On the nature of intermittency in weakly dissipative systems

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Abstract

We propose a new perspective on the intermittency in weakly dissipative systems. It is focused on the elementary (burst-like) event separating states with different properties. This event is seen as a real-space-time manifestation of an instanton connecting in an extended space-time states with distinct topology of the idealized version of the system.
I. Introduction

The intermittency is in general associated with particular statistical properties of a fluctuating quantity like the fields describing the turbulent fluid or plasma. In certain regimes the nonlinear dynamics generates quasi-coherent structures like vortices and the high correlations of these formations induce a non Gaussian statistics. The random generation and destruction of quasi-coherent vortices is one of the sources of intermittency in fluid and plasma. The slow algebraic decay of the spectrum at high wavenumbers is also considered a manifestation of the intermittency. In this case the random events consisting of rapid energy exchanges at small spatial scales should be attributed to vortex merging or island coalescence, processes which generates quasi-singular layers at reconnection. For other dynamical systems, the intermittency is manifested as a random sequence of bursts, i.e. time localised, strong increase of the amplitude of the field. These events separate states where the system preserves stationary properties, like are the regular flows or the turbulence with stationary statistical properties. During such an event, which is very short in time, the field is highly irregular, producing in general a reorganization of the flow. To simplify the terminology we will say that the intermittent events separate states with stationary properties (SWSP) of the system.

We note that the statistical perspective on the intermittency is mainly observational or attempts to characterize the statistical properties (spectrum, scaling laws of the correlations) on the basis of the general properties of the dynamical equations, like symmetry and conservation.

As an alternative to the statistical approach we propose in this paper a general model for the intermittency, regarded as a dynamical process leading to fast and significant changes in the fluctuating field. We take as the essential characteristic of the intermittency the events consisting of passages between states with stationary properties of the system. In many cases it can be recognized, for ideal systems, a SWSP as a family of configurations with the same topology. Consider for example a number of vortices moving in an ideal fluid. Because the fluid is ideal it is not possible to go from one topological configuration (for example $n$ vortices) to a different one ($n' \neq n$ vortices) since the passage requires breaking and reconnection of the flow lines which in turn depends on the presence of resistivity (in general, of dissipation). In real space-time these transitions are not allowed and the ideal system is constrained to evolve inside a family of topologically equivalent states, what we have defined as a regime of SWSP. But even for the ideal system, if the space-time is adequately extended it may be possible, under certain conditions, that the transitions can be realized. The passage taking place in an extended space-time between configurations with distinct topological content will be called instanton, in analogy with the instantons connecting states of degenerate vacua in field theory.

Compared to the ideal case, the evolution of the weakly dissipative system consists of motions which are homotopically equivalent and, from time to time, transitions between distinct configurations, the transitions being only allowed
due to the presence of the dissipation. The essential idea of the model we propose is that these dissipative transitions are not arbitrary: these transitions evolve in a way which is the manifestation in real space-time of the instantons connecting topologically distinct configurations in the extended space-time. These transitions in real space-time are only possible if it exists for the particular system a topological structure in which are embedded both the initial and the final real space configurations.

II. Intermittency and singularities in the plane of complex time

Many numerical studies have been done for systems exhibiting intermittency in the form of burst-like events. It has been found that there is a connection between the positions of these events on the time axis and the positions of the singularities of the solution in the complex time plane. This has first been shown by Frisch and Morf in a study of a nonlinear differential equation for the variable (velocity) $v$ in the presence of damping $\gamma$ and random drive $f(t)$:

$$m \dot{v} = -\gamma v - v^3 + f(t)$$

The numerical results show that the singularity is located above the burst event and that the amplitude is larger when the singularity is closer to the real time axis. The connection has been investigated using the Fourier representation of the solution.

