Robust Stability Control for a Class of Uncertain Quantum Systems Through Direct and Indirect Couplings

CHENGLI XIANG
School of Automation, Hangzhou Dianzi University, Hangzhou 310018, China
e-mail: xcd@hdu.edu.cn

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ABSTRACT In this paper, a class of uncertain quantum systems with uncertainties in the interaction Hamiltonian is connected with a coherent controller through direct and indirect couplings. The aim of the paper is to design a robust coherent stability controller to make the closed-loop system robustly mean square stable. A sufficient and necessary condition is presented to build a connection between robust control design with uncertain parameters and scaled control design without uncertain parameters. This condition is also considered for a special class of linear passive quantum systems. A numerical procedure is proposed to obtain explicit expression for parameters of the desired coherent controller using linear matrix inequality, multi-step optimization methods and physical realizability conditions.

INDEX TERMS Quantum feedback control, coherent control, robustly mean square stable, quantum control, robust control.

I. INTRODUCTION Recent development in quantum technology has drawn a wide attention from scientists and engineers all over the world focusing on quantum control research [1]–[9]. The control of quantum system is a rapid growing field and has made a great achievements in the areas, such as quantum metrology, quantum computing and quantum communication [10]–[14].

Similar as in classical control area [15]–[21], robustness is also a key issue in the quantum control theory [22]–[44]. Quantum systems need to maintain a desired performance in the presence of uncertainties and disturbances, which unavoidably exist in the modeling and control process for real quantum systems of practical applications [22]–[29]. Hence, a considerable number of robust control methods have been developed to deal with all kinds of uncertainties and disturbances in quantum systems, such as noise, unknown modeling errors, decoherence, etc. [30]–[44]. Among them, risk-sensitive control was proposed in terms of a new unnormalized conditional state to provide some robustness properties in the model and external disturbances [30]. A sampled-data approach was used in robust control of a single qubit with uncertainties and had the potential in applications such as quantum error-correction and in constructing robust quantum gates [31]. A two-step method was applied by combining feedback strategy and open-loop control technique of dynamical decoupling to solve a control problem of quantum state stabilization [32]. To deal with uncertainties in the system Hamiltonian for two-level quantum systems, a sliding mode control approach was proposed in [33]. The studies in [34] further extended these sliding mode control results to include uncertainties described as perturbations in the free Hamiltonian. For a class of linear quantum system which is subject to quadratic perturbations in the system Hamiltonian, a robust coherent guaranteed cost method was used to achieve improved control performance [35]. The work in [36] applied a sequential convex programming method for designing robust quantum gates.

Similar as in classical control systems, robust stability is also counted as one of the most important issues in quantum robust control study. The small gain theorem and Popov criterion, which are two of the most useful tests for robust stability in classical control systems [45], have been extended to the quantum domain. The small gain theorem has been used in robust stability analysis in a class of quantum systems with norm bounded uncertainties in the Hamiltonian [37]. This method has also been applied to robust stability studies of quantum systems with different perturbations and applications (e.g., see [38]). The work in [39] proposed the quantum version of Popov stability criterion for uncertain quantum systems in terms of a frequency domain condition, which is less conservative than the small gain theorem [37]. In order to
guarantee stability robustness against parameter uncertainties in quantum systems, a quantum version of strict bounded real condition was presented to build a connection between robust stability and $H^\infty$ property [41]. Robust stability analysis of a class of uncertain quantum systems in terms of annihilation operators was discussed in [42].

Most of the existing studies have only considered the robust stability analysis and have not put robust stability controller design into consideration [37], [39]. In this paper, we consider the uncertain quantum system connected with a robust coherent controller through direct and indirect couplings. The mean square stability condition without uncertainties is given through direct and indirect plant through direct and indirect couplings. The mean square stability condition without uncertainties is given through direct and indirect couplings. The mean square stability condition without uncertainties is given through direct and indirect couplings.

A sufficient and necessary condition is presented to build a connection between robust control design with uncertain parameters and scaled control design without uncertain parameters.

A numerical procedure is proposed to obtain the corresponding solution of the robust coherent controller.

The remainder of this paper is formulated as follows. In Section II, a class of uncertain quantum systems with uncertainties in the interaction Hamiltonian is introduced. The desired coherent controller is connected with the quantum plant through direct and indirect couplings. The mean square stability condition without uncertainties is given through a necessary and sufficient condition. In Section III, the scaled mean square stability condition for a class of quantum systems in terms of annihilation operators is presented. In Section IV, a numerical procedure is proposed to obtain the solution of the coherent controller using LMI and multi-step optimization methods. A conclusion is drawn in Section V.

A. NOTATION

In this paper, $I_m$ denotes an $m \times m$ identity matrix and $0_{m \times n}$ denotes an $m \times n$ zero matrix, where the subscript is omitted when $m$ and $n$ can be determined from the context. Define $J_m = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ and $n_m = n_m/2$.

