THE EULER CHARACTERISTIC OF A HECKE ALGEBRA

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Abstract. To an associative $R$-algebra $A$ equipped with an $R$-linear involution $\circlearrowleft : A \to A^{op}$, a linear representation $\lambda$, and a free $R$-basis $B$ satisfying certain conditions one can associate an Euler characteristic $\chi_A \in R$ (cf. §4.6). These algebras will be called Euler algebras.

Under a suitable condition on the distinguished parameter $q \in R$, we show that the $R$-Hecke algebra $H_q$ associated with a finitely generated Coxeter group $(W, S)$ is an $R$-Euler algebra, and its Euler characteristic coincides with $p_{(W,S)}(q)^{-1}$, where $p_{(W,S)}(t)$ is the Poincaré series associated with $(W, S)$.

1. Introduction

For a finitely generated Coxeter group $(W, S)$ one defines the Poincaré series by

$$p_{(W,S)}(t) = \sum_{w \in W} t^{\ell(w)} \in \mathbb{Z}[t],$$

where $\ell: W \to \mathbb{N}_0$ denotes the length function associated with $(W, S)$. It is well known that $p_{(W,S)}$ is a rational function in $t$ (cf. [3, Chap. IV, §1, Ex. 25 and 26]). This function is explicitly known for finite Coxeter groups, and explicitly computable for any given infinite, finitely generated Coxeter group $(W, S)$ using the recursive sum formula

$$\frac{1}{p_{(W,S)}(t)} = \sum_{I \subseteq S} (-1)^{|S|-|I|-1} \frac{1}{p_{(W,I)}(t)},$$

(cf. [8, §5.12]). The formula (1.2) suggests that one should be able to interpret $p_{(W,S)}(1)^{-1}$ as an Euler characteristic, but it is not clear in which context. This point of view is also supported by a result of J-P. Serre who showed that $p_{(W,I)}(1)^{-1}$ coincides with the Euler characteristic of the Coxeter group $W$ (cf. [13, §1.9]).

The main goal of this paper is to show that $p_{(W,S)}(q)^{-1}$ coincides with the Euler characteristic of the Hecke algebra $H_q(W, S)$ for suitable values of $q$.

Let $R$ be a commutative ring with unit. For certain augmented, associative $R$-algebras (cf. [4,5]) one can define an Euler characteristic. These algebras will be called Euler algebras (cf. §4.2). For any distinguished element $q \in R$ one may define the $R$-Hecke algebra $H = H_q(W, S)$ associated with the Coxeter group $(W, S)$. This algebra can be seen as a deformation of the $R$-group algebra of the Coxeter group $(W, S)$. It particular, it comes equipped with an antipodal map $\circlearrowleft : H \to H^{op}$, an augmentation $\varepsilon_q : H \to R$, and an $R$-basis $B = \{ Tw \mid w \in W \}$. Moreover, one has the following.

Theorem A. Let $(W, S)$ be a finitely generated Coxeter group, let $R$ be a commutative ring with unit, and let $q \in R$ be such that $p_{(W,I)}(q)$ is invertible in $R$ for any spherical parabolic subgroup $(W_I, I)$. Then $H = H_q(W, S)$ is an $R$-Euler algebra, and its Euler characteristic is given by $\chi_H = p_{(W,S)}(q)^{-1}$.

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The proof of Theorem A will be established in two steps. First one has to show that $H = (\mathcal{H}_I, b, \varepsilon_q, B)$ is an $R$-Euler algebra; then one has to compute the Hattori-Stallings rank $r_{R_q}$, where $R_q$ denotes the left $H$-module associated to $\varepsilon_q$. For both these goals we will make use of a chain complex $C = (C_*, \partial_*)$ of left $H$-modules first established by V.V. Deodhar in [3].

For the ring $R = \mathbb{Z}[q]$ the Poincaré series of $(W, S)$ can be written as

$$p_{(W,S)}(q) = p_H = \sum_{T_w \in B} \varepsilon_q(T_w) \in \mathbb{Z}[q].$$

Hence one has the identity $p_H \cdot \chi_H = 1$ in $\mathbb{Z}[q]$. A similar identity involving combinatorial data and cohomological data is known for a Koszul algebra $A$, over a field $F$; one has $h_A^*(t) \cdot h_A^*(A) = 1$ in $\mathbb{Z}[t]$, where $h_A(t)$ denotes the Hilbert series associated with a connected, graded $F$-algebra of finite type (cf. [12], p.22, Cor. 2.2). It would be interesting to know whether there exist other examples of $\mathbb{Z}[q]$-Euler algebras $A$ for which $p_A$ given by (1.3) is defined and which satisfy the identity $p_A \cdot \chi_A = 1$.

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2. Coxeter groups and Hecke algebras

2.1. Coxeter groups. A *Coxeter graph* $\Gamma$ is a finite combinatorial graph$^4$ with non-oriented edges $e$ labelled by positive integers $m(e) \geq 3$ or infinity. The Coxeter group $(W, S)$ associated to $\Gamma$ consists of the group $W$ generated by the set of involutions $S = \{ s_v \mid v \in \mathcal{V}(\Gamma) \}$ subject to the relations $(s_v s_w)^{m(e)} = 1$, where $e = \{ v, w \} \in \mathcal{E}(\Gamma)$ is an edge of label $m(e) < \infty$, and the commutation relations $s_v s_w = s_w s_v$ whenever $\{ v, w \} \notin \mathcal{E}(\Gamma)$. The *length function* $\ell$ on $W$ with respect to $S$ will be denoted by $\ell: W \rightarrow \mathbb{N}_0$. Since $S = S^{-1}$ is a set of involutions, $\ell(w) = \ell(w^{-1})$, and it is well known that a longest element $w_0 \in W$ exists if, and only if, $W$ is finite. In this case it is unique and has the property that $\ell(xv) = \ell(vx) = \ell(x)$ for all $x \in W$. A Coxeter group which is finite is called *spherical*, and *non-spherical* otherwise. For a subset $I \subseteq S$ let $W_I$ be the corresponding parabolic subgroup, i.e., $W_I$ is the subgroup of $W$ generated by $I$. It is isomorphic to the Coxeter group associated to the Coxeter subgraph $\Gamma'$ based on the vertices $\{ v \in \mathcal{V}(\Gamma) \mid s_v \in I \}$. The length function of $W$ restricted to $W_I$ coincides with the intrinsic length function of the Coxeter group $(W_I, I)$. Put

$$W^I = \{ w \in W \mid \ell(sw) > \ell(w) \text{ for all } s \in I \},$$

and let $I^W = (W^I)^{-1}$, i.e.,

$$I^W = \{ w \in W \mid \ell(sw) > \ell(w) \text{ for all } s \in I \}.$$

One has the following properties (cf. [3], §5.12).

