Schur positivity and log-concavity related to longest increasing subsequences

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Abstract. Chen proposed a conjecture on the log-concavity of the generating function for the symmetric group with respect to the length of longest increasing subsequences of permutations. Motivated by Chen’s log-concavity conjecture, Bóna, Lackner and Sagan further studied similar problems by restricting the whole symmetric group to certain of its subsets. They obtained the log-concavity of the corresponding generating functions for these subsets by using the hook-length formula. In this paper, we generalize and prove their results by establishing the Schur positivity of certain symmetric functions. This also enables us to propose a new approach to Chen’s original conjecture.

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1 Introduction

Given positive integers $m,n$ and $\lfloor \frac{n}{2} \rfloor \leq k \leq n$, let $(k^m,(n-k)^m)$ denote the partition with $m$ parts equal to $k$ and $m$ parts equal to $n-k$. Similarly, for $1 \leq k \leq n$, let $(k^m,1^{m(n-k)})$ denote the partition with $m$ parts equal to $k$ and $m(n-k)$ parts equal to 1. Given a partition $\lambda$, let $f^\lambda$ denote the number of standard Young tableaux of shape $\lambda$. The main objective of this paper is to prove the following result.

**Theorem 1.1.** Suppose that $m,n$ are two positive integers.

1. For $\lfloor \frac{n}{2} \rfloor < k < n$ we have
   \[
   (f(k^m,(n-k)^m))^2 \geq f((k+1)^m,(n-k-1)^m)f((k-1)^m,(n-k+1)^m),
   \]

2. For $1 < k < n$ we have
   \[
   (f(k^m,1^{m(n-k)})^2 \geq f((k+1)^m,1^{m(n-k-1)})f((k-1)^m,1^{m(n-k+1)}).
   \]

The roots of this paper lie in the work by Bóna, Lackner and Sagan [2], who first proved the above theorem for the case of $m = 1, 2$ by using the celebrated hook-length formula. We will present two proofs of Theorem 1.1, one of which is the same as Bóna, Lackner and Sagan’s proof for small $m$, and the other is based on some results on Schur positivity due to Lam, Postnikov, and Pylyavskyy [7].

Let us first review some backgrounds. We will adopt the notation and terminology found in Bóna, Lackner and Sagan [2]. Given a positive integer $n$, let $S_n$ be the symmetric group of all
permutations of \([n] := \{1, 2, \ldots, n\}\). For a given permutation \(\pi \in \mathfrak{S}_n\), let \(\ell(\pi)\) denote the length of a longest increasing subsequence of \(\pi\). Define \(L_{n,k}\) to be the set of permutations \(\pi \in \mathfrak{S}_n\) with \(\ell(\pi) = k\) for \(1 \leq k \leq n\). Let \(\ell_{n,k} = |L_{n,k}|\). Chen proposed the following conjecture.

**Conjecture 1.2** ([3, Conjecture 1.1]). For any fixed \(n\), the sequence \(\{\ell_{n,k}\}_{k=1}^n\) is log-concave, namely, \(\ell_{n,k}^2 \geq \ell_{n,k+1} \ell_{n,k-1}\) for \(1 < k < n\).

Bóna, Lackner and Sagan [2] further made a companion conjecture for involutions. Define \(I_{n,k}\) to be the set of involutions \(\pi \in \mathfrak{S}_n\) with \(\ell(\pi) = k\) for \(1 \leq k \leq n\). Let \(i_{n,k} = |I_{n,k}|\). They proposed the following conjecture.

**Conjecture 1.3** ([2, Conjecture 1.2]). For any fixed \(n\), the sequence \(\{i_{n,k}\}_{k=1}^n\) is log-concave.

Bóna, Lackner and Sagan showed that there is a close connection between Conjecture 1.2 and Conjecture 1.3 by using the Robinson-Schensted correspondence. It is well known that each permutation \(\pi \in \mathfrak{S}_n\), under the Robinson-Schensted correspondence, is mapped to a pair of standard Young tableaux of the same partition shape, say \(\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n\). Moreover, there holds \(\ell(\pi) = \lambda_1\). In that case, we also say that \(\pi\) is of shape \(\lambda\), denoted \(\sh \pi = \lambda\). Bóna, Lackner and Sagan proved that if there is a shape-preserving injection from \(I_{n,k-1} \times I_{n,k+1}\) to \(I_{n,k} \times I_{n,k}\), then there is a shape-preserving injection from \(L_{n,k-1} \times L_{n,k+1}\) to \(L_{n,k} \times L_{n,k}\), see [2, Theorem 2.2].

