Irreducibility criterion for the set of two matrices

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Abstract

We give the criteria for the irreducibility, the Schur irreducibility and the indecomposability of the set of two $n \times n$ matrices $\Lambda_n$ and $A_n$ in terms of the subalgebra associated with the "support" of the matrix $A_n$, where $\Lambda_n$ is a diagonal matrix with different non zeros eigenvalues and $A_n$ is an arbitrary one. The list of all maximal subalgebras of the algebra Mat($n, \mathbb{C}$) and the list of the corresponding invariant subspaces connected with these two matrices is also given. The properties of the corresponding subalgebras are expressed in terms of the oriented graphs associated with the support of the second matrix.

For arbitrary $n$ we describe all minimal subsets of the elementary matrices $E_{km}$ that generate the algebra Mat($n, \mathbb{C}$).

Key words: matrix algebra, representation, irreducible, Schur irreducible, indecomposable representation, invariant subspace, graph theory, strongly (weakly) connected directed graph (digraph),

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1 Introduction

In the representation theory of different objects (groups, rings, algebras etc.) the problem of the irreducibility of the concrete representations (modules) sometimes reduces to the irreducibility of the algebra, generated by two operators or by two matrices if the representation is finite dimensional.

In the case of the discrete group generated by two elements this is exactly the problem one need to solve. The most popular examples are the following: the free group $\mathbb{F}_2$ generated by two elements, the Artin braid group $B_3$ on three strands, the group $\text{PSL}(2, \mathbb{Z}) = \text{SL}(2, \mathbb{Z})/\pm 1$.

We give the criteria of the irreducibility and the Schur irreducibility (see below the definitions) of the set of two complex $n \times n$ matrices $\Lambda_n$ and $A_n$ in terms of the “support” of the matrix $A_n$, where $\Lambda_n$ is a diagonal matrix with distinct nonzero eigenvalues and $A_n$ is an arbitrary one (Theorem 6). The list of all invariant subspaces for this two matrices is also given (Theorem 7).

This criterion allows us to study completely in [2] the irreducibility of some family of representations depending on the parameters of the braid group $B_3$ in any dimensions.

There are three different notions connected with the irreducibility of the representations $T$ of a group $G$ in a complex vector space $V$

$$G \ni g \mapsto T_g \in \text{GL}(V),$$

where $\text{GL}(V)$ is the group of invertible linear operators on a space $V$. They are the following: 1) irreducible, 2) Schur irreducible, 3) indecomposable.
Definition 1 We say that a representation is irreducible (resp. Schur irreducible) if there are no nontrivial invariant closed subspaces for all operators of the representation (resp. there are no nontrivial bounded operators commuting with all operators of the representation). The representation is indecomposable if it can not be presented as the direct sum of the subrepresentations.

Remark 2 It is well known that the relations between the mentioned notions when the space $V$ is finite dimensional are as follows: $1) \Rightarrow 2) \Rightarrow 3)$.

Remark 3 The notions of the irreducibility and the Schur irreducibility coincide for the unitary representation of an arbitrary group $G$ (hence, for an arbitrary representation of a compact group due to the "Wayl trick" [7]).

Counterexample 1. $2) \not\Rightarrow 1)$. Let us consider the subalgebra of the algebra $\text{Mat}(2, \mathbb{C})$ consisting of matrices

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \ a, \ b, \ c \in \mathbb{C}.\)$$

This subalgebra is subspace reducible (the subspace in $\mathbb{C}^2$ generated by the vector $(1, 0)$ is invariant) but the algebra is Schur irreducible.

Counterexample 2. $3) \not\Rightarrow 2)$. The classical example of the Schur reducible but the indecomposable representation of the additive group of $\mathbb{C}$ is as follows:

$$\mathbb{C} \ni z \mapsto \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \in \text{GL}(2, \mathbb{C}).$$

(1)

1.1 Irreducibility criteria

Let $\text{Mat}(n, \mathbb{C})$ be the algebra of all complex matrices over the field of complex numbers $\mathbb{C}$ and let $\Lambda_n$ (resp. $A_n$) be a diagonal (resp. an arbitrary) matrix in $\text{Mat}(n, \mathbb{C})$:

$$\Lambda_n = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n), \ A_n = (a_{km})_{k,m=1}^n \in \text{Mat}(n, \mathbb{C}).$$

Definition 4 We call the support of the matrix $A = (a_{km})_{k,m=1}^n$ the subset of indices $(k, m)$ for which the corresponding entries $a_{km}$ are nonzero i.e.

$$\text{Supp}(A) = \{(k, m) \in \{1, 2, \ldots, n\}^2 \mid a_{km} \neq 0\}. \ \ (2)$$

Remark 5 It is well known that the algebra $\text{Mat}(n, \mathbb{C})$ acting on the space $\mathbb{C}^n$ is irreducible (and hence Schur irreducible).

