LISTENING FOR RINGING BLACK HOLES

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Abstract

The gravitational radiation produced by binary black holes during their inspiral, merger, and ringdown phases is a promising candidate for detection by the first or second generation of kilometer-scale interferometric gravitational wave antennas. Waveforms for the last phase, the quasinormal ringing, are well understood. I discuss the feasibility of detection of the quasinormal ringing of a black hole based on an analysis of the Caltech 40-meter interferometer data.

1. INTRODUCTION

Broad band interferometric gravitational wave observatories such as LIGO and VIRGO will soon be in operation. In order to analyze the data produced by these instruments, one needs to know what kind of gravitational wave signals may be present. Since these signals are expected to be very weak, the signal detection process requires that one know the form of the signals very accurately in order to distinguish them from the noise in the instrument. Additionally, one needs to understand the characteristics of the noise itself. I describe a potentially interesting source of gravitational radiation—the “ringdown” of a perturbed black hole—and discuss some of the difficulties in searching for this radiation.

In section 2, I describe what a black hole ringdown is and why it is potentially observable. In section 3, I describe a detection strategy that may be used to search for such a waveform in an interferometer output. My discussion of the statistics of the detection strategy involves assumptions about the noise present in the interferometer. The application of the detection strategy to real interferometer data obtained from the Caltech 40-meter prototype is described in section 4. I find that the observed receiver statistic has a different distribution than the expected distribution found in section 3. I give a brief discussion of the implications of this result in the final section.
2. BLACK HOLE RINGDOWN

2.1. Phases of black hole binary evolution

Black hole binaries will provide an important source of gravitational radiation for detection by LIGO and VIRGO. The detectable radiation produced by these systems occurs in three phases. The first phase is the late stages of the binary inspiral where the binary system loses energy due to gravitational radiation reaction, and the companions spiral towards each other in a quasi-stationary process. Eventually, the orbit of the companions becomes unstable, and the two black holes will plunge into each other; this merger is the second phase. The two black holes combine to form a single distorted event horizon, and the distortion is dissipated into gravitational radiation. In time, the distortion will be well described by linear perturbations of the final equilibrium black hole spacetime; such perturbations are called the quasinormal modes of the final black hole. These quasinormal modes are decaying modes: the radiation emitted from the modes is called black hole ringdown—the third phase of gravitational wave emission from the binary coalescence.

The relative importance of these three phases has been discussed in detail by Hughes and Flanagan [1]. Intermediate mass binaries with system masses of hundreds of solar masses will produce most of their observable signal in the merger and ringdown phases. Merger waveforms are not yet known, so the best way to detect such systems might be to look for the ringdown signature. Lower mass binaries will be best detected by the radiation produced by the inspiral phase; however, the very late stages of the inspiral waveforms are not yet known reliably because of the breakdown of the post-Newtonian analysis when the system enters a regime of strong gravity and high velocity motion. Thus, the information gained from the ringdown phase may be useful for binaries where the total mass is as low as fifty solar masses. Such binary systems could be the first sources detected by LIGO and VIRGO: the large system mass means that these sources will be relatively bright, which compensates for the lower event rate of black hole binary mergers compared to neutron star binary mergers in a given volume of space [2]. In addition, the search strategy I adopt here can be used in analyzing LISA data in search of gravitational waves arising from the ringdown of perturbed supermassive black holes.

2.2. The ringdown phase

The quasinormal modes of a Kerr black hole are eigenfunctions of the Teukolsky equation—which describes linear perturbations of the curvature of the Kerr spacetime—with boundary conditions corresponding to purely ingoing radiation at the event horizon and purely outgoing radiation at large distances from the black hole. The perturbation of the curvature of Kerr spacetime can be described by the components $\Psi_0$ and $\Psi_4$ of the Weyl tensor; of particular interest is $\Psi_4$ since it describes outgoing waves in the radiative zone. The $\Psi_4$ component is a function of the radius $r$, inclination $\mu = \cos \theta$, and azimuth $\phi$ of the observer; it also depends on the mass $M$ and specific angular momentum $a$ of the perturbed Kerr black hole. I often refer to the dimensionless angular momentum parameter $\hat{a} = a/M$, which must be between zero (Schwarzschild limit) and unity (extreme Kerr limit). Teukolsky [3] was able to separate the Einstein equations, linearized about Kerr spacetime, to obtain solutions of the form $\Psi_4 = (r - i\mu a)^{-4}e^{-i\omega t} - 2R_{\ell m}(r) - 2S_{\ell m}(\mu) e^{i\omega t}$ where $-2R_{\ell m}(r)$ is a solution to a radial (ordinary) differential equation, and $-2S_{\ell m}(\mu)$ is a spin-weighted spheroidal
wave function. The perturbation has the spheroidal eigenvalues $\ell$ and $m$ and a complex frequency $\omega$.

