DISPERSSIVE EFFECTS OF THE INCOMPRESSIBLE VISCOELASTIC FLUIDS

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ABSTRACT. We consider the Cauchy problem of the N-dimensional incompressible viscoelastic fluids with \( N \geq 2 \). It is shown that, in the low frequency part, this system possesses some dispersive properties derived from the one parameter group \( e^{\pm i\Lambda} \). Based on this dispersive effect, we construct global solutions with large initial velocity concentrating on the low frequency part. Such kind of solution has never been seen before in the literature even for the classical incompressible Navier-Stokes equations.

1. Introduction. At first, we introduce the incompressible Navier–Stokes system, namely,

\[
\begin{align*}
    u_t + u \cdot \nabla u - \Delta u + \nabla \Pi &= 0, \quad x \in \mathbb{R}^N, \quad t > 0, \quad N \geq 2, \\
    \text{div} u &= 0, \\
    u|_{t=0} &= u_0,
\end{align*}
\]  

(1.1)

where \( u(t, x) \) and \( \Pi(t, x) \) denote the fluid velocity and the pressure, respectively. The existence of at least one global weak solution of (1.1) is well-known since the pioneering work of Leray [31] and Hopf [22], when \( u_0 \in L^2 \). To prove the uniqueness of the solution in some time interval \( [0, T] \), we need besides the condition \( u_0 \in L^2 \) a further regularity property of the initial value \( u_0 \). The first sufficient condition on the initial value for a bounded domain seems to have been described in [25]. Then many results on sufficient conditions on \( u_0 \) to guarantee the existence and uniqueness of the solution were proved, see, e.g., [2, 5, 17, 18, 24, 26] etc. For example, when \( u_0 \in H^{N/2-1} \), the global weak solution for (1.1) is unique on some local time interval \( [0, T] \). When \( \|u_0\|_{H^{N/2-1}_x} \) is sufficiently small, the incompressible Navier–Stokes system has a unique global strong solution. Due to the energy conservation law

\[
\|u(t)\|_{L^2}^2 + 2\|\nabla u\|_{L^2}^2 \leq \|u_0\|_{L^2}^2,
\]

one sees that the key of the global well-posedness result is to control the high frequency part of the velocity \( u \). We hope to prove that when the high frequency part of \( u_0 \) is small and the low frequency part of \( u_0 \) could be very large, then...
the incompressible Navier–Stokes system (1.1) has a unique global strong solution. As an attempt, in this paper, we consider such problem to another incompressible system with elastic effect.

More precisely, we consider the so called incompressible viscoelastic fluids:

\[
\begin{aligned}
\frac{\partial}{\partial t} F + u \cdot \nabla F &= \nabla u F, \\
\frac{\partial}{\partial t} u + u \cdot \nabla u + \nabla \Pi &= \Delta u + \text{div} \left( \frac{\partial W(F)}{\partial F} F^\top \right), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N, \quad N \geq 2, \\
\text{div} u &= 0, \\
u(0, x) &= u_0(x), \quad F(0, x) = F_0(x). 
\end{aligned}
\]

(1.2)

Here \(u, F, \Pi\) are the velocity, deformation tensor, and pressure of the fluids, respectively. Moreover, \(W(F)\) is the elastic energy functional, \(\frac{\partial W(F)}{\partial F}\) is the Piola-Kirchhoff stress tensor and \(\frac{\partial W(F)}{\partial F} F^\top\) is the Cauchy-Green tensor.

Throughout this paper, we adopt the following notations:

\[
(\nabla u)_{ij} = \partial_j u^i, \quad (\nabla uF)_{ij} = \partial_k u^i F^{kj}, \quad (\text{div } F)^i = \partial_k F^{ik},
\]

and summation over repeated indices will always be understood.

For the sake of simplicity, we only focus on the case of Hookean elastic materials: \(W(F) := \frac{1}{2} |F|^2\). However, this simplify does not reduce the essential difficulties for analysis. Indeed, all the results we describe here can be generalized to more general cases. Obviously, the equilibrium \((I, 0)\) solves (1.2). In this paper, we investigate the system (1.2) near \((I, 0)\), and thus impose on (1.2) the following initial conditions:

\[
F(0, x) = I + E_0(x), \quad u(0, x) = u_0(x).
\]

(1.3)

We further assume that \(E_0(x)\) and \(u_0(x)\) satisfy the following constraints

\[
\begin{aligned}
\text{div} u_0 &= 0, \\
det(I + E_0) &= 1, \\
\text{div} E_0^\top &= 0, \\
\partial_m E_0^{ij} - \partial_j E_0^{im} &= E_0^{lj} \partial_l E_0^{im} - E_0^{lm} \partial_l E_0^{ij}.
\end{aligned}
\]

(1.4)

The first three of (1.4) are nothing but the consequences of the incompressibility condition and the last one can be understood as the consistency condition for changing variables between the Lagrangian and Eulerian coordinates [29]. We would like to remark that (1.4) is preserved for all time for sufficiently smooth solutions to (1.2), c.f. [29]. On this basis, denoting by \(E := F - I\), then the system satisfied by \((E, u)\) takes the form

\[
\begin{aligned}
\frac{\partial}{\partial t} E^{ij} + (u \cdot \nabla) E^{ij} - \partial_j u^i = \partial_k u^i E^{kj}, \\
\partial_t u^i + (u \cdot \nabla) u^i + \partial_i \Pi - \Delta u^i - \partial_j E^{ij} = E^{ik} \partial_j E^{ik}, \\
E(0, x) = E_0(x), \quad u(0, x) = u_0(x),
\end{aligned}
\]

(1.5)

with constraints

\[
\begin{aligned}
\text{div } u &= 0, \quad \text{div } E^\top = 0, \\
det(I + E) &= 0, \\
\partial_m E^{ij} - \partial_j E^{im} &= E^{lj} \partial_l E^{im} - E^{lm} \partial_l E^{ij}.
\end{aligned}
\]

(1.6)
Using the variant $d^{ij} := -\Lambda^{-1} \partial_j u^i$ introduced in [34], where $\Lambda^* f = F^{-1}(|\xi|^* \hat{f})$, the system (1.5)-(1.6) can be reformulated as

$$
\begin{align*}
\text{div} E^\top &= \text{div} d^\top = 0, \\
\partial_t E^{ij} + (u \cdot \nabla) E^{ij} + \Lambda d^{ij} &= \partial_k u^k E^{kj}, \\
\partial_t d^{ij} - \Delta d^{ij} - \Lambda E^{ij} &= \Lambda^{-1} \partial_k (E^{kj} \partial_k E^{lj} - E^{lk} \partial_k E^{lj}), \\
+ (\delta^{im} + \partial_i (-\Delta)^{-1} \partial_m) \Lambda^{-1} \partial_j ((u \cdot \nabla) u^m - E^{lk} \partial_k E^{mk}), \\
(1.7)
\end{align*}
$$

when the initial data satisfy the constraints (1.4).

The theory of viscoelastic fluids recently gained quite some attention. According to the methods used to get the global estimates, the results on viscoelastic fluids fall into two categories.

(I) Energy approach. The homogeneous linearized system of (1.5) is the following

$$
\begin{align*}
\partial_t E - \nabla u &= 0, \\
\partial_t u - \Delta u - \text{div} E &= 0. \\
(1.8)
\end{align*}
$$

From the standard energy estimate for (1.8), it is not difficult to find that the damping effect for $E$ is partial. More precisely, the damping effect is only available for $\text{div} E$, but not for $E$ itself. So the main difficulty to obtain the global solutions to system (1.5)-(1.6) is the lack of damping mechanism on $E$. To overcome this problem, in the two dimensional case, Lin, Liu and Zhang [32] write

$$
E = \left( \begin{array}{cc}
-\partial_2 \phi^1 & -\partial_2 \phi^2 \\
\partial_1 \phi^1 & \partial_1 \phi^2
\end{array} \right)
$$

(1.9)

for some $\phi = (\phi^1, \phi^2)$ due to the fact $\text{div} E^\top = 0$, and then consider the new system for $(u, \phi)$. For the 3D case, (1.9) is unavailable anymore. Zhang et al [11, 33] used the following quantities

$$
G := F^{-1}, \quad U := G - I
$$

(1.10)

introduced by Sideris and Thomases [35] to replace the standard deformation tensor $F$. One very important property of this matrix $G$ is that

$$
\partial_i G^{kj} = \partial_j G^{ki}, \quad \text{for} \quad i, j, k = 1, 2, 3,
$$

(1.11)

which is not true for $F$. Later, Liu, Lei and Zhou [29] proved that although the deformation tensor $F$ is not curl free, curl$F$ is actually of higher order nonlinear term, see (1.6)$_3$. Combining the partial damping effect possessed by (1.8) with the fine property of curl$G$ (see (1.11)) or curl$F$ (see (1.6)$_3$), global solutions to (1.2) were established in certain Sobolev space $H^s$, please refer to [11, 33, 29]. Some other results can be found in [27, 28, 30].

Besides, for the 2D case, Hu and Lin recently [23] proved the global-in-time existence of the Leray-Hopf type weak solutions to (1.2) in the physical energy space via the DiPerna-Lions theory.

(II) The scaling invariant approach. This approach goes back to the pioneering work by Fujita and Kato [17] for the classical incompressible Navier-Stokes equations. Danchin [12] first applied this approach to the compressible Navier-Stokes equations. In fact, Danchin studied carefully the following hyperbolic-parabolic
system (with convection terms)
\[
\begin{cases}
\partial_t a + \Lambda v = f, \\
\partial_t v - \Delta v - \Lambda a = g.
\end{cases}
\]  
(1.12)

This system arises when linearizing the isentropic compressible Navier-Stokes equations on the density \(\rho\) and on the potential part of the velocity \(u\). The smoothing properties for (1.12) were exploited in [12], and the different behaviors of the solution to (1.12) for low and high frequencies were revealed as well.

The difficulty to understand (1.5)-(1.6) is similar to that of the isentropic compressible Navier-Stokes equations. As mentioned above, thanks to the new variant \(d^{\beta} := -\Lambda^{-1}\partial_j u^i\) and the constraint (1.6), Qian [34] found that the system (1.5)-(1.6) is equivalent to (1.7), and the linearized system of (1.7) takes exactly the form (1.12). Accordingly, following the arguments in [12], Qian proved the global well-posedness of the system (1.5)-(1.6) if the initial data \((E_0, u_0)\) are small in
\[
\left( B_{2,1}^{N-1} \cap B_{2,1}^{N} \right) \times \dot{B}_{p,1}^{N-1}.
\]
(1.13)

Later, the first two authors [36] extended the result in [34] to the \(L^p\) setting. They showed that, for the high frequency part \((E_{0H}, u_{0H})\) of the initial data (please refer to (1.32) for the definition of \(f_L\) and \(f_H\) with \(f \in \mathcal{S}'(\mathbb{R}^N)\)), the space in (1.13) can be replaced by
\[
\dot{B}_{p,1}^{N} \times \dot{B}_{p,1}^{N-1}.
\]
(1.14)

As a result, the large highly oscillating initial velocity as in [6] for the classical incompressible Navier-Stokes equations, for example, if \(N = 3\),
\[
u_0(x) := \varepsilon^{\frac{3}{2} - 1} \sin \left(\frac{x_3}{\varepsilon}\right) (-\partial_2 \phi(x), \partial_1 \phi(x), 0), \phi \in \mathcal{S}(\mathbb{R}^3), \varepsilon > 0,
\]
(1.15)
is also allowed in [36], see [9, 10] for related results on the isentropic compressible Navier-Stokes equations. For the case that the anisotropy is taken into consideration, see the result of the second author [37]. Very recently, we [16] constructed a new class of solutions to the isentropic compressible Navier-Stokes equations with the aid of the dispersive properties of (1.12) in the low frequent part. In particular, the low frequency part of the initial velocity \(u_0\) is not necessarily small in \(B_{2,1}^{N-1}\).

Motivated by [16] and [36], in this paper, we study the global existence and uniqueness of the solutions to the incompressible viscoelastic fluids (1.5)-(1.6) with the initial data \((E_0, u_0)\) small in the following space
\[
\mathcal{E}_{0,\alpha,p} := \left\{ (\phi, \varphi) \in \mathcal{S}_h' \times \mathcal{S}_h' : (\phi_L, \varphi_L) \in \dot{B}_{2,1}^{\frac{N}{2} - 1 + \alpha}, \phi_H \in \dot{B}_{p,1}^{\frac{N}{2}}, \varphi_H \in \dot{B}_{p,1}^{\frac{N}{2} - 1} \right\},
\]
(1.16)
with some \(\alpha > 0\) and \(p \geq 2\). In the following, we denote
\[
\|(E_0, u_0)\|_{\mathcal{E}_{0,\alpha,p}} := \|(E_{0L}, u_{0L})\|_{\dot{B}_{2,1}^{\frac{N}{2} - 1 + \alpha}} + \|E_{0H}\|_{\dot{B}_{p,1}^{\frac{N}{2}}} + \|u_{0H}\|_{\dot{B}_{p,1}^{\frac{N}{2} - 1}}.
\]
(1.17)

If \(\alpha = 0\), \(\mathcal{E}_{0,0,p}\) is chosen in [36] for the initial data. Furthermore, if \(\alpha = 0\) and \(p = 2\), \(\mathcal{E}_{0,0,2}\) becomes the space in (1.13), chosen in [34] and [12]. Owing to Bernstein’s inequality and the low frequency cut-off, it is easy to verify that
\[
\mathcal{E}_{0,0,2} \hookrightarrow \mathcal{E}_{0,0,p} \hookrightarrow \mathcal{E}_{0,\alpha,p}.
\]
In what follows, we denote
\[ \mathcal{E}_0 := \mathcal{E}_{0, \alpha, p}, \]
for simplicity. Due to the extra $\alpha$-order regularity in the low frequency part, even the second component of the element in $\mathcal{E}_0$ is not scaling invariant under the transformation
\[ u_0(x) \rightarrow lu_0(lx), l > 0. \]
In that case, we will run into difficulties when estimating the nonlinear terms. However, the situation is different if we take the dispersive property of the system (1.12) into consideration. To shed some light on this property, we fall back on the eigenvalues associated with the linear system (1.12) in the low frequency case ($|\xi| < 2$):
\[ \lambda_{\pm}(\xi) = -\frac{|\xi|^2}{2} \pm i|\xi| \sqrt{1 - \frac{|\xi|^2}{4}}. \]
It is easy to see that there are two main parts $-\frac{|\xi|^2}{2}$ and $\pm i|\xi|$ in $\lambda_{\pm}(\xi)$ if $|\xi| \ll 1$. The two (semi)groups corresponding to them are $e^{\pm i \Delta/2}$ and $e^{\pm i t \lambda}$, respectively. Previous studies on the hyperbolic-parabolic system (1.12), such as [21] and [12], ignored the dispersive effect of the one-parameter group $e^{\pm i t \lambda}$. On the contrary, in this paper, $e^{\pm i t \lambda}$ plays an important role. Indeed, the proof of the global well-posedness has two main components:

- Energy estimates to control the high frequency part of $(E, u)$. The main difficulty in this part is to find the damping effect of $E_H$.
- Energy estimates and Strichartz estimates to control the low frequency part of $(E, u)$. The interplay between the dissipation and dispersive effect is present at this stage.

Different from our previous results in [16], we deal with the high frequency part of (1.12) in the critical $L^p$ framework. The energy estimates for the high frequency part in [16] does not work here. We solve this problem by means of the so called effective velocity used by Haspot in [19], without resorting to the Green matrix of (1.12) used in [36].

We shall obtain the existence and uniqueness of a solution $(E, u)$ to (1.5)-(1.6) in the following spaces.

