Pencils of geodesics in symmetric spaces, 
Karpelevich boundary, 
and associahedron-like polyhedra

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To the memory of F.I.Karpelevich

These notes on the matrix geometry are prepared for the Karpelevich memorial volume of AMS. Our standpoint is the geometric part of his treatise 'The geometry of geodesics and the eigenfunctions of the Beltrami–Laplace operator on symmetric spaces' (1966). The subject of analytical part of his work was the Dynkin's problem of description of the Martin boundary for symmetric spaces, for its solution see [Olsh], [GJT2]. We do not touch this subject.

The existence of the complicated Karpelevich boundary is well known, but in few works (I know only [Kush], [GJT1], [GJT2]) it was really discussed. We give elementary geometric descriptions of the Karpelevich boundary and of some Karpelevich-like constructions. We consider only the spaces $GL(n, \mathbb{R})/O(n)$ and use a minimal necessary language.

Boundaries of symmetric spaces are an old subject arising to the works of Chasles (1864–65), Schubert (1879), Study (1886), and Semple (1946–52) on the enumerative algebraic geometry. Later these boundaries appeared as objects and tools of the analysis on symmetric spaces. Some references are [Sem1], [Sem3], [Sat], [OS], [DCP1], [Pop], [Olsh], [Ner2], [Ner4], [Ner7], [GJT2]. For further references and for the history of the subject, see [Ner4], [GJT2], [Kle].

In the present work we start from pencils of geodesics and gluing of points at infinity as limits of pencils, i.e., we begin from the ordinary differential geometry. As a result, we obtain some elements of the geometry of angles at infinity.

Recall, that in symmetric spaces the usual distance is replaced by the so-called “complex distance” (other terms are “compound distance”, “composite distance”, “angles”, and “stationary angles”). This “distance” is a finite collection of real numbers. Not much is known about this geometrical structure. In last years, after [Klya], the problem of the triangle inequality became popular (see [Ful], [Ner6], [KT], [KLM]). Another fact of the geometry of angles is the “compression of angles” phenomenon (see [Ner1], [Ner3], VI.3, [Ner5], [Kou]).

We show, that the ‘geometry of angles at infinity’ leads to some moduli space like polyhedra as the associahedron, the permutoassociahedron, and the Karpelevich polyhedron; the associahedron was constructed by Stasheff [Sta], see also [DJS], [Kapo], it is a real form of the Deligne–Mumford moduli space of point configurations on a rational curve, the permutoassociahedron was constructed by Kapranov [Kap] (see also [RZ], [DCP2]), an algebraic-geometric counterpart of the Karpelevich polyhedron (polydiagonal blowing) was recently constructed by Ulyanov [Ulya].

Our Section 1 contains preliminaries on the symmetric spaces $GL(n, \mathbb{R})/O(n)$.

In Section 2, we give an explicit description of the Satake–Furstenberg boundary of these spaces.
In Section 3, we discuss pencils of geodesics in symmetric spaces. Following Karpelevich, we define finite pencils, null pencils and solvable pencils.

After this, we add limits of pencils as points of a symmetric space at infinity. Three types of pencils give 4 different boundaries.

The most simple case is discussed in Section 4, limits of all the finite pencils form the so-called visibility sphere at infinity.

Limits of solvable pencils form a noncompact space. This space can be compactified by two similar inductive procedures (Section 5). In these cases, the compactifications of Cartan (flat) subspaces are some combinatorial polyhedrons, namely the permutoassociahedrons and the Karpelevich polyhedrons. They are described explicitly in Section 6. In Section 7, we finish the description of the associahedral and Karpelevich boundaries.

In Section 8, we construct the sea urchin [Ner7] using null pencils.

1. Symmetric spaces GL(n, \( \mathbb{R} \))/O(n) and PGL(n, \( \mathbb{R} \))/PO(n)

1.1. The space GL(n, \( \mathbb{R} \))/O(n). Consider the space \( \mathcal{E}_n \) of positive definite real symmetric matrices of a size \( n \times n \). The general linear group GL(n, \( \mathbb{R} \)) acts on this space by the transformations \( X \mapsto gXg^\top \), where \( X \in \mathcal{E}_n, g \in \text{GL}(n, \mathbb{R}) \); and the sign \( \top \) denotes the transposition. The stabilizer of the point \( X = E \) is the orthogonal group O(n) and hence \( \mathcal{E}_n = \text{GL}(n, \mathbb{R})/\text{O}(n) \).

1.2. The space PGL(n, \( \mathbb{R} \))/PO(n). Denote by \( \mathcal{P}_n \) the space of positive definite \( n \times n \) matrices defined up to a scalar factor: \( X \sim \lambda X, \quad \text{where} \quad X \in \mathcal{E}_n, \lambda > 0 \).

Obviously, \( \mathcal{P}_n = \text{PGL}(n, \mathbb{R})/\text{PO}(n) \); where PGL(n, \( \mathbb{R} \)) is the quotient group of GL(n, \( \mathbb{R} \)) by the group \( \mathbb{R}^\ast \) of scalar matrices, and \( \text{PO}(n) = \text{O}(n)/\{\pm 1\} \).

We also can consider the space \( \mathcal{P}_n \) as the space of all positive definite matrices \( X \) such that \( \det(X) = 1 \), hence \( \mathcal{P}_n = \text{SL}(n, \mathbb{R})/\text{SO}(n) \).

1.3. Quadratic forms. For each \( X \in \mathcal{E}_n \) we define the positive definite bilinear form on \( \mathbb{R}^n \) by

\[
Q_X(v, w) = \frac{1}{2} \sum_{i,j \leq n} x_{ij} v_i w_j, \quad (1.1)
\]

where \( v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n) \in \mathbb{R}^n \), and \( x_{ij} \) are the matrix elements of the matrix \( X \).

Thus we identify \( \mathcal{E}_n \) with the space of positive definite quadratic forms and \( \mathcal{P}_n \) with the space of positive definite quadratic forms defined up to a scalar factor.

1.4. Space of ellipsoids. For any form (1.1) we consider the ellipsoid

\[
\frac{1}{2} \sum_{i,j \leq n} x_{ij} v_i v_j = 1. \quad (1.2)
\]

Thus we identify the space \( \mathcal{E}_n \) with the space of ellipsoids with center at 0. Also we identify the space \( \mathcal{P}_n \) with the space of ellipsoids defined up to a homothety \( v \mapsto \lambda v, \quad \text{where} \quad v \in \mathbb{R}^n, \lambda > 0 \).
1.5. Complex distance. Let $X, Y \in \mathbb{E}_n$. We consider the equation

$$\det(X - \lambda Y) = 0$$

and denote its solutions (they are real and positive) by

$$\lambda_1(X, Y) \geq \lambda_2(X, Y) \geq \ldots \geq \lambda_n(X, Y).$$

**Theorem 1.1.** For $X, Y, X', Y'$ the following conditions are equivalent.

(i) There exists $g \in \text{GL}(n, \mathbb{R})$ such that $gXg^\top = X'$; $gYg^\top = Y'$.

(ii) $\lambda_j(X, Y) = \lambda_j(X', Y')$ for all $j$.

We define the complex distance in $\mathbb{E}_n$ as the collection

$$\psi_j(X, Y) = \ln \lambda_j(X, Y).$$

In the space $\mathbb{P}\mathbb{E}_n$, this collection is defined up to a common additive constant

$$(\psi_1, \ldots, \psi_n) \sim (\psi_1 + \tau, \ldots, \psi_n + \tau).$$

**Theorem 1.2.** \cite{Ner6} Fix $X, Y, Z \in \mathbb{E}_n$. Let $\Psi = (\psi_1, \ldots, \psi_n)$ be the complex distance between $X$ and $Y$, $\Phi = (\varphi_1, \ldots, \varphi_n)$ be the complex distance between $Y$ and $Z$, $\Theta = (\theta_1, \ldots, \theta_n)$ be the complex distance between $X$ and $Z$. Denote by $\mathcal{H}$ the convex hull of all the vectors in $\mathbb{R}^n$ obtained from $\Phi$ by permutations of the coordinates. Then $\Theta \in \Psi + \mathcal{H}$.

1.6. Riemannian metrics. The $\text{GL}(n, \mathbb{R})$-invariant Riemannian metric on $\mathbb{E}_n$ is given by the formula

$$ds^2 = \text{tr} \left( dX \cdot X^{-1} \cdot dX \cdot X^{-1} \right). \quad (1.3)$$

1.7. Geodesics.

**Theorem 1.3.** Any geodesic in $\mathbb{E}_n$ (or $\mathbb{P}\mathbb{E}_n$) has the form

$$\gamma(t) = g \begin{pmatrix} e^{\varphi_1 t} & 0 & \ldots & 0 \\ 0 & e^{\varphi_2 t} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e^{\varphi_n t} \end{pmatrix} g^\top,$$

where $g \in \text{GL}(n, \mathbb{R})$ and $\varphi_1, \ldots, \varphi_n$ are fixed.

**Corollary 1.4.** Geodesic distance between $X, Y \in \mathbb{E}_n$ is

$$\rho(X, Y) = \left[ \sum_j \psi_j^2(X, Y) \right]^{1/2}. \quad (1.4)$$

Let normalize the complex distance in $\mathbb{P}\mathbb{E}_n$ by the condition $\sum \psi_j(X, Y) = 0$. Then the geodesic distance is given by the same formula (1.4).

1.8. Cartan subspaces. Cartan subspaces in $\mathbb{E}_n$ are subspaces of the form

$$L(t_1, \ldots, t_n) = g \begin{pmatrix} e^{t_1} & 0 & \ldots & 0 \\ 0 & e^{t_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e^{t_n} \end{pmatrix} g^\top,$$
where $g \in \text{GL}(n, \mathbb{R})$ is fixed, and $t_j$ ranges in $\mathbb{R}$. Cartan subspaces are totally geodesic submanifolds, The restriction of the Riemannian metric (1.3) to the Cartan subspace $L(t_1, \ldots, t_n)$ is $\sum dt_j^2$. In particular, $L(t_1, \ldots, t_n)$ is flat, and $t_j$ are flat coordinates.

Let us prove this. Let $\sigma$ ranges in diagonal matrices whose eigenvalues are $\pm 1$. Then the maps $X \mapsto \sigma X\sigma^\top$ are involutions, and hence sets of their fixed points are totally geodesic submanifolds. But fixed points for all the such maps are diagonal matrices. This also implies Theorem 1.3.

In the language of 1.4, a Cartan subspace consists of coaxial ellipsoids.

2. Satake–Furstenberg boundary

The object described in this section is called the Satake–Furstenberg compactification of a symmetric space (see [Sat]). Its explicit construction given below arises to Semple [Sem1], [Sem3] and Alguneid [Alg].

2.1. Semple–Satake space. A point of the Semple–Satake space $S_n$ is the following collection (i)–(iii) of the data.

(i) A subset

$$I = \{i_1, i_2, \ldots, i_p\} \subset \{1, 2, \ldots, n-1\},$$

where $p = 0, 1, \ldots, n-1$. It is convenient to assume $i_0 = 0, i_{p+1} = n$.

(ii) A flag

$$\mathbb{R}^n = W_0 \supset W_1 \supset W_2 \supset \cdots \supset W_p \supset W_{p+1} = 0,$$

where for each $k = 1, \ldots, p$

$$\text{codim } W_k = i_k.$$ 

(iii) A collection

$$Q_j; \quad j = 1, \ldots, p+1$$

of positive definite quadratic forms on $W_{j-1}/W_j$ defined up to scalar factors\(^1\).

