Fractional Matchings under Preferences: Stability and Optimality*

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Abstract

We thoroughly study a generalized version of the classic Stable Marriage and Stable Roommates problems where agents may share partners. We consider two prominent stability concepts: ordinal stability [2] and cardinal stability [7], and two optimality criteria: maximizing social welfare (i.e., the overall satisfaction of the agents) and maximizing the number of fully matched agents (i.e., agents whose shares sum up to one). After having observed that ordinal stability always exists and implies cardinal stability, and that the set of ordinally stable matchings is a lattice structure, we obtain a complete picture regarding the computational complexity of finding an optimal ordinally stable or cardinally stable matching. In the process we answer an open question raised by Caragiannis et al. [8].

1 Introduction

“A joy shared is a joy doubled!”

This is particularly prevalent in matching markets, where the market participants, jointly referred to as agents, have preferences over whom they want to have as partner. The goal is to match agents with partners so as to achieve some desirable properties, such as stability, i.e., no two agents would like to deviate from their current assignments under the matching. In its most simple form, a matching consists of disjoint pairs of agents, meaning that each agent is assigned to at most one other agent; we call such matchings integral matchings. A stable integral matching is an integral matching where no two agents would prefer to be matched to each other rather than with their assigned partners, if any.

Unfortunately, a stable integral matching does not always exist. If however the agents are allowed to share partners, i.e., to have a fractional matching, then the social welfare may increase and stability is guaranteed! Here, a fractional matching is a function which assigns each pair of agents a value between zero and one such that for each agent, the sum of the values of all pairs containing this agent is at most one. An integral matching is hence a restricted variant of fractional matchings where each fraction is either zero or one.

Fractional matchings have applications in time-sharing. For example, in a job market the agents may be partitioned into two sets, freelancers and companies. A fractional matching models the amount of time a freelancer spends working for a company. The preferences can model intensity of interest in working with the agents of the other set, and then stability models an equilibrium in such a job market. Similar scenarios are time-sharing assignments between advisors and apprentices or between workers and projects. An instance of the non-bipartite case occurs when agents (e.g., nurses) work in multiple shifts, and each shift is carried out by two workers. A fractional matching determines the fraction of shifts that each worker carries out with another worker. The preferences can model the intensity of willingness to work with each other, and then stability models the situation where no workers want to swap shifts. Fractional matchings also find application in random matching [25, 4]: By the Birkhoff-von Neumann theorem a fractional matching can be interpreted as a probability distribution over integral matchings in the bipartite case. Choosing an integral matching at random instead of deterministically enables many desirable properties such as fairness and increased expected welfare.

There are multiple natural ways to extend the notion of stability for integral matchings to fractional matchings. For an illustration, let us consider the following example with six agents, called a, b, c, d, e, f as shown in Figure 1.

The preference of an agent i towards another agent j is specified through a non-negative cardinal value (the higher the better), called satisfaction, and is depicted at the end of edge \( (i, j) \) closer to i in the graph on the left. For instance, b’s satisfaction towards a and c are 3 and 1, respec-

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*Supported by the WWTF research grant (VRG18-012).
tively. This means that \( b \) prefers \( a \) to \( c \), expressed as \( b \succ c \).

In this example, no integral matching is stable due to the cyclic preferences of the three agents \( a, b, c \). No matter how an integral matching looks like, at least two of the three would prefer to be with each other rather than with the assignment by the matching. Indeed, odd cycles with such kinds of cyclic preferences are the main obstruction to having a stable integral matching. However, in practice, odd cycles are rather the norm as social networks often have large clustering coefficients which essentially means that it is likely for three agents to form a triangle [22]. Thus it is not far-fetched to suppose that odd cycles with cyclic preferences are likely in matching markets and hence, no stable integral matchings exist.

For fractional matchings the situation is different. Consider the green fractional matching \( M \) in Figure 1 (indicated by the green edges): Agents \( a, b, c \) in the triangle are half-integrally matched with each other (i.e., each of the three pairs receives a half-integral value: 0.5), and \( d \) is integrally matched with \( e \). This green matching \( M \) is **cardinally stable** [7], i.e., no two agents could increase their utilities by being integrally matched with each other. Herein, the utility of an agent towards a fractional matching is the sum of her satisfactions towards her potential partners weighted by the respective fractional matching value. The utilities of agents \( a, b, \) and \( c \) are \( 1 \cdot 0.5 + 2 \cdot 0.5 = 1.5 \), \((1 + 3) \cdot 0.5 = 2\), and \((3 + 2) \cdot 0.5 = 2.5\), respectively.

The green matching \( M \) (shown by the green lines in Figure 1) satisfies two more fractional stability concepts which were originally defined for ordinal preferences (see the right hand side of Figure 1). More precisely, the green matching \( M \) is **ordinally stable** [2], i.e., for each pair of agents at least one agent in the pair is satisfied with \( M \) regarding the pair. Herein, an agent \( i \) is satisfied with a matching regarding a pair \( \{i, j\} \) if the fractional values assigned by the matching between \( i \) and someone she finds better or equal to \( j \) sum up to one. For instance, agent \( a \) is satisfied with the green matching \( M \) regarding \( \{a, b\} \) since she prefers \( c \) to \( b \) and the values of matching her to \( c \) and \( b \) sum up to one. However, she is not satisfied with \( M \) when regarding \( \{a, c\} \) since the values of matching her to someone better or equal to \( c \) sum up to 0.5 which is less than one. Ordinal stability models the desired property that no two agents exist who both can increase the fractional value of matching them, by possibly decreasing the fractional values of matching either of them to someone less preferred.

Lastly, the green matching \( M \) is also **linearly stable** [25, 1], meaning that each pair of agents is jointly satisfied with \( M \). A pair \( \{i, j\} \) is jointly satisfied with a matching if the fractional values of matching \( i \) to someone better or equal to \( j \) plus the fractional values of matching \( j \) to someone better or equal than \( i \) sum up to at least one. For instance, under the green matching, for pair \( \{a, c\} \) the sum is \( 0.5 + 0.5 = 1 \), and hence \( \{a, c\} \) is jointly satisfied with the green matching.

For inclusivity we may strive to maximize the total matching values. Indeed, there is another cardinality stable (fractional) matching \( M' \), indicated by the red lines in the graph in Figure 1, where everyone is **fully matched**, i.e., the matching values for each agent sum up to one. In matching \( M' \), we match both \( a \) and \( c \) each half-integrally with both \( d \) and \( b \), and match \( e \) with \( f \) integrally. Matching \( M' \) is, however, neither ordinally stable nor linearly stable since \( \{d, e\} \) is not jointly satisfied with \( M' \), implying also that neither \( d \) nor \( e \) is satisfied with \( M' \) regarding the pair \( \{d, e\} \).

In terms of social welfare, defined as the sum of the utilities of all agents, the red matching \( M' \) has a welfare of 11, making it superior to the green matching \( M \), with a welfare of 10. Indeed, matching \( M' \) has achieved maximum-welfare since this value is the maximum that any matching of the corresponding edge-weighted graph can achieve: here the weight of an edge representing a pair \( \{i, j\} \) is equal to the sum of the satisfactions of \( i \) and \( j \) towards each other.

**Our contribution.** It is fairly straightforward to see that when restricted to integral matchings all three stability concepts coincide with the classical (weak) stability concept. Aiming for a better understanding of fractional matchings under preferences, in the first part of the paper we take a structural approach to study how the three stability concepts (cardinal stability, ordinal stability, and linear stability) relate to each other. In the second part, we focus on computing stable fractional matchings that maximize the number of fully matched agents or the social welfare. Since linear stability can be formulated via linear programs, finding an optimal linearly stable matching can be solved in polynomial time whenever the objective can be formulated as a linear function of the matching values. Hence, we focus on the other two stability concepts. We investigate how the complexity of finding an optimal stable fractional matching is influenced by

- the presence of ties (i.e., an agent may have the same satisfaction towards different agents) and
- the type of matching market (i.e., in a marriage market the agents are divided into two disjoint parts such that all agents in one part have preferences over a subset of agents in the other part whereas in a roommates market there is no such division).

We highlight our findings below.

(1) Among the three stability concepts, ordinal stability is the most stringent one since it implies both cardinal stability and linear stability, even in roommates markets, whereas the latter two do not necessarily imply each other. Similar to the linear stability for strict preferences, in the marriage case the set of ordinally stable matchings admits a distributed lattice, and in the roommates case this set is closed under a median operation (see Section 3).

(2) We introduce the problem of finding an ordinally stable or cardinaly stable matching maximizing the number of fully matched agents. We show that for ordinal stability, ties make a difference: It is polynomial-time solvable when ties are not present, and NP-hard otherwise. For the marriage case, the tractability result in this dichotomy comes from the fact that each ordinally stable matching is a convex combination of integral stable matchings and hence techniques for integral stable matchings can be applied. For cardinal stability, it is NP-hard even for preferences.
without ties and for the marriage case.

(3) For maximizing the social welfare, the problem is mostly NP-hard, with only one exception: Finding a maximum-welfare ordinally stable matching for preferences without ties and the marriage case is polynomial-time solvable; the other cases remain NP-hard.

Note that the hardness result for cardinal stability behind Theorem 5.6 (also see the remark afterwards) is in stark contrast to the usual understanding of marriage problems without ties, for which most problems are solvable in polynomial time. Moreover, the result resolves an open question asked by Caragiannis et al. [8].

Our results are summarized in Table 1.

**Related work.** Roth, Rothblum, and Vate [25] studied linear stability (they called it fractional stability) in marriage markets without ties, and showed that the set of linearly stable matchings enjoys a lattice structure. Abeledo and Rothblum [1] also studied linear stability, but in roommates markets. They observed that linear stability in roommates markets does not have a lattice structure in general, but showed that linearly stable matchings are closed under the so-called median operation. Following Aziz and Klaus [4] considered multiple fractional stability concepts in marriage markets, including linear stability and ordinal stability (which they called fractional stability and ex-ante stability, respectively), but not cardinal stability. They showed that ordinal stability implies linear stability. We strengthen their result by showing the same for the roommates case.

Caragiannis et al. [7] introduced the problem of finding maximum-welfare cardinaly stable matchings in marriage markets. They showed that the problem is NP-hard and hard to approximate even if each agent has at most three different satisfaction values but may contain ties in her preferences. We improve on this result by showing NP-hardness even when no ties are present and each agent finds at most five agents acceptable. A subset of the structural results, namely the ones about cardinal stability in the marriage setting and for perfect matching (see Observation 3.1) has been observed independently in parallel in a recent journal version [8, Appendix A] of this paper [7].

Finally, Aharoni and Fleiner [2] studied ordinal stability in the hypergraphic setting where each agent has strict preferences over subsets (hyperedges) of agents which contain i, and a fractional matching is a function that gives each hyperedge a non-negative fractional value such that the sum of values of the hyperedges incident to each agent is at most one. They elaborated that the powerful Scarf lemma from game theory guarantees the existence of ordinally stable matchings. However, Kintali et al. [19] and Ishizuka and Kamiyama [18] showed that finding an ordinally stable matching in the hypergraphic setting is as hard as finding a Nash equilibrium (PPAD-hard), even when each agent finds only a constant number of hyperedges acceptable.

For an overview on integral stable matchings, we refer to the books of Gusfield and Irving [14] and Manlove [21].

## 2 Preliminaries

Given an integer \( z \), we use \( [z] \) to denote the set \( \{1, 2, \ldots, z\} \).

**Graphs with cardinal preferences, and matchings.** Let \( G=(V,E) \) be a graph and sat : \( V \times V \to \mathbb{Q}_{\geq 0} \) be a function, where

- \( V \) denotes a set of vertices (also called agents),
- \( E \) denotes a set of edges such that an edge between two vertices means that the corresponding agents find each other acceptable, and
- \( \text{sat} \) specifies the cardinal preferences (also called satisfaction) of an agent towards another agent, i.e., for all \( u, v \in V \), the value sat(\( u, v \)) specifies the satisfaction of \( u \) towards \( v \).

**Remarks.** We assume throughout that (1) \( G \) contains no isolated vertices, (2) for all \( u \in V \) it holds that sat(\( u, u \)) = 0, and (3) for all \( u, v \in V \) it holds that \( \{u, v\} \in E \) if and only if “sat(\( u, v \)) > 0 or sat(\( v, u \)) > 0”.

From the satisfaction function sat of \( G \) we derive a preference list \( \succeq_v \) over the neighborhood \( N_G(v) = \{u \mid \{v, u\} \in E\} \) of each agent \( v \in V \) as follows: Let \( \succeq_v \) denote a complete and transitive binary relation of \( N_G(v) \) such that for each two agents \( x, y \) with \( x, y \in N_G(v) \) it holds that \( x \succeq_v y \) if and only if sat(\( v, x \)) \succeq sat(\( v, y \)); we say that \( v \) weakly prefers \( x \) to \( y \).

We use \( \succ_v \) to denote the asymmetric part of \( \succeq_v \) (i.e., sat(\( v, x \)) \succ sat(\( v, y \))), meaning that \( v \) (strictly) prefers \( x \) to \( y \), and \( \sim_v \) to denote the symmetric part of \( \succeq_v \) (i.e., sat(\( v, x \)) = sat(\( v, y \))), meaning that \( x \) and \( y \) are tied by \( v \).
We use $\mathcal{P} = (\succeq_v)_{v \in V}$ to denote the collection of the preference lists from sat. We say that $x$ is a most preferred agent of $v$ if for each agent $y \in N_C(v)$ we have $x \succeq_v y$.

For each two agents $u, v \in V$, we use $\mathcal{B}_E(u)$ (resp. $\mathcal{B}_E(v)$) to denote the set of agents that $u$ weakly prefers (resp. strictly prefers) over $v$, i.e., $\mathcal{B}_E(u) := \{ w \in V \mid w \succeq_v u \}$, and $\mathcal{B}_E(v) := \{ w \in V \mid w \succ_v u \}$.

An instance $I = (G, \text{sat})$ contains (preferences with) ties if there exists $v \in V$ and two neighbors $x, y \in N_C(v)$ with $\text{sat}(v, x) = \text{sat}(v, y)$; otherwise it has strict preferences.

We extend the standard integral matching concept to fractional ones.

**Definition 2.1 (Fractional Matching).** A fractional matching $M : E \to \mathbb{R} \geq 0$ is an assignment of non-negative weights to each edge $e \in E$ such that $\sum_{(v,u) \in E} M(\{u, v\}) \leq 1$ for each agent $v \in V$.

Moreover, for each two agents $x, y$, we use $M(x, y)$ to denote the following sums:

$$M(x, y) := \sum_{y' \in \mathcal{B}_E(x)} M(x, y'), \text{ and } M(x, \succ y) := \sum_{y' \in \mathcal{B}_E(x)} M(x, y').$$

A fractional matching $M$ may satisfy one of the following properties: An agent $v$ is called fully matched (resp. matched) under $M$ if $\sum_{u \in N_C(v)} M(v, u) = 1$ (resp. $\sum_{u \in N_C(v)} M(v, u) > 0$). $M$ is called perfect if each agent is fully matched. $M$ is called integral (resp. half-integral) if $M(e) \in \{0, 1\}$ (resp. $M(e) \in \{0, 0.5, 1\}$) for each edge $e$.