**Definition 1.1.** For $T > 0$, $\alpha \geq 0$, $p \in [2, \infty)$, and $N \in \mathbb{N} \cap [2, \infty)$, let us denote by $\mathcal{E}_{\alpha}^N(T)$ the space of functions $(E, u)$ such that
\[ (E_L, u_L) \in \widehat{C}_T(\hat{B}_{2,1}^N(1+\alpha)) \cap L_T^1(\hat{B}_{2,1}^{N+1+\alpha}) \cap \hat{L}_T^{\frac{1}{2}}(\hat{B}_{\infty,1}^{2\alpha-1}); \]
\[ E_H \in \widehat{C}_T(\hat{B}_{p,1}^N) \cap L_T^1(\hat{B}_{p,1}^N), \quad u_H \in \widehat{C}_T(\hat{B}_{p,1}^{N-1}) \cap L_T^1(\hat{B}_{p,1}^{N+1}). \]

We shall endow the space with the norm:
\[ \| (E, u) \|_{\mathcal{E}_{\alpha}^N(T)} := \| (E_L, u_L) \|_{\hat{L}_T^p(\hat{B}_{2,1}^{N-1+\alpha}) \cap L_T^1(\hat{B}_{2,1}^{N+1+\alpha}) \cap \hat{L}_T^{\frac{1}{2}}(\hat{B}_{\infty,1}^{2\alpha-1})} \]
\[ +\| E_H \|_{\hat{L}_T^p(\hat{B}_{p,1}^N) \cap L_T^1(\hat{B}_{p,1}^N)} + \| u_H \|_{\hat{L}_T^p(\hat{B}_{p,1}^{N-1}) \cap L_T^1(\hat{B}_{p,1}^{N+1})}. \]

In particular, if $\alpha = 0$, $\mathcal{E}_{\alpha}^N(T)$ coincides with the space used in [36]. We use the notation $\mathcal{E}_{\alpha}^\infty$ if $T = \infty$, changing $[0, T]$ into $[0, \infty)$ in the definition above.
Remark 1.1. If $0 < T < \infty$, in view of Bernstein’s inequality and the low frequency embedding (2.3), it is easy to verify that $(E, u) \in \mathcal{E}^N_\alpha (T)$ if and only if $(E, u)$ satisfies

\[
E_L \in \mathcal{C}_T (\dot{B}^{N-1+\alpha}_{2,1}), \quad E_H \in \mathcal{C}_T (\dot{B}^N_{p,1});
\]

\[
u_L \in \mathcal{C}_T (\dot{B}^N_{2,1}) \cap \mathcal{L}_T (\dot{B}^{N+1+\alpha}_{2,1}), \quad u_H \in \mathcal{C}_T (\dot{B}^{N-1}_{p,1}) \cap \mathcal{L}_T (\dot{B}^{N+1}_{p,1}).
\]

At first, we can obtain the following local well-posedness results.

**Theorem 1.1** (local well-posedness). Assume that the initial data $(E_0, u_0)$ satisfy $E_0 \in \dot{B}^N_{p,1}$, $u_{0L} \in \dot{B}^N_{p,1}$, and $u_{0H} \in \dot{B}^{N-1}_{p,1}$ with $\text{div} E_0^\top = \text{div} u_0 = 0$.

(i): If $p \in [1, \infty)$, then there exists a positive time $T_0$ depending on $N$ and $p$, such that the system (1.5) admits a solution $(E, u)$ satisfying

\[
\begin{cases}
E \in \mathcal{C}_{T_0} (\dot{B}^N_{p,1}), & u_L \in \mathcal{C}_{T_0} (\dot{B}^{N+2}_{p,1}) \\
u_L \in \mathcal{C}_{T_0} (\dot{B}^{N-1}_{p,1}) \cap \mathcal{L}_{T_0} (\dot{B}^{N+1}_{p,1}).
\end{cases}
\]

(ii): If $p \in [1, 2N]$, then the solution $(E, u)$ is unique.

**Corollary 1.1.** If $(E_0, u_0) \in \mathcal{E}_0$, and $(p, \alpha)$ satisfies

\[
2 \leq p \leq 4, \quad 0 \leq \alpha \leq 1 - N \left( \frac{1}{2} - \frac{1}{p} \right),
\]

then the system (1.5) admits a unique solution $(E, u)$ in $\mathcal{E}^N_\alpha (T_0)$.

Remark 1.2. If $E_0$ also satisfies (1.4)_2 and (1.4)_4, then these constraints will be inherited by $E$, i.e., (1.6)_2-(1.6)_3 hold.

Remark 1.3. Theorem 1.1 and Corollary 1.1 extend the local well-posedness result in [34]. To the best of our knowledge, so far the space given by (1.18) is the largest one in which the incompressible viscoelastic fluids (1.2) is local well-posed.

Then, we can obtain the main result as follows.

**Theorem 1.2** (global well-posedness). Let (1.19) hold, and

\[
\begin{cases}
\alpha \in \left( 0, \frac{1}{2} \right], & \text{if } N \geq 4, \\
\alpha \in \left( 0, \frac{1}{2} \right), & \text{if } N = 3, \\
\alpha \in \left( 0, \frac{1}{2} \right], & \text{if } N = 2.
\end{cases}
\]

There exist positive constants $c_0$ and $M_0$ depending on $N$ and $p$, such that for all $(E_0, u_0) \in \mathcal{E}_0$ satisfying the constraints (1.4) and

\[
\| (E_0, u_0) \|_{\mathcal{E}_0} \leq c_0,
\]

the system (1.5)-(1.6) admits a unique global solution $(u, E)$ in $\mathcal{E}^N_\alpha$ with

\[
\| (E, u) \|_{\mathcal{E}^N_\alpha} \leq M_0 \| (E_0, u_0) \|_{\mathcal{E}_0}.
\]

Remark 1.4. Let $\alpha$ be as in Theorem 1.2. For all $\phi \in \mathcal{S}(\mathbb{R}^N)$ with $\text{div} \phi = 0$, $\| \phi \|_{\dot{B}^{N-1}_{p,1}} = R$, and $\hat{\phi}$ supported in a compact set, say, $\text{Supp} \, \hat{\phi} \subset B(0, 1)$, let us denote

\[
\phi_l (x) := l^{1-\beta} \phi (lx), \quad \text{with } l > 0, \quad \text{and } 0 < \beta < \alpha.
\]
Then we have
\[ \hat{\phi}_t \subset B(0,l), \quad \text{and} \quad \|\phi_t\|_{B^{N,-1}_{2,1}} = l^{-\beta} R \to \infty, \quad \text{as} \quad l \to 0. \tag{1.23} \]

Next, set
\[ l := \frac{3}{8} 2^{-Q}, \tag{1.24} \]
where \( Q \in \mathbb{N} \) will be determined below. Then it follows that
\[ \hat{\Delta}_q \phi_t = 0, \quad \text{for all} \quad q \geq -Q. \tag{1.25} \]

Choosing \( Q \in \mathbb{N} \) so large that
\[ 2^{-(\alpha - \beta)Q} \leq \left( \frac{3}{8} \right)^{\beta} c_0 R, \tag{1.26} \]
then from (1.23)-(1.25), we infer that
\[ \|\phi_t\|_{\dot{B}^{\frac{N}{2}}_{p,1}} \leq \frac{3}{8} 2^{-Q} \|\phi_t\|_{\dot{B}^{\frac{N}{2}}_{p,1}} \leq 2^{-\alpha Q} l^{-\beta} R \leq c_0. \tag{1.27} \]

Now let
\[ (E_0, u_0) := (0, \phi_t), \tag{1.28} \]
with \( l \) satisfying (1.24) and (1.26). Then it is easy to verify that
\[ \|(E_0, u_0)\|_{\mathcal{E}_0} \leq c_0, \tag{1.29} \]
and thus the data given in (1.28) apply to Theorem 1.2, even though
\[ \|(E_0, u_0)\|_{\mathcal{E}_{0,0.2}} \to \infty, \quad \text{as} \quad l \to 0. \]

**Remark 1.5.** In this paper, we take the Lebesgue index \( p = \infty \) in the Strichartz estimates of the system (1.12) to minimize the restrictions on \( \alpha \). The same cannot not be achieved in our previous result [16] for the isentropic compressible Navier-Stokes equations due to the interactions between the potential part of the velocity (with dispersive property) and the divergence free part of the velocity (without dispersive property).

**Remark 1.6.** Compared with [16], the uniform estimates for the system (1.12) in the high frequency part is extended to the \( L^p \) critical framework. This explains the restriction (1.19) for \( p \) and \( \alpha \). Clearly, if \( p = 2 \), the restriction (1.19) will disappear. Moreover, the estimates obtained in Proposition 4.1 can be used to the isentropic compressible Navier-Stokes equations.

If \( (E_0, u_0) \in \left( \dot{B}^{\frac{N}{2}}_{2,1} \cap \dot{B}^{\frac{N}{2}}_{2,1} \right) \times \dot{B}^{\frac{N}{2}}_{2,1} \), from Theorem 1.2, we immediately get the following corollary. The proof of which is similar to that of Theorem 1.2 in [16], we omit the details here.

**Corollary 1.2.** Assume that \( \alpha \) satisfies (1.20), and \( (E_0, u_0) \in \left( \dot{B}^{\frac{N}{2}}_{2,1} \cap \dot{B}^{\frac{N}{2}}_{2,1} \right) \times \dot{B}^{\frac{N}{2}}_{2,1} \) with \( (u_0, E_0) \) satisfying the constrains (1.4). Then there exist positive constants \( c_1 \), such that for \( Q \in \mathbb{N} \), if
\[
2^{-\alpha Q} \left( \|P_{< -Q} E_0\|_{\dot{B}^{\frac{N}{2}}_{2,1}} + \|P_{< -Q} u_0\|_{\dot{B}^{\frac{N}{2}}_{2,1}} \right)
+ \|P_{\geq -Q} E_0\|_{\dot{B}^{\frac{N}{2}}_{2,1} \cap \dot{B}^{\frac{N}{2}}_{2,1}} + \|P_{\geq -Q} u_0\|_{\dot{B}^{\frac{N}{2}}_{2,1}} \leq c_1, \tag{1.30} \]
then the system (1.5)-(1.6) admits a unique global solution \( (E, u) \) in \( \mathcal{C}_{0,0}^{\frac{N}{2}} \).
Remark 1.7. It can be seen from Corollary 1.2 that we construct a unique global solution to (1.5)-(1.6) with the initial data small in the high frequency part. The low frequency part of the initial data can be very large.

Remark 1.8. In view of the high frequency embedding (2.4) and Bernstein’s inequality, we find that

\[
\left(\dot{B}^{N-1+\alpha}_{2,1} \cap \dot{B}^{N+\alpha}_{2,1}\right) \times \dot{B}^{N-1+\alpha}_{2,1} \hookrightarrow E_0, \quad \alpha > 0.
\]

Therefore, if \(\alpha\) satisfies (1.20), by Theorem 1.2, the initial data \((E_0, u_0)\) (satisfying the constraints (1.4)) with small norm in \(\left(\dot{B}^{N-1+\alpha}_{2,1} \cap \dot{B}^{N+\alpha}_{2,1}\right) \times \dot{B}^{N-1+\alpha}_{2,1}\) also can generate global solutions to the system (1.5)-(1.6).

The rest part of this paper is organized as follows. In Section 2, we introduce the tools (the Littlewood-Paley decomposition and paradifferential calculus) and give some nonlinear estimates in Besov spaces. The estimates for the system of acoustics, transport equation and heat equations are given in Section 3. Section 4 is devoted to a new property of the hyperbolic-parabolic system (1.12). In Section 5, we obtain the global \emph{a priori} estimates of system (1.7), or equivalently, of system (1.5)-(1.6). In Sections 6-7, we prove the local well-posedness result Theorem 1.1 and Corollary 1.1. The proof of Theorem 1.2 is given in Section 8.

Notation.

1. For \(f \in S'(\mathbb{R}^N), Q \in \mathbb{Z}\), denote \(f_q := \Delta_q f\), and

\[
P_{<Q}f := \sum_{q<Q} f_q, \quad P_{\geq Q}f := f - P_{<Q}f = \sum_{q \geq Q} f_q. \tag{1.31}
\]

In particular,

\[
f_L := \sum_{q<q_0} f_q, \quad \text{and} \quad f_H := \sum_{q \geq q_0} f_q, \quad \text{for some} \quad q_0 \in \mathbb{N}. \tag{1.32}
\]

2. Denote \(p^* := \frac{2p}{p-2}\), i.e., \(\frac{1}{p^*} = \frac{1}{2} - \frac{1}{p}\), for \(p \geq 2\).

3. Throughout the paper, \(C\) denotes various “harmless” positive constants, which may change line by line.

2. The functional toolbox. The results of the present paper rely on the use of a dyadic partition of unity with respect to the Fourier variables, the so-called the \emph{Littlewood-Paley decomposition}. Let us briefly explain how it may be built in the case \(x \in \mathbb{R}^N\), and the readers may see more details in [3, 7]. Let \((\chi, \varphi)\) be a couple of \(C^\infty\) functions satisfying

\[
\text{Supp}\chi \subset \left\{ |\xi| \leq \frac{4}{3} \right\}, \quad \text{Supp}\varphi \subset \left\{ \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \right\},
\]

and

\[
\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1,
\]

\[
\sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) = 1, \quad \text{for} \quad \xi \neq 0.
\]
Set $\varphi_q(\xi) = \varphi(2^{-q}\xi)$, $h_q = \mathcal{F}^{-1}(\varphi_q)$, and $\tilde{h} = \mathcal{F}^{-1}(\chi)$. The dyadic blocks and the low-frequency cutoff operators are defined for all $q \in \mathbb{Z}$ by

$$\dot{\Delta}_q u = \varphi(2^{-q}D)u = \int_{\mathbb{R}^N} h_q(y)u(x-y)dy,$$

$$\dot{S}_q u = \chi(2^{-q}D)u = \int_{\mathbb{R}^N} \tilde{h}_q(y)u(x-y)dy.$$

Then

$$u = \sum_{q \in \mathbb{Z}} \dot{\Delta}_q u,$$

(2.1)

holds for tempered distributions modulo polynomials. As working modulo polynomials is not appropriate for nonlinear problems, we shall restrict our attention to the set $\mathcal{S}_h'$ of tempered distributions $u$ such that

$$\lim_{q \to -\infty} \|\dot{S}_q u\|_{L^\infty} = 0.$$

Note that (2.1) holds true whenever $u$ is in $\mathcal{S}_h'$ and that one may write

$$\dot{S}_q u = \sum_{p \leq q-1} \dot{\Delta}_p u.$$

Besides, we would like to mention that the Littlewood-Paley decomposition has a nice property of quasi-orthogonality:

$$\dot{\Delta}_p \dot{\Delta}_q u \equiv 0 \text{ if } |p-q| \geq 2 \text{ and } \dot{\Delta}_p (\dot{S}_{q-1} u \dot{\Delta}_q u) \equiv 0 \text{ if } |p-q| \geq 5. \quad (2.2)$$

One can now give the definition of homogeneous Besov spaces.

**Definition 2.1.** For $s \in \mathbb{R}$, $(p,r) \in [1, \infty]^2$, and $u \in \mathcal{S}'(\mathbb{R}^N)$, we set

$$\|u\|_{\dot{B}_{p,r}^s} = \left\| 2^{sq}\|\dot{\Delta}_q u\|_{L^p} \right\|_{\ell^r}.$$

We then define the spaces $\dot{B}_{p,r}^s := \{ u \in \mathcal{S}_h'(\mathbb{R}^N), \|u\|_{\dot{B}_{p,r}^s} < \infty \}$.

