On the domination polynomials of cactus chains

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ABSTRACT

Let $G$ be a simple graph of order $n$. The domination polynomial of $G$ is the polynomial $D(G, x) = \sum_{i=\gamma(G)}^{n} d(G, i)x^i$, where $d(G, i)$ is the number of dominating sets of $G$ of size $i$ and $\gamma(G)$ is the domination number of $G$. In this paper we consider cactus chains with triangular and square blocks and study their domination polynomials.

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1 Introduction

Let $G = (V, E)$ be a simple graph. For any vertex $v \in V(G)$, the open neighborhood of $v$ is the set $N(v) = \{u \in V(G) \mid \{u, v\} \in E(G)\}$ and the closed neighborhood of $v$ is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V(G)$, the open neighborhood of $S$ is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of $S$ is $N[S] = N(S) \cup S$. A set $S \subseteq V(G)$ is a dominating set if $N[S] = V$ or equivalently, every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in $S$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in $G$. For a detailed treatment of these parameters, the reader is referred to [9]. Let $D(G, i)$ be the family of dominating sets of a graph $G$ with cardinality $i$ and let $d(G, i) = |D(G, i)|$. The domination polynomial $D(G, x)$ of $G$ is defined as $D(G, x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G, i)x^i$, where $\gamma(G)$ is the domination number of $G$ (see [11, 3]). Obviously, the number of dominating sets of a graph $G$ is $D(G, 1)$ (see [8, 12]). Recently the number of the dominating sets of graph $G$, i.e., $D(G, 1)$ has been considered and studied in [17] with a different approach.

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Domination theory have many applications in sciences and technology (see [9]). Recently the dominating set has found application in the assignment of structural domains in complex protein structures, which is an important task in bio-informatics ([7]).

We recall that the Hosoya index $Z(G)$ of a molecule graph $G$, is the number of matching sets, and the Merrifield-Simmons index $i(G)$ of graph $G$, is the number of independent sets. The Hosoya index of a graph has application to correlations with boiling points, entropies, calculated bond orders, as well as for coding of chemical structures. The Merrifield-Simmons index is one of the most popular topological indices in chemistry. For more information of these two indices see [14, 15, 18]. Note that $Z(G)$ and $i(G)$ can be study by the value of matching polynomial and independence polynomial at 1.

In this paper we consider a class of simple linear polymers called cactus chains. Cactus graphs were first known as Husimi trees; they appeared in the scientific literature some sixty years ago in papers by Husimi and Riddell concerned with cluster integrals in the theory of condensation in statistical mechanics [8, 10, 16]. We refer the reader to papers [6, 13] for some aspects of domination in cactus graphs.

A cactus graph is a connected graph in which no edge lies in more than one cycle. Consequently, each block of a cactus graph is either an edge or a cycle. If all blocks of a cactus $G$ are cycles of the same size $i$, the cactus is $i$-uniform. A triangular cactus is a graph whose blocks are triangles, i.e., a 3-uniform cactus. A vertex shared by two or more triangles is called a cut-vertex. If each triangle of a triangular cactus $G$ has at most two cut-vertices, and each cut-vertex is shared by exactly two triangles, we say that $G$ is a chain triangular cactus. By replacing triangles in this definitions by cycles of length 4 we obtain cacti whose every block is $C_4$. We call such cacti square cacti. Note that the internal squares may differ in the way they connect to their neighbors. If their cut-vertices are adjacent, we say that such a square is an ortho-square; if the cut-vertices are not adjacent, we call the square a para-square.

In Section 2 we study the domination polynomial of the chain triangular cactus with two approach. In Section 3 we study the domination polynomials of chains of squares.
2 Domination polynomials of the chain triangular cactus

We call the number of triangles in $G$, the length of the chain. An example of a chain triangular cactus is shown in Figure 1. Obviously, all chain triangular cacti of the same length are isomorphic. Hence, we denote the chain triangular cactus of length $n$ by $T_n$. In this paper we investigate the domination polynomial of $T_n$ by two different approach.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{chain_triangular_cactus.png}
\caption{The chain triangular cactus.}
\end{figure}

2.1 Computation of $D(T_n, x)$ using recurrence relation

In the first subsection, we use results and recurrence relations of the domination polynomial of a graph to find a recurrence relation for $D(T_n, x)$.

