Self-Dual Fields and Quaternion Analyticity

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Abstract

Quaternionic formulation of $D = 4$ conformal group and of its associated twistors and their relation to harmonic analyticity is presented. Generalization of $SL(2,C)$ to the $D = 4$ conformal group $SO(5,1)$ and its covering group $SL(2,Q)$ that generalizes the euclidean Lorentz group in $R^4$ [namely $SO(3,1) \approx SL(2,C)$ which allow us to obtain the projective twistor space $CP^3$] is shown. Quasi-conformal fields are introduced in $D = 4$ and Fueter mappings are shown to map self-dual sector onto itself (and similarly for the anti-self-dual part). Differentiation of Fueter series and various forms of differential operators are shown, establishing the equivalence of Fueter analyticity with twistor and harmonic analyticity. A brief discussion of possible octonion analyticity is provided.

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1 Introduction

We give an expanded review of recent developments of Fueter’s theory as discussed in Gürsey and Tze’s book “On the Role of Division, Jordan and Related Algebras in Particle Physics.” A brief discussion to possible extension to octonionic analyticity will be presented. An appendix is added to the review on use of coherent states and coset decompositions.

There are intriguing correspondences between the rotation (or pseudorotation) groups and conformal groups in $D = 1, 2, 4, 6, 10$ Euclidean or Lorentzian space-times due to the properties of the underlying division algebras. The conformal group is infinite dimensional in $D = 2$ and trivially so in $D = 1$. This has led to the development of conformal field theories and string theories on the $D = 2$ world sheet as well as solvable lattice models in $D = 2$ statistical mechanics. These features again appear in $D = 2 + 1$ topological field theories and quantum gravity. There are also some correspondences to the self dual and anti self dual sectors of $D = 4$ gauge theories. Here we shall confine ourselves to the cases $D = 2, 4$ associated respectively with $\mathbb{C}$ and $\mathbb{H}$.

The situation is summarized in the following table:

| Dim. | Rot. | Conf. | Div. Alg. |
|------|------|-------|-----------|
| $D = 1$ | $Z_2$ | $SO(2, 1) \sim SL(2, \mathbb{R})$ | $\mathbb{R}$ |
| $D = 1 + 1$ | $O(1, 1)$ | $SO(2, 2) \sim SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ | $\mathbb{R} \otimes \mathbb{R}$ |
| $D = 2$ | $O(2) \sim U(1)$ | $SO(3, 1) \sim SL(2, \mathbb{C})$ | $\mathbb{C}$ |
| $D = 3 + 1$ | $SO(3, 1) \sim SL(2, \mathbb{C})$ | $SO(4, 2) \sim SU(2, 2)$ | $\mathbb{C}$ |
| $D = 4$ | $O(4) \sim SU(2) \times SU(2)$ | $SO(5, 1) \sim SL(2, \mathbb{Q})$ | $\mathbb{H}$ |

In the Lorentzian two dimensional space the conformal group factorizes, giving rise to left movers and right movers associated with real wave functions $f_L(x + t), f_R(x - t)$. On the other hand $O(4)$ factorizes in euclidean $D = 4$
where real three dimensional self dual fields exist associated with \((1,0)\) and \((0,1)\) representations of \(SU(2) \times SU(2)\). In \(D = 2\) instead of left and right movers we can define analytic and anti analytic functions \(f(z)\) and \(g(\bar{z})\), \(f(z)\) being anti analytic. This is analogous to the \((1,0)\) representation of \(O(4)\) in \(D = 4\) being complex and \((0,1)\) being like the complex conjugate of \((1,0)\) just like analytic and anti analytic functions. Thus some aspects of analytic functions in \(D = 2\) will have their counterparts in the self dual or anti self dual sectors in \(D = 4\) with complex numbers being replaced by quaternions. These are the possibilities we shall now explore.

2 The Conformal group in \(D = 4\), quaternions and twistors

The conformal group in Euclidean \(D = 4\) is the non-compact Lie group \(SO(5,1)\) with the covering group \(SL(2,Q)\). These groups generalize the Lie groups \(SO(3,1) \sim SL(2,C)\) in Euclidean \(D = 4\).

In \(D = 2\) we can start from the two-dimensional representations of \(SL(2,C)\) namely the complex spinor \(\psi_L\) with components \(\psi^\alpha (\alpha = 1,2)\). It transforms under the group as

\[
\psi'_L = L \psi_L, \quad (\text{Det } L = 1)
\]

where \(L\) is a \(2 \times 2\) complex unimodular matrix, \(\psi_L\) is a \((\frac{1}{2},0)\) representation. The right handed spinor \(\psi_R\) associated with the \((0,\frac{1}{2})\) representation transforms as

\[
\psi'_R = L^{\dagger -1} \psi_R
\]

Because of the identity

\[
L^{\dagger -1} = (i\sigma_2)^{-1} L^* i\sigma_2 = \sigma_2 L^* \sigma_2,
\]

the \(CP\) conjugate of \(\psi_L\), defined as

\[
\hat{\psi}_L = -i\sigma_2 \psi_L^* = \begin{pmatrix} -\psi_2^* \\ \psi_1^* \end{pmatrix}
\]

transforms like a right handed spinor \(\psi_R\) according to the law in Eq.(2).
\[ \psi'_L = L^\dagger \psi_L \] (5)

When \( SL(2, C) \) is interpreted as the Lie conformal group in \( D = 2 \), \( \psi_L \) is a twistor transforming linearly under \( SO(3, 1) \) which is generated by translations, rotations, dilatation and inversion. \( \psi^\alpha \) can be regarded as the homogeneous coordinates of a point in the projective space \( CP^1 \). The inhomogeneous coordinate is

\[ z = \frac{\psi_1}{\psi_2} \] (6)

which transforms non linearly under \( SL(2, C) \) represented by \( L \)

\[ L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (ad - bc = 1) \] (7)

Then \( z \) transforms undergoes a Möbius transformation

\[ z' = \frac{az + b}{cz + d}, \] (8)

with special cases giving:

a) dilatation

\[ z \rightarrow k \ z \quad (k = \text{real, positive}) \] (9)

b) rotation

\[ z \rightarrow e^{i\alpha} \ z \quad (\alpha = \text{real}) \] (10)

c) translation

\[ z \rightarrow z + b \quad (b = \text{complex}) \] (11)

d) inversion

\[ z \rightarrow \frac{1}{z} \] (12)

e) special conformal transformation (translation of the inverse)

\[ \frac{1}{z} \rightarrow \frac{1}{z} - s^* \quad (s = \text{complex}) \] (13)

4
or

\[ z \to \frac{z}{1 - s^* z} \]  \hspace{1cm} (14)

We now consider homogeneous functions of the spinor \( \psi_L \) (2−D twistor) of the form \( F(\psi_1, \psi_2) \) such that

\[ F(\lambda \psi_1, \lambda \psi_2) = \lambda^\delta \ F(\psi_1, \psi_2) \]  \hspace{1cm} (15)

\( \lambda \) being complex and \( \delta \) denoting the degree of homogeneity.

Choosing \( \lambda = \psi_2^{-1} \) and using Eq. \text{(6)} we obtain

\[ F(z, 1) = f(z) = \psi_2^{-\delta} \ F(\psi_1, \psi_2) \]  \hspace{1cm} (16)

Under \( SL(2, C) \) we have

\[ \psi'_1 = a \ \psi_1 + b \ \psi_2, \hspace{1cm} \psi'_2 = c \ \psi_1 + d \ \psi_2, \]  \hspace{1cm} (17)

so that we can define

\[ f'(z') = (cz + d)^\delta \ f\left(\frac{az + b}{cz + d}\right) \]  \hspace{1cm} (18)

These correspond to the \((\delta, 0)\) representation of \( SL(2, C) \) in terms of homogeneous analytic functions \( f^{(\delta)}(z) \). The \((0, \bar{\delta})\) representations are obtained through the transformation

\[ \psi \to \hat{\psi}, \hspace{1cm} \text{or} \hspace{1cm} z \to -\frac{1}{\bar{z}}, \]  \hspace{1cm} (19)

and

\[ L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to L^{-1} = \begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix}, \]  \hspace{1cm} (20)

so that

\[ G^{(\delta)}(-\bar{\psi}_2 \bar{\lambda}, \bar{\psi}_1 \bar{\lambda}) = \bar{\lambda}^\delta \ G^{(\delta)}(-\bar{\psi}_2, \bar{\psi}_1), \]  \hspace{1cm} (21)

or, taking \( \lambda = \bar{\psi}_2^{-1} \)

\[ G^{(\delta)}(-\bar{\psi}_2, \bar{\psi}_1) = (\bar{\psi}_2)^\bar{\delta} \ G^{(\delta)}(-1, \bar{z}), \]  \hspace{1cm} (22)

which gives
\[ g(z) = G^{(\delta)}(-1, \bar{z}) = (\bar{\psi}_2)^{-\delta} G^{(\delta)}(-\bar{\psi}_2, \bar{\psi}_1) \]  

(23)

Hence

\[ g'(z') = (\bar{c}z + \bar{d})^\delta g(\frac{\bar{a}z + \bar{b}}{cz + d}) \]  

(24)

For the \((\delta, \bar{\delta})\) representation, we have

\[ h'(\delta, \bar{\delta})(z', \bar{z}') = (cz + d)^\delta(\bar{c}z + \bar{d})^\bar{\delta} h^{(\delta, \bar{\delta})}(\frac{az + b}{cz + d}, \frac{\bar{a}z + \bar{b}}{\bar{c}z + \bar{d}}) \]  

(25)

Such primary fields transform like

\[ (dz)^\lambda (d\bar{z})^{\bar{\lambda}} \]  

(26)

Indeed under

\[ z' = \frac{az + b}{cz + d} \]  

(27)

we have

\[ dz' = (cz + d)^{-2} dz \]  

(28)

leading to

\[ (dz')^\lambda (d\bar{z'})^{\bar{\lambda}} = (cz + d)^{-2\lambda}(\bar{c}z + \bar{d})^{-2\bar{\lambda}} (dz)^\lambda (d\bar{z})^{\bar{\lambda}} \]  

(29)

Hence

\[ \lambda = -\frac{\delta}{2}, \quad \bar{\lambda} = -\frac{\bar{\delta}}{2} \]  

(30)

Under a more general transformation

\[ z \rightarrow w(z), \quad dz \rightarrow w'(z)dz, \]  

(31)

we obtain

\[ h^{(\lambda, \bar{\lambda})}(z, \bar{z}) \rightarrow w'(z)^\lambda \bar{w}'(\bar{z})^{\bar{\lambda}} h^{(\lambda, \bar{\lambda})}(w(z), \bar{w}(\bar{z})) \]  

(32)

The transformation (31) is still a conformal mapping in \(D = 2\) and corresponds to an infinite group that generalizes the Möbius transformation (28).
We now turn to the generalization of $SL(2, C)$ to the conformal group $SL(2, Q)$ in $D = 4$. Its $2 \times 2$ quaternionic representation is

$$
\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{Det } \Lambda = |ac^{-1}dc - bc|^2 = 1 \tag{33}
$$

$\Lambda$ acts on a two dimensional quaternionic ket $w$ in quaternionic Hilbert spaces

$$
w = \begin{pmatrix} V \\ U \end{pmatrix}, \quad w' = \Lambda w \tag{34}
$$

with the $2 \times 2$ complex representations

$$
V = \begin{pmatrix} v_1 & -v_2^* \\ v_2 & v_1^* \end{pmatrix}, \quad U = \begin{pmatrix} u_1 & -u_2^* \\ u_2 & u_1^* \end{pmatrix} \tag{35}
$$

Thus $w$ can be represented by the $4 \times 2$ complex matrix with first column $\psi$ and second column $\hat{\psi}$, where

$$
\psi = \begin{pmatrix} v_1 \\ v_2 \\ u_1 \\ u_2 \end{pmatrix}, \quad \hat{\psi} = -i\sigma_2\psi^* = \begin{pmatrix} -v_2^* \\ v_1^* \\ -u_2^* \\ u_1^* \end{pmatrix} \tag{36}
$$

The inverse $SL(2, Q)$ matrix $\Lambda^{-1}$ can also act on a $(1 \times 2)$ quaternionic row from the right, so that

$$
s^\dagger = (\bar{T}, R), \quad s'^\dagger = s^\dagger \Lambda^{-1} \tag{37}
$$

$$
\Lambda^{-1} = \begin{pmatrix} (a - bd^{-1}c)^{-1} & (c - db^{-1}a)^{-1} \\ (b - ac^{-1}d)^{-1} & (d - ca^{-1}b)^{-1} \end{pmatrix} \tag{38}
$$

where the bar denotes quaternionic conjugation.

