Asymptotic moments of spatial branching processes

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(P, G)-Branching Markov Process

- Particles will live in E a Lusin space (e.g. a Polish space would be enough)
- Let \( P = (P_t, t \geq 0) \) be a semigroup on E.
- Write \( B^+(E) \) for non-negative bounded measurable functions on E.

- Particles evolve independently according to a \( P \)-Markov process.
- In an event which we refer to as ‘branching’, particles positioned at \( x \) die at rate \( \beta \in B^+(E) \) and instantaneously, new particles are created in E according to a point process.
- The configurations of these offspring are described by the random counting measure
  \[
  Z(A) = \sum_{i=1}^{N} \delta_{x_i}(A),
  \]
  with probabilities \( P_x \), where \( x \in E \) is the position of death of the parent.
- Without loss of generality we can assume that \( P_x(N = 1) = 0 \). On the other hand, we do allow for the possibility that \( P_x(N = 0) > 0 \) for some or all \( x \in E \).
- Henceforth we refer to this spatial branching process as a \((P, G)\)-branching Markov process.
\((p, g)\)-Branching Markov Process

- Define the so-called branching mechanism

\[
G[f](x) := \beta(x)E_x \left[ \prod_{i=1}^{N} f(x_i) - f(x) \right], \quad x \in E,
\]

where we recall \(f \in B_1^+(E) := \{f \in B^+(E) : \sup_{x \in E} f(x) \leq 1\}\).

- Configuration of particles at time \(t\) is denoted by \(\{x_1(t), \ldots, x_{N_t}(t)\}\) and, on the event that the process has not become extinct or exploded,

\[
X_t(\cdot) = \sum_{i=1}^{N_t} \delta_{x_i(t)}(\cdot), \quad t \geq 0.
\]

is Markovian in \(N(E)\), the space of integer atomic measures.

- Its probabilities will be denoted \(\mathbb{P} := (\mathbb{P}_\mu, \mu \in N(E))\).

- Define,

\[
v_t[f](x) = \mathbb{E}_{\delta_x} \left[ \prod_{i=1}^{N_t} f(x_i(t)) \right], \quad f \in B_1^+(E), t \geq 0.
\]

- Non-linear evolution semigroup

\[
v_t[f](x) = \hat{P}_t[f](x) + \int_0^t \mathbb{P} \left[ G[v_{t-s}[f]](x) \right] ds, \quad t \geq 0.
\]
Our main results concern understanding the growth of the $k$-th moment functional in time

\[ T_t^{(k)}[f](x) := \mathbb{E}_{\delta_x}[\langle f, X_t \rangle^k], \quad x \in E, f \in B^+(E), k \geq 1, t \geq 0. \]

**Notational convenience:** Write $T_t$ in place of $T_t^{(1)}$

**Related historical work:** A number of papers have opened the topic of moments for branching particle systems and superprocesses, including e.g.:

- E. Dumonteil and A. Mazzolo. Residence times of branching diffusion processes. *Phys. Rev. E*, 94:012131, 2016.
- J. Fleischman. Limiting distributions for branching random fields. *Trans. Amer. Math. Soc.*, 239:353–389, 1978.
- I. Iscoe. On the supports of measure-valued critical branching Brownian motion. *Ann. Probab.*, 16(1):200–221, 1988.
- A. Klenke. Multiple scale analysis of clusters in spatial branching models. *Ann. Probab.*, 25(4):1670–1711, 1997.

**Our objective:** to show that for $k \geq 2$ and any positive bounded measurable function $f$ on $E$,

\[ \lim_{t \to \infty} g_k(t)\mathbb{E}_{\delta_x}[\langle f, X_t \rangle^k] = C_k(x,f) \]

where the constant $C_k(x,f)$ can be identified explicitly.