We need the following theorem:

**Theorem 1.**[4] If a graph $G$ consists of $k$ components $G_1, \ldots, G_k$, then $D(G, x) = \prod_{i=1}^{k} D(G_i, x)$.

The vertex contraction $G/u$ of a graph $G$ by a vertex $u$ is the operation under which all vertices in $N(u)$ are joined to each other and then $u$ is deleted (see[19]).

The following theorem is useful for finding the recurrence relations for the domination polynomials of arbitrary graphs.

**Theorem 2.**[2][11] Let $G$ be a graph. For any vertex $u$ in $G$ we have

$$D(G, x) = xD(G/u, x) + D(G - u, x) + xD(G - N[u], x) - (1 + x)p_u(G, x),$$

where $p_u(G, x)$ is the polynomial counting the dominating sets of $G - u$ which do not contain any vertex of $N(u)$ in $G$. 

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Domination polynomial satisfies a recurrence relation for arbitrary graphs which is based on the edge and vertex elimination operations. The recurrence uses composite operations, e.g. \( G - e/u \), which stands for \( (G - e)/u \).

**Theorem 3.**[1] Let \( G \) be a graph. For every edge \( e = \{u, v\} \in E \),

\[
D(G, x) = D(G - e, x) + \frac{x}{x - 1}\left[D(G - e/u, x) + D(G - e/v, x)\right] - D(G/u, x) - D(G/v, x) - D(G - N[u], x) - D(G - N[v], x) + D(G - e - N[u], x) + D(G - e - N[v], x).
\]

We use for graphs \( G = (V, E) \) the following vertex operation, which is commonly found in the literature. Let \( v \in V \) be a vertex of \( G \). A vertex appending \( G + e \) (or \( G + \{v, \cdot\} \)) denotes the graph \( (V \cup \{v'\}, E \cup \{v, v'\}) \) obtained from \( G \) by adding a new vertex \( v' \) and an edge \( \{v, v'\} \) to \( G \).

The following theorem gives recurrence relation for the domination polynomial of \( T_n \).

**Theorem 4.** For every \( n \geq 3 \),

\[
D(T_n, x) = (x^2 + 2x)D(T_{n-1}, x) + (x^2 + x)D(T_{n-2}, x),
\]

with initial condition \( D(T_1, x) = x^3 + 3x^2 + 3x \) and \( D(T_2, x) = x^5 + 5x^4 + 10x^3 + 8x^2 + x \).

**Proof.** Consider the graph \( T_n \) as shown in the following Figure 1. Since \( T_n/u \) is isomorphic to \( T_n - u \) and \( p_u(T_n, x) = 0 \), by Theorem 2 we have:

\[
D(T_n, x) = xD(T_n/u, x) + D(T_n - u, x) + xD(T_n - N[u], x) - (1 + x)p_u(T_n, x)
= (x + 1)D(T_n/u, x) + xD(T_n - N[u], x)
= (x + 1)D(T_{n-1} + e, x) +xD(T_{n-2} + e, x).
\]

(1)

Note we use Theorems 1 and 2 to obtain the domination polynomial of the graph \( T_{n-1} + e \) (see Figure 2). Suppose that \( v' \) be a vertex of degree 1 in graph \( T_{n-1} + e \) and let \( u \) be its neighbor. Note that in this case \( p_u(T_{n-1} + e, x) = 0 \). We deduce that for each \( n \in \mathbb{N} \), \( D(T_{n-1} + e, x) = \ldots \)
Figure 2: The Graph $T_{n-1} + e$.

\[ x[D(T_{n-1}, x) + D(T_{n-2} + e, x) + D(T_{n-3} + e, x)] \]. Therefore by equation (1) and this equality we have

\[ D(T_n, x) = (x^2 + x)(D(T_{n-1}, x) + D(T_{n-3} + e, x)) + (x^2 + 2x)D(T_{n-2} + e, x). \]

Now it’s suffices to prove the following equality:

\[ (x^2 + x)D(T_{n-3} + e, x) + (x^2 + 2x)D(T_{n-2} + e, x) = xD(T_{n-1}, x) + (x^2 + x)D(T_{n-2}, x). \]

For this purpose we use Theorem 2 for $D(T_{n-1}, x)$. We have

\[ xD(T_{n-1}, x) = (x^2 + x)D(T_{n-2} + e, x) + x^2D(T_{n-3} + e, x). \]

Now we use Theorem 2 for $v'$ to obtain domination polynomial of $T_{n-2} + e$, then we have

\[ D(T_{n-2} + e, x) = (1 + x)D(T_{n-2}, x) + xD(T_{n-3} + e, x) - (1 + x)D(T_{n-3} + e, x). \] Therefore the result follows.

