Low-rank matrix recovery with non-quadratic loss: projected gradient method and regularity projection oracle

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Abstract

Existing results for low-rank matrix recovery largely focus on quadratic loss, which enjoys favorable properties such as restricted strong convexity/smoothness (RSC/RSM) and well conditioning over all low rank matrices. However, many interesting problems involve more general, non-quadratic losses, which do not satisfy such properties. For these problems, standard nonconvex approaches such as rank-constrained projected gradient descent (a.k.a. iterative hard thresholding) and Burer-Monteiro factorization could have poor empirical performance, and there is no satisfactory theory guaranteeing global and fast convergence for these algorithms.

In this paper, we show that a critical component in provable low-rank recovery with non-quadratic loss is a regularity projection oracle. This oracle restricts iterates to low-rank matrices within an appropriate bounded set, over which the loss function is well behaved and satisfies a set of approximate RSC/RSM conditions. Accordingly, we analyze an (averaged) projected gradient method equipped with such an oracle, and prove that it converges globally and linearly. Our results apply to a wide range of non-quadratic low-rank estimation problems including one bit matrix sensing/completion, individualized rank aggregation, and more broadly generalized linear models with rank constraints.

1 Introduction

In this paper, we consider the problem of rank-constrained generalized linear model (RGLM), where the goal is to recover a rank-$r$ ground truth matrix $X^\natural \in \mathbb{R}^{d_1 \times d_2}$ from independent data $(y_i, A_i) \in \mathbb{R} \times \mathbb{R}^{d_1 \times d_2}, i = 1, \ldots, n$ generated as follows. Given the measurement matrix $A_i$, the response $y_i$ follows a generalized linear model [FHT10] with the exponential family distribution

$$
P(y_i \mid A_i) \propto \exp \left\{ \frac{y_i (A_i, X^\natural) - \psi ((A_i, X^\natural))}{c(\sigma)} \right\},
$$

where $\psi$ is some convex log partition function that is twice continuously differentiable and $c(\sigma)$ is a function measuring the noise level. Examples of RGLM include matrix sensing with Gaussian noise [CP11], one-bit matrix sensing (a generalization of one-bit compressed sensing [BB08]), noisy matrix completion [CP10], one-bit matrix completion [DPVDBW14], individualized rank aggregation from pairwise comparison [LN15], and so on.

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Nonconvex formulation and regularity projection oracle

The above problem can be cast as a rank-constrained nonconvex optimization problem as follows:

\[
\begin{align*}
\min_{X \in \mathbb{R}^{d_1 \times d_2}} & \quad \mathcal{L}(X) := \frac{1}{n} \sum_{i=1}^{n} \left[ \psi(\langle X, A_i \rangle) - y_i \langle A_i, X \rangle \right] \\
\text{subject to} & \quad \text{rank}(X) \leq r, X \in \mathcal{C}.
\end{align*}
\]

(2)

Here, \( \mathcal{L} \) is the negative log-likelihood function, which is convex due to the convexity of \( \psi \). Whenever the distribution of \( \{y_i\} \) is non-Gaussian, \( \mathcal{L} \) is non-quadratic in general. The convex constraint set \( \mathcal{C} \subseteq \mathbb{R}^{d_1 \times d_2} \) is usually a certain norm ball (see Section 3 for examples) playing the role of regularization. This additional constraint is crucial for the success of Problem (2) in the following two aspects:

- **Optimization landscape**: Including the constraint \( X \in \mathcal{C} \) leads to a well-behaved landscape of the loss \( \mathcal{L} \): the restricted strong convexity and restricted smoothness conditions (RSC/RSM), or their approximate versions (see Definition 1), are satisfied for all the feasible \( X \) of (2). In contrast, in the absence of the constraint \( \mathcal{C} \), RSC/RSM are no longer satisfied. See Section 3.1 and 3.2 for details.

- **Statistical error**: The regularization constraint ensures that the local minimizers of (2) are nontrivially correlated with the ground truth \( X^\natural \), even when the sample size \( n \) is small. In particular, our theoretical results provide the best known statistical recovery guarantees for many RGLM problems. Without this constraint, the solution to (2) could be no better than a trivial constant estimator, as shown in Figure 2 of Example 1.

To leverage the regularization constraint algorithmically, we introduce the regularity projection oracle

\[
P_{r,\mathcal{C}}(X) := \arg \min_{\text{rank}(V) \leq r, V \in \mathcal{C}} \| X - V \|_F
\]

(3)

for a given rank parameter \( r > 0 \). Concrete instances of the oracle are displayed in Section 2.1. As a primary example, we note that when \( \mathcal{C} \) is the Frobenius norm ball, \( P_{r,\mathcal{C}} \) is equivalent to the standard rank-\( r \) SVD followed by a projection of the singular values to the Euclidean ball. In this case, \( P_{r,\mathcal{C}}(X) \) computes the best rank-\( r \) approximation of \( X \) with a bounded Frobenius norm.

**Goal and challenges with non-quadratic losses**

Given the regularity projection oracle \( P_{r,\mathcal{C}} \), we aim to design an iterative algorithm achieving the following two properties simultaneously:

- Each iteration only requires one access to the oracle \( P_{r,\mathcal{C}} \), and one computation of the gradient;

- The algorithm converges to \( X^\natural \) globally and linearly, with a contraction factor independent of the dimension (\( d_1 \) and \( d_2 \)), up to a certain statistical error.

To this end, one might be tempted to use a natural projected gradient descent (PG) method:

\[
X_{t+1} = P_{r,\mathcal{C}}(X_t - \eta \nabla \mathcal{L}(X_t)), \quad t = 1, 2, \ldots
\]

(4)

where \( \eta > 0 \) is the step size.

When the constraint is trivial (i.e., \( \mathcal{C} = \mathbb{R}^{d_1 \times d_2} \)), the PG algorithm (4) reduces to the well-known Iterative Hard Thresholding (IHT) algorithm [JTK14, BH18], a.k.a. singular value projection [JMD10]. Existing theory [JTK14, LB18] of IHT only applies when RSC/RSM holds for all low rank matrices. Unfortunately, this is true only in the simplest settings, such as quadratic loss \( \mathcal{L} \) with Gaussian linear measurements. Even when \( \mathcal{L} \) is quadratic, for harder problems such as matrix
completion, RSC/RSM no longer holds for all low rank matrices. For such problems, existing theory is scarce and unsatisfactory. While the work by [DC18] proves that IHT does recover $X^2$ in the matrix completion problem, the analysis is complicated and tailored for quadratic loss, and leads to sub-optimal sample complexity bounds.

For non-quadratic $\mathcal{L}$ with a general constraint set $\mathcal{C}$, the best applicable result is [BH18], which only establishes local convergence from a good initialization. Moreover, the quality of initialization is measured by the concavity parameter (details in Section 1.1) of the feasible region of (2). This concavity parameter is difficult to compute except in the trivial case where $\mathcal{C} = \mathbb{R}^{d_1 \times d_2}$, and there is no known algorithm guaranteeing a good initialization for (2) beyond the quadratic loss setting. Solving the convex relaxation of (2) sometimes provides a good initialization, but the computational complexity is prohibiting with a superlinear dependence on the dimension; in particular, known algorithms need to compute full SVD in each iteration in order to simultaneously enforce the constraint $X \in \mathcal{C}$ and the other constraints of the convex relaxation.

Our Algorithm and contributions To achieve the two goals mentioned above, we introduce an algorithm called averaged projected gradient (AVPG), displayed as Algorithm 1. Inspired by [AZHHL17], AVPG is a version of PG (4) after averaging the iterates. Our contributions henceforth can be summarized as follows:

• Conceptually, we identify the importance of the regularization constraint $X \in \mathcal{C}$ and its algorithmic counterpart, the regularity projection oracle.

• Algorithmically, we design AVPG based on the regularity projection oracle and show that it converges globally and linearly, and has a low iteration complexity under standard RSC/RSM conditions (and their approximate versions); see Section 2 and Theorem 1.

• Statistically, we apply AVPG to several RGLMs with non-quadratic losses, such as one-bit matrix sensing/completion, and prove that it recover $X^2$ up to a certain statistical error, which before our work was only achievable by convex relaxation; see Section 3.

Organization The rest of the paper is organized as follows. In Section 1.1 we first review existing approaches to Problem (2), including convex relaxation, Burer-Monteiro approach, and projected gradient method. We then compare our results to existing approaches and demonstrate situations where our approach is advantageous. In Section 2 we present our main algorithm AVPG, discuss the intuition, and establish the theoretical convergence guarantees. In Section 3 we apply our theoretical guarantees to concrete examples of RGLM and show that AVPG recovers $X^2$ up to a certain statistical error. We conclude the paper in Section 4, where we discuss possible applications of AVPG beyond RGLMs as well as several intriguing questions regarding the gap between theory and practice.

1.1 Related work and comparison

In this section, we discuss some most related approaches: convex relaxation, projected gradient, and Burer-Monteiro, and why ours is advantageous in certain aspects. To facilitate the discussion, we denote the condition number of $\alpha$-RSC and $\beta$-RSM (see Definition 1) as $\kappa = \frac{\beta}{\alpha}$.

Convex relaxation Convex relaxation usually replaces the rank constraint of Problem (2) with some nuclear norm constraint, such as [SS05, RFP10, CR09, NW12, Laf15, GRG14, LN15]. We refer readers to [CC18 Section 4] and [War19 Chapter 11] for an overview of this topic. Despite...
the beautiful theory established, first order algorithm suffers from dimensional number of iterations in theory, and full SVD or at least \(\mathcal{O}(d_1d_2)\) operations in fulfilling the constraint set \(\mathcal{C}\).

**Burer-Monteiro approach**  Another natural approach is to factor the low-rank matrix as \(X = AB\), where \(A \in \mathbb{R}^{d_1 \times r}\) and \(B \in \mathbb{R}^{r \times d_2}\), then solve Problem (2) in variables \(A,B\) instead of \(X\). This approach was first proposed in [BM03] and recently gained much attention [ZL16, CW15, HLB18]. We refer readers to [CC18, CLC19] for a more comprehensive survey. Algorithms with provably quick convergence [CW15, PKCS18, CCD+19] typically require an initial solution close (measured by Frobenius norm) to the ground truth \(X^\ast\) or the optimal solution up to a small fraction of the singular value of \(X^2\). However, effective and efficient initialization is only available for the quadratic loss problems. Another line of work studies the the landscape of a penalized or constrained version of (2) with the above variables \(A,B\) and characterizes when there is no spurious local minimum [GJZ17, ZLTW18, ZWYG18]. However, the conditions for such results are quite stringent: either the loss is quadratic, or the condition number \(\kappa\) must be very close to one; otherwise, spurious local minima may exist [ZLTW18 pp. 3-4]. Note that neither of the aforementioned conditions is satisfied for the problem of one bit matrix sensing/completion if the constraint set \(X \in \mathcal{C}\) is absent; we discuss this issue in Sections 3.1 and 3.2. Even with this constraint and for favorable case of one-bit matrix completion, the condition number \(\kappa\) may not be close to one, making existing landscape results inapplicable; see Section 1 in the appendix for a detailed discussion.

**Projected gradient method**  While the PG method (4) sometimes works well empirically, its theoretical guarantees is far from satisfactory, as mentioned earlier. Here we explain in details the “concavity parameter” defined in [BH18, Equation (5)], and related convergence guarantees. Denote the set of matrices in \(\mathbb{R}^{d_1 \times d_2}\) with rank at most \(r\) as \(\mathbb{R}^{d_1 \times d_2}_r\). The concavity parameter for the set \(\mathcal{C} \cap \mathbb{R}^{d_1 \times d_2}_r\) at a point \(X \in \mathcal{C} \cap \mathbb{R}^{d_1 \times d_2}_r\) is defined as

\[
\gamma_X(\mathcal{C} \cap \mathbb{R}^{d_1 \times d_2}_r) := \sup \left\{ \frac{\langle Y - X, Z - X \rangle}{\| Z - X \| \| Y - X \|^2_F} \mid Y \in \mathcal{C} \cap \mathbb{R}^{d_1 \times d_2}_r, Z \text{ such that } P_{\mathcal{C}}(Z) = X \right\}.
\]

where \(\| \cdot \|\) is some arbitrary norm. The convergence guarantee of PG requires initialization in a neighborhood of \(X^\ast\) and that the condition \(\gamma_X(\mathcal{C} \cap \mathbb{R}^{d_1 \times d_2}_r)\|\nabla \mathcal{L}(X)\| < \frac{\alpha}{2}\) holds uniformly for all \(X\) in the neighborhood [BH18, Equation (14)]. However, it should be noted that once \(r > \text{rank}(X^\ast)\), the quantity \(\gamma_X\) is expected to approach \(+\infty\) [BH18, End of Section 2.1]. In particular, if \(\mathcal{C} = \mathbb{R}^{d_1 \times d_2}_r\), according to [BH18, Lemma 5], \(\gamma_X = \infty\) for \(r \geq r^\ast\), and hence the main result in [BH18, Theorem 3] does not apply. On the contrary, our result is still applicable when \(r \geq r^\ast\). Even assuming the rank parameter \(r\) is correctly specified, \(r = r^\ast\), there is no result for bounding \(\gamma_X(\mathcal{C} \cap \mathbb{R}^{d_1 \times d_2}_r)\) except for the trivial case \(\mathcal{C} = \mathbb{R}^{d_1 \times d_2}_r\), and \(\| \cdot \|\) being the operator norm; in particular, there lacks a bound even when \(\mathcal{C}\) is a Frobenius norm ball, let alone an infinity norm ball. We note that there is no simple monotone relation such as \(\gamma_X(\mathcal{C} \cap \mathbb{R}^{d_1 \times d_2}_r) \leq \gamma_X(\mathbb{R}^{d_1 \times d_2}_r)\) as the sets being maximized over does not simply become larger by dropping \(\mathcal{C}\). Hence for interesting constraint set \(\mathcal{C}\), to apply the results of [BH18], one would need significant additional work to estimate \(\gamma_X(\mathcal{C} \cap \mathbb{R}^{d_1 \times d_2}_r)\) and can only hope to guarantee local convergence with the rank parameter \(r\) correctly specified.

