CHARACTERISTIC CLASSES OF BUNDLES OF K3 MANIFOLDS AND
THE NIELSEN REALIZATION PROBLEM

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Abstract. Let $K$ be the K3 manifold. In this note, we discuss two methods to prove that certain generalized Miller–Morita–Mumford classes for smooth bundles with fiber $K$ are non-zero. As a consequence, we fill a gap in a paper of the first author, and prove that the homomorphism $\text{Diff}(K) \to \pi_0\text{Diff}(K)$ does not split. One of the two methods of proof uses a result of Franke on the stable cohomology of arithmetic groups that strengthens work of Borel, and may be of independent interest.

1. Introduction

In this paper $K$ denotes the $K3$ manifold, which is the underlying oriented manifold of a complex $K3$ surface. This uniquely specifies its diffeomorphism type, and one may construct it as the hypersurface in $\mathbb{CP}^3$ cut out by the homogeneous equation $z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$. For each element $c \in H^i(\text{BSO}(4); \mathbb{Q})$, there is a characteristic class $\kappa_c$ of smooth oriented manifold bundles with fiber $K$, called a generalized Miller–Morita–Mumford class: given such a bundle $E \to B$ we take the vertical tangent bundle $T_vE$ and integrate the class $c(T_vE) \in H^i(E; \mathbb{Q})$ over the fibers to get $\kappa_c(E) \in H^{i-4}(B; \mathbb{Q})$.

Let $\text{Diff}(K)$ denote the group of orientation-preserving $C^2$-diffeomorphisms, in the $C^2$-topology. Its classifying space $B\text{Diff}(K)$ carries a universal smooth manifold bundle with fiber $K$, and hence there are classes $\kappa_c \in H^*(B\text{Diff}(K); \mathbb{Q})$ which may or may not be zero. Let $L_2 = \frac{1}{45}(7p_2 - p_1^2)$ be the second Hirzebruch $L$-polynomial, then we prove the following:

**Theorem A.** The generalized Miller–Morita–Mumford-class $\kappa_{L_2} \in H^4(B\text{Diff}(K); \mathbb{Q})$ is nonzero.

The Hirzebruch $L$-polynomials are related to the signature of manifolds and as a corollary of Theorem A, there exists a smooth bundle of $K3$ manifolds over a closed stably-framed 4-manifold whose total space has nonzero signature. We shall give two proofs of Theorem A: the first is an explicit calculation for the tautological bundle over a certain moduli space of $K3$ surfaces, while the second combines the study of Einstein metrics with a general result about cohomology of arithmetic groups following work of Franke.

Either proof can be combined with the Bott vanishing theorem to prove the following result. We define the mapping class group $\text{Mod}(K)$ to be the group $\pi_0\text{Diff}(K)$ of path components of $\text{Diff}(K)$.

**Theorem B.** The surjection $p: \text{Diff}(K) \to \text{Mod}(K)$ does not split, i.e. there is no homomorphism $s: \text{Mod}(K) \to \text{Diff}(K)$ so that $p \circ s = \text{Id}$.

This is an instance of the Nielsen realization problem; see e.g. [MT18]. Theorem B first appeared in [Gia09], but the proof was flawed (see [Gia19]). However, it can be repaired with small modifications and many of the ideas in this paper derive from [Gia09].

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Table 1. The class $\kappa_{i+1}(X_{2g})$ in terms of the Hodge class $\lambda$ for $i \leq 8.$

| $i$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|-----|----|----|----|----|----|----|----|----|----|
| $\kappa_{i+1}(X_{2g})$ | $24\;\lambda^3$ | $8\lambda^2$ | $8\lambda^2$ | $\frac{16\lambda^6}{3}$ | $8\lambda^8$ | $\frac{16\lambda^{10}}{45}$ | $16\lambda^{10}$ | $16\lambda^{12}$ | $\frac{32\lambda^{14}}{24967525}$ | $8\lambda^{16}$ |

2. Quasi-polarized K3 surfaces

Suppose that $E \to B$ is an oriented manifold bundle with closed fibers of dimension $d$. This has a vertical tangent bundle $T_v E$ with corresponding characteristic classes $c(T_v E) \in H^i(E; \mathbb{Q})$ for each $c \in H^i(BSO(d); \mathbb{Q})$. The generalized Miller–Morita–Mumford classes are obtained by integration of these classes along the fibers:

$$\kappa_c(E) := \int_M c(T_v E) \in H^{i-d}(B; \mathbb{Q}).$$

Applying this construction to the universal bundle of K3 manifolds over $B \text{Diff}(K)$ results in classes $\kappa_c \in H^{i-d}(B \text{Diff}(K); \mathbb{Q})$ for each $c \in H^i(BSO(4); \mathbb{Q}) = \mathbb{Q}[e, p_1]$. These classes are natural in the bundle: for any continuous map $f : B' \to B$, $\kappa_c(f^* E) = f^* \kappa_c(E)$.

