BOUNDARY ESTIMATES FOR POSITIVE SOLUTIONS TO SECOND ORDER ELLIPTIC EQUATIONS

MIKHAIL V. SAFONOV

Abstract. Consider positive solutions to second order elliptic equations with measurable coefficients in a bounded domain, which vanish on a portion of the boundary. We give simple necessary and sufficient geometric conditions on the domain, which guarantee the Hopf-Oleinik type estimates and the boundary Lipschitz estimates for solutions. These conditions are sharp even for harmonic functions.

1. Introduction. Formulation of main results

Let $\Omega$ be a bounded open set in $\mathbb{R}^n$. Consider a second order elliptic operator

$$Lu := \sum_{i,j} a_{ij} D_{ij} u + \sum_i b_i D_i u$$

in $\Omega$, where $D_i u = \partial u / \partial x_i$, $D_{ij} u = D_i D_j u$, $a_{ij} = a_{ji} \in L^\infty(\mathbb{R}^n)$, $b_i \in L^\infty(\mathbb{R}^n)$, and $a_{ij}$ satisfy the uniform ellipticity condition

$$\nu |\xi|^2 \leq \sum_{i,j} a_{ij} \xi_i \xi_j \leq \nu^{-1} |\xi|^2 \quad \text{for all} \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n,$$

with a constant $\nu \in (0, 1]$. In 1952, E. Hopf [10] and O.A. Oleinik [19] independently proved the following boundary point lemma.

Lemma 1.1. Suppose that $\Omega$ satisfies an interior sphere condition at $x_0 \in \partial \Omega$, i.e. there exists a ball

$$B := B_{r_0}(y_0) := \{ x \in \mathbb{R}^n : |x - y_0| < r_0 \} \subset \Omega,$$

2000 Mathematics Subject Classification. 35J15, 35J67 (Primary). 35B05 (Secondary).

Key words and phrases. Second-order elliptic equations, measurable coefficients, boundary Hopf lemma, estimates for ratios of solutions.

The work was partially supported by NSF Grant DMS-9971052.
with \( x_0 \in (\partial \Omega) \cap (\partial B) \). Then for any function \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) satisfying \( u > 0 \), \( Lu \leq 0 \) in \( \Omega \), and \( u(x_0) = 0 \), we have

\[
\liminf_{t \to 0^+} \frac{u(x_0 + tl)}{t} > 0.
\]

where \( l \) is an arbitrary interior vector to \( B \) at the point \( x_0 \), which means \( x_0 + tl \in B \) for all \( t \) in an interval \((0, t_0)\).

In a particular case when \( L = \Delta \) - the Laplacian, this result was established in 1910 by M.S. Zaremba [23]. In the beginning of 1930s, G. Giraud [9] has got a similar result for domains \( \Omega \) with the boundary \( \partial \Omega \in C^{1,\alpha} \), \( 0 < \alpha < 1 \), and operators \( L \) with coefficients satisfying some continuity assumptions. See bibliographical notes in [20], Ch. 2, and [7], Ch. 3, for early references on this subject.

On the other hand, it is well known (see, e.g. [5], IV.7.3) that an exterior sphere condition at \( x_0 \in \partial \Omega \), together with the boundary condition \( u = 0 \) near \( x_0 \), guarantees the boundedness of the ratio \( u(x)/|x - x_0| \) in \( \Omega \). In a “model” case, this property can be formulated as follows.

**Lemma 1.2.** Suppose that \( \Omega \) satisfies an exterior sphere condition at a point \( x_0 \in \partial \Omega \), i.e. there exists a ball \( B := B_{r_0}(y_0) \), such that \( \Omega \cap B = \emptyset \), and \( x_0 \in (\partial \Omega) \cap (\partial B) \). Let \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) satisfy \( u > 0 \), \( Lu \geq 0 \) in \( \Omega \), and

\[
u = 0 \quad \text{on} \quad (\partial \Omega) \cap (B_{\varepsilon_0}(x_0)), \quad \text{where} \quad \varepsilon_0 = \text{const} > 0.
\]

Then

\[
\sup_{\Omega} \frac{u(x)}{|x - x_0|} < \infty.
\]

The proofs of Lemmas 1.1 and 1.2 and their generalizations are usually based on the classical comparison principle ([7], Theorem 3.3).

**Theorem 1.3** (Comparison principle). Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \), and let \( u_1, u_2 \) be functions in \( C^2(\Omega) \cap C(\overline{\Omega}) \) satisfying \( Lu_1 \geq Lu_2 \) in \( \Omega \), and \( u_1 \leq u_2 \) on \( \partial \Omega \). Then \( u_1 \leq u_2 \) in \( \Omega \).

We give short proofs of Lemmas 1.1 and 1.2 which contain some elements of the proofs of our main results, Theorems 1.8 and 1.9. For this purpose, we need the following elementary lemma, which will also be useful later, in the proof of Lemma 2.3.

**Lemma 1.4.** The functions \( v(x) := |x|^{-\lambda} \) satisfies the inequality

\[
\sum a_{ij} D_{ij} v \geq 0 \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}, \quad \text{provided the constant} \quad \lambda = \lambda(n, \nu) > 0 \quad \text{is large enough.}
\]
Proof. We have

\[ \sum_{i,j} a_{ij} D_{ij}(|x|^{-\lambda}) = \lambda |x|^{-\lambda-2} \left[ (\lambda + 2) \sum_{i,j} \frac{a_{ij} x_i x_j}{|x|^2} - \text{tr} \ a \right] \geq \lambda |x|^{-\lambda-2} \left[ (\lambda + 2) \nu - n \nu^{-1} \right] \geq 0 \quad \text{for} \quad x \neq 0 , \]

provided \( \lambda > 0 \) and \( \lambda + 2 \geq n \nu^{-2} \).

\[ \Box \]

Remark 1.5. The previous lemma says that \( L(|x|^{-\lambda}) \geq 0 \) for \( x \neq 0 \), where \( L \) is an operator in (1.1) with \( b_i \equiv 0 \). One can easily adjust the proof of this lemma to the case \( |b_i| \leq K = \text{const} \), with \( \lambda = \lambda(n, \nu, K, \text{diam} \Omega) > 0 \).

Proof of Lemma 1.1. We have \( u \geq c = \text{const} > 0 \) on the set \( \partial B_{r_0/2}(y_0) \), which is a compact subset of \( \Omega \). Following the argument in §1.3 of the book by E.M. Landis [16], consider the function

\[ u_1(x) := c_1 \left( |x - y_0|^{-\lambda} - r_0^{-\lambda} \right) \quad \text{in} \quad \Omega_1 := B_{r_0}(y_0) \setminus B_{r_0/2}(y_0) \subset \Omega , \]

where \( c_1 := (2^n - 1)^{-1} r_0^n \gamma > 0 \). Then \( u_1 = c \leq u \) on \( \partial B_{r_0/2}(y_0) \), and \( u_1 = 0 \leq u \) on \( \partial B_{r_0}(y_0) \), i.e. \( u_1 \leq u \) on \( \partial \Omega \). Moreover, \( Lu_1 \geq 0 \geq Lu \) in \( \Omega_1 \). By the comparison principle, we have \( u_1 \leq u \) in \( \Omega_1 \). It is easy to see that (1.3) holds true for the function \( u_1 \), hence it is also true for the given function \( u \).

\[ \Box \]

Proof of Lemma 1.2. We adjust the argument in §IV.7.3 of the book by R. Courant and D. Hilbert [5]. Replacing the ball \( B \) by a smaller ball if necessary, one can assume that it lies at a positive distance from \( (\partial \Omega) \setminus B_{c_0}(x_0) \). Then it is possible to choose a constant \( R_0 > r_0 \) close to \( r_0 \), such that the set \( (\partial \Omega) \cap (B_{R_0}(y_0) \setminus B_{r_0}(y_0)) \) is a subset of \((\partial \Omega) \cap B_{c_0}(x_0)\). Consider the function

\[ u_2(x) := c_2 \left( r_0^{-\lambda} - |x - y_0|^{-\lambda} \right) \quad \text{in} \quad \Omega_2 := \Omega \cap \left( B_{R_0}(y_0) \setminus B_{r_0}(y_0) \right) . \]

