Second quantization of time and energy in Relativistic Quantum Mechanics

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Abstract

Based on Lorentz invariance and Born reciprocity invariance, the canonical quantization of Special Relativity (SR) has been shown to provide a unified origin for the existence of Dirac’s Hamiltonian and a self adjoint time operator that circumvents Pauli’s objection. As such, this approach restores to Quantum Mechanics (QM) the treatment of space and time on an equivalent footing as that of momentum and energy. Second quantization of the time operator field follows step by step that of the Dirac Hamiltonian field. It introduces the concept of time quanta, in a similar way to the energy quanta in Quantum Field Theory (QFT). An early connection is found already in Feshbach’s unified theory of nuclear reactions. Its possible relevance in current developments such as Feshbach resonances in the fields of cold atom systems, of Bose-Einstein condensates and in the problem of time in Quantum Gravity is noted.

"One must be prepared to follow up the consequences of theory, and feel that one just has to accept the consequences no matter where they lead” P.A.M. Dirac

1 Introduction

Resolving the Problem of Time is still a central issue in Quantum Mecanics and Quantum Gravity[1,2,3]. Pauli’s objection[2] to the existence of a time operator

1Quoted by J. Polchinsky of UCSB in “23d Solvay Conference - The Quantum Structure of Space and Time”, World Scientific, 2007
2"In the older literature on quantum mechanics, we often find the operator equation \(Ht\cdot tH=\hbar\). It is generally not possible, however, to construct a Hermitian operator (e.g. as a function of \(P\) and \(Q\)) which satisfies this equation. This is so because, from the C.R. written above it follows that \(H\) possesses continuously all eigenvalues from \(-\infty\) to \(+\infty\), whereas on the other hand, discrete eigenvalues of \(H\) can be present. We, therefore, conclude that
canonically conjugate to the Hamiltonian did set time to remain a parameter (a c-number) while space coordinates were represented by self-adjoint operators (q-numbers)\[5\], foregoing the equal footing of space and time accorded by Special Relativity (SR), as well as questioning the existence and interpretation of a time energy uncertainty relation\[6, 7\].

However the canonical quantization of SR\[8, 9\], together with Born’s reciprocity principle\[10, 11\], has now been shown to provide a formal basis for both the Dirac Hamiltonian and the existence of a self-adjoint "time operator" in Relativistic Quantum Mechanics (RQM). As the generator of continuous momentum displacements (there is no gap in the momentum spectrum) this time operator induces consequently a shift of energy in both branches of the energy relativistic spectrum, circumventing Pauli’s objection \[9, 12, 13\]. Its eigenspinors provide an orthonormal basis alternative to the energy momentum spinor basis provided by the Dirac Hamiltonian, and consequently a different representation. Then both can be subjected to second quantization to show that a field can be equivalently expressed as a function of spacetime coordinates $\psi(x^\mu)$ or of energy–momentum coordinates $\phi(p_\mu)$, the two formulations being related by the ordinary Fourier transform.

The present paper explores whether this new basic element of RQM provides additional insights on the dynamics of many particle systems in the occupation number representation of Quantum Field Theory (QFT). To quote: "Physics does not depend on the choice of basis, but which is the most convenient choice depends on the physics"\[3\]. Section 2 reviews briefly the derivation of the time operator. Section 3 presents the second quantization of the time operator field that follows step by step that of the Dirac Hamiltonian field. Interpretation and conclusions are included in Section 4.

## 2 First quantization: configuration, momentum, time and energy representations\[8\]

Canonical quantization and factorization of the special relativity free particle invariant $p^\mu p_\mu = \pi^2 := (m_0 c)^2$ yields a constraint that is satisfied by the linear equation:

$$[\gamma^\nu \hat{p}_\nu - m_0 c] |\Psi\rangle = 0 \quad (1)$$

provided that in the Minkowski metric $\eta^{\mu\nu} = diag(1,-1,-1,-1)$ the following anticommutation and commutation relations are satisfied:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathbf{I} \quad [\hat{p}_\mu, \hat{p}_\nu] = 0 \quad (2)$$

where $\mathbf{I}$ is the $4 \times 4$ identity matrix. Thus the $\gamma$'s satisfy a Clifford algebra and are represented by matrices. Eq.1 is recognized as the Lorentz invariant Dirac
equation with:

\[ H_D = c\alpha \hat{\mathbf{p}} + \beta m_0 c^2, \quad \alpha^i = \gamma^0 \gamma^i \quad \beta = \gamma^0 \]