As noted by Aziz and Klaus [4, p. 226], by the Birkhoff-von Neumann theorem a fractional matching $M$ in a bipartite graph can be decomposed into a convex combination of integral matchings [15, Theorem 3.2.6]. (The bound $k \in O(n^2)$ below follows from the fact that Theorem 3.2.6 in [15] indeed shows that the each fractional matching is contained in an $O(n^2)$-dimensional polyhedron together with Carathéodory’s Theorem about convex hulls.)

**Proposition 2.2.** For each fractional matching $M$ of a bipartite graph $G$ over $n$ vertices, there exists an integer $k \in O(n^2)$, positive coefficients $x_1, x_2, \ldots, x_k \in \mathbb{R} > 0$, and $k$ integral matchings $M_1, M_2, \ldots, M_k$ of $G$ such that $\sum_{j \in [k]} x_j = 1$ and for each edge $e \in E$ it holds that

$$M(e) = \sum_{j \in [k]} x_j \cdot M_j(e).$$

The integral matchings $(M_j)_{j \in [k]}$ constitute a support of the matching $M$. There may be multiple supports of $M$.

**Three stability concepts wrt. fractional matchings.**

**Definition 2.3 (Utilities, blocking pairs, and stability).** Let $G$ be a graph with cardinal preferences sat. The utility of each agent $v \in V$ under a matching $M$ of $(G, \text{sat})$ is defined as $U_{\text{sat}}(M) := \sum_{(v,u) \in E(G)} \text{sat}(v, u) \cdot M(v, u)$. If sat is clear from the context, we omit it from $U_{\text{sat}}$. Given a matching $M$ of $(G, \text{sat})$, an edge $(u, v) \in E(G)$ is

- a cardinaly blocking pair (or cardinaly blocking edge) if $U_{\text{sat}}(M) < \text{sat}(u, v)$ and $U_{\text{sat}}(M) < \text{sat}(v, u)$;
- an ordinally blocking pair (or ordinally blocking edge) if $M(v, u) > 0$ and $M(v, u) < 1$;
- a linearly blocking pair (or linearly blocking edge) if $M(u, v) + M(v, u) - M(u, v) < 1$.

A matching $M$ of $(G, \text{sat})$ is cardinaly stable, ordinally stable, and linearly stable if it contains no cardinaly blocking, no ordinally blocking, and no linearly blocking pairs, respectively. The acronyms $\text{CSM}$, $\text{OSM}$, and $\text{LSM}$ stand for cardinaly stable, ordinally stable, and linearly stable fractional matching, respectively.

**Remark.** Note that for integral matchings, all three stability concepts are equivalent to the classical stability concept.

To illustrate the three stability concepts, consider the following.

**Example 2.4.** Take the following bipartite graph on vertices $U \cup W$, $U = \{a, b, \ldots, e\}$, with strict preferences.

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1: a\rightarrow b\rightarrow c, a: 2\rightarrow 1,
2: b\rightarrow a, b: 1\rightarrow 3, 2,
3: c\rightarrow b\rightarrow d, c: 1\rightarrow 3, 4,
4: d\rightarrow c\rightarrow e, d: 3\rightarrow 4\rightarrow 5,
5: d, e: 4.
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It admits three stable integral matchings $M_1, M_2, M_3$, where

1. $M_1 = M_3$, $M_2$.
2. $M_2(1, b) = M_2(2, a) = M_2(3, c) = M_2(4, d) = 1$.
3. $M_3(1, b) = M_3(2, a) = M_3(3, d) = M_3(4, c) = 1$.

In terms of linear stability, matching $M_1$ with $M_1 = 0.5 \cdot M_1 + 0.5 \cdot M_2$ is ordinally stable, cardinaly stable, and linearly stable. In terms of linear stability, matching $M_2$ with $M_2 = (1/2 + \varepsilon) \cdot M_1 + (1/2 - \varepsilon) \cdot M_3$ and $0 < \varepsilon < 1/6$ is linearly stable, but it is neither ordinally stable nor cardinaly stable: $M_2(3, b)$ is both ordinally blocking and cardinaly blocking $M_2$. In terms of cardinal stability, matching $M_3$ (marked in red), where $M_3(1, b) = M_3(2, a) = 1, M_3(3, c) = M_3(2, d) = M_3(4, c) = M_3(4, d) = M_3(5, d) = 1/3$, and all remaining edges are set to zero, is cardinaly stable. For instance, $U_{\text{sat}}(M_3) = 1$ and $U_{\text{sat}}(M_3) = 1/3$. $M_3$ is, however, neither linearly stable nor ordinally stable: Edge $\{3, d\}$ is both linearly blocking and ordinally blocking $M_3$. Observe that in $M_3$ every agent is matched although no stable integral matching can match agent 5 or agent $e$.

**2.1 Computational problems**

We focus on two types of decision problems, one aiming for maximizing the number of fully matched agents, and the other aiming for maximizing social welfare. For this, given a graph $G=(V, E)$ with satisfaction function $\text{sat} : V \times V \to \mathbb{Q}_{\geq 0}$, and given a fractional matching $M$ in $G$, let $\#\text{fully}(M)$ and $\text{welfare}_{\text{sat}}(M)$ denote the number of fully matched agents and the sum of utilities of the agents.
under $M$:

$$\#\text{fully}(M) := |\{ x \in V \mid \sum_{y \in N_G(x)} M(x, y) = 1\}|,$$

and

$$\text{welfare}_{\text{sat}}(M) := \sum_{v \in V} U_{\text{sat}, M}(v).$$

If $\text{sat}$ is clear from the context then we drop it in $\text{welfare}_{\text{sat}}$.

The problems are defined as follows, where $\Pi \in \{\text{OSM, CSM}\}$:

**MAX-FULL II MATCHING (MAX-FULL II)**

**Input:** A graph $G = (V, E)$, a satisfaction function $\text{sat} : V \times V \rightarrow \mathbb{R}_{\geq 0}$, and a non-negative integer $\tau$.

**Question:** Does $(G, \text{sat})$ admit a matching $M$ under which at least $\tau$ agents are fully matched, i.e., $\#\text{fully}(M) \geq \tau$?

**MAX-WELFARE II MATCHING (MAX-WELFARE II)**

**Input:** A graph $G = (V, E)$, a satisfaction function $\text{sat} : V \times V \rightarrow \mathbb{R}_{\geq 0}$, and a non-negative real $\gamma \in \mathbb{R}_{\geq 0}$.

**Question:** Does $(G, \text{sat})$ admit a matching $M$ with welfare at least $\gamma$, i.e., $\text{welfare}(M) \geq \gamma$?

**Proposition 2.5 (**). MAX-FULL OSM, MAX-WELFARE OSM, MAX-FULL CSM, and MAX-WELFARE CSM are contained in NP.

**Proof.** To show NP-containment, we observe that our problems can be formulated via mixed integer linear programs (MILP), which are contained in NP [24]. A similar MILP approach has already been used for cardinal stability by Caragiannis et al. [7], but they did not address the issue regarding NP-containment as they only considered the maximization variant of MAX-WELFARE CSM.

In fact, our problems reduce in polynomial time to a very restricted variant of MILP for which all integer variables have binary values. Due to this, we can directly provide a polynomial-time non-deterministic algorithm to solve our problem: We guess non-deterministically in polynomial-time the values of the integer variables and solve the resulting linear program (LP) in polynomial time. For the sake of completeness, we describe this approach here. To this end, let us first describe the MILP for our problems. Let $I = (G, \text{sat})$ be an instance with graph $G = (V, E)$ and cardinal preferences $\text{sat}$. A fractional matching $M$ of $G$ can be encoded via an LP as follows. For each edge $\{u, v\}$, we introduce a fractional variable $x_{\{u, v\}}$ to denote the matching value assigned to edge $\{u, v\}$ by a solution matching $M$.

$$\sum_{v \in N_G(u)} x_{\{u, v\}} \leq 1, \quad \forall u \in V$$  \hspace{1cm} (LP1)

$$x_{\{u, v\}} \in \mathbb{R}_{\geq 0}, \quad \forall \{u, v\} \in E$$  \hspace{1cm} (LP2)

To encode the cardinal stability (resp. ordinal stability) of $M$, we need to make sure that no edge $\{u, v\} \in E$ is cardinaly blocking (resp. ordinally blocking) $M$. To formulate these constraints, for each edge $\{u, v\}$ we introduce a binary variable $y_{\{u, v\}}$ and add the following three MILP constraints for cardinal stability:

$$\forall \{u, v\} \in E:$$

$$\sum_{w \in N_G(u)} \text{sat}(u, w) \cdot x_{\{u, w\}} \geq \text{sat}(u, v) \cdot y_{\{u, v\}}, \quad (\text{CSM1})$$

$$\sum_{w \in N_G(v)} \text{sat}(v, w) \cdot x_{\{v, w\}} \geq \text{sat}(v, u) \cdot (1 - y_{\{u, v\}}). \quad (\text{CSM3})$$

Note that the intended meaning of $y_{\{u, v\}} = 1$ is that the utility of agent $u$ under $M$ should be at least $\text{sat}(u, v)$, while $y_{\{u, v\}} = 0$ means that the utility of agent $u$ under $M$ should be at least $\text{sat}(v, u)$.

For ordinal stability, we instead add the following three MILP constraints:

$$\forall \{u, v\} \in E:$$

$$y_{\{u, v\}} \in \{0, 1\}, \quad (\text{OSM1})$$

$$\sum_{w \in N_G(u)} x_{\{u, w\}} \geq y_{\{u, v\}}, \quad (\text{OSM2})$$

$$\sum_{w \in N_G(u)} x_{\{v, w\}} \geq 1 - y_{\{u, v\}}. \quad (\text{OSM3})$$

Note that the intended meaning of $y_{\{u, v\}} = 1$ is that the sum of values of matching agent $u$ to someone better or equal to $v$ should be at least one, while $y_{\{u, v\}} = 0$ means that the sum of values of matching agent $v$ to someone better or equal to $u$ should be at least one,

To solve MAX-FULL CSM (resp. MAX-FULL OSM) with objective value $\gamma$, we introduce one more binary variable $z_u$ for each agent $u \in V$ to specify whether it will be fully matched and add the following MILP constraints:

$$\sum_{u \in V} z_u \geq \tau, \quad (\text{FULL1})$$

$$\sum_{v \in N_G(u)} x_{\{u, v\}} \geq z_u, \quad \forall u \in V, \quad (\text{FULL2})$$

$$z_u \in \{0, 1\}, \quad \forall u \in V. \quad (\text{FULL3})$$

to the constraints (LP1)–(LP2) and (CSM1)–(CSM3) (resp. to the constraints (LP1)–(LP2) and (OSM1)–(OSM3)).

To solve MAX-WELFARE CSM (resp. MAX-WELFARE OSM) with objective value $\gamma$, we only add the following constraint:

$$\sum_{\{u, v\} \in E} (\text{sat}(u, v) + \text{sat}(v, u)) \cdot x_{\{u, v\}} \geq \gamma. \quad (\text{WELFARE})$$

to the constraints (LP1)–(LP2) and (CSM1)–(CSM3) (resp. to the constraints (LP1)–(LP2) and (OSM1)–(OSM3)).

This completes the description of the MILPs for our problems. As already discussed at the beginning of the proof, since our MILPs have $O(|E| + |V|)$ binary variables $y_{\{u, v\}}$, $\{u, v\} \in E$ and $z_u$, $u \in V$, we guess their values and check in polynomial time whether the guessed values combined with the resulting LP constraints are feasible. This shows that our decision problems belong to NP.

By the above containment results, when we show NP-completeness later it suffices to prove NP-hardness.

\footnote{We omit linear stability since both problems for linear stability can be formulated as linear programs and are hence polynomial.}
3 Structural properties

We now discuss relations among and existence of fractional matchings regarding the three stability concepts, and then show that OSMs behave similarly to LSMS in terms of lattice property. First, we observe that ordinal stability is a notion stronger than linear stability and cardinal stability, while cardinal stability and linear stability are not comparable to each other (see Figure 2).

**Observation 3.1** *(i)* Every OSM of a graph with cardinal preferences is a LSM and a CSM.

**(ii)** There exists a graph $G$ with strict preferences such that $G$ admits a LSM which is neither an OSM nor a CSM and admits a CSM which is neither an LSM nor an OSM.

The implication from ordinal stability to linear stability in Statement (i) has been proved by Aziz and Klaus [4] in the marriage setting (see the statement that ex-ante weak stability implies robust ex-post weak stability in their Theorem 3.) A counterexample for the statement that linear stability implies ordinal stability (which is part of Statement (ii) above) has also been given by Aziz and Klaus [4], see their Theorem 3 as well.

**Proof of Observation 3.1.** The first of Statement (i) regarding OSMs and LSMS follows directly from the definition. Now, to show the second part of Statement (i), let $M$ be an OSM of an instance $I = (G, \text{sat})$ with graph $G$ and cardinal preferences sat. Consider an arbitrary edge $(u, v) \in E(G)$. Since $M$ is ordinally stable it follows that $(u, v)$ is not an ordinally blocking edge. That is, $M(u, v) \geq 1$ or $M(v, u) \geq 1$.

If (1) holds, then it follows that

$$\forall_{w \in BE_u(v)}sat(w) \cdot M(u, w) \geq sat(u, v).$$

If (2) holds, then it follows that

$$\forall_{w \in BE_v(u)}sat(v, w) \cdot M(v, w) \geq sat(v, u).$$

Hence, $(u, v)$ is not cardinally blocking $M$, implying that $M$ is cardinaly stable.

In the instance given in Example 2.4 matchings $M_2$ and $M_3$ show Statement (ii).

The following concept of *stable partitions*, introduced by Tan [26], turns out to be very useful for showing the existence of ordinally stable matchings.

**Algorithm 1:** Compute an OSM for a graph $G$ with cardinal preferences sat.

1. Compute the preference lists $P$ from sat
2. sat $\leftarrow$ Break ties in sat arbitrarily
3. Compute a stable partition $\pi$ for $(G, \text{sat})$ and the corresponding matching $M^*$ according to Definition 3.2
4. return $M^*$

**Definition 3.2** (Stable partitions and cycles, their corresponding matchings). A *stable partition* of $(G = (V,E), \text{sat})$ with sat being strict is a permutation $\pi : V \rightarrow V$ on the vertices, which satisfies the following two conditions for each vertex $v_i \in V$:

1. if $\pi(v_i) \neq \pi^{-1}(v_i)$, then $\{v_i, \pi(v_i)\}, \{v_i, \pi^{-1}(v_i)\} \in E$ and $\text{sat}(v_i, \pi(v_i)) > \text{sat}(v_i, \pi^{-1}(v_i))$;
2. for each vertex $v_j$ adjacent to $v_i$, if $\pi(v_j) = v_i$ or $\text{sat}(v_i, \pi(v_j)) > \text{sat}(v_i, \pi^{-1}(v_j))$, then $\text{sat}(v_j, \pi^{-1}(v_j)) > \text{sat}(v_j, v_i)$.

We call $v_i$ a *singleton* if $\pi(v_i) = v_i$. A stable partition $\pi$ can be decomposed into *cycles, singletons, and transpositions* (i.e., disjoint edges). Here, a subpermutation $\sigma$ on a subset $V' \subseteq V$ of vertices is called a cycle if the edge set $\{\{v, \sigma(v)\} \mid v \in V'\}$ forms a cycle in $G$; we define the *length* of a cycle $\sigma$ to be the size of $V'$.

