On the Existence of Unstable Bumps in Neural Networks

Vadim Kostrykin and Anna Oleynik

Abstract. We study the neuronal field equation, a nonlinear integro-differential equation of Hammerstein type. By means of the Amann three fixed point theorem we prove the existence of bump solutions to this equation. Using the Krein-Rutman theorem we show their Lyapunov instability.

Mathematics Subject Classification (2000). 47H07, 47H10, 37C75, 92B20.

Keywords. Neural field equation, fixed point theorems, stability.

1. Introduction and Main Results

The behavior of a single layer of neurons can be modeled by a nonlinear integro-differential equation of the Hammerstein type,

\[ \partial_t u(x, t) = -u(x, t) + \int_{\mathbb{R}} \omega(x - y) f(u(y, t) - h) dy. \]  

Here \( u(x, t) \) and \( f(u(x, t) - h) \) represent the averaged local activity and the firing rate of neurons at the position \( x \in \mathbb{R} \) and time \( t > 0 \), respectively. The parameter \( h \geq 0 \) is a firing threshold, and \( \omega(x - y) \) describes a coupling between neurons at positions \( x \) and \( y \).

The model described above has been studied in numerous mathematical papers (for a review see, e.g., \([3,4]\)). In particular, the global existence and uniqueness of solutions to the initial value problem for Eq. (1.1) under rather mild assumptions on \( f \) and \( \omega \) has been proven in \([12]\).

In 1977, Amari studied pattern formation in (1.1) for a model where \( f \) is the Heaviside function and \( \omega \) is assumed to be continuous, integrable and even, with \( \omega(0) > 0 \) and having exactly one positive zero. In particular, he showed the existence of stable and unstable bumps, that is, time independent spatially localized solutions to (1.1). For more general \( f \) and \( \omega \) the existence of stable solutions of this kind has been shown by Kishimoto and Amari in \([9]\).

This work is supported in part by the Deutsche Forschungsgemeinschaft, grant KO 2936/4-1.
and later generalized by Oleynik et al. in [11]. In the present work we prove the existence of unstable bumps.

Our main assumptions are as follows.

**Assumption A.** Let \( f : \mathbb{R} \to [0, 1] \) be an arbitrary continuous nondecreasing function such that \( f(x) = 0 \) for all \( x \leq 0 \) and \( f(x) = 1 \) for all \( x \geq \tau \) with \( \tau > 0 \).

In particular, \( f \) is a distribution function of a continuous probability measure supported on the interval \([0, \tau]\). As an example of such function we have

\[
  f(u) = \begin{cases} 
    0, & u \leq 0, \\
    \frac{u^p}{u^p + (\tau - u)^p}, & 0 < u < \tau, \\
    1, & u \geq \tau, 
  \end{cases}
\]

(1.2)

with \( p > 0 \) arbitrary. It is straightforward to see that \( f \in C[^\lfloor p \rfloor](\mathbb{R}) \), where \( \lfloor p \rfloor \) denotes the integer part of \( p \).

**Assumption B.** We assume that the integral kernel \( \omega \) meets the following conditions:

(i) \( \int_{-\infty}^{\infty} |\omega(x)|dx < \infty \), that is, \( \omega \in L^1(\mathbb{R}) \).

(ii) \( \omega \) is bounded and continuous.

(iii) \( \omega \) is a symmetric function, i.e., \( \omega(-x) = \omega(x) \).

(iv) There is an \( a > 0 \) such that \( \omega(x) > 0 \) for almost all \( x \in [0, 2a] \).

(v) For given \( h, \tau > 0 \)

\[
  2a \int_0^{2a} \omega(y)dy > h + \tau.
\]

The conditions (i)-(v) guarantee that there are \( 0 < \Delta_- < \Delta_+ < a \) such that

\[
  \int_0^{2\Delta_-} \omega(y)dy = h, \quad \int_0^{2\Delta_+} \omega(y)dy = h + \tau.
\]

For all \( x \in \mathbb{R} \) we define

\[
  u_\pm(x) := \int_{-\Delta_\pm}^{\Delta_\pm} \omega(x - y)dy.
\]

(1.3)

(vi) There is a \( d \in (\Delta_+, a] \) such that \( u_+(d) = h \).

(vii) \( \omega \) is decreasing on \([0, 2d]\) and \( \omega(x) \leq \omega(2d) \) for all \( x \geq 2d \).

Let \( \chi_{(\tau, \infty)} \) and \( \chi_{(0, \infty)} \) be characteristic functions of \((\tau, \infty)\) and \((0, \infty)\), respectively. Under Assumption B it is easy to show that the functions \( u_+ \) and \( u_- \) solve Eq. (1.1) with \( f = \chi_{(\tau, \infty)} \) and \( f = \chi_{(0, \infty)} \), respectively. The proof is given in Appendix, see Lemma A.1.