In the same way, canonical quantization and factorization of the special relativity invariant \( x^\mu x_\mu = s^2 := (\tau_0 c)^2 \) yields a constraint that is satisfied by the linear equation:

\[ [\gamma^\nu \hat{x}_\nu - \tau_0 c] |\Psi\rangle = 0 \quad (3) \]

provided now that:

\[ \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}1 \quad [\hat{x}^\mu, \hat{x}^\nu] = 0 \quad (4) \]

Eq.3 is a Lorentz invariant equation satisfied by the self-adjoint time operator:

\[ T = \alpha \hat{r}/c + \beta \tau_0 \quad \alpha^i = \gamma^0 \gamma^i \quad \beta = \gamma^0 \]

introduced earlier in analogy to the Dirac Hamiltonian[9]. It introduces an intrinsic time characteristic \( \tau_0 \) of the system.

Finally the purely imaginary symmetrized invariant \( O^- := (\hat{x}^\mu \hat{p}_\mu - \hat{p}_\mu \hat{x}^\mu) \) that satisfies Born’s reciprocity invariance under the transformation \( \hat{x}^\nu \rightarrow \hat{p}_\nu, \hat{p}_\nu \rightarrow -\hat{x}^\nu \) [10] suggests using Planck’s constant to accept:

\[ \hat{x}^0 \hat{p}_0 - \hat{p}_0 \hat{x}^0 = \hat{x}^i \hat{p}_i - \hat{p}_i \hat{x}^i = i\hbar \quad (5) \]

that insures the satisfaction of the constraint:

\[ [\hat{x}^\mu \hat{p}_\mu - \hat{p}_\mu \hat{x}^\mu] |\Psi\rangle = [(\hat{x}^0 \hat{p}_0 - \hat{p}_0 \hat{x}^0) - (\hat{x}^i \hat{p}_i - \hat{p}_i \hat{x}^i)] |\Psi\rangle = 0 \]

Eqs.5 complement the commutation relations \( [\hat{p}_\mu, \hat{p}_\nu] = 0 \) and \( [\hat{x}^\mu, \hat{x}^\nu] = 0 \) to yield: a) infinite continuous range of the four-space and four-momentum spectra; b) the known configuration and momentum representations of the operators \( \hat{x}^\nu \) and \( \hat{p}_\nu; \) c) the Fourier transform relation between the representations of the system state vector and d) the position-momentum uncertainty relation[8].

In the configuration representation, \( \Psi^\dagger (\mathbf{r}, \tau_0) = \langle \mathbf{x} | \Psi \rangle \), defining \( t := \tau_0/c \), Eq.1 reads:

\[ i\hbar \frac{\partial \Psi (\mathbf{r}, t)}{\partial t} = \{-i\hbar c\alpha^j \frac{\partial}{\partial x^j} + \beta m_0 c^2\} \Psi (\mathbf{r}, t) \quad (6) \]

recognized as the time dependent Dirac equation for free motion.

In the momentum representation \( \Phi (\mathbf{p}, p_0) = \langle \mathbf{p} | \Psi \rangle \), defining \( e := cp_0 \), Eq.3 reads:

\[ i\hbar \frac{\partial \Phi (\mathbf{p}, e)}{\partial e} = \{i\hbar c\alpha_j \frac{\partial}{\partial p_j} + \beta \tau_0\} \Phi (\mathbf{p}, e) \quad (7) \]

that clearly relates energy changes to momentum changes.