**We need two** fundamental assumptions.
ASSUMPTION (H1): ASMUSSEN-HERING CLASS

There exists an eigenvalue $\lambda \in \mathbb{R}$ and a corresponding right eigenfunction $\varphi \in B^+(E)$ and finite left eigenmeasure $\tilde{\varphi}$ such that, for $f \in B^+(E),$

$$\langle T_t[\varphi], \mu \rangle = e^{\lambda t} \langle \varphi, \mu \rangle \text{ and } \langle T_t[f], \tilde{\varphi} \rangle = e^{\lambda t} \langle f, \tilde{\varphi} \rangle,$$

for all $\mu \in N(E)$ if $(X, \mathbb{P})$ is a branching Markov process (resp. a superprocess). Further let us define

$$\Delta_t = \sup_{x \in E, f \in B_1^+(E)} |\varphi(x)^{-1} e^{-\lambda t} T_t[f](x) - \langle \tilde{\varphi}, f \rangle|, \quad t \geq 0.$$

We suppose that

$$\sup_{t \geq 0} \Delta_t < \infty \text{ and } \lim_{t \to \infty} \Delta_t = 0.$$

NOTE: This assumption allows us to talk about criticality of the $(\mathbb{P}, \mathbb{G})$-BMP:

$$\lambda = 0 \text{ (critical) } | \lambda > 0 \text{ (supercritical) } | \lambda < 0 \text{ (subcritical)}$$
Who lives in the AsmusSEN-HERING class?

- Branching Brownian Motion in a bounded domain
- Neutron Branching process in a Bounded domain
Assumption \((H2)_k\)

\[
\sup_{x \in E} \mathcal{E}_x(\langle 1, Z \rangle^k) < \infty.
\]
THEOREM: THE CRITICAL CASE ($\lambda = 0$)

Suppose that (H1) holds along with (H2)$_k$ for some $k \geq 2$ and $\lambda = 0$. Define

$$
\Delta_t^{(\ell)} = \sup_{x \in E, \|f\| \leq 1} \left| t^{-(\ell-1)} \varphi(x)^{-1} T_t^{(\ell)} [f](x) - 2^{-(\ell-1)} \ell! \langle f, \tilde{\varphi} \rangle^\ell \langle \nabla \varphi, \tilde{\varphi} \rangle^{\ell-1} \right|
$$

where

$$
\nabla \varphi(x) = \beta(x) E_x \left( \langle \varphi, Z \rangle^2 - \langle \varphi^2, Z \rangle \right).
$$

Then, for all $\ell \leq k$

$$
\sup_{t \geq 0} \Delta_t^{(\ell)} < \infty \text{ and } \lim_{t \to \infty} \Delta_t^{(\ell)} = 0.
$$

In short, subject to (H1) at criticality and (H2)$_k$, we have, for $f \in B_1^+(E)$,

$$
\lim_{t \to \infty} t^{-(k-1)} E_x \left[ \langle f, X_t \rangle^k \right] = 2^{-(k-1)} k! \langle f, \tilde{\varphi} \rangle^k \langle \nabla \varphi, \tilde{\varphi} \rangle^{k-1} \varphi(x)
$$

"At criticality the $k$-th moment scales like $t^{k-1}$"
IDEAS FROM THE PROOF

• The obvious starting point:

\[ T_t^{(k)}[f](x) = (-1)^k \frac{\partial^k}{\partial \theta^k} \mathbb{E}_{\delta_x}[e^{-\theta \langle f, X_t \rangle}] \bigg|_{\theta=0} \]

• Recall that

\[ v_t[f](x) = \mathbb{E}_{\delta_x}\left[ \prod_{i=1}^{N_t} f(x_i(t)) \right], \quad f \in B_1^+(E), t \geq 0. \]

• Non-linear evolution semigroup

\[ v_t[f](x) = \hat{P}_t[f](x) + \int_0^t P_s [G[v_{t-s}[f]]] (x) ds, \quad t \geq 0. \]

• Hence

\[ v_t[e^{-\theta f}](x) = \mathbb{E}_{\delta_x}[e^{-\theta \langle f, X_t \rangle}] \]