**Comparison**  For a fair comparison, we consider the case when the projection oracle (3) has comparable (or less) complexity in forming the gradient for low rank matrices. One example is that \(\mathcal{C}\) is a Frobenius norm ball, the corresponding RGLM has Gaussian measurements and the \(\mathcal{L}_n\) can be potentially non-quadratic. The projection oracle for this example reduces to \(r\)-SVD plus some scaling, and can be computed in linear time of matrix vector product of input \(X\) [AZL16].
As explained earlier, no existing algorithm provably works in this regime, as they either suffer from extraordinary polynomial costs such as the convex relaxation approach based ones, or lack guarantees on convergence such as the IHT approach or the Burer-Monteiro approach based ones. As mentioned earlier, even if the projection oracle (3) is not available, our identification of the projection oracle reveals the key and critical component in solving the rank-constrained generalized linear model.

**Notation**  We introduce the shorthand $d = \max \{d_1, d_2\}$. For a positive integer $n$, the notation $[n]$ stands for $\{1, \ldots, n\}$. We equip the linear space $\mathbb{R}^{d_1 \times d_2}$ with the trace inner product: for $A, B \in \mathbb{R}^{d_1 \times d_2}$, $\langle A, B \rangle = \text{tr}(A^T B) = \sum_{i,j} A_{ij} B_{ij}$. For a given norm $\| \cdot \|$, $B_\|\|_\beta(\xi)$ denotes the associated ball centered at origin with radius $\xi > 0$. We make use of several matrix norms, including the Frobenius norm $\| \cdot \|_F$, the operator norm (largest singular value) $\| \cdot \|_{\text{op}}$, the nuclear norm (sum of singular values) $\| \cdot \|_{\text{nuc}}$, and the infinity norm (maximum absolute value of entries) $\| \cdot \|_\infty$.

## 2 Algorithm and guarantees

In this section, we present details of the averaged projected gradient algorithm (Section 2.1) and provide convergence guarantees under an approximate RSC/RSM condition (Section 2.2).

### 2.1 Algorithm description

Given the regularity projection oracle (3), AVPG is displayed as Algorithm 1. Each iteration of AVPG consists of three steps: (i) a choice of step size, (ii) a projected gradient step, and (iii) an averaging step. Note that the initial iterate is assumed to lie in $C$, which is merely for convenience as the projection ensures this property for all future iterates.

**Algorithm 1** Averaged projected gradient method (AVPG)

| Input: | A rank estimate $r$, an initial iterate $X_0 \in C \subseteq \mathbb{R}^{d_1 \times d_2}$ with rank($X$) $\leq r$, step size parameter $\eta_0 \in [0, 1]$, RSM parameter estimate $\beta$, a period integer $t_0 \in \mathbb{Z}$ |
| for $t=1,2,\ldots, \text{do}$ |
| | Choice of step size: if $t$ is an integer multiple of $t_0$, set $\eta = 1$, otherwise, $\eta = \eta_0$. |
| | Projection step: $V_t = P_{r,C} \left( X_t - \frac{1}{\beta \eta} \nabla \mathcal{L}(X_t) \right)$ |
| | Averaging step: $X_{t+1} = (1 - \eta) X_t + \eta V_t$. |
| end for |

**The role of step size and period $t_0$** Per our choice of step size, AVPG runs in periods of length $t_0$. Within each period, we average the projected solution $V_t$ with the previous iterate $X_t$; at the end of the period, the iterate is set to $V_t$ without averaging. By sub-additivity of rank, the rank of the iterate is always bounded by $rt_0$. The boundedness of rank is desirable both computationally and theoretically. Computationally, the boundedness of the rank benefits (i) the storage of the iterate, and (ii) the time in computing the gradient and projection oracle under certain structure of $A_i$. Theoretically, the bounded rank enables us to use the approximate RSC/RSM condition to prove guarantees as those properties are restricted to low rank matrices.

It is tempting to set $t_0 = 1$, in which case there is no averaging step and AVPG reduces to PG. However, this choice of $t_0$ destroys the additional leeway provided by the averaging step, which is
The role of averaging  Compared to the naive PG method (4), AVPG has an additional averaging step. This step is crucial in establishing a linear convergence guarantee (given in Theorem 1) that is valid for a general set $C$, without relying on additional structures of $C$. To explain the intuition, let us assume that the objective function $L$ is $\alpha$-strongly convex and $\beta$-smooth, i.e., Definition 1 with $\epsilon_\alpha = \epsilon_\beta = 0$ and $r = d$ (cf. [Nes13] Definition 2.1.2). With $\Delta_t := X_t - X^2$, a critical step in our analysis involves the following chain of inequalities:

$$L(X_{t+1}) = (a) \leq L((1 - \eta)X_t + \eta V_t) \leq L(X_t) + \eta \langle \nabla L(X_t), V_t - X_t \rangle + \frac{\beta \eta^2}{2} \|X_t - V_t\|_F^2$$

$$(b) \leq L(X_t) + \eta \langle \nabla L(X_t), -\Delta_t \rangle + \frac{\beta \eta^2}{2} \|\Delta_t\|_F^2,$$

where step (a) follows from the definition of $X_{t+1}$, step (b) follows from $\beta$-smoothness, and step (c) follows from the optimality of $V_t$ in the definition (3) of the projection $P_{r,C}$.

The averaging step allows for additional leeway, provided by $\eta$, in steps (a) and (b). This in turns enables step (c), which holds without appealing to other properties of the projection oracle. Without averaging, one may replace step (b) with an application of $\beta$-smoothness to the two iterates $X_{t+1}$ and $X_t$, leading to the inequality $L(X_{t+1}) \leq L(X_t) + \langle \nabla L(X_t), X_{t+1} - X_t \rangle + \frac{\beta}{2} \|X_{t+1} - X_t\|_F^2$. To proceed at this point, one would need to analyze how the projection interplays with the difference between iterates $\|X_{t+1} - X_t\|_F^2$. Doing so typically requires exploiting the delicate properties of SVD [JTK14, LB18] and specific structures of the set $C$, such as the local concavity [BH18]. In contrast, our analysis is much simpler, while holds more generally. Such generality allows us to instantiate our convergence guarantee in a diverse range of concrete RGLM problems (see Section 3), in which the interplay between SVD and the set $C$ is non-trivial and crucial.

Computing projection oracle  In many cases, the oracle $P_{r,C}$ can be computed via rank-$r$ SVD (which gives the best rank-$r$ approximation in terms of Frobenius norm):

- If $C = \mathbb{R}^{d_1 \times d_2}$, then $P_{r,C}(X)$ is given by the rank-$r$ SVD of $X$.

- If $C = \mathbb{B}_{\|\cdot\|_F}(\xi)$ (resp. $\mathbb{B}_{\|\cdot\|_{\text{nuc}}}(\xi)$), then $P_{r,C}(X)$ is given by the rank-$r$ SVD of $X$ followed by a projection of the $r$ singular values to the $\ell_2$ (resp. $\ell_1$) norm ball in $\mathbb{R}^r$ with radius $\xi$.

- More generally, if $C$ is the ball of Schatten-$p$ norm with radius $\xi$, then $P_{r,C}(X)$ is given by the rank-$r$ SVD followed by a projection of $r$ singular values to the $\ell_p$ norm ball in $\mathbb{R}^r$ with radius $\xi$. This is true even when $0 < p < 1$; see Lemma 4 for the proof.

Note that the rank-$r$ SVD of $X$ can be computed using $m$ matrix-vector product operations of $X$, with $m$ being linear in the rank $r$ In our RGLM example, this means that the SVD can be computed in time linear in the number of the matrix-vector product with the gradient $\nabla L$.

There are other interesting choices of $C$ not defined by the singular values, e.g., the $\ell_\infty$ norm ball $\mathbb{B}_{\|\cdot\|_{\ell_\infty}}(\xi)$. In this case, the oracle $P_{r,C}$ can be computed by alternating projection [Lew14], which works well in our experiments (see Appendix C), though its convergence property and running time are more involved due to the non-convexity.

1 More precisely, achieving an $\epsilon$ error requires $m = \min \left\{ \tilde{O}\left( \frac{\epsilon}{\sigma_1^2} \right), \tilde{O}\left( \frac{1}{\sqrt{\text{gap}}} \log \frac{1}{\epsilon} \right) \right\}$ by the results in [AZL16], where $\text{gap} := [\sigma_r(X) - \sigma_{r+1}(X)]/\sigma_r(X)$ is the eigen gap of $X$. Note that the first term in the expression of $m$ is independent of $\text{gap}$. Here $\tilde{O}$ omits logarithmic factors.
2.2 Convergence guarantee under approximate RSC/RSM

To state our convergence guarantees for AVPG, we introduce notions of approximate restricted strong convexity and restricted smoothness.

**Definition 1.** The loss function $\mathcal{L}$ satisfies approximate $(\epsilon_\alpha, r, \alpha, C)$-RSC and $(\epsilon_\beta, r, \beta, C)$-RSM for some $\epsilon_\alpha, \epsilon_\beta \geq 0$ and convex $C$ if for all matrices $X, Y \in C$ with rank at most $r$, there hold the inequalities

\[
\frac{\alpha}{2} \|X - Y\|^2_F - \epsilon_\alpha \leq \mathcal{L}(X) - \mathcal{L}(Y) - \langle \nabla \mathcal{L}(X), Y - X \rangle \leq \frac{\beta}{2} \|X - Y\|^2_F + \epsilon_\beta.
\]

If $\epsilon_\alpha = \epsilon_\beta = 0$, we say that $\mathcal{L}$ satisfies $(r, \alpha, C)$-RSC and $(r, \beta, C)$-RSM.

Standard RSC/RSM assumption [JTK14] Definitions 1 and 2] corresponds to $\epsilon_\alpha = \epsilon_\beta = 0$ and $C = \mathbb{R}^{d_1 \times d_2}$. Such a strong assumption is typically required in the analysis of IHT [JMD10, JTK14, LB18]. In comparison, our definition allows for the additional constraint set $C$ and error terms $\epsilon_\alpha$ and $\epsilon_\beta$, and hence is less restrictive. In the RGLM setting of interests, these error terms account for the statistical error due to the finite sample size and the measurements structure, and vanishes to 0 at the rate $O\left(\frac{r^d \log d}{n}\right)$ (see Corollary 1.2 and 3). Moreover, the relaxation of standard RSC/RSM is essential for examples in Section 3.2 as discussed in Section 3.2.4. The constants $\alpha$ and $\beta$ in the RGLM setting of interests should be dimension independent constants (see Lemma 1.2 and 3), and hence so is the condition number $\kappa$. Note that the above definition also appears in [BHT18] for the analysis of projected gradient descent.

We now state the theoretical guarantees, whose proof is deferred to Appendix A.2. We introduce the shorthands $\kappa := \beta/\alpha$ for the condition number, $\Delta_t := X_t - X^*$ for the iterate difference to the ground truth, and $h_t := \mathcal{L}(X_t) - \mathcal{L}(X^*)$ for the objective difference.

**Theorem 1.** Suppose $\mathcal{L}$ satisfies approximate $(\epsilon_\alpha, rt_0, \alpha, C)$-RSC and $(\epsilon_\beta, rt_0, \beta, C)$-RSM for $r \geq r^*$ and $t_0 \geq \lceil 4\kappa (\log 4\kappa + 1) \rceil$. Let $\epsilon_\nabla := \|\nabla \mathcal{L}(X^*)\|_{op}, \eta_0 = \frac{1}{4\kappa}$, and $s$ be the largest integer so that $st_0 \leq t$. Also let $\tau_* = \min_{1 \leq \tau \leq t+1} h_{\tau},$ Then the iterate $X_t$ from Algorithm 1 with parameters $\eta_0, \beta, t_0,$ and $r$ satisfies the bounds

\[
h_{\tau_*} \leq \max \left\{ (1 - \frac{1}{4\kappa})^t (4\kappa)^s h_0, \epsilon_n \right\} \quad \text{and} \quad \|\Delta_{\tau_*}\|^2_F \leq \frac{4}{\alpha} \max \left\{ (1 - \frac{1}{4\kappa})^t (4\kappa)^s h_0, \epsilon_n \right\},
\]

where $\epsilon_n = \frac{4\kappa}{\alpha} (rt_0 + r^2)\epsilon^2 + \epsilon \sqrt{\frac{8\rho\epsilon_n}{\alpha}} + \epsilon \sqrt{\frac{64t^2 \rho \kappa \epsilon_\beta}{\alpha}} + 2\kappa \epsilon_\alpha + 2\epsilon_\beta$.

