TIME-DEPENDENT SYSTEMS OF GENERALIZED YOUNG MEASURES

G. DAL MASO, A. DESIMONE, M.G. MORA, AND M. MORINI

Abstract. In this paper some new tools for the study of evolution problems in the framework of Young measures are introduced. A suitable notion of time-dependent system of generalized Young measures is defined, which allows to extend the classical notions of total variation and absolute continuity with respect to time, as well as the notion of time derivative. The main results are a Helly type theorem for sequences of systems of generalized Young measures and a theorem about the existence of the time derivative for systems with bounded variation with respect to time.

Keywords: Young measures, bounded variation, absolute continuity, weak derivatives, concentration and oscillation effects

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1. Introduction

The notion of Young measure was introduced by L.C. Young in [25] to describe generalized solutions to minimum problems in the calculus of variations. Since then it has been applied to several problems in the calculus of variations, in control theory, in partial differential equations, and in mathematical economics. For the general theory of Young measures we refer to [3], [4], [7], [14, Chapters 2 and 3], [16], [19], [23], [24, Chapter IV], and [26]. Several applications are devoted to evolution problems (see, e.g., [11], [12], [17], [18], [20], and [21]).

In this paper we introduce some new tools in the theory of Young measures for the study of rate independent evolution problems. To describe the content of this paper, let us consider a problem defined on a time interval $I$, with space variable $x$ in a compact metric space $X$, and state variable $u$ in a finite dimensional Hilbert space $\Xi$. We assume that $X$ is endowed with a given nonnegative Radon measure $\lambda$ with $\text{supp} \lambda = X$. Given a sequence $u_k = u_k(t,x)$ of functions from $I \times X$ to $\Xi$, satisfying suitable estimates, it is often possible
to extract a subsequence converging, for every \( t \in I \), to a Young measure \( \mu_t \), which encodes information on the statistics of the space oscillations of \( u_k(t,x) \) at time \( t \).

To simplify the notation, the Young measure \( \mu_t \) will always be regarded as a measure on \( X \times \Xi \), whose projection on \( X \) coincides with \( \lambda \). In this introduction we will never consider the standard disintegration \((\mu^x_t)_x \in X\), which is usual in the classical presentation of the theory (see Remark 3.5).

If we want to extend some natural notions, like total variation, absolute continuity, or time derivative, from the original context of time dependent functions to the general context of time-dependent Young measures, we need to know the joint oscillations of \( u_k(t_1,x), \ldots, u_k(t_m,x) \) for every finite sequence \( t_1, \ldots, t_m \) of times. These are described by the Young measure \( \mu_{t_1 \ldots t_m} \), with state space \( \Xi^m \), generated by the sequence of \( \Xi^m \)-valued functions \((u_k(t_1,x), \ldots, u_k(t_m,x))\). It is easy to see that \( \mu_{t_1 \ldots t_m} \) cannot be derived from the measures \( \mu_{t_1}, \ldots, \mu_{t_m} \). Indeed, these measures give no information on the correlation between the oscillations at different times. The situation is similar to what happens in stochastic processes, where the knowledge of the distribution function of each single random variable is not enough to deduce their joint distribution.

This leads to the notion of system of Young measures, defined as a family \( (\mu_{t_1 \ldots t_m}) \), where \( t_1, \ldots, t_m \) run over all finite sequences of elements of \( I \), with \( t_1 < \cdots < t_m \), and each \( \mu_{t_1 \ldots t_m} \) is a Young measure on \( X \) with values in \( \Xi^m \). We assume that \((\mu_{t_1 \ldots t_m})\) satisfies the following compatibility condition, which is always satisfied when \( \mu_{t_1 \ldots t_m} \) is generated by a sequence of time-dependent functions: if \( \{s_1, \ldots, s_n\} \subset \{t_1, \ldots, t_m\} \) and \( s_1 < \cdots < s_n \), then \( \mu_{s_1 \ldots s_n} \) coincides with the corresponding projection of \( \mu_{t_1 \ldots t_m} \).

The notions of total variation (Definition 8.1), absolute continuity (Definition 10.1), or time derivative (Definition 9.4) can be easily defined in the framework of systems of Young measures in such a way that they coincide with the standard notions in the case of time-dependent functions. The main result of the paper is a version of Helly’s Theorem for systems of Young measures (Theorem 8.10): if \( (\mu^k_{t_1 \ldots t_m}) \) has uniformly bounded variation, then there exist a system \( (\mu_{t_1 \ldots t_m}) \) with bounded variation, a set \( \Theta \subset I \), with \( I \setminus \Theta \) at most countable, and a subsequence, still denoted \( (\mu^k_{t_1 \ldots t_m}) \), such that \( \mu^k_{t_1 \ldots t_m} \rightharpoonup \mu_{t_1 \ldots t_m} \) weakly* for every finite sequence \( t_1, \ldots, t_m \in \Theta \) with \( t_1 < \cdots < t_m \).

Another important result provides the existence of the time derivative \( \dot{\mu}_t \) almost every \( t \) whenever the family \( (\mu_{t_1 \ldots t_m}) \) has bounded variation (Theorem 9.7). The variation can be expressed by an integral involving the time derivatives when \( (\mu_{t_1 \ldots t_m}) \) is absolutely continuous (Theorem 10.4).

Our motivation for studying systems of Young measures stems from the analysis of quasistatic evolution problems with nonconvex energies, which arise in the study of plasticity with softening [10]. Since in these applications the energy functionals have linear growth in some directions, we have to consider the case where the generating sequence \( (u_k(t,x)) \) is bounded in \( L^r(X;\Xi) \) only for \( r = 1 \). It is well known that in this case Young measures should be replaced by more general objects, which take into account concentrations at infinity (see [12]). In [1] and [13] this is done by considering a pair \((\mu^Y, \mu^\infty)\), where \( \mu^Y \) is a Young measure on \( X \) with values in \( \Xi \) and \( \mu^\infty \), called the varifold measure, is a measure supported on \( X \times \Sigma_{\Xi} \), where \( \Sigma_{\Xi} \) denotes the unit sphere in \( \Xi \).

In the spirit of [12], we prefer to present these generalized Young measures in a different way, using homogeneous coordinates to describe the completion of \( \Xi \) obtained by adding a point at infinity for each direction. We replace the pair \((\mu^Y, \mu^\infty)\) by a single nonnegative measure \( \mu \) on \( X \times \Xi \times \mathbb{R} \) (Definition 3.9), acting only on continuous functions \( f(x,\xi,\eta) \) which are positively homogeneous of degree one in \( (\xi, \eta) \). We assume that \( \mu \) is supported on the set \( \{\eta \geq 0\} \) and that the projection of \( \eta \mu \) onto \( X \) coincides with \( \lambda \). We show that, if \( \lambda \) is nonatomic, then the space \( L^1(X;\Xi) \) can be identified (Definition 3.1) with a dense subset of the space of generalized Young measures (Theorem 5.1).
Using this approach, we are able to prove the results on total variation and time derivatives for systems of Young measures in a context that is general enough for the applications considered in [10].

2. A SPACE OF HOMOGENEOUS FUNCTIONS AND ITS DUAL

If $E$ is a locally compact space with a countable base and $\Xi$ is a finite dimensional Hilbert space, $\mathcal{M}_b(E;\Xi)$ denotes the space of bounded Radon measures on $E$ with values in $\Xi$, endowed with the norm $\|\nu\| := |\nu|(E)$, where $|\nu|$ denotes the variation of $\nu$. When $\Xi = \mathbb{R}$, the corresponding space will be denoted simply by $M_b(E)$. As usual, $M_b^+(E)$ denotes the cone of nonnegative bounded Radon measures on $E$. If $\nu \in M_b(E)$ and $f \in L^1_b(E;\Xi)$, the measure $f \nu \in M_b(E;\Xi)$ is defined by $(f \nu)(A) := \int_A f d\nu$ for every Borel set $A \subset E$.

By the Riesz Representation Theorem $M_b(E;\Xi)$ can be identified with the dual of $C_0(E;\Xi)$, the space of continuous functions $\varphi : E \to \Xi$ such that $\{|\varphi| \geq \varepsilon\}$ is compact for every $\varepsilon > 0$. The weak* topology of $M_b(E;\Xi)$ is defined using this duality.

Throughout the paper $(X,d)$ is a given compact metric space and $\lambda$ is a fixed nonnegative Radon measure on $X$ with $\text{supp} \lambda = X$. The symbol $\Xi$ will denote any finite dimensional Hilbert space. The spaces $L^r(X;\Xi)$, $r \geq 1$, will always refer to the measure $\lambda$. If $\mu \in \mathcal{M}_b(X;\Xi)$, $\mu^a$ and $\mu^s$ denote the absolutely continuous and the singular part of $\mu$ with respect to $\lambda$. Measures in $\mathcal{M}_b(X;\Xi)$ which are absolutely continuous with respect to $\lambda$ will always be identified with their densities, which belong to $L^1(X;\Xi)$. In this way $L^1(X;\Xi)$ is regarded as a subspace of $\mathcal{M}_b(X;\Xi)$.

In order to define the notion of generalized Young measure on $X$ with values in $\Xi$, it is convenient to introduce a space of homogeneous functions and to discuss some properties of its dual.

**Definition 2.1.** Let $C_\text{hom}(X \times \Xi)$ be the space of all $f : X \times \Xi \to \mathbb{R}$ such that $\xi \mapsto f(x, \xi)$ is positively homogeneous of degree one on $\Xi$ for every $x \in X$; i.e., $f(x,t\xi) = tf(x,\xi)$ for every $x \in X$, $\xi \in \Xi$, and $t \geq 0$. This space is endowed with the norm

$$\|f\|_{\text{hom}} := \max\{|f(x,\xi) : x \in X, \xi \in \Sigma_\Xi\},$$

where $\Sigma_\Xi := \{\xi \in \Xi : |\xi| = 1\}$.

We introduce now two dense subspaces of $C_\text{hom}(X \times \Xi)$ that will be useful in the proof of some properties of generalized Young measures.

**Definition 2.2.** Let $C_{L_1}^\text{hom}(X \times \Xi)$ be the space of all $f \in C_\text{hom}(X \times \Xi)$ which satisfies the following Lipschitz condition: there exists a constant $a \in \mathbb{R}$ such that

$$|f(x,\xi_1) - f(x,\xi_2)| \leq a |\xi_1 - \xi_2|$$

for every $x \in X$ and every $\xi_1,\xi_2 \in \Xi$.

**Remark 2.3.** If $f \in C_{L_1}^\text{hom}(X \times \Xi)$ and $\omega$ is the modulus of continuity of the restriction of $f$ to $X \times \Sigma_\Xi$, then (2.1) and the homogeneity of $f$ imply that

$$|f(x_1,\xi_1) - f(x_2,\xi_2)| \leq |f(x_1,\xi_1) - f(x_1,\xi_2)| + |f(x_1,\xi_2) - f(x_2,\xi_2)| \leq a |\xi_1 - \xi_2| + \omega(d(x_1, x_2)).$$

Exchanging the roles of $\xi_1$ and $\xi_2$ we obtain

$$|f(x_1,\xi_1) - f(x_2,\xi_2)| \leq a |\xi_1 - \xi_2| + \min\{|\xi_1|, |\xi_2|\} \omega(d(x_1, x_2))$$

for every $x_1, x_2 \in X$ and every $\xi_1,\xi_2 \in \Xi$.

**Lemma 2.4.** The space $C_{L_1}^\text{hom}(X \times \Xi)$ is dense in $C_\text{hom}(X \times \Xi)$.
Let us fix \( f \in C^\text{hom}(X \times \Xi) \). For every \( k > \|f\|_{\text{hom}} \) let us consider the Moreau-Yosida approximation \( f_k: X \times \Xi \to \mathbb{R} \) defined by
\[
 f_k(x, \xi) := \min_{\xi' \in \Xi} \{ f(x, \xi') + k \| \xi' - \xi \| \}.
\]
Using the standard properties of Moreau-Yosida approximations it is easy to check that \( f_k \in C^\text{hom}(X \times \Xi) \) (with constant \( k \)) and that the sequence \( f_k \) is nondecreasing and converges pointwise to \( f \) (see, e.g., [8, Remark 9.6 and Theorem 9.13]). By Dini’s Theorem we conclude that \( f_k \to f \) uniformly on \( X \times \Sigma \), hence \( f_k \to f \) in \( C^\text{hom}(X \times \Xi) \).

**Definition 2.5.** Let \( C^\text{hom}(X \times \Xi) \) be the space of all \( f \in C^\text{hom}(X \times \Xi) \) which satisfy the triangle inequality \( f(x, \xi_1 + \xi_2) \leq f(x, \xi_1) + f(x, \xi_2) \) for every \( x \in X \) and every \( \xi_1, \xi_2 \in \Xi \).

**Remark 2.6.** As \( |f(x, \xi)| \leq \|f\|_{\text{hom}} \), each \( f \in C^\text{hom}_\Delta(X \times \Xi) \) is Lipschitz continuous with respect to \( \xi \) and satisfies
\[
 |f(x, \xi_1) - f(x, \xi_2)| \leq \|\xi_1 - \xi_2\| \|f\|_{\text{hom}}
\]
for every \( x \in X \) and every \( \xi_1, \xi_2 \in \Xi \). Therefore \( C^\text{hom}_\Delta(X \times \Xi) \subset C^\text{hom}(X \times \Xi) \).

**Lemma 2.7.** The space of functions of the form \( f_1 - f_2 \), with \( f_1, f_2 \in C^\text{hom}_\Delta(X \times \Xi) \), is dense in \( C^\text{hom}(X \times \Xi) \).

**Proof.** Thanks to the obvious density in \( C^\text{hom}(X \times \Xi) \) of the space of \( f \in C^\text{hom}(X \times \Xi) \) such that \( f(x, \cdot) \) belongs to \( C^2(\Xi \setminus \{0\}) \) for every \( x \), it is enough to prove that every such function can be written as \( f = f_1 - f_2 \), with \( f_1, f_2 \in C^\text{hom}_\Delta(X \times \Xi) \). To this aim it suffices to show that there exists a constant \( c := c(f) \) such that \( f_2(x, \xi) := c \|\xi\| - f(x, \xi) \) is convex in \( \xi \) for every \( x \in X \). A simple calculation shows that the quadratic form corresponding to the Hessian matrix of \( f_2 \) with respect to \( \xi \) at a point \( (x, e) \), with \( e \in \Sigma \), is given by
\[
 D^2_{\xi} f_2(x, e) \xi \cdot \xi = c \|\xi\|^2 - c (\xi \cdot e)^2 - D^2_{\xi} f(x, e) \xi \cdot \xi.
\] (2.2)
By the Euler relation we have \( D_{\xi} f(x, \xi) \xi = f(x, \xi) \). Taking the derivative with respect to \( \xi \) we obtain \( D^2_{\xi} f(x, \xi) \xi = 0 \) for every \( \xi \), in particular \( D^2_{\xi} f(x, e) \) has an eigenvalue 0 with eigenvector \( e \). This implies that there is a constant \( b(x, e) \) such that \( D^2_{\xi} f(x, e) \xi \cdot \xi \leq b(x, e) \|\xi\|^2 \), where \( \xi_{\perp} := \xi - (\xi \cdot e) e \) is the component of \( \xi \) orthogonal to \( e \). As \( b(x, e) \) is bounded by the continuity of the second derivatives of \( f \), and \( \|\xi_{\perp}\|^2 = \|\xi\|^2 - (\xi \cdot e)^2 \), by (2.2) there exists a constant \( c \) such that \( D^2_{\xi} f_2(x, e) \) is positive definite for every \( x \in X \) and every \( e \in \Sigma \), hence \( f_2(x, \xi) \) is convex with respect to \( \xi \) for every \( x \in X \). \( \square \)

**Definition 2.8.** The dual of the space \( C^\text{hom}(X \times \Xi) \) is denoted by \( M_*(X \times \Xi) \), and the corresponding dual norm by \( \|\cdot\|_* \); the weak* topology of \( M_*(X \times \Xi) \) is defined by using this duality. It is sometimes convenient to write the dummy variables explicitly and to use the notation \( \langle f(x, \xi), \mu(x, \xi) \rangle \) for the duality product \( \langle f, \mu \rangle \). The positive cone \( M^+_*(X \times \Xi) \) is defined as the set of all \( \mu \in M_*(X \times \Xi) \) such that
\[
 \langle f, \mu \rangle \geq 0 \quad \text{for every } f \in C^\text{hom}(X \times \Xi) \text{ with } f \geq 0.
\]

**Remark 2.9.** It is easy to see that for every \( \mu \in M^+_*(X \times \Xi) \) we have
\[
 \|\mu\|_* = \langle |\xi|, \mu(x, \xi) \rangle.
\]
Strictly speaking, the elements \( \mu \) of \( M_*(X \times \Xi) \) are not measures, because they act only on homogeneous functions. However, the notion of image of \( \mu \) under a map \( \psi \) can be defined by duality, as in measure theory.

**Definition 2.10.** Let \( \Xi \) and \( \Xi' \) be two finite dimensional Hilbert spaces and let \( \psi: X \times \Xi \to X \times \Xi' \) be a continuous map of the form \( \psi(x, \xi) = (x, \varphi(x, \xi)) \), with \( \varphi: X \times \Xi \to \Xi' \) positively...
one-homogeneous in $\xi$. The image $\psi(\mu)$ of $\mu \in M_*(X \times \Xi')$ under $\psi$ is defined as the element of $M_*(X \times \Xi')$ such that

$$ \langle f, \psi(\mu) \rangle = \langle f \circ \psi, \mu \rangle = \langle f(x, \varphi(x, \xi)), \mu(x, \xi) \rangle $$

for every $f \in C^{\text{hom}}(X \times \Xi')$.

Similarly we can define the notion of support of $\mu \in M_*(X \times \Xi)$. We say that a subset $C$ of $X \times \Xi$ is a $\Xi$-cone if $(x, \xi) \in C \Rightarrow (x, t\xi) \in C$ for every $t \geq 0$.

