A PETRI THEOREM FOR RANK-2 VECTOR BUNDLES WITH CANONICAL DETERMINANT

ELISA CASINI AND HERBERT CLEMENS

ABSTRACT. This paper establishes the correctness of a conjecture of Bertram-Feinberg and Mukai for a special class of globally generated rank-two bundles with canonical determinant over a generic Riemann surface of genus at least four.

1. INTRODUCTION

1.1. The problem. Petri’s general conjecture establishes the unobstructedness of linear series on a general compact Riemann surface C of genus $g > 1$. (See, for example, [2].) Some years ago, Bertram-Feinberg [BF] and Mukai [M] independently formulated an analogous conjecture for stable rank-2 vector bundles on C with determinant $\omega_C$. The conjecture is that the natural map

$$\text{Sym}^2 H^0 (E) \to H^0 (\text{Sym}^2 E)$$

is injective for all stable E. The stronger assertion that

$$H^0 (E) \otimes H^0 (E) \to H^0 (E \otimes E)$$

is injective is known to be false for some stable bundles [1]. In this paper, over a generic Riemann surface C of genus $g > 3$, we construct examples of globally generated, semi-stable bundles E for which the stronger assertion is true. We accomplish this by employing Hitchin’s theory of spectral curves [H] to reduce the problem to an assertion about line bundles L of a spectral cover

$$\tilde{C} \to C$$

and then applying to $\left( \tilde{C}, L \right)$ the techniques used in [2] to establish Petri’s general conjecture. A critical ingredient will be a condition introduced by Beauville [B] which equates, under sufficiently general conditions, the local analytic deformation space of E with twists of L by skew-symmetric line bundles on $\tilde{C}$. The necessity of this condition is shown by an example of V. Mercat, which we explore in an Appendix to this paper.

1.2. Hitchin’s spectral curves. Let E be a stable rank-2 vector bundle over C with

$$\det E = \omega,$$

the canonical line bundle on C. We have a natural isomorphism

$$E \to \text{Hom} (E, \omega)$$

$e \mapsto e \wedge$
Following the theory of Hitchin [H], let 
\( \tilde{\varphi} \in \text{Hom} (E^\vee, E) = \text{Hom} (E^\vee, E^\vee \otimes \omega) \).

Replacing \( \varphi \) by
\[
\tilde{\varphi} - \frac{\text{tr} \tilde{\varphi}}{2}
\]
if necessary, we can (and will) assume that
\[
\tilde{\varphi} \in \Gamma \left( \text{Hom}^0 (E^\vee, E^\vee \otimes \omega) \right)
\]
where \( \text{Hom}^0 \) means homomorphisms of trace zero. We assume that \( \tilde{\varphi} \) is not the zero homomorphism. Now
\[
y^2 + \varphi
\]
is the characteristic polynomial of \( \tilde{\varphi} \) where
\[
\varphi = \det \tilde{\varphi} \in H^0 (\omega^2).
\]
and \( y \) takes values in sections of \( \omega \). Considering \( \varphi (C) \) as a curve in the geometric line bundle \( \omega^2 \), let \( \tilde{C} \) denote the inverse image of \( C \) under the squaring map 
\[
\omega \to \omega^2.
\]
We then have a double covering
\[
\pi : \tilde{C} \to C
\]
branched at
\[
\Delta = \text{zero} (\varphi),
\]
and the arithmetic genus \( \tilde{g} \) of \( \tilde{C} \) is computed from the identity
\[
2\tilde{g} - 2 = 4 (2g - 2)
\]
\[
\tilde{g} = 2 (2g - 2) + 1.
\]

1.3. Globally generated \( E \). As in §4 of [vGI], suppose that \( E \) is globally generated. We can then some construct morphisms (4) as follows. Let
\[
W \subseteq H^0 (E)
\]
be any subspace such that the evaluation map
\[
W \otimes \mathcal{O}_C \to E
\]
is surjective. Then we have the standard exact sequence
\[
0 \to F \to W \otimes \mathcal{O}_C \to E \to 0.
\]
So for each quadric 
\[
Q \in \text{Sym}^2 W
\]
we obtain a diagram
\[
0 \to E^\vee \xrightarrow{\xi^\vee} W^\vee \otimes \mathcal{O}_C \to F^\vee \to 0
\]
\[
0 \to F \to W \otimes \mathcal{O}_C \xrightarrow{\xi} E \to 0
\]
where
\[
T (w^\vee) = \langle Q, w^\vee \rangle \in W^{\vee \vee} = W.
\]
So we have an induced morphism
\[ \tilde{Q} = \varepsilon \circ T \circ \varepsilon^\vee \in \Gamma \text{Hom}^0 (E^\vee, E). \]

Rewriting
\[ \text{Hom}^0 (E^\vee, E) = \text{Hom}^0 (E^\vee, E^\vee \otimes \omega) = \text{Sym}^2 E, \]
\[ \tilde{Q} \mapsto \left( (e_1^\vee, e_2^\vee) \mapsto \left\langle \tilde{Q} (e_1^\vee) \left| e_2^\vee \right. \right\rangle \right) \]
the map
\[ \text{Sym}^2 W \to H^0 (\text{Sym}^2 E) \]
\[ Q \mapsto \tilde{Q} \]
is just the standard map induced by multiplication of sections.

On the other hand, for any section \( y \) of \( \omega \), we have
\[ y : E^\vee \to E^\vee \otimes \omega = E \]
with trace \( 2y \). For fixed \( c \in C \), let
\[ e_1, e_2 \]
be a basis for \( E_c \) and let
\[ e_1 \wedge e_2 = y_c \in \omega_c. \]

Then the mapping \( [7] \) is given by
\[ e_1^\vee \mapsto -\frac{y (c)}{y_c} e_2 \]
\[ e_2^\vee \mapsto \frac{y (c)}{y_c} e_1. \]
and \( \tilde{Q}_c \) is given by
\[ e_1^\vee \mapsto Q_{11} \cdot e_1 + Q_{12} \cdot e_2 \]
\[ e_2^\vee \mapsto Q_{21} \cdot e_1 + Q_{22} \cdot e_2. \]
where
\[ Q_{ij} = Q (e_i^\vee, e_j^\vee). \]

