An Improved Algorithm to Enhance Recovery Stability for Low-Rank Image

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\textbf{Abstract.} When recovering original image for low-rank image with noise, the effect usually has not been satisfactory through minimizing matrix nuclear norm to obtain low-rank resolution. Aiming at this problem, we introduced Frobenius norm of low-rank matrix, and combined with original low-rank matrix nuclear norm to form new regular item, and utilized an augmented lagrange multiplier method to resolve the problem after convex relaxation. Through adding $\| \mathbf{F} \|$ item on the base of original low-rank image recovery model, we can get better denoising result for low-rank image. The experimental results indicated that, by selecting proper parameters, the improved algorithm has superior recovery result compared to traditional LRMR model on wiping off impulse noise and gaussian noise.

\section*{Introduction}

Recently, sparse representation for vector data has been becoming a new form of expression for data in compressed sensing theory [1]. Low-rank matrix recovery (LRMR) generalize has generalized the data representation to matrix, and has been becoming a research hotspot for image sparse representation [2]. At present, LRMR have made significant progress in the field of artificial intelligence [3]. For example, LRMR can be used in face recognition [3]. Literature [4, 5] put forward a RAIA (Robust Associate with Image Alignment) algorithm based on LRMR which can remove obstructions on face. Literature [6] applied LRMR to analysis of texture of low-rank image, and realized an adjustment for characters in images. Literature [7] applied LRMR to background modeling for surveillance video, and distinguished active prospects from motionless background.

There are usually much noise such as Gaussian noise and impulse noise in real images, which leads bad recovery stability from original images when applied LRMR algorithms. Although LRMR can separate low-rank matrix from sparse matrix for low-rank images involved noise, sparse matrix is generally sparse in distribution and has big random amplitude. Concerning this issue, we used Frobenius norm of matrix to be recovered as new regular term, and joined original core norms of matrix to be recovered to perform regularization. In the above scheme, sparse matrix may include random impulse noise and even Gaussian noise, which enhanced denoising property. Simulation results showed, the proposed model proposed in the paper based on the above scheme had an obvious effect for all kinds of noise removing and improved recovery stability for low-rank matrix.

\section*{Solution and Model for Low-Rank Matrix Recovery}

\textbf{Model for Low-Rank Matrix Recovery}

Low-rank matrix recovery, which called low-rank sparse matrix decomposition (LRSMD) or robust principal component analysis (RPCA) [8], is to recognize the broken elements automatically and recover original matrix when some elements in matrix were broken. The prediction of recovering matrix is that matrix is low-rank or rough low-rank. Suppose matrix $\mathbf{E}$ is a sparse
matrix with noise, and $D$ is a low-rank that destroyed by $E$, the recovery of matrix $D$ may be regarded as a following optimization problem:

$$\min \text{rank}(A) + \lambda \|E\|_0 \quad \text{s.t.} \quad A + E = D$$  \hspace{1cm} (1)$$

where $\|E\|_0$ is the number of nonzero elements of sparse matrix, namely, also known as norm $\ell_0$ of $E$. Although Eq(1) can be realized in theory, calculated amount is actually very large, in other words, the calculation of Eq(1) is a NP hard problem, so, we only give approximate solution of it. In order to find suitable norm, based on the Candes’ proving, for minimized resolution, norm $\ell_1$ (the sum of absolute value of all elements in matrix) is close to norm $\ell_0$, and core norm is norm $\ell_1$ of singular value. So, the rank function of matrix of Eq(1) can be approximated with core norm, namely,

$$\min \|A\|_1 + \lambda \|E\|_1 \quad \text{s.t.} \quad A + E = D$$  \hspace{1cm} (2)$$

where

$$\|A\|_1 = \sum_{k=1}^{n} \sigma_k(A)$$

$$\lambda = 1/\sqrt{\max(m,n)}$$

In Eq.(2), $\|A\|_1$ is core norm, $\sigma_k(A)$ is the $k$-th singular value of matrix $A$, and $\lambda$ is weight. This equation converts resolution of convex optimization to resolution of traces of matrix [9].