Let $\pi$ be a stable partition. Define a matching $M^*$ for $G$ corresponding to $\pi$ as follows.

(a) For each non-singleton $v_i \in V$ (i.e., $\pi(v_i) \neq v_i$), if $\pi(v_i) = \pi^{-1}(v_i)$, meaning that $(v_i, \pi(v_i))$ forms a transposition in $\pi$, then $M^*(v_i, \pi(v_i)) := 1$; otherwise $M^*(v_i, \pi(v_i)) = M(v_i, \pi^{-1}(v_i)) := 0.5$.

(b) For each remaining edge $e$, let $M^*(e) := 0$.

**Example 3.3.** Consider the instance from the introduction. There is only one stable partition $\pi = (a, b, c)(d, e)(f)$. Since it consists of an odd cycle of length three, namely $(a, b, c)$, the instance does not admit a stable and integral matching. The matching marked in green is ordinally stable, and hence cardinaly stable and linearly stable. Note that this matching is exactly $M^*$ that we define for $\pi$ in Definition 3.2.

Singleton agents of a graph with strict preferences are unique in the following sense.

**Proposition 3.4** ([26]). Let $G$ be a graph with $n$ vertices and strict preferences sat. Then, $(G, \text{sat})$ admits a stable partition, which can be found in $O(n^2)$ time. Moreover, every stable partition of $(G, \text{sat})$ has the same set of singleton agents.

We will see that the matchings corresponding to stable partitions are ordinally stable, even when ties are present. Before we show this, we note that the case without ties is already observed by Aharoni and Fleiner [2], Biró, Cechlárová, and Fleiner [5].

**Proposition 3.5** ([2, 5]). Let $G = (V,E)$ be a graph with strict preferences sat. The matching $M^*$ as defined in Definition 3.2 is an OSM and a CSM for $(G, \text{sat})$.

In the following, we derive the same result for cardinal preferences using the notion of stable partitions.
Lemma 3.6 (*). Each graph on $n$ vertices and with cardinal preferences (and possibly ties) admits an OSM, and hence a CSM, that is half-integral and matches each matched agent fully. Algorithm 1 finds such a matching in $O(n^2)$ time.

Proof. To show the statement we first show that $(G, \text{sat})$ admits an OSM. Let $G = (V, E)$ be a graph with cardinal preferences, and let $P$ be the preferences lists derived from sat. We aim to show that $M^\pi$ as returned by Algorithm 1 on input $(G, \text{sat})$ is a half-integral OSM where each matched agent is fully matched. Let sat be a satisfaction function with strict preferences, which is derived from sat by breaking ties arbitrarily (except the values sat$(x, y)$ with $x = y$ or with $\{x, y\} \notin E$). That is, sat is a satisfaction function without ties such that for each agent $x$ and each two neighbors $y, z \in N_G(x)$ it holds that if sat$(x, y) > sat(x, z)$ then sat$(x, y) > sat(x, z)$. By Proposition 3.4, let $\pi$ be the stable partition of $(G, \text{sat})$ and $M^\pi$ be the corresponding matching computed in line 3 of Algorithm 1.

Clearly, $M^\pi$ is half-integral such that every matched agent is fully matched. It remains to show that $M^\pi$ is an OSM of $(G, \text{sat})$. First of all, we show the following claim.

Claim 1. For each edge $\{x, y\} \in E$ it holds that if $x$ is not a singleton and sat$(x, \pi^{-1}(x)) \geq \text{sat}(x, y)$, then $M(x, \geq y) = 1$.

Proof of Claim 1. We distinguish between two cases.

- If $\pi(x) = \pi^{-1}(x)$, then since $x$ is not a singleton, by the definition of $M^\pi$, it follows that $M^\pi(x, \pi^{-1}(x)) = M(x, \pi(x)) = 1$, implying our claim.

- If $\pi(x) \neq \pi^{-1}(x)$, then by the definition of $M^\pi$, it follows that $M^\pi(x, \pi^{-1}(x)) + M(x, (x, \pi(x))) = 1$. Since $x$ is not a singleton, by Definition 3.2(1), it follows that sat$(x, \pi(x)) > sat(x, \pi^{-1}(x))$. By our definition of sat it follows that sat$(x, \pi(x)) \geq sat(x, \pi^{-1}(x))$. The claim follows immediately. \hfill $\Box$

Now, we are ready to show that $M^\pi$ is an OSM of $(G, \text{sat})$. Suppose, for the sake of contradiction, that $M$ admits an ordinally blocking pair, say $e = \{u, v\}$. We distinguish between three cases, in each case obtaining a contradiction.

- If one of $u$ and $v$ is a singleton, say $u$, then by Definition 3.2(2), $v$ cannot be a singleton as otherwise, sat$(v, u) < sat(v, v) = 0$, which is not possible by the definition. Moreover, again by Definition 3.2(2), we have sat$(v, \pi^{-1}(v)) > sat(v, u)$ since $u$ is a singleton. This implies that sat$(v, \pi^{-1}(v)) \geq sat(v, u)$. By Claim 1, we have that $M(v, \geq u) = 1$, a contradiction to $\{u, v\}$ being an ordinally blocking pair of $M$.

- If neither $u$ nor $v$ is a singleton, but sat$(u, v) > sat(u, \pi^{-1}(u))$, then by the definition of sat, it follows that sat$(u, v) > sat(u, \pi^{-1}(u))$. By Definition 3.2(2) of stable partitions it follows that sat$(v, \pi^{-1}(v)) \geq sat(v, u)$. By Claim 1, we again obtain that $\{u, v\}$ is not an ordinally blocking pair, a contradiction.

Together with Observation 3.1, we know that $M^\pi$ is also ordinally stable. Hence, every graph with cardinal preferences admits an OSM.

It remains to consider the running time of Algorithm 1. Recall that $n$ denotes the number of vertices in $G$. Computing $P$ and sat in lines 1–2 can clearly be done in $O(n^2)$ time. By Proposition 3.4, $\pi$ can be computed from $(G, P')$ in $O(n^2)$. By Definition 3.2, the matching $M^\pi$ corresponding to $\pi$ can be computed $O(n^2)$ time. In total, the running time of Algorithm 1 is $O(n^2)$.

We close this section by considering the lattice property of OSMs. It is well-known by Roth, Rothblum, and Vate [25] that for bipartite graphs with strict cardinal preferences, the set of LSMs displays a certain lattice structure. Following their result, we show that the same holds for OSMs. To this end, given two fractional matchings $M_1$ and $M_2$ of a bipartite graph $G = (U \cup W, E)$ with preference lists $(\geq_x)_{x \in U \cup W}$, the join $\lor$ and meet $\land$ of $M_1$ and $M_2$ by Roth, Rothblum, and Vate [25] are defined as follows:

$$\forall x \in U, \forall y \in W :$$

$$M_1 \lor M_2((x, y)) := \max( M_1(x, \geq y) , M_2(x, \geq y))$$

$$M_1 \land M_2((x, y)) := \min( M_1(x, \geq y) , M_2(x, \geq y))$$

We say that matching $M_1$ weakly $U$-dominates (resp. $U$-dominates) matching $M_2$, written $M_1 \succeq_U M_2$ (resp. $M_1 \succ_U M_2$) if

$$\forall(x, y) \in U \times W : M_1(x, \geq y) \geq M_2(x, \geq y)$$

(resp. $M_1(x, \geq y) > M_2(x, \geq y)$).

For an illustration, consider the following example.

Example 3.7. Consider the following instance with strict preference lists; the underlying acceptability graph is a complete bipartite graph on $U \times W$ with $U = \{a, b, c\}$ and $W = \{1, 2, 3\}$.

$$1 : a \succ b \succ c, \quad 2 : b \succ c \succ a, \quad 3 : c \succ a \succ b$$

$$a : 2 \succ 3 \succ 1, \quad b : 3 \succ 1 \succ 2, \quad c : 1 \succ 2 \succ 3$$

Note that for OSMs, the exact values of the cardinal preferences in sat are not important. For the sake of completeness, we define sat as follows: sat$(1, a) = sat(2, b) = sat(3, c) = sat(a, 2) = sat(b, 3) = sat(c, 1) = 2$, sat$(1, b) = sat(2, c) = sat(3, a) = sat(a, 3) = sat(b, 1) = sat(c, 2) = 1$, and sat$(1, c) = sat(2, a) = sat(3, b) = sat(1, a) = sat(b, 2) = sat(c, 3) = 0$.

This instance admits 3 stable integral matchings $N_1, N_2$, and $N_3$ with $N_1(1, a) = N_1(2, b) = N_1(3, c) = 1$, $N_2(1, b) = N_2(3, a) = N_2(3, 2) = 1$, $N_3(1, c) = N_3(2, a) = N_3(3, b) = 1$ (unmentioned edges are set to zero). It also has two OSMs: $M_1$ and $M_2$ where each agent is fully matched: $M_1 = \frac{1}{3} \cdot N_1 + \frac{1}{3} \cdot N_2 + \frac{1}{3} \cdot N_3$ and $M_2 = \frac{1}{3} \cdot N_1 + \frac{1}{3} \cdot N_2 + \frac{1}{3} \cdot N_3$. We obtain the join and the meet of $M_1$ and $M_2$ as follows:

$$M_1 \lor M_2 = \frac{1}{3} \cdot N_1 + \frac{1}{3} \cdot N_2 + \frac{1}{3} \cdot N_3$$

and $M_1 \land M_2 = \frac{1}{3} \cdot N_1 + \frac{1}{3} \cdot N_2 + \frac{1}{3} \cdot N_3$. Note that any convex combination of $N_1$, $N_2$, and $N_3$ forms an OSM, which is not always the case for all bipartite graphs with strict preferences.
Proposition 3.8 ([25]). Let $G = (U \cup W, E)$ be a bipartite graph with strict preferences and let $\mathcal{P} = (\succeq x)_{x \in U \cup W}$ be the associated preference lists. Let $M_1$ and $M_2$ be two LSMs for $(G, \mathcal{P})$. Then $M_1 \lor M_2$ and $M_1 \land M_2$ are two LSMs for $(G, \mathcal{P})$. Moreover, for each $(x, y) \in U \times W$ it holds that

1. $M_1 \lor M_2(x, y) = \max(M_1(x, y), M_2(x, y))$,
2. $M_1 \land M_2(x, y) = \min(M_1(x, y), M_2(x, y))$.

The set of all LSMs of $(G, \mathcal{P})$ and the partial order $\succeq_U$ forms a distributive lattice, with $\lor$ and $\land$ representing the join and meet of any two LSMs.

Using the above fundamental property and by Observation 3.1(i), we show that OSMs form a lattice substructure of LSMs regarding the partial order $\succeq_U$ on the set of fractional matchings.

Proposition 3.9 (*). For each bipartite graph with strict preferences, the set of OSMs forms a distributive lattice under the partial order $\succeq_U$.

Proof. Let $G = (U \cup W, E)$ be a bipartite graph with strict preferences sat, let $\mathcal{P} = (\succeq x)_{x \in U \cup W}$ denote the induced strict preference lists, and let $\mathcal{M}_0$ denote the set of all OSMs of $(G, \mathcal{P})$. Since $\mathcal{M}_0$ is a subset of the set of all LSMs of $(G, \mathcal{P})$, to show that $(\mathcal{M}_0, \succeq_U)$ is a distributive lattice, it suffices to show that for each two OSMs $M_1, M_2 \in \mathcal{M}_0$ both $M_1 \lor M_2$ and $M_1 \land M_2$ are ordinaly stable.

Consider an arbitrary ordered pair $(x, y) \in U \times W$. We need to show that (a) $M_1 \lor M_2(x, y) \geq 1$ or $M_1 \lor M_2(y, x) \geq 1$, and (b) $M_1 \land M_2(x, y) \geq 1$ or $M_1 \land M_2(y, x) \geq 1$.

To show (a) suppose, for the sake of contradiction, that $M_1 \lor M_2(x, y) < 1$ and $M_1 \lor M_2(y, x) < 1$. By Proposition 3.8(1), it follows that $M_1(x, y) < 1$ and $M_1(y, x) < 1$. Since $M_1$ and $M_2$ are ordinaly stable, by definition, it must hold that $M_1(y, x) \geq 1$ and $M_2(y, x) \geq 1$. By Proposition 3.8(3), this means that $M_2(y, x) = \max(M_1(y, x), M_2(y, x)) \geq 1$, a contradiction to our assumption.

The reasoning for (b) is omitted because it is analogous using Proposition 3.8(4) and Proposition 3.8(2) instead of Proposition 3.8(3) and Proposition 3.8(1).

Similar to linear stability, the lattice structure of ordinal stability does not hold in the roommates setting, but the set of ordinaly stable matchings is closed under a median operation. For this, given three real values $x, y, z$, let $\text{med}(x, y, z)$ denote the second largest (or smallest) number among $x, y, z$. To show the median property for linear stability, Abeledo and Rothblum [1] extended the median notion to fractional matchings. Given a graph $G$ with preferences $(\succeq x)_{x \in V(G)}$ and three fractional matchings $M_1, M_2, M_3$ of $G$, let the median of $M_1, M_2, M_3$, denoted as $\text{med}(M_1, M_2, M_3)$, be defined as follows:

$$\forall x, y \in V: \text{med}(M_1, M_2, M_3)(x, y) := \text{med}(M_1(x, y), M_2(x, y), M_3(x, y)) = \frac{\text{med}(M_1(x, y), M_2(x, y), M_3(x, y))}{3}.$$

Proposition 3.10 ([11]). Let $\mathcal{M}_0$ denote the set of all LSMs of a graph with strict preferences. Then, for each $M_1, M_2, M_3 \in \mathcal{M}_0$ it holds that

1. $\text{med}(M_1, M_2, M_3)(x, y) = \text{med}(M_1(x, y), M_2(x, y), M_3(x, y))$ for all $x, y \in V$.
2. $\text{med}(M_1, M_2, M_3)(x, y) = \text{med}(M_1(x, y), M_2(x, y), M_3(x, y))$ for all $x, y \in V$, and
3. $\text{med}(M_1, M_2, M_3) \in \mathcal{M}_0$.

Proposition 3.10 immediately implies an analogous median property for OSMs.

Proposition 3.11 (*). Let $\mathcal{M}_0$ be the set of all OSMs of a graph with strict preferences. Then, for each $M_1, M_2, M_3 \in \mathcal{M}_0$ we have $\text{med}(M_1, M_2, M_3) \in \mathcal{M}_0$.

