GRAY CURVATURE IDENTITIES FOR ALMOST CONTACT METRIC MANIFOLDS

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Abstract. Alfred Gray introduced in [8] three curvature identities for the class of almost Hermitian manifolds. Using the warped product construction and the Boothby-Wang fibration we will give an equivalent of these identities for the class of almost contact metric manifolds.

1. Introduction

In [8], A. Gray introduced three curvature identities, $K_i$, $i = 1, 2, 3$, for almost Hermitian manifolds. These identities have important applications in geometry and topology (for example, some other geometric objects can be constructed on manifolds satisfying $K_i$, see e.g. [12, 13]).

A naturally arising question would be: which are the equivalent identities for the almost contact manifolds?

Concerning this, A. Bonome, L. M. Hervella, and I. Rozas defined in [4] $K_{i\varphi}$-curvature identities ($i = 1, 2, 3$) for an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ by using the usual Hermitian structure on $M \times \mathbb{R}$ (the product manifold). Gray proved in [8] that Kaehlerian manifolds satisfy $K_i$, $i = 1, 2, 3$ (curvature identities for almost Hermitian manifolds). In the same spirit, in [4] it is shown that cosymplectic manifolds satisfy $K_{i\varphi}$-identities.

It is known that both cosymplectic and Sasakian manifolds are natural odd-dimensional versions of Kaehlerian manifolds. We ask what happens if the analogy with the Kaehlerian manifolds is extended to Sasakian manifolds? We bear in mind the fact that a Riemannian manifold $(M, g)$ is Sasakian if the holonomy group of the metric cone on $M$: $(C(M) = \mathbb{R}_+ \times M, \tilde{g} = dt^2 + t^2 g)$

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reduces to a subgroup of $U\left(\frac{m+1}{2}\right)$, i.e., $(C(M), \tilde{g})$ is Kaehlerian. (Here $m = \dim M$.)

Inspired by this definition and by [4], we will take a different approach to Gray curvature identities for almost contact metric manifolds. For this purpose we will use the warped product and the Boothby-Wang fibration, two different constructions which lead us to the same result.

1.1. Gray curvature identities

An almost Hermitian manifold $(M, J, g)$ is said to satisfy Gray curvature identities $(K1)$, $(K2)$ and respectively $(K3)$, if its Riemann Christoffel curvature tensor satisfies

$(K1) R(X, Y, Z, W) = R(X, Y, JZ, JW),$

$(K2) R(X, Y, Z, W) = R(JX, JY, Z, W) + R(JX, Y, JZ, W) + R(JX, Y, Z, JW),$

$(K3) R(X, Y, Z, W) = R(JX, JY, JZ, JW)$

for all vector fields $X, Y, Z, W$ on $\chi(M)$. Throughout this paper, the curvature tensor is defined by $R_{XY}Z = \nabla_X \nabla_Y Z − \nabla_Y \nabla_X Z − [X, Y]Z$ for all $X, Y, Z \in \chi(M)$ while the Riemann Christoffel curvature tensor is given by $R(X, Y, Z, W) = −g(R_{XY}Z, W)$. The identity $(K1)$ is often called Kaehler identity. It is known the fact that there exist non Kaehler manifolds satisfying $(K1)$ identity (e.g. para-Kaehler manifolds, [11]).

2. Warped product manifolds

Singly warped products or simply warped products were first defined by Bishop and O’Neill in [1] in order to construct Riemannian manifolds with negative sectional curvature. Let $(B, g_B)$ and $(F, g_F)$ be Riemannian manifolds and let $b : B \rightarrow (0, \infty)$ be a smooth function. The warped product $\tilde{M} = B \times_b F$ is the product manifold $B \times F$ endowed with the metric $\tilde{g} = g_B \oplus b^2 g_F$. More precisely, if $\pi : B \times F \rightarrow B$ and $\tau : B \times F \rightarrow F$ are natural projections, the metric $g$ is defined by

$\tilde{g} = \pi^* g_B + (b \circ \pi)^2 \tau^* g_F.$

The function $b$ is called warping function. If $b \equiv 1$, then we have a product manifold.

If $X, Y$ are tangent to $B$ and $Z, W$ tangent to $F$, then the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{M}$ is given by

\[
\begin{cases}
\tilde{\nabla}_X Y = \nabla^B_X Y, & \tilde{\nabla}_X Z = X(\ln b)Z \\
\tilde{\nabla}_Z W = \nabla^F_Z W - b^2 g_F(Z, W)\nabla^B(\ln b),
\end{cases}
\]

where $\nabla^B$ and $\nabla^F$ are the Levi-Civita connections on $B$, respectively on $F$, and $\nabla^B(\ln b)$ is the gradient of $\ln b$ with respect to the metric $g_B$. 

Let \((M, \varphi, \xi, \eta, g)\) be an almost contact metric manifold. Consider the warped product manifold \(\tilde{M} = \mathbb{R}^+ \times_t M\), where \(t > 0\) is the global coordinate of \(\mathbb{R}^+\), i.e., the metric \(\tilde{g}\) of \(\tilde{M}\) is defined by
\[
\tilde{g} = dt^2 + t^2g.
\]

Let an endomorphism on \(\chi(\tilde{M})\) defined by
\[
J\partial_t = - \frac{1}{t} \xi, \quad JX = \varphi X + t\eta(X)\partial_t, \quad \forall X \in \chi(M),
\]
where \(\partial_t = \frac{dt}{t}\). For \(\tilde{X} = (a, X) \in \chi(\tilde{M}), a \in C^\infty(\mathbb{R}^+), X \in \chi(M)\) we have
\[
J\tilde{X} = J(a, X) = \left(t\eta(X), \varphi X - \frac{a}{t} \xi\right).
\]
The proofs of the following propositions are straightforward.

