A finitization of the bead process

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Received: 6 July 2009 / Revised: 9 September 2010 / Published online: 20 October 2010
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Abstract The bead process is the particle system defined on parallel lines, with underlying measure giving constant weight to all configurations in which particles on neighbouring lines interlace, and zero weight otherwise. Motivated by the statistical mechanical model of the tiling of an $abc$-hexagon by three species of rhombi, a finitized version of the bead process is defined. The corresponding joint distribution can be realized as an eigenvalue probability density function for a sequence of random matrices. The finitized bead process is determinantal, and we give the correlation kernel in terms of Jacobi polynomials. Two scaling limits are considered: a global limit in which the spacing between lines goes to zero, and a certain bulk scaling limit. In the global limit the shape of the support of the particles is determined, while in the bulk scaling limit the bead process kernel of Boutillier is reclaimed, after appropriate identification of the anisotropy parameter therein.

Mathematics Subject Classification (2010) 60G55 · 60C05 · 60B20

1 Introduction

Consider a particle system confined to equally spaced lines that are parallel to the $y$-axis and pass through the points $x = k$ for $k \in \mathbb{Z}^+$, and let this latter integer label the lines. For each line $k$ place point particles uniformly at random, with the constraint
that their positions interlace with the positions of the particles on lines \( k \pm 1 \). It has been shown by Boutillier [5] that such uniformly distributed interlaced configurations—referred to as a bead process—are naturally generated within dimer models. Furthermore, it is shown in [5] that the bead process is an example of a determinantal point process: the \( k \)-point correlation function is equal to a \( k \times k \) determinant with entries independent of \( k \). If all the particles are on the same line, this correlation function is precisely that for the eigenvalues in the bulk of a random matrix ensemble with unitary symmetry, for example the Gaussian unitary ensemble of complex Hermitian matrices (GUE).

If some of the particles are on different lines, and the anisotropy parameter (see below for this notion) is zero, the correlations between particles on different lines coincide with the correlation between bulk eigenvalues in the GUE minor process [9]. The latter is the multi-species particle system formed by the eigenvalues of a GUE matrix and its successive minors. Let \( M_n \) denote an \( n \times n \) GUE matrix, and form the principal minors \( M_t, t = 1, \ldots, n - 1 \) as the \( t \times t \) GUE matrices corresponding to the top \( t \times t \) block of \( M_n \). Let the eigenvalues of \( M_t \) be denoted by \( \{ x^{(t)}_j \}_{j=1}^{t} \) with \( x^{(t)}_1 < \cdots < x^{(t)}_t \). By a theorem of linear algebra, the successive minors \( M_{t+1}, M_t \) have the fundamental property that their eigenvalues interlace,

\[
x^{(t+1)}_t < x^{(t)}_t < x^{(t+1)}_t < \cdots < x^{(t+1)}_2 < x^{(t)}_1 < x^{(t+1)}_1
\]

(1.1)

for \( t = 1, \ldots, n - 1 \). It turns out that for \( n \times n \) GUE matrices [1], the joint distribution of all the \( \frac{1}{2}n(n + 1) \) eigenvalues \( \cup_{s=1}^{n} \{ x^{(s)}_j \}_{j=1}^{s} \) of all successive minors is proportional to

\[
\prod_{s=1}^{n-1} \chi(x^{(s)}, x^{(s+1)}) \prod_{j=1}^{n} e^{-(x^{(n)}_j)^2} \prod_{1 \leq j < k \leq n} (x^{(n)}_j - x^{(n)}_k)
\]

(1.2)

where \( \chi(x^{(s)}, x^{(s+1)}) \) denotes the interlacing (1.1) between neighbouring species. Regarding species \( (s) \) as occurring on line \( s \), one sees that the eigenvalues to the left of line \( n \) all occur uniformly within interlaced regions, precisely as required by the bead process.

The main objective of this paper is to construct and analyze a finitization of the bead process. Like in [5], we are motivated by a statistical mechanical model—in our case the tiling of an \( abc \)-hexagon by three species of rhombi. However unlike in [5] this statistical mechanical model plays no essential role in the ensuing analysis. We will see in Sect. 2 that the rhombi tiling of an \( abc \)-hexagon leads to a particle system defined on the segments \( 0 < y < 1 \) of the lines at \( x = 1, 2, \ldots, p + q - 1 \) \( (p \leq q) \) in the \( xy \)-plane. The number of particles, \( r(t) \) say, on line \( x = t \) (to be referred henceforth as line \( t \)) is given by

\[
r(t) = \begin{cases} 
  t, & t \leq p \\
  p, & p \leq t \leq q \\
  p + q - t, & q \leq t. 
\end{cases}
\]

(1.3)