On the phase diagram of quenched QCD with Wilson fermions

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In this talk, I reported on recent work on the Aoki phase diagram for quenched QCD with two flavors of Wilson fermions. Part of this work was done in collaboration with Yigal Shamir \textsuperscript{1}, and a shorter account of this part appeared recently \textsuperscript{2}. In this write-up, I will therefore limit myself to reporting on work done with Steve Sharpe and Robert Singleton, Jr. \textsuperscript{3}, which has not been published yet. We discuss the symmetries of quenched QCD, paying careful attention to non-perturbative issues. This allows us to derive an effective lagrangian which agrees with standard quenched chiral perturbation theory, but which can also be used to address questions of a non-perturbative nature.

1. Introduction

The original aim of the work I report on in this talk was to see whether a phase-diagram analysis based on the effective theory describing the Goldstone-boson physics of full QCD with two flavors of Wilson fermions \textsuperscript{4} can be extended to the quenched case as well. The unquenched theory is expected to have a so-called Aoki phase \textsuperscript{5} at non-zero lattice spacing, and Ref. \textsuperscript{4} demonstrated that this can be understood from an effective-lagrangian point of view, if one includes the effects of scaling violations in a systematic, Symanzik-like expansion in powers of the lattice spacing (see also Ref. \textsuperscript{5}).

This investigation led to two different problems which needed to be resolved in order to study the possible existence of an Aoki phase in quenched QCD. First, it was observed numerically (see for example Ref. \textsuperscript{6}) that there always appears to exist a non-zero density of near-zero modes of the (hermitian) Wilson–Dirac operator in quenched QCD if the quark mass is in the super-critical region.\textsuperscript{1} Through the Banks–Casher relation \textsuperscript{7} this would seem to lead to the conclusion that an Aoki condensate always exists in the supercritical region, in contrast to what is expected in unquenched QCD. It was shown in Ref. \textsuperscript{8} that this is not the case, if one defines the Aoki phase as that region of the phase diagram where Goldstone bosons associated with the symmetry breaking occur. It was argued that regions may exist where the condensate does not vanish because of the existence of a density of \textit{exponentially localized} near-zero modes, \textit{without} any of the corresponding long-range physics usually associated with spontaneous symmetry breaking (SSB). This phenomenon is an artifact of the quenched approximation. It follows that an effective-theory investigation of the phase structure following Ref. \textsuperscript{4} should also be possible for the quenched theory, and it is that aspect of the project that I will report on here.

However, in setting up an effective lagrangian for the quenched theory which is also suitable for non-perturbative questions such as the phase structure of the theory, one runs into the second problem. The leading-order effective lagrangian in standard quenched chiral perturbation theory \textsuperscript{9} contains a potential (the term linear in the quark-mass matrix) which has a saddle point extremum at $\Sigma = 1$, where $\Sigma$ is the $U(N|N)$ valued non-linear field describing the Goldstone multiplet of quenched QCD (with $N$ flavors). However, this saddle-point is not a minimum of the

\textsuperscript{1}Defined as the region where the Wilson–Dirac operator \textit{in principle} can have exact zero modes.

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potential. Worse, if one takes this effective lagrangian literally, one would conclude that the condensate in the ghost-quark sector is different from that in the physical quark sector. Clearly, this is nonsense, since the ghost-quark sector is only introduced to give a path-integral definition of quenched QCD - ghost quarks have no business of living a life of their own.

It turns out that a careful reanalysis of the symmetries of quenched QCD (in particular those of the ghost sector) leads to a resolution of the second problem. While chiral perturbation theory (ChPT) starting from the effective lagrangian constructed here (and in Ref. [3]) [11], these two effective lagrangians are different. The one I will discuss here makes it possible to investigate the existence of an Aoki phase in quenched QCD following the lines along which this was done for unquenched QCD [4].