As a self-adjoint operator, \( T \) is the generator of infinitesimal momentum displacements (Lorentz boosts) \( \delta \mathbf{p} = (\delta e/c)\mathbf{\alpha} = (\delta e/c^2)\mathbf{c} \mathbf{\alpha} \) and thus indirectly energy displacements, circumventing Pauli’s objection. In the same way, \( H_D \) is the generator of infinitesimal space displacements \( \delta \mathbf{r} = c\mathbf{\alpha} (\delta t) \) where \( c\mathbf{\alpha} = \mathbf{d} \mathbf{r}/dt \).
As in the non relativistic limit Eq. 6 yields the two-component positive energy Schrödinger-Pauli equation \[15\], Eq. 7 results in a two-component positive time (
\[
\begin{align*}
T \text{ also generates a phase change } \delta \varphi &= \beta (\delta e) \tau_0 / \hbar \text{ while } H_D \text{ generates a phase change } \delta \chi &= \beta (\delta t) m_0 c^2 / \hbar. \text{ These are equal provided } \delta e &= m_0 c^2 \text{ and } \delta t = \tau_0. \text{ Furthermore a common finite } 2\pi \text{ phase shift requires } \tau_0 = h / m_0 c^2. \\
\text{In conclusion, the dynamical time operator } T = \alpha \hat{\mathbf{r}} / c + \beta h / m_0 c^2 \text{ (where the parameter } \tau_0 \text{ is equated to de Broglie period } h / m_0 c^2) \text{, generates the Lorentz boost that gives rise to the de Broglie wave}[15]. \text{ This also supports the de Broglie period as an intrinsic property of matter, in agreement with experiment}[16].
\end{align*}
\]

These Dirac energy and time operators satisfy the commutation relation:
\[
[T, H_D] = i \{ I + 2 \beta K \} \hbar + 2 \beta \{ \tau_0 H_D - m_0 c^2 T \}
\]

where \( K = \beta \{ 2 s \lambda / h^2 + 1 \} \)[15] is a constant of motion. The related uncertainties are such that \( (\Delta T) / (\Delta H_D) \approx (\Delta \sigma) / (\Delta p)[9][13] \), sustaining the interpretation given by Bohr originally: the time uncertainty in the instant of passage at a certain point is given by the width of the wave packet which is complementary to the momentum uncertainty and thus to the energy uncertainty [14].

In the Heisenberg picture:
\[
\frac{dT(t)}{dt} = \frac{1}{i \hbar} [T, H_D] = \{ I + 2 \beta K \} + \frac{2}{i \hbar} \beta \{ \tau_0 H_D - m_0 c^2 T \}
\]
that upon integration yields:
\[
T(t) = \{ I + 2 \beta K + 2 \beta (1 / i \hbar) \tau_0 H_D \} t - (1 / i \hbar) m_0 c^2 \beta \int_0^t dt T
\]
as \( \hat{H}_D \) is constant. The last term introduces an oscillatory behavior (Zitterbewegung) about a linear time evolution.

The eigenspinors of the self adjoint energy \( H_D \) and time \( T \) operators (Appendix A) provide orthogonal basis additional to the continuous vector ones generated by operators \( \hat{\mathbf{x}}, \) and \( \hat{\mathbf{p}}[3] \), with the following characteristics. The energy spectrum goes from \(-\infty\) to \(+\infty\), with a \( 2m_0 c^2 \) gap at the origin. The time spectrum goes from \(-\infty\) to \(+\infty\), with a \( 2\tau_0 \) gap at the origin. As \( \tau_0 = h / m_0 c^2 \) (the deBroglie or Compton period\[^{15, 18}\]) the gaps are seen to be complementary. To a small energy gap corresponds a large time gap, and viceversa. This complementarity may provide the first indication that the electron neutrino mass has to differ from zero as this would send the time gap to infinity. Also in the same way that the energy spectrum clearly defines non relativistic \( (cp \ll m_0 c^2) \) and relativistic energy limits \( (cp \gg m_0 c^2) \) , the time spectrum recognizes short \( (r/c \ll \tau_0; \ r \ll c\tau_0) \) and long \( (r/c \gg \tau_0; \ r \gg c\tau_0) \) time or space limits. As in the non relativistic limit Eq.6 yields the two-component positive energy Schrödinger-Pauli equation\[^{15}\]. Eq. 7 results in a two-component positive time short range approximation.

\[^{15}\]This clarifies the confusion addressed by Hilgevoord\[^{21}\] between the coordinates of a point in space and the position variables of a particle (“clearly fostered by the notation \( x, y, z \) for both concepts”); as well as noting that \( t := p^0/c \) and \( H_D \) are not canonical conjugate variables.
3 Second quantization of the energy-momentum
and time-space fields