• We need a new representation of the non-linear semigroup \((v_t, t \geq 0)\) which connects us to the assumption (H1).
LINEAR TO NON-LINEAR SEMIGROUP

- Recall

\[ T_t[f](x) = T_t^{(1)}[f](x) = \mathbb{E}_{\delta_x}[\langle f, X_t \rangle], \quad t \geq 0, f \in B_1^+(E), x \in E. \]

- For \( f \in B^+(E) \), it is well known that the mean semigroup evolution satisfies

\[ T_t[f](x) = P_t[f] + \int_0^t P_s[F T_{t-s}[f]](x) ds \quad t \geq 0, x \in E, \quad (1) \]

where

\[ F[f](x) = \beta(x) \mathcal{E}_x \left[ \sum_{i=1}^{N} f(x_i) - f(x) \right], \quad x \in E. \]
LINEAR TO NON-LINEAR SEMIGROUP

We now define a variant of the non-linear evolution semigroup equation

\[ u_t[f](x) = \mathbb{E}_{\delta_x} \left[ 1 - \prod_{i=1}^{N_t} f(x_i(t)) \right], \quad t \geq 0, \ x \in E, \ f \in B_1^+(E). \]

For \( f \in B_1^+(E) \), define

\[ A[f](x) = \beta(x) \mathcal{E}_x \left[ \prod_{i=1}^{N} (1 - f(x_i)) - 1 + \sum_{i=1}^{N} f(x_i) \right], \quad x \in E. \]

\[ v_t[f](x) = \hat{P}_t[f](x) + \int_0^t P_s \left[ G[v_{t-s}[f]] \right] (x) \, ds \quad \text{and} \quad T_t[f](x) = P_t[f] + \int_0^t P_s \left[ FT_{t-s}[f] \right] (x) \, ds \]

gives us.....

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Lemma

For all \( g \in B_1^+(E) \), \( x \in E \) and \( t \geq 0 \), the non-linear semigroup \( u_t[g](x) \) satisfies

\[ u_t[g](x) = T_t[1 - g](x) - \int_0^t T_s \left[ A[u_{t-s}[g]] \right] (x) \, ds. \]
Nonlinear to k-th moment evolution equation

In terms of our new semigroup equation:

\[ T_t^{(k)} [f](x) = (-1)^{k+1} \frac{\partial^k}{\partial \theta^k} u_t [e^{-\theta f}](x) \bigg|_{\theta=0}. \]

Theorem

Fix \( k \geq 2 \). Assuming (H1) and (H2)\(_k\), with the additional assumption that

\[ \sup_{x \in E, s \leq t} T_s^{(\ell)} [f](x) < \infty, \quad \ell \leq k - 1, f \in B^+(E), t \geq 0, \tag{2} \]

it holds that

\[ T_t^{(k)} [f](x) = T_t [f^k](x) + \int_0^t T_s \left[ \beta \eta_t^{(k-1)} [f] \right](x) \, ds, \quad t \geq 0, \tag{3} \]

where

\[ \eta_t^{(k-1)} [f](x) = \mathcal{E}_x \left[ \sum_{[k_1, \ldots, k_N]^2_k} \binom{k}{k_1, \ldots, k_N} \prod_{j=1}^N T_{t-s}^{(k_j)} [f](x_j) \right], \]

and \([k_1, \ldots, k_N]^2_k\) is the set of all non-negative N-tuples \((k_1, \ldots, k_N)\) such that \( \sum_{i=1}^N k_i = k \) and at least two of the \( k_i \) are strictly positive.
**INDUCTION: \( k \mapsto k + 1 \)**