**Interpretation of Theorem 1.** To better understand the above theorem, let us assume that the AVPG algorithm is run for $k$ periods, i.e., $t = kt_0$ iterations. In this case, the bounds (7) become

\[
\min_{1 \leq \tau \leq t+1} h_{\tau} \leq \max \left\{ e^{-k} h_0, \epsilon_n \right\} \quad \text{and} \quad \|\Delta_{\tau_*}\|^2_F \leq \frac{4}{\alpha} \max \left\{ e^{-k} h_0, \epsilon_n \right\}.
\]

Each of the above bounds involves two terms. The first term $e^{-k} h_0$ corresponds to the optimization error, which shrinks geometrically at every period, i.e., every $t_0 = \lceil 4\kappa (\log 4\kappa + 1) \rceil$ iterations. This geometric convergence holds up to a statistical error given by the second term $\epsilon_n$, which is on the order $O\left(\frac{r^d \log d}{n}\right)$ for RGLM as shown in Section 3.
Comparison with IHT  Let us compare our guarantees to those for IHT, the projected gradient method \(^4\) with \(C = \mathbb{R}^{d_1 \times d_2}\). Our comparison is only for accuracy \(\epsilon > \epsilon_n\), as feasible matrices of our problem (2) are statistically equally good estimators of \(X^\natural\) once they achieve the error \(\epsilon_n\). The work \([JTK14, LB18]\) shows that the iteration complexity of IHT is \(O(\kappa \log(\frac{h_0}{\epsilon} ))\), whereas ours is \(O(\kappa \log \kappa \log(\frac{h_0}{\epsilon} ))\). Note that IHT requires computing the top \(\Omega(\kappa^2 r^2)\) singular values/vectors in each step to ensure convergence \([LB18]\), while ours only requires the top \(r^2\) ones. Therefore, to achieve \(\epsilon\)-accuracy, the total work required by IHT amounts to a number of \(O(\kappa^3 r^2 \log(\frac{h_0}{\epsilon} ))\) rank-1 SVD computation, whereas AVPG requires \(O(\kappa r^2 \log \kappa \log(\frac{h_0}{\epsilon} ))\) and is better than IHT by a factor of \(\kappa^2 \log \kappa\).

The per iteration dependence on condition number is not an artifact of the theory. In Figure \([\text{ZLTW18 pp 3-4}]\) we show the results of IHT and AVPG applied to the weighted matrix recovery problem on \([\text{ZLTW18}]\). Specifically, we consider min\(_{X \in \mathbb{R}^{d \times d}, \text{rank}(X) \leq r} \| W \odot (X - X^\natural) \|_F^2\), where \(d = 50\), and \(W\) is the matrix with all ones except the \((1,1)\)-th and \((2,2)\)-th entry whose values are 2, and \(X^\natural\) is a rank 1 matrix with all zero values except for the top left \(2 \times 2\) block which has all one entries. We start the algorithm at a point \(M_0\) with all zero values except for the top left \(2 \times 2\) block which is \(\frac{3}{5} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\), and with the choice of rank \(r = 1 = r^\natural(X^\natural)\). Here \(\odot\) denotes the Hadamard product. The stepsize of IHT is chosen to be \(O(1/\beta)\) as suggested by most theory papers \([JTK14, LB18]\). The stepsize of AVPG is chosen to be \(\eta_0 = O(\frac{1}{\kappa})\) as suggested by our theory. This problem has condition number 4. Note that IHT stays at the starting point \(M_0\), meaning that \(M_0\) is a non-optimal fixed point of iteration \([4]\), while AVPG is able to move away from this starting point.
3 Consequences for solving RGLM

In this section, we apply our general Theorem 1 to concrete settings of RGLM, by calculating the parameters of the approximate RSC/RSM and the gradient norm \( \| \nabla L_n(X) \|_\text{op} \). The results in this section explain why the regularity constraint \( C \) is crucial as claimed in the introduction, highlighting the key role of the projection oracle (3). We consider several examples based on the form of the measurement matrices \( \{A_i\} \); including Gaussian measurements (Section 3.1), entrywise sampling for matrix completion (Section 3.2), and pairwise sampling for rank aggregation (Section 3.3).

3.1 Gaussian measurements and Frobenius norm ball

Let us first explain the setup of matrix sensing and one-bit matrix sensing.

3.1.1. Problem setup Suppose that the measurement matrices \( \{A_i, i = 1, \ldots, n\} \) are independent of each other and have i.i.d. standard Gaussian entries. For the distribution of the response \( \{y_i\} \) in the RGLM (1), we are interested in the following two settings:

1. **Matrix sensing**: \( \psi(\theta) = \frac{1}{2} \theta^2 \), and \( c(\sigma) = \sigma^2 \). In this case, the distribution of \( y_i \) is Gaussian with mean \( \langle A_i, X^2 \rangle \) and variance \( \sigma^2 \).

2. **One-bit matrix sensing**: \( \psi(\theta) = \log(1 + \exp(\theta)) \), and \( c(\sigma) = 1 \). The distribution of \( y_i \) is Bernoulli with probability \( \frac{\exp(\langle A_i, X^2 \rangle)}{1 + \exp(\langle A_i, X^2 \rangle)} \), which is a logistic function of \( \langle A_i, X^2 \rangle \).

In words, in matrix sensing \( y_i \) is the linear measurement \( \langle A_i, X^2 \rangle \) corrupted by additive Gaussian noise, whereas in one-bit matrix sensing, \( y_i \) contains only binary information of \( \langle A_i, X^2 \rangle \).

Next, we explain the choice of \( C \) and why such choice is critical for the successful recovery of \( X^2 \).

3.1.2. The choice of \( C \) and its importance For Gaussian \( \{A_i\} \), the operator \( A : \mathbb{R}^{d_1 \times d_2} \to \mathbb{R}^n \) defined by \( [A(X)]_i = \langle A_i, X \rangle \), satisfies the Restricted Isometric Property (RIP); i.e., \( (1 - \frac{1}{16}) \| X \|_F \leq \| A(X) \|_2 \leq (1 + \frac{1}{16}) \| X \|_F \) for any rank-\( r \) matrix \( X \), with high probability provided that \( n \) is sufficiently large [CP11]. Accordingly, we choose the regularization constraint \( C \) to be the Frobenius norm ball \( \mathbb{B}_{\| \cdot \|_F} \). Below we discuss this choice and explain why it is crucial for the non-quadratic loss associated with one-bit matrix sensing.

The Hessian of the loss function \( L_n \) is given by \( \nabla^2 L_n(X) \Delta = \frac{1}{2n} \sum_{i=1}^n \psi''(\langle X,A_i \rangle) (\Delta_i A_i)^2 \).

For matrix sensing, for which \( L_n \) is quadratic, we have \( \psi'' = 1 \), a constant regardless of \( X \). For one bit matrix sensing, on the other hand, we have \( \psi''(\theta) \to 0 \) for \( \theta \to \pm \infty \). In this case, the condition number of \( \nabla^2 L_n(X) \) is unbounded if we consider all low-rank matrices. Restricting to matrices in the Frobenius norm ball \( C \) ensures a bounded condition number, so that \( L_n \) is well behaved due to RIP. Below we corroborate the above arguments by a numerical example.

**Example 1.** We generate a random rank-1 matrix \( X^2 \) with \( \| X^2 \|_F = 1 \), and sample 1000 data points \( \{y_i, A_i\} \) using the one-bit matrix sensing model. We then apply projected gradient (4), as well as AVPG, using the projection \( P_{r,\| \cdot \|_F} \) or \( P_{r,\| \cdot \|_F} \), i.e., with or without the regularity oracle. We consider random initialization with different Frobenius norm \( \| X_0 \|_F = \gamma \) for \( \gamma = 0, 0.5, 1, 2, 4 \). The distance to ground truth and objective value of the iterates are shown in Figure 2.

As we can see, all the methods converge to a stationary point or local minimizer, as the objective value keeps decreasing and approaches stagnancy. However, the regularity projection approaches converge to a better solution, with distance to ground truth approaching 0.77; the other approaches produce solutions worse than the trivial estimator \( X = 0 \).
(a) Comparison of relative distance to ground truth $\frac{\|X_t-X^\natural\|_F}{\|X^\natural\|_F}$, and objective $\mathcal{L}(X_t)$ for PG, projected gradient (4).

(b) Comparison of relative distance to ground truth $\frac{\|X_t-X^\natural\|_F}{\|X^\natural\|_F}$, and objective $\mathcal{L}(X_t)$ for AVPG.

Figure 2: Comparison of projected gradient and AVPG with and without the regularity projection oracle (i.e., with projection $\mathcal{P}_{r,B\|\cdot\|_F}$ (1) and with $\mathcal{P}_{r,B^d_1\times d_2}$, respectively).

Finally, we provide performance guarantees for AVPG applied to matrix sensing and one-bit matrix sensing.

3.1.3. Theoretical guarantees Let us first state the following lemma establishing the desired structural properties of the loss $\mathcal{L}$, including RSC/RSM (proved in Appendix B.3.1) and bounds on the gradient (proved in Lemma 15 in the Appendix).

Lemma 1. Suppose $C = B\|\cdot\|_F(\tau\|X^2\|_F)$ for some $\tau \geq 1$, and the measurements $A_i, i = 1, \ldots, n$ are standard Gaussian. Then there are universal constants $c, C, c_0, c_1 > 0$ such that if $n \geq c[\kappa \log \kappa + 1])r^5d$, with probability at least $1 - \exp(-c_1d)$, the following two statements hold:

- (RSC/RSM) the loss function $\mathcal{L}$ satisfies $(t_0r^5, \frac{15}{16}B, C)$-RSC and $(t_0r^5, \frac{17}{16}B, C)$-RSM;
- (Gradient bound) $\|\nabla \mathcal{L}(X^\natural)\|_{op} \leq C\sqrt{c(\sigma)B}\sqrt{\frac{d}{n}}$, where $t_0 = \left[4\kappa (\log 4\kappa + 1)\right]$ and $\kappa = 1.1B/B$ where $B = \|\psi''\|_{\infty} = \sup_{x \in \mathbb{R}} |\psi''(x)|$, and $B = \inf_{|x| \leq \sqrt{\frac{\tau\sigma}{c_0}}r\|X^2\|_F}\psi''(x)$. 

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Scaling of Lemma \[1\] To better understand Lemma \[1\], consider the scenario where the ground truth is a constant, i.e., $\|X^*\|_F = O(1)$. Such requirement is necessary for constant conditioning of non-quadratic loss due to the non-constancy of $\psi''$. We have $\kappa$ being a universal constant for both cases. For some universal constants $c_1, c_2, c_3, C$, the loss $L$ satisfies $(c_1 r^2, c_2, C)$-RSC and $(c_1 r^2, c_3, C)$-RSM, and $\|\nabla L(X^*)\|_{op} \leq C \sqrt{\frac{d c(\sigma)}{n}}$ where $c(\sigma) = \sigma^2$ for matrix sensing and $c(\sigma) \equiv 1$ for one-bit matrix sensing. Note that for matrix sensing, the requirement on constant $\|X^*\|_F$ is not needed, as $\psi''$ is constant.

With Lemma \[1\] we can bound the distance to $X^*$ by invoking the general Theorem \[1\]. We assume the input to AVPG is $r = r^2$, $\beta = \frac{17}{16} B$, $\eta_0 = \frac{1}{4r}$, and $t_0 = [4\kappa (\log 4\kappa + 1)]$.

**Corollary 1.** Instate the assumptions and notation in Lemma \[7\] and assume AVPG uses the input described above. Define $\tau_* = \arg \min_{1 \leq \tau \leq t+1} L(X_\tau)$. Then for some universal constant $c, c_1$, with probability at least $1 - \exp(-c_1 d)$, there holds the inequality

$$\|X_{\tau_*} - X^*\|_F^2 \leq cB^{-1} \max \left\{ (1 - \frac{1}{4\kappa})^t (4\kappa)^s h_0, \kappa^3 r^2 (\log \kappa) c(\sigma) \frac{d}{n} \right\},$$

where $s$ is the largest integer so that $st_0 \leq t$.

**Interpretation of Corollary \[1\]** To better interpret Corollary \[1\], we let $t \to \infty$, in which case the second RHS term in (8) dominates and corresponds to the statistical error. For matrix sensing, we have $c(\sigma) = \sigma^2$ and hence the error is $O \left( \frac{\sigma^2 r d}{n} \right)$. For one-bit matrix sensing, where $c(\sigma) \equiv 1$, the error is $O \left( \frac{r d}{n} \right)$ when the Frobenius norm of the ground truth is a universal constant. The second bound is new for non-convex methods, and matches the error bound achieved using the (more computationally expensive) convex relaxation approach [Wai19 Corollary 10.10].

### 3.2 Entrywise sampling and infinity norm ball

Let us first explain the setup of matrix completion and one-bit matrix completion.

**3.2.1. Problem setup** Let $e_i$ denote the $i$-th standard basis vector in some appropriate dimension. Entrywise sampling involves measurement matrices of the form $A_i = \sqrt{d_1 d_2} e_{k(i)}^T e_{l(i)}^T$.

Here for each $i \in [n]$, the index pair $(k(i), l(i))$ is uniformly sampled from $[d_1] \times [d_2]$ and independent of anything else. We consider the following two settings:

1. **Matrix completion:** $\psi(\theta) = \frac{1}{2} \theta^2$, and $c(\sigma) = \sigma^2$.
2. **One-bit Matrix completion:** $\psi(\theta) = \log(1 + \exp(\theta))$, and $c(\sigma) = 1$.

Analogous to the models in Section 3.1, matrix completion corresponds to a partial observation of matrix entries with Gaussian noise, and one-bit matrix completion corresponds to a binary observation.

Next we explain the choice of $C$ and its importance.

