CILIBERTO-MIRANDA DEGENERATIONS OF \( \mathbb{CP}^2 \) BLOWN UP IN 10 POINTS

THOMAS ECKL

Abstract. We simplify Ciliberto’s and Miranda’s method [CM08] to construct degenerations of \( \mathbb{CP}^2 \) blown up in several points yielding lower bounds of the corresponding multi-point Seshadri constants. In particular we exploit an asymptotic result of [Eck08a] which allows to check the non-specialty of much fewer linear systems on \( \mathbb{CP}^2 \). We obtain the lower bound \( \frac{117}{370} \) for the 10-point Seshadri constant on \( \mathbb{CP}^2 \).

0. Introduction

Conjecture 0.1 (Nagata, [Nag59]). Let \( p_1, \ldots, p_n \) be \( n \geq 10 \) points in \( \mathbb{CP}^2 \) in general position, and let \( C \) be an irreducible curve of degree \( d \) on \( \mathbb{P}^2 \), passing with multiplicity \( m_i \) through the point \( p_i \). Then

\[ d > \sqrt{n} \sum_{i=1}^{n} m_i. \]

Cast in the language of Seshadri constants, Nagata claimed in effect that \( \frac{1}{\sqrt{n}} \) is the multi-point Seshadri constant of \( p_1, \ldots, p_n \in \mathbb{CP}^2 \), for the line bundle \( O_{\mathbb{P}^2}(1) \), or equivalently, that

\[ H - \frac{1}{\sqrt{n}} \sum_{j=1}^{n} E_j \]

is a nef \( \mathbb{R} \)-divisor on \( \tilde{X} = \text{Bl}_n(\mathbb{P}^2) \), the blowup of \( \mathbb{P}^2 \) in \( n \) points, where \( H \) is the pullback of a line in \( \mathbb{P}^2 \) and \( E_j \) are the exceptional divisors over the blown up points. The best known bounds for the Seshadri constant of 10 points on \( \mathbb{P}^2 \) until very recently were \( \frac{6}{19} \) by Biran [Bir99] and \( \frac{177}{560} \) by Harbourne and Roé [HR03]. Some months ago, Ciliberto and Miranda [CM08] presented a new method to improve these bounds, and obtained \( \frac{55}{174} \).

Their approach relies on the well known fact that Nagata’s conjecture can be deduced from another conjecture on the dimension of linear systems on \( \mathbb{CP}^2 \) (see e.g. [CM01]):

Conjecture 0.2 (Harbourne-Gimigliano-Hirschowitz [Har86, Gim87, Hir89]). Let \( p_1, \ldots, p_n \) be \( n \) points in \( \mathbb{CP}^2 \) in general position, and let \( \pi : X \rightarrow \mathbb{CP}^2 \) be the blow up of these \( n \) points. Furthermore, call \( H \) the divisor class of a line on \( \mathbb{CP}^2 \), and

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denote the exceptional divisor over $p_i$ with $E_i$. Given a degree $d$ and $n$ multiplicities $m_1, \ldots, m_n$, the linear system $|d\pi^*H - \sum_{i=1}^n m_i E_i|$ has the expected dimension

$$\max(-1, \frac{d(d+3)}{2} - \sum_{i=1}^n m_i(m_i + 1)/2)$$

iff there exists no $(-1)$-curve $C$ on $X$ such that

$$C.(d\pi^*H - \sum_{i=1}^n m_i E_i) \leq -2.$$

Linear systems on $\mathbb{P}^2$ are often analysed via degenerations: If the degenerated linear system on the central fiber of the $\mathbb{P}^2$-degeneration has expected dimension, then nearby fibers inherit this property by semi-continuity. In [CM98] Ciliberto and Miranda use a degeneration of $\mathbb{P}^2$ into a union of $\mathbb{P}^2$ and the first Hirzebruch surface $F_1$ to check the Harbourne-Hirschowitz conjecture in a number of cases. Unfortunately, for Nagata’s conjecture the results do not yield interesting bounds for Seshadri constants. The failure is due to $(-1)$-curves which intersect the degenerated linear systems negatively.

In [CM08] Ciliberto and Miranda observe that the normal bundle of these "bad" $(-1)$-curves is negative. Their new idea is to flop these curves, possibly after some blow ups, thus removing them from a new degeneration. Iterating these flops of "bad" curves Ciliberto and Miranda obtain $\frac{55}{174}$ as a lower bound for the Seshadri constant of 10 points on $\mathbb{P}^2$.

The main technical difficulty in their calculations is the study of linear systems with small expected dimension. They require an intricate case-by-case analysis. To avoid this as much as possible this paper uses an approximative approach to Nagata’s conjecture developped in [Eck08a]:

**Theorem 0.3** ([Eck08a]). Let $p_1, \ldots, p_n$ be $n \geq 10$ points on $\mathbb{CP}^2$ in general position, and let $\pi : X \to \mathbb{CP}^2$ be the blow up of these $n$ points. If $(d_i, m_i), i \in \mathbb{N}$, is a sequence of integer pairs, such that the linear system $|d_i\pi^*H - m_i \sum_{j=1}^n E_j|$ is non-empty of expected dimension, and $\frac{d_i}{m_i} \xrightarrow{i \to \infty} \frac{a}{\sqrt{n}}$ then the $\mathbb{R}$-divisor

$$\pi^*H - \frac{a}{\sqrt{n}} \sum_{j=1}^n E_j$$

is nef on $X$.

A first attempt to apply this method was made in [Eck08b], using Dumnicki’s reduction algorithm [Dum07], but only led to the uninteresting bound $\frac{4}{15}$. In this paper the combination with Ciliberto-Miranda degenerations yields the new and up to now best known lower bound $\frac{117}{370} \approx 0.31621 \ldots$ compared to $\frac{55}{174} \approx 0.31609 \ldots$ this is one decimal closer to $\sqrt{10} \approx 3.1622 \ldots$.

Besides the approximative approach there are two other new ingredients in this paper:

1. We consistently use a non-specialty criterion for line bundles, which generalizes the core of Harbourne’s Criterion for linear systems on $\mathbb{P}^2$ blown up in several points, in [Har85]. It also works for Hirzebruch surfaces.

2. Instead of flopping the "bad" $(-1)$-curves we only blow them up until the exceptional divisor is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. According to the Atiyah flop
Ciliberto and Miranda continue blowing down the other fibering of \( \mathbb{P}^1 \times \mathbb{P}^1 \) thus really erasing the ”bad” curve. But in this way they produce non-normal components, and to prove non-speciality they must again pull back to the blown up components.

To avoid this extra turn we just modify the degenerated line bundle on the blow up such that its intersection with the ”bad” curves vanishes. The central fiber of our degenerations thus contain more components but we still consider this procedure to be more transparent. In particular it lead us to discover two degenerations underlying the Fifth Degeneration of Ciliberto and Miranda, and finally to the new bound \( \frac{117}{370} \).

The steps of the method and the statements proving the correctness are presented in Section 1. The calculations leading to the new bound \( \frac{117}{370} \) are the contents of Section 2. In the last paragraphs we show that the next degeneration is considerably more complicated than the previous ones, and we discuss some conditions which would guarantee that the algorithm never terminates.

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Notation. We consider smooth complex projective surfaces \( X \) and sequences of morphisms

\[
X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \ldots \xrightarrow{\pi_1} X_0 := X,
\]

where each \( \pi_i \) is the blow up of a point \( p_i \in X_{i-1} \). We also denote \( X_i \) by \( X(p_1, \ldots, p_i) \), and set \( \pi := \pi_n \circ \ldots \circ \pi_1 \).

Note that the \( p_1, \ldots, p_n \) are not assumed to be in general position. For example, the point \( p_i \) can be mapped onto the point \( p_j \) by the intermediate blow downs, if \( i > j \). Then \( p_i \) is said to be infinitely near to \( p_j \). Sometimes we emphasize this relation by brackets: \([p_1; p_2, \ldots, p_k]\) means that the points \( p_2, \ldots, p_k \) are infinitely near to \( p_1 \). Each of the \( p_2, \ldots, p_k \) can again be replaced by pairs of infinitely near points etc.

\( E_i \subset X_i = X(p_1, \ldots, p_i) \) denotes the exceptional divisor over \( p_i \), the divisor

\[
\mathcal{E}_i := \pi_n^* \cdots \pi_{i+1}^* E_i
\]

denotes the pullback on \( X_n \).

\((\mathcal{E}_1, \ldots, \mathcal{E}_n)\) is called an exceptional configuration on \( X_n \). We know

\[
\mathcal{E}_i^2 = -1, \quad \mathcal{E}_i \cdot \mathcal{E}_j = 0, i \neq j, \quad \pi^* \mathcal{F} \cdot \mathcal{E}_i = 0
\]

for all line bundles \( \mathcal{F} \) on \( X \), and \( \text{Pic}(X_n) \) is generated by \( \text{Pic}(X) \) and \( \mathcal{E}_1, \ldots, \mathcal{E}_n \).

If \( X = \mathbb{P}^2 \) we set \( \mathcal{E}_0 := \pi^* \mathcal{O}_{\mathbb{P}^2}(1) \), and

\[
\mathcal{L}(d; m_1, \ldots, m_n) := d \cdot \mathcal{E}_0 - \sum_{i=1}^n m_i \mathcal{E}_i.
\]

Sometimes, multiple occurrences of the same coefficient \( m \) is abbreviated by \( m^k \).
Later on, degree and multiplicities will linearly depend on parameters \( d, m, a \). We do not abbreviate these forms by introducing new letters thus making the notation of some line bundles quite cumbersome. But we prefer to leave the dependence of degrees and multiplicities transparent and easy to analyse.

Finally, \( F_0 \cong \mathbb{P}^1 \times \mathbb{P}^1, F_1 \cong \mathbb{P}^2(p), \ldots, F_k, \ldots \) denote the Hirzebruch surfaces, with projections \( \pi_{F_k} \) to \( \mathbb{P}^1 \). Accepting some ambiguity \( E_k \subset F_k \) denotes the curve at infinity, with self intersection \(-k\), whereas \( F_k \) denotes a fiber of the \( \mathbb{P}^1\)-bundle \( F_k \).

On \( \mathbb{P}(p_1, \ldots, p_n) \),

\[
\mathcal{L}_{\mathcal{F}_k}(d_1, d_2; m_1, \ldots, m_n) := d_1 \cdot E_k + d_2 \cdot F_k - \sum_{i=1}^{n} m_i \mathcal{E}_i.
\]

1. The Ciliberto-Miranda method

1.1. Degenerations and the Gluing Lemma. Degenerations are a well-known tool to study (complete) linear systems.

**Proposition 1.1.** Let \( f : X \to \Delta \) be a reduced family of projective complex schemes over the unit disc \( \Delta \), and let \( \mathcal{L} \) be a line bundle on \( X \). Let \( X_t \) denote the fiber of \( X \) over \( t \in \Delta \), and set \( \mathcal{L}_t := \mathcal{L}|_{X_t} \). Then:

\[
h^1(X_0, L_0) = 0 \implies h^1(X_t, L_t) = 0,
\]

for all \( t \in \Delta' \subset \Delta \), a smaller unit disc.

**Proof.** This is a consequence of upper-semicontinuity of the \( h^1 \)-function on flat families of projective schemes [Har77, Thm.III.12.8]. The flatness follows because \( X \) is reduced over a 1-dimensional smooth base [Har77, Prop.III.9.7]. □

Using this proposition on a given degeneration requires to calculate \( H^1(X_0, L_0) \).

**Lemma 1.2** (Gluing Lemma [CM08]). Let \( X = \bigcup_{i=1}^{n} V_i \) be a union of projective complex schemes, where the \( V_i \) are closed subschemes of \( X \). Set \( W_k := \bigcup_{i=1}^{k} V_i \), \( k = 1, \ldots, n \), and denote by \( C_{k-1} \) the scheme-theoretic intersection \( V_k \cap W_{k-1} \), \( k = 2, \ldots, n \).

Let \( L \) be a line bundle on \( X \) satisfying

(i) \( H^1(V_i, L|_{V_i}) = 0 \) for \( i = 1, \ldots, n \),

(ii) the difference maps \( H^0(W_{k-1}, L|_{W_{k-1}}) \oplus H^0(V_k, L|_{V_k}) \to H^0(C_{k-1}, L|_{C_{k-1}}) \) are surjective for all \( k = 2, \ldots, n \).

Then: \( H^1(X, L) = 0 \).

**Proof.** This follows inductively from the long exact cohomology sequences obtained of the short exact sequences in the next lemma, applied on \( W_k = W_{k-1} \cup V_k \) and \( L|_{W_k} \). □

**Lemma 1.3.** Let \( X = V \cup W \) be a projective complex scheme, where \( V, W \) are closed subschemes of \( X \), \( C = V \cap W \) the scheme-theoretic intersection, and \( \mathcal{L} \) an invertible sheaf on \( X \).

Then there exists an exact sequence of coherent sheaves on \( X \),

\[
0 \to \mathcal{L} \to \mathcal{L}|_{V} \oplus \mathcal{L}|_{W} \to \mathcal{L}|_{C} \to 0,
\]

where \( \mathcal{L}|_{V} \oplus \mathcal{L}|_{W} \to \mathcal{L}|_{C} \) is the difference map.
Proof. The exactness of the sequence can be checked on open affine subsets SpecA, on which \( \mathcal{L} \) is trivial. If \( I_V, I_W, I_C \subset A \) are the ideals describing the closed subschemes \( V, W, C \) in SpecA, the claim follows from \( I_V + I_W = I_C \) and \( I_V \cap I_W = (0) \).

Remark 1.4. Condition (ii) of the Gluing Lemma \([3.2]\) is already satisfied if \( H^1(W_{k-1}, L_{|W_{k-1}} \otimes \mathcal{I}_{C_{k-1}/W_{k-1}}) = 0 \) or \( H^1(V_k, L_{|V_k} \otimes \mathcal{I}_{C_{k-1}/V_k}) = 0 \). Here, \( \mathcal{I}_{C_{k-1}/W_{k-1}} \) resp. \( \mathcal{I}_{C_{k-1}/V_k} \) are the ideal sheaves of \( C_{k-1} \) in \( W_{k-1} \) resp. \( V_k \). This follows from the long exact cohomology sequence associated to

\[
0 \to L_{|W_{k-1}} \otimes \mathcal{I}_{C_{k-1}/W_{k-1}} \to L_{|W_{k-1}} \to L_{|C_{k-1}} \to 0
\]

(or the analogue sequence for \( V_k \)), because then

\[
H^0(W_{k-1}, L_{|W_{k-1}}) \to H^0(C_{k-1}, L_{|C_{k-1}}) \to H^1(W_{k-1}, L_{|W_{k-1}} \otimes \mathcal{I}_{C_{k-1}/W_{k-1}}) = 0
\]

is exact (or the analogue sequence for \( V_k \)).

1.2. Non-special linear systems. The degenerations of \( \mathbb{P}^2 \) blown up in 10 points studied later on have a central fiber \( X_0 \) consisting of irreducible components isomorphic to \( \mathbb{P}^2 \) or a Hirzebruch surface \( \mathbb{F}_k \) blown up in several points, possibly in special position, and intersecting in curves without embedded points. Then \( L_{|V_k} \otimes \mathcal{I}_{C_{k-1}/V_k} = L_{|V_k} \otimes \mathcal{O}_{V_k}(-C_{k-1}) \) is again a line bundle. In view of Remark \([3.4]\) this implies for applying the Gluing Lemma \([3.2]\) that we only need criteria for the vanishing of \( H^1 \)-groups of line bundles on such surfaces.

The first vanishing criterion is extracted from the central argument of Harbourne’s Criterion discussed afterwards.

Theorem 1.5. Let \( \mathbb{F} \) be \( \mathbb{P}^2 \) or a Hirzebruch surface \( \mathbb{F}_k \). Let \( X = \mathbb{F}(p_1, \ldots, p_n) \) be a blow up of \( \mathbb{F} \) in several points. Let \( \mathcal{F} \) be a line bundle on \( X \), and set

\[
l := \min\{k : \mathcal{F} = \pi_n^* \cdots \pi_1^* \mathcal{F}_k, \mathcal{F}_k \text{ line bundle on } X_k = \mathbb{F}(p_1, \ldots, p_n)\}.
\]

Let \( C \) be a reduced curve on \( X_l \) with irreducible components \( C_1, \ldots, C_r \). Assume that

1. \( |K_{X_l} \otimes \mathcal{O}_{X_l}(C)|.C_i < \mathcal{F}_i.C_i \) for all \( i = 1, \ldots, r \), and
2. \( H^1(X_l, \mathcal{F}_l \otimes \mathcal{O}_{X_l}(-C)) = 0 \).

Then: \( H^1(X, \mathcal{F}) = 0 \).

Proof. As \( X \) is obtained from successive blow ups of points from \( X_1 \), the cohomology groups of \( \mathcal{F} \) and \( \mathcal{F}_i \) are isomorphic \([3.4]\) Prop.V.3.4. Hence we assume \( l = n \).

