ARTICLE TYPE

Declarative Programming with Intensional Sets in Java Using JSetL

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Abstract

Intensional sets are sets given by a property rather than by enumerating their elements. In previous work, we have proposed a decision procedure for a first-order logic language which provides Restricted Intensional Sets (RIS), i.e., a sub-class of intensional sets that are guaranteed to denote finite—though unbounded—sets. In this paper we show how RIS can be exploited as a convenient programming tool also in a conventional setting, namely, the imperative O-O language Java. We do this by considering a Java library, called JSetL, that integrates the notions of logical variable, (set) unification and constraints that are typical of constraint logic programming languages into the Java language. We show how JSetL is naturally extended to accommodate for RIS and RIS constraints, and how this extension can be exploited, on the one hand, to support a more declarative style of programming and, on the other hand, to effectively enhance the expressive power of the constraint language provided by the library.

KEYWORDS:
Java; JSetL; declarative programming; set programming; logic programming; constraint programming; set theory; set constraint

1 | INTRODUCTION AND MOTIVATIONS

Set-oriented programming is a programming paradigm where the well-known mathematical notion of set plays a fundamental role in providing high-level (declarative) descriptions of problem solutions. This approach is well exemplified by specification languages such as Z and B and programming languages such as SETL. Set-oriented programming is also supported to some extent by other general-purpose programming languages, such as Claire, Miranda and Bandicoot. The main goal of set-oriented programming is to support rapid software prototyping. However, a set-oriented approach can be of great help also in other software development activities, such as program verification, since it provides a valuable tool for the development of correct-by-construction prototypes.

In the nineties many proposals for set-oriented programming have emerged in the field of declarative programming, where sets and operations on sets are added to a first-order logic language as first-class entities of the language—e.g., CLP(SET). Efforts in this direction are well attested by the two workshops on Logic/Declarative Programming with Sets held in the nineties. Among that proposals, CLP(SET) is particularly linked to our work. In effect, CLP(SET) is a constraint logic programming (CLP) language whose constraint domain is that of hereditarily finite sets, i.e., finitely nested sets that are finite at each level of nesting. CLP(SET) allows to operate with partially specified sets through a number of primitive constraints representing
all the most commonly used set-theoretic operations (e.g., union, intersection, difference). A complete constraint solver for this language is provided, capable of deciding the satisfiability of arbitrary conjunctions of primitive constraints.

These ideas and results have been implemented also in the context of more conventional programming languages, namely Java, in the form of the JSetL library\(^1\). JSetL implements the notions of logical variable, (set) unification and constraints, that are typical of constraint logic programming languages, into the Java language. JSetL supports \textit{declarative constraint programming with sets}. This style of programming is illustrated\(^3,17,18\) by a number of sample programs showing how JSetL facilities, such as partially specified sets, set unification, non-determinism, can be of great help in devising and implementing high-level declarative solutions for many, possibly complex, problems.

In the practice of mathematics, however, it is common to distinguish between sets designated via explicit enumeration (\textit{extensional sets})—e.g., \{a, b, c\}—and sets described through the use of properties and/or characteristic functions (\textit{intensional sets})—e.g., \{x : \varphi(x)\}. Intensional sets are widely recognized as a key feature to describe complex problems. Having the possibility to represent and manipulate intensional sets could constitute a valuable improvement in the expressive power of a programming language. Notwithstanding, very few programming languages provide support for intensional sets—e.g., SETL\(^1\) and Python. In these proposals, however, set operators are applied only to intensional sets that denote completely specified sets, i.e., sets where all elements have a known value. Moreover, often these proposals are mostly concerned with aggregate operations (e.g., finding the maximum or the minimum or the sum of all elements of the set), and place limited attention to other basic set-theoretical facilities. Ultimately, in all these proposals the language does not provide real direct support for \textit{reasoning} about intensional sets. Conversely, languages/libraries supporting declarative programming with sets, such as CLP(\textit{SET}) and JSetL, provide general set-theoretic operations on \textit{extensional sets}, usually in the form of (set) \textit{constraints}, that allow set objects to be manipulated even if they are represented by variables or they are only partially specified.

In previous works\(^19,2\), we have extended a first-order logic language providing extensional sets to support also intensional sets. This is done by introducing them as first-class entities of the language and providing operations on them as primitive constraints (i.e., \textit{intensional set constraints}). The proposed constraint solver is able to deal with intensional sets \textit{without} explicitly enumerating all their elements. In particular, \textit{Restricted Intensional Sets (RIS)}\(^3\) have proved to be much more effective from a programming viewpoint than general intensional sets.

RIS have similar syntax and semantics to the set comprehensions available in the Z formal specification language, i.e. \{c : D | F \cdot p(c)\}, where \(D\) is a finite set, \(F\) is a quantifier-free formula over a first-order theory \(\mathcal{X}\), and \(c\) and \(p\) are \(\mathcal{X}\)-terms. Intuitively, the semantics of \{c : D | F \cdot p(c)\} is “the set of terms \(p(c)\) such that \(c\) is in \(D\) and \(F\) holds”. We say that this class of intensional sets is \textit{restricted} because they denote \textit{finite} sets, while in Z they can be infinite. The finiteness of \(D\), along with a few restrictions on variables occurring in \(F\) and \(p\), guarantees that the RIS is a finite set, given that its cardinality is as large as \(D\)’s. Nonetheless, RIS can be not completely specified. In particular, as the domain can be a variable, RIS are finite but \textit{unbounded}.

The work in\(^3\), however, is mainly concerned with the definition of the constraint (logic) language—called \(\mathcal{L}_{\text{RIS}}\)—and its solver, and the proof of soundness and completeness of the constraint solving procedure. In this paper, our main aim is to explore \textit{programming} with (restricted) intensional sets. Specifically, we are interested in exploring the potential of using RIS in the more conventional setting of imperative O-O languages. To this purpose, we consider the Java library JSetL. First, we show how JSetL is naturally extended to accommodate for RIS. Then, we show with a number of simple examples how this extension can be exploited, on the one hand, to support a more declarative style of programming and, on the other hand, to effectively enhance the expressive power of the constraint language provided by the library. It is worth noting that although we are focusing on Java, the same considerations can be easily ported to other O-O languages, such as C++.

Our claim is that the language of RIS constraints is expressive enough to allow usual programming solutions to be encoded as formulas in that language. In particular, the fact that RIS can also be \textit{recursively} defined, together with the fact that ordered pairs can be set elements, makes it possible to define most of the classic recursive functions in a set-oriented fashion. That is, in our language, functions are sets of ordered pairs that are managed by means of classic set theoretic operators such as equality and membership.

The paper is organized as follows. Section\(^2\) introduces the theoretical framework underlying RIS. Section\(^3\) briefly reviews the JSetL library, while Section\(^4\) presents the extension of JSetL with RIS. In Section\(^5\) we start showing examples using JSetL to demonstrate the usefulness of RIS and RIS constraints to support declarative programming; in particular, Section\(^5.2\) shows how RIS can be used to define and manipulate partial functions. In Section\(^6\) we consider some extensions to RIS and we present examples showing their usefulness. In Section\(^7\) we provide some general considerations on the practical usability of RIS and set-oriented programming, in general. A comparison of our approach with other works and our conclusions are presented in Sections\(^8\) and\(^9\) respectively.
2 | A THEORY OF RIS

In this section we introduce the theoretical framework underlying the JSetL library, with special reference to the support it offers to RIS.

The language that embodies RIS, called \( \mathcal{L}_{RIS} \), is a quantifier-free first-order logic language which provides both RIS and extensional sets, along with basic operations on them, as primitive entities of the language. \( \mathcal{L}_{RIS} \) is parametric with respect to an arbitrary theory \( \mathcal{X} \), for which a decision procedure for any admissible \( \mathcal{X} \)-formula is assumed to be available. Elements of \( \mathcal{L}_{RIS} \) sets are the objects provided by \( \mathcal{X} \), which can be manipulated through the primitive operators that \( \mathcal{X} \) offers. The \( \mathcal{X} \) language, called \( \mathcal{L}_{\mathcal{X}} \), is assumed to provide at least, equality (\( =_{\mathcal{X}} \)) and inequality (\( \neq_{\mathcal{X}} \)), a not empty collection of constants, \( a_{1}, a_{2}, \ldots \), and a binary function symbol to represent ordered pairs, e.g., \( (a_{1}, a_{2}) \).

Besides, the function and predicate symbols provided by \( \mathcal{X} \), \( \mathcal{L}_{RIS} \) provides special set constructors and a handful of reserved predicate symbols endowed with a pre-designated set-theoretic meaning. Set constructors are used to build set terms.