**Remark 2.1.** The inhomogeneous Besov spaces can be defined in a similar way. Indeed, for $u \in \mathcal{S}'(\mathbb{R}^d)$, we set

$$\Delta_q u = 0 \text{ if } q < -1, \quad \Delta_{-1} u = \chi(D)u,$$

$$\Delta_q u = \varphi(2^{-q}D)u \text{ if } q \geq 0, \quad \text{and } \quad \dot{S}_q u = \sum_{p \leq q-1} \dot{\Delta}_p u.$$

Then for all $u \in \mathcal{S}'(\mathbb{R}^d)$, we have the inhomogeneous Littlewood-Paley decomposition $u = \sum_{q \in \mathbb{Z}} \Delta_q u$, and for $(p,r) \in [1, +\infty]^2$, $s \in \mathbb{R}$, we define the inhomogeneous Besov space $\dot{B}_{p,r}^s$ as

$$\dot{B}_{p,r}^s = \left\{ u \in \mathcal{S}'(\mathbb{R}^d), \|u\|_{\dot{B}_{p,r}^s} := \|2^{sq}\|\Delta_q u\|_{L^p} \|_{\ell^r} < \infty \right\}.$$

Homogeneous Besov spaces fail to have nice inclusion properties, however, if there is a cut-off in the frequency space, the situation is different. The following low and high frequency embeddings will be used frequently in this paper. The proof of which follows from the definition of Besov norm immediately.
Lemma 2.1. For all \( s \in \mathbb{R}, Q \in \mathbb{Z}, \delta > 0, 1 \leq p \leq \infty, \) there hold
\[
\|P_{<Q}u\|_{\dot{B}^s_{p,1}} \leq C2^{Q\delta}\|P_{<Q}u\|_{\dot{B}^{s-\delta}_{p,1}}, \tag{2.3}
\]
and
\[
\|P_{>Q}u\|_{\dot{B}^s_{p,1}} \leq C2^{-Q\delta}\|P_{>Q}u\|_{\dot{B}^{s+\delta}_{p,1}}. \tag{2.4}
\]

Next we recall a few nonlinear estimates in Besov spaces which may be obtained by means of paradifferential calculus. Firstly introduced by J. M. Bony in [4], the paraproduct between \( f \) and \( g \) is defined by
\[
\dot{T}f g = \sum_{q \in \mathbb{Z}} \dot{S}_{q-1} f \dot{\Delta} q g,
\]
and the remainder is given by
\[
\dot{R}(f,g) = \sum_{q \in \mathbb{Z}} \dot{\Delta} q f \dot{\Delta} q g
\]
with
\[
\dot{\Delta} q f = (\Delta_{q-1} + \Delta_q + \Delta_{q+1}) f.
\]
We have the following so-called Bony’s decomposition:
\[
fg = \dot{T}f g + \dot{T}g f + \dot{R}(f,g) = \dot{T}f g + \dot{T}g f, \tag{2.5}
\]
where \( \dot{T}f := \dot{T}g + \dot{R}(f,g) \). The paraproduct \( \dot{T} \) and the remainder \( \dot{R} \) operators satisfy the following continuous properties.

Proposition 2.1 ([3]). For all \( s \in \mathbb{R}, \sigma > 0, \) and \( 1 \leq p, p_1, p_2, r, r_1, r_2 \leq \infty, \) the paraproduct \( \dot{T} \) is a bilinear, continuous operator from \( L^\infty \times \dot{B}_p^{s,r} \rightarrow \dot{B}_p^{s,r} \) and from \( \dot{B}_p^{-s_1}_{r_1} \times \dot{B}_p^{-s_2}_{r_2} \rightarrow \dot{B}_p^{s_1+s_2}_{r_1+r_2} \) with \( \frac{1}{r} = \min\{1, \frac{1}{r_1} + \frac{1}{r_2}\} \). The remainder \( \dot{R} \) is bilinear continuous from \( \dot{B}_p^{s_1}_{r_1} \times \dot{B}_p^{s_2}_{r_2} \rightarrow \dot{B}_p^{s_1+s_2}_{r_1+r_2} \) with \( s_1 + s_2 > 0, \) \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1, \) and \( \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \leq 1. \)

In view of (2.5), Proposition 2.1 and Bernstein’s inequality, one easily deduces the following product estimates:

Corollary 2.1. Let \( p \in [1, \infty] \). If \( s_1, s_2 \leq \frac{N}{p} \) and \( s_1 + s_2 > N \max\{0, \frac{2}{p} - 1\} \), then there holds
\[
\|uv\|_{\dot{B}^{s_1+s_2}_{p,1} - \frac{N}{p}} \leq C\|u\|_{\dot{B}^{s_1}_{p,1}}\|v\|_{\dot{B}^{s_2}_{p,1}}. \tag{2.6}
\]

The following Proposition will be used to prove the uniqueness of solutions obtained in Theorem 1.1.

Proposition 2.2 ([15]). Let \( p \geq 2, s_1 \leq \frac{N}{p}, s_2 < \frac{N}{p}, \) and \( s_1 + s_2 \geq 0, \) then
\[
\|uv\|_{\dot{B}^{s_1+s_2}_{p,\infty} - \frac{N}{p}} \leq C\|u\|_{\dot{B}^{s_1}_{p,1}}\|v\|_{\dot{B}^{s_2}_{p,\infty}}.
\]

The study of non-stationary PDEs requires spaces of the type \( L^p_T(X) = L^p(0,T; X) \) for appropriate Banach spaces \( X \). In our case, we expect \( X \) to be a Besov space, so that it is natural to localize the equations through Littlewood-Paley decomposition. We then get estimates for each dyadic block and perform integration in time. But, in doing so, we obtain the bounds in spaces which are not of the type \( L^p(0,T; \dot{B}_p^{s,r}) \). That naturally leads to the following definition introduced by Chemin and Lerner in [8].
Definition 2.2. For $\rho \in [1, +\infty]$, $s \in \mathbb{R}$, and $T \in (0, +\infty)$, we set
\[
\|u\|_{\tilde{L}_T^p(B^s_{p,r})} = \left\|2^{2s} \|\Delta_q u(t)\|_{L^p_t(L^r)}\right\|_{t_r}
\]
and denote by $\tilde{L}_T^p(B^s_{p,r})$ the subset of distributions $u \in \mathcal{S}'([0,T] \times \mathbb{R}^N)$ with finite $\|u\|_{\tilde{L}_T^p(B^s_{p,r})}$ norm. When $T = +\infty$, the index $T$ is omitted. We further denote $\tilde{C}_T(B^s_{p,r}) = C([0,T]; B^s_{p,r}) \cap \tilde{L}^\infty_t(B^s_{p,r})$.

3. Preliminary. We begin this section by recalling the Strichartz estimate for the acoustic system, which is of great importance in this paper.

Proposition 3.1 ([13]). Let $(b, v)$ be a solution of the following system of acoustics:
\[
\begin{align*}
\begin{cases}
\partial_t b + \epsilon^{-1} \Lambda v &= f, \\
\partial_t v - \epsilon^{-1} \Lambda b &= g, \\
(b, v)|_{t=0} &= (b_0, v_0).
\end{cases}
\end{align*}
\]
Then, for any $s \in \mathbb{R}$ and $T \in (0, \infty)$, the following estimate holds:
\[
\begin{align*}
\| (b, v) \|_{\tilde{L}_T^p(B_{p,1}^s) \cap \tilde{L}^\infty_t(B_{p,1}^s)} &\leq C \epsilon^{\frac{1}{2}} \| (b_0, v_0) \|_{B_{p,1}^s} + \epsilon^{1 + \frac{q}{p} - \frac{N}{r}} \| (f, g) \|_{\tilde{L}_T^p(B_{p,1}^{s+\frac{N}{2}} \cap \tilde{L}^\infty_t(B_{p,1}^{s+\frac{N}{2}}))},
\end{align*}
\]
with
\[
p \geq 2, \frac{2}{r} \leq \min(1, \gamma(p)), (r, p, N) \neq (2, \infty, 3), \quad \tilde{p} \geq 2, \frac{\tilde{q}}{\tilde{r}} \leq \min(1, \gamma(\tilde{p})), (\tilde{r}, \tilde{p}, N) \neq (2, \infty, 3),
\]
where $\gamma(q) := (N-1)(\frac{1}{2} - \frac{1}{q}), \frac{1}{p} + \frac{1}{r} = 1$, and $\frac{1}{\tilde{p}} + \frac{1}{\tilde{r}} = 1$.

Next, we give the classical estimates in Besov space for the transport and heat equations. Please refer to, for example, [3] and [7] for the proofs.

Proposition 3.2. Let $\sigma \in (-N \min\{\frac{1}{p}, \frac{1}{r}\} - 1, 1 + \frac{N}{p})$ and $1 \leq p, r \leq +\infty$, or $\sigma = 1 + \frac{N}{p}$ if $r = 1$. Let $v$ be a solenoidal vector such that $\nabla v \in L_T^1(B_{p,r}^{\frac{N}{r}} \cap L^\infty)$, $f_0 \in B_{p,r}^\sigma$ and $g \in L_T^1(B_{p,r}^\sigma)$. There exists a constant $C$, such that for all solution $f \in L^\infty([0,T]; B_{p,r}^\sigma)$ of the equation
\[
\partial_t f + v \cdot \nabla f = g, \quad f|_{t=0} = f_0,
\]
we have the following a priori estimate
\[
\|f\|_{\tilde{L}_T^p(B_{p,r}^\sigma)} \leq e^{CV(T)} \left( \|f_0\|_{B_{p,r}^\sigma} + \int_0^T e^{-CV(t)} \|g(t)\|_{B_{p,r}^\sigma} \, dt \right),
\]
where $V(t) = \int_0^t \|\nabla v(\tau)\|_{B_{p,r}^{\frac{N}{r}} \cap L^\infty} \, d\tau$.

Proposition 3.3. Let $\sigma \in \mathbb{R}$ and $1 \leq \rho, p, r \leq +\infty$. Assume that $f_0 \in B_{p,r}^\sigma$, $g \in L_T^\rho(B_{p,r}^{\sigma-2+\frac{2}{r}})$, and $f$ solves
\[
\partial_t f - \nu \Delta f = g, \quad f|_{t=0} = f_0.
\]
Denote \( \rho_2 = (1 + 1/\rho_1 - 1/\rho)^{-1} \). Then there exists two positive constants \( c \) and \( C \) depending only on \( N \), such that for all \( \rho_1 \in [\rho, \infty] \), we have

\[
\| f \|_{L^q_t(B_{p,r}^s)} \leq C \left( \sum_{q \in \mathbb{Z}} 2^{q\sigma} \| \tilde{\Delta}_q f_0 \|_{L^p} \left( \frac{1 - e^{-c\nu T_2^2 \rho_1}}{c\nu \rho_1} \right)^{\frac{1}{q}} + \sum_{q \in \mathbb{Z}} 2^{q(\sigma - 2 + \frac{2}{p})} \| \tilde{\Delta}_q g \|_{L^p_t(\rho^r)} \left( \frac{1 - e^{-c\nu T_2^2 \rho_2}}{c\nu \rho_2} \right)^{\frac{1}{q}} \right). \tag{3.4}
\]

Moreover, there holds

\[
\nu \frac{1}{\rho_1} \| f \|_{L^q_t(B_{p,r}^s)} \leq C \left( \| f_0 \|_{B_{p,r}^s} + \nu^\frac{1}{\rho_1} \| g \|_{L^p_t(B_{p,r}^s)} \right). \tag{3.5}
\]

If \( r < \infty \), then \( f \) belongs to \( C_T(B_{p,r}^s) \).

4. A new property of the linear system (1.12). In this section, we focus on the following linear hyperbolic-parabolic system

\[
\begin{align*}
\partial_t a + (u \cdot \nabla) a + Ad &= f, \\
\partial_t d - \Delta d - \Lambda a &= g, \\
(a, d)(0, x) &= (a_0, d_0)(x). \tag{4.1}
\end{align*}
\]

It is worth pointing out that the convection term \( (u \cdot \nabla) a \) can not be incorporated into \( f \) since there is no smoothing effect in the high frequency part of \( a \). The purpose of this section is to establish the following Proposition:

**Proposition 4.1.** Let \( T > 0 \), \( \alpha \) and \( p \) satisfy (1.20)-(1.19), and \( (a, d) \) be a solution to the system (4.1). Then there exists a constant \( C \) depending on \( N \) and \( p \), such that

\[
\| (a, d) \|_{\mathcal{E}_\alpha^T} \leq C \| (a_0, d_0) \|_{\mathcal{E}_0} + C \| (f_L, g_L) \|_{L^p_t(\mathcal{B}_{p,1}^{\frac{N}{p}})} + C \| f_H \|_{L^p_t(\mathcal{B}_{p,1}^{\frac{N}{p}})} + C \| g_H \|_{L^p_t(\mathcal{B}_{p,1}^{\frac{N}{p}})} + C \| (a, u) \|_{\mathcal{E}_\alpha^T}^{\frac{N}{p}}. \tag{4.2}
\]

**Proof:** The proof will be divided into three steps.

**Step (I). Energy estimates in the high frequency part.** Following the ideas of Haspot [19, 20], we introduce the so called "effective velocity":

\[
w := d - \Lambda^{-1} a.
\]

Then it is easy to verify that

\[
\partial_tw - \Delta w = g - \Lambda^{-1} f + \Lambda^{-1} ((u \cdot \nabla) a) + w + \Lambda^{-1} a. \tag{4.3}
\]

Applying the operator \( P_{\geq Q} \), \( Q \in \mathbb{N} \) to be determined below, to (4.3), and then using Proposition 3.3, we are led to

\[
\| P_{\geq Q} w \|_{L^p_t(B_{p,1}^{\frac{N}{p}}) \cap L^p_t(\mathcal{B}_{p,1}^{\frac{N}{p} + 1})} \leq C \left( \| P_{\geq Q} w_0 \|_{B_{p,1}^{\frac{N}{p} - 1}} + \| P_{\geq Q} g \|_{L^p_t(\mathcal{B}_{p,1}^{\frac{N}{p} - 1})} + \| P_{\geq Q} f \|_{L^p_t(\mathcal{B}_{p,1}^{\frac{N}{p} - 2})} + \| P_{\geq Q} ((u \cdot \nabla) a) \|_{L^p_t(\mathcal{B}_{p,1}^{\frac{N}{p} - 2})} \right).
\]
By virtue of Lemma 2.1, one deduces that
\begin{align}
\left\| P_{\geq Q} w \right\|_{L^1_t(B_{p,1}^{\frac{N}{p}-1})} & \leq C2^{-2Q} \left\| P_{\geq Q} w \right\|_{L^1_t(B_{p,1}^{\frac{N}{p}+1})}, \\
\left\| P_{\geq Q} a \right\|_{L^1_t(B_{p,1}^{\frac{N}{p}-2})} & \leq C2^{-2Q} \left\| P_{\geq Q} a \right\|_{L^1_t(B_{p,1}^{\frac{N}{p}})}. \tag{4.5}
\end{align}

The convection term \((u \cdot \nabla) a\) can be estimated as follows:
\begin{align}
\left\| P_{\geq Q} (u \cdot \nabla) a \right\|_{L^1_t(B_{p,1}^{\frac{N}{p}-2})} & \leq \left\| P_{\geq Q} \left( \hat{T}_u \cdot \nabla a \right) \right\|_{L^1_t(B_{p,1}^{\frac{N}{p}-2})} + \left\| P_{\geq Q} \left( \hat{T}_{\nabla a} u \right) \right\|_{L^1_t(B_{p,1}^{\frac{N}{p}-2})} =: I_1 + I_2. \tag{4.6}
\end{align}