Notation. Denote by $E_{km}$ the matrix units i.e. the matrix in which the $(k, m)$ entry is 1 and the other are zero. Obviously $E_{km}E_{pq} = \delta_{mp}E_{kq}$, where $\delta_{mp}$ are
The Kronecker symbols.

**Theorem 6** Let all eigenvalues \( \lambda_k \) of \( \Lambda_n \) be distinct and nonzero. Then

(i) the family of two matrices \((\Lambda_n, A_n)\) is irreducible if and only if the set \( \{ E_{km} \mid (k, m) \in \text{Supp}(A_n) \} \) generates the algebra \( \text{Mat}(n, \mathbb{C}) \);

(ii) the family \((\Lambda_n, A_n)\) is Schur irreducible if and only if the set \( \{ E_{km} \mid (k, m) \in \text{Supp}(A_n) \cup \text{Supp}(A'_n) \} \) generates the algebra \( \text{Mat}(n, \mathbb{C}) \);

(iii) the family \((\Lambda_n, A_n)\) is indecomposable if and only if it is Shur irreducible.

For a pair of two complex \( n \times n \) matrices \( A \) and \( B \) the following problems have been studied. When they have 1) a common eigenvectors; 2) a common invariant subspace of dimension \( k \), \( 2 \leq k < n \)?

In 1984 Dan Shemesh [5] shows that the criteria for 1) is:

\[
\bigcap_{k,l=1}^{n-1} \ker[A^k, B^l] \neq 0.
\]

In [4], under the additional assumption that at least one of the matrix \( A \) and \( B \) has distinct eigenvalues, were given some sufficient conditions for 2) in terms of \( k \)th compound matrix \( C_k(A) \) and \( C_k(B) \) of the matrix \( A \) and \( B \) (for definition see e.f. [3], chapt. I, § 4). Namely, 2) holds if the matrix \( C_k(A) \) and \( C_k(B) \) have a common invariant vector.

The **advantage of our approach** is that in the case where one of the matrices is diagonal, we give the **criteria for 2)** in terms of the support of the second matrix. Namely, 2) holds if and only if subalgebra generated by \( \{ E_{km} \mid (k, m) \in \text{Supp}(A_n) \} \) is contained in the subalgebra \( s_i(n) \) defined by (7) where \( i = \{ i_1, i_2, \ldots, i_k \} \). The list of all invariant subspaces for this two matrices is also given (Theorem 7). In Section 3 we reformulate Theorems 6 and 7 in terms of the directed graph associated with the support of the second matrix. It allows us to make use of **graph theory** (which is well developed).

### 1.2 Irreducibility

**PROOF.** (i) The sufficiency part “ \( \Leftarrow \) ” is obvious due to Remark 5. Indeed let us denote by \( \mathfrak{A}_n \) the algebra generated by matrices \( \Lambda_n \) and \( A_n \). Since \( \lambda_k \) are distinct and nonzero we conclude that \( E_{kk} \in \mathfrak{A}_n \), \( 0 \leq k \leq n \). Further, since

\[
E_{kk}A_nE_{mm} = a_{km}E_{km},
\]

we conclude that \( E_{km} \in \mathfrak{A}_n \) if \( a_{km} \neq 0 \) i.e. if \( (k, m) \in \text{Supp}(A_n) \).

To prove the necessity part “ \( \Rightarrow \) ” for any fixed \( n = 1, 2, \ldots \), let us suppose that the set \( \{ E_{km} \mid (k, m) \in \text{Supp}(A_n) \} \) does not generate the whole algebra \( \text{Mat}(n, \mathbb{C}) \), but only some proper subalgebra \( s(n) \) of the following form

\[
s(n) = \{ x \in \text{Mat}(n, \mathbb{C}) \mid x = \sum_{(k,m) \in S(n)} x_{km}E_{km} \},
\]

5
corresponding to some subset of indices $S(n) \subseteq \{1, ..., n\}^2$. We can suppose that this subalgebra is maximal proper subalgebra of the form (3). Indeed, if we can find the invariant subspace $V$ for the maximal subalgebra hence this subspace would be also invariant one for any of its subalgebra. By Theorem 7 the list of the maximal proper subalgebras $s(n)$ in $\text{Mat}(n, \mathbb{C})$ of the form (3) is:

$$s_1(n) = \{ x = (x_{km})_{k,m=1}^n \in \text{Mat}(n, \mathbb{C}) \mid x_{km} = 0, k \in \hat{i}, m \in i \},$$

where $i = \{i_1, i_2, ..., i_k\} \subseteq \{1, 2, ..., n\}$, $k \leq n$ and $\hat{i} = \{1, 2, ..., n\} \setminus i$.