Though they could not prove Conjectures 1.2 and 1.3, Bóna, Lackner and Sagan proposed a new way to look at these problems. Given a set \(\Lambda\) of partitions of \(n\), for \(1 \leq k \leq n\) let

\[
L^\Lambda_{n,k} = \{\pi \in L_{n,k} \mid \sh \pi \in \Lambda\}, \quad \ell^\Lambda_{n,k} = |L^\Lambda_{n,k}|; \quad I^\Lambda_{n,k} = \{\pi \in I_{n,k} \mid \sh \pi \in \Lambda\}, \quad i^\Lambda_{n,k} = |I^\Lambda_{n,k}|
\]

Thus, the sequence \(\{\ell^\Lambda_{n,k}\}_{k=1}^n\) (resp. \(\{i^\Lambda_{n,k}\}_{k=1}^n\)) is just \(\{\ell^\Lambda_{n,k}\}_{k=1}^n\) (resp. \(\{i^\Lambda_{n,k}\}_{k=1}^n\)) when taking \(\Lambda\) to be the set of all partitions of \(n\). They noted that the log-concavity of \(\{\ell^\Lambda_{n,k}\}_{k=1}^n\) is equivalent to that of \(\{i^\Lambda_{n,k}\}_{k=1}^n\) provided that the set \(\Lambda\) contains at most one partition with first row of length \(k\) for each \(1 \leq k \leq n\). They further obtained the following results, see [2, Theorems 3.1, 3.2, 4.4 and 4.5].

**Theorem 1.4.** Suppose that \(n\) is a positive integer and \(m = 1, 2\).

1. For \(\Lambda = \{(j^m, (n-j)^m) \mid \left\lfloor \frac{m}{2} \right\rfloor \leq j \leq n\}\), the sequence \(\{i^\Lambda_{mn,k}\}_{k=1}^n\) is log-concave.
2. For \(\Lambda = \{(j^m, 1^{m(n-j)}) \mid 1 \leq j \leq n\}\), the sequence \(\{i^\Lambda_{mn,k}\}_{k=1}^n\) is log-concave.

The Robinson-Schensted correspondence tells that \(f^\Lambda = |\{\pi \mid \pi^2 = id \text{ and } \sh \pi = \lambda\}|\). Thus Theorem 1.1 and Theorem 1.4 are equivalent to each other for \(m = 1, 2\).

To prove the inequalities on \(f^\Lambda\), a natural way is to use the hook-length formula, as Bóna, Lackner and Sagan did in their paper [2]. Here we will propose another way based on the property of the exponential specialization. Let \(\Lambda_Q\) denote the ring of symmetric functions over the field \(\mathbb{Q}\) of rational numbers. Recall that the exponential specialization \(\text{ex} : \Lambda_Q \rightarrow \mathbb{Q}[t]\) is defined by acting on the power sums \(p_n\) as

\[
\text{ex}(p_n) = t \delta_{1n},
\]
and then extended algebraically. For any symmetric function $f$, let $\text{ex}_1(f) = \text{ex}(f)_{i=1}$. It is well known that

$$\text{ex}_1(s_\lambda) = \frac{f^\lambda}{n!}, \quad \text{or equivalently,} \quad f^\lambda = \text{ex}_1(n!s_\lambda)$$

(2)

for any $\lambda \vdash n$. For more information on the exponential specialization, see [9]. Since $\text{ex}$ is an algebra homomorphism, the inequalities on $f^\lambda$ considered in Theorem 1.1 can be deduced from the Schur positivity of the differences of products of Schur functions $s_\lambda$.

The rest of the paper is organized as follows. In Section 2, we give a proof of Theorem 1.1 by using the hook-length formula. In Section 3, we present an alternative proof of Theorem 1.1 based on the Schur positivity of certain symmetric functions.

## 2 Proof by the hook-length formula

The aim of this section is to give an proof of Theorem 1.1 by using the hook-length formula.

