A build-at-home student laboratory experiment in mechanical vibrations

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Abstract
The Covid-19 pandemic has caused many university educators to redesign their teaching to online delivery. This can be an effective approach for theoretical and conceptual teaching, but it is challenging to provide practical laboratory experiences. The objective here is to design a hands-on laboratory experience that can safely be undertaken by students remotely and that has substantial educational content. A new experiment was designed featuring a bifilar pendulum that students build themselves from readily available low-cost materials. This simple vibrating system has a surprisingly rich set of interesting physical characteristics that provide several important learning points. Initial trials indicate good student experience with the new experiment, notably an appreciation for the “do-it-yourself” aspect of the apparatus construction. The self-directed features and multiple learning features of the new student experiment make it attractive for use during Covid-19 times and beyond.

Keywords
Bifilar pendulum, student laboratory, center of percussion, vibrations laboratory, curve veering

Introduction
The Covid-19 pandemic has presented substantial challenges to the Engineering education community. Many students are not able to attend class in person, either because of personal travel restrictions or due to the closure of the entire university.
In response to this situation, the majority of universities have moved to remote course delivery, typically online. This approach can work effectively for presenting theoretical or conceptual material, but is challenging to implement for hands-on laboratory exercises.\textsuperscript{1-3} The apparatus for experiments is typically fixed within a campus laboratory and cannot be accessed remotely. In addition, the equipment is often complex and costly and requires significant supervision to be operated safely and effectively.

The author was faced with the presentation of a senior year vibrations course to a class of 150 students. The course comprises an introduction to the theory of multi-degree-of-freedom vibrating systems and includes some supporting hands-on laboratory experiments. The existing laboratory experiments involve fixed apparatus setups that can be operated only in person on site. Consequently, it was necessary to design some new experiments that would be practicable and effective within the context of a remotely presented course. Criteria for the new experiments include that:

1. they can be conducted safely without in-person supervision. 
2. they are challenging and provide substantial complementary support to the theoretical and conceptual parts of the course.
3. they are interesting and satisfying to students.
4. the needed apparatus can be constructed at home using low-cost and easily available materials.

The experiment described here features a bifilar pendulum,\textsuperscript{4,5} which is a practical multi-degree-of-freedom arrangement commonly used for measuring the radius of gyration of irregularly shaped objects. For that purpose, the most frequently used geometrical arrangement is a setup that is symmetrical around the center of mass of the measured object. However, it turns out that an unsymmetrical setup has some very interesting vibrational characteristics that can give important educational insights. These include the concepts of center of percussion and of avoided crossings of natural frequencies and mode shapes. A sequence of experiments is briefly outlined here. A more detailed laboratory manual is freely available online for educational use.\textsuperscript{6}

**Bifilar pendulum vibration**

Figure 1 schematically shows a bifilar pendulum (from Latin: “bi” = two, “filum” = a string). The general case where the two strings have different lengths and are unsymmetrically placed is considered here, as shown in the figure. The bifilar pendulum has 3 degrees-of-freedom (DOF), which are transverse motions in the x (lateral) and y (longitudinal) directions and a rotational motion around the z axis (vertical at center of mass). For the special case of two equal-length strings, these three motions occur independently as uncoupled vibration modes. The two transverse modes have the same natural frequency as a simple pendulum,
independent of string separation. The natural frequency of the rotational mode around the z axis is proportional to string separation.

The present experiment focuses on the two transverse vibration modes (lateral vibration in the x direction and rotation around the z axis). The third vibration mode, involving longitudinal motion of the suspended object, here a meter ruler, is a simple parallelogram motion, independent of the other two vibration modes. Thus, for the purposes of the experiment, the system is considered to have just the first two degrees-of-freedom.

Figure 2 shows a plan view of the bifilar pendulum, illustrating the displacement coordinate system (x, \theta) based on the center of mass. It may be shown that the matrix equation of motion of the vibrating system is:

\[
\begin{bmatrix}
    m & 0 \\
    0 & m \frac{R^2}{D^2}
\end{bmatrix}
\begin{bmatrix}
    \ddot{x} \\
    \ddot{\theta}
\end{bmatrix}
+ \begin{bmatrix}
    mgx_1x_2 \\
    \frac{mgx_1x_2}{L_1L_2}
\end{bmatrix}
\begin{bmatrix}
    \frac{L_1}{x_2} + \frac{L_2}{x_1} \\
    \frac{L_1 - L_2}{x_2L_1 + x_1L_2}
\end{bmatrix}
\begin{bmatrix}
    x \\
    D\theta
\end{bmatrix}
= \begin{bmatrix}
    0 \\
    0
\end{bmatrix}
\] (1)

where

\[
x_1 = \frac{1}{2} + s/D \quad \text{and} \quad x_2 = 1 - x_1 = \frac{1}{2} - s/D
\] (2)
and where the rotational coordinate $\theta$ has been multiplied by $D$ to make the matrices dimensionally consistent. $R$ is the radius of gyration of the suspended object and the other symbols are defined in Figure 1.

