On the robustness of solitons crystals in the Skyrme model

Gonzalo Barriga$^{1,2}$, Fabrizio Canfora$^{3,1}$, Marcela Lagos$^{4}$, Matías Torres$^{1,2,5}$, Aldo Vera$^{4}$

1Centro de Estudios Científicos (CECS), Casilla 1469, Valdivia, Chile,
2Departamento de Física, Universidad de Concepción, Casilla 160-C, Concepción, Chile
3Universidad San Sebastián, sede Valdivia, General Lagos 1163, Valdivia 5110693, Chile
4Instituto de Ciencias Físicas y Matemáticas, Universidad Austral de Chile, Casilla 567, Valdivia, Chile
5Dipartimento di Fisica “E. Pancini”, Università di Napoli Federico II - INFN sezione di Napoli, Complesso Universitario di Monte S. Angelo Edificio 6, via Cintia, 80126 Napoli, Italy

gobarriga@udec.cl, fabrizio.canfora@uss.cl, marcela.lagos@uach.cl, matiatorres@udec.cl, aldo.vera@uach.cl

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Abstract

In this work we analyze how the inclusion of extra mesonic degrees of freedom affect the finite density solitons crystals of the Skyrme model. In particular, the first analytic examples of hadronic crystals at finite baryon density in both the Skyrme $\omega$-mesons model as well as for the Skyrme $\rho$-mesons theory are constructed. These configurations have arbitrary topological charge and describe crystals of baryonic tubes surrounded by a cloud of vector-mesons. In the $\omega$-mesons case, it is possible to reduce consistently the complete set of seven coupled non-linear field equations to just two integrable differential equations; one ODE for the Skyrmion profile and one PDE for the $\omega$-mesons field. This analytical construction allows to show explicitly how the inclusion of $\omega$-mesons in the Skyrme model reduces the repulsive interaction energy between baryons. In the Skyrme $\rho$-mesons case, it is possible to construct analytical solutions using a meron-type ansatz and fixing one of the couplings of the $\rho$-mesons action in terms of the others. We show that, quite remarkably, the values obtained for the coupling constants by requiring the consistency of our ansatz are very close to the values used in the literature to reduce nuclei binding energies of the Skyrme model without vector-mesons. Moreover, our analytical results are in qualitative agreement with the available results on the nuclear spaghetti phase.
1 Introduction

A detailed description of the phase diagram of Quantum Chromodynamics (QCD henceforth) - especially at low temperature and finite baryon density- is one of the greatest open challenges in theoretical physics. It is usually assumed that it is not fruitful to analyze the complex phase diagram of QCD in this regime with analytic tools (see [1], [2], [3], [4], and references therein) and, consequently, smart numerical methods must be used. One of the most remarkable features, which manifests itself when many baryons coexist within a finite volume, is the appearance of the so-called nuclear pasta phase (see [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15] and the nice up to date review [16]). In such a phase ordered structures appear, in which most of the baryonic charge is contained in regular shapes like thick baryonic layers (called nuclear lasagna) or thick baryonic tubes (called nuclear spaghetti). Not surprisingly, due to the large number of strongly interacting particles characterizing these nice regular structures, the nuclear pasta phase is considered to be the prototype configuration where analytic approaches are expected to fail. Even more, in such phases the numerical analysis are quite challenging and, as the above references show clearly, very high computing power is required.

In this paper we will analyze the appearance of these structures at finite density as analytical solutions of the Skyrme theory (in particular when this model is coupled to vector-mesons), which at leading order in the t’ Hooft expansion represents the low energy limit of QCD.

For numerical results on multi-solitons crystals see [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27] and references therein.
The Skyrme theory \([29]\) is described by a bosonic action for a \(SU(N)\)-valued scalar field \(U\) (being the two-flavors case \(U(x) \in SU(2)\) the most frequently studied), whose small excitations represent pions while the topological solitons of the theory are interpreted as baryons \([30], [31], [32], [33], [34]\). It is worth to emphasize that the interest on the Skyrme model goes far beyond QCD since it has been applied in astrophysics \([35]\), Bose-Einstein condensates \([36]\), nematic liquids \([37]\), magnetic structures \([38]\) and condensed matter physics \([39]\) (see also \([40]\) and \([41]\)). However, as it is well known since the eighties, the predictions of the Skyrme theory on the neutron-proton mass difference, the baryon resonances, the nuclei binding energies, the electromagnetic form factors and the matrix element of the singlet axial current, are not in excellent agreement with experiments. Fortunately, there is a very natural way to improve the theoretical results mentioned above, and it is including more families of vector-mesons such as the \(\omega\)-mesons and the \(\rho\)-mesons to the Skyrme action \([42], [43], [44], [45], [46], [47], [48]\) (see also \([49]\), \([50]\) and \([51]\)). The first part of this work it is devoted to the construction of analytic solutions of the Skyrme \(\omega\)-mesons theory and the second part to the Skyrme \(\rho\)-mesons theory.

In Skyrme-like models containing vector-mesons the field equations appears to be tremendously more complicated than the ones of the Skyrme model alone, but (besides the already mentioned results in \([42], [43], [44], [45], [46], [47], [48]\) there are further sound reasons to include the \(\omega\)-mesons and \(\rho\)-mesons in the game nevertheless. First, in \([52]\), the authors pointed out that it is possible to stabilize the Skyrmions without the need to consider the Skyrme term by coupling the baryonic current to the \(\omega\)-mesons. Secondly, the “stabilizing role” of the \(\omega\)-mesons becomes more and more important as the baryonic charge increases (see \([53], [54], [55], [56], [57]\) and references therein). Consequently, as we are interested in describing configurations with high topological charge (which is a necessary condition in the formation of nuclear pasta), it is extremely important to include the effects of the \(\omega\)-mesons in our analysis. On the other hand, the Skyrme model produces nuclei binding energies that are larger than the experimental ones, and the clustering structure of light nuclei in the Skyrme model is also not optimal. However, both of these problems can be solved by including the next lightest Isospin 1 meson, namely, the \(\rho\)-mesons \([58], [59]\).

The main goal of the present paper is to construct analytic configurations of the Skyrme vector-mesons theory representing crystals of baryons surrounded by a clouds of \(\omega\)-mesons and \(\rho\)-mesons, suitable for describing the nuclear spaghetti phase. The physical motivation is to reach a deeper understanding with some analytic control on how the vector-mesons affect the complex configurations of nuclear pasta appearing at finite baryon density. An important byproduct of our analysis will be to disclose the importance of the inclusion of the vector-mesons in the nuclear pasta phase\(^2\).

In order to achieve these goals, we will generalize the methods introduced in \([61], [62], [63], [64], [65], [66], [67], [68], [69], [70], [71], [72], [73], [74]\) to the Skyrme vector-mesons theory (see also \([75], [76], [77], [78], [79] and [80]\)). In fact, the above references have allowed the construction of several analytic and topologically non-trivial solutions of the Skyrme model. From the viewpoint of the goals of the present work, the most relevant configurations analyzed in those references corresponds to ordered baryonic

\(^2\) Although the half-Skyrmion configuration \([60]\) gives a good numerical description at high density of baryonic matter, it does not describe phases in which the nucleons loose their individuality, i.e, phases like nuclear pasta, where nucleons are “melted” at finite volume forming ordered patterns.
arrays in which (most of) the topological charge and total energy are concentrated within tube-shaped regions.\textsuperscript{3} In particular, the similarity of the contour plots in \cite{69} with the spaghetti-like configurations found numerically (see the plots in \cite{5}, \cite{6}, \cite{7}, \cite{8}, \cite{9} and \cite{16}) are extremely encouraging, and strongly support the viability of the present analytic approach\textsuperscript{4}. The analytic results described in the following sections allow to compute several relevant physical quantities such as the distribution of the $\omega$-mesons around the peaks of the baryon density and the “shielding” effect of the $\omega$-mesons in the repulsive Skyrmion-Skyrmion interactions.

The case of the $\rho$-meson Skyrme theory of \cite{58}, \cite{59} is the most complex due to the very intricate non-linear interactions between the Skyrmions and the $\rho$-mesons. Nevertheless, in \cite{58} and \cite{59}, the authors showed numerically that there are (at least two) possible choices of the many coupling constants of the theory giving rise to nuclei binding energy in very good agreement with experiments. In order to deal with the fifteen non-linear coupled field equations of the $\rho$-meson Skyrme theory we use the ansatz of \cite{69} for the $U$ field and a meron-like ansatz for the $\rho$-mesons (which works extremely well in the Yang-Mills case: see \cite{83}, \cite{84}, \cite{85}, \cite{86}, \cite{87} and \cite{88}). Of course, one may wonder whether the extremely complicated non-linear field equations of the $\rho$-meson Skyrme theory of \cite{58} and \cite{59} can be dealt with a meron-like ansatz, given the fact that there is no non-Abelian gauge symmetry in the $\rho$-mesons case.

