Dynamical properties of the Rabi model

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Abstract
We study the dynamical properties of the quantum Rabi model using a systematic expansion method. Based on the observation that the parity symmetry of the Rabi model is kept during evolution of the states, we decompose the initial state and the time-dependent one into positive and negative parity parts expanded by superposition of the coherent states. The evolutions of the corresponding positive and the negative parities are obtained, in which the expansion coefficients in the dynamical equations are known from the derived recurrence relation.

Keywords: quantum Rabi model, dynamical properties, parity symmetry

1. Introduction
The interplay of light and matter is vital for understanding the magnetic field on an atom possessing a nuclear spin. Semi-classical and quantum-mechanical methods are usually used to investigate phenomena involving light and its interactions with matter. Semi-classically, the radiation field is treated as a classical field and the interactions between matter and the radiation field are considered as certain interaction terms in the Hamiltonian. Quantum mechanically, light is treated as quantized photons [1]. The simplest model which takes a full quantum-mechanical treatment of the interaction of the radiation field with matter is the quantum Rabi model [2], in which the matter is a simple two-level atom. Nowadays, this model is realized and applied in a variety of quantum systems, from quantum optics and condensed-matter physics to ultracold atomic systems. A few examples include optical-cavity quantum electrodynamics (QED) [3, 4], trapped ions [5], quantum dots, and circuit QED [6, 7]. Due to the development of ultracold atomic systems [8], precise simulation and control of the quantum Rabi model becomes possible, ranging from weak, strong and ultrastrong to deep strong coupling regimes.
Recently the dynamical nature of the quantum Rabi-like model has been addressed [9–17]. For a quantum Rabi model in the rotating-wave approximation (RWA) (neglecting the counter rotating terms corresponding to virtual light–matter excitations), i.e. the celebrated Jaynes–Cummings model [18], a dynamical description is given which predicts collapses and revivals of the population inversion. Dynamical correlation functions of the quantum Rabi model, through time-dependent correlation functions, are studied to classify the ultrastrong and deep strong coupling of the quantum Rabi model [13]. A stochastic Schrödinger equation approach was developed to access the strong-coupling limit of the Rabi model and study the effects of dissipation and ac drive in an exact manner [15].

Based on an approximation for the Rabi model in the slow-oscillator regime and intermediate coupling strength the analytical expressions for the energies and eigenstates, and thus the dynamics of the oscillator, are given by Irish et al [19]. Wolf et al obtained the dynamics properties of the Rabi model for short and long times based on the $Z_2$ invariance of the model of an arbitrarily prepared initial state, and also the representation of the oscillator degree of freedom in the Bargmann space of analytical functions without truncation of the Hilbert space [20].

Advances in ultracold atomic systems make experimental study of the dynamical properties possible [8]. A quantum simulation of the quantum Rabi model with cold atoms was proposed by loading the atoms onto a periodic lattice [21]. The effective two-level quantum system is simulated by different Bloch bands in the first Brillouin zone, and the bosonic radiation field is represented by the motion of the atomic cloud in a superimposed harmonic optical-trap potential. Hereby the time evolution of a quantum system prepared in an initial state can be simulated by a quantum quench. In this way, the dynamical properties of the quantum Rabi model are reached in the extreme parameter regime, for example in the following ultrastrong and deep strong coupling regime, which is the key interest in the present study of the quantum Rabi model.

In this paper, we will propose a systematic method to investigate the dynamical properties of the quantum Rabi model beyond the adiabatic approximation and RWA. The rest of the paper is organized as follows: in section 2, we introduce the model. In section 3, we explicitly derive the dynamic equations for the positive and negative parity and thus the recurrence equations. The conclusion is given in the last section.

2. The model

The quantum Rabi model considers a two-level atom (or spin 1/2) of transition frequency $\Delta$ coupled to a single-mode cavity field with mode frequency $\omega$ (described by a quantized harmonic oscillator $\omega a^\dagger a$), which is a minimal model for the interaction of matter and light:

$$H = \frac{\Delta}{2} \sigma_z + \omega a^\dagger a + \lambda \sigma_x (a + a^\dagger).$$

(1)

Here $\sigma_{x,y,z}$ are the Pauli matrices and $a^\dagger$ and $a$ are the bosonic creation and annihilation operators, respectively, of the cavity field. The coupling strength between light and atom is $\lambda$. The three parameters $\Delta$, $\omega$ and $\lambda$ determine all the static and distinct dynamics. We set $\hbar = 1$ throughout the whole paper.

