Non-asymptotic Optimal Prediction Error for RKHS-based 
Partially Functional Linear Models

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Abstract

Under the framework of reproducing kernel Hilbert space (RKHS), we consider the penalized least-squares of the partially functional linear models (PFLM), whose predictor contains both functional and traditional multivariate part, and the multivariate part allows a divergent number of parameters. From the non-asymptotic point of view, we focus on the rate-optimal upper and lower bounds of the prediction error. An exact upper bound for the excess prediction risk is shown in a non-asymptotic form under a more general assumption known as the effective dimension to the model, by which we also show the prediction consistency when the number of multivariate covariates $p$ slightly increases with the sample size $n$. Our new finding implies a trade-off between the number of non-functional predictors and the effective dimension of the kernel principal components to ensure the prediction consistency in the increasing-dimensional setting. The analysis in our proof hinges on the spectral condition of the sandwich operator of the covariance operator and the reproducing kernel, and on the concentration inequalities for the random elements in Hilbert space. Finally, we derive the non-asymptotic minimax lower bound under the regularity assumption of Kullback-Leibler divergence of the models.

Keywords: partially functional linear models, concentration inequality in Hilbert space, reproducing kernel Hilbert space, non-asymptotic bound, minimax rate, diverging number of covariates

1 Introduction

Statistical analysis of functional data has become an important and difficult part in modern statistics since the leading work Ramsay (1982) and pioneering paper Grenander (1950). Due to the technological innovation, the progress of the data storage enables scientists to acquire complex data sets with the structures of curves, images, or other data with functional structures (referred as functional data). There has been a large amount of works now focusing on many different non-parametric aspects of functional data such as kernel ridge regressions Cai et al. (2006); Preda (2007); Du and Wang (2014); Reincurrh et al. (2018), penalized B-spline regressions Cardot et al. (2003), functional principal component regressions Yao et al. (2005), local linear regressions Baiłlo and Grané (2009), and reader can refer to the review paper Wang et al. (2016) for more details. The prediction in functional data analysis is a crucial topic, which has a wide range of applications including chemometrics, econometrics and biomedical studies Ramsay and Silverman (2007); Kokoszka and Reincurrh (2017).

Many existing works considering the estimation and prediction problems of functional data are based on the framework of functional principal component analysis (FPCA), see Yao et al. (2005); Cai et al. (2006); Hall et al. (2007). However, the predictive power of FPCA-based methods is weakened when the functional principal components cannot form an effective basis for the slope function, which often occurs in practice and coincides with the similar phenomenon in principal component regressions, se
Jolliffe (1982). An alternative method for the functional data is based on the framework of reproducing kernel Hilbert space (RKHS), which assumes the slope function is contained in an RKHS. It is shown in Cai and Yuan (2012) that the RKHS-based method performs better than the FPCA-based method when the slope function does not align well with the eigenfunctions of the covariance kernel. In fact, FPCA strongly relies on the leading principal scores with large eigenvalues correspondingly, and the eigenfunctions for representing the slope function inevitably lose some information for the response. From the view of machine learning theory, the FPCA is essentially a non-supervise method that often performs poorly in the data analysis, for examples, the analysis of Canadian weather data mentioned in Cai and Yuan (2012) and the Section 3 of Cui et al. (2020).

In this paper, we study the partially functional linear models (PFLM) containing both functional and multivariate parts in the predictor, which is originally considered in Shin (2009). Let \( X = (X_1, \cdots, X_p)^T \) be the \( p \)-dimensional multivariate predictor, \( Y(t) \) be the functional predictor, \( \varepsilon \) be the random noise and \( Z \) be the scalar response. In our work, we consider the PFLM taking the semi-parametric form

\[
Z = X^T \alpha_0 + \int_0^T Y(t) \beta_0(t) dt + \varepsilon, \tag{1}
\]

where the \( \beta_0(t) \) is the slope function for functional predictor and the \( \alpha_0 \) is the regression coefficient for multivariate predictor. The model (1) contains both parametric and non-parametric part, which belongs to the semi-parametric statistics. We assume the predictor to be the random design where \( X, Y(t) \) and \( \varepsilon \) are independent random variables. Because the intercepts of predictor are easy to estimate by centralizing, we assume \( EX = 0 \) and \( EY(t) = 0 \) for simplicity.

In some situations, a large number of non-functional predictors are often collected for practical data analysis, and this increasing-dimensional setting has been considered in Aneiros et al. (2015); Kong et al. (2016). Moreover, our work can also be applied to deal with divergent number of parameters. Theoretically, this setting requires assuming that the number of scalar covariates grows with the sample size, i.e. \( p = p_n \rightarrow \infty \), and the convergence rate of the desired estimator becomes totally different from the case where the dimension of the non-functional predictors is fixed.

Dealing with the functional data as a stochastic process is a significant challenge in functional data analysis. Obviously, a functional covariate \( Y(t) \) has an infinite number of predictors over the time domain (observed as discrete-time points) that are all highly correlated. The covariance function characterizes the correlation of the functional covariate. The estimation of the slope function in functional regressions is connected to ill-posed inverse problems. To handle the infinite-dimensionality of \( \beta_0(t) \), people often impose certain regularity conditions on the hypothesized space of the slope function to ensure that the infinite-dimensional problems are tractable as an finite-dimensional approximation solution. Notwithstanding, the convergence rate of the slope estimators depends directly on the assumptions of the eigenvalue decay of the covariance operator and the restricted space of the slope function. Thus, the convergence rate cannot be parametric due to the infinite-dimensionality of the model (1).

Some recent developments in PFLM include Zhu et al. (2019); Cui et al. (2020). Because of the shortcoming of the FPCA-based methods, we apply penalized least squares under the framework of RKHS here. There are few efforts on the non-asymptotic upper and lower bounds in the existing literature. For FPCA-based method, Brunel et al. (2016) considers the adaptive estimation procedure of functional linear models under a non-asymptotic framework; Wahl (2018) analyses the prediction error of functional principal component regression (FPCR) and proves a non-asymptotic upper bound for the corresponding squared risk. For the RKHS-based method, many works focus on the asymptotic results, such as Cai et al. (2006); Cai and Yuan (2012). Under the framework of RKHS-based kernel ridge regressions, Liu and Li (2020) recently studies the non-asymptotic RKHS-norm error bounds (called oracle inequalities) for the estimated function \( f_0 \) in Gaussian non-parametric regression \( Y = f_0(X) + \varepsilon, \varepsilon \sim N(0, \sigma^2) \) where \( f_0 \) belongs to \( L_2 \), and this non-asymptotic approach has been early studied in ?. By applying the Matern kernel and supposing \( f_0 \) in a Holder space with the polynomial decay rate of eigenvalues \( \lambda_n = O(n^{-2a}) \), Liu and Li (2020) derives the nearly minimax optimal convergence rate \( \left( \frac{\log n}{n} \right)^{\frac{2a}{2a + d}} \) (up to a log \( n \) factor) for \( L_2 \)-norm estimation error.

To analyse the PFLM, the main innovation of our work is that we provide a non-asymptotic upper and lower bounds for the excess prediction risk under a more general assumption to the effective dimension. Tong and Ng (2018) establishes the optimal convergence rate of the excess prediction risk for the RKHS-based slope estimator of the functional linear models, but they do not consider the partially linear case. Moreover, we show the convergence rate of the excess prediction risk of the PFLM is the same as that of the functional linear model, which means the convergence of the prediction risk of the functional part
dominates the convergence of the prediction risk of the whole PFLM. We also derive a minimax lower bound for the excess prediction risk under the assumption to the Kullback-Leibler divergence of the model. The specific theoretical contributions of our work are listed as

- A significant contribution is that we obtain the non-asymptotic upper and lower bounds, which have not been well studied in the existing literature. We provide an exact non-asymptotic minimax lower bound on the excess prediction risk in FLRM. Moreover, a particular application of the proposed non-asymptotic version of the optimal prediction upper bound is that it allows to analyse the PFLM with divergent number of non-functional predictors, which leads to the prediction consistency under the setting \( p^2 \log^5(p) = o(n) \).

- We derive the non-asymptotic upper bound of the excess prediction risk for the RKHS-based least squares estimation in PFLM, and the optimal bound we obtain is more exact than that of Cui et al. (2020) which only obtains the stochastic order of the convergence rate without the definite multiplying constants relevant to the high probability events. Our derivation for the optimal bound does not need the inverse Cauchy inequality

\[
E \left( \int Y(t)f(t)dt \right)^4 \leq C \left[ E \left( \int Y(t)f(t)dt \right)^2 \right]^2, \text{ for } f \in L^2(\mathcal{T})
\]

as a moment assumption of the functional predictor. This condition is imposed in Cui et al. (2020); Cai and Yuan (2012) to attain minimax prediction bounds for (partially) functional linear regressions. Our proof does not rely on the well-known representation lemma for the smoothing splines, see Wahba (1990); Cucker and Smale (2001).

- The proof for the Theorem 1 is divided into three steps, and it relies on new non-trivial results. First, we prove the difference of the functional part between the true parameter and our least squares estimate is bounded. Second, based on the boundedness, we show the excess prediction risk contributed by the multivariate part of the predictor is convergent at \( n^{-1/2} \)-rate. Finally, according to the convergence of the multivariate part, we obtain the convergence of the prediction risk corresponding to the functional part in \( n^{-\theta} \)-rate, where \( \theta \) is related to the effective dimension in the Assumption 4. Specifically, the novelty of the proof lies in the Lemma 2, which is a crucial lemma for the Theorem 1. In the Lemma 2, to show the concentration property of the random elements in Banach space, we use the methods in functional analysis and convert the random elements in Banach space to other relevant random elements in Hilbert space.

The outline of this paper is constructed as follows. In Section 2, we provide the notations and definitions we need and a brief introduction on the RKHS and the PFLM. In Section 3, we show our main theorem about the non-asymptotic upper bound for the excess prediction risk and two relevant corollaries. In Section 4, we state the minimax lower bound for the excess prediction risk. In Section 5, we provide the proof of the Theorem 1 in Section 3. In Section 6, we show the proof the Theorem 2 in Section 4. In Section 7 and 8, we prove the lemmas we need for the proofs in Section 5 and 6. In Section 9, we summarize our conclusions and point out some future directions for research.

2 Preliminaries

2.1 Notations and Definitions

Define \( \|v\| := \left( \sum_{i=1}^{p} v_i^2 \right)^{1/2} \) to be the \( \ell_2 \)-norm of vector \( v \in \mathbb{R}^p \). Let \( \mathcal{T} \subset \mathbb{R} \) be a compact set. Denote by \( L^2(\mathcal{T}) \) the Hilbert space composed by square integrable functions on \( \mathcal{T} \), whose inner product and norm are respectively denoted by \( \langle f, g \rangle \) and \( \|f\| \) for any \( f, g \in L^2(\mathcal{T}) \).

Consider \( T \) a bounded linear operator from a Banach space \( A \) to a Banach space \( B \) respectively endowed with the norms \( \|\cdot\|_A \) and \( \|\cdot\|_B \). Define the operator norm of \( T \) as

\[
\|T\|_{op} := \sup_{x \in A, \|x\|_A = 1} \|T(x)\|_B.
\]

Let \( T^* \) be the adjoint of \( T \) from \( B^* \) to \( A^* \) defined by

\[
T^*(f)(x) := f(T(x)), \text{ for any } f \in B^*.
\]
Notice the adjoint of an operator does not change the operator norm and thus we have \( \|T^*\|_{\text{op}} = \|T\|_{\text{op}} \).

For a matrix \( E = (e_{ij})_{1 \leq i,j \leq p} \in \mathbb{R}^{p \times p} \), when writing \( \|E\|_{\text{op}} \), we actually view \( E \) as a bounded linear operator from \( \mathbb{R}^p \) to \( \mathbb{R}^p \) endowed with \( \ell_2 \)-norm defined by \( \|v\| = \|Ev\| \), which is also called the spectral norm in other literature. Let \( \|E\|_{\infty} := \max_{1 \leq i,j \leq p} |e_{ij}| \) be the \( \ell_\infty \)-norm of the matrix \( E \) and \( \lambda_{\max}(E) \) be the maximal eigenvalue of the matrix \( E \). Moreover, we have \( \|E\|_{\text{op}} \leq p\|E\|_{\infty} \) from 5.6.P23 in Horn and Johnson (2012).