The dualizing sheaf on the Cartier divisor \( C \) is \( \omega_C := [K_X \otimes \mathcal{O}(C)]|_C \) and Serre duality holds (see \([3.4]\) III.7):

\[
h^1(C, \mathcal{F}|C) = h^0(C, \omega_C \otimes \mathcal{F}^{-1}|C).
\]

Using the morphism \( \phi : C_1 \sqcup \ldots \sqcup C_r \to C \) from the disjoint union of the irreducible components \( C_1, \ldots, C_r \) on \( C \), the inclusion \( \mathcal{O}_C \subset \phi_*(\mathcal{O}_{C_1 \sqcup \ldots \sqcup C_r}) \) and the projection formula, we conclude that \( H^0(C, \omega_C \otimes \mathcal{F}^{-1}|C) \) is a subgroup of

\[
H^0(C_1 \sqcup \ldots \sqcup C_r, \phi^*(\omega_C \otimes \mathcal{F}^{-1}|C)) = \bigoplus_{i=1}^r H^0(C_i, [K_X \otimes \mathcal{O}_X(C) \otimes \mathcal{F}^{-1}|C]|_{C_i}).
\]
But since \( [K_X \otimes \mathcal{O}_X(C)] \cdot C_i \leq F.C_i \), the degree of the invertible sheaf \( [K_X \otimes \mathcal{O}_X(C) \otimes F^{-1}_{|C_i}|] \) is negative on the irreducible curve \( C_i \), hence
\[
h^0(C_i, [K_X \otimes \mathcal{O}_X(C) \otimes F^{-1}_{|C_i}|]) = 0,
\]
and by Serre duality, \( H^1(C, F_{|C}) = 0 \).

The claim follows from considering the long exact cohomology sequence associated to the short exact sequence
\[
0 \to F \otimes \mathcal{O}_X(-C) \to F \to F_{|C} \to 0.
\]

\[\square\]

Of course, this theorem just shifts the proof of vanishing to another line bundle which hopefully is simpler. For \( F = \mathbb{P}^2 \), Harbourne [Har85] developed an inductive scheme which guarantees vanishing if \( |-K_X| \) contains an irreducible and reduced section and the coefficients of \( F = \mathcal{L}(d; m_1, \ldots, m_n) \) satisfy some numerical conditions.

**Definition 1.6.** A surface \( X = \mathbb{P}^2(p_1, \ldots, p_n) \) is called **strongly anticanonical** iff the anticanonical linear system \( |-K_X| \) contains an irreducible and reduced section.

A line bundle \( F = \mathcal{L}(d; m_1, \ldots, m_n) \) on \( X \) is called **standard** iff
\[
d \geq 0, \ m_i \geq 0, \ m_i - m_{i+1} \geq 0, \ d - m_i - m_j - m_k \geq 0, \ 1 \leq i < j < k \leq n.
\]

\( F \) is called **excellent** iff \( F \) is standard and \( F.K_X < 0 \).

**Remark 1.7.** Let \( p_1, \ldots, p_n \in \mathbb{P}^2 \) be \( n \leq 8 \) points in general position on \( \mathbb{P}^2 \). Then \( X = \mathbb{P}^2(p_1, \ldots, p_n) \) is strongly anti-canonical: For 8 points in general position on \( \mathbb{P}^2 \) there always exists a smooth cubic curve passing through the points, hence pulling back to a section of the anticanonical bundle \( -K_X = \mathcal{L}(3; 1^n) \)

**Remark 1.8.** The line bundle \( \mathcal{L}(d; m_1, \ldots, m_n) \) on \( X = \mathbb{P}^2(p_1, \ldots, p_n) \) is excellent if \( \mathcal{L}(d'; m'_1, \ldots, m'_n) \) is excellent and
\[
d \geq d', \ m_i \leq m'_i, m_i \geq m_j \ \text{for all} \ 1 \leq i < j \leq n.
\]

**Theorem 1.9 (Harbourne’s Criterion [Har85]).** Let \( X = \mathbb{P}^2(p_1, \ldots, p_n) \) be strongly anti-canonical and \( F \) an excellent line bundle on \( X \). Then:
\[
H^1(X, F) = 0.
\]

**Proof.** The statement is a consequence of Theorem 1.5 and another way of writing standard line bundles:

**Claim.** A line bundle \( F \) on \( X \) is standard if, and only if, it can be written as
\[
F = a_0 \mathcal{E}_0 + a_1 (\mathcal{E}_0 - \mathcal{E}_1) + a_2 (2\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2) + \sum_{i=3}^{n} a_i (-K_{X_i}),
\]
for some integers \( a_0, \ldots, a_n \geq 0 \).

**Proof of the Claim.** This is [Har85, Lem. 1.4]. \( \square \)

Here the anticanonical line bundle \( -K_{X_i} \) on \( X_i = \mathbb{P}^2(p_1, \ldots, p_i) \) is interpreted as a line bundle on \( X \) via pullback.
The proof of the Criterion now proceeds by a double induction on
\[ l := \min\{k : \mathcal{F} = \pi^* \cdots \pi^* \mathcal{F}_k, \mathcal{F}_k \text{ line bundle on } X_k = \mathbb{P}^2(p_1, \ldots, p_k)\} \]
and \( a_i \): The induction start with \( \mathcal{F} = \mathcal{O}_X \) is trivial. For the induction step, we can apply Theorem 1.5 because

(i) all the line bundles \( \mathcal{E}_0, \mathcal{E}_0 - \mathcal{E}_1, 2\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 \) and the \( -K_X \) have an irreducible and reduced section on the \( X_l \) where they are not a pullback: a line, the strict transforms of a line through \( p_1 \) and of a conic through \( p_1, p_2 \), the images of the \( -K_X \)-section,

(ii) \( (K_X + C_l)C_l < 0 \) for \( l = 0, 1, 2 \), and \( = 0 \) for \( l \geq 3 \),

(iii) \( \mathcal{F} \mathcal{E}_0 = a_0 > 0 \) for \( l = 0 \), \( \mathcal{F} \mathcal{E}_0 - \mathcal{E}_1 = a_0 + a_1 - a_1 = a_0 \geq 0 \) for \( l = 1 \),

\( \mathcal{F} (2\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2) = 2a_0 + a_1 + 2a_2 > 0 \) for \( l = 2 \), and \( \mathcal{F} (-K_X) > 0 \) for \( l \geq 3 \).

\[ \square \]

Not all the surfaces occurring in the degenerations constructed below are strongly anticanonical blow ups of \( \mathbb{P}^2 \). In these cases, we will try to find curves on which we can iteratively apply Theorem 1.5 until we obtain a linear system for which we can show vanishing with Harbourne’s Criterion.

The next criterion will be useful for checking the surjectivity condition in the Gluing Lemma 1.2.

**Proposition 1.10.** Let \( X \) be a projective complex surface and \( \pi : \tilde{X} = X(p) \to X \) the blow up of \( X \) in \( p \), with exceptional divisor \( E \subset \tilde{X} \). Let \( \mathcal{F} \) be a line bundle on \( X \) such that \( H^1(X, \mathcal{F}) = 0 \). Then:

\[ H^1(\tilde{X}, \pi^* \mathcal{F} \otimes \mathcal{O}(E)) = 0. \]

**Proof.** From the exact sequence
\[ 0 \to \pi^* \mathcal{F} \to \pi^* \mathcal{F} \otimes \mathcal{O}(E) \to \pi^* \mathcal{F}_E \otimes \mathcal{O}_E(E) = \mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^1}(-1) \to 0 \]
we obtain the exact sequence
\[ H^1(\tilde{X}, \pi^* \mathcal{F}) \to H^1(\tilde{X}, \pi^* \mathcal{F} \otimes \mathcal{O}(E)) \to H^1(E, \mathcal{O}_E(E)), \]
and the proposition follows from \( 0 = H^1(X, \mathcal{F}) = H^1(\tilde{X}, \pi^* \mathcal{F}) \) and \( H^1(E, \mathcal{O}_E(E)) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0 \).

**1.3. Transforming exceptional configurations.** Harbourne’s Criterion requires the standardness of line bundles on \( \mathbb{P}^2(p_1, \ldots, p_n) \) which depends on the exceptional configuration. These configurations are not at all unique on a given surface, and often a major step in applying Harbourne’s Criterion is to change them appropriately, by means of Cremona transformations. Normally, Cremona transformations denote birational self-maps of \( \mathbb{P}^2 \). But in our context we instead consider the change of exceptional configurations on the desingularisation of these rational maps.

We only use compositions of quadratic Cremona transformations involving 3 base points. The following lemma describes the possible configurations of these base points:

**Lemma 1.11** ([Har85]). Let \( X = \mathbb{P}^2(p_1, \ldots, p_n) \) and \( p_i, p_j, p_k \in X \) points such that either

(i) \( p_i, p_j, p_k \) are not collinear, or
such that

\[ E_0 = 2E_0' - E_i' - E_j' - E_k', \quad E_i = E_i' \text{ for } l \neq i, j, k, \]

\[ E_i = E_0' - E_j' - E_k', \quad E_j = E_0' - E_i' - E_k', \quad E_k = E_0' - E_i' - E_j'. \]

Then there exist \( p_1', \ldots, p_n' \) and an isomorphism \( \mathbb{P}^2(p_1, \ldots, p_n) = \mathbb{P}^2(p_1', \ldots, p_n') \) such that

\[ \mathcal{E}_0 = 2\mathcal{E}_0' - \mathcal{E}_i' - \mathcal{E}_j' - \mathcal{E}_k', \quad \mathcal{E}_i = \mathcal{E}_i' \text{ for } l \neq i, j, k, \]

\[ \mathcal{E}_i = \mathcal{E}_0' - \mathcal{E}_j' - \mathcal{E}_k', \quad \mathcal{E}_j = \mathcal{E}_0' - \mathcal{E}_i' - \mathcal{E}_k', \quad \mathcal{E}_k = \mathcal{E}_0' - \mathcal{E}_i' - \mathcal{E}_j'. \]

In particular, a line bundle \( \mathcal{F} = d\mathcal{E}_0 - m_1\mathcal{E}_1 - \ldots - \mathcal{E}_n \) can be rewritten as

\[ \mathcal{F} = (2d - m_i - m_j - m_k)\mathcal{E}_0' - \sum_{l \neq i, j, k} m_i\mathcal{E}_l' - (d - m_j - m_k)\mathcal{E}_i' - (d - m_i - m_j)\mathcal{E}_j'. \]

**Proof.** Note that for all 3 configurations we can renumber the base points \( p_1, \ldots, p_k \) such that \( i = 1, j = 2, k = 3 \). In particular, the blow ups for the other points \( p_4, \ldots, p_n \) are not touched when exchanging \( p_1, p_2, p_3 \) with \( p_1', p_2', p_3' \). So we can assume w.l.o.g. that \( n = 3 \).

The proof can be read off the following diagrams. The integers denote self intersections, the arrows infinitely near points.

(i)

(ii)

(iii)

We detail the last diagram, the others being even simpler: Let \( L \) and \( L' \) denote the lines through \( p_1 \) and \( p_1' \). Furthermore, let \( \overline{L}, \overline{E}_1, \overline{E}_2 \) and \( \overline{E}_3 = E_3 \)
The claim follows from the equalities $E_0 = L + E_1 + 2E_2 + E_3$, $E_1 = E_1 + E_2 + E_3$, $E_2 = E_2 + E_3$, $E_3 = E_3$ on $\mathbb{P}^2(p_1, p_2, p_3)$ and

$\mathcal{L} = E_0 - E_1 - E_2$, $\mathcal{L} = E_1 - E_2 - E_3$, $\mathcal{L} = E_2 - E_3 - E_0$.

Remark 1.12. Even if the blow ups of $p_4, \ldots, p_n$ are not touched by the Cremona transformations, these points might become infinitely near to $p'_1, p'_2, p'_3$. See the Constructions below.

We must also know the position of the $p'_1, \ldots, p'_n$ relative to each other. The following statement is the most general one in this direction, and is almost always implicitly applied:

**Proposition 1.13.** Assume that $p_1, \ldots, p_n$ on $\mathbb{P}^2$ do not contain infinitely near points and are in general position. If $\mathbb{P}^2(p_1, \ldots, p_n) \cong \mathbb{P}^2(p'_1, \ldots, p'_n)$ by means of a quadratic Cremona transformation as described in Lemma 1.11 case (i), then the $p'_1, \ldots, p'_n$ do not contain infinitely near points and are in general position.

**Proof.** A quadratic Cremona transformation as described in Lemma 1.11 case (i), induces a birational map from $\mathbb{P}^2$ onto itself, which is an isomorphism outside the lines connecting the 3 base points. Since $p_4, \ldots, p_n$ are in general position, they do not lie on these lines. The claim follows because 3 points on $\mathbb{P}^2$ can be freely moved around by the action of $\text{PGL}(3)$, hence are always in general position.

In some situations some of the points $p_4, \ldots, p_n$ will not be in general position relative to $p_1, p_2, p_3$. We collect the configurations relevant in the constructions below:

**Cremona Transformation I.** Let $p_1, \ldots, p_5$ be points on $\mathbb{P}^2$ such that

- $p_1, p_2, p_4$ are not collinear,
- $p_3$ is infinitely near to $p_2$, directed to $p_1$ and
- $p_5$ is infinitely near to $p_4$, directed to $p_1$.

After a Cremona transformation with base points $p_2, p_3, p_4$, the new exceptional configuration is again of the type above, with base points $p'_2, p'_3, p'_4$, and the infinitely near points $p'_3, p'_4$ are directed to $p'_5$. This can be read off the following diagram:

![Diagram](https://example.com/diagram.png)

**Cremona Transformation II.** Let $p_1, \ldots, p_5$ be points on $\mathbb{P}^2$ such that
• $p_1, p_2, p_4$ are not collinear,
• $p_3$ is infinitely near to $p_2$, but not directed to $p_1$ or $p_4$ and
• $p_5$ is infinitely near to $p_4$, but not directed to $p_1$ or $p_2$.

After two Cremona transformation with base points $p_1, p_2, p_4$ and $p_1', p_3, p_5'$, the new exceptional configuration is again of the type above, with base points $p_1'', p_3', p_5'$, and the infinitely near points $p_2'', p_4''$ are not directed to $p_1'', p_4''$ resp. $p_3', p_5'$. This can be read off the following diagram:

[Diagram showing Cremona Transformation III]

Cremona Transformation III. Let $p_1, \ldots, p_5$ be points on $\mathbb{P}^2$ such that

• No three of the points $p_1, p_2, p_3, p_4$ are collinear,
• $p_5$ is infinitely near to $p_4$, directed to $p_1$.

After a Cremona transformation with base points $p_1, p_2, p_3$ the point $p_5'$ infinitely near to $p_4'$ is directed to $p_1'$. This can be read off the following diagram:

[Diagram showing Cremona Transformation III]

Finally, $p_1, \ldots, p_n$ might not be in general position because they lie on a special curve $C$. This curve can be interpreted as the section of a line bundle $\mathcal{L}(d; m_1, \ldots, m_n)$, which is transformed by a Cremona transformation — or several of them — to $\mathcal{L}(d'; m_1', \ldots, m_n')$ resp. $C'$. Thus we can translate the special position of the $p_1, \ldots, p_n$ into a special position of the $p_1', \ldots, p_n'$.

1.4. Throwing curves. Consider the setting of Prop. 1.14. Let $f : \mathcal{X} \to \Delta$ be a family of complex varieties over the unit disc $\Delta$, and let $\mathcal{L}$ be a line bundle on $\mathcal{X}$. Then the cohomology group $H^1(X_t, L_t)$ vanishes if $H^1(X_0, L_0) = 0$. To show this we want to apply the Gluing Lemma 1.14. Its application fails if $H^1(V, \mathcal{O}_V) \neq 0$ on an irreducible component $V \subset X_0$. For $\mathbb{P}^2$ or Hirzebruch surfaces $\mathbb{F}_k$ blown up in several points this is the case if there exists a $(-1)$-curve $E$ on $V$ such that $\mathcal{L}|_V.E \leq -2$ and $\mathcal{L}|_V$ has a global section:

Lemma 1.14 ([Harr85, CM98]). Let $\mathbb{F}$ be $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_k$, and $V = \mathbb{F}(p_1, \ldots, p_n)$ be a blow up of $n$ points in $\mathbb{F}$. Let $L$ be an effective line bundle on $V$. Assume that $E$ is a $(-1)$-curve on $V$ such that $L.E \leq -2$. Then:

$H^1(V, L) \neq 0$.

