E cient Cluster Algorithm for CP (N = 1) Models

B.B. Beard\textsuperscript{a}, M. Pepe\textsuperscript{bhc}, S. Riederer\textsuperscript{c}, and U.-J. Wiese\textsuperscript{c}

\textsuperscript{a} Department of Physics and Mechanical Engineering
Christian Brothers University, Memphis, TN 38104, USA.

\textsuperscript{b} Istituto Nazionale di Fisica Nucleare and
Dipartimento di Fisica, Universita di Milano-Bicocca
3 Piazza della Scienza, 20126 Milano, Italy

\textsuperscript{c} Institute for Theoretical Physics
Bern University,Sidlerstrasse 5, 3012 Bern, Switzerland

October 18, 2021

Abstract

Despite several attempts, no e cient cluster algorithm has been constructed for CP (N = 1) models in the standard Wilson formulation of lattice \textit{SU}(N) \textit{D}-theory. In fact, there is a no-go theorem that prevents the construction of an e cient \textit{SU}(N) \textit{W}-type embedding algorithm. In this paper, we construct an e cient cluster algorithm for ferromagnetic \textit{SU}(N)-symmetric quantum spin system \textit{s}. Such system \textit{s} provide a regularization for CP (N = 1) models in the framework of \textit{D}-theory. We present detailed studies of the autocorrelations and find a dynamical critical exponent that is consistent with \( z = 0 \).
1 Introduction

Cluster algorithms are among the most efficient tools for numerical simulations of nonperturbative lattice field theories and statistical mechanics models. The first cluster algorithm for discrete classical spins was developed by Swendsen and Wang for Ising and Potts models [1]. This algorithm was generalized by Wol to $O(N)$ models formulated in terms of continuous classical spins [2]. In the 2-d $O(3)$ model, the Wol cluster algorithm has been extended to a meron-cluster algorithm which has been used to solve the complex action problem at non-zero vacuum angle [3]. $CP(N-1)$ models in two dimensions [4,5,6] provide a generalization of the $O(3) = CP(1)$ model alternative to $O(N)$ models. They have a global $SU(N)$ symmetry and share several features with QCD, including asymptotic freedom, the nonperturbative dynamical generation of a mass-gap, a topological charge, and thus nontrivial vacuum. Finding efficient algorithms for $CP(N-1)$ models is difficult in the standard Wilson formulation. In particular, a Wol-type embedding cluster algorithm turned out to be inefficient for $CP(N-1)$ models [7]. In fact, there is even a no-go theorem that prevents the success of this type of algorithm [8]. Still, a rather efficient multigrid algorithm does exist [9,10] for $= 0$. However, due to a very severe complex action problem, no efficient algorithm has ever been found for non-zero vacuum angle. Consequently, in the Wilson formulation numerical studies of vacuum [11,12] have been limited to moderate volumes, or are based on additional assumptions [13,14,15,16].

D-theory is an alternative formulation of field theory in which continuous classical fields arise dynamically as collective excitations by the dimensional reduction of discrete variables such as quantum spins [17,18,19]. The discrete nature of the quantum variables allows the development of new types of cluster algorithms. The first cluster algorithm for quantum spins was developed in [20]. This algorithm works efficiently for 1-d ferro- and antiferromagnetic quantum spin chains, while the loop-cluster algorithm [21,22] works efficiently in any dimension. For example, the loop-algorithm has led to very accurate simulations of the 2-d Heisenberg quantum antiferromagnet [23], which can even be performed in continuous Euclidean time [24]. The D-theory formulation of field theory has been used, for example, to construct an efficient cluster algorithm that solves the complex action problem in the 2-d $O(3)$ model at non-zero chemical potential [25]. In this paper we construct an efficient algorithm for the D-theory formulation of $CP(N-1)$ models. The D-theory approach allows efficient simulations both at $= 0$ and . In D-theory $CP(N-1)$ models are represented by ferro- or antiferromagnetic SU($N$) quantum spin systems. Just like the SU(2) Heisenberg antiferromagnet, SU($N$) quantum antiferromagnets can be simulated with the loop-cluster algorithm [26] which was recently reviewed in [27,28]. Here we construct an SU($N$) loop-cluster algorithm for a ferron magnet to study $CP(N-1)$ models at $= 0$, and we investigate their numerical efficiency. We find a dynamical exponent for critical slowing down which is consistent with $z = 0$. The D-theory method even solves the vacuum sign
problem, and has recently allowed us to confirm the conjectured first order phase transition \[25,30\] at \( = \).