**Notation.** For each $n$ let $V_{i_1i_2...i_k}(n) := \langle e_{i_1}, e_{i_2}, ... e_{i_k} \rangle$ be the linear subspace in $\mathbb{C}^n$ generated by the vectors $e_{i_1}, e_{i_2}, ... e_{i_k}$, $1 \leq i_1 < i_2 < ... < i_k \leq n$, where $e_k = (\delta_{rk})_{r=1}^n \in \mathbb{C}^n$, $1 \leq k \leq n$.

The subspace $V_1(n) = V_{i_1i_2...i_k}(n)$ is an invariant subspace for the algebra $s_1(n)$.

**(ii)** To prove the Schur irreducibility we note that the commutant $(\Lambda_n)' := \{ B \in \text{Mat}(n, \mathbb{C}) \mid [\Lambda_n, B] = 0 \}$ of the operator $\Lambda_n$ has the following form:

$$(\Lambda_n)' = \{ B \in \text{Mat}(n, \mathbb{C}) \mid B = \text{diag}(b_k)_{k=1}^n \}$$

hence, the relation $[A_n, B] = 0$ is equivalent to

$$a_{km}b_m = b_ka_{km}, \quad 1 \leq k, m \leq n. \quad (4)$$

We say that we can weakly connect $k$ and $m$ where $k, m \in \{1, 2, ..., n\}$ if $a_{km} \neq 0$ or $a_{mk} \neq 0$, i.e. $(k, m) \in \text{Supp}(A_n)$ or $(k, m) \in \text{Supp}(A_n')$. In this case $b_k = b_m$. To show that all $b_k$ coincide (i.e. that $B = bI$) we should be able to connect step by step all $k$ and $m$ i.e. for any $(k, m) \in \{1, 2, ..., n\}^2$ we need to find a sequence $(k_r, m_r)_{r=1}^l \subseteq \text{Supp}(A_n) \cup \text{Supp}(A_n')$, such that

$$E_{km} = E_{k_1,m_1}E_{k_r,m_r}...E_{k_l,m_l}. \quad (5)$$

This proves the sufficiency part of the second part of the theorem. We say in this case that the set $\{1, 2, ..., n\}$ is weakly connected (see definition (11)).

To prove the necessity part let us suppose that the set $J = \{1, 2, ..., n\}$ is not weakly connected i.e. it consists of $l$ weakly connected components $J_r$ (i.e. $J = \bigcup_{r=1}^l J_r$). In this case $b_k = b_m$ for $k, m \in J_r$ and the operator $B = \bigoplus_{r=1}^l b_rI_r$, where $I_r = \sum_{k \in J_r} E_{kk}$, commute with $A_n$, i.e. $[A_n, B] = 0$. Hence, the representation is Schur reducible. Part (iii) is evident. $\square$

6
2 Maximal proper subalgebras of \( \text{Mat}(n, \mathbb{C}) \)

We give the list of all subsets of indices \( S(n) \subset \{1, 2, \ldots, n\}^2 \) such that the subspace \( s(n) \in \text{Mat}(n, \mathbb{C}) \) defined by

\[
 s(n) = \{ x \in \text{Mat}(n, \mathbb{C}) \mid x = \sum_{(k,m) \in S(n)} x_{km}E_{km} \}
\]

is a maximal proper subalgebra in \( \text{Mat}(n, \mathbb{C}) \).

Theorem 7 The list of all maximal proper subalgebras \( s(n) \) in \( \text{Mat}(n, \mathbb{C}) \) of the form (6) is as follows

\[
 s_i(n) = \{ x = (x_{km})_{k,m=1}^n \in \text{Mat}(n, \mathbb{C}) \mid x_{km} = 0, \; k \in \hat{i}, \; m \in i \},
\]

where \( i = \{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}, \; k < n \) is a proper subset and \( \hat{i} = \{1, 2, \ldots, n\} \setminus i \). The corresponding invariant subspace is \( V_i(n) := V_{1i_2\ldots i_k}(n) \).

