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Nonlinear differential equations satisfied by certain classical modular forms

Abstract. A unified treatment is given of low-weight modular forms on $\Gamma_0(N)$, $N = 2, 3, 4$, that have Eisenstein series representations. For each $N$, certain weight-1 forms are shown to satisfy a coupled system of nonlinear differential equations, which yields a single nonlinear third-order equation, called a generalized Chazy equation. As byproducts, a table of divisor function and theta identities is generated by means of $q$-expansions, and a transformation law under $I_0(4)$ for the second complete elliptic integral is derived. More generally, it is shown how Picard–Fuchs equations of triangle subgroups of $PSL(2, \mathbb{R})$, which are hypergeometric equations, yield systems of nonlinear equations for weight-1 forms, and generalized Chazy equations. Each triangle group commensurable with $\Gamma(1)$ is treated.

1. General introduction

In this article we systematically derive ordinary differential equations (ODEs) that are satisfied by certain elliptic modular forms and their roots. The latter are respectively single-valued holomorphic functions, and potentially multivalued ones, on the upper half plane $\mathbb{H} = \{ \Im \tau > 0 \}$. Among the classical modular groups that will appear are the full modular group $\Gamma(1) = PSL(2, \mathbb{Z})$, the Hecke congruence subgroups $I_0(N), N = 2, 3, 4$, and their Fricke extensions $I_0^+(N) < PSL(2, \mathbb{R})$.

There are two distinct sorts of ODE satisfied by forms on classical modular groups, distinguished by their independent variables.

1. If the independent variable is the period ratio $\tau \in \mathbb{H}$, the ODE will typically be nonlinear. Classical examples include Ramanujan’s coupled ODEs for Eisenstein series on $\Gamma(1)$, Rankin’s fourth-order ODE for the modular discriminant $\Delta$ (a weight-12 form on $\Gamma(1)$), and Jacobi’s third-order ODE for the theta-null functions $\vartheta_2, \vartheta_3, \vartheta_4$ (which are weight-$\frac{1}{2}$ forms on $I_0(4)$).

2. If the independent variable is a $P^1(\mathbb{C})$-valued Hauptmodul (function field generator) for a modular group $\Gamma$, the ODE will be linear [42]. Classical examples include Jacobi’s hypergeometric ODE for the first complete elliptic integral $K$ (a weight-1 form on $I_0(4)$), viewed as a function of the modulus $k$ or its square $k^2$ (a Hauptmodul for $I_0(4)$); and Picard–Fuchs equations satisfied by periods of elliptic families rationally parametrized by other Hauptmoduls.

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The stress in this article is on deriving ODEs of the first (nonlinear) class, which are less closely tied than the second to the classical theory of special functions. The computations will make heavy use of $q$-expansions and the theory of modular forms. However, special function methods such as hypergeometric transformations will prove useful in dealing with forms on the Fricke extensions $\Gamma_0^+(N)$.

The derivation of ODEs for classical modular forms has recently been considered by Ohyama (e.g., in [30]) and Zudilin [47]. Our approach differs from theirs by being extensively ‘modular,’ in that it exploits dimension formulas (coming ultimately from the Riemann–Roch theorem), explicit $q$-expansions and number-theoretic interpretations of their coefficients, etc. Also, Ohyama focuses on deriving systems of nonlinear ODEs satisfied by weight-2 quasi-modular forms, analogous to the Eisenstein series $E_2$ on $\Gamma(1)$. His coupled ODEs are of an interesting quadratic type, called Darboux–Halphen systems [1, 22]. In the present article the fundamental dependent variables are weight-1 modular forms, sometimes multivalued, which are analogous to $E_4^{1/4}$, $E_6^{1/6}$, and $\Delta^{1/12}$; and we regard the resulting differential systems as more fundamental than Darboux–Halphen ones, though quasi-modular forms play a role. Zudilin focuses on deriving systems, and also linear ODEs of the second type distinguished above, which are satisfied by forms on $I_0(N)$, $I_0^+(N)$, or on subgroups of $I_0^+(N)$. We are able to clarify the modular underpinnings of his ODEs, and derive several more such equations.

The weight-1 forms studied below include triples of forms on $I_0^+(N)$, $N = 2, 3, 4$, which we denote $A_r, B_r, C_r$, $r = 4, 3, 2$. They were introduced as functions on $\mathcal{H}$ by the Borweins [7] in their study of alternative AGM (arithmetic-geometric mean) iterations. Their work was inspired by Ramanujan’s theory of elliptic functions to alternative bases, the base being specified by the ‘signature’ $r$. (Cf. Berndt et al. [3].) Our approach places these functions firmly in a modular setting (see also [27]). As a byproduct of the analysis of these modular forms and their powers, we derive many divisor function and theta identities. A minor example is Jacobi’s Six Squares Theorem; our proof of it may be the most explicitly modular one to date. (See Thm. 3.8) We also give a modular interpretation of the second complete elliptic integral $E$, probably for the first time, by identifying it as a weight-1 form on $H_0(4)$ with an explicit, quasi-modular transformation law. (See Prop. 5.1)

The fundamental goal of this article, however, is the development of a modular theory of ‘nonlinear’ special functions, by determining which integrable nonlinear ODEs (generalized Chazy equations, in our terminology) can arise in certain well-specified modular contexts. (See Thms. 2.3 and 7.1 and the discussion in §7.4)

2. Motivation and the first theorem

As initial motivation, consider forms on the full modular group $\Gamma(1) = PSL(2, \mathbb{Z})$ and their differential relations. Van der Pol [32 §13] and Rankin [37] proved that the modular discriminant function $\Delta$ on $\mathcal{H} = \{ \tau \in \mathbb{H}, \ Im \tau > 0 \}$, which when viewed as a function of $q := \exp(2\pi i \tau), |q| < 1$, is defined by

$$\Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad (2.1)$$
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satisfies the nonlinear, fourth-order homogeneous differential equation

\[ 2 \Delta^3 \Delta''' - 10 \Delta^2 \Delta'' - 3 \Delta^2 \Delta'' - 24 \Delta \Delta' \Delta'' - 13 \Delta^4 = 0, \tag{2.2} \]

where '$' signifies the derivation \( \frac{d q}{d q} = \frac{d}{d \tau} \). (The derivation \( \frac{d}{d \tau} \) will be indicated by a dot; the primes in (2.2) can be optionally replaced by dots.) This ODE is fairly well known, as is Jacobi’s third-order one for his theta-null functions \( \vartheta_i, i = 2, 3, 4 \). (For remarks on the latter, see [15].) In the following, the parallels between them will be brought out.

One approach to understanding the rather complicated Eq. (2.2) is to treat it as a corollary of a much nicer nonlinear third-order ODE [39, 32], namely

\[ u'' - 12 u' + 18 u = 0, \tag{2.3a} \]

i.e.,

\[ 2 E''_2 - 2 E_2 E''_2 + 3 E_2^2 = 0. \tag{2.3b} \]

Here, \( u = (2 \pi i / 12) E_2 = \pi i E_2 / 6 \), and \( E_2 \) is the second (normalized) Eisenstein series on the full modular group. The Eisenstein series \( E_k = E_k^{1,1} \) on \( \Gamma(1) \) are

\[ E_k(q) = 1 + a_k \sum_{n=1}^{\infty} \sigma_k(n) q^n = 1 + a_k \sum_{n=1}^{\infty} \frac{n^{k-1} q^n}{1 - q^n}, \]

\[ \sigma_k(n) = \sum_{d|n} d^k, \quad a_k = \frac{2}{\zeta(1-k)} = \frac{2}{L(1-k,1)} = - \frac{2 k}{B_k}, \]

where \( B_k \) is the \( k \)th Bernoulli number; so \( a_2, a_4, a_6, \ldots \) are \(-24, 240, -504, \ldots\). The nonlinear ODE (2.3a) is a so-called Chazy equation, with the interesting analytic property of having solutions with a natural boundary (e.g., \( \Im \tau = 0 \) or \( |q| = 1 \)), beyond which they cannot be continued; much as is the case with a lacunary series. (See [2, pp. 342–3] and [11].) Substituting \( E_2 = \Delta' / \Delta \) into (2.3b) yields (2.2).

Equation (2.3b), in turn, follows from a result of Ramanujan. He introduced functions \( P, Q, R \) on the disk \( |q| < 1 \), defined by convergent \( q \)-series, which are identical to \( E_2, E_4, E_6 \). That is, they are respectively a quasi-modular form of weight \( 2 \) and depth \( \leq 1 \), and modular forms of weights \( 4 \) and \( 6 \). He determined the differential structure on the ring \( \mathbb{C}[E_2, E_4, E_6] \) by showing that

\[ (E_4^3)' = E_2 \cdot E_4^3 - E_4^2 E_6, \tag{2.4a} \]
\[ (E_6^2)' = E_2 \cdot E_6^2 - E_4^2 E_6, \tag{2.4b} \]
\[ \Delta' = E_2 \cdot \Delta, \tag{2.4c} \]
\[ 12 E_2' = E_2 \cdot E_2 - E_4, \tag{2.4d} \]

where Eqs. (2.4abc) are linearly dependent, since \( E_4^3 = E_6^2 + 12 \Delta \), which is an equality between weight-12 modular forms. By rewriting the system (2.4bd) into
a single third-order equation for $E_2$, one obtains Eq. (2.3b). It is worth noting for later use that the system (2.4abcd) can be rewritten as

\[
\begin{align*}
(A^{12})' &= \mathcal{A} \cdot A^{12} - A^8 B^6, \\
(B^{12})' &= \mathcal{B} \cdot B^{12} - B^8 D^6, \\
(C^{12})' &= \mathcal{C} \cdot C^{12}, \\
12 \mathcal{E}' &= \mathcal{E} \cdot \mathcal{E} - \mathcal{E}^4,
\end{align*}
\]

where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{E}$ are respectively $E_4^{1/4}, E_6^{1/6}, (12^3 \Delta)^{1/12}; E_2$. Of these, $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are formally weight-1 forms for $\Gamma(1)$; but the first two are multivalued on $\mathfrak{H}$. (Their $q$-expansions, the integer coefficients of which lack an arithmetical interpretation, do not converge on all of $|q| < 1$.)

Some of Ramanujan’s results along this line were subsequently extended by Ramamani (35); see also (36). She introduced three $q$-series somewhat similar to his $PQR$, and derived a coupled system of first-order ODEs that they satisfy. Recently, Ablowitz, Chakravarty and Hahn (1); see also (21) showed that her $q$-series define modular forms on the Hecke subgroup $\Gamma_0(2) < \Gamma(1)$, including a weight-2 quasi-modular form analogous to $E_2$, and derived a single nonlinear third-order ODE that it satisfies. This turns out to be a Chazy-like equation, of a general type first studied by Bureau (9).

One may wonder whether these results can be generalized, by extending them to other modular subgroups. The question is answered in the affirmative by Theorem 2.3 below, which provides a unified treatment of certain Eisenstein series on the subgroups $\Gamma_0(2), \Gamma_0(3), \Gamma_0(4)$. For the latter two as well as for $\Gamma_0(2)$, a nonlinear third-order ODE is satisfied by a quasi-modular form of weight 2. A unified treatment is facilitated by the fact that up to isomorphism, these are the only genus-zero proper subgroups of $\Gamma(1)$ that have exactly three inequivalent fixed points on $\mathfrak{H}$; with the exception of the principal modular subgroup $\Gamma(2)$, which is conjugated to $\Gamma_0(4)$ by the 2-isogeny $\tau \mapsto 2\tau$ in $\text{PSL}(2, \mathbb{R})$; and also the index-2 subgroup $\Gamma^2$, which is a bit anomalous. The statement of the theorem requires

**Definition 2.1.** If $u$ is a holomorphic function on $\mathfrak{H}$, define functions $u_4, u_6, u_8, \ldots$ by $u_4 := \dot{u} - u^2$ and $u_{k+2} := u_k - kuu_k$. Thus,

\[
\begin{align*}
u_4 &= \dot{u} - u^2, \\
u_6 &= \ddot{u} - 6u\dot{u} + 4u^3, \\
u_8 &= \dot{\dddot{u}} - 12u\dot{u} - 6u\dddot{u} + 48u^2\dot{u} - 24u^4.
\end{align*}
\]

A generalized Chazy equation $C_p$ for $u$ is a differential equation of the form $p = 0$, where $p \in \mathbb{C}[u_4, u_6, u_8]$ is a nonzero polynomial, homogeneous in that the weights of its monomials are equal. Here, the weight of $u_4^a u_6^b u_8^c$ is $4a + 6b + 8c$.

**Remark.** The classical Chazy equation, Eq. (2.3a), has $p = u_8 + 24u_4^2$. The so-called Chazy–XII class (11) includes equations $C_p$ with $p = u_8 + \text{const} \cdot u_4^2$. This further generalization is prefigured by the treatment of Clarkson and Olver (12).
Definition 2.2. For any \( \chi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C} \), define the \( \chi \)-weighted divisor and conjugate divisor functions
\[
\sigma_k(n; \chi) = \sum_{d \mid n} \chi(d \mod N) d^k, \quad \sigma_k^*(n; \chi) = \sum_{d \mid n} \chi((n/d) \mod N) d^k.
\]

Such weighted divisor functions, with \( \chi \) not necessarily a Dirichlet character, have been considered by Glaisher [19], Fine [17, §§32 and 33], and others. The argument \( \chi \) will usually be written out in full, as \( \chi(0), \ldots, \chi(N - 1) \).

Results attached to \( \Gamma_0(2), \Gamma_0(3), \Gamma_0(4) \) will be referred to as belonging to Ramanujan’s theories of signature \( 4, 3, 2 \), respectively. The fixed points on \( S^+ \) of each group include (the equivalence classes of) two cusps, namely the infinite cusp \( \tau = i\infty \) (i.e., \( q = 0 \)) and the cusp \( \tau = 0 \); and also a third fixed point, which for \( \Gamma_0(2) \) is the quadratic elliptic point \( \tau = i \), for \( \Gamma_0(3) \) is the cubic elliptic point \( \tau = \xi_3 := \exp(2\pi i/3) \), and for \( \Gamma_0(4) \) is an additional cusp, namely \( \tau = 1/2 \). (For a review of these facts, and for triangular fundamental domains the vertices of which are these fixed points, see, e.g., [41].)

Theorem 2.3. On each modular subgroup \( \Gamma_0(N) \), \( N = 2, 3, 4 \), i.e., for each of the corresponding signatures \( r = 4, 3, 2 \), the following are true.

1. There is a quasi-modular form \( \mathcal{E}_r \) of weight \( 2 \) and depth \( \leq 1 \), equaling unity at the infinite cusp, such that \( \alpha = (2\pi i/r)\mathcal{E}_r \) satisfies a generalized Chazy equation \( C_r \), for some polynomial \( p_r \). Namely,
\[
\mathcal{E}_4(q) = \frac{1}{6} [4E_2(q^2) - E_2(q)] = 1 + 8 \sum_{n=1}^{\infty} \sigma_1(n; -1, 1) q^n = 1 + 8 \sum_{n=1}^{\infty} \sigma_1^*(n; -3, 1) q^n,
\]
\[
\mathcal{E}_3(q) = \frac{1}{5} [9E_2(q^3) - E_2(q)] = 1 + 3 \sum_{n=1}^{\infty} \sigma_1(n; -2, 1, 1) q^n = 1 + 3 \sum_{n=1}^{\infty} \sigma_1^*(n; -8, 1, 1) q^n,
\]
\[
\mathcal{E}_2(q) = \frac{1}{4} [4E_2(q^4) - E_2(q^2)] = 1 + 4 \sum_{n=1}^{\infty} \sigma_1(n; -1, 0, 1, 0) q^n = 1 + 8 \sum_{n=1}^{\infty} \sigma_1^*(n; -3, 0, 1, 0) q^n,
\]
so that \( \mathcal{E}_2(q) = \mathcal{E}_4(q^2) \). The polynomials \( p_r \in \mathbb{C}[u_4, u_6, u_8] \) are
\[
p_4 = u_4u_8 - u_6^2 + 8u_4^3, \quad \tag{2.6}
\]
\[
p_3 = u_4^2u_8 - u_6^2u_8 + 24u_4u_6u_8 - 15u_4^2u_6^2 + 144u_4^5, \quad \tag{2.7}
\]
\[
p_2 = u_4u_8 - u_6^2 + 8u_4^3, \quad \tag{2.8}
\]
so that \( p_2 = p_4 \).

2. There is a triple of weight-1 modular forms \( \mathcal{A}_r, \mathcal{B}_r, \mathcal{C}_r \) (allowed to have non-trivial [i.e., non-Dirichlet] multiplier systems, and also allowed to be multivalued in the above sense of being roots of conventional [single-valued] modular forms), such that
functions modular, based on results of [27]. Each of plier systems; cf. [45, Lemma 5]. We give a second proof that is less explicitly functions [3]. In [27], we interpreted them as forms on introduced by the Borweins [7] as the theta functions of certain quadratic forms; C
rem 2.3(1). They too can be derived by linear algebra. (Indeed, for each ϑ, the transformation law for elliptic integrals, viewed as functions of the nome
§
In [27], we give a direct proof of the generalized Chazy equations of Theo-
rem 2.3(2) are equivalent to those of Ramamani [35], Ablowitz et al. [1], and Remark. For the subgroup Γ0(2), i.e., when r = 4, the coupled ODEs of The-
Remark. The results of van der Pol–Rankin and Ramanujan, attached to Γ(1), cannot be subsumed into Thm. 2.3; but see the more general Theorem 7.1 below. The body of this article is laid out as follows. In §3 the modular forms A
= E
, B
, C
, are defined as eta products and q-series. (These functions on |q| < 1 were introduced by the Borweins [7] as the theta functions of certain quadratic forms; see the Appendix. They play a role in Ramanujan’s alternative theories of elliptic functions [8]. In [27], we interpreted them as forms on Γ0(2), Γ0(3), Γ0(4).) In passing, we generate a table of q-expansions and divisor-function identities (Table I), of independent interest, and give a modular proof of Jacobi’s Six Squares Theo-
rem. In §3 we prove Theorem 2.3(2) by exploiting the dimensionality of spaces of modular forms, i.e., by applying linear algebra to the graded ring C[A
, B
, C
].

Section 5 is a digression. From the r = 2 system, we derive an elliptic integral transformation law, and differential relations for theta-nulls that imply Jacobi’s non-linear third-order ODE. Deriving interesting identities is facilitated by the quasi-modular form E
2(q) equaling (up to a transcendental constant factor) the even function K(q)E
(q), i.e., the product of the classical first and second complete elliptic integrals, viewed as functions of the nome q. No satisfactorily ‘modular’ transformation law for E = E(q) has previously been derived.

In §6 we give a direct proof of the generalized Chazy equations of Theo-
to be derived by linear algebra. (Indeed, for each r, the functions u
, u
, u
 are modular forms of the specified weight, with trivial multiplier systems; cf. [45] Lemma 5.) We give a second proof that is less explicitly modular, based on results of [27]. Each of A
, B
, C
 satisfies a ‘hypergeometric’
Picard–Fuchs equation, which is a linear second-order ODE with three singular points, the independent variable of which is a Hauptmodul for the corresponding group \( \Gamma_0(N) \). Moreover, \( \tau \) is a ratio of solutions of this equation (cf. [18]). These facts make possible the second proof. Theorem 6.4 is an extension of Theorem 2.3, or equivalently, a general result on solutions of Gauss hypergeometric equations. It reveals which generalized Chazy equations can arise from genus-zero subgroups of \( \text{PSL}(2, \mathbb{R}) \) with three inequivalent fixed points.

