Gauge-invariant quark and gluon fields in QCD: dynamics, topology, and the Gribov ambiguity

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We review the implementation, in a temporal-gauge formulation of QCD, of the non-Abelian Gauss’s law and the construction of gauge-invariant gauge and matter fields. We then express the QCD Hamiltonian in terms of these gauge-invariant operator-valued fields, and discuss the relation of this Hamiltonian and the gauge-invariant fields to the corresponding quantities in a Coulomb gauge formulation of QCD. We argue that a representation of QCD in terms of gauge-invariant quantities could be particularly useful for understanding low-energy phenomenology. We present the results of an investigation into the topological properties of the gauge-invariant fields, and show that there are Gribov copies of these gauge-invariant gauge fields, which are constructed in the temporal gauge, even though the conditions that give rise to Gribov copies do not obtain for the gauge-dependent temporal-gauge fields.

1. INTRODUCTION

I will review here the construction of gauge-invariant non-Abelian gauge and matter fields and the use of these fields for a discussion of QCD dynamics and of Gribov copies of gauge fields from a somewhat novel perspective. To illustrate one reason for our interest in formulating QCD in terms of gauge-invariant fields, it is helpful to first focus attention on QED. When we formulate QED in one of a number of gauges — for example, the Lorentz gauge or the temporal (Weyl) gauge — and transform to a representation in which the charged matter field and the gauge field are gauge-invariant (the former having been obtained by use of a transformation due to Dirac, the latter being just the transverse part of the gauge field) we obtain the following Hamiltonian:

$$\hat{H}_{QED} = \int d^3r \left[ \frac{1}{2} \Pi_i(r) \Pi_i(r) + \frac{1}{4} F_{ij}(r) F_{ij}(r) \right. $$

$$+ \psi^\dagger(r) \left( \beta m - i\alpha_i \partial_i \right) \psi(r) - A_i^{(T)}(r) j_i(r) \left. \right] + \int d^3r r ' j_0(r) j_0(r ') \frac{1}{8\pi |r - r'|} + H_g. $$

We can recognize this transformed Hamiltonian as the sum of the Coulomb-gauge Hamiltonian and \(H_g\), which is gauge-dependent. For the temporal gauge, \(H_g\) is given by

$$H_g = -\frac{1}{2} \int d^3r \left( \partial_i \Pi_i(r) \frac{1}{\partial^2} j_0(r) + j_0(r) \frac{1}{\partial^2} \partial_i \Pi_i(r) \right)$$

where \(\Pi_i(r)\) is the negative electric field as well as the momentum conjugate to \(A_i(r)\), where \(j_0 = e\psi^\dagger \psi\), and \(\psi\) is the gauge-invariant charged-matter field in this transformed representation. \(\partial_i \Pi_i \approx 0\) is the form that Gauss’s law takes in the transformed representation, with the charge density \(j_0\) included but not appearing explicitly because a unitary transformation very much like the one introduced in Ref. has folded it into \(\partial_i \Pi_i\), which we therefore refer to as the Abelian “Gauss’s law operator” in the transformed representation. The \(\approx\) indicates that the equality is “soft” — i.e. that it is true only on a suitably defined constraint surface, or only when applied to a set of appropriately fashioned state vectors. As was shown in Refs. for a variety of gauges, \(H_g\) plays no role whatsoever in the time-evolution of state vectors, and therefore does not affect any of the physical results obtained from the application of \(\hat{H}_{QED}\).

In the history of electrodynamics, the macro-
scopic long-range forces that dominate the classical phenomenology were very familiar long before photon-electron scattering became an important concern. But let us imagine a fictitious scenario in which photon-electron scattering phenomenology was our first experience with electrodynamics, and that we knew the Lagrangian of covariant-gauge QED and Feynman rules long before we knew about electrodynamics. If, at that point, someone had expressed that theory in terms of gauge-invariant “physical” variables, and had obtained the Hamiltonian in Eq. (3), it would have become apparent that this Hamiltonian was not very appropriate for generating a renormalizable S-matrix. But it would also have become clear that, as a theory for low-energy phenomenology such as electrostatics, it was superior to formulations that used gauge-dependent fields; that, in fact, the Coulomb interaction suffices for understanding the energy levels and wave functions of almost all atoms, and that it is very useful in the classical domain as well.