II. COHERENT FEEDBACK CONTROLLER DESIGN

In this section, we introduce linear quantum stochastic systems [41] as the system model, which are commonly arose in quantum optical systems [46]. The quantum plant under consideration is a class of uncertain quantum systems with uncertainties in the interaction Hamiltonian. The aim of the coherent controller design is to make the closed-loop system robustly stable.

A. QUANTUM SYSTEM MODEL AND CONTROLLER

The quantum plant under consideration is a class of uncertain quantum systems with uncertainties in the interaction Hamiltonian. The system is connected with the controller via interaction Hamiltonian and field couplings, generating direct and indirect couplings.

The quantum plant we consider is described by quantum stochastic differential equations (QSDEs) in the following form [29], [41]:

$$
\dot{x}(t) = A\dot{k}(t) + (B_{PK} + \Delta B_{PK})\dot{x}(t)dt + B_{eq}\dot{w}(t) + B_{d}\dot{u}(t),
$$

$$
\dot{d}(t) = C_{p}\dot{x}(t) + D_{du}\dot{u}(t),
$$

$$
\dot{d}(t) = C\dot{x}(t) + D_{d}\dot{u}(t) + D_{w}\dot{w}(t),
$$

where $A \in \mathbb{R}^{n \times n}$, $B_{PK} \in \mathbb{R}^{n \times n_k}$, $B_{w} \in \mathbb{R}^{n \times n_w}$, $B_{d} \in \mathbb{R}^{n \times n_d}$, $B_{v} \in \mathbb{R}^{n \times n_v}$, $B_{v} \in \mathbb{R}^{n \times n_v}$, $B_{v} \in \mathbb{R}^{n \times n_v}$, $C \in \mathbb{R}_{n \times n}$, $D_{v} \in \mathbb{R}^{n \times n_v}$, $D_{w} \in \mathbb{R}^{n \times n_w}$, and $n, n_k, n_v, n_w, n_n, n_d, n_v$ are positive integers and even numbers. $\dot{x}(t)$ is a vector of self-adjoint possibly non-commutative system variables, where $q(t)$ is the position operators and $p(t)$ is momentum operators on an appropriate Hilbert space. The vector $\dot{x}(t)$ is required to satisfy the canonical commutation relation in the following form [41]:

$$
[\dot{x}(t), \dot{x}(t)^T] = \dot{x}(t)(t)^T - (\dot{x}(t)(t)^T)^T = 2iJ_n.
$$

Also, $\dot{u}(t)$ is a control input, $\dot{w}(t)$ represents a quantum disturbance signal and $\dot{v}(t)$ stands for any additional quantum noise. In addition, the quantity $\dot{y}(t)$ stands for the measured output and the quantity $\dot{z}(t)$ stands for the performance output. The term $(B_{PK} + \Delta B_{PK})\dot{x}(t)$ is introduced due to the direct coupling and uncertainties of direct coupling between the quantum plant and the coherent controller.

The coherent controller to be designed is a non-stochastic system of the form:

$$
\dot{\dot{x}}_K(t) = A_K\dot{x}(t) + (B_{PK} + \Delta B_{PK})\hat{x}(t)dt + B_{du}\hat{v}(t) + B_{d}\hat{u}(t),
$$

$$
\hat{d}(t) = C_{K}\dot{x}(t) + D_{K}\hat{u}(t),
$$

where $A_K \in \mathbb{R}^{n \times n_k}$, $B_{PK} \in \mathbb{R}^{n \times n_k}$, $B_{w} \in \mathbb{R}^{n \times n_w}$, $B_{v} \in \mathbb{R}^{n \times n_v}$, $C_K \in \mathbb{R}^{n \times n_d}$, $D_{v} \in \mathbb{R}^{n \times n_v}$, $D_{w} \in \mathbb{R}^{n \times n_w}$, and $K$ is a vector of self-adjoint controller variables, $\hat{v}(t)$ is a vector of additional quantum noise. Similarly, the term $(B_{PK} + \Delta B_{PK})\hat{x}(t)$ is introduced because of the direct coupling and uncertain part of direct coupling between the system and the controller.

The direct coupling between the quantum system and the coherent controller is through interaction Hamiltonian. Also, there exist uncertainties in the interaction Hamiltonian. Hence, the interaction Hamiltonian can be described in the following form:

$$
H_{int} = \frac{1}{2}(\hat{x}(t)(t)^T + \hat{x}(t)(t)^T + K(\dot{R} + \Delta R)),
$$

where $R = -J_{n_k}K$ and $K \in \mathbb{R}^{n \times n}$ is the direct coupling parameter. $\Delta R = -J_{n_k}K$ describes uncertain part of the
interaction Hamiltonian, where $\Delta K = E_1 \Delta E_2$ and an uncertain real matrix $\Delta \in \mathbb{R}^{p \times q}$ satisfies the following norm bounded condition:

$$\Delta^T \Delta \leq I.$$  (5)