**Proposition 2.1.** \(J\) is an almost complex structure compatible with the metric \(\tilde{g}\).

**Proposition 2.2.** The Levi-Civita connection \(\tilde{\nabla}\) of \(\tilde{g}\) is given by:
\[
\begin{align*}
\tilde{\nabla}_{\partial_t} = 0, \quad \tilde{\nabla}_X \partial_t = \tilde{\nabla}_{\partial_t} X = \frac{1}{t}X \\
\tilde{\nabla}_X Y = \nabla_X Y - t\eta(Y)\partial_t, \quad X, Y \in \chi(M).
\end{align*}
\]

**Proposition 2.3.** The covariant derivative of \(J\) is given by:
\[
\begin{align*}
(\tilde{\nabla}_{\partial_t}J)\partial_t = (0, 0), \quad (\tilde{\nabla}_{\partial_t}J)X = (0, 0) \\
(\tilde{\nabla}_X J)\partial_t = \left(0, -\frac{1}{t}(\nabla_X \xi + \varphi X)\right) \\
(\tilde{\nabla}_X J)Y = \left(t(\nabla_{X\eta}(Y) - g(X, \varphi Y)), \nabla_X \varphi Y - g(X, Y)\xi + \eta(Y)X\right).
\end{align*}
\]

**Corollary 2.4.** \(J\) is parallel if and only if
\[
\begin{align*}
(\tilde{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \quad (\nabla_X \eta)(Y) = g(X, \varphi Y) \\
\nabla_X \xi = -\varphi X, \quad X, Y \in \chi(M)
\end{align*}
\]
i.e., \((\tilde{M}, J, \tilde{g})\) is Kaehler if and only if \((M, \varphi, \xi, \eta, g)\) is Sasakian.

**Proposition 2.5.** For the curvature of the manifold \(\tilde{M}\) we have
\[
\begin{align*}
\tilde{R}(\partial_t, X)\partial_t = 0, \quad \tilde{R}(X, Y)\partial_t = 0, \quad \tilde{R}(\partial_t, X)Y = 0 \\
\tilde{R}(X, Y)Z = R(X, Y)Z - g(Y, Z)X + g(X, Z)Y
\end{align*}
\]
where \(\tilde{R}\) (respectively \(R\)) are the curvature tensors for \(\tilde{g}\) (respectively for \(g\)).

Moreover, the following relations hold:
\[
\begin{align*}
\tilde{R}(\partial_t, X)(J\partial_t) = 0, \quad \tilde{R}(\partial_t, X)(JY) = 0 \\
\tilde{R}(X, Y)(J\partial_t) = -\frac{1}{2}[R(X, Y)\xi - \eta(Y)X + \eta(X)Y] \\
\tilde{R}(X, Y)(JZ) = R(X, Y)(\varphi Z) - g(Y, \varphi Z)X + g(X, \varphi Z)Y.
\end{align*}
\]
In the following we compute expressions of the form $\tilde{g}(\tilde{R}(A, B)(jc), jd)$.

The useful following expressions are obtained in the cases:

1. $\tilde{g}(\tilde{R}(X, Y)(J\partial_t), JW)$
   \[= - t\left[g(R(X, Y)\xi, \varphi W) - \eta(Y)g(X, \varphi W) + \eta(X)g(Y, \varphi W)\right].\]

2. $\tilde{g}(\tilde{R}(X, Y)(JZ), J\partial_t)$
   \[= - t\left[\eta(R(X, Y)(\varphi Z)) - \eta(X)g(Y, \varphi Z) + \eta(Y)g(X, \varphi Z)\right].\]

3. $\tilde{g}(\tilde{R}(X, Y)(JZ), JW)$
   \[= t^2 [g(R(X, Y)\varphi Z, \varphi W) - g(Y, \varphi Z)g(X, \varphi W) + g(X, \varphi Z)g(Y, \varphi W)].\]

**Theorem 2.6.** $\tilde{M}$ is (K1) if and only if

\[R(X, Y, Z, W)\]

(1) $= R(X, Y, \varphi Z, \varphi W) - g(X, \varphi Z)g(Y, \varphi W) + g(Y, \varphi Z)g(X, \varphi W)
- g(Y, Z)g(X, W) + g(X, Z)g(Y, W).$

**Proof.** $\tilde{M}$ is (K1) if and only if $\tilde{g}(\tilde{R}(A, B)(jc), jd) = \tilde{g}(\tilde{R}(A, B)c, d)$
for all $A, B, C, D \in \chi(\tilde{M})$. We have:

1. $\tilde{g}(\tilde{R}(X, Y)(J\partial_t), JW) = \tilde{g}(\tilde{R}(X, Y)\partial_t, W)$
   
   \[\Rightarrow - t\left[g(R(X, Y)\xi, \varphi W) - \eta(Y)g(X, \varphi W) + \eta(X)g(Y, \varphi W)\right] = 0\]

   \[\Rightarrow g(\varphi(R(X, Y)\xi - \eta(Y)X + \eta(X)Y), W) = 0 \text{ for every } W\]

   \[\Rightarrow R(X, Y)\xi - \eta(Y)X + \eta(X)Y \in \ker(\varphi).\]

Thus,

\[R(X, Y)\xi = \eta(Y)X - \eta(X)Y\]

since both terms have no $\xi$-component.

2. $\tilde{g}(\tilde{R}(X, Y)(JZ), J\partial_t) = \tilde{g}(\tilde{R}(X, Y)Z, \partial_t)$

   \[\Rightarrow \tilde{g}\left(R(X, Y)(\varphi Z) - g(Y, \varphi Z)X + g(X, \varphi Z)Y, -\frac{1}{t}\xi\right)\]

   \[= \tilde{g}(R(X, Y)Z - g(Y, Z)X + g(X, Z)Y, \partial_t)\]

   \[\Rightarrow -\frac{1}{t} \eta(R(X, Y)(\varphi Z) - g(Y, \varphi Z)X + g(X, \varphi Z)Y) = 0.\]

