Space Quasi-periodic Standing Waves for Nonlinear Schrödinger Equations

W.-M. Wang

CNRS and Département de Mathématique, Université Cergy-Pontoise, 95302 Cergy-Pontoise Cedex, France.
E-mail: wei-min.wang@math.cnrs.fr

Received: 16 July 2018 / Accepted: 28 April 2020
Published online: 16 July 2020 – © Springer-Verlag GmbH Germany, part of Springer Nature 2020

Abstract: We construct space quasi-periodic standing wave solutions to the nonlinear Schrödinger equations on \( \mathbb{R}^d \) for arbitrary \( d \). This is a type of quasi-periodic nonlinear Bloch-Floquet waves.

Contents

1. Introduction to the Theorem ........................................ 783
2. Green’s Function Estimates in \((\theta_1, \theta_2, \ldots, \theta_d)\) ............... 786
3. Nonlinear Construction—Proof of the Theorem ................. 795

1. Introduction to the Theorem

Consider the nonlinear Schrödinger equations (NLS) on \( \mathbb{R}^d \):

\[
i \frac{\partial}{\partial t} U = -\Delta U - |U|^{2p} U,
\]

where \( p \geq 1 \) and \( p \in \mathbb{N} \) is arbitrary; \( U \) is a complex valued function on \( \mathbb{R} \times \mathbb{R}^d \). In this paper, we seek standing wave solutions of the form

\[
U(t, x) = e^{-iEt} u(x),
\]

where \( E \in \mathbb{R} \), and \( u \) is even and quasi-periodic in each \( x_k, k = 1, 2, \ldots, d \), given by a quasi-periodic cosine series:

\[
u(x) = u(x_1, x_2, \ldots, x_d) = \sum_{j_1, j_2, \ldots, j_d} \hat{u}(j_1, j_2, \ldots, j_d) \prod_{k=1}^{d} \cos(j_k \cdot \lambda_k) x_k, \quad \text{(QP)}
\]
where for each \( k \in \{1, 2, \ldots, d\} \), \( j_k \in \mathbb{Z}^2 \) and \( \lambda_k \in (1/2, 3/2)^2 \). The \( \lambda_k \)'s are the parameters in the problem, and are assumed to be irrational, satisfying

\[
\| j_k \cdot \lambda_k \|_\mathbb{T} \neq 0,
\]
for all \( j_k \neq 0 \), where \( \| \cdot \|_\mathbb{T} \) denotes distance to the integers. We note that the quasi-periodic series (QP) reduces to a periodic cosine series if \( j_k \) and \( \lambda_k \) were one dimensional: \( j_k \in \mathbb{Z} \) and \( \lambda_k \in (1/2, 3/2) \) for \( k = 1, 2, \ldots, d \). For example, setting \( \lambda_k = 1 \) for all \( k \), leads to a periodic series with period \( 2\pi \) in each directions. In that case, solutions with more general time dependence, the time quasi-periodic solutions, are known to exist from e.g., [W1], cf. also [W2].

Substituting the Ansatz (1.2) into (1.1) yields the following stationary, nonlinear elliptic problem:

\[
-\Delta u - |u|^{2p} u = Eu. \tag{1.3}
\]

For \( u \in H^1(\mathbb{R}^d) \) with a fixed \( L^2(\mathbb{R}^d) \) norm, there is a well established variational structure under appropriate conditions on \( p \): \( E \) is a Lagrange multiplier and (1.3) are the minimizers for the energy functional:

\[
E(U) = \int_{\mathbb{R}^d} dx \left[ \frac{1}{2} \| \nabla U \|^2 - \frac{1}{2p+2} |U|^{2p+2} \right].
\]

Localized standing wave solutions are well known from the works of, for example, Cazenave and Lions [CL]. (Cf. also the references therein.) The \( u \)'s given by (QP), even though smooth, are only in \( L^\infty \) and solving (1.3) produces space quasi-periodic nonlinear Bloch-Floquet waves, which are not localized. (For quasi-periodic linear Bloch-Floquet theory in one dimension, see e.g., [DS,E], cf. also [K] for linear quasi-periodic ground states in arbitrary dimensions.)

**Remark.** For the purpose of this paper, the sign of the nonlinear term is unimportant, i.e., it can be focusing or defocusing. (See the remark after the Theorem.) Functions that are even under \( x_k \to -x_k \), for all \( k = 1, 2, \ldots, d \), form an invariant subspace for (1.3). Here we seek solutions \( u \) in this subspace given by the series in (QP).

To simplify notations, define

\[
(j \cdot \lambda)^2 := \sum_{k=1}^{d} (j_k \cdot \lambda_k)^2, \tag{1.4}
\]

where \( j = (j_1, j_2, \ldots, j_d) \in \mathbb{Z}^{2d} \) and \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in (1/2, 3/2)^{2d} \).

Let

\[
\tilde{U} = ae^{-i(j \cdot \lambda)^2 t} \prod_{k=1}^{d} \cos(\tilde{j}_k \cdot \lambda_k) x_k, \tag{1.5}
\]

where \( a \in \mathbb{R} \) and \( \tilde{j} \in \mathbb{Z}^{2d} \). (If \( \tilde{j} = 0 \), \( \tilde{U} = ae^{i|a|^{2p} t} \) trivially solves (1.1).) Then \( \tilde{U} \) satisfies the linear equation:

\[
i \frac{\partial}{\partial t} \tilde{U} = -\Delta \tilde{U}; \tag{1.6}
\]
and
\[ \tilde{u} = a \prod_{k=1}^{d} \cos(\tilde{j}_k \cdot \lambda_k) x_k \]  
(1.7)
satisfies
\[ -\Delta \tilde{u} = \tilde{E} \tilde{u}, \]  
(1.8)
with
\[ \tilde{E} = (\tilde{j} \cdot \lambda)^2. \]  
(1.9)

Our main result is

**Theorem.** For every solution to the linear equation (1.6) in the form (1.5)
\[ \tilde{U} = ae^{-i(\tilde{j} \cdot \lambda)^2 t} \prod_{k=1}^{d} \cos(\tilde{j}_k \cdot \lambda_k) x_k, \]
where \( a \in \mathbb{R} \) and \( \tilde{j} \in \mathbb{Z}^{2d} \), there is a set in \( \lambda, \Lambda \subset (1/2, 3/2)^{2d} \) satisfying
\[ \text{meas } \Lambda \geq 1 - |a|^{p/6}, \]  
(1.10)
provided \( |a| \ll 1 \). If \( \lambda \in \Lambda \), then there is a solution \( U \), bifurcating from \( \tilde{U} \), to the nonlinear equation (1.1) in the form (1.2, QP):
\[ U(t, x) = e^{-i(\tilde{j} \cdot \lambda)^2 t + O(|a|^2 p)} [a \prod_{k=1}^{d} \cos(\tilde{j}_k \cdot \lambda_k) x_k + O(|a|^p)]. \]  
(1.11)

The nonlinear eigenvalue \( E \), as a function in \( \lambda \), is \( C^1 \) on \( (1/2, 3/2)^{2d} \).

**Remark.** For notational simplicity we have taken \( \lambda_i, i = 1, 2, \ldots, d \), to be two-dimensional. The Theorem holds for higher dimensional \( \lambda_i, i = 1, 2, \ldots, d \), with essentially the same proof. The set \( \Lambda \) is a Cantor set (of positive measure). Since \( a \) is small, the same Theorem holds if the nonlinearity enters with a plus sign (defocusing).

1.1. Some background. Most of the results in the literature on (1.1) or (1.3) are for \( u \), which are fast decaying or periodic in \( \mathbb{R}^d \). To our knowledge, the above Theorem is the first such result on global in time, non-decaying solutions \( u \) which do not have an underlying translation symmetry group. (Cf. Moser [M] for an iterative method in the space periodic setting, i.e., on the quotient space \( L^2(\mathbb{R}^d / \mathbb{Z}^d) := L^2(\mathbb{T}^d) \).) It is periodic in time (with only the basic frequency), quasi-periodic in space and exists in arbitrary dimensions. The Theorem shows that under appropriate conditions, every small even generalized eigenfunctions of the linear operator in (1.8) bifurcates to an eigenfunction of the nonlinear operator in (1.3), after small deformation.

Generally speaking, due to the non-compact \( \mathbb{R}^d \) setting, there are very few known results on space quasi-periodic solutions to nonlinear partial differential equations. In one dimension, Damanik and Goldstein proved the global existence and uniqueness to Cauchy problems for the KdV equation with small quasi-periodic initial data [DG].
Their method, however, seems to hinge on the integrable structure. It is noteworthy that the Cauchy solutions in [DG] are almost-periodic in time (and quasi-periodic in space). This result in fact motivated us to seek space quasi-periodic solutions in a more general setting, albeit with simpler time dependence, as in the Theorem. However, solutions with more complicated time dependence to the NLS in (1.1), such as space-time quasi-periodic solutions can be analyzed, see [W3]. Note also that equation (1.1) is used to study Bose–Einstein condensation, cf. e.g., [LOSK], and is usually called the Gross–Pitaevskii equation, when seeking non-decaying solutions.

One may pose similar questions for nonlinear difference equations, for example, for the Frenkel–Kontorava model, studied in Aubry–Mather theory, cf. e.g., [EFRJ] for its physical origin and [SdlL,GPT] for KAM-type results in one dimension. The method proposed here should be applicable, providing (periodic) space quasi-periodic solutions in arbitrary dimensions, corresponding to sliding.

