Generation and Decay of Two-Dimensional Quantum Turbulence in a Trapped Bose-Einstein Condensate

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In a recent experiment, Kwon et al. [arXiv:1403.4658 [cond-mat.quant-gas]] generated a disordered state of quantum vortices by translating an oblate Bose-Einstein condensate past a laser-induced obstacle and studying the subsequent decay of vortex number. Using mean-field simulations of the Gross-Pitaevskii equation, we shed light on the various stages of the observed dynamics. We find that the flow of the superfluid past the obstacle leads initially to the formation of a classical-like wake, which later becomes disordered. Following removal of the obstacle, the vortex number decays due to vortices annihilating and reaching the boundary. Our results are in excellent agreement with the experimental observations. Furthermore, we probe thermal effects through phenomenological dissipation.

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Ultracold gaseous Bose-Einstein condensates (BECs) provide a unique testbed with which we can investigate the phenomenon of quantum turbulence and the more rudimentary realm of superfluid vortex dynamics [1, 2]. These systems provide an impressive degree of parameter manipulation unavailable in superfluid helium, the traditional context for studying quantum turbulence [3], with scope to control the particle interactions and potential landscape in both time and space. The typical size of these systems is only one or two orders or magnitude larger than the inter-vortex spacing, which in turn is another order of magnitude larger than the vortex core size. These compact length scales mean that the collective behaviour of vortices and their interaction with the background condensate is significant. The emergence of turbulent-like behaviour in the form of a vortex core size. These compact length scales mean that the collective behaviour of vortices and their interaction with the background condensate is significant. The emergence of turbulent-like behaviour in the form of a vortex wake, which later becomes disordered. Following removal of the obstacle, the vortex number decays due to vortices annihilating and reaching the boundary. Our results are in excellent agreement with the experimental observations. Furthermore, we probe thermal effects through phenomenological dissipation.

In the recent experiment of Kwon et al. [4, 5], a trapped, oblate BEC was translated past a stationary, laser-induced obstacle. As is characteristic of superfluids, vortices and anti-vortices were nucleated into the condensate once the relative speed exceeded a critical value [6]. The authors monitored the number of vortices, revealing the dependence on the relative speed and the thermal relaxation of the vortices. Furthermore, they directly observed vortex-antivortex annihilations, characterised by a crescent-shaped depletion in the condensate density.

In this article we elucidate these experimental findings through mean-field simulations of the two-dimensional (2D) Gross-Pitaevskii equation (GPE), both at zero-temperature and in the presence of thermal dissipation, modelled through a phenomenological damping term in the GPE. Notably, our simulations provide insight into the sign of the circulation of the vortices and the early stage evolution, not accessible experimentally. We establish the key stages of the dynamics, from the initial nucleation of vortices and formation of a quasi-classical wake, through the rapid symmetry breaking and disorganization of the vortices, to the decay of the vortices by annihilation or passage out of the condensate. Our approach gives excellent agreement with the experimental observations, despite the three-dimensional geometry of the experimental system.

In the experiment, a $^{23}$Na condensate with $N = 1.8 \times 10^6$ atoms was confined within a highly-oblate cylindrically symmetric harmonic trap $V_{\text{trap}}(x, y, z) = \frac{1}{2} m (\omega_x^2 x^2 + y^2) + \omega_z^2 z^2$, with axial frequency $\omega_z = 2\pi \times 350$ Hz and radial frequency $\omega_r = 2\pi \times 15$ Hz (corresponding to an aspect ratio parameter $\omega_z/\omega_r \approx 23$) and where $m$ denotes the atomic mass. A 2D mean-field description is strictly valid when the condition $Na l_2^2/l_z^4 \ll 1$ is satisfied, where $l_z = \sqrt{\hbar/\mu \omega_z}$ and $l_r = \sqrt{\hbar/\mu \omega_r}$ are the axial and radial harmonic oscillator lengths and $a$ is the s-wave scattering length [10, 11]. For this experiment, $Na l_2^2/l_z^4 = 8.3$, i.e. the system remains 3D in nature. Nonetheless, the dynamics of the vortices is essentially 2D because of the suppression of Kelvin waves in the $z$-direction [12]. Therefore, we adopt a 2D description throughout this work and show that it is sufficient to capture the experimental observations. It is worth noting that in the $xy$ plane the condensate closely approximates a Thomas-Fermi (inverted parabola) density profile with radius $R_{\text{TF}} \approx 70 \mu m$.