Following Amari [2] we call a stationary solution of Eq. (1.1) a bump (more precisely, 1-bump) if the support of the function \( x \mapsto f(u(x) - h) \)
is an interval. According to this definition \( u_+ \) and \( u_- \) are bumps provided
\[ f = \chi_{(\tau, \infty)} \quad \text{and} \quad f = \chi_{(0, \infty)}, \]
respectively, see Lemma A.1 in Appendix.

One of the common choices of \( \omega \) in the study of neural field models is
that of a 'Mexican hat' function, such as
\[ \omega(x) = K \exp(-kx^2) - M \exp(-mx^2), \quad K > M > 0, \quad k > m > 0, \]
see, e.g., [2,3,11]. This function satisfies Assumption B for some values of \( h \) and \( \tau \). The other common choices of \( \omega \) are the exponential function
\[ \omega(x) = e^{-|x|/2} \] and the Gaussian function \( \omega(x) = \exp(-x^2) \). It is easy to see that the
conditions of Assumption B are satisfied for these functions if \( h + \tau < 1/2 \) and \( h + \tau < \sqrt{\pi}/2 \), respectively.

The condition (ii) of Assumption B implies that \( u_\pm \) are continuous,
whereas from the conditions (iii) and (iv) the inequality \( u_-(x) < u_+(x) \) for
all \( x \in [-d,d] \) follows.

Lemma 1.1. The condition (vii) in Assumption B is fulfilled if and only if
\[ \omega(x - y) \leq \omega(d - y) \quad \text{for all} \quad x > d \quad \text{and} \quad y \in [-d,d]. \]

Proof. Assume that the condition (1.4) is fulfilled. We introduce \( \xi = d - y \).
Then we have
\[ \omega(\xi + (x - d)) \leq \omega(\xi), \quad \xi \in [0, 2d]. \]
Since \( x - d > 0 \) the inequality (1.5) implies the monotonicity of \( \omega \) on \([0, 2d]\).
Next, we set \( \xi = 2d \) in (1.5), thus obtaining
\[ \omega(\eta) \leq \omega(2d) \quad \text{with} \quad \eta = x + d \geq 2d. \]

Conversely, assume that the condition (vii) is satisfied. Let \( x \geq d \) and \( y \in [0, 2d] \) be arbitrary. If \( x - y \in [0, 2d] \), then the inequality \( \omega(x - y) \leq \omega(d - y) \) follows from the monotonicity of \( \omega \) and \( x - y \geq d - y \). If \( x - y > d \) we then obtain
\[ \omega(x - y) \leq \omega(2d) \leq \omega(d - y). \]

Our main results are as follows:

Theorem 1. Under Assumptions A and B there exists a bump solution \( \tilde{u} \) to the integro-differential Eq. (1.1), that is, a stationary solution with
\( \text{supp}(\tilde{u}(\cdot) - h) \) an interval. Moreover,
\[ u_-(x) \leq \tilde{u}(x) \leq u_+(x) \]
holds for all \( x \in [-d,d] \) and, hence, the support of \( f(\tilde{u}(\cdot) - h) \) is contained
in \([-d,d]\).

Theorem 2. Assume in addition to Assumptions A and B that
(i) \( \omega \in W^{1,\infty}(\mathbb{R}) \), the Sobolev space of almost everywhere differentiable
functions with essentially bounded derivative,
(ii) \( \omega(x) \to 0 \) as \( |x| \to \infty \),
(iii) \( f \in C^{1,\mu}(\mathbb{R}) \), that is, \( f \) is continuously differentiable and its derivative is Hölder continuous with an exponent \( \mu \in (0,1] \), \( |f'(x) - f'(y)| \leq C|x - y|^\mu \).

Then the solution \( \tilde{u} \) referred to in Theorem 1 belongs to \( C_\infty(\mathbb{R}) \). It is a Lyapunov-unstable equilibrium of the integro-differential Eq. (1.1), that is, for all sufficiently small \( \varepsilon > 0 \) there is an initial value in the ball \( B_\varepsilon(\tilde{u}) \subset C_\infty(\mathbb{R}) \) such that the corresponding solution to (1.1) leaves \( B_\varepsilon(\tilde{u}) \) in finite time.

Here \( C_\infty(\mathbb{R}) \) denotes the Banach space of continuous functions vanishing at infinity.

It is straightforward to see that the conditions of Theorem 2 are fulfilled for all three above examples of \( \omega \) and for \( f \) in (1.2) with \( p > 1 \).