For a real, symmetric, square integrable and nonnegative definite function \( R : \mathcal{T} \times \mathcal{T} \to \mathbb{R} \), let \( L_R : L^2(\mathcal{T}) \to L^2(\mathcal{T}) \) be an integral operator (also a bounded linear operator) defined by

\[
L_R(f)(t) := (R(s,t), f(s)) = \int_{\mathcal{T}} R(s,t) f(s) ds.
\]

According to the spectral theorem, there exists a set of orthonormalized eigenfunctions \( \{\phi_k^R : k \geq 1\} \) and a sequence of eigenvalues \( \theta_k^R \geq \theta_{k+1}^R \geq \cdots > 0 \) such that

\[
R(s,t) = \sum_{k=1}^{\infty} \theta_k^R \phi_k^R(s) \phi_k^R(t), \quad \forall s,t \in \mathcal{T}.
\]

Noticing the orthonormality of the eigenfunctions \( \{\phi_k^R : k \geq 1\} \), we have

\[
L_R(\phi_k^R)(s) = (R(s,t), \phi_k^R(t)) = \sum_{i=1}^{\infty} \theta_i^R \phi_i^R(s) (\phi_i^R(t), \phi_k^R(t)) = \theta_i^R \phi_i^R(s).
\]

In what follows, we let \( \{\theta_k^R, \phi_k^R : k \geq 1\} \) be the eigenvalue-eigenfunction pairs corresponding to the operator (or the equivalent bivariate function) \( R \).

Let \( L_{20}^R \) be the operator satisfying

\[
L_{20}^R(\phi_k^R) = \sqrt{\theta_k^R} \phi_k^R.
\]

Define the bivariate function \( R_{20}^R(s,t) \) by

\[
R_{20}^R(s,t) := \sum_{k=1}^{\infty} \sqrt{\theta_k^R} \phi_k^R(s) \phi_k^R(t), \quad \forall s,t \in \mathcal{T}.
\]

For two bivariate functions \( R_1, R_2 : \mathcal{T} \times \mathcal{T} \to \mathbb{R} \), define

\[
(R_1 R_2)(s,t) := \langle R_1(s,\cdot), R_2(\cdot,t) \rangle = \int_{\mathcal{T}} R_1(s,u) R_2(u,t) du.
\]

Then we have the relation and \( L_{R_1 R_2} = L_{R_1} \circ L_{R_2} \), and hence \( L_{20}^R = L_{R_{20}^R} \) where \( \circ \) means the composition of mappings. To show \( L_{R_1 R_2} = L_{R_1} \circ L_{R_2} \), we notice

\[
L_{R_1} \circ L_{R_2}(f)(t) = \int_{\mathcal{T}} R_1(t,s) L_{R_2}(f)(s) ds = \int_{\mathcal{T}} R_1(t,s) \left( \int_{\mathcal{T}} R_2(s,u) f(u) du \right) ds = \int_{\mathcal{T}} \left( \int_{\mathcal{T}} R_1(t,s) R_2(s,u) du \right) f(u) du = L_{R_1 R_2}(f)(t).
\]

Let \( H(\mathcal{T}) \) be the Hilbert space of the Hilbert-Schmidt operators on \( L^2(\mathcal{T}) \) with the inner product \( \langle A, B \rangle_H := \text{Tr}(B^* A) \) and the norm \( \|A\|_H^2 = \sum_{k=1}^{\infty} \|A(\phi_k)\|^2 \) where \( \{\phi_k : k \geq 1\} \) is an orthonormal basis of \( L^2(\mathcal{T}) \). The space \( H(\mathcal{T}) \) is a subspace of the bounded linear operators on \( L^2(\mathcal{T}) \), with the norm relations \( \|A\|_{\text{op}} \leq \|A\|_H \) and \( \|AB\|_H \leq \|A\|_{\text{op}} \|B\|_H \).

A reproducing kernel \( K : \mathcal{T} \times \mathcal{T} \to \mathbb{R} \) is a real, symmetric, square integrable and nonnegative definite function. Given a reproducing kernel \( K \), we can uniquely identify a RKHS \( \mathcal{H}(K) \) composed by a subspace of \( L^2(\mathcal{T}) \) satisfying \( K(t,\cdot) \in \mathcal{H}(K) \) for any \( t \in \mathcal{T} \), which is endowed with an inner product \( \langle \cdot, \cdot \rangle_K \) such that

\[
f(t) = \langle K(t,\cdot), f \rangle_K, \quad \text{for any } f \in \mathcal{H}(K).
\]

There is a well-known fact that \( L_{K_{\frac{1}{2}}}^2(\mathcal{L}^2(\mathcal{T})) = \mathcal{H}(K) \), i.e. the RKHS \( \mathcal{H}(K) \) can be characterized as the range of \( L_{K_{\frac{1}{2}}}^2 \) equipped with the norm \( \|L_{K_{\frac{1}{2}}}^2(f)\|_K = \|f\|_{L^2(\mathcal{T})} \), see Sun (2005) for more details. For simplicity, we let \( \mathcal{H}(K) \) be dense in \( L^2(\mathcal{T}) \), which means \( L_{K_{\frac{1}{2}}}^2 \) is injective. And let \( \kappa \) be the operator norm of \( L_{K_{\frac{1}{2}}}^{-1} \), \( \kappa := \|L_{K_{\frac{1}{2}}}^{-1}\|_{\text{op}} \).

Readers can refer to Wahba (1990); Cucker and Smale (2001) for more discussions on RKHS.
2.2 The Penalized Least Square for PFLM

The goal of prediction given the predictor \( X \) and \( Y(t) \) is to recover the prediction \( \eta_0 \), the right side of (1) without the random noise \( \varepsilon \),

\[
\eta_0(X, Y(t)) := X^T \alpha_0 + \int_T Y(t)\beta_0(t)dt.
\]

We assume that the training sample \( \{(Z_i, X_i, Y_i(t))\}_{i=1}^n \) is composed by \( n \) independent copies of \( (Z, X, Y(t)) \). To estimate the true parameter \( (\alpha_0, \beta_0) \), the penalized least squares is defined as

\[
(\hat{\alpha}_n, \hat{\beta}_n) := \arg\min_{(\alpha, \beta) \in \mathbb{R}^p \times \mathcal{H}(K)} \frac{1}{n} \sum_{i=1}^n \left( Z_i - X_i^T \alpha - \int_T Y_i(t)\beta(t)dt \right)^2 + \lambda_n \| \beta \|^2_K.
\]

(2)

Noticing \( L^p_K(L^2(\mathcal{T})) = \mathcal{H}(K) \), there exists \( f_n \in L^2(\mathcal{T}) \) such that \( L^p_K(f_n) = \beta_n \). So the (2) is replaced by

\[
(\hat{\alpha}_n, \hat{f}_n) := \arg\min_{(\alpha, f) \in \mathbb{R}^p \times L^2(\mathcal{T})} \frac{1}{n} \sum_{i=1}^n \left( Z_i - X_i^T \alpha - \langle Y_i, L^p_K(f) \rangle \right)^2 + \lambda_n \| f \|^2.
\]

(3)

Let \( \hat{\eta}_n \) be the prediction rule induced by the penalized least squares \( (\hat{\alpha}_n, \hat{\beta}_n) \)

\[
\hat{\eta}_n(X, Y(t)) := X^T \hat{\alpha} + \int_T Y(t)\hat{\beta}_n(t)dt.
\]

For a prediction rule \( \eta(X, Y(t)) \), define the prediction risk to be

\[
\mathcal{E}(\eta) := \mathbb{E}[Z^* - \eta(X^*, Y^*(t))^2],
\]

where \( (Z^*, X^*, Y^*(t)) \) is an independent copy of \( (Z, X, Y(t)) \).

We measure the accuracy of the prediction \( \hat{\eta}_n \) by the excess prediction risk

\[
\mathcal{E}(\hat{\eta}_n) - \mathcal{E}(\eta_0) = \mathbb{E}[\hat{\eta}_n(X^*, Y^*(t)) - \eta_0(X^*, Y^*(t))^2].
\]

Let \( f_0 \in L^2(\mathcal{T}) \) satisfying \( L^p_K f_0 = \beta_0 \) and rewrite \( \eta_0(X, Y(t)) \) and \( \hat{\eta}_n(X, Y(t)) \) to

\[
\eta_0(X, Y(t)) = X^T \alpha_0 + \int_T Y(t)(L^p_K f_0(t))dt \quad \text{and} \quad \hat{\eta}_n(X, Y(t)) = X^T \hat{\alpha}_n + \int_T Y(t)(L^p_K \hat{f}_n(t))dt,
\]

by which we can bound the excess prediction risk

\[
\mathcal{E}(\hat{\eta}_n) - \mathcal{E}(\eta_0) = \mathbb{E} \left[ X^T (\alpha_0 - \hat{\alpha}_n) + \int_T Y^*(t)(L^p_K (f_0 - \hat{f}_n))(t)dt \right]^2,
\]

\[
\leq 2 \mathbb{E}[X^T (\alpha_0 - \hat{\alpha}_n)]^2 + 2 \mathbb{E} \left[ \int_T Y^*(t)(L^p_K (f_0 - \hat{f}_n))(t)dt \right]^2,
\]

\[
= 2(\alpha_0 - \hat{\alpha}_n)^T \mathbb{E} X^* X^{**T} (\alpha_0 - \hat{\alpha}_n) +
\]

\[
+ 2 \int_{\mathcal{T} \times \mathcal{T}} \mathbb{E}[Y^*(t)Y^*(s)](L^p_K (f_0 - \hat{f}_n))(t)(L^p_K (f_0 - \hat{f}_n))(s)dtds.
\]

Define the empirical covariance matrix \( D_n \) and the covariance matrix \( D \) for the multivariate part of the predictor to be

\[
D_n := \frac{1}{n} \sum_{i=1}^n X_i X_i^T \quad \text{and} \quad D := \mathbb{E} (X X^T).
\]

Let \( \lambda_{\max} := \lambda_{\max}(D) \) be the maximal eigenvalue of the covariance matrix \( D \). Similarly, define the empirical covariance function \( C_n(s, t) \) and the covariance function \( C(s, t) \) for the functional part of the predictor to be

\[
C_n(s, t) := \frac{1}{n} \sum_{i=1}^n Y_i(s)Y_i(t) \quad \text{and} \quad C(s, t) := \mathbb{E}(Y(s)Y(t)).
\]
Notice the equality
\[
E \left[ \int_T Y(t)f(t)dt \right]^2 = \int_T \int_T E[Y(s)Y(t)]f(s)f(t)dsdt = \int_T f(t) \left( \int_T C(s,t)f(s)ds \right) dt = (f, L_C f).
\]
Define the sandwich operator of the covariance operator and the reproducing kernel by \( T = L_K^T \circ L_C \circ L_K^2 \) and its empirical version \( T_n = L_K^{T_n} \circ L_{C_n} \circ L_K^{T_n} \), see Cai and Yuan (2012) for details. With these definitions, we reformulate the upper bound for the excess prediction risk to
\[
\mathcal{E}(\hat{y}_n) - \mathcal{E}(y_0) \leq 2\lambda_{\text{max}}\|\hat{\alpha}_n - \alpha_0\|^2 + 2\langle L_K^T(\hat{f}_n - f_0), L_{C}L_K^2(\hat{f}_n - f_0) \rangle
\]
\[
= 2\lambda_{\text{max}}\|\hat{\alpha}_n - \alpha_0\|^2 + 2\|T_K^T(\hat{f}_n - f_0)\|^2,
\]
which is relatively easy to analysis.

For simplicity of the following discussion, we need more definitions. Define \( g_n := \frac{1}{n} \sum_{i=1}^{n} \epsilon_i L_K^2 Y_i \) and \( a_n := \frac{1}{n} \sum_{i=1}^{n} \epsilon_i X_i \), which are key quantities to derive the convergence rate of the desired estimator. Define the bounded linear operators \( G_n : L^2(T) \to \mathbb{R}^p \) and \( H_n : \mathbb{R}^p \to L^2(T) \) by
\[
G_n(f) := \frac{1}{n} \sum_{i=1}^{n} \langle Y_i, L_K^2 f \rangle X_i, \quad \forall f \in L^2(T) \quad \text{and} \quad H_n(\alpha) := \frac{1}{n} \sum_{i=1}^{n} \langle X_i^T \alpha \rangle L_K^2 Y_i, \quad \forall \alpha \in \mathbb{R}^p.
\]
Let \( \{ (\tau_k, \varphi_k) : k \geq 1 \} \) be the set of eigenvalue-eigenfunction pairs related to \( T \) as an operator on \( L^2(T) \). Define the trace of the operator \((T + \lambda I)^{-1}T\) as
\[
D(\lambda) := \text{Tr}((T + \lambda I)^{-1}T),
\]
which is also called the effective dimension in learning theory, see Zhang (2005).