We parameterize the condensate by a 2D wavefunction $\phi(x, y, t)$; the condensate density distribution follows as $n(x, y, t) = |\phi(x, y, t)|^2$. The wavefunction satisfies the 2D GPE:

$$i\hbar \frac{\partial \phi}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V(x, y, t) + g|\phi|^2 - \mu \right) \phi$$

(1)
where $\mu$ denotes the chemical potential of the condensate and $g = 2\hbar a(2\pi\omega_{\text{r}}\hbar/m)^{1/2}$ characterizes the effective 2D nonlinear interactions arising from $s$-wave atomic collisions. We solve the GPE on a 1024 × 1024 grid using a fourth-order Runge-Kutta method.

Following the experiment, the total potential acting on the condensate $V(x, y, t)$ is the above harmonic trap plus a static Gaussian-shaped obstacle potential $V_{\text{obs}}(x, y) = V_0 \exp \left[2(x^2 + y^2)/d^2\right]$, located at the origin, with $V_0 = 15\mu$ and $d = 15\mu$. The initial ground-state BEC is obtained by solving the GPE in imaginary time with enforced norm $N = 1.8 \times 10^6$. From $t = 0$ the harmonic trap is translated along $x$ at speed $v$ over a distance $37\mu$m; to smooth this speed curve we additionally include a linear acceleration/deceleration over 3.75ms at the start/end. Once the trap is at rest, the obstacle amplitude $V_0$ is ramped down to zero over 0.4s.

Following removal of the obstacle, we determine the number of vortices in the system $N_v$ (performed by identifying locations where the condensate possesses a $2\pi$ singularity in the phase). We limit our search to 75 percent of the Thomas-Fermi radius; by avoiding the low density periphery we avoid artifacts from ghost vortices and match closely what is performed experimentally (since vortices close to the edge are not detected due to low signal-to-noise [13]). In Fig. 1 we plot $N_v$ versus the translation speed $v$. We see the same qualitative form between our simulations (red circles) and the experiment (black crosses): above a critical speed $v_c \approx 0.45$mm/s vortices enter the system, nucleated by the relative motion between the obstacle and the superfluid, and for $v > v_c$ the growth in $N_v$ is initially rapid but tails off for $v \gg v_c$. Quantitatively, however, the GPE overestimates $N_v$. One can expect that thermal dissipation, not accounted for in the GPE, will act to reduce the number of vortices in the system. We introduce the effects of such dissipation into the GPE via the addition of phenomenological damping, $\gamma [14, 15]$, which enters the GPE $\mathbf{1}$ by replacing $i$ on the left hand side by $(i - \gamma)$. This term induces the decay of excitations; for single vortices this manifests in them spiraling out of the trapped condensate $[12, 16, 18]$. We choose a small value of $\gamma = 0.0003$ so as to model the experiment in its very coldest realization of $\sim 130$K and enforce the norm throughout the dissipative simulations so as to emulate the experiment (for which no significant loss of atom number was observed).

With this small amount of dissipation, the data for $N_v$ becomes reduced, bringing it closely in line with the experimental data. Experimental limitations in resolving and counting vortices may also contribute to the overestimate of $N_v$ from the GPE.

We now examine in detail the evolution of the condensate, charting its dynamics from the initial stage (when the obstacle translation begins) to the intermediate and final stages (randomization and decay of the vortices). We see the same qualitative evolution with and without dissipation, and for all velocities exceeding $v_c$. For the purposes of illustration, we focus on an example with dissipation and a translation speed $v = 1.4$mm/s. Figure 2 shows the condensate density at various times. At the start of the simulation ($t = 0$) the condensate has a smooth circular density profile, with a density depression due to the obstacle. Later vortices appear as small dots of low density; superimposed red/blue markers tag vortices of positive/negative circulation.