**Definition 1** (Set terms). A set term is any \( \mathcal{L}_{RIS} \) term of one of the following forms:

- \( \emptyset \) (empty set);
- \( \{x/A\} \) (extensional set term), where \( x \), called element part, is an \( \mathcal{X} \)-term, and \( A \), called set part, is a set term;
- \( \{c : D \mid F \cdot p(c)\} \) (RIS term), where \( c \), called control term, is an \( \mathcal{X} \)-term; \( D \), called domain, is a set term; \( F \), called filter, is an \( \mathcal{X} \)-formula; and \( p \), called pattern, is an \( \mathcal{X} \)-term containing \( c \);
- any variable belonging to a denumerable set of variables \( \mathcal{V}_{S} \) (set variables).

Intuitively, an extensional set term \( \{x/A\} \) is interpreted as \( \{x\} \cup A \). A RIS term is interpreted as follows: if \( x_{1}, \ldots, x_{n} \) \( (n > 0) \) are all the variables occurring in \( c \), then:

\[
\{c : D \mid F \cdot p(c)\}
\]

denotes the set:

\[
\{y \mid \exists x_{1}, \ldots, x_{n}(c \in D \land F \land y =_{\mathcal{X}} p(c))\}
\]

where \( x_{1}, \ldots, x_{n} \) are bound variables whose scope is the RIS term itself. Hence, \( \mathcal{L}_{RIS} \) set terms represent untyped unbounded finite hybrid sets, i.e., unbounded finite sets whose elements are of arbitrary sorts.

**Remark 1.** Notation. As a notational convenience, \( \{t_{1}/\{t_{2}/\cdots\{t_{n}/\emptyset\}\cdots\}\} \) (resp., \( \{t_{1}/\{t_{2}/\cdots\{t_{n}/\emptyset\}\cdots\}\} \)), where \( t_{1}, \ldots, t_{n} \), \( n \geq 1 \), are \( \mathcal{X} \)-terms, is written as \( \{t_{1}, t_{2}, \ldots, t_{n}/t\} \) (resp., \( \{t_{1}, t_{2}, \ldots, t_{n}/t\} \)). When useful, the domain of a RIS can be represented also as an interval \( [m, n] \), \( m \) and \( n \) integer constants, which is intended as a shorthand for \( \{m, m+1, \ldots, n\} \). When the pattern is the same as the control term, the former can be omitted (as in \( \mathcal{Z} \)). Furthermore, the following name conventions will be used throughout the paper: \( A, B, C, D, E, S \) stand for arbitrary sets (either variable or not), while \( X, Y, Z, N \) stand for variable sets; \( R \) stands for a RIS; and \( x, y, z, n \) are variables representing set elements.

It is important to observe that elements and sets in both extensional set terms and RIS terms can be variables.

**Definition 2.** If \( s \) is a set term, we say that \( s \) denotes a partially specified set if either \( s \) is a variable; or \( s \) is \( \{t_{1}, \ldots, t_{n}\} \) and at least one \( t_{i} \) is a variable or a term containing a variable; or \( s \) is \( \{t_{1}, \ldots, t_{n}/t\} \) and \( t \) is a variable; or \( s \) is a RIS term and its domain is a set term denoting a partially specified set.

The following are simple examples of RIS terms.

**Example 1.** Assume that \( \mathcal{L}_{\mathcal{X}} \) provides the constant, function and predicate symbols of the theory of the integer numbers. Let \( x, y \) and \( z \) be \( \mathcal{X} \)-variables (i.e., variables ranging over the domain of \( \mathcal{X} \)) and let \( D \) and \( S \) be set variables. The following are RIS terms:

1. \( \{x : [-2, 2] \mid x \mod 2 = 0 \cdot x\} \) (also written as \( \{x : [-2, 2] \mid x \mod 2 = 0\} \))
2. \( \{x : D \mid x > 0 \cdot (x, x \ast x)\} \), where \( D \) is a free variable in the RIS
3. \( \{(x, y) : \{z/A\} \mid y \neq 0 \cdot (x, y)\} \), where \( z \) and \( A \) are free variables.

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\(^{1}\)When \( c \) is the variable \( x \), \( \{c : D \mid F \cdot p\} \) is usually written in mathematics as \( \{p : D \mid F\} \).
Definition 3 ($\mathcal{L}_{\text{RIS}}$ constraints). A (primitive) $\mathcal{L}_{\text{RIS}}$ constraint is any $\mathcal{L}_{\text{RIS}}$ atom of one of the following forms: $A = B$, $A \neq B$, $e \in A$, $e \notin A$, un($A, B, C$) and disj($A, B$), where $A, B$ and $C$ are $\mathcal{L}_{\text{RIS}}$ set terms and $e$ is an $X$-term.

The intuitive meaning of the $\mathcal{L}_{\text{RIS}}$ constraints is: $A = B$ (resp., $A \neq B$) represents set equality (resp., inequality) between the sets denoted by $A$ and $B$; $e \in A$ (resp., $e \notin A$) represents set membership (resp., not membership); un($A, B, C$) represents set union, i.e., $C = A \cup B$; and disj($A, B$) represents set disjunction, i.e., $A \cap B = \emptyset$.

$\mathcal{L}_{\text{RIS}}$ formulas are built from $\mathcal{L}_{\text{RIS}}$ constraints using conjunction and disjunction in the usual way.

The collection of predicate symbols used for the primitive constraints turns out to be sufficient to define constraints implementing other common set operators. Specifically, the following atoms are provided by $\mathcal{L}_{\text{RIS}}$ as defined formulas: $A \subseteq B$ (interpreted as set inclusion), inters($A, B, C$) (interpreted as $C = A \cap B$), diff($A, B, C$) (interpreted as $C = A \setminus B$). As an example, $A \subseteq B$ is defined by the $\mathcal{L}_{\text{RIS}}$ formula un($A, B, B$). We will refer to these atoms as derived constraints. Whenever a formula contains a derived constraint, the constraint is replaced by its definition turning the given formula into a $\mathcal{L}_{\text{RIS}}$ formula.

The same approach is used to introduce the negative counterparts of set operators not defined as primitive constraints. Specifically, derived constraints are introduced for $\neg \cup$ and $\neg \parallel$ (called nun and ndisj, respectively), as well as for $\neg \subseteq$, $\neg \cap$ and $\neg \setminus$ (called $\notin$, ninters and ndiff, respectively). Observe that, thanks to the availability of the negated versions of set operators as derived constraints (hence, as positive atoms), classical negation is not necessary in $\mathcal{L}_{\text{RIS}}$.

$\mathcal{L}_{\text{RIS}}$ provides a complete constraint solver, for a large fragment of its input language. This solver is able to decide the satisfiability of any admissible $\mathcal{L}_{\text{RIS}}$ formula. Intuitively, non-admissible formulas are those where a variable $X$ is the domain of a RIS representing a function and, at the same, time $A$ is either a sub or a superset of that function. For example, the $\mathcal{L}_{\text{RIS}}$ formula \( \{ x : D \mid \text{true} \ast (x, y) \} \subseteq D \land D \neq \emptyset \) is non-admissible since it implies that if $z \in D$ then $(z, n_1) \in D$ and so forth, thus generating an infinite $X$-term. In the rest of this paper, we will restrict our attention to admissible $\mathcal{L}_{\text{RIS}}$ formulas.

The $\mathcal{L}_{\text{RIS}}$ constraint solver reduces any input formula $\Phi$ to either false (hence, $\Phi$ is unsatisfiable), or to an equi-satisfiable disjunction of formulas in a simplified form, called the solved form, which is guaranteed to be satisfiable (hence, $\Phi$ is satisfiable). If $\Phi$ is satisfiable, the answer computed by the solver constitutes a finite representation of all the concrete (or ground) solutions of the input formula.

The following examples show simple RIS-formulas involving RIS and their processing by the $\mathcal{L}_{\text{RIS}}$ constraint solver.

Example 2 ($\mathcal{L}_{\text{RIS}}$ constraint solving).

i. The $\mathcal{L}_{\text{RIS}}$ constraint $(5, y) \in \{ x : D \mid x > 0 \ast (x, x \ast x) \}$ is rewritten by the solver to the solved form formula $y = 25 \land D = \{5 / N_1\}$, where the second equality states that $D$ must contain 5 and something else, denoted $N_1$.

ii. The formula $S = \{2, 4, 6\} \land S = \{x : D \mid x \text{ mod } 2 = 0\}$ is rewritten by the solver to a solved form formula containing the constraint $D = \{2, 4, 6 / N_1\} \land \{x : N_1 \mid x \text{ mod } 2 = 0\} = \emptyset$, where the second equality states that $N_1$ cannot contain even numbers (note that this constraint has the obvious solution $N_1 = \emptyset$).

iii. The formula $A = \{x : D \mid x \neq 0\} \land \text{un}(A, B, C) \land A \mid C \land A \neq \emptyset$ is rewritten by the solver to false (as a matter of fact, un($A, B, C) \land A \mid C$ is satisfiable only if $A = \emptyset$); hence, the formula is unsatisfiable.