Let us first give an overview of related definitions and results. Given a partition $\lambda$, let $\ell(\lambda)$ denote the number of its nonzero parts. Each partition $\lambda$ is associated to a left justified array of cells with $\lambda_i$ cells in the $i$-th row, called the Ferrers or Young diagram of $\lambda$. Here we number the rows from top to bottom and the columns from left to right. The cell in the $i$-th row and $j$-th column is denoted by $(i,j)$. The hook-length of $(i,j)$, denoted by $h(i,j)$, is defined to be the number of cells directly to the right or directly below $(i,j)$, counting $(i,j)$ itself once. The classical hook-length formula is stated as follows, which was discovered by Frame, Robinson and Thrall [5].

**Theorem 2.1** ([5]). For any partition $\lambda \vdash n$, we have

$$f^\lambda = \frac{n!}{\prod_{(i,j) \in \lambda} h(i,j)}. \quad (3)$$

For our purpose here, it turns out to be easier to work with the following equivalent form of the hook-length formula.

**Theorem 2.2** ([6]). Given a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{\ell(\lambda)}) \vdash n$, we have

$$f^\lambda = \frac{n!}{h(1,1)!h(2,1)! \cdots h(\ell(\lambda),1)!} \prod_{1 \leq j_1 < j_2 \leq \ell(\lambda)} (h(j_1,1) - h(j_2,1)). \quad (4)$$

The equivalence between these two formulas is evident by virtue of the equality

$$\prod_{(i,j) \in \lambda} h(i,j) = \frac{h(1,1)!h(2,1)! \cdots h(\ell(\lambda),1)!}{\prod_{1 \leq j_1 < j_2 \leq \ell(\lambda)} (h(j_1,1) - h(j_2,1))}. \quad (5)$$

It should be mentioned that (4) can be taken as a direct consequence of the Frobenius character formula, see Fulton and Harris [6].

Now we can give a proof of Theorem 1.1.
Proof of Theorem 1.1. Let us first prove that for $\left\lceil \frac{n}{2} \right\rceil < k < n$

$$(f(k^m, (n-k)^m))^2 \geq f((k+1)^m, (n-k-1)^m)f((k-1)^m, (n-k+1)^m).$$

To this end, we will use the expression of $f(k^m, (n-k)^m)$ given by Theorem 2.2. It is readily to see that the hook-lengths of the first column of the partition $(k^m, (n-k)^m)$ are given by

$$h_{(i,1)} = \begin{cases} k + 2m - i, & \text{for } 1 \leq i \leq m; \\
-k + 2m - i, & \text{for } m + 1 \leq i \leq 2m. \end{cases} \quad (6)$$

Therefore,

$$f(k^m, (n-k)^m) = \frac{(mn)!}{\prod_{i=1}^m h_{(i,1)}! \times \prod_{i=m+1}^{2m} h_{(i,1)}!} \times \prod_{1 \leq i_1 < i_2 \leq m} (h_{(i_1,1)} - h_{(i_2,1)}) \times \prod_{m+1 \leq i_1 < i_2 \leq 2m} (h_{(i_1,1)} - h_{(i_2,1)}).$$

Substituting (6) into the above formula, we obtain

$$f(k^m, (n-k)^m) = \frac{(mn)!}{\prod_{i=0}^{m-1} (m + k + i)! \times \prod_{i=0}^{m-1} (n - k + i)!} \times \prod_{0 \leq i_1 < i_2 \leq m-1} (2k - n + 1 + i_2 + i_1) \times \prod_{0 \leq i_1 < i_2 \leq m-1} (i_2 - i_1) \times \prod_{0 \leq i_1 < i_2 \leq m-1} (i_2 - i_1).$$

