“Wick Rotations”: The Noncommutative Hyperboloids and Other Surfaces of Rotations

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Abstract

A “Wick rotation” is applied to the noncommutative sphere to produce a noncommutative version of the hyperboloids. A harmonic basis of the associated algebra is given. It is noted that, for the one sheeted hyperboloid, the vector space for the noncommutative algebra can be completed to a Hilbert space, where multiplication is not continuous. A method of constructing noncommutative analogues of surfaces of rotation, examples of which include the paraboloid and the $q$-deformed sphere, is given. Also given are mappings between noncommutative surfaces, stereographic projections to the complex plane and unitary representations. A relationship with one dimensional crystals is highlighted.

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1 Introduction

This letter is divided into two sections. The first is concerned with analytically continuing the algebra of the noncommutative sphere so producing the noncommutative analogue of the hyperboloid, whilst the second section uses this to produce noncommutative analogues of a vast collection of axially symmetric two dimensional surfaces.

As every school child knows \( x^2 + y^2 + z^2 = R^2 \) is the equation for a sphere \((S^2)\) of radius \( R \) embedded in \( \mathbb{R}^3 \). Likewise \( z^2 - x^2 - y^2 = R^2 \) is the equation of a two sheeted hyperboloid \((H^+_2 \cup H^-_2)\) where \( H^+_2 \) and \( H^-_2 \) are the upper and lower sheets), and \( z^2 - x^2 - y^2 = -R^2 \) is the equation for the one sheeted hyperboloid \( H_1 \). It is obvious that if one performs the “Wick rotation” \( x \rightarrow ix \) and \( y \rightarrow iy \) one passes from the sphere to the two sheeted hyperboloid, whilst the substitution \( R \rightarrow iR \) takes one from the two sheeted hyperboloid to the one sheeted hyperboloid.

The standard method of analysing the noncommutative or “fuzzy” sphere is by the use of matrices [1, chapter 7.2]. In such an approach it is not clear how one can perform a “Wick rotation”. However in [2], we present a two parameter algebra \( \mathcal{P}(\varepsilon, R) \) which may be thought of as the noncommutative sphere, since for \( \varepsilon = 0 \), \( \mathcal{P}(0, R) \) is equivalent to the algebra of complex valued functions on the sphere. For a discreet set of \( \varepsilon \), \( \mathcal{P}(\varepsilon, R) \) can be mapped into the algebra of matrices. Since the approach of that article is more algebraic it is easier to perform a “Wick Rotation” so producing noncommutative analogues of the one and two sheeted hyperboloid. In section 2 we give the details of such a rotation. The new algebra contains an extra parameter, \( \alpha \in \mathbb{C} \) with \( |\alpha| = 1 \) which gives the angle of rotation, smoothly rotating between the algebra for the sphere \( \mathcal{P}_{S^2} \) when \( \alpha = 1 \) and the algebra for the hyperboloids \( \mathcal{P}_{H_2^+} \) and \( \mathcal{P}_{H_1} \) when \( \alpha = i \). We rewrite the major expressions in [2] for general \( \alpha \). We also give a formula for the product of two basis polynomials in terms of Wigner 6j symbols.

The one sheeted hyperboloid \( H_1 \) is of particular interest to physicists since it may be considered as a globally hyperbolic spacetime in one plus one dimensions and as a two dimensional equivalent of de Sitter space. The algebra \( \mathcal{P}_{H_1} \) associated with this space may aid the construction of a noncommutative (quantum) theory of fields on de Sitter spaces. This algebra has a very useful property. The sesquilinear form on \( \mathcal{P}_{H_1} \) is positive definite and hence an inner product. It is therefore possible to complete the underlying vector space to produce a Hilbert space \( \overline{\mathcal{P}_{H_1}} \). Multiplication within \( \overline{\mathcal{P}_{H_1}} \) is not continuous and, as a result, the elements of \( \mathcal{P}_{H_1} \) may be represented by unbounded operators as they act on \( \overline{\mathcal{P}_{H_1}} \) by left (or right) multiplication. We discuss the existence or otherwise of a representation of \( su(1, 1) \) by the action of left multiplication on \( \overline{\mathcal{P}_{H_1}} \).

In the second section of this letter we construct noncommuting analogies for surfaces of rotation. Connected surfaces of rotation are either topologically equivalent to the sphere, the disc or the cylinder.

In section 3 we give a definition of noncommutative surfaces of rotation and show how to map functions on one surface to functions on another. These maps are generalisations of the Holstein and Primakoff formalism. They indicate a strong relationship between (1) the topology of the manifold (2) the Hermitian conjugation of the algebra, and (3) the unitary representations of the algebra.

In section 3.1 we show that the algebras for the noncommutative sphere and hyperboloids analysed in section 2 fit nicely into this framework and that the Heisenberg-Weil algebra can be viewed as the noncommutative paraboloid. We also show that the \( q \)-deformed
sphere may be seen as a way of continuously deforming the sphere (for \( q = 1 \)) into a cylinder with end discs (for \( q = \infty \)).

In section 3.3 we give the unitary representations of the noncommutative surfaces. Compact surfaces have finite dimensional representations, whilst noncompact surfaces have infinite dimensional representations. We highlight a relationship between representations of noncommutative surfaces and a large class of one dimensional crystal lattice problems.