The mapping
\[ \left( y_c \tilde{Q}_c - y (c) \cdot \right) : E_c^\vee \to E_c \]
drops rank if and only
\[ y (c) = \pm y_c \sqrt{\det \left( \begin{array}{cc} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{array} \right)} \]
with eigenvectors given by the zeros of the quadratic form
\[ Q |_{E_c^\vee}. \]
1.4. E as a pushforward of a line bundle. For smooth $\tilde{C}$, the eigenvector subspace
\[ \{(c, v) \in E^\vee : \tilde{\varphi} (v) = (\pm \sqrt{\varphi}) v\} \]
has the structure of a line bundle $L^\vee$ on $\tilde{C}$. The restriction map
\[ \text{Hom} (E^\vee, \mathcal{O}_C) \to \text{Hom} (L^\vee, \mathcal{O}_C) =: L \]
induces an isomorphism
\[ E = \pi_* L. \]
Since
\[ \det \pi_* L = \omega, \]
we have
\[ \deg L = 2 (2g - 2) = \tilde{g} - 1. \]
For any small deformation $L'$ of $\{L\}$ in $Pic^{g-1} \left( \tilde{C} \right)$, $E' = \pi_* L'$ is stable, but
\[ \det E' \]
may not equal $\omega$, a difficulty we can always remedy by twisting $L'$ by $\pi^* (M)$ for some $\{M\} \in Pic^0 (C)$.
Let
\[ \iota : \tilde{C} \to \tilde{C} \]
\[ y \mapsto -y \]
denote the involution on $\tilde{C}$. By considering $\pm 1$-eigenspaces, $\iota$ induces an isomorphism
\[ \pi_* \mathcal{O}_{\tilde{C}} = \mathcal{O}_C \oplus \mathcal{O}_C \left( \sqrt{-\Delta} \right) = \mathcal{O}_C \oplus \mathcal{O}_C (\omega^{-1}) = \mathcal{O}_C \oplus T_C. \]  
A first-order deformation of $L$ is given by an element of
\[ H^1 (\mathcal{O}_C) = H^1 (\mathcal{O}_C) \oplus H^1 \left( \mathcal{O}_C \left( \sqrt{-\Delta} \right) \right). \]
By the projection formula, first-order deformations in $H^1 (\mathcal{O}_C)$ change $\det \pi_* L$. Thus first-order deformations with fixed determinant are given by elements of
\[ H^1 \left( \mathcal{O}_C \left( \sqrt{-\Delta} \right) \right) \subseteq H^1 (\mathcal{O}_C). \]
The natural map
\[ \pi_* \text{End} (L) \to \text{End} (\pi_* L) \]
is an element of
\[ \mathcal{O}_C \oplus \text{Hom} \left( \mathcal{O}_C \left( \sqrt{-\Delta} \right), \text{End}^0 E \right) \]
where $\text{End}^0$ denotes those endomorphism of trace zero. The summand of this map in
\[ \text{Hom} \left( \mathcal{O}_C \left( \sqrt{-\Delta} \right), \text{End}^0 E \right) = \text{Hom}^0 \left( E^\vee, E \right) \]
is given by
\[ \tilde{\varphi} \in \Gamma \left( \text{Hom}^0 \left( E^\vee, E^\vee \otimes \omega \right) \right) \]
considered as the inclusion
\[ \psi : O_C \left( \sqrt{\Delta} \right) \to \text{End}^0 E \]
The induced mapping
\[ H^1 \left( O_C \left( \sqrt{\Delta} \right) \right) \to H^1 \left( \text{End}^0 E \right) \]
is an isomorphism for general choice of \((C, \psi)\). (See \[B\], §1.5.)

1.5. ˜\(\psi\) arising from a quadric. Returning to the exact sequence (6), the dual sequence
\[ 0 \to E^\vee \to W^\vee \otimes O_C \to F^\vee \to 0 \]
lets us view (6) as a morphism
\[ l : C \to G := \text{Gr}(2, W^\vee) \]
\[ c \mapsto P(E_c^\vee) \]
So we see that, if we choose a quadric \(Q\) on \(P(W^\vee)\), then for
\[ \psi = \hat{Q} \]
we have that \(\hat{C}\) is the inverse image of \(C\) under the double cover of \(P(W)\) branched at \(c \in C\) exactly when the line \(l(c)\) is tangent to the quadric \(Q\). So, if we choose \(Q\) generally, \(\hat{C}\) has \(2g - 2\) distinct branchpoints in \(C\) and so must be smooth. (The authors wish to thank Christian Pauly for pointing this fact out to us and providing the above proof.) Thus \(\hat{C}\) is the double cover of \(C\) induced from the standard double cover of \(G\) induced by the quadric \(Q\), that is, the double cover branched along the divisor consisting of those lines tangent to \(Q\).

Continuing in the situation (12), a section \(w \in W \subseteq H^0 (E)\)
gives a section of \(L\) which vanishes exactly when one of the two points of
\[ P(E_c^\vee) \cap Q \]
lies in the hyperplane \(w \in W = W^\vee\). So
\[ \pi_* L = O_C(2). \]

1.6. Quotients of \(C \times \mathbb{C}^3\). If \(E\) is globally generated, we can choose \(W\) in (5) such that
\[ \dim W = 3. \]
In this case, we obtain a morphism
\[ C \to P(W) = \mathbb{P}^2. \]
\[ c \mapsto P(F_c) \]
Since
\[ c_1(F) = \omega^{-1} \]
we see that (13) is a projection of the canonical mapping. Conversely suppose we take any projection of the canonical curve
\[ p : C \to \mathbb{P}^2, \]
whose center does not meet the canonical curve. Then the resulting exact sequence
\[ 0 \to p^*\mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{p^*} \mathbb{C}^3 \otimes \mathcal{O}_C \to E \to 0 \]
gives rise to a globally generated rank-2 bundle $E$ with canonical determinant. If a section of $E$ coming from $w \in \mathbb{C}^3$ has $m$ zeros, then
\[ p(C) \]
must have an $m$-fold point at $w$ and conversely. Assuming $C$ is not hyperelliptic, by a result of Lazarsfeld (see Theorem 1.1 of [G]) there are three sections $\alpha, \beta, \gamma \in H^0(\omega)$ such that the map
\[ \alpha \cdot + \beta \cdot + \gamma \cdot : H^0(\omega) \to H^0(\omega^2) \]
is surjective. Dually, for general $C$, the generic map
\[ H^1(T_C) \to H^1(\mathcal{O}_C)^{\otimes 3} \]
induced by the choice of 3 sections of $\omega$ is injective, so that
\[ (16) \quad \mathbb{C}^3 \to H^0(E) \]
is surjective for generic $C$ and $p$. Now any non-semi-stable bundle $E$ has a subbundle of degree at least $g$ and therefore a section with $g$ or more zeros. Therefore $p(C)$ has a $g$-tuple point. Thus we conclude that, for general choice of $C$ and $p$, the bundle $E$ is semi-stable if $g > 2$ since $p(C)$ has only nodes. Furthermore, for general choice of $C$ and $p$ and generic
\[ \check{\varphi} \in \Gamma Hom^0(E^\vee, E) \]
the associated spectral curve $\check{C}$ is smooth.

We also claim that the map $\Gamma$ is an isomorphism in this case. To see this last assertion, it will suffice to prove that $\Gamma$ is an isomorphism for some special choice of $C, p, \check{\varphi}$. For example let $C'$ be a general curve which has a vanishing theta-null, that is, admits a line bundle $J$ with $J^2 = \omega$ and
\[ h^0(J) = 2. \]
(This is a codimension-one condition on $C$.) Let
\[ E' = J \oplus J. \]
Since $J$ is globally generated, so is $E'$. If $g > 3$, then $C'$ is not hyperelliptic, so again by the result of Lazarsfeld there are three sections $\alpha, \beta, \gamma \in H^0(\omega)$ such that the map
\[ \alpha \cdot + \beta \cdot + \gamma \cdot : H^0(\omega) \to H^0(\omega^2) \]
is surjective. Define
\[ \check{\varphi}' : J^\vee \oplus J^\vee \to J \oplus J \]
by the matrix
\[ \begin{pmatrix} \alpha & -\beta \\ \gamma & -\alpha \end{pmatrix}. \]
The mapping
\[ \Gamma Hom^0((E')^\vee, E') \to H^0(\omega^2) \]
\[ \tilde{\psi} \mapsto Tr(\tilde{\psi} \circ \check{\varphi}) \]
is therefore surjective. So by Serre duality, the map
\[ \varphi' : H^1(T_C) \to H^1(\text{End}^0 E') \]
is injective. (Compare [B], §1.3.) Now for generic
\[ W \subseteq H^0(E'), \dim W = 3 \]
\( E' \) is a quotient as in (6) and so given by a sub-bundle \( F' \) corresponding to a projection of the canonical curve \( C' \) into \( \mathbb{P}^2 \). But the triple \((C', F', \varphi')\) can be deformed to the generic such triple \((C, F, \varphi)\), that is, to the generic projection of a general curve \( C \) of genus \( g \) to \( \mathbb{P}^2 \). The resulting quotient \( E \) is globally generated by construction and (11) for \( \varphi \) is injective by semi-continuity of rank. Thus \( \varphi \) is an isomorphism since the general quotient \( E \) has \( h^0(\text{End}^0 E) = 0 \).