2. Proof of Theorem 1

In this section we treat \( u_{\pm} \) defined in (1.3) as functions on \([-d,d]\). We define a nonlinear integral operator

\[
(Tu)(x) := \int_{-d}^{d} \omega(x - y) f(u(y) - h) dy
\]

and consider the fixed point problem

\[
u = Tu
\]

in the real Banach space \( C([-d,d]) \). The cone

\[
K := \{u \in C([-d,d]) : u(x) \geq 0 \text{ for all } x \in [-d,d] \}
\]

defines a partial order in \( C([-d,d]) \). We write \( u \geq v \) if \( u - v \in K \), \( u > v \) if \( u \geq v \) and \( u \neq v \), and \( u \gg v \) if \( u - v \) is in the interior of \( K \).

**Lemma 2.1.** Under Assumptions A and B the operator \( T : C([-d,d]) \to C([-d,d]) \) is monotone increasing and compact. Moreover, \( Tu_- \ll u_- \) and \( Tu_+ \gg u_+ \).

Recall that an operator \( T \) acting on the ordered Banach space \( X \) is called monotone increasing if \( u \leq v \) implies \( Tu \leq Tv \).

**Proof.** The linear integral operator

\[
u \mapsto \int_{-d}^{d} \omega(\cdot - y) u(y) dy
\]

is continuous and compact as a mapping in \( C([-d,d]) \). Since the integral kernel \( \omega(x - y) \) is positive for all \( x, y \in [-d,d] \), it is monotone increasing. The mapping \( u \mapsto f(u - h) \) is continuous, monotone increasing, and bounded. This
implies that $T$ is compact and monotone increasing. Since $f(t) < \chi_{(0,\infty)}(t)$ on a set of positive measure, we obtain

\[
(Tu_\pm)(x) = \int_{-d}^{d} \omega(x-y)f(u_\pm(y) - h)dy < \int_{-d}^{d} \omega(x-y)\chi_{(0,\infty)}(u_\pm(y) - h)dy
\]

\[
= \int_{-\Delta}^{\Delta} \omega(x-y)dy = u_\pm(x),
\]

which proves the first inequality. Similarly, the inequality $f(t) > \chi_{(\tau,\infty)}(t)$ holds on a set of positive measure. Therefore,

\[
(Tu_\pm)(x) = \int_{-d}^{d} \omega(x-y)f(u_\pm(y) - h)dy > \int_{-d}^{d} \omega(x-y)\chi_{(\tau,\infty)}(u_\pm(y) - h)dy
\]

\[
= \int_{-\Delta}^{\Delta} \omega(x-y)dy = u_\pm(x),
\]

which proves the second inequality. □

For any $u$ in the order interval $[u_-, u_+]$, that is,

\[
[u_-, u_+] := \{ u \in C([-d, d]) : u_- \leq u \leq u_+ \}
\]

we define the mapping

\[
(\hat{T}u)(x) := \max \{ \min \{(Tu)(x), u_+(x)\}, u_-(x) \}, \quad x \in [-d, d], \quad (2.3)
\]

or, more explicitly,

\[
(\hat{T}u)(x) = \begin{cases} 
  u_-(x) & \text{if } (Tu)(x) \leq u_-(x), \\
  (Tu)(x) & \text{if } u_-(x) \leq (Tu)(x) \leq u_+(x), \\
  u_+(x) & \text{if } u_+(x) \leq (Tu)(x). 
\end{cases}
\]

Since the r.h.s. in this definition is a continuous function satisfying

\[
u_-(x) \leq \max \{ \min \{(Tu)(x), u_+(x)\}, u_-(x) \} \leq u_+(x)
\]

for all $x \in [-d, d]$, $\hat{T}$ is a self-mapping of $[u_-, u_+]$. Furthermore, $u_\pm$ are fixed points,

\[
\hat{T}u_\pm = u_\pm.
\]

**Lemma 2.2.** The operator $\hat{T}$ is monotone increasing and compact. Moreover, for sufficiently small $\varepsilon > 0$ one has

\[
\hat{T}(u_- + \varepsilon) \ll u_- + \varepsilon \ll u_+ - \varepsilon
\]

and

\[
\hat{T}(u_+ - \varepsilon) \gg u_+ - \varepsilon \gg u_- + \varepsilon.
\]
Proof. By the monotonicity of the operator \( T \) one has \( Tu_1 \geq Tu_2 \) whenever \( u_1 \geq u_2 \). Hence,

\[
\min\{(Tu_1)(x), u_+(x)\} \geq \min\{(Tu_2)(x), u_+(x)\} \quad \text{for all} \quad x \in [-d, d],
\]

and, therefore,

\[
\max\{\min\{(Tu_1)(x), u_+(x)\}, u_-(x)\} \geq \max\{\min\{(Tu_2)(x), u_+(x)\}, u_-(x)\}.
\]

Thus, \( \hat{T} \) is monotone increasing.

