Sundman stability of natural planet satellites

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ABSTRACT

The stability of the motion of planet satellites is considered in a model of the general three-body problem (sun–planet–satellite). ‘Sundman surfaces’ are constructed, by means of which the concept of ‘Sundman stability’ is formulated. A comparison of Sundman stability with the results of Golubev’s $c^2 h$ method and with Hill’s classical stability in the restricted three-body problem is performed. The constructed Sundman stability regions in the plane of ‘energy–moment of momentum’ parameters coincides with the analogous regions obtained by Golubev’s method, with the value $(c^2 h)_c$.

Construction of Sundman surfaces in the three-dimensional space of the specially selected coordinates $xyR$ is carried out by means of the exact Sundman inequality in the general three-body problem. Determination of the singular points of surfaces and regions of possible motion and Sundman stability analysis are implemented. It is shown that the singular points of the Sundman surfaces in the coordinate space $xyR$ lie in different planes. The Sundman stability of all known natural satellites of planets is investigated. It is shown that a number of natural satellites that are stable according to Hill and also some satellites that are stable according to Golubev’s method are unstable in the sense of Sundman stability.

Key words: celestial mechanics – planets and satellites: dynamical evolution and stability.

1 INTRODUCTION

The study of the stability of the motion of planet satellites has usually been performed by means of Hill surfaces (Hill 1878) constructed either for a model of the restricted three-body problem (Proskurin 1950) or for the Hill problem (Hagihara 1952). Even since the works of Golubev (1967, 1968), Golubev & Grebenikov (1985), who developed the method sometimes referred to as the $c^2 h$ method for the general three-body problem, the stability analysis of motions has usually been carried out by Golubev’s method. This method is based on the famous Sundman inequality (Sundman 1912).

The search for regions of stable motion in the general three-body problem is divided into two tasks:

(i) determination of stability regions in the plane of two free parameters – the constant of the energy integral and constant of the moment of momentum integral;

(ii) determination of stable regions in the space of the coordinates used.

Actually, Golubev (1967) carried out the complete solution of the first task through the introduction of the ‘index of Hill stability’ $s = c^2 h$, where $h$ is the energy constant and $c$ the moment of momentum constant. Golubev showed that the curve $s = s_c$ is the boundary of the stability region in the plane of $c h$, where the value of $s_c$ is calculated at the Eulerian inner libration point $L_2$ by use of the known equation of fifth power as some function of the body masses.

The solution of the second problem is considerably more complex, since it requires the construction of Sundman surfaces in the multidimensional space of the coordinates used. For the solution of this problem, Golubev (1968) applied the simplified Sundman inequality, with the aid of which the solution of the problem leads to the construction of Hill curves in the plane of two rectangular coordinates $x$ and $y$.

Some authors (Marchal & Saari 1975; Zare 1976; Marchal & Bozis 1982; Marchal 1990) have elaborated on Golubev’s results by use of the simplified Sundman inequality.

The construction of regions of possible motion in the space of the selected coordinates by use of the exact Sundman inequality is hindered by the large number of variables. Thus, in relative or Jacobian coordinate systems the number of variables is six: three coordinates for one body and three for the other. Therefore, the construction of the Sundman surfaces can be carried out in the six-dimensional space of these coordinates.

In a large series of works by Szebehely & Zare (1977), Walker, Emslie & Roy (1980), Donnison & Williams (1983), Donnison (2009), Li, Fu & Sun (2010), Donnison (2010) and other authors, the determination of the regions of possible motion is carried out in...
the six-dimensional space of the Keplerian elements \( a_1, a_2, e_1, e_2, i_1 \) and \( i_2 \) or in the four-dimensional space \( a_1, a_2, e_1 \) and \( e_2 \) for the planar problem.

At the same time, the construction of the Sundman surfaces and thus the determination of regions of possible motion and stability regions can be conducted in the space of three variables. This substantially facilitates the readability of results and eases their application. This possibility is explained by the fact that the exact Sundman inequality depends on the coordinates only by means of three quantities: the three mutual distances between the bodies.

The article by Lukyanov & Shirmin (2007) is likely the first work that confirmed this possibility. In this work the mutual distances between the bodies are the rectangular coordinates. The existence of a Hill surfaces analogue for the general three-body problem in the space of the mutual distances is shown here using the exact Sundman inequality. The stability regions are determined in the space of the mutual distances and have the form of an infinite ‘tripod’.

Lukyanov (2011) used another choice of three coordinates. The coordinate system is determined by the accompanying triangle of mutual positions of bodies, namely the origin of the coordinate system coincides with one of the bodies, the axis \( x \) is directed towards the second body and the axis \( y \) is perpendicular to the axis \( x \) and lies in the plane of the triangle. In this coordinate system the position of all bodies is determined by three coordinates \( x, y, R \), where \( x \) and \( y \) are the coordinates of the third body and \( R \) is the distance between the first and second bodies. The system of coordinates \( xyR \) is to a certain degree similar to the rotating coordinate system in the restricted three-body problem following the motion of the basic bodies.