One of our purposes in this work is to explore whether a similarly useful role can be assigned to QCD expressed in terms of gauge-invariant field variables. In order to examine this question, we have implemented the non-Abelian Gauss’s law operator \( \hat{G}^a \) by explicitly constructing states |\( \Psi \rangle \) that are annihilated by the non-Abelian Gauss’s law operator \( \hat{G}^a \) given by

\[
\mathcal{A} \equiv \partial_\mathbf{r} \Pi_i^a - g f^{abc} A_i^b \Pi_i^c + j_0^a(\mathbf{r}),
\]

where \( j_0^a = g \bar{\psi} \gamma^a \frac{1}{2d} \gamma^\mu \gamma^\nu \partial_\mu \partial_\nu \psi \) is the quark color-charge density, \( \partial_\mathbf{r} \Pi_i^a \) is the “pure glue” form of the Gauss’s law operator, and \( j_0^a(\mathbf{r}) = g f^{abc} A_i^b \Pi_i^c \) is the color-charge density of the gauge field.

2. GAUSS’S LAW AND GAUGE INVARIANCE

The states that are annihilated by the “pure glue” Gauss’s law operator \( \mathcal{A} \) have the form |\( \Psi \rangle = |\Phi \rangle \), where |\( \Phi \rangle \) represents a state annihilated by \( \partial_\mathbf{r} \Pi_i \), the Abelian part of \( \partial_\mathbf{r} \Pi_i \), and |\( \Phi \rangle \) is given by |\( \Psi \rangle = \parallel \exp(\mathcal{A}) \parallel \). The |\( \parallel \parallel \) ordered product orders terms so that all functionals of \( A_i^a \) are to the left of all functionals of \( \Pi_i^a \), \( \mathcal{A} \) is the integral operator

\[
\mathcal{A} = i \int d\mathbf{r} \mathcal{A}_{\gamma}^a(\mathbf{r}) \Pi_i^\gamma(\mathbf{r}),
\]

and \( \mathcal{A}_{\gamma}^a(\mathbf{r}) \) is the resolvent field. In the course of this investigation, it becomes apparent that the resolvent field is central to achieving our objective.

In earlier work, we have obtained an integral equation for the resolvent field, given by

\[
\int d\mathbf{r} \mathcal{A}_{\gamma}^a(\mathbf{r}) V_j^\gamma(\mathbf{r}) = \sum_{\eta=1}^{\infty} \frac{i q_n^{\eta}}{n!} \int d\mathbf{r} \left\{ \psi_{(\eta)}^{\gamma}(\mathbf{r}) + f_{(\eta)}^{\gamma\beta} M_{(\eta)}^a(\mathbf{r}) B_{(\eta)}^\beta(\mathbf{r}) \right\} V_j^\gamma(\mathbf{r}),
\]

where

\[
\mathcal{A}_\gamma^a(\mathbf{r}), \quad M_{(\eta)}^a(\mathbf{r}) = \prod_{m=1}^{\eta} \mathcal{A}_\gamma^m(a_m(\mathbf{r})), \quad \text{and} \quad B_{(\eta)}^\beta(\mathbf{r}) = a_\beta^a(\mathbf{r}) + \left( \delta_{ij} - \frac{n}{(n+1)} a_i \right) A_i^\beta(\mathbf{r});
\]

\( a_\beta^a(\mathbf{r}) \) designates the transverse part of the gauge field. The fact that the resolvent field \( \mathcal{A}_\gamma^a(\mathbf{r}) \) appears in \( B_{(\eta)}^\beta(\mathbf{r}) \) and also appears in \( \mathcal{A}_\gamma^a(\mathbf{r}) \), which is raised to all powers in \( M_{(\eta)}^a(\mathbf{r}) \), makes Eq. (3) a nonlinear integral equation. \( f_{(\eta)}^{\gamma\beta} \) denotes the chain of structure constants

\[
f_{(\eta)}^{\gamma\beta} = f^{\alpha[1]} f^{\beta[2]} f^{\gamma[3]} \times \ldots \times f^{\delta[n-2]} a^{[n-1]} f^{[\eta-1]} a^{[\eta]};
\]

summed over repeated indices. \( \psi_{(\eta)}^{\gamma}(\mathbf{r}) \) in Eq. (2) depends only on the gauge-dependent gauge field and is understood to be an inhomogeneous source term for the nonlinear integral equation. Iterative expansions of the resolvent field are readily obtained and have been given. But our main interest will be in non-iterative representations of the resolvent field.