The inhomogeneous coordinate in the projective quaternionic space $HP^1$ is the ratio of $V$ and $U$ so that

$$
w = \begin{pmatrix} x\bar{U} \\ U \end{pmatrix}, \quad x = VU^{-1} \quad \text{or} \quad V = x\bar{U} \tag{39}
$$

The 4-spinor $\psi$ is the twistor representation of the conformal group $SO(5, 1)$. We find
The twistor $\psi$ takes the form

$$\psi = \left( \left| u \right| x_+ \right) = \left( xu \right), \quad u = \left( \begin{array}{c} u_1 \\ u_2 \end{array} \right) = U^{1+\sigma_3} \frac{1}{2}$$

where

$$x_+ = \left| u \right|^{-1} xu = (x_0 - i\vec{\sigma} \cdot \vec{x})u \left| u \right|^{-1}$$

and

$$\left| u \right| = \sqrt{\left| u_1 \right|^2 + \left| u_2 \right|^2}$$

We also have the $O(4)$ invariant quaternion

$$w = U^{-1} V = \frac{1}{\left| u_1 \right|^2 + \left| u_2 \right|^2} \left( \begin{array}{cc} u_1^* & u_2^* \\ -u_2 & u_1 \end{array} \right) \left( \begin{array}{cc} v_1 & -v_2^* \\ v_2 & v_1^* \end{array} \right)$$

$$= \frac{1}{\left| u_1 \right|^2 + \left| u_2 \right|^2} \left( \begin{array}{cc} u^i v & u^i \hat{v} \\ \hat{u}^i v & \hat{u}^i u \end{array} \right)$$

Under a conformal transformation $\Lambda \in SO(5,1)$ we have

$$V' = aV + b\hat{U} = (ax + b)\hat{U}$$

$$\hat{U}' = cV + d\hat{U} = (cx + d)\hat{U}$$

where $a$, $b$, $c$, $d$ are the quaternionic elements of $\Lambda$.

Hence, the position quaternion $x$ transforms under the Möbius transformation

$$x' = (ax + b)(cx + d)^{-1}$$

while
\[ y' = (cx + d)^{-1}(ax + b) \]  

(49)

Special cases, together with their infinitesimal forms, are

a) Translations

\[ x \rightarrow x + b \quad (\delta_x = \epsilon) \]  

(50)

b) Dilatations

\[ x \rightarrow \lambda x, \quad (\text{Vec } \lambda = 0, \lambda > 0), \quad (\delta_\kappa x = \kappa x) \]  

(51)

c) Left rotations

\[ x \rightarrow mx, \quad (| m | = 1), \quad [\delta_\mu x = \mu x, \quad (\text{Sc } \mu = 0)], \]  

(52)

d) Right rotations

\[ x \rightarrow x\bar{n}, \quad (| n | = 1), \quad [\delta_\nu x = \nu x, \quad (\text{Sc } \nu = 0)], \]  

(53)

e) Inversion

\[ x \rightarrow x^{-1}, \quad (\text{no infinitesimal form}) \]  

(54)

f) Special conformal transformations

\[ x^{-1} \rightarrow x^{-1} - \bar{c}, \quad \text{or} \quad x \rightarrow x(1 - \bar{c}x)^{-1}, \quad (\delta_c x = x\bar{c}x) \]  

(55)

Like in the case of \( SL(2, C) \) we consider homogeneous functions of the \( SO(5, 1) \) twistor. These can be regarded as functions of \( x^a \) and the spinor \( u \), or the spinors \( x_+ \) and \( u \), or again functions of quaternions \( V \) (equal to \( x\bar{U} \)) and \( \bar{U} \).

Under the \( O(4) \sim SU(2) \times SU(2) \) subgroup of \( SL(2, Q) \) we have

\[ x' = mx\bar{n}, \quad V \rightarrow mV, \quad \bar{U} \rightarrow n\bar{U}, \]  

\[ (56) \]

\[ x_+ = mx_+, \quad u' = nu \]  

(57)

We could also introduce another position vector \( y \) through the components of the left twistor \( s^\dagger \) of Eq. [37]

\[ R^{-1}\bar{T} = \bar{y}, \quad R'^{-1}\bar{T}' = \bar{y}' \]  

(58)
\[(\bar{T}, R) = (R\bar{y}, R)\]  

Under the \(O(4)\) subgroup we have \[\Lambda^{-1} = \begin{pmatrix} \bar{m} & 0 \\ 0 & \bar{n} \end{pmatrix},\]  

so that \[\bar{T}' = \bar{T}\bar{m}, \quad R' = R\bar{n}\]  

This gives the transformation law \[\bar{y}' = n\bar{y}\bar{m}, \quad \text{or} \quad y' = my\bar{n}\]  

Hence \(y\) also transforms like a 4-vector.

We find \[\omega_L = x\bar{y}, \quad \omega'_L = m\omega_L\bar{m},\]  

\[\omega_r = \bar{y}x, \quad \omega'_R = n\omega_R\bar{n}\]  

It follows that from a right twistor and a left twistor we can form the two 4-vectors \(\omega_L\) and \(\omega_R\) which transform respectively like a \((0, 0) + (1, 0)\) and \((0, 0) + (0, 1)\) representations of the \(O(4)\) group. It follows that \[\lambda_L = \text{Vec} (x\bar{y}) = \frac{1}{2}(x\bar{y} - y\bar{x}) = -i\vec{\sigma} \cdot \vec{\lambda}_L\]  

is an anti self-dual antisymmetrical tensor. On the other hand \[\text{Sc} (\omega_L) = \frac{1}{2}(x\bar{y} + y\bar{x}) = \text{Sc} (\omega_R)\]  

is an \(O(4)\) invariant scalar.

### 3 Introduction of quasi-conformal fields in \(D = 4\)

#### 4. Covariant Fueter mappings

In analogy to primary conformal fields in \(D = 2\) that are homogeneous functions of the spinor representations of \(SL(2, C) \sim SO(3, 1)\) we shall construct
certain functions of the lowest dimensional representation of $SL(2,Q) \sim SO(5,1)$ in $D = 4$. We have seen that in $D = 2$ homogeneous functions of $(\psi_1, \psi_2)$ can be regarded as analytic functions of $z = \frac{\psi_1}{\psi_2}$ once a power of $\psi_2$ is factored out. In turn, the analytic function $f(z)$ represents a conformal mapping of the $z$-plane. Let us then start from a function $F$ of the two quaternionic components $V$ and $\bar{U}$ of the fundamental representation of $SL(2,Q)$, or, equivalently a function $G$ of the first column $v$ and $u$ of $V$ and $\bar{U}$ which form the spinorial components of a twistor. Introducing the position quaternion $x = V \bar{U}^{-1}$ we have

$$F = F(V, \bar{U}) = F(x\bar{U}, \bar{U})$$

or

$$G = G(v, u) = G(xu, u) = G(x_+ , u),$$

where we have used the definition (44).

These functions are homogeneous with respect to dilatation if we can write

$$F(\lambda V, \lambda \bar{U}) = \lambda^k F(V, \bar{U}) = \lambda^k F(x\bar{U}, \bar{U})$$

$$G(\lambda v, \lambda u) = \lambda^k G(v, u) = \lambda^k G(x_+, u)$$

We can now choose

$$\lambda = |U|^{-1} = (U\bar{U})^{-\frac{1}{2}} = |u|^{-1} = (u_1 u_1^* + u_2 u_2^*)^{-\frac{1}{2}}$$

so that

$$F(V, \bar{U}) = \lambda^{-k} F(V | U|^{-1}, \bar{U} | U|^{-1}),$$

$$G(v, u) = \lambda^{-k} G(x_+, \frac{u}{|u|})$$

If we put

$$w = \frac{u}{|u|}, \quad |w| = 1, \quad W = |U|^{-1} U,$$

then $W$ is a unit quaternion denoting a point on the sphere $S^3$ parametrized by the normalized spinor $w$, so that
\[ F(V, \bar{U}) = |U|^k \ F(x\bar{W}, \bar{W}), \quad (W\bar{W} = 1), \quad (75) \]

\[ G(v, u) = |u|^k \ G(x_+, w), \quad (w^i w = 1, \ x_+ = xw) \quad (76) \]

We can go further, to see for what kind of functions the quaternion \( W \) represented by a unitary matrix can be further reduced.

Let

\[ W = Z \ e^{-i\alpha \frac{\hat{z}}{2}} \quad (77) \]

Then

\[ Z \in (SU(2)/U(1) = S^2) \quad (78) \]

where \( U(1) \) is associated with rotations around the third axis. Since the \( 2 \times 2 \) matrix representation for \( W \) is

\[ W = \begin{pmatrix} w_1 & -w_2^* \\ w_2 & w_1^* \end{pmatrix}, \quad (79) \]

its general parametrization is

\[ W = \frac{1}{\sqrt{1 + |\xi|^2}} \begin{pmatrix} 1 & -\zeta^* \\ \zeta & 1 \end{pmatrix} \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix} \quad (80) \]

so that the complex number

\[ \zeta = \frac{w_2}{w_1} = \frac{u_2}{u_1} \quad (81) \]

denotes a point on \( S^2 \).

By taking \( \lambda \) in Eq.\( (73) \) complex

\[ \lambda = u_1^{-1} \quad (82) \]

we can write

\[ G(v, u) = u_1^k G(x_+, \zeta) \quad (83) \]

where
\[ x_+ = (x_0 - i \vec{\sigma} \cdot \vec{x}) \left( \frac{1}{\zeta} \right) \frac{1}{\sqrt{1 + |\zeta|^2}}. \]  

(84)

and \( \zeta \) is a point on \( S^2 \). Since in that case the function of \( v_1, v_2, u_1, u_2 \) are reduced to functions of \( v_1/u_1, v_2/u_1 \) and \( u_2/u_1 \), we find the projective space \( CP^3 \) introduced by Penrose and Ward\(^2\). Such functions can be also interpreted as function of \( x_+ \) and a point on \( S^2 \) parametrized by \( \zeta \). This is the method of harmonic analyticity of F. Gürsey, V. Ogievetsky and their collaborators C. Devchand, M. Evans, W. Jiang, C-H. Tze and others\(^3\),\(^4\),\(^5\),\(^6\).

We shall now present a third category of functions associated with a quaternionic homogeneity corresponding to the quaternionic projective space \( HP^1 \).

Consider functions \( F(V, \bar{U}) \) that transform like some representation of \( O(4) \) subgroup of \( SO(5,1) \) with \( \mu, \nu \) being quaternionic, \( F \) is assumed to get multipliers \( \mu(\lambda), \nu(\lambda) \) such that

\[ F(V\lambda, \bar{U}\lambda) = \mu(\lambda) \, F(V, \bar{U}) \, \nu(\lambda) \]  

(85)

or

\[ F(V, \bar{U}) = \mu^{-1}(\lambda) \, F(V\lambda, \bar{U}\lambda) \, \nu^{-1}(\lambda) \]  

(86)

Taking

\[ \lambda = \bar{U}^{-1} \]  

(87)

we obtain

\[ F(V, \bar{U}) = \mu^{-1}(\bar{U}^{-1}) \, F(x, 1) \, \nu^{-1}(\bar{U}^{-1}) \]  

(88)

where we have used Eq.(39).

More generally \( V \) and \( \bar{U} \) can also undergo an \( O(4) \) transformation

\[ V \rightarrow mV, \quad \bar{U} \rightarrow n\bar{U}, \quad (|m| = |n| = 1) \]  

(89)

Hence, we can have more general multipliers \( M \) and \( N \). Indeed,

\[ F(mV\lambda, n\bar{U}\lambda) = F(mV\lambda, mp\bar{U}\lambda) \]  

(90)

where have have put
\[ n = mp \]  \hspace{1cm} (91)

Taking
\[ \lambda = \bar{U}^{-1} p^{-1} \bar{m}, \quad (\bar{m} = m^{-1}) \]  \hspace{1cm} (92)

we find
\[ F(V, \bar{U}) = M^{-1} F(mV \lambda, mp\bar{U} \lambda) N^{-1} \]
\[ = M^{-1} F(mx p^{-1} \bar{m}) N^{-1} \]  \hspace{1cm} (93)

where \( M \) and \( N \) are in general function of \( m, p \) and \( \bar{U} \). If \( F \) is a scalar, then \( M \) and \( N \) can not depend on \( m \) and \( n \), or alternatively \( m \) and \( p \). They will depend on the scalar \( |U| \). Hence in that case if the homogeneity degree is \( k \) we have
\[ F(V, \bar{U}) = |U|^k F(mx p^{-1} \bar{m}) \]  \hspace{1cm} (94)

If \( F \) is a quaternion transforming like \((0,0) + (1,0)\) under \( O(4) \), then we must have
\[ F^{(k)}(V, \bar{U}) = |U|^k m^{-1} F^{(k)}(mx p^{-1} \bar{m})m \]  \hspace{1cm} (95)

Such a function is a power series in
\[ Z = xp^{-1} \]  \hspace{1cm} (96)

so that
\[ F_L(z) = \sum_n c_n Z^n \quad (\text{Vec } c_n = 0) \]  \hspace{1cm} (97)

These are just left-right holomorphic Fueter functions of \( z \). If \( F_R \) transform like \([(0,0) + (0,1)] \) representation of \( O(4) \) then we have a function of
\[ S = p^{-1} x \]  \hspace{1cm} (98)

where \( p \) transforms like a 4-vector under \( O(4) \) as
\[ p \rightarrow mp\bar{n} \]  \hspace{1cm} (99)
so that

\[ S \rightarrow nS\bar{n} \quad (100) \]

Then

\[ F_R^{(k)}(V, U) = |U|^k \ n^{-1} \ F^{(k)}(np^{-1}x\bar{n}) \ n \quad (101) \]

If \( F \) transforms like a 4-vector, then it is of the form

\[ F = p^{-1} \sum_n c_n(xp^{-1})^n = p^{-1} \ F_L(Z) = F_R(S)p^{-1} \quad (102) \]

It transforms as

\[ F \rightarrow n \ F \ \bar{m} \quad (103) \]

which is the transformation law for \( x^{-1} \). Thus we have the mapping defined by

\[ x' = n \ (p^{-1} \sum_n c_n(xp^{-1})^n + \ell^{-1})^{-1} \ \bar{m} \quad (104) \]

where we have also used an arbitrary \( O(4) \) transformation. Note that unlike the \( D = 2 \) case this mapping is not conformal. We shall call it quasi-conformal.

