Moduli of Hyperelliptic Curves and Multiple Dirichlet Series

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Abstract
In this paper we provide an explicit construction of a distinctive multiple Dirichlet series associated to products of quadratic Dirichlet L-series, which we believe should be tightly connected to a generalized metaplectic Whittaker function on the double cover of a Kac-Moody group. To do so, we first impose a set of axioms, independent of any group of functional equations, which the aforementioned object should satisfy. As a consequence, we deduce that the coefficients of the $p$-parts of the multiple Dirichlet series satisfy certain recurrence relations. These relations lead to a family of identities, which turns out to be encoded in the combinatorial structure of certain moduli spaces of admissible double covers. Finally, via this crucial connection, we apply Deligne’s theory of weights to express inductively the coefficients of the $p$-parts in terms of the eigenvalues of Frobenius acting on the $l$-adic étale cohomology of local systems on the moduli $\mathcal{H}_g[2]$ of hyperelliptic curves of genus $g$ with level 2 structure.

Contents

1 Introduction 2
2 Preliminaries 10
3 The series $Z(T, t_{r+1}; q)$ and moments of character sums 14
4 The generating series of $a(\kappa, l; q)$ 19
5 The coefficients $\lambda(\kappa, l; q)$ for $l = 3, 4$ 24
6 Cohomological interpretation of the moment-sums 30
   6.1 Local systems on $A_g$ 30
   6.2 Moments of characteristic polynomials 32
7 The main theorem 37
   7.1 A Special Case 37
   7.2 The General Case 40
8 The series $\bar{C}(X, T, q)$ 53
   8.1 Moduli spaces of admissible double covers 54
   8.2 A homomorphism 56
   8.3 Special elements 57
   8.4 Further Computations 58
   8.5 Essential parts of admissible double covers with marked points 59
   8.6 Finishing the proof of Theorem 7.5 62

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Constructing the generating series $\Lambda_l(T,q)$

9.1 Deligne’s purity theorem .......................................................... 65
9.2 Decomposing $A_l(T,q)$ in terms of $q$-Weil numbers .................. 66

10 An application 69

A Appendix 74

B Appendix 75

C Appendix 77

D Appendix 79

1 Introduction

For a reduced root system $\Phi$ of rank $r$ and a number field $F$ containing the $2n$-th roots of unity, one associates a Weyl group multiple Dirichlet series (see [13], [14] and [15])

$$Z_\Psi(s_1,\ldots,s_r;\Phi) = \sum H(c_1,\ldots,c_r)\Psi(c_1,\ldots,c_r)N(c_1)^{-s_1}\ldots N(c_r)^{-s_r}$$

(1)

the sum being over non-zero ideals $c_1 = (c_1),\ldots,c_r = (c_r)$ of the ring $\mathcal{O}_S$ of $S$-integers for some sufficiently large set $S$ of places; it is assumed that the finite set $S$ contains all archimedean places, and that $\mathcal{O}_S$ is a principal ideal domain. The product $H\Psi$ remains unchanged if $c_i$ ($1 \leq i \leq r$) is multiplied by a unit, i.e., it is a function of ideals in $\mathcal{O}_S$. The function $H$ is very important, giving the structure of the multiple Dirichlet series; it is completely determined, via a twisted multiplicativity, by the function field analog of $Z_\Psi(s_1,\ldots,s_r;\Phi)$, which turns out to be a rational function.

Alternatively, one specifies the $p$-parts of $Z_\Psi$ for primes $(p)$, that is, the generating series

$$\sum_{k_1,\ldots,k_r \geq 0} H(p^{k_1},\ldots,p^{k_r})N(p)^{-k_1s_1-\cdots-k_rs_r}.$$ 

The factor $\Psi$ is less important, and represents a technical device chosen from a finite-dimensional vector space of functions on $F_S = \prod_{v \in S} F_v$, constant on cosets of an open subgroup, such that, together with $H$, it gives a multiple Dirichlet series possessing meromorphic continuation to $\mathbb{C}^r$ and satisfying a finite group of functional equations isomorphic to the Weyl group of $\Phi$.

Over the past ten years or so, it has emerged that these multiple Dirichlet series can be understood in the world of Eisenstein series. More precisely, let $G$ be a simply connected algebraic group over $F$ whose root system is the dual of the root system $\Phi$, i.e., $\Phi$ is the root system of the $L$-group $^L G$. Then Brubaker, Bump and Friedberg [14] made the following conjecture:

**Eisenstein Conjecture.** — The Weyl group multiple Dirichlet series $Z_\Psi(s_1,\ldots,s_r;\Phi)$ in (1) is a Whittaker coefficient of a minimal parabolic Eisenstein series on an n-fold metaplectic cover $\tilde{G}$ of $G$.

This conjecture has been established in the case of metaplectic covers of $GL_n$ by Brubaker, Bump and Friedberg [16, 17]. Shortly after [16] and [17] have been published, Chinta and Offen [28] obtained formulas for the spherical Whittaker functions on the $m$-fold metaplectic cover of $GL_n$ over a $p$-adic field. Their formulas generalize the well-known formula of Shintani for the spherical $GL_n$-Whittaker function, which corresponds to the nonmetaplectic case, i.e., $m = 1$. Furthermore, the authors provide in [28] the relationship between $p$-adic metaplectic Whittaker functions and the local parts of Weyl group multiple Dirichlet series associated to root systems of type $A_{n-1}$. Very recently, McNamara [71] extended the results of Chinta and Offen to the more general case of covers of unramified reductive groups; see also [75].
For very important applications, it is necessary, however, to establish the relevant analytic properties of Weyl group multiple Dirichlet series possessing infinite groups of functional equations. While the theory of Weyl group multiple Dirichlet series associated with classical (finite) root systems has seen great advances in the last decade, it is not at all clear how to extend this theory to the more general setting of Kac-Moody Lie algebras and their Weyl groups. Undoubtedly, such an extension will require deep foundational work. In this regard, a Casselman-Shalika type formula for Whittaker functions on metaplectic covers of Kac-Moody groups over non-archimedean local fields has been recently established by Patnaik and Puskás in their beautiful paper [76]; their work builds on [74] and [75]. However, except for some special cases of affine Kac-Moody groups, it is not at all clear, even conjecturally, how their formula is related to the local parts of Weyl group multiple Dirichlet series.

In this paper we provide an explicit construction of a distinctive multiple Dirichlet series associated to products of quadratic Dirichlet L-series, which we believe should be tightly connected to a generalized metaplectic Whittaker function on the double cover of a Kac-Moody group. To do so, we shall first impose a set of axioms, independent of any group of functional equations, which the aforementioned object should satisfy\(^1\); see also [90] and [91], where the same axiomatic approach was taken up to construct Weyl group multiple Dirichlet series associated to simply-laced affine root systems. As a consequence, we deduce that the coefficients of the \(p\)-parts of the multiple Dirichlet series we are interested in satisfy certain recurrence relations. These relations lead to a family of identities, which in turn, as shown in Sections 7 and 8, is encoded in the combinatorial structure of certain moduli spaces of admissible double covers. Finally, via this crucial connection, we apply Deligne’s theory of weights to express inductively the coefficients of the \(p\)-parts of the multiple Dirichlet series in terms of certain \(q\)-Weil numbers (the eigenvalues of Frobenius acting on the \(\ell\)-adic étale cohomology of local systems on the moduli \(\mathcal{H}_g[2]\) of hyperelliptic curves of genus \(g\) with level 2 structure).

More concretely, let \(\chi_d\) denote the real primitive Dirichlet character associated to \(\mathbb{Q}(\sqrt{d})\). For \(r \geq 1\), we take the multiple Dirichlet series attached to the \(r\)-th moment of quadratic Dirichlet L-series to be of the form (cf. [26] in the finite-dimensional case)

\[
\sum_{m_1,\ldots,m_r, \text{ d-odd}} \frac{\chi_d(m_1) \cdots \chi_d(m_r)}{m_1^{r_1} \cdots m_r^{r_r}} \cdot \frac{(d_0 d_1^2)^{s+1}}{d_0 \text{ sq.free}} \cdot b(m_1, \ldots, m_r, d) \quad \text{(with } b(1, \ldots, 1) = 1) 
\]

where \(\widehat{m}_i\), for \(i = 1, \ldots, r\), denotes the part of \(m_i\) coprime to \(d_0\). The coefficients \(b(m_1, \ldots, m_r, d)\) are assumed to be multiplicative,

\[
b(m_1, \ldots, m_r, d) = \prod_{p^k || m_i, \text{ } p^l || d} b(p^{k_1}, \ldots, p^{k_r}, p^l)
\]

the product being over primes \(p \neq 2\). The generating series

\[
\sum_{k_1, \ldots, k_r, l \geq 0} b(p^{k_1}, \ldots, p^{k_r}, p^l) p^{-k_1 s_1 - \cdots - k_r s_r - l s_{r+1}}
\]

is usually referred to as the \(p\)-part of the multiple Dirichlet series.

To specify the coefficients \(b(p^{k_1}, \ldots, p^{k_r}, p^l)\), let \(\mathbb{F}_q\) be a finite field of odd characteristic, and consider a similar multiple Dirichlet series over \(\mathbb{F}_q(x)\). In particular, for \(m_1, \ldots, m_r, d\) monic polynomials in \(\mathbb{F}_q[x]\), we have multiplicative coefficients \(a(m_1, \ldots, m_r, d)\),

\[
a(m_1, \ldots, m_r, d) = \prod_{\pi^k || m_i} a(\pi^{k_1}, \ldots, \pi^{k_r}, \pi^l)
\]

the product being taken this time over monic irreducibles \(\pi\) in \(\mathbb{F}_q[x]\). For compatibility reasons, we shall assume that \(a(\pi^{k_1}, \ldots, \pi^{k_r}, 1) = 1\) for all \(k_1, \ldots, k_r \in \mathbb{N}\), and that \(a(\pi^{k_1}, \ldots, \pi^{k_r}, \pi) = 0\) unless \(k_1 = \cdots = k_r = 0\), in which case this

\(^1\)These axioms are easily verified in the finite-dimensional case.
Furthermore, the multiple Dirichlet series can also be expressed as a power series
\[ \sum_{k_1, \ldots, k_r, l \geq 0} \lambda(k_1, \ldots, k_r, l; q) q^{-k_1s_1-\cdots-k_rs_r-ls_{r+1}} \]
in this case. We require that the coefficients \( \lambda(k_1, \ldots, k_r, l; \cdot) \) be defined at 1/q for every odd prime power q, and that
\[ a(\pi^{k_1}, \ldots, \pi^{k_r}; \pi^l) = |\pi|^{k_1+\cdots+k_r+l} \lambda(k_1, \ldots, k_r, l; 1/|\pi|) \]
for every monic irreducible \( \pi \) in \( \mathbb{F}_q[x] \). In addition, we require that
\[ b(p^{k_1}, \ldots, p^{k_r}, p^l) = p^{k_1+\cdots+k_r+l} \lambda(k_1, \ldots, k_r, l; 1/p) \]
for all primes \( p \neq 2 \). In other words, the \( p \)-part of the multiple Dirichlet series over the rationals coincides with the \( \pi \)-part of the multiple Dirichlet series over \( \mathbb{F}_p(x) \), for every monic polynomial \( \pi \) in \( \mathbb{F}_p[x] \). It may be worth remarking that the same principle applies over any number field \([26, 27] \), allowing one to obtain the corresponding multiple Dirichlet series in this context. Thus we need only discuss the rational function field case.

It turns out that any (untwisted) Weyl group multiple Dirichlet series over the rational function field and any of its own \( \pi \)-parts can be transformed into each other by a simple change of variables. This remarkable fact (which is also assumed in this work) was first noticed in \([23, 24] \) for the Weyl group of \( \Phi = A_2 \). Of course, the conditions imposed so far do not suffice to determine a Weyl group multiple Dirichlet series. In the finite-dimensional case, one starts with a reduced root system \( \Phi \), and seeks to construct a multiple Dirichlet series satisfying a group of functional equations isomorphic to the Weyl group of \( \Phi \). Over a rational function field this object can be obtained by using the Chinta-Gunnells averaging (over the Weyl group of \( \Phi \)) method, introduced in \([26] \) and further developed in \([25, 27] \). This method has been extended by Lee and Zhang \([69] \) to symmetrizable Kac-Moody root systems, but in general, the series constructed in this way fails to satisfy the required conditions. For the Weyl group multiple Dirichlet series associated to a quadratic Dirichlet L-series, for example, the underlying group of functional equations is isomorphic to the Weyl group \( W_r \) of the Kac-Moody algebra \( \mathfrak{g}(A) \) with generalized Cartan matrix (cf. \([59] \))

\[
A = \begin{pmatrix}
2 & -1 \\
2 & -1 \\
\vdots & \\
-1 & -1 & \cdots & -1 & 2
\end{pmatrix}
\]

In particular, this group is finite if \( r \leq 3 \) and infinite if \( r \geq 4 \). The Chinta-Gunnells construction gives in this context a series whose coefficients are easily seen (see \([18] \) and \([69] \)) to be polynomial functions of \( q \). On the other hand, as long as \( r \) is large enough\(^2\) this property is not compatible with the above conditions imposed on this series. What happens is that this construction is missing the precise contribution corresponding to the so-called imaginary roots of \( \mathfrak{g}(A) \), which grows in complexity as \( r \) increases.

Instead, we proceed as follows. The conditions imposed on the multiple Dirichlet series over \( \mathbb{F}_q(x) \) imply immediately the identity:
\[ \lambda(k_1, \ldots, k_r, l; q) = \sum \chi_{d_1}(\tilde{m}_1) \cdots \chi_{d_r}(\tilde{m}_r) a(m_1, \ldots, m_r, d) \quad \text{(for fixed } k_1, \ldots, k_r, l \in \mathbb{N} \text{)} \]
the sum being over all monic polynomials \( m_1, \ldots, d \) of degrees \( k_1, \ldots, l \), respectively; the coefficients \( a(m_1, \ldots, m_r, d) \) can be expressed as
\[ a(m_1, \ldots, m_r, d) = \prod_{\pi | d} q^{(k_1+\cdots+k_r+l) \deg \pi} \lambda(k_1, \ldots, k_r, l; q^{-\deg \pi}). \]

\(^2\)For instance, one can take \( r \geq 9 \).
These identities lead to a recurrence relation in the coefficients $\lambda(k_1, \ldots, k_r, l; q)$. (For this, we shall need to impose one additional constraint on these coefficients.) The main question arises whether, for any given $r$, there exists a multiple Dirichlet series satisfying all our requirements; this will be answered affirmatively by providing the explicit construction of this (uniquely determined) series.

To construct this multiple Dirichlet series, we proceed by induction on $l$. We have:

$$\lambda(k_1, \ldots, k_r, 0; q) = q^{k_1 + \cdots + k_r}$$

and

$$\lambda(k_1, \ldots, k_r, 1; q) = \begin{cases} q & \text{if } k_1 = \cdots = k_r = 0 \\ 0 & \text{otherwise; } \end{cases}$$

these follow directly from the compatibility/initial conditions. The coefficients corresponding to $l = 2$ can be read off from the identity

$$\sum_{k_1, \ldots, k_r \geq 0} \lambda(k_1, \ldots, k_r, 2; q) t_1^{k_1} \cdots t_r^{k_r} = \prod_{i=1}^r \left( 1 - qt_i \right)^{-1} \cdot \Res_{z=0} \left[ \frac{q^{z^2} - q^2}{z(z-1)} \prod_{i=1}^r \left( 1 - zt_i \right) \left( 1 - q t_i \right) \frac{1}{z} \right]$$

(see Proposition C.1, Appendix C); in particular, the coefficients $\lambda(k_1, \ldots, k_r, 2; q)$ are polynomial functions of $q$.

The first non-trivial coefficients occur when $l = 3$. To describe how these coefficients can be obtained, for $\kappa = (k_1, \ldots, k_r) \in \mathbb{N}^r$, put $|\kappa| = k_1 + \cdots + k_r$. Let $r_1 = r_1(\kappa)$ and $r_2 = r_2(\kappa)$ denote the number of components of $\kappa$ equal to 1 and 2, respectively. Then, in this case, (2) can be written as

$$\lambda(\kappa, 3; q) - q^{|\kappa|+4} \lambda(\kappa, 3; 1/q) = a(\kappa, 3; q)$$

where $a(\kappa, 3; q) = 0$, unless $r_1$ is even and $r_1 + 2r_2 = |\kappa|$, in which case,

$$a(\kappa, 3; q) = q^{2\kappa} M_3(r_1; q) + (q - 1) \cdot \sum_{k_i - k_j = 0, 1} (-1)^{|\kappa| - |\kappa'|} q^{|\kappa'| + 3} \lambda(\kappa', 2; 1/q).$$

Here $M_3(r; q)$ is the moment-sum defined by

$$M_3(r; q) := \sum_{\deg d_0 = 3} \left( - \sum_{\theta \in \mathbb{F}_q} \chi(d_0(\theta)) \right)^r$$

where $\chi$ is the non-trivial real character of $\mathbb{F}_q^*$, extended to $\mathbb{F}_q$ by setting $\chi(0) := 0$. One will notice that (4) implies the functional equation

$$a(\kappa, 3; q) = -q^{|\kappa|+4} a(\kappa, 3; 1/q)$$

which is not at all a priori clear from the above expression of $a(\kappa, 3; q)$, when $r_1$ is even and $r_1 + 2r_2 = |\kappa|$. The reason for this functional equation is, roughly, that $a(\kappa, 3; q)$ is a quantity attached to the one-point compactification $\bar{A}_1$ of the moduli space $A_1$ of elliptic curves, with $q^{2\kappa} M_3(r_1; q)$ corresponding to $A_1$ and the remaining piece of $a(\kappa, 3; q)$ corresponding to the boundary. In the general case, a similar geometric interpretation of (2) is required for our purposes, and establishing it (which is quite non-trivial) represents the main contribution of this work.

For example, take $r = 10$, $k_1 = \cdots = k_5 = 1$, and for simplicity, assume that $q \geq 3$ is a prime. One finds that

$$a(1, \ldots, 1, 3; q) = 42q^8 - 42q^6 - q^2 \tau(q) + q \tau(q)$$

where the function $n \mapsto \tau(n)$ is Ramanujan’s $\tau$-function, i.e., the $n$-th Fourier coefficient of the cusp form

$$\Delta(z) := e^{2\pi iz} \prod_{m=1}^{\infty} (1 - e^{2\pi imz})^{24} = \sum_{n=1}^{\infty} \tau(n) e^{2\pi imz} \quad (z = x + iy \text{ with } y > 0)$$
of weight $12$ for the group $SL_2(Z)$. This is an immediate consequence of (3) and a well-known identity of Bryan Birch [11]. We shall impose an additional (dominance) condition to determine the coefficients $\lambda(\kappa, l; q)$, which in this case gives

$$\lambda(1, \ldots, 1, 3; q) = 42 q^8 - q^2 \tau(q).$$

More generally, we will prove:

**Theorem.** — Let $q \geq 3$ be a prime, and let $T_{2k}(q) := \text{Tr}(T_q | S_{2k})$ denote the trace of the Hecke operator $T_q$ acting on the space of elliptic cusp forms of weight $2k$ on $SL_2(Z)$. For $\kappa = (k_1, \ldots, k_r) \in \mathbb{N}^r$, let $r_1 = r_1(\kappa)$ and $r_2 = r_2(\kappa)$ denote the number of components of $\kappa$ equal to 1 and 2, respectively. If $r_1 + 2r_2 = |\kappa|$, and $r_1 = 2R$ is even, then

$$\lambda(\kappa, 3; q) = \frac{(2R)!}{R!(R + 1)!} q^{R+r_2+3} - \frac{R}{1} \sum_{j=1}^{R} (2j + 1) \frac{(2R)!q^{R+r_2-j+2}}{(R-j)!((R+j+1)!} T_{2j+2}(q).$$

Otherwise, the coefficients $\lambda(\kappa, 3; q)$ all vanish.

When dealing with the general case, it is more convenient to work with moments defined for partitions $n = (1^{n_1}, 2^{n_2}, \ldots)$ by

$$A_n(t_1, \ldots, t_r, q) := \sum_{d \in \mathcal{P}(n,q)} \left( \prod_{k=1}^{r} P_{C_d}(t_k) \right);$$

here $\mathcal{P}(n,q) = \mathcal{P}(n,F_q) \subset F_q[x]$ is the set of all monic square-free polynomials $d$ with factorization type $n$, and $P_{C_d}(d \in \mathcal{P}(n,q))$ is the numerator of the zeta function of the (hyper)elliptic curve $C_d/F_q$ defined by the affine model $y^2 = d(x)$. The connection with the cohomology of local systems on moduli spaces of hyperelliptic curves comes by expressing the symmetric polynomial $A_n(t_1, \ldots, t_r, q)$ in terms of symplectic Schur functions [46, Appendix A, §A.3., A.45], and by applying Behrend’s Lefschetz trace formula [6] to each of the corresponding coefficients. This alternative expression and the Poincaré-Verdier duality will be used to define $A_n(t_1, \ldots, t_r, 1/q)$.

We note here that an identity of independent interest is obtained by taking a suitable linear combination of the quantities $A_n(q^{-1/2}, \ldots, q^{-1/2}, q)$ (see (50)). Concretely, for a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_g \geq 0)$ of weight $|\lambda|$, let $V_\lambda$ denote the corresponding irreducible representation of $Sp_{2g}(C)$. For $\lambda \in (r^g)$ (i.e., $\lambda_i \leq r$ for all $1 \leq i \leq g$), define $\lambda' := (g - \lambda_r' \geq \cdots \geq g - \lambda_1')$, where the non-zero integers among $\lambda'_j (j = 1, \ldots, r)$ are the parts of the conjugate partition of $\lambda$. As usual, write

$$\prod_{k=1}^{r} (2k!) = \frac{g_r}{(r+1)/2)!}$$

for the constant appearing in the moment-conjectures of Conrey-Farmer [29] and Keating-Snaith [61] for the leading-order asymptotics of the moments of L-functions within symplectic families, and let

$$P_r(x) = \sum_{i=1}^{r} (x + 2i) \prod_{1 \leq i < j \leq r} (x + i + j).$$

The polynomial $P_r(x-1)$ appears in the main term of the polynomial $Q_r(x)$ conjectured by Conrey, Farmer, Keating, Rubinstein and Snaith [30]; see also the recent work of Andrade and Keating [2, Conjecture 5] in the function-field setting. With this notation, we have

$$\frac{1}{q^{2g}(q-1)} \sum_{d \text{ monic & square-free}} \deg d = 2g+1 \sum L\left(\frac{1}{2}, \chi_d, \right)^r = \frac{g_r P_r(2g)}{(r+1)/2)!} + \sum (\dim V_\lambda) \text{Tr}(F^* c^\lambda_{\mathcal{H}_g}(w^1) \otimes_q F_q)) q^{1-2g-|\lambda|/2}$$

the sum in the right-hand side being over all non-trivial partitions $\lambda \in (r^g)$ of even weight, and the trace of the geometric Frobenius $F^*$ on the Euler characteristic $c^\lambda_{\mathcal{H}_g}(w^1) \otimes_q F_q)$ is given by

$$\text{Tr}(F^* c^\lambda_{\mathcal{H}_g}(w^1) \otimes_q F_q)) = \frac{1}{q} \sum_{d \text{ monic & square-free}} \deg d = 2g+1 s_{\lambda}(\omega(C_d)^{z_1}, \ldots, \omega(C_d)^{z_1});$$

6
Here \( s_{(\lambda)} \) is the symplectic Schur function associated to \( V_\lambda \), and for a hyperelliptic curve \( C_d, \omega_1(C_d), \ldots, \omega_d(C_d) \), \( \omega_1(C_d)^{-1}, \ldots, \omega_d(C_d)^{-1} \) are the normalized (i.e., unitary) eigenvalues of the endomorphism \( F^* \) of \( H^1_d(C_d, \mathbb{Q}_\ell) \) (with \( \ell \neq q \)). The sum in the right-hand side of the above moment-identity can be estimated using the following well-known result [60, Theorem 10.8.2] of Katz and Sarnak:

**Theorem (Katz-Sarnak).** — There exist positive constants \( A(g) \) and \( C(g) \) such that, for every partition \( \lambda \in (r^g) \), \( \lambda \neq 0 \), of even weight, we have the estimate

\[
\frac{1}{q^{2g}(q-1)} \left| \sum_{\deg d = 2g+1} s_{(\lambda)}(\omega_1(C_d)^{\pm 1}, \ldots, \omega_d(C_d)^{\pm 1}) \right| \leq \frac{2C(g)(\dim V_\lambda)}{\sqrt{q}}
\]

as long as \( q = |\mathbb{F}_q| \geq A(g) \).

Returning now to our main goal, we shall see that (2) (suitably transformed) reflects in fact certain relations among slightly more general moments of characteristic polynomials \( A_n(i,j)(t_1, \ldots, t_r, q) \), defined for partitions \( n \), \( i \) and \( j \). These moments can be obtained by applying certain differential operators to \( A_n(t_1, \ldots, q) \); in particular, we can define \( A_n(i,j)(t_1, \ldots, t_r, 1/q) \). We reiterate that the identity (2) (and implicitly the relations among \( A_n(i,j)(t_1, \ldots, t_r, q) \)) is just hypothetical at this point.

For notational convenience, we state here our main result only in the split case, i.e., for partitions \( n \) of the form \( (1^n) \). Although quite technical, the relations among the moments \( A_n(i,j)(t_1, \ldots, t_r, q) \) and their moduli interpretation in the general case, discussed in detail in Sections 7 and 8, follow the same principle as in the split case.

For \( n \geq 1 \), let \( \mathcal{P}_n \subset \mathbb{F}_q[x] \) denote the set of all monic square-free polynomials of degree \( n \) splitting in \( \mathbb{F}_q \). Define \( N_{i,j}(d, q) \) (for \( d \in \mathcal{P}_n \) and \( i, j \geq 0 \)) by

\[
N_{i,j}(d, q) = \prod_{k=0}^{i-1} \left( \frac{q - n + a_1(C_d) + \epsilon - 2k}{2} \right) \prod_{l=0}^{j-1} \left( \frac{q - n - a_1(C_d) - \epsilon - 2l}{2} \right)
\]

with \( \epsilon = 0 \) or 1 according as \( n \) is odd or even, and \( a_1(C_d) := \text{Tr}(F^*|H^1_d(C_d, \mathbb{Q}_\ell)) \). (If \( i = 0 \) or \( j = 0 \), we take the corresponding product to be 1.) Put

\[
A_n(i,j)(t_1, \ldots, t_r, q) := \sum_{d \in \mathcal{P}_n} \left( N_{i,j}(d, q) \prod_{k=1}^{r} P_{C_d}(t_k) \right).
\]

Note that if we denote \( D_{A_n,0,0}(t_1, \ldots, t_r, q) = \frac{\partial A_n,0,0}{\partial t_{r+1}}(t_1, \ldots, t_r, 0, q) \), then

\[
D_{A_n,0,0}(t_1, \ldots, t_r, q) = -\sum_{d \in \mathcal{P}_n} \left( a_1(C_d) \prod_{k=1}^{r} P_{C_d}(t_k) \right)
\]

Define \( D_{A_n,0,0}(t_1, \ldots, t_r, q) \) \((k \in \mathbb{N})\) by iterating. Then

\[
A_n(i,j)(t_1, \ldots, t_r, q) = \frac{\partial^{(r+n-i-j)}}{\partial x^{r+n-i-j}}(x^{n-1})^{\frac{1}{2}} A_{n,0,0}(t_1, \ldots, t_r, q)
\]

where the two binomial symbols are viewed as differential polynomials in \( D \). The right-hand side of this identity makes sense if we replace \( q \) by \( 1/q \), allowing us to define \( A_n(i,j)(t_1, \ldots, t_r, 1/q) \).

Now, for \( x, y \) and \( z \) algebraically independent variables, consider the (exponential) generating functions:

\[
c_{\text{odd}}(x, y, z; t_1, \ldots, t_r, q) = \sum_{n,i,j \geq 0} A_{2n+1,i,j}(t_1, \ldots, t_r, q) x^n y^i z^j
\]
where 

\[ \bar{c}(x_1, \ldots, x_r) = \left( \frac{1}{1 - t_k} \right)^{q - 1} \]

we have

\[ c_{\text{odd}}(x, z, -t_1, \ldots, -t_r, q) = c_{\text{odd}}(x, y, z; t_1, \ldots, t_r, q) \]

Finally, define

\[ c_{\text{even}}(x, y, z; t_1, \ldots, t_r, q) = c_{\text{even}}(x, z, y; -t_1, \ldots, -t_r, q), c_{\text{even}}(x, y, z; t_1, \ldots, t_r, q) \]

where \( x := (x, y, z) \). Then the relations required in this case are precisely the ones encoded in the following theorem, which itself is of independent interest.

**Theorem (The Split Case).** We have

\[ c(c(x, T, q), q T, 1/q) = x. \]

In other words, \( c(x, T, q) \) is the formal compositional inverse of \( c(x, q T, 1/q) \).

To prove this, we shall make essential use of the functional relation satisfied by the power series \( c(x, T, q) \) constructed later in Section 8. The corresponding functional relation in the general case is a simple consequence of Theorem 8.4, whose proof in turn rests on a careful analysis of the combinatorial structure of certain moduli spaces of admissible double covers.

The above result is in the same spirit as the following beautiful theorem of Getzler:

**Theorem (Getzler).** Let \( \mathcal{M}_{0,n} \) denote the moduli space of Riemann surfaces of genus zero with \( n \) ordered marked points, and let \( \overline{\mathcal{M}}_{0,n} \) denote its Deligne-Mumford compactification. Then the generating series

\[ f(x) = x - \sum_{n=0}^{\infty} \chi(\mathcal{M}_{0,n+1}) \frac{x^n}{n!} = 2x - (1 + x) \log(1 + x) \]

and

\[ g(x) = x + \sum_{n=2}^{\infty} \chi(\overline{\mathcal{M}}_{0,n+1}) \frac{x^n}{n!} \]

of Euler characteristics are compositional inverses of one another.

In fact, Getzler’s theorem is easily seen to be the limiting case of our theorem as

\[ t_1 = \cdots = t_r = 0, \quad y = z \quad \text{and} \quad q \to 1 \]

see Section 8 for details. Closely related results can be found in the work of Kisin and Lehrer [62], where it is shown how, in certain circumstances, an equivariant comparison theorem in \( \ell \)-adic cohomology may be used to convert the computation of the graded character of the induced action of a finite group (acting as a group of automorphisms of a smooth complex algebraic variety defined over a number field) on cohomology into questions about numbers of rational points of varieties over finite fields.

Having established a similar result in the general case, we shall apply Deligne’s theory of weights to determine inductively (essentially as in the case when \( l = 3 \)) the coefficients \( \lambda(k_1, \ldots, k_r, l; q) \); see Section 9 for details. The upshot is that we obtain inductively an expression for the differences

\[ \lambda(k_1, \ldots, k_r, l; q) - q^{k_1 + \cdots + k_r + l + 1} \lambda(k_1, \ldots, k_r, l; 1/q) \]
in terms of $q$-Weil numbers, from which the coefficients $\lambda(k_1, \ldots, k_r; l; q)$ are determined by imposing the dominance condition. Notice that, by applying a well-known result of Deligne [32, Proposition 4.8] which, in the notation of 6.1, gives that
\[
T_{\lambda+2}(q) = \text{Tr}(T_q | S_{\lambda+2}) = \text{Tr}(F^* | H^1(A_1 \otimes_{\mathbb{F}_q} \mathbb{F}_q, V(\lambda)))
\]
the above formula for the coefficients $\lambda(k_1, \ldots, k_r; 3; q)$ is, indeed, of this type.

The multiple Dirichlet series over $\mathbb{Q}$ constructed in this paper may be written symbolically as
\[
\sum_{d \text{ odd}} \frac{L^{(2)}(s_1, \chi_d) \cdots L^{(2)}(s_r, \chi_d)}{d^{s+1}}
\]
although, strictly speaking, the numerator is a product of $L$-series only when $d$ is square-free. When $r \leq 3$, this multiple Dirichlet series and its analytic properties have been studied in [83], [53], [19], [38] and [39].

On the other hand, it was noticed by Bump, Friedberg and Hoffstein in [19] that multiple Dirichlet series satisfying infinite groups of functional equations (e.g., (5) when $r = 4$) cannot be continued in $s_1, s_2, \ldots$ everywhere, i.e., they must have a wall of singularities; see the conjectures in loc. cit., p. 167 and p. 170. We should say that obtaining the continuation of (5), when $r \geq 4$, to a domain in $\mathbb{C}^{r+1}$ containing the point $(\frac{1}{2}, \ldots, \frac{1}{2})$ is an extremely difficult problem; the analogous problem for affine Weyl group multiple Dirichlet series over rational function fields of odd characteristic has been investigated in [18], [90] and [91] (see also [44], where an asymptotic formula for the fourth moment of quadratic Dirichlet $L$-functions in the rational function field case, summed over monic square-free polynomials, has been established). As far as we can see, when $r \geq 5$, this is a very difficult problem even in the rational function field case. We should point out here that a similar “natural boundary” phenomenon occurs when dealing with the meromorphic continuation of negative (i.e., lower triangular) parabolic Eisenstein series on loop groups, introduced in [12] in the function field case; see also [47], and the introduction of [48]. As already alluded to at the beginning of this introduction, the meromorphic continuation of similar metaplectic loop group Eisenstein series could potentially yield the relevant analytic properties of (5) when $r = 4$.

Finally, let us briefly describe the structure of the paper. In Section 2, we collect some generalities about quadratic Dirichlet $L$-functions, which will be used later in the paper. We then discuss the method introduced by Bump, Friedberg and Hoffstein in [20] to construct multiple Dirichlet series attached to moments of quadratic $L$-series. In particular, this method gives the “correct” Weyl group multiple Dirichlet series only when $r \leq 3$. In Section 3, we first list the conditions which the Weyl group multiple Dirichlet series (to be constructed) has to satisfy; these conditions imply immediately the identity (2). Next, we wish to express the left-hand side of this identity more conveniently. Probably the most efficient way to proceed is to reformulate (2) in terms of generating functions; after some preliminary considerations, we do so in Section 4. It is this form of the original identity where the moments $A_{n, i, j}(t_1, \ldots, t_r; q)$ arise. In Section 5, we combine a well-known formula for the above moment-sums $M_{d}(r; q)$ (see [11] and [58]) with Proposition B.2, Appendix B, Proposition D.1, Appendix D, and the formula given in Proposition C.2, Appendix C to illustrate how the coefficients $\lambda(k_1, \ldots, k_r; l; q)$ ($l = 3, 4$) are constructed. In Section 6, we review some facts about the cohomology of symplectic local systems on moduli of hyperelliptic curves, and discuss the cohomological interpretation of the above moment-sums $A_n(t_1, \ldots, t_r, q)$. We then use this interpretation to define $A_n(t_1, \ldots, t_r, 1/q)$ by duality. In Section 7, we prove our main result, first in the split case, and then in general, subject to a functional relation satisfied by a certain generating function, whose proof is postponed to Section 8. In particular, this ensures the compatibility of the relations generated by (2) among the coefficients of the multiple Dirichlet series. In Section 8, we study various generating series attached to moduli spaces of admissible double covers, and then show how these series are connected to the generating series introduced in the previous sections. We then use this connection to establish the functional relation needed to finish the proof of Theorem 7.5. This section, which may be of some interest in its own right, is essentially independent from the rest of the paper. In Section 9, we recall some facts from Deligne’s theory [35], which will then be used to determine inductively the coefficients $\lambda(k_1, \ldots, k_r; l; q)$. It should be pointed out that the arguments can be reversed to obtain new information about the cohomology of local systems on moduli of hyperelliptic curves, assuming full knowledge\footnote{When $r \leq 3$, these coefficients can be computed from the explicit expressions of $Z(t_1, \ldots, t_{r+1}; q) \ (r \leq 3)$ recorded in Appendix A.} of the coefficients $\lambda(k_1, \ldots, k_r; l; q)$. We illustrate this in Section 10 by computing explicitly the trace of
Frobenius on the (motivic) Euler characteristic corresponding to the Eisenstein cohomology of local systems on moduli spaces of principally polarized abelian surfaces, and thus recovering a result of Bergström, Faber and van der Geer (see [8, Corollary 4.6] and [49, Theorem 9.1]).

Here is a roadmap summarizing the key steps of our construction:

![Roadmap diagram]

For the reader’s convenience, we have also included four appendices. In the first one, we provide the (previously known) simple description of the Weyl group multiple Dirichlet series discussed in this paper, for each \( r \leq 4 \). In Appendix C, we compute explicitly the generating series of the coefficients \( \lambda(k_1, \ldots, k_r, 2; q) \), and in Appendices B and D, we discuss some elementary relations among moments of character sums used in previous sections.

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2 Preliminaries

We recall that for a smooth, projective, and geometrically connected curve \( C \) of genus \( g \geq 1 \) over a finite field \( F = \mathbb{F}_q \) with \( q \) elements, one defines the zeta function \( Z_C(t) \) of \( C \) by

\[
Z_C(t) = \exp\left(\sum_{n \geq 1} |C(\mathbb{F}_{q^n})| \frac{t^n}{n}\right) \quad \text{(for } |t| < 1/q\text{)}.
\]

It is well-known that \( Z_C(t) \) is a rational function

\[
Z_C(t) = \frac{P_C(t)}{(1-t)(1-qt)}
\]

with numerator \( P_C(t) \in \mathbb{Z}[t] \) of degree \( 2g \), and constant term \( P_C(0) = 1 \). Moreover, it satisfies the functional equation

\[
Z_C(t) = (qt^2)^{-1} Z_C(1/qt).
\]  
(6)

Although we shall not need it here, we recall for completeness that the analogue of the Riemann hypothesis, proved by André Weil [87], states that \( P_C(t) \) has all its zeros on the circle \( |t| = q^{-1/2} \).

From now on, we shall assume that \( q \) is odd, and fix once for all an algebraic closure \( \overline{F} \) of \( F \). For a non-zero polynomial \( m \in \mathbb{F}[x] \), we define its norm by \( |m| = q^{\deg m} \). Let \( \pi \in \mathbb{F}[x] \) be monic and irreducible. For a polynomial \( d \in \mathbb{F}[x] \) coprime to \( \pi \), we define the Legendre symbol \( (d/\pi) = 1 \) or \(-1\) according as \( d \) is a square modulo \( \pi \) or not; if \( d \in \mathbb{F}[x] \)
and \(\pi\) divides \(d\), we set \((d/\pi) = 0\). Finally, define \((d/m)\) for arbitrary \(d \in \mathbb{F}[x]\) and monic \(m \in \mathbb{F}[x]\) by setting \((d/m) = 1\) if \(m = 1\), and by multiplicativity for general \(m\).

The quadratic residue symbol \((d/m)\) we just defined is completely multiplicative in both \(d\) and \(m\), and satisfies the following reciprocity law [79]:

**Reciprocity.** For \(d, m \in \mathbb{F}[x]\) monic polynomials, we have

\[
\left(\frac{d}{m}\right) = (-1)^{\frac{|d| - 1}{2} \frac{|m| - 1}{2}} \left(\frac{m}{d}\right).
\]

In addition, we have the supplement

\[
\left(\frac{d}{m}\right) = \text{sgn} \deg m (d) \quad \text{(for } d \in \mathbb{F}^\times)\]

where \(\text{sgn}(d) = 1\) or \(-1\) according as \(d\) is a square in \(\mathbb{F}\) or not.

**L-functions.** For \(d \in \mathbb{F}[x]\) square-free and a monic polynomial \(m \in \mathbb{F}[x]\), let \(\chi_d(m) := (d/m)\). We define the quadratic Dirichlet L-function attached to \(\chi_d\) by

\[
L(s, \chi_d) = \sum_{m \in \mathbb{F}[x]} \chi_d(m)[m]^{-s} = \prod_{\pi}(1 - \chi_d(\pi)|\pi|^{-s})^{-1} \quad \text{(for complex } s \text{ with } \Re(s) > 1);
\]

see Artin [4]. Here the product is over all monic irreducible polynomials \(\pi \in \mathbb{F}[x]\). In particular, when \(d = 1\) (i.e., \(\chi_d\) is trivial) we obtain the zeta function of \(\mathbb{F}(x)\),

\[
\zeta_{\mathbb{F}(x)}(s) = \sum_{m=\text{monic}} |m|^{-s} = \frac{1}{1 - q^{-s}}
\]

and for \(d \in \mathbb{F}^\times \setminus (\mathbb{F}^\times)^2\)

\[
L(s, \chi_d) = \frac{1}{1 + q^{-s}}.
\]

If \(\deg d = k \geq 1\), the L-function \(L(s, \chi_d)\) is a polynomial (in \(q^{-s}\)) of degree \(k - 1\). To be more precise, let \(C_d\) denote the hyperelliptic curve defined in affine form by \(y^2 = d(x)\), and consider the numerator \(P_{C_d}(t)\) of the zeta function \(Z_{C_d}(t)\); if \(\deg d = 1, 2\), we take \(P_{C_d}(t) = 1\). Then

\[
L(s, \chi_d) = \sum_{i=0}^{k-1} q^{-is} \sum_{m=\text{monic}} \chi_d(m) = (1 + q^{-s})^{\epsilon_k} P_{C_d}(q^{-s})
\]

where the + or - sign is determined according to whether the leading coefficient of \(d\) is a square in \(\mathbb{F}\) or not, and \(\epsilon_k = 0\) or 1 according as \(k\) is odd or even. From (6), we deduce that \(L(s, \chi_d)\) satisfies the functional equation

\[
(1 + q^{-s})^{-\epsilon_k} L(s, \chi_d) = q^{(\epsilon_k + 1)(s - \frac{1}{2})} [d]^\frac{1}{2} - s (1 + q^{s - 1})^{-\epsilon_k} L(1 - s, \chi_d).
\]

(7)

**Multiple Dirichlet series attached to moments of quadratic L-functions.** Adapting the discussion from [20], this is a function of several complex variables admitting series representations of the form

\[
Z(s_1, \ldots, s_{r+1}) = \sum_{d \in \mathbb{F}[x] \atop d \text{-monic}} A(s_1, \ldots, s_r; d) \quad \text{= } \sum_{m_1, \ldots, m_r \in \mathbb{F}[x] \atop m_1, \ldots, m_r \text{-monic}} \frac{B(s_{r+1}; m_1, \ldots, m_r)}{|m_1|^{s_1} \cdots |m_r|^{s_r}}
\]

for \(s_1, \ldots, s_{r+1}\) with sufficiently large real parts, and whose coefficients are Euler products

\[
A(s_1, \ldots, s_r; d) = \prod_{\pi \mid d} A_{\pi, l}(s_1, \ldots, s_r, d_0) \quad \text{and } B(s_{r+1}; m_1, \ldots, m_r) = \prod_{\pi \mid m_0} B_{\pi, k}(s_{r+1}, M_0)
\]
over all monic irreducibles $\pi \in \mathbb{F}[x]$. Here we set $\kappa = (k_1, \ldots, k_r)$, and for $d, m_1, \ldots, m_r$ monic polynomials, we denoted by $d_0$ and $M_0$ the monic square-free parts of $d$ and $M = m_1 \cdots m_r$, respectively. Letting $\widetilde{\chi}_{M_0}(\pi) = (\pi/M_0)$ and $|\kappa| = k_1 + \cdots + k_r$, we can present the local factors as

$$A_{\pi, l}(s_1, \ldots, s_r, d_0) = \begin{cases} \prod_{r=1}^l (1 - \chi_{d_0}(\pi)|\pi|^{-s_1})^{-1} \cdot P_{\pi, l}(s_1, \ldots, s_r, \chi_{d_0}(\pi)) & \text{if } l \geq 0 \text{ is even} \\ P_{\pi, l}(s_1, \ldots, s_r) & \text{if } l \text{ is odd} \end{cases}$$

and

$$B_{\pi, \kappa}(s_{r+1}, M_0) = \begin{cases} (1 - \widetilde{\chi}_{M_0}(\pi)|\pi|^{-s_{r+1}})^{-1} \cdot Q_{\pi, \kappa}(s_{r+1}, \widetilde{\chi}_{M_0}(\pi)) & \text{if } |\kappa| \geq 0 \text{ is even} \\ Q_{\pi, \kappa}(s_{r+1}) & \text{if } |\kappa| \text{ is odd} \end{cases}$$

for a certain class of polynomials $P_{\pi, l}$ (respectively $Q_{\pi, \kappa}$) in $|\pi|^{-s_1}, \ldots, |\pi|^{-s_1}$ (respectively $|\pi|^{-s_{r+1}}$) characterized as follows.