**Proposition 2.1.** Let $(W, S)$ be a Coxeter group, let $w \in W$ and let $I \subseteq S$.

(a) There exist a unique element $w_I \in W_I$ and a unique element $w^I \in W^I$ such that $w = w^I w_I$. Moreover, $\ell(w) = \ell(w^I) + \ell(w_I)$.

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$^4$In this context the graph $\emptyset$ with empty vertex set is also considered as a Coxeter graph.
(b) There exist a unique element $iw \in W_I$ and a unique element $iw' \in IW$ such that $w = iw'w$. Moreover, $\ell(w) = \ell(iw) + \ell(w')$.

(c) $IW$ and $IW'$ are sets of coset representatives, distinguished in the sense that the decomposition is length-additive.

(d) The element $w \in IW$ is the unique shortest element in $wW_I$.

(e) Let $y \in IW$ and $u \in W_I$. Then $(yu)^I = y$, $(yu)_I = u$, and $\ell(yu) = \ell(y) + \ell(u)$.

(f) For $s \in S$ one has $W = (s)W \sqcup (s')W$, where $\sqcup$ denotes disjoint union.

(g) Let $I \subseteq J \subseteq S$. Then $W_J \subseteq IW_I$. Moreover, $W^S = \{1\}$ and $W^0 = W$.

2.2. Hecke algebras. Let $R$ be a commutative ring with unit and with a distinguished element $q \in \mathbb{R}$. The $R$-Hecke algebra $\mathcal{H} = \mathcal{H}_q(W,S)$ associated to $(W,S)$ and $q$ is the unique associative $R$-algebra which is a free $R$-module with basis $\{T_w \mid w \in W\}$ subject to the relations

$$T_wT_s = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w) \\ (q-1)T_w + qT_{sw} & \text{if } \ell(sw) < \ell(w), \end{cases}$$

for $s \in S$, $w \in W$. In particular, one has a canonical isomorphism $\mathcal{H}_I \simeq R[W]$, where $R[W]$ denotes the $R$-group algebra of $W$. The $R$-algebra $\mathcal{H}$ comes equipped with an antipodal map $\tilde{\omega}: \mathcal{H} \rightarrow \mathcal{H}^{op}$, $\tilde{\omega} = T_{w^{-1}}$, i.e., $\tilde{\omega}$ is an isomorphism satisfying $\tilde{\omega}^2 = \text{id}_\mathcal{H}$ (cf. [3, Chap. 7.3, Ex. 1]).

For $I \subseteq S$ we denote by $\mathcal{H}_I$ the corresponding parabolic subalgebra, i.e., the $\mathcal{H}$-subalgebra of $\mathcal{H}$ generated by $\{T_w \mid s \in I\}$ which coincides with the $R$-module spanned by $\{T_w \mid w \in W_I\}$. For further details see [3, Chap. 7].

2.3. $\mathcal{H}$-modules. Any $R$-algebra homomorphism $\lambda \in \text{Hom}_{R\text{-alg}}(\mathcal{H}, R)$ defines a 1-dimensional left $\mathcal{H}$-module $R_\lambda$, i.e., for $T_w \in \mathcal{H}$, $w \in W$, and $r \in R_\lambda$ one has $T_w.r = \lambda(T_w).r$. Note that the relations (2.3) force $\lambda(T_s) \in \{1\}$ for all $s \in S$. Moreover, for $s, s' \in S$ and $m(s, s')$ odd, one has $\lambda(T_s) = \lambda(T_{s'}).$ There are two particular $R$-algebra homomorphisms $\varepsilon_q, \varepsilon_1 \in \text{Hom}_{R\text{-alg}}(\mathcal{H}, R)$, given by $\varepsilon_q(T_s) = q$, $\varepsilon_1(T_s) = -1$, $s \in S$. One may consider $\varepsilon_q$ as the augmentation and $\varepsilon_1$ as the sign-character. Note that $\varepsilon_q(T_w) = q^{\ell(w)}$ and $\varepsilon_1(T_w) = (-1)^{\ell(w)}$, and therefore $\varepsilon_q(T_w) = \varepsilon_q(T_w^2)$ and $\varepsilon_1(T_w) = \varepsilon_1(T_w^2)$ for all $w \in W$. For short we put $R_q = R_{\varepsilon_q}, R_1 = R_{\varepsilon_1}$, and use also the same notation for the restriction of these modules to any parabolic subalgebra.

For $I \subseteq S$ let $\mathcal{H}^I = \text{span}_R \{T_w \mid w \in W^I\} \subseteq \mathcal{H}$. Multiplication in $\mathcal{H}$ induces a canonical map of right $\mathcal{H}_I$-modules $\mathcal{H}^I \otimes_R \mathcal{H}_I \rightarrow \mathcal{H}$. Let $y \in W^I$ and $u \in W_I$. As $\ell(yu) = \ell(y) + \ell(u)$ (cf. Prop. 2.1(e)), one has $T_yT_u = T_{yu}$. This shows that this map is an isomorphism. In particular, $\mathcal{H}$ is a projective and thus a flat right $\mathcal{H}_I$-module. This implies that

$$\text{ind}_{\mathcal{H}_I}^\mathcal{H}(\bigcup) = \text{ind}_{\mathcal{H}_I}^\mathcal{H}(\bigcup) = \mathcal{H} \otimes_{\mathcal{H}_I} \mathcal{H}_I$$

is an exact functor mapping projectives to projectives. Moreover, one has the following.

**Fact 2.2.** The canonical map $c_I: \mathcal{H}_I \rightarrow \text{ind}_{\mathcal{H}_I}^\mathcal{H}(R_q)$ given by $c_I(T_w) = T_w\eta_I$, where $\eta_I = T_1 \otimes 1 \in \text{ind}_{\mathcal{H}_I}^\mathcal{H}(R_q)$ and $w \in W^I$, is an isomorphism of $R$-modules. Moreover, for $w \in W$, one has $T_w\eta_I = \varepsilon_q(T_w)T_w^\ast I_{\eta_I}$.

