Some remarks on configurations of lines
on a certain hypersurface

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Abstract

We show that on a certain hypersurface in $\mathbb{P}^3$ there is a $(q^3 + q^2 + q + 1)q + 1$-symmetric configuration (resp. a $(q^3 + 1)(q^2 + 1)q + 1$) -configuration) made up of the rational points over $\mathbb{F}_q$ (resp. over $\mathbb{F}_{q^2}$) and the lines over $\mathbb{F}_q$ (resp. the lines over $\mathbb{F}_{q^2}$). We also determine the structure of a Lefschetz pencil made by using a line from these lines.

1 Introduction

In this paper, we consider a geometric realization of an abstract configuration. A triple $A, B, R$, where $A, B$ are non-empty finite sets and $R \subset A \times B$ is a relation, is called an abstract configuration if the cardinality of the set $R(x) = \{ B \in B \mid (x, B) \in R \}$ (resp. $R(B) = \{ x \in A \mid (x, B) \in R \}$) does not depend on $x \in A$ (resp. $B \in B$). Elements of $A$ are called points, and elements of $B$ are called blocks. Denoting by $|X|$ the number of elements in a finite set $X$, we set

$$v = |A|, b = |B|, k = |R(x)|, r = |R(B)|.$$

Then, the configuration is called a $(v_k, b_r)$-configuration. We have the relation $kv = br$. Therefore, if $v = b$, then we have $k = r$. In this case, the configuration is called a symmetric configuration. Such a symmetric configuration is called $v_k$-configuration (for details, see Dolgachev[3]).

The most typical example of a geometric realization of an abstract configuration is given by the projective plane over a finite field. Let $p$ (resp. $a$) be a prime number (resp. a positive integer) and let $\mathbb{F}_q$ be a finite field with $q = p^a$ elements. Then, in the projective plane $\mathbb{P}^2$ there are $q^2 + q + 1$ rational points over $\mathbb{F}_q$ and there are $q^2 + q + 1$ lines defined over $\mathbb{F}_q$. We see that $q + 1$ lines pass through each point, and on each line there exist $q + 1$ points. We denote the set of these points by $A$ and the set of these lines by $B$. The relation $R$ consists of the pairs of a point and a line which pass through the point. The triple $A, B, R$ gives a $(q^2 + q + 1)_{q+1}$-symmetric configuration.

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One more typical configuration is given by Kummer surfaces. Let \( C \) be a non-singular complete curve of genus 2 defined over an algebraically closed field of characteristic \( p \neq 2 \). We consider the Jacobian variety \( J(C) \). Then, \( C \) gives a principal polarization, and by a suitable translation we may assume that \( C \) is invariant under the inversion \( \iota \) of \( J(C) \). For a two-torsion point \( a \in J(C)_2 \), we denote by \( T_a \) the translation given by \( a \). Then we have 16 curves \( \{ T_a C \mid a \in J(C)_2 \} \). We consider the quotient surface \( J(C)/\langle \iota \rangle \) and let \( \pi : J(C) \to J(C)/\langle \iota \rangle \) be the projection. Then, we have the set \( \mathcal{A} \) of 16 rational double points of type \( A_1 \) on \( J(C)/\langle \iota \rangle \), and we have the set \( B = \{ \pi(T_a C) \mid a \in J(C)_2 \} \) of 16 rational curves which are conics. The relation \( R \) consists of the pairs of a point and a conic which pass through the point. The triple \( \mathcal{A}, B, R \) gives a 166-symmetric configuration.

In this note, we consider the hypersurface \( S \) in \( \mathbb{P}^3 \) which is defined by

\[
x_0 x_1^q - x_1 x_0^q + x_2 x_3^q - x_3 x_2^q = 0
\]

and show that on this surface there exists a \((q^3 + q^2 + q + 1)_{q+1}\)-symmetric configuration (resp. a \(((q^3 + 1)(q^2 + 1)_{q+1}, (q^3 + 1)(q + 1)_{q^2 + 1})\)-configuration) made up of the rational points over \( \mathbb{F}_q \) (resp. over \( \mathbb{F}_{q^2} \)) and the lines over \( \mathbb{F}_q \) (resp. the lines over \( \mathbb{F}_{q^2} \)).