Let \( (u_n) \) be an arbitrary sequence in \([u_-, u_+]\). Since \( T \) is compact, \( (Tu_n) \) has a subsequence \( (Tu_{n_k}) \) converging to some \( v \in C([-d, d]) \). For arbitrary \( \varepsilon > 0 \) let \( n_0 \in \mathbb{N} \) be so large that

\[
|Tu_{n_k}(x) - v(x)| < \varepsilon \quad \text{for all} \quad k \geq n_0 \quad \text{and} \quad x \in [-d, d].
\]

Then one has

\[
\min\{Tu_{n_k}(x), u_+(x)\} \leq \min\{v(x) + \varepsilon, u_+(x)\} \leq \min\{v(x), u_+(x)\} + \varepsilon
\]

and

\[
\min\{Tu_{n_k}(x), u_+(x)\} \geq \min\{v(x) - \varepsilon, u_+(x)\} \geq \min\{v(x), u_+(x)\} - \varepsilon,
\]

which shows that \( \min\{Tu_{n_k}(x), u_+(x)\} \) converges uniformly to \( \min\{v(x), u_+(x)\} \). Similarly, one can show that \( (\hat{T}u_{n_k}) \) converges uniformly to \( \max\{\min\{v(x), u_+(x)\}, u_-(x)\} \), thus proving that the range of \( \hat{T} \) is relatively compact.

Now assuming that the sequence \( (u_n) \) converges to some \( u \in [u_-, u_+] \) and using the continuity of \( T \), we arrive at the conclusion that \( (\hat{T}u_n) \) converges to \( (\hat{T}u) \), thus proving that \( \hat{T} \) is continuous.

Since the mapping \( u \in C([-d, d]) \mapsto \inf_{x \in [-d, d]} u(x) \) is continuous, the functional \( \rho : C([-d, d]) \to \mathbb{R} \)

\[
\rho(u) := \inf_{x \in [-d, d]} (u(x) - (Tu)(x))
\]

is continuous as well. Hence, due to \( \rho(u_-) > 0 \), there is an \( \varepsilon > 0 \) such that \( \rho(u) > 0 \) for all \( u \in B_{2\varepsilon}(u_-) \). We can choose \( \varepsilon \) so small that \( u_- + \varepsilon \ll u_+ - \varepsilon \). Thus,

\[
T(u_- + \varepsilon) \ll u_- + \varepsilon \ll u_+ - \varepsilon,
\]

from which it follows that

\[
\min\{T(u_- + \varepsilon), u_+\} = T(u_- + \varepsilon)
\]

and consequently

\[
\hat{T}(u_- + \varepsilon) = \max\{T(u_- + \varepsilon), u_-\} \ll u_- + \varepsilon.
\]

The second inequality can be proven in the same way. \( \square \)

The main tool for the proof of Theorem 1 is Amann’s theorem on three fixed points [1, Theorem 14.2 and Corollary 14.3] in the version of Zeidler [13, Theorem 7.F and Corollary 7.40].
Theorem 2.3. Let $X$ be a real Banach space with an order cone having a nonempty interior. Assume there are four points in $X$

$$p_1 \ll p_2 < p_3 \ll p_4$$

and a monotone increasing image compact operator $\hat{T} : [p_1, p_4] \to X$ such that

$$\hat{T}p_1 = p_1, \quad \hat{T}p_2 < p_2, \quad \hat{T}p_3 > p_3, \quad \hat{T}p_4 = p_4.$$ 

Then $\hat{T}$ has a third fixed point $p$ satisfying $p_1 < p < p_4$, $p \notin [p_1, p_2]$, and $p \notin [p_3, p_4]$.

Recall that the operator $\hat{T}$ is called image compact if it is continuous and its image $\hat{T}[p_1, p_4]$ is relatively compact in $X$. In the case $X = C([-d, d])$, the order cone $K$ defined in (2.2) is normal, that is, the order interval $[p_1, p_4]$ is norm bounded (see, e.g., [7]). Therefore, the operator $\hat{T}$ is image compact if and only if it is compact.

We choose $p_1 = u_-, p_2 = u_- + \varepsilon, p_3 = u_+ - \varepsilon, p_4 = u_+$, where $\varepsilon > 0$ as in Lemma 2.2. Theorem 2.3 yields the existence of a fixed point $u_*$ of the operator $\hat{T}$ satisfying $u_- \leq u_* \leq u_+$. Obviously, $u_*$ is a fixed point of the operator $T$ defined in (2.1) as well.

Lemma 2.4. If a fixed point $u$ of the operator $T$ satisfies the inequality $u(d) \leq u_+(d) = h$, then

$$\tilde{u}(x) = \int_{-d}^{d} \omega(x - y) f(u(y) - h) dy, \quad x \in \mathbb{R}.$$ 

is a bump which solves (1.1).

