Bispinor Auxiliary Fields in Duality-Invariant Electrodynamics Revisited: The $U(N)$ Case

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**ABSTRACT**

We update and detail the formulation of the duality-invariant systems of $N$ interacting abelian gauge fields with $N$ auxiliary bispinor fields added. In this setting, the self-duality amounts to $U(N)$ invariance of the nonlinear interaction of the auxiliary fields. The $U(N)$ self-dual Lagrangians arise after solving the nonlinear equations of motion for the auxiliary fields. We also elaborate on a new extended version of the bispinor field formulation involving some additional scalar auxiliary fields and study $U(N)$ invariant interactions with derivatives of the auxiliary bispinor fields. Such interactions generate higher-derivative $U(N)$ self-dual theories.

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1 Introduction

The $U(N)$ duality property is inherent to nonlinear interactions of $N$ abelian gauge field strengths $F^k_{\alpha\beta}, \bar{F}^k_{\dot{\alpha}\dot{\beta}}$, ($k = 1, \ldots, N$) [1]-[5]. The notorious examples of these duality invariant systems are multi-field generalizations of the Born-Infeld (BI) theory. So far, the construction of such generalized BI systems was based on introducing some auxiliary matrix scalar fields $\chi$ with bilinear algebraic relations between these fields and the scalar combinations of the gauge field strengths [4]. The corresponding nonlinear Lagrangians as functions of $F^k_{\alpha\beta}, \bar{F}^k_{\dot{\alpha}\dot{\beta}}$ arose after substituting the perturbative solution for $\chi$.

The recent revival of interest in the duality-invariant systems was mainly triggered by the hypothesis that the duality considerations could play the decisive role in checking the conjectured ultraviolet finiteness of the maximally extended $\mathcal{N} = 8, d = 4$ supergravity (see, e.g., [6, 7, 8]). In theories of this kind there simultaneously appear a few gauge fields, so it is just $U(N)$ duality that is of relevance to this circle of problems.

The auxiliary bispinor field formulation of the $U(N)$ duality was introduced in [9] as a natural generalization of the analogous approach to the $U(1)$ duality-symmetric (or self-dual) systems [10] [11]. The $U(N)$ self-duality is equivalent to the manifest $U(N)$ invariance of the interaction Lagrangian for the auxiliary fields.

In this paper we further detail the bispinor field formulation and consider several new examples of the $U(N)$ duality-invariant models. It is a continuation of our recent paper [12], where the bispinor auxiliary field formulation of the $U(1)$ duality was renewed and related to the latest developments in this area.

We start, in Section 2, by recalling the standard setting for the $U(N)$ self-dual theories in terms of $N$ Maxwell gauge field strengths $F^k_{\alpha\beta}$ and then turn to the $(F, V)$ representation of these theories with $N$ bispinor auxiliary fields $V^k_{\alpha\beta}$ added. By construction, the $U(N)$ self-dual $(F, V)$ Lagrangian satisfies the Gaillard-Zumino (GZ) representation with an arbitrary invariant interaction $\mathcal{E}(V)$ [9]. We study the general parametrization of the scalar $U(N)$ invariants constructed out of the auxiliary fields. The equations of motion for the auxiliary fields are in one-to-one correspondence with “the deformed twisted self-duality constraints” (for the $U(N)$ duality group) proposed in [8]. We also introduce additional scalar matrix auxiliary fields $\mu$ and consider an alternative formalism involving two types of the auxiliary fields. This $\mu$ representation simplifies solving the auxiliary-field equations and constructing the self-dual Lagrangians.

Section 3 is devoted to examples of the $U(N)$ self-dual theories, including $U(N)$ generalizations of the BI theory. The first type of the $U(N)$ BI models is the real form of the $U(N) \times U(N)$ generalization considered in Ref. [4]. We translate this model into our $\mu$ representation with the special $U(N)$ invariant auxiliary interaction. We also propose an alternative $U(N)$ generalization of the BI theory and construct the perturbative Lagrangian for this model. Some other examples of the $U(N)$ self-dual models are as well considered. One example corresponds to the simplest quartic interaction of the auxiliary fields, and another one is constructed by analogy with the new exact $U(1)$ self-dual Lagrangian given in [12].

The $U(N)$ self-dual models with higher derivatives are studied in Section 4. In the Appendix we rewrite our $(F^k, V^k)$ Lagrangian and the self-duality equation in the tensor
formalism.
Throughout the paper we basically use the notations and abbreviations of Ref. [12].

2 Auxiliary fields for $U(N)$ duality

2.1 The standard setting

Our starting point is the nonlinear Lagrangian with $N$ abelian gauge field strengths $F^i_{\alpha\beta}, \bar{F}^{i\dot{\alpha}\dot{\beta}}, (i = 1, \ldots, N)$

$$L(F^k, \bar{F}^l) = -\frac{1}{2}[(F^k F^k) + (\bar{F}^k \bar{F}^k)] + L^{int}(F^k, \bar{F}^l). \quad (2.1)$$

It is manifestly invariant under the real $O(N)$ transformation

$$\delta_\xi F^k_{\alpha\beta} = \xi^{kl} F^l_{\alpha\beta}, \quad \delta_\xi \bar{F}^{k\dot{\alpha}\dot{\beta}} = \xi^{kl} \bar{F}^{l\dot{\alpha}\dot{\beta}}, \quad \xi^{kl} = -\xi^{lk}. \quad (2.2)$$

It is convenient to define $\frac{1}{2}N(N+1)$ complex scalar variables and to consider the Lagrangian as a real function of these variables

$$\varphi^{kl} = \varphi^{lk} = (F^k F^l), \quad \bar{\varphi}^{kl} = (\bar{F}^k \bar{F}^l), \quad (2.3)$$

$$L(F^k, \bar{F}^k) = L(\varphi^{kl}, \bar{\varphi}^{kl}). \quad (2.4)$$

The nonlinear equations of motion

$$E^k_{\alpha\dot{\alpha}} = \partial_\alpha \bar{P}^k_{\alpha\dot{\beta}}(F) - \partial_{\dot{\alpha}} P^k_{\alpha\beta}(F) = 0 \quad (2.5)$$

involve the dual nonlinear field strengths

$$P^k_{\alpha\beta}(F) = i \frac{\partial L}{\partial F^{k\alpha\beta}} = 2i F^l_{\alpha\beta} \frac{\partial L}{\partial \varphi^{kl}}, \quad \text{and c.c.}. \quad (2.6)$$

The gauge field strengths $F^k_{\alpha\beta}, \bar{F}^{k\dot{\alpha}\dot{\beta}}$ obey the standard Bianchi identities

$$B^k_{\alpha\dot{\alpha}} = \partial_\alpha \bar{F}^k_{\alpha\dot{\beta}} - \partial_{\dot{\alpha}} F^k_{\alpha\beta} = 0. \quad (2.7)$$