In fact, the present approach is surprisingly effective in this case as well. One can fix one of the coupling constants of \cite{58} and \cite{59} in terms of the other coupling constants of the theory by requiring the consistency of our ansatz (namely by requiring the solvability of the field equations in topologically non-trivial sectors of high baryonic charge). At a first glance, one could think that such a requirement is a bit artificial. In particular, there is no obvious reason why the values of the coupling constants arising insisting that the present ansatz must work in the $\rho$-mesons Skyrme theory should in any way be related with the values of the coupling constants of \cite{58} and \cite{59} (chosen by the authors in order to achieve agreement with the nuclei binding energies). Nevertheless, quite remarkably, the choice of the coupling constants which makes the present framework suitable to deal with the $\rho$-mesons Skyrme theory is very close to the results in \cite{58} and \cite{59}. With the inclusions of the vector mesons, our analytical results are in qualitative agreement with the available results on the nuclear spaghetti phase.

The paper is organized as follows. In Section 2 we construct analytical solutions for the Skyrme $\omega$-mesons theory, and we show that these configurations can be interpreted as a lattice of baryonic tubes surrounded by $\omega$-mesons. Also we compute the effect of the $\omega$-mesons on the repulsive interaction energy. In Section 3, using a similar approach, we present analytical solutions of the Skyrme $\rho$-mesons theory showing that the inclusion of the $\rho$-mesons “shield” the interaction between the baryons on the

\textsuperscript{3}In \cite{81} and \cite{82}, numerical string shaped solutions in the Skyrme model with mass term have been constructed. However, those configurations have a zero topological density (and they are expected to decay into pions). The configurations analyzed in the present paper are topologically non-trivial and therefore can not decay into those of \cite{81} and \cite{82}.

\textsuperscript{4}Using this framework, in \cite{77} it has been possible to compute the shear modulus of lasagna configurations obtaining good agreement with \cite{11} and \cite{15}. Moreover, in \cite{72}, the very interesting characteristics of the electromagnetic field generated by nuclear spaghetti have been analyzed explicitly.

4
crystal. Section 4 is devoted to the conclusions.

In our convention \( c = \hbar = 1 \), Greek indices run over the space-time with mostly plus signature and Latin indices are reserved for those of the internal space.

2 Crystals of Baryons with a cloud of \( \omega \)-mesons

In this section we will construct analytical solutions describing crystals of baryonic tubes in the Skyrme \( \omega \)-mesons theory.

2.1 The Skyrme \( \omega \)-mesons theory

The action for the \( SU(2) \)-Skyrme model coupled with the \( \omega \)-mesons \([52]\) is given by

\[
I(U, \omega) = \int \sqrt{-g} d^4x \left[ \frac{K}{4} \text{Tr} \left( R_\mu R^\mu + \frac{\lambda}{8} F_{\mu\nu} F^{\mu\nu} \right) - \frac{M_\pi^2}{2} \text{Tr} \left( U + U^{-1} \right) \right.

\left. - \frac{1}{4} S_{\mu\nu} S^{\mu\nu} - \frac{1}{2} M_\omega^2 \omega_\mu \omega^\mu + g_0 J_\mu \omega^\mu \right],
\]

where \( U(x) \in SU(2) \), \( \omega_\mu \) is a 4-vector, \( g \) is the metric determinant, \( \nabla_\mu \) is the Levi-Civita covariant derivate and \( t_a = i \sigma_a \) are the generators of the \( SU(2) \) Lie group, being \( \sigma_a \) the Pauli matrices. The Skyrme couplings \( K \) and \( \lambda \) as well as the \( \omega \)-mesons coupling \( g_0 \) are positive constant fixed experimentally, while \( M_\pi \) and \( M_\omega \) correspond to the pions and \( \omega \)-mesons mass, respectively. Here the parameters \( K \) and \( \lambda \) are related to the meson decay coupling constant \( F_\pi \) and the Skyrme coupling \( e \) via \( F_\pi = 2\sqrt{K} \) and \( K \lambda e^2 = 1 \), where \( F_\pi = 141 \text{MeV} \) and \( e = 5.45 \). Thus, the energy, \( \int d^3x \sqrt{-g} T^{00} \), is in units of \( F_\pi/e \text{ MeV} \) (all the energy plots of the present paper will be given in terms of these units).

The pions and \( \omega \)-mesons interact in Eq. (1) through a term proportional to the topological current, \( J_\mu \), which is defined as

\[
J^\mu = \epsilon^{\mu\alpha\beta} \text{Tr}(R_\nu R_\alpha R_\beta) .
\]

Integrating the temporal component of the topological current defined above on a space-like hypersurface one obtain the topological charge

\[
B = \frac{1}{24\pi^2} \int J^0 , \quad J^0 = \epsilon^{ijk} \text{Tr} [R_i R_j R_k] ,
\]

which in the Skyrme model is identified as the baryonic number.

At this point is it important to emphasize the relations between the Skyrme \( \omega \)-mesons theory and the Walecka model \([89]\). In the study of compact stars, the Walecka model is a very useful theory of nucleons and two mesons: the scalar meson \( \sigma \) and the \( \omega \)-vector meson (see \([89], [90] \) and \([91]\)).
Lagrangian density of the Walecka model is given by,

\[ \mathcal{L}_W = \bar{\psi} [i \gamma_\mu (\partial^\mu + ig_\omega \omega^\mu) - (m_0 - g_\sigma \sigma)] \psi + \frac{1}{2} (\partial_\mu \sigma \partial^\mu \sigma - m_0^2 \sigma^2) - \frac{1}{4} \omega_\mu \omega^{\mu\nu} + \frac{1}{2} m_\omega^2 \omega_\mu \omega^\mu. \]  (4)

Here the \( \omega \)-vector meson is coupled to the nucleons via a minimal coupling, while in the case of the Skyrme model is via the topological current. The present inhomogeneous topologically non-trivial configurations manifest clear similarities with the ones appearing in the nonhomogeneous phase of the Walecka model (see Chapter 5 in [90]). Indeed, despite the fact that our analytic solutions live in three spatial dimensions, one of the profiles, namely \( \alpha \), can be expressed in terms of inverse elliptic functions, similar to what happens with the solutions in the Walecka model [90]. Thus, in a sense, the presence of the Skyme field supports the inhomogeneous phase of the Walecka model (see [92], [93], [94] and references therein).

In the next section we will see that the boundary conditions for the soliton profile emerge naturally by requiring that the topological charge be an integer.

The variation of the action in Eq. (1) w.r.t the fundamental fields \( U \) and \( \omega_\mu \) leads to the following field equations

\[ \nabla_\mu \left( R^\mu + \frac{\lambda}{4} [R, F^{\mu\nu}] \right) + \frac{M_\pi^2}{K} (U - U^{-1}) + \frac{6g_0}{K} g^{\alpha\lambda\rho \nu} \nabla_\nu (\omega_\alpha) R_\lambda R_\rho = 0, \]  \( \nabla_\nu S^{\nu\mu} - M_\omega^2 \omega_\mu + g_0 J^\mu = 0. \]  (6)

Eqs. (5) and (6) are, in general, a set of seven coupled non-linear partial differential equations. One of the main results of the present work is that, despite the complexity of the above system, this can be solved analytically using an appropriate ansatz, as we will see below.

The energy-momentum tensor of the theory corresponds to

\[ T_{\mu\nu} = -\frac{K}{2} \text{Tr} \left[ R_\mu R_\nu - \frac{1}{2} g_{\mu\nu} R_\alpha R_\alpha + \lambda \left( g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} F^{\sigma\rho} F_{\sigma\rho} \right) \right] + \frac{M_\pi^2}{K} g_{\mu\nu} (U + U^{-1}) \]

\[ + S_\mu S_\nu - \frac{1}{4} S_\alpha S^{\alpha\beta} g_{\mu\nu} + M_\omega^2 \left( \omega_\mu \omega_\nu - \frac{1}{2} g_{\mu\nu} \omega_\alpha \omega^\alpha \right) - g_0 \left( J_\mu \omega_\nu + J_\nu \omega_\mu - g_{\mu\nu} J_\alpha \omega^\alpha \right). \]  (7)

### 2.2 The Ansatz

We will construct analytical solutions that describe states of multi-solitons at finite density, so we consider as a starting point the metric of a box whose line element\(^5\) is

\[ ds^2 = -dt^2 + L^2 (dr^2 + d\theta^2 + d\phi^2), \]  (8)\(^5\)Here the coordinates \( \{r, \theta, \phi\} \) represent Cartesian coordinates and they must not be confused with spherical coordinates.
where $L$ is a constant representing the length of the box where the solitons are confined. The adimensional coordinates $\{r, \theta, \phi\}$ have the following ranges

$$0 \leq r \leq 2\pi, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi,$$

so that the volume available for the solitons is $V = 4\pi^3L^3$.