Finally in the situation (14) and (12), given \( c \in C \), we can choose three conics \( Q_1, Q_2, Q_3 \) such that
\[ Q_i|_{E^\vee} \]
form a basis for \( \text{Sym}^2 E_c \). Thus for \( \varphi_i = Q_i \) in (10), the map
\[ \varphi_1 + \varphi_2 + \varphi_3 : \mathcal{O}_C(\sqrt{-\Delta})^3 \to \text{End}^0 E \]
is generically surjective. Thus the same is true for generic choice of \( \varphi_i \in \Gamma \text{Hom}^0( E^\vee, E) \).

1.7. The theorem. In what follows we shall treat only those rank-2 bundles \( E \) such that:

\textbf{Condition 1:} \( E \) is simple, globally generated, \( \det E = \omega \).

\textbf{Condition 2:} The map (11) is an isomorphism for generic \( \varphi \in \Gamma \text{Hom}^0( E^\vee, E) \).

Notice that, since \( \mathcal{O}_C(\sqrt{-\Delta}) = T_C \), Condition 2) implies that
\[ h^0(\text{End}^0 E) = 3g - 3 + 0 + 3(1 - g) = 0, \]
that is, \( E \) is \textit{simple}. Since \( H^0(E) = H^0(L) \), global generation in Condition 1) requires that
\[ h^0(L) \geq 3. \]
So we are restricting attention to a locally closed subvariety of relatively high codimension in \( \mathfrak{M}(C) \).

Let
\[ W^r_{2, \omega} \subseteq \mathfrak{M}(C) \]
denote the scheme defined by those \( E \) such that \( h^0(E) \geq r + 1 \). Bertram-Feinberg [BF] and Mukai [M] show that \( W^r_{2, \omega} \) has the natural structure of a determinantal scheme whose cok tangent space at \((C, E)\) is the cokernel of the map
\[ \text{Sym}^2 H^0(E) \to H^0(\text{Sym}^2 E) \]
and that, if (18) is injective, then \( W^r_{2, \omega} \) is smooth and reduced and so of dimension
\[ \rho(2, r, \omega) = 3g - 3 - \frac{(r + 1)(r + 2)}{2} \]
at \((C, E)\). The purpose of this paper is to prove:
**Theorem 1.1.** If $C$ is of general moduli and $E$ satisfies Condition 1 and Condition 2 above, the map

$$H^0(E) \otimes H^0(E) \to H^0(E \otimes E)$$

is injective, so that (18) is also. If $g \geq 4$, there exist bundles $E$ satisfying Conditions 1 and 2 so that, by semi-continuity, these conditions are satisfied for a generic globally generated $E$ with canonical determinant.

For $C$ is a curve of genus $g$ of general moduli, this result has been conjectured for all stable rank-2 $E$ with canonical determinant by Bertram-Feinberg and by Mukai (loc. cit.). A corollary in the special case in which $E$ satisfies Condition 1 and Condition 2 above is that the map

$$\bigwedge^2 H^0(E) \to H^0\left(\bigwedge^2 E\right) = H^0(\omega)$$

is also injective. So, in particular

$$3 \leq \frac{h^0(E)(h^0(E) - 1)}{2} \leq g.$$  

1.8. **Analogy with proof of Petri conjecture.** Our proof will develop analogously to the proof of the classical Petri conjecture for line bundles in [C2]. In particular we employ Kuranishi theory as in [C1] and [C2] to relate $n$-th order deformation theory of $(C, \iota, L)$ to the first cohomology of sheaves of (holomorphic) differential operators

$$D_n(L)$$

of order $\leq n$ on sections of $L$.

2. **The space $P(E^\vee)$**

Let

$$X = P(E^\vee).$$

The natural inclusion

$$L^\vee \subseteq E^\vee$$

induces an imbedding

$$(19) \quad \breve{C} \to X$$

which assigns to an eigenvalue $\left(c, \sqrt{\varphi(c)}\right)$ in $\breve{C}$ the corresponding eigenvector in $E_c^\vee$. Abusing notation we will let

$$\pi : X \to C$$

also denote the projection map extending $\pi : \breve{C} \to C$.

Thus we have the exact sequences

$$(20) \quad 0 \to T_\pi \to T_X \to \pi^*T_C \to 0$$

and

$$0 \to \pi_*T_\pi \to \pi_*T_X \to T_C \to 0.$$
Notice that the Euler sequence
\[ 0 \to \mathcal{O}_X \to \pi^* E^\vee \otimes \mathcal{O}_X (1) \to T_\pi \to 0 \]
gives
\[ T_\pi = \pi^* T_C \otimes \mathcal{O}_X \]
and
\[ \pi_* T_\pi = \operatorname{End}^0 (E^\vee) = \operatorname{End}^0 E = T_C \otimes \operatorname{Sym}^2 E. \]
Since
\[ \operatorname{Ext}^1 (\pi^* T_C, T_\pi) = H^1 (\pi^* \omega \otimes T_\pi) \]
\[ = H^1 (\omega \otimes \pi_* T_\pi) \]
\[ = H^1 (\omega \otimes \operatorname{End}^0 E) \]
\[ = H^0 (\operatorname{End}^0 E)^\vee \]
\[ = 0 \]
the sequence (20) is split and hence so is its push-forward. Thus non-canonically
(21) \[ T_X = T_\pi \oplus \pi^* T_C \]
(22) \[ \pi_* T_X = \operatorname{End}^0 E \oplus T_C. \]

The Euler sequence also gives
\[ 0 \to \mathcal{O}_X \to \pi^* \operatorname{End} (E^\vee) \to \pi^* \pi_* T_\pi \to 0. \]
The kernel of the mapping
\[ \pi^* \operatorname{End} (E^\vee) \to \pi^* E^\vee \otimes \mathcal{O}_X (1) \]
\[ (c, [v], \varepsilon) \mapsto (c, [v], \varepsilon |_{[v]}) \]
at \((c, [v])\) is those endomorphisms \(\varepsilon\) which vanish on \(v\) and so the kernel of
(23) \[ \pi^* \operatorname{End} (E^\vee) \to T_\pi \]
is those \(\varepsilon\) such that \(v\) is an eigenvector of \(\varepsilon\). Thus the inclusion
\[ \pi^* T_C \to \pi^* \operatorname{End} (E^\vee) \]
in (10) given by
\[ \tilde{\varphi} \in \operatorname{Hom} (E^\vee, E^\vee \otimes \omega) \]
composes with (23) to give a map
(24) \[ \pi^* T_C \to T_\pi \]
of line bundles on \(X\) which vanishes exactly along \(\tilde{C}\). Thus
\[ T_\pi = \mathcal{O}_X (\tilde{C}) \otimes \pi^* T_C. \]
By the Euler sequence
\[ \pi^* \det (E^\vee) \otimes \mathcal{O}_X (2) = T_\pi \]
so that
(25) \[ \mathcal{O}_X (\tilde{C}) = \mathcal{O}_X (2). \]
3. The higher $\nu$-maps

The isomorphism (25) and the resulting exact sequence
$$0 \to \mathcal{O}_X (-1) \to \mathcal{O}_X (1) \to L \to 0$$
gives, by applying $R\pi_*$, isomorphisms
$$H^i (E) = H^i (\mathcal{O}_X (1)) \to H^i (L)$$
for all $i \geq 0$. The tangent space to the deformation space of $(X, \mathcal{O}_X (1))$ is
$$H^1 (\mathcal{D}_1 (\mathcal{O}_X (1))) = H^1 (\pi_* \mathcal{D}_1 (\mathcal{O}_X (1)))$$
and the element $\alpha \in H^0 (L) = H^0 (E)$ deforms to first-order under $\zeta \in H^1 (\mathcal{D}_1 (\mathcal{O}_X (1)))$ if and only if
$$\zeta \cdot \alpha \in H^1 (\mathcal{O}_X (1)) = H^1 (E)$$
vanishes. (See [AC] or [C2].)

