On Estimates for the Fourier-Bessel Transform in the Space $L^p(\mathbb{R}^2_+, x^{2\alpha_1 + 1} y^{2\alpha_2 + 1} dxdy)$

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Abstract. In this paper, we prove two estimates useful in applications for the Fourier-Bessel transform in the space $L^p(\mathbb{R}^2_+, x^{2\alpha_1 + 1} y^{2\alpha_2 + 1} dxdy)$, $(1 < p \leq 2)$, as applied to some classes of functions characterized by a generalized modulus of continuity.

1. Introduction and preliminaries

In [2], Abilov and Kerimov proved two estimates for the Fourier-Bessel transform in the space $L^2(\mathbb{R}^2_+)$ characterized by the generalized modulus of continuity. In this paper, we prove of these estimates in the space $L^p(\mathbb{R}^2_+, x^{2\alpha_1 + 1} y^{2\alpha_2 + 1} dxdy)$, $(1 < p \leq 2)$. We point out that similar results have been established in the context of Bessel transform in the space $L^p(\mathbb{R}^n)$, for the Dunkl transform, for the Cherednik-Opdam transform, for the Fourier transform and etc (for example see [3–6]).

Assume that $L^p(\mathbb{R}^2_+) = L^p(\mathbb{R}^2_+, x^{2\alpha_1 + 1} y^{2\alpha_2 + 1} dxdy)$, $(1 < p \leq 2$ and $\alpha_1, \alpha_2 > -\frac{1}{2}$), is the space of $p$-power integrable two-variables functions $f : \mathbb{R}^2_+ \to \mathbb{R}$ with the norm

$$||f||_p = ||f||_{L^p(\mathbb{R}^2_+)} = \left(\int_{\mathbb{R}^2_+} |f(x, y)|^p x^{2\alpha_1 + 1} y^{2\alpha_2 + 1} dxdy\right)^{\frac{1}{p}}$$

For $\alpha > -\frac{1}{2}$, we introduce the normalized spherical Bessel function $j_\alpha$ defined by

$$j_\alpha(z) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(k + \alpha + 1)} \sum_{k=0}^{\infty} \frac{(-1)^k (z^2)^k}{2^{2k} k!}$$

where $\Gamma(x)$ is the gamma-function.
From [1], we have
\[ 1 - j_a(u) = O(1), \quad u \geq 1. \]  
(1)

\[ 1 - j_a(u) = O(u^2), \quad 0 \leq u \leq 1. \]  
(2)

\[ j_a(u) = O(u^{a-\frac{1}{2}}) \]  
(3)

**Definition 1.1.** The Fourier-Bessel transform for two-variable functions is defined on \( L^1(\mathbb{R}_2^2) \) by
\[
\hat{f}(\xi, \eta) = \int_{\mathbb{R}_2^2} f(x, y) j_{a_1}(\xi x) j_{a_2}(\eta y) x^{2a_1+1} y^{2a_2+1} \, dx \, dy
\]

**Proposition 1.2.** Let \( f \) be in \( D_2(\mathbb{R}^2) \), then we have inversion formula
\[
f(x, y) = \frac{1}{2^{(a_1+a_2)}\Gamma^2(a_1+1)\Gamma^2(a_2+1)} \int_{\mathbb{R}_2^2} \hat{f}(\xi, \eta) j_{a_1}(\xi x) j_{a_2}(\eta y) \xi^{2a_1+1} \eta^{2a_2+1} \, d\xi \, d\eta,
\]
where \( D_2(\mathbb{R}^2) \) the space of \( C^\infty \)-function on \( \mathbb{R}^2 \), with compact support and even with respect to each variable.

The Fourier-Bessel transform above extends to a bounded linear map \( f \to \hat{f} \) from \( L^p(\mathbb{R}_2^2) \) to \( L^q(\mathbb{R}_2^2) \). We have the Hausdorff Young inequality
\[
\|\hat{f}\|_q \leq A\|f\|_p, \quad \forall f \in L^p(\mathbb{R}_2^2),
\]
where \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( A \) is a positive constant.

In \( L^p(\mathbb{R}_2^2) \), consider the following generalized translation operator defined by
\[
T_h f(x, y) = c_{a_1, a_2} \int_{[0, \pi]^2} f(\sqrt{x^2+h^2-2hx \cos u}, \sqrt{y^2+h^2-2hy \cos v}) \sin^{2a_1}(u) \sin^{2a_2}(v) \, dudv,
\]
which corresponds to the Bessel operator for two-variable functions
\[
D = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{2a_1+1}{x} \frac{\partial}{\partial x} + \frac{2a_2+1}{y} \frac{\partial}{\partial y}
\]
and with
\[
c_{a_1, a_2} = \Gamma(a_1+1)\Gamma(a_2+1) \frac{\Gamma(a_1+\frac{1}{2})\Gamma(a_2+\frac{1}{2})}{\pi \Gamma(a_1+a_2+1)}
\]