2.2 Computation of $D(T_n, x)$ by counting the number of dominating sets

In this section we shall obtain a recurrence relation for the domination polynomial of $T_n$. For this purpose we count the number of dominating sets of $T_n$ with cardinality $k$. In other words, we first find a two variables recursive formula for $d(T_n, k)$.

Recently by private communication, we found that the following result also appear in [5] but were proved independently.

**Theorem 5.** The number of dominating sets of $T_n$ with cardinality $k$ is given by

\[ d(T_n, k) = 2d(T_{n-1}, k - 1) + d(T_{n-1}, k - 2) + d(T_{n-2}, k - 1) + d(T_{n-2}, k - 2). \]
Proof. We shall make a dominating set of $T_n$ with cardinality $k$ which we denote it by $T_n^k$. We consider all cases:

Case 1. If $T_n^k$ contains both of $v$ and $w$, then we have $T_n^k = T_{n-1}^{k-2} \cup \{v, w\}$. In this case we have $d(T_n, k) = d(T_{n-1}, k - 2)$.

Case 2. If $T_n^k$ contains only $v$ or $w$ (say $v$), then we have $T_n^k = T_{n-1}^{k-1} \cup \{v\}$. In this case we have $d(T_n, k) = 2d(T_{n-1}, k - 1)$.

Case 3. If $T_n^k$ contains none of $v$ and $w$, then we can construct $T_n^k$ by $T_{n-2}^{k-2}$ or $T_{n-2}^{k-1}$ as shown in Figure 3. In this case we have $d(T_n, k) = d(T_{n-2}, k - 1) + d(T_{n-2}, k - 2)$. By adding all contributions we obtain the recurrence for $d(T_n, k)$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{recurrence.png}
\caption{Recurrence relation for $d(T_n, k)$.}
\end{figure}

Corollary 1. For every $n \geq 3$,

$$D(T_n, x) = (x^2 + 2x)D(T_{n-1}, x) + (x^2 + x)D(T_{n-2}, x).$$

Proof. It follows from Theorem 5 and the definition of the domination polynomial.
We mention here the Hosoya index of a graph $G$ is the total number of matchings of $G$ and the Merrifield-Simmons index is the total number of its independent sets. Motivation by these indices, we are interested to count the total number of dominating set of a graph which is equal to $D(G, 1)$. Here we present a recurrence relation to the total number of the chain triangular cactus.

**Theorem 6.** The enumerating sequence $\{t_n\}$ for the number of dominating sets in $T_n$ $(n \geq 2)$ is

$$t_n = 3t_{n-1} + 2t_{n-2}$$

with initial values $t_0 = 2$, $t_1 = 7$.

**Proof.** Since $t_n = D(T_n, 1)$, it follows from Corollary 1.

3 Domination polynomials of chains of squares

By replacing triangles in the definitions of triangular cactus, by cycles of length 4 we obtain cacti whose every block is $C_4$. We call such cacti, square cacti. An example of a square cactus chain is shown in Figure 4. We see that the internal squares may differ in the way they connect to their neighbors. If their cut-vertices are adjacent, we say that such a square is an ortho-square; if the cut-vertices are not adjacent, we call the square a para-square.

3.1 Domination polynomial of para-chain square cactus graphs

![Figure 4: Para-chain square cactus graphs.](image)

In this subsection we consider a para-chain of length $n$, $Q_n$, as shown in Figure 4. We shall obtain a recurrence relation for the domination polynomial of $Q_n$. As usual we denote the
number of dominating sets of $Q_n$ by $d(Q_n, k)$. The following theorem gives a recurrence relation for $D(Q_n, x)$.

We need the following Lemma for finding domination polynomial of the $Q_n$.