Proof. Due to condition (vii) of Assumption B and Lemma 1.1, we have $\omega(x - y) \leq \omega(d - y)$ for all $x > d$. Hence, $\tilde{u}(x) \leq \tilde{u}(d) \leq h$. This implies that $\tilde{u}(x)$ solves the equation

$$\tilde{u}(x) = \int_{-\infty}^{\infty} \omega(x - y) f(\tilde{u}(y) - h) dy, \quad x \in \mathbb{R}. \quad (2.4)$$

We note that $\tilde{u}$ is not an isolated solution of (2.4). Indeed, $\tilde{u}(\cdot - c)$ is again a solution for any $c \in \mathbb{R}$.

3. Proof of Theorem 2

The proof of Theorem 2 heavily relies on the Krein-Rutman theorem (see, e.g., [13, Proposition 7.26] or [10, Theorem 6.1]):

Theorem 3.1. Let $X$ be a real Banach space with the order cone $K$ having a nonempty interior. Suppose that $T : X \to X$ is linear, compact, and positive, with the spectral radius $r(T) > 0$. Then $r(T)$ is an eigenvalue of $T$ with all eigenvectors in $K$. 

The second tool is a classical result on the instability of equilibrium solutions of differential equations [5, Theorem VII.2.3] (cf. also Corollary 5.1.6 in [8]).

**Theorem 3.2.** Let $X$ be a Banach space, $A$ be a linear continuous operator on $X$, $F : X \to X$ a nonlinear Lipschitz continuous operator. If

(i) $v_0 = 0$ satisfies $Av_0 + Fv_0 = 0$,
(ii) the operator $F$ obeys the estimate
$$
\|Fv\| \leq C\|v\|^{1+\mu}, \quad C > 0, \quad \mu > 0
$$
for all $u \in X$ with $\|v\| < \varepsilon$ for some $\varepsilon > 0$,
(iii) the spectrum $\sigma(A)$ contains a point $\lambda$ with $\text{Re}\lambda > 0$,

then $v_0$ is an unstable equilibrium of the differential equation
$$
v_t = Av + Fv, \quad t > 0.
$$

Let $u_* \in C([-d, d])$ denote the fixed point of the operator $T$ (2.1) referred to in the previous section. From the condition (i) of Theorem 2 it follows that $u_*$ belongs to $C^1([-d, d])$. For the proof see Lemma A.2 in Appendix. Due to (1.6), one has $u_*(-d) \leq h$, $u_*(0) \geq u_-(0) > h$, and $u_*(d) \leq h$. Thus, $u_*$ is not monotone.

We observe that under the conditions of Theorem 2 the operator $T$ is Fréchet differentiable with
$$
(T'(u)v)(x) = \int_{-d}^{d} \omega(x - y)f'(u(y) - h)v(y)dy, \quad v \in C([-d, d]).
$$

It is a linear, compact, and positive operator with respect to the cone defined by (2.2).

Since $u_* (\pm d) \leq h$, integrating by parts we obtain
$$
u'_*(x) = \int_{-d}^{d} \omega'(x - y)f(u_*(y) - h)dy - \int_{-d}^{d} \frac{\partial}{\partial y} \omega(x - y)f(u_*(y) - h)dy$$
$$= \int_{-d}^{d} \omega(x - y)f'(u_*(y) - h)u'_*(y)dy.
$$

Hence, $u'_*$ is an eigenfunction of the operator $T'(u_*)$ with eigenvalue 1. Thus, the spectral radius $r(T'(u_*))$ is not smaller than 1.

Assume that $r(T'(u_*)) = 1$. Applying the Krein-Rutman theorem with $X = C([-d, d])$, the cone $K$ defined in (2.2), and the operator $T'(u_*)$, we obtain that $u'_*(x) \geq 0$ for all $x \in [-d, d]$, which is a contradiction. Thus, $r(T'(u_*)) > 1$. Again by the Krein-Rutman theorem $r(T'(u_*))$ is an eigenvalue of $T'(u_*)$. Hence, we arrive at the conclusion that the Fréchet derivative of $u \mapsto -u + Tu$ at the point $u_*$ has a strictly positive eigenvalue.

Denote by $\tilde{T}$ the nonlinear integral operator defined via
$$
(\tilde{T}u)(x) := \int_{\mathbb{R}} \omega(x - y)f(u(y) - h)dy.
$$
Observe that under the condition (ii) of Theorem 2, $\tilde{T}$ maps $C_\infty(\mathbb{R})$ into itself. Hence, the bump $\tilde{u}$ referred to in Theorem 1 belongs to $C_\infty(\mathbb{R})$.