When the correct boundary conditions are imposed, the quasinormal modes of the black hole are found. These modes have a spectrum of complex eigenfrequencies $\omega_n$. I shall only consider the fundamental ($n = 0$) mode because the harmonics of this mode have shorter lifetimes; for the same reason, I shall only consider the quadrupole ($\ell = 2$ and $m = 2$) mode because this mode is the longest lived. I assume that the observer is at a large distance from the black hole; in this case one can approximate the perturbation as

$$
\Psi_4 \approx \frac{A}{r} e^{-i\omega t_{\text{ret}}} - 2S_{22}(\mu) e^{2i\phi}.
$$

Here, $t_{\text{ret}} = t - r^* \times$ is the retarded time, where $r^*$ is the “tortoise” radial coordinate. The parameter $A$ represents the amplitude of the perturbation. The eigenfrequency spectrum can be computed for any mode ($\ell, m, n$): it depends on the mass and angular momentum of the black hole. Because the eigenfrequency is complex, the perturbation corresponds to an exponentially damped sinusoid with a central frequency $f = \frac{2\pi \Re \omega}{\omega}$ and a quality $Q = -\frac{1}{2\Re \omega/\Im \omega}$. For the fundamental quadrupolar mode, the eigenfrequency is well approximated by the analytic form found by Echeverria [4]:

$$
f \approx 32 \text{ kHz} \times [1 - 0.63(1 - \hat{a})^{3/10}] \left(\frac{M_\odot}{M}\right),
$$

$$
Q \approx 2(1 - \hat{a})^{-9/20}.
$$

For a Kerr black hole with $M = 50M_\odot$ and $\hat{a} = 0.98$, $f \approx 515$ Hz and $Q \approx 12$.

### 2.3. Gravitational radiation

From the curvature perturbation $\Psi_4$, one can extract useful physical quantities. The first quantity is the gravitational strain of the radiation. The “$+$” and “$\times$” polarizations of the strain induced by the gravitational radiation are found from

$$
h_+ - ih_\times = \frac{2\Psi_4}{|\omega|^2}.
$$

The quantity $h_+ = h_{\hat{\theta}\hat{\theta}} = h_{\hat{\phi}\hat{\phi}}$ is the metric perturbation representing the linear polarization state along the unit vectors $\hat{e}_\hat{\theta}$ and $\hat{e}_\hat{\phi}$, and the quantity $h_\times = h_{\hat{\theta}\hat{\phi}}$ is the metric perturbation representing the linear polarization state along $\hat{e}_\hat{\theta} \pm i\hat{e}_\hat{\phi}$.

The second useful quantity that can be obtained from $\Psi_4$ is the power radiated (per unit solid angle) towards the observer:

$$
\frac{d^2E}{dt d\Omega} = \lim_{r \to \infty} \frac{r^2 |\Psi_4|^2}{4\pi |\omega|^2}.
$$

Given equation (1) for the gravitational perturbation in the far field zone, we can integrate equation (5) over the entire sphere and the interval $0 \leq t_{\text{ret}} < \infty$ to obtain an expression for the total energy radiated as a function of the perturbation amplitude. It will be useful to characterize the amplitude of the perturbation in terms of the fractional mass loss in the perturbation $\epsilon = E/M$.

I now translate the metric perturbation of a quasinormal mode into an experimentally more useful quantity: the strain produced in an interferometric gravitational wave antenna. This strain is given by $h(t_{\text{ret}}) = F_+ h_+(t_{\text{ret}}) + F_\times h_\times(t_{\text{ret}})$ where $F_+$ and $F_\times$ are the antenna response patterns of the interferometer. These response patterns depend on the the altitude and azimuth of the source of the
radiation as well as on the angle of polarization of the radiation. The exact form of these patterns can be found in reference [5]. For a source at a given distance, the “typical” waveform can be obtained by rms averaging over these angles as well as over the inclination of the source and azimuth of the perturbation. When this is done, one finds

\[
    h_{\text{ave}}(t_{\text{ret}}) \approx 6.825 \times 10^{-21} \eta(\hat{a}) \left( \frac{\text{Mpc}}{r} \right) \left( \frac{M}{M_\odot} \right) \left( \frac{\epsilon}{0.01} \right)^{1/2} \times e^{-\pi f t_{\text{ret}}/Q} \cos(2\pi f t_{\text{ret}} + \psi_0). \tag{6}
\]

where \(\psi_0\) is the initial phase of the waveform and \(\eta(\hat{a})\) is an efficiency factor which monotonically decreases from unity at \(\hat{a} = 0\) to zero at \(\hat{a} = 1\) (note that for large values of \(\hat{a}\), \(Q\) is also large and the duration of the ringdown is long). The efficiency remains relatively large until the black hole has nearly extreme spin; even for a value of \(\hat{a} = 0.98\), the efficiency is reasonably high with \(\eta(0.98) \approx 0.29\).