Next let
$$\mathcal{D}_n \subseteq \mathcal{D}_n (E)$$
denote the image of $\pi_* \mathcal{D}^n (\mathcal{O}_X (1))$ under the natural map
$$\pi_* \mathcal{D}_n (\mathcal{O}_X (1)) \to \mathcal{D}_n (\pi_* \mathcal{O}_X (1))$$.
Using (21) we have the (symbol) exact sequences
$$0 \to \mathcal{O}_C \to \mathcal{D}_1 \sigma \to T_C \oplus \text{End}^0 E \to 0$$
and
$$0 \to \mathcal{D}_{n-1} \to \mathcal{D}_n \sigma \to T^n_C \oplus \left( T_{n-1}^p \otimes \text{End}^0 E \right) \to 0.$$ So we conclude from the symbol sequence
$$0 \to \mathcal{D}_{n-1} (E) \to \mathcal{D}_n (E) \sigma \to T^n_C \otimes \text{End} E \to 0$$
that
$$\mathcal{D}_{n-1} (E) \subseteq \mathcal{D}_n.$$ (26)

From (10), we have the inclusion
$$T_C \oplus T_C \to T_C \oplus \text{End}^0 E$$
defined by $\tilde{\varphi}$. We define
$$\tilde{\mathcal{D}}_1 = \sigma^{-1} (T_C \oplus T_C)$$
resulting in the (symbol) exact sequence
$$0 \to \mathcal{O}_C \to \tilde{\mathcal{D}}_1 \sigma \to T_C \oplus T_C \to 0.$$ Since
$$\tilde{\mathcal{D}}_1 \subseteq \pi_* \mathcal{D}_1 (\mathcal{O}_X (1)) \to \mathcal{D}_1 (\pi_* \mathcal{O}_X (1))$$
is injective, we can (and will) consider $\tilde{\mathcal{D}}_1$ as a subsheaf of $\mathcal{D}_1$. In fact
$$\tilde{\mathcal{D}}_1 \cap \mathcal{D}_0 (E) = \mathcal{O}_C \oplus T_C \subseteq \mathcal{D}_0 (E) = \text{End} E$$
where the containment is given by sending $\mathcal{O}_C$ to scalar endomorphisms and using the map
$$\tilde{\varphi} : T_C \to \text{End}^0 E.$$
Also, (11) implies that
\[ H^1(\tilde{D}_1) \]
is the tangent space to the deformation space of pairs \((C, E)\). The subspace
\[ S_\omega \subseteq H^1(\tilde{D}_1) \]
of those deformations such that \(\text{det} E\) remains the canonical bundle maps isomorphically onto \(H^1(T_C \oplus T_C)\).

Let
\[ S = \{ \zeta \in S_\omega : (\zeta : H^0(E) \to H^1(E)) = 0 \} . \]
The assumption that \(C\) is generic implies that the map
\[
(27) \quad S \to H^1\left(\frac{\tilde{D}_1}{\tilde{D}_1 \cap \mathcal{D}_0(E)}\right) = H^1(T_C)
\]
is surjective.

We let
\[ \tilde{D}_n \subseteq \mathcal{D}_n \]
be the image of the natural map
\[ \tilde{D}_1^\otimes n \to \mathcal{D}_n \]
induced by composition of operators. We have the morphism of exact sequences
\[
(28) \quad 0 \to \tilde{D}_{n-1} \to \tilde{D}_n \to T_n^C \otimes \text{Sym}^n (O^{\oplus 2}_C) \to 0
\]
\[ \quad 0 \to \mathcal{D}_{n-1} \to \mathcal{D}_n \to T_n^C \oplus (T_{n-1}^C \otimes \text{End}^0 E) \to 0 \]

As with the Petri problem for line bundles (see [C2]), we want to consider the maps
\[ \tilde{\nu}^n : H^1(\tilde{D}_n) \to \text{Hom}(H^0(E), H^1(E)) \]
and the induced maps
\[ \nu^n : H^1(\tilde{D}_n) \to \text{Hom}(H^0(E), H^1(E)) \]
\[ \nu^n \to \text{image}(H^1(\text{End}E) \to \text{Hom}(H^0(E), H^1(E))) . \]

The surjectivity of (11) shows that
\[
(29) \quad \tilde{\nu}^1 \left( H^1(\tilde{D}_1) \right) = \text{image}(H^1(\text{End}E) \to \text{Hom}(H^0(E), H^1(E))) . \]

Analogously to the solution Petri problem for line bundles, we wish to show that, for general \(C\) and \(E\) as above,
\[
\nu^n = 0 \quad \text{for } n > 1 . \]

The next two sections are devoted to a proof of this fact. In the next section, we construct elements of \(H^1(\tilde{D}_n)\) whose symbols span \(H^1(\tilde{T}_n^C \oplus (T_{n-1}^C \otimes \text{Sym}^n (O^{\oplus 2}_C)))\)
for \(n > 1\). In the following section we show that these elements lie in the kernel of \(\tilde{\nu}^n\).
4. Generators of the domains of the higher $\nu$-maps

To construct the requisite elements of $H^1 \left( \tilde{Q}_n \right)$, we need to choose divisors along which to make (Schiffer) deformations of our general curve $C$. First recall that

$$H^1(\pi_* TX) = H^1(TX) = H^1(T\pi \oplus \pi^*TC) = H^1(End^0 E) \oplus H^1(TC) = H^1(TC) \oplus H^1(TC)$$

is the tangent space to the deformations of $X$. The first factor corresponds to deformations of the bundle (with canonical determinant) and the second summand corresponds to first-order deformations of the base curve $C$. Suppose now that $A$ is a sufficiently ample simple divisor on $C$ and let $U$ be a small analytic neighborhood of (the support of) $A$. The (non-canonical) splitting

$$TX = T\pi \oplus \pi^*TC,$$

whose summands we will call the vertical and horizontal tangent spaces respectively) allows us to fix an isomorphism

$$\pi^{-1}(U) = U \times \mathbb{P}^1.$$ such that

$$\pi^*T_C|_{\pi^{-1}(U)} = T_U \boxtimes \mathcal{O}_{\mathbb{P}^1}.$$ The inclusion

$$\pi^*T_C \to T\pi$$

which vanishes along $\tilde{C}$ also realizes $\pi^*T_C$ as a sheaf of (vertical) vector fields on $X$. Let

$$\alpha, \beta$$

be $C^\infty$-sections of $T_C$ with support inside $U$ which are meromorphic some neighborhood of $A$ with poles only above the points of $A$. By the above remarks, we have distinguished liftings of $\alpha, \beta$ to vector fields on $X$ supported in $\pi^{-1}(U)$ which are respectively horizontal and vertical. We denote the lifted vector fields again as $\alpha, \beta$. Then, as in [C1] and [C2],

$$\partial_t e^{-tL_{\alpha, \beta}}$$

determines a deformation of $X$. There is a lifting of $\alpha + \beta$ to a vector field $\tilde{\alpha} + \tilde{\beta}$ on the total space of $O_X(-1)$ with the property that

$$[\chi, \tilde{\alpha}] = 0 = [\chi, \tilde{\beta}]$$

where $\chi$ is the Euler vector field on $O_X(-1)$. Any two liftings differ by an element $$(a \circ \pi) \chi$$
where \( a \) is a function supported on \( U \) and meromorphic near \( A \) with poles only above the points of \( A \). The tangent space to Pic\(X\) is \( H^1(\mathcal{O}_C) \) and therefore as in Lemma 3.2 of [2] we can arrange that a deformation

\[
\partial, e^{-\left(tL_{\alpha_1 + \beta_1 + a_1 x} + t^2L_{a_2 x} + t^3L_{a_3 x} + \ldots \right)}
\]

of the bundle \( \mathcal{O}_X(-1) \) has the property that

\[
\det \pi_* \mathcal{O}_X(1)
\]

remains canonical.