For arbitrary odd prime power $q$, $\kappa = (k_1, \ldots, k_r) \in \mathbb{N}^r$ and $l \in \mathbb{N}$, there exist (see [20]) polynomials $P_l(t_1, \ldots, t_r; q)$ and $Q_\kappa(t_{r+1}; q)$ in $t_1, \ldots, t_r$ (respectively $t_{r+1}$) with coefficients depending on $l$ and $q$ (respectively $\kappa$ and $q$) such that:

1. For sufficiently small $|t_1|, \ldots, |t_{r+1}|$, we have

$$\left(1 - t_1\right)^{-1} \cdots \left(1 - t_r\right)^{-1} \sum_{l - \text{even}} P_l(t_1, \ldots, t_r; q)t_{r+1}^l + \sum_{l - \text{odd}} P_l(t_1, \ldots, t_r; q)t_{r+1}^l = \left(1 - t_{r+1}\right)^{-1} \sum_{|\kappa| - \text{even}} Q_\kappa(t_{r+1}; q)t_1^{s_1} \cdots t_r^{s_r} + \sum_{|\kappa| - \text{odd}} Q_\kappa(t_{r+1}; q)t_1^{s_1} \cdots t_r^{s_r}$$

i.e., the generating series of the polynomials $P_l(t_1, \ldots, t_r; q)$ and $Q_\kappa(t_{r+1}; q)$ coincide. We take this series normalized by assuming that $P_0(0, \ldots, 0; q) = 1$.

2. The power series in $t_1, \ldots, t_{r+1}$ obtained by expanding

$$\sum_{l \geq 0} P_l(t_1, \ldots, t_r; q)t_{r+1}^l$$

is absolutely convergent for arbitrary $t_1, \ldots, t_r \in \mathbb{C}$, provided $|t_{r+1}|$ is sufficiently small, and for arbitrary $t_{r+1} \in \mathbb{C}$, provided all $|t_1|, \ldots, |t_r|$ are sufficiently small.

3. The polynomials $P_l(t_1, \ldots, t_r; q)$ are symmetric in the variables $t_1, \ldots, t_r$, and they satisfy the functional equation

$$P_l(t_1, t_2, \ldots, t_r; q) = \begin{cases} \left(q^{-l}\right)^{l/2} P_l(1/qt_1, t_2, \ldots, t_r; q) & \text{if } l \text{ is even} \\ \left(q^{-l}\right)^{(l-1)/2} P_l(1/qt_1, t_2, \ldots, t_r; q) & \text{if } l \text{ is odd} \end{cases}$$

4. The polynomials $Q_\kappa(t_{r+1}; q)$ satisfy the functional equation

$$Q_\kappa(t_{r+1}; q) = \begin{cases} \left(q^{-r+1}\right)^{|\kappa|/2} Q_\kappa(1/qt_{r+1}; q) & \text{if } |\kappa| \text{ is even} \\ \left(q^{-r+1}\right)^{(|\kappa|-1)/2} Q_\kappa(1/qt_{r+1}; q) & \text{if } |\kappa| \text{ is odd} \end{cases}$$

5. If $l$ is odd, $P_l(t_1, \ldots, t_r; q) = P_l(-t_1, \ldots, -t_r; q)$.

Defining

$$P_{\pi, l}(s_1, \ldots, s_r, \chi_{d_0}(\pi)) = P_l(\chi_{d_0}(\pi)|\pi|^{-s_1}, \ldots, \chi_{d_0}(\pi)|\pi|^{-s_r}; |\pi|) \quad \text{if } l \geq 0 \text{ is even}$$

$$P_{\pi, l}(s_1, \ldots, s_r) = P_l(|\pi|^{-s_1}, \ldots, |\pi|^{-s_r}; |\pi|) \quad \text{if } l \text{ is odd}$$
the Coxeter group where for the latter, we use in addition the quadratic reciprocity. The group of functional equations is isomorphic to

\[ Q_{\pi, \kappa}(s_{r+1}, \delta_{\mu_0}(\pi)) = Q_{\kappa}(\delta_{\mu_0}(\pi)|\pi|^{-s_{r+1}}; |\pi|) \quad \text{if } |\kappa| \geq 0 \text{ is even} \]
\[ Q_{\pi, \kappa}(s_{r+1}) = Q_{\kappa}(|\pi|^{-s_{r+1}}; |\pi|) \quad \text{if } |\kappa| \text{ is odd} \]

one checks that, indeed, the identity (8) holds. Note that \( P(t_1, \ldots, t_r; q) = 1 \) if \( l = 0, 1 \), and hence \( A(s_1, \ldots, s_r; 1) = \zeta_{\infty}(s_1) \cdots \zeta_{\infty}(s_r) \), and \( A(s_1, \ldots, s_r; d) = L(s_1, \chi_{d}) \cdots L(s_r, \chi_{d}) \) if \( d = d_0 \) is monic and square-free; we also have \( B(s_{r+1}; m_1, \ldots, m_r) = L(s_{r+1}, \delta_{m_1}, \ldots, m_r) \) if the product \( m_1 \cdots m_r \) is monic and square-free. Accordingly, it makes sense to denote the coefficients \( A(s_1, \ldots, s_r; d) \) and \( B(s_{r+1}; m_1, \ldots, m_r) \), for arbitrary monomials \( d, m_1, \ldots, m_r \), by \( L(s_1, \chi_{d}) \cdots L(s_r, \chi_{d}) \) and \( L(s_{r+1}, \delta_{m_1}, \ldots, m_r) \), respectively. We hope that our notation is not causing any confusion, especially with a product of \( r \) imprimitive \( L \)-functions in case \( d \) is not a square-free polynomial.

It can be observed that the multiple Dirichlet series (8) can be expressed as a power series in \( q^{-s_1}, i = 1, \ldots, r + 1 \), for \( s_1, \ldots, s_{r+1} \) with sufficiently large real parts. In what follows, we shall substitute \( t_i = q^{-s_i} \) for \( i = 1, \ldots, r + 1 \), set \( T = (t_1, \ldots, t_r) \), and denote the power series of (8) by \( Z(T, t_{r+1}; q) \). This function satisfies a group of functional equations generated by the \( r + 1 \) involutions \( \alpha_i : (T, t_{r+1}) \to (t_1, \ldots, 1/qt_i, \ldots, t_r, \sqrt{q} t_{r+1}) \) for \( 1 \leq i \leq r \), and \( \alpha_{r+1} : (T, t_{r+1}) \to (\sqrt{q} t_{r+1} T, 1/q t_{r+1}) \). Indeed, from the definition of \( Z(T, t_{r+1}; q) \) and (7), one finds that:

\[
Z(T, t_{r+1}; q) = \frac{1}{\sqrt{q} t_{r+1}} \left( \frac{Z(\alpha_i(T, t_{r+1}); q) - Z(\alpha_i(-T, t_{r+1}); q)}{2} \right)
- \frac{1 - q t_i}{t_i(1 - q t_i)} \left( \frac{Z(\alpha_i(T, t_{r+1}); q) + Z(\alpha_i(-T, t_{r+1}); q)}{2} \right)
\]

(9) for \( 1 \leq i \leq r \), and

\[
Z(T, t_{r+1}; q) = \frac{1}{\sqrt{q} t_{r+1}} \left( \frac{Z(\alpha_{r+1}(T, t_{r+1}); q) - Z(\alpha_{r+1}(-T, t_{r+1}); q)}{2} \right)
- \frac{1 - t_{r+1}}{t_{r+1}(1 - q t_{r+1})} \left( \frac{Z(\alpha_{r+1}(T, t_{r+1}); q) + Z(\alpha_{r+1}(-T, t_{r+1}); q)}{2} \right)
\]

(10) where for the latter, we use in addition the quadratic reciprocity. The group of functional equations is isomorphic to the Coxeter group \( W_r \) mentioned in the introduction. In particular, it is finite if \( r \leq 3 \), and infinite if \( r \geq 4 \). It is straightforward to check that the functional equations (9) and (10) are equivalent to certain recurrence relations among the coefficients of the power series \( Z(T, t_{r+1}; q) \). Fortunately, when \( r \leq 3 \), one can apply these recurrence relations as in the proof of Theorem 3.7 in [18] to determine all the coefficients of \( Z(T, t_{r+1}; q) \); we refer the reader to Appendix A for an expression of \( Z(T, t_{r+1}; q) \) as a rational function in each case. When \( r \geq 4 \), we just remark for the moment that additional information is required to completely characterize the power series \( Z(T, t_{r+1}; q) \); the case \( r = 4 \) discussed in [18] is still very special in many respects.

On the other hand, letting \( P(T, t_{r+1}; q) \) denote the generating series in (1), the properties (3) and (4) satisfied by the polynomials \( P_i \) and \( Q_{\pi, \kappa} \), respectively, translate into the functional equations:

\[
P(T, t_{r+1}; q) = \frac{1}{\sqrt{q} t_{r+1}} \left( \frac{P(\alpha_i(T, t_{r+1}); q) - P(\alpha_i(-T, t_{r+1}); q)}{2} \right)
- \frac{1 - q t_i}{q t_i(1 - t_i)} \left( \frac{P(\alpha_i(T, t_{r+1}); q) + P(\alpha_i(-T, t_{r+1}); q)}{2} \right)
\]

for \( i = 1, \ldots, r \), and

\[
P(T, t_{r+1}; q) = \frac{1}{\sqrt{q} t_{r+1}} \left( \frac{P(\alpha_{r+1}(T, t_{r+1}); q) - P(\alpha_{r+1}(-T, t_{r+1}); q)}{2} \right)
- \frac{1 - t_{r+1}}{q t_{r+1}(1 - t_{r+1})} \left( \frac{P(\alpha_{r+1}(T, t_{r+1}); q) + P(\alpha_{r+1}(-T, t_{r+1}); q)}{2} \right)
\]
As in [26], we call $P(T, t_{r+1}; q)$, for $q = |\pi|$, the $\pi$-part of $Z(T, t_{r+1}; q)$. We note that there is a slight difference, as explained in [26, Remark 4.3], between this definition of the $\pi$-part of our multiple Dirichlet series, and the generating series of the $p$-part-coefficient $H(p^{k_1}, \ldots, p^{k_r})$ in [13, 14, 15] and [27].

Just as for $Z(T, t_{r+1}; q)$, the power series $P(T, t_{r+1}; q)$ is also uniquely determined by its properties, when $r \leq 3$. Then, we must have that

$$P(T, t_{r+1}; q) = Z(qT, qt_{r+1}; 1/q) \quad \text{(for } r \leq 3)$$

as the power series in the right-hand side satisfies the required properties. The reader might want to verify the relevant facts about $Z(qT, qt_{r+1}; 1/q)$, when $r \leq 3$, using the expression of $Z(T, t_{r+1}; q)$ given in Appendix A. Unfortunately, when $r \geq 4$ the properties (1) – (5) are insufficient to determine $P(T, t_{r+1}; q)$, the obvious reason being that in this case there exist non-trivial power series in $t_1, \ldots, t_{r+1}$ invariant under the group of functional equations. For example, if $r = 4$ a power series in the variable $t_1 \cdots t_4 t_5^2$ is, indeed, invariant under this group. Very shortly, we shall propose a way to remedy this deficiency, but before doing so, let us conclude this section with the following remark.

**Remark.** As indicated in the introduction, every choice of the generating series (1) gives rise, as above (with just some standard adjustments), to the corresponding number field version of (8). When $r \leq 3$, this Weyl group multiple Dirichlet series is a Whittaker coefficient of a minimal parabolic Eisenstein series on the double cover of a split, semisimple, simply-connected algebraic group. To investigate the possibility of extending this result, it is important (as previous experience shows) to have the potential candidates for the generating series $P(T, t_{r+1}; q)$ ($r \geq 4$) available beforehand.

### 3 The series $Z(T, t_{r+1}; q)$ and moments of character sums

To correct the deficiency indicated at the end of the previous section, we begin by imposing more stringent conditions on the multiple Dirichlet series (8). It is interesting to note that our assumptions make no reference to the group $W_r$ of functional equations.

As in [24], we first require $Z(s_1, \ldots, s_{r+1})$ to be of the form

$$Z(s_1, \ldots, s_{r+1}) = \sum_{m_1, \ldots, m_r, d \in \mathbb{F}[x]} \frac{\chi_{d_0}(\hat{m}_1) \cdots \chi_{d_r}(\hat{m}_r)}{|m_1|^{s_1} \cdots |m_r|^{s_r} |d|^{s_{r+1}}} \cdot a(m_1, \ldots, m_r, d)$$

the sum taken over all $(r+1)$-tuples of monic polynomials, with $\hat{m}_i$ the part of $m_i$ coprime to $d_0$ for $i = 1, \ldots, r$. The coefficients $a(m_1, \ldots, m_r, d)$ are assumed to be multiplicative, i.e.,

$$a(m_1, \ldots, m_r, d) = \prod_{\pi^{k_i} | m_i, \pi^{k_r} \nmid d} a(\pi^{k_i}, \ldots, \pi^{k_r}, \pi^l)$$

the product being taken over monic irreducibles $\pi \in \mathbb{F}[x]$. Now, express the series (12) as

$$Z(T, t_{r+1}; q) = \sum_{k_1, \ldots, k_r, l \geq 0} \lambda(k_1, \ldots, k_r, l; q) t_1^{k_1} \cdots l_r^{k_r} t_{r+1}^{l}$$

where, as before, we substituted $t_i = q^{-a_i}$, for $i = 1, \ldots, r + 1$, and set $T = (t_1, \ldots, t_r)$. We require $a(\pi^{k_1}, \ldots, \pi^{k_r}, \pi^l)$ and $\lambda(k_1, \ldots, k_r, l; q)$ be such that:

(i) **Initial Conditions.** The subseries

$$\sum_{k_1, \ldots, k_r \geq 0} \lambda(k_1, \ldots, k_r, 0; q) t_1^{k_1} \cdots l_r^{k_r} = \prod_{i=1}^{r} \frac{1}{1 - qt_i}$$

14
i.e., is a product of \( r \) zeta functions. In addition,

\[
\sum_{k_1, \ldots, k_r \geq 0} \lambda(k_1, \ldots, k_r; q) t_1^{k_1} \cdots t_r^{k_r} = \sum_{\text{deg} d = 1} L(s_1, \chi_d) \cdots L(s_r, \chi_d) = q
\]

and

\[
\sum_{l \geq 0} \lambda(0, \ldots, 0; l; q) t_{r+1}^l = \frac{1}{1 - qt_{r+1}}.
\]

In particular, \( a(1, \ldots, 1, 1) = \lambda(0, \ldots, 0; q) = 1 \).

(ii) **Extension of (11).** Fix nonnegative integers \( k_1, \ldots, k_r, l \). Then, for each odd prime \( p \), there exists a finite number of complex numbers \( c_j \) and \( w_j = p^{a_j + b_j} (a_j, b_j \in \mathbb{R}) \) such that

\[
\lambda(k_1, \ldots, k_r; p^n) = \sum_j c_j (w_j^n + \overline{w_j}^n) \quad \text{(for all } n \in \mathbb{Z})
\]

all data in the right-hand side depending on \( k_1, \ldots, k_r, l \) and \( p \). Moreover,

\[
a(\pi^{k_1}, \ldots, \pi^{k_r}, \pi^l) = |\pi|^{k_1 + \cdots + k_r + l} \lambda(k_1, \ldots, k_r, l; 1/|\pi|)
\]

for every monic irreducible \( \pi \in \mathbb{F}[x] \).

(iii) **Dominance.** For every \((r+1)\)-tuple \((k_1, \ldots, k_r, l) \in \mathbb{N}^{r+1} \) with \( k_1 + \cdots + k_r + l \geq 2 \), the coefficient \( \lambda(k_1, \ldots, k_r, l; q) \) represents the dominant half (as in the example below) of

\[
\lambda(k_1, \ldots, k_r, l; q) - q^{k_1 + \cdots + k_r + l + 1} \lambda(k_1, \ldots, k_r, l; 1/q)
\]

that is, \( \min\{a_j\} > (k_1 + \cdots + k_r + l + 1)/2 \).

**Example.** When \( r = 10 \), \( k_1 = \cdots = k_r = 1 \), \( l = 3 \), and \( q \) is an odd prime, it turns out (see Section 5) that

\[
\lambda(1, \ldots, 1, 3; q) - q^{14} \lambda(1, \ldots, 1, 3; 1/q) = 42q^8 - 42q^6 - q^2 \tau(q) + q\tau(q)
\]

where \( \tau(q) \) is Ramanujan’s tau function; we recall, see [32], that \( \tau(q) = \alpha_q + \overline{\alpha_q} \) with \( |\alpha_q| = q^{1/2} \). By (iii), it follows that \( \lambda(1, \ldots, 1, 3; q) = 42q^8 - q^2 \tau(q) \). (Here we set \( \tau(1/q) = q^{-11} \tau(q) \).

To obtain concrete information about the coefficients of the multiple Dirichlet series (12), fix \( k_1, \ldots, k_r, l \in \mathbb{N} \). Comparing the coefficients \( (k_1, \ldots, k_r, l) \) of (12) and (14), we see that

\[
\lambda(k_1, \ldots, k_r, l; q) = \sum \chi_{d_0}(\tilde{m}_1) \cdots \chi_{d_0}(\tilde{m}_r) a(m_1, \ldots, m_r, d)
\]

with the sum over all monic polynomials \( m_1, \ldots, m_r, d \) of degrees \( k_1, \ldots, k_r, l \), respectively. By (13) and (ii), the dependence of the coefficients \( a(m_1, \ldots, m_r, d) \) on monic irreducible polynomials \( \pi \) is only upon their degrees \( \mu_\pi \), and the exact powers \( \nu_1, \pi, \ldots, \nu_r, \pi \) to which \( \pi \) is dividing \( m_1, \ldots, m_r, d \). This simple observation will be used to express the right-hand side of (15) in terms of classical moments of (projective) character sums defined for partitions \( \mu = (\mu_1 \geq \cdots \geq \mu_n \geq 1) \) and \( \gamma = (1^{\gamma_1}, \ldots, m^{\gamma_m}) \) by

\[
M_{\mu, \gamma}(q) = \sum_{d_0 \in \mathcal{P}(\mu)} \prod_{j=1}^{m} \left( -\frac{1 + (-1)^{\deg d_0}}{2} - \sum_{\theta \in \mathcal{F}_j} \chi_j(d_0(\theta)) \right)^{\gamma_j}.
\]

Here the sum is over the subset \( \mathcal{P}(\mu) \in \mathbb{F}[x] \) of all monic square-free polynomials \( d_0 \) having factorization type (over \( \mathbb{F} \)) into irreducibles \( d_0 = \prod_{j=1}^{m} \pi_j \), with \( \deg \pi_j = \mu_j \); for \( j \geq 1 \), we put \( \mathbb{F}_j = \mathbb{F}_{q^j} \), and \( \chi_j : = \chi \circ \mathbb{N}_{\mathbb{F}_{q^j}, \mathbb{F}} \) is the non-trivial real character of \( \mathbb{F}_{q^j}^* \), extended to \( \mathbb{F}_j \) by setting \( \chi_j(0) = 0 \). The connection to moments of character sums will play a major role in this work. It will become apparent that the conditions (i)–(iii) together with the identity (15) allow a recursive
computation of the coefficients \( \lambda(k_1, \ldots, k_r; l; q) \), provided certain precise information about the moments \( M_{\mu, \gamma}(q) \) could be obtained.

We begin by first summing over all \( m_1, \ldots, m_r, d \) with prescribed factorizations into irreducibles by giving a partition \( \mu = (\mu_1, \ldots, \mu_n) \) together with a collection of \( (r + 1) \)-tuples \((\nu_1, \ldots, \nu_{r+1}, \nu) \in \mathbb{N}^{r+1} \), with \( 1 \leq j \leq n \), such that

\[
\sum_{j=1}^{n} \nu_j \mu_j = k_i \quad \text{and} \quad \sum_{j=1}^{n} \nu_j \mu_j = l \quad (\text{for } i = 1, \ldots, r).
\]  

(16)

To \( \mu \) and \( N := (\nu_1, \ldots, \nu_r, \nu) \) as above, we associate the coefficients

\[
\lambda_\mu(N; q) := \prod_{j=1}^{n} q^{\nu_j(\mu_1 \cdots \mu_j \cdots \mu)} \lambda(\nu_1, \ldots, \nu_r, \nu; 1/q^\nu)
\]  

(17)

and certain character sums \( S_{\mu, N} \) defined as follows. Split the set of indices \( 1 \leq j \leq n \) into two parts \( J_0 \) and \( J_1 \) according as \( \nu_j \) is odd or even. Let \( \nu := (\nu_1, \ldots, \nu_n) \), and let \( \mathcal{P}_v(\mu) \subset \mathbb{F}[x] \) denote the set of all monic square-free polynomials \( d_0 \) having factorization type into irreducibles, \( \prod_{j \in J_0} \pi_j \) with \( \deg \pi_j = \mu_j \). With this notation, we define

\[
S_{\mu, N} := C_{\mu, \nu} \sum_{d_0 \in \mathcal{P}_v(\mu)} \sum_{\theta_j \in J_1} \prod_{j \in J_1} \chi_{\mu_j}(d_0(\theta_j))^{\nu_j + \cdots + \nu_r}
\]

where \( \sum^* \) indicates that we are summing over all tuples \((\theta_j)_{j \in J_1}\) such that \( \mathbb{F}(\theta_j) = \mathbb{F}_{\mu_j} \) for every \( j \in J_1 \), and \( \sigma(\theta_j) \neq \theta_j \), for all \( \sigma \in \text{Gal}(\mathbb{F}/\mathbb{F}) \), and all \( j \neq j' \). The normalizing constant is given by

\[
C_{\mu, \nu} = \prod_{i=1}^{m} n_i!^m n_i!
\]

where \( \mu_{j_1}, \ldots, \mu_{j_m} \) are the distinct components of \((\mu_j)_{j \in J_0}\), and \( n_i \), for \( i = 1, \ldots, m \), is the multiplicity of \( \mu_{j_i} \) in \( J_1 \), i.e., the cardinality of the set \( J_1^{(i)} = \{ j \in J_1 : \mu_j = \mu_{j_i} \} \).

Recalling that \( \chi_{d_0}(\pi) = \chi_\pi(d_0(\theta)) \) (see [79]) for monic \( d_0, \pi \in \mathbb{F}[x] \), with \( d_0 \) square-free and \( \pi \) irreducible of degree \( j \), and \( \theta \in \mathbb{F} \) any root of \( \pi \), we see that the identity (15) can be presented in the equivalent form

\[
\lambda(\kappa; l; q) = \sum \lambda_\mu(N; q) S_{\mu, N} \quad ((\kappa, l) := (k_1, \ldots, k_r, l))
\]  

(18)

with the sum over all pairs \((\mu, N)\) satisfying (16) and the following simple condition: for all \( j \in J_0 \) and \( j' \in J_1 \) such that \( \mu_j = \mu_{j'} \), we shall always assume that \( j < j' \).

From the initial conditions (i), we compute

\[
\lambda(k_1, \ldots, k_r, 0; q) = q^{k_1 + \cdots + k_r}.
\]  

(19)

for all \( k_1, \ldots, k_r \in \mathbb{N} \), and

\[
\lambda(k_1, \ldots, k_r, 1; q) = \begin{cases} q & \text{if } k_1 = \cdots = k_r = 0 \\ 0 & \text{otherwise.} \end{cases}
\]  

(20)

Thus, it suffices to focus our investigation on the coefficients \( \lambda(\kappa; l; q) \) with \( l \geq 2 \).

The following proposition computes the part of the sum in (18) corresponding to coefficients \( \lambda_\mu(N; q) \) containing a factor \( \lambda(\kappa', l'; 1/q) \) with \( l' = l \).

**Proposition 3.1.** For \( \kappa = (k_1, \ldots, k_r) \in \mathbb{N}^r \) and \( l \geq 2 \), let \( a(\kappa; l; q) \) denote the part of the sum in (18) over all pairs \((\mu, N)\) for which \( \lambda_\mu(N; q) \) does not contain a factor of the form \( \lambda(\kappa', l; 1/q) \). Then,

\[
\lambda(\kappa; l; q) - a(\kappa; l; q) = \begin{cases} q^{\kappa+l+1} & \text{if } l \text{ is odd} \\ \sum_{\kappa' \leq \kappa} q^{\kappa'-l+r(\kappa'-\kappa)+1} (q-1)^{r-k} \lambda(\kappa', l; 1/q) & \text{if } l \text{ is even} \end{cases}
\]

where \( |\kappa| = k_1 + \cdots + k_r \), and \( r(\kappa - \kappa') \) is the number of non-zero components of \( \kappa - \kappa' \).
Proof. Let $\lambda_\mu(N'; q)$ be a coefficient containing a factor of the form $\lambda(k', l; 1/q)$. Since
\[ \lambda_\mu(N'; q) = \sum_{j=1}^{n} q^{\mu_j (\nu_{j_1} + \cdots + \nu_{j_r} + \nu_{j})} \lambda(\nu_{j_1}, \ldots, \nu_{j_r}, \nu_j; 1/q^{\mu_j}) \] with $\sum_{j=1}^{n} \nu_j \mu_j = l$

it follows that $\nu_{j_0} = l$ and $\mu_{j_0} = 1$ for some $1 \leq j_0 \leq n$, and $\nu_j = 0$ for $j \neq j_0$. By (19), we see that $\lambda_\mu(N'; q) = q^{k'|l} \lambda(k', l; 1/q)$. The contribution of the part in (18) attached to this coefficient comes then from $(r + 1)$-tuples of monic polynomials
\[ ((x - \theta)^{k_1}, m_1, \ldots, (x - \theta)^{k_r}, m_r, (x - \theta)^l) \] (with $\theta \in \mathbb{F}$ and $m_i(\theta) \neq 0$ for $i = 1, \ldots, r$)

and is obtained immediately from the coefficient $\kappa - \kappa' = (k_1 - k_1', \ldots, k_r - k_r')$ of
\[ \sum_{\theta \in \mathbb{F}} \sum_{m_1, \ldots, m_r} \frac{X_{\theta - q}(m_1) \cdots X_{\theta - q}(m_r)}{m_1^{s_1} \cdots m_r^{s_r}} = \sum_{\theta \in \mathbb{F}} \prod_{i=1}^{r} L(s_i, X_{\theta - q}) = q \quad \text{(for } l \text{ odd)} \]

and
\[ q \cdot \prod_{i=1}^{r} \frac{1 - q^{-s_i}}{1 - q} \quad \text{(for } l \text{ even).} \]

The coefficient $(k_1 - k_1', \ldots, k_r - k_r')$ of the last product is $q^{k'|l - (\kappa - \kappa') + 1}(q - 1)^{r(\kappa - \kappa')}$. The proposition now follows by summing over all $\kappa' \leq \kappa$. \square

Remark. One can use a similar argument as in the proof of Proposition 3.1 to obtain the part of the sum in $a(\kappa, l; q)$ corresponding to an arbitrary coefficient $\lambda_\mu(N'; q)$. Indeed, recall that to a given pair $(\mu, N)$, we associated a set of polynomials $\mathcal{P}_\nu(\mu)$, and we split the set of indices $1 \leq j \leq n$ into two parts $J_0$ and $J_1$. By (17) and (19), we have
\[ \lambda_\mu(N'; q) = \prod_{j \in J_1^0} q^{\mu_j (\nu_{j_1} + \cdots + \nu_{j_r} + \nu_j)} \lambda(\nu_{j_1}, \ldots, \nu_{j_r}, \nu_j; 1/q^{\mu_j}) \]

where $J_1^0 := \{ j \in J_1 : \nu_j = 0 \}$. Set $\mu_0 := (\mu_j)_{j \in \{1, \ldots, n\}\setminus J_1^0}$ and $N_0 := (\nu_{j_1}, \ldots, \nu_{j_r}, \nu_j)_{j \in \{1, \ldots, n\}\setminus J_1^0}$, i.e., $(\mu_0, N_0)$ is the pair obtained by removing the components of $\mu$ and $N$ corresponding to all elements in $J_1^0$. For convenience, let us refer to $(\mu_0, N_0)$ as a primitive pair, and to $(\mu, N)$ as a derived pair from $(\mu_0, N_0)$. Since $\lambda_\mu(N'; q) = \lambda_{\mu_0}(N_0; q)$, we can express the identity (18) as
\[ \lambda(k, l; q) = \sum_{(\mu_0, N_0) - \text{primitive}} \lambda_{\mu_0}(N_0; q) \sum_{(\mu, N) - \text{derived from } (\mu_0, N_0)} S_{\mu, N}. \quad (21) \]

To achieve our goal, consider $\mu_0 := (\mu_1, \ldots, \mu_{n_0})$ and $N_0 := (\nu_{j_1}, \ldots, \nu_{j_r}, \nu_j)_{1 \leq j \leq n_0}$ giving a primitive pair $(\mu_0, N_0)$. Let $\nu_0 = (\nu_1, \ldots, \nu_{n_0})$, and let $J_0$ and $J_1$ be, as before, the sets of indices $1 \leq j \leq n_0$ associated to $\nu_0$. If $k' = (k_1', \ldots, k_r')$, with $k_i' = \sum_{1 \leq j \leq n} \nu_j \mu_j$ for $i = 1, \ldots, r$, we see that the inner sum in (21) (corresponding to $(\mu_0, N_0)$) is
\[ C_{\mu_0, \nu_0} \sum_{d_0 \in \mathcal{P}_{\nu_0}(\mu_0)} \left( \prod_{j \in J_1} \chi_{\nu_j}(d_0(\theta_j))^{\nu_{j_1} + \cdots + \nu_{j_r}} \right) b_{d_0}^{(\theta_j)}(k - k'). \quad (22) \]

Here,
\[ b_{d_0}^{(\theta_j)}(k - k') = \text{Coefficient}_{k_1 - k_1', \ldots, k_r - k_r'} \left[ \prod_{i=1}^{r} (1 - q t_i)^{-1} \right] \left[ \prod_{j \in J_1} (1 - t_i^{\nu_j}) \right] \] (if $d_0 = 1$). \quad (23)

When $d_0$ is non-constant,
\[ b_{d_0}^{(\theta_j)}(k - k') = \text{Coefficient}_{k_1 - k_1', \ldots, k_r - k_r'} \left[ \prod_{i=1}^{r} (1 - t_i)^{\epsilon_{d_0}} P_{C_{d_0}}(t_i) \prod_{j \in J_1} (1 - \chi_{\nu_j}(d_0(\theta_j)) t_i^{\nu_j}) \right] \]

where $P_{C_{d_0}}(t)$ denotes, as in Section 2, the numerator of the zeta function $Z_{C_{d_0}}(t)$, and $\epsilon_{d_0} = 0$ or 1 according as $\deg d_0$ is odd or even. As $\deg d_0 \equiv l \mod 2$, the exponent $\epsilon_{d_0} = \epsilon_l$ depends, in fact, only on the parity of $l$.
Example. Let us take \( l = 2 \). By Proposition 3.1, we know that

\[
\lambda(\kappa, 2; q) = a(\kappa, 2; q) + \sum_{\kappa' \leq \kappa} q^{[\kappa'-r(\kappa-\kappa')]+3(q-1)^{\nu(\kappa-\kappa')}} \lambda(\kappa', 2; 1/q).
\]  

(25)

Note that every coefficient \( \lambda_\mu(N'; q) \) appearing in \( a(\kappa, 2; q) \) is either 0 or 1; the product (17) involves only coefficients \( \lambda(\nu_1, \ldots, \nu_r; q) \) with \( \nu_j \in \{0, 1\} \), from which, by (19) and (20), one deduces that \( \lambda_\mu(N'; q) \in \{0, 1\} \).

Moreover, the above remark implies that

\[
a(\kappa, 2; q) = \sum_{d_0 \text{ monic and square-free}} b_{d_0}(\kappa)
\]

where \( b_{d_0}(\kappa) \) is the coefficient \((k_1, \ldots, k_r)\) of the product

\[
\prod_{i=1}^{r} (1-t_i) P_{c_{d_0}}(t_i) = \prod_{i=1}^{r} (1-t_i).
\]

Hence,

\[
a(\kappa, 2; q) = \begin{cases} (-1)^{[\kappa]} q(q-1) & \text{if } k_i = 0 \text{ or } 1 \text{ for all } i = 1, \ldots, r \\ 0 & \text{otherwise} \end{cases}
\]

The identity (25) allows a recursive computation of the coefficients \( \lambda(\kappa, 2; q) \). To see this, it is convenient to simplify notation by setting \( \lambda_2(\kappa; q) := \lambda(1, \ldots, 1; 1; q) \). Since \( \lambda(\kappa, 2; q) = \lambda_2(\kappa; q) \) when \( \kappa = (k_1, \ldots, k_r) \) with \( k_i = 0 \) or 1 for all \( i = 1, \ldots, r \), and \( |\kappa| = j \), we deduce that

\[
\lambda_2(r; q) - q^{r+3} \lambda_2(1/r; 1/q) = (-1)^r q(q-1) + \sum_{j=0}^{r-1} \binom{r}{j} q^{r+3}(q-1)^{r-j} \lambda_2(j/1/q).
\]

To put things in perspective, one should think of the quantity \((-1)^r q(q-1)\), representing \( a(\kappa, 2; q) \) for \( \kappa = (1, \ldots, 1) \), as obtained from the simple identity

\[
\sum_{d_0 \text{ monic and square-free}} \left( \sum_{\theta \in \mathbb{F}} \chi(d_0(\theta)) \right)^r = (-1)^r q(q-1) \quad \text{for } r \in \mathbb{N}.
\]  

(26)

For example, take \( r = 6 \). By the method we are just describing, one finds inductively that:

\[
\lambda_2(0; q) = q^2 \quad \lambda_2(1; q) = 0 \quad \lambda_2(2; q) = q^3 \quad \lambda_2(3; q) = q^4 \quad \lambda_2(4; q) = 2q^4 + q^5 \quad \lambda_2(5; q) = 5q^5 + q^6.
\]

(An alternative way of finding these coefficients, but only when \( r \leq 3 \), comes by expressing the rational function \( Z(T, t_4; q) \) into a power series.) Replacing \( q \) by \( 1/q \) in these values of \( \lambda_2(j; q) \) \((0 \leq j \leq 5)\), we obtain:

\[
\lambda_2(6; q) - q^9 \lambda_2(6; 1/q) = q^7 + 9q^6 + 5q^5 - 5q^4 - 9q^3 - q^2.
\]

By condition (iii), we identify \( \lambda_2(6; q) \) with the dominant half of the right-hand side, obtaining

\[
\lambda_2(6; q) = q^7 + 9q^6 + 5q^5.
\]

In Appendix C we shall see that

\[
\lambda_2(r; q) = \sum_{j=1}^{[r/2]} \frac{1}{(r-j+1)(r-j)} \frac{r!}{j!(j-1)!(r-2j)!} q^{r+2-j}
\]

for \( r \geq 1 \)

where \([x]\) denotes the integer part of a real number \( x \).
Presumably, all coefficients $\lambda(\kappa, l; q)$ can be determined by a similar procedure using an induction over $l$ and $r$, should one be able to get some information about the moment-sums $M_{\mu, r}(q)$ defined in this section. More precisely, we need a different expression for $M_{\mu, r}(q)$ playing the same role as (26) did in the above example. To gain more insight into the general problem, consider a coefficient $\lambda(\kappa, l; q)$ with $l \geq 3$. For simplicity, let us assume that $l$ is odd. By Proposition 3.1, we have

$$\lambda(\kappa, l; q) - q^{\frac{1}{2}[\kappa + \frac{1}{2}]} \lambda(\kappa, l; 1/q) = a(\kappa, l; q).$$

The right-hand side involves only coefficients $\lambda(\kappa, l'; q)$ with $l' < l$ (which are supposed to be known by the induction hypothesis), and hence, $a(\kappa, l; q)$ is an explicit combination of moments of character sums. Moreover, the left-hand side of the identity suggests that $a(\kappa, l; q)$ should satisfy the functional equation

$$a(\kappa, l; q) = -q^{\frac{1}{2}[\kappa + \frac{1}{2}]} a(\kappa, l; 1/q).$$

Accordingly, one needs to find an alternative expression of $a(\kappa, l; q)$, allowing to identify (by condition (iii)) the two parts corresponding to $\lambda(\kappa, l; q)$ and $-q^{\frac{1}{2}[\kappa + \frac{1}{2}]} \lambda(\kappa, l; 1/q)$. This will be completely clarified in Sections 7 and 8.

### 4 The generating series of $a(\kappa, l; q)$

For $l \in \mathbb{N}$ and algebraically independent variables $t_1, \ldots, t_r$, consider the generating series

$$\Lambda_l(T, q) = \sum_{\kappa \in \mathbb{N}^r} \lambda(\kappa, l; q) T^\kappa$$

and

$$A_l(T, q) = \sum_{\kappa \in \mathbb{N}^r} a(\kappa, l; q) T^\kappa$$

where we set $T^\kappa := t_1^{k_1} \cdots t_r^{k_r}$. By Proposition 3.1, $\Lambda_l(T, q)$ should satisfy:

$$\Lambda_l(T, q) = A_l(T, q) + q^{l+1} \frac{E(q T)^{s_1}}{E(T)^{s_1}} \Lambda_l(q T, 1/q)$$

(for $l \geq 2$) (27)

where

$$E(T) = \prod_{i=1}^r (1 - t_i)^{-1}$$

and $\epsilon_i = 0$ or 1 according as $l$ is odd or even. The product $E(T)$ should be interpreted as the gamma factor of $\Lambda_l(T, q)$ when $l$ is even. The identity (27) implies that $A_l(T, q)$ should satisfy the functional equation

$$E(T)^{s_1} A_l(T, q) = -q^{l+1} E(q T)^{s_1} A_l(q T, 1/q).$$

(28)

Note that $\Lambda_l(T, q)$ is just what

$$\sum_{\deg d = l} L(s_1, \chi_d) \cdots L(s_r, \chi_d)$$

(with $q^{-s_i}$ replaced by $t_i$ for $i = 1, \ldots, r$) should be.

To gain a better understanding of what these generating functions should be, fix a partition $\mu = (\mu_1 \geq \cdots \geq \mu_n \geq 1)$ and an $n$-tuple of positive integers $\nu = (\nu_1, \ldots, \nu_n)$ such that

$$\sum_{j=1}^n \nu_j \mu_j = l.$$

We shall assume throughout the section that $l \geq 2$. Let $A_{\mu, \nu}(T, q)$ denote the generating series whose $\kappa$-th coefficient is the part of the sum in (21) corresponding to all primitive pairs $(\mu_0, N_0)$ for which $\mu_1 = \mu$ and $\nu_0 = \nu$. For instance, if $\mu = (1)$ and $\nu = (l)$ we have

$$A_{\mu, \nu}(T, q) = q^{l+1} \frac{E(q T)^{s_1}}{E(T)^{s_1}} \Lambda_l(q T, 1/q).$$
Note that

$$A_i(T, q) + q^{|i+1|} E(qT)^{e_i} A_i(qT, 1/q) = \sum_{(\mu, \nu)} A_{\mu, \nu}(T, q).$$  

(29)

This reduces the study of $A_i(T, q)$ to the study of the generating series $A_{\mu, \nu}(T, q)$, for arbitrary $\mu$ and $\nu$.

The following special case is an immediate consequence of (21), (22) and (23).

**Proposition 4.1.** — Let $\mu_{j_1}, \ldots, \mu_{j_m}$ be the distinct components of $\mu$, each $\mu_{j_i}$ occurring with multiplicity $n_i$. If all components $\nu_j$, $j = 1, \ldots, n$, of $\nu$ are even, then

$$A_{\mu, \nu}(T, q) = q^\sum_{j \in J_0} \sum_{j \in J_1} A_{\nu_j}(q^{|\mu_j|T^{\mu_j}, 1/q^{|\mu_j|}} \cdot \frac{E(qT) \prod_{j \in J} A_{\nu_j}(q^{|\mu_j|T^{\mu_j}, 1/q^{|\mu_j|}})}{E(T^{|\mu_j|})}$$

where $\text{Irr}_q(\mu_{j_i})$ denotes the number of irreducible polynomials of degree $\mu_{j_i}$ over $\mathbb{F}$.

For general $\mu$ and $\nu$, split, as before, the set of indices $1 \leq j \leq n$ into two parts $J_0$ and $J_1$ according as $\nu_j$ is odd or even. For all $j \in J_0$ and $j' \in J_1$ such that $\mu_{j} = \mu_{j'}$, we shall assume that $j < j'$. For $\varepsilon = (\varepsilon_j)_{j \in J_1}$ with $\varepsilon_j \in \{1, 2\}$, let

$$A'_{\mu, \nu}(T, q) = q^\sum_{j \in J_0} \sum_{j \in J_1} A_{\nu_j}(q^{|\mu_j|T^{\mu_j}, 1/q^{|\mu_j|}}) \cdot \frac{E(qT) \prod_{j \in J} A_{\nu_j}(q^{|\mu_j|T^{\mu_j}, 1/q^{|\mu_j|}})}{E(T^{|\mu_j|})}$$

and

$$U_{\mu, \nu}(T, q) = \frac{C_{\mu, \nu}}{E(T)^{e_i}} \sum_{d_0 \in \mathcal{P}_e(\mu)} \prod_{(\theta_j)_{j \in J_1}} \chi_{\nu_j}(d_0(\theta_j))^{e_j} \cdot \prod_{j \in J_1} \prod_{t \in T} \left(1 - \chi_{\nu_j}(d_0(\theta_j)) t_j \right).$$

Here $C_{\mu, \nu}$ is the normalizing constant defined in the previous section. By (21), (22) and (24), it is not hard to see that the contribution to $A_i(T, q)$ corresponding to $(\mu, \nu)$ is

$$A_{\mu, \nu}(T, q) = \sum_{\varepsilon} U_{\mu, \nu}(T, q) A'_{\mu, \nu}(T, q)$$

the sum being taken over all $\varepsilon$ as above.

To allow more flexibility in these expressions, introduce a sum in $U_{\mu, \nu}(T, q)$ over all tuples $\delta = (\delta_j)_{j \in J_1}$ with $\delta_j \in \{-1, 1\}$, and sum over $d_0 \in \mathcal{P}_e(\mu)$ and $(\theta_j)_{j \in J_1}$ such that $\chi_{\nu_j}(d_0(\theta_j)) = \delta_j$. Note that for every $d_0 \in \mathcal{P}_e(\mu)$ and $\delta$, the number $N_{\mu, \nu}(d_0, \delta)$ of such tuples $(\theta_j)_{j \in J_1}$ can be expressed as

$$N_{\mu, \nu}(d_0, \delta) = \sum_{(\theta_j)_{j \in J_1}} \left( \prod_{j \in J_1} \delta_j \chi_{\nu_j}(d_0(\theta_j)) \right) = \frac{1}{2^{\sum_{j \in J_1}} \sum_{\delta \in \delta_0} \delta_S b_{\mu}^S(d_0)$$

with

$$\delta_S := \prod_{j \in S} \delta_j$$ and

$$b_{\mu}^S(d_0) := \sum_{(\theta_j)_{j \in J_1}} \left( \prod_{j \in S} \chi_{\nu_j}(d_0(\theta_j)) \right).$$

Moreover, for every $\delta = (\delta_j)_{j \in J_1}$, we have

$$\sum_{\varepsilon} \left( \prod_{j \in J_1} \delta_j \right) A'_{\mu, \nu}(T, q) = q^\sum_{j \in J_0} \sum_{j \in J_1} A_{\nu_j}(q^{|\mu_j|T^{\mu_j}, 1/q^{|\mu_j|}}) \cdot \prod_{j \in J_1} \Lambda_{\nu_j}(\delta_j q^{|\mu_j|T^{\mu_j}, 1/q^{|\mu_j|}}).$$

20
Now write \( A_{\mu,\nu}(T, q) = \frac{C_{\mu,\nu}}{E(T)^{r}} \sum_{\delta} \sum_{d_0 \in \mathbb{P}^*_{\mu}} N_{\mu,\nu}(d_0, \delta) \prod_{i=1}^{r} P_{C_{d_0}}(t_i) \cdot \Lambda_{\delta,\mu,\nu}(T, q) \) (30)

with

\[
A_{\delta,\mu,\nu}(T, q) := q^{j} \prod_{j \in J_0} \Lambda_{\mu_j}(q^{\mu_j} T^{\mu_j}, 1/q^{\mu_j}) \cdot \prod_{j \in J_1} \frac{\Lambda_{\nu_j}(\delta_j q^{\mu_j} T^{\mu_j}, 1/q^{\mu_j})}{E(\delta_j T^{\mu_j})}.
\] (31)

Our next objective is to obtain an explicit expression for the character sum \( b_{\mu}^{(s,t)}(d_0) \) in \( N_{\mu,\nu}(d_0, \delta) \) in terms of the coefficients of the characteristic polynomial \( P_{C_{d_0}}(t) \). For this purpose, let us first fix some notation.

