Global existence, asymptotic behavior, and pattern formation driven by the parametrization of a nonlocal Fisher-KPP problem

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Abstract

The global boundedness and the hair trigger effect of solutions for the nonlinear nonlocal reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u + \mu u^\alpha (1 - \kappa J * u^\beta), \quad \text{in } \mathbb{R}^N \times (0, \infty), \quad N \geq 1$$

with $\alpha, \beta \geq 1$, $\mu, \kappa > 0$ and $u(x,0) = u_0(x)$ are investigated. Under appropriate assumptions on $J$, it is proved that for any nonnegative and bounded initial condition, if $\alpha \in [1, \alpha^*) \cup [1, \frac{1+\beta}{2}]$ with $\alpha^* = 1 + \beta$ for $N = 1$ and $\alpha^* = 1 + \frac{2\beta}{N}$ for $N \geq 2$, then the problem has a global bounded classical solution. Under further assumptions on the initial datum, the solutions satisfying $0 \leq u(x,t) \leq \kappa^{-\frac{1}{\beta}}$ for any $(x,t) \in \mathbb{R}^N \times [0, +\infty)$ are shown to converge to $\kappa^{-\frac{1}{\beta}}$ uniformly on any compact subset of $\mathbb{R}^N$, which is known as the hair trigger effect. 1D numerical simulations of the above nonlocal reaction-diffusion equation are performed and the effect of several combinations of parameters and convolution...
kernels on the solution behavior is investigated. Namely, for the supercritical case, the unboundedness of the solution is numerically tested. For the subcritical case, i.e. the case for which the boundedness of solutions has been proved, the hair trigger effect is revealed for small $\kappa$ values. For relatively large $\kappa$, take $\kappa = 1$ for example, the hair trigger effect is numerically confirmed for small $\mu$’s, while for relatively large $\mu$ values, different patterns appear with different choices of convolution kernels. These motivate a discussion about some conjectures arising from this model and further issues to be studied in this context.

A formal deduction of the model from a mesoscopic formulation is provided as well.

Mathematical Subject Classifications: 35K65, 35K40.

Keywords: Global boundedness; hair trigger effect; nonlocal reaction-diffusion equation.

1 Introduction

In this work we study the nonlinear nonlocal reaction-diffusion equation$^1$

$$\frac{\partial u}{\partial t} = \Delta u + \mu u^\alpha (1 - \kappa J * u^\beta), \quad (x,t) \in \mathbb{R}^N \times (0, \infty),$$

$$(1.1)$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^N,$$

$$(1.2)$$

where $\alpha, \beta \geq 1$, $\mu, \kappa > 0$, $N \geq 1$, $J(x)$ is a competition kernel with

$$0 \leq J \in L^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} J(x)dx = 1, \quad \text{and} \quad \inf_{B(0,\delta_0)} J > \eta \quad \text{for some} \ \delta_0 > 0, \eta > 0,$$

$$(1.3)$$

where $B(0,\delta_0) = \{x \in \mathbb{R}^N : |x_i| \leq \delta_0, i = 1, 2, \cdots, N\}$ and

$$J * u^\beta(x,t) = \int_{\mathbb{R}^N} J(x-y)u^\beta(y,t)dy.$$

This problem can be seen to characterize the evolution of a population of density $u$, whose individuals are moving by diffusion and interaction. Their interaction modus determines the faith of the population with respect to growth or decay: the reaction term describes the joint influence of a nonlinear growth accounting for a weak Allee effect and of concurrence for available resources (prevention of overcrowding). The latter takes a nonlocal form; several individuals interact in a space/phenotypic trait/etc. domain, thereby sampling all occupancy information therein. Such problems arise e.g., when modeling emergence and evolution of a biological

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$^1$Here we consider the equation to be already nondimensionalized.
species \([6, 12, 13, 17, 39, 45]\). Thereby the respective population is structured by a phenotypical trait and its individuals infer two essential interactions: mutation and selection. In this context \(u(x,t)\) represents the density of a population having phenotype \(x\) at time \(t\). The mutation process, which acts as a diffusion operator on the trait space, is modeled by a classical diffusion operator, whereas the selection process is described by the nonlocal term \(u^\alpha(1 - J * u^\beta)\). Similar nonlocal reaction terms also occur in describing natural selection of cancer cells, which leads to the emergence of therapy-resistant clones \([35, 36]\).

Equation (1.1) is a particular case of a more general monospecies setting which can be deduced in various ways. The perhaps simplest one (see e.g. \([10]\)) starts from the local reaction-diffusion equation

\[
    u_t = D \Delta u + r(u)u 
\]

and lets the growth rate \(r(u)\) depend not only on the population density at a certain location \(x\), but also at the other points in some domain of interest (which can be the whole space):

\[
    r(u) = ru^{\alpha-1}(1 - J * u^\beta). 
\]

Indeed, a spatially heterogeneous population can exceed locally its carrying capacity, which in the usual Fisher equation (logistic growth) would simply lead to decay. By letting, however, the population use resources/signals available at more or less proximal sites such decay is not necessary to happen. It is known, for instance, that cells in a tissue are able to communicate with each other by way of thin protrusions (lamellipodia, filopodia, cytonemes, nanotubes) which can reach at long distances with respect to the cell size, see \([25, 43]\) and references therein. Moreover, clustering together or organizing in groups may even provide advantages, depending on the competition strength; this can apply to cells \([32]\), but also to animals cooperating for hunt, associating in schools or swarms, or simply undergoing sexual reproduction (case \(\alpha = 2\)).

Another way to obtain a PDE of the type (1.1), with or without diffusion, is by relying on individual-based formulations involving stochastic processes and performing some appropriate upscaling, see e.g. \([14]\); we also refer to \([29]\) for an instance of deducing a reaction-diffusion system with nonlocality in the reaction terms by using master equations and mean field limits.

Yet another approach \([5]\) uses kinetic transport equations to derive by a hydrodynamic limit PDEs for which an equation of the form (1.1) is a particular case. We shortly illustrate in the Appendix its concrete application.
When the interaction kernel $J$ in (1.1) is replaced by the Dirac delta function the so called generalized Fisher-KPP equation is obtained as a local reaction-diffusion equation. In [22] and [31], the Fisher-KPP equation

$$\frac{\partial u}{\partial t} = \Delta u + u(1 - u)$$

was introduced to model the spreading of some advantageous gene in a population. It is well known that any solution $u(x,t)$ with a nonnegative and nontrivial initial data, tends to 1 as $t \to \infty$, locally uniformly in $x \in \mathbb{R}^N$. This is referred to as the hair trigger effect [3]. When accounting for a weak Allee effect the above equation becomes

$$\frac{\partial u}{\partial t} = \Delta u + u^\alpha(1 - u)$$

for $\alpha > 1$. As stated in [37], an immediate difficulty arises when trying to apply standard comparison methods, since the equilibrium $u = 0$ is degenerate. It turns out that the dynamics of solutions is much more complicated and interesting than that for $\alpha = 1$. In [3], Aronson and Weinberger showed that for $N \geq 2$, the hair trigger effect remains valid as long as $1 \leq \alpha \leq 1 + \frac{2}{N}$, whereas some small initial data may lead to extinction, or quenching, when $\alpha > 1 + \frac{2}{N}$.

For $J(x) = 1$, which corresponds to the situation of blind competition, with general $\alpha \geq 1$ and $\beta \geq 1$, the problem has been studied in [8, 7] in terms of the existence of solutions both in bounded and unbounded domains, respectively. Moreover, from the analysis of [16], the positive solution of

$$\frac{\partial u}{\partial t} = \Delta u + u \left( 1 - \int_{\mathbb{R}^N} u(t,y)dy \right)$$

converges uniformly to 0, which is actually the only non-negative stationary solution.

For $J$ satisfying (1.3), the consumption of resources at the space/phenotypic trait point $x$ depends on individuals located in some area around this point. As stated in [1], introducing nonlocal intraspecific competition for resources changes the properties of solutions of this equation. Some progress has recently been attained in this direction for the so called nonlocal Fisher-KPP equation

$$\frac{\partial u}{\partial t} = \Delta u + \mu u(1 - J * u), \quad (1.4)$$

for which $u = 1$ is a stationary solution. The latter is stable in the case of the local equation, but it can lose its stability for the nonlocal one. If it becomes unstable, then a periodic in space stationary solution bifurcates from it [11, 24, 26]. This phenomenon is observed in the study of
travelling wave solutions. If the Fourier transform of $J$ is everywhere positive or if $\mu$ is small enough, then it is known that travelling waves necessarily connect 0 to 1 (see [2, 6, 21, 27, 38]), while if $\mu$ is large, then $u = 1$ can indeed become unstable and Turing patterns appear [38, 40]. Similar results were obtained for the bistable case

$$\frac{\partial u}{\partial t} = \Delta u + \mu u^2(1 - J * u) - du,$$  \hspace{1cm} (1.5)

in our previous work [33], where $-du$ is the mortality term and $d$ is the death rate. For the long time behavior of solutions, in a recent work [42], it was proved that for the nonlocal Fisher-KPP equation (1.4) and under the assumption

$$\forall f \in L^2(\Omega), \quad \int_{\Omega \times \Omega} J(x - y)f(x)f(y)dxdy \geq 0,$$

the solution converges to 1 uniformly in a bounded domain $\Omega$. Moreover, [16] considered (1.1) for the case $\alpha = 1$ and $\beta \geq 1$ on a bounded domain with a Neumann boundary condition. For $J(x - y) = J_0(y) + \varepsilon J_1(x - y)$ and upon relying on non-linear relative entropy identities and an orthogonal decomposition, it was proved that for a small $\varepsilon$, there exists a unique steady state which is positively asymptotically stable. For unbounded domains, however, whether the hair trigger effect will occur or not for such nonlocal problems is an issue far from obvious, as also mentioned in [16, 42].

In this paper, depending on the balance between the weak Allee effect and the overcrowding avoidance effect, we find sufficient conditions for the global boundedness of solutions for (1.1) and the hair trigger effect in long time behavior. The solution bounds are influenced (in a certain way) by the source strength $\mu$ and/or the competition parameter $\kappa$. The main results of this paper are the following.

**Theorem 1.1.** Suppose $\alpha \in [1, \alpha^*) \cup [1, \frac{\beta+1}{2}]$ with

$$\alpha^* = \begin{cases} 
1 + \beta, & N = 1, 2, \\
1 + \frac{2\beta}{N}, & N > 2,
\end{cases}$$

and (1.3) holds. Then for every initial data $0 \leq u_0 \in L^\infty(\mathbb{R}^N)$, the nonnegative solution of (1.1)–(1.2) exists and is globally bounded in time, that is, there exists $M > 0$ such that

$$0 \leq u(x,t) \leq M, \quad \forall \ (x,t) \in \mathbb{R}^N \times [0, +\infty).$$  \hspace{1cm} (1.6)

Furthermore, we have:
\(\sum\)ption

In Theorem 1.1, we obtain the global boundedness of solutions under the assumption

\[ N = 1, 2, \]

where \(s = +\infty\) for \(N = 1, 2\) and \(s = \frac{2N}{N - 2}\) for \(N > 2\), and \(G(s, N)\) is the constant that appears in Sobolev’s inequality, then for any \(K > 1\), there exists \(\mu^* > 0\) such that for \(\mu \in (0, \mu^*)\), (1.6) holds with \(M = K \max \{1, \|u_0\|_{L^\infty(\mathbb{R}^N)}\}\).

(ii) If \(\beta > 1\) and \(\max\{1, \frac{(N+2\beta)(N-2)}{N^2+4}\}\) \(\leq \alpha < \alpha^{**}\) with

\[
\alpha^{**} = \begin{cases}
\frac{\beta+1}{2}, & N = 1, 2, \\
\frac{N+2\beta}{N+2}, & N > 2,
\end{cases}
\]

then there exists \(\alpha^{**} > 0\) such that for any \(\kappa \in (0, \alpha^{**})\), (1.6) holds with \(M = \kappa^{-\frac{1}{\beta}}\).

**Remark 1.1.** In Theorem 1.1, we obtain the global boundedness of solutions under the assumption \(\alpha \in [1, \alpha^*] \cup [1, \frac{\beta+1}{2}]\). Besides, for small steady states, which correspond to “large” \(\kappa\) values, a “quasi”-maximum principle holds in the case of weak sources, i.e. of small \(\mu\)’s. Moreover, under the assumption \(\max\{1, \frac{(N+2\beta)(N-2)}{N^2+4}\}\) \(\leq \alpha < \alpha^{**}\), a maximum principle holds in the case of large steady states, i.e. for small \(\kappa\)’s, without any requirement about the source strength.

**Theorem 1.2.** Let \(u(x, t)\) be a global solution of (1.1)–(1.2) with 0 \(\leq u(x, t) \leq \kappa^{-\frac{1}{\beta}}\) for any \((x, t) \in \mathbb{R}^N \times [0, +\infty)\). If in addition \(\int_{B(x, \delta)} u_0^{-\alpha}(s) ds \in L^\infty(\mathbb{R}^N)\) holds for some \(\delta > 0\), then

\[
\lim_{t \to \infty} u(x, t) = \kappa^{-\frac{1}{\beta}}
\]

locally uniformly in \(\mathbb{R}^N\).

