A spectral condition for odd cycles in graphs

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February 1, 2008

Abstract

Let $G$ be a graph of sufficiently large order $n$, and let the largest eigenvalue $\mu(G)$ of its adjacency matrix satisfies $\mu(G) > \sqrt{\lfloor n^2/4 \rfloor}$. Then $G$ contains a cycle of length $t$ for every $t \leq n/320$.

This condition is sharp: the complete bipartite graph $T_2(n)$ with parts of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ contains no odd cycles and its largest eigenvalue is equal to $\sqrt{\lfloor n^2/4 \rfloor}$.

This condition is stable: if $\mu(G)$ is close to $\sqrt{\lfloor n^2/4 \rfloor}$ and $G$ fails to contain a cycle of length $t$ for some $t \leq n/321$, then $G$ resembles $T_2(n)$.

Keywords: odd cycle; triangle; graph spectral radius; stability
AMS classification: 05C50, 05C35..

Introduction

This note is part of an ongoing project aiming to build extremal graph theory on spectral grounds, see, e.g., [3] and [6, 13].

It is known ([9], [14]) that if $G$ is a graph of order $n$ and the largest eigenvalue $\mu(G)$ of its adjacency matrix satisfies $\mu(G) > \sqrt{\lfloor n^2/4 \rfloor}$, then a triangle exists in $G$.

Here we show that the same premises imply the existence of other cycles as well.

Theorem 1 Let $G$ be a graph of sufficiently large order $n$ with $\mu(G) > \sqrt{\lfloor n^2/4 \rfloor}$. Then $G$ contains a cycle of length $t$ for every $t \leq n/320$.

Write $T_2(n)$ for the complete bipartite graph with parts of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$. Note that $T_2(n)$ contains no odd cycles and $\mu(T_2(n)) = \sqrt{\lfloor n^2/4 \rfloor}$; thus, Theorem 1 gives a sharp spectral condition for the existence of odd cycles.

Moreover, there is stability in this condition: if $\mu(G)$ is close to $\sqrt{\lfloor n^2/4 \rfloor}$ and $G$ fails to contain a cycle of length $t$ for some $t \leq n/321$, then $G$ resembles $T_2(n)$. Here is a precise form of this statement.
Theorem 2 Let \(0 < \theta < 2^{-16}\) and \(n\) be sufficiently large. For every graph \(G\) of order \(n\) with \(\mu (G) > (1/2 - \theta) n\), one of the following conditions holds:

(i) \(G\) contains a cycle of length \(t\) for every \(t \leq n/321\);
(ii) there exists an induced bipartite subgraph \(G_0 \subset G\) satisfying \(|G_0| > (1 - 4\theta^{1/3}) n\) and \(\delta (G_0) > (1/2 - 7\theta^{1/3}) n\).

The proofs of Theorems 1 and 2 are based on three results of independent interest.

Lemma 3 Let \(G\) be a graph of order \(n\) with minimum degree \(\delta\) and \(\mu (G) = \mu\). If \((x_1, \ldots, x_n)\) is a unit eigenvector to \(\mu\), then
\[
\min \{x_1, \ldots, x_n\} \leq \sqrt{\frac{\delta}{\mu^2 + \delta n - \delta^2}}.
\]

Lemma 4 Let \(G\) be a graph of order \(n\) with \(\mu (G) = \mu\). If \((x_1, \ldots, x_n)\) is a unit eigenvector to \(\mu\) and \(u\) is a vertex satisfying \(x_u = \min \{x_1, \ldots, x_n\}\), then
\[
\frac{\mu (G - u)}{n - 1} > \frac{\mu (G)}{n} \left(1 + \frac{1}{n - 1} \left(1 - nx_u^2 - \frac{1}{n - 1}\right)\right).
\]

Combining these two lemmas, we get Theorem 5 below. We hope that this technical statement can be used in other spectral extremal problems.

Theorem 5 Let \(0 < 4\alpha \leq 1, 0 < 2\beta \leq 1, 1/2 - \alpha/4 \leq \gamma < 1, K \geq 0, \) and \(n \geq (42K + 4)/\alpha^2\beta\). If \(G\) is a graph of order \(n\) with
\[
\mu (G) > \gamma n - K/n \quad \text{and} \quad \delta (G) \leq (\gamma - \alpha) n,
\]
then there exists an induced subgraph \(H \subset G\) with \(|H| \geq (1 - \beta) n\), satisfying one of the following conditions:

(i) \(\mu (H) > \gamma (1 + \beta \alpha/2) |H|\);
(ii) \(\mu (H) > \gamma |H|\) and \(\delta (H) > (\gamma - \alpha) |H|\).