For equal length strings, $L_1 = L_2 = L$. Substituting $x_1 + x_2 = 1$ gives:

$$\begin{bmatrix} m & 0 \\ 0 & m \frac{R^2}{D^2} \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \frac{D}{D}\dot{\theta} \end{bmatrix} + \frac{mg}{L} \begin{bmatrix} 1 & 0 \\ 0 & x_1 x_2 \end{bmatrix} \begin{bmatrix} x \\ D\dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The matrices in equation (3) are both diagonal, indicating that $x$ and $\theta$ are Principal Coordinates. The equations are uncoupled and may be solved individually.

Vibration mode 1 is a pure translation in the $x$ direction. The bifilar pendulum oscillates from side to side without rotation as if it were a simple pendulum. The natural frequency $\omega_1 = \sqrt{g/L}$ corresponds to that of a simple pendulum.

Vibration mode 2 is a pure rotation around the center of mass for all values of the string offset $s$. After substituting equation (1), the natural frequency is:

$$\omega_2 = \frac{D}{R} \sqrt{\frac{g}{L}} \sqrt{\frac{1}{4} - \left(\frac{s}{D}\right)^2}$$

For the symmetrical case where $s = 0$:

$$\omega_2 = \frac{D}{2R} \sqrt{\frac{g}{L}}$$

**Center of percussion**

In general, the two natural frequencies of the bifilar pendulum are different, each with its own specific mode shape. Vibration can be in one mode or other, or in both simultaneously. When combined vibration occurs, the motion deviates from simple harmonic, and appears irregular. An interesting special case occurs when $\omega_1 = \omega_2$, for which:

$$D^2 = \frac{R^2}{x_1 x_2}$$

In this case, the two same-frequency vibrations come together synchronously to form a combined mode shape that retains simple harmonic motion. If the translational and rotational mode shapes shown in Figure 3(a) and (b) are added, they
form the combined mode shape shown in Figure 3(c). The combined mode shape has a nodal point at the first string position. If the second mode shape were subtracted from the first, a mode shape with a nodal point at the second string position would be obtained. All these vibration modes have natural frequency \( \omega = \sqrt{g/L} \).

The mode shape in Figure 3(c) is remarkable because the presence of a nodal point at one of the string positions means that there is no string motion there and hence no resultant force. All the reaction force from the vibration is taken by the second string. The position of the second string is said to be at the “Center of Percussion” for rotations around the first string position. The center of percussion concept is familiar to baseball and tennis players. There is a “sweet spot” on the bat or racquet at which the ball hits smoothly without producing any reaction force on the player’s hand. This “sweet spot” corresponds to the center of percussion.

Figure 4 shows a baseball bat that is swung around point 1 and that impacts a ball at point 2. It may be shown\(^6\) that zero reaction force is felt at point 1 when:

\[
D^2 = \frac{R^2}{\frac{x_1}{x_2}}
\]  

which is the same as equation (6). Thus, the equal natural frequency condition \( \omega_1 = \omega_2 \) occurs when each string of the bifilar pendulum is at the center of percussion of the other.
The center of percussion is an important concept beyond its use in sports. For example, the wheels of many cars are designed so that the front and back wheels are at the centers of percussion relative to each other. In that way, the forces from road bumps felt by the front wheels are absorbed smoothly and do not transfer to the back wheels, and vice versa.

Unequal string lengths

The case of unequal string lengths introduces coupling between the vibration modes and reveals some interesting vibration behaviors. Useful insight can be obtained by first considering the case where the bifilar pendulum strings are at the centers of percussion, which for a symmetrical pendulum corresponds to $D = 2R$. For a uniform rod of length $\ell$, the radius of gyration $R = \ell/\sqrt{12}$. For this special case, a vibrational force at one string produces no reaction force at the other string. Thus, the bifilar pendulum can vibrate as if it comprised two independent simple pendulums, each involving movement of one string without movement of the other. The resulting vibrations are said to be “uncoupled.” The individual simple pendulum natural frequencies are:

$$f_1 = \frac{1}{2\pi} \sqrt{\frac{g}{L_1}} \quad \text{and} \quad f_2 = \frac{1}{2\pi} \sqrt{\frac{g}{L_2}} \quad (8)$$

Figure 5 shows a graph of $f_1$ and $f_2$ vs. length ratio $L_2/L_1$. The first natural frequency $f_1$ is independent of $L_2/L_1$, and so remains constant. The second natural frequency decreases with $L_2/L_1$, according to an inverse square-root relationship. The two lines cross at $L_1 = L_2$.