**The Resolution Based on Alternate Augmented Lagrange Multiplier**

After proved possibility of low-rank matrix recovery in theory by Recht et al, many scholars have deeply studied on recovery algorithms. For the resolution of Eq. (2), literature [10] adopted iterative threshold algorithm, which has simple calculation but low rate of convergence and difficult choice for step size. To solve the question, a quick threshold algorithm with less iteration time was proposed in literature [10]. Literature [11] proposed a gradient algorithm with accelerating adjacent areas, although this algorithm is similar to the quick threshold iterative algorithm, it reduced iterative times. Literature [12] proposed an alternating direction multiplier algorithm, which is similar to augmented lagrange multiplier algorithm (ALM). ALM is a low-rank matrix recovery algorithm with less memory space, higher speed and better precision [13]. In this paper, we mainly studied optimization procedure of ALM.

Augmented lagrange function need to be constructed for applying ALM to low-rank matrix recovery, as shown in eq. (3)

$$L(A, E, Y, \mu) = \|A\|_1 + \lambda \|E\|_1 + \langle Y, D - A - E \rangle + \frac{\mu}{2} \|D - A - E\|_F^2$$  \hspace{1cm} (3)$$

where $Y$ is lagrange multiplier, and $\mu$ is penalty parameter for convergent of algorithm.

Based on the proof of literatures [13, 14], $A$ and $E$ need to be updated only once, we can get an approximate solution, which make the algorithm to converge to optimal solution. Hereby, we can get a quicker algorithm. An inexact augmented lagrange multiplier is shown below:

$$A_{k+1} = \arg \min L(A_{k+1}, E_k, Y_k, \mu_k) = D - E_k + \mu_k^{-1} Y_k$$  \hspace{1cm} (7)$$

$$E_{k+1} = \arg \min L(A_k, E_{k+1}, Y_k, \mu_k) = S_{\mu_k}^{-1} (D - A_k + \mu_k^{-1} Y_k)$$  \hspace{1cm} (8)$$
Algorithm flow is shown in Fig. 1. IALM algorithm has been largely successful in denoising of low-rank image, such as faster than APG, and same precision, which made the study and application of LRMR to be got considerable progress. However, the processing of Gaussian noise of low-rank image is not very satisfactory. To address this problem, we improved LRMR model to process Gaussian noise and impulse noise very well.

The Improvement of Denoising Algorithm for Low-Rank Matrix

Stability of Low-Rank Recovery

The model of low-rank matrix recovery has usually special requirements to sparse matrix, which leads to instability of denoising, and limits practical application of model of low-rank matrix recovery. Literature [15] explained that sparsity and stability of solution of model cannot be satisfied simultaneously, and the balance of the both needs to be considered under regarding both of Frobenius norm and L1 norm as regularization term. The model in Literature [16] combined L1 norm with L2 norm as penalty function, and combined with least square regression, which got the balance between stability and sparsity.

Usually, L1 norm can generate sparse solution, but has some disadvantages such as super compression or unstable solution while L2 norm has better stability but difficult sparse solution. Known from previous analysis, due to the only considering of regulation term of L1 norm in model, for the image with stronger correlation, it is possible for low-rank recovery problem we cannot get stable solution. Concerning this issue, in the recovery model for low-rank image in this paper, we combined core function of matrix that will be recovered with Frobenius norm, and utilized core function to control sparsity and uniqueness of matrix that will be recovered, and utilized Frobenius norm to control stability of solution, to achieve goal of enhancing image recovery stability. Improved recovery model is shown as Eq. (9).

\[
\text{min} \| A \|_F^\gamma + \lambda \|E\|_1 + \gamma \|A\|_F^\delta \quad \text{s.t.} \quad A + E = D
\]  

(9)
**Improved Algorithm**

The basic idea of the improved algorithm is to use augmented Lagrangian to solve recovery problem of matrix which including $\|A\|_F^2$ term. The function of augmented Lagrangian including $\|A\|_F^2$ is as following:

$$L(A,E,Y,\mu) = |A| + \lambda \|E\|_F + \mu |A| + \langle Y, D - A - E \rangle + \frac{\mu}{2} \|D - A - E\|_F^2$$  \hspace{0.5cm} (10)

Adopt variable separating method to resolve eq.(10), the flow chart of the improved algorithm is shown in Fig.2.