The on-shell duality transformations are realized as

$$\delta_\eta F^k_{\alpha\beta} = \eta^{kl} P^l_{\alpha\beta}, \quad \delta_\eta P^k_{\alpha\beta} = -\eta^{kl} F^l_{\alpha\beta}. \quad (2.8)$$

\footnote{In the tensor notation one deals with the self-dual field $\sigma_{mn}^{+k}$,

$$P^k_{\alpha\beta}(F) = \frac{1}{8} (\sigma^m \sigma^n - \sigma^n \sigma^m)_{\alpha\beta} \sigma_{mn}^{+k}. \quad (2.6)$$}
where $\eta^{kl} = \eta^{lk}$ are $\frac{1}{2}N(N + 1)$ real parameters. These transformations extend the $O(N)$ group \((2.2)\) to the group $U(N)$ and so reside in the coset $U(N)/O(N)$. The equations of motion \((2.5)\) together with the Bianchi identities \((2.7)\) are covariant under \((2.8)\),

$$
\delta_\eta E^k_{\alpha\beta} = -\eta^{kl}B^k_{\alpha\beta}, \quad \delta_\eta B^k_{\alpha\beta} = \eta^{kl}E^k_{\alpha\beta},
$$

provided that the generalized consistency conditions hold:

$$
(P^k P^l) + (F^k F^l) - \text{c.c.} = 0, \quad (F^k P^l) - (F^l P^k) - \text{c.c.} = 0.
$$

These conditions are $U(N)$ covariant on their own right. The symmetric conditions can be rewritten as the matrix differential equations for the Lagrangian

$$
\phi - 4L_\phi\bar{\phi}L_\phi = \bar{\phi} - 4L_\bar{\phi}\phiL_\phi.
$$

### 2.2 The $(F, V)$ representation

The generalized auxiliary field representation of the $O(N)$ invariant Lagrangian \([9]\) contains $N$ complex auxiliary $O(N)$ vector fields $V^k_{\alpha\beta}$, $\bar{V}^k_{\alpha\beta}$

$$
\mathcal{L}(F^k, V^k) = \mathcal{L}_2(F^k, V^k) + E[(V^k V^l), (\bar{V}^k \bar{V}^l)],
$$

$$
\mathcal{L}_2(F^k, V^k) = \frac{1}{2}[(F^k F^l) + (\bar{F}^k \bar{F}^l)] - 2[(F^k V^l) + (\bar{F}^k \bar{V}^l)] + (V^k V^k) + (\bar{V}^k \bar{V}^k).
$$

Here, $E$ is an $O(N)$ invariant real interaction depending on $\frac{1}{2}N(N + 1)$ scalar complex variables

$$
\nu^{kl} = (V^k V^l), \quad \bar{\nu}^{kl} = (\bar{V}^k \bar{V}^l).
$$

The interaction $E(\nu^{kl}, \bar{\nu}^{kl})$ is assumed to be regular at the origin, so that it admits expansion in power series. The dynamical equation of motion following from this Lagrangian has the form

$$
\partial^\alpha \tilde{P}^k_{\alpha\beta}(F, V) - \partial^\beta \tilde{P}^k_{\alpha\beta}(F, V) = 0,
$$

where

$$
P^k_{\alpha\beta}(F, V) = i(F^k - 2V^k)_{\alpha\beta}.
$$

The $U(N)/O(N)$ duality transformations are implemented as

$$
\delta_\eta F^k_{\alpha\beta} = \eta^{kl}P^l_{\alpha\beta} + i\eta^{kl}(F^l - 2V^l)_{\alpha\beta}, \quad \delta_\eta P^k_{\alpha\beta} = -\eta^{kl}E^l_{\alpha\beta}.
$$

The corresponding $\eta^{kl}$ transformations of the auxiliary fields follow from \((2.17)\) and the definition \((2.16)\). The full $U(N)$ transformations of the auxiliary fields can be written as

$$
\delta V^k_{\alpha\beta} = (\xi^{kl} - i\eta^{kl})V^l_{\alpha\beta}, \quad \delta \bar{V}^k_{\alpha\beta} = (\xi^{kl} + i\eta^{kl})\bar{V}^l_{\alpha\beta}.
$$
The $SU(N)$ subgroup is singled out by the condition $\eta^{kk} = \text{Tr} \eta = 0$.

The $U(N)$ transformations of the scalar variables $\nu^{kl}$ and $\bar{\nu}^{kl}$ can be presented in the matrix form as

$$
\delta \nu = [\xi, \nu] - i\{\eta, \nu\}, \quad \delta \bar{\nu} = [\xi, \bar{\nu}] + i\{\eta, \bar{\nu}\}.
$$

Then, we define the Hermitian matrix variables

$$
a^{kl} = (\bar{\nu} \nu)^{kl}, \quad a^{kl} = (\nu \bar{\nu})^{kl} = (\bar{\nu} \nu)^{lk},
$$

$$
\delta a = i[\xi, a] + i[\eta, a], \quad \delta \bar{a} = i[\xi, \bar{a}] - i[\eta, \bar{a}].
$$

We also define the matrix monomials $a^n$ with the following properties

$$
(a^n \bar{\nu})^{kl} = (\bar{\nu} a^n)^{kl}, \quad (\nu a^n)^{kl} = (a^n \nu)^{kl}.
$$

From these monomials one can construct $N$ independent real $U(N)$ invariants $A_n$, $(n = 1, 2, \ldots, N)$:

$$
A_n = \frac{1}{n} \text{Tr} a^n, \quad dA_n = \text{Tr} (da a^{n-1}), \quad \frac{\partial A_n}{\partial a^{kl}} = (a^{n-1})^{kl}.
$$

An alternative choice of the $U(N)$ invariants is connected with the spectrum $\lambda_1(A_1, \ldots, A_N), \ldots, \lambda_N(A_1, \ldots, A_N)$ of the Hermitian matrix $a^{13, 14}$. This spectrum can be found by solving the characteristic equation

$$
A(a) = (a - \lambda_1)(a - \lambda_2) \cdots (a - \lambda_N) = 0.
$$

Like in the $U(1)$ case $[10, 11, 12]$, the dynamical equations of motion (2.15), together with the Bianchi identities (2.7) and the algebraic equations of motion for the auxiliary fields $V^{k}_{\alpha \beta}, \bar{V}^{k}_{\bar{\alpha} \bar{\beta}}$ following from (2.12), are covariant under the $U(N)$ duality transformations, provided that the interaction function $E(\nu^{kl}, \bar{\nu}^{kl})$ in (2.12) is $U(N)$ invariant $[9]$, 

$$
E(\nu^{kl}, \bar{\nu}^{kl}) \Rightarrow \mathcal{E}(A_1, \ldots, A_N),
$$

with $\mathcal{E}(A_1, \ldots, A_N)$ being an analytic function.