Thus we have obtained,

(2) $R(X, Y)(\varphi Z) - g(Y, \varphi Z)X + g(X, \varphi Z)Y \in \ker(\eta).$

3. $\tilde{g}(\tilde{R}(X, Y)(JZ), JW) = \tilde{g}(\tilde{R}(X, Y)Z, W)$

   \[\Rightarrow \tilde{g}(R(X, Y)(\varphi Z) - g(Y, \varphi Z)X + g(X, \varphi Z)Y, \varphi W + t\eta(W)\partial_t)\]

   \[= \tilde{g}(R(X, Y)Z - g(Y, Z)X + g(X, Z)Y, W).\]
We obtain directly
\[ g(R(X,Y)(\varphi Z), \varphi W) - g(Y, \varphi Z)g(X, \varphi W) + g(X, \varphi Z)g(Y, \varphi W) = g(R(X,Y)Z, W) - g(Y, Z)g(X, W) + g(X, Z)g(Y, W) \]
and consequently
\[ R(\varphi W, \varphi Z, X, Y) - R(W, Z, X, Y) = g(Y, \varphi Z)g(X, \varphi W) - g(X, \varphi Z)g(Y, \varphi W) + g(X, Z)g(Y, W) - g(Y, Z)g(X, W). \]
Note that (2) implies (3) and (4) implies (2). □

As consequences we have
\[ R(\xi, Y, \xi, W) = g(Y, W), \]
\[ R(\xi, Y, Z, W) = R(\xi, Y, \varphi Z, \varphi W) = 0, \]
\[ R(X, Y, Z, W) - g(Y, W)g(X, Z) + g(X, W)g(Y, Z) = R(X, Y, \varphi Z, \varphi W) - g(Y, \varphi W)g(X, \varphi Z) + g(X, \varphi W)g(Y, \varphi Z), \]
where \(X, Y, Z\) and \(W\) are orthogonal to \(\xi\).

**Definition 2.7.** We say that an almost contact metric manifold satisfies (G1)-identity if its curvature tensor fulfills (1).

**Proposition 2.8.** The curvature tensor of a Sasakian manifold satisfies (G1) (see also Lemma 7.1 in [3]).

**Proposition 2.9.** Any contact manifold satisfying (G1) is Sasakian.

**Proof.** It is known (e.g. Proposition 7.6 from [3]) that a contact manifold is Sasakian if and only if \(R(X,Y)\xi = \eta(Y)X - \eta(X)Y\) for all \(X\) and \(Y\). □

Back to the cone manifold \(\tilde{M}\). We give:

**Theorem 2.10.** \(\tilde{M}\) is \((K2)\) if and only if
\[ R(X, Y, Z, W) \]
\[ = R(\varphi X, Y, Z, \varphi W) + R(X, \varphi Y, Z, \varphi W) + R(X, Y, \varphi Z, \varphi W) \]
\[ + g(X, Z)\eta(W)\eta(Y) - g(Z, Y)\eta(X)\eta(W). \]

**Proof.** \(\tilde{M}\) is \((K2)\) if and only if \(\tilde{R}(A, B, C, D) = \tilde{R}(JA, B, C, JD) + \tilde{R}(A, JB, C, JD) + \tilde{R}(A, B, JC, JD), \)
where \(A, B, C\) and \(D\) are arbitrary vector fields on \(\tilde{M}\).

Three cases are essential:
1) \(A = \partial_t, B = Y, C = \partial_t, D = W\) which is trivial.
2) \(A = \partial_t, B = Y, C = Z, D = W. \)
One has:

\[\tilde{R}(J\partial_t, Y, Z, JW) = -\frac{1}{t} \tilde{R}(\xi, Y, Z, \varphi W),\]

\[\tilde{R}(\partial_t, JY, Z, JW) = 0,\]

\[\tilde{R}(\partial_t, Y, JZ, JW) = 0.\]

It follows that the right side is equal to:

\[tg(\xi, R(Z, \varphi W)Y - g(\varphi W, Y)Z + g(Z, Y)\varphi W).\]

Since the left side vanishes, in this case we obtain

\[(6) \quad R(\xi, Y, Z, \varphi W) = \eta(Z)g(\varphi W, Y) \text{ for every } Y, Z, W \in \chi(M).\]

3) \(A = X, \ B = Y, \ C = Z, \ D = W.\) One has

\[\tilde{R}(JX, Y, Z, JW) = \tilde{R}(\varphi X, Y, Z, \varphi W),\]

\[\tilde{R}(X, JY, Z, JW) = \tilde{R}(X, \varphi Y, Z, \varphi W),\]

\[\tilde{R}(X, Y, JZ, JW) = \tilde{R}(X, Y, \varphi Z, \varphi W).\]

It follows that the right side is equal to

\[t^2[R(\varphi X, Y, Z, \varphi W) + R(X, \varphi Y, Z, \varphi W) + R(X, Y, \varphi Z, \varphi W)] + t^2[-g(\varphi W, \varphi Y)g(X, Z) + g(Z, Y)g(\varphi X, \varphi W)]\]

while the left side is equal to:

\[t^2R(X, Y, Z, W) + t^2[-g(W, Y)g(X, Z) + g(Z, Y)g(X, W)],\]

and hence we have (5). Since (5) implies (6), we get the statement. \(\square\)

As consequences one has

\[R(\xi, Y, \xi, W) = g(Y, W),\]

\[R(\xi, Y, Z, W) = 0,\]

\[R(X, Y, Z, W) = R(\varphi X, Y, Z, \varphi W) + R(X, \varphi Y, Z, \varphi W) + R(X, Y, \varphi Z, \varphi W)\]

for all \(X, Y, Z, W \in \chi(M).\)

**Definition 2.11.** We say that an almost contact metric manifold satisfies (G2)-identity if its curvature tensor fulfills (5).

Let us focus our attention to the third identity of Gray.

**Theorem 2.12.** The manifold \(\tilde{M}\) is (K3) if and only if

\[(7) \quad R(X, Y, Z, W) = R(\varphi X, \varphi Y, \varphi Z, \varphi W) + g(X, Z)\eta(W)\eta(Y) - g(Z, Y)\eta(X)\eta(W)\]

\[+ g(Y, W)\eta(X)\eta(Z) - g(X, W)\eta(Y)\eta(Z)\]

for all \(X, Y, Z, W \in \chi(M).\)
Proof. $\overline{M}$ is $(K3)$ if and only if $\overline{R}(A,B,C,D) = \overline{R}(JA,JB,JC,JD)$ for all $A, B, C, D \in \chi(\overline{M})$.

The essential cases are:

1) $A = \partial_t, B = Y, C = \partial_t, D = W$.