Proof. Set $L.E = -k$, $k \geq 2$ an integer. From the short exact sequence

$0 \to L \otimes \mathcal{O}(-E) \to L \to L|_E \to 0$
we obtain the following part of the long exact cohomology sequence:

\[ H^1(V, L) \rightarrow H^1(E, L_{|E}) \rightarrow H^2(V, L \otimes O(-E)). \]

Next, we calculate

\[ H^1(E, L_{|E}) \cong H^1(\mathbb{P}^1, O(-k)) \cong H^0(\mathbb{P}^1, O(-2 + k))^\vee \neq 0. \]

Since \( L \) has a global section, \( L \cdot E < 0 \) implies that \( E \) is a fixed divisor in the associated linear system. Consequently, \( L - E \) is also effective, and \( (L - E).F \geq 0 \) for every nef divisor \( F \). For \( F = \mathbb{P}^2 \) let \( F \) be the pull back of a line on \( \mathbb{P}^2 \), for \( F = \mathbb{F}_k \) let \( F \) be the pull back of a fiber in the \( \mathbb{P}^1 \)-bundle \( \mathbb{F}_k \). In both cases \( K_V.F < 0 \), and this implies

\[ (K_V \otimes L^\vee \otimes O(E)).F < 0, \]

hence \( K_V \otimes L^\vee \otimes O(E) \) cannot be effective. Consequently,

\[ H^2(V, L \otimes O(-E)) \cong H^0(V, K_V \otimes L^\vee \otimes O(E))^\vee = 0. \]

The lemma follows.

The new idea in [CM08] is to change the degeneration \( f : \mathcal{X} \rightarrow \Delta \) and the line bundle \( L \), whenever such a "bad" \((-1)\)-curve as in the lemma occurs in one of the components of \( X_0 \), by flopping it. Such a flop certainly exists if the \((-1)\)-curve \( E \) has normal bundle \( N_{E/\mathcal{X}} \cong O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}(-1) \); it is the Atiyah flop (see [Mat02, Ex.3-4-3]). This is not always the case, but the following lemma shows that the normal bundle is at least always negative. Hence we can improve the normal bundle by blowing up \( \mathcal{X} \) several times along \( E \) resp. its strict transforms, until it is possible to flop \( E \). The flop contracts \( E \) on the component \( V \), but other curves pop up on different components. Therefore this operation is called a "throw".

**Lemma 1.15** (Three-point formula). Let \( f : \mathcal{X} \rightarrow \Delta \) be a projective fibration from a smooth 3-fold \( \mathcal{X} \) such that \( X_0 = \bigcup V_i \) is a union of smooth projective surfaces. Suppose that \( C \subset V_i \) is a \((-1)\)-curve on \( V_i \) not contained in any other component \( V_j \), and set \( s := \sum_{i \neq j} C.V_j \). Then

\[ N_{C/\mathcal{X}} \cong N_{C/V_i} \oplus N_{V_i/\mathcal{X}|C} \cong O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}(-s). \]

**Proof.** Since \( f \) is a fibration, \( O_{\mathcal{X}} = O_{\mathcal{X}}(X_0) = \bigotimes O_{\mathcal{X}}(V_j) \). Hence \( O_C = \bigotimes O_C(V_j) \), and

\[ O_C(V_i) = \bigotimes_{i \neq j} O_C(-V_j) \cong O_{\mathbb{P}^1}(-s). \]

The claim follows because there is a natural bundle surjection

\[ N_{C/\mathcal{X}} \rightarrow N_{V_i/\mathcal{X}|C} \cong O_C(V_i) \]

whose kernel is \( T_{V_i|C}/T_C = N_{C/V_i} \cong O_{\mathbb{P}^1}(-1). \)

After the flop the strict transforms of other components besides \( V \) can be singular. To analyse line bundles on these singular surfaces, Ciliberto and Miranda use the desingularization given by the blow up part of the Atiyah flop. To avoid this additional technical difficulty we present the throwing procedure as a sequence of blow ups only.
Construction 1.16 (Throwing \((-1\)-curves).) Let \(f : X \rightarrow \Delta\) be a projective fibration from a smooth complex 3-fold \(X\) to the unit disc \(\Delta\) such that the central fiber \(X_0\) is a union \(\bigcup V_i\) of smooth projective surfaces. Let \(L\) be a line bundle on \(X\). Assume that \(C\) is a \((-1\)-curve on a component \(V_i\) such that for \(j \neq i\) the intersection \(C \cap V_j\) consists of \(s_j\) points with multiplicity 1, and \(C \cap V_j \cap V_{j'} = \emptyset\) for \(j, j' \neq i\). Set \(n := \sum_{j \neq i} s_j\) and \(l := -L|_{V_i} \cdot C\).

Then we construct a sequence of blow ups

\[\tilde{X} = X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 = X\]

such that the center of \(\pi_1\) is \(C \subset X\) and for \(k = 2, \ldots, n\), the center \(C_k \cong C\) of \(\pi_k\) is the intersection of the strict transform \(V_i^{(k-1)} \cong V_i\) of \(V_i\) on \(X_{k-1}\) with the exceptional divisor \(T_{k-1}\) of \(\pi_{k-1}\).

Setting \(\phi_k := \pi_k \circ \cdots \circ \pi_n\) we denote by \(\tilde{T}_k\) the strict \((\phi_{k+1})\)-transform of \(T_k \subset X_k\), for \(k = 1, \ldots, n-1\). Note that \(\tilde{T}_k \cong T_k\), by construction.

Finally we define a new line bundle on \(\tilde{X}\):

\[\tilde{L} := \phi^*_n L \otimes O_{\tilde{X}}(\sum_{i=1}^n a_i \tilde{T}_i).\]

This construction is called an \(n\)-throw. It has the following properties:

1. \(T_k \cong \mathbb{F}_{n-k}\), the \((n-k)\)th Hirzebruch surface.
2. \(\tilde{V}_j\) is the blow up of sequences of \(n\) infinitely near points \(p_1, \ldots, p_n\) over each point \(p = p_i \in C \cap V_j\), where \(p_i\) is infinitely near to \(p_{i-1}\) but not to \(p_{i-2}, i \geq 3\). In particular the choice of \(C\) fixes these points.
3. \(\tilde{T}_k \cap \tilde{T}_{k'} = \emptyset\) if \(k < k' - 1\), whereas \(\tilde{T}_k \cap \tilde{T}_{k+1}\) is the curve at infinity on \(\tilde{T}_k \cong \mathbb{F}_{n-k}\), and is a section of \(\tilde{T}_{k+1}\) not intersecting its curve at infinity \(E_{n-k-1}\) and linearly equivalent to \(E_{n-k-1} + (n-k-1)F_{n-k-1}\) (see the notation in the introduction).
4. \(\tilde{T}_k \cap \tilde{V}_i = \emptyset\) if \(k < n\), whereas \(T_n \cap \tilde{V}_i\) is \(C\) on \(\tilde{V}_i \cong V_i\), and a horizontal \(\mathbb{P}^1\)-fiber on \(T_n \cong \mathbb{P}^1 \times \mathbb{P}^1\).
5. For \(j \neq i\) the intersection \(\tilde{T}_k \cap \tilde{V}_j\) consists of the exceptional divisors of the \(k\)th blow ups over points in \(C \cap V_j\) on \(\tilde{V}_j\), which are \(\mathbb{P}^1\)-fibers of \(\tilde{T}_k \cong \mathbb{F}_{n-k}\) on \(\tilde{T}_k\).
6. The strict transform of an irreducible intersection curve in \(V_i \cap V_j\) is linearly equivalent to the pullback of the intersection curve minus the exceptional divisors over points in \(V_j \cap C\) on this curve.
7. \(\tilde{E}_{\tilde{V}_i} = \phi^*_n L_{\tilde{V}_i} \otimes O_{\tilde{V}_i}(a_n C_n)\), and this line bundle is trivial on \(C_n\) iff \(a_n = -k\).
8. If \(E_1, \ldots, E_n\) is the configuration of exceptional divisors over an intersection point in \(V_j \cap C\) for \(j \neq i\), the divisor \(E_m\) occurs in \(\tilde{L}_{\tilde{V}_j}\) with multiplicity \(-a_m + a_{m-1} - a_n\) (in the notation of the introduction).
9. \(\tilde{L}_{\tilde{T}_n} \cong O_{\mathbb{P}^1 \times \mathbb{P}^1}(-k - a_n, a_{n-1} - a_n)\).
10. For \(k < n\),

\[\tilde{L}_{\tilde{T}_k} \cong O_{\mathbb{F}_{n-k}}((-a_{k+1} - a_k)E_{n-k} + (-l - a_k (n-k+1) + a_{k-1} (n-k))F_{n-k}).\]

Proof. The Three-point Formula \([1.13]\) yields

\[N_{C/X} = N_{C/V_i} \oplus N_{V_i/X} |_{C} \cong O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}(-n).\]
If we projectivize \( N_{C/X} \) the intersection curve of this \( \mathbb{P}^1 \)-bundle with \( V^{(1)}_i \) comes from the summand \( O_{\mathbb{P}^1}(-1) \) hence is the curve at infinity on \( T_1 \cong \mathbb{F}_{n-1} \).

\[
N_{C_k/X_{k-1}} = N_{C_k/V^{(k-1)}_i} \oplus N_{C_k/T_{k-1}},
\]

because \( V^{(k-1)}_i \) and \( T_{k-1} \) intersect transversally in \( C_k \cong \mathbb{P}^1 \),

\[
N_{C_k/T_{k-1}} \cong O_{\mathbb{P}^1}(-n + k - 1),
\]

because by induction \( C_k \) is the curve at infinity of \( T_{k-1} \cong \mathbb{F}_{n-k+1} \). Consequently,

\[
N_{C_k/X_{k-1}} \cong O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}(-n + k - 1),
\]

which yields (1).

(2), (3), (4), (5) follow from construction, (6) is true because \( C \) intersects the components \( V_j \), \( j \neq i \), transversally.

(7) is obvious from the definition of \( \tilde{L} \), and the intersection configurations described in (4).

(8) follows from (6) and the fact that \( E_m \) contains every exceptional divisor \( E_m' \), (resp. its strict transform) exactly once, if \( m' \geq m \).

(9) is the result of the following calculation:

\[
\tilde{L}|_{T_n} = \phi^*_n \mathcal{L}|_{T_n} \otimes \mathcal{T}_n(-a_n T_n) \otimes \mathcal{O}_{T_n}(a_n T_{n-1})
\]

\[
\cong \mathcal{O}_{p_1 \times \mathbb{P}^1}^{(-1)}(\mathcal{L}|_{|V_i|}, C, 0) \otimes \mathcal{O}_{p_1 \times \mathbb{P}^1}^{(-1)}(-a_n, -a_n) \otimes \mathcal{O}_{p_1 \times \mathbb{P}^1}^{(-1)}(0, a_{n-1})
\]

because

\[
\mathcal{O}_{T_n}(T_n) = N_{T_n/X} = \mathcal{O}_{\mathbb{P}^1(N_{C_n/X_{n-1}})} = \mathcal{O}_{\mathbb{P}^1(\mathcal{O}_{\mathbb{P}^1}^{(1)} \oplus \mathcal{O}_{\mathbb{P}^1}^{(1)})(-1) = \mathcal{O}_{\mathbb{P}^1(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1})(-1) \otimes p_1^* \mathcal{O}_{\mathbb{P}^1}(-1) = \mathcal{O}_{p_1 \times \mathbb{P}^1}^{(-1), -1}).
\]

(10) is a consequence of

\[
\tilde{L}|_{T_k} = \phi^*_n \mathcal{L}|_{T_k} \otimes \mathcal{T}_k(a_k T_{k+1}) \otimes \mathcal{T}_k(a_k T_{k-1}) \otimes \mathcal{T}_k(a_k T_{k-1}),
\]

\[
\phi^*_n \mathcal{L}|_{T_k} \otimes \mathcal{T}_k(a_k T_{k+1}) \otimes \mathcal{T}_k(a_k T_{k-1}) \cong \mathcal{O}_{\mathbb{P}^1}^{(-1)(-lF_{n-k})} \otimes \mathcal{O}_{\mathbb{P}^1}^{(-1)(a_k E_{n-k})} \otimes \mathcal{O}_{\mathbb{P}^1}^{(-1)(a_k E_{n-k} + (n - k)F_{n-k})},
\]

\[
\tilde{T}_k = \phi^*_k T_k - T_{k+1} - \ldots - T_n
\]

and

\[
\mathcal{O}_{T_k}(T_k) \cong \mathcal{O}_{\mathbb{P}^1}^{(-1)(-E_{n-k} - (n - k + 1)F_{n-k})}.
\]

\[\Box\]

**Remark 1.17.** Note that in our description of a throw, we do not contract the "bad" \((-1)\)-curve \( C \) and push down the line bundle \( \tilde{L} \), but we only change the line bundle until it is trivial on \( C \).

**Remark 1.18.** In the examples of throws below we choose \( a_1, \ldots, a_{n-1} \) such that the restrictions of the line bundle \( \tilde{L} \) to the exceptional divisors \( \tilde{T}_k \) become minimal.

**Remark 1.19.** Ciliberto and Miranda [CM08] only need 1- and 2-throws. But we will see in Section [3] that more blow ups can be necessary.
1.5. Bounds from linear inequalities. Applying the Gluing Lemma requires the vanishing of $H^1(V_i, L_{|V_i})$ on components $V_i$ of $X_0$. In the Ciliberto-Miranda degenerations constructed in section 2 below, $V_i$ is always a blow up of $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_k$ in points $p_1, \ldots, p_n$, and $L_{|V_i} \cong L(d; m_1, \ldots, m_n)$ resp. $L_{|V_i} \cong L(d_1, d_2; m_1, \ldots, m_n)$ is linearly depending on parameters $d$ resp. $d_1, d_2$, $m_1, \ldots, m_n$. After possibly performing some Cremona transformations, we would like to use the criteria in section 1.2 to deduce the vanishing of the first cohomology group.

It turns out in section 2 that this is possible on the occurring varieties whenever the integers $d$ resp. $d_1, d_2, m_1, \ldots, m_n$ satisfy a set of linear inequalities. Together with the Gluing Lemma, this observation can be used to find $d, m$ arbitrarily big such that $L(d; m^n)$ is non-special on $\mathbb{P}^2(p_1, \ldots, p_n)$, with $p_1, \ldots, p_n$ in general position:

**Theorem 1.20.** Let $f : X \to \Delta$ be a projective fibration from a smooth complex 3-fold such that for $t \neq 0$, $X_t \cong \mathbb{P}^2(p_1, \ldots, p_n)$, $n > 9$, with $p_1, \ldots, p_n$ in general position, and $X_0 \cong \bigcup V_i$, all the $V_i \cong \mathbb{P}^2(q_1^{(i)}, \ldots, q_{n_i}^{(i)})$ or $\cong \mathbb{F}_k(q_1^{(i)}, \ldots, q_{n_i}^{(i)})$. Furthermore, denote by $C_i$ the intersection curve $\bigcup_{j < i} V_j \cap V_i$.

Suppose that there exists $k \in \mathbb{N}$ such that for every $d, m, a \in k \cdot \mathbb{N}$, we can construct a line bundle $L = L(d, m, a)$ satisfying the following conditions:

- $L_{|X_t} \cong L(d; m^n)$ for $t \neq 0$,
- $L_{|V_i} \cong L(d_1^{(i)}, d_2^{(i)}; m_1^{(i)} \ldots, m_{n_i}^{(i)})$ resp. $L(d_1^{(i)}, d_2^{(i)}; m_1^{(i)} \ldots, m_{n_i}^{(i)})$, the $d^{(i)}$ resp. $d_1^{(i)}, d_2^{(i)}$, $m_1^{(i)} \ldots, m_{n_i}^{(i)}$ depending linearly on $d, m, a$, and
- if the $d_1^{(i)}$ resp. $d_2^{(i)}$, $m_1^{(i)} \ldots, m_{n_i}^{(i)}$ satisfy a finite set of weak linear inequalities then

$$H^1(V_i, L_{|V_i}) = H^1(V_i, L_{|V_i} \otimes \mathcal{O}(-C_i)) = 0.$$

Substituting $d, m, a$ in the $d^{(i)}$ resp. $d_1^{(i)}, d_2^{(i)}$, $m_1^{(i)} \ldots, m_{n_i}^{(i)}$, we can consider the closed convex polyhedron $P \subset \mathbb{R}^3$ described by the resulting weak linear inequalities in $d, m, a$, and its projection $P' \subset \mathbb{R}^2$ onto the $d - m$-coordinates. Set

$$\mu := \inf \left\{ \frac{d}{m} : (d, m) \in P' \right\}.$$

If $P'$ is unbounded, both in $d$ and in $m$, then there exist $\epsilon > 0$, $M > 0$, such that for all integers $d, m > M$ with $d, m \in k \cdot \mathbb{N}$ and $0 \leq \frac{d}{m} - \mu \leq \epsilon$, the line bundle $L(d; m^n)$ is a non-special line bundle on the blow up of $\mathbb{P}^2$ in $n$ points in general position.

If $\mu > \sqrt{n}$ then the line bundle $L(d; m^n)$ is furthermore effective, for $M > 0$.

