Functional Keldysh Theory of Spin Torques

R.A. Duine,1 A. S. Núñez,2 Jairo Sinova,3 and A.H. MacDonald4

1Institute for Theoretical Physics, Utrecht University, Leuvenlaan 4, 3584 CE Utrecht, The Netherlands
2 Instituto de Física, PUCV Av. Brasil 2950, Valparaíso, Chile
3Department of Physics, Texas A&M University, College Station, TX 77843-4242, USA
4The University of Texas at Austin, Department of Physics, 1 University Station C1600, Austin, TX 78712-0264, USA

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We present a microscopic treatment of current-induced torques and thermal fluctuations in itinerant ferromagnets based on a functional formulation of the Keldysh formalism. We find that the nonequilibrium magnetization dynamics is governed by a stochastic Landau-Lifschitz-Gilbert equation with spin transfer torques. We calculate the Gilbert damping parameter α and the non-adiabatic spin transfer torque parameter β for a model ferromagnet. We find that β ≠ α, in agreement with the results obtained using imaginary-time methods of Kohno, Tatara and Shibata [J. Phys. Soc. Japan 75, 113706 (2006)]. We comment on the relationship between s – d and isotropic-Stoner toy models of ferromagnetism and more realistic density-functional-theory models, and on the implications of these relationships for predictions of the β/α ratio which plays a central role in domain wall motion. Only for a single-parabolic-band isotropic-Stoner model with an exchange splitting that is small compared to the Fermi energy does β/α approach one. In addition, our microscopic formalism incorporates naturally the fluctuations needed in a nonzero-temperature description of the magnetization. We find that to first order in the applied electric field, the usual form of thermal fluctuations via a phenomenological stochastic magnetic field holds.

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I. INTRODUCTION

Phenomena related to order-parameter manipulation by transport currents have recently received a great deal of attention in magnetic metals and magnetic semiconductors. Spin transfer torques, which lead to current-driven nanomagnet reversal and to domain wall motion in narrow wires, have been at the center of this activity.

In spin-transfer-torque theory, the usual Landau-Lifschitz-Gilbert (LLG) equation of motion for the magnetization direction \( \hat{\Omega} \) acquires terms corresponding to so-called adiabatic and non-adiabatic spin transfer torques which are both proportional to current. Both torques can be constructed from symmetry arguments by requiring that they be orthogonal to the magnetization direction \( \hat{\Omega} \), and by realizing that the current essentially causes the nonzero current implies that the system is out of equilibrium. Arguing that both \( \alpha \) and \( \beta \) correspond to dissipative processes and the fluctuation-dissipation theorem need not be valid because the nonzero current implies that the system is out of equilibrium.

As noted in the literature on current-driven domain wall motion, the case of \( \beta = \alpha \) is special because both sides of Eq. (1) then contain the “co-moving” derivative \( D/Dt = \partial/\partial t + \mathbf{v}_s \cdot \nabla \), so that the equation of motion admits solutions \( \Omega_0(t) = \Omega_0(\mathbf{x} - \mathbf{v}_s t) \) where \( \Omega_0(\mathbf{x}) \) is a time-independent solution of the LLG equation in the absence of currents. The solution \( \Omega_0(t) \) corresponds to “drift” of static magnetization textures with velocity \( \mathbf{v}_s \). Arguing that these solutions must exist Barnes and Maekawa claim that \( \beta = \alpha \). However, in realistic systems there
is no Galilean invariance that requires the existence of such solutions and therefore in general $\beta \neq \alpha$. Instead, the non-adiabatic spin transfer torque acquires contributions from all microscopic processes that violate spin conservation and therefore correspond to terms in the microscopic Hamiltonian that are not invariant under spin rotations. Such processes also contribute to the Gilbert damping term and therefore in principle any nonzero Gilbert damping parameter $\alpha$ implies nonzero non-adiabatic spin transfer torques, as we show in our specific microscopic model calculations.

The LLG equation [Eq. (1)] is motivated mainly by symmetry considerations and contains four different quantities whose meaning can be specified precisely only by a microscopic theory which details, at least in principle, precisely how they should be evaluated given the full system Hamiltonian. These quantities are: $i)$ the effective magnetic field $H$, $ii)$ the transport spin velocity $v_s$, $iii)$ the Gilbert damping parameter $\alpha$, and $iv)$ the non-adiabatic spin-transfer torque parameter $\beta$. The effective magnetic field $H$ includes the external magnetic field and additional contributions due to magnetostatic interactions and magnetocrystalline anisotropy. The physics of $H$ is well understood and not the subject of this paper. The three remaining quantities emerge in a microscopic theory from the slow (up to first order in time derivatives or frequency $\omega$) smooth (up to first order in space derivatives or wavevector $\mathbf{q}$) response of the magnetization direction to an external magnetic field, in the presence of an external electric field which drives a transport current. The coefficient $\alpha$ then emerges as the ratio of the reactive and dissipative contributions that appear at first order in $\omega$ in this response function. When spin-orbit interactions are neglected, it is easy to verify that the coefficient of the reactive term in the total spin-response is the unperturbed spin-density, explaining the unit value of this coefficient in Eq. (1). The two first order space derivative terms in this equation reflect respectively the change in the reactive and dissipative response due to an external electric field. Like its zero-current counterpart, the current-related reactive terms can be understood in quite general terms based only on spin-conservation considerations, while the dissipative term is sensitive to microscopic details.