Proofs

We start with some notation and results needed for our proofs.

Our graph-theoretical notation follows [2]. Specifically, given a graph \(G\), we write:

- \(|G|\) for the number of vertices of \(G\);
- \(E (G)\) for the edge set of \(G\);
- \(k_3 (G)\) for the number of triangles of \(G\);
- \(d (u)\) for the degree of a vertex \(u\);
- \(\Gamma (u)\) for the set of neighbors of a vertex \(u\);
- \(\delta (G)\) for the minimum degree of \(G\).

The following fact is a reduced version of Theorem 1 of [5].
Fact 6 Let $G$ be a nonbipartite graph of sufficiently large order $n$, and let $\delta(G) \geq n/3$. Then $C_t \subset G$ for every integer $t \in [4, \delta(G) + 1]$. □

The following facts are particular cases of Theorems 2 and 4 in [3].

Fact 7 If $G$ is a graph of order $n$, then $k_3(G) \geq (\mu(G)/n - 1/2) n^3/12$. □

Fact 8 Let $0 < \theta < 2^{-16}$ and let $G$ be a triangle-free graph of order $n$ with $\mu(G) \geq (1/2 - \theta)n$. Then there exists an induced bipartite graph $H \subset G$ satisfying $|H| > (1 - 3\theta^{1/3})n$ and $\delta(H) > (1/2 - 6\theta^{1/3})n$. □

Proof of Lemma 3

Set $\sigma = \min \{x_1, \ldots, x_n\}$. If $\sigma = 0$, the assertion holds trivially, so we assume that $\sigma > 0$. This implies also that $\delta > 0$. Taking $u \in V(G)$ to satisfy $d(u) = \delta$, we have

$$\mu^2\sigma^2 \leq \mu^2x_u = \left(\sum_{i \in \Gamma(u)} x_i\right)^2 \leq \delta \sum_{i \in \Gamma(u)} x_i^2 \leq \delta \left(1 - \sum_{i \in V(G) \setminus \Gamma(u)} x_i^2\right) \leq \delta \left(1 - (n - \delta)\sigma^2\right) = \delta - (\delta n - \delta^2)\sigma^2,$$

implying that $(\mu^2 + \delta n - \delta^2)\sigma^2 \leq \delta$, and the desired inequality follows. □

Proof of Lemma 4

Set for short $c = 1 - nx_u^2$ and $\mu = \mu(G)$. We have

$$\mu x_u = \sum_{v \in \Gamma(u)} x_v \quad \text{and} \quad \mu = 2 \sum_{vw \in E(G)} x_v x_w.$$ 

Hence, by Rayleigh’s principle, we obtain

$$\mu = 2 \sum_{vw \in E(G-U)} x_v x_w + 2x_u \sum_{v \in \Gamma(u)} x_v \leq \mu(G - u) \left(1 - x^2\right) + 2x_u^2 \mu(G),$$

implying that

$$\frac{\mu}{n-1} \geq \frac{\mu(G)}{n-1} \cdot \frac{1 - 2x_u^2}{1 - x_u^2} = \frac{\mu(G)}{n-1} \cdot \left(\frac{n-2+2c}{n-1+c}\right). \quad (1)$$

On the other hand, in view of $0 \leq c \leq 1$, we find that

$$n\left(\frac{n-2+2c}{n-1+c}\right) - n + 1 - c + \frac{1}{n-1} = -\frac{1 + c + cn}{n-1 + c} - c + \frac{1}{n-1}$$

$$= -\frac{(1-c)^2}{n-1+c} + \frac{1}{n-1} \geq 0.$$
Hence, inequality (1) implies that
\[
\frac{\mu (G - u)}{n - 1} \geq \frac{\mu (G)}{n - 1} \cdot \left( \frac{n - 2 + 2c}{n - 1 + c} \right) \geq \frac{\mu (G)}{n} \cdot \left( 1 + \frac{c}{n - 1} - \frac{1}{(n - 1)^2} \right),
\]
completing the proof. □