Note that the last two factors on the right hand side are independent of $k$. Denote the second factor by $a_k$, namely,

$$a_k = \prod_{0 \leq i_1 < i_2 \leq m-1} (2k - n + 1 + i_2 + i_1).$$

It is easy to verify that

$$\frac{a_k^2}{a_{k-1}a_{k+1}} = \prod_{0 \leq i_1, i_2 \leq m-1} \frac{(2k - n + 1 + i_2 + i_1)^2}{(2k - n - 1 + i_2 + i_1)(2k - n + 3 + i_2 + i_1)} \geq 1,$$

since, for any $\left\lceil \frac{n}{2} \right\rceil < k < n$ and $0 \leq i_1, i_2 \leq m - 1$, there holds

$$(2k - n + 1 + i_2 + i_1)^2 \geq (2k - n - 1 + i_2 + i_1)(2k - n + 3 + i_2 + i_1)$$

by the inequality of arithmetic and geometric means. Thus, for $\left\lceil \frac{n}{2} \right\rceil < k < n$, we have

$$\frac{(f(k^m, (n-k)^m))^2}{f((k+1)^m, (n-k-1)^m)f((k-1)^m, (n-k+1)^m)} = \frac{2m + k}{m + k} \times \frac{n - k + m}{n - k} \times \frac{a_k^2}{a_{k-1}a_{k+1}} \geq 1,$$

as desired.

We proceed to prove the second part of the theorem, namely, for $1 < k < n$,

$$(f(k^m, 1^{(n-k)})^2 \geq f((k+1)^m, 1^{(n-k-1)})f((k-1)^m, 1^{(n-k+1)}).$$
Let us first give an expression of \( f(\lambda, \nu) \) by using Theorem 2.1. Note that, for \( 1 \leq k \leq n \), the hook-lengths of the partition \((\lambda, \nu)\) are given by

\[
\begin{align*}
\begin{cases}
  m(n-k) + m + k - i, & \text{for } j = 1 \text{ and } 1 \leq i \leq m; \\
  m(n-k) + m + 1 - i, & \text{for } j = 1 \text{ and } m + 1 \leq i \leq m(n-k) + m; \\
  h'_{(i,j-1)}, & \text{for } 2 \leq j \leq k \text{ and } 1 \leq i \leq m
\end{cases}
\end{align*}
\]

where \( h'_{(i,j)} \) denotes the hook-length of the cell \((i,j)\) in partition \(((k-1)^m)\). Therefore,

\[
f(\lambda, \nu) = \frac{(mn)!}{\prod_{1 \leq i \leq m} h_{(i,1)} \times \prod_{m+1 \leq i \leq m(n-k)+m} h_{(i,1)} \times \prod_{1 \leq i \leq m, 2 \leq j \leq k} h_{(i,j)}}.
\]

Substituting (7) into the above formula, we obtain

\[
f(\lambda, \nu) = \frac{(mn)!}{[m(n-k)]! \times \prod_{i=0}^{m-1} [m(n-k) + k + i] \times \prod_{1 \leq i \leq m, 2 \leq j \leq k} h'_{(i,j-1)}}.
\]

While, we see that

\[
\prod_{1 \leq i \leq m, 2 \leq j \leq k} h'_{(i,j-1)} = \prod_{(i,j) \in ((k-1)^m)} h'_{(i,j)} = \frac{\prod_{i=0}^{m-1} (k-1+i)!}{\prod_{0 \leq i_1 < i_2 \leq m-1} (i_2 - i_1)}.
\]

where the second equality is obtained by applying (5) to the partition \( \lambda = ((k-1)^m) \). Thus, we have

\[
f(\lambda, \nu) = \frac{(mn)!}{[m(n-k)]! \times \prod_{i=0}^{m-1} [m(n-k) + k + i] \times \prod_{0 \leq i_1 < i_2 \leq m-1} (i_2 - i_1)} \times \frac{\prod_{i=0}^{m-1} (k-1+i)!}{\prod_{0 \leq i_1 < i_2 \leq m-1} (k-1+i)!}.
\]

Let

\[
b_k = [m(n-k)]! \prod_{i=0}^{m-1} [m(n-k) + k + i] = \frac{(mn-mk)!(mn-mk+k+1)!}{(mn-mk+k+m-1)!}.
\]

Then, for \( 1 < k < n \), we have

\[
\frac{(f(\lambda, \nu))^2}{f((k+1)^m, \nu) f((k-1)^m, \nu)} = \frac{k + m - 1}{k - 1} \times \frac{b_{k-1}b_{k+1}}{b_k^2}.
\]