To prove $\kappa_c \neq 0 \in H^*(B \text{Diff}(K); \mathbb{Q})$, it therefore suffices to find a single bundle $E \to B$ such that $\kappa_c(E) \neq 0$.

We shall use the moduli space $\mathcal{M}_{2g}$ of quasi-polarized K3 surfaces of genus $2g$. This is actually a stack with finite automorphism groups of bounded order, but since we are interested in its rational cohomology we may ignore these technical details. We shall not go into the details of its construction, but recall some facts from [vdGK05, Pet16]. There is a universal family $\pi : \mathcal{X}_{2g} \to \mathcal{M}_{2g}$ of K3 surfaces. As this is a bundle of complex surfaces, its vertical tangent bundle has Chern classes $c_i := c_i(T_v \mathcal{X}_{2g}) \in H^{2i}(\mathcal{X}_{2g}; \mathbb{Q})$. The first Chern class is the pullback of the Hodge class $\lambda \in H^2(\mathcal{M}_{2g}; \mathbb{Q})$. The main result of [vdGK05] is that $\lambda^{17} \neq 0$ but $\lambda^{18} = 0$, in the Chow ring of $\mathcal{M}_{2g}$. Petersen proved the corresponding result in rational cohomology [Pet16]. We shall use this to prove the following improvement of Theorem A:

**Proposition 1.** The generalized Miller–Morita–Mumford-class $\kappa_{i+1} \in H^{4i}(B\text{Diff}(K); \mathbb{Q})$ is non-zero for $i \leq 8$.

**Proof.** It suffices to prove that $\kappa_{i+1}(X_{2g}) \neq 0$. Since the K3 manifold is 4-dimensional, $p_1$, $p_2$ are the only non-zero Pontryagin classes of the vertical tangent bundle. These can be expressed in terms of the Chern classes using [MS74, Corollary 15.5]:

$$p_1(T_v \mathcal{X}_{2g}) = t_1^2 - t_2 \quad\text{and}\quad p_2(T_v \mathcal{X}_{2g}) = t_2^2.$$

We substitute these into the first nine Hirzebruch $L$-polynomials, as computed by McTague [McT14]. Since integration along fibers is linear, it suffices to compute $\int_M t_1^4 t_2^2$. By the push-pull formula, this is $\lambda^2 \int_M t_2^2$, and [vdGK05, Section 3] used Grothendieck–Riemann–Roch to determine that $\int_M t_2^2 = a_{j-1} \lambda^{2j-2}$ for particular integers $a_{j-1}$. Using this, we compute that $\kappa_{i+1}(X_{2g})$ is a non-zero multiple of $\lambda^{2i}$ for $1 \leq i \leq 8$ and hence non-zero, cf. Table 1. \hfill \square

**Example 2.** Let us do the computation for $i = 3$ as an example:

$$L_4(T_v \mathcal{X}_{2g}) = \frac{-3t_1^4 + 24t_1^2 t_2^2 - 50t_2^4 + 8t_1^4 t_2^2 + 21t_2^4}{14175}.$$

Using this, we compute that $\kappa_{i+1}(X_{2g})$ is a non-zero multiple of $\lambda^{2i}$ for $1 \leq i \leq 8$ and hence non-zero, cf. Table 1. \hfill \square
Remark 3. The classes $\kappa_{\mathcal{L}_{i+1}}$ remain non-zero when pulled back to $H^{4i}(B \text{Diff}(K \text{ rel } *) ; \mathbb{Q})$, because $H^*(B \text{Diff}(K) ; \mathbb{Q}) \to H^*(B \text{Diff}(K \text{ rel } *) ; \mathbb{Q})$ is injective: its composition with the Becker–Gottlieb transfer is given by multiplication with $\chi(K) = 24$. We do not know whether $\kappa_{\mathcal{L}_{i+1}}$ remains non-zero when pulled back to $H^{4i}(B \text{Diff}(K \text{ rel } D^4) ; \mathbb{Q})$.