1.2. Ideas of the proof. Since \( \tilde{u} \) is real, we may seek real solutions \( u \) to (1.3). Use diag \( \cdot \) to denote a diagonal matrix. Substituting (QP) into (1.3) leads to the nonlinear matrix equation on \( \ell^2(\mathbb{Z}^d) \):

\[
\text{diag} \left( \sum_{k=1}^{d} (j_k \cdot \lambda_k)^2 - E \right) \hat{u} - (\hat{u})^{2p} \ast \hat{u} = 0;
\]

with the linearized operator being

\[
H = \text{diag} \left( \sum_{k=1}^{d} (j_k \cdot \lambda_k)^2 - E \right) - (2p + 1)(\hat{u})^{2p} \ast .
\]

To fix ideas, set \( \hat{u} = \hat{u}^{(0)} \). \( H \) is then quasi-periodic in \( d \)-dimensions on diagonal plus a convolution operator. The issue is to control the inverse of \( H \). The main difficulty here is that for \( d > 1 \), Diophantine conditions on \( \lambda \) do not suffice. The problem is more geometric, and we use the semi-algebraic set technique developed by Bourgain in the study of Anderson localization [B3] to do the linear analysis. (Such techniques first appeared in [BGS] on quasi-periodic Anderson localization in \( \mathbb{Z}^2 \).) This is different from the space periodic setting in [W1], cf. also [W2], where the quasi-periodicity is in time only, which is one dimensional. Diophantine conditions together with eigenvalue variations suffice for the linear analysis. (The main work in [W1, 2] is to extract parameters from the nonlinear term, in order to deal with the original (fixed) nonlinear equations such as that in (1.1), and not merely a family of parameter-dependent equations as in [B1, 2]. The method of extraction is algebraic.)

Once we have good control on the inverse of \( H \), the nonlinear analysis proceeds using a Newton iteration, based on Chap. 18 of [B2], cf. [BW, W1, 2]. This part is rather standard, and shares many common features with other KAM-type schemes. It is, in fact, simpler here, since the “dynamical variables” are the space variables \( j \in \mathbb{Z}^d \) and there is no modulation to the frequency \( \lambda \).

2. Green’s Function Estimates in \( (\theta_1, \theta_2, \ldots, \theta_d) \)

Returning to the problem at hand, we seek solutions \( U \) close to \( \tilde{U} \), in the form (1.2) and (QP), leading to the nonlinear matrix equation on \( \ell^2(\mathbb{Z}^d) \):
\[
\text{diag} \left( \sum_{k=1}^{d} (j_k \cdot \lambda_k)^2 - E \right) \hat{u} - (\hat{u})^{2p} \ast \hat{u} = 0,
\]
and the linearized operator
\[
H = \text{diag} \left( \sum_{k=1}^{d} (j_k \cdot \lambda_k)^2 - E \right) - (2p + 1)(\hat{u})^{2p} \ast .
\]

To fix ideas, we give an example of the convolution matrix. Let \(d = 2\) and \(p = 1\); write \((x, y)\) for \((x_1, y_1)\) and set \(\hat{u} = \hat{\tilde{u}}\), as in (1.7); set \(a = 1\). We compute:
\[
\cos^2(\tilde{j}_1 \cdot \lambda_1) x \cos(j_1 \cdot \lambda_1)x = 1/2[\cos(j_1 \cdot \lambda_1)x] + 1/4[\cos(j_1 + 2\tilde{j}_1) \cdot \lambda_1 x + \cos(j_1 - 2\tilde{j}_1) \cdot \lambda_1 x];
\]
and similarly in the \(y\) variable, i.e., with the subindex \(1 \leftrightarrow 2\) and \(x \leftrightarrow y\).

It follows that \(\hat{\tilde{u}}^{2p}\) is a convolution matrix on \(\mathbb{Z}^4\), with the non-vanishing entries:
- \(1/4\) on diagonal;
- \(1/8\) on the \((\pm 2\tilde{j}_1, 0)\) and \((0, \pm 2\tilde{j}_2)\) off-diagonals (4 in total); and
- \(1/16\) on the \((\pm 2\tilde{j}_1, \pm 2\tilde{j}_2)\) off-diagonals (4 in total).

We now proceed to the analysis.

2.1. Lyapunov–Schmidt decomposition. We use a Newton scheme to solve (2.1), using as initial approximation \(u^{(0)} = \tilde{u}\) in (1.7) and \(E^{(0)} = \tilde{E} = (\tilde{j} \cdot \lambda)^2\) in (1.9). In the matrix notation of (2.1), \(\hat{u}\) is a column vector and \(\hat{u}^{(0)}(j) = a/2^d\), if \(j_k = \pm \tilde{j}_k, k = 1, 2, \ldots, d\), and 0 otherwise. Below, since we only work with \(\hat{u}\), we slightly abuse the notation and write \(u\) for \(\hat{u}\). We may also assume \(a > 0\) as, if \(u\) is a solution, then so is \(-u\).

Let
\[
S = \{ \pm \tilde{j}_k, k = 1, 2, \ldots, d \}.
\]
Writing (2.1) as
\[
F(u) = 0,
\]
the equations are divided into the \(Q\)-equations:
\[
F(u)|_S = 0;
\]
and the \(P\)-equations:
\[
F(u)|_{\mathbb{Z}^{2d}\setminus S} = 0.
\]
The amplitudes on the set \(S\) are held fixed:
\[
u|_S = a/2^d;\]
while the $Q$-equations are used to solve for $E$. Due to symmetry, the $2^d$ equations in (2.4) are the same, yielding
\[ E = (\vec{j} \cdot \lambda)^2 - (2^d/a)(u)^{2p+1} |_{\vec{j}}. \] (2.7)
So, for example, the first iteration gives
\[ E^{(1)} = (\vec{j} \cdot \lambda)^2 - (2^d/a)(u^{(0)})^{2p+1} |_{\vec{j}}. \] (2.8)
Substituting the result in (2.7) into the $P$-equations (2.5), we use a Newton scheme to solve for $u$ on $\mathbb{Z}^{2d} \setminus S$. For simplicity, omitting the subindex $\mathbb{Z}^{2d} \setminus S$ from now on, we have formally, (note that the $\Delta$ below denotes increment),
\[ \Delta u = -[F'(u)]^{-1} F(u), \] (2.9)
where $F'(u)$ is the linearized operator:
\[ F'(u) = \text{diag} \left( \sum_{k=1}^{d} (j_k \cdot \lambda_k)^2 - E - (2p+1)(u)^{2p} \right). \] (2.10)

Generally speaking, the idea is to start with the initial approximation $(u^{(0)}, E^{(0)})$ as in (1.7, 1.9) and to iterate the Newton scheme, with each iteration $i$ resulting in an approximate solution $(u^{(i)}, E^{(i)})$, after appropriate excisions in $\lambda$; and as $i \to \infty$, $(u^{(i)}, E^{(i)})$ converges to a solution $(u, E)$ to (1.3). Hence $U$ in (1.2)-(QP) is a solution to (1.1) for a subset of $\lambda$.

Remark. We note that the above $P$ and $Q$-equations are decomposed according to the Fourier support of $u, S$, and uses the condition (D). The $Q$-equations are resonant, as the diag in (2.10) is 0 on $S$, when $E = \tilde{E}$; while the $P$-equations are non-resonant.

2.2. Invertibility of the linearized operators. From (2.9), the invertibility of $F'$ is central to the Newton iteration. Since we seek solutions close to $u^{(0)}$, which is only supported on $S$, we adopt a multiscale Newton scheme. The idea is as follows.

At each iteration $i$, choose an appropriate scale $N$ and estimate $[F'_N]^{-1}$, where $F'_N$ is $F'$ restricted to
\[ [-N, N]^{2d} \subset \mathbb{Z}^{2d}. \] (2.11)
We call the $[F'_N]^{-1}$, the Green’s functions. To facilitate the estimates, add $d$ auxiliary variables
\[ \theta_1, \theta_2, \ldots, \theta_d, \]
to $F'$ and define:
\[ F'(\theta_1, \theta_2, \ldots, \theta_d) := \text{diag} \left( \sum_{k=1}^{d} (j_k \cdot \lambda_k + \theta_k)^2 - E - (2p+1)(u)^{2p} \right). \] (2.12)
Denote $(\theta_1, \theta_2, \ldots, \theta_d)$ by $\theta \in \mathbb{R}^d$. We first make estimates on $F'_N(\theta)$ in $\theta$ and then use the covariance with respect to the $\mathbb{Z}^{2d}$ action on $\mathbb{R}^d$:
\[ (\theta_1, \theta_2, \ldots, \theta_d) \mapsto (\theta_1 + j_1 \cdot \lambda_1, \theta_2 + j_2 \cdot \lambda_2, \ldots, \theta_d + j_d \cdot \lambda_d), \] (2.13)
to deduce estimates for
\[ [F'_N(\theta = 0)]^{-1} := [F'_N]^{-1}, \]
the Green’s functions used in the Newton scheme (2.9).
2.3. The \((\theta_1, \theta_2, \ldots, \theta_d)\) estimates. Denote the linearized operator \(F'\) by \(T\); and \(F'_N, T_N\). The goal of this section is to estimate the Green’s functions \(T^{-1}_N(\theta)\) for all \(N\), away from a set in \(\theta \in \mathbb{R}^d\) of small sectional measure, after appropriate excisions in \(\lambda \in (1/2, 3/2)^{2d}\). To apply the covariance in (2.13) for the Green’s function analysis, it is essential that the excised set in \(\lambda\) is independent of the starting point, i.e., at \(j_1 = j_2 = \cdots = j_d = 0\), of \(\theta \in \mathbb{R}^d\), which we denote by \(\tilde{\theta} \in \mathbb{R}^d\). This is, in essence, accomplished by variable reduction, eliminating the variable \(\tilde{\theta} \in \mathbb{R}^d\). (See Lemmas 2.2 and 2.3 below).

Since \(d\) is arbitrary, the geometry of the sets in \(\theta = (\theta_1, \theta_2, \ldots, \theta_d)\) comes into play. Diophantine conditions, i.e., quantitative versions of \((D)\), generally do not suffice, and we shall use the semi-algebraic set technique developed by Bourgain in [B3], cf. Chap. 9 [B2]. For that purpose, we need that \(u^{(i)}\) and \(E^{(i)}\) are algebraic in \(\lambda\) and control their degrees. To begin with, \(u^{(0)}\) does not depend on \(\lambda\) (recall that \(u^{(0)}\) now stands for \(\hat{u}^{(0)}\)), and from (1.9), (2.8), \(E^{(0)}\) and \(E^{(1)}\) are both quadratic polynomials in \(\lambda\).

Since \(u^{(i)}\) and \(E^{(i)}\) depend on the scale \(N\), we denote them by \(u_N\) and \(E_N\) in this section. We assume what is needed on \(u_N\) and \(E_N\) in (2.10) from the nonlinear analysis, in order to estimate the Green’s functions. Later in Sect. 3, we verify these assumptions.