For every \( \omega \geq 1 \), let \( \mathbb{F}^*_\omega \) denote the set of all elements in \( \mathbb{F}_q^* \) of degree \( \omega \) over \( \mathbb{F} \). For \( s, t \geq 0 \) with \( s + t \geq 1 \), and a monic square-free polynomial \( d_0 \) over \( \mathbb{F} \), we set

\[
b_{\mu}^{(s,t)}(d_0) := \sum_{(\theta_1, \ldots, \theta_s, \ldots, \theta_t)} \left( \prod_{j=1}^{s} \chi_{\omega}(d_0(\theta_j)) \cdot \prod_{j=s+1}^{s+t} \chi_{\omega}(d_0(\theta_j))^2 \right)
\]

where, as before, the sum is over all tuples \( (\theta_j)_{j=1}^{s+t} \) with \( \theta_j \in \mathbb{F}^*_\omega \) for every \( 1 \leq j \leq s + t \), and \( \sigma(\theta_j) \neq \theta_{j'} \) for all \( \sigma \in \text{Gal}(\mathbb{F}/\mathbb{F}) \), and all \( j \neq j' \). Finally, let

\[
a_{\mu}^{*}(d_0) = \sum_{\theta \in \mathbb{F}^*_\omega} \chi_{\omega}(d_0(\theta)) \quad \text{and} \quad c_{\omega}(d_0) = \sum_{\theta \in \mathbb{F}^*_\omega} \chi_{\omega}(d_0(\theta))^2.
\]

Note that \( c_{\omega}(d_0) \) is constant on \( \mathbb{P}^*_\mu \).

Now write \( b_{\mu}^{S}(d_0) \) in \( N_{\mu,\nu}(d_0, \delta) \) as

\[
b_{\mu}^{S}(d_0) = \sum_{(\theta_1, \ldots, \theta_s)} \left( \prod_{j \in S} \chi_{\mu_j}(d_0(\theta_j)) \cdot \prod_{j \in S^*} \chi_{\mu_j}(d_0(\theta_j))^2 \right)
\]

where \( S^* \) denotes the complement of the subset \( S \) in \( J_1 \). Let \( \mu_{j_1}, \ldots, \mu_{j_m} \) be, as before, the distinct components of \( (\mu_j)_{j \in J_1} \), and for \( i = 1, \ldots, m \), set \( J_1^{(i)} := \{ j \in J_1 : \mu_j = \mu_{j_i} \} \). With this notation, we can further write

\[
b_{\mu}^{S}(d_0) = \prod_{i=1}^{m} \sum_{(\theta_1, \ldots, \theta_{s_i})} \left( \prod_{j \in S \cap J_1^{(i)}} \chi_{\mu_j}(d_0(\theta_j)) \cdot \prod_{j \in S^* \cap J_1^{(i)}} \chi_{\mu_j}(d_0(\theta_j))^2 \right) = \prod_{i=1}^{m} b_{\mu_{j_i}}^{(s_i, t_i)}(d_0)
\]

where \( s_i = |S \cap J_1^{(i)}| \) and \( t_i = |S^* \cap J_1^{(i)}| \) (so that \( |J_1^{(i)}| = s_i + t_i \)) for \( i = 1, \ldots, m \). Moreover, it is easy to see that

\[
b_{\omega}^{(s,t)}(d_0) = b_{\omega}^{(s,0)}(d_0) \cdot \prod_{j=0}^{t-1} (c_{\omega}(d_0) - \omega s - \omega j) \quad \text{if} \quad t \geq 1
\]

and hence, we can restrict ourselves to the character sums

\[
b_{\omega}^{(s,t)}(d_0) = b_{\omega}^{(s,0)}(d_0) = \sum_{(\theta_1, \ldots, \theta_s)} \left( \prod_{j=1}^{s} \chi_{\omega}(d_0(\theta_j)) \right).
\]

The following lemma expresses \( b_{\omega}^{(s,t)}(d_0) \) in terms of \( a_{\mu}^{*}(d_0) \) and \( c_{\omega}(d_0) \).

**Lemma 4.2.** — For \( \omega, s \geq 1 \), and \( d_0 \) a monic square-free polynomial over \( \mathbb{F} \), we have

\[
b_{\omega}^{(s,t)}(d_0) = s! \sum_{i} \frac{(-\omega)^{s-|i|}}{z(i)} \cdot a_{\omega}^{*}(d_0)^{i_{\text{odd}}} c_{\omega}(d_0)^{i_{\text{even}}}.
\]
where the sum is over all $s$-tuples $i = (i_1, \ldots, i_s)$ of nonnegative integers with $i_1 + 2i_2 + \cdots + si_s = s$, $z(i) = i_1!i_1 \cdot i_2!i_2 \cdots i_s!i_s$, and $|i| = |i_{\text{odd}}| + |i_{\text{even}}|$; here $|i_{\text{odd}}| = i_1 + i_3 + \cdots$ (respectively $|i_{\text{even}}| = i_2 + i_4 + \cdots$) is the sum of the odd (respectively even) components of $i$. Equivalently, we have the identity:

$$1 + \sum_{s \geq 1} \frac{b^s(0)}{s!} X^s = (1 + \omega X) \frac{c_0(0) + a^s(0)}{2} (1 - \omega X) \frac{c_0(0) - a^s(0)}{2}.$$  

**Proof.** Consider the $j$-th elementary symmetric and power sum polynomials in $k$ variables,

$$e_j(x_1, \ldots, x_k) = \sum_{1 \leq i_1 < \cdots < i_j \leq k} x_{i_1} \cdots x_{i_j} \quad \text{and} \quad p_j(x_1, \ldots, x_k) = x_1^j + \cdots + x_k^j,$$

respectively. For $i = (i_1, \ldots, i_s)$ an $s$-tuple of nonnegative integers with $i_1 + 2i_2 + \cdots + si_s = s$, set $p^{(i)} := p_1^{i_1} p_2^{i_2} \cdots p_s^{i_s}$.

By [46, Appendix A, §A.1., Exercise A.32 (vi)], we have the identity

$$e_s = \sum_{i=1}^{k} (-1)^{|l|} \frac{e_l}{z(i)} p^{(i)}$$  

(32)

where the sum is over all $i = (i_1, \ldots, i_s)$ as above, $|i| = i_1 + i_2 + \cdots + i_s$, and $z(i)$ is the product of the statement of this lemma. This identity is encoded in the well-known identity of formal power series:

$$1 + \sum_{s \geq 1} (-1)^s e_s t^s = \exp \left(- \sum_{j \geq 1} \frac{t^j}{j} \right).$$  

(33)

We apply (32) and (33) in our context as follows. Let $O_{\omega}^\text{Gal} = \{O_1, \ldots, O_{\omega} \}$ denote the set of Galois orbits in $\mathbb{P}_\omega'$, and set $\chi_\omega(d_0(\theta)) := \chi_\omega(\omega(d_0(\theta)))$ for $O \in O_{\omega}^\text{Gal}$ and $\theta \in \mathcal{O}$. Clearly the value of $\chi_\omega(d_0(\theta))$ is independent of the choice of $\theta \in \mathcal{O}$. Since

$$\frac{1}{s!} \frac{1}{\omega^s} b^s_\omega(d_0) = \sum_{1 \leq i_1 < \cdots < i_s \leq \omega} \chi_\omega(d_0(O_{i_1}, \ldots, O_{i_s})) = e_s(\chi_\omega(d_0(O_1), \ldots, \chi_\omega(d_0(O_{\omega}))))$$

as it can be easily checked, the lemma follows by applying (32) and (33). \(\square\)

Using this lemma, we can write

$$N_{\mu,\nu}(d_0, \delta) = \frac{1}{2^{\mu_1}} \sum_{S \subseteq J_1} \delta_S \prod_{i=1}^{m} \frac{b^s_{\mu_{j_i}}(d_0)}{\mu_{j_i}} \prod_{j=0}^{s-1} \left( c_{\mu_{j_i}}(d_0) - \mu_{j_i}s_j - \mu_{j_i}j \right)$$  

(34)

with

$$\delta_S = \prod_{j \in S} \delta_j \quad \text{and} \quad b^s_{\mu_{j_i}}(d_0) = (s_i)! \sum_{l} \left( -\mu_{j_i} \right)^{s_i - l} \frac{a^s_{\mu_{j_i}}(d_0) |_{\text{odd}}} {z(l)} \cdot c_{\mu_{j_i}}(d_0) |_{\text{even}}.$$

The sum in the expression of $b^s_{\mu_{j_i}}(d_0)$ is over all $s_i$-tuples $i = (t_1, \ldots, t_{s_i})$ of nonnegative integers with $t_1 + 2t_2 + \cdots + t_{s_i} = s_i$.

The following proposition gives a simpler expression for $N_{\mu,\nu}(d_0, \delta)$.

**Proposition 4.3.** For $i = 1, \ldots, m$, let $n_i$ denote the cardinality of the set $J_1^{(i)}$. If $\delta = (\delta_j)_{j \in J_1}$ with $\delta_j \in \{-1, 1\}$ and $d_0 \in \mathbb{P}_\omega(\mu)$, we have

$$N_{\mu,\nu}(d_0, \delta) = \prod_{i=1}^{m} \frac{c_{\mu_{j_i}}(d_0) + a^s_{\mu_{j_i}}(d_0)}{2\mu_{j_i}} \cdot \left( \frac{c_{\mu_{j_i}}(d_0) - a^s_{\mu_{j_i}}(d_0)}{2\mu_{j_i}} \right)^{n_i} \cdot \left( \frac{c_{\mu_{j_i}}(d_0) + a^s_{\mu_{j_i}}(d_0)}{2\mu_{j_i}} \right)^{n_i^\delta},$$

where $\delta^{(i)} = (\delta_j)_{j \in J_1^{(i)}}$, and $n_i^{\delta^{(i)}} = (n_i \pm |\delta^{(i)}|)/2$.  

22
Proof. We observe from (34) that \( N_{\mu, \nu}(d_0, \delta) \) factors as

\[
N_{\mu, \nu}(d_0, \delta) = \prod_{i=1}^{m} N_{\mu, \nu}^{(i)}(d_0, \delta^{(i)})
\]

where

\[
N_{\mu, \nu}^{(i)}(d_0, \delta^{(i)}) = \frac{1}{2n_i} \sum_{s_i=0}^{n_i} b_{\mu j_i}^{(s_i, n_i-s_i)}(d_0) \cdot \sum_{h=0}^{s_i} \delta_{s_i-h}.
\]

The inner sum is

\[
\epsilon_{s_i}(\delta^{(i)}) = \text{Coefficient}_{t^s}(1 + t)^{(n_i+|\delta^{(i)}|)/2} (1 - t)^{(n_i-|\delta^{(i)}|)/2}
\]

and hence,

\[
N_{\mu, \nu}^{(i)}(d_0, \delta^{(i)}) = \frac{1}{2n_i} \sum_{s_i=0}^{n_i} b_{\mu j_i}^{(s_i, n_i-s_i)}(d_0) \cdot \sum_{h=0}^{s_i} (-1)^{s_i-h}\left(\frac{(n_i + |\delta^{(i)}|)}{h}\right)\left(\frac{(n_i - |\delta^{(i)}|)}{s_i-h}\right).
\]

On the other hand, by the second identity in the lemma, we deduce easily that the generating polynomial \( F_\omega(X, Y) = F_{\omega, \sigma_0^+(d_0)}(X, Y) \) of \( b_{\sigma_0^+(d_0)}(d_0)/(s!) \) is

\[
F_\omega(X, Y) = \sum_{s, t \geq 0} b_{\sigma_0^+(d_0)}(d_0) \frac{X^s Y^t}{s! t!} = (1 + \omega X + \omega Y) \frac{\omega(d_0)}{2\omega} (1 - \omega X + \omega Y) \frac{\omega(d_0)-\omega^2(d_0)}{2\omega^2}.
\]

One verifies that

\[
N_{\mu, \nu}^{(i)}(d_0, \delta^{(i)}) = \text{Coefficient}_{X^s Y^t} \left[ \left(\frac{n_i}{s!}\right)! \left(\frac{\partial}{\partial Y} - \frac{\partial}{\partial X}\right)^n F_{\mu j_i}(X, Y) \right]_{X=Y}\]

It is also easy to see that for a power series in two variables \( F(X, Y) = \sum_{n, t \geq 0} b_{n, t} \frac{X^n Y^t}{n! t!} \), we have the formal series identity

\[
\sum_{n \geq 0} \left( \frac{\partial}{\partial Y} - \frac{\partial}{\partial X} \right)^n F(X, Y) \bigg|_{X=Y+U} \frac{U^n}{n!} = F(V-U, V+U).
\]

From these simple observations, it follows (as stated in the proposition) that for \( d_0 \in \mathcal{P}_\nu(\mu) \) and \( \delta = (\delta_j)_{j \in J} \), with \( \delta_j \in \{-1, 1\} \), we have

\[
N_{\mu, \nu}(d_0, \delta) = \prod_{i=1}^{m} \left( \frac{\epsilon_{\mu j_i}(d_0) + \alpha_{\mu j_i}^+(d_0)}{2\mu j_i} \right)^{n_{\mu j_i}} \left( \frac{\epsilon_{\mu j_i}(d_0) - \alpha_{\mu j_i}^+(d_0)}{2\mu j_i} \right)^{n_{\nu j_i}^+} \left( \frac{\epsilon_{\mu j_i}(d_0) + \alpha_{\mu j_i}^-(d_0)}{2\mu j_i} \right)^{n_{\mu j_i}^-} \left( \frac{\epsilon_{\mu j_i}(d_0) - \alpha_{\mu j_i}^-(d_0)}{2\mu j_i} \right)^{n_{\nu j_i}^-}.
\]

Now, for a hyperelliptic curve \( C_{d_0} \) of genus \( g \) corresponding to a monic square-free polynomial \( d_0 \in \mathbb{F}[x] \) of degree \( 2g + 1 \) or \( 2g + 2 \), and a prime \( \ell \) different from the characteristic \( p \) of \( \mathbb{F} \), we recall that

\[
P_{C_{d_0}}(t) = \det(1 - F^* t | H_{\ell}^1(C_{d_0}, \mathbb{Q}_\ell))
\]

where \( \bar{C}_{d_0} \) is obtained from \( C_{d_0} \) by extending the scalars from \( \mathbb{F} \) to \( \bar{\mathbb{F}} \), and \( F^* \) is the endomorphism of the \( \ell \)-adic étale cohomology induced by the Frobenius morphism \( F : C_{d_0} \to \bar{C}_{d_0} \). Let \( \alpha_1(C_{d_0}), \ldots, \alpha_{2g}(C_{d_0}) \) denote the eigenvalues of \( F^* \), ordered in such a way that \( \alpha_k(C_{d_0}) \alpha_{k+g}(C_{d_0}) = q \), and for every positive integer \( j \), put

\[
a_j(C_{d_0}) := -\frac{1 + (-1)^{\deg d_0}}{2} - \sum_{\theta \in \mathcal{F}_j} \chi_j(d_0(\theta)) = \text{Tr}(F^{*j} | H_{\ell}^1(\bar{C}_{d_0}, \mathbb{Q}_\ell)).
\]

(35)
Writing \(P_{c_{d_0}}(t)\) as
\[
P_{c_{d_0}}(t) = 1 + \sum_{k=1}^{2g} (-1)^k \lambda_k(C_{d_0}) t^k
\]
it follows from (32) that
\[
\lambda_k(C_{d_0}) = e_k(\alpha_1(C_{d_0}), \ldots, \alpha_{2g}(C_{d_0})) = \sum_{z(1_k)} \frac{(-1)^{|i_k|}}{z(1_k)} a^{(i_k)}(C_{d_0})
\]
the sum being over all \(k\)-tuples \(i_k = (i_1, \ldots, i_k)\) of nonnegative integers such that \(i_1 + \cdots + ki_k = k\). Here we set \(a^{(i_k)}(C_{d_0}) := a_1(C_{d_0})a_2(C_{d_0}) \cdots a_k(C_{d_0})^{i_k}\) (with the understanding that \(a_j(C_{d_0})^{i_j} = 1\) if \(i_j = 0\) for some \(j\)).

One can easily express \(a^{\omega}_\omega(d_0)\) (\(\omega \geq 1\)) in terms of the classical character sums (35) and the elementary quantities \(c_{\omega/2^u}(d_0)\), with \(u = 1, 2, \ldots\), as follows. We can write
\[
a_k(C_{d_0}) = -\frac{1 + (-1)^{\deg_d}}{2} - \sum_{\omega | k} \sum_{\theta \in \mathbb{Z}_H} \chi_k(d_0(\theta)) \quad \text{ (for } k \geq 1).\]

Recalling that for \(\theta \in \mathbb{H}_\omega\), we have \(\chi_k(d_0(\theta)) = \chi_\omega(d_0(\theta))\) or \(\chi_\omega(d_0(\theta))^2\) according as \(k/\omega\) is odd or even, we can rewrite the above equality as
\[
\frac{1 + (-1)^{\deg_d}}{2} - a_k(C_{d_0}) = \sum_{k/\omega \text{ odd}} a^{\omega}_\omega(d_0) + \sum_{k/\omega \text{ even}} c_{\omega}(d_0).
\]

By setting \(\rho_\omega = 1\) or \(0\) according as \(\omega\) is a power of two or not, and \(\epsilon_{d_0} := (1 + (-1)^{\deg_d})/2\), it follows using the Möbius inversion that
\[
a^{\omega}_\omega(d_0) = -\rho_\omega \epsilon_{d_0} - \sum_{k/\omega \text{ odd}} \mu(\omega/k) a_k(C_{d_0}) - \sum_{u=1}^{a} c_{\omega/2^u}(d_0) \quad \text{ (for } 2^u \parallel \omega).\]

Note that we can express \(P_{c_{d_0}}(t)\) as
\[
P_{c_{d_0}}(t) = (1 - t)^{-\epsilon_{d_0}} \cdot \prod_{m=1}^{\infty} \left(1 - t^m \right)^{\frac{e_m(d_0) + a^{\omega}_\omega(d_0)}{2m}} \left(1 + t^m \right)^{\frac{e_m(d_0) - a^{\omega}_\omega(d_0)}{2m}}.
\]

5 The coefficients \(\lambda(\kappa, l; q)\) for \(l = 3, 4\)

To determine the coefficients \(\lambda(\kappa, 3; q)\), we follow the strategy outlined at the end of Section 3. By Proposition 3.1,
\[
\lambda(\kappa, 3; q) - q^{\kappa} \lambda(\kappa, 3; 1/q) = a(\kappa, 3; q)
\]
with \(a(\kappa, 3; q)\) computed using Remark 2 in Section 3. Explicitly,
\[
a(\kappa, 3; q) = \sum_{\deg_d = 3} b_{d_0}(\kappa) + \sum_{\kappa' \leq \kappa} \left( \sum_{\deg_d = 1} \sum_{\kappa'' = \kappa} \chi(d_0(\theta))^{\kappa''} b_{d_0}^{(\theta)}(\kappa - \kappa') \right) q^{\kappa''} \lambda(\kappa, 2; 1/q)
\]
where \(b_{d_0}(\kappa)\) is the coefficient \(b_{d_0}(k_1, \ldots, k_r)\) of
\[
\prod_{i=1}^{r} P_{c_{d_i}}(t_i) = \prod_{i=1}^{r} (1 - a(C_{d_i}) t_i + q t_i^2)
\]
and
\[ b_{d_0}^{(\theta)}(\kappa - \kappa') = \begin{cases} (-\chi(d_0(\theta)))^{\lvert \kappa - \kappa' \rvert} & \text{if } k_i - k_i' = 0 \text{ or } 1 \text{ for all } i = 1, \ldots, r \\
0 & \text{otherwise}. \end{cases} \]

For every \( \kappa' = (k'_1, \ldots, k'_r) \in \mathbb{N}^r \) whose components satisfy \( k_i - k'_i = 0 \) or \( 1 \) for all \( i = 1, \ldots, r \), we have
\[ \sum_{\deg d_0 = 1} \sum_{\theta \in F} \chi(d_0(\theta))^{\lvert \kappa \rvert} b_{d_0}^{(\theta)}(\kappa - \kappa') = \begin{cases} (-1)^{\lvert \kappa - \kappa' \rvert} q(q - 1) & \text{if } \lvert \kappa \rvert \text{ is even} \\
0 & \text{if } \lvert \kappa \rvert \text{ is odd}. \end{cases} \]

It follows that the triple sum in the expression of \( a(\kappa, 3; q) \) is
\[ q(q - 1) \cdot \sum_{\kappa' \leq \kappa} (-1)^{\lvert \kappa - \kappa' \rvert} q^{\lvert \kappa' \rvert + 2} \lambda(\kappa', 2; 1/q) \]
or zero according as \( \lvert \kappa \rvert \) is even or odd.

To deal with the remaining part of \( a(\kappa, 3; q) \), let
\[ M_3(r; q) := \sum_{\deg d_0 = 3} d_0 - \text{monic} \cdot a(C_{d_0})^r = \sum_{\deg d_0 = 3} d_0 - \text{monic} \cdot \left( -\sum_{\theta \in F} \chi(d_0(\theta)) \right)^r. \]

Notice that a simple substitution \( \theta \to \theta_0 \theta \), with \( \chi(\theta_0) = -1 \), implies immediately that \( M_3(r; q) = 0 \) for \( r \) odd. We normalize \( M_3(r; q) \) by setting \( M_3^*(r; q) := (q(q - 1))^{-1} M_3(r; q) \). Using the Eichler-Selberg trace formula [81] (see also [65, Appendix by D. Zagier, pp. 44–54]), Birch [11] and Ihara [58] proved independently the following beautiful theorem:

**Theorem (Birch-Ihara).** — If \( \mathbb{F} = \mathbb{F}_q \) is a finite field of odd characteristic \( p \), then
\[ \sum_{r=0}^{k-1} \binom{k + r - 1}{2r} (-q)^{k-1-r} M_3^*(2r; q) = -T_{2k}(q) - 1 \quad (\text{for } k \geq 1) \]
with \( T_{2k}(q) = \text{Tr}(T_q | S_{2k}) \) if \( q = p \), and \( T_{2k}(q) = \text{Tr}(T_q | S_{2k}) - p^{2k-1} \text{Tr}(T_{q/2} | S_{2k}) \) if \( q \neq p \), where \( \text{Tr}(T_n | S_{2k}) \) is the trace of the Hecke operator \( T_n \) acting on the space of elliptic cusp forms of weight \( 2k \) on the full modular group. Equivalently, for \( r \geq 0 \) we have the identity:
\[ M_3^*(2r; q) = \frac{(2r)!}{r!(r + 1)!} q^{r+1} - \sum_{k=1}^{r} \frac{(2k + 1)!}{(r-k)!(r+k+1)!} (T_{2k+2}(q) + 1). \]

The above result is the main ingredient in the determination of the coefficients \( \lambda(\kappa, 3; q) \) which is given in the following

**Theorem 5.1.** — For \( \kappa = (k_1, \ldots, k_r) \in \mathbb{N}^r \), let \( r_1 = r_1(\kappa) \) and \( r_2 = r_2(\kappa) \) denote the number of components of \( \kappa \) equal to 1 and 2, respectively. If \( r_1 + 2r_2 = \lvert \kappa \rvert \), and \( r_1 = 2R \) is even, then
\[ \lambda(\kappa, 3; q) = \frac{(2R)!}{R!(R+1)!} q^{R+r_2+3} - \sum_{j=1}^{R} \frac{(2R)!}{(R-j)!(R+j+1)!} (T_{2j+2}(q) + 1). \]

Otherwise, the coefficients \( \lambda(\kappa, 3; q) \) all vanish.

\[ \text{25} \]
\textbf{Proof.} From the above considerations and the discussion in Appendix C, it is easy to see that \(a(\kappa, 3; q) = 0\), unless \(r_1\) is even, and \(r_1 + 2r_2 = |\kappa|\). Moreover,

\[
(q(q - 1))^{-1}a(\kappa, 3; q) = q^{r_2}M^*_3(r_1; q) + \sum_{\kappa' \leq \kappa \atop k_i, -k_i' = 0, 1} (-1)^{|\kappa - \kappa'|} q^{|\kappa'|+2} \lambda(\kappa', 2; 1/q)
\]

in the remaining case. The sum in the right is

\[
\sum_{j=0}^{R} (2j + 1) \frac{(2R)! q^{R+r_2-j}}{(R-j)!(R+j+1)!}
\]

see Appendix C, which combined with Birch-Ihara identity gives

\[
a(\kappa, 3; q) = \frac{(2R)!}{R!(R+1)!} q^{R+r_2+3} - \sum_{j=1}^{R} (2j + 1) \frac{(2R)! q^{R+r_2-j+2}}{(R-j)!(R+j+1)!} T_{2j+2}(q)
\]

\[
- \frac{(2R)!}{R!(R+1)!} q^{R+r_2+1} + \sum_{j=1}^{R} (2j + 1) \frac{(2R)! q^{R+r_2-j+1}}{(R-j)!(R+j+1)!} T_{2j+2}(q).
\]

(Notice the cancellation that occurs in the process of obtaining the last expression of \(a(\kappa, 3; q)\).) Letting

\[
T_{2j+2}(1/q) := q^{-2j-1} T_{2j+2}(q)
\]

we can present the second sum as

\[
\sum_{j=1}^{R} (2j + 1) \frac{(2R)! q^{R+r_2+j+2}}{(R-j)!(R+j+1)!} T_{2j+2}(1/q).
\]

To finish the proof, we notice that the coefficient \(\lambda(\kappa, 3; q)\) in the statement of the theorem is just the dominant half of \(a(\kappa, 3; q)\).  

\(\square\)

\textbf{The case} \(l = 4\). To find the coefficients \(\lambda(\kappa, 4; q)\), we apply Proposition 3.1 with \(l = 4\), and thus

\[
\lambda(\kappa, 4; q) = a(\kappa, 4; q) + \sum_{\kappa' \leq \kappa} q^{(r-l)|\kappa'|-r} \lambda^{r}\kappa' = \lambda(\kappa', 4; 1/q).
\]

The right-hand side of the identity consists of distinct contributions corresponding to pairs \(x = (x_1 \geq \cdots \geq x_m)\) and \(y = (y_1, \ldots, y_m)\), with \(x_i, y_i\) positive integers for all \(i = 1, \ldots, m\), such that

\[
\sum_{i=1}^{m} y_i x_i = 4.
\]

(Notice that the contribution corresponding to \(4 \cdot 1\) is precisely the sum in the right-hand side of the identity.)

To compute \(a(\kappa, 4; q)\), recall the notation introduced in Section 3. By (17), (19) and (20), for every partition \(\mu = (\mu_1 \geq \cdots \geq \mu_m \geq 1)\) and \(\mathcal{N} = (\nu_{1j}, \ldots, \nu_{rj}, \nu_j)_{1 \leq j \leq m}\), subject to \(\nu_j = 0\) or 1 for all \(j = 1, \ldots, n\), and

\[
\sum_{j=1}^{n} \nu_{ij} \mu_j = k_i, \quad \sum_{j \in J_i^0} \mu_j = 4 \quad (\text{for } i = 1, \ldots, r)
\]

we have

\[
\lambda_{\mu}(\mathcal{N}; q) = \prod_{j \in J_i^0} q^{\mu_j} (\nu_{1j} + \cdots + \nu_{rj} + 1) \lambda(\nu_{1j}, \ldots, \nu_{rj}, 1; 1/q^{\mu_j}) = 1
\]
if $\nu_j = \cdots = \nu_{r_j} = 0$ for all $j \notin J_1^0$, and zero otherwise. By Remark 3 in Section 3, we see that for every fixed $\kappa = (k_1, \ldots, k_r) \in \mathbb{N}^r$, the total contribution to $a(\kappa, 4; q)$ corresponding to all such pairs $(\mu, N)$ is

$$
\sum_{\deg d_0 = 4} b_{d_0}(\kappa)
$$

where $b_{d_0}(\kappa)$ is the coefficient $(k_1, \ldots, k_r)$ of

$$
\prod_{i=1}^r (1 - t_i)(1 - a(C_{d_0})t_i + qt_i^2)
$$

with $a(C_{d_0}) = -1 - \sum_{\theta \in \mathcal{S}} \chi(d_0(\theta)).$

Note that this contribution corresponds to the five partitions $x = (x_1 \geq \cdots \geq x_m), 1 \leq m \leq 4$, of 4. (For every partition $x$, we choose $y_1 = \cdots = y_m = 1$ in (37).) In what follows, we shall refer to the expression

$$
\sum_{\deg d_0 = 4} \left( \prod_{i=1}^r (1 - t_i)(1 - a(C_{d_0})t_i + qt_i^2) \right)
$$

(38)

as the non-degenerate part of $A_4(T, q)$.

There are four additional (degenerate) contributions to $a(\kappa, 4; q)$ which can be computed similarly using Remark 3 in Section 3. They are:

$$
\frac{1}{2}(q - 1) \cdot \sum_{2\kappa' \leq \kappa} q^{\kappa'(|\kappa| + 5)}(1 - q^{-2})^{-r_{(\kappa - 2\kappa')}} \lambda(\kappa'; 2; 1/q^2)
$$

with generating series

$$
\frac{q^3(q - 1)}{2} \frac{E(qT)\Lambda_2(q^2T^2, 1/q^2)}{E(T)E(-T)}
$$

(39)

corresponding to $2 \cdot 2$ in (37),

$$
\frac{1}{2}q^{\kappa'(5+\kappa)}(q - 1) \cdot \sum_{\kappa'' \leq \kappa} \left( \frac{q - 1}{q} \right)^{2r_{(\kappa - \kappa'')}} \left( \frac{q(q - 2)}{(q - 1)^2} \right)^{r_{(\kappa - \kappa'')}} \lambda(\kappa', 2; 1/q) \lambda(\kappa'', 2; 1/q)
$$

with generating series

$$
\frac{q^5(q - 1)}{2} \frac{E(qT)\Lambda_2(qT, 1/q)q^2}{E(T)^2}
$$

(40)

corresponding to $2 \cdot 1 + 2 \cdot 1$,

$$
\sum_{\kappa' \leq \kappa} \frac{q(q - 1)^2}{2} \cdot \sum_{\kappa'' \leq \kappa} \left( -2 \right)^{r_{(\kappa - \kappa'')}} \cdot q^{\kappa'\kappa''} \lambda(\kappa', 2; 1/q)
$$

with generating series

$$
\frac{q^3(q - 1)(q - 2)}{2} \frac{\Lambda_2(qT, 1/q)}{E(T)^2} + \frac{q^4(q - 1)}{2} \frac{\Lambda_2(-qT, 1/q)}{E(T)E(-T)}
$$

(41)
corresponding to $1 \cdot 2 + 2 \cdot 1$ or $2 \cdot 1 + 1 \cdot 1 + 1 \cdot 1$, and finally

$$(q - 1) \sum_{\kappa' \leq \kappa \atop k_i - k'_i = 0, 1} (-1)^{|\kappa - \kappa'|} q^{|\kappa'|+4} \lambda(\kappa'; 3; 1/q)$$

with generating series

$$q^4(q - 1) \frac{\Lambda_3(q T, 1/q)}{E(T)}$$

(42)

corresponding to $3 \cdot 1 + 1 \cdot 1$. Notice that in the above degenerate contributions we have evaluated the elementary character sums that occur.

Now set

$$\mathcal{M}_4(r; q) := \sum_{\deg d_0 = 4 \atop d_0 \text{monic \\& square-free}} a(C_{d_0})^r = \sum_{\deg d_0 = 4 \atop d_0 \text{monic \\& square-free}} \left( -1 - \sum_{\theta \in \mathbb{F}} \chi(d_0(\theta)) \right)^r.$$

We have:

$$\mathcal{M}_4(r; q) = \begin{cases} -\mathcal{M}_3(r + 1; q) & \text{if } r \text{ is odd} \\ q \mathcal{M}_3(r; q) & \text{if } r \text{ is even} \end{cases}$$

see Appendix D for details. Accordingly, we can express the non-degenerate contribution (38) as

$$\frac{1}{E(T)} \sum_{r_1(\kappa) = 1 \atop \kappa \in \mathfrak{S}} \mathcal{M}_3(r_1(\kappa); q)q^{r_2(\kappa)+1}T^\kappa + \frac{1}{E(T)} \sum_{r_1(\kappa) = 1 \atop \kappa \in \mathfrak{S}} \mathcal{M}_3(r_1(\kappa); 1; q)q^{r_2(\kappa)}T^\kappa$$

where, as before, $r_1(\kappa) = \{ j : k_j = i \}$ for $i = 1, 2$. With this last piece of information, we are now in the position to justify the functional equation satisfied by $A_4(T, q)$.

**Proposition 5.2.** — The generating series $A_4(T, q)$ satisfies the functional equation

$$E(T)A_4(T, q) = -q^5 E(q T)A_4(q T, 1/q).$$

**Proof.** The idea of the proof is to identify the parts of $E(T)A_4(T, q)$ satisfying the functional equation in the statement of the proposition. Indeed, from the $l = 3$ case we know that

$$\Lambda_3(T, q) - q^4 \Lambda_3(q T, 1/q) = A_3^{(0)}(q T) + \frac{q^3(q - 1)}{2} \left( \frac{\Lambda_2(q T, 1/q)}{E(T)} + \frac{\Lambda_2(-q T, 1/q)}{E(-T)} \right)$$

where

$$A_3^{(0)}(q T) := \sum_{r_1(\kappa) = 1 \atop \kappa \in \mathfrak{S}} \mathcal{M}_3(r_1(\kappa); q)q^{r_2(\kappa)}T^\kappa$$

is the non-degenerate part of $A_3(T, q)$. Using this identity together with (41), (42), and the above expression of (38), compute

$$E(T)A_4(T, q) - q \Lambda_3(T, q) + q^4 \Lambda_3(q T, 1/q)$$

which should clearly satisfy the correct functional equation. Moreover, employing the identity

$$-q^2 + E(T) \Lambda_2(T, q) = -q + q^3 E(q T) \Lambda_2(q T, 1/q)$$

corresponding to the $l = 2$ case, it follows that both expressions:

$$\frac{q(q - 1)}{2E(T)E(q T)} - q^3(q - 1) \frac{\Lambda_2(q T, 1/q)}{E(T)} + \frac{q^5(q - 1)}{2} \frac{E(q T)\Lambda_2(q T, 1/q)}{E(T)}$$

28
and
\[-\frac{q(q-1)}{2E(-T)E(-qT)} + \frac{q^5(q-1)}{2} \frac{E(qT)\Lambda_2(q^2T,1/q^2)}{E(-T)}\]
satisfy the desired functional equation. A simple calculation reduces then the problem to showing the identity
\[A_3^{(1)}(T,q) + q^5A_3^{(1)}(qT,1/q) = \frac{q(q-1)(q-1)}{2} \left( \frac{1}{E(-T)E(-qT)} - \frac{1}{E(T)E(qT)} \right)\]  (43)
where we set
\[A_3^{(1)}(T,q) := \sum_{r_1(\kappa)+2r_2(\kappa)=|\kappa|} M_3(r_1(\kappa) + 1; q) q^{r_2(\kappa)} T^\kappa.\]

To prove this, we add an additional variable \(t_{r+1}\) (i.e., pass from \(r\) to \(r+1\)) and split the sum defining \(A_3^{(0)}(t_1, \ldots, t_r, t_{r+1}, q) = A_3^{(0)}(T, t_{r+1}, q)\) into three parts according to whether \(k_{r+1} = 0, 1, 2\) to obtain:
\[A_3^{(0)}(T, t_{r+1}, q) = (1 + qt_{r+1}^2)A_3^{(0)}(T, q) + t_{r+1}A_3^{(1)}(T, q)\]  (44)
Replacing \(t_i\) by \(qt_i\) for all \(i = 1, \ldots, r+1, q\) by \(1/q\), and then multiplying by \(q^4\), we obtain
\[q^4 A_3^{(0)}(qT, qt_{r+1}, 1/q) = q^4(1 + qt_{r+1}^2)A_3^{(0)}(qT, 1/q) + q^5 t_{r+1} A_3^{(1)}(qT, 1/q).\]  (45)
Add (44) and (45), and apply the identity
\[A_3^{(0)}(T, q) = \Lambda_3(T, q) - q^4 \Lambda_3(qT, 1/q) - \frac{q^3(q-1)}{2} \left( \frac{\Lambda_2(qT, 1/q)}{E(T)} + \frac{\Lambda_2(-qT, 1/q)}{E(-T)} \right)\]
to express \(A_3^{(0)}(T, t_{r+1}, q), A_3^{(0)}(qT, qt_{r+1}, 1/q), A_3^{(0)}(T, q)\) and \(A_3^{(0)}(qT, 1/q)\) in terms of \(\Lambda_2\). Now (43) follows by applying the identity corresponding to the \(l = 2\) case in the form
\[\frac{\Lambda_2(T, q)}{E(qT)} - q^3 \frac{\Lambda_2(qT, 1/q)}{E(T)} = \frac{q(q-1)}{E(T)E(qT)}\]
and completes the proof of the proposition. \(\square\)

Using the relation in Proposition 3.1 rewritten in the equivalent form
\[\lambda(\kappa, 4; q) - q^{|\kappa|+5} \lambda(\kappa, 4; 1/q) = a(\kappa, 4; q) + \sum_{\kappa' < \kappa} q^{|\kappa|-r(\kappa-\kappa')+5}(q-1)^{r(\kappa-\kappa')} \lambda(\kappa', 4; 1/q)\]  (46)
we can find recursively the coefficients \(\lambda(\kappa, 4; q)\); the sum in (46) can be expressed as
\[\sum_{\kappa' \leq \kappa} \left( a(\kappa', 4; q) - \lambda(\kappa', 4; q) + q^{|\kappa|+5} \lambda(\kappa', 4; 1/q) \right)\]
and hence, the right-hand side of (46) satisfies the same symmetry, as \(q \to 1/q\), as the \(\kappa\)-coefficient
\[\sum_{\kappa' \geq \kappa} a(\kappa', 4; q)\]
of \(E(T) A_4(T, q)\) does (cf. Proposition 5.2). The coefficients of \(A_4(T, q)\) are determined by expressing the moments \(\mathcal{M}_q(r; q)\) in (38) using Birch-Ihara identity (see also Appendix D), and by (39), (40), (41) (combined with the computations in Appendix C), (42) and Theorem 5.1.
Now starting from $\lambda(0, \ldots, 0, q) = q^4$ and using the above information, one obtains the coefficients $\lambda(\kappa, 4; q)$ in the form

$$\lambda(\kappa, 4; q) = P_0(q) + \sum_{j=5}^{[(r_1(\kappa)+1)/2]} P_j(q) T_{2j+2}(q)$$

where $P_0(q) = P_0(\kappa, q), P_5(q) = P_5(\kappa, q), \ldots$ are polynomials with integer coefficients, independent of $q$, such that:

$$\deg P_0 \leq |\kappa| + 4 \quad \text{deg } P_j \leq |\kappa| - 2j + 3 \quad \text{(for } j = 5, \ldots, [(r_1(\kappa)+1)/2])$$

and

$$q^{[(|\kappa|+1)/2]+3} | P_0(q) \quad q^{[(|\kappa|+2)/2]-j+3} | P_j(q) \quad \text{(for } j = 5, \ldots, [(r_1(\kappa)+1)/2]).$$

In the general case we shall use the same idea to determine the coefficients $\lambda(\kappa, l; q)$, but for this we need (28) for arbitrary $l$, and a suitable substitute for Birch-Ihara identity; such an identity is available, unfortunately, only in very special cases.

### 6 Cohomological interpretation of the moment-sums

For the purposes of this section, it is more convenient to work with the slightly modified moment-sums $\tilde{M}_{\nu, \gamma}(q)$ defined for partitions $\nu = (1^{\nu_1}, \ldots, (2g+2)^{\nu_{2g+2}})$ of $2g + 2$, and $\gamma = (1^{\gamma_1}, \ldots, g^{\gamma_g}), g \geq 2$, by

$$\tilde{M}_{\nu, \gamma}(q) = |GL_2(\mathbb{F})|^{-1} \sum_{d_0 \in \mathcal{P}_g(\nu,\mathbb{F})} \prod_{j=1}^{g} a_j(C_{d_0})\gamma_j$$

where $\mathcal{P}_g(\nu, \mathbb{F}) \subset \mathbb{F}[x]$ is the subset of all square-free polynomials of degree $2g + 1$ or $2g + 2$ defining hyperelliptic curves whose ramification points have fields of definition given by $\nu$, and where $a_j(C_{d_0}) = \text{Tr}(F^{d_0} | H^1_{\text{et}}(C_{d_0}, \mathbb{Q}_l))$. The normalizing factor $|GL_2(\mathbb{F})|$ represents the number of $\mathbb{F}$-isomorphisms between curves $C_{d_0}$ ($d_0 \in \mathcal{P}_g(\nu, \mathbb{F})$), see [51, Section 3] and [7, Section 3]. When $g = 2$ these sums appear in [8, eqn. (5.1)].

Notice that $\tilde{M}_{\nu, \gamma}(q)$ vanishes if its weight $|\gamma| := \gamma_1 + 2\gamma_2 + \cdots + g\gamma_g$ is odd. Note also that $\tilde{M}_{\nu, \gamma}(q)$ can be expressed in terms of the sums $M_{\mu, \gamma}(q)$ introduced in Section 3 and vice versa. (To see it the other way around, apply the functional equation (2.1) of $Z_{G_{d_0}}(t), d_0 \in \mathcal{P}(\mu)$, and Propositions B.1, B.2 in Appendix B.)

It turns out that the moment-sums $\tilde{M}_{\nu, \gamma}(q)$ arise naturally when studying the cohomology of local systems on certain classical moduli spaces. To make this precise, let $A_g$ denote the moduli stack of principally polarized abelian schemes of relative dimension $q$; this is a smooth (but not proper) Deligne-Mumford stack defined over $\text{Spec}(\mathbb{Z})$. We shall need to work with the moduli stack $A_{g,2}$ over $\text{Spec}(\mathbb{Z}[1/2])$ classifying principally polarized abelian schemes of relative dimension $g$ with principal symplectic level-2 structure. (The standard reference on these moduli and their compactification is [43].)

#### 6.1 Local systems on $A_g$

Let $GSp_{2g}$ denote the Chevalley group scheme over $\mathbb{Z}$ of symplectic similitudes on the symplectic space $\mathbb{Z}^{2g}$ with its standard non-degenerate alternating form defined by $(u,v) \times (u',v') \mapsto u \cdot ^t u' - v \cdot ^t v'$ for $u,v,u',v' \in \mathbb{Z}^g$. We can write

$$GSp_{2g}(R) = \left\{ \delta = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A \cdot ^t B - B \cdot ^t A = C \cdot ^t D - D \cdot ^t C = 0 \right\}$$

for any commutative ring $R$ with identity. The homomorphism $\eta : GSp_{2g} \to G_m$ is called the multiplier representation of $GSp_{2g}$. The kernel of $\eta$ is by definition the group scheme $Sp_{2g}$.
For $\lambda = (\lambda_1 \geq \cdots \geq \lambda_g \geq 0)$, we have natural $\mathbb{Q}_\ell$-adic smooth étale sheaves $V(\lambda)$ on $A_g \otimes \mathbb{Z}[1/\ell]$ corresponding to irreducible algebraic representations $V_\lambda$ of $GSp_{2g}(\mathbb{Q})$. Specifically, the $\mathbb{Q}_\ell$-sheaf $V = V(1, \ldots, 0) = R^1\pi_*\mathbb{Q}_\ell$, defined using the universal family $\pi : \mathcal{X} \to A_g$, corresponds to the contragredient of the standard representation $V$ of $GSp_{2g}(\mathbb{Q})$. We have a non-degenerate alternating pairing

$$V \times V \to \mathbb{Q}_\ell(-1).$$

For general $\lambda$, the $\mathbb{Q}_\ell$-sheaf $V(\lambda)$ occurs in the decomposition of

$$\text{Sym}^{\lambda_1-\lambda_g}V \otimes \cdots \otimes \text{Sym}^{\lambda_{g-1}-\lambda_g}V \otimes \text{Sym}^{\lambda_g}V$$

into irreducibles; it is the $\ell$-adic coefficient system of weight $|\lambda| = \lambda_1 + \cdots + \lambda_g$ corresponding to the irreducible representation $V_\lambda$ with dominant weight $(\lambda_1 - \lambda_2)\omega_1 + \cdots + (\lambda_{g-1} - \lambda_g)\omega_{g-1} + \lambda_g\omega_g - |\eta|$. Here $\omega_1, \ldots, \omega_g$ is the set of fundamental weights. If $\lambda_1 > \cdots > \lambda_g > 0$, the local system $V(\lambda)$ will be called regular. Finally, the Tate sheaf $\mathbb{Q}_\ell(1)$ corresponds to $\eta$.

In a series of papers, Bergström, Faber, and van der Geer investigated the motivic Euler characteristic

$$e_c(A, V(\lambda)) = \sum_{i=0}^{g(g+1)/2} (-1)^i[H^i_c(A, V(\lambda))]$$

when $A = A_g$, $g = 2, 3$ (see [41], [9]), and $A = A_{2, g}$ (see [8]). This expression is taken in the Grothendieck group $K_0(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ of $\ell$-adic $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representations, or in $K_0(\text{MHS})$, the Grothendieck group of the category of mixed Hodge structures. (Here and throughout the section, we shall use the same notation $V(\lambda)$ for the corresponding local systems obtained by pullback to $A_{g, 2}$.)

**Remark.** As the compactly supported cohomology of $V(\lambda)$ on $A = A_g$ (or $A_{g, 2}$) always vanishes if $|\lambda|$ is odd, one can only consider local systems $V(\lambda)$ of even weights.