The more convenient representation for the interaction Lagrangian $\mathcal{E}(A_n)$ is through the matrix function $E(a)$

$$
\mathcal{E} = \text{Tr} E(a).
$$

The derivative matrix function $E_a$ is defined as follows

$$
d\mathcal{E} = dA_n \mathcal{E}_n = \text{Tr} (da E_a) = (da_l^k E_a^{kl}), \quad \mathcal{E}_n = \frac{\partial \mathcal{E}}{\partial A_n},
$$

whence

$$
(E_a)^{kl} = \mathcal{E}_1 \delta^{kl} + \mathcal{E}_2 a^kl + \mathcal{E}_3 a^{kj} a^{jl} + \ldots.
$$
Using this representation and the relations (2.22), we define the holomorphic derivatives

\[ \mathcal{E}^{kl} := \frac{\partial \mathcal{E}}{\partial \nu^{kl}} = (E_a)^{kr} \bar{\nu}^r = \mathcal{E}_1 \bar{\nu}^{kl} + \mathcal{E}_2 a^{kr} \bar{\nu}^r + \mathcal{E}_3 a^{kj} a^{jr} \bar{\nu}^r + \ldots, \]  

(2.29)

\[ \bar{\mathcal{E}}^{kl} := \frac{\partial \mathcal{E}}{\partial \bar{\nu}^{kl}} = \nu^{kr} (E_a)^{rl} = \mathcal{E}_1 \nu^{kl} + \mathcal{E}_2 \nu^{kr} a^{rl} + \mathcal{E}_3 \nu^{kr} a^{jr} a^{jl} + \ldots. \]  

(2.30)

The basic algebraic equations of the \( U(N) \) duality-invariant models are obtained by varying, with respect to \( V_{\alpha\beta}^k \) and \( \bar{V}_{\dot{\alpha}\dot{\beta}}^k \), the Lagrangian (2.12), in which the general function \( E \) is substituted by the \( U(N) \) invariant one \( \mathcal{E} \) defined in (2.25):

\[ (F^k - V^k)_{\alpha\beta} = \mathcal{E}^{kl} V_{\alpha\beta}^l = (E_a)^{kr} \bar{\nu}^r V_{\alpha\beta}^l, \quad (\bar{F}^k - \bar{V}^k)_{\dot{\alpha}\dot{\beta}} = \bar{\mathcal{E}}^{kl} \bar{V}_{\dot{\alpha}\dot{\beta}}^l = \nu^{kr} (E_a)^{rl} \bar{V}_{\dot{\alpha}\dot{\beta}}^l. \]  

(2.31)

Equations of motion (2.31) are equivalent to the nonlinear twisted self-duality constraints which were postulated in [8, 15]. The important corollaries of (2.31) are the scalar matrix algebraic equation

\[ \varphi^{kl} = [\delta^{kr} + \mathcal{E}^{kr}] \nu^{rl}[\delta^{sl} + \mathcal{E}^{sl}] \]  

(2.32)

and its conjugate.

By analogy with the \( U(1) \) case [9, 12], the general solution of the algebraic equations (2.31) can be written in the following concise form:

\[ V_{\alpha\beta}^k = F_{\alpha\beta}^l G^{kl}(\varphi, \bar{\varphi}), \]  

(2.33)

\[ G^{kl} = [\delta^{kl} + \mathcal{E}^{kl}]^{-1} = \frac{1}{2} \delta^{kl} - \frac{\partial L}{\partial \varphi^{kl}}, \]  

(2.34)

\[ P_{\alpha\beta}^k = 2i F_{\alpha\beta}^l \frac{\partial L}{\partial \varphi^{kl}} = i F_{\alpha\beta}^l [\delta^{kl} - 2G^{kl}]. \]  

(2.35)

Using (2.33), (2.34), we can uniquely restore the Lagrangian \( L(\varphi^{kl}, \bar{\varphi}^{kl}) \) by its holomorphic derivatives:

\[ dL = d\varphi^{lk} \frac{\partial L}{\partial \varphi^{kl}} + \text{c.c.}. \]  

(2.36)

Note that Eqs. (2.31), (2.32) are simplified under the particular choices of \( \mathcal{E} \), e.g., for \( \mathcal{E} = \mathcal{E}(A_1) \):

\[ F^k_{\alpha\beta} = V_{\alpha\beta}^r [\delta^{kr} + \mathcal{E}_1 \bar{\nu}^{kr}], \]  

(2.37)

\[ \varphi^{kl} = \nu^{kl} + \mathcal{E}_1 [\bar{\nu}^{kr} \nu^r + \nu^{ks} \bar{\nu}^s] + \mathcal{E}_2 \bar{\nu}^{kr} \nu^{rs} \bar{\nu}^s. \]  

(2.38)

The famous GZ representation of the \( U(N) \) self-dual Lagrangians has the following form in the \((F,V)\) representation

\[ \mathcal{L}(F^k, V^k) = \frac{i}{2} [\bar{F}^k (F, V) \bar{F}^k - P^k (F, V) F^k] + [(V^k V^k) - (F^k V^k)] \]

\[ + [(V^k V^k) - (\bar{F}^k \bar{V}^k)] + \mathcal{E}, \]  

(2.39)
where \((V^k V^k) - (F^k V^k)\) is the complex bilinear \(U(N)\) invariant. Using Eqs. (2.31), we can also prove that the \(U(N)\) self-duality conditions (2.10) in the \((F, V)\) representation are none other than the conditions of \(U(N)\) invariance of the auxiliary interaction \(\mathcal{E}\) [9]

\[
(P^k P^l) + (F^k F^l) - c.c. = \left( V^l \frac{\partial \mathcal{E}}{\partial V^k} \right) + \left( V^k \frac{\partial \mathcal{E}}{\partial V^l} \right) - c.c. = 0,
\]

\[
(F^k P^l) - (F^l P^k) - c.c. = i \left( V^k \frac{\partial \mathcal{E}}{\partial V^l} \right) - i \left( V^l \frac{\partial \mathcal{E}}{\partial V^k} \right) + c.c. = 0.
\]

### 2.3 The \(\mu\) representation

In the \(U(1)\) case, there is a more convenient parametrization of the duality-invariant auxiliary interaction, the “\(\mu\) representation” [11, 12]. Making use of it essentially simplifies the road from the auxiliary-field equations to the final nonlinear duality-invariant Lagrangian.