Consider the special case

\[ c_n = -c \quad (105) \]

with \( n \) going from zero to infinity. Then the mapping takes the form

\[ x' = m \ (\frac{c}{x - p} + \ell^{-1})^{-1} \ \bar{n} \quad (106) \]

which is equivalent to a quaternionic Möbius transformation that represents a conformal transformation. Hence the infinite parameter mapping of Eq.(104) admits the conformal group as a subgroup.

This is the generalization to \( D = 4 \) of the infinite parameter holomorphic mapping in \( D = 2 \),

\[ z' = f(z) = \sum_n b_n z^n \quad (107) \]
where $b_n, z^n$ are complex. The mapping admits as a finite parameter subgroup

$$z' = \frac{az + b}{cz + d}, \quad (ad - bc = 1) \quad (108)$$

which is the Möbius transformation that is a nonlinear realization of $SO(3, 1) \sim SL(2, C)$ on the coset

$$SL(2, C)/\Delta \times E_2 \quad (109)$$

$\Delta$ being the dilatation and $E_2$ the Euclidean group in $D = 2$.

In the case of $D = 2$ the holomorphic group and its Möbius subgroup are both conformal, since

$$ds^2 = dz' d\bar{z}' = |f'(z)|^2 \, dz \, d\bar{z} \quad (110)$$

This is not true in $D = 4$. The infinite group Eq.(106) does not lead to a conformally flat metric, but its Möbius subgroup does. Indeed we have

$$dx' = -m \left( \frac{c}{x - p} + \ell^{-1} \right)^{-1} dx \left( \frac{c}{x - p} + \ell^{-1} \right)^{-1} \bar{n} \quad (111)$$

so that

$$dx' \, d\bar{x}' = \left( \frac{c}{x - p} + \ell^{-1} \right)^{-2} \, dx \, d\bar{x} \quad (112)$$

The quaternionic holomorphic mapping of Eq.(106) can be called quasi conformal. If we make the $O(4)$ transformation

$$X = mx'\bar{n}, \quad x' = mx\bar{n}, \quad P = mp\bar{n}, \quad L = m\ell\bar{n} \quad (113)$$

it takes the form

$$X'^{-1} = \sum_n c_n (P^{-1}x)^n \, P^{-1} + L^{-1} \quad (114)$$

or, putting

$$Z' = P^{-1}X', \quad Z = P^{-1}X = Z_0 - i\vec{\sigma} \cdot \vec{Z} \quad (115)$$

$$Z'^{-1} = \sum_n c_n Z^n + L^{-1}P \quad (116)$$
In this last form the mapping is recognized as being a left-right holomorphic Fueter transformation.

If we write

\[ ds^2 = dZ' d\bar{Z}' = g_{\mu\nu} \, dZ^\mu \, dZ^\nu \]  \hspace{1cm} (117)

then it is easily shown that in 4-dimensional polar coordinates we have\(^{[4]}\)

\[ g_{0n} = 0, \quad (n = 1, 2, 3) \]  \hspace{1cm} (118)

In fact, putting

\[ z = Z_0 + i |\vec{Z}| \]  \hspace{1cm} (119)

it takes the form\(^{[5]}\)

\[ ds^2 = \Phi^2(z, \bar{z}) \, dz \, d\bar{z} + \rho^2(z, \bar{z}) \, d\Omega^2(\theta, \phi), \]  \hspace{1cm} (120)

where \(d\Omega^2\) is the line element on \(S^2\). Then, this Kruskal form that generalizes the conformal metric merits the name: quasi conformal.

Another similarity with \(D = 2\) is provided by differential equations satisfied by the mappings.

In \(D = 2\) we have

\[ (\partial_x + i\partial_y) \, f(z) = \frac{\partial}{\partial\bar{z}} \, f(z) = 0 \]  \hspace{1cm} (121)

which imply the harmonicity of \(f(z)\), i.e.:

\[ \Delta f(z) = \frac{\partial}{\partial z} \frac{\partial}{\partial\bar{z}} \, f(z) = 0 \]  \hspace{1cm} (122)

In \(D = 4\) the mapping

\[ x' = F(x) \]  \hspace{1cm} (123)

defined by Eq.(104) satisfies

\[ \Box \Box F(x) = 0 \]  \hspace{1cm} (124)

showing that \(F(x)\) is biharmonic. We have seen that in a special \(O(4)\) frame this mapping takes the Fueter form.

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\[ Z' = \sum_n c_n Z^n = \Phi(Z) \] (125)

which satisfies

\[ \Box D\Phi(z) = DDD\Phi(z) = 0 \] (126)

where

\[ D = \frac{\partial}{\partial z^0} - i\sigma \cdot \frac{\partial}{\partial \vec{z}} \quad \bar{D} = \frac{\partial}{\partial \bar{z}^0} + i\sigma \cdot \frac{\partial}{\partial \bar{z}} \] (127)

In other words the function

\[ G(z^0, \bar{z}) = \Box \Phi(Z) \] (128)

satisfies

\[ D G = 0 \] (129)

which is equivalent to the massless 2-component Dirac equation

\[ \left( \partial_0 - i\vec{\sigma} \cdot \vec{\nabla} \right) \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = 0 \] (130)

where we have used the following \(2 \times 2\) representation for \(G\):

\[ G = \begin{pmatrix} g_1 & -g_2^* \\ g_2 & g_1^* \end{pmatrix} \] (131)

The Eq. (130) generalizes the Cauchy-Riemann Eq. (121).

Here we must note that instead of \(Z\) standing for \(P^{-1}X\) which transforms like \((0, 0) + (0, 1)\) representation of \(O(4)\) we could have taken \(U\) representing \(XP^{-1}\). Also writing \(U'\) for \(X'P^{-1}\) that transforms like \((0, 0) + (1, 0)\) we have an alternative form of the pseudo conformal mapping that reads

\[ U' = \sum_n c_n U^n \] (132)

Under the \(O(4)\) transformation Eq. (113) we have

\[ Z \rightarrow nZ\bar{n}, \quad Z' \rightarrow nZ'\bar{n}, \] (133)
This shows that the self-dual sector is mapped into itself, the same being
valid for the anti self-dual sector.

4 Geometric Interpretation - Functions on elements of the coset \( SO(5, 1)/SO(4) \times O(1, 1) \)

The covariant Fueter functions can be shown to be elements of a function of
a coset element

\[
\phi = SL(2, Q)/Sp(1) \times Sp(1) \times O(1, 1)
\]

represented by a \( 2 \times 2 \) quaternionic matrix. The dilation \( O(1, 1) \) and the
\( O(4) \sim Sp(1) \times Sp(1) \) groups act linearly on \( \phi \). We parametrize the \( SL(2, Q) \)
matrix \( \Lambda \) in the following way:

\[
\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ p^{-1} & 1 \end{pmatrix} \begin{pmatrix} \kappa^2 m & 0 \\ 0 & \kappa^{-2} n \end{pmatrix}
\]

or

\[
\Lambda = F H
\]

where

\[
H = \begin{pmatrix} \kappa^2 m & 0 \\ 0 & \kappa^{-2} n \end{pmatrix}, \quad (\kappa \text{ real, } |m| = |n| = 1)
\]

represents the subgroup \( O(4) \times O(1, 1) \) and

\[
F = \begin{pmatrix} 1 + xp^{-1} & x \\ p^{-1} & 1 \end{pmatrix}, \quad (\text{Det } F = 1)
\]

is the coset element. In order to eliminate the subgroup \( H \) we introduce the element
\[ \eta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \eta^2 = I, \quad [H, \eta] = 0, \quad (140) \]

which commutes with \( H \) and form the matrix
\[
\phi = \Lambda \eta \Lambda^{-1} \eta^{-1} = F \eta F^{-1} \eta, \quad (\text{Det } \phi = 1) \quad (141)
\]

We have
\[
F^{-1} = \begin{pmatrix} 1 & -x \\ -p^{-1} & 1 + p^{-1}x \end{pmatrix} \quad (142)
\]

Hence \( \phi \) has the form
\[
\phi = \begin{pmatrix} 1 + v x \\ p^{-1} 1 
\end{pmatrix} \begin{pmatrix} 1 & x \\ p^{-1} 1 + u \end{pmatrix}
= \begin{pmatrix} 1 + 2v x & 2(1 + v)x \\ 2p^{-1} 1 + 2u \end{pmatrix} \quad (143)
\]

where
\[
u = p^{-1}x, \quad v = xp^{-1} \quad (144)
\]

\[
xu = vx = xp^{-1}x \quad (145)
\]

We can also write
\[
\phi = 1 + 2W \quad (146)
\]

where
\[
W(p, x) = \begin{pmatrix} v & (1 + v)x \\ p^{-1} & u \end{pmatrix} = \begin{pmatrix} v & v(1 + v)p \\ p^{-1} & u \end{pmatrix} \quad (147)
\]

Under the subgroup \( R \in H \) acting on \( M \) from the left, we have
\[
M \rightarrow RM, \quad \phi \rightarrow R\phi R^{-1}, \quad W \rightarrow RWR^{-1} \quad (148)
\]
or
\[
u \rightarrow nu\bar{n}, \quad v \rightarrow mv\bar{m}, \quad (149)
\]
\[ x \rightarrow kmx\bar{n}, \quad p \rightarrow kmp\bar{n} \tag{150} \]

Consider now a power series in \( \phi \), or a function of \( W \)

\[ G(W) = \sum_n c_n W^n \tag{151} \]

under \( R \) we have

\[ G(W) \rightarrow RG(W)R^{-1} \tag{152} \]

The function \( G(W) \) is biharmonic when \( G(W) \) is regarded as a function of \( x \)

\[ \Box_x \Box_x G(W(x)) = 0 \tag{153} \]

since we have

\[ G^{(n)} = W^n(x) = \begin{pmatrix} G_{11}(v) & h^{(n)}(v)p \\ k^{(n)}(u)p^{-1} & g_{22}(u) \end{pmatrix} \tag{154} \]

Introduce the \( 2 \times 2 \) Dirac operators

\[ D_x = \partial_0 - i\vec{\sigma} \cdot \vec{\nabla}, \quad D_u = pD_x, \quad D_v = D_xp \tag{155} \]

We have

\[ D_u \Box G_{22}(u) = 0, \quad \text{or} \quad D_x \Box_x G_{22}(p^{-1}x) = 0, \tag{156} \]

\[ \Box G_{11}(v) \vec{D}_v = 0, \quad \text{or} \quad \Box_x G_{11}(xp^{-1}) \vec{D}_x = 0, \tag{157} \]

\[ D_v \Box G_{11}(v) = 0, \quad \Box G_{22}(u) \vec{D}_v = 0 \tag{158} \]

Hence, if we define

\[ \mathcal{D} = \begin{pmatrix} D_u & 0 \\ 0 & D_v \end{pmatrix} \tag{159} \]

we have

\[ \mathcal{D} \Box G(W) = \mathcal{D} \vec{D} \mathcal{D} G(W) = 0, \tag{160} \]

which is the condition of Fueter analyticity for the function
\( g(x, p) = \Box G(W) \)  \hspace{1cm} (161)

The elements of \( G \) have definite tensorial properties under \( O(4) \):

\[
G_{11} \sim (0, 0) + (1, 0), \quad G_{22} \sim (0, 0) + (0, 1) \quad (162)
\]

\[
G_{12} \sim (\frac{1}{2}, \frac{1}{2}), \quad G_{21} \sim (\frac{1}{2}, \frac{1}{2}) \quad (163)
\]

We now turn to the transformation properties under the remaining transformations of \( SO(5, 1)/SO(4) \times O(1, 1) \) under which both \( x \) and \( p \) transform non linearly.

We have, from Eq.(136)

\[
\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (1 + xp^{-1})\kappa^{\frac{1}{2}}m & x\kappa^{-\frac{1}{2}}n \\ p^{-1}\kappa^{\frac{1}{2}}m & \kappa^{-\frac{1}{2}}n \end{pmatrix} \quad (164)
\]

Under left action of \( SL(2, Q) \) we have

\[
\Lambda' = K\Lambda, \quad (165)
\]

where

\[
K = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad (Det K = 1) \quad (166)
\]

Solving for \( x \) and \( p \) we find

\[
x = bd^{-1}, \quad p + x = ac^{-1} \quad (167)
\]

Hence

\[
x' = b'd^{-1}, \quad p' = a'c'^{-1} - b'd'^{-1} \quad (168)
\]

giving

\[
x' = (\alpha x + \beta)(\gamma x + \delta)^{-1}, \quad (169)
\]

\[
p' = \left[ \alpha(p + x) + \beta \right][\gamma(p + x) + \delta]^{-1} - (\alpha x + \beta)(\gamma x + \delta)^{-1} \quad (170)
\]
It follows that $p$ transforms like the difference of two position vectors, or like $\Delta x$. This implies that under the translation subgroup

$$x' = x + \beta \quad (171)$$

$$p' = p \quad (172)$$

Hence $p$ transforms like a momentum under the Poincaré subgroup of the conformal group.

We note that

$$\begin{pmatrix} b \\ d \end{pmatrix} \quad (173)$$

and

$$\begin{pmatrix} a \\ c \end{pmatrix} \quad (174)$$

behave like quaternionic twistors with respect to left multiplication of $\Lambda$ by $K \in SL(2, Q)$.