In [43, Theorem 5.5, p. 233], Faltings and Chi provided Hodge filtrations of mixed Hodge structures on the cohomology groups $H^i_c(A_g \otimes \mathbb{C}, V(\lambda))$ and $H^i(\mathcal{X}, V(\lambda))$ (and also on $H^i_c(A_{g, 2} \otimes \mathbb{C}, V(\lambda))$ and $H^i(A_{g, 2} \otimes \mathbb{C}, V(\lambda))$) of weights $\leq |\lambda| + i$ and $\geq |\lambda| + i$, respectively. (There is also an analogue of this for the $\ell$-adic cohomology.) A main ingredient in their theory is the construction in [42], using some ideas of Bernstein, Gelfand and Gelfand, of a complex $K^*_\lambda$ of vector bundles, called the dual BGG-complex for $\lambda$, which is a direct summand in the de Rham complex of $V(\lambda)^\vee$; this complex is a filtered resolution of $V(\lambda)^\vee$. The steps in the Hodge filtrations are given (see [9]) by the sums of the elements of any of the $2^g$ subsets of $\{\lambda_g + 1, \lambda_g + 2, \ldots, \lambda_g + g\}$. It was also proved by Faltings [42] that, for $V(\lambda)$ regular, the inner cohomology groups $H^i_c(A_g, V(\lambda))$ (or $A = A_{g, 2}$), i.e., the image of the natural map $H^i_c(A_g, V(\lambda)) \to H^i(A, V(\lambda))$, vanish for $i + g(g + 1)/2$. The middle inner cohomology group $H^{g(g+1)/2}_c(A_g, V(\lambda))$ is pure of weight $|\lambda| + g(g + 1)/2$, and by [43, Section 6, p. 237], the step of the Hodge filtration of $H^{g(g+1)/2}_c(A_g, V(\lambda))$ corresponding to the full set $\{\lambda_g + 1, \lambda_g + 2, \ldots, \lambda_g + g\}$ can be identified with the complex vector space $S_{n}(\lambda)(\Gamma(A))$ of Siegel modular cusp forms of weight $n(\lambda) = (\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{g-1} - \lambda_g, \lambda_g + g + 1)$ on $\Gamma(A)$, with $\Gamma(A) = Sp_{2g}(\mathbb{Z})$ if $A = A_g$, and $\Gamma(A) = \Gamma_g[2]$ is the kernel of $Sp_{2g}(\mathbb{Z}) \to Sp_{2g}(\mathbb{Z}/2)$ if $A = A_{g, 2}$. For basic facts about the theory of Siegel modular forms, we refer the reader to [3], [43] and [45]. It is expected (see [9] for more details) that

$$e_c(A, V(\lambda)) = \left(-1\right)^{\frac{g(g+1)}{2}}S[\Gamma(A), n(\lambda)] + e_{g, \text{endo}}(A, \lambda) + e_{g, \text{Eis}}(A, \lambda)$$

for $\lambda \neq (0, \ldots, 0)$ with $|\lambda| \equiv 0 \mod 2$, where $S[\Gamma(A), n(\lambda)]$ is the conjectural motive in [9] associated to $S_{n}(\lambda)(\Gamma(A))$, $e_{g, \text{endo}}(A, \lambda)$ is the Euler characteristic of the remaining part of the inner cohomology, presumably consisting of contributions connected to the endoscopic groups, and $e_{g, \text{Eis}}(A, \lambda)$ denotes the Euler characteristic corresponding to the kernel of the natural map $H^i_c(A, V(\lambda)) \to H^i(A, V(\lambda))$, called the Eisenstein cohomology. The most successful

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5In this instance, we are taking the compactly supported cohomology of the Betti version of $V(\lambda)$, constructed from $V = R^1\pi_*\mathbb{Q}_\ell$ on $A \otimes \mathbb{C}$. 

31
The connection to the sums $\hat{M}_{r,\nu}(q)$ comes by considering the $\ell$-adic Euler characteristic $e_c(\mathcal{H}_g[2] \otimes \mathbb{F}, \nu'(\lambda))$ in $K_0(\text{Gal}_q)$ of the hyperelliptic Jacobian locus $\mathcal{H}_g[2]$ in $\mathcal{A}_g$. Here $\nu'(\lambda)$ denotes the restriction of $\nu(\lambda)$ to $\mathcal{H}_g[2]$. Let $[\mathcal{H}_g[2](\mathbb{F})]$ denote the set of isomorphism classes of the category $\mathcal{H}_g[2](\mathbb{F})$. By Torreli’s theorem [68, Appendice, Théorème 1], we can identify the elements of $[\mathcal{H}_g[2](\mathbb{F})]$ by isomorphism classes of tuples $(C, w_1, \ldots, w_{2g+2})$ of hyperelliptic curves of genus $g$ together with their $2g + 2$ marked Weierstrass points; we shall also identify the local system $\nu'(\lambda)$ and its pullback under the Torreli morphism.

The natural action of the symmetric group $S_{2g+2}$ on $\mathcal{H}_g[2]$ induces a decomposition of the representation $H^i(\mathcal{H}_g[2] \otimes \mathbb{F}, \nu'(\lambda))$ into a direct sum of factors $H^i_{c,\mu}(\mathcal{H}_g[2] \otimes \mathbb{F}, \nu'(\lambda))$ indexed by the partitions $\mu$ of $2g + 2$, and by [46, eqn. (2.32)], the weighted sum

$$
\frac{\chi_\mu(id)}{(2g + 2)!} \sum_{\sigma \in S_{2g+2}} \chi_\mu(\sigma^{-1}) \cdot \sigma : H^i(\mathcal{H}_g[2] \otimes \mathbb{F}, \nu'(\lambda)) \to H^i(\mathcal{H}_g[2] \otimes \mathbb{F}, \nu'(\lambda))
$$

is the projection of $H^i(\mathcal{H}_g[2] \otimes \mathbb{F}, \nu'(\lambda))$ onto $H^i_{c,\mu}(\mathcal{H}_g[2] \otimes \mathbb{F}, \nu'(\lambda))$. Here $\chi_\mu$ is the character of the irreducible representation $R_\mu$ of $S_{2g+2}$ corresponding to $\mu$. For a conjugacy class $[\nu]$ of $S_{2g+2}$ determined by a partition $\nu = (1^{\nu_1}, \ldots, (2 + g)_{\nu_{2g+2}})$ of $2g + 2$, we shall denote the value $\chi_\mu(\nu)$ by $\chi_\mu(\nu)$.

Now for any partition $\lambda$ of length at most $g$, let $s_{(\lambda)}$ denote the symplectic Schur function (see [46, Appendix A, §A.3, A.45]) associated to the irreducible representation $V_\lambda$. If we write $s_{(\lambda)} = \sum_{|\gamma| \leq |\lambda|} r^{\gamma}_{\lambda} p^{\gamma}$ for some $r^{\gamma}_{\lambda} \in \mathbb{Q}$, where the sum is over $\gamma = (1^{\gamma_1}, \ldots, g^{\gamma_g})$, and $p^{\gamma} = p_1^{\gamma_1} \cdots p_g^{\gamma_g}$ is the power sum polynomial associated to $\gamma$, then by Behrend’s Lefschetz trace formula (see [6] or [67, Thm. 19.3.4]), the trace of the geometric Frobenius on $e_{c,\mu}(\mathcal{H}_g[2] \otimes \mathbb{F}, \nu'(\lambda))$ is given by

$$
\text{Tr}(F^*| e_{c,\mu}(\mathcal{H}_g[2] \otimes \mathbb{F}, \nu'(\lambda))) = q^{\frac{|\lambda|}{2}} \sum_{\nu \in S_{2g+2}} \chi_\mu(\nu) \sum_{d \in \mathbb{P}_g(\nu, \mathbb{F})} s_{(\lambda)}(\omega_1(C_d), \ldots, \omega_g(C_d)).
$$

Here for a hyperelliptic curve $C_d$, $d \in \mathbb{P}_g(\nu, \mathbb{F})$, $\omega_1(C_d), \ldots, \omega_g(C_d)$ denote the normalized (i.e., unitary) eigenvalues of $F^*$: one can also see this by applying the Lefschetz trace formula to the correspondence on $(\mathcal{H}_g[2], p^\nu \nu'(\lambda))$ defined by composing the action of an element $\sigma^{-1} \in [\nu]$ with Frobenius (i.e., the relative Frobenius with respect to the $\mathbb{F}$-structure induced by $\sigma^{-1}$), where $H_g[2]$ denotes the coarse moduli space of $\mathcal{H}_g[2] \otimes \mathbb{F}$, and $p : \mathcal{H}_g[2] \otimes \mathbb{F} \to H_g[2]$ is the natural map.

In terms of the moment-sums $\hat{M}_{\nu,\lambda}(q)$,

$$
\text{Tr}(F^*| e_{c,\mu}(\mathcal{H}_g[2] \otimes \mathbb{F}, \nu'(\lambda))) = \chi_\mu(id) \sum_{\nu} \sum_{|\gamma| \leq |\lambda|} r^{\gamma}_{\lambda} q^{\frac{|\lambda|-|\gamma|}{2}} \hat{M}_{\nu,\lambda}(q)
$$

see also [7], [8] and [9]. (Notice that $|\gamma|$ stands here for the sum $\gamma_1 + 2\gamma_2 + \cdots + g\gamma_g$.) The last equation can be inverted to express $\hat{M}_{\nu,\lambda}(q)$, and hence $M_{\nu,\lambda}(q)$, in terms of $\text{Tr}(F^*| e_{c,\mu}(\mathcal{H}_g[2] \otimes \mathbb{F}, \nu'(\lambda)))$. It would be very interesting to identify the contribution of the Euler characteristic $e_c(\mathcal{H}_g[2], \nu'(\lambda))$ to (47) when $A = A_g, 2$.

### 6.2 Moments of characteristic polynomials

If $\lambda = (\lambda_1 \geq \cdots \geq \lambda_g \geq 0)$ is a partition, we shall write $\lambda \in (r^g)$ to indicate that $\lambda_i \leq r$ for all $1 \leq i \leq g$. For $\lambda \in (r^g)$, let $\lambda' = (g - \lambda'_1 \geq \cdots \geq g - \lambda'_r)$, where the non-zero integers among $\lambda'_j$, $j = 1, \ldots, r$, are the parts of the conjugate partition $\lambda'$ of $\lambda$.
Lemma 6.1. — For independent variables $t_1, \ldots, t_r, z_1, \ldots, z_g$, we have the identity
\[
\prod_{i=1}^{q} \prod_{j=1}^{r} (z_i + z_i^{-1} + t_j + t_j^{-1}) = \sum_{\lambda \in \mathcal{P}(r)} s_{(\lambda)}(z_1^{\pm 1}, \ldots, z_g^{\pm 1}) s_{(\lambda')}(t_1^{\pm 1}, \ldots, t_r^{\pm 1}).
\]

Proof. See [21, Section 5].

Lemma 6.2. — With notations as above, we have
\[
\sum_{d_0 \in \mathcal{P}_g(\nu, F)} s_{(\lambda)}(\omega_1(C_{d_0})^{\pm 1}, \ldots, \omega_g(C_{d_0})^{\pm 1}) = 0 \quad (\text{if } |\lambda| \text{ is odd}).
\]

Proof. Write as above
\[
s_{(\lambda)}(z_1^{\pm 1}, \ldots, z_g^{\pm 1}) = \sum_{|\gamma| \leq |\lambda|} r_{\gamma}^{\lambda} (z_1^{\pm 1}, \ldots, z_g^{\pm 1}).
\]
Replacing $z_i$ by $-z_i$ for $i = 1, \ldots, g$, we deduce that $r_{\gamma}^{\lambda} = 0$ unless $|\lambda|$ and $|\gamma|$ have the same parity. In particular, if $|\lambda|$ is odd the only terms that contribute are those corresponding to partitions $\gamma$ of odd weights.

Now take $z_i = \omega_i(C_{d_0})$ ($i = 1, \ldots, g$) for $d_0 \in \mathcal{P}_g(\nu, F)$, and recall that if $C_{d_0}$ is the hyperelliptic curve corresponding to $d_0$, then
\[
a_k(C_{d_0}) = \text{Tr}(F^{*k} | H^1_{\text{ét}}(\mathcal{C}_{d_0}, \mathbb{Q}_\ell)) = - \sum_{\theta \in \mathcal{P}^2(\mathcal{P}_k)} \chi_k(d_0(\theta)).
\]

Note that
\[
\sum_{d_0 \in \mathcal{P}_g(\nu, F)} p^{(\gamma)}(\omega_1(C_{d_0})^{\pm 1}, \ldots, \omega_g(C_{d_0})^{\pm 1}) = \sum_{d_0 \in \mathcal{P}_g(\nu, F)} q^{\frac{g\gamma}{2}} \prod_{k=1}^{g} a_k(C_{d_0})^{\gamma_k} = 0 \quad (\text{if } |\gamma| \text{ is odd})
\]
and the lemma follows.

Theorem 6.3. — For any partition $\mu$ of $2g + 2$ and independent variables $t_1, \ldots, t_r$, let $S_\mu(T, q) = S_\mu(t_1, \ldots, t_r, q)$ be defined by
\[
S_\mu(T, q) = \frac{\chi_\mu(id)}{|\GL_2(F)|} \sum_{\nu} \chi_\mu(\nu) \sum_{d_0 \in \mathcal{P}_g(\nu, F)} \left( \prod_{k=1}^{g} P_{C_{d_0}}(t_k) \right).
\]

Then
\[
S_\mu(T, q) = (t_1 \cdots t_r)^g \sum_{\lambda \in \mathcal{P}(r)} \sum_{|\lambda| \text{ even}} \left( \prod_{k=1}^{g} P_{C_{d_0}}(t_k) \right) \left( \prod_{\lambda \in \mathcal{P}(r)} s_{(\lambda)}(q^{\frac{g\lambda_1}{2}} t_1^{\pm 1}, \ldots, q^{\frac{g\lambda_r}{2}} t_r^{\pm 1}) q^{\frac{g\lambda}{2}} \right).
\]

Proof. For any $d_0 \in \mathcal{P}_g(\nu, F)$, we apply Lemma 6.1 with $t_k \rightarrow -q^{1/2} t_k$ for $k = 1, \ldots, r$, and $z_1^{\pm 1}, \ldots, z_g^{\pm 1}$ chosen to be the normalized (by $\sqrt{q}$) eigenvalues of the Frobenius $F^*$ on $H^1_{\text{ét}}(C_{d_0}, \mathbb{Q}_\ell)$, with $\ell$ a prime different from the characteristic of $F$. Multiplying both sides of the identity by $(-1)^{g\tau} (\sqrt{q} t_1 \cdots \sqrt{q} t_r)^g \chi_\mu(\nu)$ and summing over $d_0$ and $\nu$, the theorem follows at once from (48) and Lemma 6.2.

Remark. By applying Lemma 6.1 and the identity
\[
\prod_{i=1}^{m} \prod_{j=1}^{q} (1 - t_i z_j)^{-1} (1 - t_i z_j^{-1})^{-1} = \prod_{1 \leq i < j \leq m} (1 - t_i t_j)^{-1} \sum_{\lambda \in \mathcal{P}(g)} s_{(\lambda)}(Z^{\pm 1}) s_\lambda(T)
\]
(see [21, Section 5, Proof of Theorem 4]) one can obtain in a similar fashion the variant of Theorem 6.3 for sums of ratios of characteristic polynomials; here $s_\lambda$ is the ordinary Schur polynomial, $Z^{\pm 1} = (z_1^{\pm 1}, \ldots, z_g^{\pm 1})$, and $T = (t_1, \ldots, t_m)$.  

33
Let \( S = (\chi_{\mu}(\nu))_{\mu,\nu}, \) where \( \mu, \nu \) indicate respectively the rows and columns of \( S \). This is the transpose of the transition matrix from the basis of the \( \mathbb{Q} \)-vector space of homogeneous symmetric polynomials of degree \( 2g + 2 \) in \( 2g + 2 \) variables consisting of the sum polynomial basis to the basis consisting of the ordinary Schur polynomials. If \( \nu = (1^\nu_1, 2^\nu_2, \ldots, (2g + 2)^\nu_{2g+2}), \) put, as before, \( z(\nu) = 1^\nu_11^\nu_21^\nu_32^\nu_2 \cdots 2^\nu_{2g+2}((2g + 2)^\nu_{2g+2}) \). Then by [46, Appendix A, §A.1, Exercise A.29], we have \( S^{-1} = (\chi_{\mu}(\nu)/z(\nu))_{\nu,\mu}. \) Accordingly, if we denote the components of

\[
\langle S \cdot (\chi_{\mu}(id)^{-1} e_{c,\mu}(\mathcal{H}_g[2], \mathcal{V}'(\lambda)) \rangle_{\mu}
\]

by \( \tilde{e}_{c,\nu}(\mathcal{H}_g[2], \mathcal{V}'(\lambda)) \), then Theorem 6.3 implies that, for every \( \nu \), we have

\[
\frac{z(\nu)}{|\text{GL}_2(\mathbb{F})|} \sum_{d \in \mathbb{N}} \left( \prod_{k=1}^r P_{c_{d_0}}(t_k) \right) (t_1 \cdots t_r)^g \sum_{\lambda \in \ell (r^g)} \prod_{|\lambda|} ((\mu_i - \mu_j + j - i)) \text{ if } \mu = (\mu_1, \ldots, \mu_k) \geq 0
\]

(49)

Recalling that \( \chi_{\mu} \) is the character of the irreducible representation \( R_{\mu} \) of \( S_{2g+2} \) corresponding to \( \mu \), note that

\[
\chi_{\mu}(id) = \dim R_{\mu} = \frac{(2g + 2)!}{\prod_{i=1}^k (\mu_i + k - i)!} \prod_{1 \leq i < j \leq k} (\mu_i - \mu_j + j - i)
\]

(see, for instance, [46, §4.1, eqn. (4.11)]).

By letting

\[
e^\lambda_{c}(\mathcal{H}_g(\nu)) = \sum_{\nu = (1^{\nu_1}, \ldots, (2g + 2)^{\nu_{2g+2}})} \frac{\nu_1!}{\mu_1!} \tilde{e}_{c,\nu}(\mathcal{H}_g[2], \mathcal{V}'(\lambda))
\]

(50)

one obtains the following

**Theorem 6.4.** For \( g \geq 1 \), we have

\[
\frac{1}{q(q-1)} \sum_{d \text{ square-free}} \left( \prod_{k=1}^r P_{c_{d_0}}(t_k) \right) (t_1 \cdots t_r)^g \sum_{\lambda \in \ell (r^g)} \prod_{|\lambda|} ((\mu_i - \mu_j + j - i)) \text{ if } \mu = (\mu_1, \ldots, \mu_k) \geq 0
\]

(51)

the first sum being over monic polynomials in \( \mathbb{F}[x] \). When \( g = 1 \), we define \( \mathcal{H}_1(\nu^1) := A_1 \).

**Proof.** When \( g = 1 \) this is clear, and so we can assume that \( g \geq 2 \). By (49) and (50), the right-hand side clearly equals

\[
\frac{1}{|\text{GL}_2(\mathbb{F})|} \sum_{\nu = (1^{\nu_1}, \ldots, (2g + 2)^{\nu_{2g+2}})} \nu_1! \sum_{d \in \mathbb{N}} \left( \prod_{k=1}^r P_{c_{d_0}}(t_k) \right).
\]

Note that \( |\text{GL}_2(\mathbb{F})| = q(q + 1)(q - 1)^2 \). We split the sum in the left-hand side of (51) according to the factorization types \( \mu = (1^{\mu_1}, \ldots, (2g + 1)^{\mu_{2g+1}}) \), with \( |\mu| = 2g + 1 \), of \( d \).

Now for a hyperelliptic curve \( C_{d_0} \), express the product of \( P_{c_{d_0}}(t_k) \) over \( k = 1, \ldots, r \) in the form

\[
\prod_{k=1}^r P_{c_{d_0}}(t_k) = \exp \left( - \sum_{j=1}^\infty \frac{a_j(C_{d_0})}{j} \cdot \sum_{k=1}^r t_k^j \right)
\]

with

\[
a_j(C_{d_0}) = \text{Tr}(F^*|_{H_e^1(C_{d_0}, \mathbb{Q}_\ell)}) = - \sum_{\theta \in \overline{\mathbb{P}^1(\mathbb{F}_l)}} \chi_j(d_0(\theta)).
\]

\(^6\)This can probably be better formulated using \( S_{2g+2} \)-equivariant Euler characteristics.
Applying this, we see that it suffices to show that
\[
\sum_{\mu=(1^{\nu_1},\ldots,(2g+1)^{\nu_{2g+1}})} \sum_{d \in \mathcal{P}(\mu)} \prod_{j=1}^{m} a_j(C_d)^{\gamma_j} = \frac{1}{q^2 - 1} \sum_{\mu=(1^{\nu_1},\ldots,(2g+2)^{\nu_{2g+2}})} \nu_1 \prod_{d_0 \in \mathcal{P}(\nu, \emptyset)} \prod_{j=1}^{m} a_j(C_d)^{\gamma_j}.
\]
for any partition \( \gamma = (1^{\nu_1}, \ldots, m^{\nu_m}) \). Here \( \mathcal{P}(\mu) \subset \mathbb{P}[x] \) denotes, as in Section 3, the set of all monic square-free polynomials \( d \) with factorization type \( \mu \). It is easy to see that both sides vanish when \( \gamma \) has odd weight (see also the beginning of the proof of Lemma 7.2). When \( |\gamma| \) is even, the identity is easily reduced to
\[
\sum_{\mu=(1^{\nu_1},\ldots,(2g+1)^{\nu_{2g+1}})} \sum_{d \in \mathcal{P}(\mu)} \prod_{j=1}^{m} a_j(C_d)^{\gamma_j} = \frac{1}{q + 1} \sum_{\mu=(1^{\nu_1},\ldots,(2g+2)^{\nu_{2g+2}})} \nu_1 \prod_{d_0 \in \mathcal{P}(\nu, \emptyset)} \prod_{j=1}^{m} a_j(C_d)^{\gamma_j}.
\]
(52)

By Proposition B.1 in Appendix B, for any partition \( \mu = (1^{\nu_1}, \ldots, (2g+1)^{\nu_{2g+1}}) \) of \( 2g + 1 \), we have
\[
\sum_{d \in \mathcal{P}(\mu)} \prod_{j=1}^{m} a_j(C_d)^{\gamma_j} = \frac{\mu_1 + 1}{q + 1} \left( \sum_{d_0 \in \mathcal{P}(\mu)} \prod_{j=1}^{m} a_j(C_{d_0})^{\gamma_j} + \sum_{d_0 \in \mathcal{P}(\mu')} \prod_{j=1}^{m} a_j(C_{d_0})^{\gamma_j} \right)
\]
with \( \mu' = (1^{\nu_1+1}, \ldots, (2g+1)^{\nu_{2g+1}}) \). Clearly this implies (52), and completes the proof. \( \square \)

**Corollary 6.5.** — Notation being as in Theorem 6.4, we have
\[
\frac{1}{q(q - 1)} \sum_{\text{deg } d = 2g + 1} L(\frac{1}{r}, \chi_d)^{r} = \left( \dim V_{\lambda'} \right) \text{Tr} (F^\ast c_\lambda^\prime (H_q(w^1) \otimes \overline{\mathbb{F}})) q^{-\frac{1}{2r}}.
\]

**Proof.** We recall from Section 2 that \( L(s, \chi_d) = P_{c_d}(q^{-s}) \). The identity stated is just the limiting case of the identity in Theorem 6.4 as \( t_i \to 1/\sqrt{q} \), for \( i = 1, \ldots, r \). \( \square \)

When \( g = 1, 2 \), one can use Theorem (Birch-Ihara), and the results in [56, 78] to express the traces of Frobenius on \( \varepsilon_\ast (A_g \otimes \mathbb{F}, V(\lambda)) \) in terms of traces of Hecke operators on spaces of Siegel (vector-valued) modular forms. We give here the simplest version of this trace comparison in the following corollary.

**Corollary 6.6.** — Notation being as in Section 5, if \( g = 1 \) we have
\[
\frac{1}{q(q - 1)} \sum_{\text{deg } d = 3} L(\frac{1}{2}, \chi_d)^{r} = \sum_{\lambda \text{ even } \lambda \neq 0} \left( \dim V_{\lambda'} \right) (-1 - T_{\lambda + 2}(q)) q^{-\frac{1}{2}}
\]
where we take \( T_{\lambda + 2}(q) = -q - 1 \) if \( \lambda = 0 \).

**Proof.** Expressing the traces of Frobenius in (51) in terms of moment-sums, our assertion follows at once by combining Corollary 6.5 and Theorem (Birch-Ihara). \( \square \)

**Remark.** By [46, §24.2, Exercise 24.20], the dimension of the irreducible representation \( V_{\lambda'} \) is
\[
\dim V_{\lambda'} = \frac{\prod_{i=1}^{r}(g + r - \lambda'_i - i + 1) \cdot \prod_{1 \leq i < j \leq r} (\lambda'_j - \lambda'_i - j + 1)(2g + 2r + 2 - \lambda'_i + \lambda'_j - i - j)}{1! \cdots (2r - 3)!(2r - 1)!}
\]
In particular, if \( \lambda = 0 \) we can write
\[
\dim V_0 = \prod_{k=1}^{r} \frac{k!}{(2k)!} \cdot \prod_{i=1}^{r} (2g + 2i) \prod_{1 \leq i < j \leq r} (2g + i + j).
\]
Note that the first product is precisely the constant \( g_r/(r(r+1)/2)! \) appearing in the moment-conjectures of Conrey-Farmer [29] and Keating-Snaith [61] for the leading-order asymptotics of the moments of L-functions within symplectic families. Moreover, if we let

\[
P_r(x) = \prod_{i=1}^{r} (x + 2i) \prod_{1 \leq i < j \leq r} (x + i + j)
\]

then the polynomial \( P_r(x-1) \) appears in the main term of the polynomial \( Q_r(x) \) conjectured by Conrey, Farmer, Keating, Rubinstein and Snaith [30]; see also the recent work of Andrade-Keating [2, Conjecture 5] in the function-field setting. In particular, \( P_1(x-1) = 1 + x \), \( P_2(x-1) = 6 + 11x + 6x^2 + x^3 \) and

\[
P_3(x-1) = 360 + 942x + 949x^2 + 480x^3 + 130x^4 + 18x^5 + x^6;
\]

compare this to eqns. (5.16), (5.21) and (5.26), respectively, in [2].

By counting points over \( \mathbb{F} \), one finds easily that \( \text{Tr}(F^*| e^0_c(\mathcal{H}_g(w^1) \otimes \mathbb{F})) = q^{2g-1} \), and therefore

\[
(\dim V_\nu^0) \text{Tr}(F^*| e^0_c(\mathcal{H}_g(w^1) \otimes \mathbb{F})) = \frac{g_r P_r(2g)}{(r(r+1)/2)!} q^{2g-1}.
\]

If \( |\nu| \neq 0 \), we have the well-known result [60, Theorem 10.8.2] of Katz and Sarnak: there exist positive constants \( A(g) \) and \( C(g) \) such that the estimate

\[
\left| \sum_{\text{deg } \Delta = 2g+1} s_{(\nu)}(\omega_1(\mathcal{C}_d)^{\pm 1}, \ldots, \omega_d(\mathcal{C}_d)^{\pm 1}) \right| \leq 2C(g)(\dim V_\lambda)q^{2g-\frac{3}{2}} \quad (\text{if } q \geq A(g)).
\]

The (normalized) sum in the left-hand side is precisely \( q^{-|\nu|/2} \text{Tr}(F^*| e^0_c(\mathcal{H}_g(w^1) \otimes \mathbb{F})) \); this follows from (51) and Lemma 6.1, or by combining the Grothendieck trace formula [54], [55] (see also the end of Section 9) with [60, Theorem 10.1.18.3].

Of great importance for us is that Theorem 6.3 allows us to make the following definition:

**Definition 6.1.** For any partition \( \mu \) of \( 2g+2 \), we define \( q^{\dim \mathcal{H}_g^2} s_\mu(q T, 1/q) = q^{2g-1} s_\mu(q T, 1/q) \) by simply replacing the compactly supported cohomology in Theorem 6.3 with \( H^r_\nu(\mathcal{H}_g[2] \otimes \mathbb{F}, \mathcal{V}^0(\mathcal{V}) \otimes \mathbb{Q}_\ell(|\nu|/2)) \), that is

\[
q^{2g-1} s_\mu(q T, 1/q) = (t_1 \cdots t_r)^g \sum_{\lambda \in \mathcal{P}(\mathcal{V})} \text{Tr} \left( \Phi_q^{-1} | e_\mu(\mathcal{H}_g[2] \otimes \mathbb{F}, \mathcal{V}^0(\mathcal{V}) \otimes \mathbb{Q}_\ell(|\lambda|/2)) \right) s_\lambda(q^{1/2} t_1^{\pm 1}, \ldots, q^{1/2} t_r^{\pm 1}) q^{|\lambda|/2}.
\]

Here \( \Phi_q \) is the arithmetic Frobenius endomorphism relative to \( \mathbb{F} \) acting, by transport of structures, on the \( \ell \)-adic cohomology.

Note that this definition is made according to the duality between cohomology and ordinary cohomology. We recall that for a Deligne-Mumford stack \( \mathcal{X} \), assumed to be smooth over some base \( S \) and purely \( d \)-dimensional, a map \( f: \mathcal{X} \to S \) and a \( \mathbb{Q}_\ell \)-local system \( \mathcal{F} \) on \( \mathcal{X} \) (with \( \ell \) invertible on our base), the Poincaré-Verdier duality gives a natural isomorphism (see [10]):

\[
R\text{Hom}_S(Rf_!\mathcal{F}, \mathcal{Q}_\ell) \cong Rf_! (\mathcal{F}^\vee [-d][-2d]).
\]

In the next section, we shall need to work with the sums of products of characteristic polynomials \( A_\nu(T, q) \) defined for partitions \( \nu = (1^{\nu_1}, 2^{\nu_2}, \ldots) \) of \( 2g+1 \) or \( 2g+2 \) by

\[
A_\nu(T, q) = \sum_{d \in \mathcal{P}(\nu)} \left( \prod_{k=1}^{r} P_{c_d}(t_k) \right).
\]

Here \( \mathcal{P}(\nu) = \mathcal{P}(\nu, q) \subset \mathbb{F}[x] \) stands, as before, for the set of all monic square-free polynomials with factorization type \( \nu \). Since the moment-sums \( M_{\mu, \gamma}(q) \) can be expressed in terms of \( M_{\nu, \gamma}(q) \) (by Propositions B.1 and B.2 in Appendix B), we can express \( A_\nu(T, q) \), analogously, as a sum of traces of Euler characteristics. (This will be made precise in 9.2.) Accordingly, we can define \( A_\nu(q T, 1/q) \) similarly.
7 The main theorem

We recall that our main goal is to construct inductively a sequence of generating series \( (\Lambda_l(T, q))_{l \geq 2} \) satisfying (27). In particular, for every fixed \( l \geq 2 \), the generating series

\[
A_l(T, q) = \sum_{(\mu, \nu) \in \{(1, 1), (1, l)\}} A_{\mu, \nu}(T, q)
\]

with \( A_{\mu, \nu}(T, q) \) as in Proposition 4.1 and (30), has to satisfy the functional equation

\[
E(T)^{\epsilon_l} A_l(T, q) + q^{l+1} E(q T)^{\epsilon_l} A_l(q T, 1/q) = 0
\]  

(53)

with \( \epsilon_l = 0 \) or 1 according as \( l \) is odd or even.

To see how (53) can be explained, consider

\[
q^{l+1} E(q T)^{\epsilon_l} A_l(q T, 1/q) = q^{l+1} E(q T)^{\epsilon_l} \cdot \sum_{(\mu, \nu) \not\in \{(1, 1), (1, l)\}} A_{\mu, \nu}(q T, 1/q)
\]

and express \( A_{\mu, \nu}(q T, 1/q) \), for each \( (\mu, \nu) \), by (30) (or by the formula in Proposition 4.1). The other term in (53) has a similar expression, and it is related to that of \( E(q T)^{\epsilon_l} A_l(q T, 1/q) \) by replacing \( q \to 1/q \) and \( T \to q T \). Assuming for the moment that the identity (27) holds for all \( 2 \leq l' < l \), and applying it to transform each factor \( \Lambda_{\mu, \nu}(\pm T^{l''}, q^{l''}) \) of each term in \( q^{l+1} E(q T)^{\epsilon_l} A_l(q T, 1/q) \) into \( \Lambda_{\mu, \nu}(\pm q^{l''} T^{l''}, 1/q^{l''}) \), one finds that

\[
E(T)^{\epsilon_l} A_l(T, q) + q^{l+1} E(q T)^{\epsilon_l} A_l(q T, 1/q)
\]

can be expressed as a sum over monomials of the form

\[
\prod_{j \in J_0} \Lambda_{\nu_j}(q^{\mu_j} T^{l''_j}, 1/q^{l''_j}) \cdot \prod_{j \in J_1} \frac{\Lambda_{\nu_j}(\delta_j q^{\mu_j} T^{l''_j}, 1/q^{l''_j})}{E(\delta_j T^{l''_j})}.
\]

(54)

We are omitting here the factors corresponding to all \( j \in J_0 \) for which \( \nu_j = 1 \) as \( q \Lambda_1(q T, 1/q) = 1 \) by (20). Henceforth, we shall slightly abuse notation and continue to use \( \Lambda_{\delta, \mu, \nu}(T, q) \) to denote the monomial (54). If we put

\[
\Sigma_{\delta, \mu, \nu}(T, q; l) := \text{Coefficient}_{\delta, \mu, \nu(T, q)} \left[ E(T)^{\epsilon_l} A_l(T, q) + q^{l+1} E(q T)^{\epsilon_l} A_l(q T, 1/q) \right]
\]

then we will show that

\[
\Sigma_{\delta, \mu, \nu}(T, q; l) = 0
\]

(55)

for all \( \delta, \mu, \nu \) and \( l \geq 2 \). It is clear that this strengthening of (53) makes no reference whatsoever to any of the generating series \( \Lambda_l(T, q) (l \geq 2) \). Simply put, showing (55) is a completely independent problem. In this section and the next, we shall see how the family of identities (55) is encoded in the combinatorial structure of moduli spaces of admissible double covers. Once this is established, we shall apply Deligne’s theory of weights to construct the sequence \( (\Lambda_l(T, q))_{l \geq 2} \).

7.1 A Special Case

For greater clarity, let us begin by first investigating the identities (55) corresponding to partitions of the form \( (1^n) \), that is, corresponding to

\[
\Sigma_n(T, q) := \text{Coefficient}_{\Lambda_{\delta, (q T, 1/q)^n}} \left[ E(T)^{\epsilon_n} A_{3n}(T, q) + q^{3n+1} E(q T)^{\epsilon_n} A_{3n}(q T, 1/q) \right]
\]

(56)

for \( n \geq 2 \).

Although the main ideas involved in the general case are essentially the same, the overall discussion of it may seem quite technical.
For $n \geq 1$, let $\mathcal{P}_n \subset \mathbb{F}_q[x]$ denote the set of all monic square-free polynomials of degree $n$ splitting in $\mathbb{F}_q$. For $i, j \geq 0$ and $d \in \mathcal{P}_n$, define

$$N_{i,j}(d, q) = \prod_{k=0}^{i-1} \left( \frac{q - n + a_1(C_d) + \epsilon - 2k}{2} \right) \prod_{l=0}^{j-1} \left( \frac{q - n - a_1(C_d) - \epsilon - 2l}{2} \right)$$

where $\epsilon = 0$ or $1$ according as $n$ is odd or even. Here if $i = 0$ or $j = 0$, we take the corresponding product to be $1$. (Recall that $a_1(C_d) = \text{Tr}(F^*|H^1_{d}(\bar{C}_d, \mathbb{Q}_x))$ with $\ell$ a prime different from the characteristic of $\mathbb{F}_q$.) Set

$$A_{n,i,j}(T, q) := \sum_{d \in \mathcal{P}_n} \left( N_{i,j}(d, q) \prod_{k=1}^{r} P_{c_d}(t_k) \right).$$

Note that if we denote $D_{A_{n,0,0}}(T, q) = \frac{\partial A_{n,0,0}}{\partial t_{n+1}}(t_1, \ldots, t_r, 0, q)$, we can write

$$D_{A_{n,0,0}}(T, q) = - \sum_{d \in \mathcal{P}_n} \left( a_1(C_d) \prod_{k=1}^{r} P_{c_d}(t_k) \right).$$

Define $D_{A_{n,0,0}}(T, q)$ for $k \in \mathbb{N}$ by iterating. Then, it is clear that

$$A_{n,i,j}(T, q) = (t)^{(q^n - q q^n)}(q^n - q^n T)^{2n,1} A_{n,0,0}(T, q) \quad (\text{for } n \geq 1).$$

Here the two binomial symbols are viewed as differential polynomials in $D$. Note that the right-hand side makes sense if we replace $T$ by $q T$ and $q$ by $1/q$, allowing us to define $A_{n,i,j}(q T, 1/q)$.

For $x, y$ and $z$ algebraically independent variables, consider the (exponential) generating functions

$$c_{\text{odd}}(x, y, z; T, q) = \sum_{n,i,j \geq 0} A_{2n+1,i,j}(T, q) x^{2n+1} y^i z^j$$

and

$$c_{\text{even}}(x, y, z; T, q) = \tilde{E}(T) \left[ (1 + z/\tilde{E}(T))^q - 1 \right] + \sum_{n \geq 1, i,j \geq 0} A_{2n+1,i,j}(T, q) x^{2n} y^i z^j$$

where $\tilde{E}(T) := E(T) E(q T)$. Since $A_{2n+1,i,j}(-T, q) = A_{2n+1,j,i}(T, q)$ (see Lemma 7.2), we see that

$$c_{\text{odd}}(x, z, y; -T, q) = c_{\text{odd}}(x, y, z; T, q).$$

Let $c(x, T, q) = (c_{\text{odd}}(x, y, z; T, q), c_{\text{even}}(x, z, y; -T, q), c_{\text{even}}(x, y, z; T, q))$, where we set $\underline{x} := (x, y, z)$.

**Theorem 7.1.** — With notation as above, we have

$$c(c(\underline{x}, T, q), q T, 1/q) = x.$$

In other words, $c(\underline{x}, T, q)$ is the formal compositional inverse of $c(\underline{x}, q T, 1/q)$.

**Proof.** Expand

$$c_{\text{odd}}(c(\underline{x}, T, q); q T, 1/q) = \sum_{n,i,j \geq 0} C_{2n+1,i,j}(T, q) x^{2n+1} y^i z^j$$

and

$$c_{\text{even}}(c(x, T, q); q T, 1/q) = \sum_{n,i,j \geq 0} C_{2n,i,j}(T, q) x^{2n} y^i z^j.$$

Note that $C_{1,0,0}(T, q) = A_{1,0,0}(q T, 1/q) A_{1,0,0}(T, q) = 1$, and $C_{1,i,j}(T, q) = 0$ if either $i \neq 0$ or $j \neq 0$. Similarly, $C_{0,0,1}(T, q) = 1$, and $C_{0,i,j}(T, q) = 0$ otherwise. Moreover, one checks that

$$\Sigma_N(T, q) = q C_{N,0,0}(T, q) \quad (\text{for } N \geq 2)$$
with $\Sigma_N(T, q)$ given by (56).

We shall first prove that if $C_{N,0,0}(T, q) = 0$ for an $N \geq 2$, then $C_{N,I,J}(T, q) = 0$ for all $I, J \geq 0$. Indeed, note that for any $N$, we can write

$$C_{N,0,0}(T, q) = \sum_{\alpha} c_{\alpha} M_{\alpha}(T, q)$$

where the sum is over tuples $\alpha = (\alpha_1, \ldots, \alpha_n, \ldots, \alpha_{n+1}, \ldots, \alpha_{n+i+1}, \ldots, \alpha_{n+i+j})$ $\in \mathbb{N}^{n+i+j}$ ($|\alpha| = \alpha_1 + \cdots + \alpha_{n+i+j} = N$), with $\alpha_1, \ldots, \alpha_n$ all odd, and $\alpha_k$'s all even, and where

$$M_{\alpha}(T, q) := \left( \prod_{m=1}^{n} A_{\alpha_m,0,0}(T, q) \right) \cdot \prod_{k=n+1}^{n+i} A_{\alpha_k,0,0}(-T, q) \cdot \prod_{l=n+i+1}^{n+i+j} A_{\alpha_l,0,0}(T, q) \frac{A_{0,i,j}(q T, 1/q)}{E(-T)^{i}E(T)^{j}}.$$

Here we allow $n = 0$ if we put

$$A_{0,i,j}(T, q) = \begin{cases} \frac{q^{i} E(T)}{(q^{-j})!} & \text{if } i = 0 \text{ and } j \geq 1 \\ 0 & \text{if } i = j = 0 \text{ or } i \neq 0. \end{cases}$$

(Of course, $\alpha = (\alpha_1^+, \ldots, \alpha_n^+)$ in this case.) To express the coefficient $c_{\alpha}$, it is rather convenient to represent $\alpha = (1^{a_1}, 3^{a_2}, 5^{a_3}, \ldots, 2^{b_2}, 4^{b_4}, 6^{b_6}, \ldots, 2^{b_2}, 4^{b_4}, 6^{b_6}, \ldots)$. With this notation, one checks that

$$c_{\alpha} = \frac{(e_1 \cdots e_n c_{a_1} \cdots c_{a_n} \cdots)}{\kappa_1 \cdots \kappa_2 \cdots \kappa_{n+1} \cdots}.$$

Now assume that $C_{N,0,0}(T, q) = 0$ for some $N \geq 2$. Pick arbitrary $I, J \geq 0$, and express $C_{N,I,J}(T, q)$, as above, in terms of the coefficients $A_{n,i,j}$. We can relate the expression of $C_{N,I,J}(T, q)$ to the corresponding expression of $C_{N,0,0}(T, q)$ in the following way. Every term, say, $c_{\alpha} M_{\alpha}(T, q)$, occurring in $C_{N,I,J}(T, q)$ is attached to a tuple $\tilde{\alpha} = (\alpha_1, i, j_1; \ldots; \alpha_n, i_n, j_n; \alpha_{n+1}, i_{n+1}, j_{n+1}; \ldots)$ by

$$M_{\tilde{\alpha}}(T, q) = A_{\alpha_1, i_1, j_1}(T, q) \cdots A_{\alpha_n, i_n, j_n}(T, q) \frac{A_{0,i,j}(q T, 1/q)}{E(-T)^{i}E(T)^{j}}.$$

Here $\alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \ldots$ are exactly as before. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots)$ be obtained by taking the corresponding non-zero components of $\tilde{\alpha}$. We refer to $\alpha$ as derived from $\tilde{\alpha}$, and denote this by $\tilde{\alpha} / \alpha$. It is clear that $c_{\alpha} M_{\alpha}(T, q)$ occurs in the expression of $C_{N,0,0}(T, q)$. Furthermore, one checks that the contribution of all terms $c_{\alpha} M_{\alpha}(T, q)$, with $\tilde{\alpha} / \alpha$ for a fixed $\alpha$ in $C_{N,I,J}(T, q)$ is

$$\sum_{\tilde{\alpha} / \alpha} c_{\alpha} M_{\tilde{\alpha}}(T, q) = c_{\alpha} \frac{(1+e q^{-N-D})/2(1-e q^{-N-D})/2}{E(-T)^{i}E(T)^{j}} M_{\alpha}(T, q)$$

with $\epsilon$ as above. The last identity holds even if $\alpha$ is such that $n = 0$. Summing now over all $\tilde{\alpha}$, it follows that

$$C_{N,I,J}(T, q) = \frac{(1+e q^{-N-D})/2(1-e q^{-N-D})/2}{E(-T)^{i}E(T)^{j}} C_{N,0,0}(T, q) = 0.$$

Now, suppose that $C_{n,i,j}(T, q) = 0$ for $2 \leq n \leq N - 1$, and all $i, j \geq 0$, and let us show that $C_{N,0,0}(T, q) = 0$. (It is easy to check that $C_{2,0,0}(T, q) = 0$, hence $C_{2,i,j}(T, q) = 0$ for $i, j \geq 0$ by what we just proved.) We shall assume that $N$ is odd since the argument is completely analogous if $N$ is even. Accordingly, we can write

$$c_{\text{odd}}(c(\varepsilon, T, q); q T, 1/q) = x + \sum_{n \geq (N-1)/2} \sum_{i,j \geq 0} C_{2n+1,i,j}(T, q) x^{2n+1} y^{i} z^{j}.$$
To this we apply $c(x, T, q) \coloneqq (\bar{A}_{\text{odd}}(x, T, q), \bar{A}_{\text{even}}(x, -T, q), \bar{A}_{\text{even}}(x, T, q))$ given in Section 8, and satisfying
\[
c(c(\bar{c}(x, T, q), T, q), q T, 1/q) = \bar{c}(x, T, q).
\] (57)

This implies that
\[
\sum_{n \geq (N-1)/2} \sum_{i,j \geq 0} C_{2n+1,i,j}(T, q) \bar{A}_{\text{odd}}(x, T, q)^{2n+1} \bar{A}_{\text{even}}(x, -T, q)^i \bar{A}_{\text{even}}(x, T, q)^j = 0.
\]

Taking the coefficient of $x^N$, it follows that $C_{N,0,0}(T, q) = 0$.