**Remark 1.2.** The hair trigger effect in Theorem 1.2 is proved for those solutions staying between 0 and \(\kappa^{-\frac{1}{\beta}}\). The last result in Theorem 1.2 shows that such solutions exist. Furthermore, the condition \(\int_{B(x, \delta)} u_0^{-\alpha}(s) ds \in L^\infty(\mathbb{R}^N)\) on the initial data is necessary for the hair trigger effect, at least for the case \(\max\{1 + \frac{2}{N}, \frac{(N+2\beta)(N-2)}{N^2+4}\}\) \(\leq \alpha < \alpha^{**}\), with \(\beta > 1 + \frac{2}{N}\) for \(N = 1, 2\) and \(\beta > 2 + \frac{2}{N}\) for \(N > 2\). In fact, for any \(\alpha > 1 + \frac{2}{N}\), the choice of small initial data decaying at infinity leads to extinction of solutions. Indeed, from results of Fujita [23], denote

\[
\omega(x, t; d, \alpha) := \left(\frac{N}{2} - \frac{1}{\alpha - 1}\right)^{1/(\alpha - 1)} \left(\frac{\mu t + d}{\mu t + d}\right)^{1/(\alpha - 1)} e^{-\frac{\mu}{\mu t + d}},
\]
where $\alpha > 1 + \frac{2}{N}$, $d > 0$. Then

$$\lim_{t \to +\infty} \omega(x, t; d, \alpha) = 0$$

uniformly for $x \in \mathbb{R}^N$ and

$$\frac{\partial \omega}{\partial t} - \Delta \omega - \mu \omega^\alpha \geq 0 \text{ in } \mathbb{R}^N \times \mathbb{R}^+,$$

while

$$\frac{\partial u}{\partial t} - \Delta u - \mu u^\alpha \leq 0 \text{ in } \mathbb{R}^N \times \mathbb{R}^+.$$ 

If the initial data satisfy $0 \leq u_0(x) \leq \omega(x, 0; d, \alpha)$ for some suitable value of $d$, then by the comparison principle for parabolic equations we have $0 \leq u(x, t) \leq \omega(x, t; d, \alpha)$ in $\mathbb{R}^N \times \mathbb{R}^+$ and $u(x, t) \to 0$ as $t \to +\infty$ uniformly in $\mathbb{R}^N$. That is, for $\alpha > 1 + \frac{2}{N}$, a uniform lower bound for initial data is necessary to get the hair trigger effect.

**Remark 1.3.** The results obtained in this paper by using the mentioned localized technique solve the problems addressed in [16, 42] to obtain results on $\mathbb{R}^N$.

Next we summarize the main methods used in this paper. To prove the global boundedness, the main difficulty comes from the degeneracy of the source and the nonlocal effect of the competition. Here we introduce a localization method motivated by the ideas from [6].

Let $\delta_0, \eta > 0$ be as in (1.3) and $x \in \mathbb{R}^N$ fixed. For any $0 < \delta \leq \frac{1}{2} \delta_0$, we obtain (see (2.3)) for $p > 1$ that

$$\frac{\partial}{\partial t} \int_{B(x, \delta)} u^p dy + p\mu \int_{B(x, \delta)} u^p dy \leq \Delta \int_{B(x, \delta)} u^p dy - \frac{4(p-1)}{p} \int_{B(x, \delta)} |\nabla u|^2 dy$$

$$+ p\mu \int_{B(x, \delta)} u^p dy + p\mu \int_{B(x, \delta)} u^{p+\alpha-1} dy \left(1 - \kappa \eta \int_{B(x, \delta)} u^\beta dy\right).$$

For the case $1 \leq \alpha < \alpha^*$, the idea is to use diffusion and nonlocal reaction to control the nonlinear growth of the reaction, where $\alpha^*$ comes directly from the Sobolev embedding theorem. We can derive

$$\int_{B(x, \delta)} u^p dy + \int_{B(x, \delta)} u^{p+\alpha-1} dy \leq \frac{2(p-1)}{p^2 \mu} \int_{B(x, \delta)} |\nabla u|^2 dy + \frac{\kappa \eta}{2} \int_{B(x, \delta)} u^{p+\alpha-1} dy \int_{B(x, \delta)} u^\beta dy + C.$$ 

As it is well-known that an $L^p$ estimate will not directly provide by itself an $L^\infty$ evaluation, a modified Moser type estimate is needed here. More precisely, we define the sequence $p_k := b^k + h$.
with \( h \) carefully chosen and \( k \geq m \), where \( m \) indexes the initial step of the iteration. Then (1.8) gives the global uniform boundedness of \( \|u\|_{L^p(B(x,\delta))} \). The \( L^\infty \) estimate of \( u \) on \( \mathbb{R}^N \times [0, +\infty) \) will be obtained by an iterative procedure after taking \( k \to \infty \).

At this moment, we will keep the free choice of \( b \) in order to obtain a maximum estimate. Our goal is to look for the possible parameters \( \mu \) and \( \kappa \) so that the solution can be estimated pointwisely by \( \max\{1, \|u_0\|_{L^\infty(\mathbb{R}^N)}\} \) or \( \max\{\kappa^{-\beta}, \|u_0\|_{L^\infty(\mathbb{R}^N)}\} \). In the end, two cases can be handled. Namely, in the case when the equation has a small constant stationary state, a “quasi” maximum estimate can be obtained for a small source, while for the case with a large constant stationary state, the maximum estimate can be achieved for arbitrary sources. The latter shows that the assumption for the hair trigger effect result is valid.

For the case \( 1 \leq \alpha \leq \frac{1}{2}(\beta + 1) \), by choosing \( p = \alpha \) in (1.8), we can derive the global uniform boundedness of the term

\[
\int_{B(x,\delta)} u^\alpha dy + \int_{B(x,\delta)} u^{2\alpha - 1} dy \left( 1 - \kappa \eta \int_{B(x,\delta)} u^\beta dy \right),
\]

which together with the comparison principle imply the global uniform boundedness of \( \|u\|_{L^\alpha(B(x,\delta))} \) and \( \|u\|_{L^1(B(x,\delta))} \). Then by the formula

\[
u(x, t) \leq \int_{\mathbb{R}^N} e^{-\frac{|y|^2}{4(t-t_0)}} u(x - y, t - 1) dy + \mu \int_{t-1}^t \int_{\mathbb{R}^N} e^{-\frac{|y|^2}{4(t-\tau)}} u^\alpha(x - y, \tau) dy d\tau,
\]

one can write the integral on the whole space as a summation of integrals on boxes. With the boundedness of these two integrals and the space decay due to the heat kernel, one can obtain the global boundedness of \( u \) itself.

Furthermore, for the long time behavior of the solution, Weinberger’s classical result [3] fails, as the comparison principle does not hold. The problem with nonlocal competition requires a new method. Our approach relies on localized estimates and the fact that if \( 0 \leq u(x, t) \leq \kappa^{-\frac{\beta}{2}} \) for \( (x, t) \in \mathbb{R}^N \times [0, +\infty) \), then

\[
\int_{B(x,\delta)} \int_{\mathbb{R}^N} (u^\beta(z, t) - \kappa^{-1})(\kappa^{-1} - u^\beta(y, t)) J(z - y) dz dy \leq - (1 - \varepsilon) \eta (2\delta)^N \int_{B(x,\delta)} (\kappa^{-1} - u^\beta(z, t))^2 dz + \frac{2}{3(\kappa^{2+(\alpha-1)/\beta})} C(\varepsilon) \beta^2 (\kappa^{-1} - u^\beta(y, t)) |\nabla u(y, t)|^2 dy
\]

with \( 0 < \varepsilon < 1 \). We introduce a nonnegative functional \( F(x, t) \) with \( F(x, 0) \in L^\infty(\mathbb{R}^N) \), which
is shown to satisfy
\[
\frac{\partial}{\partial t} F(x,t) \leq \Delta F(x,t) - C \int_{B(x,\delta)} \left( \kappa^{-1} - u^\beta(y,t) \right)^2 \, dy. \tag{1.9}
\]
Then for any \( x \in \mathbb{R}^N \), as \( t \to \infty \), \( \int_{B(x,\delta)} \left( \kappa^{-1} - u^\beta(y,t) \right)^2 \, dy \to 0 \), which implies that \( u \to \kappa^{-\frac{1}{\beta}} \) as \( t \to \infty \) locally uniformly in \( \mathbb{R}^N \).

The structure of the paper is the following. Sections 2 and 3 are devoted to prove Theorems 1.1 and 1.2, respectively. Numerical simulations together with a discussion of the results are presented in Section 4. A formal derivation of the model from a mesoscopic setting in the framework of kinetic transport equations is provided in the attachment.

## 2 Global existence of a solution

The global existence of a solution will be obtained by a local wellposedness result and a uniform in time \( L^\infty \) estimate.

The local existence, uniqueness, and nonnegativity of solutions for this parabolic problem can be readily obtained upon using the maximum principle and standard parabolic estimates. This issue has been extensively studied for many different problems, see [23] for an example. For convenience, we list the corresponding result below.

**Proposition 2.1.** Assume the initial data \( 0 \leq u_0(x) \in L^\infty(\mathbb{R}^N) \). Then there is a maximal existence time \( T_{\text{max}} \in (0, \infty) \) and \( u \in C([0,T_{\text{max}}), L^\infty(\mathbb{R}^N)) \cap C^{2,1}(\mathbb{R}^N \times (0,T_{\text{max}})) \) such that \( u \) is the unique non-negative classical solution of problem (1.1)–(1.2). Furthermore, if \( T_{\text{max}} < +\infty \), then

\[
\lim_{t \to T_{\text{max}}} \| u(\cdot,t) \|_{L^\infty(\mathbb{R}^N)} = \infty.
\]

The following extended version of the Gagliardo–Nirenberg inequality will be used in our later analysis.

**Lemma 2.1.** (see [41]) Let \( \Omega \) be an open subset of \( \mathbb{R}^N \). Assume that \( r \in (0,p) \), \( 1 \leq p, q \leq \infty \), and \( (N-q)p < Nq \). Then for any \( \psi \in W^{1,q}(\Omega) \cap L^r(\Omega) \), there exists a constant \( C_{GN} > 0 \) only depending on \( N, q, r \) and \( \Omega \) such that

\[
\| \psi \|_{L^p(\Omega)} \leq C_{GN} \left( \| \nabla \psi \|_{L^r(\Omega)}^{\lambda^*_r} \| \psi \|_{L^q(\Omega)}^{1-\lambda^*_r} + \| \psi \|_{L^r(\Omega)} \right),
\]
holds with \( \lambda^* \in (0, 1) \) satisfying
\[
\lambda^* = \frac{N_q - N_p}{1 - N_q + N_p}.
\]

The following result is an application of Ghidaglia’s lemma (see [44], Lemma 5.1). It is a generalized version of Lemma 4.1 in [9]. For completeness, we also give the proof.

**Lemma 2.2.** Assume \( y_k(t) \geq 0, k = 0, 1, 2, \ldots \) are \( C^1 \) functions for \( t > 0 \), satisfying
\[
y_k'(t) + c_k y_k(t) \leq c_k A_k \max\{1, \sup_{t \geq 0} y_{k-1}^b(t)\}, \tag{2.1}
\]
where \( c_k > 0, A_k = \overline{a} b^D_k \geq 1 \) with \( \overline{a}, D, b > 1 \) being positive constants. Assume also that there exists a constant \( K > 0 \) such that \( y_k(0) \leq K^{\ell b} \). Then for all \( m \geq 1 \), it holds
\[
y_k(t) \leq (2\pi)^{\frac{b^{k-m+1}}{b-1}} b^D \left( \frac{\lambda(k-m-1)}{(b-1)^2} + \frac{mb^{k+1-m}}{b-1} \right) \max\{\sup_{t \geq 0} y_{m-1}^{b^{k-m+1}}(t), K^{\ell b}, 1\}.
\]

**Proof.** From (2.1) we obtain
\[
(e^{c_k t} y_k(t))' \leq c_k A_k e^{c_k t} \max\{1, \sup_{t \geq 0} y_{k-1}^b(t)\}
\]
and then
\[
y_k(t) \leq (1 - e^{-c_k t}) A_k \max\{1, \sup_{t \geq 0} y_{k-1}^b(t)\} + e^{-c_k t} y_k(0)
\]
\[
\leq 2A_k \max\{1, \sup_{t \geq 0} y_{k-1}^b(t), y_k(0)\}
\]
\[
\leq 2A_k \max\{\sup_{t \geq 0} y_{k-1}^b(t), K^{\ell b}, 1\}.
\]

By an iterative procedure we obtain
\[
y_k(t) \leq 2A_k (2A_{k-1})^{\frac{b}{b}} (2A_{k-2})^{\frac{b}{b}} (2A_{k-3})^{\frac{b}{b}} \cdots (2A_m)^{\frac{b}{b}} \max\{\sup_{t \geq 0} y_{m-1}^{b^{k-m+1}}(t), K^{\ell b}, 1\}
\]
\[
= (2\pi)^{\sum_{i=0}^{m-k-1} b^i} b^D \sum_{i=0}^{m-k-1} (k-i)^{b^i} \max\{\sup_{t \geq 0} y_{m-1}^{b^{k-m+1}}(t), K^{\ell b}, 1\}
\]
\[
= (2\pi)^{\frac{b^{k+m+1}}{b-1}} b^D \left( \frac{\lambda^{(k-m-1)}}{(b-1)^2} + \frac{mb^{k+1-m}}{b-1} \right) \max\{\sup_{t \geq 0} y_{m-1}^{b^{k-m+1}}(t), K^{\ell b}, 1\}.
\]

Now we are ready to proceed with the global \( L^\infty \) estimates.