The paper is organized as follows. We present the standard Wilson formulation of lattice \( N \) models in section 2 and the D-theory formulation in section 3. Section 4 describes the path integral formulation for \( N \) models in terms of ferromagnetic quantum spins transforming in the fundamental representation of \( SU(N) \). The corresponding loop-cluster algorithm is constructed in section 5, and its efficiency is investigated in section 6. Finally, section 7 contains our conclusions.

2 Standard Formulation of \( N \) Models

The manifold \( CP(N = U(N) \) is a \((2N - 2)\)-dimensional coset space relevant in the context of the spontaneous breakdown of an \( SU(N) \) symmetry to a \( U(N = 1) \) subgroup. In particular, in more than two space-time dimensions \( (d > 2) \) the corresponding Goldstone bosons are described by an \( N \) matrix-valued field \( P(x) \) which obeys

\[
\begin{align*}
P(x)^2 &= P(x); \quad P(x)^\dagger = P(x); \quad \text{Tr}P(x) = 1; \quad (2.1)
\end{align*}
\]

For \( d = 2 \) the Hohenberg-Mermin-Wagner-Coleman theorem implies that the \( SU(N) \) symmetry cannot break spontaneously. Correspondingly, similar to 4-d non-Abelian gauge theories, the field \( P(x) \) develops a mass-gap nonperturbatively. Motivated by these observations, D'Adda, DiVecchia, and Luscher [4] introduced \( N \) models as interesting toy models for QCD. The corresponding Euclidean action is given by

\[
S[P] = \frac{1}{g^2} \int \text{d}^2x \text{Tr}[P @ P @ P] i \Omega[P]; \quad (2.2)
\]

where \( g^2 \) is the dimensionless coupling constant. The topological charge

\[
\begin{align*}
Q[P] &= \frac{1}{2} \int \text{d}^2x \text{Tr}[P @ P @ P] 2 \text{CP}(N = 1) = \mathbb{Z}; \quad (2.3)
\end{align*}
\]

takes integer values in the second homotopy group of \( \text{CP}(N = 1) \), and is the corresponding vacuum angle. Note that both the action and the topological charge are invariant under global transformations \( 2 SU(N) \)

\[
\begin{align*}
P(x)^0 &= P(x)^\dagger; \quad (2.4)
\end{align*}
\]

The topological term explicitly breaks the charge conjugation symmetry \( C \) which acts as \( CP(x) = P(x) \), such that \( Q[F[P] = Q[P]. \) It should be noted that \( C \) is not explicitly broken at \( = 0 \) or \( = \). In particular, for \( = \) the Boltzmann weight

\[
\text{Note that the factor in eq. (2.3) is necessary for obtaining a real-valued quantity.}
\]
$\exp(i \mathcal{Q} \mathcal{P}) = (1^\mathcal{P})$ is C-invariant. For $N = 3$ it has been conjectured that there is a 1st order phase transition at $\beta = \beta_c$, at which charge conjugation is broken spontaneously [29]. Recently, this has been confirmed numerically using a cluster algorithm similar to the one described below [31]. For CP$(1) = O(3)$, on the other hand, charge conjugation remains unbroken, but there is still a second order phase transition at $\beta = \beta_c$ [32, 33, 34]. At that point the model corresponds to the $k = 1$ Wess-Zumino-Novikov-Witten conformal field theory [34, 35, 36].

It is straightforward to define CP$(N = 3)$ models beyond perturbation theory in the framework of Wilson's lattice regularization. Then the CP$(2)$ model is defined on the sites $x$ of a quadratic lattice and the standard lattice action is given by

$$S[\mathcal{P}] = \frac{2^N}{g^2}\sum_{x} \text{Tr}[\mathcal{P}_x \mathcal{P}_x^\dagger];$$

where $\dagger$ is the unit-vector in the $-\text{direction}$.