We parameterize the Skyrme field $U(x)$ as usual for an element of the $SU(2)$ group, namely

$$U^{\pm 1}(x^\mu) = \cos(\alpha)\mathbf{1}_{2\times 2} \pm \sin(\alpha)n^a t_a,$$

where $\mathbf{1}_{2\times 2}$ is the $2 \times 2$ identity matrix and $n^1 = \sin \Theta \cos \Phi$, $n^2 = \sin \Theta \sin \Phi$, $n^3 = \cos \Theta$, $\alpha = \alpha(x^\mu)$, $\Theta = \Theta(x^\mu)$ and $\Phi = \Phi(x^\mu)$ are the three degrees of freedom of the $U(x)$ field. With the parameterization in Eqs. (10) and (11) the topological charge density in Eq. (3) becomes

$$J^0 = -12 \left( \sin^2 \alpha \sin \Theta \right) d\alpha \wedge d\Theta \wedge d\Phi.$$

From the above expression it follows that, in order to have non-trivial topological configurations, we must demand that $d\alpha \wedge d\Theta \wedge d\Phi \neq 0$. This implies the necessary (but not sufficient) condition that $\alpha$, $\Theta$ and $\Phi$ must be three independent functions. The existence of arbitrary topological charge of our solutions will be revealed later when appropriate boundary conditions be imposed.

The strategy introduced in [69] and [70] for the Skyrme-Maxwell case provides with a very efficient ansatz which reduces the complete set of seven coupled non-linear field equations to just two integrable equations (one ODE for the Skyrmion profile and one PDE for the Maxwell potential) keeping alive the topological charge. Remarkably, such a strategy can be extended to the Skyrme $\omega$-mesons theory as follows. Firstly, the functions $\alpha$, $\Theta$ and $\Phi$ must be chosen as

$$\alpha = \alpha(r), \quad \Theta = q\theta, \quad \Phi = p\left(\frac{t}{L} - \phi\right), \quad q = 2v + 1, \quad v \in \mathbb{Z}.$$

The above ansatz is a very convenient choice for (at least) two reasons. The first one is a practical reason, since it is straightforward to verify that Eq. (13) satisfies the following identities

$$\nabla_\mu \Phi \nabla^\mu \alpha = \nabla_\mu \alpha \nabla^\mu \Theta = \nabla_\mu \Phi \nabla^\mu \Phi = \nabla_\mu \Theta \nabla^\mu \Phi = 0,$$

which greatly simplify the field equations in Eq. (5). The second reason is because Eq. (13) allows to avoid the Derrick’s theorem [95] due to the time dependence of the $U$ field. It is also possible to verify that although the $U$ field depends explicitly on time the energy density is static, in such a way that the solutions constructed here are in fact topological solitons with finite energy.

The construction in [69] and [70] also suggest the following ansatz for the $\omega$-mesons:

$$\omega_\mu = (u, 0, 0, -Lu), \quad u = u(r, \theta).$$
The above expression in Eq. (14) is also very convenient because it satisfies the following relations

\[ \epsilon^{\alpha \nu \lambda \rho} \nabla_\nu (\omega_\alpha) \text{Tr}[R_\lambda R_\rho] = 0, \quad \omega_\mu \omega^\mu = 0, \]

which not only significantly reduce the \( \omega \)-mesons field equations in Eq. (6), but also allows to decouple the contribution of the \( \omega \)-mesons from the Skyrme field equations in Eq. (5).

### 2.3 Analytical solutions

Replacing the ansatz introduced in Eqs. (8), (10), (11), (13) and (14) into Eq. (5), the set of three non-linear differential Skyrme equations are reduced to only one first order ODE for the profile \( \alpha \), namely

\[ \partial_r \left[ Y(\alpha) \left( \frac{\alpha'}{2} \right)^2 - V(\alpha) - E_0 \right] = 0, \]

being \( \alpha' = \frac{\partial \alpha}{\partial r} \), \( E_0 \) an integration constant (fixed by the boundary conditions, as we will see below) and where we have defined the functions \( Y(\alpha) \) and \( V(\alpha) \) as follows

\[ Y(\alpha) = 2(q^2 \lambda \sin^2(\alpha) + L^2) \quad , \quad V(\alpha) = -\frac{1}{2} L^2 q^2 \cos(2\alpha) + \frac{4L^4 M_0^2 \cos(\alpha)}{K}. \]

The equation for the profile \( \alpha \), that can be conveniently written as

\[ \frac{d\alpha}{\eta(\alpha, E_0)} = \pm dr \quad , \quad \eta(\alpha, E_0) = \frac{[2(E_0 + V(\alpha))]^{1/2}}{Y(\alpha)^{1/2}}, \]

can be solved analytically in terms of Elliptic Functions. Note that Eq. (15) does not depend on the potential \( u(r, \theta) \), i.e, it is completely decoupled from the \( \omega \)-mesons due to the good properties of the ansatz described above. In fact, the ansatz in Eqs. (10), (11) and (13) (that can be called “generalized hedgehog ansatz”) has advantages with respect to the original spherical hedgehog ansatz introduced by Skyrme in [29]. Firstly, it allows to reduce the complete set of Skyrme equations to a first order equation which can be solved analytically instead of numerically. Secondly, it leads to configurations with arbitrary topological charge and, moreover, is not restricted to spherical symmetry.

Similarly, the four field equations for the \( \omega \)-mesons in Eq. (6) are also reduced to just one equation but, in this case, a partial differential equation for the function \( u(r, \theta) \) introduced in Eq. (14). Then, the \( \omega \)-mesons profile satisfies a two-dimensional Poisson equation with a source term provided by the Skyrmion profile, that is

\[ \left( -\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial \theta^2} + M_0^2 L^2 \right) u = J_{\text{eff}}, \quad J_{\text{eff}} = 12g_0 pq L^2 \sin(q\theta) \sin^2(\alpha) \alpha'. \]

At this point it is important to highlight a crucial difference between this reduction and what happens in the case where the non-linear sigma model (NLSM) or the Skyrme model are coupled to the Maxwell theory. While in the Skyrme-Maxwell (or NLSM-Maxwell) case the equations for the electromagnetic...
field are reduced to a two-dimensional periodic Schrödinger equation that requires a numerical treatment (see [70] and [71]), in the present case the coupling with the ω-mesons is simpler since Eq. (18) can also be directly solved. In fact, the solution of Eq. (18) can be obtained as

\[ u(\vec{x}) = \int d\vec{x}' G(\vec{x},\vec{x}') J_{\text{eff}}(\vec{x}') , \quad (-\nabla_{\vec{x}}^2 + \omega^2 L^2) G(\vec{x},\vec{x}') = \delta(\vec{x} - \vec{x}') , \]

where \( x = (r, \theta) \).

Summarizing, using the generalized hedgehog ansatz for the Skyrme field in addition to a null-vector as ansatz for the ω-field, we are able to reduce the total set of seven non-linear coupled differential field equations to only two integrable equations.

### 2.4 Topological charge and energy density

Plugging the ansatz in Eqs. (10), (11) and (13) into Eq. (3), the topological charge density of the matter field reads

\[ J^0 = 3pq \sin(q\theta) \frac{\partial}{\partial r} (\sin(2\alpha) - 2\alpha) . \]

Since the baryon number must be an integer, it is straightforward to verify that for this purpose one needs to impose the following boundary condition on the soliton profile \( \alpha(r) \):

\[ \alpha(2\pi) - \alpha(0) = n\pi , \]

with \( n \) an integer number. Therefore, integrating in the ranges defined in Eq. (9), the topological charge takes the value

\[ B = -np , \]

where we have used the fact that \( q \) is an odd number, as specified in the ansatz in Eq. (13). From Eqs. (9), (17) and (20) it follows that the integration constant \( E_0 \) is determined by the relation

\[ n \int_{0}^{\pi} \frac{d\alpha}{\eta(\alpha, E_0)} = 2\pi . \]

Equation (22) is an equation for \( E_0 \) in terms of \( n \) that always has a real solution when \( E_0 + V(\alpha) > 0 \), which implies that \( E_0 \) is bounded from below

\[ E_0 > \frac{L^2q^2}{2} + 4L^4M_\pi^2 \frac{K}{K} , \]

so that, for given values of \( q \) and \( L \), the integration constant \( E_0 \) determines the value of the \( \alpha \) profile for the boundary conditions defined in Eq. (20).

