Matching on a line

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Abstract: Matching is a method of the design of experiments. If we had an even number of patients and wanted to form pairs of patients such that their ages, for example, in each pair be as close as possible, we would use nonbipartite matching. Not only do we present a fast method to do this, we also extend our approach to triples, quadruples, etc.

In part 1 a matching algorithm uses \(kn\) points on a line as vertices, pairs of vertices as edges, and either absolute values of differences or the squares of differences as weights or distances. It forms \(n\) of \(k\)-tuples with the minimal sum of distances within each \(k\)-tuple in \(O(n \log n)\) time.

In part 2 we present a trivial algorithm for bipartite matching with absolute values or squares of differences as weights and a generalisation to tripartite matching on tripartite graphs.

Introduction

Further references about the use of nonbipartite matching in experimental design are in survey papers Beck (2015) or Lu (2011). Imagine we have something like 300 patients and we want to form pairs of patients such that the ages, as an example of a confounding variable, of the patients in each pair are as close as possible. Our goal is to make applications of treatment A and treatment B comparable. In our paper we also show how to form triples that are convenient for an application of placebo, treatment A, or treatment B. Quadruples may be used for an application of placebo, treatment A, treatment B, and interaction of A and B combined.

We picked age as an example of a trivial confounding variable but there are sophisticated ways of defining such a variable, propensity score being one of them.

Matching is used when we want to avoid the effect of a confounding variable. If there are more such variables, it is customary to aggregate them to get just one variable typically called a scale or score. Applications vary in the fields of medicine, social sciences, psychology, or education.

Should we want to use an \(n\)-dimensional space for \(n\) confounding variables we would have to multiply each of these variables by some constant to take care of their importance, units, etc, and we would have to derive those constants somehow only to find out that scores are a better choice.

The repeatability of matching is important because, unlike randomization, matching gives the same result each time it is repeated, save for ties.
In such a setting, each individual becomes a vertex of a simple complete graph and the weight of each edge is defined as the absolute value $A$ of the difference between their scores or it is defined as the square $S$ of this difference.

In part 1 we want to show that the calculation of nonbipartite matching becomes trivial. Also triangle matching, termed 3-matching, becomes an easy task and so does 4-matching, and generally $n$-matching so far for $n \leq 16$ in the case of absolute values of differences as weights or $n \leq 8$ when the sum of squares of differences is used.

We consider a complete simple graph $G$ with an even number $|V|$ of vertices $V$. The set of edges is denoted as $E$. Let $M$ be a subset of $E$. $M$ is called a matching if no edges in $M$ are adjacent in $G$. A matching $M$ is called perfect if each vertex in $V$ is incident to some edge in $M$. We assume a weight $w(e) \geq 0$ is attached to every edge $e \in E$. We are then looking for a perfect matching $M$ for which the sum of weights is maximal. Equivalently, we may look for a perfect matching with a minimal sum of weights by picking some upper bound $u$ of weights and form new weights as $u - w(e)$ for each $e \in E$.

The Edmonds method for finding a maximal weighted matching in a general weighted graph is presented in Papadimitriou (1982). The time necessary for calculation is polynomial, $O(|V|^3)$, in the number of vertices. Writing the program would be tedious but we recommend Beck (2015) or an internet address through which the solution may be obtained:

http://biostat.mc.vanderbilt.edu/wiki/Main/NonbipartiteMatching

Even though the running time is polynomial, the degree 3 may turn out to be too high for practical calculations for a large number of vertices $|V|$ requiring a large $|V|$ by $|V|$ matrix of distances.

If the number of vertices is divisible by three and a constant $B$ is given, the decision problem whether there is a perfect 3-matching such that the sum over all the triples of the distances between the three points in each triple is less than some constant $B$ is known to be $NP$-complete. We may refer to the problem exact cover by 3-sets in Garey (1979) or Papadimitriou (1982). That is why the problems we study seem to be so discouraging.

In the second part of the paper we show a trivial method of calculating a minimal perfect matching on a regular complete bipartite graph with weights on edges being absolute values of differences of weights on vertices. This is only a stepping stone to the design of a method of calculus of a minimal perfect matching on a regular complete tripartite graph with the same definition of weights.

**Part 1**

1.1. Matching on a line

We study a complete graph the vertices of which are points on a real $x$-axis. These points are denoted as $x_i$.

The edges are the line segments between these points. The weight of each edge $(x_i, x_j)$ is defined as the distance of its two endpoints $|x_i - x_j|$.

Since the possibility of repeated observations is common in statistics, we do not use sets, we use the notion of a $k$-tuple. We call $(1, 1, 2)$ a triple whereas a set would consist of two elements, 1 and 2.
Definition 1.1. We define a distance \( A \) within a \( k \)-tuple as the sum of the distances of all the pairs formed of the elements of the \( k \)-tuple.

\[
A(x_1, x_2, \ldots, x_k) = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} |x_j - x_i|.
\]

There are \( k(k-1)/2 \) summands in this formula. Our calculations will be simplified by the following.

Definition 1.2. A \( k \)-tuple \((x_1, x_2, \ldots, x_k)\) is sorted if \( x_1 \leq x_2 \leq \ldots \leq x_k \).