Then, in the space of coordinates \( xyR \) the Sundman surfaces are constructed, the singular points of surfaces (coinciding with the Euler and Lagrange libration points) are located and the regions of possible motion and Sundman stability are determined. The stability region of any body relative to another body in the space \( xyR \) has a form similar to an infinite ‘spindle’. No restriction on the masses of bodies or their mutual positions is assumed in this case.

In the present work, a method of constructing the regions of possible motion (Sundman lobes) in the general three-body problem is presented. Sundman stability analysis of the natural satellites of planets is carried out, using the high-precision ephemerides of the natural satellites of planets MULTI-SAT (Emelyanov & Arlot 2008) available on the websites http://www.sai.msu.ru/neb/nss/index.htm and http://www.imcce.fr/sat.

2 SUNDMAN SURFACES

The regions of possible motion of bodies in the general three-body problem are determined by the Sundman inequality

\[
(U - C) J \geq B,
\]

where the force function \( U \) and barycentric moment of inertia \( J \) are determined by the expressions

\[
U = \frac{G m_1 m_2}{R_{12}} + \frac{G m_2 m_3}{R_{23}} + \frac{G m_3 m_1}{R_{31}},
\]

\[
J = \frac{m_1 m_2 R_{12}^2 + m_2 m_3 R_{23}^2 + m_3 m_1 R_{31}^2}{m},
\]

\[
C = -h \text{ is the analogue of the Jacobi constant, } h \text{ is the energy constant, } B = c^2/2 \text{ is the Sundman constant and } c \text{ is the constant of the integral of area.}
\]

Here \( G \) is the universal gravitational constant, \( m_1, m_2, m_3 \) are the masses of bodies, \( m = m_1 + m_2 + m_3 \) is the total mass of the system and \( R_{12}, R_{23}, R_{31} \) are the mutual distances between the bodies.

Constants \( C \) and \( B \) are determined by the initial conditions in the barycentric coordinate system from the relationships

\[
C = -U = m_1 \frac{V_1^2}{2} - m_2 \frac{V_2^2}{2} - m_3 \frac{V_3^2}{2},
\]

\[
B = \frac{c^2}{2} = \frac{1}{2}(m_1 r_1 \times V_1 + m_2 r_2 \times V_2 + m_3 r_3 \times V_3)^2,
\]

where \( r_i, V_i (i = 1, 2, 3) \) are the barycentric state and speed vectors of the bodies.

The boundary of the region of possible motions can be established if in (1) inequality is replaced with equality:

\[
(U - C) J = B.
\]

This equality determines the equation of the Sundman surface. In the general case, the mutual distances in (5) depend on nine coordinates of three moving bodies, which substantially hampers the construction of Sundman surfaces. The transformation to the relative coordinate system makes it possible to reduce the number of coordinates to six. However, in this case the construction of the Sundman surfaces should be conducted in six-dimensional space. Furthermore, the number of coordinates can be reduced to three if the positions of bodies are determined by the following special coordinates.

The position of the body \( M_2 \) relative to the body \( M_1 \) will be characterized by the abscissa \( R \) on the axis \( M_1 X \). We will define the position of the body \( M_3 \) relative to \( M_1 \) by the rectangular coordinates \( X \) and \( Y \) in the system \( M_1 XY \), which always lies in the plane that passes through all three bodies. The positions of the bodies in the coordinate system \( M_1 XYR \) are defined by three quantities, coordinates \( X, Y \) and \( R \), which allows us to construct the Sundman surfaces in three-dimensional space.

We will use a dimensionless system of coordinates \( M_1 xyR \), making the substitution

\[
X = Rx, \quad Y = Ry.
\]

Then the mutual distances between the bodies can be expressed in terms of three quantities \( x, y, R \). The Sundman-surface equation transforms to the form of functions of three variables:

\[
S(x, y, R) = (U - C) J
\]

\[
= \frac{G}{R} \left( \frac{m_2 m_3}{\sqrt{(x - 1)^2 + y^2}} + \frac{m_3 m_1}{\sqrt{x^2 + y^2}} \right) - C
\]

\[
\times \frac{R^2}{m} \left\{ m_1 m_2 + m_2 m_3 \left[ (x - 1)^2 + y^2 \right] 
+ m_3 m_1 (x^2 + y^2) \right\}
\]

\[
= \frac{c^2}{2} = B.
\]

Equation (7) allows us to conduct the construction of the Sundman surface in the three-dimensional Cartesian space of variables \( xyR \).
The singular points of the Sundman surfaces are determined from the system of three algebraic equations

\[
\frac{\partial S}{\partial x} = \frac{2R^2(U-C)}{m} [m_2m_3(x_1 - 1) + m_3m_1x] - \frac{GJ}{R} \left[ \frac{m_2m_3(x_1 - 1)}{r_{23}^3} + \frac{m_1m_2x}{r_{31}^3} \right] = 0,
\]

\[
\frac{\partial S}{\partial y} = \frac{2R^2(U-C)}{m} (m_2m_3 + m_3m_1) - \frac{GJ}{R} \left[ \frac{m_1m_2}{r_{23}^3} + \frac{m_1m_1}{r_{31}^3} \right] = 0,
\]

\[
\frac{\partial S}{\partial R} = J(U - 2C) = 0.
\]  (8)