Proof. Let $G = (V, E)$ be a graph with strict preferences sat, let $\mathcal{P} = (\succeq x)_{x \in V}$ denote the induced preference lists. Let $\mathcal{M}_0$ be the set of OSMs of $(G, \mathcal{P})$, and let $M_1, M_2, M_3 \in \mathcal{M}_0$ as be defined in the statement. By Observation 3.1(i), $M_1, M_2, M_3$ are LSMs. Hence, by Proposition 3.10(3), $\text{med}(M_1, M_2, M_3)$ is a fractional matching of $G$, since it is an LSM of $(G, \mathcal{P})$. To show the membership in $\mathcal{M}_0$, we need to show that for each pair $(x, y) \in V$ of agents, $\text{med}(M_1, M_2, M_3)(x, y) \geq 1$ or $\text{med}(M_1, M_2, M_3)(y, x) \geq 1$. Consider an arbitrary pair $(x, y) \in V$, if $\text{med}(M_1, M_2, M_3)(x, y) \geq 1$, then we are done. Hence, let us assume that $\text{med}(M_1, M_2, M_3)(x, y) < 1$. By Proposition 3.10(1), we have that $\text{med}(M_1(x, y), M_2(x, y), M_3(x, y)) < 1$. This means that at least two of the three real values $M_1(x, y), M_2(x, y), M_3(x, y)$ are strictly smaller than one. Without loss of generality by symmetry, assume that $M_1(x, y) < 1, M_2(x, y) < 1$. Then, since $M_1, M_2 \in \mathcal{M}_0$, it follows that $M_1(x, y) \geq 1, M_2(x, y) \geq 1$. We distinguish between two cases for $M_3(x, y)$.

If $M_3(y, x) \geq 1$, then $\text{med}(M_1, M_2, M_3)(y, x) \geq 1$. Otherwise, $\text{med}(M_1(y, x), M_2(y, x), M_3(y, x)) = \min(M_1(y, x), M_2(y, x)) \geq 1$. In both cases, we obtain that $\text{med}(M_1, M_2, M_3)(y, x) \geq 1$. Hence, $\text{med}(M_1, M_2, M_3) \in \mathcal{M}_0$.

4 Algorithmic results

The structural properties from Section 3 give rise to efficient algorithms for finding optimal stable matchings.

Bipartite graphs with strict preferences. To describe efficient algorithms for this case, we first observe that for the case of bipartite graphs, each support of each OSM consists of integral stable matchings. We remark that this has also been proved by Aziz and Klaus (2019, Theorem 3) in their study of random matchings.3 For self-containedness and since our proof is short, instructive, and more direct than theirs, we include it here:

3The relevant statement is that an ex-ante weakly stable random matching $p$ is also robust ex-post weakly stable.
Lemma 4.1 ([4, Theorem]). Let \( G \) be a bipartite graph with satisfaction sat, \( M \) be an OSM for \((G, \text{sat})\), and \((M_j)_{j \in [k]}\) be a support for \( M \). Then, each \( M_j, j \in [k] \), is (integ rally) stable.

Proof. Let \( x_1, x_2, \ldots, x_k \in \mathbb{R}_{>0} \) such that \( \sum_{j \in [k]} x_j = 1 \) and for each edge \( e \in E(G) \) we have \( M(e) = \sum_{j \in [k]} x_j M_j(e) \). Fix an arbitrary edge \( \{u, v\} \in E(G) \). We show that for each \( j \in [k] \) edge \( \{u, v\} \) does not (integ rally) block \( M_j \). It suffices to show that \( M_j(u, v) = 1 \) or \( M_j(v, u) = 1 \) since \( M_j \) is an integral matching.

By the ordinal stability of \( M \) we have \( M(u, v) = 1 \) or \( M(v, u) = 1 \).

Say the latter holds:

\[
M(v, u) = 1; \tag{3}
\]

the proof if the former holds is analogous. If we can show that \( M_j(v, u) = 1 \) for each \( j \in [k] \), then we achieve what we wanted to show, namely that each \( M_j, j \in [k] \), is stable.

Thus, it remains to show that \( M_j(v, u) = 1 \) for each \( j \in [k] \). By the definition of supports and by (3), we have

\[
1 = \sum_{u \in B \varepsilon, (u)} \sum_{j \in [k]} x_j \cdot M_j(v, u) = \sum_{j \in [k]} x_j \cdot \left( \sum_{u \in B \varepsilon, (u)} M_j(v, u) \right) = \sum_{j \in [k]} x_j \cdot M_j(v, u). \tag{4}
\]

Since each \( M_j \) is an integral matching, it holds that \( M_j(v, u) \in \{0, 1\} \). Since each \( x_j \) is a positive real value with \( \sum_{j \in [k]} x_j = 1 \), it must hold that \( M_j(v, u) = 1 \) as otherwise inequality (4) could not hold. Thus, for each \( j \in [k] \) we have \( M_j(v, u) = 1 \), which implies that \( \{u, v\} \) does not (integ rally) block \( M_j \), as required.

The reverse of Lemma 4.1 does not hold, i.e., a convex combination of stable integral matchings is not necessarily ordinally stable or cardinally stable. This is shown in Example 2.4, where \( M_1 \) is a convex combination of stable integral matchings, but it is neither cardinally nor ordinally stable. This has also been observed for ordinal stability by Aziz and Klaus [4]; see their Theorem 3.

Now a replacement argument shows that the maximum achievable welfare and the maximum number of fully matched agents for integral stable matchings are also the maximum for OSMs, due to the following.

Lemma 4.2 (*). Let \( G \) be a bipartite graph with satisfactions sat, \( M \) be a matching for \((G, \text{sat})\), and \((M_j)_{j \in [k]}\) be a support of \( M \) with the coefficients \( x_1, x_2, \ldots, x_k \in \mathbb{R}_{>0} \). Then, the following hold: \( \text{welfare}(M) = \sum_{j \in [k]} \{ x_j \cdot \text{welfare}(M_j) \} \) and \#fully(M) \leq \max_{j \in [k]} \#fully(M_j).

Proof. Let \( G, \text{sat}, M, (M_j)_{j \in [k]}, x_1, \ldots, x_k \) be as defined in the statement. By the definition of supports, we have

\[
\text{welfare}(M) = \sum_{v \in V} U_M(v) = \sum_{v \in V} \sum_{u \in \text{NG}(v)} \text{sat}(v, u) \cdot M(u, v)
\]

\[
= \sum_{v \in V} \sum_{u \in \text{NG}(v)} \text{sat}(v, u) \cdot \left( \sum_{j \in [k]} x_j \cdot M_j(u, v) \right)
\]

\[
= \sum_{v \in V} \sum_{u \in \text{NG}(v)} x_j \cdot \text{sat}(v, u) \cdot M_j(u, v)
\]

\[
= \sum_{j \in [k]} x_j \cdot \sum_{v \in \text{NG}(v)} \text{sat}(v, u) \cdot M_j(u, v)
\]

\[
= \sum_{j \in [k]} x_j \cdot \#\text{fully}(M_j).
\]

This shows the first statement.

As for the the number of fully matched agents, again, by the definition of supports, we have

\[
\#\text{fully}(M) \leq \sum_{j \in [k]} x_j \cdot \#\text{fully}(M_j)
\]

\[
= x_r \cdot \#\text{fully}(M_r) + \left( \sum_{j \in [k] \setminus \{r\}} x_j \cdot \#\text{fully}(M_j) \right)
\]

\[
\leq x_r \cdot \#\text{fully}(M_r) + \left( \sum_{j \in [k] \setminus \{r\}} x_j \cdot \#\text{fully}(M_j) \right)
\]

\[
= \#\text{fully}(M_r).
\]

Using Lemma 4.2 we may assume that any optimal ordinally stable matching is an optimal integral stable matching since using a simple exchange argument, we may swap out matchings in the support of a fractional matching for integral matchings with maximum welfare or with maximum number of fully matched agents in order to decrease the number of matchings in the support until only one remains. Since finding an optimal stable integral matching for bipartite graphs with strict preferences is polynomial-time solvable [17], we immediately obtain the same for ordinal stability.

Theorem 4.3 (*). For bipartite graphs with strict preferences, MAX-WELFARE OSM and MAX-FULL OSM are polynomial-time solvable.

Proof. To show the statement, we show that for bipartite graphs with strict preferences, the maximization variants
of MAX-FULL OSM and MAX-WELFARE OSM can be solved in polynomial time. To this end, let \( G = (U \cup W, E) \) be a bipartite graph with strict preferences \( \text{sat} \) and let \( P = (\succ_v)_{v \in V} \).

We first consider the maximization variant of MAX-FULL OSM. By Lemma 4.2, it suffices to find an integral stable matching of \((G, P)\) which has the maximum number of matched agents among all stable integral matchings. Now, observe that for strict preferences, every stable integral matching matches the same set of agents [14]. This means that every stable integral matching of \((G, P)\) fulfills our requirements. Hence, we can simply use Gale and Shapley’s extended algorithm to find a stable integral matching.

Next, we consider the maximization variant of MAX-WELFARE OSM. Let \( \text{optfrac} \) be the maximum welfare of an OSM for \((G, \text{sat})\) and let \( \text{optint} \) be the maximum welfare of a stable integral matching for \((G, P)\). We first claim that \( \text{optfrac} = \text{optint} \). It is clear that \( \text{optfrac} \geq \text{optint} \). For the other direction, let \( M_{\text{frac}} \) be an OSM for \((G, \text{sat})\) that has welfare \( \text{optfrac} \). By Proposition 2.2, let \( (M_j)_{j \in [k]} \), together with \( x_1, x_2, \ldots, x_k \in \mathbb{R}_{>0} \) such that \( \sum_{j \in [k]} x_j = 1 \), be a support of \( M_{\text{frac}} \). By Lemma 4.1 for each \( j \in [k] \) we have that \( M_j \) is stable. Let \( r \in [k] \) such that \( M_r \) has maximum welfare among \( \{ M_j \mid j \in [k] \} \). By Lemma 4.2 we have

\[
\text{welfare}(M_r) = x_r \cdot \text{welfare}(M_r) + (1 - x_r) \cdot \text{welfare}(M_r) 
\geq \sum_{j \in [k]} x_j \cdot \text{welfare}(M_j) = \text{welfare}(M_{\text{frac}}).
\]

Since \( M_r \) is stable, we have \( \text{optfrac} = \text{optint} \).

Now, to find an OSM with maximum welfare it suffices to find a stable integral matching with maximum welfare. Since a stable integral matching with maximum welfare has also achieved the minimum egalitarian cost, and since a stable integral matching with minimum egalitarian cost can be found in \( O((|U| + |W|)^4) \) time [17], we can find an OSM with maximum welfare in \( O((|U| + |W|)^4) \) time.

Remark. The proof for Theorem 4.3 also shows that for bipartite graphs without ties, finding an OSM with maximum sum of matching values can be done in polynomial time.

Non-bipartite graphs with strict preferences. The tractability result of MAX-FULL OSM for bipartite graphs with strict preferences heavily utilizes the fact that each fractional matching of a bipartite graph is a convex combination of integral matchings. This fact, however, does not hold for non-bipartite graphs. Nevertheless, we can extend the polynomial-time result to the non-bipartite case, using the first phase of Irving’s polynomial-time algorithm for finding a stable integral matching (1985) (see Algorithm 2). The correctness is based on the following.

Lemma 4.4 (*). Let \( G = (V, E) \) be a graph with cardinal and strict preferences \( \text{sat} \), and let \( M \) be an OSM of \((G, \text{sat})\). The following hold for each agent \( y \in V \):

1. \( M(x, \succeq y) = 1 \), where \( x \) is the most preferred agent of \( y \).
2. The following three statements are equivalent:
   - \( y \) is matched in \( M \).
   - \( y \) is not a singleton in a stable partition of \((G, \text{sat})\).
   - \( y \) is fully matched in \( M \).

Proof. Let \( G = (V, E) \), \( \text{sat} \), and \( y \) be as in the statement. To show (1), suppose, for the sake of contradiction, that \( M(x, \succeq y) < 1 \), where \( x \) is the most-preferred agent of \( y \). This implies \( M(y, \preceq x) = M(y, x) < 1 \), a contradiction to \( M \) being ordinally stable regarding edge \( \{x, y\} \). In order to show the equivalence of the statements in (2), we need to know which agents are matched by \( M \). To obtain this, let us first utilize Statement (1) to repeatedly “delete” pairs from \( P \) that will not be part of and will not block any OSM. We need some more notations. Since we will modify the preference lists in \( P \), we use \( 1_{SP}(x) \) and \( \text{last}_P(x) \) to refer to the most-preferred agent and the least-preferred agent in the preference list \( \succ x \in P \) of \( x \). Now, if \((G, P)\) admits an OSM, say \( M \), then \( M \) cannot match some pair \( \{x, z\} \) for which there exists an agent \( y \) with \( x = 1_{SP}(y) \) such that \( x \) prefers \( y \) to \( z \) as this violates Statement (1) regarding \( \{x, y\} \). This means that we can repeatedly delete such pairs. A pseudocode description of the above approach is given in the while loop of Algorithm 2. We aim to show that no matched pair of \( M \) is deleted in Algorithm 2. Clearly, after the first iteration in the while loop of Algorithm 2 (see lines 2–4), no pair \( e = \{i, j\} \) with \( M(e) > 0 \) are deleted; we call such pairs matched pairs. Using the above reasoning successively, we know that in every iteration, no matched pairs of \( M \) are deleted. In other words, after the while loop, no matched pairs of \( M \) are deleted. This also means that after the while loop, every agent matched under \( M \) must contain at least one agent in her preference list. Hence, if an agent’s preference list becomes empty after the while loop, then no OSM will match her. Since an agent is a singleton if and only if its preference list becomes empty after the while loop Tan [26], no OSM will match a singleton agent.

This shows “Statement (2i) \( \Rightarrow \) (2ii)”. It remains to show that “Statement (2ii) \( \Rightarrow \) (2iii)” since “Statement (2ii) \( \Rightarrow \) (2ii)” clearly holds. To this end, let \( S \) denote the set of agents returned from Algorithm 2 on input \((G, \text{sat})\), and let \( P \) denote the modified preference lists after the execution of Algorithm 2. Clearly, for each \( x \in V \setminus S \) there exists an agent \( y \) with \( x = 1_{SP}(y) \) and \( y = \text{last}_P(x) \). By Statement (1) it must hold that \( M(x, \succeq y) = 1 \).

In other words, each agent \( x \in V \setminus S \) must be fully matched under \( M \). Since by Tan [26], each agent \( x \in V \) is a non-singleton if and only if \( x \in V \setminus S \), it follows that each non-singleton is fully matched under \( M \). This completes the proof of the equivalence of the statements in (2).

Using Lemma 4.4 we can show that the OSM returned from Algorithm 1 achieves the maximum number of fully matched agents whenever no ties are present.

Lemma 4.5 (*). For graphs with \( n \) vertices and with strict preferences, MAX-FULL OSM can be solved in \( O(n^2) \) time.

Proof. Let \((G, \text{sat}, \tau)\) be an instance of MAX-FULL OSM with \( G = (V, E) \) and \( \text{sat} \) having strict preferences, and let \( P = (\succ_v)_{v \in V} \) denote the strict preference lists derived from \( \text{sat} \).
Algorithm 2: Irving’s phase 1 algorithm on input \((G = (V, E)), \text{sat}\)

\[
\begin{align*}
1 & \text{ Compute the preference lists } \mathcal{P} \text{ from sat} \\
2 & \text{ while } \exists y \in V \text{ with non-empty pref. s.t. last}_\mathcal{P}(\text{1st}_\mathcal{P}(y)) \neq y \\
3 & \quad \text{ do } \\
4 & \quad \quad D \leftarrow \{\{\text{1st}_\mathcal{P}(y), z\} \mid \text{1st}_\mathcal{P}(y) \text{ prefers } y \text{ to } z \text{ in } \mathcal{P}\} \\
5 & \quad \mathcal{P} \leftarrow \mathcal{P} - D \\
6 & \text{ return } \{v \in V \mid v \text{ has empty pref. list in } \mathcal{P}\}
\end{align*}
\]

We aim to show that on input \((G, \text{sat})\), Algorithm 1 returns an optimal OSM \(M^\pi\) which maximizes the number of fully matched agents. To this end, let \(S\) the set of singleton agents according to Proposition 3.4 and let \(M\) denote the set of (fully) matched agents in \((G, \text{sat})\). By Lemma 4.4(2), we know that no agent in \(S\) is matched under \(M\). In other words, every agent matched under \(S\) comes from \(V \setminus S\). Since by Lemma 3.6, every agent from \(V \setminus S\) is fully matched under \(M^\pi\), we obtain that \(M^\pi\) maximizes the number of fully matched agents.