In order to allow the solver to act as a decision procedure for a large part of its input language, the control term $c$ and the pattern $p$ of a RIS are restricted to be of specific forms. All RIS shown above, meet these restrictions.

Definition 4 (Admissible control terms and patterns). If $x$ and $y$ are $X$-variables, then an admissible control term $c$ is either $x$ or $(x, y)$, while an admissible pattern $p$ is either $c$ or $(c, s)$, where $s$ is any $X$-term, possibly involving the variables in $c$.

As it will be evident in Section 6, these restrictions could be often relaxed in practice.

2.1 An instance of $\mathcal{L}_{\text{RIS}}$

$\mathcal{L}_{\text{RIS}}$ is parametric with respect to the theory $X$. In the rest of this paper we will consider a specific instance of $\mathcal{L}_{\text{RIS}}$, indicated as $\mathcal{L}_{\text{RIS}}(\mathcal{S}ET)$, where $X$ is the theory $\mathcal{S}ET$.

$\mathcal{S}ET$ is basically the theory of hereditarily finite sets defined in [], augmented with the theory underlying CLP(FD), that is integer arithmetic over finite domains. The constraint language of this theory, here simply called $\mathcal{L}_{\mathcal{S}ET}$, provides the same function symbols as $\mathcal{L}_{\text{RIS}}$ for building extensional set terms (namely, $\emptyset$ and $\{ \cdot | \cdot \}$), along with a collection of predicate
symbols including those of $\mathcal{L}_{\text{RIS}}$, with the same interpretation. In addition, $\mathcal{L}_{\text{SET}}$ provides the usual function symbols representing operations over integer numbers (e.g., $+, -, \text{mod}$, etc.), as well as the predicate symbols $\text{size}$, representing set cardinality, and $\leq$, representing the order relation on the integers. One notable difference w.r.t. $\mathcal{L}_{\text{RIS}}$ is that set elements can be either finite sets or non-set elements of any sort (i.e., nested sets are allowed).

The theory $\mathcal{SET}$ is endowed with a constraint solver that combines the (set) constraint solving technique of $\mathcal{L}_{\text{RIS}}$ with those of CLP(FD)\(^{21}\), namely integer constraint solving over finite domains. The constraint solver for $\mathcal{SET}$ is proved to be a decision procedure for its formulas, provided a finite domain is associated to each integer variable occurring in the input formula.

**Example 3** ($\mathcal{L}_{\text{RIS}}(\mathcal{SET})$ formulas). The following formula written in $\mathcal{L}_{\text{RIS}}(\mathcal{SET})$ states the equality between an extensional set and a RIS computing all the subsets of cardinality 2 of a given set and:

$$\{X : \{|\{1, 3\}, \{2\}, \{1\}\} | \text{size}(X, 2) = \{|1, 3\}\}$$

This formula is (correctly) proved by the $\mathcal{L}_{\text{RIS}}(\mathcal{SET})$ solver to be true.\(\square\)

Note that we are using the same external notation for both $\mathcal{L}_{\text{RIS}}$ set terms and $\mathcal{L}_{\text{SET}}$ set terms. However, which kind of set terms we are actually referring to is automatically inferred from the context where the terms occur.

## 3 | AN INFORMAL INTRODUCTION TO JSETL

In this section, we introduce JSetL, a Java library that supports declarative (constraint) programming in an O-O framework\(^{3}\). To this end, JSetL combines the object-oriented programming paradigm of Java with valuable concepts of CLP languages, such as logical variables, unification, constraint solving and non-determinism.

Specifically, JSetL implements $\mathcal{L}_{\text{SET}}$ in Java. As such it provides, among others, very general forms of (possibly, partially specified) extensional sets, along with most of the usual set-theoretical operators (e.g., set equality, membership, union, inclusion, etc.) as constraints. The use of sets and set constraints has proved to be a powerful tool to support a real declarative programming style even in an O-O programming context\(^{13,17,18}\).

The fundamental data abstraction to support declarative constraint programming in JSetL is that of *logical object*. Basically, logical objects occur in the form of logical variables and logical collections.

*Logical variables* represent “unknowns”. Differently from ordinary programming language variables, logical variables have no modifiable value stored in them. In fact, values are associated to logical variables through constraints, representing relations over some specific domains. In JSetL, logical variables are instances of the class `LVar`. When created, an `LVar` object can be either *uninitialized* (i.e., its value is unknown) or *initialized* (i.e., its value is bound to some specific value). Moreover, each `LVar` object can have an optional external name (namely, a string) which can be useful when printing the variable and the possible constraints involving it (see Example 4). `LVar` objects can be manipulated through constraints, namely equality (`eq`), inequality (`neq`), set membership (`in`) and not membership (`nin`). The library provides also utility methods to test whether a variable is initialized or not, to get the value of a initialized variable (but not to modify it), to get/set its external name, and so on.

**Example 4** (Logical variables).

```java
LVar x = new LVar(); // an uninitialized logical variable
LVar y = new LVar("y",1); // an initialized logical variable
  // with external name "y" and value 1
x.setName("x"); // set the external name of x to "x"
y.output(); // print the value bound to y
```

Executing `y.output()` will print on the standard output `_y = 1`, i.e., the external name of the logical variable followed by its value (or `unknown` if the variable is uninitialized)\(^3\)\(\square\)

Values associated with generic logical variables can be of any type. For some specific domains, however, JSetL offers specializations of the `LVar` data type, which provide further specific constraints. In particular, for the domain of integers, JSetL offers the class `IntLVar`, which extends `LVar` with a number of new methods and constraints specific for integers. An *integer*
logical expression is created using methods implementing arithmetic operations, such as \texttt{mul}, \texttt{mod}, etc., applied to \texttt{IntLVar} objects, and returning \texttt{IntLVar} objects. Moreover, \texttt{IntLVar} provides integer comparison constraints, such as \texttt{< \texttt{lt}}, \texttt{\leq \texttt{le}}, etc. to relate integer logical expressions to each other.

Another important kind of logical objects are logical collections, namely, \textit{logical sets} and \textit{lists}. Hereafter we will focus on logical sets, but most of the following considerations apply also to logical lists. The value of a logical set is a collection of elements of any type, including other logical objects. Logical sets can be \textit{partially specified}, in that they can contain uninitialized logical objects as elements, as well as an uninitialized logical set as the rest of the set.

In JSetL, logical sets are instances of the class \texttt{LSet}, which in turn is a subclass of the class \texttt{LCollection}. Values of \texttt{LSet} objects are objects of the \texttt{java.util} class \texttt{Set}.

\textit{Remark 2.} For the sake of clarity, we will adopt here the same syntactic notations for names used in $\mathcal{L}_{RIS}$—see Remark 1—although it may sometimes conflict with the conventions usually adopted in Java.

\textbf{Example 5 (Logical sets).}

\begin{verbatim}
LSet S1 = new LSet("S1"); // an uninitialized logical set
     // with external name "S1"
LSet S2 = LSet.empty().ins(1,2); // the set \{1,2\}
LVar x = new LVar("x");
LSet S3 = S1.ins(x); // the set \{x\} $\cup$ S1
\end{verbatim}

\texttt{empty()} is a static method of the class \texttt{LSet} returning the empty set. \texttt{ins} is the \textit{element insertion} method for \texttt{LSet} objects: \texttt{S.ins(o\textsubscript{1}, ..., o\textsubscript{n})}, \texttt{n \geq 1}, returns the new logical set whose elements are those of the set \texttt{S} $\cup \{o\textsubscript{1}, ..., o\textsubscript{n}\}$. In particular, the last statement in Example 5 creates a partially specified set \texttt{S3} containing an unknown element \texttt{x} and an unknown rest \texttt{S1} (i.e., \texttt{\{x / S1\}}, using the abstract notation of Remark 1).

A number of constraints are provided to work with \texttt{LSet} objects, which extend those provided by \texttt{LVar}. In particular, \texttt{LSet} provides equality and inequality constraints that account for the semantic properties of sets (namely, irrelevance of order and duplication of elements); moreover it provides constraints for many of the standard set-theoretical operations, such as union (\texttt{union}), intersection (\texttt{inters}), inclusion (\texttt{subset}), and so on.

Constraints are instances of the library class \texttt{Constraint}. They are solved using a \textit{constraint solver} that implements the $\mathcal{L}_{SET}$ solver in Java.

A constraint solver in JSetL is an instance of the class \texttt{Solver}. Basically, it provides methods for adding constraints to its \textit{constraint store} (e.g., the method \texttt{add}) and to prove constraint satisfiability (e.g., the method \texttt{solve}). If \texttt{solver} is a solver, \texttt{Γ} is the collection of constraints stored in its constraint store (possibly empty), and \texttt{C} is a constraint, then \texttt{solver.solve(C)} checks whether \texttt{Γ $\land$ C} is satisfiable or not, i.e., whether there exists an assignment of values to the logical variables of \texttt{Γ $\land$ C} that makes this formula true in the intended interpretation. The order in which constraints are posted to the solver is irrelevant.