Let us first define a semi-algebraic set.

**Definition.** A set \(S\) is called semi-algebraic if it is a finite union of sets defined by a finite number of polynomial equalities and inequalities. More specifically, let \(\mathcal{P} = \{P_1, P_2, \ldots, P_s\} \subset \mathbb{R}[x_1, x_2, \ldots, x_n]\) be a family of \(s\) real polynomials of degree bounded by \(\kappa\). A (closed) semi-algebraic set \(S\) is given by an expression

\[
S = \bigcup_j \bigcap_{\ell \in \mathcal{L}_j} \{P_\ell s_{j\ell} \leq 0\},
\]

where \(\mathcal{L}_j \subset \{1, 2, \ldots, s\}\) and \(s_{j\ell} \in \{\geq, =, \leq\}\) are arbitrary. We say that \(S\) as introduced above has degree at most \(sk\) and its degree \(B\) is the minimum \(sk\) over all representations \((S)\) of \(S\).

The following is a special case of Theorem 1 in [Ba], cf. Theorem 9.3 in Chap. 9 [B2].

**Lemma 2.1.** Let \(S \subset \mathbb{R}^n\) be as in \((S)\). Then the number of connected components of \(S\) does not exceed \(O(sk)^n\).

The two properties of semi-algebraic sets that play a central role here are the Tarski-Seidenberg principle, which states that the projection of a semi-algebraic set of \(\mathbb{R}^n\) onto \(\mathbb{R}^{n-1}\) is semi-algebraic; and the Yomdin-Gromov triangulation theorem of these sets. They are both stated in [B3], cf. the references therein. (For the complete proof of the Yomdin-Gromov triangulation theorem, see [BiN], cf., also the earlier paper [Bu].) We do not repeat them here, except their consequences for thin sets.

Below we call connected open sets intervals. Our main goal is to prove the following.

**Main Lemma.** Let \(I\) be an interval in \((1/2, 3/2)^{2d}\), \(u_N\) and \(E_N\) two sequences of real rational functions in \(\lambda\),

\[
u_N : I \mapsto \ell^2(\mathbb{Z}^{2d}),
\]

satisfying

\[
u_N(j) = 0, \ j \notin [-N^K, N^K]^{2d},
\]
for some $K > 1$; and

$$E_N : I \mapsto \mathbb{R}.$$  

For $0 < a \ll 1$, assume that there exists $N_0 = N_0(a) \gg 1$, such that for $N \geq N_0$, the following conditions are satisfied:

\begin{align*}
\deg u_N &\lesssim e^{(\log N)^3}, \\
|u_N(j)| &\leq e^{-\gamma |j|}, \quad j \in \mathbb{Z}^d (\gamma > 0), \\
\|u_N - u_{N+1}\|_{\ell^2(\mathbb{Z}^d)} &\leq e^{-\tilde{\gamma} N} (\tilde{\gamma} > \gamma > 0);
\end{align*}

and

\begin{align*}
\deg E_N &\lesssim e^{(\log N)^3}. \\
E_N &= O(1) \\
|E_N - E_{N+1}| &\leq e^{-\tilde{\gamma} N}.
\end{align*}

For all $N \geq N_0$, there exists $A_N \subset I$, a semi-algebraic set of

\begin{align*}
\deg A_N &\leq N^{8d}, \\
\text{meas } (A_{N-1} \setminus A_N) &\leq N^{-c}, \quad c > 0,
\end{align*}

such that for any $\lambda \in A_N$, there exists a subset $\Theta_1 \subset \mathbb{R}^d$, whose sectional measures satisfy

$$\text{meas } [\theta_i | \forall \text{ fixed } \theta_k, k \neq i; \theta \in \Theta_N] \leq e^{-N^\tau} (\tau > 0),$$

for all $i = 1, 2, \ldots, d$. If $\theta \notin \Theta_N$, the linearized operator $F' := T$ in (2.12), after truncations, satisfy the estimates

$$\|[T_N(u_N, E_N)(\theta)]^{-1}\|_{\text{op}} \leq e^{N^\sigma} (1 > \sigma > \tau > 0),$$

and

$$\|[T_N(u_N, E_N)(\theta)]^{-1}(j, j')\| \leq e^{-\beta |j-j'|} (0 < \beta < \gamma), \forall |j - j'| > N/10.$$

The nonlinear construction in Sect. 3 will verify (2.14)–(2.19) by using the double exponential convergence of the Newton scheme. Note that the algebraic in $\lambda$ requirements on $u_N$ and $E_N$ for the linear analysis is due to quasi-periodicity in space, this is different from Chaps. 19 and 20 in [B2] and [W1, 2], which are space periodic and the algebraic dependence is only used in the nonlinear analysis.
2.4. Proof of the main lemma. The proof is an application of Proposition 2.2 in [B3], complemented as Theorem 4.1 in [JLS]. The algebraic arguments rely on Lemmas 1.18 and 1.20 in [B3], and are stated below as Lemmas 2.2 and 2.3. The analysis arguments after formula (2.42), pp 696–699 in Sect. 2 [B3] have been dissected and clarified in Theorems 3.6, 4.1 and their proofs in [JLS].

Lemma 2.2. Let $A \subset [0, 1]^{n+r}$ be semi-algebraic of degree $B$ and such that
\[ \text{for each } t \in [0, 1], \text{ meas}_n A(\cdot, t) < \eta, \eta > 0. \] (2.25)

Then
\[ \mathcal{A} := \{(x_1, x_2, \ldots, x_{2r}) | A(x_1) \cap \cdots \cap A(x_{2r}) \neq \emptyset \} \subset [0, 1]^{n+2r} \] (2.26)
is semi-algebraic of degree at most $B^C$ and measure at most
\[ \eta_r = B^C \eta^{n-r} 2^{r(r-1)} \] (2.27)
with $C = C(r) > 1$.

Lemma 2.2 is a variable reduction lemma, eliminating the $r$-dimensional variable $t$. It is worth noting that $2^r$ copies of $A$ are used. The measure in (2.27), however, is in $n+2r$ dimensions; while we need the measure of a $n$-dimensional section of $\mathcal{A}$. Lemma 1.20 in [B3] serves this purpose, and is stated below.

Lemma 2.3. Let $A \subset [0, 1]^{n\rho}$ be a semi-algebraic set of degree $B$ and
\[ \text{meas}_{n\rho} A < \eta. \]

Let $\omega_i \in [0, 1], i = 1, 2, \ldots, n,$ and
\[ \omega = (\omega_1, \omega_2, \ldots, \omega_n) \in [0, 1]^n. \]

Let $k_i \in \mathbb{Z}, i = 1, 2, \ldots, n,$ and
\[ k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n. \]

Denote by $\{ \cdot \}$, the fractional part of a real number in $[0, 1)$, and
\[ k\omega := ([k_1\omega_1], [k_2\omega_2], \ldots, [k_n\omega_n]). \] (2.28)

Let $K_1, K_2, \ldots, K_{\rho-1} \subset \mathbb{Z}^n$ be finite sets with the following properties:
\[ \min_{1 \leq \ell \leq n} |k_\ell| > \left[ B \max_{1 \leq i \leq \nu \leq n} |m_{\nu}| \right]^C, \] (2.29)

if $k \in K_i$ and $m \in K_{i-1}, i = 2, \ldots, \rho - 1,$ and where $C = C(n, \rho)$. Assume also
\[ \frac{1}{\eta} > \max_{k \in K_{\rho-1}} |k|^C. \] (2.30)

Then
\[ \text{meas} \{ \omega \in [0, 1]^n | (\omega, k^{(1)}\omega, \ldots, k^{(\rho-1)}\omega) \in A \text{ for some } k^{(i)} \in K_i \} < B^C \delta, \] (2.31)

where
\[ \frac{1}{\delta} = \min_{k \in K_1} \min_{1 \leq \ell \leq n} |k_\ell|. \] (2.32)
Remark. As noted in the first paragraph in Sect. 2.3, Lemmas 2.2 and 2.3 are the tools to eliminate the variable θ (the starting point). The inequalities in (2.29) are steepness conditions. Even though we will not make use of it in proving the Main Lemma, we mention that using the special structure of the $\mathbb{Z}^{2d}$ action in (2.13), these conditions could, in fact, be relaxed to steepness in half of the directions, i.e., in $d$-dimensions only.

We use Lemma 2.3 to prove the Main Lemma, where we will set $n = 2d$ and $\rho = 2^{2d} + 1$. We first state a direct corollary of Proposition 2.2 in [B3], by assuming $u_N$ and $E_N$ are fixed, instead of varying with $N$. (The “$E$” in [B3] is set to be 0 here.)

Lemma 2.4. Let I be an interval in $(1/2, 3/2)^{2d}$ as in the Main Lemma, and $u_N = u^{(0)}$, $E_N = E^{(1)}$ (in (2.8)) for all $N$. There exists $N_0 = N_0(a)$, such that for all $N \geq N_0$, there exists $A_N \subset I$ satisfying (2.20), and

$$\text{meas } (I \setminus \cap_{N \geq N_0} A_N) \to 0,$$

as $a \to 0$. On $A_N$, (2.22)–(2.24) hold.

Proof. Choose $N_0 = |\log a|^s$ for some $s > 1$. Set

$$D = \sum_{k=2}^{d} (j_k \cdot \lambda_k + \theta_k)^2 - (\tilde{j} \cdot \lambda)^2.$$  \hfill (2.33)

From (2.7) and (2.10), to prove (2.23) and (2.24) at $N = N_0$, it suffices that

$$|(j_1 \cdot \lambda_1 + \theta_1)^2 + D| \geq a^{p+1}$$  \hfill (2.34)

for all $j = (j_1, j_2, \ldots, j_d) \in [-N_0, N_0]^{2d}$. This leads to excise a set in $\theta_1$ of measure satisfying (2.22), if $0 < s \tau < 1$ and $s \sigma > 1$. No excision in $\lambda$ is needed for this step, so (2.20) is trivially satisfied.