3. MILLER–MORITA–MUMFORD CLASSES AND THE ACTION ON HOMOLOGY

One can also approach the group of diffeomorphisms of $K$ through its action on $H_3(K; \mathbb{Z})$. In particular, we shall explain a relationship between the generalized Miller–Morita–Mumford classes and the arithmetic part of the mapping class group.

The middle-dimensional homology group $H_3(K; \mathbb{Z}) \cong \mathbb{Z}^{22}$ has intersection form given by $M = H \oplus H \oplus H \oplus -E_8 \oplus -E_8$, with $H$ the hyperbolic form and $-E_8$ the negative of the $E_8$-form. This is equivalent over $\mathbb{Q}$ to the symmetric $(22 \times 22)$-matrix $B$

$$B = \begin{pmatrix} I_3 & 0 \\ 0 & -I_9 \end{pmatrix},$$

where $I_n$ is the $(n \times n)$ identity matrix. In particular, we can consider $\text{Aut}(M)$ as a subgroup of the Lie group $O(3, 19)$.

The action of $\text{Mod}(K)$ on $H_3(K; \mathbb{Z})$ preserves the intersection form and hence induces a homomorphism $\alpha: \text{Mod}(K) \to \text{Aut}(M)$, whose image is the index 2 subgroup of $\text{Aut}(M)$ of those elements such that the product of the determinant and the spinor norm equals 1, cf. [Gia09, §4.1].

The generalized MMM-classes associated to the Hirzebruch $L$-polynomials $L_i \in H^{4i}(B \text{SO}; \mathbb{Q})$, whose pullback to $H^{4i}(B \text{SO}(4); \mathbb{Q})$ we shall denote in the same manner, can be obtained from the arithmetic group $\Gamma_K$ as follows. There are homomorphisms

$$\Gamma_K \to \text{Aut}(M) \to O(3, 19) \to O(3) \times O(19) \to U(3) \times U(19),$$

the right map given by complexification. Thus we get, up to homotopy, a map $w: B\Gamma_K \to B\text{SO}(3) \times B\text{SO}(19)$. We define a class $x_{4i} := w^*(\text{ch}_{4i} \otimes 1 - 1 \otimes \text{ch}_{4i}) \in H^{4i}(B\Gamma_K; \mathbb{Q})$, where $\text{ch}_{4i}$ denotes the degree $4i$ component of the Chern character. Atiyah proved that $x_{4i}$ pulls back to $\kappa_{\mathcal{L}_{i+1}} \in H^{4i}(B \text{Diff}(K); \mathbb{Q})$ [Ati69, §4].

Let $\Gamma_{\text{Ein}} < \Gamma_K$ be the index 2 subgroup of those elements such that both the determinant and the spinor norm are 1; it has index 4 in $\text{Aut}(M)$ and is the unique maximal subgroup contained in the identity component of $O(3, 19)$. There are maps

$$B\Gamma_{\text{Ein}} \to B\text{SO}(3, 19) \xrightarrow{\cong} B\text{SO}(3) \times B\text{SO}(19) \xrightarrow{\pi_1} B\text{SO}(3).$$

To understand the induced map $H^*(B\text{SO}(3); \mathbb{Q}) \to H^*(B\Gamma_{\text{Ein}}; \mathbb{Q})$, we introduce the space

$$X_u = \frac{\text{SO}(22)}{\text{SO}(3) \times \text{SO}(19)}.$$

In Section 4 we shall discuss the Matsushima homomorphism

$$(1) \quad H^*(X_u; \mathbb{C}) \to H^*(B\Gamma_{\text{Ein}}; \mathbb{C}).$$

The principal $\text{SO}(3)$-bundle $\text{SO}(22)/\text{SO}(19) \to X_u$ is classified by a map $X_u \to B\text{SO}(3)$ which factors over the map $X_u \to B\text{SO}(3) \times B\text{SO}(19)$. By [Gia09, Lemma 3.4] (a special case of [Bor77, Proposition 7.2]), the Matsushima homomorphism fits in a commutative diagram

$$\begin{array}{ccc}
H^*(B\text{SO}(3) \times B\text{SO}(19); \mathbb{C}) & \to & H^*(X_u; \mathbb{C}) \\
\downarrow & & \downarrow (1) \\
H^*(B\text{SO}(3); \mathbb{C}) & \to & H^*(B\Gamma_{\text{Ein}}; \mathbb{C}).
\end{array}$$

