Sojourn time in an union of intervals for diffusions

Aimé LACHAL *

Université de Lyon, CNRS
INSA-Lyon, ICJ, UMR5208, F-69621, France

Abstract

We give a method for computing the iterated Laplace transform of the sojourn time in an union of intervals for linear diffusion processes. This random variable comes from a model occurring in biology concerning the clustering of membrane receptors. The way used hinges on solving differential equations. We finally have a look on the particular case of Brownian motion and we provide a representation for the Laplace transform of its local time in a finite set.

AMS 2000 subject classifications: primary 60G50; 60J22; secondary 60J10; 60E10.

Key words: sojourn time, Laplace transform, linear system.

1 Introduction

We consider a diffusion process \((X_t)_{t \geq 0}\) evolving on the real line \(\mathbb{R}\) with infinite lifetime (that is without any killing). Let \(u_1, v_1, \ldots, u_n, v_n\) be real numbers such that \(u_1 < v_1 < \cdots < u_n < v_n\). Set also \(v_0 = -\infty\) and \(u_{n+1} = +\infty\). Let us now introduce the union of intervals

\[ E = \bigcup_{i=1}^{n} [u_i, v_i] \]

together with the sojourn time that the process \((X_t)_{t \geq 0}\) spends in \(E\) up to a fixed time \(t\):

\[ T_t = \int_0^t \mathbb{1}_E(X_s) \, ds. \]

Of course, introducing the sojourn times of \((X_t)_{t \geq 0}\) in \([u_i, v_i], i \in \{1, \ldots, n\}\),

\[ T_i = \int_0^t \mathbb{1}_{[u_i, v_i]}(X_s) \, ds, \]

it is plain that time \(T_t\) can be decomposed into

\[ T_t = \sum_{i=1}^{n} T_i. \]

The functional \(T_i\) is one of the most classical and historical random variables introduced in stochastic processes theory. It has been of great interest for many researchers in particular when the set \(E\) is an infinite interval \([v, +\infty)\) or \((-\infty, u]\)–cases leading to the famous arcsine law when \((X_t)_{t \geq 0}\) is Brownian motion for instance–or a bounded interval \([u, v]\), or when \(E\) is reduced to a point \(\{u\}\). In this last case, it is related to the famous local time after some normalization.

*Postal address: INSTITUT NATIONAL DES SCIENCES APPLIQUÉES DE LYON
Pôle de Mathématiques/Institut Camille Jordan
Bâtiment Léonard de Vinci, 20 avenue Albert Einstein
69621 Villeurbanne Cedex, FRANCE
E-mail: aime.lachal@insa-lyon.fr
Web page: http://maths.insa-lyon.fr/~lachal
The case of an union of intervals for $E$ has appeared to us in a biological context that we describe here. Consider a cellular medium. The plasmic membrane is a place where many interactions occur between the cell and its direct external medium. In classical models, the membrane is described as a fluid mosaic made of different constituents; certain, the so-called ligands which possibly come from the extracellular environment, randomly roam along the membrane while other group into still clusters (the so-called rafts viewed as receptors) which form inhomogeneities on the membrane. Few functional properties of these inhomogeneities are known. A particular phenomenon is the binding mechanism between ligands and receptors. As a matter of fact, a quantity of great interest for biologists is the docking-time, this is the time that ligands and receptors bind up to a finite time (the duration of the experience). Indeed, this quantity is an important indicator of affinity/sensitivity of the ligands for receptors. For a more accurate description of this biological context, we refer the reader to the paper by Caré & Soula [2].

In [3], we proposed a random walk model for simulating this problem: the membrane is viewed as the integer line $\mathbb{Z}$ (or a finite torus $\mathbb{Z}/\mathbb{Z}$), the receptors are fixed numbers $a_1, \ldots, a_r$ in $\mathbb{Z}$ and the ligands evolve along the membrane like independent random walks $(S_i(i))_{i \in \mathbb{N}}, (S_2(i))_{i \in \mathbb{N}}, \ldots$ on $\mathbb{Z}$. Hence, the movement of the family of $\ell$ ligands is modeled by the $\ell$-dimensional random walk $(S(i))_{i \in \mathbb{N}}$ defined, for any $i \in \mathbb{N}$, by $(S(i) = (S_1(i), \ldots, S_\ell(i))$. The docking-time is proportional to the sojourn time spent by the random walk $(S(i))_{i \in \mathbb{N}}$ in the set $\bigcup_{j=1}^{\ell-1} \mathbb{Z}^{\ell-1} \times \{a_1, \ldots, a_r\} \times \mathbb{Z}^{\ell-j}$. We provided a methodology for computing the probability distribution of this sojourn time. At the end of [3], we addressed the continuous counterpart to this problem: replace the $\ell$-dimensional random walk by an $\ell$-dimensional Brownian motion (or a more general diffusion process) and compute the distribution of the time spent by Brownian motion in a set of the form $\bigcup_{j=1}^{\ell-1} \mathbb{R}^{\ell-1} \times E \times \mathbb{R}^{\ell-j}$ where $E$ is a set modeling rafts, that is, $E$ is an union of intervals.

In this paper, we consider the case of the dimension $\ell = 1$: this is the evolution of one ligand roaming as a diffusion process along a linear membrane (viewed as $\mathbb{R}$). In this case, the docking-time is quantified by the sojourn time of the diffusion process in an union of intervals, namely $T_1$. Our aim is to describe the probability distribution of $T_1$ as well as the joint distribution of $(T_1, X_1)$. For this, we provide a representation of the iterated Laplace transforms of these distributions which could be easily implemented on a computer. Actually, our method allows us to compute the joint distributions of $(T_1^1, \ldots, T_1^n)$ and $(T_1^1, \ldots, T_1^n, X_i)$. We finally have a look on the particular case of Brownian motion and we provide a representation for the Laplace transform of its local time in a finite set.

2 Settings

Let us introduce the transition densities of the process $(X_t)_{t \geq 0}$: for any $t \geq 0$ and $x,y \in \mathbb{R}$,
\[ p(t; x, y) = \mathbb{P}_x \{ X_t \in \mathrm{dy} \} / dy = \mathbb{P} \{ X_t \in \mathrm{dy} | X_0 = x \} / dy \]
together with its $\lambda$-potential: for any $\lambda > 0$ and $x,y \in \mathbb{R}$,
\[ \rho_\lambda(x, y) = \int_0^\infty e^{-\lambda t} p(t; x, y) \, dt. \]

Let $\mathcal{D}$ be the infinitesimal generator of the diffusion $(X_t)_{t \geq 0}$. This is a second-order differential operator. Since the lifetime of $(X_t)_{t \geq 0}$ is infinite, the killing rate vanishes and then $\mathcal{D}$ has the form
\[ \mathcal{D}_x = \frac{1}{2} \sigma(x)^2 \frac{\partial^2}{\partial x^2} + \tau(x) \frac{\partial}{\partial x}. \]
The notation $\mathcal{D}_x$ means that the operator $\mathcal{D}$ acts on the variable $x$. It is well-known that the density $p$ solves the Kolmogorov backward equation
\[ \mathcal{D}_x p(t; x, y) = \frac{\partial p}{\partial t}(t; x, y), \quad p(0^+; x, y) = \delta(x - y) \]
and that the potential solves
\[ \mathcal{D}_x \rho_\lambda(x, y) = \lambda \rho_\lambda(x, y) - \delta(x - y). \]
The potential $\rho_\lambda$ is the Green function of the operator $\mathcal{D} - \lambda \mathrm{id}$.

Let us introduce a basis of fundamental solutions $\{c, d\}$ of the linear second-order differential equation $\mathcal{D} u = \lambda u$ such that $c$ is increasing and $d$ is decreasing. Then the potential $\rho_\lambda$ admits the following representation (see, e.g., [1] p. 19):
\[ \rho_\lambda(x, y) = \frac{1}{w(y)} c(\min(x, y)) \, d(\max(x, y)) \]
where \( w(y) = [c'(y) d(y) - c(y) d'(y)] / \kappa(y) \) with \( \kappa(y) = 2/\sigma(y)^2 \).

We see that the potential fulfills the following properties. The function \( x \mapsto \rho_x(x, y) \) is continuous on \( \mathbb{R} \) and, in particular, at \( y \). Moreover, it is differentiable on \( \mathbb{R} \setminus \{y\} \) and its derivative admits a jump at \( y \). More precisely, we have the conditions below at \( y \):

\[
\begin{cases}
  \rho_x(y^+, y) = \rho_x(y^-, y), \\
  \frac{\partial \rho_x}{\partial x}(y^+, y) - \frac{\partial \rho_x}{\partial x}(y^-, y) = -\kappa(y).
\end{cases}
\]  

(2.1)

We also need to introduce the first hitting time of any level \( u \):

\( \tau_u = \inf\{t \geq 0 : X_t = u\} \).

The Laplace transform of the distribution of \( \tau_u \) is given by (II p. 18)

\[
\mathbb{E}_x(e^{-\lambda \tau_u}) = \begin{cases}
  \frac{c(x)}{c(u)} & \text{if } x \leq u, \\
  \frac{d(x)}{d(u)} & \text{if } x > u.
\end{cases}
\]  

(2.2)

In this paper we compute the iterated Laplace transform of the random vector \( T_t = (T_1^t, \ldots, T_n^t) \):

\[
\varphi_{\lambda, \mu}(x) = \int_0^\infty e^{-\lambda t} \mathbb{E}_x(e^{-<\mu, T_t>}) \, dt
\]

as well as that of the random vector \((T_t, X_t)\):

\[
\psi_{\lambda, \mu}(x, y) = \int_0^\infty e^{-\lambda t} \left[ \mathbb{E}_x(e^{-<\mu, T_t>}, X_t \in dy) / dy \right] dt.
\]

In the foregoing quantities, \( \mu \) is a vectorial argument: \( \mu = (\mu_1, \ldots, \mu_n) \) and \( <\mu, T_t> \) is the inner product of the vectors \( \mu \) and \( T_t \):

\[
<\mu, T_t> = \sum_{i=1}^n \mu_i T_i^t.
\]

Notice that we use bold symbols for vectorial variables while unbold symbols are related to scalar variables. The iterated Laplace transform of the random variable \( T_t \) can be immediately deduced from \( \varphi_{\lambda, \mu}(x) \) by identifying all the components of \( \mu \) if \( \mu_1 = \cdots = \mu_n = \mu \), then \( <\mu, T_t> = \mu T_t \).

For this, we begin by writing integral equations satisfied by the functions \( \varphi_{\lambda, \mu} \) and \( \psi_{\lambda, \mu} \) which lead to ordinary linear differential equations associated with some regularity conditions. Of course, we have

\[
\varphi_{\lambda, \mu}(x) = \int_{-\infty}^{+\infty} \psi_{\lambda, \mu}(x, y) \, dy.
\]

So, because of this relationship, it should be enough to compute \( \psi_{\lambda, \mu} \) and next integrate it with respect to \( y \) for deriving \( \varphi_{\lambda, \mu} \). Actually, this way appears to be untractable from the obtained formulas, so we shall evaluate separately both functions \( \varphi_{\lambda, \mu} \) and \( \psi_{\lambda, \mu} \).