The left member vanishes and the right member is equal to $R(\xi, \varphi Y, \xi, \varphi W) - g(\varphi W, \varphi Y)$. We get

$$R(\xi, \varphi Y, \xi, \varphi W) = g(\varphi W, \varphi Y).$$

2) $A = \partial_t, B = Y, C = Z, D = W$.

The left hand vanishes and on the right hand we have $R(\xi, \varphi Y, \varphi Z, \varphi W)$. We get

$$R(\xi, \varphi Y, \varphi Z, \varphi W) = 0.$$

3) $A = X, B = Y, C = Z, D = W$.

Again, the left part is equal to

$$I^2[R(X, Y, Z, W) - g(Z, X)g(W, Y) + g(Y, Z)g(X, W)]$$

and the right part is equal to

$$I^2[R(\varphi X, \varphi Y, \varphi Z, \varphi W) - g(\varphi W, \varphi Y)g(\varphi X, \varphi Z) + g(\varphi Y, \varphi Z)g(\varphi X, \varphi W)].$$

Hence (7) is proved. Note that (7) implies both (8) and (9).

Consequently

$$R(\xi, Y, \xi, W) = g(Y, W),$$

$$R(\xi, Y, Z, W) = 0,$$

$$R(X, Y, Z, W) = R(\varphi X, \varphi Y, \varphi Z, \varphi W)$$

for all $X, Y, Z, W \in \chi(M)$ orthogonal to $\xi$.

**Definition 2.13.** We say that an almost contact metric manifold satisfies $(G3)$-identity if its curvature fulfills the relation (7).

### 3. The Boothby-Wang fibration

In this section we use the Boothby-Wang fibration [5] in order to strengthen previous results.

Let $M$ be a $(2n + 1)$-dimensional smooth manifold. A contact form on $M$ is a 1-form $\eta$ satisfying

$$\eta \wedge (d\eta)^n \neq 0.$$ We say that $\eta$ endows $M$ with a contact structure. It is clear that $\eta$ induces an orientation on $M$ and hence there is a global non vanishing vector field $\xi$ on $M$ so that $\eta(\xi) = 1$. If $\xi$ is regular in the sense of Palais (see [10]), then the contact structure (and also $M$) is called regular. If moreover $M$ is compact, one can consider the space of all orbits of $\xi$, i.e., $N = M/\xi$ thus obtaining a smooth manifold. We have:
Theorem A (5). Let \((M, \eta)\) be a compact, regular, contact manifold. Then \(M\) is a principal circle bundle over \(N\) and \(\eta\) is a connection form of this bundle. The curvature form \(\Theta\) of \(\eta\) defines a symplectic form on \(N\).

This fibration \(\mathbb{S}^1 \to M \xrightarrow{\pi} N\) is called the Boothby-Wang fibration.

Let \(\Omega\) the symplectic 2-form of \(N\). We denote by \(G\) the associated metric, i.e., \(\Omega(X, Y) = G(X, JY)\) with \(J\) the almost complex structure.

In the following, by \(X^\uparrow\) we denote the lift of a vector field \(X \in \chi(N)\). \(X^\uparrow\) is a horizontal vector field of \(M\). On \(M\) a \((1, 1)\) tensor field \(\varphi\) can be defined, namely

\[
\varphi X^\uparrow = (JX)^\uparrow, \quad \varphi \xi = 0.
\]

We can easily see that

\[
\varphi^2 = -I + \eta \otimes \xi.
\]

In this way, \((\varphi, \xi, \eta)\) becomes an almost contact structure. The metric \(G\) can be lifted and hence one defines \(g\) on \(M\) as follows:

\[
g = \pi^* G + \eta \otimes \eta.
\]

The metric \(g\) is compatible with the contact structure and \(\xi = \eta^\#\).

Without loss of generality one can suppose \(d\eta = \pi^* \Omega\) and thus we have

\[
g(X^\uparrow, \varphi Y^\uparrow) = G(X, JY) \circ \pi = \Omega(X, Y) \circ \pi = \pi^* \Omega(X^\uparrow, Y^\uparrow) = d\eta(X^\uparrow, Y^\uparrow).
\]

In this way, \((\varphi, \xi, \eta, g)\) becomes a contact metric structure on \(M\).

If the symplectic structure of \(N\) derives from a Kaehlerian structure \((J, G)\), the obtained structure on \(M\) is Sasakian (i.e., contact and normal manifold). See e.g. [3]. But generally, a symplectic structure need not come from a Kaehlerian one. Yet, one can always find an almost Kaehlerian structure inducing it. In this case, the contact structure on the total space of a Boothby-Wang fibration is K-contact, i.e., the vector field \(\xi\) is Killing, namely \(\mathcal{L}_\xi g = 0\). It easily follows that the integral curves of \(\xi\) are geodesics.

It is easy to prove the relation

\[
[X^\uparrow, Y^\uparrow] = [X, Y]^\uparrow - 2G(X, JY)\xi
\]

for all \(X, Y \in \chi(N)\).

Denote by \(\nabla^M\) and \(\nabla^N\) the Levi-Civita connections on \(M\) and \(N\), respectively. We have

\[
g(\nabla^M_{X^\uparrow} Y^\uparrow, Z^\uparrow) \circ \pi = G(\nabla^N_X Y, Z)
\]

for any \(X, Y, Z \in \chi(N)\). For the vertical part we shall compute \(g_{\nabla^M}: Y^\uparrow\):

\[
2g(\nabla^M_X Y^\uparrow, \xi) = X^\uparrow g(Y^\uparrow, \xi) + Y^\uparrow g(X^\uparrow, \xi) - \xi g(X^\uparrow, Y^\uparrow) + g([X^\uparrow, Y^\uparrow], \xi)
+ g([\xi, X^\uparrow], Y^\uparrow) + g(X^\uparrow, [\xi, Y^\uparrow])
= \eta([X^\uparrow, Y^\uparrow]) - (\mathcal{L}_\xi g)(X^\uparrow, Y^\uparrow)
= -2d\eta(X^\uparrow, Y^\uparrow).
\]
We obtain
\[ \eta(M \nabla^M_X X^\uparrow Y^\uparrow) \circ \pi = -G(X, JY). \]