Constraint solving is based on the reduction of any conjunction of constraints to a simplified form—called the \textit{solved form}—which is proved to be satisfiable. The success of this reduction allows one to conclude the satisfiability of the original collection of constraints. On the other hand, the detection of a failure (logically, the reduction to \texttt{false}) implies the unsatisfiability of the original constraint. Solved form constraints are \textit{irreducible} constraints. As such, they are left in the constraint store and possibly passed ahead to a new invocation of the constraint solver. A successful computation, therefore, can terminate with a collection of solved form constraints in the final constraint store; conversely, if a failure is detected, the method \texttt{solve} raises the exception \texttt{Failure}.

\textbf{Example 6 (Constraint solving).}

\begin{verbatim}
LSet S1 = LSet.empty().ins(1,new LVar("z")); // the set \{1, z\}
LVar x = new LVar("x"), y = new LVar("y");
LSet S2 = LSet.empty().ins(x,y); // the set \{x, y\}
Solver solver = new Solver();
solver.add(S1.eq(S2).and(x.neq(1))); // the constraint S1=S2 $\land$ x$\neq$1
solver.solve();
y.output();
solver.showStore();
\end{verbatim}
The method `showStore` prints all the constraints stored in the constraint store. Executing this code fragment will output: \_y = 1, Store: \_x neq 1.

The following is an example that illustrates the declarative programming style supported by JSetL. It exploits the nondeterminism embedded in set operations. Solving equalities, as well as other basic set-theoretical operations, over partially specified sets yields, in general, multiple solutions. The JSetL solver is able to nondeterministically compute all these solutions, by means of backtracking. In this and in all other examples in the paper, `solver` represents an instance of the class `Solver`.

**Example 7 (Permutations).** Print all permutations of an array \(A\) of distinct integer numbers. The problem can be modelled as a set unification problem, where the set \(E\) of all elements of \(A\) is unified with a (partially specified) set of \(|E|\) logical variables, i.e., \(E = \{x_1, \ldots, x_{|E|}\}\). Each solution to this problem, that is, each assignment of values to variables \(x_1, \ldots, x_{|E|}\), represents a possible permutation of the integers in \(A\). Note that set unification computes all such permutations as the order of set elements is immaterial to establish equality between two sets.

```java
public static void allPermutations(Integer[] A) {
    LSet E = LSet.empty().insAll(A);
    LSet S = LSet.mkSet(A.length);
    solver.add(E.eq(S));
    solver.forEachSolution(i -> {System.out.print(i + " ");
                                S.printElems(' ');});
}
```

The method `allPermutations` takes an array of integers \(A\) and calls the JSetL method `printElems` for each permutation of the input array. The first line creates a logical set \(E\) out of the elements of array \(A\). The second line creates a logical set \(S\) which contains as many logical variables as the length of \(A\). The third line adds the set equality constraint \(E = S\) to the constraint store. The solutions of this constraint will non-deterministically assign all the values in \(A\) to the variables in \(S\). The last line asks the solver to execute the given statements for each solution of the constraint added above. \(i\) represents the index of the computed solution; `printElems` prints all elements of the set \(S\), separated by the specified character. The following is an example of usage of the method `allPermutations`.

```java
Integer[] elems = {1, 2, 3};
allPermutations(elems);
```

The output produced by executing this code is:

```
1) 1 2 3 
2) 1 3 2 
3) 2 1 3 
4) 2 3 1 
5) 3 1 2 
6) 3 2 1 
```

## 4 RIS IN JSETL

In this section we show how JSetL can be naturally extended to implement \(\mathcal{L}_{RIS}(SE\mathcal{T})\), that is to provide RIS and constraints over RIS in conjunction with the set abstractions which are already available in JSetL.

Actually, JSetL implements an extended version of the language of RIS described in Section\[^2\] In this section, we will focus on the simpler version of RIS (basically that presented in Section\[^2\]), while extended RIS will be discussed in detail in Section\[^6\].

### 4.1 RIS Data Abstraction

**Definition 5 (RIS).** A *Restricted Intensional Set* (RIS) is an instance of the class `Ris`, created by the `Ris` constructor:
\textbf{Example 8} (Ris object creation). The RIS \{x : [-2,2] | x \ mod 2 = 0 \cdot x\} (see Example 1) is created in JSetL as follows:

\begin{verbatim}
IntLVar x = new IntLVar();
Ris R = new Ris(x,new IntLSet(-2,2),x.mod(2).eq(0));
\end{verbatim}

where \texttt{IntLSet(-2,2)} represents the closed (integer) interval \([-2,2]\).

Logically, variables in \(c\) and in \texttt{dummyVars} are existentially quantified variables, inside the RIS. Operationally, they are treated as dummy variables, i.e., a new instance of the variables in \(c\) and in \texttt{dummyVars} is created for each application of \(F\) and \(p\). The use of variables in \texttt{dummyVars} will be discussed in Section 6.2.

Given that \texttt{Ris} extends \texttt{LSet}, \texttt{Ris} objects can be used as logical sets, and all methods of \texttt{LSet} are inherited by \texttt{Ris}. Some of these methods, however, are suitably adapted to work with RIS. For example, the utility method \texttt{isBound()} returns true iff the domain of the \texttt{Ris} object is bound to some value. \texttt{RIS} can be expanded into the corresponding extensional sets under certain conditions.

\textbf{Definition 6} (Expandable RIS). The RIS \(\{c : D \mid F \cdot p\}\) is \textit{expandable} if and only if either \(c\) is empty or \(c\) contains at least a ground element and the filter \(F\) does not contain free variables.

If \(R\) is an expandable RIS, then the method \(R\).\texttt{expand()} returns the \texttt{LSet} object containing the result of the application of the pattern to each element of the domain that is ground and satisfies the filter. In particular, if the domain is empty the expansion of the RIS is the empty \texttt{LSet}. If \(R\) is not expandable, \(R\).\texttt{expand()} raises an exception.

\textbf{Example 9}. If \(R\) is the \texttt{Ris} object created in Example 8 then the corresponding extensional set \(S\) is computed and printed as follows:

\begin{verbatim}
LSet S = R.expand().setName("S");
S.output();
\end{verbatim}

whose execution yields \(S = \{0,-2,2\}\).

The following are two more examples of RIS that can be written using JSetL (note that in these examples the RIS patterns are omitted since they are the same as the corresponding control terms).

\textbf{Example 10 (Ris objects)}.

\begin{enumerate}
  \item The set of sets, belonging to \(D\), containing a given set \(A\) (i.e., \(\{S : D \mid A \subseteq S\}\)):

\begin{verbatim}
LSet A = new LSet("A");
LSet S = new LSet(), D = new LSet();
Ris R = new Ris(S,D,A.subset(S));
\end{verbatim}

\item The set of ordered pairs \((s,m)\) belonging to \(D\), where \(S\) is a set and \(m\) is its cardinality, provided \(m\) is greater than 1 (i.e., \(\{(S,m) : D \mid m = |S| \land m > 1\}\)):

\begin{verbatim}
LSet S = new LSet(), D = new LSet();
IntLVar m = new IntLVar();
Ris R = new Ris(new LPair(S,m),D,S.size(m).and(m.gt(1)));
\end{verbatim}

where the intuitive meaning of \(S\).\texttt{size(m)} is \(m = |S|\).
\end{enumerate}
4.2 RIS constraints

In this section we show how the atomic set constraints provided by JSetL are extended to work with RIS as well. A complete list of all JSetL constraint methods can be found in [23].

**Definition 7 (Atomic RIS constraints).** An atomic RIS constraint is an expression of one of the forms:

- \( o \cdot op (R) \), where \( R \) is a Ris and \( o \) any logical object;
- \( S1 \cdot op (S2) \), where \( S1 \) and \( S2 \) are either LSet objects or objects of the Java class Set, and at least one of them is a Ris object.
- \( S1 \cdot op (S2,S3) \), where \( S1 \), \( S2 \) and \( S3 \) are either LSet objects or objects of the Java class Set, and at least one of them is a Ris object.

where \( o \) and \( op \) stand for equality and inequality; \( subset \) and \( nsubset \), for set inclusion and not inclusion, and so on.

Atomic constraints can be combined using the methods and and or, whose intuitive meaning is logical conjunction and disjunction, respectively.

**Definition 8 (JSetL constraints).** A JSetL constraint is either an atomic constraint (in particular, an atomic RIS constraint), or an expression of one of the forms:

- \( C1 \cdot and (C2) \)
- \( C1 \cdot or (C2) \)

where \( C1 \) and \( C2 \) are (recursively) JSetL constraints. Both atomic and general JSetL constraints are instances of the class Constraint.