As explained above the condition $\beta = \alpha$ corresponds to Galilean invariance at a macroscopic level. Since the dissipative terms emerge from spin-dependent disorder (or spin-independent disorder when spin-orbit interactions are included in the crystal band structure), it is clear that Galilean invariance does not hold microscopically. Our calculations show that $\beta \approx \alpha$ can occur in models with very specific properties, but does not occur in general. For example $\beta \approx \alpha$ occurs in the specific toy model that we study below only when the ferromagnetism is weak in the sense that the exchange splitting is much smaller than the Fermi energy. We believe that we obtain this result only because the model has isotropic parabolic bands and that $\beta = \alpha$ (macroscopic Galilean invariance) can occur only accidentally in systems with either realistic bands or realistic disorder. In the important transition metal ferromagnet spintronic materials in particular, we will argue that orbitals which have dominant $d$-character contribute more strongly to the magnetization than to transport and that $\beta$ will tend to be larger than $\alpha$ as a consequence.

In this paper we present a microscopic derivation of the equation of motion in Eq. (1) of the direction of magnetization in the presence of current and at finite temperature. We use a functional formulation of the Keldysh non-equilibrium formalism which leads in a natural way to the path-integral formulation of stochastic differential equations. Within our microscopic treatment the dissipative nature of $\alpha$ and $\beta$ is explicit because, as briefly discussed above and to be shown in more detail, they follow from the dissipative part of the spin-density spin-density response function and photon two-magnon interaction vertex, respectively. We focus on the simple microscopic toy model used in previous work that is intended to provide a qualitative description of a generic ferromagnet and includes disorder and short-range repulsive electron-electron interactions. The model’s ferromagnetism is treated at the level of Stoner mean-field theory. For the disorder we use the same model as in Ref. and, where applicable, our results for $\alpha$ and $\beta$ agree with theirs. Our random phase approximation treatment of Stoner quasiparticle fluctuations evinces the equivalence of Stoner and $s-d$ models, in the sense that in both models the quantities $\alpha$ and $\beta$ are determined by the same response function. In particular $\alpha \neq \beta$ for both models in general.

One benefit of the concinnity of the functional formulation of the Keldysh formalism is that it enables a natural determination of the thermal fluctuations without explicitly appealing to the fluctuation-dissipation theorem. We find that to lowest order in the applied electric field the form usually assumed for the strength of the fluctuations holds and that there is no contribution to the white-noise thermal fluctuations that is related to the non-adiabatic torque. We emphasize that this formalism, which has been reviewed in other publications and applied to other problems, is similar in structure to the functional formulation of standard equilibrium Green’s functions for linear response theory, but is more powerful for non-equilibrium and non-linear problems.

Since the formalism we use may not be familiar to most readers, we first present the model and main results in a separate section, namely Sec. In Sec. we present the formalism and outline the calculations. In the appendix we carry out a typical calculation in more detail. Both Sec. and the appendix may be skipped by readers who are familiar with the formalism or who may be more interested in the results obtained. We end in Sec. with our conclusions.
II. MODEL AND SUMMARY OF RESULTS

We model the disordered itinerant ferromagnet as electrons with delta-function-like repulsive interactions, using the Hamiltonian,

\[ H[\hat{\psi}, \hat{\psi}] = \int d^3x \left\{ \frac{\hbar^2 \nabla^2}{2m} - \frac{\Delta_{\text{ext}}}{2} \tau_z + V_0(x) \right. \]

\[ + V_a(x) \tau_a \hat{\psi}(x,t) + \frac{1}{c} \mathbf{J}(x,t) \cdot \mathbf{A}(x,t) \]

\[ + U \hat{\psi}^\dagger_a(x,t) \hat{\psi}^\dagger_b(x,t) \hat{\psi}_b(x,t) \hat{\psi}_a(x,t) \} , \quad (2) \]

where for notational convenience we have introduced the spinor

\[ \hat{\psi}(x,t) = \left( \begin{array}{c} \hat{\psi}_\uparrow(x,t) \\ \hat{\psi}_\downarrow(x,t) \end{array} \right) . \quad (3) \]