Theorem 1.1. If the \( k \)-tuple is sorted, the distance within the \( k \)-tuple is

\[
A(x_1, x_2, \ldots, x_k) = \sum_{i=1}^{k} (2i - k - 1)x_i
\]

Proof.

\[
A(x_1, x_2, \ldots, x_k) = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} (x_j - x_i) = \sum_{i=1}^{k-1} x_j - \sum_{i=1}^{k-1} x_i
\]

\[
= x_2 + 2x_3 + \ldots + (k-1)x_k - \sum_{i=1}^{k-1} (k-i)x_i = \sum_{i=1}^{k} (i-1)x_i - \sum_{i=1}^{k-1} (k-i)x_i
\]

\[
= \sum_{i=1}^{k} (2i - k - 1)x_i.
\]

For example, if a sorted pair \((x_1, x_2)\), \( x_1 \leq x_2 \), is given, the distance is defined as \( A(x_1, x_2) = x_2 - x_1 \). For a sorted triple \((x_1, x_2, x_3)\), \( x_1 \leq x_2 \leq x_3 \), we define the distance within as \( A(x_1, x_2, x_3) = 2(x_3 - x_1) \).

Definition 1.3. Let a \( kn \)-tuple be given. A partition of this \( kn \)-tuple into \( n \) of \( k \)-tuples is called a \( k \)-tuple partition of a \( kn \)-tuple.

Definition 1.4. Let a \( kn \)-tuple be given. A \( k \)-tuple partition of this \( kn \)-tuple is called minimal if the sum of the distances within taken over all the \( n \) of \( k \)-tuples is less than or equal to the sum of distances within taken over \( k \)-tuples of any other \( k \)-tuple partition.

We may assume there may be more than one minimal partition. For a \( kn \)-tuple there are \((kn)!/(k!)^n\) \( k \)-tuple partitions from which we want to find those with minimal sum of distances within \( k \)-tuples. They are too many for the brute force method to work for a large \( n \). But it will work for \( n = 2 \) if \( k \) is small.

The greedy method will not work either. That can be shown by way of an example \((1, 3, 4, 5, 8, 9)\) in which the triple with the smallest sum of distances within is \((3, 4, 5)\), \( A(3, 4, 5) = 4 \), the distance within the remaining items is \( A(1, 8, 9) = 16 \). The sum is \( A(3, 4, 5) + A(1, 8, 9) = 20 \). We get a smaller sum of distances within if we take \((1, 3, 4)\) and \((5, 8, 9)\) yielding \( A(1, 3, 4) + A(5, 8, 9) = 6 + 8 = 14 \).
First we want to show what a minimal partition for $2k$-tuples looks like. If we can do that, we will use induction to show it works for any $n > 2$.

**Theorem 1.2.** Let a sorted 4-tuple $(x_1, x_2, x_3, x_4)$ be given. Then the two pairs $(x_1, x_2)$ and $(x_3, x_4)$ have a minimal sum of distances defined as absolute values of differences.

**Proof.** The sum of distances of $(x_1, x_2)$ and $(x_3, x_4)$ is $x_2 - x_1 + x_4 - x_3$. We form other possible partitions into pairs, calculate the sum of their distances, and compare them with the sum of distances of $(x_1, x_2)$ and $(x_3, x_4)$.

Other possible sorted pairs are:

1) $(x_1, x_3)$ and $(x_2, x_4)$. The difference is

$$A(x_1, x_3) + A(x_2, x_4) - (A(x_1, x_2) + A(x_3, x_4)) = 2(x_3 - x_2) \geq 0.$$  

2) $(x_1, x_4)$ and $(x_2, x_3)$. The difference is

$$A(x_1, x_4) + A(x_2, x_3) - (A(x_1, x_2) + A(x_3, x_4)) = 2(x_3 - x_2) \geq 0.$$  

**Theorem 1.3.** Let a sorted 6-tuple $(x_1, x_2, \ldots, x_6)$ be given. Then the two sorted triples $(x_1, x_2, x_3)$ and $(x_4, x_5, x_6)$ have a minimal sum of distances within defined as the sum of absolute values of differences.

**Proof.** We form all the other possible triples, calculate the sum of their distances within, and subtract from them the sum of distances $A(x_1, x_2, x_3) + A(x_4, x_5, x_6)$.

Other possible triples are listed in such a way that $x_1$ appears in the first triple because the order does not matter in this case:

1) $(x_1, x_2, x_4)$ and $(x_3, x_5, x_6)$

$$A(x_1, x_2, x_4) + A(x_3, x_5, x_6) - A(x_1, x_2, x_3) - A(x_4, x_5, x_6) = 4(x_4 - x_3) \geq 0.$$  

2) $(x_1, x_2, x_5)$ and $(x_3, x_4, x_6)$

$$A(x_1, x_2, x_5) + A(x_3, x_4, x_6) - A(x_1, x_2, x_3) - A(x_4, x_5, x_6) = 2(x_5 + x_4 - 2x_3) \geq 0.$$  

3) $(x_1, x_2, x_6)$ and $(x_3, x_4, x_5)$

$$A(x_1, x_2, x_6) + A(x_3, x_4, x_5) - A(x_1, x_2, x_3) - A(x_4, x_5, x_6) = 2(x_5 + x_4 - 2x_3) \geq 0.$$  