This combined with the previous step completes an induction process and the proof of the theorem. \quad \Box

In particular, we have
\[
\Sigma_N(T, q) = 0 \quad \text{(for } N \geq 2)\]
which is just (55) corresponding to $1^N$.

## 7.2 The General Case

In what follows, we shall extend the above considerations in a way that will allow us to treat all partitions at once. Before doing so, let us fix some notation.

**Notation.** Throughout this subsection $n, i, j, \ldots$ will denote partitions written as $n = (1^{n_1}; 2^{n_2}, \ldots)$, where for all but finitely many $j \geq 1$, $n_j = 0$. As usual, $|n| = n_1 + 2n_2 + \cdots$ stands for the weight of $n$. For every $n = (1^{n_1}; 2^{n_2}, \ldots)$, let $\mathcal{P}(n, q) = \mathcal{P}(n, \mathbb{F}_q) \subset \mathbb{F}_q[x]$ denote the set of all monic square-free polynomials $d$ with factorization type $n$. Let
\[
A_n(T, q) \coloneqq \sum_{d \in \mathcal{P}(n, q)} \left( \prod_{k=1}^r P_{C_d}(t_k) \right).
\]
(If $n_1, n_2, \ldots$ are all zero, we denote the corresponding partition by $0$, and take $A_0(T, q)$ to be zero.) Note that $A_n(T, q)$ corresponds to the sum $A_{n,0,0}(T, q)$ introduced at the beginning of the previous subsection. In addition, for $n \neq 0$ as above, and $i = (1^{i_1}; 2^{i_2}, \ldots), j = (1^{j_1}; 2^{j_2}, \ldots)$, we define $A_{n,i,j}(T, q)$ by
\[
A_{n,i,j}(T, q) \coloneqq \sum_{d \in \mathcal{P}(n, q)} \left( \prod_{m \geq 1} \left( \frac{c_m(d, q) + a_m^*(d, q)}{2m} \right) \right) \left( \frac{c_m(d, q) - a_m^*(d, q)}{2m} \right)^j \left( \frac{c_m(d, q) - a_m^*(d, q)}{2m} - i_m \right)^i \prod_{k=1}^r P_{C_d}(t_k)
\]
\[(59)\]
where $c_m(d, q) = c_m(d)$ and $a_m^*(d, q) = a_m^*(d)$ are as defined before. It is understood that $A_{n,0,0}(T, q) = A_n(T, q)$.

Notice that the quantity
\[
c_m(d, q) = \sum_{\theta \in \mathcal{P}_m} \chi_m(d(\theta))^2 = m(\text{Irr}_q(m) - n_m) \quad \text{(for } m \geq 1)\]
\[\text{(60)}\]
does not depend on $d \in \mathcal{P}(n, q)$. We shall also need the following transform of $A_{n,i,j}(T, q)$ defined by
\[
\hat{A}_{n,i,j}(T, q) \coloneqq \sum_{d \in \mathcal{P}(n, q)} \left( \prod_{m \geq 1} \left( \frac{c_m(d, q) + (-1)^m a_m^*(d, q)}{2m} \right) \right) \left( \frac{c_m(d, q) - (-1)^m a_m^*(d, q)}{2m} \right)^j \left( \frac{c_m(d, q) - (-1)^m a_m^*(d, q)}{2m} - i_m \right)^i \prod_{k=1}^r P_{C_d}(-t_k).
\]

**Lemma 7.2.** — If $|n|$ is odd, then $\hat{A}_{n,i,j}(T, q) = A_{n,i,j}(T, q)$ for arbitrary partitions $i$ and $j$.

40
**Proof.** For \( i = (1^i, 2^{i_2}, \ldots) \), consider the moment-sum

\[
M_{n,i}(q) = \sum_{d \in \mathcal{P}(n,q)} a^{(i)}_d(C_d) = \sum_{d \in \mathcal{P}(n,q)} \prod_j \left( -\sum_{\theta_j \in F_q} \chi_j(d(\theta_j)) \right)^{i_j}.
\]

As before, by changing \( \theta_j \to \xi \theta_j \), with \( \chi(\xi) = -1 (\xi \in F_q) \), one sees that \( M_{n,i}(q) = 0 \) whenever \(|i|\) is odd. Expressing \( A_n(T, q) \) in the form

\[
A_n(T, q) = \sum_{d \in \mathcal{P}(n,q)} \left( \prod_{k=1}^r P_{C_d}(t_k) \right) = \sum_{d \in \mathcal{P}(n,q)} \exp \left( -\sum_{j=1}^\infty \frac{a_j(C_d)}{j} \sum_{k=1}^r t_k^j \right)
\]

it follows that \( A_n(-T, q) = A_n(T, q) \). This identity holds for arbitrary \( r \). Accordingly, by adding more variables \( t_{r+1}, t_{r+2}, \ldots, t_{r+s} \) to \( t_1, \ldots, t_r \), we see that

\[
(-1)^{\alpha_1 + \cdots + \alpha_s} \frac{\partial^{\alpha_1 + \cdots + \alpha_s} A_n}{\partial t_{r+1}^{\alpha_{r+1}} \cdots \partial t_{r+s}^{\alpha_{r+s}}}(t_1, \ldots, -t_r, 0, \ldots, 0, q) = \frac{\partial^{\alpha_1 + \cdots + \alpha_s} A_n}{\partial t_{r+1}^{\alpha_{r+1}} \cdots \partial t_{r+s}^{\alpha_{r+s}}}(t_1, \ldots, t_r, 0, \ldots, 0, q)
\]

for arbitrary \( \alpha_1, \ldots, \alpha_s \in \mathbb{N} \). Dividing both sides by \( \alpha_1! \cdots \alpha_s! \), and expressing

\[
P_{C_d}(t) = 1 + \sum_{\alpha=1}^{2q} (-1)^\alpha \lambda_\alpha(C_d) t^\alpha
\]

we deduce that

\[
(-1)^{\alpha_1 + \cdots + \alpha_s} \sum_{d \in \mathcal{P}(n,q)} \left( \lambda_{\alpha_1}(C_d) \cdots \lambda_{\alpha_s}(C_d) \prod_{k=1}^r P_{C_d}(-t_k) \right) = \sum_{d \in \mathcal{P}(n,q)} \left( \lambda_{\alpha_1}(C_d) \cdots \lambda_{\alpha_s}(C_d) \prod_{k=1}^r P_{C_d}(t_k) \right).
\]

Now recall that we expressed the quantities \( a_m^*(d, q) \) by

\[
a_m^*(d, q) = -\sum_{k \mid m} \mu(m/k) a_k(C_d) - \sum_{\nu_2(m)} c_{m/2^\nu}(d, q) \quad (\text{for } m \geq 1 \text{ and } d \in \mathcal{P}(n,q))
\]

where \( \nu_2 \) denotes the 2-adic order. By applying Girard’s formula (more commonly known as Girard-Waring formula),

\[
ak(C_d) = k \sum (\frac{(l_1 + \cdots + l_k - 1)!}{l_1! \cdots l_k!}) \lambda_{l_1}^1(C_d) \cdots \lambda_{l_k}^k(C_d)
\]

summed over all nonnegative integers \( l_1, \ldots, l_k \) such that \( l_1 + 2l_2 + \cdots + kl_k = k \), it follows that

\[
\sum_{d \in \mathcal{P}(n,q)} (-1)^m a_m^*(d, q) \prod_{k=1}^r P_{C_d}(-t_k) = \sum_{d \in \mathcal{P}(n,q)} a_m^*(d, q) \prod_{k=1}^r P_{C_d}(t_k) \quad (\text{for } m \geq 1).
\]

This identity extends to

\[
\sum_{d \in \mathcal{P}(n,q)} H(-a_1^*(d, q), a_2^*(d, q), \ldots) \prod_{k=1}^r P_{C_d}(-t_k) = \sum_{d \in \mathcal{P}(n,q)} H(a_1^*(d, q), a_2^*(d, q), \ldots) \prod_{k=1}^r P_{C_d}(t_k)
\]

for arbitrary polynomial expressions \( H \) in \( a_1^*(d, q), a_2^*(d, q), \ldots \), and the lemma follows. \( \square \)

**Remark.** We can express \( A_{n,i,j}(T, q) = \nu_{n,1,1} a_{n,1,1}(q T, 1/q) \), an identity which allows us to define \( A_{n,i,j}(q T, 1/q) \). Clearly \( A_{n,i,j}(q T, 1/q) = A_{n,i,j}(q T, 1/q) \) when \(|n|\) is odd.

---

\( \nu_{n,1,1} \) is the differential operator \( \mathcal{D}_{n,1,1} \) as follows. For \( l_1, \ldots, l_k \in \mathbb{N} \), let

\[
\partial_{l_1, \ldots, l_k} = \left( \frac{1}{l_1!^2 l_2! \cdots k! l_k!} \right) \frac{\partial^{l_1+2l_2+\cdots+kl_k}}{\partial t_{r+1}^{l_{r+1}} \partial t_{r+2}^{l_{r+2}} \partial t_{r+3}^{l_{r+3}} \cdots \partial t_{r+s}^{l_{r+s}} \partial \theta_{r+1}^{l_{r+1}} \partial \theta_{r+2}^{l_{r+2}} \cdots \partial \theta_{r+s}^{l_{r+s}} \partial \_1^{l_1} \partial \_2^{l_2} \cdots \partial \_k^{l_k} }.
\]

41
Let $X_1, \ldots, X_n, \ldots, Y_1, \ldots, Z_1, \ldots$ be algebraically independent variables. Let $X = (X_n)_{n \geq 1}, Y = (Y_n)_{n \geq 1}, Z = (Z_n)_{n \geq 1}$, and set $X := (X, Y, Z)$. For $k \geq 1$ and a partition $n$, let $k n$ and $X^{k n}$ denote, respectively, the partition $(k^{n_1}, (2k)^{n_2}, \ldots)$ and the product $X_1^{k_1} X_2^{n_2} \cdots$.

Now, for $k \geq 1$, let $\tilde{E}(T^k, q^k) = E(T^k) E(q^k T^k)$, where we recall that $E(T) = \prod_{i=1}^{\infty} (1 - t_i)^{-1}$. Define the generating series

$$C_{k, \text{odd}}(X, T, q) = \sum_{n, i, j} A_{n, i, j}(T^k, q^k) X^{n Y^j i Z^j} \frac{1}{i! j!} E(T^k, q^k)^{1} E(T^k, q^k)^{1}$$

$$C_{k, \text{even}}^+(X, T, q) = \sum_{n, i, j} A_{n, i, j}(T^k, q^k) X^{n Y^j i Z^j} \frac{1}{i! j!} E(T^k, q^k)^{1} E(T^k, q^k)^{1}$$

and

$$C_{k, \text{even}}^-(X, T, q) = \sum_{n, i, j} A_{n, i, j}(T^k, q^k) X^{n Y^j i Z^j} \frac{1}{i! j!} E(T^k, q^k)^{1} E(T^k, q^k)^{1}$$

here $A_{0, i, j}(T, q)$ and $A'_{0, i, j}(T, q)$ are taken such that:

$$f_0(Y, Z, T, q) := \sum_{i, j} A_{0, i, j}(T, q) Y^i Z^j = E(T) E(q T) \left[ -1 + \prod_{m=1}^{\infty} (1 + Z_m)^{\text{irr}_q(m)} \right]$$

and

$$\sum_{i, j} A'_{0, i, j}(T, q) Y^i Z^j = E(-T) E(-q T) \left[ -1 + \prod_{m=\text{odd}} (1 + Y_m)^{\text{irr}_q(m)} \cdot \prod_{m=\text{even}} (1 + Z_m)^{\text{irr}_q(m)} \right].$$

Also, $E^-(T^k, q^k)^i$ and $E^+(T^k, q^k)^i$ are defined by

$$E^-(T^k, q^k)^i = \tilde{E}(-T^k, q^k)^i \cdots \tilde{E}(-T^{mk}, q^{mk})^{i m}$$

and

$$E^+(T^k, q^k)^i = \tilde{E}(T^k, q^k)^j \cdots \tilde{E}(T^{mk}, q^{mk})^{j m}.$$ 

Let $C_{\text{odd}}(X, T, q) = (C_{k, \text{odd}}(X, T, q))_{k \geq 1}$, $C_{\text{even}}^+(X, T, q) = (C_{k, \text{even}}^+(X, T, q))_{k \geq 1}$, and set

$$C(X, T, q) := (C_{\text{odd}}(X, T, q), C_{\text{even}}^+(X, T, q), C_{\text{even}}^+(X, T, q)).$$

Note that by a passage from $q$ to $q^k$ we understand a base change to $\mathbb{F}_{q^k}$. Accordingly, $C_{k, \text{odd}}(X, T, q)$, for example, is just $C_{1, \text{odd}}(X, T, q^k)$ in which we replace

$$t_1, \ldots, t_r, \ldots, X_m, \ldots, Y_m, \ldots, Z_m, \ldots$$

Define

$$D_m^* = \sum_{k \mid m, \mu(m/k) \text{ odd}} \sum_{\mu(k)} \sum_{\mu(\mu)} \frac{(-1)^{l_1 + \cdots + l_k + 1}}{l_1! \cdots l_k!} \partial_{l_1, \ldots, l_k}$$

the inner sum being over all $l_1, \ldots, l_k \in \mathbb{N}$ satisfying $l_1 + 2l_2 + \cdots + kl_k = k$. By (60), (62) and (63)

$$\sum_{d \in \mathbb{Z}_{\geq 1}(n, q)} a^*_m(d, q) \cdot \prod_{k=1}^{r} P_{C_d}(t_k) = D_m^* A_n(T, q) - \left[ \rho_m (1 + (-1)^{[m]}) / 2 + m \cdot \sum_{u=1}^{\nu_2(m)} (\text{irr}_q(m/2^u) - n_{m/2^u}) / 2^u \right] A_n(T, q)$$

with $\rho_m = 1$ or $0$ according as $m$ is a power of two or not. Then $D_{m, i, j, k} A_n(T, q)$ is obtained by replacing $a^*_m(d, q)$, for each $m$, by

$$D_m^* - \rho_m (1 + (-1)^{[m]}) / 2 - m \cdot \sum_{u=1}^{\nu_2(m)} (\text{irr}_q(m/2^u) - n_{m/2^u}) / 2^u$$

in (59).
by

\[
t_1^k, \ldots, t_r^k, \ldots, X_{km}, \ldots, Y_{km}, \ldots, Z_{km}, \ldots
\]

respectively. One defines \(C(X, qT, 1/q)\) as follows. For a partition \(n = (1^{n_1}, 2^{n_2}, \ldots)\), put

\[
f_n(Y, Z, T, q) = \sum_{i,j} A_{n,i,j}(T, q) \frac{Y^i Z^j}{i! j!}.
\]

Note that, for \(n \neq 0\), we can write

\[
f_n(Y, Z, T, q) = \sum_{d \in \mathcal{P}(n, q)} \left[ \prod_{m=1}^{\infty} (1 + Y_m) \frac{c_m(d, q) - a_m(d, q)}{2m} (1 + Z_m) \frac{c_m(d, q) + a_m(d, q)}{2m} \cdot \prod_{k=1}^{r} P_{c_d}(t_k) \right].
\]

Replacing \(c_m(d, q)\) by \(m(\text{Irr}_q(m) - n_m)\) and \(a_m(d, q)\) by

\[
D^*_m(n, q) = D_m - \rho_m(1 + (-1)^{m})/2 - m \cdot \sum_{u=1}^{\nu_2(m)} (\text{Irr}_q(m/2^u) - n_{m/2^u})/2^u
\]

where \(D^*_m\) is the differential operator defined above, we can further write

\[
f_n(Y, Z, T, q) = \prod_{m=1}^{\infty} (1 + Y_m)^{\text{Irr}_q(m) - n_m} / (1 + Z_m)^{\text{Irr}_q(m) - n_m} A_n(T, q).
\]

Here \((1 + U)^D\), for a variable \(U\) and a (differential) operator \(D\), stands for the formal power series

\[
1 + \sum_{n=1}^{\infty} \frac{D(D-1) \cdots (D-n+1)}{n!} U^n.
\]

In particular, if \(n = 1\) (i.e., \(n = (1^{n_1}, 2^{n_2}, \ldots)\) with \(n_1 = 1\) and \(n_m = 0\) for \(m > 2\)), then

\[
f_1(Y, Z, T, q) = q \prod_{m=1}^{\infty} (1 + Y_m)^{c_m/2^{\nu_2(m)}} (1 + Z_m)^{c_m - c_m/2^{\nu_2(m)}} / (2m)
\]

where

\[
c_m(q) = \begin{cases} q - 1 & \text{if } m = 1, \\ m \text{ Irr}_q(m) & \text{if } m > 2. \end{cases}
\]

We define

\[
f_n(Y, Z, qT, 1/q) = \prod_{m=1}^{\infty} (1 + Y_m)^{\text{Irr}_q(m) - n_m} / (1 + Z_m)^{\text{Irr}_q(m) - n_m} A_n(qT, 1/q)
\]

and \(C_{k, odd}(X, qT, 1/q)\), for all integers \(k \geq 1\), by

\[
C_{k, odd}(X, qT, 1/q) = \sum_{n \equiv 0 \pmod{2}} f_n((Y_{mk}/E(-T^mk, q^{nk}))_{m \geq 1}, (Z_{mk}/E(T^mk, q^{nk}))_{m \geq 1}, q^k T^k, q^{-k}) X^{kn}.
\]

Define \(C_{k, even}(X, qT, 1/q)\) similarly, and let \(C_{odd}(X, qT, 1/q)\), \(C_{even}(X, qT, 1/q)\) and \(C(X, qT, 1/q)\) as defined above.

Using the generating function \(C(X, qT)\), we can reinterpret (27) and the relations (55) as follows. Let

\[
Z_{odd}(T, t_{r+1}: q) = [Z(T, t_{r+1}: q) - Z(T, -t_{r+1}: q)]/2 = \sum_{l \equiv 0 \pmod{2}} A_l(T, q)t_{r+1}^l
\]

and

\[
Z_{even}(T, t_{r+1}: q) = [Z(T, t_{r+1}: q) + Z(T, -t_{r+1}: q)]/2 = \sum_{l \equiv 0 \pmod{2}} A_l(T, q)t_{r+1}^l
\]
denote the odd (respectively even) part of the series $Z(T, t_{r+1}; q)$ in (14). Put

$$Z_{\text{odd}}(T, t_{r+1}; q) = (Z_{\text{odd}}(T^{k}, t_{r+1}; q))_{k \geq 1} \quad \text{and} \quad Z_{\text{even}}^{+}(T, t_{r+1}; q) = (E(\pm T^{k})Z_{\text{even}}(\pm T^{k}, t_{r+1}; q))_{k \geq 1}.$$ 

If we set

$$Z(T, t_{r+1}; q) := (Z_{\text{odd}}(T, t_{r+1}; q), Z_{\text{even}}(T, t_{r+1}; q), Z_{\text{even}}^{+}(T, t_{r+1}; q))$$

then by (29), Proposition 4.1 and (30), the relation (27) can be expressed as

$$Z(T, t_{r+1}; q) = C(Z(q T, q t_{r+1}; 1/q), T, q). \quad (68)$$

For the purpose of interpreting the relations (55), let $C_{0}(X, T, q) = C(C(X, T, q) - q \cdot X$, where for $X$ as above, $q \cdot X = ((q^n X_n)_{n \geq 1}, (q^n Y_n)_{n \geq 1}, (q^n Z_n)_{n \geq 1})$ is the linear part of $C(X, T, q)$. Then (68) can be rewritten as

$$Z(T, t_{r+1}; q) - q \cdot Z(q T, q t_{r+1}; 1/q) = C_{0}(Z(q T, q t_{r+1}; 1/q), T, q).$$

Replacing $q \to 1/q, T \to q T, t_{r+1} \to q t_{r+1}$, and substituting $Z(T, t_{r+1}; q)$ by using (68), we see that $Z(q T, q t_{r+1}; 1/q)$ has to satisfy

$$Z(q T, q t_{r+1}; 1/q) - (q^{-1}) \cdot C(Z(q T, q t_{r+1}; 1/q), T, q) = C_{0}(C(Z(q T, q t_{r+1}; 1/q), T, q), q T, 1/q).$$

The last identity prompts us to wonder if we have in fact:

$$X - (q^{-1}) \cdot C(X, T, q) = C_{0}(C(X, T, q), q T, 1/q) \quad (69)$$

or what amounts to the same,

$$C(C(X, T, q), q T, 1/q) = X. \quad (70)$$

Clearly (69) and (70) are equivalent to the relations (55).

Shortly, we will show (see Theorem 7.5) that, indeed, (70) holds. Its proof depends chiefly upon understanding how (70) is encoded in the combinatorial structure of certain moduli spaces, see Section 8.

We shall need the following well-known identities:

**Lemma 7.3.** — For $m \in \mathbb{N}^{*}$, we have

$$\sum_{h|m} \text{Irr}_{q_1}(h)\text{Irr}_{q_2}(m/h) = \text{Irr}_{q_1 q_2}(m)$$

where for an indeterminate $t$, we put

$$\text{Irr}_{t}(m) = m^{-1} \sum_{k|m} \mu(m/k) t^{k}.$$ 

In particular, if $m \geq 2$, then

$$\sum_{h|m} \text{Irr}_{q^{-1}}(h)\text{Irr}_{q^{2}}(m/h) = 0$$

and

$$\sum_{h|m} h \text{Irr}_{q}(h) = q^{m} \quad (\text{for } m \geq 1).$$

**Proof.** Indeed, the left-hand side of the identity can be written as

$$\frac{1}{m} \sum_{h|m} \left( \sum_{k|h} \mu(h/k) q_{1}^{k} \right) \left( \sum_{l|m/h} \mu(m/hl) q_{2}^{l} \right) = \frac{1}{m} \sum_{k|m} q_{1}^{k} \sum_{l|m/k} \mu\left(\frac{m}{lk}\right) q_{2}^{l} \sum_{n|l} \mu(n).$$
Since \( \sum_{n \mid \ell} \mu(n) = 0 \) if \( \ell > 1 \), it follows that

\[
\sum_{h \mid m} \text{Irr}_{q_1}(h) \text{Irr}_{q_2}(m/h) = \frac{1}{m} \sum_{h \mid m} \mu\left(\frac{m}{k}\right) (q_1 q_2)^k = \text{Irr}_{q_1 q_2}(m).
\]

The second identity in the statement of the lemma follows by replacing \( q_1 \) and \( q_2 \) by \( q^{-1} \) and \( q \), respectively. By equating the coefficients of \( t^m \), i.e., the leading coefficients, on both sides of the identity

\[
\sum_{h \mid m} \text{Irr}_q(h) \text{Irr}_{q^h}(m/h) = \text{Irr}_q(m)
\]

we deduce that \( \sum_{h \mid m} h \text{Irr}_q(h) = q^m \) for \( m \geq 1 \). \( \square \)

For \( N \geq 1 \), let \( D_N^* \) be the differential operator defined by

\[
D_N^* = \sum_{k \mid N} \mu(N/k) k \sum_{(N/k)-\text{odd}} (-1)^{l_1 + \cdots + l_k + l_1 + \cdots + l_k - 1} l_1! \cdots l_k! \partial_{l_1, \ldots, l_k}
\]  

(71)

the inner sum being over all \( l_1, \ldots, l_k \in \mathbb{N} \) satisfying \( l_1 + \cdots + kl_k = k \). We shall need some elementary properties of the action of this operator on the \( \mathbb{C} \)-algebra \( \mathcal{B} \) consisting of linear combinations

\[
\sum_{\alpha} c_\alpha F_\alpha(T) \quad \left( c_\alpha \in \mathbb{C} \text{ and } F_\alpha(T) = \prod_{k=1}^n f_\alpha(t_k) \right)
\]

with \( f_\alpha(t) \) of the form

\[
f_\alpha(t) = \exp\left( \sum_{j=1}^\infty \frac{a_\alpha(j)}{j^t} \right) \quad (a_\alpha(j) \in \mathbb{C} \text{ for all } j \geq 1)
\]

(viewed as a composition of formal power-series), for all \( \alpha \). In particular, \( A_{n,i,j}(T, q) \) is in \( \mathcal{B} \), and by using the results of Section 8 (specifically, see (91)), one can express inductively \( A_{n,i,j}(q T, 1/q) \) as a linear combination (independent of \( r \)) of derivatives \( a_{l_1} \cdots a_{l_k} A_{m,p}(T, q) \) \( (l_1, \ldots, l_k \in \mathbb{N}) \), hence \( A_{n,i,j}(q T, 1/q) \) is also in \( \mathcal{B} \) for any partitions \( n, i \) and \( j \).

It is understood that, for \( F(T) \in \mathcal{B} \), its formal derivative \( \partial_N F(T) \) is taken as in the right-hand side of (61). Clearly \( D_N^* \) is \( \mathbb{C} \)-linear, and it satisfies the Leibniz rule \( \partial_N(FG)(T) = G(T) \partial_N F(T) + F(T) \partial_N G(T) \), for \( F(T) \) and \( G(T) \) in \( \mathcal{B} \). In addition, if \( F_\alpha(T) \) is as above, then

\[
\partial_N F_\alpha(T) = F_\alpha(T) \cdot \sum_{k \mid N} \mu(N/k) a_\alpha(k).
\]

In particular, we have

**Lemma 7.4.** — For sequences \( \{\gamma_l\}_{l \geq 1} \) and \( \{\eta_l\}_{l \geq 1} \) of complex numbers, define

\[
f(t) = \prod_{l=1}^\infty \left( 1 - q^l t^{2l} \right)^{\gamma_l/2} \quad \text{and} \quad g(t) = \prod_{l=1}^\infty \left( \frac{1 + t^l}{1 - t^l} \right)^{\eta_l/2l}.
\]

Put \( F(T) = \prod_{k=1}^N f(t_k) \) and \( G(T) = \prod_{k=1}^N g(t_k) \). Then, for any \( N \in \mathbb{N}^* \), we have

\[
\partial_N F(T) = \left( \sum_{\substack{l m = N \\text{m-even} \atop m/h \text{even}}} l \gamma_l \sum_{h \mid m} \mu\left(\frac{m}{h}\right)^{q_1 q_2} \right) F(T)
\]  

(72)

and

\[
\partial_N G(T) = \eta_N G(T)
\]

(73)

with the understanding that the double sum in the right-hand side of (72) vanishes if \( N \) is odd.
Proof. Writing
\[
    f(t) = \exp \left( \sum_{i=1}^{\infty} \frac{t^i}{i} \log \left( 1 - q^i t^{2i} \right) \right)
\]
and expanding \( \log \left( 1 - q^i t^{2i} \right) = - \sum_{m=1}^{\infty} \left( q^{im} t^{2im} / m \right) \), the identity (72) follows after a simple calculation. Similarly,
\[
    \varphi_{\infty}(G(T)) = \left( \sum_{M=1}^{\infty} \mu(M) \sum_{k=1}^{M - \text{odd}} \eta_k \right) G(T)
\]
and (73) follows by applying the M"{o}bius inversion. This completes the proof.

The remaining of this section is devoted to the proof of our main theorem in the general case, subject to a functional relation, whose proof will be given in Section 8.

**Theorem 7.5.** — With the above notation, we have
\[
    C(C(X, T, q), qT, 1/q) = X.
\]
In other words, \( C(X, T, q) \) is the formal compositional inverse of \( C(X, qT, 1/q) \).

Proof. As in the previous subsection, we shall reduce the assertion of the theorem to the existence of a generating function satisfying a functional relation, see Section 8.

First, notice that it is enough to show that
\[
    C_{1, \text{odd}}(C(X, T, q), qT, 1/q) = X_1
\]
\[
    C_{1, \text{even}}(C(X, T, q), qT, 1/q) = Y_1 \quad \text{and} \quad C_{1, \text{even}}(C(X, T, q), qT, 1/q) = Z_1.
\]
By Lemma 7.2, the expansion of \( A_n(T, q) \) in terms of the symplectic Schur functions \( s_{1,1}(\eta^{1/2} T^{1/2}) \) is only over partitions of even weight whenever \( |n| \) is odd; therefore, \( A_n(qT, 1/q) = A_n(T, 1/q) \), and from the definition of \( A_{n,i,j}(qt, 1/q) \),
\[
    A_{n,i,j}(qt, 1/q) = A_{n,1,1}(qt, 1/q)
\]
whenever \( |n| \) is odd (see also the above remark). Replacing \( T \) by \( -T \) (i.e., \( t_k \) by \( -t_k \) for \( 1 \leq k \leq r \)), and interchanging \( Y_1 \) and \( Z_1 \), \( Y_3 \) and \( Z_3 \), \ldots, it follows that we have the implication
\[
    C_{1, \text{even}}(C(X, T, q), qT, 1/q) = Z_1 \quad \Longrightarrow \quad C_{1, \text{even}}(C(X, T, q), qT, 1/q) = Y_1.
\]
Now expand
\[
    C_{1, \text{odd}}(C(X, T, q), qT, 1/q) = \sum_{n,i,j \in \text{[n] - odd}} C_{n,i,j}(T, q) X^n Y^i Z^j
\]
and
\[
    C_{1, \text{even}}(C(X, T, q), qT, 1/q) = \sum_{n,i,j \in \text{[n] - even}} C_{n,i,j}(T, q) X^n Y^i Z^j.
\]
By (66), the subseries of \( C_{1, \text{odd}}(C(X, T, q), qT, 1/q) \) corresponding to \( n = 1 \) can be expressed by
\[
    X_1 \sum_{i,j} C_{1,i,j}(T, q) Y^i Z^j = X_1 \prod_{m=1}^{\infty} \left( 1 + Y_m / \tilde{E}(-T^m, q^m) \right) \omega_m(q) \left( 1 + Z_m / \tilde{E}(T^m, q^m) \right) \omega_m(q)
\]

46
where for \( m \geq 1 \) with \( 2^b \parallel m \),
\[
\omega_m^-(q) = \frac{c_m(q^2b)}{2m} + \sum_{\substack{l \mid m \text{ odd} \\
\ell I}} \text{Irr}_q'(m/l) \frac{\text{Irr}_{q^b}(q^{-2^b})}{2l}.
\]

and
\[
\omega_m^+(q) = \frac{2c_m(q) - c_m(q^2b)}{2m} + \sum_{\substack{k \mid m \\
\text{m}/k \text{ even} \text{ odd}}} c_k(1/q) \frac{\text{Irr}_{q^b}(m/k)}{k} + \sum_{\substack{l \mid m \text{ odd} \\
\ell I}} \frac{\text{Irr}_q'(m/l)(2c_l(1/q) - c_l(q^2b))}{2l}.
\]

Put \( m = 2^b m' \). If \( m' = 1 \), then, by (67),
\[
\omega_m^-(q) = 2^{-b-1} \left( c_1(q^{2b}) + q^{2b} c_1(q^{-2b}) \right) = 0
\]
and if \( m' > 1 \), we can write
\[
\omega_m^-(q) = \frac{c_{m'}(q^{2b})}{2^{b+1} m'} + \sum_{l' \mid m'} \frac{c_{l'}(q^{-2b}) \text{Irr}_{q^{b' l'}}(m'/l')}{2^{b+1} l'} = 2^{-b-1} \sum_{l' \mid m'} \text{Irr}_{q^{2b}}(l') \text{Irr}_{q^{b' l'}}(m'/l').
\]

The last sum vanishes by Lemma 7.3. Similarly, \( \omega_1^+(q) = 0 \), and if \( m > 1 \), one can reduce \( \omega_m^+(q) \) to
\[
\omega_m^+(q) = -\frac{c_{m'}(q^{2b})}{2^{b+1} m'} + \frac{\text{Irr}_{q^{b'}}(m') - \text{Irr}_1(m')}{2^{b+1}}.
\]

Thus \( \omega_m^+(q) = 0 \).

Next, notice that
\[
\sum_{i,j} C_{0,i,j}(T, q) Y^i Z^j = E(T) E(qT) \left[ \prod_{k,l=1}^{\infty} \left( 1 + \frac{Z_{kl}}{E(T^{kl}) E(q^{kl} T^{kl})} \right)^{\text{Irr}_{q^{-1}(k)} \text{Irr}_{q^{b}(l)}} \right] = 1.
\]

By Lemma 7.3
\[
\sum_{k,l=m} \text{Irr}_{q^{-1}(k)} \text{Irr}_{q^b}(l) = 0 \quad \text{(if } m > 1 \text{)}
\]
and so
\[
\sum_{i,j} C_{0,i,j}(T, q) Y^i Z^j = E(T) E(qT) \left[ \left( 1 + \frac{Z_1}{E(T) E(qT)} \right)^{\text{Irr}_{q^{-1}(1)} \text{Irr}_{q^{b}(1)}} \right] = Z_1.
\]

Our next step is to show that if \( C_{\mathcal{N},0,0}(T, q) = 0 \) for a partition \( \mathcal{N} \) of weight \( |\mathcal{N}| \geq 2 \), then \( C_{\mathcal{N},l,j}(T, q) = 0 \) for arbitrary partitions \( \mathcal{J} \) and \( \mathcal{N} \). As in the proof of Theorem 7.1, we shall deduce this from the fact that \( C_{\mathcal{N},l,j}(T, q) = * C_{\mathcal{N},0,0}(T, q) \) for a certain differential operator \( * \); the precise definition of this differential operator is given in (80).

Indeed, as in the proof of Theorem 7.1, we first write
\[
C_{\mathcal{N},0,0}(T, q) = \sum_S c_S M_S(T, q)
\]
summed over tuples \( S = (S_1, \ldots, S_1^-, \ldots, S_t^+, \ldots) \) with each \( S_k \), or \( S^+_k \), itself a tuple of partitions; for \( k \geq 1 \), \( S_k = (n_{k1}, n_{k2}, \ldots, n_{kn_k}) \) with \( n_{kh} \) odd for all \( 1 \leq h \leq n_k \), and for \( l \geq 1 \), \( S^+_l = (n_{l1}^+, n_{l2}^+, \ldots, n_{ln_l}^+) \) with \( n_{lm}^+ \) even for all \( 1 \leq m \leq n_l^+ \); we also require that
\[
\mathcal{N} = n_{11}^+ \cup \ldots \cup n_{1n_1}^+ \cup 2n_{21}^- \cup \ldots \cup 2n_{2n_2}^- \cup \ldots \cup n_{11}^- \cup \ldots \cup n_{1n_1}^- \cup \ldots \cup n_{n_1}^- \cup \ldots \cup \ldots \cup n_{n_l}^- \cup \ldots .
\]
If we define $M_{S_k}(T,q)$ and $M_{S_f}(T,q)$ by

$$M_{S_k}(T,q) = A_{n_k}(T,q) \cdots A_{n_{k_n}}(T,q) \quad \text{and} \quad M_{S_f}(T,q) = A_{n_k^+}(\pm T,q) \cdots A_{n_{k_n}^+}(\pm T,q)$$

then $M_S(T,q)$ above is given by

$$M_S(T,q) = \frac{A_{n,i,j}(qT,1/q)}{E(T,q)E(T,q)^i} \prod_{k,l,l'} M_{S_k}(T^k,q^k) M_{S_f}(T^l,q^l) M_{S_f^+}(T'^l,q'^l)$$

with $n = (1^{n_1}, 2^{n_2}, \ldots)$, $i = (1^{n_1}, 2^{n_2}, \ldots)$ and $j = (1^{n_1}, 2^{n_2}, \ldots)$. The constant $c_S$ is given in (75).

Define

$$\mathcal{D}^\ast(Y,Z) = \sum_{N=1}^{\infty} \left( \frac{1}{1+\tilde{Y}_N} \right)^{\mathcal{D}_N^\ast}$$

where, for notational convenience, we set $\tilde{Y}_N := Y_N/\tilde{E}(T^N,q^N)$ and $\tilde{Z}_N := Z_N/\tilde{E}(T^N,q^N)$, and where $\mathcal{D}_N^\ast$ for $N \geq 1$ is the differential operator defined by (71). The idea is to compare $\mathcal{D}^\ast(Y,Z)$ with the contribution of all the terms derived from, or corresponding to $M_S(T,q)$, for fixed $S$, in the composition $C_{1,odd}(C(X,T,q),qT,1/q)$. We will show that these two factors agree up to a normalizing factor, depending only upon the fixed partition $g_l$.

First, the contribution of all the terms derived from $M_S(T,q)$ in $C_{1,odd}(C(X,T,q),qT,1/q)$ can be easily described as follows. Consider the subseries $X^{n}f_n((\tilde{Y}_m)_{m \geq 1},(\tilde{Z}_m)_{m \geq 1},T,q,1/q)$ of $C_{1,odd}(X,T,q)$, where we recall that

$$f_n(Y,Z,T,q,1/q) = \prod_{m=1}^{\infty} (1+Y_m)^{l_{m+1}/l_m} (1+Z_m)^{l_{m+1}/l_m} (1+T_m)^{l_{m+1}/l_m};$$

for simplicity, let us assume that $|n|$ is odd. In $X^{n}f_n((\tilde{Y}_m)_{m \geq 1},(\tilde{Z}_m)_{m \geq 1},T,q,1/q)$, replace $X_k, Y_l$ and $Z_{l'}$ for all $k, l, l' \geq 1$ by the subseries of $C_{k,odd}(X,T,q), C_{l,even}(X,T,q)$ and $C_{l',even}(X,T,q)$ given by

$$\sum X^{k_{n_{kh}}}f_{n_{kh}}((\tilde{Y}_{km})_{m \geq 1},(\tilde{Z}_{km})_{m \geq 1},T^k,q^k)$$

and

$$\sum X^{l_{n_{kh}}^+}f_{n_{kh}^+}((\tilde{Y}_{lm})_{m \geq 1},(\tilde{Z}_{lm})_{m \geq 1},T^l,q^l)$$

respectively, the sums being over all the distinct components of $S_k$ and $S_f$. Here, for a partition $n$ of even weight, we put: $f^+_n(Y,Z,T,q) = f_n(Y,Z,T,q)$, and

$$f_n(Y,Z,T,q) = \sum_{i,j} ^t A_{n,i,j}(T,q) \frac{Y_i Z_j}{l_i l_j}$$

the generating series $f_n^+(Y,Z,T,q) = f_n(Y,Z,T,q)$ (resp. $f_n^-(Y,Z,T,q)$) is defined by (64) (resp. (65)). This gives an expression, in which the coefficient of $X^{n} := X^{n_1} \cdots X^{n_{ni}} X^{2n_2} \cdots X^{n_{ni}^+} \cdots$ is precisely the contribution of all the terms corresponding to the monomial $M_S(T,q)$.

To write this contribution explicitly, define

$$f_{S_k}(Y,Z,T,q) = (m(n_{k_1}), \ldots, m(n_{k_{p_k}})) f_{n_{k_1}}(Y,Z,T^k,q^k) \cdots f_{n_{k_{p_k}}}(Y,Z,T^k,q^k)$$

where $n_{k_1}, \ldots, n_{k_{p_k}}$ are the distinct components of $S_k$, and $m(n_{k_{kh}})$ denotes the multiplicity (i.e., the number of occurrences) of $n_{k_{kh}}$ in $S_k$, for all $1 \leq h \leq p_k$. Similarly, define

$$f_{S_f^+}(Y,Z,T,q) = (m(n_{1_1}^+), \ldots, m(n_{1_{p_1}^+}^+)) f_{n_{1_1}^+}(Y,Z,T^l,q^l) \cdots f_{n_{1_{p_1}^+}}(Y,Z,T^l,q^l)$$

$$f_{S_f^+}(Y,Z,T,q) = (m(n_{1_1}), \ldots, m(n_{1_{p_1}^+})) f_{n_{1_1}^+}(Y,Z,T^l,q^l) \cdots f_{n_{1_{p_1}^+}}(Y,Z,T^l,q^l)$$
where we put \( g_0^*(Y, Z, T, q) = 1 + (f_0^*(Y, Z, T, q)/\hat{E}(z T, q)) \), and
\[
\epsilon_t^n = (\text{Irr}_{1/q}(l) - n_l)/2 \pm D_l^n(1/q)/2!
\]
are precisely the exponents in the product appearing in the above expression of \( f_0(Y, Z, qT, 1/q) \). If we set
\[
f_\sigma(Y, Z, T, q) := \prod_{k, l, t} f_{\sigma}((\tilde{Y}_{k\alpha}, (Z_{k\alpha}, \alpha, T, q) f_{\sigma}((\tilde{Y}_{l\alpha}, (Z_{l\alpha}, \alpha, T, q) f_{\sigma}((\tilde{Y}_{t\alpha}, (Z_{t\alpha}, \alpha, T, q)
\]
then the contribution we are interested in is expressed by
\[
E^\sigma(T, q) = \epsilon_{\sigma}(T, q)^1 E^\sigma(T, q)^1 A_n(q T, 1/q). \quad (74)
\]
In this expression, the differential operators commute with each other, and we have
\[
A_{n, \sigma}(q T, 1/q) = \prod_i \epsilon_t^n(1/q) (\epsilon_t^n(1/q) - 1) - m(n_1^n) \cdots - m(n_\ell^n) A_n(q T, 1/q).
\]
The remaining parts of the multinomial coefficients make up the constant \( c_S \), that is,
\[
c_S = \prod_{k} \left( m(n_{k1}), \ldots, m(n_{k\ell_k}) \right) \prod_{l} \left( m(n_{1l}^--) \cdots m(n_{\ell_l}^--) \right)^{-1}. \quad (75)
\]
To simplify things, using Lemma 7.3, one can write the products in (74) as
\[
\prod_{l=1}^{\infty} \prod_{m=1, m-\text{odd}} \left( 1 + \tilde{Z}_{lm} \right)^{l_{\sigma}^*(n_{l1}^-, \ldots, n_{l\ell_l}^-)} 1 + \tilde{Y}_{lm} \right)^{l_{\sigma}^*(n_{l1}^+, \ldots, n_{l\ell_l}^+)} \prod_{m=1, m-\text{even}} \left( 1 + \tilde{Z}_{lm} \right)^{l_{\sigma}^*(n_{l1}^-, \ldots, n_{l\ell_l}^-)} \prod_{m=1, m-\text{even}} \left( 1 + \tilde{Y}_{lm} \right)^{l_{\sigma}^*(n_{l1}^+, \ldots, n_{l\ell_l}^+)} \quad (76)
\]
with \( \epsilon_1 = 1 \) and \( \epsilon_N = 0 \) if \( N \geq 2 \). We combine the factor \( \prod_{l, m=1}^{\infty} \left( 1 + \tilde{Y}_{lm} \right) \left( 1 + \tilde{Z}_{lm} \right) \) in (76) with \( f_\sigma(Y, Z, T, q) \). Note that the products in (76) not involving any of the \( D_l^n(1/q) \) \((l = 1, 2, \ldots)\) act on \( A_n(q T, 1/q) \) by scalar multiplication. Let
\[
G = \prod_{l=1}^{\infty} \prod_{m=1, m-\text{odd}} \left( 1 + \tilde{Z}_{lm} \right)^{l_{\sigma}^*(n_{l1}^-, \ldots, n_{l\ell_l}^-)} \prod_{m=1, m-\text{even}} \left( 1 + \tilde{Y}_{lm} \right)^{l_{\sigma}^*(n_{l1}^+, \ldots, n_{l\ell_l}^+)} \prod_{m=1, m-\text{even}} \left( 1 + \tilde{Z}_{lm} \right)^{l_{\sigma}^*(n_{l1}^-, \ldots, n_{l\ell_l}^-)} \prod_{m=1, m-\text{even}} \left( 1 + \tilde{Y}_{lm} \right)^{l_{\sigma}^*(n_{l1}^+, \ldots, n_{l\ell_l}^+)}.
\]

**Lemma 7.6.** — With notations as above, we have
\[
G = \prod_{N=1}^\infty \left( 1 + \tilde{Z}_N \right)^{(B_N^* + R_N)/2N}
\]
with
\[
B_N = \sum_{k \mid N} q^k \mu(N/k) k \sum_{l_1, \ldots, l_k} (-1)^{l_1 + \ldots + l_k + 1} (l_1 + \ldots + l_k - 1)! l_1! \cdots l_k! \quad (N/k)^{-}\text{odd}
\]
the inner sum being over all \( l_1, \ldots, l_k \in \mathbb{N} \) satisfying \( l_1 + 2l_2 + \ldots + k l_k = k \), and
\[
R_N = \sum_{l=1}^{\infty} \sum_{m=1}^{N} l (\text{Irr}_{q^{-1}}(l) - n_l) \cdot \sum_{h=1}^{m} \mu\left( \frac{m}{h} \right) q^{l h}.
\]
Proof. In
\[(2N)^{-1} \sum_{\substack{lm=N \atop m-\text{odd}}} m \operatorname{Irr}_{q^l}(m) D^*_l(n, 1/q)\]
replace \(D^*_l(n, 1/q)\) by
\[D^*_l(n, 1/q) \equiv D^*_l - i \sum_{u=1}^{\nu_2(l)} (\operatorname{Irr}_{q^{-1}}(l/2^u) - n_{l/2^u})/2^u\]
(recall that we are assuming that \(|n|\) is odd) to get
\[(2N)^{-1} \sum_{\substack{lm=N \atop m-\text{odd}}} m \operatorname{Irr}_{q^l}(m) D^*_l - (2N)^{-1} \sum_{\substack{lm=N \atop m-\text{odd}}} m \operatorname{Irr}_{q^l}(m) l \sum_{u=1}^{\nu_2(l)} (\operatorname{Irr}_{q^{-1}}(l/2^u) - n_{l/2^u})/2^u.\] (77)
Since
\[D^*_l = \sum_{\substack{k|l \atop (l/k)\text{-odd}}} \mu(l/k) k \sum_{\substack{m|l \atop (m/k)\text{-odd}}} (-1)^{l_1 + \cdots + l_k + 1} (l_1 + \cdots + l_k - 1)! \frac{l_1! \cdots l_k!}{m!} \partial_{l_1, \ldots, l_k}\]
we can use the Möbius inversion, in the form
\[x(l) = \sum_{\substack{k|l \atop (l/k)\text{-odd}}} \mu(l/k) y(k) \iff y(m) = \sum_{\substack{h|m \atop (m/h)\text{-odd}}} x(h)\]
to express the first sum in (77) as
\[(2N)^{-1} \sum_{\substack{k|l \atop (N/k)\text{-odd}}} q^k \mu(N/k) k \sum_{\substack{m|l \atop (m/k)\text{-odd}}} (-1)^{l_1 + \cdots + l_k + 1} (l_1 + \cdots + l_k - 1)! \frac{l_1! \cdots l_k!}{m!} \partial_{l_1, \ldots, l_k}.\]
Similarly, if we put \(x(l) = \sum_{u=1}^{\nu_2(l)} (\operatorname{Irr}_{q^{-1}}(l/2^u) - n_{l/2^u})/2^u\), then
\[y(m) = \sum_{h|m \atop (m/h)\text{-odd}} x(h) = \sum_{l|(m/2)} l(\operatorname{Irr}_{q^{-1}}(l) - n_l)\]
and
\[(2N)^{-1} \sum_{\substack{lm=N \atop m-\text{odd}}} m \operatorname{Irr}_{q^l}(m) x(l) = (2N)^{-1} \sum_{\substack{lm=N \atop m-\text{odd}}} m \operatorname{Irr}_{q^l}(m) \sum_{\substack{k|l \atop (l/k)\text{-odd}}} \mu(l/k) y(k).\] (78)
Summing first over \(k\) and using the definition of \(\operatorname{Irr}_{q^l}(m)\), we can express the right-hand side of (78) as
\[(2N)^{-1} \sum_{\substack{k|l \atop (N/k)\text{-odd}}} \mu(k) \sum_{\substack{m|(N/k) \atop (m/k)\text{-odd}}} \mu(m) q^k = (2N)^{-1} \sum_{\substack{k|l \atop (N/k)\text{-odd}}} \mu(N/k) y(k) q^k.\]
This further equals to
\[(2N)^{-1} \sum_{\substack{lm=N \atop m-\text{even}}} l(\operatorname{Irr}_{q^{-1}}(l) - n_l) \cdot \sum_{\substack{h|m \atop (m/h)\text{-odd}}} \mu(m/h) q^h.\]
Our assertion follows at once from the fact that
\[\prod_{l=1}^{\infty} \prod_{m-\text{even}}^{} \frac{1 + \tilde{Z}_{lm}^{(m)\text{-even}}}{1 + Y_{lm}^{\text{even}}} = \prod_{N=1}^{\infty} \left(1 + \tilde{Z}_N \right)^{\mathcal{Q}_N/2N}\]
with
\[\mathcal{Q}_N = \sum_{\substack{lm=N \atop m-\text{even}}} l(\operatorname{Irr}_{q^{-1}}(l) - n_l) \sum_{\substack{h|m \atop (m/h)\text{-odd}}} \mu(m/h) q^h.\]
Now for any partition \( n = (n_1, 2n_2, \ldots) \) of odd weight, express
\[
f_n(Y, Z, T, q) = \prod_{m=1}^{\infty} (1 + Y_m)^{(1 + q)_{2m}(n_m - n_m)/2 - D^*_m(n_m) + 2m} (1 + Z_m)^{(1 + q)_{2m}(n_m + n_m)/2 + D^*_m(n_m) - 2m} A_n(T, q).
\]

Note that by (64) and (65), for any partition \( n \neq 0 \) of even weight, we can also write
\[
f_n(Y, Z, T, q) = \prod_{m=1}^{\infty} (1 + Y_m)^{(1 + q)_{2m}(n_m - n_m)/2 - D^*_m(n_m) + 2m} (1 + Z_m)^{(1 + q)_{2m}(n_m + n_m)/2 + D^*_m(n_m) - 2m} A_n(T, q)
\]
and
\[
f_{-n}(Y, Z, T, q) = \prod_{m=1}^{\infty} (1 + Y_m)^{-(1 + q)_{2m}(n_m - n_m)/2 - D^*_m(n_m) + 2m} (1 + Z_m)^{-(1 + q)_{2m}(n_m + n_m)/2 + D^*_m(n_m) - 2m} (A_n(T, q)).
\]

One can combine the definition of \( f_{\delta}(Y, Z, T, q) \) with the above identities to express
\[
f_{\delta}(Y, Z, T, q) \prod_{l, m=1}^{\infty} [(1 + \bar{Y}_{lm})(1 + \bar{Z}_{lm})]^{-n_{l1}(\bar{1} + \bar{T}_{lm})/2}. \tag{79}
\]
Indeed, by applying the first identity to \( \prod_{l \geq 1} f_{\delta}(Y, Z, T, q)_l \), note that the product in (79) cancels out. Furthermore, by putting \( X_{\delta(Y)} = X_{(n_1, \ldots, 1, \ldots, 2, \ldots)}/\text{exponent of } X_{\delta(Y)} \) for \( l \geq 1 \), we can also separate out the product
\[
\prod_{l=1}^{\infty} \prod_{m=1}^{\infty} [(1 + \bar{Y}_{lm})(1 + \bar{Z}_{lm})]^{-\text{exponent of } X_{\delta(Y)}}/2
\]
which clearly depends only upon our original fixed partition \( \delta \). The remaining part of \( f_{\delta}(Y, Z, T, q) \), which acts on \( A_n(qT, 1/q) \) by scalar multiplication, can be expressed using (58). (Recall that the differential operator \( D^*_m(n, d) \), for \( m \geq 1 \), acts on a product \( \prod_{k=1}^{\infty} P_{c_k}(t_k) \), with \( d \in \mathcal{P}(n, q) \), as scalar multiplication by \( a_{m}^*a_{m}^*(d, d) \).