In section 3.4 we show that the algebra for a noncommutative surface may also be regarded as the algebra for a noncommutative complex variable. That is, in certain situations, we can construct a one parameter algebra in which a complex variable \( z \) does not commute with its complex conjugate \( \bar{z} \). In the commutative limit this algebra is equivalent to the algebra of complex valued functions on a domain of \( \mathbb{C} \). We give explicit maps between \( P_{S^2} \) and \( P_{H^{\pm}_2} \) and the noncommutative complex plane. These maps may be regarded as the noncommutative analogue of stereographic projections.

Finally in section 4, we discuss how one might use the results in this letter to develop a quantum theory of gravity, some of the problems that are likely to arise, and some of their possible solutions.

## 2 A "Wick rotation" of the Noncommutative Sphere to the Noncommutative Hyperboloid

Let us write \( J_0 = z \) and \( J_{\pm} = x \pm iy \) then \( J_0^2 + \frac{1}{2}(J_+ J_- + J_- J_+) = R^2 \) is the equation of a sphere, whilst \( J_0^2 - \frac{1}{2}(J_+ J_- + J_- J_+) = \pm R^2 \) are the equations of the hyperboloids. The "Wick rotation" from a sphere to the hyperboloids may be made by mapping \( J_- \mapsto \pm J_- \) and allowing \( R \) to be complex. This mapping may also be continued to the noncommutative case. To make the rotation more explicit we consider \( J_+ \mapsto \alpha J_+ \) where \( \alpha \) may be any complex nonzero number. However, since we can rescale \( J_+ \) we shall set \( |\alpha| = 1 \).

For the case of the noncommutative sphere there exists an algebra of polynomials \( \mathcal{P} \) given in [2]. This algebra now becomes the algebra of polynomials generated by \( \{ J_+, J_-, J_0 \} \) where

\[
[J_0, J_+] = \varepsilon J_+ \quad [J_0, J_-] = -\varepsilon J_- \quad [J_+, J_-] = 2\varepsilon \alpha^2 J_0 \quad J_0^2 + \frac{1}{2\alpha^2}(J_+ J_- + J_- J_+) = R^2 \quad (2.1)
\]

The algebra \( \mathcal{P} \) thus depends on \( \varepsilon, R, \alpha \in \mathbb{C} \) which are all independent.

The only way this algebra is distinguished from a simple complexification of the case when \( \alpha = 1 \) is by the choice of Hermitian conjugate. This is given by \( \dagger : \mathcal{P} \mapsto \mathcal{P} \)

\[
J_0^\dagger = J_0, \quad J_+^\dagger = J_-, \quad J_-^\dagger = J_+, \quad (ab)^\dagger = b^\dagger a^\dagger \quad \lambda^\dagger = \bar{\lambda} \quad \forall a, b \in \mathcal{P}, \lambda \in \mathbb{C} \quad (2.2)
\]

Clearly this conjugation is consistent with (2.1) if and only if \( \varepsilon, R^2, \alpha^2 \in \mathbb{R} \). There are six cases when \( \varepsilon, R^2, \alpha^2 \in \mathbb{R} \):

\[
\begin{align*}
\alpha^2 = 1 & \quad R^2 > 0 \quad \mathcal{P} = \mathcal{P}_{S^2} \quad \text{Sphere} \\
\alpha^2 = 1 & \quad R^2 = 0 \quad \mathcal{P} \quad \text{Point} \\
\alpha^2 = 1 & \quad R^2 < 0 \quad \mathcal{P} \quad \text{No Manifold} \\
\alpha^2 = -1 & \quad R^2 > 0 \quad \mathcal{P} = \mathcal{P}_{H^{\pm}_2} \quad \text{Two-sheeted Hyperboloid} \\
\alpha^2 = -1 & \quad R^2 = 0 \quad \mathcal{P} \quad \text{Two cones} \\
\alpha^2 = -1 & \quad R^2 < 0 \quad \mathcal{P} = \mathcal{P}_{H^1} \quad \text{One-sheeted Hyperboloid}
\end{align*}
\]
The algebra $\mathcal{P}$ is still valid even when it does not correspond to a manifold. In this letter we shall let $\alpha$, $\varepsilon$ and $R$ be formal, self Hermitian ($\alpha^\dagger = \alpha$ etc.) parameters in the centre of $\mathcal{P}$. Thus we can still do manipulations involving conjugation without requiring them to be real numbers.

The sesquilinear form is defined in the same way as in [3], that is $\langle f, g \rangle = \pi_0(f^\dagger g)$ where $\pi_0(f)$ is the coefficient of unity when $f$ is written as a formally tracefree symmetric polynomial. For the commutative sphere ($\alpha^2 = 1$, $\varepsilon = 0$) this is the standard inner product calculated by integrating over the sphere; $\langle f, g \rangle = \int_{S^2} f \overline{g} \, d\mu$. It is also the trace with respect to (2.3); the finite dimensional representation of $sl(2, \mathbb{C})$. With respect to this inner product there is an orthogonal (but unnormalised) basis of $\mathcal{P}$ given by $\{ P^m_n(\varepsilon, R) \mid n, m \in \mathbb{Z}, n \geq 0, |m| \leq n \}$ where

$$P^m_n(\varepsilon, R) = \alpha^{m-n} \varepsilon^{m-n} \left( \frac{(n+m)!}{(2n)!} \frac{(n-m)!}{(n-m)!} \right)^{1/2} (\text{ad}_J)^{n-m}(J^n_+).$$