The ratio of the coupling strength $\lambda$ and the mode frequency $\omega$, $g = \lambda \omega$, determines the different properties of the model and restricts the corresponding theoretical methods. Typical quantum optical values give $g \sim 10^{-6}$, and the present circuit QED gives $g \sim 10^{-2}$. This is the weak-coupling regime, where the RWA applies and continuous $U(1)$-symmetry results in the exactly solvable Jaynes–Cummings model [18, 22]. Nowadays, superconducting circuits have
reached the so-called ultrastrong (UTS) coupling regime with $g \sim 0.1$ [23, 24]. Recently, a deep strong coupling regime with $g \gtrsim 1$ was considered, where the dynamics can be explained as a photon number wavepacket propagating along defined parity chains of the Hilbert space [25]. In the latter two cases, the RWA fails and more elegant methods are required.

Analytical solutions of the quantum Rabi model beyond the RWA have recently been explored [26–31]. It has been shown that the Rabi model undergoes a second-order quantum phase transition (QPT) in the joint limit of $\Delta \omega \to \infty$ and $\lambda \to \infty$ where the control parameter $2\lambda\sqrt{\omega}\Delta$ is kept constant [32]. These results show that the quantum Rabi model has a discrete $Z_2$ (parity) symmetry [26, 33], that is, $[P, H] = 0$, where the parity operator $P = \sigma_z e^{i\pi a^+a}$ satisfies $P|p\rangle = p|p\rangle$ and $p = \pm 1$. The Hilbert space splits into two infinite-dimensional invariant spaces, the positive (+) and negative (−) parity parts. The following dynamical evolution of the Rabi model will be built from these two Hilbert spaces.

### 3. Dynamical evolution of the Rabi model

In the following, we will show that how the system dynamics evolves inside the Hilbert space split in two unconnected subspaces or parity chains.

The time-dependent problem can be solved either through a time-dependent Schrödinger equation $i\partial_t |t\rangle = H(t)$ by evolving a state vector $|t\rangle = e^{-iH(t-t_0)|t_0\rangle}$ or a Heisenberg equation $\partial_t \rho(t) = -i[H, \rho(t)]$ by evolving a state density matrix $\rho(t) = e^{-iH(t-t_0)|t_0\rangle\langle t_0| e^{iH(t-t_0)}}$ [34]. For both cases, the numerical simulation of state evolution has to be done through the functional integral via Trotterization of the time interval:

$$e^{iH(t-t_0)} = \lim_{N \to \infty} (1 + i\delta_t H)^N, \quad \delta_t = \frac{t-t_0}{N}.$$ 

To simplify the numerical calculation of the Rabi model, an adiabatic approximation can be taken by substituting a full Hamiltonian $H$ into an adiabatic one $H_{\text{adia}}$ [10] and $|t\rangle = e^{-iH_{\text{adia}}(t-t_0)|t_0\rangle}$. The validity of the adiabatic approximation is judged from a full wavepacket simulation by the split operator procedure [35], which relies on the separation of the evolution operator. For the open system in the UTS regime, the second-order time-convolutionless projection operator method is used to correctly describe the dynamics of the system [37] and $|t\rangle = e^{-iH_{\text{UTS}}(t-t_0)|t_0\rangle}$. Thus, in many cases, the evolution of the wavepacket is given using perturbation theory or computed numerically.

Based on a Fulton–Gouterman transformation and the density matrix calculation, the analytical and numerical results for the charge dynamics of a two-site system interacting with an oscillator (equivalent Rabi model) over a very large time range are provided (also at finite temperature) by means of exact diagonalization, the quantum-classical dynamics approach and the static approximation (see details in [34]).