The iteration to larger scales, $N > N_0$, uses Lemmas 2.2 and 2.3. In order to import directly the proof of Proposition 2.2 in [B3], we shall not use the special structure of the $\mathbb{Z}^{2d}$ action in (2.13). Therefore we double the dimension and introduce

$$\tilde{\theta} = (\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_d) = (\tilde{\theta}_{1,1}, \tilde{\theta}_{1,2}, \tilde{\theta}_{2,1}, \tilde{\theta}_{2,2}, \ldots, \tilde{\theta}_{d,1}, \tilde{\theta}_{d,2}) \in \mathbb{R}^{2d}.$$  \hfill (2.35)

The covariance of the $\mathbb{Z}^{2d}$ action on $\mathbb{R}^{2d}$ of the corresponding linearized operator $F'(\tilde{\theta})$ is then:

$$(\tilde{\theta}_{1,1}, \tilde{\theta}_{1,2}, \tilde{\theta}_{2,1}, \tilde{\theta}_{2,2}, \ldots, \tilde{\theta}_{d,1}, \tilde{\theta}_{d,2}) \mapsto (\tilde{\theta}_{1,1} + \tilde{\theta}_{1,2} + j_1 \cdot \lambda_1, \tilde{\theta}_{2,1} + \tilde{\theta}_{2,2} + j_2 \cdot \lambda_2, \ldots, \tilde{\theta}_{d,1} + \tilde{\theta}_{d,2} + j_d \cdot \lambda_d).$$  \hfill (2.36)

We note that the right hand side is independent of

$$\theta_i^- = \tilde{\theta}_{i,1} - \tilde{\theta}_{i,2}, \text{ for all } i = 1, 2, \ldots, d.$$

It follows that

$$\theta_i = \tilde{\theta}_{i,1} + \tilde{\theta}_{i,2}, \text{ for } i = 1, 2, \ldots, d,$$  \hfill (2.37)

to return to (2.13).

Let $\tilde{\Theta}_N \subset \mathbb{R}^{2d}$ be the set, on the complement of which, (2.23) and (2.24) (in the argument $\tilde{\theta}$) hold. From the above discussion, the set $\tilde{\Theta}_N$ is independent of $\theta_i^-$, $i = 1, 2, \ldots, d$, etc.
1, 2, \ldots, d. Clearly, for the initial estimate at scale $N_0$, we may proceed as in (2.33)–(2.34) and obtain $\tilde{\Theta}_{N_0}$, satisfying

\[
\text{meas} \left[ \tilde{\theta}_{i,1} | \forall \text{ fixed } \tilde{\theta}_{i,2}, \tilde{\theta}_k, k \neq i; \tilde{\theta} \in \tilde{\Theta}_{N_0} \right] \leq e^{-N_0^2} (\tau > 0),
\]

\[
\text{meas} \left[ \tilde{\theta}_{i,2} | \forall \text{ fixed } \tilde{\theta}_{i,1}, \tilde{\theta}_k, k \neq i; \tilde{\theta} \in \tilde{\Theta}_{N_0} \right] \leq e^{-N_0^2} (\tau > 0),
\]

for all $i = 1, 2, \ldots, d$. Since $a$ is fixed, $N_0$ is fixed; $u^{(0)}$ and $E^{(0)}$ are fixed, $T$ is a fixed operator; using $\tilde{\Theta}_{N_0}$, we are in exactly the same setting as [B3] and [JLS] in $2d$ dimensions.

Proposition 2.2 [B3] and Theorem 4.1 [JLS] are directly applicable. With $2d$ replacing $d$, we obtain that for all scales $N \geq N_0$, there is a good frequency set $A_N$, deg $A_N \leq N^{8d}$, such that the conclusions in (2.22)–(2.24) hold, with $\tilde{\Theta}_N$ replacing $\Theta_N$, and

\[
\text{meas} \left[ \tilde{\theta}_{i,1} | \forall \text{ fixed } \tilde{\theta}_{i,2}, \tilde{\theta}_k, k \neq i; \tilde{\theta} \in \tilde{\Theta}_N \right] \leq e^{-N^2} (\tau > 0),
\]

\[
\text{meas} \left[ \tilde{\theta}_{i,2} | \forall \text{ fixed } \tilde{\theta}_{i,1}, \tilde{\theta}_k, k \neq i; \tilde{\theta} \in \tilde{\Theta}_N \right] \leq e^{-N^2} (\tau > 0),
\]

for all $i = 1, 2, \ldots, d$. Using (2.37), (2.38)–(2.39) become

\[
\text{meas} \left[ \tilde{\theta}_i | \forall \text{ fixed } \tilde{\theta}_k, k \neq i; \tilde{\theta} \in \tilde{\Theta}_N \right] \leq e^{-N^2} (\tau > 0).
\]

Since $\tilde{\Theta}_N$ is independent of

\[
\theta_i^- = \tilde{\theta}_{i,1} - \tilde{\theta}_{i,2}, \quad \text{for all } i = 1, 2, \ldots, d,
\]

this with (2.37) lead to (2.22). The measure estimate in the Lemma also follows. \hfill \Box

**Proof of the Main Lemma.** Scale $N = N_0$ is already proved in Lemma 2.4. To obtain (2.23) and (2.24) at larger scales, we use induction. Below we keep to the [JLS] notations.

The induction uses 3 scales: $N_1$, $N_2 = N_1^{2/c_1}$, and $N_3 = e^{N_1}$, where $c_1 > 0$, and $c_1 = \tau$ here. Assume that the Main Lemma holds at scales $N_1$ and $N_2$, we shall show that it holds at $N_3$.

At scale $N_1$, if $\lambda \in A_{N_1}$, then for $\tilde{\theta} \notin \tilde{\Theta}_{N_1}$:

\[
\left\| [T_{N_1}(u_{N_1}, E_{N_1})(\tilde{\theta})]^{-1} \right\|_{\text{Op}} \leq e^{N_1^2} (1 > \sigma > \tau > 0),
\]

\[
\left\| [T_{N_1}(u_{N_1}, E_{N_1})(\tilde{\theta})]^{-1} (j, j') \right\| \leq e^{-\beta|j - j'|} (0 < \beta < \gamma), \quad \forall |j - j'| > N_1/10Q
\]

Due to geometric reasons in the induction (paving) process, aside from cubes:

\[
Q_N = [-N, N]^{2d},
\]

we also need to consider regions of the form:

\[
Q_N = [-N, N]^{2d} \setminus \{ n \in \mathbb{Z}^{2d} : n_i \xi_i 0, 1 \leq i \leq 2d \},
\]

where for $i = 1, 2, \ldots, 2d$, $\xi_i \in \{<, >, \emptyset\}^{2d}$ and at least two $\xi_i$ are not $\emptyset$. We assume that (2.40) and (2.41) hold for $Q_{N_1}$ as well.

Theorem 2.7 in [JLS] is applicable and gives the following: There is a semi-algebraic set $A_3 \subset A_{N_1}$, with deg $A_3 \leq N_3^{8d}$ and

\[
\text{meas} \left( A_{N_1} \setminus A_3 \right) \leq N_3^{-C_3},
\]
with \( c_3 = 8d c_1 = 8d \tau \), such that if \( \lambda \in \mathcal{A}_3 \), then for any \( \tilde{\theta} \in \mathbb{R}^{2d} \), there exists \( \tilde{N} \in [N_3^c, N_3^{c_3}] \) and annulus

\[
\Gamma := [-\tilde{N}, \tilde{N}]^{2d} \setminus [-\tilde{N} \frac{1}{\tau}, \tilde{N} \frac{1}{\tau}]^{2d},
\]

such that for all \( k \in \Gamma \), \( Q_{N_1}(u_{N_1}, E_{N_1})(\tilde{\theta}) + k \) satisfy (2.40) and (2.41). Using (2.15), (2.16), (2.18) and (2.19) between scales \( N_1 \) and \( N \geq N_3 \), we obtain

\[
\|[T_{N_1}(u_N, E_N)(\tilde{\theta})]^{-1}\|_{op} \leq (1 + e^{-\gamma N_1}) e^{N_1^q} (1 > \sigma > \tau > 0), \tag{2.45}
\]

\[
\|[T_{N_1}(u_N, E_N)(\tilde{\theta})]^{-1}(j, j')| \leq (1 + e^{-\gamma N_1}) e^{-\beta |j-j'|} (0 < \beta < \gamma), \quad \forall |j - j'| > N_1/10, \tag{2.46}
\]

where \( T_{N_1} \) now denotes restrictions to \( Q_{N_1} + k \), \( k \in \Gamma \).

We are now at the point to apply Theorem 3.6 in [JLS], which uses Cartan Theorem. For this purpose, we need the estimates (2.23) and (2.24) at scale \( N_2 \). Using (2.15), (2.16), (2.18) and (2.19) between scales \( N_2 \) and \( N \geq N_3 \), yields:

\[
\|[T_{N_2}(u_N, E_N)(\tilde{\theta})]^{-1}\|_{op} \leq (1 + e^{-\gamma N_2}) e^{N_2^q} (1 > \sigma > \tau > 0), \tag{2.47}
\]

\[
\|[T_{N_2}(u_N, E_N)(\tilde{\theta})]^{-1}(j, j')| \leq (1 + e^{-\gamma N_2}) e^{-\beta |j-j'|} (0 < \beta < \gamma), \quad \forall |j - j'| > N_2/10. \tag{2.48}
\]

Applying Theorem 3.6 [JLS], we then obtain that for all \( N \in [N_3, N_3^2] \), (2.20), (2.22)–(2.24) hold at scale \( N \), with

\[
\beta_N = \beta - \mathcal{O}(1)/N_1^q
\]

replacing \( \beta \) in (2.24), provided

\[
\lambda \in \mathcal{A}_N := \mathcal{A}_3 \cap \mathcal{A}_{N_2}. \tag{2.49}
\]