In the final section, we examine a Lefschetz pencil of the surface \( S \), and determine the singular fibers and sections of the fiber space. As a corollary, we give a proof of the fact that the Fermat surface of degree \( q + 1 \) is unirational, which was long ago proved by Shioda [9] (also see Rudakov-Shafarevich [6]). For \( p = 3 \), this surface is a K3 surface. In fact, it is known that in this case the surface is a supersingular K3 surface with Artin invariant 1. The Lefschetz pencil is a quasi-elliptic surface with 10 singular fibers of type IV (for the existence of such a quasi-elliptic surface, see H. Ito [4]), and have a \((280_4, 112_{10})\)-configuration on this K3 surface. Such a structure is related to the theory of Leech lattice and these 112 lines correspond with Leech roots in the theory. We examined the lattice structure of these lines in [5].

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2 Rational points over a finite field

We consider the hypersurface \( S \) in the 3-dimensional projective space \( \mathbb{P}^3 \) which is defined by

\[
(1) \quad x_0 x_1^q - x_1 x_0^q + x_2 x_3^q - x_3 x_2^q = 0
\]

It is easy to show that over \( \mathbb{F}_{q^2} \) this surface is isomorphic to the Fermat surface defined by

\[
x_0^{q+1} + x_1^{q+1} + x_2^{q+1} + x_3^{q+1} = 0.
\]
However, since the number of rational points over $\mathbb{F}_q$ of $S$ is different from the one of the Fermat surface, we see that $S$ is not isomorphic to the Fermat surface over $\mathbb{F}_q$. By the result in Weil [11], the number of $\mathbb{F}_{q^2}$-rational points of the Fermat surface is known. Therefore, the number of $\mathbb{F}_{q^2}$-rational points of $S$ is also known. However, to make this paper as self-contained as possible, we give here a direct calculation of the number of $\mathbb{F}_{q^2}$-rational points.

Suppose $x_0 \neq 0$. To calculate the rational points, we may assume $x_0 = 1$. Then, we have the equation

$$x_1^q - x_1 = x_3 x_2^q - x_2 x_3^q.$$

We have the following exact sequence of $\mathbb{F}_q$-vector spaces:

$$0 \rightarrow \mathbb{F}_q \rightarrow \mathbb{F}_{q^2} \xrightarrow{F-\text{id}} \mathbb{F}_{q^2}.$$

Here, $F$ is the Frobenius morphism over $\mathbb{F}_q$ and $\text{id}$ is the identity mapping. We set

$$V = \{ \alpha \in \mathbb{F}_{q^2} | \alpha^q = -\alpha \}.$$

$V$ is a vector space over $\mathbb{F}_q$, and we have

$$\text{Im}(F-\text{id}) \subset V.$$

Since $\dim_{\mathbb{F}_q} V = \dim_{\mathbb{F}_q} \text{Im}(F-\text{id}) = 1$, we see that $V = \text{Im}(F-\text{id})$.

Now, assume $x_2, x_3 \in \mathbb{F}_{q^2}$. Then, $x_2^q = x_2$ and $x_3^q = x_3$. Therefore, we see $x_3 x_2^q - x_2 x_3^q \in V$. Hence, for each $x_2$ and $x_3 \in \mathbb{F}_{q^2}$, we can find $q$ numbers of $x_1$, using the exact sequence (2) with $V = \text{Im}(F-\text{id})$. Hence, in this affine open set, the surface defined by the equation (1) has $q \times q^2 \times q^2 = q^5$ rational points over $\mathbb{F}_{q^2}$.

Suppose now $x_0 = 0$. Then, the equation (1) becomes

$$x_2 x_3^q - x_3 x_2^q = 0.$$

Factorizing the left hand side, we have

$$x_2 x_3 (x_3^{q-1} - x_2^{q-1}) = x_2 x_3 \prod_{a \in \mathbb{F}_q^*} (x_3 - a x_2).$$

Here, $\mathbb{F}_q^*$ is the multiplicative group of non-zero elements of $\mathbb{F}_q$. If $x_2 = x_3 = 0$, then we have only one rational point $(0, 1, 0, 0)$. If $x_2 = 0$ and $x_3 \neq 0$, then the rational points are of the form $(0, *, 0, 1)$. Therefore, we have $q^3$ rational points. If $x_2 \neq 0$ and $x_3 = 0$, then the rational points are of the form $(0, *, 1, 0)$. Therefore, we have $q^2$ rational points. If $x_2 \neq 0$ and $x_3 \neq 0$, then the rational points are of the form $(0, b, a, 0)$ with $b \in \mathbb{F}_{q^2}, a \in \mathbb{F}_q^*, c \in \mathbb{F}_{q^2}^*$. Moreover, if $b = 0$, the rational points are of the form $(0, 0, 1, a)$. Therefore, we have
$q - 1$ rational points. If $b \neq 0$, the rational points are of the form $(0, 1, \alpha, a\alpha)$. Therefore, we have $(q - 1)(q^2 - 1)$ rational points.