Objective function is strong convex. Combined $|A|$, with $\|A\|_F$, uniqueness and stability of solution can all be enhanced? The following experiment deeply demonstrated the performance of improved model.

**Experimental Results and Analysis**

**Comparison of Performance of Algorithms**

Comparative result between improved algorithm and IALM is shown in Table 1. Matrix is generated randomly by triad of $\{m, \text{rank}(A), \|E\|\}$, where $m$ is dimensions of matrix. In experiment, a matrix $A^T = LR^T$ which rank equals $r$ was generated, where $L$ and $R$ are $m \times r$ random matrices. Meanwhile, a sparse matrix $E^*$ with number of nonzero elements of $\|E\|$, was generated, and its nonzero elements are even distributed in area of [-500,500], and at last, an object matrix was generated according to $D = A^* + E^*$.

We applied the above algorithms to resolve $\hat{E}$ and $\hat{A}$, and calculated run time and iteration time of each algorithm. Simulation results indicated that improved model has higher precision, but consumed more time than LRMR in de-nosing. With increase of dimensions, calculating speed of the improved model increased substantially. For processing Gaussian noise, the improved model was less than LRMR one percent point in relative error, but its precision and calculating speed were superior to LRMR obviously.

| $m$ | rank | method     | Uniform distribution(sparse rate:0.05) | Gaussian distribution(mean value:0, variance:0.01) |
|-----|------|------------|----------------------------------------|-----------------------------------------------|
|     |      |            | Time(s) | iterations | Time(s) | iterations |
| 100 | 10   | Improved algorithm | 0.5923 | 31 | 0.6722 | 37 |
|     |      | LRMR      | 0.5142 | 24 | 0.6843 | 36 |
| 20  | 20   | Improved algorithm | 0.8324 | 32 | 0.8746 | 40 |
|     |      | LRMR      | 0.8102 | 33 | 0.9102 | 39 |
| 200 | 20   | Improved algorithm | 2.3247 | 29 | 3.5867 | 41 |
|     |      | LRMR      | 2.1358 | 31 | 4.1203 | 39 |
| 40  | 40   | Improved algorithm | 4.5726 | 30 | 4.8923 | 40 |
|     |      | LRMR      | 4.6781 | 31 | 5.2314 | 38 |
| 500 | 50   | Improved algorithm | 17.3242 | 31 | 21.4356 | 39 |
|     |      | LRMR      | 18.9756 | 29 | 24.7238 | 39 |
| 100 | 100  | Improved algorithm | 26.4327 | 32 | 29.3201 | 39 |
|     |      | LRMR      | 28.9435 | 30 | 32.1208 | 39 |
Samples for Image Denoising

In order to verify the denoising performance of the improved model, taking monochrome bar code images with resolution of 512×512 as examples, we tested the improved model proposed in this paper under increasing impulse noise and Gaussian noise in images from 10% up to 50%. Image recovery results are shown as Table 2.

Fig.3 shows the comparation of recovery error rate of barcode images with resolution of 512×512 between improved model and LRMR model, and Fig.4 shows the comparation of signal to noise ratio of the two models.

Table 2. Recovery results of noise image using improved model.

| Noise/% | SNR/dB | PSNR/ dB | Time/s | Error noise variance | SNR/ dB | PSNR/ dB | Time/s | Error |
|---------|--------|----------|--------|----------------------|---------|----------|--------|-------|
| 10      | 67     | 82       | 6.89   | 0.00011              | 0.00625 | 43       | 45     | 6.11  | 0.00423 |
| 20      | 54     | 75       | 7.97   | 0.00019              | 0.0125  | 40       | 42     | 7.12  | 0.00581 |
| 30      | 48     | 60       | 8.12   | 0.00033              | 0.025   | 34       | 39     | 7.18  | 0.01296 |
| 40      | 38     | 48       | 9.83   | 0.00075              | 0.05    | 31       | 36     | 7.92  | 0.02432 |
| 50      | 25     | 32       | 11.28  | 0.00283              | 0.1     | 28       | 30     | 8.46  | 0.10794 |