Since the map $X_u \to B\text{SO}(3)$ is 39-connected, $x_{4i}$ for $i \leq 9$ can also be obtained by pulling back $\text{ch}_{4i}$ along the map $B\Gamma_{\text{Ein}} \to B\text{SO}(3) \to B\text{U}(3)$, or equivalently the Pontryagin character $\text{ph}_{4i} \in H^{4i}(B\text{SO}(3); \mathbb{Q})$ along the map $B\Gamma_{\text{Ein}} \to B\text{SO}(3)$. 

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4. Results of Franke and Grobner

In this section we explain a result about the Matsushima homomorphism (1), which implies:

**Proposition 4.** The homomorphism $H^*(BSO(3); \mathbb{C}) \to H^*(B\Gamma_{\text{Eis}}; \mathbb{C})$ is injective in degrees $* \leq 20$.

Let $G$ be a connected semi-simple algebraic group over $\mathbb{Q}$. The real points $G(\mathbb{R})$ form a semi-simple Lie group. Fix maximal compact subgroups $K < G(\mathbb{R})$ and $U < G(\mathbb{C})$ with $K \subset U$, let $Y_\infty := G(\mathbb{R})/K$ be the symmetric space of $G$, and $X_u := U/K$ be the compact dual symmetric space of $G$ (the construction of $X_u$ in the previous section is the case $G = SO(3, 19)$). Fixing an arithmetic lattice $\Gamma < G(\mathbb{Q})$, by work of Matsushima and Borel [Mat62, Bor74] there is a homomorphism $H^*(X_u; \mathbb{C}) \to H^*(\Gamma\backslash Y_\infty; \mathbb{C})$, and we shall call the composition

$$H^*(X_u; \mathbb{C}) \to H^*(\Gamma\backslash Y_\infty; \mathbb{C}) \simeq H^*(B\Gamma; \mathbb{C})$$

the **Matsushima homomorphism**. It may be helpful to point out that this homomorphism in general is not induced by a map of spaces, since it doesn’t preserve the rational cohomology [Bor77, Oku01]. Borel [Bor74] proved that it is an isomorphism in a range of degrees, and by work of Franke [Fra08] it is injective in a larger range.

**Theorem 5** (Franke). The homomorphism (2) is injective in degrees

$$* \leq \min_R \dim N_R,$$

where $R$ ranges over maximal parabolic subgroups of $G$ over $\mathbb{Q}$, and $N_R \subset R$ is the unipotent radical.

This is not stated explicitly in [Fra08] but a similar statement is given in [Gro13], as we now explain. For more information, see [FS98], [LS04] or [Har19, §6,8]. We require the following additional setup. Define the **adelic symmetric space** $Y^A$ and the **adelic locally symmetric space** $X^A$ by

$$Y^A := Y_\infty \times G(\mathbb{A}_f) \quad \text{and} \quad X^A := G(\mathbb{Q}) \backslash Y^A,$$

where $\mathbb{A}_f$ is the ring of finite adeles of $\mathbb{Q}$. The (sheaf) cohomology $H^*(X^A; \mathbb{C})$ can be identified with the colimit

$$H^*(G; \mathbb{C}) := \lim_{\to} H^*(X^A/K_f; \mathbb{C}),$$

where $K_f \subset G(\mathbb{A}_f)$ ranges over open compact subgroups. Each $X/K_f$ is a finite disjoint union $\bigsqcup_i \Gamma_i \backslash Y_\infty$ with $\Gamma_i < G(\mathbb{Q})$ an arithmetic lattice. Furthermore, there is a decomposition

$$H^*(G; \mathbb{C}) = H^*_{\text{cusp}}(G; \mathbb{C}) \oplus H^*_{\text{Eis}}(G; \mathbb{C})$$

coming from the theory of automorphic forms.

