Determination of quantum phases via continuous measurements

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We derive how the quantum phase of a complex system can be deduced from the record of a continuous and diffusive quantum measurement. We show that weak continuous probing of operators commuting with parts of the Hamiltonian defining different quantum phases lead to qualitatively distinct power spectra of the probe signals. This may be exploited to determine and define quantum phases of open systems based on the measurement record alone. This is in contrast to existing methods that use repeated destructive measurement of a single state, requiring very low entropy. We test the resulting phase criterion in a numerical simulation of the Bose-Hubbard model under two different, experimentally feasible measurements that satisfy complementary commutation relations. At low measurement strength, our criterion yields a critical point for the superfluid to Mott-Insulator transition in reasonable agreement with the quantum phase transition in the ground state of the closed system in the thermodynamic limit. At higher measurement strengths, the system’s response enters a Zeno regime and becomes dominated by whichever phase is being measured.

The notion of different quantum phases allows faithful descriptions of complex systems in simpler terms than a microscopic description. Distinct phases span wide areas in parameter space characterized by the structure of fundamental excitations, which govern the system’s equilibrium properties and response to perturbations. Quantum phases are defined via their transitions at zero temperature in the thermodynamic limit. Despite this unrealistic setting, their study is experimentally relevant because the phases often extend beyond zero temperature and apply to finite sized systems.

Quantum phases are used to investigate a wide range of physical phenomena, including electronic, magnetic and optical properties of solid state systems, nuclear physics, and cosmological topological defects. However, for open and finite quantum systems, quantum phases remain ill defined. This is particularly unsatisfying because such systems become increasingly relevant with the advent of experiments that offer quantum limited access to the system with high controllability and low entropy.

Since the seminal realization of the Bose-Hubbard model, ensembles of optically trapped, ultracold gases have emerged as a powerful tool for the experimental study of quantum phases. In a quantum simulator, a desired Hamiltonian is realized by exploiting interactions of light with the internal structure of trapped atoms, tuning their interactions using a Feshbach resonance or adjusting the tunneling between sites of an optical lattice via the depth of the optical potentials. The system is cooled close to its ground state and released from the trap. The resulting expansion reveals the populations of momentum modes and projectively measures phase-coherence, a hallmark of long-range order. Alternatively, the system’s dynamics are frozen in a quantum gas microscope and a site-resolved measurement of atomic populations determines the quantum phase.

In both cases, the final measurement destroys the system and the experiment needs to be performed many times to acquire signal or statistics. Such an averaging introduces variances into otherwise well defined parameters, e.g. particle numbers. More importantly, such a measurement determines the properties of a particular quantum state of the system, but not of that system’s quantum phase. Phase coherence is a property of a superfluid’s ground state, but not of the superfluid phase. To illustrate this distinction, consider interacting bosons hopping on a lattice in a superfluid. It is straightforward to find pure states in this system which do not exhibit phase coherence, e.g. any state with a zero variance in population on each site. The preparation of such a state represents a quench of the system and initiates non-trivial unitary evolution. Nevertheless, the dynamics are best described as a highly excited state of a superfluid. More generally, the expectation value of any operator is not suited to define a quantum phase, because expectation values are properties of states, while the quantum phase is a property of the system’s entire spectrum.

Although modifications of complex systems by measurement have been studied, the fundamental question if the quantum phase of a complex system can be determined from the record of a continuous measurement alone, remains to the best of our knowledge unanswered. If this is possible, the resulting procedure may serve as a defining feature of quantum phases for open and finite-sized quantum systems.

In this Letter, we answer this question in the affirmative and give a criterion for the determination, and indeed the definition, of quantum phases of open systems. The criterion relies only on the obtained measurement record of a single experimental run, is based on fundamental physical principles, and relevant to current state-of-the-art experiments. The fundamental mechanism is a continuous measurement that is not a quantum non-demolition measurement. Such a measure-
ment reveals the time-average of an operator’s expectation value and simultaneously probes the system’s dynamical response via the measurement back-action. We exploit this mechanism to disturb the system and record its reaction to the disturbance. We numerically verify the effectiveness of our criterion in the Mott-Insulator to superfluid transition of the 1D Bose-Hubbard model, and demonstrate how the measurement strength itself becomes a parameter of an open system’s phase diagram.