Here \( c_2 > 0 \) is a large enough constant, such that

\[ u \leq c_2 \left( r_0^{-\lambda} - R_0^{-\lambda} \right) = u_2 \quad \text{on} \quad \Omega \cap \partial B_{R_0}(y_0) . \]

On the remaining part of \( \partial \Omega_1 \), which is a subset of \((\partial \Omega) \cap B_{c_0}(x_0)\), we have \( u = 0 \leq u_2 \). This means \( u \leq u_2 \) on \( \partial \Omega_2 \). Moreover, \( Lu \geq 0 \geq Lu_2 \) in \( \Omega_2 \). By the comparison principle, we have \( u \leq u_2 \) in \( \Omega_2 \). Since \( u_2 \) is a Lipschitz function on \( \Omega_2 \), and \( u_2(x_0) = 0 \), the ratio

\[ \frac{u(x)}{|x - x_0|} \leq \frac{u_2(x)}{|x - x_0|} \leq N = \text{const} \quad \text{in} \quad \Omega_2 . \]

On the complementary set \( \Omega \setminus \Omega_2 \), the function \( u \in C(\overline{\Omega}) \) is bounded, and \(|x - x_0| \geq R_0 - r_0 > 0 \). This implies the desired estimate (1.4).

\[ \Box \]
In the formulations of Lemmas 1.1 and 1.2, one cannot replace an exterior or interior sphere condition by a corresponding cone condition, as the following simple example shows.

**Example 1.6.** (i) Fix a constant $\theta_1 \in (0, \pi/2)$ and denote

$$\Omega_1 := \{ x = (x_1, x_2) \in \mathbb{R}^2 : |x| < 1, x_2 > K \cdot |x_1| \},$$

where $K := \cot \theta_1 > 0$. In the polar coordinates $x_1 = \rho \sin \theta$, $x_2 = \rho \cos \theta$, we have

$$\Omega_1 := \{ 0 < \rho < 1, |\theta| < \theta_1 \}, \quad \text{and} \quad z := ix_1 + x_2 = \rho e^{i\theta}.$$  

The function

$$u_1(x_1, x_2) := \text{Re} \left( z^{\gamma_1} \right) = \rho^{\gamma_1} \cos(\gamma_1 \theta),$$

where $\gamma_1 := \frac{\pi}{2\theta_1} > 1$,

belongs to $C^\infty(\Omega_1) \cap C(\mathring{\Omega}_1)$ and satisfies $u_1 > 0$, $\Delta u_1 = 0$ in $\Omega_1$, and $u_1(0) = 0$. It is easy to see that $u_1$ does not satisfy the strict inequality (1.3) (we have an equality) at the point $x_0 = 0 \in \partial \Omega_1$, where $l$ is an arbitrary interior vector to $\Omega_1$.

(ii) The set

$$\Omega_2 := \{ x = (x_1, x_2) \in \mathbb{R}^2 : |x| < 1, x_2 > -K \cdot |x_1| \},$$

can be described in a similar way with $\theta_2 := \pi - \theta_1 \in (\pi/2, \pi)$ in place of $\theta_1$. The function

$$u_2(x_1, x_2) := \text{Re} \left( z^{\gamma_2} \right) = \rho^{\gamma_2} \cos(\gamma_2 \theta),$$

where $\gamma_2 := \frac{\pi}{2\theta_2} \in (0, 1),$

belongs to $C^\infty(\Omega_2) \cap C(\mathring{\Omega}_2)$ and satisfies $u_2 > 0$, $\Delta u_2 = 0$ in $\Omega_2$, and $u_2 \equiv 0$ on $(\partial \Omega_2) \cap B_1(0)$. Obviously, the ratio $u_2(x)/|x|$ is unbounded on $\Omega_2$, i.e. (1.4) fails at the point $x_0 = 0 \in \partial \Omega_2$.

Now consider a more general situation, when a ball $B$ in Lemmas 1.1 and 1.2 is replaced by a body of rotation $Q$.

**Definition 1.7.** Let a constant $r_0 > 0$ be given, and let $\psi(r)$ be a non-negative, non-decreasing function on $[0, r_0]$, with $\psi(r_0) < r_0$. Define

$$Q := \{ x = (x', x_n) \in \mathbb{R}^n : |x'| < r_0, 0 < x_n - \psi(|x'|) < r_0 \}.$$

(i) We say that an open set $\Omega \subset \mathbb{R}^n$ satisfies an **interior Q-condition** at a point $x_0 \in \partial \Omega$ if $\Omega$ contains a body which is congruent with $Q$ with vertex at $x_0$. This means that in an appropriate coordinate system, we have $Q \subset \Omega$, and $x_0 = 0 \in (\partial \Omega) \cap (\partial Q)$.

(ii) We say that an open set $\Omega \subset \mathbb{R}^n$ satisfies an **exterior Q-condition** at a point $x_0 \in \partial \Omega$ if its complement $\Omega^c := \mathbb{R}^n \setminus \overline{\Omega}$ satisfies an interior $Q$-condition at $x_0$. 

Our main results are contained in Theorems 1.8–1.11 below. Theorems 1.8 and 1.9 can be considered as generalizations of Lemmas 1.1 and 1.2 correspondingly, when instead of (exterior or interior) sphere conditions we impose $Q$-conditions with

$$I(\psi) := \int_0^{r_0} \frac{\psi(r) \, dr}{r^2} < \infty.$$  

Without loss of generality, we assume that the coordinate system is chosen in such a way that $x_0 = 0 \in \partial \Omega$, $Q \subset \Omega$ if $\Omega$ satisfies an interior $Q$-condition, and $-Q := \{ x \in \mathbb{R}^n : -x \in Q \} \subset \Omega^c := \mathbb{R}^n \setminus \Omega$ if $\Omega$ satisfies an exterior $Q$-condition. Note that sphere conditions are equivalent to $Q$-condition with $\psi(r) = cr^2$, $c = \text{const} > 0$. In this case $I(\psi) < \infty$ automatically. We prove Theorems 1.8 and 1.9 in Section 3. Another two theorems, Theorems 1.10 and 1.11, are given here just for completeness, without proofs. They claim that the assumption $I(\psi) < \infty$ is sharp: if $I(\psi) = \infty$, then the estimates in Lemmas 1.1 and 1.2 fail. Example 1.6 can serve as a clear demonstration of this fact for $\psi(r) = Kr$.

In Theorems 1.8–1.11, we assume that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a positive solution of the inequality $Lu \leq 0$ or $Lu \geq 0$ in $\Omega$, where $Lu := \sum a_{ij} D_{ij} u$ has the form (1.1), (1.2), with $b_i \equiv 0$. Combining our techniques with others, especially those in the paper by O. A. Ladyzhenskaya and N. N. Ural’tseva [15], one can extend the results in Theorems 1.8–1.11 to more general operators $L$ in (1.1) with $b_i \in L^q$, $q > n$. We plan to do it in our subsequent work. In particular, proofs of Theorems 1.10 and 1.11 will be presented in a more general setting. On the other hand, Example 1.12 below shows that in the case $b_j \in L^n$ all the estimates under considerations fail even for flat boundary, when $\psi \equiv 0$. Here we restrict ourselves to the case $b_j \equiv 0$ in order to expose our method in its “pure” form.

**Theorem 1.8.** Suppose that $\Omega$ satisfies an interior $Q$-condition: $Q \subset \Omega$, with $I(\psi) < \infty$, and $0 \in \partial \Omega$. Then for any function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying $u > 0$, $Lu \leq 0$ in $\Omega$, and $u(0) = 0$, we have

$$\liminf_{t \to 0^+} t^{-1} u(t l) > 0 \quad \text{for each} \quad l \in \mathbb{R}^n_+ := \{ x \in \mathbb{R}^n : x_n > 0 \}.$$

Note that from $I(\psi) < \infty$ it follows that $t l \in Q \subset \Omega$ for small $t > 0$ (Corollary 3.3 below), so that $u(t l)$ in (1.7) is well defined.