In this framework, there is a map [Gro13, pg. 1062]

$$H^*(\mathfrak{g}, K; \mathbb{C}) \to H^*(G; \mathbb{C}) \to H^*_{\text{Eis}}(G; \mathbb{C}),$$

where $H^*(\mathfrak{g}, K; \mathbb{C})$ is relative Lie algebra cohomology, and the second map is the projection with respect to the decomposition (5). The construction of (2) passes through the isomorphism $H^*(X_u; \mathbb{C}) \simeq H^*(\mathfrak{g}, K; \mathbb{C})$ [Oku01, §4], [Bor74, §10] and fits in a commutative diagram

$$\begin{array}{ccc}
H^*(\mathfrak{g}, K; \mathbb{C}) & \longrightarrow & H^*(G; \mathbb{C}) \\
\simeq \downarrow & & \downarrow \\
H^*(X_u; \mathbb{C}) & \longrightarrow & H^*(B\Gamma; \mathbb{C}).
\end{array}$$

According to [Gro13, §7.4], the composition (6) is injective in degrees $* \leq \min_R \dim N_R$ as in the statement of Theorem 5. Grobner attributes this result to [Fra08, (7.2) pg. 59], although Franke does not state it in this form; independently, Grobner [Gro13, Cor. 17] proves that (6) is injective in a smaller range $* \leq \frac{1}{2} \min_R \dim N_R$. 
Since $H^*(g, K; \mathbb{C})$ is degree-wise finite-dimensional, in the desired range (6) factors over a finite stage of the colimit $H^*(G; \mathbb{C})$ and thus the injectivity of (6) provides an injective map $H^*(g, K; \mathbb{C}) \to H^*(B\Gamma'; \mathbb{C})$ for some arithmetic lattice $\Gamma' \leq G(\mathbb{Q})$. Any arithmetic lattice $\Gamma \leq G(\mathbb{Q})$ is commensurable to $\Gamma'$, and hence $\Gamma$ and $\Gamma'$ have a common finite index subgroup $\Gamma''$.

Consider the commutative diagram

\[
\begin{array}{ccc}
H^*(g, K; \mathbb{C}) & \longrightarrow & H^*(B\Gamma'; \mathbb{C}) \\
\downarrow & & \downarrow \\
H^*(B\Gamma; \mathbb{C}) & \longrightarrow & H^*(B\Gamma''; \mathbb{C}).
\end{array}
\]

By a transfer argument the top composition is injective in the desired range, and hence so is $H^*(g, K; \mathbb{C}) \to H^*(B\Gamma; \mathbb{C})$, proving that (6) and hence (2) is injective in the desired range.

In the remainder of this section we compute Franke’s constant for $G = \text{SO}(p, q)$ and finish the proof of Theorem B. We also compute Franke’s constant for $G = \text{Sp}_{2g}$ and $G = \text{SL}_n$, since these are examples of common interest.

4.1. Special orthogonal groups. Fix $1 \leq p \leq q$, set $d = q - p$, and consider the algebraic group

$$
\text{SO}(B) := \{ g \in \text{SL}_{p+q} \mid g^tBg = B \},
$$

where $B$ is the $(p + q) \times (p + q)$-matrix given by

$$
B = \begin{pmatrix}
I_p & 0 \\
0 & -I_q
\end{pmatrix}.
$$

The associated compact dual symmetric space is $X_u = \text{SO}(p + q)/(\text{SO}(p) \times \text{SO}(q))$, whose cohomology $H^*(X_u; \mathbb{C})$ can be computed using [McC01, Theorem 8.2].

**Proposition 6.** Fix a finite-index subgroup $\Gamma \leq \text{SO}(B; \mathbb{Z})$. Then the Matsushima homomorphism $H^*(X_u; \mathbb{C}) \to H^*(B\Gamma; \mathbb{C})$ is injective in degrees $* \leq p + q - 2$.

**Proof.** By the preceding discussion, it suffices to prove

$$
\min R \dim N_R = p + q - 2,
$$

where $R$ ranges over a maximal parabolic subgroups over $\mathbb{Q}$, and $N_R$ is the unipotent radical. Parabolic subgroups of $\text{SO}(B; \mathbb{R})$ are stabilizers of isotropic flags in $(\mathbb{R}^{p+q}, B)$. A maximal parabolic subgroup is specified by a single non-trivial isotropic subspace. Let $e_1, \ldots, e_p, f_1, \ldots, f_q$ be the standard basis for $\mathbb{R}^{p+q}$ (whose Gram matrix is $B$). Denoting $u_i = e_i + f_i$, let $R_k < \text{SO}(B; \mathbb{R})$ be the stabilizer of $W = \mathbb{R}\{u_1, \ldots, u_k\}$ for $1 \leq k \leq p$. Every maximal parabolic subgroup is conjugate to some $R_k$.