In the following, we will ignore \( \pi \), due to the isomorphism between the horizontal distribution of \( T(M) \) and \( T(N) \). Hence
\[ M \nabla^M_X X^\uparrow Y^\uparrow = (N \nabla_X Y)\uparrow - G(X, JY)\xi. \]

In the same way, one can show
\[ M \nabla^M_X X^\uparrow \xi = -\varphi X^\uparrow. \]

Denote by \( R^M \) and \( R^N \) the curvature tensors of \( M \) and \( N \), respectively. Then
\[ R^M(X^\uparrow, Y^\uparrow)Z^\uparrow = (R^N(X, Y)Z)^\uparrow + g(Y^\uparrow, \varphi Z^\uparrow)\varphi X^\uparrow - g(X^\uparrow, \varphi Z^\uparrow)\varphi Y^\uparrow - 2g(x^\uparrow, \varphi Y^\uparrow)\varphi Z^\uparrow \]
\[ + \left\{ g(X^\uparrow, (M \nabla^M_Y \varphi)Z^\uparrow) - g(Y^\uparrow, (M \nabla^M_X \varphi)Z^\uparrow) \right\} \xi \]

and hence
\[ R^M(W^\uparrow, Z^\uparrow, X^\uparrow, Y^\uparrow) = R^N(W, Z, X, Y) \circ \pi - 2g(X^\uparrow, \varphi Y^\uparrow)g(W^\uparrow, \varphi Z^\uparrow) \]
\[ + g(Y^\uparrow, \varphi Z^\uparrow)g(W^\uparrow, \varphi X^\uparrow) - g(X^\uparrow, \varphi Z^\uparrow)g(W^\uparrow, \varphi Y^\uparrow). \]

Suppose that the base manifold \( N \) satisfies Gray identities. What are the corresponding curvature identities for the upstairs manifold \( M \)?

If \( N \) is \( (K_1) \), then
\[ R^M(\varphi X^\uparrow, Y^\uparrow, Z^\uparrow, W^\uparrow) = R^N(X^\uparrow, Y^\uparrow, Z^\uparrow, W^\uparrow) \]
\[ - g(Y^\uparrow, W^\uparrow)g(Z^\uparrow, X^\uparrow) - g(Y^\uparrow, \varphi W^\uparrow)g(Z^\uparrow, \varphi X^\uparrow) \]
\[ + g(X^\uparrow, W^\uparrow)g(Z^\uparrow, Y^\uparrow) + g(X^\uparrow, \varphi W^\uparrow)g(Z^\uparrow, \varphi Y^\uparrow). \]

If \( N \) is \( (K_2) \), then
\[ R^M(\varphi X^\uparrow, Y^\uparrow, Z^\uparrow, W^\uparrow) + R^M(X^\uparrow, \varphi Y^\uparrow, Z^\uparrow, W^\uparrow) \]
\[ + R^M(X^\uparrow, Y^\uparrow, \varphi Z^\uparrow, W^\uparrow) + R^M(X^\uparrow, Y^\uparrow, Z^\uparrow, \varphi W^\uparrow) = 0. \]

If \( N \) is \( (K_3) \), then
\[ R^M(\varphi X^\uparrow, \varphi Y^\uparrow, \varphi Z^\uparrow, \varphi W^\uparrow) - R^M(X^\uparrow, Y^\uparrow, Z^\uparrow, W^\uparrow) = 0. \]

These relations are exactly the defined Gray identities for almost contact metric manifolds for vector fields orthogonal to \( \xi \).
4. Properties and examples

In their paper [9], D. Janssens and L. Vanhecke have studied curvature tensors for almost contact metric structures and defined \(C(\alpha)\)-manifolds, namely those almost contact metric manifolds whose curvature tensor satisfies the following property:

\[
\exists \alpha \in \mathbb{R} \text{ such that for all } X, Y, Z, W \in \chi(M)
\]

\[
R(X, Y, Z, W) = R(X, Y, \varphi Z, \varphi W) + \alpha \left( g(X, Z)g(Y, W) - g(X, W)g(Y, Z) \right) + g(X, \varphi Z)g(Y, \varphi W) + g(X, \varphi W)g(Y, \varphi Z) \}
\]

(In the original paper [9] different signs appear due to the fact that the Riemann Christoffel curvature tensor is defined with the opposite sign.) This means that manifolds satisfying the first Gray identity \((K_1\varphi)\) in the sense of Bonome et al. are in fact \(C(0)\)-manifolds, while manifolds satisfying \((G1)\) are \(C(1)\)-manifolds. Note that cosymplectic, Sasakian and Kenmotsu manifolds are respectively \(C(0)\), \(C(1)\) and \(C(-1)\) manifolds (see Theorem 2.3, in [9]).

Let us come back to Gray identities for an almost Hermitian manifold.

It is known that \(K1 \Rightarrow K2 \Rightarrow K3\) (see [8], §5). Consequently we have:

**Proposition 4.1.** For a class \(L\) of almost contact metric manifolds, denote by \(L_i\) the subclass of manifolds whose curvature satisfies \(G_i\), \(i = 1, 2, 3\). Then we have the following inclusions

\[\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \mathcal{L}_3 \subseteq \mathcal{L}.\]

As Gray remarked for Kaehlerian manifolds, we can say that as \(i\) decreases, a manifold in \(\mathcal{L}_i\) resembles Sasakian manifold more closely.

**Proposition 4.2.** Let \((M, \varphi, \xi, \eta, g)\) be a K-contact manifold satisfying \(G1\) curvature identity. Then the manifold \(M\) is Sasakian.

**Proof.** By using Proposition 7.5 in [3, p. 94], a K-contact manifold whose curvature satisfies \(R_{XY} \xi = \eta(Y)X - \eta(X)Y\) is Sasakian. But this last relation is a consequence of \(G1\) identity. See also Proposition 2.9. \(\square\)

**Proposition 4.3.** Let \(M\) be a contact metric manifold for which \(\xi\) belongs to the \((\kappa, \mu)\)-nullity distribution, namely its curvature satisfies

\[
R_{XY} \xi = \kappa \left( \eta(Y)X - \eta(X)Y \right) + \mu \left( \eta(Y)hX - \eta(X)hY \right),
\]

where \(h = \frac{1}{2} \mathcal{L}_\xi \varphi\) and \(\kappa, \mu\) are constants. Suppose \(M\) satisfies \((G1)\) identity. Then \(M\) is Sasakian.