**Vortex wake formation:** The harmonic trap is translated to the left sufficiently rapidly that the condensate does not adiabatically follow the trap minimum, but rather begins a sloshing motion in the trap, in which the centre-of-mass of the BEC oscillates at the trap frequency and the BEC undergoes a quadrupolar shape oscillation. As the BEC sloshes first to the left, its speed increases. When the local fluid velocity exceeds the speed of sound, vortices nucleate at the poles of the obstacle (where the local fluid velocity is the greatest) and are washed downstream (to the left). The pattern of vortices nucleated by a moving obstacle in a superfluid depends, in general, on the speed, shape and size of the obstacle [19–21]. During the initial evolution vortices of negative and positive circulation are created near each pole in an irregular manner, sometimes with alternating circulation; other times several vortices of the same circulation appear. In our case, the rate of vortex nucleation is sufficiently high that the vortices interact strongly with each other, collectively forming macroscopic wakes of negative and positive vortices downstream of the object ($t = 43$ms). During this early stage, vortices of opposite circulation may become very close and annihilate (i.e. undergo a 2D reconnection), leaving behind density (sound) waves. The condensate then sloshes to the right; this motion not only carries the existing vortices to the opposite (right) side of the obstacle but nucleates further vortices. As the condensate’s sloshing mode is damped by the dissipation, the relative speed of the obstacle decreases and the vortex nucleation pattern changes: like-signed vortices are
generated near each pole, forming symmetric classical-like wakes \[21\]. This effect leads to further clustering of like-signed vortices \((t = 69\text{ms})\). As the condensate continues to slosh, more vortices nucleate into the system. It must be stressed that, up to these early times \((t = 191\text{ms})\), the vortex distribution remains symmetric about the \(x\) axis, and that without the dissipation term in the GPE, the sloshing mode persists over time rather than decaying.

![Snapshots of the condensate density](image)

**FIG. 2:** Snapshots of the condensate density, for a translational speed \(v = 1.4\text{mm/s}\) and in the presence of dissipation \((\gamma = 0.0003)\). The obstacle is completely removed at 0.43s. The field of view is of size \([170\mu\text{m}]^2\) and centered on the centre-of-mass of the condensate. Vortices with positive (negative) circulation are highlighted by red circles (blue triangles).

**Vortex randomization:** In the presence of the obstacle and the sloshing mode, vortices continually nucleate and their spatial distribution remains approximately symmetric about the \(x\) axis. At later times \((t > 318\text{ms})\) this symmetry breaks and the vortices evolve into a completely disorganised, apparently random configuration with no significant clustering of like-signed vortices. Besides vortices, the condensate contains also an energetic, disordered sound field, indicative of two-dimensional quantum turbulence \[5, 7\].

It is interesting to note the obstacle is still in the system at this point, nucleating vortices in a symmetrical manner. The disorganised vortices already in the system create a velocity field which quickly mixes newly created vortices nucleated at the poles of the obstacle. Visual inspection, confirmed by a clustering-detection algorithm \[22, 23\], shows no significant clusters beyond this stage of the evolution. By the time the obstacle is removed the vortex configuration is essentially random, but the number of positive and negative vortices stays approximately equal. It is important to remark that, without detecting the sign of the vortex circulation, we could not reach these conclusions.