As you can see in Fig.3, when impulse noise reaches 40%, the error rate of the improved model is on the level of $10^{-4}$, which is very ideal. When impulse noise increase to 50%, the error rate of the improved model increase to the level of $10^{-3}$, which still lower than LRMR model with the error rate of level of $10^{-2}$. Fig.4 includes recovery SNR and PSNR of barcode images, and shows that, with the increasement of impulse noise, the SNR of improved model reduce gradually, and when noise reaches 50%, the SNR of improved model is lower than 30dB, but is higher than LRMR. For Gaussian noise, the PSNR of improved model is higher than LRMR up 10dB, which shows greater advantage. Especially, when variance reaching 0.05, PSNR even reach up to 30dB.

Through adding scratch, 40% impulse noise and Gaussian noise with variance of 0.05 into original low-rank images, we verified the performance of LRMR and the improved algorithm, as shown in Fig.5. It can be seen from Fig.5 that satisfied effects are obtained for the recovery of the three kinds of images using improved algorithm proposed in the paper, especially, signal to noise rate reaches up to 20dB due to removing Gaussian noise, and higher than LRMR nearly 10dB, and error rate is lower than LRMR roughly two point of percent. On recovery time, the improved algorithm is also quicker than LRMR two seconds, while LRMR has few effect for Gaussian noise, and has poor satisfied result for scratch besides impulse noise.
Figure 5. The comparison of de-noising effect between improved model and LRMR.

Fig.6 shows the processing results for the real low-rank images with 40% of impulse noise using improved model.
Figure 6. Comparison of de-noising effect between LRMR and improved model.

Fig.6 shows the effect of removing impulse noise. It can be seen from Fig.6 that, for CT images and wall images, the recovery effect of LRMR is not good due to some impulse noise, while the improved model proposed in the paper has better effect for removing impulse noise, which indicates better image recovery effect. Table 3 shows the comparing result of improved model and LRMR for removing impulse noise.

Table 3 gives the results of two kinds of algorithms in recovery time, error rate and SNR. As shown in Table 3, for images with weak correlation such as cloth, wallpaper, the proposed model has slower recovery time but better error rate and SNR than LRMR algorithm and for images with strong correlations such as wall space, CT, the propose algorithm not only has better performance in error rate and SNR, but also has less recovery time (faster more than 2s than LRMR). This is because the images with strong correlation have more dimensions and bigger ranks in recovered matrix. Taken together, the algorithm we proposed in this study has better recovery performance for the images with strong correlation, which makes the algorithm to own bigger real meaning.

Table 3. The comparation of recovery performance between improved model and LRMR in Fig.6.

| Images  | Algorithms | Time/s | SNR/dB | PSNR/dB | Error |
|---------|------------|--------|--------|---------|-------|
| Cloth   | LRMR       | 7.214  | 21     | 24      | 0.0039|
|         | Improved model | 10.345 | 23     | 25      | 0.0031|
| Wallpaper| LRMR       | 8.782  | 19     | 22      | 0.0047|
|         | Improved model | 11.079 | 22     | 14      | 0.0042|
| Wall space| LRMR     | 10.823 | 13     | 17      | 0.0202|
|         | Improved model | 8.791  | 16     | 20      | 0.0107|
| CT      | LRMR       | 12.565 | 9      | 12      | 0.0225|
|         | Improved model | 10.259 | 12     | 15      | 0.0178|

Conclusion

In this paper, we proposed an improved denoising algorithm which can recover effectively original images corrupted by impulse noise or gaussian noise. By improving LRMR model, the proposed model enhanced inhibiting effect to gaussian noise and increased stability of recovery for images with strong correlation. By selecting parameters, the improved model can remove impulse noise and gaussian noise to the hilt, which makes the improved algorithm to be superior to LRMR obviously in recovery effect for images, especially for real images with strong correlation. Besides the above advantages, the improved model can also increase speed of image recovery. Experimental results show that the model proposed in the paper is an effective recovery method for noisy low-rank images.
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