3 The Analysis and Main Results

3.1 The Analysis

According to the definition of the penalized least squares \((\hat{\alpha}_n, \hat{f}_n)\), which minimize (3), the difference between the penalized least squares \((\hat{\alpha}_n, \hat{f}_n)\) and the true parameter \((\alpha_0, f_0)\) can be represented as
\[
\begin{align*}
\hat{f}_n - f_0 &= -\lambda_n(T_n + \lambda_n I)^{-1}f_0 - (T_n + \lambda_n I)^{-1}H_n(\hat{\alpha}_n - \alpha_0) + (T_n + \lambda_n I)^{-1}g_n, \\
\hat{\alpha}_n - \alpha_0 &= -D_n^{-1}G_n(\hat{f}_n - f_0) + D_n^{-1}a_n,
\end{align*}
\]
where \( T_n, H_n, G_n, D_n, g_n \) and \( a_n \) are given in the previous section. The derivation of (6) and (7) is left to the Subsection 7.1, where we use the method of the calculus of variations.

Some common regularity assumptions are needed to ensure our main results.

**Assumption 1.** \( X \) satisfies the growth of moments condition for each component \( X_j \) (1 ≤ j ≤ p): there exist \( M_1, \nu > 0 \) such that
\[
E(|X_j|^l) \leq \frac{M_1^2}{2} \nu^{l-2}l!, \quad \text{for all integer } l \geq 2, 1 \leq j \leq p.
\]
And we assume \( D = E(XX^T) \) and \( D^{-1} \) to be positive definite.

**Assumption 2.** \( Y(t) \) is a bounded square integrable stochastic process: there exists \( M_2 > 0 \) such that \( \|Y(t)\| \leq M_2 \) (a.s.).

**Assumption 3.** The random noise \( \epsilon \) has zero mean and finite variation: \( E\epsilon = 0 \) and \( E\epsilon^2 = \sigma^2 < \infty \).

**Assumption 4.** The effective dimension of \( T \) satisfies: \( D(\lambda) = \text{Tr}((T + \lambda I)^{-1}T) \leq c\lambda^{-\theta} \) for constants \( c, \theta > 0 \).

**Assumption 5.** The number of the multivariate covariates \( p = p_n \) can be increasing as a function of sample size \( n \).
The Assumption 1 has been adopted in Ai et al. (2020), which is to serve for the Bernstein’s inequality. We assume that $Y(t)$ is bounded almost surely in the Assumption 2. Define a random variable $\xi$ taking values in $\text{HS}(T)$ by
\[
\xi(f) := (T + \lambda_n I)^{-\frac{1}{2}}(L_K^f Y, f) L_K^f Y.
\]
Actually, we can weaken the Assumption 2 by assuming the Bernstein’s growth of moments condition of $\xi$ in $\text{HS}(T)$: there exist $M_2, \nu > 0$ such that
\[
\mathbb{E}(|\xi - \mathbb{E}\xi|_{\text{HS}}^l) \leq \frac{D(\lambda_n) M_2^2}{2} \nu^{l-2} l!, \quad \text{for all integer } l \geq 2.
\] (8)
Then using the Lemma 7, we obtain the same result as in the Lemma 1, by which we have the same result for the non-asymptotic optimal prediction error as in the Theorem 1. But the condition (8) is difficult to verify, a similar condition is also provided for the FPCA method in Brunel et al. (2016). Here, we do not provide the complete proof under the condition (8). The Assumption 3 follows the general assumptions of zero mean and finite variance on random noise. The Assumption 4 on the effective dimension has been adopted in Tong and Ng (2018), which reflects the convergence of eigenvalues of $L_C$ and $L_K$ and how their eigenfunctions align. The Assumption 4 also contains the common assumption on the decay rate of the eigenvalues of $T$ (see Remark 2).

**Remark 1.** For the simplicity of later proof and statement of lemmas, we further assume $\kappa$, $M_2 > 1$. These assumptions make no essential difference compared with the original assumptions because of the boundedness.

Before getting to our main results, we need two important lemmas, of which the proofs are left to the Section 7 and 8. The following Lemma 1 and Lemma 2 can be viewed as the concentration inequalities for the operator-valued random variable $T_n, G_n$ and $H_n$. The concentration inequalities for the random variable taking values in a Hilbert space as stated in the Lemma 6 and 7 play an important role in the proofs of lemmas.

**Lemma 1.** Under the Assumption 2, for any $\delta_1 \in (0, 2e^{-1})$, with probability at least $1 - \delta_1$, there exists
\[
\|(T + \lambda_n I)^{-\frac{1}{2}}(T_n - T)\|_{\text{op}} \leq c_1 \log(\frac{2}{\delta_1}) B_n,
\]
where $c_1 := 2\kappa^2 M_2^2$ and $B_n := \frac{1}{n\lambda_n} + \sqrt{\frac{D(\lambda_n)}{\nu}}$.

**Lemma 2 (\|G_n\|_{\text{op}} = \|H_n\|_{\text{op}}).** Under the Assumptions 1 and 2, for any $\delta_2 \in (0, 1)$, with probability at least $1 - \delta_2$, we have
\[
\|G_n\|_{\text{op}} = \|H_n\|_{\text{op}} \leq \frac{c_2 \log(\frac{2}{\delta_2})}{\sqrt{n}},
\]
where $c_2 := 2p(c + M_1) M_2$.

### 3.2 Main Results

With all the preparations we establish above, we enable to state the main result of this paper. The following theorem provides a non-asymptotic upper bound for the excess prediction risk.

**Theorem 1.** Under the Assumptions 1-4, for any $\delta_1, \delta_3, \delta_4, \delta_5 \in (0, 1)$ and $\delta_2 \in (0, 2e^{-1})$, by taking $\lambda_n = \omega n^{-\frac{\nu}{\kappa}}$, there exists an integer given by $n_0 = \lceil \max\{N_1, N_2\} \rceil$, where
\[
N_1 = 48e^2 p \|D^{-1}\|_{\text{op}} (48p \|D^{-1}\|_{\text{op}} M_2^2 + 1) \log(\frac{2p}{\delta_5}) \quad \text{and}
\]
\[
N_2 = \left( \frac{12p^2 \kappa^2 (c + M_1)^2 M_2^2 \|D^{-1}\|_{\text{op}}}{\omega} \log^3 \left( \frac{2p}{\delta_2} \log^2 \frac{2p}{\delta_5} \right) \right)^{\frac{1}{3 + \nu}},
\]
such that for $n \geq n_0$, we have with probability at least $1 - \sum_{i=1}^5 \delta_i$
\[
\mathcal{E}(\hat{\eta}_n) - \mathcal{E}(\eta_0) \leq \left( 2\lambda_{\text{max}}(2c_4 c_6 + c_5)^2 + 2c_5^2 \right) n^{-1} + \left( 4(c_7 + c_8) c_9 \sqrt{\omega} \right) n^{-\frac{3 + \nu}{3}} + \left( 2(c_7 + c_8)^2 \right) n^{-\frac{1 + \nu}{3}},
\] (9)
where $c_i$ ($4 \leq i \leq 9$) are specific constants given in the proof that depend on the true parameters and the assumptions, and can be written as

\[
c_4 = 3p\kappa (v + M_1)M_2 \| D^{-1} \|_{op} \log \left( \frac{2p}{\delta_2} \right), \quad c_5 = 3\sqrt{\pi} \sigma M_1 \| D^{-1} \|_{op} / 2\sqrt{\delta_4},
\]

\[
c_6 = \| f_0 \| + 2p\kappa (v + M_1)M_2c_5 \log \left( \frac{2p}{\delta_2} \right) + \sigma (\omega^{-1} + \omega^{-\frac{1+5}{2}} \sqrt{\delta_3}) / \sqrt{\delta_3},
\]

\[
c_7 = \| f_0 \| \left( 2\kappa^2 M_2^2 (\omega^{-1} + \omega^{-\frac{1+5}{2}} \sqrt{\delta_3}) \log \left( \frac{2}{\delta_1} \right) + 1 \right),
\]

\[
c_8 = \left( 2\kappa^2 M_2^2 (\omega^{-1} + \omega^{-\frac{1+5}{2}} \sqrt{\delta_3}) \log \left( \frac{2}{\delta_1} \right) + 1 \right) 2 \sigma (\omega^{-1} + \omega^{-\frac{1+5}{2}} \sqrt{\delta_3}) \text{ and}
\]

\[
c_9 = \frac{2p\kappa (v + M_1)M_2 (2c_4 c_6 + c_8)}{\sqrt{\omega}} \left( 2\kappa^2 M_2^2 (\omega^{-1} + \omega^{-\frac{1+5}{2}} \sqrt{\delta_3}) \log \left( \frac{2}{\delta_1} \right) + 1 \right) \log \left( \frac{2p}{\delta_2} \right).
\]

Equation (9) presents an exact convergence rate of the excess prediction risk with all precise constants determined by the regularity conditions. The first term in the right side of (9) is ascribed to the parametric part of the PFLM. The second term is a mixed rate consisting of both the parametric and the functional part since the prediction risk is a square function composed by both the functional and non-functional predictors. The last term is a dominated term, which reveals that the signal strength of the functional part since the prediction risk is a square function composed by both the functional and non-functional predictors. The last term is a dominated term, which reveals that the signal strength is difficult to derive an analogy of central limit theorem for the functional parameter.

Remark 2. The assumption that the eigenvalues $\tau_k$ decay as $\tau_k \leq c^l k^{-2r}$ ($r > \frac{1}{2}$) is a special case of our more general assumption 4 with regard to the effective dimension once we notice

\[
D(\lambda_n) = \sum_{k=1}^{+\infty} \frac{\tau_k}{\tau_k + \lambda_n} \leq \sum_{k=1}^{+\infty} \frac{c^l k^{-2r}}{c^l k^{-2r} + \lambda_n} = \sum_{k=1}^{+\infty} c^l / c^l + \lambda_n k^{2r} \leq \int_{0}^{+\infty} \frac{c^l}{c^l + \lambda_n k^{2r}} dt = \int_{0}^{1} \frac{c^l}{c^l + s^{2r}} ds \lesssim \lambda_n^{\frac{1}{2r}} \approx n^{\frac{1}{2r} - 1}
\]

\[
(\lambda_n = \omega n^{-\frac{2r}{1+2r}}).
\]

And we state this special case as the corollary below.
Corollary 2. Suppose the Assumptions 1-3 are satisfied. Assume the eigenvalues $\tau_k$ decay as $\tau_k \leq c_k^{r-2r}$ for some $c' > 0$ and $r > \frac{1}{2}$. For any $\delta_1, \delta_3, \delta_4, \delta_5 \in (0,1)$ and $\delta_2 \in (0,2e^{-1})$, by taking $\lambda_n = \omega n^{-\frac{2}{2r} - \varepsilon}$, there exists an integer $n_0$ such that for $n > n_0$, we have with probability at least $1 - \sum_{i=1}^{5} \delta_i$

$$\mathcal{E}(\hat{\eta}_n) - \mathcal{E}(\eta_0) \leq (2\lambda_{\max}(2c_4c_6 + c_8)^2 + (c_4^2)n^{-1} + (4(c_7 + c_8)c_9\sqrt{\omega})n^{-\frac{2+\varepsilon}{2r}} + (2(c_7 + c_8)^2\omega)n^{-\frac{2+\varepsilon}{2r}}$$

where $c_i (4 \leq i \leq 9)$ and $n_0$ are the same as those of the Theorem 1 except replacing $\theta$ by $\frac{1}{2}$ and $c$ by a constant relevant to $c'$ and $r$.

An useful and insightful application of the Theorem 1 is that we can consider the situation where the number of multivariate covariates $p$ increases as a function of $n$. According to the definition of $c_i (4 \leq i \leq 9)$ and $N_i (i = 1, 2)$ in the proof, we reveal that

$$c_4 = O(p \log(p)), \quad c_5 = O(p^\frac{3}{2}), \quad c_6 = O(p^\frac{3}{2} \log(p)), \quad c_7 = c_8 = O(1),$$

$$c_9 = O(p^2 \log^3(p)), \quad N_1 = O(p^2 \log(p)) \quad \text{and} \quad N_2 = O(p^{(2+\varepsilon)} \log^2(p))$$

From these orders, it implies

$$(2\lambda_{\max}(2c_4c_6 + c_8)^2 + 2c_9^2)n^{-1} = O(p^\frac{7}{2} \log^6(p)n^{-1}), \quad (4(c_7 + c_8)c_9\sqrt{\omega})n^{-\frac{2+\varepsilon}{2r}} = O(p^\frac{3}{2} \log^3(p)n^{-\frac{2+\varepsilon}{2r}})$$

and

$$(2c_7 + c_8)^2\omega)n^{-\frac{2+\varepsilon}{2r}} = O(n^{-\frac{2+\varepsilon}{2r}}).$$

When letting $p^\frac{7}{2} \log^6(p)n^{-1} \to 0$, i.e. $n \gg O(p^\frac{7}{2} \log^6(p))$. The term $O(p^\frac{3}{2} \log^3(p)n^{-\frac{2+\varepsilon}{2r}})$ can be dominated by the term $p^\frac{3}{2} \log^3(p)n^{-\frac{2+\varepsilon}{2r}}$ so we have as $n, p \to \infty$

$$p^\frac{3}{2} \log^3(p)n^{-\frac{2+\varepsilon}{2r}} \ll O(n^{\frac{5}{2}}) \to 0.$$ 

Notice $n \gg O(p^\frac{7}{2} \log^6(p)) \gg O(p^2 \log(p)) = N_1$. If we let $\frac{(2+\varepsilon)}{\theta} < 7 \Leftrightarrow \theta > \frac{7}{2}$, we have $n \gg N_2$ under the condition $p^\frac{7}{2} \log^6(p)n^{-1} \to 0$ after noticing $p \gg \log^\varepsilon(p)$ for any $\varepsilon \in \mathbb{R}$. Therefore we have the following prediction consistency for the increasing dimension situation of non-functional parameters.