4) $(x_1, x_3, x_4)$ and $(x_2, x_5, x_6)$

$$A(x_1, x_3, x_4) + A(x_2, x_5, x_6) - A(x_1, x_2, x_3) - A(x_4, x_5, x_6) = 2(2x_4 - x_3 - x_2) \geq 0.$$  

5) $(x_1, x_3, x_5)$ and $(x_2, x_4, x_6)$

$$A(x_1, x_3, x_5) + A(x_2, x_4, x_6) - A(x_1, x_2, x_3) - A(x_4, x_5, x_6) = 2(x_5 + x_4 - x_3 - x_2) \geq 0.$$  

6) $(x_1, x_3, x_6)$ and $(x_2, x_4, x_5)$

$$A(x_1, x_3, x_6) + A(x_2, x_4, x_5) - A(x_1, x_2, x_3) - A(x_4, x_5, x_6) = 2(x_5 + x_4 - x_3 - x_2) \geq 0.$$  

7) $(x_1, x_4, x_5)$ and $(x_2, x_3, x_6)$
Theorem 1.4. If for any sorted 2k-tuple \((x_1, x_2, \ldots, x_{2k})\) the two sorted k-tuples \((x_1, x_2, \ldots, x_k)\) and \((x_{k+1}, x_{k+2}, \ldots, x_{2k})\) are the minimal solution of the k-matching problem, then the minimal solution of the k-matching problem for a sorted kn-tuple, \(n > 0\), is given by \(n\) sorted k-tuples

\[
(x_{(i-1)k+1}, x_{(i-1)k+2}, \ldots, x_{(i-1)k+k})
\]

for \(i = 1, \ldots, n\).

Proof. We prove the theorem by induction. If \(n = 1\), the theorem is obvious. If \(n = 2\), the theorem follows directly from its assumption. We assume the theorem is true if \(n - 1 > 1\) and show it is true for \(n\).

We consider all the possible minimal k-tuple partitions. If there is an k-tuple partition such that for some sorted k-tuple \((y_1, y_2, \ldots, y_k)\) we have \(x_i = y_i\) for all
\[ i = 1, 2, \ldots, k, \text{ we are done and we may also exclude the case that the smallest sub-script of discordance is not defined in the following.} \]

If \((x_1, x_2, \ldots, x_k) \neq (y_1, y_2, \ldots, y_k)\), we will show a contradiction. We will compare \(k\)-tuples with \((x_1, x_2, \ldots, x_k)\). Out of all the minimal \(k\)-tuple partitions we pick the one containing the \(k\)-tuple \((y_1, y_2, \ldots, y_k)\) for which the smallest subscript of discordance \(j\) is the highest. It is obvious that for such a \(k\)-tuple \(y_1 = x_1\) holds for otherwise the smallest subscript of discordance \(j\) would be 1. If \(y_1 = x_1\), we have \(j > 1\).

Since \(j\), where \(1 < j \leq k\), is the lowest subscript for which \(y_j \neq x_j\), this \(x_j\) must be in some other \(k\)-tuple \((z_1, z_2, \ldots, z_k)\) in the partition. We concatenate these two \(k\)-tuples to obtain a \(2k\)-tuple \((y_1, y_2, \ldots, y_k, z_1, z_2, \ldots, z_k)\) and apply the assumption of the theorem to obtain a minimal solution of the \(k\)-matching problem on this \(2k\)-tuple as \((t_1, t_2, \ldots, t_j, \ldots, t_k)\) and \((u_1, u_2, \ldots, u_k)\) where \(t_i = x_i\) for \(i = 1, 2, \ldots, j\).

We have two cases. We obtain a \(k\)-tuple partition containing \((x_1, x_2, \ldots, x_k)\) which is a contradiction.

If we do not obtain a partition containing \((x_1, x_2, \ldots, x_k)\), the smallest subscript of discordance is \(i\), where \(j < i < k\), when \((t_1, t_2, \ldots, t_k)\) is compared with \((x_1, x_2, \ldots, x_k)\). We have obtained a \(k\)-tuple partition for which the smallest subscript of discordance is higher than \(j\), contradicting the assumption.

The proof is finished by removing \(x_1, x_2, \ldots, x_k\) from the original \(kn\)-tuple obtaining a \((n - 1)k\)-tuple.

**Corollary.** If the assumption in theorem 1.4 holds, then the minimal solution of the \(k\)-matching problem for a not necessarily sorted \(kn\)-tuple, \(n > 0\), is obtained in the running time necessary for sorting the \(kn\)-tuple.

It means the matching problem is solved in \(O(N \log N)\) time where \(N = kn\) is the number of items to be matched.

### 1.2 Sum of squares of differences

We all know that statisticians would prefer the sum of squares of all differences to evaluate the distance within a \(k\)-tuple. Let \((x_1, x_2, \ldots, x_k)\) be a \(k\)-tuple the distance within will be defined as

\[
S(x_1, x_2, \ldots, x_k) = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} (x_j - x_i)^2
\]

Some avid statisticians would even require the minimization of the sum of variances but this is equivalent to the sum of squares of all the differences as explained in the appendix.

First we want to show what a minimal partition for \(2k\)-tuples looks like.

**Theorem 1.5.** Let a sorted 4-tuple \((x_1, x_2, x_3, x_4)\) be given. Then the two pairs \((x_1, x_2)\) and \((x_3, x_4)\) have a minimal sum of squares of all differences.