This study presented a certain resemblance with the problem of integrability of second-order ordinary differential equation, which has been formulated by Painlevé in the form of a precise criterion: a second order differential equation is integrable if the only movable singularities of the solution, in the complex plane of the variable (e.g. time), are poles. The analogy between the intermittency/singularity and integrability/singularity problems has inspired a considerable effort of characterizing by numerical methods the nature and the position in the complex plane of the singularities, eventually leading to the extension of the Painlevé criterion to larger classes of equations. Part of these studies have been devoted to understanding chaos (via Melnikov integral) as opposed to the integrability. More connected with the problem of intermittency are the numerical studies of the following system:

$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} + m^2 \varphi + g\varphi^3 = 0$$

By numerical integration the position in the complex plane of the singularities, denoted $x_s + iy_s$, has been obtained. In Bassetti et al. the Fourier representation has been used to express the function $\varphi$ by its singularities closest to the real time axis: $\text{Im} \varphi \sim \exp(-ky_s)$ and a graph of $y_s(t)$ is obtained numerically.

From the perspective explained above we have to see the two minima of the potential in Eq. as topologically distinct states. The transition between
these minima is forbidden for a classical particle, in the absence of negative dissipation (i.e., contact with a thermal bath). But in an extension of the space-time (here simply: imaginary time) there are instantons (or topological solitons) connecting the two states. In this particular case we can precisely identify them, since the equation describes a $\varphi^4$ theory and the solution is ($u$ is the velocity and $t_0$ is the initial time)

$$\varphi(x,t) = \frac{m}{\sqrt{g}} \tanh \left( \frac{m}{\sqrt{2}} \frac{(t-t_0) - ux}{\sqrt{1-u^2}} \right)$$

This kink is an instanton connecting the state $i\varphi = -m/\sqrt{g}$ at $t = -\infty$ with $i\varphi = +m/\sqrt{g}$ at $t = +\infty$. We take $u = 0$ and $t_0 = 0$ and note that the tanh has singularities when the argument is $(i\pi/2)(2n+1)$, $n \in \mathbb{N}$. Using the approximation $\tanh \left( \frac{i\pi}{2} + x \right) \sim \frac{1}{x}$ we find the imaginary part $y_s$ of the position in the complex $t$ plane of the singularity:

$$y_s = -\frac{1}{k} \ln \left( \frac{\sqrt{g} t}{2} \right)$$

This formula reproduces the Fig.2 of Bassetti et al. We have an example where the knowledge of the instanton explains the connection between complex singularities and the intermittent bursts.

III. The conjecture

We formulate once more the basic idea: the burst-like intermittency is a real-space/time manifestation of an instanton transition between topologically distinct configurations generalizing states with stationary properties (SWSP). The projection on real-space/time is only allowed in the presence of dissipation.

This is nothing but a conjecture and we have to gather solid arguments to support it. However this idea already suggests a series of steps to be taken in examining the model of an intermittent system.

1. First, we have to find an ideal version of the system, e.g., by suppressing the dissipation part. Actually this is often invoked in the examination of the onset of chaos in systems weakly perturbed around integrability.

2. Then the ideal model which is closest to the real one must be examined for identifying the distinct topological classes to which the solutions belong.

3. Next we have to extend the system: not only the space-time must be extended (to larger dimensionality and/or complex variables) but the equation of the model must be embedded into a larger system (e.g., the simple pendulum equation is reduced from a self-dual Yang-Mills model).

4. In this extended theory one should look for instantons connecting configurations of distinct topological classes. The existence of these solutions is the exclusive condition for the possibility of real space/time intermittency.
5. One should be able to find also the nature of singularities of these instantons, whose signature could still be identifiable after returning to the original real-space/time system. This will help to localize the bursts by their relation with the extended space/time singularities.

The extension of the theory and of the space-time means the inclusion of the theory into a much larger context. For the ideal system it is necessary to represent its SWSP as states with nontrivial topological content. The instantons are transitions from one topological (SWSP) state to another.

In general one should expect that in arbitrary extension of the theory instantons do not exist, due to topological constraints. The instanton must be a solution of the extended system having as initial condition a configuration with a particular topology and as target a configuration with a different topology. Most frequently, the initial and final configurations are sections of fibre bundles, as is the case for the $O(n)$ model. Connecting two configurations first requires to embed both homotopy classes to which belong the two sections into one single object.