**Proof.** If \(M\) is \((G1)\), then \(R_{XY} \xi = \eta(Y)X - \eta(X)Y\) for all \(X, Y \in \chi(M)\). Combining with the fact that \(\xi\) belongs to the \((\kappa, \mu)\)-nullity distribution we obtain

\[
(\kappa - 1)(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) = 0
\]
for all $X, Y \in \chi(M)$. If $\mu \neq 0$ this implies $hY = \frac{1-\kappa}{\mu} Y$ for all $Y \in \ker(\eta)$. We know that $h$ anticommutes with $\varphi$ and hence one gets $\kappa = 1$. But using [3, Theorem 7.7, p. 103], it follows that $M$ is a Sasakian manifold. If $\mu = 0$ we immediately have $\kappa = 1$.

**Proposition 4.4.** Let $(M, \varphi, \eta, \xi, g)$ be a contact metric manifold satisfying $(G3)$ identity. Then $M$ is K-contact.

**Proof.** Choose a $\varphi$-adapted local orthonormal frame on $M$, namely $\{X_i, \varphi X_i, \xi\}$, $i = 1, \ldots, n$. Since $M$ is $(G3)$ the relation $R(X, \xi, Y, \xi) = g(X, Y)$ holds for all $X, Y \in \ker(\eta)$. Taking $X = Y = X_i$ (respectively $X = Y = \varphi X_i$) one immediately obtains $\text{Ric}(\xi, \xi) = 2n$, where $\text{Ric}$ is the Ricci tensor on $M$. Now we use the fact that a contact metric manifold is K-contact if and only if the Ricci tensor in the direction of the characteristic vector field $\xi$ is equal to $2n$ ([2, Theorem, p. 65]). □

As consequence, we can state:

**Proposition 4.5.** If $(M, \varphi, \eta, \xi, g)$ is a contact metric manifold satisfying $(G1)$ identity, then it is Sasakian.

**Proof.** The statement follows from Propositions 4.2 and 4.4. □

### 4.1. The generalized Heisenberg group $H(p, 1)$

It is defined as the set of matrices of real numbers having the form

$$a = \begin{bmatrix} 1 & A & c \\ 0 & I_p & tB \\ 0 & 0 & 1 \end{bmatrix},$$

where $I_p$ is the identity $p \times p$ matrix, $A = (a_1, \ldots, a_p)$, $B = (b_1, \ldots, b_p) \in \mathbb{R}^p$ and $c \in \mathbb{R}$. (Cf. [7].) $H(p, 1)$ is a connected, simply connected nilpotent Lie group of dimension $2p + 1$. We will consider $p = 2$. A global system of coordinates $(x^1, x^2, y^1, y^2, z)$ on $H(2, 1)$ is defined by $x^i(a) = a_i, y^i(a) = b_i$ for $i = 1, 2$ and $z(a) = c$. The global vector fields

$$X_i = 2 \frac{\partial}{\partial x^i}, \quad Y_i = 2 \left( \frac{\partial}{\partial y^i} + x^i \frac{\partial}{\partial z} \right)$$

for $i = 1, 2$, and $\xi = 2 \frac{\partial}{\partial z}$ are left invariant. We consider $\eta = \frac{1}{2}(dz - x^1 dy^1 - x^2 dy^2)$ and the metric

$$g = \frac{1}{4}(dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dy^1 \otimes dy^1 + dy^2 \otimes dy^2) + \eta \otimes \eta.$$

By direct computations we obtain that $d\eta = -\frac{1}{2}(dx^1 \wedge dy^1 + dx^2 \wedge dy^2)$ and $\xi$ is the characteristic vector field, namely $\eta(\xi) = 1$ and $i_\xi d\eta = 0$. Moreover, the basis defined above is orthonormal: $g(X_i, X_j) = g(Y_i, Y_j) = \delta_{ij}, g(\xi, \xi) = 1$ and $g(X_i, Y_j) = g(X_i, \xi) = g(Y_i, \xi) = 0$. One has $[X_i, Y_i] = 2\xi$ for $i = 1, 2$.
and the other brackets are equal to zero. Therefore it is easy to verify that the 
Levi-Civita connection is given by the following formulas:

\[ \nabla_\xi X_i = -Y_i = \nabla_X \xi, \]
\[ \nabla_\xi Y_i = X_i = \nabla_Y \xi, \]
\[ \nabla_X Y_i = -\nabla_Y X_i = \xi \]

for \( i = 1, 2 \), the other derivatives being zero. We compute also the Riemann-
Christoffel curvature tensor field:

\[ R(X_1, X_2, Y_1, Y_2) = -1, \quad R(X_1, Y_2, X_2, Y_1) = -1, \]
\[ R(X_1, Y_1, X_2, Y_2) = -2, \quad R(X_i, Y_i, X_i, Y_i) = -3, \]
\[ R(X_i, \xi, X_i, \xi) = 1 \quad R(Y_i, \xi, Y_i, \xi) = 1 \quad \text{for} \quad i = 1, 2. \]
The other values are zero or can be obtained from these ones.

Define \( \varphi \) by:

\[ \varphi X_1 = \cos \theta Y_1 + \sin \theta Y_2, \quad \varphi X_2 = \varepsilon \sin \theta Y_1 - \varepsilon \cos \theta Y_2, \]
\[ \varphi Y_1 = -\cos \theta X_1 - \varepsilon \sin \theta X_2, \quad \varphi Y_2 = -\sin \theta X_1 + \varepsilon \cos \theta X_2, \quad \varphi \xi = 0, \quad \varepsilon = \pm 1. \]

Hence \((H(2,1), \varphi, \xi, \eta, g)\) is an almost contact metric manifold.

**Proposition 4.6.** The structure is quasi Sasakian. Moreover, it is K-contact 
if and only if \( \varepsilon = -1 \) and \( \theta = 0 \). In this case \( H(2,1) \) becomes a Sasakian 
manifold.