**Theorem 1.9.** Suppose that $\Omega$ satisfies an exterior $Q$-condition: $-Q \subset \Omega^c$, with $I(\psi) < \infty$, and $0 \in \partial \Omega$. Then for any function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying $u > 0$, $Lu \geq 0$ in $\Omega$, and $u = 0$ on $(\partial \Omega) \cap B_{r_0}(0)$, we
have

\begin{equation}
M(r_0) := \sup_{x \in B_{r_0}(0)} \frac{u(x)}{|x|} < \infty.
\end{equation}

The notation \(M(r)\) is also used in the following

**Theorem 1.10.** Suppose that \(\Omega \cap B_{r_0}(0) \subset Q\), with \(I(\psi) = \infty\), and \(0 \in \partial\Omega\). Then for any function \(u \in C^2(\Omega) \cap C(\Omega)\) satisfying \(u > 0\), \(Lu \geq 0\) in \(\Omega\), and \(u = 0\) on \((\partial\Omega) \cap B_{r_0}(0)\), we have \(M(r) \to 0\) as \(r \to 0^+\). Obviously, in this case the estimate (1.7) fails.

**Theorem 1.11.** Suppose that \(\Omega^c \cap B_{r_0}(0) \subset -Q\), with \(I(\psi) = \infty\), and \(0 \in \partial\Omega\). Then for any function \(u \in C^2(\Omega) \cap C(\Omega)\) satisfying \(u > 0\), \(Lu \leq 0\) in \(\Omega\), and \(u = 0\) on \((\partial\Omega) \cap B_{r_0}(0)\), we have

\begin{equation}
\liminf_{t \to 0^+} t^{-1} u(t l) = \infty \quad \text{for all} \quad l \in \mathbb{R}^n_+.
\end{equation}

In 1969–1970, similar facts were established by B.N. Khimchenko, first in the case \(L = \Delta\) [12], and then for general elliptic operators \(L\) [9], under the additional assumption \(\psi'' \geq 0\) (in these two papers, the same author’s name is spelled slightly differently). Further, in a series of joint papers by L.I. Kamynin and B.N. Khimchenko (see [11] and references therein), these results were extended to the parabolic and degenerate elliptic equations, under a different assumption \(\psi'(r) = r\psi_1'(r)\) with \(\psi'_1 \geq 0\), \(\psi''_1 \leq 0\). Each of these assumptions, as well as our assumption (1.6), holds true for \(\psi(r) := r^{1+\alpha}\), \(0 < \alpha < 1\), so that the above mentioned result by G. Giraud [9] for \(\partial\Omega \in C^{1,\alpha}\) is extended to general operators \(L\) with bounded measurable coefficients. This case is also covered in the paper [17] by Gary M. Lieberman, in which \(\partial\Omega\) has a Dini continuous normal.

In the papers [10], [19], [12], [9], [11], [17], and many others, the estimates of such kind are proved by means of special comparison functions, which are constructed in a more or less explicit form. Our method is quite different: it does not use any explicit expressions for comparison functions, and it does not require additional assumptions on the functions \(\psi(r)\) in Definition (1.7). Instead, we use the estimates for quotients \(u_2/u_1\) of positive solutions of \(Lu = 0\) in a Lipschitz domain \(\Omega\), which vanish on a portion of \(\partial\Omega\). These estimates were proved by Patricia Bauman in 1982 in her PhD thesis [2], and published a bit later in [3]. Note that some estimates in her paper depend on the modulus of continuity of coefficients \(a_{ij}\). However, it is easy to get rid of this additional dependence. In a more general parabolic case, this was done in [6], Theorem 4.3.
We essentially use the fact that $u(x) \equiv x_n$ is a solution to the elliptic equation $Lu := \sum a_{ij} D_{ij} u = 0$; this is why we assume $b_i \equiv 0$ in (1.1).

Note that the estimates for the quotients $u_2 / u_1$ are also true for solutions to the equations in the divergence form $Lu := \sum D_i (a_{ij} D_j u) = 0$ (see [4]), but they are not helpful here, because linear functions do not satisfy such equations in general, and in fact, the Hopf-Oleinik estimate (1.3) fails even when the boundary is flat (see [7], Problem 3.9).

**Example 1.12.** Consider the functions

$$u_1(x) := \frac{x_n}{|\ln |x||} \quad \text{and} \quad u_2(x) := x_n \cdot |\ln |x||$$

in the cylinder $Q := \{ x = (x', x_n) \in \mathbb{R}^n : |x'| < 1/2, 0 < x_n < 1/2 \}$, extended as $u_1 = u_2 = 0$ on $(\partial Q) \cap \{ x_n = 0 \}$. Then each of these two functions can be considered as a solution to the equation

$$\Delta u + b \cdot Du := \Delta u + \sum b_i D_i u = 0 \quad \text{in} \quad Q,$$

where the vector function $b := -\Delta u \cdot |Du|^{-2} Du$ satisfies

$$|b| = \frac{\Delta u}{|Du|} \leq \frac{\text{const} \cdot |Du|}{|x| \cdot |\ln |x||} \in L^n(Q) \quad \text{for} \quad n \geq 2.$$

However, the left side of (1.7) is 0 for $u = u_1$, and the left side of (1.8) is $\infty$ for $u = u_2$.

In Section 2, we bring together, in a convenient form, some basic facts, including the estimated for the quotients $u_2 / u_1$ of positive solutions, which are essential for our approach. Finally, in Section 3, we prove Theorems 1.8 and 1.9.

**Notations.** We use notations $N$ and $c$ for various positive constants depending only on the prescribed constants, such as $n$, $\nu$, etc., which do not depend on smoothness of coefficients $a_{ij}$. These constants may be different in different expression. The expression $A := B \text{ or } B =: A$ means “$A = B$ by definition”.

$B_r(x_0) := \{ x \in \mathbb{R}^n : |x - x_0| < r \}$ is a ball of radius $r > 0$ centered at $x_0 \in \mathbb{R}^n$. $\mathbb{R}^n_+ := \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0 \}$.

**Acknowledgements.** The author thanks N. V. Krylov, N. N. Ural’tseva, and H. F. Weinberger for very useful discussion of results in this paper.

2. AUXILIARY STATEMENTS

In the rest of this paper, $Lu := \sum a_{ij} D_{ij} u$ with $a_{ij} = a_{ji} \in L^\infty$ satisfying the ellipticity condition (1.2) with a constant $\nu \in (0, 1]$. Note that the results in this section are valid for more general operators $L$
in \([1,1]\), which include the lower order terms \(\sum b_i D_i u\) with \(b_i \in L^\infty\). In this case, the constants \(N\) and \(c\) depend also on the upper bounds for \(|b_i|\).

The following theorem was proved by N. V. Krylov and the author \([14, 21]\) (see also \([13]\), Theorem IV.2.8, and \([7]\), Corollary 9.25).

**Theorem 2.1** (Interior Harnack inequality). Let \(\Omega\) be a bounded domain in \(\mathbb{R}^n\), such that the set

\[
\Omega^\delta := \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \delta \}
\]

is connected, where \(\delta = \text{const} > 0\). Then

\[
\sup_{\Omega^\delta} u \leq N \cdot \inf_{\Omega^\delta} u,
\]

with a constant \(N\) depending only on \(n, \nu\), and \(\delta/\text{diam } \Omega\).