Of particular importance is to correctly infer the extended theory whose equation must reduce to the original ones when we return to the real space-time. There is however a series of deep connections that have been revealed in recent years between classical integrable or topological differential equations and the Self-Dual Yang-Mills (SDYM) field theory. Reducing equations from SDYM has been done for many well known differential equations, as will be discussed bellow. On the other hand there are precise situations where the construction of instantons as geometric-algebraic objects can be done systematically, using twistor theory. While these instruments are very useful in investigating the conjecture, they do not automatically lead to successful determination of the extended theory.

IV. Arguments supporting the conjecture

The requirement to enlarge the theoretical framework of the ideal version of the system to a more complex theory can be formulated conversely: the ideal system should result by reduction from a more general theory where instantons can be found connecting the states of distinct topology. There is a large number of ideal systems with exceptional integrability properties, which can be derived by reduction from the Self-Dual Yang-Mills theory. For example the integrable hierarchies (KdV, Nonlinear Schrodinger Equation, $sine$-Gordon, etc.), the Painlevé transcendents etc. The self duality equations have solutions with nontrivial topology inherited from the structure of fibre bundle defined by the base space and the group of automorphism of the typical fibre, most frequently (for principal fibre bundles) the algebra of symmetry group. Self duality (equality of the curvature two form with its Hodge dual) provides nonzero Chern class. It is in this framework that the first examples of instantons (in particular the 't Hooft-Polyakov instanton) have been found. We shall see bellow that one can
construct instantons as Riemann surfaces, at least for simple topology of the SWSP configurations.

V. An example

The simplest example of topological nontrivial state is represented by a closed line on a torus surface. It is specified by two integer numbers $(m, n)$ which means that $m$ turns must be made in the toroidal direction and respectively $n$ in the poloidal direction for the line to close in itself. Any homotopic deformation preserves $(m, n)$. A line with a different pair $(m', n')$ cannot be deformed into the first one, i.e. the homotopy classes are labeled by the two integers. There are homotopic deformations which makes that a line from one family becomes close on a finite space region to a deformed line belonging to other family. Then the presence of dissipation can allow for reconnection, which produces a transition of the line from one family to another. This occurs in real space-time. According to the conjecture, this transition is the manifestation in real space-time of the existence of a solution connecting in extended space-time solutions belonging to the two families. We have to look for this supersolution.

To examine this example we simplify taking $m = 1$. Any solution represents a section in a fibred space whose base space is the circle (the axis of the torus) and the space of internal symmetry (the fibre) is also a circle (a phase variable, represented by an arrow from the current point on basis to a point of a circle, here the poloidal section). A configuration (helical line) is a map

$$S^1 \rightarrow S^1$$

characterized by an integer number $n$ representing the degree of the map. This means how many times a circle covers a circle, or, how many times the internal-space phase variable $\theta$ varies between 0 and $2\pi$ for a single turn along the circle representing the base space.

![Figure 1: The mapping $S^1 \rightarrow S^1$ with two topological degrees, $n = 1$ (left) and $n = 2$ (right)](image)

We will look for configurations allowing a transition between a line with (for example) $n = 1$ to another line with $n = 2$. 
First the model is extended to one where the homotopical deformations are given by the time dynamics, a much more complex problem, where the simple lines represent the instantaneous positions of the masses of a chain of pendulae. This is the sine-Gordon equation.

A. Extending the sine-Gordon equation to a larger system

When we start from an ideal system (like Korteweg-deVries, sine-Gordon, Nonlinear Schrodinger, Painlevé transcendents, etc.) and try to look at it as a reduced form of a larger and more complex system, it systematically appears the Self-Dual Yang-Mills system.