3 Some equations

In this section, we derive some integral equations (Subsection 3.1) and next deduce some differential equations (Subsection 3.2) satisfied by the functions \( \varphi_{\lambda, \mu} \) and \( \psi_{\lambda, \mu} \). The method we use is standard, this is the famous Feynman-Kac approach. Nonetheless, in order to facilitate the reading of the paper, we provide some details. Our results come from solving differential equations (3.6), (3.7) below associated with the boundary values (3.8), (3.9), (3.10). The aforementioned equations as well as the additional conditions directly come from integral equations (3.1) and (3.2).
3.1 Integral equations

Proposition 3.1 The functions $\varphi_{\lambda, \mu}$ and $\psi_{\lambda, \mu}$ solve the following integral equations:

$$\varphi_{\lambda, \mu}(x) = \frac{1}{\lambda} + \sum_{i=1}^{n} \mu_{i} \int_{u_{i}}^{v_{i}} \rho_{i}(x, z) \varphi_{\lambda, \mu}(z) \, dz, \quad (3.1)$$

$$\psi_{\lambda, \mu}(x, y) = \rho_{\lambda}(x, y) - \sum_{i=1}^{n} \mu_{i} \int_{u_{i}}^{v_{i}} \rho_{i}(x, z) \psi_{\lambda, \mu}(z, y) \, dz. \quad (3.2)$$

**Proof**

Let us introduce the quantity

$$\chi_{\mu}(t; x, y) = \mathbb{E}_{x} \left( e^{-\mu T_{\lambda}} \right), X_{t} \in dy \big/ dy.$$  

We write that

$$1 - e^{-\mu T_{\lambda}} = \int_{0}^{t} \left( \sum_{i=1}^{n} \mu_{i} \mathbb{I}_{[u_{i}, v_{i}]}(X_{s}) \right) \exp \left( -\sum_{i=1}^{n} \mu_{i} \int_{s}^{t} \mathbb{I}_{[u_{i}, v_{i}]}(X_{u}) \, du \right) \, ds$$

$$= \int_{0}^{t} \left( \sum_{i=1}^{n} \mu_{i} \mathbb{I}_{[u_{i}, v_{i}]}(X_{s}) \right) e^{-\mu T_{\lambda} - \theta s} \, ds = \int_{0}^{t} \left( \sum_{i=1}^{n} \mu_{i} \mathbb{I}_{[u_{i}, v_{i}]}(X_{s}) \right) e^{-\mu T_{\lambda} - \theta s} \, ds,$$

where $(\theta s)_{s \geq 0}$ is the usual shift operator defined by $X_{t} \circ \theta s = X_{s+t}$ for any $s, t \geq 0$. Next, we apply the Markov property:

$$\chi_{\mu}(t; x, y) - p(t; x, y) = -\mathbb{E}_{x} \left( 1 - e^{-\mu T_{\lambda}} \right), X_{t} \in dy \big/ dy$$

$$= -\sum_{i=1}^{n} \mu_{i} \int_{0}^{t} \left[ \mathbb{E}_{x} \left( \mathbb{I}_{[u_{i}, v_{i}]}(X_{s}) \right) e^{-\mu T_{\lambda} - \theta s}, X_{t-s} \in dy \big/ dy \right] \, ds$$

$$= -\sum_{i=1}^{n} \mu_{i} \int_{0}^{t} \int_{u_{i}}^{v_{i}} \mathbb{P}_{x} \{ X_{s} \in dz \} \left[ \mathbb{E}_{x} \left( e^{-\mu T_{\lambda} - \theta s}, X_{t-s} \in dy \big/ dy \right) \right] \, ds$$

$$= -\sum_{i=1}^{n} \mu_{i} \int_{0}^{t} \int_{u_{i}}^{v_{i}} \rho_{i}(s; x, z) \chi_{\mu}(t-s; z, y) \, ds \, dz. \quad (3.3)$$

By applying the Laplace transform to (3.3), we get Eq. (3.2) which in turn yields Eq. (3.1) by integrating with respect to $y$ on $\mathbb{R}$ and using

$$\int_{-\infty}^{\infty} \rho_{\lambda}(x, y) \, dy = \int_{0}^{\infty} e^{-\lambda t} \mathbb{P}_{x} \{ X_{t} \in \mathbb{R} \} \, dt = \frac{1}{\lambda}.$$  

We also state the following result which will be useful.

**Proposition 3.2** For $x \in (-\infty, u_{1})$, we have

$$\varphi_{\lambda, \mu}(x) = \mathbb{E}_{x} \left( e^{-\lambda \tau_{u_{1}}} \right) \left( \varphi_{\lambda, \mu}(u_{1}) - \frac{1}{\lambda} \right) + \frac{1}{\lambda}, \quad (3.4)$$

$$\psi_{\lambda, \mu}(x, y) - \rho_{\lambda}(x, y) = \mathbb{E}_{x} \left( e^{-\lambda \tau_{u_{1}}} \right) \left[ \psi_{\lambda, \mu}(u_{1}, y) - \rho_{\lambda}(u_{1}, y) \right]. \quad (3.5)$$

**Proof**

Suppose $x < u_{1}$. When $\tau_{u_{1}} > t$, the process $(X_{s})_{s \geq 0}$ remains confined in $(-\infty, u_{1})$ up to time $t$; we have $T_{i} = 0$ for all $i \in \{1, \ldots, n\}$ and then $1 - e^{-\mu T_{\lambda}} = 0$. So, we can write that

$$\mathbb{E}_{x} \left( 1 - e^{-\mu T_{\lambda}}, X_{t} \in dy \big/ dy \right) = \mathbb{E}_{x} \left( 1 - e^{-\mu T_{\lambda}}, X_{t} \in dy, \tau_{u_{1}} \leq t \big/ dy \right)$$

$$= \mathbb{E}_{x} \left( 1 - e^{-\mu T_{\lambda} - \tau_{u_{1}} \circ \theta \tau_{u_{1}}}, X_{t-\tau_{u_{1}}} \in dy, \tau_{u_{1}} \leq t \big/ dy \right)$$

$$= \int_{0}^{t} \mathbb{P}_{x} \{ \tau_{u_{1}} \in ds \} \mathbb{E}_{u_{1}} \left( 1 - e^{-\mu T_{\lambda}}, X_{t-s} \in dy \big/ dy \right).$$
Moreover, due to (2.1) and (3.2), we have the following conditions at points

\[ u \quad \text{and for } \quad b \]

The functions \( \varphi_{\lambda,\mu} \) and \( \psi_{\lambda,\mu} \) solve the following differential equations:

\[
D_x \varphi_{\lambda,\mu}(x) = \left\{ \begin{array}{ll}
(\lambda + \mu_i) \varphi_{\lambda,\mu}(x) - 1 & \text{if } x \in (u_i, u_i) \text{ and } i \in \{1, \ldots, n\}, \\
\lambda \varphi_{\lambda,\mu}(x) - 1 & \text{if } x \in (v_i, u_{i+1}) \text{ and } i \in \{0, 1, \ldots, n\},
\end{array} \right.
\]  
\[ \text{(3.6)} \]

and for \( x \in \mathbb{R} \setminus \{y\} \),

\[
D_x \psi_{\lambda,\mu}(x, y) = \left\{ \begin{array}{ll}
(\lambda + \mu_i) \psi_{\lambda,\mu}(x, y) & \text{if } x \in (u_i, v_i) \text{ and } i \in \{1, \ldots, n\}, \\
\lambda \psi_{\lambda,\mu}(x, y) & \text{if } x \in (v_i, u_{i+1}) \text{ and } i \in \{0, 1, \ldots, n\}.
\end{array} \right.
\]  
\[ \text{(3.7)} \]

Additionally, we see by (3.1) and the regularity properties of \( \rho_{\lambda} \) that \( \varphi_{\lambda,\mu} \) is differentiable on \( \mathbb{R} \). So, we get the following conditions at points \( u_i, v_i, i \in \{1, \ldots, n\} \):

\[
\begin{align*}
\varphi_{\lambda,\mu}(u_i^+) &= \varphi_{\lambda,\mu}(u_i^-), & \varphi'_{\lambda,\mu}(u_i^+) &= \varphi'_{\lambda,\mu}(u_i^-), \\
\varphi_{\lambda,\mu}(v_i^+) &= \varphi_{\lambda,\mu}(v_i^-), & \varphi'_{\lambda,\mu}(v_i^+) &= \varphi'_{\lambda,\mu}(v_i^-). 
\end{align*}
\]  
\[ \text{(3.8)} \]

In the same way, we see by (3.2) that the function \( x \mapsto \psi_{\lambda,\mu}(x, y) \) is differentiable on \( \mathbb{R} \setminus \{y\} \). So, we get the following conditions at points \( u_i, v_i, i \in \{1, \ldots, n\} \):

\[
\begin{align*}
\psi_{\lambda,\mu}(u_i^+, y) &= \psi_{\lambda,\mu}(u_i^-, y), & \psi'_{\lambda,\mu}(u_i^+, y) &= \psi'_{\lambda,\mu}(u_i^-, y), \\
\psi_{\lambda,\mu}(v_i^+, y) &= \psi_{\lambda,\mu}(v_i^-, y), & \psi'_{\lambda,\mu}(v_i^+, y) &= \psi'_{\lambda,\mu}(v_i^-, y).
\end{align*}
\]  
\[ \text{(3.9)} \]

Moreover, due to (2.1) and (3.2), we have the following conditions at point \( y \):

\[
\begin{align*}
\psi_{\lambda,\mu}(y^+, y) &= \psi_{\lambda,\mu}(y^-, y), \\
\frac{\partial \psi_{\lambda,\mu}}{\partial y}(y^+, y) &= \frac{\partial \psi_{\lambda,\mu}}{\partial y}(y^-, y) = -\kappa(y).
\end{align*}
\]  
\[ \text{(3.10)} \]

Different cases for \( y \) must be distinguished: \( y \in [u_{i_0}, v_{i_0}] \) for a certain \( i_0 \in \{1, \ldots, n\} \) or \( y \in (v_{i_0}, u_{i_0} + 1) \) for a certain \( i_0 \in \{0, 1, \ldots, n\} \).

In the forthcoming sections, we solve Eq. (3.6) with conditions (3.8) and Eq. (3.7) with conditions (3.9)–(3.10). Let us introduce a basis of fundamental solutions \( \{a_i, b_i\} \) of the linear second-order differential equation \( D_u(x) = (\lambda + \mu_i) u(x) \) for any \( i \in \{1, \ldots, n\} \) as well as a basis of fundamental solutions \( \{c, d\} \) of the equation \( D_u(x) = \lambda u(x) \). These functions are chosen such that \( a_i, c \) are increasing and \( b_i, d \) are decreasing.
4 Solving differential equations (3.6) and (3.7)

4.1 Solving Eq. (3.6) with conditions (3.8)

In view of the differential operator $\mathcal{D}$, we see that the form of the solution of (3.6) is

$$
\varphi_{\lambda, \mu}(x) = \begin{cases} 
\alpha_i a_i(x) + \beta_i b_i(x) + \frac{1}{\lambda + \mu_i} & \text{for } x \in (u_i, v_i) \text{ and } i \in \{1, \ldots, n\}, \\
\gamma_i c(x) + \delta_i d(x) + \frac{1}{\lambda} & \text{for } x \in (v_i, u_{i+1}) \text{ and } i \in \{0, 1, \ldots, n\}.
\end{cases}
$$

We have to determine the unknown coefficients $\gamma_0, \delta_0, \alpha_1, \beta_1, \gamma_1, \delta_1, \ldots, \alpha_n, \beta_n, \gamma_n, \delta_n$. Put $v_i = \frac{\mu_i}{\lambda + \mu_i}$. 

For large enough negative $x$ (so that $x < u_1$), we have $\varphi_0(x) = \frac{c(x)}{c(u_1)}$ and (3.4) supplies

$$
\varphi_{\lambda, \mu}(x) = \left(1 - \frac{c(x)}{c(u_1)}\right) + \frac{1}{\lambda}.
$$

This clearly implies that $\delta_0 = 0$. Similarly, considering $\varphi_{\lambda, \mu}(x)$ for large positive $x$, we see that $\gamma_n = 0$.

Next the regularity conditions (3.8) at $u_i$ and $v_i$ yield

$$
\begin{align*}
\alpha_i a_i(u_i) + \beta_i b_i(u_i) - \gamma_{i-1} c(u_i) - \delta_{i-1} d(u_i) &= \nu_i, \\
\alpha_i a_i'(u_i) + \beta_i b_i'(u_i) - \gamma_{i-1} c'(u_i) - \delta_{i-1} d'(u_i) &= 0, \\
\alpha_i a_i(v_i) + \beta_i b_i(v_i) - \gamma_{i} c(v_i) - \delta_{i} d(v_i) &= \nu_i, \\
\alpha_i a_i'(v_i) + \beta_i b_i'(v_i) - \gamma_{i} c'(v_i) - \delta_{i} d'(v_i) &= 0.
\end{align*}
$$

Let us introduce the matrices

$$
A_i = \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}, \quad B_i = \begin{pmatrix} \gamma_i \\ \delta_i \end{pmatrix}, \quad C_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad M_i(x) = \begin{pmatrix} a_i(x) & b_i(x) \\ a_i'(x) & b_i'(x) \end{pmatrix}, \quad N(x) = \begin{pmatrix} c(x) & d(x) \\ c'(x) & d'(x) \end{pmatrix}.
$$

Notice that $M_i(x)$ depends on $\lambda + \mu_i$ and $N(x)$ depends on $\lambda$. The system (4.1) can be rewritten into a matrix form as

$$
\begin{align*}
M_i(u_i)A_i - N(u_i)B_{i-1} &= \nu_i C_0, \\
M_i(v_i)A_i - N(v_i)B_i &= \nu_i C_0.
\end{align*}
$$

We extract from the first equation of (4.2) the relationship $A_i = M_i(u_i)^{-1} N(u_i) B_{i-1} + \nu_i M_i(u_i)^{-1} C_0$. Plugging this equality into the second equation of (4.2), we derive the recursive identity for $B_i$

$$
B_i = [N(v_i)^{-1} M_i(v_i)^{-1} M_i(u_i)^{-1} N(u_i)] B_{i-1} + \nu_i [N(v_i)^{-1} M_i(v_i)^{-1} M_i(u_i)]^{-1} - N(v_i)^{-1}] C_0.
$$