Now it suffices to show that \( b_k^2 \leq b_{k-1}b_{k+1} \) for \( 1 < k < n \). Let

\[
b(z) = \frac{\Gamma(mn-mz+1)\Gamma(mn-mz+z)}{\Gamma(mn-mz+z+m)}
\]
be the continuous function on \([1, n]\), where \(\Gamma(z)\) is the Gamma function. Hence, for \(1 \leq k \leq n\), we have \(b_k = b(k)\), the value of \(b(z)\) evaluated at \(z = k\). To prove \(b_k^2 \leq b_{k-1}b_{k+1}\) for \(1 < k < n\), it suffices to show that \((\ln b(z))'' \geq 0\) for \(z \in [1, n]\). To this end, we first compute the logarithmic derivative of \(b(z)\) as follows:

\[
(\ln b(z))' = -mv'(mn - mz + 1) - (m - 1)v(mn - mz + z) + (m - 1)v(mn - mz + z + m)
\]

where \(v(z) = (\ln \Gamma(z))'\) is the digamma function. Then we obtain that

\[
(\ln b(z))'' = m^2v''(mn - mz + 1) + (m - 1)^2v'(mn - mz + z) - (m - 1)^2v'(mn - mz + z + m).
\]

It is known that \(v'(z) = \sum_{k=0}^{\infty} \frac{1}{(z+k)^2}\) over \(z \in (0, +\infty)\), and hence it is positive and decreasing, see \([1]\). Thus, \((\ln b(z))'' \geq 0\) for any \(z \in [1, n]\). This completes the proof. \(\square\)

As an immediate consequence of Theorem 1.1, we obtain the following result, which shows that Theorem 1.4 is true for any positive integer \(m\).

**Corollary 2.3.** Suppose that \(m, n\) are two positive integers.

1. For \(\Lambda = \{(j^m, (n - j)^m) \mid \left\lceil \frac{n}{2} \right\rceil \leq j \leq n\}\), both \(\{\ell^\Lambda_{mn,k}\}_{k=1}^m\) and \(\{i^\Lambda_{mn,k}\}_{k=1}^m\) are log-concave.
2. For \(\Lambda = \{(j^m, 1^{m(n-j)}) \mid 1 \leq j \leq n\}\), both \(\{\ell^\Lambda_{mn,k}\}_{k=1}^m\) and \(\{i^\Lambda_{mn,k}\}_{k=1}^m\) are log-concave.

### 3 Proof by Schur positivity

The aim of this section is to give another proof of Theorem 1.1 by using Schur positivity. Recall that a symmetric function is said to be Schur positive if it can be written as non-negative integer linear combination of Schur functions. By (2), Theorem 1.1 is implied by the following result.

**Theorem 3.1.** Suppose that \(m\) and \(n\) are two positive integers.

1. For \(\left\lceil \frac{n}{2} \right\rceil < k < n\), the difference

\[
1\text{\(s_{(k^m,(n-k)^m)}^2 - s_{((k+1)^m,(n-k-1)^m)}s_{((k-1)^m,(n-k+1)^m)}\)
}

is Schur positive.

2. For \(1 < k < n\), the difference

\[
1\text{\(s_{(k^m,1^{m(n-k)})}^2 - s_{((k+1)^m,1^{m(n-k-1)})}s_{((k-1)^m,1^{m(n-k+1)})}\)
}

is Schur positive.

Our proof of Theorem 3.1 is based on some Schur positivity results due to Lam, Postnikov and Pylyavaskyy \([7]\). For vectors \(v, w\) and a positive integer \(n\), we assume that the operations \(v + w, \frac{v}{n}, [v]\) and \([v]\) are performed coordinate-wise. In particular, we have well-defined operations \(\lfloor \lambda + \mu \rfloor\) and \(\lceil \lambda + \mu \rceil\) on pairs of any partitions. If \(\lambda, \mu\) are partitions with \(\lambda_i \geq \mu_i\) for all \(i \geq 1\), then the skew diagram \(\lambda/\mu\) is the diagram of \(\lambda\) with the diagram of \(\mu\) removed from its upper left-hand corner. Lam, Postnikov and Pylyavaskyy obtained the following result, which answered a conjecture of Okounkov \([8]\).
Theorem 3.2 ([7, Theorem 11]). Given any two skew partitions $\lambda/\mu$ and $\nu/\rho$, the difference
\[ s_{(\lambda^\mu)} - s_{(\nu/\rho)} \]
is Schur positive.