We can modify (32) over a given deformation of \( C \) by modifying \( \beta_1 \) as follows.

\[
H^0 \left( \pi^*\mathcal{T}_C(A) \right) \rightarrow H^1 (\pi^*\mathcal{T}_C) = H^1 (End^0 E)
\]

is surjective. Recalling that \( H^1 (End^0 E) \) is the tangent space to \( \mathfrak{M}(C) \) at \( \{ E \} \), we can therefore choose a \( C^\infty \)-section \( \varepsilon_1 \) of \( \pi^*\mathcal{T}_C \subseteq \mathcal{T}_\pi \) which is supported over \( U \) and meromorphic at \( \pi^{-1} (A) \) so that, applying \( \partial \) to the element, we can realize any given element of \( H^1 (End^0 E) \). Thus

\[
\partial, e^{tL_{\alpha_1 + \beta_1 + \varepsilon_1}}
\]

(34)

gives any first-order deformation of \( X \) over the given deformation of \( C \). Again using Lemma 3.2 of [2] above we can arrange that a deformation

\[
\partial, e^{-\left(tL_{\alpha_1 + \beta_1 + a_1 x} + t^2L_{a_2 x} + t^3L_{a_3 x} + \ldots \right)}
\]

(35)

of the bundle \( \mathcal{O}_X(-1) \) has the property that

\[
\det \pi_* \mathcal{O}_X(1)
\]

remains canonical.

However the sections of

\[
E = \pi_* \mathcal{O}_X(1)
\]

may not all deform under the deformation of \( (X, \mathcal{O}_X(1)) \) determined by (33). We will call a deformation of \( X \) admissible if all sections of \( \pi_* \mathcal{O}_X(1) \) do indeed deform. To modify (34) to produce an admissible deformation, we proceed as follows. Choose a sufficiently ample divisor \( B \) supported in an open set \( V \subseteq C \) disjoint from \( U \). Using the surjectivity of

\[
H^0 \left( \pi^*\mathcal{T}_C(B) \right) \rightarrow H^1 (\pi^*\mathcal{T}_C) = H^1 (End^0 E)
\]

we can choose a \( C^\infty \)-section \( \delta_1 \) of \( \pi^*\mathcal{T}_C \) which is supported on \( \pi^{-1} (V) \) and meromorphic at \( \pi^{-1} (V) \) and, applying \( \partial \) to the element, we can realize any given element of \( H^1 (End^0 E) \). And we can pick \( \delta_1 \) so that

\[
\partial, e^{tL_{\alpha_1 + \beta_1 + \varepsilon_1 + \delta_1}}
\]

(36)

gives any first-order deformation of \( X \) over the given deformation of \( C \), in particular, the admissible one which must exist by the assumption that \( C \) is generic. Similarly we can choose \( \delta_2 \) so that

\[
\partial, e^{tL_{\alpha_1 + \beta_1 + \varepsilon_1 + \delta_1 + t^2L_{\delta_2}}}
\]
gives the admissible second-order deformation over the given deformation of \( C \), etc. Repeating this argument we achieve an admissible formal deformation

\[
\partial, e^{tL_{\alpha_1+\beta_1+\epsilon_1+\delta_1+t^2L_{\alpha_2+\beta_2+t^3L_{\alpha_3+\beta_3}}+...}}
\]

of \( X \) over the deformation (31) of \( C \). Again using Lemma 3.2 of [C2] above we can arrange that a deformation

\[
\partial, e^{-\left(tL_{\tilde{\alpha}}+\tilde{\beta}+\tilde{\epsilon}+\tilde{\delta}+a\chi+t^2L_{\alpha_2+\beta_2+t^3L_{\alpha_3+\beta_3}}+...\right)}
\]

of the bundle \( \mathcal{O}_X (-1) \) has the property that

\[
\det \pi_* \mathcal{O}_X (1)
\]

remains canonical.

Now we are ready to look at the symbols of the operators we have just constructed. In

\[
H^1 (T^n_C) \otimes H^0 (\text{Sym}^n (\mathcal{O}_C)) = H^1 (T^n_C \otimes \text{Sym}^n (\mathcal{O}_C))
\]

we consider the subspace

\[
H^1 (T^n_C) \otimes H^0 (\text{Sym}^n (\mathcal{O}_C \oplus 0))
\]

corresponding to the \( n \)-th power of vertical tangent vectors, that is, those lying in \( T_\pi \).

**Lemma 4.1.** i) Elements (37) are of the form

\[
\sum_{n>0} D_n t^n
\]

with

\[
D_n \in H^1 \left( \tilde{\mathcal{D}}_n \right).
\]

ii) For \( n > 1 \), the maps \( \nu^n_1 \) factor through

\[
H^1 \left( \frac{T^n_C \otimes \text{Sym}^n (\mathcal{O}_C)}{T^n_C \otimes \text{Sym}^n (\mathcal{O}_C \oplus 0)} \right)
\]

iii) For \( n > 1 \), the collection of such elements \( D_n \) obtained by varying \( \tilde{\alpha}_1, \tilde{\epsilon}_1 \) and the divisor \( A \) have symbols

\[
\sigma (D_n)
\]

which span \( H^1 \left( \frac{T^n_C \otimes \text{Sym}^n (\mathcal{O}_C)}{T^n_C \otimes \text{Sym}^n (\mathcal{O}_C \oplus 0)} \right) \).

**Proof.** i) Consider the (direct limit) exact sequence

\[
0 \to \tilde{\mathcal{D}}_n \to \mathcal{O}_C (\infty \cdot (A + B)) \otimes \tilde{\mathcal{D}}_n \to \frac{\mathcal{O}_C (\infty \cdot (A + B)) \otimes \tilde{\mathcal{D}}_n}{\tilde{\mathcal{D}}_n} \to 0.
\]

The coefficient of \( t^n \) in \( e^{-\left(tL_{\tilde{\alpha}_1+\tilde{\beta}_1+\tilde{\epsilon}_1+\tilde{\delta}_1+t^2L_{\alpha_2+\beta_2+t^3L_{\alpha_3+\beta_3}}+...\right)} \) lies in \( H^0 \left( \frac{\mathcal{O}_C (\infty \cdot (A + B)) \otimes \tilde{\mathcal{D}}_n}{\tilde{\mathcal{D}}_n} \right) \).

ii) Suppose first that we restrict the elements (37) to those for which \( \alpha_1 = 0 \). Then the symbols of the coefficients of \( t^n \) lie in the “subspace of vertical symbols”

\[
H^1 (T^n_C) \otimes H^0 (\text{Sym}^n (\mathcal{O}_C \oplus 0)) \subseteq H^1 (T^n_C) \otimes H^0 (\text{Sym}^n (\mathcal{O}_C)) \).
A PETRI THEOREM FOR RANK-2 VECTOR BUNDLES WITH CANONICAL DETERMINANT

Letting the subscript "n" denote the coefficient of $t^n$, we have by Lemma 4.1 of [C2] that we can vary $\varepsilon_1$ and $A$ to obtain elements

$$D_n = \left[ \overline{\partial}, e^{-(\beta_0 + \beta_1 + \alpha_{1+a_1} + t^2L_{a_2} + t^3L_{a_3} + \ldots)} \right]_n$$

whose symbols generate

$$H^1(T^n_C) \otimes H^0(Sym^n(\mathcal{O}_C \oplus 0)) \subseteq H^1(T^n_C) \otimes H^0(Sym^n(\mathcal{O}^{\otimes 2}_C)).$$

Furthermore these elements $D_n$ have the property that

$$\tilde{\nu}^n(D_n) \in \text{image} \left( H^1(End^0E) \rightarrow \text{Hom} \left( H^0(E), H^1(E) \right) \right) \subseteq \text{image} \left( \nu^{n-1} \right).$$

Thus for $n > 1$ we can consider $\nu^n$ as a map

$$\nu^n : H^1 \left( \frac{T^n_C \otimes Sym^n(\mathcal{O}^{\otimes 2}_C)}{T^n_C \otimes Sym^n(\mathcal{O}_C \oplus 0)} \right) \rightarrow \text{Hom} \left( H^0(E), H^1(E) \right) \bigg/ \text{image} \left( \nu^{n-1} \right).$$

iii) Notice that our restriction on the support of the $\delta_j$ insures that, for $n > 1$, the coefficient of $t^n$ in (37) has the same symbol in $H^1(T^n_C) \otimes H^0(Sym^n(\mathcal{O}^{\otimes 2}_C))$ as

$$\left\{ \left[ \overline{\partial}, e^{\alpha_1 + \alpha_{1+a_1}} \right] \right\}$$

does. So it suffices to show that the symbols of the coefficients of $t^n$ in elements of the form (38) generate

$$H^1(T^n_C) \otimes H^0(Sym^n(\mathcal{O}^{\otimes 2}_C)) = H^1(T^n_C \otimes Sym^n(\mathcal{O}^{\otimes 2}_C)).$$

But $Sym^nH^0(\mathcal{O}^{\otimes 2}_C)$ generates $H^0(Sym^n(\mathcal{O}^{\otimes 2}_C))$, so it suffices to show that elements of the form

$$[\overline{\partial}, \alpha^n_1]$$

generate $H^1(T^n_C)$. But this is again the content of Lemma 4.1 of [C2].