We note the important property of the Fourier-Bessel transform: If \( f \in L^p(\mathbb{R}_2^2) \)
\[
(D\hat{f})(\xi, \eta) = -(\xi^2 + \eta^2)\hat{f}(\xi, \eta)
\]
(5)

The following relation connect the generalized translation operator and the Fourier-Bessel transform
\[
(T_h f)(\xi, \eta) = j_{a_1}(\xi h) j_{a_2}(\eta h) \hat{f}(\xi, \eta)
\]
(6)

Note some its properties (see [1, 7])

1. \( T_h \) is a linear operator
2. \( T_h(j_{a_1}(\lambda x) j_{a_2}(\mu y)) = \lambda^{a_1} j_{a_1}(\lambda h) j_{a_2}(\mu h) j_{a_1}(\lambda x) j_{a_2}(\mu y) \)
3. \( \|T_h f - f\|_p \to 0 \) as \( h \to 0^+ \)
The first-and higher order finite differences of $f(x, y)$ as defined as follows

$$
\Delta_h f(x, y) = T_h f(x, y) - f(x, y) = (T_h - I) f(x, y)
$$

where $I$ is the identity operator in the space $L^2(\mathbb{R}^2)$ and $k = 1, 2, \ldots$.

The $k$th-order generalized modulus of continuity of a function $f \in L^p(\mathbb{R}^2)$ is defined as

$$
\Omega_k(f, \delta) = \sup_{0<h<\delta} \| \Delta_h^k f(x, y) \|_p
$$

where $\Delta_h^k f(x, y) = \Delta_h(\Delta_h^{k-1} f(x, y)) = (T_h - I)^k f(x, y)$ (7)

2. Estimates for the Fourier-Bessel transform for two-variable functions

In this section, we estimate the integral

$$
\int \int_{\xi^2 + \eta^2 > N^2} |\widehat{f}(\xi, \eta)|^q \xi^{2a_1+1} \eta^{2a_2+1} d\xi d\eta
$$

in some classes of two-variable functions.

**Lemma 2.1.** For $f \in L^p(\mathbb{R}^2)$

$$
\int \int_{\mathbb{R}^2} |\widehat{f}(\xi, \eta)|^q (\xi^2 + \eta^2)^p |1 - j_{a_1}(\xi h) j_{a_2}(\eta h)|^q \xi^{2a_1+1} \eta^{2a_2+1} d\xi d\eta \leq A^q \| \Delta_h^k D^f f(x, y) \|_p
$$

**Proof.** From formula (5), we obtain

$$
\widehat{D^f f}(\xi, \eta) = (-1)^r (\xi^2 + \eta^2)^r \widehat{f}(\xi, \eta)
$$

(8)

We use the formulas (6) and (8), we conclude

$$
(\widehat{T^n_h D^f f})(\xi, \eta) = (-1)^r j_{a_1}(\xi h) j_{a_2}(\eta h)(\xi^2 + \eta^2)^r \widehat{f}(\xi, \eta), \ 1 \leq i \leq k.
$$

(9)

It follows from the definition of finite difference (7) and formula (9) the image $\Delta_h^k D^f f(x, y)$ under the Fourier-Bessel transform has the form

$$
\Delta_h^k D^f f(x, y) = (-1)^r (\xi^2 + \eta^2)^r (j_{a_1}(\xi h) j_{a_2}(\eta h) - 1)^r \widehat{f}(\xi, \eta)
$$

then, using the Hausdorff Young inequality (4), we have the result. □
Theorem 2.2. For functions \( f(x, y) \in L^p(\mathbb{R}^2) \) in the space \( W^{r,k}_{p,q}(D) \)

\[
\sup_{W^{r,k}_{p,q}(D)} \int_{\xi^2 + \eta^2 \geq N^2} |f(\xi, \eta)|^q \xi^{2r+1} \eta^{2s+1} d\xi d\eta = O \left( N^{-2r/q} \phi(N) \right)
\]

where \( r = 0, 1, \ldots; k = 1, 2, \ldots; c > 0 \) is a fixed constant, \( \phi \) is any nonegative function defined on \([0, \infty)\) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