**Proof of Theorem 1.1.** Fix \( x_0 \triangleq (x_0^0, x_0^1, \ldots, x_0^n) \in \mathbb{R}^N \), choose \( 0 < \delta \leq \frac{1}{2} \delta_0 \), and denote \( B(x_0, \delta) \triangleq \{x \triangleq (x_1, x_2, \ldots, x_n) \in \mathbb{R}^N \mid |x_i - x_i^0| \leq \delta, i = 1, 2, \ldots, N\} \) with \( |B(x_0, \delta)| = (2\delta)^N\).
For any \(x \in \mathbb{R}^N\), multiply (1.1) by \(pu^{p-1}\varphi_\varepsilon\) with \(p \geq 1\), \(\varphi_\varepsilon \in C_0^\infty(B(x, \delta))\), and \(\varphi_\varepsilon(y) \to 1\) locally uniformly in \(B(x, \delta)\) as \(\varepsilon \to 0\). Integrating by parts over \(B(x, \delta)\) we obtain

\[
\frac{\partial}{\partial t} \int_{B(x, \delta)} u^p \varphi_\varepsilon dy + \frac{4(p-1)}{p} \int_{B(x, \delta)} |\nabla u^\frac{p}{2}|^2 \varphi_\varepsilon dy = \int_{B(x, \delta)} \Delta u^p \varphi_\varepsilon dy + p\mu \int_{B(x, \delta)} u^{p+\alpha-1}(1 - \kappa J * u^\beta) \varphi_\varepsilon dy. \tag{2.2}
\]

Taking \(\varepsilon \to 0\), we obtain

\[
\frac{\partial}{\partial t} \int_{B(x, \delta)} u^p dy + \frac{4(p-1)}{p} \int_{B(x, \delta)} |\nabla u^\frac{p}{2}|^2 dy = \int_{B(x, \delta)} \Delta u^p dy + p\mu \int_{B(x, \delta)} u^{p+\alpha-1}dy \left(1 - \eta \kappa \int_{B(x, \delta)} u^\beta dy\right), \tag{2.3}
\]

where we have used the fact that for any \(y \in B(x, \delta)\), \(J(z - y) \geq \eta\) for \(z \in B(y, 2\delta)\) and \(B(x, \delta) \subset B(y, 2\delta)\), then

\[
J * u^\beta(y, t) \geq \eta \int_{B(y, 2\delta)} u^\beta(z, t) dz \geq \eta \int_{B(x, \delta)} u^\beta(y, t) dy
\]

and

\[
\int_{B(x, \delta)} \Delta u^p dy = \int_{B(0, \delta)} \Delta u^p(y + x, t) dy = \Delta \int_{B(0, \delta)} u^p(y + x, t) dy = \Delta \int_{B(x, \delta)} u^p dy.
\]

Now we proceed to estimate the term \(\int_{B(x, \delta)} u^{p+\alpha-1} dy\) in two cases.

**Case 1:** \((1 \leq \alpha < \alpha^*)\). The proof in this case is divided into four steps, namely: the \(L^p\) estimates, the \(L^\infty\) estimates, the "quasi"-maximum principle when the equation has a small constant steady state and a small source, and the maximum principle when the equation has a large constant steady state.

**Step 1. \(L^p\) estimates.** Firstly by Hölder’s inequality and the Sobolev embedding theorem, for

\[
p > \max\left\{\left(\frac{N}{2} - 1\right)(\alpha - 1), 1\right\} \tag{2.4}
\]

and

\[
\max\left\{\frac{N(\alpha - 1)}{p}, \frac{2(\alpha - 1)}{p}, 1\right\} < r < \min\left\{\frac{2(p + \alpha - 1)}{p}, s\right\} \tag{2.5}
\]
with
\[
s = \begin{cases} 
+\infty, & N = 1, 2, \\
\frac{2N}{N-2}, & N > 2, 
\end{cases}
\]  

we get
\[
\int_{B(x,\delta)} u^{p+\alpha-1} dy = \|u^\frac{p}{p-1}\|_{L^\frac{2(p+\alpha-1)}{p-1}(B(x,\delta))}^{2(p+\alpha-1)} \leq \|u^\frac{p}{p-1}\|_{L^\frac{2(1-\lambda)(p+\alpha-1)}{p}(B(x,\delta))}^{2(1-\lambda)(p+\alpha-1)} \leq \left(S(N, \delta)\|u^\frac{p}{p-1}\|_{W^{1,2}(B(x,\delta))}\right)^\frac{2(1-\lambda)(p+\alpha-1)}{p} \|u^\frac{p}{p-1}\|_{L^\frac{2(1-\lambda)(p+\alpha-1)}{p}(B(x,\delta))}^{2(1-\lambda)(p+\alpha-1)} \leq \left(2S(N, \delta)\|\nabla u^\frac{p}{p-1}\|_{L^2(B(x,\delta))}\right)^\frac{2(1-\lambda)(p+\alpha-1)}{p} \|u^\frac{p}{p-1}\|_{L^\frac{2(1-\lambda)(p+\alpha-1)}{p}(B(x,\delta))}^{2(1-\lambda)(p+\alpha-1)} + \left(2S(N, \delta)\|u^\frac{p}{p-1}\|_{L^2(B(x,\delta))}\right)^\frac{2(1-\lambda)(p+\alpha-1)}{p} \|u^\frac{p}{p-1}\|_{L^\frac{2(1-\lambda)(p+\alpha-1)}{p}(B(x,\delta))}^{2(1-\lambda)(p+\alpha-1)}
\]

where
\[
S(N, \delta) = \sqrt{2} \max\{(2\delta)^N(\frac{1}{2} - \frac{1}{p}), (2\delta)^{1-\frac{N}{2}} + \frac{N}{2}\} G(s, N)
\]
is the Sobolev embedding constant (see Theorem 2.1 and Theorem 3.1 in [46]) and
\[
\lambda = \begin{cases} 
1 - \frac{pr}{2(p+\alpha-1)}, & N = 1, 2, \\
\frac{N}{p} - \frac{2(p+\alpha-1)}{1-\frac{N}{2}} + \frac{N}{2}, & N > 2,
\end{cases}
\]  

then it is easy to verify from (2.4) and (2.5) that
\[
\lambda \in (0, 1), \quad 2\lambda(p + \alpha - 1)/p \in (0, 2).
\]  

On the other hand, by Poincaré’s inequality, we have
\[
\|u^\frac{p}{p-1} - \overline{u}\|_{L^2(B(x,\delta))} \leq P(N, \delta)\|\nabla u^\frac{p}{p-1}\|_{L^2(B(x,\delta))},
\]
where \(P(N, \delta) = C(N)\delta\) is the Poincaré constant (see Theorem 3.3 in [46]) and \(\overline{u}\) represents the average of \(u\) over \(B(x, \delta)\). Then
\[
\|u^\frac{p}{p-1}\|_{L^2(B(x,\delta))} \leq \|\overline{u}\|_{L^2(B(x,\delta))} + P(N, \delta)\|\nabla u^\frac{p}{p-1}\|_{L^2(B(x,\delta))} = \|u^\frac{p}{p-1}\|_{L^1(B(x,\delta))}\|B(x, \delta)\|^{-\frac{1}{2}} + P(N, \delta)\|\nabla u^\frac{p}{p-1}\|_{L^2(B(x,\delta))} \leq \|u^\frac{p}{p-1}\|_{L^r(B(x,\delta))}\|B(x, \delta)\|^{\frac{r}{2} - \frac{1}{2}} + P(N, \delta)\|\nabla u^\frac{p}{p-1}\|_{L^2(B(x,\delta))}.
\]  

Inserting (2.10) into (2.7) we obtain
\[
\int_{B(x,\delta)} u^{p+\alpha-1} dy
\]
\[
\begin{align*}
&\leq \left(2S(N, \delta)\|\nabla u^2\|_{L^2(B(x, \delta))}\right)^{\frac{2\lambda(p+\alpha-1)}{p}} \|u^2\|_{L^p(\delta)}^{\frac{2(1-\lambda)(p+\alpha-1)}{p}} + 2\left[2S(N, \delta)\left(\|u^2\|_{L^p(\delta)} B(x, \delta)\right)^{\frac{1}{p}-\frac{1}{\beta}} + P(N, \delta)\|\nabla u^2\|_{L^2(B(x, \delta))}\right)\right]^{\frac{2\lambda(p+\alpha-1)}{p}} \|u^2\|_{L^p(\delta)}^{\frac{2(1-\lambda)(p+\alpha-1)}{p}} \\
&\leq 2\left(C_1(N, \delta)\|\nabla u^2\|_{L^2(B(x, \delta))}\right)^{\frac{2\lambda(p+\alpha-1)}{p}} \|u^2\|_{L^p(\delta)}^{\frac{2(1-\lambda)(p+\alpha-1)}{p}} + \left(C_2(N, \delta, r)\|u^2\|_{L^p(\delta)}^{\frac{2(p+\alpha-1)}{p}}, \right),
\end{align*}
\]

where

\[
C_1(N, \delta) = 2S(N, \delta)(1 + 2P(N, \delta)), \quad C_2(N, \delta, r) = 4S(N, \delta)(2\delta)^{\frac{N(r-2)}{2r}}.
\]

Denote

\[
Q := \frac{2(1-\lambda)(p+\alpha-1)}{p - \lambda(p+\alpha-1)}.
\]

Then from (2.9) we have

\[
Q = 2 \left(1 + \frac{\alpha - 1}{p - \lambda(p+\alpha-1)}\right) \geq \frac{2(p+\alpha-1)}{p},
\]

which together with Young’s inequality leads to

\[
\begin{align*}
&2\left(C_1(N, \delta)\|\nabla u^2\|_{L^2(B(x, \delta))}\right)^{\frac{2\lambda(p+\alpha-1)}{p}} \|u^2\|_{L^p(\delta)}^{\frac{2(1-\lambda)(p+\alpha-1)}{p}} + \left(C_2(N, \delta, r)\|u^2\|_{L^p(\delta)}^{\frac{2(p+\alpha-1)}{p}}, \right) \\
&\leq \frac{p-1}{\mu p^2}\|\nabla u^2\|_{L^2(B(x, \delta))}^2 + C_3(p, \mu, N, \delta)\|u^2\|_{L^p(\delta)}^Q + \left(C_2(N, \delta, r)\|u^2\|_{L^p(\delta)}^{\frac{2(p+\alpha-1)}{p}}, \right) \\
&\leq \frac{p-1}{\mu p^2}\|\nabla u^2\|_{L^2(B(x, \delta))}^2 + \left(C_3(p, \mu, N, \delta) + C_2\left(2\lambda(p+\alpha-1)\frac{2(p+\alpha-1)}{p}, \right)\|u^2\|_{L^p(\delta)}^Q, \right)
\end{align*}
\]

where \(C_3(p, \mu, N, \delta) = 2\left(\frac{2\mu p^2C_1(N, \delta)}{p-1}\right)^{\lambda(p+\alpha-1)/p-\lambda(p+\alpha-1)}.\)

Next we proceed to estimate the term \(\|u^2\|_{L^p(\delta)}^Q.\) Notice that for \(p > \max\{\beta - \alpha + 1, 1\},\)

\[
\|u^2\|_{L^p(\delta)}^Q \leq \|u^2\|_{L^\infty(B(x, \delta))}^\theta \|u^2\|_{L^p(B(x, \delta))}^{(1-\theta)Q} \\
= \left(\|u^2\|_{L^\infty(B(x, \delta))}^{\frac{2}{\beta}}, \|u^2\|_{L^p(B(x, \delta))}^{\frac{2(p+\alpha-1)}{p}}, \right) \|u^2\|_{L^p(B(x, \delta))}^{(1-\theta)Q - (p+\alpha-1)\theta Q/\beta} = 0,
\]

where \(\theta = \frac{\beta}{p+\alpha - 1 + \beta}.\) Choose

\[
r = \frac{p + \alpha - 1 + \beta}{p},
\]

which obviously satisfies (2.5) and then \(\theta = \frac{\beta}{p+\alpha - 1 + \beta}.\) Then we have

\[
(1-\theta)Q - (p+\alpha - 1)\theta Q/\beta = 0.
\]
and
\[ \|u^p\|_{L^q(B(x,\delta))}^Q \leq \left( \|u^p\|_{L^\frac{2p}{p+\alpha-1}(B(x,\delta))}^{\frac{2p}{p+\alpha-1}} \|u^{\frac{2(p+\alpha-1)}{p}}\|_{L^\frac{2(p+\alpha-1)}{p}(B(x,\delta))}^{\frac{2(p+\alpha-1)}{p}} \right)^{\frac{Q\theta p}{2\beta}}. \] (2.17)

By using the definition of \(\lambda\) in (2.8) and the above choices of \(r\) and \(\theta\), we have that
\[
Q\theta p \frac{2\beta}{2} = \begin{cases} 
\frac{p}{p+\beta+1-\alpha}, & N = 1, 2 \\
\frac{p+\alpha-1-N(\alpha-1)}{p+\alpha-1-N(\alpha-1)+\beta}, & N > 2.
\end{cases}
\] (2.18)