Setting $P_i = N(v_i)^{-1} M_i(v_i)^{-1} N(u_i)$, $Q_i = N(v_i)^{-1} M_i(v_i)^{-1} - N(v_i)^{-1}$, we write (4.3) more concisely as

$$
B_i = P_i B_{i-1} + \nu_i Q_i C_0.
$$

By iterating this recursive equality, we get for $i \in \{0, 1, \ldots, n\}$

$$
B_i = P_i P_{i-1} \cdots P_1 P_0 + (\nu_i Q_i + \nu_{i-1} P_i Q_{i-1} + \nu_{i-2} P_i P_{i-1} Q_{i-2} + \cdots + \nu_i P_i P_{i-1} \cdots P_2 P_{1} Q_1) C_0
$$

with the conventions that $P_i P_{i-1} \cdots P_2 = O$ and $P_i P_{i-1} \cdots P_1 = I$ if $i = 0$, and $P_i P_{i-1} \cdots P_2 = I$ if $i = 1$. The condition $\delta_0 = 0$ gives $B_0 = \gamma_0 C_0$. Put, for $i \in \{0, 1, \ldots, n\}$,

$$
R_i = P_i P_{i-1} \cdots P_1, \quad S_i = \nu_i Q_i + \nu_{i-1} P_i Q_{i-1} + \nu_{i-2} P_i P_{i-1} Q_{i-2} + \cdots + \nu_i P_i P_{i-1} \cdots P_2 Q_1.
$$

By the aforementioned conventions, we have $R_0 = I$, $S_0 = O$, $S_1 = \nu_1 Q_1$. With these settings at hand, we get an simple representation for $B_1$

$$
B_1 = [\gamma_0 R_i + S_i] C_0.
$$

Now, we use the condition $\gamma_n = 0$. By observing that

$$
\gamma_n = (1 \ 0) B_n = \gamma_0 (1 \ 0) R_n C_0 + (1 \ 0) S_n C_0,
$$

we get

$$
\gamma_n = \nu_n Q_n + \nu_{n-1} P_n Q_{n-1} + \nu_{n-2} P_n P_{n-1} Q_{n-2} + \cdots + \nu_n P_n P_{n-1} \cdots P_2 Q_1.
$$

Therefore, we have

$$
\gamma_n = \nu_n Q_n + \nu_{n-1} P_n Q_{n-1} + \nu_{n-2} P_n P_{n-1} Q_{n-2} + \cdots + \nu_n P_n P_{n-1} \cdots P_2 Q_1.
$$

where $\nu_n = \nu_i$.
we deduce the value of the coefficient \( \gamma_0 \):
\[
\gamma_0 = -\frac{(1 \ 0)S_0C_0}{(1 \ 0)R_0C_0}.
\]
Let us point out that \((1 \ 0)R_nC_0\) (resp. \((1 \ 0)S_nC_0\)) is nothing but the first entry of the matrix \(R_n\) (resp. \(S_n\)).

Finally, we express \(A_i\) by means of \(B_i\): due to (4.2), we have
\[
A_i = M_i(v_i)^{-1}N(v_i)B_i + \nu_iM_i(v_i)^{-1}C_0 = [\gamma_0M_i(v_i)^{-1}N(v_i)R_i + M_i(v_i)^{-1}N(v_i)S_i + \nu_iM_i(v_i)^{-1}]C_0.
\]
We sum up the results obtained in this section in the statement below.

**Theorem 4.1** The iterated Laplace transform of the probability distribution of \(T_t\) is given by
\[
\int_0^\infty e^{-\lambda t}E_n(e^{-\mu<T_t>})dt = \begin{cases} (1 \ 0)M_i(x)A_i + \frac{1}{\lambda + \mu_i}M_i(v_i)B_i + \frac{1}{\lambda}R_nC_i & \text{for } x \in [u_i, v_i] \text{ and } i \in \{1, \ldots, n\}, \\ (1 \ 0)N(x)B_i + \frac{1}{\lambda}R_nC_i & \text{for } x \in (v_i, u_{i+1}] \text{ and } i \in \{0, 1, \ldots, n\}, \end{cases}
\]
with, for \(i \in \{1, \ldots, n\},
\[
A_i = M_i(v_i)^{-1}N(v_i)\left[ S_i\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{(1 \ 0)S_0\begin{pmatrix} 1 \\ 0 \end{pmatrix}R_i\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{(1 \ 0)R_n\begin{pmatrix} 1 \\ 0 \end{pmatrix}}\right] + \frac{\mu_i}{\lambda(\lambda + \mu_i)}M_i(v_i)^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix},
\]
for \(i \in \{0, 1, \ldots, n\},
\[
B_i = S_i\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{(1 \ 0)S_0\begin{pmatrix} 1 \\ 0 \end{pmatrix}R_i\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{(1 \ 0)R_n\begin{pmatrix} 1 \\ 0 \end{pmatrix}},
\]
where the matrices \(R_i, S_i, i \in \{0, 1, \ldots, n\}\), are defined by (4.4) and (4.7).

**Remark 4.2** The relationships \(B_n = [\gamma_nR_n + S_n]C_0\) and \(B_n = \delta_nD_0\) with \(D_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\) imply
\[
\delta_n = (0 \ 1)B_n = \gamma_0(0 \ 1)R_0C_0 + (0 \ 1)S_0C_0
\]
which gives the following expression of the coefficient \(\delta_n\):
\[
\delta_n = (0 \ 1)S_nC_0 - \frac{(1 \ 0)S_0C_0}{(1 \ 0)R_0C_0}(0 \ 1)R_nC_0.
\]
Another expression can be derived by reversing the sense of the algorithm we used for solving the system (4.2). Indeed, let us rewrite (4.5) as \(B_{i-1} = P_i^{-1}B_i - \nu_iP_i^{-1}Q_iC_0\). Set
\[
\tilde{P}_i = P_i^{-1} = N(u_i)^{-1}M_i(u_i)M_i(v_i)^{-1}N(v_i), \quad \tilde{Q}_i = -P_i^{-1}Q_i = N(u_i)^{-1}M_i(u_i)M_i(v_i)^{-1} - N(u_i)^{-1}.
\]
The matrices \(\tilde{P}_i\) and \(\tilde{Q}_i\) can be easily deduced from \(P_i\) and \(Q_i\) by interchanging \(u_i\) and \(v_i\). We have the following recurrence:
\[
B_{i-1} = \tilde{P}_iB_i + \nu_i\tilde{Q}_iC_0
\]
which can be successively iterated from \(i = 1\) to \(n\); this yields
\[
B_0 = \tilde{R}_nB_n + \tilde{S}_nC_0
\]
where
\[
\tilde{R}_n = \tilde{P}_1\tilde{P}_2\ldots\tilde{P}_n, \quad \tilde{S}_n = \nu_1\tilde{Q}_1 + \nu_2\tilde{P}_1\tilde{Q}_2 + \nu_3\tilde{P}_1\tilde{P}_2\tilde{Q}_3 + \cdots + \nu_n\tilde{P}_1\tilde{P}_2\ldots\tilde{P}_{n-1}\tilde{Q}_n.
\]
Since \(B_0 = \gamma_0C_0\) and \(B_n = \delta_nD_0\), we get
\[
\gamma_0C_0 = \delta_n\tilde{R}_nD_0 + \tilde{S}_nC_0
\]
and we extract the simple expression
\[
\delta_n = -\frac{(0 \ 1)\tilde{S}_nC_0}{(0 \ 1)\tilde{R}_nD_0}.
\]
4.2 Solving Eq. (3.7) with conditions (3.9) and (3.10)

We have several cases to consider depending on the location of $y$ inside or outside the intervals of the set $E$.

First case: $y \in [u_{i_0}, v_{i_0}]$ for a certain $i_0 \in \{1, \ldots, n\}$

The form of the solution of (3.7) is

$$
\psi_{\lambda, \mu}(x, y) =
\begin{cases}
\alpha_i(y) a_i(x) + \beta_i(y) b_i(x) & \text{for } x \in (u_i, v_i) \text{ and } i \in \{1, \ldots, n\} \setminus \{i_0\}, \\
\gamma_i(y) c(x) + \delta_i(y) d(x) & \text{for } x \in (v_i, u_{i+1}) \text{ and } i \in \{0, \ldots, n\}.
\end{cases}
$$

If $y = u_{i_0}$ (resp. $v_{i_0}$), we consider that the interval $(u_{i_0}, v_{i_0})$ (resp. $(y, v_{i_0})$) is empty. We have to determine the unknown coefficients $\gamma_0(y), \delta_0(y), \alpha_1(y), \beta_1(y), \gamma_1(y), \delta_1(y), \ldots, \alpha_{i_0}(y), \beta_{i_0}(y), \gamma_{i_0}(y), \delta_{i_0}(y)$.

For large enough negative $x$ (so that $x < \min(y, u_1)$), we have $\rho_n(x, y) = \frac{1}{w(y)} c(x) d(y)$ and $E_x (e^{-\lambda x}) = c(x)/c(u_1)$. Thus (3.8) gives

$$
\psi_{\lambda, \mu}(x, y) = \frac{1}{c(u_1)} [\psi_{\lambda, \mu}(u_1, y) - \rho_n(u_1, y)] + \frac{1}{w(y)} d(y) c(x).
$$

This clearly implies that $\delta_0(y) = 0$. Similarly, considering $\psi_{\lambda, \mu}(x, y)$ for large positive $x$, we see that $\gamma_n(y) = 0$.

Next, conditions (3.9) at $u_i, v_i, i \in \{1, \ldots, n\} \setminus \{i_0\}$, yield

$$
\begin{align*}
\alpha_i(y) a_i(u_i) + \beta_i(y) b_i(u_i) &= \gamma_{i-1}(y) c(u_i) + \delta_{i-1}(y) d(u_i), \\
\alpha_i(y) a_i(v_i) + \beta_i(y) b_i(v_i) &= \gamma_i(y) c(v_i) + \delta_i(y) d(v_i).
\end{align*}
$$

Similarly, conditions (3.9) at $u_{i_0}, v_{i_0}$ give

$$
\begin{align*}
\alpha_{i_0}(y) a_{i_0}(u_{i_0}) + \beta_{i_0}(y) b_{i_0}(u_{i_0}) &= \gamma_{i_0-1}(y) c(u_{i_0}) + \delta_{i_0-1}(y) d(u_{i_0}), \\
\alpha_{i_0}(y) a_{i_0}(v_{i_0}) + \beta_{i_0}(y) b_{i_0}(v_{i_0}) &= \gamma_{i_0}(y) c(v_{i_0}) + \delta_{i_0}(y) d(v_{i_0}).
\end{align*}
$$

Additionally, conditions (3.10) at $y$ yield

$$
\begin{align*}
\alpha_{i_0}(y) a_{i_0}(y) + \beta_{i_0}(y) b_{i_0}(y) - \alpha_{i_0}(y) a_{i_0}(y) - \beta_{i_0}(y) b_{i_0}(y) &= 0, \\
\alpha_{i_0}(y) a_{i_0}(y) + \beta_{i_0}(y) b_{i_0}(y) - \alpha_{i_0}(y) a_{i_0}(y) - \beta_{i_0}(y) b_{i_0}(y) &= \sigma(y).
\end{align*}
$$

Using the matrices introduced in the foregoing subsection and setting also

$$
A_{i_0,1}(y) = \begin{pmatrix} \alpha_{i_0}(y) \\ \beta_{i_0}(y) \end{pmatrix}, \quad A_{i_0,2}(y) = \begin{pmatrix} \alpha_{i_0,2}(y) \\ \beta_{i_0,2}(y) \end{pmatrix}, \quad D_{i_0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
$$

we rewrite all these equations into a matrix form: for $i \in \{1, \ldots, n\} \setminus \{i_0\}$,

$$
\begin{align*}
M_i(u_i) A_i(y) &= N(u_i) B_{i-1}(y), \\
M_i(v_i) A_i(y) &= N(v_i) B_i(y),
\end{align*}
$$

for $i = i_0$,

$$
\begin{align*}
M_{i_0}(u_{i_0}) A_{i_0,1}(y) &= N(u_{i_0}) B_{i_0-1}(y), \\
M_{i_0}(v_{i_0}) A_{i_0,2}(y) &= N(v_{i_0}) B_{i_0}(y),
\end{align*}
$$

for $i = i_0$.
and finally

\[ M_{i_0}(y)A_{i_02}(y) - M_{i_0}(y)A_{i_01}(y) = -\kappa(y)D_0. \]  