**Proof.** In its standard form, the Harnack inequality is formulated for two concentric balls in place of \(\Omega^\delta\) and \(\Omega\), e.g. for \(B_{R/8}^\delta\) and \(B_R\) in \([21]\), Theorem 3.1. In general case, fix \(x, y \in \Omega^\delta\), and choose a sequence \(x \rightarrow y\) in \(\Omega^\delta\) such that \(|x(k-1) - x(k)| < \delta/8\) for \(k = 1, 2, \ldots, m\). One can do it in such a way that \(m\) does not exceed a constant \(m_0\) depending only on \(n\) and \(\delta/\text{diam } \Omega\). Then applying the “standard” Harnack inequality with \(R := \delta\), we get

\[
u(x(k-1)) \leq N_1 \nu(x(k)) \quad \text{for } k = 1, 2, \ldots, m,
\]

where \(N_1 = N_1(n, \nu) \geq 1\). Therefore,

\[
u(x) = \nu(x(0)) \leq N_1 \nu(x(1)) \leq \cdots \leq N_1^m \nu(x(m)) = N_1^m \nu(y),
\]

and the desired estimate \((2.2)\) follows with \(N := N_1^{m_0}\). \(\square\)

The following lemma will help us to reduce the proofs of our main results for operators \(Lu := \sum a_{ij} D_{ij} u\) to the case \(a_{ij} \in C^\infty\). We can assume that \(a_{ij}\) are defined on the whole space \(\mathbb{R}^n\). Consider the convolutions \(a_{ij}^\varepsilon := a_{ij} * \eta^\varepsilon\) with kernels \(\eta^\varepsilon\) such that

\[
0 \leq \eta^\varepsilon \in C^\infty(\mathbb{R}^n), \quad \eta^\varepsilon(x) \equiv 0 \quad \text{for } |x| \geq \varepsilon, \quad \text{and} \quad \int \eta^\varepsilon(x) \, dx = 1.
\]

Then \(a_{ij}^\varepsilon \in C^\infty(\mathbb{R}^n), \ a_{ij}^\varepsilon = a_{ij}^\varepsilon\) satisfy \((1.2)\), and moreover,

\[
a_{ij}^\varepsilon \rightarrow a_{ij} \quad \text{as } \varepsilon \rightarrow 0^+ \quad \text{a.e. in } \Omega.
\]

This convergence follows from the properties of the Lebesgue sets (see \([22]\), Sec. I.1.8).
Lemma 2.2 (Approximation lemma). Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ satisfying an exterior cone condition at each point $x_0 \in \partial \Omega$, i.e. an exterior $Q$-condition in Definition 1.7 with
\begin{equation}
Q := \{ x = (x', x_n) : |x| < r_0, x_n > K \cdot |x'| \}
\end{equation}
with constants $K > 0$ and $r_0 > 0$. Let $u$ be a function in $C^2(\Omega) \cap C(\Omega)$ satisfying $Lu := \sum a_{ij} D_{ij} u \leq 0$ in $\Omega$. For $\varepsilon > 0$, consider the above approximations of $a_{ij}$ by functions $a_{ij}^\varepsilon \in C^\infty$, which satisfy (1.2) and (2.2), and let $u^\varepsilon$ be a unique solution to the problem
\begin{equation}
L^\varepsilon u^\varepsilon := \sum_{i,j} a_{ij}^\varepsilon D_{ij} u^\varepsilon = 0 \text{ in } \Omega, \quad u^\varepsilon = u \text{ on } \partial \Omega,
\end{equation}
in the class $C^\infty(\Omega) \cap C(\Omega)$. Then
\begin{equation}
\sup_{\Omega} (u^\varepsilon - u) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0^+.
\end{equation}
If $Lu = 0$ in $\Omega$, then $u^\varepsilon \rightarrow u$ as $\varepsilon \rightarrow 0^+$ uniformly on $\Omega$.

Note that the existence of a solution $u^\varepsilon \in C^\infty(\Omega) \cap C(\Omega)$ to the problem (2.5) (under an exterior cone condition) follows from the results by K. Miller [18].

Proof. From the arguments in the proof of Theorem 3 in [18] it follows that
\begin{equation}
\sup_{\Omega \cap B_\delta(x_0)} |u^\varepsilon(x) - u(x_0)| \leq \omega(\delta) \rightarrow 0 \text{ as } \delta \rightarrow 0^+,
\end{equation}
uniformly with respect to $x_0 \in \partial \Omega$ and $\varepsilon > 0$. Since $u \in C(\Omega)$, this property also holds true for $u(x)$ in place of $u^\varepsilon(x)$. By the triangle inequality, we get
\begin{equation}
\sup_{\Omega \setminus \Omega^\delta} |u^\varepsilon - u| \leq 2\omega(\delta),
\end{equation}
where $\Omega^\delta$ is defined in (2.1). Moreover, since $L^\varepsilon u^\varepsilon = 0 \geq Lu$, we also have
\begin{equation}
L^\varepsilon(u^\varepsilon - u) \geq f^\varepsilon := (L - L^\varepsilon)u := \sum_{i,j} (a_{ij} - a_{ij}^\varepsilon) D_{ij} u.
\end{equation}
Now we can use the A.D. Aleksandrov type estimate (see [1] or [7], Theorem 9.1):
\begin{equation}
\sup_{\Omega^\delta} (u^\varepsilon - u) \leq \sup_{\partial \Omega^\delta} (u^\varepsilon - u) + N \cdot ||f^\varepsilon||_{L^\infty(\Omega^\delta)}.
\end{equation}
By virtue of (2.7), this yields
\begin{equation}
\sup_{\Omega} (u^\varepsilon - u) \leq 2\omega(\delta) + N \cdot ||f^\varepsilon||_{L^\infty(\Omega^\delta)}.
\end{equation}
Since $D_{ij}u$ are bounded on $\Omega^\delta$, and $a^{ij}_\varepsilon \to a^{ij}$ a.e., the last term converges to 0 as $\varepsilon \to 0^+$. Hence

$$0 \leq \limsup_{\varepsilon \to 0^+} \sup_{\Omega} (u^\varepsilon - u) \leq 2\omega(\delta).$$

The desired property (2.6) follows by sending $\delta$ to 0.

In the case $Lu = 0$, we can apply (2.6) to both functions $u$ and $-u$, which gives the uniform convergence of $u^\varepsilon$ to $u$ on $\Omega$. $\square$

We also need a lower estimate for positive supersolutions in $\Omega$, which are strictly positive on a Lipschitz portion of the boundary $\partial \Omega$. For the proof of this estimate, it is convenient to replace the Lipschitz property of $\partial \Omega$ by a weaker assumption (2.9) below.

**Lemma 2.3.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy $u > 0$, $Lu \leq 0$ in $\Omega$. Suppose that

$$u \geq \mu = \text{const} \quad \text{on} \quad (\partial \Omega) \cap B_{r_0}(x_0),$$

where $x_0 \in \partial \Omega$ and $r_0 > 0$ is a given constant. Moreover, let $\delta > 0$ be a constant such that the set $\Omega^\delta$ in (2.1) is connected, and there are balls

$$B_{\delta}(y_0) \subset \Omega^\delta \cap B_{r_0/2}(x_0) \quad \text{and} \quad B_{\delta}(z_0) \subset \Omega \cap B_{r_0/2}(x_0).$$

Then

$$u \geq c\mu \quad \text{in} \quad \Omega^\delta, \quad \text{where} \quad c = c(n, \nu, \delta/\text{diam } \Omega) > 0.$$

**Proof. Step 1.** From (2.9) it follows that $\delta \leq r_0/4$, and the balls $B_{3\delta}(y_0)$ and $B_{3\delta}(z_0)$ are contained in $B_{r_0}(x_0)$. Therefore, same is true for $B_{3\delta}(y)$, and by (2.8), $u \geq \mu$ on $(\partial \Omega) \cap B_{3\delta}(y)$ for each $y$ in the segment $[y_0, z_0]$.

Next, choose a sequence of points $y_0, y_1, \ldots, y_m = z_0$ in $[y_0, z_0]$, such that $|y_{k+1} - y_k| \leq \delta$ for all $k = 0, 1, \ldots, m-1$. Obviously, we can assume that $m$ does not exceed a constant $m_1$ depending only on $\delta/\text{diam } \Omega$.