PROOF. For \( n = 2 \) we have just two subsets \( S(2) \), namely \( \{(1, 1), (1, 2), (2, 2)\} \) and \( \{(1, 1), (2, 1), (2, 2)\} \). Using the notation (7) the corresponding subalgebras are

\[
 s_1(2) = \left( \begin{array}{cc}
 * & * \\
 0 & 0
\end{array} \right), \quad s_2(2) = \left( \begin{array}{cc}
 * & 0 \\
 0 & *
\end{array} \right).
\]

The mentioned subalgebras \( s_1(2) \) and \( s_2(2) \) have respectively the invariant subspaces: \( V_1(2) = \langle e_1 = (1, 0) \rangle \) and \( V_2(2) = \langle e_2 = (0, 1) \rangle \).

For \( n = 3 \) the list of all maximal proper subalgebras is

\[
 s(3) : \quad \left( \begin{array}{ccc}
 * & * & *
\end{array} \right), \quad \left( \begin{array}{cc}
 * & * \\
 0 & *
\end{array} \right), \quad \left( \begin{array}{cc}
 * & 0 \\
 * & *
\end{array} \right), \quad \left( \begin{array}{cc}
 0 & 0 \\
 * & *
\end{array} \right), \quad \left( \begin{array}{cc}
 * & 0 \\
 0 & *
\end{array} \right), \quad \left( \begin{array}{cc}
 * & *
\end{array} \right).
\]

The mentioned subalgebras \( s(3) \) have respectively the following invariant subspaces: \( V_1(3) \), \( V_2(3) \), \( V_3(3) \); \( V_{23}(3) \), \( V_{13}(3) \) and \( V_{12}(3) \).

To obtain the list of subalgebras \( s(n+1) \) from the list of \( s(n) \) we consider two projectors \( P^{(0)}_{n,n+1} \) and \( P^{(1)}_{n,n+1} \) defined as follows

\[
 P^{(r)}_{n,n+1} : \text{Mat}(n+1, \mathbb{C}) \mapsto \text{Mat}(n, \mathbb{C}),
\]

\[
 \sum_{1 \leq k,m \leq n+1} x_{km}E_{km} = x \mapsto P^{(r)}_{n,n+1}(x) = \sum_{r+1 \leq k,m \leq n+r} x_{km}E_{km},
\]

\[
 \left( \begin{array}{ccc}
 * & * & * \\
 * & * & *
\end{array} \right) P^{(0)}_{n,n+1} \rightarrow \left( \begin{array}{ccc}
 * & * & 0 \\
 * & 0 & *
\end{array} \right), \quad \left( \begin{array}{ccc}
 * & * & * \\
 * & * & *
\end{array} \right) P^{(1)}_{n,n+1} \rightarrow \left( \begin{array}{ccc}
 0 & 0 & 0 \\
 0 & * & *
\end{array} \right).
\]

Notation. For an arbitrary subset of indices \( i = \{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\} \) let us denote by \( s_i^{(r)}(n) = (P^{(r)}_{n,n+1})^{-1}(s_i(n)) \) the corresponding subspace in the
algebra $\text{Mat}(n + 1, \mathbb{C})$, where we denote by $A^{-1}(H_0) = \{ x \in H_1 \mid Ax \in H_0 \}$ the preimage of the subset $H_0 \subset H_2$ for an operator $A : H_1 \rightarrow H_2$.

Let us show how to obtain the list $s(3)$ from the list $s(2)$. Since the algebra $s(3)$ is contained in the space $s^{(r)}(2) = (P_{2,3}^{(r)})^{-1}(s(2))$ for $r = 0, 1$ we get

$$s_1(2) = \left( \begin{smallmatrix} * \times \times \\ 0 \times \times \\ \times \times \times \end{smallmatrix} \right) \left( \begin{smallmatrix} * \times \times \\ 0 \times \times \\ \times \times \times \end{smallmatrix} \right) ; \quad s_2(2) = \left( \begin{smallmatrix} * \times \times \\ 0 \times \times \\ \times \times \times \end{smallmatrix} \right).$$