Fix $1 \leq k \leq p$. An element of $R_k$ preserves the flag $0 < W \subset W^\perp \subset \mathbb{R}^{p+q}$. The unipotent radical $N_k \subset R_k$ is the subgroup that acts trivially on each of the quotients $W/0, W^\perp/W, \mathbb{R}^{p+q}/W^\perp$. To determine $\dim N_k$, denote $v_i = e_i - f_i$ for $1 \leq i \leq p$, and work in the ordered basis

$$
u_1, \ldots, u_k, u_{k+1}, \ldots, u_p, f_{p+1}, \ldots, f_q, v_{k+1}, \ldots, v_p, v_1, \ldots, v_k.
$$

Then $g \in N_k$ can be expressed as a block matrix

$$
g = \begin{pmatrix}
I_k & y & z \\
0 & I_{p+q-2k} & x \\
0 & 0 & I_k
\end{pmatrix},
$$

where $y = -x^tQ$ and $z + z^t = x^tQx$ and $Q$ is the $(p + q - 2k) \times (p + q - 2k)$ matrix

$$
Q = \begin{pmatrix}
0 & 0 & I_{p-k} \\
0 & I_{q-p} & 0 \\
I_{p-k} & 0 & 0
\end{pmatrix}.
$$
The homomorphism \( N_k \ni g \mapsto x \in \mathbb{R}^{k(p+q-2k)} \) has kernel the space of skew-symmetric matrices, so \( \dim N_k = k(p+q-2k) + \frac{k(k-1)}{2} \). For \( 1 \leq k \leq p \), this number is smallest when \( k = 1 \), which gives the constant claimed in the theorem. \( \square \)

**Proof of Proposition 4.** Since \( M \) is equivalent to \( B \) with \( p = 3 \) and \( q = 19 \) over \( \mathbb{Q} \), \( \Gamma_{\text{Ein}} \) is commensurable to \( \text{SO}(3,19;\mathbb{Z}) \). In particular they have a common finite index subgroup \( \Gamma \). By Proposition 6, the map \( H^*(X_u;\mathbb{C}) \rightarrow H^*(B\Gamma_{\text{Ein}};\mathbb{C}) \rightarrow H^*(B\Gamma;\mathbb{C}) \) is injective for \( * \leq 20 \) and hence so is (1). \( \square \)

### 4.2. Symplectic groups.

We next specialize Theorem 5 to finite index subgroups of symplectic groups. Take \( G = \text{Sp}_{2n} \) to be the algebraic group defined by

\[
\text{Sp}_{2n} := \{ g \in \text{SL}_{2n} \mid g^t J_n g = J_n \},
\]

where \( J_n \) is the \( 2n \times 2n \) matrix given by

\[
J_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.
\]

The associated compact dual symmetric space is \( X_u = \text{Sp}(n)/U(n) \), whose cohomology in the range below is the free polynomial algebra on generators \( c_1, c_3, c_5, \ldots \) with \( |c_i| = 2i \).

**Proposition 7.** For any finite-index subgroup \( \Gamma \leq \text{Sp}_{2n}(\mathbb{Z}) \) the Matsushima homomorphism \( H^*(X_u;\mathbb{C}) \rightarrow H^*(B\Gamma;\mathbb{C}) \) is injective in degrees \( * \leq 2n-1 \).

**Remark 8.** This should also follow from van der Geer’s computation of the tautological ring in the Chow ring of the moduli space of principally polarized abelian varieties \( [vdG99] \). It is suggested in the literature that his argument can be adapted to cohomology, but we do not know a reference for this.

