FLAG HIGHER NASH BLOWUPS

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Abstract. In his previous paper [5], the author has defined a higher version of the Nash blowup and considered it a possible candidate for the one-step resolution. In this paper, we will introduce another higher version of the Nash blowup and prove that it is compatible with products and smooth morphisms. We will also prove that the product of curves can be desingularized via both versions.

1. Introduction

Let $X$ be a variety over an algebraically closed field $k$ of characteristic 0, and $X_{sm}$ its smooth locus. For $x \in X$ and a non-negative integer $n$, put $x^{(n)} := \text{Spec} \mathcal{O}_X / m^{n+1}_x$. The author [5] has defined the $n$-th Nash blowup of $X$, denoted $\text{Nash}_n(X)$, to be the closure of the set \{x^{(n)} | x \in X_{sm}\} in the Hilbert scheme $\text{Hilb}(X)$. In this paper, we call it the simple $n$-th Nash blowup, distinguished from what we will introduce below. Unfortunately, as we will see (Example 5.9), it is not generally compatible with either products or smooth morphisms. This would be explained as follows: Let $Y$ be another variety, and let $U \subset X$ and $V \subset Y$ be clusters (that is, 0-dimensional closed subschemes) with $U \in \text{Nash}_n(X)$ and $V \in \text{Nash}_n(Y)$. Then $U \times V$ seems the only natural cluster in $X \times Y$ constructed from $U$ and $V$. But it is not of the expected length $(\dim X + \dim Y + n)$, so $U \times V \notin \text{Nash}_n(X \times Y)$. In particular, for $x \in X_{sm}$ and $y \in Y_{sm}$, $x^{(n)} \times y^{(n)} \neq (x, y)^{(n)}$.

For instance, the ideal $(u, v)^{n+1} \subset k[u, v]$ is not identical to either $(u^{n+1}, v^{n+1})$ or $(u^{n+1})(v^{n+1})$, but to $\bigcap_{i=0}^{n+1}(u^i) + (v^{n+2-i})$, $\bigcap_{i=0}^{n+1}(u^i) \cap (v^{n+1-i})$ and $\bigcap_{i=0}^{n+1}(u^i)(v^{n+1-i})$. This observation suggests considering a collection $(x^{(0)}, \ldots, x^{(n)})$ rather than a single $x^{(n)}$. We now define the flag $n$-th Nash blowup, denoted $\text{fNash}_n(X)$, to be the closure of the set \{(x^{(0)}, \ldots, x^{(n)}) | x \in X_{sm}\} in $(\text{Hilb}(X))^{n+1}$. This is also a higher version of the classical Nash blowup. Namely $\text{fNash}_1(X)$ is canonically isomorphic to the classical Nash blowup. By definition, every point of $\text{fNash}_n(X)$ is the collection of $n + 1$ clusters in $X$.

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For \( U_\ast = (U_0, \ldots, U_n) \in \text{fNash}_n(X) \) and \( V_\ast = (V_0, \ldots, V_n) \in \text{fNash}_n(Y) \) and for \( 0 \leq i \leq n \), put
\[
\{U_\ast, V_\ast\}_i := \bigcup_{j=0}^i U_j \times V_{i-j} \subset X \times Y,
\]
and \( \{U_\ast, V_\ast\}_\ast := (\{U_\ast, V_\ast\}_0, \ldots, \{U_\ast, V_\ast\}_n) \). We will see \( \{U_\ast, V_\ast\}_\ast \in \text{fNash}_n(X \times Y) \). The main theorem of this paper is the following.

**Theorem 1.1.** Let \( X \) and \( Y \) be varieties. Then there is a canonical isomorphism,
\[
\text{fNash}_n(X) \times \text{fNash}_n(Y) \cong \text{fNash}_n(X \times Y)
\]
\( (U_\ast, V_\ast) \mapsto \{U_\ast, V_\ast\}_\ast \).

Using this, we will prove also that the flag higher Nash blowup is compatible with smooth morphisms. In general, it is expected that a good resolution is compatible with smooth morphisms. Resolution with this property was first constructed by Villamayor \[3, 4\] (see also \[1, 2\]). This property enables us to even construct resolution of Artin stacks.

The second half of the paper will be devoted to the study of simple and flag higher Nash blowups of curves and products of curves. Consider a formal irreducible curve \( X = \text{Spec} A \) (that is, \( A \) is the complete Noetherian local domain of dimension 1 with coefficient field \( k \)). We can also define simple and flag higher Nash blowups of such “formal varieties”. Fix an embedding \( A \rightarrow k[[x]] \) so that \( k[[x]] \) is the integral closure of \( A \). Then the associated numerical monoid of \( X \) (of \( A \)) is
\[
S := \{s \in \mathbb{Z}_{\geq 0} | \exists f \in A, \ \text{ord} \ f = s\} = \{0 = s_0 < s_1 < \cdots \} \subset \mathbb{Z}_{\geq 0}.
\]
Then a theorem in \[5\] says that \( \text{Nash}_n(X) \) is normal if and only if \( s_{n+1} - 1 \in S \). (Notice that the indices of \( s_i \) are shifted by one from those in \[5\].) For the flag higher Nash blowup, we will obtain a similar result; \( \text{fNash}_n(X) \) is normal if and only if \( s_{m-1} \in S \) for some \( m \leq n+1 \).

Next consider formal curves \( X_i, 1 \leq i \leq l \). Let
\[
S_i = \{0 = s_{i,0} < s_{i,1} < \cdots \} \subset \mathbb{Z}_{\geq 0}
\]
be their associated numerical monoids respectively. It follows from Theorem \[14\] that \( \text{fNash}_n(\prod_i X_i) \) is regular if and only if \( \forall i, \exists m_i \leq n+1, \ s_{i,m_i} - 1 \in S_i \). Quite strangely, when \( l \geq 2 \), \( \text{Nash}_n(\prod_i X_i) \) is regular exactly when so is \( \text{fNash}_n(\prod_i X_i) \), though its proof is more involved.

Let \( X \) be a variety whose analytic branches all have the singularity type of the product of curves, \( C \subset X \) its conductor subscheme, \( Z \in \text{Nash}_n(X) \) and \( Z_\ast \in \text{fNash}_n(X) \). Then it is a corollary of the preceding
results that if $Z \not\subset C$ (resp. $Z_n \not\subset C$), then $\text{Nash}_n(X)$ (resp. $\text{fNash}_n(X)$) is smooth at $Z$ (resp. $Z_n$).

