SEMIGLOBAL RESULTS FOR $\overline{\partial}$ ON A COMPLEX SPACE WITH ARBITRARY SINGULARITIES

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Abstract. We obtain some $L^2$-results for the $\overline{\partial}$ operator on forms that vanish to high order near the singular set of a complex space.

1. Introduction

Let $X$ be a pure $n$-dimensional reduced Stein space, $A$ a lower dimensional complex analytic subset with empty interior containing $X_{\text{sing}}$. Let $\Omega$ be an open relatively compact Stein domain in $X$ and $K = \overline{\Omega}$ be the holomorphic convex hull of $\Omega$ in $X$. Since $X$ is Stein and $K = \overline{K}$, $K$ has a neighborhood basis of Oka-Weil domains in $X$ (see Theorem 11, in [8], Volume III, page 102). Let $X_0$ be an Oka-Weil neighborhood of $K$ in $X$, $X_0 \subset \subset X$. Then $X_0$ can be realized as a holomorphic subvariety of an open polydisk in some $\mathbb{C}^N$.

Set $\Omega^* = \Omega \setminus A$. Let $d_A$ be the distance to $A$, relative to an embedding of $X_0$ in $\mathbb{C}^N$ and $| |$ and $dV$ denote the induced norm on $\Lambda^{p,q}(\Omega^*)$, resp. the volume element (different embeddings of neighborhoods of $\Omega$ in $\mathbb{C}^N$ give rise to equivalent distance functions and norms). For a measurable $(p,q)$ form $u$ on $\Omega^*$ set

$$\|u\|_{N,\Omega}^2 = \int_{\Omega^*} |u|^2 dA dV.$$

In this paper we address the question of whether we can solve the equation $\overline{\partial}u = f$ in $\Omega^*$ for a $\overline{\partial}$-closed $(p,q)$ form $f$ on $\Omega^*$ that vanishes to "high order" on $A$. Our main result is the following theorem:

**Theorem 1.1.** Let $X, \Omega$ be as above. For every $N_0 \geq 0$, there exists $N \geq 0$ such that if $f$ is a $\overline{\partial}$-closed $(p,q)$-form on $\Omega^*, q > 0$, with $\|f\|_{N,\Omega} < \infty$, there is $v \in L^{2,\text{loc}}_{p,q-1}(\Omega^*)$ solving $\overline{\partial}v = f$, with $\|v\|_{N_0,\Omega'} < \infty$ for every $\Omega' \subset \subset \Omega$. For each $\Omega' \subset \subset \Omega$, there is a solution of this kind satisfying $\|v\|_{N_0,\Omega'} \leq C\|f\|_{N,\Omega}$, where $C$ is a positive constant that depends only on $\Omega', N, N_0$.

When $A \cap \overline{\Omega}$ is a finite subset of $\overline{\Omega}$ with $b\Omega \cap A = \emptyset$, $\Omega$ is Stein and $\overline{\Omega}$ has a Stein neighborhood, we obtain the following corollary of Theorem 1.1:

**Corollary 1.2.** With $N_0, N$ as in Theorem 1.1 and for $f$ a $\overline{\partial}$-closed $(p,q)$-form on $\Omega^*, q > 0$, with $\|f\|_{N,\Omega} < \infty$, there is a solution $u$ of $\overline{\partial}u = f$ on $\Omega^*$ with $\|u\|_{N,N_0} \leq c\|f\|_{N,\Omega}$, $c$ independent of $f$. In other words, we obtain a weighted $L^2$ estimate for $u$ on all of $\Omega$.

Theorem 1.1 extends to the case when $\Omega$ is just holomorphically convex and contains a maximal compact subvariety $B$ that is contained in $A$. It also extends to the case of $(p,q)$ forms on $X^*$ with values in a holomorphic vector bundle $E$ over $X$. Theorem 1.1 and Corollary 1.2 can be used to construct analytic objects with prescribed behaviour on the
maximal, positive dimensional compact subvariety $B$ of a holomorphically convex manifold. We also expect them to be useful in studying the obstructions to solving $\overline{\partial}$ on a deleted neighborhood of an isolated singular point of a complex analytic set of dimension bigger than $2$. The power series arguments that were used in the surface case in [3], [4] might be replaced by the solution of a Cousin problem with $L^2$ bounds which exist by our results when a finite number of obstructions vanish.

We have not managed to find a proof for Theorem 1.1 using transcendental $L^2$ methods. Instead, our arguments are based on resolution of singularities combined with cohomological arguments in the spirit of Grauert [5]. In particular, there exists a proper, holomorphic surjection $\pi : \tilde{X} \to X$ with the following properties:

i) $\tilde{X}$ is an $n$-dimensional complex manifold.

ii) $\tilde{A} = \pi^{-1}(A)$ is a hypersurface in $\tilde{X}$ with only “normal crossing singularities”, i.e. near each $x_0 \in \tilde{A}$ there are local holomorphic coordinates $(z_1, \ldots, z_n)$ in terms of which $\tilde{A}$ is given by $z_1 \cdots z_m = 0$, where $1 \leq m \leq n$

iii) $\pi : \tilde{X} \setminus \tilde{A} \to X \setminus A$ is a biholomorphism.