In the \(U(N)\) case, the \(\mu\) representation is set up in terms of the matrix variables

\[
\mu^{kl} = \frac{\partial \mathcal{E}(a)}{\partial \nu^{kl}} = (E_a)^{kr} \bar{\nu}^r l, \quad \bar{\mu}^{kl} = \nu^{kr} (E_a)^{rl},
\]

\[
b^{kl} = \mu^{ks} \bar{\mu}^{sl} = (E_a \bar{\nu} \nu E_a)^{kl} = (aE_a^2)^{kl}, \quad b^{lk} = \bar{b}^{kl} = \bar{\mu}^{kr} \mu^{rl},
\]

where the relations (2.22) were used. These newly defined matrix variables possess the following transformation laws:

\[
\delta \mu = [\xi, \mu] + i \{\eta, \mu\}, \quad \delta \bar{\mu} = [\xi, \bar{\mu}] - i \{\eta, \bar{\mu}\}, \quad \delta b = [\xi, b] + i [\eta, b]
\]

and reveal the properties

\[
(b^n \mu)^{kl} = (\mu \bar{b}^n)^{kl} = (\bar{b}^n \mu)^{lk}, \quad (\bar{\mu} b^n)^{kl} = (\bar{b}^n \bar{\mu})^{kl} = (\bar{b} \bar{b} b^k)^{lk}.
\]

From them one can construct \(N\) independent \(U(N)\) invariants

\[
B_n = \frac{1}{n} \text{Tr} b^n,
\]

which are going to be the arguments of the \(\mu\) representation analog of the invariant function \(\mathcal{E}(A_n)\).

The connection with the basic objects of the original \(\nu\) representation is established through the Legendre transformation

\[
\mathcal{I}(B_n) = \text{Tr} I(b) := \mathcal{E} - \nu^{kl} \mu^{kl} - \bar{\nu}^{kl} \bar{\mu}^{kl} = \text{Tr} [E - 2a E_a],
\]

\[
\nu^{kl} = - \frac{\partial \mathcal{I}}{\partial \mu^{kl}} = - (\bar{\mu} I_b)^{kl} = - (\bar{I}_b \bar{\mu})^{kl},
\]

\[
d\mathcal{I} = \text{Tr} (db I_b) = \text{Tr} (d\bar{b} \bar{I}_b),
\]

where \(\mathcal{I}(B_n)\) is a real analytic invariant function which is the interaction in the \(\mu\) representation. We introduced the covariant matrix functions \(I(b)\) and \(I_b\) which are representable...
as formal series over the powers of the matrix $b$, with the coefficients being functions of the invariants $B_n$, e.g.,

$$I(b) = \sum_k \frac{1}{k!} I^{(k)}(B_n) b^k, \quad \delta I(b) = [(\xi + i\eta), I(b)].$$

(2.50)

They are related to the matrix functions $E(a)$ and $E_a$ by the covariant matrix equations

$$I(b) = E(a) - 2a E_a, \quad E(a) = I(b) - 2b I_b, \quad E_a = -I_b^{-1}, \quad a = bI_b^2, \quad b = aE_a^2.$$  

(2.51)

Eqs. (2.32) can be rewritten in the $\mu$ representation as

$$\varphi^{kl} = [\delta^{kr} + \mathcal{E}_{kr}] \nu^{rs} [\delta^{sl} + \mathcal{E}_{sl}] = -(\delta^{kr} + \mu^{kr}) \frac{\partial I}{\partial \nu^{rs}} (\delta^{sl} + \mu^{sl})$$

(2.52)

or

$$\varphi = -(1 + \mu) I_b (1 + \tilde{\mu}), \quad \varphi = -(1 + \tilde{\mu}) I_b (1 + \mu),$$

(2.53)

where the relations (2.45) are used and 1 denotes the unit matrix.

In the particular representation, $I_b$ can be chosen as the matrix power series expansion with the numerical coefficients $i_k$

$$(I_b)^{kl} = -2 \delta^{kl} + i_2 b^{kl} + \frac{1}{2} i_3 (b^2)^{kl} + \ldots.$$  

(2.54)

Then we can write the following recursive complex matrix equation for $\mu$:

$$\mu = \frac{1}{2} \tilde{\varphi} - \mu \tilde{\mu} - \tilde{\mu} \mu - \mu \bar{\mu} \bar{\mu} + \frac{1}{2} i_2 \mu \bar{\mu} \mu + \frac{1}{2} i_2 \bar{\mu} \mu \mu$$

$$+ \frac{1}{2} i_2 \bar{\mu} \mu \mu \mu + \frac{1}{2} i_2 \mu \bar{\mu} \mu \mu + \frac{1}{4} i_3 \mu \bar{\mu} \mu \mu + O(\mu^6).$$

(2.55)

Solving it, e.g., for $\mu = \mu^{kl} (\varphi, \bar{\varphi})$, we can reconstruct the holomorphic derivatives of the Lagrangian from Eqs. (2.34) and (2.42),

$$\left( \frac{1}{1 + \mu} \right)^{kl} = \frac{1}{2} \delta^{kl} - \frac{\partial L}{\partial \varphi^{kl}},$$

(2.56)

and finally restore the nonlinear perturbative Lagrangian in the $F$-representation

$$L = \frac{1}{2} \text{Tr} \left[ - (\varphi + \tilde{\varphi}) + \varphi \tilde{\varphi} - \frac{1}{2} \varphi^2 \tilde{\varphi} - \frac{1}{2} \varphi^2 \tilde{\varphi} \right]$$

$$+ \frac{1}{8} \text{Tr} \left[ \varphi^3 \varphi + \varphi \varphi^3 + (2 + \frac{1}{4} i_2) \varphi \tilde{\varphi} \varphi + 2 \varphi^2 \tilde{\varphi} \right] + O(\varphi^5),$$

(2.57)

where the single-trace matrix terms of higher orders are omitted.
Like in the $U(1)$ case [12], one can define a combined $(F, V, \mu)$ off-shell representation for the $U(N)$ self-dual Lagrangians, treating $\mu^{ik}, \bar{\mu}^{ik}$ as independent auxiliary fields

$$L(V^k, F^k, \mu^{kl}) = \frac{1}{2}[(F^k F^l) + (\bar{F}^k \bar{F}^l)] - 2 [(V^k \cdot F^l) + (\bar{V}^k \cdot \bar{F}^l)]$$

$$+ (V^k V^l) (\delta^{kl} + \mu^{kl}) + \bar{V}^k \bar{V}^l (\delta^{kl} + \bar{\mu}^{kl}) + \mathcal{I}(B_n),$$

where $B_n$ are the invariants (2.46). Eliminating the $V^k$ variables from this Lagrangian,

$$V^k_{\alpha\beta} = \left[(1 + \mu) - 1\right]^{kl} F^l_{\alpha\beta}, \quad \text{and c.c.},$$

we arrive at the $(F, \mu)$ representation of the Lagrangian:

$$\bar{L}(F^k, \mu^{kl}) = \frac{1}{2}(F^k F^l) [(\mu - 1)(1 + \mu)^{-1}]^{kl} + \text{c.c.} + \mathcal{I}(B_n).$$

Varying this Lagrangian with respect to $\mu^{kl}$ we obtain the matrix auxiliary equation which is equivalent to Eq. (2.53).