(4.9)

We extract from (4.7) the relationships

\[ B_i(y) = N(v_i)^{-1}M_i(v_i)A_i(y) \]

and

\[ A_i(y) = M_i(u_i)^{-1}N(u_i)B_{i-1}(y) \]

which entail for \( i \in \{1, \ldots, n\} \setminus \{i_0\} \)

\[ B_i(y) = P_iB_{i-1}(y). \]

(4.10)

Therefore, since \( \delta_0(y) = 0 \) and then \( B_0(y) = \gamma_0(y)C_0 \),

\[ B_i(y) = \begin{cases} 
\gamma_0(y)P_iP_{i-1}\ldots P_1C_0 & \text{if } i \in \{0, 1, \ldots, i_0 - 1\}, \\
P_iP_{i-1}\ldots P_{i_0+1}B_{i_0}(y) & \text{if } i \in \{i_0 + 1, \ldots, n\}, 
\end{cases} \]

with, in view of (4.8) and (4.9),

\[ B_{i_0}(y) = N(v_{i_0})^{-1}M_{i_0}(v_{i_0})A_{i_02}(y) = N(v_{i_0})^{-1}M_{i_0}(v_{i_0})[A_{i_01}(y) - \kappa(y)M_{i_0}(y)^{-1}D_0] = N(v_{i_0})^{-1}M_{i_0}(v_{i_0})[M_{i_0}(u_{i_0})^{-1}N(u_{i_0})B_{i_0-1}(y) - \kappa(y)M_{i_0}(y)^{-1}D_0] = N(v_{i_0})^{-1}M_{i_0}(v_{i_0})[\gamma_0(y)M_{i_0}(u_{i_0})^{-1}N(u_{i_0})P_{i_0-1}P_{i_0-2}\ldots P_1C_0 - \kappa(y)M_{i_0}(y)^{-1}D_0] = \gamma_0(y)P_{i_0}P_{i_0-1}\ldots P_1C_0 - \kappa(y)N(v_{i_0})^{-1}M_{i_0}(v_{i_0})M_{i_0}(y)^{-1}D_0. \]

As a byproduct, recalling that we set \( R_i = P_iP_{i-1}\ldots P_1 \) and putting, for \( i \in \{0, 1, \ldots, i_0 - 1\} \),

\[ U_i = P_iP_{i-1}\ldots P_{i_0+1}N(v_{i_0})^{-1}M_{i_0}(v_{i_0}) \]

(4.11)

with the conventions that \( P_iP_{i-1}\ldots P_{i_0+1} = I \) if \( i = i_0 \) and \( P_iP_{i-1}\ldots P_{i_0+1} = O \) if \( i < i_0 \), we obtain the following expression for \( B_i(y) \): for any \( i \in \{0, 1, \ldots, n\} \),

\[ B_i(y) = \gamma_0(y)R_iC_0 - \kappa(y)U_iM_{i_0}(y)^{-1}D_0. \]

(4.12)

Now it remains to determine \( \gamma_0(y) \). For this, we invoke the condition \( \gamma_0(y) = 0 \). Since

\[ \gamma_n(y) = (1 \ 0)B_n(y) = \gamma_0(y)(1 \ 0)R_nC_0 - \kappa(y)(1 \ 0)U_nM_{i_0}(y)^{-1}D_0, \]

we deduce the value of the coefficient \( \gamma_0(y) \):

\[ \gamma_0(y) = \kappa(y) \frac{(1 \ 0)U_nM_{i_0}(y)^{-1}D_0}{(1 \ 0)R_nC_0}. \]

(4.13)

Finally, due to (4.7) and (4.12), we can express \( A_i(y) \) by means of \( B_i(y) \), for \( i \in \{1, \ldots, n\} \setminus \{i_0\} \), as

\[ A_i(y) = M_i(v_i)^{-1}N(v_i)B_i(y) = M_i(v_i)^{-1}N(v_i)[\gamma_0(y)R_iC_0 - \kappa(y)U_iM_{i_0}(y)^{-1}D_0]. \]

Moreover, by (4.8), (4.9) and (4.12),

\[ A_{i_01}(y) = M_{i_0}(u_{i_0})^{-1}N(u_{i_0})B_{i_0-1}(y) = \gamma_0(y)M_{i_0}(u_{i_0})^{-1}N(u_{i_0})R_{i_0-1}C_0, \]

\[ A_{i_02}(y) = M_{i_0}(v_{i_0})^{-1}N(v_{i_0})B_{i_0}(y) = \gamma_0(y)M_{i_0}(u_{i_0})^{-1}N(u_{i_0})R_{i_0-1}C_0 - \kappa(y)M_{i_0}(y)^{-1}D_0. \]

Remark 4.3 As in Remark 4.2, we can obtain two expressions of the coefficient \( \delta_n(y) \). Because of the relationships

\[ B_n(y) = \gamma_0(y)R_nC_0 - \kappa(y)U_nM_{i_0}(y)^{-1}D_0 \]

and

\[ B_n(y) = \delta_n(y)D_0, \]

we can see that

\[ \delta_n(y) = \kappa(y) \frac{(1 \ 0)U_nM_{i_0}(y)^{-1}D_0}{(1 \ 0)R_nC_0} \]

(4.14)

Another expression can be derived by reversing the sense of the algorithm we used for solving the system (4.7). Indeed, let us rewrite (4.5) as \( \tilde{B}_{i-1}(y) = \tilde{P}_iB_i(y) \) for \( i \in \{1, \ldots, n\} \setminus \{i_0\} \) (recall that \( \tilde{P}_i = P_i^{-1} \)). We have

\[ B_0(y) = \tilde{P}_1\ldots \tilde{P}_{i_0-1}B_{i_0-1}(y), \]

\[ B_{i_0}(y) = \tilde{P}_{i_0+1}\ldots \tilde{P}_nB_n(y). \]
Moreover, by (4.8) and (4.9), we successively have
\[ B_{i_0-1}(y) = N(u_{i_0})^{-1}M_{i_0}(u_{i_0})A_{i_0-1}(y) = N(u_{i_0})^{-1}M_{i_0}(u_{i_0})[A_{i_0-1}(y) + \kappa(y)M_{i_0}(y)^{-1}D_0] \]
\[ = N(u_{i_0})^{-1}M_{i_0}(u_{i_0})[M_{i_0}(v_{i_0})^{-1}N(v_{i_0})B_{i_0}(y) + \kappa(y)M_{i_0}(y)^{-1}D_0] \]
\[ = \tilde{P}_{i_0}B_{i_0}(y) + \kappa(y)N(u_{i_0})^{-1}M_{i_0}(u_{i_0})M_{i_0}(y)^{-1}D_0 \]
\[ = \tilde{P}_{i_0}\tilde{P}_{i_0+1} \ldots \tilde{P}_n B_n(y) + \kappa(y)N(u_{i_0})^{-1}M_{i_0}(u_{i_0})M_{i_0}(y)^{-1}D_0 \]
and then
\[ B_0(y) = \tilde{P}_1 \ldots \tilde{P}_n B_n(y) + \kappa(y)\tilde{P}_1 \ldots \tilde{P}_{i_0-1}N(u_{i_0})^{-1}M_{i_0}(u_{i_0})M_{i_0}(y)^{-1}D_0. \]

We rewrite this last equality as
\[ \gamma_0(y)C_0 = \delta_n(y)\tilde{R}_nD_0 + \kappa(y)\tilde{U}_0M_{i_0}(y)^{-1}D_0 \]
with \( \tilde{R}_n = \tilde{P}_1 \ldots \tilde{P}_n \) and \( \tilde{U}_0 = \tilde{P}_1 \ldots \tilde{P}_{i_0-1}N(u_{i_0})^{-1}M_{i_0}(u_{i_0}) \). We finally get the following expression of \( \delta_n(y) \):
\[ \delta_n(y) = -\kappa(y) \frac{(0 \ 1)\tilde{U}_0M_{i_0}(y)^{-1}D_0}{(0 \ 1)\tilde{R}_nD_0}. \]

**Second case:** \( y \in (v_{i_0}, u_{i_0+1}) \) for a certain \( i_0 \in \{0, \ldots, n\} \)

Since the computations are very analogous, we briefly outline the corresponding matrix equations.

Set
\[ B_{i_01}(y) = \begin{pmatrix} \gamma_{i_01}(y) \\ \delta_{i_01}(y) \end{pmatrix}, \quad B_{i_02}(y) = \begin{pmatrix} \gamma_{i_02}(y) \\ \delta_{i_02}(y) \end{pmatrix}. \]

The equations write as follows:
\[
\begin{align*}
M_i(u_i)A_i(y) &= N(u_i)B_{i-1}(y) \quad \text{for } i \in \{1, \ldots, n\} \setminus \{i_0 + 1\}, \\
M_i(v_i)A_i(y) &= N(v_i)B_i(y) \quad \text{for } i \in \{1, \ldots, n\} \setminus \{i_0\}, \\
M_i(u_i)A_i(y) &= N(u_i)B_{i+1}(y), \\
N(y)B_{i+2}(y) - N(y)B_{i+1}(y) &= -\kappa(y)D_0.
\end{align*}
\]

We deduce that
\[ B_i(y) = \begin{cases} 
\gamma_0(y)P_iP_{i-1} \ldots P_1C_0 & \text{if } i \in \{0, 1, \ldots, i_0 - 1\}, \\
P_iP_{i-1} \ldots P_{i+2}B_{i+1}(y) & \text{if } i \in \{i_0 + 1, \ldots, n\},
\end{cases} \]
with
\[ B_{i_0+1}(y) = \begin{pmatrix} 1 \ 0 \end{pmatrix}N(v_{i_0+1})^{-1}M_{i_0+1}(v_{i_0+1})A_{i_0+1}(y) \]
\[ = \begin{pmatrix} 1 \ 0 \end{pmatrix}N(v_{i_0+1})^{-1}M_{i_0+1}(v_{i_0+1})M_{i_0+1}(u_{i_0+1})^{-1}N(u_{i_0+1})B_{i_02}(y) \]
\[ = P_{i_0+1}B_{i_02}(y). \]

We need to compute \( B_{i_02}(y) \): we have
\[ B_{i_02}(y) = B_{i_01}(y) - \kappa(y)N(y)^{-1}D_0 \]
with
\[ B_{i_01}(y) = \begin{pmatrix} 1 \ 0 \end{pmatrix}N(v_{i_0})^{-1}M_{i_0}(v_{i_0})A_{i_0}(y) = \begin{pmatrix} 1 \ 0 \end{pmatrix}N(v_{i_0})^{-1}M_{i_0}(v_{i_0})M_{i_0}(u_{i_0})^{-1}N(u_{i_0})B_{i_01}(y) \]
\[ = P_{i_0}B_{i_0-1}(y) = \gamma_0(y)P_{i_0}P_{i_0-1} \ldots P_1C_0. \]

Thus
\[ B_{i_0+1}(y) = P_{i_0+1} [\gamma_0(y)P_{i_0} \ldots P_1C_0 - \kappa(y)N(y)^{-1}D_0] \]
and for \( i \in \{i_0 + 1, \ldots, n\} \),
\[ B_i(y) = \gamma_0(y)P_iP_{i-1} \ldots P_1C_0 - \kappa(y)P_iP_{i-1} \ldots P_{i+1}N(y)^{-1}D_0. \]
Finally, we can express $A_i(y)$ by means of $B_i(y)$: for $i \in \{0,1,\ldots,n\}$,

$$A_i(y) = M_i(v_i)^{-1} N(v_i) B_i(y) = M_i(v_i)^{-1} N(v_i) \left[ \gamma_0(y) R_i C_0 - \kappa(y) V_i N(y)^{-1} D_0 \right]$$

and, for $i = i_0$,

$$A_{i_0}(y) = M_{i_0}(u_{i_0})^{-1} N(u_{i_0}) B_{i_0}(y) = M_{i_0}(u_{i_0})^{-1} N(u_{i_0}) \left[ \gamma_0(y) R_{i_0} C_0 - \kappa(y) V_{i_0} N(y)^{-1} D_0 \right].$$

We sum up the results obtained in this section in the following statement.

**Theorem 4.5** The iterated Laplace transform of the joint probability distribution of $(T_1, X_1)$ is given by the formulas below.