Corollary 3. Under the Assumptions 1-5, if the constant $\theta > \frac{7}{2}$ in the Assumption 4 and $p^\frac{7}{2} \log^6(p) = o(n)$ in the Assumption 5, we have the consistency for the excess prediction risk:

$$\mathcal{E}(\hat{\eta}_n) - \mathcal{E}(\eta_0) = o_p(1).$$

Remark 3. If we assume the eigenvalues $\tau_k$ decay as $\tau_k \leq c_k^{r-2r}$ ($r > \frac{1}{2}$) in the increasing-dimensional setting, by applying the Corollary 3 and noticing $\theta = \frac{1}{2}$, we need to further assume $r < \frac{1}{2}$ to obtain the prediction consistency, which means the convergence rate of eigenvalues can not be too fast. Intuitively, when $p$ increases, $r$ can not be too large or equivalently, the effective dimension $D(\lambda_n) \approx n^{\frac{2}{2r}}$ can not be too small. It implies we need to find a trade-off between the number of non-functional predictors and the effective dimension to get the prediction consistency.

The prediction consistency theory has been well-established for the estimators in non-parametric statistics and high-dimensional statistics, see Zhuang and Lederer (2018) for the recent development for general regularized maximum likelihood estimators. However their works specially aim for non-parametric or high-dimensional models, they do not cover the semi-parametric case as studied in our paper.

4 Minimax Lower Bound

In this section, we derive a minimax lower bound for the excess prediction risk in the following Theorem 2. To verify the optimality of the upper bound of the prediction risk for the proposed estimator, the result on minimax lower bound below shows the prediction risk of our estimator achieves the theoretical lower bound caused by the intrinsic limitation of the PFLM. Let $P_{\alpha_0, \beta_0}$ be the probability taken over the space $(Z, X, Y)$ where $Z$ is generated by the true parameter $Z = X^T \alpha_0 + (Y, \beta_0) + \varepsilon$. Before stating the main result, we need a regularity assumption relevant to the variance of the random noise.
\section*{5 Proof of the Theorem 1}

When setting \(\lambda_n = \omega n^{-1+\eta}\), we have \(\frac{1}{\delta} < n^{-1+\eta} = \frac{\lambda_n}{\delta} = \sqrt{\frac{\omega}{\delta}} \sqrt{\lambda_n}\) and

\[
D(\lambda_n) \leq c(\omega n^{-1+\eta})^{-\theta} \leq c\omega^{-\theta} n^{-1+\eta} = c\omega^{-(1+\theta)}\lambda_n,
\]

by which we have \(B_n \leq (\omega^{-1} + \omega^{-1+\eta} \sqrt{c}) \sqrt{\lambda_n}\) in Lemma 1 and \(n\lambda_n = \omega n^{-1+\eta} \geq \omega\).

Applying Lemmas 2, 3 and 5 to (7), when \(n > n_0\), we have with probability at least \(1 - \delta_2 - \delta_4 - \delta_5\)

\[
\|\alpha_n - \alpha_0\| \leq \|D^{-1}_{n}\|\|G_n\|\|\hat{f}_n - f_0\| + \|D_n^{-1}\|\|a_n\|
\leq \frac{3}{2}\|D^{-1}\|\|c_2 \log \frac{2p}{\delta_2} \sqrt{n}\| \|\hat{f}_n - f_0\| + \frac{3}{2}\|D^{-1}\|\|c_3 \sqrt{\delta_4} \sqrt{n}\| \|\hat{f}_n - f_0\| + \frac{c_5}{\sqrt{n}},
\]

for some constants \(c_2, c_3, c_4, c_5\).
where we let $c_4 := \frac{3c_3D^{-\frac{1}{2}}\|\hat{\omega}\|_2 \log\left(\frac{2p}{\delta_5}\right)}{2\sqrt{\lambda_n}}$ and $c_5 := \frac{3c_3D^{-\frac{1}{2}}\|\omega\|_2}{2\sqrt{\lambda_n}}$.

First, we want to bound $\|\hat{f}_n - f_0\|$. According to (6), it gives

$$
\|\hat{f}_n - f_0\| \leq \lambda_n\|(T_n + \lambda_n I)^{-\frac{1}{2}}\|_{op}\|f_0\| + \|(T_n + \lambda_n I)^{-\frac{1}{2}}\|_{op}\|H_n\|_{op}\|\hat{\alpha}_n - \alpha_0\| + \|(T_n + \lambda_n I)^{-\frac{1}{2}}g_n\|
=: I_1 + I_2 + I_3.
$$

For the term $I_1$, we have $I_1 \leq \|f_0\|$ because $\|(T_n + \lambda_n I)^{-\frac{1}{2}}\|_{op} \leq \frac{1}{\lambda_n}$.

For the term $I_2$, combining (14) we have with probability at least $1 - \delta_2 - \delta_3 - \delta_4 - \delta_5$

$$
I_2 \leq \frac{c_2c_4\log\left(\frac{2p}{\delta_5}\right)}{n\lambda_n}\|\hat{f}_n - f_0\| + \frac{c_2c_5\log\left(\frac{2p}{\delta_5}\right)}{n\lambda_n}\|\hat{f}_n - f_0\| + \frac{c_2c_5}{\omega}\log\left(\frac{2p}{\delta_5}\right),
$$

where we use $n\lambda_n \geq \omega$.

For the term $I_3$, we obtain with probability at least $1 - \delta_3$ by Lemma 8

$$
I_3 \leq \|(T_n + \lambda_n I)^{-\frac{1}{2}}\|_{op}\|(T_n + \lambda_n I)^{-\frac{1}{2}}g_n\| \leq \frac{1}{\sqrt{\lambda_n}}\sqrt{\delta_3}B_n \leq \frac{\sigma(\omega^{-1} + \omega^{-\frac{1+\theta}{2}}\sqrt{\delta_3})}{\sqrt{\delta_3}},
$$

where we use $B_n \leq (\omega^{-1} + \omega^{-\frac{1+\theta}{2}}\sqrt{\delta_3})\sqrt{\lambda_n}$ in the last step.

Thus we can bound $\|\hat{f}_n - f_0\|$ by $I_1$, $I_2$ and $I_3$

$$
\|\hat{f}_n - f_0\| \leq \|f_0\| + \frac{c_2c_4\log\left(\frac{2p}{\delta_5}\right)}{n\lambda_n}\|\hat{f}_n - f_0\| + \frac{c_2c_5}{\omega}\log\left(\frac{2p}{\delta_5}\right) + \frac{\sigma(\omega^{-1} + \omega^{-\frac{1+\theta}{2}}\sqrt{\delta_3})}{\sqrt{\delta_3}}.
$$

where we define $c_6 := \|f_0\| + \frac{c_2c_4\log\left(\frac{2p}{\delta_5}\right)}{n\lambda_n}\|\hat{f}_n - f_0\| + \frac{c_2c_5}{\omega}\log\left(\frac{2p}{\delta_5}\right)$.

Notice when $\lambda_n = \omega n^{-\frac{1+\theta}{2}}, n\lambda_n = \omega n^{1+\theta}$ and when $n > \left(\frac{2c_2c_4\log\left(\frac{2p}{\delta_5}\right)}{\omega}\right)^{\frac{1+\theta}{\theta}},$ we have

$$
\frac{c_2c_4\log\left(\frac{2p}{\delta_5}\right)}{n\lambda_n} \leq \frac{1}{2}.
$$

Therefore, when $n > N_1$, we obtain with probability at least $1 - \delta_2 - \delta_3 - \delta_4 - \delta_5$

$$
\frac{1}{2}\|\hat{f}_n - f_0\| \leq \|\hat{f}_n - f_0\| - \frac{c_2c_4\log\left(\frac{2p}{\delta_5}\right)}{n\lambda_n}\|\hat{f}_n - f_0\| \leq c_6,
$$

which equals to $\|\hat{f}_n - f_0\| \leq 2c_6$.

Next, we turn to bound $\|\hat{\alpha}_n - \alpha_0\|$. According to (14), we have with probability at least $1 - \delta_2 - \delta_3 - \delta_4 - \delta_5$

$$
\|\hat{\alpha}_n - \alpha_0\| \leq \frac{2c_4c_6 + c_5}{\sqrt{n}}.
$$

Then we find a way to bound $\|T_n^\frac{1}{2}(\hat{f}_n - f_0)\|$. According to (6), we have

$$
\|T_n^\frac{1}{2}(\hat{f}_n - f_0)\| \leq \lambda_n\|T_n^\frac{1}{2}(T_n + \lambda_n I)^{-\frac{1}{2}}\|_{op}\|f_0\| + \|T_n^\frac{1}{2}(T_n + \lambda_n I)^{-\frac{1}{2}}\|_{op}\|H_n\|_{op}\|\hat{\alpha}_n - \alpha_0\| + \|T_n^\frac{1}{2}(T_n + \lambda_n I)^{-\frac{1}{2}}g_n\|
=: E_1 + E_2 + E_3.
$$

For the term $E_1$, we have with probability at least $1 - \delta_1$ by Lemma 1

$$
E_1 \leq \lambda_n\|(T_n + \lambda_n I)^{\frac{1}{2}}(T_n + \lambda_n I)^{-\frac{1}{2}}\|_{op}\|(T_n + \lambda_n I)^{-\frac{1}{2}}\|_{op}\|f_0\|
\leq \lambda_n\left(\frac{1}{\sqrt{\lambda_n}}\|(T_n + \lambda_n I)^{-\frac{1}{2}}(T_n - T)\|_{op} + 1\right)\frac{1}{\sqrt{\lambda_n}}\|f_0\|
\leq \frac{\sqrt{\lambda_n}}{\|f_0\|}\left(c_1\log\left(\frac{\omega^{\frac{1}{2}}}{\delta_1}\frac{B_n}{\sqrt{\lambda_n}} + 1\right) \leq \|f_0\|\left(c_1(\omega^{-1} + \omega^{-\frac{1+\theta}{2}}\sqrt{\delta_3})\log\left(\frac{\omega^{\frac{1}{2}}}{\delta_1}\right) + 1\right)\sqrt{\lambda_n} = c_7\sqrt{\lambda_n},
$$

\text{for some constant } c_7 > 0.\tag{15}
where we use the Inequality 2 in the second inequality and the fact $B_n \leq (\omega^{-1} + \omega^{-\frac{1}{2}} \sqrt{c}) \sqrt{\lambda_n}$ in the last inequality and define

$$c_7 := \|f_0\| \left( c_1 (\omega^{-1} + \omega^{-\frac{1}{2}} \sqrt{c}) \log(\frac{2}{\delta_1}) + 1 \right).$$

For the term $E_3$, by applying the Inequality (2) and the Lemma 8, we have with probability at least $1 - \delta_1 - \delta_3$

$$E_3 \leq \|(T + \lambda_n I)^\frac{1}{2} (T + \lambda_n I)^{-\frac{1}{2}} \|_{op} \|(T + \lambda_n I)^{-\frac{1}{2}} (T + \lambda_n I)^{\frac{1}{2}} g_n\|
\leq \left( \frac{1}{\sqrt{\lambda_n}} \|(T + \lambda_n I)^{-\frac{1}{2}} (T - T)\|_{op} + 1 \right)^2 \frac{\sigma}{\sqrt{\delta_3}} B_n
\leq \left( c_1 \log(\frac{2}{\delta_1}) \frac{B_n}{\sqrt{\lambda_n}} + 1 \right)^2 \frac{\sigma}{\sqrt{\delta_3}} B_n
\leq \left( c_1 (\omega^{-1} + \omega^{-\frac{1}{2}} \sqrt{c}) \log(\frac{2}{\delta_1}) + 1 \right)^2 \frac{\sigma (\omega^{-1} + \omega^{-\frac{1}{2}} \sqrt{c})}{\sqrt{\delta_3}} \sqrt{\lambda_n} = c_8 \sqrt{\lambda_n},$$
where in the last inequality we let

$$c_8 := \left( c_1 (\omega^{-1} + \omega^{-\frac{1}{2}} \sqrt{c}) \log(\frac{2}{\delta_1}) + 1 \right)^2 \frac{\sigma (\omega^{-1} + \omega^{-\frac{1}{2}} \sqrt{c})}{\sqrt{\delta_3}}.$$