It has been proved that all the equations mentioned above are reduced forms of the SDYM equations. The derivation consists of several steps:

- definition of the space-time; for the Painlevé transcendents this is the complexified four dimensional space \((\xi, \hat{\xi}, \tau, \hat{\tau})\) with metric \(ds^2 = d\xi d\hat{\xi} - d\tau d\hat{\tau}\).

- the fibre bundle structure, where the basis is the space-time and the fibre is a vector space with the group of automorphism \(SL(2, \mathbb{C})\). The gauge potential is a connection one-form and the self-duality expresses the equality of the curvature two-form with its Hodge dual.

- imposing the invariance to a group of symmetries. The generators of the algebra of the symmetry group are constructed using the normal form of the matrix that combines the basis vector fields in the projective coordinates.

- the connection one-form is contracted with these generators and particular expressions are obtained. Requiring self-duality generates the nonlinear equations

The abstract procedure can be expressed in the language of twistors.

After finding that the idealized form of the initial system can be seen as a reduced form of the SDYM system, we can look for a sufficiently general framework for the SDYM: larger dimensionality, larger symmetry group, etc. The aim is to reach the form of the theory where the solution connecting the two topological structures is available. In this idea, the extension of the SDYM system to the Hitchin system will prove to be essential for identifying the passage between the two \(S^1 \rightarrow S^1\) states characterized by different topological degrees \(n\) as the Riemann surface representing interaction of two strings.
B. String theory realisation of the instanton as a Riemann surface

Looking for systematic methods to construct instantons connecting configurations with different topologies, we must start with the cases where the initial and final states have the simplest nontrivial homology. These are essentially the classes of states of a system representing the mapping \( \varphi: S^1 \rightarrow S^1 \) whose first homotopy group is \( \pi_1(S^1) = \mathbb{Z} \). Such a system is the sine-Gordon equation. The states are classified by the integer \( n \) and we ask how to connect, for example, a state \( n \) with a state \( n' \neq n \). We can find a suggestive algebraic-geometric construction in the Matrix theory.

In general terms the Matrix theory is the non-perturbative description of the \( M \)-theory which, in turn, is the 11-dimensional theory of the strong coupling limit of type \( IIA \) strings. The string theory can be obtained from the \( M \)-theory by compactification on a circle and identifying the string coupling constant with the radius of this circle. By compactification it results a supersymmetric Yang-Mills theory (SYM) which describes non-perturbatively the type \( IIA \) strings. The duality of these theories (strings and SYM) is underlined by the relation between their coupling constants \( g_s = \left( \sqrt{\alpha'} g_{SYM} \right)^{-1} \) and means that examining the strong coupling limit of SYM we get the weakly interacting, i.e., perturbative, limit of the string theory. In Ref.\,[11] it is shown that the deep infrared limit (equivalent to strong coupling in SYM) is a theory describing strings propagating freely. The supersymmetric Yang-Mills theory offers two models equally interesting for their topological properties. In the absence of interaction (a phase with completely broken \( U(N) \) symmetry), the strings are multiply wound along the compact dimension of the cylindrical base space. When a weak interaction is allowed we have separation and connection of strings. At this point a Riemann surface is introduced and this is the construction we want to examine. However, in order to reach that point we have to say few words about the matrix theory. Everything that follows can be found in the papers related to the \( M \)-theory, in particular in Refs.\,[12,13], which we strongly recommend to be read for more detailed explanation of this framework.

The starting point is the supersymmetric Yang-Mills theory in 10 dimensions. This is dimensionally reduced to two spatial dimensions yielding a SYM theory with gauge group \( U(N) \) defined on the \( 1+1 \) dimensional Minkowski space, with the action

\[
S = \frac{1}{2\pi} \int d\tau d\sigma Tr \left[ \frac{1}{2} (D_\alpha X^I)^2 + i \Theta^\alpha \mathcal{D}_\Theta - \frac{g_s^2}{2} F_{\alpha\beta}^2 \right] + \frac{1}{2g_s^2} \left[ X^I, X^J \right]^2 + \frac{1}{g_s} \Theta^\gamma_i \left[ X^I, \Theta \right] \tag{3}
\]