Here we consider the dynamical properties for the system prepared in some initial state $|t = 0\rangle$ and evolving according to the full quantum mechanical Hamiltonian. A generic quench protocol can realize this procedure [21, 36]. We should mention that the initial state is not necessarily an eigenstate of equation (1). At $t > 0$, the system evolves with the Hamiltonian $H$:

$$|t\rangle = e^{-iH(t)}|t = 0\rangle$$

$$= |t = 0\rangle - iH|t = 0\rangle + \frac{1}{2!}(-iH)^2|t = 0\rangle + \ldots$$

$$+ \frac{1}{j!}(-iH)^j|t = 0\rangle + \ldots.$$ (2)
Due to the parity symmetry of the Rabi model, the system keeps the same parities during the evolution of the states. Therefore the state vector can at any time be divided into positive and negative parities:

\[ |r| = |+, \tau| + |-, \tau|, \tag{3} \]

This is also true for any initial state \( |\tau = 0| \)

\[ |\tau = 0| = |+, 0| + |-, 0|. \tag{4} \]

By substituting equations (3) and (4) into equation (2), we obtain alternatively the time-dependent states for the even and odd parities, respectively:

\[ |+, \tau| = |+, 0| - i H|+, 0| - \frac{1}{2} H^2 \tau^2|+, 0| + \ldots \tag{5} \]

\[ |-, \tau| = |-, 0| - i H|-, 0| - \frac{1}{2} H^2 \tau^2|-, 0| + \ldots \tag{6} \]

The states \( |+, \tau| \) and \( |-, \tau| \) can be represented by the two-component spinors which are further assumed as the superposition of the coherent state, where \( \psi_{1,2} \) and \( \varphi_{1,2} \) are the corresponding expansion coefficients and \( |\zeta + i \eta| \) is the coherent state defined as,

\[ |\zeta + i \eta\rangle = \exp\{(\zeta + i \eta)a^\dagger\}|0\rangle. \tag{9} \]

As a result, solution of equations (5) and (6) is equivalent to deriving the recurrence formula for the time-dependent coefficients. For this purpose, we need to make use of the operation of the bosonic operators \( a, a^\dagger, a^\dagger a \) contained in the Hamiltonian \( H \) on the coherent state \( |\zeta + i \eta\rangle \). Thus, the relation between the Fock state \( |n\rangle \) and the coherent state \( |\zeta + i \eta\rangle \) is important (see the derivation in appendix A).

\[ |n\rangle = \int \int e^{-\zeta^2 - \eta^2} \sum_{l=0}^{n} \frac{\sqrt{n!}}{l!(n-l)!} (\zeta)^{n-l} (-i \eta)^l |\zeta + i \eta\rangle \frac{d\zeta d\eta}{\pi}. \tag{10} \]

In order to calculate equation (5), we first need to know what \( H|+, \tau = 0\rangle \), \( H^2|+, \tau = 0\rangle \), and \( H|+, \tau = 0\rangle \) \( (j \geq 3) \) are, \( |+, \tau = 0\rangle \) is defined by equation (7) at \( \tau = 0 \). We set the result of \( H^j \) \( (j = 1, 2, 3, \ldots) \) on \( |+, \tau = 0\rangle \) as

\[ H^j|+, 0\rangle = \left( \begin{array}{c} \int \int A_j(\zeta, \eta) [|\zeta + i \eta\rangle - |\zeta - i \eta\rangle] \frac{d\zeta d\eta}{\pi} \\ \int \int B_j(\zeta, \eta) [|\zeta + i \eta\rangle + |\zeta - i \eta\rangle] \frac{d\zeta d\eta}{\pi} \end{array} \right) \tag{11} \]
By making use of the operation of the bosonic operators $a, a^\dagger, a^4$ on the coherent state derived in appendix B, we first obtain the corresponding coefficients $A_i(\zeta, \eta)$ and $B_i(\zeta, \eta)$ in $H^1_+, t = 0$,

$$A_i(\zeta, \eta) = \frac{\Delta}{2} \psi_i(\zeta, \eta; 0) + \lambda \pi \psi_i(\zeta, \eta; 0)(\zeta + i\eta)
\quad + \sum_{m=0}^{\infty} \sum_{l=0}^{2m+1} \frac{2m+1}{l! (2m+1-l)!} e^{-\zeta^2 - \eta^2} \zeta^{2m+1-l-1} (-i\eta)^l \int \int d\zeta d\eta
\times \left[ \lambda \psi_i(\zeta, \eta; 0)(\zeta_1 + i\eta_1)^{2m} + \omega \psi_i(\zeta, \eta; 0)(\zeta_1 + i\eta_1)^{2m+1} \right],$$