Proof. Let \( f \in W^{r,k}_{p,q}(D) \). Taking in to account the Holder inequality

\[
\int_{\xi^2 + \eta^2 \geq N^2} |f(\xi, \eta)|^q \xi^{2r+1} \eta^{2s+1} d\xi d\eta = \int_{\xi^2 + \eta^2 \geq N^2} (1 - j_{k_1}(\xi h)j_{k_2}(\eta h)) |f(\xi, \eta)|^q \xi^{2r+1} \eta^{2s+1} d\xi d\eta
\]

\[
= \int_{\xi^2 + \eta^2 \geq N^2} (1 - j_{k_1}(\xi h)j_{k_2}(\eta h)) |\hat{f}(\xi, \eta)|^q \xi^{2r+1} \eta^{2s+1} d\xi d\eta
\]

\[
\leq \left( \int_{\xi^2 + \eta^2 \geq N^2} |\hat{f}(\xi, \eta)|^q \xi^{2r+1} \eta^{2s+1} d\xi d\eta \right)^{\frac{1}{q}} \times \left( \int_{\xi^2 + \eta^2 \geq N^2} (1 - j_{k_1}(\xi h)j_{k_2}(\eta h)) |\hat{f}(\xi, \eta)|^q \xi^{2r+1} \eta^{2s+1} d\xi d\eta \right)^{\frac{1}{q}}
\]

\[
= \left( \int_{\xi^2 + \eta^2 \geq N^2} |\hat{f}(\xi, \eta)|^q \xi^{2r+1} \eta^{2s+1} d\xi d\eta \right)^{\frac{1}{q}} \times \left( \int_{\xi^2 + \eta^2 \geq N^2} \frac{1}{(\xi^2 + \eta^2)r} (\xi^2 + \eta^2)^{q/2} |1 - j_{k_1}(\xi h)j_{k_2}(\eta h))|^q |\hat{f}(\xi, \eta)|^q \xi^{2r+1} \eta^{2s+1} d\xi d\eta \right)^{\frac{1}{q}}
\]

\[
\leq N^{-\frac{r}{q}} \left( \int_{\xi^2 + \eta^2 \geq N^2} |\hat{f}(\xi, \eta)|^q \xi^{2r+1} \eta^{2s+1} d\xi d\eta \right)^{\frac{1}{q}} \times \left( \int_{\xi^2 + \eta^2 \geq N^2} (\xi^2 + \eta^2)^{q/2} |1 - j_{k_1}(\xi h)j_{k_2}(\eta h))|^q |\hat{f}(\xi, \eta)|^q \xi^{2r+1} \eta^{2s+1} d\xi d\eta \right)^{\frac{1}{q}}
\]

From Lemma 2.1, we have the inequality

\[
\int_{\mathbb{R}^2} |\hat{f}(\xi, \eta)|^q (\xi^2 + \eta^2)^{q/2} |1 - j_{k_1}(\xi h)j_{k_2}(\eta h))|^q |\hat{f}(\xi, \eta)|^q \xi^{2r+1} \eta^{2s+1} d\xi d\eta \leq A^q \|D^r \hat{f}(x, y)\|^q_p
\]

Thus
\[
\int \int_{\mathcal{E}_2^2+\mathcal{E}_2^2} |\tilde{f}(\xi, \eta)|^q \xi^{2n+1} \eta^{2n+1} d\xi d\eta \\
\leq \int \int_{\mathcal{E}_2^2+\mathcal{E}_2^2} |j_{n_1}(\xi h)| |j_{n_2}(\eta h)| |\tilde{f}(\xi, \eta)|^q \xi^{2n+1} \eta^{2n+1} d\xi d\eta \\
+ A^2 N^{-\frac{2}{q}} \left( \int \int_{\mathcal{E}_2^2+\mathcal{E}_2^2} |\tilde{f}(\xi, \eta)|^q \xi^{2n+1} \eta^{2n+1} d\xi d\eta \right)^{\frac{q-1}{q}} \|\Delta_x^a D_y^b f(x, y)\|_{L^q}^\frac{1}{q}
\]

Now we estimate the integral
\[
I = \int \int_{\mathcal{E}_2^2+\mathcal{E}_2^2} |j_{n_1}(\xi h)| |j_{n_2}(\eta h)| |\tilde{f}(\xi, \eta)|^q \xi^{2n+1} \eta^{2n+1} d\xi d\eta
\]
which is divided into two
\[
I = \int \int_{B_1} + \int \int_{B_2}
\]
where \( B_1 = \{(\xi, \eta); \xi^2 + \eta^2 \geq \mathcal{E}_2^2, \xi \geq \eta\} \) and \( B_1 = \{(\xi, \eta); \xi^2 + \eta^2 \geq \mathcal{E}_2^2, \xi < \eta\} \).