All fields are hermitian matrices of order \( N \times N \). The indices \( \alpha \) and \( \beta \) take values \((0,1)\) and \( I \) takes values between 1 and 8. The covariant derivative is
\[ D_\alpha X^I = \partial_\alpha X^I + i [A_\alpha, X^I] \]

The operator of covariant derivative is contracted with the \textit{gamma} matrices in 2D, which verify the relations: \( \{ \rho_\alpha, \rho_\beta \} = -2\eta_{\alpha\beta} \) where \( \eta_{\alpha\beta} \) is the flat Minkowski metric. The \( \Theta \) fields consists of 8 matrices \( N \times N \) having as elements the 2-spinors \( \Theta^T = (\theta_s^-, \theta_c^+) \) where the \( \pm \) sign correspond to the chirality in 2 dimensions and \( \theta_s^-, \theta_c^+ \) are spinors in the representations \( 8_s \) and \( 8_c \) of \( SO(8) \). The matrices \( \gamma_i \) are 16 \( \times \) 16 gamma matrices of \( SO(8) \). The coupling constant is \( g_s \).

For small \( g_s \) the strings are weakly coupled. At the limit \( g_s = 0 \) there is no interaction and all matrices commute. In this case the matrices \( X \)'s and the fermionic fields \( \Theta \) can be written

\[
X^I = U x^I U^\dagger \\
\Theta = U \theta U^\dagger
\]

where \( x^I \) and \( \theta \) are diagonal matrices and the matrix \( U \) is unitary.

1. Multiply wound strings

It is possible to find field configurations corresponding to strings multiply wound around the compact direction \( \sigma \). To make them more explicit we have to take the matrix \( U \) of the form

\[
U (\sigma + 2\pi) = U (\sigma) g
\]

which means

\[
x (\sigma + 2\pi) = g x (\sigma) g^\dagger
\]

where \( g \) is an element of the Weyl group of the group \( U(N) \). One can see that the variation around the compact coordinate \( \sigma \) yields an interchange of the eigenvalues which form cycles of different lengths. Considering a cycle of length \( n \), it implies \( n \) eigenvalues \( x_1(\sigma, \tau), x_2(\sigma, \tau), ..., x_n(\sigma, \tau) \) with the cyclying property

\[
x_i(\sigma + 2\pi, \tau) = x_{i+1}(\sigma, \tau) \text{ et } x_{n+1} = x_1
\]

In the infrared limit there are sectors corresponding to different ways to divide the total number of eigenvalues in cyclic groups

\[
N = \sum_n n N_n
\]

where \( N_n \) is the number of cycles of length \( (i.e. \) number of eigenvalues in the cycle) \( n \). For these sectors the original non-abelian symmetry is broken to the discrete symmetric group \( S_N \), which has two consequences: (1) it permutes the different cycles of the same length \( n \); and (2) it performs cyclic permutations inside every cycle.
The string interaction appears when two eigenvalues, as functions of \((\sigma, \tau)\), come close and are interchanged. In the point \((\sigma, \tau)\) where they are touching a group \(U(2)\) (a subgroup of the original gauge symmetry \(U(N)\) which was completely broken) is restored.

In conclusion the strings do not interact in the infrared limit, where the gauge symmetry is completely broken down to the maximal torus \(U(1)^r\) where \(r\) is the rank of the gauge group.

### 2. String interaction

As we have said, leaving the infrared limit and allowing a weak interaction it appears in the SYM theory that certain non-abelian subgroups are restored in some region of the space-time. This corresponds to the fact that, around a particular point \((\sigma, \tau)\) two eigenvalues of \(X\) become equal

\[
x^I = x^J
\]

restoring a \(U(2)\) symmetry out of two \(U(1)\). This means that for a nonzero \(g_s\) it occurs a transition between states characterized by the transposition of these two eigenvalues. This is an elementary process of separation or of connection of two strings.