#### 2.4.1 A necessary condition for stability

When the field equations reduce to a single equation for the profile in a topologically non-trivial sector (as in the present case) one says that “the hedgehog property holds”. In many situations (although
not always, see for detailed discussions [31], [32] and references therein) the perturbations which could more easily lead to a decrease in the energy of the system are those perturbations of the profile which keep the hedgehog property. In the present case, these dangerous perturbations are of the following form:

\[ \alpha \rightarrow \alpha + \varepsilon \xi (r), \quad 0 < \varepsilon \ll 1, \]  

where \( \alpha \) is the Skyrmion profile of the background solution, which do not change the \( SU(2) \) Isospin degrees of freedom. It is easy to see that the linearized Skyrme field equations for \( \xi (r) \) in Eq. (24) always has the following zero-mode: \( \xi (r) = \partial_r \alpha \). From here one can deduce the constraint in Eq. (23). If the above condition is satisfied the zero mode \( \xi (r) = \partial_r \alpha \) has no node, hence the solution is stable under these perturbations.

### 2.4.2 Interacting Baryons surrounded by \( \omega \)-mesons

Evaluating the ansatz defined in Eqs. (8), (10), (11), (13) and (14) in Eq. (7), the energy density (\( E = T_{00} \)) of the configurations constructed above turns out to be

\[
E = E_{\text{Sk}} + E_\omega + E_{\text{int}},
\]

where we have defined

\[
E_{\text{Sk}} = \frac{K \alpha'^2}{2L^4} \left( L^2 + \lambda \sin^2(\alpha) \left[ q^2 + 2p^2 \sin^2(q\theta) \right] \right),
\]

\[
+ \frac{K \sin^2(\alpha)}{2L^4} \left( L^2 \left[ q^2 + 2p^2 \sin^2(q\theta) \right] + 2\lambda p^2 q^2 \sin^2(q\theta) \sin^2(\alpha) \right),
\]

\[
E_\omega = \frac{(\partial_\theta u)^2 + (\partial_r u)^2}{L^2} + M_\omega^2 u^2,
\]

\[
E_{\text{int}} = -24g_0 pq \sin(q\theta) \sin^2(\alpha) \alpha' u.
\]

Note that the energy density (in general, the energy-momentum tensor) and the the topological charge density do not depend on the coordinate \( \phi \) while they do depend on \( r \) and \( \theta \); that is why these configurations describe the nuclear spaghetti phase.

Fig. 1 shows the energy density of the configurations. One can see that, in fact, the system describes a lattice of baryonic tubes with a crystalline order where the peaks of the energy density are localized where the baryonic density takes its maximum values and vanishes outside the tubes. In fact, the similarity of the contour plots with the spaghetti-like configurations found (numerically) in the nuclear pasta phase in [5], [6], [7], [8] and [9] is quite remarkable.

For this type of solutions the inclusion of the \( \omega \)-mesons in the Skyrme model causes the energy density distribution to blur due to the presence of this cloud of mesons, as can be seen comparatively in Fig. 1. In Fig. 2 one can see the total energy per baryon as a function of the density, \( \rho = 1/V = 1/(4\pi^3 L^3) \), of the box containing the topological solitons. The energy curves have the characteristic “u-shape” of this kind of configurations: At low density the energy of the system is a decreasing function of \( \rho \), then there is a critical point from which the behavior reverses in such a way that at
Figure 1: Energy densities for Skyrmions without $\omega$-mesons (left column) and for Skyrmions with a cloud of $\omega$-mesons (right column) for $B = 1$, $q = 1$ and $B = 3$, $q = 3$ (top to bottom). The peaks of the energy densities correspond with the peaks of the topological charge density. It can be seen that the energy density is blurred in the presence of the $\omega$-mesons. Here the values of the couplings constants have been setting to $p = L = 1$, $M_\pi = M_\omega = 0$, $K = 2$, $\lambda = 1$ and $g_0 = 0.17$. It is important to observe that the tightness of the spaghetti is related to the box size and shape, as well as to $B/V$ where $B$ is the topological charge and $V$ is the volume of the box.

Figure 2: Total energy per baryon as a function of the density for different values of the baryonic charge of the Skyrmions with $\omega$-mesons. The energy curves have the characteristic “u-shape” of nuclear pasta. Here the coupling constants have been set as $K = 2$, $\lambda = 1$, $g_0 = 0.001$, $p = q = 1$, $M_\pi = M_\omega = 0$.

higher densities the energy of the system increases with $\rho$. In fact, Fig. 2 is in qualitative agreement with what has been obtained in numerical simulations of nuclear pasta (see [16]).

Now we introduce an important quantity, $\Delta = \Delta(B)$, which is a measure of the interaction energy
between baryons

\[ \Delta(B) = \frac{E(B+1) - (E(B) + E(1))}{(B + 1) E(1)}, \tag{29} \]

where \( B \) is the number of baryons of the system and \( E(i) \) the total energy of a configuration containing \((i)\) baryons. As it is well known, Skyrmions have a strong short range repulsion. This is reflected, for instance, in the growth of the function \( \Delta(B) \) with \( B \) (see, for instance, the plots in [16] of \( E(B)/B \)). On the other hand, due to the stabilizing role of the \( \omega \)-mesons (see [52]), it is natural to expect that the quantity \( \Delta(B) \) when the \( \omega \)-mesons are turned on (\( \Delta_{\text{Full}} \)) should grow, as function of \( B \), slower than when the \( \omega \)-mesons are turned off (\( \Delta_{\omega=0} \)). One of the main results of the present paper is that we can test this intuition with exact analytic solutions of the Skyrme \( \omega \)-mesons theory. In fact, the results in Fig. 3 confirm that for any value of \( B \) it is true that \( \Delta_{\text{Full}} < \Delta_{\omega=0} \), therefore one can say that the \( \omega \)-mesons partially “shield” the baryon-baryon repulsion. The conclusion is that it is crucial to take the effects of the \( \omega \)-mesons into account in the analysis of complex configurations such as the ones of the nuclear pasta phase. Also, from Fig. 3 we see the expected fact that the stabilization of the Skyrmions is also achieved without the need to consider the Skyrme term by coupling the baryonic current to the \( \omega \)-mesons (see the curve \( \Delta_{\nu=0} \)).

Figure 3: \( \Delta(B) \) for the following cases: The Skyrme theory coupled with the \( \omega \)-mesons (\( \Delta_{\text{Full}} \)), the Skyrme theory without the \( \omega \)-mesons (\( \Delta_{\omega=0} \)) and the NLSM coupled to the \( \omega \)-mesons (\( \Delta_{\lambda=0} \)). We can see that the inclusion of the \( \omega \)-mesons stabilize the Skyrmions. Here we have set \( K = 2, \lambda = 1, g_0 = 0.1, p = q = 1, L = 1, M_\pi = M_\omega = 0. \)

3 Crystals of Baryons with a cloud of \( \rho \)-mesons

The Skyrme model produces nuclei binding energies larger than the experimental ones. This problem can be solved by including the next lightest Isospin 1 mesons, the \( \rho \)-mesons (see [58] and [59] and references therein). At a first glance, the combined \( \rho \)-mesons Skyrme theory able to fix the nuclei binding energy appears to be hopelessly complicated to be studied with analytical methods. However, in this section we will analyze relevant configurations of the family of actions introduced in [58] and [59] in which a full analytic treatment is possible.
3.1 The Skyrme $\rho$-mesons theory

The Skyrme $\rho$-mesons theory (see [58] and [59]) is described by the action

$$I[U, A_\mu] = \int \sqrt{-g} d^4x \left( L_{\text{Skyrme}} + L_{\rho} + L_{\text{int}} \right),$$

where

$$L_{\text{Skyrme}} = \text{Tr} \left\{ \frac{c_1}{2} R_{\mu} R^\mu + \frac{c_2}{16} F_{\mu\nu} F^{\mu\nu} - \frac{M^2}{2} (U + U^{-1}) \right\},$$

$$L_{\rho} = \text{Tr} \left\{ a_1 S_{\mu\nu}^2 + a_2 A_{\mu}^2 + c_3 S_{\mu\nu} G^{\mu\nu} + c_4 G_{\mu\nu}^2 \right\},$$

$$L_{\text{int}} = \text{Tr} \left\{ c_5 L_{\mu\nu}^2 - c_6 F_{\mu\nu} S_{\mu\nu} + c_7 F_{\mu\nu} G^{\mu\nu} + a_8 F_{\mu\nu} L^{\mu\nu} + a_9 L_{\mu\nu} S_{\mu\nu} + a_{10} L_{\mu\nu} G^{\mu\nu} \right\},$$

and

$$G_{\mu\nu} = [A_\mu, A_\nu], \quad L_{\mu\nu} = [R_\mu, A_\nu] - [R_\nu, A_\mu], \quad S_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

being $A_\mu$ the $SU(2)$-valued one-form that characterizes the $\rho$-mesons

$$A_\mu = (A_\mu)^a t_a.$$

In this section we have decided to denote as $a_i$ and $c_i$ the coupling constants instead of $K$ and $\lambda$ in order to keep the value of these as general as possible and also to compare our results with those of the references [58] and [59]. In fact, we get to the action considered in [58] and [59] if we set the $a_i$ couplings in Eqs. (31) and (32) as follows

$$a_1 = \frac{1}{8} c_8, \quad a_2 = \frac{1}{4} m_\rho^2, \quad a_8 = \frac{1}{2} c_6, \quad a_9 = -\frac{1}{8} c_8, \quad a_{10} = -\frac{1}{2} c_3.$$