(12)

and

$$B_i(\zeta, \eta) = -\frac{\Delta}{2} \psi_i(\zeta, \eta; 0) + \lambda \pi \psi_i(\zeta, \eta; 0)(\zeta + i\eta)
\quad + \sum_{m=0}^{\infty} \sum_{l=0}^{2m+1} \frac{2m+2}{l! (2m+2-l)!} e^{-\zeta^2 - \eta^2} \zeta^{2m+2-l-1} (-i\eta)^l \int \int d\zeta d\eta
\times \left[ \lambda \psi_i(\zeta, \eta; 0)(\zeta_1 + i\eta_1)^{2m+1} + \omega \psi_i(\zeta, \eta; 0)(\zeta_1 + i\eta_1)^{2m+2} \right].$$

(13)

Similarly, we can get the corresponding coefficients $A_j(\zeta, \eta)$ and $B_j(\zeta, \eta)$ in $H^2_+, t = 0$. Comparison with equations (12) and (13) shows that what we need to do is to substitute $\psi(\zeta, \eta; t = 0) \rightarrow A(\zeta, \eta)$ and $\psi(\zeta, \eta; t = 0) \rightarrow B(\zeta, \eta)$. By repeating the above procedure, the recurrence formulae are given for all the coefficients $A_j(\zeta, \eta)$ and $B_j(\zeta, \eta)$ ($j = 0, 1, 2, ...$),

$$A_{j+1}(\zeta, \eta) = \frac{\Delta}{2} A_j(\zeta, \eta) + \lambda \pi B_j(\zeta, \eta)(\zeta + i\eta)
\quad + \sum_{m=0}^{\infty} \sum_{l=0}^{2m+1} \frac{2m+1}{l! (2m+1-l)!} e^{-\zeta^2 - \eta^2} \zeta^{2m+1-l-1} (-i\eta)^l \int \int d\zeta d\eta
\times \left[ \lambda B_j(\zeta, \eta)(\zeta_1 + i\eta_1)^{2m} + \omega A_j(\zeta, \eta)(\zeta_1 + i\eta_1)^{2m+1} \right],$$

(14)

and

$$B_{j+1}(\zeta, \eta) = -\frac{\Delta}{2} B_j(\zeta, \eta) + \lambda \pi A_j(\zeta, \eta)(\zeta + i\eta)
\quad + \sum_{m=0}^{\infty} \sum_{l=0}^{2m+1} \frac{2m+2}{l! (2m+2-l)!} e^{-\zeta^2 - \eta^2} \zeta^{2m+2-l-1} (-i\eta)^l \int \int d\zeta d\eta
\times \left[ \lambda A_j(\zeta, \eta)(\zeta_1 + i\eta_1)^{2m+1} + \omega B_j(\zeta, \eta)(\zeta_1 + i\eta_1)^{2m+2} \right].$$

(15)

Here we let $A_0(\zeta, \eta) \equiv \psi(\zeta, \eta; t = 0)$ and $B_0(\zeta, \eta) \equiv \psi(\zeta, \eta; t = 0)$.

In summary, given the initial states and thus the coefficients $\psi(\zeta, \eta; t = 0)$ and $\psi(\zeta, \eta; t = 0)$, we can obtain all the other coefficients $A_{n+1}$ and $B_{n+1}$ according to the recurrence relations (14) and (15), and thus equation (5) is solved in a systematic way.
The same can be done for equation (6), with

\[ |-, t \rangle = \left\{ \int \int \varphi (\zeta, \eta; 0) [|\zeta + i\eta| + |\zeta - i\eta|] d\zeta d\eta \right\} \]

\[- i t \left( \int \int A_1 (\zeta, \eta; 0) [|\zeta + i\eta| - |\zeta - i\eta|] d\zeta d\eta \right) + \ldots \]

\[ + \frac{(-i)^{j}}{j!} \left( \int \int A_j (\zeta, \eta; 0) [|\zeta + i\eta| - |\zeta - i\eta|] d\zeta d\eta \right) + \ldots \quad (16) \]

The coefficients are determined by the following recurrence relations for \( C_j \) and \( D_j \) with \( j = 0, 1, 2, \ldots \):

\[ C_{j+1} (\zeta, \eta) = \pi \frac{\Delta}{2} C_j (\zeta, \eta) + \lambda \pi D_j (\zeta, \eta)(\zeta + i\eta) \]

\[ + \sum_{m=0}^{\infty} \sum_{l=0}^{m+2} \frac{2m + 2}{l! \, (2m + 2 - l)!} e^{-\zeta^2 - \eta^2} \zeta^{2m+2-l} (-i\eta)^l \int d\zeta d\eta \]