This follows from the following two facts: a) every reduced, complex space can be desingularized and, b) every reduced, closed complex subspace of a complex manifold admits an embedded desingularization. The exact statements and proofs can be found in [1], [2].

Let $\tilde{\Omega} := \cup_{x \in \tilde{A}} \mathcal{O}_{\tilde{X}}(x)$ be the ideal sheaf of $\tilde{A}$ in $\tilde{X}$ and $\Omega^p$ the sheaf of holomorphic $p$ forms on $\tilde{X}$. We shall consider the following sheaves on $\tilde{X}$ that are defined by:

$$\mathcal{L}_{p,q}(U) = \{ u \in L^2_{p,q}(U); \overline{\partial} u \in L^2_{p,q+1}(U) \},$$

for every $U$ open subset of $\tilde{X}$ and the obvious restriction maps $r^U_V : \mathcal{L}_{p,q}(U) \to \mathcal{L}_{p,q}(V)$, where $V \subset U$ are open subsets of $\tilde{X}$. Then $u \to \overline{\partial} u$ defines an $\mathcal{O}_{\tilde{X}}$-homomorphism $\overline{\partial} : \mathcal{L}_{p,q} \to \mathcal{L}_{p,q+1}$ and the sequence

$$0 \to \Omega^p \to \mathcal{L}_{p,0} \to \mathcal{L}_{p,1} \to \cdots \to \mathcal{L}_{p,n} \to 0$$

is exact by the local Poincaré lemma for $\overline{\partial}$. Since each $\mathcal{L}_{p,q}$ is closed under multiplication by smooth cut-off functions we have a fine resolution of $\Omega^p$. In the same way, since $J$ is locally generated by one function, the sequence

$$0 \to J^k \Omega^p \to J^k \mathcal{L}_{p,0} \to \cdots \to J^k \mathcal{L}_{p,n} \to 0$$

is a fine resolution of $J^k \Omega^p$. Here, $u \in (J^k \mathcal{L}_{p,q})_x$ if it can locally be written as $h^k u_0$ where $h$ generates $J_x$ and $u_0 \in (\mathcal{L}_{p,q})_x$. It follows that

$$H^q(\tilde{\Omega}, (J^k \Omega^p)_x) \cong \frac{\ker(\overline{\partial} : J^k \mathcal{L}_{p,q}(\tilde{\Omega}) \to J^k \mathcal{L}_{p,q+1}(\tilde{\Omega}))}{\overline{\partial}(J^k \mathcal{L}_{p,q-1}(\tilde{\Omega}))}.$$  

Here is an outline of the proof of Theorem 1.1: The pullback $\pi^* f$ satisfies

$$\int_{\tilde{\Omega}} |\pi^* f|^2 dA d\bar{\sigma} \leq C \int_{\Omega^*} |f|^2 dA dV,$$
for a suitable $0 < N_1 < N$ and $\overline{\partial} \pi^* f = 0$ on $\tilde{\Omega}$. Suppose for the moment that we could prove the following proposition:

**Proposition 1.3.** For $q > 0$ and $k \geq 0$ given, there exists a natural number $\ell$, $\ell \geq k$ such that the map 

$$i_* : H^q(\tilde{\Omega}, J^t \cdot \Omega^p) \to H^q(\tilde{\Omega}, J^k \cdot \Omega^p),$$

induced by the inclusion $i : J^t \cdot \Omega^p \to J^k \cdot \Omega^p$, is the zero map.

Using (2) we can show that $\pi^* f \in J^l L^p,q(\tilde{\Omega})$ if $l \leq \frac{N_1}{2n}$. Assuming Proposition 1.3 this means that $\overline{\partial}_v = \pi^* f$ has a solution in $J^k L^p,q-1(\tilde{\Omega})$. Since $|h(x)| \leq Cd(x)$ on compacts in the set where $h$ generates $J$ it follows that

$$\int_{\tilde{\Omega}} |(\pi^{-1})^*v|^2 d_A^{-N_0} dV \leq c \int_{\tilde{\Omega}} |v|^2 d_A^{-2k} d\tilde{V}_\sigma$$

when $k$ is big enough.

**Remark:** Proposition 1.3 was inspired by Grauert [5] (Satz 1, Section 4). Grauert’s result corresponds to the case where $A$ is a finite set.

The paper is organized as follows: In section 2 we prove Proposition 1.3. Section 3, contains the estimates for the pullback of forms under $\pi$ and $\pi^{-1}$. In Section 4 we prove Theorem 1.1. The proof of Corollary 1.2 is contained in section 5. Last but not least, in section 6 we discuss some generalizations to Theorem 1.1, Corollary 1.2.