There are, at least, two sets of values for the constants $c_i$ that allow to reduce the binding energy, namely

$$c_1 \rightarrow \frac{1}{4 \sqrt{\pi}}, \quad c_2 \rightarrow 0.198, \quad c_3 \rightarrow \frac{1}{10}, \quad c_4 \rightarrow \frac{1}{10}, \quad c_5 \rightarrow 0.038,$$

$$c_6 \rightarrow \frac{\pi^2}{12 \sqrt{2}}, \quad c_7 \rightarrow 0.049, \quad c_8 \rightarrow 1, \quad m_\rho \rightarrow \frac{1}{\sqrt{2}}.$$
and

\begin{equation}
\begin{aligned}
c_1 &\rightarrow \frac{1}{6},
 c_2 &\rightarrow \frac{1}{6},
 c_3 &\rightarrow \frac{\sqrt{3}}{10},
 c_4 &\rightarrow \frac{9}{140},
 c_5 &\rightarrow \frac{3}{80},
 c_6 &\rightarrow \frac{\sqrt{3}}{24},
 c_7 &\rightarrow \frac{1}{20},
 c_8 &\rightarrow 1,
 m_\rho &\rightarrow \frac{2}{\sqrt{5}}.
\end{aligned}
\end{equation}

However, it is very likely that there are even more choices of the coupling constants of the theory which allow to get good results for the binding energies. Due to the large number of coupling constants, it makes a lot of sense to explore such parameters space in order to identify a smaller set of coupling constants which allows to simplify the analysis without loosing the good features arising from the inclusion of the ρ-mesons. A good and effective criterion which will be adopted here is to select the subset of couplings which allows a complete analytic solutions of the field equations keeping, at the same time, good physical properties (such as a reasonable behavior of the \(\Delta(B)\) defined in Eq. (29) when the ρ-mesons are turned on). We will show that the difference between our choices and the ones in [58] and [59] is small.

The fifteen non-linear differential equations (see Appendix A for the explicit derivation of these field equations) for the Skyrme ρ-mesons are obtained by taking the variation of the action w.r.t the fields \(U\) and \(A_\mu\):

\begin{align}
\nabla_\mu \left( c_1 R^\mu + \frac{c_2}{4} [R_\nu, F^{\mu\nu}] - E_1^\mu \right) - [R_\mu, E_1^\mu] - M_\pi^2 (U - U^{-1}) &= 0, \\
\nabla_\nu O_1^{\mu\nu} + 2 a_2 A^\mu - E_2^\mu &= 0,
\end{align}

where

\begin{align}
E_1^\mu &= [S^{\mu\nu}, 2a_9 A_\nu - 2c_6 R_\nu] + [G^{\mu\nu}, 2a_{10} A_\nu - 2c_7 R_\nu] + 2a_8 [F^{\mu\nu}, A_\nu] + [L^{\mu\nu}, 4c_5 A_\nu + 2a_8 R_\nu], \\
E_2^\mu &= [S^{\mu\nu}, 2c_3 A_\nu + 2a_9 R_\nu] + [G^{\mu\nu}, 4c_4 A_\nu + 2a_{10} R_\nu] - [F^{\mu\nu}, 2c_7 A_\nu - 2a_8 R_\nu] \\
&\quad + [L^{\mu\nu}, 2a_{10} A_\nu + 4c_5 R_\nu], \\
O_1^{\mu\nu} &= 2 \left( 2a_1 S^{\mu\nu} + c_3 G^{\mu\nu} - c_6 F^{\mu\nu} + a_9 L^{\mu\nu} \right).
\end{align}

### 3.2 The ansatz

In order to constuct analytical solutions of the Skyrme ρ-mesons theory we will use the same ansatz for the \(U\) field presented in Eqs. [10], [11] and [13] together with a meron-like ansatz for the ρ-mesons field (see [83], [84], [85], [86], [87] and [88]) in the form:

\begin{equation}
\rho_\mu = \bar{\lambda}(r)U^{-1}\partial_\mu U,
\end{equation}

\footnote{The condition \(c_8 = 1\) is required canonically because it is the unit residue of the propagator at the ρ-meson mass.}

\footnote{We thanks C. Naya for this remark.}
where \( \bar{\lambda} \) will be called the “profile” of the \( \rho \)-mesons. It is worth to emphasize that, due to the fact that the \( \rho \)-mesons Lagrangian is not gauge invariant one should not expect great simplifications using an ansatz inspired from the Yang-Mills theory, however we will show that the above ansatz works nevertheless. Here the metric it is also the finite box defined in Eq. \([8]\).

The energy density corresponding to the ansatz in Eqs. \([8]\), \([10]\), \([11]\), \([13]\) and \([41]\) takes the form:

\[
\mathcal{E} = \mathcal{E}_{\text{Sk}} + \mathcal{E}_{\rho} + \mathcal{E}_{\text{int}} ,
\]

where

\[
\mathcal{E}_{\text{Sk}} = \frac{c_2 \sin^2(\alpha) (2p^2 \sin^2(q\theta)(q^2 \sin^2(\alpha) + \alpha'^2) + q^2 \alpha'^2)}{L^4} \quad + \quad 2M^2_\pi \cos(\alpha) + \frac{c_1 (\sin^2(\alpha)(2p^2 \sin^2(q\theta) + q^2) + \alpha'^2)}{L^2} ,
\]

\[
\mathcal{E}_\rho = \frac{\sin^2(\alpha)(p^2 + q^2 - p^2 \sin(2q\theta)) \left( \bar{\lambda}^2 (8a_1 + a_2L^2 + 8\bar{\lambda}\alpha'^2(c_4\bar{\lambda} - c_3)) + 2a_1\bar{\lambda}^2 \right)}{L^4} \quad + \quad \frac{16\rho^2 q^2 \sin^2(q\theta) \sin^4(\alpha)\bar{\lambda}^2(a_1 - c_3\bar{\lambda} + c_4\bar{\lambda}^2) + a_2L^2\bar{\lambda}^2\alpha'^2}{L^4} ,
\]

\[
\mathcal{E}_{\text{int}} = \frac{16\bar{\lambda}\sin^2(\alpha)(\bar{\lambda}(2a_{10}\bar{\lambda} - 2a_9 + 4c_5 - c_7) + 2a_8 + c_6)(2p^2 \sin^2(q\theta)(q^2 \sin^2(\alpha) + \alpha'^2) + q^2 \alpha'^2)}{L^4} .
\]

### 3.3 Analytical solutions with a constant meron profile

A direct way to construct analytical solutions of the Skyrme \( \rho \)-mesons theory is to assume a constant \( \rho \)-mesons profile\(^8\), that is \( \bar{\lambda}(r) = \lambda_0 \), in the ansatz in Eqs. \([8]\), \([10]\), \([11]\), \([13]\) and \([41]\) (in Yang-Mills terminology this would be a proper meron ansatz). Remarkably, with this assumption the three Skyrme field equations can be reduced to just one integrable equation for the soliton profile \( \alpha(r) \) while the twelve non-linear \( \rho \)-mesons field equations are reduced to three polynomial equations for \( \lambda_0 \) provided that two extra constraints on the coupling constants of the \( \rho \)-mesons action are satisfied. Indeed, with \( \bar{\lambda}(r) = \lambda_0 \) and the \( U \) field as in Eqs. \([10]\), \([11]\) and \([13]\) the field equations read

\[
a_8 + \lambda_0 \left( 3a_{10}\lambda_0 - a_9 - c_3\lambda_0 + 2c_4\lambda_0^2 + 4c_5 - c_7 \right) = 0 ,
\]

\[
c_6 - \lambda_0(-2a_1 + 2a_9 + c_3\lambda_0) = 0 ,
\]

\[
a_2 = 0 ,
\]

\[
\frac{\partial}{\partial r} \left[ \frac{1}{2} I(\alpha)\alpha'^2 - \bar{V}(\alpha) - E_0 \right] = 0 ,
\]

where \( E_0 \) is an integration constant, and we have defined

\[
I(\alpha) = 2\bar{a}_1q^2 \sin^2(\alpha) - 2c_1L^2 , \quad \bar{V}(\alpha) = \frac{1}{2} c_1L^2q^2 \cos(2\alpha) + 4L^4M^2_\pi \cos(\alpha) ,
\]