Combining this with (1) gives
\[
I = O \left( \int \int_{B_1} |j_{n_1}(\xi h)| |\tilde{f}(\xi, \eta)|^q \xi^{2n+1} \eta^{2n+1} d\xi d\eta + \int \int_{B_2} |j_{n_2}(\eta h)| |\tilde{f}(\xi, \eta)|^q \xi^{2n+1} \eta^{2n+1} d\xi d\eta \right)
\]
It follows that from (3) that
\[
j_{n_1}(\xi h) = O \left( (\xi h)^{-\alpha_1 - \frac{1}{2}} \right); \quad j_{n_2}(\xi h) = O \left( (\xi h)^{-\alpha_2 - \frac{1}{2}} \right)
\]
Therefore
\[
I = O \left( h^{-\alpha_1 - \frac{1}{2}} \int \int_{B_1} \xi^{-\alpha_1 - \frac{1}{2}} |\tilde{f}(\xi, \eta)|^q \xi^{2n+1} \eta^{2n+1} d\xi d\eta + h^{-\alpha_2 - \frac{1}{2}} \int \int_{B_2} \eta^{-\alpha_2 - \frac{1}{2}} |\tilde{f}(\xi, \eta)|^q \xi^{2n+1} \eta^{2n+1} d\xi d\eta \right)
\]
Then
\[
I = O \left( N^{-\alpha_1 - \frac{1}{2}} h^{-\alpha_1 - \frac{1}{2}} \int \int_{B_1} |\tilde{f}(\xi, \eta)|^q \xi^{2n+1} \eta^{2n+1} d\xi d\eta \right) \\
+ O \left( N^{-\alpha_2 - \frac{1}{2}} h^{-\alpha_2 - \frac{1}{2}} \int \int_{B_2} |\tilde{f}(\xi, \eta)|^q \xi^{2n+1} \eta^{2n+1} d\xi d\eta \right)
\]
Now let \( h = \frac{\mathcal{E}_2}{c} \), where \( c > 0 \) is an arbitrary constant, then
\[
I = O \left( \max(c^{-\alpha_1 - \frac{1}{2}}, c^{-\alpha_2 - \frac{1}{2}}) \right) \int \int_{\mathcal{E}_2^2+\mathcal{E}_2^2} |\tilde{f}(\xi, \eta)|^q \xi^{2n+1} \eta^{2n+1} d\xi d\eta
\]
We obtain
\[
\int\int_{\xi^2+\eta^2\geq N^2} |\mathcal{F}(\xi,\eta)|^p |\xi^{2n_1+1}\eta^{2n_2+1}| \,d\xi d\eta \leq A^\frac{1}{p} N^{-\frac{2r}{q}} \left( \int\int_{\xi^2+\eta^2\geq N^2} |\mathcal{F}(\xi,\eta)|^q |\xi^{2n_1+1}\eta^{2n_2+1}| \,d\xi d\eta \right)^{\frac{1}{q}}
\]
\[
\times \|\Delta^k_D f(x,y)\|_p^\frac{1}{p} + O \left( \max(c^{-n_1-\frac{1}{2}},c^{-n_2-\frac{1}{2}}) \right) \int\int_{\xi^2+\eta^2\geq N^2} |\mathcal{F}(\xi,\eta)|^q |\xi^{2n_1+1}\eta^{2n_2+1}| \,d\xi d\eta
\]

Now choosing the necessary constant \(c > 0\), such that \(1 - A \left( \max(c^{-n_1-\frac{1}{2}},c^{-n_2-\frac{1}{2}}) \right) \geq \frac{1}{2}\), where \(A\) is a positive constant.

\[
\int\int_{\xi^2+\eta^2\geq N^2} |\mathcal{F}(\xi,\eta)|^p |\xi^{2n_1+1}\eta^{2n_2+1}| \,d\xi d\eta = O(N^{-\frac{2r}{q}} \|\Delta^k_D f(x,y)\|_p^\frac{1}{p})
\]

It followos that
\[
\left( \int\int_{\xi^2+\eta^2\geq N^2} |\mathcal{F}(\xi,\eta)|^q |\xi^{2n_1+1}\eta^{2n_2+1}| \,d\xi d\eta \right)^{\frac{1}{q}} = O(N^{-\frac{2r}{q}} \|\Delta^k_D f(x,y)\|_p^\frac{1}{p})
\]
and this ends the proof.

\[\blacksquare\]

Corollary 2.3. Let \( f(x, y) \in W^{r,k}_{p,r} (D) \), \((\nu > 0)\), then
\[
\int\int_{\xi^2+\eta^2\geq N^2} |\mathcal{F}(\xi, \eta)|^q |\xi^{2n_1+1}\eta^{2n_2+1}| \,d\xi d\eta = O(N^{-2\nu q_k})
\]
where \(r = 0, 1, \ldots; k = 1, 2, \ldots \) and \(\frac{1}{p} + \frac{1}{q} = 1\).

Proof. Let \( f \in W^{r,k}_{p,r} (D) \) and \( \phi(t) = t^\nu \). Then from Theorem 2.2, we have
\[
\int\int_{\xi^2+\eta^2\geq N^2} |\mathcal{F}(\xi, \eta)|^q |\xi^{2n_1+1}\eta^{2n_2+1}| \,d\xi d\eta = O(N^{-2\nu q_k})
\]

Thus, the proof is finished.

\[\blacksquare\]

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