Numerical solution of inverse problems in hydrodynamics

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Abstract. The paper is devoted to inverse problems for the unsteady viscous incompressible fluid flow. In the problems under consideration we estimate the velocity vector on a part of the boundary using velocity measurements in a certain observation subdomain. Optimal control approach reduces this problem to the constrained minimization. For unsteady flow and the discrete measurement data we use discrete optimization methods to solve this problem. New numerical algorithm based on the finite dimensional minimization is proposed and numerical results are discussed.

1. Introduction
Inverse and optimization problems in hydrodynamics have a large number of applications in science and engineering. In inverse problems we must find distributed or boundary sources, coefficients or initial conditions using some information about the state. Usually these problems are formulated as minimization problems for suitable cost functionals and solved by applying a unified approach based on the constrained optimization theory [1]. Boundary control problems for the Navier-Stokes equations were considered in [2, 3, 4]. Optimization problems for heat convection equations were solved numerically in [5, 6, 7, 8, 9]. The same approach can be applied for inverse and parameter estimation problems (see for example [10, 11]).

In this paper we consider an unsteady channel flow of the viscous incompressible fluid over a forward facing step (see Figure 1). We assume that flow rate on the inlet and the main outlet are not equal. The bottom boundary of the channel contains unknown outlets. In our inverse problem we use velocity measurements in the downstream observation subdomain to find these outlets and estimate the boundary velocity. More precisely, the channel bottom wall has several holes. The positions of these holes are determined by the boundary velocity that was found numerically as the solution of the inverse problem under consideration.

To solve this problem we propose new numerical algorithm based on the finite dimensional minimization. The algorithm has several advantages over previously used methods. It does not use iterations and the first order necessary optimality conditions. The state and control are calculated simultaneously at the same time step. The vortex reduction problem for heat and electrically conducting fluid with temperature boundary control was solved by similar approach in [12]. In computational experiments we verify the efficiency of the algorithm and estimate the influence of the problem parameters (Reynolds number, size of the observation domain and a number of basis functions) on the accuracy of the inverse problem solution.
2. Problem statement
In the bounded two-dimensional domain $\Omega$ on the time interval $(0, t_{\text{max}})$ we consider the following dimensionless system of partial differential equations
\[
\frac{\partial v}{\partial t} + (v \cdot \nabla)v - \frac{1}{Re} \Delta v + \nabla p = 0 \text{ in } \Omega, \quad (1)
\]
\[
\text{div } v = 0 \text{ in } \Omega, \quad v|_{t=0} = v_0 \text{ in } \Omega, \quad (2)
\]
\[
v = v_{\text{in}} \text{ on } \Gamma_{\text{in}}, \quad v = 0 \text{ on } \Gamma_0, \quad v = g \text{ on } \Gamma_c, \quad p n - \frac{1}{Re} \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_{\text{out}}, \quad (3)
\]
describing the unsteady viscous incompressible fluid flow in a channel. Here $v$ and $p$ are the dimensionless velocity and pressure, $n$ is the unit outward normal vector. Reynolds number $Re$ is dimensionless parameter of this problem. The boundary $\Gamma$ consists of four parts: inlet section $\Gamma_{\text{in}}$, outlet section $\Gamma_{\text{out}}$, solid walls $\Gamma_0$ and control section $\Gamma_c$ (see Figure 1). For the velocity vector $v$ we prescribe the inflow parabolic profile $v_{\text{in}}$ on $\Gamma_{\text{in}}$, no-slip boundary condition on $\Gamma_0$ and "do nothing" boundary condition on the outlet $\Gamma_{\text{out}}$. The velocity boundary value $g$ on the control section $\Gamma_c$ will play the role of control.

3. Time discretization
In our inverse problem we want to obtain the velocity field $v$ close to a given vector field $v_d$ measured in the observation subdomain $\Omega_d$ using velocity boundary control $g$ on $\Gamma_c$. We assume that values of the vector $v_d$ are given only at fixed time points $t_1, t_2, \ldots, t_N = t_{\text{max}}$. So we have a discrete set of data and need to solve the discrete minimization problem.