**Example 11 (RIS constraints).** If \( R \) is the Ris object created in Example 8 then the following are possible RIS constraints posted on \( R \):

```java
LSet S = LSet.empty().ins(-2,0,2);
solver.add(R.eq(S)); // {x:[-2,2] | x mod 2 = 0 • x} = {-2,0,2}
LVar y = new LVar(1);
solver.add(y.nin(R)); // 1 nin {x:[-2,2] | x mod 2 = 0 • x}
```

The same can be obtained by posting a conjunction of the two atomic constraints:

```java
solver.add(R.eq(S).and(y.nin(R)));
```

4.3 Constraint solving with RIS

RIS constraints are solved by the JSetL solver using the same technique adopted for all other constraints. Basically, RIS constraints are reduced to a solved form using the rewrite rules developed for the theory of RIS presented in Section 2. In order to account for RIS, the solved form returned by the solver is extended accordingly. The following notion is crucial in the definition of solved form for RIS.

**Definition 9 (Variable-RIS).** A Ris is a variable-RIS if its domain is an uninitialized logical set; otherwise it is a non-variable-RIS.

**Definition 10.** (RIS constraints in solved form) Let \( R \), \( R1 \), \( R2 \) be variable-RIS, \( X \) an uninitialized LSet object, \( V1 \), \( V2 \) either variable-RIS or uninitialized LSet objects, \( o \) any logical object, and \( \emptyset \) either a logical collection representing the empty set or a Ris object whose domain is the empty set. An atomic RIS constraint of a JSetL constraint \( C \) is in solved form if it has one of the following forms:

- \( X \cdot eq (R) \), and \( X \) does not occur in the other constraints of \( C \)
- \( R \cdot eq (\emptyset) \) or \( \emptyset \cdot eq (R) \)
Note that all RIS occurring in a RIS constraint in solved form are variable-RIS.

Intuitively, the key idea behind the rewriting rules for RIS is a sort of lazy partial evaluation of RIS. That is, a RIS object is treated as a block until it is necessary to identify one of its elements. When that happens, the RIS is transformed into an extensional set whose element part is the identified element and whose set part is the rest of the RIS. At this point, classic set constraint rewriting (in particular set unification) can be applied. For example, the RIS \{x : \{z/D \mid F ∘ p\}\} will be rewritten, in general, to the extensional set \{p(z)/\{x : D ∘ F ∘ p\}\}, provided \(F(z)\) holds.

According to [2], an admissible \(L_{RIS}\) constraint where all its atomic constraints are in solved form is satisfiable w.r.t. the interpretation structure (i.e., there exists an assignment of values to all variables of the constraint that makes it true in the considered interpretation). Since the rewriting rules applied by the solver to its input constraint are proved to preserve the set of solutions of the input formula, then the ability of the solver to produce a solved form constraint guarantees the satisfiability of the original constraint. Conversely, if the solver detects a failure, then the original constraint is unsatisfiable.

It is important to observe that if the input constraint is satisfiable, then the generated solved form constraint constitutes a finite representation of all the concrete (or ground) solutions of the given constraint. In JSetL all the computed constraints in solved form can be displayed using some utility methods. Specifically, equalities of the form \(x eq(t)\), where \(x\) is an uninitialized logical object and \(t\) any value, can be displayed by calling \(x .output()\), while all other solved form constraints computed by the solver, which are stored in the solver’s constraint store, can be displayed by calling the method \showStore\ on the current solver.

The following are two examples of RIS constraints along with the answer computed by the JSetL constraint solver.

**Example 12** (RIS constraint solving).

i. Executing the code (cf. the second formula of Example [2]):

```java
  IntLVar x = new IntLVar("x");
  LSet D = new LSet("D");
  Ris R = new Ris(x,D,x.mod(2).eq(0));
  LSet S = LSet.empty().ins(2,4,6); // S = \{2,4,6\}
  solver.solve(S.eq(R));
  D.output();
  solver.showStore();
```

will produce the output:

\_D = \{2,4,6/_N1\}

Store: \{ _x : _N1 \mid _N2 = 0 AND _N2 = _x mod 2 \cdot _x \} = {} 

meaning that the given constraint is satisfiable, with \(D\) bound to \{2,4,6/_N1\} and the constraint store containing a solved form RIS constraint involving \(N1\), where \(N1\) and \(N2\) are fresh uninitialized logical objects of the proper type.

ii. Executing the code (cf. the third formula of Example [2]):

```java
  LSet A = new LSet(), B = new LSet(), C = new LSet();
  IntLVar x = new IntLVar();
  solver.add(A.eq(new Ris(x,new LSet(),x.neq(0))));
  solver.add(A.union(B,C).and(A.disj(C)));
  solver.add(A.neq(LSet.empty()));
  solver.solve();
```

causes the solver to detect a failure, raising the exception Failure.
5 | DECLARATIVE PROGRAMMING WITH RIS

Intensional sets represent a powerful tool for supporting a declarative programming style, as pointed out for instance in [16]. In this (and the next) section we provide some evidence for this claim by showing a number of simple programming examples, using JSetL’s facilities for RIS creation and manipulation.

5.1 | Using RIS to define Restricted Universal Quantifiers

A first interesting application of RIS to support declarative programming is to represent Restricted Universal Quantifiers (RUQ). The RUQ:
\[ \forall x \in D : F(x) \]
can be easily implemented by using a RIS as follows:
\[ D \subseteq \{ x : D \mid F(x) \} \]
Intuitively, solving this formula amounts to check whether \( F(x) \) holds for all \( x \) in \( D \).

RUQ are made available in JSetL by exploiting the JSetL constraint subset applied to RIS. The next two examples are Java programs that solve simple—though not trivial—problems using JSetL with RIS. Basically, their solution is expressed declaratively as a formula using RUQ.

Example 13. Compute and print the minimum of a set of integers \( S \).

```java
public static LVar minValue(LSet S) throws Failure {
    IntLVar x = new IntLVar(), m = new IntLVar();
    Ris R = new Ris(x, S, m.le(x));
    solver.add(m.in(S).and(S.subset(R)));
    solver.solve();
    return m;
}
```
The method `minValue` posts the constraint \( m \in S \land S \subseteq \{ x : S \mid m \leq x \} \). The solver, non-deterministically binds a value from \( S \) to \( m \) and then it checks if the property \( m \leq x \) is true for all elements \( x \) in \( S \). If this is not the case, the solver backtracks and tries a different choice for \( m \). A possible call to this method is:

```java
Integer[] sampleSetElems = {8,4,6,2,10,5};
LSet A = LSet.empty().insAll(sampleSetElems);
LVar min = minValue(A).setName("min");
min.output();
```
and the printed answer is \( \text{min} = 2 \).

It is important to observe that operations on logical sets, including RIS, are dealt with as constraints. This implies, among others, that it is possible to compute even with partially specified sets [17]. For example, the set passed to the method `minValue` can be \( \{8, z, 4, 6\} \), where \( z \) is an uninitialized logical variable, or it can contain an unknown part, e.g., \( \{8, 4/S\} \) where \( S \) is an uninitialized LSet object, or even it can be simply an uninitialized LSet object. In all cases the JSetL solver is able to check the given constraints and possibly to find a solution for them. For instance, if \( A \) is the set \( \{8, z, 4, 6\} \), then the call `minValue(A)` will non-deterministically generate two distinct answers, one with \( \text{min} = z, z \leq 4 \), and another with \( \text{min} = 4, z \geq 4 \).

Another example that shows the use of RIS to define a universal quantification in a declarative way is the following simple instance of the well-known map coloring problem.

Example 14. Given a set of \( n \) regions \( R_g \), a cartographic map \( M_p \) of regions in \( R_g \), and a set \( C_l \) of \( m \) colors, \( n, m \geq 1 \), find an assignment of colors to the regions such that no two neighboring regions have the same color. Each region in the set \( R_g \) can be represented as a distinct logical variable and a map as a set of unordered pairs (hence, sets) of variables representing neighboring

\[ \forall x \in D : P(x) \equiv \forall x \in D \Rightarrow P(x) \equiv \forall x \in D \Rightarrow x \in D \land P(x) \equiv \forall x \in D \Rightarrow x \in \{ x : D \mid P(x) \} \equiv D \subseteq \{ x : D \mid P(x) \}. \]
regions. An assignment of colors to regions is represented by an assignment of values (i.e., the colors) to the logical variables representing the different regions.

```java
public static void coloring(LSet Rg, LSet Mp, LSet Cl)
throws Failure {
    solver.add(Rg.subset(Cl));
    LSet P = new LSet();
    Ris R = new Ris(P, Mp, P.size(2));
    solver.add(Mp.subset(R));
    solver.solve();
}
```

The method `coloring` posts the constraint \( R_g \subseteq C_l \land M_p \subseteq \{ P : |P| = 2 \} \). The first conjunct exploits the subset constraint to non-deterministically assign a value to all variables in `regions`. The second conjunct requires that all pairs of regions in the map have cardinality equal to 2, i.e., all pairs have distinct components. If `coloring` is called, for instance, with \( R_g = \{r_1, r_2, r_3\}, r_1, r_2, r_3 \) uninitialized logical variables, \( M_p = \{\{r_1, r_2\}, \{r_2, r_3\}\} \), and \( C_l = \{"red", "blue"\} \), the invocation terminates with success, and \( r_1, r_2, r_3 \) are bound to "red", "blue", "red", respectively (actually, also the other solution which binds \( r_1, r_2, r_3 \) to "blue", "red", "blue", respectively, can be computed through backtracking).