- Suppose the result is true for the first \( k \) moments.
- Recall \( T_t[f](x) \to \langle f, \varphi \rangle \varphi(x) \) so that, for \( k \geq 2 \),
  \[
  \lim_{t \to \infty} t^{-k} T_t[f^{k+1}](x) \to 0
  \]
- Hence:
  \[
  \lim_{t \to \infty} t^{-k} \left[ T_t^{(k+1)}[f](x) \right] \\
  = \lim_{t \to \infty} t^{-k} \int_0^t T_s \left[ \sum_{[k_1, \ldots, k_N]_{k+1}^2} \binom{k + 1}{k_1, \ldots, k_N} \prod_{j=1}^N T_{t-s}[f](x_j) \right] (x) \, ds \\
  = \lim_{t \to \infty} t^{-(k-1)} \int_0^1 T_{ut} \left[ \sum_{[k_1, \ldots, k_N]_{k+1}^2} \binom{k + 1}{k_1, \ldots, k_N} \prod_{j=1}^N T_{t(1-u)}^{(k_j)}[f](x_j) \right] (x) \, du \\
  = \lim_{t \to \infty} \int_0^1 T_{ut} \left[ \sum_{[k_1, \ldots, k_N]_{k+1}^2} \binom{k + 1}{k_1, \ldots, k_N} \frac{(t(1-u))^{k+1-\#\{j:k_j>0}\} t^{k-1}}{\prod_{j=1}^N \frac{T_{t(1-u)}^{(k_j)}[f](x_j)}{(t(1-u))^{k_j-1}}} \right] (x) \, du
ROUGH OUTLINE OF THE INDUCTION: $k \mapsto k + 1$

- From the last slide:

$$\lim_{t \to \infty} t^{-k} T_t^{(k+1)} [f](x)$$

$$= \lim_{t \to \infty} \int_0^1 T_{ut} \left[ \mathcal{E} \left[ \sum_{[k_1, \ldots, k_N]_{k+1}^2} \binom{k+1}{k_1, \ldots, k_N} \frac{(t(1-u))^{k+1-\#\{j: k_j > 0\}}}{t^{k-1}} \prod_{j=1}^N \frac{T_{t(1-u)}^{(k_j)} [f](x_j)}{(t(1-u))^{k_j-1}} \right] \right] (x) du$$

- Largest terms in blue correspond to those summands for which $\#\{j: k_j > 0\} = 2$
- The induction hypothesis plus $\sum_{i=1}^N k_j = k + 1$ ensures that the product term is asymptotically a constant
- The simple identity

$$\sum_{[k_1, \ldots, k_N]_{k+1}^2} \binom{k+1}{k_1, \ldots, k_N} \leq N^{k+1}$$

shows us where the need for the hypothesis (H2) comes in.
- We need an ergodic limit theorem that reads (roughly): If

$$F(x, u) := \lim_{t \to \infty} F(x, u, t), \quad x \in E, u \in [0, 1],$$

"uniformly" for $(u, x) \in [0, 1] \times E$, then

$$\lim_{t \to \infty} \int_0^1 T_{ut}[F(\cdot, u, t)](x) du = \int_0^1 \langle \tilde{\varphi}, F(\cdot, u) \rangle du$$

"uniformly" for $x \in E$. 
YAGLOM LIMITS

Theorem
Suppose that

\begin{itemize}
  \item (H1) holds (mean-semigroup ergodicity),
  \item the number of offspring is uniformly bounded by a constant \( N_{\text{max}} \),
  \item for all \( t \) sufficiently large \( \sup_{x \in E} \mathbb{P}_{\delta_x}(t < \zeta) < 1 \),
  \item there exists a constant \( C > 0 \) such that for all \( g \in B^+(E) \),
\end{itemize}
\[
\langle \tilde{\varphi}, \beta V[g] \rangle \geq C \langle \tilde{\varphi}, g \rangle^2,
\]
where \( V[g](x) = \beta(x) \mathbb{E}_x \left[ \langle g, \mathcal{Z} \rangle^2 - \langle g^2, \mathcal{Z} \rangle \right] \).