The simple $(n+1)$-th Nash blowup is not generally isomorphic to the $n$-th even if the $n$-th is smooth. So the simple $n$-th Nash blowup is not generally isomorphic to the $n$-times iteration of classical Nash blowups. On the other hand, at least for curves and products of curves, which are the only cases computed up to now, the flag $(n+1)$-th Nash blowup is isomorphic to the $n$-th if the $n$-th is smooth. Moreover the flag higher Nash blowup and the iteration of classical Nash blowups are both compatible with products and smooth morphisms. However, we compute various blowups of $\text{Spec} \, k[[x^5, x^7]]$ in §5.1.1, which shows that the flag $n$-th Nash blowups and the $n$-times iteration of the classical blowups are not generally isomorphic to each other.

1.1. Convention. Throughout the paper, we work over an algebraically closed base field $k$ of characteristic zero. The product of $k$-schemes, denoted $\times$, means the fiber product over $k$, and the tensor product of $k$-algebras, denoted $\otimes$, means the tensor product over $k$. A point of a scheme means a $k$-point. A variety means a separated integral scheme of finite type over $k$. A cluster means a zero-dimensional closed subscheme. For a variety $X$, we denote by $\text{Hilb}(X)$ the Hilbert scheme of $X$ of clusters. For a cluster $Z \subset X$, we write $Z \in \text{Hilb}(X)$ for the corresponding point. Similarly, if $Z_\ast := (Z_0, \ldots, Z_n)$ is a collection of clusters, then we write $Z_\ast \in \text{Hilb}(X)^{n+1}$.

2. Simple and flag higher Nash blowups

In this section, we define the simple and flag higher Nash blowups and show their basic properties. For more details of simple higher Nash blowup, we refer the reader to [5].

Let $X$ be a variety of dimension $d$. For a point $x \in X$ with defining ideal $m \subset \mathcal{O}_X$, its $n$-th infinitesimal neighborhood, denoted $x^{(n)}$, is the cluster defined by $m^{n+1}$. We denote by $X_{sm}$ the smooth locus of $X$.

**Definition 2.1.** The simple $n$-th Nash blowup, denoted $\text{Nash}_n(X)$, is the closure of the set $\{x^{(n)}|x \in X_{sm}\}$ in $\text{Hilb}(X)$ (endowed with the reduced scheme structure).

**Remark 2.2.** In [5], $\text{Nash}_n(X)$ was defined to be the closure of $\{(x, x^{(n)})|x \in X_{sm}\} \subset X \times \text{Hilb}(X)$, which is however canonically isomorphic to $\text{Nash}_n(X)$ in Definition 2.1 in characteristic zero.

Every point $Z \in \text{Nash}_n(X)$ is a cluster whose coordinate ring is a $k$-vector space of dimension $\binom{n+d}{n}$. Moreover as a set, $Z$ is a point, say
\( x \in X \). We have \( Z \subset x^{(n)} \). But the equality does not generally hold unless \( x \in X_{\text{sm}} \). There is a natural morphism

\[ \text{Nash}_n(X) \to X, \ Z \mapsto x, \]

which is the morphism forgetting the scheme structure of \( Z \). The morphism is projective and an isomorphism over \( X_{\text{sm}} \).

**Definition 2.3.** The *flag n-th Nash blowup*, denoted \( \text{fNash}_n(X) \), is the closure of the set \( \{(x^{(0)}, x^{(1)}, \ldots, x^{(n)}) | x \in X_{\text{sm}} \} \) in \( \text{Hilb}(X)^{n+1} \).

Every point of \( \text{fNash}_n(X) \) is a collection \( Z_* = (Z_0, Z_1, \ldots, Z_n) \) such that \( Z_i \) is a cluster of length \( \binom{i+d}{i} \) and \( Z_i \subset Z_{i+1}, 0 \leq i \leq n-1 \). In particular, \( Z_0 \) is a reduced point and we have a map

\[ \text{fNash}_n(X) \to X, \ Z_* \mapsto Z_0. \]

Again this is projective and an isomorphism over \( X_{\text{sm}} \). Moreover for \( m \leq n \), there are natural projections

\[ \text{fNash}_n(X) \to \text{fNash}_m(X), (Z_0, \ldots, Z_n) \mapsto (Z_0, \ldots, Z_m), \]

\[ \text{fNash}_n(X) \to \text{Nash}_m(X), (Z_0, \ldots, Z_n) \mapsto Z_m, \]

which are projective and birational. It is easy to see that \( \text{fNash}_n(X) \) is isomorphic to the irreducible component of

\[ \text{Nash}_n(X) \times_X \text{Nash}_{n-1}(X) \times_X \cdots \times_X \text{Nash}_1(X) \times_X \text{Nash}_0(X) \]

which dominates \( X \). Indeed, they are the same subscheme of \( (\text{Hilb}(X))^{n+1} \).

There is a slightly different construction of higher Nash blowups via relative Hilbert scheme. Let \( \Delta \subset X \times X \) be the diagonal and \( \Delta^{(n)} \) its \( n \)-th infinitesimal neighborhood. Namely if \( I_\Delta \) is the defining ideal sheaf of \( \Delta \), then \( \Delta^{(n)} \) is the closed subscheme of \( X \times X \) defined by \( I_\Delta^{n+1} \). We think of \( \Delta^{(n)} \) as an \( X \)-scheme via the first projection. If \( \text{Hilb}_{\binom{n+d}{n}}(\Delta^{(n)}/X) \) is the relative Hilbert scheme of clusters of length \( \binom{n+d}{n} \), then \( \text{Nash}_n(X) \) is canonically isomorphic to the irreducible component of \( \text{Hilb}_{\binom{n+d}{n}}(\Delta^{(n)}/X) \) dominating \( X \).

We adopt this construction as the definition for formal varieties. Let \( A \) be a complete local Noetherian reduced ring with coefficient field \( k \) and put \( X \) := \( \text{Spec} \ A \). Let \( \Delta^{(n)} \subset X \times X := \text{Spec} \ A \widehat{\otimes} A \) be the \( n \)-th infinitesimal neighborhood of the diagonal. We regard this as an \( X \)-scheme via the first projection. We make the following assumption:

**Assumption 2.4.** \( X \) has pure dimension \( d \) and there exists an open dense subscheme of \( X \) over which \( \Delta^{(n)} \) is flat and finite of degree \( \binom{n+d}{d} \).
The author does not know whether the second condition in Assumption 2.3 always holds under the first. But it holds at least in the following case; $X'$ is the completion of a variety $X$ at a point, ${X_i}_{i \in I}$ a subcollection of irreducible components of $X$ and $X = \cup_{i \in I} X_i$. In particular, in the case where $\dim X = 1$.