In these expressions the Heisenberg operators \( \hat{\psi}_\sigma(x,t) \) annihilate an electron in the spin state labelled by \( \sigma \in \{ \uparrow, \downarrow \} \), and obey the usual equal-time commutation relations. These spin states have their quantization axis parallel to an external Zeeman magnetic field in the \( z \)-direction which contributes \( \Delta_{\text{ext}} \) to the energy difference between minority and majority spins. Note that in Eq. (2) the Pauli matrices are indicated by \( \tau_a \), and that a sum over the repeated index \( a \in \{ x, y, z \} \) is implied. The free-electron dispersion at momentum \( \hbar \mathbf{k} \), given by \( \epsilon_k = \hbar^2 k^2/2m \), is parabolic with an effective mass \( m \) (\( \hbar \) denotes Planck’s constant).

We choose a delta-function interaction with strength \( U \) because then the field-theoretic procedure to introduce the magnetization direction as a dynamic variable is easier to implement. This so-called Hubbard-Stratonovich transformation \( ^{25, 25} \) can also be generalized to spatially non-local interactions. \( ^{25} \) This procedure, to be discussed in more detail in the next section, also yields the mean-field, i.e., Stoner, saddle-point equation for the exchange-interaction contribution to the spin splitting

\[ \Delta = U \int \frac{d^3k}{(2\pi)^3} \left\{ N_F \left[ \epsilon_k - \frac{(\Delta + \Delta_{\text{ext}})}{2} - \mu \right] \right. \]

\[ \left. - N_F \left[ \epsilon_k + \frac{(\Delta + \Delta_{\text{ext}})}{2} - \mu \right] \right\} = U \rho_s , \quad (4) \]

where \( N_F(x) = [\exp(x/k_B T) + 1]^{-1} \) is the Fermi distribution function with \( k_B T \) the thermal energy, \( \mu \) is the chemical potential that includes a Hartree mean-field shift, and \( \rho_s \) is the magnetization density. In practice we do not explicitly determine the exchange splitting from this equation, but simply assume \( \Delta \) is a solution whose value may be determined from experiment if needed. This is another reason for simply using a delta function interaction.

For the disorder we use the same model as Kohno et al. \( ^{25} \) in which the spin-dependent disorder potentials are characterized by

\[ V_a(x) V_b(x') = \sigma_a \delta(x - x') \delta_{ab} , \quad (5) \]

where \( \langle \sigma \rangle \) indicates averaging over different realizations of the disorder. For randomly distributed scatterers

\[ \sigma_{x,y} = n_s u_s^2 S_x^2 + \sigma_z = n_s u_s^2 S_z^2 , \quad \sigma_0 = n_i u_i^2 , \quad (6) \]

where \( u_i \) \( (u_s) \) and \( n_i \) \( (n_s) \) are the strength and density of the scatterer charge (spin) component, respectively, and \( S_{\sigma}^2 \) denotes the average scatterer field orientation. Within the self-consistent Born approximation the decay rate \( \gamma_\sigma \) and lifetime \( \tau_\sigma \) of a plane wave with spin state \( |\sigma \rangle \) are determined from

\[ \hbar \gamma_\sigma = \frac{\hbar}{2\tau_\sigma} = \pi n_s u_s^2 \nu_\sigma + \pi n_s u_s^2 \left( 2 S_z^2 \nu_\sigma - S_z^2 \nu_\sigma \right) , \quad (7) \]

where the density of states per spin at the Fermi level \( \nu_\sigma = m k_F^2/2\pi^2 \hbar^2 \), and the Fermi wave number \( k_F \) \( (\hbar \nu_F = \sqrt{2m(\epsilon_F + \sigma M)/\hbar^2}) \), where \( M = (\Delta + \Delta_{\text{ext}})/2 \) is the total spin splitting.

Finally, the current in our theory is induced by an external homogeneous electric field \( \mathbf{E} \) that, in the London gauge, is related to the vector potential by

\[ \mathbf{A}(t) = \frac{e}{i \omega_p} e^{-i \omega t} \mathbf{E} , \quad (8) \]

where \( \omega_p \) is the frequency of the electric field, to be taken to zero eventually, and \( e \) is the speed of light. In the Hamiltonian [Eq. (2)] the vector potential is minimally coupled to the electrons via the charge current-density operator

\[ \mathbf{J}(x,t) = \frac{i|e|}{2m} \left[ \hat{\psi}^\dagger(x,t) \nabla \hat{\psi}(x,t) - \left( \nabla \hat{\psi}^\dagger(x,t) \right) \hat{\psi}(x,t) \right] \quad (9) \]

with \(-|e|\) the electron charge. In the above expression we have omitted the diamagnetic contribution as it plays no role in the following.