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## 2. Proof of Proposition 1.3

Following Grauert [5], we consider more generally the coherent analytic sheaves $S$ on $\tilde{X}$ that are torsion free i.e. sheaves with the property

$$T(S)_x = 0 \ \text{for all} \ x \in \tilde{X}$$

where $T(S)_x = \{g_x \in S_x : f_x \cdot g_x = 0 \ \text{for some} \ f_x \neq 0, f_x \in \mathcal{O}_x \}.$

We shall show (Lemma 2.1) that when $S$ is coherent and torsion free and $i : J^t S \to S$ is the inclusion homomorphism, then the induced map $i_{\tilde{\Omega}}^* : H^q(\tilde{\Omega}, J^t S) \to H^q(\tilde{\Omega}, S)$ is zero when $q > 0$ and $t$ is big enough. In order to exploit the idea that analytic sheaf cohomology on $\tilde{\Omega}$ is concentrated over $\tilde{A}$, the exceptional set of the resolution, we need to introduce the higher direct image sheaves, denoted by $R^q \pi_* S$, of an analytic sheaf $S$ on $\tilde{X}$, $q \geq 0$. 
and recall some basic facts about them. For $q \geq 0$ and $S$ an $\mathcal{O}_X$-module, the higher direct image sheaves of $S$ are the sheaves on $X$, associated to the presheaf
\[ P : U \to H^q(\pi^{-1}(U), S) \]
where $U$ open in $X$. When $\phi : S \to S'$ is an $\mathcal{O}_X$-homomorphism the induced maps $\phi_* : H^q(\pi^{-1}(U), S) \to H^q(\pi^{-1}(U), S')$, $U$ open in $X$, determine a sheaf homomorphism $\phi^# : R^q\pi_*S \to R^q\pi_*S'$ on $X$. For future reference, we recall the $\mathcal{O}_X$-module structure on $R^q\pi_*S$. Given $U$ an open subset of $X$, $f \in \mathcal{O}_X(U)$, we define a map $f_U^\bullet : S_{|\pi^{-1}(U)} \to S_{|\pi^{-1}(U)}$ described by $(f_U^\bullet)s_x = (f \circ \pi)_x \cdot s_x$, $x \in \pi^{-1}(U)$, $s_x \in S_x$ and let and $(f_U^\bullet)_*: H^q(\pi^{-1}(U), S) \to H^q(\pi^{-1}(U), S)$ be the induced map on cohomology. We can then define a map $\mathcal{O}_X(U) \times H^q(\pi^{-1}(U), S) \to H^q(\pi^{-1}(U), S)$ that sends $(f, c) \in \mathcal{O}_X(U) \times H^q(\pi^{-1}(U), S)$ to $(f_U^\bullet)_*c$. It is easy to check that it is a morphism of presheaves $\mathcal{O}_X(-) \times H^q(\pi^{-1}(-), S) \to H^q(\pi^{-1}(-), S)$ which extends naturally to a morphism on the associated sheaves $\mathcal{O}_X \times R^q\pi_*S \to R^q\pi_*S$.

The main theorem in Grauert [6], says that the direct image sheaves $R^q\pi_*S$ are coherent $\mathcal{O}_X$-modules, when $S$ is a coherent $\mathcal{O}_X$-module and $q \geq 0$. Since $\Omega$ is a Stein domain, Satz 5, Section 2 in [6], gives that the natural map $\pi_\Omega : H^q(\tilde{\Omega}, J^tS) \to \Gamma(\Omega, R^q\pi_*S_{|\tilde{\Omega}})$ is an isomorphism. This fact and the following lemma will enable us to finish the proof of Proposition 1.3.

**Lemma 2.1.** For each $q > 0$ and for each coherent, torsion free $\mathcal{O}_X$-module $S$ there exists a $t \in \mathbb{N}$ such that $i_{\tilde{\Omega},*} : H^q(\tilde{\Omega}, J^tS) \to H^q(\tilde{\Omega}, S)$ is the zero map, where $i : J^tS \hookrightarrow S$ is the inclusion map.

**Proof.** We shall prove the lemma using downward induction on $q > 0$. Observe that $\tilde{\Omega}$ is an $n$-dimensional complex manifold with no compact $n$-dimensional connected components since it is obtained by blow-ups from a pure $n$-dimensional Stein space $\Omega$. It follows from the Main Theorem in Siu [12] that $H^q(\tilde{\Omega}, S) = 0$ for every coherent $\mathcal{O}_\tilde{\Omega}$-module $S$. Hence, the statement is true for $q = n$ and any $t \in \mathbb{N}$.