We claim that

$$u \geq \theta^k \mu \quad \text{in} \quad \Omega \cap B_{\delta}(y_k) \quad \text{for} \quad k = 0, 1, \ldots, m,$$

with a constant $\theta = \theta(n, \nu) \in (0, 1)$, to be specified later. Here we impose a natural agreement that (2.11) is true automatically if $\Omega \cap B_{\delta}(y_k)$ is empty, which is the case if $k = 0$. In order to use induction, we only need to prove (2.11) with $k + 1$ in place of $k$, based on the assumption that it is true for some $k < m$. For this purpose, we compare the function $u(x)$ with

$$v_k(x) := \theta^k \mu \cdot \frac{|x - y_k|^{-\gamma} - (3\delta)^{-\gamma}}{\delta^{-\gamma} - (3\delta)^{-\gamma}} \quad \text{in} \quad \Omega_k := \Omega \cap (B_{3\delta}(y_k) \setminus B_{\delta}(y_k)),$$
where $\gamma = \gamma(n, \nu) > 0$ is a constant in Lemma 1.4. Of course, we can skip this part if $\Omega_k$ is empty. By this lemma, $Lv_k \geq 0 \geq Lu$ in $\Omega_k$. Moreover, (2.11) implies $u \geq \theta^k \mu = v_k$ on $(\partial \Omega_k) \cap \partial B_\delta(y_k)$. We also have $u \geq 0 = v_k$ on $(\partial \Omega_k) \cap \partial B_{3\delta}(y_k)$, and by (2.8), $u \geq \mu \geq v_k$ on the remaining part of $\partial \Omega_k$. By the comparison principle, $u \geq v_k$ in $\Omega_k$. Together with (2.11), this gives us

$$u \geq \theta^{k+1} \mu \quad \text{in} \quad \Omega \cap B_{2\delta}(y_k), \quad \text{if} \quad \theta := \frac{(3/2)^\gamma - 1}{3^\gamma - 1} \in (0, 1).$$

Finally, $|y_{k+1} - y_k| \leq \delta$ implies that the set $\Omega \cap B_\delta(y_{k+1})$ is contained in $\Omega \cap B_{2\delta}(y_k)$, so that the inequality in (2.11) holds true for $k + 1$. By induction, the proof of (2.11) is complete. In particular, taking $k = m \leq m_1$, we get

(2.12) \hspace{1cm} u \geq c_1 \mu \quad \text{on} \quad B_\delta(z_0), \quad \text{where} \quad c_1 := \theta^{m_1} > 0.

*Step 2.* For an arbitrary point $z \in \Omega^\delta$, and choose a sequence of points $z_0, z_1, \ldots, z_m = z$ in $\Omega^\delta$, such that $|z_{k+1} - z_k| \leq \delta_1 := \delta/3$ for all $k = 0, 1, \ldots, m - 1$. Here we can assume that $m \leq m_2 = m_2(n, \delta / \text{diam } \Omega)$. Similarly to (2.11), with $z_k$ in place of $y_k$ and $\delta_1$ in place of $\delta$, and some simplifications because of the property $B_{3\delta_1}(z_k) = B_\delta(z_k) \subset \Omega$, we obtain

$$u \geq \theta^k c_1 \mu \quad \text{in} \quad B_\delta(z_k) \quad \text{for} \quad k = 0, 1, \ldots, m.$$

In particular, $u(z) = u(z_m) \geq \theta^m c_1 \mu \geq \theta^{m_2} c_1 \mu$. Since $z$ is an arbitrary point in $\Omega^\delta$, the desired estimate (2.10) is proved with $c := \theta^{m_2} c_1 = \theta^{m_1 + m_2}$. \hfill \Box

The following theorem, which is due P. Bauman (see [3], Theorem 2.1), is the main tool in our approach.

**Theorem 2.4 (Comparison theorem).** Let $\varphi$ be a Lipschitz continuous function on $\mathbb{R}^{n-1}$:

$$|\varphi(x') - \varphi(y')| \leq K \cdot |x' - y'| \quad \text{for all} \quad x', y' \in \mathbb{R}^{n-1},$$

with $K = \text{const} \geq 0$, and $\varphi(0) = 0$. For $r > 0$, define

$$\Omega_r := \{x = (x', x_n) \in \mathbb{R}^n : |x'| < r, 0 < x_n - \varphi(x') < r\},$$

and $\Gamma_r := (\partial \Omega_r) \cap \{x_n = \varphi(x')\}$. Let $u, v$ be functions in $C^2(\Omega_{2r}) \cap C(\overline{\Omega_{2r}})$ satisfying

$$u > 0, \quad v > 0, \quad Lu = Lv = 0 \quad \text{in} \quad \Omega_{2r},$$

and $u = v = 0$ on $\Gamma_{2r}$. Then

(2.13) \hspace{1cm} \sup_{\Omega_r} \frac{u}{v} \leq N \cdot \frac{u(0, r)}{v(0, r)}, \quad \text{where} \quad N = N(n, \nu, K) > 0.
Corollary 2.5. Under the assumptions of the previous theorem, we also have

\begin{equation}
(2.14) \quad \frac{u(0, r)}{v(0, r)} \leq N \cdot \inf_{\Omega_r} \frac{u}{v}, \quad \text{where} \quad N = N(n, \nu, K) > 0.
\end{equation}

Proof. Obviously, we can interchange \( u \) and \( v \) in (2.13), and then (2.14) follows from an elementary relation \( \inf(\frac{u}{v}) = \left( \sup(\frac{v}{u}) \right)^{-1} \).

\[ \square \]

Remark 2.6. In [3], this theorem was proved with \( \Omega_{8r}, \Gamma_{8r} \) in place of \( \Omega_r, \Gamma_r \) correspondingly. In order to apply this fact to the proof of (2.13), consider separately each of two possible cases for \( x = (x', x_n) \in \Omega_r \): (i) \( x_n - \varphi(x') < r/8 \) and (ii) \( x_n - \varphi(x') \geq r/8 \). In the case (i), from [3], after obvious change of notations, it follows

\[ \frac{u(x)}{v(x)} \leq N_1(n, \nu, K) \cdot \frac{u(x', \varphi(x') + r/8)}{v(x', \varphi(x') + r/8)}, \]

and then by the Harnack inequality, Theorem 2.1

\[ \frac{u(x', \varphi(x') + r/8)}{v(x', \varphi(x') + r/8)} \leq N_2(n, \nu, K) \cdot \frac{u(0, r)}{v(0, r)}, \]

so that \( u/v(x) \leq N \cdot u/v(0, r) \) with \( N := N_1N_2 \). In the case (ii), we get this estimate with \( N := N_2 \) by the Harnack inequality directly. Therefore, (2.13) holds true.

The above argument also shows that in the formulation of Theorem 2.4 one can replace \( 2r \) by \( cr \) with any absolute constant \( c > 1 \). We will use this observation with \( c = 3/2 \) in order to get the estimate (2.15) below.

Corollary 2.7. The estimate (2.13) in Theorem 2.4 remains valid if the condition \( v = 0 \) on \( \Gamma_{2r} \) is omitted.

Proof. Having in mind the approximation lemma (Lemma 2.3), we can assume that \( a_{ij} \) are smooth. Take a continuous function \( g \) on \( \partial\Omega_{3r/2} \) such that \( 0 \leq g \leq v \) on \( \partial\Omega_{3r/2} \) and \( g \equiv 0 \) on \( \Gamma_{3r/2} \) and \( g \equiv v \) on

\[ \Gamma^*_{3r/2} := \{ x = (x', x_n) \in \mathbb{R}^n : |x'| \leq 3r/2, x_n - \varphi(x') = 3r/2 \}. \]

Since \( a_{ij} \) are smooth, there exists a solution \( v_0 \in C^2(\Omega_{3r/2}) \cap (\overline{\Omega_{3r/2}}) \) of the problem

\[ Lv_0 = 0 \quad \text{in} \quad \Omega_{3r/2}, \quad v_0 = g \quad \text{on} \quad \partial\Omega_{3r/2}. \]

By Theorem 2.4 applied to the functions \( u \) and \( v_0 \) in \( \Omega_{3r/2} \),

\begin{equation}
(2.15) \quad \sup_{\Omega_r} \frac{u}{v_0} \leq N \cdot \frac{u(0, r)}{v_0(0, r)}.
\end{equation}
Moreover, by the comparison principle, $0 \leq v_0 \leq v$ in $\Omega_{3r/2} \supset \Omega_r$, hence we can replace $v_0$ by $v$ in the left side. In the right side, we first apply Lemma 2.3 to the function $v_0$ in $\Omega_{3r/2}$ with $r_0 := 3r/2$ and $x_0 := (0,3r/2) \in \Gamma_{3r/2}$, and then the Harnack inequality to the function $v$ in $\Omega_{2r}$. As a result, we get

$$v_0(0,r) \geq c_1\mu, \quad \text{where} \quad \mu := \inf_{\Gamma_{3r/2}} v_0 = \inf_{\Gamma_{3r/2}} v \geq c_2 \cdot v(0,r),$$

with positive constants $c_1$ and $c_2$ depending only on $n$, $\nu$ and $K$. Therefore, from (2.15) it follows the desired estimate (2.13). \hfill \Box

3. Proof of Theorems 1.8 and 1.9

First of all, we write the integral condition $I(\psi) < \infty$ in (1.6) in an equivalent “discrete” form.