Notice we have (15) with probability at least $1 - \delta_2 - \delta_3 - \delta_4 - \delta_5$. Therefore, for the term $E_2$ we obtain with probability at least $1 - \sum_{i=1}^{5} \delta_i$, by using the Inequality 2, $\|(T + \lambda_n I)^{-\frac{1}{2}} \|_{op} \leq \frac{\sqrt{\lambda_n}}{\sqrt{n}}$. Lemma 2 and (15),

$$E_2 \leq \|(T + \lambda_n I)^\frac{1}{2} (T + \lambda_n I)^{-\frac{1}{2}} \|_{op} \|(T + \lambda_n I)^{-\frac{1}{2}} H_n\|_{op} \|\hat{\alpha}_n - \alpha_0\|
\leq \left( c_1 (\omega^{-1} + \omega^{-\frac{1}{2}} \sqrt{c}) \log(\frac{2}{\delta_1}) + 1 \right) \frac{1}{\sqrt{\lambda_n}} \frac{c_2 \log(\frac{2}{\delta_1})}{\sqrt{n}} \frac{2c_4 c_6 + c_5}{\sqrt{n}}
\leq \frac{c_2 (2c_4 c_6 + c_5)}{\sqrt{\omega}} \left( c_1 (\omega^{-1} + \omega^{-\frac{1}{2}} \sqrt{c}) \log(\frac{2}{\delta_1}) + 1 \right) \log(\frac{2p}{\delta_2}) \frac{1}{\sqrt{n}} = \frac{c_9}{\sqrt{n}},$$
where in the last step we use $n\lambda_n \geq \omega$ and define

$$c_9 := \frac{c_2 (2c_4 c_6 + c_5)}{\sqrt{\omega}} \left( c_1 (\omega^{-1} + \omega^{-\frac{1}{2}} \sqrt{c}) \log(\frac{2}{\delta_1}) + 1 \right) \log(\frac{2p}{\delta_2}).$$

Thus, we bound $\|T^{\frac{1}{2}}(\hat{f}_n - f_0)\|$ by

$$\|T^{\frac{1}{2}}(\hat{f}_n - f_0)\| \leq E_1 + E_2 + E_3 \leq (c_7 + c_8) \sqrt{\lambda_n} + \frac{c_9}{\sqrt{n}}.$$}

Recall the excess prediction risk can be bounded by

$$\mathcal{E}(\hat{\eta}_n) - \mathcal{E}(\eta_0) \leq 2\lambda_{max} \|\hat{\alpha}_n - \alpha_0\|^2 + 2\|T^{\frac{1}{2}}(\hat{f}_n - f_0)\|^2,$$

based on which we further have

$$\mathcal{E}(\hat{\eta}_n) - \mathcal{E}(\eta_0) \leq 2\lambda_{max} \left( \frac{2c_4 c_6 + c_5}{\sqrt{n}} \right)^2 + 2 \left( (c_7 + c_8) \sqrt{\lambda_n} + \frac{c_9}{\sqrt{n}} \right)^2 = C_1 n^{-1} + C_2 \sqrt{\frac{\lambda_n}{n}} + C_3 \lambda_n,$$

where $C_1 := 2\lambda_{max} (2c_4 c_6 + c_5)^2 + 2c_9^2$, $C_2 := 4(c_7 + c_8) c_9$ and $C_3 := 2(c_7 + c_8)^2$.

Finally we get the desired conclusion after we notice $\sqrt{\frac{\lambda_n}{n}} = \sqrt{\frac{\lambda_n}{n}}$ in the above proof. □
6 Proof of the Theorem 2

Let $M$ be the smallest integer greater than $b_0\sqrt{\frac{1}{n}}$, where $b_0$ will be defined in later proof. For a binary sequence $\theta = (\theta_{M+1}, \ldots, \theta_{2M}) \in \{0,1\}^M$, define

$$\beta_0 = M^{-\frac{1}{2}} \sum_{k=M+1}^{2M} \theta_k L^T_K \varphi_k.$$ 

By applying $\langle L^T_K \varphi_j, L^T_K \varphi_k \rangle_K = \langle \varphi_j, L_K \varphi_k \rangle_K = \langle \varphi_j, \varphi_k \rangle = \delta_{jk}$, we can show $\beta_0 \in \mathcal{H}(K)$, because

$$\|\beta_0\|^2_K = \|M^{-\frac{1}{2}} \sum_{k=M+1}^{2M} \theta_k L^T_K \varphi_k\|^2_K = \sum_{k=M+1}^{2M} M^{-1} \theta^2_k\|L^T_K \varphi_k\|^2_K \leq M^{-1} \sum_{k=M+1}^{2M} \|L^T_K \varphi_k\|^2_K = 1.$$

Using the Lemma 10, there exist a set $\Theta = \{\theta^i\}_{i=0}^N \subset \{0,1\}^M$ such that

(i) $\theta^0 = (0,\cdots,0)$, (ii) $H(\theta^i, \theta^j) > \frac{M}{8}$ for all $i \neq j$, (iii) $N \geq 2^{2^M}.$

For $\eta_0 = (\alpha_0, \beta_0)$, let $P_{\alpha_0, \beta_0}$ be the joint distribution on the product space of training samples $\{(Z_i, X_i, Y_i)\}_{i=1}^n$ generated by the true parameter $(\alpha_0, \beta_0)$, where $Z_i = X_i^T \alpha_0 + (Y_i, \beta_0) + \epsilon_i$, and $P_{\alpha_0, \beta_0}$ be the distribution on a single sample $(Z, X, Y)$, where $Z = X^T \alpha_0 + (Y, \beta_0) + \epsilon$.

By the independence of the training samples, for fixed $\alpha^*_0 \in \mathbb{R}^p$ and different $\theta, \theta' \in \Theta$, we have

$$\log \left( \frac{dP_{\theta, \theta'}^{n}}{dP_{\alpha_0^*, \beta_0}^{n}}(Z_i, X_i, Y_i) \right) = \sum_{i=1}^n \log \left( \frac{dP_{\theta, \theta'}^{n}}{dP_{\alpha_0^*, \beta_0}^{n}}(Z_i, X_i, Y_i) \right).$$

Using the Assumption 6, we can bound the Kullback-Leibler distance between $P_{\theta, \theta'}^{n}$ and $P_{\alpha_0^*, \beta_0}^{n}$

$$K(P_{\theta, \theta'}^{n} \parallel P_{\alpha_0^*, \beta_0}^{n}) = \sum_{i=1}^n E_{\alpha_0^*, \beta_0} \log \left( \frac{dP_{\theta, \theta'}^{n}}{dP_{\alpha_0^*, \beta_0}^{n}}(Z_i, X_i, Y_i) \right) \leq nK_{\tau^2} E((Y, \beta_{\theta'} - \beta_0)^2).$$

Noticing $\langle L^T_K \varphi_j, L_C L^T_K \varphi_k \rangle = \langle \varphi_j, T \varphi_k \rangle = \tau_k \delta_{jk}$, we have

$$E((Y, \beta_{\theta'} - \beta_0)^2) = \langle \beta_{\theta'} - \beta_0, L_C(\beta_{\theta'} - \beta_0) \rangle$$

$$= \left< M^{-\frac{1}{2}} \sum_{k=M+1}^{2M} (\theta'_k - \theta_k) L^T_K \varphi_k, M^{-\frac{1}{2}} \sum_{k=M+1}^{2M} (\theta'_k - \theta_k) L_C L^T_K \varphi_k \right>$$

$$= M^{-1} \sum_{k=M+1}^{2M} (\theta'_k - \theta_k)^2 \tau_k \leq M^{-1} sM \sum_{k=M+1}^{2M} (\theta'_k - \theta_k)^2 = M^{-1} sM H(\theta', \theta) \leq sM \leq b_2 M^{-2r},$$

from which we have $K(P_{\theta, \theta'}^{n} \parallel P_{\alpha_0^*, \beta_0}^{n}) \leq b_2 nK_{\tau^2} M^{-2r}$.

If we let $b_0 := (\frac{b_2 K_{\tau^2}}{\log 2})^{-\frac{1}{2}} \rho^{-\frac{1}{2}}$, then for any $\rho \in (0, \frac{1}{3})$, we have

$$\frac{1}{N} \sum_{j=1}^N K(P_{\alpha_0^*, \beta_0}^{n} \parallel P_{\alpha_0^*, \beta_0}^{n}) \leq b_2 nK_{\tau^2} M^{-2r} \leq \rho \log(2^M) \leq \rho \log(N).$$

For $\theta \in \Theta$ and a fixed $\alpha^*_0 \in \mathbb{R}^p$, let the prediction rule $\eta_0$ be $\eta_0(X, Y) := X^T \alpha^*_0 + (Y, \beta_0)$. For different $\theta, \theta' \in \Theta$, when the true parameter is $(\alpha^*_0, \beta_0)$, the excess prediction risk for the prediction rule $\eta_0$ is

$$E(\eta_0) - E(\eta) = E \left[ X^T (\alpha^*_0 - \alpha_0) + (Y, \beta_0 - \beta_0) \right]^2 = E((Y, \beta_0 - \beta_0)^2)$$

$$= M^{-1} \sum_{k=M+1}^{2M} (\theta'_k - \theta_k)^2 \tau_k \geq M^{-1} \tau_{2M} \sum_{k=M+1}^{2M} (\theta'_k - \theta_k)^2 = M^{-1} \tau_{2M} H(\theta', \theta) \geq M^{-1} b_1(2M)^{-2} \frac{M}{8} = b_1 2^{-(2r+3)} M^{-2r}.$$
Notice $M$ is the smallest integer greater than $b_0n^{\frac{1}{2r}}$, thus when $b_0n^{\frac{1}{2r}} \geq 1 \Leftrightarrow n \geq \frac{\rho \log 2}{b_2K_{2r}}$. Thus we obtain the lower bound for $E(\eta_0) - E(\eta)$

$$E(\eta) - E(\eta_0) \geq b_12^{-(2r+3)(2b_0n^{\frac{1}{2r}})-2r} = b_12^{-(3+4r)(\frac{8b_2K_{2r}^2}{\log 2})-\frac{2r}{1+2r}} \rho^{-\frac{2r}{1+2r}} n^{-\frac{2r}{1+2r}}.$$ 

For fixed $\alpha_n^* \in \mathbb{R}^p$, consider the set $\Xi := \{(\alpha_n^*, \beta) : \theta \in \Theta\}$. By the Lemma 9, we have

$$\inf_{\tilde{\eta}} \sup_{\eta_0 \in \Xi} P(\mathcal{E}(\tilde{\eta}) - E(\eta_0) \geq \cdots) \leq \sup_{\eta_0 \in \mathbb{R}^p \times \mathcal{H}(K)} P(\mathcal{E}(\tilde{\eta}) - E(\eta_0) \geq \cdots) \text{ and } \log(N) \geq \frac{\log 2}{8} M,$$

we have the desired conclusion. 

\section{Proofs of the Key Lemmas}

\subsection{The Derivation of Equation (6) and Equation (7)}

Recall that

$$Z_i = X_i^T \alpha_0 + (Y_i, L_k^\frac{1}{2} f_0) + \varepsilon_i,$$

where $(X_i, Y_i, \varepsilon_i) (1 \leq i \leq n)$ are independent copies of $(X, Y, \varepsilon)$ in (1). Thus the right side of (3) can be written as

$$F_n(\alpha, f) := \frac{1}{n} \sum_{i=1}^{n} \left( X_i^T (\alpha - \alpha_0) + (Y_i, L_k^\frac{1}{2}(f - f_0)) - \varepsilon_i \right)^2 + \lambda_n \| f \|^2.$$

Notice $(\hat{\alpha}_n, \hat{f}_n)$ is the minimum of $F_n(\alpha, f)$, therefore

$$\frac{\partial F_n(\alpha, f)}{\partial \alpha} = 0,$$

from which we have

$$0 = \frac{1}{n} \sum_{i=1}^{n} X_i \left( X_i^T (\hat{\alpha}_n - \alpha_0) + (Y_i, L_k^\frac{1}{2}(\hat{f}_n - f_0)) - \varepsilon_i \right)$$

$$= \frac{1}{n} \sum_{i=1}^{n} X_i X_i^T (\hat{\alpha}_n - \alpha_0) + \frac{1}{n} \sum_{i=1}^{n} (Y_i, L_k^\frac{1}{2}(\hat{f}_n - f_0)) X_i - \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i X_i := D_n(\hat{\alpha}_n - \alpha_0) + G_n(\hat{f}_n - f_0) - a_n,$$

thus $\hat{\alpha}_n - \alpha_0 = -D_n^{-1} G_n(\hat{f}_n - f_0) + D_n^{-1} a_n$.