**Proof of Theorem 5**

Let \( \alpha, \beta, \gamma, K, n, \) and the graph \( G \) satisfy the conditions of the theorem. We immediately see that
\[
n \geq \frac{42K + 4}{\alpha^2 \beta} > \max \left\{ \frac{15}{(1 - \beta) \alpha}, \frac{1}{\beta}, \frac{84K}{\alpha^2} \right\}.
\]

Define a sequence of graphs \( G_0, \ldots, G_k \) by the following procedure \( \mathcal{P} \):

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begin
set \( G_0 = G \);
set \( k = 0 \);
while \( \delta (G_i) \leq (\gamma - \alpha) (n - k) \) and \( k < \lfloor \beta n \rfloor \) do
begin
select a unit eigenvector \( (x_1, \ldots, x_{n-k}) \) to \( \mu (G_k) \);
select a vertex \( u_k \in V(G_k) \) such that \( x_{u_k} = \min \{ x_1, \ldots, x_{n-k} \} \);
set \( G_{k+1} = G_k - u_k \);
add 1 to \( k \);
end;
end.
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Let \( H = G_k \) and note that
\[
|H| = n - k \geq n - \lfloor \beta n \rfloor \geq (1 - \beta) n.
\]

We shall show that
\[
\mu (H) > \gamma \left( 1 + \frac{4k\alpha}{7n} \right) |H|.
\]
(2)

To this end, we first prove by induction on \( i \) that
\[
\frac{\mu (G_i)}{n - i} \geq \left( 1 + \frac{3i\alpha}{5n} \right) \frac{\mu (G)}{n}
\]
(3)

for every \( i = 0, \ldots, k \).

The assertion is trivially true for \( i = 0 \). Let \( 0 \leq i \leq k - 1 \) and assume that (3) holds for \( i \); we shall prove that it also holds for \( i + 1 \). Set \( \delta = \delta (G_i), \mu = \mu (G_i), \) and note first that
\[
\delta \leq (\gamma - \alpha) (n - i),
\]
(4)
\[
\mu \geq (n - i) \left( 1 + \frac{3i\alpha}{5n} \right) \frac{\mu (G)}{n} > (n - i) \left( \gamma - \frac{K}{n^2} \right).
\]
(5)
Let \((x_1, \ldots, x_{n-i})\) be a unit eigenvector to \(\mu\), and let \(u \in V(G_i)\) satisfy \(x_u = \min \{x_1, \ldots, x_{n-i}\}\). Then Lemma 3 implies that

\[
x_u^2 \leq \frac{\delta}{\mu^2 + (n-i) \delta - \delta^2}.
\]

Noting that the right-hand side increases with \(\delta\) and decreases with \(\mu\), in view of (4) and (5), we find that

\[
x_u (n-i) \leq \frac{(\gamma - \alpha)(n-i)^2}{(n-i)^2 (\gamma - K/n^2)^2 + (n-i) (\gamma - \alpha) (n-i) - (\gamma - \alpha)^2 (n-i)^2}
\]

\[
< \frac{\gamma - 2\gamma K/n^2 + \gamma - \alpha + 2\gamma \alpha - \alpha^2}{\gamma - \alpha} \leq \frac{\gamma (\gamma + 2\alpha) - \alpha - 2K/n^2 + \gamma - \alpha^2}{\gamma - \alpha}
\]

\[
< \frac{\gamma - \alpha}{-2K/n^2 + \gamma - \alpha^2} < \frac{\gamma - \alpha}{\gamma - 2\alpha^2} \leq 1 - \frac{2\alpha}{3}.
\]

In the above derivation we used the inequalities

\[
\alpha \leq 1/4, \quad 2K/n^2 < \alpha^2, \quad 1 \geq \gamma \geq 1/2 - \alpha/4 \geq 3\alpha/4 > 2\alpha^2.
\]

Next, Lemma 4 implies that

\[
\frac{\mu(G_{i+1})}{n-i-1} \geq \frac{\mu(G_i)}{n-i} \left(1 + \frac{1}{n-i-1} \left(\frac{2\alpha}{3} - \frac{1}{n-i-1}\right)\right) \geq \frac{\mu(G_i)}{n-i} \left(1 + \frac{3\alpha}{5n}\right).
\]

Therefore,

\[
\frac{\mu(G_{i+1})}{n-i-1} \geq \left(1 + \frac{3\alpha}{5n}\right) \left(1 + \frac{3i\alpha}{5n}\right) \frac{\mu(G)}{n} \geq \left(1 + \frac{3(i+1)\alpha}{5n}\right) \frac{\mu(G)}{n},
\]

completing the induction step and the proof of (3).