**Lemma 3.1.** Let $\psi(r)$ be a non-negative, non-decreasing function on $[0,r_0]$, where $r_0 = \text{const} > 0$. Then $I(\psi) < \infty$ if and only if

$$\sum_{k=0}^{\infty} \frac{h_k}{r_k} < \infty \quad \text{where} \quad r_k := 4^{-k}r_0, \quad h_k := \psi(r_k). \quad (3.1)$$

**Proof.** Since $h_{k+1} \leq \psi(r) \leq h_k$ on $[r_{k+1}, r_k]$, and $r_k - r_{k+1} = 3r_{k+1} = 3r_k/4$, we obtain

$$I(\psi) = \sum_{k=0}^{\infty} \int_{r_{k+1}}^{r_k} \frac{\psi(r)}{r^2} \, dr \geq \sum_{k=0}^{\infty} \frac{3r_{k+1} h_{k+1}}{r_k^2} = \frac{3}{16} \sum_{k=0}^{\infty} \frac{h_{k+1}}{r_{k+1}}.$$

On the other hand, $I(\psi) \leq \sum 3r_{k+1} h_k / r_{k+1}^2 = 12 \sum h_k / r_k$. Therefore, $I(\psi) < \infty$ if and only if $\sum h_k / r_k < \infty$. \hfill \Box

**Corollary 3.2.** If $I(\psi) < \infty$, then for arbitrary constant $K_0 > 0$, there is a constant $0 < R_0 \leq \min(r_0, h_0)$ such that the set

$$V_0 := \{ x = (x', x_n) : |x| < R_0, x_n > K_0 |x'| \}$$

is contained in $Q$.

**Proof.** From $I(\psi) < \infty$ it follows $\sum h_k / r_k < \infty$, hence $h_k / r_{k+1} = 4h_k / r_k \to 0$ as $k \to \infty$. Choose an integer $k_0 \geq 1$ such that $h_k / r_{k+1} \leq K_0$ for all $k \geq k_0$, and set $R_0 := \min(r_{k_0}, h_{k_0})$. We claim that that each $x = (x', x_n) \in V_0$ belongs to $Q$. This is obvious if $x' = 0$, so we can assume $x' \neq 0$. Then there is an integer $k \geq k_0$ (depending on $x$) such that $r_{k+1} \leq |x'| < r_k \leq R_0$. This implies

$$\psi(|x'|) \leq \psi(r_k) =: h_k \leq K_0 r_{k+1} \leq K_0 |x'| < x_n,$$

which means $x \in Q$. \hfill \Box
The next lemma can be considered as a very special case of Theorem 1.8. However, this “model” case contains the main difficulties, so that Theorem 1.8 in full generality follows easily by the comparison principle.

**Lemma 3.3.** Let \( Q \) be a set defined in (1.5), where \( r_0 = \text{const} > 0 \), and \( \psi(r) \) is a non-negative, non-decreasing function on \([0, r_0]\), satisfying the condition \( I(\psi) < \infty \) in (1.6). Let \( v \) be a function in \( C^2(Q) \cap C(\overline{Q}) \), such that

\[
v > 0, \quad Lv := \sum_{i,j=1}^{n} a_{ij} D_{ij}v = 0 \quad \text{in} \quad Q,
\]

and \( v = 0 \) on \( \Gamma := (\partial Q) \cap \{ x_n \in \psi(|x'|) \} \). Then

\[
\inf_{0 < x_n \leq r_0/2} \frac{v(0, x_n)}{x_n} > 0.
\]

Note that the non-decreasing function \( \psi(r) \) may be discontinuous. In order to guarantee that the set \( \Gamma \) is connected, we define \( \psi(r) := [\psi(r-), \psi(r+)] \) - the segment whose ends are one-sided limits of \( \psi(r') \) as \( r' \to r \), subject to restriction \( r' < r \) or \( r' > r \). Obviously, if \( \psi \) is continuous at some point \( r \), then this segment is reduced to the corresponding point \( \psi(r) \).

**Proof.** We assume that the coefficients \( a_{ij} \) are smooth functions on \( \mathbb{R}^n \). The general case follows from the approximation lemma (Lemma 2.2), because all the estimates in the proof do not depend on this smoothness. Using notations in Lemma 3.1, denote \( \theta_k := h_k/r_k \). By this lemma, we have \( \sum \theta_k < \infty \). We can start our considerations with large enough \( k \geq 1 \). Therefore, without loss of generality, we assume that \( 0 \leq \theta_k \leq \varepsilon_0 < 1 \) for all \( k \geq 1 \), where \( \varepsilon_0 = \varepsilon_0(n, \nu) \) is a small constant in \((0, 1)\), which will be specified later.

For integers \( k \geq 1 \), denote \( Q_k := Q \cup C_{r_k} \), where

\[
C_r := \{ x = (x', x_n) \in \mathbb{R}^n : |x'| < r, \ 0 < x_n < r \}.
\]

We will approximate the given function \( v \) by solutions \( v_k \in C^2(Q_k) \cap C(\overline{Q_k}) \) of the Dirichlet problem

\[
Lv_k = 0 \quad \text{in} \quad Q_k, \quad v_k = g_k \quad \text{on} \quad \partial Q_k,
\]

where \( g_k \) is a continuous function on \( \partial Q_k \), defined as \( g_k \equiv v \) on \( (\partial Q_k) \cap (\partial Q) \), and \( g_k \equiv 0 \) on the remaining part of \( \partial Q_k \). Note that \( Q_k \) are Lipschitz domains, hence the existence of such solutions for equations with smooth coefficients is known. It is easy to see that \( Q_k \searrow Q \), and by the comparison principle \( v_k \searrow v \) in \( Q \) as \( k \to \infty \).
The following estimate is an important step in our proof:

\[(3.3) \quad \sup_{C_{r_k}} \frac{v_k}{x_n} \leq N \cdot \frac{v_k(0, r_k)}{r_k}, \text{ where } N = N(n, \nu) \geq 1.\]

Here both functions \(v_k\) and \(x_n\) are positive and satisfy the equation \(Lv = 0\) in the domain \(\Omega_{2r} := Q_r \cap C_{2r}\), and \(v_k = 0\) on the set \(\Gamma_{2r} := (\partial Q_r) \cap (\partial \Omega_{2r})\) with \(r = r_k\). However, we cannot apply Corollary 2.7 directly, because \(\Gamma_{2r}\) is not represented as the graph of a Lipschitz function. In order to fix this gap, note that \(\Gamma_{2r}\) is a surface of rotation, and the function \(\psi(r)\) is non-decreasing. Therefore, \(\Gamma_{2r}\) is still the graph of a Lipschitz function locally with an absolute constant \(K\) in a neighborhood of each of its point \(x_0\), in a rotated coordinate system centered at \(x_0\). This allows us to estimate the ratio \(v_k/x_n\) near \(x_0\) by the same ratio at a point strictly inside of \(\Omega_{2r}\), an then use the Harnack inequality in order to get (3.3) with a constant \(N = N(n, \nu) \geq 1\). This argument is similar to that in Remark 2.6. In the rest of the proof, \(N\) denotes different positive constants depending only on \(n\) and \(\nu\).