Hence, in total the number of rational points over $\mathbb{F}_q$ is equal to

$$q^5 + 1 + q^2 + (q - 1) + (q - 1)(q^2 - 1) = q^5 + q^3 + q^2 + 1 = (q^3 + 1)(q^2 + 1)$$

Since the equation (1) contains all $\mathbb{F}_q$-rational points of $\mathbb{P}^3$, we see that the number of rational points over $\mathbb{F}_q$ is equal to

$$q^3 + q^2 + q + 1$$

3 Lines defined over a finite field

Now, we will count the number of lines on the surface $S$ (also see Tate[10], Segre[8], Shioda[9] and Schütt-Shioda-van Luijk[7]), and we examine how these lines sit on our surface $S$ to show the existence of our configuration.

Suppose there exists a line $\ell$ defined over $\mathbb{F}_q$ (resp. $\mathbb{F}_q$) on the surface (1). Then, on $\ell$ we have $q^3 + 1$ (resp. $q + 1$) rational points defined over $\mathbb{F}_q$ (resp. $\mathbb{F}_q$). Therefore, any such line on (1) can be obtained by connecting two rational points on $S$.

Take two rational points $P' = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, $Q' = (\beta_0, \beta_1, \beta_2, \beta_3)$ on the surface (1) defined over $\mathbb{F}_q$ (resp. $\mathbb{F}_q$), and assume that the line $\ell$ which connects $P'$ with $Q'$ lies on the surface (1). Then, for any $t \in k$, points $(\alpha_0 + t\beta_0, \alpha_1 + t\beta_1, \alpha_2 + t\beta_2, \alpha_3 + t\beta_3)$ lie on the surface (1). Substitute these points into (1).

Since $P'$ and $Q'$ are points on the surface (1), we have

$$\alpha_0\beta_1t^3 + \beta_0\alpha_1t - \beta_0\alpha_0t = \alpha_3\beta_2t^3 + \beta_3\alpha_2t - \alpha_2\beta_3t^3 - \beta_2\alpha_3t.$$ 

Since $t$ is arbitrary, we have

$$\alpha_0\beta_1 - \alpha_1\beta_0 = \alpha_3\beta_2 - \alpha_2\beta_3,$$

$$\beta_0\alpha_1 - \alpha_0\beta_1 = \beta_3\alpha_2 - \beta_2\alpha_3.$$

These two equations have same solutions over $\mathbb{F}_q$ (resp $\mathbb{F}_q$). Hence, the condition becomes

$$\alpha_0\beta_1 - \alpha_1\beta_0 = \alpha_3\beta_2 - \alpha_2\beta_3$$

Now, we consider the hyperplane $H'$ defined by

$$H' : \beta_0x_0 - \beta_0x_1 + \beta_0x_2 - \beta_0x_3 = 0$$

This hyperplane is nothing but the tangent space of the surface (1) at the point $Q'$. By (3), we see that $H'$ passes through the point $P'$. Hence, any line defined over $\mathbb{F}_q$ (resp $\mathbb{F}_q$) on the surface (1) is obtained as the lines cut by a tangent hyperplane at the rational points over $\mathbb{F}_q$ (resp. $\mathbb{F}_q$).
Now, take a rational point $P = (\alpha, \beta, \gamma, \delta)$ on the surface (1) defined over $F_{q^2}$ (resp. $F_q$). Then, the tangent space $H$ of the surface (1) at $P$ is given by

\begin{equation}
\beta^q x_0 - \alpha^q x_1 + \delta^q x_2 - \gamma^q x_3 = 0.
\end{equation}

Changing to inhomogeneous coordinates, without loss of generality we may assume the case $\gamma = 1$. Then, we have

\begin{equation}
x_3 = \beta^q x_0 - \alpha^q x_1 + \delta^q x_2
\end{equation}

Substituting this into (1) and using $-\alpha^q \beta + \beta^q \alpha = \delta^q - \delta$, we have an equation

\begin{equation}
(x_0 - \alpha x_2)(x_1 - \beta x_2) \prod_{\epsilon \in F_{q^2}^*} \{(x_1 - \beta x_2) - \epsilon(x_0 - \alpha x_2)\} = 0.
\end{equation}