5. VANISHING HIGHER $\nu$-MAPS

**Lemma 5.1.** For generic $C$ and for $E$ satisfying Condition 1 and Condition 2, the maps

$$\nu^n : H^1 \left( T^n_C \right) \otimes \left( T^{n-1}_C \otimes Sym^n(\mathcal{O}^{\otimes 2}_C) \right) \rightarrow \frac{\text{Hom} \left( H^0(E), H^1(E) \right)}{\text{image} \left( \tilde{\nu}^{n-1} \right)}$$

vanish for $n > 1$.

**Proof.** By Lemma 4.1 therefore, it suffices to show that, for each expression (37), there are operators

$$\tilde{D}_n \in H^1(\tilde{D}_n)$$

such that

1) $\tilde{D}_n$ has the same symbol in

$$H^1(T^n_C \otimes Sym^n(\mathcal{O}^{\otimes 2}_C))$$
as the coefficient of $t^n$ in

$$\left[ \partial, e^{-\left(tL_{\tilde{\alpha}} + \tilde{\beta}L_1 + \tilde{\epsilon}L_2 + L_3 + \ldots \right)} \right]$$

(39)

2)

$$\hat{\nu}^n \left( \hat{D}_n \right) = 0.$$ 

The proof of this fact is identical to the proof given in §3.4-3.5 of \cite{C2}. The notational dictionary is

\begin{align*}
\text{reference paper} & \leftrightarrow \text{this paper} \\
X_0 & \leftrightarrow X \\
L_0 & \leftrightarrow O_X (1) \\
S^n (T_{X_0}) & \leftrightarrow T^n_C \otimes \text{Sym}^n (O_C^{\oplus 2}).
\end{align*}

In the proof we consider the line bundle

$$\hat{E} (1) = \mathcal{O}_{\mathbb{P}(H^0 (E))} (1) \boxtimes O_X (1)$$

on the product

$$\mathbb{P} \left( H^0 (O_X (1)) \right) \times X.$$ 

Just as in §3.4-3.5 of \cite{C2} we use Lemma 2.9 and Lemma 3.1iii) of loc. cit. to twist the Kuranishi data (39) above to obtain

$$\left[ \partial, e^{-\left(tL_{\tilde{\alpha}} + \tilde{\beta}L_1 + \tilde{\epsilon}L_2 + L_3 + \ldots \right)} e^{-L_{\tilde{\beta}T(t)}} \right]$$

(40)

for which the condition that the tautological section extends under deformation implies that the map

$$H^0 \left( \hat{E} (1) \right) \to \sum_{n>0} H^1 \left( \hat{E} (1) \right) t^n$$

given by (40) is 0.

Next notice that the generic surjectivity of (17) implies that the maps

$$\tilde{\varphi}_1 + \tilde{\varphi}_2 + \tilde{\varphi}_3 : \left( \mathcal{O}_C \left( \sqrt{-\Delta} \right)^{\oplus 3} \right) \otimes T^{n-1}_C \to \left( \text{End}^0 E \right) \otimes T^{n-1}_C$$

are all generically surjective. Thus the maps

$$H^1 \left( \left( \mathcal{O}_C \left( \sqrt{-\Delta} \right) \right) \otimes T^{n-1}_C \right)^{\oplus 3} \to H^1 \left( \left( \mathcal{O}_C \left( \sqrt{-\Delta} \right)^{\oplus 3} \right) \otimes T^{n-1}_C \right)$$

(41)

are all surjective. For $i = 1, 2, 3$, let

$$\hat{D}^{(i)}_n$$

be the sheaf $\hat{D}_n$ on $C$ derived from the spectral curve associated to $\tilde{\varphi}_i$. Now define a map

$$F : \hat{\mathcal{D}}^{(1)}_n \oplus \hat{\mathcal{D}}^{(2)}_n \oplus \hat{\mathcal{D}}^{(3)}_n \to \mathcal{D}_n$$

by adding the three maps

$$\hat{\mathcal{D}}^{(i)}_n \to \mathcal{D}_n$$
associated to the spectral curves for $\tilde{\chi}_i$ for $i = 1, 2, 3$. We have a commutative diagram

$$
0 \to \sum_i \hat{\mathcal{D}}_{n-1}^{(i)} \to \sum_i \hat{\mathcal{D}}_n^{(i)} \to (T^n_C \otimes \text{Sym}^n (\mathcal{O}_C^{\otimes 2}))^{\otimes 3} \to 0
$$

and the generic surjectivity of (41) and induction shows that

$$
\text{image} (\sum_i \hat{\mathcal{D}}_n^{(i)})
$$

is surjective for each $n$.

**Corollary 5.2.** Letting $\tilde{\mu}^n : H^1 (\mathfrak{D}_n (E)) \to \text{Hom} (H^0 (E), H^1 (E))$ denote the natural maps induced by the action of $\mathfrak{D}_n (E)$ on sections of $E$, the induced maps

$$
\mu^n : H^1 (T^n_C \otimes \text{End} E) \to \frac{\text{Hom} (H^0 (E), H^1 (E))}{\text{image} (\tilde{\mu}^{n-1})}
$$

are all zero for $n \geq 1$.

**Proof.** Referring to (26), consider the chain

$$
\mathfrak{D}_0 = \mathcal{O}_C \subseteq \mathfrak{D}_1 (E) = \text{End} E \subseteq \mathfrak{D}_2 \subseteq \mathfrak{D}_3 \subseteq \mathfrak{D}_4 \subseteq \ldots
$$

of subsheaves and the image of $H^1$ of each in $\text{Hom} (H^0 (E), H^1 (E))$. By (29), Lemma 5.1, and the surjectivity of (42), the induced maps on $H^1$ of the graded quotients of this chain into respective quotients of $\text{Hom} (H^0 (E), H^1 (E))$ are all zero from

$$
H^1 \left( \mathfrak{D}_1 \bigg/ \text{End} E \right) \to \frac{\text{Hom} (H^0 (E), H^1 (E))}{\text{image} (H^1 (\text{End} E) \to \text{Hom} (H^0 (E), H^1 (E)))}
$$
on.

\square

**6. The BFM-conjecture for globally generated bundles**

In this final section we prove the injectivity of the map

$$
\mu_0 : H^0 (E) \otimes H^0 (E) \to H^0 (E \otimes E)
$$

in the case of a generic curve $C$ of genus $g > 1$ and a vector bundle $E$ satisfying Condition 1 and Condition 2 of §1. (Notice that we have only established the existence of such $E$ when $g > 3$ since we relied on the existence of a non-hyperelliptic $E$ with one vanishing theta-null.)