The apparatus we developed for implementing Gauss’s law also enables us to construct gauge-invariant matter (quark) and gauge (gluon) fields.
The basic idea is that the complete Gauss’s law operator $\tilde{G}^a(r)$ and the “pure glue” Gauss’s law operator $\mathcal{G}^a(r)$ are unitarily equivalent; and that, $U_G$, the unitary operator that implements the transformation $\tilde{G}^a(r) = U_G \mathcal{G}^a(r) U_G^{-1}$, is given by

$$U_G = e^{C_0} e^{\bar{C}}$$

where

$$C_0 = i \int \! d\tilde{r} \lambda^a(\tilde{r}) j^a_0(\tilde{r})$$

and

$$\bar{C} = i \int \! d\tilde{r} \bar{\mathcal{G}}^a(\tilde{r}) j^a_0(\tilde{r}),$$

the last equation showing the role of the resolvent form the matter field by the gauge-transformation, in fact is operator-valued; which are gauge invariant in the SU(2) case, the relation among these quantities is that of angles in rigid-body rotations. In this new representation, the quark field $\psi$ and the current density $g\bar{\psi} \lambda^a \gamma^\mu \psi$ are gauge-invariant because they commute with $\mathcal{G}^a$. This unitary equivalence can then be used to construct operator-valued fields that are gauge invariant in the original representation:

$$\psi_G(r) = U_G \psi(r) U_G^{-1} \quad \text{and} \quad \psi^\dagger_G(r) = U_G \psi^\dagger(r) U_G^{-1}.$$ 

With the Baker-Hausdorff-Campbell theorem, we obtain

$$\psi_G(r) = V_C(r) \psi(r) \quad \text{and} \quad \psi_G^\dagger(r) = \psi^\dagger(r) V_C^{-1}(r),$$

where

$$V_C(r) = \exp \left( -i g \mathcal{A}^a(r) \frac{\lambda^a}{2} \right) \exp \left( -i g \lambda^a(r) \frac{\lambda^a}{2} \right).$$

Because the commutator algebra of the $\lambda^a$ matrices is closed, $V_C(r)$ can be expressed as

$$V_C(r) = \exp \left[ -igZ^a(r) \frac{\lambda^a}{2} \right],$$

where $Z^a(r)$ is a functional of $\lambda^a(r)$ and $\mathcal{A}^a(r)$; in the SU(2) case, the relation among these quantities is that of angles in rigid-body rotations. In the form given by Eq. (5), $V_C(r)$ has the formal structure of an operator that gauge-transforms a charged field. However, $Z^a$, which would have to be a $c$-number valued field for $V_C$ to be such a gauge-transformation, in fact is operator-valued; and under a gauge transformation which transforms the matter field by the $c$-number function $\omega^\gamma(r)$, the matter field and $V_C$ transform as

$$\psi \to \exp(-i\omega^\gamma \frac{\lambda^a}{2}) \psi \quad \text{and} \quad V_C \to V_C \exp(i\omega^\gamma \frac{\lambda^a}{2})$$

so that $V_C(r)\psi$ remains gauge-invariant. Exploiting the formal similarity of the structure of the gauge-invariant matter field to a gauge transformation of that field enables us to also construct gauge-invariant gauge fields in the form

$$A_{GI,i}(r) = V_C(r) A_i(r) V_C^{-1}(r) + \frac{1}{2} V_C(r) \partial_\gamma V_C^{-1}(r),$$

where $A_i(r) = A_i^b(r) \frac{\lambda^b}{2}$ or, equivalently,

$$A_{GI,i}(r) = A_i^b(r) \left( \frac{\lambda^b}{2} + [\delta_{ij} - \frac{\partial_j}{\partial_i}] A_i^b(r) \right).$$