**Proof.** The sum of distances of \((x_1, x_2)\) and \((x_3, x_4)\) is \(S(x_1, x_2) + S(x_3, x_4) = (x_2 - x_1)^2 + (x_4 - x_3)^2\). We form other possible partitions into pairs, calculate
the sum of the differences squared, and compare them with the sum of distances \( S(x_1, x_2) + S(x_3, x_4) \). Other possible sorted pairs are:

1) \((x_1, x_3)\) and \((x_2, x_4)\). The difference is

\[
S(x_1, x_3) + S(x_2, x_4) - (S(x_1, x_2) + S(x_3, x_4)) = 2(x_4 - x_1)(x_3 - x_2) \geq 0.
\]

2) \((x_1, x_4)\) and \((x_2, x_3)\). The difference is

\[
S(x_1, x_4) + S(x_2, x_3) - (S(x_1, x_2) + S(x_3, x_4)) = 2(x_3 - x_1)(x_4 - x_2) \geq 0.
\]

**Theorem 1.6.** Let a sorted 6-tuple \((x_1, x_2, \ldots, x_6)\) be given. Then the two sorted triples \((x_1, x_2, x_3)\) and \((x_4, x_5, x_6)\) have a minimal sum of squares of all differences.

**Proof.** We form all the other possible triples, calculate the sum of their distances, and subtract from them the sum of distances \( S(x_1, x_2, x_3) + S(x_4, x_5, x_6) \).

Other possible triples are listed in such a way that \(x_1\) appears in the first triple because the order does not matter in this case:

1) \(S(x_1, x_2, x_4) + S(x_3, x_5, x_6) - S(x_1, x_2, x_3) - S(x_4, x_5, x_6) = 2(x_4 - x_3)(x_6 + x_5 - x_2 - x_1) \geq 0\)

2) \(S(x_1, x_2, x_5) + S(x_3, x_4, x_6) - S(x_1, x_2, x_3) - S(x_4, x_5, x_6) = 2(x_3 - x_5)(x_1 + x_2 - x_4 - x_6) \geq 0\)

3) \(S(x_1, x_2, x_6) + S(x_3, x_4, x_5) - S(x_1, x_2, x_3) - S(x_4, x_5, x_6) = 2(x_6 - x_1)(x_3 + x_4 - x_2 - x_1) \geq 0\)

4) \(S(x_1, x_3, x_4) + S(x_2, x_5, x_6) - S(x_1, x_2, x_3) - S(x_4, x_5, x_6) = 2(x_4 - x_2)(x_6 + x_5 - x_3 - x_1) \geq 0\)

5) \(S(x_1, x_3, x_5) + S(x_2, x_4, x_6) - S(x_1, x_2, x_3) - S(x_4, x_5, x_6) = 2(x_5 - x_2)(x_6 + x_4 - x_3 - x_1) \geq 0\)

6) \(S(x_1, x_3, x_6) + S(x_2, x_4, x_5) - S(x_1, x_2, x_3) - S(x_4, x_5, x_6) = 2(x_6 - x_2)(x_3 + x_4 - x_2 - x_1) \geq 0\)

7) \(S(x_1, x_4, x_5) + S(x_2, x_3, x_6) - S(x_1, x_2, x_3) - S(x_4, x_5, x_6) = 2(x_5 - x_1)(x_3 + x_4 - x_3 - x_2) \geq 0\)

8) \(S(x_1, x_4, x_6) + S(x_2, x_3, x_5) - S(x_1, x_2, x_3) - S(x_4, x_5, x_6) = 2(x_6 - x_1)(x_3 + x_4 - x_3 - x_2) \geq 0\)

9) \(S(x_1, x_5, x_6) + S(x_2, x_3, x_4) - S(x_1, x_2, x_3) - S(x_4, x_5, x_6) = 2(x_6 - x_1)(x_3 + x_5 - x_3 - x_2) \geq 0\)

That finishes the proof.

It is interesting to see that the factorization of all the quadratic forms could be done. We actually wrote a program that checked the faktorizatation for \(2 \leq k \leq 8\) and verified the nonnegativity of each factor.

The final step is the use of theorem 1.4 to show how to calculate \(k\)-matching for \(2 \leq k \leq 8\). Now we see that it does not matter which of the two mentioned distances within, either the sum of absolute values of all the differences or the sum of squares of all the differences, we use, we get the same \(k\)-matching.

### 1.3 Statistical applications

The \(k\)-tuples obtained by our algorithm are sorted. That could have an unpleasant effect on statistical procedures because of the inequality of the means.
of the first entries of the \(k\)-tuples as compared with the means of the last entries of the \(k\)-tuples. We know they are different, as long as \(x_1, x_2, \ldots, x_{kn}\) are not all equal.

We want to avoid randomization in the spirit of our paper. What we are trying to achieve is the rearrangement of the items in \(k\)-tuples in such a way that all the means over the \(i\)-th items, \(i = 1, \ldots, k\), are as close as possible. The minimization process reminds us of an NP-complete optimization partition problem even though typically the numbers \(x_1, x_2, \ldots, x_{kn}\) are not integers.