**Proof.** Note that for some positive integers $c^{(i)}$, resp. $c_1^{(i)}, c_2^{(i)}$, $n_1^{(i)} \ldots, n_{n_i}^{(i)}$ not depending on $d, m, a$ the intersection curve $C_i$ is a section of the line bundle $L(c^{(i)}; n_1^{(i)} \ldots, n_{n_i}^{(i)})$ on $\mathbb{P}^2(q_1^{(i)}, \ldots, q_{n_i}^{(i)})$. resp. $L(c_1^{(i)}, c_2^{(i)}; n_1^{(i)} \ldots, n_{n_i}^{(i)})$ on $\mathbb{F}_k(q_1^{(i)}, \ldots, q_{n_i}^{(i)})$. Consequently the vanishing of $H^1(V_i, L_{|V_i} \otimes \mathcal{O}(-C_i))$ can also be deduced from a set of weak linear inequalities depending on $d, m, a$.

The unboundedness implies that there is a line with slope $\mu$ bounding the convex polytope $P'$ from below in the region $\{m \geq M\}$, and an $\epsilon > 0$ such that all pairs $(d, m)$ with $d, m > M$, $0 \leq \frac{d}{m} - \mu \leq \epsilon$ lie in $P'$. For such pairs $(d, m) \in k \cdot \mathbb{N}^2$, the assumptions tell us

$$H^1(V_i, L_{|V_i}) = H^1(V_i, L_{|V_i} \otimes \mathcal{O}(-C_i)) = 0,$$
hence both conditions of the Gluing Lemma 1.2 are satisfied (use Remark 1.4 for the surjectivity of the difference map). Consequently \( H^1(X_t, \mathcal{L}_t) = 0 \) for \( t \in \Delta \) general, by Prop. 1.1.

Since \( \mathcal{E}_0(X_t \otimes L_t^\gamma) = -3 - d < 0 \), the divisor \( K_{X_t} \otimes L_t^\gamma \) cannot be effective, and \( h^2(X_t, \mathcal{L}_t) = h^0(X_t, K_{X_t} \otimes L_t^\gamma) = 0 \). Furthermore, \( p_a = h^0(X_t, \mathcal{O}_{X_t}) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 1 \) because \( X_t \) is a blow up of \( \mathbb{P}^2 \) (Hartshorne Prop.V.3.4). Consequently, Riemann-Roch implies

\[
h^0(X_t, \mathcal{L}_t) = \frac{1}{2} L_t \cdot (L_t - K_{X_t}) = \frac{1}{2} (d(d + 3) - n \cdot m(m + 1)) + 1 > 0
\]

for \( d > \sqrt{nm} \) and \( d, m \gg 0 \).

**Remark 1.21.** The \( k \) must be introduced because the \( d^{(i)} \) resp. \( d_1^{(i)}, d_2^{(i)} \), \( m_1^{(i)}, \ldots, m_n^{(i)} \) can linearly depend on \( d, m, a \) with rational coefficients. Then \( k \) is a common denominator for all occurring fractions.

**Remark 1.22.** We can use \( H^1(V_1, \mathcal{L}_{V_1} \otimes \mathcal{O}(-C_2)) = 0 \) instead of \( H^1(V_2, \mathcal{L}_{V_2} \otimes \mathcal{O}(-C_2)) = 0 \) to show the surjectivity of the first difference map.

We can use the information obtained from the last theorem to deduce lower bounds for Seshadri constants:

**Proposition 1.23.** Assume that for all \( \epsilon > 0 \), \( M \gg 0 \) there exist \( d, m > M \) with \( 0 \leq \frac{d}{m} - \mu \leq \epsilon \) such that \( \mathcal{L}(d; m^n) \) is non-empty and non-special. Then the multi-point Seshadri constant for \( n \) points in general position on \( \mathbb{P}^2 \) is bounded by

\[
\epsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1); p_1, \ldots, p_n) \geq \frac{1}{\mu}.
\]

**Proof.** From the assumptions we construct a sequence \( (d_k, m_k) \) of monotonely increasing integers with \( d_k / m_k \to \infty \) for \( k \to \infty \) such that \( \mathcal{L}(d_k; m_k^n) \) is non-empty and non-special. Since also \( \frac{d_k}{m_k - 1} \to \mu \), we can apply Theorem 0.3 from [Eck08a]. \( \square \)

2. **Degenerations of \( \mathbb{C}P^2 \) blown up in 10 points**

As Ciliberto and Miranda in [CM08] we exemplify their method on \( \mathbb{P}^2 \) blown up in 10 points, thus being able to compare the arguments. But of course, it can also be applied to \( \mathbb{P}^2 \) blown up in more than 10 points.

2.1. **The First Degeneration.** The starting point is a degeneration constructed by Ciliberto and Miranda in [CM08]: Blow up \( \mathbb{P}^2 \times \Delta \) in a point \( p \in \mathbb{P}^2 \times \{0\} \), and obtain the projective fibration \( \pi : \mathcal{X} \to \Delta \). Its central fibre decomposes into the exceptional divisor \( P_p \cong \mathbb{P}^2 \) and \( F_p \cong \mathbb{P}^2 \), the strict transform of \( \mathbb{P}^2 \times \{0\} \).

Choose 10 sections \( p_1, \ldots, p_{10} : \Delta \to \mathcal{X} \) such that \( p_1(0), \ldots, p_4(0) \in P_p \) resp. \( p_5(0), \ldots, p_{10}(0) \in F_p \) are 4 resp. 6 points in general position. In particular, \( p, p_5(0), \ldots, p_{10}(0) \) are 7 points in general position on \( \mathbb{P}^2 \). By possibly shrinking \( \Delta \) we can assume w.l.o.g. that for all \( t \in \Delta \) the points \( p_1(t), \ldots, p_{10}(t) \) are in general position on \( \mathbb{P}^2 \).

Blowing up the images \( p_1(\Delta), \ldots, p_{10}(\Delta) \) of the sections yields a projective fibration \( \pi_1 : \mathcal{X}_1 \to \Delta \) such that

- for all \( t \in \Delta \), the fibre \( X_{1,t} \cong \mathbb{P}^2(p_1, \ldots, p_{10}) \) with \( p_1, \ldots, p_{10} \) in general position, and
\( X_0 = P_1 \cup F_1 \) with \( P_1 \cong \mathbb{P}^2(p_1, \ldots, p_4) \) and \( F_1 \cong \mathbb{P}^2(p, p_5, \ldots, p_{10}) \), all these points in general position.

Denote the exceptional divisor over \( p_i(\Delta) \) by \( E_i \).

\( C_1 = F_1 \cap F_1 \) is the pullback of a line on \( P_1 \), that is a section of \( \mathcal{L}(1; 0^4) \), and the exceptional divisor over \( p \) on \( F_1 \), that is a section of \( \mathcal{L}(0; -1, 0^6) \).

From the construction of \( X_1 \) we obtain a projection \( f : X_1 \to \mathcal{L} \to \mathbb{P}^2 \times \Delta \to \mathbb{P}^2 \).

For \( d, m, a \in \mathbb{N} \) define a line bundle on \( X_1 \) by

\[
\mathcal{L}_1 := f^* \mathcal{O}_{\mathbb{P}^2}(d) \otimes \mathcal{O}(-m \sum_{i=1}^{10} E_i) \otimes \mathcal{O}((2m + a)F_1).
\]

Then \( \mathcal{L}_{1|X_1} \cong \mathcal{L}(d; m^{10}) \) for \( t \neq 0 \),

\[
\mathcal{L}_{1|P_1} \cong \mathcal{L}(2m + a; m^4),
\]

where the 4 points are in general position on \( \mathbb{P}^2 \), and

\[
\mathcal{L}_{1|F_1} \cong \mathcal{L}(d; 2m + a, m^6),
\]

because \( \mathcal{O}_{F_1}(F_1) \cong \mathcal{O}_{P_1}(-P_1) \), by the Three-point formula applied to \( X_{1,0} = P_1 \cup F_1 \). The 7 points on \( \mathbb{P}^2 \) can be assumed to lie in general position.

We assume \( d > \sqrt{10m} \). To apply Theorem 1.20 we need \( H^1(P_1, \mathcal{L}_{1|P_1}) = 0 \), \( H^1(F_1, \mathcal{L}_{1|F_1}) = 0 \) and \( H^1(F_1, \mathcal{L}_{1|F_1} \otimes \mathcal{O}(-C_1)) = 0 \).

For the vanishing on \( P_1 \) we choose an irreducible conic \( C \) through the 4 points blown up in \( P_1 \). The strict transform of \( C \) on \( P_1 \) is a section of \( \mathcal{L}(2; 1^4) \). Since \( \mathcal{L}(2; 1^4)(K_{P_1} \otimes \mathcal{L}(2; 1^4)) = \mathcal{L}(2; 1^4), \mathcal{L}(-1; 0^4) = -2 < 0 \) and \( \mathcal{L}(2; 1^4)(\mathcal{L}_{1|P_1} \otimes \mathcal{O}(-iC)) = 2a \geq 0 \), we can deduce \( H^1(P_1, \mathcal{L}_{1|P_1} = 0 \) from \( H^1(P_1, \mathcal{L}_{1|P_1} \otimes \mathcal{O}(-mC)) = H^1(P_1, \mathcal{L}(2a; 0^4)) = 0 \), by Theorem 1.5.

For the vanishing on \( F_1 \) we note first that \( F_1 \) is strongly anticanonical, as a blow up of \( \mathbb{P}^2 \) in less than 9 points in general position. Next, \( \mathcal{L}_{1|F_1} \cdot K_{F_1} = -3d + (2m + a) + 6m < 0 \) for \( a \) small enough. We perform Cremona transformations on \( \mathcal{L}_{1|F_1} \) changing the degree and multiplicities as follows:

\[
\begin{array}{c|c|c|c}
\text{d;} & 2m + a; & (d - m - a)_2; & m^6 \\
2d - 4m - a; & d - 2m; & (d - 3m - a)_2; & m^6 \\
3d - 8m - 2a; & 2d - 6m - a; & (d - 3m - a)_4; & m^6 \\
4d - 12m - 3a; & 3d - 10m - 2a; & (d - 3m - a)_6. \\
\end{array}
\]

Here, the underlinings indicate which 3 points are used for the transformation.

After the Cremona transformations the intersection curve \( C_1 \) with \( P_1 \) on \( F_1 \) is a section of

\[
\mathcal{L}(0; -1, 0^6) \cong \mathcal{L}(1; 0, 1^2, 0^4) \cong \mathcal{L}(2; 1^4, 0^2) \cong \mathcal{L}(3; 2, 1^6).
\]

Consequently, \( \mathcal{L}_{1|F_1} \otimes \mathcal{O}(-C_1) \cong \mathcal{L}(4d - 12m - 3a - 3; 3d - 10m - 2a - 2, (d - 3m - a - 1)_6) \).

Both transformed line bundles are standard if the following inequalities are satisfied:

\[
\begin{align*}
4d - 12m - 3a - 3 & \geq 0, \quad 3d - 10m - 2a - 2 \geq 0, \quad d - 3m - a - 1 \geq 0, \\
4d - 12m - 3a & \geq (3d - 10m - 2a) + 2(3d - 3m - a) = 5d - 16m - 4a, \\
4d - 12m - 3a & \geq 3(d - 3m - a) = 3d - 9m - 3a.
\end{align*}
\]

The inequalities imply \( d > \frac{10}{3}m \). On the other hand they are satisfied if \( 4m > d > \frac{10}{3}m \) and \( a = 0 \). For such values of \( d, m, a \) Harbourne’s Criterion implies the vanishing of the two \( H^1 \)-groups. Consequently we can apply Theorem 1.20 with \( \mu = \frac{10}{3} \).
Proposition 2.1. The multi-point Seshadri constant of 10 points in general position is bounded from below by
\[ \epsilon(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2}(1); p_1, \ldots, p_{10}) \geq \frac{3}{10}. \]

Remark 2.2. We could also standardize the line bundle on \( P_1 \). But doing so we would lose the symmetry of the blown up points on \( P_1 \) thus creating further difficulties when detecting curves to throw later on.

Remark 2.3. Tensorizing \( f^*\mathcal{O}_{\mathbb{P}^2}(d) \otimes \mathcal{O}(-m \sum_{i=1}^{10} E_i) \) with \( \mathcal{O}((2m + a)F) \) is necessary for providing enough positivity on the line bundle restricted to \( P_1 \). The multiple \( 2m \) would be the minimal possible, but the additional \( a \) helps in later degenerations. We will also use this type of modification again, to ensure enough positivity for the line bundle on certain components.

Remark 2.4. The 4 points on \( P_p \) and the 6 points on \( F_p \) can be freely chosen. Any considerations on general position later on must backtrack to this choice. When transforming exceptional configurations this is done by the arguments in section 1.2 without much effort. When discussing the intersection points of curves to throw with other components we invert the Cremona transformation on the component containing the curve to throw, and argue on \( P_p \) and \( F_p \).

2.2. The Second Degeneration. Still assuming \( d > \sqrt{10m} \) we discuss what happens when \( d < \frac{10}{3}m \).

2.2.1. Identification of curves to throw. We look for curves to throw among the exceptional divisors associated to multiplicities of line bundles on components of the last degeneration. These multiplicities must be negative when \( d < \frac{10}{3}m \). This is the case for the first multiplicity \( 3d - 10m - 2a \) of \( \mathcal{L}_{1,F_1} \). Hence we want to throw the exceptional divisor \( E_1 \subset F_1 \) associated to this multiplicity. \( E_1 \) is a section of \( \mathcal{L}(0; -1, 0^6) \).

2.2.2. Intersection of curve to throw with other components. Since
\[ E_1.C_1 = \mathcal{L}(0; -1, 0^6).\mathcal{L}(3; 2, 1^6) = 2, \]
we expect two intersection points with \( P_1 \), and want to perform a 2-throw. Since for 7 points in general position on \( \mathbb{P}^2 \) the only section of \( \mathcal{L}(3; 2, 1^6) \) is the strict transform of a cubic curve with a node in the first point, we indeed get 2 different intersection points on the exceptional divisor over the node. On \( P_1 \) these 2 points together with the 4 blown up points can be assumed to lie in general position.

2.2.3. Throwing the curve: Components and their intersections. In the Throwing Construction \[ \text{we identify } C \text{ with } E_1, V_1 \text{ with } F_1 \text{ and } V_2 \text{ with } P_2, \text{ and perform a 2-throw. Call} \]
\[ X_2 := X, \quad F_2 := F_1, \quad P_2 := P_1, \quad T_1^{(2)} := T_1, \quad T_2^{(2)} := T_2. \]
Then \( F_2 \cong F_1, \quad P_2 \cong P_1([p_1, p_2], [p_3, p_4]) \) where \( p_1, p_2, p_3, p_4 \) all lie on the intersection curve with \( F_2, \quad T_1^{(2)} \supseteq \mathbb{P}^2(5) \) and \( T_2^{(2)} \supseteq \mathbb{P}^1 \times \mathbb{P}^1 \).

Next, we describe the configuration of intersection curves on each component:

- on \( F_2: \) (a section of) \( \mathcal{L}(3; 2, 1^6) \) with \( P_2, \mathcal{L}(0; -1, 0^6) \) with \( T_2^{(2)} \),
Applying the Gluing Lemma.

2.2.5. 18 THOMAS ECKL and only consider $L_1$ with $P_2$,

- on $T_1^{(2)}$: 2 sections of $L(1;1)$ with $P_2$,
  (the section of) $L(0; -1)$ with $T_2^{(2)}$,
- on $T_2^{(2)}$: a (horizontal) section of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,1)$ with $F_2$ and with $T_2^{(2)}$, 2 (vertical) sections of $\mathcal{O}(1,0)$ with $P_2$.

2.2.4. Throwing the curve: The line bundle and its restrictions. In the Throwing Construction identifying $L$ with $L_1$. Since $L_1, E_1 = L_{1|F_2}, E_1 = 3d - 10m - 2a$, we set

$$a_1 := \frac{3}{2}d - 5m - a, \ a_2 := 3d - 10m - 2a$$

and only consider $d, m \in 2 \cdot \mathbb{N}$. Call $L_2 := \tilde{L}$. Then

$$L_{2|F_2} \cong L(4d - 12m - 3a; 0, (d - 3m - a)^6),$$
$$L_{2|P_2} \cong L(2m + a; m^4, [-a_1, -a_2 + a_1^2])$$

$$= L(2m + a; m^4, [a + 5m - \frac{3}{2}d, a + 5m - \frac{3}{2}d]^2),$$
$$L_{2|T_1^{(2)}} \cong L(a_2 - 2a_1; a_2 - 2a_1 - (a_2 - 2a_1)) = L(0; 0),$$
$$L_{2|T_2^{(2)}} \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, a_1 - a_2) = \mathcal{O}(0, a + 5m - \frac{3}{2}d).$$

Note that for $\sqrt{10}m < d < \frac{4m}{9}$ there are no negative multiplicities.

2.2.5. Applying the Gluing Lemma. In the setting of Gluing Lemma we identify $V_1$ with $T_1^{(2)}$, $V_2$ with $T_2^{(2)}$, $V_3$ with $F_2$ and $V_4$ with $P_2$. Then we check when the relevant cohomology groups vanish.