Under the assumption
\[ 1 \leq \alpha < \alpha^* = \begin{cases} 
1 + \beta, & N = 1, 2 \\
1 + \frac{2\beta}{N}, & N > 2.
\end{cases} \]

(2.18) implies
\[ Q\theta p \frac{2\beta}{2} < 1. \]

Furthermore, from (2.17) and Young’s inequality, we have
\[
(C_3(p, \mu, N, \delta) + C_2 \frac{2(p+\alpha-1)}{p} (N, \delta, r))\|u^p\|_{L^q(B(x,\delta))}^Q \leq \frac{\eta\kappa}{4} \|u^p\|_{L^\frac{2p}{p+\alpha-1}(B(x,\delta))}^{\frac{2p}{p+\alpha-1}} + C_4(p, \mu, N, \delta, r, \eta, \kappa),
\]
(2.19)

where \(C_4(p, \mu, N, \delta, r, \eta, \kappa) = \left(C_3(p, \mu, N, \delta) + C_2 \frac{2(p+\alpha-1)}{p} (N, \delta, r)\right)^\frac{2\beta}{2\beta-\frac{Q^p}{2}} \left(\frac{4}{\eta\kappa}\right)^\frac{Q^p}{2\beta-\frac{Q^p}{2}}, \) where \(\eta\) is the constant in (1.3). Then from (2.11), (2.14) and (2.19), we obtain
\[
2p\mu \int_{B(x,\delta)} u^{p+\alpha-1} dy \leq \frac{2(p-1)}{p} \int_{B(x,\delta)} |\nabla u^p|^2 dy + \frac{\eta\mu\kappa}{2} \int_{B(x,\delta)} u^{p+\alpha-1} dy \int_{B(x,\delta)} u^\beta dy + 2p\mu C_5(p, \mu, N, \delta, r, \eta, \kappa)
\]
(2.20)

with \(C_5(p, \mu, N, \delta, r, \eta, \kappa) = C_4(p, \mu, N, \delta, r, \eta, \kappa) + C_2 \frac{2(p+\alpha-1)}{p} (N, \delta, r). \) From (2.3), (2.20) and the fact that \(\int_{B(x,\delta)} u^p dy \leq \int_{B(x,\delta)} u^{p+\alpha-1} dy + (2\delta)^N\) we obtain
\[
\frac{\partial}{\partial t} \int_{B(x,\delta)} u^p dy + p\mu \int_{B(x,\delta)} u^p dy \leq \Delta \int_{B(x,\delta)} u^p dy + \left(2C_5(p, \mu, N, \delta, r, \eta, \kappa) + (2\delta)^N\right) p\mu. \] (2.21)

Next we compare (2.21) with the following ordinary differential equation
\[
\begin{cases} 
\frac{d}{dt} w + p\mu w = \left(2C_5(p, \mu, N, \delta, r, \eta, \kappa) + (2\delta)^N\right) p\mu, \\
w(0) = (2\delta)^N\|u_0\|_{L^\infty(\mathbb{R}^N)}.
\end{cases} \] (2.22)
By the comparison principle, we obtain
\[
\int_{B(x,\delta)} u^p \, dy \leq w = w(0)e^{-p\mu t} + (2C_5(p, \mu, N, \delta, r, \eta, \kappa) + (2\delta)^N)(1 - e^{-p\mu t}) \\
\leq 2C_5(p, \mu, N, \delta, r, \eta, \kappa) + (2\delta)^N(1 + \|u_0\|_{L^\infty(\mathbb{R}^N)}^p)
\]
for any \((x, t) \in \mathbb{R}^N \times [0, \infty)\). Then for any \(p > \max\{\beta - \alpha + 1, 1\}\), with the explicit representation of the constant \(C_5\), we obtain
\[
\|u\|_{L^p(B(x,\delta))} \leq \left[ 2 \left( \frac{2\mu^2 C_2^2(N, \delta)}{p - 1} \right)^{\lambda(p+\alpha-1)/p} + C_2^{2\lambda(p+\alpha-1)/p} (N, \delta, r) \right]^{\frac{2\beta}{2\beta - Q\theta p}} (\frac{4}{\eta\kappa})^{Q\theta p/2\beta - Q\theta p} + 2C_2^{2\lambda(p+\alpha-1)/p} (N, \delta, r) + (2\delta)^N(1 + \|u_0\|_{L^\infty(\mathbb{R}^N)}^p) \right]^{\frac{1}{p}},
\]
which tends to \(+\infty\) as \(p \to \infty\). More precisely, by (2.8) and (2.18), the exponents have the following explicit representations:
\[
\lambda(p + \alpha - 1) = \left\{ \begin{array}{ll}
\frac{p+\alpha-1-\beta}{p - \alpha + 1 + \beta}, & N = 1, 2, \\
\frac{N(p+\alpha-1-\beta)}{p + \alpha - 1 + \beta - N(a-1)}, & N > 2,
\end{array} \right.
\]
\[
\frac{2\lambda(p + \alpha - 1)}{p} = \left\{ \begin{array}{ll}
\frac{p+\alpha-1-\beta}{p}, & N = 1, 2, \\
\frac{N(p+\alpha-1-\beta)}{(1 + \frac{\alpha}{2})(a-1)}, & N > 2
\end{array} \right.
\]
\[
\frac{2\beta}{2\beta - Q\theta p} = \left\{ \begin{array}{ll}
\frac{p+\beta+1-\alpha}{p+\alpha - 1 + \beta - a}, & N = 1, 2, \\
\frac{p+\alpha - 1 - N(a-1) + \beta}{p + \alpha - 1 + \beta - N(a-1)}, & N > 2
\end{array} \right.
\]
and
\[
\frac{Q\theta p}{2\beta - Q\theta p} = \left\{ \begin{array}{ll}
\frac{p}{p+\alpha - 1 + \beta - a}, & N = 1, 2, \\
\frac{p+\alpha - 1 - N(a-1) + \beta}{p + \alpha - 1 + \beta - N(a-1)}, & N > 2
\end{array} \right.
\]

**Step 2.** \(L^\infty\) estimate via Moser iteration
Now we advance to deduce the \(L^\infty\) estimates for \(u\) by an iterative procedure. Denote \(p_k := b^k + h\) with any fixed \(b > 1\) and
\[
h := \left\{ \begin{array}{ll}
\frac{b(a-1)}{b-1}, & N = 1, 2, \\
(\alpha - 1) \left( \frac{N}{2} + \frac{1}{b-1} \right), & N > 2
\end{array} \right.
\]
is chosen to verify (2.28). Then the estimate (2.23) in Step 1 for
\[
p = p_{m-1} = b^{m-1} + h \quad (m \geq 1)
\]
gives the starting point of the iteration. By taking \( p = p_k \) in (2.3), we have
\[
\frac{\partial}{\partial t} \int_{B(x,\delta)} u^{p_k} \, dy + \frac{4(p_k - 1)}{p_k} \int_{B(x,\delta)} |\nabla u|^{\frac{p_k}{2}} \, dy + p_k \mu \eta_k \int_{B(x,\delta)} u^{p_k + \alpha - 1} \, dy \int_{B(x,\delta)} u^\beta \, dy \\
\leq \Delta \int_{B(x,\delta)} u^{p_k} \, dy + p_k \mu \int_{B(x,\delta)} u^{p_k + \alpha - 1} \, dy.
\]
(2.25)

Let
\[
r_k := \frac{2p_{k-1}}{p_k}
\]
and correspondingly in view of (2.8) and (2.13), denote
\[
\lambda_k := \begin{cases} 
1 - \frac{p_{k-1}}{2p_k}, & N = 1, 2, \\
\frac{Np_k}{4p_k - 1} - \frac{Np_k}{4p_k - 1 - \frac{p_{k-1}}{2}}, & N > 2,
\end{cases}
\]
(2.26)
\[
Q_k = \frac{2(1 - \lambda_k)(p_k + \alpha - 1)}{p_k - \lambda_k(p_k + \alpha - 1)}.
\]

Observe that
\[
p_k > \max \left\{ \left( \frac{N}{2} - 1 \right)(\alpha - 1), 1 \right\}
\]
and
\[
\max \left\{ \frac{N(\alpha - 1)}{p_k}, \frac{2(\alpha - 1)}{p_k} \right\} < r_k < \min \left\{ \frac{2(p_k + \alpha - 1)}{p_k}, s \right\}
\]
for all \( k \geq 1 \). Then \( 0 < \lambda_k < 1 \) and \( 0 < \frac{2\lambda_k(p_k + \alpha - 1)}{p_k} < 2 \). By the Gagliardo-Nirenberg inequality in Lemma 2.1 (here the main difference from Step 1 is that we restrict \( r_k > 0 \) instead of \( r > 1 \)) we obtain
\[
\|u\|^p \|L^{p_k + \alpha - 1} \|_{L^{p_k + \alpha - 1}(B(x,\delta))} = \|u^{p_k}\|^{\frac{2(p_k + \alpha - 1)}{p_k}} L^{p_k + \alpha - 1}(B(x,\delta)) \\
\leq C_{GN} \frac{2(p_k + \alpha - 1)}{p_k} \left\| \nabla u \right\|^{\frac{p_k}{2}} \left\| \lambda_k \right\|_{L^2(B(x,\delta))} \|u^{p_k}\|^{\frac{1 - \lambda_k}{p_k}} L^{p_k + \alpha - 1}(B(x,\delta)) + \|u^{p_k}\|^{\frac{2(1 - \lambda_k)(p_k + \alpha - 1)}{p_k}} L^{p_k + \alpha - 1}(B(x,\delta)) \\
\leq \frac{p_k - 1}{p_k} \left\| \nabla u \right\|^{2} L^2(B(x,\delta)) + \left( \frac{2C_{GN}}{2} \right)^{2} \mu \rho_k^2 \left( \frac{\lambda_k(p_k + \alpha - 1)}{p_k - \lambda_k(p_k + \alpha - 1)} \right) \left( \int_{B(x,\delta)} u^{p_k - 1} \, dy \right)^\frac{Q_k}{r_k}
\leq \frac{2C_{GN}}{2} \frac{2(p_k + \alpha - 1)}{p_k} \left( \int_{B(x,\delta)} u^{p_k - 1} \, dy \right)^\frac{Q_k}{r_k}.
\]
(2.27)

By the definition of \( h \) in (2.24) and \( \lambda_k \) in (2.26), for \( N = 1, 2 \), we obtain
\[
\frac{Q_k}{r_k} = \frac{p_k(1 - \lambda_k)(p_k + \alpha - 1)}{p_k - \lambda_k(p_k + \alpha - 1)}.
\]
\[ = \frac{p_k p_{k-1}}{p_{k-1}(p_k - (p_k + \alpha - 1) + p_{k-1})} \]
\[ = \frac{p_k}{p_{k-1} - (\alpha - 1)} \]
\[ = \frac{b^k + h}{b^{k-1} + h - (\alpha - 1)} = b. \]

While for \( N > 2 \), we obtain
\[
\frac{Q_k}{r_k} = \frac{p_k(1 - \lambda_k)(p_k + \alpha - 1)}{p_{k-1}(p_k - \lambda_k(p_k + \alpha - 1))} \]
\[ = \frac{p_k}{p_{k-1} \left[ p_k \left( 1 - \frac{N}{2} \right) + \frac{N p_k}{2p_k - 1} \right] - \frac{N p_k}{2p_k - 1} (p_k + \alpha - 1) + \frac{N p_k}{2}} \]
\[ = \frac{p_k}{p_{k-1} \left[ p_k + (1 - \frac{N}{2})(\alpha - 1) \right]} \]
\[ = \frac{p_k + (1 - \frac{N}{2})(\alpha - 1)}{p_{k-1} - \frac{N}{2}(\alpha - 1)} \]
\[ = \frac{b^k + h + (1 - \frac{N}{2})(\alpha - 1)}{b^{k-1} + h - \frac{N}{2}(\alpha - 1)} = b. \]

Thus we have
\[
\frac{Q_k}{r_k} = b. \tag{2.28} \]

On the other hand, we have
\[
\frac{2(p_k + \alpha - 1)}{r_k p_k} = \frac{p_k + \alpha - 1}{p_{k-1}} = \frac{b^k + h + \alpha - 1}{b^{k-1} + h} < b, \]
which implies
\[
\left( \int_{B(x, \delta)} u^{p_k-1} dy \right)^{\frac{2(p_k + \alpha - 1)}{r_k p_k}} \leq \left( \int_{B(x, \delta)} u^{p_k-1} dy \right)^b + 1. \tag{2.29} \]

Therefore, by Young’s inequality, we obtain a further estimate for (2.27),
\[
\|u\|_{L^{p_k+\alpha-1}(B(x, \delta))} \leq \frac{p_k - 1}{p_k^2 \mu} \|\nabla u\|_{L^2(B(x, \delta))}^2 + C_6 \left( \int_{B(x, \delta)} u^{p_k-1} dy \right)^b + C_6, \tag{2.30} \]
where
\[
C_6 = \left( \frac{2C_G N}{p_k - 1} \right)^\frac{\lambda_k(p_k + \alpha - 1)}{p_k - \lambda_k(p_k + \alpha - 1)} + \left( \frac{2C_G N}{p_k} \right)^{2(p_k + \alpha - 1)} \frac{1}{p_k}. \tag{2.31} \]

Notice the fact that
\[
\int_{B(x, \delta)} u^{p_k} dy \leq \int_{B(x, \delta)} u^{p_k+\alpha-1} dy + (2\delta)^N. \tag{2.32} \]
Substituting (2.30) and (2.32) into (2.25) leads to
\[
\frac{\partial}{\partial t} \int_{B(x,\delta)} u^{p_k} dy + p_k \mu \int_{B(x,\delta)} u^{p_k} dy \leq \Delta \int_{B(x,\delta)} u^{p_k} dy + 2p_k \mu C_6 \left( \int_{B(x,\delta)} u^{p_{k-1}} dy \right)^b + 2p_k \mu (C_6 + (2\delta)^N).
\]

Further estimates for the constants that appear in (2.33) are given in the following for \( b \geq 2 \) and \( k \geq 1 \). They are handled in two cases: \( N = 1, 2 \) and \( N > 2 \).

