) has a jump deformation to every element in the families \( d_{75}(x : y) \) and \( d_{76}(x : y) \). As a consequence, it is difficult to say which element in the family \( d_{86}(p : q) \) really should be the \((0 : 0)\) element in those families. Of course, for the choice of algebras
representing \(d_{75}(p : q)\), the corresponding member of the family is precisely the element \(d_{86}(1 : 1)\), but this can be an artifact of the form of the algebra structure.

To illustrate this point, we consider the following family of algebras, given by
\[
d(p : q) = \psi_1^3 + \psi_2^3 + p\psi_4^2 + q\psi_5^2.
\]
It is easily checked that \(d(q : p) \sim d(p : q)\), and that the family is given by projective coordinates. In fact, \(d(p : q) \sim d_{78}(x : y)\) where \(\frac{2}{p} = \frac{xy}{x^2 + xy + y^2}\), so in general, the family \(d(p : q)\) and \(d_{78}(x : y)\) determine the same collection of algebras. However, \(d(0 : 0)\) is isomorphic to \(d_{86}(1 : 0)\), not \(d_{86}(1 : \alpha)\) (where \(\alpha\) is a primitive 6th root of unity), so we see that there may be no natural way to identify the generic element in a projective family as a specific algebra.

What is important is that there always is an algebra corresponding to the 'so called' generic element of the \(\mathbb{P}^1\) which parameterizes a family.

There are two additional features of this moduli space that did not arise in the lower dimensional moduli spaces of ordinary complex associative algebras. The first is that we were unsuccessful in describing the family \(d_{75}(p : q)\) with a single family of algebras with parameters \(p\) and \(q\). (A similar difficulty arose with the moduli space of 2|2-dimensional \(\mathbb{Z}_2\)-graded associative algebras,\cite{6}, so this pattern has been seen before, just not in the context of ordinary associative algebras.)

The second feature we saw was that for the algebras \(d_{73}\) and \(d_{75}(1 : 1)\), there were deformations in a neighborhood of elements of another family, without there being a jump to the point in whose neighborhood the smooth deformation occurred. This was a bit surprising to us, because we had not observed this behavior before.

The authors and collaborators have written a series of articles describing moduli spaces of Lie, associative, \(L_\infty\), and \(A_\infty\) algebras. It first became clear when studying the 3-dimensional complex Lie algebras that there was a stratification of the moduli space by projective orbifolds, each of the form \(\mathbb{P}^n\), possibly with an action of \(\Sigma_{n+1}\), given by interchanging coordinates. Our example of 4-dimensional complex associative algebras fits this pattern nicely. The authors have been conjecturing that this type of stratification by projective orbifolds ought to hold in general, but as of yet, do not have the tools to prove this.

---

| Level | Algebras |
|-------|----------|
| 1     | 1,2,3,4,5,10,11,12,13,14,15,16,17,18,19,20,39,53,54,55,56 |
| 2     | 6,7,21,22,23,24,25,26,27,28,29,30,40,41,42,43,44,45,57 |
| 3     | 33,34,35,36,49,50,66,67,69,70 |
| 4     | 37(0 : 0), 38(0 : 0), 51,52,73,75(\(p : q\)) |
| 5     | 37(0 : 0), 38(0 : 0), 51,52,73,75(\(p : q\)) |
| 6     | 71,72,76,77,78(\(p : q\)), 80,82,85 |
| 7     | 79,81,83(\(p : q\)) |
| 8     | 75(0 : 0), 78(0 : 0), 86(\(p : q\)) |
| 9     | 83(0 : 0), 86(0 : 0), 87 |

**Table 3.** The levels of the algebras
References

[1] W. De Graaf, Classification of nilpotent associative algebras of small dimension, preprint arXiv:1009.5339v1, 2010.
[2] A. Fialowski and M. Penkava, Extensions of associative algebras, to appear in Comm. Contemp. Math. 2013.
[3] A. Fialowski and M. Penkava, Formal deformations, contractions and moduli spaces of Lie algebras, Intern J. Theor. Physics 47 (2008), 561–582.
[4] ———, The moduli space of 4-dimensional non-nilpotent complex associative algebras, Forum Mathematicum (March 2013), Appeared Online.
[5] A. Fialowski, M. Penkava, and M. Phillipson, Deformations of complex 3-dimensional associative algebras, Journal of Generalized Lie Theory and Applications 5 (2011), 22.
[6] J. Frinak, A. Ott, and M. Penkava, The moduli space of 2|2-dimensional complex associative algebras, preprint, 2011.
[7] O.C. Hazlett, On the classification and invariantive characterization of nilpotent algebras, American Journal of Mathematics 38 (1916), no. 2, 109–138.
[8] R. Horn and V. Sergeichuk, Canonical matrices of bilinear and sesquilinear forms, Linear Algebra and its Applications 428 (2008), 193–223.
[9] G. Mazzola, Generic finite schemes and Hochshild cocycles, Commentari Mathematici Hevetici 55 (1980), no. 2, 267–293.

Alice Fialowski, Eötvös Loránd University, Budapest, Hungary
E-mail address: fialowsk@cs.elte.hu

Michael Penkava, University of Wisconsin-Eau Claire, Eau Claire, WI 54702-4004
E-mail address: penkavmr@uwec.edu