3 D -Theory Formulation of CP$(N = 1)$ Models

In this section we describe an alternative formulation of CP$(N = 1)$ models in which the CP$(N = 1)$ model emerges from the dimensional reduction of discrete variables in this case SU$(N)$ quantum spins in $d+1$ space-time dimensions. The dimensional reduction of discrete variables is the key ingredient of D -theory, which provides an alternative nonperturbative lattice regularization of CP$(N = 1)$ models. In D -theory we start from a fermion magnetic system of SU$(N)$ quantum spins located at the sites $x$ of a $d$-dimensional periodic hypercubic lattice of size $L^d$. The SU$(N)$ spins are represented by Hermitian operators $T^a_x = \frac{1}{2} \sigma^a x$ that generate the group SU$(N)$ (e.g., the Gell-Mann matrices for the triplet representation of SU$(3)$) and thus they obey

$$[T^a_x, T^b_y] = i g_{xy} \varepsilon_{abc} T^c_x; \quad \text{Tr}(T^a_x T^b_y) = \frac{1}{2} \delta^{ab};$$

In principle, these generators can be taken in any irreducible representation of SU$(N)$. However, not all representations lead to spontaneous symmetry breaking from SU$(N)$ to U$(1)$ and thus to CP$(N = 1)$ models. The Hamiltonian operator for an SU$(N)$ fermion magnet takes the form

$$H = \sum_{x,j} T^a_{x \to x+j};$$

where $J > 0$ is the exchange coupling. By construction, the Hamiltonian operator is invariant under the global SU$(N)$ symmetry, i.e., it commutes with the total spin given by

$$T^a = \sum_{x} T^a_x;$$
The Hamiltonian $H$ describes the evolution of the quantum spin system in an extra dimension of finite extent. In D-theory this extra dimension is not the Euclidean time of the target theory, which is part of the d-dimensional lattice. Instead, it is an additional compacted dimension which ultimately disappears via dimensional reduction. The quantum partition function

$$Z = \text{Tr} \exp (H) \quad (3.4)$$

(with the trace extending over the Hilbert space) gives rise to periodic boundary conditions in the extra dimension.

For $d = 2$ the ground state of the quantum spin system has a broken global $SU(N)$ symmetry. The choice of the $SU(N)$ representation determines the symmetry breaking pattern. We choose a totally symmetric $SU(N)$ representation corresponding to a Young tableau with a single row containing $n$ boxes. It is easy to construct the ground states of the $SU(N)$ ferromagnet, and one finds spontaneous symmetry breaking from $SU(N)$ to $U(N-1)$. Consequently, there are $(N^2 - 1) (N - 1)/2$ massless Goldstone bosons described by a field $P(x)$ in the coset space $SU(N)/U(N-1) = CP(N-1)$. In the leading order of chiral perturbation theory the Euclidean action for the Goldstone boson field is given by

$$S[P] = \int_0^\infty dt \int d^d x \text{Tr} \left[ \frac{2n}{a^2} \frac{Z_1}{Z} \partial_\tau P @ P \right]; \quad (3.5)$$

Here $s = Jn^2/4$ is the spin stiffness which is analogous to the pion decay constant in QCD. The second term in eq. (3.5) is a Wess-Zumino term which involves an integral over an interpolation parameter $\tau$. The point $\tau = 1$ corresponds to the physical space-time, such that $P(x; \tau = 1) = P(x; t)$. At $\tau = 0$, on the other hand, the field takes the constant value $P(x; \tau = 0) = \text{diag}(1; 0; \ldots; 0)$.

At intermediate values of the field, it is smoothly interpolated between the two limiting cases. There is no topological obstruction against such an interpolation because $1[CP(N-1)]$ is trivial. The integrand in the Wess-Zumino term is a total derivative. Hence, the integral depends only on the boundary values at $\tau = 0$, i.e., on the values of the field in the physical space-time. However, since $2[CP(N-1)] = Z$, there is an integer ambiguity of the integral depending on which particular interpolation is chosen for $P(x; \tau ; t)$. In order to make sure that this ambiguity does not affect the physics, the prefactor of the Wess-Zumino term must be quantized. Remarkably, this prefactor is just determined by the number of boxes $n$ of the chosen $SU(N)$ representation.