Consider now \( D^{*\delta}(Y, Z)M_{\delta}(T, q) \). By the properties of the differential operator \( D^{*\delta}_N \), we can write
\[
D^{*\delta}(Y, Z)M_{\delta}(T, q) = D^{*\delta}(Y, Z)A_{n,1}(qT, 1/q) \cdot D^{*\delta}(Y, Z) \left( \prod_{l=1}^{\infty} (T_q^{(l, \ell)} \cdot M_{\delta}(T_q^{(l, \ell)}, q)) \cdot E'(T, q) \cdot E''(T, q) \right)
\]
where, for all \( N \geq 1 \), \( D^{*\delta}_N \) acts on \( A_{n,1}(qT, 1/q) \) by \( \bar{D}^N \) (see Lemma 7.6). The second part
\[
D^{*\delta}(Y, Z) \left( \prod_{k} M_{\delta}(T_q^{(k, \ell)}, q^k) \cdot M_{\delta}(T_q^{(l, \ell)} \cdot M_{\delta}^{*\ell}(T_q^{(l, \ell)}, q^k)) \cdot E'(T, q) \cdot E''(T, q) \right)
\]
can be easily handled as follows. Express first the factors \( A_{n_{kh}}(T_q^{(l, \ell)}, q^k) \) (resp. \( A_{n_{kh}}(\pm T_q^{(l, \ell)}, q^k) \)) of \( M_{\delta}(T_q^{(l, \ell)}, q^k) \) (resp. \( M_{\delta}(T_q^{(l, \ell)}, q^k) \)) by their definition (58); also, write explicitly \( E'(T, q) \cdot E''(T, q) \). For \( n_{kh} = (n_{11}(n_{kh}), 2n_{22}(n_{kh}), \ldots) \), express each characteristic polynomial \( P_{c_d}(t_d^k) \), for \( d \in \mathcal{P}(n_{kh}, q^k) \) and \( \alpha = 1, \ldots, r \), by
\[
P_{c_d}(t_d^k) = (1 - q^k t_d^k)^{n_{kh}(n_{kh})/2} \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \left( 1 + t_d^i - t_d^j \right)^{n_{kh}(n_{kh})/2}.
\]
For \( n_{kh} = (n_{11}(n_{kh}), 2n_{22}(n_{kh}), \ldots) \), we express
\[
(1 - e_{\alpha}) = \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \left( e_{\alpha}^i - e_{\alpha}^j \right)^{n_{kh}(n_{kh})/2}.
\]
Here $\epsilon = \pm 1$, the sign agreeing with that on $n_{\ell,h}$. Notice that the product

\[ \prod_{\alpha=1}^{r} \prod_{k=1}^{\infty} (1 - q^{k_i} t_{\alpha}^{2k})^{-n_k/2} \]

factors out. Since

\[ (1 - t^2)^{1/2} = \prod_{i=1}^{\infty} (1 - q^{i} t^{2i})^{1/2} \]

it follows easily, using Lemma 7.4 (72), that

\[ \left( \prod_{\alpha=1}^{r} (1 - t_{\alpha}^{2})^{1/2} \prod_{k=1}^{\infty} (1 - q^{k_i} t_{\alpha}^{2k})^{-n_k/2} \right) E(T^2)^{1/2} \]

is precisely the contribution of the remaining part of the product $G$ in Lemma 7.6, i.e., $\prod_{N=1}^{\infty} \left( \frac{1 + \tilde{Z}_N}{1 + \tilde{Y}_N} \right)^{\frac{R_N}{2}}$ to (74). Then to each of the remaining terms, one applies Lemma 7.4 (73) to match each of them with the remaining contributions in (79).

Putting everything together, it follows that we can express (74) as

\[ c_{\beta} K_{\eta}(Y, Z) \cdot \mathcal{D}(Y, Z) [E(T^2)^{-1/2} M_{\beta}(T, q)] E(T^2)^{1/2} \]

where we set

\[ K_{\eta}(Y, Z) := \frac{\prod_{l=1}^{\infty} \prod_{m=1}^{\infty} (1 + \tilde{Y}_{lm})(1 + \tilde{Z}_{lm})^{(\nu_{lm} - \text{exponent of } X_{\ell m} \text{ in } X_{n/2})/2}}{\prod_{l=1}^{\infty} \prod_{m=1}^{\infty} \prod_{\nu=1}^{\infty} \left( \frac{1 + \tilde{Z}_{2\nu}}{1 + \tilde{Y}_{2\nu}} \right)^{-\text{exponent of } X_{\ell m} \text{ in } X_{n/2}/2^{\nu+1}}} \cdot \]

Accordingly, if we define

\[ \mathcal{D}_{\eta, \gamma, \beta}(T, q) = E(T^2)^{1/2} \text{ Coefficient}_{Y^3 Z^2} [K_{\eta}(Y, Z) \cdot \mathcal{D}(Y, Z) E(T^2)^{-1/2} \mathcal{D}(Y, Z)] \]  

\[ \mathcal{D}_{\eta, \gamma, \beta}(T, q) = E(T^2)^{1/2} \text{ Coefficient}_{Y^3 Z^2} [K_{\eta}(Y, Z) \cdot \mathcal{D}(Y, Z) E(T^2)^{-1/2} \mathcal{D}(Y, Z)] \]  

(80)

then it is clear that $C_{\eta, \gamma, \beta}(T, q) = \mathcal{D}_{\eta, \gamma, \beta}(T, q) C_{\eta, 0, 0}(T, q)$. Hence $C_{\eta, \gamma, \beta}(T, q) = 0$ if $C_{\eta, 0, 0}(T, q) = 0$. Notice that by Lemma 7.4 (72), applied with $\gamma_1 = 1$, $\gamma_1 = 0$ for $l \geq 2$, and $q = 1$, one could express

\[ \mathcal{D}(Y, Z) E(T^2)^{-1/2} \cdot E(T^2)^{1/2} = \prod_{\nu=1}^{\infty} \left( \frac{1 + \tilde{Z}_{2\nu}}{1 + \tilde{Y}_{2\nu}} \right)^{-2^{\nu-1}} \]

which could be absorbed into $K_{\eta}(Y, Z)$.

The argument is completely analogous if $|n|$ is even.

In the next section, we shall construct $\tilde{C}(X, T, q)$ such that

\[ C(C(\tilde{C}(X, T, q), T, q), T, 1/q) = \tilde{C}(X, T, q). \]

(81)

As in the proof of Theorem 7.1, one deduces easily that $C_{\eta, 0, 0}(T, q) = 0$ if one assumes that $C_{\eta, 0, 0}(T, q) = 0$, for all partitions $\mathcal{N}$ of weights $2 \leq |N| < |\mathcal{N}|$. (One checks directly that $C_{\eta, 0, 0}(T, q) = 0$ when $\mathcal{N} = (1^2)$, or $\mathcal{N} = (2^1)$.)

This completes the proof of the theorem subject to the construction of $\tilde{C}(X, T, q)$ satisfying the required properties. \[ \Box \]
8 The series $\bar{C}(X, T, q)$

In this section, we shall construct the general series $\bar{C}(X, T, q)$ satisfying the relation (81), and thus concluding the proof of Theorem 7.5; the series $\bar{c}(x, T, q)$ used in the final step of the proof of Theorem 7.1 is just a specialization of $\bar{C}(X, T, q)$. Most of what we will say here is independent from the rest of the paper.

To put things in perspective, we begin by discussing the occurrence of a special case of the series $\bar{C}(X, T, q)$ in the context of a well-known result of Ezra Getzler.

Let $\mathcal{M}_{0,n}$ ($n \geq 3$) denote the moduli space of irreducible, non-singular, projective curves of genus 0 with $n$ distinct marked points. Let $\overline{\mathcal{M}}_{0,n}$ denote its Deligne-Mumford compactification, consisting of the stable curves of genus 0 with $n$ distinct marked points. It is well-known (see [36] and [63]) that the corresponding Deligne-Mumford stacks $\mathcal{M}_{0,n}$ and $\overline{\mathcal{M}}_{0,n}$ are smooth and defined over $\mathbb{Z}$.

We recall that for a Deligne-Mumford stack $\mathcal{X}$ of finite type over $\mathbb{Z}$, one defines its number of points over a finite field $\mathbb{F}_q$ to be

$$\#\mathcal{X}(\mathbb{F}_q) = \sum_{\xi \in [\mathcal{X}(\mathbb{F}_q)]} \frac{1}{#\text{Aut}_\mathbb{F}_q(\xi)}$$

the sum being taken over the isomorphism classes of the category $\mathcal{X}(\mathbb{F}_q)$. Here, for an isomorphism class $\xi$, $\text{Aut}_\mathbb{F}_q(\xi)$ denotes the finite group of automorphisms of any object representing $\xi$. In particular, if $\mathcal{X} = \mathcal{M}_{0,n}$, then, by counting points, one sees that

$$\#\mathcal{M}_{0,n}(\mathbb{F}_q) = \#\overline{\mathcal{M}}_{0,n}(\mathbb{F}_q) = \prod_{k=2}^{n-2} (q-k)$$

$(n \geq 3)$.

The number of points $\#\overline{\mathcal{M}}_{0,n}(\mathbb{F}_q)$ $(n \geq 3)$ can be obtained from the following theorem of Getzler [50]; see also McMullen [70].

**Theorem (Getzler).** — The generating functions

$$f(x) = x - \sum_{n=2}^{\infty} \#\mathcal{M}_{0,n+1}(\mathbb{F}_q) \frac{x^n}{n!} = \frac{1 + q^2 x - (1 + x)^q}{q(q - 1)}$$

and $g(x) = x + \sum_{n=2}^{\infty} \#\overline{\mathcal{M}}_{0,n+1}(\mathbb{F}_q) \frac{x^n}{n!}$

are formal compositional inverses of one another.

Note that $g(x) = g(x, q)$ satisfies the functional equation

$$q^2 g(x, q) = q(q - 1)x + g(qx, 1/q)$$

(82)

which can be verified directly.

The connection between the function $g(x)$ in Getzler’s theorem and $\bar{c}(x, T, q)$ used in the proof of Theorem 7.1 can be easily seen as follows. First, set $t_1 = \cdots = t_r = 0$ and $y = z$ in $c_{\text{odd}}(x, y, z; T, q)$ and $c_{\text{even}}(x, y, z; T, q)$ introduced in 7.1. Denoting $c_{\text{odd}}(x, z, 0, \ldots, 0, q)$ (resp. $c_{\text{even}}(x, z, 0, \ldots, 0, q)$) by $c_{\text{odd}}(x, z; q)$ (resp. $c_{\text{even}}(x, z; q)$), one can write, for example,

$$c_{\text{odd}}(x, z; q) = \sum_{n,k \geq 0} x^{2n+1} z^k \sum_{i+j=k} A_{2n+1,i,j}(0, \ldots, 0, q) \frac{1}{i!j!}.$$  

For fixed $n \geq 1$ and $k \geq 0$, we have

$$\sum_{i+j=k} A_{n,i,j}(0, \ldots, 0, q) \frac{1}{i!j!} = \sum_{d \in \mathbb{Z}_n} \sum_{i+j=k} N_{i,j}(d, q) \frac{1}{i!j!} = \sum_{d \in \mathbb{Z}_n} \sum_{i+j=k} \left( \frac{q^{-d} - 1}{d} \right) \left( \frac{q^{-d} - 1}{d} \right).$$

8We hope that some of the readers will find the material included in this section of independent interest.

9This series is obtained by specializing $X_1 = x$, $X_n = 0$, for $n \geq 2$, and taking $t_1 = \cdots = t_r = 0$.  

53
The last double sum is just \((q)_k (q - n)_k\). Hence

\[
c_{\text{odd}}(x, z; q) = \frac{(1 + x + z)^q - (1 - x + z)^q}{2}.
\]

Similarly,

\[
c_{\text{even}}(x, z; q) = -1 + \frac{(1 + x + z)^q + (1 - x + z)^q}{2}.
\]

Next, consider

\[
\check{c}(x, q) = \check{c}(x, T, q)|_{t_2 = \cdots = t_k = 0} = (\check{A}_{\text{odd}}(x, q), \check{A}_{\text{even}}(x, q), \check{A}_{\text{even}}(x, q))
\]

where \(\check{c}(x, T, q)\) will be given in 7.5. The components \(\check{A}_{\text{odd}}(x, q)\) and \(\check{A}_{\text{even}}(x, q)\) are formal power series of the form

\[
\check{A}_{\text{odd}}(x, q) = x + \sum_{m \geq 3} a_m(q)x^m \quad \text{and} \quad \check{A}_{\text{even}}(x, q) = \sum_{m \geq 2} a_m(q)x^m
\]

and (as it turns out in this special case) these series can also be obtained as the unique solution of the system

\[
c_{\text{odd}}(\check{A}_{\text{odd}}(x, q), \check{A}_{\text{even}}(x, q); q) = -q(q - 1)x + q^2 \check{A}_{\text{odd}}(x, q)
\]

and

\[
c_{\text{even}}(\check{A}_{\text{odd}}(x, q), \check{A}_{\text{even}}(x, q); q) = q^2 \check{A}_{\text{even}}(x, q).
\]

Setting \(A := \check{A}_{\text{odd}} + \check{A}_{\text{even}}\), it follows that

\[
(c_{\text{odd}} + c_{\text{even}})(\check{A}_{\text{odd}}(x, q), \check{A}_{\text{even}}(x, q); q) = -1 + (1 + A(x, q))q = -q(q - 1)x + q^2 A(x, q)
\]

and thus

\[
A(x, q) = g(x, q) = x + \sum_{n=2}^{\infty} \# \mathcal{M}_{0,n+1}(\mathbb{F}_q) \frac{x^n}{n!}.
\]

Note that the functional equation (82) implies the relation (57).

As we shall shortly see, Getzler’s theorem will be used, in fact, in the initial step of the construction of the more general series \(\check{C}(X, T, q)\).

### 8.1 Moduli spaces of admissible double covers

We first need to introduce certain moduli spaces of admissible double covers, and some generating series attached to them.

For an even positive integer \(n\) and a nonnegative integer \(m\) such that \(m + n \geq 3\), let \(\text{Adm}_{n,m}\) denote the moduli space of admissible double covers over stable curves of genus 0 with \(n\) branched points and additional \(m\) marked points. These moduli spaces are smooth, proper Deligne-Mumford stacks defined over \(\text{Spec} \mathbb{Z}[\frac{1}{2}]\); the standard references for stacks of admissible covers are [57] and [1].

Let \(\mathbb{F}_q\) be, as before, a finite field of odd characteristic, and let \(\overline{\mathbb{F}}_q\) be a fixed algebraic closure of it. We shall consider the moduli space \(\text{Adm}_{n,m}\) over \(\overline{\mathbb{F}}_q\).

The group \(G := G_{n,m} := S_n \times (S_2 \wr S_m)\) acts on the \(\overline{\mathbb{F}}_q\)-points of the elements of \(\text{Adm}_{n,m}\) by permuting the marked points of the target curves and (eventually) by switching the points in the corresponding fibers. It is mentioned in [22] (p. 2), with reference to [84] and [92], that, for the hyperoctahedral group \(S_2 \wr S_m\) of order \(2^m \cdot m!\), the conjugacy classes are parametrized by pairs of partitions \((i, j)\) such that \(|i| + |j| = m\). Thus the conjugacy classes of the group \(G_{n,m}\) are parametrized by triples of partitions \((n, i, j)\) with \(|n| = n\) and \(|i| + |j| = m\).

To be more precise, an element \(\sigma\) of the hyperoctahedral factor \(S_2 \wr S_m\) can be viewed as a signed permutation acting on the set \(\mathcal{I} := \{\pm 1, \ldots, \pm m\}\) with the property that \(\sigma(a) = -\sigma(-a)\) for all \(a \in \mathcal{I}\). The orbit of an element \(a \in \mathcal{I}\) is either
even (i.e., if $b$ is in the orbit of $a$ then $-b$ is also in the orbit of $a$) or, alternatively, the orbit of $a$ is disjoint from the orbit of $-a$. The two partitions $(i,1)$ corresponding to the conjugacy class of $\sigma$ are determined by these orbits. Here $i$ corresponds to the even orbits and $j$ to the remaining ones.

Let $D \to C$ be an admissible double cover. Let $P \in C(\overline{F}_q)$ be a marked point which is not branched, and let $(P_1, P_2)$ be the points above $P$. Let $\sigma \in S_2 : S_m$, and assume that $D \to C$ is invariant under $F \sigma$. If the orbit of $[(P_1, P_2) \to P]$ is even, there exists a smallest positive integer $k$ such that $\sigma^k([(P_1, P_2) \to P]) = [(P_2, P_1) \to P]$. It follows that $P \in C(\overline{F}_q)$ and $F^k(P_1) = P_2$; thus $P_1$ and $P_2$ are not defined over $\overline{F}_q$. If the orbit of $[(P_1, P_2) \to P]$ is disjoint from the orbit of $[(P_2, P_1) \to P]$, then $F^k(P_1) = P_1$ and $F^k(P_2) = P_2$ for all $k$ such that $\sigma^k(P) = P$, and thus the points above $P$ are defined over the same field as $P$.

For any $\tau \in G$, consider the set $Adm_{n,m}^F(\tau)$ of fixed points of $F \tau$ on $Adm_{n,m}$. Note that there exists a unique moduli $Adm_{n,m}^F(\tau)$ such that

$$Adm_{n,m}^F(\tau) \otimes_{\mathbb{F}_q} \overline{F}_q \simeq Adm_{n,m} \otimes_{\mathbb{F}_q} \overline{F}_q$$

and $F \tau$ becomes the relative Frobenius endomorphism induced by this $\overline{F}_q$-structure on $Adm_{n,m}$.

For partitions $n = (1^{n_1}, 2^{n_2}, \ldots)$, $i = (1^{i_1}, 2^{i_2}, \ldots)$ with $n$ of even weight and $|n| + |i| + |j| \geq 3$, let $\mathcal{D}_{n,i,j}^F$ denote the category of admissible double covers over $\mathbb{F}_q$ of genus 0 stable curves with branching given by $n$, i.e., with $n_k$ branched $\mathbb{F}_q$-Galois orbits of cardinality $k$ (for $k = 1, 2, \ldots$), and additional marked points of the base given by the partitions $i$ and $j$ such that:

* each point in the set of marked points corresponding to $j$ and the points in its fiber are defined over the same field,

* the points in the fiber of any marked point corresponding to $i$ are defined over an (necessarily quadratic) extension of the field of definition of the base point.

By convention, we extend this notation to the case $|n| = n = 0$, where trivial (unramified) admissible double covers can be identified with elements of the moduli space $\overline{M}_{0,|i|}$. Moreover, let $\mathcal{D}_{n,i,j}$ denote the subcategory of $\mathcal{D}_{n,i,j}$ corresponding to the smooth locus.

For independent variables $X = (X_1, X_2, \ldots)$, $Y = (Y_1, Y_2, \ldots)$ and $Z = (Z_1, Z_2, \ldots)$, we introduce the generating series $\mathcal{D}(X, Y, Z; q)$ and $\overline{\mathcal{D}}(X, Y, Z; q)$ defined by

$$\mathcal{D}(X, Y, Z; q) := \sum_{n,i,j} \#\mathcal{D}_{n,i,j}(\mathbb{F}_q) X^n Y^i Z^j$$

and

$$\overline{\mathcal{D}}(X, Y, Z; q) := \sum_{n,i,j} \#\overline{\mathcal{D}}_{n,i,j}(\mathbb{F}_q) X^n Y^i Z^j.$$  

In what follows, we shall identify $Adm_{n,m}^F(\tau)$ with $[\overline{\mathcal{D}}_{n,i,j}(\mathbb{F}_q)]$, where $\tau$ is any element in the conjugacy class represented by $(n,i,j)$.

By the Grothendieck-Lefschetz trace formula [5, 6],

$$\#\overline{\mathcal{D}}_{n,i,j}(\mathbb{F}_q) = q^{\dim Adm_{n,m}} \sum_{k} (-1)^k \text{Tr} (\Phi_q^{-1} | H^k(Adm_{n,m} \otimes_{\mathbb{F}_q} \overline{F}_q, \mathbb{Q}_\ell))$$

where $\ell$ is a prime different from the characteristic of $\mathbb{F}_q$, $\Phi_q$ is the arithmetic Frobenius endomorphism (relative to $\mathbb{F}_q$), and $\Phi_q^{-1}$ acts, by transport of structures, on the $\ell$-adic cohomology. We also define

$$\overline{\mathcal{D}}(X, Y, Z; 1/q) := \sum_{n,i,j} \left( \sum_{k} (-1)^k \text{Tr} (\Phi_q^{-1} | H^k(Adm_{n,m} \otimes_{\mathbb{F}_q} \overline{F}_q, \mathbb{Q}_\ell)) \right) X^n Y^i Z^j.$$
8.2 A homomorphism

To make the transition from \( \mathcal{D}(X, Y, Z; q) \) and \( \overline{\mathcal{D}}(X, Y, Z; q) \) to generating functions of products of characteristic polynomials, closely related to \( C(X, T, q) \), and to construct the function \( C(X, T, q) \), which is the main goal of this section, it is convenient to introduce two rings of formal power series equipped with additional structure, and a homomorphism between them compatible with the (additional) structure of the rings. As we shall shortly see, this homomorphism enjoys additional properties.

Let \( \mathcal{F} = \mathbb{Q}(q) \) be the field of rational functions in a variable \( q \); we assume that \( q \) takes values in the set of prime powers. Let \( \mathcal{R} := \mathcal{R}_{\mathcal{F}} \) denote the ring of formal power series in infinitely many variables \( X := (X_n)_{n \geq 1} \), \( Y := (Y_n)_{n \geq 1} \) and \( Z := (Z_n)_{n \geq 1} \) with coefficients in \( \mathcal{F} \). We also consider \( \mathcal{S} := \mathcal{S}_{\mathcal{F}} \), the ring of formal power series in infinitely many variables \( X := (X_n)_{n \geq 1} \) and \( p := (p_n)_{n \geq 1} \) with coefficients in \( \mathcal{F} \).

On each \( \mathcal{R} \) and \( \mathcal{S} \), we define:

1. A family of ring homomorphisms (Adams operations), \( \{ \psi^n \}_{n \in \mathbb{Z}_{\geq 0}} \) such that:
   - (a) \( \psi^n(X_m) = X_{mn} \)
   - (b) \( \psi^n(Y_m) = Y_{mn} \)
   - (c) \( \psi^n(Z_m) = Z_{mn} \)
   - (d) \( \psi^n(p_m) = p_{mn} \)
   - (e) \( \psi^n(f(q)) = f(q^n) \) for \( f \in \mathcal{F} \)

2. The involution \( \iota \) characterized by:
   - (a) \( \iota(Y_n) = Z_n \) if \( n \) is odd
   - (b) \( \iota(Z_n) = Y_n \) if \( n \) is odd
   - (c) \( \iota(p_n) = -p_n \) if \( n \) is odd, respectively
   - (d) All the other variables (and constants) are fixed by \( \iota \)

3. The involution \( \delta \) given by:
   - (a) \( \delta f(X, Y, Z; q) := f(q \ast X, q \ast Y, q \ast Z; 1/q) \)
   - (b) \( \delta h(X, p; q) := h(q \ast X, q \ast p; 1/q) \), respectively

Here \( q \ast (U_n)_{n \geq 1} := (q^n U_n)_{n \geq 1} \) for any set of variables \((U_n)_{n \geq 1}\).

Using (83), (84) and (85), one can extend \( \psi^n, \iota \) and \( \delta \) to the generating series \( \mathcal{D}(X, Y, Z; q) \) and \( \overline{\mathcal{D}}(X, Y, Z; q) \) (or similar such series) in an obvious way. For example,

\[
\psi^n \overline{\mathcal{D}}(X, Y, Z; q) = \sum_{n, i, j} \overline{\mathcal{D}}_{n, i, j}(\overline{\mathcal{F}}_{q^m}) \psi^m(X^n) \psi^m(Y^i) \psi^m(Z^j) \quad \text{for } m \geq 1
\]

and

\[
\delta \overline{\mathcal{D}}(X, Y, Z; q) := \overline{\mathcal{D}}(q \ast X, q \ast Y, q \ast Z; 1/q) = \sum_{n, i, j} q^{[n+[i+j]+]} \left( \sum_k (-1)^k \text{Tr}(\Phi_q \tau^{-1} \mid H^k(\text{Adm}_{n,m} \otimes_{\mathbb{Q}_p} \overline{\mathcal{F}}, \mathbb{Q}_\ell)) \right) X^n Y^i Z^j.
\]

Since \( \dim \text{Adm}_{n,m} = n + m - 3 \), note that we have the relation

\[
\delta \overline{\mathcal{D}}(X, Y, Z; q) = q^{3} \overline{\mathcal{D}}(X, Y, Z; q).
\]

We remark that one may choose to work more generally with formal power series over the Grothendieck ring of the abelian category \( \text{Rep}_{\mathbb{Q}_p}(\text{Gal}([\overline{\mathbb{Q}}/\mathbb{Q}])) \) of \( \ell \)-adic Galois representations, where one can define natural operations \( \{ \psi^n \}_{n \in \mathbb{Z}} \), cf. [82, Chapters 5, 6].
For two elements \(f \in \mathcal{R}\) and \(g = (g_1, g_2, g_3) \in \mathcal{R}^3\) such that \(g_i (i = 1, 2, 3)\) has no constant term, we define the *plethystic* substitution \(f \circ g\) by replacing \(X_n \mapsto \psi^n (g_1), Y_n \mapsto \psi^n (g_2)\) and \(Z_n \mapsto \psi^n (g_3)\) \((n \geq 1)\) in the expression of \(f\).

### 8.3 Special elements

In each of the rings \(\mathcal{R}\) and \(\mathcal{S}\) there exists a special element defined as follows.

On \(\mathcal{R}\) consider the element

\[
\partial_{Z_1} D_0 (Z; q) = \frac{-1 - qZ_1 + \prod_{n \geq 1} (1 + Z_n)^{1/n_q(n)}}{q(q - 1)} \in \mathcal{R}
\]

where

\[
D_0 (Z; q) = \frac{-1 + (1 + Z_1) \prod_{n \geq 1} (1 + Z_n)^{1/n_q(n)}}{q(q^2 - 1)} - \frac{Z_1^2}{2(q - 1)} - \frac{Z_1}{q(q - 1)} - \frac{Z_2}{2(q + 1)}
\]

is the constant term in \(X = (X_n)_{n \geq 1}\) of \(\mathcal{D}(X, Y, Z; q)\), and \(\partial_{Z_1}\) denotes the partial derivative with respect to \(Z_1\). The expression of \(D_0 (Z; q)\) can be obtained by direct computation; see also [52, Theorem 7.17] and [52, Theorem 7.15 (c)], or [37, Corollary 4.2], where \(D_0 (Z; q)\) is denoted by \(\chi_n (V) (p)\). Now, the special element we consider is the unique solution \(S(Z; q)\) of the equation

\[
\partial_{Z_1} D_0 (Z; q) \circ (Z_1 + S(Z; q)) = S(Z; q)
\]

where “\(\circ\)” is the plethystic substitution\(^{10}\). Letting \(\bar{D}_0 (Z; q)\) denote the constant term in \(X = (X_n)_{n \geq 1}\) of \(\bar{\mathcal{D}}(X, Y, Z; q)\), we have

\[
S(Z; q) = \partial_{Z_1} \bar{D}_0 (Z; q).
\]

This identity can be easily deduced from [52, Theorem 7.17] and [52, Theorem 7.15 (c)], or [37, Corollary 4.2].

The special element on \(\mathcal{S}\) is

\[
E(p) := \exp \left( \sum_{n \geq 1} \frac{p_n}{n} \right) = 1 + \sum_{k \geq 1} \frac{1}{k!} \left( \sum_{n \geq 1} \frac{p_n}{n} \right)^k.
\]

With the above notation and terminology, we have:

**Proposition 8.1.** — There exists a unique homomorphism \(\pi : \mathcal{R} \rightarrow \mathcal{S}\) with the following properties:

\[
\pi|_{\mathcal{F}} = \text{Id}_{\mathcal{F}}, \quad \pi (X_1) = X_1, \quad \pi (1 + Z_1 + S(Z; q)) = E(p),
\]

\[
\pi \circ \psi^n = \psi^n \circ \pi \quad \forall n \geq 1 \quad \text{and} \quad \pi \circ \varepsilon = \varepsilon \circ \pi.
\]

Moreover, we have

\[
\pi(Z_1) = \frac{1 - q^2 + q^2 E(p) - \delta E(p) q(q - 1)}{q(q - 1)} \quad \text{and} \quad \pi \circ \delta = \delta \circ \pi.
\]

**Proof.** By setting \(\deg(X_n) = \deg(Y_n) = \deg(Z_n) = \deg(p_n) = n\), we obtain a grading on each of the rings \(\mathcal{R}\) and \(\mathcal{S}\). Then, via Adams operations, each of the rings is freely generated by the degree one elements. From the definition of \(\partial_{Z_1} D_0 (Z; q)\) and (87), we see that every monomial in \(S(Z; q)\) has (weighted) degree \(\geq 2\). Since

\[
E(p) = 1 + p_1 + \text{higher degree terms}
\]

and \(Y_1 = \varepsilon(Z_1)\), the existence and uniqueness of \(\pi\) follows.

To compute \(\pi(Z_1)\), we first express (87) in the equivalent form

\[
(Z_1 - \partial_{Z_1} D_0 (Z; q)) \circ (Z_1 + S(Z; q)) = Z_1.
\]

\(^{10}\)This is defined by replacing \(Z_0 \mapsto (Z_0 + \psi^n S(Z; q))\) \((n \geq 1)\) in the expression of \(\partial_{Z_1} D_0 (Z; q)\).
Thus, in particular, let Lemma 8.3.

Writing (87) explicitly and differentiating with respect to \( \pi \), we have the formula

\[
\pi(Z_1) = \frac{1 - q^2 + q^2 E(p) - \prod_{n \geq 1} (\psi^n E(p))^{\text{Ir}(n)}}{q(q - 1)}.
\]

We have the formula

\[
\prod_{n \geq 1} (\psi^n E(p))^{\text{Ir}(n)} = \delta E(p)
\]

which can be easily verified by matching the logarithms of the two sides, and the expression of \( \pi(Z_1) \) stated follows.

Finally, to prove that \( \pi \) commutes with \( \delta \), we note that the formula for \( \pi(Z_1) \) implies that \( \pi(\delta(Z_1)) = \delta(\pi(Z_1)) \). The remaining relations can be verified by simple calculations.

8.4 Further Computations

The formulas given in the next two lemmas will play a key role in our argument.

Lemma 8.2. — Let \( \partial_{z_1} S(Z; q) \) denote the partial derivative of \( S(Z; q) \) with respect to \( Z_1 \). Then

\[
\pi(1 + \partial_{z_1} S(Z; q)) = \frac{(q - 1)E(p)}{qE(p) - \delta E(p)}.
\]

Proof. Writing (87) explicitly and differentiating with respect to \( Z_1 \), we find easily that

\[
\partial_{z_1} S(Z; q) = (1 + \partial_{z_1} S(Z; q)) \cdot \left\{ \frac{-(1 + Z_1 + S(Z; q)) + \prod_{n \geq 1} (1 + Z_1 + \psi^n S(Z; q))^{\text{Ir}(n)}}{(q - 1)(1 + Z_1 + S(Z; q))} \right\}.
\]

To this we apply \( \pi \). Then Proposition 8.1 and (88) yield

\[
\pi(1 + \partial_{z_1} S(Z; q)) - 1 = \pi(1 + \partial_{z_1} S(Z; q)) \cdot \frac{-E(p) + \delta E(p)}{(q - 1)E(p)}
\]

and the lemma follows.

Lemma 8.3. — Let \( \partial_{x_1} D_2(X, Y, Z; q) \) be defined by

\[
\partial_{x_1} D_2(X, Y, Z; q) = \frac{2X_1 \left( \prod_{m} (1 + Y_m)^{c_{m/2^{r_2(m)}}(q^{2^{r_2(m)}}))(2m)(1 + Z_m)^{c_m(q) - c_m/2^{r_2(m)}}(q^{2^{r_2(m)}})}{q(q - 1)} \right)}{\epsilon \in \mathcal{R}}.
\]

Define \( K(X, Y, Z; q) \) as the unique solution of the equation

\[
\partial_{x_1} D_2(X, Y, Z; q) \circ (X_1 + K(X, Y, Z; q), Y_1 + \iota(S(Z; q)), Z_1 + S(Z; q)) = K(X, Y, Z; q)
\]

where \( \circ \) is the plethystic substitution. Then

\[
\pi(K(X, Y, Z; q)) = 0.
\]

In particular, \( \pi(\partial_{x_1} K(X, Y, Z; q)) = 0 \).

58
Proof. As before, using Proposition 8.1 and (88), we write

\[ \pi(K(X, Y, Z; q)) = \frac{2(X_1 + K(X, Y, Z; q))}{q(q - 1)} \prod_{m \geq 1} \left( \frac{\nu^m E(p)}{\nu^m E(p)} \right)^{c_m/z_2(m)} \left( q^{z_2(m)} \right)^{(2m)} \cdot \frac{\delta E(p)}{E(p) - 1}. \]

The lemma follows from the identity

\[ \prod_{m \geq 1} \left( \frac{\nu^m E(p)}{\nu^m E(p)} \right)^{c_m/z_2(m)} \left( q^{z_2(m)} \right)^{(2m)} = \frac{E(p)}{\delta E(p)}. \]

\[ \Box \]

Remark. It can be verified that \( K(X, Y, Z; q) \) in the last lemma is precisely

\[ K(X, Y, Z; q) = \partial_{x_1} \partial_{x_2}(X, Y, Z; q) \]

where \( \partial_{x_2}(X, Y, Z; q) \) is the coefficient of \( X_1^2 \) in \( \partial \mathcal{D}(X, Y, Z; q) \).

### 8.5 Essential parts of admissible double covers with marked points

For a function \( F(X, Y, Z; q) \), define

\[ \partial F(X, Y, Z; q) := (\partial_{x_1} F(X, Y, Z; q), \partial_{x_2} F(X, Y, Z; q), \partial_{x_3} F(X, Y, Z; q)) \]

where \( \partial_{x_1}, \partial_{x_2} \) and \( \partial_{x_3} \) denote the partial derivatives of \( F \) with respect to \( X_1, Y_1 \) and \( Z_1 \), respectively.

The main purpose of this section is to obtain a recursive relation for \( \pi(\partial \mathcal{D}(X, Y, Z; q)) \). To do so, we begin by introducing some useful notation and terminology.

Let \( C \) be an admissible double cover with a distinguished smooth point \( P_0 = P_0(C) \) sometimes called the root of \( C \); we shall always assume that the point of the base curve corresponding to \( P_0 \) is defined over the field of definition of \( C \). The irreducible component of \( C \) containing \( P_0 \) is called the root component of \( C \). To each point \( P \) of \( C \), by which we mean a point of the base curve together with its fiber, we associate a color, or type, \( \kappa(P) \) as follows:

(i) \( \kappa(P) = \text{white} \), or \( X_1 \)-type, if the fiber of the corresponding point of the base curve consists of one point.

(ii) \( \kappa(P) = \text{blue} \), or \( Z_1 \)-type, if the fiber of the corresponding point of the base curve consists of two points, and both these points are defined over the same field as the corresponding point of the base curve.

(iii) \( \kappa(P) = \text{red} \), or \( Y_1 \)-type, if the fiber of the corresponding point of the base curve consists of two points, and these points are not defined over the same field as the corresponding point of the base curve.

For a type \( \varnothing \in \{ X_1, Y_1, Z_1 \} \), we shall denote by \( E^\varnothing(p) \) the function

\[ E^\varnothing(p) = \begin{cases} 
1 & \text{if } \varnothing = X_1 \\
\nu E(p) & \text{if } \varnothing = Y_1 \\
E(p) & \text{if } \varnothing = Z_1.
\end{cases} \]

In what follows, we shall consider only rooted admissible double covers. For \( \varnothing \in \{ X_1, Y_1, Z_1 \} \), let \( \partial_{x_i} \mathcal{F}_{n,i} \) denote the category of rooted admissible double covers \( C \) over \( \mathbb{F}_q \), whose distinguished points \( P_0(C) \) are of type \( \varnothing \), and the corresponding points of the base curves are marked points, i.e., the point corresponding to \( P_0(C) \) is one of the \( |n| \)
branch points if \( \varpi = X_1 \), or one of the \(|j| \) (resp. \(|j|\)) marked points if \( \varpi = Z_1 \) (resp. \( \varpi = Y_1 \)). Note that the point of the base curve corresponding to \( P_0(C) \) is defined over \( \mathbb{F}_q \). We also define

\[
\partial_\varpi \overline{\mathbb{D}}_n = \begin{cases} 
\partial_{X_1} \overline{\mathbb{D}}_{n,0,0} & \text{if } \varpi = X_1 \\
\partial_{Y_1} \overline{\mathbb{D}}_{n,1,0} & \text{if } \varpi = Y_1 \\
\partial_{Z_1} \overline{\mathbb{D}}_{n,0,1} & \text{if } \varpi = Z_1.
\end{cases}
\]

From now on, a point \( P \) of an admissible double cover \( C \) will be called \textit{ramified} or \textit{unramified} according as the corresponding point of the base curve is branched or not. We shall also refer to a point of an admissible double cover as being a node (or marked) if the corresponding point of the base is so. Finally, for each point \( P \in C \), let \( n_P \) stand for the degree of \( P \), that is, \( n_P = [k(P) : k] \), where \( k \) is the field of definition of \( C \).

Let \( C \) be an admissible double cover with a distinguished point \( i \), which belongs to \( \mathbb{F}_q \). We define its \textit{essential} part, denoted by \( \text{ess}(C) \), to be the admissible double cover obtained from \( C \) by repeating the process of deleting all the unramified markings and stabilizing it. The isomorphism class of the resulting cover is an element of \( \partial_\varpi \overline{\mathbb{D}}_n(\mathbb{F}_q) \). Note that \( \partial_\varpi \overline{\mathbb{D}}(X, Y, Z; q) \) can be expressed as

\[
\partial_\varpi \overline{\mathbb{D}}(X, Y, Z; q) = \sum_{[C_0] \in \partial_\varpi \overline{\mathbb{D}}_n(\mathbb{F}_q)} \sum_{[\text{ess}(C)] = [C_0]} \frac{1}{\#\text{Aut}_k(C)} X^{n(C)} Y^{i(C)} Z^{j(C)}
\]

where, for \( \varpi \in \{X_1, Y_1, Z_1\} \) and \( [C] \in \partial_\varpi \overline{\mathbb{D}}_{n,1,1}(\mathbb{F}_q) \), \( n(C), i(C) \) and \( j(C) \) are such that:

\[
\varpi \cdot X^{n(C)} Y^{i(C)} Z^{j(C)} = X^n Y^i Z^j.
\]

The process of constructing \( \text{ess}(C) \) can be reversed, and one has the following explicit reconstruction of all admissible double covers that have the same essential part:

(i) Consider a finite set \( S \) of Galois orbits of points that are not nodes, ramified points, or the root. Decompose the set \( S = S_1 \cup S_2 \) into two disjoint, possibly empty, subsets. We mark the orbits in \( S_1 \), and in every Galois orbit in \( S_2 \) of a point, say, \( P \), we insert a Galois orbit of an unramified admissible double cover defined over \( \mathbb{F}_q(P) \) with marked points and a root of the same type as \( P \), i.e., the point \( P \) is identified with the root.