(1) $H^1(T_1^{(2)}, L_{2|T_1^{(2)}}) = 0$: obvious.

(2) $H^1(T_2^{(2)}, L_{2|T_2^{(2)}}) = 0$ and $H^1(T_2^{(2)}, L_{2|T_2^{(2)}} \otimes \mathcal{O}(0, -1)) = 0$, for the intersection with $T_2^{(2)}$: true because $a + 5m - \frac{3}{4}d > 0$.

(3) $H^1(F_2, L_{2|F_2}) = 0$ and $H^1(F_2, L_{2|F_2} \otimes L(0; 1, 0^6)) = 0$, for the intersection with $T_2^{(2)}$: $F_2$ is strongly anti-canonical because it is the blow up of $\mathbb{P}^2$ in less than 9 points in general position, by Remark Since

$$K_{F_2}, (L_{2|F_2} \otimes L(0; 1, 0^6)) = (-3) \cdot (4d - 12m - 3a) + 1 + 6(d - 3m - a)$$

$$= -6d + 18m + 3a + 1.$$ 

is negative if $a < 2d - 6m$ and $L(4d - 12m - 3a; 1, (d - 3m - a)^6)$ is standard if $0 \leq a < d - 3m$. We can apply Harbourne’s Criterion [1.3] if $0 \leq a < d - 3m$.

(4) $H^1(P_2, L_{2|P_2}) = 0$: $P_2$ is only anti-canonical because every section of $-K_{P_2}$ decomposes into the line $L$, a section of $L(1; 0^4, [1, 1]^2)$, as the fixed part, and a conic $C$ in $L(2; 1^4, [0, 0]^2)$ as the moving part. We want to apply Theorem using the curves $L$ and $C$, but first we perform several Cremona transformations on $L_{2|P_2}$:

$$2m + a; \ m^4, \quad [5m - \frac{3}{2}d + a, 5m - \frac{3}{2}d + a]^2$$
$$m + 2a; \ m, \ a^3, \quad [5m - \frac{3}{2}d + a, 5m - \frac{3}{2}d + a]^2$$
Since the line \( L(1; 0^4, [1, 1]^2) \) is transformed to a conic \( L(2; 0, 1^3, [1, 1]^2) \) the infinitely near points are tangent to this conic and not directed to one of the three base points of the next Cremona transformation indicated by the underscores. We are in the setting of Cremona transformation II:

\[
\begin{align*}
3d - 9m + 2a & : 3d - 9m, \quad a^3, \quad (\frac{2}{3}d - 5m + a)^2, (5m - \frac{2}{3}d + a)^2 \\
6d - 19m + 2a & : 6d - 19m, \quad a^3, \quad [a - (5m - \frac{2}{3}d), a - (5m - \frac{2}{3}d)]^2
\end{align*}
\]

In particular, the non-infinitely near points remain in general position. A final Cremona transformation yields the more symmetric configuration

\[
\begin{align*}
L(12d - 38m + a; (6d - 19m)^4, [a - (5m - \frac{3}{2}d), a - (5m - \frac{3}{2}d)]^2).
\end{align*}
\]

After all these Cremona transformations, \( L \) and \( C \) are again sections of \( L(1; 0^4, [1, 1]^2) \) and \( L(2; 1^4, [0, 0]^2) \). Now, \( (K_{P_2} + C).C = K_{P_2}.C = -2 < 0 \), and \( (L_{21}.P_2 - iC).C = L_{21}.P_2.C = 2a \geq 0 \). Hence \( H^1(P_2, L_{21}|P_2) = 0 \) follows from \( H^1(P_2, L(a; 0^4, [a - (5m - \frac{3}{2}d), a - (5m - \frac{3}{2}d)]^2)) = 0 \).

Next, \( (K_{P_2} + L).L = -2 < 0 \) and if \( i > 0 \) and \( a < \frac{20}{9}m - 2d \),

\[
(L(a; 0^4, [a - (5m - \frac{3}{2}d), a - (5m - \frac{3}{2}d)]^2) - iL).L = a - 4(a - (5m - \frac{3}{2}d)) + 3i > 20m - 6d - 3a > 0.
\]

Hence the vanishing of \( H^1(P_2, L(a; 0^4, [a - (5m - \frac{3}{2}d), a - (5m - \frac{3}{2}d)]^2)) \) follows from \( H^1(P_2, L(5m - \frac{3}{2}d; 0^8)) = 0 \).

(5) \( H^1(P_2, L_{21}|P_2 \otimes \mathcal{O}(-C_2)) = 0 \) where \( C_2 = P_2 \cap (F_2 \cup T_1^{(2)} \cup T_2^{(2)}) \): The Cremona transformations above do not change the description of intersection curves on \( P_2 \) with \( F_2 \) and \( T_1^{(2)} \) whereas the 2 intersection curves with \( T_2^{(2)} \) become sections of \( L(2; 1^4, [1, 0], [0, 0]) \) and \( L(2; 1^4, [0, 0], [1, 0]) \). These curves add up to a section of \( L(5; 2^4, [1, 2]^2) \). As above we conclude that the first cohomology group of the resulting line bundle

\[
\begin{align*}
L(12d - 38m + a - 5; (6d - 19m - 2)^4, [a - (5m - \frac{3}{2}d) - 1, a - (5m - \frac{3}{2}d) - 2]^2.
\end{align*}
\]

vanishes if \( H^1(P_2, L(5m - \frac{3}{2}d + 1; 0^4, [1, 0]^2)) = 0 \). Projecting from \( P_2 \) onto \( \mathbb{P}^2 \) blown up in 2 points we obtain an excellent line bundle on a strongly anti-canonical surface, hence the vanishing.

2.2.6. Bounds. We can apply the Gluing Lemma [1.2] if the following inequalities are satisfied:

\[
\sqrt{10}m < d < \frac{3}{10}m, \quad 0 \leq a < d - 3m, \quad d > \frac{6}{19}m, \quad a > 5m - \frac{3}{2}d, \quad a < \frac{20}{3}m - 2d.
\]

These inequalities imply \( 5m - \frac{3}{2}d < d - 3m \Leftrightarrow d > \frac{16}{5}m \). Vice versa they are satisfied if

\[
\frac{16}{5}m < d < \frac{29}{9}m \text{ and } 5m - \frac{3}{2}d < a < d - 3m,
\]

because \( d - 3m < \frac{29}{3}m - 2d \Leftrightarrow d < \frac{29}{9}m \). Consequently we can apply Theorem [1.20] with \( \mu = \frac{16}{5} \).
Proposition 2.5. The multi-point Seshadri constant of 10 points in general position is bounded from below by

$$\epsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1); p_1, \ldots, p_{10}) \geq \frac{5}{16}.$$ 

2.3. The Third Degeneration. Still assuming $d > \sqrt{10}m$ we discuss what happens when $d < \frac{16}{5}m$.

2.3.1. Identification of curves to throw. This is more subtle than in the Second Degeneration: $d < \frac{16}{5}m$ implies that $5m - \frac{4}{5}d > d - 3m$, and we cannot choose $a$ such that $5m - \frac{4}{5}d < a < d - 3m$. In the following we assume

$$a < d - 3m < 5m - \frac{3}{2}d.$$

Then multiplicities in the Cremona-transformed line bundle $\mathcal{L}_2|P_2$ become negative. Before identifying the curves to throw we modify $\mathcal{L}_2$, for the reasons discussed in Remark 2.4.

$$\mathcal{L}'_2 := \mathcal{L}_2 \otimes \mathcal{O}_{\mathcal{X}_2}((a - (5m - \frac{3}{2}d)T_1^{(2)}).$$

In the proof of Construction 1.10 we showed

$$\mathcal{O}_{T_1^{(2)}}(T_1^{(2)}) \cong \mathcal{O}_{T_1}(-E_1 - 2F_1 - E_1) \cong \mathcal{L}(-2; 0).$$

Furthermore, $\mathcal{O}_{T_2} (T_1^{(2)}) \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, 1)$, $\mathcal{O}_{T_2} (T_1^{(2)}) \cong \mathcal{L}(0; 0^4, [-1, 1]^2)$ and $\mathcal{O}_{T_2}(T_1^{(2)}) \cong \mathcal{O}_{T_2}$. Consequently,

$$\mathcal{L}'_2|P_2 = \mathcal{L}_2|P_2, \mathcal{L}'_{2|P_2} \cong \mathcal{L}(10m - 3d - 2a; 0), \mathcal{L}'_{2|P_2} \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, 2a)$$

and

$$\mathcal{L}'_{2|P_2} \cong \mathcal{L}(12d - 38m + a; (6d - 19m)^4, [0, 2a - 10m + 3d]^2).$$

We throw the two $(-1)$-curves $E_{2, 1}$ of $\mathcal{L}(0; 0^4, [0, -1], [0, 0])$ and $E_{2, 2}$ of $\mathcal{L}(0; 0^4, [0, 0], [0, -1])$ simultaneously. This is possible because they do not intersect on $P_2$:

$$\mathcal{L}(0; 0^4, [0, -1], [0, 0]), \mathcal{L}(0; 0^4, [0, 0], [0, -1]) = 0.$$ 

2.3.2. Intersection of curves to throw with other components. The intersection of $E_{2, 1}$ with the other components can be computed on $P_2$, using the intersection curves of the other components with $P_2$:

- With $F_2$, there exists for both curves exactly

$$\mathcal{L}(1; 0^4, [1, 1], [1, 1]), \mathcal{L}(0; 0^4, [0, -1], [0, 0]) = \mathcal{L}(1; 0^4, [1, 1], [1, 1]), \mathcal{L}(0; 0^4, [0, 0], [0, -1]) = 1$$

intersection point. On $P_2$ these two points $p_1, p_2$ lie on the intersection curve $C_1 = P_2 \cap F_2$, a section of $\mathcal{L}(1; 0^4, [1, 1]^2)$. Backtracking through the Cremona transformations on $P_2$ it is still a section of $\mathcal{L}(1; 0^4, [1, 1]^2)$, hence the (strict transform of the) line through the 2 intersection points with the curve on $P_2$ thrown in the Second Degeneration. On the other hand, $E_{2, 1}$ and $E_{2, 2}$ become sections of $\mathcal{L}(2; 1^4, [0, 0], [0, 0])$ and $\mathcal{L}(2; 1^4, [0, 0], [0, 1])$. Hence the second intersection point of these conics with the line varies freely on the line when varying the 4 points on $P_2$. Consequently, the points $p_1, p_2$ are in general position on $C_1$, in particular with respect to the 7 points blown up on $P_2$ determining $C_1$ as a section of $\mathcal{L}(3; 2, 1^6)$. (See also Remark 2.4)

- We easily calculate $E_{2, 1}T_2^{(2)} = E_{2, 2}T_2^{(2)} = 0$. 
• $P_2$ intersects $T_1^{(2)}$ in a section $C_1$ of $\mathcal{L}(0; 0^4, [-1, 1], [0, 0])$ and a section $C_2$ of $\mathcal{L}(0; 0^4, [0, 0], [-1, 1])$. We easily calculate $E_{2, i}, T_1^{(2)} = 1$, and $E_{2, i}$ only intersects $C_i$. Since the intersection points do not lie on $T_2^{(2)}$, they are not collinear with the point blown up on $T_1^{(2)}$, hence in general position.

2.3.3. Throwing the curve: Components and their intersections. In the Throwing Construction \ref{construction:1.10} we identify $E_{2, 1}$ resp. $E_{2, 2}$ with $E_1$, $P_2$ with $V_1$, $F_2$ with $V_2$, $T_1^{(2)}$ with $V_3$ and $T_2^{(3)}$ with $V_4$, and simultaneously perform two 2-t喔s. Call

\[ X_3 := \tilde{X}, \quad P_3 := \tilde{V}_1, \quad F_3 := \tilde{V}_2, \quad T_1^{(2, 3)} := \tilde{V}_3, \quad T_2^{(3, 3)} := \tilde{V}_4, \]

\[ T_1^{(3)} := \tilde{T}_{1, 1}, \quad T_{1, 2}^{(3)} := \tilde{T}_{1, 2}, \quad T_{2, 1}^{(3)} := \tilde{T}_{2, 1}, \quad T_{2, 2}^{(3)} := \tilde{T}_{2, 2}. \]

Then $P_3 \cong P_2, F_3 \cong F_2([p_1, p_2], [p_3, p_4]),$

\[ T_1^{(2, 3)} \cong T_2^{(2)}(p_1, p_2, [p_3, p_4]) \cong \mathbb{P}^2(p_1, p_2, [p_3, p_4]), \]

where $p, p_1, p_2$ are not collinear and the infinitely near points $p_2, p_4$ are directed to $p, T_1^{(2, 3)} \cong T_2^{(2)}, T_1^{(3)} \cong T_2^{(3)} \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Next, we describe the configuration of intersection curves on each component:

• On $P_3$: (the only section of) $\mathcal{L}(1; 0^4, [1, 1], [1, 1])$ with $F_3,$
\[ \mathcal{L}(0; 0^4, [-1, 1], [0, 0]) \text{ and } \mathcal{L}(0; 0^4, [0, 0], [-1, 1]) \text{ with } T_1^{(2, 3)}, \]
\[ \mathcal{L}(2; 1^4, [1, 0], [0, 0]) \text{ and } \mathcal{L}(2; 1^4, [0, 0], [1, 0]) \text{ with } T_2^{(2, 3)}, \]

no intersections with $T_{1, 1}^{(3)}$ and $T_{1, 2}^{(3)},$
\[ \mathcal{L}(0; 0^4, [0, -1], [0, 0]) \text{ resp. } \mathcal{L}(0; 0^4, [0, 0], [0, -1]) \text{ with } T_{2, 1}^{(3)} \text{ and } T_{2, 2}^{(3)}.$

• On $F_3$: $\mathcal{L}(3; 2, 1^6, [1, 1], [1, 1])$ with $P_3,$
\[ \text{no intersection with } T_1^{(2, 3)}, \quad \mathcal{L}(0; -1, 0^6, [0, 0], [0, 0]) \text{ with } T_2^{(2, 3)}, \]
\[ \mathcal{L}(0; 0, 0^6, [-1, 1], [0, 0]) \text{ with } T_1^{(3)} \text{ and } \mathcal{L}(0; 0, 0^6, [0, 0], [-1, 1]) \text{ with } T_2^{(3)}, \]
\[ \mathcal{L}(0; 0, 0^6, [0, -1], [0, 0]) \text{ with } T_1^{(3)} \text{ and } \mathcal{L}(0; 0, 0^6, [0, 0], [0, -1]) \text{ with } T_2^{(3)}.$

• On $T_1^{(2, 3)}$: $\mathcal{L}(1; 1, [1, 1], [0, 0])$ and $\mathcal{L}(1; 1, [0, 0], [1, 1])$ with $P_3,$
\[ \text{no intersection with } F_3, \quad \mathcal{L}(0; -1, [0, 0], [0, 0]) \text{ with } T_2^{(2, 3)}, \]
\[ \mathcal{L}(0; 0, [-1, 1], [0, 0]) \text{ with } T_1^{(3)} \text{ and } \mathcal{L}(0; 0, [0, 0], [-1, 1]) \text{ with } T_2^{(3)}, \]
\[ \mathcal{L}(0; 0, [0, -1], [0, 0]) \text{ with } T_1^{(3)} \text{ and } \mathcal{L}(0; 0, [0, 0], [0, -1]) \text{ with } T_2^{(3)}.$

• On $T_2^{(2, 3)}$: $\mathcal{O}(0, 1)$ with $F_3$ and $T_1^{(2, 3)},$ two sections of $\mathcal{O}(1, 0)$ with $P_3,$
\[ \text{no intersection with } T_{1, i}^{(3)} \text{ and } T_{2, i}^{(3)}, \quad i = 1, 2.$

• On $T_1^{(3)}$: $\mathcal{L}(1; 1)$ with $F_3$ and $T_1^{(2, 3)}, \quad \mathcal{L}(0; -1)$ with $T_2^{(3)},$
\[ \text{no intersection with } P_3, \quad T_2^{(2, 3)} \text{ and } T_1^{(3)}, \quad j = 1, 2.$

• On $T_2^{(3)}$: $\mathcal{O}(0, 1)$ with $F_3$ and $T_1^{(2, 3)}, \quad \mathcal{O}(1, 0)$ with $P_3$ and $T_1^{(3)},$
\[ \text{no intersection with } T_2^{(2, 3)} \text{ and } T_{j, 2}^{(3)}, \quad j = 1, 2.$

2.3.4. Throwing the curve: The line bundle and its restrictions. In the Throwing Construction \ref{construction:1.10} identify $\mathcal{L}$ with $\mathcal{L}'_2.$ Since $\mathcal{L}'_2, \mathcal{E}_{2, i} = \mathcal{L}'_{2; 0, 0} \mathcal{E}_{2, i} = 3d - 10m + 2a,$ we set

\[ a_1 := \frac{3}{2} d - 5m + a, \quad a_2 := 3d - 10m + 2a. \]
2.3.5. Applying the Gluing Lemma. In the setting of Gluing Lemma \[\text{we identify} \ V_1 \text{ with } T^{(3)}_{1,1} \cup T^{(3)}_{1,2}, \ V_2 \text{ with } T^{(2,3)}_{1,1} \cup T^{(3)}_{2,2}, \ V_3 \text{ with } T^{(3)}_{1,3} \cup T^{(3)}_{2,1}, \ V_4 \text{ with } T^{(2,3)}_{1,2}, \ V_5 \text{ with } F_3 \text{ and } V_6 \text{ with } F_3. \] Then we check when the relevant cohomology groups vanish.