In §7 a comparable extension of Theorem 2.3 is obtained. Theorem 7.1 derived using ODE manipulations like those of Ohyama [30] presents the system of nonlinear first-order ODEs, satisfied by a triple of weight-1 modular forms \( \mathcal{A}, \mathcal{B}, \mathcal{C} \), that arises from any specified triangle subgroup of \( \text{PSL}(2, \mathbb{R}) \); i.e., from its Picard–Fuchs equation. As examples, we treat the nine triangle groups corresponding to index-2 subgroups of these three groups, is derived as well.

3. Modular forms and divisor function identities

The modular forms \( \mathcal{A}_r, \mathcal{B}_r, \mathcal{C}_r, r = 4, 3, 2 \), of which only \( \mathcal{A}_4 \) is multivalued on \( \mathcal{H} \), will be defined here in terms of the Dedekind eta function, rather than univariate or multivariate theta functions. In the Appendix, several of the original definitions of the Borweins [17] are reproduced, as are AGM identities these forms satisfy.

Being a (single-valued) form has its usual meaning. On \( \mathcal{H}^\tau = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q}) \), i.e., \( \mathcal{H} \cup \mathbb{Q} \cup \{i\infty\} \), a holomorphic function \( f \) is modular of integral weight \( k \) on some \( \Gamma < \Gamma(1) \) if \( f(\frac{a\tau + b}{c\tau + d}) = \tilde{f}(a,b,c,d)(c\tau + d)^k f(\tau) \) for all \( \pm \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma \). Here, \( \tilde{f} \) is a \( \mathbb{C}^\tau \)-valued multiplier system, with \( \tilde{f}(-a,-b,-c,-d) = (-1)^k \tilde{f}(a,b,c,d) \).

The simplest case, occurring if \( \Gamma < \Gamma_0(N) \) for some \( N \), is when \( \tilde{f}(a,b,c,d) = f(d) = 0 \) if \( (d,N) > 1 \), where \( \langle \cdot, \cdot \rangle \) is the g.c.d. The notation \( \mathbf{1}_\tau \) for the principal character mod \( N \), satisfying \( \mathbf{1}_\tau(d) = 1 \) if \( (d,N) = 1 \), will be used. The trivial character of period 1 will be denoted \( \mathbf{1} \).

In terms of \( g_2 \), the eta function \( \eta(\tau) = \frac{e^{\pi i g_2(\tau)/24}}{\sqrt{2\pi i \tau}} \). On \( \Gamma(1) \), it transforms as [34]

\[
\eta(\frac{a\tau+b}{c\tau+d}) = \begin{cases} \left( \frac{d}{c} \right) \frac{\zeta(1-c) + bd(1-c^2) + ce(a+d)}{\sqrt{24}} \left[ -i(c\tau + d) \right]^{1/2} \eta(\tau), & \text{c odd,} \\ \left( \frac{c}{d} \right) \frac{brd + ac(1-d^2) + (b-c)(-i(c\tau + d))}{\sqrt{24}} \left[ -i(c\tau + d) \right]^{1/2} \eta(\tau), & \text{d odd,} \end{cases}
\]

if \( c > 0 \), where \( \zeta_{24} := \exp(2\pi i/24) \), and the Jacobi symbol is taken to satisfy \( \left( \frac{d}{c} \right) = \left( \frac{c}{d} \right) \). Fine’s notation \([\delta]\) for the function \( \tau \mapsto \eta(\delta \tau) \) on \( \mathcal{H}^\tau \) will be used, so that, e.g., \( \Delta = [1]^{24} \). At any cusp \( s = \frac{a}{b} \in \mathbb{Q} \cup \{i\infty\} \) (in lowest terms, with \( \frac{1}{b} \) signifying \( i\infty \)), the order of vanishing of \( \eta(\delta \tau) \), denoted \( \text{ord}_s([\delta]) \), is given by a well-known formula stated in Ref. [28],

\[
\text{ord}_s([\delta]) = \frac{1}{24} (\delta, d)^2 / \delta.
\]
Here, \( \text{ord}_s(\cdot) \) is computed with respect to a local parameter on the quotient curve \( X(1) = \Gamma(1) \setminus \mathcal{H}^* \), such as the Klein–Weber \( j \)-invariant (which equals \( E_4^3/\Delta = 12^3 E_4^3/(E_4^3 - E_6^3) \) and is a Hauptmodul for \( \Gamma(1) \)). As usual, \( \text{ord}_q(f) \) is the lowest power of \( q \) in the Fourier expansion of \( f \).

If \( f \) is a modular form on \( \Gamma \), its order of vanishing at a cusp \( s \in \mathcal{H}^* \), computed with respect to a local parameter for \( \Gamma \) (i.e., on the quotient curve \( X = \Gamma \setminus \mathcal{H}^* \)) is

\[
\text{Ord}_{s, \Gamma}(f) := h_{\Gamma}(s) \cdot \text{ord}_s(f),
\]

(3.3)

Here, \( h_{\Gamma}(s) \) is the multiplicity with which the image of \( s \) in \( X \) is mapped to \( X(1) \), i.e., the width of the cusp \( s \). If \( s \in \mathcal{H}^* \) is not a cusp but rather a quadratic or cubic elliptic fixed point of \( \Gamma \) (implying that \( s \in \mathcal{H} \)), then by definition \( s \) will be mapped doubly, resp. triply to \( X \). In this case,

\[
\text{ord}_s(f) = (2, \text{resp. } 3) \cdot \text{Ord}_{s, \Gamma}(f),
\]

(3.4)

where \( \text{ord}_s(f) \) is the order of vanishing of \( f \) at the point \( s \in \mathcal{H} \) in the conventional sense of analytic functions. If \( f \) has no poles and is single-valued at \( \mathcal{H} \), i.e., has no branch points, then this order must be a non-negative integer.

In the case \( \Gamma = \Gamma_0(N) \), the inequivalent cusps \( \tau = \frac{\alpha}{\beta} \) on \( \mathcal{H}^* \) may be taken to be the fractions \( \frac{\alpha}{\beta} \in \mathbb{Q} \) with \( d | N, 1 \leq a \leq N \), and with \( a \) reduced modulo \( (d, N/d) \) while remaining coprime to \( d \). (E.g., the cusps of \( \Gamma_0(2) \) would be \( \frac{1}{2} \) if \( N = 2 \); \( \frac{1}{3}, \frac{1}{4} \) if \( N = 3 \); and \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \) if \( N = 4 \). Note that \( \frac{1}{N} \sim 0 \) and \( \frac{1}{N} \sim \infty \) under \( \Gamma_0(N) \).)

If this convention is adhered to, then each inequivalent cusp \( \frac{\alpha}{\beta} \) will have width \( h_{\Gamma_0(N)}(\frac{\alpha}{\beta}) = e_{d, N} := N/(d, N/d) \).

**Definition 3.1.** \( A_r, B_r, C_r, r = 4, 3, 2, \) are certain functions on \( \mathcal{H}^* \), defined to have the eta-product representations

\[
\begin{align*}
A_4 &= (2^5 \cdot [2]^{24} + [1]^{24})^{1/4} / [1][2]^2, \\
B_4 &= [1]^4 / [2]^2, \\
C_4 &= 2^{3/2} \cdot [2]^4 / [1]^2; \\
A_3 &= (3^3 \cdot [3]^{12} + [1]^{12})^{1/3} / [1][3], \\
B_3 &= [1]^3 / [3], \\
C_3 &= 3 \cdot [3]^{1/3} / [1]; \\
A_2 &= (2^4 \cdot [4]^{8} + [1]^{8})^{1/2} / [2]^2, \\
B_2 &= [1]^4 / [2]^2, \\
C_2 &= 2^2 \cdot [4]^4 / [2]^2,
\end{align*}
\]

so that by definition, \( A_r, B_r, C_r, \) at the infinite cusp (i.e., at \( q = 0 \)) each \( A_r \) and \( B_r \) equals unity, and each \( C_r \) vanishes. The \( A_r \), defined as roots of single-valued modular forms, are potentially multivalued, but it will be shown that \( A_2, A_3 \) are single-valued. One notes that \( B_4 = B_2 \) and \( C_4(q) = 2^{-1/2} \cdot C_2(q^{1/2}) \).

**Remark.** Connections to theta functions, such as Jacobi’s theta-nulls \( \vartheta_2, \vartheta_3, \vartheta_4 \), will be discussed in §5 (Also, see the Appendix.) For the moment, observe that by theta identities first proved by Euler, or alternatively by the Jacobi triple product formula, \( A_2, B_2, C_2 \) equal \( \vartheta_3^2, \vartheta_4^2, \vartheta_2^2 \). Similarly, \( A_4^2 = \vartheta_2^4 + \vartheta_4^4, B_4 = \vartheta_2^2, \) and \( C_4 = \sqrt{\sum A_2 C_2} = 2^{1/2} \cdot \vartheta_2 \vartheta_3 \).
Proposition 3.2. \( A_r, B_r, C_r, r = 4, 3, 2 \) are weight-1 modular forms on the subgroups \( \Gamma_0(N), N = 2, 3, 4 \), respectively, with each being single-valued save for \( A_4 \), the square of which is single-valued. Each has exactly one equivalence class of zeroes on \( \mathcal{H} \), at which its order of vanishing is \( 1/r \) (computed with respect to a local parameter for \( \Gamma_0(N) \)). Under the Fricke involution \( W_N : \tau \mapsto -1/N \tau \) for \( \Gamma_0(N) \), \( B_r \) and \( C_r \) are interchanged in the sense that \( B_r^2 W_N = -C_r^2 \), and \( A_r^2 \) is negated. There is an alternative, explicitly single-valued representation for \( A_2 \), namely \( A_2 = [2]^2 \cdot [1]^4 \cdot [4]^4 \).

Proof. It follows from (3.2) that for each \( r \), \( \text{ord}_{\infty}(B_r) = 0 \), \( \text{ord}_{\infty}(C_r) = 1/r \) and \( \text{ord}_0(C_r) = 0 \); and for \( r = 4, 3, 2 \), that \( \text{ord}_0(B_r) = 1/9 \cdot 1/8 \cdot 1/9 \). Also, \( \text{ord}_{1/2}(B_2) = \text{ord}_{1/2}(C_2) = 0 \).

The cusps \( \tau = 0, i\infty \) of \( \Gamma_0(2), \Gamma_0(3) \) have widths \( 2, 1 \) and \( 3, 1 \), and the cusps \( \tau = 0, 1/2 \infty \) of \( \Gamma_0(4) \) have widths \( 4, 1, 1 \). It follows from (3.3) that the order of \( B_r, C_r \) at each cusp is zero, except at \( \tau = 0, i\infty \) respectively, where the big-O order in each case equals \( 1/r \), as claimed.

To prove the claim about the zeroes of \( A_r \), note that \( t_2 = 2^{12} \cdot [2]^2 \cdot [1]^{24} \), \( t_3 = 3^6 \cdot [3]^2 \cdot [1]^{12} \), \( t_4 = 2^8 \cdot [4]^8 \cdot [1]^8 \) are Hauptmoduls for \( \Gamma_0(2), \Gamma_0(3), \Gamma_0(4) \), i.e., rational parameters for the associated quotient curves \( X_0(N) \). Each vanishes at the cusp \( \tau = i\infty \) and has a pole at the cusp \( \tau = 0 \). (See [27]; the normalization factors are unimportant here.) By construction, \( A_4/B_4 = (1 + t_2/2^6)^{1/4}, \ A_3/B_3 = (1 + t_3/3^3)^{1/3} \), and \( A_2/B_2 = (1 + t_4/2^4)^{1/2} \). Hence,

\[
\text{Ord}_{0, \Gamma_0(N)}(A_r) = \text{Ord}_{0, \Gamma_0(N)}(B_r) + \text{Ord}_{0, \Gamma_0(N)}(A_r/B_r) = 1/r - 1/r = 0,
\]
i.e., each \( A_r \) must be regular and nonzero at the cusp \( \tau = 0 \). Also, each of these quotients \( A_r/B_4 \) is zero at the third fixed point; see [27] Table 2]. It follows that \( A_r \) must have big-O order at the third fixed point equal to \( 1/r \). The third fixed point is quadratic, resp. cubic, for \( r = 4 \), resp. \( r = 3 \); hence by (3.2), the small-o order of vanishing there will be \( 2 \cdot (1/4) = 1/2 \), resp. \( 3 \cdot (1/3) = 1 \). One concludes that \( A_4 \) has quadratic branch points on \( \mathcal{H} \), but its square and \( A_3 \) are single-valued.

The statements about the Fricke involution follow readily from the transformation law \( \eta(-1/\tau) = (-\tau)^{1/2} \eta(\tau) \) and the definitions of \( A_r, B_r, C_r \). To prove that \( A_2 = [2]^4 \cdot [1]^4 \cdot [4]^4 \), observe that \( A_2 / ([2]^4 \cdot [1]^4 \cdot [4]^4) \) has zero order of vanishing at each of the three inequivalent cusps of \( \Gamma_0(4) \). \( \square \)

Proposition 3.3.

1. On \( \Gamma_0(2) \), \( A_4^2 \) and \( A_4^4 \) have trivial character \( 1_2(d) \), which takes \( d \equiv 1 \) (mod 2) to 1.
2. On \( \Gamma_0(3) \), \( A_3 \) and \( A_3^3 \) have quadratic character \( \chi_{-3}(d) := (d^3)^{1/3} \), which takes \( d \equiv 1, 2 \) (mod 3) to 1, -1, and \( A_3^2 \) has trivial character \( 1_1(d) \), which takes \( d \equiv 1, 2 \) (mod 3) to 1.
3. On \( \Gamma_0(4) \), \( A_2 \) has quadratic character \( \chi_{-4}(d) := (d^2)^{1/2} \), which takes \( d \equiv 1, 3 \) (mod 4) to 1, -1, and \( A_2^2 \) have trivial character \( 1_4(d) \), which takes \( d \equiv 1, 3 \) (mod 4) to 1.
Similarly, trivial character, are well known \([13,14]\). For generating set \(\Gamma_6(3)\) has (minimal) generating set \(\pm \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right), \pm \left( \begin{smallmatrix} 1 & -1 \\ 3 & 1 \end{smallmatrix} \right)\), and for each of the associated maps \(\tau \mapsto \frac{a \tau + b}{c \tau + d}\), the power of \(\zeta_3\) appearing in the transformation law for \(\mathcal{B}_3^3\), deduced from \(33\), is consistent with the Dirichlet character \(\chi_{-3}\). The same is true for \(\zeta_3^3\); hence for \(A_3^3\) as well, since \(A_3^3 = \mathcal{B}_3^3 + \zeta_3^3\). Hence, the claim involving \(A_3^3, \mathcal{B}_3^3, \zeta_3^3\) is proved. Further details are left to the reader.

The formulas for \(\dim M_k(\Gamma_0(N))\) and \(\dim S_k(\Gamma_0(N))\), the dimensions of the vector spaces of all modular forms and of cusp forms on \(\Gamma_0(N)\) of weight \(k\), with trivial character, are well known \([13,14]\). For \(\Gamma_6(2), \Gamma_6(3), \Gamma_6(4)\), the spaces \(M_2, M_4\) have dimensions 1, 1, 2, 2, 3 respectively; and there are no cusp forms of weight 2 or 4. Also, \(\dim M_6(\Gamma_0(2)) = 2\), and there are no cusp forms of weight 6 on \(\Gamma_6(2)\).

Similarly, \(M_1(\Gamma_6(3), \chi_{-3}), M_3(\Gamma_6(3), \chi_{-3}), M_5(\Gamma_6(3), \chi_{-3})\) have dimensions 1, 2, 2, and \(M_1(\Gamma_6(4), \chi_{-4}), M_3(\Gamma_6(4), \chi_{-4})\) have dimensions 1, 2, cusp forms being absent in all cases. In the absence of cusp forms, all modular forms in the preceding spaces are combinations of Eisenstein series.

**Proposition 3.4.** The following spanning relations hold.

1. \(M_2(\Gamma_6(2)) = \langle A_2^2 \rangle, M_4(\Gamma_6(2)) = \langle A_4^4, B_4^4 \rangle, M_6(\Gamma_6(2)) = \langle A_6^6, A_6^2 B_4^4 \rangle\).
2. \(M_1(\Gamma_6(3), \chi_{-3}) = \langle A_3 \rangle, M_2(\Gamma_6(3)) = \langle A_2^2 \rangle, M_3(\Gamma_6(3), \chi_{-3}) = \langle A_3^3, B_3^3 \rangle, M_4(\Gamma_6(3)) = \langle A_2^4, A_4^2 B_2^2 \rangle\).
3. \(M_1(\Gamma_6(4), \chi_{-4}) = \langle A_2 \rangle, M_2(\Gamma_6(4)) = \langle A_2^2, B_2^2 \rangle, M_3(\Gamma_6(4), \chi_{-4}) = \langle A_4^4, B_4^4 A_2^2 B_2^2 \rangle, M_4(\Gamma_6(4)) = \langle A_4^4, B_4^4 A_2^2 B_2^2 \rangle\).

**Proof.** Immediate, by Proposition 3.3 and dimension considerations. \(\square\)

Let \(M_{\text{even}}(\Gamma)\) denote the graded ring of even-weight modular forms on \(\Gamma\). By exploiting the valence formula one can prove the following generalization.

**Proposition 3.5.**

1. \(M_{\text{even}}(\Gamma_6(2)) = C[A_2^2, B_4^4] = C[A_2^2, B_4^4 - \zeta_3^4]\).
2. \(M_{\text{even}}(\Gamma_6(3)) = C[A_3^3, A_2^3 B_3^3] = C[A_3^3, A_2^3 B_3^3 - \zeta_3^3]\).
3. \(M_{\text{even}}(\Gamma_6(4)) = C[A_2^2, B_2^2] = C[A_2^2, B_2^2 - \zeta_3^2]\).

In the sequel, some standard Eisenstein machinery will be used. (Cf. \([14\ Thms. 4.5.2, 4.6.2, 4.8.1]\) Let a subgroup \(\Gamma_0(N), N \geq 2\), and an integer weight \(k \geq 1\) be specified. Let a Dirichlet character \(\chi: (\mathbb{Z}/N\mathbb{Z})^\times \to C^\times\), extended to \(\mathbb{Z}\), satisfying \(\chi(-1) = (-1)^k\), also be given. The conductor (primitive one) \(\chi\) will divide \(N\). For each pair \(\psi, \phi\) of Dirichlet characters, the conductors \(\nu, \nu\) of which satisfy \(\nu | N\) and for which \(\psi \phi = \chi\) (the equality being one of characters mod \(N\)), there is an Eisenstein series \(E_k^\psi = \psi(\Gamma_0(N), \chi)\), namely

\[
E_k^\psi(q) := \begin{cases} 
1 + \sum_{c=1 \atop \gcd(c, \psi) = 1} \frac{2}{c} \cdot \hat{E}_k(1, \phi), & \text{if } \psi = 1, \\
2 \cdot \hat{E}_k(\psi, \phi), & \text{if } \psi \neq 1,
\end{cases}
\]
then these series are of the form
\[ E_n^\psi \phi (z) = \sum_{n=1}^{\infty} \sum_{0 < d | n} \psi(n/d) \phi(d) d^{k-1} \] (3.7)
and are of the form
\[ \psi \phi \] with the cusps of \( \Gamma \) subject to \( \psi(z) = \psi(|z|) \).

In the case when \( \psi = 1 \), these Eisenstein series are of the form \( E_k^\psi \psi^{-1} \), where \( \psi \) ranges over the characters mod \( N \) with conductor \( u \), subject to \( u^2 \mid N \). The subcase \( \psi = 1 \) is special: \( E_k^1 \) reduces to \( E_k \), the \( k \)-th Eisenstein series on \( \Gamma(1) \).