2. Rigorous definition of quenched QCD

We will begin with a discussion of the lagrangian for quenched QCD in the continuum. The euclidean lagrangian for quenched QCD is

\[ \mathcal{L} = \bar{q} D q + \bar{q} M q + \bar{\mathcal{q}}^\dagger D \mathcal{q} + \bar{\mathcal{q}}^\dagger M \mathcal{q} + \bar{\mathcal{q}}^\dagger \mathcal{q} \]  

(1)

Here the Grassmann variables \( q \) and \( \mathcal{q} \) describe the physical (valence) quarks, and the complex c-number field \( \bar{q} \) describes the ghost quarks. Note that, unlike \( \mathcal{q}, \bar{\mathcal{q}} \) is not an independent variable. The path integral over \( \bar{q} \) and \( \bar{\mathcal{q}} \) is only well defined if \( \bar{\mathcal{q}} \) is the hermitian conjugate of \( \mathcal{q} \), and if the eigenvalues of the quark-mass matrix \( M \) have a positive real part (in euclidean space, \( D \), the Dirac operator, has purely imaginary eigenvalues). It follows that the kinetic term of the ghost quarks couples \( \bar{\mathcal{q}} L = \frac{1}{2} (1 + \gamma_5) \bar{q} \) to \( \bar{\mathcal{q}} R = \bar{\mathcal{q}}^\dagger \) (1 - \( \gamma_5 \)). This is unlike the physical sector, where the left- and right-handed projections of \( q \) and \( \bar{q} \) maybe defined independently.

Having a convergent integral defining the ghost sector of the theory is a step closer to defining the full path integral for quenched QCD non-perturbatively. However, the path integral still needs to be regularized, and we will do so here employing Wilson fermions. This is done by replacing \( D \) by the naive nearest-neighbor covariant Dirac operator on the lattice, and by adding the Wilson mass term

\[ W q = \bar{q}^\dagger W \bar{q}, \]  

(2)

with

\[ (W q)(x) = -\frac{r}{2} \sum_{\mu} (U_{\mu}(x) q(x + \mu) \]  

(3)

\[ + U_{\mu}^\dagger(x - \mu) q(x - \mu) - 2q(x)). \]

Now a new problem arises: the eigenvalues of \( D + W + M \) may have negative real parts for certain gauge fields if the quark masses in \( M \) are close to their critical values. This would make the ghost-quark path integral ill-defined. We may fix this problem as follows. We start over with unquenched lattice QCD with Wilson fermions. The path integral for this theory exists, irrespective of the properties of the Wilson–Dirac operator. In this theory, we rotate each quark field:

\[ q \to e^{i \frac{\pi}{2} \gamma_5} q, \quad \mathcal{q} \to \mathcal{q} e^{i \frac{\pi}{2} \gamma_5}. \]  

(4)

This leads to a quark lagrangian

\[ \mathcal{L}_{\text{quark}} = \mathcal{L}(D + i \gamma_5(W + M)) q. \]  

(5)

Note that the operator \( D + i \gamma_5(W + M) \) is anti-hermitian, and also that the axial rotation of Eq. (4) is not anomalous (since it rotates both \( W \) and \( M \)). We may now define quenched QCD with Wilson fermions by adding for the ghost sector the lagrangian

\[ \mathcal{L}_{\text{ghost}} = \bar{\mathcal{q}}^\dagger(D + i \gamma_5(W + M)) \bar{q} + \epsilon \bar{\mathcal{q}} \]  

(6)

The lagrangian \( \mathcal{L}_{\text{glue}} + \mathcal{L}_{\text{quark}} + \mathcal{L}_{\text{ghost}} \) defines quenched QCD with Wilson fermions in the limit \( \epsilon > 0 \).

3. Symmetries of quenched QCD

In order to construct an effective theory, we need to identify the symmetries of the theory first. Because of the fact that \( \bar{\mathcal{q}} \) is not independent of \( \mathcal{q} \), it turns out that the symmetries of quenched QCD are in fact different from those discussed in Ref. [9].
It is instructive to first consider the physical and ghost sectors separately. Ignoring the Wilson mass term \( W \), the symmetries of the physical sector are

\[
q_{L,R} \rightarrow V_{L,R} q_{L,R}, \quad \overline{q}_{L,R} \rightarrow \overline{q}_{L,R} V_{L,R}^{-1}, \quad V_{L,R} \in GL(N).
\]

The symmetry group is \( GL(N)_L \times GL(N)_R \) (for \( N \) flavors of physical quarks).