**Definition 2.11.** The support $\text{supp} \mu$ of $\mu \in M_*(X \times \Xi)$ is defined as the smallest closed $\Xi$-cone $C \subset X \times \Xi$ such that $\langle f, \mu \rangle = 0$ for every $f \in C^{\text{hom}}(X \times \Xi)$ vanishing on $C$.

**Remark 2.12.** For every $\mu \in M_*(X \times \Xi)$ there exists a measure $\check{\mu} \in M_b(X \times \Xi)$ with compact support such that

$$ \langle f, \mu \rangle = \int_{X \times \Xi} f \, d\check{\mu} \tag{2.3} $$

for every $f \in C^{\text{hom}}(X \times \Xi)$. A measure with this property can be constructed by considering the continuous linear map on $C(X \times \Xi)$ defined by $g \mapsto \langle |\xi|g(x, \xi/|\xi|), \mu(x, \xi) \rangle$. By the Riesz Representation Theorem there exists $\check{\mu} \in M_b(X \times \Xi)$ such that

$$ \langle |\xi|g(x, \xi/|\xi|), \mu(x, \xi) \rangle = \int_{X \times \Xi} g \, d\check{\mu} $$

for every $g \in C(X \times \Xi)$. Regarding $\check{\mu}$ as a measure on $X \times \Xi$ supported by $X \times \Xi$, we obtain (2.3). This construction suggests that the measure $\check{\mu}$ satisfying (2.3) is not unique; indeed, we can repeat the same construction with $\Sigma_\Xi$ replaced by any other concentric sphere.

For the applications it is convenient to extend some of the previous results to a suitable space of possibly discontinuous functions.

**Definition 2.13.** Given two finite dimensional Hilbert spaces $\Xi$ and $\Xi'$, let $B^{\text{hom}}_{\infty}(X \times \Xi; \Xi')$ be the space of Borel functions $f : X \times \Xi \to \Xi'$ such that

(a) for every $x \in X$ the function $\xi \mapsto f(x, \xi)$ is positively homogeneous of degree one on $\Xi$,

(b) there exists a constant $a \in \mathbb{R}$ such that $|f(x, \xi)| \leq a|\xi|$ for every $(x, \xi) \in X \times \Xi$.

The smallest constant $a$ satisfying the previous inequality is denoted by $\|f\|_{\text{hom}}$. When $\Xi' = \mathbb{R}$, the corresponding space will be denoted simply by $B^{\text{hom}}_{\infty}(X \times \Xi)$.

**Definition 2.14.** For every $f \in B^{\text{hom}}_{\infty}(X \times \Xi)$ and every $\mu \in M_*(X \times \Xi)$ the duality product $\langle f, \mu \rangle$ is defined by

$$ \langle f, \mu \rangle := \int_{X \times \Xi} f \, d\hat{\mu}, $$

where $\hat{\mu}$ is any measure satisfying the conditions of Remark 2.12. By homogeneity the value of $\langle f, \mu \rangle$ does not depend on the particular measure $\hat{\mu}$ chosen in (2.3). The same definition (with values in $\mathbb{R} \cup \{+\infty\}$ this time) is adopted if $\mu \in M^*_c(X \times \Xi)$ and $f : X \times \Xi \to \mathbb{R} \cup \{+\infty\}$ is a Borel function such that $f(x, \xi)$ is positively homogeneous of degree one in $\xi$ and $f(x, \xi) \geq -c|\xi|$ for some constant $c \geq 0$.

Let $\pi_X : X \times \Xi \to X$ be the projection onto $X$. We now define the image under $\pi_X$ of the product $h \mu$ of an element $\mu$ of $M_*(X \times \Xi)$ by a homogeneous function $h$.

**Definition 2.15.** Let $\Xi$ and $\Xi'$ be two finite dimensional Hilbert spaces, let $\mu \in M_*(X \times \Xi)$, and let $h \in B^{\text{hom}}_{\infty}(X \times \Xi; \Xi')$. The measure $\pi_X(h\mu)$ is the element of $M_b(X; \Xi')$ such that

$$ \int_X \varphi \cdot d\pi_X(h\mu) = \langle \varphi(x) \cdot h(x, \xi), \mu(x, \xi) \rangle $$

for every $\varphi \in C(X; \Xi')$, where the dot denotes the scalar product in $\Xi'$.
Remark 2.16. For every $\hat{\mu} \in M_b(X \times \Xi)$ satisfying (2.3) we can consider the Radon measure $h\hat{\mu} \in M_b(X \times \Xi; \Xi')$ having density $h$ with respect to $\hat{\mu}$. It is easy to check that the measure $\pi_X(h\hat{\mu})$ defined by (2.4) coincides with the image under $\pi_X$ of the measure $h\hat{\mu}$. Note that the measure $h\hat{\mu}$ depends on the choice of $\hat{\mu}$ satisfying (2.3), while, by (2.4), its projection $\pi_X(h\hat{\mu})$ does not.

Remark 2.17. It follows from the definition that we have the estimate
\[ \|\pi_X(h\mu)\| \leq \|h\|_{\text{hom}} \|\mu\|, \]
for every $\mu \in M_*(X \times \Xi)$ and every $h \in B_{\text{hom}}(X \times \Xi; \Xi')$.

3. Generalized Young measures

As mentioned in the introduction, the notion of generalized Young measure is used to describe oscillation and concentration phenomena for sequences which are bounded in $L^r(X; \Xi)$ only for $r = 1$. To study concentration phenomena, where the sequences tend to infinity along given directions in the space $\Xi$, it is useful to introduce homogeneous coordinates. This is done by replacing the space $\Xi$ by $\Xi \times \mathbb{R}$, whose generic point is denoted by $(\xi, \eta)$; the set of points with $\eta = 1$ is identified with $\Xi$, while points with $\eta = 0$ are interpreted as directions at infinity.

In our presentation the space of generalized Young measures will be a subset of the space $M^+_b(X \times \Xi \times \mathbb{R})$, where $\Xi \times \mathbb{R}$ plays the role of the Hilbert space $\Xi$ of the previous section. Before describing this set, we first consider generalized Young measures associated with functions.

Definition 3.1. Given $u \in L^1(X; \Xi)$, the **generalized Young measure associated with** $u$ is defined as the element $\delta_u$ of $M^+_b(X \times \Xi \times \mathbb{R})$ such that
\[ \langle f, \delta_u \rangle = \int_X f(x, u(x), 1) \, d\lambda(x) \]
for every $f \in C^\text{hom}(X \times \Xi \times \mathbb{R})$.

In the spirit of [15] and [22] we extend this definition to measures $p \in M_b(X; \Xi)$.

Definition 3.2. Given $p \in M_b(X; \Xi)$, the **generalized Young measure associated with** $p$ is defined as the element $\delta_p$ of $M^+_b(X \times \Xi \times \mathbb{R})$ such that for every $f \in C^\text{hom}(X \times \Xi \times \mathbb{R})$
\[ \langle f, \delta_p \rangle = \int_X f(x, \frac{dp}{d\sigma}(x), \frac{d\lambda}{d\sigma}(x)) \, d\sigma(x), \]
where $\sigma$ is an arbitrary nonnegative Radon measure on $X$ with $\lambda << \sigma$ and $p << \sigma$.

The homogeneity of $f$ implies that the integral does not depend on $\sigma$ and that the definitions coincide when $p = u \in L^1(X; \Xi)$. The norm of $\delta_p$ is given by the following lemma.

Lemma 3.3. Let $p \in M_b(X; \Xi)$. Then
\[ \|\delta_p\|_* = \int_X \sqrt{1 + |p^\ast|^2} \, d\lambda + |p^\ast|(X) \leq \lambda(X) + |p|(X). \]

Proof. Let us consider the Borel partition $X = X^a \cup X^s$ with $\lambda(X^s) = 0 = |p^\ast|(X^s)$ and let $\sigma := \lambda + |p^\ast|$, so that $\sigma = \lambda$ on $X^a$ and $\sigma = |p^\ast|$ on $X^s$. By Remark 2.9 we have
\[ \|\delta_p\|_* = \langle \sqrt{|\xi|^2 + |\eta|^2}, \delta_p(x, \xi, \eta) \rangle = \int_X \sqrt{\left|\frac{dp}{d\sigma}\right|^2 + \left|\frac{d\lambda}{d\sigma}\right|^2} \, d\sigma = \]
\[ = \int_{X^s} \sqrt{1 + |p^\ast|^2} \, d\lambda + |p^\ast|(X^s) = \int_X \sqrt{1 + |p^\ast|^2} \, d\lambda + |p^\ast|(X) , \]
which concludes the proof. \[\square\]
We recall the definition of Young measure.

**Definition 3.4.** A Young measure on $X$ with values in $\Xi$ is a measure $\nu \in M^+_b(X \times \Xi)$ such that $\pi_X(\nu) = \lambda$. The space of Young measures on $X$ with values in $\Xi$ is denoted by $Y(X; \Xi)$. For every $r \geq 1$ let $Y^r(X; \Xi)$ be the space of all $\nu \in Y(X; \Xi)$ whose $r$-moment

$$\int_{X \times \Xi} |\xi|^r \, d\nu(x, \xi)$$

is finite.

**Remark 3.5.** By the Disintegration Theorem (see, e.g., [23, Appendix A2]) for every $\nu \in Y(X; \Xi)$ there exists a measurable family $(\nu^x)_{x \in X}$ of probability measures on $\Xi$ such that

$$\int_{X \times \Xi} g(x, \xi) \, d\nu(x, \xi) = \int_X \left( \int_{\Xi} g(x, \xi) \, d\nu^x(\xi) \right) \, d\lambda(x)$$

for every bounded Borel function $g: X \times \Xi \to \mathbb{R}$. The probability measures $\nu^x$ are uniquely determined for $\lambda$-a.e. $x \in X$.

**Definition 3.6.** Given $\nu \in Y^1(X; \Xi)$, the generalized Young measure associated with $\nu$ is defined as the element $\varphi$ of $M^+_b(X \times \Xi \times \Xi)$ such that

$$\langle f, \varphi \rangle := \int_{X \times \Xi} f(x, \xi, 1) \, d\nu(x, \xi)$$

for every $f \in C^{\text{hom}}(X \times \Xi \times \Xi)$.

**Remark 3.7.** It follows from Remark 2.9 that

$$\|\varphi\|_* = \int_{X \times \Xi} \sqrt{1 + |\xi|^2} \, d\nu(x, \xi) \leq \lambda(X) + \int_{X \times \Xi} |\xi| \, d\nu(x, \xi).$$

**Remark 3.8.** If $\mu = \delta_p$ for some $p \in M_b(X; \Xi)$, the following properties hold:

$$\text{supp } \mu \subset X \times \Xi \times [0, +\infty), \quad \pi_X(\eta \mu) = \lambda.$$ (3.1)

We will refer to (3.2) as the projection property. According to (2.4), it is equivalent to

$$\langle \varphi(x) \eta, \mu(x, \xi, \eta) \rangle = \int_X \varphi \, d\lambda \quad \text{for every } \varphi \in C(X). \quad (3.3)$$

Properties (3.1) and (3.2) continue to hold if $\mu = \varphi$ for some $\nu \in Y^1(X; \Xi)$.

This motivates the following definition.

**Definition 3.9.** The space $GY(X; \Xi)$ of generalized Young measures on $X$ with values in $\Xi$ is defined as the set of all $\mu \in M^+_b(X \times \Xi \times \Xi)$ satisfying (3.1) and (3.2). On $GY(X; \Xi)$ we consider the norm and the weak* topology induced by $M_b(X \times \Xi \times \Xi)$.

**Remark 3.10.** By approximation we can prove that (3.3) holds for every $\mu \in GY(X; \Xi)$ and for every bounded Borel function $\varphi: X \to \mathbb{R}$.

The sequential compactness of every bounded subset of $GY(X; \Xi)$ is given by the following theorem.

**Theorem 3.11.** Every bounded sequence in $GY(X; \Xi)$ has a subsequence which converges weakly* to an element of $GY(X; \Xi)$.

**Proof.** Since $GY(X; \Xi)$ is closed in the weak* topology of $M_b(X \times \Xi \times \Xi)$, the result follows from the Banach-Alaoglu Theorem.

**Remark 3.12.** If $\mu_k$ is a sequence in $GY(X; \Xi)$ which converges weakly* to $\mu \in GY(X; \Xi)$, then $\|\mu_k\|_* \to \|\mu\|_*$ by Remark 2.9.
Remark 3.13. If \( f : X \times \Xi \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\} \) is lower semicontinuous, \( (\xi, \eta) \mapsto f(x, \xi, \eta) \) is positively homogeneous of degree one, and \( f(x, \xi, \eta) \geq -c\sqrt{|\xi|^2 + |\eta|^2} \) for some constant \( c \geq 0 \), then
\[
\langle f, \mu \rangle \leq \liminf_{k \to \infty} \langle f, \mu_k \rangle
\]
for every sequence \( \mu_k \) in \( G(Y; \Xi) \) which converges weakly* to \( \mu \in G(Y; \Xi) \), where \( \langle f, \cdot \rangle \) is defined by \( (2.3) \). Indeed, any such \( f \) is the supremum of a family of functions in \( C^{\hom}(X \times \Xi \times \mathbb{R}) \).

In the case of a generalized Young measure \( \mu \in G(Y; \Xi) \) the duality product \( \langle f, \mu \rangle \) can be defined for every \( f \) in the space \( B^{\hom}_{\infty,1}(X \times \Xi \times \mathbb{R}) \) introduced by the following definition, which is slightly larger than the space \( B^{\hom}_{\infty}(X \times \Xi \times \mathbb{R}) \) considered in the previous section.

Definition 3.14. Given two finite dimensional Hilbert spaces \( \Xi \) and \( \Xi' \), we consider the space \( B^{\hom}_{\infty,1}(X \times \Xi \times \mathbb{R}; \Xi') \) of all Borel functions \( f : X \times \Xi \times \mathbb{R} \to \Xi' \) such that
\[
\text{(a) for every } x \in X \text{ the function } (\xi, \eta) \mapsto f(x, \xi, \eta) \text{ is positively homogeneous of degree one on } \Xi \times \mathbb{R},
\]
\[
\text{(b) there exist a constant } a \in \mathbb{R} \text{ and a function } b \in L^1(X) \text{ such that }
\]
\[
|f(x, \xi, \eta)| \leq a|\xi| + b|\eta|
\]

(3.4)

for every \( (x, \xi, \eta) \in X \times \Xi \times \mathbb{R} \).

When \( \Xi' = \mathbb{R} \), the corresponding space will be denoted simply by \( B^{\hom}_{\infty,1}(X \times \Xi \times \mathbb{R}) \).

Lemma 3.15. Let \( \mu \in G(Y; \Xi) \) and let \( \tilde{\mu} \in M_b(X \times \Xi \times \mathbb{R}) \) be a measure with compact support satisfying \( (2.3) \). Then every \( f \in B^{\hom}_{\infty,1}(X \times \Xi \times \mathbb{R}) \) is \( \tilde{\mu} \)-integrable.

Proof. Let us fix \( f \in B^{\hom}_{\infty,1}(X \times \Xi \times \mathbb{R}) \). For every \( k \) let \( f_k \) be defined by \( f_k(x, \xi, \eta) := f(x, \xi, \eta) \) if \( |f(x, \xi, \eta)| \leq a|\xi| + k|\eta| \), and by \( f_k(x, \xi, \eta) := 0 \) otherwise. Then we have
\[
\int_{X \times \Xi \times \mathbb{R}} |f_k| d\tilde{\mu} \leq \int_{X \times \Xi \times \mathbb{R}} [a|\xi| + (b(x) \wedge k)|\eta|] d\tilde{\mu} = \langle a|\xi| + (b(x) \wedge k)|\eta|, \mu(x, \xi, \eta) \rangle \leq \langle a|\xi|, \mu(x, \xi, \eta) \rangle + \int_X b d\lambda.
\]

It follows from Fatou's Lemma that \( f \) is \( \tilde{\mu} \)-integrable. \( \square \)

Definition 3.16. Given \( f \in B^{\hom}_{\infty,1}(X \times \Xi \times \mathbb{R}) \) and \( \mu \in G(Y; \Xi) \), the duality product \( \langle f, \mu \rangle \) is defined by
\[
\langle f, \mu \rangle := \int_{X \times \Xi \times \mathbb{R}} f d\tilde{\mu},
\]

(3.5)

where \( \tilde{\mu} \in M_b(X \times \Xi \times \mathbb{R}) \) is any measure with compact support satisfying \( (2.3) \), with \( \Xi \) replaced by \( \Xi \times \mathbb{R} \).

Remark 3.17. The integral in \( (3.5) \) is well defined by Lemma 3.15. It is easy to see that the value of this integral does not depend on the choice of \( \tilde{\mu} \) satisfying \( (2.3) \), and that
\[
|\langle f, \mu \rangle| \leq a \|\mu\|_* + \|b\|_1,
\]

where \( a \) and \( b \) satisfy \( (3.4) \) and \( \|b\|_1 \) denotes the \( L^1 \) norm of \( b \).

We now consider the image of a generalized Young measure.