If \( k \geq 3 \), the collection of all \( E_k^\psi \psi^{-1} (q^a) \), where \( E_k^\psi \psi^{-1} \) is of the above form and \( 0 < \ell \mid N/(uv) \), is a basis for \( M_k(I_0(N), \chi)/S_k(I_0(N), \chi) \). For instance, if \( \chi = 1_N \), then these series are of the form \( E_k^\psi \psi^{-1} (q^a) \) with \( 0 < \ell \mid N/u \), and are equinumerous with the cusps of \( \Gamma(1) \), of which there are \( \sum_{0 < d | N} \phi(dN/d) \) in all. (Here, \( (\cdot, \cdot) \) and \( \phi(\cdot) \) are the g.c.d. and totient functions.) But if \( k \leq 2 \), the preceding statements must be modified. When \( k \leq 2 \), the Eisenstein series \( E_k^\psi \psi^{-1} (q^a) \) are quasi-modular but in general are not modular, so the quotient \( M_k(I_0(N), \chi)/S_k(I_0(N), \chi) \) is a proper subspace of their span.

**Proposition 3.6.** One has the Eisenstein series and divisor-function representations shown in Table 11 for monomials in \( A_r, B_r, r = 4, 3, 2 \), with multiplier systems of quadratic Dirichlet-character type. (The ones involving \( C_r \) are included for completeness; they follow from \( A_r, B_r, C_r \).)

**Proof.** For each of the monomials in Proposition 3.4 by working out the first few coefficients in its \( q \)-expansion one determines the Eisenstein representation given in the rightmost column, and hence the full \( q \)-expansion. This is a matter of linear algebra, since
\[
\begin{align*}
M_k(I_0(2)) &\subseteq (E_k(q^2), E_k(q)) , & k = 2, 4, 6, \\
M_k(I_0(3)) &\subseteq (E_k(q^3), E_k(q)) , & k = 2, 4, \\
M_k(I_0(4)) &\subseteq (E_k(q^4), E_k(q^2), E_k(q)) , & k = 2, 4, \\
M_k(I_0(3), \chi_{-3}) &\subseteq (E_k^{1, \chi_{-3}, E_k^{1, \chi_{-3}^A}}) , & k = 1, 3, 5, \\
M_k(I_0(4), \chi_{-4}) &\subseteq (E_k^{1, \chi_{-4}, E_k^{1, \chi_{-4}^A}}) , & k = 1, 3,
\end{align*}
\]
where \( \subseteq \) signifies \( \subset \) if \( k \leq 2 \) and \( = \) if \( k > 2 \). The \( L \)-series values \( L(1 - k, \chi_{-3}) \) and \( L(1 - k, \chi_{-4}) \) are computed from \( L(1 - k, \phi) = -B_k \phi/k \) and the generalized Bernoulli formula for any Dirichlet character \( \phi \) to the modulus \( N \).

\[
\sum_{k=0}^{\infty} B_k \phi \frac{x^k}{k!} = x \frac{\phi(a)}{e^{ax} - 1} \sum_{a=0}^{N-1} \phi(a) e^{ax}.
\]
\[ \square \]
Table 1.

| $M_2(I_0(2))$ | $A_4^r = 1 + 24 \sum \sigma_1(n; 0, 1)q^n = 2E_2(q^2) - E_2(q)$ |
| $M_4(I_0(2))$ | $A_4^r = 1 + 24 \sum \sigma_3(n; 3, 2)q^n = \frac{1}{2} [4E_4(q^3) + E_4(q)]$ |
|              | $B_4^r = 1 - 16 \sum \sigma_3(n; -1, 1)q^n = \frac{1}{16} [16E_4(q^2) - 2E_4(q)]$ |
|              | $C_4^r = 8 \sum \sigma_3(n; 7, 8)q^n = - \frac{2}{9} [E_2(q^2) - E_2(q)]$ |
| $M_6(I_0(2))$ | $A_4^6 = 1 + 18 \sum \sigma_3(n; 3, 4)q^n = \frac{1}{4} [8E_6(q^3) - E_6(q)]$ |
| $A_4^2B_4^2$ | $1 + 8 \sum \sigma_5(n; -1, 1)q^n = \frac{1}{16} [64E_6(q^3) - E_6(q)]$ |
| $A_4^2C_4^2$ | $2 \sum \sigma_5(n; 31, 32)q^n = \frac{1}{4} [E_4(q^7) - E_4(q)]$ |

| $M_1(I_0(3), \chi_3)$ | $A_3 = 1 + 6 \sum \sigma_0(n; 0, 1, -1)q^n = E_3^{2\chi_3}(q)$ |
| $M_2(I_0(3))$ | $A_3^2 = 1 + 12 \sum \sigma_1(n; 0, 1, 1)q^n = \frac{1}{3} [3E_3(q^2) - E_3(q)]$ |
| $M_3(I_0(3), \chi_3)$ | $A_3^3 = B_3^3 + C_3^3$ (see below) = $E_3^{3\chi_3}(q) + \frac{3}{2}E_3^{\chi_3+1}(q)$ |
| $B_3^3$ | $1 - 9 \sum \sigma_2(n; 0, 1, -1)q^n = E_3^{\chi_3}(q)$ |
| $C_3^3$ | $27 \sum \sigma_3(n; 0, 1, -1)q^n = \frac{3}{2} E_3^{\chi_3+1}(q)$ |
| $M_4(I_0(3))$ | $A_3^4 = 1 + 8 \sum \sigma_3(n; 4, 3, 3)q^n = \frac{1}{9} [9E_4(q^3) + E_4(q)]$ |
| $A_3B_3^3$ | $1 - 3 \sum \sigma_3(n; -2, 1, 1)q^n = \frac{1}{9} [81E_4(q^3) - E_4(q)]$ |
| $A_3C_3^3$ | $\sum \sigma_3(n; 26, 27, 27)q^n = - \frac{9}{4} [E_4(q^3) - E_4(q)]$ |
| $M_5(I_0(3), \chi_3)$ | $A_3^5 = A_3^2B_3^3 + A_3^2C_3^3$ (see below) = $E_3^{5\chi_3}(q) + \frac{4}{3}E_3^{\chi_3+1}(q)$ |
| $A_3^2B_3^3$ | $1 + 3 \sum \sigma_0(n; 0, 1, -1)q^n = E_3^{3\chi_3}(q)$ |
| $A_3^2C_3^3$ | $27 \sum \sigma_0(n; 0, 1, -1)q^n = \frac{3}{2} E_3^{\chi_3+1}(q)$ |

| $M_1(I_0(4), \chi_4)$ | $A_4 = 1 + 4 \sum \sigma_0(n; 0, 1, 0, -1)q^n = E_4^{\chi_4}(q)$ |
| $M_2(I_0(4))$ | $A_4^2 = 1 + 8 \sum \sigma_1(n; 0, 1, 1, 1)q^n = \frac{1}{4} [4E_4(q^2) - E_4(q)]$ |
| $B_4^2$ | $1 - 8 \sum \sigma_2(n; 0, 1, -2, 1)q^n = \frac{1}{4} [8E_4(q^2) - 6E_3^2(q^2) + E_3(q)]$ |
| $C_4^2$ | $8 \sum \sigma_3(n; 0, 2, -1, 2)q^n = - \frac{3}{4} [2E_4(q^2) - 3E_3^2(q^2) + E_3(q)]$ |
| $M_3(I_0(4), \chi_4)$ | $A_4^3 = A_2B_2^2 + A_2C_2^2$ (see below) = $E_4^{3\chi_4}(q) + 8E_4^{\chi_4+1}(q)$ |
| $A_4B_2^2$ | $1 + 4 \sum \sigma_0(n; 0, 1, 0, -1)q^n = E_4^{\chi_4}(q)$ |
| $A_4C_2^2$ | $16 \sum \sigma_0(n; 0, 1, 0, -1)q^n = 8E_4^{\chi_4+1}(q)$ |
| $M_4(I_0(4))$ | $A_4^4 = 1 + 4 \sum \sigma_3(n; 4, 4, 3, 4)q^n = \frac{1}{15} [16E_4(q^3) - 2E_4(q^2) + E_4(q)]$ |
| $B_4^3$ | $1 + 16 \sum \sigma_3(n; -1, 1)q^n = \frac{1}{15} [16E_4(q^3) - E_4(q)]$ |
| $C_4^3$ | $4 \sum \sigma_3(n; 7, 0, 8, 0)q^n = - \frac{16}{15} [E_4(q^3) - E_4(q)]$ |
| $A_2B_2^2$ | $1 - 2 \sum \sigma_3(n; -1, 0, 1, 0)q^n = \frac{1}{15} [16E_4(q^3) - E_4(q)]$ |
| $A_2C_2^2$ | $2 \sum \sigma_3(n; 7, 8)q^n = - \frac{2}{15} [E_4(q^3) - E_4(q)]$ |
| $B_2^2C_2^2$ | $2 \sum \sigma_3(n; -7, 8, -9, 8)q^n = \frac{1}{15} [16E_4(q^3) - 17E_4(q^2) + E_4(q)]$ |
Remark. Each $q$-expansion in Table 1 of a modular form of even weight $k$ can alternatively be written in terms of a $\sigma_{k-1}$ conjugate divisor function, rather than a $\sigma_k$ divisor function. For instance,

\[ A_4^2 = \vartheta_2^4 + \vartheta_3^4 = 1 + 24 \sum_{n=1}^{\infty} \sigma_1(n; 0, 1)q^n = 1 + 24 \sum_{n=1}^{\infty} \sigma_1^n(n; -1, 1)q^n, \quad (3.8a) \]

\[ C_4^4 = 4 \vartheta_2^4 \vartheta_3^4 = 8 \sum_{n=1}^{\infty} \sigma_3(n; 7, 8)q^n = 64 \sum_{n=1}^{\infty} \sigma_3^n(n; 0, 1)q^n. \quad (3.8b) \]

Using (3.8b), one can check that $A_4^2, C_4^4/64$ are identical to the forms $C, D$ used by Kaneko and Koike [23] as generators of $M_{\text{even}}(I_0(2))$.

Remark. The modular form $E_{1,3}^2(q) = 1 + 6 \sum_{n=1}^{\infty} \sigma_0(n; 0, 1, -1)q^n$ figured in Wiles’ proof of the Modularity Theorem; for a sketch, see [14, Ex. 9.6.4]. Table 1 reveals that this modular form is identical to $A_3$, the Borweins’ cubic theta function in the spirit of Ramanujan. This observation may be new.

Remark. Each representation in Table 1 can be rewritten as a Lambert series identity. Of the resulting identities, several were recorded by Ramanujan and have been given non-modular proofs by Berndt and others [45].

Remark. The difficulty in extending Table 1 to higher-degree monomials in the triples $A_r, B_r, C_r$, i.e., in deriving simple expressions for their Fourier coefficients in terms of divisor functions, is of course that one begins to encounter cusp forms. To some extent one can work around this. For instance, Van der Pol [33] expressed the coefficients of $A_3^{12}, B_3^{12}, C_3^{12}$, i.e., $\vartheta_3^{24}, \vartheta_4^{24}, \vartheta_2^{24}$, with the aid of Ramanujan’s tau function. Recently Hahn [21, Thm. 2.1], for each even $k \geq 4$, worked out the combination of the basis monomials $\{ A_4 2^a B_4 4^b, 2a + 4b = k \}$ of $M_k(I_0(2))$, i.e., the theta polynomials $\{ (\vartheta_2^4 + \vartheta_3^4)^a \vartheta_4^{8b}, 2a + 4b = k \}$, which equals

\[ E_k(q) := \frac{1}{2^{k-1}}[2^k E_k(q^2) - E_k(q)] = 1 + \frac{2k}{(2^k - 1)B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n; -1, 1)q^n. \]

This is a weight-$k$ Eisenstein form on $I_0(2)$ which vanishes at the cusp $\tau = 0$ and is nonzero at $\tau = i\infty$, like $B_4^k$. In effect, her combination of monomials (unlike the single monomial $B_4^k$ for even $k \geq 6$) has no cusp-form component, and therefore has a $q$-expansion with coefficients expressible in terms of divisor functions.

**Proposition 3.7.** One has the supplementary $q$-expansions shown in Table 2 for certain powers of $B_r, C_r$, $r = 4, 3, 2$, the multiplier systems of which are not of Dirichlet-character type. (In each, $k$ denotes the weight.)

**Proof.** Each $q$-expansion comes from an Eisenstein representation computed by linear algebra, like those of Table 1. The starting points are

\[ B_3(q), C_3(q^3) \in M_1(I_0(9), \chi_{-3}), \quad (3.9) \]

\[ B_2(q), C_2(q^2) \in M_1(I_6(8), \chi_{-4}), \quad (3.10) \]

which follow from the definitions of $B_3, C_3$ and $B_2, C_2$, like Proposition 5.3. (The statements about $C_3(q^3), C_2(q^2)$ here are equivalent to $C_3 \in M_1(\Gamma(3), \chi_{-3})$ and

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\[ \mathcal{E}_2 \in M_1(I_0(4) \cap \Gamma(2), \chi_{-4}). \] To derive the given expansions of \( \mathcal{B}_3, \mathcal{C}_4 \) and \( \mathcal{B}_4, \mathcal{C}_4^2 \), one simply uses the facts that \( \mathcal{B}_4 = \mathcal{B}_2 \) and \( \mathcal{C}_4 = 2^{-1/2} \cdot \mathcal{E}_2(q^{1/2}). \) (The latter fact incidentally implies that \( \mathcal{E}_2 \in M_1(\Gamma(4), \chi_{-4}). \)

The \( k = 1 \) expansions in Table 2 have previously been been derived by non-modular methods, in [17, §§ 32 and 33] and [7]. The final two expansions, of weight-3 forms on \( I_0(4) \), may possibly be classical (since \( \mathcal{B}_2 = \Theta_2 \) and \( \mathcal{C}_2 = \Theta_2 \)), but are more likely to be new. They come from

\[
\begin{align*}
\mathcal{B}_2(q) & = -E_1^{1/4}(q) + 2E_1(q^2) - 8E_3^{1/4}(q^2) + 64E_3(q^2), \\
\mathcal{C}_2(q) & = E_3^{1/4}(q) - E_3(q^2) + 2E_3^{1/4}(q^2) - 8E_3(q^2),
\end{align*}
\]

in which the four \( E_3 \)'s span \( M_3(I_0(8), \chi_{-4}). \)

The (formal!) weight-1 modular form \( \mathcal{A}_4 = \sqrt{\Theta_3^4 + \Theta_4^4} \) on \( I_0(2) \) does not fit into the preceding Eisenstein framework, since it is multivalued. This is why \( \mathcal{A}_4 \) and its odd powers are not expanded in Table 1 or 2. A bit of computation yields

\[ \mathcal{A}_4 = 1 + 12 [q - 5q^2 + 64q^3 - 917q^4 + 14850q^5 + \cdots], \]

but there is no obvious arithmetical interpretation of the (integral, see [23]) coefficients of this \( q \)-expansion, any more than there is for the \( q \)-expansions

\[
\begin{align*}
E_4^{1/4} & = 1 + 60 [q - 81q^2 + 11008q^3 - 1751057q^4 + \cdots], \\
E_6^{1/6} & = 1 - 84 [q + 243q^2 + 78784q^3 + 29826307q^4 + \cdots].
\end{align*}
\]

of the multivalued weight-1 forms \( E_4^{1/4}, E_6^{1/6} \) on \( \Gamma(1) \), introduced in §2. It should be noted that the form \( \mathcal{A}_4^2 \in M_2(I_0(2)) \) is the theta function of the \( D_4 \) lattice.
The divisor-function representations of Tables 1 and 2 can be viewed as theta identities; including even the \( r = 3 \) ones, since \( A_3, B_3, C_3 \) too can be expressed in terms of \( \vartheta_2, \vartheta_3, \vartheta_4 \). (See \[7\] and \$5\) below.) They imply, inter alia,

**Theorem 3.8.** Let \( r_2(n), n \geq 1 \), resp. \( r_4(n), n \geq 0 \), denote the number of ways of representing an integer \( n \) as the sum of 2 squares, resp. triangles. (These terms signify \( m^2 \), resp. \( m(m+1)/2 \), with \( m \) ranging over \( \mathbb{Z} \).) Then in terms of divisor and conjugate divisor functions,

\[
egin{align*}
 r_2(n) &= 4 \sigma_0(n; 0, 1, 0, -1), \\
r_4(n) &= 8 \sigma_1(n; 0, 1, 1, 1) = 8 \sigma_1^2(n; -3, 1, 1, 1), \\
r_6(n) &= 16 \sigma_2^2(n; 0, 1, 0, -1) - 4 \sigma_2(n; 0, 1, 0, -1), \\
r_8(n) &= 4 \sigma_3(n; 4, 4, 3, 4) = 16 \sigma_3^2(n; 15, 1, -1, 1); \\
t_2(n) &= 4 \sigma_0(4n + 1; 0, 1, -1, -1, 0, 1, 1, -1) \\
 &= 4 \sigma_0(8n + 2; 0, 1, 0, -1), \\
t_4(n) &= 8 \sigma_1(2n + 1; 0, 2, -1, 2) = 16 \sigma_1^2(2n + 1; 0, 1, -2, 1) \\
 &= 16 \sigma_1(2n + 1; 1) = 16 \sigma_1^2(2n + 1; 1), \\
t_6(n) &= \sigma_2(4n + 3; 0, -4, 1, 4, 0, -4, -1, 4) + 4 \sigma_2^2(4n + 3; 0, 1, -4, -1, 0, 1, 4, -1) \\
 &= 8 \sigma_2(4n + 3; 0, -1, 0, 1), \\
t_8(n) &= 4 \sigma_3(2n + 2; 7, 0, 8, 0) = 256 \sigma_3^2(2n + 2; 0, 0, 1, 0) \\
 &= 32 \sigma_3(n + 1; 7, 8) = 256 \sigma_3^2(n + 1; 0, 1). 
\end{align*}
\]

**Proof.** \( A_2 = \vartheta_3^2 \) and \( \vartheta_3(q) = \sum_{m \in \mathbb{Z}} q^m \); hence \( r_2(n) \) is the coefficient of \( q^n \) in the \( q \)-expansion of \( A_2^2 \). Similarly, \( C_2 = \vartheta_5^2 \) and \( \vartheta_5(q) = q^{1/4} \sum_{m \in \mathbb{Z}} q^{m(m+1)} \); hence \( t_2(n) \) is the coefficient of \( q^{2n} \) in the \( q \)-expansion of \( (q^{-1/2}C_2)^2 \).

Each \( r_2(n) \) formula is taken directly from Table 1 or 2 and if possible, rewritten in an alternative form based on a conjugate divisor function. The same is true of the first line of each of the \( t_2(n) \) formulas. The second, simpler lines of the latter follow by elementary arithmetic arguments. \( \Box \)

**Theorem 3.8** is a restatement of Jacobi’s Two, Four, Six, and Eight Squares Theorems, and the known formulas for \( t_2, t_4, t_6, t_8 \) \[31\]. But the present modular proof of the formulas for \( r_6(n), t_6(n) \), in particular, is illuminating. (For the history of these difficult formulas, see \[29\] p. 80.) The present proof, unlike previous arithmetical or elliptic ones, makes it clear for the first time how the two terms in the rather awkward formula for \( r_6(n) \) come from \( E_3^{X-1}, E_3^{X-1} \in M_3(\Gamma_0(4), \chi_4) \). In contrast, a modular derivation of the seemingly simple formula for \( t_6(n) \) has already been given by Ono et al. \[31\], but the present derivation, based on Eq. (3.12), reveals its complicated underpinnings.

Difficulties arise in extending any Eisenstein approach to \( s > 4 \), of course. As Rankin \[38\] showed, the power \( \vartheta_3^{2s} \) (i.e., \( A_2^{2s} \)) for each \( s > 4 \) has a nonzero cusp-form component.
4. Proof of Theorem 2.3(2)

Using the results obtained in the last section, one can derive the differential systems of Theorem 2.3(2) as an exercise in linear algebra, as follows.