As before we can introduce functions of $x$ and $p$ that acquire quaternionic multipliers under an $SL(2, Q)$ transformations. They are generalized quasi-conformal fields as discussed in the previous section.

5 The Quadratic Schwarz Differentials in $D = 4$. Differentiation of Fueter Series

In the case of the complex Möbius transformation

$$w = \frac{az + b}{cz + d} \quad (175)$$

the Schwarz derivative

$$\{w, z\} = \frac{d^3 w}{dz^3} (\frac{dw}{dz})^{-1} - \frac{3}{2} \left[ \frac{d^2 w}{dz^2} (\frac{dw}{dz})^{-1} \right]^2 \quad (176)$$

vanishes, so that
\[ \left\{ \frac{az + b}{cz + d}, z \right\} = 0 \quad (177) \]

We can also define the quadratic Schwarz differential

\[ w(w, z) = \{w, z\} dz^2 = d^3 w(dw)^{-1} - \frac{3}{2} [d^2 w(dw)^{-1}]^2 \quad (178) \]

which also vanishes for the Möbius transformation.

The Schwarz differential can also be obtained as the limit of the cross-ratio of 4 points when they all tend to a common point.

Consider the successive mappings

\[ Z \longrightarrow u(z) \longrightarrow w(u(z)) \quad (179) \]

Then

\[ \omega(w, z) = \omega(w, u) + \omega(u, z) \quad (180) \]

which is equivalent to, but simpler then the well known rule

\[ \{w, z\} = \{w, u\} \left(\frac{du}{dz}\right)^2 + \{u, z\} \quad (181) \]

In the special case

\[ u = \frac{az + b}{cz + d} \quad (182) \]

we have

\[ \omega(w, z) = \omega(w, \frac{az + b}{cz + d}) \quad (183) \]

or

\[ \{w, z\} = \{w, \frac{az + b}{cz + d}\} \frac{1}{(cz + d)^4} \quad (184) \]

Thus, under \( SL(2, C) \), \( \{w, z\} \) transforms like \((dz)^2\) and \( \{w, z\}^k \) like \(dz^k\) so that it has conformal weight \(k\). However, under \( z \rightarrow u(z) \) the conformal field transforms inhomogeneously, acquiring an additional piece that corresponds to the central extension of the infinite group of homogeneous analytic transformations.

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We now turn to the $D = 4$ case and the quasi-holomorphic mappings $x \to x'$ of quaternions. It is much easier to generalize differentials than derivatives in the non commutative case. Hence we define two kinds of quadratic Schwarz differentials, both being cross ratios of four nearby points in euclidean space-time.

\[ \Omega_L(w, x) = d^3w \, dw^{-1} - \frac{3}{2}(d^2w \, dw^{-1})^2, \quad (185) \]

\[ \Omega_R(w, x) = dw^{-1} \, d^3w - \frac{3}{2}(dw^{-1} \, d^2w)^2 \quad (186) \]

If $x$ and $w$ are 4-vectors, then

\[ \Omega = Sc \, \Omega_L = Sc \, \Omega_R \quad (187) \]

is $O(4)$ invariant, while $Vec \, \Omega_L$ and $Vec \, \Omega_R$ transform respectively like a self dual and an anti self dual tensor. Now we have

\[ \Omega_L \left[ (ax + b)(cx + d)^{-1}, x \right] = 0, \quad (188) \]

\[ \Omega_R \left[ (ax + b)(cx + d)^{-1}, x \right] = 0, \quad (189) \]

for a quaternionic Môbius transformation, generalizing Eq.(177).

In the case of the more general pseudo-conformal mapping, we seek a generalization of Eq.(180). To this end we introduce the infinite matrix $S$ with elements equal to unity in the upper line parallel to the diagonal, with all other elements being zero.

\[ S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \rightarrow \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \downarrow & & & & \ddots \end{pmatrix} \quad (190) \]

We now consider the quaternionic matrix

\[ \frac{1}{1 - SZ} = 1 + SZ + SZ^2 + \cdots \]
Define the kets

\[
| \alpha > = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \end{pmatrix}, \quad | 0 > = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}
\]  

(192)

The basis for this kets is provided by the harmonic oscillator operators \( a, \ a^\dagger \) which obey

\[
[a, a^\dagger] = 1
\]  

(193)

Then we have

\[
\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = a^\dagger | 0 > = | 1 >, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} = \frac{a^2}{\sqrt{2!}} | 0 > = | 2 >
\]  

(194)

d. With these notations it is easy to check that

\[
S = \sum_{n=0}^{\infty} | n > < n + 1 | = \sum_n \frac{a^n}{\sqrt{n!}} | 0 > < 0 | \frac{a^{n+1}}{\sqrt{(n + 1)!}}
\]  

(195)

\[
U = | 0 > < (1 - SZ)^{-1} | \alpha > = \sum_n \alpha_n Z^n = F(Z)
\]  

(196)

displaying the Fueter mapping as a matrix element of the operator \((1 - SZ)^{-1}\). Its other matrix elements are also Fueter functions. We can now calculate \(dU\):
\[ dU = <0 | (1 - SZ)^{-1} S dZ (1 - SZ)^{-1} | \alpha > \] (197)

If we have

\[ W = W(U), \quad U = U(Z), \] (198)

we can write

\[ dW = <0 | \frac{1}{1 - SU} S dU \frac{1}{1 - SU} | \beta >, \] (199)

\[ d^2W = <0 | \left\{ \frac{2}{1 - SU} S dU \frac{1}{1 - SU} S dU \frac{1}{1 - SU} \\
+ \frac{1}{1 - SU} S d^2U \frac{1}{1 - SU} \right\} | \beta >, \quad \text{etc.} \] (200)

while

\[ d^2U = 2 <0 | \frac{1}{1 - SZ} S dZ \frac{1}{1 - SZ} S dZ \frac{1}{1 - SZ} | \alpha > \] (201)

These formulae allow us to evaluate quaternionic quadratic Schwarz differentials for Fueter mappings.

Note that the operator

\[ H(Z) = (1 - SZ)^{-1} \] (202)

satisfies

\[ \Omega_L (H(Z), Z) = 0, \quad \Omega_R (H(Z), Z) = 0 \] (203)

while this is not true for the matrix elements of \( H(Z) \). Thus

\[ \Omega_L (<0 | H(Z) | \alpha >, Z) \neq 0 \] (204)

The possibility now arises to define a left or right pseudo-conformal field as a quaternionic Fueter function that transforms like a power of \( \Omega_L \) or \( \Omega_R \). Due to the inhomogeneous term in the transformation that vanishes for Möbius transformation such fields will represent a centrally extended Fueter algebra, the extension being quaternionic in general. The determination
of the exact nature of the extension is still an open problem. The general problem first investigated by Gelfand and Fuchs [9] has been recently discussed by Cardy [10], Fradkin and his collaborators [11], Nair and Schiff [13] and others from various angles (see for example references [14, 15, 16, 17, 18, 19]). Many of the formulas of ADHM construction of solutions to self-dual Yang-Mills are most conveniently written in terms of quaternions [21].

6 Basis of Fueter analytic functions. Relation to harmonic analyticity.

We have seen that the left-right holomorphic quaternionic Fueter series of the form

\[ F(x) = \sum_n a_n x^n, \quad (\text{Vec } a_n = 0) \quad (205) \]

cannot be used in physics if \( x \) is a 4-vector transforming under \( O(4) \) as in Eq.(56) since powers of \( x \) transform like tensor elements, so that \( F \) will not have a definite tensorial property. However functions of \( Z \) or \( U \), where

\[ Z = x p^{-1}, \quad U = p^{-1} x \quad (206) \]

with \( a_n, b_n \) being \( O(4) \) scalars lead to

\[ F_L(Z) = \sum_n a_n Z^n, \quad F_R(U) = \sum_n b_n U^n \quad (207) \]

such that \( \text{Sc } F_L, \text{Sc } F_R \) are \( O(4) \) scalars, \( \text{Vec } F_L \) is self-dual and \( \text{Vec } F_R \) is anti self-dual.

Then the functions \( G_L, G_R \) defined by

\[ G_L = \Box F_L, \quad G_R = \Box F_R \quad (208) \]

are called left-right analytic by Fueter. Under \( O(4) \) we have, using the definition Eq.(127)

\[ G_L \rightarrow m G_L \bar{m}, \quad G_R \rightarrow n G_R \bar{n}, \quad (209) \]

\[ D \rightarrow m D \bar{n}, \quad \bar{D} \rightarrow n \bar{D} \bar{m} \quad (210) \]
Thus we have

$$\bar{D}G_L \rightarrow n\bar{D}G_L\bar{m}, \quad G_L \bar{\bar{D}} \rightarrow mG_L \bar{\bar{D}}\bar{n}$$

(211)

$$DG_R \rightarrow mDG_R\bar{n}, \quad G_R \bar{\bar{D}} \rightarrow nG_R \bar{\bar{D}}\bar{m}$$

(212)

By direct differentiation one finds

$$DG_R = 0, \quad G_L \bar{\bar{D}} = 0$$

(213)

Each term of the series $F_R$ satisfies the analytic equation. Thus

$$Dg_n^R(x) = 0, \quad g_n^R = -\frac{|p|^2}{4} \Box (p^{-1}x)^n$$

(214)

Putting

$$p^{-1}x = u, \quad \frac{u}{|u|} = \xi$$

(215)

we find

$$g_n^R = -\frac{|p|^2}{4} \Box u^n = (\bar{u})^{n-2} \frac{d}{d\xi}(1 + \xi + \ldots + \xi^{n-1})$$

$$= \bar{u}^{n-2} \frac{d}{d\xi} \left( \frac{1 - \xi^n}{1 - \xi} \right)$$

(216)

for $n > 1$. We have

$$g_0 = g_1 = 0$$

(217)

For negative $n$ we have

$$g_{-n}^R = (\bar{u})^{-n-2} \frac{d}{d\xi}(1 + \xi^{-1} + \xi^{-2} + \ldots + \xi^{-n})$$

(218)

We also have the orthogonality property

$$(g_n^L, g_m^R) = \int \bar{g}_n^L d\Sigma g_m^R$$

$$= \int d^4x \left( \bar{g}_n^L Dg_m^R + \bar{g}_n^L \bar{\bar{D}}g_m^R \right) = 0$$

(219)
for $n \neq m$. Here

$$d\Sigma = d\Sigma_0 - i\vec{\sigma} \cdot d\vec{\Sigma} \quad (220)$$

where $d\Sigma_\mu$ is the surface element. The 3-dimensional surface integral can be taken over $S^3$.

In this way a solution $f$ of the massless Dirac equation

$$Df = 0 \quad (221)$$

can be expanded in the basis functions $g^R_n$. Now a solution of the Dirac equation is also a solution of the equation

$$\bar{x}Df = 0 \quad (222)$$

We can write

$$\bar{x}D = x^\mu \partial_\mu - \vec{\sigma} \cdot \vec{L} = \Delta - \vec{\sigma} \cdot \vec{L}, \quad (223)$$

where

$$\vec{L} = -i(\vec{x} \times \vec{\nabla} + \vec{x} \partial_0 - x_0 \vec{\nabla}) \quad (224)$$

is the self dual part of the angular momentum tensor associated with the $SU(2)_L$ subgroup of $O(4)$. The operator $\Delta$ is the dilatation. Hence $f$ satisfies

$$\Delta f(x) = \vec{\sigma} \cdot \vec{L} f(x) \quad (225)$$

If $f^{(j)}(x)$ is homogeneous of degree $j$, we have

$$\Delta f^{(j)}(x)f(x) = j f^{(j)}(x) \quad (226)$$

Iterating and using the commutation relation

$$\vec{L} \times \vec{L} = i \vec{L} \quad (227)$$

we find

$$(\vec{\sigma} \cdot \vec{L})^2 = \vec{L} \cdot \vec{L} - \vec{\sigma} \cdot \vec{L} \quad (228)$$

so that

$$\vec{L} \cdot \vec{L} f^{(j)}(x) = j(j + 1) f^{(j)}(x) \quad (229)$$
Hence \( f^{(j)}(x) \) is proportional to a Wigner \( D_{mm'}^{jj} \) function on \( S^3 \) (or \( SU(2) \)). The formulas (213)-(218) give the relation between the \( S^3 \) functions and the basis functions of Fueter analytic mappings.