At the beginning we split the time interval $(0, t_{\text{max}})$ into $N$ parts $(t_{n-1}, t_n)$ of length $\tau_n = t_n - t_{n-1}$, $n = 1, 2, \ldots, N$ and write the following semidiscrete approximation for the problem (1)–(3):
\[
\frac{v^n - v^{n-1}}{\tau_n} + (v^{n-1} \cdot \nabla)v^n - \frac{1}{Re} \Delta v^n + \nabla p^n = 0 \text{ in } \Omega, \quad (4)
\]
\[
\text{div } v^n = 0 \text{ in } \Omega, \quad v^0 = v_0 \text{ in } \Omega, \quad (5)
\]
\[
v^n = v_{\text{in}}(t_n) \text{ on } \Gamma_{\text{in}}, \quad v^n = 0 \text{ on } \Gamma_0, \quad v = g(t_n) \text{ on } \Gamma_c, \quad p^n n - \frac{1}{Re} \frac{\partial v^n}{\partial n} = 0 \text{ on } \Gamma_{\text{out}}, \quad (6)
\]
Here $v^n, p^n$ ($n = 1, 2, \ldots, N$) are the new unknown functions that depend only on the space variables $x, y$. Let us note that we use an implicit difference scheme for time discretization. It ensures the stability of our numerical solutions even for large time intervals.

Multiplying equations in (4), (5) by the corresponding test functions, integrating over $\Omega$ and using Green's formula for certain terms we obtain a variational formulation of the problem (4)–(6). It consists in finding functions $v^n, p^n$ ($n = 1, 2, \ldots, N$) satisfying equations

$$\frac{(v^n - v^{n-1}, w)}{\tau_n} + ((v^{n-1} \cdot \nabla)v^n, w) + \frac{1}{Re}(\nabla v^n, \nabla w) - (p^n, \text{div } w) = 0 \quad \forall w \in W,$$

$$(\chi, \text{div } v^n) = 0 \quad \forall \chi \in X,$$

$$v^n = v^n_{\text{in}} \quad \text{on } \Gamma_{\text{in}}, \quad v^n = 0 \quad \text{on } \Gamma_0, \quad v^n = g^n \quad \text{on } \Gamma_c.$$  

Here and below $W$ and $X$ are the test function spaces; the function $v^0 = v_0$ is determined by the initial condition in (2); $v^n_{\text{in}} = v_{\text{in}}(t_n), g^n = g(t_n); (\cdot, \cdot)$ and $(\cdot, \cdot)_{\Gamma_c}$ are the scalar products in $L^2(\Omega)$ and $L^2(\Gamma_c)$.

If we already know all values at time $t = t_{n-1}$ then this problem is linear for unknowns $(v^n, p^n)$. We will use this linearity in the construction of our numerical algorithm. Applying the finite element method for space discretization we can solve the problem (7)–(9) numerically.

4. Discrete minimization

Let $\Omega_d \subset \Omega$ be the subset of $\Omega$. It will play the role of the observation subdomain in our control problem for the system (7)–(9). At time step $t = t_n$ the problem of finding the boundary function $g^n$ for the vector field $v^n_d = v_d(t_n)$ is reduced to the minimization of the quality functional

$$J(v^n, g^n) = \frac{1}{2} \int_{\Omega_d} |v^n - v^n_d|^2 d\Omega + \frac{\mu}{2} \int_{\Gamma_c} |g^n|^2 d\Gamma,$$

that depends on the function $g^n$ and corresponding velocity vector $v^n$. Here $\mu = \text{const} \geq 0$ is the small regularization parameter.

The following method is based on the main idea of [2] where an optimal boundary control problem for the stationary Navier-Stokes equations was solved using the finite dimensional minimization approach. Here we develop an algorithm for the nonstationary problem.

We write the unknown control function $g^n$ in the following form:

$$g^n = \sum_{i=1}^{M} k_i g_i$$

where $\{g_i\}_{i=1}^{M}$ are the given basic functions on $\Gamma_c$ while $k_i, i = 1, 2, \ldots, M$ are the unknown coefficients that must be found at each time step. The corresponding velocity vector then can be written as

$$v^n = w^n + \sum_{i=1}^{M} k_i v^n_i.$$  

Here $w^n$ is the solution of the problem (7)–(9) for $g^n = 0$ in (9) while $v^n_i$ ($i = 1, \ldots, M$) is the solution of the problem

$$\frac{(v^n_i, w)}{\tau_n} + ((v^{n-1} \cdot \nabla)v^n_i, w) + \frac{1}{Re}(\nabla v^n_i, \nabla w) - (p^n_i, \text{div } w) = 0 \quad \forall w \in W,$$

$$(\chi, \text{div } v^n_i) = 0 \quad \forall \chi \in X,$$
\[ v^n_i = 0 \text{ on } \Gamma_{in}, \quad v^n_i = 0 \text{ on } \Gamma_0, \quad v^n_i = g_i \text{ on } \Gamma_c. \]