From the third equation of this system, it is possible to determine the mutual distance \( R \) between the bodies \( M_1 \) and \( M_2 \) in the form of a function of unknowns \( x, y \) and constant \( C \):

\[
R = \frac{G}{2C} \left( m_1m_2 + \frac{m_2m_3}{\sqrt{(x-1)^2 + y^2}} + \frac{m_3m_1}{\sqrt{x^2 + y^2}} \right).  \]  (9)

Substituting this expression for \( R \) into the first two equations of set (8), we obtain a system of two equations with two unknowns \( x \) and \( y \):

\[
\left( m_1m_2 + \frac{m_2m_3}{r_{23}^3} + \frac{m_1m_1}{r_{31}^3} \right) [m_2m_3(x_1 - 1) + m_3m_1x] - \frac{m_3m_1}{r_{31}^3} \left[ \frac{m_2m_3(x_1 - 1)}{r_{23}^3} + \frac{m_1m_2x}{r_{31}^3} \right] \times (m_1m_2 + m_2m_3r_{23}^2 + m_3m_1r_{31}^2) = 0,
\]

\[
y \left( m_1m_2 + \frac{m_2m_3}{r_{23}^3} + \frac{m_1m_1}{r_{31}^3} \right) (m_2m_3 + m_3m_1) - \frac{m_3m_1}{r_{31}^3} \times (m_1m_2 + m_2m_3r_{23}^2 + m_3m_1r_{31}^2) = 0.  \]  (10)

The second equation in set (10) can be satisfied in two ways: by setting \( y = 0 \) or by considering as zero the entire coefficient in the brackets beside \( y \). The first possibility \( (y = 0) \) leads to collinear singular points, the second to triangular.

For \( y = 0 \) we obtain from set (10) one equation for the determination of coordinate \( x \) of the collinear singular points:

\[
\varphi(x) = \left( m_1m_2 + \frac{m_2m_3}{\sqrt{(x-1)^2 + y^2}} + \frac{m_1m_1}{\sqrt{x^2 + y^2}} \right) \times [m_2m_3(x_1 - 1) + m_3m_1x] - \left[ \frac{m_2m_3}{(x-1)^2 + y^2} + \frac{m_1m_1}{x^2 + y^2} \right] \times [m_1m_2 + m_2m_3(x_1 - 1)^2 + m_3m_1x^2] = 0.  \]  (11)

The derivative \( \varphi'(x) \) is always positive, and the following limits occur:

\[
\lim_{x \to \pm \infty} \varphi(x) = \mp \infty, \quad \lim_{x \to 0} \varphi(x) = \pm \infty, \quad \lim_{x \to \pm \infty} \varphi(x) = \pm \infty.  \]  (12)

This proves the existence of three real solutions of the equation \( \varphi(x) = 0 \), which, in their turn, determine three collinear singular points of the family of Sundman surfaces in space \( xyR \):

\[
L_i = \left( x_i, 0, \frac{Gm_im_2}{2C^2} + \frac{Gm_im_3}{2\sqrt{(x_1 - 1)^2 + y^2}} + \frac{Gm_im_1}{2\sqrt{x^2 + y^2}} \right)  \]  (13)

where the coordinates \( x_i \) are determined by the numerical solution of equation (11).

However, if \( y \neq 0 \), then after simple conversions we obtain two triangular solutions of set (10),

\[
R_{23} = R_{31} = R,  \quad L_{4.5} = \left( \frac{1}{2}, \pm \frac{1}{2}, \frac{Gm_im_2 + m_2m_3 + m_3m_1}{2C} \right).  \]  (15)

The obtained collinear and triangular singular points correspond to the collinear Euler and triangular Lagrange solutions known in the general three-body problem.

Collinear singular points in the space \( xyR \) lie in different planes \( R = R_i, \) i.e. \( R_i \neq R_j \), and for triangular singular points the following equality is fulfilled: \( R_i = R_j \).

Knowing the coordinates of singular points and constant \( C \), from formula (7) the values of Sundman constants \( B_1, B_2, B_3 \) and \( B_{4.5} \) at all singular points \( L_i (i = 1, 2, \ldots, 5) \) are calculated. Constants \( C \) and \( B_i \) are connected by reciprocal proportion:

\[
B_i = \frac{G^2}{4mC} \left[ m_1m_2 + m_2m_3 \left( \frac{r_{23}^2}{r_{23}^2} \right) + m_3m_1 \left( \frac{r_{31}^2}{r_{31}^2} \right) \right],
\]

\[
\times \left[ m_1m_2 + m_2m_3 \left( \frac{r_{23}^2}{r_{23}^2} \right) + m_3m_1 \left( \frac{r_{31}^2}{r_{31}^2} \right) \right].  \]  (16)

where \( (r_{23}) \) and \( (r_{31}) \) are calculated at the singular point \( L_i \).