We take an interval of initial scales \( N_0 \in ((\log |\log a|)^{1/\tau}, |\log a|^s] \), \( 0 < s \tau < 1 \). Clearly (2.22)–(2.24) hold for all such \( N_0 \), and \( \mathcal{A}_{N_0} = I \). For example, for the scale \( (\log |\log a|)^{1/\tau} \), modifying (2.34) to require

\[
|(j_1 \cdot \lambda_1 + \theta_1)^2 + D| \geq 2|\log a|^{-1},
\]

for all \( j = (j_1, j_2, \ldots, j_d) \in [-N_0, N_0]^{2d} \), leads to the desired estimates. Denote the sub-exponential induction in scales by the function \( f \): e.g., \( N_3 = f(N_1) = e^{N_1^{\tau}} \), \( \tau > 0 \), since \( f^2(x) > f(x + 1) \), e.g., \( N_3^2 = e^{2N_1^{\tau}} > e^{(N_1^{\tau}+1)^\tau} \), the iterates therefore generate all possible scales. Consequently, we obtain that (2.22)–(2.24) hold for all \( N \) with exponential rate of decay \( \beta_N \) bounded below by

\[
\beta_\infty = \beta_{N_0} - \sum_{i=0}^{\infty} \frac{\mathcal{O}(1)}{[f^{(i)}(N_0)]^\sigma} > \frac{\beta_{N_0}}{2}, \tag{2.50}
\]

where \( f^{(i)} \) is the \( i \)th iterate of \( f \). Set

\[
\beta = \beta_\infty. \tag{2.51}
\]
We are only left to prove (2.21). From construction, if both $N - 1$ and $N \in [N_3, N_3^2]$, for some $N_3$, then $A_{N-1} \setminus A_N = \emptyset$, meas $A_{N-1} \setminus A_N = 0$. Otherwise, if $N = N_3^2 + 1$, then from (2.44) one needs to make an additional excision of measure less than
$$(N_3^2 + 1)^{-c_3} = N^{-8d \tau}$$
from the set $A_{N_1'}$, satisfying the inclusions,
$$A_{N_1'} \supset A_{N_2'} \supset A_{N_3},$$
where $N_1' = (\log N)^{1/\tau} \ll N_3$ and $N_2' = N_1'^2/\tau \ll N_3$, are the two scales in the induction to reach scale $N$. Since $A_{N_3} = A_{N-1}$, the inclusion in (2.50) gives (2.21) with $c = 8d \tau$. □

2.5. How to use the main lemma. We anticipate in the next few lines the application of the Main Lemma to the nonlinear analysis in Sect. 3.

From the Newton scheme (2.9), and its multiscale realizations, the $u_N$, hence $T_N$, in the Main Lemma are defined only on intervals $I$ such that an appropriate restricted $F' = T$ is invertible. (This is one of the main differences with linear theory, where the operators are given, and therefore defined a priori on all of the parameter space.) So in the nonlinear application in Sect. 3, the intervals “$I$” in the Main Lemma, will vary with $N$, and we shall apply the Lemma to each interval $I$ in $\lambda$, on which $T_N$ is defined. Note that the measure estimate in (2.21) is per interval. To control the total excised measure, the nested properties of different generations of intervals, which we already had a glimpse of in the proof of the Main Lemma, shall come into play.

3. Nonlinear Construction—Proof of the Theorem

Our goal is to seek space quasi-periodic solutions in the form (QP) to the NLS (1.1). Recall that it leads to the nonlinear matrix equation (2.1), which we denote by $F(u) = 0$ in (2.3). (Recall also that $u$ now stands for $\hat{u}$.) The equations are divided into the $Q$-equations in (2.4), leading to (2.7), which yields $E$,
$$E = (\tilde{j} \cdot \lambda)^2 - (2^d/a)(u)^{2p+1}\tilde{j},$$
and the $P$-equations (2.5), which are used to solve for $u$. The initial approximation $u^{(0)}$ is given by $\hat{u}$ (1.7). Substituting $u^{(0)}$ into (3.0) gives
$$E^{(1)} = (\tilde{j} \cdot \lambda)^2 - (2^d/a)(u^{(0)})^{2p+1}\tilde{j}.$$

The $P$-equations are solved using a Newton scheme and iteration in scales. Let $M$ be a large integer, and $M'$, $r = 1, 2, \ldots$, the geometric sequence of scales. Denote by $u^{(r)}$ the $r$th approximation, and the increment
$$\Delta u^{(r)} = u^{(r)} - u^{(r-1)}.$$
We define
$$\Delta u^{(r)} = -[F_N(u^{(r-1)}, E^{(r)})]^{-1} F(u^{(r-1)}),$$
(3.1)
where $N = M^r$ and $F_N'(u^{(r-1)}, E^{(r)})$ is the restriction of the linearized operator,

$$F_N'(u^{(r-1)}, E^{(r)}) = \text{diag} \left( \sum_{k=1}^{d} (j_k \cdot \lambda)^2 - E^{(r)} \right) - (2p+1)(u^{(r-1)})^{2p*},$$

to the cube $[-N, N]^d$, and where

$$E^{(r)} = (\tilde{j} \cdot \lambda)^2 - (2d/a)(u^{(r-1)})^{2p+1}|_{\tilde{j}}.$$

Equations (3.2) and (3.1) together with $u^{(0)}$ iteratively solve the $Q$ and the $P$-equations, provided (3.1) is well-defined for all $r$ and the resulting series converges. The Main Lemma is pivotal in estimating $[F_N'(u^{(r-1)}, E^{(r)})]^{-1}$ in (3.1), which ensures double exponential convergence of the Newton scheme. We first lay down the induction hypothesis.

Let $M$ be a large positive integer. As earlier, one may assume $a > 0$. It consists in showing that the following are satisfied for all $r > 0$ and fixed small $a$:

On the entire $\lambda$ space, namely $(1/2, 3/2)^d$:

(Hi) $\sup u^{(r)} \subseteq B(0, M^r)$ ($\sup u^{(0)} \subseteq B(0, M)$).

(Hiii) $\|\Delta u^{(r)}\| < \delta_r$, $\|\partial \Delta u^{(r)}\| < \tilde{\delta}_r$ with $\delta_{r+1} \ll \delta_r$ and $\tilde{\delta}_{r+1} \ll \tilde{\delta}_r$, where $\partial$ denotes $\partial_\lambda$ and $\|\| := \sup_{\lambda} \| \|_ell^2(\mathbb{Z}^d)$.

(Hiii) $|u^{(r)}(j)| < ae^{-|j|}(a > 0)$.

Using (3.6) and (Hi-i), the nonlinear eigenvalue $E^{(r)}$ is $C^1$ in $\lambda$ on $(1/2, 3/2)^d$. Moreover by (Hiii),

$$|E^{(r)} - E^{(r-1)}| \lesssim \|u^{(r)} - u^{(r-1)}\| < \delta_r,$$

so that $E^{(r-1)}$ is a $\delta_r$ approximation of $E^{(r)}$.

Below we continue with the assumptions on the restricted intervals in $\lambda$ on $(1/2, 3/2)^d$, where one could construct approximate solutions.

(Hiv) There is a collection $\Lambda_r$ of intervals of size $a^{p+2}M^{-rC}$, $C > 7$, such that

(a) On $I \in \Lambda_r$, $u^{(r)}(\lambda)$ is given by a rational function in $\lambda$ of degree at most $M^{Cr^3}$.

(Consequently, $E^{(r)}$ is rational of degree at most $M^{(2p+1)Cr^3}$ from (3.2).)

(b) For $\lambda \in \bigcup_{I \in \Lambda_r} I$,

$$\|F(u^{(r)})\| < \kappa_r, \|\partial F(u^{(r)})\| < \tilde{\kappa}_r$$

with $\kappa_{r+1} \ll \kappa_r$ and $\tilde{\kappa}_{r+1} \ll \tilde{\kappa}_r$.

(c) Let $N = M^r$. For $\lambda \in \bigcup_{I \in \Lambda_r} I$, $T = T(u^{(r-1)}) := F'(u^{(r-1)})$ satisfies

$$\|T_N^{-1}\| < a^{-(p+2)}M^C,$$

$$|T_N^{-1}(j, j')| < a^{-p+2}e^{-\alpha j-j'}, \text{ for } |j - j'| > r^C,$$

where $T_N$ is $T$ restricted to $[-N, N]^d$.

(d) Each $I \in \Lambda_r$ is contained in an interval $I' \in \Lambda_{r-1}$ and

$$\text{meas} \left( \bigcup_{I' \in \Lambda_{r-1}} I' \setminus \bigcup_{I \in \Lambda_r} I \right) < a^{p/5}r^{-5}.$$

The iteration holds with

$$\delta_r < a^p M^{-(\frac{3}{2})r}, \tilde{\delta}_r < a^p M^{-(\frac{3}{2})r}, \kappa_r < a^{2p} M^{-(\frac{3}{2})r+2}, \tilde{\kappa}_r < a^{2p} M^{-(\frac{3}{2})r+2}.$$

(W)
We remark that the approximate solutions \( u^{(r)} \) are defined, a priori, on \( \Lambda_r \), but using the derivative estimates in (Hiii) together with (W), as \( C^1 \) functions they can be, and are extended to \((1/2, 3/2)^{2d}\), by using a standard argument.

3.1. About the induction hypothesis. Let us provide some intuitions to the hypothesis in (Hiv); (Hi-iii) follow by the construction defined in (3.1), (3.2) and the Newton scheme.

First of all, since \( u^{(0)} \) is independent of \( \lambda \) and \( E^{(0)} = \tilde{E} \) in (1.9) is quadratic in \( \lambda \), \( E^{(r)} \) defined in (3.2) and \( u^{(r)}(\lambda) \) defined using (3.1), are (formally) clearly rational functions, for \( r = 1, 2, \ldots \), which provide the basis for our analysis.

- Size of the intervals: controlled by the bounds in (Hiv, c), as one may perturb \( \lambda \) and retain essentially the same bound.
- Number of intervals: given by the inverse of the size of the intervals. Note that it is obtained by analytic arguments, and not topological ones.
- Pointwise estimates in (Hiv, c): available at scales \( r C \ll N = Mr \), this means that the Main Lemma is used at much smaller scales \( r C \), cf. (2.24).
- The intervals in \( \Lambda_r \): the smaller the \( r \), the smaller number of intervals, hence complexity; moreover there is the nested property exhibited in (Hiv, d). This will be essential when applying the Main Lemma.
- The induction: consists of the initial steps and the general steps. The initial steps are direct perturbations using small amplitude \( a \); the general steps use the Main Lemma and then convert the \( \theta \)-estimates into \( \lambda \) estimates, leading to successive generations of \( \Lambda_r \).