**3.2.2. The choice of $C$ and its importance** Here the regularity constraint is taken to be the $\ell_\infty$ norm ball, $C = B_{\|\cdot\|_\infty}(\psi)$. This constraint ensures that the matrix is incoherent/non-spiky, well known to be necessary for matrix completion. For the one-bit setting the constraint $C$ is even more crucial, without which the condition number becomes unbounded for reasons similar to before—see
Appendix [D] for further discussion. Within $C$, Lemma [2] below shows that the desired approximate RSC/RSM properties hold, which in turn allows us to establish the statistical guarantees in Corollary 2.

An intriguing question is whether the constraint $C$ should be imposed explicitly in practice. In previous work, this constraint is sometimes ignored in the experiments [LN15, DPVDBW14] and relegated as an artifact of analysis [KU20, pp.9]. For problems with quadratic loss, the work in [CCF+19, MWCC19, DC18] proves that the iterates of the algorithm stay in $C$ automatically, though their sample complexity requirement is substantially larger than optimal.

Here we argue for explicit enforcement of $C$. Our experiment result for projected gradient [4] and AVPG in Figure 3, whose setting presented in details in Appendix [C], shows that doing so is very beneficial, especially when the sample size is limited and when the loss is non-quadratic as in one-bit matrix completion. Imposing $C$ enhances algorithm stability and reduces statistical errors. Without $C$, the estimation error is sometimes worse than a trivial constant estimator, a similar situation as in Example [1]. Indeed, all the iterates converge to some stationary point or local minimizer, as the objective value keeps decreasing and approaches stagnancy. As mentioned, the distances of the iterates with regularity projection approach 0.39 while the others approaching a number larger than 1 (not shown here), worse than the trivial estimator 0.

Finally, we provide theoretical guarantees.

3.2.3. Theoretical guarantees As is standard in the matrix completion literature, we introduce the following spikeness measure $\alpha_{sp,z} := \sqrt{\frac{d_x d_z \|X^z\|_2}{\|X^z\|_F^2}}$ of the true matrix $X^z$. The following lemma verifies the desired structural properties of the loss $\mathcal{L}$, including approximate RSC/RSM (proved in Appendix [B.3.2]) and bounds on the gradient (proved in Lemma 16 in the appendix).

Lemma 2. Consider the RGLM with matrix completion or one-bit matrix completion setting. Let the constraint set $C = B_{\|\cdot\|_2}(\frac{\alpha'}{\sqrt{d_x d_z}} \|X^z\|_F)$ for some $\alpha' \geq \alpha_{sp}(X^z)$. Then there are universal constants $c, c_0, c_1, c_2 > 0$ such that for any $n \geq c d r^2 \log d$, with probability at least $1 - \exp(-c_1 n) - c_2 d^{-2}$, the following two statements hold:

1. (RSC/RSM) $\mathcal{L}$ satisfies $(c_0 B_{\alpha'^2} \epsilon_n, t_0 r^2, \frac{15}{16} B, C)$-RSC and $(c_0 B \epsilon_n, t_0 r^2, \frac{15}{16} B, C)$-RSM;
2. (Gradient bound) $\|\nabla \mathcal{L}(X^z)\|_{op} \leq C \sqrt{\frac{c(\sigma) d \log d}{n}}$.

Here $\epsilon_n = \frac{\kappa \log d \log d}{n}$, $t_0 = [4\kappa (\log 4\kappa + 1)]$, and $\kappa = 1.1 \frac{B}{B}$, where $B := \inf_{|x| \leq \alpha' \|X^z\|_F} \psi''(x)$ and $\overline{B} := \sup_{|x| \leq \alpha' \|X^z\|_F} \psi''(x)$.

Scaling in Lemma [2] To better understand the scaling in Lemma [2] consider the scenario $\alpha' = \alpha_{sp,z}$ and $\|X^z\|_F$ are both universal constants, then $\kappa, \overline{B}$, and $\overline{B}$ are also universal constants. For some universal $c_1, c_2, c_3, c_4, C > 0$, the loss $\mathcal{L}$ satisfies $(c_1 r^2 d \log d, c_2 r^2, c_3, C)$-RSC and $(c_1 r^2 d \log d, c_2 r^2, c_4, C)$-RSM; and the gradient $\|\nabla \mathcal{L}(X^z)\|_{op} \leq C \sqrt{\frac{c(\sigma) d \log d}{n}}$ where $c(\sigma) = \sigma^2$ for matrix completion and $c(\sigma) \equiv 1$ for one-bit matrix completion.

Suppose that the input of AVPG is $r = r^2, \beta = \frac{15}{16} \overline{B}, \eta_{t_0} = \frac{1}{\sqrt{t_0}}$, and $t_0 = [4\kappa (\log 4\kappa + 1)]$. The following corollary is immediate from combining Theorem 1 and Lemma 2.

Corollary 2. Instate the assumptions and notation in Lemma 3 and suppose AVPG uses the input described above. Define $\tau_* = \arg \min_{1 \leq \tau \leq t+1} \mathcal{L}(X_t)$. Then there exist some universal constants $c, c'$

\footnote{Note that if one is willing to early stop the algorithm, then the generalization error in terms of the distance to the ground truth $X^z$ is actually better. The distance for PG and AVPG without the the regularity projection gets very close to $X^z$ in the beginning. However, determining the stopping time is a nontrivial issue.}
(a) Comparison of relative distance to ground truth $\frac{\|X_t - X^*\|_F}{\|X^*\|_F}$, and objective $\mathcal{L}(X_t)$ for PG, projected gradient

(b) Comparison of relative distance to ground truth $\frac{\|X_t - X^*\|_F}{\|X^*\|_F}$, and objective $\mathcal{L}(X_t)$ for AVPG.

Figure 3: Comparison of projected gradient and AVPG with and without the regularity projection oracle (i.e., with projection $\mathcal{P}_{r,B\|\cdot\|_F}$ and with $\mathcal{P}_{r,Bd_1 \times d_2}$, respectively) for the one-bit matrix completion problem. The horizontal such that for any $n \geq cr^2d$, with probability at least $1 - c'd^{-2}$

$$\|X_{t_n} - X^*\|_F^2 \leq \frac{c}{B} \max \left\{ \left(1 - \frac{1}{4\kappa}\right)^t (4\kappa)^s h_0, \kappa \log \kappa (\kappa^2 c(\sigma) + \alpha' \sqrt{c(\sigma)B\kappa + (\alpha')^2B}) \frac{r^2d \log d}{n} \right\}.$$ (9)

Here $s$ is the largest integer so that $st_0 \leq t$.

**Interpretation of Corollary 2** Recall that $c(\sigma) = \sigma$ for matrix completion and $c(\sigma) \equiv 1$ for one-bit matrix completion. To better interpret Corollary 2, we let $t \to \infty$ and focus on the second RHS term of statistical error in the bound $\|X_{t_n} - X^*\|_F^2 \leq \frac{c}{B} \max \left\{ \left(1 - \frac{1}{4\kappa}\right)^t (4\kappa)^s h_0, \kappa \log \kappa (\kappa^2 c(\sigma) + \alpha' \sqrt{c(\sigma)B\kappa + (\alpha')^2B}) \frac{r^2d \log d}{n} \right\}$. Also assume that we take $\alpha' = \alpha_{sp,\natural}$ in AVPG. For matrix completion, the statistical error is $\mathcal{O}(\sigma^2 + \alpha_{sp,\natural}^2 \frac{r^2d \log d}{n})$. For one-bit matrix completion, further assume that $\alpha_{sp,\natural} = \mathcal{O}(1)$ and $\|X^*\|_F = \mathcal{O}(1)$, then we have the statistical error bound $\mathcal{O}(\frac{r^2d \log d}{n})$. Both bounds match the those achieved by convex relaxation methods; cf. [Wai19, Corollary 10.18].

3Since the first RHS term is geometrically decreasing, we actually only need $t = \mathcal{O}(\log(\text{second RHS term}))$ to balance the two term.
3.2.4. Essentiality of approximate RSC/RSM  Here we would like to point out that it is essential to consider the approximate version instead of the standard version $\epsilon_\alpha = \epsilon_\beta = 0$ for entrywise sampling. To see this, suppose $d_1 = d_2 = d$ for simplicity and consider the matrix $X = 0$ and $Y = \frac{1}{2}e_1e_1^\top$, a matrix with all zero entries except the top left entry being 1/d. Then the standard RSC/RSM fails for the scaling described above. But the approximate version does hold still. Indeed, for both matrix completion and one-bit matrix completion, the middle term in the RSC/RSM condition in (1) is $\mathcal{L}(X) - \mathcal{L}(Y) - \langle \nabla \mathcal{L}(X), Y - X \rangle = 0$ with high probability whenever $n = \Theta(r^2d\log d)$. However, the RSC term, the left term of (6), $\frac{2}{n} \| X - Y \|_F^2 - \epsilon_\alpha$ is nonzero for $\epsilon_\alpha = 0$. Hence the strict RSC cannot hold in this case. However, if we allow $\epsilon_\alpha = C_{\text{RSC/RSM}}^{r^2d\log d}$ for some numerical constant $C$ (recall the scaling in the last paragraph), then $\frac{2}{n} \| X - Y \|_F^2 - \epsilon_\alpha < 0$ and our approximate RSC does hold still. The reason why the approximate RSC/RSM is enough for our purpose is that we shall choose $X = X_t$ and $Y = X^z$ in our analysis. And we only need to consider the case when $\| X_t - X^z \|_F$ is larger than the statistical error (measured by the Frobenius norm).

3.3 Pairwise sampling and infinity norm ball
In this section, we consider individualized rank aggregation (IRA) setting studied in [LN15].

3.3.1. Problem setup  The measurement matrix $A_i$ in this setting satisfies that $A_i = \sqrt{d_1d_2}e_{k(i)}(e_{l(i)} - e_{j(i)})^\top$. Here for each $i \in [n]$, the number $k(i) \in [d_1]$ is uniformly distributed on $[d_1]$ independent of anything else, and $(l(i), j(i))$ is uniformly distributed over $[d_2]^2$ independent of anything else. We call such sampling "pairwise" because it always picks two entries in the same row as a pair. The response $y_i$ is Bernoulli, meaning that $\psi(\theta) = \log(1 + \exp(\theta))$ and $c(\sigma) \equiv 1$.

This model can be considered as users’ responses when giving a pair of items in a recommendation system. Each row of $X^z$ represents a user’s score for different items. In each sample $(y_i, k(i), l(i), j(i))$, the $k(i)$-th user gives a response $y(i)$, representing whether she prefers item $l(i)$ to item $j(i)$. The value $y(i) = 1$ means that she prefers $l(i)$-th item to $j(i)$-th term, otherwise, she prefers the other way. Let us now introduce the constraint set $C$.

3.3.2. The choice of $\mathcal{C}$  For pairwise sampling, apart from the infinity norm ball (which is imposed for similar reasons of matrix completion) the constraint set in $\mathcal{C}$ has an additional constraint that $\mathcal{F} := \{ X \mid \sum_{1 \leq \ell \leq d_1} X_{k\ell} = 0, \text{ for all } 1 \leq k \leq d_1 \}$ compared to entrywise sampling. This constraint eliminates identification issue due to the difference in the measurements $A_i$ and the modeling of the probability that $y_i = 1$, see [LN15] Section 2.1 for more information on this condition. Finally, we provide the theoretical guarantees.

3.3.3. Theoretical guarantees  The proof of RSC/RSM condition can be found in Section B.3.3 in the appendix. The gradient norm condition is proved in Lemma 16 in the appendix. We summarize the two in the following lemma. The scaling of the parameters of approximate RSC/RSM and the bound of $\| \nabla \mathcal{L}(X^z) \|_{\infty}$ under the condition, constant $\alpha' = \alpha_{sp, k}$ and $\| X^z \|_F$, follow the same behavior as those for one-bit matrix completion.

Lemma 3 (RSC/RSM and small gradient for pairwise measurement). Consider the RGLM with individualized rank aggregation setting. Let the constraint set $\mathcal{C} = \{ X \mid \sum_{1 \leq \ell \leq d_1} X_{k\ell} = 0, \text{ for all } 1 \leq k \leq d_1 \} \cap \mathcal{F} = \{ X \mid \frac{\alpha'}{\sqrt{d_1d_2}} \| X^z \|_F \}$ for some $\alpha' \geq \alpha_{sp}(X^z)$. Then there is a universal constant $c, C, c_0, c_1, c_2 > 0$ such that for any $n \geq c_1r^2d\log d$, with probability at least $1 - \exp(-c_1n) - c_2d^{-2}$, the following two hold.
1. The loss function $\mathcal{L}$ satisfies $(c_0 B^2 \epsilon_n, t_0 r^2, \frac{31}{10} B, C)$-RSC and $(c_0 B^2 \epsilon_n, t_0 r^2, \frac{31}{10} B, C)$-RSM.

2. The gradient satisfies that $\|\nabla \mathcal{L}(X^*)\|_p \leq C \sqrt{c(\sigma) B \frac{d \log d}{n}}$.

Here $\epsilon_n = \frac{\kappa \log \kappa d \log d}{n}$, $t_0 = [4\kappa (\log 4\kappa + 1)]$, and $\kappa = 1.1 \frac{B}{B}$ where $B := \inf_{|x| \leq \alpha' \|X^*\|_p} \psi''(x)$ and $\overline{B} := \sup_{|x| \leq \alpha' \|X^*\|_p} \psi''(x)$.