In the specific examples we can exploit the similarity between the $U(1)$ interaction function $I(b)$ [12] and the matrix function $I(b)$ of the $U(N)$ case, although solving the matrix equations is the much more difficult task. For the simple particular $U(N)$ interaction presented by a one-argument function $\mathcal{E}(A_1), A_1 = a^{kk} = \nu^{kl}\bar{\nu}^{kl}$, we find, e.g.,

$$\mu^{kl} = \mathcal{E}_1 b^{kl}, \quad \bar{\mu}^{kl} = \mathcal{E}_1 \bar{b}^{kl}, \quad \nu^{kl} = -\mathcal{I}_1 \bar{\mu}^{kl},$$

$$b^{kl} = \mathcal{E}_1^2 a^{kl}, \quad B_1 = b^{kk} = \mathcal{E}_1^2 A_1, \quad \mathcal{E}_1 = -\mathcal{I}_1^{-1}.$$  \hspace{1cm} (2.61)

The corresponding interaction function in the $\mu$-representation involves only the trace $B_1$,

$$\mathcal{I}(B_1) = \mathcal{E}(A_1) - 2A_1 \mathcal{E}_1, \quad \nu^{kl} = -\mathcal{I}_1 \bar{\mu}^{kl}, \quad a^{kl} = \mathcal{I}_1^2 b^{kl}, \quad \mathcal{E}_1 = -\mathcal{I}_1^{-1}. \hspace{1cm} (2.62)$$

The equation (2.53) has the following form in this case:

$$\varphi = -\mathcal{I}_1 (1 + \mu) \bar{\mu} (1 + \mu). \hspace{1cm} (2.64)$$

We consider the representation

$$\mathcal{I}(B_1) = -2B_1 + \frac{1}{2}j_2 B_1^2 + \frac{1}{6}j_3 B_1^3 + \ldots, \quad \mathcal{I}_1 = -2 + j_2 B_1^2 + \frac{1}{2}j_3 B_1^2 + \ldots$$  \hspace{1cm} (2.65)

($j_2, j_3, \ldots$ are some constants) and the corresponding recursion relations for $\mu(\varphi, \bar{\varphi})$. The 4-th order term in the corresponding self-dual Lagrangian

$$L^{(4)}(\varphi, \bar{\varphi}) = \frac{1}{8} \text{Tr} \left[ \varphi^3 \bar{\varphi} + \varphi \bar{\varphi}^3 + 2\varphi \bar{\varphi} \varphi \bar{\varphi} + 2\varphi^2 \bar{\varphi}^2 \right] + \frac{1}{32} j_2 [\text{Tr}(\varphi \bar{\varphi})]^2. \hspace{1cm} (2.66)$$

contains the double-trace term. The subsequent recursions give terms with several traces.
3 Examples of the $U(N)$ self-dual models

Here we present some examples of $U(N)$ self-dual models with actions involving no higher derivatives. Basically, these are generalizations of the $U(1)$ examples considered in [12]. Similarly to the $U(1)$ case, the corresponding interactions written in terms of the auxiliary variables can be chosen in a closed form, while the equivalent on-shell expressions, with the auxiliary variables being eliminated in terms of the Maxwell field strengths, can be given only as infinite series in powers of the field strengths. An important difference from the $U(1)$ case is that there exist several inequivalent $U(N)$ duality-invariant models which are reduced to the same $U(1)$ model in the one field-strength limit. For instance, there are few $U(N)$ duality-invariant extensions of the standard BI theory. As distinct from the latter, the Lagrangians of such generalized BI theories seem not to admit a closed representation in terms of the Maxwell field strengths (even for the simplest non-trivial $U(2)$ case).

3.1 $U(N)$ generalizations of the Born-Infeld model

The $U(N) \times U(N)$ generalization of the BI theory proposed in [4] deals with the Hermitian scalar matrix fields

$$\alpha^{kl} = \frac{1}{4}(F^k F^l), \quad \beta^{kl} = \frac{1}{4}(\bar{F}^k \bar{F}^l)$$

constructed out of $N$ complex field-strengths $F^k_{mn}$ and their conjugates $\bar{F}^k_{mn}$ (with $\bar{F}^k_{mn} = \frac{1}{2} \varepsilon_{mnst} F^k_{st}$). The basic complex scalar auxiliary field $\chi^{kl} \neq \chi^{lk}$ of this model satisfies the matrix equation

$$\chi^{kl} + \frac{1}{2} \chi^{kr} \chi^r l = \alpha^{lk} + i \beta^{kl} = \phi^{kl}, \quad (3.2)$$

with the solution representable as the matrix power series.

The $U(N)$ generalization of the BI theory we are interested in corresponds to imposing the reality condition $F^k = \bar{F}^k$ and using the symmetric matrices

$$\hat{\alpha}^{kl} \to t^{kl} = \frac{1}{4} \eta^{mr} \eta^{ns} F^k_{mn} F^l_{rs}, \quad \hat{\beta}^{kl} \to z^{kl} = \frac{1}{8} \varepsilon^{mnrs} F^k_{mn} F^l_{rs},$$

$$\hat{\phi}^{kl} \to \phi^{kl} = (F^k F^l). \quad (3.3)$$

In this notation, we consider the following nonlinear Lagrangian of $N$ abelian gauge field strengths $F^k_{\alpha\beta}$

$$L_{\text{ABMZ}}(\varphi, \bar{\varphi}) = \text{Tr} \left[ -\frac{1}{2} (\varphi + \bar{\varphi}) + \frac{1}{2} \varphi \bar{\varphi} - \frac{1}{4} (\varphi^2 \bar{\varphi} + \varphi \bar{\varphi}^2) \right]$$

$$+ \frac{1}{8} \text{Tr} \left[ \varphi^3 \bar{\varphi} + 2 \varphi^2 \bar{\varphi}^2 + \varphi \bar{\varphi} \varphi \bar{\varphi} + \varphi \bar{\varphi}^3 \right] + O(\varphi^5). \quad (3.4)$$

The $U(N)$ duality condition [2.11] can be directly proved for this Lagrangian.
Our interpretation of this model makes use of the following exact invariant interaction in the matrix $\mu$ representation (2.60):

$$I = \text{Tr} I(b), \quad I(b) = \frac{2b}{(b-1)}, \quad I_b = -\frac{2}{(b-1)^2}. \quad (3.5)$$

We can calculate $L(\varphi, \bar{\varphi})$ as the power series, based on the expansion

$$I(b) = -2b - 2b^2 - 2b^3 - 2b^4 - \ldots. \quad (3.6)$$

For the proper choice of the numerical coefficients in (2.57), $i_2 = -4, \ldots$, we can reproduce the Lagrangian (3.4).