As mentioned before, the kinetic term of the ghost quarks couples left- and right-handed ghost quarks, and thus the transformation of the right-handed fields determines that of the left-handed fields:

\[
\tilde{q}_R \rightarrow V \tilde{q}_R \Rightarrow \tilde{q}_L \rightarrow V^{-1} \tilde{q}_L, \quad V \in GL(N).
\]

In other words, the symmetry group in the ghost sector is just the group \( GL(N) \). The vector subgroup in the physical sector (defined as the symmetries which leave the Wilson mass term invariant) is the diagonal subgroup \( GL(N)_L \times GL(N)_R \); in the ghost sector the Wilson term \( \tilde{q}^\dagger W \tilde{q} = \tilde{q}_L^\dagger W \tilde{q}_L + \tilde{q}_R^\dagger W \tilde{q}_R \) is invariant when \( VV^{-1} = 1 \), and the vector subgroup is the group \( U(N) \subset GL(N) \). One may introduce spurion fields for the mass terms in order to trace the quark-mass dependence of the effective theory in the usual way.

Taking into account also transformations between physical and ghost quarks, the full symmetry group is

\[
G = \left\{ (V_L, V_R) \in GL(N|N)_L \times GL(N|N)_R \mid V_{Lgg}\text{body} = V_{Rgg}\text{body} \right\}.
\]

\[
V_L = \begin{pmatrix} V_{Lqq} & V_{Lqg} \\ V_{Lgq} & V_{Lgg} \end{pmatrix}.
\]

Here \( qq, \ gq, \ eq \) denote the quark-quark, quark-ghost, etc. \( N \times N \) blocks of the \( 2N \times 2N \) matrix \( V_L \), and the subscript “body” refers to the \( c \)-number parts of the matrix elements of the \( gg \) block.

4. Effective theory for quenched QCD

We start with the construction of the effective theory in the continuum limit, using the symmetries of quenched QCD without \( W \). As usual, we introduce a non-linear field \( \Sigma = \exp(\Phi) \) transforming like \( \Psi_L \overline{V}_R \) (with \( \Psi_L = (q_L, \overline{q}_L) \), \( \overline{V}_R = (\overline{q}_R^L, \overline{q}_L) \), etc.):

\[
\Sigma \rightarrow V_L \Sigma V_R^{-1} \Rightarrow \Sigma^{-1} \rightarrow V_R \Sigma^{-1} V_L^{-1}.
\]

The field \( \Phi \) can be written as

\[
\Phi = \begin{pmatrix} i\phi_1 + \phi_2 \\ \chi \phi \end{pmatrix},
\]

where \( i\phi_1 + \phi_2 \) parameterizes the coset \( GL(N)_L \times GL(N)_R/GL(N) \), and \( \phi \) parameterizes the coset \( GL(N)/U(N) \). The order \( p^2 \) invariant chiral lagrangian is then given by

\[
\mathcal{L}_{\text{eff}} = \frac{1}{8} f^2 \text{str} (\partial_\mu \Sigma \partial_\mu \Sigma^{-1}) - v \text{ str } M (\Sigma + \Sigma^{-1}).
\]

In order to arrive at this result, one also makes use of a parity symmetry.

Note that the lagrangian depends only on the field \( \phi \equiv \phi_1 - i\phi_2 \), and not on its hermitian conjugate. There is therefore no redundancy in the fields describing the Goldstone sector, but in order to set up a path integral for the effective theory, we need to specify the contour along which this field is integrated. We specify the contour by taking \( \phi \) to be real, i.e. we set \( \phi_2 = 0 \). While we do not have a “derivation” of this choice, it is a sensible choice for the following reasons. First, the same issue arises for the euclidean path integral in the unquenched case. There, after continuation to Minkowski space, one imposes the condition \( \mathcal{7} = \bar{q} \gamma_0 \), which restricts all \( GL(N) \) groups to their unitary subgroups, and \( \Sigma \) then describes the coset \( U(N)_L \times U(N)_R/U(N) \). Our choice in the quenched case is consistent with this.\(^2\) One can in fact demonstrate that the quenched group integral relevant in the so-called \( c \)-regime (see for example Ref.\(^3\)) is independent of the quark mass.