**Definition 2.5.** We define the simple (resp. flag) $n$-th Nash blowup of $X$, denoted $\text{Nash}_n(X)$ (resp. $\text{fNash}_n(X)$), to be the union of those irreducible components of $\text{Hilb}_{(d+n)}(\Delta(n)/X)$ (resp. $\text{Hilb}_{(d+n)}(\Delta(n)/X) \times_X \text{Hilb}_{(d+n-1)}(\Delta(n-1)/X) \times_X \cdots \times_X \text{Hilb}_1(\Delta/X)$) which dominate irreducible components of $X$.

If $X$ is a variety and $X$ is its completion at a point, and if $\Delta \subset X \times X$ and $\Delta \subset X \times X$ are the diagonals respectively, then $\Delta(n) \times_X X \cong \Delta(n)$, so $\text{Hilb}(\Delta/X) \times_X X \cong \text{Hilb}(\Delta/X)$. Hence if $X_i, i \in \Lambda$, are the irreducible components of $X$, then

$$\text{Nash}_n(X) \times_X X \cong \text{Nash}_n(X) = \bigcup_{i \in \Lambda} \text{Nash}_n(X_i)$$

$$\text{fNash}_n(X) \times_X X \cong \text{fNash}_n(X) = \bigcup_{i \in \Lambda} \text{fNash}_n(X_i)$$

3. **Compatibility with products**

Let $X$ and $Y$ be varieties of dimension $d$ and $e$ respectively, and $U_0 \in \text{fNash}_n(X), V_0 \in \text{fNash}_n(Y)$. We follow the convention that $U_{-1} := \emptyset$ and $V_{-1} := \emptyset$. For closed subschemes $A, B \subset C$ defined by the ideal sheaves $I, J \subset \mathcal{O}_C$ respectively, define $A \vee B \subset C$ to be the closed subscheme defined by $IJ$.

**Proposition 3.1.** For each $0 \leq i \leq n$, we have the following identification of clusters of $X \times Y$,

$$\bigcup_{j=0}^i U_j \times V_{i-j} = \bigcap_{j=1}^i (U_j \times Y) \cup (X \times V_{i-1-j}) = \bigcap_{j=-1}^i (U_j \times Y) \vee (X \times V_{i-1-j}).$$

Moreover this cluster is of length $\binom{d+e+i}{i}$. When $U_\ast = (x^{(0)}, \ldots, x^{(n)})$ and $V = (y^{(0)}, \ldots, y^{(n)})$ with $x \in X_{sm}$ and $y \in Y_{sm}$, this cluster is identical to $(x, y)^{(i)}$.

**Proof.** We may suppose that $X$ and $Y$ are affine, say $X = \text{Spec} R$, $Y = \text{Spec} S$. Let $m_{i+1} \subset R$ and $n_{i+1} \subset S$ be the defining ideals of $U_i$ and $V_i$ ($i = 0, 1, \ldots, n$) respectively. Put $\tilde{m}_i := m_i (R \otimes S)$ and $\tilde{n}_i := n_i (R \otimes S)$. Then the left side of the equation in the proposition is
defined by the ideal $\bigcap_{j=0}^{i+1} m_j + \tilde{n}_{i+1-j}$, the middle by $\sum_{j=0}^{i+1} m_j \cap \tilde{n}_{i+1-j}$, and the right by $\sum_{j=0}^{i+1} m_j \tilde{n}_{i+1-j}$.

Take bases $A \subset R$ (resp. $B \subset S$) of $R$ (resp. $S$) as $k$-vector spaces such that for each $i$, $\{a \in A \mid a \in m_i\}$ (resp. $\{b \in B \mid b \in n_i\}$) is a basis of $m_i$ (resp. $n_i$). Then $\{a \otimes b \mid a \in A, b \in B\}$ is a basis of $R \otimes S$. For $f \in S$ and $g \in R$, define

$$\text{ord } f := \text{max}\{i \mid f \in m_i\} \in \{0, 1, \ldots, n+1\},$$

$$\text{ord } g := \text{max}\{i \mid g \in n_i\} \in \{0, 1, \ldots, n+1\},$$

$$\text{ord } f \otimes g := \text{ord } f + \text{ord } g \in \{0, 1, \ldots, 2(n+1)\}.$$ For $h = \sum_{i=1}^{l} c_i a_i \otimes b_i$ ($c_i \in k \setminus \{0\}, a_i \in A, b_i \in B$), $\text{ord } h := \min_i (\text{ord } a_i \otimes b_i)$. We claim that

$$\{h \mid \text{ord } h \geq i + 1\} = \bigcap_{j=0}^{i+1} \tilde{m}_j \tilde{n}_{i+1-j} = \bigcap_{j=0}^{i+1} m_j \cap \tilde{n}_{i+1-j} = \bigcap_{j=1}^{i+1} m_j + \tilde{n}_{i+2-j}.$$ It is easy to see that

$$\{h \mid \text{ord } h \geq i + 1\} \subset \bigcap_{j=0}^{i+1} \tilde{m}_j \tilde{n}_{i+1-j} \subset \bigcap_{j=0}^{i+1} m_j \cap \tilde{n}_{i+1-j} \subset \bigcap_{j=1}^{i+1} m_j + \tilde{n}_{i+2-j}.$$ Since $\tilde{m}_j + \tilde{n}_{i+2-j} = \langle a \otimes b \mid a \in A \cap m_j \text{ or } b \in B \cap n_{i+2-j} \rangle$, we have

$$\bigcap_{j=1}^{i+1} \tilde{m}_j + \tilde{n}_{i+2-j} = \langle a \otimes b \mid \forall j, a \in A \cap m_j \text{ or } b \in B \cap n_{i+2-j} \rangle = \langle a \otimes b \mid \forall j, \text{ord } a \geq j \text{ or } \text{ord } b \geq i + 2 - j \rangle = \langle a \otimes b \mid \text{ord } b \geq i + 2 - (\text{ord } a + 1) \rangle = \langle a \otimes b \mid \text{ord } a \otimes b \geq i + 1 \rangle = \{h \mid \text{ord } h \geq i + 1\}.$$ This proves our claim and the first assertion of the proposition.

The length of our cluster is equal to $\sharp \{a \otimes b \mid a \in A, b \in B, \text{ord } a \otimes b \leq i\}$, which depends only on $d, e$ and $i$. So the second assertion follows from the last.