The isomorphism $[3]$ and Serre duality gives

$$
H^1 (E) = H^1 (E^\vee \otimes \omega) = H^0 (E)^\vee
$$

so that we have a decomposition of

$$
\text{Hom} (H^0 (E), H^1 (E)) = H^1 (E) \otimes H^1 (E)
$$

into the direct sum

$$
\text{Sym}^2 H^1 (E) \oplus \bigwedge^2 H^1 (E).
$$
Looking at the ±1-eigenspaces of the natural action given by reversing factors, the map

$$H^1(\text{End}E) \to \text{Hom}(H^0(E), H^1(E))$$

is the direct sum of the maps

$$H^1(\text{End}^0E) \to \text{Sym}^2H^1(E)$$

and

$$H^1(\mathcal{O}_C) \to \bigwedge^2 H^1(E).$$

Dually the multiplication map $\mu_0$ is the direct sum of the maps

$$\text{Sym}^2(H^0(E)) \to H^0(\text{Sym}^2E)$$

and

$$\bigwedge^2 H^0(E) \to H^0\left(\bigwedge^2 E\right) = H^0(\omega).$$

This final argument generalizes a similar one in §9.14 of [ACGH]. Let $p_i : C \times C \to C$ for $i = 1, 2$ denote the two projections and let $D \subseteq C \times C$ denote the diagonal and $\mathcal{I}$ its ideal in $\mathcal{O}_{C \times C}$. We have

$$D_n(E) = \text{Hom}\left(\left(\left(p_1\right)_* \frac{p_2^* E}{\mathcal{I}^n + 1} p_2^* E, E\right), \left(\left(p_1\right)_* \frac{p_2^* E}{\mathcal{I}^n + 1} p_2^* E, E^\vee \otimes \omega\right)\right).$$

This gives a perfect pairing

$$D_n(E) \otimes \left(\left(p_1\right)_* \frac{E \otimes E}{\mathcal{I}^n + 1} (E \otimes E)\right) \to \omega$$

under which the exact sequence

$$0 \to D_{n-1}(E) \to D_n(E) \xrightarrow{\sigma} T^n_{(-)} \otimes \text{End}E \to 0$$

pairs with the exact sequence

$$0 \to \left(\left(p_1\right)_* \frac{T^n E \otimes E}{\mathcal{I}^{n+1}} (E \otimes E)\right) \xrightarrow{\sigma^\vee} \left(\left(p_1\right)_* \frac{E \otimes E}{\mathcal{I}^{n+1}} (E \otimes E)\right) \xrightarrow{\left(\left(p_1\right)_* \frac{T^n E \otimes E}{\mathcal{I}^{n+1}} (E \otimes E)\right)} 0$$

where

$$\left(\left(p_1\right)_* \frac{T^n E \otimes E}{\mathcal{I}^{n+1}} (E \otimes E)\right) = \left(\left(p_1\right)_* \frac{T^n E}{\mathcal{I}^{n+1}}\right) \otimes E \otimes E.$$

So we have induced a perfect pairing

$$D_n(E) \otimes \left(\left(p_1\right)_* \frac{T^n E \otimes E}{\mathcal{I}^{n+1}} (E \otimes E)\right) \to \omega.$$

Now the natural maps

$$\hat{\mu}_n : H^0(E) \otimes H^0(E) = H^0(E \otimes E) \to H^0\left(\left(\left(p_1\right)_* \frac{E \otimes E}{\mathcal{I}^{n+1}} (E \otimes E)\right)\right)$$
are the adjoints of the maps
\[ \tilde{\mu}^n : H^1(\mathcal{D}_n(E)) \to \text{Hom}(H^0(E), H^1(E)) = H^1(E) \otimes H^1(E) \]
and the induced maps
\[ \mu_n : \ker \tilde{\mu}^{-1} = H^0((p_1)_* \mathcal{T}_n(E \boxtimes E)) \to H^0 \left( \left( p_1 \right)_* \frac{\mathcal{T}_{n+1}(E \boxtimes E)}{\mathcal{T}_n(E \boxtimes E)} \right) \]
are the adjoints of the maps
\[ \mu^n : H^1(T^n_C \otimes \text{End}E) \to \text{Hom}(H^0(E), H^1(E)). \]
Thus the vanishing of \( \tilde{\mu}^n \) for \( n \geq 1 \) implies the vanishing of the \( \mu_n \) for \( n \geq 1 \). But this in turn implies that the maps
\[ H^0 \left( \mathcal{T}^{n+1}_n( E \boxtimes E ) \right) \to H^0 \left( \mathcal{T}_n(E \boxtimes E) \right) \]
are isomorphisms for all \( n \geq 1 \). Since
\[ h^0 ( \mathcal{T}_n (E \boxtimes E)) = 0 \]
for large \( n \) we conclude that
\[ \ker \tilde{\mu}_0 = 0. \]
But
\[ \tilde{\mu}_0 = \mu_0. \]

7. Appendix: Mercat’s example

7.1. Necessity of Condition 2. Theorem 1.1, even injectivity on the symmetric summand, is definitely false without Condition 2. We are indebted to Vincent Mercat for pointing out a counterexample. Suppose \( g \) is odd and \( \mathbb{P}(H^0(B)) \) is a general (basepoint-free) \( g_{1,3}^{1,3} \)'s on a general curve \( C \). Since \( C \) is generic, the injectivity of the Petri map
\[ H^0(B^2) \otimes H^0(\omega \otimes B^{-2}) \to H^0(\omega) \]
and the injectivity of
\[ \text{Sym}^2 H^0(B) \to H^0(B^2) \]
imply that \( h^0(\omega \otimes B^{-2}) = 0 \). So
\[ \dim \text{Ext}^1(B, \omega \otimes B^{-1}) = 4 \]
so that, up to non-zero scalar, there is a unique non-trivial \( \varepsilon \in \text{Ext}^1(B, \omega \otimes B^{-1}) \) such that the (symmetric) map
\[ \varepsilon \cdot : H^0(B) \to H^1(\omega \otimes B^{-1}) \]
is zero. In fact \( \varepsilon \) is a hyperplane in \( H^0(B^2) \) given by the image of the injective map
\[ \text{Sym}^2 H^0(B) \to H^0(B^2), \]
then \( \varepsilon \) gives an extension
\[ 0 \to \omega \otimes B^{-1} \to E \to B \to 0 \]
such that the induced map
\[ H^0(E) \to H^0(B) \]
is surjective. Thus \( E \) is globally generated. By degree, every endomorphism of \( E \) acts as multiplication by a scalar on \( \omega \otimes B^{-1} \) and it follows easily that
\[ h^0(\text{End}^0 E) = 0, \]
that is, \( E \) is simple. Also, since \( h^0(\omega \otimes B^{-1}) = \frac{g-1}{2} \),
\[ h^0(E) = \frac{g+3}{2}. \]
So, for large enough \( g \),
\[ \dim \text{Sym}^2 H^0(E) > h^0(\text{Sym}^2 E). \]

7.2. Geometry of the spectral curve in Mercat’s example. In the presentation
\[ 0 \to F \to H^0(E) \otimes \mathcal{O}_C \xrightarrow{\varepsilon} E \to 0 \]
in Mercat’s example, the sections of \( E \) coming from sections of \( \omega \otimes B^{-1} \) all have
\[ 2g - 2 - \frac{g+3}{2} = \frac{3g-7}{2} \]
zeros on \( C \). For
\[ W = H^0(E) \]
the subspace
\[ W' := H^0(\omega \otimes B^{-1}) \subseteq H^0(E) \]
gives rise to an exact sequence
\[ 0 \to F' \to W' \otimes \mathcal{O}_C \to \omega \otimes B^{-1} \to 0 \]
and so a commutative diagram
\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & B^\vee & (W')^\vee \otimes \mathcal{O}_C & B \\
\downarrow & \downarrow & \downarrow & \\
0 & E^\vee & W^\vee \otimes \mathcal{O}_C & F^\vee \\
\downarrow & \downarrow & \downarrow & \\
0 & \omega^{-1} \otimes B & (W')^\vee \otimes \mathcal{O}_C & (F')^\vee \\
\downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0 \\
\end{array}
\]
with exact rows and columns. The sheaf of linear functionals on \( E \) which vanish on \( \omega \otimes B^{-1} \) lie in the intersection of \( (W')^\vee \otimes \mathcal{O}_C \) and \( E^\vee \) in \( W^\vee \otimes \mathcal{O}_C \). Thus, for the mapping
\[ p : C \to G, \]
\[ c \mapsto \mathbb{P}(\varepsilon^\vee (E^\vee_c)) \]
\( p(C) \) lies in the Schubert cycle of lines incident to the line
\[ l_0 := \mathbb{P}\left(\left(\frac{W}{W'}\right)^\vee\right) \subseteq \mathbb{P}(W^\vee). \]
In fact, from (45) it is easy to see that the $g_2$ on $C$ is given by assigning to each point $x \in l_0$ the divisor of points $c \in C$ such that $x \in \mathbb{P}(E'_c)$.