The definition of quasi-modular form used here is standard. On $\mathfrak{H}^\ast$, a holomorphic function $f$ is quasi-modular of weight 2 and depth $\leq 1$ on a subgroup $\Gamma < \Gamma(1)$, with trivial multiplier system, if

$$f \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^2 f(\tau) + (s/2\pi i)c(c\tau + d), \quad (4.1)$$

for all $\pm \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma$ and some $s \in \mathbb{C}$. One writes $f \in M^\perp_2(\Gamma)$. The constant $s$ is called the coefficient of affinity of $f$.

**Lemma 4.1.** If $\mathcal{F} \in M_2(\Gamma, \mathcal{F})$ and $\mathcal{G} \in M_1(\Gamma, \mathcal{G})$, i.e., $\mathcal{F}, \mathcal{G}$ are modular forms on $\Gamma$ with multiplier systems not required to be of Dirichlet-character type, and $\mathcal{F}$ vanishes only at cusps, then

1. $\mathcal{E} := \mathcal{F}/\mathcal{F} \in M^\perp_2(\Gamma)$, and $\mathcal{E}$ has coefficient of affinity $k$.
2. $k \mathcal{E}' - \mathcal{E} \cdot \mathcal{E} \in M_4(\Gamma)$.
3. $k \mathcal{G}' - \ell \mathcal{E} \cdot \mathcal{G} \in M_4(\Gamma, \mathcal{G})$.

**Proof.** By differentiation of the transformation laws for $\mathcal{F}, \mathcal{G}$ and $\mathcal{E}$. $\square$

**Proof of Theorem 2.3(2).** Given $A_r, B_r, E_r, r = 4, 3, 2$, as in Definition 3.1 define the function $E_r$ of Theorem 2.3 as $(E_r')'/E_r$. By part (1) of the lemma, it is quasi-modular of weight 2 and depth $\leq 1$ on $I_0(2), I_0(3), I_0(4)$, respectively, with trivial multiplier system. The space of such quasi-modular forms is spanned respectively by $E_2(q^2), E_2(q)$; by $E_2(q^3), E_2(q)$; and by $E_2(q^4), E_2(q^3), E_2(q)$. By working out the first few Fourier coefficients of $E_r$ and $(E_r')'/E_r'$, and comparing them with those of these basis functions, one derives the Eisenstein and divisor-function representations of $E_r$ stated in the theorem.

By part (2) of the lemma, $r E_r' - E_r \cdot E_r$ must lie in $M_4(I_0(2)), M_4(I_0(3)), M_4(I_0(4))$ for $r = 4, 3, 2$. By working out the first two Fourier coefficients of $r E_r' - E_r \cdot E_r$, and comparing them with the $q$-expansions of the basis monomials of these vector spaces, given in Table II one proves that in each case this form equals $-A_r q^{-r} B_r$, as claimed.

A single example (the $r = 3$ case) will suffice. By direct computation,

$$3 E_3' - E_3 \cdot E_3 = -1 + 3q + \ldots, \quad (4.2)$$

and according to the table, $M_4(I_0(3))$ is spanned by

$$A_3^4 = 1 + 24q + \ldots, \quad (4.3a)$$
$$A_3 B_3^3 = 1 - 3q + \ldots. \quad (4.3b)$$

The identification of $3E_3' - E_3 \cdot E_3$ with $-A_3 B_3^3$ is justified by the agreement to first order in $q$. 
By part (3) of the lemma, \((A_r',r) - E_r \cdot A_r'\) must lie in the spaces \(M_\delta(I_0(2))\), \(M_\delta(I_0(3), \chi_{-1}), M_4(I_0(4))\), for \(r = 4, 3, 2\). By expanding in \(q\) again, and comparing coefficients with the \(q\)-expansions of the spanning monomials listed in Table I one proves that this form equals \(\neg A_r^2 B_r'\), as claimed. The details are elementary. \(\Box\)

One can derive the generalized Chazy equations of Theorem 2.3(1) by eliminating \(A_r, B_r, C_r\) from the differential systems satisfied by \(A_r, B_r, C_r\). But the computations are undesirably lengthy, especially for \(r = 3\). Alternative, more structured proofs of Theorem 2.3(1) will be given in § 6.

5. Elliptic integral and differential theta identities

This section is a digression, in which the systems of Theorem 2.3(2) are employed to derive an elliptic integral transformation formula and differential identities involving Jacobi’s theta-nulls. The latter are defined on \(\mathcal{H} \ni \tau\), i.e., on \(|q| < 1\), by

\[
\begin{align*}
\vartheta_2(q) &= \sum_{m \in \mathbb{Z}} q^{(m+\frac{1}{2})^2}, \\
\vartheta_3(q) &= \sum_{m \in \mathbb{Z}} q^{m^2}, \\
\vartheta_4(q) &= \sum_{m \in \mathbb{Z}} (-1)^m q^{m^2},
\end{align*}
\]

the given eta representations following from classical \(q\)-series identities. Each \(\vartheta_i\) is a weight \(\frac{1}{2}\) modular form on \(I_0(4)\) with a non-Dirichlet multiplier system \([34, \S 81]\). They satisfy \(\vartheta_3^4 = \vartheta_2^4 + \vartheta_4^4\). As noted in \([34]\), \(A_2, B_2, C_2\) equal \(\vartheta_3^2, \vartheta_4^2, \vartheta_2^2\), and moreover \([7]\), e.g., \(A_3(q) = \vartheta_3(q)(q^2) + \vartheta_2(q)(q^3)\).

The theta-nulls \(\vartheta_2, \vartheta_3, \vartheta_4\) vanish respectively at \(q = 0, -1, 1\), i.e., at the points \(\tau = i\infty, 1/2, 0\), which are the inequivalent cusps of \(I_0(4)\). Informally, each \(\vartheta_i\) has a simple zero at the respective cusp, and is nonzero and regular elsewhere. This does not mean that in the conventional analytic sense, \(\vartheta_i\) is bounded as either of the other two cusps is approached. For instance, \(\vartheta_3(q) \to \infty\) logarithmically as \(q \to 1^-\), i.e., as \(\tau \to 0\) along the positive imaginary axis. Having zero order of vanishing at a finite cusp does not preclude a logarithmic divergence.

The reader is cautioned that in the classical literature, and in the applied mathematics literature to this day, the argument of each \(\vartheta_i\) is taken to be \(q_2 := e^{\pi i \tau} = \exp(\pi i \tau)\) rather than \(q = \exp(2\pi i \tau)\). Using \(q_2\) rather than \(q\) is equivalent to viewing the theta-nulls as modular forms on \(I'(2)\) rather than \(I_0(4)\), since the two subgroups of \(I'(1)\) are conjugates under the 2-isogeny \(\tau \mapsto 2\tau\) in \(PSL(2, \mathbb{R})\). In this article the \(I_0(4)\) convention is adhered to.

The following is a brief review of how theta-nulls arise from elliptic integrals. Consider the parametric family \(\mathcal{E}\) of elliptic plane curves \(E_x / \mathbb{C}\) defined by the equation \(y^2 = (1-x^2)(1-\alpha x^2)\), where \(\alpha \in \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}\). The (first) complete elliptic integral \(K = K(\alpha)\) is defined by

\[
K(\alpha) = \frac{1}{2} \int_0^1 x^{-1/2}(1-x)^{-1/2}(1-\alpha x)^{-1/2} \, dx
\]

which makes sense if 0 \(\leq\) \(\alpha < 1\), and can be continued to a holomorphic function on \(\mathbb{P}^1(\mathbb{C})\), slit between \(\alpha = 1\) and \(\alpha = \infty\) to ensure single-valuedness. The
fundamental periods of the curve $E_\alpha$ are proportional to $K(\alpha), iK(1-\alpha)$, so its period ratio $\tau = \tau_1/\tau_2 \in \mathbb{H}$ is $iK(1-\alpha)/K(\alpha)$. Since $K(0) = \pi/2$, it is convenient to normalize by defining $K = K/(\pi/2)$.

One can show (e.g., by comparing $q$-series) that if $K$ is regarded as a function of the nome $q = \exp(2\pi i \tau)$, i.e., $\hat{K} = \hat{K}(q)$, then $\hat{K}(q)$ equals $\Theta^2(q)$, which is holomorphic and single-valued on $\mathbb{H}$. The reason for this equality is that in modern language, $\Theta$ is the elliptic family attached to $\Gamma_0(4)$. The parameter $\alpha$ can also be viewed as a function of $q$, i.e., as a $\Gamma_0(4)$-stable holomorphic function on $\mathfrak{H}$, with a zero at $\tau = i\infty$ and a pole at $\tau = 0$; it is a Hauptmodul for $\Gamma_0(4)$.

So, $\hat{K} = \Theta^2 = A_2$. By Table 1, $\hat{K} \in M_1(\Gamma_0(4); \chi_{-4})$, and $\hat{K}$ has the Eisenstein series representation

$$\hat{K}(q) = 1 + 4 \sum_{n=1}^{\infty} \sigma_0(n; 0, 1, 0, -1) q^n = E_1^{\chi_{-4}}(q). \quad (5.2)$$

This expansion is well known, as is the presence of the character $\chi_{-4}$ in the transformation law of $\hat{K}$ under $\tau \mapsto \frac{a\tau + b}{c\tau + d}$ with $(a, b, c, d) \in \Gamma_0(4)$. But, analogous expansions and transformation properties for the second complete elliptic integral are not. This function $E = E(\alpha)$ is defined locally (on $\alpha \leq 0 < 1$) by

$$E(\alpha) = \frac{1}{2} \int_0^1 x^{-1/2} (1 - x)^{-1/2} (1 - \alpha x)^{1/2} \, dx. \quad (5.3)$$

Since $E(0) = \pi/2$ also, one normalizes by letting $\tilde{E} = E/(\pi/2)$. As with $\hat{K}, \tilde{E}$ can be viewed as $\tilde{E}(q)$, a holomorphic and single-valued function on $\mathfrak{H}$. It is a classical result (see [16] p. 218 and [20] § 31]) that

$$\hat{K}(q)\tilde{E}(q) = 1 + 8 \sum_{n=1}^{\infty} \frac{q^{2n}}{(1 + q^{2n})^2} = 1 + 4 \sum_{n=1}^{\infty} \sigma_1(n; -1, 0, 1, 0) q^n = \frac{1}{2} \left[ 4E_2(q^4) - E_2(q^2) \right]. \quad (5.4)$$

Hence $\tilde{E} = (\hat{K}\tilde{E})/\hat{K}$, i.e.,

$$\tilde{E}(q) = \frac{1 + 4 \sum_{n=1}^{\infty} \sigma_1(n; -1, 0, 1, 0) q^n}{1 + 4 \sum_{n=1}^{\infty} \sigma_0(n; 0, 1, 0, -1) q^n} = \frac{4E_2(q^4) - E_2(q^2)}{3E_1^{\chi_{-4}}(q)} \quad (5.5)$$

(cf. [20] § 38]). Remarkably, the divisor-function representation of (5.4) is identical to that of the quasi-modular form $\varepsilon_2 \in M_2^{\chi_{-1}}(\Gamma_0(4))$, given in Theorem 2.3. So,

$$\varepsilon_2 = (\varepsilon_2^2)'/\varepsilon_2^2 = (\theta_2^4)'/\theta_2^4 = \hat{K}\tilde{E} \quad (5.6)$$

and $\hat{K}\tilde{E} \in M_2^{\chi_{-1}}(\Gamma_0(4))$. Also, one can write $\tilde{E} = \varepsilon_2/A_2$.

**Proposition 5.1.** The forms $\hat{K}, \tilde{E}, \hat{K}\tilde{E}$ have the transformation laws

$$\hat{K}(q_1) = \chi_{-4}(d)(c\tau + d)\hat{K}(q),$$

$$\tilde{E}(q_1) = \chi_{-4}(d) \left[ (c\tau + d)\tilde{E}(q) + (\pi i)^{-1}c\hat{K}(q) \right],$$

$$\hat{K}\tilde{E}(q_1) = (c\tau + d)^2 \hat{K}\tilde{E}(q) + (\pi i)^{-1}c(c\tau + d),$$

for all $((a, b, c, d)) \in \Gamma_0(4)$. Here, $q = \exp(2\pi i \tau_1)$, $q_1 = \exp(2\pi i \tau_1)$ with $\tau_1 = \frac{a\tau + b}{c\tau + d}$. 

Remark. This transformation law under \( \Gamma_0(4) \) for the (normalized) second complete elliptic integral \( E(q) \) is arguably the most informative obtained to date. Tri-comi [46, Chap. IV, § 2] has some related results, but it is difficult to compare them, since he (i) used homogeneous modular forms, i.e., functions of \( \tau_1, \tau_2 \) rather than \( \tau \), (ii) worked in terms of \( K(\alpha), E(\alpha) \), and especially, (iii) treated only \( \left( \frac{a}{c} \frac{b}{d} \right) = \left( \frac{1}{0} \frac{0}{1} \right) \), which are generators of \( \Gamma(1) \) rather than \( \Gamma_0(4) \) (or \( \Gamma(2) \)).

**Remark.** One can similarly define quasi-modular forms \( \hat{K}\hat{G}, \hat{K}\hat{I} \in \mathcal{M}^{\leq 1}(\Gamma_0(4)) \) by

\[
\begin{align*}
(A_2^2)' / A_2^2 &= (\vartheta_3^4)' / \vartheta_3^4 =: \hat{K}\hat{G}, \\
(B_2^2)' / B_2^2 &= (\vartheta_4^4)' / \vartheta_4^4 =: \hat{K}\hat{I}
\end{align*}
\]

(cf. Glaisher [20]), and work out the transformation laws of \( \hat{G}, \hat{I} \). Each of \( \hat{G}, \hat{I} \) has a representation as a complete elliptic integral, analogous to (5.3) for \( \hat{E} \).

**Proposition 5.2.** The theta-nulls \( \vartheta_2, \vartheta_3, \vartheta_4 \), together with \( \hat{K}\hat{E} \), satisfy a differential system on the half-plane \( \Im \tau \), namely

\[
\begin{align*}
4 \vartheta_2'/\vartheta_2 &= \hat{K}\hat{E}, \\
2(\hat{K}\hat{E})' &= (\hat{K}\hat{E})^2 - \vartheta_3^4 \vartheta_4^4, \\
4 \vartheta_3'/\vartheta_3 &= \hat{K}\hat{E} - \vartheta_4^4, \\
4 \vartheta_4'/\vartheta_4 &= \hat{K}\hat{E} - \vartheta_3^4,
\end{align*}
\]

where ' signifies \( dq/dq = (2\pi i)^{-1}d/d\tau \).

**Proof.** Substitute \( \vartheta_2^2, \vartheta_3^2, \vartheta_4^2, \hat{K}\hat{E} \) for \( A_2, B_2, C_2, \bar{E}_2 \) in the \( r = 2 \) system of Theorem 2.

**Remark.** This system of coupled ODEs may be new, though it can be deduced from identities of Glaisher and of the Borweins [6, § 2,3]. For \( i = 2, 3, 4 \), one can derive from it a nonlinear third-order ODE satisfied by \( \vartheta_i \), by eliminating the other three dependent variables. For each \( \vartheta_i \) this turns out to be

\[
(\vartheta^2 \vartheta'' - 15 \vartheta \vartheta' \vartheta'' + 30 \vartheta^2 \vartheta^2) = 32(\vartheta \vartheta'' - 3 \vartheta^2)^3 = \vartheta^{10}(\vartheta \vartheta'' - 3 \vartheta^2)^2.
\]

This is the 1847 equation of Jacobi [24], which was mentioned in § 2. His derivation used differentiation with respect to Hauptmoduls for \( \Gamma_0(4) \) (his \( k^2 \) and \( k'^2 \)).

For an easy proof that each of \( \vartheta_2, \vartheta_3, \vartheta_4 \) must satisfy the same third-order ODE, reason as follows. First, work out the differential systems for \( \vartheta_2, \vartheta_3, \vartheta_4; \hat{K}\hat{G} \) and \( \vartheta_2, \vartheta_3, \vartheta_4; \hat{K}\hat{I} \) that are analogues of the system for \( \vartheta_2, \vartheta_3, \vartheta_4; \hat{K}\hat{E} \) in Proposition 5.2. Then notice that up to cyclic permutations of the ordered pairs \( (\vartheta_2, \hat{K}\hat{E}), (\vartheta_3, \hat{K}\hat{G}), (\vartheta_4, \hat{K}\hat{I}) \), the three systems are the same. Hence, eliminating all dependent variables except a single \( \vartheta_i \) must yield the same equation, irrespective of \( i \).

Brezhnev [8, § 7] has recently derived a different but related differential system, symmetric and elegant, in which the dependent variables are \( \vartheta_2, \vartheta_3, \vartheta_4 \), and (in the notation used here) the element \( (\hat{K}\hat{E} + \hat{K}\hat{G} + \hat{K}\hat{I})(q) \) of \( \mathcal{M}^{\leq 1}(\Gamma_0(4)) \), which by examination is proportional to \( E_2(q^2) \). His system can be obtained by averaging together the three preceding ones; and this averaging ensures symmetry.
6. Proofs of Theorem 2.3(1); Hypergeometric identities

Direct derivations of the generalized Chazy equations of Theorem 2.3(1) will now be given. They will not employ, except superficially, the differential systems satisfied by the weight-1 modular forms \( A_r, B_r, C_r \).

Two proofs of Theorem 2.3(1) are supplied. The first is an explicitly modular, linear-algebraic one. It is modeled after Resnikoff’s proof \([40]\) of Eq. (5.8), and relies on results of \([27]\). It is based on a sort of nonlinear hypergeometric identity, stated as Proposition 6.3, which holds for certain very special parameter values and is based on elementary arithmetic methods. One may speculate that the generalized Chazy equations can also be derived by such methods.

6.1. A modular proof of Theorem 2.3(1)

Define \( A_r, B_r, C_r \) as in (33) and let \( E_r = (E_r^r)'/E_r^r \), as in the proof of Theorem 2.3(2). For \( r = 4, 3, 2 \), \( E_r \) is quasi-modular of weight 2 and depth \( \leq 1 \) on \( \Gamma_0(N) \), \( N = 2, 3, 4 \), respectively. For \( k = 4, 6, 8, \ldots \), define \( u_k^{(r)} \) by

\[
u_k^{(r)} = rE_r^r - E_r^r \cdot E_r, \quad u_k^{(r)} = u_k^{(r)} - (k/r)E_r \cdot u_k^{(r)}.
\]

By Lemma 2.7, \( u_k^{(r)} \in M_k(\Gamma_0(N)) \). If \( u := (2\pi i/r)E_r \) and \( u_k \) is defined in terms of \( u \) as in (33) then one has that \( u_k = (2\pi i)^k u_k/r^2 \) for all \( k \).

By the last differential equation in Theorem 2.3(2c), \( u_4^{(r)} = -A_{2r}^{-r}B_r^r \).

By Theorem 2.3(2b), \( \text{Ord}_{0}(A_r) = 0 \) and \( \text{Ord}_{0}(\Gamma_0(N)) = 1/r \); hence one has that \( \text{Ord}_{\Gamma_0(N)}(u_4^{(r)}) = 1 \). It is evident that \( \text{Ord}_{\Gamma_0(N)}(u_k^{(r)}) \geq 1 \) for \( k \geq 4 \).