Given two partitions $\lambda$ and $\mu$, let $\lambda \cup \mu = (\nu_1, \nu_2, \nu_3, \ldots)$ be the partition obtained by rearranging all parts of $\lambda$ and $\mu$ in the weakly decreasing order. Let $\text{sort}_1(\lambda, \mu) := (\nu_1, \nu_3, \nu_5, \ldots)$ and $\text{sort}_2(\lambda, \mu) := (\nu_2, \nu_4, \nu_6, \ldots)$. Lam, Postnikov and Pylyavaskyy also obtained the following result, which was first conjectured by Fomin, Fulton, Li and Poon [4].

Theorem 3.3 ([7, Corollary 12]). For any two partitions $\lambda$ and $\mu$, the difference
\[ s_{\text{sort}_1(\lambda, \mu)} s_{\text{sort}_2(\lambda, \mu)} = s_{\lambda} s_{\mu} \]
is Schur positive.

We are now in the position to give a proof of Theorem 3.1.

Proof of Theorem 3.1. For $\lceil \frac{n}{2} \rceil < k < n$, taking $\lambda = ((k+1)^m, (n-k-1)^m)$, $\nu = ((k-1)^m, (n-k+1)^m)$, and $\mu = \rho = \emptyset$ in Theorem 3.2, we obtain the Schur positivity of
\[ s_{(k^m, (n-j)^m)} - s_{((k+1)^m, (n-k-1)^m)} s_{((k-1)^m, (n-k+1)^m)}. \]

For $1 < k < n$, taking $\lambda = ((k+1)^m, 1^{m(n-k-1)})$, $\nu = ((k-1)^m, 1^{m(n-k+1)})$ and $\mu = \rho = \emptyset$ in Theorem 3.2, we obtain the Schur positivity of
\[ s_{(k^m, 1^{m(n-k+1)})} s_{(k^m, 1^{m(n-k-1)})} - s_{((k+1)^m, 1^{m(n-k-1)})} s_{((k-1)^m, 1^{m(n-k+1)})}. \] (8)

Taking $\lambda = (k^m, 1^{m(n-k+1)})$ and $\mu = (k^m, 1^{m(n-k-1)})$ in Theorem 3.3, we obtain the Schur positivity of
\[ s_{(k^m, 1^{m(n-k+1)})}^2 - s_{(k^m, 1^{m(n-k-1)})}^2 s_{(k^m, 1^{m(n-k-1)})}. \] (9)

Combining (8) and (9), we obtain the Schur positivity of
\[ s_{(k^m, 1^{m(n-k+1)})}^2 - s_{((k+1)^m, 1^{m(n-k-1)})} s_{((k-1)^m, 1^{m(n-k+1)})}. \]

This completes the proof. \hfill \Box

As we mentioned at the end of Section 2, Theorem 1.4 implies the log-concavity of certain sequences concerning longest increasing subsequences. The approach of this section to Theorem 1.4 inspired us to study Conjecture 1.2 and Conjecture 1.3 from the viewpoint of Schur positivity. Note that for a fixed integer $n$, the Robinson-Schensted correspondence shows that
\[ \ell_{n,k} = \sum_{\lambda \vdash n, \lambda_1 = k} (f^\lambda)^2 \quad \text{and} \quad i_{n,k} = \sum_{\lambda \vdash n, \lambda_1 = k} f^\lambda \] (10)
for $1 \leq k \leq n$. We have the following conjectures.
Conjecture 3.4. For $1 \leq k \leq n$, let
\[ f_{n,k} = \sum_{\lambda \vdash n, \lambda_1 = k} s_\lambda, \]
then $f_{n,k}^2 - f_{n,k+1} f_{n,k-1}$ is Schur positive with the convention that $f_{n,0} = f_{n,n+1} = 0$.

Conjecture 3.5. For $1 \leq k \leq n$, let
\[ g_{n,k} = \sum_{\lambda \vdash n, \lambda_1 = k} s_\lambda, \]
then $g_{n,k}^2 - g_{n,k+1} g_{n,k-1}$ is Schur positive with the convention that $g_{n,0} = g_{n,n+1} = 0$.