In case that $I \subseteq S$ generates a finite group, one has the following.

**Proposition 2.3.** Let $I$ be a subset of $S$ such that $W_I$ is finite. Put $\tau_I = \sum_{w \in W_I} T_w$. Then one has the following:

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\footnote{For certain types it is also possible to consider multiple parameter Hecke algebras. This is discussed in [13].}
Moreover if \( p(W_I) \in R^\times \) is invertible in \( R \), let \( e_I = (p(W_I))^{-1} e_I \). Then

(b) the element \( e_I \) is a central idempotent in \( H_I \) satisfying \( e_I^2 = e_I \).

(c) the left ideal \( H e_I \) is a finitely generated, projective, left \( H \)-module isomorphic to \( \text{ind}_I^S(R_q) \).

(d) \( T_w e_I = \varepsilon_q(T_{w1})T_w e_I \).

Proof. For \( s \in I \), put \( X_s = \sum_{w \in \tau_i(W_I)} T_w \). Then \( \tau_I = (T_1 + T_s)X_s \) (cf. Prop. 2.1(f)) and therefore

\[
T_s \tau_I = T_s(T_1 + T_s)X_s = [T_s + qT_1 + (q - 1)T_s]X_s = q(T_1 + T_s)X_s = \varepsilon_q(T_s)\tau_I.
\]

This shows (a). Part (b) is an immediate consequence of (a), and the first part of (c) follows from the decomposition of the regular module \( H = H e_I \oplus H(T_1 - e_I) \).

The canonical map \( \pi: H \to \text{ind}_I^S(R_q) \), \( \pi(T_w) = T_w \eta_I \), is a surjective morphism of \( H \)-modules with \( \ker(\pi) = H(T_1 - e_I) \). This yields the second part of (c). Part (d) follows from part (b) and Proposition 2.1(a). \( \square \)

3. The Deodhar complex

There is a chain complex of left \( H \)-modules \( C = (C_*, \partial_*) \) which can be seen as the module-theoretic analogue of the Coxeter complex associated with a Coxeter group \((W, S)\). This chain complex has been introduced first by V.V. Deodhar in [5]. For spherical Coxeter groups it was studied in more detail by A. Mathas in [10]. Recently, M. Linckelmann and S. Schroll introduced a two-sided version of this complex for spherical Coxeter groups (cf. [9]). The definition of this chain complex is quite technical and depends on the choice of a sign function, and one may speculate that this is the reason why it has not found its way in the standard literature yet. In this section we recall its definition and basic properties for the convenience of the reader.

3.1. Sign maps. Let \( H = H_q(W, S) \) be an \( R \)-Hecke algebra, and let \( \mathcal{P}(S) \) denote the set of subsets of \( S \). A sign map for \( H \) is a function \( \text{sgn}: S \times \mathcal{P}(S) \to \{\pm 1\} \) satisfying

\[
\text{sgn}(s, I) \text{sgn}(t, I \cup \{s\}) + \text{sgn}(t, I) \text{sgn}(s, I \cup \{t\}) = 0
\]

for all \( I \subseteq S \) and \( s, t \in S \setminus I \), \( s \neq t \). E.g., if \( \langle \cdot, \cdot \rangle \) is a total order on the finite set \( S \), the function \( \text{sgn}(s, I) = (-1)^{|\{t \in S \setminus I \mid s < t\}|} \) is a sign map.

3.2. The Deodhar complex. Let \( I \) and \( J \) be subsets of \( S \) satisfying \( I \subseteq J \subseteq S \). The canonical injection \( H_I \to H_J \) is a morphism of augmented \( R \)-algebras. Hence it induces a morphism of left \( H \)-modules \( d_I^J: \text{ind}_I^J(R_q) \to \text{ind}_I^J(R_q) \) given by

\[
d_I^J(T_w \otimes_{H_I} r) = T_w \otimes_{H_J} r.
\]

For a subset \( I \subseteq S \) put \( \text{deg}(I) = |S| - |I| - 1 \). Thus \( \text{deg}(I) \in \{-1, \ldots, |S| - 1\} \). For a non-negative integer \( k \) let \( C_k \) be the left \( H \)-module

\[
C_k = \prod_{I \subseteq S \atop \text{deg}(I) = k} \text{ind}_I^S(R_q),
\]

and let \( \partial_k: C_k \to C_{k-1} \) be the morphism of left \( H \)-modules given by

\[
\partial_k = \sum_{I, J \subseteq S \atop \text{deg}(I) = k, \text{deg}(J) = k - 1} \partial_{I, J},
\]

where

\[
\tau_I^2 = p(W_I)(q)\tau_I.
\]
where

\[
\partial_{i,J} = \begin{cases} 
\text{sgn}(s,I) d^I_J & \text{if } J = I \cup \{s\} \\
0 & \text{if } J \not\supseteq I,
\end{cases}
\]

and \(d^I_J\) is given as in (3.2). Note that \(C_k = 0\) for \(k > |S| - 1\). The following properties have been established in [5, Thm. 5.1].

**Theorem 3.1** (V.V. Deodhar). Let \(H = H_q(W,S)\) be an \(R\)-Hecke algebra, and let \(C = (C_\bullet, \partial_\bullet)\) be as described in (3.5). Then

(a) \(\partial_h \circ \partial_{h+1} = 0\), i.e., \(C\) is a chain complex.

(b) If \(W\) is finite, then

\[
H_k(C) \simeq \begin{cases} 
R_q & \text{for } k = 0 \\
R_{|S| - 1} & \text{for } k = |S| - 1 \\
0 & \text{otherwise.}
\end{cases}
\]

(c) If \(W\) is infinite, then \(C\) is acyclic with \(H_0(C) \simeq R_q\).

From now on we will refer to the complex \(C = (C_\bullet, \partial_\bullet)\) as the Deodhar complex of \(H\).

**Remark 3.2.** Let \(C = (C_\bullet, \partial_\bullet)\) be the Deodhar complex of \(H\).

(a) In degree \(|S| - 1\), \(C_{|S| - 1} = \text{ind}_{\mathbb{S}}^R(R_q) \simeq H\) coincides with the regular left \(H\)-module.

(b) Let \(\varepsilon : C_0 \rightarrow R_q = \text{ind}_{\mathbb{S}}^R(R_q)\) denote the canonical map given by (3.6). Then the chain complex

\[
\ldots \rightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} R_q \rightarrow 0
\]

has trivial homology for \(W\) infinite.