We denote the piece of $S_n$ corresponding to the collection (2.1) by

$$S_n(I) = S_n(i_1, i_2, \ldots, i_p).$$

Thus,

$$S_n = \bigcup_{I \subset \{1, 2, \ldots, n-1\}} S_n(I).$$

We also denote by $F(I)$ the set of all the flags (2.2) satisfying (2.3).

**Remark.** The set $S_n(\emptyset)$ is $\mathbb{P}\mathbb{E}_n$. A set $F(I)$ is a fiber bundle whose base is the space $F(I)$ and fibers are

$$\prod_{k=1}^{p+1} \mathbb{P}\mathbb{E}_{i_k-i_k-1}. \quad (2.4)$$

\(^1\)equivalently, we have an ellipsoid defined up to a homothety in each subquotient $W_{j-1}/W_j$. 

4
Each expert in semisimple groups can easily translate this form of definition into the root language.

**Remark.** In particular, for \( I = \{1, 2, \ldots, n - 1\} \), the fibers are points, and hence \( S_n(I) = \mathcal{F}(I) \) is the space of complete flags.

A simple calculation shows that
\[
\dim S_n(i_1, \ldots, i_p) = n^2 - 1 - p = \dim \mathbb{P}E_n - p.
\]

Now we will define the topology of a compact metrizable space on \( S_n \), this topology satisfies the property:
\[
\text{Closure of } S_n(I) = \bigcup_{J \supset I} S_n(J).
\]

In particular, the closure of \( S_n(\emptyset) = \mathbb{P}E_n \) is the whole space \( S_n \).

### 2.2. Inductive definition of convergence in \( S_n \)

Assume that the convergence is defined in all the spaces \( S_m \) for \( m < n \). Consider a sequence of \( n \)-dimensional ellipsoids
\[
Q^{(j)} : \frac{1}{2} \sum_{k,l} x_{kl}^{(j)} v_k v_l = 1
\]
defined up to a homothety. We can assume that the shortest semiaxis of each ellipsoid \( Q^{(j)} \) equals 1.

The first necessary condition of convergence is the convergence of ellipsoids (normalized in this way) in the human sense, i.e., the convergence of the corresponding matrices \( X^{(j)} \). Denote the limit by \( Y \).

If \( Y \) is a nondegenerate matrix, then \( Y \) is the a limit in the sense of the space \( S_n \) (and \( Y \in \mathbb{P}E_n \)).

Assume that the matrix \( Y \) is degenerate. Consider the kernel \( \ker Y \), denote its dimension by \( m \); geometrically \( \ker Y \) is the directing subspace of the cylinder
\[
\frac{1}{2} \sum y_{kl} v_k v_l = 1.
\]
In particular, we obtain the ellipsoid (defined up to a homothety) in the quotient space \( \mathbb{R}^n / \ker Y \) and the sequence of ellipsoids \( Q^{(j)} \cap \ker Y \) in the subspace \( \ker Y \). Now, the sufficient and necessary condition of the convergence is the convergence of the sequence \( Q^{(j)} \cap \ker Y \) of ellipsoids in the sense of the space \( S_m \) (where \( m = \dim \ker Y \)).

### 2.3. Examples

Let \( n = 3 \), i.e., we have a sequence of ellipsoids in \( \mathbb{R}^3 \).

**Example 1**, see Fig. 1. Consider the sequence of ellipsoids
\[
Q^{(j)} : x^2 + y^2/j^2 + z^2 = 1.
\]
If \( j \to \infty \), this family of surfaces converges to the cylinder \( x^2 + z^2 = 1 \). Its axis is \( Oy \). Thus we obtain the flag \( \mathbb{R}^3 \supset Oy \supset 0 \) and the circle \( x^2 + z^2 = 1 \) in the quotient \( \mathbb{R}^3/Oy \).

**Example 2.** Consider the sequence of ellipsoids
\[
Q^{(j)} : x^2 + y^2/j^4 + z^2/j^2 = 1.
\]
The limit is the pair of planes \( z^2 = 1 \). The directing plane of \( x^2 = 1 \) is \( yOz \).
The section of \( Q^{(j)} \) by the plane \( x = 0 \) is the ellipse \( y^2/j^4 + z^2/j^2 = 1 \). We
consider quadrics up to a homothety, hence we can replace our equation by
\[ \frac{y^2}{j^2} + z^2 = 1 \]
This sequence of ellipses converges to the pair of lines \( z^2 = 1 \). Its directing line is \( z = 0 \). Finally, we obtain the flag \( \mathbb{R}^3 \supset yOz \supset Oy \) as the limit of the sequence of ellipsoids.

### 2.4. Noninductive definition of convergence in \( \mathbb{S}_n \) (it is used only in 3.9)

Consider a sequence of positive definite matrices \( X^{(j)} \in \mathbb{P}E_n \). Denote their eigenvalues by
\[
\lambda_1^{(j)} \geq \lambda_2^{(j)} \geq \ldots \geq \lambda_n^{(j)}.
\]  
(2.5)

Now we present the conditions for the sequence \( X^{(j)} \) be convergent.

**Condition A.** There exists a separation of the set of eigenvalues \( (2.5) \) into the ‘packets’
\[
(\alpha_{s}^{(j)}), (\beta_{t}^{(j)}), (\gamma_{r}^{(j)}), (\delta_{p}^{(j)}), \ldots
\]
such that \( s, t, r, \text{ etc. are independent on } j \) and the following list of the conditions
\[ 1^\circ - 3^\circ \] is satisfied.

1°. For sufficiently large values of \( j \),
\[
\alpha_{1}^{(j)} \geq \ldots \geq \alpha_{s}^{(j)} \geq \beta_{1}^{(j)} \geq \ldots \geq \beta_{t}^{(j)} \geq \gamma_{1}^{(j)} \geq \ldots
\]

2°. For the eigenvalues from one packet we have
\[
\forall k, m \lim_{j \to \infty} \frac{\alpha_{k}^{(j)}}{\alpha_{m}^{(j)}} \text{ is finite and nonzero}
\]
\[
\forall u, v \lim_{j \to \infty} \frac{\beta_{u}^{(j)}}{\beta_{v}^{(j)}} \text{ is finite and nonzero}
\]
etc.

3°. For the eigenvalues from different packets we have
\[
\forall k, u \lim_{j \to \infty} \frac{\alpha_{k}^{(j)}}{\beta_{u}^{(j)}} = \infty, \quad \forall u, v \lim_{j \to \infty} \frac{\beta_{u}^{(j)}}{\gamma_{v}^{(j)}} = \infty
\]
\[
\forall v, w \lim_{j \to \infty} \frac{\gamma_{v}^{(j)}}{\delta_{w}^{(j)}} = \infty, \quad \text{etc.}
\]
a) The sequence of ellipsoids with semiaxes 1, \(j\), \(j^2\).

b) The pair of planes \(x^2 = 1\).

c) The limit in the Semple–Satake space is the flag: the line \(Oy\) and the plane \(yOz\).

**Condition B.** Denote by the \(V_{\alpha}^{(j)} \subset \mathbb{R}^n\) the subspace spanned by the eigenvectors corresponding to the eigenvalues \(\alpha_1^{(j)}, \ldots, \alpha_s^{(j)}\) of \(X^{(j)}\); in the same way we define \(V_\beta^{(j)}, V_\gamma^{(j)}, \text{etc.}^2\). Consider the subspaces

\[
W_1^{(j)} = V_\beta^{(j)} \oplus V_\gamma^{(j)} \oplus V_\delta^{(j)} \oplus \ldots
\]

\[
W_2^{(j)} = V_\gamma^{(j)} \oplus V_\delta^{(j)} \oplus \ldots
\]

\[
W_3^{(j)} = V_\delta^{(j)} \oplus \ldots
\]

etc. Our requirement is

*for each \(q\) the sequence of subspaces \(W_q^{(j)}\) converges to some \(W_q\).*

Thus, we obtain the flag \(\mathbb{R}^n = W_0 \supset W_1 \supset W_2 \supset \ldots\) and our next purpose is to obtain a quadratic form in each subquotient \(W_q/W_{q+1}\).

**Condition C.** Denote by \(R_q^{(j)}\) the restriction of the bilinear form

\[
Q_X(v, w) = \frac{1}{2} \sum x_{kl} v_k w_l
\]

to the subspace \(W_q^{(j)}\). These forms are defined up to scalar factors, we fix these factors from the condition: the shortest semiaxis of \(R_q^{(j)}\) is 1.

Our last requirement is:

*for each \(q\) the sequence \(R_q^{(j)}\) of bilinear forms converges as \(j \to \infty\).*

\^2For sufficiently large values of \(j\) we have \(\alpha_2^{(j)} > \beta_1^{(j)}\), and hence \(V_\alpha\) is well-defined.
We must say this more carefully, since the forms $R_q^{(j)}$ are defined on different subspaces. For this, consider a sequence of orthogonal operators $h^{(j)} \in \text{SO}(n)$ such that $h^{(j)}$ converges to $E$ and $h^{(j)}W_q^{(j)} = W_q$. Thus we identify $W_q^{(j)}$ and $W_q$. After this we can tell about convergence of bilinear forms on $W_q$.

Denote by $R_q \times q$ the limit of the sequence $R_q^{(j)}$. Evidently, $R_q \times (v, w) = 0$ for $v \in W_q$, $w \in W_{q+1}$. Hence, we obtain the well-defined bilinear form $R_q \times$ on the quotient space $W_q/W_{q+1}$.

Thus, the convergence of a sequence in $\mathbb{P}E_n$ to a point of $S_n$ is defined.

2.5. Convergence on the boundary. As we have seen, each set $S_n(I)$ is a bundle over $F(I)$ with fibers (2.4). Let us compactify each factor $\mathbb{P}E_{i_k-i_{k+1}}$ as $S_{i_k-i_{k+1}}$. Thus we obtain a compactification $\mathbb{S}_n(I)$ of $S_n(I)$.

But we have the obvious embedding $\mathbb{S}_n(I) \to S_n$. This remark also defines the convergence on the boundary.

2.6. Result. **Theorem 2.1.** $S_n$ is a compact metrizable topological space.

3. Pencils of geodesics

The term ‘geodesic’ here and below means a directed geodesic.

Following Karpelevich, in 3.2–3.6 we define and describe explicitly 3 types of pencils of geodesics in $\mathbb{E}_n$ and $\mathbb{P}E_n$. In 3.7-3.9 we define limit points of pencils at infinity. In Section 5, we also need in description of pencils in products of the type $\mathbb{P}E_{k_1} \times \ldots \times \mathbb{P}E_{k_m}$. The necessary modification of the constructions of pencils is given in 3.10-3.12.

A. Definitions and canonical forms of pencils

3.1. Velocities of geodesics. Consider a directed geodesic

$$
\mu(t) = g \begin{pmatrix}
0 & e^{\varphi_1 t} & 0 & \cdots \\
0 & 0 & e^{\varphi_2 t} & \cdots \\
\vdots & \vdots & \ddots & \ddots \\
\end{pmatrix} g^T
$$

with

$$
\varphi_1 \geq \varphi_2 \geq \ldots \geq \varphi_n.
$$

(3.2)

Its velocity in $\mathbb{E}_n$ is the collection of numbers (3.2) defined up to a joint positive factor; this freedom corresponds to the substitution $t = at'$ to (3.1).