When written as a formally tracefree symmetric polynomial in $(J_0, J_+, J_-)$, $P^m_n$ is homogeneous of order $n$ and is independent of $R$ and $\varepsilon$ (but not necessarily $\alpha$). Each $P^m_n$ is an eigenvector of the operators $\text{ad}_{J_0}$ and $\Delta = \text{ad}_{J_0}^2 + \frac{1}{\varepsilon^2}(\text{ad}_{J_+} \text{ad}_{J_-} + \text{ad}_{J_-} \text{ad}_{J_+})$:

$$\text{ad}_{J_0} P^m_n = \varepsilon m P^m_n$$
$$\Delta P^m_n = \varepsilon^2 n(n+1) P^m_n.$$ (2.4)

The ladder operators $\text{ad}_{J_+}, \text{ad}_{J_-}$ increase or decrease $m$:

$$\text{ad}_{J_\pm} P^m_n = \alpha \varepsilon (n \mp m)^{1/2} (n \pm m + 1)^{1/2} P^{m \pm 1}_n$$ (2.6)

and the normal of $P^m_n$ is given by

$$\|P^m_n\|^2 = \alpha^{2n} \left( \frac{(n!)^2}{(2n+1)!} \right) \prod_{r=1}^{n} (4R^2 + \varepsilon^2(1 - r^2)).$$ (2.7)

If we require $\varepsilon$, $\alpha^2, R^2 \in \mathbb{R}$ then, in general, $\|P^m_n\|^2$ may be positive negative or zero. However, for the 1-sheeted hyperboloid ($\alpha^2 = -1, 4R^2 < -\varepsilon^2$) $\|P^m_n\|^2 > 0$ for all $n$. This enables us to complete $\mathcal{P}_{\mathcal{H}_1}$ into a Hilbert space denoted $\overline{\mathcal{P}_{\mathcal{H}_1}}$. It is clear that the action of left or right multiplication by $J_0$ or $J_\pm$ on the $\overline{\mathcal{P}_{\mathcal{H}_1}}$ are given by unbounded operators. This is examined at the end of this section.

The finite dimensional representation of $sl(2, \mathbb{C})$ are given for $2k \in \mathbb{Z}$, $k \geq 0$ by

$$J_0|k, j\rangle = \varepsilon j|k, j\rangle \quad \quad J_\pm|k, j\rangle = \alpha \varepsilon (k \mp j)^{1/2} (k \pm j + 1)^{1/2}|k, j \pm 1\rangle.$$ (2.8)

This representation is unitary when $\alpha = 1$. It is easy to see that any other representation is unitary only when $\alpha^2, \varepsilon, R^2 \in \mathbb{R}$. As a result the only other unitary representations are the classical unitary representations of $su(1, 1).

As before the projection $\pi_0 : \mathcal{P} \to \mathbb{C}$ is given by the trace: $\pi_0(f) = \frac{1}{2k+1} \sum_{j=-k}^{k} \langle k, j| f |k, j\rangle$. This is used to calculate (2.7) using $R^2 = \varepsilon^2 k(k+1)$. It is not clear how one can use the unitary representations of $su(1, 1)$ to generate a formula for $\pi_0(f)$.

**Theorem 1.** As operators on a Hilbert space, $P^m_n$ can be viewed as a Wigner operator:

$$P^m_n|k, j\rangle = (-1)^n \|P^m_n\| (2n+1)^{1/2} \left( \begin{array}{c} 2n \\ n + m \end{array} \right) |k, j\rangle$$ (2.9)
We can use this to write the formula for the product of two basis elements in terms of Wigner 6j symbols:

\[ P_{m_1}^{n_1} P_{m_2}^{n_2} = \sum_{n=|n_1-n_2|}^{n_1+n_2} C_{m_1 m_2 m_1 + m_2}^{n_1 n_2 n} R_{n_1 n_2 n}^{n_1 n_2 n} P_{m_1 + m_2}^{n_1 + m_2} \]  \hspace{1cm} (2.10)

where \( C_{m_1 m_2 m_1 + m_2}^{n_1 n_2 n} \) is the Clebsh-Gordan coefficient, and the reduced matrix element \( R_{n_1 n_2 n}^{n_1 n_2 n} \) is given by

\[ R_{n_1 n_2 n}^{n_1 n_2 n} = (-1)^{2k+n_1+n_2} \frac{\|P_{m_1}^{n_1}\|\|P_{m_2}^{n_2}\|}{\|P_{n_1+n_2}^{n_1+m_2}\|} (2k+1)^{1/2}(2n_1+1)^{1/2}(2n_2+1)^{1/2} \binom{k}{n_2} \binom{n_1}{k} \binom{k}{n} \]  \hspace{1cm} (2.11)

where the symbol in the curly brackets is Wigner’s 6-j coefficient.

**Proof.** By application of the Wigner-Eckart theorem we have

\[ P_{n}^{m} |k, j\rangle = D_{nk} C_{j, mj+m}^{k, k} |k, j+m\rangle \]

where \( D_{nk} \in \mathbb{C} \) is the associated reduced matrix element. To calculate this put \( m = n \). In this case \( C_{j, mj+m}^{k, k} \) has only one term. Substituting this into the definition of the Wigner operator \cite{4}, eqn (3.341) gives (2.9). One then uses the product law given by \cite{3} eqn (3.350).

Here we define \( \|P_{n}^{m}\| \equiv \alpha^{n} \left( \alpha^{-2n}\|P_{n}^{m}\|^{2} \right)^{1/2} \) which is well defined since \( \alpha^{-2n}\|P_{n}^{m}\|^{2} > 0 \).

Because of the defining equations for the algebra (2.4), one can use \( R^{2} = \varepsilon^{2} k (k + 1) \) to remove \( k \) from (2.11) to give an expression for the reduced matrix element \( R_{n_1 n_2 n}^{n_1 n_2 n} \) which is a polynomial in \( R^{2} \).