The relations (16) have been established by Golubev (1967) in the \( c^2h \) method.

Singular points are points of bifurcation, at which a qualitative change in the shape of the Sundman surface occurs. The curves (16) on plane \( BC \) are the boundaries of topologically different regions of possible motion. The curve \( L_2 \) limits the Sundman-stable region of body \( M_3 \), and this stable region is shaded in Fig. 1.

The general form of the Sundman surfaces for three bodies with mass ratios in proportion 9:3:1 is shown in Fig. 2; a section of the Sundman surfaces at planes \( R = R_i \) and \( y = 0 \) is presented in Fig. 3.

In the general three-body problem, as in the restricted problem, the concept of Hill stability is conserved. However, to distinguish it from the restricted problem, we will call this stability in the general three-body problem Sundman stability.

We will call the motion of body \( M_1 \) in the general three-body problem ‘stable on Sundman’ if there exist regions of possible motion, limited by the appropriate Sundman surfaces, inside which body \( M_1 \) will always (at any instant of time) be located at a finite distance from one of the bodies \( M_i \) or \( M_2 \). In other words, body \( M_1 \) will be an eternal satellite of one of bodies \( M_i \) or \( M_2 \), while bodies \( M_1 \) or \( M_2 \) can be at any distance from each other, including an infinite one.

The criterion of Sundman stability is the inequality

\[
B_i \geq B_2,  \quad (17)
\]

where \( B_2 \) is the value of the Sundman constant at the inner Euler libration point \( L_2 \). The fulfillment of this condition guarantees that the body \( M_1 \) can on some ‘spindly’ surfaces (see Figs 2, 3) remain the eternal satellite of body \( M_1 \), or on other ‘spindly’ surfaces remain the satellite of body \( M_2 \), or else lie in a remote open oval
area when the distance between bodies $M_1$ and $M_2$ remains finite, not exceeding $G m_1 m_2/C$. This last case can be treated as Sundman stability of the relative motion of bodies $M_1$ and $M_2$.

Thus, for (17) any pair of bodies will have Sundman stability if at the initial instant the bodies forming this pair are in one of these regions of stability. The loss of stability (body $M_3$ leaving the ‘spindly’ area) occurs if the value $R$ is close enough to its value $R_5$ at the libration point $L_5$.

### 3 Sundman Stability of Planet Satellite Motion

Analysis of Sundman stability of the motion of all known natural planet satellites of the Solar system is undertaken with the theory presented here. The ephemerides of all planet satellites are calculated with the most up-to-date theories implemented on the NSDC website, constructed by Emelyanov & Arlo (2008). From these ephemerides, constants $C$, $B$ and $B_2$ were calculated in the barycentric coordinate system. Sundman stability was determined from formula (17).

For each satellite the construction of Sundman-surface sections using the coordinate plane $xy$ was also conducted. Sections are given for the Jovian satellites J6 Himalia and J9 Sinope (Fig. 4). Himalia’s coordinate curve is within the Sundman stability region, while Sinope’s curve is outside it. In spite of the location of Sinope’s orbit inside the Sundman lobe, which corresponds to Sundman constant value $B = B_2$, its energy is sufficiently high that it has the potential capability of leaving this lobe (see the dashed curve). However, this does not mean that the satellite will leave the vicinity of the planet in all circumstances. Sundman instability means that the Sundman surfaces are open and allow the satellite to leave the vicinity of planet, they do not tell whether this will occur or not. The same is the case for Hill stability.

Lukyanov (2011) showed the Sundman stability of the Moon’s motion. The Sundman stability results for the satellites of other planets are given in Tables 1–5. The tables also list the results of classical Hill stability. All satellites are listed in the order of increasing semimajor axes of their orbits around the planet. The relative masses of distant planet satellites obtained from satellite photometric observations (Emelyanov & Uralskaya, 2011) are taken from the NSDC website (http://www.sai.msu.ru/neb/nss/index.htm).

The Martian satellites (Phobos and Deimos) show Hill and Sundman stability (Table 1).

The main and the inner satellites of Jupiter show Hill stability and Sundman stability and are not included in the tables. The distant satellites, which have prograde and retrograde orbits, are of special interest. All prograde satellites of Jupiter have Hill and Sundman stability (Table 2). All retrograde satellites with $a > 18.34 \times 10^6$ km are unstable according to Hill and Sundman, independent of their masses. An exception is the satellite S/2003 J12 ($a = 19 \times 10^6$ km, $i = 145.8$, $e = 0.376$) with a relatively small mass, which has Hill stability and Sundman stability.