4. Euler algebras

Let \(R\) be a commutative ring with unit. In this section we will investigate a class of associative \(R\)-algebras for which one has a natural notion of *Euler characteristic*. For this reason we call such algebras *Euler algebras*.

Let \(A\) be an associative \(R\)-algebra (with unit \(1 \in A\)). An \(R\)-linear isomorphism \(\Lambda^3 : A \rightarrow A^{op}\) will be called an *antipode*, if \(\Lambda^3 = \text{id}_A\). If \((A, \Lambda^3)\) is an \(R\)-algebra with antipode then \(\lambda \in \text{Hom}_{R-alg}(A, R)\) will be called an *augmentation* if \(\lambda(a) = \lambda(a^2)\) for all \(a \in A\). Note that \(\lambda\) defines a left \(A\)-module \(R_{\lambda}\) which is as set equal to \(R\) and satisfies \(a \cdot r = \lambda(a) r\) for \(a \in A\) and \(r \in R\).  

**4.1. Trace functions.** Let \(A\) be an associative \(R\)-algebra. A homomorphism of \(R\)-modules \(\tau : A \rightarrow R\) satisfying \(\tau(ab) = \tau(ba)\) for all \(a, b \in A\) is called a *trace function* on \(A\). Let \([A, A] = \text{span}_R(\{ab - ba \mid a, b \in A\})\), and let \(\overline{A}\) denote the \(R\)-module \(A/[A, A]\). Then \(A^* = \text{Hom}_R(\overline{A}, R)\) is the \(R\)-module of all trace functions of \(A\). The following elementary property will be useful for our purpose.

**Lemma 4.1.** Let \((A, \Lambda^3, \lambda)\) be an augmented, associative \(R\)-algebra with antipode, and let \(B \subseteq A\) be a free generating system of the \(R\)-module \(A\) with the following properties:

(i) \(1 \in B\);

(ii) \(B^2 = B\).

\[3\]In the standard literature (cf. [2], [1], [4]) this \(R\)-module is denoted by \(T(A)\).
(iii) the symmetric \( R \)-bilinear form

\[
\langle \_ , \_ \rangle : A \times A \rightarrow R, \quad \langle a, b \rangle = \delta_{a,b}\lambda(a), \quad a, b \in B;
\]

where \( \delta_{\_ , \_} \) denotes Kronecker’s \( \delta \)-function, satisfies

\[
\langle ab, c \rangle = (b, a^2c) \quad \text{for all} \quad a, b, c \in A.
\]

Then \( \tilde{\mu} \in \text{Hom}_R(A, R) \) given by \( \tilde{\mu}(a) = \langle 1, a \rangle, \quad a \in A, \) is a trace function.

Proof. By definition, one has for all \( a, b \in A \) that \( \langle a^2, b^2 \rangle = \langle a, b \rangle. \) Hence

\[
\tilde{\mu}(ab - ba) = \langle 1, ab \rangle - \langle 1, ba \rangle = \langle a^2, b \rangle - \langle b^2, a \rangle = 0.
\]

for all \( a, b \in A. \) This yields the claim. \( \square \)

If \( (A, \Delta, \lambda, \mathcal{B}) \) satisfies the hypothesis of Lemma \( \text{1.1} \) the map \( \mu \in \text{Hom}_R(A, R) \)
induced by \( \tilde{\mu} \) can be seen as the canonical trace function associated with \( (A, \Delta, \lambda, \mathcal{B})). \)

4.2. Euler algebras. Let \( A \) be an associative \( R \)-algebra. A left \( A \)-module \( M \) is called of type \( FP \), if it has a finite, projective resolution \( (P_\bullet, \partial^P_\bullet, \varepsilon_M), \) \( \varepsilon : P_0 \rightarrow M, \)
by finitely generated projective left \( A \)-modules, i.e., there exists a positive integer \( m \) such that \( P_k = 0 \) for \( k > m \) or \( k < 0, \) and \( P_k \) is finitely generated for all \( k. \)
An augmented, associative \( R \)-algebra with antipode \( \Delta = (A, \Delta^2, \lambda, \mathcal{B}) \) is called of type \( FP \), if the left \( A \)-module \( \Delta \lambda \), is of type \( FP. \) We call an augmented, associative \( R \)-algebra with antipode \( (A, \Delta^2, \lambda, \mathcal{B}) \) of type \( FP \) with a free \( R \)-basis \( \mathcal{B} \) satisfying the hypothesis of Lemma \( \text{1.1} \) an Euler algebra.

4.3. Hattori-Stallings trace maps. For a finitely generated, projective, left \( A \)-module \( P \) let \( P^* = \text{Hom}_A(P, A) \). Then \( P^* \) carries canonically the structure of a right \( A \)-module, and it is also finitely generated and projective. One has a canonical isomorphism \( \gamma_P : P^* \otimes_A P \rightarrow \text{End}_A(P) \) given by \( \gamma_P(p^* \otimes p)(q) = p^*(q)p, \quad p^*, p \in P^*, \)
\( p, q \in P. \) (cf. \[14\] Chap. I, Prop. 8.3]). The evaluation map \( ev_P : P^* \otimes_A P \rightarrow A \)
is given by \( ev_P(p^* \otimes p) = p^*(p) + [A, A]. \) The map

\[
\text{tr}_P = ev_P \circ \gamma_P^{-1} : \text{End}_A(P) \rightarrow A
\]
is called the Hattori-Stallings trace map on \( P \) and \( r_P = \text{tr}_P(\text{id}_P) \in A \) the Hattori-
Stallings rank of \( P \) (cf. \[14\], \[4\] Chap. IX.2)). In particular, \( \text{tr}_P \) is \( R \)-linear, and for \( f, g \in \text{End}_A(P) \) one has

\[
\text{tr}_P(f \circ g) = \text{tr}_P(g \circ f).
\]

From the elementary properties of the evaluation map one concludes that if \( P_1 \) and \( P_2 \)
are two finitely generated projective left \( A \)-modules, one has

\[
r_{P_1 \oplus P_2} = r_{P_1} + r_{P_2}.
\]

Let \( e \in A, \quad e = e^2, \) be an idempotent in the \( R \)-algebra \( A. \) Then \( Ae \) is a finitely generated, projective, left \( A \)-module, and

\[
r_{Ae} = e + [A, A].
\]