1) If $y \in (u_{i_0}, v_{i_0})$ for a certain $i_0 \in \{1,\ldots,n\}$,

$$\int_0^\infty e^{-\lambda t} \left[ \mathbb{E}_x \left( e^{-\theta Y}, Y < T_1 \right), X_1 \in dy \right] / dy \ dt = \left\{ \begin{array}{ll}
(1 \ 0) M_1(x) A_1(y) & \mbox{for } x \in (u_i, v_i) \mbox{ and } i \in \{1,\ldots,n\} \setminus \{i_0\}, \\
(1 \ 0) N(x) B_1(y) & \mbox{for } x \in (v_i, u_{i+1}) \mbox{ and } i \in \{0,\ldots,n\}, \\
(1 \ 0) M_{i_0}(x) A_{i_01}(y) & \mbox{for } x \in (u_{i_0}, v_{i_0}) \\
(1 \ 0) M_{i_0}(x) A_{i_02}(y) & \mbox{for } x \in [v_{i_0}, y], \\
(1 \ 0) R_{i_0} \left( \begin{array}{c} 0 \\ 1 \end{array} \right) & \mbox{for } x \in (v_{i_0}, y). 
\end{array} \right.$$
and, for \( i \in \{0,1,\ldots,n\}, \)

\[
B_i(y) = \kappa(y) \left[ \frac{(1 0) U_n M_{i0}(y)^{-1} (0 1)}{(1 0) R_n (\begin{array}{c} 1 \\ 0 \end{array})} R_i (\begin{array}{c} 1 \\ 0 \end{array}) - U_i M_{i0}(y)^{-1} (0 1) \right],
\]

where the matrices \( R_i, U_i \) are defined by (4.4), (4.6) and (4.11).

2) If \( y \in (v_{i_0}, u_{i_0+1}) \) for a certain \( i_0 \in \{0,\ldots,n\}, \)

\[
\int_0^\infty e^{-\lambda t} \left[ \mathbb{E}_x \left( e^{-\mu^T \mathbf{T}_t} \right), X_t \in dy \right] / dy \ dt = \begin{cases} 
(1 0) M_i(x) A_i(y) & \text{for } x \in [u_i, v_i] \text{ and } i \in \{1,\ldots,n\}, \\
(1 0) N(x) B_i(y) & \text{for } x \in (v_i, u_{i+1}) \text{ and } i \in \{0,\ldots,n\} \setminus \{i_0\}, \\
(1 0) N(x) B_{i+1}(y) & \text{for } x \in (v_{i_0}, y), \\
(1 0) N(x) B_{i+2}(y) & \text{for } x \in [y, u_{i_0+1}),
\end{cases}
\]

with, for \( i \in \{1,\ldots,n\} \setminus \{i_0\}, \)

\[
A_i(y) = \kappa(y) M_i(v_i)^{-1} N(v_i) \left[ \frac{(1 0) V_n N(y)^{-1} (0 1)}{(1 0) R_n (\begin{array}{c} 1 \\ 0 \end{array})} R_i (\begin{array}{c} 1 \\ 0 \end{array}) - V_i N(y)^{-1} (0 1) \right],
\]

\[
B_i(y) = \kappa(y) \left[ \frac{(1 0) V_n N(y)^{-1} (0 1)}{(1 0) R_n (\begin{array}{c} 1 \\ 0 \end{array})} R_i (\begin{array}{c} 1 \\ 0 \end{array}) - V_i N(y)^{-1} (0 1) \right],
\]

and

\[
A_{i0}(y) = \kappa(y) M_{i0}(u_{i0})^{-1} N(u_{i0}) \left[ \frac{(1 0) V_n N(y)^{-1} (0 1)}{(1 0) R_n (\begin{array}{c} 1 \\ 0 \end{array})} R_{i0} (\begin{array}{c} 1 \\ 0 \end{array}) - V_{i0} N(y)^{-1} (0 1) \right],
\]

\[
B_{i01}(y) = \kappa(y) \left[ \frac{(1 0) V_n N(y)^{-1} (0 1)}{(1 0) R_n (\begin{array}{c} 1 \\ 0 \end{array})} R_{i0} (\begin{array}{c} 1 \\ 0 \end{array}) \right],
\]

\[
B_{i02}(y) = \kappa(y) \left[ \frac{(1 0) V_n N(y)^{-1} (0 1)}{(1 0) R_n (\begin{array}{c} 1 \\ 0 \end{array})} R_{i0} (\begin{array}{c} 1 \\ 0 \end{array}) - N(y)^{-1} (0 1) \right],
\]

where the matrices \( R_i, V_i \) are defined by (4.4), (4.6) and (4.11).

5 Examples

5.1 Case of one bounded interval

In this part, we focus on the set \( E \) made of one two-sided interval \( E = [u, v] \). This classical case corresponds to the values of the parameters \( n = 1 \) and \( u_1 = u, v_1 = v, T_1 = T_i \). We relabel the argument \( \mu_i \) into \( \mu \), the functions \( a_1, b_1 \) into \( a, b \) and the related matrix \( M_1 \) into \( M \). The settings write here \( R_0 = I, S_0 = O, R_1 = P_1 = N(v)^{-1} M(v) M(v)^{-1} N(u), Q_1 = N(v)^{-1} M(v) M(u)^{-1} - N(v)^{-1}, S_1 = \frac{a_1}{M(v)^{-1} Q_1}, \hat{R}_1 = \hat{P}_1 = N(u)^{-1} M(u) M(v)^{-1} N(v), \hat{Q}_1 = N(u)^{-1} M(u) M(v)^{-1} - N(u)^{-1}, \hat{S}_1 = \frac{a_1}{M(v)^{-1} Q_1} \).

Probability distribution of \( T_i \)
The distribution of $T_t$ is characterized by

$$
\int_0^\infty e^{-\lambda t} E_x(e^{-\mu T_t}) \, dt = \begin{cases}
(1 \ 0) N(x) B_0 + \frac{1}{\lambda} & \text{for } x \in (-\infty, u], \\
(1 \ 0) M(x) A_1 + \frac{1}{\lambda + \mu} & \text{for } x \in [u, v], \\
(1 \ 0) N(x) B_1 + \frac{1}{\lambda} & \text{for } x \in [v, +\infty).
\end{cases}
$$

Observing that $M(v)^{-1}N(v)P_1 = M(u)^{-1}N(u)$ and $M(v)^{-1}N(v)Q_1 + M(v)^{-1} = M(u)^{-1}$, the matrix $A_1$ can be simplified into

$$
A_1 = \gamma_0 M(u)^{-1}N(u) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\mu}{\lambda(\lambda + \mu)} M(u)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}
$$

and the other matrices are given by $B_0 = \gamma_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $B_1 = \delta_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with

$$
\gamma_0 = -\frac{\mu}{\lambda(\lambda + \mu)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} Q_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \delta_1 = -\frac{\mu}{\lambda(\lambda + \mu)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tilde{P}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
$$

Moreover,

$$
(1 \ 0) N(x) B_0 = \gamma_0 c(x), \quad (1 \ 0) N(x) B_1 = \delta_1 d(x).
$$

**Joint probability distribution of $(T_t, X_t)$**

1) Assume that $y \in (-\infty, u]$. This case corresponds to $i_0 = 0$ and we have $V_0 = I, V_1 = P_1$. The distribution of $(T_t, X_t)$ is characterized by

$$
\int_0^\infty e^{-\lambda t} \left[ E_x(e^{-\mu T_t}, X_t \in dy) / dy \right] \, dt = \begin{cases}
(1 \ 0) N(x) B_{01}(y) & \text{for } x \in (-\infty, y], \\
(1 \ 0) N(x) B_{02}(y) & \text{for } x \in [y, u], \\
(1 \ 0) M(x) A_1(y) & \text{for } x \in [u, v], \\
(1 \ 0) N(x) B_1(y) & \text{for } x \in [v, +\infty).
\end{cases}
$$

Since $M(v)^{-1}N(v)P_1 = M(u)^{-1}N(u)$, we have for $A_1(y)$

$$
A_1(y) = \gamma_0(y) M(v)^{-1}N(v) \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \kappa(y) M(u)^{-1}N(u) N(y)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix},
$$

and the other matrices are given by

$$
B_{01}(y) = \gamma_0(y) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B_{02}(y) = \gamma_0(y) \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \kappa(y) N(y)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix},
$$

$$
B_1(y) = P_1 \left[ \gamma_0(y) \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \kappa(y) N(y)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]
$$

with

$$
\gamma_0(y) = \kappa(y) \begin{pmatrix} 1 \\ 0 \end{pmatrix} P_1 N(y)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
$$

2) Assume that $y \in [u, v]$. This case corresponds to $i_0 = 1$ and we have $U_0 = O, U_1 = N(v)^{-1}M(v)$. The distribution of $(T_t, X_t)$ is characterized by

$$
\int_0^\infty e^{-\lambda t} \left[ E_x(e^{-\mu T_t}, X_t \in dy) / dy \right] \, dt = \begin{cases}
(1 \ 0) N(x) B_0(y) & \text{for } x \in (-\infty, u], \\
(1 \ 0) M(x) A_1(y) & \text{for } x \in [u, y], \\
(1 \ 0) M(x) A_1(y) & \text{for } x \in [y, v], \\
(1 \ 0) N(x) B_1(y) & \text{for } x \in [v, +\infty).
\end{cases}
$$
The matrices are given by

\[ A_{11}(y) = \gamma_0(y)M(u)^{-1}N(u) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad A_{12}(y) = \gamma_0(y)M(u)^{-1}N(u) \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \kappa(y)M(y)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \]

\[ B_0(y) = \gamma_0(y) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B_1(y) = \gamma_0(y)P_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \kappa(y)U_1M(y)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

with

\[ \gamma_0(y) = \kappa(y) \frac{(1 \ 0)U_1M(y)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}{(1 \ 0)P_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}}. \]

3) Assume that \( y \in [v, +\infty) \). This case corresponds to \( i_0 = 1 \) and we have \( V_0 = O, V_1 = I \). The distribution of \((T_i, X_i)\) is characterized by

\[ \int_0^\infty e^{-\lambda t} \left[ \mathbb{E}_x \left( e^{-\mu T_i}, X_i \in dy \right) /dy \right] dt = \begin{cases} (1 \ 0)N(x)B_0(y) & \text{for } x \in (-\infty, u], \\
(1 \ 0)M(x)A_1(y) & \text{for } x \in [u, v], \\
(1 \ 0)N(x)B_11(y) & \text{for } x \in [v, y], \\
(1 \ 0)N(x)B_12(y) & \text{for } x \in [y, +\infty). \end{cases} \]

The matrices are given by

\[ A_1(y) = \gamma_0(y)M(v)^{-1}N(v) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B_0(y) = \gamma_0(y) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \]

\[ B_{11}(y) = \gamma_0(y)P_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B_{12}(y) = \gamma_0(y)P_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \kappa(y)N(y)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

with

\[ \gamma_0(y) = \kappa(y) \frac{(1 \ 0)N(y)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}{(1 \ 0)P_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}}. \]

### 5.2 Brownian motion

In this example, we take for \((X_t)_{t \geq 0}\) linear Brownian motion rescaled such that \( \mathbb{E}(X_t^2) = 2t \), that is \( X_t = \sqrt{2} B_t \) where \((B_t)_{t \geq 0}\) is the standard Brownian motion satisfying \( \mathbb{E}(B_t^2) = t \). This choice which corresponds to \( \sigma(x) = \sqrt{2} \) and \( \tau(x) = 0 \) is done for simplifying the forthcoming settings. We take for \( a, b, c, d \) the functions

\[ a(x) = e^{\sqrt{\lambda+\mu}x}, \quad b(x) = e^{-\sqrt{\lambda+\mu}x}, \quad c(x) = e^{\sqrt{\lambda}x}, \quad d(x) = e^{-\sqrt{\lambda}x}. \]