The situation is different for the satellites of Saturn. The main, inner and distant prograde satellites of Saturn, which belong to the Gallic ($i = 34^\circ$) and Inuit ($i = 45^\circ$) groups, have Hill stability and Sundman stability (Table 3). The retrograde satellites with $a < 18.6 \times 10^6$ km have Hill and Sundman stability, while those with $a > 18.6 \times 10^6$ km have Hill stability but Sundman instability. Furthermore, the satellite S LI Greip with semimajor axis $a = 18.1 \times 10^6$ km is also Sundman-unstable.

The main and inner satellites of Uranus have Hill stability and Sundman stability. The stability results coincide for all distant satellites, except for the most distant satellite U XXIV Ferdinand ($a = 20.9 \times 10^6$ km), which has Sundman instability (Table 4).

Triton and the Neptune inner satellites have Hill and Sundman stability. Two distant Neptune satellites have Sundman instability: N X Psamathe ($a = 46 \times 10^6$ km) and N XIII Neso ($a = 48 \times 10^6$ km). The rest of Neptune’s satellites have Hill stability and Sundman stability (Table 5).

The comparison of the results for Sundman stability and Hill stability shows that Hill stability always follows from Sundman
stability, but the reverse assertion is not correct. It is caused by the fact that, in contrast to Hill’s model, in the Sundman model the satellite masses are not zero but finite. Therefore, each satellite of any planet has an individual value of Sundman constant $B_2$, while in Hill’s model all satellites of any planet have the same value of the Hill constant $C_2$.

Comparison of the results obtained with the Golubev $c^2h$ method is carried out in two forms:

Table 1. Martian satellites. Here $a$ is the semimajor axis of the satellite orbit, $i$ is the inclination, $e$ is the eccentricity and $m/M_P$ is the ratio of the satellite mass to the planet mass.

| Satellite | $a$ (km) | $e$ | $i$ (deg) | $m/M_P$ $10^{-8}$ | Stability |
|-----------|----------|-----|-----------|--------------------|-----------|
| M1 Phobos | 9380     | 0.0151 | 1.1       | 1.6723             | yes       |
| M2 Deimos | 23460    | 0.0002 | 0.9–2.7   | 0.2288             | yes       |
Table 2. The irregular Jovian satellites (notation as in Table 1).