In the next section we derive, starting from the Hamiltonian in Eq. (2), the equations of motion for long-wavelength deviations \( \delta \mathbf{\Omega} \) of the magnetization direction from the collinear ground state, defined by \( \mathbf{\Omega} = \hat{z} + \delta \mathbf{\Omega} \). We find that these transverse deviations obey the stochastic equations of motion

\[ \left( \frac{\partial}{\partial t} + \mathbf{v}_s \cdot \nabla \right) \delta \mathbf{\Omega}_a = \epsilon_{ab} \frac{\Delta_{\text{ext}}}{\hbar} \delta \mathbf{\Omega}_b \]

\[ + \alpha \left( \frac{\partial}{\partial t} + \beta \mathbf{v}_s \cdot \nabla \right) \delta \mathbf{\Omega}_b - \hbar b \right] , \quad (10) \]

where a sum over repeated transverse indices \( a, b \in \{ x, y \} \) is implied and \( \epsilon_{ab} \) is the two-dimensional Levi-Civita tensor. The Gilbert damping parameter is given by

\[ \alpha = \frac{2\pi}{\rho_s} \left\{ n_s u_s^2 S_z^2 \left( \nu_s^2 + \nu_f^2 \right) + 2 S_z^2 \nu_f n \right\} , \quad (11) \]
with the magnetization density $\rho_s = \Delta / U$. (For the $s - d$ model $\rho_s$ corresponds to the carrier spin polarization density.) The velocity $v_s$ is related to the electric field by

$$v_s = -\frac{eE}{m\rho_s} (n_+\tau_+^{sc} - n_-\tau_-^{sc}) ,$$

in terms of the density of majority and minority electrons, denoted by $n_+$ and $n_-$. Using the fact that to linear order in the electric field the current densities of the majority and minority electrons are determined from $j_\sigma = n_\sigma |e|^2 \tau_\sigma^{sc} E / m$, we observe that the expression for $v_s$ reduces to the usual expression $v_s = (j_1 - j_i) / (|e|\rho_s)$. Our result for the $\beta$-parameter reads

$$\beta = \frac{2\pi n_s \nu s^2}{M} \left[ \frac{n_+\tau_+^{sc} (S_s^\nu_1 + S_s^\nu_1) - n_-\tau_-^{sc} (S_s^\nu_1 + S_s^\nu_1)}{(n_+\tau_+^{sc} - n_-\tau_-^{sc})} \right].$$

Notice that, as expected, only spin-dependent scattering contributes to the non-adiabatic torque parameter $\beta$ and the Gilbert damping parameter $\alpha$.

In addition, we find that the thermal fluctuations via the stochastic magnetic field $h$ are determined by

$$\langle h_a(x,t)h_b(x',t') \rangle_{\text{noise}} = \frac{2\alpha h q T}{\hbar (\rho_s/2)} \delta(x-x') \delta(t-t') \delta_{ab} ,$$

where the average is over different realizations of the noise. We stress that this form for the strength of the fluctuations is derived explicitly, without appealing to the fluctuation-dissipation theorem. This is important since it is not a priori obvious that in the current-carrying situation the fluctuation-dissipation theorem holds. The form of the strength of the fluctuations in Eq. (14) is however of the usual form, i.e., it is of the same form as inferred by the equilibrium fluctuation-dissipation theorem. This result comes about because shot noise contributions to the magnetization noises enter as higher-order terms in the applied electric field than the linear response in electric field considered here.

The linear response result in Eq. (10) is consistent up to $O(\delta\Omega)$ with the Landau-Lifschitz-Gilbert equation that includes both non-adiabatic and adiabatic spin transfer torques and thermal fluctuations

$$\left( \frac{\partial}{\partial t} + v_s \cdot \nabla \right) \hat{\Omega} - \hat{\Omega} \times (H + h)$$

$$= -\alpha \hat{\Omega} \times \left( \frac{\partial}{\partial t} + \frac{\beta v_s}{\alpha} \nabla \right) \hat{\Omega} ,$$

where $h$ is a stochastic magnetic field that obeys the correlations given by Eq. (13).