For \( N = 1, 2 \) we have
\[
\lambda_k = 1 - \frac{p_{k-1}}{p_k + \alpha - 1} = \frac{b^{k-1}(b-1) + \alpha - 1}{b^k + \frac{b(\alpha-1)}{b-1} + \alpha - 1} = \frac{(b-1)(b^{k-1}(b-1) + \alpha - 1)}{b^k(b-1) + (2b-1)(\alpha - 1)} > \frac{(b-1)(b^{k-1}(b-1) + \alpha - 1)}{2b^k(b-1) + 2b(\alpha - 1)} = \frac{b-1}{2b} \geq \frac{1}{4}
\]

and
\[
\frac{\lambda_k(p_k + \alpha - 1)}{p_k - \lambda_k(p_k + \alpha - 1)} = \frac{p_k + \alpha - 1 - p_{k-1}}{p_k - (p_k + \alpha - 1 - p_{k-1})} = \frac{p_{k-1} - (\alpha - 1)}{b^k - b^{k-1} + \alpha - 1} = \frac{b^{k-1} + \frac{b}{b-1}(\alpha - 1) - (\alpha - 1)}{b^{k-1} + \frac{b}{b-1}(\alpha - 1) - (\alpha - 1)} = b - 1.
\]

For \( N > 2 \) we have
\[
\lambda_k = \frac{Np_k - \frac{Np_kp_{k-1}}{p_k + \alpha - 1}}{2p_{k-1}(1 - \frac{N}{2}) + Np_k} = \frac{p_k(N(p_k + \alpha - 1) - Np_{k-1})}{(p_k + \alpha - 1)(2p_{k-1} + N(p_k - p_{k-1}))} = \frac{Np_k(p_k - p_{k-1}) + Np_k(\alpha - 1)}{Np_k(\alpha - 1) + 2p_kp_{k-1} + (2 - N)p_{k-1}(\alpha - 1)} = \frac{Np_k(b^{k-1}(b-1) + \alpha - 1) + 2\frac{p_kp_{k-1}}{b-1}(b^{k-1}(b-1) + (\alpha - 1))}{Np_k(b^{k-1}(b-1) + \alpha - 1) + \frac{2p_kp_{k-1}}{b-1}(b^{k-1}(b-1) + (\alpha - 1))}
\]
\[
\begin{align*}
\frac{Np_k(b-1)}{Np_k(b-1) + 2bp_{k-1}} & > \frac{N}{N(b-1) + 2b} \geq \frac{N}{N+4} > \frac{1}{4} \\
\text{and} \\
\frac{\lambda_k(p_k + \alpha - 1)}{p_k - \lambda_k(p_k + \alpha - 1)} &= \frac{\frac{Np_k}{2p_{k-1}}(p_k + \alpha - 1) - \frac{N}{2}p_k}{p_k(1 - \frac{N}{2} + \frac{Np_k}{2p_{k-1}}) - \frac{Np_k}{2p_{k-1}}(p_k + \alpha - 1) + \frac{Np_k}{2}} \\
&= \frac{Np_k(p_k + \alpha - 1) - Np_kp_{k-1}}{2p_kp_{k-1} - Np_k(p_k + \alpha - 1)} \\
&= \frac{N(p_k + \alpha - 1) - Np_{k-1}}{2p_{k-1} - N(p_k + \alpha - 1)} \\
&= \frac{N(b^k - b^{k-1}) + N(b - 1)}{2(b^{k-1} + (\alpha - 1)(\frac{N}{2} + \frac{1}{b-1})) - N(\alpha - 1)} \\
&= \frac{N}{2}(b - 1).
\end{align*}
\]

Thus we obtain

\[
\lambda_k \geq \frac{1}{4}
\]

and

\[
\frac{\lambda_k(p_k + \alpha - 1)}{p_k - \lambda_k(p_k + \alpha - 1)} = D(b, N) := \left\{ \begin{array}{ll}
\ b - 1, & N = 1, 2, \\
\frac{N}{2}(b - 1), & N > 2. \end{array} \right.
\]

Noticing for \( b \geq 2 \) and \( k \geq 1, \)

\[
\frac{p_k}{p_k - 1} = 1 + \frac{1}{b^k + h - 1} \leq 1 + \frac{1}{b - 1} \leq 2,
\]

\[
\frac{2(p_k + \alpha - 1)}{p_k} \leq 2 + \frac{2(\alpha - 1)}{b} \leq \alpha + 1,
\]

\[
h < (1 + N)(\alpha - 1).
\]

Then w.l.o.g., let \( C_{GN} > \frac{1}{2} \) to obtain

\[
C_6 + (2\delta)^N = \left( \frac{((2C_{GN})^2 \mu p_k^2)^{\frac{\lambda_k(p_k + \alpha - 1)}{p_k - \lambda_k(p_k + \alpha - 1)}}}{p_k - 1} \right) + (2C_{GN}) \frac{2(p_k + \alpha - 1)}{p_k} + (2\delta)^N
\]
\[
\begin{align*}
&\leq \left( (2C_{GN})^2 \mu p_k \right)^{D(b, N)} + (2C_{GN})^{\alpha+1} + (2\delta)^N \\
&\leq \left( \left( (2C_{GN})^2 \mu (1 + h) \right)^{D(b, N)} + (2C_{GN})^{\alpha+1} + (2\delta)^N \right) b^{D(b, N)k} \\
&\leq \left( (2C_{GN})^2 \mu (\alpha + N(\alpha - 1)) + (2C_{GN})^{\alpha+1} + (2\delta)^N \right) b^{D(b, N)k}.
\end{align*}
\]

Denote
\[
a_0 := (2C_{GN})^2 \mu (\alpha + N(\alpha - 1)) + (2C_{GN})^{\alpha+1} + (2\delta)^N > 1, \tag{2.35}
\]
then the iteration estimate for \( \int_{B(x, \delta)} u^p \, dy \) is given by
\[
\frac{\partial}{\partial t} \int_{B(x, \delta)} u^p \, dy + p_k \mu \int_{B(x, \delta)} u^p \, dy \\
\leq \Delta \int_{B(x, \delta)} u^p \, dy + 4p_k \mu a_0 b^{D(b, N)k} \max \left\{ 1, \sup_{t \geq 0} \left( \int_{B(x, \delta)} u^{p_k-1} \, dy \right)^b \right\}.
\]

For \( k \geq m \geq 1 \), let \( y_k(t) \) be the solution of the following iterating ordinary differential equation
\[
y_k'(t) + p_k \mu y_k(t) = 4p_k \mu a_0 b^{D(b, N)k} \max \left\{ \sup_{t \geq 0} y_k^{b-1} (t), 1 \right\},
\]

\[
y_k(0) = \| u_0 \|_{L^\infty(\mathbb{R}^N)}^{p_k} (2\delta)^N
\]

with
\[
y_{m-1}(t) = \sup_{(x, t) \in \mathbb{R}^N \times [0, \infty)} \int_{B(x, \delta)} u^{p_{m-1}} \, dx
\]

being the starting point of the iteration. Now we are ready to use Lemma 2.2. For \( \overline{a} = 4a_0 \), it is obvious that \( \overline{a} b^{D(b, N)k} \geq 1 \) for all \( k \geq 1 \). Furthermore, \( \delta \) can be chosen such that \( (2\delta)^N < \frac{1}{\| u_0 \|_{L^\infty(\mathbb{R}^N)}} \), therefore
\[
y_k(0) = \| u_0 \|_{L^\infty(\mathbb{R}^N)}^{p_k} (2\delta)^N \leq \| u_0 \|_{L^\infty(\mathbb{R}^N)}^{p_k}.
\]

From Lemma 2.2 with \( c_k = p_k \mu, D = D(b, N) \), we obtain
\[
y_k(t) \leq (2\overline{a})^{\frac{k-m+1}{b-1}} b^{D(b, N) \left( \frac{k-m+1}{(b-1)^2} + \frac{mk-m+1}{b-1} \right)} \max \left\{ \sup_{t \geq 0} y_{m-1}^{k-m+1} (t), \| u_0 \|_{L^\infty(\mathbb{R}^N)}^{p_k} \right\}
\]

and then by the comparison principle, we have
\[
\| u(\cdot, t) \|_{L^{p_k}(B(x, \delta))} \leq \left( (8a_0)^{\frac{k-m+1}{b-1}} b^{D(b, N) \left( \frac{k-m+1}{(b-1)^2} + \frac{mk-m+1}{b-1} \right)} \right)^\frac{1}{p_k} \max \left\{ \sup_{t \geq 0} y_{m-1}^{p_k} (t), \| u_0 \|_{L^\infty(\mathbb{R}^N)}^{p_k} \right\}.
\]
By letting $k \to \infty$, we obtain
\[
\|u(\cdot, t)\|_{L^\infty(B(x, \delta))} \leq (8a_0)^{\frac{1-m}{b-1}} b^D(b,N)b^{1-m}(\frac{1}{(a-1)^2} + \frac{m}{2}) \cdot \max \left\{ \sup_{(x,t) \in \mathbb{R}^N \times [0,\infty)} \left( \int_{B(x,\delta)} u^{b^{m-1}+h} dy \right)^{\frac{1}{b^{m-1}}}, \|u_0\|_{L^\infty(\mathbb{R}^N)}, 1 \right\}.
\]

Since $x \in \mathbb{R}^N$ and $t \in [0, \infty)$ are arbitrary and the boundedness of
\[
\sup_{(x,t) \in \mathbb{R}^N \times [0,\infty)} \left( \int_{B(x,\delta)} u^{b^{m-1}+h} dy \right)^{\frac{1}{b^{m-1}}}
\]
is verified from Step 1, we have
\[
\|u\|_{L^\infty(\mathbb{R}^N \times [0,\infty))} \leq M
\]
with
\[
M = (8a_0)^{\frac{1-m}{b-1}} b^D(b,N)b^{1-m}(\frac{1}{(a-1)^2} + \frac{m}{2}) \cdot \max \left\{ \sup_{(x,t) \in \mathbb{R}^N \times [0,\infty)} \left( \int_{B(x,\delta)} u^{b^{m-1}+h} dy \right)^{\frac{1}{b^{m-1}}}, \|u_0\|_{L^\infty(\mathbb{R}^N)}, 1 \right\}.
\]

Therefore, the global boundedness of $u$ is obtained.

**Step 3. ”Quasi”-maximum principles when the equation has small constant steady state and small source**

We optimize $M$ in the following by using the flexibility of $b$ and $m$. Specifically, in this part we will prove that for large $\kappa$, the maximum $M$ introduced in (2.37) can be optimized almost to $\max \{1, \|u_0\|_{L^\infty(\mathbb{R}^N)}\}$ for small $\mu$.