Since $E_{21} = E_{23}E_{31}$ and $E_{32} = E_{31}E_{12}$ we have just two subalgebras in $s_1^{(0)}(2)$ and two subalgebras in $s_1^{(1)}(2)$:

$$\left( \begin{smallmatrix} * \times \times \\ 0 \times \times \\ \times \times \times \end{smallmatrix} \right) \rightarrow \left( \begin{smallmatrix} * \times \times \\ 0 \times \times \\ \times \times \times \end{smallmatrix} \right) ; \quad \left( \begin{smallmatrix} * \times \times \\ 0 \times \times \\ \times \times \times \end{smallmatrix} \right) \rightarrow \left( \begin{smallmatrix} * \times \times \\ 0 \times \times \\ \times \times \times \end{smallmatrix} \right); \quad \left( \begin{smallmatrix} * \times \times \\ 0 \times \times \\ \times \times \times \end{smallmatrix} \right) \rightarrow \left( \begin{smallmatrix} * \times \times \\ 0 \times \times \\ \times \times \times \end{smallmatrix} \right).$$

and since $E_{12} = E_{13}E_{32}$ and $E_{23} = E_{21}E_{13}$ we have only two subalgebras in $s_2^{(0)}(2)$ and two subalgebras in $s_2^{(1)}(2)$:

$$\left( \begin{smallmatrix} * \times \times \\ 0 \times \times \\ \times \times \times \end{smallmatrix} \right) \rightarrow \left( \begin{smallmatrix} * \times \times \\ 0 \times \times \\ \times \times \times \end{smallmatrix} \right) ; \quad \left( \begin{smallmatrix} * \times \times \\ 0 \times \times \\ \times \times \times \end{smallmatrix} \right) \rightarrow \left( \begin{smallmatrix} * \times \times \\ 0 \times \times \\ \times \times \times \end{smallmatrix} \right).$$

Finally we obtain the list (8) of subalgebras $s(3)$. We see that

$$s_1^{(0)}(2) \rightarrow s_1^{(0)}(3), \quad s_1^{(1)}(2) \rightarrow s_1^{(1)}(3), \quad s_2^{(0)}(2) \rightarrow s_2^{(0)}(3), \quad s_2^{(1)}(2) \rightarrow s_2^{(1)}(3).$$

The list of subalgebra $s(4)$ is as follows:

$$s_i^{(4)} : \left( \begin{smallmatrix} * \times \times \times \\ 0 \times \times \times \\ 0 \times \times \times \\ \times \times \times \times \end{smallmatrix} \right) ; \quad \left( \begin{smallmatrix} * \times \times \times \\ 0 \times \times \times \\ 0 \times \times \times \\ \times \times \times \times \end{smallmatrix} \right) ; \quad \left( \begin{smallmatrix} * \times \times \times \\ 0 \times \times \times \\ 0 \times \times \times \\ \times \times \times \times \end{smallmatrix} \right) ; \quad \left( \begin{smallmatrix} * \times \times \times \\ 0 \times \times \times \\ 0 \times \times \times \\ \times \times \times \times \end{smallmatrix} \right).$$

The corresponding invariant subspaces are $V_i^{(4)}$, $1 \leq i \leq 4$; $V_{i1i2i3}^{(4)}$, $1 \leq i_1 < i_2 < i_3 \leq 4$; and $V_{i1i2}^{(4)}$, $1 \leq i_1 < i_2 \leq 4$.

To get $s(4)$ from $s(3)$ we show how this works only for two subalgebras $s_1^{(3)}$ and $s_{13}^{(3)}$

$$\left( \begin{smallmatrix} * \times \times \\ 0 \times \times \\ \times \times \times \end{smallmatrix} \right) \left( \begin{smallmatrix} * \times \times \\ 0 \times \times \\ \times \times \times \end{smallmatrix} \right) ; \quad \left( \begin{smallmatrix} * \times \times \\ 0 \times \times \\ \times \times \times \end{smallmatrix} \right) \left( \begin{smallmatrix} * \times \times \\ 0 \times \times \\ \times \times \times \end{smallmatrix} \right).$$

Since we have only one possibility to obtain $E_{21}$ and $E_{31}$, namely $E_{21} = E_{24}E_{41}$ and $E_{31} = E_{34}E_{41}$, we have only two subalgebras in $s_1^{(0)}(3)$ (case (a)). The other cases are treated similarly. In the case (b) $s_1^{(1)}(3)$ we have $E_{32} = E_{31}E_{12}$ and $E_{42} = E_{41}E_{12}$; in the case (c) $s_{13}^{(0)}(3)$ we have $E_{21} = E_{24}E_{41}$ and $E_{23} = E_{24}E_{41}$. 

$E_{24}E_{43}$: in the case (d) $s_{13}^{(1)}(3)$ we have $E_{32} = E_{31}E_{12}$ and $E_{34} = E_{31}E_{14}$.