To show the last assertion, we may assume that $R = k[x_1, \ldots, x_d]$ and $S = k[y_1, \ldots, y_e]$, and that $m_i = (x_1, \ldots, x_d)^i$ and $n_i = (y_1, \ldots, y_e)^i$. Then for $h \in R \otimes S = k[x_1, \ldots, x_d, y_1, \ldots, y_e]$, the order defined above, $\text{ord } h_i$, is equal to the usual order of polynomial. So $\{h \mid \text{ord } h \geq i + 1\} = (x_1, \ldots, x_d, y_1, \ldots, y_e)^{i+1}$, which completes the proof. □
Definition 3.2. We denote the cluster in Proposition 3.1 by \( \{U_*, V_*\}_i \) and put \( \{U_*, V_*\}_s := (\{U_*, V_*\}_1, \ldots, \{U_*, V_*\}_n) \).

Lemma 3.3. There exists a morphism that takes \( (U_*, V_*) \) to \( \{U_*, V_*\}_s \). Moreover it is birational and surjective.

Proof. Let \( \mathcal{U}_s := (\mathcal{U}_1, \ldots, \mathcal{U}_{n+1}) \), \( \mathcal{U}_i \subset \mathfrak{fNash}_n(X) \times X \) (resp. \( \mathcal{V}_s := (\mathcal{V}_1, \ldots, \mathcal{V}_{n+1}) \), \( \mathcal{V}_i \subset \mathfrak{fNash}_n(Y) \times Y \)) be the universal collection of clusters over \( \mathfrak{fNash}_n(X) \) (resp. \( \mathfrak{fNash}_n(Y) \)). Set

\[
\{\mathcal{U}_s, \mathcal{V}_s\}_i := \bigcap_{j=-1}^{i} (\mathcal{U}_j \times \mathfrak{fNash}_n(Y) \times Y) \vee (\mathfrak{fNash}_n(X) \times X \times \mathcal{V}_{i-1-j}).
\]

Then the fiber of the projection \( \{\mathcal{U}_s, \mathcal{V}_s\}_i \to \mathfrak{fNash}_n(X) \times \mathfrak{fNash}_n(Y) \) over \( (U_*, V_*) \) is \( \{U_*, V_*\}_i \). So from Proposition 3.1 \( \{\mathcal{U}_s, \mathcal{V}_s\}_i \) is flat over \( \mathfrak{fNash}_n(X) \times \mathfrak{fNash}_n(Y) \) and generically the family of \( z^{(i)} \), \( z \in X_{sm} \times Y_{sm} \). From the universality, there exists a morphism \( \mathfrak{fNash}_n(X) \times \mathfrak{fNash}_n(Y) \to \mathfrak{fNash}_n(X \times Y) \) corresponding to \( \{\mathcal{U}_s, \mathcal{V}_s\}_s \), which takes \( (U_*, V_*) \) to \( \{U_*, V_*\}_s \). It is an isomorphism over \( X_{sm} \times Y_{sm} \). Since \( \mathfrak{fNash}_n(X) \times \mathfrak{fNash}_n(Y) \) is proper over \( X \times Y \), the morphism is surjective. \( \square \)

Let \( p : X \times Y \to X \) and \( q : X \times Y \to Y \) be the projections. For \( Z_s = (Z_0, \ldots, Z_n) \in \mathfrak{fNash}_n(X \times Y) \), we denote by \( p(Z_i) \) (resp. \( q(Z_i) \)) the scheme-theoretic image of \( Z_i \) by \( p \) (resp. \( q \)), and set \( p(Z_s) := (p(Z_0), \ldots, p(Z_n)) \) and \( q(Z_s) := (q(Z_0), \ldots, q(Z_n)) \).

Theorem 3.4. We have a canonical isomorphism

\[
\mathfrak{fNash}_n(X) \times \mathfrak{fNash}_n(Y) \cong \mathfrak{fNash}_n(X \times Y)
\]

\[
(U_*, V_*) \mapsto \{U_*, V_*\}_s
\]

\[
(p(Z_s), q(Z_s)) \mapsto Z_s.
\]

Proof. Since \( \mu \) in Lemma 3.3 is surjective, any \( Z_s \in \mathfrak{fNash}_n(X \times Y) \) is \( \{U_*, V_*\}_s \) for some \( (U_*, V_* \in \mathfrak{fNash}_n(X) \times \mathfrak{fNash}_n(Y) \). Then it is clear that \( p(Z_s) = U_* \) and \( q(Z_s) = V_* \). Thus the map \( \mathfrak{fNash}_n(X \times Y) \to \mathfrak{fNash}_n(X) \times \mathfrak{fNash}_n(Y), Z_s \mapsto (p(Z_s), q(Z_s)) \) is the inverse of \( \mu \). We will show this map is a morphism of schemes.

We have a natural morphism \( \mathfrak{fNash}_n(X \times Y) \to X \times Y \to Y \), which induces a closed embedding

\[
\mathfrak{fNash}_n(X \times Y) \times X \hookrightarrow \mathfrak{fNash}_n(X \times Y) \times X \times Y.
\]
We denote the image of this embedding by $W$. Let $Z_\ast \subset f\text{Nash}_n(X \times Y)$ be the universal collection of clusters. Then the scheme-theoretic intersection $Z_i \cap W$ is a family of clusters in $X$ over $f\text{Nash}_n(X \times Y)$. If $F$ is the fiber of the projection $f\text{Nash}_n(X \times Y) \times X \times Y \to f\text{Nash}_n(X \times Y)$ over $Z_\ast$, then the fiber of $Z_i \cap W \to f\text{Nash}_n(X \times Y)$ over $Z_\ast$ is

$$Z_i \cap W \cap F = (Z_i \cap F) \cap (W \cap F) = Z_i \cap (X \times q(Z_0)) = p(Z_i).$$

So $Z_i \cap W$ is a flat family, generically of $x^{(i)}$, $x \in X_{sm}$. From the universality, there exists the corresponding morphism $f\text{Nash}_n(X \times Y) \to f\text{Nash}_n(X)$. Similarly we obtain $f\text{Nash}_n(X \times Y) \to f\text{Nash}_n(Y)$, and $f\text{Nash}_n(X \times Y) \to f\text{Nash}_n(X) \times f\text{Nash}_n(Y)$. The last morphism is clearly the inverse of $\mu$. \hfill \blackslug

It is straightforward to generalize the theorem above to the product of an arbitrary number of varieties. Let $X_i$, $i = 1, \ldots, m$, be varieties. For $U_{i,s} \in f\text{Nash}_n(X_i)$, $i = 1, \ldots, m$, we set

$$\{U_{1,s}, \ldots, U_{m,s}\}_j := \{\cdot \cdot \cdot \{U_{1,s}, U_{2,s}\}_s, U_{3,s}\}_s, \ldots\}_s, U_{m,s}\}_j = \bigcup \prod U_{i,j_i}.$$

Let $p_l : \prod X_i \to X_l$ be the $l$-th projection.