According to the interaction Hamiltonian (4), $B_{PK}$, $B_{KP}$, $\Delta B_{PK}$, $\Delta B_{KP}$ can be described in the following form [29]:

$$B_{PK} = 2J_n K^T J_{nk}, \quad B_{KP} = 2K,$$

$$\Delta B_{PK} = 2J_n E_2^T \Delta^T E_1^T J_{nk}, \quad \Delta B_{KP} = 2E_1 \Delta E_2.$$  (6)

From (6), the relationship between $B_{PK}$ and $B_{KP}$ can be described as:

$$B_{KP} = J_{nk} B_{pk}^T J_n.$$  (7)

Suppose that $\bar{x}(t)$ commutes with $\bar{x}_k(t)$ at time $t = 0$. The closed-loop feedback quantum system consisting of plant (1) and controller (3) through direct and indirect couplings can be described in the following form:

$$d\xi(t) = (\bar{A} + \Delta \bar{A}) \xi(t) dt + \bar{B}_w d\bar{w}(t) + \bar{B}_v d\bar{v}(t),$$

$$d\bar{z}(t) = \bar{C}_v \xi(t) dt + \bar{D} d\bar{v}(t),$$

where

$$\xi(t) = \begin{bmatrix} \bar{x}(t) \\ \bar{x}_k(t) \end{bmatrix}, \quad \tau(t) = \begin{bmatrix} \bar{v}(t) \\ \bar{v}_k(t) \end{bmatrix},$$

$$\bar{A} = \begin{bmatrix} A & B_n C_k + B_{PK} \\ B_k C + B_{KP} & A_k \end{bmatrix},$$

$$\bar{B}_w = \begin{bmatrix} B_w \\ B_k D_w \end{bmatrix}, \quad \bar{B}_v = \begin{bmatrix} B_v \\ B_k D_v + B_{KV} \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} C_p \\ D_n C_k \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} 0 \\ D_n D_K \end{bmatrix},$$

$$\Delta \bar{A} = \begin{bmatrix} \Delta B_{PK} \\ \Delta B_{KP} \end{bmatrix},$$

$$\bar{H}_1 = \begin{bmatrix} 0 \\ 2E_1 \end{bmatrix}, \quad \bar{H}_2 = \begin{bmatrix} E_2 \\ 0 \end{bmatrix},$$

$$\bar{F} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta^T \end{bmatrix}.$$  (9)

B. THE DESIRED CONTROLLER PROPERTY

In this section, we introduce a robustly mean square stable definition for the closed-loop system (8). A scaled mean square stable condition without uncertain parameters is proposed to be used in the robust stability controller design, which is explained in Theorem 1.

Definition 1: The quantum system (8) is said to be robustly mean square stable if there exists a real positive definite matrix $P > 0$ and a constant $\lambda_1 > 0$ such that

$$\langle \xi(t)^T P \xi(t) \rangle + \int_0^t \langle \xi(s)^T \xi(s) \rangle ds \leq \langle \xi(0)^T P \xi(0) \rangle + \lambda_1 t,$$

$$\forall t > 0$$  (10)

for all Gaussian states $\rho$. Here, the notation $\langle \cdot \rangle$ stands for expectation over all initial variables and noises.

Lemma 1: (see [41]) The quantum system (8) is robustly mean square stable if and only if the matrix $\bar{A} + \Delta \bar{A}$ is Hurwitz.

In the following theorem, A sufficient and necessary condition is presented to build a connection between robust control design with uncertain parameters and scaled control design without uncertain parameters.

Theorem 1: The quantum system (1) is robustly mean square stable via a given linear dynamic output feedback controller (3) if and only if there exists a constant $\epsilon > 0$ and a positive definite real symmetric matrix $P$ such that

$$\bar{A}^T P + P \bar{A} + \epsilon P \bar{H}_1 \bar{H}_1^T P + \frac{1}{\epsilon} \bar{H}_2^T \bar{H}_2 < 0,$$  (11)

or equivalently,

$$\begin{bmatrix} \bar{A}^T P + P \bar{A} + \frac{1}{\epsilon} \bar{H}_2^T \bar{H}_2 & \frac{1}{\epsilon} \bar{H}_1^T P \\ \frac{1}{\epsilon} \bar{H}_1 P & -I \end{bmatrix} < 0.$$  (12)

In order to prove the theorem, a number of lemmas are required.

Fact 1: (see [43]) For any two matrices $X \in \mathbb{R}^{n \times m}$ and $Y \in \mathbb{R}^{n \times m}$, we have that

$$X^T Y + Y^T X \leq \epsilon X^T X + \frac{1}{\epsilon} Y^T Y.$$  (13)

where $\epsilon > 0$ is a free parameter.

Lemma 2: (see [47]) For $S \in \mathbb{R}^{n \times n}$, we have that $S^T S \leq k_n$ holds if and only if $S S^T \leq k_m$ holds.