**Proof of Proposition 7.** The proof follows from Theorem 5 similar to Proposition 6. Let \( e_1, \ldots, e_n, f_1, \ldots, f_n \) be the standard symplectic basis for \( \mathbb{R}^{2n} \). Let \( R_k \) be the maximal parabolic subgroup of \( \text{Sp}_{2n} \) as defined in Theorem 5 for \( 1 \leq k \leq n \). Working in the basis \( e_1, \ldots, e_k, e_k+1, \ldots, e_n, f_k+1, \ldots, f_n, f_1, \ldots, f_k \), an element of the unipotent radical \( N_k \) can be expressed as a block matrix

\[
g = \begin{pmatrix} I_k & y & z \\ 0 & I_{2n-2k} & x \\ 0 & 0 & I_k \end{pmatrix},
\]

where \( y = x^t J' \) and \( z = z^t J' y \) and \( J' = J_{n-k} \). It follows that \( \dim N_k = 2k(n-k) + k + \frac{k(k-1)}{2} \). For \( 1 \leq k \leq n \), this number is smallest when \( k = 1 \). \( \square \)

### 4.3. Special linear groups.

Finally, we specialize Theorem 5 to finite-index subgroups of special linear groups. Now we have \( G = \text{SL}_n \) and \( X_u = \text{SU}(n)/\text{SO}(n) \), whose cohomology in the range below is the free polynomial algebra on generators \( c_3, c_5, c_7, \ldots \) with \( |c_i| = 2i - 1 \).

**Proposition 9.** For any finite-index subgroup \( \Gamma \leq \text{SL}_n(\mathbb{Z}) \) the Matsushima homomorphism \( H^*(X_u;\mathbb{C}) \rightarrow H^*(B\Gamma;\mathbb{C}) \) is injective in degrees \( * \leq n-1 \).

The proof is similar to the proof of Propositions 6 and 7, but simpler; one identifies the maximal parabolic subgroups over \( \mathbb{Q} \) as the stabilizers of a non-trivial subspace \( W \) and observes that the stabilizers of 1-dimensional subspaces have the smallest unipotent radical, of dimension \( n-1 \).

**Remark 10.** Proposition 9 resolves a question in [EVGS13, Remark 7.5]; the non-zero classes in \( H^5(\text{SL}_n(\mathbb{Z});\mathbb{Q}) \) for \( n = 6,7 \) found by computer calculation are multiples of the Borel classes.

**Remark 11.** [Lee78] announced that the range in Proposition 9 can be improved to \( * \leq 2n-3 \). However, as far as we know this claim remains unproven.
5. Moduli of Einstein metrics

To apply our knowledge of the cohomology of arithmetic groups, we use the global Torelli theorem to study the moduli space $\mathcal{M}_{\text{Ein}}$ of Einstein metrics on the $K3$ manifold. Following [Gia09, §4], for us this shall mean the homotopy quotient

$$\mathcal{M}_{\text{Ein}} := \mathcal{T}_{\text{Ein}} \sslash \Gamma_{\text{Ein}}$$

of a moduli space $\mathcal{T}_{\text{Ein}}$ of marked Einstein metrics by the subgroup $\Gamma_{\text{Ein}} \leq \Gamma_K$. The space $\mathcal{T}_{\text{Ein}}$ admits a description as a hyperplane complement, but we only use a pair of consequences of this.

Assume $\Gamma' \leq \Gamma_K$, and assume $\Gamma'$ is contained in the identity component of $\text{O}(3,19)$. Take $\text{Mod}' := \alpha^{-1}(\Gamma')$.

**Proposition 12.** The homomorphism $H^*(B\Gamma'; \mathbb{C}) \rightarrow H^*(B\text{Mod}; \mathbb{C})$ is injective for any $\Gamma' \leq \Gamma_{\text{Ein}}$.

**Proof.** We will use that the surjection $\text{Mod}(\text{Ein}) \rightarrow \Gamma_K$ splits over $\Gamma_{\text{Ein}}$ by Giansiracusa’s work: there is a map

$$\epsilon: \mathcal{M}_{\text{Ein}} \rightarrow B\text{Diff}(K) \rightarrow B\text{Mod}(K) \rightarrow B\Gamma_K.$$  

The induced homomorphism $\pi_1(\mathcal{M}_{\text{Ein}}) \rightarrow \Gamma_K$ is injective with finite cokernel by the global Torelli theorem [Gia09, §4-5]. Thus, $\text{Mod}(\text{Ein}) \rightarrow \Gamma_K$ splits over the image of $\epsilon_\ast: \pi_1(\mathcal{M}_{\text{Ein}}) \rightarrow \Gamma_K$, which is $\Gamma_{\text{Ein}}$. This proves the case $\Gamma' = \Gamma_{\text{Ein}}$; for general $\Gamma'$ one restricts the splitting to $\Gamma'$.