**Phase Determination** – Consider an ergodic system that exhibits a single quantum phase transition, such that we may write the Hamiltonian as
\[
\hat{H} = \hat{H}_0 + \alpha \hat{H}_1.
\]  
(1)
The critical value \(\alpha_c\) separates two quantum phases of \(\hat{H}\). For \(\alpha < \alpha_c\), the ground state and linear response of \(\hat{H}\) are well approximated by that of \(\hat{H}_0\), while for \(\alpha > \alpha_c\) the ground state and linear response has the characteristics of \(\hat{H}_1\). Let \(\hat{M}_0\) be a hermitian operator, satisfying \([\hat{H}_0, \hat{M}_0] = 0\), \([\hat{H}_1, \hat{M}_0] \neq 0\). A probe, which diffusively measures \(\hat{M}_0\) with measurement strength \(\gamma\), will disturb the system through measurement back-action and result in a measurement record \(I(t)\). The measurement signal \(I(t)\) is governed by the stochastic rule
\[
dI(t) = \gamma \langle \hat{M}_0(t) \rangle dt + \sqrt{\gamma} dW,
\]  
(2a)
where \(\langle \cdot \rangle\) denotes the expectation value and \(dW\) is a Wiener increment. The state of the system conditioned on the measurement outcome evolves according to
\[
d|\tilde{\psi}(t)\rangle = \left[ -i\hat{H} - \frac{\gamma}{2} \hat{M}_0^2 + I(t)\hat{M}_0 \right] dt|\tilde{\psi}(t)\rangle,
\]  
(2b)
with \(\hbar = 1\) and \(\tilde{\psi}\) denotes a non-normalized vector. The first term in Eq. \([2]\) describes the unitary evolution. The second and third term include dissipation associated with the measurement and drive the state towards an eigenstate of the measurement operator in accordance with the recorded result.

Two-time correlations of the measurement record are expressed in the power spectral density (PSD) \(S(\omega) = (\gamma T)^{-1} \int_0^T e^{-i\omega t} I(t) dt|^2\). In a stationary state and in the weak measurement regime, the PSD can be calculated from perturbation theory \([2] [25]\)
\[
S(\omega) \propto \sum_{ij} \frac{\Gamma_{ij}(j|\hat{M}_0|i)}{(\omega_{ij} - \omega)^2 + \Gamma_{ij}^2},
\]  
(3)
with \(|i\rangle\) the Hamiltonian’s eigenstates with eigenvalues \(E_i\), \(\omega_{ij} = E_i - E_j\) and \(\Gamma_{ij} = \langle i|\hat{M}_0^2|i\rangle + \langle j|\hat{M}_0^2|j\rangle - 2\langle i|\hat{M}_0|i\rangle \langle j|\hat{M}_0|j\rangle\). The PSD is a sum of Lorentzians, each centered at a transition in the Hamiltonian with height proportional to the amplitude of that transition being caused by the measurement. For \(\alpha = 0\), the Hamiltonian’s eigenstates are also eigenstates of \(\hat{M}_0\) and the PSD is given by a sum of delta functions at \(\omega = 0\). For a weakly disturbing measurement with \(0 < \alpha \ll \alpha_c\), the PSD broadens due to two effects. The diagonal elements in \([3]\) acquire non-zero widths and \(\hat{M}_0\) now causes transitions between different eigenstates. To first non-vanishing order in \(\alpha\), these transitions are favored to occur between states with low energy difference. Summarizing both of these effects, the measurement of an operator whose commutator with the system Hamiltonian is small, will lead to a PSD with a narrow, Lorentzian line-shape. Such a line-shape indicates that the measurement excites states that are well described as perturbations of the Hamiltonian defining the phase the system is in. For \(\alpha > \alpha_c\) the system is in the phase defined by \(\hat{H}_1\), and the states excited by the back-action of measuring \(\hat{M}_0\) cannot be considered perturbations of the system’s eigenstates. In that case, the PSD will have significant contributions at non-zero frequencies. These elemental concepts lead to a quantitative and measurable criterion for the phase of the measured quantum system: A system is found in a particular quantum phase, if a measurement of an operator commuting with the Hamiltonian defining that phase results in a PSD that is well described by a narrow Lorentzian distribution.