Let us start from the state where there are only free strings. This means that the fields \(X^I\) are diagonal having on the diagonal groups of eigenvalues forming cycles of various lengths. The fields are

\[
A_\mu = 0
\]

\[
X(\sigma, \tau) = \text{diag}(x_1(\sigma, \tau), \ldots, x_n(\sigma, \tau))
\]

with the cycling condition

\[
x_i(\sigma + 2\pi, \tau) = x_{i+1}(\sigma, \tau)
\]

The matrix \(X\) verifies the “free” equation \(\partial_\mu \partial^\mu X = 0\).

When we turn around the interaction point \((\sigma, \tau)\) the eigenvalues are interchanged and the fields are gauge-transformed

\[
X = U(\sigma, \tau) \text{diag}(x_1(\sigma, \tau), \ldots, x_n(\sigma, \tau)) U^\dagger(\sigma, \tau)
\]

\[
A_\mu = igU^\dagger(\partial_\mu U), \quad A_\tau = 0
\]

The fields now verify the non-abelian equations

\[
D_\mu D^\mu X = 0
\]

The gauge matrix \(U\) has the condition

\[
U(\sigma + 2\pi, \tau) = U(\sigma, \tau) g
\]
where \( g \) is the cyclic shift matrix
\[
g = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix}
\] (8)

We conclude that the presence of cycles, i.e. multiply wound strings is connected to the existence of a non-nul pure gauge field \( A_\mu \). On the basis cylinder we can change to complex coordinates \((\sigma, \tau) \rightarrow (z, \bar{z})\) where \( z = \exp\left[\frac{i}{2} (\tau + i\sigma)\right] \). Then the field \( X(\sigma, \tau) \) can be seen as a covering of the complex plane. If there are no cycles, i.e. all eigenvalues of \( X \) are distinct (cycles have length 1) then \( X \) consists of a \( N \) distinct sheets covering of the complex \( z \) plane. When multiply wound strings exists there are branching points of different order placed in the origin. The order of the branching is the length of the cycle and the length of the string. The interactions arise when the branching points are different from the origin.

The explicit determination of the matrix \( X \) is based on the fact that it is a Riemann surface realizing a \( N \)-sheet covering of the complex plane, which means that it is the solution of a polynomial equation of degree \( N \)
\[
\sum_{j=0}^{N} a_j(z) X^j = 0
\] (9)

The essential observation is that the equation (9) is satisfied by the matrix
\[
M = \begin{pmatrix}
-a_{N-1} & -a_{N-2} & \cdots & -a_1 & -a_0 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}
\] (10)

This matrix can be diagonalized by a Vandermonde matrix \( S \)
\[
M = S \, \text{diag}(x_1, x_2, \ldots, x_N) \, S^{-1}
\]
\[
S = \begin{pmatrix}
x_1^{N-1} & x_2^{N-1} & \cdots & x_N^{N-1} \\
x_1^{N-2} & x_2^{N-2} & \cdots & x_N^{N-2} \\
\vdots & \vdots & & \vdots \\
1 & 1 & \cdots & 1
\end{pmatrix}
\] (11)

where \( x_i, i = 1, \cdots, N \) are the eigenvalues of \( M \) and are scalar holomorphic functions of \( z \).

Turning around a ramification point the matrix is multiplied by \( g \)
\[
S \rightarrow Sg
\]
an element of the Weyl group of $U(N)$. The solution for $S$ can be found explicitly when the surface is known (i.e. the coefficients $a_i(z)$) since the eigenvalues $x_i(z)$ can be determined. Then $X$ and $A$ can be calculated (an explicit example for the $\mathbb{Z}_N$ covering is given in [12 and 13]). It is the nontrivial gauge field configuration $A_\mu = igU^\dagger (\partial_\mu U)$ which interpolates between the winding sector in the past (here: $z = 0$) and the winding sector in the infinite future ($z = \infty$).