\(^2\)Most likely, it makes little sense to try continue the quenched theory to Minkowski space.
only for this choice of the contour [14,15], as the quenched partition function should be. Finally, we note that, expanding around \( \Sigma = 1 \), ChPT does not depend on the choice of contour, and in fact coincides with standard quenched ChPT as developed in Ref. [9].

The restriction to \( \phi \) real reduces the symmetry group of the effective theory, because this condition reduces the field space available. However, one can show that this does not affect the chiral symmetry \( [3] \), which changes the properties of parity in the first term is again a consequence of the rotation \( [4] \). Carrying out this shift, the effective potential becomes (for degenerate quark masses)

\[
V_{\text{eff}} = -ic_1 \text{str} (\Sigma - \Sigma^{-1}) + c_2 \left( (\text{str} \Sigma)^2 + (\text{str} \Sigma^{-1})^2 \right) + c_3 \text{str} \Sigma \text{str} \Sigma^{-1} + c_4 \left( (\text{str} \Sigma)^2 + (\text{str} \Sigma^{-1})^2 \right),
\]

where

\[
c_1 \sim \Lambda^3 + a \Lambda^5, \quad c_{2,3,4} \sim m^2 \Lambda^2 + ma^2 \Lambda^4 + a^2 \Lambda^6,
\]

with \( \Lambda \) of order the QCD scale. The minus sign in the first term is again a consequence of the rotation \( [4] \), which changes the properties of parity symmetry \( [3] \). I am leaving out terms which appear as a consequence of the anomaly, as they will not change the results discussed in the following.

5. Analysis of the effective potential

While we will be interested in the two-flavor theory, let us look at the one-flavor theory as a warm up. Substituting

\[
\Sigma = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix},
\]

the effective potential becomes, including the \( \epsilon \) term of Eq. \( [6] \)

\[
V_{\text{eff}} = 2c_1 \sin \phi + 2c_1 \sinh \hat{\phi} + 2\epsilon \cosh \hat{\phi} + \ldots ,
\]

where the dots refer to the higher order terms in Eq. \( [14] \). We see that the effective potential is complex! It looks like the usual procedure of minimizing a potential to find the vacuum structure breaks down. However, in the context of euclidean field theory, “finding the vacuum” is equivalent to performing a saddle-point approximation, and this is what we propose to do in this case as well. For the leading-order effective potential shown in Eq. \( [17] \), the saddle-point solution is \( [3] \)

\[
\phi = -\text{sign}(c_1) \frac{\pi}{2}, \quad \hat{\phi} = -i \text{sign}(c_1) \frac{\pi}{2}.
\]

It is straightforward to check that shifting the fields \( \phi \) and \( \hat{\phi} \) by this solution precisely “undoes” the axial rotation \( [4] \). Carrying out this shift, the effective potential becomes (for \( \epsilon \to 0 \))

\[
V_{\text{eff}} = 2|c_1| (-\cos \phi + \cosh \hat{\phi}).
\]

(\text{which has a minimum at } \phi = \hat{\phi} = 0).\text{ Note that the solution } [18], \text{ translated in terms of quark condensates, means that}

\[
\langle \text{tr } \bar{q}q \rangle = \langle \text{tr } \hat{\bar{q}}\hat{q} \rangle,
\]

as one would expect: the ghost condensate reproduces the physical condensate.