| Satellite   | $a$ (10^6 km) | $i$ (deg) | $e$ | $m/M_p$ $10^{-9}$ | Stability       |
|-------------|---------------|-----------|-----|-------------------|----------------|
|             | 2             | 3         | 4   |                   | Hill Sundman    |
| XVIII Themisto | 7.507         | 43.08     | 0.242 | 3.4889            | yes yes        |
| XIII Leda    | 11.165        | 27.46     | 0.164 | 5.76              | yes yes        |
| VI Himalia   | 11.461        | 27.50     | 0.162 | 22101.8            | yes yes        |
| X Lysithea   | 11.717        | 28.30     | 0.112 | 331.5             | yes yes        |
| VII Elara    | 11.741        | 26.63     | 0.217 | 4578.2            | yes yes        |
| XLVI Carpo   | 16.889        | 51.4      | 0.430 | 0.3394            | yes yes        |
| S/2003 J3    | 18.340        | 143.7     | 0.241 | 0.1263            | no no          |
| S/2003 J12   | 19.002        | 145.8     | 0.376 | 0.0631            | yes yes        |
| XXXIV Euporie| 19.302        | 145.8     | 0.144 | 0.2447            | no no          |
| S/2003 J18   | 20.700        | 146.5     | 0.119 | 0.2920            | no no          |
| XXXV Orthosie| 20.721        | 145.9     | 0.281 | 0.3431            | no no          |
| XXIX Thymoe  | 20.940        | 148.5     | 0.229 | 0.6946            | no no          |
| S/2003 J16   | 21.000        | 148.6     | 0.270 | 0.1342            | no no          |
| XL Mneme     | 21.069        | 148.6     | 0.227 | 0.3315            | no no          |
| XXII Harpalyke| 21.105       | 148.6     | 0.226 | 0.8367            | no no          |
| XXX Hermippe | 21.131        | 150.7     | 0.210 | 1.4919            | no no          |
| XXVII Praxidike| 21.147       | 149.0     | 0.230 | 2.8495            | no no          |
| XLII Thelxinoe| 21.162       | 151.4     | 0.221 | 0.3473            | no no          |
| XXIV Iocaste | 21.269        | 149.4     | 0.216 | 1.3971            | no no          |
| XII Ananke   | 21.276        | 148.9     | 0.244 | 157.9             | no no          |
| S/2003 J15   | 22.000        | 140.8     | 0.110 | 0.1342            | no no          |
| S/2003 J4    | 23.258        | 144.9     | 0.204 | 0.0947            | no no          |
| L Herse      | 22.000        | 163.7     | 0.190 | 0.2526            | no no          |
| S/2003 J9    | 22.442        | 164.5     | 0.269 | 0.0947            | no no          |
| S/2003 J19   | 22.800        | 162.9     | 0.334 | 0.1263            | no no          |
| XLIII Arche  | 22.931        | 165.0     | 0.259 | 0.2842            | no no          |
| XXXVIII Pasithee| 23.096       | 165.1     | 0.267 | 0.1658            | no no          |
| XIX Chaldene | 23.179        | 165.2     | 0.251 | 0.7499            | no no          |
| XXXVII Kale  | 23.217        | 165.0     | 0.260 | 0.2447            | no no          |
| XXVI Isophoe | 23.217        | 165.2     | 0.246 | 0.6157            | no no          |
| XXXI Aitne    | 23.231        | 165.1     | 0.264 | 0.4026            | no no          |
| XXV Erinome  | 23.279        | 164.9     | 0.266 | 0.3789            | no no          |
| XX Taygete   | 23.360        | 165.2     | 0.252 | 1.1445            | no no          |
| XI Carme     | 23.404        | 164.9     | 0.253 | 694.6             | no no          |
| XXIII Kalyke | 23.583        | 165.2     | 0.245 | 1.5471            | no no          |
| XLVII Eukelade| 23.661       | 165.5     | 0.272 | 0.7104            | no no          |
| XLIV Kallichore| 24.043       | 165.5     | 0.264 | 0.2289            | no no          |
| S/2003 J5    | 24.084        | 165.0     | 0.210 | 0.9788            | no no          |
| S/2003 J10   | 24.250        | 164.1     | 0.214 | 0.3947            | no no          |
| XLV Helike   | 21.263        | 154.8     | 0.156 | 0.7183            | no no          |
| XXXII Eurydome| 22.865       | 150.3     | 0.276 | 0.4262            | no no          |
| XXVIII Autonoe| 23.039       | 152.9     | 0.334 | 0.7814            | no no          |
| XXXVI Spone  | 23.487        | 151.0     | 0.312 | 0.2763            | no no          |
| VIII Pasiphae| 23.624        | 151.4     | 0.409 | 1.5787            | no no          |
| XIX Megaleite| 23.806        | 152.8     | 0.421 | 2.1312            | no no          |
| IX Sinope    | 23.939        | 158.1     | 0.250 | 394.7             | no no          |
| XXXIX Hegemone| 23.947       | 155.2     | 0.328 | 0.3394            | no no          |
| XLI Aoede    | 23.981        | 158.3     | 0.432 | 0.6473            | no no          |
| S/2003 J23   | 24.055        | 149.2     | 0.309 | 0.0947            | no no          |
| XVII Callirhoe | 24.102      | 147.1     | 0.283 | 5.3044            | no no          |
| XLVIII Cyllene| 24.349        | 149.3     | 0.319 | 0.2368            | no no          |
| XLIX Kore    | 24.543        | 145.0     | 0.325 | 0.3947            | no no          |
| S/2003 J2    | 28.570        | 151.8     | 0.380 | 0.1500            | no no          |

(i) comparison of the stability criteria used;
(ii) comparison of the obtained regions of possible motion.

Analytical forms of the stability criterion in our work, $B \geq B_2$, and in Golubev’s method, $c^2 h \leq (c^2 h)_c$, are the same. However, the calculation of the constants on the left- and right-hand sides of the inequalities is carried out using different formulae. This leads to some differences in the numerical results. Comparison with the results of the work of Walker et al. (1980) for the satellites J1–J13 shows that the Sundman stability or instability of these satellites obtained in our work agrees with the results of Walker et al. (1980) for all satellites, except for four satellites of Jupiter with retrograde motion.
Table 3. The irregular Saturnian satellites (notation as in Table 1).

| Satellite | $a$ (10$^6$ km) | $i$ (deg) | $e$ | $m/M_p$ 10$^{-11}$ | Stability Hill Sundman |
|-----------|----------------|-----------|-----|---------------------|------------------------|
| XXIV Kiviuq | 11.111 | 45.71 | 0.334 | 0.8629 | yes yes |
| XXII Ijiraq | 11.124 | 46.44 | 0.316 | 0.3248 | yes yes |
| IX Phoebe | 12.944 | 174.8 | 0.164 | 1458.957 | yes yes |
| XX Paaliaq | 15.200 | 176.7 | 0.218 | 0.0248 | yes yes |
| XXVII Skathi | 15.541 | 152.6 | 0.270 | 0.0588 | yes yes |
| S/2007 S2 | 15.600 | 176.7 | 0.218 | 0.0248 | yes yes |
| XXVII Bebhionn | 17.119 | 35.01 | 0.469 | 0.0261 | yes yes |
| XXVIII Erriapus | 17.343 | 34.62 | 0.474 | 0.2294 | yes yes |
| XXIX Siarnaq | 17.531 | 45.56 | 0.295 | 0.0430 | yes yes |
| XLVII Skoll | 17.665 | 161.2 | 0.464 | 0.0237 | yes yes |
| LII Greip | 18.105 | 172.7 | 0.374 | 0.0158 | yes no |
| XLV Kari | 22.118 | 156.3 | 0.478 | 0.0409 | yes no |
| XLVI Loge | 22.707 | 177.5 | 0.451 | 0.0127 | yes no |
| XLII Fornjot | 25.108 | 170.4 | 0.215 | 0.0211 | yes no |