4.4. Finite, projective chain complexes. A chain complex \( P = (P_\bullet, \partial^P_\bullet) \) of left \( A \)-modules will be called finite if \( \{ k \in Z \mid P_k \neq 0 \} \) is finite and \( P_k \) is finitely generated for all \( k \in Z. \) Moreover, \( P \) will be called projective, if \( P_k \) is projective for all \( k. \)

For \( P = (P_\bullet, \partial^P_\bullet) \) and \( Q = (Q_\bullet, \partial^Q_\bullet) \) finite, projective chain complexes of left \( A \)-modules we denote by \( \text{Hom}_A(P, Q)_\bullet, \partial_\bullet \) the chain complex of right \( A \)-modules

\[
\text{Hom}_A(P, Q)_k = \prod_{j=i+k} \text{Hom}_A(P_i, Q_j),
\]
with differential given by
\[
(4.9) \quad (d_k(f_k))_{i,j-1} = \partial_j^P \circ f_{i,j} - (-1)^k f_{i-1,j-1} \circ \partial_i^P,
\]
for \(f_k = \sum_{j=i+k} f_{i,j}\). In particular, \(f_0 = \sum_{i\in\mathbb{Z}} f_{i,i} \in \text{Hom}_A(P,Q)_0\) is a chain map of degree 0 if, and only if, \(f_0 \in \ker(d_0)\), and \(f_0\) is homotopy equivalent to the 0-map if, and only if, \(f_0 \in \text{im}(d_1)\) (cf. [4, Chap. I]). Put \(\text{Ext}_A^0(P,Q) = H_0(\text{Hom}_A(P,Q))\).

Let \(B = (B_\bullet, \partial^B)\) be a finite, projective chain complex of right \(A\)-modules. Then \((B \otimes_A P, \partial^P)\) denotes the complex
\[
(4.10) \quad (B \otimes_A P)_k = \prod_{i+j = k} B_i \otimes_A P_j,
\]
\[
\partial^P_{i+j}(b_i \otimes p_j) = \partial^P_i(b_i) \otimes p_j + (-1)^i b_i \otimes \partial^P_j(p_j).
\]
Let \(A[0]\) denote the chain complex of left \(A\)-modules concentrated in degree 0 with \(A[0]_0 = A\), and let \(A[0]\) denote the chain complex of \(R\)-modules concentrated in degree 0 with \(A[0]_0 = A\). Then \(P^\otimes = (P^\otimes, \partial^P) = (\text{Hom}_A(P,A[0])_\bullet, d_\bullet)\),
\[
(4.11) \quad \partial^P_k = \text{Hom}_A(P_{-k}, A),
\]
is a finite, projective complex of right \(A\)-modules. Note that the differential of the complex is chosen in such a way that the standard evaluation mapping
\[
(4.12) \quad \text{ev}_P: P^\otimes \otimes_A P \rightarrow A[0],
\]
is a mapping of chain complexes. However, the natural isomorphism
\[
(4.13) \quad \gamma: \text{Hom}_A(-, A[0]) \otimes_A - \rightarrow \text{Hom}_A(-, -)
\]
\[
\gamma_{s,t}(p^s \otimes_A q^t)(x,s) = (-1)^st p^s_s(q^t_t(x,s)q^t_t)
\]
comes equipped with a non-trivial sign (cf. [4, Chap. I, Prop. 8.3(b) and Chap. VI, §6, Ex. 1]). In this context the Hattori–Stallings trace map is given by
\[
(4.14) \quad \text{tr}_P = H_0(\text{ev}_P \circ \gamma^{-1}_{P,P}): \text{Ext}_A^0(P,P) \rightarrow A.
\]
It has the following properties:

**Proposition 4.2.** Let \(P = (P_\bullet, \partial^P)\) be a finite, projective complex of left \(A\)-modules, and let \([f], [g] \in \text{Ext}_A^0(P,P)\), \(f = \sum_{k\in\mathbb{Z}} f_k\), be homotopy classes of chain maps of degree 0. Then
(a) \(\text{tr}_P([f]) = \sum_{k\in\mathbb{Z}} (-1)^k \text{tr}_{P_k}(f_k)\);
(b) \(\text{tr}_P([f] \circ [g]) = \text{tr}_P([g] \circ [f])\).
(c) Let \(Q = (Q_\bullet, \partial^Q)\) be another finite, projective complex of left \(A\)-modules which is homotopy equivalent to \(P\), i.e., there exist chain maps \(\phi: P \rightarrow Q\), \(\psi: Q \rightarrow P\), which composites are homotopy equivalent to the respective identity maps. Let \([h] \in \text{Ext}_A^0(Q,Q)\) such that \([\phi] \circ [f] = [h] \circ [\phi]\). Then
\[
\text{tr}_P([f]) = \text{tr}_Q([h]).
\]

**Proof.** Part (a) is a direct consequence of (4.13), and (b) follows from (a) and (4.14). The left hand side quadrangle in the diagram
\[
\begin{array}{ccc}
\text{Hom}_A(P,P) & \xrightarrow{\gamma} & P^\otimes \otimes_A P \\
\downarrow \text{ev}_P & & \downarrow \text{ev}_P \\
\text{Hom}_A(Q,Q) & \xrightarrow{\gamma} & Q^\otimes \otimes_A Q
\end{array}
\]

commutes, and the right hand side quadrangle commutes up to homotopy equivalence. This yields claim (c).

Let \( P = (P_\bullet, \partial_\bullet^P) \) be a finite, projective complex of left \( A \)-modules. Then one defines the Hattori–Stallings rank of \( P \) by

\[
(4.16) \quad r_P = \text{tr}_P([\text{id}_P]) = \sum_{k \in \mathbb{Z}} (-1)^k r_{P_k} \in \mathbb{A}
\]

Proposition 4.2 implies that if \( Q = (Q_\bullet, \partial_\bullet^Q) \) is another finite, projective, complex of left \( A \)-modules which is homotopy equivalent to \( P \) then \( r_P = r_Q \).