Definition 3.18. Let \( \Xi \) and \( \Xi' \) be two finite dimensional Hilbert spaces and let \( \psi : X \times \Xi \times \mathbb{R} \to X \times \Xi' \times \mathbb{R} \) be a map of the form \( \psi(x, \xi, \eta) = (x, \varphi(x, \xi, \eta), \eta) \), with \( \varphi \in B^{\hom}_{\infty,1}(X \times \Xi \times \mathbb{R}; \Xi') \). The image \( \psi(\mu) \) of \( \mu \in G(Y; \Xi) \) under \( \psi \) is defined as the element of \( G(Y; \Xi') \) such that
\[
\langle f, \psi(\mu) \rangle = \langle f \circ \psi, \mu \rangle = \langle f(x, \varphi(x, \xi, \eta), \eta), \mu(x, \xi, \eta) \rangle
\]

(3.6)

for every \( f \in B^{\hom}_{\infty,1}(X \times \Xi \times \mathbb{R}) \).
Remark 3.19. Under the assumptions of the definition the function \( f \circ \psi \) belongs to \( B_{\text{hom}}^{\text{Y}}(X \times \Xi \times \mathbb{R}) \), so that the duality product \( \langle f \circ \psi, \mu \rangle \) is well defined. Moreover, by the particular form of the map \( \psi \), the element of \( M^+_b(X \times \Sigma \times \mathbb{R}) \) defined by (3.6) satisfies (3.1) and (3.2), therefore it belongs to \( GY(X; \Xi') \).

4. Comparison with other presentations of the theory

In this section we show that every \( \mu \in GY(X; \Xi) \) can be represented by a unique Young measure-varifold pair \((\mu^Y, \mu^\infty)\), where \( \mu^Y \in Y^1(X; \Xi) \) and \( \mu^\infty \in M^+_b(X \times \Sigma) \). To introduce this representation, we recall that \( Y^1(X; \Xi) \) can be identified with a suitable subset of \( GY(X; \Xi) \) (Definition 3.6). The following definition identifies the measures in \( M^+_b(X \times \Sigma) \) with particular elements of \( M_b(X \times \Sigma) \).

Definition 4.1. For every \( \nu \in M^+_b(X \times \Sigma) \) let \( \tilde{\nu} \) be the element of \( M_b(X \times \Sigma) \) defined by
\[
\langle f, \tilde{\nu} \rangle = \int_{X \times \Sigma} f(x, \xi, 0) \, d\nu(x, \xi)
\]
for every \( f \in C^{\text{hom}}(X \times \Xi \times \mathbb{R}) \).

Remark 4.2. It follows from the definition that \( \pi_X(\eta \tilde{\nu}) = 0 \) for every \( \nu \in M^+_b(X \times \Sigma) \). Since \( \tilde{\nu} \) does not satisfy the projection property (3.2), it does not belong to \( GY(X; \Xi) \).

The main result of this section is the following theorem.

Theorem 4.3. Let \( \mu \in GY(X; \Xi) \). Then there exists a unique pair \((\mu^Y, \mu^\infty)\), with \( \mu^Y \in Y^1(X; \Xi) \) and \( \mu^\infty \in M^+_b(X \times \Sigma) \), such that
\[
\mu = \tilde{\mu}^Y + \hat{\mu}^\infty,
\]
which is equivalent to
\[
\langle f, \mu \rangle = \int_{X \times \Xi} f(x, \xi, 1) \, d\mu^Y(x, \xi) + \int_{X \times \Sigma} f(x, \xi, 0) \, d\mu^\infty(x, \xi)
\]
for every \( f \in B_{\text{hom}}^{\text{Y}}(X \times \Xi \times \mathbb{R}) \).

Remark 4.4. The converse of Theorem 4.3 is also true: if \( \mu^Y \in Y^1(X; \Xi) \) and \( \mu^\infty \in M_b(X \times \Sigma) \), then formula (4.1) defines an element of \( GY(X; \Xi) \).

Remark 4.5. Let \( \lambda^\infty := \pi_X(\mu^\infty) \). Since \( \lambda = \pi_X(\mu^Y) \), by the Disintegration Theorem (see, e.g., [23, Appendix A2]) there exist a measurable family \((\mu^{x,Y})_{x \in X}\) of probability measures on \( \Xi \) and a measurable family \((\mu^{x,\infty})_{x \in X}\) of probability measures on \( \Sigma \) such that
\[
\langle f, \mu \rangle = \int_{X \times \Xi} f(x, \xi, 1) \, d\mu^Y(x, \xi) + \int_{X \times \Sigma} f(x, \xi, 0) \, d\mu^\infty(x, \xi) = \\
= \int_X \left( \int_{\Xi} f(x, \xi, 1) \, d\mu^{x,Y}(\xi) \right) \, d\lambda(x) + \int_X \left( \int_{\Sigma} f(x, \xi, 0) \, d\mu^{x,\infty}(\xi) \right) \, d\lambda^\infty(x)
\]
for every \( f \in B_{\text{hom}}^{\text{Y}}(X \times \Xi \times \mathbb{R}) \).

Remark 4.6. Thanks to Remark 2.9, if we apply (4.1) to \( f(x, \xi, \eta) = \sqrt{|\xi|^2 + |\eta|^2} \), we obtain
\[
\|\mu\|_* = \int_{X \times \Xi} \sqrt{1+|\xi|^2} \, d\mu^Y(x, \xi) + \mu^\infty(X \times \Sigma) \leq \\
\leq \lambda(X) + \int_{X \times \Xi} |\xi| \, d\mu^Y(x, \xi) + \mu^\infty(X \times \Sigma).
\]
Proof of Theorem 4.3. For every Borel function \( g : X \times \Xi \to \mathbb{R} \) with
\[
\kappa_g := \sup_{(x, \xi) \in X \times \Xi} \frac{|g(x, \xi)|}{\sqrt{1 + |\xi|^2}} < +\infty ,
\]
we consider the Borel function \( \varphi_g : X \times \Xi \times \mathbb{R} \to \mathbb{R} \) defined by
\[
\varphi_g(x, \xi, \eta) := \begin{cases} 
\eta g(x, \xi/\eta) & \text{if } \eta > 0 , \\
0 & \text{if } \eta \leq 0 .
\end{cases}
\]
Since \( \varphi_g \in B_{\text{hom}}^0(X \times \Xi \times \mathbb{R}) \), we can consider the duality product \( \langle \varphi_g, \mu \rangle \). The function \( g \mapsto \langle \varphi_g, \mu \rangle \) is linear, bounded, and positive on \( C_0(X \times \Xi) \). By the Riesz Representation Theorem there exists \( \mu^Y \in M^+_b(X \times \Xi) \) such that
\[
\langle \varphi_g, \mu \rangle = \int_{X \times \Xi} g(x, \xi) \, d\mu^Y(x, \xi)
\]
for every \( g \in C_0(X \times \Xi) \). As \( \|g\|_{\text{hom}} \leq \kappa_g \), we have
\[
|\int_{X \times \Xi} g(x, \xi) \, d\mu^Y(x, \xi)| \leq \kappa_g \|\mu\|_*
\]
for every \( g \in C_0(X \times \Xi) \). By approximation we can prove that
\[
\int_{X \times \Xi} |\xi| \, d\mu^Y(x, \xi) \leq \|\mu\|_*,
\]
and that (4.3) holds for every Borel function \( g : X \times \Xi \to \mathbb{R} \) satisfying (4.2). By (3.2) we have \( \pi_X(\mu^Y) = \lambda \), which, together with (4.4), gives \( \mu^Y \in Y^1(X; \Xi) \).

For every bounded Borel function \( h : X \times \Sigma \Xi \to \mathbb{R} \) we consider the Borel function \( \psi_h : X \times \Sigma \Xi \times \mathbb{R} \to \mathbb{R} \) defined by
\[
\psi_h(x, \xi, \eta) := \begin{cases} 
|\xi| h(x, \xi/|\xi|) \quad & \text{if } \eta = 0, \xi \neq 0 , \\
0 & \text{otherwise}.
\end{cases}
\]
Since \( \psi_h \in B_{\text{hom}}^0(X \times \Sigma \Xi \times \mathbb{R}) \), we can consider the duality product \( \langle \psi_h, \mu \rangle \). The function \( h \mapsto \langle \psi_h, \mu \rangle \) is linear, bounded, and positive on \( C(X \times \Sigma \Xi) \). By the Riesz Representation Theorem there exists \( \mu^\infty \in M^+_b(X \times \Sigma \Xi) \) such that
\[
\langle \psi_h, \mu \rangle = \int_{X \times \Sigma \Xi} h \, d\mu^\infty
\]
for every \( h \in C(X \times \Sigma \Xi) \). By approximation we can prove that the previous equality holds for every bounded Borel function \( h : X \times \Sigma \Xi \to \mathbb{R} \).

Given any \( f \in B_{\text{hom}}^0(X \times \Xi \times \mathbb{R}) \), we consider the functions \( g : X \times \Xi \to \mathbb{R} \) and \( h : X \times \Sigma \Xi \to \mathbb{R} \) defined by
\[
g(x, \xi) := f(x, \xi, 1) , \quad h(x, \xi) := f(x, \xi, 0).
\]
By homogeneity we have \( f = \varphi_g + \psi_h \) on \( X \times \Xi \times [0, +\infty) \). Then (4.1) follows from (3.1), (4.3), and (4.5). The result can be extended to \( f \in B_{\text{hom}}^0(X \times \Xi \times \mathbb{R}) \) by approximation.

The uniqueness of the pair \( (\mu^Y, \mu^\infty) \) can be deduced from the fact that, if (4.1) is satisfied, then (4.3) holds for every \( g \in C_0(X \times \Xi) \), while (4.5) holds for every \( h \in C(X \times \Sigma \Xi) \). \( \square \)

Remark 4.7. It is easy to prove, by approximation, that (4.1) continues to hold when \( f : X \times \Xi \times [0, +\infty) \to \mathbb{R} \) is a Borel function such that \( f(x, \xi, \eta) \) is positively homogeneous of degree one in \( (\xi, \eta) \) and satisfies the inequality \( f(x, \xi, \eta) \geq -c\sqrt{|\xi|^2 + |\eta|^2} \) for some constant \( c \geq 0 \). This allows to characterize the generalized Young measures associated with Young
measures with finite $r$-moment, $r > 1$, using the homogeneous functions $\{P_r\}: \Xi \times \mathbb{R} \to [0, +\infty]$ defined by

$$\{P_r\}(\xi, \eta) := \begin{cases} \|\xi\|^r/\eta^{r-1} & \text{if } \eta > 0, \\ +\infty & \text{if } \eta \leq 0. \end{cases} \quad (4.6)$$

Indeed, for $r > 1$ formula (4.1) implies that $\mu = \overline{\mu}^Y$ with $\mu^Y \in Y^r(X; \Xi)$ if and only if $\langle \{P_r\}, \mu \rangle < +\infty$.

Let $\psi^\Xi_0: X \times \Xi \times \mathbb{R} \to X \times \Xi \times \mathbb{R}$ be the Borel map defined by

$$\psi^\Xi_0(x, \xi, \eta) = \begin{cases} (x, \xi, \eta) & \text{if } \eta \neq 0, \\ (x, 0, 0) & \text{if } \eta = 0. \end{cases}$$

Note that $\psi^\Xi_0$ satisfies the conditions of Definition 3.18.

For every $f \in B_{\text{hom}}(X \times \Xi \times \mathbb{R})$ we have

$$(f \circ \psi^\Xi_0)(x, \xi, \eta) = \begin{cases} f(x, \xi, \eta) & \text{if } \eta \neq 0, \\ 0 & \text{if } \eta = 0. \end{cases}$$

From (4.1) it follows that for every $\mu \in GY(X; \Xi)$

$$\langle f, \overline{\mu}^Y \rangle = \langle f \circ \psi^\Xi_0, \overline{\mu}^Y \rangle = \langle f \circ \psi^\Xi_0, \mu \rangle,$$

hence $\overline{\mu}^Y = \psi^\Xi_0(\mu)$.

**Lemma 4.8.** Let $\Xi$ and $\Xi'$ be two finite dimensional Hilbert spaces, let $\mu \in GY(X; \Xi)$, let $\psi: X \times \Xi \times \mathbb{R} \to X \times \Xi' \times \mathbb{R}$ be a map as in Definition 3.18, and let $\nu := \psi(\mu)$. Then

$$\overline{\nu}^Y = \psi(\overline{\mu}^Y), \quad \nu^\infty = \psi(\mu^\infty).$$

**Proof.** The former equality follows from the fact that $\psi \circ \psi^\Xi_0 = \psi^\Xi' \circ \psi$. The latter follows now from Theorem 4.3 by the linearity of the map $\mu \mapsto \psi(\mu)$. \qed

Combining the compactness property (Theorem 3.11) and the representation formula (Theorem 4.3) we recover the following result, originally proved in [1] (see Remark 4.5).

**Theorem 4.9.** Let $u_k$ be a bounded sequence in $L^1(X; \Xi)$. Then there exist a subsequence, still denoted $u_k$, a Young measure $\mu^Y \in Y^1(X; \Xi)$, and a measure $\mu^\infty \in M_b(X \times \Sigma_\Xi)$, such that

$$\int_X g(x, u_k(x)) \, d\lambda(x) \to \int_{X \times \Xi} g(x, \xi) \, d\mu^Y(x, \xi) + \int_{X \times \Sigma_\Xi} g^\infty(x, \xi) \, d\mu^\infty(x, \xi) \quad (4.7)$$

for every continuous function $g: X \times \Xi \to \mathbb{R}$ such that for every $(x_0, \xi_0) \in X \times \Xi$ the limit

$$g^\infty(x_0, \xi_0) := \lim_{\eta \to 0^+} \eta g(x_0, \xi/\eta)$$

exists and is finite.

**Proof.** Let us consider the sequence $\delta_{u_k}$ in $GY(X; \Xi)$ introduced in Definition 3.1. By Lemma 3.3 we have $\|\delta_{u_k}\|_* \leq \sup_j \|u_j\|_1 + \lambda(X) < +\infty$. By Theorem 3.11 there exists a subsequence, still denoted $u_k$, such that $\delta_{u_k}$ converge weakly* to an element $\mu$ of $GY(X; \Xi)$. Let $g$ be as in the statement of the theorem and let $f: X \times \Xi \times \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x, \xi, \eta) := \begin{cases} \eta g(x, \xi/\eta) & \text{if } \eta > 0, \\ g^\infty(x, \xi) & \text{if } \eta \leq 0. \end{cases}$$

It is easy to check that $f$ is continuous in $(x, \xi, \eta)$ and homogeneous of degree one in $(\xi, \eta)$.

Therefore, the weak* convergence of $\delta_{u_k}$ to $\mu$ implies that

$$\int_X g(x, u_k(x)) \, d\lambda(x) = \int_X f(x, u_k(x), 1) \, d\lambda(x) \to \langle f, \mu \rangle. \quad (4.8)$$
By Theorem 4.3, taking into account the definition of $f$, we obtain that there exists a pair $(\mu^Y, \mu^\infty)$, with $\mu^Y \in Y^1(X; \Xi)$ and $\mu^\infty \in M^+_B(X \times \Sigma_\Xi)$, such that

$$\langle f, \mu \rangle = \int_{X \times \Xi} g(x, \xi) \, d\mu^Y(x, \xi) + \int_{X \times \Sigma_\Xi} g^\infty(x, \xi) \, d\mu^\infty(x, \xi).$$

(4.9)

The conclusion follows from (4.8) and (4.9).

5. A DENSITY RESULT

In this section we prove that, if $\lambda$ is nonatomic, then the generalized Young measures of the form $\delta_u$ associated with functions $u \in L^1(X; \Xi)$ are dense in $GY(X; \Xi)$. The main result is the following approximation theorem.

**Theorem 5.1.** Assume that $\lambda$ is nonatomic and let $\mu \in GY(X; \Xi)$. Then there exists a sequence $u_n$ in $L^1(X; \Xi)$ such that $\delta_{u_n} \rightharpoonup \mu$ weakly* in $GY(X; \Xi)$.

**Proof.** We consider the decomposition

$$\mu = \bar{\mu}^Y + \bar{\mu}^\infty$$

of Theorem 4.3 and we fix a sequence $\sigma_n$ converging to 0 with $0 < \sigma_n < \min\{1, \lambda(X)\}$. For every $n$ we consider two countable partitions $\Xi = \bigcup_j B_j^n, Y$ and $\Sigma_\Xi = \bigcup_j B_j^n, \infty$, where the sets $B_j^n, Y$ and $B_j^n, \infty$ satisfy

$$\text{diam } B_j^n, Y \leq \sigma_n \quad \text{and} \quad \text{diam } B_j^n, \infty \leq \sigma_n.$$  \hspace{1cm} (5.1)

Let $\lambda^Y := \pi_X(\sqrt{1 + \|\xi\|^2} \mu^Y)$ and $\lambda^\infty := \pi_X(\mu^\infty) \colon i.e.,$

$$\lambda^Y(A) = \int_{A \times \Xi} \sqrt{1 + \|\xi\|^2} \, d\mu^Y(x, \xi) \quad \text{and} \quad \lambda^\infty(A) = \mu^\infty(A \times \Sigma_\Xi)$$

for every Borel set $A \subset X$. As $\lambda = \pi_X(\mu^Y)$, the measure $\lambda^Y$ is absolutely continuous with respect to $\lambda$.

Let us consider the partition $X = X^a \cup X^s$, with $\lambda(X^s) = 0 = \lambda^\infty_s(X^a)$, where $\lambda^\infty_s$ is the singular part of $\lambda^\infty$ with respect to $\lambda$. To prove the theorem, for every $n$ we will consider a new partition $X = X^{n,a} \cup X^{n,s}$, where $X^{n,a}$ and $X^{n,s}$ are suitable approximations of $X^a$ and $X^s$ such that $\lambda(X^{n,a}) > 0$ and $\lambda(X^{n,s}) > 0$. We will construct the approximating sequence $u_n$ by defining it separately on $X^{n,a}$ and $X^{n,s}$.

**Step 1. Definition of $u_n$ on $X^{n,s}$.** We begin by constructing $X^{n,s}$. For every $n$ we can find a countable Borel partition $X^s = \bigcup_i X^{n,s}_i \cup N^{n,s}$, where each $X^{n,s}_i$ is closed,

$$\text{diam } X^{n,s}_i \leq \sigma_n/2, \quad \text{and} \quad \lambda^\infty(N^{n,s}) = 0.$$  \hspace{1cm} (5.2)

In the following, given a subset $E \subset X$ and a radius $r > 0$, the $r$-neighbourhood of $E$ will be denoted by $(E)_r := \{x \in X : d(x, E) < r\}$.