(Note There are many books on quantum field theory. In this section the
presentation of F. Schawbl, "Advanced Quantum Mechanics", Chapter 13[19] is
followed, where the representation of the field is given as a superposition of free
solutions in a finite volume $V$. The passage to infinite volume is achieved with
\[
\sum_{k} \left( \frac{m}{V k_0} \right)^{1/2} \Rightarrow \int \frac{d^3k}{(2\pi)^3} \frac{\sqrt{m}}{k_0} \]

where the factor $\sqrt{m}$ is chosen in order to cancel
the factor $1/\sqrt{m}$ in the spinors, so that the limit $m \to 0$ exists).

a) The time-space representation

The second quantization of the $\Psi(r, t)$ field considers its expansion in terms
of the spinor eigenvector basis $\{|e^q\}\$ (Appendix A) and transforms the expan-
sion coefficients into creation and annihilation (particle and antiparticle) operators:
\[
\langle r | \hat{\Psi} \rangle = \hat{\Psi}(r) = \sum_{p,q} \left( \frac{m_0 c^2}{V e_p} \right)^{1/2} (\hat{b}_{q,p} u^e_{q,p}(p)e^{-ipr/h} + \hat{d}^\dagger_{q,p} w^e_{q,p}(p)e^{ipr/h})
\]

where the "energy spinors" are:
\[
|e^q\rangle = u^e_{q,p}(p) |p\rangle \quad e_p > m_0 c^2 \text{ and } \quad |e^q\rangle = w^e_{q,p}(p) |p\rangle \quad e < m_0 c^2
\]

\[
u^e_{q,p}(p) = \left( \frac{e_p + m_0 c^2}{2 m_0 c^2} \right)^{1/2} \left( \frac{\chi_q}{e_p + m_0 c^2} \chi_p \right)
\]
\[
w^e_{q,p}(p) = -\left( \frac{e_p + m_0 c^2}{2 m_0 c^2} \right)^{1/2} \left( \frac{\chi_p}{e_p + m_0 c^2} \chi_q \right)
\]

with $p = |p|$, $e_p = +\sqrt{(cp)^2 + (m_0 c^2)^2}$ and for $q = 1, 2$ $\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,
$\chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus $u_{1,2}(p)$ correspond to positive energy and up and down
spin, while $w_{1,2}(p)$ correspond to negative energy and up and down spin, as
can be seen clearly in the rest frame where $p = 0$. The minus sign in $w^e_{q,p}(p)$
insures that the charge conjugation operation $C$ transforms the spinors $u_q(p)$
into $w_q(p)$ and viceversa.

It then follows:
\[
P^\mu = i \int d^3x \{ \bar{\psi} \gamma_0 \partial^\mu \psi \} = \sum_{p,q} p^\mu (\hat{b}^\dagger_{q,p} \hat{b}_{q,p} - \hat{d}_{q,p} \hat{d}^\dagger_{q,p})
\]

where:
\[
P^0 = \sum_{p,q} c p_0 (\hat{b}^\dagger_{q,p} \hat{b}_{q,p} - \hat{d}_{q,p} \hat{d}^\dagger_{q,p}) = \sum_{p,q} e_p (\hat{b}^\dagger_{q,p} \hat{b}_{q,p} - \hat{d}_{q,p} \hat{d}^\dagger_{q,p})
\]
Anticommutation of the field operators to satisfy Fermi-Dirac statistics and normal ordering to avoid zero point terms yields:

\[ P^0 := H = \sum_{p,q} e_p (\hat{b}_{qp} \hat{b}_{qp} + \hat{d}_{qp} \hat{d}_{qp}) > 0 \]  

(17)

i.e., the total energy as the sum of positive energy quanta \( e_p \) for both states, now interpreted as representing electrons and positrons respectively. The total momentum is given as:

\[ P = \sum_{p,q} p (\hat{b}_{qp} \hat{b}_{qp} + \hat{d}_{qp} \hat{d}_{qp}) \]  

(18)

As components of a four vector, \( P^0 \) and \( P^i \) are part of the energy-momentum tensor (stress-energy tensor):

\[ T^{\mu\nu} = i \bar{\psi} \gamma^{\mu} \partial^{\nu} \psi \]  

(19)

whose other components yield the total (orbital plus spin) angular momentum of the system.