A Galactic black hole ringdown with mass \(M = 50 M_\odot\), spin \(\hat{a} = 0.98\), fractional mass loss of \(\epsilon = 1\%\) and distance of 10 kpc should be easily detected by the Caltech 40-meter prototype interferometer.

3. SIGNAL DETECTION

In order to detect the ringdown from a black hole, one must pass the interferometer output through a receiver that will perform a test of the hypotheses “there is a signal present in the data” and “there is no signal present in the data.” In order for the receiver to conclude that there is a signal present, it constructs some statistic and compares the statistic to a pre-assigned threshold. The problem of reception, then, is two-fold: one must find an optimal statistic, and one must select some threshold. My exploration of these problems (below) follows the method presented in reference [6].

In designing the optimal statistic, it is customary to assume that the noise in the detector is stationary and Gaussian. These assumptions simplify the statistical analysis. I shall also make these assumptions about the noise in this section; however, they are known to be poor assumptions in the case of the Caltech 40-meter prototype interferometer. The effect of the non-stationary and non-Gaussian noise components will be evident in the observations presented in section [6].

3.1. The optimal filter

Suppose that the detector output, \(h\), contains either noise alone, \(h = n\), or both a signal and noise, \(h = s + n\). The optimal receiver (i.e., the optimal data analysis process) is one that returns the quantity \(P(s \mid h)\): the probability of a signal being present given the output. Using Bayes’ law, this probability can be expressed in terms of the a posteriori probabilities of obtaining the output given that a signal is or is not present, \(P(h \mid s)\) and \(P(h \mid \neg s)\), and the a priori probability of a signal being present \(P(s)\) and its converse \(P(\neg s) = 1 - P(s)\). One finds

\[
    \frac{P(s \mid h)}{P(\neg s \mid h)} = \frac{P(h \mid s)}{P(h \mid \neg s)} \frac{P(s)}{P(\neg s)} = \Lambda \frac{P(s)}{P(\neg s)}. \tag{7}
\]

In general, there is no universal way of evaluating the a priori probabilities \(P(s)\) and \(P(\neg s)\), so one often adopts the maximum likelihood receiver which returns the likelihood ratio \(\Lambda = P(h \mid s)/P(h \mid \neg s)\). Notice that as \(\Lambda\) grows larger, the
probability of a signal increases, so the maximum likelihood receiver can be used to test the hypotheses as follows: If $\Lambda$ is greater than some threshold $\Lambda_*$ then one decides that there is a signal present; otherwise, one decides that there is no signal present. Lacking any *a priori* information about whether there is a signal present, the threshold should be chosen by setting a desired probability for a false alarm and/or a false dismissal; these probabilities are computed in subsection 3.2 below.

Consider the case in which one is searching for a ringdown waveform of some fixed frequency and quality, so it has an exactly known form. Assume that the noise samples are drawn from a stationary Gaussian distribution with correlations amongst the noise events (coloured noise). The noise correlations can be expressed in terms of the one-sided noise power spectrum,

$$\frac{1}{2} S_h(|f|) \delta(f - f') = \langle \tilde{n}(f) \tilde{n}^*(f') \rangle,$$

where $\tilde{n}(f)$ is the Fourier transform of the noise $n(t)$, and $\ast$ denotes complex conjugation. Because the noise is Gaussian, the probability of obtaining an instance of noise, $n(t)$, is $P(n) \propto \exp[-\frac{1}{2}(n | n)]$, where the inner product $(\cdot | \cdot)$ is defined by

$$ (a | b) = \int_{-\infty}^{\infty} df \frac{\tilde{a}^*(f) \tilde{b}(f) + \tilde{a}(f) \tilde{b}^*(f)}{S_h(|f|)}. \quad (8) $$