By analogy with the $U(1)$ case, we can come back to the original $(F, V)$ formulation, defining the matrix variable $a$ by the relation

$$a = \frac{4b}{(1-b)^4}. \quad (3.7)$$

The auxiliary interaction has the single-trace form $E(a) = \text{Tr} E(a)$. The matrix relations between various quantities in the $(F, V)$ and $\mu$ representations are similar to those valid in the $U(1)$ duality case for the BI theory [11, 12]

$$E(a) = \frac{2b(a)[1 + b(a)]}{[1 - b(a)]^2} = 2[2t^2(a) + 3t(a) + 1], \quad (3.8)$$

$$t^4 + t^3 - \frac{1}{4}a = 0, \quad t = \frac{1}{b-1}, \quad (3.9)$$

$$2E_a = [1 - aE_a^2]^2, \quad E_a = \frac{1}{2}[b(a) - 1]^2. \quad (3.10)$$

Solving these equations, one can find closed expressions for both $t(a)$ and the single-trace interaction $E(a)$. They look rather bulky and so it is not too illuminating to present them here. Up to the 3-d order in $a$:

$$E = \text{Tr} \left[ \frac{1}{2}a - \frac{1}{8}a^2 + \frac{3}{32}a^3 + O(a^4) \right]. \quad (3.11)$$

An alternative $U(N)$ generalization of the BI Lagrangian proceeds from the $\mu$ representation with the invariant auxiliary interaction of the simple form

$$I(B_1) = \frac{2B_1}{B_1 - 1}, \quad B_1 = \text{Tr} \mu \bar{\mu}. \quad (3.12)$$

This interaction corresponds to the choice $j_2 = -4, \ldots$ in (2.65). We can obtain the self-dual nonlinear Lagrangian in the $F$ representation, using the recursion equation (2.64). The $(F, V)$ representation of the same model deals with the invariant interaction which is a function of the single $U(N)$ invariant variable $A_1 = (V^k V^l)(\bar{V}^l \bar{V}^k)$:

$$E(A_1) = \frac{1}{2}A_1 - \frac{1}{8}A_1^2 + \frac{3}{32}A_1^3 + O(A_1^4). \quad (3.13)$$
3.2 Other examples of $U(N)$ self-dual theories

The simplest quartic $U(N)$ invariant interaction

$$\mathcal{E}_{SI} = \frac{1}{2} (V^k V^l) (\bar{V}^k \bar{V}^l) = \frac{1}{2} \text{Tr} \ a = \frac{1}{2} A_1$$  \hspace{1cm} (3.14)

produces the self-dual model, which is $U(N)$ generalization of the “simplest interaction $U(1)$ self-dual model” of refs. \[11, 12\] (it was rediscovered in \[8, 15\]). In this case, the basic polynomial auxiliary equation

$$F^k_{\alpha\beta} = [\delta^{kl} + \frac{1}{2} (\bar{V}^k \bar{V}^l)] V^l_{\alpha\beta}$$  \hspace{1cm} (3.15)

has the perturbative solution for the function $G_{kl} = \frac{1}{2} \delta_{kl} - \partial L_{SI} / \partial \varphi^{kl}$ (2.34). The corresponding power-series Lagrangian reads

$$L_{SI} = \text{Tr}[\frac{1}{2} (\varphi + \bar{\varphi}) + \frac{1}{2} \varphi \bar{\varphi} - \frac{1}{4} (\varphi^2 \bar{\varphi} + \varphi \bar{\varphi}^2) + \frac{1}{8} (\varphi^3 \bar{\varphi} + 2 \varphi^2 \bar{\varphi}^2 + 2 \varphi \bar{\varphi} \varphi \bar{\varphi} + \varphi \bar{\varphi}^3) + \ldots .$$  \hspace{1cm} (3.16)

We can also consider the $\mu$ representation for this model

$$\mathcal{I}(B_1) = -2B_1.$$  \hspace{1cm} (3.17)

Some other $U(1)$ examples considered in \[11, 12\] also admit extensions to the $U(N)$ duality case. For instance, we can choose the interaction

$$\mathcal{I}(B_1) = 2 \ln(1 - B_1) = -2(B_1 + \frac{1}{2} B_1^2 + \frac{1}{2} B_1^3 + \ldots ), \quad \mathcal{I}_1 = \frac{2}{B_1 - 1}$$  \hspace{1cm} (3.18)

and construct the self-dual Lagrangian in the $F$-representation, using the perturbative solution of Eq. (2.64) specialized to this case.

4 Interactions with higher derivatives

The generalized $U(N)$ self-dual Lagrangians with higher derivatives \[16\] in the formulation through bispinor auxiliary fields are constructed in the close analogy with the $U(1)$ case \[12\]. They involve the same bilinear term (2.13) and the $U(N)$ invariant interaction $\mathcal{E}^K_{der}$ containing derivatives of the auxiliary fields

$$\mathcal{L}_{der}(F^k, V^k) = \mathcal{L}_2(F^k, V^k) + \mathcal{E}^K(V, \partial V, \partial^2 V, \ldots \partial^K V),$$  \hspace{1cm} (4.1)

where $K$ denotes the maximal total degree of derivatives. Terms with derivatives in $\mathcal{E}^K$ contain the coupling constant $c$ of dimension $-2$ and additional dimensionless coupling constants. This generalized Lagrangian admits the same GZ-representation (2.39).
The \( U(N) \)-covariant local equations of motion for the auxiliary fields in this case contain the Lagrangian derivative of the invariant auxiliary interaction

\[
(V^k - F^k)_{\alpha\beta} + \frac{1}{2} \frac{\Delta \xi^K}{\Delta V^{k\alpha\beta}} = 0,
\]

where we consider the Lagrange derivative

\[
\frac{\Delta \xi^K}{\Delta V^k} = \frac{\partial \xi^K}{\partial V^k} - \frac{\partial m}{\partial \partial_m \partial n \partial \xi^K} + \frac{\partial \xi^K}{\partial \partial_m \partial n V^k} + \ldots.
\]

The equivalent twisted self-duality relations for higher-derivative theories were postulated in \([8, 16]\). The consistency condition

\[
(P^k P^l) + (F^k F^l) = 2(V^k V^l) + 2(F^l V^k) - 4(V^k V^l) = \left( V^l \frac{\Delta \xi^K}{\Delta V^k} \right) + \left( V^k \frac{\Delta \xi^K}{\Delta V^l} \right)
\]

\[
= \left[ \left( V^k \frac{\partial \xi^K}{\partial V^l} \right) \right] + \left[ \left( \frac{\partial m}{\partial \partial_m \partial n \partial \xi^K} \right) \right] + \left[ \left( \frac{\partial \xi^K}{\partial \partial_m \partial n V^k} \right) \right] + \ldots + [k \leftrightarrow l]
\]

+ total derivatives

is evidently valid for the derivative interaction \( \xi^K \). Then the \( \eta^{kl} \) invariance of the interaction \( \xi^K \) is equivalent to the following integral form of the self-duality condition

\[
\eta^{kl} \int d^4x [(P^k P^l) + (F^k F^l) - (\bar{P}^k \bar{P}^l) - (\bar{F}^k \bar{F}^l)] = \int d^4x [\delta_\eta \xi^K + \text{derivatives}] = 0. \quad (4.4)
\]

The \( O(N) \) \( \xi \)-invariance of \( \mathcal{L}_{\text{der}}(F^k, V^k) \) is manifest, as in the case without derivatives.