Then the functional (10) can be written as

\[
J(v^n, g^n) = \frac{1}{2} \int_{\Omega_d} |v^n - v_d^n|^2 d\Omega + \frac{\mu}{2} \int_{\Gamma_c} |g^n|^2 d\Gamma + \frac{1}{2} \int_{\Omega_d} |w^n - w^n_d + \sum_{i=1}^M a_i v^n_i|^2 d\Omega
\]

\[
+ \frac{\mu}{2} \int_{\Gamma_c} \left( \sum_{i=1}^M k_i g_i \right)^2 d\Gamma = \frac{1}{2} \int_{\Omega_d} \sum_{j=1}^M a_{ij} k_i k_j \sum_{j=1}^M b_{ij} k_j + \frac{1}{2} c.
\]

Here the coefficients

\[
a_{ij} = (v^n_i, v^n_j)_{\Omega_d} + \mu (g_i, g_j)_{\Gamma_c}, \quad b_j = (v^n_d - w^n, v^n_j)_{\Omega_d}, \quad c = \|v^n_d - w^n\|^2_{\Omega_d} \quad (13)
\]

can be easily calculated by known functions \(g_i, v^n_i, v^n_d\) and \(w^n\).

As a result, we have the finite dimensional minimization problem for variables \(k_i, i = 1, 2, \ldots, M\). Solution of this problem can be found by solving the following system of linear algebraic equations:

\[
\sum_{i=1}^M a_{ij} k_i = b_j, \quad j = 1, 2, \ldots, M. \quad (14)
\]

Substituting \(a_{ij}\) and \(b_j\) in this system and using (12) we obtain the following equalities

\[
(v^n - v^n_d, v^n_j)_{\Omega_d} + \mu (g^n, g_j)_{\Gamma_c} = 0, \quad j = 1, 2, \ldots, M.
\]

that have a clear meaning. If \(\mu = 0\) then the difference \(v^n - v^n_d\) must be orthogonal to all functions \(v^n_i\) in the observation subdomain \(\Omega_d\). Solving the system (14) and substituting \(k_i\) in (11) we find the boundary control \(g^n\) on the \(n\)-th time step.

5. Numerical algorithm

Proposed numerical algorithm can be written as follows.

Step 0. Choose \(M\) basic functions \(g_i\) on \(\Gamma_c\). Set initial value \(v^0\). Set \(n = 1\).

Step 1. Assuming that \(v^{n-1}\) is already known, solve \(M+1\) linearized boundary value problems to find \(v^n_i, i = 1, \ldots, M\) and \(w^n\).

Step 2. Calculate \(a_{ij}\) and \(b_j\) by (13). Solve linear system (14) to find \(k_i, i = 1, \ldots, M\).

Step 3. Calculate \(g^n\) by (11). Solve linear boundary value problem (7)–(9) to find \(v^n, p^n\).

Step 4. If \(n < N\) then set \(n := n + 1\) and go to step 1.

Let us note that this algorithm does not use the first order necessary optimality conditions (see [5, 6, 7]) and can be simply implemented. It can be efficiently parallelized because \(M+1\) boundary value problems to find \(w^n\) and \(v^n_i, i = 1, \ldots, M\) are solved independently. It must be noted that at each time step \(t = t_n\) we find the control \(g^n\) and the state \((v^n, p^n)\) simultaneously. A similar idea was used to optimize the viscous heat-conducting fluid flow by optimal heating or cooling of the boundary in [9].

6. Computational results

Now let us discuss some computational results for the inverse problem under consideration. In this problem we find the boundary velocity \(g\) on the control boundary section \(\Gamma_c\) to create the velocity \(v\) closed to the given vector field \(v_d\) in the subdomain \(\Omega_d\). The initial boundary value problem (1)–(3) is considered in the channel with a forward facing step (see Figure 1). All quantities are dimensionless. The size of the channel is determined by values \(|x| < 4, |y| < 1\). The
step is placed at \( x = 2 \) and \( \Gamma_c = \{(x, y) : 0 \leq x \leq 2, y = -1\} \). The dimensionless time interval \((0, t_{\text{max}})\) was chosen as \((0, 20)\) with \( N = 100 \). In these computations we used different Reynolds numbers \( \text{Re} \) from 50 to 500. The inflow velocity \( v_{\text{in}} \) is prescribed as \( v_{\text{in}} = (1 - y^2, 0) \). The observation subdomain \( \Omega_d \) is defined by the inequality \( x > x_d \) where \( x_d \) is a variable parameter of the inverse problem. In order to obtain “the measured velocity” \( v_d \) we solve numerically the initial boundary value problem (1)–(3) for a given boundary velocity \( g_d \) on \( \Gamma_c \). Then we use this field \( v_d \) in our inverse problem and compare the obtained numerical solutions \( g = (0, g(x)) \) and \( v = (v_x, v_y) \) with the original functions \( g_d = (0, g_d(x)) \) and \( v_d = (v_{dx}, v_{dy}) \). We assume that the boundary \( \Gamma_c \) contains two outlets and boundary velocity \( g_d(x) \) is defined as

\[
g_d(x) = \begin{cases} 
  x(x - 0.5), & \text{if } 0 \leq x \leq 0.5 \\
  0, & \text{if } 0.5 < x < 1.5 \\
  2(x - 1.5)(x - 2), & \text{if } 1.5 \leq x \leq 2 
\end{cases}
\]