According to the valence formula \([41]\) Chap. V], the total number of zeroes of a nonzero element \( f \in M_k(\Gamma_0(N)) \), counted with respect to local parameters, is equal to \( (k/12)[\Gamma(1) : \Gamma_0(N)] \). It follows that at any \( s \in \mathfrak{f}^{\ast} \), it is the case that \( \text{Ord}_{\Gamma_0(N)}(f) > (k/12)[\Gamma(1) : \Gamma_0(N)] \) if \( f = 0 \). Here, the subgroup index \( [\Gamma(1) : \Gamma_0(N)] \) equals 3, 4, 6 when \( N = 2, 3, 4 \).

In the following analyses, the superscript \(^{(r)}\) will be omitted for readability.

- \( r = 4 \), \( \Gamma_0(N) = \Gamma_0(2) \). One sets \( k = 12 \), i.e., uses linear algebra on \( M_{12}(\Gamma_0(2)) \).
  - For each \( g \in \mathfrak{W} = \{ u_4 u_8, u_8^2, u_4^3 \} \), it is the case that \( g \in M_{12}(\Gamma_0(2)) \) and \( \text{Ord}_{\Gamma_0(2)}(g) \geq 2 \). There is a linear combination \( f \) of the three monomials in \( \mathfrak{W} \) for which \( \text{Ord}_{\Gamma_0(2)}(f) \geq 4 \). But if \( f \in M_{12}(\Gamma_0(2)) \) vanishes with order greater than \( (k/12)[\Gamma(1) : \Gamma_0(2)] = 3 \), then \( f = 0 \).
This combination can be found by direct computation, using $q$-series (even though $q$-series are expansions at the infinite cusp, not at $\tau = 0$). To $O(q^4)$,

$$u_4 u_8 = \frac{3}{4} - 8q + \ldots,$$
$$u_6^2 = 1 + 16q + \ldots,$$
$$u_4^3 = -1 + 48q + \ldots.$$

There is a unique combination (up to scalar multiples) that is zero to this order, and must therefore vanish identically; namely, $2u_4 u_8 - 2u_6^2 + u_4^3$. Its vanishing is equivalent to $u_4 u_8 - u_6^2 + 8u_4^3 = 0$.

- $r = 3, \Gamma_0(N) = \Gamma_0(3)$. One sets $k = 20$, i.e., uses linear algebra on $M_{20}(\Gamma_0(3))$. For each $g \in \mathbb{V} = \{u_4 u_8, u_6^2 u_8, u_4^3 u_8, u_4^2 u_6^2, u_4^3\}$, it is the case that $g \in M_{20}(\Gamma_0(3))$ and $\text{Ord}_{\Gamma_0(3)}(g) \geq 3$. There is a unique combination $f$ of the five monomials in $\mathbb{V}$ for which $\text{Ord}_{\Gamma_0(3)}(g) \geq 7$. But if $f \in M_{20}(\Gamma_0(3))$ vanishes with order greater than $(k/12)[\Gamma(1):\Gamma_0(3)] = 20/3$, then $f = 0$.

As in the $r = 4$ case, this combination can be found by a direct computation (a tedious one). To $O(q^3)$,

$$u_4 u_8^2 = -\frac{64}{7} - \frac{115}{9} q + 23q^2 - \frac{2123}{9} q^3 + \ldots,$$
$$u_6^2 u_8 = -\frac{128}{27} - \frac{368}{9} q - \frac{244}{9} q^2 + \frac{7381}{9} q^3 + \ldots,$$
$$u_4^3 u_8 = \frac{8}{3} - 13q - 201q^2 + 2075q^3 + \ldots,$$
$$u_4^2 u_6^2 = \frac{16}{9} - \frac{8}{3} q - 71q^2 - \frac{2654}{3} q^3 + \ldots,$$
$$u_4^5 = -1 + 15q + 45q^2 - 2145q^3 + \ldots.$$

There is a unique combination (up to scalar multiples) that is zero to this order, and therefore must vanish identically; namely,

$$9u_4 u_8^2 - 9u_6^2 u_8 + 24u_4^3 u_8 - 15u_3^2 u_6^2 + 16u_4^5.$$

Its vanishing is equivalent to $u_4 u_8^2 = u_6^2 u_8 + 24u_4^3 u_8 - 15u_3^2 u_6^2 + 144u_4^5 = 0$.

- $r = 2, \Gamma_0(N) = \Gamma_0(4)$. No linear algebra is needed, since as noted in the statement of Theorem 2.3.1, $E_2(q) = E_2(q^2)$. By comparing

$$u_4^{(4)} = 4E_4' - E_4 \cdot E_4$$
$$u_{k+1}^{(4)} = u_k^{(4)} - (k/4)E_4 \cdot u_k^{(4)},$$

one deduces that

$$u_k^{(2)}(q) = 2^{(k-4)/2}u_k^{(4)}(q^2).$$

But in the $r = 4$ case,

$$2u_4 u_8 - 2u_6^2 + u_4^3 = 0$$

(see the treatment above). Hence, for $r = 2$,

$$u_4 u_8 - u_6^2 + 2u_4^3 = 0.$$

This is equivalent to $u_4 u_8 - u_6^2 + 8u_4^3 = 0$. 

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6.2. A hypergeometric proof of Theorem 2.3(1)

This proof is in the spirit of Jacobi, since it employs Hauptmoduls and derivatives with respect to them. It uses the results of [27], which were inspired by the following standard theorem on subgroup actions of $\text{PSL}(2, \mathbb{R})$ [18 § 44, Thm. 15].

**Theorem 6.1.** Let $\Gamma < \text{PSL}(2, \mathbb{R})$ be a Fuchsian group of Mobius transformations of $\mathfrak{H}$ (of the first kind) that has a Hauptmodul $\tau = \tau(\tau)$, i.e., a non-constant simple automorphic function with a single simple zero on a fundamental region of $\Gamma$. Then $\tau$ can be expressed as a (multivalued) function of $t$ as $f_1/t_2$, a ratio of independent solutions $f_1, f_2$ of some second-order differential equation

$$\mathcal{L}(\Gamma) f := \left[ D_t^2 + P(t) \cdot D_t + Q(t) \right] f = 0 \quad (6.1)$$

on $\mathbb{P}^1(\mathbb{C})$, in which $P, Q \in \mathbb{C}(t)$.

Equation (6.1) is called a Picard–Fuchs equation (the term being historically most accurate when $\Gamma < \Gamma'(1)$). It is an ODE on the genus-zero curve $\mathbb{P}^1(\mathbb{C})$, which is essentially the fundamental region of $\Gamma$ with boundary identifications, i.e., the (compactified) quotient of $\mathfrak{H}$ by $\Gamma$. It follows from a second theorem on automorphic functions [13 § 110, Thm. 6] that Eq. (6.1) must be a ‘Fuchsian’ ODE, i.e., all its singular points on $\mathbb{P}^1(\mathbb{C})$ must be regular. These points are bijective with the vertices of the fundamental region of $\Gamma$. The difference of the two characteristic exponents of the operator $\mathcal{L}(\Gamma)$ will be 0 at a cusp, and $1/n$ at an order-$n$ elliptic fixed point. That is, it will be the reciprocal of the order of the associated stabilizing subgroup.

The Picard–Fuchs equation has solution space $\mathbb{C}f_1 \oplus \mathbb{C}f_2$, i.e., $(\mathbb{C} \tau \oplus \mathbb{C})f_2$. It will shortly be of interest to determine whether the logarithmic derivative $u := f_2/f_1$ also satisfies an ODE, in this case with respect to $\tau \in \mathfrak{H}$. (As always, the dot signifies differentiation with respect to $\tau$.) For this, the following will be useful. Let $u_k, k = 4, 6, \ldots$, be defined as in Theorem 2.3, i.e., $u_4 = \dot{u} - u_2^2$ and $u_{k+2} = \dot{u}_k - ku_2$, and let differentiation with respect to $\tau$ be denoted by a subscripted $\dot{\tau}$.

**Lemma 6.2.** One can write $u_k = \dot{u}_k t^k / 2$, where the sequence $\dot{u}_4, \dot{u}_6, \ldots$ follows from $\dot{u}_4 = -Q$ and $\dot{u}_{k+2} = (\dot{u}_k) + (k/2)Pu_k$. Thus

$$u_4 = -Qt^2,$$
$$u_6 = -(Q + 2PQ)t^3,$$
$$u_8 = -(Q + 5PQ + 2QP^2 + 6P^2Q)t^4.$$

**Proof.** $t = (d\tau/d\tau)^{-1} = 1/(f_1/f_2) = f_2/w$, where $w = w(f_1, f_2)$ is the Wronskian. Similarly, $\dot{t} = Pt^2 + 2ut$ comes from (6.1) by differential calculus. The recurrence $\dot{u}_{k+2} = (\dot{u}_k) + (k/2)Pu_k$ comes from $u_{k+2} = \dot{u}_k - ku_2$ by substituting $d\tau/d\tau = 1/dt$, and exploiting these facts. □

Picard–Fuchs differential operators $\mathcal{L}(\Gamma)$ that illustrate Theorem 6.1 were obtained in [27] for the groups $\Gamma = \Gamma_0(N), N = 2, 3, 4$, among others. The corresponding Hauptmoduls $t = t_N = t_N(\tau)$ were chosen to be

$$t_2(\tau) = 2^{12} \cdot [2]^{24}/[1]^{24}, \quad t_3(\tau) = 3^6 \cdot [3]^{12}/[1]^{12}, \quad t_4(\tau) = 2^8 \cdot [4]^{16}/[1]^8,$$
where the prefactors are of arithmetical significance but are not important here (they could equally well be set equal to unity). In each of these three cases, \( t_N = 0 \) corresponds to the infinite cusp, and \( t_N = \infty \) to the cusp \( \tau = 0 \). For \( N = 2, 3, 4 \), the operators \( \mathcal{L}_N = D_N^2 + P(t_N) \cdot D_N + Q(t_N) \) were computed to be

\[
\begin{align*}
\mathcal{L}_2 &= D_2^2 + \frac{1}{t_2^2} + \frac{1}{2(t_2 + 64)} D_2 + \frac{1}{16t_2(t_2 + 64)}, \quad (6.2a) \\
\mathcal{L}_3 &= D_3^2 + \frac{1}{t_3^2} + \frac{2}{3(t_3 + 27)} D_3 + \frac{1}{9t_3(t_3 + 27)}, \quad (6.2b) \\
\mathcal{L}_4 &= D_4^2 + \frac{1}{t_4^2} + \frac{1}{t_4 + 16} D_4 + \frac{1}{4t_4(t_4 + 16)}, \quad (6.2c)
\end{align*}
\]

Each is a Gauss hypergeometric operator, up to a scaling of the independent variable. That is, each has three (regular) singular points, located at \( t = t_N^\alpha, \infty, 0 \), where \( t_N^\alpha = -64, -27, -16 \) is the third fixed point of \( I_0(N) \) on the quotient curve \( X_0(N) = I_0(N) \setminus \mathcal{S}_f \cong \mathbb{P}^1(C)_{t_N} \). It is respectively a quadratic elliptic point, a cubic one, and a third cusp (the image of \( \tau = 1/2 \)), as mentioned in \[3\).

For each \( N \), there is a solution \( f = h_N(t_N) \) of \( \mathcal{L}_N f = 0 \) that is holomorphic and equal to unity at \( t_N = 0 \). It was shown in \[27\] that if \( h_N(\tau) := (h_N \circ t_N)(\tau) \), then

\[
\begin{align*}
h_2(\tau) &= [1]^{1/2}, \\
h_3(\tau) &= [1]^{1/3}, \\
h_4(\tau) &= [1]^{1/4}.
\end{align*}
\]

That is, the holomorphic local solution of \((6.1)\) at the infinite cusp, in each case, can be continued to a weight-1 modular form on \( \mathcal{S}_f \). In fact, \( h_2, h_3, h_4 \) are respectively equal to \( B_2, B_3, B_2 \) in the notation of the present article (see Definition \[3\]).

For \( N = 2, 3, 4 \), a weight-1 modular form \( \bar{h}_N(\tau) = (h_N \circ t_N)(\tau) \) that vanishes at the infinite cusp, and has zero order of vanishing at the cusp \( \tau = 0 \), is obtained by multiplying \( h_N(\tau) \) by an appropriate power of the Hauptmodul \( t_N(\tau) \). Let

\[
\begin{align*}
\bar{h}_2(t_2) &= 2^{-3/2} t_2^{1/4} h_2(t_2), \\
\bar{h}_3(t_3) &= 3^{-1} t_3^{1/3} h_3(t_3), \\
\bar{h}_4(t_4) &= 2^{-1} t_4^{1/4} h_4(t_4).
\end{align*}
\]

Then by Definition \[3\], \( \bar{h}_2(\tau), \bar{h}_3(\tau), \bar{h}_4(\tau) \) are identical to \( \bar{c}_4, \bar{c}_3, \bar{c}_2 \). It follows by changing (dependent) variables in the equations \( \mathcal{L}_N h_N = 0 \) that \( \bar{h}_N \) satisfies the slightly modified Picard–Fuchs equation \( \mathcal{L}_N \bar{h}_N = 0 \), where

\[
\begin{align*}
\mathcal{L}_2 &= D_2^2 + \left[ \frac{1}{2t_2} + \frac{1}{2(t_2 + 64)} \right] D_2 + \frac{4}{t_2^2(t_2 + 64)}, \quad (6.3a) \\
\mathcal{L}_3 &= D_3^2 + \left[ \frac{2}{3t_3} + \frac{2}{3(t_3 + 27)} \right] D_3 + \frac{3}{t_3^2(t_3 + 27)}, \quad (6.3b) \\
\mathcal{L}_4 &= D_4^2 + \left[ \frac{0}{t_4} + \frac{1}{t_4 + 16} \right] D_4 + \frac{4}{t_4^2(t_4 + 16)}, \quad (6.3c)
\end{align*}
\]

The fixed points of each \( I_0(N) \) on the corresponding quotient \( X_0(N) \cong \mathbb{P}^1(C)_{t_N} \) are visible in \((6.3)\), just as in \((6.2)\).

Each modified equation \( \mathcal{L}^N f = 0 \) is of the form \( \mathcal{L}_{\alpha, \beta, \gamma} f = 0 \), where

\[
\mathcal{L}_{\alpha, \beta, \gamma} := D_t^2 + \left[ \frac{\alpha + \beta}{t} + \frac{1 - \alpha}{t - t^*} \right] D_t + \frac{1}{4t^2(t - t^*)} \left[ t^2 - (1 - \alpha - \beta)^2 t^* \right]. \quad (6.4)
\]
The operator $\mathcal{L}_{\alpha, \beta, \gamma}$ is the general second-order Fuchsian operator on $\mathbb{P}^1(\mathbb{C})$, that has singular points at $t = t^*, \infty$ with respective exponent differences $\alpha, \beta, \gamma$, and with one exponent at each of $t = t^*, \infty$ constrained to be zero. It is of hypergeometric but not Gauss-hypergeometric type. The solutions of $\mathcal{L}_{\alpha, \beta, \gamma}f = 0$ include

$$t^{(1-\alpha-\beta-\gamma)/2} {}_2F_1\left(\frac{1-\alpha-\beta-\gamma}{2}, \frac{1-\alpha+\beta-\gamma}{2}; 1-\gamma; t/t^*\right),$$

which is the local solution at $t = 0$ associated to the exponent $(1 - \alpha - \beta - \gamma)/2$. (Here, ${}_2F_1(\lambda, \mu; \nu; x)$ is the Gauss hypergeometric function, defined and single-valued on the disk $|x| < 1$.) This is the representation of the form $\hat{h}_N = \hat{c}_r$ as a (multivalued) function of the Hauptmodul $t = t_N$. For $N = 2, 3, 4$, the parameters $(\alpha, \beta, \gamma)$, which are the reciprocals of the orders of elements of $\Gamma_0(N)$ that stabilize the corresponding fixed points, are respectively $(\frac{1}{4}, 0, 0), (\frac{1}{6}, 0, 0), (0, 0, 0)$.

By Theorem 2.3(2), $u = (2\pi i/r) \hat{c}_r$ equals $\hat{c}_r/\hat{c}_r$. Hence, if one takes the operator $\mathcal{L}^{(\tau)} = \mathcal{L}^{(\hat{c}_r)}$ to equal $\mathcal{L}_N$ rather than $\mathcal{L}_N$, the function $u$ of Theorem 2.3(1) will agree with the function $u = f_2/f_1$, in the notation of Theorem 6.3 and Lemma 6.2. Therefore Theorem 2.3(1) will follow immediately from

**Proposition 6.3.** Let $\tau = f_1/f_2$, a ratio of independent solutions of the hypergeometric ODE $\mathcal{L}_{\alpha, \beta, \gamma}f = 0$ on $\mathbb{P}^1(\mathbb{C})$. Let $u = f_2/f_1$, and let $u_k, k = 4, 6, \ldots$ be defined by $u_4 = \hat{u} - \hat{u}^2$ and $u_{k+2} = u_k - k\hat{u}u_k$. Then $u$, regarded as a function of $\tau$, will satisfy a nonlinear third-order ODE: the generalized Chazy equation

$$
\begin{align*}
&u_4u_8 - u_6^2 + 8u_4^3 = 0, & \text{if } (\alpha, \beta, \gamma) = (\frac{1}{4}, 0, 0), \\
&u_4u_8 - u_6^2 + 24u_4u_8 - 15u_4^2u_6 + 144u_4^2 = 0, & \text{if } (\alpha, \beta, \gamma) = (\frac{1}{6}, 0, 0), \\
&u_4u_8 - u_6^2 + 8u_4^3 = 0, & \text{if } (\alpha, \beta, \gamma) = (0, 0, 0).
\end{align*}
$$

**Proof.** By direct computation, using the expressions of Lemma 6.2 for $u_4, u_6, u_8$ in terms of the coefficient functions $P, Q \in \mathbb{C}(t)$, which can be read off from the formula (6.4) for $\mathcal{L}_{\alpha, \beta, \gamma}$. \(\square\)

So, in each of the cases $N = 2, 3, 4$, i.e., $r = 4, 3, 2$, the quasi-modular form $u = \hat{c}_r/\hat{c}_r$, satisfies a generalized Chazy equation. This proof of Theorem 2.3(1) is more analytic than the proof given in 5.1 and less explicitly modular.

The reader may wonder whether this second, alternative proof was necessary. It required extra machinery, such as the Picard–Fuchs equations $\mathcal{L}_N^hN = 0$ and $\mathcal{L}_N^N = 0$, and the Hauptmoduls $\tau_N$, the $q$-expansions of which are relatively complicated and are not discussed here. Also, Proposition 6.3 is restricted to very special triples of parameter values.

In fact, Proposition 6.3 is the tip of an interesting iceberg. Following the is its extension to arbitrary triples $(\alpha, \beta, \gamma)$.

**Theorem 6.4.** Let $\tau = f_1/f_2$, a ratio of independent solutions of the hypergeometric ODE $\mathcal{L}_{\alpha, \beta, \gamma}f = 0$ on $\mathbb{P}^1(\mathbb{C})$. Let $u = f_2/f_1$, and let $u_k, k = 4, 6, \ldots$ be defined by $u_4 = \hat{u} - \hat{u}^2$ and $u_{k+2} = u_k - k\hat{u}u_k$. Then $u$, regarded as a function of $\tau$, will satisfy a nonlinear third-order ODE: a generalized Chazy equation

$$
C_{68}u_4^2u_8^2 + C_{56}u_4u_8u_6^2 + C_{94}u_4^2u_8 + C_6u_4^2 + C_{64}u_4u_6^2 + C_{44}u_8^2 = 0
$$

with coefficients symmetric under $\alpha \leftrightarrow \beta$, namely
\[
\begin{align*}
C_{88} &= (2\alpha - 1)(2\beta - 1)(\alpha + \beta - \gamma - 1)^2, \\
C_{86} &= -[(2\alpha - 1)(3\beta - 1) + (3\alpha - 1)(2\beta - 1)](\alpha + \beta - \gamma - 1)^2(\alpha + \beta + \gamma - 1)^2, \\
C_{84} &= -16(2\alpha - 1)(2\beta - 1)(\alpha + \beta - 1)(\alpha + \beta - \gamma - 1)(\alpha + \beta + \gamma - 1), \\
C_{66} &= (3\alpha - 1)(3\beta - 1)(\alpha + \beta - \gamma - 1)^2(\alpha + \beta + \gamma - 1)^2, \\
C_{64} &= 4 \left[2(2\alpha - 1)^2(3\beta - 1) + 2(3\alpha - 1)(2\beta - 1)^2 - 3(\alpha - \beta)^2\right] \\
&\quad \times (\alpha + \beta - \gamma - 1)(\alpha + \beta + \gamma - 1), \\
C_{44} &= 64(2\alpha - 1)(2\beta - 1)(\alpha + \beta - 1)^2.
\end{align*}
\]

Proof. With the aid of a computer algebra system, eliminate $t$ from the expressions for $u_8/u_3$ and $u_6/u_4$ that follow from Lemma 6.2. As in the proof of Proposition 6.3, $P, Q \in \mathbb{C}[t]$ come from Eq. (6.4).