Using the high frequency embedding (2.4), Bernstein’s inequality and Proposition 2.1, we obtain
\begin{align}
I_1 & \leq C \left\| P_{\geq Q} \left( \hat{T}_u \cdot \nabla P_{\geq Q} a \right) \right\|_{L^1_t(B_{p,1}^{\frac{N}{p}+1})} + C \left\| P_{\geq Q} \left( \hat{T}_u \cdot \nabla P_{\geq Q} a \right) \right\|_{L^1_t(B_{p,1}^{\frac{N}{p}-2})} \\
& \leq C \|u\|_{L^\infty_{T}(B_{p,1}^{\frac{N}{p}-1})} \left( \|\nabla P_{\geq Q} a\|_{L^1_T(B_{p,1}^{\frac{N}{p}-1})} + \|\nabla P_{\geq Q} a\|_{L^1_T(B_{p,1}^{\frac{N}{p}-1-2\alpha})} \right) \\
& \leq C \|a\|_{L^\infty_{T}(B_{p,1}^{\frac{N}{p}-1})} \left( \|P_{\geq Q} a\|_{L^1_T(B_{p,1}^{\frac{N}{p}+1})} + \|P_{\geq Q} a\|_{L^1_T(B_{p,1}^{\frac{N}{p}+1})} \right), \tag{4.7}
\end{align}
and
\begin{align}
I_2 & \leq C \left\| P_{\geq Q} \left( \hat{T}_{\nabla a} P_{\geq Q} u \right) \right\|_{L^1_t(B_{p,1}^{\frac{N}{p}+1})} + C \left\| P_{\geq Q} \left( \hat{T}_{\nabla a} P_{\geq Q} u \right) \right\|_{L^1_t(B_{p,1}^{\frac{N}{p}})} \\
& \leq C \|\nabla a\|_{L^\infty_{T}(B_{p,1}^{\frac{N}{p}})} \left( \|P_{\geq Q} u\|_{L^1_T(B_{p,1}^{\frac{N}{p}+1})} + \|P_{\geq Q} u\|_{L^1_T(B_{p,1}^{\frac{N}{p}+1})} \right) \\
& \leq C \|a\|_{L^\infty_{T}(B_{p,1}^{\frac{N}{p}})} \left( \|P_{\geq Q} u\|_{L^1_T(B_{p,1}^{\frac{N}{p}+1})} + \|P_{\geq Q} u\|_{L^1_T(B_{p,1}^{\frac{N}{p}+1})} \right). \tag{4.8}
\end{align}

Substituting (4.5)-(4.8) into (4.4), and taking \(Q\) sufficiently large, we arrive at
\begin{align}
\left\| P_{\geq Q} w \right\|_{L^\infty_{T}(B_{p,1}^{\frac{N}{p}-1}) \cap L^1_{T}(B_{p,1}^{\frac{N}{p}+1})} & \leq C \left\| P_{\geq Q} w_0 \right\|_{L^1_{T}(B_{p,1}^{\frac{N}{p}-1})} + C \left\| P_{\geq Q} g \right\|_{L^1_{T}(B_{p,1}^{\frac{N}{p}-1})} + C \left\| P_{\geq Q} f \right\|_{L^1_{T}(B_{p,1}^{\frac{N}{p}-2})} \\
& + C \|u\|_{L^\infty_{T}(B_{p,1}^{\frac{N}{p}-1})} \left( \|P_{\geq Q} a\|_{L^1_T(B_{p,1}^{\frac{N}{p}+1})} + \|P_{\geq Q} a\|_{L^1_T(B_{p,1}^{\frac{N}{p}+1})} \right) \\
& + C \|a\|_{L^\infty_{T}(B_{p,1}^{\frac{N}{p}})} \left( \|P_{\geq Q} u\|_{L^1_T(B_{p,1}^{\frac{N}{p}+1})} + \|P_{\geq Q} u\|_{L^1_T(B_{p,1}^{\frac{N}{p}+1})} \right) \\
& + C2^{-2Q} \left\| P_{\geq Q} a \right\|_{L^1_t(B_{p,1}^{\frac{N}{p}})}, \tag{4.9}
\end{align}
where the term \(2^{-Q} \left\| P_{\geq Q} w \right\|_{L^1_{T}(B_{p,1}^{\frac{N}{p}+1})}\) appearing in (4.5) has been adsorbed by the left hand side. To close the estimate of \(w\), we turn to bound the high frequency
part of \( a \). In fact, by the definition of the effective velocity \( w \), the equation of \( a \) in (4.1) can be rewritten as

\[
\partial_t a + (u \cdot \nabla) a + a = f - \Lambda w.
\]

Applying the dyadic blocks \( \Delta_q \), \( q \geq Q \) to the above equation yields

\[
\partial_t a_q + \dot{S}_{q-1} u \cdot \nabla a_q + a_q = f_q - \Lambda w_q - \Delta_q \left( \dot{T}^e_{\nabla_a} u \right) + R_q,
\]

with \( R_q := \dot{S}_{q-1} u \cdot \nabla a_q - \Delta_q \left( \dot{T}^e_{\nabla_a} u \right) \). Arguing as in Theorem 3.14 in [3], we obtain for all \( t \in [0, T] \),

\[
\|a_q(t)\|_{L^p} + \|a_q\|_{L^1_t(L^p)} \leq \|a_q(0)\|_{L^p} + \frac{1}{p} \int_0^t \|\text{div} \dot{S}_{q-1} u\|_{L^\infty} \|a_q\|_{L^p} dt' + \int_0^t \|f_q - \Lambda w_q\|_{L^p} dt' + \int_0^t \|R_q\|_{L^p} dt'.
\]

Next, following the computations in [16], we have

\[
\|\nabla \dot{S}_{q-1} u\|_{L^\infty} \leq C(m_Q(u) + \|\nabla P_{\geq Q} u\|_{L^\infty}),
\]

where

\[
m_Q(u) := \min \left\{ 2^{q(2-2\alpha)} \|\nabla P_{< Q} u\|_{B^2_{\infty, \infty}}, \|\nabla P_{< Q} u\|_{L^\infty} \right\},
\]

and

\[
\|R_q\|_{L^p} \leq C(m_Q(u) + \|\nabla P_{\geq Q} u\|_{L^\infty}) \sum_{|q' - q| \leq 4} \|a_{q'}\|_{L^p}.
\]

Multiplying (4.11) by \( 2^q \), summing up with respect to \( q \) over \( \{Q, Q + 1, Q + 2, \cdots\} \), and using (4.12)-(4.13), we find that

\[
\|P_{\geq Q} a\|_{L^\infty_t(B^N_{\frac{\infty}{p}, 1}(B^N_{\frac{\infty}{p}, 1}))} \leq \|P_{\geq Q} a_0\|_{B^N_{\frac{\infty}{p}, 1}} + C\|P_{\geq Q} f\|_{L^1_t(B^N_{\frac{\infty}{p}, 1})} + C\|P_{\geq Q} u\|_{L^1_t(B^N_{\frac{\infty}{p+1}, 1})} + C \int_0^T \sum_{q \geq Q} 2^q \left( m_Q(u) + \|\nabla P_{\geq Q} u\|_{L^\infty} \right) \sum_{|q' - q| \leq 4} \|a_{q'}\|_{L^p} dt' + C\left\|P_{\geq Q} \left( \dot{T}^e_{\nabla_a} u \right) \right\|_{L^1_t(B^N_{\frac{\infty}{p+1}, 1})}.
\]

Clearly,

\[
\int_0^T \sum_{q \geq Q} 2^q \|\nabla P_{\geq Q} u\|_{L^\infty} \sum_{|q' - q| \leq 4} \|a_{q'}\|_{L^p} dt' \leq C\|\nabla P_{\geq Q} u\|_{L^1_t(B^\infty_{\frac{\infty}{p}, 1})} \|a\|_{L^\infty_t(B^N_{\frac{\infty}{p}, 1})},
\]

and

\[
\int_0^T \sum_{q \geq Q} 2^q \left( m_Q(u) \right) \sum_{|q' - q| \leq 4} \|a_{q'}\|_{L^p} dt'.
\]
Moreover, \( \|P_2Q(T_v^0u)\|_{L^p_t(B_{p,1}^{\infty})} \) can be bounded in the same way as \( I_2 \). Then, substituting (4.8), (4.15)-(4.16) into (4.14), one deduces that

\[
\|P_2Qa\|_{L^p_T(B_{p,1}^{\infty})} \leq \|P_2Qa_0\|_{L^p_t(B_{p,1}^{\infty})} + C\|P_2Qf\|_{L^p_t(B_{p,1}^{\infty})} + C\|P_2Qa\|_{L^p_T(B_{p,1}^{\infty})} + C\|P_2Qu\|_{L^p_T(B_{p,1}^{\infty})}
\]

\[
+C\|P_2Qa\|_{L^p_T(B_{p,1}^{\infty})} \leq C\left(\|P_2Qa_0\|_{L^p_t(B_{p,1}^{\infty})} + \|P_2Qa_0\|_{L^p_t(B_{p,1}^{\infty})} + \|P_2Qf\|_{L^p_t(B_{p,1}^{\infty})} + \|P_2Qa\|_{L^p_T(B_{p,1}^{\infty})}\right)
\]

\[
+C\|P_2Qa\|_{L^p_T(B_{p,1}^{\infty})} + C\|P_2Qa\|_{L^p_T(B_{p,1}^{\infty})} + C\|P_2Qa\|_{L^p_T(B_{p,1}^{\infty})} + C\|P_2Qa\|_{L^p_T(B_{p,1}^{\infty})}
\]

From now on, we take \( Q = q_0 \), and exploit the notation in (1.32). Thanks to the decomposition \( u = u_L + u_H \) and the interpolation, we have

\[
\|u\|_{L^p_t(B_{p,1}^{\infty})} \leq \|u_L\|_{L^p_t(B_{p,1}^{\infty})} + C\|u_H\|_{L^p_t(B_{p,1}^{\infty})}^{1-\alpha} \|u_H\|_{L^p_t(B_{p,1}^{\infty})}^\alpha.
\]

Similarly, using the low frequency embedding (2.3), we find that

\[
\|a\|_{L^p_t(B_{p,1}^{\infty})} \leq C\|a_L\|_{L^p_t(B_{p,1}^{\infty})} + C\|a_H\|_{L^p_t(B_{p,1}^{\infty})} + C\|a_0\|_{L^p_t(B_{p,1}^{\infty})}
\]
Moreover, interpolating between $L_T^\infty(\dot B_{2,1}^{\frac{\alpha}{2}+1})$ and $L_T^1(\dot B_{2,1}^{\frac{\alpha}{2}+1})$ yields
\begin{equation}
\|a_L\|_{L_T^\infty(\dot B_{2,1}^{\frac{\alpha}{2}+1})} \leq C\|a_L\|_{L_T^\infty(\dot B_{2,1}^{\frac{\alpha}{2}+1})}^{1-\alpha}\|a_L\|_{L_T^1(\dot B_{2,1}^{\frac{\alpha}{2}+1})}^{\alpha},
\end{equation}
(4.21) and
\begin{equation}
\|u_L\|_{L_T^\infty(\dot B_{2,1}^{\frac{\alpha}{2}+1})} \leq C\|u_L\|_{L_T^\infty(\dot B_{2,1}^{\frac{\alpha}{2}+1})}^{\frac{2-\alpha}{\alpha}}\|u_L\|_{L_T^1(\dot B_{2,1}^{\frac{\alpha}{2}+1})}.
\end{equation}
(4.22)

Noticing that $d = w + \Lambda^{-1}a$, using the high frequency embedding again, we infer from (4.18)-(4.22) that
\begin{align}
\|a_H\|_{\dot B_{p,1}^{\frac{\alpha}{2}}(\dot B_{p,1}^{\frac{\alpha}{2}+1})} + \|d_H\|_{\dot B_{p,1}^{\frac{\alpha}{2}}(\dot B_{p,1}^{\frac{\alpha}{2}+1})} \\
\leq C\left(\|a_0H\|_{\dot B_{p,1}^{\frac{\alpha}{2}}(\dot B_{p,1}^{\frac{\alpha}{2}+1})} + \|d_0H\|_{\dot B_{p,1}^{\frac{\alpha}{2}}(\dot B_{p,1}^{\frac{\alpha}{2}+1})} + \|f_H\|_{L_T^1(\dot B_{p,1}^{\frac{\alpha}{2}})}
\right) \\
+ \|g_H\|_{L_T^1(\dot B_{p,1}^{\frac{\alpha}{2}+1})} + C\|(a, u)\|_{\dot E_T^F(B_{p,1}^1)}. (4.23)
\end{align}

**Step (II). Energy estimates in the low frequency part.** According to the energy estimates for the hyperbolic-parabolic system (4.1) obtained by Danchin [12], we get
\begin{align}
\|(a_L, d_L)\|_{L_T^\infty(\dot B_{2,1}^{\frac{\alpha}{2}+1})} \\
\leq C\|(a_0L, d_0L)\|_{\dot B_{2,1}^{\frac{\alpha}{2}+1}} + C\|f - (u \cdot \nabla)a, g\|_{L_T^1(\dot B_{2,1}^{\frac{\alpha}{2}+1})},
\end{align}
(4.24)

In view of Bony’s decomposition, there holds
\begin{align}
\|((u \cdot \nabla)a)_L\|_{L_T^1(\dot B_{2,1}^{\frac{\alpha}{2}+1})} \\
\leq \left\|\hat{T}_u \cdot \nabla a\right\|_{L_T^1(\dot B_{2,1}^{\frac{\alpha}{2}+1})} + \left\|\hat{T}_u \cdot \nabla a\right\|_{L_T^1(\dot B_{2,1}^{\frac{\alpha}{2}+1})} =: J_1 + J_2.
\end{align}
(4.25)