When $q > 0$, $\text{Supp} R^q\pi_*S$ is contained in $A$. The annihilator ideal $A'$ of $R^q\pi_*S$ is coherent and by Cartan’s Theorem A there exist functions $f_1, \ldots, f_M \in \mathcal{A}(X)$ that generate each stalk $A'_x$ in a neighborhood of $\overline{\Omega}$. Let $\mathcal{A}$ be the $\mathcal{O}_\tilde{\Omega}$-ideal generated by $\tilde{f}_j = f_j \circ \pi$, $1 \leq j \leq M$. A crucial observation which will be useful later, is that $(\tilde{f}_j)_{\tilde{\Omega},*} : H^q(\tilde{\Omega}, S_{|\tilde{\Omega}}) \to H^q(\tilde{\tilde{\Omega}}, S_{|\tilde{\tilde{\Omega}}})$ are zero for all $j$, $1 \leq j \leq M$, $q > 0$. To see this, consider the following commutative diagram
\[
\begin{array}{cccc}
H^q(\tilde{\Omega}, S_{|\tilde{\Omega}}) & \xrightarrow{(\tilde{f}_j)_{\tilde{\Omega},*}} & H^q(\tilde{\tilde{\Omega}}, S_{|\tilde{\tilde{\Omega}}}) \\
\downarrow & & \downarrow \\
R^q\pi_*S(\Omega) & \xrightarrow{(f_j)_{\Omega,#}} & R^q\pi_*S(\Omega)
\end{array}
\]
The vertical maps are isomorphisms, due to Satz 5, Section 2, in [6]. Recalling the way $\mathcal{O}_X$ acts on $R^q\pi_*S$ and using the fact that the $f_j$’s are in the annihilator ideal of $R^q\pi_*S$ we conclude that $(f_j)_{\Omega,#} = 0$. Hence, due to the commutativity of the above diagram $(\tilde{f}_j)_{\tilde{\Omega},*}$ is zero.
Let $Z(A)$ (resp. $Z(A')$) denote the zero variety of $A$ (resp. $A'$). Since $Z(A') = \text{Supp}R^q\pi_*S$ is contained in $A$, we have that $Z(A)$ is contained in $\tilde{A}$ near $\tilde{\Omega}$. Thus by Rückert’s Nullstellenzatz for ideal sheaves, (see Theorem, page 82 in [4]), we have $J^\mu \subset A$ on $\tilde{\Omega}$ for some $\mu \in \mathbb{N}$. Consider the surjection $\phi : S^M \to A \cdot S$ given by $(s_1, \cdots, s_M) \to \sum_j f_j s_j$ and set $K = \ker\phi$. Clearly, $K$ is torsion free, whenever $S$ is. By definition the sequence

$$0 \to K \xrightarrow{i} S^M \xrightarrow{\phi} A \cdot S \to 0$$

is exact, and it follows from [4] and the fact that $J$ is locally generated by one element that

$$0 \to J^a \cdot K \xrightarrow{i} J^a \cdot S^M \xrightarrow{\phi} J^a \cdot A \cdot S \to 0$$

is also exact for any $a \in \mathbb{N}$.

Taking all the above into consideration we obtain the following commutative diagram:

$$
\begin{array}{ccc}
H^q(\tilde{\Omega}, J^a+\mu \cdot S) & \xrightarrow{\phi_{\tilde{\Omega},*}} & H^q(\tilde{\Omega}, A \cdot S) \\
\downarrow & & \downarrow \\
H^q(\tilde{\Omega}, J^a \cdot A \cdot S) & \xrightarrow{\delta} & H^{q+1}(\tilde{\Omega}, J^a \cdot K) \\
\downarrow \phi_{\tilde{\Omega},*} & & \downarrow i_1 \\
H^q(\tilde{\Omega}, S)^M & \xrightarrow{\delta} & H^{q+1}(\tilde{\Omega}, K) \\
\downarrow \chi & & \downarrow i_2 \\
H^q(\tilde{\Omega}, S) & & \\
\end{array}
$$

where the third row is exact (as part of the long exact cohomology sequence that arises from [5]) and the vertical maps are induced by sheaf inclusions. The map $\chi$ is defined to be $\chi := i_2 \circ \phi_{\tilde{\Omega},*}$ and we can show that $\chi(c_1, \cdots, c_M) = \sum_{j=1}^M (f_j)_{\tilde{\Omega},*} c_j$, where $c_j \in H^q(\tilde{\Omega}, S)$, $1 \leq j \leq M$. It follows from the induction hypothesis for $(q+1)$ applied to the coherent, torsion-free sheaf $K$ that there exists an integer $a$ large enough such that $i_1 = 0$. Then, for an element $\sigma \in H^q(\tilde{\Omega}, A \cdot S)$ that comes from $H^q(\tilde{\Omega}, J^a+\mu \cdot S)$, we have $\delta \sigma = 0$, so $\sigma = \phi_{\tilde{\Omega},*}(\sigma_1, \cdots, \sigma_M)$, $\sigma_j \in H^q(\tilde{\Omega}, S)$, $1 \leq j \leq M$. By the crucial observation above and the way $\chi$ is defined, we conclude that $\chi$ is the zero map. Hence $i_2(\sigma) = i_2 \circ \phi_{\tilde{\Omega},*}(\sigma_1, \cdots, \sigma_M) = \sum_{j=1}^M (f_j)_{\tilde{\Omega},*} \sigma_j = 0$. Thus, for $i : J^{a+\mu} \cdot S \hookrightarrow S$ the inclusion map, we have that $i_2(\sigma) = H^q(\tilde{\Omega}, J^{a+\mu} \cdot S) \to H^q(\tilde{\Omega}, S)$ is the zero map. 