Even though partitioning is beyond the scope of this paper, one heuristic way of handling the problem is sorting the \(k\)-tuples with respect to the distances within in a descending order and keeping track of the subtotals starting from the first \(k\)-tuple and rearranging each consecutive \(k\)-tuple to keep the differences as small as possible at each step. This approach obviously does not guarantee we obtain the smallest possible differences among the means.

Part 2

2.1 Bipartite graphs

Even though algorithms for finding optimal bipartite matching are so well known that they are presented in introductory textbooks, such as Bondy(1976), we present another approach because it will find applications in tripartite matching.

The regular complete bipartite graph consists of two disjoint vertex sets \(A\) and \(B\), \(|A| = |B| = n\), and edges \(A \times B\). We assume that to each of the vertices in \(A\) and in \(B\) respectively a real numbers \(x_i\) and \(y_i\) are assigned as their values. The weight associated with each edge \((a_i, b_j)\) is defined as either \(w_{abs}(a_i, b_j) = |x_i - y_j|\) for all \(1 \leq i, j \leq n\) or \(w_{sq}(a_i, b_j) = (x_i - y_j)^2\) for all \(1 \leq i, j \leq n\).

**Definition 2.1.** A perfect matching in a regular complete bipartite graph with vertex set \(A \cup B\) is a subset of edges \(M_{A,B}\) such that each vertex of \(A\) is connected by an edge to one vertex of \(B\) and each vertex of \(B\) is connected to one vertex of \(A\).

We use the notation \(M_{A,B}\) to indicate that we are dealing with the vertex set \(A \cup B\).

**Definition 2.2.** The weight of a perfect matching is

\[
w(M_{A,B}) = \sum_{(a,b) \in M_{A,B}} w(a, b)
\]

**Definition 2.3.** A perfect matching \(M_{A,B}^{\min}\) is minimal if its weight is less than or equal to that of any other perfect matching, \(w(M_{A,B}^{\min}) \leq w(M_{A,B})\).

To avoid any trouble, we mention we do not make any distinction between the vertices and the values they are assigned, we again consider \(n\)-tuples of real numbers. Sorted \(n\)-tuples are described in definition 1.2.
Definition 2.4. Let two sorted $n$-tuples, $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ be given. Let a weight of each edge be defined as the absolute value of the difference between $x_i$ and $y_j$, $w_{ij}(a_i, b_j) = |x_i - y_j|$, for each $1 \leq i, j \leq n$, then the minimal perfect matching consists of edges with values $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

Theorem 2.1. Let two sorted $n$-tuples, $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ be given. If the weight of each edge is defined as the absolute value of the difference between $x_i$ and $y_j$, $w_{ij}(a_i, b_j) = |x_i - y_j|$, for each $1 \leq i, j \leq n$, then the minimal perfect matching consists of edges with values $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

Proof. The theorem is true for $n = 1$. If $n > 1$, we assume it is true for $n - 1$. In a minimal perfect matching there is an edge with one endpoint value $x_1$. If the other endpoint value of this edge is $y_1$, we are done. If not, the other endpoint value is $y_j$ for some $j > 1$. Another edge must have $y_1$ as its endpoint value, this edge has some $x_i, i > 1$ as its other endpoint value.

Let the required $x_1, x_i, y_1, y_j$ be given. We first consider the three cases when $x_1$ is less than the rest of the points, $x_i, y_1, y_j$. Three cases are listed depending on the position of $x_i$.

Two options are possible in each case. The one containing $x_1, y_1$ is subtracted from the other one.

1. Let $x_1 \leq x_i \leq y_j$. We subtract $|x_1 - y_1| + |x_i - y_j| = y_1 - x_1 + y_j - x_i$ from $|x_1 - y_j| + |x_i - y_1|$.

2. Let $x_1 \leq y_1 \leq x_i \leq y_j$. Then $|x_1 - y_1| + |x_i - y_j| = y_1 - x_1 + y_j - x_i$ is subtracted from $|x_1 - y_j| + |x_i - y_1| = y_j - x_1 + x_i - y_1$, the difference is $y_i - x_1 + x_i - y_1 = 2y_i - y_1 = 2(x_1 - y_1) \geq 0$.

3. Let $x_1 \leq y_1 \leq y_j \leq x_i$. Then $|x_1 - y_1| + |x_i - y_j| = y_1 - x_1 + x_i - y_1$ is subtracted from $|x_1 - y_j| + |x_i - y_1| = y_j - x_1 + x_i - y_1$, the difference is $y_j - x_1 + x_i - y_1 = 2(x_1 - y_1) \geq 0$.

In the case that $y_1$ is the smallest number we just swap $x$’s with $y$’s.

We conclude that the minimal matching contains an edge with endpoint values $(x_1, y_1)$ and induction makes sense.

Theorem 2.2. Let two sorted $n$-tuples, $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ be given. Let a weight of each edge be defined as $w_{ij} = (x_i - y_j)^2$, for each $1 \leq i, j \leq n$, then the minimal perfect matching consists of edges with values $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

Proof. The theorem is true for $n = 1$. If $n > 1$, we assume it is true for $n - 1$. In a minimal perfect matching there is an edge with one endpoint value $x_1$. If the other endpoint value of this edge is $y_1$, we are done. If not, the other endpoint value is $y_j$ for some $j > 1$. Another edge must have $y_1$ as its endpoint value, this edge has some $x_i, i > 1$ as its other endpoint value.