Table 4. The irregular Uranian satellites (notation as in Table 1).

| Satellite | $a$ (10$^6$ km) | $e$ | $i$ (deg) | $m/M_p$ 10$^{-9}$ | Stability Hill Sundman |
|-----------|----------------|-----|-----------|---------------------|------------------------|
| XXII Francisco | 4.2760 | 0.1425 | 147.613 | 0.0518 | yes yes |
| XVI Caliban | 7.1689 | 0.0823 | 139.681 | 8.1305 | yes yes |
| XX Stephano | 7.9424 | 0.1459 | 141.538 | 0.3494 | yes yes |
| XXI Trinculo | 8.5040 | 0.2078 | 166.332 | 0.0200 | yes yes |
| XVII Sycorax | 12.2136 | 0.5094 | 152.669 | 46.6790 | yes yes |
| XXIII Margaret | 14.3450 | 0.7827 | 50.651 | 0.0609 | yes yes |
| XVIII Prospero | 16.1135 | 0.3274 | 146.340 | 1.1306 | yes yes |
| XIX Ymir | 23.040 | 145.2 | 0.521 | 0.0430 | yes no |
| XXIV Ferdinand | 20.9010 | 0.4262 | 167.278 | 0.0874 | yes no |

motion: J VIII, J IX, J XI and J XII. For these satellites we obtained instability, while in the work cited these satellites were indicated as being stable. This is likely to be due to the approximation of the three-body problem by two problems of two bodies and also by the neglect of orbit inclinations.

We conducted the construction of regions of possible motion in the three-dimensional space $xyR$, while in all works of other authors the value of $R$ is excluded from the examination and the construction of regions of possible motion is conducted in the $xy$ plane. For this reason, in the $c^2h$ method it is not possible to obtain a number of important results. For example, it cannot be shown that the loss of stability (withdrawal of body $M_3$ from the stability region) can occur only for a certain distance between bodies $M_1$ and $M_2$. Generally, Sundman curves in the plane $R = $ constant with a change in $R$ can
differ sharply and qualitatively from Hill’s curves, as shown by Lukyanov (2011).

4 DISCUSSION

The famous Sundman inequality in the general three-body problem takes the form
\[
(U - C)J - B \geq \frac{j^2}{8}.
\]
For the material motions of bodies, i.e. with the fulfilment of conditions \( J^2 \geq 0 \), it determines the regions of possible motion satisfying the inequality
\[
(U - C)J \geq B.
\]

The boundaries of the region of possible motion are determined by the equation
\[
(U - C)J = B,
\]
which we call the equation of the Sundman surface, while the stability in Hill’s sense for the three-body problem is denoted as Sundman stability. By analogy with surfaces of zero speed in the restricted three-body problem, we may call the Sundman surfaces in the general three-body problem the surfaces of zero rate of change of the barycentric moment of inertia of bodies \( J = 0 \).

The determination of Sundman stability and the construction of Sundman curves in the plane of parameters \( C \) and \( B \) (see Fig. 1) has been completely solved by Golubev 1 (1967) in his \( c^2h \) method (in our designations \( c^2 = 2B, h = -C \)). Now this method is called Golubev’s method.

Golubev’s method determines the surfaces but the Sundman curves located in the plane of the triangle formed by the mutual distances between the bodies. The mutual distances between the bodies \( R_{12} \) and \( R_{23} \) are substituted by the relative values \( R_{12}/R \) and \( R_{23}/R \) and the value of \( R = R_{12} \) is generally excluded from examination.

The equation of the ‘current’ Sundman curve in Golubev’s method has the form of a hyperbola \( CB = \text{constant} \). If in this case the constants \( C \) and \( B \) are expressed in terms of any other variables then, in its turn, the task of construction of the Sundman curves in the space of these variables arises. Thus, in the large series of works of Szebehely & Zare (1976), Walker (1983), Donnison (2010) and many other authors, the task of constructing Hill–Sundman curves and determination of stability regions in the general three-body problem is solved by Golubev’s method in the space of six quantities: semimajor axes \( a_1, a_2 \), eccentricities \( e_1, e_2 \) and inclinations \( i_1, i_2 \). For calculation of the constants \( C \) and \( B \), the approximation of the three-body problem by two problems of two bodies is used. This introduces a certain error to the solution of the problem. Additionally, the value of \( R \) remains unknown.

For the representation of Sundman curves on the plane \( xy \), Golubev (1968) considered another method. He used the simplified Sundman inequality instead of the exact inequality (19):
\[
U^2J \geq BC,
\]
which is the consequence of inequality (19) and is obtained after the multiplication of inequality (19) by \( U \), taking into account inequalities \( C > 0 \) and \( U > C \). Inequality (21) does not reflect the entire diversity of the Sundman surfaces.