The potential \( \rho_{ij} \) writes \( \rho_{ij}(x, y) = \frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}|x-y|} \) and then \( w(x) = 2\sqrt{\lambda} \) and \( \kappa(x) = 1 \). The related matrices \( M \) and \( N \) write

\[ M(x) = \begin{pmatrix} e^{\sqrt{\lambda+\mu}x} & e^{-\sqrt{\lambda+\mu}x} \\
\sqrt{\lambda+\mu} e^{\sqrt{\lambda+\mu}x} & -\sqrt{\lambda+\mu} e^{-\sqrt{\lambda+\mu}x} \end{pmatrix}, \]

\[ N(x) = \begin{pmatrix} e^{\sqrt{\lambda}x} & e^{-\sqrt{\lambda}x} \\
\sqrt{\lambda} e^{\sqrt{\lambda}x} & -\sqrt{\lambda} e^{-\sqrt{\lambda}x} \end{pmatrix}, \]

and their inverse are given by

\[ M(x)^{-1} = \frac{1}{2\sqrt{\lambda+\mu}} \begin{pmatrix} \sqrt{\lambda+\mu} e^{-\sqrt{\lambda+\mu}x} & e^{-\sqrt{\lambda+\mu}x} \\
\sqrt{\lambda+\mu} e^{\sqrt{\lambda+\mu}x} & -e^{\sqrt{\lambda+\mu}x} \end{pmatrix}, \]

\[ N(x)^{-1} = \frac{1}{2\sqrt{\lambda}} \begin{pmatrix} \sqrt{\lambda} e^{-\sqrt{\lambda}x} & e^{-\sqrt{\lambda}x} \\
\sqrt{\lambda} e^{\sqrt{\lambda}x} & -e^{\sqrt{\lambda}x} \end{pmatrix}. \]
We have
\[
M(y)M(x)^{-1} = \begin{pmatrix}
\cosh(\sqrt{\lambda + \mu}(y - x)) & \sinh(\sqrt{\lambda + \mu}(y - x)) \\
\sqrt{\lambda + \mu} \sinh(\sqrt{\lambda + \mu}(y - x)) & \cosh(\sqrt{\lambda + \mu}(y - x))
\end{pmatrix}
\begin{pmatrix}
\cosh(z) & \sinh(z) \\
\sqrt{\lambda + \mu} \sinh(z) & \cosh(z)
\end{pmatrix},
\]
where we set \(z = \sqrt{\lambda + \mu}(y - x)\) for lightening the text. Next,
\[
N(y)^{-1}M(y)M(x)^{-1} = \frac{1}{2} \begin{pmatrix}
e^{-\sqrt{\lambda}y} \left[ \cosh(z) + \sqrt{\frac{\lambda + \mu}{\lambda}} \sinh(z) \right] & e^{\sqrt{\lambda}y} \left[ \cosh(z) + \sqrt{\frac{\lambda + \mu}{\lambda}} \sinh(z) \right] \\
e^{\sqrt{\lambda}y} \left[ \cosh(z) - \sqrt{\frac{\lambda + \mu}{\lambda}} \sinh(z) \right] & e^{-\sqrt{\lambda}y} \left[ \cosh(z) - \sqrt{\frac{\lambda + \mu}{\lambda}} \sinh(z) \right]
\end{pmatrix}.
\]
Making use of the elementary identity
\[
\cosh x + a \sinh x = 1 + 2 \sinh^2 \frac{x}{2} + 2a \cosh \frac{x}{2} \sinh \frac{x}{2} = 1 + 2 \sinh \frac{x}{2} \left( a \cosh \frac{x}{2} + \sinh \frac{x}{2} \right),
\]
we get
\[
N(y)^{-1}M(y)M(x)^{-1} - N(y)^{-1} = \sinh(\frac{z}{2}) \begin{pmatrix}
e^{-\sqrt{\lambda}y} \left[ \sqrt{\frac{\lambda + \mu}{\lambda}} \cosh(\frac{z}{2}) + \sinh(\frac{z}{2}) \right] & e^{\sqrt{\lambda}y} \left[ \sqrt{\frac{\lambda + \mu}{\lambda}} \cosh(\frac{z}{2}) + \sinh(\frac{z}{2}) \right] \\
e^{\sqrt{\lambda}y} \left[ - \sqrt{\frac{\lambda + \mu}{\lambda}} \cosh(\frac{z}{2}) + \sinh(\frac{z}{2}) \right] & e^{-\sqrt{\lambda}y} \left[ - \sqrt{\frac{\lambda + \mu}{\lambda}} \cosh(\frac{z}{2}) - \sinh(\frac{z}{2}) \right]
\end{pmatrix}.
\]
Now,
\[
N(y)^{-1}M(y)M(x)^{-1}N(x) = \frac{1}{2} \begin{pmatrix}
e^{\sqrt{\lambda}(x-y)} \left[ 2 \cosh(z) + \left( \sqrt{\frac{\lambda + \mu}{\lambda}} + \sqrt{\frac{\lambda + \mu}{\lambda}} \right) \sinh(z) \right] \\
e^{\sqrt{\lambda}(x+y)} \left( \sqrt{\frac{\lambda + \mu}{\lambda}} - \sqrt{\frac{\lambda + \mu}{\lambda}} \right) \sinh(z)
\end{pmatrix}
\begin{pmatrix}
e^{-\sqrt{\lambda}(x+y)} \left( \sqrt{\frac{\lambda + \mu}{\lambda}} - \sqrt{\frac{\lambda + \mu}{\lambda}} \right) \sinh(z) \\
e^{\sqrt{\lambda}(y-x)} \left[ 2 \cosh(z) - \left( \sqrt{\frac{\lambda + \mu}{\lambda}} + \sqrt{\frac{\lambda + \mu}{\lambda}} \right) \sinh(z) \right]
\end{pmatrix}.
\]
Observing that
\[
2 \cosh x + \left( a + \frac{1}{a} \right) \sinh x = 2 \left( \cosh^2 \frac{x}{2} + \sinh^2 \frac{x}{2} \right) + 2 \left( a + \frac{1}{a} \right) \cosh \frac{x}{2} \sinh \frac{x}{2} = \frac{2}{a} \left( a \cosh \frac{x}{2} + \sinh \frac{x}{2} \right) \left( \cosh \frac{x}{2} + a \sinh \frac{x}{2} \right),
\]
we obtain
\[
N(y)^{-1}M(y)M(x)^{-1}N(x) = \frac{1}{\sqrt{\lambda}(\lambda + \mu)} \begin{pmatrix}
e^{\sqrt{\lambda}(x-y)} \left[ \sqrt{\lambda} \cosh(\frac{z}{2}) + \sqrt{\lambda + \mu} \sinh(\frac{z}{2}) \right] \left[ \sqrt{\lambda + \mu} \cosh(\frac{z}{2}) + \sqrt{\lambda} \sinh(\frac{z}{2}) \right] \\
&- \mu e^{\sqrt{\lambda}(x+y)} \cosh(\frac{z}{2}) \sinh(\frac{z}{2}) \\
&\mu e^{-\sqrt{\lambda}(x+y)} \cosh(\frac{z}{2}) \sinh(\frac{z}{2}) \\
e^{\sqrt{\lambda}(y-x)} \left[ \sqrt{\lambda} \cosh(\frac{z}{2}) - \sqrt{\lambda + \mu} \sinh(\frac{z}{2}) \right] \left[ \sqrt{\lambda + \mu} \cosh(\frac{z}{2}) - \sqrt{\lambda} \sinh(\frac{z}{2}) \right]
\end{pmatrix}.
\]
5.3 Case of one bounded interval for Brownian motion

We now consider the sojourn time of Brownian motion in the interval $[u, v]$. In order to lighten the paper, we only compute the distribution of $T_u$, that of $(T_t, X_t)$ being more cumbersome.

For evaluating $\gamma_0$, we compute, with $w = \frac{1}{2}\sqrt{\lambda + \mu}(v - u)$,

\[
(1 \ 0)P_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{e^{\sqrt{\lambda}(u - v)}}{\sqrt{\lambda}(\lambda + \mu)}[\sqrt{\lambda}\cosh(w) + \sqrt{\lambda + \mu}\sinh(w)][\sqrt{\lambda + \mu}\cosh(w) + \sqrt{\lambda}\sinh(w)]
\]

and

\[
(1 \ 0)Q_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{e^{-\sqrt{\lambda}v}}{\sqrt{\lambda}} \sinh(w)[\sqrt{\lambda + \mu}\cosh(w) + \sqrt{\lambda}\sinh(w)]
\]

from which we deduce

\[
\gamma_0 = -\frac{\mu}{\lambda(\lambda + \mu)} \frac{(1 \ 0)Q_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{(1 \ 0)P_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}} = -\frac{\mu}{\lambda\sqrt{\lambda + \mu} \sqrt{\lambda}\cosh(w) + \sqrt{\lambda + \mu}\sinh(w)} e^{-\sqrt{\lambda}v} \sinh(w).
\]

For evaluating $\delta_1$, we compute

\[
(0 \ 1)\tilde{P}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{e^{\sqrt{\lambda}(u - v)}}{\sqrt{\lambda}(\lambda + \mu)}[\sqrt{\lambda}\cosh(w) + \sqrt{\lambda + \mu}\sinh(w)][\sqrt{\lambda + \mu}\cosh(w) + \sqrt{\lambda}\sinh(w)]
\]

and

\[
(0 \ 1)Q_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{e^{\sqrt{\lambda}u}}{\sqrt{\lambda}} \sinh(w)[\sqrt{\lambda + \mu}\cosh(w) + \sqrt{\lambda}\sinh(w)]
\]

from which we deduce

\[
\delta_1 = -\frac{\mu}{\lambda(\lambda + \mu)} \frac{(0 \ 1)Q_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{(0 \ 1)\tilde{P}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}} = -\frac{\mu}{\lambda\sqrt{\lambda + \mu} \sqrt{\lambda}\cosh(w) + \sqrt{\lambda + \mu}\sinh(w)} e^{\sqrt{\lambda}v} \sinh(w).
\]

This yields

\[
(1 \ 0)N(x)B_0 = \gamma_0(1 \ 0)N(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{\mu}{\lambda\sqrt{\lambda + \mu} \sqrt{\lambda}\cosh(w) + \sqrt{\lambda + \mu}\sinh(w)} \sinh(w)e^{\sqrt{\lambda}(x - u)}
\]

\[
(1 \ 0)N(x)B_1 = \delta_1(1 \ 0)N(x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{\mu}{\lambda\sqrt{\lambda + \mu} \sqrt{\lambda}\cosh(w) + \sqrt{\lambda + \mu}\sinh(w)} \sinh(w)e^{\sqrt{\lambda}(v - x)}.
\]

On the other hand,

\[
M(u)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} e^{\sqrt{\lambda}u} \begin{pmatrix} e^{-\sqrt{\lambda + \mu}u} \\ e^{\sqrt{\lambda + \mu}u} \end{pmatrix},
\]

\[
M(u)^{-1}N(u) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{e^{\sqrt{\lambda}u}}{2\sqrt{\lambda + \mu}} \begin{pmatrix} \sqrt{\lambda + \mu} e^{-\sqrt{\lambda + \mu}u} + e^{-\sqrt{\lambda + \mu}u} \sqrt{\lambda + \mu} e^{\sqrt{\lambda + \mu}u} \\ \sqrt{\lambda + \mu} e^{\sqrt{\lambda + \mu}u} - e^{-\sqrt{\lambda + \mu}u} \sqrt{\lambda + \mu} e^{-\sqrt{\lambda + \mu}u} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{e^{\sqrt{\lambda}u}}{2\sqrt{\lambda + \mu}} \begin{pmatrix} \sqrt{\lambda + \mu} + \sqrt{\lambda} \sinh(w) e^{-\sqrt{\lambda + \mu}u} \\ \sqrt{\lambda + \mu} - \sqrt{\lambda} \sinh(w) e^{\sqrt{\lambda + \mu}u} \end{pmatrix}.
\]
where characterized by

The distribution of the sojourn time

\[ \text{Theorem 5.1} \]

where, for any \( i \) and then

Let us apply our results to the following set \((1.4)\) local time in a finite set for Brownian motion

We retrieve the well-known distribution (1.7.1), p. 140 of [1].

Now,

\[ \int_0^\infty e^{-\lambda t} E_x (e^{-\mu T_i}) dt = \begin{cases} 
\frac{1}{\lambda} & -\mu \cosh(\sqrt{\lambda + \mu}(x - (u + v)/2)) \\
\frac{1}{\lambda + \mu} & \cosh(\sqrt{\lambda + \mu}(x - (u + v)/2)) \\
\frac{1}{\lambda} & \cosh(\sqrt{\lambda + \mu}(v - x)) 
\end{cases} \]

for \( x \in (-\infty, u] \),

for \( x \in [u, v) \),

for \( x \in [v, +\infty) \),

where \( w = \frac{1}{2} \sqrt{\lambda + \mu} (v - u) \).

We retrieve the well-known distribution (1.7.1), p. 140 of [1].

5.4 Local time in a finite set for Brownian motion

Let us apply our results to the following set \((u_1, \ldots, u_n)\) are real numbers such that \( u_1 < \cdots < u_n \):

\[ E_\varepsilon = \bigcup_{i=1}^n [u_i - \varepsilon, u_i + \varepsilon] \]

where \( \varepsilon > 0 \) is subject to tend to 0 and denote \( T_{t, \varepsilon} = \int_0^t 1_{E_\varepsilon}(X_s) ds \). Set also \( u_0 = -\infty \) and \( u_{n+1} = +\infty \).