Choose $m > 1$ and notice that
\[
C_2(N, \delta, r) = 4\sqrt{2} \max\{(2\delta)^N(\frac{1}{r} - \frac{1}{s})^\frac{1}{2}, (2\delta)^{1-N} + \frac{N}{s} G(s, N)\}
\]
with $r$ given in (2.16) and $s$ in (2.6). Moreover,
\[
\lim_{b \to \infty} C_2 \left( N, \delta, \frac{b^{m-1} + h + \alpha - 1 + \beta}{b^{m-1} + h} \right) = \begin{cases} 
\frac{4\sqrt{2}}{(2\delta)^\gamma} \max \{1,2\delta G(s, N)\}, & N = 1,2, \\
\frac{4\sqrt{2}}{(2\delta)^{\frac{1}{1+s}}} \max \{1,2\delta G(s, N)\}, & N > 2.
\end{cases}
\]
From (2.23), if $\mu < \frac{1}{2C_1^b(b^{m-1}+h)}$, then for $N = 1, 2$ we have
\[
\int_{B(x,\delta)} u^{b^{m-1}+h} dy
\]
while for $N > 2$, we have

$$\int_{B(x,\delta)} u_b^{m-1+h} \, dy \leq 2 \left( \frac{b^{m-1} + h}{b^{m-1} + h - 1} \right)^{\frac{\alpha}{\beta + 1 + \alpha}} \left( \frac{4}{\kappa \eta} \right) \left( \frac{b^{m-1} + h}{b^{m-1} + h - 1} \right)^{\frac{\alpha}{\beta + 1 + \alpha}} + 2C_2 \left( \frac{b^{m-1} + h}{b^{m-1} + h - 1} \right)^{\frac{\alpha}{\beta + 1 + \alpha}} \left( \frac{4}{\kappa \eta} \right) \left( \frac{b^{m-1} + h}{b^{m-1} + h - 1} \right)^{\frac{\alpha}{\beta + 1 + \alpha}} + (2\delta)^N (1 + \|u_0\|_{L^\infty(\mathbb{R}^N)})$$

$$:= M_2(b).$$

Notice that

$$\lim_{b \to \infty} (M_1(b))^{\frac{1}{m-1}} = \max \left\{ 2 + \frac{4\sqrt{2}}{(2\delta)^N} \max \{1, 2\delta G(\gamma, N)\} \right\} \left( \frac{4}{\kappa \eta} \right)^{\frac{1}{\beta + 1 + \alpha}}, \|u_0\|_{L^\infty(\mathbb{R}^N)}, 1 \right\}$$

and

$$\lim_{b \to \infty} (M_2(b))^{\frac{1}{m-1}} = \max \left\{ 2 + \frac{4\sqrt{2}}{(2\delta)^{1+\frac{N}{2}}} \max \{1, 2\delta G(\gamma, N)\} \right\} \left( \frac{4}{\kappa \eta} \right)^{\frac{1}{\beta + 1 + \alpha}}, \|u_0\|_{L^\infty(\mathbb{R}^N)}, 1 \right\}.$$
\[
\lim_{b \to \infty} (M_2(b))^{\frac{1}{b-1}} = \max \left\{ 1, \left( \frac{\kappa^*}{\kappa} \right)^{\frac{1}{\beta - \frac{2}{(\alpha-1)}}}, \|u_0\|_{L^\infty(\mathbb{R}^N)} \right\}
\]
\[
= \max \left\{ 1, \|u_0\|_{L^\infty(\mathbb{R}^N)} \right\}.
\]

On the other hand, by recalling the definition of \(D(b, N)\) in (2.34), for \(m > 1\), we obtain
\[
\lim_{b \to \infty} (8a_0)^{\frac{1}{b-1}} b^{D(b, N)h^1 - m} \left( \frac{1}{(b-1)^2} + \frac{m}{b-1} \right) = 1.
\]

According to the above discussion, if we let \(b\) go to infinity there will be no positive \(\mu\) such that the maximum principle holds. However, we can get the following relaxed version of maximum principle. Namely, for arbitrary \(K > 1\), from (2.36), due to (2.38), (2.39) and (2.40), there exists a large \(b\) (which depends only on \(K\)) such that for \(\mu \in (0, \mu^*)\) with \(\mu^* = \frac{1}{2C_1^2(b^m - 1 + h)}\), we have
\[
\|u\|_{L^\infty(\mathbb{R}^N \times [0, \infty))} \leq K \max \left\{ 1, \|u_0\|_{L^\infty(\mathbb{R}^N)} \right\}.
\]

**Step 4. Maximum estimate when the equation has a large, constant stationary solution**

In this part, we will prove that if \(\max \{1, \frac{(N+2\beta)(N-2)}{N^2+4}\} \leq \alpha < \alpha^{**}\), for small \(\kappa\), then the maximum \(M\) can be optimized to \(\left( \frac{1}{\kappa} \right)^{\frac{1}{2}}\).

Now if we choose \(m = 1\) in (2.37), it holds
\[
M = (8a_0)^{\frac{1}{b-1}} b^{D(b, N)h^1 - m} \cdot \max \left\{ \sup_{(x,t) \in \mathbb{R}^N \times [0, \infty)} \int_{B(x, \delta)} u^{1+h} dy, \|u_0\|_{L^\infty(\mathbb{R}^N)}, 1 \right\}.
\]

From (2.23), for any \((x, t) \in \mathbb{R}^N \times [0, +\infty)\) by choosing \(\delta\) small such that \(2\delta G(s, N) < 1\), we get \(S(N, \delta) = \sqrt{2}(2\delta)^{N\left(\frac{1}{2} - \frac{1}{2}\right)}\).

For \(N = 1, 2\) we obtain
\[
\int_{B(x, \delta)} u^{1+h} dy \leq 2 \left( 2\mu(1 + h)^2 C_1^2(N, \delta) \right)^{\frac{h+\alpha-\beta}{h+1}} + C_2^{\frac{h+\alpha-\beta}{h+1}} \left( N, \delta, \frac{h + \alpha + \beta}{h + 1} + \frac{h^{\frac{h+\alpha-\beta}{h+1}}(N, \delta, \frac{h + \alpha + \beta}{h + 1}) + (2\delta)^N (1 + \|u_0\|_{L^\infty(\mathbb{R}^N)}^{h+1})}{\kappa \eta} \right)^{\frac{4}{h+1}}.
\]
with $h = \frac{b(a-1)}{b-1}$, $D(b, N) = b - 1$, and
\[
C_1(N, \delta) \leq 2\sqrt{2}(1 + 2C(N)\delta)(2\delta)^{-\frac{N}{2}}, \quad C_2 \left( N, \delta, \frac{h + \alpha + \beta}{h + 1} \right) \leq 4\sqrt{2}(2\delta)^{\frac{(h+1)N}{h+\alpha+\beta}}.
\]
Likewise, for $N > 2$ we have
\[
\int_{B(x, \delta)} u^{1+h} dy 
\leq 2 \left( 2 \left( \frac{2\mu(1+h)^2C^2_1}{h} \right)^{\frac{N}{h+\alpha+\beta} - N(a-1)} + C \left( \frac{\int_{B(x, \delta)} u \, dy}{\left( h + \frac{h + \alpha + \beta}{h + 1} \right)^{\frac{N}{h+\alpha+\beta}}} \right)^{\frac{h + \alpha - N(a-1) + \beta}{\beta - \frac{N}{2}(a-1)}} \right)
\]
\[
\cdot \left( \frac{4}{\kappa \eta} \right)^{\frac{h + \alpha - \frac{2}{N}a}{\beta - \frac{N}{2}(a-1)}} + 2C_2 \left( \frac{\int_{B(x, \delta)} u \, dy}{\left( h + \frac{h + \alpha + \beta}{h + 1} \right)^{\frac{N}{h+\alpha+\beta}}} \right)^{\frac{h + \alpha - N(a-1) + \beta}{\beta - \frac{N}{2}(a-1)}} (2\delta)^N \left( 1 + \|u_0\|_{L^\infty(\mathbb{R}^N)} \right)^2
\]
with $h = (\alpha - 1) \left( \frac{N}{2} + \frac{1}{b-1} \right)$, $D(b, N) = \frac{N}{2}(b - 1)$, and
\[
C_1(N, \delta) = 2\sqrt{2}(1 + 2C(N)\delta)(2\delta)^{-1}, \quad C_2 \left( N, \delta, \frac{h + \alpha + \beta}{h + 1} \right) = 4\sqrt{2}(2\delta)^{\frac{N-2}{2} - \frac{(h+1)N}{h+\alpha+\beta}}.
\]
For any $\beta > 1$ and $N > 2$ with $h = \frac{b(a-1)}{b-1}$, and
\[
0 < \kappa^* \leq \left( \frac{1}{(8a_0)^{\frac{1}{p-1}} b^{D(b, N)} \left( \frac{1}{(b-1)^p + \frac{1}{p-1}} \right) \max\{\|u_0\|_{L^\infty(\mathbb{R}^N), 1}} \right)^{\beta}
\]
such that for any $0 < \kappa \leq \kappa^*$, we can choose small $\delta$ in (2.42) and (2.43) such that for any $(x,t) \in \mathbb{R}^N \times [0, +\infty)$, we have

$$\int_{B(x,\delta)} u^{1+h} dy \leq \frac{1}{\kappa^{\frac{1}{b}}} (8a_0)^{\frac{1}{b-N}} b^{D(b,N)} \left( \frac{1}{(\beta-1)^2} + \frac{1}{\eta \kappa^*} \right)$$

and then

$$\|u\|_{L^\infty(\mathbb{R}^N \times [0,\infty))} \leq (8a_0)^{\frac{1}{b-N}} b^{D(b,N)} \left( \frac{1}{(\beta-1)^2} + \frac{1}{\eta \kappa^*} \right) \max \left\{ \sup_{(x,t) \in \mathbb{R}^N \times [0,\infty)} \int_{B(x,\delta)} u^{1+h} dy, \|u_0\|_{L^\infty(\mathbb{R}^N)} \right\} \leq \left( \frac{1}{\kappa^*} \right)^{\frac{1}{b-N}}.$$

**Case 2:** $(1 \leq \alpha \leq \frac{1}{2}(\beta + 1))$. Choosing $p = \alpha$ in (2.3), we obtain

$$\frac{\partial}{\partial t} \int_{B(x,\delta)} u^\alpha dy \leq \Delta \int_{B(x,\delta)} u^\alpha dy - \frac{4(\alpha - 1)}{\alpha} \int_{B(x,\delta)} |\nabla u^{\alpha/2}|^2 dy + \alpha \mu \int_{B(x,\delta)} u^{2\alpha-1} dy \left( 1 - \eta \kappa \int_{B(x,\delta)} u^{\beta} dy \right).$$

(2.45)

For the case $1 \leq \alpha \leq \frac{1}{2}(\beta + 1)$, that is $\beta \geq 2\alpha - 1 \geq 1$, using Young’s inequality, we have

$$\int_{B(x,\delta)} u^\alpha dy + \int_{B(x,\delta)} u^{2\alpha-1} dy \left( 1 - \eta \kappa \int_{B(x,\delta)} u^{\beta} dy \right) \leq \left( \int_{B(x,\delta)} u^\beta dy + (2\delta)^N \right) \left( 2 - \eta \kappa \int_{B(x,\delta)} u^{\beta} dy \right) \leq \int_{B(x,\delta)} u^\beta dy \left( 2 - \eta \kappa \int_{B(x,\delta)} u^\beta dy \right) + 2(2\delta)^N \leq \frac{1}{\eta \kappa} + 2(2\delta)^N,$$

(2.46)

which together with (2.45) implies

$$\frac{\partial}{\partial t} \int_{B(x,\delta)} u^\alpha dy + \alpha \mu \int_{B(x,\delta)} u^\alpha dy \leq \Delta \int_{B(x,\delta)} u^\alpha dy + \alpha \mu \left( \frac{1}{\eta \kappa} + 2(2\delta)^N \right).$$

(2.47)

Considering the initial value problem

$$V_t + \alpha \mu V = \alpha \mu \left( \frac{1}{\eta \kappa} + 2(2\delta)^N \right),$$

$$V(0) = (2\delta)^N \|u_0\|_{L^\infty(\mathbb{R}^N)}^\alpha,$$

by the parabolic comparison principle, we obtain

$$\int_{B(x,\delta)} u^\alpha dy \leq V \leq M,$$

(2.48)
where \( M = (2\delta)^N \| u_0 \|_{L^\infty(\mathbb{R}^N)}^\alpha + \frac{1}{\| u_0 \|_{L^\infty(\mathbb{R}^N)}} + 2(2\delta)^N \). Moreover, from
\[
\int_{B(x,\delta)} u(y,t) dy \leq C_\delta \left( \int_{B(x,\delta)} u^\alpha(y,t) dy \right)^{\frac{1}{\alpha}} \leq C_\delta M^{\frac{1}{\alpha}},
\]
where \( C_\delta \) is a positive constant depending on \( \delta \), it follows that for all \((x,t) \in \mathbb{R}^N \times [0, +\infty)\),
\[
0 \leq u(x,t) \leq \int_{\mathbb{R}^N} \frac{e^{-\frac{|y|^2}{4(t-t_\tau)}}}{(4\pi(t-t_\tau))^\frac{N}{2}} u^\alpha(x-y, t_\tau) d\tau + \mu \int_{t-1}^t \int_{\mathbb{R}^N} \frac{e^{-\frac{|y|^2}{4(t-t_\tau)}}}{(4\pi(t-t_\tau))^\frac{N}{2}} u^\alpha(x-y, t_\tau) d\tau d\tau \leq C_\delta M^{\frac{1}{\alpha}},
\]
with \( C_\delta \) depending on \( \delta \).

The global boundedness of \( u \) is obtained. As a summary, we have proved that the solution \( u \) is globally bounded in time for the cases \( 1 \leq \alpha < \alpha^* \) and \( 1 \leq \alpha \leq \beta + \frac{1}{2} \) respectively. Then the blow-up criterion in Proposition 2.1 shows that \( u \) is the unique classical solution of (1.1)-(1.2) on \( \mathbb{R}^N \times (0, +\infty) \).

3 Long time behavior (hair trigger type effect)

Now we consider the long time behavior of the solution of (1.1)-(1.2).

**Proposition 3.1.** Under the assumptions of Theorem 1.2, for all \( t > 0 \), the function
\[
F(x,t) = \int_{B(x,\delta)} h(u^\beta(y,t)) dy
\]
is nonnegative and satisfies
\[
\partial_t F(x,t) \leq \Delta F(x,t) - D(x,t)
\]
with
\[
h(s) = \begin{cases}
\frac{s}{2} - \frac{1}{\kappa s} \ln s - \frac{1}{\kappa s} (1 + \ln \kappa) & , \quad \alpha = 1, \\
\frac{s}{2} \left( \frac{1}{\beta - \alpha} - \frac{1}{\beta} \right) + \kappa^{\alpha-\beta} \left( \frac{1}{1-\alpha} - \frac{1}{1+\beta-\alpha} \right) & , \quad 1 < \alpha < \alpha^*.
\end{cases}
\]
and \( D(x, t) = \frac{1}{2} \eta \mu \kappa (2\delta)^N \int_{B(x, \delta)} (\kappa^{-1} - u^\beta(y, t))^2 dy. \)

**Proof.** Noticing that

\[
h'(s) = \begin{cases} 
\frac{1}{\beta} - \frac{1}{\kappa^s}, & \alpha = 1, \\
\frac{1}{\beta} \left( \frac{1-\alpha}{\kappa^s} - \kappa^{-1} \frac{1-\alpha-\beta}{\kappa^s} \right), & 1 < \alpha < \alpha^* 
\end{cases}
\]

and \( h'(s) < 0 \) for \( 0 < s < \kappa^{-1} \) and \( h'(s) > 0 \) for \( s > \kappa^{-1} \), we obtain that \( h(s) \geq h(\kappa^{-1}) = 0 \) and \( F(x, t) \) is nonnegative.