Then
\[
\lim_{t \to \infty} t \mathbb{P}_{\delta_x}(\zeta > t) = \frac{2\varphi(x)}{\langle \tilde{\varphi}, \beta V[g] \rangle},
\]
\[
\lim_{t \to \infty} \mathbb{E}_{\delta_x} \left[ \left( \frac{\langle f, X_t \rangle}{t} \right)^k \right]_{\zeta > t} = k! \langle \tilde{\varphi} \rangle^k \left( \frac{\langle \tilde{\varphi}, \beta V[g] \rangle}{2} \right)^k
\]
and hence
\[
\text{Law} \left( \frac{\langle f, X_t \rangle}{t} \right| \zeta > t \right) \to \exp \left( \frac{2}{\langle \tilde{\varphi}, \beta V[g] \rangle \langle f, \tilde{\varphi} \rangle} \right).
\]
**Theorem: SuperCritical \((\lambda > 0)\)**

Suppose that \((H1)\) holds along with \((H2)_k\) for some \(k \geq 2\) and \(\lambda > 0\). Redefine

\[
\Delta_t^{(\ell)} = \sup_{x \in E, ||f|| \leq 1} \left| \varphi(x)^{-1} e^{-\ell \lambda t} T_t^{(\ell)}[f](x) - \ell! \langle f, \tilde{\varphi} \rangle L_\ell(x) \right|
\]

where \(L_1(x) = 1\) and we define iteratively for \(k \geq 2\),

\[
L_k(x) = \int_0^\infty e^{-\lambda_* ks} \varphi(x)^{-1} \psi_s \left[ \gamma \mathcal{E} \left[ \sum_{[k_1, \ldots, k_N]^2 \atop j=1} \prod_{j: k_j > 0} \varphi(x_j)L_{k_j}(x_j) \right] \right](x) ds,
\]

Then, for all \(\ell \leq k\)

\[
\sup_{t \geq 0} \Delta_t^{(\ell)} < \infty \text{ and } \lim_{t \to \infty} \Delta_t^{(\ell)} = 0.
\]

"At subcriticality the \(k\)-th moment scales like \(e^{\lambda kt}\) (i.e. the first moment to the power \(k\))"
**Theorem: Subcritical** ($\lambda < 0$)

Suppose that (H1) holds along with (H2) for some $k \geq 2$ and $\lambda < 0$. Redefine

$$\Delta_t^{(\ell)} = \sup_{x \in E, ||f|| \leq 1} \left| \varphi(x)^{-1} e^{-\lambda t} T_t^{(\ell)}[f](x) - L_\ell \right|,$$

where we define iteratively $L_1 = \langle f, \tilde{\varphi} \rangle$ and for $k \geq 2$,

$$L_k = \tilde{\varphi}[f^k] + \int_0^\infty e^{-\lambda_* s} \tilde{\varphi} \left[ \gamma \mathcal{E} \left[ \sum_{[k_1, \ldots, k_N]^2_k} \binom{k}{k_1, \ldots, k_N} \prod_{j=1}^N \psi_s^{(k_j)}[f](x_j) \right] \right] ds.$$

Then, for all $\ell \leq k$

$$\sup_{t \geq 0} \Delta_t^{(\ell)} < \infty \text{ and } \lim_{t \to \infty} \Delta_t^{(\ell)} = 0.$$

"At subcriticality the $k$-th moment scales like $e^{\lambda t}$ (i.e. like the first moment)"
**What about the occupation measure?**

- Let us define the running occupation of the branching particle system via
  \[ \int_0^t X_s(\cdot)ds, \quad t \geq 0. \]

- What can we say about its moments?
  \[ M_t^{(k)}[g](x) := E_{\delta_x} \left[ \left( \int_0^t \langle g, X_s \rangle ds \right)^k \right], \quad x \in E, g \in B^+(E), k \geq 1, t \geq 0. \]