Next define the function $\varphi_n(t; \alpha, f, g) := F_n(\alpha, f + tg)$, and the fact that $(\hat{\alpha}_n, \hat{f}_n)$ minimizes $F_n(\alpha, f)$ implies

$$\frac{d\varphi(t; \hat{\alpha}_n, \hat{f}_n, g)}{dt} \bigg|_{t=0}, \quad \forall g \in L^2(T),$$

from which we have

$$0 = \frac{1}{n} \sum_{i=1}^{n} \left( X_i^T (\hat{\alpha}_n - \alpha_0) + (Y_i, L_k^\frac{1}{2}(\hat{f}_n - f_0)) - \varepsilon_i \right) (Y_i, L_k^\frac{1}{2}g) + \lambda_n (\hat{f}_n, g).$$

(16)

From (16), we have

$$\frac{1}{n} \sum_{i=1}^{n} X_i^T (\hat{\alpha}_n - \alpha_0) L_k^\frac{1}{2} Y_i + \frac{1}{n} \sum_{i=1}^{n} (Y_i, L_k^\frac{1}{2}(\hat{f}_n - f_0)) L_k^\frac{1}{2} Y_i - \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i L_k^\frac{1}{2} Y_i + \lambda_n \hat{f}_n = 0.$$

(17)

Notice

$$L_{C_n} f = \frac{1}{n} \sum_{i=1}^{n} Y_i(s) Y_i(t), f(t)) = \frac{1}{n} \sum_{i=1}^{n} (Y_i, f) Y_i,$$
thus (17) can be reformulated as
\[(T_n + \lambda_n I)\hat{f}_n - T_n f_0 + H_n(\hat{\alpha}_n - \alpha_0) - g_n = 0.\]

Finally we have
\[
\hat{f}_n - f_0 = (T_n + \lambda_n I)^{-1} T_n f_0 - f_0 - (T_n + \lambda_n I)^{-1} H_n(\hat{\alpha}_n - \alpha_0) + (T_n + \lambda_n I)^{-1} g_n
= -\lambda_n(T_n + \lambda_n I)^{-1} f_0 - (T_n + \lambda_n I)^{-1} H_n(\hat{\alpha}_n - \alpha_0) + (T_n + \lambda_n I)^{-1} g_n.
\]

\[\Box\]

7.2 Proof of Lemma 2
We first prove \(H_n = G_n^*\), which shows \(\|G_n\|_{op} = \|H_n\|_{op}\). Notice for any \(\alpha \in \mathbb{R}^p\) and \(f \in L^2(T)\), we have
\[\alpha^T G_n(f) = \frac{1}{n} \sum_{i=1}^n (Y_i, L_K^\perp f) (\alpha^T X_i) = \frac{1}{n} \sum_{i=1}^n (X_i^T \alpha)^L (L_K^\perp Y_i, f) = \langle H_n(\alpha), f \rangle.\]

Now we turn to bound \(\|G_n\|_{op}\), for \(1 \leq j \leq p\), define the operator \(G_{n,j} : L^2(T) \rightarrow \mathbb{R}\) by
\[G_{n,j}(f) := \frac{1}{n} \sum_{i=1}^n (Y_i, L_K^\perp f) X_{i,j},\]
which can also be viewed as a random variable taking values in a Hilbert space \(L^2(T)^* = L^2(T)\).

Notice \(G_n = (G_{n,1}, \ldots, G_{n,p})\), thus we have
\[\|G_n(f)\|^2 = \sum_{j=1}^p |G_{n,j}(f)|^2 \leq \sum_{j=1}^p \|G_{n,j}\|_{op}^2 \|f\|^2 \leq (\sum_{j=1}^p \|G_{n,j}\|_{op})^2 \|f\|^2,
\]
which means \(\|G_n\|_{op} \leq \sum_{j=1}^p \|G_{n,j}\|_{op}\).

Define the operator \(\xi_{i,j} : L^2(T) \rightarrow \mathbb{R}\) and \(\xi_j : L^2(T) \rightarrow \mathbb{R}\) by
\[\xi_{i,j}(f) := (Y_i, L_K^\perp f) X_{i,j} \quad \text{and} \quad \xi_j(f) := (Y, L_K^\perp f) X_j,
\]
from which we rewrite \(G_{n,j} = \frac{1}{n} \sum_{i=1}^n \xi_{i,j}\) and notice \(\{\xi_{i,j}\}_{i=1}^n\) are independent copies of \(\xi_j\).

By the definition of \(\xi_j\), after noticing the isomorphism between \(L^2(T)^*\) and \(L^2(T)\), we have
\[\|\xi_j\|_{op}^\perp = \|X_j L_K^\perp Y\|^\perp \leq |X_j|^\perp \|L_K^\perp \|_{op} \|Y\|^\perp
\]
by which we have \(\|\xi_j\|_{op}^\perp \leq \kappa^j M_2^j |X_j|^\perp\). After taking expectation and using the growth of moments condition for \(X_j\), we have
\[E(\|\xi_j\|_{op}^\perp) \leq \kappa^j M_2^j E(|X_j|^\perp) \leq \frac{M_2^j}{2} \kappa^j M_2^j |X_j|^\perp = \frac{(\kappa M_2^j)^2}{2} |X_j|^\perp.
\]
Because \(X\) and \(Y\) are independent with zero mean, we have
\[E(\xi_j)(f) = (E Y_i, L_K^\perp f) \cdot E X_j = 0,
\]
which means \(E \xi_j = 0\).

Taking \(B = \kappa^j M_1^2 M_2\) and \(M = \kappa M_2^v\) in the Lemma 7, we have for any \(\delta_2 \in (0, 1)\), with probability at least \(1 - \frac{\delta_2}{p}\), there exists
\[
\|G_{n,j}\|_{op} = \left| \frac{1}{n} \sum_{i=1}^n \xi_{i,j} \right|_{op} \leq \frac{2\kappa M_2^v \log(\frac{2p}{\delta_2})}{n} + \kappa M_1 M_2 \sqrt{\frac{2 \log(\frac{2p}{\delta_2})}{n}} \leq \frac{2\kappa (v + M_1) M_2 \log(\frac{2p}{\delta_2})}{\sqrt{n}}.
\]
Notice \( \|G_n\|_{\text{op}} \leq \sum_{j=1}^{p} \|G_{n,j}\|_{\text{op}} \), thus we have the following relation for the events

\[
\left\{ \|G_n\|_{\text{op}} \geq \frac{2p\kappa(v + M_1)M_2 \log(\frac{2p}{\delta_2})}{\sqrt{n}} \right\} \subseteq \bigcup_{1 \leq i \leq p} \left\{ \|G_{n,i}\|_{\text{op}} \geq \frac{2\kappa(v + M_1)M_2 \log(\frac{2p}{\delta_2})}{\sqrt{n}} \right\},
\]

from which we have

\[
P\left( \|G_n\|_{\text{op}} \geq \frac{2p\kappa(v + M_1)M_2 \log(\frac{2p}{\delta_2})}{\sqrt{n}} \right) \leq \sum_{j=1}^{p} P\left( \|G_{n,j}\|_{\text{op}} \geq \frac{2\kappa(v + M_1)M_2 \log(\frac{2p}{\delta_2})}{\sqrt{n}} \right) \leq \sum_{j=1}^{p} \delta_2 = \delta_2.
\]

Therefore we conclude with probability at least \( 1 - \delta_2 \), there exists

\[
\|G_n\|_{\text{op}} \leq \frac{2p\kappa(v + M_1)M_2 \log(\frac{2p}{\delta_2})}{\sqrt{n}}.
\]

where we let \( c_2 := 2p\kappa(v + M_1)M_2 \).

### 7.3 Lemma 3

The Lemma 3 shows the fact \( \|a_n\| = O_p(n^{-\frac{1}{2}}) \), and its proof is based on the Markov’s inequality.

**Lemma 3.** Under the Assumptions 1 and 3, for any \( \delta_4 \in (0, 1) \), with probability at least \( 1 - \delta_4 \), we have

\[
\|a_n\| \leq \frac{c_3}{\sqrt{\delta_4 n}} \quad \text{with} \quad c_3 := \sqrt{\varphi \sigma} M_1.
\]

**Proof.** Define \( \xi := \varepsilon X \) and \( \xi_i := \varepsilon_i X_i \) (\( 1 \leq i \leq n \)), by which we rewrite \( a_n = \frac{1}{n} \sum_{i=1}^{n} \xi_i \).

Notice for \( i \neq j \), we have \( \text{E}(\xi_i \xi_j) = \text{E}(\varepsilon_i \varepsilon_j X_i^T X_j) = \varepsilon_i \varepsilon_j \text{E}(X_i^T X_j) = 0 \). And

\[
\text{E}(\|\xi\|^2) = \text{E}(\varepsilon^2 \|X\|^2) = \varepsilon^2 \text{E} \|X\|^2 = \sigma^2 \sum_{j=1}^{p} \text{E} |X_j|^2 \leq \varphi \sigma M_1^2,
\]

where we use the growth of moments condition of \( X \) in the Assumption 1. Therefore, we have

\[
\text{E}(\|a_n\|^2) = \text{E}(\|\frac{1}{n} \sum_{i=1}^{n} \xi_i\|^2) = \frac{\text{E}(\|\xi\|^2)}{n} \leq \frac{\varphi \sigma M_1^2}{n},
\]

from which we obtain the Markov’s inequality for \( a_n \),

\[
P(\|a_n\| \geq t) \leq \frac{\varphi \sigma M_1^2}{nt^2}.
\]

Therefore, we can conclude for \( \delta_4 \in (0, 1) \), we have with probability at least \( 1 - \delta_4 \)

\[
\|a_n\| \leq \frac{c_3}{\sqrt{\delta_4 n}} \quad \text{with} \quad c_3 := \sqrt{\varphi \sigma} M_1.
\]

### 7.4 Lemma 4

The Lemma 4 is the concentration inequality for the empirical covariance matrix \( D_n \).

**Lemma 4.** Under the Assumption 1, for \( t > 0 \), we have

\[
P(\|D_n - D\|_{\infty} \geq t) \leq 2p^2 \exp \left( -\frac{nt^2}{16v^2(16M_1^2 + t)} \right).
\]


Proof. For $1 \leq j, k \leq p$, put
\[ d_{jk} := (D)_{jk} \quad \text{and} \quad d_{jk}^n := (D_n)_{jk} = \frac{1}{n} \sum_{i=1}^n X_{i,j} X_{i,k} = \frac{1}{n} \sum_{i=1}^n d_{jk,i}^n \quad \text{with} \quad d_{jk,i}^n := X_{i,j} X_{i,k}. \]
Let $e_{jk,i}^n$ be the centralization of $d_{jk,i}^n$,
\[ e_{jk,i}^n := d_{jk,i}^n - E d_{jk,i}^n = d_{jk,i} - d_{jk}. \]
Thus we have
\[ \{ \| D_n - D \|_{\infty} \geq t \} = \bigcup_{1 \leq j,k \leq p} \{ |d_{jk}^n - d_{jk}| \geq t \} = \bigcup_{1 \leq j,k \leq p} \{ \frac{1}{n} \sum_{i=1}^n e_{jk,i}^n \geq t \}. \]
By the Cauchy-Schwarz inequality, we have
\[ E(d_{jk,i}^n) = E(|X_{i,j}|^2 | X_{i,k}|) \leq (E |X_{i,j}|^2)^{\frac{1}{2}} (E |X_{i,k}|^2)^{\frac{1}{2}} \leq M_2 \sqrt{2} \cdot 4^{l-2} 2l!, \]
where we use the assumption of the growth of moments condition of $X$ in the last step. Using $(2l)! \leq 4^l l!$, we have
\[ E(d_{jk,i}^n) \leq \frac{M_2^2}{2} \sqrt{2} \cdot 4^{l-2} 4^l l! = \frac{(4vM_2)^2}{2} (4v^2)^{l-2} 2l!. \]
Using $|a - b| \leq 2(|a| + |b|)$, we have
\[ |e_{jk,i}^n|^l = |d_{jk,i}^n - E d_{jk,i}^n|^l \leq 2^l (|d_{jk,i}^n|^l + |E d_{jk,i}^n|^l). \]
Take expectation and use the Jensen’s inequality $|E d_{jk,i}^n|^l \leq E |d_{jk,i}^n|^l$, we have
\[ E(|e_{jk,i}^n|^l) \leq 2^{l+1} E |d_{jk,i}^n|^l \leq 2^{l+1} \frac{(4vM_2)^2}{2} (4v^2)^{l-2} 2l! = \frac{(8v^2M_2)^2}{2} (8v^2)^{l-2} 2l!. \]
For the independent random variables $\{e_{jk,i}^n\}_{i=1}^n$, by the Bernstein’s inequality with the growth of moments condition, we have
\[ P \left( \left\| \frac{1}{n} \sum_{i=1}^n e_{jk,i}^n \right\| \geq t \right) \leq 2 \exp \left( -\frac{n^2 t^2}{256v^2 M_2^2 + 16v^2 M_2 t} \right) = 2 \exp \left( -\frac{nt^2}{16v^2 (16M_2^2 + t)} \right). \]
Thus we conclude
\[ P(\| D_n - D \|_{\infty} \geq t) \leq \sum_{1 \leq j,k \leq p} P \left( \left\| \frac{1}{n} \sum_{i=1}^n e_{jk,i}^n \right\| \geq t \right) \leq 2p^2 \exp \left( -\frac{nt^2}{16v^2 (16M_2^2 + t)} \right). \]

\[ \Box \]

7.5 Lemma 5

The Lemma 5 shows we can use $\frac{3}{2} \| D^{-1} \|_{op}$ to bound $\| D_n^{-1} \|_{op}$ from above.