Inequality (3) implies that

\[
\frac{\mu(|H|)}{|H|} = \frac{\mu(G_k)}{n-k} \geq \left(1 + \frac{3k\alpha}{5n}\right) \frac{\mu(G)}{n} \geq \left(1 + \frac{3k\alpha}{5n}\right) \left(\frac{\gamma - K}{n^2}\right)
\]

\[
= \gamma \left(1 + \frac{3k\alpha}{5n}\right) - \frac{K}{n^2} \left(1 + \frac{3k\alpha}{5n}\right) > \gamma \left(1 + \frac{4k\alpha}{7n}\right) + \frac{\alpha}{42n} - \frac{2K}{n^2}
\]

\[
> \gamma \left(1 + \frac{4k\alpha}{7n}\right),
\]

as claimed.

To complete the proof of the theorem, note that, after the procedure \(P\) stops, we have either \(k = \lfloor \beta n \rfloor\) or \(\delta(H) > (\gamma - \alpha) |H|\). If \(k = \lfloor \beta n \rfloor\), then

\[
\mu(H) \geq \gamma \left(1 + \frac{4\lfloor \beta n \rfloor \alpha}{7n}\right) |H| > \gamma \left(1 + \frac{\beta \alpha}{2}\right) |H|;
\]
hence, condition (i) holds.

If $k < \lfloor \beta n \rfloor$, then $\delta (H) > (\gamma - \alpha) |H|$, and, in view of (2), we find that

$$\mu (H) > \gamma \left( 1 + \frac{k\alpha}{2n} \right) |H| > \gamma |H|;$$

hence, condition (ii) holds, completing the proof. \(\square\)

### Proof of Theorem 1

Let $G$ be a graph of order $n$ with $\mu (G) > \sqrt{\lceil n^2/4 \rceil}$. Assume first that $\delta (G) > 2n/5$. Since $G$ contains a triangle, it is nonbipartite; hence, for $n$ sufficiently large, Fact 6 implies that $C_t \subset G$ for every $t \leq \delta (G) + 1$, completing the proof.

Thus, we shall assume that $\delta (G) \leq 2n/5$. Let

$$\alpha = 1/10, \quad \beta = 1/2, \quad \gamma = 1/2, \quad K = 1.$$

We have $\delta (G) \leq (\gamma - \alpha) n$ and

$$\mu (G) \geq \sqrt{\lceil n^2/4 \rceil} \geq n/2 - 1/n = \gamma n - K/n.$$

Hence, Theorem 5 implies that, for $n$ sufficiently large, there exists an induced subgraph $H \subset G$ with $|H| > n/2$, satisfying one of the following conditions:

(i) $\mu (H) > (1/2 + 1/80) |H|$;

(ii) $\mu (H) > |H|/2$ and $\delta (H) > 2 |H|/5$.

Assume first that condition (i) holds. Then, by Fact 7 we obtain

$$k_3 (H) > \left( \frac{\mu (H)}{|H|} - \frac{1}{2} \right) \frac{1}{12} |H|^3 \geq \frac{1}{80 \cdot 12} |H|^3 = \frac{1}{960} |H|^3.$$

Thus, there is a vertex $u \in V (H)$ contained in at least $3k_3 (H) / |H| \geq |H|^2/320$ triangles in $H$, and so the neighborhood of $u$ induces more than $|H|^2/320$ edges. By a theorem of Erdős and Gallai [4], the neighborhood of $u$ contains a path $P$ longer than

$$\frac{2}{320} |H| \geq \frac{1}{320} n.$$

Clearly, the path $P$ and the vertex $u$ form a cycle $C_t$ for every $t \leq n/320$, completing the proof in this case.

If condition (ii) holds then, by $\mu (H) > |H|/2$, the graph $H$ contains a triangle; thus, by Fact 6 $C_t \subset H$ for every $t \leq \delta (H) + 1$, completing the proof. \(\square\)
Proof of Theorem 2

Let $G$ be a graph of order $n$ with $\mu (G) > (1/2 - \theta) n$. If $G$ is triangle-free, the proof is completed by Fact 8 so we shall assume that $G$ contains a triangle.