Hence, to solve MAX-FULL OSM, we only need to compare whether \(\tau \leq |M^\pi|\), and return yes if and only if this is the case. The running time of solving MAX-FULL OSM comes from using Algorithm 1 which can be done in \(O(|V|^2)\) time due to Lemma 3.6.

By Lemma 4.4, we also obtain the same structural property for ordinal stability as for integral stability.

Observation 4.6. For graphs with strict cardinal preferences, the set of agents is partitioned into two subsets, those that are (fully) matched in every OSM and those that are matched in none.

We close this section by observing that the OSM that we obtain from Algorithm 1 is also a 2-approximate solution for maximizing fully matched agents for both cardinal stability and ordinal stability, even for preferences with ties.

Proposition 4.7 (*). Algorithm 1 is a 2-approximation algorithm for the maximization variants of MAX-FULL OSM and MAX-FULL CSM.

Proof. Let \(G = (V, E)\) denote a graph with cardinal preferences sat with ties. We only show the case of cardinal stability since the case of ordinal stability is analogous. Let \(M^C\) denote an optimal CSM of \((G, \text{sat})\) with maximum number of fully matched agents. By Lemma 3.6, let \(M^\pi\) denote the OSM returned by Algorithm 1 on input \((G, \text{sat})\), and let \(\mathcal{P}\) be the strict preference lists that are used to compute the corresponding stable partition \(\pi\) (see lines (2)–(3)). Recall that in \(M^\pi\) every matched agent is fully matched. Let \(A^\pi\) denote the set of (fully) matched agents in \(M^\pi\); note that \(|A^\pi| = \#\text{fully}(M^\pi)|\).

Since \(M^\pi\) is a CSM, to show that Algorithm 1 is a 2-approximation algorithm for cardinal stability, we only need to show that \(\#\text{fully}(M^C) \leq 2|A^\pi|\). We first show that each acceptable pair must include some agent from \(A^\pi\), i.e.,

\[
\forall \{x, y\} \in E : x \in A^\pi \text{ or } y \in A^\pi.
\]

Consider an arbitrary pair \(\{x, y\} \in E\). Since \(M^\pi\) is an OSM of \((G, \text{sat})\), it follows that \(M^\pi(x, \geq) = 1\) or \(M^\pi(y, \geq x) = 1\). In other words, \(x \in A^\pi\) or \(y \in A^\pi\) since an agent is fully matched under \(M^\pi\) if and only if she is in \(A^\pi\) (see Lemma 3.6).

Altogether, we derive that

\[
\#\text{fully}(M^C) \leq \sum_{x \in V \times V} M(x, y) = \sum_{x \in A^\pi, y \in N_G(x)} M(x, y) + \sum_{y \in N_G(x)} M(x, y)
\]

\[
\leq \sum_{y \in V \setminus A^\pi} M(x, y) + \sum_{y \in V \setminus A^\pi} M(x, y)
\]

\[
\leq 2 \cdot |A^\pi|.
\]

Since Algorithm 1 runs in \(O(|V|^2)\) (see Lemma 3.6), it is a 2-approximation for the maximization variant of MAX-FULL CSM.

5 Hardness results

We first give our results for ordinally stable matchings and then turn to cardinaly stable matchings.

Hardness for optimal ordinally stable matchings The structural property that for the marriage case (i.e., bipartite graphs), ordinally stable matchings can be decomposed into a convex combination of stable integral matchings (Lemmas 4.1 and 4.2) implies that MAX-FULL OSM is equivalent to finding a stable integral matching with maximum cardinality (MAX-CARD SMTI), and MAX-WELFARE OSM is equivalent to finding a stable integral matching with minimum egalitarian cost (MIN-EGAL SMTI); the egalitarian cost and the welfare of an integral matching are dual to each other. Since both problems are known to be NP-hard Manlove et al. [20], we obtain the following.

Theorem 5.1 (*). When the preferences have ties, MAX-FULL OSM and MAX-WELFARE OSM become NP-complete, even for bipartite graphs.

Proof. We first tackle MAX-FULL OSM by reducing from the NP-hard problem MAX-CARD SMTI [20]. MAX-CARD SMTI is the problem of deciding whether, given a bipartite graph \(G = (U \cup W, E)\) with preference lists \(\mathcal{P} = (\geq_x)_{x \in U \cup W}\) such that each agent \(u \in U\) (resp. \(w \in W\)) has a preference list over \(N_G(u)\) (resp. \(N_G(w)\)), and a non-negative integer \(\tau\), there exists a stable integral matching of cardinality at least \(\tau\). Recall that an integral matching \(M\) is stable if no two agents form a blocking pair, and two agents \(u, w\) form a blocking pair of \(M\) if

(i) \(\{u, w\} \in E\) and

(ii) \(M(u, \geq w) + M(w, \geq u) = 0\).

Let \(I = (G, \mathcal{P}, \pi)\) be an instance of MAX-CARD SMTI and let sat be the cardinal preferences derived from \(\mathcal{P}\), i.e., for each \(x \in U \cup W\) and each \(y \in N_G(x)\) define

\[
\forall (x, y) \in (U \cup W) \times (U \cup W):
\]

\[
\text{rank}_\mathcal{P}(x, y) := \begin{cases} 
|\{y' : y' \geq x\}|, & \text{if } \{x, y\} \in E \\
|N_G(x)|, & \text{otherwise},
\end{cases}
\]

\[
\text{sat}_\mathcal{P}(x, y) := |N_G(x)| - \text{rank}_\mathcal{P}(x, y).
\]
Note that this definition satisfies Condition (PREF) so that the preference lists derived from \( \text{sat}_{\mathcal{P}} \) are equivalent to \( \mathcal{P} \).

We aim to show that \((G, \text{sat}_{\mathcal{P}})\) admits a stable integral matching with cardinality \( \tau \) if and only if \((G, \text{sat}_{\mathcal{P}})\) admits an OSM where at least \( \tau \) agents are fully matched. The “only if” direction is clear since every stable integral matching of \((G, \mathcal{P})\) is an OSM of \((G, \text{sat}_{\mathcal{P}})\). For the “if” direction, let \( M \) be an OSM of \((G, \text{sat}_{\mathcal{P}})\) with \( \#\text{fully}(M) \geq \tau \). By Proposition 2.2, let \((M_j)_{j \in [k]}\) be a support of \( M \). By Lemma 4.2, \( \max_{j \in [k]} \#\text{fully}(M_j) \geq \#\text{fully}(M) \geq \tau \). Since every \( M_j \), \( j \in [k] \), is stable by Lemma 4.1, there exists an integral stable matching of \((G, \text{sat}_{\mathcal{P}})\) with cardinality at least \( \tau \). This completes the proof for showing that MAX-FULL OSM is NP-hard.

The proof for MAX-WELFARE OSM works similarly. Instead of reducing from MAX-CARD SMTI, we reduce from the NP-hard MIN-EQUAL SMT problem [20]. MIN-EQUAL SMT is the problem of deciding whether, given a complete bipartite graph \( G = (U \cup W, E) \) with complete preference lists \( \mathcal{P} = (\{x\}, x \in U \cup W) \) such that each agent \( u \in U \) (resp. \( w \in W \)) has a preference list over all agents from \( W \) (resp. \( U \)), and a non-negative integer \( \gamma \), there exists a stable integral matching of egalitarian cost at most \( \gamma \). Here, the egalitarian cost of an integral matching \( M \) is defined as

\[
\text{egal}_{\mathcal{P}}(M) := \sum_{x \in U \cup W} \text{rank}_{\mathcal{P}}(x).
\]

Note that for complete preference lists, each stable integral matching must be perfect.

Now, we observe that if we use the satisfaction function \( \text{sat}_{\mathcal{P}} \) given in (7), then each each integral matching \( N \) of \((G, \text{sat}_{\mathcal{P}})\) satisfies the following duality:

\[
\text{egal}_{\mathcal{P}}(N) + \text{welfare}_{\text{sat}_{\mathcal{P}}}(N) = 2|E| \quad (8)
\]

Hence, a stable integral matching has maximum social welfare if and only if it has minimum egalitarian cost (among all stable integral matchings). This means that finding a stable integral matching with maximum social welfare is NP-hard. Hence, to show NP-hardness for MAX-WELFARE OSM, it suffices to show that \((G, \mathcal{P})\) admits a stable integral matching with welfare \( \gamma \) if and only if \((G, \text{sat}_{\mathcal{P}})\) admits an OSM with welfare \( \gamma \). The “only if” direction is clear since every integral matching of \((G, \mathcal{P})\) is an OSM of \((G, \text{sat}_{\mathcal{P}})\).

For the “if” direction, let \( M \) be an OSM of \((G, \text{sat}_{\mathcal{P}})\) with welfare \( M \geq \gamma \). By Proposition 2.2, let \((M_j)_{j \in [k]}\) be a convex combination of \( M \). To show the “if” direction, we only need to show that \( \max_j \text{welfare}(M_j) \geq \text{welfare}(M) \).

Similarly to the proof of Theorem 4.3, by Lemma 4.2 and by the property of convex combinations \( \text{welfare}(M) = \sum_{j \in [k]} \text{welfare}(M_j) \leq \max_{j \in [k]} \text{welfare}(M_j) \).

This completes the proof for MAX-WELFARE OSM.

\[\square\]

Remark. The proof for Theorem 5.1 also implies NP-hardness when we instead aim to find an ordinally stable matching with maximum sum of matching values.

Feder (1994) showed that finding a maximum-welfare stable integral matching in non-bipartite graphs is APX-hard, even if no ties are present. In the next theorem we show that the idea behind the NP-hardness reduction of Feder with some additional analysis yields the same inapproximability result for ordinal stability.

**Theorem 5.2** (⋆). The maximization variant of MAX-WELFARE OSM is APX-hard and MAX-WELFARE OSM is NP-complete, even if no ties are present.

**Proof.** To show the inapproximability, we reduce from the maximization variant of INDEPENDENT SET, the MAX INDEPENDENT SET problem, which is defined as follows.

**MAX INDEPENDENT SET (MAX-IS)**

**Input:** A graph \( G = (V, E) \).

**Task:** Find a maximum-cardinality independent set of \( G \); here, an independent set of \( G \) is a vertex subset \( X \subseteq G \) such that \( G[X] \) is edgeless.

**MAX INDEPENDENT SET** is APX-hard and hence NP-hard, even for cubic graphs [3]. Let \( G \) be a cubic graph with vertex set \( \{v_1, v_2, \ldots, v_n\} \). The basic idea is to construct an instance of MAX-WELFARE OSM where for each vertex \( v_i \) of \( G \), there are essentially two possible matchings, say \( M_{1i} \) and \( M_{2i} \), which each may lead to some ordinaly stable matching. However, \( M_{1i} \) has a higher welfare than \( M_{2i} \), but it is not possible to include \( M_{1i} \) for two adjacent vertices as they will induce an ordinaly blocking pair.

We create our instance \( I' \) of MAX-WELFARE OSM as follows. For each vertex \( v_i \in V \), we create four agents \( u_i, w_i, x_i, y_i \). For ease of notation, define \( L := 2n + 2m \). We describe the cardinal preferences of the agents in Figure 3, where for each vertex \( v_i \in V \), let \( d_i \) and \( N(u_i) \) denote the degree of \( v_i \) in \( G \) and the set of agents \( u_\nu \) which correspond to the neighbors \( v_\nu \) of \( v_i \) in \( G \), respectively, and let \( \pi_i : N(u_i) \rightarrow [d_i] \) denote an arbitrary but fixed enumeration of the agents in \( N(u_i) \); the intent of \( \pi_i \) is to give each “adjacent” agent a unique rank so that no ties are present. Observe that the constructed preferences indeed contain no ties.

This completes the construction of our instance \( I' \). Clearly, \( I' \) can be constructed in linear time and every agent
has strict preferences.

We first show the “equivalence” in terms of solutions.

**Claim 2.** $G$ has an independent set $V'$ with $|V'| = k$ if and only if the constructed instance $I'$ admits an OSM $M$ with welfare($M$) = $L \cdot (k + 1)$.

**Proof of Claim 2.** For the “only if” part, assume that $G$ has an independent set of cardinality $k$ and let $V' = \{v_i\}$ denote such an independent set. We form an integral matching $M$ as follows.

(i) For each $v_i \in V'$, let $M(u_i, y_i) = M(x_i, w_i) = 1$.
(ii) For each $v_i \in V \setminus V'$, let $M(u_i, w_i) = M(x_i, y_i) = 1$.
(iii) For each remaining acceptable edge $e = \{a, b\}$ not mentioned in (i)–(ii), let $M(a, b) = 0$.

By construction, matching $M$ has the following social welfare:

$$
\text{welfare}(M) = \sum_{v_i \in V'} (\text{sat}(u_i, y_i) + \text{sat}(x_i, w_i)) + \sum_{v_i \in V \setminus V'} (\text{sat}(u_i, w_i) + \text{sat}(x_i, y_i)) = \sum_{v_i \in V'} (d_i + 2 + L) + \sum_{v_i \in V \setminus V'} (d_i + 2) = 2n + \sum_{v_i \in V'} (d_i + 2) + 2m = \sum_{v_i \in V'} (d_i + 2 + L) + \sum_{v_i \in V \setminus V'} (d_i + 2) = L \cdot (k + 1).
$$

It remains to show that $M$ is ordinally stable.

Clearly, for each $v_i \in V \setminus V'$, neither $u_i$ nor $x_i$ is involved in an ordinally blocking pair since they already receive their most preferred agents, respectively. Analogously, for each $v_i \in V'$, neither $w_i$ nor $y_i$ is involved in an ordinally blocking pair. Hence, for each $i \in [n]$, no two agents from $\{u_i, w_i, x_i, y_i\}$ form an ordinally blocking pair.

Finally, suppose that $\{u_i, u_{i'}\}$ is an ordinally blocking pair of $M$. By the definition of $M$, $u_i$ and $u_{i'}$ appear in each other’s preference lists and $M(u_i, y_i) = M(u_{i'}, y_{i'}) = 1$. By the cardinal preferences and by the definition of $M$, $v_i$ and $v_{i'}$ are neighbors in $G$. Hence, $v_i, v_{i'} \in V'$, a contradiction to $V'$ being an independent set.

For the “if” part, assume that $I'$ admits an OSM with welfare($M$) = $L \cdot (k + 1)$, and let $M$ be such a matching. We claim that the following vertex set $V'' := \{v_i \in V \mid M(u_i, y_i) > 0\}$ is an independent set of size at least $k$.