To find the relation with the harmonic analyticity of Ogievetsky and his collaborators\(^3\) let us rewrite Eq. (222) in \( 2 \times 2 \) matrix form. We have

\[
x = \begin{pmatrix} \bar{u}_2 & u_1 \\ -\bar{u}_1 & u_2 \end{pmatrix}, \quad \bar{x} = \begin{pmatrix} u_2 & -u_1 \\ \bar{u}_1 & \bar{u}_2 \end{pmatrix}
\]

(230)

where

\[
u_1 = -(x^2 + ix^1), \quad u_2 = x^0 + ix^3
\]

(231)

With these new coordinates \( D \) takes the form

\[
\partial_0 - i \vec{\sigma} \cdot \vec{\nabla} = D = \begin{pmatrix} \partial/\partial u_2 & \partial/\partial \bar{u}_1 \\ -\partial/\partial u_1 & \partial/\partial \bar{u}_2 \end{pmatrix},
\]

(232)

\[
i\bar{x}D = \Delta + \vec{\sigma} \cdot \vec{L} =
\]

\[
i\begin{pmatrix} u_1 \partial/\partial u_1 + u_2 \partial/\partial u_2 & -u_2 \partial/\partial \bar{u}_1 - u_1 \partial/\partial \bar{u}_2 \\ \bar{u}_1 \partial/\partial u_2 - \bar{u}_2 \partial/\partial u_1 & \bar{u}_1 \partial/\partial \bar{u}_1 + \bar{u}_2 \partial/\partial \bar{u}_2 \end{pmatrix}
\]

(233)

We can rewrite this as

\[
i\bar{x}D = i \begin{pmatrix} D^{++} & -D^{+-} \\ D^{--} & D^{-+} \end{pmatrix}
\]

(234)

Let

\[
f = \begin{pmatrix} f_1 & -\bar{f}_2 \\ f_2 & \bar{f}_1 \end{pmatrix}
\]

(235)

We find

\[
D^{++} = L_2 + iL_1
\]

(236)

or if

\[
L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)
\]

(237)
\[ D^{++} = u_1 \partial / \partial \tilde{u}_2 - u_2 \partial / \partial \tilde{u}_1 \]
\[ = L_{20} + L_{31} + i (L_{23} + L_{10}) \]  

(238)

an operator associated with the self-dual angular momentum generating left rotations. The \( U(1) \) subgroup is generated by

\[ L_3 = \frac{i}{2} (D^{+-} - D^{-+}) \]  

(239)

whereas the dilatations giving the degree of homogeneity is given by

\[ \Delta = \frac{i}{2} (D^{+-} + D^{-+}) \]  

(240)

On \( S^2 \) for a given representation \( L_3 \) and \( \Delta \) have given eigenvalues so that the self-dual analyticity equation \[ (222) \] reduces to an equation in \( D^{++} \). This operator and the variable \( x_+ \) defined in Eq.(44) are the cornerstones of harmonic analyticity which involve functions of \( x_+ \) and \( u \), where \( u \) represents a spinor or a unit quaternion. If \( u \) is defined up to a phase it represents a point on \( S^2 \) and functions of \( x_+ \) and \( u \) are equivalent to functions of a twistor or functions of two quaternions. It follows that twistor analyticity, harmonic analyticity and quaternionic Fueter analyticity are all related.

The operator \( \bar{x}D \) represents the dilatation and the left rotation subgroups of the conformal group. Similarly

\[ Dx = x_0 D - i \bar{x} \cdot D \bar{\sigma} \]  

(241)

represents the combination of dilatations and right rotations. \( D \) represents the translations. In a special conformal transformation we have from Eq.(55)

\[ \delta_c x = c_\mu x \bar{e}^\mu x, \quad \delta_c x^\nu = c_\mu \text{Sc}(\bar{e}^\nu x \bar{e}^\mu x) \]  

(242)

where

\[ e^0 = \bar{e}^0 = 1, \quad e^n = -\bar{e}^n = -i\sigma^n \]  

(243)

Hence for a function \( \phi(x) \) we have

\[ \delta_c \phi(x) = \phi(x + \delta_c x) - \phi(x) = c_\mu \text{Sc}(x \bar{e}^\mu x \bar{D}) \phi(x), \]  

(244)
This equation gives for the generators of special conformal transformation the operators

\[ S^\mu = Sc(x\bar{e}^\mu x\bar{D}) \]  \hspace{0.5cm} (245)

which in quaternion form read

\[ S = e_\mu S^\mu = x\bar{D}x = x\bar{e}^\mu x\partial_\mu. \]  \hspace{0.5cm} (246)

Thus, the \(2 \times 2\) quaternionic matrix operator for \(SL(2, Q)\) takes the form

\[ \mathcal{L} = \begin{pmatrix} D\bar{x} & D \\ \bar{x}D\bar{x} & -\bar{x}D \end{pmatrix} \]  \hspace{0.5cm} (247)

7 Various forms of quaternionic differential operators

Consider the quaternion

\[ U = e^\nu \xi_\nu = \xi_0 - i\vec{\sigma} \cdot \vec{\xi}, \quad (\bar{U} = \xi_0 + i\vec{\sigma} \cdot \vec{\xi}) \]  \hspace{0.5cm} (248)

Written as a \(2 \times 2\) matrix it has the form

\[ U = \begin{pmatrix} \xi_0 - i\xi_3 & -i\xi_1 - \xi_2 \\ i\xi_1 + \xi_2 & \xi_0 + i\xi_3 \end{pmatrix} = \rho \ u \]  \hspace{0.5cm} (249)

where

\[ \rho = (U\bar{U})^{\frac{1}{2}} = \sqrt{\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2} = \sqrt{\text{Det} U} \]  \hspace{0.5cm} (250)

and

\[ u\bar{u} = uu^\dagger = 1 \]  \hspace{0.5cm} (251)

so that \(u\) is unitary and has the form

\[ u = \frac{1}{\rho} \begin{pmatrix} \bar{u}_2 & u_1 \\ -\bar{u}_1 & u_2 \end{pmatrix} \]  \hspace{0.5cm} (252)

with

\[ u_1 = - (\xi_2 + i\xi_1), \quad u_2 = \xi_0 + i\xi_3 \]  \hspace{0.5cm} (253)
The unitary matrix \( u \) is an element of \( SU(2) \). Its subgroup \( U(1) \) can be chosen as the diagonal subgroup

\[
h(\psi) = \exp(-i\sigma_3 \psi)
\] (254)

Thus

\[
u = Z(z) h(\psi)
\] (255)

where the complex \( z \) parametrizes the coset \( Z \in SU(2)/U(1) \) which has the form

\[
Z(z) = (1 + z\bar{z})^{-\frac{1}{2}} \begin{pmatrix} 1 & z \\ -\bar{z} & 1 \end{pmatrix}
\] (256)

and represents the sphere \( S^2 = S^3/S^1 \).

Thus, instead of the real 4-coordinate \( \xi_\mu \) we can use the two complex coordinates \( u_1, u_2 \) or the coordinates \( \rho, z \) and \( \psi \) with \( \rho \) real and positive, \( \psi \) real and in the interval \((0, \pi)\) and \( z \) complex.

The relations are given by Eqs. (253) and (250) and

\[
z = \frac{\xi_2 + i\xi_1}{\xi_0 + i\xi_3} = \frac{u_1}{u_2}
\] (257)

\[
e^{2i\psi} = \frac{\xi_0 + i\xi_3}{\xi_0 - i\xi_3} = \frac{u_2}{\bar{u}_2}
\] (258)

or

\[
\psi = \frac{1}{2i} \ln \left( \frac{\xi_0 + i\xi_3}{\xi_0 - i\xi_3} \right)
\] (259)

Also

\[
u_1 = -(\xi_2 + i\xi_1) = \rho \frac{ze^{i\psi}}{\sqrt{1 + z\bar{z}}}
\] (260)

\[
u_2 = \xi_0 + i\xi_3 = \rho \frac{e^{i\psi}}{\sqrt{1 + z\bar{z}}}
\] (261)

\[
\rho^2 = u_1\bar{u}_1 + u_2\bar{u}_2 = \xi_\mu \xi^\mu
\] (262)

We now construct the quaternionic differential operators \( D \) and \( \bar{D} \) given by
\[ D = e_\mu \frac{\partial}{\partial \xi_\mu} = \frac{\partial}{\partial \xi_0} - i \vec{\sigma} \cdot \frac{\partial}{\partial \xi} \]  
\[ (263) \]

\[ D = \frac{\partial}{\partial \xi_0} + i \vec{\sigma} \cdot \frac{\partial}{\partial \xi} \]  
\[ (264) \]

\( D \) has the matrix form
\[ D = \begin{pmatrix} \partial_0 - i \partial_3 & -i \partial_1 - \partial_2 \\ -i \partial_1 + \partial_2 & \partial_0 + i \partial_3 \end{pmatrix} \]  
\[ (265) \]

where
\[ \partial_\mu = \frac{\partial}{\partial \xi_\mu} \]  
\[ (266) \]

In terms of \( u_1, u_2 \) we have
\[ D = 2 \begin{pmatrix} \partial/\partial u_2 & \partial/\partial \bar{u}_1 \\ -\partial/\partial u_1 & \partial/\partial \bar{u}_2 \end{pmatrix} \]  
\[ (267) \]

\[ \bar{D} = 2 \begin{pmatrix} \partial/\partial \bar{u}_2 & -\partial/\partial \bar{u}_1 \\ \partial/\partial u_1 & \partial/\partial u_2 \end{pmatrix} \]  
\[ (268) \]

We also have
\[ \frac{\partial}{\partial \bar{u}_1} = e^{i\psi} \left( \frac{1}{2} \frac{z}{\sqrt{1 + zz}} \frac{\partial}{\partial \rho} + \frac{1}{\rho} \sqrt{1 + zz} \frac{\partial}{\partial \bar{z}} \right) \]  
\[ (269) \]

\[ \frac{\partial}{\partial u_2} = e^{i\psi} \left( \frac{1}{2} \frac{z}{\sqrt{1 + zz}} \frac{\partial}{\partial \rho} - \frac{\bar{z}}{\rho} \sqrt{1 + zz} \frac{\partial}{\partial \bar{z}} \right) - \frac{1}{2i \rho} \sqrt{1 + zz} \frac{\partial}{\partial \psi} \]  
\[ (270) \]

Consider the dilatation operator
\[ \Delta = \xi^\mu \frac{\partial}{\partial \xi^\mu} = \frac{1}{2} \left( u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} + \bar{u}_1 \frac{\partial}{\partial \bar{u}_1} + \bar{u}_2 \frac{\partial}{\partial \bar{u}_2} \right) \]  
\[ (271) \]

and the covariant angular momentum operator
\[ L_{\mu\nu} = i(\xi_\mu \partial_\nu - \xi_\nu \partial_\mu) \]  
\[ (272) \]
Its self dual part is
\[ \ell_{\mu\nu} = \frac{1}{2} (L_{\mu\nu} + \tilde{L}_{\mu\nu}) \] (273)

where
\[ \tilde{L}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} L^{\alpha\beta} \] (274)

Let
\[ \ell_1 = \ell_23 = \ell_{01}, \quad \ell_2 = \ell_{31} = \ell_{02}, \quad \ell_3 = \ell_{12} = \ell_{03} \] (275)

We find
\[ S_c \, \bar{U} \, D = \frac{1}{2} Tr \, \bar{U} \, D = \Delta \] (276)

\[ \text{Vec} \, \bar{U} \, D = \bar{U} \, D - \frac{1}{2} I Tr(\bar{U} \, D) = -i \vec{\sigma} \cdot \vec{\ell} \] (277)

where \( \bar{U} \) is given by Eq. (248). Let
\[ \Omega = \frac{1}{2} \bar{U} \, D = \frac{1}{2} (\Delta - i \vec{\sigma} \cdot \vec{\ell}) = \frac{1}{2} \begin{pmatrix} D^{+-} & -D^{++} \\ D^{--} & D^{-+} \end{pmatrix} \] (278)

We find
\[ D^{+-} = \Delta - i \ell_3 = u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} \] (279)

\[ D^{--} = \Delta + i \ell_3 = \bar{u}_1 \frac{\partial}{\partial \bar{u}_1} + \bar{u}_2 \frac{\partial}{\partial \bar{u}_2} \] (280)

\[ D^{--} = \bar{u}_1 \frac{\partial}{\partial u_2} - \bar{u}_2 \frac{\partial}{\partial u_1} \] (281)

Using Eqs. (260) to (262), (269) and (270) we find
\[ D^{++} = e^{2i\psi}[-(1 + z \bar{z}) \frac{\partial}{\partial \bar{z}} + i \frac{z}{2 \bar{z}} \frac{\partial}{\partial \psi}] \] (282)

\[ D^{--} = \rho \frac{\partial}{\partial \rho} + \frac{i}{\rho} \frac{\partial}{\partial \psi} \] (283)
\[ D^{+-} = \rho \frac{\partial}{\partial \rho} - \frac{i}{2} \frac{\partial}{\partial \psi} = (D^{-+})^* \]  

(284)

\[ D^{-+} = -(D^{++})^* \]  

(285)

or

\[ \Delta = \xi^\mu \partial_\mu = \rho \frac{\partial}{\partial \rho} \]  

(286)

\[ \ell_3 = \frac{i}{2} \frac{\partial}{\partial \psi} \]  

(287)

\[ \ell_1 - i\ell_2 = i e^{2i\psi}[-(1 + z\bar{z}) \frac{\partial}{\partial \bar{z}} + \frac{i}{2} z \frac{\partial}{\partial \psi}] \]  

(288)

Note that \( \Delta \) depends only on \( \rho \) and \( \bar{\ell} \) does not depend on \( \rho \), while \( \ell_3 \) depends only on \( \psi \) as it should be.

8 Factorization of the Energy-Momentum tensor as the analog of the Sommerfield-Sugawara factorization in \( D = 2 \). Generalizations.

In a large class of \( D = 2 \) chiral field theories we are led to the Sommerfield-Sugawara\[22, 23\], form of the traceless energy-momentum tensor

\[ T_{\alpha\beta} = J^i_\alpha (x) J_{\beta i} (x) - \frac{1}{2} \delta_{\alpha\beta} J^i_{\gamma} J_{\gamma i}, \]  

(289)

where \( J^i_\alpha \) and their duals \( \tilde{J}^i_\alpha \) are conserved currents of the theory

\[ \partial_\alpha J^{\alpha i} = 0, \quad \partial_\alpha \tilde{J}^{\alpha i} = \epsilon^{\alpha\beta} \partial_\alpha J^{\beta i} = 0 \]  

(290)

Then

\[ \partial_\alpha T^{\alpha}_\beta = (\partial_\alpha J^{i\alpha}) J_{i\beta} + J^{\alpha i} \partial_\alpha J^i_\beta - J^{\alpha i} \partial_\beta J^i_{\alpha i} = 0 \]  

(291)

It is known that the moments of \( J^{\alpha i} \) generate a Kac-Moody algebra, whereas those of \( T^{\alpha}_\beta \) satisfy a Virasoro algebra\[24\].