If, however, a signal is present, then $h(t) = As(t) + n(t)$ where I have assumed that the signal $s(t)$ is normalized to some fiducial distance so that the amplitude $A$ represents the inverse distance of the source relative to this fiducial distance. The likelihood ratio is the ratio of the probabilities $P(h | As) = p(h - As)$ and $P(h | -s) = p(h)$:

$$ \Lambda = e^{Ax - A^2 \sigma^2 / 2} \quad (9) $$

where $x = (h | s)$ and $\sigma^2 = (s | s)$. Notice that the likelihood ratio is a monotonically increasing function of $x$ and that the output $h$ appears only in the construction of $x$; therefore, one can set a threshold on the value of $x$ obtained rather than on the likelihood ratio. In fact, it will be useful to consider the signal-to-noise ratio, which I define as $\rho = |x|/\sigma$ (the absolute value is taken because it is not known whether the signal has a positive or a negative amplitude).

### 3.2. Properties of the optimal filter

It is straightforward to compute false alarm and false dismissal probabilities for any choice of threshold and signal amplitude. Suppose that a threshold $\rho_*$ for the signal-to-noise ratio is chosen. Then the false alarm probability is the probability that $\rho \geq \rho_*$ when no signal is present:

$$ P(\text{false alarm}) = P(\rho \geq \rho_* \ | -s) = \text{erfc}(\rho_* / \sqrt{2}) \quad (10) $$

where the complementary error function is defined by $\text{erfc}(x) = (2 / \sqrt{\pi}) \int_{x}^{\infty} e^{-t^2} dt$.

When a signal is present with amplitude $A$, then the converse of the false dismissal probability is the probability of a true detection:

$$ P(\text{true detection}) = P(\rho \geq \rho_* \ | As) = \frac{1}{2} \text{erfc}[(\rho_* - A\sigma) / \sqrt{2}] + \frac{1}{2} \text{erfc}[(\rho_* + A\sigma) / \sqrt{2}]. \quad (11) $$

Notice that the signal shifts the signal-to-noise probability distribution by $A\sigma$. Using these equations, one can compute the threshold required for a choice of false alarm probability or false dismissal probability.

In the above discussion, I have made the implicit assumption that we know the arrival time of the signal. Since this will not be known in general, it is necessary
to obtain the signal-to-noise ratio maximized over all possible signal arrival times. This can be done simply by replacing the single value of $x$ used above by the time series obtained by the correlation

$$x(t) = \int_{-\infty}^{\infty} df e^{-2\pi i ft} \frac{\tilde{a}^*(f) \tilde{b}(f) + \tilde{a}(f) \tilde{b}^*(f)}{S_h(|f|)}.$$ \hspace{1cm} (12)

The signal-to-noise ratio is then constructed by finding the maximum absolute value of the time series $x(t)$:

$$\rho = \sigma^{-1} \max_t |x(t)|.$$

Unfortunately, the determination of the false alarm and true detection probabilities are greatly complicated because of correlations that are present in the time series $x(t)$. An overestimate of the false alarm probability can be made by assuming that $x(t)$ and $x(t+\Delta)$ are independent where $\Delta^{-1}$ is the sampling rate. In an observation time $T$ consisting of $N = T\Delta^{-1}$ samples, the probability of a false alarm is approximately

$$P(\text{false alarm}) \approx N \text{erfc}(\rho_* / \sqrt{2}) \quad (\rho_* \gg 1).$$ \hspace{1cm} (13)

for sufficiently short observation times (so that $P(\text{false alarm}) \ll 1$). In order to calculate the actual false alarm rate, it is necessary to use a Monte Carlo analysis in which a large number of noise samples is simulated, and the fractional number of samples in which the signal-to-noise ratio exceeds a threshold is calculated.

Since the different possible waveforms depend on the mass and spin of the black hole (or equivalently on the central frequency and the quality of the damped sinusoid), and since these parameters are continuous, it is necessary to discretize the waveforms to form a “mesh” that covers the parameter space sufficiently finely. By “sufficiently finely,” I mean that the degradation in the signal-to-noise ratio due to having a filter with slightly incorrect parameters should be small. The number of templates that will be needed to search for all ringdown waveforms of interest with very little loss of signal-to-noise ratio will be a few thousand for the Caltech 40-meter prototype: comparable to the number needed for the binary inspiral searches. For simplicity, I assume hereafter that all parameters of the signal, apart from its time of arrival, are known.