We can take this formal similarity further, by noting that for $A_0(r) = 0$,

$$A_{GI,0}(r) = \frac{1}{2} V_C(r) \partial_\gamma V_C^{-1}(r).$$

With these results, we can identify the gauge-invariant negative chromoelectric field as

$$\Pi^a_{GI,i} = \frac{1}{2} \text{Tr} [V_C^{-1} \lambda^a V_C^0] \Pi^b_i.$$ (8)

Finally, the gauge-invariant chromomagnetic field is

$$F^a_{GI,ij} = \partial_j A^a_{GI,i} - \partial_i A^a_{GI,j} - g \epsilon^{abc} A^b_{GI,i} A^c_{GI,j}.$$ (9)

3. GAUGE-INvariant QCD DYNAMICS

In this section, we will make use of earlier work, in which the temporal-gauge Hamiltonian was expressed entirely in terms of the gauge-invariant quantities that we introduced in earlier sections. In this form, the Hamiltonian is

$$\hat{H}_{GI} = \int \! d^3r \left[ \frac{1}{2} \Pi^a_{GI,i}(r) \Pi^a_{GI,i}(r) + \frac{1}{2} F^a_{GIlij}(r) F^a_{GIlij}(r) - \psi^\dagger(r) \left( \beta \gamma - i \alpha \gamma \partial_i \right) \psi(r) \right] + \hat{H}' + \hat{H}_G$$

with

$$\hat{H}' = \int \! d^3r \left( \frac{1}{2} J^a_{\partial 0}(r) \frac{1}{\partial^2} \mathcal{K}^a_\partial(r) + \frac{1}{2} \mathcal{K}^a_\partial(r) \frac{1}{\partial^2} J^a_{\partial 0}(r) \right)$$

and

$$\hat{H}_G = -\frac{1}{2} \int \! d^3r \left[ \mathcal{G}^a_0 \frac{1}{\partial^2} \mathcal{K}^a_0(r) + \mathcal{K}^a_0(r) \frac{1}{\partial^2} \mathcal{G}^a_0 \right].$$

(10)
\[ K_0^a + ge^{abc} A_{Gi} \frac{\partial}{\partial x^c} K_0^b = -J_0^a, \]

and \( J_0^a(Gi) \) is the gauge-invariant “glue” color-charge density \( J_0^a(Gi) = gf^{abc} A_{Gi}^b \Pi_{Gi}^2 \). We observe that \( \tilde{H}' \) manifests interesting similarities to the QED Hamiltonian shown in Eq. (3). One of its terms describes the interaction of the gauge-invariant (transverse) gauge field with the transverse color-current density, which is also gauge-invariant, and, as is true for a cognate term in QED, not likely to make important contributions at low energies. \( \tilde{H}' \) also contains terms describing Coulomb interactions between gauge-invariant quark-quark and quark-gluon color-charge densities.

\( \tilde{H}_G \) has some features in common with expressions obtained by other investigators who have formulated QCD in the Coulomb gauge. But \( \tilde{H}_G \) also differs from Hamiltonians in Coulomb-gauge formulations of QCD in a number of ways, for example in the presence of \( H_G \), which is the term that “remembers” that this formulation is specific to the temporal gauge, but which, as was shown in Ref. [1], cannot affect any of the physical consequences obtained with \( \tilde{H}_G \). The gauge-invariant fields resemble those of the Coulomb gauge, and have equal-time commutation rules very much like those obtained by Schwinger for that gauge,[1] but differ from them in operator order. The situation in QCD is therefore very similar to the one we described for QED in connection with Eq. (3). The nonlocality of \( K_0^a \), and its interactions with itself and with the gauge-invariant gluon color-charge density, provide an incentive to examine the long-range properties of the interaction described by \( \tilde{H}' \) — in particular, whether it might describe a confining force acting on color-bearing objects. In Ref. [1], we also argued that in the regime in which QCD variables describe hadronic interactions, the form of \( K_0^a \) is suggestive of color transparency for combinations of quarks in a color-singlet configuration.