The open source software freeFEM++ (www.freefem.org) was used to solve the boundary value problems (7)–(9) by the finite element method. The main goal of the computational experiments was to determine the dependence of the solution accuracy on the choice of the problem parameters. The accuracy of the solution depends on the number of basis functions \( M \). We use linear splines as basis functions \( g_i(x) \). Original boundary velocity \( g_d(x) \) and computed controls \( g(x) \) for \( M = 11 \) and \( M = 22 \) at time point \( t = 20 \) with \( \text{Re} = 100, x_d = 0, \mu = 10^{-6} \) are shown in Figure 2. It can be seen that this numerical solution has a good agreement with the original boundary velocity and allows us to find the location of the holes in the wall.

In order to estimate the accuracy of the inverse problem solution we use three errors

\[
E_g = \|g - g_d\|/\|g_d\|, \quad E_{v_x} = \|v_x - v_{dx}\|/\|v_{dx}\|, \quad E_{v_y} = \|v_y - v_{dy}\|/\|v_{dy}\|.
\]

The values of these errors for different numbers of basis functions \( M \) and Reynolds numbers \( \text{Re} \) are given in tables 1 and 2 respectively. It is clearly seen that in order to decrease the errors we must increase the number \( M \) and decrease the Reynolds number \( \text{Re} \).

**Figure 2.** Original and estimated boundary velocity \( g(x) \) on \( \Gamma_c \).
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Numerical experiments for different values of \( x_d \) show that the solution of inverse problem is more accurate if the observation subdomain \( \Omega_d \) is located closer to the control boundary \( \Gamma_c \). Based on the analysis of computational results we can choose optimal values of problem parameters and develop some recommendations for future applications.

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Table 1. Error values for different numbers of basis functions \( M (\text{Re}=100, x_d = 0) \).

| \( M = 10 \) | \( M = 15 \) | \( M = 20 \) | \( M = 25 \) | \( M = 30 \) | \( M = 35 \) |
|---|---|---|---|---|---|
| \( E_q = 4.1 \times 10^{-3} \) | \( 3.1 \times 10^{-3} \) | \( 1.2 \times 10^{-3} \) | \( 9.9 \times 10^{-4} \) | \( 8.7 \times 10^{-4} \) | \( 7.9 \times 10^{-4} \) |
| \( E_{v_s} = 2.2 \times 10^{-4} \) | \( 1.7 \times 10^{-4} \) | \( 1.4 \times 10^{-4} \) | \( 1.1 \times 10^{-4} \) | \( 5.6 \times 10^{-5} \) | \( 4.1 \times 10^{-5} \) |
| \( E_{v_y} = 1.2 \times 10^{-4} \) | \( 7.6 \times 10^{-5} \) | \( 5.2 \times 10^{-5} \) | \( 4.1 \times 10^{-5} \) | \( 2.6 \times 10^{-5} \) | \( 1.9 \times 10^{-5} \) |

Table 2. Error values for different Reynolds numbers \( \text{Re} (M = 20, x_d = 0) \).

| \( \text{Re}=50 \) | \( \text{Re}=100 \) | \( \text{Re}=200 \) | \( \text{Re}=300 \) | \( \text{Re}=400 \) | \( \text{Re}=500 \) |
|---|---|---|---|---|---|
| \( E_q = 1.2 \times 10^{-3} \) | \( 1.2 \times 10^{-3} \) | \( 1.3 \times 10^{-3} \) | \( 1.6 \times 10^{-3} \) | \( 1.9 \times 10^{-3} \) | \( 2.3 \times 10^{-3} \) |
| \( E_{v_s} = 1.2 \times 10^{-4} \) | \( 1.4 \times 10^{-4} \) | \( 1.9 \times 10^{-4} \) | \( 4.1 \times 10^{-4} \) | \( 8.4 \times 10^{-4} \) | \( 1.3 \times 10^{-3} \) |
| \( E_{v_y} = 4.6 \times 10^{-5} \) | \( 5.2 \times 10^{-5} \) | \( 8.8 \times 10^{-5} \) | \( 1.6 \times 10^{-4} \) | \( 2.6 \times 10^{-4} \) | \( 3.7 \times 10^{-4} \) |

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