We end this section by sketching how the various results come about. In the theory to be discussed in more detail in the next section, the two quantities of interest are the transverse spin-density spin-density response function (or magnon propagator), which determines the Gilbert damping parameter, and the photon two-magnon interaction vertex which gives rise to spin transfer torques. (Note that the photons simply correspond to the external electric field in this case.) Feynman diagrams for both of these functions are given in Fig. 1 (a) and (b). Quite generally, the response function in Fig. 1 has reactive and dissipative parts. In the long time and length scale expansion corresponding to the LLG equations, the small-frequency zero-momentum part of the reactive contribution gives rise to the time-derivative on the right-hand side of Eq. (10). (Note that after Fourier transformation frequencies turn into time derivatives.) The small-frequency zero-momentum part of the dissipative contribution to the same response function determines the Gilbert damping term on the right-hand side of Eq. (10). Physically, this dissipative contribution comes from spin waves that decay into particle-hole excitations. Energy conservation then leads a delta-function-like, i.e., dissipative, contribution of the form $\delta(h\omega - \epsilon_1 + \epsilon_2)$ where $h\omega$ is the energy of the spin wave and $\epsilon_1 - \epsilon_2$ the energy of the particle-hole pair. Summing over all possible particle-hole pair energies and performing a zero-momentum low-frequency expansion then leads to the Gilbert damping term on the right-hand side of Eq. (10). Similarly, the spatial derivatives on the left-hand side of Eq. (10) are the re-
result of the reactive contribution to the zero-frequency small-momentum behavior of the photon spin-wave interaction vertex and give rise to the adiabatic spin transfer torque. The non-adiabatic torque, proportional to $\beta$ on the right-hand side of Eq. (10), then emerges from the dissipative part of the interaction vertex, and gets contributions from physical processes in which a spin-wave interacts with the current and subsequently decays into an incoherent particle-hole excitation.

### III. NONEQUILIBRIUM MAGNETIZATION DYNAMICS

In this section we derive the stochastic equation of motion for the transverse magnetization in the presence of current. We start out by deriving the general equations, and subsequently give the results for the long-wavelength low-frequency limit. We discuss the equilibrium situation, i.e., the case without electric field, and the nonequilibrium situation separately.

#### A. Stochastic Equations of Motion

Our starting point is the path-integral expression for the coherent-state probability distribution, written as a functional integral

$$P[\phi^*, \phi; t] = \int d[\psi_1^*] d[\psi_1] d[\psi_2^*] d[\psi_2] \exp \left\{ \frac{i}{\hbar} S[\psi^*, \psi] \right\} .$$

Roughly speaking, this distribution specifies the probability for the system to be in the Grassman coherent state $\phi(x, t)$. The action is expressed in terms of the fermionic fields $\psi$ and $\psi^*$ by

$$S[\psi^*, \psi] = \int_{C^t} dt' \int d\xi \left\{ \psi_2^*(x, t') i \hbar \frac{\partial \psi_2(x, t')}{\partial t'} - H[\psi_2^*(x, t'), \psi(x, t')] \right\} .$$

The functional integration in Eq. (16) is over all fields evolving forward in time from $-\infty$ to $t$, and back, thereby defining the time integration in the action in Eq. (17) to be over the Keldysh contour $C^t$.

We rewrite the interaction term as

$$U \psi_1^* \psi_1 \psi_1^* \psi_1 = \frac{U}{4} (\psi^* \psi)^2 - \frac{U}{4} (\psi^* \tau \cdot \dot{\hat{n}} \psi)^2 ,$$

with $\hat{n}(x, t)$ an arbitrary unit vector that determines the spin quantization axis. Functional integration over the latter enforces rotation invariance. The interaction terms on the right-hand side of Eq. (18) are decoupled by writing them as a Gaussian functional integral over a density field $\langle \rho(x,t) \rangle = \langle \psi^* \psi \rangle$, and spin-density field $\langle \Delta(x,t) \hat{n}(x,t) \rangle = U (\psi^* \tau \cdot \psi^*)/2$, respectively. The precise meaning of the brackets $\langle \cdots \rangle$ is defined below Eq. (23).

This Hubbard-Stratonovich transformation then introduces the density and spin density as dynamical variables in the path-integral in Eq. (16). Density and spin-density amplitude fluctuations are gapped and can be approximated at low temperatures and energies by their saddle-point values. For the density we then find a Hartree-Fock equation, giving rise to a mean-field Hartree shift which we absorb in the chemical potential. For the spin-density amplitude we find the saddle-point equation for $\Delta$ in Eq. (11).

After these steps we ultimately find that the probability distribution is given by

$$P[\phi^*, \phi, \hat{\Omega}; t] = \int d[\psi_1^*] d[\psi_1] d[\psi_2^*] d[\psi_2] \exp \left\{ \frac{i}{\hbar} S'[\psi^*, \psi, \hat{n}] \right\} ,$$

where the unit vector $\hat{\Omega}$ enters as the boundary condition at $t' = t$ on the functional integration over the fluctuating magnetization orientation $\hat{n}$. We do not explicitly indicate the boundary condition on the fermion fields, because, as we shall see, the quantity that enters is the fermion Green’s function which is determined without explicitly referring to the boundary conditions. The action $S'[\psi^*, \psi, \hat{n}]$ is, using the same notation as for the Hamiltonian in Eq. (2), explicitly given by

$$S'[\psi^*, \psi, \hat{n}] = \int_{C^t} dt' \int d\xi \left\{ \psi_2^*(x, t') \left[ i \hbar \frac{\partial}{\partial t'} + \frac{\hbar^2 \nabla^2}{2m} + \frac{\Delta}{2} \hat{n}(x, t') \cdot \tau + \frac{\Delta_{ext}}{2} \tau_z - V_0(x) - V_a(x) \tau_a \right] \psi(x, t') - \frac{1}{e} J(x, t') \cdot \mathbf{A}(t') \right\} .$$

At this point we note that, if we would add a separate Berry-phase term in this action to enforce the angular-momentum like quantization of $\hat{n}$, the resulting action would be the starting point for treating the $s - d$ model.