Using the low frequency embedding (2.3), Bernstein’s inequality and Proposition 2.1, we find that
\begin{align}
J_1 &\leq \left\|\hat{T}_u \cdot \nabla a_L\right\|_{L_T^1(\dot B_{2,1}^{\frac{\alpha}{2}+1})} + C2^\theta \left\|\hat{T}_u L \cdot \nabla a_H\right\|_{L_T^1(\dot B_{2,1}^{\frac{\alpha}{2}+1})} \\
&+ C2^\theta \left\|\hat{T}_u L \cdot \nabla a_H\right\|_{L_T^1(\dot B_{2,1}^{\frac{\alpha}{2}+1})} \\
&\leq C\|u\|_{L_T^\infty(\dot B_{2,1}^{\frac{\alpha}{2}+1})} \|\nabla a_L\|_{L_T^1(\dot B_{2,1}^{\frac{\alpha}{2}+1})} \\
&+ C2^\theta \left\|\hat{T}_u L \cdot \nabla a_H\right\|_{L_T^1(\dot B_{2,1}^{\frac{\alpha}{2}+1})} \\
&\leq C\|u\|_{L_T^\infty(\dot B_{2,1}^{\frac{\alpha}{2}+1})} \|a_L\|_{L_T^1(\dot B_{2,1}^{\frac{\alpha}{2}+1})} \\
&+ C2^\theta \left\|\hat{T}_u L \cdot \nabla a_H\right\|_{L_T^1(\dot B_{2,1}^{\frac{\alpha}{2}+1})} \\
&\leq C\|u\|_{L_T^\infty(\dot B_{2,1}^{\frac{\alpha}{2}+1})} \|a_L\|_{L_T^1(\dot B_{2,1}^{\frac{\alpha}{2}+1})} \\
&+ C2^\theta \left\|\hat{T}_u L \cdot \nabla a_H\right\|_{L_T^1(\dot B_{2,1}^{\frac{\alpha}{2}+1})} + C2^\theta \left\|\hat{T}_u L \cdot \nabla a_H\right\|_{L_T^1(\dot B_{2,1}^{\frac{\alpha}{2}+1})}.
\end{align}
(4.26)
\[ J_2 \leq \left\| (T_{V^a} a_{UL}) \right\|_{L_1^1(B_{z_1}^{\frac{N}{2}-1+\alpha})} + \left\| (T_{V^a} a_{UL}) \right\|_{L_1^1(B_{z_1}^{\frac{N}{2}-1+\alpha})} + \left\| (T_{V^a} u_H) \right\|_{L_1^1(B_{z_1}^{\frac{N}{2}-1+\alpha})} \]

\[ \leq C\left\| \nabla a \right\|_{L_1^L(B_{z_1}^{\frac{N}{2}-2+\alpha})} + \left\| \nabla a_H \right\|_{L_1^L(B_{z_1}^{\frac{N}{2}-2+\alpha})} \]

\[ + C\left( \left\| \nabla a_L \right\|_{L_1^L(B_{z_1}^{\frac{N}{2}-1+\alpha})} + \left\| \nabla a_H \right\|_{L_1^L(B_{z_1}^{\frac{N}{2}-1+\alpha})} \right) \left\| u_H \right\|_{L_1^L(B_{z_1}^{\frac{N}{2}+1})} \]

\[ \leq C\left( \left\| a \right\|_{L_1^L(B_{z_1}^{\frac{N}{2}-1+\alpha})} + \left\| a_H \right\|_{L_1^L(B_{z_1}^{\frac{N}{2}+1})} \right) \left\| u_H \right\|_{L_1^L(B_{z_1}^{\frac{N}{2}+1})} \]

(4.27)

Owing to the decomposition \( a = a_L + a_H \) and the high frequency embedding, we have

\[ \left\| a \right\|_{L_1^L(B_{z_1}^{\frac{N}{2}-1+\alpha})} \leq \left\| a_L \right\|_{L_1^L(B_{z_1}^{\frac{N}{2}-1+\alpha})} + C\left\| a_H \right\|_{L_1^L(B_{z_1}^{\frac{N}{2}+1})} \]

(4.28)

From (4.24)-(4.28), (4.19) and (4.21) with \( a_L \) replaced by \( u_L \), we infer that

\[ \left\| (a_L, d_L) \right\|_{L_1^L(B_{z_1}^{\frac{N}{2}-1+\alpha})} \leq C\left( \left\| a_L \right\|_{L_1^L(B_{z_1}^{\frac{N}{2}-1+\alpha})} + \left\| f \right\|_{L_1^L(B_{z_1}^{\frac{N}{2}+1})} + \left\| g \right\|_{L_1^L(B_{z_1}^{\frac{N}{2}+1})} \right) \]

(4.29)

**Step (III).** Dispersive estimates in the low frequency part. Applying Proposition 3.1 to the system (4.1) with \( s = \frac{N}{2} - 1 + \alpha, \; p = \infty, \; r = \frac{1}{\alpha}, \; \bar{p} = 2, \; \bar{r} = \infty, \; \epsilon = 1 \), we obtain

\[ \left\| (a_L, d_L) \right\|_{L_1^L(B_{z_1}^{\frac{N}{2}-1+\alpha})} \leq C\left( \left\| a_L \right\|_{L_1^L(B_{z_1}^{\frac{N}{2}-1+\alpha})} + \left\| f \right\|_{L_1^L(B_{z_1}^{\frac{N}{2}+1})} + \left\| g \right\|_{L_1^L(B_{z_1}^{\frac{N}{2}+1})} \right) \]

(4.30)

By (4.29) and (4.25)-(4.27), one easily deduces that

\[ \left\| (a_L, d_L) \right\|_{L_1^L(B_{z_1}^{\frac{N}{2}-1+\alpha})} \leq C\left( \left\| a_L \right\|_{L_1^L(B_{z_1}^{\frac{N}{2}-1+\alpha})} + \left\| f \right\|_{L_1^L(B_{z_1}^{\frac{N}{2}+1})} + \left\| g \right\|_{L_1^L(B_{z_1}^{\frac{N}{2}+1})} \right) \]

(4.31)

It follows from (4.23), (4.29) and (4.31) that (4.2) holds. This completes the proof of Proposition 4.1.

**Remark 4.1.** In Proposition 4.1, we do not impose the assumption that \( \text{div} u = 0 \), so the result in Proposition 4.1 can be used for compressible fluids, such as the isentropic compressible Navier-Stokes equations.
5. A priori estimates. The purpose of this section is to give the global estimates of solutions \((E, u)\) to the system (1.5)-(1.6).

**Proposition 5.1.** Let \(T > 0, p\) and \(\alpha\) satisfy (1.19)-(1.20). Assume that \((E, u)\) is a solution to system (1.5)-(1.6) in \(\mathcal{E}_N^T(\tau)\). Then there exists a constant \(C_1\) depending on \(N\) and \(p\), such that

\[
\|\|(E, u)\|_{\mathcal{E}_N^T(\tau)} \leq C_1\|(E_0, u_0)\|_{\mathcal{E}_N^T(\tau)} + C_1\|\|(E, u)\|_{\mathcal{E}_N^T(\tau)}^{\frac{2}{N}}.
\]

(5.1)

To prove Proposition 5.1, it suffices to apply Proposition 4.1 to the system (1.7) with \(a = E^j, d = d^j,\) and

\[
f := \partial_k u E^{kj},
g := L^{-1}\partial_k(E^{ij}\partial_l E^{ik} - E^{lk}\partial_i E^{ij})
+ (\partial^m + \partial_i(\nabla)^{-1} \partial_m) L^{-1}\partial_i ((u \cdot \nabla) u^m - E^{lk}\partial_l E^{mk}).
\]

The nonlinear estimates for \(f\) and \(g\) will be given in the following three lemmas.

**Lemma 5.1.** Let \(T > 0, p\) and \(\alpha\) satisfy (1.19)-(1.20). Assume that \((E, u) \in \mathcal{E}_N^T(\tau)\) satisfying

\[
\text{div} u = \text{div}^T = 0,
\]

then we have

\[
\|\|\partial_k u E^{kj}\|_{L^1_t(B_{x_1}^{\infty \alpha})} + \|\|\partial_k u E^{kj}\|_{L^1_t(B_{x_1}^{\infty \alpha})} \leq C\|(E, u)\|_{\mathcal{E}_N^T(\tau)}^{\frac{2}{N}}.
\]

(5.2)

**Proof.** The condition \(\text{div} E^T = 0\) implies that \(\partial_k u E^{kj} = \partial_k(u^k E^j)\). Then by the definition of paraproduct, Bernstein’s inequalities and Proposition 2.1, we have

\[
\|\||u E^{kj}\|_{L^1_t(B_{x_1}^{\infty \alpha})}
\leq C\|u\|_{L^1_t(B_{x_1}^{\infty \alpha})} + C\|\|\partial_k u E^{kj}\|_{L^1_t(B_{x_1}^{\infty \alpha})} + C\|\|\partial_k u E^{kj}\|_{L^1_t(B_{x_1}^{\infty \alpha})},
\]

and

\[
\|\|u E^{kj}\|_{L^1_t(B_{x_1}^{\infty \alpha})}
\leq C\|\|\partial_k u E^{kj}\|_{L^1_t(B_{x_1}^{\infty \alpha})} + C\|\|\partial_k u E^{kj}\|_{L^1_t(B_{x_1}^{\infty \alpha})} + C\|\|\partial_k u E^{kj}\|_{L^1_t(B_{x_1}^{\infty \alpha})}.
\]

(5.3)

Moreover, using the low frequency embedding (2.3), we are led to

\[
\|\|u E^{kj}\|_{L^1_t(B_{x_1}^{\infty \alpha})}
\leq C\|\|\partial_k u E^{kj}\|_{L^1_t(B_{x_1}^{\infty \alpha})} + C\|\|\partial_k u E^{kj}\|_{L^1_t(B_{x_1}^{\infty \alpha})} + C\|\|\partial_k u E^{kj}\|_{L^1_t(B_{x_1}^{\infty \alpha})}.
\]

(5.4)
Under the conditions of Lemma 5.1, we have
\[ \left\| \left( u^H_E \right)_H \right\| \leq C \left\| \left( \tilde{T} u^H_E \right)_H \right\| + C \left\| \left( \tilde{T} E^k \right)_H \right\|, \]
and
\[ \left\| u_L \right\| \leq C \left\| u_L \right\| + C \left\| u_H \right\|. \]

Next, to bound \( \left\| (\partial_t u^j E^k) H \right\| \), we take some precautions. Indeed, according to the low frequency embedding (2.3) and the interpolations, we find that
\[ \left\| u_L \right\| \leq C \left\| u_L \right\| + C \left\| u_H \right\|. \]

Then using the high frequency embedding (2.4) and Corollary 2.1, we arrive at
\[ \left\| (\partial_t u^j E^k)_H \right\| \leq C \left\| (\partial_t u^j E^k)_H \right\|, \]
and
\[ \left\| E \right\| \leq C \left\| E \right\|. \]

Collecting the estimates above, the proof of Lemma 5.1 is completed.
Similarly, we have

\[
\| \left( E_H^{i,j} E_L^{l,k} \right)_L \|_{L^1_{1/2}(B_{2/1}^{N-3})} \leq C\| \hat{T}_{E_H} E_L \|_{L^1_{1/2}(B_{2/1}^{N+3})} + C\| \hat{T}'_{E_L} E_H \|_{L^1_{1/2}(B_{2/1}^{N-1})} \\
\leq C\| E_H \|_{L^2_{1/2}(B_{1/1}^{N+3})} \| E_L \|_{L^2_{1/2}(B_{1/1}^{N+3})} + C\| E_L \|_{L^2_{1/2}(B_{1/1}^{N-1})} \| E_H \|_{L^2_{1/2}(B_{1/1}^{N})} \\
\leq C\| E_H \|_{L^2_{1/2}(B_{1/1}^{N+3})} \| E_L \|_{L^2_{1/2}(B_{1/1}^{N+3})} + C\| E_L \|_{L^2_{1/2}(B_{1/1}^{N-1})} \| E_H \|_{L^2_{1/2}(B_{1/1}^{N})} \\
+ C\| E_L \|_{L^2_{1/2}(B_{1/1}^{N-1})} \| E_H \|_{L^2_{1/2}(B_{1/1}^{N})},
\]

(5.11)

and

\[
\| \left( E_H^{i,j} E_L^{l,k} \right)_H \|_{L^1_{1/2}(B_{1/1}^{N+3})} \leq C\| \hat{T}_{E_H} E_L \|_{H^1_{1/2}(B_{1/1}^{N+3})} + C\| \hat{T}'_{E_L} E_H \|_{H^1_{1/2}(B_{1/1}^{N-1})} \\
\leq C\| E_H \|_{H^2_{1/2}(B_{1/1}^{N+3})} \| E_L \|_{H^2_{1/2}(B_{1/1}^{N+3})} + C\| E_L \|_{H^2_{1/2}(B_{1/1}^{N-1})} \| E_H \|_{H^2_{1/2}(B_{1/1}^{N})} \\
\leq C\| E_H \|_{H^2_{1/2}(B_{1/1}^{N+3})} \| E_L \|_{H^2_{1/2}(B_{1/1}^{N+3})} + C\| E_L \|_{H^2_{1/2}(B_{1/1}^{N-1})} \| E_H \|_{H^2_{1/2}(B_{1/1}^{N})} \\
+ C\| E_H \|_{L^2_{1/2}(B_{1/1}^{N+3})} \| E_L \|_{L^2_{1/2}(B_{1/1}^{N+3})},
\]

(5.12)

Finally, it is not difficult to verify that

\[
\| \left( E_H^{i,j} E_L^{l,k} \right)_L \|_{L^1_{1/2}(B_{1/1}^{N+3})} \leq C\| E_H \|_{L^1_{1/2}(B_{1/1}^{N+3})} \| E_L \|_{L^1_{1/2}(B_{1/1}^{N+3})} + C\| \hat{R}(E_H, E_H) \|_{L^1_{1/2}(B_{1/1}^{N+3})} \\
\leq C\| E_H \|_{L^2_{1/2}(B_{1/1}^{N+3})} \| E_H \|_{L^2_{1/2}(B_{1/1}^{N+3})} + C\| E_L \|_{L^2_{1/2}(B_{1/1}^{N+3})} \| E_H \|_{L^2_{1/2}(B_{1/1}^{N+3})} \\
\leq C\| E_H \|_{L^2_{1/2}(B_{1/1}^{N+3})} \| E_H \|_{L^2_{1/2}(B_{1/1}^{N+3})} + C\| E_L \|_{L^2_{1/2}(B_{1/1}^{N+3})} \| E_H \|_{L^2_{1/2}(B_{1/1}^{N+3})},
\]

(5.13)

and

\[
\| \left( E_H^{i,j} E_L^{l,k} \right)_H \|_{L^1_{1/2}(B_{1/1}^{N+3})} \leq C\| E_H \|_{L^1_{1/2}(B_{1/1}^{N+3})} \| E_H \|_{L^1_{1/2}(B_{1/1}^{N+3})}.
\]

(5.14)

This completes the proof of Lemma 5.2.

\[ \square \]

**Lemma 5.3.** Under the conditions of Lemma 5.1, we have

\[
\| ((u \cdot \nabla)u)_L \|_{L^1_{1/2}(B_{1/1}^{N-1})} + \| ((u \cdot \nabla)u)_H \|_{L^1_{1/2}(B_{1/1}^{N-1})} \leq C\| (E, u) \|_{H^2_{1/2}(T)}^{\frac{N}{2}},
\]

(5.15)

**Proof.** Similar to (5.10), we easily have

\[
\| ((u_L \cdot \nabla)u_L)_L \|_{L^1_{1/2}(B_{1/1}^{N-1})} + \| ((u_L \cdot \nabla)u_L)_H \|_{L^1_{1/2}(B_{1/1}^{N-1})} \leq C\| u_L \|_{L^2_{1/2}(B_{2/1}^{N-3})} \| u_L \|_{L^2_{1/2}(B_{2/1}^{N-3})},
\]

(5.16)
6. Local well-posedness. In this section, we prove Theorem 1.1. We will proceed in two steps. First we prove the existence of the solution. The second part is devoted to the proof of uniqueness of the solution.