**Vortex decay:** It is clear from Fig. 2 that, following the removal of the obstacle, the number of vortices \((N_v)\) depletes. Indeed, one expects that the condensate will decay towards its vortex-free, time-independent ground state. To quantify the vortex decay, Fig. 3 plots \(N_v\) versus time. The onset of vortex nucleation is at around \(t = 0.02\text{ms}\); this is the time taken for accelerating condensate to exceed the speed of sound at the poles of the object. \(N_v\) then grows steeply, as vortices are rapidly driven into the system. At this point \(N_v\) grows slowly and fluctuates: more vortices nucleate from the obstacle, but existing vortices annihilate or move into low density regions of the condensate where they are not detected. Once the obstacle is removed, \(N_v\) decays monotonically with time. Kwon et al. \[8\] argued that there are two mechanisms by which vortices decay, (i) thermal dissipation (resulting in loss of vortices at the edge of the condensate), and (ii) vortex-antivortex annihilation events, and proposed that the vortex decay takes the form:

\[
\frac{dN_v}{dt} = -\Gamma_1 N_v - \Gamma_2 N_v^2,
\]

where the linear and nonlinear terms, parameterized by the positive coefficients \(\Gamma_1\) and \(\Gamma_2\), respectively model vortices which move out of the condensate and vortices which annihilate. Our simulations support their findings. Fitting our \(N_v(t)\) data (with the constraint \(\Gamma_1, \Gamma_2 > 0\)) to Eq. 2, we find \(\Gamma_1 = 0\) for the dissipation-free GPE, meaning that the only significant decay is through annihilations, and \(\Gamma_1 = 0.113, \Gamma_2 = 0.00439\) in the presence of dissipation, which compares well with the coldest experimental data in \[8\].

In the experiment, Kwon et al. observed the occa-
FIG. 4: Phase (left) and density (right) just before (a), immediately following (b) and a later time after (c) a vortex-antivortex annihilation event. The field of view is $[4 \, \mu\text{m}]^2$, centered on the vortex pair/sound pulse (highlighted by a circle in the phase).

A cleaner and more efficient means to generate vortices may be provided by employing a laser-induced obstacle with elliptical, rather than circular, cross-section (attainable through cylindrical beam focussing). Repeating our simulations with such an elliptical obstacle $V_{\text{obs}}(x, y) = V_0 \exp[2(\epsilon x^2 + y^2)/d^2]$ with arbitrary ellipticity $\epsilon = 3$ (the short/long axis being parallel/perpendicular to the flow) confirms the same qualitative behaviour as for homogeneous systems [21]: the ellipticity acts to reduce the critical superfluid velocity and, for a given flow speed, increase the rate of vortex nucleation. For example, using this elliptical obstacle we can generate the same number of vortices as depicted in Fig. 3 (circular obstacle, $v = 1.4 \, \text{mm s}^{-1}$) with a translation speed of only $v = 0.8 \, \text{mm s}^{-1}$. At this reduced speed, the disruption (centre of mass mode, surface oscillations, sound waves) of the condensate is vastly reduced. What’s more, the elliptical obstacle promotes the formation of clusters of same-signed vortices, and thus may facilitate future exploitation of coherent vortex structures.

In conclusion, we have shown that the recent experimental creation and decay of vortices within a BEC [8] is well described by simulations of the 2D GPE with phenomenological dissipation (despite the 3D nature of the system). Theoretical access to the condensate phase, and thus the circulation of the vortices, promotes our understanding of the dynamics. In the early stages of translation of the obstacle, a quasi-classical wake of vortices forms behind it, before symmetry breaking causes disorganisation of the vortices. After the obstacle is removed, the vortices decay in a manner which is both qualitatively and quantitatively consistent with the two mechanisms proposed by Kwon et al., i.e. loss of vortices at the condensate edge due to thermal dissipation and vortex-antivortex annihilation events within the condensate. We confirm the occasional appearance of crescent-shaped density features, resulting either from the proximity of vortex cores or from the sound radiation pulse which follows a vortex-antivortex reconnection. Finally, we propose that a moving elliptical obstacle may provide a cleaner and more efficient means to generate two-dimensional quantum turbulence.

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[1] A. C. White, B. P. Anderson, and V. S. Bagnato, Proceedings of the National Academy of Sciences 111, 4719 (2014).
[2] C. F. Barenghi, L. Skrbek, and K. R. Sreenivasan, Proceedings of the National Academy of Sciences 111, 4647 (2014).