There is more to say about the one-flavor case, especially if one also takes into account the
anomaly. However, instead, I will now discuss the two-flavor case, assuming that no non-trivial condensates form in the singlet sector. First, we shift the singlet fields $\phi_0$ and $\phi_1$ by the saddle-point solution \[13\], in order to undo the axial rotation \[14\]. We then set the shifted singlet fields equal to zero, in accordance with the assumption that no non-trivial singlet condensate occurs. Then, for the two-flavor theory, we substitute

$$
\Sigma = \left( \begin{array}{cc} \exp(i\sigma_3\phi/2) & 0 \\ 0 & \exp(\sigma_3\hat{\phi}/2) \end{array} \right),
$$

pointing the condensate in the third isospin direction without loss of generality. The effective potential can now be written as

$$
V_{\text{eff}} = 4|c_1| \left( -\cos(\phi/2) + \cosh(\hat{\phi}/2) \right) -4(2c_2 + c_3) \left( \cos(\phi/2) - \cosh(\hat{\phi}/2) \right)^2 + 4c_4 \left( -\cos \phi + \cosh \hat{\phi} \right).
$$

First, consider the case $2c_2 + c_3 = 0$. In that case the physical ($\phi$) and ghost ($\hat{\phi}$) sectors decouple. In the physical sector, the potential maybe minimized, and the solution is that found already in Ref. \[14\] for the unquenched case. This solution is

$$
\phi = 0, \quad c_4 > -|c_1|/4,
$$

$$
\phi \neq 0, \quad c_4 < -|c_1|/4.
$$

The solution $\phi \neq 0$ corresponds to a non-zero $\bar{q}\gamma_5\sigma_3 q$ condensate \[^3\] and corresponds to an Aoki phase. From Eq. \[24\], it is clear that, in order to find a non-trivial condensate, $c_4$ and $c_1$ have to be of the same order of magnitude. Referring to Eq. \[16\], this implies that an Aoki phase can only occur if $am \sim a^3\Lambda^3$. In other words, approaching the continuum limit, the Aoki phase is a narrow “finger” of width $a^3$ in the $am-g$ phase diagram \[1].

As expected, this solution is completely reproduced in the ghost sector, if one performs a saddle-point extremization on the $\hat{\phi}$ part of the effective potential. When one turns on the coupling $2c_2 + c_3$, the solution actually remains the same, as can be checked from the combined saddle-point equations for the fields $\phi$ and $\hat{\phi}$ in that case.

There is one embarrassment in the ghost sector which should be pointed out. For $c_4 < 0$, the effective potential is unbounded from below for large values of the non-compact field $\hat{\phi}$. However, one can check that the saddle-point is still locally stable, and we believe that therefore our physics conclusions are valid, despite this embarrassment. The effective theory describes the theory only well below the chiral-symmetry breaking scale, and clearly should not be trusted at values of the fields above this scale. It is because of this that we believe that local stability of the saddle point solution is sufficient to conclude that the solution in the physical sector is reproduced in the ghost sector, as it should.

6. Conclusions

In this talk, I have shown how quenched lattice QCD with Wilson fermions can be defined rigorously. This allows us to decide what the symmetries of this theory are, which then may be used to construct the effective chiral lagrangian for quenched QCD, including lattice spacing effects, in a systematic expansion in the quark mass and the lattice spacing. This precise analysis exposes a flaw in the original construction \[2\] of the quenched effective theory, but this does not change the perturbative expansion based on the effective lagrangian. All ChPT results based on the lagrangian of Ref. \[2\] therefore remain valid.

If one wants to investigate a non-perturbative issue such as the possible existence of an Aoki phase, the lagrangian described here must be used. I sketched the arguments leading to the conclusion that the two-flavor quenched theory exhibits the same possible phase structure as the unquenched two-flavor theory, which was investigated using an effective-theory approach before \[10\]. Of course, whether the Aoki “fingers” are realized in two-flavor QCD depends on the values of the coefficients multiplying the powers of $m$ and $a$ in $c_4$, and can thus not be answered within the framework of effective field theory. Note also that the relevant low-energy constants

\[^3\]In terms of the original quark fields before the axial rotation \[\ref{4}\].

\[^4\]Subtracting first the $a\Lambda^5$ term in $c_1$ (cf. Eq. \[15\]); see Ref. \[4\] for more detail.
could turn out to be different in the quenched and unquenched cases, so that the appearance of Aoki fingers in the quenched case does not imply their existence in the unquenched case, and vice versa.

Finally, I skipped over many technical details, including in particular a more thorough discussion of the anomaly and the singlet sector. For a more complete analysis I refer to Ref. 3, which I hope will appear soon.

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