4. OBSERVATIONS

Having reviewed a possible detection strategy, and derived the expected distribution of the detection statistic in the presence of stationary Gaussian noise, I now examine the results of applying the detection strategy to real interferometer data. In November of 1994, the Caltech 40-meter prototype interferometer was used to collect approximately 46 hours of data. In my analysis of this data, I implemented the detection strategy discussed in the previous section (for a single filter only) using routines which are provided in the GRASP data analysis software package. The single filter I used corresponded to the fundamental quadrupole quasi-normal mode of a Kerr black hole with a mass $M = 50M_\odot$ and a spin $\hat{a} = 98\%$ of the extreme spin. This mode is a damped sinusoid with a central frequency of $f \simeq 510$ Hz and a quality of $Q \simeq 12$. The central frequency of the filter is within frequency band of the instrument: between approximately 300 and 3000 Hz. The filter was cutoff when the waveform was attenuated by 30 dB in amplitude; the filter was about 30 ms in duration. The sampling rate of the interferometer was $\Delta^{-1} \simeq 9.868$ kHz. The data were analyzed in segments of $2^{16}$ points, of which
The observed false alarm distributions from the November 1994 data run of the Caltech 40-meter prototype interferometer with segments lengths of 57,344 points (∼5.8 s). The “observed” curve was obtained from the actual interferometer data, while the “simulated” curve is the expected false alarm rate for stationary Gaussian noise. The solid “theoretical” curve corresponds to the theoretical false alarm distribution, equation (13), under the assumption that all points in the correlation are independent.

The first and last $2^{12}$ points of the correlation were discarded in order to remove the effects of wrap-around from the numerical correlation algorithm. Thus, each data segment corresponded to approximately 5.8 s of actual data or 57,344 points. Only 21 h (about 13,000 segments) of data were used: these were the segments in which the instrument was in lock and “well behaved” in the sense that there were no outlier data points of more than five times the sample standard deviation for the segment.

Ringdowns produced by black hole mergers within our Galaxy may have sufficient brightness to be detected by the prototype interferometer: a ringdown with the same parameters as the above filter occurring near the Galactic centre (10 kpc) that radiates 1% of its mass would be seen with a signal-to-noise ratio of about 50. However, such events would be extremely rare, so I assume that the data obtained are representative of interferometer noise alone. It is important to understand the properties of the noise in developing data analysis techniques. In particular, it is interesting to see how well the assumptions that the noise is stationary and Gaussian made in the previous section apply. To this end, I estimate the false alarm probability as a function of the threshold applied to the segments of 57,344 points based on: (a) the signal-to-noise ratio output for all of the segments analyzed, and (b) the signal-to-noise ratio output for segments generated using simulated stationary Gaussian noise with the prototype interferometer power spectrum.

These two curves are presented in figure 1. In addition to these curves, figure 1 includes the false alarm distribution (solid line) expected under the assumption that all of the 57,344 points in the correlation are independent given by equation (13). Notice that the expected false alarm distribution (b) lies just to the left of the solid line corresponding to equation (13), as expected. However, the observed false alarm distribution differs greatly from the expected curve: there are an excess of high signal-to-noise ratio events produced by the non-stationary and non-Gaussian noise components.
5. DISCUSSION

Black hole ringdown waveforms will be an important potentially observable source of gravitational radiation for the interferometric gravitational wave detectors now being constructed. However, the data analysis techniques to be used to detect a black hole ringdown are just now being developed. I have implemented the optimal filter technique to search for a single black hole ringdown waveform—corresponding to a black hole with mass $M = 50\ M_\odot$ and spin $\dot{\theta} = 98\%$ of the extreme spin—in the data obtained from the Caltech 40-meter prototype interferometer in November 1994. I find that there is an excess of high signal-to-noise ratio events, even when the obviously poor data segments (those which fail an outlier test) are excluded.

The results of the previous section indicate that the detector noise cannot be approximated as a stationary Gaussian process for the purposes of false alarm rate calculations. However, there may be some method for improving the data analysis technique that will improve the statistical properties of the observed signal-to-noise ratios. Of particular concern is the present technique for estimating the power spectrum $S_h(\nu)$ when the noise is non-stationary. In addition, there may be improved methods for rejecting particularly noisy segments. Such methods for improving the data analysis technique are presently under investigation.

It is not surprising that the false alarm rate should be higher than anticipated from the assumption of stationary Gaussian noise: it is known that the prototype interferometer data contains many instrumental effects (e.g., servo-mechanism excitations, etc.), which efficiently trigger ringdown filters. In order to reduce the number of false alarms, it will be necessary to develop a set of tests to discriminate between instrumental effects and actual signals of astronomical origin. The simplest discriminant would be to reject signals found in a single detector but not in other detectors in operation at that time. More elaborate tests will likely be needed in order to successfully veto all the non-stationary instrumental effects; such tests are under development.

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