The method `coloring` uses a pure “generate & test” approach; hence it quickly becomes very inefficient as soon as the map becomes more and more complex. However, it may represent a first “prototype” whose implementation can be subsequently refined, without having to change its usage. For example, the coloring problem can be, alternatively, modelled in terms of Finite Domain (FD) constraints, and the method `coloring` can be implemented by exploiting the more efficient FD solver provided by JSetL. On the other hand, as already noted for Example 13, the general formulation presented here allows the method `coloring` to be immediately exploitable also to solve other related problems, such as, for instance, given a map and a set of unknown colors (actually, uninitialized logical variables), find whether the colors are enough to obtain an admissible coloring of the map and, if this is the case, which constraints the colors must obey.

Solving an equality such as \( \{ \{ x \cdot D | \varphi \cdot p \} = 0 \} \), where \( D \) is bound to a nonempty set, requires to check that the filter \( \varphi \) is false for all elements in \( D \), i.e., \( \forall x \in D : \neg \varphi \). The next program illustrates the use of RIS to exploit this kind of universal quantification.

**Example 15.** Check whether \( n \) is a prime number or not.

```java
public static Boolean isPrime(int n) {
    if (n <= 1) return false;
    IntLVar x = new IntLVar();
    Ris R = new Ris(x,new IntLSet(2,n/2),new IntLVar(x).mod(x).eq(0));
    return solver.check(R.eq(LSet.empty()));
}
```

The method `isPrime` posts the constraint \( \{ x \cdot [2, n/2] | n \mod x = 0 \} = 0 \). The equality between the RIS and the empty set ensures that there is no \( x \) in the interval \([2, n/2]\) such that \( n \mod x = 0 \) holds. If, for instance, \( n = 101 \), then the call to `isPrime` returns `true`.

### 5.2 Using RIS to define partial functions

Another notable application of RIS is using them to represent *partial functions* as sets. In general, a RIS of the form \( \{ x : D | F \cdot (x, f(x)) \} \), where \( f \) is any function definable in the underlying language, represents a partial function with domain \( D \). In fact, such a RIS denotes a set of ordered pairs as its pattern is an ordered pair; besides, it is a (partial) function because each of its first components never appears twice, since they belong to the set \( D \).

---

4. `s.check()` differs from `s.solve()` in that the latter raises an exception if the constraint in the constraint store of `s` is unsatisfiable, whereas the former returns a Boolean value indicating whether the constraint is satisfiable or not.
Given that RIS are sets, and partial functions can be represented as RIS, then partial functions can be evaluated, compared and point-wise composed through standard set operators; moreover, the inverse of a function can also be computed by means of constraint solving. The following examples illustrate these ideas in the context of Java, using JSetL.

**Example 16.** The square function of an integer \( n \).

```java
IntLVar x = new IntLVar();
LSet D = new LSet();
Ris sqr = new Ris(x,D,Constraint.truec(),new LPair(x,x.mul(x)));
```

where `Constraint.truec()` is a static method of the class `Constraint` returning an always true constraint. `sqr` defines the set of all ordered pairs \((x, x \times x)\), with \( x \) belonging to a set \( D \). This function can be “evaluated” in a point \( n \), and the result sent to the standard output, by executing the following code:

```java
IntLVar y = new IntLVar("y");
solver.solve(new LPair(n,y).in(sqr));
y.output();
```

that is, \( y \) is the image of \( n \) through function `sqr`. If, for instance, \( n \) has value 5, then the printed result is \( _y = 25 \). Note that the RIS domain, \( D \), is left underspecified as a variable.

As usual in declarative programming, there is no real distinction between inputs and outputs. Therefore, the same RIS of Example 16 can be used also to calculate the inverse of the square function, that is the square root of a given number. To obtain this, it is enough to replace the call to `solve` in Example 16 with the following new call:

```java
solver.solve(new LPair(y,n).in(sqr));
```

If, for instance, \( n \) has value 25, then the computed result is \( _y = \text{unknown} - \text{Domain: {-5, 5}} \), stating that the possible values for \( _y \) are -5 and 5.

The interesting aspect of using RIS for defining functions is that RIS are sets and sets are data. Thus, we have a simple way to deal with functions as data. In particular, since Ris objects can be passed as arguments to a function, we can use RIS to write generic functions that take other functions as their arguments. The following is an example illustrating this technique.

**Example 17.** The following method takes as its arguments an array of integers \( A \) and a function \( f \) and updates \( A \) by applying \( f \) to all its elements:

```java
public static void mapList(Integer[] A,LSet f) throws Failure {
    for(int i=0; i<A.length; i++) {
        IntLVar y = new IntLVar();
        solver.solve(new LPair(A[i],y).in(f));
        A[i] = y.getValue();
    }
}
```

If, for instance, the array passed to `mapList` is \{3, 5, 7\} and \( f \) is the Ris object `sqr` of Example 16 then the modified array is \{9, 25, 49\}.

As a simple variant of this program, replacing the last two statements of `mapList` with the single statement:

```java
if (solver.check(new LPair(y,A[i]).in(f)))
    System.out.println(A[i]);
```

provides an easy solution to the problem of printing all numbers in the given array \( A \) that are the squares of some integer number.

### 5.3 Using JSetL as a theorem prover

As already observed, in JSetL operations on logical objects are dealt with as constraints. Thus it is possible to compute with logical objects, such as LSet and Ris objects, even if they are only partially specified or completely unknown. In particular, we can use the JSetL solver to check satisfiability of very general formulas involving both extensional and intensional sets, in a similar way to what is done with theorem provers.
Example 18. Check the property \( C = A \cap B \iff C = \{ x : A \mid x \in B \} \). This is proved in JSetL by showing that the formula 
\[
\text{inters}(A, B, C) \land R = \{ x : A \mid x \in B \cdot x \} \land R \neq C
\]
is false.

```java
LSet A = new LSet(), B = new LSet(), C = new LSet();
solver.add(A.inters(B,C)); // the constraint C = A \cap B
LVar x = new LVar();
Ris R = new Ris(x,A,x.in(B)); // R = \{ x:A \mid x \in B \}
solver.add(R.neq(C)); // the constraint R \neq C
```

Calling `solver.solve()` causes the exception `Failure` to be thrown (i.e., the formula is found to be false).

The next example shows that JSetL can be used as a prover for a non-trivial fragment of first-order logic with quantifiers.

Example 19. The formula \((\forall x \in S : x > 0) \land -1 \in S\) can be written in JSetL with RIS as follows:

```java
LSet S = new LSet();
IntLVar x = new IntLVar();
Ris R = new Ris(x,S,x.gt(0));
Constraint C = S.subset(R).and(new IntLVar(-1).in(S));
```

and can be proved to be unsatisfiable by posting and solving the constraint \( C \) by using the JSetL solver:

```java
solver.solve(C);
```

6 | EXTENDED RIS

To guarantee that the constraint solver is indeed a decision procedure a number of restrictions are imposed on the form of RIS\(^2\). Specifically: (i) the control term and pattern of RIS are restricted to be of specific forms—see Definition\(^4\) (ii) the filter of RIS cannot contain “local” variables, i.e., existentially quantified variables declared inside the RIS, besides those in the control term; and (iii) recursively defined RIS such as \( X = \{ x : D \mid F(X) \cdot x \} \) are not allowed\(^5\).

Although compliance with these restrictions is important from a theoretical point of view, in practice there are many cases in which they can be (partially) relaxed without compromising the correct behavior of programs using RIS.

In this section we show how JSetL extends the language of RIS presented in Section\(^2\) by relaxing all the above mentioned restrictions. We also show, through a number of simple examples, that the availability of these new features can considerably enhance the expressive power of the language.