Let us next consider the $d = 2$ case. For $d = 1$ the system then has a spontaneously broken global symmetry and thus massless Goldstone bosons. However, as soon as $d$ becomes finite, due to the Hohenberg-Mermin-Wagner-Coleman theorem, the symmetry can no longer be broken, and, consequently, the Goldstone bosons pick up a small mass nonperturbatively. As a result, the corresponding correlation length $\xi = m$ becomes finite and the $SU(N)$ symmetry is restored over that length scale. The question arises if $d$ is bigger or smaller than the extent of
the extra dimension. When the Goldstone boson field is essentially constant along the extra dimension and the system undergoes dimensional reduction. Since the \( \text{Wess-Zumino term} \) contains \( \partial \phi \), it simply vanishes after dimensional reduction. Correspondingly, the action dimensionally reduces to

\[
S[\phi] = \int d^2x \text{Tr}[\phi \partial \phi];
\]

(3.6)

which is just the action of the 2-d target CP\((N - 1)\) model at \( \ell = 0 \). The coupling constant of the 2-d model is determined by the extent of the extra dimension and is given by

\[
\frac{1}{g^2} = s:
\]

(3.7)

Due to asymptotic freedom of the 2-d CP\((N - 1)\) model, for \( \ell = 0 \), the correlation length is exponentially large, i.e.

\[
/ \exp(4sN):
\]

(3.8)

Here \( N = 4 \) is the 1-loop coefficient of the perturbative -function. Indeed, one sees that as long as itself is sufficiently large. In particular, somewhat counter-intuitively, dimensional reduction happens in the large \( \ell \) limit because then grows exponentially. In D-theory one approaches the continuum limit not by varying a bare coupling constant but by increasing the extent of the extra dimension. This mechanism of dimensional reduction of discrete variables is generic and occurs in all asymptotically free D-theory models [17, 18, 19]. It should be noted that (just like in the standard approach) no fine-tuning is needed to approach the continuum limit.

In order to show that D-theory indeed defines the same \( \ell \) theory in the continuum limit as the standard Wilson approach, we have evaluated a universal physical quantity in both regularizations [31]. A convenient quantity is the universal finite-size scaling function \( F(y) = (2L) = \text{L}. \) Here \( L \) is the correlation length (obtained with the second moment method) in a finite system of size \( \text{L} \), and \( y = \frac{\text{L}}{\text{L}} \) is a finite-size scaling variable that measures the size of the system in physical units. Figure 1 shows Monte Carlo data for \( F(y) \) obtained both from D-theory and from the standard Wilson approach. Up to small scaling violations, the agreement between the two data sets confirms that, after dimensional reduction, the \( (2 + 1) \)-d SU\( (N) \) quantum ferromagnet indeed provides the correct continuum limit of the 2-d CP\((N - 1)\) model. Thanks to the cluster algorithm, the D-theory framework allows calculations that are much more accurate than the ones using Wilson’s approach.

Let us also discuss the \( d > 2 \) case. Then dimensional reduction still occurs, but the Goldstone bosons remain massless even at finite \( \ell \). However, if \( \ell \) is decreased sufficiently, the SU\( (N) \) symmetry is restored and the Goldstone modes are replaced by massive excitations. By fine-tuning to the critical value (again, as in the standard approach) one can reach a continuum limit of the target theory, provided that the corresponding phase transition is second order.
Figure 1: Monte Carlo data for the universal finite-size scaling function $F(y)$ of the 2-d CP(2) model. The filled circles represent D-theory data from a (2 + 1)-d SU(3) quantum ferromagnet at $J = 6$, while the open squares correspond to data obtained with the standard Wilson 2-d lattice field theory at $\frac{1}{\epsilon^2} = 2.25$.