These formulae extend naturally to the algebra of deformed rotation matrices given in \cite{4}.

**A possible unitary representation of \( su(1, 1) \) by action on \( \overline{P_{H_1}} \)**

Since \( \overline{P_{H_1}} \) is a Hilbert space upon which the generators of \( su(1, 1) \) \{\( J_{0}, J_{+}, J_{-} \)\} act by left multiplication as unbounded operators we can ask whether there exists a subspace of \( \overline{P_{H_1}} \) for which they are bounded operators.

We propose the subspace \( Q_{\lambda} \) given by

\[ Q_{\lambda} = \text{span}\{Q_{\lambda}^{m} \mid m \in \mathbb{Z}\} \]  \hspace{1cm} (2.12)

where \( \text{ad}_{J_{0}} Q_{\lambda}^{m} = \varepsilon m Q_{\lambda}^{m} \) and \( \|Q_{\lambda}^{m}\| = 1 \). Left multiplication by the generators of \( su(1, 1) \) on \( Q_{\lambda} \) is given by

\[ J_{0} Q_{\lambda}^{m} = (\lambda + \varepsilon m) Q_{\lambda}^{m} \quad J_{\pm} Q_{\lambda}^{m} = (\lambda + \varepsilon m \pm \frac{1}{2} \pm i \hat{R}) Q_{\lambda}^{m \pm 1} \]  \hspace{1cm} (2.13)

where \( \hat{R}^{2} = -R^{2} - \frac{1}{4} \varepsilon^{2} \geq 0 \). These expressions are similar to the standard continuous series of representation of \( su(1, 1) \).

The problem, which is still unsolved, is whether there exists \( \lambda \in \mathbb{C} \) for which \( Q_{\lambda}^{m} \) has finite norm and hence can by normalised.

By setting \( Q_{\lambda}^{m} = \sum_{n=0}^{\infty} c_{n} P_{n}^{m} / \|P_{n}^{m}\| \), it is necessary to show that \( |c_{n}|^{2} \) is a convergent series.
Either by the manipulations of theorem [1] or by manipulation of the Hahn Polynomials one can show that
\[ J_0 P_n^m + P_n^m J_0 = -\beta_{n+1}((n+1)^2 - m^2)^{1/2} P_{n+1}^m - \beta_n(n^2 - m^2)^{1/2} P_{n-1}^m \] (2.14)
where
\[ \beta_n = \frac{1}{2\alpha(2n)^{1/2}(2n-1)^{1/2}}, \quad \tilde{\beta}_n = \frac{\alpha(4R^2 + \varepsilon^2(1-n^2))}{4(2n+1)(2n-1)^{1/2}(2n)^{1/2}} \] and \[ \|P_n^m\|^2 \beta_n = \|P_{n-1}^m\|^2 \tilde{\beta}_n \]
After further manipulation we can show that the \( c_n \) satisfy the recursive relation
\[ \gamma_{n+1} c_{n+1} + i(\lambda + \frac{1}{2}\varepsilon m) c_n + \gamma_n c_{n-1} = 0 \] (2.15)
where
\[ \gamma_n = -i\beta_n(n^2 - m^2)^{1/2} \frac{\|P_n^m\|}{\|P_{n-1}^m\|} \left( \frac{(n^2 - m^2)(4R^2 + \varepsilon^2 n^2)}{16(4n^2 - 1)} \right)^{1/2} \]
Substituting \( c_n = n^n + O(n^{n-1}) \) into (2.13) above we have \( a = -\frac{1}{2} + i(\lambda + m\varepsilon/2) \). Thus the first term in the expansion of \( |c_n|^2 \) is convergent if \( \text{Im}(\lambda) > 0 \). This shows that the representation (2.13) cannot be a representation of the Lie group for which \( \lambda \) must be a real integer multiple of \( \varepsilon \).

Further analysis is necessary to establish whether there is a \( \lambda \in \mathbb{C} \) for which \( |c_n|^2 \) is convergent series.

## 3 Noncommutative Surfaces of Rotation

As stated in the introduction, we would now like to consider what other axially symmetric surfaces have noncommutative analogues. Here we give a definition of an algebra \( \mathcal{A}(\rho, \varepsilon) \) where \( \rho \) is an analytic function and \( \varepsilon \in \mathbb{C} \), and show that when \( \varepsilon = 0 \) it is the commutative algebra of functions on a surface of rotation. In subsection [3.1] we give examples of the sphere, the hyperboloids, the paraboloid, and the \( q \)-deformed sphere. We then show how to map between noncommutative surfaces (subsection [3.2]), and whether they have unitary representations (subsection [3.3]). Finally we show how to interpret \( \mathcal{A}(\rho, \varepsilon) \) as the noncommutative complex plane, and give noncommutative analogues of the stereographic projection of \( S^2 \) and \( \mathcal{H}_2^+ \) (subsections [3.4] and [3.3]).