This criterion is based on the system’s response to the disturbance brought about by the measurement, as measured by the two-time correlation of the measurement output \([26]\). For small \(\gamma\), the effect of a long measurement is identical to acting on the system with the measurement operator \([26] [27]\). If the operator excites higher lying modes, this corresponds to a quench of the system \([22]\), indicating that the measurement operator does not commute with the Hamiltonian determining the system’s linear response. Based on fundamental properties common to all quantum systems and on the system’s response to perturbations, the criterion does not depend on the system being in a particular state or at low temperature. This remains true independent of the particular operator being measured. That is, the criterion finds the same phase transition points if we consider a different measurement \(\hat{M}_1\) with \([\hat{M}_1, \hat{H}_0] \neq 0\) and \([\hat{M}_1, \hat{H}_1] = 0\).

**Probed Bose-Hubbard Model** – We demonstrate the validity of our criterion in the experimentally relevant 1D Bose-Hubbard model. This model provided the first demonstration of a quantum phase transition in ultracold atoms \([8]\) and remains important for studies of dissipative and driven quantum systems \([8] [17] [18]\). The Hamiltonian reads
\[
\hat{H}_{BH} = -J \sum_{<j,k>} \left( \hat{b}_j^\dagger \hat{b}_k + \hat{b}_k^\dagger \hat{b}_j \right) + \frac{U}{2} \sum_j \hat{b}_j^\dagger \hat{b}_j (\hat{b}_j^\dagger \hat{b}_j - 1),
\]  
(4)
where the bosonic field operator is expanded in Wannier functions \(\Psi(x,t) = \sum_j \hat{b}_j(t) w_j(x)\) and \(J\) and \(U\) are the tunneling strength and on-site interaction respectively. For \(U/J\) below the critical value, the system’s ground
state exhibits long range phase-coherence and is a super-fluid. Above that critical value, the ground state features Fock-states of particle numbers on each site and the system is in the Mott-Insulator state.

Let us now dispersively probe this system with an optical cavity aligned with the trapping lattice. The probe light is described as

$$\hat{a}(t)f_a(x, \omega_L)e^{-i\omega_L t},$$

with $\omega_L$ the probe frequency and $f_a(x, \omega_L)$ the spatial mode function. We ignore the dimensions perpendicular to the cavity axis and treat the system in 1D. The probe light is detuned by $\Delta$ from the closest atomic transition, to which it couples with strength $g$. After adiabatically eliminating the excited atomic state we find the atom-light coupling

$$\mathcal{H}_{\text{int}} = \sum_{jk} M_{jk} \hat{b}_j \hat{b}_k \hat{a}^\dagger \hat{a},$$

with the measurement matrix elements $M_{jk} = \frac{g^2}{\Delta} \int dx |f_a(x, \omega_L)|^2 w_j^*(x) w_k(x)$. Assuming a large cavity decay rate compared to atomic timescales, we eliminate the measurement field and the dynamics of a single realization of an experiment are governed by (2b), with the measurement operator $\hat{M} = \sum_{jk} M_{jk} \hat{b}_j \hat{b}_k$. For a Fabry-Pérot cavity we have $f_a(x, \omega_L) \propto \cos(k_L x)$ and it is straight-forward to calculate $M_{jk}$ for a given $\omega_L$.

We focus on two relevant cases, namely where the probe has twice the period of the trapping potential and when the probe and the lattice have the same periodicity, but a $\pi/2$ phase shift.