(ii) In the Galois orbit of a node \( P \) (or the root \( P_0 \)), we can either leave it as it is, or insert a Galois orbit of an admissible double cover defined over \( \mathbb{F}_q(P) \) (or \( \mathbb{F}_q(P_0) \)) with marked points, the root \( Q_0 \), and one additional distinguished point\(^\text{11}\) \( Q_1 \) defined over \( \mathbb{F}_q(P) \) (or \( \mathbb{F}_q(P_0) \)). With no marked points, other than possibly \( Q_0 \) and \( Q_1 \), in the case of a node, consider the partial normalization of the base curve at the point \( p \) corresponding to \( P \). Since \( p \) is a separating node, the corresponding points \( \{p_0, p_1\} \) of the normalization at \( p \) (i.e., the normalized double point) can be chosen such that \( p_0 \) and the point corresponding to the root lie in the same connected component of the partial normalization, whereas \( p_1 \) lies in the other connected component. This yields two admissible double covers, with base curves the connected components of the partial normalization at \( p \), each with a point \( \tilde{p}_j \) (\( j = 0, 1 \)) — namely, that corresponding to \( p_j \). We identify \( \tilde{p}_0 \) with \( Q_0 \) and \( \tilde{p}_1 \) with \( Q_1 \). In the remaining case, we identify the root \( P_0 \) with \( Q_0 \), and \( Q_1 \) becomes the root of the resulting cover.

(iii) In the Galois orbit of a ramified point \( P \), different from the root, we can either just leave the orbit marked, or, alternatively, insert a Galois orbit of an admissible double cover defined over \( \mathbb{F}_q(P) \) with marked points and with two marked points \( Q_0 \) and \( Q_1 \) defined over \( \mathbb{F}_q(P) \), the point \( Q_0 \) being also the root of the cover. In this case, we identify \( P \) with \( Q_0 \).

Starting with \( C \) (\( [C] \in \partial_\varpi \overline{\mathbb{D}}_n(\mathbb{F}_q) \)), one applies (i), (ii) and (iii) (only once) to obtain an admissible double cover whose essential part is \( C \).

\(^{11}\)In the sense that we choose it always to be the last marked point defined over the field of definition of the admissible double cover.
Finally, for an admissible double cover \( C \) defined over \( \mathbb{F}_q \), we define

\[
P_C(p) = P_C(p, q) := \prod_{p \in C^+ \cap G_{q_i}} \psi^{n_p} E(p) \prod_{p \in C^- \cap G_{q_i}} \psi^{n_q} E(p)
\]

where \( C^+ \) (resp. \( C^- \)) is the set of \( \mathbb{F}_q \)-points of \( C \) of blue (resp. red) color, and \( G_{q_i} = \text{Gal}(\mathbb{F}_q/\mathbb{F}_{q_i}) \). If \( C \) is a hyperelliptic curve over \( \mathbb{F}_q \) then, under the specialization \( p_n \rightarrow p_n(T) = \sum_{i \geq 1} t_i^n \) for all \( n \geq 1 \), we have by (36) that

\[
P_C(p(T)) = E(T)^{q(C)} \cdot \prod_{m \geq 1} E(T^m) \frac{e_m(C) + a_{-m}(C)}{2m} E(-T^m) \frac{e_m(C) - a_{-m}(C)}{2m} = \prod_{i \geq 1} P_C(t_i)
\]

where \( E(T) = \prod_{i \geq 1} (1 - t_i)^{-1} \).

We now prove the following

**Theorem 8.4.** — Consider the series \( H(X, Y, Z; p, q) \) defined by

\[
H(X, Y, Z; p, q) = \left( X_1, -\frac{1}{iE(p)(qE(p) - \delta E(p))} Y_1, \frac{q - 1}{E(p)(qE(p) - \delta E(p))} Z_1 \right)
\]

\[
\circ \partial \left( \sum_{n, i, j} \frac{P_C(p)}{\# \text{Aut}_q(C)} \cdot X^n Y^i Z^j \right).
\]

Letting

\[
\bar{D}^*(X, Y, Z; q) := \bar{D}(X, Y, Z; q) - \bar{D}_0(Z; q) = \sum_{n, i, j} \# \bar{D}_{n, i, j} \cdot X^n Y^i Z^j
\]

we have

\[
H(X, Y, Z; p, q) \circ \left( X_1 + \pi(\partial_{X, \bar{D}^*})(X, p; q), \frac{\pi(\partial_{Y, \bar{D}^*})(X, p; q)}{iE(p)}, \frac{\pi(\partial_{Z, \bar{D}^*})(X, p; q)}{E(p)} \right)
\]

\[
= \left( \pi(\partial_{X, \bar{D}^*})(X, p; q), \frac{\pi(\partial_{Y, \bar{D}^*})(X, p; q)}{iE(p)}, \frac{\pi(\partial_{Z, \bar{D}^*})(X, p; q)}{E(p)} \right).
\]

**Proof.** For any \( \vartriangle \in \{X_1, Y_1, Z_1\} \), we first write \( \pi(\partial_{\vartriangle, \bar{D}^*})(X, p; q) \) as

\[
\pi(\partial_{\vartriangle, \bar{D}^*})(X, p; q) = \sum_{\left| n \right| = \text{even}} \sum_{\left[ C \right] \in \left\{ \partial_{\vartriangle, \bar{D}^*}(\mathbb{F}_q) \right\}} \frac{\alpha^\vartriangle(p, q(C))}{\# \text{Aut}_q(C)} X^{n(C)} \prod_{p \in C^+ \cap G_{q_i}} \psi^{n_p} E(p) \prod_{p \in C^- \cap G_{q_i}} \psi^{n_q} E(p)
\]

\[
\cdot \prod_{p \in N(C) \cap G_{q_i}} \psi^{n_q} \alpha^\vartriangle(p, q(C))
\]

where \( N(C) \) denotes the set of nodes of \( C \),

\[
\alpha^\vartriangle(p) = \alpha^\vartriangle(p, q) = \begin{cases} 1 & \text{if } \vartriangle = X_1 \\ \frac{q - 1}{qE(p) - \delta E(p)} & \text{if } \vartriangle = Y_1 \\ \frac{q - 1}{qE(p) - \delta E(p)} & \text{if } \vartriangle = Z_1 \end{cases}
\]

and \( G_{q_i} = \text{Gal}(\mathbb{F}_q/\mathbb{F}_{q_i}) \).

Indeed, one follows the recipe to reconstruct all the admissible double covers from a fixed \( C \) with \( [C] \in \left\{ \partial_{\vartriangle, \bar{D}^*}(\mathbb{F}_q) \right\} \).

Counting the number of curves that can be inserted by the process, it follows from Proposition 8.1, Lemma 8.2 and
Lemma 8.3 that, after applying \( \pi \), the contribution corresponding to the Galois orbit of a node (or the root) \( P \) is 
\[
\psi^{n,p}(\alpha(p) \cdot E^{n,p}(p)),
\]
while the contribution corresponding to the Galois orbit of a point \( P \) which is neither a node nor the root is 
\[
\psi^{n,p}E^{n,p}(p).
\]

Next, one considers all the curves \( C \) with the same root component \( C_0 \). Note that \( C_0 \) can be represented by a hyperelliptic curve with marked points given by the nodes of \( C \) belonging to \( C_0 \), and with a distinguished point given by the root. In particular, \([C_0]\) is an element of \([D_{n,i}(\mathbb{F}_q)]\) for some partitions \( n, i \) and \( j \). Now the contribution to the first two products in (89) coming from the points of \( C_0 \) is \( C_0(p) \). Finally, notice that the subcurves of \( C \) in a Galois orbit of a node \( P \) contributes the same as in 
\[
\psi^{n,p}\left(\frac{\pi(\partial_{X,Y,Z}D^*)(X,p,q)}{E^{n,p}(p)}\right)
\]
if \( \kappa(p) = Y_1 \) or \( Z_1 \), and as in 
\[
\psi^{n,p}(X_1 + \pi(\partial_{X,Y,Z}D^*)(X,p,q))
\]
if \( \kappa(p) = X_1 \), and the theorem follows.

**Corollary 8.5.** — Define \( B(X,Y,Z;p,q) \) by

\[
B(X,Y,Z;p,q) = (qX_1, qY_1, qZ_1) + q(q - 1) \cdot \left( \frac{qE(p) - \delta E(p)}{q - 1} \cdot \delta E(p) \right) Y_1, \frac{E(p)}{(q - 1) \cdot \delta E(p) Z_1} \circ H(X,Y,Z;p,q).
\]

Then

\[
B(X,Y,Z;p,q) = \left( X_1 + \pi(\partial_{X,Y,Z}D^*)(X,p,q) \cdot \frac{\pi(\partial_{X,Y,Z}D^*)(X,p,q)}{\delta E(p)} \right) \cdot \frac{E(p)}{(q - 1) \cdot \delta E(p) Z_1} \circ H(X,Y,Z;p,q).
\]

**Proof.** A simple calculation using the relation in Theorem 8.4 implies that the left-hand side of the identity in the corollary is

\[
qX_1 + q^2 \pi(\partial_{X,Y,Z}D^*)(X,p,q), \frac{q^2 \pi(\partial_{X,Y,Z}D^*)(X,p,q)}{\delta E(p)} \cdot \frac{E(p)}{(q - 1) \cdot \delta E(p) Z_1} \circ H(X,Y,Z;p,q).
\]

From the definition of \( \bar{D}_0(Z;q) \), we have that

\[
\delta \bar{D}_0(Z;q) = q^3 \bar{D}_0(Z;q).
\]

Differentiating (86) and (90), one finds the relations

\[
\delta(\partial_{X,Y,Z}D^*)(X,Y,Z;q) = q^3 \partial_{X,Y,Z}D^*(X,Y,Z;q)
\]
\[
\delta(\partial_{X,Y,Z}D^*)(X,Y,Z;q) = q^3 \partial_{X,Y,Z}D^*(X,Y,Z;q)
\]
\[
\delta(\partial_{X,Y,Z}D^*)(X,Y,Z;q) = q^3 \partial_{X,Y,Z}D^*(X,Y,Z;q)
\]

and thus we can write

\[
q^2 \pi(\partial_{X,Y,Z}D^*)(X,p,q) = \pi(q^2 \partial_{X,Y,Z}D^*(X,Y,Z;q)) = \pi(\delta(\partial_{X,Y,Z}D^*)(X,Y,Z;q))
\]

(and similarly for the other partial derivatives of \( \bar{D}^*(X,Y,Z;q) \)).

The corollary follows now from the commutativity of \( \pi \) and \( \delta \), see Proposition 8.1.

**8.6 Finishing the proof of Theorem 7.5**

From now on we specialize \( p = (p_n)_{n \geq 1} \) to be the power sums,

\[
p_n = p_n(T) = \sum_{i \geq 1} t_i^n
\]
for independent variables $T = (t_i)_{i \geq 1}$. For compatibility, put $\psi^n(t_i) = t_i^n$, $\iota(t_i) = -t_i$ and $\delta(t_i) = qt_i$, for all $i \geq 1$. We define the plethystic substitution for functions in the variables $X, Y, Z, T$ and $q$ in the obvious way.

We first establish the connection between the function

$$C_1(\bar{X}, T, q) := (C_{1, \text{odd}}(\bar{X}, T, q), C_{1, \text{even}}^-(\bar{X}, T, q), C_{1, \text{even}}^+(\bar{X}, T, q))$$

(see 7.2) and the function

$$B(\bar{X}, T, q) = B(\bar{X}, T, q)|_{(p_n - p_n(T), n \geq 1)}.$$  

To do so, consider temporarily the function $C_1^*(\bar{X}, T, q)$ defined by

$$C_1^*(\bar{X}, T, q) = - (qX_1, q\tilde{E}(-T, q)Y_1, q\tilde{E}(T, q)Z_1) + C_1(\bar{X}, T, q) \circ (X_1, \tilde{E}(-T, q)Y_1, \tilde{E}(T, q)Z_1)$$

where, as before, $\tilde{E}(T, q) = E(T)E(qT)$.

Let $M(\bar{X}, T, q)$ denote the part of the function $H$ introduced in Theorem 8.4 (normalized by $q(q-1)$) given by

$$M(\bar{X}, T, q) = q(q-1) \cdot \partial \left( \sum_{n,i,j} \sum_{[C] \in \{D_{n, i, j}(F_q)\}} \prod_{i \geq 1} P_{C}(t_i) \cdot X^nY^iZ^j \right).$$

Then, by a simple counting argument, we see that

$$C_1^*(\bar{X}, T, q) = M(\bar{X}, T, q).$$

From the definition of $B(\bar{X}, T, q)$, it follows easily that

$$C_1(\bar{X}, T, q) = (X_1, \tilde{E}(-T, q)Y_1, \tilde{E}(T, q)Z_1) \circ B(\bar{X}, T, q) \circ \left( X_1, \frac{Y_1}{\tilde{E}(-T, q)}, \frac{Z_1}{\tilde{E}(T, q)} \right)$$

that is, $C_1(\bar{X}, T, q)$ and $B(\bar{X}, T, q)$ are conjugated under a linear plethystic transformation. Thus we can rewrite the relation in Corollary 8.5 as

$$C_1(\bar{X}, T, q) \circ \tilde{C}_1(\bar{X}, T, q) = \delta(\tilde{C}_1(\bar{X}, T, q)) = \tilde{C}_1(q \ast X, qT, 1/q)$$

(91)

where

$$\tilde{C}_1(\bar{X}, T, q) := \left( X_1 + \pi(\partial_{x_1}\tilde{D}^*)(X, T, q), E(-qT) \cdot \pi(\partial_{y_1}\tilde{D}^*)(X, T, q), E(qT) \cdot \pi(\partial_{z_1}\tilde{D}^*)(X, T, q) \right).$$

Notice that

$$X_1 + \pi(\partial_{x_1}\tilde{D}^*)(X, T, q) = X_1 + \text{higher degree terms}$$

while

$$\pi(\partial_{x_1}\tilde{D}^*)(X, T, q) = \sum_{n,i,j} \#\tilde{D}_{n,i,j}(F_q)X^n \cdot \pi(Y^iZ^j)(T, q).$$

Applying $C_1(\bar{X}, qT, 1/q) \circ -$ to both sides, we obtain

$$C_1(\bar{X}, qT, 1/q) \circ C_1(\bar{X}, T, q) \circ \tilde{C}_1(\bar{X}, T, q) = C_1(\bar{X}, qT, 1/q) \circ \tilde{C}_1(q \ast X, qT, 1/q) = \tilde{C}_1(\bar{X}, T, q)$$

which implies the functional relation (81) used to finish the proof of Theorem 7.5.

Finally, the function $\tilde{c}(x, T, q)$ used to complete the induction process in the proof of Theorem 7.1 is obtained by specializing $X_1 = x$ and $X_n = 0$, for $n \geq 2$, in $\tilde{C}_1(\bar{X}, T, q)$.
9 Constructing the generating series $\Lambda_l(T, q)$

To determine the coefficients $\lambda(\kappa, l; q)$ of $\Lambda_l(T, q)$, for $l \geq 5$ and $\kappa = (k_1, \ldots, k_r) \in \mathbb{N}^r$, one proceeds as in the case $l = 4$. We show that every coefficient $\lambda(\kappa, l; q)$ is a finite sum

$$
\lambda(\kappa, l; q) = \sum_j P_j(\kappa, l)(q)\alpha_j
$$

(92)

where $P_j(\kappa, l)(x) \in \mathbb{Q}[x]$ is independent of $q$ for all $j$, and $\alpha_j$ are distinct $q$-Weil algebraic integers of weights $m_j \in \mathbb{N}$; every $\alpha_j$ occurs in the sum together with all its complex conjugates, and $P_j(\kappa, l)(x) = P_j(\kappa, l)(x)$ if the algebraic numbers $\alpha_j$ and $\alpha_j'$ are conjugates over $\mathbb{Q}$. Moreover, for each $j$, we have

$$
\deg P_j(\kappa, l) + m_j \leq |\kappa| + l \quad \text{and} \quad P_j(\kappa, l)(x)^2 \equiv 0 \pmod{x^{|\kappa|+l-m_j+2}}
$$

(93)

with $|\kappa| = k_1 + \cdots + k_r$. As we shall see, the numbers $\alpha_j$ are the (suitably) normalized eigenvalues of Frobenius acting on the components $H^*_{c, \mu}(\mathcal{H}_g[2] \otimes \overline{F}_q, \mathcal{V}'(\lambda))$ of the $\mathbb{S}_{2g-2}$-isotypic decomposition of $H^*_{c, \mu}(\mathcal{H}_g[2] \otimes \overline{F}_q, \mathcal{V}'(\lambda))$.

Note that

$$
\lambda(\kappa, l; 1/q) = \sum_j P_j(\kappa, l)(1/q)q^{-m_j}\alpha_j;
$$

(94)

compare also with [82, (*), p. 6]. Accordingly, we must have that

$$
\lambda(\kappa, l; q) = \left( P_j(\kappa, l)(q) - q^{|\kappa|+l-m_j+1}P_j(\kappa, l)(1/q) \right)\alpha_j;
$$

(95)

note that $Q_j(\kappa, l)(x) \equiv x^{|\kappa|+l-m_j}P_j(\kappa, l)(1/x) \equiv x^{|\kappa|+l-m_j}P_j(\kappa, l)(x)$ for all $j$, which says that (iii) in Section 3 is satisfied. As $\overline{F}_q$ is an arbitrary finite field of odd characteristic, the condition (ii) is also satisfied.

Now let us justify our assertions. From the initial conditions (i) in Section 3, we have $\lambda(0, \ldots, 0; l; q) = q^l$. We argue by induction on $\kappa$ and $l$. Assume that (92) (together with the corresponding conditions on the polynomials and $q$-Weil integers involved) holds for all coefficients $\lambda(\kappa', l'; *)$ with $l' < l$, and all coefficients $\lambda(\kappa', l; q)$ with $\kappa' < \kappa$. Consider (27) in the symmetric form

$$
E(T)^{\xi} \Lambda_l(T, q) = E(T)^{\xi} A_l(T, q) + q^{l+1}E(T)^{\xi} \Lambda_l(qT, l; 1/q).
$$

By equating the coefficients of $T^\kappa = t_1^{k_1} \cdots t_r^{k_r}$ on both sides, we find that the left-hand side of (95) can be expressed as

$$
\sum_{\kappa' \leq \kappa} a(\kappa', l; q) - \sum_{\kappa' < \kappa} \left( \lambda(\kappa', l; q) - q^{|\kappa'|+l+1}\lambda(\kappa', l; 1/q) \right)
$$

(96)

if $l$ is even, and $a(\kappa, l; q)$ if $l$ is odd.

We recall that the generating series $A_l(T, q)$ satisfies the functional equation (53); as explained at the beginning of Section 7, (53) is implied by the identities (55), or equivalently (70) (proved in Theorem 7.5). Accordingly, if $l$ is odd, $a(\kappa, l; q)$ satisfies the functional equation

$$
a(\kappa, l; q) = -q^{|\kappa|+l+1}a(\kappa, l; 1/q)
$$

(97)

for each $\kappa = (k_1, \ldots, k_r) \in \mathbb{N}^r$. It is clear that both sums in (96) also satisfy the functional equation (97); the first sum in (96) is the coefficient of $T^\kappa$ in $E(T)A_l(T, q)$.

Next, we shall apply Deligne’s theory of weights to express the first sum in (96) (respectively $a(\kappa, l; q)$ when $l$ is odd) in terms of $q$-Weil numbers. The coefficient $\lambda(\kappa, l; q)$ will then be easily determined from the corresponding expression of the sums (96). To this end, let us first recall some facts from Deligne’s theory that we shall need. We follow the standard reference [35].
Let $\mathbb{F}_q$ be a finite field with $q$ elements. Fix $\overline{\mathbb{F}}_q$ an algebraic closure of $\mathbb{F}_q$. Let $\mathcal{F}$ be a constructible $\overline{\mathbb{Q}}_l$-sheaf $(\ell + q)$ on a scheme $X$ of finite type over $\mathbb{F}_q$. Set $\overline{X} := X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$, and denote the pullback of $\mathcal{F}$ to $\overline{X}$ by $\overline{\mathcal{F}}$. The Frobenius morphism $F : \overline{X} \to \overline{X}$ induces a natural isomorphism $F^* : F^*\overline{\mathcal{F}} \to \overline{\mathcal{F}}$. The finite morphisms $F$ and $F^*$ define an endomorphism

$$F^* : H^i_c(\overline{X}, \overline{\mathcal{F}}) \to H^i_c(\overline{X}, F^*\overline{\mathcal{F}}) \to H^i_c(\overline{X}, \overline{\mathcal{F}})$$

of cohomology groups with compact support. For a closed point $y \in [X]$, $F^*$ defines a morphism

$$F^*_y : \overline{\mathcal{F}}(y) \to \overline{\mathcal{F}}(y).$$

Let $x \in [X]$ be a closed point, and let $\bar{x}$ be a geometric point of $X$ over $x$. (The closed point of $\overline{X}$ corresponding to $\bar{x}$ is also denoted by $\bar{x}$.) Let $F^*_x$ denote the endomorphism $F^*_{\deg x}$ of $\overline{\mathcal{F}}_{\bar{x}}$, where $\deg x := [\mathbb{F}_q(x) : \mathbb{F}_q]$. Up to isomorphism, $(\overline{\mathcal{F}}_{\bar{x}}, F^*_x)$ does not depend on the choice of $\bar{x}$. If we put

$$\det(I - F^*_x t | \overline{\mathcal{F}}_{\bar{x}}) = \det(I - F^*_x t | \overline{\mathcal{F}}_{\bar{x}}),$$

then the Grothendieck-Lefschetz trace formula (see [54], [55] and [34]) gives the following identity of formal power series:

$$L(X, \mathcal{F}, t) := \prod_{x \in [X]} \det(I - F^*_x t^{\deg x} | \mathcal{F})^{-1} = \prod_i \det(I - F^*_x t | H^i_c(\overline{X}, \overline{\mathcal{F}}))^{(-1)^{i+1}}.$$

### 9.1 Deligne’s purity theorem

Let notations be as above.

**Definition 9.1.** Let $K$ be a field of characteristic 0, and let $n \in \mathbb{Z}$. An element $\lambda \in K$ is called a $q$-Weil number of weight $n$ if $\lambda$ is algebraic over $\mathbb{Q}$, and all its complex conjugates have absolute value $q^{n/2}$.

**Definition 9.2.** Let $\mathcal{F}$ be a constructible $\overline{\mathbb{Q}}_l$-sheaf on the $\mathbb{F}_q$-scheme $X$. The sheaf $\mathcal{F}$ is said to be pure of weight $n$ if, for every closed point $x \in [X]$, the $\overline{\mathbb{Q}}_l$-eigenvalues of the endomorphism $F^*_x$ are $N(x) := [\mathbb{F}_q(x) : \mathbb{F}_q]$-Weil numbers of weight $n$. The sheaf $\mathcal{F}$ is said to be mixed if it admits a finite increasing filtration by constructible $\overline{\mathbb{Q}}_l$-sheaves with successive quotients that are pure. The weights of the successive quotients are the weights of $\mathcal{F}$.

**Theorem (Deligne).** Let $f : X \to S$ be a separated map between finite-type $\mathbb{F}_q$-schemes, and let $\mathcal{F}$ be a constructible $\overline{\mathbb{Q}}_l$-sheaf mixed of weights $\leq n$ on $X$. Then the sheaf $R^i f_* \mathcal{F}$ on $S$ is mixed of weights $\leq n + i$, for all $i$. In particular, every eigenvalue $\alpha$ of Frobenius on $H^i_c(\overline{X}, \overline{\mathcal{F}})$ is a $q$-Weil number of weight $m$, for some integer $m \leq n + i$.

Now let $H^i_g[2]$ $(g \geq 2)$ denote the coarse moduli space of $\mathcal{H}_g[2] \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_g$, and let $p : \mathcal{H}_g[2] \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_g \to H^i_g[2]$ be the natural map. The direct image $\mathcal{V}'_\lambda := p_* \mathcal{V}'(\lambda)$ on $H^i_g[2]$ is a constructible sheaf, pure of weight $|\lambda|$. We recall from 6.1 that $\mathcal{V}'(\lambda)$ is the $\ell$-adic local system (for a prime $\ell + q$) on $\mathcal{H}_g[2]$ corresponding to the irreducible symplectic representation $V_\lambda$; as in loc. cit., the pullback of this local system to $\mathcal{H}_g[2] \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_g$ will still be denoted by $\mathcal{V}'(\lambda)$. If we put

$$e_c(\mathcal{H}_g[2], \mathcal{V}'_\lambda) = \sum_{i \geq 0} (-1)^i[H^i_c(\mathcal{H}_g[2], \mathcal{V}'_\lambda)]$$

then the trace of Frobenius on $e_c(\mathcal{H}_g[2] \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \mathcal{V}'(\lambda))$ is the same as the trace of the corresponding Frobenius on $e_c(\mathcal{H}_g[2], \mathcal{V}'_\lambda)$ (cf. Section 2 in [10]). By Deligne’s theorem, every eigenvalue of Frobenius on $H^i_c(\mathcal{H}_g[2], \mathcal{V}'_\lambda)$ is a $q$-Weil number of weight at most $|\lambda| + i$. Since $\mathcal{V}'_\lambda$ is in particular integral, every eigenvalue is in fact a $q$-Weil algebraic integer (cf. Corollaire (3.3.4) in [35]). Of course, the cohomology group $H^i_c(\mathcal{H}_g[2], \mathcal{V}'_\lambda)$ vanishes for $i >
2\dim(H_g[2]) = 4g - 2. The natural action of the symmetric group $S_{2g+2}$ on $H_g[2]$ induces an isotypic decomposition of the Galois representation $H^*_c(H_g[2] \otimes \mathbb{F}_q, V'(\lambda))$ as

$$H^*_c(H_g[2] \otimes \mathbb{F}_q, V'(\lambda)) = \bigoplus_{\mu \vdash 2g+2} H^*_c(H_g[2] \otimes \mathbb{F}_q, V'(\lambda))$$

where for an irreducible representation $R_\mu$ of $S_{2g+2}$ indexed by the partition $\mu$ of $2g + 2$,

$$H^*_c(H_g[2] \otimes \mathbb{F}_q, V'(\lambda)) = R_\mu \otimes \text{Hom}_{S_{2g+2}}(R_\mu, H^*_c(H_g[2] \otimes \mathbb{F}_q, V'(\lambda))).$$

By Theorem 6.3 and (49), for each $\nu = (\nu_1, \nu_2, \ldots, (2g + 2)\nu_{2g+2})$, the sum

$$\frac{1}{|\text{GL}_2(\mathbb{F}_q)|} \sum_{\nu \in \text{P}_\nu} \left( \prod_{k=1}^r P_c(t_k) \right)$$

can be expressed in terms of traces of $F^*$ on $H^*_c, (\mathcal{H}_g[2] \otimes \mathbb{F}_q, V'(\lambda))$ with $\lambda \in (r^o)$ and $|\lambda|$ even; the $q$-Weil algebraic integers appearing in $\text{Tr}(F^*:\bar{\epsilon}_{c,\nu}(\mathcal{H}_g[2] \otimes \mathbb{F}_q, V'(\lambda)))$ are of weights at most $|\lambda| + 4g - 2$. Moreover, for every $\lambda$, $q^{\frac{|\lambda|}{2}} \left( q^{\frac{1}{2}} t_1 \cdots q^{\frac{1}{2}} t_r \right) s_\lambda(q^{\frac{1}{2}} t_1, \ldots, q^{\frac{1}{2}} t_r)$ in (49) is easily seen to be a polynomial in $t_1, \ldots, t_r$, the weight $|\kappa|$ of any monomial $t_1^{k_1} \cdots t_r^{k_r}$ occurring in this polynomial is an even integer in the set $\{|\lambda|, \ldots, 2rg - |\lambda|\}$. Hence the powers of $q$ occurring in this polynomial are all nonnegative and $\leq rg - |\lambda|$.

We express (98) in terms of $q$-Weil numbers as follows. Fix $\lambda \in (r^o)$ a partition with $|\lambda|$ even. Let $\beta$ be an eigenvalue of $F^*$ on $H^*_c, (\mathcal{H}_g[2] \otimes \mathbb{F}_q, V'(\lambda))$ of weight $m_\beta \leq |\lambda| + 2d$, where $d = 2g - 1 = \dim H_g[2]$. By duality, $q^{|\lambda|+d-\beta} = q^{|\lambda|+d-m_\beta} \bar{\beta}$ is an eigenvalue of Frobenius on the ordinary cohomology. We normalize $\beta$ by setting

$$\alpha := \begin{cases} \beta & \text{if } m_\beta \leq |\lambda| + d \\ q^{m_\beta - |\lambda|/2} \alpha & \text{if } m_\beta > |\lambda| + d \end{cases}$$

and denote by $m_\alpha$ the weight of $\alpha$. Notice that $m_\alpha + m_\beta \leq 2(|\lambda| + d)$, with equality if $m_\beta \geq |\lambda| + d$.

Let $\kappa = (k_1, \ldots, k_r) \in \mathbb{N}^r$ be such that $|\kappa|$ is even and $|\lambda| \leq |\kappa| \leq 2rg - |\lambda|$, and consider the coefficient of the monomial $t_1^{k_1} \cdots t_r^{k_r}$ in the sum (98) corresponding to $\lambda$. Note that $t_1^{k_1} \cdots t_r^{k_r}$ comes together with a $q^{(|\kappa| - |\lambda|)/2}$ in the polynomial (99). By taking this into account, we see that the relevant part of the contribution of an eigenvalue $\beta$ to the coefficient we are interested in is $q^{(|\kappa| - |\lambda|)/2} \beta$. Notice that

$$q^{(|\kappa| - |\lambda|)/2} \beta = q^{m_\beta - d + (|\kappa| - 3|\lambda|)/2} \alpha \quad \text{(if } m_\beta > |\lambda| + d).$$

Referring to the right-hand side of this identity, consider the sum between the degree of the coefficient of $\alpha$ and the weight $m_\alpha$ of $\alpha$. We shall need the following simple estimate of this quantity:

$$m_\alpha + m_\beta - d + (|\kappa| - 3|\lambda|)/2 = 2(|\lambda| + d) - d + (|\kappa| - 3|\lambda|)/2 = d + (|\kappa| + |\lambda|)/2 \leq |\kappa| + d$$

as $|\kappa| \geq |\lambda|$. The fact that the normalized eigenvalues $\alpha$ of $F^*$ on the compactly supported cohomology are algebraic integers follows from [33, §5, Appendice, Théorème 5.2.2]; see also [40, Appendix, Theorem 0.2].

### 9.2 Decomposing $A_1(T, q)$ in terms of $q$-Weil numbers

We shall now combine our induction assumptions and the above considerations with (29), Proposition 4.1, (30) and (31) to express the coefficient of $T^\kappa = t_1^{k_1} \cdots t_r^{k_r}$ in $E(T)^q A_1(T, q)$ in terms of traces of Frobenius on the cohomology of the local systems $V'(\lambda)$ on $\mathcal{H}_g[2] \otimes \mathbb{F}_q$. 

66
Consider first the sum $A_n(T, q) \,(n = (1^{n_1}, 2^{n_2}, \ldots))$ introduced at the beginning of 7.2. We can express it in terms of the sum (98) as follows. For $d \in \mathcal{P}(n, q)$, we write
\[
\prod_{k=1}^{r} P_{c_d}(t_k) = \exp\left(-\sum_{j=1}^{\infty} \frac{a_j(C_d)}{j} \cdot \sum_{k=1}^{r} t_k^j\right)
\]
with
\[
a_j(C_d) = \text{Tr}(F^{*j} | H_{\theta}^1(C_d, \mathbb{Q}_l)) = -\sum_{\theta \in \mathcal{P}_n(F_{\theta})} \chi_j(d(\theta)).
\]
Using this identity, express $A_n(T, q)$ as
\[
A_n(T, q) = \sum_{d \in \mathcal{P}(n, q)} \exp\left(-\sum_{j=1}^{\infty} \frac{a_j(C_d)}{j} \cdot \sum_{k=1}^{r} t_k^j\right).
\]
We distinguish two cases according as $|n|$ is even or odd. If $|n|$ is odd, then, by Proposition B.1, Appendix B, we can further express
\[
A_n(T, q) = \frac{(n_1 + 1)|GL_2(F_q)|}{q^2 - 1} \left[ \frac{1}{|GL_2(F_q)|} \sum_{d \in \mathcal{P}_{n', (n_2, \ldots, n_r)}(n_1, \bar{q}_q)} \left( \prod_{k=1}^{r} P_{c_d}(t_k) \right) \right]
\]
where $n' := (1^{n_1+1}, 2^{n_2}, \ldots)$. If $|n|$ is even, we write
\[
A_n(T, q) = \frac{1}{2} \sum_{\epsilon = 0, 1} \sum_{d \in \mathcal{P}(n, q)} \exp\left(-\sum_{j=1}^{\infty} \frac{a_j(C_d)}{j} \cdot \sum_{k=1}^{r} t_k^j\right) + (-1)^{r} \exp\left(-\sum_{j=1}^{\infty} \frac{a_j(C_d)}{j} \cdot \sum_{k=1}^{r} (-t_k)^j\right).
\]
Again, by Proposition B.1, the even ($\epsilon = 0$) part is just
\[
\frac{(q + 1 - n_1)|GL_2(F_q)|}{q^2 - 1} \left[ \frac{1}{|GL_2(F_q)|} \sum_{d \in \mathcal{P}_{n', (n_2, \ldots, n_r)}(n_1, \bar{q}_q)} \left( \prod_{k=1}^{r} P_{c_d}(t_k) \right) \right]
\]
Note that this holds even when $n_1 = 0$. The remaining part can be expressed using Proposition B.2, Appendix B as
\[
\frac{|GL_2(F_q)|}{q^2 - 1} \left[ \frac{1}{|GL_2(F_q)|} \sum_{d \in \mathcal{P}_{n, (n_2, \ldots, n_r)}(n_1, \bar{q}_q)} \left( \prod_{k=1}^{r} P_{c_d}(t_k) \right) \right] dP_{c_d}(0).
\]
Accordingly, we obtain the desired expression for the coefficients of $A_n(T, q)$,
\[
\text{Coefficient}_{\tau^\alpha} A_n(T, q) = \sum_{\alpha} p_{\alpha}^{(\kappa, n)}(q) \alpha \quad \text{for} \, \kappa \in \mathbb{N}^r
\]
in terms of normalized eigenvalues of Frobenius on the cohomology of the local systems $\mathcal{V}(\lambda) \,(|\lambda| \leq |\kappa| + 1)$ on $X_p[2] \otimes_{\mathbb{Q}_l} \bar{F}_q$. Here $p_{\alpha}^{(\kappa, n)}(q)$ are polynomials in $q$ with rational coefficients, and since $|GL_2(F_q)| = q(q + 1)(q - 1)^2$, $p_{\alpha}^{(\kappa, n)}(0) = 0$, for every $\alpha$; if $\alpha$ and $\alpha'$ are conjugates, $p_{\alpha}^{(\kappa, n)} = p_{\alpha'}^{(\kappa, n)}$. Furthermore, by (100) (applied also with $r$ replaced by $r + 1$), we see that
\[
\deg p_{\alpha}^{(\kappa, n)} + m_{\alpha} \leq |\kappa| + |n|.
\]
Remark. By Poincaré duality and Definition 6.1 we find that
\[
\text{Coefficient}_{\tau^\alpha} A_n(T, 1/q) = \sum_{\alpha} p_{\alpha}^{(\kappa, n)}(1/q) \alpha^{-1}.
\]
As in the proof of Lemma 7.2, by increasing \( r \) to \( r + r' \) in (101) with \( r' \) sufficiently large, differentiating accordingly, and then setting \( t_{r+1} = \cdots = t_{r+r'} = 0 \), we see that (101) yields a similar expression for the coefficients of a sum of the form

\[
\sum_{d \in \mathcal{P}(n,q)} \left( H(a_1^*(d,q), a_2^*(d,q), \ldots) \cdot \prod_{k=1}^{r} P_{C_d}(t_k) \right).
\]

Here \( H(\cdots) \) is any given polynomial expression in \( a_1^*(d,q), a_2^*(d,q), \ldots \). This applies in particular to (59), and for partitions \( n, i \) and \( j \) (\( n \neq 0 \)), note that the degree of the coefficient of any eigenvalue \( \alpha \) occurring in the expression of \( \text{Coefficient}_{r} \cdot A_{n,i,j}(T, q) \) is at most \( |\alpha| + |n| + |i| + |j| - m_\alpha \).

We can now prove the following.

**Theorem 9.1.** — For \( \kappa = (k_1, \ldots, k_r) \in \mathbb{N}^r \), the coefficient of \( t_1^{k_1} \cdots t_r^{k_r} \) in \( E(T)^{\kappa} A_1(T, q) \) can be expressed as

\[
\sum_{\alpha} R_{\alpha}^{(\kappa,i)}(q) \alpha
\]

the sum being over the distinct (normalized) eigenvalues of Frobenius acting on \( \left( \mathcal{H}_{\alpha}^{\ast}, \mathcal{H}_{\alpha}^{\ast} \right) \otimes \mathbb{F}_q, \mathcal{V}(\lambda) \right)_{g, \lambda, \mu} \) with \( g \leq \left[ (l-1)/2 \right] \), and \( R_{\alpha}^{(\kappa,i)}(x) \in x\mathbb{Q}[x] \). Moreover, we have \( \deg R_{\alpha}^{(\kappa,i)}(x) + m_\alpha \leq |\kappa| + l \).

**Proof.** By (29), it suffices to investigate the coefficients of \( \mathcal{E}(\lambda) \cdot A_{\mu, \nu}(T, q) \) for \( \mu \neq (1) \) and \( \nu \neq (1) \). We recall that \( \mu = (\mu_1 \geq \cdots \geq \mu_n \geq 1) \) and \( \nu = (\nu_1, \ldots, \nu_n) \) are such that \( \nu_1 \mu_1 + \cdots + \nu_n \mu_n = l \). By the induction assumptions (see (92), (93) and (94)), the coefficients of \( q^{m_\mu} \cdot \mathcal{E}_{\nu}(q^{m_\nu} T^{n_\nu}, 1/q^{m_\nu}) \) can be expressed as

\[
q^{\mu_j (|\kappa| + \nu_j - m_\alpha)} P_{\kappa, \nu_j}^{(\kappa, \nu_j)}(1/q^{m_\nu}) \alpha^{\mu_j} \quad (\kappa \in \mathbb{N}^r)
\]

where \( P_{\alpha}^{(\kappa, \nu_j)}(x) := \sum_{\alpha} q^{m_\mu (|\kappa| + \nu_j - m_\alpha)} P_{\kappa, \nu_j}^{(\kappa, \nu_j)}(1/q^{m_\nu}) \alpha^{\mu_j} \) for all \( \alpha \) and for \( 1 \leq j \leq n \); then, clearly, \( \deg P_{\alpha}^{(\kappa, \nu_j)}(x) \leq (|\kappa| + \nu_j - m_\alpha - 2)/2 \), and so

\[
\deg Q_{\alpha}^{(\kappa, \nu_j)} + m_\alpha \leq (|\kappa| + \nu_j + m_\alpha - 2)/2 \leq |\kappa| + \nu_j - 1.
\]

It follows that for \( \kappa_1 \in \mathbb{N}^r \), the coefficients of the \( q \)-Weil numbers \( \alpha_1 \) occurring in \( \text{Coefficient}_{r} \cdot A_{\mu, \nu}(T, q) \), with \( \Lambda_{\mu, \nu}(T, q) \) defined by (31), are polynomials of degree at most \( |\kappa_1| + l - |\mu| - m_{\alpha_1} \) with rational coefficients.

To handle the other part of \( \mathcal{E}(\lambda) \cdot A_{\mu, \nu}(T, q) \) in (30), note that the sum

\[
\sum_{d_0 \in \mathcal{D}_\mu(d, \delta)} \left( N_{\mu, \nu}(d_0, \delta) \prod_{i=1}^{r} P_{C_{d_0}}(t_i) \right)
\]

is (up to a constant) just \( A_{n,i,j}(T, q) \) with \( n = (\mu_1 \geq \cdots \geq \mu_n \geq 1) \), and \( i, j \) determined by splitting \( (\mu_j)_{j \in J_1} \) (depending on \( \delta \)) using Proposition 4.3. As \( |n| + |i| + |j| = |\mu| \), the coefficient of any eigenvalue \( \alpha_2 \) in \( \text{Coefficient}_{r+2} \cdot A_{n,i,j}(T, q) \) \( (\alpha_2 \in \mathbb{N}^r) \) is a polynomial of degree at most \( |\kappa_2| + |\mu| - m_{\alpha_2} \) with rational coefficients. Our assertions follow now for contributions to \( \mathcal{E}(\lambda) \cdot A_{\mu, \nu}(T, q) \) given by (30).

To complete the proof, note that, for \( A_{\mu, \nu}(T, q) \) given by Proposition 4.1, the same estimates apply. \( \square \)

We can further assume that the sum in the theorem is reduced in the sense that \( \alpha' = q^n \alpha \) for some \( n \in \mathbb{N} \) if and only if \( \alpha' = \alpha \). Since

\[
\sum_{\alpha} R_{\alpha}^{(\kappa, l)}(q^n \alpha) = -\sum_{\alpha} q^{n(|\kappa| + |l| - m_\alpha + 1)} R_{\alpha}^{(\kappa, l)}(1/q^n \alpha)
\]

(obtained from (53) applied over a finite extension \( \mathbb{F}_{q^n} \) of \( \mathbb{F}_q \)), it follows that

\[
R_{\alpha}^{(\kappa, l)}(x) = -x^{(|\kappa| + |l| - m_\alpha + 1)} R_{\alpha}^{(\kappa, l)}(1/x)
\]

(for all \( \alpha \)).

Using this combined with (96), one defines \( \lambda(\kappa, l; q) \) by cutting off the coefficient of each \( q \)-Weil algebraic integer in the expression of (96) accordingly.
10 An application

For $0 \leq i \leq 6$, let $A_2(w^i)$ denote the quotient stack $A_{2,2}/S_{6-i}$, where $S_{6-i}$ is the subgroup of $S_6$ fixing $\{1, \ldots, i\}$ pointwise. Here we identify $\text{GSp}_4(\mathbb{Z}/2)$ with $S_6$, as in [8, Section 2], under the isomorphism defined by an embedding of the stack $\mathcal{H}_4[2]$ (denoted by $M_2(\mu^{36})$ in [8]) into $A_{2,2}$. Take $\nu = 1$, and consider the Euler characteristic

$$
eu.{A}(A_2(w^i) \otimes \overline{\mathbb{F}_q}, \mathcal{V}(\lambda)) = \sum_{j=0}^{6} (-1)^j \left[ H^i_{\text{Eis}}(A_2(w^j) \otimes \overline{\mathbb{F}_q}, \mathcal{V}(\lambda)) \right]$$

where $\lambda = (\lambda_1 \geq \lambda_2 \geq 0)$ with $|\lambda| = \lambda_1 + \lambda_2 \equiv 0 \pmod{2}$.