Lemma 3: (see [48]) For any $\eta \in \mathbb{R}^n$, the following relation holds:

$$\max \{ (\eta^T P D D^T \eta)^2 : F^T F \leq I \} = \eta^T P D D^T \eta^T E \eta.$$  (14)

Lemma 4: (see [49]) Let real symmetric matrices $X \in \mathbb{R}^{n \times n}, Y \in \mathbb{R}^{n \times n}$ and $Z \in \mathbb{R}^{n \times n}$ be given with the properties $X \succeq 0, Y \prec 0$ and $Z \succeq 0$. Moreover, suppose that $(\eta^T Y \eta^T)^2 - 4(\eta^T X \eta^T Z \eta) > 0$ for all non-zero $\eta \in \mathbb{R}^n$. Then, we have that there exists a constant $\epsilon > 0$ such that the inequality $\epsilon^2 X + \epsilon Y + Z < 0$ holds.

Proof of Theorem 1: It follows from Lemma 1 that the quantum system (1) is robustly mean square stable via a given linear dynamic output feedback controller (3) if and only if $\bar{A} + \Delta \bar{A}$ is Hurwitz, that is,

$$(\bar{A} + \Delta \bar{A})^T P + P(\bar{A} + \Delta \bar{A}) < 0.$$  (15)

First, we prove the sufficient part of the theorem. Assume that (11) is satisfied. According to (5) and Lemma 2, we know that $\Delta \bar{A}^T \leq I$ is satisfied. It is known from (9) that

$$\Delta \bar{A} = \bar{H}_1 \bar{F} \bar{H}_2, \quad \bar{F} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta^T \end{bmatrix}.$$  (16)

It is straightforward to show that

$$\bar{F}^T \bar{F} \leq I.$$  (17)

Using the Fact 1 and (17), it follows that for any $\epsilon > 0$, the following inequalities are satisfied:

$$\Delta \bar{A}^T P + P \Delta \bar{A} \leq \epsilon P \bar{H}_1 \bar{H}_1^T P + \frac{1}{\epsilon} \bar{H}_2^T \bar{H}_2,$$

$$\leq \epsilon P \bar{H}_1 \bar{H}_1^T P + \frac{1}{\epsilon} \bar{H}_2^T \bar{H}_2.$$  (18)
Therefore, it follows from (11) and (18) that (15) is satisfied. That is, the closed-loop system (8) is robustly mean square stable.

The necessary part of the theorem is shown as below. Assume (15) is satisfied. Then the following inequality is achieved:

$$\bar{X}^T P + P \bar{X} < -\Delta \bar{X}^T P - P \Delta \bar{A}. \quad (19)$$

Thus, for given any $\eta \in \mathbb{R}^n$, $\eta \neq 0$ and $\bar{F}$ satisfying (17), the following inequality is obtained:

$$\eta^T (\bar{X}^T P + P \bar{A}) \eta < -2\eta^T P \bar{F} \bar{F}^T \eta. \quad (20)$$

From (20), we have

$$\eta^T (\bar{X}^T P + P \bar{A}) \eta < -2\max\{\eta^T P \bar{F} \bar{F}^T \eta\} \leq 0. \quad (21)$$

Hence, it follows from (21) that

$$(\eta^T (\bar{X}^T P + P \bar{A}) \eta)^2 > 4\max\{\eta^T P \bar{F} \bar{F}^T \eta\}^2 : \bar{F}^T \bar{F} \leq I. \quad (22)$$

for any given $\eta \in \mathbb{R}^n$, $\eta \neq 0$. According to Lemma 3, for any $\eta \in \mathbb{R}^n$, $\eta \neq 0$, we have that

$$\eta^T (\bar{X}^T P + P \bar{A}) \eta > 4\max\{\eta^T P \bar{F} \bar{F}^T \eta\}^2 : \bar{F}^T \bar{F} \leq I. \quad (23)$$

Hence, using Lemma 4, it follows that there exists a constant $\epsilon > 0$ such that

$$\epsilon^2 \eta^T P \bar{F} \bar{F}^T \eta + \eta^T (\bar{X}^T P + P \bar{A}) \eta > 0. \quad (24)$$

It follows from (24) that (11) is obtained and the proof is complete. \(\square\)

### III. ROBUST STABILITY CONTROL FOR A CLASS OF ANNIHILATION OPERATOR SYSTEMS

In Section II, the uncertain quantum system and the coherent controller are modeled in terms of quadrature form which is used to describe a general class of linear quantum systems. This class of quantum systems often include active systems that can be represented by both annihilation and creation operators. In this section, a special class of linear quantum system is considered and can be modeled purely in terms of annihilation operators and not the creation operators. This kind of quantum systems often refers to quantum passive systems from quantum optics field and is relatively easy to implement from an experimental point of view [8], [44]. The dimension of the quantum passive system is half of the general linear quantum system in Section II, due to lack of quantum creation operator. Hence, the computation complexity is greatly reduced, especially for systems with high dimensions.