Next, note that \(0 \leq x_n \leq h_k := \psi(r_k)\) on the set \((\partial Q) \cap C_k\), hence by (3.3), \(0 \leq v \leq N\theta_k v_k(0, r_k)\) on this set. We also have \(v = v_k\) on the rest of \(\partial Q\). By the comparison principle, this yields

\[(3.4) \quad 0 \leq v_k - v_{k+1} \leq v_k - v \leq N\theta_k v_k(0, r_k) \quad \text{in} \quad Q.\]

Combining the Harnack inequality with Corollary 2.5 we get

\[(3.5) \quad 0 < \frac{v_k(0, r_k)}{r_k} \leq \frac{N v_k(0, r_{k+1})}{r_{k+1}} \leq N \mu_k, \quad \text{where} \quad \mu_k := \inf_{C_{r_{k+1}}} \frac{v_k}{x_n}.\]

Further, from an elementary inequality \(\inf A_k - \inf B_k \leq \sup(A_k - B_k)\) and \(C_{r_{k+2}} \subset C_{r_{k+1}}\) it follows

\[
\mu_k - \mu_{k+1} \leq \inf_{C_{r_{k+2}}} \frac{v_k}{x_n} - \inf_{C_{r_{k+2}}} \frac{v_{k+1}}{x_n} \leq \sup_{C_{r_{k+2}}} \frac{v_k - v_{k+1}}{x_n}.
\]

Here the right side can be estimated by Theorem 2.4 In combination with (3.4) and (3.5), this gives us

\[
\mu_k - \mu_{k+1} \leq \frac{N(v_k - v_{k+1})(0, r_{k+2})}{r_{k+2}} \leq \frac{N\theta_k v_k(0, r_k)}{r_k} \leq N\theta_k \mu_k.
\]

As we noticed in the beginning of the proof, we can assume that \(\theta_k := h_k/r_k \leq \varepsilon_0\) for all \(k\), with a convenient choice of the constant \(\varepsilon_0 = \varepsilon_0(n, \nu) \in (0, 1)\). Choose \(\varepsilon_0\) such that in the previous expression, \(\alpha_k := N\theta_k \leq N\varepsilon_0 \leq 1/2\) for all \(k\). By iteration, we obtain

\[
\mu_{k+1} \geq (1 - \alpha_k) \mu_k \geq (1 - \alpha_k)(1 - \alpha_{k-1}) \cdots (1 - \alpha_1) \mu_1.
\]
Finally, we use the fact that convergence of the series $\sum \alpha_j = N \sum \theta_j$ is equivalent to convergence of the product $\prod (1 - \alpha_j)$. More specifically, from convexity of the function $f(\alpha) := -\ln(1 - \alpha)$ it follows that its values lie between $\alpha$ and $2\ln 2 \cdot \alpha$ for all $\alpha \in [0, 1/2]$. Hence

$$-\ln \mu_{k+1} \leq -\ln \mu_1 - \sum_{j=1}^{k} \ln (1 - \alpha_j) \leq -\ln \mu_1 + 2\ln 2 \sum_{j=1}^{\infty} \alpha_j < \infty$$

for all $k$. Then $v_k(0, r_{k+1})/r_{k+1} \geq \mu_k \geq \text{const} > 0$ for all $k$, and by the Harnack inequality, same is true for the sequence $v_k(0, r_k)/r_k$. We can also assume that $N\theta_k \leq 1/2$ in (3.4), hence $v(0, r_k)/r_k \geq v_k(0, r_k)/2r_k \geq \text{const} > 0$ for all $k$.

Now we see that the ratio $v(0, x_n)/x_n$ is separated from 0 for $x_n = r_k := 4^{-k}, k \geq 1$. By the Harnack inequality, this is also true for $r_{k+1} \leq x_n \leq r_k$, and (3.2) follows. \qed

**Proof of Theorem 1.8** As in the preceding proof, we can assume that $a_{ij}$ are smooth. Replacing $r_0 > 0$ in (1.5) by a smaller number if necessary, we can also assume that $u$ is not identically 0 on $\partial Q$. Choose an arbitrary function $g \in C(\partial Q)$, such that $0 \leq g \leq u$ on $\partial Q$, $g \equiv 0$ on $\Gamma := (\partial Q) \cap \{x_n = \psi(|x'|)\}$, and $g$ is not identically 0. Then define $v \in C^2(Q) \cap C(\overline{Q})$ as a solution of the equation $Lu = 0$ in $Q$ with the boundary data $v = g$ on $\partial Q$. This function $v$ automatically satisfies all the assumptions of Lemma 3.3, and moreover, by the comparison principle, $u \geq v > 0$ in $Q$. Therefore, for the proof of (1.7), it suffices to establish a similar property for the function $v$.

Fix an arbitrary vector $l = (l', l_n) \in \mathbb{R}^n$, choose a constant $K_1 > 0$ such that $l_n > K_1|l'|$, and another constant $K_0 \in (0, K_1)$. Finally, take a constant $R_0 \in (0, r_0)$ according to Corollary 3.2. This guarantees that $Q$ contains the set $V_0 := \{x | < R_0, x_n > K_0|x'|\}$. In turn, by our construction $V_0$ contains the set $V_1 := \{x | < R_0/2, x_n > K_1|x'|\}$, and $tl \in V_1$ for all $t$ in an interval $(0, t_0)$. By the Harnack inequality, $v(0, tl_n) \leq Nv(tl)$ for all $t \in (0, t_0)$. Now the desired estimate follows from (3.2) with $x_n = tl_n$. \qed

In the rest of the paper, we skip some details of proofs which are similar to those in the proofs of Lemma 3.3 and Theorem 1.8. In particular, we assume that $a_{ij}$ are smooth, so that the Dirichlet problem $Lu := \sum a_{ij}D_{ij}u = 0$ in $\Omega$ with the boundary condition $u = g$ on $\partial \Omega$ has a classical solution for any bounded Lipschitz domain $\Omega$ and any function $g \in C(\partial \Omega)$. The following lemma covers a “model” case for the proof of Theorem 1.9.
Lemma 3.4. Let $\psi(r)$ be a non-negative, non-decreasing function on $[0, r_0]$, with $I(\psi) < \infty$. Define

$$Q^* := \{ |x'| < r_0, -\psi(|x'|) < x_n < r_0 \},$$
$$\Gamma^* := (\partial Q^*) \cap \{ -x_n \in \psi(|x'|) \}.$$  

(3.6)

Let $w$ be a function in $C^2(Q^*) \cap C(\overline{Q^*})$, such that

$$w > 0, \quad Lw = 0 \text{ in } Q^*; \quad w = 0 \text{ on } \Gamma^*.$$  

Then the ratio $w(x)/|x|$ is bounded on $Q^*$. As in Lemma 3.3, we assume $\psi(r) = [\psi(r-), \psi(r+)]$ for $0 < r < r_0$.

Proof. We approximate $Q^*$ by a sequence of domains $Q^*_k$, $k \geq 1$, with flat boundaries in the $r_k$-neighborhood of the origin. Namely, set

$$Q^*_k := \{ x = (x', x_n) : |x'| < r_0, -\psi_k(|x'|) < x_n < r_0 \},$$

where $\psi_k(r) \equiv 0$ on $[0, r_k]$, and $\psi_k(r) \equiv \psi(r)$ on $(r_k, r_0]$. Correspondingly, the given function $w$ will be approximated by solutions $w_k \in C^2(Q^*_k) \cap C(\overline{Q^*_k})$ of the Dirichlet problems

$$Lw_k = 0 \text{ in } Q^*_k, \quad w_k = g_k \text{ on } \partial Q^*_k,$$

where the functions $g_k \in C(\partial Q^*_k)$ are defined by the equalities $g_k \equiv w$ on $(\partial Q^*_k) \cap (\partial Q^*)$, and $g_k \equiv 0$ on $(\partial Q^*_k) \setminus (\partial Q^*)$. We have $Q^*_k \nearrow Q^*$, and by the comparison principle $w_k \nearrow w$ in $Q^*$ as $k \to \infty$, if we formally extend $w_k \equiv 0$ on $Q^* \setminus Q^*_k$. As in the proof of Lemma 3.3, we can assume that $\theta_k := h_k/r_k \leq \varepsilon_0 = \varepsilon_0(n, \nu)$ - a small constant in $(0, 1)$.