This means that the intersection of the surface (1) and the tangent space $H$ splits into $q + 1$ lines defined over $F_{q^2}$ (resp. $F_q$). Since there exist $q^2 + 1$ (resp. $q + 1$) rational points over $F_{q^2}$ (resp. $F_q$) on each line defined over $F_{q^2}$ (resp. $F_q$), we conclude that on the surface (1) there are

\begin{equation}
(q^3 + 1)(q^2 + 1) \times (q + 1) \div (q^2 + 1) = (q^3 + 1)(q + 1)
\end{equation}

lines defined over $F_{q^2}$. We also see that on the surface (1) there exist

\begin{equation}
(q^3 + q^2 + q + 1) \times (q + 1) \div (q + 1) = q^3 + q^2 + q + 1
\end{equation}

lines defined over $F_q$.

Hence, considering rational points and lines over $F_{q^2}$ (resp. $F_q$) on the surface (1), we have the following theorem.

**Theorem 3.1** On the hypersurface $S$ in $\mathbb{P}^3$ which is defined by

\begin{equation}
x_0x_1^q - x_1x_0^q + x_2x_3^q - x_3x_2^q = 0,
\end{equation}

there exist a $((q^3 + 1)(q^2 + 1)_{q+1}, (q^3 + 1)(q + 1)_{q+1})$-configuration and a $(q^3 + q^2 + q + 1)_{q+1}$-symmetric configuration.

**Remark 3.2** In case $q = p = 3$, the surface $S$ given by (1) is the supersingular K3 surface with Artin invariant 1. In this case, our configuration is a $(2804, 112_{10})$-configuration. We showed in [5] that 112 lines correspond with Leech roots in the Picard lattice $\text{Pic}(S)$.

**Remark 3.3** In case $q = p$ our surface $S$ is related to the locus of supersingular K3 surfaces with Artin invariant $\sigma \leq 3$ in the moduli space of supersingular K3 surface (cf. Rudakov-Shafarevich[6], p1520 and p1522, Theorem 2).
Remark 3.4 Let $A$ and $B$ be two sets, and $R$ be a relation between $A$ and $B$. The elements of $A$ are called points and the elements of $B$ are called blocks. A triple $\{A, B, R\}$ is called a $t$-$(v, k, \lambda)$ design if the following three conditions hold.

(i) $|A| = v$;
(ii) Every block $B \in B$ relates to precisely $k$ points;
(iii) Every $t$ distinct points together relates to precisely $\lambda$ blocks.

Using this notion, our $((q^3 + 1)(q^2 + 1)q + 1, (q^3 + 1)(q + 1)q^2 + 1)$-configuration is a $1$-$(q^3 + 1)(q^2 + 1), q + 1, q + 1)$ design.

4 Lefschetz pencil

On the surface $S$ defined by (1), we have $(q^3 + 1)(q + 1)$ lines. We take any line $\ell$ from these and make the Lefschetz pencil $f : S \rightarrow \mathbf{P}^1$ by using this line. Firstly, we consider the following special case.

Lemma 4.1 Let $\ell$ be a line on $S \subset \mathbf{P}^3$ given by $(0, 1, t, 0)$ with parameter $t$, and $f : S \rightarrow \mathbf{P}^1$ be the Lefschetz pencil by using the line $\ell$. Then, the general fiber is a rational curve with one singularity and we have singular fibers on the points $(t, 1) \in \mathbf{P}^1(F_{q^2})$.

Proof The hyperplanes which contain $\ell$ is given by

$$tx_1 - x_2 = 0,$$

and the Lefschetz pencil is given by

$$(5) \quad x_0x_1^{q-1} - x_0^q + tx_3^q - t^qx_1^{q-1}x_3 = 0$$

with $t \in \mathbf{P}^1$. The results follow from this equation. 

Remark 4.2 With the notation in Lemma 4.1, we consider the change of base given by $t = s^q$. To go to an inhomogeneous coordinate, we set $x_1 = 1$. Then, we have

$$x_0 - x_0^q + s^qx_3^q - s^qx_3 = 0.$$ 

Setting $x_0 - sx_3 = y$, we have

$$(s - s^q)x_3 + y - y^q = 0,$$

which shows $k(x_0, x_3, s) = k(s, y)$. Therefore, the surface $S$ is unirational. This gives a proof of the theorem which was proved by Shioda [9] (also see Rudakov-Shafarevich [6]). Namely, since $S$ is isomorphic to the Fermat surface

$$x_0^{q+1} + x_1^{q+1} + x_2^{q+1} + x_3^{q+2} = 0,$$

the Fermat surface is unirational over an algebraically closed field in characteristic $p > 0$. 