The momentum operator \( P^\mu \) is furthermore shown to be the generator of space-time displacements, i.e.:

\[ e^{ia}_\mu P^\mu \Psi(x) e^{-ia}_\mu P^\mu = \Psi(x + a) \]  

(20)

b) The energy-momentum representation

The above procedure can also be applied to the energy-momentum representation \( \Phi(p) = \langle p | \Psi = \Phi(p, p_0) \rangle \) and its adjoint. Expanding the field \( \Phi(p, E) \) in the spinor \( \{|t_r\} \) basis (Appendix A) yields:

\[ \langle p | \hat{\Psi} = \hat{\Phi}(p) = \sum_{r,q} \left( \frac{\tau_0}{V_{tr}} \right)^{1/2} \left( \delta_{qr} u_{t r}^{t q}(r) e^{-ip r / \hbar} + \tilde{c}_{qr} w_{t r}^{t q}(r) e^{ip r / \hbar} \right) \]  

(21)

with eigenvectors, now "time spinors":

\[ u_{t r}^{t q}(r) = \left( \frac{t_r + \tau_0}{2\tau_0} \right)^{1/2} \left( \frac{\chi_q}{\sqrt{\tau_r + \tau_0}} \chi_q \right) \quad t > 0 \]  

(22)

\[ w_{t r}^{t q}(r) = \left( \frac{t_r + \tau_0}{2\tau_0} \right)^{1/2} \left( \frac{\sigma_{r/t} x_{tr}}{\sqrt{\tau_r + \tau_0}} \chi_q \right) \quad t < 0 \]  

(23)

with \( t_r = +\sqrt{(r/c)^2 + \tau_0^2} \) and again \( q = 1,2 \), \( \chi_1 = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \), \( \chi_2 = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \).

Thus \( u_{t r}^{t q}(r) \) correspond to positive time and up and down spin, while \( w_{t r}^{t q}(r) \) correspond to negative time and up and down spin.

\[ \text{Here a finite momentum volume is considered, yielding an unfamiliar discretization of space. However in the infinite volume limit } \sum_{r} \text{ goes to the familiar } \int dr. \]
One now obtains:

\[ T^\mu = \sum_{r,q} (x^\mu/c)(\hat{a}^\dagger_{qr} \hat{a}_{qr} - \hat{c}_{qr} \hat{c}^\dagger_{qr}) \]  \hspace{1cm} (24)

As before, requiring anticommutation and normal ordering yields:

\[ T^0 = \sum_{r,q} t_r (\hat{a}^\dagger_{qr} \hat{a}_{qr} + \hat{c}^\dagger_{qr} \hat{c}_{qr}) > 0 \quad t_r = +\sqrt{(r/c)^2 + \tau_0^2} > 0 \] \hspace{1cm} (25)

i.e., the time operator field contains only positive times. This development introduces **time quanta** \( t_r = +\sqrt{(r/c)^2 + \tau_0^2} > 0 \) that are created and destroyed by the operators \( a, a^\dagger \) and \( c, c^\dagger \). In analogy to Eq.17, Eq.25 represents a total intrinsic time associated with the system.

One also obtains from Eq.23 the vector relation:

\[ T = \sum_{r,q} (r/c)(\hat{a}^\dagger_{qr} \hat{a}_{qr} + \hat{c}^\dagger_{qr} \hat{c}_{qr}) \] \hspace{1cm} (26)

or:

\[ cT = \sum_{r,q} r (\hat{a}^\dagger_{qr} \hat{a}_{qr} + \hat{c}^\dagger_{qr} \hat{c}_{qr}) \] \hspace{1cm} (27)

In a similar way \( T^0 \) and \( T^i \) as components of a four vector are part of a time-space tensor,

\[ \tilde{T}^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi \] \hspace{1cm} (28)

whose other components yield the space boosts of the system as in analogy with Eq.20 the time operator \( T^\mu \) can be shown to be the generator of energy-momentum displacements, i.e.:

\[ e^{iq\mu T^\mu} \Phi(p)e^{-iq\mu T^\mu} = \Phi(p + q) \] \hspace{1cm} (29)

### 4 Interpretation and Conclusion

The inclusion of the self-adjoint time operator and associated representation restores in QM the equal footing of time and energy accorded by SR, circumventing Pauli’s objection and providing an extended insight on the problem of time. The time space representation exhibits a total energy which is the sum of positive energy quanta for both particles and antiparticles present (Eq.17). Particles with negative energies are interpreted as antiparticles with positive energies. On the other hand, the energy-momentum representation exhibits a total intrinsic time as the sum of positive time quanta for both particles and antiparticles (Eq.25), giving a formal support to Feynman’s extraordinary identification of the negative energy solutions evolving backward as positrons evolving forward in time.