In the former case, we find the dominant contributions to give $\hat{M} = \hat{M}_{\text{pop}} = m_{\text{pop}} \sum_k \hat{b}_k \hat{b}_k$, where $m_{\text{pop}}$ is a constant calculated from the mode overlaps. The probe sums atomic populations on all even lattice sites. This operator commutes with the interaction term in (4), but not the hopping. In the second case, the dominant contributions are $M_{jk} = u \hat{b}_j \hat{b}_j + v \hat{b}_j \hat{b}_{j+1} + H.C.$, where $u$ and $v$ are again calculated from the overlap. The probe measures the total population and the sum of all nearest-neighbor coherences. The total population is conserved and leads to a constant offset to the expectation value without back-action. This effect can be absorbed into a redefinition of the Hamiltonian. What remains is the sum over coherences, $\hat{M}_{\text{coh}} = v \sum_j \hat{b}_j \hat{b}_{j+1} + H.C.$, which commutes with the hopping term in the Hamiltonian but does not commute with the interaction. The two operators $\hat{M}_{\text{pop}}$ and $\hat{M}_{\text{coh}}$ are continuous measurement analogues of a quantum gas microscope and a time-offlight image respectively and we can test our criterion in the Mott-Insulator and superfluid phases.

We numerically simulated the described Bose-Hubbard system with 6 atoms on 6 sites under the influence of both measurement operators. To compare across different ratios of $U/J$, we rescale the Hamiltonian such that its spectrum always spans the same frequency (20 in dimensionless units). Before we move on to the dynamics of a measurement, let us consider the weak measurement regime perturbatively. Measurement back-action acts as a broad-band heat-bath at infinite temperature, and we may assume that all states are populated equally in the long-time limit. The perturbative expressions for the PSDs are given by Eq. (3) and plotted in Fig. 1 for the two different measurements.

As expected, the PSDs acquire Lorentzian line-shapes for each measurement in regimes dominated by the phase commutating with the measurement operator, while distinct peaks appear at higher energies in the converse regime, indicating transitions induced by back-action. The region of criticality leaves a broad-line shape lacking resolved peaks in the PSD. These results support our criterion identifying quantum phases, but are based on a perturbative treatment of a highly mixed state. To test an experimentally relevant situation, we numerically simulate the stochastic dynamics of the system starting from the ground state.

To quantify the phase, we normalize the obtained PSD and calculate its overlap with a Lorentzian distribution centered at $\omega = 0$. Weak measurement of either operator for sufficiently long time reveals the system’s phases via the line-shape of the measurement record’s PSD as shown in Fig. 2 (a). The measurement with both opera-
FIG. 2. (a) Maximum overlap of the measurement records with a Lorentzian distribution at \( \omega = 0 \) for measurement of \( \hat{\mathcal{M}}_{\text{pop}} \) with \( \gamma = 0.1 \) (crosses) and \( \hat{\mathcal{M}}_{\text{coh}} \) with \( \gamma = 0.001 \) (circles). (b) Phase coherence before (crosses) and after (circles) continuous measurement of the coherences for weak measurement strength \( \gamma = 0.01 \), normalized to the expectation value in the superfluid state. Values after the measurement are obtained as time-averages exploiting the system’s ergodicity. Dotted line gives the phase transition point in the thermodynamic limit and lines are added to guide the eye.

FIG. 3. Phase diagrams depicting the overlap of the PSD of a continuous measurement with a Lorentzian distribution. (a) measurement of \( \hat{\mathcal{M}}_{\text{coh}} \). (b) measurement of \( \hat{\mathcal{M}}_{\text{pop}} \). The initial state is the ground state of the system and the measurement time is \( T = 2000 \). We verified that the results remain identical when cutting off initial transients. Dashed line indicates the phase transition point in the thermodynamic limit [32].