Finally we get

\[(a) \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \rightarrow \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad (b) \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \rightarrow \begin{pmatrix} * & 0 & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad (c) \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \rightarrow \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad (d) \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \rightarrow \begin{pmatrix} * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.\]

So we have the following relations using (9) and the latter considerations:

\[
\begin{align*}
s_1^{(0)}(2) & \rightarrow s_1(3), \quad s_{13}(3), \quad s_1^{(0)}(3) \rightarrow s_1(4), \quad s_{14}(4), \\
s_1^{(1)}(2) & \rightarrow s_2(3), \quad s_{12}(3), \quad s_1^{(1)}(3) \rightarrow s_2(4), \quad s_{12}(4), \\
s_2^{(0)}(2) & \rightarrow s_2(3), \quad s_{23}(3), \quad s_{13}^{(0)}(3) \rightarrow s_{13}(4), \quad s_{134}(4), \\
s_2^{(1)}(2) & \rightarrow s_3(3), \quad s_{13}(3), \quad s_{13}^{(1)}(2) \rightarrow s_{24}(4), \quad s_{124}(4).
\end{align*}
\]

To guess the general formula for an arbitrary $n$ we note that

\[s_{14}^{(0)}(4) \rightarrow s_{14}(5), \quad s_{145}(5).\]

The similar considerations explains us how to describe all the subalgebras $s(n+1)$ starting from the subalgebras $s(n)$. Namely we have

\[s_i(n) \leftarrow s_i^{(0)}(n), \quad s_i^{(1)}(n),\]

\[s_i^{(0)}(n) \rightarrow s_i(n+1), \quad s_{i0}(n+1), \quad s_i^{(1)}(n) \rightarrow s_{i1}(n+1), \quad s_{i1}(n+1),\]

or

\[
\begin{tikzpicture}
  \node (A) at (0,0) {$s_i^{(0)}(n)$};
  \node (B) at (2,0) {$s_i^{(1)}(n)$};
  \node (C) at (0,-2) {$s_i(n+1)$};
  \node (D) at (2,-2) {$s_{i1}(n+1)$};
  \node (E) at (2,-4) {$s_{i0}(n+1)$};
  \node (F) at (0,-4) {$s_i(n+1)$};

  \draw[->] (A) -- (B);
  \draw[->] (A) -- (C);
  \draw[->] (B) -- (D);
  \draw[->] (B) -- (E);
  \draw[->] (C) -- (F);
  \draw[->] (D) -- (F);

\end{tikzpicture}
\]

(10)

where for $i = \{i_1, i_2, ..., i_k\}$ we write $i + 1 = \{i_1 + 1, i_2 + 1, ..., i_k + 1\}$, $i_0 = i \cup \{n + 1\}$ and $i_1 = i + 1 \cup \{1\}$. \[\square\]

3 Generating sets, maximum subalgebra and the graph theory

**Definition 8** We say that a subset $G \subset \{1, 2, ..., n\}^2$ is generating subset if the set of matrices

\[\{E_{km} \mid (k, m) \in G\}\]

generates the algebra $\text{Mat}(n, \mathbb{C})$. 

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We would like to describe the *minimal* generating subsets \( G \) in terms of the graphs. It would be nice also to find the complete list \( G(n) \) of the *minimal* generating subsets in \( \{1, 2, \ldots, n\}^2 \). We remind several definitions from the graph theory.

**Definition 9** We associate with any subset \( G \subset \{1, 2, \ldots, n\}^2 \) an oriented graph (digraph) \( \Gamma \) on \( n \) vertices in the usual way: if \( (k, m) \in G \) we draw the edge (arrow, arc) from the vertex \( k \) to the vertex \( m \) on the graph.

**Definition 10** The adjacency matrix of a graph \( \Gamma \) is the \( n \times n \) matrix \( A_\Gamma \), where \( n \) is the number of vertices in the graph. If there is an edge from some vertex \( x \) to some vertex \( y \), then the element \( a_{xy} \) is 1 (or in general the number of \( xy \) edges), otherwise it is 0.