The velocity of $\mu(t)$ in $\mathbb{P}E_n$ is the vector (3.2) defined up to transformations

$$
(\varphi_1, \varphi_2, \ldots) \mapsto (a\varphi_1 + b, a\varphi_1 + b, \ldots).
$$

(3.3)

We need in an overfilled system of notation for velocity vectors.

Fix positive integers $\alpha_1, \ldots, \alpha_m$ such that $\sum \alpha_j = n$. Fix real numbers

$$
\psi_1 > \psi_2 > \cdots > \psi_m.
$$

(3.4)

For such data we compose the velocity vector

$$
(\underbrace{\psi_1, \ldots, \psi_1}_\alpha, \underbrace{\psi_2, \ldots, \psi_2}_\beta, \underbrace{\psi_3, \ldots, \psi_3}_\gamma).
$$

(3.5)
In this notation, the geodesic (3.1) can be written in the form

\[
\mu(t) = g \begin{pmatrix}
  e^{\psi_1 t} E_{\alpha_1} & 0 & \cdots \\
  0 & e^{\psi_2 t} E_{\alpha_2} & \cdots \\
  \vdots & \vdots & \ddots
\end{pmatrix} g^\top,
\]

where \( E_{\alpha} \) denotes the unit \( \alpha \times \alpha \) matrix.

We also define the subset

\[
I = \{ i_0, i_1, \ldots, i_m \} \subset \{ 0, 1, 2, \ldots, n \}
\]

by

\[
i_0 = 0,
\]

\[
i_k = \alpha_1 + \cdots + \alpha_k \quad \text{for } k = 1, \ldots, m - 1
\]

\[
i_m = \alpha_1 + \cdots + \alpha_m = n
\]

For any subset \( I = \{ 0, i_1, \ldots, i_{m-1}, n \} \subset \{ 0, 1, 2, \ldots, n \} \) denote by \( \Delta(I) \) the simplex consisting of collections (3.4) defined up to a positive factor; by \( \Delta_o(I) \) denote the set of all collections (3.4) defined up to the equivalence (3.3).

We defined the velocity of a geodesic in the terms of its canonical form. Let us define it in terms of complex distance.

**Lemma 3.1.** Denote by \( e^{\tau_1(t)} \geq e^{\tau_2(t)} \geq \cdots \) the eigenvalues of the matrix \( \mu(t) \) given by (3.1). For all \( j \), we have \( \lim_{t \to +\infty} \tau_j(t)/t = \varphi_j \).

**Corollary 3.2.** Fix \( A \in \mathbb{E}_n \). Denote by \( \sigma_1(t) \geq \sigma_2(t) \geq \cdots \) the complex distance between \( A \) and \( \mu(t) \). Then for all \( j \) we have \( \lim_{t \to +\infty} \sigma_j(t)/t = \varphi_j \).

These statements follow from Theorem 1.2.

**3.2. Null pencils and finite pencils.** Consider a directed geodesic \( \mu(t) \) (in \( \mathbb{E}_n \) or \( \mathbb{PE}_n \)). The corresponding null pencil \( \Pi^{mul}_\mu \) is the set of all the geodesics \( \nu(t) \) such that

\[
\lim_{t \to +\infty} \text{dist}(\mu(t), \nu) = 0,
\]

where the distance between a point \( X \) and a geodesic \( \nu \) is

\[
\text{dist}(X, \nu) := \min_{s \in \mathbb{R}} \rho(X, \nu(s))
\]

We also define the finite pencil \( \Pi^{fin}_\mu \) as the set of all the geodesics \( \nu(t) \) such that there exists a finite limit

\[
\lim_{t \to +\infty} \text{dist}(\mu(t), \nu)
\]

**3.3. Canonical forms of null pencils and finite pencils.** Consider a geodesic \( \gamma \) given by

\[
\gamma(t) = \begin{pmatrix}
  e^{\psi_1 t} E_{\alpha_1} & 0 & \cdots \\
  0 & e^{\psi_2 t} E_{\alpha_2} & \cdots \\
  \vdots & \vdots & \ddots
\end{pmatrix},
\]

(3.9)
numbers \( \psi_j \) satisfy (3.4).

**Theorem 3.3.** a) The finite pencil \( \Pi^\text{fin}_\gamma \) consists of all the geodesics that can be represented in the form

\[
h\gamma(t)h^\top, \tag{3.10}
\]

where \( h \) ranges in \((\alpha_1 + \alpha_2 + \ldots) \times (\alpha_1 + \alpha_2 + \ldots)\) block matrices of the shape

\[
h = \begin{pmatrix}
H_{11} & 0 & 0 & \ldots \\
H_{12} & H_{22} & 0 & \ldots \\
H_{13} & H_{23} & H_{33} & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}. \tag{3.11}
\]

b) The null pencil \( \Pi^\text{nul}_\gamma \) consists of all the geodesics having the form (3.10), where \( h \) ranges in \((\alpha_1 + \alpha_2 + \ldots) \times (\alpha_1 + \alpha_2 + \ldots)\) block matrices of the shape

\[
h = \begin{pmatrix}
E_{\alpha_1} & 0 & 0 & \ldots \\
H_{12} & E_{\alpha_2} & 0 & \ldots \\
H_{13} & H_{23} & E_{\alpha_3} & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}. \tag{3.12}
\]

**Proof.** Let \( \mu \in \Pi^\text{fin}_\gamma \). Theorem 1.2 implies coincidence of the velocities of \( \mu, \gamma \), hence \( \mu(t) = g\gamma(t)g^\top \) for some \( g \in \text{GL}(n, \mathbb{R}) \). By the same theorem, for large \( |t - s| \) the points \( \mu(s) \) and \( \nu(t) \) are far.

Let \( \lambda_k(t) = e^{\psi(t)} \) be the solutions of the equation \( \text{det}(\lambda\gamma(t) - g\gamma(t)g^\top) = 0 \), i.e., \( \psi_k(t) \) is the complex distance between \( \gamma(t) \) and \( \mu(t) \). Equivalently, \( \lambda_k(t) \) are the eigenvalues of \( g\gamma(t)g^\top \). Thus, \( \lambda_k(t)^{1/2} \) are the singular values of \( Z(t) := \gamma(t)^{-1/2}g\gamma(t)^{1/2} \). But the numbers \( \psi_k(t) \) are bounded, hence matrix elements of \( Z(t) \) are bounded. Therefore, \( g \) is triangular.

All geodesics lying in a given pencil have the same velocity. Thus the velocity of a pencil is well defined.

**3.4. Decomposition of finite pencils into null pencils.** Denote by \( P_\gamma \) the group of all the matrices (3.11), and by \( N_\gamma \) the group of all the matrices (3.12). Evidently, \( N_\gamma \) is a normal subgroup in \( P_\gamma \).

The group \( P_\gamma \) acts on \( \Pi^\text{fin}_\gamma \) by the transformations \( h : \mu(t) \mapsto h\mu(t)h^\top \) and the subgroup \( N_\gamma \) transfers the null pencil \( \Pi^\text{nul}_\gamma \) to itself. Moreover, \( N_\gamma \) transfers each null subpencil of \( \Pi^\text{nul}_\gamma \) to itself.

Let \( \mu_1, \mu_2 \in \Pi^\text{fin}_\gamma \). We say that \( \mu_1 \sim \mu_2 \) if they lie in one null pencil. We denote by \( \tilde{\Pi}_\gamma \) the quotient of \( \Pi^\text{fin}_\gamma \) by this equivalence relation. The group

\[
P_\gamma/N_\gamma \simeq \text{GL}(\alpha_1, \mathbb{R}) \times \text{GL}(\alpha_2, \mathbb{R}) \times \ldots
\]

acts on \( \tilde{\Pi}_\gamma \) in a natural way, and we obtain

\[
\tilde{\Pi}_\gamma \simeq \text{GL}(\alpha_1, \mathbb{R})/O(\alpha_1) \times \text{GL}(\alpha_2, \mathbb{R})/O(\alpha_2) \times \cdots = E_{\alpha_1} \times E_{\alpha_2} \times \ldots \tag{3.13}
\]
For $\nu_1, \nu_2 \in \Pi_1^{\text{fin}}$ we can define the distance at infinity
\[
\text{dist}(\nu_1, \nu_2) = \lim_{t \to \infty} \inf_{s \in \mathbb{R}} \rho[\nu_1(t), \nu_2(s)].
\]
Then this distance is the geodesic distance in the symmetric space (3.13).

3.5 Solvable pencils. Finite pencils and null pencils have sense for any space of nonpositive curvature. For symmetric spaces there exists a natural intermediate equivalence of geodesics.

In the previous subsection, we constructed the map
\[
\Pi_1^{\text{fin}} \to \prod_j \text{GL}(\alpha_j, \mathbb{R})/\text{O}(\alpha_j).
\]

The symmetric space $\prod \text{GL}(\alpha_j, \mathbb{R})/\text{O}(\alpha_j)$ is not semisimple. Consider its natural projection to the semisimple space $\prod_j \text{PGL}(\alpha_j, \mathbb{R})/\text{PO}(\alpha_j) \simeq \prod_j \mathbb{E}_{\alpha_j}$. Thus, we obtain the canonical map
\[
\Pi_1^{\text{fin}} \to \prod_j \text{PGL}(\alpha_j, \mathbb{R})/\text{PO}(\alpha_j) \simeq \prod_j \mathbb{E}_{\alpha_j}. \tag{3.14}
\]

We say that $\nu_1, \nu_2$ are elements of one solvable pencil if their images under this map coincide.

For any geodesic $\gamma$, we denote by $\Pi_1^{\text{solv}}$ the corresponding solvable pencil. Obviously, we have
\[\Pi_1^{\text{fin}} \supset \Pi_1^{\text{solv}} \supset \Pi_1^{\text{nul}}.\]

3.6 Canonical forms of solvable pencils.

Proposition 3.4. Let a geodesic $\gamma$ has the form (3.9). Then the corresponding solvable pencil $\Pi_1^{\text{solv}}$ consists of all the geodesics having the form $h\gamma(t)h^\top$, where $h$ is an $(\alpha_1 + \alpha_2 + \ldots) \times (\alpha_1 + \alpha_2 + \ldots)$ block matrix of the shape
\[
h = \begin{pmatrix}
\tau_1 E_{\alpha_1} & 0 & 0 & \ldots \\
H_{12} & \tau_2 E_{\alpha_2} & 0 & \ldots \\
H_{13} & H_{23} & \tau_3 E_{\alpha_3} & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad \tau_j \in \mathbb{R}. \tag{3.15}
\]

The set of solvable pencils in a given finite pencil (3.10)–(3.11) is parametrized by the collection of diagonal blocks $H_{jj}H_{jj}^\top$ of the matrix (3.10), these blocks are defined up to a multiplication by positive scalars.

B. Boundary data for pencils

3.7 Boundary data for solvable pencils.

Lemma 3.5. a) Each geodesic $\mu(t)$ has a limit in $\mathbb{S}_n$ as $t \to \infty$.

b) If $\mu_1, \mu_2$ lie in one solvable pencil, then their limits in $\mathbb{S}_n$ coincide.

In fact, evaluation of the limit of a geodesic in $\mathbb{S}_n$ is reduced to evaluation of limit of a family of coaxial ellipsoids. For instance, let us describe explicitly the limit of a geodesic $\gamma$ given by (3.9). Denote by $e_j$ the standard basis in
Denote by $W_k$ the subspace $W_k = \oplus_{p>i} \mathbb{R}^p$. Thus we obtain the flag $\mathbb{R}^n = W_0 \supset W_1 \supset \ldots$. The ellipsoids in the quotients $W_{k-1}/W_k$ are spheres.