3.2. The initial steps: \( r \leq R \). We start with the initial steps. In the Lemma below, for simplicity, \( T_N, N = M^r \), stand for \( T_N(\lambda, u^{(r-1)}), E^{(r)}(\lambda) \).

**Lemma 3.1.** There is a set \( B_N \) in \( \lambda \), with meas \( B_N < a^{p/5} \), such that on \((1/2, 3/2)^{2d}\setminus B_N \),

\[
\| T_N^{-1} \| < a^{-(p/2)}, \quad | T_N^{-1}(j, j')| < a^{-(p/2)} e^{-| \log a ||j-j'||},
\]

for all \( N \leq e^{| \log a |^{5/6}} \).

**Proof.** This follows from perturbation of the diagonals. From (3.2),

\[
E^{(1)} = \sum_{k=1}^{d} (\tilde{j}_k \cdot \lambda)^2 + O(a^{2p}) = (\tilde{j} \cdot \lambda)^2 + O(a^{2p}),
\]

it suffices if

\[
| \sum_{k=1}^{d} (j_k \cdot \lambda_k)^2 - (\tilde{j} \cdot \lambda)^2 | > 2a^{p/2},
\]

for all \( j = (j_1, \ldots, j_k, \ldots, j_d) \in [-N, N]^{2d}\setminus S \). For each \( j \in [-N, N]^{2d}\setminus S \), it is easy to see that
\[
\sum_{k=1}^{d} (j_k \cdot \lambda_k)^2 - (j \cdot \lambda)^2 = \sum_{k=1}^{d} [(j_k \cdot \lambda_k)^2 - (\tilde{j}_k \cdot \lambda_k)^2]
\]

\[
= \sum_{k=1}^{d} [(j_k - \tilde{j}_k) \cdot \lambda_k][(j_k + \tilde{j}_k) \cdot \lambda_k] \neq 0,
\]
by setting \( \lambda = (1, 0, \ldots, 0) \) and the Diophantine condition (D). It is a quadratic polynomial in \( \lambda \). Summing over \( j \) then gives the measure estimate for (3.4) to hold. The norm estimate in (3.3) follows from (3.4) by simple perturbation; while the pointwise estimate by resolvent series expansion. \( \square \)

**Corollary.** Set \( R = | \log a |^{3/4} \), (Hi-iv) and (W) hold for \( 1 \leq r \leq R \).

**Proof.** We first address (Hiv). Using Lemma 3.1 for the first \( R \), \( R = | \log a |^{3/4} \), steps of the induction, (Hiv, c) is verified, with

\( \alpha = \mathcal{O}(| \log a |) \),

for all scales \( N \),

\( N \in [M, M^{1 \log a |^{3/4}}] \),

with corresponding sets of intervals \( \Lambda_r \), \( r = (\log N/ \log M) \leq R \):

\[ \bigcup_{I \in \Lambda_r} I \subseteq (1/2, 3/2)^{2d} \setminus \mathcal{B}_N. \]

The nested property (Hiv, d) is manifest. On each \( I \), (3.3) is satisfied. Clearly (3.4) and hence (3.3) are stable under perturbations of size \( a^{p+2} \). So the intervals \( I \) are of size \( \mathcal{O}(a^{p+2}) \).

To prove (Hiv, a), we use induction. When \( r = 0 \), \( u^{(0)} \) is independent of \( \lambda \). Assume that it holds at stage \( r \),

\( \deg u^{(r)} \leq M^{Cr^3} \).

Appealing to the definition (3.1) and using the expression below it, we obtain

\( \deg u^{(r+1)} \lesssim 2p \deg u^{(r)} M^{2d(r+1)} < M^{C(r+1)^3} \),

where the volume factor \( M^{2d(r+1)} \) stems from the determinant used to compute the inverse. The above argument evidently holds for all \( r = 1, 2, \ldots \).

We are left with (Hiv, b). When \( r = 0 \),

\( F(u^{(0)}) = \mathcal{O}(a^{2p}) \),

and

\( \partial F(u^{(0)}) = 0. \)

On \( \Lambda_r \), \( \Delta u^{(r)} \) is constructed using (3.1):

\( \Delta u^{(r)} = -[F'_N(u^{(r-1)})]^{-1} F(u^{(r-1)}) \).
Using (3.3), this gives, when \( r = 1 \),
\[
\|\Delta u^{(1)}\| = O(a^{3p/2}),
\]
more over (Hi)ii is satisfied with \( \alpha \) satisfying (3.5). Similarly
\[
\|\partial \Delta u^{(1)}\| = O(a^p).
\]
So the first two expressions in (W) at \( r = 1 \) are satisfied.

To verify the other two expressions, we write
\[
F(u^{(1)}) = F(u^{(0)}) + F'(u^{(0)})\Delta u^{(1)} + O((\Delta u^{(1)})^2)
\]
\[
= (T - T_N)\Delta u^{(1)} + O((\Delta u^{(1)})^2)
\]
\[
= -[(T - T_N)T_N^{-1}F(u^{(0)}) + O(\|T_N^{-1}\|^2F(u^{(0)})^2)]
\]
\[
< a^2p+1,
\]
using (Hi-i)ii at \( r = 1 \), (3.1) and (3.3). This verifies the third expression in (W) at \( r = 1 \). Similarly we may verify the fourth expression.

The extension argument in Sect. 10, (10.33–10.37) in [B1], then proves (Hi-iii) on the entire \( \lambda \) space, moreover (W) is satisfied at \( r = 1 \). Iterating the above arguments, we prove (Hi-iv) and (W) for all \( r \leq R \) with \( \alpha \) satisfying (3.5). (For details of the iteration to prove (W), see Lemma 5.2 and its proof in [W1]. This is rather routine and clearly holds for all \( r = 1, 2, \ldots \)). \( \square \)

**Remark.** As a side, we mention that the semi-algebraic sets \( B_N \) can be described by the violation of \((2N + 1)^{2d}\) quadratic polynomial inequalities in (3.4), Lemma 2.1 gives that the number of connected components in \( B_N \) is bounded above by \( O(N^{4d^2}) \). The set \( B_N \) is, moreover, independent of \( u^{(0)} \), in fact, all \( u^{(k)} \) for \( k \leq R - 1 \).

Unlike the first \( R \) steps, however, the iterations to subsequent scales use the \( \theta \) estimates and the Main Lemma to make excisions in \( \lambda \), in order to fulfill (Hiv, c, d). Afterwards, the same induction arguments used in the first \( R \) steps, will validate (Hi-iii, Hiv, a, b) and (W) for all \( r > R \).

### 3.3. The general steps: \( r \geq R \)

Assume (Hi-iv) hold at stage \( r \). To construct \( u^{(r+1)} \), we need to control
\[
T_N^{-1}(u^{(r)}) \text{ with } N = M^{r+1}.
\]
This requires another excision in \( \lambda \), which will lead to the next set of intervals \( \Lambda_{r+1} \).

To simplify notations, given two sets of intervals \( Z_1 \) and \( Z_2 \), we say that
\[
Z_2 \subset Z_1,
\]
if for all \( I \in Z_2 \), there exists \( I' \in Z_1 \), such that \( I \subset I' \). We also define
\[
\text{meas} (Z_1 \setminus Z_2) = \text{meas} \left( \bigcup_{I' \in Z_1} I' \setminus \bigcup_{I \in Z_2} I \right)
\]
Cover \([-M^{r+1}, M^{r+1}]^{2d}\) by \([-M^r, M^r]^{2d}\) and smaller cubes \([-M_0, M_0]^{2d} + J\), with \( M^r/2 < |J| < M^{r+1} \) and \( M_0 \ll N \) to be specified shortly. Let \( r \geq R \). For simplicity, we
drop the prefactors in $a$ in (Hi-iv), since they are fixed, and only keep track of variations in $r$.

- **$\theta$-estimates of $T_{M_0}^{-1}(\theta)$**

The following Lemma provides $\theta$ estimates on the $M_0$-cube centered at the origin.

**Lemma 3.2.** Assume (Hi-iv) hold at stage $r$, scale $N = M^r$. Set

$$ M_0 = (\log N)^C = r^C (\log M)^C, \quad (3.6) $$

with $C > 7/c$, and $c$ as in (2.21); define

$$ r_0 = \frac{\log M_0}{\log M}, \quad (3.7) $$

and

$$ \tilde{r}_0 := r_0 \frac{\log M}{\log 4/3} < 2C \frac{\log r}{\log 4/3} \ll r. \quad (3.8) $$

Then there is $\Lambda'_{r+1} \subset \Lambda_r$, so that on $\Lambda'_{r+1}$, the following estimates hold:

$$ \| T_{M_0}^{-1}(u^{(\tilde{r}_0)}), E^{(\tilde{r}_0+1)}(\theta) \| < e^{M_0^\sigma}, \quad 0 < \sigma < 1, $$

$$ |T_{M_0}^{-1}(u^{(\tilde{r}_0)}), E^{(\tilde{r}_0+1)}(\theta)(x, y)| < e^{-\alpha|x-y|}, \quad \alpha > 0, $$

for all $x, y$ such that $|x - y| > M_0/10$, provided $\theta$ is in the complement of a set $\Theta_{M_0}$, whose sectional measures satisfy

$$ \text{meas} [\theta_i|\forall \text{fixed } \theta_k, k \neq i; \theta \in \Theta_N] \leq e^{-M_0^\tau} (\tau > 0). \quad (3.10) $$

**Remark.** The expressions in (3.6)–(3.8), if not manifestly integers, are understood to be the integer part.

**Proof.** For the first $r \leq R$ steps, direct perturbation in $a$ proves that (3.9) and (3.10) hold on $\Lambda_r$ with $\alpha = O(\log a)$, without additional excisions. (Here one may assume $M_0 \geq M$.)