In this section, the quantum plant and the coherent controller are linear passive quantum systems and interact with each other through direct and indirect couplings. The uncertainties exist in the interaction Hamiltonian. The correspondence between the matrices in general linear quantum systems and matrices in passive systems with annihilation operators only is described in [8], [42]. The dynamic evolution of the system is described by the QSDEs in terms of annihilation operators in the following form:

$$\begin{align*}
\dot{a}(t) &= Aa(t)dt + (Bp_K + \Delta Bp_K)a(t)dt + B_n dw(t) + B_d du(t) + B_v dv(t), \\
\dot{z}(t) &= C\alpha(t)dt + D_d du(t), \\
\dot{y}(t) &= Ca(t)dt + D_d du(t) + D_v dv(t).
\end{align*} \quad (25)$$

where $A \in \mathbb{C}^{n \times n}$, $Bp_K \in \mathbb{C}^{\frac{n}{2} \times n_K}$, $B_w \in \mathbb{C}^{\frac{n}{2} \times n_w}$, $B_n \in \mathbb{C}^{\frac{n}{2} \times n_n}$, $B_v \in \mathbb{C}^{\frac{n}{2} \times n_v}$, $C_p \in \mathbb{C}^{\frac{n}{2} \times n_p}$, $C_d \in \mathbb{C}^{\frac{n}{2} \times n_d}$, $D_d \in \mathbb{C}^{\frac{n}{2} \times n_d}$, and $n_i$, $n_i^*$, $n_i$, $n_i$, $n_i$, $n_i^*$ are half of the integers $n$, $n_K$, $n_w$, $n_n$, $n_v$, $n_v$, $n_v$. Here, $a(t)$ is a vector of annihilation operators on an underlying Hilbert space. The relationship between the operator $\eta \in \mathbb{R}$ and $a_i$ in (25) is that

$$\eta = \begin{bmatrix} a + a^* \\ -i(a - a^*) \end{bmatrix}.$$ Here, the notation * represents the adjoint in the case of operators. The corresponding coherent controller is described in terms of annihilation operators in the following form:

$$\begin{align*}
\eta(t) &= (A + \Delta \bar{A})\eta(t)dt + (Bp_K + \Delta Bp_K)a(t)dt \\
\eta_a(t) &= C\eta(t)dt + D_d du(t) + D_v dv(t), \\
\eta_y(t) &= C\eta(t)dt + D_d du(t).
\end{align*} \quad (26)$$

The closed-loop system between the system (25) and the controller (26) connected through direct and indirect couplings is shown as follows:

$$\begin{align*}
\dot{\xi}(t) &= (A + \Delta \bar{A})\xi(t)dt + \overline{B}_w dw(t) + \overline{B}_v dv(t), \\
\eta(t) &= \overline{C}\xi(t)dt + \overline{D}_d du(t) + \overline{D}_v dv(t). \quad (27)
\end{align*}$$

where

$$\begin{align*}
\xi(t) &= \begin{bmatrix} a(t) \\ \alpha(t) \end{bmatrix}, \\
\overline{A} &= \begin{bmatrix} A & Bp_K + \Delta Bp_K \\ B_K C + B_{Kp} & \bar{A}_K \end{bmatrix}, \\
\overline{B}_w &= \begin{bmatrix} B_{wK} \\ B_{wK} \end{bmatrix}, \\
\overline{B}_v &= \begin{bmatrix} B_v \\ B_{vK} \end{bmatrix}, \\
\overline{C} &= \begin{bmatrix} C_p \\ D_{dK} \end{bmatrix}, \\
\overline{D} &= \begin{bmatrix} D_{vK} \\ D_{vK} \end{bmatrix}, \\
\Delta \bar{A} &= \begin{bmatrix} 0 & \Delta Bp_K \\ \Delta Bp_K & 0 \end{bmatrix} = \mathcal{H}_1 \mathcal{F} \mathcal{H}_2. \quad (28)
\end{align*}$$

Now we introduce a robustly mean square stable definition for the closed-loop system (27).

**Definition 2:** (see [42]) The quantum system (27) is said to be robustly mean square stable if there exists a positive definite Hermitian matrix $P > 0$ and a constant $\lambda_2 > 0$ such that

$$\begin{align*}
\mathbb{E}<\xi(t)\xi(t)^* + \xi(t)^* P^\ast \xi(t)^*> &> \lambda_2 t \\
&\geq \mathbb{E}<\xi(0)\xi(0)^* + \xi(0)^* P^\ast \xi(0)^*> + \lambda_2 t,
\end{align*} \quad (29)$$

for all Gaussian states $\rho$.

**Lemma 5:** (see [42]) The complex quantum system (27) is robustly mean square stable if and only if the matrix $\bar{A} + \Delta \bar{A}$ is Hurwitz.
Theorem 2: The quantum system (25) is robustly mean square stable via a given linear dynamic output feedback controller (26) if and only if there exists a constant $\epsilon > 0$ and a complex positive-definite Hermitian matrix $P$ such that
\[
\mathcal{A}^T P + P \mathcal{A} + \epsilon \mathcal{H}_1 \mathcal{H}_1^T P + \frac{1}{\epsilon} \mathcal{H}_2 \mathcal{H}_2^T P < 0,
\] (30)
or equivalently,
\[
\left[ \begin{array}{ccc} \text{Re}(O_{11}) - i \text{Im}(O_{11}) & \text{Re}(O_{12}) - i \text{Im}(O_{12}) \\ \text{Re}(O_{21}) - i \text{Im}(O_{21}) & \text{Re}(O_{22}) - i \text{Im}(O_{22}) \end{array} \right] < 0.
\] (31)

In order to prove the theorem, the following lemma is needed.