To show that $V''$ is an independent set, let us consider an arbitrary vertex $v_i \in V''$. We aim to show that no neighbor of $v_i$ belongs to $V''$. By applying Lemma 4.4 (1) three times (setting $y = u_i$, $x = x_i$, $y = y_i$, respectively), we obtain that

$$
M(w_i, x_i) + M(w_i, u_i) = M(y_i, u_i) + M(y_i, x_i) = M(x_i, y_i) + M(x_i, w_i) = 1.
$$

This means that

$$
M(u_i, w_i) + M(u_i, y_i) = 1.
$$

In other words, $M(u_i, u_{i'}) = 0$ for each $u_{i'}$ with $v_{i'}$ being the neighbor of $v_i$, and $V''$ is indeed an independent set.

It remains to consider the size $|V'|$. To ease notation, for each $v_i \in V$, let $\alpha_i = M(u_i, y_i)$ with $0 \leq \alpha_i \leq 1$. Then, by (9)–(10), we have that $M(u_i, y_i) = M(x_i, w_i) = \alpha_i$ while $M(u_i, w_i) = M(x_i, y_i) = 1 - \alpha_i$. Hence,

$$
\text{welfare}(M) = \sum_{v_i \in V'} (\text{sat}(u_i, y_i) + \text{sat}(x_i, w_i)) + \sum_{v_i \in V \setminus V'} (\text{sat}(u_i, w_i) + \text{sat}(x_i, y_i)) = \sum_{v_i \in V'} (d_i + 2 + L) + \sum_{v_i \in V \setminus V'} (d_i + 2) = L \cdot (k + 1).
$$

Now, suppose that we have a polynomial-time algorithm which approximates the maximization variant of MAX-WELFARE OSM within factor $\varepsilon$. That is, on input $(G, \text{sat})$ and a positive approximation error $\varepsilon$ with $0 < \varepsilon < 1$, the algorithm returns an OSM $M$ such that welfare($M$) $\geq \varepsilon \cdot M^\ast$, where $M^\ast$ denotes the maximum welfare of all OSMs of $(G, \text{sat})$. If the maximum-cardinality independent set of $G$ has at least $k^\ast$ vertices, then by Claim 2, our instance $I'$ has an OSM $M^\ast$ with welfare($M^\ast$) = $L \cdot (k^\ast + 1)$. Hence, the approximation algorithm finds an OSM $M$ with

$$
\text{welfare}(M) \geq \varepsilon \cdot L \cdot (k^\ast + 1).
$$

Again, by Claim 2, the approximation algorithm also finds an independent set $V'$ for $G$ with $|V'| = k$ and

$$
k = \frac{\text{welfare}(M)}{L} - 1 \geq \varepsilon k^\ast + \varepsilon - 1 > \varepsilon k^\ast - 1.
$$

Since $k$ is an integer, it follows that $k \geq \varepsilon \cdot k^\ast$. In other words, the approximation algorithm can also be used to approximate MAX INDEPENDENT SET to an arbitrary factor, a contradiction.

Since MAX-IS is also NP-hard, the same reduction shows that MAX-WELFARE OSM is NP-hard, even for strict preferences.

**Remark.** Note that in the instance created in the reduction for Theorem 5.2 the number of agents acceptable to each agent is bounded by five. This implies that the both APX-hardness and NP-hardness remain even for this restricted case.

**Hardness for optimal cardinally stable matchings.** We now prove that MAX-WELFARE CSM and MAX-FULL CSM are NP-complete even when the input graph $G$ is bipartite and the values of sat($v$, $\cdot$) are distinct for each vertex $v$ in $G$, i.e., sat has no ties. For each of the problems we give a many-to-one polynomial-time reduction from the well-known NP-complete problem INDEPENDENT SET [13].
**INDEPENDENT SET (IS)**

**Input:** A graph $G$ and a non-negative integer $k$.

**Question:** Is there a size-at-least-$k$-vertex independent set $X$ in $G$, that is, a vertex subset $X$ such that $G[X]$ is edgeless?

The gadgets used in and the correctness proof of the two reductions have some similar parts. In fact, we use the same edge gadgets, which we now describe.

**Construction 1 (Edge gadget for the cardinal stability).** Let $(G = (V, E), k)$ be an instance of IS, where $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$ denote the vertex set and the edge set, respectively. The **edge gadgets** are contained in a bipartite graph $G_E$ with preference function set. The vertex set $V(G_E)$ is the union of two disjoint sets $U_E$ and $V_E$; we call the vertices in $G_E$ agents to distinguish them from the vertices in $G$. For each $e_j \in E$ with $e_j = \{v_i, v_{i'}\}$ do the following:

- Add to $U_E$ three agents $e_j, f_j$, and $g_j$; add to $V_E$ an agent $h_j$.
- Add to $V_E$ two agents $e'_j$ and $e''_j$.
- Add to $U_E$ two extra agents $u_i$ and $u_{i'}$ to $U_E$ if not already present. They represent the connections of the edge gadget to the vertex gadgets.

For each $j \in [m]$ by the **edge gadget for edge $e_j$** (of $G$) we refer to the subgraph of $G_E$ induced by the agents $\{e_j, f_j, g_j\} \cup \{h_j, e'_j, e''_j\}$. Figure 4 illustrates the edge gadget for edge $e_j \in E$. More precisely, it depicts the acceptability (sub)graph corresponding to edge $e_j$, $e_j \in E$, labeled with the cardinal preferences set, and the derived preferences lists. The cardinal preferences set of the agents $u_i$ and $u_{i'}$ from the vertex gadgets towards the agents $e_j$ and $e_j$ are bounded by $5m$. The cardinal preferences not mentioned until now are set to zero. The precise values are irrelevant for now and will be defined in the hardness constructions when we use the edge gadgets.

Note that although we use the symbol $e_j$ for both an edge in $G$ from the IS instance and an agent in the edge gadget, the precise meaning will be clear from the context.

The edge gadget by Construction 1 has two crucial properties for a CSM, one regarding social welfare and another regarding fully matched agents. We will exploit both properties for the hardness proofs. The desired behavior is essentially that, in every feasible matching $M$ for $G_E$, that is, in every CSM that has sufficiently large social welfare or sufficiently many fully matched agents, at least one of $\{e'_j, e''_j\}$ is unsatisfied with respect to $M$, i.e., $e'_j$ or $e''_j$ would rather like to be integrally matched with $u_i$ or $u_{i'}$, respectively. We then use this property to force the matching to assign $u_i$ or $u_{i'}$ to partners so that at most one of the two assignments signifies that $v_i$ or $v_{i'}$ is supposed to be in the independent set.

We first summarize this property regarding social welfare.

**Lemma 5.3 (⋆).** Let $M$ be a CSM for $G_E$. Then, the total welfare $\omega$, received from $M$ by the vertices of the edge gadgets is at most $3m(2m^2 + 9)$. Moreover, if there is an edge $e_j \in E$ with $e_j = \{v_i, v_{i'}\}$ such that for both $\nu \in \{i, i'\}$ we have $\nu_M(v_\nu) < \text{sat}(e'_j, e''_j)$, then $\omega < 3m(2m^2 + 9) - n$.

**Proof.** Fix $j \in [m]$ and assume that $e_j = \{v_i, v_{i'}\}$ is an edge from $E$. For the sake of readability, define $U_j := \{e_j, g_j, f_j\}$ and $W_j := \{e'_j, e''_j, h_j\}$. We use $G_j$ to refer to the edge gadget corresponding to $e_j$, where the edges are incident to some agent from $U_j \cup W_j$. Note that the total welfare $\omega_j$ received from $M$ by the agents of the edge gadget $G_j$ is $\omega_j = \sum_{x \in U_j} U_M(x) + \sum_{y \in W_j} U_M(y)$.

We first show that the maximum welfare given by $M$ to the agents in $G_j$ is at most $3(2m^2 + 9)$, i.e.,

$$\omega_j \leq 3(2m^2 + 9).$$

(12)

Observe that for each edge $\{x, y\} \in E(G_j)$ from gadget $G_j$ we have sat$(x, y) + \text{sat}(y, x) = 2m^2 + 9$, i.e.,

$$\forall(x, y) \in U_j \times W_j \text{ with } \{x, y\} \in E(G_j):$$

$$\text{sat}(x, y) + \text{sat}(y, x) = 2m^2 + 9.$$ 

(13)

Moreover, each edge in $G_j$ is incident with $e_j, g_j$, or $f_j$, and no edge is incident with two of them since $G_j$ is bipartite. Thus, the welfare $\omega_j$ achieved by the agents in $G_j$ is bounded as follows:

$$\omega_j = \left( \sum_{(y, x) \in W_j \times U_j} M(\{y, x\}) \cdot \text{sat}(x, y) + \text{sat}(y, x) \right)$$

+ sat$(e'_j, u_i) \cdot M(e'_j, u_i) + \text{sat}(e'_j, u_{i'}) \cdot M(e''_j, u_{i'})$

$$\leq (2m^2 + 9) \sum_{(y, x) \in W_j \times U_j} M(\{y, x\}) + \sum_{(y, x) \in E(G_j)} \text{sat}(e'_j, u_i) \cdot M(e'_j, u_i)$$

$$+ \text{sat}(e''_j, u_{i'}) \cdot M(e''_j, u_{i'})$$

$$= (2m^2 + 9) \left( \sum_{x \in U_j \cup \{u_i\}} M(\{h_j, x\}) + \sum_{x \in U_j \cup \{u_i\}} M(\{e'_j, x\}) \right)$$

$$+ \sum_{x \in U_j \cup \{u_{i'}\}} M(\{e''_j, x\})$$

$$\leq 3(2m^2 + 9).$$
in \( W_j \) is at most one, and \( \text{sat}(e_{ij}', u_i) \) and \( \text{sat}(e_{ij}', u_{v'}) \) are bounded by 7 which are strictly smaller than than \( 2m^2 + 9 \). This shows Inequality (12). Indeed, the equation hold only when the matching values assigned to the pairs \( \{e_{ij}', u_i\} \) and \( \{e_{ij}', u_{v'}\} \) are both zero.

To complete the proof, we show that if \( \mathcal{U}_M(u_i) < \text{sat}(u_i, e_{ij}') \) and \( \mathcal{U}_M(u_{v'}) < \text{sat}(u_{v'}, e_{ij}') \), then the total welfare received by \( M \) for the agents in \( G_j \) has \( \omega \leq 3(2m^2 + 9) - n \). Since \( \mathcal{U}_M(u_i) < \text{sat}(u_i, e_{ij}') \) and \( M \) is a CSM, we have
\[
\mathcal{U}_M(e_{ij}') \geq \text{sat}(e_{ij}', u_i) = 5. \tag{14}
\]
Similarly, since \( \mathcal{U}_M(u_{v'}) < \text{sat}(u_{v'}, e_{ij}') \) and \( M \) is a CSM, we have
\[
\mathcal{U}_M(e_{ij}') \geq \text{sat}(e_{ij}', u_{v'}) = 7. \tag{15}
\]
Consider the two edges incident with \( f_j \) in \( G_j \). In order for Equation (14) to hold, we must have \( 7(1 - M(e_{ij}', f_j)) + 3M(e_{ij}', f_j) \geq 5 \), i.e., \( M(e_{ij}', f_j) \leq 0.5 \). In order for Equation (15) to hold, we must have \( 9(1 - M(e_{ij}', f_j)) + 4M(e_{ij}', f_j) \geq 7 \), i.e., \( M(e_{ij}', f_j) \leq 0.4 \). Combined, this implies the desired upper bound for the welfare \( \omega_j \), as follows:
\[
\omega_j = \left( \sum_{(y,x) \in W_j \times U_j; \{y,x\} \in E(G_j)} M(\{y, x\}) \cdot (\text{sat}(x, y) + \text{sat}(y, x)) \right) + \text{sat}(e_{ij}', u_i) \cdot M(e_{ij}', u_i) + \text{sat}(e_{ij}', u_{v'}) \cdot M(e_{ij}', u_{v'}) \\
\leq (2m^2 + 9) \cdot \sum_{(y,x) \in W_j \times U_j; \{y,x\} \in E(G_j)} M(\{y, x\}) + 12 \\
= (2m^2 + 9) \cdot \left( \sum_{(x,y) \in \{e_{ij}, g_j\} \times W_j} M(\{x, y\}) \right) \\
+ (2m^2 + 9) \cdot (M(\{f_j, e_{ij}'\}) + M(\{f_j, e_{ij}'\})) + 12 \\
\leq 2.9(2m^2 + 9) + 12 \\
< 3(2m^2 + 9) - n,
\]
where the first inequality holds since \( \text{sat}(e_{ij}', u_i) \leq 5 \) and \( \text{sat}(e_{ij}', u_{v'}) \leq 7 \), the second last inequality holds since the sum of values of the matching for each agent is at most one, and the last inequality holds since we assume without loss of generality that \( 0.2 \cdot m^2 - 11.1 > n \).

The following lemma summarizes the desired property regarding the fully matched agents.

**Lemma 5.4.** Let \( M \) be a fractional matching of \( G_E \) (see Construction 1). For each edge \( e_{ij} \in E(G) \) with \( e_{ij} = \{v_i, v_{v'}\} \) it holds that if agent \( f_j \) is fully matched, then we have \( \mathcal{U}_M(e_{ij}') < \text{sat}(e_{ij}', u_i) \) or \( \mathcal{U}_M(e_{ij}') < \text{sat}(e_{ij}', u_{v'}) \).

**Proof.** Let fractional matching \( M \), edge \( e_{ij} = \{v_i, v_{v'}\} \in E(G) \), and agent \( f_j \) be defined. Since \( f_j \) is fully matched, \( M(f_j, e_{ij}) + M(f_j, e_{ij}') = 1. \) We distinguish between two cases (see Figure 4 for the cardinal preferences):

- If \( M(f_j, e_{ij}') > 2/5 \), then \( \mathcal{U}_M(e_{ij}') \leq \text{sat}(e_{ij}', e_{ij}) \cdot (1 - M(f_j, e_{ij}')) + \text{sat}(e_{ij}', f_j) \cdot M(e_{ij}', f_j) = 9 - 5 \cdot M(f_j, e_{ij}') < 7 \), implying that \( \mathcal{U}_M(e_{ij}') < \text{sat}(e_{ij}', u_{v'}) \).
- If \( M(f_j, e_{ij}') \leq 2/5 \), meaning that \( M(f_j, e_{ij}') \geq 3/5 > 1/2 \), then \( \mathcal{U}_M(e_{ij}') \leq \text{sat}(e_{ij}', e_{ij}) \cdot (1 - M(f_j, e_{ij}')) + \text{sat}(e_{ij}', f_j) \cdot M(e_{ij}', f_j) = 7 - 1/2 \cdot M(f_j, e_{ij}') < 5 \), implying that \( \mathcal{U}_M(e_{ij}') < \text{sat}(e_{ij}', u_i) \).

Finally, to ease the hardness proofs for both optimality criteria, we show a technical result regarding some specific fractional matchings, which is straightforward to verify.

**Lemma 5.5.** Let \( M \) be a fractional matching for edge gadget \( G_E \). Consider an edge \( e_{ij} = \{v_i, v_{v'}\} \) of \( G \) with \( i < i' \).