\(^8\text{In Appendix B we will include a possible choice of the coupling constants which allows an analytic solution with } \bar{\lambda} = \bar{\lambda}(r) \text{ non-constant.} \)
First of all, it is worth to emphasize that the effects of the presence of the $\rho$-mesons into the Skyrme field equations manifest themselves through the effective coupling $\tilde{a}_1$ here above. Hence, in a sense, the presence of the $\rho$-mesons manifests itself in a renormalization of the coupling constants which would appear in the Skyrme theory alone. The comparison between the equation for the profile $\alpha$ in Eqs. (15) and (16) without $\rho$-mesons and the Eqs. (49) and (50) that include the effects of the $\rho$-mesons is very instructive. It is quite amusing that in such a complicated system it is possible to read off explicitly the effects of the $\rho$-mesons on the Skyrmion profile $\alpha$. Secondly, this choice of the coupling (in which the mass term of the $\rho$-mesons vanishes, $a_2 = 0$) is very appropriate for large baryon number, in which case the mass of the $\rho$-mesons (as well as the mass of the pions $M_\pi$) can be neglected with respect to the mass of the nuclear pasta configuration: obviously, the mass of solitonic configurations with large baryonic charge is many orders of magnitude larger than the $\rho$-mesons and pions masses (thus, in all the plots here below we will assume that $a_2 = M_\pi = 0$).

Resuming, in order for the ansatz $\lambda(r) = \lambda_0$ together with the one in Eqs. (10), (11) and (13) to reduce the complete set of fifteen coupled non-linear field equations to just one integrable equation for $\alpha$ plus an algebraic equation for $\lambda_0$, one only needs two constraints on the coupling constant of the $\rho$-mesons action: one constraint is $a_2 = 0$ while the second constraint on the coupling constant can be deduced by solving Eq. (47) for $\lambda_0$ and then replacing this solution into Eq. (46). We will discuss such a constraint in the next subsections. Also in this case it is possible to integrate the equation for the soliton profile in Eq. (49) analytically as follows

$$
\frac{d\alpha}{\eta_\rho(\alpha, E_0)} = \pm dr , \quad \eta_\rho(\alpha, E_0) = \frac{\sqrt{2(E_0 + \tilde{V}(\alpha))}}{I(\alpha)^{1/2}},
$$

(51)

Figure 4: Comparison between the profile of the Skyrmion with and without the contribution of the $\rho$-mesons for $B = 1$. We see that the inclusion of the $\rho$-mesons smooths out the behavior of the soliton profile.

The larger is the baryonic charge of the solitonic configuration, the more accurate approximation $a_2 = 0$ becomes. Thus, in the present context we can safely assume it.
where $\bar{E}_0$ is fixed by the boundary conditions, namely
\[
\int_0^{\pi} \frac{d\alpha}{\eta_{\rho}(\alpha, \bar{E}_0)} = n \int_0^{\pi} \frac{d\alpha}{\eta_{\rho}(\alpha, \bar{E}_0)} = 2\pi.
\]

### 3.4 Binding energy on the crystal

As it has been already remarked, there are two remaining equations, Eqs. (46) and (47), which will fix $\lambda_0$ in terms of the couplings of the theory and will give an extra constraint on the couplings. A bound on the coupling constants arises by requiring that the solution $\lambda_0$ should be real, so the discriminant of Eq. (47) has to be positive, that is
\[
a_1^2 - 2a_1a_9 + a_9^2 + c_3c_6 > 0.
\]

Interestingly enough, this inequality is satisfied by both set of parameters in Eqs. (34) and (35) (see [58] and [59]). It is worth emphasizing here that the above constraint in Eq. (52) on the coupling constants of the $\rho$-mesons action arises if one insists that our approach must reduce consistently the complete set of field equations of the Skyrme $\rho$-mesons theory to just one integrable ODE for $\alpha$ plus an algebraic equation for $\lambda_0$. On the other hand, the two choices of coupling constants analyzed in [58] and [59] arise with the physically well-motivated requirement to reduce the nuclei binding energies of the Skyrme model. Obviously, there is, a priori, no reason why one should expect that the constraint in Eq. (52) (which is a necessary condition in order to ensure that our ansatz works) should be satisfied by the two choices of coupling constants in [58] and [59] (motivated by the need to improve the prediction of the nuclei binding energy of the Skyrme model). Nevertheless, the constraint in Eq. (52) is satisfied by both choices of coupling constants in [58] and [59] (see Eqs. (34) and (35)).

From Eqs. (46) and (47) we can choose the coupling $c_4$ as the “dependent one”: namely, we will solve $c_4$ in terms of the other coupling constants while the values of the other coupling constants will be as in Eqs. (34) and (35). In Table 1 we show the four values of the dependent coupling $c_4$ that solves the constraints.

| $c_4$ | $\lambda_0$ positive root | $\lambda_0$ negative root |
|-------|--------------------------|--------------------------|
| Eq. (34) | 0.0198233 | -0.182293 |
| Eq. (35) | 0.0586335 | -0.598634 |

Table 1: The four values of $c_4$ that solves the constraints in Eqs. (46) and (47), according to the coupling constants in Eqs. (34) and (35).

The differences between the values of $c_4$ taking into account Eqs. (46) and (47) and the set of values as in Eqs. (34) and (35) are shown in Table 2.

One can see that the positive root for $c_4$ is very close to the choice in Eq. (35). This is a quite non-trivial result since there is no obvious relation between the condition that “the present ansatz should work” for the Skyrme $\rho$-mesons theory and the physically well-motivated condition in references [58].
Table 2: Difference between the coupling constant $c_4$ in Eqs. (34) and (35) (found in references [58] and [59]) and the two values of $c_4$ that came from the two branches that solve Eqs. (46) and (47). One can see that the positive root is extremely close to one of the values that reduces the binding energy.

and [59] to reduce the nuclei binding energies. Notwithstanding this, among the infinitely many values that $c_4$ could have, the mathematical consistency of the present approach produces a value for $c_4$ which is very close to the one in Eq. (35). These results clearly show that the present analytic approach is very well suited to describe multi-solitonic solutions of the Skyrme-vector mesons theory.

Figure 5: Energy densities for Skyrmions without $\rho$-mesons (left column) and for Skyrmions with a cloud of $\rho$-mesons (right column) for $B = 1$, $q = 1$ and $B = 2$, $q = 1$ (top to bottom). The peaks of the energy densities correspond with the peaks of the topological charge density. The presence of the $\rho$-mesons turns the transverse sections of the tubes into a circular shape. Here the values of the coupling constants have been setting to $p = L = 1$, $M_\pi = a_2 = 0$, $c_4 = -0.5986$ and the remaining $c_i$ are fixed as in Eq. (35).

Fig. 5 shows the energy density for Skyrmionic configurations with and without $\rho$-mesons. In the absence of $\rho$-mesons the tubes of Skyrmions have the shape of ellipses (even when the theory is coupled to $\omega$-mesons, as we showed in the previous section), however, here we can see that this characteristic is modified due to the cloud of $\rho$-mesons. In fact, the presence of these vector mesons turns the transverse sections of the tubes into a circular shape (in the $r - \theta$ plane).

In Fig. 6 we show the energy of the solitons coupled with $\rho$-mesons in the low density sector, where the curves for different values of the topological charge can be clearly distinguished. In the high
density sector the energy becomes an increasing function in such a way that the energy (per baryon) versus density has the characteristic “u-shape” of nuclear pasta. A relevant fact that can be seen from the energy plots is that, for the range of parameters that we have chosen here, the presence of the $\rho$-mesons does not spoil the good properties of the Skyrme model: in particular, the energy conditions are satisfied in agreement with Wald theorem.

Figure 6: Total energy per baryon as a function of the density for different values of the baryonic charge for the Skyrmions with $\rho$-mesons. The coupling constants have been set as $p = q = 1, M_\pi = a_2 = 0, c_4 = -0.1823$ and the remaining $c_i$ are fixed as in Eq. (34).

Finally, we show a measure of the interaction energy between baryons using Eq. (29). In Fig. 7 it is shown the case with $c_4$ being the closest value to the results of [58] and [59]. We can see that the inclusion of the $\rho$-mesons reduces the binding energy between the Skyrmions for small values of baryonic charge. Above a critical value of $B$, the behavior changes and the Skyrmionic binding energy increases due to the presence of the $\rho$-mesons. In Fig. 8 we have $\Delta$ for the remaining allowed values of $c_4$. Here it is clear that the inclusion of the $\rho$-meson reduces the binding energy between the Skyrmions, for all the values of the baryonic charge. These cases are very relevant since it is very likely that there exist more good choices of the coupling constants of the theory, besides the ones found in [58] and [59], as we already mentioned above.