Since $\lambda(X^{n,s}) = \lambda^Y(X^{n,s}) = 0$ and $\lambda((X^{n,s})_r) > 0$ for every $r > 0$, we can construct inductively a decreasing sequence $r^n_i$ such that $0 < r^n_i \leq \sigma_n/2$,

$$\lambda((X^{n,s}_i)_r) \leq \sigma_n \lambda^\infty(X^{n,s}_i), \quad \lambda^Y((X^{n,s}_i)_r) \leq 2^{-i} \sigma_n, \quad \lambda^\infty((X^{n,s}_i)_r) \leq 2^{-i} \sigma_n, \quad \lambda((X^{n,s}_i)_r) \leq \frac{1}{2} \lambda((X^{n,s}_i)_r). \quad \text{and} \quad \text{and} \quad \lambda((X^{n,s}_i)_r) \leq \frac{1}{2} \lambda((X^{n,s}_i)_r).$$

(5.3) \hspace{1cm} (5.4) \hspace{1cm} (5.5) \hspace{1cm} (5.6)

We define

$$A^{n,s}_i := (X^{n,s}_i)_r \setminus \bigcup_{j > i} (X^{n,s}_j)_r.$$ \hspace{1cm} (5.6)

By (5.6) we have

$$\lambda(A^{n,s}_i) \geq \frac{1}{2} \lambda((X^{n,s}_i)_r) > 0.$$
while (5.2), together with the inequality $0 < r_i^n \leq \sigma_n/2$, yields
\begin{equation}
\text{diam } A_i^{n,s} \leq \sigma_n.
\end{equation}
(5.7)

By (5.3) we have
\begin{equation}
0 < \lambda(A_i^{n,s}) \leq \sigma_n \lambda^\infty(X_i^{n,s}) = \sigma_n \mu^\infty(X_i^{n,s} \times \Sigma) = \sigma_n \sum_j \mu^\infty(X_i^{n,s} \times B_j^{n,\infty}).
\end{equation}
Since $\lambda$ is nonatomic we can find a countable Borel partition $A_i^{n,s} = \bigcup_j A_i^{n,j,s}$ such that
\begin{equation}
0 < \lambda(A_i^{n,s}) \leq \sigma_n \mu^\infty(X_i^{n,s} \times B_j^{n,\infty}).
\end{equation}
(5.8)

For every $n$ we define
\begin{equation}
X^{n,s} := \bigcup_{ij} A_i^{n,j,s} \cup X^s.
\end{equation}
(5.9)

Note that by (5.4) and (5.5) we have
\begin{equation}
\lambda^Y(X^{n,s}) \leq \sigma_n \quad \text{and} \quad \lambda^\infty(X^{n,s} \setminus \cup^s) \leq \sigma_n.
\end{equation}
(5.10)

We define
\begin{equation}
u_n(x) := c_{ij}^n \xi_{n,\infty} \quad \text{for } x \in A_i^{n,j,s},
\end{equation}
where $\xi_{n,\infty}$ are arbitrary points of $B_j^{n,\infty}$ and
\begin{equation}
c_{ij}^n := \frac{\mu^\infty(X_i^{n,s} \times B_j^{n,\infty})}{\lambda(A_i^{n,j,s})}.
\end{equation}
(5.12)

By (5.8) we have that
\begin{equation}c_{ij}^n \geq 1/\sigma_n.
\end{equation}
(5.13)

By (5.12) and (5.13) we have
\begin{equation}\int_{X^{n,s}} \sqrt{1 + |u_n|^2} d\lambda \leq \lambda^\infty(X^s) \sqrt{1 + \sigma_n^2}.
\end{equation}
(5.14)

**Step 2. Definition of $u_n$ on $X^{n,a}$**

We set
\begin{equation}
X^{n,a} := X \setminus X^{n,s}.
\end{equation}

In order to define $u_n$ on $X^{n,a}$ we consider a countable Borel partition $X^{n,a} = \bigcup_i A_i^{n,a}$, with $A_i^{n,a}$ satisfying
\begin{equation}0 < \text{diam } A_i^{n,a} \leq \sigma_n.
\end{equation}
(5.15)

As $X^{n,a} \subset X^a$ by (5.9), $\lambda^\infty$ is absolutely continuous with respect to $\lambda$ on $X^{n,a}$. Since $\lambda$ is nonatomic, for every $i$ we may choose $0 < \varepsilon_i^n \leq \sigma_n$ and two disjoint Borel sets $A_i^{n,Y}$ and $A_i^{n,\infty}$ in such a way that $A_i^{n,a} = A_i^{n,Y} \cup A_i^{n,\infty}$
\begin{equation}\lambda(A_i^{n,a}) = \varepsilon_i^n \lambda^\infty(A_i^{n,a}) \leq \sigma_n \lambda(A_i^{n,a}).
\end{equation}
(5.16)

Since $\lambda$ is nonatomic and
\begin{equation}\varepsilon_i^n \lambda^\infty(A_i^{n,a}) = \varepsilon_i^n \mu^\infty(A_i^{n,a} \times \Sigma) = \sum_j \varepsilon_i^n \mu^\infty(A_i^{n,a} \times B_j^{n,\infty}),
\end{equation}
we can also find a countable Borel partition $A_i^{n,\infty} = \bigcup_{ij} A_i^{n,j,\infty}$ such that
\begin{equation}\lambda(A_i^{n,\infty}) = \varepsilon_i^n \mu^\infty(A_i^{n,a} \times B_j^{n,\infty}).
\end{equation}
(5.17)

Note also that by (5.16) we have
\begin{equation}\lambda(A_i^{n,Y}) = \lambda(A_i^{n,a}) - \lambda(A_i^{n,\infty}) \geq (1 - \sigma_n) \lambda(A_i^{n,a})
\end{equation}
and so there exists $0 < \delta_i^n \leq \sigma_n$ such that $\lambda(A_i^{n,Y}) = (1 - \delta_i^n) \lambda(A_i^{n,a})$.

As $\lambda = \pi_X(\mu_Y)$, arguing as before we may find a countable Borel partition $A_i^{n,Y} = \bigcup_{ij} A_i^{n,j,Y}$ such that
\begin{equation}\lambda(A_i^{n,Y}) = (1 - \delta_i^n) \mu_Y(A_i^{n,a} \times B_j^{n,Y}).
\end{equation}
(5.18)

We are ready to define $u_n$ on $X^{n,a}$ by setting
\begin{equation}u_n(x) := \xi_{ij}^{n,Y} \quad \text{for } x \in A_i^{n,Y} \quad \text{and} \quad u_n(x) := \frac{1}{\varepsilon_i^n} \xi_{ij}^{n,\infty} \quad \text{for } x \in A_i^{n,\infty}.
\end{equation}
(5.19)
where $\xi_j^{n,Y}$ are arbitrary points in $B_j^0$ and $\xi_j^{n,\infty}$ are the points of $B_j^\infty$ chosen in (5.11). Using (5.17) and (5.18) it is easy to check that

$$\int_{X^{n,a}} \sqrt{1 + |u_n|^2} \, d\lambda \leq \sigma_n \lambda(X) + \lambda^Y(X) + \lambda^\infty(X^{n,a}) \sqrt{1 + \sigma_n^2}. \quad (5.20)$$

By (5.14) and (5.20) we have

$$\int_{X} \sqrt{1 + |u_n|^2} \, d\lambda \leq \sigma_n \lambda(X) + \lambda^Y(X) + \lambda^\infty(X) \sqrt{1 + \sigma_n^2}, \quad (5.21)$$

which implies that $u_n$ is bounded in $L^1(X; \Xi)$. It follows from Lemma 3.3 that $||\delta_n||_*$ is uniformly bounded.

**Step 3. Proof of the convergence.** Thanks to Lemma 2.4, to prove the weak* convergence of $\delta_n$, it is enough to show that

$$\langle f, \delta_n \rangle \to \langle f, \mu \rangle, \quad (5.22)$$

for every function $f \in C^0_L(X \times \Xi \times \mathbb{R})$. Let us fix $f \in C^0_L(X \times \Xi \times \mathbb{R})$. By Remark 2.3 there exist a constant $a \in \mathbb{R}$ and a continuous function $\omega: [0+\infty) \to [0+\infty)$, with $\omega(0) = 0$, such that

$$|f(x_1, \xi_1, \eta_1) - f(x_2, \xi_2, \eta_2)| \leq a \sqrt{|\xi_1 - \xi_2|^2 + |\eta_1 - \eta_2|^2 + \omega(d(x_1, x_2)) \min\{\sqrt{|\xi_1|^2 + |\eta_1|^2}, \sqrt{|\xi_2|^2 + |\eta_2|^2}\}} \quad (5.23)$$

for every $x_1, x_2 \in X$, $\xi_1, \xi_2 \in \Xi$, $\eta_1, \eta_2 \in \mathbb{R}$.

By definition we have

$$\langle f, \delta_n \rangle = \int_{X^{n,a}} f(x, u_n, 1) \, d\lambda + \int_{X^{n,a}} f(x, u_n, 1) \, d\lambda. \quad (5.24)$$

By (5.19) the first integral in the right-hand side can be written as

$$\int_{X^{n,a}} f(x, u_n, 1) \, d\lambda = \sum_{ij} \int_{A_{ij}^{n,a}} f(x, \xi_i^{n,Y}, 1) \, d\lambda + \sum_{ij} \int_{A_{ij}^{n,a}} f(x, \xi_i^{n,\infty}/\varepsilon_i^n, 1) \, d\lambda = \sum_{ij} f(x_i^{n,a}, \xi_i^{n,Y}) \lambda(A_{ij}^{n,Y}) + \sum_{ij} f(x_i^{n,a}, \xi_i^{n,\infty}/\varepsilon_i^n, 1) \lambda(A_{ij}^{n,\infty}) + r_{n,1}^{a,1}, \quad (5.25)$$

where $x_i^{n,a}$ are arbitrary points in $A_{ij}^{n,a}$ and the remainder $r_{n,1}^{a,1}$ tends to 0 as a consequence of (5.15), (5.17), (5.18), and (5.23), which lead to the estimate

$$|r_{n,1}^{a,1}| \leq \omega(\sigma_n) \sum_{ij} \sqrt{1 + |\xi_i^n|^2} \lambda(A_{ij}^{n,Y}) + \omega(\sigma_n) \sum_{ij} \frac{1}{\varepsilon_i^n} \sqrt{\left(\frac{\varepsilon_i^n}{\xi_i}\right)^2 + |\xi_i^{n,\infty}|^2} \lambda(A_{ij}^{n,\infty}) \leq \sigma_n \omega(\sigma_n) \lambda(X^{n,a}) + \omega(\sigma_n) \lambda^Y(X^{n,a}) + \omega(\sigma_n) \lambda^\infty(X^{n,a}) \sqrt{1 + \sigma_n^2}. \quad (5.26)$$

On the other hand by (5.17) and (5.18) we have

$$\sum_{ij} f(x_i^{n,a}, \xi_j^{n,Y}, 1) \lambda(A_{ij}^{n,Y}) + \sum_{ij} f(x_i^{n,a}, \xi_j^{n,\infty}/\varepsilon_i^n, 1) \lambda(A_{ij}^{n,\infty}) = \sum_{ij} \lambda(A_{ij}^{n,a} \times B_j^{n,Y}) + \lambda(A_{ij}^{n,a} \times B_j^{n,\infty}) = \int_{X \times \Xi} f(x, \xi, 1) \, d\mu^Y(x, \xi) + \int_{X \times \Xi} f(x, \xi, 0) \, d\mu^\infty(x, \xi) + r_{n,2}^{a,2}, \quad (5.26)$$

where the remainder $r_{n,2}^{a,2}$ tends to 0 as a consequence of (5.1), (5.10), (5.15), and (5.23), which lead to the estimate

$$|r_{n,2}^{a,2}| \leq (2a\sigma_n + \omega(\sigma_n)) \lambda(Y) + \lambda^\infty(X^{n,a}) \sqrt{1 + \sigma_n^2} + 2a\sigma_n.$$
From (5.25) and (5.26) we obtain
\[
\int_{X^{n,a}} f(x, u_n, 1) d\lambda = \int_{X \times \Xi} f(x, \xi, 1) d\mu^Y(x, \xi) + \int_{X^{n} \times \Sigma} f(x, \xi, 0) d\mu^\infty(x, \xi) + r_n^a,
\]
where \( r_n^a := r_n^{a,1} + r_n^{a,2} \) tends to 0.

By (5.11) and (5.12) the second integral in the right-hand side of (5.24) can be written as
\[
\int_{X^{n,s}} f(x, u_n, 1) d\lambda = \sum_{ij} f(x^{n,s}_i, c_{ij}^{n,\infty}, 1) \lambda(A^{n,s}_{ij}) + r_n^{s,1},
\]
where \( x^{n,s}_i \) are arbitrary points in \( X^{n,s}_i \) and the remainder \( r_n^{s,1} \) tends to zero as a consequence of (5.2), (5.7), (5.12), (5.13), and (5.23), which lead to the estimate
\[
|r_n^{s,1}| \leq \omega(\sigma_n) \lambda^\infty(X^s) \sqrt{1 + \sigma_n^2}.
\]

On the other hand by (5.12)
\[
\sum_{ij} f(x^{n,s}_i, c_{ij}^{n,\infty}, 1) \lambda(A^{n,s}_{ij}) = \sum_{ij} f(x^{n,s}_i, \xi^{n,\infty}, 1/c_{ij}^{n}) \mu^\infty(X^{n,s}_i \times B^{n,\infty}_j) = \int_{X^s \times \Sigma} f(x, \xi, 0) d\mu^\infty(x, \xi) + r_n^{s,2},
\]
where the remainder \( r_n^{s,2} \) tends to 0 as a consequence of (5.1), (5.2), (5.13), and (5.23), which lead to the estimate
\[
|r_n^{s,2}| \leq (a\sigma_n + \omega(\sigma_n) \sqrt{1 + \sigma_n^2}) \lambda^\infty(X^s).
\]

From (5.28) and (5.29) we obtain
\[
\int_{X^{n,s}} f(x, u_n, 1) d\lambda = \int_{X^s \times \Sigma} f(x, \xi, 0) d\mu^\infty(x, \xi) + r_n^s,
\]
where \( r_n^s := r_n^{s,1} + r_n^{s,2} \) tends to 0.

From (4.1), (5.24), (5.27), and (5.30) we obtain (5.22), which concludes the proof of the theorem. \( \Box \)

**Remark 5.2.** If \( u_n \) is a sequence in \( L^1(X; \Xi) \) such that \( \delta_{u_n} \rightharpoonup \mu \) weakly* in \( GY(X; \Xi) \), then
\[
\int_X \sqrt{1 + |u_n|^2} d\lambda \rightharpoonup \|\mu\|_*
\]
by Lemma 3.3 and Remark 3.12.

6. **The Notion of Barycentre**

In this section we study some properties of the barycentre of a generalized Young measure.

**Definition 6.1.** The barycentre of a generalized Young measure \( \mu \in GY(X; \Xi) \) is the measure \( \text{bar}(\mu) \in M_b(X; \Xi) \) defined by
\[
\text{bar}(\mu) = \pi_X(\xi \mu).
\]

**Remark 6.2.** By Definition 2.15 a measure \( p \in M_b(X; \Xi) \) coincides with \( \text{bar}(\mu) \) if and only if
\[
\int_X \varphi \cdot dp = \langle \varphi(x) \cdot \xi, \mu(x, \xi, \eta) \rangle
\]
for every \( \varphi \in C(X; \Xi) \). By approximation we can prove that the same equality holds for every bounded Borel function \( \varphi : X \to \Xi \).
Remark 6.3. Let \( \mu = \pi_Y + \hat{\mu}^\infty \) be the decomposition of Theorem 4.3, let \( (\mu^x)_x \in X \) and \( (\mu^{x,\infty})_x \in X \) be the families of probability measures introduced in Remark 4.5, and let \( \lambda^\infty := \pi_X(\hat{\mu}^\infty) \). We consider the functions \( u^Y : \Omega \to \Xi \) and \( u^\infty : \Omega \to \Xi \) defined by

\[
  u^Y(x) := \int_{\Xi} \xi \, d\mu^x\langle \xi \rangle \quad \text{and} \quad u^\infty(x) := \int_{\Sigma_\Xi} \xi \, d\mu^{x,\infty}(\xi) .
\]

Then \( \text{bar}(\mu) = u^Y + u^\infty \lambda^\infty \). In particular, if \( \mu = \pi_Y \), then \( \text{bar}(\mu) = u^Y \in L^1(X; \Xi) \).

Therefore, \( \text{bar}(\delta_u) = u \) for every \( u \in L^1(X; \Xi) \). It follows immediately from Definition 3.2 and (6.1) that we have also \( \text{bar}(\delta_p) = p \) for every \( p \in M_b(X; \Xi) \).

Remark 6.4. From Remark 2.17 we obtain

\[
  \| \text{bar}(\mu) \| \leq \| \mu \|_* .
\]

If \( \mu_k \rightharpoonup \mu \) weakly* in \( GY(X; \Xi) \), then \( \text{bar}(\mu_k) \rightharpoonup \text{bar}(\mu) \) weakly* in \( M_b(X; \Xi) \). If \( \mu = \pi_Y \) with \( \mu^Y \in Y^r(X; \Xi) \) for some \( r > 1 \), then Remark 6.3 implies that \( \text{bar}(\mu) \in L^r(X; \Xi) \) and

\[
  \| \text{bar}(\mu) \|_r \leq \left( \int_{X \times \Xi} |\xi|^r \, d\mu^Y(x, \xi) \right)^{1/r} = \langle \{ P_r \}, \mu \rangle^{1/r} ,
\]

where \( \| \cdot \|_r \) denotes the norm in \( L^r(X; \Xi) \) and \( \{ P_r \} \) is the homogeneous function defined in (4.6).