We now do perturbation theory around the collinear state by writing

$$\hat{n}(x, t) \approx \left( \frac{\delta n_x(x, t)}{\delta n_y(x, t)} \right) \left( \frac{\delta n_y(x, t)}{1 - \frac{1}{2} (\delta n_x(x, t))^2 - \frac{1}{2} (\delta n_y(x, t))^2} \right) ,$$

and integrate out the electronic fields using second-order perturbation theory in $\delta n_a(x, t)$, and first-order perturbation theory in $\mathbf{A}(x, t)$. (Note that to find an equation of motion for $\delta n_a$ that is valid up to first-order the action needs to be determined up to quadratic terms in $\delta n_a$.) To this order the effective action for the magnetization is written in the form

$$S_{eff}[\delta \mathbf{n}] = \int_{C^t} dt' \int d\xi \left\{ - \frac{\Delta}{4} \rho_a \delta n^2_a(x, t') \right\}$$

$$+ \int_{C^t} dt'' \int d\xi' \delta n_a(x, t') \Pi_{ab}(x - x'; t'', t') \delta n_b(x', t'')$$
\[ + \int_{C_t} dt'' \int dx' \delta n_a(x, t') K_{ab}(x - x', t', t'') \delta n_b(x', t'') \}, \tag{22} \]

where a sum over repeated transverse indices \(a, b \in \{x, y\}\) is again implied. In Eq. (22), the function \(\Pi_{ab}(x; t, t')\) is determined by the transverse spin-density spin-density response function

\[ \Pi_{ab}(x - x'; t, t') = \frac{i\Delta^2}{8\hbar} \langle \psi^\dagger (x, t) \tau_a \psi(x, t) \psi(x', t') \psi^\dagger (x', t') \tau_b \psi(x', t') \rangle, \tag{23} \]

shown diagrammatically in Fig. 1 (a). In this expression the brackets \(\langle \cdot \cdot \cdot \rangle \equiv \text{Tr} \langle \hat{\rho} (\infty) \cdot \cdot \cdot \rangle\) denote an average with respect to the density matrix \(\hat{\rho} (\infty)\) of a system of electrons with the action in Eq. (20), with \(\hat{n}(x, t) = \hat{d}^\dagger \hat{d}\) and \(\mathbf{A} = 0\), that is in equilibrium. The function \(K_{ab}(x; t, t')\) is given by

\[ K_{ab}(x - x'; t, t') = \int_{C_t} dt'' \int dx'' \left[ \frac{\Delta^2}{8\hbar^2 c} x_{ab}(x, x'', t, t', t'') \cdot A(t'') \right], \tag{24} \]

where the photon two-magnon vertex function

\[ \Lambda_{ab}(x, x', x''; t, t', t'') \]

\[ S_{\text{eff}}[\delta \Omega, \xi] = \int_{-\infty}^{t} dt' \int dx \left\{ -\frac{\Delta p_b}{2} \delta \Omega_a(x, t') \xi_a(x, t') \right\} + \int_{-\infty}^{t} dt' \int dx \int_{-\infty}^{t} dt'' \int dx' \left\{ \delta \Omega_a(x, t') \left[ \Pi_{ab}^{(-)}(x - x'; t' - t'') + K_{ab}^{(-)}(x - x'; t' - t'') \right] \xi_b(x', t'') \right\} + \int_{-\infty}^{t} dt' \int dx \int_{-\infty}^{t} dt'' \int dx' \left\{ \xi_a(x, t') \left[ \Pi_{ab}^{(+)}(x - x'; t - t'') + K_{ab}^{(+)}(x - x'; t - t'') \right] \delta \Omega_b(x', t'') \right\} + \int_{-\infty}^{t} dt' \int dx \int_{-\infty}^{t} dt'' \int dx' \left\{ 2 \xi_a(x, t') \left[ \Pi_{ab}^{K}(x - x'; t - t'') + K_{ab}^{K}(x - x'; t - t'') \right] \xi_b(x', t'') \right\}, \tag{27} \]

where the time integrations are now over the real axis from minus infinity to \(t\).