Combined the above lemma and Theorem 1, Corollary 3 is immediate. Let $r = r^2$, $\beta = \frac{33}{10} B$, $\eta_0 = \frac{1}{4\kappa}$, and $t_0 = [4\kappa (\log 4\kappa + 1)]$ to be the input of AVPG.

Corollary 3 (Distance to $X^*$). Instate the assumptions and notation in Lemma 3, and suppose AVPG uses the input described above. Define $\tau_* = \arg \min_{1 \leq \tau \leq t} \mathcal{L}(X_t)$. Then there are some universal constants $c, c', C$ such that for any $n < d^2 \log d$ and $r^2 n \geq C d \log d$, with probability at least $1 - c'd^{-2}$

$$\|X_{\tau_*} - X^*\|_p^2 \leq \frac{c}{d^2} \max \left\{ \left( 1 - \frac{1}{4\kappa} \right)^4 (4\kappa)^s h_0, \kappa \log \kappa (\kappa^2 + \alpha' \sqrt{\overline{B} k} + (\alpha')^2 B) \frac{r^2 d \log d}{n} \right\}.$$ 

Here $s$ is the largest integer so that $s t_0 \leq t$.

Interpreting Corollary 3. Same as the case of one-bit matrix completion, for $\alpha' = O(1)$ and $\|X^*\|_p = O(1)$, the bound reduces to $O\left( \frac{r^2 d \log d}{n} \right)$ for $t \to \infty$ and matches the bound of convex relaxation in [LN15].

4 Discussion

In this paper, we identify the regularity projection oracle as the key component of solving many interesting problems under RGLM. We develop efficient algorithm that converges linearly and globally. Furthermore, we show state-of-art statistical recovery bounds in concrete RGLM problems. Here we lay out a few interesting future directions.

- **Models beyond RGLM.** We expect that our algorithm and theoretical framework are broadly applicable to other low-rank problems with non-quadratic loss, such as matrix completion with general exponential family response [Laf15], and multinomial sampling scheme as those in [KU20, OTX15].

- **The choice of $t_0$.** Even though our theory requires $t_0$ to be finite, we found that setting $t_0$ to be infinity still works in our experiments. Empirically, this is due to the fact that the iterate becomes very low rank even though the averaging step is performed in every iteration. Also, empirically, we found that setting $t_0 = 1$, i.e., AVPG is reduced to PG, does not affect the performance of the algorithm, even though our theory requires $t_0 = O(\kappa \log \kappa)$. It is interesting to theoretically explain why PG is still effective in the RGLM setting.

- **Overcoming stationary point by larger stepsize.** We found that in the experiment performed in Figure 1, even though IHT stays at the nonoptimal stationary point while AVPG is able to get rid of it. Empirically, IHT with a larger stepsize is actually able to escape the fixed point $M_0$.

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A Proofs and Lemmas for Section 2

A.1 Lemmas for projection oracle (3)

We denote the Shatten $p$ norm ball with radius $\xi$ as $\mathbb{B}_p(\xi)$.

Lemma 4. Let $(u_i^t, v_i^t, \sigma_i) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times \mathbb{R}, i = 1, \ldots, r$ be the top $r$ left and right singular vectors, and singular values of $X$. The solution $V^*$ to the problem $\min_{\text{rank}(V) \leq r} \|X - V\|_F$ is of the following form: $V^* = \sum_{i=1}^{r} a_i^t u_i^t (v_i^t)^\top$ where $a_i^t \geq 0$. Here the numbers $a_i^t, i = 1, \ldots, i = r$ are the solution to $\min_{a \in \mathbb{N}^r, \|a\|_p \leq \xi} \|a - \sigma\|_2$ where $\sigma = (\sigma_1, \ldots, \sigma_r)$.

Proof. We first note that the solution to the problem $\min_{\text{rank}(V) \leq r} \|X - V\|_F$ is of the form $V^* = \sum_{i=1}^{k} a_i^t u_i^t (v_i^t)^\top$ by using [AZHHL17, Lemma 3.1 and its proof]. This means we only need to choose $a_i, i = 1, \ldots, r$. It is then immediate $a = (a_1, \ldots, a_r)$ should be the solution to $\min_{a \in \mathbb{N}^r, \|a\|_p \leq \xi} \|a - \sigma\|_2$ and our proof is complete.

A.2 Proof of Theorem 1

Proof. We fix $t$ and do a one-step analysis. Consider the following two inequalities for the pair $(X_t, X^2)$:

\[ \frac{\alpha}{2} \|X_t - X^2\|_F^2 \geq 2 \epsilon_{\alpha}, \]  
\[ \frac{\beta \eta^2}{2} \|X_t - X^2\|_F^2 \geq \epsilon_{\beta}, \]  

where $\eta = 1$ or $\frac{1}{4\alpha}$ depending on whether $t$ is a multiple of $t_0$. Suppose that inequality (10) does not hold, i.e., $\|X_t - X^2\|_F \leq \frac{4}{\alpha} \epsilon_{\alpha}$. In this case, using the $(\epsilon_{\beta}, rt_0, \alpha, C)$-RSM property, we find that

\[ h_t = \mathcal{L}_n(X_t) - \mathcal{L}_n(X^2) \leq \langle \nabla \mathcal{L}_n(X^2), X_t - X^2 \rangle + 2 \kappa \epsilon_{\alpha} + \epsilon_{\beta} \]
\[ \leq (a) \in \epsilon_{\nabla} \sqrt{\frac{8 \eta \alpha \epsilon_{\alpha}}{\alpha}} + 2 \kappa \epsilon_{\alpha} + \epsilon_{\beta}, \]  

where in step (a) we use H"older’s inequality and the fact that $X_t, X^2$ has rank no more than $t_0 r$. Therefore, we have the desired bound (7). By a similar argument, we can show that if inequality (11) does not hold, then we again have the desired bound (7).

We henceforth assume that both inequalities (10) and (11) hold. Using the approximate RSM property and the update rule of $X_{t+1}$, we find that

\[ \mathcal{L}(X_{t+1}) \leq \mathcal{L}(X_t) + \eta \langle \nabla \mathcal{L}_n(X_t), V_t - X_t \rangle + \frac{\beta \eta^2}{2} \|V_t - X_t\|_F^2 + \epsilon_{\beta} \]
\[ \leq (a) \in \mathcal{L}(X_t) + \eta \langle \nabla \mathcal{L}_n(X_t), X^2 - X_t \rangle + \frac{\beta \eta^2}{2} \|X^2 - X_t\|_F^2 + \epsilon_{\beta} \]
\[ \leq (b) \in \mathcal{L}(X_t) - \eta \left( \mathcal{L}_n(X_t) - \mathcal{L}_n(X^2) \right) + \eta \left( \beta \eta - \frac{\alpha}{\alpha} \right) \|X^2 - X_t\|_F^2 \]
\[ \leq (c) \in \mathcal{L}(X_t) - \eta h_t + \frac{4 \eta}{\alpha} \left( \beta \eta - \frac{\alpha}{\alpha} \right) \max \left\{ h_t, \frac{4}{\alpha} (rt_0 + r^2) \right\}. \]  

Here in step (a), we use the optimality of $V_t$; in step (b), we use the approximate RSC and the inequalities (10) and (11); in the last step (c) we use Lemma 5 and (10). Now, we subtract $\mathcal{L}_n(X^2)$
from both sides of (13). Doing so and using the choice of η for t is a multiple of t₀ and the cast t is not, we obtain the inequality
\[ h_{t+1} \leq \begin{cases} (4κ - 1) \max \{ h_t, \frac{4}{κ} \left( r[κt₀ + r^2]ε₂ \right) \}, & \exists k \in \mathbb{Z} : t = kt₀, \\ (1 - \frac{4}{κ})h_t, & \text{otherwise}. \end{cases} \] (14)

Applying the inequality (14) inductively proves the desired bound (7) on the objective value.

Finally, combining the approximate RSC property with the bound on \( h_t \) in (7) we just proved, we immediately obtain the desired distance bound on \( ∥Δ_t∥_F \) in (7).

Lemma 5. Given a rank r matrix \( X \in \mathbb{R}^{d_1 \times d_2} \) with \( r \geq r^2 \), and suppose that \( L \) satisfies \((r, α)\)-RSC. Let \( ∥∇L(X^2)∥_op = ε_2 \). Then we have
\[ ∥X - X^2∥_F^2 \leq \frac{4}{α} \max \{ L_n(X) - L_n(X^2), \frac{4}{α}(r + r^2)ε^2_2 \} \].

Proof. First if \( 2 \left| ∑∇L_n(X^2), X - X^2 \right| \geq \frac{2}{α}∥X - X^2∥_F^2 \), using Hölder’s inequality in the following step (a) and \( ∥X - X^2∥_{nuc} \leq √r + r^2∥X - X^2∥_F \) in step (b), we have that
\[ \frac{α}{2}∥X - X^2∥_F^2 \leq 2⟨∇L_n(X^2), X - X^2⟩ \leq 2∥∇L(X^2)∥_op∥X - X^2∥_nuc \leq 2\sqrt{r + r^2}ε_2∥X - X^2∥_F \] (a)
\[ \leq \frac{α}{2}∥X - X^2∥_F^2 \leq 4\sqrt{r + r^2}ε_2. \] (b)

The step (c) is due canceling the term \( ∥X - X^2∥_F \) from both sides of the inequality.

Otherwise, we should have \( 2 \left| ∑∇L_n(X^2), X - X^2 \right| \leq \frac{2}{α}∥X - X^2∥_F^2 \). Using the \((r, α)\)-RSC of \( L \) in the following step (a), and \( 2 \left| ∑∇L_n(X^2), X - X^2 \right| \leq \frac{2}{α}∥X - X^2∥_F^2 \) in step (b), we have
\[ L_n(X) \geq L_n(X^2) + ⟨∇L_n(X^2), X - X^2⟩ + \frac{α}{2}∥X - X^2∥_F^2 \] (a)
\[ \geq L_n(X^2) + \frac{α}{4}∥X - X^2∥_F^2. \] (b)

By combining the inequalities (15) and (16), we achieve the desired inequality
\[ ∥X - X^2∥_F^2 \leq \frac{4}{α} \max \{ L_n(X) - L_n(X^2), \frac{4}{α}(r + r^2)ε^2_2 \} \].

B Lemmas and proofs for Section 3

To establish the RSC/RSM and its approximate version for the loss of RGLM, let us first introduce the linear operators relating to \( A_i, i = 1, \ldots, n \). For any \( S \subset [n] \), we define the linear operator \( A_S : \mathbb{R}^{d_1 \times d_2} \to \mathbb{R}^{|S|} \) with
\[ [A_S(X)]_i = ⟨A_i, X⟩ \] for \( i \in S \).

In particular, if \( S = [n] \), we denote \( A = A_{[n]} \). The corresponding quadratic function of \( A_S \) is defined as
\[ L_{A_S}(X) := \frac{1}{2n} ∥A_S(X)∥_2^2. \]
We shall first show that RSC/RSM, its approximate version of \( \mathcal{L} \) holds whenever certain deterministic conditions of the map \( \mathcal{A}_S \). We then verify these two conditions holds, as well as small gradient norm condition \( \| \nabla \mathcal{L}(X^2) \|_{op} \leq \epsilon \gamma \), with high probability for random \( \mathcal{A} \) so that Theorem 1 can be applied.

B.1 Deterministic condition of \( \mathcal{A}_S \) for (approximate) RSC/RSM

We require the linear map \( \mathcal{A} \) to satisfy one of the following properties:

- The function \( \mathcal{L}_{\mathcal{A}_S} \) satisfies \((r, \beta, C)\)-RSC and \((r, \alpha, C)\)-RSM for each \( |S| \geq (1 - \gamma_0)n \) for some universal \( \gamma_0 > 0 \).

- The function \( \mathcal{L}_\mathcal{A} \) satisfies approximate \((\epsilon_\alpha, r, \alpha, C)\)-RSC and \((\epsilon_\beta, r, \beta, C)\)-RSM.

Since the Hessian of \( \mathcal{L}_{\mathcal{A}_S} \) is actually constant, the quadratic form is simply \( \nabla^2 \mathcal{L}_{\mathcal{A}_S}(Z)[X][X] = 2\mathcal{L}_{\mathcal{A}_S}(X) \) for any \( Z \in \mathbb{R}^{d_1 \times d_2} \). The above properties henceforth are easier to establish using techniques in high dimensional probabilities. Specifically, we might consider iid Gaussian sensing matrix for the first case, and entrywise type sampling scheme in matrix completion or aggregate individual ranking for the second case.

We shall show that the RSC and RSM of \( \mathcal{L}_{\mathcal{A}_S} \) implies the RSC and RSM of \( \mathcal{L} \) with different parameters. Similarly, we show that approximate version of RSC and RSM will imply the same properties of \( \mathcal{L} \) with different parameters. However, due to the nonlinearity of \( \psi' \), we need to restrain our attention to certain bounded set instead of the full space \( \mathbb{R}^{d_1 \times d_2} \), and impose certain boundedness assumption on \( \psi'' \).

**Lemma 6.** Suppose the function \( \mathcal{L}_{\mathcal{A}_S} \) satisfies \((r, \alpha, C)\)-RSC and \((r, \beta, C)\)-RSM for any \( |S| \geq (1 - \gamma_0)n \) for some universal \( \gamma_0 > 0 \) and \( 0 \in \mathcal{C} \). Then the loss \( \mathcal{L} \) satisfies \((B\alpha, r, B\|\cdot\|_{sf}(\xi_0) \cap \mathcal{C})\) RSC and \((r, B\beta, B\|\cdot\|_{sf}(\xi_0) \cap \mathcal{C})\) RSM where \( B = \|\psi''\|_{\infty} := \sup_{x \in \mathbb{R}} |\psi''(x)| \), and \( B = \inf_{|x| \leq \sqrt{\frac{2.2\beta}{\gamma \gamma_0}} \xi_0} \psi''(x) \) for an \( \mathcal{A} \) independent \( \xi_0 > 0 \).