We now prove the Jensen inequality for generalized Young measures.

Theorem 6.5. Let \( f : X \times \Xi \times \mathbb{R} \to \mathbb{R} \cup \{ +\infty \} \) be a Borel function such that \( (\xi, \eta) \mapsto f(x, \xi, \eta) \) is positively one-homogeneous, convex, and lower semicontinuous for every \( x \in X \) and satisfies the inequality

\[
  f(x, \xi, \eta) \geq -c \sqrt{\langle \xi \rangle^2 + \langle \eta \rangle^2}
\]

for some constant \( c \). Then

\[
  \langle f, \delta_{\text{bar}(\mu)} \rangle \leq \langle f, \mu \rangle \quad (6.2)
\]

for every \( \mu \in GY(X; \Xi) \).

Proof. Let us fix \( \mu \in GY(X; \Xi) \), let \( p := \text{bar}(\mu) \), and let \( \sigma \in M_b^+(X) \) be such that \( p \ll \sigma \) and \( \lambda \ll \sigma \). We consider an increasing sequence of functions \( f_k \) converging to \( f \) such that each \( f_k \) has the form

\[
  f_k(x, \xi, \eta) = \sup_{1 \leq i \leq k} \{ a_i(x) \cdot \xi + b_i(x) \eta \}
\]

with \( a_i : X \to \Xi \) and \( b_i : X \to \mathbb{R} \) bounded \( \sigma \)-measurable functions (see, e.g., [6, Theorem 2.2.4]). For every \( k \) there exists a Borel partition \( (B_i^k)_{1 \leq i \leq k} \) such that

\[
  \int_X f_k(x, \frac{dp}{d\sigma}, \frac{d\lambda}{d\sigma}) \, d\sigma = \sum_{i=1}^k \left\{ \int_{B_i^k} a_i \, dp + \int_{B_i^k} b_i \, d\lambda \right\} .
\]

By (3.2) and (6.1) we obtain

\[
  \int_{B_i^k} a_i \, dp + \int_{B_i^k} b_i \, d\lambda = \langle (a_i(x) \cdot \xi + b_i(x) \eta) 1_{B_i^k}(x), \mu(x, \xi, \eta) \rangle \leq \langle f(x, \xi, \eta) 1_{B_i^k}(x), \mu(x, \xi, \eta) \rangle .
\]

Summing over \( i \) we get

\[
  \int_X f_k(x, \frac{dp}{d\sigma}, \frac{d\lambda}{d\sigma}) \, d\sigma \leq \langle f(x, \xi, \eta), \mu(x, \xi, \eta) \rangle ,
\]

and taking the limit with respect to \( k \) gives inequality (6.2). \( \square \)
Remark 6.6. Let $f : X \times \Xi \times \mathbb{R} \to [0, +\infty]$ be a Borel function such that $(\xi, \eta) \mapsto f(x, \xi, \eta)$ is positively one-homogeneous for every $x \in X$, and let $\overline{\overline{f}}$ be the lower semicontinuous convex envelope of $f$ with respect to $(\xi, \eta)$. By applying (6.2) to $\overline{\overline{f}}$ we obtain
$$\langle \overline{\overline{f}}, \delta_{\text{bar}(\mu)} \rangle \leq \langle f, \mu \rangle$$
for every $\mu \in GY(X; \Xi)$.

The opposite inequality requires special conditions on $f$ and $\mu$, as shown in the following lemma, that will be used in [10].

Lemma 6.7. Let $\mu \in GY(X; \Xi)$, let $f : X \times \Xi \times \mathbb{R} \to [0, +\infty]$ be a Borel function such that $(\xi, \eta) \mapsto f(x, \xi, \eta)$ is positively one-homogeneous for every $x \in X$, and let $\overline{\overline{f}}$ be the lower semicontinuous convex envelope of $f$ with respect to $(\xi, \eta)$. Assume that $\langle f, \mu \rangle \leq \langle \overline{\overline{f}}, \delta_{\text{bar}(\mu)} \rangle < +\infty$. Then $\text{supp} \, \mu$ is contained in the closure of $\{f = \overline{\overline{f}}\}$.

Proof. Using the hypothesis and (6.2) we obtain
$$\langle \overline{\overline{f}}, \delta_{\text{bar}(\mu)} \rangle \leq \langle \overline{\overline{f}}, \mu \rangle \leq \langle \overline{\overline{f}}, \delta_{\text{bar}(\mu)} \rangle,$$

hence, $\langle f - \overline{\overline{f}}, \mu \rangle = 0$. Since $f - \overline{\overline{f}}$ and $\mu$ are nonnegative, we conclude that $\text{supp} \, \mu$ is contained in the closure of $\{f = \overline{\overline{f}}\}$. \hfill $\Box$

7. Compatible systems of generalized Young measures

Let $A \subset \mathbb{R}$ and let $t \mapsto p(t)$ be a function from $A$ into $M_b(X; \Xi)$. For every finite sequence $t_1 < t_2 < \cdots < t_m$ in $A$ we consider the measure $(p(t_1), \ldots, p(t_m)) \in M_b(X; \Xi^m)$ and the corresponding generalized Young measure
$$\mu_{t_1, \ldots, t_m} := \delta_{(p(t_1), \ldots, p(t_m))} \in GY(X; \Xi^m) \quad (7.1)$$
introduced in Definition 3.2, with $\Xi$ replaced by $\Xi^m$. To describe an important property of this family of generalized Young measures it is convenient to introduce the following definition.

Definition 7.1. If $\{s_1, s_2, \ldots, s_n\} \subset \{t_1, t_2, \ldots, t_m\} \subset \mathbb{R}$, with $s_1 < s_2 < \cdots < s_n$ and $t_1 < t_2 < \cdots < t_m$, we define the projection $\pi_{s_1, \ldots, s_n}^{t_1, \ldots, t_m} : X \times \Xi^m \times \mathbb{R} \to X \times \Xi^m \times \mathbb{R}$ by
$$\pi_{s_1, \ldots, s_n}^{t_1, \ldots, t_m}(x, \xi_{t_1}, \ldots, \xi_{t_m}, \eta) = (x, \xi_{s_1}, \ldots, \xi_{s_n}, \eta).$$

Remark 7.2. It is easy to see that the family of generalized Young measures (7.1) satisfies the compatibility condition
$$\mu_{s_1, \ldots, s_n} = \pi_{s_1, \ldots, s_n}^{t_1, \ldots, t_m}(\mu_{t_1, \ldots, t_m}) \quad (7.2)$$
whenever $\{s_1, s_2, \ldots, s_n\}$ and $\{t_1, t_2, \ldots, t_m\}$ are as in Definition 7.1.

This motivates the following definition.

Definition 7.3. A compatible system of generalized Young measures on $X$, with values in a finite dimensional Hilbert space $\Xi$ and with time set $A \subset \mathbb{R}$, is a family $\mu = (\mu_{t_1, \ldots, t_m})$ of generalized Young measures $\mu_{t_1, \ldots, t_m} \in GY(X; \Xi^m)$ satisfying the compatibility condition (7.2), with $t_1, \ldots, t_m$ running over all finite sequences of elements of $A$ with $t_1 < t_2 < \cdots < t_m$. The space of all such systems is denoted by $GY(A, X; \Xi)$ and is equipped with the weakest topology such that the maps $\mu \mapsto \mu_{t_1, \ldots, t_m}$ from $GY(A, X; \Xi)$ into $GY(X; \Xi^m)$, endowed with the weak* topology, are continuous for every $m$ and every finite sequence $t_1, \ldots, t_m$ in $A$ with $t_1 < t_2 < \cdots < t_m$. Although this topology is not induced by duality, we shall refer to it as the weak* topology of $GY(A, X; \Xi)$.

Definition 7.4. Given a function $t \mapsto p(t)$ from $A \subset \mathbb{R}$ into $M_b(X; \Xi)$, the family $(\mu_{t_1, \ldots, t_m})$ defined in (7.1) is called the compatible system of generalized Young measures associated with $t \mapsto p(t)$. 

The compatibility condition (7.2) implies that the barycentre of $\mu_{t_1} \ldots t_m$ is completely determined by the barycentres of $\mu_{t_1}, \ldots, \mu_{t_m}$.

**Proposition 7.5.** Let $\mu \in SGY(A, X; \Xi)$. Then

$$\bar{\mu}(\mu_{t_1} \ldots t_m) = (\bar{\mu}(\mu_{t_1}), \ldots, \bar{\mu}(\mu_{t_m}))$$

for every finite sequence $t_1, \ldots, t_m$ in $A$ with $t_1 < t_2 < \cdots < t_m$.

**Proof.** Let $(p_1, \ldots, p_m) := (\bar{\mu}(\mu_{t_1} \ldots t_m))$ and $q_i = (\bar{\mu}(\mu_{t_i}))$ for $i = 1, \ldots, m$. Using (6.1) for every $(\varphi_1, \ldots, \varphi_m) \in C(X; \Xi^m)$ we have

$$\sum_{i=1}^{m} \int_X \varphi_i \cdot dp_i = \sum_{i=1}^{m} \langle \varphi_i(x) \cdot \xi_i, \mu_{t_1} \ldots t_m(x, \xi_1, \ldots, \xi_m, \eta) \rangle, \quad (7.3)$$

$$\int_X \varphi_i \cdot dq_i = \langle \varphi_i(x) \cdot \xi_i, \mu_{t_i}(x, \xi_i, \eta) \rangle \quad \text{for } i = 1, \ldots, m. \quad (7.4)$$

The compatibility condition (7.2) implies that

$$\langle \varphi_i(x) \cdot \xi_i, \mu_{t_1} \ldots t_m(x, \xi_1, \ldots, \xi_m, \eta) \rangle = \langle \varphi_i(x) \cdot \xi_i, \mu_{t_i}(x, \xi_i, \eta) \rangle,$$

hence (7.3) and (7.4) yield

$$\sum_{i=1}^{m} \int_X \varphi_i \cdot dp_i = \sum_{i=1}^{m} \int_X \varphi_i \cdot dq_i$$

for every $(\varphi_1, \ldots, \varphi_m) \in C(X; \Xi^m)$. This gives $p_i = q_i$ for $i = 1, \ldots, m$. \qed

The notion of left continuity, introduced in the next definition, is very useful in the applications.

**Definition 7.6.** A system $\mu \in SGY(A, X; \Xi)$ is said to be **left continuous** if for every finite sequence $t_1, \ldots, t_m$ in $A$ with $t_1 < \cdots < t_m$ the following continuity property holds:

$$\mu_{s_1} \ldots s_m \rightharpoonup \mu_{t_1} \ldots t_m \quad \text{weakly}^* \quad \text{in } GY(X; \Xi^m) \quad (7.5)$$

as $s_i \to t_i$, with $s_i \in A$ and $s_i \leq t_i$.

The following theorem proves the weak* compactness of the subsets of $SGY(A, X; \Xi)$ defined by imposing bounds on the norms of $\mu_t$ for every $t \in A$.

**Theorem 7.7.** For every function $C : A \to [0, +\infty)$ the set

$$\{ \mu \in SGY(A, X; \Xi) : \|\mu_t\| \leq C(t) \text{ for every } t \in A \} \quad (7.6)$$

is weakly* compact in $SGY(A, X; \Xi)$.

To prove the theorem we need the following lemma which provides an estimate of the norm $\|\mu_{t_1} \ldots t_m\|$ in terms of the norms $\|\mu_{t_i}\|$.

**Lemma 7.8.** For every $\mu \in SGY(A, X; \Xi)$ we have

$$\|\mu_{t_1} \ldots t_m\| \leq \sum_{i=1}^{m} \|\mu_{t_i}\|$$

for every finite sequence $t_1, \ldots, t_m$ in $A$ with $t_1 < t_2 < \cdots < t_m$.

**Proof.** By Remark 2.9 and by the compatibility condition (7.2) we have

$$\|\mu_{t_1} \ldots t_m\| = \langle (\xi_1, \ldots, \xi_m, \eta), \mu_{t_1} \ldots t_m(x, \xi_1, \ldots, \xi_m, \eta) \rangle \leq \sum_{i=1}^{m} \langle (\xi_i, \eta), \mu_{t_1} \ldots t_m(x, \xi_1, \ldots, \xi_m, \eta) \rangle = \sum_{i=1}^{m} \| (\xi_i, \eta) \cdot \mu_{t_i}(x, \xi_i, \eta) \| = \sum_{i=1}^{m} \| \mu_{t_i} \|,$$

which concludes the proof. \qed
Proof of Theorem 7.7. By Lemma 7.8 for every function \( C: A \to [0, +\infty) \) the set defined in (7.6) is contained in the set of all \( \mu \in SGY(A, X; \Xi) \) such that

\[
\|\mu_{t_1 \ldots t_m}\|_* \leq \sum_{i=1}^m C(t_i)
\]

for every finite sequence \( t_1, \ldots, t_m \) in \( A \) with \( t_1 < t_2 < \cdots < t_m \). As the topology in \( SGY(A, X; \Xi) \) is induced by the product of the weak* topologies of the spaces \( GY(X; \Xi^m) \) corresponding to the projections \( \mu_{t_1 \ldots t_m} \), the set (7.6) is compact in the weak* topology of \( SGY(A, X; \Xi) \) by Tychonoff's Theorem.

□

Remark 7.9. If \( A = \{a_0, a_1, \ldots, a_k\} \), with \( a_0 < a_1 < \cdots < a_k \), then for every \( \mu \in GY(X; \Xi^{k+1}) \) there exists a unique system \( \mu^A \in SGY(A, X; \Xi) \) such that \( \mu^A_{a_0 \ldots a_k} = \mu \).

This system is defined by

\[
\mu^A_{t_1 \ldots t_m} = \pi_{t_1 \ldots t_m}^A(\mu)
\]

for every \( \{t_1, t_2, \ldots, t_m\} \subset \{a_0, a_1, \ldots, a_k\} \) with \( t_1 < t_2 < \cdots < t_m \).

The notion of piecewise constant interpolation will be useful in the application to evolution problems.

Definition 7.10. Let \( A = \{a_0, a_1, \ldots, a_k\} \), with \( a_0 < a_1 < \cdots < a_k \). For every \( t_1, \ldots, t_m \) in \( [a_0, a_k] \) with \( t_1 < t_2 < \cdots < t_m \) let \( \rho_{t_1 \ldots t_m} : X \times \Xi^{k+1} \times \mathbb{R} \to X \times \Xi^m \times \mathbb{R} \) be defined by

\[
\rho_{t_1 \ldots t_m}(x, \xi_{a_0}, \ldots, \xi_{a_k}, \eta) := (x, \xi_{t_1}, \ldots, \xi_{t_m}, \eta),
\]

with \( \xi_{t_i} = \xi_{a_j} \), where \( j \) is the largest index such that \( a_j \leq t_i \). For every \( \mu \in GY(X; \Xi^{k+1}) \) the piecewise constant interpolation \( \mu^A \) of \( \mu \) is the element of \( SGY([a_0, a_k], X; \Xi) \) defined by

\[
\mu^A_{t_1 \ldots t_m} := \rho_{t_1 \ldots t_m}(\mu) \quad (7.7)
\]

for every \( t_1, \ldots, t_m \) in \( [a_0, a_k] \) with \( t_1 < t_2 < \cdots < t_m \).

Remark 7.11. It is easy to check that \( \rho_{s_1 \ldots s_n} = \pi_{s_1 \ldots s_n}^A \circ \rho_{t_1 \ldots t_m} \) whenever \( \{s_1, s_2, \ldots, s_n\} \subset \{t_1, t_2, \ldots, t_m\} \subset [a_0, a_k] \), with \( s_1 < s_2 < \cdots < s_n \) and \( t_1 < t_2 < \cdots < t_m \). Therefore the family of generalized Young measures \( (\mu^A_{t_1 \ldots t_m}) \) defined by (7.7) satisfies the compatibility condition (7.2).

8. THE NOTION OF VARIATION

In this section we study the notion of variation on a time interval of a compatible system of generalized Young measures, and prove a compactness theorem which extends Helly's Theorem.

Definition 8.1. Given a set \( A \subset \mathbb{R} \), the variation of \( \mu \in SGY(A, X; \Xi) \) on the time interval \( [a, b] \), with \( a, b \in A \), is defined as

\[
\text{Var}(\mu; a, b) := \sup \sum_{i=1}^k (|\xi_i - \xi_{i-1}|, \mu_{t_0 \ldots t_k}(x, \xi_0, \ldots, \xi_k, \eta)),
\]

where the supremum is taken over all finite families \( t_0, t_1, \ldots, t_k \) in \( A \) such that \( a = t_0 < t_1 < \cdots < t_k = b \) (with the convention \( \text{Var}(\mu; a, b) = 0 \) if \( a = b \)).

Remark 8.2. If \( \mu \) is the compatible family of generalized Young measures associated with a function \( t \mapsto p(t) \) from \( A \) into \( M_b(X; \Xi) \) according to (7.1), then \( \text{Var}(\mu; a, b) \) reduces to the variation of \( t \mapsto p(t) \) on \( [a, b] \cap A \).
Remark 8.3. Returning to the general case, the compatibility condition (7.2) yields
\[
\text{Var}(\mu; a, b) = \sup_k \sum_{i=1}^k \langle |\xi_i - \xi_{i-1}|, \mu_{a,t_i} (x, \xi_{i-1}, \xi_i, \eta) \rangle ,
\]
where the supremum is taken over all finite families \( t_0, t_1, \ldots, t_k \) in \( A \) such that \( a = t_0 < t_1 < \cdots < t_k = b \).