Before we proceed, we make some general statements about dealing with functions on the Keldysh contour.\(^\text{21}\) A general function \(A(t, t')\), with time arguments on the Keldysh contour, can be decomposed into its analytic pieces by means of

\[ A(t, t') \equiv \theta(t, t') A^>(t, t') + \theta(t', t) A^<(t, t') \], \tag{28} \]

with \(\theta(t, t')\) the Heaviside step function on the Keldysh contour. Generally there can also be a piece \(A^\delta(t, t')\), but such a general decomposition is not needed here. Retarded and advanced functions, distinguished by the superscripts \((+)\) and \((-)\), respectively, are related to the analytic pieces by

\[ A^{(\pm)}(t, t') \equiv \pm \theta(\pm(t - t')) \left[ A^>(t, t') - A^<(t, t') \right]. \tag{29} \]

In addition, the Keldysh part, which, as we shall see, determines the strength of the fluctuations, is defined by

\[ A^K(t, t') \equiv \left[ A^>(t, t') + A^<(t, t') \right]. \tag{30} \]

Note that in the effective action [Eq. (27)] the retarded, advanced, and Keldysh parts of the various functions depend only on the difference of time arguments (we have implicitly taken the limit \(\omega_p \to 0\)).
To derive the equation of motion that is obeyed by the magnetization $\delta \Omega_a$ we perform another Hubbard-Stratonovich transformation, and write the part of the action that is quadratic in the fluctuations $\xi_a$ as a Gaussian functional integral over an auxiliary field $\eta_a$ which will turn out to correspond, up to prefactors, to the stochastic magnetic field $\mathbf{h}$. Explicitly, we then have for the probability distribution that

$$P[\phi^*, \phi, \hat{\Omega}; t] = \int d\delta \Omega d\xi d[\eta]$$

with the effective action

$$S_{\text{eff}}[\delta \Omega, \xi, \eta] = \int_{-\infty}^{t} dt' \int dx \left\{ -\frac{\Delta \rho_s}{2} \delta \Omega_a(x, t') \xi_a(x, t') + \eta_a(x, t) \xi_a(x, t) \right\}$$

$$+ \int_{-\infty}^{t} dt' \int dx \int_{-\infty}^{t} dt'' \int dx' \left\{ \delta \Omega_a(x, t') \left[ \Pi_{ab}^{(-)}(x - x'; t' - t'') + K_{ab}^{(-)}(x - x'; t' - t'') \right] \xi_b(x', t'') \right\}$$

$$+ \int_{-\infty}^{t} dt' \int dx \int_{-\infty}^{t} dt'' \int dx' \left\{ \xi_a(x, t') \left[ \Pi_{ab}^{(+)}(x - x'; t' - t'') + K_{ab}^{(+)}(x - x'; t' - t'') \right] \delta \Omega_b(x', t'') \right\} .$$

This action is now linear in the fluctuations $\xi$, and the functional integration over these fluctuations leads to a constraint that is precisely the equation of motion for the magnetization $\delta \Omega$. We find that

$$- \left[ \Pi_{ab}^{(+)} \left( -i \nabla, i \frac{\partial}{\partial t} \right) + \Pi_{ba}^{(-)} \left( i \nabla, -i \frac{\partial}{\partial t} \right) + K_{ab}^{(+)} \left( -i \nabla, i \frac{\partial}{\partial t} \right) + K_{ba}^{(-)} \left( i \nabla, -i \frac{\partial}{\partial t} \right) \right] \delta \Omega_b(x, t)$$

$$+ \frac{\Delta \rho_s}{2} \delta \Omega_a(x, t) = \eta_a(x, t) ,$$

(33)

with $\Pi_{ab}(q, \omega)$ and $K_{ab}(q, \omega)$ denoting the Fourier transforms of $\Pi_{ab}(x - x'; t - t')$ and $K_{ab}(x - x'; t - t')$, respectively. Note that the procedure used in Eq. (26) and (27), which leads ultimately to the above equation of motion, circumvents the usual difficulties of deriving an equation of motion with dissipative terms from an action. The probability distribution for the noise is given by

$$P[\eta] = \exp \left\{ \frac{i}{\hbar} \int_{-\infty}^{t} dt' \int dx \int_{-\infty}^{t} dt'' \int dx' \left\{ 2\eta_a(x, t') \left[ \Pi_{ab}^{K}(x - x'; t' - t'') + K_{ab}^{K}(x - x'; t' - t'') \right]^{-1} \eta_b(x', t'') \right\} \right\} ,$$

(34)

so that the correlation function of the stochastic magnetic field follows as

$$\langle \eta_a(x, t) \eta_b(x', t') \rangle_{\text{noise}} = \frac{\hbar}{i} \left[ \Pi_{ab}^{K}(x - x'; t - t') + K_{ab}^{K}(x - x'; t - t') \right] .$$

(35)

The results in Eqs. (33) and (35) are the main results of this subsection. Clearly, our main tasks are now to determine the long-wavelength low-frequency behavior of the non-equilibrium spin-density spin-density response function and the photon two-magnon vertex function. These calculations will be outlined in the next two subsections. We start out with the equilibrium situation in which the electric field is zero and we only need to consider the spin-density spin-density response function.
where the trace is over spin space and $\tau_0$ denotes the $2 \times 2$ identity matrix. The first term in this equation corresponds to the lowest-order diagram in Fig. 2a) and the second term to the diagram with the vertex correction in Fig. 2b).