Like the \( c^2h \) criterion (obtained from the condition of positivity of the discriminant of the quadratic trinomial for \( R \) from the left-hand side of the Sundman inequality), simplified inequality (21) does not contain the mutual distance \( R_{12} = R \). Therefore, by means of inequality (21) it is possible to construct not the surfaces but the Sundman curves in the plane of relative coordinates \( xy \). The construction of these curves was subsequently conducted in the works of Marchal & Saari (1975), Marchal & Bozis (1982) and other authors.

Thus, the task of constructing the Sundman surfaces in the space of the coordinates used remained incomplete before the publications of Lukyanov & Shirmin (2007) and Lukyanov (2011) appeared. Lukyanov & Shirmin (2007) used the mutual distances between the bodies as the coordinates. This made it possible to construct exact Sundman surfaces in the three-dimensional space of mutual distances. Lukyanov (2011) used the more convenient rectangular coordinate system \( xyR \), determined by the accompanying triangle of mutual positions of the three bodies.

In these works the exact Sundman inequality (19) is used and, therefore, the value of \( R \) is not excluded from the examination. In this case no simplifications or assumptions are applied. The construction of the Sundman surfaces is implemented in the three-dimensional space of the coordinates used with the determination of the singular points of surfaces, regions of possible motion and Sundman stability regions.

Regions of possible motion constructed by means of the exact Sundman inequalities differ from the analogous regions defined according to the simplified Sundman inequality, both quantitatively and qualitatively.

The stability regions determined by the simplified Sundman inequality (21) have larger sizes than those calculated by exact inequality (19). Therefore, the stability obtained by means of (21) can turn to instability, when using exact inequality (19).

It is easy to derive by means of exact Sundman surfaces that the loss of Sundman stability for body \( M_1 \) can occur only when a certain distance \( R \) between the bodies \( M_1 \) and \( M_2 \) is reached, so that the ‘passage’ through the neighbourhood of the singular point \( L_2 \) is open. It is caused by the fact that the singular points of the Sundman

\[\begin{array}{|c|c|c|c|c|c|}
\hline
Satellite & a (10^6 km) & e & i (deg) & m/M_p \times 10^{-6} & Stability \\
\hline
II Nereid & 5.5134 & 0.7512 & 7.232 & 301.38 & yes & yes \\
IX Halimede & 15.728 & 0.5711 & 134.101 & 3.0835 & yes & yes \\
XI Sao & 22.422 & 0.2931 & 48.511 & 0.6445 & yes & yes \\
XII Laomedea & 23.571 & 0.4237 & 34.741 & 0.5606 & yes & yes \\
X Psmatthe & 46.695 & 0.4499 & 137.391 & 0.9244 & yes & no \\
XIII Neso & 48.387 & 0.4945 & 132.585 & 1.3423 & yes & no \\
\hline
\end{array}\]

\footnote{In English-language literature, the surname Golubev is frequently written incorrectly.}

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Table 5. The irregular Neptunian satellites (notation as in Table 1).
surfaces are determined by three coordinates \(L_i(x_i, y_i, R_i)\) and in
the space \(xyR\) they lie, generally speaking, in different planes. This result cannot be established with the aid of inequality (21), since it
does not depend on \(R\).

The construction of exact Sundman surfaces allows us to define the regions of possible motion for any of the three bodies and for
any values of \(C\) and \(B\). Using the Sundman surfaces yields, for example, that with the fulfilment of the stability criterion the body
\(M_3\) (it can be any body) for any time \(−∞ < t < ∞\) will be located at
a finite distance from one of bodies \(M_1\) or \(M_2\) or at a large distance
from these bodies. Qualitatively, the analogous result is known for
Hill surfaces in the restricted three-body problem as well. If body
\(M_3\) is located, for example, in the stability region near \(M_1\), then the
Sundman surfaces admit the possibility of the retreat of body \(M_2\)
to any large distance from the pair \(M_1, M_3\). For Hill surfaces, this
situation is not possible.

By means of Sundman surfaces, it is possible to establish the sta-
bility of only one pair of bodies, and the third body will in this case be unstable in the Sundman sense. Sundman surfaces do not establish the simultaneous stability of three bodies, i.e. the guaranteed
location of all bodies in a certain finite region of space (Lagrange
stability), although these surfaces do not exclude this case. Sund-
man instability does not mean that a body will necessarily leave the
neighbourhood of another body. The Sundman surfaces do not allow us to determine whether this retreat will actually occur. This
result is analogous to that of Hill stability. The determination of
Sundman stability of the planet satellites of the Solar system con-
ducted in this study shows the effectiveness of the use of Sundman
surfaces in coordinate form.

We believe that our results are of particular interest for celestial
mechanics and for astronomy as a whole.

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