The authors give the same transition and the agreement with its value in the thermodynamic limit is reasonable, considering the small system size [32]. Measuring \( \hat{\mathcal{M}}_{\text{coh}} \) yields higher contrast, but even for the measurement of \( \hat{\mathcal{M}}_{\text{pop}} \) the overlap jumps by more than five standard deviations of its value within each phase across the phase boundary. This signature can be directly compared to the commonly used metric, the expectation value of \( \hat{\mathcal{M}}_{\text{coh}} \) in the system’s ground state plotted in Fig. 2 (b).

Even a weak measurement leaves the system in a highly excited state. The phase coherences before and after the measurement are plotted in Fig. 2 (b) and show that for all but the weakest interactions, back-action has destroyed phase coherence in the system. The ground state population integrated over the measurement time is in the single percent range for \( \gamma = 0.01 \). Remarkably, the continuous measurement still gives a clear answer as to the phase of the system. This illustrates the robustness of our proposal to define a system’s quantum phase through our criterion and a continuous measurement, rather than a property of its ground state in the thermodynamic limit.

Strong measurement – While our criterion is motivated by the weak measurement regime, it may be used to define quantum phases for strong measurement strengths too. The measurement strength becomes a free parameter and additional dimension of the phase diagram, which now depends on the operator being measured. The two phase diagrams in Fig. 3 show how strong measurements force the system to evolve through states with well-defined values of the operator being probed. Due to the commutation relations, this regime resembles the corresponding phase of the Hamiltonian. Such regimes have been identified previously as a dynamical phase transition into a Zeno regime [25, 33] and our criterion allows to connect these results to the zero temperature regime. In the given example, we demonstrate how a strong measurement of coherences turns a Mott-Insulator into a superfluid, and conversely a strong measurement of populations turns a superfluid into a Mott-Insulator, if the dynamic response of the system is used as the metric defining different phases.

Summary and Outlook – We have investigated the notion of quantum phases in an open, continuously measured quantum system. We found a criterion based on commutation relations between system and measurement operators that allows to determine the phase from such a measurement. Our criterion relies solely on the obtained measurement record, is based on physical considerations, applicable to many experimentally relevant situations and does not rely on a particular preparation of the system. We verify the validity via a numerical simu-
lation of a continuously measured Bose-Hubbard system. Our methods allow to lift the measurement strength to an independent system parameter and confirm the existence of Zeno dynamics for quantum phases.

We have focused on the situation where the system Hamiltonian is known, but in many situations of interest, this is not the case. The measurement operator acting on a system depends on the probe and its interaction with some of the system’s constituents, e.g., electrons interacting with a light field. The probe-system interaction may well be better known than the system Hamiltonian. It is a promising avenue to investigate what can be deduced about a system’s Hamiltonian from its response to the back-action of a measurement. Furthermore, our criterion may be generalized to topological \[34\] and dynamical phase transitions \[35\], which also have been simulated successfully. Beyond quantum simulators, applying our results to condensed matter systems may pave a way towards extending the parameter space of quantum phases with desirable properties.

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Supplementary Material

Determination of Quantum Phases via Continuous Measurements

IMPLEMENTATION OF MEASUREMENT OPERATORS IN THE BOSE-HUBBARD MODEL

In this section we discuss how to implement a measurement of $\hat{M}_{\text{pop}}$ and $\hat{M}_{\text{coh}}$. In the dipole approximation, the full atom-light Hamiltonian is given by

$$\hat{H} = \hat{H}_A + \hat{H}_{\text{BH}} + \hat{H}_a + g\hat{a} \int \hat{\Psi}_+(x,t)f_a(x,\omega_L)\hat{\Psi}_-(x,t)dx + g\hat{a}^\dagger \int \hat{\Psi}_-(x,t)f_a(x,\omega_L)\hat{\Psi}_+(x,t)dx,$$

(S1)

where $\hat{\Psi}_{+(-)}$ is the bosonic field operator for the excited (ground) state of the atoms and we have neglected fast-rotating terms. The probe field has frequency $\omega_L$ and $g$ is the coupling strength. The optical lattice and interactions between atoms are captured in $\hat{H}_{\text{BH}}$, see e.g. [S1 S2]. Moreover, we have $\hat{H}_A = \omega_{eg} \int d^3x \hat{\Psi}_+(x,t)\hat{\Psi}_+(x,t)$ the energy of the excited state and $\hat{H}_a = \omega_L\hat{a}^\dagger\hat{a}$ the free Hamiltonian of the cavity light field with real spatial mode function $f_a(x,\omega_L)$.