Solving Eq. (4.2), we obtain the perturbative solution \( V^k_{\alpha\beta}(F^k) \), which involves both the field strengths and their derivatives.

The simplest bilinear invariant with two derivatives

\[
\mathcal{E}(V, \partial V) = c a_1 \partial^\beta \bar{V}^{k\alpha\beta} \partial^\alpha \partial^k \bar{V}^k \xi
\]

makes the fields \( V \) and \( \bar{V} \) propagating.

An example of the nonlinear interaction with two derivatives (still with the standard bilinear terms) corresponds to the choice

\[
\mathcal{E}^2 = b_1 c \partial^m (V^k V^l) \partial_m (\bar{V}^k \bar{V}^l).
\]

The basic auxiliary equation in this case reads

\[
F^k_{\alpha\beta} = V^l_{\alpha\beta} [\delta^{kl} - c b_1 \Box (V^k V^l)]
\]

and it can be recursively solved for \( V^l_{\alpha\beta} \) in terms of \( F^k_{\alpha\beta} \) and its derivatives. The corresponding Lagrangian in the \( F \)-representation is given by the formal series

\[
L = -\frac{1}{2} (\varphi^{kk} + \bar{\varphi}^{kk}) + c b_1 \partial^m \varphi^{kl} \partial_m \varphi^{kl} - c^2 b_1^2 (\varphi^{kl} \Box \varphi^{lr} \Box \varphi^{rk} + \varphi^{kl} \Box \varphi^{lr} \Box \varphi^{rk}) + O(c^3). \quad (4.8)
\]
The number of derivatives increases with each recursion.

An example of the $U(N)$ invariant auxiliary interaction involving four derivatives is

$$E^4 = gc^2(\partial^m V^k \partial^n V^l)(\partial_m \bar{V}^k \partial_n \bar{V}^l), \tag{4.9}$$

where $g$ is a dimensionless coupling constant. The corresponding auxiliary field equation reads

$$F^k_{\alpha \beta} = V^k_{\alpha \beta} - 2gc^2 \partial^m [\partial^l V^l_{\alpha \beta}(\partial_m \bar{V}^k \partial_n \bar{V}^l)]. \tag{4.10}$$

It is not difficult to recursively solve this equation too and to construct the corresponding self-dual Lagrangian with higher derivatives.

All these examples are $U(N)$ generalizations of the $U(1)$ self-dual Lagrangians with higher derivatives presented in [12]. Like in the case without higher derivatives, the set of inequivalent $U(N)$ duality-invariant models of this kind is much richer compared to their $U(1)$ prototypes due to the proliferation of the Maxwell field strengths and the associated auxiliary tensorial fields.

5 Conclusions

In this paper, we further elaborated on the formalism with the bispinor (tensor) auxiliary fields for the $U(N)$ self-dual abelian gauge theories initiated in [9]. The general Lagrangian of the $U(N)$ self-dual model is parametrized by the invariant interaction of the auxiliary fields. The $U(N)$ covariant local twisted self-duality condition arises in this formulation as the equation of motion for the bispinor auxiliary fields. As compared to [9], we presented an alternative formulation of the $U(N)$ self-dual theories which makes use of the matrix scalar auxiliary fields in parallel with the bispinor ones, discussed a few new examples and showed how to generate $U(N)$ self-dual theories with higher derivatives in the considered setting.

In a recent paper [17] we gave basic elements of $\mathcal{N} = 1$ supersymmetric generalization of the $U(N)$ self-dual bosonic actions, using the auxiliary chiral superfields. The $\mathcal{N} = 1$ supersymmetrization of the bispinor auxiliary field formalism for the $U(1)$ case, through enhancing this field to an auxiliary chiral $\mathcal{N} = 1$ superfield, was earlier accomplished by Kuzenko [18] within a more general framework of the superfield $\mathcal{N} = 1$ supergravity.

The $U(N)$ self-duality formulations detailed here seem to admit rather straightforward supersymmetric extensions (with both rigid and local supersymmetries) along the lines of these works.

We also note that the bispinor auxiliary field formulation can be set up as well for self-dual abelian gauge theories in the $d = 4$ Euclidean space and the space with the signature $(2, 2)$. The complex fields $V^i_{\alpha \beta}$ and $\bar{V}^i_{\dot{\alpha} \dot{\beta}}$ are substituted by two sets of real independent fields transforming as $SO_L(3)$ and $SO_R(3)$ vectors in the Euclidean case, or as $SO_L(1, 2)$ and $SO_R(1, 2)$ vectors for the signature $(2, 2)$. The relevant duality group

\footnote{In [18], $\mathcal{N} = 2$ supersymmetrization was also considered.}
is non-compact, and it is the general linear group $GL(N)$ (it is reduced to dilatations $L(1) \sim SO(1,1)$ in the $N = 1$ case).

Finally, it is worthwhile to mention that the notion of self-duality can be defined for theories with $p$-form gauge fields, not only for $p = 1$, and in diverse dimensions, not only for $d = 4$ (see, e.g., [3] and references therein). It would be tempting to introduce the appropriate tensorial auxiliary fields for this web of generalized self-dualities and to see how they could help in constructing the relevant actions and understanding the interrelations between various types of such dualities \[3\].

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**A. Spinor and tensor notations in self-dual theories**

Our bispinor $U(1)$ formalism translated to the tensor notation was considered in [12]. Here we present the basic formulas of the tensor reformulation of this approach for the case of $U(N)$ self-dual theories.

The vectors in the spinor and tensor notations are related as

\[ A^k = (\sigma^m)_{\alpha\dot{\beta}} A^k_m, \quad (A.1) \]

where $k = 1, 2, \ldots, N$. The same correspondence for $N$ abelian field strengths is given by the relations

\[ F^k_{\alpha\beta}(A) = \frac{1}{4} (\sigma^m \sigma^n)_{\alpha\beta} F^k_m = \frac{1}{8} (\sigma^m \sigma^n - \sigma^n \sigma^m)_{\alpha\beta} F^{+k}_m, \quad (A.2) \]

\[ \bar{F}^{k\dot{\alpha}}_{\dot{\beta}} = \frac{1}{4} (\tilde{\sigma}^n)_{\dot{\beta}\dot{\alpha}} (\sigma^m)_{\beta\alpha} F^k_m = -\frac{1}{8} (\sigma^m \sigma^n - \sigma^n \sigma^m)_{\dot{\beta}\dot{\alpha}} \bar{F}^{-k}_m, \quad (A.3) \]

where

\[ F^k_{mn} = \partial_m A^k_n - \partial_n A^k_m, \quad \bar{F}^k_{mn} = \frac{1}{2} \varepsilon_{mrs} F^{krs}_m, \quad F^{+k}_{mn} = \frac{1}{2} F^k_{mn} + \frac{i}{2} \bar{F}^k_{mn}, \quad (A.4) \]

\[ \bar{F}^{+k}_{mn} = -i F^{+k}_{mn}, \quad F^{-k}_{mn} = \frac{1}{2} F^k_{mn} - i \bar{F}^k_{mn}, \quad \bar{F}^{-k}_{mn} = i F^{-k}_m. \quad (A.5) \]

Thus, $F^k_{\alpha\beta}$ is the equivalent bispinor notation for the self-dual tensor field $F^{+k}_{mn}$, and $\bar{F}^k_{\dot{\alpha}\dot{\beta}}$ amounts to the anti-self-dual tensor $F^{-k}_{mn}$.