**Proof.** It can be proved that \( \nabla_\xi \varphi = 0 \). Then for every \( X, Y \in \chi(M) \) we have

\[ g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = 0, \]

which means that \( \xi \) is Killing. The following relation holds on \( H(2,1) \):

\[ g(Y, (\nabla_X \varphi)Z) = \eta(Y)(\nabla_\varphi \eta)(Z) + \eta(Z)(\nabla_Y \eta)(\varphi X) \]

which characterizes quasi Sasakian manifolds. Direct computations yield the 
second part of the statement. \( \square \)

Moreover, we can give the following:

**Proposition 4.7.** On \( H(2,1) \)

1. the G3 identity holds if and only if \( \cos \theta = 0 \) or \( \sin \theta = 0 \) or \( \varepsilon = 1 \);
2. the G2 identity holds if and only if \( \varepsilon = 1 \) or \( \sin \theta = 0 \);
3. the G1 identity holds if and only if \( \varepsilon = -1 \) and \( \sin \theta = 0 \).

**Proof.** Straightforward computations. \( \square \)
4.2. Other examples

Let \((N, \bar{g}, J)\) be an almost Hermitian manifold. Consider the warped product manifold \(M = \mathbb{R} \times_f N\), where \(f = f(\theta)\) is the warping function and \(\theta\) is the global parameter on \(\mathbb{R}\). Denote by \(g = d\theta^2 + f^2(\theta)\bar{g}\) the Riemannian metric on \(M\). Define the global vector field \(\xi = \frac{\partial}{\partial \theta}\) and the 1-form \(\eta = d\theta\). Define also the \((1,1)\) tensor field \(\varphi\) by \(\varphi X = JX\) if \(X\) is tangent to \(N\) and \(\varphi \frac{\partial}{\partial \theta} = 0\).

Thus \((\varphi, \xi, \eta, g)\) is an almost contact metric structure on \(M\). If \(\bar{\nabla}\) and \(\nabla\) are the Levi-Civita connections on \(N\), respectively on \(M\), we have

\[
\nabla_{\xi}X = \nabla_X\xi = f' f X, \quad \nabla_{\xi}\xi = 0, \quad \nabla_X Y = \bar{\nabla}_X Y - f f' \bar{g}(X, Y) \xi
\]

for all \(X, Y\) tangent to \(N\).

The Riemann Christoffel curvature tensor is given by

\[
R(W, \xi, X, Y) = 0, \quad R(W, \xi, X, \xi) = -\frac{f''}{f} \bar{g}(X, W),
\]

\[
R(W, Z, X, Y) = f^2 \left[ \bar{R}(W, Z, X, Y) + (f')^2 (\bar{g}(X, Z)\bar{g}(Y, W) - \bar{g}(Y, Z)\bar{g}(X, W)) \right].
\]

In order to have one of the three curvature identities we directly have

\[
\frac{f''}{f} = -1
\]

which implies that \(f = \alpha \cos \theta + \beta \sin \theta\) with \(\alpha\) and \(\beta\) real constants. Now one can state the following:

**Proposition 4.8.** The manifold \(M\) is \(G_2\) (respectively \(G_3\)) if and only if the almost Hermitian manifold \(N\) is \(K_2\) (respectively \(K_3\)).

**Proof.** One has the following relations:

\[
R(\varphi W, Z, X, \varphi Y) + R(W, \varphi Z, X, \varphi Y) + R(W, Z, \varphi X, \varphi Y) = f^2 \left[ \bar{R}(JW, Z, X, JY) + \bar{R}(W, JZ, X, JY) + \bar{R}(W, Z, JX, JY) \right]
\]

and

\[
R(W, Z, X, Y) - R(\varphi W, \varphi Z, \varphi X, \varphi Y) = f^2 \left[ \bar{R}(W, Z, X, Y) - \bar{R}(JW, JZ, JX, JY) \right].
\]

Hence the statement. \(\square\)

**Remark 4.9.** If \(\dim N \geq 4\), then the manifold \(M\) cannot be \(G1\).

**Proof.** Suppose \(M\) satisfies \(G1\) identity. A straightforward computation gives

\[
\bar{R}(W, Z, JX, JY) - \bar{R}(W, Z, X, Y) = (1 + (f')^2) \left[ \bar{g}(JX, W)\bar{g}(JY, Z) - \bar{g}(JX, Z)\bar{g}(JY, W) + \bar{g}(Y, W)\bar{g}(X, Z) - \bar{g}(Y, Z)\bar{g}(X, W) \right].
\]
Since $f$ depends on $\theta$ (and it is not linear) while $\bar{g}$ and $\bar{R}$ do not, it follows that $N$ is $K1$ and
\[
\bar{g}(JX, W)\bar{g}(JY, Z) - \bar{g}(JX, Z)\bar{g}(JY, W) + \bar{g}(Y, W)\bar{g}(X, Z) - \bar{g}(Y, Z)\bar{g}(X, W) = 0
\]
for all $X, Y, Z, W$ tangent to $N$. This yields
\[
\bar{g}(JY, Z)JX - \bar{g}(JX, Z)JY + \bar{g}(X, Z)Y - \bar{g}(Y, Z)X = 0.
\]
If $\dim N \geq 4$ we can choose $X$ and $Y$ so that $X, Y, JX$ and $JY$ are linearly independent, so, the previous equality is impossible. □

Example 4.10. On $\mathbb{R}^4$ consider the global coordinates $x, y, u$ and $v$ respectively. Let $z \in I = (0, \pi/2)$. Define the warped product $M = I \times f \mathbb{R}^4$, where the warping function $f : I \mapsto \mathbb{R}$ is given by $f(z) = \cos z$. More precisely, consider the Riemannian metric $g = dz^2 + \cos^2 z (dx^2 + dy^2 + du^2 + dv^2)$. Let us define the almost contact structure by:
\[
\xi = \partial_z, \eta = dz, \varphi \partial_x = \partial_y, \varphi \partial_y = -\partial_x, \varphi \partial_u = \partial_v, \varphi \partial_v = -\partial_u \text{ and } \varphi \partial_z = 0.
\]
Then $M$ is $G2$ but not $G1$.