Choosing as $S := J^{k}\Omega^p$ we obtain Proposition 1.3.

3. Pointwise estimates for the pull back of forms under $\pi, \pi^{-1}$

Let $\sigma$ be a metric on $\tilde{X}$, $| \cdot |_{x, \sigma}$ denote the pointwise norm of an element of $\wedge^r T_x\tilde{\Omega}$ or $\wedge^r T_x\tilde{\tilde{\Omega}}$ for some $r > 0$ with respect to the metric $\sigma$ and $d_{\tilde{\tilde{\Omega}}, \tilde{\Omega}}$ the distance to $\tilde{\tilde{\Omega}}$ in $\tilde{\tilde{\Omega}}$. Let $d_{\tilde{\Omega}, \tilde{\tilde{\Omega}}}$ denote the distance to $\tilde{\Omega}$ relative to an embedding of a neighborhood $X_0$ of $\tilde{\Omega}$ in $\mathbb{C}^N$ and let $| \cdot |_y$ denote the pointwise norm of an element in $\wedge^r T_y(X_0 \setminus X_{\text{sing}})$ for some $r > 0$, with respect to the
restriction of the pull back of the euclidean metric in $\mathbb{C}^N$ to $X_0 \setminus X_{\text{sing}}$. Let $dV$, $d\tilde{V}_\sigma$ denote the volume forms on $X_0 \setminus X_{\text{sing}}$, and $\tilde{X}$. The map $\pi : \tilde{X} \setminus \tilde{A} \to X \setminus A$ is a biholomorphism of complex manifolds. It induces a linear isomorphism $\pi_* : \wedge^r T_x(\tilde{X} \setminus \tilde{A}) \to \wedge^r T_{\pi(x)}(X \setminus A)$ for $x \notin \tilde{A}$.

**Lemma 3.1.** We have for $x \in \tilde{\Omega} \setminus \tilde{A}$, $v \in \wedge^r T_x(\tilde{\Omega})$

\begin{align}
(6) &\quad c' \, d^c_A(x) \leq d_A(\pi(x)) \leq C' \, d^c_A(x), \\
(7) &\quad c \, d^M_A(x) \, |v|_{x,\sigma} \leq |\pi_* (v)|_{\pi(x)} \leq C \, |v|_{x,\sigma}.
\end{align}

for some positive constants $c', c, C', C, t, M$, where $c, C, M$ may depend on $r$.

For an $r$-form $a$ in $\Omega^*$ set $|\pi^* a|_{x,\sigma} := \max \{|a_{\pi(x)}, \pi_* v| : |v|_{x,\sigma} \leq 1, v \in \wedge^r T_x(\tilde{\Omega} \setminus \tilde{A})\}$, where by $<,>$ we denote the pairing of an $r$-form with a corresponding tangent vector. Using (7) we obtain:

\begin{equation}
(8) \quad c \, d^M_A(x) \, |a|_{\pi(x)} \leq |\pi^* a|_{x,\sigma} \leq C \, |a|_{\pi(x)}
\end{equation}

on $\tilde{\Omega}$, for some positive constant $M$.

**Proof.** The right hand side inequalities in the above estimates are obvious consequences of the differentiability of $\pi$, while the left hand side inequalities are consequences of the Lojasiewicz inequalities (see for example [10], or [11] Chapter 4, Theorem 4.1) in the following form:

**Lemma 3.2.** Let $S$ be a real analytic subvariety of some open subset $V$ of $\mathbb{R}^d$ and let $f$ be a real analytic, real-valued function in $V$. Let $Z_f = \{x \in V ; f(x) = 0\}$. Then, for every compact $K \subset S$, there exist positive constants $c, m$ such that

$$|f(x)| \geq c \, d(x, Z_f)^m$$

when $x \in K$.

Lemma 3.2 generalizes easily to the case when $S$ lies in a real analytic manifold and the distance is defined by a Riemannian metric.

To prove the left hand side inequality in (6) let $f : \tilde{X} \times A \to \mathbb{R}$ be given by $f(x, z) = |\pi(x) - z|^2$ and $K := \overline{\Omega} \times (\text{compact neighborhood of } \overline{\Omega} \cap A)$. Clearly $Z_f \subset \tilde{A} \times A$. When $x \in \overline{\Omega}$ and $z$ is the nearest point to $\pi(x)$ in $A$, we have:

$$f(x, z) = |\pi(x) - z|^2 = d(\pi(x), A)^2 \geq c \, d((x, z), Z_f)^m \geq c \, d_A(x)^m.$$ 

If we write $m = 2t$ for some $t > 0$ constant, then we obtain from this last estimate the left hand side inequality in (6).