For the global solution \( u \) satisfying \( 0 \leq u(x, t) \leq \left( \frac{1}{\kappa} \right)^\frac{1}{\beta} \) for all \( (x, t) \in \mathbb{R}^N \times [0, \infty) \), the positivity follows from the fact that

\[
u(\cdot, t) = G(\cdot, t) * u_0 + \mu \int_0^t (\nu^\alpha(\cdot, s)(1 - \kappa J * u^\beta(\cdot, s))) * G(\cdot, t - s) ds 
\geq G(\cdot, t) * u_0 > 0,
\]

with \( G(x, t) = \frac{e^{-|x|^2}}{(4\pi t)^{\frac{N}{2}}} \) the heat kernel.

Test (1.1) by \( (u^\beta - \kappa^{-1} u^\alpha) \varphi_\varepsilon \) with \( \varphi_\varepsilon \in C_0^\infty(B(x, \delta)) \), \( \varphi_\varepsilon \to 1 \) in \( B(x, \delta) \) as \( \varepsilon \to 0 \). Integrating by parts over \( B(x, \delta) \) we obtain

\[
\frac{\partial}{\partial t} \int_{B(x, \delta)} h(u^\beta) \varphi_\varepsilon dy 
= \int_{B(x, \delta)} \Delta u(u^\beta - \kappa^{-1} u^\alpha) \varphi_\varepsilon dy + \mu \kappa \int_{B(x, \delta)} (u^\beta - \kappa^{-1})(\kappa^{-1} - J * u^\beta) \varphi_\varepsilon dy 
= \int_{B(x, \delta)} \Delta h(u^\beta) \varphi_\varepsilon dy - \frac{4(\beta - \alpha)}{(\beta - \alpha + 1)^2} \int_{B(x, \delta)} |\nabla u|^2 \varphi_\varepsilon dy - \frac{\alpha}{\kappa} \int_{B(x, \delta)} u^{-\alpha-1} |\nabla u|^2 \varphi_\varepsilon dy 
+ \mu \kappa \int_{B(x, \delta)} \int_{\mathbb{R}^N} (u^\beta(y, t) - \kappa^{-1})(\kappa^{-1} - u^\beta(z, t)) J(z - y) \varphi_\varepsilon(y) dz dy. \quad (3.2)
\]

Taking \( \varepsilon \to 0 \), we obtain

\[
\frac{\partial}{\partial t} F(x, t) = \Delta F(x, t) - \frac{4(\beta - \alpha)}{(\beta - \alpha + 1)^2} \int_{B(x, \delta)} |\nabla u|^{\alpha-\frac{\alpha+1}{2}} dy - \frac{\alpha}{\kappa} \int_{B(x, \delta)} u^{-\alpha-1} |\nabla u|^2 dy 
+ \mu \kappa \int_{B(x, \delta)} \int_{\mathbb{R}^N} (u^\beta(y, t) - \kappa^{-1})(\kappa^{-1} - u^\beta(z, t)) J(z - y) dz dy. \quad (3.3)
\]

Then for \( \delta < \frac{\delta_0}{2} \), noticing \( 0 < u \leq \kappa^{-\frac{1}{\beta}} \), then

\[
\int_{B(x, \delta)} \int_{\mathbb{R}^N} (u^\beta(y, t) - \kappa^{-1})(\kappa^{-1} - u^\beta(z, t)) J(z - y) dz dy 
\leq \int_{B(x, \delta)} \int_{B(x, \delta)} (u^\beta(y, t) - \kappa^{-1})(\kappa^{-1} - u^\beta(z, t)) J(z - y) dz dy 
= - \int_{B(x, \delta)} \int_{B(x, \delta)} (u^\beta(y, t) - \kappa^{-1})^2 J(z - y) dz dy
\]
For any $\theta \in (0,1]$, we have

$$F(x,t) \leq \Delta F(x,t) - (1-\varepsilon)\eta(2\delta)^N\mu \kappa \int_{B(x,\delta)} \kappa^{-1} - u^3(y,t)^2 \, dy.$$  

(3.1)
we can also verify (3.7) by choosing $0 < \mu < \min\{\mu^*, \frac{3\kappa^{(\alpha-1)/\beta}}{2C(t)\delta^\alpha}\}$ in (3.6). Then taking $\varepsilon = \frac{1}{2}$ in (3.7), we obtain (3.1).

Now we are in a position to prove Theorem 1.2.

**Proof of Theorem 1.2.** From (3.1), we have

\[
F(x, t) \leq \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4(t-s)}} F_0(y)dy - \int_0^t \frac{1}{(4\pi(t-s))^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4(t-s)}} D(y, s)dyds,
\]

from which we obtain

\[
\int_0^t \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4(t-s)}} D(y, s)dyds \leq \|F_0\|_{L^\infty(\mathbb{R}^N)}.
\]

Due to the fact that $u$ is a classical solution, we have that

\[
\int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4(t-s)}} D(y, s)dy \in C^{2,1}(\mathbb{R}^N \times (0, \infty]),
\]

which implies that for all $x \in \mathbb{R}^N$, the following limit holds:

\[
\lim_{t \to \infty} \lim_{s \to t} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4(t-s)}} D(y, s)dy = 0,
\]

or equivalently,

\[
\lim_{t \to \infty} \lim_{s \to t} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4(t-s)}} \int_{B(y, \delta)} (\kappa^{-1} - u^\beta(z, s))^2dzdy = 0.
\]

Together with the fact that the heat kernel converges to delta function as $s \to t$, we have that for any $x \in \mathbb{R}^N$,

\[
\lim_{t \to \infty} \int_{B(x, \delta)} (\kappa^{-1} - u^\beta(y, t))^2dy = 0,
\]

from which we can obtain the uniform convergence of solutions in $B(x, \delta)$, namely

\[
\|u^\beta - \kappa^{-1}\|_{L^\infty(B(x, \delta))} \to 0
\]

as $t \to \infty$. Then for any compact set in $\mathbb{R}^N$, by finite covering, we obtain that $u$ converges to 1 uniformly in that compact set, which means that $u$ converges locally uniformly to $\kappa^{-\frac{1}{\beta}}$ in $\mathbb{R}^N$ as $t \to \infty$. The proof is complete.
4 Numerical simulations and discussion

In Sections 2 and 3, we established global boundedness and the hair trigger effect of solutions to the nonlinear nonlocal reaction-diffusion initial value problem (1.1), (1.2). The obtained results provide information about the relationship between the exponents $\alpha$ (weak Allee effect) and $\beta$ (overcrowding effect) in (1.1). The deduction of (1.1) performed in the Appendix suggests that the whole dynamics is controlled by the interaction strengths $\alpha, \beta$, the spatial dimension $N$, the kernel $J$, and the population carrying capacity encoded via nondimensionalization in the strength of the source term, thus on the constants $\mu$ and $\kappa$ below (for simplicity we assumed the speed of individuals to be constant).

The constant $\alpha^*$ offering an upper bound for the exponent $\alpha$ was found here to depend on $\beta$ and $N$; moreover, it is uniform with respect to the kernel $J$. By introducing the nonlocal competition term $J * u$, the $\alpha$-interval $(1, 1 + \frac{2}{N})$ leading to blow-up for $\frac{\partial u}{\partial t} = \Delta u + u^\alpha$ is turned into an interval providing global existence for $\frac{\partial u}{\partial t} = \Delta u + \mu u^\alpha(1 - \kappa J * u)$ with $N \geq 2$. Furthermore, by introducing the exponent $\beta$ to the nonlocal competition term, the $\alpha$-interval $(1, 1 + \frac{2}{N})$ ensuring global existence is further enlarged to $\alpha \in [1, 1 + \frac{2\beta}{N}) \cup [1, 1 + \frac{1+2\beta}{2}]$, for which the upper bound is increasing with $\beta$.

Next, by numerical simulations, we also provide some clues for further investigations on the effect of the kernel $J$ on the solution behavior. Throughout this section we consider the following $(\alpha, \beta, \mu, \kappa)$-parametrized problem:

$$
\begin{align*}
    u_t &= u_{xx} + \mu u^\alpha(1 - \kappa J * u^\beta), \quad x \in \mathbb{R}, \quad t > 0 \\
    u(0, x) &= u_0(x), \quad x \in \mathbb{R},
\end{align*}
$$

4.1 Simulations related to the effect of the kernel on the global boundedness

Following the algorithm in [38] we perform numerical simulations in 1D for the initial value problem (4.1) and test several combinations of $\mu, \alpha$. For $J$ we choose either the uniform kernel $J(x) = \frac{1}{2}1_{[-1,1]}$ or the so-called logistic kernel $J(x) = \frac{1}{2 + e^{x} + e^{-x}}$, see e.g. [34]. In order to handle the problem on the whole $\mathbb{R}$ we consider as in [38] a bounded interval $(x_l, x_r) \subset \mathbb{R}$ and set $u \equiv 1$ on $(-\infty, x_l]$, and $u \equiv 0$ on $[x_r, \infty)$. We take the initial condition
\[ u_0(x) = \begin{cases} 
1, & \text{for } x \leq x_l \\
e^{-(x-x_l)^2}, & \text{for } x_l < x \leq 0 \\
e^{-x_l^2}(1 - \frac{x}{x_r}), & \text{for } 0 < x \leq x_r \\
0, & \text{for } x > x_r.
\end{cases} \]

In our simulations $x_l = -5$, $x_r = 5$, and $\beta = \kappa = 1$. Figure 1 shows solution profiles of $u$ for $J$ being the uniform kernel with several different values of $\mu$ and $\alpha$.

Figure 1: Simulation results for (4.1) with $\beta = \kappa = 1$, $\mu = 1$ (upper row), $\mu = 10$ (middle row), $\mu = 100$ (lower row), $J$ uniform kernel, and different values of $\alpha$. Subfigures 1c, 1e, 1f, 1h and 1i show the solution at the time moments just before it blows up.
Our numerical experiments show that the solution stays bounded for any values of $\alpha$ when $\mu$ is rather small, e.g. for $\mu = 1$. The corresponding solution profiles look like those in Subfigures 1a, 1b; the only change noticed for different concrete values of $\alpha$ is in the curve connecting the levels $u = 1$ and $u = 0$ at $x = x_l$ and $x = x_r$, respectively: For small $\alpha^2$ this is a shoulder curve which can slightly and transiently exceed the level $u = 1$ (see Subfigure 1a), while for $\alpha$ large it becomes a straight line, as in Subfigure 1b. For $\mu = 10$ the solution is still bounded, even if $\alpha$ exceeds the upper bound (here $\alpha^* = 2$) obtained in our analysis. It first became unbounded for $\alpha = 6$. We also explored (still for $\mu = 10$) the global boundedness of the solution in the case with no diffusion, for which there are no theoretical results: In this situation the solution was found to blow up already for $\alpha = 1.9$, see Subfigure 1f. This indicates that neglecting diffusion leads to insufficient dampening of the growth, which triggers unboundedness even for smaller values of $\alpha$. Moreover, in this case the combination $\mu = 1$ and $\alpha \geq 2.8$ leads to blow-up as well, see Subfigure 1c.

Simulation results for $\mu = 100$ are shown in Subfigures 1g-1i. This increase of the interaction rate reduces the range of $\alpha$ in which there is no blow-up occurring. Our tests showed that the latter already happens for $\alpha = 2.56$, while the solution stays bounded for $\alpha$ below that value, although it can exhibit a highly oscillatory behavior, occasionally with very large and locally concentrated aggregates, as shown in Subfigures 1g, 1i. Such peaks of the solution can suddenly emerge, grow very fast, and then stabilize or drop back to small values. Neglecting diffusion has for $\mu = 100$ the same effect as above for $\mu = 10$: It leads to blow-up of the solution, and this already for even smaller values of $\alpha$.

Figure 2 illustrates solution profiles of $u$ for $J$ being the logistic kernel, $\beta = \kappa = 1$, and different $\mu$ and $\alpha$ combinations. The solution behavior is similar to that for $J$ being uniform, but now blow-up occurs (for the same values of $\mu$) at lower values of $\alpha$: For $\mu = 10$ the solution explodes for $\alpha \geq 4.23$, and in case $\mu = 100$ for $\alpha \geq 2.3$. Subfigures 2a and 2b show solution profiles for $\mu = 10$ and $\alpha = 4.22$ (no blow-up) and for $\mu = 100$ and $\alpha = 2.3$ (at the time just before blow-up), respectively. The shapes of the solutions are quite alike, just with higher maxima for $\mu$ increasing. In all simulations with this type of kernel there are far less oscillations and peaks, which get damped rather fast, only one dominant aggregate remaining. Interestingly, for $D = 0$ and $\mu = 1$ the first exponent for which blow-up occurs is $\alpha = 3.1$,\footnote{less than approximately $\alpha = 15$}
which is larger than in the case where $J$ was uniform, compare Subfigure 1c.