**Theorem 3.5.** We have a canonical isomorphism

$$\prod f\text{Nash}_n(X_i) \cong f\text{Nash}_n(\prod X_i)$$

$$(U_{1,s}, \ldots, U_{m,s}) \mapsto \{U_{1,s}, \ldots, U_{m,s}\}_s$$

$$(p_1(Z_\ast), \ldots, p_m(Z_\ast)) \leftrightarrow Z_\ast.$$

**4. Compatibility with smooth morphisms**

If $f : Y \to X$ is an etale morphism of varieties, then there is a natural isomorphism $\Delta_X^{(n)} \cong \Delta_X^{(n)} \times_X Y$, where $\Delta_Y \subset Y \times Y$ and $\Delta_X \subset X \times X$ are the diagonals. So both simple and flag higher Nash blowups are compatible with etale morphism, that is, there are canonical isomorphisms,

$$\text{Nash}_n(Y) \cong \text{Nash}_n(X) \times_X Y$$

$$f\text{Nash}_n(Y) \cong f\text{Nash}_n(X) \times_X Y.$$

The composite $f\text{Nash}_n(Y) \to f\text{Nash}_n(X) \times_X Y \to f\text{Nash}_n(X)$ takes $Z_\ast$ to $f(Z_\ast)$. 
Next if $f : Y \to X$ is a smooth morphism, then there is an open covering $Y = \bigcup Y_e$ such that for each $e$, $f|_{Y_e}$ factors as

$$Y_e \xrightarrow{h} X \times \mathbb{A}^c \xrightarrow{p} X,$$

where $h$ is etale and $p$ is the projection. The morphisms

$$f_{\text{Nash}}(Y_e) \to f_{\text{Nash}}(X \times \mathbb{A}^c), \ Z_s \mapsto h(Z_s), \text{ and}$$

$$f_{\text{Nash}}(X \times \mathbb{A}^c) \to f_{\text{Nash}}(X), \ W_s \mapsto p(W_s)$$

are well-defined, and so is

$$f_{\text{Nash}}(Y_e) \to f_{\text{Nash}}(X), \ Z_s \mapsto f(Z_s).$$

Gluing them yeild the morphism

$$f_{\text{Nash}}(Y) \to f_{\text{Nash}}(X), \ Z_s \mapsto f(Z_s).$$

**Corollary 4.1.** Let $f : Y \to X$ be a smooth morphism of varieties. Then there is a canonical isomorphism

$$f_{\text{Nash}}(Y) \cong f_{\text{Nash}}(X) \times_X Y$$

$$Z_s \mapsto f(Z_s).$$

**Proof.** Let the $Y_e$ be as above. Then

$$f_{\text{Nash}}(Y_e) \cong f_{\text{Nash}}(X \times \mathbb{A}^c) \times_{X \times \mathbb{A}^c} Y_e$$

$$\cong (f_{\text{Nash}}(X) \times \mathbb{A}^c) \times_{X \times \mathbb{A}^c} Y_e$$

$$\cong f_{\text{Nash}}(X) \times_X Y_e.$$

Since the isomorphisms $f_{\text{Nash}}(Y_e) \cong f_{\text{Nash}}(X \times \mathbb{A}^c) \times_{X \times \mathbb{A}^c} Y_e$ are canonical, we can glue them and obtain $f_{\text{Nash}}(Y) \cong f_{\text{Nash}}(X) \times_X Y$. □

### 5. Curves and products of curves

#### 5.1. Curves.** Let $X = \text{Spec} A$ be a formal irreducible curve. Namely $A$ is a complete Noetherian local domain with coefficient field $k$. Fix an embedding $A \hookrightarrow k[[x]]$ so that $k[[x]]$ is the integral closure of $A$. We define the associated numerical monoid of $X$ (and $A$) to be $S := \{ s \in \mathbb{Z}_{\geq 0} | \exists f \in A, \text{ord} f = s \}$ and write

$$S = \{ 0 = s_0 < s_1 < s_2 < \cdots \}.$$  

(Caution: The indices of $s_i$ differ by 1 from those in [3].)

**Theorem 5.1 ([3]).** $f_{\text{Nash}}(X)$ is normal if and only if $s_{n+1} - 1 \in S$. 

Sketch of the proof. Let \( \iota : k[[x]] \to k[[y]] \) be an isomorphism defined by \( x \mapsto -y \). The composite \( A \xrightarrow{\iota^{-1}} k[[x]] \xrightarrow{\nu} k[[y]] \) corresponds to a morphism \( \nu : Y := \text{Spec} k[[y]] \to X \), which is the normalization of \( X \). For each \( n \), there exists a natural factorization of \( \nu \) as follows; \( Y \xrightarrow{\phi_n} \text{Nash}_n(X) \to X \). Then \( \phi_n \) corresponds to a family of clusters over \( Y \), say \( Z_n \subset X \times Y := \text{Spec} A[[y]] \).

Let \( I \subset A[[y]] \) be the prime ideal defining the graph \( \Gamma \subset X \times Y \) of \( \nu \).

Then the defining ideal of \( Z_n \) is the \((n+1)\)-th symbolic power \( I^{(n+1)} \) of \( I \). Let \( \epsilon : \text{Spec} k[y]/(y^2) \to \text{Spec} k[[y]] \) be the natural morphism and set \( Z_{n,\epsilon} := Z_n \times_y \text{Spec} k[y]/(y^2) \subset \text{Spec} A[y]/(y^2) \) and \( Z_n := Z_n \times_Y \text{Spec} k \).

Put \( a_{n+1} := I^{(n+1)}(A[y]/(y)) \subset A \), which is the defining ideal of \( Z_n \). Then \( \text{Nash}_n(X) \) is normal if and only if \( \phi_n \circ \epsilon \) is a nonzero tangent vector if and only if \( Z_{n,\epsilon} \) is a non-trivial family. We can construct polynomials \( h_m \in k[x,y], m \in \mathbb{Z}_{\geq 0} \) such that

1. \( h_m \) is divisible by \((x+y)^m\),
2. \( h_m \) lies in \( A[[y]] \),
3. \( I^{(n+1)} \) is generated by \( h_m, m \geq n + 1 \),
4. If we write
   \[
   h_m = h_{m,0} + h_{m,1}y + h_{m,2}y^2 + \cdots, \quad h_{m,i} \in A,
   \]
   then \( \text{ord} h_{m,0} = s_m \) and \( \text{ord} h_{m,1} \geq s_m - 1 \). Moreover \( \text{ord} h_{m,1} = s_m - 1 \) if and only if \( s_m - 1 \in \mathcal{S} \).