Theorem 6.4 is a nonlinear hypergeometric identity, relevant even to hypergeometric equations without modular applications. It belongs to the theory of special functions, but as one sees, in a loose sense it is a relation of linear dependence among modular forms of weight 24 (since each monomial has that weight).

Rational exponent differences $\alpha, \beta, \gamma$ that are not members of $\{0, \frac{1}{4}, \frac{1}{2}\}$ occur in the theory of automorphic functions on subgroups of $PSL(2, \mathbb{R})$ that are not subgroups of $\Gamma(1) = PSL(2, \mathbb{Z})$. This will be illustrated in the next section.

7. Differential systems for weight-1 forms

The systems of Theorem 7.1, satisfied by triples $A_r, B_r, C_r$, will now be greatly generalized. Associated to any first-kind Fuchsian subgroup $\Gamma < PSL(2, \mathbb{R})$ that is a triangle group, i.e., that has a hyperbolic triangular fundamental domain in $\mathcal{H}$ and a Hauptmodul, there are weight-1 modular forms $\mathcal{A}, \mathcal{B}, \mathcal{C}$ (possibly multivalued) that vanish respectively at the three vertices. The forms will satisfy a system of coupled nonlinear first-order ODEs with independent variable $\tau$.

The key result on this is Theorem 7.1, which is proved by hypergeometric manipulations related to those of Ohyama [53]. It deals with the case when $\Gamma$ has at least one cusp, which without loss of generality can be taken to be $\tau = i\infty$.

In §§7.2 and 7.3 as illustrations, the triangle subgroups $\Gamma < PSL(2, \mathbb{R})$ that are commensurable with $\Gamma(1) = PSL(2, \mathbb{Z})$ are examined. These include $\Gamma(1)$ itself; $H_0(N), N = 2, 3$; the Fricke extensions $H_0^+(N), N = 2, 3$; the index-2 subgroup $\Gamma^2$ of $\Gamma(1)$; and two others [43]. For each such $\Gamma$, the forms $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are worked out explicitly, as are the hypergeometric representation of $\mathcal{A}$, the differential system the forms satisfy, and the generalized Chazy equation that the system implies.

7.1. Hypergeometric manipulations

Let $\Gamma < PSL(2, \mathbb{R})$ be a Fuchsian subgroup (of the first kind), regarded as a group of Möbius transformations of $\mathcal{H} \ni \tau$. If it has a Hauptmodul $t = t(\tau)$, i.e., is
of genus zero, then \( \Gamma \setminus \mathcal{S}^* \cong \mathbb{P}^1(\mathbb{C}) \), and \( \tau \) can be expressed as a ratio \( \tau_1/\tau_2 \) of two solutions \( f = \tau_1/\tau_2 \) of a Picard–Fuchs equation \( \mathcal{L}(\Gamma)f = 0 \) on \( \mathbb{P}^1(\mathbb{C}) \), as stated in Theorem 6.11. Its solution space will be \( \mathbb{C}\tau_1 \oplus \mathbb{C}\tau_2 = (\mathbb{C}\tau \oplus \mathbb{C})\tau_2 \). The solution \( \tau_2 \) can be viewed as a weight-1 form on \( \Gamma \). This follows by considering the homogeneous counterpart \( \Gamma^* < \text{SL}(2, \mathbb{R}) \) to \( \Gamma \), which acts on vectors \( \{\tau_i\} \), and the associated homogeneous forms, which are functions of \( \tau_1, \tau_2 \).

Suppose that \( \Gamma \) is a triangle group, i.e., is of genus zero with a triangular fundamental domain and hence with three inequivalent classes of fixed points on \( \mathcal{S}^* \), say classes A,B,C. They correspond to three conjugacy classes of stabilizing elements of \( \Gamma \), either elliptic or parabolic. The group \( \Gamma \) is specified up to conjugacy by their orders, i.e., by a signature \((n_{\alpha}, n_{\beta}, n_{\gamma})\) such as the signature \((3,2,\infty)\) of \( \Gamma(1) \). It will be assumed that at least one of these orders is \( \infty \), i.e., at least one of the three classes is parabolic, corresponding to a cusp. Without loss of generality one can take \( n_{\gamma} = \infty \), and the cusp to be \( \tau = i\infty \). This infinite cusp will be fixed by some \( \tau \rightarrow \tau + v \), where by definition, \( v \in \mathbb{R}^+ \) is its width.

By a Möbius transformation the Hauptmodul \( t \) can be redefined, if necessary, so that \( t = 0 \) at the infinite cusp, and \( t = t^*, \infty \), for some \( t^* \in \mathbb{C} \setminus \{0\} \), on the fixed points in the classes A,B. The Picard–Fuchs equation, being hypergeometric, will then have \( t = t^*, \infty, 0 \) as its (regular) singular points. Their respective exponent differences \( (\alpha, \beta, \gamma) \) will equal \((1/n_{\alpha}, 1/n_{\beta}, 1/n_{\gamma})\). These are vertex angles in terms of \( \pi \) radians, and necessarily \( \alpha + \beta + \gamma < 1 \). (If the convention of the last paragraph is adopted then \( n_{\gamma} = \infty \) and \( \gamma = 1/\infty = 0 \), but for reasons of symmetry \( n_{\gamma} \) and \( \gamma \) will be kept as independent parameters.)

Fuchs’s relation on characteristic exponents implies that the six exponents of any second-order ODE of hypergeometric type must sum to unity. This leaves two degrees of freedom in the choice of \( \mathcal{L}(\Gamma) \), as in §6.2. (Cf. \( \mathcal{L}_N \) vs. \( \mathcal{L}_\bar{N} \).) Let \( \mathcal{L}(\Gamma) \) be chosen to have exponents

\[
\{0, \alpha\} \at \t = \t^*, \quad \{0, \beta\} \at \t = \infty, \quad \frac{1 - \alpha - \beta - \gamma}{2}, \frac{1 - \alpha - \beta + \gamma}{2} \at \t = 0, \tag{7.1}
\]

i.e., so that there is a zero exponent at each of \( t = t^*, \infty \). With this choice,

\[
\mathcal{L}(\Gamma) = \mathcal{L}_{\alpha, \beta, \gamma} := D_t^2 + \left[ \frac{\alpha + \beta}{t} + \frac{1 - \alpha}{t - t^*} \right] D_t + \frac{[\gamma^2 - (1 - \alpha - \beta)^2]t^*}{4t^2(t - t^*)}, \tag{7.2}
\]

defined as in §6.3. Let the local solution of \( \mathcal{L}(\Gamma)f = \mathcal{L}_{\alpha, \beta, \gamma}f = 0 \) at the singular point \( t = 0 \) (i.e., the infinite cusp), associated to the exponent \((1 - \alpha - \beta - \gamma)/2\), be denoted \( C = C(t) \). Then the lifted function \( \mathcal{C}(\tau) := C(t(\tau)) \) will be a weight-1 form on \( \Gamma \), which vanishes at cusps in class \( C \) because \( \alpha + \beta + \gamma < 1 \).

Also, define (potentially multivalued) functions \( \mathcal{A}(\tau), \mathcal{B}(\tau) \) that vanish at the fixed points in the classes A,B, at which \( t = t^*, \text{resp.} \ t = \infty \), by

\[
\mathcal{A} = [(t - t^*)/t]^{1/\rho} \mathcal{C}, \tag{7.3a}
\]

\[
\mathcal{B} = [-t^*/t]^{1/\rho} \mathcal{C}, \tag{7.3b}
\]

where

\[
\rho := \frac{2}{1 - \alpha - \beta - \gamma} = \frac{2}{1 - n_{\alpha}^{-1} - n_{\beta}^{-1} - n_{\gamma}^{-1}}. \tag{7.4}
\]
With these definitions,  
\[ \mathcal{A}^0 = \mathcal{B}^0 + \mathcal{C}^0. \]  
(7.5)

The corresponding quotients
\[ t/(t-t^*) = \mathcal{C}^0 / \mathcal{A}^0, \]  
(7.6a)
\[ t/t^* = -\mathcal{C}^0 / \mathcal{B}^0, \]  
(7.6b)
are normalized Hauptmoduls, the respective values of which on the classes A,B,C are \( \infty, 1, 0 \) and \( 1, \infty, 0 \). The \( \Gamma \)-specific quantity \( \rho \in \mathbb{Q}^+ \) of (7.4) generalizes the ‘signature’ \( r \) that parametrizes Ramanujan’s alternative theories of elliptic integrals. (Recall that \( r = 4, 3, 2 \) correspond to \( \Gamma = I_0(2), I_0(3), I_0(4) \), i.e., to \( (n, \alpha, \beta, \epsilon) = (2, \infty, \infty), (3, \infty, \infty), (\infty, \infty, \infty) \).

The functions \( \mathcal{A}(\tau), \mathcal{B}(\tau) \) could also be defined as \( A(t(\tau)) \) and \( B(t(\tau)) \), where \( A(t), B(t) \) are solutions of Picard–Fuchs equations having appropriately modified exponents, but the same exponent differences as \( L^{(\Gamma)}_\alpha, \gamma \) i.e., \( \alpha, \beta, \gamma \). (Cf. the relation between \( \mathcal{L}_N, \mathcal{L}'_N \) in §6.2) Their respective exponents would be
\[ \{1 - \frac{\alpha - \beta - \gamma}{2}, 1 + \frac{\alpha - \beta - \gamma}{2}\} \text{ at } t = t^*, \quad \{0, \beta\} \text{ at } t = \infty, \quad \{0, \gamma\} \text{ at } t = 0, \]  
(7.7a)
\[ \{0, \alpha\} \text{ at } t = t^*, \quad \{1 - \frac{\alpha - \beta - \gamma}{2}, 1 + \frac{\alpha - \beta - \gamma}{2}\} \text{ at } t = \infty, \quad \{0, \gamma\} \text{ at } t = 0. \]  
(7.7b)

Gauss-hypergeometric representations of \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) in terms of \( \Phi_1 \) are
\[ \mathcal{A}(\tau) = A(t(\tau)) = \Phi_1 \left( \frac{1 - \alpha - \beta - \gamma}{2}, 1 - \gamma; 1 - \gamma; t(\tau)/t(t^*) \right), \]  
(7.8a)
\[ \mathcal{B}(\tau) = B(t(\tau)) = \Phi_1 \left( \frac{1 - \alpha - \beta - \gamma}{2}, 1 + \gamma; 1 - \gamma; t(\tau)/t(t^*) \right), \]  
(7.8b)
\[ \mathcal{C}(\tau) = C(t(\tau)) = \left( -t/t^* \right)^{1/2} \Phi_2 \left( \frac{1 - \alpha - \beta - \gamma}{2}, 1 + \gamma; 1 - \gamma; t(\tau)/t(t^*) \right), \]  
(7.8c)
in which the normalizations of \( \mathcal{A}, \mathcal{B}, \mathcal{C} \), not previously specified, have been set by requiring that \( \mathcal{A}, \mathcal{B} \) equal unity at the infinite cusp, at which \( t = 0 \). The parameters and arguments of the \( \Phi_1 \)'s are determined by \( \Phi_1 (\lambda, \mu; v; x) \) having exponents \( \{0, 1 - v\} \) at \( x = 0 \), \( \{0, v - \lambda - \mu\} \) at \( x = 1 \), and \( \{\lambda, \mu\} \) at \( x = \infty \).

The representations
\[ \mathcal{A} = \Phi_1 \left( \frac{1 - \alpha - \beta - \gamma}{2}, 1 - \gamma; \mathcal{C}^0 / \mathcal{A}^0 \right), \]  
(7.9a)
\[ \mathcal{B} = \Phi_2 \left( \frac{1 - \alpha - \beta - \gamma}{2}, 1 - \gamma; -\mathcal{C}^0 / \mathcal{B}^0 \right) \]  
(7.9b)
follow from (7.8a) with the aid of (7.6a). The identities (7.9b) are equivalent: they are related by Pfaff’s transformation of \( \Phi_1 \). The function \( \Phi_1 (\lambda, \mu; v; x) \) is defined on the disk \( |x| < 1 \), so (7.9b) hold in a neighborhood of the infinite cusp, at which \( \mathcal{C} = 0 \). In fact each will hold near any cusp in the class C, if an appropriate constant of proportionality is included. Similarly, there follow
\[ \mathcal{C} \propto \Phi_1 \left( \frac{1 - \alpha - \beta - \gamma}{2}, 1 - \beta; \mathcal{A}^0 / \mathcal{C}^0 \right), \]  
(7.10a)
\[ \mathcal{C} \propto \Phi_2 \left( \frac{1 - \alpha - \beta - \gamma}{2}, 1 - \beta; -\mathcal{C}^0 / \mathcal{B}^0 \right), \]  
(7.10b)
which hold near any fixed point in the class A, resp. B, with the constant of proportionality dependent on the choice of fixed point.
As defined, \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) are formally weight-1 modular forms on \( \Gamma \), with some multiplier systems; and they vanish respectively on the classes A,B,C of fixed points of \( \Gamma \) on \( \mathcal{S} \). However, if A, resp. B comprises elliptic fixed points, then \( \mathcal{A} \), resp. \( \mathcal{B} \) may be a multivalued function of \( \tau \). This is because of the fractional powers in their definitions (7.3b).

The test for multivaluedness on \( \mathcal{S} \) is as follows. By construction, each of \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) has order of vanishing (computed with respect to a local parameter for \( \Gamma \); e.g., \( t \)) equal to \( 1/\rho \). Fixed points in classes A,B,C are mapped to \( \Gamma \) \( \Setminus \mathcal{S} \cong \mathbb{P}^1(C) \), with multiplicities \( n_{\mathcal{A}}, n_{\mathcal{B}}, n_{\mathcal{C}} \). If \( n_{\mathcal{A}} < \infty \), resp. \( n_{\mathcal{B}} < \infty \), signaling ellipticity, then the order of vanishing of \( \mathcal{A}, \mathcal{B} \) at the associated elliptic points on \( \mathcal{S} \), in classes A,B, will be \( n_{\mathcal{A}}/\rho, n_{\mathcal{B}}/\rho \in \mathbb{Q}^+ \). If this is not an integer then \( \mathcal{A} \), resp. \( \mathcal{B} \) will be multivalued, i.e., the \( k \)th root of a true modular form (of weight \( k \)), where \( k \) equals the numerator of the fraction \( \rho/n_{\mathcal{A}} = \rho \alpha, \rho/n_{\mathcal{B}} = \rho \beta \), expressed in lowest terms.

The generalization of Theorem 6.3(2) to arbitrary triangle groups can now be stated and proved. As always, \(' \) signifies \( qd/dq = (2\pi i)^{-1}d/d\tau \).

**Theorem 7.1.** Let \( \Gamma < \text{PSL}(2, \mathbb{R}) \) be a triangle group with signature \( (n_{\mathcal{A}}, n_{\mathcal{B}}, n_{\mathcal{C}}) \) and exponents \( (\alpha, \beta, \gamma) = (1/n_{\mathcal{A}}, 1/n_{\mathcal{B}}, 1/n_{\mathcal{C}}) \), with \( \alpha + \beta + \gamma < 1 \), and let \( \rho := \frac{1}{2 - \alpha - \beta - \gamma} \). Assume that \( \gamma = 0 \), i.e., that the third vertex is a cusp, and define the formal weight-1 modular forms \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) as above, vanishing at the classes A,B,C of fixed points of \( \Gamma \) on \( \mathcal{S} \) (the last containing the infinite cusp, of width \( \nu \)), and satisfying \( \mathcal{A}^0 = \mathcal{B}^0 + \mathcal{C}^0 \). Then \( \mathcal{A}^0, \mathcal{B}^0, \mathcal{C}^0 \), along with the weight-2, depth-\( \leq 1 \) quasi-modular form \( \mathcal{E} := \nu(\mathcal{E}^0)/(\mathcal{E}^0) \) that is associated with class C, satisfy the coupled system of nonlinear first-order equations

\[
\begin{align*}
\nu(\mathcal{A}^0)' = & \mathcal{E} \cdot \mathcal{A}^0 - \mathcal{A}^{0(1-\alpha)} \mathcal{B}^{0(1-\beta)}, \\
\nu(\mathcal{B}^0)' = & \mathcal{E} \cdot \mathcal{B}^0 - \mathcal{A}^{0(1-\alpha)} \mathcal{B}^{0(1-\beta)}, \\
\nu(\mathcal{C}^0)' = & \mathcal{E} \cdot \mathcal{C}^0, \\
\nu \mathcal{E}'' = & \mathcal{E} \cdot \mathcal{E} - \mathcal{A}^{0(1-2\alpha)} \mathcal{B}^{0(1-2\beta)},
\end{align*}
\]

from which a generalized Chazy equation \( C_{\rho} \) for \( u = (2\pi i/\nu \mathcal{E}) \mathcal{E} \), parametrized by \( \alpha, \beta, \gamma \) as in Theorem 6.3(2) can be derived by elimination. (The third equation says that \( u = \mathcal{E}/\mathcal{E} \).)

**Proof.** Equation (7.11c) is true by definition, and (7.11a) is implied by (7.11b), (7.11c) and \( \mathcal{A}^0 = \mathcal{B}^0 + \mathcal{C}^0 \). It remains to prove (7.11b), (7.11d). They will come from a useful formula for \( i = dt/d\tau \), deduced as follows.