A change in string separation away from $D = 2R$ disrupts the exact center of percussion relationship of Figure 5 and causes each string motion to affect the other. The string motions are then said to be “coupled.” Figure 6(a) shows the natural frequency vs. string length ratio for a bifilar pendulum with $D = 2.05R$. A remarkable thing happens; instead of the two lines crossing over each other, as in Figure 5, they start to approach, but then veer away to follow the path of the other line. This is called an “avoided crossing,” also sometimes described as “curve veering.” This phenomenon occurs quite often in coupled vibrating systems, with many different examples reported in the literature.8–10
Before the avoided crossing, the vibration mode mostly consists of the corresponding vibration mode from Figure 5. However, while traversing the avoided crossing there is a rapid transformation to the second vibration mode on the other side. Within the avoided crossing, the vibration comprises a mixture of the two vibration modes. The small circle at $L_2/L_1 = 1$ corresponds to the case where the entire bifilar pendulum oscillates side to side as a simple pendulum with natural frequency $f = \sqrt{g/L_1}/2\pi$. One of the two lines always passes through this point; for $D > 2R$ it is the lower line, for $D < 2R$ it is the upper line. For string separations further removed from $D = 2R$, the avoided crossing effect increases and the curves in Figure 6 move further apart.

**Apparatus setup**

In preparation for the laboratory experiment, students are asked to build their own apparatus. Figure 7 shows an example bifilar pendulum assembled in a home setting. It can be seen that the setup is very simple and can be constructed by students of ordinary skill. A Failure Mode and Effect Analysis was carried out to assess potential risks and their consequences. No significant safety concerns were identified.

The required materials for the apparatus setup are commonly available domestic items. Many items students will already have at home and the remainder can be obtained at low cost from local retailers. The required materials are:

1. A meter ruler or a uniform wooden stick ~ 1m long.
2. A tape measure.
3. String or dental floss.
4. A sturdy table with >1 m between the legs.
5. An empty cardboard box approximately 25 x 20 x 15 cm$^3$.
6. Two pieces of stiff cardboard, each ~50 x 6 cm$^2$.
7. A medium size binder clip.
8. Packing tape to secure strings and cardboard pieces.
9. A pencil.
10. A timer device, for example a stopwatch or phone.

Figure 6. Natural frequency vs. string length ratio for a bifilar pendulum with symmetrical string separation $D = 2.05R$. String length $L_1 = 0.70$ m.

Figure 7. Example bifilar pendulum assembled in a home setting.
These specifications are flexible and students are encouraged to improvise to fit their own circumstances. Most items can be assembled quickly; the only significant construction is for a vertical cardboard rail attached to a box to allow variation of the second string length $L_2$. Figure 8 shows a prototype version. Before proceeding further, students are asked to complete and submit a “pre-lab” exercise, in which they respond to questions about the theoretical background of the experiment and the constructional details. These responses are reviewed and the student is given feedback, with suggestions as needed.

Students are then asked to proceed with a series of experiments, initially specifically defined and later more open-ended to encourage student initiative and creativity. The experiment instructions are:

1. **Vibration mode observation.** For a symmetrical pendulum, observe the lateral and rotational vibration modes, also in this case the longitudinal (parallelogram) vibration mode. Use a timer to measure the vibration periods and hence find the natural frequencies.

2. **Effect of string separation.** For a symmetrical pendulum, observe the lateral and rotational vibration modes and natural frequencies for a range of string
separations $D$. Use the rotational natural frequencies to find the radius of gyration of the suspended stick.

3. **Effect of stick offset.** Keeping the string lengths equal, observe the lateral and rotational vibration modes and natural frequencies for a range of stick offsets $s$. Compare the measured natural frequencies with the theoretical expectations from equation (4).

4. **Center of Percussion suspension.** For a symmetrical pendulum suspended at its centers of percussion ($D = \text{stick length}/3$), tap the stick at various points along its length and observe the resulting vibration modes and natural frequencies. In particular, observe that when the stick is tapped at one string location, there is a nodal point at the other.

5. **Unequal string lengths.** Keeping the center of percussion suspension, observe the lateral and rotational vibration modes and natural frequencies for a range of string lengths $L_2$. A natural frequency graph similar to Figure 5 should be observed.

6. **Avoided crossing observation.** Move the strings slightly apart, away from the center of percussion suspension. Again observe the lateral and rotational vibration modes and natural frequencies for a range of string lengths $L_2$. A natural frequency graph similar to Figure 6 should be observed. Observe how the resulting vibrations vary and how combination vibration modes are excited when the stick is tapped at points other than the nodal points.