Given an analytic function \( \rho : \mathbb{C} \mapsto \mathbb{C} \) and a constant \( \varepsilon \in \mathbb{C} \) we define the algebra \( \mathcal{A}(\rho, \varepsilon) \) to be the set of polynomials generated by the elements
\[ \{X_0, X_+, X_-\} \cup \{\rho(X_0 + r\varepsilon) \mid r \in \mathbb{Z}\} \] (3.1)
quotiented by the ideal generated by
\[ [X_0, X_+] = \varepsilon X_+ \quad [X_0, X_-] = -\varepsilon X_- \quad X_+ X_- = \rho(X_0) \quad X_- X_+ = \rho(X_0 + \varepsilon) \] (3.2)
We say \( \rho \) is real if \( \rho|_\mathbb{R} : \mathbb{R} \mapsto \mathbb{R} \). If \( \rho \) is real then there is a conjugation on \( \mathcal{A}(\rho, \varepsilon) \) given by \( X_0^\dagger = X_0, \ X_+^\dagger = X_-, \ X_-^\dagger = X_+ \). Also if \( \rho \) is real, let \( I_\rho \subseteq \mathbb{R} \) be the set \( I_\rho = \{u \in \mathbb{R} \mid \rho(u) > 0\} \). This set is important for three reasons: (1) When \( \varepsilon = 0 \) it determines the topology of the surface of rotation. (2) It determines the nature of the unitary representation of \( \mathcal{A}(\rho, \varepsilon) \). (3) If \( \rho|_{I_\rho} \) is an invertible function then there exists an interpretation of \( \mathcal{A}(\rho, \varepsilon) \) in terms of noncommutative complex numbers.

If \( I_\rho \) is connected then let \( |I_\rho| \) be the size of \( I_\rho \). That is \( |I_\rho| \) is the difference between the two endpoints if \( I_\rho \) is bounded and infinity otherwise.
Theorem 2. If \( \rho \) is real, \( I_\rho \neq \emptyset \) and \( \varepsilon = 0 \) then \( \mathcal{A}(\rho, 0) \) is the commutative algebra of polynomials in \((x, y, z)\) restricted to the surface

\[
\mathcal{M}_\rho = \{(x, y, z) \in \mathbb{R} \mid x^2 + y^2 = \rho(z)\}
\]

(3.3)

where \( X_0 = z \) and \( X_\pm = x \pm iy \). The limit of the commutator as \( \varepsilon \to 0 \) gives \( \mathcal{M}_\rho \) a Poisson structure, given by

\[
\{f, g\} = \lim_{\varepsilon \to 0} \left( \frac{1}{\varepsilon} [f, g] \right) = i \left( \frac{\partial f}{\partial \phi} \frac{\partial g}{\partial z} - \frac{\partial g}{\partial \phi} \frac{\partial f}{\partial z} \right)
\]

(3.4)

Furthermore, if \( I_\rho \) is connected then one of the following three is true:

- \( I_\rho \) is bounded and \( \mathcal{M}_\rho \) is topologically equivalent to the sphere
- \( I_\rho \) is bounded only from one sided and \( \mathcal{M}_\rho \) is topologically equivalent to the disc.
- \( I_\rho = \mathbb{R} \) and \( \mathcal{M}_\rho \) is topologically equivalent to the cylinder.

Proof. From (3.2), \( \mathcal{A}(\rho, 0) \) is a commutative algebra and \( X_+X_- = x^2 + y^2 = \rho(z) \).

For (3.4) we note that both forms of the Poison bracket are bi-differentials, that is they obey Leibniz rule with respect to both variables. Therefore, it is only necessary to check the products of the generators: \( \{X_0, X_\pm\} \) and \( \{X_+, X_-\} \).

The topology classes for \( \mathcal{M}_\rho \) are obvious.

There exist more complicated situations if \( I_\rho \) is not connected. For instance, the surface may be locally topologically equivalent to the intersections of two cones. These situations will not be considered here.

3.1 Examples

The Sphere and Hyperboloids

We can see instantly that the noncommutative sphere \( \mathcal{P}_{S^2} \) and hyperboloids \( \mathcal{P}_{H_\pm^2} \) and \( \mathcal{P}_{H_1} \) are examples of noncommutative surfaces with

\[
\rho(u) = \alpha^2(R^2 - u^2 + \varepsilon u)
\]

(3.5)

where \( \alpha^2, R^2, \varepsilon \in \mathbb{R} \).

The paraboloid

Let \( \rho(u) = u \) then \( \mathcal{M}_\rho \) is a paraboloid. From (3.2) we have \( [X_+, X_-] = \varepsilon \) making \( X_+ \) and \( X_- \) the creation an annihilation operators for the Heisenberg-Weil algebra. Thus we can view the Heisenberg-Weil algebra as the noncommutative paraboloid.

The \( q \) deformed Sphere, \( su_q(2) \)

The algebra \( su_q(2) \) is generated by \( \{X_0, X_+, X_-\} \) which satisfy

\[
[X_0, X_\pm] = \pm X_\pm \quad [X_+, X_-] = \frac{q^{2X_0} - q^{-2X_0}}{q - q^{-1}}
\]

(3.6)

There are many ways of extending this to a set of algebras, which are parameterised by \( \varepsilon \) and which are continuous when \( \varepsilon = 0 \). One possibility is

\[
[X_0, X_\pm] = \pm \varepsilon X_\pm \quad [X_+, X_-] = \frac{\sinh(\varepsilon \kappa) \sinh(2\kappa X_0)}{\sinh(\kappa)^2}
\]

(3.7)
where $e^\kappa = q$. To write this as a noncommutative surface of rotation, let

$$\rho(u) = \frac{\cosh(3\kappa u - \varepsilon \kappa)}{2 \sinh(\kappa)^2} + \frac{1}{2\kappa^2} + R^2 + \frac{\varepsilon^2}{4} - \frac{1}{6} + C(\kappa, \varepsilon)$$

(3.8)

where $C(\kappa = 0, \varepsilon) = 0$. The constant (with respect to $u$) in $\rho(u)$ is set by requiring that $\rho(u) \to R^2 - u^2 + \varepsilon u$ as $\kappa \to 0$.