(1) \[H^1(T^{(3)}_{1,1}, L_{3|T^{(3)}_{1,1}}) = H^1(\mathbb{P}^2(x), L(0; 0)) = 0: \text{ obvious. For the surjectivity on} \ V_2 \cap W_1 \text{ Prop. 1.10 implies } H^1(\mathbb{P}^2(x), L(-1; -1)) = H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1)) = 0. \]

(2) \[H^1(T^{(2,3)}_{1,1}, L_{3|T^{(2,3)}_{1,1}}) = 0: \text{ First, } T^{(2,3)}_{1,1} \text{ is strongly anti-canonical since we can find a smooth cubic curve passing through a configuration of 5 points as blown up on } T^{(2,3)}_{1,1}. \] Using the Cremona transformation I in section 1.3.

\[L_{3|T^{(2,3)}_{1,1}} \cong L(10m - 3d - 2a; 0, [5m - \frac{3}{2}d - a, 5m - \frac{3}{2}d - a]^2) \]

can be standardized to

\[L(5m - 3d - a; [0, 0]^2, 5m - \frac{3}{2}d - a). \]

Then we can apply Harbourne’s Criterion 1.9.

After the Cremona transformation the intersection curves are sections of the following line bundles:

\[L(0; [0, 0], [-1, 1], 0) \text{ and } L(1; [0, 0], [1, 1], 1) \text{ with } P_3, \]

no intersection with \( F_3, \) \( L(0; [0, 0], [-1, 1], 0) \) with \( T^{(2,3)}_{2,1}, \)

\( L(0; [-1, 1], [0, 0], 0) \) with \( T^{(3)}_{1,1} \) and \( L(1; [1, 1], [0, 0], 1) \) with \( T^{(3)}_{1,2}, \)

\( L(1; [1, 0], [1, 0], 0) \) with \( T^{(3)}_{2,1} \) and \( L(0; [0, 0], [0, 0], -1) \) with \( T^{(3)}_{2,2}. \)

(3) \[H^1(T^{(3)}_{2,i}, L_{3|T^{(3)}_{2,i}}) = 0 \text{ and } H^1(T^{(3)}_{2,i}, L_{3|T^{(3)}_{2,i}} \otimes \mathcal{O}_{T^{(3)}_{2,i}}(-T^{(3)}_{1,i} - T^{(2,3)}_{1,i})) = 0, \text{ for the intersection with } W_2: \text{ true if } 5m - \frac{3}{2}d > a, \text{ because} \]

\[\mathcal{O}_{T^{(3)}_{2,i}}(-T^{(3)}_{1,i} - T^{(2,3)}_{1,i}) \cong \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(-1, -1). \]

(4) \[H^1(T^{(2,3)}_{2,2}, L_{3|T^{(2,3)}_{2,2}}) = 0 \text{ and } H^1(T^{(2,3)}_{2,2}, L_{3|T^{(2,3)}_{2,2}} \otimes \mathcal{O}_{T^{(3)}_{2}}(-T^{(3)}_{1} - T^{(2,3)}_{1})) = 0, \text{ for the intersection with } W_3: \text{ true if } a > 0 \text{ because} \]

\[\mathcal{O}_{T^{(3)}_{2}}(-T^{(3)}_{1} - T^{(2,3)}_{1}) \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, -1). \]

(5) \[H^1(F_3, L_{3|F_3}) = 0 \text{ and } H^1(F_3, L_{3|F_3} \otimes \mathcal{O}_{F_3}(-W_4) = 0, \text{ for the intersection with } W_4: \text{ First,} \]

\[L_{3|F_3} \cong L(4d - 12m - 3a; 0, (d - 3m - a)^6, [5m - \frac{3}{2}d - a, 5m - \frac{3}{2}d - a]^2) \]
is not standard if \( d < \frac{54}{17} m \) because
\[
4d - 12m - 3a < 3 \cdot (5m - \frac{3}{2} d - a) \Leftrightarrow d < \frac{54}{17} m.
\]

Note that \( \frac{19}{5} < \frac{54}{17} \).

Next, the infinitely near points are not directed to any of the other points. Hence we can perform Cremona Transformation I without specifying the third point, and obtain
\[
\mathcal{L} \left( \frac{25}{2} d - 39m - 3a; 0, (d-3m-a)6, [7d-22m-a, 7d-22m-a], 7d-22m-a, 5m-\frac{3}{2} d-a \right)
\]

Note that now the infinitely near point is directed to the last point blown up. Under our assumption on \( a \), this line bundle is standard if \( \frac{19}{5} m < d < \frac{54}{17} m \) because then \( d - 3m - a < 7d - 22m - a < 5m - \frac{3}{2} d - a \), and
\[
\frac{25}{2} d - 39m - 3a = 2 \cdot (7d - 22m - a) + (5m - \frac{3}{2} d - a).
\]

After the Cremona transformation the intersection curves of \( F_3 \) with the other components are sections of the following line bundles:
\[
\mathcal{L}(3; 2, 1^6, [1, 1], 1, 1) \text{ with } P_3, \text{ no intersection with } T_{1}^{(2,3)},
\mathcal{L}(0; -1, 0^6, [0, 0], 0, 0) \text{ with } T_{2}^{(2,3)},
\mathcal{L}(0; 0, 0^6, [-1, 1], 0, 0) \text{ with } T_{3}^{(3)} \text{ and } \mathcal{L}(1; 0, 0^6, [1, 1], 0, 1) \text{ with } T_{1}^{(3)},
\mathcal{L}(1; 0, 0^6, [1, 0], 1, 0) \text{ with } T_{2,1}^{(3)} \text{ and } \mathcal{L}(0; 0, 0^6, [0, 0], 0, -1) \text{ with } T_{2,2}^{(3)}.
\]

When showing \( H^1(F_3, \mathcal{L}_{3|F_3}) = 0 \) we can forget the point with multiplicity 0 and study the line bundle
\[
\mathcal{L} := \mathcal{L} \left( \frac{25}{2} d - 39m - 3a; (d-3m-a)6, [7d-22m-a, 7d-22m-a], 7d-22m-a, 5m-\frac{3}{2} d-a \right)
\]
on \( \tilde{F} = \mathbb{P}^2(p_1, \ldots, p_6, [p_7, p_8], p_9, p_{10}) \). \( \tilde{F} \) is strongly anti-canonical because the image of the cubic in \( \mathcal{L}(3; 2, 1^6, [1, 1], 1, 1) \) on \( F_3 \) is a section of \( -K_{\tilde{F}} \). Furthermore,
\[
\mathcal{L}.K_{\tilde{F}} = 3(\frac{25}{2} d - 39m - 3a) + 6(d - 3m - a) + 3(7d - 22m - a) + (5m - \frac{3}{2} d - a)
\]
\[
= -12d + 38m - a < 0
\]
if \( d > \frac{19}{6} m \). Consequently we can apply Harbourne’s Criterion \[1.9\]

Finally all the intersection curves of \( F_3 \) with components of \( W_4 \) add up to
\( \mathcal{L}(2; -1, 0^6, [1, 2], 1, 0) \). Since \( \mathcal{L}' := \mathcal{L}_{3|F_3} \otimes \mathcal{L}(-2; 1, 0^6, [-1, -2], -1, 0) \) has no vanishing multiplicity we cannot work directly on \( \tilde{F} \). But we can apply Theorem \[1.5\] on \( \mathcal{L}' \) and the strict transform \( C \) of the cubic in \( \mathcal{L}(3; 2, 1^6, [1, 1], 1, 1) \), because \( (K_{F_3} + C).C = -2 \) and
\[
\mathcal{L}'.C = 12d - 38m + a - 4 > 0
\]
if \( d > \frac{19}{6} m \) (and \( a > 2 \)). Consequently we only have to show
\[
H^1(F_3, \mathcal{L}_{3|F_3} \otimes \mathcal{L}(-5; -1, (-1)^6, [-2, -3], -2, -1)) = 0,
\]
and after applying Prop. \[1.10\] this follows as above, working on \( \tilde{F} \).
are satisfied:

\[ 2.3.6. \]

\[ 2.4.2. \]

None of the intersection points on \( C \) are collinear. Therefore we can perform a Cremona transformation on 3 of them and obtain

\[ \tilde{L} \cong \mathcal{L}(12d - 38m + a; (6d - 19m)^4). \]

This is a standard line bundle, and \( \tilde{L}.K_{\tilde{P}} = -12d + 38m - 3a \) if \( d \geq \frac{19}{6}m \) and \( a < 6d - 19m \). Hence we can apply Harbourne’s Criterion. Finally, the sum of all intersection curves of \( P_3 \) with components of \( W_5 \) is a section of \( \mathcal{L}(3; [1,1]^2) \). By Prop. 2.6.10

\[ H^1(P_3, \mathcal{L}_{3|P_5} \otimes \mathcal{L}(-3; (1,1)^2)) = H^1(\tilde{P}; \tilde{L} \otimes \mathcal{L}(-3; (1,1)^2)). \]

We can standardize as above and apply Harbourne’s Criterion.

We do not use the above Cremona transformation in later degenerations.

2.3.6. **Bounds.** We can apply the Gluing Lemma if the following inequalities are satisfied:

\[ a < 6d - 19m, \quad \frac{19}{6}m < d < \frac{54}{17}m. \]

Consequently we can apply Theorem 1.20 with \( \mu = \frac{19}{6} \).

**Proposition 2.6.** The multi-point Seshadri constant of 10 points in general position is bounded from below by

\[ \epsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1); p_1, \ldots, p_{10}) \geq \frac{6}{19}. \]

2.4. **The Fourth Degeneration.** Still assuming \( d > \sqrt{10}m \) we discuss what happens when \( d < \frac{19}{6}m \).

2.4.1. **Identification of curves to throw.** If \( d < \frac{19}{6}m \) the \((-1\)-curves \( E_{3,1}, E_{3,2}, E_{3,3}, E_{3,4} \) in \( \mathcal{L}(0; -1,0^3,[0,0]^2), \mathcal{L}(0; 0,-1,0^2,[0,0]^2), \mathcal{L}(0; 0^2,-1,0,[0,0]^2), \mathcal{L}(0; 0^3,-1,[0,0]^2) \) intersect \( \mathcal{L}_{3|P_5} \cong \mathcal{L}(12d - 38m + a; (6d - 19m)^4) \) negatively, and they do not intersect each other.

We want to throw simultaneously the 4 curves \( E_{3,1}, E_{3,2}, E_{3,3}, E_{3,4} \).

2.4.2. **Intersection of curves to throw with other components.** There is no intersection of \( E_{3,1}, E_{3,2}, E_{3,3}, E_{3,4} \) with \( F_3, T^{(2,3)}_1, T^{(2,3)}_j, T^{(2,3)}_j \).

The intersection curve of \( P_3 \) with \( T^{(2,3)}_2 \) consists of a conic \( C_1 \) in \( \mathcal{L}(2; 1^4, [1,0], [0,0]) \) and a conic \( C_2 \) in \( \mathcal{L}(2; 1^4, [0,0], [1,0]) \). Both sections intersect each of the \( E_{3,j} \) exactly once. Call the intersection points \( p_j \) and \( p_{j+4}, j = 1, \ldots, 4 \).

None of the intersection points on \( C_2 \) lies on the same horizontal fiber of \( T^{(2,3)}_2 \) as one of the intersection points with \( C_1 \); Reversing the Cremona transformations applied on \( P_3 \cong P_3 \) in the second degeneration, it turns out that the \( E_{3,i} \) can also be interpreted as quartics in

\[ \mathcal{L}(4; 1,2^3,[1,1]^2), \mathcal{L}(4; 2,1,2^2,[1,1]^2), \mathcal{L}(4; 2^2,1,2,[1,1]^2), \mathcal{L}(4; 2^3,1,[1,1]^2). \]

The intersection curves \( C_1 \) and \( C_2 \) turn into sections of \( \mathcal{L}(0; 0^4,[0,-1],[0,0]) \) and \( \mathcal{L}(0; 0^4,[0,0],[0,-1]) \). These \((-1\)-curves are identified by the horizontal projection...
of $T_2^{(2,3)}$ on $\mathbb{P}^1$, and the identification is not affected by different choices of the 4 points blown up on $P_2$. On the other hand, moving the 4 points with a pulled back $\mathbb{C}^*$-action fixing all points on $C_1$ and only 2 points on $C_2$ varies the quartic $E_{3,i}$ in such a way that the intersection points with $C_1$ are fixed, and those with $C_2$ vary.

2.4.3. **Throwing the curve: Components and their intersections.** In the Throwing Construction [1,16] we identify the curves $E_{3,j}$ with $E_{1}, P_3$ with $V_1$, $T_2^{(2,3)}$ with $V_2$, $F_3$, $T_1^{(2,3)}$, $T_{1,i}^{(3)}$, $T_{2,i}^{(3)}$ with $V_3, \ldots, V_8$, and simultaneously perform four 2-throws. Call

$$X_4 := \tilde{X}, \quad P_4 := \tilde{V}_1, \quad T_2^{(2,4)} := \tilde{V}_2, F_4, T_1^{(2,3)}, T_{1,i}^{(3)}, T_{2,i}^{(4)} := \tilde{V}_3, \ldots, \tilde{V}_8.$$ 

Then $P_4 \cong P_3, F_4, T_1^{(2,4)}, T_{1,i}^{(3)}, T_{2,i}^{(4)}$ are isomorphic to $F_3, T_1^{(2,4)}, T_{1,i}^{(3)}, T_{2,i}^{(3)}$, and

$$T_2^{(2,4)} \cong T_2^{(2,3)}([p_1, q_1], \ldots, [p_8, q_8]),$$

where $p_1, q_1, \ldots, p_4, q_4$ are on one vertical fiber, $p_5, q_5, \ldots, p_8, q_8$ are on another vertical fiber, and no 2 points $p_i, p_j$ are on the same horizontal fiber. Finally,

$$T_{1,j}^{(4)} \cong \mathbb{P}^1, \quad T_{2,j}^{(4)} \cong \mathbb{P}^1 \times \mathbb{P}^1.$$ 

Next, we describe the configuration of intersection curves on each component.

- On $F_4, T_1^{(2,4)}, T_{1,i}^{(3)}, T_{2,i}^{(4)}$: as on $F_3, T_1^{(2,4)}, T_{1,i}^{(3)}, T_{2,i}^{(3)}$ in the Third Degeneration.
- On $P_4$: with $F_4, T_1^{(2,4)}, T_{1,i}^{(3)}, T_{2,i}^{(4)}$ as on $P_3$ with $F_3, T_1^{(2,4)}, T_{1,i}^{(3)}, T_{2,i}^{(3)}$ in the Third Degeneration,

$$T_2^{(2,4)}$$

with no intersections with $T_{1,j}^{(4)}$,

- a section of $\mathcal{L}(0; -1, 0, 0, 0, 0, 0, 0)$, that is $E_{3,1}$, with $T_{2,1}^{(4)}$,

- similarly with the other $T_{2,j}^{(4)}$.

- On $T_2^{(2,4)}$: $\mathcal{O}(1, 0)([0, 0]^4, [1, 1]^4)$ and $\mathcal{O}(1, 0)([1, 1]^4, [0, 0]^4)$ with $P_4$, $\mathcal{O}(1, 0)([0, 0]^8)$ with $F_4$ and $T_1^{(2,4)}$,

$$\mathcal{O}(0, 0)([-1, 1], [0, 0)^3, [0, 0]^4)$$

and $\mathcal{O}(0, 0)([0, 0]^4, [-1, 1], [0, 0]^3)$ with $T_{1,j}^{(4)}$,

- similarly with $T_{1,j}^{(4)}$, $j = 2, 3, 4$,

$$\mathcal{O}(0, 0)([0, -1], [0, 0]^3, [0, 0]^4)$$

and $\mathcal{O}(0, 0)([0, 0]^4, [0, -1], [0, 0]^3)$ with $T_{2,1}^{(4)}$,

- similarly with $T_{2,j}^{(4)}$, $j = 2, 3, 4$.

- On $T_{1,j}^{(4)}$: 2 sections of $\mathcal{L}(1; 1)$ with $T_{2,1}^{(2,4)}$ and one of $\mathcal{L}(0; -1)$ with $T_{2,j}^{(4)}$.

- On $T_{2,j}^{(4)}$: 2 sections of $\mathcal{O}(1, 0)$ with $T_{2,1}^{(2,4)}$ and one of $\mathcal{O}(0, 1)$ with $P_4, T_1^{(4)}$.