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Theorem 4.3 Let $F_q$ be a finite field with $q = p^a$ elements. Take any line $\ell$ on $S$ and consider the Lefschetz pencil $f: S \rightarrow P^1$ with respect to $\ell$. Then, the general fiber is a rational curve with one singular point and it has $q^2 + 1$ singular fibers. Each singular fiber consists of $q$ lines which all intersect transversely two by two at the same point.

Proof The general unitary group $GU_4(q)$ acts naturally on the surface $S$, and the order of $GU_4(q)$ is equal to

$$(q + 1)q^6(q^4 - 1)(q^3 + 1)(q^2 - 1).$$

(cf. [2]). It is easy to see that the order of the stabilizer of $GU_4(q)$ at the line defined by $x_0 = x_2 = 0$ is equal to $q^6(q^4 - 1)(q^2 - 1)$ (cf. Appendix). Therefore, $GU_4(q)$ acts $(q + 1)(q^3 + 1)$ lines transitively. This means that to show the first part of this theorem it suffices to show it for a line. Therefore, the first statement follows from Lemma 4.1.

By the calculation of the previous section, singular fibers exist over the $F_q$-rational points of the base curve $P^1$. Therefore, we have $q^2 + 1$ singular fibers. Again, by the calculation of the previous section, each singular fiber consists of $q$ nonsingular rational curves which all intersect each other at the same point $P$ mutually transversely. Therefore, we have in total $q \times (q^2 + 1)$ lines in singular fibers. The closure of the singular locus of general fibers is given by $\ell$. Therefore, it is a rational curve which is purely inseparable covering of degree $q$ over the base curve.

Theorem 4.4 Under the same notation as in Theorem 4.3, the group of the sections of the group scheme $S \setminus \ell \rightarrow P^1$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{\oplus 4a}$.

Proof Take a section $C$ of the group scheme $S \setminus \ell \rightarrow P^1$. Then, it intersects each irreducible component of a singular fiber with multiplicity one. Therefore, it intersects a line $\ell'$ in a singular fiber with multiplicity one. Since any singular fiber is given by an intersection of $S$ and a tangent space. Therefore the section $C$ intersect the tangent space which is in $P^3$ with multiplicity one. Hence, $C$ is a line on $S$. Therefore, by the consideration in the previous section, the hyperplane which is spanned by $C$ and $\ell'$ is a tangent space of $S$ at the intersection point of $C$ and $\ell'$. Therefore, $C$ is one lines of $(q^3 + 1)(q + 1)$ lines which we already had. Considering all singular fibers and $\ell$, we know that the number of sections is equal to

$$(q^3 + 1)(q + 1) - q \times (q^2 + 1) - 1 = q^4 = p^{4a}.$$  

Since the general fiber of $S \setminus \ell \rightarrow P^1$ is an additive group scheme $G_a$ and any non-trivial torsion of $G_a$ is of order $p$, we know that these sections form a group isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{\oplus 4a}$.

\[\blacksquare\]
Finally, we give a remark on a special case where the characteristic of the field \( k \) is equal to 3. Since it is known that the surface \( S \):
\[
x_0x_1^3 - x_1x_0^3 + x_2x_3^3 - x_3x_2^3 = 0.
\]
is a supersingular K3 surface with Artin invariant 1, we summarize our results in this interesting case. By the consideration above, we have 112 lines defined over \( \mathbb{F}_{p^2} \) on \( S \). Take any line among these 112 curves and make the Lefschetz pencil \( f : S \to \mathbb{P}^1 \) by using the line. Then, we have a quasi-elliptic fibration over the rational curve \( \mathbb{P}^1 \) with 10 singular fibers of type IV. We have just 10 \( \mathbb{F}_q \)-rational points on \( \mathbb{P}^1 \) on which the singular fibers lie. Hence, we have 30 lines in the singular fibers and one line as the cusp locus which we use to make the Lefschetz pencil. The other lines are the sections of this quasi-elliptic surface. Therefore, we have the following theorem.