Assume first that $\delta (G) > 2n/5$. Since $G$ is nonbipartite, for $n$ sufficiently large, Fact 8 implies that $C_t \subset G$ for every $t \leq \delta (G) + 1$, completing the proof.

Thus, we shall assume that $\delta (G) \leq 2n/5$. Let

$$\alpha = 1/10 + \theta, \quad \beta = 40\theta, \quad \gamma = 1/2 - \theta, \quad K = 0.$$ 

We have $\delta (G) \leq (\gamma - \alpha) n$ and $\mu (G) > 1/2 - \theta = \gamma n$. Hence, Theorem 5 implies that, for $n$ sufficiently large, there exists an induced subgraph $H \subset G$ with $|H| > (1 - \beta)n$, satisfying one of the following conditions:

(i) $\mu (H) > \gamma (1 + \alpha/2) |H|$;

(ii) $\mu (H) > \gamma |H|$ and $\delta (H) > (\gamma - \alpha) |H|$.

Assume first that condition (i) holds. Then,

$$\mu (H) \geq \gamma (1 + \alpha/2) |H| = (1/2 - \theta + (1/2 - \theta) (1/10 + \theta) 20\theta) |H| = (1/2 - \theta - \theta + 8\theta^2 - 20\theta^3) |H| > |H|/2,$$

and so, by Theorem 1 $C_t \subset H \subset G$ for every $t < |H|/320$. This completes the proof in view of

$$|H|/320 \geq (1 - 40\theta)n/320 > n/321.$$ 

If condition (ii) holds then, in view of $\delta (H) > (\gamma - \alpha) |H| = 2 |H|/5$, Fact 6 implies that $C_t \subset H$ for all $t \leq \delta (H) + 1$, unless $H$ is bipartite. To complete the proof we have to consider case of bipartite $H$. Since $H$ is triangle-free and $\mu (H) > \gamma |H| = (1/2 - \theta) |H|$, Fact 8 implies that there exists an induced bipartite subgraph $G_0 \subset H$ satisfying

$$|G_0| > (1 - 3\theta^{1/3}) |H| \geq (1 - 3\theta^{1/3}) (1 - \beta) n = (1 - 3\theta^{1/3}) (1 - 40\theta) n > (1 - 4\theta^{1/3}) n$$

and

$$\delta (G_0) > (1 - 6\theta^{1/3}) |H| \geq (1 - 6\theta^{1/3}) (1 - \beta) n = (1 - 3\theta^{1/3}) (1 - 40\theta) n > (1 - 7\theta^{1/3}) n,$$

completing the proof. 

Concluding remarks

It is clear that the constant 1/320 in Theorem 1 can be increased even with the present methods; thus, the following question arises:

**Question** What is the maximum $C$ such that for all positive $\varepsilon < C$ and sufficiently large $n$, every graph $G$ of order $n$ with $\mu (G) > \sqrt{[n^2/4]}$ contains a cycle of length $t$ for every $t \leq (C - \varepsilon)n$. 

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It is known ([1], p. 150) that if \( G \) is a graph of order \( n \) with \( e(G) > \lfloor n^2/4 \rfloor \), then \( G \) contains a cycle of length \( t \) for every \( 3 \leq t \leq \lceil n/2 \rceil \). Thus, one can conjecture that \( C = 1/2 \). However, this is not true: taking the join of a complete graph of order \( k = \lceil (3 - \sqrt{5}) n/4 \rceil \) and an empty graph of order \( n - k \), we obtain a graph \( H \) of order \( n \) with \( \mu(H) > n/2 \geq \sqrt{\lfloor n^2/4 \rfloor} \), but having no cycles longer than \( 2k \sim (3 - \sqrt{5}) n/2 \).

Finally, a word about the project mentioned in the introduction: in this project we try to follow the following principles:

- give results that can be used as wide-range tools, like Lemmas 3 and 4, Theorem 5, and Facts 7 and 8;
- give explicit conditions for the parameters in statements, like the conditions for \( \alpha, \beta, \gamma, K, n \) in Theorem 5;
- prefer simple to optimal bounds, like the factor \( 1/320 \) in Theorem 11.

We aim to give results that can be used further, hoping to add more integrity to spectral extremal graph theory.

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