Now let $\mu(t)$ be another geodesic of the same finite pencil, i.e., $\mu(t) = h\gamma(t)h^\top$ with $h$ given by (3.11). Then the limit flag is the same, and the quadratic forms in the quotients $W_{k-1}/W_k$ are $Q_k = H_{kk}H_{kk}^\top$, they are defined up to a multiplication by a scalar factor.

The following statement becomes obvious.

**Theorem 3.6.** A solvable pencil is uniquely determined by its velocity and its limit in the space $\mathbb{S}_n$.

Consider a geodesic $\mu(t)$, whose velocity is contained in the simplex $\Delta(I)$, see 3.1. As we have explained above, $\lim\mu(t)$ in the sense of $\mathbb{S}_n$ belongs $\mathbb{S}_n(I)$; the strata $\mathbb{S}_n(I)$ were defined in 2.1.

**Corollary 3.7.** Fix a set $I$. Denote by $\mathcal{R}(I)$ the space of all the solvable pencils in $\mathbb{P}\mathbb{E}_n$, whose velocities have the form (3.5). Then

$$\mathcal{R}(I) \simeq \Delta_0(I) \times \mathbb{S}_n(I)$$

and

$$\dim \mathcal{R}(I) = n^2 - 2 = \dim \mathbb{P}\mathbb{E}_n - 1 \quad (3.16)$$

**3.8. Boundary data for finite pencils.** As we have seen, each geodesic $\mu(t)$ has a limit in $\mathbb{S}_n$. In particular, we have a canonically defined flag in $\mathbb{R}^n$, we call it by the limit flag.

**Proposition 3.8.** Two geodesics lie in one finite pencil iff their limit flags and their velocities coincide.

Thus, for any finite pencil we associate the following collection of data.

1) A set $I = \{0, i_1, \ldots, i_k - 1, n\} \subset \{0, 1, 2, \ldots, n\}$, where $k > 1$.
2) A point of the simplex $\Delta(I)$ (or $\Delta_0(I)$ for $\mathbb{P}\mathbb{E}_n$).
3) A flag lying in $\mathcal{F}(I)$ (see Subsection 2.1).

**3.9. Boundary data for null pencils in $\mathbb{E}_n$.** Obviously, in this case we must remember more than in the case of solvable pencils.

Consider a geodesic $\mu \subset \mathbb{E}_n$, whose velocity has the form (3.5). Let us fix a parameter $t$ on $\mu$. This means that we fix an origin $X_0 \in \mu$ and we fix the velocity vector $(\psi_1, \psi_2, \ldots)$ literally (without any equivalence).

Preserving the notation of 2.1, we change the construction of 2.4 in one place.

Thus, we have the family of quadratic forms $Q(t)$ on $\mathbb{R}^n$ corresponding to points of the geodesic $\mu(t)$. After obtaining the one-parametric family of flags $\mathbb{R}^n = W_0(t) \supset W_1(t) \supset W_2(t) \supset \ldots$, we consider the restriction $R_p(t)$ of the form $Q(t)$ to the subspace $W_{p-1}(t)$. Then we consider

$$R_p^\infty := \lim_{t \to +\infty} e^{-\psi t}R_p(t)$$

This is a well defined nondegenerate symmetric bilinear form on $W_{p-1}$. The subspace $W_p$ is the kernel of this form, and finally we obtain a nondegenerate symmetric bilinear form $R_p^\infty$ on each subquotient $W_{p-1}/W_p$. 

12
We emphasize, that in Section 2 the forms $R_p^\square$ on subquotients were defined up to positive factors. Now they are defined literally.

But we started from a geodesic with a fixed parametrization.

If we multiply the velocity $(\psi_1, \psi_2, \ldots)$ by a scalar and leave the origin, then our limit data (the flag $W_p$ and the forms $R_p^\square$) do not change.

If we move the origin $X_0$ along the geodesic, then the collection $R_p^\square$ changes in the following way

$$(R_1^\square, R_2^\square, \ldots) \mapsto (e^{\psi_1 s}R_1^\square, e^{\psi_2 s}R_2^\square, \ldots) \quad \text{for some } s \in \mathbb{R}. \quad (3.17)$$

Thus for any null pencil, we associate the following boundary data

1) A set $I = \{0, i_1, \ldots, i_{m-1}, n\} \subset \{0, 1, 2, \ldots, n\}$, where $m > 1$.
2) A point of the simplex $\Delta(I)$.
3) A flag $W_1 \supset W_2 \supset \cdots \supset W_{m-1}$ lying in $F(I)$.
4) The family $(R_1^\square, R_2^\square, \ldots, R_k^\square)$ of positive definite quadratic forms on subquotients $W_{p-1}/W_p$ defined up to the equivalence (3.17).

**Theorem 3.9.** The space of all null pencils is in one to one correspondence with the collections of data 1-4.

### C. Pencils in products of symmetric spaces

**3.10. Abstract definition of solvable pencils.** Let us give a definition of a solvable pencil in a semisimple Riemannian symmetric spaces $G/K$; $K$ is a maximal compact subgroup in $G$. The only case interesting for us is $G/K = \prod_{\tau=1}^{\beta} E_{m_\tau}$.

First, consider the group $P_\gamma$ of all the isometries of $G/K$ mapping a finite pencil $\Pi_{\gamma}^{fin}$ to itself. It is a parabolic subgroup [Kar], and it acts transitively on $\Pi_{\gamma}^{fin}$. Denote by $L_\gamma$ the Levi factor in $P_\gamma$ (maximal reductive subgroup) Denote by $K_\gamma$ the maximal compact subgroup in $L_\gamma$.

As in 3.5, the space of all null-pencils in $\Pi_{\gamma}^{fin}$ can be identified with $L_\gamma/K_\gamma$. It is a reductive symmetric space, hence it is a product $S_\gamma \times L_\gamma$, where $S_\gamma$ is a semisimple symmetric space and $L_\gamma$ is an Euclidean space. We say that two elements of $\Pi_{\gamma}^{fin}$ lie in one solvable pencil if their images under the map $\Pi_{\gamma}^{fin} \rightarrow L_\gamma/K_\gamma \rightarrow S_\gamma$ coincide.

**3.11. Vel-geodesics.** Consider a Riemannian manifold $M$. We say, that a parameter $t$ on a geodesic $\gamma$ is semi-natural, if the Riemannian length of the tangent vector $\|\gamma'(t)\| = c$ is independent on $t$. We say, that a vel-geodesic is a directed geodesic with a fixed constant $c$.

**3.12. Pencils in products of spaces $E_{m_\tau}$.** Consider a vel-geodesic $\gamma(t)$ in the space

$$\prod_{\tau=1}^{\beta} P^E_{m_\tau}. \quad (3.18)$$

Let $\gamma_\tau(t)$ be projections of $\gamma(t)$ to $P^E_{m_\tau}$.

We define a solvable vel-pencil of geodesics as a solvable pencil with fixed constant $c$ as in 3.11.

**Lemma 3.10.** Geodesics $\gamma, \mu$ lie in one solvable vel-pencils, iff for each $\tau$ the geodesics $\gamma_\tau, \mu_\tau$ lie in one solvable vel-pencil.
In particular, the space of all solvable pencils in our space is parametrized by the following collections of data

A. Family $I_1, \ldots, I_\beta$ of subsets

$$I_\tau = \{0, i_1^{(\tau)}, i_2^{(\tau)}, \ldots, i_{j_\tau}^{(\tau)}, m_\tau\} \subset \{0, 1, 2, \ldots, m_\tau - 1, m_\tau\}$$

B. A collection of numbers

$$\psi_1^{(1)} > \cdots > \psi_{j_1+1}^{(1)}; \quad \psi_1^{(2)} > \cdots > \psi_{j_2+1}^{(2)}; \quad \ldots.$$  (3.19)

defined up to the equivalence

$$\{[\psi_1^{(1)}, \ldots, \psi_{j_1+1}^{(1)}], [\psi_1^{(2)}, \ldots, \psi_{j_2+1}^{(2)}], [\psi_1^{(3)}, \ldots, \psi_{j_3+1}^{(3)}], \ldots\} \sim \{[a\psi_1^{(1)} + b_1, \ldots, a\psi_{j_1+1}^{(1)} + b_1], [a\psi_1^{(2)} + b_2, \ldots, a\psi_{j_2+1}^{(2)} + b_2],$$

$$[a\psi_1^{(3)} + b_3, \ldots, a\psi_{j_3+1}^{(3)} + b_3], \ldots\}. \quad \text{(3.20)}$$

C. A point of $S_{m_1}(I_1) \times \cdots \times S_{m_\beta}(I_\beta)$.

We denote by $\Delta_{\circ}(I_1, \ldots, I_\tau)$ the set of all the collections (3.19) defined up to the equivalence (3.20).

4. Finite pencils. Matrix sky and its tiling

Now we want to construct an ideal boundary of $E_n$ or $PE_n$ as a set of limits of pencils of geodesics. All the three types of pencils are available for this purpose, but the final results in these three cases are essentially distinct.

In this section, there is no difference between $E_n$ and $PE_n$. For definiteness, we discuss $E_n$.

4.1. Sphere at infinity.

PROPOSITION 4.1. For each point $X \in E_n$ and each finite pencil $\Pi^{fin}_\gamma$, there exists a unique geodesic $\mu \in \Pi^{fin}_\gamma$ passing the point $X$.

PROOF. Consider the geodesic $\gamma(t)$ given by (3.9). Denote by $P_\gamma$, the group of all the matrices (3.11). First, each positive matrix can be represented in the form $hh^\top$, where $h \in P_\gamma$. Therefore a finite pencil $\Pi^{fin}_\gamma$ sweep all space $E_n$.

Second, it is easy to check, that a geodesic $hh^\top$, where $h \in P_\gamma$, has no intersections with $\gamma$ or coincide with $\gamma$. \qed

Fix a point $X_0 \in E_n$ (to be concrete, let $X_0 = E$). Denote by $T$ the tangent space to $E_n$ at $X_0$. Evidently, we can consider $T$ as the space of all symmetric matrices. By $PT$ we denote the set of all rays in $T$: a ray is a set of the form $\theta v$, where a nonzero vector $v \in T$ is fixed and $\theta$ ranges in positive numbers.

For each ray $\xi$, we consider the geodesic $\gamma_\xi$ passing through $X_0$ in the direction $\xi$.

By Proposition 4.1, the set of all finite pencils is in one-to-one correspondence with the space $PT$.

Now we are ready to glue the sphere $S^{far}$ at infinity to $E_n$. Points $A_\xi$ of the sphere $S^{far}$ are enumerated by rays $\xi \in PT$. It remains to define the convergence.
Consider a sequence \( Z_1, Z_2, \ldots \in \mathbb{E}_n \). Consider the geodesics \( \gamma_{\xi_j} \) connecting \( X_0 = E \) with \( Z_j \). The sequence \( Z_j \) converges to a point \( A_\xi \) iff \( \lim_{j \to \infty} \rho(E, Z_j) = \infty \) and \( \lim_{j \to \infty} \xi_j = \xi \).

4.2. Tilling of \( S^{\text{far}} \). Thus we identified the space of finite pencils with the sphere \( S^{\text{far}} \). Another parametrization of the same space was given above in Subsection 3.8. This parametrization gives a canonical tilling of the sphere \( S^{\text{far}} \) by a continual family of open simplices.