**Notation.** We shall use the same notation for the subset \( G(n) \subseteq \{1, 2, \ldots, n\}^2 \) and for the corresponding adjacency matrix \( A_G(n) \) namely, \( G(n) = (g_{km})_{k,m=1}^n \)

\[
g_{km} = \begin{cases} 
1, & \text{if } (k, m) \in G(n), \\
0, & \text{if } (k, m) \notin G(n). 
\end{cases}
\]

For \( n = 2 \) we have only one subset \( G(2) := \{(1, 2), (2, 1)\} \):

\[
G(2) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

For \( n = 3 \) the list of all subsets \( G(3) \) is:

\[
G(3) : \quad G_1(3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad G_2(3) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad G_3(3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad G_4(3) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad G_5(3) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\]

There are two distinct notions of connectivity in a digraph.

**Definition 11** A digraph is weakly connected if there is an undirected path between any pair of vertices, and strongly connected if there is a directed path between every pair of vertices ([9], Skiena 1990, p. 173).

**Lemma 12** The subset \( G \) is minimal generating if and only if the corresponding graph \( \Gamma \) is minimal strongly connected.

**PROOF.** Use (5). \( \square \)

Now we can reformulate Theorem 6 in terms of the graph \( \Gamma_{A_n} \) associated with the support of \( G_{A_n} = \text{Supp}(A_n) \) of the matrix \( A_n \).
**Theorem 13** (i) The family \((\Lambda_n, A_n)\) is irreducible if and only if the graph \(\Gamma_{A_n}\) is strongly connected;
(ii) the family \((\Lambda_n, A_n)\) is Schur irreducible if and only if the graph \(\Gamma_{A_n}\) is weakly connected;
(iii) the family \((\Lambda_n, A_n)\) is indecomposable if and only if it is Schur irreducible.

Let us denote by \(A_\Gamma = A_G\) the adjacency matrix of the graph \(\Gamma\) associated with the \(A_G\). We have the following correspondence:

\[
\text{set } G \leftrightarrow \text{graph } \Gamma \leftrightarrow \text{adjacency matrix } A_G = A_\Gamma. \tag{12}
\]

**Definition 14** For two subsets \(G_1, G_2 \subseteq \{1, 2, ..., n\}^2\) define the product \(G_3 = G_1 \circ G_2\) by

\[
G_1 \circ G_2 = \{(k, m) \mid (k, p) \in G_1, (p, m) \in G_2 \text{ for some } p\}. \tag{13}
\]

Let us denote for any subset \(G \subset \{1, ..., n\}^2\) by \(g\) the corresponding subspace

\[
g = \{x \in \text{Mat}(n, \mathbb{C}) \mid x = \sum_{(k, m) \in G} x_{km}E_{km}, \ x_{km} \in \mathbb{C}\}. \tag{14}
\]

Let \(g_1\) and \(g_2\) be the subspaces corresponding (via (14)) to two subsets \(G_1\) and \(G_2\). We define the product \(g_1g_2\) as follows: \(g_1g_2 = z = xy \mid x \in g_1, y \in g_2\). Obviously, we have

\[
g_1g_2 = \{x \in \text{Mat}(n, \mathbb{C}) \mid x = \sum_{(k, m) \in G_3} x_{km}E_{km}\},
\]

where \(G_3 = G_1 \circ G_2\).

To define correctly the product of two adjacency matrices \(A_{G_1}\) and \(A_{G_2}\) corresponding to the subsets \(G_1\) and \(G_2\), we assume that the entries of the matrix \(A_{G_i}\), which are equal to 0 and 1, are in the semiring \(R\) defined below.

**Definition 15** Denote by \(R\) the semiring consisting of two elements 0 and 1 with operations (see [1])

\[
0 + 0 = 0, \quad 0 + 1 = 1, \quad 1 + 1 = 1, \quad 0 \times 0 = 0, \quad 0 \times 1 = 0, \quad 1 \times 1 = 1, \tag{15}
\]

\[
\text{We have } A_{G_1}A_{G_2} = A_{G_1 \circ G_2}. \tag{16}
\]

**Lemma 16** The set \(G\) generates the algebra \(\text{Mat}(n, \mathbb{C})\) if and only if the powers \(G^k = G \circ ... \circ G, \ k = 1, 2, ..., n\) cover the set \(\{1, 2, ..., n\}^2\).

Using Theorem 7 and Lemma 12 we get
Lemma 17  (i) The number $\sharp(s(n))$ of the maximal proper subalgebras is equal to
\[ \sharp(s(n)) = \sum_{r=1}^{n-1} C_n^r = 2^n - 2, \]
the number of ordered subsets of the set \{1, 2, ..., n\} of the length between 1 and \(n - 1\);
(ii) the number $\sharp(G(n))$ of the generating subset $G(n)$ is equal to the number of the minimal strongly connected graphs with $n$ labeled vertices.