Remark 8.4. If \( t_1, t_2, t_3 \in A \) and \( t_1 < t_2 < t_3 \), by the compatibility condition (7.2) and by the triangle inequality we have
\[
\langle |\xi_3 - \xi_1|, \mu_{t_1, t_3} (x, \xi_1, \xi_3, \eta) \rangle = \langle |\xi_3 - \xi_2|, \mu_{t_1, t_2} (x, \xi_1, \xi_2, \xi_3, \eta) \rangle + \langle |\xi_2 - \xi_1|, \mu_{t_2, t_3} (x, \xi_1, \xi_2, \xi_3, \eta) \rangle = \langle |\xi_3 - \xi_2|, \mu_{t_2, t_3} (x, \xi_2, \xi_3, \eta) \rangle + \langle |\xi_2 - \xi_1|, \mu_{t_1, t_2} (x, \xi_1, \xi_2, \xi_3, \eta) \rangle.
\]
Using this inequality it is easy to deduce from Remark 8.3 that
\[
\text{Var}(\mu; a, c) = \text{Var}(\mu; a, b) + \text{Var}(\mu; b, c) \tag{8.1}
\]
for every \( a, b, c \in A \) with \( a \leq b \leq c \). This implies in particular that the function \( t \mapsto \text{Var}(\mu; a, t) \) is nondecreasing on \( A \cap [a, +\infty) \).

Remark 8.5. If \( A = \{a_0, \ldots, a_k\} \subset \mathbb{R} \) is a finite set, with \( a_0 < a_1 < \cdots < a_k \), \( \mu \in GY(X; \Xi^{k+1}) \), and \( \mu^A \in SGY(A, X; \Xi) \) is the associated system defined in Remark 7.9, it follows from (8.1) that
\[
\text{Var}(\mu^A; a_0, a_k) = \sum_{i=1}^k \langle |\xi_i - \xi_{i-1}|, \mu^A_{a_{i-1}, a_i} (x, \xi_{i-1}, \xi_i, \eta) \rangle = \sum_{i=1}^k \langle |\xi_i - \xi_{i-1}|, \mu (x, \xi_0, \ldots, \xi_k, \eta) \rangle .
\]
It is easy to see that, if \( \mu^{[A]} \in SGY([a_0, a_k], X; \Xi) \) is the piecewise constant interpolation of \( \mu \) defined by (7.7), then
\[
\text{Var}(\mu^{[A]}; a_0, a_k) = \text{Var}(\mu^A; a_0, a_k) = \sum_{i=1}^k \langle |\xi_i - \xi_{i-1}|, \mu_{a_{i-1}, a_i} (x, \xi_{i-1}, \xi_i, \eta) \rangle = \sum_{i=1}^k \langle |\xi_i - \xi_{i-1}|, \mu (x, \xi_0, \ldots, \xi_k, \eta) \rangle .
\]

Definition 8.6. Let \( h: \Xi \to [0, +\infty) \) be a positively one-homogeneous function satisfying the triangle inequality. Given a set \( A \subset \mathbb{R} \), the \( h \)-variation of \( \mu \in SGY(A, X; \Xi) \) on the time interval \( [a, b] \), with \( a, b \in A \), is defined as
\[
\text{Var}_h (\mu; a, b) := \sup \sum_{i=1}^k \langle h(\xi_i - \xi_{i-1}), \mu_{t_{i-1} t_i} (x, \xi_0, \ldots, \xi_k, \eta) \rangle ,
\]
where the supremum is taken over all finite families \( t_0, t_1, \ldots, t_k \) in \( A \) such that \( a = t_0 < t_1 < \cdots < t_k = b \) (with the convention \( \text{Var}_h (\mu; a, a) = 0 \) if \( a = b \)).

Remark 8.7. It is well known that every positively one-homogeneous function \( h: \Xi \to [0, +\infty) \) satisfying the triangle inequality is continuous and satisfies an estimate of the form \( h(\xi) \leq c |\xi| \) for some constant \( c \). It follows that \( \text{Var}_h (\mu; a, b) \leq c \text{Var} (\mu; a, b) \). It is easy to see that all properties of \( \text{Var}(\mu; a, b) \) proved so far can be extended to \( \text{Var}_h (\mu; a, b) \).

Using the compatibility condition it is easy to prove the following lemma.

Lemma 8.8. Let \( T > 0 \) and let \( \mu \in SGY([0, T], X; \Xi) \) with \( \text{Var}(\mu; 0, T) < +\infty \). For every \( f \in C^\text{hom}_L (X \times \Xi \times \mathbb{R}) \) the function \( t \mapsto \langle f, \mu_t \rangle \) has bounded variation on \( [0, T] \).
The proof is omitted, since it is similar to the proof of the following lemma, which will be used in Theorem 9.7.

**Lemma 8.9.** Let \( T > c > 0 \) and let \( \mu \in SGY([0,T], X; \Xi) \) with \( \text{Var}(\mu; 0, T) < +\infty \). For every \( f \in C^1_\text{hom}(X \times \Xi^2 \times \mathbb{R}) \) the function \( \Phi^f_c(t) := \langle f, \mu_{t+c} \rangle \) has bounded variation on \([0,T-c]\).

**Proof.** Let \( V(t) := \text{Var}(\mu; 0, t) \) for every \( t \in [0,T] \). Let us fix \( f \in C^1_\text{hom}(X \times \Xi^2 \times \mathbb{R}) \) and let \( a \) be a constant satisfying (2.1). Let \( t_1, t_2 \) with \( 0 \leq t_1 < t_1 + c < t_2 < t_2 + c \leq T \). Using the compatibility condition (7.2) and (8.1), we obtain

\[
|\Phi^f_c(t_2) - \Phi^f_c(t_1)| = |\langle f(x, \xi_1, \xi_2, \eta) - f(x, \xi_1, \xi_1, \eta), \mu_{t_1, t_1+c, t_2, t_2+c}(x, \xi_1, \xi_2, \xi_2, \eta) \rangle|
\leq a(\|\xi_2 - \xi_1\| + \|\xi_2 - \xi_1\|, \mu_{t_1, t_1+c, t_2, t_2+c}(x, \xi_1, \xi_1, \xi_2, \xi_2, \eta)) =
 a(\|\xi_2 - \xi_1\|, \mu_{t_1, t_2}(x, \xi_1, \xi_2, \eta)) + a(\|\xi_2 - \xi_1\|, \mu_{t_1+c, t_2+c}(x, \xi_1, \xi_2, \xi_2, \eta)) \leq |V(t_2) - V(t_1)| + V(t_2 + c) - V(t_1 + c).
\]

The same inequality can be proved if \( 0 \leq t_1 < t_2 < t_1 + c < t_2 + c \leq T \). As \( V \) is nondecreasing, we conclude that the total variation of \( \Phi^f_c \) on \([0,T-c]\) is less than or equal to \( V(T-c) + V(T) \). \( \square \)

The following result can be considered as a version of Helly’s Theorem for compatible systems of generalized Young measures. Note that this is a sequential compactness result, in contrast with Theorem 7.7.

**Theorem 8.10.** Let \( T > 0 \) and let \( \mu^k \) be a sequence in \( SGY([0,T], X; \Xi) \) such that

\[
\sup_k \text{Var}(\mu^k; 0, T) \leq C, \quad \text{(8.2)}
\]

\[
\sup_k \|\mu^k_t\|_* \leq C_* \quad \text{for every } t \in [0,T], \quad \text{(8.3)}
\]

for some \( t_0 \in [0,T] \) and some finite constants \( C \) and \( C_* \). Then there exist a subsequence, still denoted \( \mu^k \), a set \( \Theta \subset [0,T] \), containing 0 and with \([0,T] \setminus \Theta \) at most countable, and a left continuous \( \mu \in SGY([0,T], X; \Xi) \), with

\[
\text{Var}(\mu; 0, T) \leq C, \quad \text{(8.4)}
\]

\[
\|\mu_t\|_* \leq C_* + C \quad \text{for every } t \in [0,T], \quad \text{(8.5)}
\]

such that

\[
\mu^k_{t_1, \ldots, t_m} \rightharpoonup \mu_{t_1, \ldots, t_m} \quad \text{weakly* in } GY(X; \Xi^m) \quad \text{(8.6)}
\]

for every finite sequence \( t_1, \ldots, t_m \) in \( \Theta \) with \( 0 \leq t_1 < \cdots < t_m \leq T \).

**Proof.** The proof is divided in several steps.

**Step 1. Boundedness of \( \mu^k_{t_1, \ldots, t_m} \).** We begin by proving that \( \|\mu^k_t\|_* \) is bounded uniformly with respect to \( t \in [0,T] \) and \( k \). Let us fix \( t < t_0 \). By the compatibility condition (7.2),

\[
\langle |x|, \mu^k_t(x, \xi, \eta) \rangle - \langle |\xi_0|, \mu^k_t(x, \xi, \eta) \rangle = \langle |x|, \mu^k_{t_0}(x, \xi, \xi_0, \eta) \rangle - \langle |\xi_0|, \mu^k_{t_0}(x, \xi, \xi_0, \eta) \rangle \leq \langle |\xi - \xi_0|, \mu^k_{t_0}(x, \xi, \xi_0, \eta) \rangle \leq \text{Var}(\mu^k; t, t_0) \leq C.
\]

Thanks to Remark 2.9, from (8.2) and (8.3) we obtain that

\[
\sup_k \|\mu^k_t\|_* \leq C_* + C \quad \text{(8.7)}
\]

for every \( t \in [0,t_0) \). A similar argument proves (8.7) when \( t \in [t_0,T] \).

By Lemma 7.8 and (8.7) we obtain

\[
\|\mu^k_{t_1, \ldots, t_m}\|_* \leq m(C_* + C) \quad \text{(8.8)}
\]

for every finite sequence \( t_1, \ldots, t_m \) with \( t_1 < \cdots < t_m \).
Step 2. Choice of the subsequence. Let $D$ be a countable dense subset of $[0, T]$ containing 0. By the compactness Theorem 3.11, using (8.8) and a diagonal argument, we can extract a subsequence, still denoted $\mu^k$, such that for every $s_1, \ldots, s_m$ in $D$ with $0 \leq s_1 < \cdots < s_m \leq T$, the sequence $\mu^{k}_{s_1 \cdots s_m}$ converges weakly* in $G Y (X; \Xi^m)$.

Step 3. Choice of $\Theta$. Let $V^k(t) := V a r (\mu^k; 0, t)$. Since $V^k$ is nondecreasing, by (8.2) and by Helly’s Theorem there exists a subsequence, still denoted $V^k$, such that, for every $t \in [0, T]$, $V^k(t) \to V(t)$, where $V$ is a nondecreasing function on $[0, T]$ with values in $[0, C]$. Let

$$\Theta := \{0\} \cup \{t \in (0, T] : \lim_{s \to t-} V(s) = V(t)\}. \tag{8.9}$$

Step 4. Convergence and left continuity on $\Theta$. Let us fix two finite sequences $t_1, \ldots, t_m$ and $s_1, \ldots, s_m$ in $[0, T]$ such that $0 \leq s_1 < t_1 < \cdots < s_m < t_m \leq T$. We want to estimate the difference $\mu^{k}_{t_1 \cdots t_m} - \mu^{k}_{s_1 \cdots s_m}$. Let $f \in C_L^{\text{hom}}(X \times \Xi^m \times \mathbb{R})$. Then there exists a constant $a$ such that

$$|f(x, \xi_1, \ldots, \xi_m, \eta) - f(x, \xi_1, \ldots, \xi_m, \eta)| \leq a \sum_{i=1}^{m} |\xi_i - \xi_i| \tag{8.10}$$

By the compatibility condition (7.2) we have the estimate

$$|\langle f, \mu^{k}_{t_1 \cdots t_m} \rangle - \langle f, \mu^{k}_{s_1 \cdots s_m} \rangle| =$$

$$= |\langle f(x, \xi_1, \ldots, \xi_m, \eta) - f(x, \xi_1, \ldots, \xi_m, \eta), \mu^{k}_{s_1 t_1 \cdots s_m t_m} (x, \xi_1, \xi_1, \ldots, \xi_m, \xi_1, \xi_m, \eta) \rangle| \leq$$

$$\leq a \sum_{i=1}^{m} |\xi_i - \xi_i|, \mu^{k}_{s_1 t_1 \cdots s_m t_m} (x, \xi_1, \xi_1, \ldots, \xi_m, \xi_1, \xi_m, \eta)| =$$

$$= a \sum_{i=1}^{m} |\xi_i - \xi_i|, \mu^{k}_{s_1 t_1 \cdots s_m t_m} (x, \xi_1, \xi_1, \ldots, \xi_m, \xi_1, \xi_m, \eta)|,$$

which by (8.1) gives

$$|\langle f, \mu^{k}_{t_1 \cdots t_m} \rangle - \langle f, \mu^{k}_{s_1 \cdots s_m} \rangle| \leq a \sum_{i=1}^{m} (V^k(t_i) - V^k(s_i)) \tag{8.11}$$

A simple modification of the proof shows that (8.11) holds even if $0 = s_1 = t_1 < s_2 \leq t_2 < \cdots < s_m \leq t_m \leq T$.

If $t_1, \ldots, t_m \in \Theta$ with $0 \leq t_1 < \cdots < t_m \leq T$, for every $\varepsilon$ we can choose $s_1, \ldots, s_m \in D$, with $0 \leq s_1 \leq t_1 < s_2 \leq t_2 < \cdots < s_m \leq t_m \leq T$, such that $a \sum_{i=1}^{m} (V(t_i) - V(s_i)) < \varepsilon$. Using (8.11) we deduce that $|\langle f, \mu^{k}_{t_1 \cdots t_m} \rangle - \langle f, \mu^{k}_{s_1 \cdots s_m} \rangle| < \varepsilon$ for $k$ large enough. Since the sequence $\langle f, \mu^{k}_{s_1 \cdots s_m} \rangle$ converges, we have $|\langle f, \mu^{k}_{s_1 \cdots s_m} \rangle - \langle f, \mu^{k}_{s_1 \cdots s_m} \rangle| < \varepsilon$ for $k, k'$ large enough. It follows that $|\langle f, \mu^{k}_{t_1 \cdots t_m} \rangle - \langle f, \mu^{k}_{t_1 \cdots t_m} \rangle| < 3\varepsilon$ for $k, k'$ large enough, hence $\langle f, \mu^{k}_{t_1 \cdots t_m} \rangle$ is a Cauchy sequence for every $f \in C_L^{\text{hom}}(X \times \Xi^m \times \mathbb{R})$. By (8.8) we deduce from Lemma 2.4 that, for every $t_1, \ldots, t_m \in \Theta$ with $0 \leq t_1 < \cdots < t_m \leq T$, the sequence $\mu^{k}_{t_1 \cdots t_m}$ converges weakly* to some element $\mu_{t_1 \cdots t_m}$ of $G Y (X; \Xi^m)$ satisfying

$$||\mu_{t_1 \cdots t_m}||_* \leq m(C_* + C). \tag{8.12}$$

We observe that, given $t_1, \ldots, t_m$ and $s_1 \ldots s_m$ in $\Theta$, we can pass to the limit in (8.11) and obtain

$$|\langle f, \mu_{t_1 \cdots t_m} \rangle - \langle f, \mu_{s_1 \cdots s_m} \rangle| \leq a \sum_{i=1}^{m} (V(t_i) - V(s_i)) \tag{8.13}$$

for every $f$ satisfying (8.10) and every pair of finite sequences $t_1, \ldots, t_m$ and $s_1, \ldots, s_m$ in $\Theta$ such that $s_1 \leq t_1 < s_2 \leq t_2 < \cdots < s_m \leq t_m$. Using the definition (8.9) of $\Theta$ and Lemma 2.4, we deduce from (8.12) and (8.13) that, for every $t_1, \ldots, t_m$ in $\Theta$ with
Whenever \( t \) satisfies a Cauchy condition for every \( t^* \) in \( V(7.5) \). To this purpose, it is enough to observe that, since \( s \) satisfies the compatibility conditions, inequalities (8.4) and (8.5), and the continuity property (7.5). To this purpose, it is enough to observe that, since \( V \) has a finite limit from the left at each point, we have
\[
\lim_{k,k' \to \infty} \sum_{i=1}^m (V(s_i^k) - V(s_i^{k'})) = 0 \tag{8.14}
\]
for every sequence \( (s_1^k, \ldots, s_m^k) \) in \( \Theta^m \) with \( s_i^k \to t_i, s_i^k \leq t_i \). Indeed, if \( t_i \notin \Theta \), we have
\[
V(s_i^k) \to V^-(t_i) := \lim_{s \downarrow t_i} V(s).
\]
For these sequences \( (s_1^k, \ldots, s_m^k) \) we can deduce from estimate (8.13) that \( \langle f, \mu_{s_1^k \ldots s_m^k} \rangle \) satisfies a Cauchy condition for every \( f \) satisfying (8.10). By (8.12) we deduce from Lemma 2.4 the existence of the weak* limit of \( \mu_{s_1 \ldots s_m} \) as \( s_i \to t_i, s_i \in \Theta \), and \( s_i \leq t_i \). We take such a weak* limit as the definition of \( \mu_{t_1 \ldots t_m} \). Clearly \( \mu_{t_1 \ldots t_m} \) satisfies (8.12) and, by construction, from (8.13), we deduce that for every \( f \) satisfying (8.10) and every pair of finite sequences \( t_1, \ldots, t_m \) and \( s_1, \ldots, s_m \) in \([0, T]\), with \( 0 \leq s_1 \leq t_1 < s_2 \leq t_2 < \cdots < s_m \leq t_m \leq T \), there holds
\[
|\langle f, \mu_{t_1 \ldots t_m} \rangle - \langle f, \mu_{s_1 \ldots s_m} \rangle| \leq \sum_{i=1}^m (V^- (t_i) - V^- (s_i)), \tag{8.15}
\]
where \( V^- \) is the left-continuous representative of \( V \) defined by (8.14). The continuity property (7.5) follows easily from (8.15) and from Lemma 2.4.