**Figure 5:** Graphs $Q^	riangle_n$, $Q'_n$ and $Q_n(2)$, respectively.

**Figure 6:** Graphs $(Q_n + e)/w$ and $Q_n + e$, respectively.

**Lemma 1.** For graphs in figures 5 and 6 have:

(i) $D(Q^	riangle_n, x) = (1 + x)D(Q_n + e, x) + xD(Q'_n, x)$, where $D(Q^\triangle_0, x) = x^3 + 3x^2 + 3x$.

(ii) $D(Q_n(2), x) = x(D(Q_n + e, x) + D(Q_n, x) + D(Q'_n, x))$, where $D(Q_0(2), x) = x^3 + 3x^2 + x$.

(iii) $D(Q'_n, x) = (1 + x)D(Q_n + e, x) - xD(Q'_n-1, x)$, where $D(Q'_0, x) = x^3 + 3x^2 + x$.

(iv) $D(Q_n + e, x) = x(D(Q_n, x) + D(Q_n-1, x)) + xD(Q'_n-1, x) + 2x^2 D(Q'_n-2, x)$, where $D(Q_1 + e, x) = x^5 + 5x^4 + 9x^3 + 4x^2$.

**Proof.** The proof of parts (i) and (ii) follow from Theorems 1 and 2 for vertex $u$ in graphs $Q_n^\triangle$ and $Q_n(2)$, respectively. Note that in these cases $p_u(G, x) = 0$.

(iii) We use Theorems 1 and 2 for vertex $u$ to obtain domination polynomial of $Q'_n$, then we have

$$D(Q'_n, x) = (1 + x)D(Q_n + e, x) + x^2 D(Q'_n-1, x) - (1 + x)x D(Q'_n-1, x)$$

$$= (1 + x)D(Q_n + e, x) - x^2 D(Q'_n-1, x).$$
Therefore by parts (iv) we use Theorems 1 and 2 for vertex $w$ to obtain domination polynomial of $Q_n + e$, as shown in Figure 6 then we have \[ D(Q_n + e, x) = xD((Q_n + e)/w, x) + xD(Q'_{n-1}, x) + xD(Q_n, x). \] Now consider the graph $(Q_n + e)/w$ as shown in figure 6. We use Theorems 1 and 3 for $e = \{u, v\}$ to obtain $D((Q_n + e)/w, x)$, then we have

\[
D((Q_n + e)/w, x) = D(Q_n, x) + \frac{x}{x-1} [D(Q_{n-1}^\triangle, x) + D(Q_{n-1}^\triangle, x) - (Q_{n-1}^\triangle, x)]
- D(Q_{n-1}^\triangle, x) - D(Q_{n-2}' - x) - xD(Q_{n-2}' - x) + xD(Q_{n-2}' - x)]
= D(Q_n, x) + 2x D(Q_{n-2}' - x).
\]

Therefore the result follows. □

**Theorem 7.** The domination polynomial of para-chain $Q_n$ is given by

\[
D(Q_n, x) = (x^3 + 2x^2 + x)D(Q_{n-1}, x) + (x^3 + 2x^2)D(Q_{n-2}, x)
+ (x^3 + 3x^2)D(Q_{n-2}', x) + (2x^4 + 4x^3)D(Q_{n-3}', x),
\]

with initial conditions $D(Q_1, x) = x^4 + 4x^3 + 6x^2$ and $D(Q_2, x) = x^7 + 7x^6 + 21x^5 + 29x^4 + 15x^3$.

**Proof.** Consider the labeled $Q_n$ as shown in Figure 4. We use Theorems 1 and 2 for vertex $u_n$ to obtain the domination polynomial of $Q_n$. We have

\[
D(Q_n, x) = xD(Q_{n-1}^\triangle, x) + D(Q_{n-1}(2), x) + x^2 D(Q_{n-2}' - x) - (1 + x)xD(Q_{n-2}' - x)
= xD(Q_{n-1}^\triangle, x) + D(Q_{n-1}(2), x) - xD(Q_{n-2}' - x).
\] (2)