Figure 7: $\Delta(B)$ for the following cases: The Skyrme theory coupled with the $\rho$-mesons ($\Delta_{\text{Full}}$) with the solution $c_4 = 0.0586335$ and the Skyrme theory without the $\rho$-mesons ($\Delta_{\rho=0}$). The inclusion of the $\rho$-mesons reduces the binding energy between the Skyrmions for low values of $B$.

Finally, in order to respect chiral symmetry one can add axial chiral mesons to the theory as, in references [96] and [97], or to eliminate the coupling constant $a_1$ in Eq. (31). In both cases, our asantz
Figure 8: $\Delta(B)$ for the following cases: The Skyrme theory coupled with the $\rho$-mesons ($\Delta_{\text{Full}}$) with $c_4 = 0.0198233$, $-0.182293$ and $-0.598634$ (from left to right) and the Skyrme theory without the $\rho$-mesons ($\Delta_{\rho=0}$). The inclusion of the $\rho$-mesons reduces the binding energy between the Skyrmions at every value of the baryonic charge.

still solves the field equations, and the soliton configurations maintain their physical properties.

4 Conclusions

In the present manuscript we have constructed the first analytic examples of hadronic crystals at finite baryon density for the Skyrme $\omega$-mesons model and for the Skyrme $\rho$-mesons theory. These multi-Skyrmionic configurations represent crystals of baryonic tubes surrounded by a cloud of vector-mesons at finite baryon density. For the Skyrme $\omega$-mesons case a suitable ansatz for the Skyrmions and the $\omega$-mesons reduce consistently the complete set of seven coupled non-linear field equations to just two integrable differential equations, one for the Skyrme profile and the other for the $\omega$-mesons (which is actually a two-dimensional Poisson equation in which the Skyrmion profile acts as a source term). This analytical construction allows to show explicitly how the inclusion of $\omega$-mesons in the Skyrme action reduces the repulsive interaction energy between baryons. It is possible to analyze explicitly the effects of the $\omega$-mesons on various observables relevant for the nuclear pasta phase: the present results strongly suggest that the $\omega$-mesons can be important in the nuclear spaghetti phase. Using a similar approach, it is also possible to include the $\rho$-mesons. A priori, the Skyrme $\rho$-mesons system introduced in [58] and [59] is even more complicated than the Skyrme $\omega$-mesons system since the $\rho$-mesons are described as a one-form taking values in the Lie algebra of $SU(2)$ and, moreover, the family of actions in [58] and [59] contains many very complicated interaction terms. Nevertheless, if one insists in using the same ansatz for the Skyrme field together with a meron-like ansatz for the $\rho$-mesons, one can reduce the complete set of fifteen non-linear field equations to just one integrable equation for the Skyrmion profile and an algebraic equation for the $\rho$-mesons profile provided a suitable relation between the coupling constants of the $\rho$-mesons theory holds. Quite remarkably, if one expresses one of the coupling constant (say, $c_4$) in terms of the others (using as input the values in [58] and [59]) the constraint that arises requiring the consistency of the ansatz when the $\rho$-mesons are included provides a value for $c_4$ very close to the ones in [58] and [59]. These analytic results on the Skyrme $\rho$-mesons system show that a proper analytic description of the nuclear pasta phase should include these vector-mesons.
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Appendix A: Field equations for the Skyrme \( \rho \)-mesons theory

The variation of the Lagrangian defined in Eq. (1) w.r.t. the fundamental fields \( U \) and \( A_{\mu} \) leads to

\[
\delta \mathcal{L} = \text{Tr} \left\{ c_1 \delta R_{\mu} R^\mu + \frac{c_2}{8} \delta F_{\mu \nu} F^{\mu \nu} - \frac{m_\pi}{2c_2} (\delta U + \delta U^{-1}) \right\}
+ \text{Tr} \left\{ 2a_2 \delta A_{\mu} A^\mu + \frac{1}{2} \left( \delta S_{\mu \nu} O^{\mu \nu}_1 + \delta G_{\mu \nu} O^{\mu \nu}_2 + \delta L_{\mu \nu} O^{\mu \nu}_3 + \delta F_{\mu \nu} O^{\mu \nu}_4 \right) \right\},
\]

where we have defined the antisymmetric tensors \( O^{\mu \nu}_i \) as follow:

\[
\begin{align*}
    \frac{1}{2} O^{\mu \nu}_1 &= 2a_1 S^{\mu \nu} + c_3 G^{\mu \nu} - c_6 F^{\mu \nu} + a_9 L^{\mu \nu}, \\
    \frac{1}{2} O^{\mu \nu}_2 &= c_3 S^{\mu \nu} + 2c_4 G^{\mu \nu} - c_7 F^{\mu \nu} + a_{10} L^{\mu \nu}, \\
    \frac{1}{2} O^{\mu \nu}_3 &= a_{9} S^{\mu \nu} + a_{10} G^{\mu \nu} + a_8 F^{\mu \nu} + 2c_5 L^{\mu \nu}, \\
    \frac{1}{2} O^{\mu \nu}_4 &= -c_6 S^{\mu \nu} - c_7 G^{\mu \nu} + a_8 L^{\mu \nu}.
\end{align*}
\]

It is direct to verify that

\[
\begin{align*}
    \text{Tr}(\delta S_{\mu \nu} O^{\mu \nu}_1) &= 2 \text{Tr}(\delta A_{\mu} \nabla_{\nu} O^{\mu \nu}_1), \\
    \text{Tr}(\delta G_{\mu \nu} O^{\mu \nu}_2) &= 2 \text{Tr}(\delta A_{\mu} [A_{\nu}, O^{\mu \nu}_2]), \\
    \text{Tr}(\delta L_{\mu \nu} O^{\mu \nu}_3) &= 2 \text{Tr}(\delta R_{\mu} [A_{\nu}, O^{\mu \nu}_3] - \delta A_{\mu} [O^{\mu \nu}_3, R_{\nu}])), \\
    \text{Tr}(\delta F_{\mu \nu} O^{\mu \nu}_4) &= 2 \text{Tr}(\delta R_{\mu} [R_{\nu}, O^{\mu \nu}_4]).
\end{align*}
\]

Now, replacing the above in the variation of the Lagrangian and grouping terms we obtain

\[
\delta \mathcal{L} = \text{Tr} \left\{ c_1 \delta R_{\mu} R^\mu + \frac{c_2}{4} [\nabla_{\nu}, F^{\mu \nu}] - [O^{\mu \nu}_3, A_{\nu}] - [O^{\mu \nu}_4, R_{\nu}] \right\} - \frac{m_\pi c^2}{2c_2} (\delta U + \delta U^{-1}) \\
+ \text{Tr} \left\{ \delta A_{\mu} (\nabla_{\nu} O^{\mu \nu}_1 + 2a_2 A^\mu - [O^{\mu \nu}_2, A_{\nu}] - [O^{\mu \nu}_3, R_{\nu}]) \right\}.
\]
Taking into account that

\[ \text{Tr}(\delta R_\mu C^\mu) = \text{Tr}(U^{-1} \delta U [(C^\mu, R_\mu) - \nabla_\mu C^\mu]) , \]

for an arbitrary tensor \( C^\mu \), it follows that

\[ \delta L = \text{Tr} \left\{ U^{-1} \delta U \left( -c_1 \nabla_\mu R^\mu - \frac{c_2}{4} \nabla_\mu [R_\nu, F^\mu\nu] - \frac{m^2 c_1^2}{2 c_2} (U - U^{-1}) + \nabla_\mu E_1^\mu - [E_1^\mu, R_\mu] \right) \right\} + \text{Tr} \left\{ \delta A_\mu (\nabla_\nu O_1^\mu + 2 a_2 A^\mu - E_2^\mu) \right\} , \]

where \( E_i^\mu \) have been defined as

\[ E_1^\mu = [O_3^{\mu\nu}, A_\mu] + [O_4^{\mu\nu}, R_\nu] \]
\[ = [S^{\mu\nu}, 2 a_9 A_\nu - 2 c_6 R_\nu] + [G^{\mu\nu}, 2 a_{10} A_\nu - 2 c_7 R_\nu] + 2 a_8 [F^{\mu\nu}, A_\nu] + \left[ L^{\mu\nu}, 4 c_5 A_\nu + 2 a_8 R_\nu \right] , \]
\[ E_2^\mu = [O_2^{\mu\nu}, A_\nu] + [O_3^{\mu\nu}, R_\nu] \]
\[ = [S^{\mu\nu}, 2 c_3 A_\nu + 2 a_9 R_\nu] + [G^{\mu\nu}, 4 c_4 A_\nu + 2 a_{10} R_\nu] + [F^{\mu\nu}, -2 c_7 A_\nu + 2 a_8 R_\nu] + \left[ L^{\mu\nu}, 2 a_{10} A_\nu + 4 c_5 R_\nu \right] . \]