The local time in the set \( \{u_1, \ldots, u_n\} \) of Brownian motion \( (X_t)_{t \geq 0} \) up to time \( t \) is defined by

\[ L_t = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} T_{t, \varepsilon}. \]

Of course, as previously, we can decompose \( L_t \) into the sum

\[ L_t = \sum_{i=1}^n L_t^i \]

where, for any \( i \in \{1, \ldots, n\} \),

\[ L_t^i = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t 1_{[u_i - \varepsilon, u_i + \varepsilon]}(X_s) ds. \]
Let us introduce the vector of local times at each point \( u_i: \mathbf{L}_t = (L^1_t, \ldots, L^n_t) \). Set, for \( i \in \{1, \ldots, n\} \),

\[
\nu_i, \varepsilon = \frac{\mu_i / \varepsilon}{\lambda + \mu_i / \varepsilon},
\]

\[
P_{i, \varepsilon} = N(u_i + \varepsilon)^{-1} M_i(u_i + \varepsilon) M_i(u_i - \varepsilon)^{-1} N(u_i - \varepsilon),
\]

\[
Q_{i, \varepsilon} = N(u_i + \varepsilon)^{-1} M_i(u_i + \varepsilon) M_i(u_i - \varepsilon)^{-1} - N(u_i - \varepsilon)^{-1},
\]

\[
R_{i, \varepsilon} = P_{i, \varepsilon} P_{i-1, \varepsilon} \cdots P_{1, \varepsilon},
\]

\[
S_{i, \varepsilon} = \nu_{i, \varepsilon} Q_{i, \varepsilon} + \nu_{i-1, \varepsilon} P_{i, \varepsilon} Q_{i-1, \varepsilon} + \nu_{i-2, \varepsilon} P_{i, \varepsilon} P_{i-1, \varepsilon} Q_{i-2, \varepsilon} + \cdots + \nu_{1, \varepsilon} P_{i, \varepsilon} P_{i-1, \varepsilon} \cdots P_{2, \varepsilon} Q_{1, \varepsilon},
\]

\[
B_{i, \varepsilon} = S_{i, \varepsilon} C_0 - \left( \frac{1}{0} \right) S_{n, \varepsilon} C_0 R_{i, \varepsilon} C_0.
\]

As usual, we set \( R_{0, \varepsilon} = I , S_{0, \varepsilon} = O \) and \( B_{0, \varepsilon} = - \left( \frac{1}{0} \right) S_{n, \varepsilon} C_0 \).

In this part, we compute the following limit, for \( x \in (u_i, u_{i+1}) \) (which entails that \( x \in (u_i + \varepsilon, u_{i+1} - \varepsilon) \) for small enough \( \varepsilon \)) and \( i \in \{0, 1, \ldots, n\} \),

\[
\int_{0}^{\infty} e^{-\lambda t} \mathbb{P}_x (e^{-<\mu \mathbf{L}_t>}) dt = \lim_{\varepsilon \to 0} (1, 0) N(x) B_{i, \varepsilon} + \frac{1}{\lambda}
\]

where

\[
N(x) = \begin{pmatrix} e^{\sqrt{\lambda} x} & e^{-\sqrt{\lambda} x} \\ \sqrt{\lambda} e^{\sqrt{\lambda} x} & -\sqrt{\lambda} e^{-\sqrt{\lambda} x} \end{pmatrix}.
\]

Set \( \varepsilon = \varepsilon \sqrt{\lambda + \mu_i / \varepsilon} \). Using the results of Section 5.2, we have for \( i \in \{1, \ldots, n\} \)

\[
P_{i, \varepsilon} = \frac{1}{\sqrt{\lambda (\lambda + \mu_i / \varepsilon)}} \left[ e^{2\sqrt{\lambda}} \left[ \sqrt{\lambda} \cosh(\varepsilon) + \sqrt{\lambda + \mu_i / \varepsilon} \sinh(\varepsilon) \right] \left[ \sqrt{\lambda + \mu_i / \varepsilon} \cosh(\varepsilon) + \sqrt{\lambda} \sinh(\varepsilon) \right] \right. \\
\left. - (\mu_i / \varepsilon) e^{2u_i \sqrt{\lambda}} \cosh(\varepsilon) \sinh(\varepsilon) \right] \\
\left. e^{2\sqrt{\lambda}} \left[ \sqrt{\lambda} \cosh(\varepsilon) - \sqrt{\lambda + \mu_i / \varepsilon} \sinh(\varepsilon) \right] \left[ \sqrt{\lambda + \mu_i / \varepsilon} \cosh(\varepsilon) - \sqrt{\lambda} \sinh(\varepsilon) \right] \right)
\]

and

\[
Q_{i, \varepsilon} = \sinh(\varepsilon) \left( e^{-\sqrt{\lambda} (u_{i+1} + \varepsilon)} \left[ \sqrt{\lambda + \mu_i / \varepsilon} \cosh(\varepsilon) + \sinh(\varepsilon) \right] - e^{\sqrt{\lambda} (u_{i+1} + \varepsilon)} \left[ \sqrt{\lambda + \mu_i / \varepsilon} \cosh(\varepsilon) + \sinh(\varepsilon) \right] \right) \\
\left. - e^{-\sqrt{\lambda} (u_i + \varepsilon)} \left[ \sqrt{\lambda + \mu_i / \varepsilon} \cosh(\varepsilon) + \sinh(\varepsilon) \right] + e^{\sqrt{\lambda} (u_i + \varepsilon)} \left[ \sqrt{\lambda + \mu_i / \varepsilon} \cosh(\varepsilon) - \sinh(\varepsilon) \right] \right)
\]

By the elementary asymptotics for \( \varepsilon \to 0^+ \)

\[
\sqrt{\lambda} \cosh(\varepsilon) + \sqrt{\lambda + \mu_i / \varepsilon} \sinh(\varepsilon) \sim \sqrt{\lambda} + \mu_i,
\]

\[
\sqrt{\lambda + \mu_i / \varepsilon} \cosh(\varepsilon) + \sqrt{\lambda} \sinh(\varepsilon) \sim \sqrt{\mu_i / \varepsilon},
\]

\[
\cosh(\varepsilon) \sinh(\varepsilon) \sim \sqrt{\mu_i / \varepsilon},
\]

we get, for \( i \in \{1, \ldots, n\} \), \( \lim_{\varepsilon \to 0^+} P_{i, \varepsilon} = \tilde{P}_i \), \( \lim_{\varepsilon \to 0^+} Q_{i, \varepsilon} = \tilde{Q}_i \), \( \lim_{\varepsilon \to 0^+} R_{i, \varepsilon} = \tilde{R}_i \), \( \lim_{\varepsilon \to 0^+} S_{i, \varepsilon} = \tilde{S}_i \) with

\[
\tilde{P}_i = \frac{1}{\sqrt{\lambda}} \left( \begin{array}{cc} \sqrt{\lambda} + \mu_i & \mu_i e^{-2\sqrt{\lambda} u_i} \\ -\mu_i e^{2\sqrt{\lambda} u_i} & \sqrt{\lambda} - \mu_i \end{array} \right),
\]

\[
\tilde{Q}_i = \frac{\mu_i}{\sqrt{\lambda}} \left( \begin{array}{cc} e^{-\sqrt{\lambda} u_i} & 0 \\ -e^{\sqrt{\lambda} u_i} & 0 \end{array} \right),
\]

\[
\tilde{R}_i = P_i \tilde{P}_{i-1} \cdots \tilde{P}_1, \quad \tilde{S}_i = \frac{1}{\lambda} \left[ \tilde{Q}_i + \tilde{P}_i \tilde{Q}_{i-1} + \tilde{P}_i \tilde{P}_{i-1} \tilde{Q}_{i-2} + \cdots + \tilde{P}_i \tilde{P}_{i-1} \cdots \tilde{P}_2 \tilde{Q}_1 \right].
\]

We also have, for \( i \in \{0, 1, \ldots, n\} \), \( \lim_{\varepsilon \to 0^+} B_{i, \varepsilon} = \tilde{B}_i \) where

\[
\tilde{B}_i = \tilde{S}_i \left( \begin{array}{c} 1 \\ 0 \end{array} \right) - \tilde{\gamma}_0 \tilde{R}_i \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \quad \text{with} \quad \tilde{\gamma}_0 = \frac{(1, 0) \tilde{S}_n (1, 0)}{(1, 0) \tilde{R}_n (1, 0)}.
\]

We can now state the following result.
Theorem 5.2 The iterated Laplace transform of the vector of Brownian local times $L_t$ at points $u_1, \ldots, u_n$ is given, for $x \in (u_i, u_{i+1})$ and $i \in \{0, 1, \ldots, n\}$, by

$$
\int_0^\infty e^{-\lambda t} \mathbb{E}_x(e^{-\lambda L_t}) \, dt = (1 \ 0) N(x) \hat{B}_i + \frac{1}{\lambda}
$$

where the matrices $\hat{B}_i$, $i \in \{0, 1, \ldots, n\}$, are given by (5.7), (5.8) and (5.9). Additionally, this formula holds also for $x = u_i$, $i \in \{1, \ldots, n\}$. In particular, the iterated Laplace transform of the Brownian local time $L_t$ in $(u_1, \ldots, u_n)$, namely $\int_0^\infty e^{-\lambda t} \mathbb{E}_x(e^{-\lambda L_t}) \, dt$, can be deduced from the previous formula by choosing $\mu = (\mu, \ldots, \mu)$.

**Proof**

It remains to prove the assertion concerning the case $x = u_i$. Observing that we have

$$(1 \ 0) N(u_i) \hat{P}_i = \frac{1}{\sqrt{\lambda}} \left( e^{\sqrt{\lambda} u_i} \ e^{-\sqrt{\lambda} u_i} \right) \left( \frac{\sqrt{\lambda} + \mu_i}{-\mu e^{2\sqrt{\lambda} u_i} \sqrt{\lambda} - \mu} \ \mu_i e^{-2\sqrt{\lambda} u_i} \sqrt{\lambda} - \mu \right) = \left( e^{\sqrt{\lambda} u_i} \ e^{-\sqrt{\lambda} u_i} \right) = (1 \ 0) N(u_i),$$

and

$$(1 \ 0) N(u_i) \hat{Q}_i = \frac{\mu_i}{\sqrt{\lambda}} \left( e^{\sqrt{\lambda} u_i} \ e^{-\sqrt{\lambda} u_i} \right) \left( 0 \ e^{\sqrt{\lambda} u_i} \ e^{-\sqrt{\lambda} u_i} \ 0 \right) = (0 \ 0),$$

we deduce that

$$(1 \ 0) N(u_i) \hat{R}_i = (1 \ 0) N(u_i) \hat{P}_i \hat{P}_{i-1} \ldots \hat{P}_1 = (1 \ 0) N(u_i) \hat{P}_{i-1} \ldots \hat{P}_1 = (1 \ 0) N(u_i) \hat{R}_{i-1}
$$

and

$$(1 \ 0) N(u_i) \hat{S}_i = \frac{1}{\lambda} \left[ (1 \ 0) N(u_i) \hat{Q}_i + (1 \ 0) N(u_i) \hat{P}_i (\hat{Q}_{i-1} + \cdots + \hat{P}_{i-1} \cdots \hat{P}_2 \hat{Q}_1) \right]
= \frac{1}{\lambda} (1 \ 0) N(u_i) (\hat{Q}_{i-1} + \cdots + \hat{P}_{i-1} \cdots \hat{P}_2 \hat{Q}_1) = (1 \ 0) N(u_i) \hat{S}_{i-1}.$$

As a result, since $\hat{B}_i$ is linear combination of the matrices $\hat{R}_i$ and $\hat{S}_i$, we have

$$(1 \ 0) N(u_i) \hat{B}_i = (1 \ 0) N(u_i) \hat{B}_{i-1}.$$

This proves that

$$\lim_{x \to u_i} \int_0^\infty e^{-\lambda t} \mathbb{E}_x(e^{-\lambda L_t}) \, dt = \lim_{x \to u_i} \int_0^\infty e^{-\lambda t} \mathbb{E}_x(e^{-\lambda L_t}) \, dt$$

and then, by continuity with respect to $x$,

$$\int_0^\infty e^{-\lambda t} \mathbb{E}_x(e^{-\lambda L_t}) \, dt = \lim_{x \to u_i} \int_0^\infty e^{-\lambda t} \mathbb{E}_x(e^{-\lambda L_t}) \, dt = (1 \ 0) N(u_i) \hat{B}_i + \frac{1}{\lambda}.$$ 

We end up this part be considering the particular cases $n = 1$ and $n = 2$.