It is readily to see that Conjecture 3.4 implies Conjecture 1.2, and Conjecture 3.5 implies Conjecture 1.3 by (2). We have verified Conjecture 3.4 for $n \leq 9$ and Conjecture 3.5 for $n \leq 20$.

Chen [3] also put forward some log-concavity conjecture about perfect matchings, which was turned into the form of Conjecture 3.6 by Bóna, Lackner and Sagan. For any fixed $n$, let $\Theta$ be the set of partitions of $n$ all of whose column lengths are even. Chen’s conjecture can be stated as follows.

Conjecture 3.6 ([3, Conjecture 1.5]). For any fixed $n$, the sequence $\{i^{\Theta}_{n,k}\}_{k=1}^n$ is log-concave.

Inspired by Conjectures 3.4 and 3.5, we propose the following conjecture, which implies Conjecture 3.6.

Conjecture 3.7. For $1 \leq k \leq n$, let
\[ g^{\Theta}_{n,k} = \sum_{\lambda \in \Theta, \lambda_1 = k} s_\lambda, \]
then $(g^{\Theta}_{n,k})^2 - g^{\Theta}_{n,k+1} g^{\Theta}_{n,k-1}$ is Schur positive with the convention that $g^{\Theta}_{n,0} = g^{\Theta}_{n,n+1} = 0$.

This conjecture has been verified for $n \leq 30$. Bóna, Lackner and Sagan further proposed a companion conjecture to Conjecture 3.6.

Conjecture 3.8 ([2, Conjecture 4.3]). For any fixed $n$, the sequence $\{f^{\Theta}_{n,k}\}_{k=1}^n$ is log-concave.

However, Conjecture 3.8 does not admit a similar conjecture as Conjecture 3.7 as illustrated below. For $1 \leq k \leq n$, let
\[ f^{\Theta}_{n,k} = \sum_{\lambda \in \Theta, \lambda_1 = k} s_\lambda^2. \]
In general, the difference $(f^{\Theta}_{n,k})^2 - f^{\Theta}_{n,k+1} f^{\Theta}_{n,k-1}$ is not Schur positive. For instance, when $n = 10$, we have
\[
\begin{align*}
  f^{\Theta}_{10,2} &= s^2_{(2,2,2,2,1,1)} + s^2_{(2,2,1,1,1,1,1,1)}, \\
  f^{\Theta}_{10,3} &= s^2_{(3,3,2,2)} + s^2_{(3,3,1,1,1,1)}, \\
  f^{\Theta}_{10,4} &= s^2_{(4,4,1,1,1)}.
\end{align*}
\]
However, the symmetric function $(f_{10,3}^\ominus)^2 - f_{10,2}^\ominus f_{10,4}^\ominus$ is not Schur positive by computer exploration using the open-source mathematical software Sage [10] and its algebraic combinatorics features developed by the Sage-Combinat community [11].

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References

[1] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables (10th ed.), 1972, Dover, New York.

[2] M. Bóna, M.-L. Lackner, and B.E. Sagan, Longest increasing subsequences and log concavity, Ann. Combin., to appear.

[3] W.Y.C. Chen, Log-concavity and $q$-log-convexity conjectures on the longest increasing subsequences of permutations, arXiv:0806.3392.

[4] S. Fomin, W. Fulton, C.-K. Li and Y.-T. Poon, Eigenvalues, singular values, and Littlewood-Richardson coefficients, Amer. J. Math. 127 (2005), 101–127.

[5] J. Frame, G. Robinson and R. Thrall, The hook graphs of the symmetric group, Canad. J. Math. 6 (1954), 316–325.

[6] W. Fulton and J. Harris, Representation Theory: A First Course. Springer-Verlag, New York, 1991.

[7] T. Lam, A. Postnikov, and P. Pylyavskyy, Schur positivity and Schur log-concavity, Amer. J. Math. 129 (2007), 1611–1622.

[8] A. Okounkov, Log-concavity of multiplicities with applications to characters of $U(\infty)$, Adv. Math. 127 (1997), 258–282.

[9] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge University Press, New York, Cambridge, 1999.

[10] SageMath, the Sage Mathematics Software System (Version 7.5.1), The Sage Developers, 2017, http://www.sagemath.org.

[11] The Sage-Combinat community, Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics, http://combinat.sagemath.org, 2008.