**Lemma 3.3.** Let the conditions of Theorem 2 be satisfied. Then the Fréchet derivative $\tilde{T}'(\tilde{u}) : C_\infty(\mathbb{R}) \to C_\infty(\mathbb{R})$ of the operator $\tilde{T}$,

$$\left(\tilde{T}'(\tilde{u})v\right)(x) = \int_{\mathbb{R}} \omega(x - y) f'(\tilde{u}(y) - h)v(y)dy,$$

is compact.

**Proof.** The proof is based on the following compactness criterion [6, Theorem IV.6.5]:

- A bounded subset $S \subset C(\mathbb{R})$ is relatively compact if and only if for every $\varepsilon > 0$ there is a finite collection of sets $E_i \subset \mathbb{R}, i = 1, \ldots, n$, $\bigcup_{i=1}^n E_i = \mathbb{R}$, and points $x_i \in E_i$ such that

$$\sup_{\varphi \in S} \sup_{x \in E_i} |\varphi(x_i) - \varphi(x)| < \varepsilon$$

for all $i = 1, \ldots, n$.

Consider a set

$$S := \{\varphi \in C_\infty(\mathbb{R}) : \varphi = \tilde{T}'(\tilde{u})v, \quad v \in B_1(0) \subset C_\infty(\mathbb{R})\}.$$ 

Using the mean value theorem we obtain

$$|\varphi(x)| \leq \int_{-d}^{d} |\omega(x - y)| f'(\tilde{u}(y) - h)dy$$

$$\leq |\omega(x - \eta)| \cdot f'(\tilde{u}(\eta) - h)$$

for some $\eta \in [-d, d]$ and any $\varphi \in S$. Hence,

$$|\varphi(x)| \leq C \sup_{y \in [-d, d]} |\omega(x - y)|$$

for all $\varphi \in S$. Therefore, by the condition (ii) of Theorem 2, for an arbitrary $\varepsilon > 0$ we can choose $R > d$ so large that

$$\sup_{\varphi \in S} |\varphi(x)| < \varepsilon / 2 \quad \text{for all} \quad |x| > R.$$ 

Thus, we obtain

$$\sup_{\varphi \in S} \sup_{x \in E_\pm} |\varphi(x) - \varphi(\pm 2R)| < \varepsilon,$$

where $E_\pm := \{x \in \mathbb{R} : \pm x > \pm R\}$.

Let $S_0$ be the set in $C([-R, R])$ consisting of all functions in $S$ restricted to the interval $[-R, R]$,

$$S_0 := \{\varphi_0 = \varphi|_{[-R, R]} : \varphi \in S\}.$$ 

This set is the range of the compact integral operator

$$v \mapsto \int_{-R}^{R} \omega(\cdot - y) f'(\tilde{u}(y) - h)v(y)dy,$$
mapping \(C([-R, R])\) into itself. Thus, \(S_0\) is relative compact.

By the compactness criterion above, there is a finite collection \((E_i)_{i=1}^n\) of subsets in \([-R, R]\) and points \(x_i \in E_i\) such that
\[
\sup_{\varphi \in S_0} \sup_{x \in E_i} |\varphi(x_i) - \varphi(x)| < \varepsilon
\]
for all \(i = 1, \ldots, n\). Combining this with (3.2), we arrive at the conclusion that the collection \((E_1, \ldots, E_n, E_+, E_-)\) with points \((x_1, \ldots, x_n, 2R, -2R)\) satisfies the condition of the compactness criterion, thus, proving that \(S\) is a relative compact set. Hence, \(\tilde{T}'(\tilde{u})\) is a compact operator. \(\square\)

Now we show that the linear operators \(\tilde{T}'(\tilde{u})\) and \(T'(u_\star)\) have the same spectra. Since both operators are compact, it suffices to prove that they have the same eigenvalues. Assume that \(\lambda \neq 0\) is an eigenvalue of \(T'(u_\star)\) with an eigenfunction \(v \in C([-d, d])\). We set
\[
\tilde{v}(x) := \frac{1}{\lambda} \int_{-d}^{d} \omega(x - y)f'(u_\star(y) - h)v(y)dy, \quad x \in \mathbb{R}.
\]
It is easy to check that \(\tilde{v}\) is an eigenfunction of \(\tilde{T}'(\tilde{u})\) corresponding to the eigenvalue \(\lambda\). Conversely, assume that \(\tilde{v} \in C(\mathbb{R})\) is an eigenfunction of \(\tilde{T}'(\tilde{u})\) corresponding to the eigenvalue \(\lambda \neq 0\). Then
\[
\lambda \tilde{v}(x) = \int_{-d}^{d} \omega(x - y)f'(\tilde{u}(y) - h)\tilde{v}(y)dy = \int_{-d}^{d} \omega(x - y)f'(u_\star(y) - h)\tilde{v}(y)dy
\]
holds for all \(x \in [-d, d]\). This implies that \(\lambda\) is an eigenvalue of \(T'(u_\star)\) with an eigenfunction \(v := \tilde{v}|_{[-d, d]}\).