**Proof.** Given any \( X, Y \in B_{\|\cdot\|_{sf}}(\xi_0) \cap \mathcal{C} \) with their ranks not exceeding \( r \) and Frobenius norms not exceeding \( \xi_0 \), define \( \Delta := Y - X \). The Taylor expansion of \( \mathcal{L} \) gives

\[
\mathcal{L}(Y) - \mathcal{L}_n(X) - \langle \nabla \mathcal{L}_n(X), Y - X \rangle = \frac{1}{2n} \sum_{i=1}^n \psi''((X + t_i \Delta, A_i))(\Delta, A_i)^2 \\
\leq B \frac{1}{2n} \|A(\Delta)\|_2^2 \\
\leq B \beta \frac{1}{2} \|\Delta\|_F^2,
\]

where \( t_i \in [0, 1] \). Here we use the assumption \( B \geq \|\psi''\|_{\infty} \) in step (a), and the \((r, \beta, C)\) RSM of \( \mathcal{L}_A \) in step (b).

To prove restricted strong convexity, we claim that for any \( \gamma_0 \in (0, 1) \), there is at most \( \frac{1}{2} \gamma_0 \in (0, 1) \) fraction of the \{\( (X, A_i)^2 \)\}_{i=1}^n\) satisfying \( (X, A_i)^2 \geq \frac{2.2\beta}{\gamma_0} \|X\|_F^2 \). Indeed, otherwise, we will have \( \frac{1}{n} \|A(X)\|_2^2 \geq \frac{\gamma_0}{2} \frac{2.2\beta}{\gamma_0} \|X\|_F^2 > \beta \|X\|_F^2 \), a contradiction to \((r, \beta, C)\) RSM of \( \mathcal{L}_A \) (This is where \( 0 \in \mathcal{C} \) is used). Similarly, we know that at most \( \frac{\gamma_0}{2} \) of the \{\( (Y, A_i)^2 \)\}_{i=1}^n\) satisfying \( (Y, A_i)^2 \geq \frac{2.2\beta}{\gamma_0} \|Y\|_F^2 \). Hence we can find a set \( \mathcal{S} \subset [n] \) with cardinality at least \( n(1 - \gamma) \) such that \( (X, A_i)^2 \leq \frac{2.2\beta}{\gamma_0} \|X\|_F^2 \leq \frac{2.2\beta}{\gamma_0} \xi_0^2 \) for every \( i \in \mathcal{S} \) and the same inequality holds for \( (Y, A_i)^2 \). By
choosing \( \gamma_0 = c_0 \), we see that \( \mathcal{L}_{A_S} \) also satisfies \( \alpha \)-RSC by our assumption. Combining pieces, we have
\[
\mathcal{L}(Y) - \mathcal{L}_n(X) - \langle \nabla \mathcal{L}_n(X), Y - X \rangle = \frac{1}{2n} \sum_{i=1}^{n} \psi''((X + t_i \Delta, A_i)) \langle \Delta, A_i \rangle^2
\]
\[
\geq B \frac{1}{2n} \|A_S(\Delta)\|_2^2
\]
\[
\leq \frac{B}{2} \|\Delta\|_2^2,
\]
where \( t_i \in [0, 1], \) and \( B = \inf_{|x| \leq \sqrt{\frac{2\alpha}{\gamma_0}}} \psi''(x) \). Here we use the construction of \( S \subset [n] \) in step (a), and the \( \alpha \) RSC of \( \mathcal{L}_{A_S} \) in step (b).

**Lemma 7.** Suppose the function \( \mathcal{L}_A \) satisfies \((\epsilon_\alpha, r, \alpha, C)\)-RSC and \((\epsilon_\beta, r, \beta, C)\)-RSM. If \( X \in \mathcal{C} \) and \( \text{rank}(X) \leq r \) implies that \( |\langle X, A_i \rangle| \leq \xi_1 \) for some \( A \) independent \( \xi_1 > 0 \). Then \( \mathcal{L}_n \) satisfies \((\mathcal{B}_1, \epsilon_\alpha, r, \mathcal{B}_1, \alpha, C)\)-RSC and \((\mathcal{B}_2, \epsilon_\beta, r, \mathcal{B}_2, \beta, C)\)-RSM, where \( \mathcal{B}_1 := \inf_{|x| \leq \xi_1} \psi''(x) > 0 \) if \( \psi \) is strongly convex in any bounded domain, and \( \mathcal{B}_2 := \sup_{|x| \leq \xi_1} \psi''(x) \).

**Proof.** Given any \( X, Y \in \mathcal{C} \) with their rank not exceeding \( r \), define \( \Delta := Y - X \). The Taylor expansion of \( \mathcal{L} \) gives
\[
\mathcal{L}(Y) - \mathcal{L}_n(X) - \langle \nabla \mathcal{L}_n(X), Y - X \rangle = \frac{1}{2n} \sum_{i=1}^{n} \psi''((X + t_i \Delta, A_i)) \langle \Delta, A_i \rangle^2
\]
where \( t_i \in [0, 1] \). Using the assumption that \( X, Y \in \mathcal{C} \) implies that \( |\langle X, A_i \rangle| \leq \xi_1 \) and \( |\langle Y, A_i \rangle| \leq \xi_1 \) for every \( i \in [n] \), we see
\[
|\langle X + t_i \Delta, A_i \rangle| \in [\mathcal{B}_1, \mathcal{B}_2],
\]
where \( \mathcal{B}_1 := \inf_{|x| \leq \xi_1} \psi''(x) > 0 \) as \( \psi \) is strongly convex in any bounded domain, and \( \mathcal{B}_2 := \sup_{|x| \leq \xi_1} \psi''(x) \). Hence we can combine this inequality with (19) and reach that
\[
\frac{B_1}{n} \|A(\Delta)\|_2^2 \leq \mathcal{L}(Y) - \mathcal{L}_n(X) - \langle \nabla \mathcal{L}_n(X), Y - X \rangle \leq \frac{B_1}{n} \|A(\Delta)\|_2^2.
\]
Using the approximate RSC and RSM properties of the function \( \mathcal{L}_A \), we achieved the approximate RSC and RSM properties of \( \mathcal{L} \).

**B.2 Random \( A_S \) satisfying (approximate) RSC/RSM with high probability**

**B.2.1. Gaussian measurements \( A_i \)**

**Lemma 8.** If the measurements \( A_i \) have iid standard Gaussian entries, then for some universal constant \( c, c_0, C > 0 \), so long as \( n \geq Cr \), with probability at least \( 1 - \exp(-nc) \), simultaneously for all \( S \subset [n] \) with \( |S| \geq (1 - c_0)n \), the loss \( \mathcal{L}_{A_S} \) satisfies \((\mathcal{B}_1, \epsilon_\alpha, r, \mathbb{R}^{d_1 \times d_2})\)-RSC and \((\mathcal{B}_2, \epsilon_\beta, r, \mathbb{R}^{d_1 \times d_2})\)-RSM.

**Proof.** A standard result [CP11, Theorem 2.3] shows that we have \((r, 1 - \delta, \mathbb{R}^{d_1 \times d_2})\) RSC and \((r, 1 + \delta, \mathbb{R}^{d_1 \times d_2})\) RSM of \( \mathcal{L}_{A_S} \) with probability \( 1 - \exp(-c|S|) \) for each \( S \subset [n] \) if \( |S| \geq c'rd \) for some universal \( c, c' > 0 \). Let \( |S| \geq (1 - \epsilon)n \) for some \( \epsilon \) to be determined, there are at most \( \epsilon n \left(\frac{n}{1-\epsilon}\right) \)
many $S$. The number $\epsilon n \binom{n}{(1-\epsilon)n}$ is bounded by
\[
\epsilon n \binom{n}{(1-\epsilon)n} \leq \exp \left( \log \epsilon + \log n \right) \binom{n}{\epsilon n} \leq \exp \left( \log \epsilon + \log n \right) \left( \frac{\epsilon}{\epsilon - c} \right)^{\epsilon n} \leq \exp \left( \log n + (\epsilon - \epsilon \log \epsilon) n \right).
\]

Here in step (a) we use the fact that $\binom{n}{(1-\epsilon)n} \leq \left( \frac{en}{k} \right)^{k}$. Since $-\epsilon \log \epsilon \to 0$ for $\epsilon \to 0$, we know there are universal constant $c_1 > 0$, and $c_0, c_3$ depends only on $c, c', c_1$ such that for every $n > c_1$, and $S$ with $|S| \geq (1-c_0)n$ that
\[
\exp(-c|S|) \exp(\log n + (\epsilon - \epsilon \log \epsilon)n) \leq \exp(-c_3n).
\]
The proof is then complete. \hfill \square

B.2.2. Entrywise sampling $A_i$. Recall entrywise sampling is defined as follows: the measurement matrix $A_i$ satisfies that $A_i = \sqrt{d_1 d_2} \epsilon_{k(i)}^{T} e_{l(i)}$. Here for each $i \in [n]$, the number $k(i) \in [d_1]$ is uniformly distributed on $[d_1]$ independent of anything else, and $l(i)$ is uniformly distributed over $[d_2]$ and is independent of anything else. Recall the collection of measurement matrices $A_i, i = 1, \ldots, n$ defines our entrywise sampling operator $A : \mathbb{R}^{d_1 \times d_2} \to \mathbb{R}^n$ with $[A_S(X)]_i = \langle A_i, X \rangle$.

We have the following lemma from [Wai19, Theorem 10.17].

**Lemma 9.** For the random entrywise sampling operator $A : \mathbb{R}^{d_1 \times d_2} \to \mathbb{R}^n$, let $d = \max\{d_1, d_2\}$. There are universal constants $c_1, c_2$ that
\[
\left| \frac{1}{n} \left\| A(X) \right\|^2_2 - \left\| X \right\|^2_2 \right| \leq c_1 \alpha_{sp}(X) \left\| X \right\|_{sp} \sqrt{\frac{d \log d}{n}} + c_2 \alpha_{sp}^2 \frac{d \log d}{n}
\]
for all $X \in \mathbb{R}^{d_1 \times d_2}$ with probability at least $1 - 2e^{-\frac{1}{2}d \log d}$.

**Lemma 10.** Under the same setting as Lemma 9, we have for with probability at least $1 - 2e^{-\frac{1}{2}d \log d}$ that for any $\delta \in (0, 1)$, and all $X$ with rank no more than $r$ and $\left\| X \right\|_\infty \leq \frac{\alpha}{\sqrt{d_1 d_2}}$ simultaneously that
\[
\left| \frac{1}{n} \left\| A(X) \right\|^2_2 - \left\| X \right\|^2_2 \right| \leq 2\delta \left\| X \right\|^2_2 + \frac{1}{\delta} c \alpha^2 \frac{r \delta \log d}{n}.
\]

for some universal $c > 0$.

**Proof.** Using Lemma 9 we found that
\[
\left| \frac{1}{n} \left\| A(X) \right\|^2_2 - \left\| X \right\|^2_2 \right| \leq c_1 \sqrt{d_1 d_2} \left\| X \right\|_\infty \left\| X \right\|_{sp} \sqrt{\frac{d \log d}{n}} + c_2 d_1 d_2 \left\| X \right\|_\infty \frac{d \log d}{n}.
\]
Combining with $\left\| X \right\|_\infty \leq \frac{\alpha}{\sqrt{d_1 d_2}}$ and rank($X$) $\leq r$, we have
\[
\left| \frac{1}{n} \left\| A(X) \right\|^2_2 - \left\| X \right\|^2_2 \right| \leq c_1 \alpha \left\| X \right\|_F \sqrt{\frac{r \delta \log d}{n}} + c_2 \alpha^2 \frac{d \log d}{n}.
\]
Now if \( \|X\|_F^2 \leq \frac{1}{\delta} c_1 \alpha \|X\|_F \sqrt{\frac{rd \log d}{n}} \), or \( \|X\|_F^2 \leq \frac{1}{\delta^2} c_2 \alpha^2 \frac{d \log d}{n} \), then we always have for some universal \( c_3 \) that
\[
\|X\|_F \leq \frac{1}{\delta} c_3 \alpha \sqrt{\frac{rd \log d}{n}}.
\]
Combining with (23), the lemma is immediate. Otherwise, we shall have
\[
\|X\|_F \geq \max \left\{ \frac{1}{\delta} c_1 \alpha \sqrt{\frac{rd \log d}{n}}, \frac{1}{\delta^2} \alpha \sqrt{\frac{d \log d}{n}} \right\}.
\]
The lemma is again immediate by combining the above inequality with (23).

\[\square\]

### B.2.3. Pairwise sampling \( A_i \)
We consider the sampling scheme described in [LN15]. Recall the measurement matrix \( A_i \) satisfies that \( A_i = \sqrt{d_1 d_2} e_{k(i)} (e_{l(i)} - e_{j(i)})^\top \). Here for each \( i \in [n] \), the number \( k(i) \in [d_1] \) is uniformly distributed on \([d_1]\) independent of anything else, and \((l(i), j(i))\) is uniformly distributed over \([d_2]^2\) and is independent of anything else. We shall establish the following lemma.