Being generated by \( h_{m,0}, m \geq n + 1 \), the ideal \( a_{n+1} \) is identical to the set \( \{ f \in A | \text{ord} f \geq s_{n+1} \} \). Since \( Z_{n,\epsilon} \) corresponds to the homomorphism \( a_{n+1} \to A/a_{n+1} \), which takes \( h_{m,0} \) to \( h_{m,1} \), \( Z_{n,\epsilon} \) is non-trivial if and only if \( h_{m,1} \notin a_{n+1} \) for some \( m \geq n + 1 \) if and only if \( s_{n+1} - 1 \in \mathcal{S} \). This completes the proof. \( \square \)

Definition 5.2. Let \( X \) be a reduced scheme and \( \nu : Y \to X \) the normalization. Then the conductor ideal \( \mathfrak{c} \subset \mathcal{O}_X \) is the annihilator of \( \nu_*\mathcal{O}_Y/\mathcal{O}_X \). The conductor subscheme of \( X \) is the closed subscheme defined by \( \mathfrak{c} \).

By definition, the conductor subscheme is the non-normal locus endowed with a suitable scheme structure.

Corollary 5.3. Let \( X \) be a variety of dimension 1, \( C \) the conductor subscheme of \( X \) and \( Z \in \text{Nash}_n(X) \) with \( Z \notin C \). Then \( \text{Nash}_n(X) \) is normal around \( Z \).

Sketch of the proof. By \([\mathbf{5}] \text{ Prop. } 2.5\), we only need to show the same assertion for formal irreducible curve. In this case, with the notations
as above, the conductor ideal \( c \subset A \) is \( \{ f \in A | \text{ord} f \geq c \} \), where \( c := \min \{ s \in S | \forall s' \geq s, s' \in S \} \). Since \( a_{n+1} = \{ f \in A | \text{ord} f \geq s_{n+1} \} \),

\[
a_{n+1} \not\supset c \iff s_{n+1} > c \Rightarrow s_{n+1} - 1 \in S.
\]

By Theorem 5.1, Nash\(_n\)(X) is normal. □

Similar arguments apply to the flag higher Nash blowup.

**Corollary 5.4.** Let the notations be as in Theorem 5.1. Then \( \text{fNash}_n(X) \) is normal if and only if \( s_m - 1 \in S \) for some \( 1 \leq m \leq n + 1 \).

**Proof.** We keep the notation above. The normalization \( Y \rightarrow X \) factors as \( Y \xrightarrow{\psi_n} \text{fNash}_n(X) \rightarrow X \). It is clear that \( \text{fNash}_n(X) \) is normal if and only if \( \psi_n \) is an isomorphism. Since \( \psi_n \) corresponds to \( (Z_0, \ldots, Z_n) \), the last condition is equivalent to that for some \( m \leq n + 1 \), \( Z_{m,e} \) is non-trivial, equivalently, for some \( m \leq n + 1 \) \( s_m - 1 \in S \). □

**Corollary 5.5.** Let \( X \) be a variety of dimension 1, \( C \) its conductor subscheme and \( Z_\ast \in \text{fNash}_n(X) \) with \( Z_n \not\supset C \). Then \( \text{fNash}_n(X) \) is normal around \( Z_\ast \).

**Proof.** Arguments similar to the proof of Corollary 5.3 apply also to this corollary. □

### 5.1.1 Various blowups of \( \text{Spec} k[[x, y]] \)

Let \( X := \text{Spec} k[[x, y]] \). Then

\[
S = \{0, 5, 7, 10, 12, 14, 15, 17, 19, 20, 21, 22\} \cup \{n | n \geq 24\}.
\]

We compute \( h_m \) (\( m \leq 6 \)) in the proof of Theorem 5.1 to be

\[
\begin{align*}
h_1 &= x^5 + y^5, \\
h_2 &= x^7 - (7/5)x^5y^2 - (2/5)y^7, \\
h_3 &= x^{10} + (25/7)x^7y^3 - 3x^5y^5 - (3/7)y^{10}, \\
h_4 &= x^{12} - (14/5)x^{10}y^2 - 4x^7y^5 + (12/5)x^5y^7 + (1/5)y^{12}, \\
h_5 &= x^{14} - (21/5)x^{12}y^2 + (147/25)x^{10}y^4 + (24/5)x^7y^7 \
& \quad - (56/25)x^5y^9 - (3/25)y^{14}, \text{ and} \\
h_6 &= x^{15} + (125/49)x^{14}y - (25/7)x^{12}y^3 + 3x^{10}y^5 \
& \quad + (75/49)x^7y^8 - (4/7)x^5y^{10} - (1/49)y^{15}.
\end{align*}
\]

(\text{Check that for every } i \leq 6, h_i \in k[[x^i, y]] \text{ and } h_i \text{ is divisible by } (x + y)^i.) If \( A_n \) is the coordinate ring of \( \text{Nash}_n(X) \), then \( A_n \) is (isomorphic to) the least complete \( k \)-subalgebra \( B \subset k[[y]] \) such that \( k[[y^7, y^7]] \subset B \) and for some generators \( f_\lambda, \lambda \in \Lambda, \) of \( I^{(n+1)} \), for every \( \lambda \in \Lambda, f_\lambda \in B \).
If $A'_n$ is the coordinate ring of $fNash_n(X)$, then $A'_n$ is isomorphic to the least complete $k$-subalgebra of $k[[y]]$ which contains $A_m$, $m \leq n$.

Now $I^{(2)}$ is generated by $h_2$ and $h_2^2$. So $A_1 = A'_1 = k[[y^2, y^5]]$. Then $I^{(3)}$ is generated by $h_3$, $h_4 + (14/5)y^2h_3$, and $h_1h_3$. So $A_2 = k[[y^3, y^5, y^7]]$ and $A'_2 = k[[y^2, y^3]]$. Next $I^{(4)}$ is generated by $h_4$, $h_5$ and $h_6 - (125/49)yh_5 - (50/7)y^3h_4 = x^{15} + 8x^{10}y^5 + (125/7)x^{7}y^8 - 12x^5y^{10} - 8/7y^{15}$.

So $A_3 = k[[y^2, y^5]]$ and $A'_3 = k[[y^2, y^3]]$. Finally $I^{(5)}$ is generated by $h_5$, $h_6 - (25/49)yh_5$ and $h_2h_3$. So $A_4 = A'_4 = k[[y^2, y^3]]$.