6.1 | RIS with general patterns

As noted in\(^2\), a condition for patterns to guarantee correctness and completeness of the constraint solving procedure is for patterns to be bijective functions. All the admissible patterns of \( \mathcal{L}_{\text{RIS}} \) are bijective patterns. Besides these, however, other terms can be bijective patterns. For example, \( x + n \cdot n \) constant, is also a bijective pattern, though it is not allowed in \( \mathcal{L}_{\text{RIS}} \). Conversely, \( x \cdot x \) is not bijective as \( x \) and \(-x\) have \( x \cdot x \) as image, although \( (x, x \cdot x) \) is indeed a bijective pattern allowed in \( \mathcal{L}_{\text{RIS}} \).

Unfortunately, the property for a term to be a bijective pattern cannot be easily syntactically assessed. Thus in\(^2\) a more restrictive definition of admissible pattern is adopted. However, from a practical point of view, as in JSetL, we can admit also more general patterns. If the expression used in the RIS pattern defines a bijective function, then dealing with the RIS should be safe (i.e., the answer computed by the solver is fully reliable); otherwise, it is not safe in general, but it may work correctly in many cases.

Specifically, RIS patterns in JSetL can be any logical object. In particular, we allow patterns to be integer logical expressions involving variables occurring in the RIS control term. The following is an example using a RIS whose pattern is not a bijective function, but despite this is dealt with correctly by the solver.

Example 20. Compute the set of squares of all even numbers in \([1,10]\).

---

\(^5\)Note that, on the contrary, a formula such as \( X = \{ D(X) \mid F \cdot p \} \) is an admissible constraint, and it is suitably handled by the JSetL solver.
### 6.2 RIS with dummy variables

Allowing existentially quantified variables in RIS raises major problems when the formula representing the RIS filter has to be negated during RIS constraint solving (basically, negation of the RIS filter is necessary to assure that any element that does not satisfy the filter does not belong to the RIS itself). In fact, this would require that the solver is able to deal with possibly complex universally quantified formulas, which is usually not the case (surely, it is not the case for the JSetL solver). Thus, to avoid such problems *a priori*, in $L_{RIS}$ the RIS filter cannot contain any explicit existentially quantified variable.

However, as already observed for RIS patterns, in practice there are cases in which we can relax restrictions on RIS without losing the ability to correctly deal with more general RIS constraints.

Thus, in JSetL, we allow the user to specify that some (logical) variables in the RIS filter are indeed local variables. This is achieved by using the fifth argument of the `Ris` constructor, which accepts a sequence of logical objects that the user wants to be treated as existentially quantified (or dummy) variables.

**Example 21.** If $S$ is a set of ordered pairs and $D$ is a set, then the subset of $S$ where all the first components belong to $D$ can be defined as \{ $x : D \mid \exists y ((x, y) \in S \cdot (x, y))$ \}, where $D$ and $S$ are free variables, while $x$ and $y$ are existentially quantified variables. This is implemented in JSetL by the following declarations:

```java
LSet S = new LSet("S"), D = new LSet("D");
LVar x = new LVar(), y = new LVar();
Ris R = new Ris(x,D,new LPair(x,y).in(S),new LPair(x,y),y);
```

If we execute:

```java
solver.solve(new LPair(1,2).in(R).and(new LPair(3,4).in(R)));
D.output(); S.output();
```

then the program terminates with success, printing:

```
_D = {1,3/_N1}
_S = {(1,2),(3,4)/_N2}
```

where $N1$ and $N2$ are fresh uninitialized logical sets.

In the above example, $y$ is a dummy variable. If $y$ is not declared as dummy, then the same call to `solver.solve` will terminate with failure, since $y$ is dealt with as a free variable and the first constraint, $(1, 2) \in R$, binds $y$ to $2$ so that the second constraint $(3, 4) \in R$ fails.

It is worth noting that many uses of dummy variables can be avoided by a proper use of the control term and pattern of a RIS. For example, the RIS of Example 21 can be replaced by the RIS without dummy variables \{ $(x, y) : S \mid x \in D \cdot (x, y)$ \}. Similarly, the last RIS of Example 10 (without dummy variables) is equivalent to the RIS (with dummy variables) \{ $S : D \mid \exists c \text{size}(S, c) \wedge c > 1 \cdot S$ \}. Hence, allowing control terms and patterns for RIS to be any logical object can also be useful to alleviate the problem of existentially quantified variables in RIS filters.

### 6.3 Recursive RIS

The class `Ris` extends the class `LSet`. Hence it is possible to use `Ris` objects inside the RIS filter formula in place of `LSet` objects. This allows, among other things, to define *recursive restricted intensional sets* (RRIS).

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6Variables occurring in the control term are also (implicitly) existentially quantified. However, since they are required to take their values from the RIS domain, negation of the RIS filter for such variables turns out to be a form of restricted universal quantification which is conveniently implemented through recursion, by extracting one element at a time from the RIS domain.
The presence of recursive definitions may compromise the finiteness of RIS and hence the decidability of the formulas involving them. Therefore RRIS are prohibited in the base language of RIS, $\mathcal{L}_{RIS}$. In practice, however, their availability can considerably enhance the expressive power of the language and hence RRIS are allowed in the extended version of $\mathcal{L}_{RIS}$ implemented in JSetL. Programmers are responsible for guaranteeing termination.

As shown in Section 5.2, a function $f$ can be defined as a set of ordered pairs $G_f = \{x : D \mid F \cdot (x, f(x))\}$, for some filter $F$ and domain $D$. A call to $f$, e.g., $y = f(x)$, is simply expressed as a set membership predicate over the set defining $f$, i.e., $(x, y) \in G_f$. A call to $f$ in the filter of the RIS defining $f$ itself is a recursive call to $f$. For example, the well known factorial function $\text{fact}(x)$ can be defined as a recursive RIS as follows:

$$\text{fact} = \{(0, 1)/\{x : D \mid \exists z (x \geq 0 \land (x - 1, z) \in \text{fact} \cdot (x, x \cdot z))\}.$$  

Note that the domain of the RIS is left underspecified, and recursion is simply expressed as $(x - 1, z) \in \text{fact}$, meaning that $z$ is the factorial of $x - 1$. Also note that the base case of the recursive definition of $\text{fact}$ is simply added as a known element, $(0, 1)$, to the set being defined.

Such kind of recursive definitions are directly supported by the implementation of RIS in JSetL.

**Example 22** (Factorial of a number $x$).

```java
LSet fact = new LSet();
IntLVar x = new IntLVar(), z = new IntLVar();
Constraint C = x.gt(0).and(new LPair(x.sub(1),z).in(fact));
Ris R_fact = new Ris(x,new LSet(),C,new LPair(x,z.mul(x)),z);
solver.add(fact.eq(R_fact.ins(new LPair(0,1))));
```

where $z$ is a dummy variable which is used to contain $\text{fact}(x - 1)$. If we conjoin, for example, the constraint new LPair(5,ff).in(fact), where ff is an uninitialized IntLVar, and ask the solver to solve the current constraint, then the solver will return $ff = 120$. Conversely, if we conjoin the constraint new LPair(n,120).in(fact), where $n$ is an uninitialized IntLVar, then the solver will return $n = 5$.

The following is another non-trivial example using both recursive RIS and dummy variables.

**Example 23** (Reachable nodes). Given a directed acyclic graph $G = (N, E)$, where $N$ is the set of nodes and $E$ the set of directed edges, and a node $n \in N$, compute the set $R$ of all nodes reachable from $n$ (including $n$ itself) in an arbitrary long number of steps. Using RIS it is possible to compute $R$ as follows:

```java
public static LSet reachable(LSet N, LRel E, LVar n){
    LVar x = new LVar("x");
    LVar y = new LVar("y");
    LSet R = new LSet();
    Ris R_R = new Ris(x,N,x.eq(n)
                       .or(y.in(R).and(new LPair(y,x).in(E))),x,y);
    R.eq(R_R).check();
    return R_R.expand();
}
```

where LRel is a JSetL class, extending LSet, that provides the data abstraction of binary relation (i.e., sets of ordered pairs). The Ris object $R_R$ represents the RIS $\{x : N \mid x = n \lor \exists y (y \in R \land (y, x) \in E)\}$. The statement $R.eq(R_R).check()$ forces $R = R_R$ to hold, thus making $R_R$ a recursive RIS. Finally, the last line of reachable returns an extensional set containing the set of nodes reachable from $n$. The code below shows an example of the usage of the method transitiveClosure.

```java
Integer[] nodesArray = {1,2,3,4,5,6};
LPair[] edgesArray = {new LPair(1,2), new LPair(1,3),
                        new LPair(2,5), new LPair(4,6)};
LSet Nodes = LSet.empty().insAll(nodesArray);
LRel Edges = LRel.empty().insAll(edgesArray);
LVar start = new LVar(1);
reachable(Nodes,Edges,start).setName("Reachable").output();
```
7 | SET-ORIENTED PROGRAMMING IN PRACTICE

The main goal of JSetL is to allow more readable and reliable programs to be obtained through the use of set theory.