To prove the left hand side inequality in (7), we consider the unit sphere bundle $S^r(\tilde{X})$ in $\wedge^r T \tilde{X}$. We give $\tilde{X}$ a real analytic metric such that $S^r(\tilde{X})$ becomes a real analytic manifold. We choose a metric on $S^r(\tilde{X})$ such that the projection $p : S^r(\tilde{X}) \to \tilde{X}$ is distance
Using the fact that
choose appropriate orientations on \( \Omega \)
\( (10) \)
This result applies in particular to the volume form in \( \Omega \)
\( \mathbb{M}, \mathbb{M}' \)

Given \( N _ { 0 } \in \mathbb{N} \), choose \( k \geq M + t \frac{N _ { 0 } }{2} \geq 0 \), with \( t, M \) as in Lemma 3.1. Then by Proposition 1.3, there exists \( \ell \geq k \) such that \( H ^ { q } ( \bar{\Omega}, J ^ { t } \Omega ^ { p } ) \to H ^ { q } ( \bar{\Omega}, J ^ { k } \Omega ^ { p } ) \) is the zero homomorphism. Choose \( N \in \mathbb{N} \) such that \( N \geq 2n \ell + M _ { 1 } \), where \( M _ { 1 } \) is as in \( [2] \).

The proof of theorem 1.1 will be based on the following change of variables result:

**Lemma 4.1.** Let \( M, M' \) be orientable, Riemannian manifolds and \( F : M \to M' \) an orientation preserving diffeomorphism. Let \( dV, dV' \) denote the corresponding volume elements of \( M, M' \) respectively. For \( f \in L ^ { 1 } ( M', dV' ) \) we have:

\[
(10) \quad \int _ { M ' } f dV' = \int _ { M } ( f \circ F ) F ^ { * } ( dV' ).
\]

Since \( \pi : \bar{\Omega} \setminus \bar{A} \to \Omega \setminus A \) is a biholomorphism & orientation-preserving map-as-long as we choose appropriate orientations on \( \Omega \setminus A \), \( \bar{\Omega} \setminus \bar{A} \), for any \( f \) satisfying \( \| f \| _ { N, \Omega ^ { * } } < \infty \) we have (by applying Lemma 4.1):

\[
\int _ { \Omega \setminus A } | f | ^ { 2 } d_{ \bar{A} } ^ { - N } dV = \int _ { \bar{\Omega} \setminus \bar{A} } | f | _ { \pi ( x ) } ^ { 2 } d_{A} ( \pi ( x ) ) ^ { - N } ( \pi ^ { * } dV ) _ { x }.
\]

Using the fact that
\[ |f|_{\pi(x)} \geq C^{-1}|\pi^*f|_{x,\sigma} \quad \text{(right hand side of (8)),} \]
\[ d_A(\pi(x))^{-1} \geq C'^{-1}d_A^{-1}(x) \quad \text{(right hand side of (6)),} \]
\[ (\pi^*dV)_{x,\sigma} \geq c_1d_A^{M_1}(x)d\tilde{V}_{x,\sigma} \quad \text{(left hand side of (9)),} \]
we obtain
\[ \|f\|^2_{N,\Omega^*} \geq c'' \int_{\tilde{\Omega}_tA} |\pi^*f|_{x,\sigma}^2 d_A^{M_1-N}d\tilde{V}_{x,\sigma} \]

for some \( c'' > 0 \) constant. Since \( N \) was chosen such that \( N \geq M_1 \), we see that \( \tilde{\sigma}\pi^*f = 0 \) on \( \tilde{\Omega} \). It is not hard to show that \( \pi^*f \in J^0L_{p,q}(\tilde{\Omega}) \). By proposition 1.3 we know that there exists \( v \in J^kL_{p,q-1}(\tilde{\Omega}) \) such that \( \tilde{\sigma}v = \pi^*f \) in \( \tilde{\Omega} \). Set \( u := (\pi^{-1})^*v \). Then \( \tilde{\sigma}u = f \) in \( \Omega^* \) and for any \( \Omega' \subset \subset \Omega \) we have:
\[
\int_{\Omega'} |u|^2 d_A^{-N_0}dV = \int_{\tilde{\Omega}_{tA}} |u|_{\pi(x)}^2 d_A^{-N_0}(\pi(x)) \pi^*(dV) \\
\leq \int_{\tilde{\Omega}_{tA}} |v|_{x,\sigma}^2 d_A^{-1-N_0-2M} d\tilde{V}_{x,\sigma} \\
\leq \int_{\tilde{\Omega}_{tA}} |v|_{x,\sigma}^2 d_A^{-2k} d\tilde{V}_{x,\sigma} < \infty.
\]

To pass from the 1st line to the 2nd one we use the fact that \( |u|_{\pi(x)} \leq c^{-1}d_A^{-M}(x)|v|_{x,\sigma} \)
\( d_A^{-N_0}(\pi(x)) \leq c^{-N_0}d_A^{-tN_0}(x) \) and that \( (\pi^*dV)_{x,\sigma} \leq C_1 d\tilde{V}_{x,\sigma} \).