For $X' := \text{Spec } k[[x^2, x^5]] \cong \text{Nash}(X)$, we similarly define $I' \subset k[[x^2, x^5, y]]$. Then $I'^{(2)}$ is generated by

$$x^4 - 2x^2y^2 + y^4$$ and $$x^5 + (5/2)x^2y^3 - (3/2)y^5.$$ 

So if $\text{Nash}^n(\cdot)$ denotes the $n$-times iteration of the classical Nash blowup, then $\text{Nash}^2(X) = \text{Nash}(X') \cong \text{Spec } k[[x^2, x^3]]$, and $\text{Nash}^3(X) \cong \text{Spec } k[[x]]$.

If $B(\cdot)$ denotes the blowup with respect to the reduced special point and $B^n(\cdot)$ is its $n$-times iteration, then it is easy to see that

$$B(X) \cong \text{Spec } k[[x^2, x^5]]$$

$$B^2(X) \cong \text{Spec } k[[x^2, x^3]]$$

$$B^3(X) \cong \text{Spec } k[[x]].$$

Thus the flag $n$-th Nash blowup differs also from the $n$-times iteration of blowups with respect to the reduced special point.

Every blowup considered here is the spectrum of the complete algebra associated to a numerical monoid. Table 1 shows the correspondence between monoids and blowups.

**Table 1.** The numerical monoids associated to various blowups of $\text{Spec } k[[x^5, x^7]]$. Here $\langle a_1, \ldots, a_l \rangle$ is the numerical monoid generated by the natural numbers $a_1, \ldots, a_l$.

| $n$ | 1   | 2   | 3   | 4   | 5   |
|-----|-----|-----|-----|-----|-----|
| Nash$_n(X)$ | $\langle 2, 5 \rangle$ | $\langle 3, 5, 7 \rangle$ | $\langle 2, 5 \rangle$ | $\langle 2, 3 \rangle$ | $\langle 1 \rangle$ |
| $fNash_n(X)$ | $\langle 2, 5 \rangle$ | $\langle 2, 3 \rangle$ | $\langle 2, 3 \rangle$ | $\langle 2, 3 \rangle$ | $\langle 1 \rangle$ |
| Nash$^n(X)$ | $\langle 2, 5 \rangle$ | $\langle 2, 3 \rangle$ | $\langle 1 \rangle$ | $\langle 1 \rangle$ | $\langle 1 \rangle$ |
| $B^n(X)$ | $\langle 2, 5 \rangle$ | $\langle 2, 3 \rangle$ | $\langle 1 \rangle$ | $\langle 1 \rangle$ | $\langle 1 \rangle$ |

**Conjecture 5.6.** Let $X$ be a formal curve with the associated numerical monoid $S = \{0 = s_0 < s_1 < \cdots \}$. Then the associated numerical monoid of $\text{Nash}_n(X)$ is generated by $s_m - s_l$, $m > n$, $l \leq n$. That of $f\text{Nash}_n(X)$ is generated by $s_m - s_l$, $m > l$, $l \leq n$. 
5.2. **Product of curves.** Let $X_i = \text{Spec} \ A_i$, $i = 1, 2, \ldots, l$, be formal irreducible curves. As above, we fix embeddings $A_i \hookrightarrow k[[x_i]]$, define their associated numerical monoids $S_i$ and write

$$S_i = \{0 = s_{i,0} < s_{i,1} < s_{i,2} < \cdots\}.$$ 

Similarly we define normalizations $Y_i := \text{Spec} \ k[[y_i]] \to X_i$. Since formal curves are algebraizable, by Theorem 3.4, $\hat{\text{Nash}}_n(\prod_i X_i) \cong \text{fnash}_n(\prod_i X_i)$. So there are natural morphisms

$$\hat{\prod}_i Y_i \to \hat{\prod}_i \text{Nash}_n(X_i) \cong \text{fnash}_n(\hat{\prod}_i X_i) \to \text{Nash}_n(\hat{\prod}_i X_i) \to \hat{\prod}_i X_i.$$ 

By Corollary 5.4, we obtain:

**Theorem 5.7.** $\text{fnash}_n(\hat{\prod}_i X_i)$ is regular if and only if for every $i$, there exists $1 \leq m_i \leq n + 1$ such that $s_{i,m_i} - 1 \in S_i$.

Strangely the same statement holds for the simple higher Nash blowup if $l \geq 2$.

**Theorem 5.8.** Suppose $l \geq 2$. Then $\text{Nash}_n(\hat{\prod}_i X_i)$ is regular if and only if for every $i$, there exists $m_i \leq n + 1$ with $s_{i,m_i} - 1 \in S_i$.

**Proof.** For each $i$, we define $I_i \subseteq A_i[[y_i]]$ and $h_{i,m} \in A_i[[y_i]]$ $(m = 0, 1, \ldots)$ as in the proof of Theorem 5.1. We denote by $\psi_n$ the natural morphism $\hat{\prod}_i Y_i \to \text{Nash}_n(\hat{\prod}_i X_i)$. Then $\text{Nash}_n(\hat{\prod}_i X_i)$ is regular if and only if $\psi_n$ is an isomorphism. Consider standard tangent vectors

$$\epsilon_i : \text{Spec} \ k[y_i]/(y_i^2) = \text{Spec} \ k[y_1, \ldots, y_i]/(y_1, \ldots, y_{i-1}, y_i^2, y_{i+1}, \ldots, y_l) \to \hat{\prod}_i Y_i.$$ 

Then $\psi_n$ is an isomorphism if and only if the tangent vectors $\psi_n \circ \epsilon_i$ are linearly independent.

Let $B := A_1 \otimes \cdots \otimes A_l \subset k[[x_1, \ldots, x_l]]$ and $I_i^{(m_i)} := I_i^{(m_i)} B[[y_1, \ldots, y_l]]$. Then the family corresponding to $\psi_n$ is defined by the ideal

$$J^{(n+1)} = \sum_{m_i = n+1} \prod_i I_i^{(m_i)} \subset B[[y_1, \ldots, y_l]],$$

which is generated by $\prod_i h_{i,m_i}$, $\sum_i m_i \geq n + 1$. Set

$$T_{n+1} := \{(s_{1,m_1}, \ldots, s_{l,m_l}) \mid \sum_i m_i \geq n + 1\} \subseteq S_1 \times \cdots \times S_l \subseteq \mathbb{Z}_{\geq 0}^l.$$ 

Let $k[[T_{n+1}]] \subset k[[x_1, \ldots, x_l]]$ be the complete semigroup algebra (without unit) associated to $T_{n+1}$. For $f \in k[[x_1, \ldots, x_l]]$, we denote by $\text{in}(f)$
its initial form, that is, the nonzero homogeneous part of the lowest degree. Then the special point \(W_n \in \text{Nash}_n(\prod_i X_i)\) is defined by the ideal

\[(1) \quad b_{n+1} := J^{(n+1)}B = \{ f \in B \mid \text{in}(f) \in k[[T_{n+1}]] \}.
\]