### 4.2 Simulations for the influence of the kernel on the hair trigger effect and pattern formation with a relatively large $\kappa$

In order to test the hair trigger effect and to get some insight into the qualitative behavior of the solution we also performed numerical simulations for different values of $\mu$ and two different kernels.

We start with the case $\kappa = \beta = 1$ and several combinations of the parameters $\alpha$ and $\mu$. As before, $J$ is taken to be the uniform or the logistic kernel. The results are shown in Figures 3 and 4 for the uniform kernel and for the logistic kernel, respectively. From Subfigure 3b we notice that for $\mu = 1$ and $\alpha < \alpha^*$, the solution converges locally uniformly to 1, which is the hair trigger effect. Subfigures 3c, 3d show that for $\mu = 50$ and 150, respectively, the solution forms different patterns, larger $\mu$ values leading to more oscillatory patterns. These facts suggest that the smallness assumption on $\mu$ in Theorem 1.2 is necessary. A similar behavior is observed for $\alpha$ coinciding with or being slightly beyond $\alpha^*$, while the solution explodes for $\alpha \geq \hat{\alpha} > \alpha^* = 2$, the critical value $\hat{\alpha}$ depending as before on the choice of $\mu$, compare Subfigures 3e and 3f.

Allowing for more frequent oscillations in the initial condition leads to the same behavior, however with singularities occurring at later times. Simulations with the same initial condition, but with $J$ being the logistic kernel are illustrated in Figure 4. The hair trigger effect for small $\mu$ values is similar, however the solution exhibits less oscillations, which means that the shape of patterns is strongly influenced by the choice of the kernel. Furthermore, for the logistic kernel
the solution blows up earlier, and for smaller values of $\alpha$: Subfigure 4d shows the solution profile for $\alpha = 3.8$ shortly before blow-up; compare with Subfigure 3f.

Next we consider the case (ii) in Theorem 1.1. We choose $\beta = 2$, which leads to $\alpha^{**} = 1.5$ and try a small $\kappa$-value, then a relatively large one. We take $\alpha \geq 1.5$, i.e. go over the upper margin prescribed for $\alpha$ in the theoretical result. The initial condition is still that given in Figure 3a. The results for $\mu = 100$ and $\kappa = 0.1$ are shown in Figure 5. Subfigures 5a and 5b illustrate the case with $\alpha = \alpha^{**}$ and $\beta = 2$. The solution remains bounded (in the sense of Theorem 1.1 (ii) and Theorem 1.2) even for this critical $\alpha$-value, and even for very large values of $\mu$ (we tested up to $\mu = 700$). This applies to both choices of $J$ (logistic, uniform). Increasing $\alpha$ well beyond $\alpha^{**}$, but below $\alpha^*$, eventually drives the solution into the regime that it is still bounded but not be able to dominated by the steady state. Subfigures 5c and 5d show the solution for $\alpha = 2$ and $\beta = 1.5$ for the logistic and uniform kernel $J$, respectively.

4.3 Discussion

The simulation-based observations in 4.1 and 4.2 suggest that the solution behavior w.r.t. to global boundedness and patterning is influenced not only by the values of $\alpha$, $\beta$, but also by $\mu$, $\kappa$ and the shape of the convolution kernel $J$. Moreover, it seems that the ($\beta$-dependent) bounds established for $\alpha$ in this paper could be non-sharp. It would be interesting to investigate the conditions under which the solution ceases to remain bounded. Concerning the form of the convolution kernel, we expect that there is some $\hat{\alpha}(J, \mu) \geq \alpha^*$, such that for any initial data, the solution still exists and stays bounded for $\alpha^* < \alpha < \hat{\alpha}(J, \mu)$, whereas for $\alpha \geq \hat{\alpha}(J, \mu)$, there exists initial data such that blow-up occurs. Whether and when this model can exhibit pattern formation remains unsolved.

We handled in this paper a PDE describing the dynamics of a single species under linear diffusion and nonlocal intrapopulation interactions. In many applications, however, it turns out that other types of diffusion might be more appropriate to characterize the behavior of a certain population. For instance, the so-called myopic diffusion $\nabla \nabla : (\mathbb{D}(x)u) = \nabla \cdot (\nabla \cdot \mathbb{D}(x)u + \mathbb{D}(x)\nabla u)$ has been obtained in connection with the anisotropic spread of brain tumor cells in the surrounding tissue (with or without proliferation), see e.g. [20, 19, 28] and references therein. This kind of diffusion corresponds to the cells perceiving their surroundings right where they are; the respective PDE is most often obtained either from master equations written for
Figure 3: Initial condition and simulation results for (4.1) with $J$ uniform kernel, $\beta = \kappa = 1$. 

(a) Initial condition

(b) $\alpha = 1.5$, $\mu = 1$

(c) $\alpha = 1.5$, $\mu = 50$

(d) $\alpha = 1.5$, $\mu = 150$

(e) $\alpha = 2$, $\mu = 150$

(f) $\alpha = 4$, $\mu = 150$
Figure 4: Simulation results for (4.1) with $J$ logistic kernel, $\beta = \kappa = 1$.

position jumps between the sites of a lattice (where the transition probabilities depend on the information available at the current cell locations) or from velocity jump processes on the mesoscale, with a subsequent, adequate upscaling (usually of the parabolic type). Further types of diffusion lead to quasilinear equations where the diffusion coefficients depend on the solution in a more or less complicated way. The analysis of such equations with nonlocality is itself a nontrivial problem. Yet other interesting issues relate to nonlocal interactions between at least two different populations or between a population of individuals performing a certain type of tactic motion towards or away from some diffusing or nondiffusing signal. For a review on related nonlocal models we refer to [15] and for more comprehensive reviews in a broader context to e.g., [18, 30].
Appendix: deduction of an equation of type (1.1) from a mesoscopic formulation

We start with the kinetic transport equation

\[ p_t + v \cdot \nabla_x p = \lambda \mathcal{L}[p] + \tilde{\mu} \mathcal{I}[p, p], \tag{4.2} \]

where \( p(x, t, v) \) represents the distribution function of individuals being at time \( t \) in position \( x \) and having velocity \( v \in V \). The velocity space \( V \) is assumed to be bounded, e.g. of the form \( V = [s_1, s_2] \times \mathbb{S}^{N-1} \), where \( s_1, s_2 \) denote the minimal, respectively maximal speed of an individual.

Figure 5: Simulation results for (4.1) with \( \mu = 100, \kappa = 0.1 \).
The right hand side operators describe reorientations of individuals and growth/decay of $p$ due to proliferative/competitive interactions. The coefficients $\lambda, \tilde{\mu} > 0$ represent the turning frequency and the interaction rate, respectively, and are assumed here to be constants. By an appropriate rescaling we get $\lambda = \frac{1}{\varepsilon}, \tilde{\mu} = \varepsilon$, and the above equation (4.2) becomes

$$
\varepsilon p_t + v \cdot \nabla x p = \frac{1}{\varepsilon} \mathcal{L}[p] + \varepsilon \mathcal{I}[p, p].
$$ (4.3)

We define the integral operators as follows:

$$
\mathcal{L}[p](x, t, v) = \int_V \left( T(v, v') p(x, t, v') - T(v', v) p(x, t, v) \right) dv',
$$ (4.4)

$$
\mathcal{I}[p, p](x, t, v) = \frac{p^\alpha(x, t, v)}{\int_V M^\alpha(v) dv} - \frac{\tilde{\kappa}}{\int_V M^{\alpha+\beta}(v) dv} \int \Omega J(x, x') p^\beta(x', t, v) dx'.
$$ (4.5)

Thereby, $\alpha, \beta \geq 1$ and $\tilde{\kappa} > 0$ are constants, $J(x, x')$ is a function weighting the interactions between (groups of) individuals having the same velocity regime within $\Omega \subseteq \mathbb{R}^N$. We assume that $J$ depends on the distance between the interacting (clusters of) individuals and take here $J(x, x') = J(x-x')$, also requiring $J$ to satisfy conditions (1.3). Further, $T(v, v') \geq 0$ is a turning kernel giving the likelihood of an individual having velocity $v'$ to assume the new velocity $v$. The operator $\mathcal{L}$ and its turning kernel are supposed to satisfy the following assumptions:

- $\int_V T(v, v') dv = 1$;
- $\int_V \mathcal{L}[\phi] dv = 0$, for all $\phi$;
- There exists a bounded velocity distribution $M(v) > 0$, not depending on $t, x$, such that the detailed balance condition $T(v, v') M(v') = T(v', v) M(v)$ holds and $\int_V M(v) dv = 1, \int_V v M(v) dv = 0$.
- There exist $c, C > 0$ constants such that $cM(v) \leq T(v, v') \leq CM(v)$, for all $v, v' \in V$, $x \in \Omega$, and $t > 0$.

The following result holds:

---

3We assume that the turning time, i.e. $\frac{1}{\lambda}$ is $\varepsilon$-small when compared to the characteristic time $\tau$ of the mesoscopic dynamics described by (4.2). Moreover, $\tilde{\mu}$ is assumed to be much smaller than $\lambda$: the individuals have a high preference of changing direction rather than interacting and crowding.

4Correspondingly normalized if $\Omega$ is bounded.
Lemma 4.1. (see e.g., [5]) Under the above assumptions the operator $\mathcal{L}$ has the properties:

- $\mathcal{L}$ is self-adjoint in the weighted space $L^2(V, \frac{dv}{M(v)})$;

- For $\psi \in L^2$ there is a unique $\phi \in L^2(V, \frac{dv}{M(v)})$ such that $\mathcal{L}[\phi] = \psi$, which satisfies
  \[ \int_V \phi(v) dv = 0 \quad \text{iff} \quad \int_V \psi(v) dv = 0; \]

- $\text{Ker } \mathcal{L} = <M(v)>$, the vector space spanned by $M(v)$;

- There exists a unique function $\theta(v)$ satisfying $\mathcal{L}[\theta(v)] = vM(v)$.

Specifically, we consider $T(v, v') = M(v)$, which obviously has the required properties. This gives $\theta(v) = -vM(v)$.

Let $u(x, t) = \int_V p(x, t, v) dv$, with $p$ being a solution of (4.3). We decompose $p(x, t, v) = M(v)u(x, t) + \varepsilon g(x, t, v)$, which gives $\int_V g(x, t, v) dv = 0$ and

\[ \partial_t (M(v)u) + \varepsilon \partial_v g + \frac{1}{\varepsilon} vM(v) \cdot \nabla_x u + v \cdot \nabla_v g = \frac{1}{\varepsilon} \mathcal{L}[g] + \mathcal{I}[p, p]. \quad (4.6) \]

Integrating (4.6) with respect to $v$ yields

\[ u_t + \nabla_x \cdot \int_V v g(x, t, v) dv = \int_V \mathcal{I}[p, p] dv. \quad (4.7) \]

Observe that $\mathcal{I}[M(v)u + \varepsilon g, M(v)u + \varepsilon g] = \mathcal{I}[M(v)u, M(v)u] + O(\varepsilon)$.

Then considering as in [4] the orthogonal projection operator onto $\text{Ker } \mathcal{L}$ we have

\[ P_M(h)(v) = M(v) \int_V h(v) dv, \quad h \in L^2(V, \frac{dv}{M(v)}) \]

\[ (I - P_M)(M(v)u) = P_M(g) = 0 \]

\[ (I - P_M)(vM(v) \cdot \nabla_x u) = vM(v) \cdot \nabla_x u. \]

Apply $I - P_M$ to (4.6) to obtain

\[ \varepsilon \partial_t g + \frac{1}{\varepsilon} vM(v) \cdot \nabla_x u + (I - P_M)(v \cdot \nabla_v g) = \frac{1}{\varepsilon} \mathcal{L}[g] + (I - P_M)(\mathcal{I}[p, p]), \quad (4.8) \]

from which

\[ g = \mathcal{L}^{-1}(vM(v) \cdot \nabla_x u) + O(\varepsilon). \quad (4.9) \]

Plugging this into (4.7) leads to

\[ u_t + \int_V v \cdot \nabla_x (\mathcal{L}^{-1}(vM(v) \cdot \nabla_x u)) dv = \int_V \mathcal{I}[M(v)u, M(v)u] dv + O(\varepsilon). \quad (4.10) \]
Now observe that
\[
\int_{V} v \cdot \nabla_x (L^{-1} (v M(v) \cdot \nabla_x u)) dv = \nabla_x \cdot \left( \int_{V} v \otimes \theta(v) dv \cdot \nabla_x u \right),
\]
therefore from (4.10) we formally obtain in the limit $\varepsilon \to 0$ the macroscopic nonlocal PDE
\[
u_t - D \Delta u = u^\alpha \left( 1 - \tilde{\kappa} J \ast u^\beta \right),
\]
where $D = \int_{V} v \otimes v M(v) dv$. In particular, if we consider the uniform velocity distribution $M(v) = \frac{1}{|V|}$ and assume that the individuals can have different orientations, but all preserve the same constant speed $s$, so that $V = s S^{N-1}$, then we have $|V| = \omega_0 s^{N-1}$, with $\omega_0 = |S^{N-1}|$, leading to $D = s^2 / N$. A nondimensionalization leads to the PDE having the form (1.1).

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