Let \( \mathcal{K}(A) \) denote the additive category the objects of which are finite, projective chain complexes of left \( A \)-modules. Morphisms \( \text{Hom}_{\mathcal{K}(A)}(P, Q) = \text{Ext}^0_A(P, Q) \) are given by the homotopy classes of chain maps of degree 0. In particular, \( \mathcal{K}(A) \) is a triangulated category and distinguished triangles are triangles isomorphic to the cylinder/cone triangles (cf. [7], [16, Chap. 10]). Thus, if

\[
(4.17) \quad A \longrightarrow B \longrightarrow C \longrightarrow A[1]
\]

is a distinguished triangle in \( \mathcal{K}(A) \), one has \( r_B = r_A + r_C \).

Let \( M \) be a left \( A \)-module of type FP, and let \( (P_\bullet, \partial_\bullet, \varepsilon_M) \) be a finite, projective resolution of \( C \). In particular, \( P = (P_\bullet, \partial_\bullet) \) is a finite, projective chain complex of left \( A \)-modules. One defines the Hattori–Stallings rank of \( M \) by \( r_M = r_P \in \mathbb{A} \). The comparison theorem in homological algebra implies that this element is well defined. The following property will be useful for our purpose.

**Proposition 4.3.** Let \( C = (C_\bullet, \partial_\bullet^C) \) be a chain complex of left \( A \)-modules concentrated in non-negative degrees with the following properties:

(a) \( C \) is acyclic, i.e., \( H_k(C) = 0 \) for \( k \in \mathbb{Z}, k \neq 0 \);

(b) \( C \) is finitely supported, i.e., \( C_k = 0 \) for almost all \( k \in \mathbb{Z} \);

(c) \( C_k \) is of type FP for all \( k \in \mathbb{Z} \).

Then \( H_0(C) \) is of type FP, and one has

\[
(4.18) \quad r_{H_0(C)} = \sum_{k \geq 0} (-1)^k r_{C_k} \in \mathbb{A}
\]

**Proof.** Let \( \ell(C) = \min \{ n \geq 0 \mid C_{n+j} = 0 \text{ for all } j \geq 0 \} \) denote the length of \( C \). We proceed by induction on \( \ell(C) \). For \( \ell(C) = 1 \), there is nothing to prove. Suppose the claim holds for chain complexes \( D, \ell(D) \leq \ell - 1 \), satisfying the hypothesis (a)–(c), and let \( C \) be a complex satisfying (a)–(c) with \( \ell(C) = \ell \). Let \( C^\wedge \) be the chain complex coinciding with \( C \) in all degrees \( k \in \mathbb{Z} \setminus \{0\} \) and satisfying \( C^\wedge_0 = 0 \). Then \( C^\wedge[-1] \) satisfies (a)–(c) and \( \ell(C^\wedge[-1]) \leq \ell - 1 \). Then by induction, \( M = H_1(C^\wedge) = H_0(C^\wedge[-1]) \) is of type FP, and \( r_M = \sum_{k \geq 1} (-1)^{k+1} r_{C_k} \). By construction, one has a short exact sequence of left \( A \)-modules \( 0 \rightarrow M \rightarrow C_0 \rightarrow H_0(C) \rightarrow 0 \). Let \( (P_\bullet, \partial_\bullet^P, \varepsilon_M) \) be a finite, projective resolution of \( M \), and let \( (Q_\bullet, \partial_\bullet^Q, \varepsilon_C) \) be a finite, projective resolution of \( C_0 \). By the comparison theorem in homological algebra, there exists a chain map \( \alpha_\bullet: P_\bullet \rightarrow Q_\bullet \) inducing \( \alpha \). Let \( \text{Cone}(\alpha_\bullet) \) denote the mapping cone of \( \alpha_\bullet \). Then \( \text{Cone}(\alpha_\bullet), \partial_\bullet, \varepsilon_\bullet \) is a finite, projective resolution of \( H_0(C) \), i.e., \( H_0(C) \) is of type FP. Moreover, by the remark following (4.17) one has

\[
(4.19) \quad r_{H_0(C)} = r_{\text{Cone}(\alpha_\bullet)} = r_Q = r_P = r_{C_0} = r_M.
\]

This yields the claim. \( \square \)

4.5. **The Euler characteristic of an Euler algebra.** Let \( A = (A_-, \lambda, B) \) be an Euler \( R \)-algebra with canonical trace function \( \mu \in \text{Hom}_R(\underline{A}, R) \). We define the Euler characteristic of \( A \) by

\[
(4.20) \quad \chi_A = \chi_{(A_-, \lambda, B)} = \mu(r_{R_\lambda}) \in R.
\]
4.6. Induction. Let \( \mathcal{B} \subseteq \mathcal{A} \) be an \( R \)-subalgebra of \( \mathcal{A} \). The canonical injection \( j : \mathcal{B} \to \mathcal{A} \) induces a canonical map
\[
\text{tr}_{\mathcal{B}/\mathcal{A}} : \mathcal{B} \to \mathcal{A}.
\]
Induction \( \text{ind}_{\mathcal{B}}^{\mathcal{A}} = \mathcal{A} \otimes_{\mathcal{B}} \mathcal{B} \) is a covariant additive right-exact functor mapping finitely generated projective left \( \mathcal{B} \)-modules to finitely generated projective left \( \mathcal{A} \)-modules. Moreover, if \( \mathcal{A} \) is a flat right \( \mathcal{B} \)-module, then \( \text{ind}_{\mathcal{B}}^{\mathcal{A}} \) is exact. Let \( P \) be a finitely generated left \( \mathcal{B} \)-module, and let \( Q = \text{ind}_{\mathcal{B}}^{\mathcal{A}}(P) \). Then one has a canonical map \( \iota : P \to Q \), \( \iota(p) = 1 \otimes p \), which is a homomorphism of left \( \mathcal{B} \)-modules. As induction is left adjoint to restriction, every map \( f \in \text{End}_{\mathcal{B}}(P) \) induces a map \( \iota_0(f) = (\iota \circ f)_* \in \text{End}_{\mathcal{A}}(Q) \).

Let \( P^* = \text{Hom}_{\mathcal{B}}(P, \mathcal{B}) \) and \( Q^* = \text{Hom}_{\mathcal{A}}(Q, \mathcal{A}) \). Then for \( f \in \text{Hom}_{\mathcal{B}}(P, \mathcal{B}) \) one has an induced map \( \iota_0(f) = (j \circ f)_* \in Q^* \) making the diagram
\[
\begin{array}{ccc}
\text{End}_{\mathcal{B}}(P) & \xrightarrow{\gamma_P} & P^* \otimes_{\mathcal{B}} P \\
\downarrow{\iota_*} & & \downarrow{\iota_* \otimes 1} \\
\text{End}_{\mathcal{A}}(Q) & \xrightarrow{\gamma_Q} & Q^* \otimes_{\mathcal{A}} Q \\
\end{array}
\]
commute. This shows the following.