Consider an arbitrary flag \( W \in F(I) \), see 2.1. Consider the set \( \Delta(W) \) of all finite pencils, whose limit flag (see Subsection 3.8) is \( W \). By 3.8, \( \Delta(W) \simeq \Delta(I) \).

**Proposition 4.2.** The closure of \( \Delta(W) \) is \( \bigcup W' \subset W \Delta(W') \) where \( \{W'\} \) ranges in all subflags of \( W \).

**Remark.** This structure is called **Tits building at infinity**. Its abstract definition for an arbitrary space of nonpositive curvature is contained in [BGS].

5. Solvable pencils. Karpelevich and associahedral boundaries

In this section we consider the symmetric spaces \( PE_n \).

5.1. **The inductive definition of the associahedral boundary.** We intend to construct the associahedral compactification \( \text{Ass}(PE_n) \) of the spaces \( PE_n \) (see [Ner4]). First, we will describe these compactifications as disjoint unions of sets (as it was done above for \( S_n \)).

The existence of a natural topology on \( \text{Ass}(PE_n) \) is claimed in Theorem 5.1, the explicit construction is contained in Section 7; before this, in Section 6 we describe the closure of a Cartan subspace in the associahedral compactification.

The construction of the compactification is inductive. Assume that \( \text{Ass}(PE_k) \) is constructed for all \( k < n \).

For any solvable pencil \( \Pi^\text{solv}_s \), we define its limit point at infinity as the corresponding collection of the boundary data from Subsection 3.7.

By Corollary 3.7, the boundary obtained in this way is the union of \( 2^{n-1} - 1 \) disjoint pieces \( \mathcal{R}(I) \) having the same dimension \( n^2 - 2 = \dim PE_n - 1 \). Each piece has the form

\[
\mathcal{R}(I) = \Delta_c(I) \times S_n(I).
\]

(5.1)

The space \( S_n(I) \) is a bundle, whose base is the space of (noncomplete) flags \( F_n(I) \) (defined above in 2.1) and fibers are the symmetric spaces

\[
\prod_s PE_{i_s-i_{s-1}}.
\]

(5.2)

We will call these fibers by **boundary symmetric spaces**.

Now we assume that each fibre (5.2) already is compactified as

\[
\prod_s \text{Ass}(PE_{i_s-i_{s-1}}),
\]

(5.3)

all the spaces \( \text{Ass}(PE_{i_s-i_{s-1}}) \) are defined by the inductive hypothesis.

Thus, the boundary is constructed.
Remark. We compactified the factor $S_n(I)$ in (5.1). We emphasis that the simplexes $\Delta_s(I)$ are not compact and hence a topology of a compact space is yet not defined.

5.2. Existence of topology.

Theorem 5.1. There exists a topology of a compact metrizable space on each $Ass(\mathbb{P}\mathbb{E}_n)$ such that for any geodesic $\gamma$ in the space $\mathbb{P}\mathbb{E}_n$ or in any boundary symmetric space the limit $\lim_{t \to +\infty} \gamma(t)$ with respect to this topology coincides with the limit in the sense defined above.

5.3. Inductive construction of the Karpelevich boundary. Now we intend to construct the Karpelevich compactification $Karp(\mathbb{P}\mathbb{E}_n)$ of $\mathbb{P}\mathbb{E}_n$. Its inductive construction given below involves the Karpelevich compactifications of all the spaces $\prod \mathbb{P}\mathbb{E}_n$. Thus, assume that the compactifications $Karp(\mathbb{P}\mathbb{E}_{k_1} \times \cdots \times \mathbb{P}\mathbb{E}_{k_l})$ are already constructed for all the collections $(k_1, \ldots, k_l)$ such that $k_1 + \cdots + k_l < n$.

Consider a space $E := \mathbb{P}\mathbb{E}_{m_1} \times \cdots \times \mathbb{P}\mathbb{E}_{m_\beta}$, where $m_1 + \cdots + m_\beta = n$.

For each solvable pencil $\Pi^{\text{sol}}$ in $E$, we add formally a corresponding point at infinity. Thus the set of all such points is a disjoint union of strata $\mathcal{R}(I_1, \ldots, I_\beta)$ described above in Subsection 3.12.

Each stratum is a product of some polyhedron $\Delta_s(I_1, I_2, \ldots)$ and the set $S_{m_1}(I_1) \times \cdots \times S_{m_\beta}(I_\beta)$. The latter set is a bundle, whose base is $\prod \mathcal{F}_{m_\tau}(I_\tau)$ and fibers have the form

$$\prod_{\tau=1}^{\beta} \prod_{k=1}^{j_\tau} \mathbb{P}\mathbb{E}_{(r)}_{k-1}^{(r)}.$$ 

After this, we replace each fibre by its compactification

$$Karp\left(\prod_{\tau=1}^{\beta} \prod_{k=1}^{j_\tau} \mathbb{P}\mathbb{E}_{(r)}_{k-1}^{(r)}\right).$$

These space are constructed by the inductive hypothesis.

5.4. Existence theorem.

Theorem 5.2. The exists a topology of a compact metrizable space on each set $Karp(\mathbb{P}\mathbb{E}_{m_1} \times \cdots \times \mathbb{P}\mathbb{E}_{m_\beta})$ and limits of geodesics with respect to this topology coincide with limits constructed above.

6. Permutoassociahedron and Karpelevich polyhedron.

A. Definition of permutoassociahedron and karpelevich-hedron

6.1. Spaces $\Xi(I)$. Let $I$ be a finite set, denote by $#(I)$ the number of its elements; the basic example is the set $I = J$:

$$J = J_n := \{1, 2, \ldots, n\}.$$

Denote by $\Xi(I)$ the set of all functions $I \to \mathbb{R}$ defined up to an addition of a constant function; we also denote

$$\Xi_n := \Xi(J).$$
The space $\Xi_n$ consists of vectors $(\varphi_1, \ldots, \varphi_n) \in \mathbb{R}^n$ defined up to the equivalence

$$(\varphi_1, \ldots, \varphi_n) \sim (\varphi_1 + a, \ldots, \varphi_n + a).$$

We also can consider elements of $\Xi_n$ as ordered collections of points on $\mathbb{R}$ defined up to a translation.

For a subset $K \subset I$, we have the natural map

$$\Xi(I) \to \Xi(K)$$

(we forget part of coordinates).

Consider a partition $\alpha$ of the set $I$, denote by $I/\alpha$ the corresponding quotient and by $\pi : I \to I/\alpha$ the natural projection. We have a natural embedding

$$\Xi(I/\alpha) \to \Xi(I).$$

(6.2)

i.e., to a function $f : I/\alpha \to \mathbb{R}$ we assign the function $f \circ \pi$.

Again, consider a partition $\alpha$ of $I$, let $I_1, \ldots, I_s$ be its elements. Denote by $C[I; \alpha]$ the space of functions $I \to \mathbb{R}$ that are constants on each subset $I_m$. Consider the quotient linear space

$$\Xi[I; \alpha] := \Xi(I)/C[I; \alpha].$$

(6.3)

We have natural projection map

$$\Xi(I) \to \Xi[I; \alpha].$$

(6.4)

Also, we have the obvious identification

$$\Xi[I; \alpha] \simeq \bigoplus_{m=1}^s \Xi(I_m).$$

(6.5)

If a partition $\beta$ is a subdivision of $\alpha$, then we have the map

$$\Xi[I; \alpha] \to \Xi[I; \beta]$$

(6.5a)

We call by walls the hyperplanes $f(a) = f(b)$, where $a, b \in I$. These hyperplanes divide the space $\Xi(I)$ into $\#(I)!$ simplicial cones, which are called Weyl chambers.

**Example.** Tilings of $\Xi_3$, $\Xi_4$ by Weyl chambers are presented on Fig. 3.

Now assume that the set $I$ is an ordered set with the order $\prec$. Then we have the positive Weyl chamber $\Lambda^+(I)$ defined by the inequalities

$$a \prec b \Rightarrow f(a) \geq f(b)$$

We also denote $\Lambda_n^+ := \Lambda^+(J)$.

**6.2. Compactification of the spaces** $\Xi(I)$. Let $V$ be a linear space. A ray is a subset in $V$ having the form $\lambda v$, where $v \neq 0$ is a fixed vector in $V$ and $\lambda$ ranges in positive numbers.
a) The space $\Xi_3 \simeq \mathbb{R}^2$, the lines $\varphi_i = \varphi_j$, and the simplex $\partial \Lambda$ at infinity

b) $24$ Weyl chambers in $\Xi_4 \simeq \mathbb{R}^3$. Intersections of planes $\varphi_i = \varphi_j$ and the surface of cube.

Fig. 3.

We compactify each ray by a point at infinity. Denote the set of all such points at infinity by $\partial V$ (sphere at infinity). By $\overline{V}$ we denote $V \cup \partial V$. We define a topology on $\overline{V}$ in the obvious way.

In particular, we obtain the spaces

$$
\Xi(I) = \Xi(I) \cup \partial \Xi(I), \quad \Xi[I; a] = \Xi[I; a] \cup \partial \Xi[I; a].
$$

We emphasize (compare with (6.5)), that

$$
\Xi[I; a] \neq \prod \Xi(I_k).
$$

For an ordered set $I$ we denote by $\Lambda^+(I)$ the closure of the positive Weyl chamber $\Lambda^+(I)$ in $\Xi(I)$ and by $\partial \Lambda^+(I) := \Lambda^+(I) \setminus \Lambda^+(I)$ its boundary.

This allows to consider $\Xi(I)$ as a polyhedron; the space $\Xi(I)$ is its interior and the boundary $\partial \Xi(I)$ is divided into simplexes of the type $\partial \Lambda^+$. This point of view is also represented on Fig. 3.

6.3. Definition of the permutoassoocihedron. For each subset $I \subset J$ consider the 'forgetting' map $\Xi_n \to \Xi(I)$, see (6.1). Consider the diagonal embedding

$$
\iota_n : \Xi_n \to \prod_{I \subset J} \Xi(I).
$$

We also have the inclusion

$$
\prod_{I \subset J} \Xi(I) \subset \prod_{I \subset J} \Xi(I). \quad \text{(6.6)}
$$

DEFINITION. The permutoassociahedron $\text{Pass}_n$ (see Kap) is the closure of image $\iota_n(\Xi_n)$ in $\prod_{I \subset J} \Xi(I)$.

6.4. Definition of the Karpelevich polyhedron. For each partition $a$ of $J$, we have the map $\Xi_n \to \Xi[J; a]$, see (6.4). Consider the diagonal embedding

$$
\Xi_n \to \prod_a \Xi[J; a] \subset \prod_a \Xi[J; a]. \quad \text{(6.7)}
$$
where the product is given over all the partitions \( a \) of \( J \).

**Definition.** The Karpelevich polyhedron \( \text{Karp}_n \) is the closure of the image of \( \Xi_n \) in the space \( \prod \Xi[J; a] \).

**Remark.** Assume that \( a \) consists of a subset \( I \) and single-element sets. Then \( \Xi[J; a] = \Xi(I) \). Thus each factor of the product (6.6) is a factor of the product (6.7), and hence we obtain the natural projection \( \text{Karp}_n \rightarrow \text{Pass}_n \).