Projection with center $l_0$ maps

$$\tilde{C} \subseteq \mathbb{P}(W')$$

$2 - 1$ to the embedding

$$C \subseteq \mathbb{P}((W')^\vee)$$

induced by the bundle inclusion

$$\omega^{-1} \otimes B \subseteq (W')^\vee \otimes \mathcal{O}_C,$$

that is, by the sections of the line bundle $\omega \otimes B^{-1}$. Also, via intersection, the line $l_0$ gives a section

$$S := \mathbb{P}(B')$$

of the bundle

$$\pi : \mathbb{P}(E') \to C$$

so that (44) is obtained by applying

$$\pi_* \circ (\mathcal{O}_{\mathbb{P}(E')} (1) \otimes )$$

to the sequence

$$0 \to \mathcal{O}_{\mathbb{P}(E')} (-S) \to \mathcal{O}_{\mathbb{P}(E')} \to \mathcal{O}_S \to 0.$$

We next choose a general

$$\tilde{\phi} \in \text{Hom}^0 (E', E)$$

with spectral double cover

$$\tilde{C} \subseteq \mathbb{P}(E') \to C.$$

For the line bundle

$$L = \mathcal{O}_{\mathbb{P}(E')} (1)|_{\tilde{C}}$$

and involution

$$\iota : \tilde{C} \to \tilde{C},$$

we have

$$\pi_* (\iota^* L) = \pi_* L. \tag{46}$$

The inclusion

$$\omega \otimes B^{-1} \to \pi_* L$$

induces a non-trivial map

$$\pi^* (\omega \otimes B^{-1}) \to L \tag{47}$$

so that

$$L = \pi^* (\omega \otimes B^{-1}) \otimes \mathcal{O}(\tilde{D}_0)$$

$$\iota^* L = \pi^* (\omega \otimes B^{-1}) \otimes \mathcal{O}(\iota^* \tilde{D}_0)$$
for some effective divisor $\tilde{D}_0$ on $\tilde{C}$ with

$$\omega = \det \pi_* L = \omega \otimes B^{-2} \otimes \mathcal{O} \left( \pi_* \tilde{D}_0 \right)$$

$$\mathcal{O} \left( \pi_* \tilde{D}_0 \right) = B^2.$$ 

In fact, since

$$\mathcal{O}_{\mathbb{P}(W^\vee)} (1) \big|_C = \pi^* (\omega \otimes B^{-1}) \otimes \mathcal{O} \left( \tilde{D}_0 \right)$$

$$\mathcal{O}_{\mathbb{P}(W')^\vee} (1) \big|_C = \omega \otimes B^{-1}$$

the support of $\tilde{D}_0$ must equal the intersection of $\varepsilon^\vee \left( \tilde{C} \right)$ with the center $l_0$ of the projection

$$\mathbb{P} (W^\vee) \dashrightarrow \mathbb{P} \left( (W')^\vee \right).$$

Now suppose, in Mercat’s example,

$$\tilde{\varphi} = \tilde{\Omega}$$

for some quadric

$$Q \in Sym^2 \left( H^0 \left( E \right) \right).$$

For example, suppose that

$$Sym^2 H^0 \left( \mathcal{O}_{\mathbb{P}(E^\vee)} (1) \right) = Sym^2 H^0 \left( E \right) \rightarrow H^0 \left( Sym^2 E \right) = H^0 \left( \mathcal{O}_{\mathbb{P}(E^\vee)} (2) \right)$$

is surjective. Since

$$\varepsilon^\vee \left( \tilde{C} \right) = Q \cap \varepsilon^\vee \left( E^\vee \right),$$

and the $g^1_{2+3}$ on $C$ is given by assigning to each point $x \in l_0$ the divisor of points $c \in C$ such that $x \in \mathbb{P} \left( E^\vee_c \right)$, we conclude that

$$\tilde{D}_0 = \tilde{D}_1 + \tilde{D}_2,$$

where the divisors

$$\pi_* \tilde{D}_1, \pi_* \tilde{D}_2 \in g^1_{2+3}$$

are those parametrized by the two points of

$$l_0 \cap Q.$$

Then

$$H^0 \left( \mathcal{O}_{\mathbb{P}(E^\vee)} (2) \right) \rightarrow H^0 \left( \mathcal{O}_{S} (2) \right) = H^0 \left( B^2 \right)$$

is not surjective since it has image $Sym^2 H^0 \left( B \right) \not\subseteq H^0 \left( B^2 \right)$.

By §1.3.2 of [B] Condition 2 is equivalent to the condition

$$h^0 \left( \pi^* \omega \otimes \mathcal{O} \left( \iota^* \tilde{D}_0 - \tilde{D}_0 \right) \right) = 0.$$ 

It follows from the main theorem of this paper that Condition 2 must fail for Mercat’s example. It would be reassuring to have an independent proof of this fact, but the authors have not found an independent method for deciding whether or not Condition 2 holds in the case of Mercat’s example.
A PETRI THEOREM FOR RANK-2 VECTOR BUNDLES WITH CANONICAL DETERMINANT

REFERENCES

[AC] Arbarello, E., Cornalba, M. “Su una congettura di Petri.” Comment. Math. Helvetici 56 (1981) 1-38.

[ACGH] Arbarello, E., Cornalba, M., Griffiths, P., Harris, J. “Special divisors on algebraic curves.” Lecture notes: Regional Algebraic Geometry Conference, Athens, Georgia, May, 1979.

[B] Beauville, A. “Fibrés de rang 2 sur une courbe, fibré déterminant et fonctions thêta.” Bull. Soc. math. France, 116 (1988), 431-448.

[BF] Bertram, A., Feinberg, B. “On stable rank-2 vector bundles with canonical determinant and many sections.” Lecture notes in pure and applied mathematics. Dekker, 200 (1998), 259-269.

[C1] Clemens, H. “Cohomology and Obstructions I: On the geometry of formal Kuranishi theory.” Preprint, math.AG/9901084 (1999).

[C2] Clemens, H. “A local proof of Petri’s conjecture at the general curve.” J. Diff. Geom. 54 (2000), no.1, 139-176.

[G] Gieseker, D. “A lattice version of the KP equation.” Acta Math. 168 (1992), no. 3-4, 219-248.

[vGI] van Geeman, B., Izadi, E. “The tangent space to the moduli space of vector bundles on a curve and the singular locus of the theta divisor of the Jacobian.” J. Alg. Geom. 10 (2001), 133-177.

[H] Hitchin, N.J. “Stable bundles and integrable systems.” Duke Math. J. 54 (1987), no. 1, 91-114.

[M] Mukai, S. “Vector bundles and Brill-Noether theory.” Complex Algebraic Geometry. MSRI Publ. 28 (1992), 255-263.

[T] Teixidor i Bigas, M. “Brill-Noether theory for stable vector bundles.” Duke Math. J. 62 (1991), 385-400

E-mail address: casini@science.unitn.it, clemens@math.utah.edu