Lemma 5. Under the Assumption 1, for any $\delta_5 \in (0, 1)$, let
\[ N_1 = 48v^2 p \| D^{-1} \|_{op} (4p \| D^{-1} \|_{op} M_1^2 + 1) \log \left( \frac{2p^2}{\delta_5} \right), \]
we have
\[ P(\| D_n^{-1} \|_{op} \leq \frac{3}{2} \| D^{-1} \|_{op}) \geq 1 - \delta_5 \]
when $n > N_1$. 

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Proof. Notice the fact that if $A, B \in \mathbb{R}^{p \times p}$ are invertible and $\|A^{-1}\|_{\text{op}} \|A - B\|_{\text{op}} < 1$, then

$$\|A^{-1} - B^{-1}\|_{\text{op}} \leq \frac{\|A^{-1}\|_{\text{op}}^2 \|A - B\|_{\text{op}}}{1 - \|A^{-1}\|_{\text{op}} \|A - B\|_{\text{op}}}.$$  

(see Lemma E.4 in Sun et al. (2017)). Let $A = D$ and $B = D_n$, when $\|D^{-1}\|_{\text{op}} \|D - D_n\|_{\text{op}} \leq \frac{1}{3}$, we have

$$\|D^{-1} - D_n^{-1}\|_{\text{op}} \leq \frac{\frac{3}{2} \|D^{-1}\|_{\text{op}}}{1 - \frac{1}{3}} = \frac{3}{2} \|D^{-1}\|_{\text{op}},$$

from which we have

$$\|D_n^{-1}\|_{\text{op}} \leq \|D^{-1}\|_{\text{op}} + \|D^{-1} - D_n^{-1}\|_{\text{op}} \leq \frac{3}{2} \|D^{-1}\|_{\text{op}}.$$ 

Therefore, it gives

$$P(\|D_n^{-1}\|_{\text{op}} \leq \frac{3}{2} \|D^{-1}\|_{\text{op}}) \geq P(\|D^{-1}\|_{\text{op}} \|D - D_n\|_{\text{op}} \leq \frac{1}{3}).$$

Recall that we have $\|A\|_{\text{op}} \leq p \|A\|_{\infty}$ for $A \in \mathbb{R}^{p \times p}$, by the Lemma 4, we have

$$P(\|D_n^{-1}\|_{\text{op}} \geq \frac{3}{2} \|D^{-1}\|_{\text{op}}) \leq P(\|D^{-1}\|_{\text{op}} \|D - D_n\|_{\text{op}} \geq \frac{1}{3} \leq P(p \|D^{-1}\|_{\text{op}} \|D - D_n\|_{\infty} \geq \frac{1}{3}) \leq 2p^2 \exp\left(-\frac{n}{48 \pi^2 p \|D^{-1}\|_{\text{op}} (48 p \|D^{-1}\|_{\text{op}} M_2^2 + 1)}\right),$$

Let $N_1$ be defined in (18), as $n > n_1$, we get $P(\|D_n^{-1}\|_{\text{op}} \geq \frac{3}{2} \|D^{-1}\|_{\text{op}}) \leq \delta_5.$

\section{The Auxiliary Lemmas}

The Lemma 1, Lemma 8, Inequality (1) and (2) are from the proof of Tong and Ng (2018). We provide all the complete proofs in this section for integrity.

\subsection{Proof of Lemma 1}

Define the random variable

$$\xi(f) := (T + \lambda_n I)^{-\frac{1}{2}}(L_{\text{HS}}^\frac{1}{2} Y, f)L_{\text{HS}}^\frac{1}{2} Y$$ and $\xi_i(f) := (T + \lambda_n I)^{-\frac{1}{2}}(L_{\text{HS}}^\frac{1}{2} Y_i, f)L_{\text{HS}}^\frac{1}{2} Y_i$ \ (1 \leq i \leq n).

Then $\xi_i$ are independent copies of $\xi$, which takes values in $\text{HS}(T)$.

Recall we define $\{(\tau_k, \varphi_k) : k \geq 1\}$ to be the set of eigenvalue-eigenfunction pairs of the operator $T$, we have

$$\|\xi\|^2_{\text{HS}} = \sum_{k=1}^{+\infty} \|((T + \lambda_n I)^{-\frac{1}{2}}(L_{\text{HS}}^\frac{1}{2} Y, \varphi_k)L_{\text{HS}}^\frac{1}{2} Y\|^2 \leq \|((T + \lambda_n I)^{-\frac{1}{2}}\|^2_{\text{op}}\|L_{\text{HS}}^\frac{1}{2} Y\|^2 \sum_{k=1}^{+\infty} \|L_{\text{HS}}^\frac{1}{2} Y, \varphi_k\|^2$$

$$= \|((T + \lambda_n I)^{-\frac{1}{2}}\|^2_{\text{op}}\|L_{\text{HS}}^\frac{1}{2} Y\|^4 \leq \frac{\kappa^4 M_2^4}{\lambda_n^2},$$

where we use the fact $\sum_{k=1}^{+\infty} |(L_{\text{HS}}^\frac{1}{2} Y, \varphi_k)^2 = \|L_{\text{HS}}^\frac{1}{2} Y\|^2$,

$$\|(T + \lambda_n I)^{-\frac{1}{2}}\|^2_{\text{op}} \leq \frac{1}{\sqrt{\lambda_n}} \text{ and } \|L_{\text{HS}}^\frac{1}{2} Y\| \leq \|L_{\text{HS}}^\frac{1}{2} Y\|_{\text{op}} \leq \kappa M_2.$$

Notice $L_{\text{HS}}^\frac{1}{2} Y$ can be expanded to $\sum_{l=1}^{+\infty} (L_{\text{HS}}^\frac{1}{2} Y, \varphi_l)\varphi_l$, we have

$$\|\xi\|^2_{\text{HS}} = \sum_{k=1}^{+\infty} \left| \sum_{l=1}^{+\infty} (L_{\text{HS}}^\frac{1}{2} Y, \varphi_l) (T + \lambda_n I)^{-\frac{1}{2}} \varphi_l \right|^2$$

$$= \|L_{\text{HS}}^\frac{1}{2} Y\|^2 \left| \sum_{l=1}^{+\infty} \frac{1}{\sqrt{\tau_l + \lambda_n}} (L_{\text{HS}}^\frac{1}{2} Y, \varphi_l) \varphi_l \right|^2$$

$$= \|L_{\text{HS}}^\frac{1}{2} Y\|^2 \sum_{l=1}^{+\infty} \frac{1}{\sqrt{\tau_l + \lambda_n}} (L_{\text{HS}}^\frac{1}{2} Y, \varphi_l)^2 \leq \kappa^2 M_2^2 \sum_{l=1}^{+\infty} \frac{1}{\tau_l + \lambda_n} (L_{\text{HS}}^\frac{1}{2} Y, \varphi_l)^2.$$
Using (4) we have
\[ E[|L^+_K Y, \varphi_i|^2] = E[(Y, L^+_K \varphi_i)]^2 = (L^+_K \varphi_i, L^+_K \varphi_i) = \tau_i, \]
from which we get
\[ E(||\xi||^2_{\mathcal{H}}) \leq \kappa^2 M_2^2 \sum_{i=1}^{\infty} \frac{\tau_i}{\tau_i + \lambda_n} = \kappa^2 M_2^2 D(\lambda_n). \]

Notice
\[ E((L^+_K Y, f)(L^+_K Y, g)) = E((Y, L^+_K f)(Y, L^+_K g)) = E \int_T Y(s)Y(t)(L^+_K f)(s)(L^+_K g)(t)dsdt \]
\[ = \int_T \left( \int_T C(s, t)(L^+_K f)(s)ds \right)(L^+_K g)(t)dt = (L_C L^+_K f, L^+_K g) = (T f, g), \]
from which we have \( E((L^+_K Y, f)L^+_K Y) = T(f) \). Therefore \( (E \xi)(f) = (T + \lambda_n I)^{-\frac{1}{2}} T(f) \).

Taking \( ||\xi||_{\mathcal{H}} \leq \frac{\kappa^2 M_2}{\sqrt{\lambda_n}} \) and \( E(||\xi||^2_{\mathcal{H}}) \leq \kappa^2 M_2^2 D(\lambda_n) \) in the Lemma 6, we have for any \( \delta_1 \in (0, 2e^{-1}) \), with probability at least \( 1 - \delta_1 \)
\[ (T + \lambda_n)^{-\frac{1}{2}}(T_n - T) \|_{op} \leq \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\xi_i - E \xi_i), \]
\[ \leq \frac{2\kappa^2 M_2^2 \log(e)}{n \sqrt{\lambda_n}} + \sqrt{\frac{2\kappa^2 M_2^2 D(\lambda_n) \log(e)}{n}} \leq c_1 \log \left( \frac{2}{\delta_1} \right) B_n, \]
where we let \( c_1 := 2\kappa^2 M_2^2 \) and \( B_n := \frac{1}{n \sqrt{\lambda_n}} + \sqrt{\frac{D(\lambda_n)}{n}}. \)

8.2 Lemma 6 and Lemma 7

The Lemma 6 and Lemma 7 can be seen as the concentration inequalities for the random variables taking values in a Hilbert space. In the Lemma 6, we assume the random variables to be bounded with regard to the norm in the Hilbert space, while we assume that the random variables satisfy the growth of moments condition in the Lemma 7.

**Lemma 6.** Let \( \mathcal{H} \) be a Hilbert space endowed with norm \( \| \cdot \|_{\mathcal{H}} \) and \( \xi \) be a random variable taking values in \( \mathcal{H} \). Let \( \{\xi_i\}_{i=1}^{n} \) be a sequence of \( n \) independent copies of \( \xi \). Assume that \( \|\xi\|_{\mathcal{H}} \leq M \) (a.s.), then for any \( \delta \in (0, 1) \),
\[ \left\| \frac{1}{n} \sum_{i=1}^{n} (\xi_i - E \xi_i) \right\|_{\mathcal{H}} \leq \frac{2 M \log(e)}{n} + \sqrt{\frac{2 E(\|\xi\|_{\mathcal{H}})^2 \log(e)}{n}} \]
with probability at least \( 1 - \delta \).

**Proof.** We refer readers to Pinelis (1994) for the proof of this lemma.

**Lemma 7.** Let \( \mathcal{H} \) be a Hilbert space endowed with norm \( \| \cdot \|_{\mathcal{H}} \). Let \( \{\xi_i\}_{i=1}^{n} \) be a sequence of \( n \) independent random variables in \( \mathcal{H} \) with zero mean. Assume there exist \( B, M > 0 \) such that for all integers \( l \geq 2 \):
\[ E(||\xi_i||_{\mathcal{H}}^{2l}) \leq \frac{B^2}{2} M^{l-2}, \]
then for any \( \delta \in (0, 1) \), we have
\[ P \left( \left\| \frac{1}{n} \sum_{i=1}^{n} \xi_i \right\|_{\mathcal{H}} \geq \frac{2 M \log(e)}{n} + \sqrt{\frac{2 \log(e)}{n} B} \right) \leq \delta. \]

**Proof.** We refer readers to Yurinsky (2006) for the proof of this lemma.
8.3 Lemma 8

The Lemma 8 shows the concentration property of \((T + \lambda_n I)^{-\frac{1}{2}}g_n\).