For every finite sequence \( t_1, \ldots, t_m \) in \( \Theta \) with \( t_1 < \cdots < t_m \) we have
\[
\sum_{i=1}^m \langle |f(t) - f(s)|, \mu_{t_1 \ldots t_i} \rangle \leq C.
\]
Passing to the limit as \( k \to \infty \), we obtain
\[
\sum_{i=1}^m \langle |f(t) - f(s)|, \mu_{t_1 \ldots t_i} \rangle \leq C
\]
whenever \( t_1, \ldots, t_m \in \Theta \). This restriction can be removed by an approximation argument, and this proves (8.4).

The compatibility condition (7.2) for \( \mu^k \) implies that
\[
\langle f, \mu_{s_1 \ldots s_n}^k \rangle = \langle f \circ \pi_{s_1 \ldots s_n}^k, \mu_{t_1 \ldots t_m}^k \rangle
\]
for every \( f \in C^{\text{hom}}(X \times \Xi^h \times \mathbb{R}) \) and every pair of finite sequences \( s_1, \ldots, s_n \) and \( t_1, \ldots, t_m \) in \([0, T]\) with \( s_1 < \cdots < s_n, t_1 < \cdots < t_m \), and \( \{s_1, \ldots, s_n\} \subset \{t_1, \ldots, t_m\} \). Passing to the limit as \( k \to \infty \), we obtain
\[
\langle f, \mu_{s_1 \ldots s_n} \rangle = \langle f \circ \pi_{s_1 \ldots s_n}, \mu_{t_1 \ldots t_m} \rangle
\]
whenever \( s_i \) and \( t_j \) belong to \( \Theta \). This restriction can be removed by an approximation argument, therefore \( \mu \in \text{SGY}([0, T], X; \Xi) \).

We conclude this section by proving the lower semicontinuity of the \( h \)-variation.

**Theorem 8.11.** Let \( T > 0 \) and let \( \mu^k \) be a sequence in \( \text{SGY}([0, T], X; \Xi) \). Suppose that there exist a dense set \( D \subset [0, T] \), and a left continuous \( \mu \in \text{SGY}([0, T], X; \Xi) \) such that
\[
\mu_{t_1 \ldots t_m}^k \to \mu_{t_1 \ldots t_m} \text{ weakly}^* \text{ in } \text{GY}(X; \Xi^m)
\]
for every finite sequence $t_1, \ldots, t_m$ in $D$ with $t_1 < \cdots < t_m$. Then
\[
\operatorname{Var}_h(\mu; 0, T) \leq \liminf_{k \to \infty} \operatorname{Var}_h(\mu^k; 0, T)
\]
for every positively one-homogeneous function $h : \Theta \to [0, +\infty)$ satisfying the triangle inequality.

**Proof.** Let us fix $h$. For every finite sequence $t_1, \ldots, t_m$ in $D$ with $t_1 < \cdots < t_m$ we have
\[
\sum_{i=1}^m \langle h(\xi_i - \xi_{i-1}), \mu_{t_{i-1}t_i}(x, \xi_{i-1}, \xi_i) \rangle \leq \operatorname{Var}_h(\mu^k; 0, T).
\]
Since $h$ is continuous (Remark 8.7), passing to the limit as $k \to \infty$ we obtain
\[
\sum_{i=1}^m \langle h(\xi_i - \xi_{i-1}), \mu_{t_{i-1}t_i}(x, \xi_{i-1}, \xi_i) \rangle \leq \liminf_{k \to \infty} \operatorname{Var}_h(\mu^k; 0, T)
\]
whenever $t_1, \ldots, t_m \in D$. The same inequality can be proved when $t_1, \ldots, t_m \in [0, T]$ by an approximation argument, thanks to left continuity. The conclusion is obtained by taking the supremum with respect to $t_1, \ldots, t_m$. \qed

9. Weak* derivatives of systems with bounded variation

In this section we introduce the notion of weak* derivative of a compatible system of generalized Young measures on the time interval $[0, T]$, with $T > 0$, and prove that, if $\operatorname{Var}(\mu; 0, T) < +\infty$, then the weak* derivative exists at almost every $t \in [0, T]$.

**Definition 9.1.** Given $\mu \in \Sigma_Y([0, T]; X; \Theta)$, the difference quotient of $\mu$ between times $t_1$ and $t_2$, with $0 \leq t_1 < t_2 \leq T$, is the element of $\Gamma(Y; \Theta)$ defined as the image
\[
q_{t_1t_2}(\mu_{t_1t_2})
\]
of $\mu_{t_1t_2}$ under the map
\[
q_{t_1t_2} : X \times \Theta \times \Theta \times \mathbb{R} \to X \times \Theta \times \Theta \times \mathbb{R}
\]
defined by
\[
q_{t_1t_2}(x, \xi_1, \xi_2, \eta) = (x, \frac{\xi_2 - \xi_1}{t_2 - t_1}, \eta).
\]

**Remark 9.2.** It follows from Definition 3.18 that the difference quotient is characterized by the equality
\[
\langle f(x, \xi, \eta), q_{t_1t_2}(\mu_{t_1t_2})(x, \xi, \eta) \rangle = \langle f(x, \frac{\xi_2 - \xi_1}{t_2 - t_1}, \eta), \mu_{t_1t_2}(x, \xi_1, \xi_2, \eta) \rangle
\]
for every $f \in C^{\text{hom}}(X \times \Theta \times \mathbb{R})$.

**Remark 9.3.** It follows from the definition of barycentre that
\[
\operatorname{bar}(q_{t_1t_2}(\mu_{t_1t_2})) = \frac{\operatorname{bar}(\mu_{t_1}) - \operatorname{bar}(\mu_{t_2})}{t_2 - t_1}.
\]
(see Definition 3.2).

**Definition 9.4.** We say that $\mu \in \Sigma_Y([0, T]; X; \Theta)$ has a weak* derivative $\bar{\mu}_{t_0}$ at time $t_0 \in [0, T]$ if $q_{t_0t}(\mu_{t_0}) \rightharpoonup \bar{\mu}_{t_0}$ weakly* in $\Gamma(Y; \Theta)$ as $t \to t_0^+$ and $q_{t_1t_0}(\mu_{t_1}) \rightharpoonup \bar{\mu}_{t_0}$ weakly* in $\Gamma(Y; \Theta)$ as $t \to t_0^-$, which is equivalent to
\[
\lim_{t \to t_0^+} \langle f(x, \xi_0, \eta), \mu_{t_0t}(x, \xi_0, \eta) \rangle = \lim_{t \to t_0^-} \langle f(x, \xi_0, \eta), \mu_{t_0t}(x, \xi_0, \eta) \rangle
\]
for every $f \in C^{\text{hom}}(X \times \Theta \times \mathbb{R})$. 

\[
\eqref{9.3}.
\]
Remark 9.5. It follows from (9.2) that, if \( \mu \) is the compatible system of generalized Young measures associated with a function \( t \mapsto p(t) \) from \([0,T]\) into \( M_b(X;\Xi)\) according to (7.1) and
\[
p(t) - p(t_0) \rightarrow \dot{p}(t_0)
\]
strongly in \( M_b(X;\Xi) \) as \( t \rightarrow t_0 \), then \( \mu \) has a weak* derivative at \( t_0 \) and
\[
\dot{\mu}_{t_0} = \delta_{\dot{p}(t_0)}.
\]
This is not true if
\[
p(t) - p(t_0) \rightarrow \dot{p}(t_0)
\]
only in the weak* topology of \( M_b(X;\Xi) \). However, using Remark 9.3, in this case we obtain
\[
\bar{\text{bar}}(\dot{\mu}_{t_0}) = \dot{p}(t_0),
\]
if the weak* derivative of \( \mu_{t_1,...,t_m} := \delta_{\dot{p}(t_1),...,\dot{p}(t_m)} \) exists at \( t_0 \).
An example where (9.4) holds but \( \dot{\mu}_{t_0} \neq \delta_{\dot{p}(t_0)} \) can be constructed in the following way. Let \( T = 2, X = [-1,1] \), \( \Xi = \mathbb{R} \), let \( \lambda \) be the Lebesgue measure, let \( w: \mathbb{R} \rightarrow \mathbb{R} \) be the 2-periodic function defined by
\[
w(x) := \begin{cases} 1 & \text{if } 2k \leq x < 2k + 1 \text{ for some } k \in \mathbb{Z}, \\ -1 & \text{if } 2k - 1 \leq x < 2k \text{ for some } k \in \mathbb{Z}. \end{cases}
\]
For every \( t \in [0,2] \) let \( u(t) \in L^1(X) \) be the function defined by
\[
u(t,x) := \begin{cases} (t-1)w(\frac{x}{t}) & \text{if } t \neq 1, \\ 0 & \text{if } t = 1. \end{cases}
\]
As \( t \rightarrow 1 \) we have
\[
\frac{u(t) - u(1)}{t - 1} \rightarrow 0 \quad \text{weakly* in } M_b(X;\Xi),
\]
while
\[
\delta_{\dot{u}(t)-\dot{u}(1)} \rightarrow \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1} \quad \text{weakly* in } GY(X;\Xi),
\]
which implies \( \dot{\mu}_1 = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1} \).

Remark 9.6. If \( \mu \in SGY([0,T],X;\Xi) \) has a weak* derivative \( \dot{\mu}_{t_0} \) at time \( t_0 \in [0,T] \), then
\[
\frac{\bar{\text{bar}}(\mu_t) - \bar{\text{bar}}(\mu_{t_0})}{t - t_0} \rightarrow \bar{\text{bar}}(\dot{\mu}_{t_0})
\]
weakly* in \( M_b(X;\Xi) \) as \( t \rightarrow t_0 \). This follows from (9.1) and Remark 6.4.

The following theorem is the main result of this section.

Theorem 9.7. Let \( T > 0 \) and let \( \mu \in SGY([0,T],X;\Xi) \) with \( \text{Var}(\mu;0,T) < +\infty \). Then the weak* derivative \( \dot{\mu}_t \) exists for a.e. \( t \in [0,T] \). Moreover, for every \( f \in C^{hom}(X \times \Xi \times \mathbb{R}) \) the function \( t \mapsto \langle f, \dot{\mu}_t \rangle \) is integrable on \([0,T]\). Finally, if \( h: \Xi \rightarrow [0,\infty) \) is a positively one-homogeneous function satisfying the triangle inequality, then
\[
\int_a^b \langle h(\xi), \dot{\mu}_t(x,\xi,\eta) \rangle \, dt \leq \text{Var}_h(\mu;a,b)
\]
for every \( a,b \in [0,T] \) with \( a \leq b \).

Proof. Some ideas of this proof are borrowed from the proof of [2, Theorem 4.1.1] on the existence of the metric derivative of a Lipschitz curve.

Step 1. Boundedness of the difference quotients. By Remark 2.9 and by (3.1) for every \( t_1, t_2 \in [0,T] \), with \( t_1 \leq t_2 \), we have
\[
\|q_{t_1,t_2}(\mu_{t_1,t_2})\|_* \leq \frac{1}{t_2-t_1} \langle |\xi_2 - \xi_1|, \mu_{t_1,t_2}(x,\xi_1,\xi_2,\eta) \rangle + \langle \eta, \mu_{t_1,t_2}(x,\xi_1,\xi_2,\eta) \rangle.
\]
Let $V : [0, T] \to [0, +\infty)$ be the nondecreasing function defined by
\[
V(t) := \text{Var}(\mu; 0, t).
\]
By (3.2) and (8.1) we conclude that
\[
\|q_{t_1 t_2}(\mu_{t_1 t_2})\|_* \leq \frac{V(t_2) - V(t_1)}{t_2 - t_1} + \lambda(X).
\]
Let $t_0 \in [0, T]$ be a point where the derivative of $V$ exists. By the previous inequality we have that
\[
\|q_{t_0 t}(\mu_{t_0 t})\|_* \quad \text{and} \quad \|q_{t_0 t}(\mu_{t_0 t})\|_*
\]
are bounded uniformly with respect to $t$. By the separability of $C^\text{hom}(X \times \Xi \times \mathbb{R})$ there exists a countable dense subset $\mathcal{F}$ of the set $C^\text{hom}(X \times \Xi \times \mathbb{R})$ introduced in Definition 2.5. Therefore, since $\mathcal{F}$ is dense in $C^\text{hom}(X \times \Xi \times \mathbb{R})$ (see Lemma 2.7), to prove the existence of the weak* derivative of $\mu$ at $t_0$ it is enough to show that
\[
\lim_{t \to t_0^-} (f(x, \xi, \eta), \mu_{t_0 t_0}(x, \xi, \eta)) = \lim_{t \to t_0^+} (f(x, \xi, \eta), \mu_{t_0 t_0}(x, \xi, \eta))
\]
for every $f \in \mathcal{F}$.

Step 2. Some auxiliary functions. In order to prove (9.7), let us fix $f \in \mathcal{F}$ and let $\tau_i$ be a countable dense sequence in $[0, T]$. For every $i$ we define
\[
\phi_i(t) := \begin{cases} 
(f(x, \xi - \zeta_i, (t - \tau_i)\eta), \mu_{t \tau_i}(x, \xi, \zeta_i, \eta)) & \text{if } t < \tau_i, \\
0 & \text{if } t = \tau_i, \\
(f(x, \xi - \zeta_i, (t - \tau_i)\eta), \mu_{t \tau_i}(x, \xi, \zeta_i, \eta)) & \text{if } t > \tau_i.
\end{cases}
\]

Let us prove that $\phi_i$ has bounded variation. Let us fix $t_1, t_2 \in [0, T]$, with $t_1 < t_2$. We consider first the case $t_1 < \tau_i < t_2$. By the compatibility condition (7.2) we have
\[
|\phi_i(t_2) - \phi_i(t_1)| \leq (|f(x, \xi_2 - \zeta_i, (t_2 - \tau_i)\eta) - f(x, \xi_1 - \zeta_i, (t_1 - \tau_i)\eta)|, \mu_{t_1 \tau_i}(x, \xi_1, \zeta_i, \xi_2, \eta)) .
\]
Since, by Remark 2.6,
\[
|f(x, \xi_2 - \zeta_i, (t_2 - \tau_i)\eta) - f(x, \xi_1 - \zeta_i, (t_1 - \tau_i)\eta)| \leq (|\xi_2 - \xi_1| + (t_2 - t_1)|\eta|) \|f\|_\text{hom},
\]
using again (7.2) we obtain
\[
|\phi_i(t_2) - \phi_i(t_1)| \leq (|\xi_2 - \xi_1|, \mu_{t_1 \tau_i}(x, \xi_1, \xi_2, \eta)) + (t_2 - t_1) (|\eta|, \mu_{t_1 \tau_i}(x, \xi_1, \xi_2, \eta)) \|f\|_\text{hom} .
\]
The same inequality can be proved when $\tau_i \leq t_1$ or $\tau_i \geq t_2$. By (3.1), (3.2), (8.1), and (9.6) we conclude that
\[
|\phi_i(t_2) - \phi_i(t_1)| \leq (V(t_2) - V(t_1) + (t_2 - t_1)\lambda(X)) \|f\|_\text{hom} .
\]
We now prove that for every $t_1, t_2 \in [0, T]$, with $t_1 < t_2$, we have
\[
\phi_i(t_2) - \phi_i(t_1) \leq (f(x, \xi_2 - \zeta_i, (t_2 - t_1)\eta), \mu_{t_1 \tau_i}(x, \xi_1, \xi_2, \eta)) .
\]
We consider first the case $t_1 < \tau_i < t_2$. By (7.2) and (9.8) we have
\[
\phi_i(t_2) = (f(x, \xi_2 - \zeta_i + \zeta_i - \zeta_i, (t_2 - t_1 + t_1 - \tau_i)\eta), \mu_{t_1 \tau_i}(x, \zeta_1, \zeta_2, \eta)) .
\]
From the triangle inequality and from (7.2) we get
\[
\phi_i(t_2) \leq (f(x, \xi_2 - \zeta_i, (t_2 - t_1)\eta), \mu_{t_1 \tau_i}(x, \xi_1, \xi_2, \eta)) + (f(x, \xi_1 - \zeta_i, (t_1 - \tau_i)\eta), \mu_{t_1 \tau_i}(x, \xi_1, \zeta_i, \eta)) ,
\]
which gives (9.10) by (9.8). The proof in the cases $\tau_i \leq t_1$ and $\tau_i \geq t_2$ is similar.

Let $W : [0, T] \to \mathbb{R}$ be the increasing function defined by
\[
W(t) := V(t) + t \lambda(X)
\]
and let $\sigma : [0, W(T)] \to [0, T]$ be the nondecreasing function defined by
\[
\sigma(s) := \inf \{t \in [0, T] : W(t) \geq s\} .
\]
It is easy to see that
\[ \sigma(W(t)) = t \quad \text{for every } t \in [0, T]. \] (9.13)
As \( W(t_2) - W(t_1) \geq (t_2 - t_1)\lambda(X) \) for every \( t_1 < t_2 \), we have
\[ 0 \leq \sigma(s_2) - \sigma(s_1) \leq (s_2 - s_1)/\lambda(X) \] (9.14)
for every \( s_1 < s_2 \), hence \( \sigma \) is Lipschitz continuous.

By (9.9) and (9.13) we have
\[ |(\varphi_i^f \circ \sigma)(s_2) - (\varphi_i^f \circ \sigma)(s_1)| \leq |s_2 - s_1| \|f\|_{\text{hom}} \]
for every \( s_1, s_2 \in W([0, T]) \). Therefore, there exists a function \( \psi_i^f : [0, W(T)] \to \mathbb{R} \) such that \( \psi_i^f(s) = (\varphi_i^f \circ \sigma)(s) \) for every \( s \in W([0, T]) \) and
\[ |\psi_i^f(s_2) - \psi_i^f(s_1)| \leq |s_2 - s_1| \|f\|_{\text{hom}} \] (9.15)
for every \( s_1, s_2 \in [0, W(T)] \). For every \( s_0 \in [0, W(T)] \) let
\[ \hat{\psi}_i^f(s_0) = \limsup_{s \to s_0} \frac{\psi_i^f(s) - \psi_i^f(s_0)}{s - s_0}. \]

By (9.15) we have \( |\hat{\psi}_i^f(s_0)| \leq \|f\|_{\text{hom}} \), and by Lebesgue’s Differentiation Theorem the limsup is a limit for a.e. \( s_0 \in [0, W(T)] \). Finally, let \( \omega^f : [0, W(T)] \to \mathbb{R} \) be the function defined by
\[ \omega^f(s) := \sup_i \hat{\psi}_i^f(s). \] (9.16)

By the bound on \( \hat{\psi}_i^f \) we have
\[ |\omega^f(s)| \leq \|f\|_{\text{hom}} \] (9.17)
for every \( s \in [0, W(T)] \).