**Proposition 4.4.** Let \( \mathcal{B} \subseteq \mathcal{A} \) be an \( R \)-subalgebra of \( \mathcal{A} \) such that \( \mathcal{A} \) is a flat right \( \mathcal{B} \)-module, and let \( M \) be a left \( \mathcal{B} \)-module of type \( \mathcal{F} \mathcal{P} \). Then \( \text{ind}_{\mathcal{B}}^{\mathcal{A}}(M) \) is of type \( \mathcal{F} \mathcal{P} \), and one has
\[
\text{tr}_{\mathcal{B}/\mathcal{A}}(\mathcal{B}^M) = \text{tr}_{\mathcal{B}/\mathcal{A}}(\mathcal{A}^M).
\]

Let \( (\mathcal{A}, \mathcal{B}, \lambda, \mathcal{E}) \) be an augmented, associative, \( R \)-algebra with antipode and a distinguished \( R \)-basis \( \mathcal{B} \) satisfying the hypothesis of Lemma [4.1]. Let \( \mathcal{B} \subseteq \mathcal{A} \) be an \( R \)-subalgebra of \( \mathcal{A} \) such that

(i) \( \mathcal{A} \) is a flat right \( \mathcal{B} \)-module;

(ii) \( \mathcal{B}^2 = \mathcal{B} \);

(iii) \( \mathcal{C} = \mathcal{B} \cap \mathcal{B} \) is an \( R \)-basis of \( \mathcal{B} \).

Then the augmented, associative \( R \)-algebra \( (\mathcal{A}, \mathcal{B}, \lambda, \mathcal{E}) \) satisfies the hypothesis of Lemma [4.1]. Let \( \mu_{\mathcal{A}} : \mathcal{A} \to R \) and \( \mu_{\mathcal{B}} : \mathcal{B} \to R \) denote the associated canonical traces. Then one has a commutative diagram
\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\text{tr}_{\mathcal{B}/\mathcal{A}}} & \mathcal{A} \\
\downarrow{\mu_{\mathcal{B}}} & & \downarrow{\mu_{\mathcal{A}}} \\
R & & R
\end{array}
\]
From this one concludes the following direct consequence of Proposition [4.3].

**Corollary 4.5.** Let \( (\mathcal{A}, \mathcal{B}, \lambda, \mathcal{E}) \) be an augmented, associative, \( R \)-algebra with antipode and a distinguished \( R \)-basis \( \mathcal{B} \) satisfying the hypothesis of Lemma [4.1] and let \( \mathcal{B} \subseteq \mathcal{A} \) be an \( R \)-subalgebra satisfying (i)-(iii). Let \( M \) be a left \( \mathcal{B} \)-module of type \( \mathcal{F} \mathcal{P} \). Then \( \mu_{\mathcal{B}}(\mathcal{B}^M) = \mu_{\mathcal{A}}(\text{ind}_{\mathcal{B}}^{\mathcal{A}}(\mathcal{B}^M)). \)

5. The Euler characteristic of a Hecke algebra

5.1. The canonical trace of a Hecke algebra. Let \( \mathcal{H} = \mathcal{H}_q(W, S) \) be the \( R \)-Hecke algebra associated to the finitely generated Coxeter group \( (W, S) \), and let \( \mathcal{B} = \{ T_w \mid w \in W \} \). Then \( \mathcal{H} = \mathcal{H}_q^{\text{op}}, T_w = T_{w^{-1}} \), is an anti-automorphism of \( \mathcal{H} \) satisfying \( T_w^2 = \text{id}_{\mathcal{H}} \) (cf. [8, Chap. 7.3, Ex. 1]) and \( \varepsilon_q(a^2) = \varepsilon_q(a) \) for all \( a \in \mathcal{H} \). One has the following property.
Proposition 5.1. Let \( \mathcal{H} \) be the Hecke algebra associated to the finitely generated Coxeter group \((W, S)\). Then the \( R \)-bilinear map \((\_, \_): \mathcal{H} \times \mathcal{H} \to R \) associated to \((\mathcal{H}, \mathbb{Z}, \varepsilon, q, B)\) satisfies \((\mathcal{H}, \mathbb{Z}, \varepsilon, q, B)\). In particular, \( \mu_B = \langle T_1, \_ \rangle \) is a trace function.

Proof. By Lemma 4.1, one has to show that

\[
\langle T_u T_v, T_w \rangle = \langle T_v, T_u^{-1} T_w \rangle \quad \text{for all } u, v, w \in W.
\]

Using induction one easily concludes that it suffices to show (5.1) in the case that \( u = s \in S \). In this case one has:

\[
(5.2) \quad \lambda = \langle T_s T_v, T_w \rangle = \begin{cases} 
\delta_{s,v,w} \varepsilon_q(T_w) & \text{if } \ell(sv) > \ell(v) \\
(q-1)\delta_{s,v,w} \varepsilon_q(T_w) + q \delta_{s,v,w} \varepsilon_q(T_{sv}) & \text{if } \ell(sv) < \ell(v)
\end{cases}
\]

and

\[
(5.3) \quad \rho = \langle T_v, T_s T_w \rangle = \begin{cases} 
\delta_{v,s,w} \varepsilon_q(T_v) & \text{if } \ell(sw) > \ell(w) \\
(q-1)\delta_{v,s,w} \varepsilon_q(T_v) + q \delta_{v,s,w} \varepsilon_q(T_{sv}) & \text{if } \ell(sw) < \ell(w)
\end{cases}
\]

We proceed by a case-by-case analysis.

**Case 1:** \( \ell(sv) > \ell(v) \) and \( \ell(sw) > \ell(w) \). Suppose that \( \lambda \neq 0 \). Then \( sv = w \), but \( \ell(w) = \ell(sv) > \ell(v) = \ell(sw) \), a contradiction. Hence \( \lambda = 0 \). Reversing the rôles of \( v \) and \( w \) yields \( \lambda = \rho = 0 \) and thus the claim.