**Theorem 4.5** Let \( f : S \to \mathbb{P}^1 \) be the Lefschetz pencil as above. Then, it forms a quasi-elliptic surface with 10 singular fibers of type IV and the Mordell-Weil group of this quasi-elliptic surface is isomorphic to \( (\mathbb{Z}/3\mathbb{Z})^\oplus 4 \).

We note that the existence of quasi-elliptic surfaces with such singular fibers in characteristic 3 is shown by H. Ito. He also examines, in details, the structure of Mordell-Weil groups of quasi-elliptic surfaces (cf. Ito[4]).

## 5 Appendix

We denote by \( GL_n(q^2) \) the general linear group which consists of all the regular \( n \times n \)-matrices with entries in \( \mathbb{F}_{q^2} \). For \( x \in \mathbb{F}_{q^2} \), let \( x \mapsto \bar{x} = x^q \) be the automorphism of \( \mathbb{F}_{q^2} \) whose fixed field is \( \mathbb{F}_q \). We consider the non-singular Hermitian form given by
\[
x_1\bar{x}_3 + x_3\bar{x}_1 + x_2\bar{x}_4 + x_4\bar{x}_2.
\]
The general unitary group \( GU_4(q) \) is the subgroup of all elements of \( GL_4(q^2) \) that fix the non-singular Hermitian form. We consider the hypersurface \( S' \) defined by
\[
x_1\bar{x}_3 + x_3\bar{x}_1 + x_2\bar{x}_4 + x_4\bar{x}_2 = 0.
\]
in the 3-dimensional projective space \( \mathbb{P}^3 \). It is clear that \( S' \) is isomorphic to the surface \( S \) defined by the equation \( 1 \) and \( GU_4(q) \) acts on \( S' \). Let \( \ell \) be the line defined by \( x_1 = x_2 = 0 \). Since we cannot find a suitable reference, in this appendix we prove the following proposition.

**Proposition 5.1** The order of the stabilizer of \( GU_4(q) \) at \( \ell \) is equal to
\[
q^6(q^4 - 1)(q^2 - 1).
\]
We denote by $M_2(q^2)$ the set of all the $2 \times 2$-matrices with entries in $F_{q^2}$, and we first show the following lemma.

**Lemma 5.2** We set $M = \{ X \in M_2(q^2) \mid ^t\bar{X} = -X \}$. Then we have $|M| = q^4$.

**Proof** We set $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $^t\bar{X} = -X$, we have $a = -a^q, b = -c^q, c = -b^q, d = -d^q$.

The number of solutions of $a = -a^q$ (resp. $d = -d^q$) in $F_{q^2}$ is equal to $q$, and the number of common solutions of $b = -c^q$ and $c = -b^q$ in $F_{q^2}$ is equal to $q^2$.

Hence, we have $|M| = q^4$. \[\Box\]

Now, we prove Proposition 5.1. Since the general unitary group $GU_4(q)$ fixes the Hermitian form $x_1\bar{x}_3 + x_3\bar{x}_1 + x_2\bar{x}_4 + x_4\bar{x}_2$, the element $A \in GU_4(q)$ satisfies $AJ^t\bar{A} = J$, where

$$J = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$$

with $2 \times 2$-identity matrix $E$. Setting

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

with $2 \times 2$-matrix $A_i$ ($i = 1, 2, 3, 4$), we have

$$A_1^t\bar{A}_2 + A_2^t\bar{A}_1 = 0,$$
$$A_1^t\bar{A}_4 + A_2^t\bar{A}_3 = E,$$
$$A_3^t\bar{A}_4 + A_4^t\bar{A}_3 = 0.$$

Assume $A$ fixes the line $\ell$. This means that $A_2 = 0$. Therefore, we have

$$A_1^t\bar{A}_4 = E, A_3^t\bar{A}_4 + A_4^t\bar{A}_3 = 0.$$  

Therefore, $A_4 \in GL_2(q^2)$, and the number of such matrices is equal to $(q^4 - 1)(q^4 - q^2)$. Since

$$^t(A_3^t\bar{A}_4) = -A_3^t\bar{A}_4,$$

for each $A_4 \in GL_2(q^2)$ we have, by Lemma 5.2, $q^4$ matrices in $M_2(q^2)$ which satisfy this equation. Hence, we conclude that the order of the stabilizer at the line $\ell$ is equal to $q^6(q^4 - 1)(q^2 - 1)$. 

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