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In conformally invariant $D = 4$ theories the energy momentum tensor $T_{\mu\nu}$ is traceless and symmetric. It can be represented by a traceless symmetric $4 \times 4$ matrix $T$ associated with the representation $(1, 1)$ of $O(4)$.

Hence it should be possible to factorize it into $(1, 0)$ self-dual fields $F_{\alpha\beta}^i$ and $(0, 1)$ anti self-dual fields $G_{\alpha\beta}^i$ so that

$$T_{\alpha\beta} = F_{\alpha\beta}^i(x) G_{\beta\alpha}^i(x) \quad (292)$$

Now quaternions have two real $4 \times 4$ matrix representations associated respectively with left and right quaternionic multiplications. Since these two operations commute so do the matrix representations. For purely vectorial quaternions one corresponds to a self-dual antisymmetric matrix, while the other is anti self dual and antisymmetric. They have the forms

$$F = \vec{e}_L \cdot \vec{f} = \begin{pmatrix} 0 & -f_3 & f_2 & f_1 \\ f_3 & 0 & -f_1 & f_2 \\ -f_2 & -f_1 & 0 & f_3 \\ -f_1 & -f_2 & -f_3 & 0 \end{pmatrix} = \vec{e} \cdot \vec{f} \quad (293)$$

and

$$G' = \vec{e}_R \cdot \vec{g} = \begin{pmatrix} 0 & -g_3 & g_2 & -g_1 \\ g_3 & 0 & -g_1 & -g_2 \\ -g_2 & g_1 & 0 & -g_3 \\ g_1 & g_2 & g_3 & 0 \end{pmatrix} = \vec{e}' \cdot \vec{g} \quad (294)$$

and we have

$$[F, G'] = 0, \quad F^2 = -\vec{f} \cdot \vec{f} I, \quad G'^2 = -\vec{g} \cdot \vec{g} I \quad (295)$$

In the case of one internal degree of freedom we have

$$T = F(x) G'(x) = G'(x) F(x) \quad (296)$$

or

$$T = T^{T} =$$

38
\[
\begin{pmatrix}
2f_1g_1 - \vec{f} \cdot \vec{g} & f_1g_2 + f_2g_1 & f_3g_1 + f_1g_3 & -(\vec{f} \times \vec{g})_1 \\
 f_1g_2 + f_2g_1 & 2f_2g_2 - \vec{f} \cdot \vec{g} & f_2g_3 + f_3g_2 & -(\vec{f} \times \vec{g})_2 \\
 f_3g_1 + f_1g_3 & f_2g_3 + f_3g_2 & 2f_3g_3 - \vec{f} \cdot \vec{g} & -(\vec{f} \times \vec{g})_3 \\
-(\vec{f} \times \vec{g})_1 & -(\vec{f} \times \vec{g})_2 & -(\vec{f} \times \vec{g})_3 & \vec{f} \cdot \vec{g}
\end{pmatrix}
\]

(297)

T is symmetric and traceless. On putting
\[
\vec{f} = \frac{1}{2}(\vec{E} + \vec{B}), \quad \vec{g} = \frac{1}{2}(\vec{E} - \vec{B})
\]

(298)
one obtains the familiar form of Maxwell’s tensor in euclidean space-time. When \(\vec{f}\) and \(\vec{g}\) are labeled by an internal symmetry index \(i\) they can be interpreted as the self dual and anti self dual part of an anti-symmetric Yang-Mills field tensor in euclidean space-time. Then
\[
\vec{f}^i = 0, \quad (F^i = 0)
\]

(299)
becomes the self-duality equation.

Note that \(\vec{f}^i\) are not currents but self-dual fields. Currents \(J^i_\mu\) are represented by \(4 \times 4\) quaternions with trace \(4J^i_0\) which transform like vectors under \(O(4)\). In the Maxwell case we have
\[
J = F \vec{D} = D G' = \frac{1}{2}(D G' + F \vec{D})
\]

(300)
The homogeneous set of Maxwell’s equations reads
\[
D G' - F \vec{D} = 0
\]

(301)
The current is given by
\[
J = J_0 + \vec{\sigma}_R \cdot \vec{J}
\]

(302)
and it transforms like the \((\frac{1}{2}, \frac{1}{2})\) representation of \(O(4)\).

The conservation of \(J\) is given by
\[
Sc \vec{D}G' = Sc \Box G' = 0
\]

(303)
since \(G'\) is traceless. In the anti self-dual case \(G' = 0\) so that \(J = 0\) as expected. The same result follows from self duality.

Thus, the decomposition of \(T\) in terms of currents is
\[ T = (D^{-1}J)(\vec{D}^{-1}) \]  

where \( D^{-1} \) is the Dirac Green’s function in the massless case.

When \( J = 0 \) we can obtain \( F \) and \( G' \) as quaternionic analytic functions, since for example

\[ D G' = 0 \]  

Then

\[ G' = \Box \sum_n c_n (p^{-1}x)^n \]  

which is also a solution of

\[ \bar{x} D G' = 0 \]  

associated with harmonic analyticity as we have seen earlier.

The energy momentum tensor \( T \) is not the only tensor that can be constructed from \( F \) and \( G' \). In fact we can obtain a scalar \((0, 0)\), a \( s = 2 \) for \( SU_L(2) \) \([(2, 0) \text{ of } O(4)]\) represented by self-dual Weyl tensor \( W_L \) and \( s = 2 \) for \( SU_R(2) \) \([(0, 2) \text{ of } O(4)]\) represented by an anti self-dual Weyl tensor \( W_R \). The \((2, 0)\) field is represented by a traceless \( 3 \times 3 \) symmetric matrix. It can be combined with the \((0, 0)\) scalar \( \ell \) into a \( 4 \times 4 \) matrix \( L \) while a scalar \( r \) is combined with the \((0, 2)\) into another \( 4 \times 4 \) matrix \( R \). Introducing the diagonal matrix \( \Delta \)

\[ \Delta = \begin{pmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & -2/3 \end{pmatrix} \]  

We find

\[ L = L^T = \begin{pmatrix} W_L & 0 \\ 0 & \ell \end{pmatrix} = F \Delta F = \]

\[ \begin{pmatrix} f_1^2 - \frac{1}{3} \vec{f} \cdot \vec{f} & f_1 f_2 & f_3 f_1 & 0 \\ f_1 f_2 & f_2^2 - \frac{1}{3} \vec{f} \cdot \vec{f} & f_2 f_3 & 0 \\ f_3 f_1 & f_2 f_3 & f_3^2 - \frac{1}{3} \vec{f} \cdot \vec{f} & 0 \\ 0 & 0 & 0 & -\frac{1}{3} \vec{f} \cdot \vec{f} \end{pmatrix} \]
and similarly for $R$

$$R = \begin{pmatrix} W_R & 0 \\ 0 & r \end{pmatrix} = G' \Delta G' \quad (310)$$

We have

$$-3\ell = \vec{f} \cdot \vec{f} = \frac{1}{4} (\vec{E} + \vec{B})^2 = \frac{1}{4} (\vec{E}^2 + \vec{B}^2) + \frac{1}{2} \vec{E} \cdot \vec{B} \quad (311)$$

$$-3r = \vec{g} \cdot \vec{g} = \frac{1}{4} (\vec{E} - \vec{B})^2 = \frac{1}{4} (\vec{E}^2 + \vec{B}^2) - \frac{1}{2} \vec{E} \cdot \vec{B} \quad (312)$$

giving the two invariants of Maxwell’s theory in the euclidean case.

Here we note that the curvature tensor in a Riemannian euclidean tensor can be decomposed into 5 pieces transforming under $O(4)$ as follows:

$$R = (0, 0), \quad \text{scalar curvature} \quad (313)$$

$$R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R = (1, 1), \quad \text{traceless Ricci tensor} \quad (314)$$

$$C_{\alpha\beta\mu\nu}^L = (2, 0), \quad \text{self dual Weyl tensor} \quad (315)$$

$$C_{\alpha\beta\mu\nu}^R = (2, 0), \quad \text{anti self dual Weyl tensor} \quad (316)$$

In the case these 20 functions can be expressed quadratically in terms of the six components $\vec{f}$ and $\vec{g}$ of an antisymmetrical tensor $\phi_{\mu\nu}$, its pieces can be factorized into $F' G', F' \Delta F$ and $G' \Delta G'$. If $G' = 0$ (self dual $\phi$) then the Ricci tensor as well as $W_R$ vanish and we have a half-flat space. All the curvature can be expressed in terms of $F$ which has the quaternion analytic form

$$F(x) = \Box \sum_n c_n (x p^{-1})^n \quad (317)$$

When $F$ is meromorphic we have a gravitational instanton solution. These results generalize to the Yang-Mills case with $F' \Delta F \to F' \Delta F_i$, etc.

This shows that the factorization of the self-dual Weyl tensor is a better analog to the Sommerfield-Sugawara factorization of $T$ in $D = 2$, since both
factors $F$ can be analytic, whereas $T$ in $D = 4$ requires one analytic factor $(1, 0)$ and one anti analytic factor $(0, 1)$.

For an operator product expansion of $L$ for example we would have

$$L(x)L(x') = \begin{pmatrix} W_L(x) & W_L(x') \\ 0 & \ell(x) \ell(x') \end{pmatrix}$$

(318)

$W_L(x) W_L(x')$ is a $3 \times 3$ matrix that decomposes into a multiple of unity ($s = 0$), an antisymmetrical matrix ($s = 1$) and a symmetric traceless matrix. Thus we can write

$$W_L(x) W_L(x') = K(x - x') I + u(x - x') F(x) + v(x - x') W_L(x)$$

(319)

with singular $c$-number coefficients that can be determined from dimensional arguments. The product $T(x) T(x')$ is more complicated and bears less resemblance to the $D = 2$ case. We conclude that the analog of $T(z)$ in $D = 2$ is $W_L(x)$ which has the simplest operator product expansion.

To test its true usefulness and power, this quaternionic formalism must still await further progress in the representation theory of $D = 4$ diffeomorphism and current algebras and the discovery of $D = 4$ counterpart of the Wess-Zumino-Novikov-Witten model.

Understanding of geometry of supersymmetry through harmonic analyticity and harmonic superspace led into wide variety of applications in physics and geometry$^{26}$. It has potentially interesting application to some on-shell supermultiplets which arise naturally in string theory in the context of Maldacena conjecture$^{27}$ (which relates supergravity theories on anti-de Sitter backgrounds to conformal field theories on the boundary, i.e. super-Minkowski space).

The search for exceptional structures associated with eleven dimensions has lead us into description of space-time by means of exceptional groups$^{28}$. A quest toward a unified theory has taken us through a progression from particle world lines of quantum field theory ($\mathcal{R}$ analyticity) to string worldsheet in order to incorporate gravity ($\mathcal{C}$ analyticity) into $M$-brane worldvolumes in order to realize duality ($\mathcal{H}$ analyticity). Next step would be the search for octonionic analyticity ($\Omega$ analyticity). $M$-theory might turn out to be supersymmetric in eleven dimensions.
Octonionic planes may possibly provide the geometrical foundation for the existence of internal symmetries like color and flavor. These octonionic geometries allow us to construct new finite Hilbert spaces with unique properties. The non-Desargues’ian geometric property makes them non-embeddeble in higher spaces, hence essentially finite. It also leads to peculiarities in the superposition principle in the color sector of the Hilbert space. This theory of the charge space, if correct, suggests a new geometric picture for the substructure of the material world, that of an octonionic geometry attached at each point of Einstein’s Riemannian manifold for space time with local symmetry that leaves the properties of charge space invariant. Extension of the formalism we presented to octonionic structures may help to understand geometry, duality and non perturbative physics in a deeper way.