In what follows, we shall be interested in computing the trace of Frobenius

$$\text{Tr}(F^*| e_{\text{Eis}}(A_2(w^1) \otimes \overline{\mathbb{F}_q}, \mathcal{V}(\lambda))) = \text{Tr}(F^*| e_{\text{Eis}}(A_2(w^3) \otimes \overline{\mathbb{F}_q}, \mathcal{V}(\lambda)))$$

see also [49] and [8]. To do so, we first recall (see Theorem 6.4) that

$$\frac{1}{q(q-1)} \sum_{d_0 - \text{monic \\ & square-free}} \left( \prod_{k=1}^{r} P_{C_{d_0}}(t_k) \right) = (t_1 \cdots t_r)^2 \sum_{\lambda \in \mathbb{Z}^2} \sum_{|\lambda| \text{ even}} \text{Tr}(F^*| e_{\text{c}}(\mathcal{H}_2(w^1) \otimes \overline{\mathbb{F}_q})) s_{\lambda(1)}(q^{\frac{1}{2}}T^{e_1}) q^{\frac{e_1 - 1}{2}}.$$

Put

$$A_5^{(0)}(T, q) := \sum_{d_0 - \text{monic \\ & square-free}} \left( \prod_{k=1}^{r} P_{C_{d_0}}(t_k) \right)$$

and note that we can express

$$A_5^{(0)}(T, q) = \frac{q(q-1)}{|\text{GL}_2(\mathbb{F}_q)|} \sum_{\nu \in \{1^{m_1}, \ldots, 6^{m_6}\}} \nu_1 \cdot \sum_{d_0 \in \mathcal{P}_2(\nu, \mathbb{F}_q)} \left( \prod_{k=1}^{r} P_{C_{d_0}}(t_k) \right).$$

For each partition $\nu = (1^{m_1}, \ldots, 6^{m_6})$, the (normalized) inner sum

$$\sum_{d_0 \in \mathcal{P}_2(\nu, \mathbb{F}_q)} \left( \prod_{k=1}^{r} P_{C_{d_0}}(t_k) \right)$$

(102)

can be thought of as the contribution corresponding to Jacobians of smooth projective irreducible algebraic curves of genus two in the moduli of principally polarized abelian surfaces. As we are interested in the Euler characteristics of $A_2(w^i)$, to (102) we add the remaining contribution

$$\frac{1}{2} \frac{1}{|\text{G}(\mathbb{F}_q)|^2} \sum_{\nu = \sigma \cup \tau} \sum_{f \in \mathcal{P}_1(\sigma, \mathbb{F}_q)} \sum_{h \in \mathcal{P}_1(\tau, \mathbb{F}_q)} \left( \prod_{k=1}^{h} P_{E_f}(t_k) P_{E_{h}}(t_k) \right) + \frac{1}{2} \frac{1}{|\text{G}(\mathbb{F}_q)|} \sum_{f \in \mathcal{P}_1(\mu, \mathbb{F}_q)} \left( \prod_{k=1}^{h} P_{E_f}(t_k) \right)$$

(103)

corresponding to pairs of elliptic curves joined at the origin. Here the first sum is over all ordered choices of partitions $\sigma$ and $\tau$ of $3$ such that $\nu = \sigma \cup \tau$, and $P_1(\mu, \mathbb{F}_q)$ (for a partition $\mu$ of $3$ and a finite field $\mathbb{F}_q$ of odd characteristic) denotes the subset of $\mathbb{F}_q[x]$ consisting of all square-free cubic polynomials with factorization type $\mu$: each element $f \in P_1(\mu, \mathbb{F}_q)$ defines an elliptic curve $E_f$ given by $y^2 = f(x)$ with $x = \infty$ as origin. The normalizing constant

$$|\text{G}(\mathbb{F}_q)| = q(q-1)^2$$

represents the number of $\mathbb{F}_q$-isomorphisms among the elliptic curves $E_f$ ($f \in P_1(\mu, \mathbb{F}_q$)). The second contribution in (103), corresponding to elliptic curves defined over $\overline{\mathbb{F}_q}$, each curve being joined at the origin with its Frobenius

\footnote{We are taking $i = 1$ for simplicity.}
corresponding to the partitions occurring only if \( \nu \) is of the form \((2^r, 4^s, 6^t)\), in which case we set \( \nu^{1/2} := (1^{r}, 2^s, 3^t)\) (see [8, Section 5]). Accordingly, if \( A_{(2,1)}(T, q) \) and \( A_{(3,3)}(T, q) \) are defined by (58), the generating function

\[
(t_1 \cdots t_r)^2 \sum_{\lambda \in \{x, y\}} \text{Tr}(F^x e_c(A_2(w^1) \otimes \overline{\Upsilon}(\lambda))) s_{\lambda'}(q^{\frac{1}{2}} T^{\pm 1}) q^{r - |\lambda|}
\]

is just

\[
A_3^{(0)}(T, q) + A_3^{(1)}(T, q) (A_{(2,1)}(T, q) + 3A_{(3,3)}(T, q))
\]

with

\[
A_3^{(0)}(T, q) = \sum_{\text{f} - \text{monic} \& \text{square-free}} \left( \prod_{k=1}^{r} P_{E_k}(t_k) \right).
\]

We recall now that the full Eisenstein cohomology is the difference between the compactly supported and the usual cohomology, and the corresponding Euler characteristic is

\[
e_{\text{full Eis}}(A_2(w^1), \mathcal{V}(\lambda)) = e_c(A_2(w^1), \mathcal{V}(\lambda)) - e(A_2(w^1), \mathcal{V}(\lambda)).
\]

Letting

\[
A_3^{(2)}(T, q) = A_{(2,1)}(T, q) + 3A_{(3,3)}(T, q)
\]

it follows that the generating function (104) corresponding to \( e_{\text{full Eis}}(A_2(w^1), \mathcal{V}(\lambda)) \) equals

\[
A_3^{(0)}(T, q) q(q-1) + A_3^{(0)}(T, q) A_3^{(2)}(T, q) \frac{q^2(q-1)^2}{q(q-1)} - \frac{q^4 A_5^{(0)}(q T, 1/q)}{q-1} - \frac{q^5 A_3^{(0)}(q T, 1/q) A_3^{(2)}(q T, 1/q)}{(q-1)^2}.
\]

On the other hand, the generating series \( A_5(T, q) \) introduced at the beginning of Section 4 can be decomposed as

\[
A_5(T, q) = A_5^{(0)}(T, q) + \text{degenerate part}
\]

where the degenerate part consists of the following contributions:

\[
\frac{q^2(q A_3^{(0)}(T, q) - A_3^{(2)}(T, q) + A_3^{(1)}(T, q)) A_2(q T, 1/q)}{E(T)} + \frac{q^2(q A_3^{(0)}(-T, q) - A_3^{(2)}(-T, q) + A_3^{(1)}(-T, q)) A_2(-q T, 1/q)}{E(-T)}
\]

(106)

corresponding to the partitions \( 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 2 \cdot 1, 1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1, 1 \cdot 3 + 2 \cdot 1, \)

\[
q^3 A_3^{(2)}(T, q) A_5(q T, 1/q)
\]

(107)

corresponding to \( 1 \cdot 1 + 1 \cdot 1 + 3 \cdot 1 \) and \( 1 \cdot 2 + 3 \cdot 1, \)

\[
\frac{q^5(q-1)^2 A_2(q^2 T^2, 1/q^2)}{E(T^2)} + \frac{q^5(q^2-1) A_2(-q^2 T^2, 1/q^2)}{E(-T^2)}
\]

(108)

corresponding to \( 2 \cdot 2 + 1 \cdot 1, \)

\[
\frac{q^5(q-1)^2 A_2(q T, 1/q) A_2(-q T, 1/q)}{E(T) E(-T)} + \frac{q^5(q-1)(q-3)}{8} \left( \frac{A_2(q T, 1/q)^2}{E(T)^2} + \frac{A_2(-q T, 1/q)^2}{E(-T)^2} \right)
\]

(109)

70
corresponding to $2 \cdot 1 + 2 \cdot 1 + 1 \cdot 1$,  
\[
\frac{q^6(q-1)}{2} \Lambda_3(qT, 1/q) \left( \frac{\Lambda_2(qT, 1/q)}{E(T)} + \frac{\Lambda_2(-qT, 1/q)}{E(-T)} \right)
\]  
(110)
corresponding to $3 \cdot 1 + 2 \cdot 1$, and finally  
\[
\frac{q^5(q-1)}{2} \left( \frac{\Lambda_4(qT, 1/q)}{E(T)} + \frac{\Lambda_4(-qT, 1/q)}{E(-T)} \right)
\]  
(111)
corresponding to $4 \cdot 1 + 1 \cdot 1$. We recall that  
\[
A_5(T, q) = -q^6 A_5(qT, 1/q).
\]
Dividing both sides by $q^2 - q$, and using the trivial identity  
\[
q^2 - q = -q^3 \left( \frac{1}{q^2} - \frac{1}{q} \right)
\]
we see that  
\[
\frac{A_5(T, q)}{q^2 - q} - q^3 A_5(qT, 1/q) = 0.
\]  
(112)
Notice that the results of Section 8 (specifically, see (91)) give, in particular, a formula for the trace of Frobenius on $e_{w}^{\lambda}(\mathcal{H}_q(w^1) \otimes \overline{\mathbb{F}}_p)$ ($g \geq 2$); when $g = 2$, the corresponding formula can be made explicit. In what follows, we shall illustrate this in the case of $A_2(w^1)$, thus recovering the formula in [8, Corollary 4.6], and proved in [49, Theorem 9.1].

**Theorem (Bergström-Faber-van der Geer).** — For every $\lambda = (\lambda_1 \geq \lambda_2 \geq 0)$, the trace of Frobenius $F^*$ on $e_{w}^{\lambda}(\mathcal{A}_2(w^1) \otimes \overline{\mathbb{F}}_p, \mathcal{V}(\lambda))$ is given by  
\[
\left[ \frac{\lambda_1 - \lambda_2 - 1}{4} \right] - \left[ \frac{\lambda_1 + \lambda_2 + 1}{4} \right] \ q^{\lambda_2 + 1} + \begin{cases} 
2 + 2T_{\lambda_2 + 2}(q) & \text{if } \lambda_2 \text{ is even}
\end{cases}
\]  
and  
\[
\left[ \frac{\lambda_1 - \lambda_2 - 1}{4} \right] - \left[ \frac{\lambda_1 + \lambda_2 + 1}{4} \right] \ q^{\lambda_2 + 1} + \begin{cases} 
-2T_{\lambda_2 + 3}(q) & \text{if } \lambda_2 \text{ is odd}.
\end{cases}
\]

**Proof.** First, express the left-hand side of (112) using (106)–(111), and then apply (27) to transform each $\Lambda_l(\pm q^m T^m, q^{-m})$, for $2 \leq l \leq 4$, into $\Lambda_l(\pm q^m T^m, q^{-m})$; we also write the corresponding contribution $A_l(\pm q^m T^m, q^{-m})$ explicitly by recalling that $E(T) A_2(T, q) = q^2 - q$, that is,  
\[
A_3(T, q) = A_3^{(0)}(T, q) + \frac{q^3(q-1)}{2} \left( \frac{\Lambda_2(qT, 1/q)}{E(T)} + \frac{\Lambda_2(-qT, 1/q)}{E(-T)} \right)
\]
and, by (38)–(42),  
\[
E(T) A_4(T, q) = qA_3^{(0)}(T, q) + A_3^{(1)}(T, q) + \frac{q^5(q-1)}{2} E(qT)\frac{\Lambda_2(q^2T^2, 1/q^2)}{E(-T)}
\]
\[
+ q^3(q-1) \left[ \frac{q^2 E(qT)\Lambda_2(qT, 1/q)^2}{2E(T)} + \frac{(q-2)\Lambda_2(qT, 1/q)}{2E(T)} + \frac{q\Lambda_2(-qT, 1/q)}{2E(-T)} + q\Lambda_3(qT, 1/q) \right]
\]
with $A_3^{(1)}(T, q)$ defined as in the proof of Proposition 5.2. To simplify things somewhat, we can use the identities:  
\[
A_3(-qT, 1/q) = \Lambda_3(qT, 1/q) \quad A_3^{(0)}(-T, q) = A_3^{(0)}(T, q) \quad \text{and} \quad A_3^{(2)}(-T, q) = A_3^{(2)}(T, q).
\]
In the expression obtained, the coefficient of every monomial  
\[
A_4(\pm q^m T^m, q^{-m}) K_1 \cdots K_n A_5(\pm q^m T^m, q^{-m}) K_n \quad (k_1, \ldots, k_n \geq 0)
\]
\
\[71\]
has to vanish. By taking the coefficient of \( \Lambda_3(q T, 1/q) \) we find, in particular, that
\[
A_3^{(2)}(T, q) + q^4 A_3^{(2)}(q T, 1/q) = q(q - 1)^2 \left( \frac{1}{E(T) E(q T)} + \frac{1}{E(-T) E(-q T)} \right).
\]

We also need (43) together with the identity
\[
A_3^{(0)}(T, q) + q^4 A_3^{(0)}(q T, 1/q) = \frac{q(q - 1)^2}{2} \left( \frac{1}{E(T) E(q T)} + \frac{1}{E(-T) E(-q T)} \right)
\]
which follows easily by combining the above expression of \( A_3(T, q) \) with the functional equations (28) and (27), applied for \( l = 3 \) and \( l = 2 \), respectively.

Now subtract the constant term of the expression obtained for the left-hand side of (112) (i.e., all the terms free of factors \( \Lambda_l(\pm q^m T^m, q^{-m}) \)) with \( 2 \leq l \leq 4 \), from (105). By using the last two identities and (43), we can express each \( A_3^{(j)}(q T, 1/q) \) \((j = 0, 1, 2)\) in terms of \( A_3^{(j)}(T, q) \), giving the following alternative expression for the generating function of the trace of Frobenius on \( \text{c}_{\text{nil}, \text{Eu}}(A_2(w^1) \otimes \mathbb{F}_q, \mathcal{V}(\lambda)) \):
\[
\frac{(q + 1) A_3^{(0)}(T, q) + A_3^{(1)}(T, q)}{q E(T) E(q T)} + \frac{(q + 1) A_3^{(0)}(-T, q) + A_3^{(1)}(-T, q)}{q E(-T) E(-q T)}
- \frac{(q - 1)^3}{8} \left( \frac{1}{E(T)^2 E(q T)^2} + \frac{1}{E(-T)^2 E(-q T)^2} \right) + \frac{1}{4} \frac{(q - 1)(q + 1)^2}{E(-T)^2 E(-q T)^2} \cdot (q + 1)(q - 1)^2
\]

Note that
\[
(q + 1) A_3^{(0)}(T, q) + A_3^{(1)}(T, q) = \sum_{\deg f = 3 \atop f \text{monic & square-free}} \left( (q + 1 - a(E_f)) \cdot \prod_{k=1}^r P_{E_f}(t_k) \right).
\]

Expressing \( P_{E_f}(t_k) \) for all \( f \) as
\[
P_{E_f}(t_k) = 1 - a(E_f) t_k + q t_k^2 = (1 - \sqrt{q} \alpha_{E_f} t_k)(1 - \sqrt{q} \bar{\alpha}_{E_f} t_k) \quad \text{with } |\alpha_{E_f}| = 1
\]
it follows from Lemma 6.1 that we can write
\[
\frac{(q + 1) A_3^{(0)}(T, q) + A_3^{(1)}(T, q)}{E(T) E(q T)}
= \sum_f \left[ (q + 1 - a(E_f)) \cdot \prod_{k=1}^r (1 - \sqrt{q} \alpha_{E_f} t_k)(1 - \sqrt{q} \bar{\alpha}_{E_f} t_k)(1 - t_k)(1 - q t_k) \right]
= q^r (t_1 \cdots t_r)^2 \sum_{\lambda \subseteq (r^2)} (-1)^{|\lambda|} s_{(\lambda)}(\lambda^1_{1/2} q^{1/2} t_1^{1/2}, \ldots, q^{1/2} t_r^{1/2}) \cdot \sum_{\lambda \subseteq (r^2)} (-1)^{|\lambda|} s_{(\lambda)}(\lambda^1_{1/2} q^{1/2} t_1^{1/2}, \ldots, q^{1/2} t_r^{1/2})
\]

By applying the Weyl character formula, for each \( \lambda \subseteq (r^2) \),
\[
s_{(\lambda)}(\alpha_{E_f}^{1/2}, q^{1/2}) = \frac{[\alpha_{E_f}^{1/2} - \alpha_{E_f}^{1/2}]_{q^{1/2}}}{q^{1/2} - q^{-1/2}} \cdot \frac{[\alpha_{E_f}^{1/2} - \alpha_{E_f}^{1/2}]_{q^{-1/2}}}{q^{1/2} - q^{-1/2}}
= -\sqrt{q} s_{(\lambda_1)}(\alpha_{E_f}^{1/2}) s_{(\lambda_2)}(\alpha_{E_f}^{1/2}) - s_{(\lambda_1 + 1)}(\alpha_{E_f}^{1/2}) s_{(\lambda_2 + 1)}(\alpha_{E_f}^{1/2})
\]

72
Substituting this into (113), summing over \( f \) and applying Birch-Ihara identity, one sees that

\[
(q + 1)A^0_q(T, q) + A^1_q(T, q) = \frac{qE(T)E(qT)}{qE(T)E(qT)} \quad \text{and} \quad (q + 1)A^0_q(-T, q) + A^1_q(-T, q) = \frac{qE(-T)E(-qT)}{qE(-T)E(-qT)}
\]

\[
= 2(t_1 \cdots t_r)^2 \sum_{\lambda \in (r^2) \atop \lambda_1, \lambda_2 \text{odd}} (1 + T_{\lambda_1+3}(q)) q^{r_+ \lambda_1} \left(q^{\frac{\lambda_2+1}{2}} - q^{-\frac{\lambda_2+1}{2}}\right)s_{(\lambda_1)}(q^{\frac{r_+}{2}T^{\lambda_1}})
\]

\[
+ 2(t_1 \cdots t_r)^2 \sum_{\lambda \in (r^2) \atop \lambda_1, \lambda_2 \text{even}} (-1 - T_{\lambda_2+2}(q)) q^{r_+ \lambda_1} \left(q^{\frac{\lambda_2+1}{2}} - q^{-\frac{\lambda_2+1}{2}}\right)s_{(\lambda_1)}(q^{\frac{r_+}{2}T^{\lambda_1}}).
\]

(114)

On the other hand, we can write

\[
- \frac{(q - 1)^3}{8} \left( \frac{1}{(E(T)^2E(qT)^2)} + \frac{1}{E(-T)^2E(-qT)^2} \right) + \frac{1}{4} \frac{(q - 1)(q + 1)^2}{E(-T^2)E(-q^2T^2)} - \frac{(q + 1)(q - 1)^2}{E(T)^2E(qT)^2}
\]

\[
= q^r(t_1 \cdots t_r)^2 \sum_{\lambda \in (r^2) \atop |\lambda| \text{even}} \left[ - \frac{1}{4}(q - 1)^3 s_{(\lambda)}(q^{\frac{r_+}{2}}, q^{\frac{r_+}{2}}) + \frac{1}{4}(q - 1)(q + 1)^2 s_{(\lambda)}(iq^{\frac{r_+}{2}}, -iq^{\frac{r_+}{2}}) \right.
\]

\[
- (q + 1)(q - 1)^2 s_{(\lambda)}(q^{\frac{r_+}{2}}, -q^{\frac{r_+}{2}}) \left] s_{(\lambda)}(q^{\frac{r_+}{2}}T^{\lambda_1}).
\]

(115)

Using again the Weyl character formula, we have

\[
s_{(\lambda)}(q^{\frac{r_+}{2}}, q^{\frac{r_+}{2}}) = \frac{(\lambda + 2)(q^{2r_+\lambda_1} + 1)(q^{1+\lambda_2} - 1) - (\lambda_2 + 1)(q^{2r_+\lambda_1} - 1)(q^{1+\lambda_2} + 1)}{(q - 1)^3} q^{\frac{|\lambda|}{2}}
\]

\[
s_{(\lambda)}(iq^{\frac{r_+}{2}}, -iq^{\frac{r_+}{2}}) = \frac{(-1)^{\lambda_1 - \lambda_2}/2 \left(q^{2r_+\lambda_1} - (-1)^{\lambda_1}(q^{1+\lambda_2} + (-1)^{\lambda_2})\right)}{(q^2 - 1)(q + 1)} q^{-\frac{|\lambda|}{2}}
\]

and

\[
s_{(\lambda)}(q^{\frac{r_+}{2}}, -q^{\frac{r_+}{2}}) = \frac{(-1)^{\lambda_1}(q^{2r_+\lambda_1} - 1)(q^{1+\lambda_2} - 1)}{(q^2 - 1)(q - 1)} q^{-\frac{|\lambda|}{2}}.
\]

Fix \( \lambda \subseteq (r^2) \) of even weight. By adding the coefficients of \( q^{r_+\lambda_1 - \frac{|\lambda|}{2}} \) \((t_1 \cdots t_r)^2 s_{(\lambda)}(q^{\frac{r_+}{2}}T^{\lambda_1})\) in (114) and (115), we obtain the expression

\[
\frac{\lambda_1 - \lambda_2 - 1}{4} - \left[ \frac{\lambda_1 - \lambda_2 + 1}{4} \right] q^{\lambda_1 + 1} + \begin{cases} 2 + 2T_{\lambda_2+2}(q) & \text{if } \lambda_2 \text{ is even} \\ -2T_{\lambda_1+3}(q) & \text{if } \lambda_2 \text{ is odd} \end{cases}
\]

\[
- \left( \frac{\lambda_1 - \lambda_2 - 1}{4} - \left[ \frac{\lambda_1 + \lambda_2 + 1}{4} \right] q^{-\lambda_2 - 1} + \begin{cases} 2 + 2T_{\lambda_2+2}(1/q) & \text{if } \lambda_2 \text{ is even} \\ -2T_{\lambda_1+3}(1/q) & \text{if } \lambda_2 \text{ is odd} \end{cases} \right) q^{3+|\lambda|}.
\]

We recall that, by convention, we take \( T_2(q) = -q - 1 \).

The full Eisenstein cohomology is anti-invariant under Poincaré duality, and is determined by the Eisenstein cohomology by antisymmetrizing. By [77, Theorem 3.5], the terms in the above expression that come from the compactly supported Eisenstein cohomology are precisely those of weights < \(|\lambda|+3\) which completes the proof of the theorem.
A Appendix

In this appendix, we give, for each $r \leq 3$, the explicit expression of the power series $Z(T,t_{r+1};q)$ in Section 2 as a rational function. One way to obtain these expressions is by using the averaging technique in [26]. In fact, the expression of (11) as a rational function is given in [26, Example 3.6] when $r = 1$, and [26, Example 3.7] when $r = 2$. Accordingly, we have

$$Z(t_1, t_2; q) = \frac{1 - q^2 t_1 t_2}{(1 - q t_1)(1 - q t_2)(1 - q^3 t_1^2 t_2^2)}$$

when $r = 1$, and

$$Z(t_1, t_2, t_3; q) = \frac{1 - q^2 t_1 t_3 - q^2 t_2 t_3 + q^3 t_1 t_2 t_3 + q^3 t_1 t_2 t_3^2 - q^4 t_1^2 t_2 t_3^2 - q^4 t_1^2 t_2^2 t_3^2 + q^6 t_1^2 t_2^2 t_3^2}{(1 - q t_1)(1 - q t_2)(1 - q t_3)(1 - q^3 t_1^2 t_2^2 t_3^2)}$$

when $r = 2$. For $r = 3$, we used instead the recurrence relations in the proof of Theorem 3.7 in [18]. If we put

$$Z(t_1, t_2, t_3, t_4; q) = \frac{N(t_1, t_2, t_3, t_4; q)}{D(t_1, t_2, t_3, t_4; q)}$$

then by using Mathematica, one finds:

$$N(t_1, t_2, t_3, t_4; q) = 1 - q^2 t_1 t_4 - q^2 t_2 t_4 + q^3 t_1 t_2 t_4 - q^2 t_3 t_4 + q^3 t_1 t_2 t_3 t_4 - q^4 t_1 t_2 t_3 t_4 + q^5 t_1^2 t_2^2 t_3 t_4$$

and

$$D(t_1, t_2, t_3, t_4; q) = q^4 t_1 t_2 t_3 - q^4 t_1^2 t_2^2 t_3^2 - q^6 t_1^2 t_2^2 t_3^2 + q^3 t_1^3 t_2^2 t_3^2 - 2q^4 t_1 t_2 t_3 t_4 - q^5 t_1^2 t_2^2 t_3 t_4$$

The corresponding denominator is easily seen to be

$$D(t_1, t_2, t_3, t_4; q) = (1 - q t_1)(1 - q t_2)(1 - q t_3)(1 - q t_4)(1 - q^3 t_1^2 t_2^2 t_3 t_4)(1 - q^3 t_1^2 t_2^2 t_3 t_4)$$

A simple comparison with the corresponding (transformed) rational function (11) shows in each case that our conditions
(i) – (iii) (see Section 3) are satisfied.

74
We conclude this appendix by remarking that Ian Whitehead established in his doctoral thesis that the unique multiple Dirichlet series associated to the fourth moment of quadratic Dirichlet L-series over \( \mathbb{F}_q(x) \) satisfying the conditions (i) – (iii) is given by

\[
Z_4(t_1, \ldots, t_5; q) = \prod_{j=0}^{\infty} \left( 1 - q^{6j+6} \left( t_1 t_2 t_3 t_4 t_5 \right)^{2j+2} \right)^{-1} \cdot Z(t_1, \ldots, t_5; q) \tag{116}
\]

where \( Z(t_1, \ldots, t_5; q) \) is such that its residue at \( t_5 = 1/q \) corresponds to [18, eqn. (4.29)] by setting \( t_j = q^{-s_j} \), for \( j = 1, \ldots, 4 \); the series \( Z_4(t_1, \ldots, t_5; q) \) can be made quite explicit by expressing it in terms of the average [18, eqn. (4.16)]. There is also a number field version of this multiple Dirichlet series (obtained from (116) as explained in the introduction). The series (116) (or its number field counterpart) is precisely the multiple Dirichlet series that we expect to occur in a Fourier-Whittaker coefficient of an Eisenstein series on the double cover of a loop group (with the corresponding affine Weyl group isomorphic to \( W_\delta \)).

B Appendix

Throughout this appendix, \( \mathbb{F} \) will be a fixed finite field of odd characteristic. For partitions \( \mu = (1^{\mu_1}, 2^{\mu_2}, \ldots, n^{\mu_n}) \) and \( \gamma = (1^{\gamma_1}, 2^{\gamma_2}, \ldots, m^{\gamma_m}) \), we shall denote \( \mu' = (1^{\mu_1+1}, 2^{\mu_2}, \ldots, n^{\mu_n}) \) and \( \gamma' = (1^{\gamma_1+1}, 2^{\gamma_2}, \ldots, m^{\gamma_m}) \). Let \( M_{\mu, \gamma}(q) \) (for \( \mu \) and \( \gamma \) as above) be defined as in Section 3.

We have the following

**Proposition B.1.** — For \( |\mu| = 2g + 1 \), we have

\[
\frac{\mu_1 + 1}{q + 1} \left( M_{\mu, \gamma}(q) + M_{\mu', \gamma}(q) \right) = M_{\mu, \gamma}(q).
\]

**Proof.** The identity we need to prove is equivalent to

\[
(\mu_1 + 1)M_{\mu', \gamma}(q) = (q - \mu_1)M_{\mu, \gamma}(q). \tag{117}
\]

For any partition \( \nu = (1^{\nu_1}, 2^{\nu_2}, \ldots) \), let \( \mathcal{P}(\nu) \subset \mathbb{F}[x] \) denote the set of all monic square-free polynomials with factorization type \( \nu \). For each \( d \in \mathcal{P}(\mu) \) and \( \alpha \in \mathbb{F} \) such that \( d(\alpha) \neq 0 \), let \( d_\alpha \in \mathcal{P}(\mu') \) be defined by

\[
d_\alpha(x) = \frac{(x - \alpha)^{2g+2} d(\alpha)}{d(\alpha) (\alpha + 1) \left( x - \alpha \right)}. \]

Notice that for each \( d \) there are exactly \( q - \mu_1 \) choices of \( \alpha \), and for each such choice, we must have \( d_\alpha(\alpha) = 0 \). Conversely, let \( \tilde{d} \in \mathcal{P}(\mu') \), and let \( \alpha \in \mathbb{F} \) be a root of \( \tilde{d} \). (The polynomial \( \tilde{d} \) has exactly \( \mu_1 + 1 \) distinct roots in \( \mathbb{F}_q \)) If we define

\[
d(x) = \frac{(x - \alpha)^{2g+2} \tilde{d}(\alpha)}{\tilde{d}(\alpha) (\alpha + 1) \left( x - \alpha \right)},
\]

then \( d \in \mathcal{P}(\mu) \), \( d(\alpha) \neq 0 \) and \( \tilde{d} = d_\alpha \), where \( d_\alpha \) is defined as above.

Since the hyperelliptic curves \( C_d \) and \( C_{d_\alpha} \) corresponding to \( d \in \mathcal{P}(\mu) \) and \( d_\alpha \) are isomorphic, the identity (117) follows by a simple counting argument. \( \square \)

We remark that both sides of the identity in the above proposition vanish if the partition \( \gamma \) has odd weight. When \( \gamma = (1^{\gamma_1}, \ldots, g^{\gamma_g}) \) ( \( g \geq 2 \) ) has even weight, we can express the identity in the equivalent form

\[
(\mu_1 + 1)\tilde{M}_{\mu', \gamma}(q) = \frac{1}{q(q - 1)} M_{\mu, \gamma}(q)
\]

where \( \tilde{M}_{\mu', \gamma}(q) \) is the moment-sum introduced at the beginning of Section 6.
Proposition B.2. — For $|\mu| = 2g + 1$ and $|\gamma|$ odd, we have the identity

$$M_{\mu',\gamma'}(q) + M_{\mu,\gamma}(q) = -(q + 1)M_{\mu',\gamma}(q).$$

Moreover, if $\mu = (2^{\mu_2}, 3^{\mu_3}, \ldots)$ is a partition of weight $2g + 2$, and $|\gamma|$ is odd, then

$$M_{\mu,\gamma}(q) = -(q + 1)M_{\mu',\gamma}(q).$$

Proof. We first write

$$M_{\mu',\gamma'}(q) = -M_{\mu',\gamma}(q) - \sum_{\theta \in \mathbb{F}} \sum_{d \in \mathcal{P}(\mu')} \chi(d(\theta)) \left( \prod_{j=1}^{m} a_j(C_d)^{\gamma_j} \right)$$

and let $\mathcal{P}_0(\mu')$ (respectively $\mathcal{P}^0(\mu')$) denote the set of polynomials in $\mathcal{P}(\mu')$ which do not vanish (respectively vanish) at 0. Replacing $d(x)$ by $d(x - \theta)$ (for each $\theta \in \mathbb{F}$), we see that

$$M_{\mu',\gamma'}(q) = -M_{\mu',\gamma}(q) - q \sum_{d \in \mathcal{P}_0(\mu')} \chi(d(0)) \left( \prod_{j=1}^{m} a_j(C_d)^{\gamma} \right).$$

Since $|\gamma|$ is odd, we can write

$$\chi(d(0)) = \chi(d(0))^{|\gamma|} = \prod_{j=1}^{m} a_j(d(0))^{\gamma_j} \quad \text{(for } d \in \mathcal{P}_0(\mu')).$$

Moreover, one can express $\chi_j(d(0))a_j(C_d)$, for $d \in \mathcal{P}_0(\mu')$ and $j = 1, \ldots, m$, as

$$-1 - \chi_j \left( \frac{1}{d(0)} \right) - \sum_{\theta \in \mathbb{F}} \chi_j \left( \theta^{2q+2} \frac{d(1/\theta)}{d(0)} \right).$$

By using the transformation $d(x) \mapsto \frac{x^{2q+2}}{d(0)} d(1/x)$ on $\mathcal{P}_0(\mu')$, it follows that

$$\sum_{d \in \mathcal{P}_0(\mu')} \chi(d(0)) \left( \prod_{j=1}^{m} a_j(C_d)^{\gamma_j} \right) = \sum_{d \in \mathcal{P}^0(\mu')} \prod_{j=1}^{m} a_j(C_d)^{\gamma_j}$$

and hence

$$M_{\mu',\gamma'}(q) = -(q + 1)M_{\mu',\gamma}(q) + q \sum_{d \in \mathcal{P}^0(\mu')} \prod_{j=1}^{m} a_j(C_d)^{\gamma_j}.$$  \hspace{1cm} (118)

Similarly,

$$M_{\mu,\gamma}(q) = -\sum_{\theta \in \mathbb{F}} \sum_{d \in \mathcal{P}(\mu')} \chi(d(\theta)) \left( \prod_{j=1}^{m} a_j(C_d)^{\gamma_j} \right) = -q \sum_{d \in \mathcal{P}_0(\mu')} \chi(d(0)) \left( \prod_{j=1}^{m} a_j(C_d)^{\gamma_j} \right)$$

from which we deduce, as before, that

$$M_{\mu,\gamma}(q) = -q \sum_{d \in \mathcal{P}^0(\mu')} \prod_{j=1}^{m} a_j(C_d)^{\gamma_j}. \hspace{1cm} (119)$$

Now from (118) and (119), it follows that

$$M_{\mu',\gamma'}(q) + M_{\mu,\gamma}(q) = -(q + 1)M_{\mu',\gamma}(q)$$

which proves our first assertion.

The second assertion is proved by a similar argument. \qed
C Appendix

Let us consider (25), written in the equivalent form
\[
\lambda(\kappa, 2; q) - q^{[\kappa] + 3} \lambda(\kappa, 2; 1/q) = a(\kappa, 2; q) + \sum_{\kappa'_<\kappa} q^{[\kappa] - r(\kappa - \kappa')} (q - 1)^{r(\kappa - \kappa')} \lambda(\kappa', 2; 1/q)
\]
for \(\kappa = (k_1, \ldots, k_r) \in \mathbb{N}^r\) \((r \geq 1)\), where
\[
a(\kappa, 2; q) = \begin{cases} (-1)^{|\kappa|} q(q - 1) & \text{if } k_i = 0 \text{ or } 1 \text{ for all } i = 1, \ldots, r \\ 0 & \text{otherwise.} \end{cases}
\]
This is equivalent to (27) (with \(l = 2\)), which can be written explicitly as
\[
\Lambda_2(T, q) = \frac{q(q - 1)}{E(T)} + q^3 \frac{E(qT)}{E(T)} \Lambda_2(qT, 1/q).
\]
We recall that \(T = (t_1, \ldots, t_r), qT = (qt_1, \ldots, qt_r)\), and
\[
E(T) = \prod_{i=1}^r (1 - t_i)^{-1}.
\]
As explained in Section 5, in the case \(l = 4\), starting from \(\lambda(0, \ldots, 0; 2; q) = q^2\), we can find recursively the coefficients \(\lambda(\kappa, 2; q) = P(\kappa, q)\) in polynomial form; the polynomials \(P(\kappa, q)\) have integer coefficients, are independent of \(q\), \(\deg P(\kappa, q) \leq |\kappa| + 2\) (condition ensuring that \(q^{[\kappa] + 3}\lambda(\kappa, 2; 1/q)\) is a polynomial with no constant term), and \(q^{|(\kappa| + 1)/2|} P(\kappa, q)\) (implying that \(\deg (q^{[\kappa] + 3}\lambda(\kappa, 2; 1/q)) < ((|\kappa| + 1)/2) + 2\)).

The generating series of \(P(\kappa, q)\). Let notations be as above, and take
\[
\Lambda_2(T, q) = \sum_{\kappa \in \mathbb{N}^r} P(\kappa, q) T^\kappa
\]
where, for \(\kappa = (k_1, \ldots, k_r)\), we put (as before) \(T^\kappa := t_1^{k_1} \cdots t_r^{k_r} \).

Proposition C.1. — We have
\[
\Lambda_2(T, q) = q E(qT) \frac{\text{Res}_{z=0} [z^2 - q/1/E(zT)E(qT/z)]}{E(zT)E(qT/z)}.
\]
Proof. Put \(f(z, T, q) = z^2 q^{-1} \frac{1}{E(qzT)E(T/z)} \). Then (120) amounts to:
\[
\text{Res}_{z=1} f(z, T, q) + \text{Res}_{z=0} f(z, T, q) = \text{Res}_{z=0} [f(1/z, T, q)/z].
\]
Since \(f(\cdot, T, q)\), as a function of \(z\) is rational, the left-hand side of (121) equals
\[
- \text{Res}_{z=\infty} f(z, T, q) = \text{Res}_{z=0} [f(1/z, T, q)/z^2]
\]
and thus we are left to show that
\[
\text{Res}_{z=0} [f(1/z, T, q)/z^2] = \text{Res}_{z=0} [f(1/z, T, q)/z].
\]
Now
\[
\left(\frac{z-1}{z^2}\right) f(1/z, T, q) = \frac{1}{E(zT)E(qT/z)} - \frac{q}{z^2} \frac{1}{E(zT)E(qT/z)}
\]
77
and note that
\[
\text{Res}_{z=0} \left[ \frac{1}{E(z) E(qT/z)} \right] = -\text{Res}_{z=\infty} \left[ \frac{1}{E(z) E(qT/z)} \right] = \text{Res}_{z=0} \left[ \frac{1}{z^2 E(T/z) E(qzT)} \right].
\]

Replacing \( z \) by \( z/q \) in the expression of the last residue, we have
\[
q^2 \text{Res}_{z=0} \left[ \frac{1}{z^2 E(qT/z) E(zT)} \right] = \text{Res}_{z=0} \left[ \frac{1}{z^2 E(T/z) E(qzT)} \right]
\]
and (122) follows. As
\[
q^2 \Lambda_2(q, 1/q) = \sum_{\kappa \in \mathbb{N}^r} q^{\mid \kappa \mid + 2} P(\kappa, 1/q) T^\kappa = E(T) \text{Res}_{z=0} \left[ \frac{z^2 - q}{z-1} \frac{z + q}{(z+q)^2} \right]
\]
clearly \( q^{\mid \kappa \mid + 2} P(\kappa, 1/q) \) is a polynomial in \( q \), or, what amounts to the same, \( \deg \, P(\kappa, q) \leq |\kappa| + 2 \).

Finally, let us fix \( \kappa \in \mathbb{N}^r \). Then
\[
P(\kappa, q) = \sum_{k_j \leq \kappa} (-1)^{r_1(\kappa')} q^{\mid \kappa' \mid + r_2(\kappa')} R_0 \left[ \frac{z^2 - q}{z-1} \frac{z + q}{(z+q)^2} \right]
\]
where, as before, \( r_j(\kappa') \) \( (j = 1, 2) \) is the number of components of \( \kappa' \) equal to \( j \). For each \( \kappa' \), the expression
\[
q^{\mid \kappa - \kappa' \mid + r_2(\kappa')} R_0 \left[ \frac{z^2 - q}{z-1} \frac{z + q}{(z+q)^2} \right]
\]
is trivially a polynomial in \( q \), and it is easy to see that the exponent of the largest power of \( q \) dividing this polynomial is at least
\[
|\kappa - \kappa'| + r_2(\kappa') + \frac{r_1(\kappa')}{2} + 2 = \frac{|\kappa - \kappa'|}{2} + \frac{|\kappa|}{2} + 2.
\]
Then clearly \( q^{(|\kappa|+1)/2} \mid P(\kappa, q) \), and our assertion follows now from the uniqueness of a polynomial solution \( \Lambda_2(T, q) \) to (120) satisfying the above properties.

Notice that by taking \( \kappa = (1, 1, \ldots, 1) \in \mathbb{N}^r \) in (123), we can write
\[
\lambda_2(r; q) := \lambda(1, 1, \ldots, 1; q) = \sum_{k_j \leq \kappa} (-1)^{\mid \kappa \mid} q^{-|\kappa'|} R_0 \left[ \frac{z^2 - q}{z-1} \frac{z + q}{(z+q)^2} \right]
\]
from which it follows that
\[
\lambda_2(r; q) = \sum_{j=1}^{[\frac{r}{2}]} \frac{1}{(r-j+1)(r-j)!} \frac{r!}{j!(j-1)!(r-2j)!} q^{r+2j}
\]
as claimed in Section 3, Example 3.

The following identity was used in the proof of Theorem 5.1.

**Proposition C.2.** — If \( \kappa = (k_1, \ldots, k_r) \in \mathbb{N}^r \) is such that \( r_1(\kappa) = 2R \) is even, and \( |\kappa| = r_1(\kappa) + 2r_2(\kappa) \) then
\[
\sum_{k_j \leq \kappa} (-1)^{|\kappa'|-k'} q^{|\kappa'|+2} \lambda(\kappa', 2; 1/q) = \sum_{j=0}^{R} 2j + 1 \left( \frac{(2R)! q^{r_1(\kappa')-j}}{(R-j)! (R+j+1)!} \right)
\]
Proof. The left-hand side of the identity is the coefficient of $T^\kappa = t_1^{k_1} \cdots t_r^{k_r}$ in

$$q^2 \Lambda_2(qT, 1/q) = \text{Res}_{z=0} \left[ \frac{q z^2 - 1}{z(z-1)} \cdot \frac{1}{E(qz T) E(T/z)} \right];$$

it vanishes, unless $|\kappa| = r_1(\kappa) + 2r_2(\kappa)$. If $|\kappa| = r_1(\kappa) + 2r_2(\kappa)$, and $r_1(\kappa) = 2R$, then we can express this coefficient by

$$q^{r_2(\kappa)} \cdot \text{Res}_{z=0} \left[ \frac{q z^2 - 1}{z(z-1)} \left( \frac{qz + 1}{z} \right)^{2R} \right]$$

from which our assertion follows. \qed

\section{Appendix}

As before, let $F = \mathbb{F}_q$ be a fixed finite field of odd characteristic, and let $\chi$ denote the non-trivial real character of $F^\times$, extended to $F$ by $\chi(0) = 0$. For $r \in \mathbb{N}$, let

$$M_3(r; q) = \sum_{\begin{subarray}{c} \deg d = 3 \\ d \text{--monic \& square-free} \end{subarray}} \left( - \sum_{\theta \in F} \chi(d(\theta)) \right)^r$$

and

$$M_4(r; q) = \sum_{\begin{subarray}{c} \deg d = 4 \\ d \text{--monic \& square-free} \end{subarray}} \left( -1 - \sum_{\theta \in F} \chi(d(\theta)) \right)^r.$$

be the moment-sums introduced in Section 5.

The purpose of this appendix is to discuss the identity

$$M_4(r; q) = \begin{cases} -M_3(r + 1; q) & \text{if } r \text{ is odd} \\ qM_3(r; q) & \text{if } r \text{ is even} \end{cases}$$

used in the proof of Proposition 5.2. Notice that Proposition B.2, applied for all partitions of 4, implies easily the equivalence:

$$M_4(r; q) = -M_3(r + 1; q) \iff M_4(r; q) = qM_3(r; q) \text{ if } r \text{ is even.}$$

Accordingly, it suffices to prove the following

\begin{proposition}
For $r \in \mathbb{N}$ even, we have $M_4(r; q) = qM_3(r; q)$.
\end{proposition}

Proof. Recall that every complete, non-singular curve of genus one over $F$ has an $F$-rational point, and so, it is isomorphic to its Jacobian; the Jacobian is an elliptic curve defined over $F$. Let

$$E(r; q) = \sum_{[E]} \frac{a(E)^r}{\# \text{Aut}_F(E)}$$

the sum being over all $F$-isomorphism classes of elliptic curves over $F$, and where $a(E) = q + 1 - \#E(F)$. We shall compare $M_4(r; q)$ and $M_3(r; q)$ with $E(r; q)$. To do so, let $P$ denote the set of square-free polynomials of degree 3 or 4 with coefficients in $F$. Let $C_d$ denote the curve corresponding to $d \in P$. The group $G = (\text{GL}_2(F) \times F^\times) / D$, where

$$D = \left\{ \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) : \alpha, \beta, \gamma, \delta \in F^\times \right\},$$

acts on $P$ by

$$g \cdot d(x) = \frac{(\gamma x + \delta)^4}{\eta^2} \cdot \frac{(\alpha x + \beta)^r}{(\gamma x + \delta)^r} \quad \left( \text{for } g = \left[ \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right), \eta \right] \in G \text{ and } d \in P \right).$$
the corresponding curves $C_d$ and $C_{gd}$ are clearly $\mathbb{F}$-isomorphic. As explained in [7, pp. 268-269], we can write

$$E(r; q) = \sum_{[d] \in \mathbb{F}/G} \frac{a(C_d)^r}{\#Stab_G(d)} = \frac{1}{\#G} \sum_{d \in \mathbb{F}} a(C_d)^r.$$ 

Since $\#G = |GL_2(\mathbb{F})| = q(q + 1)(q - 1)^2$ and $r$ is even, this identity can be written in the form

$$E(r; q) = \frac{1}{q^3 - q} \left( M_3(r; q) + M_4(r; q) \right). \quad (124)$$

Now, let $E/\mathbb{F}$ be an elliptic curve. It is well-known that the Riemann-Roch theorem gives an isomorphism of $E$ onto a curve given by a Weierstrass equation of the form $y^2 = d(x)$, for some monic square-free cubic polynomial $d \in \mathbb{F}[x]$. Moreover, any two such Weierstrass equations for $E$ are related by a linear change of variables of the form

$$x = \alpha^2 x' + \beta \quad \text{and} \quad y = \alpha^3 y' \quad \text{(with } \alpha \in \mathbb{F}^* \text{ and } \beta \in \mathbb{F}).$$

(Recall that we are assuming that $\text{char}(\mathbb{F}) > 2$.) Identifying the group of these transformations with the corresponding subgroup (say $G_0$) of $G$, it follows that

$$E(r; q) = \sum_{\{d \in \mathbb{F}[x]: \text{monic and square-free, } \deg d = 3\}/G_0} \frac{a(C_d)^r}{\#Stab_{G_0}(d)} = \frac{1}{\#G_0} M_3(r; q).$$

Thus

$$E(r; q) = \frac{1}{q^3 - q} M_3(r; q)$$

which, combined with (124), gives the identity in the proposition. \qed

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