Including the free evolution of the excited state and the light field into their field operators, $\hat{a} \rightarrow e^{i\omega_Lt}\hat{a}$ and $\hat{\Psi}_+ \rightarrow \hat{\Psi}_+e^{i\omega_{eg}t}$, we find the equations of motion

$$\partial_t\hat{\Psi}_+(x,t) = -ig\int \hat{\Psi}_-(x,t)f_a(x,\omega_L)\hat{\Psi}_+(x,t)e^{i\Delta t}dx$$

(S2a)

$$\partial_t\hat{\Psi}_-(x,t) = -ig\hat{a}^\dagger(t)\hat{\Psi}_+(x,t)f_a(x,\omega_L)e^{i\Delta t}$$

(S2b)

$$\partial_t\hat{\Psi}_+(x,t) = -ig\hat{a}(t)\hat{\Psi}_-(x,t)f_a(x,\omega_L)e^{-i\Delta t},$$

(S2c)

where $\Delta = \omega_L - \omega_{eg}$ is the detuning of the probe from the atomic transition. We assume $\Delta \gg g$ and the initial population state of the atoms is their ground state. In that case, the last equation is dominated by the rotating term and we write

$$\hat{\Psi}_+(x,t) = \frac{g}{\Delta} \hat{a}(t)\hat{\Psi}_-(x,t)f_a(x,\omega_L)e^{-i\Delta t},$$

(S3)

which allows us to adiabatically eliminate the excited atomic state

$$\partial_t\hat{a}(t) = -\frac{ig^2}{\Delta} \hat{a}(t) \int |f_a(x,\omega_L)|^2 \hat{\Psi}_-(x,t)\hat{\Psi}_-(x,t)dx$$

(S4)

$$\partial_t\hat{\Psi}_-(x,t) = -\frac{ig^2}{\Delta} \hat{a}^\dagger(t)\hat{a}(t)\hat{\Psi}_-(x,t)f_a(x,\omega_L)^2.$$  

(S5)

We can drop the subscript for the atomic field operator and obtain a new effective Hamiltonian

$$\hat{\mathcal{H}} = \hat{H}_{\text{BH}} + \frac{g^2}{\Delta} \int |f_a(x,\omega_L)|^2 \hat{\Psi}_+(x,t)\hat{\Psi}_-(x,t)\hat{a}^\dagger(t)\hat{a}(t)dx.$$  

(S6)

The localized Wannier functions form a complete set for the first band of the system and we expand the atomic field operators in terms of these functions

$$\hat{\Psi}(x,t) = \sum_j \hat{b}_j(t)w_j(x).$$

(S7)

By construction, $\hat{H}_{\text{BH}}$ becomes the Bose-Hubbard Hamiltonian in terms of these functions with

$$J = \int w_j(x)^* \left( -\frac{\hbar^2}{2m} + V_l(x) \right) w_{j+1}(x)dx$$

(S8)

$$U = g_s \int |w_j(x)|^4 dx,$$

(S9)

with $V_l$ the lattice potential and $g_s$ the s-wave scattering length in one-dimension.
FIG. S1. Entries of $M_{jk}$ for probe wavelength twice the lattice period, leading to $\hat{M}_{\text{pop}}$ on the left and identical probe and lattice periodicity but a $\pi/2$ phase shift, leading to $\hat{M}_{\text{coh}}$ on the right. Lattice depth equals five recoil energies.