Lemma 6: Assume a square complex matrix $O$ is in the following form:
\[
O = \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix},
\] (32)
where the matrices $O_{11}$ and $O_{22}$ are also square complex matrices. Let $T$ be a corresponding real matrix defined by
\[
T = \begin{bmatrix} \text{Re}(O_{11}) - i \text{Im}(O_{11}) & \text{Re}(O_{12}) - i \text{Im}(O_{12}) \\ \text{Re}(O_{21}) - i \text{Im}(O_{21}) & \text{Re}(O_{22}) - i \text{Im}(O_{22}) \end{bmatrix}.
\] (33)

Then the eigenvalues of the matrix $T$ are equal to the eigenvalues of the matrix $O$ as well as the eigenvalues of the matrix $O^*$.

Proof of Lemma 6: By algebraic computation, we have that
\[
TT_1 = T_1 \begin{bmatrix} O^* & 0 \\ 0 & O \end{bmatrix},
\] (34)
where
\[
T_1 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.
\]

Since $T_1$ is nonsingular matrix, the result is achieved. \(\square\)

Proof of Theorem 2: The relationship between the closed-loop system (8) and the closed-loop system (27) is described as follows [8], [42]:
\[
\xi = \begin{bmatrix} \xi + \xi^* \\ -i(\mathcal{A} - \mathcal{A}^*) \end{bmatrix},
\]
\[
\mathcal{A} = \frac{1}{2} \begin{bmatrix} \mathcal{A} + \mathcal{A}^* & i(\mathcal{A} - \mathcal{A}^*) \\ -i(\mathcal{A} - \mathcal{A}^*) & \mathcal{A} + \mathcal{A}^* \end{bmatrix},
\]
\[
\mathcal{H}_1 = \frac{1}{2} \begin{bmatrix} \mathcal{H}_1 + \mathcal{H}_1^* & i(\mathcal{H}_1 - \mathcal{H}_1^*) \\ -i(\mathcal{H}_1 - \mathcal{H}_1^*) & \mathcal{H}_1 + \mathcal{H}_1^* \end{bmatrix},
\]
\[
\mathcal{H}_2 = \frac{1}{2} \begin{bmatrix} \mathcal{H}_2 + \mathcal{H}_2^* & i(\mathcal{H}_2 - \mathcal{H}_2^*) \\ -i(\mathcal{H}_2 - \mathcal{H}_2^*) & \mathcal{H}_2 + \mathcal{H}_2^* \end{bmatrix}.
\] (35)

Assume that the closed-loop system (27) is mean square stable. Then there exists a positive definite Hermitian matrix $P > 0$ and a constant $\lambda > 0$ such that (29) is satisfied. We let
\[
P = \frac{1}{2} \begin{bmatrix} P + P^* & i(P - P^*) \\ -i(P - P^*) & P + P^* \end{bmatrix},
\] (36)
which is a real positive definite matrix. It follows that
\[
\xi^T P \xi = 2\xi^T \mathcal{P} \xi + 2\xi^T \mathcal{P} \xi.
\] (37)

Also, we have that
\[
\xi^T \xi = 2\xi^T \mathcal{P} \xi.
\] (38)

Thus, multiplying (29) by two and using equations (37), (38), we have the following form:
\[
\langle \xi(t)^T P \xi(t) \rangle + \int_0^t \langle \xi(s)^T \xi(s) \rangle ds \leq \langle \xi(0)^T P \xi(0) \rangle + \lambda_1 t,
\] (39)
where $\lambda_1 = 2\lambda_2$. Based on Definition 1, the system (8) is robustly mean square stable, which implies (12) is satisfied according to Theorem 1. Based on the parameter relationship (35), we have that
\[
\mathcal{A}^T P + P \mathcal{A} + \frac{1}{\epsilon} \mathcal{H}_1 \mathcal{H}_1^T P + \frac{1}{\epsilon} \mathcal{H}_2 \mathcal{H}_2^T P < 0.
\] (40)

where $G = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$. Let $Z = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$. Then we pre-multiply the matrix (40) with $Z$ and post-multiply the matrix (40) with $Z^T$. It follows from (12) and (40) that (31) is satisfied.