The Zariski tangent space of \(\text{Nash}_n(\prod_i X_i)\) at \(W_n\) is identified with a \(k\)-subspace of

\[\text{Hom}_{B\text{-modules}}(b_{n+1}, B/b_{n+1}).\]

Writing

\[h_{i,m} = h_{i,m,0} + h_{i,m,1}y_i + h_{i,m,2}y_i^2 + \cdots, \quad h_{i,m,j} \in A_i,\]

we have

\[\prod_i h_{i,m_i} = \prod_i h_{i,m_i,0} + \sum_i \left( h_{i,m_i,1} \prod_{j \neq i} h_{j,m_j,0} \right) y_i + \cdots, \quad \text{and}\]

\[b_{n+1} = (\prod_i h_{i,m_i,0} | \sum_i m_i \geq n + 1).\]

So the tangent vector \(\psi_n \circ \epsilon_i\) is identified with the homomorphism \(\xi_i : b_{n+1} \rightarrow B/b_{n+1}\) that takes \(\prod_j h_{j,m_j,0}\) to \(h_{i,m_i,1} \prod_{j \neq i} h_{j,m_j,0}\). From the expression (1), if \(\sum m_j > n + 1\), we have \(\xi_i(\prod_j h_{j,m_j,0}) = 0\). Moreover when \(\sum m_j = n + 1\),

\[\xi_i(\prod_j h_{j,m_j,0}) \neq 0 \quad \Leftrightarrow \quad \text{in}(h_{i,m_i,1}) = cy_i^{s_{m_i}-1}, \quad c \in k \setminus \{0\} \quad \Leftrightarrow \quad s_{i,m_i} - 1 \in S_i.
\]

To prove the “only if” in the theorem, we now suppose that for some \(i\) and every \(m \leq n + 1\), \(s_{i,m} - 1 \notin S_i\). Then for any \((m_1, \ldots, m_l)\) with \(\sum m_j = n + 1\), we have \(\xi_i(\prod_j h_{j,m_j,0}) = 0\), so \(\xi_i = 0\). As a consequence, \(\psi_n\) is not an isomorphism and \(\text{Nash}_n(\prod_i X_i)\) is not regular.

Let us now prove the “if”. Let \(m_i \leq n + 1, \ i = 1, \ldots, l\), be such that \(s_{m_i} - 1 \in S_i\). Then for each \(i\), there exist \(n_j\) \((1 \leq j \leq l)\) such that \(n_i = m_i\) and \(\sum_j n_j = n + 1\). (Here the assumption \(l \geq 2\) is necessary.) We have that

\[\xi_{j'}(\prod_j h_{j,n_j,0}) = \text{the class of } h_{j,n_j,1} \prod_{j \neq j'} h_{j,n_j,0} \text{ modulo } b_{n+1},\]

which is nonzero if \(j' = i\). It follows that \(\xi_i(\prod_j h_{j,n_j,0})\) is not any \(k\)-linear combination of \(\xi_{j'}(\prod_j h_{j,n_j,0})\), \(j' \neq i\). So if a linear relation
$\sum_j r_j \xi_j$ ($r_j \in k$) holds, then $r_i = 0$. Since this holds for every $i$, the $\xi_i$'s are linearly independent, which completes the proof. \qed

**Example 5.9.** Suppose that $l = 2$, $S_1 = \mathbb{Z}_{\geq 0}$ and

$$S_2 := \langle 3, 4 \rangle = \{0, 3, 4, 6, 7, 8, 9, \ldots \}.$$  

Then $\text{Nash}_n(X_1) \times \text{Nash}_n(X_2)$ is regular if and only if $\text{Nash}_n(X_2)$ is regular if and only if $n \neq 0, 2$. On the other hand, $\text{Nash}_n(X_1 \times X_2)$ is regular if and only if $n > 0$. So

$$\text{Nash}_2(X_1) \times \text{Nash}_2(X_2) \not\cong \text{Nash}_2(X_1 \times X_2).$$

In particular, this example says that the simple higher Nash blowup is not generally compatible with either products or smooth morphisms.

**Corollary 5.10.** Let $X$ be a variety such that every analytic branch of $X$ at every point has the same singularity type as the product of curves, and $C \subset X$ the conductor subscheme. Then for every $n$, the normalization $Y \to X$ factors as $Y \to \text{fNash}_n(X) \to \text{Nash}_n(X) \to X$. Moreover for $Z \in \text{Nash}_n(X)$ with $Z \not\subset C$, (resp. $Z_s \in \text{fNash}_n$ with $Z_n \not\subset C$), $\text{Nash}_n(X)$ (resp. $\text{fNash}_n(X)$) is smooth at $Z$ (resp. $Z_s$).

**Proof.** From [5] Prop. 2.5, it suffices to show the same assertion for the product of formal irreducible curves. Let the notations be as in the proof of Theorem 5.8. We may suppose that $l \geq 2$. Then the special points of $\text{Nash}_n(\prod_i X_i)$ and $\text{fNash}_n(\prod_i X_i)$ correspond to the same ideal $a_{n+1}$. Let $c_i$, $i = 1, \ldots, l$, be the conductor ideals of $A_i$, and $\tilde{c}_i := c_iB$. Then the conductor ideal of $B$ is $c := \prod_{i} \tilde{c}_i$. Let $U := \{(m_1, \ldots, m_l) | m_i \geq c_i \} \subset \mathbb{Z}_{\geq 0}^l$. Then $c = k[[U]] \subset k[[x_1, \ldots, x_l]]$.

Now the assumption $a_{n+1} \not\subset c$ is equivalent to that $T_{n+1} \not\subset U$. If it is the case, then for every $i$, $a_i + 1 \leq s_{i,n+1}$. From Theorems 5.7 and 5.8, $\text{Nash}_n(\prod_i X_i)$ and $\text{fNash}_n(\prod_i X_i)$ are regular. \qed

Lastly we raise two problems.

**Problem 5.11.** Is the correspondence of Theorems 5.7 and 5.8 only a coincidence? Or, could it be that $\text{fNash}_n(X) \cong \text{Nash}_n(X)$ for any variety $X$ of dimension $\geq 2$?

**Problem 5.12.** Let $X$ be a variety, $C$ its conductor subscheme, $Z \in \text{Nash}_n(X)$ with $Z \not\subset C$ and $Z_s \in \text{fNash}_n(X)$ with $Z_n \not\subset C$. Then are $\text{fNash}_n(X)$ and $\text{fNash}_n(X)$ normal at $Z$ and $Z_s$ respectively?

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