Second quantization, also referred to as occupation number representation, is a formulation that allows to describe states with varying numbers of particles...
either free or bound by an external or a self consistent Hartree-Fock potential arising from their interactions. In the each case the energy quanta are the single particle energies, which comprise the discrete and continuum energy spectra in the single particle potential \( \{e_i, e(p)\} \). Then Eq.17 represent the total energy of the independent particle approximation, where the energy quanta are the single particle energies. In a description where the ground state is the full Fermi sea, this energy accounts for the number of particle-hole (electrons and positrons or protons and neutrons) states that define an excited state of the system. In the same way, Eq.25 represents a total intrinsic time summing up the corresponding individual time quanta expectation values, which are \( \pm \tau_0 \) for bound single particle states. Thus the intrinsic time increases with the number of particle-hole pairs involved in an excited state of the system.

An early connection is found in Feshbach’s unified theory of nuclear reactions\[25, 26\] where sharp energy resonances are shown to be due to compound nucleus states. These are typically many particle-hole configurations of mainly single particle bound states in the common potential. Unbound single particle states connecting to the open reaction channels will appear with very small amplitudes, leading to long delay times and long lifetimes\[26, 28, 27\], whereas in the present formulation they are shown to carry large intrinsic times.

To conclude, the purpose of this paper stresses the extension of basic RQM with the existence a self-adjoint time operator and the additional basis it provides. Some applications have already been identified. Observable effects in experiments that simulate the Dirac equation\[22\] and in tunneling in attosecond optical ionization\[23\] have already been associated with the existence of the time operator in the first quantization of SR. The time operator also provides support for the conditional interpretation of time in QG\[24\]. However its full impact in the extensive development and applications of QFT remains to be explored, both theoretically and experimentally. To be noted is that Feshbach resonances are currently subject of extensive research in the fields of cold atom systems\[29\] (where they provide the essential tool to control the interaction between atoms), and of Bose-Einstein condensates\[30, 31\]. The many facets of the problem of time in QG\[2, 3\] are also to be considered in the present context.

\[\text{In the RQM description of atom and nuclear bound states appear as a consequence of attractive potentials. If these depend solely on position, e.g., the Coulomb potential in atoms, the shell model self consistent potential in nuclei, the time operator } T = \alpha r/c + \beta \tau_0 \text{ satisfies the same commutation relation as with the free particle Hamiltonian}\[6\]. The expectation value of } T \text{ in a bound state } n, \text{ is:} \]

\[\langle T \rangle_n = \langle n | T | n \rangle = \langle n | \alpha r/c + \beta \tau_0 | n \rangle = \pm \tau_0 \]

as the first term is zero, \( \alpha \) being a non diagonal matrix.
5 Appendix A Time operator eigenvalues and eigenvectors

A plane wave solution of the form \( \Psi(r, x_0) = e^{-ipx/\hbar}\psi(r) = e^{-i(p_0x_0 - p \cdot r/\hbar)}\psi(r) \) in Eq.8 yields the eigenvalue equation:

\[
\{-i\hbar\alpha^i \frac{\partial}{\partial x^i} + \beta m_0 c^2\} \psi(r) = e\psi(r) \tag{A.1}
\]

with positive and negative eigenvalues:

\[
e = \pm \sqrt{(c p)^2 + (m_0 c^2)^2} \tag{A.2}
\]

and eigenvectors ("energy" spinors):

\[
|e^q\rangle = u_q(p) |p\rangle \quad e > m_0 c^2 \quad \text{and} \quad |e^q\rangle = w_q(p) |p\rangle \quad e < -m_0 c^2 - \tag{A.3}
\]