6.1. Existence of a local solution. First of all, we give a local version of Proposition 5.1.

**Proposition 6.1.** Let $T > 0$, $p \in [1, \infty)$. Assume that $(E, u)$ is a solution to (1.5) with $\text{div}E_0^T = \text{div}u_0 = 0$. Denote
\[
A(T) := \|E\|_{L_T^{q+2}(B_{p+1}^1)} + \|u_L\|_{L_T^{q+2}(B_{p+1}^1) \cap L_T^{q+2}(B_{p+1}^1)} + \|u_H\|_{L_T^{q+2}(B_{p+1}^1)} ,
\]
\[
A_0 := \|E_0\|_{B_{p+1}^1} + \|u_0L\|_{B_{p+1}^1} + \|u_0H\|_{B_{p+1}^1} ,
\]
and
\[
B(T) := \|u_H\|_{L_T^{q+1}(B_{p+1}^1)} + \|u_H\|_{L_T^{q+1}(B_{p+1}^1)} .
\]
Then there exists a positive constant $C_2$ depending on $p$ and $N$, such that
\[
A(T) \leq C_2 \left( A_0 + T(1 + A(T))A(T) + (1 + A(T))B(T) \right) ,
\]
and
\[
B(T) \leq C_2 \left( \sum_{q \in \mathbb{Z}} 2^q \|\hat{\Delta}_q u_0H\|_{L^p} \left( 1 - e^{-c2^qT} \right) \right) \right)^{\frac{1}{2}} + T \left( 1 + A(T) \right) A(T) + B^2(T) .
\]
Proof. First of all, from (1.5), similar to (4.14), we have
\[
\|E\|_{L^\infty_p(B_{p,1}^q)} \\
\leq \|E_0\|_{B_{p,1}^q} + C \|u\|_{L^1_p(B_{p,1}^{q+1})} + C \|\nabla u E\|_{L^1_p(B_{p,1}^q)} \\
+ C \|T_{\nabla E} u\|_{L^1_p(B_{p,1}^q)} + C \int_0^T \sum_{q \in \mathbb{Z}} 2^q \|\nabla u\|_{L^{\infty_q}} \sum_{|q'-q| \leq 4} \|E_{q'}\|_{L^p} dt'.
\]

(6.3)

In view of Proposition 2.1 and Corollary 2.1, one easily deduces that
\[
\|\nabla u E\|_{L^1_p(B_{p,1}^q)} + \|T_{\nabla E} u\|_{L^1_p(B_{p,1}^q)} \leq C \|u\|_{L^1_p(B_{p,1}^{q+1})} \|E\|_{L^\infty_p(B_{p,1}^q)}.
\]

(6.4)

Obviously,
\[
\int_0^T \sum_{q \in \mathbb{Z}} 2^q \|\nabla u\|_{L^{\infty_q}} \sum_{|q'-q| \leq 4} \|E_{q'}\|_{L^p} dt' \leq C \|u\|_{L^1_p(B_{p,1}^{q+1})} \|E\|_{L^\infty_p(B_{p,1}^q)}.
\]

(6.5)

According to the low frequency embedding (2.3), we find that
\[
\|u\|_{L^1_p(B_{p,1}^{q+1})} \leq \|u_{L}\|_{L^1_p(B_{p,1}^{q+1})} + \|u_{H}\|_{L^1_p(B_{p,1}^{q+1})} \\
\leq CT \|u_{L}\|_{L^\infty_p(B_{p,1}^q)} + \|u_{H}\|_{L^1_p(B_{p,1}^{q+1})}.
\]

(6.6)

Substituting (6.4)-(6.6) into (6.3), we obtain
\[
\|E\|_{L^\infty_p(B_{p,1}^q)} \leq \|E_0\|_{B_{p,1}^q} + C \left( T \|u_{L}\|_{L^\infty_p(B_{p,1}^q)} + \|u_{H}\|_{L^1_p(B_{p,1}^{q+1})} \right) \\
\times \left( 1 + \|E\|_{L^\infty_p(B_{p,1}^q)} \right).
\]

(6.7)

To bound $u$, in view of $\text{div} u = \text{div} E^T = 0$, applying the Leray operator $\mathbb{P} := \mathbb{I} + \nabla (-\Delta)^{-1} \text{div}$ to the equation of $u$ yields
\[
\partial_t u - \Delta u = \text{div} E + \mathbb{P} \text{div} \left( -u \otimes u + EE^T \right).
\]

(6.8)

Then using Proposition 3.3, we find that
\[
\|u_{L}\|_{L^\infty_p(B_{p,1}^{q+2})} + \|u_{H}\|_{L^1_p(B_{p,1}^{q+1})} \\
\leq C \left( \|u_{0L}\|_{B_{p,1}^q} + \|u_{0H}\|_{B_{p,1}^{q+1}} + \|E_{L}\|_{L^1_p(B_{p,1}^{q+1})} + \|E_{H}\|_{L^1_p(B_{p,1}^q)} \right) \\
+ C \left( \|(-u \otimes u + EE^T)_L\|_{L^1_p(B_{p,1}^{q+1})} \right) \\
+ \|(-u \otimes u + EE^T)_H\|_{L^1_p(B_{p,1}^q)}.
\]

(6.9)
and

\[ \| u_H \|_{L^1_T(B_{p,1}^{\frac{N}{p}+1})} + \| u_H \|_{L^2_T(B_{p,1}^{\frac{N}{p}})} \leq C \left( \sum_{q \in \mathbb{Z}} 2^{q \frac{N}{p}} \| \Delta_q u_{0H} \|_{L^p} (1 - e^{-c2^{2q}T})^{\frac{1}{2}} + \| E_H \|_{L^1_T(B_{p,1}^{\frac{N}{p}})} \right) + \| (u \otimes u + EE^T) \|_{L^2_T(B_{p,1}^{\frac{N}{p}})}. \]  

(6.10)

From the low frequency embedding (2.3), Corollary 2.1, and Hölder’s in equality, we have

\[ \| (u \otimes u + EE^T) \|_{L^1_T(B_{p,1}^{\frac{N}{p}+1})} + \| (u \otimes u + EE^T) \|_{L^1_T(B_{p,1}^{\frac{N}{p}})} \leq \| u \|_{L^p}^2 \| E \|_{L^p}^2 \leq C \left( \| u \|_{L^p}^2 \| E \|_{L^p}^2 + \| u_L \|_{L^p}^2 \right) + C \| u_H \|_{L^p}^2. \]  

(6.11)

From (6.9) and (6.11), one deduces that

\[ \| u_L \|_{L^p_T(B_{p,1}^{\frac{N}{p}+2})} + \| u_L \|_{L^p_T(B_{p,1}^{\frac{N}{p}+1})} \leq C \left( \| u_{0L} \|_{L^p(B_{p,1}^{\frac{N}{p}})} + \| u_{0H} \|_{L^p(B_{p,1}^{\frac{N}{p}-1})} \right) + CT \| E \|_{L^p_T(B_{p,1}^{\frac{N}{p}})} + \| u_L \|_{L^p_T(B_{p,1}^{\frac{N}{p}})} \]  

(6.12)

Similarly, (6.10) reduces to

\[ \| u_H \|_{L^1_T(B_{p,1}^{\frac{N}{p}+1})} + \| u_H \|_{L^2_T(B_{p,1}^{\frac{N}{p}})} \leq C \left( \sum_{q \in \mathbb{Z}} 2^{q \frac{N}{p}} \| \Delta_q u_{0H} \|_{L^p} (1 - e^{-c2^{2q}T})^{\frac{1}{2}} + \| E \|_{L^1_T(B_{p,1}^{\frac{N}{p}})} \right) + CT \| u_L \|_{L^p_T(B_{p,1}^{\frac{N}{p}})} + \| u_L \|_{L^p_T(B_{p,1}^{\frac{N}{p}})} + C \| u_H \|_{L^2_T(B_{p,1}^{\frac{N}{p}})}. \]  

(6.13)

It follows from (6.7) and (6.12) that (6.1) holds. Clearly, (6.2) is a consequence of (6.13). This completes the proof Proposition 6.1.

Next, we recall the following local well-posedness theorem.

**Theorem 6.1** ([34]). Let \( s > \frac{N}{2} \) be any real number, \( (E_0, u_0) \in H^s(\mathbb{R}^N) \) with \( \text{div} E_0^0 = \text{div} u_0 = 0 \). Then there exists a positive time \( T^* \) such that (1.5) has a unique classical solution \((E, u)\) on \([0, T^*)\), with \((E, u) \in C_{T}(H^s(\mathbb{R}^N))\) and \( \nabla u \in \)
by virtue of Propositions 3.2 and 3.3, it is not difficult to verify that

\[ \int_0^T \| \nabla u(t) \|_{L^\infty} \, dt = \infty. \]  

(6.14)

**Step (I). Approximate solutions.**

In order to apply Theorem 6.1, we first construct approximate initial data. As a matter of fact, thanks to Lemma 4.2 in [1], we can choose \((E_0^n, u_0^n) \in H^\infty(\mathbb{R}^N)\) such that

\[
\begin{aligned}
E_0^n &\to E_0 \quad \text{in} \quad \dot{B}_{p,1}^{\infty}, \quad \text{as} \quad n \to \infty, \\
u_{0L}^n \to u_{0L} &\quad \text{in} \quad \dot{B}_{p,1}^{\infty}, \quad \text{as} \quad n \to \infty, \\
u_{0H}^n \to u_{0H} &\quad \text{in} \quad \dot{B}_{p,1}^{-1}, \quad \text{as} \quad n \to \infty, \\
\| E_0^n \|_{\dot{B}_{p,1}^{\infty}} + \| u_{0L}^n \|_{\dot{B}_{p,1}^{\infty}} + \| u_{0H}^n \|_{\dot{B}_{p,1}^{-1}} &\leq 2A_0, \quad \text{for all} \quad n \in \mathbb{N}, \\
\text{div}(E_0^n) &\to \text{div}u_0^0 = 0, \quad \text{for all} \quad n \in \mathbb{N}.
\end{aligned}
\]

(6.15)

Then Theorem 6.1 ensures that system (1.5) with the initial data \((E_0^n, u_0^n)\) admits a solution \((E^n, u^n) \in L_{T_n}^\infty(H^\infty)\) for some \(T_n > 0\) and sufficiently large \(s_0\). Moreover, by virtue of Propositions 3.2 and 3.3, it is not difficult to verify that

\[ E^n \in \dot{L}_{T_n}^\infty(\dot{B}_{p,1}^{\infty}), \quad u_L^n \in \dot{L}_{T_n}^\infty(\dot{B}_{p,1}^{\infty}) \cap L_{T_n}^1(\dot{B}_{p,1}^{\infty+2}), \quad u_H^n \in \dot{L}_{T_n}^\infty(\dot{B}_{p,1}^{-1}) \cap L_{T_n}^1(\dot{B}_{p,1}^{\infty+1}). \]

Furthermore, the properties of continuity with respect to time can be proved as those of Theorem 10.2 in [3]. Consequently, we actually have \((E^n, u^n) \in \mathcal{E}_T^n(T_n)\), here and in what follows we denote by \(\mathcal{E}_T^n(T_n)\) the solution space given by (1.18).

Next, we will show that there exists a uniform time \(T_0 > 0\), such that

\[ (E^n, u^n) \] is uniformly bounded in \(\mathcal{E}_T^n(T_0)\).  

(6.16)

To this end, let us denote by \(T_n^1\) the maximal existence time of \((E^n, u^n)\), and define \(T_n^1\) be the supremum of all time \(T \in [0, T_n^1]\) such that

\[ A^n(T) \leq 4C_2A_0, \quad \text{and} \quad B^n(T) \leq \eta, \]  

(6.17)

with \(\eta\) to be determined below. Then from Proposition 6.1 and (6.15), for all \(T \in [0, T_n^1]\), we infer that

\[
\begin{align*}
A^n(T) &\leq 2C_2 \left( A_0 + T(1 + A^n(T))A^n(T) + (1 + A^n(T))B^n(T) \right) \\
&\leq 2C_2 \left( A_0 + 4C_2A_0(1 + 4C_2A_0)T + (1 + 4C_2A_0)\eta \right),
\end{align*}
\]

(6.18)

and

\[
\begin{align*}
B^n(T) &\leq 2C_2 \left( \sum_{q \in \mathbb{Z}} 2^q \| \tilde{\Delta}_q u_{0H} \|_{L^p}(1 - e^{-c2^qT})^{\frac{1}{2}} + T(1 + A^n(T))A^n(T) + (B^n)^2(T) \right) \\
&\leq 2C_2 \left( \sum_{q \in \mathbb{Z}} 2^q \| \tilde{\Delta}_q u_{0H} \|_{L^p}(1 - e^{-c2^qT})^{\frac{1}{2}} + 4C_2A_0(1 + 4C_2A_0)T + \eta^2 \right).
\end{align*}
\]

(6.19)
Clearly, owing to Bernstein’s inequality, there holds
\[ L^\infty \{ (\Delta u_n) \} \leq \frac{A_0}{4}, \]
\[ 8C_2^2 A_0 (1 + 4C_2 A_0) T_0 \leq \frac{\eta}{4}, \]
\[ 2C_2 \sum_{n \in \mathbb{Z}} 2^n \| \Delta u_n \|_{L^p} (1 - e^{-c2^2 T_n})^{\frac{1}{2}} \leq \frac{\eta}{4}. \]

Then (6.18) and (6.19) reduce to
\[ A^n(T) \leq 3C_2 A_0, \quad (6.20) \]
and
\[ B^n(T) \leq \frac{3\eta}{4}, \quad (6.21) \]
respectively. Therefore, \( T_n \geq T_{n+1} \geq T_0 \) for all \( n \in \mathbb{N} \), and hence (6.16) holds.

**Step (II). Compactness.**

Using the classical compactness method, we will prove that the approximate sequence \((E_n, u^n)\) tends to some function \((E, u)\) which satisfies the system (1.5). In fact, for any fixed \( T \in [0, T_0] \), from (6.16), we see that \( u_n \) is uniformly bounded in \( \tilde{L}_T^\infty (\dot{B}_{p,1}^\infty) \hookrightarrow L_T^2 (\dot{B}_{p,1}^\infty) \), and \( u^n \) is uniformly bounded in
\[ \left( \tilde{L}_T^\infty (\dot{B}_{p,1}^{\infty-1}) \cap L_T^1 (\dot{B}_{p,1}^{\infty+1}) \right) \hookrightarrow L_T^2 (\dot{B}_{p,1}^{\infty}). \]

Therefore,
\[ u^n \text{ uniformly bounded in } L_T^2 (\dot{B}_{p,1}^{\infty}). \quad (6.22) \]

Moreover, (6.16) also implies that \( E^n \) is uniformly bounded in \( \tilde{L}_T^\infty (\dot{B}_{p,1}^{\infty}) \). Then using Corollary 2.1, we obtain
\[ \left\| (u^n, \nabla u^n) \right\|_{L_T^2 (\dot{B}_{p,1}^{\infty-1})} + \left\| \nabla u^n \right\|_{L_T^2 (\dot{B}_{p,1}^{\infty-1})} \leq C \left\| u^n \right\|_{L_T^2 (\dot{B}_{p,1}^{\infty})}, \quad (6.23) \]
where we have used the fact \( \text{div} u^n = \text{div}(E^n)^T = 0 \). It follows from (6.22) and (6.23) that \( \partial_t E^n \) is uniformly bounded in \( L_T^2 (\dot{B}_{p,1}^{\infty-1}) \). Let us denote \( \bar{E}^n := E^n - E_0^n \). Then from above estimates we get that
\[ \bar{E}^n \text{ uniformly bounded in } C_T (\dot{B}_{p,1}^{\infty}) \cap C_T^1 (\dot{B}_{p,1}^{\infty-1}). \quad (6.24) \]

As regards \( \{u^n\}_{n \geq 1} \), we first denote \( u^\ell := e^{\ell \Delta} u_0 \), and \( \bar{u}^n := u^n - u^\ell \). Then from Proposition 3.3 and (6.20)-(6.21), we infer that
\[ \bar{u}^n \text{ uniformly bounded in } C_T (\dot{B}_{p,1}^{\infty} + \dot{B}_{p,1}^{\infty-1}) \cap L^1 (\dot{B}_{p,1}^{\infty+2} + \dot{B}_{p,1}^{\infty+1}). \quad (6.25) \]