For $r \geq R$, set

$$ N_1 = (\log M_0)^{1/\tau} < (2C \log r)^{1/\tau} $$

from (3.6), and

$$ \tilde{r} = \frac{\log N_1}{\log M} $$

and

$$ \tilde{\tilde{r}} = \tilde{r} \frac{\log M}{\log 4/3} < (10/\tau) \log \log r \ll \tilde{r}_0 \ll r. \quad (3.11) $$

(In the language of the Proof of the Main Lemma, $M_0 := N_3$.) To apply the Main Lemma, fix $I \in \Lambda_{\tilde{r}}$. By the choice of $\tilde{r}$ and using (W), on $\Lambda_{\tilde{r}} \cap \Lambda_{\tilde{r}_0}$,

$$ \| T_{N_1}(u^{(\tilde{r})}, E^{(\tilde{r}+1)}(\theta) - T_{N_1}(u^{(\tilde{r}_0)}), E^{(\tilde{r}_0+1)}(\theta) \| \lesssim \delta_{\tilde{r}_0} \leq e^{-\tilde{\alpha}N_1}, \quad \tilde{\alpha} > \alpha > 0. $$
(Here “∩” is in the sense of intersections of the intervals in the two sets; note from (Hiv, d) that each interval in $\Lambda_{\tilde{r}_0}$ is contained in an interval in $\Lambda_{\tilde{r}}$.) There are at most

$$M^{(\tilde{r})_C} \approx M^{(\log \log r)_C}$$

such intervals in $\Lambda_{\tilde{r}}$, by using (Hiv) at stage

$$\tilde{r} \sim \log \log r \ll r.$$ 

Denote the intersection over $I$ of the good sets by $\tilde{A}_{M_0}$ and let $\Lambda_{r+1} = \Lambda_r \cap \tilde{A}_{M_0}$, then

$$\text{meas } \Lambda_r \setminus \Lambda_{r+1} < M^{(\log \log r)_C}/M_0^C < M^{(\log \log r)_C}/r^{Cc} < 1/r^6,$$

if $Cc > 7$, using (2.21). Here we appealed again to (Hiv, d), but at stage $r$, namely that each interval in $\Lambda_r$ is contained in an interval in $\Lambda_{\tilde{r}_0}$. On $\Lambda_{r+1}$, (3.9)–(3.10) hold. □

- **Invertibility of $T^{-1}_{M_0}(\theta = 0)$**

The projection lemma below, stated as (1.5) in [B3], converts the $\theta$ estimates in (3.9)–(3.10) for the $M_0$-cube centered at the origin to $M_0$-cubes centered at large $J \in \mathbb{Z}^{2d}$ at $\theta = 0$.

**Lemma 3.3.** Let $S \subset [0, 1]^{n_1} \times [0, 1]^{n_2} := [0, 1]^n$, be a semi-algebraic set of degree $B$ and $\text{meas}_n S < \eta$, $\log B \ll \log 1/\eta$. Denote by $(x, y) \in [0, 1]^{n_1} \times [0, 1]^{n_2}$ the product variable. Fix $\epsilon > \eta^{1/n}$. Then there is a decomposition

$$S = S_1 \cup S_2,$$

with $S_1$ satisfying

$$\text{meas}_{n_1}(\text{Proj}_x S_1) < B^K \epsilon \quad (K > 0),$$

and $S_2$ the transversality property

$$\text{meas}_{n_2}(S_2 \cap L) < B^K \epsilon^{-1} \eta^{1/n} \quad (K > 0),$$

for any $n_2$-dimensional hyperplane $L$ in $[0, 1]^{n_1+n_2}$ such that

$$\max_{1 \leq j \leq n_1} |\text{Proj}_L(e_j)| < \frac{1}{100} \epsilon,$$

where $e_j$ are the basis vectors for the $x$-coordinates.

**Remark.** Lemma 2.3 is, in fact, also derived from Lemma 3.3, cf., the Proof of Lemma 1.20 in [B3].

**Lemma 3.4.** There exists $\Lambda_{r+1} \subset \Lambda_{r+1}' \subset \Lambda_r$, satisfying

$$\text{meas } \Lambda_r \setminus \Lambda_{r+1} < 1/r^5,$$

provided $C > \max(1/\tau, 7/c)$. On the intervals in the set $\Lambda_{r+1}$, $T^{-1}_{[-M_0, M_0]^{2d}+J}(u(\tilde{r}_0))$ satisfy the upper bounds in (3.9) for all $J$ with $M^r/2 < |J| < M^{r+1}$. 
Proof. We first make estimates on

\[ T_{[-M_0, M_0]^{2d} + J(u(\tilde{r}_0))}, \]

with \( M'/2 < |J| < M'^{r+1} \), on the set \( \Lambda'_{r+1} \). Fix \( I \in \Lambda_{r_0} \). To apply Lemma 3.3, identify the set \( \Theta_{M_0} \subset \mathbb{R}^d \) with the set

\[ \tilde{\Theta}_{M_0} = \Theta_{M_0} \times \{0\} \subset \mathbb{R}^{2d}. \]

One may assume

\[ \Theta_{M_0} \subset [-4M_0, 4M_0]^{d}, \]

as otherwise \( T_{M_0}(\theta) \) is invertible. Make the partition:

\[ \tilde{\Theta}_{M_0} = \bigcup K \left\{ [-1, 1]^d + K \right\} \times \{0\} = \bigcup K \mathcal{I}_K, \]

where \( K \in \mathbb{Z}^d \), satisfying

\[ 0 \leq |K| \leq 4dM_0. \quad (3.12) \]

Fix a \( K \) and let

\[ S_K(\lambda, \tilde{\theta}) = I \times \{\tilde{\Theta}_{M_0} \cap \mathcal{I}_K\} \subset \mathbb{R}^{4d}; \]

\[ \text{meas } S_K \leq e^{-M_0^2}. \quad (3.13) \]

Below for notational simplicity, we generally write \( S \) for \( S_K \). The set \( S \) is described by the opposite of (3.9). Replacing the \( \ell^2 \) norm by the Hilbert–Schmidt norm and since the matrix elements of the inverse is the division of two determinants, (3.9) can be expressed as algebraic inequalities in the matrix elements of degree at most \( M_0^{C} \). Since each matrix element is quadratic in \( \tilde{\theta} \) and at most of degree \( e^{(C \log M_0)^3} \) in \( \lambda \), \( S \) is of degree at most

\[ \deg S \leq M_0^{C} e^{(C \log M_0)^3} \lesssim e^{(\log M_0)^4}. \quad (3.14) \]

Let

\[ J\lambda = (J_1 \lambda_1, J_2 \lambda_2, \ldots, J_d \lambda_d) \in \mathbb{R}^d, \]

where

\[ J_i \lambda_i = \max \{ J_{i,1} \lambda_{i,1}, J_{i,2} \lambda_{i,2}, \ldots, J_{i,d} \lambda_{i,d} \}, \]

and “maxv” denotes maximum in absolute value. Define the set \( \mathcal{L}_I \) to be

\[ \mathcal{L}_I = \bigcup_{K} \bigcup_{J} S_K(\lambda, \theta = J\lambda), \]

where the union is over \( K \in \mathbb{Z}^d \) satisfying (3.12), and \( J \in \mathbb{Z}^{2d} \), with \( M'/2 < |J| \leq M'^{r+1} \). In the complement of \( \mathcal{L}_I \),

\[ T_{[-M_0, M_0]^{2d} + J} \]

satisfies (3.9), for all \( J \) satisfying \( M'/2 < |J| \leq M'^{r+1} \).
The measure of $L_I$ is estimated using Lemma 3.3. From (3.13) and (3.14), the set $S$ satisfies
\[
\begin{align*}
\deg S &\lesssim e^{(\log M_0)^4} \lesssim e^{(\log \log N)^4}, \\
\meas S &\leq e^{-M_0^r} = e^{-(\log N)^{C_\tau}}.
\end{align*}
\]
Since $J$ satisfies
\[
M^r/2 < |J| \leq M^{r+1},
\]
equivalently
\[
N/2 < |J| \leq MN,
\]
we have
\[
\log \deg S \ll \log |J| \ll -\log \meas S.
\]
(3.16)

So the proof in (3.6)–(3.26) in [B3] remains valid, leading to the conclusion (3.9). Alternatively, one may view this as a generalization to arbitrary dimensions $d$, of Proposition 5.1 in [BGS], and its proof. We therefore have
\[
\meas L_I \leq N^{-c'} = M^{-c'r},
\]
for some $c' = c'(d) > 0$, provided $C_\tau > 1$.

Using (Hiv) at stage $\tilde{r}_0$, the number of intervals at stage $\tilde{r}_0$ is bounded above by
\[
M^{\tilde{r}_0} \simeq M^{(\log M_0)^C} \simeq M^{(\log r)^C}.
\]
Let
\[
\mathcal{L} = \bigcup_I L_I,
\]
then
\[
\meas \mathcal{L} \leq M^{(\log r)^C} \cdot M^{-c'r} \leq M^{-c'r/2}, \ c' > 0.
\]
(3.17)

Define
\[
\Lambda_{r+1} = \Lambda_{r+1}' \setminus \mathcal{L}.
\]
The set $\Lambda_{r+1}$ satisfies (Hiv, d) at stage $r + 1$, adding (3.12) and (3.17) verifies the measure estimate. On the set, $T_{[-M_0,M_0]^{2d} + J(u(\tilde{r}_0))}^{r+1}$ satisfy the upper bounds in (3.9) for all $J$ with $M^r/2 < |J| < M^{r+1}$. □

**Invertibility of $T_{M^{r+1}}^{r+1}(u^{(r)})$**

To verify (Hiv, c) at stage $r + 1$, we use resolvent expansion and cover $[-N, N]^{2d} = [-M^{r+1}, M^{r+1}]^{2d}$ by the big cube $[-M^r, M^r]^{2d}$ and smaller $M_0$-cubes, and use the estimates on $T_{M^r}^{r+1}(u^{(r)})$ and $T_{[-M_0,M_0]^{2d} + J(u(\tilde{r}_0)), M^r/2 < |J| < M^{r+1}}$. We have, using (W),
\[ \| T_{M'} (u^{(r)}) - T_{M'} (u^{(r-1)}) \| \leq \delta_{r-1} \leq M^{{-\frac{2}{3}}^{r-1}} , \]

and

\[ \| T_{[-M_0, M_0]^{2d+J}} (u^{(r)}) - T_{[-M_0, M_0]^{2d+J}} (\tilde{u}^{(r)}) \| \leq \delta_{r_0} \leq e^{-\tilde{\alpha} M_0} < M^{-r} C, \] (3.18)

for sufficiently large \( M \). (Recall that \( \tilde{r}_0 \) is constructed so that the first inequality in (3.18) holds.) From (Hiv, c) at stage \( r \),

\[ \| T_{M'}^{-1} (u^{(r-1)}) \| \leq M^r C, \] (3.19)

so

\[ \| T_{M'}^{-1} (u^{(r)}) \| \leq M^r C + M^{-\frac{2}{3}}^{r-1} < (1 + M^{-\frac{2}{3}}^{r-1}) M^r C. \] (3.20)