To conclude the proof of Theorem 1.1 we shall need the following lemma:

**Lemma 4.2.** Let \( M \) be a complex manifold and let \( E \) and \( F \) be Frechet spaces of differential forms (or currents) of type \( (p, q-1), (p, q) \), whose topologies are finer than the weak topology of currents. Assume that for every \( f \in F \), the equation \( \tilde{\sigma}u = f \) has a solution \( u \in E \). Then, for every continuous seminorm \( p \) on \( E \), there is a continuous seminorm \( q \) on \( F \) such that the equation \( \tilde{\sigma}u = f \) has a solution with \( p(u) \leq q(f) \) for every \( f \in F \), \( q(f) > 0 \).

**Proof.** Set \( G = \{(u, f) \in E \times F : \tilde{\sigma}u = f\} \). Then \( G \) is closed in \( E \times F \). To see this, let \((u_\nu, f_\nu) \in G \) with \( u_\nu \to u \) in \( E \), \( f_\nu \to f \) in \( F \). For test forms \( \phi \in C_0^\infty(n-p,n-q)(X) \) we get
\[
\int_M f \wedge \phi = \lim_{\nu \to \infty} \int_M f_\nu \wedge \phi \\
= \lim_{\nu \to \infty} (-1)^{p+q} \int_M u_\nu \wedge \tilde{\sigma}\phi = (-1)^{p+q} \int_M u \wedge \tilde{\sigma}\phi
\]
so \( \tilde{\sigma}u = f \) weakly.

Thus, \( G \) is a Frechet space and the bounded surjection \( \pi_2 : G \to F; (u, f) \to f \) must be open. The set \( \pi_2(\{(u, v) \in G : p(u) < 1\}) \) is an open neighborhood of 0 in \( F \), and contains \( \{f : q(f) \leq 1\} \) for some continuous seminorm \( q \). Let \( f \in F \), \( 0 < q(f) = c. \)
Then \( q(c^{-1}f) = 1 \), so by the previous argument there exists a solution \( c^{-1}u \) satisfying \( \overline{\partial}(c^{-1}u) = c^{-1}f \) with \( p(c^{-1}u) < 1 \), i.e. \( p(u) < c = q(f) \). \( \square \)

When \( F \) is a Banach space with norm \( \| . \| \), we conclude that, given a seminorm \( p \), there is a constant \( C > 0 \) such that \( \{ f : \| f \| \leq C^{-1} \} \subset \overline{\partial}(\{ u : p(u) \leq 1 \}) \), so \( \overline{\partial}u = f \) has a solution \( u \) with \( p(u) \leq C\| f \| \). Applying this result to our situation, we see that if \( \overline{\partial}f = 0 \), \( \| f \|_{\Omega,N} < \infty \) and \( \Omega_0 \subset \subset \Omega \), we have a solution \( u \) of \( \overline{\partial}u = f \) in \( L^2_{p,q-1}(\Omega^*) \) with \( \| u \|_{\Omega_0,N_0} \leq c\| f \|_{\Omega,N} \).

### 5. Applications of Theorem 1.1

We apply Theorem 1.1 to the case where \( A \cap \overline{\Omega} \) is a finite subset of \( \overline{\Omega} \) with \( b\Omega \cap A = \emptyset \), \( \Omega \subset \subset X \) is Stein and \( \overline{\Omega} \) has a Stein neighborhood \( \Omega' \).

**Proposition 5.1.** With \( N_0, N \) as in Theorem 1.1 and \( \overline{\partial}f = 0 \) on \( \Omega^* \) and \( \| f \|_{\Omega,N} < \infty \), there is a solution \( u \) of \( \overline{\partial}u = f \) on \( \Omega^* \) with \( \| u \|_{\Omega_0,N_0} \leq c\| f \|_{\Omega,N} \), \( c \) independent of \( f \). In other words, we obtain a weighted \( L^2 \) estimate for \( u \) on all of \( \Omega \).

**Proof.** Choosing \( \Omega_0 \subset \subset \Omega \) containing \( A \cap \Omega \), we have a solution \( u_0 \) in \( L^2_{p,q-1}(\Omega^*) \) with \( \| u_0 \|_{\Omega_0,N_0} \leq c\| f \|_{\Omega,N} \). We introduce a cut-off function \( \chi \in C^\infty(X) \) such that \( \chi = 1 \) on \( X \setminus \Omega_0 \) but \( \chi = 0 \) near \( A \cap \Omega \). Set \( f_1 = \overline{\partial}(\chi u_0) \). Clearly, \( \| f_1 \|_{\Omega^*} \leq c\| f \|_{\Omega,N} \) and \( f_1 = 0 \) near \( \Omega \cap A \).