**B. More notation**

6.5. Some functorial properties of spheres at infinity. This is used only in Subsections 6.9 and 6.15.

1) Fix a subset \( K \subset I \). Denote by \( C(K) \) the space of functions on \( I \) which are constants on \( K \). The map (6.1) induces the continuous map
\[
\partial \Xi(I) \\setminus \partial C(K) \rightarrow \partial \Xi(K).
\] (6.8)

2) Let \( a \) be a partition of \( I \) with elements \( I_k \). The map (6.2) induces the embedding
\[
\Xi(I/a) \rightarrow \Xi(I).
\] (6.9)

3) Let \( C[I; a] \) be the same as in 6.1. The map (6.4) induces the continuous map
\[
\partial \Xi(I) \\setminus \partial C[I; a] \rightarrow \Xi[I; a].
\]

4) Let \( a \) be a partition of \( I \), let \( b \) be a subdivision of \( a \). The map (6.5.a) induces the map
\[
\partial \Xi[I; a] \\setminus \partial C[I; b] \rightarrow \partial \Xi[I; b]
\] (6.9.a)

6.6. Notation for sphere at infinity outside walls. For each wall \( f(a) = f(b) \) in \( \Xi(I) \) denote by \( S_{a,b} \) its intersection with \( \partial \Xi(I) \). The \( (#(I) - 3) \)-dimensional spheres \( S_{a,b} \) divide the \( (#(I) - 2) \)-dimensional sphere \( \partial \Xi(I) \) into \( #(I)! \) simplexes. Denote
\[
\partial \Xi(I)\text{gen} := \partial \Xi(I) \setminus \cup S_{a,b};
\]
see Fig. 3b, it is the surface of cube without edges and diagonals of faces.

Also for the Weyl chamber \( \Lambda^+(I) \) we denote by
\[
\partial \Lambda^+(I)\text{gen} := \partial \Lambda^+(I) \cap \partial \Xi(I)\text{gen}.
\]

Now, let \( a \) be a partition of \( I \), let \( I_k \) be its elements. We say that a ray \( tf \), where \( t > 0 \), \( f \in \Xi[I; a] \) is generic, if for each \( I_k \) and each \( a, b \in I_k \) we have \( f(a) \neq f(b) \). We define the set \( \partial \Xi[I; a]\text{gen} \subset \partial \Xi[I; a] \) as the set of limits of generic rays.

6.7. Combinatorial partition-like structures.

**Partitions.** Consider a finite set \( M \). Its partition is a representation of \( M \) as a disjoint union of subsets.

**Tree-partitions.** A system \( A \) of subsets of \( M \) is a tree-partition if the following conditions are hold
The tree-partition is (1234567), (123), (4567), (23), (45), (67), (1), (4), (5), (6), (7). The leveled tree-partition is (1234567); (123) (4567); (1) (23) (4567); (1) (23) (45) (67); (1) (23) (4) (5) (6) (7).

a) \( M \in \mathcal{A} \)
b) For \( I_1, I_2 \in \mathcal{A} \), we have either \( I_1 \cap I_2 = \emptyset \), or \( I_1 \supset I_2, I_1 \subset I_2 \).
c) Let \( I \supset K \) be elements of \( \mathcal{A} \). Then there exists a collection \( K_1 = K, K_2, \ldots, K_n \in \mathcal{A} \) such that

\[
I = \bigcup K_j, \quad K_i \cap K_j = \emptyset \text{ for } i \neq j
\]

A subset \( I \in \mathcal{A} \) is irreducible, if there is no \( K \in \mathcal{A} \) such that \( K \subset I \).

For a reducible subset \( I \in \mathcal{A} \) there exists its unique minimal decomposition (6.10) such that for \( L \in \mathcal{A} \) satisfying \( I \supset L \supset K_j \) we have \( L = I \) or \( L = K_j \).

Another definition of tree-partitions. Consider a set \( M \). Consider a partition \( \pi \) of \( M \). For each element \( K_j \in \pi \), consider a partition \( \eta_j \) of \( K_j \). Then we repeat the same with elements of partitions \( \eta_j \), etc. Obviously, we obtain a tree-partition of \( M \).

Leveled tree-partitions. Consider a finite set \( M \). Its leveled tree-partition \( \mathcal{A} \) is a family of partitions

\[
a_0, a_1, \ldots, a_k
\]

satisfying the conditions

a) \( a_0 \) consists of the set \( M \) itself.
b) \( a_{m+1} \) is a subdivision of \( a_m \)
c) \( a_{m+1} \neq a_m \) for all \( m \).

For a leveled tree-partition \( \mathcal{A} \) of \( M \) consider \( \cup a_m \) (i.e., we consider all the elements of all the partitions \( a_m \)). Obviously, we obtain a tree-partition of \( M \).

Segmental partitions. Let \( M \) be an ordered set. Segments \([a, b] \subset M \) are subsets having the form \( a \prec j \prec b \). A segmental partition (tree-partition, leveled tree-partition) is a partition, all whose elements are segments.

Perfect tree-partitions. A tree-partition is perfect if all its irreducible elements are singletons.

C. Description of permutoassociahedron

6.8. Stratification of the permutoassociahedron. The permutoassociahedron \( \text{Pass}_n \) was defined as a subset in the polyhedron \( \prod \Xi(I) \), see (6.6). Considering the intersections of \( \text{Pass}_n \) with faces of \( \prod \Xi(I) \), we obtain a natural stratification of \( \text{Pass}_n \).
Fix a tree-partition \( \mathfrak{A} \) of \( J \). First, for any element \( K \in \mathfrak{A} \), we intend to define a set \( U_{\mathfrak{A}}(K) \)

a) For an irreducible \( K \), we assume \( U_{\mathfrak{A}}(K) := \Xi(K) \).

b) Let \( K \) be reducible. Let \( \tau \) be its minimal decomposition, and \( h(K) \) be the number of its elements. Then \( U_{\mathfrak{A}}(K) := \partial\Xi(K/\tau)^{\text{gen}} \), see Fig.5.

REMARK. In the reducible case, the set \( U_{\mathfrak{A}}(K) \) is a union of \( h(K)! \) of disjoint \((h(K) - 2)\)-dimensional open simplexes. If \( h(K) = 2 \), then \( U_{\mathfrak{A}}(K) \) is a two-point set.

Now we define the stratum \( \text{Str}(\mathfrak{A}) \) as the product

\[
\text{Str}(\mathfrak{A}) = \text{Str}^{\text{Pass}}(\mathfrak{A}) := \prod_{K \in \mathfrak{A}} U_{\mathfrak{A}}(K).
\]

(6.12)

THEOREM 6.1. The permutoassociahedron is

\[
\text{Pass}_n = \bigcup_{\mathfrak{A}} \text{Str}(\mathfrak{A}),
\]

(6.13)

where the union is given over all the tree-partitions \( \mathfrak{A} \) of \( J \).

We emphasize, that a set \( \text{Str}(\mathfrak{A}) \) is disconnected; the number of its components is \( \prod h(K)! \), the product is given other all reducible elements of \( \mathfrak{A} \). These components are (open) faces of the polyhedron \( \text{Pass}_n \).

6.9. Identification of definitions 6.3 and 6.8 of permutoassociahedron. It is sufficient to write a map

\[
\text{Str}(\mathfrak{A}) \to \Xi(L)
\]

(6.14)

for a given tree-partition \( \mathfrak{A} \) of \( J \) and for any subset \( L \subset J \). This will define the canonical map from (6.13) to \( \text{Pass}_n \).

Denote by \( K \) the minimal element of the tree-partition \( \mathfrak{A} \) containing \( L \). Obviously, this element exists, since \( J \in \mathfrak{A} \). The image of a point \( u \in \text{Str}(\mathfrak{A}) \) under (6.14) will be completely determined by its projection to the factor \( U_{\mathfrak{A}}(K) \) in (6.12). There are two cases: \( K \in \mathfrak{A} \) is irreducible and \( K \in \mathfrak{A} \) is reducible.

First, let \( K \) be an irreducible element of \( \mathfrak{A} \). Then \( U_{\mathfrak{A}}(K) = \Xi(K) \to \Xi(L) \) is the canonical map (6.1).

Second, let \( K \) be reducible. Let \( \tau \) be its canonical decomposition. Then our map is the composition of the canonical maps (see (6.9), (6.8))

\[
\partial\Xi(K/\tau)^{\text{gen}} \to \partial\Xi(K) \to \partial\Xi(L)
\]

The required map is constructed.

6.10. Convergence in the permutoassociahedron. Consider a sequence \( x_1, x_2, \ldots \) in \( \Xi_n \).
The first necessary condition of the convergence is the convergence in $\Xi$. If the limit belongs to $\Xi_n$, then it is the limit in $\text{Pass}$ (the corresponding tree-partition consists of one set $J$).

Otherwise, let $t(\mu_1, \ldots, \mu_n)$ be the limit ray. We construct a partition $p$ of $J$ by the following equivalence relation

$$k \sim l \quad \text{if and only if } \mu_k = \mu_l.$$ (6.15)

Denote the elements of the partition $p$ by $I_1, I_2, \ldots$. The sequence $x_j$ induces sequences $x_{I_s}^j$ in each space $\Xi(I_s)$.

Our next necessary condition is the convergence of each sequence $x_{I_s}^j$ in each $\Xi(I_s)$, etc. etc.

**Example.** Consider the sequence in $\Xi_6$ given by

$$(n^3 + 2n, n^3 + n, n^3, 3n, 2n + 1, 2n)$$ (6.16)

Its limit is contained in the set $\text{Str}(A)$ for the tree-partition

$$A : ((1)(2)(3)) \quad ((4) \quad (56))$$

Indeed, the limit ray for (6.16) in $\Xi_6$ is $t(1,1,1,0,0,0)$. This gives the partition (123)(456).

In $\Xi\{1,2,3\}$, we have the sequence $(n^3 + 2n, n^3 + n, n^3) \sim (2n, n, 0)$. Its limit ray is $t(1,1/2,0) \in \partial\Xi\{1,2,3\}$. This gives the partition (1)(2)(3) of (123).

In $\Xi\{4,5,6\}$, we have the sequence $(3n, 2n + 1, 2n) \sim (n, 1, 0)$. Its limit ray is $t(1,0,0) \in \partial\Xi\{4,5,6\}$, and this gives the subpartition of (456) to (4)(56).

In the space $\Xi\{5,6\}$, we have $(2n + 1, 2n) \sim (1,0)$, therefore, in $\Xi\{5,6\}$, we have the constant sequence $(1,0)$. Its limit is $(1,0) \in \Xi\{5,6\}$.

**6.11. Closures of strata.** The closure of a set $\text{Str}(A)$ is $\cup_{B} \text{Str}(B)$, the union is given over all refinements $B$ of the tree-partition $A$ (i.e., each element of $A$ is an element of $B$).

**6.12. Closure of the Weyl chamber in the permutoassociahedron.** Consider the Weyl chamber $\Lambda^+_n = \Lambda^+(J)$, i.e., the set of vectors $\varphi_1 \geq \varphi_2 \geq \ldots \geq \varphi_n$ defined up to addition of a vector $(t,t,t,\ldots)$. Let us describe its closure $\Lambda^\text{Pass}_n$ in $\text{Pass}_n$.

a) **Formal description.** Denote $[j,k] = \{j, j+1, j+2, \ldots, k\} \subset J$.