To prove Theorem A we must prove that $p_\ast x_1 \neq 0 \in H^4(B\text{Diff}(K); \mathbb{Q})$. To do so, it suffices to prove that is non-zero when pulled back to $\mathcal{M}_{\text{Ein}}$:

**Proposition 13.** For the map $\epsilon$ defined in (7), $\epsilon_\ast x_1 \neq 0 \in H^4(\mathcal{M}_{\text{Ein}}; \mathbb{Q})$.

**Proof.** We will prove that $\epsilon_\ast: H^4(B\Gamma_K; \mathbb{Q}) \rightarrow H^4(\mathcal{M}_{\text{Ein}}, \mathbb{Q})$ is injective. In [Gia09, §5], one finds a description of the Serre spectral sequence for the fibration sequence

$$\mathcal{T}_{\text{Ein}} \rightarrow \mathcal{M}_{\text{Ein}} = \mathcal{T}_{\text{Ein}} \sslash \Gamma_{\text{Ein}} \rightarrow B\Gamma_K.$$  

Its $E^2$-page is given by

$$E^2_{p,q} = \begin{cases} 0 & \text{if } q \text{ is odd}, \\ \prod_{\sigma \in \Delta_{q/2}/\Gamma_{\text{Ein}}} H^p(B\text{Stab}(\sigma); \mathbb{Q}) & \text{if } q \text{ is even}. \end{cases}$$

The description of $\Delta_{q/2}/\Gamma_{\text{Ein}}$ is not important here, as we shall only use the rows $0 \leq q \leq 3$. Of these, the following are non-zero: for $q = 0$ we get $H^0(B\Gamma_K; \mathbb{Q})$, and for $q = 2$ we get a product of the cohomology groups of groups $\Gamma$ commensurable with $\text{O}(2,19; \mathbb{Z})$ or $\text{O}(3,18; \mathbb{Z})$. For such groups $H^4(\Gamma; \mathbb{Q})$ vanishes [Mar91, Corollary 7.6.17], and thus there can not be any non-zero differential into the entry $E^2_{4,0}$.

6. Nielsen realization

We now deduce Theorem B from either Proposition 1 or 4.

**Proof of Theorem B.** Let $\text{Mod}_{\text{Ein}} \leq \text{Mod}(K)$ be the inverse image of $\Gamma_{\text{Ein}}$ under $\text{Mod}(K) \rightarrow \text{Aut}(M)$. We will show that $\text{Diff}_{\text{Ein}} \rightarrow \text{Mod}_{\text{Ein}}$ does not split, where $\text{Diff}_{\text{Ein}} := p^{-1}(\text{Mod}_{\text{Ein}})$. This is proven by contradiction, so we assume there is a splitting $s: \text{Mod}_{\text{Ein}} \rightarrow \text{Diff}_{\text{Ein}}$, which necessarily factors over the discrete group $\text{Diff}_{\text{Ein}}^\delta$ as

$$\text{Mod}_{\text{Ein}} \xrightarrow{s^\delta} \text{Diff}_{\text{Ein}}^\delta \xrightarrow{p^\delta} \text{Diff}_{\text{Ein}}.$$  

Observe that $x_8 \in H^8(B\Gamma_{\text{Ein}}; \mathbb{Q})$ is non-zero as a consequence of either Proposition 1 or 4. By Proposition 12 its pullback to $H^8(B\text{Mod}_{\text{Ein}}; \mathbb{Q})$, which we denote by $c$, is also non-zero. Its pullback under

$$B\text{Mod}_{\text{Ein}} \xrightarrow{s^\delta} B\text{Diff}_{\text{Ein}}^\delta(K) \xrightarrow{p^\delta} B\text{Diff}_{\text{Ein}}^\delta(K) \xrightarrow{p} B\text{Mod}_{\text{Ein}}$$


is $c$ and hence non-zero. By Section 3 we get $p^*c = \kappa L$ and $(p^*)_\delta \kappa L \in H^*(\text{Diff}^\text{Ein}_E(K))$ vanishes by the Bott vanishing theorem [Bot70]. This contradicts $c \neq 0$.

\[
\square
\]

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