**Case 2:** \( \ell(sv) > \ell(v) \) and \( \ell(sw) < \ell(w) \). Then, \( v \neq w \). If \( \lambda \neq 0 \), then \( sv = w \). Hence \( \ell(w) = \ell(sv) = \ell(v) + 1 \), and \( \lambda = \varepsilon_q(T_w) = \varepsilon_q(T_v) = \varepsilon_q(T_{sv}) \lambda \). On the other hand, \( \rho = (q-1)\delta_{v,s,w} \varepsilon_q(T_v) + q \delta_{v,s,w} \varepsilon_q(T_{sv}) = q \varepsilon_q(T_v) = \lambda \). If \( \lambda = 0 \), then \( sv \neq w \). Hence \( \rho = (q-1)\delta_{v,s,w} \varepsilon_q(T_v) + q \delta_{v,s,w} \varepsilon_q(T_{sv}) = 0 = \lambda \).

**Case 3:** \( \ell(sv) < \ell(v) \) and \( \ell(sw) < \ell(w) \). Reversing the rôles of \( v \) and \( w \) one can transfer the proof for Case 2 verbatim.

**Case 4:** \( \ell(sv) < \ell(v) \) and \( \ell(sw) < \ell(w) \). Suppose that \( sv = w \), or, equivalently, \( v = sw \). Then \( \ell(w) = \ell(sv) < \ell(v) < \ell(sw) < \ell(w) \), a contradiction. Hence \( sv \neq w \) and \( v \neq sw \). Thus \( \lambda = \rho \). This completes the proof. \( \square \)

Remark 5.2. The trace function \( \tilde{\mu}: \mathcal{H} \to R \) can be seen as the canonical trace function on \( \mathcal{H} \). It is straight forward to verify that for Hecke algebras of type \( A_n \), \( B_n \) or \( D_n \), this trace function coincides with the Jones–Ocneanu trace evaluated in 0 (cf. \([8]\)).

5.2. Properties of the Deodhar complex. Let \((W, S)\) be a finite Coxeter group, and let \( q \in R \) be such that \( p(W, S)(q) \in R^\times \). Then \( R_q \simeq \mathcal{H} \varepsilon S \) (cf. Prop. 2.3); in particular, \( R_q \) is a projective left \( \mathcal{H} \)-module. This shows that for any Coxeter group \((W, S)\) and \( I \subseteq S \), \( W_I \) finite, \( \text{ind}_{\mathcal{H}_I}^\mathcal{H}(R_q) \) is a finitely generated, projective, left \( \mathcal{H} \)-module. As a consequence one has the following (cf. \([8]\) §6.8):

**Proposition 5.3.** Let \((W, S)\) be a finitely generated Coxeter group, which is either affine, or co-compact hyperbolic (cf. \([8]\) Ch. 6), and let \( q \in R \) be such that \( p(W, S)(q) \in R^\times \) for any proper parabolic subgroup \((W_I, I)\). Then the Deodhar complex \((\mathcal{C}_*, \partial_*, \varepsilon)\) together with the map \( \varepsilon : \mathcal{C}_0 \to R_q \) (cf. Rem. 6.2), is a finite, projective resolution of \( R_q \).

5.3. The Euler characteristic of a Hecke algebra.

Proof of Theorem A. As \( p(W, J)(q) \in R^\times \) for any finite parabolic subgroup \((W_I, I)\), \( \text{ind}_{\mathcal{H}_I}^\mathcal{H}(R_q) \) is a finitely generated projective \( \mathcal{H} \)-module for any finite parabolic subgroup \((W_I, I)\). First we show that \( \mathcal{H}_I \) is an \( R \)-Euler algebra. We proceed by induction on \( d = |S| \). For \( |S| \leq 2 \), \((W, S)\) is spherical or affine. Hence there is nothing to prove (cf. Prop. 5.3). Assume that the claim holds for all Coxeter groups \((W_J, J)\) with \(|J| < d\), and that \( |S| = d \). By induction, for \( K \subseteq S \), \( R_q \) is a left
$\mathcal{H}_K$-module of type FP. Hence $\text{ind}_{\mathcal{H}}^\mathcal{H}_K(R_q)$ is a left $\mathcal{H}$-module of type FP. Thus $C_k$ is a left $\mathcal{H}$-module of type FP for $0 \leq k \leq d-1$. If $(W, S)$ is spherical, then by the previously mentioned remark $R_q$ is a finitely generated, projective, left $\mathcal{H}$-module (with $I = S$). If $(W, S)$ is non-spherical, then $(C_\ast, \partial_\ast)$ is acyclic. Hence $R_q$ is a left $\mathcal{H}$-module of type FP by Proposition 11.2. This shows that $\mathcal{H}_q$ is an $R$-Euler algebra (cf. Prop. 5.1).

If $(W, S)$ is spherical, then one has $R_q \simeq \mathcal{H}e_S$, where $e_S$ is given as in Proposition 2.3. Hence $r_{R_q} = e_S + [\mathcal{H}, \mathcal{H}]$ (cf. (4.7)), and thus $\chi_\mathcal{H} = \mu(r_{R_q}) = p(W, S)(q)^{-1}$.

In case that $(W, S)$ is non-spherical, one obtains for the Hattori–Stallings rank of $R_q$ using (4.13) and (4.23) that

\begin{equation}
(5.4) \quad r_{R_q} = \sum_{0 \leq k < |S|} (-1)^k r_{C_k} = \sum_{I \subseteq S} (-1)^{|S \setminus I| - 1} r_{\text{ind}_I^S(r_{R_q})}.
\end{equation}

We proceed by induction on $|S|$. Proposition 11.4, Corollary 11.5 and (5.4) imply that

\begin{align*}
\chi_\mathcal{H} = \mu_\mathcal{H}(r_{R_q}) &= \sum_{I \subseteq S} (-1)^{|S \setminus I| - 1} \mu_\mathcal{H}(r_{\text{ind}_I^S(r_{R_q})}) \\
&= \sum_{I \subseteq S} (-1)^{|S \setminus I| - 1} \mu_\mathcal{H}(r_{R_q}) = \sum_{I \subseteq S} (-1)^{|S \setminus I| - 1} \chi_\mathcal{H}_I,
\end{align*}

and thus by induction

\begin{equation}
(5.5) \quad \sum_{I \subseteq S} (-1)^{|S \setminus I| - 1} p(W, S)(q)^{-1}.
\end{equation}

It is well-known that the alternating sum (5.5) is equal to $p(W, S)(q)^{-1}$ (cf. [8, Prop. 5.12]). This yields the claim. \qed

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