In simplified terms this is a generalized form of the $\varphi^4$ theory or of the model of a particle in a two well potential. Both models can be embedded into this large framework and their equations can be derived by reduction from the equations obtained from the action $S$. The states corresponding in the simple models to mapping the circle onto the circle a fixed $n$ number of times can be regarded as a cross section of the full field $X(\sigma, \tau)$ at the $g_s = 0$ limit, although the content of $X$ can be vastly richer.

Figure 2: Riemann surface corresponding to the process where two eigenvalues $X_1(\sigma, \tau)$ and $X_2(\sigma, \tau)$ become equal at a certain $\tau$.

Figure 3: The mapping $\sigma \rightarrow X(\sigma, \tau)$ is a representation of the topological map $S^1 \rightarrow S^1$. In the left figure, for times $\tau$ before interaction, the topological degree is $n = 1$; in the right figure (after interaction) it is $n = 2$. 
It has been proved above that to the solution $X$ of the equations of motion with weak interaction corresponds a Riemann surface. This allows to identify the instanton with this surface and to identify the two topological configurations of different $n$'s with multiply wound strings in the incoming and respectively in the outgoing state of the string interaction.

C. Transitions between regimes

The example of the kink connecting in imaginary time the equilibrium positions of a classical particle in the two-well potential can be generalized. The minima of the two-well potential is replaced, in general cases, by states with stationary properties corresponding, for the idealized system, to families with definite topological content. To the instanton connecting the two particle positions it corresponds in general a solution connecting these distinct families of states.

Consider the sine-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sin u = 0$$

In Ref. [14] it is found the following solution

$$u(x, t) = 4 \arctan \left( \frac{a \sin (\tau) + \exp (\Lambda x)}{1 + a \sin (\tau) \exp (\Lambda x)} \right)$$

where $a^2 = k \leq 1$, $\Lambda = (1 - k)/(1 + k) \leq 1$, $\tau = t/(1 + k)$. This solution interpolates between two particular solutions of the sine-Gordon equation (being in this sense equivalent to an instanton connecting SWSP's)

$$\lim_{x \to -\infty} u(x, t) = u_{-\omega}$$

$$\lim_{x \to \infty} u(x, t) = u_{\omega} + 2\pi$$

where $u_{-\omega} = 4 \arctan (\exp x)$. We can see this example as a particular form of the general structure discussed above.

D. Other possible examples

**Solitons as homoclinic curves and intermittency of the Nonlinear Schroedinger Equation.** The intermittency in the case of weakly dissipative, driven, Nonlinear Schrodinger Equation has been examined numerically [10]. It has been found that there are jumps between two kinds of solitonic solutions on a periodic domain. This has been explained by the analogy between the soliton and the homoclinic curve of the simple system like the pendulum, separating two distinct types of behaviour: finite oscillations and free rotation.

Transitions have been identified between states of complex dynamics, for systems like sine-Gordon. In some cases, fast changes of the systems between
states of different symmetry patterns have been observed in experiments with burning gas in porous media, oscillations in Belousov-Zhabotinsky reactions, etc. Under the same perspective should be examined the models of the free-force type, in particular the ABC flow. They are known to exhibit intermittency and also are known to have a self-dual structure.

VI. Discussion

There are many possible extensions and developments arising from this idea, some of them being interesting challenges. For example, if this idea is proved correct and a systematic technical procedure will be available, one of the most important application will be the description of the reconnection of vortices in fluids and plasmas (and similar, of the magnetic structures in weakly resistive plasmas). The approach proposed here naturally explains why the mere presence of the resistivity is required in such systems and why its magnitude is less important: most of the time the fluid performs homotopic deformations and from time to time rapid reconnection events changes the topological degree. The amount of energy implied in this event is not significant while the simple presence of the resistivity is required if we want reconnections to be possible.

A challenging problem which is closely related to our model is the generalization of the Painlevé criterion. It is clear that the singularities of an instanton defined in a much larger theory than the original system cannot be simply reduced to singularities in the complex plane and can possibly appear as singularities of Riemann surfaces or as vanishing cycles. A precise connection between the integrability of a model and the singularity structure of the instanton would represent a generalization of the Painlevé theory.
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