To estimate \( \{\partial_t \bar{u}^n\}_{n \geq 1} \), we write the equation for \( \bar{u}^n \) as
\[ \partial_t \bar{u}^n = \Delta \bar{u}^n + \text{div} E^n + \text{div} \left( -u^n \otimes u^n + E^n (E^n)^T \right). \]
Clearly, owing to Bernstein’s inequality, there holds
\[ \Delta \bar{u}^n \text{ uniformly bounded in } L_T^\infty (\dot{B}_{p,1}^{\infty-2} + \dot{B}_{p,1}^{\infty-3}) \cap L^1 (\dot{B}_{p,1}^{\infty} + \dot{B}_{p,1}^{\infty-1}), \quad (6.26) \]
and
\[ \left\| \text{div} E^n \right\|_{L_T^\infty (\dot{B}_{p,1}^{\infty-1})} \leq C \left\| E^n \right\|_{L_T^\infty (\dot{B}_{p,1}^{\infty})}. \quad (6.27) \]
Since $\tilde{B}^N_{p,1}$ is an algebra, we have
\[ \| \text{div} \left( E^n (E^n)^T \right) \|_{L^p_T(\tilde{B}^N_{p,1})} \leq C \| E^n \|_{L^p_T(\tilde{B}^N_{p,1})}^2. \] (6.28)

Thanks to the decomposition $u^n = u^n_L + u^n_H$, there holds
\[
\text{div}(u^n \otimes u^n) = \text{div}(u^n_L \otimes u^n_L) + \text{div}(u^n_L \otimes u^n_H + u^n_H \otimes u^n_L) + \text{div}(u^n_H \otimes u^n_H).
\]

Similarly, we get
\[ \| \text{div}(u^n_L \otimes u^n_L) \|_{L^p_T(\tilde{B}^N_{p,1})} \leq C \| u^n_L \|_{L^p_T(\tilde{B}^N_{p,1})}^2 \] (6.29)
and
\[ \| \text{div}(u^n_L \otimes u^n_H + u^n_H \otimes u^n_L) \|_{L^p_T(\tilde{B}^N_{p,1})} \leq C \| u^n_L \|_{L^p_T(\tilde{B}^N_{p,1})} \| u^n_H \|_{L^p_T(\tilde{B}^N_{p,1})}. \] (6.30)

By interpolation, it is easy to verify that $u^n_H$ are uniformly bounded in $L^p_T(\tilde{B}^N_{p,1})$ for $p \in [1, \infty]$. Then in view of Proposition 2.1, one easily deduces that
\[ \| \text{div}(u^n_L \otimes u^n_H) \|_{L^p_T(\tilde{B}^N_{p,1})} \leq C \| u^n_H \|_{L^p_T(\tilde{B}^N_{p,1})} \| u^n_L \|_{L^p_T(\tilde{B}^N_{p,1})}. \] (6.31)
with $1 < \rho < \frac{2p}{2p-N}$ if $p \geq \frac{N}{2}$, and $\rho > 1$ if $p < \frac{N}{2}$. Thus, (6.29)-(6.31) imply that
div$u^n \otimes u^n$ is uniformly bounded in $L^p_T(\tilde{B}^N_{p,1} + \tilde{B}^N_{p,1})$. (6.32)

Similarly, we get
\[ \Delta \bar{v}^n \text{ is uniformly bounded in } L^2_T(\tilde{B}^N_{p,1}) + L^p_T(\tilde{B}^N_{p,1} + \tilde{B}^N_{p,1}). \] (6.33)

It follows from (6.27), (6.28), (6.32), and (6.33) that
\[ \partial_t \bar{u}_n \text{ is uniformly bounded in } L^p_T(\tilde{B}^N_{p,1} + \tilde{B}^N_{p,1}), \] (6.34)
which, together with (6.25), implies that $\bar{v}^n$ is uniformly bounded in
\[ C_T(\tilde{B}^N_{p,1} + \tilde{B}^N_{p,1}) \cap C^1_T(\tilde{B}^N_{p,1} + \tilde{B}^N_{p,1}). \] (6.35)

Now from (6.24) and (6.35), following the arguments in [12], we can deduce that there exist some matrix $\tilde{E}$ and vector $\bar{u}$, such that for all $\phi \in C^\infty_0(\mathbb{R}^N)$, there hold
\[ \phi \tilde{E} \to \phi \tilde{E} \text{ in } C_T(\tilde{B}^N_{p,1}), \] (6.36)
and
\[ \phi \bar{u} \to \phi \bar{u} \text{ in } C_T(\tilde{B}^N_{p,1} + \tilde{B}^N_{p,1}), \] (6.37)
as $n$ tends to $\infty$ (up to subsequence). Let $(E, u) := (E_0 + \tilde{E}, u_0 + \bar{u})$. Then one can easily verify that $(E, u)$ satisfies the system (1.5) in the sense of distribution. Thanks to (6.20)-(6.21), and arguing as in [12], we actually have $(E, u) \in \mathcal{E}^N(T_0)$, and there holds
\[ \| E \|_{L^p_T(\tilde{B}^N_{p,1})} + \| u_L \|_{L^p_T(\tilde{B}^N_{p,1} \cap L^2_T(\tilde{B}^N_{p,1} + \tilde{B}^N_{p,1}))} + \| u_H \|_{L^p_T(\tilde{B}^N_{p,1} \cap L^4_T(\tilde{B}^N_{p,1}))} \]
\[ \leq C \left( \| E_0 \|_{\tilde{B}^N_{p,1}} + \| u_{0L} \|_{\tilde{B}^N_{p,1}} + \| u_{0H} \|_{\tilde{B}^N_{p,1}} \right), \] (6.38)
Thus, part (i) of Theorem 1.1 is proved.
6.2. Uniqueness. For any fixed $T \in [0, T_0]$, assume that $(E_i, u_i) \in \mathcal{E}_T^{\infty}(T)$, $i = 1, 2$, are two solutions of the system (1.5) with the same initial data $(E_0, u_0)$. Set $\delta u = u_1 - u_2$, $\delta E = E_1 - E_2$. Then $(\delta E, \delta u)$ satisfies the following system,

$$
\left\{ \begin{array}{l}
\partial_t \delta E + (u_1 \cdot \nabla) \delta E + \delta u \cdot \nabla E_2 - \nabla \delta u = \nabla u_1 \delta E + \nabla \delta u E_2, \\
\partial_t \delta u + (u_1 \cdot \nabla) \delta u + \delta u \cdot \nabla u_2 + \nabla \delta E
\end{array} \right. 
- \Delta \delta u - \text{div} \delta E = \text{div}(E_1 E_1^\top - E_2 E_2^\top),
$$

(6.39)

To simplify the presentation, let us denote

$$V_i(t) := \int_0^t \| \nabla u_i(\tau) \|_{B_{p,1}^{\infty}} d\tau, i = 1, 2. $$

By using the low frequency embedding (2.3) and the uniform estimate (6.38), it is easy to verify that, for $i = 1, 2$, there holds

$$
\| u_i \|_{L^1_T(B_{p,1}^{\infty+1})} \leq C \| (u_i)_{L^1_T(B_{p,1}^{2})} + \| (u_i)_{H^1_T(B_{p,1}^{2})} \|_{L^1_T(B_{p,1}^{\infty+1})} \leq C(T, A_0).
$$

(6.40)

Case 1. $p \in [1, 2N)$. Applying Proposition 3.2 to (6.39)$_1$, and using Corollary 2.1, (6.40), and (6.38), we have

$$
\begin{align*}
\| \delta E(t) \|_{B_{p,1}^{\infty+1}} \\
& \leq e^{C V_1(t)} \int_0^t \| \nabla u_1 \delta E + \nabla \delta u E_2 - \delta u \cdot \nabla E_2 + \nabla \delta u \|_{B_{p,1}^{\infty}} ds
\leq C e^{C V_1(t)} \int_0^t \left( \| u_1 \|_{B_{p,1}^{\infty+1}} \| \delta E \|_{B_{p,1}^{\infty+1}} + \| \delta u \|_{B_{p,1}^{\infty+1}} \| E_2 \|_{B_{p,1}^{\infty+1}} + \| \delta u \|_{B_{p,1}^{\infty+1}} ds \right)
\leq C(T, A_0) \left( \| \delta u \|_{L^1_T(B_{p,1}^{\infty})} + \int_0^t \| u_1 \|_{B_{p,1}^{\infty+1}} \| \delta E \|_{B_{p,1}^{\infty+1}} ds \right),
\end{align*}
$$

(6.41)

for all $t \in [0, T]$. Next, applying first the operator $\mathbb{P}$ to (6.39)$_2$, and then using Proposition 3.3 and Corollary 2.1 again, we are led to

$$
\begin{align*}
\| \delta u \|_{L^1_T(B_{p,1}^{\infty+1})} + \| \delta u \|_{L^2_T(B_{p,1}^{\infty+1})}
& \leq C \int_0^t \left( \| \delta E \|_{B_{p,1}^{\infty+1}} + \| (u_1, u_2) \|_{B_{p,1}^{\infty+1}} \| \delta u \|_{B_{p,1}^{\infty+1}} + \| (E_1, E_2) \|_{B_{p,1}^{\infty+1}} \| \delta u \|_{B_{p,1}^{\infty+1}} ds
\leq C(A_0) \int_0^t \| \delta E \|_{B_{p,1}^{\infty+1}} ds + C \| (u_1, u_2) \|_{L^2_T(B_{p,1}^{\infty})} \| \delta u \|_{L^2_T(B_{p,1}^{\infty+1})}
\end{align*}
$$

for all $t \in [0, T]$. Similar to (6.22), we easily have

$$
\| (u_1, u_2) \|_{L^2_T(B_{p,1}^{\infty})} \leq C(T, A_0).
$$

Therefore, we can choose $\bar{T}$ so small that

$$
C \| (u_1, u_2) \|_{L^2_T(B_{p,1}^{\infty})} \leq \frac{1}{2}.
$$
Then for all $t \in [0, \bar{T}]$, there holds
\[
\|\delta u\|_{L^1_t(B^\infty_{p,1})} \leq C(T, A_0) \int_0^t \|\delta E\|_{\tilde{B}^\infty_{p,1}} ds.
\] (6.42)

Substituting (6.42) into (6.41) yields
\[
\|\delta E(t)\|_{\tilde{B}^\infty_{p,1}} \leq C(T, A_0) \left( \int_0^t \left( 1 + \|u_1\|_{B^\infty_{p,1}} \right) \|\delta E\|_{\tilde{B}^\infty_{p,1}} ds \right).
\]

Gronwall’s inequality and (6.40) imply then that
\[
\|\delta E(t)\|_{\tilde{B}^\infty_{p,1}} = 0, \quad \text{for all} \quad t \in [0, \bar{T}].
\]

Hence, $\delta u = \delta E = 0$ on $[0, \bar{T}]$. Then, step by step, we can obtain that the solution is unique on $[0, T_0]$.

**Case 2.** $p = 2N$. Since the product law in $\tilde{B}^\infty_{p,1}$ does not work for $p = 2N$, next we will work in $\tilde{B}^\infty_{p,\infty}$ as in [15]. However, our case is more complex than that of [15] due to the fact that the higher regularity of $\delta u_L$ precludes the possibility of using a logarithmic interpolation inequality on $\|\delta u\|_{L^1_t(\tilde{B}^\frac{2}{N}_{2N,1})}$. We have to deal with $\delta u$ in a new way. Firstly, similar to (6.41), thanks to Proposition 2.2, we get
\[
\|\delta E(t)\|_{\tilde{B}^{-\frac{1}{2}}_{2N,\infty}} \leq C(T, A_0) \int_0^t \|\nabla u_1 \delta E + \nabla \delta u E_2 - \delta u \cdot \nabla E_2 + \nabla \delta u\|_{\tilde{B}^{-\frac{1}{2}}_{2N,\infty}} ds
\]
\[
\leq C(T, A_0) \left( \int_0^t \|u_1\|_{\tilde{B}^{\frac{3}{2}}_{2N,1}} \|\delta E\|_{\tilde{B}^{-\frac{1}{2}}_{2N,\infty}} + \|\delta u\|_{\tilde{B}^{\frac{3}{2}}_{2N,\infty}} \|E_2\|_{\tilde{B}^{\frac{3}{2}}_{2N,\infty}} + \|\delta u\|_{\tilde{B}^{\frac{3}{2}}_{2N,\infty}} \right) ds
\]
\[
\leq C(T, A_0) \left( \|\delta u\|_{L^1_t(\tilde{B}^{\frac{2}{N}}_{2N,1})} + \int_0^t \|u_1\|_{\tilde{B}^{\frac{2}{N}}_{2N,1}} \|\delta E\|_{\tilde{B}^{-\frac{1}{2}}_{2N,\infty}} ds \right),
\] (6.43)

In view of (6.38), using Gronwall’s inequality, we have
\[
\|\delta E(t)\|_{\tilde{B}^{-\frac{1}{2}}_{2N,\infty}} \leq C(T, A_0) \|\delta u\|_{L^1_t(\tilde{B}^{\frac{2}{N}}_{2N,1})},
\] (6.44)

Next, to bound $\delta u$, we estimate $\delta u_L$ and $\delta u_H$ separately with different norms. In fact, thanks to Proposition 3.3, we are led to
\[
\|\delta u_L\|_{L^1_t(\tilde{B}^{\frac{2}{N}}_{2N,1}) \cap L^\infty_t(\tilde{B}^{\frac{2}{N}}_{2N,1})} + \|\delta u_H\|_{L^1_t(\tilde{B}^{\frac{2}{N}}_{2N,\infty}) \cap L^\infty_t(\tilde{B}^{\frac{2}{N}}_{2N,\infty})}
\]
\[
\leq C \int_0^t \left( \|\text{div}\delta E_L(s)\|_{\tilde{B}^{-\frac{1}{2}}_{2N,1}} + \|\mathbb{P}(u_1 \cdot \nabla \delta u)_L\|_{\tilde{B}^{-\frac{1}{2}}_{2N,1}} + \|\mathbb{P}(\delta u \cdot \nabla u_2)_L\|_{\tilde{B}^{-\frac{1}{2}}_{2N,1}} \right. \\
+ \left. \|\text{div}\delta E^T_1 + E_2\delta E^T_2\|_{\tilde{B}^{-\frac{1}{2}}_{2N,1}} \right) ds
\]
\[
+ C \left( \|\text{div}\delta E_H\|_{L^1_t(\tilde{B}^{\frac{1}{2}}_{2N,\infty})} + \|\mathbb{P}(u_1 \cdot \nabla \delta u)_H\|_{L^1_t(\tilde{B}^{\frac{1}{2}}_{2N,\infty})} + \|\mathbb{P}(\delta u \cdot \nabla u_2)_H\|_{L^1_t(\tilde{B}^{\frac{1}{2}}_{2N,\infty})} + \|\text{div}\delta E^T_1 + E_2\delta E^T_2\|_{H^1_t(\tilde{B}^{\frac{1}{2}}_{2N,\infty})} \right).
\]
By the low frequency and high frequency embeddings, Proposition 2.2, we easily have

\[ \| \text{div} \delta E_L(t) \|_{B_{2N,1}^{-\frac{1}{2}}} \leq C \| \delta E_L(t) \|_{B_{2N,\infty}^{-\frac{1}{2}}} , \]

\[ \| \text{Pdiv}(\delta EE_1^T + E_2 \delta E_1^T) \|_{B_{2N,1}^{-\frac{1}{2}}} + \| \text{Pdiv}(\delta EE_1^T + E_2 \delta E_1^T) \|_{B_{2N,\infty}^{-\frac{1}{2}}} \]

\[ \leq C \| \delta EE_1^T + E_2 \delta E_1^T \|_{B_{2N,\infty}^{-\frac{1}{2}}} \leq C \| (E_1, E_2) \|_{B_{2N,1}^{\frac{1}{2}}} \| \delta E \|_{B_{2N,\infty}^{-\frac{1}{2}}} , \]

and

\[ \| \text{P}(u_1 \cdot \nabla \delta u) \|_{B_{2N,1}^{-\frac{1}{2}}} + \| \text{P}(u_1 \cdot \nabla \delta u) \|_{B_{2N,\infty}^{-\frac{1}{2}}} \]

\[ \leq C \left( \| (u_1 \cdot \nabla \delta u_L) \|_{B_{2N,1}^{-\frac{1}{2}}} + \| (u_1 \cdot \nabla \delta u_L) \|_{B_{2N,\infty}^{-\frac{1}{2}}} \right) \]

\[ + \| (u_1 \cdot \nabla \delta u_H) \|_{B_{2N,1}^{-\frac{1}{2}}} + \| (u_1 \cdot \nabla \delta u_H) \|_{B_{2N,\infty}^{-\frac{1}{2}}} \]

\[ \leq C \left( \| u_1 \cdot \nabla \delta u_L \|_{B_{2N,1}^{-\frac{1}{2}}} + \| u_1 \cdot \nabla \delta u_L \|_{B_{2N,\infty}^{-\frac{1}{2}}} \right) \]

\[ \leq C \| u_1 \|_{B_{2N,1}^{\frac{1}{2}}} \left( \| \delta u_L \|_{B_{2N,1}^{\frac{1}{2}}} + \| \delta u_H \|_{B_{2N,\infty}^{-\frac{1}{2}}} \right) . \]