\[ \times \left[ \lambda D_j (\zeta, \eta)(\zeta + i\eta)^{2m+1} + \omega C_j (\zeta, \eta)(\zeta + i\eta)^{2m+2} \right] \quad (18) \]

and

\[ D_{j+1} (\zeta, \eta) = -\pi \frac{\Delta}{2} D_j (\zeta, \eta) + \lambda \pi C_j (\zeta, \eta)(\zeta + i\eta) \]

\[ + \sum_{m=0}^{\infty} \sum_{l=0}^{m+1} \frac{2m + 1}{l! \, (2m + 1 - l)!} e^{-\zeta^2 - \eta^2} \zeta^{2m+1-l} (-i\eta)^l \int d\zeta d\eta \]

\[ \times \left[ \lambda C_j (\zeta, \eta)(\zeta + i\eta)^{2m} + \omega D_j (\zeta, \eta)(\zeta + i\eta)^{2m+1} \right] \quad (19) \]

Here we let \( C_0 (\zeta, \eta) \equiv \varphi (\zeta, \eta; t = 0) \) and \( D_0 (\zeta, \eta) \equiv \varphi (\zeta, \eta; t = 0) \). We note that, due to the factor \( e^{-\zeta^2 - \eta^2} \) in the coefficients \( A (\zeta, \eta) \sim D (\zeta, \eta) \), the integrations on \( \zeta \) and \( \eta \) in equations (16) and (17) are Gaussian type and will always be convergent.
Given an initial state, we can decompose it into a positive and a negative parity part, and hence \( \psi_\zeta(\zeta, \eta; 0) \) and \( \phi_\zeta(\zeta, \eta; 0) \) \((n = 1, 2)\) are known. Within equations (14), (15), (18) and (19), all the expansion coefficients in the dynamical equations (16) and (17) can be systematically obtained. Thus, the full dynamical properties of the physical observables are known.

4. Conclusions

In summary, we have studied the dynamical properties of the quantum Rabi model. The quantum Rabi model has parity symmetry. As a result, the system keeps the same parities during the evolution of states. Therefore the state vector at any time can be decomposed into a positive and a negative parity, which are expanded by the superposition of the coherent states. The evolution equations for the positive and negative parities are obtained, and all the expansion coefficients can be systematically obtained through the recurrence relation.

We would like to mention two advantages of this method. One is that we can systematically derive the time evolution of the dynamic properties without using the Trotterization of the time interval. The other is that our method starts from the full quantum Hamiltonian of the Rabi model and does not rely on the adiabatic approximation or RWA on the Hamiltonian. The full dynamic properties of the wavepacket, and thus the occupation number in the up and down atom level, will be addressed in future work.

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Appendix A. Derivation of equation (10)

\[
|n\rangle = \int \int e^{-\zeta^2 - \eta^2} |\zeta + in\rangle \langle \zeta + in|n\rangle \frac{d\zeta d\eta}{\pi} \\
= \int \int e^{-\zeta^2 - \eta^2} |\zeta + in\rangle \left( \sum_{m=0}^{\infty} \frac{(\zeta - in)^m}{\sqrt{m!}} |m\rangle \right) \frac{d\zeta d\eta}{\pi} \\
= \int \int e^{-\zeta^2 - \eta^2} \frac{\sqrt{n!}}{\sqrt{n!}} |\zeta + in\rangle \frac{d\zeta d\eta}{\pi} \\
= \int \int e^{-\zeta^2 - \eta^2} \sum_{l=0}^{n} \frac{\sqrt{n!}}{l! (n-l)!} (\zeta)^{l} (-in)^{l} |\zeta + in\rangle \frac{d\zeta d\eta}{\pi}. \quad (A.1)
\]

When \( n = 2m \),

\[
|2m\rangle = \frac{1}{2} \int \int e^{-\zeta^2 - \eta^2} \sum_{l=0}^{2m} \frac{\sqrt{2m!}}{l! (2m-l)!} (\zeta)^{2m-l} (-in)^{l} \\
\times (|\zeta + in\rangle + |\zeta - in\rangle) \frac{d\zeta d\eta}{\pi}, \quad (A.2)
\]

and, when \( n = 2m + 1 \),
\[ |2m + 1| = \frac{1}{2} \int \int e^{-\zeta^2 - \eta^2} \sum_{l=0}^{2m+1} \frac{\sqrt{(2m + 1)!}}{l! (2m + 1 - l)!} (\zeta)^{2m+1-l} (-i \eta)^l \times (|\zeta + i \eta| - |-\zeta - i \eta|) \, d\zeta d\eta. \] (A.3)