\textit{Step 3. The exceptional set.} Let \( \mathcal{L}^1 \) be the Lebesgue measure on \( \mathbb{R} \). By (9.14) and (9.15) there exists a measurable set \( N \subset [0, W(T)] \), with \( \mathcal{L}^1(N) = 0 \), such that each point of \( [0, W(T)] \setminus N \) is a Lebesgue point of \( \omega^f \) for every \( f \in \mathcal{F} \) and a differentiability point for \( \psi_i^f \) for every \( f \in \mathcal{F} \) and for every \( i \). Let \( N_W \) be the set of points of \( [0, T] \) where the derivative \( \dot{W} \) of \( W \) does not exist. By Lebesgue’s Differentiation Theorem we have \( \mathcal{L}^1(N_W) = 0 \). Since \( \sigma \) is Lipschitz continuous and \( W^{-1}(N) = \sigma(N \cap W([-T, T])) \) by (9.13), we have that \( \mathcal{L}^1(W^{-1}(N)) = 0 \), hence
\[ \mathcal{L}^1(N_W \cup W^{-1}(N)) = 0. \] (9.18)

\textit{Step 4. Estimate from below.} Let us fix \( t_0 \notin N_W \cup W^{-1}(N) \), with \( 0 < t_0 < T \), and let \( s_0 = W(t_0) \). As \( \varphi_i^f(t) = \psi_i^f(W(t)) \), from (9.10) we obtain
\[ \psi_i^f(W(t_2)) - \psi_i^f(W(t_1)) \leq \langle f(x, \xi_2 - \xi_1, t_2 - t_1), \mu_{i(t_2)}(x, \xi_1, \xi_2, \eta) \rangle \]
for every \( t_1, t_2 \in [0, T] \) with \( t_1 < t_2 \). This implies
\[ \hat{\psi}_i^f(W(t_0)) \dot{W}(t_0) \leq \liminf_{t \to t_0} \langle f(x, \xi, \eta, \mu_{i(t)}(x, \xi_0, \eta) \rangle, \]
\[ \hat{\psi}_i^f(W(t_0)) \dot{W}(t_0) \leq \liminf_{t \to t_0} \langle f(x, \xi, \eta, \mu_{i(t)}(x, \xi_0, \eta) \rangle \]
for every \( i \), which by (9.16) gives
\[ \omega^f(W(t_0)) \dot{W}(t_0) \leq \liminf_{t \to t_0} \langle f(x, \xi, \eta, \mu_{i(t)}(x, \xi_0, \eta) \rangle, \] (9.19)
\[ \omega^f(W(t_0)) \dot{W}(t_0) \leq \liminf_{t \to t_0} \langle f(x, \xi, \eta, \mu_{i(t)}(x, \xi_0, \eta) \rangle. \] (9.20)
Step 5. Estimate from above. To prove the opposite inequality we show that

\[ \langle f(x, \xi_2 - \eta_1, (t_2 - T_1)\eta), \mu_{t_1, t_2}(x, \xi_1, \xi_2, \eta) \rangle \leq \int_{W(t_1)}^{W(t_2)} \omega^f(s) \, ds \]  

(9.21)

for every \( t_1, t_2 \), with \( 0 < t_1 < t_2 < T \), such that \( W \) is continuous at \( t_1 \) or \( t_2 \). We prove (9.21) only when \( W \) is continuous at \( t_1 \), the other case being analogous. For every \( \varepsilon > 0 \) there exists \( \varepsilon \) such that \( \tau_1 < t_1 \) and

\[ W(t_1) - W(\tau_1) < \varepsilon. \]  

(9.22)

As \( \psi^f \) is Lipschitz, from the compatibility condition (7.2) we get

\[ \int_{W(t_1)}^{W(t_2)} \omega^f(s) \, ds \geq \int_{W(t_1)}^{W(t_2)} \psi^f(s) \, ds = \psi^f(W(t_2)) - \psi^f(W(t_1)) = \]  

(9.23)

\[ = \varphi^f(t_2) - \varphi^f(t_1) = \langle f(x, \xi_2 - \zeta_1, (t_2 - T_1)\eta), \mu_{t_1, t_2}(x, \zeta_1, \xi_2, \eta) \rangle - \]  

\[ - \langle f(x, \xi_1 - \zeta_1, (t_1 - T_1)\eta), \mu_{t_1, t_2}(x, \zeta_1, \xi_2, \eta) \rangle. \]

Using Remark 2.6 we obtain

\[ f(x, \xi_2 - \zeta_1, \xi_1, (t_2 - T_1)\eta) \geq f(x, \xi_2 - \zeta_1, (t_2 - T_1)\eta) - \langle |\zeta_1 - \zeta_1|, \mu_{t_1, t_2}(x) \rangle \| f \|_{\text{hom}}, \]

\[ - f(x, \xi_1 - \zeta_1, (t_1 - T_1)\eta) \geq - \langle |\zeta_1 - \zeta_1|, \mu_{t_1, t_2}(x) \rangle \| f \|_{\text{hom}}, \]

so that, using again (3.2) and (7.2), inequality (9.23) and the definition of \( W \) give

\[ \int_{W(t_1)}^{W(t_2)} \omega^f(s) \, ds \geq \langle f(x, \xi_2 - \zeta_1, (t_2 - T_1)\eta), \mu_{t_1, t_2}(x, \zeta_1, \xi_2, \eta) \rangle - 2(W(t_1) - W(T_1)) \| f \|_{\text{hom}}. \]

By (9.22) we conclude that

\[ \int_{W(t_1)}^{W(t_2)} \omega^f(s) \, ds \geq \langle f(x, \xi_2 - \zeta_1, (t_2 - T_1)\eta), \mu_{t_1, t_2}(x, \zeta_1, \xi_2, \eta) \rangle - 2\varepsilon \| f \|_{\text{hom}}. \]

As \( \varepsilon > 0 \) is arbitrary, this proves (9.21).

Since \( W \) is differentiable at \( t_0 \) and \( W(t_0) \) is a Lebesgue point of \( \omega^f \), inequality (9.21) implies

\[ \limsup_{t \to t_0} \langle f(x, \xi_2 - \xi_0, \eta), \mu_{t_0}(x, \xi_0, \xi_2, \eta) \rangle \leq \omega^f(W(t_0)) \hat{W}(t_0), \]

\[ \limsup_{t \to t_0} \langle f(x, \xi_2 - \xi_0, \eta), \mu_{t_0}(x, \xi_0, \xi_2, \eta) \rangle \leq \omega^f(W(t_0)) \hat{W}(t_0), \]

which, together with (9.19) and (9.20), give

\[ \lim_{t \to t_0} \langle f(x, \xi_2 - \xi_0, \eta), \mu_{t_0}(x, \xi_0, \xi_2, \eta) \rangle = \omega^f(W(t_0)) \hat{W}(t_0), \]

\[ \lim_{t \to t_0} \langle f(x, \xi_2 - \xi_0, \eta), \mu_{t_0}(x, \xi_0, \xi_2, \eta) \rangle = \omega^f(W(t_0)) \hat{W}(t_0). \]

By (9.18) this proves (9.7) and concludes the proof of the existence of the weak* derivative \( \mu_{t_0} \) for a.e. \( t_0 \in [0, T] \). Moreover it shows that

\[ \langle f(x, \xi, \eta), \mu_{t_0}(x, \xi, \eta) \rangle = \omega^f(W(t_0)) \hat{W}(t_0) \]  

(9.24)

for every \( f \in \mathcal{F} \) and for a.e. \( t_0 \in [0, T] \).

Step 6. Integrability of \( t \mapsto \langle f, \mu_t \rangle \). To prove the measurability of this function for every \( f \in C_{\text{hom}}^0(X \times \Xi \times \mathbb{R}) \), we fix a sequence \( \varepsilon_k \) of positive numbers converging to 0 and a function \( f \in C_{\text{hom}}^0(X \times \Xi \times \mathbb{R}) \). By Lemma 8.9 the function

\[ t \mapsto \langle f(x, \xi_2 - \xi_0, \eta), \mu_{t_0}(x, \xi_0, \xi_2, \eta) \rangle \]
is measurable on \([0, T - \varepsilon_k]\). Since it converges to \(t \mapsto \langle f, \hat{\mu}_t \rangle\) for a.e. \(t \in [0, T]\), we conclude that this function is measurable on \([0, T]\). The same property can be proved for an arbitrary \(f \in C^{hom}(X \times \Xi \times \mathbb{R})\) by approximation, thanks to Lemma 2.4.

By (9.17) and (9.24) we have
\[
|\langle f, \hat{\mu}_t \rangle| \leq \hat{W}(t) \|f\|_{\text{hom}}
\]
for every \(f \in \mathcal{F}\). The same inequality holds for any \(f \in C^{hom}(X \times \Xi \times \mathbb{R})\) by the density of \(\mathcal{F}\) (see Lemma 2.7). Since \(\hat{W}\) is integrable, this concludes the proof of the integrability of \(t \mapsto \langle f, \hat{\mu}_t \rangle\) on \([0, T]\).

**Step 7. Estimate for \(\text{Var}_h(\mu; a, b)\).** Let \(h: \Xi \to [0, +\infty)\) be a positively one-homogeneous function satisfying the triangle inequality. Since the function \(t \mapsto \text{Var}_h(\mu; a, t)\) is non-decreasing on \([a, b]\), by Lebesgue Differentiation Theorem it is differentiable for a.e. \(t \in [a, b]\) and
\[
\int_a^b \frac{d}{dt} \text{Var}_h(\mu; a, t) \, dt \leq \text{Var}_h(\mu; a, b).
\]  
(9.25)

Let \(t_0 \in (a, b)\) be a point where \(t \mapsto \text{Var}_h(\mu; a, t)\) is differentiable and the weak* derivative \(\mu_{t_0}\) exists. By the definition of \(\text{Var}_h\) for every \(t \in (t_0, b)\) we have
\[
\text{Var}_h(\mu; a, t_0) + \langle h(\xi - \xi_0), \mu_{t_0, t}(x, \xi_0, \xi, \eta) \rangle \leq \text{Var}_h(\mu; a, t).
\]
Since \(h\) is positively homogeneous of degree one, we obtain
\[
\langle h(\xi - \xi_0), \mu_{t_0, t}(x, \xi_0, \xi, \eta) \rangle \leq \frac{\text{Var}_h(\mu; a, t) - \text{Var}_h(\mu; a, t_0)}{t - t_0}.
\]

From (9.3) we deduce that
\[
\langle h(\xi), \mu_{t_0}(x, \xi, \eta) \rangle \leq \left. \frac{d}{dt} \text{Var}_h(\mu; a, t) \right|_{t = t_0},
\]

Since this inequality holds for a.e. \(t_0 \in [a, b]\), from (9.25) we obtain
\[
\int_a^b \langle h(\xi), \hat{\mu}_t(x, \xi, \eta) \rangle \, dt \leq \text{Var}_h(\mu; a, b),
\]
which concludes the proof of (9.5).

\[\square\]

10. **Absolute continuity**

In this section we introduce the notion of absolutely continuous system of generalized Young measures on the time interval \([0, T]\), with \(T > 0\), and prove that for these systems the \(h\)-variation can be computed using the weak* derivative by the formula
\[
\text{Var}_h(\mu; a, b) = \int_a^b \langle h(\xi), \mu_t(x, \xi, \eta) \rangle \, dt
\]  
(10.1)

for every \(a, b \in [0, T]\), with \(a < b\).

**Definition 10.1.** We say that a compatible system of generalized Young measures \(\mu \in \text{SGY}([0, T], X; \Xi)\) is absolutely continuous on \([0, T]\) if for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that
\[
\sum_{i=1}^k \langle |\xi_2 - \xi_1|, \mu_{a_i,b_i}(x, \xi_1, \xi_2, \eta) \rangle \leq \varepsilon
\]  
(10.2)

for every finite family \((a_1, b_1), \ldots, (a_k, b_k)\) of nonoverlapping open intervals in \([0, T]\) with
\[
\sum_{i=1}^k (b_i - a_i) \leq \delta.
\]
Remark 10.2. It follows from Definition 3.2 that, if $\mu$ is the compatible system of generalized Young measures associated with a function $t \mapsto p(t)$ from $[0,T]$ into $M_b(X;\Xi)$ according to (7.1), then $\mu$ is absolutely continuous on $[0,T]$ if and only if $t \mapsto p(t)$ is absolutely continuous on $[0,T]$ in the usual sense of functions with values in a Banach space.

If $t \mapsto u(t)$ is an absolutely continuous function from $[0,T]$ into $L^r(X;\Xi)$ for some $r > 1$, then the derivative $u'(t)$, defined as the strong $L^r$ limit of the difference quotients, exists at a.e. $t \in [0,T]$ (see, e.g., [5, Appendix]). By Remark 9.5 it follows that, if $\mu$ is the compatible system of generalized Young measures associated with $t \mapsto u(t)$ according to (7.1), then $\hat{\mu}_t = \delta_{\hat{u}(t)}$ for a.e. $t \in [0,T]$, and (10.1) follows from the classical theory (see, e.g., [5, Appendix]).

If $t \mapsto p(t)$ is an absolutely continuous function with values in $M_b(X;\Xi)$, then the derivative $\hat{p}(t)$, defined as the weak* limit of the difference quotients, exists at a.e. $t \in [0,T]$ (see [9, Appendix]). This is not enough to guarantee that $\hat{\mu}_t = \delta_{\hat{p}(t)}$ for a.e. $t \in [0,T]$ when $\mu$ is the compatible system of generalized Young measure associated with $t \mapsto p(t)$ (see Remark 9.5). Therefore, in this case (10.1) cannot be obtained directly from known results.

Remark 10.3. As in the classical case, it is easy to see that, if $\mu \in SGY([0,T],X;\Xi)$ is absolutely continuous on $[0,T]$, then $\text{Var}(\mu;[0,T]) < +\infty$. In this case, if $V : [0,T] \to [0, +\infty)$ is the nondecreasing function defined by

$$V(t) := \text{Var}(\mu;0,t),$$

then for every $\varepsilon > 0$

$$\sum_{i=1}^{k} (V(b_i) - V(a_i)) \leq \varepsilon$$

for every finite family $(a_1, b_1), \ldots, (a_k, b_k)$ of nonoverlapping open intervals in $[0,T]$ with

$$\sum_{i=1}^{k} (b_i - a_i) \leq \delta,$$

where $\delta$ is the constant in the definition of the absolute continuity of $\mu$. In particular, $V$ is absolutely continuous on $[0,T]$.

Theorem 10.4. Suppose that $\mu \in SGY([0,T],X;\Xi)$ is absolutely continuous on $[0,T]$ and that $h : \Xi \to [0, +\infty)$ is positively one-homogeneous and satisfies the triangle inequality. Then

$$\text{Var}_h(\mu; a, b) = \int_{a}^{b} \langle h(\xi), \hat{\mu}_t(x,\xi,\eta) \rangle \, dt$$

for every $a, b \in [0,T]$ with $a \leq b$.

Proof. Let $W$ be defined by (9.12). By Remark 10.3 $W$ is absolutely continuous on $[0,T]$. By Remark 8.7 the function $f(x,\xi,\eta) := h(\xi)$ belongs to $C_{\text{hom}}(X \times \Xi \times \mathbb{R})$. Therefore, we can add this function to the set $\mathcal{F}$ introduced in Step 1 of the proof of Theorem 9.7 and we can consider the corresponding function $\omega^h : [0,W(T)] \to \mathbb{R}$ defined by (9.16). By (9.21) and (9.24) we have

$$\langle h(\xi_2 - \xi_1), \mu_t(x,\xi_1,\xi_2,\eta) \rangle \leq \int_{W(t_1)}^{W(t_2)} \omega^h(s) \, ds, \quad (10.3)$$

$$\langle h(\xi), \hat{\mu}_t(x,\xi,\eta) \rangle = \omega^h(W(t)) W(t) \quad (10.4)$$

for every $t_1, t_2 \in [0,T]$, with $t_1 < t_2$, and for a.e. $t \in [0,T]$.

By the definition of $\text{Var}_h(\mu; a, b)$, inequality (10.3) implies that

$$\text{Var}_h(\mu; a, b) \leq \int_{W(a)}^{W(b)} \omega^h(s) \, ds \quad (10.5)$$
for every \(a, b \in [0, T]\), with \(a \leq b\). On the other hand, since \(W\) is absolutely continuous on \([0, T]\), we have
\[
\int_{W(a)}^{W(b)} \omega^h(s) \, ds = \int_a^b \omega^h(W(t)) W'(t) \, dt = \int_a^b \langle h(\xi), \dot{\mu}_t(x, \xi, \eta) \rangle \, dt,
\]
where the last equality follows from (10.4). The conclusion follows now from (9.5), (10.5), and (10.6).

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(G. Dal Maso, A. DeSimone, M.G. Mora, and M. Morini) SISSA, Via Beirut 4, 34014 Trieste, Italy
E-mail address, Gianni Dal Maso: dalmaso@sissa.it
E-mail address, Antonio DeSimone: desimone@sissa.it
E-mail address, Maria Giovanna Mora: mora@sissa.it
E-mail address, Massimiliano Morini: morini@sissa.it