(1) If matching \( M \) fulfills the following:
- \( M(e_j, h_j) = 0.3 \), \( M(e_{ij}, e_{ij}') = 0.1 \), \( M(e_{ij}, e_{ij}') = 0.6 \),
- \( M(f_j, e_{ij}') = 0.8 \), \( M(f_j, e_{ij}') = 0.2 \), and
- \( M(g_j, e_{ij}') = 0.1 \), \( M(g_j, h_j) = 0.7 \), then no cardinality blocking pair of \( M \) involves an agent from \( \{e_{ij}, f_j, g_j, e_{ij}', h_j\} \).

(2) If matching \( M \) satisfies \( M(e_{ij}, e_{ij}') = M(f_j, e_{ij}') = M(g_j, h_j) = 1 \) (see the green edges in Figure 4), then no cardinality blocking pair of \( M \) involves an agent from \( \{e_{ij}, f_j, g_j, e_{ij}', h_j\} \).

**Proof.** Let \( M \) and \( e_{ij} = \{v_i, v_{v'}\} \) be as defined. For Statement (1), assume that \( M \) have the values stated in the if-condition. We first compute the utilities of the agents from \( \{e_{ij}, f_j, g_j, e_{ij}', h_j\} \):
\[
\mathcal{U}_M(e_{ij}) = 2m^2 + 2.6, \quad \mathcal{U}_M(f_j) = 2m^2 + 5.8, \\
\mathcal{U}_M(g_j) = 2m^2 + 5, \quad \mathcal{U}_M(e_{ij}') = 7.2, \quad \mathcal{U}_M(h_j) = 3.1.
\]
It is straightforward to verify that no cardinality blocking pair of \( M \) involves agents \( e_{ij} \) or \( h_j \) as they both are integrally matched with their most preferred agents, respectively. Hence, no cardinality blocking pair of \( M \) involves agent \( h_j \). This completes the proof for Statement (1).

For Statement (2), assume that \( M \) have the values stated in the if-condition. We compute the utilities of the agents from \( \{e_{ij}, f_j, g_j, e_{ij}', h_j\} \):
\[
\mathcal{U}_M(e_{ij}) = 2m^2 + 2, \quad \mathcal{U}_M(f_j) = 2m^2 + 5, \\
\mathcal{U}_M(g_j) = 2m^2 + 5, \quad \mathcal{U}_M(e_{ij}') = 7, \quad \mathcal{U}_M(h_j) = 4.
\]
It is also straightforward to verify that no cardinality blocking pair of \( M \) involves agent \( e_{ij}' \) or \( h_j \) as they both are integrally matched with their most preferred agents, respectively. Hence, no cardinality blocking pair of \( M \) involves agent \( h_j \). This completes the proof for Statement (2).

We are now ready to prove hardness for finding stable matchings with maximum social welfare.

**Theorem 5.6.** MAX-Welfare CSM is NP-complete, even for bipartite graphs with strict preferences.

**Proof.** Let \( G = (V, E) \) be a graph with \( V = \{v_1, v_2, \ldots, v_n\} \) and \( E = \{e_1, e_2, \ldots, e_m\} \), and let \( (G, k) \)
denote an instance of INDEPENDENT SET. We will indeed reduce from IS in cubic graphs [3], i.e., each vertex in $G$ has degree three. We construct an instance $(G', \text{sat}, \gamma)$ of MAX-WELFARE CSM where $\gamma = (3n + 7)n + k + 3m(2m^2 + 9)$ and $G' = (U \cup W, E')$ is a bipartite graph with partite sets $U$ and $W$ such that $U = U_E \cup \{u_i, x_i \mid v_i \in V\}$ and $W = W_E \cup \{w_i, y_i \mid v_i \in V\}$. Note that $U_E = \{e_j, z_j \mid e_j \in E\}$ and $W_E = \{e_j, e_j', h_j \mid e_j = \{u_i, v_i\} \in E\}$. In other words, $U$ (resp. $W$) includes the agents from $U_E$ (resp. $W_E$) of the edge gadgets (see Construction 1) and four vertices $u_i, x_i$ (resp. $v_i, y_i$) for each vertex $v_i \in V$. In total, we have $|U| = |W| = 6m + 4n$.

Similarly to the proof of Theorem 5.2, we will construct the cardinal preferences for the vertex agents to ensure the following. When combined with Lemma 5.3, there are essentially two possible ways of fractionally matching the vertex agents $u_i, x_i, w_i, y_i$ corresponding to the same vertex $v_i \in V$. The first one will have a higher welfare than the second one, but it is not possible to use the first one for two adjacent agents as this will induce a cardinally blocking pair.

Figure 5 illustrates the cardinal preferences of the vertex agents for each vertex $v_i \in V$. The cardinal preferences that are not depicted in the figure are set to zero. We call the subgraph induced by the vertex agents $u_i, x_i, u_i, y_i$ a **vertex gadget for** $v_i$, and use $H_i$ to denote it.

Note that $G'$ is a bipartite graph since all the introduced edges are between $U$ and $W$. This completes the construction of $(G', \text{sat}, \gamma)$ which clearly takes polynomial time.

Next, we prove the correctness, that is, $G$ has an independent set of size at least $k$ if and only if $(G', \text{sat})$ has a cardinally stable matching with welfare at least $\gamma$.

For the “if” direction, given a $k$-vertex independent set $V'$ of $G$, construct a fractional matching $M$ as follows:

- For each $z \in [n]$, if $v_z \in V'$, then set $M(u_z, y_z) = M(x_z, u_z) = 1$; else set $M(u_z, w_z) = M(x_z, y_z) = 1$.
- For each edge $e_j = \{v_i, v_{i'}\} \in E$ with $i < i'$, do:
  1. If $v_i \notin V'$, then set $$M(e_j, h_j) = 0.3, M(e_j, e_j') = 0.1, M(e_j, e_j') = 0.6, M(f_j, e_j') = 0.8, M(f_j, f_j') = 0.2$$ $$M(g_j, e_j') = 0.1, M(g_j, h_j) = 0.7, M(g_j, e_j') = 0.2,$$

Note that, $M$ satisfies the if-condition for $e_j$ stated in Lemma 5.5(1).

2. If $v_i \in V'$, then set $$M(e_j, e_j') = M(f_j, f_j') = M(g_j, h_j) = 1;$$

note that this corresponds to the if-condition stated in Lemma 5.5(2).

- For every pair $e$ of agents in $G'$ not assigned above, set $M(e) = 0$.

First, observe that $M$ is a fractional matching, that is, for each vertex, the values assigned by $M$ to edges incident with that vertex sum to at most one. Indeed, this matching is a perfect matching, i.e., every agent is fully matched. A simple calculation shows that

$$\text{welfare}(M) = \sum_{i=1}^{n} \sum_{\beta \in \{u_i, w_i, x_i, y_i\}} U_M(\beta) = (3n + 7)n + k$$

and that the total welfare received from $M$ by the agents in the edge gadgets is $3m(2m^2 + 9)$, giving the overall required welfare of $\gamma$.

It remains to prove that $M$ is cardinally stable.

**The edge gadget.** We begin by showing that there is no agent of any edge gadget is involved in a cardinally blocking pair. Consider the edge gadget of an edge $e_j = \{v_i, v_{i'}\} \in E(G)$ with $i < i'$. We distinguish between two cases.

1. If $v_i \notin V'$, then $M$ fulfills the if-condition given in Lemma 5.5(1). Hence, no cardinal blocking pair involves an agent from $\{e_j, f_j, g_j, e_j', h_j\}$. To show that neither does a cardinal blocking pair involve agent $e_j$, it remains to consider pair $\{e_j, u_i\}$. Since $v_i \notin V'$, by definition, we have that $M(u_i, u_i) = 1$, meaning that $u_i$ is integrally matched with her most preferred agent. Therefore, $\{e_j, u_i\}$ is *not* cardinal blocking $M$.

2. If $v_i \in V'$, then $M$ fulfills the if-condition given in Lemma 5.5(2). Hence, no cardinal blocking pair involves an agent from $\{e_j, f_j, g_j, e_j', h_j\}$. To show that neither does a cardinal blocking pair involve agent $e_j$, it remains to consider pair $\{e_j, u_i\}$. Since $v_i \in V'$ and $V'$ is an independent set, we have that $v_i \notin V'$ by definition, it must hold that $M(u_i, u_i) = 1$, meaning that $u_i$ is integrally matched with her most preferred agent. Therefore, $\{e_j, u_i\}$ is *not* cardinal blocking $M$.

In both cases, we have shown that no cardinal blocking pair involves an agent from the edge gadgets.

**The vertex gadget.** We now show that cardinal blocking pair of $M$ involves an agent from the vertex gadgets. This will show that $M$ is cardinally stable. Let $i \in [n]$ and consider the vertex gadget for vertex $v_i \in V$. Again, we distinguish between two cases:

1. If $v_i \in V'$, then no cardinal blocking pair of $M$ involves agents $u_i$ or $y_i$ since they are integrally matched with their most preferred agents, respectively. Hence, no cardinal blocking pair involves $x_i$. Since no cardinal blocking pair involves an agent from the edge gadget, neither is $y_i$ involved in a cardinal blocking pair.
(2) If \( v_i \in V' \), then no cardinally blocking pair of \( M \) involves agents \( u_i \) or \( x_i \); since they are integrally matched with their most preferred agents, respectively. Hence, no cardinally blocking pair involves \( w_i \) or \( x_i \). Hence, no cardinally blocking pair of \( M \) involves an agent from the vertex gadget. This concludes the proof of the “if” direction of the correctness.

For the “only if” direction, let \( M \) be a cardinally stable matching for \((G', \text{sat})\) with \( \text{welfare}(M) \geq \gamma \). We define a vertex subset \( V' = \{ u_i \in V \mid \text{sat}(u_i, y_i) \geq 1/n \} \). We show that \( V' \) is an independent set in \( G \) and \( |V'| \geq k \).

We first show that \( |V'| \geq k \). For brevity we introduce the following notation. For each \( i \in [n] \), define

\[
m_i^{uw} := M(u_i, w_i), \quad m_i^{we} := \sum_{e \in E(v_i)} M(u_i, e), \quad m_i^{wy} := M(u_i, y_i),
\]

where \( E(v_i) = \{ e^j : v_i \in e \} \) for some \( e^j \in E(G) \).

Then we have,

\[
\sum_{i=1}^{n} \sum_{j \in \{ u, w, x, y \}} U_M(z) = \sum_{i=1}^{n} (3n+4)m_i^{uw} + (3n+2)m_i^{we} + (3n+5)m_i^{wy} + 3m_i^{xy} \leq \sum_{i=1}^{n} \left( (3n+4) (m_i^{uw} + m_i^{we} + m_i^{wy}) + 3(m_i^{wy} + m_i^{xy}) \right) + \sum_{i=1}^{n} m_i^{wy}.
\]

Since \( M \) is a fractional matching, meaning that the sum of the matching values for \( u_i \) (resp. \( w_i \)) is at most one, the right-hand side of Inequality (16) is upper-bounded by

\[
(3n+4)n + 3n + \sum_{i=1}^{n} m_i^{uw} = (3n+7)n + \sum_{i=1}^{n} m_i^{uw}. \tag{17}
\]

To show that \( |V'| \geq k \), we first show that \( |V'| \geq \left\lceil \sum_{i=1}^{n} m_i^{uw} \right\rceil \). To this end, define

\[
k := \left\lceil \sum_{i=1}^{n} m_i^{uw} / k \right\rceil.
\]

Then we have \( \sum_{i=1}^{n} m_i^{uw} \leq k_1/n + |V'| \). Since \( k_1/n \leq 1 \) thus \( |V'| \geq \left\lceil \sum_{i=1}^{n} m_i^{uw} \right\rceil \). To see that \( |V'| \geq k \) it thus suffices to show \( \sum_{i=1}^{n} m_i^{uw} \geq k \). For this, out of the welfare \( \text{welfare}(M) \) at most \( 3n(2m^2 + 9) \) stems from the agents of the edge gadgets (see Lemma 5.3). Hence, at least \( (3n+4)n + 3n + k \) must stem from the agents of the vertex gadgets. By the upper bound on the welfare of these agents derived in eq. (17), thus instead \( \sum_{i=1}^{n} m_i^{uw} \geq k \), as required.

To conclude the proof, we show that \( V' \) is an independent set in \( G \). Towards a contradiction, suppose that there is an edge \( e_j \in E \) with \( e_j = \{ v_i, v_i' \} \) and \( i' > i \) such that \( e_j \subseteq V' \). By the definition of \( V' \), for each \( v \in \{ i, i' \} \) the utility of \( u_v \) has \( U_M(u_v) \leq 1/n + (n-1) \cdot (3n+3)/n < 3n \).

That is, \( U_M(u_v) < \text{sat}(u_v, e'_v) \). By Lemma 5.3, the total welfare received from \( M \) by the agents of the edge gadgets is at most \( 3m(2m^2 + 9) - n \). By Inequality (17), the total welfare received from the agents in the vertex gadgets is at most \( (3n+4)n + 3n + n \), meaning that welfare(M) < welfare, a contradiction. Thus, \( V' \) is indeed an independent set in \( G \).

\[
\square
\]

Remark. Note that even if we require the preference lists to be complete, Theorem 5.6 still holds: We can add a sufficiently large value to sat(u, v) and sat(v, u) for each acceptable pair \{ u, v \} in the constructed instance, and assign small but distinct positive values to each un-mentioned pair in the instance. Summarizing, Theorem 5.6 does not rely on edges with zero values and holds even if we have complete preferences without ties.

Next, we prove that MAX-FULL CSM is NP-complete even when the graph \( G \) is bipartite and the values of sat(v, ·) are distinct, for each vertex \( v \) in \( G \).

Theorem 5.7 (+). MAX-FULL CSM is NP-complete, even for bipartite graphs with strict preferences.

Proof. Let \( G = (V, E) \) be a graph with \( V = \{ v_1, v_2, \ldots, v_n \} \) and \( E = \{ e_1, e_2, \ldots, e_m \} \) and let \( (G, k) \) be an instance of IS. We construct an instance \((G', \text{sat})\) of MAX-FULL CSM where \( G' = (U \cup W, E') \) is a bipartite graph with partite sets \( U \) and \( W \). The basic idea is similar to Theorem 5.6: We have a vertex gadget for each vertex \( v \) of \( G \) that has two possibilities of being matched, signifying whether \( v \) is supposed to be in the independent set. If \( v \) is in the independent set, then an agent in \( v \)'s vertex gadget will be unsatisfied with respect to all agents in the edge gadgets that correspond to the edges incident to \( v \). In order to ensure that at least \( k \) vertices are selected in the independent set, we use a selector gadget. In this gadget there are \( k \) “selector” agents which, in order to be fully matched, need to be matched to agents in the vertex gadget. If an agent \( u_i \) in a vertex gadget is matched with large-enough value to a selector agent, this makes him unsatisfied with respect to the agents in his edge gadgets, thus signifying that the vertex \( v_i \in V \) corresponding to \( u_i \) is selected into the independent set. The properties of the edge gadget then ensure that the selected vertices indeed form an independent set.

We construct \( G' = (U \cup W, E') \) and sat as follows. For each \( i \in [n] \) let \( d_i \) denote the degree of the vertex \( v_i \) in \( G \).