Setting $\varepsilon = 0$, we have a deformed sphere for small $\kappa$, whilst for large $\kappa$, $\mathcal{M}_\rho$ tends to a cylinder (including the discs at the top and bottom). The cylinder has radius $(R^2 - \frac{1}{6} + C(\kappa, 0))^{1/2}$, and length 2.

### 3.2 Homomorphism between noncommutative surfaces

We give here a description for mapping between two noncommutative surfaces of rotation. These mappings are a generalisation of the Holstein and Primakoff formalism [3].

**Theorem 3.** Given algebras $\mathcal{A}(\rho_1, \varepsilon_1)$ generated by $\{X_0, X_+, X_-$ and $\mathcal{A}(\rho_2, \varepsilon_2)$ generated by $\{Y_0, Y_+, Y_-$ and given analytic functions $\sigma_\pm : \mathbb{C} \mapsto \mathbb{C} \setminus \{0\}$ and $\lambda \in \mathbb{C}$ such that

$$\rho_1\left(\frac{\xi_1}{\xi_2}u + \lambda\right) = \rho_2(u)\sigma_+(u)\sigma_-(u)$$

(3.9)

there exists an homomorphism of algebras

$$\mathcal{A}(\rho_1, \varepsilon) \mapsto \mathcal{A}(\rho_2, \sigma_+, \sigma_-, \varepsilon)$$

$$X_0 \mapsto \frac{\xi_1}{\xi_2}Y_0 + \lambda \quad X_+ \mapsto \sigma_+(Y_0)Y_+ \quad X_- \mapsto Y_-\sigma_-(Y_0)$$

(3.10)

where $\mathcal{A}(\rho_2, \sigma_+, \sigma_-, \varepsilon)$ is the enlarged algebra generated by

$$\mathcal{A}(\rho_2, \sigma_+, \sigma_-, \varepsilon) = \mathcal{A}(\rho_2, \varepsilon) \cup \{\sigma_+(Y_0 + r\varepsilon), \sigma_-(Y_0 + r\varepsilon) \mid r \in \mathbb{Z}\}$$

(3.11)

This mapping is injective but not necessarily surjective. If $\rho_1$ and $\rho_2$ are real then this mapping preserves conjugation if and only if $\sigma_+(u) = \sigma_-(u)$ and $\lambda \in \mathbb{R}$. If the mapping preserves conjugation then $\mathcal{M}_{\rho_1}$ is topologically equivalent to $\mathcal{M}_{\rho_2}$ and

$$\frac{|I_{\rho_1}|}{\varepsilon_1} = \frac{|I_{\rho_2}|}{\varepsilon_2}$$

(3.12)

**Proof.** This simply consists of substituting (3.10) into each equation of (3.2). Injectivity comes from the uniqueness of polynomials. The topology comes from looking at the zeros of $\rho$ which can only be shifted or rescaled.

In many cases we allow $\sigma_\pm$ to contain poles, zeros and branch cuts (since they often contain a square root). This allows the mapping from one topology to another, such as the stereographic projection of the sphere and the bosonic representation of spin. The latter can be viewed as a mapping between the paraboloid and the sphere.

### 3.3 Representations of noncommutative surfaces; Crystals

For any algebra $\mathcal{A}(\rho, \varepsilon)$, there exists many non unitary representations of this algebra: Given the functions $C, D : \mathbb{Z} \mapsto \mathbb{C}$ such that $C(m)D(m) = \rho(\varepsilon m)$, then a representation of $\mathcal{A}(\rho, \varepsilon)$ on the vector space $\{|m\}_{m \in \mathbb{Z}}$ is given by

$$X_0|m\rangle = \varepsilon m|m\rangle \quad X_+|m\rangle = C(m + 1)|m + 1\rangle \quad X_-|m\rangle = D(m)|m - 1\rangle$$

(3.13)
The situation is more interesting if we wish our representation to be unitary. The existence of such a representation implies that \( \rho \) is real and that \( I_\rho \) is non empty. In the following we considered only connected \( I_\rho \).

**Theorem 4.** If \( \rho|_\mathbb{R} \) is real and \( I_\rho \) is connected and non empty then there exist a (unique up to phase) unitary representation of \( \mathcal{A}(\rho, \varepsilon) \) on the vector space \( V \) with basis \( \{|m\}\}_{m \in \mathbb{M}} \), where \( \mathbb{M} \subset \mathbb{Z} \). This is given by

\[
X_0|m\rangle = \varepsilon(m+\lambda)|m\rangle \quad X_+|m\rangle = \overline{D(m+1)|m+1\rangle} \quad X_-|m\rangle = D(m)|m-1\rangle
\]  

(3.14)

where \( \lambda \in \mathbb{R}, 0 \leq \lambda < 1 \) and \( |D(m)|^2 = \rho(\varepsilon m + \varepsilon \lambda) \). Only one of the following three must occur:

- If \( |I_\rho| \) is finite then \( \mathbb{M} \) has finite range of \( \mathbb{Z} \) and \( \varepsilon \) is constrained by

\[
|I_\rho|/\varepsilon = \text{dim } V \in \mathbb{Z}
\]  

(3.15)

- If \( I_\rho \) is bounded from one side then so is \( \mathbb{M} \) and \( \text{dim } \mathbb{M} = \infty \).

- If \( I_\rho = \mathbb{R} \) then \( \mathbb{M} \in \mathbb{Z} \) and \( \text{dim}(V) = \infty \).