We arrive at the conclusion that the linear operator \(\tilde{T}'(\tilde{u})\) has an eigenvalue \(\lambda > 1\), and, thus, the condition (iii) of Theorem 3.2 is fulfilled.

Plugging \(u(x, t) = \tilde{u}(x) + w(x, t)\) into the equation \(u_t = -u + \tilde{T}u\) we obtain
\[
w_t = Aw + Fw,
\]
where
\[
Av = -v + \tilde{T}'(\tilde{u})v \quad \text{and} \quad Fv = \tilde{T}(\tilde{u} + v) - \tilde{T}\tilde{u} - \tilde{T}'(\tilde{u})v
\]
for any \(v \in C(\mathbb{R})\).

From the continuous differentiability of \(f\) it easily follows that \(\tilde{T}\) is Lipschitz continuous. By the mean value theorem one has
\[
\tilde{T}(\tilde{u} + v)(x) - (\tilde{T}\tilde{u})(x) = \int_{\mathbb{R}} \omega(x - y)(f(\tilde{u}(y) + v(y) - h) - f(\tilde{u}(y) - h))dy
\]
\[
= \int_{\mathbb{R}} \omega(x - y)f'(a(y) - h)v(y)dy,
\]
where \( a(y) \) is a point between \( \tilde{u}(y) \) and \( \tilde{u}(y) + v(y) \), \( y \in \mathbb{R} \). Hence,
\[
\tilde{T}(\tilde{u} + v)(x) - (\tilde{T}\tilde{u})(x) - (\tilde{T}'(\tilde{u})v)(x) = \int_{\mathbb{R}} \omega(x - y) (f'(a(y) - h) - f'(\tilde{u}(y) - h)) v(y) dy.
\]

From the Hölder continuity of \( f' \) it follows that

\[
|f'(a(y) - h) - f'(\tilde{u}(y) - h)| \leq C |a(y) - \tilde{u}(y)|^\mu \leq C |v(y)|^\mu.
\]

Thus,
\[
|\tilde{T}(\tilde{u} + v)(x) - (\tilde{T}\tilde{u})(x) - (\tilde{T}'(\tilde{u})v)(x)| \leq C \int_{\mathbb{R}} |\omega(x - y)||v(y)|^{1+\mu} dy \leq C \|v\|_\infty^{1+\mu} \|\omega\|_{L^1(\mathbb{R})},
\]
which implies that the condition (ii) of Theorem 3.2 is fulfilled. Consequently, by Theorem 3.2 with \( X = C_\infty(\mathbb{R}) \), \( \tilde{u} \) is an unstable equilibrium of the equation

\[
u_t = -u + \tilde{T}u. \tag{3.3}
\]

This completes the proof of Theorem 2.

**Acknowledgements**

The authors thank H.-P. Heinz for useful remarks. A.O. is grateful to the Institute for Mathematics for its kind hospitality during her stay at the Johannes Gutenberg-Universität Mainz. Her work has been supported in part by the Deutsche Forschungsgemeinschaft, grant KO 2936/4-1, and by the Inneruniversitären Forschungsförderung of the Johannes Gutenberg-Universität Mainz.

**Appendix**

**Lemma A.1.** Under Assumption B the function \( u_- \) given by (1.3) is a stationary solution of (1.1) with \( f = \chi_{(0,\infty)} \). Similarly, \( u_+ \) is a solution of (1.1) with \( f = \chi_{(\tau,\infty)} \). In particular, \( \text{supp} \chi_{(0,\infty)}(u_-(\cdot) - h) = [-\Delta_-, \Delta_-] \) and \( \text{supp} \chi_{(\tau,\infty)}(u_+(\cdot) - h) = [-\Delta_+, \Delta_+] \).

**Proof.** To prove that \( u_- \) solves (1.1) with \( f = \chi_{(0,\infty)} \) it suffices to show that

\[
u_- (x) \geq h \quad \text{for all} \quad x \in [0, \Delta_-]
\]

and

\[
u_- (x) < h \quad \text{for all} \quad x \in (\Delta_-, \infty).
\]
If $x \in [0, \Delta_-]$ we represent $u_-$ as follows

$$u_-(x) = \int_{x-\Delta_-}^{x+\Delta_-} \omega(z)dz = \int_0^{2\Delta_-} \omega(z)dz - \int_{x+\Delta_-}^{2\Delta_-} \omega(z)dz + \int_{x-\Delta_-}^0 \omega(z)dz.$$  

Observe that the first integral equals $h$. Using the symmetry of $\omega(z)$ we, thus, obtain

$$u_-(x) = h + \int_0^{\Delta_--x} (\omega(z) - \omega(z+x+\Delta_-))dz \geq h$$

by the condition (vii) of Assumption B.