Therefore by parts (i), (ii) and (iv) of Lemma 1 and equation 2, we have

\[
D(Q_n, x) = x((1 + x)D(Q_{n-1} + e, x) + xD(Q_{n-2}' - x)) + x(D(Q_{n-1} + e, x)
+ xD(Q_{n-1}, x) + D(Q_{n-2}' - x)) - xD(Q_{n-2}' - x)
= (x^2 + 2x)D(Q_{n-1} + e, x) + x^2 D(Q_{n-2}' - x) + xD(Q_{n-1}, x)
= (x^2 + 2x)[x(D(Q_{n-1} + e, x) + D(Q_{n-2}, x)) + xD(Q_{n-2}', x)
+ 2x^2 D(Q_{n-3}', x)] + x^2 D(Q_{n-2}' - x) + xD(Q_{n-1}, x)
= (x^3 + 2x^2 + x)D(Q_{n-1} + e, x) + (x^3 + 2x^2)D(Q_{n-2}, x)
+ (x^3 + 3x^2)D(Q_{n-2}' - x) + (2x^4 + 4x^3)D(Q_{n-3}', x). □
\]
3.2 Domination polynomial of ortho-chain square cactus graphs

In this subsection we consider a ortho-chain of length \( n \), \( O_n \), as shown in Figure 7. We shall obtain a recurrence relation for the domination polynomial of \( O_n \).

![Figure 7: Labeled ortho-chain square \( O_n \).](image)

We need the following Lemma for finding domination polynomial of the \( O_n \).

![Figure 8: Graphs \( O_n^\triangle \), \( O_n(2) \), \( O'_n \) and \( O_n + e \), respectively.](image)

**Lemma 2.** For graphs in figure 8 we have:

(i) \( D(O_n^\triangle, x) = (1 + x)D(O_n + e, x) + xD(O_n - 1(2), x) \), where \( D(O_0^\triangle, x) = x^3 + 3x^2 + 3x \).

(ii) \( D(O_n(2), x) = x(D(O_n + e, x) + D(O_n, x) + D(O_n - 1(2), x)) \), where \( D(O_0(2), x) = x^3 + 3x^2 + x \).

(iii) \( D(O'_n, x) = (1 + x)D(O_n^\triangle, x) - xD(O_n - 1(2), x) \), where \( D(O_0', x) = x^4 + 4x^3 + 6x^2 + 2x \).

(iv) \( D(O_n + e, x) = xD(O'_n, x) + xD(O_n - 1(2), x) + x^2D(O_n - 2(2), x) \), where \( D(O_1 + e, x) = x^5 + 5x^4 + 9x^3 + 4x^2 \).

**Proof.** The proof of parts (i), (ii) and (iv) follow from Theorems 1 and 2 for vertex \( u \) in graphs \( O_n^\triangle \), \( O_n(2) \) and \( O_n + e \), respectively. Note that in these cases \( p_u(G, x) = 0 \).
We use Theorems 1 and 2 for $u$ in graphs $O'_n$. Since $O'_n/u$ is isomorphic to $O'_n - u$ and $p_u(G, x) = xD(O_{n-1}(2), x)$. So we have the result.

(iii) We use Theorems 1 and 2 for $u$ in graphs $O'_n$. Since $O'_n/u$ is isomorphic to $O'_n - u$ and $p_u(G, x) = xD(O_{n-1}(2), x)$. So we have the result.

Theorem 8. The domination polynomial of para-chain $O_n$ is given by

$$D(O_n, x) = xD(O_{n-1}, x) + (x^2 + 2x)D(O_{n-1} + e, x) + x^2D(O_{n-2}(2), x),$$

with initial condition $D(O_1, x) = x^4 + 4x^3 + 6x^2$.

Proof. Consider the labeled $O_n$ as shown in Figure 7. We use Theorems 1 and 2 for vertex $u_n$ to obtain domination polynomial of $O_n$, then we have

$$D(O_n, x) = xD(O_{n-1}^\Delta, x) + D(O_{n-1}(2), x) + x^2D(O_{n-2}(2), x) - (1 + x)xD(O_{n-2}(2), x)$$

$$= xD(O_{n-1}^\Delta, x) + D(O_{n-1}(2), x) - xD(O_{n-2}(2), x).$$

Therefore by parts (i) and (ii) of Lemma 2 and this equation we have

$$D(O_n, x) = x((1 + x)D(O_{n-1} + e, x) + xD(O_{n-2}(2), x)) + x(D(O_{n-1} + e, x)$$

$$+ D(O_{n-1}, x) + D(O_{n-2}(2), x)) - xD(O_{n-2}(2), x)$$

$$= (x^2 + 2x)D(O_{n-1} + e, x) + x^2D(O_{n-2}(2), x) + xD(O_{n-1}, x).$$

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