Four dimensional conformal field theory can also be formulated on Kulkarni 4-folds leading to a formalism that parallels that of 2D conformal field theory on Riemann surfaces. In this framework the notion of Fueter analyticity, the quaternionic analogue of complex analyticity plays an essential role. For other approaches where symmetry of some field equations were investigated, see for example paper by Kruglov.
Appendix: Use of Coherent States in $SU(2)$

Let

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad a^\dagger = \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix}$$

and

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad \xi^\dagger = \begin{pmatrix} \xi_1^\dagger \\ \xi_2^\dagger \end{pmatrix}$$

with

$$[a_i, a_j^\dagger] = \delta_{ij}$$

and defining the vacuum state $c = |0\rangle$, we have

$$a |0\rangle = 0, \quad <0| a^\dagger = 0$$

We can construct a superposition state (coherent state) out of the vacuum state

$$|\xi\rangle = e^{a^\dagger \xi} |0\rangle, \quad <\xi| = <0| e^{\xi^\dagger a}$$

that has the following properties:

(1.)

$$<0| \xi >= <\xi| 0 >= <0| 0 >= 1$$

(2.)

$$|\xi\rangle = e^{a^\dagger \xi_1} e^{a^\dagger \xi_2} |0\rangle$$

$$= \sum_{j=0}^{\infty} \sum_{m=-j}^{j} a_{1}^{j+m} \frac{\xi_{1}^{j+m}}{\sqrt{(j+m)!}} a_{2}^{j-m} \frac{\xi_{2}^{j-m}}{\sqrt{(j-m)!}}$$

or

$$|\xi\rangle = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \frac{\xi_{1}^{j+m}}{\sqrt{(j+m)!}} \frac{\xi_{2}^{j-m}}{\sqrt{(j-m)!}} |j, m\rangle$$
\[
< \xi | = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} j, m | \frac{\xi_1^{* j+m}}{\sqrt{(j+m)!}} \frac{\xi_2^{* j-m}}{\sqrt{(j-m)!}} (329)
\]

(3.)

\[
e^{-a^\dagger \xi} a_1 e^{a^\dagger \xi} = e^{-a_1^\dagger \xi_1} a_1 e^{a_1^\dagger \xi_1} = e^{\xi_1^\dagger \xi_1 \xi_1} a_1 = a_1 + \xi_1
\]

(330)

\[
e^{-a^\dagger \xi} a_2 e^{a^\dagger \xi} = a_2 + \xi_2
\]

(331)

so that

\[
e^{-a^\dagger \xi} a_i e^{a^\dagger \xi} = a_i + \xi_i
\]

(332)

(4.)

\[
a_i \ | \xi > = \xi_i | \xi >
\]

so that \(| \xi >\) is an eigenstate of \(a_i\). We see that

\[
a_i \ | \xi > = a_i e^{a^\dagger \xi} | 0 > = a^{a^\dagger \xi} e^{-a^\dagger \xi} a_i e^{a^\dagger \xi} | 0 > = e^{a^\dagger \xi}(a_i + \xi_i) | \xi >
\]

(334)

and similarly

\[
< \xi | a_i^\dagger =< \xi | \xi^*
\]

(335)

(5.)

\[
< \xi | j, m > = \sum_{j'} \sum_{m'} \frac{\xi_1^{* j'+m'}}{\sqrt{(j'+m')!}} \frac{\xi_2^{* j'-m'}}{\sqrt{(j'-m')!}} < j', m' | j, m >
\]

\[
= \sum_{j'} \sum_{m'} \frac{\xi_1^{* j'+m'}}{\sqrt{(j'+m')!}} \frac{\xi_2^{* j'-m'}}{\sqrt{(j'-m')!}} \delta_{j'j} \delta_{m'm} (336)
\]

or

\[
< \xi | j, m > = \frac{\xi_1^{* j+m}}{\sqrt{(j+m)!}} \frac{\xi_2^{* j-m}}{\sqrt{(j-m)!}} (337)
\]
is homogeneous in $\xi_1, \xi_2$ of degree $2j$ so that

$$< \lambda \xi \mid j, m > = \lambda^{2j} < \xi \mid j, m >$$ (338)

and

$$< j, m \mid \lambda \xi > = \lambda^{2j} < j, m \mid \xi >$$ (339)

(6.)

$$\frac{1}{\sqrt{(j+m)!}} \frac{1}{\sqrt{(j-m)!}} \left( \frac{\partial}{\partial \xi_1^*} \right)^{j+m} \left( \frac{\partial}{\partial \xi_2^*} \right)^{j-m} < \xi \mid j, m > =$$

$$= \frac{1}{(j+m)!} \frac{1}{(j-m)!} (\xi_1^{*j+m}) (\xi_2^{*j-m}) = 1$$ (340)

(7.)

$$a_i^\dagger \mid \xi > = a_i^\dagger e^{a_i^\dagger \xi_1 + a_i^\dagger \xi_2} \mid 0 > = \frac{\partial}{\partial \xi_i} e^{a_i^\dagger \xi} \mid 0 > = \frac{\partial}{\partial \xi_i} \mid \xi >$$ (341)

Also

$$< \xi \mid a_i = \frac{\partial}{\partial \xi_i} < \xi \mid$$ (342)

so that on $\mid \xi >$ we have $a_i \leftrightarrow \xi_i, a_i^\dagger \leftrightarrow \frac{\partial}{\partial \xi_i}$.

(8.)

$$< \xi \mid a_1^{j+m} a_2^{j-m} \frac{1}{\sqrt{(j+m)!}} \frac{1}{\sqrt{(j-m)!}} \mid j, m > = 1$$

$$=< j, m \mid j, m > = < 0 \mid a_1^{j+m} a_2^{j-m} \frac{1}{\sqrt{(j+m)!}} \frac{1}{\sqrt{(j-m)!}} \mid j, m >$$ (343)
The D functions of $SU(2)$

Since

$$J_i = a^\dagger \frac{\sigma_i}{2} a$$

and the relations

$$J_1 + iJ_2 = a_1^\dagger a_2, \quad J_1 - iJ_2 = a_2^\dagger a_1, \quad J_3 = \frac{1}{2} (a_1^\dagger a_1 - a_2^\dagger a_2)$$

we have

$$e^{iJ_3^\alpha} a^\dagger e^{-iJ_3^\alpha} = a^\dagger e^{i\sigma_3^\frac{\alpha}{2}}$$

and

$$e^{iJ_2^\beta} a^\dagger e^{-iJ_2^\beta} = a^\dagger e^{i\sigma_2^\frac{\beta}{2}}.$$ 

Now let

$$U(\omega) | j, m > = e^{iJ_3^\alpha} e^{iJ_2^\beta} e^{iJ_3^\gamma} | j, m >$$

$$= U \frac{a_1^{ij+m}}{\sqrt{(j+m)!}} \frac{a_2^{ij-m}}{\sqrt{(j-m)!}} U^{-1} U | 0 >$$

with

$$U | 0 > = | 0 >, \quad U a_1^\dagger U^{-1} = a_1^\dagger, \quad U a_2^\dagger U^{-1} = a_2^\dagger.$$ 

where

$$a^\dagger = (a_1^\dagger \quad a_2^\dagger)$$
so that
\[
\begin{align*}
a^\dagger U^{-1} a U = & \ e^{iJ_3\alpha} e^{iJ_2\beta} e^{iJ_2\gamma} a^\dagger e^{-iJ_3\gamma} e^{-iJ_2\beta} e^{-iJ_3\alpha} \\
= & \ a^\dagger e^{i\frac{\sigma_3}{2}} e^{i\frac{\sigma_2}{2}} e^{i\frac{\sigma_1}{2}} \\
= & \ a^\dagger \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_1^* \end{pmatrix} \\
= & \ (a^\dagger \psi \ a^\dagger \hat{\psi}) = \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix}
\end{align*}
\] (353)

where
\[
\psi_1 = e^{i(\alpha+\gamma)/2} \cos \left( \frac{\beta}{2} \right), \quad \psi_2 = -e^{i(\gamma-\alpha)/2} \sin \left( \frac{\beta}{2} \right)
\] (354)

so that \( U(\omega) = U(\psi) \), and \( \psi^\dagger \psi = 1 \).

Also
\[
U(\omega) | j, m' > = \frac{a_1^\dagger j+m'}{\sqrt{(j+m')!}} \frac{a_2^\dagger j-m'}{\sqrt{(j-m')!}} | 0 > \\
= \frac{(a^\dagger \psi)^{j+m'} (a^\dagger \hat{\psi})^{j-m'}}{\sqrt{(j+m')!} \sqrt{(j-m')!}} | 0 >
\] (355)

Now we can write
\[
D^j_{mm'}(\omega) = D^j_{mm'}(\psi) = \langle j, m | U(\omega) | j, m' > \\
= \frac{(a^\dagger \psi)^j m'}{\sqrt{(j+m')!}} \frac{(a^\dagger \hat{\psi})^j m'}{\sqrt{(j-m')!}} | 0 >
\]

Now to get the expression for \( D^j_{mm'}(\psi) \), consider
\[
\langle \xi | U(\psi) | j, m' > = \langle \xi \ | \frac{(a^\dagger \psi)^j m'}{\sqrt{(j+m')!}} \frac{(a^\dagger \hat{\psi})^j m'}{\sqrt{(j-m')!}} | 0 >
\] (357)

Using
\[ <\xi| a^\dagger = <\xi| \xi^\dagger \] (358)

and \( <\xi| 0 > = 1 \) we get

\[ <\xi| U(\psi)| j, m' > = \frac{(\xi^\dagger \psi)^{j+m'}}{\sqrt{(j+m')!}} \frac{(\xi^\dagger \hat{\psi})^{j-m'}}{\sqrt{(j-m')!}} \] (359)

which is homogeneous of degree \( 2j \) in \( \xi_1^*, \xi_2^* \). Hence the expression

\[
\frac{1}{\sqrt{(j+m')!}} \frac{1}{\sqrt{(j-m')!}} (\frac{\partial}{\partial \xi_1})^{j+m} (\frac{\partial}{\partial \xi_2})^{j-m} \frac{(\xi^\dagger \psi)^{j+m'}}{\sqrt{(j+m')!}} \frac{(\xi^\dagger \hat{\psi})^{j-m'}}{\sqrt{(j-m')!}}
\] (360)

does not depend on \( \xi \). Thus

\[
= <\xi| \frac{a_1^{j+m}}{\sqrt{(j+m)!}} \frac{a_2^{j-m}}{\sqrt{(j-m)!}} U(\psi)| j, m' >
\] (361)

does not depend on \( \xi \). Hence we have

\[
<\xi| \frac{a_1^{j+m}}{\sqrt{(j+m)!}} \frac{a_2^{j-m}}{\sqrt{(j-m)!}} U(\psi)| j, m' > = <0| \frac{a_1^{j+m}}{\sqrt{(j+m)!}} \frac{a_2^{j-m}}{\sqrt{(j-m)!}} U(\psi)| j, m' > = D^j_{mm'}(\psi)
\] (362)

Thus

\[
D^j_{mm'}(\psi) = \frac{1}{\sqrt{(j+m')!}} \frac{1}{\sqrt{(j-m')!}} (\frac{\partial}{\partial \xi_1})^{j+m} (\frac{\partial}{\partial \xi_2})^{j-m} <\xi| U(\psi)| j, m' >
\] (363)

Using
\[ x \leq \xi \mid U(\psi) \mid j, m' = \frac{(\xi^\dagger \psi)^{j+m'}}{\sqrt{(j+m')!}} \frac{(\xi^\dagger \hat{\psi})^{j-m'}}{\sqrt{(j-m')!}} \]  

(364)

we finally get

\[
D_{mm'}^{j}(\psi) = \frac{1}{\sqrt{(j+m')!}} \frac{1}{\sqrt{(j-m')!}} (\frac{\partial}{\partial \xi_1^*})^{j+m} (\frac{\partial}{\partial \xi_2^*})^{j-m} \times \\
\left\{ \frac{(\xi^\dagger \psi)^{j+m'}}{\sqrt{(j+m')!}} \right\} \left\{ \frac{(\xi^\dagger \hat{\psi})^{j-m'}}{\sqrt{(j-m')!}} \right\} 
\]

(365)

We note that \( m \) numbers rows, \( m' \) numbers columns. We have

\[
D_{\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2} = \frac{\partial}{\partial \xi_1^*} (\xi_1^* \psi_1 + \xi_2^* \psi_2) = \psi_1 = e^{i(\alpha+\gamma)/2} \cos(\beta/2) 
\]

(366)

\[
D_{\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2} - \frac{1}{2} = \frac{\partial}{\partial \xi_2^*} (-\xi_1^* \psi_1 + \xi_2^* \psi_2^*) = -\psi_2^* 
\]

(367)

\[
D_{\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2} = \frac{\partial}{\partial \xi_2^*} (\xi_1^* \psi_1 + \xi_2^* \psi_2) = \psi_2 = -e^{i(\gamma-\alpha)/2} \sin(\beta/2) 
\]

(368)

\[
D_{\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2} - \frac{1}{2} = \frac{\partial}{\partial \xi_2^*} (-\xi_1^* \psi_2^* + \xi_2^* \psi_1^*) = \psi_1^* 
\]

(369)

so that

\[
D_{\frac{1}{2}}^{\frac{1}{2}}(\psi) = \left( \begin{array}{cc}
D_{\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2} & D_{\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2} - \frac{1}{2} \\
D_{\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2} - \frac{1}{2} & D_{\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2}
\end{array} \right) = \left( \begin{array}{cc}
\psi_1 & -\psi_2^* \\
\psi_2 & \psi_1^*
\end{array} \right) = \\
= e^{i\sigma_3 \frac{\alpha}{2}} e^{i\sigma_2 \frac{\beta}{2}} e^{i\sigma_3 \frac{\gamma}{2}} = \left( \begin{array}{c}
e^{\frac{\alpha+\gamma}{2}} \cos \frac{\beta}{2} \quad e^{\frac{-\gamma-\alpha}{2}} \sin \frac{\beta}{2} \\
-e^{\frac{\alpha+\gamma}{2}} \sin \frac{\beta}{2} \quad e^{\frac{-\gamma-\alpha}{2}} \cos \frac{\beta}{2}
\end{array} \right) 
\]

(370)

Similarly

\[
D_{1}(\psi) = \left( \begin{array}{cc}
\psi_1^2 & -\sqrt{2} \psi_1 \psi_2^* \\
\psi_2^2 & \sqrt{2} \psi_1 \psi_2^*
\end{array} \right) = e^{i\Sigma_3 \alpha} e^{i\Sigma_2 \beta} e^{i\Sigma_3 \gamma} 
\]