The solution space of the Picard–Fuchs equation \( \mathcal{L}_{\alpha, \beta, \gamma} \mathcal{f} = 0 \) is \( C f_1 \oplus C f_2 \oplus \mathcal{C} \mathcal{T} \mathcal{C} \oplus \mathcal{C} \mathcal{C} \cong (\mathcal{C} \mathcal{T} \oplus \mathcal{C}) \mathcal{C} \), where \( \tau \) is viewed as a (multivalued) function on the quotient curve \( \mathbb{P}^1(C) \), and \( \mathcal{C} \) is defined in terms of the \( \mathcal{C} \mathcal{C} \) by (7.8c). From the expression for \( \mathcal{L}_{\alpha, \beta, \gamma} \) given in (7.2), the Wronskian \( \mathcal{w} = \mathcal{w}(f_1, f_2) = \mathcal{w}(\tau \mathcal{C}, \mathcal{C}) \) must equal a multiple of \( 1/t^{\alpha+\beta}(t^{1-\alpha}) \). The constant of proportionality can be calculated by taking the \( t \to 0 \) limit, in which the infinite cusp is approached. In this limit,

\[
\tau \sim \left( \frac{\nu}{2\pi i} \right) \log t, \quad C \sim (−t/t^*)^{1/\rho},
\]

(7.12)
the former coming from \( t \sim \text{const} \cdot q^{1/\nu} \), which is true since \( \nu \) is the width of the infinite cusp. One readily deduces that

\[
1/w(t) = \left[ 2\pi i (-t^*)^{-\beta}/\nu \right] \cdot t^{\alpha+\beta} (t-t^*)^{1-\alpha}.
\]

(7.13)

As in the proof of Lemma 6.2, \( t = f_2^2/w \), i.e., \( t = \mathcal{C}^2(\tau)/w(t(\tau)) \) on \( \mathfrak{H} \). Taking account of (7.3b), one can rewrite this in the useful form

\[
i = [2\pi i (-t^*)/\nu] \mathcal{E}^\rho(1-\alpha) \mathcal{E}^{-\rho(1+\beta)} \mathcal{E}^\rho.
\]

(7.14)

Now consider the logarithmic derivative of the equality \( \mathcal{E}^\rho = (-t^*/t) \mathcal{E}^\rho \), i.e.,

\[
(\mathcal{E}^\rho)'/\mathcal{E}^\rho = (\mathcal{E}^\rho)'/\mathcal{E}^\rho - t'/t.
\]

(7.15)

By employing (7.14) to expand \( t' = i/2\pi i \), one obtains Eq. (7.11b). Equation (7.11d) follows by similar manipulations. \(\square\)

Let \( \mathcal{E}_A \) and \( \mathcal{E}_B \) denote the weight-2, depth-\( \leq 1 \) quasi-modular forms associated with classes A and B, i.e., \( \nu(\mathcal{E}^\rho)'/\mathcal{E}^\rho \) and \( \nu(\mathcal{E}^\rho)'/\mathcal{E}^\rho \), just as the form \( \mathcal{E}_C = \mathcal{E} = \nu(\mathcal{E}^\rho)'/\mathcal{E}^\rho \) is associated with class C. Moreover, let \( u_A, u_B, u_C \) denote the normalized forms \( (2\pi i/\nu)p_A, (2\pi i/\nu)p_B, (2\pi i/\nu)p_C \), so that \( u_A = \mathcal{E}_A/\mathcal{E}_A, u_B = \mathcal{E}_B/\mathcal{E}_B, u_C = \mathcal{E}_C/\mathcal{E}_C \). Then a bit of calculus applied to Eqs. (7.11abcd) yields

**Corollary 7.2. The weight-2, depth-\( \leq 1 \) quasi-modular forms \( u_A, u_B, u_C \) satisfy**

\[
\begin{align*}
\dot{u}_A &= u_A^2 - (1 + \rho \alpha)(u_A - u_B)(u_A - u_C), \\
\dot{u}_B &= u_B^2 - (1 + \rho \beta)(u_B - u_C)(u_B - u_A), \\
\dot{u}_C &= u_C^2 - (1 + \rho \gamma)(u_C - u_A)(u_C - u_B).
\end{align*}
\]

This is a so-called generalized Darboux–Halphen (gDH) system of ODEs \[1, 10, 22\]. It is evident that the gDH system with \( (\alpha, \beta, \gamma) = (1/n_\mathcal{E}, 1/n_\mathcal{E}, 1/n_\mathcal{E}) \) and \( \rho = 1 - \alpha - \beta - \gamma \) arises naturally from the unique (up to conjugacy) triangle subgroup of \( \mathrm{PSL}(2, \mathbb{R}) \) with signature \( (n_\mathcal{E}, n_\mathcal{E}, n_\mathcal{E}) \). Examples of gDH systems coming from modular subgroups have appeared in the literature; e.g., the ones coming from the six (up to conjugacy) triangle subgroups of \( \Gamma(1) = \mathrm{PSL}(2, \mathbb{Z}) \), which are incidentally the only gDH systems for which some linear combination of \( u_A, u_B, u_C \) satisfies the classical Chazy equation \[10\]. However, the general statement is new.

### 7.2. Triangle groups commensurable with \( \Gamma(1) \)

The triangle subgroups of \( \mathrm{PSL}(2, \mathbb{R}) \) commensurable with \( \Gamma(1) = \mathrm{PSL}(2, \mathbb{Z}) \) are well known. (Subgroups \( \Gamma_1, \Gamma_2 \) are said to be commensurable if \( \Gamma_1 \cap \Gamma_2 \) is of finite index in both.) Up to conjugacy there are exactly nine \[43\], listed in Table 3. Each is hyperbolic with at least one cusp. They are of three types, and it will be shown that to each type there is associated a differential system, parametrized by \( \rho \), which is satisfied by weight-1 forms \( \mathcal{E}, \mathcal{B}, \mathcal{C} \). For the first type the system will
be that of Theorem 7.12, in which \( \rho \) equals the signature of Ramanujan’s elliptic theories (i.e., \( \rho = r = 4, 3, 2 \)).

Type I comprises \( \Gamma_0(N) \), \( N = 2, 3, 4 \), and Type II comprises \( \Gamma(1) \) and the Fricke extensions \( \Gamma_0^+(N) \), \( N = 2, 3 \), which are not subgroups of \( \Gamma(1) \). Type III comprises three groups that will be called \( 2a'\), \( 4a'\), \( 6a'\). The group \( 2a'\) is the index-2 subgroup \( \Gamma^2 < \Gamma(1) \), but the latter two are not subgroups of \( \Gamma(1) \). These names are taken from Harnad and McKay [22]. It is known that the intersections of the groups \( 4a', 6a' \) with \( \Gamma(1) \) are \( \Gamma_0(4) \cap \Gamma(2) \), \( \Gamma_0(3) \cap \Gamma(2) \), which are conjugates of \( \Gamma_0(8) \), \( \Gamma_0(12) \) under \( \tau \mapsto 2\tau \).

For each triangle group \( \Gamma \), the exponents \( (\alpha, \beta, \gamma) = (1/n, 1/n, 1/n) \) are given; as is \( \rho = 2/(1 - \alpha - \beta - \gamma) \), which subsumes the signature of Ramanujan’s theories. For concreteness, generators \( a, b, c \in \Gamma \) of corresponding stabilizing subgroups are given as well. (What are given are homogeneous versions \( \tilde{a}, \tilde{b}, \tilde{c} \in GL(2, R) \). To convert them to \( a, b, c \in PSL(2, R) \), satisfying \( cab = \pm I \), divide each by its determinant and prepend \( \pm \).) These generators are adapted from [22 Table 1]. By convention, \( n_{\tilde{a}} = \infty \) and \( \gamma = 0 \), and the corresponding class \( C \) of fixed points (cusps) includes the infinite cusp. The width of the infinite cusp is denoted \( \upsilon \), as above.

The expressions for the formal (i.e., potentially multivalued) weight-1 modular forms \( \mathcal{A}, \mathcal{B}, \mathcal{C} \), which in each case satisfy \( \mathcal{A}^0 = \mathcal{B}^0 + \mathcal{C}^0 \), come as follows. Starting with the ones for \( \Gamma \) of Type I and for \( \Gamma = \Gamma(1) \), which have already been discussed, they are obtained as consequences of the index-2 subgroup relations [44]

\[
\Gamma_0(2) <_2 \Gamma_0^+(2), \quad \Gamma_0(3) <_2 \Gamma_0^+(3);
\]

\[
2a' <_2 \Gamma(1), \quad 4a' <_2 \Gamma_0^+(2), \quad 6a' <_2 \Gamma_0^+(3).
\]

For instance, suppose that \( \tilde{\Gamma} <_2 \Gamma \) and that the respective signatures satisfy \( 2\rho = \rho \). The goal is to relate the associated triples \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) and \( \mathcal{A}, \mathcal{B}, \mathcal{C} \). Let the corresponding classes of fixed points be \( \tilde{A}, \tilde{B}, \tilde{C} \) and \( A, B, C \). Suppose that \( \tilde{B}, \tilde{C} \) are cusp classes under \( \Gamma \), which merge into a single class \( C \) under \( \Gamma \), but that \( \tilde{A} = A \). This is precisely what happens when \( (\tilde{\Gamma}, \Gamma) = (\Gamma_0(N), \Gamma_0^+(N)) \) for \( N = 2, 3 \).

Under these assumptions, one will have \( \mathcal{A} = \mathcal{A} \). Expressions for \( \mathcal{B}, \mathcal{C} \) in terms of \( \mathcal{B}, \mathcal{C} \) come from a Hauptmodul relation. Hauptmoduls for \( \tilde{\Gamma}, \Gamma \), i.e., rational parameters on the quotient curves \( \tilde{\mathfrak{H}}^* \backslash \tilde{\Gamma}, \mathfrak{H}^* \backslash \Gamma \), will be \( \lambda = \mathcal{B}^0 / \mathcal{A}^0 \) resp. \( \lambda = \mathcal{C}^0 / \mathcal{A}^0 \). The index-2 relation \( \tilde{\Gamma} <_2 \Gamma \) induces a double covering \( \tilde{\mathfrak{H}}^* \backslash \tilde{\Gamma} \to \mathfrak{H}^* \backslash \Gamma \), i.e., \( \mathbb{P}^1(C) \mapsto \mathbb{P}^1(C)_\lambda \), i.e., a quadratic rational map \( \lambda \mapsto \lambda \). Since \( \lambda = 1, 0 \) (corresponding to classes \( \tilde{B}, \tilde{C} \) must be mapped to \( \lambda = 0 \) (corresponding to class \( C \)), and \( \lambda = \infty \) (corresponding to class \( \tilde{A} \)) must be mapped to \( \lambda = \infty \) (corresponding to class \( A \)), the map must be

\[
\lambda = 4\lambda(1 - \lambda) = 1 - (1 - 2\lambda)^2.
\]
Table 3. For each triangle subgroup $\Gamma < \text{PSL}(2, \mathbb{R})$ commensurable with $\Gamma(1) = \text{PSL}(2, \mathbb{Z})$, the basic data and the triple $\mathcal{A}, \mathcal{B}, \mathcal{C}$ of (possibly multivalued) weight-1 modular forms, satisfying $\mathcal{A}\mathcal{P} = \mathcal{B}\mathcal{P} + \mathcal{C}\mathcal{P}$. The nine subgroups are partitioned into Types I, II, III. If $n_{\mathcal{A}} < \infty$, resp. $n_{\mathcal{B}} < \infty$, then the minimum power of $\mathcal{A}$, resp. $\mathcal{B}$, which is single-valued, equals the numerator of $\rho_{\mathcal{A}}$, resp. $\rho_{\mathcal{B}}$, expressed in lowest terms. The forms $\mathcal{A}, \mathcal{B}$ on $2\mathcal{A}' = \Gamma\mathcal{A}'$ can alternatively be written as $(B_2^2 - C_2^2)^{1/2}(q^{1/2})$ and $(B_2^2 - \zeta_3 C_2^2)^{1/2}(q^{1/2})$, where $\zeta_3 = \exp(2\pi i/3)$; and the forms $\mathcal{A}, \mathcal{B}$ on $4\mathcal{A}'$ as $A_2 \pm i C_2$.

| $\Gamma$ | $(n_{\mathcal{A}}, n_{\mathcal{B}}, n_{\mathcal{C}})$ | $(x, y, z)$ | $\alpha$ | $\beta$ | $\gamma$ | $\rho$ | $\mathcal{A}$ | $\mathcal{B}$ | $\mathcal{C}$ |
|----------|---------------------------------|-------------|---------|---------|---------|-------|-------|-------|-------|
| $\Gamma_0(2)$ | $(2, \infty, \infty)$ | $(\frac{1}{2}, 0, 0)$ | 4 | $(\frac{1}{2}, -1)$ | $(\frac{1}{2}, 0)$ | $(\frac{1}{2}, 1)$ | $\mathcal{A}_4$ | $\mathcal{B}_4$ | $\mathcal{C}_4$ |
| $\Gamma_0(3)$ | $(3, \infty, \infty)$ | $(\frac{1}{3}, 0, 0)$ | 3 | $(\frac{1}{3}, -1)$ | $(\frac{1}{3}, 0)$ | $(\frac{1}{3}, 1)$ | $\mathcal{A}_3$ | $\mathcal{B}_3$ | $\mathcal{C}_3$ |
| $\Gamma_0(4)$ | $(\infty, \infty, \infty)$ | $(0, 0, 0)$ | 2 | $(\frac{1}{4}, -1)$ | $(\frac{1}{4}, 0)$ | $(\frac{1}{4}, 1)$ | $\mathcal{A}_2$ | $\mathcal{B}_2$ | $\mathcal{C}_2$ |
| $\Gamma(1)$ | $(3, 2, \infty)$ | $(\frac{1}{3}, 2, 0)$ | 12 | $(\frac{1}{3}, -1)$ | $(\frac{1}{3}, 0)$ | $(\frac{1}{3}, 1)$ | $E_{6}^{1/4}$ | $E_{6}^{1/6}$ | $(12\Delta)^{1/12}$ |
| $\Gamma_0^+(2)$ | $(4, 2, \infty)$ | $(\frac{1}{4}, 2, 0)$ | 8 | $(\frac{1}{4}, -1)$ | $(\frac{1}{4}, 0)$ | $(\frac{1}{4}, 1)$ | $\mathcal{A}_4$ | $[B_2^2 - C_2^2]^{1/4}$ | $2^{1/4}\sqrt{B_2 C_2}$ |
| $\Gamma_0^+(3)$ | $(6, 2, \infty)$ | $(\frac{1}{6}, 2, 0)$ | 6 | $(\frac{1}{6}, -1)$ | $(\frac{1}{6}, 0)$ | $(\frac{1}{6}, 1)$ | $\mathcal{A}_3$ | $[B_3^3 - C_3]^{1/3}$ | $2^{1/3}\sqrt{B_3 C_3}$ |

$2\mathcal{A}' = \Gamma\mathcal{A}'$ (3, 3, 0) | $(\frac{1}{3}, \frac{2}{3}, 0)$ | 6 | $(\frac{1}{3}, -1)$ | $(\frac{1}{3}, 0)$ | $(\frac{1}{3}, 1)$ | $E_{6} + (12\Delta)^{1/6}$ | $E_{6} - (12\Delta)^{1/6}$ | $(2\mathcal{A})^{1/6}(12\Delta)^{1/12}$ |

$4\mathcal{A}'$ $(4, 4, \infty)$ | $(\frac{1}{4}, \frac{3}{4}, 0)$ | 4 | $(\frac{1}{4}, -1)$ | $(\frac{1}{4}, 0)$ | $(\frac{1}{4}, 1)$ | $[B_2^2 + i C_2^2]^{1/2}$ | $[B_2^2 - i C_2^2]^{1/2}$ | $(4\mathcal{A})^{1/4}\sqrt{B_2 C_2}$ |

$6\mathcal{A}'$ $(6, 6, \infty)$ | $(\frac{1}{6}, \frac{5}{6}, 0)$ | 3 | $(\frac{1}{6}, -1)$ | $(\frac{1}{6}, 0)$ | $(\frac{1}{6}, 1)$ | $[B_3^3/2 + i C_3^{3/2}]^{1/3}$ | $[B_3^3/2 - i C_3^{3/2}]^{1/3}$ | $(4\mathcal{A})^{1/3}\sqrt{B_3 C_3}$ |
Here, the proportionality constant (i.e., 4) is determined by the condition that 
\( \lambda = 1 \), corresponding to the class B of fixed points under \( \Gamma \), should be a critical value of the map. If it were not, then B would also be such a class under \( \tilde{\Gamma} \); which would violate the assumption that \( \tilde{\Gamma} \) is a triangle group, with only three such classes.

Using the above expressions for \( \lambda, \rho \), and also the identities \( A = \tilde{A} \), \( B = \tilde{B} + \hat{C} \rho \), \( \mathcal{A}^2 = \mathcal{B}^2 + \hat{C} \rho \), with \( \rho = 2 \tilde{\rho} \), one immediately obtains

\[
\begin{align*}
\mathcal{A} &= \mathcal{A}, \\
\mathcal{B} &= \sqrt{\mathcal{B}^2 - \hat{C} \rho}, \\
\mathcal{C} &= 2^{1/3} \sqrt{\mathcal{B} \mathcal{C}}. 
\end{align*}
\]

Applying to the pairs \((\tilde{\Gamma}, \Gamma)\) of (7.19), these yield the triples \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) for \( \Gamma = \Gamma_{0}^+(2), \Gamma_{0}^-(3) \) that are shown in Table 3. A similar but ‘reversed’ procedure, applied to the \((\tilde{\Gamma}, \Gamma)\) of (7.17), allows the triples \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) for the Type-III groups \( 2\alpha', 4\alpha', 6\alpha' \), to be computed in terms of those for the corresponding Type-II groups \( \Gamma(1), \Gamma_{0}^+(2), \Gamma_{0}^-(3) \). The resulting triples are given in the table.

Alternative representations for the forms \( \mathcal{A}, \mathcal{B} \) on the group \( 2\alpha' = \Gamma^2 \) are supplied in the caption, and are derived as follows. Although these forms are not single-valued, their squares \( \mathcal{A}^2, \mathcal{B}^2 \) are single-valued, by the test for single-valuedness mentioned immediately before Theorem 7.1 (and reproduced in the caption). Each of \( \mathcal{A}^2, \mathcal{B}^2 \) has a \( \{1, \zeta_3, \zeta_3^2\} \)-valued multiplier system, and since \( \Gamma^2 \) has an index-3 subgroup the principal modular subgroup \( \Gamma'(2) \), each of them lies in \( M_2 (\Gamma'(2)) \). But \( \Gamma'(2) \) is is conjugated to \( \Gamma_0(4) \) by \( \tau \mapsto 2\tau \). Since \( M_2(\Gamma_0(4)) \) span \( M_2(\Gamma_0(4)) \), the forms \( \mathcal{A}^2, \mathcal{B}^2 \) must be combinations of \( \zeta_2(\tau)^2(\zeta_2^2(\tau)^2), \zeta_2(\tau)^2(\zeta_2^2(\tau)^2) \), i.e., of \( \zeta_3^2(q^{1/2}), \zeta_3^2(q^{1/2}) \). The combinations are easily worked out by linear algebra, if one expands to second order in \( q_2 = q^{1/2} \).

### 7.3. Explicit systems and Chazy equations

For each of the nine (conjugacy classes of) triangle groups \( \Gamma \) commensurable with \( \Gamma(1) = \text{PSL}(2, \mathbb{Z}) \), the associated differential system and generalized Chazy equation are computed below. They come respectively from Theorems 7.1 and 6.3. For each \( \Gamma \), a hypergeometric (i.e., elliptic-integral) representation of the corresponding weight-1 form \( A \), coming from Eq. (7.19b), is given as well. As was explained in §7.2, these triangle subgroups are of three types, denoted I, II, III. From a classical-analytic rather than a modular point of view, they differ in the dependence of the exponent differences \( (\alpha, \beta, \gamma) \) on the signature \( \rho \).

- **Type I**, for which \( (\alpha, \beta, \gamma) = (1 - 2, 0, 0) \). It comprises \( \Gamma = \Gamma_0(2), \Gamma_0(3), \Gamma_0(4) \), for which \( \rho = 4, 3, 2 \); in each case the infinite cusp has width \( v = 1 \). For each of these groups the associated triple \( \mathcal{A}_\rho, \mathcal{B}_\rho, \mathcal{C}_\rho \) of weight-1 forms equals \( A_\rho, B_\rho, \hat{C}_\rho \), and the weight-2 quasi-modular form \( \mathcal{E}_\rho := v(\mathcal{E}_\rho^2)/\mathcal{E}_\rho^2 \) equals \( E_\rho \). The system satisfied by \( A_\rho, B_\rho, C_\rho \) and the generalized Chazy equation satisfied by \( u = (2\pi i/\rho)\mathcal{E}_\rho \) were given in Theorem 2.3.
By Eq. (7.29), the hypergeometric representation for \( \omega_\rho = A_\rho \) is
\[
\hat{K}_\rho^1(\lambda_\rho) := _2F_1\left( \frac{1}{4}, 1 - \frac{4}{\rho}; 1; \lambda_\rho \right)
\] (7.20)
is the (normalized) Type-I complete elliptic integral. These cases of Type I correspond to Ramanujan’s elliptic theories of signature \( \rho \), for \( \rho = 4, 3, 2 \) (see [33, 27]). The classical (Jacobi) case is \( \rho = 2 \), and \( \hat{K}_2^1 \) is the (normalized) complete integral \( \hat{K} \), which was introduced in [35].