2.4.4. **Throwing the curve: The line bundle and its restrictions.** In the Throwing Construction [1,16] identify $\mathcal{L}$ with $L_3$. Since $\mathcal{L}_3, \mathcal{E}_{3,j} = \mathcal{L}_{3\mid P_3}, \mathcal{E}_{3,j} = 6d - 19m$, we set

$$a_1 := 3d - \frac{19}{2}m, \quad a_2 := 6d - 19m$$

and only consider $d, m \in 2 \cdot \mathbb{N}$. Call $L_4 := \tilde{L}$. Then

$$\mathcal{L}_{4\mid P_3} \cong \mathcal{L}_{3\mid P_3} \otimes \mathcal{O}(0; -1)^4, [0, 0]^2)^{\otimes a_2} \cong \mathcal{L}(12d - 38m + a; 0^4, [0, 0]^2),$$

$$\mathcal{L}_{4\mid F_4} \cong \mathcal{L}_{3\mid F_3}, \quad \mathcal{L}_{4\mid T_1^{(2,4)}} \cong \mathcal{L}_{3\mid T_1^{(2,3)}}, \quad \mathcal{L}_{4\mid T_1^{(3)}} \cong \mathcal{L}_{3\mid T_1^{(3)}}, \quad \mathcal{L}_{4\mid T_2^{(3)}} \cong \mathcal{L}_{3\mid T_2^{(3)}},$$

$$\mathcal{L}_{4\mid T_2^{(4)}} \cong \mathcal{L}(0; 0), \quad \mathcal{L}_{4\mid T_2^{(4)}} \cong \mathcal{O}(0, \frac{19}{2}m - 3d), \quad j = 1, \ldots, 4,$$
\[ \mathcal{L}_{4[T_2^{(2,4)}]} \cong \mathcal{O}(0, 2a)(\frac{19}{2}m - 3d, \frac{19}{2}m - 3d)^8. \]

2.5.1. **Applying the Gluing Lemma not possible.** Consider the strict transforms \( E_{4,k}, \ k = 1, \ldots, 8 \) of horizontal fibers through one of the 8 points blown up on \( T_2^{(2,4)} \). The \( E_{4,k} \) are sections of \( \mathcal{O}(0,1)([0,0]^{k-1}, [1,0], [0,0]^{8-k}) \), and

\[ E_{4,k} \cdot \mathcal{L}_{4[T_2^{(2,4)}]} = -(\frac{19}{2}m - 3d) \leq -2 \]

if \( 6d \leq -4 + 19m \). Lemma 1.14 implies that for these \( d, m \) the line bundle \( \mathcal{L}_{4[T_2^{(2,4)}]} \) is special.

Consequently, we cannot apply the Gluing Lemma, and we must perform further throws to obtain new bounds for the Seshadri constant.

2.5. **The Fifth Degeneration.** Without changing the assumption \( \sqrt{10}m < d < \frac{19}{6}m \) we want to throw the 8 curves \( E_{4,k} \) on \( T_2^{(2,4)} \subset \mathcal{X}_4 \) simultaneously. This is possible because the \( E_{4,k} \) are pairwise disjoint. Before throwing them we modify the line bundle \( \mathcal{L}_4 \), for the reasons discussed in Remark 2.3.

The restrictions of \( \mathcal{L}_4 \) are the same as those of \( \mathcal{L}_4 \) on all components besides those intersecting one of the \( T_{1,j} \), that is \( T_{2,j}^{(2,4)}, T_{2}^{(2,4)} \) and \( T_{1,j}^{(4)} \) itself.

In the proof of Construction 1.16 we showed

\[ \mathcal{O}_{T_1^{(4)}(T_{1,j}^{(4)})} \cong \mathcal{O}_{T_1}(-E_1 - 2F_1 - E_1) \cong \mathcal{L}(-2; 0). \]

Furthermore,

\[ \mathcal{O}_{T_2^{(2,4)}}(T_{1,j}^{(4)}) \cong \mathcal{O}_{P^1 \times P^1}(0, 1), \quad \mathcal{O}_{T_2^{(2,4)}}(\sum_{j=1}^4 T_{1,j}^{(4)}) \cong \mathcal{O}(0, 0)([-1, 1]^8). \]

Consequently,

\[ \mathcal{L}_{4[T_2^{(2,4)}]}' = \mathcal{L}(19m - 6d; 0), \quad \mathcal{L}_{4[T_2^{(2,4)}]}' = \mathcal{O}(0, 0), \quad \mathcal{L}_{4[T_2^{(2,4)}]}' = \mathcal{O}(0, 2a)(19m - 6d, 0)^8. \]

2.5.1. **Intersection of curves to throw with other components.** The curves \( E_{4,1}, \ldots, E_{4,4} \) intersect \( P_4 \) in exactly one point, on the component not containing \( p_1, \ldots, p_4 \). The curves \( E_{4,5}, \ldots, E_{4,8} \) intersect \( P_4 \) in exactly one point, on the component not containing \( p_5, \ldots, p_8 \). Finally, each of the \( E_{4,k} \) intersects \( T_{1,j}^{(4)} \) in exactly one point iff \( k \equiv j \mod 4 \).

2.5.2. **Throwing the curve: Components and their intersections.** In the Throwing Construction 1.16 we identify the curves \( E_{4,k} \) with \( E_1, T_2^{(2,4)} \) with \( V_1, P_4 \) with \( V_2, T_{1,j}^{(4)}, j = 1, \ldots, 4 \) with \( V_3, \ldots, V_6, F_4, T_{1,j}^{(2,4)}, T_{1,j}^{(3,4)}, T_{2,i}^{(3,4)}, i = 1, 2, T_{2,j}^{(4)}, j = 1, \ldots, 4 \) with \( V_7, \ldots, V_{16} \), and simultaneously perform eight 2-throws. Call

\[ \mathcal{X}_5 := \tilde{\mathcal{X}}, T_{2}^{(2,5)} := \tilde{V}_1, P_5 := \tilde{V}_2, T_{1,j}^{(4,5)} := \tilde{V}_{2+j}, j = 1, \ldots, 4, \]
\[ F_5, T_{1,j}^{(3,5), T_{1,i}^{(5)}, T_{2,i}^{(3,5)}, T_{2,j}^{(4,5)}}, j = 1, \ldots, 4, \]
\[ T_{1,k}^{(5)} := \tilde{T}_{1,k}, T_{2,k}^{(5)} := \tilde{T}_{2,k}, k = 1, \ldots, 8. \]
Then $T_{2}^{(2,5)} \cong T_{2}^{(2,4)}$, $P_5 \cong P_4([q_1, q_2], \ldots, [q_8, q_9])$, where $q_1, \ldots, q_4$ lie on the first intersection curve of $P^4$ with $T_{2}^{(2,4)}$, and $q_5, \ldots, q_9$ on the second intersection curve, $T_{1,j}^{(4,5)} \cong T_{1,j}^{(4)}([q_1,j, q_4, j, [q_1,4+j, q_4,4+j]], T_{2}^{(5)}) \cong F_1$, $T_{2}^{(5)} \cong P^1 \times P^1$, $k = 1, \ldots, 8$, and finally $V_l \cong V_{l, l = 7, \ldots, 10}$.

Next, we describe the configuration of intersection curves on each component.

- On $F_5, T_{1}^{(2,5)}, T_{2}^{(2,5)}, T_{2}^{(4,5)}$ as on $F_4, T_{1}^{(2,4)}, T_{2}^{(4,4)}, T_{2}^{(4,4)}$ in the Fourth Degeneration.

- On $T_{2}^{(2,5)}$: with $P_5, F_5, T_{1}^{(2,5)}, T_{2}^{(5)}$ as with $P_4, F_4, T_{1}^{(2,4)}, T_{2}^{(4)}$, $T_{2}^{(4)}$ in the Fourth Degeneration, with $T_{2}^{(5)}$: $E_{5,k}$, $k = 1, \ldots, 8$.

- On $P_5$: with $F_5, T_{1}^{(2,5)}, T_{1}^{(3,5)}, T_{2}^{(3,5)}, T_{2}^{(4,5)}, T_{2}^{(4,5)}$ as with $F_4, T_{1}^{(2,4)}, T_{2}^{(4,4)}, T_{2}^{(4,4)}$.

- On $T_{1,j}^{(4,5)}$: with $T_{2}^{(4,5)}$ as with $T_{2}^{(4)}$ in the Fourth Degeneration,
  
  with $T_{2}^{(5)}$: $L(2; 1^4, [1, 0], [0, 0], [1, 1]^4, [0, 0]^4)$ and $L(2; 1^4, [0, 0], [1, 0], [0, 0]^4, [1, 1]^4)$,

  with $T_{2}^{(5)}$: $L(0; 0^4, [0, 0]^2, [1, 1]^4, [0, 0]^6-1, [-1, 1], [0, 0]^8-k)$,

  with $T_{2}^{(5)}$: $L(0; 0^4, [0, 0]^2, [1, 1]^4, [0, 0]^6-1, [0, -1], [0, 0]^8-k)$.

- On $T_{1,j}^{(5)}$: with $P_5$: $L(1; 1)$, with $T_{1,j}^{(5)}$: $L(1; 1)$ if $j \equiv k \mod 4$, with $T_{1,k}^{(5)}$: $L(0; -1)$.

- On $T_{2,k}^{(5)}$: with $P_5$: $O(0, 1)$, with $T_{1,j}^{(5)}$: $O(0, 1)$ if $j \equiv k \mod 4$, with $T_{2,k}^{(5)}$: $O(1, 0)$.

2.5.3. **Throwing the curve**: The line bundle and its restrictions. In the Throwing Construction we identify $L$ with $L'$. Since $L'_{4,4,k} = L'_{4,4,k}$, $L'_{4,4,k} = 6d - 19m$, we set

$$a_1 := 3d - \frac{19}{2}m, \quad a_2 := 6d - 19m$$

and only consider $d, m \in 2 \cdot \mathbb{N}$. Call $\mathcal{L}_5 := \tilde{L}$. Then:

$$L_{5}|_{F_5} \cong L_{4}|_{F_4}, \quad L_{5}|_{T_{1}^{(2,5)}} \cong L_{4}^{(1)}|_{T_{1}^{(2,4)}}, \quad L_{5}|_{T_{1}^{(3,5)}} \cong L_{4}^{(1)}|_{T_{1}^{(3,4)}}, \quad L_{5}|_{T_{2}^{(4,5)}} \cong L_{4}^{(1)}|_{T_{2}^{(4,4)}}, \quad i = 1, 2,$$

$$L_{5}|_{T_{2}^{(5)}} \cong L_{4}^{(1)}|_{T_{2}^{(4,4)}} = O(0, 0), \quad j = 1, 2, 3, 4,$$

$$L_{5}|_{T_{2}^{(5)}} = O(0, 2a - 8(19m - 6d)), \quad L_{5}|_{T_{2}^{(5)}} = L_{4}^{(1)}|_{T_{2}^{(4,4)}} \left(\frac{19}{2}m - 3d, \frac{19}{2}m - 3d\right)^{8},$$

$$L_{5}|_{T_{1}^{(4,5)}} \cong O(19m - 6d; 0, [\frac{19}{2}m - 3d, \frac{19}{2}m - 3d]^{8}), \quad j = 1, 2, 3, 4,$$

$$L_{5}|_{T_{1}^{(5)}} \cong O(0, \frac{19}{2}m - 3d), \quad k = 1, \ldots, 8.$$ 2.5.4. **Applying the Gluing Lemma**. In the setting of Gluing Lemma we identify $V_1$ with $T_{1,1}^{(4,5)} \cup T_{1,2}^{(4,5)}$, $V_2$ with $T_{2}^{(4,5)}$, $V_3$ with $T_{2}^{(3,5)} \cup T_{2}^{(3,5)}$, $V_4$ with $\bigcup_{i=1}^{8} T_{2,i}^{(4,5)}$, $V_5$ with $T_{2}^{(5)}$, $V_6$ with $\bigcup_{j=1}^{8} T_{1,j}^{(4,5)}$, $V_7$ with $\bigcup_{k=1}^{8} T_{1,k}^{(5)}$, $V_8$ with $\bigcup_{k=1}^{8} T_{1,k}^{(5)}$, $V_9$ with $F_5$ and $V_{10}$ with $P_5$. Then we check when the relevant cohomology groups vanish.
intersection with $T_{1,i}$: true because $T_{1,i}^{(3,5)} \cong \mathbb{P}^2(p)$, $\mathcal{L}_{5,[T_{1,i}]} \cong \mathcal{L}(0;0)$ and $\mathcal{O}_{T_{1,i}}^{(3,5)}(-T_{1}^{(2,5)}) \cong \mathcal{L}(-1;-1)$.

(2) $H^1(T_{1}^{(2,5)}, \mathcal{L}_{5,[T_{1}]}^{(2,5)}) = 0$: Since $\mathcal{L}_{5,[T_{1}]}^{(2,5)} \cong \mathcal{L}(5m - \frac{3}{2}d - a; 5m - \frac{3}{2}d - a, [0,0]^2)$ this follows from Harbourne’s Criterion [1.3] if $a < 5m - \frac{3}{2}d$.

(3) $H^1(T_{1}^{(3,5)}, \mathcal{L}_{5,[T_{1}]}^{(3,5)}) = 0$ and $H^1(T_{2}^{(3,5)}, \mathcal{L}_{5,[T_{2}]}^{(3,5)} \otimes \mathcal{O}_{T_{2}}^{(3,5)}(−W_{1})) = 0$, for the intersection with $W_{2}$: Since $\mathcal{L}_{5,[T_{1}]}^{(3,5)} \cong \mathcal{O}(0,5m - \frac{3}{2}d - a)$ and the intersection curves with $W_{2}$ add up to a section of $\mathcal{O}(1,1)$ the vanishing follows if $a < 5m - \frac{3}{2}d$, using Prop. [1.10] for the second cohomology group.

(4) $H^1(T_{2}^{(4,5)}, \mathcal{L}_{5,[T_{2}]}^{(4,5)}) = 0$: true because $\mathcal{L}_{5,[T_{2}]}^{(4,5)} = \mathcal{O}(0,0)$. Note that $V_{4} \cap W_{3} = \emptyset$.

(5) $H^1(T_{2}^{(2,5)}, \mathcal{L}_{5,[T_{2}]}^{(2,5)}) = 0$ and $H^1(T_{2}^{(2,5)}, \mathcal{L}_{5,[T_{2}]}^{(2,5)} \otimes \mathcal{O}_{T_{2}}^{(2,5)}(−W_{1})) = 0$, for the intersection with $W_{4}$: Since $\mathcal{L}_{5,[T_{2}]}^{(2,5)} \cong \mathcal{O}(0,2a - 8(19m - 6d))([0,0]^8)$, the first vanishing holds if $a \geq 4(19m - 6d)$.

For $\mathcal{L}_{5,[T_{2}]}^{(2,5)} \otimes \mathcal{O}_{T_{2}}^{(2,5)}(−W_{4}) \cong \mathcal{O}(0,2a - 8(19m - 6d) - 1)([0,1]^8)$ we apply Theorem [1.3] with $C = C_{1} \cup C_{2}$,

$C_{1}$ section of $\mathcal{O}(0,1)([1,1]^4,[0,0]^4)$, $C_{2}$ section of $\mathcal{O}(0,1)([0,0]^4,[1,1]^4)$:

$$\begin{bmatrix}
K_{T_{2}^{(2,5)}} \otimes \mathcal{O}_{T_{2}}^{(2,5)}(C)
\end{bmatrix} \cdot C_{i} = \mathcal{O}(-2,0)([0,0]^8).C_{i} = 0,$$

$$\begin{bmatrix}
\mathcal{L}_{5,[T_{2}^{(2,5)}]} \otimes \mathcal{O}_{T_{2}}^{(2,5)}(−W_{4})
\end{bmatrix} \cdot C_{i} = 2a - 8(19m - 6d) - 1 - 4 > 0$$

if $a > 4(19m - 6d) + 2$,

$$H^1(T_{2}^{(2,5)}, \mathcal{L}_{5,[T_{2}^{(2,5)}]} \otimes \mathcal{O}_{T_{2}}^{(2,5)}(−W_{4}) \otimes \mathcal{O}_{T_{2}}^{(2,5)}(−C)) =$$

$$= H^1(T_{2}^{(2,5)}, \mathcal{O}(0,2a - 8(19m - 6d) - 1 - 2)([−1,0]^8) = 0,$$

if $a > 4(19m - 6d) + 2$, using Prop. [1.10].