where

\[
u_q(p) = \left(\frac{e_p + m_0 c^2}{2m_0 c^2}\right)^{1/2} \left(\begin{array}{c}
\chi_q \\
\frac{\sigma \cdot \mathbf{p}}{e_p + m_0 c^2} \chi_q
\end{array}\right)
\]

\[
w_q(p) = -\left(\frac{e_p + m_0 c^2}{2m_0 c^2}\right)^{1/2} \left(\begin{array}{c}
\chi_q \\
\frac{\sigma \cdot \mathbf{p}}{e_p + m_0 c^2} \chi_q
\end{array}\right)
\tag{A.4}
\]

where \( e_p = \sqrt{(c p)^2 + (m_0 c^2)^2} \) and for \( q = 1, 2 \), \( \chi_1 = \left(\begin{array}{c}1 \\ 0\end{array}\right) \) and \( \chi_2 = \left(\begin{array}{c}0 \\ 1\end{array}\right) \). Note that the negative energy spinors have opposite spin projection to the corresponding positive energy spinors. ("Given this definition, the charge conjugation operation \( C \) transforms the spinors \( u_q(p) \) into \( w_q(p) \) and vice versa").

For a particle in an attractive \( r \)-dependent potential with bound states, these spinors satisfy the equation

\[
\{-i\hbar\alpha^i \frac{\partial}{\partial x^i} + V(r) + \beta m_0 c^2\} \psi(r) = e\psi(r)
\]

where \( \{e\} = \{E_i, E\} \), the \( E_i \) being the bound states \( \varphi_i^q(r) \) eigenvalues and \( E \) the energy continuum of distorted eigenfunctions.

The expectation value of the time operator in a bound state is then

\[
\langle T \rangle_i = \int dr \varphi_i^{q\ast}(r)\{\alpha \cdot r/c + \beta \tau_0\}\varphi_i^q(r) = \int dr \varphi_i^{q\ast}(r)\beta \tau_0\varphi_i^q(r) = \pm \tau_0
\]

as \( \alpha \) is not a diagonal operator in spin space.

In the same way, a plane wave solution of the form \( \Phi(p, p_0) = e^{-ipx/\hbar}\phi(p) = e^{-i(p_0x_0 - p \cdot x/\hbar)}\phi(p) \) in Eq.15 yields the eigenvalue equation:

\[
\{i(\hbar/c)\alpha^i \frac{\partial}{\partial p_i} + \beta \tau_0\}\phi(p) = t\phi(p) \tag{A.5}
\]
with positive and negative eigenvalues:

\[ t = \pm \sqrt{\left(\frac{r}{c}\right)^2 + (\tau_0)^2} \]  \hspace{1cm} (A.6)

and eigenvectors ("time" spinors):

\[ |t^q\rangle = u(r)_{q} |r\rangle \quad t > \tau_0 \quad \text{and} \quad |t^q\rangle = w(r) |r\rangle \quad t < -\tau_0 \]  \hspace{1cm} (A.7)

with

\[ u_q(r) = \left(\frac{t_r + \tau_0}{2\tau_0}\right)^{1/2} \begin{pmatrix} \chi_q \\ \frac{\sigma \cdot r/c}{t_r + \tau_0} \chi_q \end{pmatrix} \]

\[ w_q(r) = -\left(\frac{t_r + \tau_0}{2\tau_0}\right)^{1/2} \begin{pmatrix} \frac{\sigma \cdot r/c}{t_r + \tau_0} \chi_q \\ \chi_q \end{pmatrix} \]  \hspace{1cm} (A.8)

where \( t_r = +\sqrt{\left(\frac{r}{c}\right)^2 + (\tau_0)^2} \) and for \( q = 1, 2 \), \( \chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), \( \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \).

As eigenvectors of self-adjoint energy and time operators, energy and time spinors constitute complete orthogonal sets \( \{|e^q\rangle\}, \{|t^q\rangle\} \) as: (eq. 6.3.15)

\[ \bar{u}_q^e(p) u_q^e(p) = \delta_{qs} \]
\[ \bar{u}_q^e(p) w_q^e(p) = 0 \]
\[ \bar{w}_q^e(p) u_q^e(p) = 0 \]
\[ \bar{w}_q^e(p) w_q^e(p) = 0 \]  \hspace{1cm} (A.9)

and similar for the time spinors. They both provide representations for a system state vector.

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