**Lemma 11.** For the random pairwise comparison operator \( A : \mathbb{R}^{d_1 \times d_2} \to \mathbb{R}^n \), let \( d = \max\{d_1, d_2\} \). If \( n < d^2 \log d \), There are universal constants \( c_1, c_2 \) that
\[
\frac{1}{n} \frac{\|A(X)\|_2^2}{\|X\|_F^2} - 1 \leq c_1 \alpha_{sp}(X) \frac{\|X\|_{auc}}{\|X\|_F} \sqrt{\frac{d \log d}{n}} + c_2 \alpha_{sp}^2(X) \frac{d \log d}{n}
\]
for all \( X \in \mathbb{R}^{d_1 \times d_2} \) with probability at least \( 1 - 2e^{-\frac{1}{2} d \log d} \).

**Proof.** Let us first define a few notions to ease our proof presentation. As the inequality is homogeneous in \( X \), we only need to consider \( \|X\|_F = 1 \). Define the set
\[
\mathcal{B}(D, \alpha) = \left\{ X \in \mathbb{R}^{d_1 \times d_2} \mid \|X\|_F = 1, \|X\|_{\infty} \leq \frac{\alpha}{\sqrt{d_1 d_2}}, \text{ and } \|X\|_{auc} \leq D \right\}.
\]
Let \( F_X(A) = \langle X, A \rangle^2 \). \( M(D) = \sup_{X \in \mathcal{B}(D)} \|X\|_{auc} \). Note that
\[
\mathbb{E}[F_X(A)] = d_1 d_2 \mathbb{E} \left[ \sum_{i=1}^{n} \left( X_{k(i)l(i)} - X_{k(i)j(i)} \right)^2 \right] = \frac{1}{d_2} \sum_{1 \leq k \leq d_1, 1 \leq j, l \leq d_2} (X_{kl} - X_{kj})^2 \\
= \frac{1}{d_2} \left( \sum_{1 \leq k \leq d_1, 1 \leq j, l \leq d_2} X_{kl}^2 + \sum_{1 \leq k \leq d_1, 1 \leq j, l \leq d_2} X_{kj}^2 - \sum_{1 \leq k \leq d_1, 1 \leq j, l \leq d_2} 2X_{kj} X_{kl} \right) \tag{24}
\]
\[
\leq 2\|X\|_F^2 - \frac{1}{d_1} \sum_{1 \leq k \leq d_1, 1 \leq j \leq d_2} X_{kj} \sum_{1 \leq l \leq d_2} X_{kl}.
\]

Here in step (a), we use the fact that \( X \in \mathcal{B}(D) \) implies that the row sum of \( X \) is zero for each row.

We shall now prove the lemma via the standard argument: concentration around the mean, bounding expectation, and the peeling argument. Denote \( \|B\|_1 = \sum_{1 \leq i \leq d_1, 1 \leq j \leq d_2} |B_{ij}| \) be the vector \( \ell_1 \) norm on any matrix \( B \in \mathbb{R}^{d_1 \times d_2} \).
For the concentration around the mean, we first find that
\[ |F_X(A_i)| \leq \|X\|_\infty^2 \|A_i\|_1 \leq \alpha^2. \tag{25} \]
Here we use the Hölder’s inequality in step (a) and the \( \|X\|_\infty \leq \frac{\alpha}{\sqrt{d_1 d_2}} \), and the definition of \( A_i \) in step (b). For the variance of \( F_X(A_i) \), we have
\[ \text{var}(F_X(A_i)) \leq \mathbb{E}[F_X(A_i)]^2 \leq \alpha^2 \mathbb{E}[F_X(A_i)] = 2\alpha^2. \tag{26} \]
Here in step (a), we use (25). Combining the inequalities (25) and (26), using the Talagrand concentration for empirical process in Lemma 13 with \( \epsilon = 1 \) and \( t = \frac{d \log d}{n} \), we conclude that there are some universal constants \( c_1, c_2 \) such that
\[ \mathbb{P} \left[ M(D, \alpha) \geq 2\mathbb{E}M(D, \alpha) + \frac{c_1}{8} \alpha^2 \sqrt{\frac{d \log d}{n}} + \frac{c_2}{4} \alpha^2 d \log d \right] \leq \exp(-d \log d). \tag{27} \]

For bounding the expectation \( \mathbb{E}M(D, \alpha) \), by following the proof in [LN15, Lemma 3] with minor modification (this is where the condition \( n < d^2 \log d \) used), we find that
\[ \mathbb{E}M(D, \alpha) \leq \frac{c_1}{16} \alpha D \sqrt{\frac{d \log d}{n}}. \tag{28} \]
for some appropriate chosen universal constant \( c_1 \).

Finally, we shall use a peeling argument to prove the bound for all \( \alpha \) and \( D \). Note our bounds (26), and (27) match exactly the ones in [Wai19, Proof of Theorem 10.17], using the step [Wai19, Extension via peeling] there, we conclude our lemma.

Using Lemma 11 and the proof of Lemma 10, the following lemma for approximate RSC/RSM is immediate.

**Lemma 12.** Under the same setting as Lemma 11, we have for with probability at least \( 1 - 2e^{-\frac{1}{2} d \log d} \) that for any \( \delta \in (0, 1) \), and all \( X \) with rank no more than \( r \) and \( \|X\|_\infty \leq \frac{\alpha}{\sqrt{d_1 d_2}} \) simultaneously that
\[ \left| \frac{1}{n} \|A(X)\|_2^2 - 2\|X\|_F^2 \right| \leq 2\delta \|X\|_F^2 + \frac{1}{\delta} \alpha^2 r d \log d. \tag{29} \]
for some universal \( c > 0 \).

**Lemma 13** (Talagrand concentration for empirical process). [Wai19, Theorem 3.27, and Equation (3.86)] Consider a countable class of functions \( F : \mathcal{X}\to \mathbb{R} \) uniformly bounded by \( b \), where \( \mathcal{X} \subset \mathbb{R}^d \) for some \( d \). For a series of i.i.d. random variable \( X_i \) follows probability distribution \( \mathbb{P}_X \) supported on \( \mathcal{X} \). Define \( \sigma^2 = \sup_{f \in F} \mathbb{E} f(X) \). Then for any \( \epsilon, t > 0 \), tje random variable \( Z = \sup_{f \in F} \frac{1}{n} \sum_{i=1}^n f(X_i) \) satisfies the upper tail bound
\[ \mathbb{P}[Z \geq (1 + \epsilon)\mathbb{E}Z + c_0 \sigma \sqrt{t} + (c_1 + c_0^2 / \epsilon)bt] \leq e^{-nt}, \]
for some universal constant \( c_0 > 0 \).
B.3 Proof of approximate RSC/RSM for Gaussian measurements, entrywise sampling, and pairwise sampling

B.3.1 Proof of approximate RSC/RSM for Gaussian measurements of Lemma 1
The approximate RSC and RSM condition listed in Lemma 1 is immediate by combining Lemma 8 and 6.

B.3.2 Proof of approximate RSC/RSM for entrywise measurements of Lemma 2
The approximate RSC and RSM condition listed in Lemma 2 is immediate by combining Lemma 10 and 7.

B.3.3 Proof of approximate RSC/RSM for entrywise measurements of Lemma 3
The approximate RSC and RSM condition listed in Lemma 3 is immediate a simple consequence of combining Lemma 7 and 12.

B.4 Small gradient norm \( \| \nabla L(X) \|_{op} \)

B.4.1 Gaussian measurements \( A_i \)
Our first lemma draws the connection between the gradient \( \nabla L(X) \) and the map \( A \).

Lemma 14. For the exponential family noise model in \( \text{(1)} \), we have

\[
\nabla L(X^z) = \frac{1}{n} \sum_{i=1}^{n} \left( \psi'(\langle X^z, A_i \rangle) - y_i \right) A_i.
\]

If \( \bar{B} := \| \Psi'' \|_{\infty} < \infty \), then each \( w_i := \psi'(\langle X^z, A_i \rangle) - y_i \) is subgaussian conditional on \( A_i \) with \( E(\exp(tw_i) \mid A_i) \leq \exp(\frac{t^2\bar{B}^2c}{2c(\sigma)}) \).

Proof. The formula for \( \nabla L(X^z) \) is immediate given the definition of \( L \). To show \( w_i \) is subgaussian, denote the shorthand that \( \theta_i = \langle A_i, X^z \rangle \). Then

\[
\log E(\exp(tw_i) \mid A_i) = t\psi'(\theta_i) + \frac{1}{c(\sigma)} (\psi(\theta_i - tc(\sigma)) - \psi(\theta_i)) \\
\leq \frac{1}{2c(\sigma)} t^2c^2(\sigma)\psi''(\theta_i - \tilde{t}c(\sigma)) \\
\leq \frac{1}{2c(\sigma)} t^2c^2(\sigma)\bar{B}.
\]

Lemma 15. Suppose the sensing scheme is Gaussian where each \( A_i \) has i.i.d. standard Gaussian entries. Let \( d = \max\{d_1, d_2\} \). Then the following bound holds

\[
P \left( \| \nabla L(X^z) \|_{op} \leq \sqrt{\frac{d}{n}c(\sigma)\bar{B}} \right) \leq 1 - \exp(-cd),
\]

where \( c \) is some universal constant, and \( \bar{B} := \| \Psi'' \|_{\infty} \).
Proof. Let \( Q = \nabla \mathcal{L}_n(X^i) \). Consider \( [u^1, \ldots, u^M] \) and \([v^1, \ldots, v^N]\) be \( 1/4 \)-covers in Euclidean norm of the spheres \( \mathbb{S}^{d_1-1} \) and \( \mathbb{S}^{d_2-1} \), respectively. By Lemma [Wai19, Lemma 5.7], we know we can make \( M \leq 9^d \) and \( N \leq 9^d \). Standard covering argument (see for example [Wai19, page 324]) shows that

\[
\|Q\|_{\text{op}} \leq 2 \max_{1 \leq j \leq M, 1 \leq l \leq N} Z^{i,l}, \quad \text{where} \quad Z^{i,l} = \langle u^j, Qv^l, \rangle
\]  

We can decompose \( Z^{i,l} \) as \( Z^{i,l} = \frac{1}{n} \sum_{i=1}^n w_i Y^{i,j} \) where \( w_i = \psi'(\langle X^i, A_i \rangle) - y_i \) and \( Y^{i,j} = \langle u^j, A_i v^l \rangle \). Since \( w_i \) and \( Y^{i,j} \) are subgaussian with parameter \( \sqrt{c(\sigma)B} \) and 1 respectively [Ver18, Definition 2.5.6], we know that \( w_i Y^{i,j} \) is subexponential with parameter \( K := \sqrt{c(\sigma)B} \) [Ver18, Definition 2.7.5, Lemma 2.7.7]. Using the Bernstein’s inequality [Ver18, Corollary 2.8.3], we have that

\[
\mathbb{P} \left\{ \frac{1}{n} \sum_{i=1}^n w_i Y^{i,j} \geq t \right\} \leq 2 \exp \left( -nc \min \left( \frac{t^2}{K^2}, \frac{t}{K} \right) \right).
\]  

Taking \( t = CK \sqrt{\frac{d}{n}} \) for some universal constant \( C > 0 \), we find that with probability at least

\[
\mathbb{P} \left( |Z^{i,l}| = \frac{1}{n} \sum_{i=1}^n w_i Y^{i,j} \leq CK \sqrt{\frac{d}{n}} \right) \leq 1 - 2 \exp(-9d).
\]  

A union bound on all \( u^j \) and \( v^l \) shows that previous inequality holds with probability at least \( 1 - 2 \exp(-cd) \) for some universal \( c > 0 \) simultaneously for all \( u^j, v^l, j = 1, \ldots, M, \) and \( l = 1, \ldots, N \). Hence, combining inequalities (33) and (31), we find that with probability at least \( 1 - \exp(-cd) \), we have

\[
\|\nabla \mathcal{L}(X^i)\|_{\text{op}} \leq C \sqrt{c(\sigma)B} \sqrt{\frac{d}{n}}.
\]

Let us consider the noise family is Gaussian, \( \psi(\theta) = \frac{1}{2} \theta^2 \) and \( c(\sigma) = \sigma^2 2 \), or is Bernoulli, \( \psi(\theta) = \log(1 + e^\theta) \) and \( c(\sigma) = 1 \):

- Gaussian noise: \( B = 1 \), and w.h.p.

\[
\|\nabla \mathcal{L}(X^i)\|_{\text{op}} \leq c\sigma \sqrt{\frac{d}{n}}.
\]  

- Bernoulli noise: \( B \leq 2 \), and w.h.p.

\[
\|\nabla \mathcal{L}(X^i)\|_{\text{op}} \leq C \sqrt{\frac{d}{n}}.
\]

B.4.2. Entrywise and pairwise sampling \( A_i \) Here we assume RGLM is either Bernoulli response, \( c(\sigma) = 1, \psi(\theta) = \log(1 + e^\theta) \), or Gaussian response, \( \psi(\theta) = \frac{1}{2} \theta^2 \) and \( c(\sigma) = \sigma^2 \). We show the following Lemma for Entrywise and pairwise sampling.