To prove the sufficient part of the theorem, suppose there exists a constant $\epsilon > 0$ and a complex positive-definite Hermitian matrix $P$ such that (31) is satisfied. Based on Lemma 6, inequalities (31) and (40) imply that the system (8) with relationship (35) satisfies
\[
\mathcal{A}^T P + P \mathcal{A} + \frac{1}{\epsilon} \mathcal{H}_1 \mathcal{H}_1^T P + \frac{1}{\epsilon} \mathcal{H}_2 \mathcal{H}_2^T P < 0,
\] (41)
where $P$ is in the form of (36) and is a positive definite symmetric matrix. It follows from Theorem 1 that the closed-loop system (8) is robustly mean square stable. Hence, we have
\[
\langle \xi(0)^T P \xi(0) \rangle + \int_0^t \langle \xi(s)^T \xi(s) \rangle ds \leq \langle \xi(0)^T P \xi(0) \rangle + \lambda_1 t.
\] (42)

By applying (37) and (38), we can rewrite (42) as follows:
\[
\langle \xi(t)^T \mathcal{P} \xi(t) + \xi(t)^T \mathcal{P} \xi(t) \rangle
\]
\[
+ \int_0^t \langle \xi(s)^T \xi(s) \rangle ds
\]
\[
\leq \langle \xi(0)^T \mathcal{P} \xi(0) + \xi(0)^T \mathcal{P} \xi(0) \rangle + \lambda_2 t,
\] (43)
\forall t > 0.

Therefore, it follows from Definition 2 and (43) that the system (27) is robustly mean square stable and the proof is complete. \(\square\)
IV. NUMERICAL PROCEDURE OF COHERENT ROBUST STABILITY CONTROLLER

In Section II, a sufficient and necessary condition is proposed to build a relationship between robust stability control design with uncertain parameters and scaled control design without uncertain parameters. Since a class of annihilation operator systems discussed in Section III is a special class of general quantum systems discussed in Section II. In this section, we are mainly based on the results in Section II to achieve detailed expression for parameters in robust coherent stability controller in Section II.

A. NUMERICAL METHOD

In this section, we determine the parameters $A_K$, $B_K$, $C_K$ of the robust coherent controller (3) such that the closed-loop system (8) is robustly mean square stable. Based on Theorem 1, the inequality (12) needs to be satisfied for closed-loop system (8) to be mean square stable. By applying Schur complement [50], (12) is equivalent to the following inequality:

$$
\begin{bmatrix}
A^T P + P A^T & \bar{H}_2^T & P \bar{H}_1 \\
\bar{H}_2 & -\epsilon I & 0 \\
\bar{H}_1 & 0 & -\frac{1}{\epsilon} I
\end{bmatrix} < 0. \tag{44}
$$

To solve the inequality (44), we use a change of variables method [51] by introducing auxiliary variables $X$, $Y$, $M$, $N$. Here, $MN^T + XY = I$, $N$, $M$ are invertible and $X$, $Y$ are symmetric. Suppose that $P = \begin{bmatrix} Y & N \\ N^T & * \end{bmatrix}$, $\Pi = \begin{bmatrix} X & I \\ I & Y \end{bmatrix}$ with $*$ referring to symmetric blocks, then $P \Pi = \begin{bmatrix} Y & N \\ N^T & \epsilon I \end{bmatrix}$ [51]. Also, we can define the changes of controller variables as follows:

$$
\tilde{A} = N(A_K M^T + B_K CX) + Y(B_n C_K M^T + AX),
\tilde{B} = NB_K,
\tilde{C} = C_K M^T. \tag{45}
$$

Then, we use a congruence transformation by pre-multiplying the matrix (44) with $\Gamma = \text{diag}(\Pi, I, I)$ and post-multiplying the matrix (44) with $\Gamma^T$ and get the following inequality:

$$
\begin{bmatrix}
\Pi^T(\bar{A}^T P + P \bar{A}^T) \Pi & \Pi^T \bar{H}_2^T & \Pi^T P \bar{H}_1 \\
\bar{H}_2 \Pi & -\epsilon I & 0 \\
\bar{H}_1 \Pi & 0 & -\frac{1}{\epsilon} I
\end{bmatrix} < 0. \tag{46}
$$

By substituting parameters of (46) using (9) and (45), (46) can be rewritten as (48), shown at the bottom of the page. Therefore, we obtain that (44) satisfies if and only if

$$
\begin{bmatrix} X & I \\ I & Y \end{bmatrix} < 0 \tag{47}
$$

and (48) hold.

After inequalities (47) and (48) solved, the corresponding parameters of the controller can be constructed based on (45) as follows:

$$
A_K = N^{-1}\tilde{A} - BU_K CX - Y(B_n C_K M^T + AX)M^{-T},
B_K = N^{-1}\tilde{B},
C_K = \tilde{C}(M^T)^{-1}. \tag{49}
$$

Due to the terms $NB_K P X$ and $YB_K M^T$, (48) is still not a linear matrix inequality. Then, a multi-step optimization method in Section IV of [29] is adopted to solve the above difficulty. The method is presented as follows:

Algorithm 1

Initialization. Set $B_{PK} = 0$, $B_{PK} = 0$ and $E_1 = 0$.