The energy-momentum tensor of the theory reads

\[ T_{\mu\nu} = T_{\mu\nu}^{\text{Sk}} + T_{\mu\nu}^\rho + T_{\mu\nu}^{\text{int}} , \]

where

\[ T_{\mu\nu}^{\text{Sk}} = \text{Tr} \left[ -c_1 \left( R_\mu R_\nu - \frac{1}{2} g_{\mu\nu} R^\alpha R_\alpha \right) - \frac{c_2}{4} \left( F^\alpha_\mu F_{\nu\alpha} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) \right] , \]
\[ T_{\mu\nu}^\rho = \text{Tr} \left[ -4 a_1 \left( S^\alpha_\mu S_{\nu\alpha} - \frac{g_{\mu\nu}}{4} S_{\alpha\beta} S^{\alpha\beta} \right) - 2 a_2 \left( A_\mu A_\nu - \frac{1}{2} g_{\mu\nu} A^\alpha A_\alpha \right) \right. \]
\[ -2 c_3 \left( S^\alpha_\mu G_{\nu\alpha} + S_{\nu\sigma} G^\alpha_\sigma - \frac{g_{\mu\nu}}{2} S_{\alpha\beta} G^{\alpha\beta} \right) \left. - 4 c_4 \left( G^\mu_\sigma G_{\nu\sigma} - \frac{g_{\mu\nu}}{4} G_{\alpha\beta} G^{\alpha\beta} \right) \right] , \]
\[ T_{\mu\nu}^{\text{int}} = \text{Tr} \left[ -2 \left( L^\alpha_\mu M_{\nu\alpha} + L_\nu^\sigma M_{\mu\sigma} - \frac{g_{\mu\nu}}{2} L_{\alpha\beta} M^{\alpha\beta} \right) + 2 c_6 \left( F^\sigma_\mu S_{\nu\sigma} + F_{\nu\sigma} S_\mu^\sigma - \frac{g_{\mu\nu}}{2} F_{\alpha\beta} S^{\alpha\beta} \right) \right. \]
\[ + 2 c_7 \left( F^\alpha_\mu G_{\nu\sigma} + F_{\nu\sigma} G^\alpha_\mu - \frac{g_{\mu\nu}}{2} F_{\alpha\beta} G^{\alpha\beta} \right) \] ,

with

\[ M_{\mu\nu} = c_5 L_{\mu\nu} + a_8 F_{\mu\nu} + a_9 S_{\mu\nu} + a_{10} G_{\mu\nu} . \]

For the meron-like ansatz

\[ A_\mu = \lambda(x) R_\mu , \]

the tensors \( S_{\mu\nu}, G_{\mu\nu}, L_{\mu\nu} \) are reduced to

\[ S_{\mu\nu} = \lambda F_{\nu\mu} + \nabla_\mu \lambda R_\nu - \nabla_\nu \lambda R_\mu , \quad G_{\mu\nu} = \lambda^2 F_{\mu\nu} , \quad L_{\mu\nu} = 2 \lambda F_{\mu\nu} . \]
Meanwhile, $O_1$ and $E_i$ become

\[
O_1^{\mu\nu} = P_1(\lambda) F^{\mu\nu} + 4a_1 (\nabla^\mu \lambda R^{\nu} - \nabla^{\nu} \lambda R^{\mu}) ,
\]

\[
E_1^\mu = 2 \left[ \lambda \frac{O_3^{\mu\nu}}{2} + \frac{O_4^{\mu\nu}}{4} , R_{\nu} \right] ,
\]

\[
= -(P_2(\lambda) - c_2/4) [R_{\nu}, F^{\mu\nu}] - 2(a_9 \lambda - c_6) \nabla_{\nu} \lambda F^{\mu\nu}
\]

\[
E_2^\mu = 2 \left[ \lambda \frac{O_2^{\mu\nu}}{2} + \frac{O_4^{\mu\nu}}{4} , R_{\nu} \right] ,
\]

\[
= -P_3(\lambda) [R_{\nu}, F^{\mu\nu}] - 2(c_3 \lambda + a_9) \nabla_{\nu} \lambda F^{\mu\nu} ,
\]

where

\[
P_1(\lambda) = 2c_3 \lambda^2 + 4(a_9 - a_1) \lambda - 2c_6 ,
\]

\[
P_2(\lambda) = 2a_{10} \lambda^3 + 2(4c_5 - a_9 - c_7) \lambda^2 + 2(c_6 + a_8 + 2a_{10}) \lambda + \frac{c_2}{4} ,
\]

\[
P_3(\lambda) = 4c_4 \lambda^3 + 2(3a_{10} - c_3) \lambda^2 + 2(a_9 - c_7 + 2c_5) \lambda + 2a_8 .
\]

Therefore, varying the action of the Skyrme $\rho$-mesons system w.r.t the fields $A_\mu$ and $U$, the fields equations turns out to be

\[
P_1(\lambda) \nabla_{\nu} F^{\mu\nu} + 2a_2 \lambda R^{\mu} + P_3(\lambda) [R_{\nu}, F^{\mu\nu}] + (P_1' + 2c_3 \lambda + 2a_9) \nabla_{\nu} \lambda F^{\mu\nu}
\]

\[
+ 4a_1 (\nabla_{\nu} \nabla^\mu \lambda R^{\nu} + \nabla^{\mu} \lambda \nabla_{\nu} R^{\nu} - \Box \lambda R^{\mu} - \nabla^{\mu} \lambda \nabla_{\nu} R^{\mu}) = 0 ,
\]

\[
c_1 \nabla_{\mu} R^{\mu} + P_2(\lambda) \nabla_{\mu} [R_{\nu}, F^{\mu\nu}] - \frac{m_\pi c_1^2}{2c_2} (U + U^{-1})
\]

\[
+(P_2' - 2(a_9 \lambda - c_6)) \nabla_{\mu} \lambda [R_{\nu}, F^{\mu\nu}] + 2(a_9 \lambda - c_6) (\nabla_{\mu} \nabla_{\nu} \lambda F^{\mu\nu} + \nabla_{\nu} \lambda \nabla_{\mu} F^{\mu\nu}) = 0 .
\]

**Appendix B: Configurations with non-constant $\tilde{\lambda}$**

Here we will show a possible choice of the coupling constants of the $\rho$-mesons action which allows to construct analytic solutions with a non-constant profile $\lambda(r)$ for the $\rho$-mesons.

One can check directly that when the coupling constants are

\[
c_3 = c_6 = c_4 = a_8 = a_{10} = 0 , \quad a_1 = a_9 = 4c_5 - c_7 ,
\]
the dynamical variables $\alpha(r)$ and $\lambda(r)$ satisfy a system of three coupled ODEs, namely

\[
2(c_1 L^2 + c_2 q^2 \sin^2(\alpha))\alpha'' + 4(c_7 - 4c_5)q^2 \sin^2(\alpha)(\lambda^2)'\alpha' \\
+ c_2 q^2 \sin(2\alpha)\alpha'^2 - 2L^2(2L^2 M_\pi^2 + c_1 q^2 \cos(\alpha)) \sin(\alpha) = 0 , \\
\lambda'' + \frac{a_2 L^2(q^2 - \csc^2(\alpha)\alpha'^2)}{2q^2(c_7 - 4c_5)} = 0 , \\
\alpha' + \frac{q^2(c_7 - 4c_5) \sin(2\alpha)\lambda'}{a_2 L^2 \lambda} = 0 .
\]

From the last equation we find an analytic expression for $\lambda$ in terms of $\alpha$, this is

\[
\bar{\lambda}(r) = l_0 \tan(\alpha)^{\frac{a_0}{a_2}} ,
\]

where $l_0$ is an integration constant and $t_0 = a_2 L^2/(4c_5 - c_7)q^2$. Once one replaces this expression for $\lambda$ in the above equations two ODEs for $\alpha$ arise. The compatibility of these equations demands that

\[
c_2 = M_\pi = 0 , \quad t_0 = -2 ,
\]

which in turn implies

\[
L = q \sqrt{\frac{2(c_7 - 4c_5)}{a_2}} , \quad l_0 = \sqrt{c_1}/\sqrt{2a_2} .
\]

Under this conditions, the resulting equation for $\alpha$ turns out to be

\[
\alpha'' - \alpha'^2 \cot(\alpha) - q^2 \sin(\alpha) \cos(\alpha) = 0 ,
\]

while $\lambda$ is

\[
\bar{\lambda}(r) = \sqrt{c_1/2a_2} \cot \alpha . \quad (53)
\]

Finally, using the change of variables

\[
\alpha(r) = \arccos(\tanh(H(r))) ,
\]

the equation for $\alpha$ can be leads to

\[
H'' + q^2 \tanh(H) = 0 .
\]

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