4 Path Integral Representation of SU(N) Quantum Spin Systems

Let us construct a path integral representation for the partition function $Z$ of the SU(N) quantum spin ferromagnet introduced above. In an intermediate step we introduce a lattice in the Euclidean time direction, using a Trotter decomposition of the Hamiltonian. However, since we are dealing with discrete variables, the path integral is completely well-defined even in continuous Euclidean time. Also the cluster algorithm to be described in the following section can operate directly in the Euclidean time continuum [29]. Hence, the final results are completely independent of the Trotter decomposition. In $d$ spatial dimensions (with an even extent $L$) we decompose the Hamiltonian operator into 2d terms

$$H = H_1 + H_2 + \cdots + H_{2d}; \quad (4.1)$$
with
\[ H_1 = \sum_{x} h_{x;1}; \quad H_{i=d} = \sum_{x} h_{x;i}; \]  
(4.2)

The individual contributions
\[ h_{x;i} = J \left( T_{x,i}^e + h_{x;1} \right); \]  
(4.3)
to a given \( H_1 \) commute with each other, but two different \( H_i \) do not commute. Using the Trotter formula, the partition function then takes the form
\[ Z = \lim_{M \to 1} \text{Tr} \left( \exp(\sum h_i) \right): \]  
(4.4)

We have introduced \( M \) Euclidean time-slices with \( \tau = M \) being the lattice spacing in the Euclidean time direction.

Inserting complete sets of spin states \( \{ 2 \} \) \( f \) \( u; d; s; \ldots \) \( g \) the partition function takes the form
\[ Z = \sum_{\{ g \}} \exp( S [g]); \]  
(4.5)
The sum extends over configurations \( \{ g \} \) of spins \( q(x,t) \) on a \((d+1)\)-dimensional space-time lattice of points \((x,t)\). The Boltzmann factor
\[ \exp( S [g]) = \prod_{x,t}^\{ n \} \frac{1}{Y} \sum_{u,d} \exp( s[q(x,t);q(x+\hat{t};t);q(x;1);q(x+\hat{t};t+1)]) \]  
(4.6)
is a product of space-time plaquette contributions with
\[ \exp( s[u;d;u;d]) = \exp( s[d;d;u;u]) = 1; \]
\[ \exp( s[u;d;u;d]) = \exp( s[u;u;u;d]) = \frac{1}{2} [1 + \exp( J)]; \]
\[ \exp( s[u;d;d;u]) = \exp( s[u;u;d;u]) = \frac{1}{2} [1 + \exp( J)]; \]  
(4.7)
In these expressions the avors \( u \) and \( d \) can be permuted to other values. All the other Boltzmann factors are zero, which implies several constraints on allowed configurations.

In the following we will consider the uniform magnetization
\[ M = \sum_{x,t}^\{ q(x,t);q(x+\hat{t};t) \}; \]  
(4.8)
which represents one component of \( T_x^a \) as well as the corresponding susceptibility
\[ = \frac{1}{L^d} \sum_{i=1}^d \left( M_i^2 \right); \]  
(4.9)
where \( L^d \) is the spatial volume of the system.
5 Cluster Algorithm for SU(N) Quantum Ferrromagnets

Let us now discuss the cluster algorithm for the SU(N) quantum ferrromagnet. Just like the original SU(2) loop-cluster algorithm [21, 23], the SU(N) cluster algorithm builds a closed loop connecting neighboring lattice points with the spin in the same quantum state, and then changes the state of all spins in that cluster to a different randomly chosen common value. To begin cluster growth, an initial lattice point \((x; t)\) is picked at random. The spin located at that point participates in two plaquette interactions, one before and one after \(t\). One picks one interaction arbitrarily and considers the states of the other spins on that plaquette. One of the corners of this interaction plaquette will be the next point on the loop. For configurations \(C_1 = [u; d; u; d]\) or \([d; u; u; d]\) the next point is the time-like neighbor of \((x; t)\) on the plaquette, while for configurations \(C_2 = [u; d; d; u]\) or \([d; u; u; d]\) the next point is the diagonal neighbor. If the states are all the same, i.e., for \(C_3 = [u; u; u; u]\) or \([d; d; d; d]\), with probability

\[
p = \frac{1}{2}[1 + \exp(-J)] \tag{5.1}
\]

the next point on the loop is again the time-like neighbor, and with probability \((1 - p)\) it is the diagonal neighbor. The next point on the loop belongs to another interaction plaquette on which the same process is repeated. In this way the loop grows until it finally closes. The cluster rules are consistent with detailed balance, i.e.

\[
p(C_i)w(C_i \rightarrow C_j) = p(C_j)w(C_j \rightarrow C_i) \tag{5.2}
\]

where \(p(C_i) = \exp\left(\frac{s[C_i]}{T}\right)\) is the Boltzmann weight of the plaquette configuration and \(w(C_i \rightarrow C_j)\) is the transition probability to go from \(C_i\) to \(C_j\).