[3] C. Barenghi, R. Donnelly, W. Vinen, and eds., Quantized Vortex Dynamics and Superfluid Turbulence (Springer, Berlin, 2001).

[4] E. A. L. Henn, J. A. Seman, G. Roati, K. M. F. Magalhães, and V. S. Bagnato, Phys. Rev. Lett. 103, 045301 (2009).

[5] N. G. Parker and C. S. Adams, Phys. Rev. Lett. 95, 145301 (2005).

[6] S. Middelkamp, P. J. Torres, P. G. Kevrekidis, D. J. Frantzeskakis, R. Carretero-González, P. Schmelcher, D. V. Freilich, and D. S. Hall, Phys. Rev. A 84, 011605 (2011).

[7] T. W. Neely, A. S. Bradley, E. C. Samson, S. J. Rooney, E. M. Wright, K. J. H. Law, R. Carretero-González, P. G. Kevrekidis, M. J. Davis, and B. P. Anderson, Phys. Rev. Lett. 111, 235301 (2013).

[8] W. J. Kwon, G. Moon, J. Choi, S. W. Seo, and Y. Shin (2014), arXiv:1403.4658 [cond-mat.quant-gas], 1403.4658.

[9] T. Frisch, Y. Pomeau, S. Rica, Phys. Rev. Lett. 69, 1644 (1992).

[10] A. Muñoz Mateo and V. Delgado, Phys. Rev. A 74, 065602 (2006).

[11] N. G. Parker and D. H. J. O'Dell, Phys. Rev. A 78, 041601 (2008).

[12] B. Jackson, N. P. Proukakis, C. F. Barenghi, and E. Zaremba, Phys. Rev. A 79, 053615 (2009).

[13] Y. Shin, Private communication.

[14] S. Choi, S. A. Morgan, and K. Burnett, Phys. Rev. A 57, 4057 (1998).

[15] M. Tsubota, K. Kasamatsu, and M. Ueda, Phys. Rev. A 65, 023603 (2002).

[16] E. Madarassy and C. Barenghi, Journal of Low Temperature Physics 152, 122 (2008), ISSN 0022-2291.

[17] A. J. Allen, E. Zaremba, C. F. Barenghi and N. P. Proukakis, Phys. Rev. A 87, 013630 (2013).

[18] D. Yan, R. Carretero-González, D. J. Frantzeskakis, P. G. Kevrekidis, N. P. Proukakis, and D. Spirn, Phys. Rev. A 89, 041613 (2014).

[19] B. Jackson, J. F. McCann, and C. S. Adams, Phys. Rev. A 61, 051603 (2000).

[20] K. Sasaki, N. Suzuki, and H. Saito, Phys. Rev. Lett. 104, 150404 (2010).

[21] G. W. Stagg, N. G. Parker, and C. F. Barenghi, Journal of Physics B: Atomic, Molecular and Optical Physics 47, 095304 (2014).

[22] A. C. White, C. F. Barenghi, and N. P. Proukakis, Phys. Rev. A 86, 013635 (2012).

[23] M. T. Reeves, T. P. Billam, B. P. Anderson, and A. S. Bradley, Phys. Rev. Lett. 110, 104501 (2013).

[24] S. Nazarenko and M. Onorato, Journal of Low Temperature Physics 146, 31 (2007), ISSN 0022-2291.

[25] C. Rozai, K. R. Sreenivasan and M. E. Fisher, Phys. Rev. B 88, 134522 (2013).

[26] S. Prabhakar, R. P. Singh, S. Gautam, and D. Angom, Journal of Physics B: Atomic, Molecular and Optical Physics 46, 125302 (2013).

[27] M. Leadbeater, T. Winiecki, D.C. Samuels, C.F. Barenghi, and C.S. Adams, Phys. Rev. Lett. 86, 1410 (2001).

[28] S. Zuccher, M. Caliari, A.W. Baggaley, and C.F. Barenghi, Phys. of Fluids 24, 125108 (2012).