**Proof.** Clearly (3.14) is consistent with (3.2). Let \( I_\rho \) be the range \( -\infty \leq u_{\min} \leq u \leq u_{\max} \leq \infty \) and \( \mathbb{M} \) be the range \( -\infty \leq m_{\min} \leq m \leq m_{\max} \leq \infty \). Then from (3.14) we have \( \varepsilon m_{\min} = u_{\min} \) and \( \varepsilon (m_{\max} + 1) = u_{\max} \). So \( m_{\min} \) (or \( m_{\max} \)) is finite if \( u_{\min} \) (or \( u_{\max} \)) is finite. If both are finite then (3.15) is obvious.

When \( \rho \) is given by (3.3) these representation correspond to the standard unitary representation of the Lie algebra \( su(2) \) and \( su(1,1) \). Further restrictions must be imposed to produce the unitary representation of the Lie group \( SU(1,1) \).

There is a connection with one dimensional crystals of either finite or infinite size. If the atoms are labelled by \( |m\rangle \) and the self energy is proportional to \( m \) (as could be the case for a simple magnetic field) and the transition energy proportional to \( D(m) \), then we have a Hamiltonian of the form

\[
H = X_0 + X_+ + X_-
\]

\[
= \sum_m \left( \varepsilon(m+\lambda)|m\rangle\langle m| + \overline{D(m+1)|m+1\rangle\langle m+1|} + D(m)|m-1\rangle\langle m-1| \right)
\]  

(3.16)

This is an example of a combination of a Stark effect with a hopping term. On would like find the energy states for this Hamiltonian. Clearly if \( \rho = \alpha^2(R^2 - u^2 + \varepsilon u) \) then we can perform a \( su(2) \) or \( su(1,1) \) rotation to produce the standard representation of these groups. For general \( \rho \) it may be possible to diagonalises (3.16) using first a Holstein-Primakoff transformation and then the appropriate rotation.

**3.4 The noncommutative complex plane and stereographic projections**

There is an alternative way of writing noncommutative surfaces such that they look more like noncommutative domains in the complex plane. Given the algebra \( \mathcal{A}(\rho, \varepsilon) \) assume that \( \rho \) is invertible and that \( \tau = \rho^{-1} \) then (3.2) is equivalent to the single equation

\[
\tau(z_- z_+) - \tau(z_+ z_-) = \varepsilon
\]  

(3.17)
where \( z_\pm = X_\pm \) and \( X_0 = \tau(z_+z_-) \).

If \( \rho \) is real, then when \( \varepsilon = 0 \) we reproduce the commutative algebra of functions in \((z, \tau)\) on the domain \( \{|z|^2 = \rho(u) \text{ for some } u \in I_\rho\} \subset \mathbb{C} \). We can see this by the substitution \( z = z_- = e^{-i\phi}(\rho(z_0))^{1/2}, \) \( z_+ = \bar{z}. \)

We can rewrite the projection given in theorem 3. This projection is a noncommutative analogue of the stereographic projection. Let \( \mathcal{A}(\rho_1, \varepsilon) \) be a another surface of rotation generated by \( \{X_0, X_\pm\} \) and let \( \mathcal{A}(\tau^{-1}, \sigma_\pm, \varepsilon) \) be the extension of \( \mathcal{A}(\tau^{-1}, \varepsilon) \) as before. From \( (3.10) \) we have the mapping : \( \mathcal{A}(\rho_1, \varepsilon) \mapsto \mathcal{A}(\tau^{-1}, \sigma_\pm, \varepsilon) \) given by

\[
X_+ \mapsto \tilde{\sigma}_+(z_+z_-)z_+ \quad X_- \mapsto z_-\tilde{\sigma}_-(z_+z_-) \quad X_0 \mapsto \tau(z_+z_-) + \lambda \quad (3.18)
\]

where \( \rho_1(\tau(x)) = x\tilde{\sigma}_+(x)\tilde{\sigma}_-(x) \) and \( \tilde{\sigma}_\pm = \sigma_\pm \circ \tau \) for the functions \( \sigma_\pm \) in theorem 3.

### 3.5 Example: The Stereographic Projection of \( S^2 \) and \( H^+_2 \)

We know that \( \mathcal{A}(\rho, \varepsilon) \) with \( \rho \) given by \( (3.3) \), \( R > 0 \) and \( \varepsilon, \alpha^2 \in \mathbb{R} \) corresponds to either \( \mathcal{P}_{S^2} \) or \( \mathcal{P}_{H^+_2} \) depending on the sign of \( \alpha^2 \). The following map may be considered the noncommutative analogue of a stereographic projection:

\[
J_0 \mapsto \hat{R}\frac{4\hat{R}^2 - \alpha^2 x}{4\hat{R}^3 + \alpha^2 x} - \frac{\varepsilon}{2} \quad J_+ \mapsto i\frac{4\hat{R}^2 \alpha^2}{4\hat{R}^2 + \alpha^2 x} z_+ \quad J_- \mapsto -iz_-\frac{4\hat{R}^2 \alpha^2}{4\hat{R}^2 + \alpha^2 x} \quad (3.19)
\]

where \( x = z_-z_+ \) and \( \hat{R}^2 = R^2 + \frac{1}{2}\varepsilon^2 \). When \( \varepsilon = 0 \) this map becomes the stereographic projection of \( S^2 \) to \( \mathbb{C} \) for \( \alpha^2 = 1 \) and the stereographic projection of \( H^+_2 \) to the disc \( \{|z| < 2R\} \subset \mathbb{C} \) for \( \alpha^2 = -1 \). For \( \alpha^2 = -1 \) and \( R^2 \leq 0 \) this is not a stereographic projection.