We subtract $(x_1 - y_1)^2 + (x_i - y_j)^2 = x_1^2 + y_1^2 - 2x_1y_1 + x_i^2 + y_j^2 - 2x_iy_j$ from $(x_1 - y_j)^2 + (x_i - y_1)^2 = x_1^2 + y_j^2 - 2x_1y_j + y_1^2 + x_i^2 - 2x_iy_1$. The difference is $-2x_1y_j - 2x_iy_1 + 2x_1y_1 + 2x_iy_j = 2(x_1 - x_i)(y_j - y_1) \geq 0$.

The property described in theorems 2.1 or 2.2 not only allows us to calculate minimal matching quickly, it will be used in the construction of tripartite matching. The following definition will allow us to formulate the results in a bit more general but simple setting.

Definition 2.4. Let two sorted $n$-tuples, $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ be given. Let a weight of each edge be defined as $w(x_i, y_j)$, for each $1 \leq i, j \leq n$, if
the minimal perfect matching consists of edges with values \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\) for any \((x_1, x_2, \ldots, x_n)\) and \((y_1, y_2, \ldots, y_n)\), then the weight \(w\) is called line matching or LM.

Counterexample: Let a weight be defined as a product \(w_p(x_i, y_j) = x_i y_j\). If we consider \(x = (1, 2, 3)\) and \(y = (1, 2, 3)\), as an example, then the sum of products is \(1 \times 1 + 2 \times 2 + 3 \times 3 = 14\). If we use the reverse order \(z = (3, 2, 1)\), then the sum of products is \(1 \times 3 + 2 \times 2 + 3 \times 1 = 10 < 14\). As a result we can say that the weight \(w_p\) defined as a product is not LM.

We will not study which weights are LM and which are not. It suffices to see that the weights \(w_{ab}\) and \(w_{sq}\) are the ones with LM property and those are the ones that would be used in practice.

2.2 Tripartite graphs

A regular complete tripartite graph is the union of three disjoint vertex sets \(A, B,\) and \(C\), for which \(|A| = |B| = |C| = n\), and edges in \(A \times B, B \times C,\) and \(C \times A\). For any \(a \in A, b \in B,\) and \(c \in C\) the edges are denoted as \((a, b), (b, c),\) and \((c, a)\) respectively.

We define a matching \(M_{A,B,C} \subset A \times B \times C\) as a set of triples of vertices in \(M_{A,B,C}\) such that if for any two distinct triples \((a_i, b_j, b_k) \in M_{A,B,C}\) and \((a_{i_2}, b_{j_2}, c_{k_2}) \in M_{A,B,C}\), we have \(a_{i_1} \neq a_{i_2}, b_{j_1} \neq b_{j_2},\) and \(c_{k_1} \neq c_{k_2}\).

Definition 2.5. A matching \(M_{A,B,C}\) is called perfect if the number of triples in \(M_{A,B,C}\) is \(n = |A| = |B| = |C|\).

We assume a nonnegative weight of each of the edges is defined for each edge \(w(a_i, b_j), w(b_i, c_j),\) and \(w(c_i, a_j)\) for any \(1 \leq i \leq n\) and \(1 \leq j \leq n\).

Definition 2.6. If a perfect matching \(M_{A,B,C}\) is given, we define its weight \(w(M_{A,B,C})\) as

\[
  w(M_{A,B,C}) = \sum_{(a,b,c) \in M_{A,B,C}} (w(a,b) + w(b,c) + w(c,a)).
\]

This definition is in accordance with Definition 1.1 where all the weights of edges in a complete graph with vertices \(a, b,\) and \(c\) are included in the sum.

Definition 2.7. A perfect matching \(M_{A,B,C}^{\text{min}}\) is called minimal if its weight is minimal, that is, \(w(M_{A,B,C}^{\text{min}}) \leq w(M_{A,B,C})\) for any other perfect matching \(M_{A,B,C}\).

Theorem 2.3. Let \(A, B,\) and \(C\) be the vertex sets of the same cardinality of a complete tripartite graph \(A \cup B \cup C\) with edges in \(A \times B, B \times C,\) and \(C \times A\). Then

\[
  w(M_{A,B}^{\text{min}}) + w(M_{B,C}^{\text{min}}) + w(M_{C,A}^{\text{min}}) \leq w(M_{A,B,C}^{\text{min}}).
\]
Proof. We check that

\[ w(M_{A,B}^{\text{min}}) \leq \sum_{(a,b,c) \in M_{A,B,C}^{\text{min}}} w(a,b), \]

\[ w(M_{B,C}^{\text{min}}) \leq \sum_{(a,b,c) \in M_{A,B,C}^{\text{min}}} w(b,c), \]

\[ w(M_{C,A}^{\text{min}}) \leq \sum_{(a,b,c) \in M_{A,B,C}^{\text{min}}} w(c,a). \]

Due to definitions 2.2 through 2.7 the sum of these three inequalities yields the result.