- We know that the pair
  \[ \left( X_t, \int_0^t X_s ds \right) \]
  is Markovian and that its semigroup
  \[ v_t[f, g] = E_{\delta_x} \left[ e^{-\langle f, X_t \rangle - \int_0^t \langle g, X_s \rangle ds} \right], \quad t \geq 0, x \in E, f, g \in B^+(E), \]
  solves
  \[ v_t[f, g](x) = \hat{P}_t[e^{-f}](x) + \int_0^t P_s \left[ G[v_{t-s}[f, g]] - g v_{t-s}[f, g] \right](x) ds. \]
Define a variant of the non-linear evolution equation associated with \((X_t, \int_0^s X_s ds)\) via

\[
    u_t[f, g](x) = \mathbb{E}_{\delta_x} \left[ 1 - e^{-\langle f, X_t \rangle - \int_0^t \langle g, X_s \rangle \, ds} \right], \quad t \geq 0, \; x \in E, \; ||f|| < \infty, ||g|| < \infty.
\]

For \(f \in B^+_1(E)\), define

\[
    A[f](x) = \beta(x) \mathcal{E}_x \left[ \prod_{i=1}^N (1 - f(x_i)) - 1 + \sum_{i=1}^N f(x_i) \right], \quad x \in E.
\]

A re-arrangement of the joint semigroup of \((X_t, \int_0^t X_s ds)\) is captured by:

**Lemma**

For all \(f, g \in B^+(E), x \in E\) and \(t \geq 0\), the non-linear semigroup \(u_t[f, g](x)\) satisfies

\[
    u_t[f, g](x) = T_t[1 - e^{-f}](x) - \int_0^t T_s \left[ A[u_{t-s}[f, g]] - g(1 - u_{t-s}[f, g]) \right] (x) ds.
\]
Theorem: Critical Case ($\lambda = 0$)

Suppose that (H1) holds along with (H2) for $k \geq 2$ and $\lambda = 0$. Define

$$
\Delta_t^{(\ell)} = \sup_{x \in E, ||g|| \leq 1} \left| t^{-(2\ell-1)} \varphi(x)^{-1} M_t^{(\ell)} [g](x) - 2^{-(\ell-1)} \ell! \langle g, \bar{\varphi} \rangle \ell \langle \nabla[\varphi], \bar{\varphi} \rangle^{\ell-1} L_{\ell} \right|
$$

where $L_1 = 1$ and $L_k$ is defined through the recursion $L_k = (\sum_{i=1}^{k-1} L_i L_{k-i}) / (2k - 1)$. Then, for all $\ell \leq k$

$$
\sup_{t \geq 0} \Delta_t^{(\ell)} < \infty \text{ and } \lim_{t \to \infty} \Delta_t^{(\ell)} = 0.
$$
**Theorem: Supercritical Case ($\lambda > 0$)**

Suppose that (H1) holds along with (H2) for some $k \geq 2$ and $\lambda > 0$. Redefine

$$\Delta^{(\ell)}_t = \sup_{x \in E, ||g|| \leq 1} \left| \varphi(x)^{-1}e^{-\ell t \lambda} M^{(\ell)}_t[g](x) - \ell! \langle g, \tilde{\varphi} \rangle^\ell L_\ell(x) \right|,$$

where $L_1 = 1/\lambda$ and for $k \geq 2$ we define iteratively,

$$L_k(x) = \int_0^\infty e^{-\lambda s} \varphi(x)^{-1} \psi_s \left( \gamma \mathcal{E} \left[ \sum_{[k_1, \ldots, k_N]_k^2} \prod_{j=1}^N \varphi(x_j) L_{k_j}(x_j) \right] \right)(x) ds,$$

Then, for all $\ell \leq k$

$$\sup_{t \geq 0} \Delta^{(\ell)}_t < \infty \text{ and } \lim_{t \to \infty} \Delta^{(\ell)}_t = 0.$$
**Theorem: Subcritical Case ($\lambda < 0$)**