- For each \( i \in [n] \) we introduce a vertex gadget for vertex \( v_i \in V \) which contains an agent \( u_i \) in \( U \) and an agent \( w_i \) in \( W \).
- We introduce a selector gadget which contains the following agents:
  - We add \( 2k \) agents, called \( t_1, t_2, \ldots, t_k \) and \( c_1, c_2, \ldots, c_k \), to \( U \).
  - We add \( 2k \) agents, called \( s_1, s_2, \ldots, s_k \) and \( a_1, a_2, \ldots, a_k \), to \( W \).
- Let \( G_E \) be the graph with vertex set \( \{ U_E \cup W_E \} \) that contains the edge gadgets as defined in Construction 1. That is, \( U_E = \{ e_j, f_j, g_j : e_j \in E \} \) and \( W_E = \{ e'_j, f'_j, h_j : e'_j \in E \} \). We add each vertex in \( U_E \) to \( U \) (if not already
Figure 6: The vertex and selector gadget used in the proof of Theorem 5.7. Top: The cardinal preferences of the two agents $u_i$ and $w_i$ which correspond to vertex $v_i$ and four agents $s_j, t_j, a_j, c_j, j \in [k]$. Here, $E(v_i) = \{e_j \mid e_j \in E$ and $v_i$ is incident to $e_j\}$ and $|E(v_i)|$ denotes the sequence resulting from ordering $E(v_i)$ in increasing order of the indices of the edge agents in $E(v_i)$. The red edges indicated in the matching signifies that $v_i$ is in the independent set. Bottom: The induced preference lists.

In total, we have introduced $6m + 2n + 4k$ agents. To completes the construction, we define $\tau = |U| + |W| = 6m + 2n + 4k$ and obtain the instance $(G', \text{sat}, \tau)$ of MAX-FULL CSM. Clearly, it can be carried out in polynomial time. Moreover, $G'$ is bipartite since all the introduced edges are between $U$ and $W$. Since $\tau = |U| + |W|$, searching for an CSM with #fully fully matched agents means searching for a perfect CSM. Next, we prove the correctness. We will show that $G$ has an independent set of size at least $k$ if and only if $(G', \text{sat})$ has a perfect CSM.

For the forward direction, given an independent set of $G$ of size at least $k$, we construct a fractional matching $M$ as follows. Let $V'$ be a subset of the independent set of size exactly $k$. Denote the vertices of $V'$ as $v_{\ell_1}, \ldots, v_{\ell_k}$, where $\ell_1 < \ell_2 < \cdots < \ell_k$ (this ordering will be crucial to make sure that the agents $s_j$ will not be involved in a cardinally blocking pair).

- For each $j \in [k]$, we set $M(u_{\ell_j}, s_j) = 1$ and set
  $M(w_{\ell_j}, c_j) = M(c_j, a_j) = M(a_j, t_j) = M(t_j, w_{\ell_j}) = 1/2$.

This matching is indicated by the red lines in Figure 5.

- For each $v_i \in V \setminus V'$, we set $M(u_i, w_i) = 1$.
- For each pair of agents in the edge gadget the values set by $M$ are the same as in Theorem 5.6.
- For each edge $e_j = \{u_i, v_i\} \in E$ with $i < i'$, do:

  1. If $v_i \notin V'$, then set $M(e_j, h_j) = 0.3, M(e_j, e_j') = 0.1, M(e_j, e_j') = 0.6, M(f_j, e_j') = 0.8, M(f_j, e_j') = 0.2
     $M(g_j, e_j') = 0.1, M(g_j, h_j) = 0.7, M(g_j, e_j') = 0.2$.

     Note that, $M$ satisfies the if-condition for $e_j$ stated in Lemma 5.5(1).

  2. If $v_i \in V'$, then set $M(e_j, e_j') = M(f_j, e_j') = M(g_j, h_j) = 1$;

     note that this corresponds to the if-condition stated in Lemma 5.5(2).

- For every pair $e$ of agents in $G'$ not assigned above, set $M(e) = 0$.

First, observe that $M$ is a matching. Indeed, $M$ is perfect: The agents in the edge gadgets are fully matched by a direct calculation. The agents in the vertex and selector gadgets are also fully matched: For each $i \in [n]$, agent $u_i$ is matched integrally to either some $s_j$ or some $w_j$. As to the agents $w_i, i \in [n]$, if $v_i \in V \setminus V'$, then agent $w_i$ is also matched integrally. Otherwise, if $v_i \in V'$, then $w_i$ is matched half-integrally with both $c_j$ and $t_j$, for some $j \in [k]$. Similarly, for each $j \in [k]$, the vertices $c_j, a_j, t_j$ are matched half-integrally with two vertices. Hence, every vertex is fully matched. Therefore, $M$ is a perfect matching.

It remains to prove that $M$ is cardinally stable. We consider the agents from the edge gadgets and the vertex and selector gadgets separately.

**The edge gadget.** Consider an arbitrary edge $e_j \in E$ with $e_j = \{v_i, v_i'\}$ for $i < i'$ and consider the edge gadget for $e_j$. Assume that $v_i \notin V'$; the case $v_i \in V'$ is analogous. By the definition of $M$ in the edge gadgets in combination with Lemma 5.5(1), we have that every cardinally blocking pair that involves an agent from the edge gadget for $e_j$ must involve both $e_j$ and $u_i$. However, since $v_i \notin V'$ and by the definition of $M$, we have $U_{M}(u_i) = \text{sat}(u_i, w_i) > \text{sat}(u_i, e_j')$. Thus, there is no cardinally blocking pair involving the agents of the edge gadgets.

**The vertex and selector gadget.** It remains to show that no cardinally blocking pair involves an agent from a vertex gadget or a selector gadget in $U$. Note that each pair of agents from the vertex or selector gadget involves an agent from the following set $V' = \{u_i \mid i \in [n]\} \cup \{c_j, t_j \mid j \in [k]\}$. It thus suffices to show that the agents in this set $V'$ are not involved in any cardinally blocking pairs.

Consider an arbitrary agent $u_i$ with $i \in [n]$. Since there is no cardinally blocking pair involving $u_i$ and an agent of the edge gadgets, each cardinally blocking pair involving $u_i$ must involve either $w_i$ or $s_j$ for some $j \in [k]$. Note that if $M(u_i, w_i) = 1$, then there is no cardinally blocking pair involving $u_i$, because $w_i$ is $u_i$’s most preferred agent. Otherwise, we have $i = \ell_p$ for some $p \in [k]$ and $M(u_i, s_p) = 1$. Then $\{u_i, w_i\}$ is not cardinally blocking: This is because $M(u_i, w_i) = 1/2$ and hence $U_{M}(u_i) = \text{sat}(u_i, w_i) = k = \text{sat}(w_i, u_i)$. Moreover, none of the agents $s_j$ with $j < p$ can form a cardinally blocking pair with $u_i$ since $M(u_i, s_p) = 1$ and hence $U_{M}(u_i) = \text{sat}(u_i, s_p) = p > k$.
Note that the last inequality holds since $T$.

Towards a contradiction assume that for both $(U, V)$ and $(U, V')$ the cardinality of $V$ is $\ell_p$. Hence, no blocking pair involves $u_i$.

Next, let an arbitrary agent $c_j$ with $j \in [k]$. Since for each $i' \in [n]$, we have $U_M(w_{ij}) \geq k > sat(w_{ij}, c_j)$, pair $(c_j, w_{ij})$ is not cardinal blocking. Since $U_M(c_j) \geq 1/2 > sat(c_j, a_j)$, neither is pair $(c_j, u_i)$ cardinal blocking. Hence, no blocking pair involves $c_j$.

Finally, consider an arbitrary agent $t_j$ with $j \in [k]$. Since $sat(a_j, t_j) = 0$, pair $(t_j, a_j)$ is never a cardinal blocking pair. Moreover, since for each $i' \in [n]$, sat$(t_j, w_{ij}) \leq n - 1$, $(t_j, w_{ij})$ is not cardinal blocking. Hence, no blocking pair involves $t_j$. Therefore, $M$ is cardinal stable. This proves the forward direction.

For the other direction, let $M$ be a cardinal blocking matching in the instance $(G^*, sat)$ where every vertex is fully matched in $M$. We define the vertex set

$$V' = \{v_i \in V \mid \sum_{j \in [k]} M(u_i, s_j) \geq 1/n\}.$$

We show that $V'$ is an independent set in $G$ and $|V'| \geq k$.

We first show that $|V'| \geq k$. Call $i \in [n]$ good if $\sum_{j \in [k]} M(s_j, u_i) \geq 1/n$ and bad otherwise. Observe that $|V'|$ is equal to the number of good indices in $[n]$. Since for each $j \in [k]$ agent $s_j$ is fully matched by $M$, we have $k = \sum_{i \in [n]} \sum_{j \in [k]} M(u_i, s_j)$. Thus,

$$k = \sum_{i \in [n]} \sum_{j \in [k]} M(s_j, u_i) + \sum_{i \in [n]} \sum_{j \in [k]} M(s_j, u_i)$$

$$\leq n - |V'| + |V'|.$$ 

Thus, since $(n - |V'|)/n \leq 1$ and $k$ is an integer, we have $|V'| \geq k$, as required.

It remains to show that $V'$ is an independent set. Consider an arbitrary edge $e_j \in E$ with $e_j = \{v_i, w_{ij}\}$ where $i < i'$. Towards a contradiction assume that both $v \in \{i, i'\}$ we have $w_{ij} \in V'$. Then, since $\sum_{j \in [k]} M(u_i, s_j) \geq 1/n$ we have

$$U_M(u_{i}) \leq (2n + d_{i}) \cdot \frac{n - 1}{n} + \frac{k}{n}$$

$$= \frac{1}{n} (2n^2 + d_{i} \cdot n - k - 2n^2 - d_{i}) < 2n^2.$$

Note that the last inequality holds since $k \leq n$ and $0 < d_{i} < n$. Thus, $U_M(u_{i}) < sat(u_{i}, e_{ij})$. However, by Lemma 5.4 we have $U_M(e_{ij}) < sat(e_{ij}, u_i) \leq M(e_{ij}, u_i)$. Thus, $(e_{ij}, u_i)$ or $(e_{ij}, w_{ij})$ forms a blocking pair, a contradiction. Thus indeed, $V'$ is an independent set. As shown above, $|V'| \geq k$, as required.

Since IS is W[1]-hard wrt. the solution size [10], the reduction for Theorem 5.7 implies that MAX-FULL CSM is W[1]-hard wrt. parameter "#fully($M^\pi$) - #fully($M^\pi$)" where $M$ is a perfect CSM and $M^\pi$ is as defined in Definition 3.2.

**Corollary 5.8.** MAX-FULL CSM for bipartite graphs $G'$ = $(U \cup W, E')$ with strict preferences sat is W[1]-hard wrt. to the parameter "$\tau - #fully(M^\pi)$", where $M^\pi$ denotes a CSM of $(G', sat)$ returned by Algorithm 1.

**Proof.** We use the same reduction as the one given in the proof of Theorem 5.7. Let $I = (G, k)$ denote an instance of IS with $G$ being a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$, and let $I' = (G^*, sat, \tau)$ be the constructed instance in the proof with $G^* = (U \cup W, E')$ and $U = \{e_j, g_j, j \in E\} \cup \{u_i | v_i \in V\} \cup \{v_i | j \in [k]\}$. \$W = \{e_j, e'_j, h_j | e_j = \{v_i, v_i\} \text{ for some } e_j \in E \cup \{u_i | v_i \in V\} \cup \{v_i | j \in [k]\} \cup \{s_j, a_j | j \in [k]\}\$. \$\tau = |U| + |W| = 6m + 2n + 4k$.

Since IS parameterized by the independent set size $k$ is W[1]-hard [9] and since we have shown in the proof of Theorem 5.7 that the reduction from IS runs in polynomial time and is correct, to show W[1]-hardness for MAX-FULL CSM, it suffices to show that $\tau - #fully(M^\pi) \leq 2k$. Since $M^\pi$ is a matching computed by Algorithm 1 on input $(G', sat)$.

For this, let $N$ be an arbitrary stable integral matching for $G'$, sat. Observe that $N$ exists, because $G'$ is bipartite. Moreover, since sat is strict, Tan [26] proved in his Proposition 2.1 that the stable matching $N$ induces a stable partition in which the transpositions (i.e., the edges) one-to-one correspond to the assignments in $N$. Moreover, by Proposition 3.4 the singletons in any two stable partitions are the same, and by Lemma 4.4(2) an agent is fully matched in $M^\pi$ if and only if she is a non-singleton. Thus, $#fully(M^\pi) = #fully(N)$. Again, since $(G, sat)$ has strict preferences, every stable integral matchings matches the same set of agents. It thus suffices to prove that there exists a stable integral matching $M$ for $G'$ such that $\tau - #fully(M^\pi) \leq 2k$. Since $\tau = |U| + |W|$ we hence only need to show that an arbitrary stable integral matching $M$ of $(G', sat)$ (fully) matches all at most $2k$ agents. One way to show this is to check which agents must be matched under $M$. We show this by carrying out the propose-and-reject algorithm by Gale and Shapley [12].

We let the agents in $U$ propose to the agents in $W$ with the following specific order of propositions:

1. For each $j \in [n]$: (a) Agent $f_j$ proposes to her most-preferred partner $e_j$. (b) Agent $g_j$ proposes to her next most-preferred partner $h_j$.
2. For each $i \in [n]$ agent $u_i$ proposes to her most-preferred partner $w_i$.
3. For each $j \in [k]$ agent $c_j$ proposes to her next most-preferred partner $a_j$ (and gets accepted).
4. For $j = k, k - 1, \ldots, 1$, agent $t_j$ proposes to her next
most-preferred partner $w_{n+k-j}$. Observe that $w_{n+k-j}$ accepts, leaving $u_{n+k-j}$ without partner. Observe that all agents in $W$, except those from $\{s_j \mid j \in \{k\}\}$, receive at least one proposal from some agent from $U$. Since in bipartite graph with strict preferences, an agent from $W$ never becomes unmatched once she got a proposal from some agent from $U$, we know that at most $k$ agents from $W$ will remain unmatched under $M$. Since $M$ is integral and $|U| = |W|$, at most $k$ agents remain unmatched under $M$. Hence, at most $2k$ agents will be unmatched under $M$, as required.

6 Conclusion and outlook
Motivated by the benefits of fractional matchings under preferences, we studied three natural stability concepts (linear stability, ordinal stability, and cardinal stability) and two optimization criteria, from structural and algorithmic perspectives. We obtained a comprehensive picture of the algorithmic complexity of computing a stable fractional matching which maximizes either the number of fully matched agents or the social welfare, taking into account whether the preferences may contain ties and whether the underlying market is a marriage market or a roommates market.

We conclude with some challenges for future work. First, it would be interesting to know whether the set of cardinally stable matchings has some form of lattice structure. Second, studying optimal stable and fractional matchings using the framework of parameterized algorithmics [23, 9] may provide more insights into the fine-grained complexity of the problem. Promising parameters are the number of fully matched agents and the social welfare of the fractional matchings in the solution. Finally, regarding preference restrictions [6], it would be interesting to know whether assuming a special preference structure can help in finding tractable cases for optimal fractional stable matchings.
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