**Local time in $\{u\}$**

Suppose that $n = 1$ and set $u_1 = u$, $\mu_1 = \mu$. We have $\hat{R}_0 = I$, $\hat{S}_0 = O$, $\hat{R}_1 = \hat{P}_1$, $\hat{S}_1 = \frac{1}{\lambda} \hat{Q}_1$. Therefore,

$$
\hat{R}_1 = \frac{1}{\sqrt{\lambda}} \left( \frac{\sqrt{\lambda} + \mu \mu e^{-2\sqrt{\lambda} u}}{-\mu e^{-2\sqrt{\lambda} u} \sqrt{\lambda} - \mu} \right), \quad \hat{S}_1 = \frac{\mu}{\lambda^{3/2}} \left( \begin{array}{cc} e^{-\sqrt{\lambda} u} & 0 \\ 0 & 0 \end{array} \right),
$$

$$
\hat{B}_0 = \hat{S}_0 \left( \begin{array}{c} 1 \\ 0 \end{array} \right) - \frac{(1 \ 0) \hat{S}_1}{(1 \ 0) \hat{R}_1} \hat{R}_0 \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = -\frac{\mu e^{-\sqrt{\lambda} u}}{\lambda (\sqrt{\lambda} + \mu)} \left( \begin{array}{c} 1 \\ 0 \end{array} \right)
$$

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and
\[ B_1 = S \left( \frac{1}{0} \right) - \frac{1}{(1 \ 0) R_1 \left( \frac{1}{0} \right)} R_1 \left( \frac{1}{0} \right) - \mu \frac{e^{\sqrt{\lambda} u}}{\lambda^{3/2}} \left[ \frac{e^{-\sqrt{\lambda} u}}{\sqrt{\lambda} + \mu} - \frac{e^{-\sqrt{\lambda} u}}{\sqrt{\lambda} + \mu} \right] = -\frac{\mu e^{\sqrt{\lambda} u}}{\lambda^{3/2}} \left[ \frac{1}{\lambda^{3/2}} \right]. \]

Next,
\[ (1 \ 0) N(x) B_0 = -\frac{\mu}{\lambda(\sqrt{\lambda} + \mu)} e^{\sqrt{\lambda} (x-u)}, \quad (1 \ 0) N(x) B_1 = -\frac{\mu}{\lambda(\sqrt{\lambda} + \mu)} e^{\sqrt{\lambda} (u-x)}. \]

As a result, the iterated Laplace transform of the local time \( L_t \) in \( \{ u \} \) is given, for any \( x \in \mathbb{R} \), by
\[
\int_0^\infty e^{-\lambda t} e^{\mu t} dt = \frac{1}{\lambda} \left[ 1 - \frac{\mu}{\sqrt{\lambda} + \mu} e^{-\sqrt{\lambda} |x-u|} \right].
\]

We retrieve formula (1.3.1), p. 126 of [1].

**Local time in \( \{ u,v \} \)**

Suppose that \( n = 2 \) and set \( u_1 = u, u_2 = v, \mu_1 = \mu, \mu_2 = \nu \). We have \( \tilde{R}_0 = I, \tilde{S}_0 = O, \tilde{R}_1 = \tilde{P}_1, \tilde{S}_1 = \frac{1}{\lambda} \tilde{Q}_1, \tilde{R}_2 = \tilde{P}_2 \tilde{P}_1, \tilde{S}_2 = \frac{1}{\lambda} (\tilde{Q}_2 + \tilde{P}_2 \tilde{Q}_1), \gamma_0 = \left[ (1 \ 0) \tilde{S}_2 \left( \frac{1}{0} \right) \right]/\left[ (1 \ 0) \tilde{R}_2 \left( \frac{1}{0} \right) \right]. \]

Explicitly,
\[ \tilde{R}_1 = \frac{1}{\lambda} \left( \sqrt{\lambda} + \nu \mu e^{-2\sqrt{\lambda} u} \right), \quad \tilde{S}_1 = \frac{\mu}{\lambda^{3/2}} \left( e^{-\sqrt{\lambda} u} \right), \quad \tilde{S}_2 = \frac{\mu}{\lambda^{3/2}} \left( e^{-\sqrt{\lambda} v} \right) \]

\[ \tilde{R}_2 = \frac{1}{\lambda} \left( \sqrt{\lambda} + \nu \mu e^{-2\sqrt{\lambda} u} \right) \left( \sqrt{\lambda} + \nu \mu e^{-2\sqrt{\lambda} v} \right) \left( \sqrt{\lambda} - \mu \right) \]

\[ = \frac{1}{\lambda} \left( \nu (\sqrt{\lambda} + \nu) \mu e^{\sqrt{\lambda} (u-v)} \right) \left( \sqrt{\lambda} + \nu \mu e^{\sqrt{\lambda} (u-v)} \right) \left( \sqrt{\lambda} - \mu \right) \]

\[ = \frac{1}{\lambda} \left( \nu (\sqrt{\lambda} + \nu) \mu e^{\sqrt{\lambda} (u-v)} \right) \left( \sqrt{\lambda} + \nu \mu e^{\sqrt{\lambda} (u-v)} \right) \left( \sqrt{\lambda} - \mu \right) \]

Hence
\[ \gamma_0 = \frac{\nu \left( \sqrt{\lambda} + \nu \right) \mu e^{\sqrt{\lambda} (u-v)} + \left( \sqrt{\lambda} + \nu \right) \mu e^{-\sqrt{\lambda} u}}{\lambda \left( \sqrt{\lambda} + \nu \right) \mu e^{\sqrt{\lambda} (u-v)} + \left( \sqrt{\lambda} + \nu \right) \mu e^{-\sqrt{\lambda} u}} \]

\[ = \frac{\mu \left( \sqrt{\lambda} + \nu \right) \mu e^{\sqrt{\lambda} (u-v)} + \left( \sqrt{\lambda} + \nu \right) \mu e^{-\sqrt{\lambda} u}}{\lambda \left( \sqrt{\lambda} + \nu \right) \mu e^{\sqrt{\lambda} (u-v)} + \left( \sqrt{\lambda} + \nu \right) \mu e^{-\sqrt{\lambda} u}} \]

Therefore, \( \tilde{B}_0 = -\gamma_0 \left( \frac{1}{0} \right) \), we get
\[ (1 \ 0) N(x) \tilde{B}_0 = \frac{\mu \left( \sqrt{\lambda} + \nu \right) \mu e^{\sqrt{\lambda} (u-v)} + \left( \sqrt{\lambda} + \nu \right) \mu e^{-\sqrt{\lambda} u}}{\lambda \left( \sqrt{\lambda} + \nu \right) \mu e^{\sqrt{\lambda} (u-v)} + \left( \sqrt{\lambda} + \nu \right) \mu e^{-\sqrt{\lambda} u}} \]

Now,
\[ B_1 = S \left( \frac{1}{0} \right) - \gamma_0 \vec{R}_1 \left( \frac{1}{0} \right) = \frac{1}{\lambda^{3/2}} \left( \mu e^{-\sqrt{\lambda} u} - \gamma_0 \left( \sqrt{\lambda} + \mu \right) \right) \]

\[ - \mu e^{-\sqrt{\lambda} u} + \gamma_0 e^{-\sqrt{\lambda} u} \]

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Straightforward computations show that numerators of the entries of $\bar{B}_1$ can be simplified into:

$$\mu e^{-\sqrt{\lambda}u} + \tilde{\gamma}_0 \mu e^{2\sqrt{\lambda}u} = -\frac{\nu \sqrt{\lambda} \left[ \sqrt{\lambda} + \mu (1 - e^{-\sqrt{\lambda}(u-v)}) \right] e^{-\sqrt{\lambda}v}}{(\sqrt{\lambda} + \mu)(\sqrt{\lambda} + \nu) - \mu \nu e^{2\sqrt{\lambda}(u-v)}},$$

$$-\mu e^{\sqrt{\lambda}u} + \tilde{\gamma}_0 (\sqrt{\lambda} - \mu) = -\frac{\nu \sqrt{\lambda} \left[ \sqrt{\lambda} + \nu (1 - e^{-\sqrt{\lambda}(u-v)}) \right] e^{\sqrt{\lambda}u}}{(\sqrt{\lambda} + \mu)(\sqrt{\lambda} + \nu) - \mu \nu e^{2\sqrt{\lambda}(u-v)}},$$

Thus

$$\bar{B}_1 = \frac{1}{\lambda \left[ (\sqrt{\lambda} + \mu)(\sqrt{\lambda} + \nu) - \mu \nu e^{2\sqrt{\lambda}(u-v)} \right]} \left( \frac{\nu \left[ \sqrt{\lambda} + \mu (1 - e^{-\sqrt{\lambda}(u-v)}) \right] e^{-\sqrt{\lambda}v}}{\mu \left[ \sqrt{\lambda} + \nu (1 - e^{-\sqrt{\lambda}(u-v)}) \right] e^{\sqrt{\lambda}u}} \right),$$

and then

$$(1 \ 0) N(x) \bar{B}_1 = - \frac{\nu \left[ \sqrt{\lambda} + \mu (1 - e^{-\sqrt{\lambda}(u-v)}) \right] e^{\sqrt{\lambda}(x-v)} + \mu \left[ \sqrt{\lambda} + \nu (1 - e^{-\sqrt{\lambda}(u-v)}) \right] e^{\sqrt{\lambda}(u-x)}}{\lambda \left[ (\sqrt{\lambda} + \mu)(\sqrt{\lambda} + \nu) - \mu \nu e^{2\sqrt{\lambda}(u-v)} \right]}.$$

Furthermore,

$$\bar{B}_2 = \tilde{S}_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \tilde{\gamma}_0 \tilde{R}_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\lambda^{3/2}} \left( \nu \left[ \sqrt{\lambda} e^{-\sqrt{\lambda}v} - \mu \nu e^{-\sqrt{\lambda}(2v-u)} + (\sqrt{\lambda} + \nu) \mu e^{-\sqrt{\lambda}u} - \mu \nu e^{2\sqrt{\lambda}(u-v)} \right] \right),$$

and then

$$\bar{B}_2 = - \frac{1}{\lambda^{3/2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with

$$\tilde{\delta}_2 = \frac{\mu \left[ \sqrt{\lambda} + \nu (1 - e^{-\sqrt{\lambda}(u-v)}) \right] e^{\sqrt{\lambda}u} + \nu \left[ \sqrt{\lambda} + \mu (1 - e^{-\sqrt{\lambda}(u-v)}) \right] e^{\sqrt{\lambda}v}}{\lambda \left[ (\sqrt{\lambda} + \mu)(\sqrt{\lambda} + \nu) - \mu \nu e^{2\sqrt{\lambda}(u-v)} \right]}.$$

and then

$$(1 \ 0) N(x) \bar{B}_2 = \frac{\mu \left[ \sqrt{\lambda} + \nu (1 - e^{-\sqrt{\lambda}(u-v)}) \right] e^{\sqrt{\lambda}(u-x)} + \nu \left[ \sqrt{\lambda} + \mu (1 - e^{-\sqrt{\lambda}(u-v)}) \right] e^{\sqrt{\lambda}(v-x)}}{\lambda \left[ (\sqrt{\lambda} + \mu)(\sqrt{\lambda} + \nu) - \mu \nu e^{2\sqrt{\lambda}(u-v)} \right]}.$$

As a result, we can see that the iterated Laplace transform of the couple of local times $(L^u_t, L^v_t)$ at $u$ and $v$ can be expressed by the unified formula, for any $x \in \mathbb{R}$,

$$\int_0^\infty e^{-\lambda t} \mathbb{E}_x \left( e^{-\mu L^u_t - \nu L^v_t} \right) dt = \frac{1}{\lambda} \left[ 1 - \frac{\mu \left[ \sqrt{\lambda} + \nu (1 - e^{-\sqrt{\lambda}(u-v)}) \right] e^{-\sqrt{\lambda}(x-u)} + \nu \left[ \sqrt{\lambda} + \mu (1 - e^{-\sqrt{\lambda}(u-v)}) \right] e^{-\sqrt{\lambda}(x-v)}}{(\sqrt{\lambda} + \mu)(\sqrt{\lambda} + \nu) - \mu \nu e^{2\sqrt{\lambda}(u-v)}} \right].$$

We retrieve formula (1.18.1), p. 150 of [I]. Consequently, the iterated Laplace transform of the local time $L^u_t + L^v_t$ in $(u, v)$ is given, for any $x \in \mathbb{R}$, by

$$\int_0^\infty e^{-\lambda t} \mathbb{E}_x \left( e^{-\mu L^u_t - \nu L^v_t} \right) dt = \frac{1}{\lambda} \left[ 1 - \frac{\mu \left[ \sqrt{\lambda} + \nu (1 - e^{-\sqrt{\lambda}(u-v)}) \right] [e^{-\sqrt{\lambda}(x-u)} + e^{-\sqrt{\lambda}(x-v)}]}{(\sqrt{\lambda} + \mu)^2 - \mu \nu e^{2\sqrt{\lambda}(u-v)}} \right].$$

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