Similarly,

\[ \| \text{P}(\delta u \cdot \nabla u_2) \|_{B_{2N,1}^{-\frac{1}{2}}} + \| \text{P}(\delta u \cdot \nabla u_2) \|_{B_{2N,\infty}^{-\frac{1}{2}}} \]

\[ \leq C \| u_2 \|_{B_{2N,1}^{\frac{1}{2}}} \left( \| \delta u_L \|_{B_{2N,1}^{\frac{1}{2}}} + \| \delta u_H \|_{B_{2N,\infty}^{-\frac{1}{2}}} \right) . \]

Collecting these estimates, we find that

\[ \| \delta u_L \|_{L^1_t(B_{2N,1}^{\frac{1}{2}}) \cap L^\infty_t(B_{2N,1}^{-\frac{1}{2}})} + \| \delta u_H \|_{L^1_t(B_{2N,1}^{\frac{1}{2}}) \cap L^\infty_t(B_{2N,1}^{-\frac{1}{2}})} \]

\[ \leq C \| (u_1, u_2) \|_{L^2_t(B_{2N,1}^{\frac{1}{2}})} \left( \| \delta u_L \|_{L^2_t(B_{2N,1}^{\frac{1}{2}})} + \| \delta u_H \|_{L^2_t(B_{2N,1}^{-\frac{1}{2}})} \right) \]

\[ + C \int_0^t \| \delta E(s) \|_{B_{2N,\infty}^{-\frac{1}{2}}} \, ds . \]

Then there exists some \( \hat{T} > 0 \), so small that for all \( t \in [0, \hat{T}] \),

\[ \| \delta u_L \|_{L^1_t(B_{2N,1}^{\frac{1}{2}})} + \| \delta u_H \|_{L^1_t(B_{2N,1}^{\frac{1}{2}})} \leq C(\hat{T}, A_0) \int_0^t \| \delta E(s) \|_{B_{2N,\infty}^{-\frac{1}{2}}} \, ds . \tag{6.45} \]

Substituting (6.44) into (6.45), we arrive at

\[ \| \delta u_L \|_{L^1_t(B_{2N,1}^{\frac{1}{2}})} + \| \delta u_H \|_{L^1_t(B_{2N,1}^{\frac{1}{2}})} \]

\[ \leq C \int_0^t \| \delta u_L \|_{L^1_t(B_{2N,1}^{\frac{1}{2}})} + \| \delta u_H \|_{L^1_t(B_{2N,1}^{\frac{1}{2}})} \, ds , \tag{6.46} \]
for all \( t \in [0, \tilde{T}] \). Gronwall’s inequality then implies that
\[
\|\delta u_L\|_{L_t^1(B^{\frac{5}{2}N,1})} + \|\delta u_H\|_{L_t^1(B^{\frac{5}{2},\infty})} \leq C \int_0^t \|\delta u_H\|_{L_t^1(B^{\frac{1}{2}N,1})} ds, \quad \text{for all} \quad t \in [0, \tilde{T}].
\] (6.47)

From Proposition 2.8 in [14], we obtain
\[
\|\delta u_H\|_{L_t^1(B^{\frac{5}{2}N,1})} \leq C \int_0^t \|\delta u_H\|_{L_t^1(B^{\frac{1}{2}N,\infty})} \log \left( e + V_3(s)\|\delta u_H\|^{-1}_{L_t^1(B^{\frac{1}{2},\infty})} \right) ds,
\]
where
\[
V_3(t) := \|\delta u_H\|_{L_t^1(B^{\frac{1}{2}N,\infty})} + \|\delta u_H\|_{L_t^1(B^{\frac{1}{2}N,\infty})},
\]
for some \( 0 < \epsilon < 1 \). From Lemma 2.1 and (6.38), it is easy to see that
\[
\|(u_i)_H\|_{L_t^1(B^{\frac{1}{2}N,1})} + \|\delta u_H\|_{L_t^1(B^{\frac{1}{2}N,\infty})} \leq C\|(u_i)_H\|_{L_t^1(B^{\frac{1}{2}N,1})} \leq C(\tilde{T}, A_0),
\]
for \( i = 1, 2 \). Consequently,
\[
V_3(t) \leq C(\tilde{T}, A_0), \quad \text{for all} \quad t \in [0, \tilde{T}].
\]

Since
\[
\int_0^1 \frac{ds}{s \ln(e + V_3(T)s^{-1})} = +\infty,
\]
by virtue of Osgood’s lemma, we get \( \delta u_H = 0 \), and hence \( \delta u = \delta E = 0 \) for all \( t \in [0, \tilde{T}] \). Then the uniqueness follows by standard arguments. This completes the proof of Theorem 1.1. \( \square \)

7. Local well-posedness for \( (E_0, u_0) \in \mathcal{E}_0 \). Proof of Corollary 1.1. Obviously, by embedding, the facts \( (E_0, u_0) \in \mathcal{E}_0 \) and (1.19) imply that \( E_0 \in \dot{B}_{p,1}^{\frac{N}{p}} \), \( u_{0L} \in \dot{B}_{p,1}^{\frac{N}{p}} \) and \( u_{0H} \in \dot{B}_{p,1}^{\frac{N}{p} - 1} \). Thus, Theorem 1.1 ensures that there this a unique solution \( (E, u) \in \dot{E}^\mathcal{N}_{\alpha}(T_0) \). In order to show that \( (E, u) \in \dot{E}_\alpha^\mathcal{N}(T_0) \), it suffices to recover Proposition 6.1 with data lying in \( \mathcal{E}_0 \). To avoid unnecessary duplication, we only give the estimates of the low frequency part \( (E_L, u_L) \) of the solution below. Firstly, we rewrite the equation of \( E \) as follows:
\[
\partial_t E = \nabla u - u \cdot \nabla E + \nabla u E.
\]

Then standard computations yield
\[
\|E_L\|_{\dot{E}_\alpha^\mathcal{N}(B_{2^1}^{\frac{N}{2} - 1 + \alpha})} \leq \|E_{0L}\|_{\dot{B}_{2^1}^{\frac{N}{2} - 1 + \alpha}} + C\|u_{0L}\|_{L_t^1(B_{2^1}^{\frac{N}{2} + \alpha})} + C\|u \cdot \nabla E\|_{L_t^1(B_{2^1}^{\frac{N}{2} - 1 + \alpha})} + C\|\nabla u E\|_{L_t^1(B_{2^1}^{\frac{N}{2} - 1 + \alpha})}. \quad (7.1)
\]
Noting that \((p, \alpha)\) satisfies (1.19), similar to (4.25)-(4.27), and using \(\text{div} u = \text{div} E^T = 0\), we arrive at

\[
\| (u \cdot \nabla E) \|_{L^1_t(B_{p,1}^{\infty})} + \| (\nabla u E) \|_{L^1_t(B_{p,1}^{\infty})} \\
\leq C \|u\|_{L^p_t(L^\infty)} \|E\|_{L^p_t(B_{p,1}^{\infty})} + C \|E\|_{L^p_t(L^\infty)} \|u\|_{L^p_t(B_{p,1}^{\infty})}
\]

\[
\|
\|u\|_{L^\infty_t(B_{p,1}^{\infty})} \leq C \left( \|u\|_{L^p_t(B_{p,1}^{\infty})} + \|u\|_{L^p_t(B_{p,1}^{\infty})} \right)
\]

\[
\|E\|_{L^\infty_t(B_{p,1}^{\infty})} \leq C \left( \|E\|_{L^p_t(B_{p,1}^{\infty})} + \|E\|_{L^p_t(B_{p,1}^{\infty})} \right)
\]

By virtue of the low frequency embedding (2.3), we have

\[
\|u\|_{L^\infty_t(B_{p,1}^{\infty})} \leq C \left( \|u\|_{L^p_t(B_{p,1}^{\infty})} + \|u\|_{L^p_t(B_{p,1}^{\infty})} \right)
\]

\[
\|E\|_{L^\infty_t(B_{p,1}^{\infty})} \leq C \left( \|E\|_{L^p_t(B_{p,1}^{\infty})} + \|E\|_{L^p_t(B_{p,1}^{\infty})} \right)
\]

It follows from (7.1)-(7.4) that

\[
\|E\|_{L^p_t(B_{p,1}^{\infty})} \leq C T^\frac{1}{2} \left( \|E\|_{L^p_t(B_{p,1}^{\infty})} \right)
\]

\[
\|E\|_{L^p_t(B_{p,1}^{\infty})} \leq C T^\frac{1}{2} \left( \|E\|_{L^p_t(B_{p,1}^{\infty})} \right)
\]

To bound \(u\), applying Proposition 3.3 to (6.8), we find that

\[
\|u\|_{L^p_t(B_{p,1}^{\infty}) \cap L^1_t(B_{p,1}^{\infty})} \leq C \left( \|u0\|_{L^p_t(B_{p,1}^{\infty})} + \|E\|_{L^p_t(B_{p,1}^{\infty})} \right)
\]

\[
\|u\|_{L^p_t(B_{p,1}^{\infty}) \cap L^1_t(B_{p,1}^{\infty})} \leq C \left( \|u0\|_{L^p_t(B_{p,1}^{\infty})} + \|E\|_{L^p_t(B_{p,1}^{\infty})} \right)
\]
Following the computations in (4.25)-(4.27) again, we infer that

\[
\| (u \otimes u + EE^T) \|_{L^T(B^{-\infty}_{p,1})} \leq C\| u \|_{L^T(B^{-\infty}_{p,1})} + C\| E \|_{L^T(B^{-\infty}_{p,1})} + C\| u \|_{L^T(B^{-\infty}_{p,1})} + C\| E \|_{L^T(B^{-\infty}_{p,1})}
\]

\[
= C\| u \|_{L^T(B^{-\infty}_{p,1})} + C\| E \|_{L^T(B^{-\infty}_{p,1})}.
\]

From (7.6), (7.7), and (7.3)-(7.4), one deduces that

\[
\| u_p \|_{L^T(B^{-\infty}_{p,1})} + C\| E \|_{L^T(B^{-\infty}_{p,1})}
\]

\[
= C\| u_0 \|_{L^T(B^{-\infty}_{p,1})} + C\| E \|_{L^T(B^{-\infty}_{p,1})} + C\| u \|_{L^T(B^{-\infty}_{p,1})} + C\| E \|_{L^T(B^{-\infty}_{p,1})}
\]

\[
+ C\| u_p \|_{L^T(B^{-\infty}_{p,1})} + C\| E \|_{L^T(B^{-\infty}_{p,1})} + C\| u \|_{L^T(B^{-\infty}_{p,1})} + C\| E \|_{L^T(B^{-\infty}_{p,1})}.
\]

Combining (7.5), (7.8) with the estimates of \((E_H, u_H)\) obtained in Proposition 6.1, it is not difficult to verify that, there exists a positive constant \(C_2\), such that

\[
A'(T) \leq C_2 \| (E_0, u_0) \|_{E_0} + T(1 + A'(T)) A'(T) + (1 + A'(T)) B(T),
\]

and

\[
B(T) \leq C_2 \left( \sum_{q \in \mathbb{Z}} 2^{2q} \| \Delta_q u_0 \|_{L^p} (1 - e^{-2^{2q} T})^{\frac{1}{2}} + T(1 + A'(T)) A'(T) + B^2(T) \right),
\]

where

\[
A'(T) := \| E_L \|_{L^T(B^{-\infty}_{p,1})} + \| E_H \|_{L^T(B^{-\infty}_{p,1})} + \| u_L \|_{L^T(B^{-\infty}_{p,1})} + \| u_H \|_{L^T(B^{-\infty}_{p,1})}.
\]

and \(B(T)\) is the same as in Proposition 6.1. From (7.9) and (7.10), following the proof of part (i) of Theorem 1.1 line by line, we obtain \((E, u) \in E^{-\alpha}_{\alpha} (T_0)\). This completes the proof of Corollary 1.1. \(\square\)
8. Proof of Theorem 1.2. Now we are in a position to prove Theorem 1.2. Let \((E_0, u_0)\) be the initial data in Theorem 1.2. According to Corollary 1.1, there is a unique solution \((E, u) \in \mathcal{E}_N^{\alpha} (T_0)\) to (1.5). Moreover, since \((E_0, u_0)\) satisfies the constrains (1.4), we actually obtain that \((E, u)\) also satisfies (1.6). Let \(T^*\) be the maximal existence time of this solution \((E, u)\). Define \(T_1\) be the supremum of all time \(T \in [0, T^*)\) such that

\[
\| (E, u) \|_{\mathcal{E}_N^{\alpha} (T)} \leq 4C_1 \| (E_0, u_0) \|_{\mathcal{E}_0},
\]

where \(C_1\) is the constant appearing in Proposition 5.1. Then from Proposition 5.1, one deduces that

\[
\| (E, u) \|_{\mathcal{E}_N^{\alpha} (T_1)} \leq C_1 \| (E_0, u_0) \|_{\mathcal{E}_0} + 16C_1 \| (E_0, u_0) \|^2_{\mathcal{E}_0}
\]

\[
\leq C_1 \| (E_0, u_0) \|_{\mathcal{E}_0} \left(1 + 16C_1 \| (E_0, u_0) \|_{\mathcal{E}_0}\right)
\]

\[
\leq 2C_1 \| (E_0, u_0) \|_{\mathcal{E}_0},
\]

provided the initial data \((E_0, u_0)\) satisfy

\[
\| (E_0, u_0) \|_{\mathcal{E}_0} \leq \frac{1}{16C_1^2}.
\]

Thus \(T_1 = T^*\), and (8.1) holds true on the interval \([0, T^*)\) provided \(\| (E_0, u_0) \|_{\mathcal{E}_0} \leq c_0\) with \(c_0 := \frac{1}{16C_1} \). Consequently, together with (4.20) and (4.22), we obtain

\[
\| E \|_{L_N^\infty (B_N^{2,1})} + \int_0^{T^*} \| \nabla u_H \|_{L^\infty} dt + \left( \int_0^{T^*} \| \nabla u_L \|_{L^\infty} dt \right)^{\frac{\alpha}{2}} \leq C.
\]

Then following the proof of Theorem 3.1 in [34], we conclude that (8.4) implies \(T^* = +\infty\). Taking \(M_0 := 4C_1\), we complete the proof of Theorem 1.2.

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