1) An application of this theorem in a general setting like this may be found in estimating the accuracy of some heuristic for finding a perfect matching. If we obtain a perfect matching \( M_{A,B,C}^{\text{heu}} \) in a complete tripartite graph, we may use this theorem 2.2 to estimate the accuracy of \( M_{A,B,C}^{\text{heu}} \) as

\[ \frac{w(M_{A,B,C}^{\text{heu}})}{w(M_{A,B,C}^{\text{min}})} \leq \frac{w(M_{A,B,C}^{\text{min}})}{w(M_{A,B,C}^{\text{min}}) + w(M_{B,C}^{\text{min}}) + w(M_{C,A}^{\text{min}})}. \]

2) If an inequality in this theorem 2.3 is satisfied as an equality for some perfect matching \( M_{A,B,C} \), that is, \( w(M_{A,B,C}) = w(M_{A,B,C}^{\text{min}}) + w(M_{B,C}) + w(M_{C,A}) \), we have a minimal solution.

3) The technique of the proof of theorem 2.2 may be used in other situations, such as 4-partite matching or \( k \)-partite matching.

### 2.3 Minimal matching on tripartite graphs

Let \( A, B, \) and \( C \) be the vertex sets of the same number of vertices. We form a complete tripartite graph \( A \cup B \cup C \) with edges in \( A \times B, B \times C, \) and \( C \times A \).

We assume that to each of the vertices in \( A, B, \) and \( C \) real numbers \( x_i, y_i, \) and \( z_i \) are assigned respectively. A weight of each of the edges is defined as \( w(a_i,b_j) = |x_i - y_j|, w(b_i,c_j) = |y_i - z_j|, \) and \( w(c_i,a_j) = |z_i - x_j| \) for any \( 1 \leq i \leq n \) and \( 1 \leq j \leq n \). Another way to define the weights is \( w(a_i,b_j) = (x_i - y_j)^2, w(b_i,c_j) = (y_i - z_j)^2, \) and \( w(c_i,a_j) = (z_i - x_j)^2 \). In general the weight has to have property LM. Without any loss of generality we assume the \( n \)-tuples \((x_1,x_2,\ldots,x_n)\), \((y_1,y_2,\ldots,y_n)\), and \((z_1,z_2,\ldots,z_n)\), are sorted. If not, we sort them together with \( a_i, b_j, \) and \( c_k \). Obtaining sorted \( n \)-tuples can be done in \( O(n \log n) \) time.

**Theorem 2.4.**

Let \( A, B, \) and \( C \) be the vertex sets, \(|A| = |B| = |C| = n\), of a complete tripartite graph \( A \cup B \cup C \) with edges in \( A \times B, B \times C, \) and \( C \times A \). If the vertices are assigned real values corresponding to sorted \( n \)-tuples \((x_1,x_2,\ldots,x_n)\), \((y_1,y_2,\ldots,y_n)\), and \((z_1,z_2,\ldots,z_n)\), then the minimal matching, with respect to weights with property LM, is given by \((a_1,b_1,c_1), (a_2,b_2,c_2), \ldots, (a_n,b_n,c_n)\).  

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Proof. We claim the matching \((a_1, b_1, c_1), (a_2, b_2, c_2), \ldots, (a_n, b_n, c_n)\) is the minimal one. When we form \(w(x_i, y_i) + w(y_i, z_i) + w(z_i, x_i)\) for each triple separately and add them up over \(i\), we get the same sum as when we calculate 
\[
\sum_{i=1}^{n} w(x_i, y_i) + \sum_{i=1}^{n} w(y_i, z_i) + \sum_{i=1}^{n} w(z_i, x_i).
\]
It shows we get an equality sign in the inequality in theorem 2.3 which, in turn, means that we have obtained a minimal matching.

We would proceed in the same way in the case of weights defined as squares of differences.

We recall that \(|x_i - y_i| + |y_i - z_i| + |z_i - x_i|\) is the distance within this triple \(D(x_i, y_i, z_i)\) introduced in definition 1.1.

Conclusion

Results in part 1 may be used as a starting value for finding an \(n\)-matching in a Euclidean space. We may fit a line to data to provide a starting \(n\)-tuple partition followed by a local search. One way to do the local search is the concatenation of pairs of \(n\)-tuples to obtain \(2n\)-tuples and enumeration of all the pairs of \(n\)-tuples. One element may be fixed so that we have \(\binom{2n-1}{n-1}\) to generate.

The matching algorithm on a line may provide a test for a general heuristic algorithm for if a general matching heuristic works, it should work on a line. A simple heuristic may be designed if the vertices are points in a Euclidean space, edges are the line segments connecting the vertices, and weights are the distances between the end points of those line segments. If the number of vertices is \(2^n\), we find the nonbipartite 2-matching that minimizes the sum of the lengths of line segments. There are \(2^{n-1}3\) line segments in this matching. We form a new graph by taking midpoints of the line segments in the matching keeping track of what original vertices the line segments came from. We repeat this process until we get three vertices. Now we work our way back forming a graph with six vertices and find optimal triples by enumerating all the pairs of triples of vertices. We continue until we get a graph with \(2^n3\) vertices. When we use this algorithm on vertices on a line, we see it gives the correct result.

In part 2 of the paper theorem 2.2 may be used in the case that the weights assigned to edges of a tripartite graph satisfy the triangle inequality. Let \((a_i, b_j, c_k)\) be given \(a_i \in A, b_j \in B, c_k \in C\), where \(A, B, C\) are disjoint, \(|A| = |B| = |C| = n\), then \(w(c_k, a_i) \leq w(a_i, b_j) + w(b_j, c_k)\). Let \((a_i, b_j) \in M_{A,B}^{\text{min}}\) and \((b_j, c_k) \in M_{B,C}^{\text{min}}\), then \((c_k, a_i)\) does not have to be in \(M_{C,A}^{\text{min}}\). Matching on a tripartite graph actually asks for 3-cycles \(a_i, b_j, c_k, a_i\).