We have the obvious projection $\Lambda^+_n \rightarrow \Lambda^+[j,k]$ and hence we have diagonal embedding

$$\Lambda^+_n \rightarrow \prod_{1 \leq j < k \leq n} \Lambda^+[j,k] \subset \prod_{1 \leq j < k \leq n} \Lambda^+[j,k].$$

The set $\Lambda^\text{Pass}_n$ is the closure of $\Lambda^+_n$ in $\prod_{1 \leq j < k \leq n} \Lambda^+[j,k]$.

b) **List of strata.** Strata are enumerated by segmental tree-partitions $\mathfrak{A}$ of $J$. A stratum has the form

$$\prod_{[k,l] \in \mathfrak{A}} U_{\mathfrak{A}}([k,l])$$ (6.17)
Fig 6. The set $W_j[\mathfrak{A}]$ consists of collections $u = (a_1 : a_2 : a_3 : b_1 : c_1 : c_2 : c_3 : \ldots)$ defined up to a common positive factor and up to an addition of $(t : t : t : s : r : r : r : \ldots)$. The numbers $a_1, a_2, a_3$ are pairwise different; $b_1, b_2, b_3$ are pairwise different, etc. In our case the variable $b_1$ is fake. The map $\text{Karp}_n \to \text{Pass}_n$ takes $u$ to the collection $[(a_1 : a_2 : a_3), (c_1 : c_2 : c_3), \ldots]$. In each bracket $(\ldots)$, the numbers are defined up to a common positive factor and addition of $(t : t : \ldots)$.

and the factors $U_\mathfrak{A}([k,l])$ are described in the following way:

— If $[k,l]$ is an irreducible element of $\mathfrak{A}$, then $U_\mathfrak{A}([k,l]) := \Lambda^+[k,l]$.

— Let $[k,l]$ be reducible. Denote its minimal decomposition by $c$. Then $U_\mathfrak{A}([k,l]) := \Lambda^+([k,l]/c)^{\text{gen}}$.

6.13. Stasheff associahedron. Consider the tree-partition $\mathfrak{A}_0 : (1)(2) \ldots (n)$ of $J$ and the corresponding open face of $\Lambda^+_{\text{pass}}$. The associahedron $\text{Ass}_n$ is its closure in $\text{Pass}_n$. Strata of $\text{Ass}_n$ are enumerated by perfect segmental tree-partitions of $J$; they are described in the previous subsection.

D. Description of the karpelevich-hedron

6.14. Stratification of the karpelevich-hedron. Strata $\text{Str}(\mathfrak{A})$ of the karpelevich-hedron are enumerated by leveled tree-partitions

$$\mathfrak{A} : a_0, a_1, \ldots, a_\tau$$

of the set $J$. Each stratum has the form

$$\text{Str}(\mathfrak{A}) = \text{Str}^{\text{Karp}}(\mathfrak{A}) = \prod_{j=0}^{\tau} W_j[\mathfrak{A}],$$

where the factors $W_j[\mathfrak{A}]$ are described in the following way.

a) Let $j < \tau$. Consider the quotient set $J/a_{j+1}$. The partition $a_j$ induces a partition of $J/a_{j+1}$. We denote this partition by $a_j/a_{j+1}$. We assume

$$W_j[\mathfrak{A}] := \partial \Xi[J/a_{j+1}; a_j/a_{j+1}]^{\text{gen}}$$

On Fig. 4, Fig.6, the set $J/a_j$ is the set of of edges coming to the dotted line from below. The set $J/a_{j+1}$ is the set of of edges coming to the dotted line above. The quotient-partition $a_j/a_{j+1}$ is the partition of the set $J/a_{j+1}$ into churn-staffs.

b) For $j = \tau$, we assume $W_\tau[\mathfrak{A}] := \Xi(J; a_\tau)$.

The karpelevich-hedron is a disjoint union

$$\text{Karp}_n = \bigcup_{\mathfrak{A}} \text{Str}(\mathfrak{A})$$

given over all the leveled tree-partitions of $J$. 
6.15. Identification of definitions 6.4 and 6.14 of karpelevich-hedron. For each leveled tree-partition $\mathfrak{A}$ and each partition $b$ of $J$, we must construct a map

$$\text{Str}(\mathfrak{A}) \rightarrow \Xi[J; b]$$

(6.21)

Consider the maximal $j$ such that $b$ is a refinement of $a_j$. Such $j$ exists since $a_0$ is the trivial partition. We consider the projection (see (6.19))

$$\text{Str}(\mathfrak{A}) \rightarrow W_j[\mathfrak{A}].$$

(6.22)

A) For $j < \tau$, the map (6.21) is the composition of the maps

$$\text{Str}(\mathfrak{A}) \rightarrow W_j[\mathfrak{A}] = \partial \Xi[J; a_{j+1}; a_j/a_{j+1}] \cong \partial \Xi[J; a_j] \rightarrow \partial \Xi[J; b],$$

the first map is the projection to a factor in (6.19), the second map is (6.9), the third map is (6.9.a).

B) Let $j = \tau$. Then $b$ is a refinement of $a_\tau$ and we have the canonical map

$$\text{Str}(\mathfrak{A}) \rightarrow W_{\tau+1}[\mathfrak{A}] = \Xi[J; a_{\tau}] \rightarrow \Xi[J; b];$$

(6.23)

the second map is (6.5.a).

6.16. Convergence in the karpelevich-hedron. Consider a sequence $x_j \in \Xi_n$. Beginning of the definition of the convergence is the same as in 6.10 until formula (6.15).

Then we obtain a sequence $x^s_j$ in each $\Xi(I_s)$, or equivalently, a sequence in $\Xi[J; p]$. Our next condition is: the sequence $x_j$ converges in

$$\Xi[J; p] \cong \prod \Xi(I_s).$$

If the limit is contained in $\prod \Xi(I_s)$, then it is the limit in $\text{Karp}_n$. Otherwise, let $t(\mu_1, \ldots, \mu_n)$ be the limit ray. For $k, l \in J$, we say $k \sim l$ iff $k, l$ lie in one $I_s$, and $\mu_k = \mu_l$.

Thus, we obtain a subpartition of each element $I_s$, hence we obtain a new partition $q$ of the whole set $J$.

Our next condition of convergence is: the sequence $x^s_j$ converges in the space $\Xi[J; q]$, etc., etc., etc.

EXAMPLE. For sequence (6.16), the corresponding leveled tree-partition is

$$a_0 = (123456); \quad a_1 = (123)(456); \quad a_2 = (1)(2)(3)(4)(56)$$

Indeed, we obtain the limit ray $t(1, 1, 1, 0, 0, 0) \in \Xi_n$, this gives the partition $a_1 = (123)(456)$.

In $\Xi[J; a_1] \cong \Xi\{1, 2, 3\} \times \Xi\{4, 5, 6\}$ we have the sequence

$$\{(n^3 + 2n, n^3 + n, n^3) \times (3n, 2n + 1, 2n)\} \sim \{(2n, n) \times (n, 1, 0)\}$$

Its limit ray in $\partial \Xi[J; a_1]$ is $t\{(2, 1, 0) \times (1, 0, 0)\}$. This gives the partition $a_2 = (1)(2)(3)(4)(56)$.
Next, \( \Xi[J; a_2] \simeq \Xi \{1\} \times \Xi \{2\} \times \Xi \{3\} \times \Xi \{4\} \times \Xi \{5, 6\} \simeq \Xi \{5, 6\} \), in \( \Xi \{5, 6\} \), we have \( (2n + 1, 2n) \simeq (1, 0) \), it is a constant sequence. Its limit is the point \( (1, 0) \in \Xi \{5, 6\} \).

6.17. Closures of strata. Let \( \mathcal{A} : a_0, \ldots, a_p \) and \( \mathcal{B} : b_0, \ldots, b_q \) be leveled tree-partitions. This say that \( \mathcal{B} \) is a refinement of \( \mathcal{A} \), if each partition \( a_j \) is contained in the list \( b_0, b_2, \ldots \).

The closure of the face \( \text{Str}(\mathcal{A}) \) is \( \bigcup \text{Str}(\mathcal{B}) \) over all the refinements \( \mathcal{B} \) of \( \mathcal{A} \).

6.18. Closure of Weyl chamber in the karpelevich-hedron, see also [GJT2]. Now let us describe the closure \( \Lambda^+_n \text{Karp} \) of the positive Weyl chamber \( \Lambda^+_n \) in the karpelevich-hedron.

Abstract description. Let \( a \) ranges in segmental partitions of \( J \). Consider the natural map \( \Lambda^+_n \subset \Xi_n \rightarrow \Xi[J; a] \) and the corresponding diagonal map

\[
\Lambda^+_n \rightarrow \prod_a \Xi[J; a] \subset \prod_a \Xi[J; a].
\]

The set \( \Lambda^+_n \text{Karp} \) coincides with the closure of the image of \( \Lambda^+_n \) in \( \prod_a \Xi[J; a] \).

Stratification. Strata are enumerated by segmental leveled tree-partitions \( \mathcal{A} : a_0, \ldots, a_p \) of \( J \). Each stratum is the product

\[
\prod_{j=0}^p Y_k[\mathcal{A}],
\]

(6.24)

where the factors have the following form

— If \( k < p \), then \( Y_k[\mathcal{A}] = \Lambda^+[J/a_j+1; a_j/a_j+1]^{\text{gen}} \). In other words, we consider collections of real numbers \( \theta(\mu) \), where \( \mu \) ranges in edges coming above to a dotted line, see Fig.4, Fig.6. These numbers are strictly increasing in each churn-staff (if we move to right along the dotted line), and they are defined up to a common positive factor and an addition of a function that is constant on each churn-staff.

— If \( k = p \), then \( Y_p[\mathcal{A}] = \prod_{[k,l] \in a_p} \Lambda^+[k,l] \).

6.19. Map \( \text{Karp}_n \rightarrow \text{Pass}_n \). Now we describe the map \( \pi : \text{Karp}_n \rightarrow \text{Pass}_n \) defined in 6.4. Fix the notation of 6.8 and 6.14. Let \( \mathcal{A} \) be a leveled tree-partition of \( J \). Let \( \mathcal{A}^\downarrow \) be the corresponding tree-partition.

First, \( \pi(\text{Str}(\mathcal{A}^\downarrow)) = \text{Str}(\text{Pass}(\mathcal{A}^\downarrow)) \).

Consider a partition \( a_j \) lying in the leveled tree-partition \( \mathcal{A} \). Let \( K^{(j)}_\alpha \) be its elements. It is sufficient to describe the map

\[
W_j[\mathcal{A}] \rightarrow \prod_{\alpha} U_{\lambda}(K^{(j)}_\alpha)
\]

(compare (6.12) and (6.19)). Consider two cases.

— Let \( j = \tau \). Then \( W_\tau \simeq \Xi[J; a_\tau] \) coincides with \( \prod_{K \in a_\tau} \Xi(K) \), and our map is the identical map.

— For \( j < \tau \), the map is described in Fig.6.

6.20. Picture. Karpelevich polyhedron. \( \text{Karp}_4 \) is a 3-dimensional polyhedron. We can imagine surface of the polyhedron as a picture on a sphere (or on
the cube from Fig. 3.b. We have 24 triangles on the surface of cube (sphere), one of these triangles PQT is drawn on Fig. 7. We present division of this triangle into faces.

One of the 2-faces ABCDEF of Karp4 is completely contained in the triangle PQT, other faces have intersections with adjacent triangles. The faces {...C’CDD’...} and {...E’EFF’...} are 12-gons. The list of all faces (2-faces, edges, vertices) having intersection with PQT is presented on Fig. 8.

**Permutoassociahedron.** For obtaining the permutoassociahedron Pass4 from the karpelevich-hedron Karp4, it is sufficient to contract the edge AB on Fig. 7 and 23 corresponding edges in other Weyl chambers.