With this single-cluster algorithm, the uniform susceptibility of eq. (4.5) has an improved estimator

\[
\chi = h[C_j]; \tag{5.3}
\]

given in terms of the cluster size

\[
\chi_j = \frac{x}{2dM} \sum_{x, t < N} 1; \tag{5.4}
\]

This shows explicitly that the clusters are physical objects whose size is directly related to a physical quantity.
Figure 2: History of the magnetization of an SU (3) ferromagnet on a $320^2$ lattice at \( J = 6.75 \) as a function of the Monte Carlo time (measured in units of sweeps with the multi-cluster algorithm).

6 Efficiency of the Algorithm in the Continuum Limit

In order to quantify the efficiency of our numerical method, we have used a multi-cluster algorithm for a $(2+1)$-dimensional SU (3) ferromagnet in discrete Euclidean time to investigate the autocorrelation times of the uniform magnetization which gives the cleanest signal. A Monte Carlo history of this observable is shown in Figure 2. This already indicates that autocorrelations are very much suppressed.

| L=a  | 20   | 40   | 80   | 160  | 320  | 640  |
|------|------|------|------|------|------|------|
| 4.5  | 5.0  | 5.55 | 6.2  | 6.75 | 7.45 |
| (L)=a| 8.78 (1) | 16.76 (1) | 32.26 (3) | 64.6 (1) | 123.4 (2) | 253 (1) |
| 0.721 (3) | 0.719 (4) | 0.724 (6) | 0.719 (1) | 0.715 (2) |       |

Table 1: Numerical data for the correlation length \( L \) and the autocorrelation time for an SU (3)-symmetric ferromagnet corresponding to a CP (2) model in D-theory.
We have analyzed the autocorrelation function for the magnetization which is illustrated in Figure 3. In order to estimate the dynamical exponent $z$ of critical slowing down, we have considered several lattices with decreasing lattice spacing, keeping the ratio $y = \frac{L}{\langle L \rangle}$ of the correlation length $\langle L \rangle$ (obtained with the continuous time cluster algorithm using the second moment method) and the size $L$ fixed. Our data are summarized in Table 1. Since the autocorrelation time does not show any dependence on $\langle L \rangle$, critical slowing down is practically eliminated. In particular, our data are consistent with $z = 0$.

7 Conclusions

We have constructed an efficient cluster algorithm for CP($N$ = 1) models within the theoretical framework of $D$-theory. The continuous classical CP($N$ = 1) ...
emerges dynamically from the dimensional reduction of discrete SU(N) quantum spins. Our cluster algorithm is a generalization of the loop algorithm for SU(2) quantum spins. We have investigated the algorithm's efficiency by estimating its autocorrelation times. Our Monte Carlo data show that critical slowing down is practically absent, i.e., the dynamical critical exponent is consistent with $z = 0$.

This is thus another example where the D-theory formulation allows us to construct efficient numerical methods which are not available using the standard Wilson lattice field theory. In particular, the use of the discrete D-theory variables evades the no-go theorem that prevents the construction of efficient Wilson-type embedding algorithms for CP(N-1) models in the standard Wilson formulation. Interestingly, using spin ladders with both ferro- and antiferromagnetic couplings, CP(N-1) models can even be simulated at vacuum angle $\theta = \frac{3\pi}{4}$ with no additional numerical cost. In particular, the sign problem that affects the standard approach is completely eliminated. Using the cluster algorithm described in this paper one can perform accurate numerical simulations of the CP(N-1) mass gap for which no analytic results are available. Also the approach to the large N limit can now be investigated in detail. It is also interesting to ask if D-theory allows us to construct efficient cluster algorithms for other field theories such as, e.g., SU(N) SU(N) chiral models with N$=3$ to which the no-go theorem also applies. Of course, the most challenging application of D-theory would be to non-Abelian gauge theories and, in particular, to QCD [17,18,19].

Acknowledgements

This work was supported in part by the Schweizerischer Nationalfonds.