Let \( \pi : \tilde{X} \to X \) be a desingularization of \( X \) and consider the equation \( \overline{\partial}v = \pi^*f_1 \) on \( \tilde{\Omega} \). Let \( \tilde{\Omega}_0 := \pi^{-1}(\Omega_0) \). The equation \( \overline{\partial}v = \pi^*f_1 \) is solvable in \( L^2_{p,q-1}(\tilde{\Omega}_0) \). We can assume that \( \tilde{\Omega} \) can be exhausted by smoothly bounded strongly pseudoconvex domains \( \tilde{\Omega}_j := \{ z \in \tilde{\Omega} ; \phi < c_j \} \) where \( c_j \) are real numbers, \( \phi \) is an exhaustion function for \( \tilde{\Omega} \), of class \( C^3(\tilde{\Omega}) \), strictly plurisubharmonic outside a compact subset and also that \( b\tilde{\Omega}_j \) is smooth and strongly pseudoconvex and contained in each \( \tilde{\Omega}_j \). To each \( \tilde{\Omega}_j \) we apply Theorem 3.4.6 in [9] and we obtain a solution \( v_j \) to the equation \( \overline{\partial}v_j = \pi^*f_1 \) in \( \tilde{\Omega}_j \) with

\[
\int_{\tilde{\Omega}_j} |v_j|^2 e^{-\phi} \, d\tilde{V}_\sigma \leq C \int_{\tilde{\Omega}} |\pi^*f_1|^2 \, d\tilde{V}_\sigma
\]

where \( C \) is a positive constant independent of \( j, f \) (this follows from a careful inspection of the proof of Theorem 3.4.6 in [9]).

Consider the trivial extensions \( v_j^o \) of \( v_j \) outside \( \tilde{\Omega}_j \). Let \( v \) be a weak limit of \( v_j^o \). Then

\[
\int_{\tilde{\Omega}} |v|^2 e^{-\phi} \, d\tilde{V}_\sigma \leq C \int_{\tilde{\Omega}} |\pi^*f_1|^2 \, d\tilde{V}_\sigma
\]

and \( \overline{\partial}v = \pi^*f_1 \) in \( \tilde{\Omega} \). So there is a solution \( v \) satisfying \( \| v \|_{L^2(\tilde{\Omega})} \leq c\| f_1 \| \). Then \( w := (\pi^{-1})_*v \) satisfies \( \overline{\partial}w = f_1 \) in \( \Omega^* \) but we have no longer control of its \( L^2 \)-norm near \( A \cap \Omega \). Choose another cut-off function \( \chi_0 \) such that \( \chi_0 = 1 \) on \( \text{supp} \chi \) but \( \chi_0 = 0 \) near \( \Omega \cap A \). Then
Finally we may solve $\overline{\partial} v_1 = \overline{\partial} \chi_0 \wedge (\pi^{-1})^* v$ in $\Omega^*$ (apply Theorem 1.1 to the trivial extension of $\overline{\partial} \chi_0 \wedge (\pi^{-1})^* v$ in $\Omega'$):

$$
\|v_1\|_{\Omega, N} \leq c \|\overline{\partial} \chi_0 \wedge (\pi^{-1})^* v\|_{\Omega', N} \leq c' \|\overline{\partial} \chi_0 \wedge (\pi^{-1})^* v\|_{L^2(\Omega)} \leq C \|f\|_{\Omega, N}
$$

since $\overline{\partial} \chi = 0$ near $A$. Thus, $u := (1 - \chi) u_0 + \chi_0 (\pi^{-1})^* v - v_1$ is a solution with the required estimate. □

6. Generalizations

Theorem 1.1 and Corollary 1.2 also extend to the case when $\Omega$ is a relatively compact domain in a complex space $X$ of pure dimension $n$ with strictly pseudoconvex boundary. We know that $\Omega$ contains a maximal positive dimensional compact variety $B$ and let $A$ be a nowhere open analytic subvariety of $X$ containing $X_{\text{sing}}$ and $B$. Then theorem 1.1 carries over verbatim to the case described above. The proof needs the following modifications: Let $\Omega \subset X_0$ be a neighborhood with strictly pseudoconvex boundary and maximal positive dimensional compact subvariety $B$. Take the Remmert reduction $\phi : X_0 \to X_1$ so that $X_1$ is Stein, $\phi(B) = B_1$ is finite and $\phi : X_0 \setminus B_0 \to X_1 \setminus B_1$ is a biholomorphism. Let $\pi : \tilde{X}_0 \to X_0$ be a desingularization of $X_0$ such that $\pi^{-1}(A)$ is a hypersurface with normal crossings. To obtain a proof of Proposition 1.3 (vanishing cohomology), we need to consider direct images $R^q (\phi \circ \pi)_* S$ on the Stein space $X_1$ and their annihilator ideal $\mathcal{A}$ for $S$ coherent on $\tilde{X}$. Then, the proof carries over.

Corollary 1.2, for the case when $X_{\text{sing}} \cap b \Omega$ is empty, with $A = B \cup (X_{\text{sing}} \cap \Omega)$ follows exactly as above.

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