Step 1. Use LMI method to solve the inequalities (47) and (48). Choose the nonsingular matrices $M$ and $N$ to make the relationship $MN^T = I - XY$ hold. Hence, we can get indirect coupling parameters $A_K$, $B_K$ and $C_K$ through (49).

Step 2. Pertaining to Step 1. Direct coupling parameters $B_{PK}$ and $B_{PK}$ can be obtained by applying inequality (48) and the relationship (7).

Step 3. Use the values $B_{PK}$ and $B_{PK}$ obtained in Step 2, and use the values $M$ and $N$ obtained in Step 1. Restart from Step 1 and parameters $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $X$, $Y$ can be obtained. Follow the procedure, go to Step 2.

Hence, by applying change of variables technique and applying multi-step optimization as stated in Algorithm 1,
the inequalities (47) and (48) can be solved using LMI method with Matlab toolbox. Consequently, we can get parameters $A_K, B_K, C_K, B_{PK}, B_{KP}$ of the robust coherent controller.

**B. PHYSICAL REALIZABLE CONDITIONS FOR THE ROBUST COHERENT STABILITY CONTROLLER**

By using the numerical procedure explained in Section IV-A, the explicit expressions for parameters $A_K, B_K, C_K, B_{PK}$ and $B_{KP}$ have been obtained. However, for the coherent controller (3) to represent a physically realizable quantum system, it is also required to satisfy the following conditions [29], [41], [43]:

$$A_K J_{nk} + J_{nk} A_K^T + B_K J_{n_i} B_K^T + B_K J_{n_i} B_K^T = 0,$$

$$B_{K_i} \left[ \begin{array}{c}
I_{nu}
\end{array} \right] = J_{nk} C_K^T J_{nu}.$$  \tag{50}

According to Theorem 5.5 of [41] and Section IV of [29], by applying the numerical method presented in Section IV-A, it is always possible to make the controller with direct and indirect couplings physically realizable through finding appropriate parameters $B_{K_i}$ and $D_K$ [52].

**C. AN ILLUSTRATIVE EXAMPLE**

The example we consider is a quantum plant with uncertainties in the interaction Hamiltonian. The quantum system is connected with a coherent controller through direct and indirect couplings. The plant which corresponds to a physically realizable quantum system is in the following form

$$A = \begin{bmatrix}
-2.8 & 0 \\
0 & 2
\end{bmatrix}, \quad B_v = \begin{bmatrix}
-2.6 & 0 \\
0 & -0.5
\end{bmatrix},$$

$$B_w = \begin{bmatrix}
-\sqrt{2} & 0 \\
0 & -\sqrt{2}
\end{bmatrix}, \quad B_u = \begin{bmatrix}
-0.1 & 0 \\
0 & -0.1
\end{bmatrix},$$

$$C = \begin{bmatrix}
\sqrt{2} & 0 \\
0 & \sqrt{2}
\end{bmatrix}, \quad C_p = \begin{bmatrix}
0.1 & 0 \\
0 & 0.1
\end{bmatrix},$$

$$D_w = I, \quad D_u = I, \quad D_p = I.$$  \tag{52}

The uncertain part of the interaction Hamiltonian $\Delta B_{PK}$ has the uncertainty $\Delta$ satisfying (5).

By applying Theorem 1 and solving corresponding inequalities (47) and (48) with Algorithm 1, we can obtain the desired quantum controller in the form (3) with the following coefficients:

$$A_K = \begin{bmatrix}
-8.4188 & 0.0008 \\
-0.0009 & -5.4652
\end{bmatrix},$$

$$B_K = \begin{bmatrix}
-1.1654 & 0.0001 \\
-0.0002 & -4.0936
\end{bmatrix},$$

$$C_K = \begin{bmatrix}
-8.5995 & 0 \\
0.0035 & -9.7521
\end{bmatrix}.$$  \tag{53}

For the controller (3) to be a fully quantum system, it is also required to satisfy the physical realizability conditions (IV-B). Hence, by adopting the method in Section IV-B, the rest coefficients of the desired controller are as follows:

$$B_{K_v} = \begin{bmatrix}
9.7521 & 0 \\
0.0035 & 8.5995
\end{bmatrix},$$

$$D_K = \begin{bmatrix}
1 & 0
\end{bmatrix}.$$  \tag{54}

Therefore, by applying the method in this paper, we can obtain the robust coherent controller so that the closed-loop system consisting of the uncertain quantum system connected with the desired controller is robustly stable.

**V. CONCLUSION**

This paper has designed a robust coherent stability controller for a class of uncertain quantum systems through direct and indirect couplings. Uncertainties exist in the interaction Hamiltonian. A special class of quantum systems in terms of annihilation operators only was also considered. A sufficient and necessary condition was provided to develop a relationship between the robust stability control problem and scaled stability control problem. To obtain the detailed representation of controller parameters, a numerical problem was proposed using LMI technique, multi-step optimization and physical realizability conditions. Thus, a physically realizable robust stability controller through direct and indirect couplings was obtained. Future work would study about robust stability control for quantum systems with other kinds of uncertainties.

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