**Lemma 16.** For Bernoulli response and Gaussian response of RGLM with entrywise sampling or Bernoulli response of RGLM with pairwise sampling, there exists universal constant \( c > 0 \) such that with probability at least \( 1 - d^{-2} \), there holds \( \|\nabla \mathcal{L}(X^i)\|_{\text{op}} = \|\frac{1}{n} \sum_{i=1}^n (\psi'(\langle X^i, A_i \rangle) - y_i) A_i\|_{\text{op}} \leq c\sqrt{c(\sigma)^d \log d} \frac{1}{n} \).
and define \( \sigma \) that with probability at least 1 \(-\frac{2}{d} \):

\[
\|\nabla L(X^2)\|_{op} = \frac{1}{n} \sum_{i=1}^{n} \left( \psi'((X^2, A_i)) - y_i \right) A_i \|_{op} \leq 8 \sqrt{\frac{d \log d}{n}},
\]

where \( d = \max(d_1, d_2) \).

- For entrywise or pairwise sampling scheme with Bernoulli noise, a direct application of Lemma 17 yields with probability at least 1 \(-\frac{2}{d} \):

\[
\|\nabla L(X^2)\|_{op} = \frac{1}{n} \sum_{i=1}^{n} \left( \psi'((X^2, A_i)) - y_i \right) A_i \|_{op} \leq 8 \sqrt{\frac{d \log d}{n}},
\]

where \( d = \max(d_1, d_2) \).

- For entrywise sampling scheme with Gaussian noise, utilizing [Wai19, Example 6.18], we find that with probability at least 1 \(-\frac{2}{d} \),

\[
\|\nabla L(X^2)\|_{op} = \frac{1}{n} \sum_{i=1}^{n} \left( \psi'((X^2, A_i)) - y_i \right) A_i \|_{op} \leq 8\sigma \sqrt{\frac{d \log d}{n}},
\]

where \( d = \max(d_1, d_2) \).

\( \square \)

**Lemma 17.** [Tro12, Theorem 1.6] Let \( W_i \) be independent \( d_1 \times d_2 \) zero-mean random matrices such that \( \|W_i\|_{op} \leq M \), and define \( \sigma_i^2 := \max\{\|\mathbb{E}[W_i W_i^\top]\|_{op}, \|\mathbb{E}[W_i^\top W_i]\|_{op}\} \) as well as \( \sigma = \sum_{i=1}^{n} \sigma_i^2 \). We have

\[
\mathbb{P} \left[ \sum_{i=1}^{n} W_i \|_{op} \geq t \right] \leq (d_1 + d_2) \max \left\{ \exp(-\frac{t^2}{4\sigma^2}), \exp\left(-\frac{t}{2M}\right) \right\}.
\]

The following lemma seems to be convenient for Poisson and exponential case.

**Lemma 18.** [Lafl15, Proposition 21] [KLT14, Proposition 11] [KLT14, Theorem 4] Consider a finite sequence of independent random matrices \( (Z_i)_{1 \leq i \leq n} \in \mathbb{R}^{d_1 \times d_2} \) satisfying \( \mathbb{E}[Z_i] = 0 \). For some \( U > 0 \), assume

\[
\inf\{\delta > 0 \mid \mathbb{E}[\exp(\|Z\|_{op}/\delta)] \leq e\} \leq U \quad \text{for} \quad i = 1, \ldots, n,
\]

and define \( \sigma_Z \) as \( \sigma_Z^2 = \max\{\frac{1}{n} \mathbb{E}[Z_i^\top Z_i] \|_{op}, \frac{1}{n} \mathbb{E}[Z_i Z_i^\top] \|_{op}\} \). Then for any \( t > 0 \) with probability at least 1 \(-e^{-t} \),

\[
\frac{1}{n} \sum_{i=1}^{n} Z_i \|_{op} \leq c_U \max \left\{ \sigma_Z \sqrt{\frac{t + \log d}{n}}, U \log \left( \frac{U}{\sigma_Z} \right) \sqrt{\frac{t + \log d}{n}} \right\},
\]

with \( d = \max\{d_1, d_2\} \) and \( c_U \) a constant which depends only on \( U \).

### C. Additional numerics

Here we described the experiments for one-bit matrix completion.

**Problem simulation setup** We simulate the ground truth via \( X^2 = \frac{M_1 M_2}{0.5|\mathbb{M}|_{\infty}} \) where each entry of \( M_1 \in \mathbb{R}^{d_1 \times r} \) and \( M_2 \in \mathbb{R}^{d_2 \times r} \) is drawn from uniform distribution on \([-0.5, 0.5]\) independently. Instead of the entrywise sampling scheme described in Section 3.2 we use Bernoulli sampling. Given a number \( p \in [0, 1] \), for each index \((i, j) \in [d_1] \times [d_2] \), we observe \((y_{ij}, w_{ij}) \in \{0, 1\}^2 \) where \( y_{ij} = z_{ij} w_{ij} \) where \( z_{ij} \sim \text{Bernoulli}(\frac{1}{1 + \exp(-X_{ij})}) \) independent of anything else and \( w_{ij} \sim \text{Bernoulli}(p) \) independent of anything else. The Bernoulli sampling is mainly a convenience of the implementation and actually the one originally studied by [DPVDBW14]. We should consider the sample size as \( n = pd_1d_2 \) here. We set \( d_1 = d_2 = 100, p = 0.5, \) and \( r = 1 \) in our experiment.
The PG algorithm and heuristic setup. Next, we perform the PG algorithm\(^4\) with the regularity oracle \( \mathcal{P}_{r,\|\cdot\|_\infty(\|X_0\|_\infty)} \) and the simple r-SVD \( \mathcal{P}_{r,\mathbb{R}^{d_1\times d_2}} \) starting at different random initialization \( X_0 \). More specifically, we simulate \( X_{-1} = \frac{M_1 M_2}{0.5 \|M_1 M_2\|_\infty} \) where each entry of \( M_i, i = -1, -2 \), is drawn from uniform distribution on \([-0.5, 0.5] \) independently, and then set \( X_0 = \gamma X_{-1} \) with \( \gamma = 0, 1, 2, 4 \). Note that \( \mathcal{P}_{r,\|\cdot\|_\infty(\|X_0\|_\infty)} \) cannot be computed efficiently to our best knowledge. Hence we use the heuristic \( \mathcal{P}_{r,a_u,a_v} \), a modified version of alternate projection, described as follows: Choose two positive number \( a_u \) and \( a_v \). Given input \( X \), we first compute the r-SVD of \( X \) as \( U \Sigma V^\top \). Then we compute \( U^1 = U \sqrt{\Sigma} \) and \( V^1 = V \sqrt{\Sigma} \) where the square root is applied entrywisely. Next, for each \( i = 1, \ldots, d_1 \) we perform the following operation for the \( i \)-th row \( U^1_i \) of \( U^1 \):

\[
U^1_i \leftarrow \begin{cases} U^1_i / \|U^1_i\|_2 & \text{if } \|U^1_i\|_2 \leq a_u, \\ a_u & \text{otherwise.} \end{cases}
\]

Similarly, we perform the following operation for each row of \( V^1 \):

\[
V^1_j \leftarrow \begin{cases} V^1_j / \|V^1_j\|_2 & \text{if } \|V^1_j\|_2 \leq a_v, \\ a_v & \text{otherwise.} \end{cases}
\]

We then set the output of \( \mathcal{P}_{r,a_u,a_v} \) as \( U^1 V^1 = \mathcal{P}_{r,a_u,a_v}(X) \). We use this heuristic \( \mathcal{P}_{r,a_u,a_v} \) to replace \( \mathcal{P}_{r,\|\cdot\|_\infty(\|X_0\|_\infty)} \) in our PG algorithm. The choice of \( a_u \) and \( a_v \) is chosen according the r-SVD of \( X^2 = U^2 \Sigma^2 (V^2)^\top \). We choose \( a_u = \max_{1 \leq i \leq d_1} \|U^2_{i,:}\|_2 \) and \( a_v = \max_{1 \leq j \leq d_2} \|V^2_{j,:}\|_2 \), where \( U^2_{i,:} = U^2 \sqrt{\Sigma^2} \) and \( V^2_{j,:} = V^2 \sqrt{\Sigma^2} \).

D Discussion on condition number of one-bit matrix completion

We consider the condition number of the loss \( \mathcal{L} \) in the one-bit matrix completion setting. We first show that condition number is unbounded without \( \mathcal{L} \) being an infinity norm ball and hence theoretical guarantees of IHT does not apply. We next argue that even in certain favorable setting with \( \mathcal{L} \), where convex relaxation [DPVDBW14] and our AVPG succeed, the results for Burer-Monteiro approach in [GJZ17, ZLTW18, ZWYG18] are still not applicable.

Let us consider the population loss

\[
\bar{\mathcal{L}} = \mathbb{E} \mathcal{L} = \frac{1}{d_1 d_2} \sum_{1 \leq i \leq d_1, 1 \leq j \leq d_2} \psi(1 + \exp(\sqrt{d_1 d_2} X_{i,j})) - \frac{\sqrt{d_1 d_2} X_{i,j}}{1 + \exp(-X_{i,j}^2 \sqrt{d_1 d_2})}.
\]

The loss has unbounded condition number over all matrices as \( \psi''(\theta) \to 0 \) as \( \theta \to \pm \infty \) using the equation (39).

Now we argue the results for Burer-Monteiro approach in [GJZ17, ZLTW18, ZWYG18] are not applicable to one-bit matrix completion even if \( \mathcal{L} \) is present. First, if the results in [GJZ17, ZLTW18, ZWYG18] is not applicable to the population loss, one should not expect they can be applied to the sample version \( \mathcal{L} \). Next, it is fairly obvious from (38), the expected loss is not a quadratic and hence result in [GJZ17] don’t apply. Let us now explain what we mean by favorable setting. Consider the Hessian for any \( X, \Delta \in \mathbb{R}^{d_1 \times d_2} \)

\[
\mathbb{E}(\nabla^2 \mathcal{L}_n(X)[\Delta, \Delta]) = \sum_{1 \leq i \leq d_1, 1 \leq j \leq d_2} \psi''(\sqrt{d_1 d_2} X_{i,j}) \Delta_{i,j}^2.
\]
We would like to have $\psi''(\sqrt{d_1 d_2} X_{ij}) \in [c_1, c_2]$ for some universal positive $c_1$ and $c_2$, so that $\mathbb{E}(\nabla^2 L_n(X)(\Delta, \Delta)) \in [c_1 \|\Delta\|_F^2, c_2 \|\Delta\|_F^2]$. That is, the Hessian behaves like a quadratic up to some constants. Note that the condition number is simply $c_2/c_1$ here. This scenario is actually implied from $\|X^2\|_F \asymp 1$ and $\alpha_{sp}(X^2) \asymp 1$ when the constraint $\mathcal{C}$ is $\mathcal{B}_{\|\cdot\|_\infty}(\frac{\alpha_{sp}(X^2)}{\sqrt{d_1 d_2}})$. Hence we define the favorable case to be the setting of one-bit matrix completion with $\|X^2\|_F \asymp 1$, $\alpha_{sp}(X^2) \asymp 1$, and the constraint $\mathcal{C}$ is $\mathcal{B}_{\|\cdot\|_\infty}(\frac{\alpha_{sp}(X^2)}{\sqrt{d_1 d_2}})$.

We recall our interpretation after Corollary 2 that AVPG produces the distance bound $O\left(\frac{(r^*)^2 d \log d}{n}\right)$ under these favorable conditions. Results in [ZLTW18, ZWYG18] require the condition number very close to 1 and more concretely 1.5 in [ZLTW18] and $\frac{16}{15}$ in [ZWYG18]. Also such constant is not an artifact of the proof as shown in [ZLTW18, pp. 3-4]. Once the condition number $\kappa > 3$, it is possible to have spurious local minima. Does $\bar{L}$ satisfies the condition number less than or equal to 1.5 in the favorable setting? The answer is no in general, even if $\|X^2\|_F \asymp 1$, $\alpha_{sp}(X^2) \asymp 1$. To see this, consider $X^2 = \gamma \sqrt{d_1 d_2} J$ for some $\gamma \geq 1$, $\alpha_{sp}(X^2) = 1$ and $\mathcal{C} = \mathcal{B}_{\|\cdot\|_\infty}(\frac{\gamma}{\sqrt{d_1 d_2}})$, where $J \in \mathbb{R}^{d_1 \times d_2}$ is the all one matrix with rank one. For constant $\gamma$ independent of the dimension, the one-bit matrix completion is in the favorable case. And $\psi''(\sqrt{d_1 d_2} X_{ij}) \in [c_1, c_2]$ for some dimension dependent constant $c_1, c_2$ for any $X \in \mathcal{C}$. However, this number $c_2/c_1$ can be larger than 1.5 when $\gamma \geq 10$. More concretely, set $\gamma = 10$, and consider $X_1 = 0.1 \frac{1}{\sqrt{d_1 d_2}} e_1 e_1^\top$ and $X_2 = 5 \frac{1}{\sqrt{d_1 d_2}} e_1 e_1^\top$, and $\Delta = \frac{1}{\sqrt{d_1 d_2}} e_1 e_1^\top$. For these $X_i$, $i = 1, 2$ and $\Delta$, they all belong to the set $\mathcal{C}$, and the equation (39) reduces to

$$\mathbb{E}(\nabla^2 L_n(X_1)(\Delta, \Delta)) \in [0.24 \|\Delta\|_F^2, 0.25 \|\Delta\|_F^2], \quad \text{and} \quad (40)$$

$$\mathbb{E}(\nabla^2 L_n(X_2)(\Delta, \Delta)) \in [0.006 \|\Delta\|_F^2, 0.007 \|\Delta\|_F^2]. \quad (41)$$

Thus the condition number is at least $\frac{0.24}{0.007} > 1.5$. Hence the results in [ZLTW18, ZWYG18] don’t really apply to this favorable case.