If $x > \Delta_-$ we represent $u_-$ as

$$u_-(x) = \int_{I_1} \omega(x-y)dy + \int_{I_2} \omega(x-y)dy,$$

where

$$I_1 := \{y \in [-\Delta_-, \Delta_-] : 0 < x - y < 2d\}$$

and

$$I_2 := \{y \in [-\Delta_-, \Delta_-] : 2d < x - y\}.$$ 

For any $y \in I_1$ we have $\omega(x-y) < \omega(\Delta_- - y)$ since $\omega$ is decreasing on $[0, 2d]$ by the condition (vii) of Assumption B. If $y \in I_2$, then again by the condition (vii) of Assumption B we obtain the inequality

$$\omega(x-y) < \omega(2d) < \omega(\Delta_- - y).$$

Hence, in both cases the inequality

$$u_-(x) < \int_{-\Delta_-}^{\Delta_-} \omega(\Delta_- - y)dy = h$$

is valid.

That $u_+$ is a solution to (1.1) with $f = \chi_{(\tau, \infty)}$ can be proved in the same way. $\Box$

**Lemma A.2.** If the integral kernel $\omega$ satisfies Assumption B and the condition (i) of Theorem 2, then the stationary solution $\tilde{u}$ referred to in Theorem 1 is continuously differentiable.

**Proof.** From (2.4) it follows that the derivative $\tilde{u}'(x)$ exists for each $x \in \mathbb{R}$ and is given by

$$\tilde{u}'(x) = \int_{\mathbb{R}} \omega'(x-y)f(\tilde{u}(y) - h)dy = \int_{\mathbb{R}} \omega'(y)f(\tilde{u}(x-y) - h)dy.$$
Hence, since \( \| \omega' \|_{L^\infty(\mathbb{R})} < \infty \), for any \( \delta > 0 \) we have
\[
|\tilde{u}'(x + \delta) - \tilde{u}'(x)| \leq \int_{\mathbb{R}} |\omega'(y)||f(\tilde{u}(x + \delta - y) - h) - f(\tilde{u}(x - y) - h)|dy \\
\leq \| \omega' \|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |f(\tilde{u}(x + \delta - y) - h) - f(\tilde{u}(x - y) - h)|dy \\
= \| \omega' \|_{L^\infty(\mathbb{R})} \| f(\cdot + \delta) - f(\cdot - h) \|_{L^1(\mathbb{R})}.
\]

By the continuity of translations in \( L^1(\mathbb{R}) \) we obtain that \( |\tilde{u}'(x + \delta) - \tilde{u}'(x)| \rightarrow 0 \) as \( \delta \rightarrow 0 \), thus proving that \( \tilde{u}' \in C(\mathbb{R}) \). \( \square \)

References

[1] Amann, H.: Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. SIAM Rev. 18, 20–709 (1976)
[2] Amari, S.: Dynamics of pattern formation in lateral-inhibition type neural fields. Biol. Cybernet. 27, 77–87 (1977)
[3] Coombes, S.: Waves, bumps, and patterns in neural field theories. Biol. Cybernet. 93, 91–108 (2005)
[4] Ermentrout, B.: Neural networks as spatio-temporal pattern-forming systems. Rep. Prog. Phys. 61, 353–430 (1998)
[5] Dalec’ki˘ı Ju, L., Kre˘ın, M. G.: Stability of Solutions of Differential Equations in Banach Space. Translations of Mathematical Monographs, vol. 43. Amer. Math. Soc., Providence, R.I. (1974)
[6] Dunford, N., Schwartz, J.T.: Linear Operators Part I: General Theory. Interscience, New York (1958)
[7] Guo, D., Lakshmikantham, V.: Nonlinear Problems in Abstract Cones. Academic Press, Boston (1988)
[8] Henry, D.: Geometric Theory of Semilinear Parabolic Equations. Lecture Notes in Mathematics, vol 840. Springer, Berlin (1981)
[9] Kishimoto, K., Amari, S.: Existence and stability of local excitations in homogeneous neural fields. J. Math. Biol. 7, 303–318 (1979)
[10] Kre˘ın, M.G., Rutman, M.A.: Linear operators leaving invariant a cone in a Banach space, Uspekhi Mat. Nauk, 3, 3–95 (1948) (in Russian) MR 27128
[11] Oleynik, A., Ponosov, A., Wyller, J.: Iterative schemes for bump solutions in a neural field model, submitted
[12] Potthast, R., beim Graben, P.: Existence and properties of solutions for neural field equations. Math. Meth. Appl. Sci. 33, 935–949 (2010)
[13] Zeidler, E.: Nonlinear functional analysis, vol.1. In: Fixed-Point Theorems. Springer, Berlin (1986)