The basic version of JSetL has been shown to be effectively usable in practice through a number of simple—though often not trivial—programming examples that are available on-line at the JSetL web page. The same has been done for JSetL with RIS in this paper.

Generally speaking, JSetL shows reasonable execution times on problems involving relatively “small” sets. This limitation can be partly relaxed whenever completely specified sets are involved. For example, the JSetL set-theoretic implementation of the method for computing the minimum of a set of integers (cf. Example 13) turns out to have a computational complexity which is at worst in $O(n^2)$ if all elements in the set are just constants. Hence, in these cases, we can use the JSetL implementation even for non-trivial data without making the program prohibitively inefficient. Certainly, if one wants to exploit the flexibility of JSetL which allows to operate even with partially specified sets, as well as without distinguishing between input and output parameters, then one must be willing to accept significant reductions in efficiency.

An interesting aspect of using very powerful set abstractions such as those provided by JSetL (in particular RIS) is that the program that one can write can be very close to a formal set-based specification, e.g., written in Z. In this sense, we can see programs based on JSetL as executable set-based specifications. This has the advantage that the specification language is the same as the design and programming language. Moreover, formal specification languages are typically non-executable or, if they are executable, can only be used for prototypes; conversely, in cases where the efficiency is not excessively low the JSetL implementation can be used as a correct-by-construction concrete implementation also for practical development.

In order to further increase these capabilities JSetL has been recently extended with new classes to support also the notions of binary relations (class LRel) and partial functions (class LMap) which are crucial for representing set-based specifications. These classes extends the class LSet, and objects created out of them can be manipulated through the usual set-theoretic constraints as well as through new ad-hoc relational constraints (e.g., the constraint comp for relation composition).

It is undeniable, however, that there are cases in which the set-theoretical implementations turn out to be inherently less efficient than the corresponding imperative, fine-tuned implementations in pure Java. As a matter of fact, set unification, which is a fundamental modeling tool in JSetL and which underlies many other set operations, has been shown to be an NP-hard problem in the general case.

Notwithstanding, the purely set-theoretical solutions using JSetL can be seen as a first executable prototype which can be subsequently replaced by more efficient implementations, possibly still using JSetL, but closer to classical imperative solutions. By exploiting the general abstraction mechanisms of the Java language, these refinements can be confined to those (single) methods that use logical set constraints and whose execution turns out to be too inefficient. The rest of the program can continue to use these methods in the same way, with no concern about their actual implementation. Moreover, if the refinements, as well as the first prototypes, are expressed in pure set-theoretical terms, then it would be straightforward to obtain from them the corresponding logical $L_{RIS}$ formulas. In this way, one could use a tool like $\{\log\}$, which implements $L_{RIS}$ within a CLP framework, to prove that the refinement implies the more abstract version. The precise definition of these refinement techniques, and how to use them in conjunction with $\{\log\}$, are left for future work.

8 | RELATED WORK

Relatively few (general-purpose) programming languages provide support for intensionally defined sets, usually in the form of list/set comprehension constructs—e.g., SETL, Python, Haskell, Miranda, Scala. In all these proposals, list/set comprehension constructs denote completely specified sets, i.e., sets where all elements have a known value, and expressions containing

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2 JSetL has been used also as one of the first six implementations for the standard Java Constraint Programming API defined in the Java Specification Request JSR-331 (see for instance http://openrules.com/jsr331/JSR331_UserManual.pdf).

3 The worst case is $S = \{k,k-1,\ldots,1,0\} \land y \in S \land S \subseteq \{x : S \mid y \leq x\}$ where the number of comparisons is $(k^2(k+1))/2$. 

4 Notwithstanding, the purely set-theoretical solutions using JSetL can be seen as a first executable prototype which can be subsequently replaced by more efficient implementations, possibly still using JSetL, but closer to classical imperative solutions. By exploiting the general abstraction mechanisms of the Java language, these refinements can be confined to those (single) methods that use logical set constraints and whose execution turns out to be too inefficient. The rest of the program can continue to use these methods in the same way, with no concern about their actual implementation. Moreover, if the refinements, as well as the first prototypes, are expressed in pure set-theoretical terms, then it would be straightforward to obtain from them the corresponding logical $L_{RIS}$ formulas. In this way, one could use a tool like $\{\log\}$, which implements $L_{RIS}$ within a CLP framework, to prove that the refinement implies the more abstract version. The precise definition of these refinement techniques, and how to use them in conjunction with $\{\log\}$, are left for future work.
list/set comprehensions are always evaluated, as encountered (though possibly using some form of “lazy” evaluation). Also Prolog, as well as extended logic programming languages such as Godel and \( \{ \log \} \), offer some form of intensionally defined lists/sets (e.g., the built-in predicate \texttt{setof} of Prolog). These facilities are basically based on set-grouping, i.e., the ability to collect into an extensional list/set all the elements satisfying the property characterizing the given intensional definition.

A form of set-grouping is offered also in the first version of JSetL by the method \texttt{setof}. Specifically, \( C \texttt{.setof}(x) \), where \( x \) is an \texttt{LVar} and \( C \) is a \texttt{Constraint} object, returns an \texttt{LSet} object whose elements are all possible solutions for \( x \) which satisfy \( C \).

Though set-grouping works fine in many cases, it may incur in a number of problems if the formula characterizing the intensional definition contains unbound variables (other than the control variable) and/or if the set of values to be collected is not completely determined. Generally speaking, all the above mentioned proposals lack the ability to perform high-level reasoning on general formulas involving intensional sets. For instance, these proposals cannot deal with general formulas like those in Examples 18 and 19.

\( \mathcal{L}_{\text{RIS}} \) is a proposal aiming at providing such capabilities in the context of CLP languages. Using the language of RIS, in \( \mathcal{L}_{\text{RIS}} \) we can express very general logic formulas involving intensional sets, and using the RIS constraint solver we can check their satisfiability and possibly compute (a finite representation) of all their solutions.

The RIS and the constraints introduced in that context are the same considered in the current paper. The purposes and methods of that work, however, are quite different from those considered in JSetL. As a matter of fact, in JSetL we are moving within the conventional setting of imperative O-O languages and we are mainly interested in exploring the potential of using RIS on programming.

As far as we know, this is the first proposal for a conventional programming language offering support for reasoning about intensional sets. Using RIS and the other JSetL facilities for constraint solving (including the \texttt{setof} method) we can deal with intensional set definitions in the same way as the other languages can; but by using RIS constraint solving we can exploit intensional sets for programming in a more general and original way.

Finally, it is worth noting that in JSetL we adopt a library-based approach, where set data abstractions are implemented on top of a high-level language (specifically, Java) by exploiting the language abstraction mechanisms. An alternative approach is defining a new language, or extending an existing one, where the desired abstractions are provided as first-class citizen of the language, such as for instance in SETL. Compared to the library-based solution, the new language approach may allow, in general, a stronger integration between the added abstractions and the rest of the language, and a greater ease of program writing thanks to ad-hoc syntactic constructs. On the other hand, the library-based approach has the undeniable advantages of (i) being easier to develop; (ii) having no impact on the host language, e.g., in terms of execution efficiency; and (iii) not requiring one to introduce any new formalism, which makes it easy to test the approach on different languages and communities.

9 | CONCLUSION AND FUTURE WORK

In this paper we have presented an extension of the Java library JSetL to support RIS, and we have shown the usefulness of this extension from the programming point of view through a number of simple examples.

The new version of the JSetL library can be downloaded from the JSetL’s home page. All sample Java programs shown in previous sections have been tested using that version and are available on-line.

The advantages of having RIS in JSetL can be summarized as follows:

- RIS represent a powerful data and control abstraction facility which integrates and enriches those provided by extensional logical sets, and as such can be of great help in achieving the goal of making program creation easier and faster.

- Since intensional sets often play a fundamental role in formal set-based specifications (e.g., in Z), their presence in JSetL contributes significantly to the ability to code abstract set-theoretical formulas directly into (executable) Java+JSetL programs.

- The JSetL solver can be used as a theorem-prover for a non-trivial quantifier-free fragment of set theory; the addition of RIS allows this fragment to be significantly enlarged, including also sets defined by properties, as well as (a restricted form of) universally quantified formulas.
Finally, the fact that RIS are objects, and that partial functions can be coded as RIS, provides an elegant solution to the problem of dealing with functions as data in an O-O language; in particular, it is possible to code also functions as recursive RIS.

As future work, it would certainly be interesting extending the set of atomic constraints that deal with RIS to the relational operators recently added to CLP($SET$)$^{22}$.

This extension could be useful also to improve the possibilities of using JSetL as a tool for writing executable set-based specifications in Java. Closely connected with this, a future work could be the precise definition of the refinement techniques mentioned at the end of Section[7] Another interesting line of work would be to explore how to use JSetL as Eiffel’s contract specification language.

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