Finally, it is also worth noting that, in the transformation to a representation in terms of gauge-invariant fields, Faddeev-Popov ghosts have not been introduced into the QCD Hamiltonian. Our procedure for arriving at a representation of QCD in terms of “physical” fields does not require the introduction of Faddeev-Popov ghost fields.

4. TOPOLOGY AND GRIBOV COPIES

In this section, I will review investigations into the topological properties of the resolvent field for two-color QCD, in which the Pauli spin matrices \( \tau^a \) replace the Gell-Mann matrices \( \lambda^a \). By representing the resolvent field \( \overline{A}^a(r) \) as a function of spatial variables that are second-rank tensors in the combined spatial and SU(2) indices, we have obtained and solved a nonlinear differential equation for \( \overline{N} \),[2] which is related to the resolvent field by

\[ \overline{N} = \left( \frac{\partial}{\partial r} \overline{A}^a \frac{\partial}{\partial r} \overline{A}^a \right)^{1/2}. \]

This equation,

\[ \frac{d^2 \overline{N}}{du^2} + \frac{d \overline{N}}{du} + 2 \left[ N \cos(\overline{N} + \overline{N}) - \sin(\overline{N} + \overline{N}) \right] + 2gr_0 \exp(u) \left( T_A \left[ \cos(\overline{N} + \overline{N}) - 1 \right] - T_G \sin(\overline{N} + \overline{N}) \right) = 0 \]

where \( u = \ln(r/r_0) \), can also describe a driven, damped, pendulum with the important proviso that \( \overline{N} \), given in Eq. (13), must be bounded in \( u \) in the entire interval \((-\infty, \infty)\), whereas the pendulum equation only applies to the interval \((0, \infty)\). In Ref. [2], we graphically display numerical solutions of Eq. (14) and show that, for the same choice of “source” terms, there are a number of bounded solutions in the interval \((-\infty, \infty)\) and that these not only differ from each other, but that they also can have different asymptotic values as \( u \to \infty \); and that different asymptotic values of \( \overline{N} \), in that limit, correspond to a variety of winding numbers, many half-integer valued, or fractional valued. We also pointed out in this work that the solutions of Eq. (14), and the asymptotic limits of these solutions as
\( u \to \infty \), are not strongly dependent on the functional forms of the source terms \( N \), \( T_A \), and \( T_C \). Eq. (14) is of the general form of an equation given by Gribov to document the existence of multiple copies of Coulomb-gauge fields.

\[
\frac{d^2 \phi}{du^2} + \frac{d \phi}{du} - 2\sin(\phi) (1 - f(u)) = 0,
\]

and the multiple solutions for \( \vec{A}_i^J(r) \) correspond to Gribov copies of the gauge-invariant gauge field. This fact has led us to the following observations about Gribov copies in the temporal and Coulomb gauges.

- When QCD is quantized in the Coulomb gauge, the quantization procedure is impeded by the existence of Gribov copies, because the non-uniqueness of the inverse of the Faddeev-Popov operator prevents the inversion of the Dirac constraint commutator matrix.

- When the quantization is carried out in the temporal gauge (or another algebraic gauge) no impediments to the inversion of the commutator matrix arise, and the procedure can be carried out consistently, without any concern about nonunique inverses of that matrix. It is in this sense that the statement that there are no Gribov copies of the gauge-dependent temporal-gauge field can be understood. But, in contrast to the Coulomb gauge (or in other gauges in which Gauss’s law is a secondary constraint) Gauss’s law remains to be implemented after quantum rules have been imposed on the operator-valued temporal-gauge fields.

- Gribov copies arise in the temporal gauge when Gauss’s law is implemented, and they are a feature of the gauge-invariant, but not the gauge-dependent fields. It is the imposition of gauge invariance that produces gauge fields that have Gribov copies in QCD. This is consistent with a proof given by Singer, and with remarks in his paper about the absence of Gribov copies in axial gauge formulations in which \( n^\nu A_\nu^c = 0 \) defines the gauge.

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