**Lemma 8.** Under the Assumption 3, for any \(\delta_3 \in (0, 1)\), with probability at least \(1 - \delta_3\), there exists

\[
\| (T + \lambda_n I)^{-\frac{1}{2}} g_n \| \leq \frac{\sigma}{\sqrt{\delta_3}} B_n.
\]

**Proof.** Define random variables \(\xi\) and \(\xi_i\) \((1 \leq i \leq n)\) taking values in the Hilbert space \(L^2(T)\) by

\[
\xi := \varepsilon(T + \lambda_n I)^{-\frac{1}{2}} L^\frac{1}{2}_K Y \quad \text{and} \quad \xi_i := \varepsilon_i(T + \lambda_n I)^{-\frac{1}{2}} L^\frac{1}{2}_K Y_i,
\]

where we notice \(\{\xi_i\}_{i=1}^n\) are independent copies of \(\xi\) and \((T + \lambda_n I)^{-\frac{1}{2}}g_n = \frac{1}{n} \sum_{i=1}^n \xi_i\).

With the independence of \(Y\) and \(\varepsilon\), we have

\[
E\xi = (E\varepsilon) \cdot E((T + \lambda_n I)^{-\frac{1}{2}} L^\frac{1}{2}_K Y),
\]

which shows \(E\xi = 0\).

Expanding \(\xi\) by the basis \(\{\varphi_k : k \geq 1\}\) of the operator \(T\), and we have

\[
E(\|\xi\|^2_H) = E \left( \sum_{k=1}^{+\infty} \langle \xi(T + \lambda_n I)^{-\frac{1}{2}} L^\frac{1}{2}_K Y, \varphi_k \rangle \varphi_k \right)^2
\]

\[
= E \left( \varepsilon^2 \sum_{k=1}^{+\infty} \left( \frac{1}{\sqrt{\tau_k + \lambda_n}} \langle L^\frac{1}{2}_K Y, \varphi_k \rangle \varphi_k \right)^2 \right)
\]

\[
= \sigma^2 \frac{\sum_{k=1}^{+\infty} E(\|L^\frac{1}{2}_K Y, \varphi_k \|^2)}{\tau_k + \lambda_n} = \sigma^2 \frac{\sum_{k=1}^{+\infty} \langle T \varphi_k, \varphi_k \rangle}{\tau_k + \lambda_n} = \sigma^2 D(\lambda_n).
\]

Hence we have

\[
E(\|T + \lambda_n I)^{-\frac{1}{2}} g_n\|^2) = E(\|\frac{1}{n} \sum_{i=1}^n \xi_i\|^2) = \frac{E(\|\xi\|^2)}{n} = \frac{\sigma^2 D(\lambda_n)}{n}.
\]

Using the Markov inequality, we get \(P(\|T + \lambda_n I)^{-\frac{1}{2}} g_n\| \geq t) \leq \frac{\sigma^2 D(\lambda_n)}{nt^2} \) from which we conclude with probability at least \(1 - \delta_3\), we have

\[
\|T + \lambda_n I)^{-\frac{1}{2}} g_n\| \leq \frac{\sigma}{\sqrt{\delta_3}} \sqrt{\frac{D(\lambda_n)}{n}} \leq \frac{\sigma}{\sqrt{\delta_3}} B_n.
\]

\(\square\)

8.4 Two Crucial Inequalities

The following two inequalities play an important role in the proof of the Theorem 1, which shows \(\|(T + \lambda_n I)(T_n + \lambda_n I)^{-1}\|_{op}\) and \(\|(T + \lambda_n I)^{\frac{1}{2}}(T_n + \lambda_n I)^{-\frac{1}{2}}\|_{op}\) can be bounded by \(\|(T + \lambda_n I)^{-\frac{1}{2}}(T_n - T)\|_{op}\).

**Inequality 1.**

\[
\|T + \lambda_n I)(T_n + \lambda_n T)^{-1}\|_{op} \leq \left( \frac{1}{\sqrt{n}} \|(T + \lambda_n I)^{-\frac{1}{2}}(T_n - T)\|_{op} + 1 \right)^2
\]

**Proof.** Using the following decomposition of the operator product

\[
BA^{-1} = (B - A)B^{-1}(B - A)A^{-1} + (B - A)B^{-1} + I
\]

with \(A = T_n + \lambda_n I\) and \(B = T + \lambda_n I\), we have

\[
(T + \lambda_n I)(T_n + \lambda_n T)^{-1} = (T - T_n)(T + \lambda_n I)^{-1}(T - T_n)(T_n + \lambda_n I)^{-1} + (T - T_n)(T + \lambda_n I)^{-1} + I
\]

\[=: F_1 + F_2 + I.\]
For the operator $F_1$, we have

$$\|F_1\|_{op} \leq \|(T - T_n)(T + \lambda_n I)^{-\frac{1}{2}}\|_{op}(T + \lambda_n I)^{-\frac{1}{2}}(T - T_n)\|_{op} \cdot \frac{1}{\lambda_n} = \frac{1}{\lambda_n} \|(T + \lambda_n I)^{-\frac{1}{2}}(T_n - T)\|_{op},$$

where we use the fact $\|AB\|_{op} = \|(AB)^*\|_{op} = \|B^*A\|_{op} = \|BA\|_{op}$ for any self-adjoint operators $A$ and $B$, and the bound $\|(T_n + \lambda_n I)^{-1}\|_{op} \leq \frac{1}{\lambda_n}$.

For the operator $F_2$, applying $\|(T + \lambda_n I)^{-\frac{1}{2}}\|_{op} \leq \frac{1}{\sqrt{\lambda_n}}$, we have

$$\|F_2\|_{op} \leq \|(T - T_n)(T + \lambda_n I)^{-\frac{1}{2}}\|_{op}(T + \lambda_n I)^{-\frac{1}{2}}(T - T_n)\|_{op} \leq \frac{1}{\sqrt{\lambda_n}} \|(T + \lambda_n I)^{-\frac{1}{2}}(T_n - T)\|_{op}.$$  

Thus we obtain

$$\|(T + \lambda_n I)(T_n + \lambda_n T)^{-1}\|_{op} \leq \frac{1}{\lambda_n} \|(T + \lambda_n I)^{-\frac{1}{2}}(T_n - T)\|_{op}^2 + \frac{1}{\sqrt{\lambda_n}} \|(T + \lambda_n I)^{-\frac{1}{2}}(T_n - T)\|_{op} + 1 \leq \left(\frac{1}{\sqrt{\lambda_n}}\|(T + \lambda_n I)^{-\frac{1}{2}}(T_n - T)\|_{op} + 1\right)^2. \hfill \Box$$

**Inequality 2.**

$$\|(T_n + \lambda_n I)^{-\frac{1}{2}}(T + \lambda_n I)^{\frac{1}{2}}\|_{op} = \|(T + \lambda_n I)^{\frac{1}{2}}(T_n + \lambda_n I)^{-\frac{1}{2}}\|_{op} \leq \frac{1}{\sqrt{\lambda_n}} \|(T + \lambda_n I)^{-\frac{1}{2}}(T_n - T)\|_{op} + 1$$

**Proof.** Applying the fact that $\|A^\gamma B^\gamma\|_{op} \leq \|AB\|_{op}^\gamma, \gamma \in (0, 1)$ for positive operators $A$ and $B$ defined on Hilbert space (see Lemma A.7 in Blanchard and Krämer (2010)), we have

$$\|(T_n + \lambda_n I)^{-\frac{1}{2}}(T + \lambda_n I)^{\frac{1}{2}}\|_{op} = \|(T + \lambda_n I)^{\frac{1}{2}}(T_n + \lambda_n I)^{-\frac{1}{2}}\|_{op} \leq \|(T + \lambda_n I)(T_n + \lambda_n T)^{-1}\|_{op} \leq \frac{1}{\sqrt{\lambda_n}} \|(T + \lambda_n I)^{-\frac{1}{2}}(T_n - T)\|_{op} + 1,$$

where in the last step we use the Inequality (1).  

**8.5 Lemma 9**

The Lemma 9 is helpful in constructing the lower bound which is based on the testing multiple hypothesis.

**Lemma 9.** Assume that $N \geq 2$ and suppose there exists $\Theta = \{\theta_i\}_{i=0}^N$ such that the following conditions are satisfied:

1. 2r-separated condition: $d(\theta_j, \theta_k) \geq 2r > 0, \forall 0 \leq j < k \leq N$

2. Kullback-Leibler average condition: if $P_j \ll P_0$ for $1 \leq j \leq N$ and

$$\frac{1}{N} \sum_{j=1}^{N} K(P_j, P_0) \leq \rho \log N$$

for some $0 < \rho < \frac{1}{4}$ and $P_j = P_{\theta_j}$ ($0 \leq j \leq N$).

Then for all possible random variables $\hat{\theta}$, we have

$$\inf_{\hat{\theta}} \sup_{\theta \in \Theta} P_0(d(\hat{\theta}, \theta) \geq r) \geq \frac{\sqrt{N}}{1 + \sqrt{N}} \left(1 - 2\rho - \sqrt{\frac{2\rho}{\log N}}\right) > 0$$

**Proof.** We refer readers to Theorem 2.5 in Tsybakov (2008) for the proof of this lemma.  

\hfill \Box
8.6 Varhsamov-Gilbert Lemma

We need the Varhsamov-Gilbert lemma in the proof of the Theorem 2 to construct the analogy of Θ in Lemma 9.

Lemma 10. Let $H(\theta, \theta') = \sum_{k=1}^{M} 1(\theta_k \neq \theta'_k)$ be the Hamming distance between elements $\theta, \theta'$ in $\{0, 1\}^M$. For any integer $M \geq 8$, there exist vectors $(\theta^i)_{i=0}^{N} \subset \{0, 1\}^M$ such that

(i) $\theta^0 = (0, \cdots, 0)$, (ii) $H(\theta^i, \theta^j) > \frac{M}{2}$ for all $i \neq j$, (iii) $N \geq 2^M$.

Proof. We refer readers to P104 in Tsybakov (2008) for the proof of this lemma. □

8.7 Derivation of the Equality (12)

Let $f_1(X, Y)$ be the density of $(X, Y)$ and $f_2(\varepsilon) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{\varepsilon^2}{2\sigma^2}}$ be the density of $\varepsilon$ since we assume $\varepsilon \sim N(0, \sigma^2)$, then the density of $P_{\alpha^*_0, \beta_1}$ can be written as

$$\frac{dP_{\alpha^*_0, \beta_1}}{d\mu}(Z, X, Y) = f_1(X, Y) f_2(Z - X^T \alpha^*_0 - (Y, \beta_1)),$$

where $\mu$ is a dominant measure on the space $\mathbb{R} \times \mathbb{R}^p \times L^2(T)$.

Therefore, under the assumption of $f_2(\varepsilon) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{\varepsilon^2}{2\sigma^2}}$, we have

$$\log \left( \frac{dP_{\alpha^*_0, \beta_1}}{dP_{\alpha^*_0, \beta_1}} \right) = \log \left( \frac{f_2(Z - X^T \alpha^*_0 - (Y, \beta_1))}{f_2(Z - X^T \alpha^*_0 - (Y, \beta_2))} \right) = \frac{1}{2\sigma^2} (Z - X^T \alpha^*_0 - (Y, \beta_1))^2 - (Z - X^T \alpha^*_0 - (Y, \beta_2))^2$$

Notice when the true parameter is $(\alpha^*_0, \beta_1)$, we have $Z = X^T \alpha^*_0 + (Y, \beta_1) + \varepsilon$, based on which we obtain

$$E_{\alpha^*_0, \beta_1} (Z - X^T \alpha^*_0 - (Y, \beta_1))(Y, \beta_1 - \beta_2) = E \varepsilon E(Y, \beta_1 - \beta_2) = 0.$$

Thus the Kullback-Leibler distance $K(P_{\alpha^*_0, \beta_1} | P_{\alpha^*_0, \beta_2}) = E_{\alpha^*_0, \beta_1} \log \left( \frac{dP_{\alpha^*_0, \beta_1}}{dP_{\alpha^*_0, \beta_2}} \right) = \frac{1}{2\sigma^2} E((Y, \beta_1 - \beta_2))^2$.

□

9 Conclusions and Future Studies

Recently, the PFLM has raised a sizable amount of challenging problems in functional data analysis. Numerous studies focus on the asymptotic convergence rate. However, we analyze the kernel ridge estimator for the RKHS-based PFLM and obtain the non-asymptotic upper bound for the corresponding excess prediction risk. Our work to drive the optimal upper bound weakens the common assumptions in the existing literature on (partially) functional linear regressions. The optimal bound reveals that the prediction consistency holds under the setting where the number of non-functional parameters $p$ slightly increases with the sample size $n$. For fixed $p$, the convergence rate of the excess prediction risk attains the optimal minimax convergence rate under the eigenvalue decay assumption of the covariance operator.

More works could be done to study the non-asymptotic upper bound for the double penalized partially functional regressions. The penalization for the non-functional parameters could be Lasso, Elastic-net, or their generalizations. The proposed non-asymptotic upper bound is novel and substantially beneficial. It is also of interest to do non-asymptotic testing based on large deviation bounds for $\|\alpha_n - \alpha_0\|^2$ and $\|T^{\pm}(f_n - f_0)\|^2$. 

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