(6) $H^1(T_{1,j}^{(4,5)}, \mathcal{L}_{5,[T_{1,j}]}^{(4,5)}) = 0$ and $H^1(T_{1,j}^{(4,5)}, \mathcal{L}_{5,[T_{1,j}]}^{(4,5)} \otimes \mathcal{O}_{T_{1,j}}^{(4,5)}(−W_{5})) = 0$, for the intersection with $W_{5}$: By Cremona Transformation I we can write $\mathcal{L}_{5,[T_{1,j}]}^{(4,5)} \cong \mathcal{L}(19m - 6d;0,[\frac{19}{2}m - 3d, \frac{19}{2}m - 3d])$ as

$$\mathcal{L}_{5,[T_{1,j}]}^{(4,5)} \cong \mathcal{L}(\frac{19}{2}m - 3d; [0,0]^2, \frac{19}{2}m - 3d).$$

Similarly,

$$\mathcal{L}_{5,[T_{1,j}]}^{(4,5)} \otimes \mathcal{O}_{T_{1,j}}^{(4,5)}(−W_{5})) \cong \mathcal{L}(\frac{19}{2}m - 3d - 1; [0,0], [0,−1], \frac{19}{2}m - 3d - 1)$$

because the intersection curves on $T_{1,j}^{(4,5)}$ with $W_{5}$ add up to $\mathcal{L}(2,1,[1,1]^2)$ which is written as $\mathcal{L}(1;[0,0],[0,1],1)$ after the Cremona transformation.

Then both vanishing follow from Harbourne’s Criterion [1.9] and Prop. [1.10] if $19m \geq 6d$.

(7) $H^1(T_{2,k}^{(5)}, \mathcal{L}_{5,[T_{2,k}]}^{(5)}) = 0$ and $H^1(T_{2,k}^{(5)}, \mathcal{L}_{5,[T_{2,k}]}^{(5)} \otimes \mathcal{O}_{T_{2,k}}^{(5)}(−W_{6})) = 0$, for the intersection with $W_{6}$: Since $\mathcal{L}_{5,[T_{2,k}]}^{(5)} \cong \mathcal{O}(0,\frac{19}{2}m - 3d)$ and the intersection curves with $W_{6}$ add up to a section of $\mathcal{O}(1,1)$, this is true if $\frac{19}{2}m - 3d \geq 0$. 


(8) $H^1(T^{(5)}_{1,k}, L_{5|T^{(5)}_{1,k}}) = 0$ and $H^1(T^{(5)}_{1,k}, L_{5|T^{(5)}_{1,k}} \otimes O_{T^{(5)}_{1,k}}(-W_7)) = 0$, for the intersection with $W_7$. Since $L_{5|T^{(5)}_{1,k}} \cong L(0; 0)$ and the intersection curves with $W_7$ add up to a section of $L(1; 0)$, this is true.

(9) $H^1(F_5, L_{5|F_5}) = 0$ and $H^1(F_5, L_{5|F_5} \otimes O_{F_5}(-W_8)) = 0$, for the intersection with $W_8$. We start with four Cremona transformations on $L_{5|F_5} \cong L_{5|F_3}$:

\[
\begin{align*}
\frac{25}{2}d - 39m - 3a &\quad 0, \quad (d - 3m - a)^6, [7d - 22m - a, 7d - 22m - a], \\
\frac{49}{2}d - 77m - 3a &\quad 0, \quad (13d - 41m - a)^2, (d - 3m - a)^4, [7d - 22m - a, 7d - 22m - a], \\
\frac{73}{2}d - 115m - 3a &\quad 0, \quad (13d - 41m - a)^4, (d - 3m - a)^2, [7d - 22m - a, 7d - 22m - a], \\
\frac{97}{2}d - 153m - 3a &\quad 0, \quad (13d - 41m - a)^6, [7d - 22m - a, 7d - 22m - a, 7d - 22m - a, \frac{45}{2}d - 71m - a], \\
76d - 240m - 3a &\quad 0, \quad (13d - 41m - a)^6, \frac{69}{2}d - 109m - a, \frac{69}{2}d - 109m - a^2).
\end{align*}
\]

These transformations are possible because before and after the first three Cremona transformations the infinitely near point is directed to the third base point: this situation is described in Cremona transformation III. In the last transformation, the last point blown up becomes infinitely near, as described in Cremona transformation I.

After the Cremona transformations the intersection curves of $F_5$ with the other components can be written as sections of the following line bundles:

$L(3; 2, 1^6, [1, 1], [1, 1])$ with $P_5$, $L(0; -1, 0^6, [0, 0], [0, 0])$ with $T^{(2,5)}_2$, no intersection with $T^{(1,5)}_1, T^{(2,5)}_{1,2}, T^{(3,5)}_2, T^{(5)}_{1,2}, T^{(5)}_{2,2}$, $L(0; 0, 0^6, [-1, 1], [0, 0])$ with $T^{(1,5)}_1$ and $L(0; 0, 0^6, [0, 0], [-1, 1])$ with $T^{(3,5)}_1$, $L(6; 0, 1^6, [3, 2], [3, 3])$ with $T^{(3,5)}_{2,2}$ and $L(6; 0, 1^6, [3, 3], [3, 2])$ with $T^{(3,5)}_{2,2}$.

As in 2.3.3.5 we can forget the point with multiplicity 0 and study the line bundle

\[
\tilde{L} := L(76d - 240m - 3a; (13d - 41m - a)^6, (\frac{69}{2}d - 109m - a, \frac{69}{2}d - 109m - a^2))
\]

on $\bar{F} = \mathbb{P}^2(p_1, \ldots, p_6, [p_7, p_8], [p_9, p_{10}])$. As in 2.3.3.5 the surface $\bar{F}$ is strongly anti-canonical, and

\[
\tilde{L}.K_{\bar{F}} = -12d + 38m - a < 0
\]

if $a > 2(19m - 6d)$. Finally, $\tilde{L}$ is standard if $d < \frac{136}{47}m$ and $a < \frac{69}{2}d - 109m$.

Then, $0 \leq \frac{69}{2}d - 109m - a < 13d - 41m - a, \frac{69}{2}d - 109m - a < \frac{76}{3}d - 80m$ and

\[
76d - 240m - 3a > 39d - 123m - 3a \iff 37d > 117m
\]

holds because $\frac{117}{37} < \sqrt{10}$. Hence we can apply Harbourne’s Criterion 1.9

As in 2.3.3.5 the intersection curves with $W_8$ add up to a section of $L(2; -1, 0^6, [1, 2], 1, 0)$. After the four Cremona transformations this line bundle can be written as

\[
O_{F_5}(W_8) \cong L(12; -1, 2^6, [5, 6], [5, 6]).
\]

We want to argue as in 2.3.3.5 and apply Theorem 1.3 on

\[
\mathcal{L'} := L_{5|F_5} \otimes L(-12; 1, (-2)^6, [-5, -6], [-5, -6])
\]
and the cubic $C$ in $\mathcal{L}(3; 2, 1^6, [1, 1]^2)$. This is possible because 
$(K_{F_5} + C).C = -2$, and $\mathcal{L}'.C = 12d - 38m + a + 4 > -2$ if 
$$a > 2(19m - 6d) + 2.$$ 
Under this assumption we only have to show 
\[ H^1(F_5, \mathcal{L}' \otimes \mathcal{O}_{F_5}(-C)), \]
and this can be done on $\widetilde{F}$ as above, after using Prop. 1.10 and assuming the same inequalities.

(10) $H^1(P_5, \mathcal{L}_{5/P_5} = 0$ and $H^1(P_5, \mathcal{L}_{5/P_5} \otimes \mathcal{O}_{P_5}(-W_9)) = 0$, for the intersection with $W_9$:
Since $\mathcal{L}_{5/P_5}$ is isomorphic to 
\[ \mathcal{L}_{4/P_4}(\frac{19}{2}m - 3d, \frac{19}{2}m - 3d^8) \cong \mathcal{L}(a - 2(19m - 6d); 0^4, [0, 0]^2, \frac{19}{2}m - 3d, \frac{19}{2}m - 3d^8), \]
we can forget all points with multiplicity 0 and work with the line bundle $\tilde{\mathcal{L}} := \mathcal{L}(a - 2(19m - 6d); \frac{19}{2}m - 3d, \frac{19}{2}m - 3d^8)$ on $\mathbb{P} := \mathbb{P}^2([p_1, q_1], \ldots, [p_8, q_8])$. Here $p_1, \ldots, p_4$ lie on the strict transform of a conic $C_1$ in $\mathcal{L}(2; [1, 1]^4, [0, 0]^4)$, whereas $p_5, \ldots, p_8$ lie on the strict transform of a conic $C_2$ in $\mathcal{L}(2; [0, 0]^4, [1, 1]^4)$. These two conics intersect in 4 points distinct from any point blown up on $\mathbb{P}$.
The infinitely near points $q_1, \ldots, q_8$ are tangent to $C_1$ resp. $C_2$. Set $C = C_1 \cup C_2$. Then:
\[ \tilde{\mathcal{L}}_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(-iC) \cong \mathcal{L}(a - 2(19m - 6d) - 4i; \frac{19}{2}m - 3d - i, \frac{19}{2}m - 3d - i^8) \]
and 
\[ \left[ \tilde{\mathcal{L}}_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(-iC) \right].C_1 = 2 \cdot (a - 2(19m - 6d)) - 8i - 8 \cdot (\frac{19}{2}m - 3d - i) \]
\[ = 2a - 8(\frac{19}{2}m - 3d - i) = \left[ \tilde{\mathcal{L}}_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(-iC) \right].C_2. \]
On the other hand, 
\[ \left[ K_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(C) \right].C_i = \mathcal{L}(1; [0, 0]^8).C_i = 2. \]
Consequently we can apply Theorem 1.5 iteratively for $i = 0, 1, \ldots, \frac{19}{2}m - 3d$, if $a > 4(19m - 6d) + 1$. Finally 
\[ \tilde{\mathcal{L}}_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(-\frac{19}{2}m - 3dC) \cong \mathcal{L}(a - 4(19m - 6d); [0, 0]^8) \]
is non-special.

For the surjectivity on $V_{10} \cap W_9$ the intersection curves on $P_5$ add up to a section of $\mathcal{L}(5; 1^4, [1, 1]^2, [1, 1]^8)$. Prop. 1.10 tells us that it will be enough to show 
\[ H^1(\mathbb{P}, \tilde{\mathcal{L}}_{\mathbb{P}} \otimes \mathcal{L}(-5; [0, 0]^8)) = 0. \]
Repeating the calculations above shows that we can still apply Theorem 1.5 iteratively for $i = 0, 1, \ldots, \frac{19}{2}m - 3d$ and obtain a non-special line bundle if $a > 4(19m - 6d) + 5$.

2.5.5. Bounds. We can apply the Gluing Lemma 1.2 if the following inequalities are satisfied:
\[ d > \sqrt{10m}, a < 5m - \frac{3}{2}d, a > 4(19m - 6d) + 2, 19m - 6d \geq 0, d < \frac{136}{43}m, \quad a \leq \frac{69}{2}d - 109m. \]
Since \( \frac{69}{2} d - 109 m < 5m - \frac{3}{2} d \Leftrightarrow 36 d < 114 m \), \( \frac{136}{43} < \frac{10}{6} \) and

\[
4(19m - 6d) + 2 < \frac{69}{2} d - 109 m \Leftrightarrow 185 m + 2 < \frac{370}{117} m + \frac{4}{117} < d,
\]

we can apply Theorem 1.20 with \( \mu = \frac{330}{117} \).

**Proposition 2.7.** The multi-point Seshadri constant of 10 points in general position is bounded from below by

\[
\epsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1); p_1, \ldots, p_{10}) \geq \frac{117}{330}.
\]

### 3. Algorithmic aspects

We do not stop with the Fifth Degeneration because a new idea is needed but because the amount of data we need to keep track of becomes unmanageable by hand. We illustrate this by identifying the next candidates of curves to throw.

#### 3.1. The next degeneration: Curves to throw

Assume that \( \sqrt{10} m < d < \frac{370}{117} m \). Then \( \frac{69}{2} d - 109 m < 4(19m - 6d) \) and we cannot choose \( a \) such that

\[
4(19m - 6d) < a < \frac{69}{2} d - 109 m.
\]

We also have \( 4(19m - 6d) < 13d - 41m \Leftrightarrow 117m < 37d \) because \( \frac{117}{37} < \sqrt{10} \). So let us assume from now on

\[
\frac{69}{2} d - 109 m < 4(19m - 6d) < a < 13d - 41m.
\]

Furthermore we modify the line bundle \( \mathcal{L} \) to

\[
\mathcal{L}' := \mathcal{L} \otimes \mathcal{O}_{\mathcal{X}_5}(\frac{69}{2} d - 109 m - a) \sum T_{1,1}^{(3,5)},
\]

for the reasons discussed in Remark 2.3. Using

\[
\mathcal{O}_{\mathcal{F}_5}(T_{1,1}^{(3,5)}) \cong \mathcal{L}(0; 0, 0^6, [1, 1], [0, 0]) \quad \text{and} \quad \mathcal{O}_{\mathcal{F}_5}(T_{1,2}^{(3,5)}) \cong \mathcal{L}(0; 0, 0^6, [0, 0], [-1, 1])
\]

we obtain

\[
\mathcal{L}' \cong \mathcal{L}(76d - 240m - 3a; (13d - 41m - a)^6, [0, 69d - 218m - 2a]^2).
\]

Consequently, the two \((-1\)-curves \( E_{5,1} \) in \( \mathcal{L}(0; 0, 0^6, [1, 1], [0, 0]) \) and \( E_{5,2} \) in \( \mathcal{L}(0; 0, 0^6, [0, 0], [-1, 1]) \) are our next candidates for curves to throw. We can throw them simultaneously because they do not intersect on \( F_5 \).

The intersection curves of \( F_5 \) with the other components add up to a section of \( \mathcal{L}(15; 1, 3^6, [6, 7], [6, 7]) \). Hence \( E_{5,1} \) resp. \( E_{5,2} \) intersect the other components in 7 points (if they are different). So we must perform two 7-throws.

#### 3.2. Non-termination of algorithm

The increasing amount of bookkeeping might be tedious to cope with by hand but would not pose any difficulties for a computer, at least in the next steps. On the other hand it is also interesting to prove general statements which ensure that the algorithm never terminates. In the following we specify and shortly discuss some issues related to that aim.
3.2.1. Existence of curves to be thrown. If a line bundle $L$ is special on $\mathbb{P}^2$ blown up in several points in general position, the existence of a $(-1)$-curve intersecting $L$ sufficiently negative is predicted by the Harbourne-Hirschowitz Conjecture. But in the degenerations constructed above we already observe components of the central fiber which are isomorphic to $\mathbb{P}^2$ blown up in points in rather special positions. In particular, we must deal with omnipresent infinitely near points. Nevertheless we always found curves to throw among the $(-1)$-curves of the exceptional configuration in which the restriction of $L$ is described. A better understanding of why they exist would be desirable.

3.2.2. Transversal intersections. The curves to throw should intersect the other components of the central fiber transversally. Otherwise, the Throwing Construction 1.16 is not applicable, or must be extended to a much more complicated situation. In the above degenerations transversality is always a consequence of sufficiently general position of blown up points. But when continuing the algorithm more intricate configurations might occur.

3.2.3. Modification of degenerated line bundle. We modified the line bundle on the central fiber in the First, Third and Fifth Degeneration, and we will also need to do it in a possible Sixth Degeneration, see the section before. The modifications can always be justified as in Remark 2.3 and use analogous components.

3.2.4. Position of points. Even if the blown up points on a component of the central fiber are not in general position they should not lie in a too special configuration. In the above degenerations the necessary generality can always be deduced from the general position of the 10 points blown up in the beginning.

3.2.5. Verifying non-specialty. In all cases in which Harbourne’s Criterion 1.9 does not work we were able to simplify the situation with Criterion 1.5. This was possible because lots of the blown up points in the considered components of the central fiber lie on simple curves. This is inherent to the algorithm, because new points always occur on intersection curves with other components. When we applied Harbourne’s Criterion 1.9 we did not motivate the choice of Cremona transformations to standardize the line bundle. Harbourne [Har85] developed an algorithm for standardization, for fixed degree and multiplicities. But in our case, degree and multiplicities depend on the parameters $d, m, a$, and which Cremona transformations lead to a standardized line bundle, depends on linear inequalities between these parameters. On the other hand these linear inequalities are exactly what we want to find. Therefore, a more systematic approach tries different inequalities, their effect on the standardization, and finally decide which set of linear inequalities gives the best bound in the end. But this is very tedious.

3.3. Future prospects. Besides trying to find bounds for the Seshadri constant of 10 points on $\mathbb{P}^2$ we could also start the algorithm to find bounds for the Seshadri constant of 11, 12, . . . points in general position on $\mathbb{P}^2$. But after some steps we will encounter the difficulties described above in all these cases. On the other hand overcoming these difficulties only requires careful bookkeeping and systematic trial-and-error. These are tasks perfectly fit to a computer. So if we want to find new bounds for Seshadri constants, we should first program a package
of tools which allow us to navigate through the data accumulated by the algorithm, without too much effort.

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Thomas Eckl, Department of Mathematical Sciences, The University of Liverpool, Mathematical Sciences Building, Liverpool, L69 7ZL, England, U.K.

E-mail address: thomas.eckl@liv.ac.uk

URL: http://pcwww.liv.ac.uk/~eckl/