7. **Radius of Gyration exploration.** Select a household object of irregular shape, but preferably approximately symmetrical left-to-right. Figure 9 shows some example objects. Find the balance point of the object and identify this as the center of mass. Suspend the object with the center of mass midway between the strings, measure the natural frequency for rotational vibrations and hence identify the radius of gyration of the object. Estimate the radius of gyration based on its geometry and compare the result with your measurement.

8. **Open-Ended Question.** Although not asked during the first run of the experiment, future sessions will include an open-ended question such as to ask...

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**Figure 9.** Some example irregular-shaped objects that could be a suitable specimen for a radius of gyration measurement.
students to consider the case where the pendulum strings are non-parallel. In courses where non-linear vibrations are considered, a question about the effects of large amplitude vibrations could be posed.

**Practical observations**

The objective of preparing the bifilar pendulum experiment described here was to meet an urgent need for a meaningful laboratory experience for students taking a course that had to be adapted at short notice from in-person to online presentation to meet Covid-19 circumstances. The achievement of significant learning outcomes was the primary objective, but their quantification was not the immediate priority. Thus of necessity, the observations here are descriptive only.

At the time of writing, the first cohort of students has just finished their laboratory work and submitted their reports. Casual conversations with several students after regular online lectures indicated a positive reception of the experiment, which was arranged as the one “Capstone” laboratory within the course. They seemed very intrigued about the idea of improvising their own equipment from materials that they had at home or could acquire nearby. This gave the activity a very personal feel and the students a sense of ownership, both very attractive features. Many students remarked that they enjoyed “making” their apparatus and consequently found doing it more engaging than other laboratory exercises where unfamiliar equipment suddenly appeared on the laboratory bench at school.

Students were asked to work individually, not in the groups customary when working with conventional laboratory equipment. In an effort to be “time efficient,” students typically use the group arrangement to divide the tasks among themselves, often resulting in no one student understanding the whole experiment. Alternatively, one active student does most of the work while the others “watch.” Here, every student does everything. While this is certainly good for learning, it also places more burden on students. Thus, the laboratory work was counted as equivalent to two “labs.” The remote learning format places more burden also on the instructor. The absence of nearby assistance to students requires the laboratory manual to be much more detailed and explicit than usual, here reaching 21 pages! In addition, there is substantially more marking to do because reports are produced from every individual rather than from every group.

The laboratory reports of the first cohort of students are of a generally good quality, showing a substantial degree of care and effort in their production. No specific page length was specified in the report instructions, the range was 10–30 pages, most around 20 pages. It was clear that the majority of students had taken the laboratory work seriously and it was very interesting and gratifying to see the photographs of their home-made apparatus. The choice of domestic items used to construct their apparatus and test specimens illustrated a very attractive degree of imagination and improvisation.

A few students mentioned having some fear of “making a mistake.” This response appeared to be a reaction to not having a teaching assistant nearby to
give immediate feedback. Teaching assistants were provided throughout the laboratory period both during assigned online office hours and by email, but they reported very modest traffic. At the end, it seems that students eventually worked out what was needed, so perhaps the struggle to find the answer themselves also became a useful learning experience.

The self-contained character of the experiment greatly simplified organizational arrangements because no individualized schedule was required, neither was it necessary to assign student groups. A four-week time period was assigned for doing the laboratory exercise during the middle third of the term. This time was chosen to allow coverage of the corresponding theory in class during the first third of the term. The multi-week assignment period turned out to be non-ideal because most students were concurrently taking several other courses, each with its own short-deadline activities. Thus, the longer deadline for the laboratory exercise caused the task to be continually deferred until it too became a short deadline. Then the occurrence of mid-term exams in other courses caused requests for extensions, eventually amounting to an extra two weeks. Fortunately, this did not cause any real harm, so it was forgiven. A “pre-lab” exercise was required after one week to encourage students to get started; evidently some further encouragement is needed to maintain momentum. In future years, an intermediate deliverable will be added to keep students on target.

While it has several significant desirable features, the laboratory exercise approach described here has a specific range of applicability; it may be used only for experiments that students can safely construct and conduct at home. The bifilar pendulum experiment worked well because it has several interesting features to explore, perhaps more than may generally be expected with home-made apparatus. However, in the same way as the construction of the experiment demands some imagination from the students, the formulation of the experiment also demands some imagination from the teacher. At the start of the formulation of the experiment, only the traditional elements of bifilar pendulum experiments were anticipated, notably the natural frequency and radius of gyration measurements. It was the desire to enrich and deepen the experiment that produced the more interesting Center of Percussion and Avoided Crossings components. Certainly, the same conceptual approach can be used by others to enrich other seemingly simple homemade experiments.

Additional information
A copy of the laboratory manual for this experiment is available for non-commercial educational use at http://mech-rrl.sites.olt.ubc.ca/files/2021/03/MECH463-Lab-Description-2020.pdf providing that the author and source are cited.6

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