Actually without knowing or caring what the weights of \((c_k, a_i)\) are, the use of the triangle inequality gives us an upper bound, \(w(c_k, a_i) \leq w(a_i, b_j) + w(b_j, c_k)\).

Thus, if we form a matching like this, denoted as \(M_{A,B,C}^{\triangle}\), we have
\[
M_{A,B,C}^{\triangle} \leq 2\left( \sum_{(a,b) \in M_{A,B}^{\text{min}}} w(a, b) + \sum_{(b,c) \in M_{B,C}^{\text{min}}} w(b, c) \right) = 2\left( w(M_{A,B}^{\text{min}}) + w(M_{B,C}^{\text{min}}) \right)
\]

Thus
\[
\frac{w(M_{A,B,C}^{\triangle})}{w(M_{A,B,C}^{\text{min}})} \leq \frac{w(M_{A,B,C}^{\triangle})}{w(M_{A,B}^{\text{min}}) + w(M_{B,C}^{\text{min}}) + w(M_{C,A}^{\text{min}})} \leq
\]
\[
\frac{2\left(w(M_{A,B}^{\min}) + w(M_{B,C}^{\min})\right)}{w(M_{A,B}^{\min}) + w(M_{B,C}^{\min}) + w(M_{C,A}^{\min})} \leq \frac{2\left(w(M_{A,B}^{\min}) + w(M_{B,C}^{\min})\right)}{w(M_{A,B}^{\min}) + w(M_{B,C}^{\min})} = 2.
\]

**Appendix**

We may try to define a measure of variability in a way different from the usual approach. We assume there are \(N\) real numbers \(x_1, x_2, \ldots, x_N\). The usual measure of variability, the variance \(S^2\), is based on the sum of squares of differences from the mean,

\[
S^2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})^2.
\]

The way we will define the measure of variability without any reference to the mean is based on the sum of squares of all the differences

\[
\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} (x_j - x_i)^2.
\]

We may check what happens if \(y_i = a + bx_i\),

\[
\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} (y_j - y_i)^2 = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} (a + bx_j - a - bx_i)^2 = b^2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} (x_j - x_i)^2.
\]

It means we have the same property for the sum of all the differences squared and the sum of differences from the mean squared and it means it is a reasonable characteristic of variability.

Before we show what relation there is between the sum of squares of all differences and the sum of squares of differences from the mean we write the sum of squares of all differences as

\[
2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} (x_j - x_i)^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} (x_j - x_i)^2.
\]

This is easy to see when we write the difference \((x_j - x_i)\) in a different order as \((x_i - x_j)\). When the subscripts are the same, we get \(x_i - x_i = 0\).

Now we review the formula for \((a + b)^2\). We usually say that \((a + b)^2 = a^2 + 2ab + b^2\) because we use commutativity \(ab = ba\) therefore \(ab + ba = 2ab\). When we don’t, we get \((a + b)^2 = aa + ab + ba + bb\). We will use this idea as

\[
\left(\sum_{j=1}^{N} x_j\right)^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j.
\]

**Theorem** Let \(N > 1\) and real numbers \(x_1, x_2, \ldots, x_N\) be given. Then

\[
\sum_{i=1}^{N} \sum_{j=1}^{N} (x_j - x_i)^2 = 2N \sum_{i=1}^{N} (x_i - \bar{x})^2.
\]
Proof. We expand the formula for twice the sum of squares of all differences
\[
\sum_{i=1}^{N} \sum_{j=1}^{N} (x_j - x_i)^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} (x_j^2 + x_i^2 - 2x_ix_j) = \\
\sum_{i=1}^{N} \sum_{j=1}^{N} x_i^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} x_j^2 - 2 \sum_{i=1}^{N} \sum_{j=1}^{N} x_ix_j = 2N \sum_{i=1}^{N} x_i^2 - 2 \sum_{i=1}^{N} \sum_{j=1}^{N} x_ix_j.
\]

We multiply the formula for the sum of squares of differences from the mean by two
\[
2N \sum_{i=1}^{N} \left( x_i - \frac{1}{N} \sum_{j=1}^{N} x_j \right)^2 = 2N \sum_{i=1}^{N} x_i^2 - 4 \sum_{i=1}^{N} x_i \sum_{j=1}^{N} x_j + 2 \frac{N}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} x_j^2 = \\
2N \sum_{i=1}^{N} x_i^2 - 4 \sum_{i=1}^{N} \sum_{j=1}^{N} x_ix_j + 2 \sum_{j=1}^{N} \sum_{j=1}^{N} x_j^2 = \\
2(N \sum_{i=1}^{N} x_i^2 - 4 \sum_{i=1}^{N} \sum_{j=1}^{N} x_ix_j + 2 \sum_{i=1}^{N} \sum_{j=1}^{N} x_ix_j) = \\
2N \sum_{i=1}^{N} x_i^2 - 2 \sum_{i=1}^{N} \sum_{j=1}^{N} x_ix_j.
\]

This proves the desired equality.

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