From Lemma 3.4 and (3.9)

\[ \| T_{[-M_0, M_0]^{2d+J}} (\tilde{u}^{(r)}) \| \leq e^{M_0^\sigma} \leq M^{\sigma C} \ll M^r C, \] (3.21)

so using (3.18),

\[ \| T_{[-M_0, M_0]^{2d+J}} (u^{(r)}) \| \leq M^{\sigma C} + M^{-r} C < 2 M^{\sigma C}. \] (3.22)

The bound in (3.18) shows, moreover, that the pointwise estimates in (3.9) holds for \( T_{[-M_0, M_0]^{2d+J}} (u^{(r)}) \) with the prefactor \((1 + e^{-\tilde{\alpha} - \alpha} M_0)\) in lieu of 1.

The pointwise estimates on \( T_{M'}^{-1} (u^{(r-1)}) \) also essentially hold for \( T_{M'}^{-1} (u^{(r)}) \). This is verified as follows. (Hiv, c) at stage \( r \) gives

\[ \| [T_{M'} (u^{(r-1)})]^{-1} \| \leq M^r C, \]

\[ \| [T_{M'} (u^{(r-1)})]^{-1} (k, k') \| \leq e^{\alpha |k-k'|} (|k-k'| > r C). \] (3.23)

We write

\[ T_{M'} (u^{(r)}) = T_{M'} (u^{(r-1)}) + [T_{M'} (u^{(r)}) - T_{M'} (u^{(r-1)})] \]

\[ := A + B \] (3.24)

To obtain pointwise estimate on \( T_{M'}^{-1} (u^{(r)}) \), we use (3.24) and resolvent series. \( A^{-1} \) has off-diagonal decay from (3.23), \( B \) has off-diagonal decay from (Hiii) at stage \( r \). Iterating the resolvent series and using (3.20), we obtain

\[ \| [T_{M'} (u^{(r)})]^{-1} (k, k') \| \leq e^{-\alpha' |k-k'|} (|k-k'| > r C), \] (3.25)

with \( \alpha' = \alpha - M^{-\delta'} (\delta' > 0) \).

Consequently, this leads to

**Lemma 3.5.** On the set of intervals in \( \Lambda_{r+1} \), there are the following estimates:

\[ \| [T_{M'} (u^{(r)})]^{-1} \| < M^{2(r+1)d} M^r C \ll M^{(r+1) C} \]

\[ \| [T_{M'} (u^{(r)})]^{-1} (k, k') \| < e^{-\tilde{\alpha} |k-k'|} \text{ for } |k-k'| > (r+1)^C, \]

with \( \tilde{\alpha} = \alpha - M^{-(r+1)\tilde{\delta}}, \tilde{\delta} > 0. \)
**Proof.** This is by an application of Lemma 5.1 in [BW], using (3.20), (3.22), the corresponding pointwise estimates as in (3.9) and (3.25). □

Lemma 3.5 reproduces the estimates in (Hiv, c) at stage $r + 1$ with a possible lowering of $\alpha$. Since $u^{(r+1)}$ is constructed using $T_{Mr+1}^{-1}(u^{(r)})$, this also represents a lowering of $\alpha$ in (Hiii). However, similar to (2.48) and (2.49), the “final”

$$\alpha := \alpha_\infty = \mathcal{O}(|\log a|) > 0.$$ 

Since the estimates (3.20) and (3.22) are stable under perturbation of size $M^{-(r+1)C}$, this produces the next set $\Lambda_{r+1}$ of intervals of size $M^{-(r+1)C}$. (Hiv, d) follows by construction. □

To summarize, we have proved

**Lemma 3.6.** Assume (Hi-iv) hold at stage $r$, there exists $\Lambda_{r+1} \subset \Lambda'_r \subset \Lambda_r$, satisfying

$$\text{meas } \Lambda_r \setminus \Lambda_{r+1} < 1/r^S,$$

provided $C > \max(1/\tau, 7/c)$, such that (Hiv, c, d) hold at stage $r + 1$.

- **Construction of $u^{(r+1)}$**
  
  By definition (3.1),
  
  $$\Delta u^{(r+1)} = -T_{Mr+1}^{-1}(u^{(r)})F(u^{(r)}),$$

  and $u^{(r+1)} = u^{(r)} + \Delta u^{(r+1)}$. ($E^{(r+1)}$ is already known from (3.2), using $u^{(r)}$). Using the same argument as in the proof of the Corollary, verifies (Hiv, a) at stage $r + 1$; (Hiv, b) follows by direct computation. The derivative estimate in (Hiv, b) leads to the derivative estimate in (Hii). The extension argument mentioned earlier then shows that (Hi-iii) hold at stage $r + 1$ as well. The induction from step $r$ to step $r + 1$ is thus complete. (Cf. [BW], sect. 6, (6.1)–(6.20).) □

**Proof of the Theorem.** The induction process above solves iteratively the $Q$ and the $P$-equations, with the convergence estimates in (W). The measure estimate (1.10) follows from (Hiv, d):

$$\text{meas } \Lambda \geq 1 - a^{p/5} \sum_{r \geq 1} r^{-5} > 1 - a^{p/6}.$$

The estimates in (1.11) follow from (Hii), (3.2) and (W), and prove the Theorem. □

**Acknowledgements.** It is a pleasure to thank T. Spencer and R. de la Llave for discussions.

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.
References

[Ba] Basu, S.: On bounding the Betti numbers and computing the Euler characteristics of semi-algebraic sets. Disc. Comput. Geom. 22(1), 1–18 (1999)

[BiN] Binyamini, G., Novikov, D.: Complex cellular structures. Ann. Math. 190, 145–248 (2019)

[B1] Bourgain, J.: Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equations. Ann. Math. 148, 363–439 (1998)

[B2] Bourgain, J.: Green’s Function Estimates for Lattice Schrödinger Operators and Applications, Annals of Mathematics Studies, vol. 158. Princeton University Press, Princeton (2005)

[B3] Bourgain, J.: Anderson localization for quasi-periodic lattice Schrödinger operators on $\mathbb{Z}^d$, d arbitrary. Geom. Funct. Anal. 17, 682–706 (2007)

[BGS] Bourgain, J., Goldstein, M., Schlag, W.: Anderson localization for Schrödinger operators on $\mathbb{Z}^2$ with quasi-periodic potential. Acta Math. 188, 41–86 (2002)

[BW] Bourgain, J., Wang, W.-M.: Quasi-periodic solutions of nonlinear random Schrödinger equations. J. Eur. Math. Soc. 10, 1–45 (2008)

[Bu] Burguet, D.: A proof of Yomdin-Gromov’s algebraic Lemma. Israel J. Math. 168, 291–316 (2008)

[CL] Cazenave, T., Lions, P.L.: Orbital stability of standing waves for some nonlinear Schrödinger equations. Commun. Math. Phys. 85, 549–561 (1982)

[DG] Damanik, D., Goldstein, M.: On the existence and uniqueness of global solutions for the KdV equation with quasi-periodic initial data. J. Am. Math. Soc. 29, 825–856 (2016)

[DS] Dinaburg, E.I., Sinai, Y.G.: The one dimensional Schrödinger equation with quasi-periodic potential. Funkt. Anal. i. Priloz 9, 8–21 (1975)

[E] Eliasson, L.H.: Floquet solutions for the 1-dimensional quasi-periodic Schrödinger equation. Commun. Math. Phys. 146, 447–482 (1992)

[EFRJ] van Erp, T.S., Fasolino, A., Radulescu, O., Janssen, T.: Pinning and phonon localization in Frenkel?Kontorova models on quasiperiodic substrates. Phys. Rev. B 60, 6522–6528 (1999)

[GPT] Garibaldi, E., Petite, S., Theullen, P.: Calibrated configurations for Frenkel?Kontorova type models in almost-periodic environments. Annales Henri Poincaré 18, 2905–2943 (2017)

[JLS] Jitomirskaya, S., Liu, W., Shi, Y.: Anderson Localization for Multi-frequency Quasi-periodic Operators on $\mathbb{Z}^d$. Geom. Func. Anal. (to appear) arXiv:1908.03805

[K] Kozlov, S.M.: Reducibility of quasiperiodic differential operators and averaging. Trudy Moskov. Mat. Obsch. 46, 99–123 (1983)

[LOSK] Louis, P.J.Y., Ostrovskaya, E.A., Savage, C.M., Kivshar, Y.S.: Bose?Einstein condensates in optical lattices: band gap structure and solitons. Phys. Rev. A 67, 013602 (2003)

[M] Moser, J.: A rapidly convergent iteration method and nonlinear partial differential equations I. Ann. Scuola Norm. Sup. Pisa 3(20), 265–315 (1966)

[SdIL] Su, X., de la Llave, R.: KAM theory for quasi-periodic equilibria in 1-D quasi-periodic media. SIAM J. Math. Anal. 14, 3901–3927 (2012)

[W1] Wang, W.-M.: Energy supercritical nonlinear Schrödinger equations: quasi-periodic solutions. Duke Math. J. 165(6), 1129–1192 (2016)

[W2] Wang, W.-M.: Quasi-periodic Solutions for Nonlinear Klein-Gordon Equations arXiv:1609.00309v2

[W3] Wang, W.-M.: Infinite Energy Quasi-periodic Solutions to Nonlinear Schrödinger Equations on $\mathbb{R}$ (2020) arXiv:1908.11627v2

Communicated by W. Schlag