References

[1] R.H. Swendsen and J.-S. Wang, Phys. Rev. Lett. 58 (1987) 86.
[2] U. Wolf, Phys. Rev. Lett. 62 (1989) 361.
[3] W. Bietenholz, A. Pochinsky, and U.-J. Wiese, Phys. Rev. Lett. 75 (1995) 4524.
[4] A. D'Adda, P. DiVecchia, and M. Luscher, Nucl. Phys. B146 (1978) 63; Nucl. Phys. B152 (1979) 125.
[5] H. Eichenherr, Nucl. Phys. B146 (1978) 215.
[6] V. L. Golov and A. M. Perelemov, Phys. Lett. 79B (1978) 112.
[7] K. Jansen and U.-J. Wiese, Nucl. Phys. B370 (1992) 762.
[8] S. Caracciolo et al., Nucl. Phys. B 403 (1993) 475.

[9] M. Hasenbusch and S. Meyer, Phys. Rev. Lett. 68 (1992) 435.

[10] M. Hasenbusch and S. Meyer, Phys. Rev. D 45 (1992) 4376.

[11] U. -J. Wiese, Nucl. Phys. B 318 (1989) 153.

[12] R. Burkhalter, M. Imachi, Y. Shinno, and H. Yoneyama, Prog. Theor. Phys. 106 (2001) 613.

[13] V. Antonio, G. Di Carlo, A. Galante, and V. Laliena, Phys. Rev. Lett. 89 (2002) 141601.

[14] V. Antonio, A. Galante, and V. Laliena, Prog. Theor. Phys. 109 (2003) 843.

[15] V. Antonio, G. Di Carlo, A. Galante, and V. Laliena, Phys. Rev. D 69 (2004) 056006.

[16] M. Imachi, Y. Shinno, and H. Yoneyama, Prog. Theor. Phys. 111 (2004) 387.

[17] S. Chandrasekharan and U. -J. Wiese, Nucl. Phys. B 492 (1997) 455.

[18] R. Brower, S. Chandrasekharan, and U. -J. Wiese, Phys. Rev. D 60 (1999) 094502.

[19] R. Brower, S. Chandrasekharan, S. Riederer, and U. -J. Wiese, Nucl. Phys. B 693 (2004) 149.

[20] U. -J. Wiese and H. -P. Ying, Phys. Lett. B 318 (1997) 475.

[21] H. G. Evertz, G. Lana, and M. Marcu, Phys. Rev. Lett. 70 (1993) 875.

[22] H. G. Evertz, Adv. Phys. 52 (2003) 1.

[23] U. -J. Wiese and H. -P. Ying, Z. Phys. B 93 (1994) 147.

[24] B. B. Beard and U. -J. Wiese, Phys. Rev. Lett. 77 (1996) 5130.

[25] S. Chandrasekharan, B. Scarlet, and U. -J. Wiese, Comput. Phys. Commun. 147 (2002) 388.

[26] K. Harada, N. Kawashima, and M. Troyer, Phys. Rev. Lett. 90 (2003) 117203.

[27] N. Kawashima and K. Harada, J. Phys. Soc. Jpn. 73 (2004) 1379.

[28] M. Troyer, F. Ale, S. Trebst, and S. Wessel, AIP Conf. Proc. 690 (2003) 156.

[29] N. Seiberg, Phys. Rev. Lett. 53 (1984) 637.

[30] I. Ack, Nucl. Phys. B 305 [FS23] (1988) 582; Phys. Rev. Lett. 66 (1991) 2429.
[31] B. B. Beard, M. Pepe, S. Riedler, and U. J. Wiese, Phys. Rev. Lett. 94 (2005) 010603.

[32] F. D. M. Haldane, Phys. Rev. Lett. 50 (1983) 1153.

[33] I. Auck and E. Lieb, Lett. Math. Phys. 12 (1986) 57.

[34] J. Wess and B. Zumino, Phys. Lett. B 37 (1971) 95.

[35] S. P. Novikov, Sov. Math. Dokl. 24 (1981) 222; Usp. Math. Nauk. 37 (1982) 3.

[36] E. Witten, Commun. Math. Phys. 92 (1984) 455.