The relevant matrix determining the spatial influence of the measurement is obtained by expressing the integral in Hamiltonian (S6) in terms of the Wannier functions. The resulting Hamiltonian reads

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{BH} + \sum_{j,k} M_{jk} \hat{b}_j^\dagger \hat{b}_k \hat{a}^\dagger \hat{a},$$

(S10)

with

$$M_{jk} = \frac{g^2}{\Delta} \int |f_a(x, \omega_L)|^2 w_j^*(x) w_k(x) dx.$$  

(S11)

The magnitude of the off-diagonal entries in $M$ decay exponentially with their distance to the diagonal, but the nearest-neighbor contribution is of the same order as the hopping term $J$ and should not be neglected. This is especially significant for the case when the spatial mode $|f_a|^2$ has twice the period of the optical lattice. For a Fabry-Pérot cavity, the spatial mode function is a harmonic function and we can easily calculate the integrals for $M_{jk}$. For the settings used to arrive at the measurements of the main text, see Fig. S1.

From Hamiltonian (S10) we have the Heisenberg-Langevin equation

$$\dot{\hat{a}} = -i \sum_{jk} M_{jk} \hat{b}_j^\dagger \hat{b}_k \hat{\alpha} - \frac{\gamma}{2} \hat{\alpha} + \eta + \sqrt{\gamma} \hat{a}_{\text{in}},$$

(S12)

with $\hat{a}_{\text{in}}$ the input vacuum noise and $\eta$ the pump power. We displace $\hat{a}$ by its coherent expectation value

$$\hat{a} \rightarrow \alpha + \hat{a}$$

(S13)

such that we have

$$\dot{\hat{\alpha}} + \hat{\alpha} = -i \sum_{jk} M_{jk} \hat{b}_j^\dagger \hat{b}_k \alpha - i \sum_{jk} M_{jk} \hat{b}_j^\dagger \hat{b}_k \hat{\alpha} - \frac{\gamma}{2} \alpha - \frac{\gamma}{2} \hat{\alpha} + \eta + \sqrt{\gamma} \hat{a}_{\text{in}}.$$  

(S14)

We choose $\alpha$ to be the steady-state of

$$\dot{\alpha} = -\frac{\gamma}{2} \alpha + \eta.$$  

(S15)
such that we have for \( \hat{a} \)

\[
\dot{\hat{a}} = -i\alpha \sum_{j,k} M_{jk} \hat{b}_j^\dagger \hat{b}_k - \frac{\gamma}{2} \hat{a} + \sqrt{\gamma} \hat{a}_{in}
\] (S16)

where we neglected the term coupling the quantum operators directly compared to the term enhanced by \( \alpha \). This equation points to the effective coupling Hamiltonian

\[
\hat{H} = \hat{H}_{BH} + \hat{H}_a + \alpha \sum_{j,k} M_{jk} \hat{b}_j^\dagger \hat{b}_k (\hat{a} + \hat{a}^\dagger),
\] (S17)

from which the stochastic Schrödinger equation used in the main text can be derived using methods from e.g. [S3] [S4].

**LORENTZIAN FILTERING**

To quantify the fit of the probe’s spectrum to a Lorentzian, we calculate the quantity

\[
F(\Gamma) = \int L_\Gamma(\omega) S'(\omega) d\omega, \quad L_\Gamma(\omega) = C/(\Gamma^2 + \omega^2).
\] (S18)

\( C \) is a normalization constant such that \( \int |L_\Gamma(\omega)|^2 d\omega = 1 \). In the measurements of \( \mathcal{M}_{coh} \), we remove the expectation value by subtracting the time average from the measurement record. This amounts to the removal of a Dirac delta function at \( \omega = 0 \) from the PSD. We renormalize the remaining distribution such that \( \int |S'(\omega)|^2 d\omega = 1 \), such that \( F(\Gamma) \leq 1 \), with equality if and only if \( S'(\omega) = L_\Gamma(\omega) \).

For each measurement run, we found \( F(\Gamma) \) to exhibit a unique maximum at a particular \( \Gamma_{max} \). At this width, the overlap quantifies how well the PSD is described by a Lorentzian distribution.

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