The same result holds if on $M$ consider the warped metric $g = dz^2 + \sin^2 z (dx^2 + dy^2 + du^2 + dv^2)$.

This kind of structure is called *sine-cone* and gives way to construct many geometric objects (e.g. nearly Kaehler structures starting from a 5-dimensional Sasaki Einstein manifold). Cf. [6].

Proposition 4.11. Let $N$ be a surface and consider the warped product manifold $M = I \times N$, where $I$ is an open interval. Then $M$ satisfies $G1$.

Proof. Being a surface, $N$ is automatically Kaehler. The almost contact structure is defined as in the beginning of this section. The statement follows from the fact that a Kaehler manifold is $K1$ and the equation (10) is satisfied in dimension 2. □

4.3. Hypersurfaces of almost Hermitian manifolds

Let $(\bar{M}, J, \bar{g})$ a $(2n+2)$-dimensional Kaehler manifold, and let $M$ be a totally umbilical (real) hypersurface in $\bar{M}$. Denote by $N$ the unit normal on $M$ and let $A, h$ be the Weingarten operator and the scalar-valued second fundamental form, respectively. As $M$ is totally umbilical, we have that $AX = \beta X$ for all $X$ tangent to $M$, with $\beta \in C^\infty(M)$.

It is well known the fact that on $M$ we can define an almost contact metric structure (see e.g. [3]). More precisely, we take $\xi = -JN$ and for $X \in \mathfrak{X}(M)$ we decompose $JX$ as:
\[
JX = \varphi X + \eta(X)N.
\]
Let $g$ be the restriction of the metric $\bar{g}$ on $M$. Denote by $\bar{\nabla}$ (respectively $\nabla$) the Levi-Civita connection on $\bar{M}$ (respectively on $M$). Then, by the formula of Gauss, one has
\[
\bar{\nabla}_X \xi = \nabla_X \xi + h(X, \xi)N.
\]
On the other hand, we have \( \tilde{\nabla}_X \xi = \tilde{J} \nabla_X N = JAX = \varphi AX + \eta(AX)N \). Hence
\[
\nabla_X \xi = \varphi AX \quad \text{and} \quad h(X, \xi) = \eta(AX).
\]
Suppose now that \( M \) satisfies the \((G3)\) identity. This implies
\[
R(X, \xi, Y, \xi) = g(X, Y) \quad \forall X, Y \in \ker(\eta).
\]
We should compute \( R(X, \xi) \xi = \nabla_X \nabla_\xi \xi - \nabla_\xi \nabla_X \xi - \nabla_{[X, \xi]} \xi = \phi AX + \eta(AX)N \). Since \( M \) is totally umbilical, we have \( \nabla_X \xi = \beta \varphi X \). Thus \( \mathcal{V}_X \xi = 0 \). Then
\[
\nabla_\xi \nabla_X \xi = (\beta \varphi) X + \beta (\phi \xi) X + \beta \varphi \nabla_\xi X.
\]
But
\[
\nabla_\xi X = \beta \varphi X - [X, \xi]
\]
and so
\[
\nabla_\xi \nabla_X \xi = (\beta \varphi) X + \beta (\phi \xi) X + \beta^2 \varphi^2 X - \beta \phi [X, \xi].
\]
It follows that
\[
R(X, \xi) \xi = -\xi(\beta \varphi) X - \beta (\phi \xi) X + \beta^2 X.
\]
Now, due to the fact \( M \) is Kähler, we have
\[
\tilde{\nabla}(JY) = J\tilde{nabla}_X Y = J(\nabla_X Y + h(X, Y)N) = \varphi \nabla_X Y + \eta(\nabla_X Y)N - h(X, Y)\xi.
\]
On the other hand
\[
\tilde{\nabla}(JY) = \tilde{\nabla}_X (\varphi Y + \eta(Y)N) = \nabla(\varphi Y) + h(X, \varphi Y)N + X \eta(Y)N - \eta(Y)\beta X.
\]
Identifying the tangent and the normal parts of \( \tilde{\nabla}(JY) \) we obtain
\[
(\nabla_X \varphi) Y = \beta \eta(Y) X - \beta g(X, Y) \xi,
\]
\[
(\nabla_X \eta)(Y) = -\beta g(X, \varphi Y),
\]
respectively.

Putting \( X = \xi \) in (12) we have \( (\nabla_\xi \varphi) Y = \beta \eta(Y) \xi - \beta g(\xi, Y) \xi = 0 \) which implies
\[
\nabla_\xi \varphi = 0.
\]
Then
\[
R(X, \xi) \xi = -\xi(\beta \varphi) X + \beta^2 X.
\]
From (11) we have
\[
g(\beta^2 X - \xi(\beta \varphi) X, Y) = 0, \quad \forall Y \in \ker(\eta).
\]
As \( X \) and \( \varphi X \) are linearly independent (and belong to \( \ker(\eta) \)), we obtain \( \beta = \pm 1 \).

Consequently
\[
AX = \pm X \quad \text{and} \quad h(X, Y) = \pm g(X, Y).
\]
For \( \beta = -1 \) it follows that
\[
(\nabla_X \varphi) Y = g(X, Y) \xi - \eta(Y) X.
\]
According to Theorem 6.14 in [3] this implies that \( M \) is Sasakian.
Proposition 4.12. Let $M$ be a totally umbilical hypersurface of a Kaehler manifold $\tilde{M}$ endowed with the usual almost contact metric structure. If $M$ satisfies $G_3$ identity, then $M$ is a Sasakian manifold and hence $M$ satisfies all $G_i$ for $i = 1, 2, 3$.

More generally, if the second fundamental form of $M$ is given by
$$h(X, Y) = \lambda \eta(X)\eta(Y) + \mu g(X, Y), \quad \forall X, Y \in \mathfrak{X}(M)$$
with $\lambda$ and $\mu$ smooth functions on $M$, i.e., $M$ is totally quasi umbilical, and if $M$ satisfies $(G_3)$ identity, then it is Sasakian. As consequence, there is no cylindrical submanifold satisfying $(G_3)$ and whose second fundamental form is $h(X, Y) = \lambda \eta(X)\eta(Y)$.

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