Problem 1. To find the number $\sharp(G(n))$ of the minimal strongly connected digraphs with $n$ labeled vertices for any $n$ (see appendix).

4 Appendix, some examples

Notation. For the sake of shortness we use as before the same notations for the set $G(n)$ and for the corresponding adjacency matrix $A_{G(n)}$. We shall denote both by $G(n)$.

Example 1. We show, using Lemma 16, that the set $G(2)$ and sets $G(3)$ from the list (11) are generating. We use firstly Lemma 16 (recall the definition 15). For $n = 2$ the set $G(2)$ is obviously unique. We get
\[ G(2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad G^2(2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow G(2) \cup G(2)^2 = \{1, 2\}^2. \]

For $n = 3$ we have $G_1(3) \cup G^2_3(3) \cup G^3_1(3) = \{1, 2, 3\}^2$. Indeed
\[ G_1(3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad G^2_1(3) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad G^3_1(3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \]
\[ G_2(3) = G^2_1(3), \quad G^2_2(3) = G^4_1(3) = G_1(3), \quad G^3_2(3) = G^6_1(3) = G^4_1(3), \]
\[ G_3(3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad G^2_3(3) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow G_3(3) \cup G^2_3(3) = \{1, 2, 3\}^2, \]
\[ G_4(3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad G^2_4(3) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow G_4(3) \cup G^2_4(3) = \{1, 2, 3\}^2, \]
\[ G_5(3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad G^2_5(3) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow G_5(3) \cup G^2_5(3) = \{1, 2, 3\}^2. \]

If we consider the corresponding directed graphs the proof becomes evident.

Example 2. The list of the non-isomorphic graphs and the corresponding adjacency matrices for $n = 1, 2, 3$:
\[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \]
Example 3. The list of $G(n)$ and $s(n)$ for $n \leq 4$ (for $n = 4$ only some $G(n)$).

$$
G(2): \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad s(2): \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

$$
G(3): \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad s(3): \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
$$

$$
G(4): \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad s(4): \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

Example 4. The number of the minimal strongly connected digraphs on $n$ labeled vertices for $n \leq 12$.

We recall that a directed graph is strongly connected if there exists a directed path from any vertex to any other, and minimal strongly connected if removing any edge destroys this property. Using [8] we get:

| n   | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
|-----|----|----|----|----|----|----|----|----|----|
| A130756 | 1  | 1  | 2  | 5  | 15 | 63 | 288|1526|8627|
| A130768 | 1  | 1  | 5  | 58 |1069|27816|943669|39757264|2010923289|

| 10  | 11  | 12  | 13  |
|-----|-----|-----|-----|
| 52021| 328432| 2160415|
| 119153235520| 8118839891161| 627023347399296| 54258093698028037|
A130756 Number of minimally strongly connected digraphs on \( n \) vertices, up to isomorphism: 1, 1, 2, 5, 15, 63, 288, 1526, 8627, 52021, 328432, 2160415,...

More terms from Vladeta Jovovic (vladeta(AT)Eunet.yu), Jul 13 2007.

A130768 Number of minimally strongly connected digraphs on \( n \) labeled vertices: 1, 5, 58, 1069, 27816, 943669, 39757264, 2010923289, 119153235520, 8118839891161, 627023347399296, 54258093698028037,...

**Example 5.** The list of the maximal subalgebras and the corresponding graphs \( n = 2, 3 \):

\[
s(2) : \quad ( \star \star \star ) , \quad ( \star \star \star ) ,
\]

\[
s(3) : \quad ( \star \star \star \star ) , \quad ( \star \star \star \star ) , \quad ( \star \star \star \star ) , \quad ( \star \star \star \star ) , \quad ( \star \star \star \star ) , \quad ( \star \star \star \star )
\]

**Example 6.** The list of the non-isomorphic maximal subalgebras and the corresponding graphs \( n = 4 \):

\[
s(4)_1 = \begin{pmatrix} \star \star \star \star \\ \star \star \star \star \\ \star \star \star \star \end{pmatrix} , \quad s(4)_{234} = \begin{pmatrix} \star \star \star \star \\ \star \star \star \star \\ \star \star \star \star \end{pmatrix} \quad s(4)_{12} = \begin{pmatrix} \star \star \star \star \\ \star \star \star \star \\ \star \star \star \star \\ \star \star \star \star \\ \star \star \star \star \\ \star \star \star \star \end{pmatrix}
\]

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