Suppose that (H1) holds along with (H2) for some $k \geq 2$ and $\lambda < 0$. Redefine

$$\Delta_t^{(\ell)} = \sup_{x \in E, ||g|| \leq 1} \left| \varphi(x)^{-1} m_t^{(\ell)}[g](x) - \ell! \langle g, \tilde{\varphi} \rangle^\ell L_\ell(x) \right|,$$

where $||g|| < \infty$, $L_1 = 1/|\lambda|$ and for $k \geq 2$, the constants $L_k$ are defined recursively via

$$L_k(x) = \int_0^\infty \varphi(x)^{-1} \psi_s \left[ \gamma \mathcal{E} \left[ \sum_{[k_1, \ldots, k_N]}^k \left( \begin{array}{c} k \\ k_1, \ldots, k_N \end{array} \right) \prod_{j=1}^N \varphi(x_j) L_{k_j}(x_j) \right] \right](x) \, ds$$

$$- k \int_0^\infty \varphi(x)^{-1} \psi_s \left[ g \varphi L_{k-1} \right](x) \, ds.$$

Then, for all $\ell \leq k$

$$\sup_{t \geq 0} \Delta_t^{(\ell)} < \infty \quad \text{and} \quad \lim_{t \to \infty} \Delta_t^{(\ell)} = 0.$$
**What about Superprocesses?**

- A Markov process $X := (X_t : t \geq 0)$ on $M(E)$, the space of finite measures on Lusin space $E$, with $\mathbb{P} := (\mathbb{P}_\mu, \mu \in M(E))$.

- Transition semigroup

  $$
  \mathbb{E}_\mu \left[ e^{-\langle f, X_t \rangle} \right] = e^{-\langle \mathcal{V}_t[f], \mu \rangle}, \quad \mu \in M(E), f \in B^+(E),
  $$

  where

  $$
  \mathcal{V}_t[f](x) = \mathbb{P}_t[f](x) - \int_0^t \mathbb{P}_s \langle \psi(\cdot, \mathcal{V}_{t-s}[f])(\cdot) \rangle + \phi(\cdot, \mathcal{V}_{t-s}[f]) | (x)ds.
  $$

- Here $\psi$ denotes the local branching mechanism

  $$
  \psi(x, \lambda) = -b(x)\lambda + c(x)\lambda^2 + \int_{(0, \infty)} \left( e^{-\lambda y} - 1 + \lambda y \right) \nu(x, dy), \quad \lambda \geq 0,
  $$

  where $b \in B(E), c \in B^+(E)$ and $(x \wedge x^2)\nu(x, dy)$ is a bounded kernel from $E$ to $(0, \infty)$, and $\phi$ is the non-local branching mechanism

  $$
  \phi(x,f) = \beta(x)f(x) - \beta(x)\gamma(x,f) - \beta(x) \int_{M(E)^\circ} (1 - e^{-\langle f, \nu \rangle})\Gamma(x, d\nu),
  $$

  where $\beta \in B^+(E), \gamma(x,f)$ is a bounded function on $E \times B^+(E)$ and $\nu(1)\Gamma(x, d\nu)$ is a bounded kernel from $E$ to $M(E)^\circ := M(E) \setminus \{0\}$ with

  $$
  \gamma(x,f) + \int_{M(E)^\circ} \langle 1, \nu \rangle \Gamma(x, d\nu) \leq 1.
  $$
What about superprocesses?

• Keep the same notation e.g.

\[ T_t^{(k)}[f](x) := \mathbb{E}_{\delta_x}[\langle f, X_t \rangle^k], \quad x \in E, f \in B^+(E), k \geq 1, t \geq 0. \]

• Under the same first ergodic moment assumption (H1) and (H2), \( k \) replaced by

\[ \sup_{x \in E} \left( \int_0^\infty |y|^k \nu(x, dy) + \int_{M(E)} \langle 1, \nu \rangle^k \Gamma(x, d\nu) \right) < \infty. \]

• A different proof is needed because we cannot work under the expectation with individual particles.

• Instead an approach using Faa di Bruno’s formula can be used taking advantage of the smoother branching mechanism than in the particle setting.

• The same conclusions hold for the critical, supercritical and subcritical setting as for the branching particle setting, albeit the constants in the limit are slightly different.
Thank you!