We can apply Corollary 2.7 to the functions

$$u := w, \quad v := x_n + h_{k-1} \quad \text{in} \quad D_k := \{ |x'| < r_k, -\psi(|x'|) < x_n < r_k \}$$

in the same way as we did it in the proof of (3.3). These functions are positive, satisfy $Lu = Lv = 0$ in a larger domain $D_{k-1}$, and $u := w = 0$ on its “bottom” $(\partial D_{k-1}) \cap \{ -x_n \in \psi(|x'|) \}$. Therefore,

$$\sup_{D_k} \frac{w}{x_n + h_{k-1}} \leq \frac{Nw(0, r_k)}{r_k},$$

From this estimate it follows

$$0 = w_k \leq w \leq N\theta_{k-1}w(0, r_k) \quad \text{on} \quad (\partial Q^*_k) \cap \overline{D_k}.$$  

On the rest of $\partial Q^*_k$, we have $w_k = w$. By the comparison principle,

$$0 \leq w_{k+1} - w_k \leq w - w_k \leq N\theta_{k-1}w(0, r_k) \quad \text{in} \quad Q^*_k.$$  

(3.7)

In particular, assuming $N\theta_{k-1} \leq N\varepsilon_0 \leq 1/2$, we get $w(0, r_k) \leq 2w_k(0, r_k)$.
Further, we apply Corollary 2.7 once again, with \( v \equiv 1 \), and then use the Harnack inequality. This implies

\[
\sup_{D_k} w \leq N w(0, r_k) \leq N r_{k+1} M_k, \text{ where } M_k := \sup_{C_{k+1} x_n} \frac{w_k}{x_n}
\]

Using inequality \( \sup A_k - \sup B_k \leq \sup (A_k - B_k) \) and Theorem 2.4 with \( u := w_{k+1} - w_k, v := x_n \) in \( C_{r_{k+2}} \subset C_{r_{k+1}} \), we obtain

\[
M_{k+1} - M_k \leq \sup_{C_{r_{k+2}}} \frac{w_{k+1} - w_k}{x_n} \leq \frac{N (w_{k+1} - w_k)(0, r_{k+2})}{r_{k+2}}.
\]

Together with (3.7) and (3.8), this implies

\[
M_{k+1} - M_k \leq N \theta_{k-1} w(0, r_k)/r_{k+2} \leq N \theta_{k-1} M_k,
\]

so that \( M_{k+1} \leq (1 + N \theta_{k-1}) M_k \). Iterating this estimate and using the fact that from convergence of the series \( \sum \theta_k \) it follows convergence of the product \( \prod (1 + N \theta_{k-1}) \), we get the estimate \( M_k \leq NM_1 \) for all \( k \geq 1 \). Finally, in order to prove the boundedness of \( w(x)/|x| \), it suffices to show that its supremum over the set \( Q^* \cap \{ r_{k+1} < |x| \leq r_k \} \), which is a subset of \( D_k \), does not exceed a constant uniformly for all \( k \). This is an immediate consequence of (3.8): for each \( x \) in this set,

\[
\frac{w(x)}{|x|} \leq \frac{1}{r_{k+1}} \cdot \sup_{D_k} w \leq N M_k \leq NM_1 < \infty.
\]

Lemma is proved.

**Proof of Theorem 1.9.** From our assumptions it follows that the set is a subset of \( Q^* \) defined in (3.6). Replacing \( r_0 > 0 \) by a smaller number if necessary, we can assume that \( u = 0 \) on \( (\partial \Omega) \cap Q^* \). Then the function \( g \) on \( \partial Q^* \) defined by the equalities \( g \equiv u \) on \( (\partial Q^*) \cap \Omega \), and \( g \equiv 0 \) on \( (\partial Q^*) \setminus \Omega \), belongs to \( C(\partial Q^*) \). Assuming that \( a_{ij} \) are smooth, we can define \( w \in C^2(Q^*) \cap C(Q^*) \) as a solution to the equation \( Lu = 0 \) in \( Q^* \) with the boundary condition \( w = g \) on \( \partial Q^* \). By the comparison principle, \( 0 < u \leq w \) in \( Q^* \cap \Omega \). Therefore, \( u(x)/|x| \) is bounded in \( \Omega \cap B_{r_0}(0) \) by Lemma 3.4.

**References**

[1] A.D. Aleksandrov, *Majorization of solutions of second-order elliptic equations*, Vestnik Leningrad Univ. 21, no. 1 (1966), 5–25 (in Russian). English transl. in Amer. Math. Soc. Transl. (2) 68 (1968), 120–143.

[2] P.E. Bauman, *Properties of nonnegative solutions of second-order elliptic equations and their adjoints*, Ph. D. Thesis, University of Minnesota, 1982.

[3] P.E. Bauman, *Positive solutions of elliptic equations in non-divergence form and their adjoints*, Arkiv för Mathematik, 22 (1984), 153–173.
[4] L.A. Caffarelli, E.B. Fabes, S. Mortola and S. Salsa, *Boundary behavior of nonnegative solutions of elliptic operators in divergence form*, Indiana J. of Math., 30 (1981), 621–640.

[5] R. Courant and D. Hilbert, *Methods of Mathematical Physics. Volume II*, Interscience, New York, 1983.

[6] E.B. Fabes, M.V. Safonov and Yu Yuan, *Behavior near the boundary of positive solution of second order parabolic equations. II*, Trans. Amer. Math. Soc. 351, no. 12 (1999), 4947–4961.

[7] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, 1983.

[8] G. Giraud, *Problèmes de valeurs à la frontière relatifs à certaines données discontinues*, Bull. de la Soc. Math. de France, 61 (1933), 1–54.

[9] B.N. Himchenko, *On the behavior of solutions of elliptic equations near the boundary of a domain of type $A^{(1)}_1$*, Dokl. Akad. Nauk SSSR 193 (1970), 304–305 (in Russian). English transl. in Soviet Math. Dokl. 11 (1970), 943–944.

[10] E. Hopf, *A remark on linear elliptic differential equations of second order*, Proc. Amer. Math. Soc., 3 (1952), 791–793.

[11] I.I. Kaminin and B.N. Khimchenko, *Development of Aleksandrov’s theory of the isotropic extremum principle*, Differents. Uravn. 16 (1980), 280–292 (in Russian). English transl. in Differential Equations 16 (1980), 181–189.

[12] B.N. Khimchenko, *The behavior of the superharmonic function near the boundary of a domain of type $A^{(1)}_1$*, Differents. Uravn. 5 (1969), 1845–1853 (in Russian). English transl. in Differential Equations 5 (1969), 1371–1377.

[13] N.V. Krylov, *Nonlinear Elliptic and Parabolic Equations of Second Order*, Nauka, Moscow, 1985 (in Russian). English transl.: Reidel, Dordrecht, 1987.

[14] N.V. Krylov and M.V. Safonov, *A certain property of solutions of parabolic equations with measurable coefficients*, Izvestia Akad. Nauk SSSR, ser. Matem., 44 (1980), 102–125 (in Russian). English transl. in Math. USSR Izvestija, 16 (1981), 109–135.

[15] O.A. Ladyzhenskaya and N.N. Ural’tseva, *Estimates on the boundary of a domain for the first derivatives of functions satisfying an elliptic or parabolic inequality*, Trudy Mat. Inst. Steklov 179 (1988), 102–125 (in Russian). English transl. in Proc. Steklov Inst. Math. 179 (1989), 109–135.

[16] E.M. Landis, *Second Order Equations of Elliptic and Parabolic Type*, Nauka, Moscow, 1971 (in Russian). English transl. in Transl. Matem. Monographs 171, Amer. Math. Soc., Providence, RI, 1998.

[17] G.M. Lieberman, *Regularized distance and its applications*, Pacific J. Math. 117 (1985), 329–352.

[18] K. Miller, *Barriers on cones for uniformly elliptic operators*, Ann. Mat. Pura Appl. (4) 76 (1967), 93–105.

[19] O.A. Oleinik, *On properties of solutions of certain boundary problems for equations of elliptic type*, Mat. Sb. (N. S.) 30 (1952), 695–702 (in Russian).

[20] M.H. Protter and H.F. Weinberger, *Maximum Principles in Differential Equations*, Englewood Cliffs, N.J., Prentice-Hall, 1967.

[21] M.V. Safonov, *Harnack inequality for elliptic equations and the Hölder property of their solutions*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI), 96 (1980), 272–287 (in Russian). English transl. in J. Soviet Math., 96 (1983), 851–863.
[22] E. Stein, *Singular Integrals and Differentiability Properties of Functions*. Princeton, Princeton University Press, 1970.

[23] M.S. Zaremba, *Sur un problème mixte relatif à l’équation de Laplace*, Bull. Intern. de l’ Acad. Sci. de Cracovie. Série A, Sci. Math. (1910), 313–344.

School of Mathematics, University of Minnesota

E-mail address: safonov@math.umn.edu