Equation \( (3.17) \) in this case is equivalent to

\[
x - y = \frac{-\varepsilon}{8\hat{R}^3 \alpha^2}(4\hat{R}^2 + \alpha^2 x)(4\hat{R}^2 + \alpha^2 y) \quad (3.20)
\]

where \( x = z_-z_+ \) and \( y = z_+z_- \), or the Möbius transformation

\[
y = \frac{(1 + \varepsilon/2\hat{R})x + 2\varepsilon \hat{R}/\alpha^2}{(-\varepsilon \alpha^2/8\hat{R})x + (1 - \varepsilon/2\hat{R})} \quad (3.21)
\]

We note that if \( 2\hat{R} = 1 \) and \( \alpha^2 = -1 \) this is equivalent to the algebra given in [3] and used latter in [5] to give noncommutative version of surfaces with higher genus.

The image of \( P_n^m \) under this map may be written

\[
P_n^m = \begin{cases} (z_+)^m p_n^m(x)(\hat{R}^2 + \alpha^2 x)^n & m \geq 0 \\ (z_-)^{-m} p_n^m(x)(\hat{R}^2 + \alpha^2 x)^n & m < 0 \end{cases} \quad (3.22)
\]

where \( p_n^m(x) \) is a polynomial of degree less than \( n + 1 \), related to the Hahn polynomials.

### 4 Discussion and Outlook

The most interesting case from section 3 is that of the noncommutative one sheeted hyperboloid. This is a globally hyperbolic spacetime and a two dimensional de Sitter space. As such it may be related to inflation in the early universe. In order to do quantum
functional field theory (second quantisation) one must first construct a Klein-Gordon inner product to distinguish positive and negative frequency states. This may have the form \( \langle f, g \rangle = \pi_0(f^\dagger, \text{ad}_{iu_0}g) \). One could then go on to construct the Fock space. Clearly we would also like to finish the calculation for the existence of unitary representation of \( su(1,1) \) given by (2.13).

The new results about the noncommutative disc given as the image of the two sheeted hyperboloid \( H^+_2 \) may give further insight of higher genus surfaces using the analysis of Klimek and Lesniewski [6, 7].

The product in \( \mathcal{P}_{S^2} \) is equivalent to that discussed by Cahen [8], who showed that was not a \( \star \)-product in the sense of Flato et al. [9]. Since \( \mathcal{P}_{H^+_1} \) and \( \mathcal{P}_{H^+_2} \) are algebraically equivalent to \( \mathcal{P}_{S^2} \) these also cannot be \( \star \)-product algebras. It would be useful to have an explicit formula for this product in terms of an expansion in \( \varepsilon \).

A principle objective of noncommutative geometry is the establishment of a theory of quantum gravity. Starting from a noncommutative algebra we would like to set up noncommutative analogues of concepts such as vector fields, spinors, connections, curvature and ultimately Einstein’s equations and gravity.

Even deciding what is the analogue a vector field presents problems. Vector fields have two properties which cannot both be required in noncommutative geometry:

1. they are derivatives of the algebra of functions, and
2. that form a module over the algebra of functions.

For a matrix geometry all derivatives are inner and the space of inner derivatives do not form a module of the algebra of matrices. This result is also true for the algebra \( \mathcal{P}_{S^2} \).

Choosing vector fields to be derivatives [10] then, for matrix representations, there is a way of defining a three dimensional space for 1-forms \( \Omega^1(\mathcal{A}(\rho, \varepsilon)) \). These are dual to \( \{ \text{ad}_{X_0}, \text{ad}_{X^+}, \text{ad}_{X^-} \} \). However, [10] also shows that the dimension of the space of 2-forms \( \Omega^2(\mathcal{A}(\rho, \varepsilon)) \) depends on the number of “symmetries” of the underlying space. For the sphere and hyperboloids, we can choose \( \Omega^2(\mathcal{P}_{S^2}), \Omega^2(\mathcal{P}_{H^+_1}) \) and \( \Omega^2(\mathcal{P}_{H^+_2}) \) to have up to four dimension, but in general \( \Omega^2(\mathcal{A}(\rho, \varepsilon)) \) has at most two dimension.

Alternatively one could try and extend the approach of [4] and find “fields” which form a module over the algebra of functions but which are derivatives only in the commutative limit. An important step in this direction would be to establish a basis for the algebra \( \mathcal{A}(\rho, \varepsilon) \). One may start with functions of the form \( \{ X^a_+, X^b_+, X^a_-X^b_+ \mid a, b \in \mathbb{Z}^+ \} \) but this set would not include \( \rho(X_0) \) unless it were a polynomial. Also, one would like to establish which polynomials where harmonic (like \( P^m_n \) in section 2). One would thus generalise the Laplace operator. Its eigenstates would be the harmonic (2.5), and should, for the finite dimensional representations, also be orthogonal.

There is still no agreement on how to define connections, curvature, etc. and there is much research in this area. However, having noncommutative analogues of a large collection of manifolds with different non constant curvatures will enable one to examine many possible ideas.

Further problems will also be encountered when one wishes to construct noncommutative analogues of spacetimes without a natural Poisson structure. This includes the four dimensional spacetimes studied in general relativity. One might have to consider alternative approaches such as adding additional dimensions or, more radically, considering non-associative algebras.

As well as applications in the theory of crystals, noncommutative surfaces of rotation may also have an interpretation in the theory of strings, membranes, and higher d-branes.
The function $\rho^{1/2}$ might correspond to some kind of vibration on a closed circular string which would not interact but may be created and annihilated.

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