(371)
where $\Sigma_i$ are the $s = 1$ (isotopic spin) matrices. They are

$$
\Sigma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Sigma_2 = \frac{1}{2i} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix}
$$

$$
\Sigma_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}
$$

(372)

and we have

$$
\Sigma_+ = \Sigma_1 + i\Sigma_2 = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
\Sigma_- = \Sigma_1 - i\Sigma_2 = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}
$$

(373)

These satisfy $[\Sigma_1, \Sigma_2] = i\Sigma_3$ and cyclic permutations. Now using angles $\phi = -\alpha, \theta = -\beta$ and $\psi = -\gamma$ we can write

$$
D^\frac{1}{2}(\psi) = \begin{pmatrix} \psi_1 & -\psi_2^* \\ \psi_2 & \psi_1^* \end{pmatrix} = e^{-i\sigma_3 \frac{\phi}{2}} e^{-i\sigma_2 \frac{\theta}{2}} e^{-i\sigma_3 \frac{\psi}{2}}
$$

$$
= \begin{pmatrix} e^{-\frac{i}{2}(\phi+\psi)} \cos \frac{\theta}{2} & -e^{-\frac{i}{2}(\phi-\psi)} \sin \frac{\beta}{2} \\ e^{\frac{i}{2}(\phi-\psi)} \sin \frac{\theta}{2} & e^{\frac{i}{2}(\phi+\psi)} \cos \frac{\theta}{2} \end{pmatrix}
$$

(374)

so that

$$
\psi_1 = \cos \frac{\theta}{2} e^{-i\frac{\phi+\psi}{2}}
$$

(375)

and

$$
\psi_2 = \sin \frac{\theta}{2} e^{i\frac{\phi-\psi}{2}}
$$

(376)

Using these we can now write $D^1$ as
\[
D^1(\psi) = \begin{pmatrix}
\frac{1 + \cos \theta}{2} e^{-i(\phi + \psi)} & -\frac{1}{\sqrt{2}} \sin \theta e^{-i \phi} & \frac{1 - \cos \theta}{2} e^{-i(\phi - \psi)} \\
\frac{1}{\sqrt{2}} \sin \theta e^{-i \phi} & \cos \theta & -\frac{1}{\sqrt{2}} \sin \theta e^{i \phi} \\
\frac{1 - \cos \theta}{2} e^{i(\phi + \psi)} & \frac{1}{\sqrt{2}} \sin \theta e^{i \phi} & \frac{1 + \cos \theta}{2} e^{i(\phi - \psi)}
\end{pmatrix}
\] (377)

We have shown that we have formed \( D \) functions with \( \psi_j^+ \psi_j^- \) in first columns.

**Euler Angles and \( SU(2)/U(1) \) Coset Decomposition**

A group element for \( SU(2) \) can be taken as

\[
g_\omega = A(\psi) = D^\frac{1}{2}(\psi)
\] (378)

where

\[
\psi = \begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
\] (379)

with \( \psi^\dagger \psi = 1 \), or

\[
\psi_1^* \psi_1 + \psi_2^* \psi_2 = 1
\] (380)

We can rewrite Eq. (374) as

\[
D^\frac{1}{2} = \begin{pmatrix}
\psi_1 & -\psi_2^*
\psi_2 & \psi_1^*
\end{pmatrix} = e^{-i \frac{\phi_0}{2}} \phi e^{-i \frac{\phi_0}{2}} \theta e^{-i \frac{\phi_0}{2}} \psi
\]

\[
= \begin{pmatrix}
e^{-i \phi_0} & 0 \\
0 & e^{i \phi_0}
\end{pmatrix} \begin{pmatrix}
\cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\
\sin \frac{\theta}{2} & \cos \frac{\theta}{2}
\end{pmatrix} \begin{pmatrix}
e^{-i \phi_0} & 0 \\
0 & e^{i \phi_0}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\cos \frac{\theta}{2} e^{-i \phi_0} & -\sin \frac{\theta}{2} e^{-i \phi_0} \\
\sin \frac{\theta}{2} e^{i \phi_0} & \cos \frac{\theta}{2} e^{i \phi_0}
\end{pmatrix}
\] (381)

Let

\[
z = \frac{\psi_2}{\psi_1} = \tan \frac{\theta}{2} e^{i \phi}
\] (382)

and

52
\[ \frac{\psi_1}{|\psi_1|} = e^{-i\frac{(\phi + \psi)}{2}} = e^{-i\frac{\eta}{2}} \]  

(383)

where \( z \) and \( \eta \) are the parameters of the coset decomposition

\[ SU(2) = (SU(2)/U(1)) \times U(1) . \]  

(384)

We can write \( A = D^\frac{1}{2} \) as \( A = BC \), where \( C \in U(1) \) and \( U(1) \) is the \( U(1) \) subgroup of \( SU(2) \) (locally isomorphic to \( SO(2) \) subgroup of \( SO(3) \)). Now

\[ D^\frac{1}{2} = \begin{pmatrix} \psi_1 & -\psi_2^* \\ \psi_2 & \psi_1^* \end{pmatrix} \]

\[ = \begin{pmatrix} \psi_1 & -\psi_2^* \\ \psi_2 & \psi_1^* \end{pmatrix} \begin{pmatrix} |\psi_1| & 0 \\ 0 & |\psi_1| \end{pmatrix} \begin{pmatrix} |\psi_1| & 0 \\ 0 & |\psi_1| \end{pmatrix} \]

\[ = \begin{pmatrix} |\psi_1| & -\frac{(\psi_2/\psi_1)^*}{|\psi_1|} \\ \frac{(\psi_2/\psi_1)^*}{|\psi_1|} & |\psi_1| \end{pmatrix} \begin{pmatrix} e^{-i\frac{\eta}{2}} & 0 \\ 0 & e^{i\frac{\eta}{2}} \end{pmatrix} \]

(385)

Since

\[ |\psi_1|^2 + |\psi_2|^2 = |\psi_1|^2 (1 + \frac{\psi_2}{\psi_1})^2 = 1 \]  

(386)

we have

\[ |\psi_1| = \frac{1}{\sqrt{1 + |z|^2}} \]  

(387)

Thus

\[ D^\frac{1}{2} = \frac{1}{\sqrt{1 + |z|^2}} \begin{pmatrix} 1 & -z^* \\ z & 1 \end{pmatrix} \begin{pmatrix} e^{-i\frac{\eta}{2}} & 0 \\ 0 & e^{i\frac{\eta}{2}} \end{pmatrix} \]  

(388)

Now

\[ B = \begin{pmatrix} 1 & -\frac{(\psi_2/\psi_1)^*}{|\psi_1|} \\ \frac{(\psi_2/\psi_1)^*}{|\psi_1|} & 1 \end{pmatrix} = \frac{1}{\sqrt{1 + |z|^2}} \begin{pmatrix} 1 & -z^* \\ z & 1 \end{pmatrix} \]  

(389)

labels cosets \( SU(2)/U(1) \). Therefore
\[ A = \frac{1}{\sqrt{1+|z|^2}} \begin{pmatrix} 1 & -z^* \\ z & 1 \end{pmatrix} \begin{pmatrix} |\psi_1|_\psi_1^* \\ 0 \\ 0 \end{pmatrix} \]  

with \( \psi_1 = z \psi_2 \) and \( z \) is the coset label for \( SU(2)/U(1) \). Under a rotation

\[ \Omega = \begin{pmatrix} \omega_1 & \omega_2 \\ -\omega_2^* & \omega_1^* \end{pmatrix} \]  

with

\[ |\omega_1|^2 + |\omega_2|^2 = 1 \]

we have

\[ A' = \Omega A = \begin{pmatrix} \psi_1' & -\psi_2' \\ \psi_2' & \psi_1' \end{pmatrix} \]  

or

\[ \begin{pmatrix} \psi_1' \\ \psi_2' \end{pmatrix} = \begin{pmatrix} \omega_1 & -\omega_2^* \\ \omega_2 & \omega_1^* \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \omega_1\psi_1 - \omega_2^*\psi_2 \\ \omega_2\psi_1 + \omega_1\psi_2 \end{pmatrix} \]

We can write

\[ A' = \frac{1}{\sqrt{1+|z'|^2}} \begin{pmatrix} 1 & -z'^* \\ z' & 1 \end{pmatrix} \begin{pmatrix} |\psi_1'|_\psi_1^* \\ 0 \\ 0 \end{pmatrix} \]

with \( \psi_2' = z'\psi_1' \).

Thus, for the coset label \( z \) we have the transformation law

\[ T_{\Omega z} = z' = \frac{\psi_2'}{\psi_1'} = \frac{\omega_2\psi_1 + \omega_1^*\psi_2}{\omega_1\psi_1 - \omega_2^*\psi_2} = \frac{\omega_2 + \omega_1^*z}{\omega_1 - \omega_2^*z} \]

which is a Möbius transformation determined by \( \Omega \).

We also have

\[ \psi_1' = (\omega_1 - \omega_2^*z)\psi_1 \]

Thus far we have shown \( A = A(z, \psi_1) \) and under \( SU(2) \)

\[ A \rightarrow T_{\Omega}A = A(T_{\Omega}z, T_{\Omega}\psi_1) \]
\[ \Omega A = A \frac{\omega_2 + \omega_1 z}{\omega_1 - \omega_2^* z} (\omega_1 - \omega_2^* z) \psi_1 \]  

(399)

where \( z \) is the label for \( SU(2)/U(1) \) and \( \frac{\psi_1}{|\psi_1|} = e^{-i\eta/2} \) where \( \eta \) is the parameter for the \( U(1) \) subgroup.

Using Eqs. (381) and (383) we arrive at

\[ A = BC = e^{-i\frac{\sigma_3}{2} \phi} e^{-i\frac{\sigma_2}{2} \theta} e^{-i\frac{\sigma_3}{2} \phi} e^{-i\frac{\sigma_3}{2} \eta} \]  

(400)

where

\[ B = e^{-i\frac{\sigma_3}{2} \phi} e^{-i\frac{\sigma_2}{2} \theta} e^{-i\frac{\sigma_3}{2} \phi} \]  

(401)

has the form

\[
B = \begin{pmatrix}
e^{-i\frac{\phi}{2}} & 0 \\
0 & e^{i\frac{\phi}{2}}
\end{pmatrix}
\begin{pmatrix}
\cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\
\sin \frac{\theta}{2} & \cos \frac{\theta}{2}
\end{pmatrix}
\begin{pmatrix}
e^{i\frac{\phi}{2}} & 0 \\
0 & e^{-i\frac{\phi}{2}}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\cos \frac{\theta}{2} & -e^{-i\phi} \sin \frac{\theta}{2} \\
e^{i\phi} \sin \frac{\theta}{2} & \cos \frac{\theta}{2}
\end{pmatrix}
= \cos \frac{\theta}{2}
\begin{pmatrix}
1 & -e^{-i\phi} \tan \frac{\theta}{2} \\
e^{i\phi} \tan \frac{\theta}{2} & 1
\end{pmatrix}
\]  

(402)

If we put

\[ z = -e^{-i\phi} \tan \frac{\theta}{2} \]  

(403)

then

\[ \cos \frac{\theta}{2} = \frac{1}{\sqrt{1 + |z|^2}} \]  

(404)

On the other hand, if

\[ \frac{\psi_1}{|\psi_1|} = e^{i\frac{\phi}{2}(\psi + \phi)} \]  

(405)

we have

\[ C = e^{-i\frac{\sigma_3}{2}(\psi + \phi)} = \begin{pmatrix}
\frac{|\psi_1|}{\psi_1} & 0 \\
0 & \frac{|\psi_1|}{\psi_1}
\end{pmatrix} \]  

(406)
Thus the formulas Eqs. (403) and (405) give the relation between Euler angles and the coset decomposition with respect to $U(1)$. Note that $z$ involves only the two Euler angles $\phi$ and $\theta$.

For $j = 1$ we find

\[
D^{(1)}(\psi) = \left( \begin{array}{ccc}
\sqrt{2}\psi_1^2 & -\sqrt{2}\psi_1\psi_2^* & -\sqrt{2}\psi_1^*\psi_2 \\
\psi_2^2 & |\psi_1|^2 - |\psi_2|^2 & -\sqrt{2}\psi_1^*\psi_2 \\
|\psi_1|^2 & 0 & 0
\end{array} \right) \times \\
\left( \begin{array}{ccc}
|\psi_1|^2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & |\psi_1|^2
\end{array} \right) \left( \begin{array}{ccc}
\psi_2 & -\sqrt{2}\psi_1^* & 0 \\
1 - |\psi_1\psi_2|^2 & -\sqrt{2}\psi_1^* & 0 \\
|\psi_1\psi_2|^2 & \sqrt{2} & 1
\end{array} \right) \left( \begin{array}{ccc}
e^{-i\eta} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{i\eta} \end{array} \right)
\]

or

\[
D^{(1)} = \frac{1}{1 + |z|^2} \left( \begin{array}{ccc}
1 & -\sqrt{2}z^* & z^2 \\
\sqrt{2}z & 1 - z^2 & -\sqrt{2}z^* \\
z^2 & \sqrt{2} & 1
\end{array} \right) \left( \begin{array}{ccc}
e^{-i\eta} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{i\eta} \end{array} \right)
\]

In general the elements of

\[
(1 + |z|^2)^{2j} D^{(j)} e^{i\eta} = Z
\]

are polynomials in $z$ and $z^*$. The first column is analytic and the last column is antianalytic.

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