---

3It was shown, e.g., in [20] that the $d = 4$ BI theory can be obtained by dimensional reduction from a self-dual theory in $d = 6$. 

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The scalar variables in the spinor formalism are related to the analogous variables in the tensor notation as
\[
\varphi^{k\ell} = F^{k\alpha\beta}F_{\alpha\beta}^\ell = t^{k\ell} + iz^{k\ell} = \frac{1}{2}(F^{k-}F^{+}\ell),
\]
\[
\bar{\varphi}^{k\ell} = \bar{F}^{k\dot{\alpha}\dot{\beta}}\bar{F}_{\dot{\alpha}\dot{\beta}}^\ell = t^{k\ell} - iz^{k\ell} = \frac{1}{2}(F^{-k}F^{-\ell}).
\] (A.6)

The bispinor and tensor representations of the dual field strengths appearing in the nonlinear equations of motion are related as
\[
\bar{P}^{k\beta}_{\alpha}(F) = \frac{1}{8}(\sigma^m\sigma^n - \sigma^n\sigma^m)\beta_g^{km}\sigma_{mn}, \quad \bar{F}^{k\dot{\beta}}_{\dot{\alpha}} = -\frac{1}{8}(\sigma^m\sigma^n - \sigma^n\sigma^m)\beta_g^{-km},
\] (A.7)
\[
G^{\pm k}_{mn} = \frac{1}{2}a^{k}_{mn} \pm \frac{i}{2}a^{k}_{mn}, \quad \bar{a}_{mn}^k = 2 \frac{\Delta L}{\Delta F^k_{mn}},
\] (A.8)
where we employed the Lagrange derivative.

The similar relations are valid for the auxiliary fields
\[
V^{k\beta}_{\alpha} = \frac{1}{8}(\sigma^m\sigma^n - \sigma^n\sigma^m)\beta_g^{km}\sigma_{mn}V^{k\ell}_{mn}, \quad \bar{V}^{k\dot{\beta}}_{\dot{\alpha}} = -\frac{1}{8}(\sigma^m\sigma^n - \sigma^n\sigma^m)\beta_g^{-km}V^{-k}_{mn}.
\] (A.9)

The scalar variable \(\nu\) can be expressed as
\[
\nu^{k\ell} = V^{k\alpha\beta}V_{\alpha\beta}^\ell = \frac{1}{2}(V^{+k}V^{+\ell}), \quad \bar{\nu}^{k\ell} = \bar{V}^{k\dot{\alpha}\dot{\beta}}\bar{V}_{\dot{\alpha}\dot{\beta}}^\ell = \frac{1}{2}(V^{-k}V^{-\ell}).
\] (A.10)

The one-to-one correspondence between the specific variables used in [8, 15, 16] and our variables in the tensor notation is as follows:\footnote{Sometimes, for brevity, we omit the antisymmetric tensor indices.}
\[
T^k = F^k - i\sigma^k, \quad \bar{T}^k = \bar{F}^k - i\sigma^k,
\]
\[
T^{*k} = F^k + i\sigma^k, \quad \bar{T}^{*k} = \bar{F}^k + i\sigma^k,
\]
\[
\delta_\omega T^k = i\omega^{kl}T^l, \quad \delta_\omega \bar{T}^k = i\omega^{kl}\bar{T}^l, \quad \delta_\omega T^{*k} = -i\omega^{kl}T^{*l}, \quad \delta_\omega \bar{T}^{*k} = -i\omega^{kl}\bar{T}^{*l},
\] (A.11)
where \(\omega^{kl} = \xi^{kl} + i\eta^{kl}\) are the \(U(N)\) parameters. The relations between the self-dual (and anti-self-dual) parts of these two sets of complex variables can be listed as
\[
T^{+k} = \frac{1}{2}(T + i\bar{T})^k = (F - V)^k + i(F - \bar{V})^k = 2(F^{+k} - V^{+k}),
\] (A.13)
\[
T^{-k} = \frac{1}{2}(T - i\bar{T})^k = 2V^{-k},
\] (A.14)
\[
T^{+k} \equiv \bar{T}^{+k} = (T^{-})^{*k} = \frac{1}{2}(T^{*} + i\bar{T}^{*})^k = 2V^{+k},
\] (A.15)
\[
T^{-k} \equiv \bar{T}^{-k} = (T^{+})^{*k} = \frac{1}{2}(T^{*} - i\bar{T}^{*})^k = 2(F^{-k} - V^{-k}).
\] (A.16)

Being cast in the tensor notations, our Lagrangian \([2, 12]\) becomes:
\[
\mathcal{L}(F, V) = -\frac{1}{4}[(F^{k+}F^{+k}) + (F^{-k}F^{-k})] + \frac{1}{2}[(V^{+k} - F^{+k})^2 + (V^{-k} - F^{-k})^2]
\]
\[+ \mathcal{E}(A_n),\]
(A.17)
where the invariant interaction $\mathcal{E}$ is defined in terms of the matrix

$$a^{kl} = \frac{1}{4} (V^{-k} V^{-r}) (V^{+r} V^{+l}), \quad A_n = \frac{1}{n} \text{Tr}^A.$$

(A.18)

The tensor form of our algebraic equation of motion (2.31) reads

$$(F^+ - V^+)^k_{mn} = \frac{\partial \mathcal{E}}{\partial V^{+kmn}}.$$

(A.19)

After passing to the $T$-tensor notation according to Eqs. (A.14), (A.15), we can rewrite the same equation as

$$T^{+k}_{mn} = 4 \frac{\partial \mathcal{E}}{\partial T^{+kmn}},$$

(A.20)

that precisely coincides with the general twisted self-duality condition postulated in [8, 15].

Our interaction function $\mathcal{E}$ proves to be identical to the “deformation function” $\mathcal{I}^{(1)}$ of this approach. Note that the variables $T^k, T^{*k}$ are entirely equivalent to our variables on mass shell, when the auxiliary fields are traded for the Maxwell field strengths by their equations of motion, while off shell it is more convenient to deal with the variables $F^k, V^k$.

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