**E. Root language**

6.21. **Permutoassociahedrons associated with root systems.** Consider an irreducible root system Δ in a linear space $V(\Delta)$. For each irreducible root subsystem $\Gamma \subset \Delta$ consider its linear span $V(\Gamma)$, and the corresponding compactification $\overline{V(\Gamma)}$. The sphere $\partial V(\Gamma)$ at infinity has the natural structure of a simplicial complex.

Consider the orthogonal projection $\pi_\Gamma : V(\Delta) \to \prod_{\Gamma \subset \Delta} \overline{V(\Gamma)}$.

The permutoassociahedron Pass(Δ) is the closure of $V(\Delta)$ in $\prod \overline{V(\Gamma)}$.

6.22. **Karpelevich-hedrons associated with root systems.** The definition of the karpelevich-hedron is the same, we only omit two times the term irreducible from the definition (and replace ‘simplicial’ by ‘polyhedral’).

7. **Existence of associahedral and Karpelevich boundaries**

7.1. **Hybrids of compactifications.** Let $A$ be a metrizable space. Let $X, Y$ be compact metrizable spaces and $\xi : A \to X, v : A \to Y$ be continuous maps; assume that the images of $A$ in $X$ and $Y$ are dense.

Then we have the diagonal map $A \to X \times Y$ given by $a \mapsto (\xi(a), v(a))$.

Consider the closure $Z \subset X \times Y$ of the image of $A$. We say that $Z$ is a hybrid of compactifications $X$ and $Y$.

7.2. **Velocity compactifications.** Consider the positive Weyl chamber $\Lambda_+^n$ described in 6.1. Let $\Lambda_+^n$ be a compact space containing $\Lambda_+^n$ as a dense open subset. Denote $\partial \Lambda_+^n = \Lambda_+^n \setminus \Lambda_+^n$.

We define a structure of a compact space on the disjoint union $\mathbb{P}E_+^n := \mathbb{P}E_n \cup \partial \Lambda_+^n$. Let $X^{(j)}$ be a sequence in $\mathbb{P}E_n$. Let $\Phi^{(j)} : \varphi_1^{(j)} \geq \varphi_2^{(j)} \geq \ldots \geq \varphi_n^{(j)}$ be the eigenvalues of $X^{(j)}$. We say that the sequence $X^{(j)}$ converges to a point $\Psi \in \partial \Lambda_+^n$ if the sequence $\Phi^{(j)} \in \Lambda_+^n$ converges to $\Psi$.

We say that $\mathbb{P}E_+^n$ is a velocity compactification of $\mathbb{P}E_n$. This construction is an analog of one-point compactification of a locally compact space.

7.3. **Example. Martin boundary.** The geometric object described below appears as the solution of the problem of Martin boundary for symmetric spaces, see [Osh], [GJT2].
Fig. 8. The list of faces in a given Weyl chamber.

1) The unique 3-dimensional cell. On Fig. 7 it is under the sheet of the paper.

(1234)

2) 2-dimensional faces

(123) (4) (12) (34) (1) (234) (1) (2) (3) (4)

...E′EFF′... ...A′ABB′... ...C′CDD′... ABCDEF

(12) (3) (4) (1) (23) (4) (1) (2) (34)

F′FAA′ D′DEE′ B′BCC′

3) Edges

(12) (3) (4) (123) (4) (1) (23) (4) (1) (23) (4) (1) (2) (34) (12) (3) (4)

FF′ EE′ DD′ CC′ AA′

(1) (2) (34) (12)(3)(4) (1) (2) (3)(4) (1) (2) (3)(4) (1) (2) (3)(4)

BB′ AB AF DE BC

(1) (2)(3) (4) (1) (2)(3) (4)

EF CD

4) Vertices

(12) (3) (4) (1) (2) (3)(4) (1) (2) (3)(4) (1) (2) (3) (4) (1) (2) (3) (4)

...... A B F E

...... D C
Let $\Lambda_n^{\ominus} = \Lambda_n^+ \cup \partial \Lambda_n^+$ defined in 6.2 The Martin compactification of $\mathbb{PE}_n$ is the hybrid of the velocity compactification associated with $\Lambda_n^+$ and the Satake–Furstenberg compactification.

It is easy to describe it explicitly. A point of the Martin compactification is a following collection of data.

a) Subset $I = \{0, i_1, \ldots, i_k, n\} \subset \{0, 1, \ldots, n\}$.

b) A point of $\Delta(I)$, see 3.1.

c) A point of $S(I)$, see 2.1.

7.4. Construction of the associahedral and Karpelevich boundaries.

Consider the completions $\Lambda_n^{Pass}$, $\Lambda_n^{Karp}$ of the Weyl chamber described in 6.12 and 6.18. Consider the associated velocity compactifications of $\mathbb{PE}_n$. The associahedral and Karpelevich compactifications of $\mathbb{PE}_n$ are the hybrids of these velocity compactifications with the Satake–Furstenberg compactification.

7.5. Stratification of associahedral compactification.

A point of the associahedral compactification of $\mathbb{PE}_n$ is the following collection of data A-C

A. A segmental tree-partition $\mathfrak{A}$ of the set $J$. Denote by $[1, i_1], [i_1 + 1, i_2], \ldots, [i_s + 1, n]$ its irreducible elements.

B. A point of the stratum (6.17) of $\Lambda_n^{Pass}$. We emphasis, that the product (6.17) contains the factors $\prod_k \Lambda^+([i_{k-1} + 1, i_{k+1}])$. A point of a factor $\Lambda^+([i_{k-1} + 1, i_{k+1}])$ is a collection of numbers

$$\psi_{i_k+1} \geq \psi_{i_{k+2}} \geq \ldots \geq \psi_{i_{k+1}}$$

(7.1)

defined up to addition of a common constant.

C. A point of the set $S\{i_1, \ldots, i_s\}$, i.e. a flag $\mathbb{R}^n = W_0 \supset W_1 \supset \ldots \supset W_{i_1 + 1} = 0$ and an ellipsoid $Q_k$ in each subquotient $W_{k-1}/W_k$ of the flag. These ellipsoids are not arbitrary. Our additional requirement is: for each $k$ the principal semiaxes of $Q_k$ are $e^{\psi_{i_k-1}}, \ldots, e^{\psi_{i_k}}$, where $\psi_{\ldots}$ are already defined (7.1).

7.6. Stratification of the Karpelevich compactification.

A point of the Karpelevich compactification of $\mathbb{PE}_n$ is the following collection of data A-C

A. A segmental leveled tree-partition $\mathfrak{A}$ of the set $J$. Denote by $[1, i_1], [i_1 + 1, i_2], \ldots, [i_s + 1, n]$ its irreducible elements.

B. A point of the stratum (6.24) of $\Lambda_n^{Karp}$. We emphasis, that the product (6.24) contains the factor $Y_p(\mathfrak{A}) = \prod_k \Lambda^+([i_{k-1} + 1, i_{k+1}])$. A point of $k$-th factor of the last product is a collection (7.1).

C. A point of the set $S\{i_1, \ldots, i_s\}$. The ellipsoids $Q_k$ in the subquotients $W_{k-1}/W_k$ of the flag are not arbitrary. Our additional requirement is: for each $k$ the principal semiaxes of $Q_k$ are $e^{\psi_{i_k-1}}, \ldots, e^{\psi_{i_k}}$, where $\psi_{\ldots}$ are already defined (7.1).

8. Sea urchin

A simple calculation shows that the dimension of the space of all null-pencils is larger than $\dim \mathbb{PE}_n$. Nevertheless it is possible to define a boundary related to null-pencils.

8.1. Definition. We define the sea urchin boundary as the set of limits of null-pencils whose velocities $(\varphi_1, \ldots, \varphi_n)$ consist of integer numbers.
Explicit description of the sea urchin can be easily obtained from 3.9.

8.2. Limits of meromorphic curves in sea urchin. Let $z$ ranges in a small interval $(0, \varepsilon)$. We say, that a map $z \mapsto X(z)$ is a meromorphic curve in $E_n$ if each matrix $x_{ij}$ element admits a Laurent decomposition

$$x_{ij}(z) = z^{-k}(a_0 + a_1z + a_2z^2 + \ldots).$$

**Lemma 8.1.** Each meromorphic curve $X(z) \in E_n$ admits a representation

$$X(z) = g(z) \begin{pmatrix} z^{-k_1} & 0 & \ldots \\ 0 & z^{-k_2} & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix} g(z)^\top$$

where $k_1 \geq k_2 \geq \ldots$, the function $g(z)$ is analytic in a neighborhood of 0, and $g(0)$ is invertible.

(We reduce the positive definite quadratic form $X(z)$ to a sum of squares in the usual way).

For the curve (8.1), consider the geodesic

$$\gamma(t) = g(0) \begin{pmatrix} e^{k_1t} & 0 & \ldots \\ 0 & e^{k_2t} & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix} g(0)^\top,$$

we substitute $z = e^{-t}$ to the middle factor in (8.1), and $z = 0$ to the first and last factors. We define the limit of the meromorphic curve (8.1) in the sea urchin as the limit of the geodesic (8.2).

**Lemma 8.2.**

a) A limit of a meromorphic curve does not depend on choice of the representation (8.2).

b) The limit of a meromorphic curve does not depend on a parametrization of the curve.

**Remark.** The sea urchin is not a compact space in the usual sense. For instance, the sequence $X_k = \begin{pmatrix} e^k & 0 \\ 0 & k \end{pmatrix}$ has no limit (and no limit points) in the sea urchin.

8.3. Projective compactifications. Consider a polynomial representation $\rho_\mathbf{m}$ of $\text{GL}(n, \mathbb{R})$ with a highest weight $\mathbf{m} : m \geq m_2 \geq \ldots \geq m_n$, where $m \in \mathbb{Z}$. It is well known, that the representation $\rho_\mathbf{m}$ contains a nonzero $O(n)$-invariant vector iff all the numbers $m_j$ are even. In this case, an $O(n)$-invariant vector $q_\mathbf{m}$ is unique up to a scalar factor.

Consider a direct sum $\theta$ of several representations $\rho_\mathbf{m}^\tau$ of $\text{GL}(n, \mathbb{R})$ with even signatures $\mathbf{m}^\tau$. Denote by $H$ the space of the representation $\theta$. Denote by $h$ the $O(n)$-invariant vector $\oplus q_{\mathbf{m}^\tau}$ in $H$.

Consider the projective space $\mathbb{P}H$ and the $\text{GL}(n, \mathbb{R})$-orbit $O$ of the vector $h$ in $\mathbb{P}H$. The projective compactification $[\mathbb{P}E_n]_\theta$ of $\mathbb{P}E_n$ is the closure of the orbit $O$ in $\mathbb{P}H$. 

30
8.5. Universality of the sea urchin. Obviously, each meromorphic curve $X(z)$ has a limit in each projective compactification.

**Theorem 8.3.** a) For each point $A$ of each projective compactification $[E_n]_\theta$, there exists a meromorphic curve $X(z) \in E_n$, whose limit in $[E_n]_\theta$ is $A$.

b) If the limits of two meromorphic curves $X_1(z)$ and $X_2(z)$ in the sea urchin coincide, then their limits in any projective compactification coincide.

c) If the limits of $X_1(z)$ and $X_2(z)$ in the sea urchin are different, then their limits in some projective compactification $[E_n]_\theta$ are different.

Thus, for each projective compactification $[E_n]_\theta$, we have the canonical surjective map from the sea urchin to $[E_n]_\theta$; an explicit variant of this construction is contained in [Ner7].

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