**Appendix B. Operation of the bosonic operators on the coherent state**

In order to derive the recurrence formula for the time-dependent coefficients, we need to make use of the operation of the bosonic operators on the coherent state,

\[ a \int \int \psi(\zeta, \eta)(|\zeta + i \eta| + |-\zeta - i \eta|) \, d\zeta d\eta = \int \int \psi(\zeta, \eta)(|\zeta + i \eta| + |-\zeta - i \eta|) \, d\zeta d\eta, \] (B.1)

and

\[ a^\dagger \int \int \psi(\zeta, \eta)(|\zeta + i \eta| + |-\zeta - i \eta|) \, d\zeta d\eta = a \sum_n \int \psi(\zeta, \eta)|n\rangle \langle |\zeta + i \eta| + |-\zeta - i \eta|| \, d\zeta d\eta = \sum_n \int \psi(\zeta, \eta)\sqrt{n + 1} |n + 1\rangle \langle n| \left[ \sum_m \frac{(\zeta + i \eta)^m}{\sqrt{m!}} |m\rangle + \sum_m \frac{(-\zeta - i \eta)^m}{\sqrt{m!}} |m\rangle \right] d\zeta d\eta = \sum_n \int \psi(\zeta, \eta)\sqrt{n + 1} |n + 1\rangle \left\{ \frac{(\zeta + i \eta)^n}{\sqrt{n!}} + \frac{(-\zeta - i \eta)^n}{\sqrt{n!}} \right\} d\zeta d\eta = \sum_n \int \psi(\zeta, \eta)\sqrt{2m + 1} |2m + 1\rangle d\zeta d\eta \]

\[ = \int \int \sum_{m=0}^{\infty} \frac{2m + 1}{l!(2m + 1 - l)!} \int \int \psi(\zeta, \eta)(\zeta + i \eta)^{2m} \, d\zeta d\eta \times e^{-\zeta^2 - \eta^2}(\zeta)^{2m+1-l} (-i \eta)^l (|\zeta + i \eta| - |-\zeta - i \eta|) \, d\zeta d\eta. \] (B.2)

Similarly,

\[ a^\dagger \int \int \psi(\zeta, \eta)(|\zeta + i \eta| - |-\zeta - i \eta|) \, d\zeta d\eta = \int \int \psi(\zeta, \eta)(\zeta + i \eta)^{2m+1-l} (-i \eta)^l (|\zeta + i \eta| - |-\zeta - i \eta|) \, d\zeta d\eta. \] (B.3)

With equations (B.1)–(B.3), we obtain...
\[ a^1 a \int \int \psi(\zeta, \eta)(|\zeta + i\eta\rangle + |\zeta - i\eta\rangle) d\zeta d\eta \]

\[ = a^1 a \int \int \psi(\zeta, \eta)(|\zeta + i\eta\rangle - |\zeta - i\eta\rangle) d\zeta d\eta \]

\[ = \sum_{m=0}^{\infty} \sum_{l=0}^{2m+2} \frac{2m + 2}{l! (2m + 2 - l)!} \int \int \psi(\zeta, \eta)(|\zeta + i\eta\rangle)^{2m+2} d\zeta d\eta 
\times e^{-\xi^2 - \eta^2}(\zeta)^{2m+2-l}(-i\eta)^l(|\zeta + i\eta\rangle + |\zeta - i\eta\rangle) \frac{d\zeta d\eta}{\pi}, \quad (B.4) \]

and

\[ a^1 a \int \int \psi(\zeta, \eta)(|\zeta + i\eta\rangle - |\zeta - i\eta\rangle) d\zeta d\eta \]

\[ = \sum_{m=0}^{\infty} \sum_{l=0}^{2m+1} \frac{2m + 1}{l! (2m + 1 - l)!} \int \int \psi(\zeta, \eta)(|\zeta + i\eta\rangle)^{2m+1} d\zeta d\eta 
\times e^{-\xi^2 - \eta^2}(\zeta)^{2m+1-l}(-i\eta)^l(|\zeta + i\eta\rangle - |\zeta - i\eta\rangle) \frac{d\zeta d\eta}{\pi}. \quad (B.5) \]

The above equations will be used in the derivation of equation (11).

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