- Type II, for which \((\alpha, \beta, \gamma) = (\frac{1}{2} - \rho, \frac{1}{2}, 0)\). It comprises \( \Gamma = \Gamma(1), \Gamma_0^+(2), \Gamma_0^+(3) \), for which \( \rho = 12, 8, 6; \) in each case the infinite cusp has width \( u = 1 \). The associated triples \( \omega_\rho, \beta_\rho, \gamma_\rho \) of weight-1 forms are in Table 3. By direct computation, the weight-2 quasi-modular forms \( \beta_\rho := \nu(\beta_\rho^\rho)/\nu(\omega_\rho^\rho) \) are
\[
\begin{align*}
\delta_{12}(q) &= E_2(q) \\
&= 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n; 1) q^n \\
&= 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n; 1) q^n, \\
\delta_8(q) &= \frac{1}{2} \left[ 2 E_2(q^2) + E_2(q) \right] \\
&= 1 - 8 \sum_{n=1}^{\infty} \sigma_1(n; 2, 1) q^n \\
&= 1 - 8 \sum_{n=1}^{\infty} \sigma_1(n; 3, 1) q^n, \\
\delta_6(q) &= \frac{1}{3} \left[ 3 E_2(q^3) + E_2(q) \right] \\
&= 1 \sum_{n=1}^{\infty} \sigma_1(n; 2, 1) q^n \\
&= 1 \sum_{n=1}^{\infty} \sigma_1(n; 4, 1, 1) q^n.
\end{align*}
\] (7.21)

They lie respectively in \( M_2^{\leq 1}(\Gamma(1)), M_2^{\leq 1}(\Gamma_0^+(2)), M_2^{\leq 1}(\Gamma_0^+(3)) \). By Theorem 7.1 the differential system parametrized by \( \rho \) is
\[
\begin{align*}
(\omega_\rho^\rho)' &= \delta_\rho \omega_\rho^\rho - \omega_\rho^\rho^{1/2} \beta_\rho^\rho^{1/2}, \\
(\beta_\rho^\rho)' &= \delta_\rho \beta_\rho^\rho - \omega_\rho^\rho^{1/2} \beta_\rho^\rho^{1/2}, \\
(\gamma_\rho^\rho)' &= \delta_\rho \gamma_\rho^\rho, \\
\rho \beta_\rho' &= \delta_\rho \beta_\rho - \omega_\rho^\rho. 
\end{align*}
\] (7.24a-c)

This is an extension of Ramanujan’s \( P-Q-R \) system (2.5), to which it reduces when \( \rho = 12 \) and \( \Gamma = \Gamma(1) \).

The generalized Chazy equation satisfied by \( u = (2 \pi i / \rho p) \beta_\rho = \beta_\rho / \omega_\rho \), according to Theorem 6.4 is the nonlinear third-order ODE \( C_{p_0} \), i.e., \( p_0 = 0 \), in which the polynomial \( p_0 \in \mathbf{C}[u_4, u_6, u_8] \) is defined by
\[
\begin{align*}
p_{12} &= u_8 + 24 u_4^2, \\
p_8 &= 2 u_4 u_8 - u_6^2 + 32 u_4^3, \\
p_6 &= 4 u_4 u_8 - 3 u_6^2 + 48 u_4^3.
\end{align*}
\] (7.25-27)
and $u_4, u_6, u_8$ were given in Definition 2.7. The differential equation $C_{p\phi}$, associated to $\Gamma(1)$, coming from (7.25), is the classical Chazy equation (2.3a) that is satisfied by $u = (2\pi i/12)E_2$. The polynomials (7.26), (7.27) yield the generalized Chazy equations associated to $\Gamma(1)$, (7.28), (7.29), which are new.

By Eq. (7.9b), the hypergeometric representation for the weight-1 form $\omega_\rho$ is $\omega_\rho = \hat{N}_\rho(\lambda_\rho)$, in which $\lambda_\rho := \rho^p / \omega_\rho$ is a Hauptmodul for $\Gamma$ and

$$\hat{N}_\rho(\lambda_\rho) := 2F_1\left(\frac{1}{3}, \frac{1}{3}; 1; \lambda_\rho\right) = \frac{\cos(\pi/\rho)}{\pi} \int_0^1 x^{-1/2-1/\rho}(1-x)^{-1/2+1/\rho}(1-\lambda_\rho x)^{-1/\rho} \, dx \tag{7.28a}$$

is the (normalized) Type-II complete elliptic integral. Equivalently,

$$\left[\hat{N}_\rho(\lambda_\rho)\right]^2 = 3F_2\left(\frac{1}{2}, \frac{1}{2}; 1, 1; \lambda_\rho\right). \tag{7.28b}$$

Such representations, when $\rho = 12$ and $\Gamma = \Gamma(1)$, are fairly well known. By Table 3, the $\rho = 12$ versions of (7.28ab) are

$$E_4^{1/4} = 2F_1\left(\frac{1}{12}; \frac{1}{12}; 1; 12^3/j\right), \tag{7.29a}$$

$$E_2^{1/2} = 3F_2\left(\frac{1}{4}, \frac{1}{4}; 1, 1; 12^3/j\right), \tag{7.29b}$$

where $j = E_4^3/\Delta = 12^3E_3^3/(E_4^3 - E_6^2)$ is the Klein–Weber invariant, the canonical Hauptmodul for $\Gamma(1)$, so that $12^3/j = (E_4^3 - E_6^2)/E_4^3$. Equation (7.29a) was known to Dedekind and was rediscovered by Stiller [42]. These identities hold in a neighborhood of the infinite cusp, at which $j = \infty$ and $12^3/j = 0$. In the same way, Eq. (7.9b) yields

$$E_6^{1/6} = 2F_1\left(\frac{1}{12}; \frac{7}{12}; 1; 12^3/(12^3 - j)\right). \tag{7.30}$$

From (7.10a) one also has

$$\Delta^{1/12} \propto 2F_1\left(\frac{1}{3}; \frac{1}{3}; \frac{1}{3}; j/12^3\right), \tag{7.31a}$$

$$\Delta^{1/6} \propto 3F_2\left(\frac{1}{4}, \frac{1}{4}; \frac{1}{4}, \frac{1}{4}; j/12^3\right), \tag{7.31b}$$

which hold near any cubic elliptic fixed point, where $j/12^3 = 0$. (E.g., near $\tau = \zeta_3 = \exp(2\pi i/3)$.) The constants of proportionality depend on the choice of fixed point.

The $\rho = 8, 6$ representations, for $\Gamma = \Gamma_0^+(2), \Gamma_0^+(3)$, were derived by Zudilin [47, Eqs. (23bc)]. The corresponding differential systems that he obtained [47, Props. 6,7] are equivalent to the $\rho = 8, 6$ cases of the system (7.23abcd), but are more complicated as they are not expressed in terms of weight-1 forms.

- **Type III**, for which $(\alpha, \beta, \gamma) = (\frac{1}{2}, -\frac{1}{3}, -\frac{1}{3}, 0)$. It comprises $\Gamma = (2a' = \Gamma^2)$, $4a', 6a'$, for which $\rho = 6, 4, 3$; in each case the infinite cusp has width $\nu = 2$. The associated triples $\omega_\rho, \beta_\rho, \gamma_\rho$ of weight-1 forms are in Table 3. The weight-2 quasi-modular forms $\delta_\rho := u(\omega_\rho^p)/(\omega_\rho^p), \rho = 6, 4, 3$, are identical to the Type-II forms $\delta_{12}, \delta_8, \delta_6$, given in Eqs. (7.21), (7.22), (7.23).
By Theorem 7.1 the differential system parametrized by $\rho$ is

\begin{align}
2(\mathcal{A}_\rho)' &= \mathcal{E}_\rho \cdot \mathcal{A}_\rho - \mathcal{A}_\rho^{\rho/2+1/2} \mathcal{B}_\rho^{\rho/2+1}, \\
2(\mathcal{B}_\rho)' &= \mathcal{E}_\rho \cdot \mathcal{B}_\rho - \mathcal{A}_\rho^{\rho/2+1} \mathcal{B}_\rho^{\rho/2+1}, \\
2(\mathcal{C}_\rho)' &= \mathcal{E}_\rho \cdot \mathcal{C}_\rho, \\
2\rho \mathcal{E}_\rho &= \mathcal{E}_\rho \cdot \mathcal{E}_\rho - \mathcal{A}_\rho^{\rho/2} \mathcal{B}_\rho^2.
\end{align}

(7.32a) (7.32b) (7.32c) (7.32d)

Although this system is significantly different from (7.24abcd), the system of Type II, the resulting generalized Chazy equations $C_p$ are identical to the Type-II equations for $\rho = 12, 8, 6$ (see (7.25), (7.26), (7.27)). By Eq. (7.9a), the hypergeometric representation for the weight-1 form $\mathcal{A}_\rho$ is

$$
\mathcal{A}_\rho = \hat{K}_{III}^{\rho}(\lambda_{\rho}),
$$

(7.33)

which is the (normalized) Type-III complete elliptic integral. These representations are new. The case $\rho = 4$, i.e., $\Gamma = 4\lambda'$, is especially noteworthy. It follows from the formulas

$$
\mathcal{A}_4, \mathcal{B}_4 = \mathcal{A}_2 \pm i \mathcal{C}_2 = \theta_3^2 \pm i \theta_2^2
$$

(7.34)

$$
\mathcal{A}_4^4 = \mathcal{B}_4^4 + \mathcal{C}_4^4
$$

(7.35)

that when $\rho = 4$, the equation $\mathcal{A}_\rho = \hat{K}_{III}^{\rho}(\lambda_{\rho})$ specializes to

$$
\theta_3^2 \pm i \theta_2^2 = 2F_1\left(\frac{1}{4}, \frac{1}{2}; 1; \frac{(\theta_3^2 \pm i \theta_2^2)^4 - (\theta_3^2 \pm i \theta_2^2)^4}{(\theta_3^2 \pm i \theta_2^2)^4}\right).
$$

(7.36)

It is unclear whether this remarkable theta identity has a non-modular proof.

### 7.4. Discussion

The results of §7.3 suggest that elliptic integrals of Types II and III (parametrized by the signature $\rho$) deserve further study, much like the elliptic integrals of Type I, which are those of Ramanujan’s alternative theories [3]. His theories fit into a larger framework: one that is larger by a factor of three, at least.

The new generalized Chazy equations $C_p$ are especially interesting, since they open a ‘modular window’ into the space of nonlinear third-order ODEs. Each of the generalized Chazy equations derived in this article can be integrated in closed form in terms of modular functions. This has taxonomic ramifications. The classical Chazy equation $u_8 + 24u_4^2 = 0$ has the Painlevé property, in that its solutions have no ‘movable branch points’ [2, §7.1.5 and Ex. 6.5.14]. The nonlinear third-order ODEs in $u$ which have this property, and in which $\ddot{u}$ is polynomial in $\dot{u}, u, \dot{u}$ and rational in $x$, were classified by Chazy [11] into classes numbered
I through XIII. But to date, there has been no extension of his scheme to third-order ODEs with the property, in which \( \ddot{u} \) is non-polynomial but rational in \( u, \dot{u}, u_\dot{\phi} \).

The equations \( C_\mu \) of this article provide examples. (For the defining polynomials \( p \in \mathbb{C}[u_4, u_6, u_8] \), see Eqs. (2.6)–(2.8) and (7.25)–(7.27).) Each equation \( p = 0 \) is a nonlinear third-order ODE satisfied by \( u = \mathcal{C}/\mathcal{C}' \), where \( \mathcal{C} \) is the weight-1 modular form that vanishes on the third class of fixed points of a triangle subgroup \( \Gamma' < \text{PSL}(2, \mathbb{R}) \) with signature \( (n_0, n_\phi, n_\psi) \). Other than \( C_{p_1} \), the rather complicated weight-20 ODE coming from \( \Gamma' = \Gamma_0(3) \), these nonlinear ODEs lie in a single new class. With one seeming exception, each is of the form

\[
(M - 2)u_4u_8 - (M - 3)u_6^2 + 8Mu_4^3 = 0. \tag{7.37}
\]

Equation (7.37) is of weight 12 unless \( M = 3 \), in which case the \( u_4^2 \) term drops out and it reduces to the classical Chazy equation, of weight 8. By Theorem 6.4, Eq. (7.37) comes from the triangle groups with signatures \( (M, 2; \infty) \) and \( (M, M; \infty) \), i.e., with vertex angles \( (\alpha, \beta; \gamma) \) (expressed in terms of \( \pi \) radians) equal to \( (\varphi_3, \varphi_2, 0) \) or \( (\varphi_2, \varphi_3, 0) \). The abovementioned seeming exception is the generalized Chazy equation attached to \( \Gamma_0(2) \) and \( \Gamma_0(4) \), which must be obtained from Eq. (7.37) by taking a formal \( M \to \infty \) limit. But such limits are familiar from Chazy’s analysis. For instance, the classical Chazy equation, which is attached to the groups \( \Gamma(1) \) and \( \Gamma^2 \), with respective signatures \( (3, 2; \infty) \) and \( (3, 3; \infty) \), is also the formal \( N \to \infty \) limit of the Chazy-XII equation

\[
(N^2 - 36)u_8 + 24N^2u_4^3 = 0, \tag{7.38}
\]

which is of weight 8 and comes from the triangle groups with signatures \( (3, 2; N) \) and \( (3, 3; N/2) \). To date, the Chazy-XII class is the only one that has been given a modular interpretation, e.g., expressed in terms of the forms \( u_4, u_6, u_8 \).

One expects that when an extended Chazy classification is finally constructed by non-modal, classical-analytic techniques, the equation (7.37), parametrized by integer \( M \), will belong to an additional ‘modular’ class. Interestingly, it is the limiting \( (N \to \infty) \) case of a two-parameter generalized Chazy equation,

\[
[(M - 2)^2N^2 - 4M^2] \cdot [(M - 2)u_4u_8 - (M - 3)u_6^2] + 8M(M - 2)^2N^2u_4^3 = 0, \tag{7.39}
\]

which is also of weight 12 (generically). By Theorem 6.4, Eq. (7.39) comes from the triangle groups with signatures \( (M, 2; N) \) and \( (M, M; N/2) \), i.e., with \( (\alpha, \beta; \gamma) \) equal to \( (\varphi_3, \varphi_2, \pi) \) or \( (\varphi_2, \varphi_3, \pi) \). The fundamental domains of the latter triangle groups are hyperbolic isosceles triangles. Equation (7.39) reduces to Eq. (7.38), the weight-8 Chazy–XII equation, when \( M = 3 \).

In a similar way, one can derive a two-parameter generalized Chazy equation extending \( C_{p_1} \), the weight-20 ODE that comes from \( \Gamma_0(3) \). The extension, also of weight 20 (generically), is

\[
[(2M - 3)^2N^2 - 9M^2] \cdot [(M - 2)u_4u_8 - (M - 3)u_6^2] u_8 + 12MN^2[(2M - 3)^2N^2 - 9M^2] \cdot u_4^2[4(M - 2)(2M - 3)u_4u_8 - (M - 3)(5M - 9)u_6^2] + 576M^2(M - 2)(2M - 3)^2N^4u_4^3 = 0. \tag{7.40}
\]
By Theorem 6.4 this nonlinear ODE comes from the triangle group with signature $(3,M;N)$, i.e., with $(\alpha, \beta; \gamma)$ equal to $(\frac{1}{3}, \frac{1}{M}, \frac{1}{N})$. It reduces to $C_{32}$ when $M \to \infty$ and $N \to \infty$, to the Chazy–XII equation when $M = 2$, and to the classical Chazy equation when $M = 2$ and $N \to \infty$, or when $M = 3$ and $N \to \infty$.

Writing $u_3, u_6, u_4 \text{ in terms of } \bar{u}, \bar{u}, \dot{u}, \dot{u}$, one sees that like $C_{32}$ itself, the ODE (7.40), when $M \neq 2$, expresses $\bar{u}$ as a degree-2 algebraic function of $\bar{u}, \dot{u}, \dot{u}$, rather than a rational function. It is of a more general type than the ODE (7.39).

Appendix: Theta representations and AGM identities

The functions $A_r, B_r, C_r$, $r = 2, 3, 4$, satisfying $A_r^\prime = B_r^\prime + C_r^\prime$, on the disk $|q| < 1$, were originally defined by the Borweins [7] as the sums of theta series of certain quadratic forms, occurring in Ramanujan’s theories of elliptic functions to alternative bases [3]. To facilitate comparison, several of their results are restated below in the notation of the present article. They defined $(A_2, B_2, C_2) = (\vartheta_3^2, \vartheta_4^2, \vartheta_6^2)$, $(A_4^2, B_4^2, C_4^2) = (\vartheta_2^4 + \vartheta_4^4, \vartheta_4^4, 2\vartheta_2^2\vartheta_4^2)$, and also

\[
A_3(q) = \sum_{n,m \in \mathbb{Z}} q^{n^2+nm+m^2}, \quad (A.1a)
\]

\[
B_3(q) = \sum_{n,m \in \mathbb{Z}} q^{r-n-m} q^{n^2+nm+m^2}, \quad (A.1b)
\]

\[
C_3(q) = \sum_{n,m \in \mathbb{Z}} q^{(n+\frac{1}{3})^2+(n+\frac{1}{3})(m+\frac{1}{3})+(m+\frac{1}{3})^2}, \quad (A.1c)
\]

where $\zeta_3$ is a primitive third root of unity. Their AGM identities include the quadratic signature-2 identities

\[
A_2(q^2) = \frac{[A_2+B_2]/2}{(q), \quad (A.2a)}
\]

\[
B_2(q^2) = \sqrt{A_2B_2}(q), \quad (A.2b)
\]

\[
C_2(q^2) = \frac{[A_2-B_2]/2}{(q), \quad (A.2c)}
\]

which originated with Jacobi, the quartic signature-2 identities

\[
\vartheta_3(q^4) = \sqrt{[\vartheta_3 + \vartheta_4]/2}(q), \quad (A.3a)
\]

\[
\vartheta_4(q^4) = \sqrt{[\vartheta_3^2 + \vartheta_4^2]/2}(\vartheta_3 \vartheta_4)/2(q), \quad (A.3b)
\]

\[
\vartheta_2(q^4) = \sqrt{[\vartheta_3 - \vartheta_4]/2}(q), \quad (A.3c)
\]

the cubic signature-3 identities

\[
A_3(q^3) = \frac{[A_3 + 2B_3]/3}{(q), \quad (A.4a)}
\]

\[
B_3(q^3) = \sqrt{[A_3^2 + A_3B_3 + B_3^2]/3B_3}(q), \quad (A.4b)
\]

\[
C_3(q^3) = \frac{[A_3 - B_3]/3}{(q), \quad (A.4c)}
\]
and the quadratic signature-4 identities

\[
\begin{align*}
A_4^2(q^2) &= \left[\left(A_4^2 + 3B_4^2\right)/4\right](q), & (A.5a) \\
B_4^2(q^2) &= \sqrt{\left(A_4^2 + B_4^2\right)B_4^2/2}(q), & (A.5b) \\
C_4^2(q^2) &= \left[\left(A_4^2 - B_4^2\right)/4\right](q). & (A.5c)
\end{align*}
\]

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