The Green’s function is defined as

$$iG_{\sigma \sigma'}(x - x', t, t') \equiv \langle \psi_{\sigma}(x, t) | \psi_{\sigma'}^\dagger(x', t') \rangle$$

$$= \theta(t, t') \text{Tr} \left[ \hat{\rho}(-\infty) \psi_{\sigma}(x, t) \psi_{\sigma'}^\dagger(x', t') \right]$$

$$- \theta(t', t) \text{Tr} \left[ \hat{\rho}(-\infty) \psi_{\sigma'}^\dagger(x', t') \psi_{\sigma}(x, t) \right],$$

so that the Fourier transforms of its analytic pieces read

$$-iG^<_{\sigma}(k, \omega) = A(k, \omega) N_F(\hbar \omega - \mu);$$

$$iG^>_{\sigma}(k, \omega) = A(k, \omega) \left[ 1 - N_F(\hbar \omega - \mu) \right],$$

where the spectral function $A(k, \omega)$ is defined by

$$A(k, \omega) = i \left[ G^{(+)}(k, \omega) - G^{(-)}(k, \omega) \right].$$

Finally, the retarded and advanced Green’s functions are given by

$$G_{\sigma \sigma'}^{(\pm)}(k, \omega) = \frac{\delta_{\sigma \sigma'}}{\hbar \omega \pm i \epsilon_k + M \sigma \pm i \hbar \gamma_\sigma},$$

where $\hbar \omega = \hbar \omega \pm 10$ as usual.

With these ingredients, the calculation of the retarded, advanced, and Keldysh components of the response function in Eq. (36), is, in principle, straightforward. Some details of these calculations are described in the appendix. Here we directly present the results. For the retarded and advanced components we find that

$$\Pi_{xx}^{(\pm)}(q, \omega) =\Pi_{yy}^{(\pm)}(q, \omega) =$$

$$\frac{\Delta^2 \rho_s}{8M} \pm i \pi \Delta^2 \hbar \omega \left\{ \frac{n_s u_s^2 \left[ v_x^2 + v_y^2 + 2 v_x v_y \right]}{16M^2} \right\},$$

$$\Pi_{xy}^{(\pm)}(q, \omega) = -\Pi_{yx}^{(\pm)}(q, \omega) = \frac{\Delta^2 \rho_s}{16M^2} i \hbar \omega,$$ (41)

with the Keldysh parts given by

$$\Pi_{xx}^K(q, \omega) = \Pi_{yy}^K(q, \omega) =$$

$$i \pi \Delta^2 k_B T \left\{ \frac{n_s u_s^2 \left[ v_x^2 + v_y^2 + 2 v_x v_y \right]}{2M^2} \right\},$$

$$\Pi_{xy}^K(q, \omega) = \Pi_{yx}^K(q, \omega) = 0.$$ (42)

In order to obtain the Gilbert damping coefficient it is sufficient to perform a zero-momentum small-frequency expansion of this response function. The first term that enters in a long-wavelength expansion is quadratic and determines the spin stiffness that is not of interest to use here. In addition, in order to determine the fluctuations it turns out to be sufficient to obtain the zero-moment zero-frequency part of the Keldysh response function. Inserting these results into the full equations of motion in Eqs. (33) and (35) straightforwardly leads to the results in Eqs. (39) and (40), with $v_x = 0$. (In arriving at these final results we have taken the limit $\Delta_{ext} \ll \Delta$.) In the next section we consider the situation with an external electric field which leads to a nonzero spin-transfer velocity $v_s$. We end this subsection by noting that, from
a phenomological viewpoint, Eqs. (10) and (14) are under debate\cite{33,35}, even for $v_s = 0$. We hope that the microscopic derivation presented here sheds new light on this controversy. Finally, we note that the temporal delta function in Eq. (14) arises by taking the zero-

C. With Current

Our next task is to evaluate the function $K_{ab}(x; t, t')$ that is proportional to the photon two-magnon interaction vertex and hence, from a microscopic point of view, ultimately gives rise to spin transfer torques. The relevant Feynman diagrams are given in Fig. 3 and correspond to the expression

\[
K_{ab}(q; t, t') = \frac{|e|^2}{4m\hbar} \int dt'' \int \frac{d^3p_1}{(2\pi)^3} \frac{e^{-i\omega t''}}{\omega_p} \left\{ \text{Tr} [\tau_a G(p_1 + q; t, t') \tau_b G(p_1; t', t'') G(p_1; t'', t)] (p_1 \cdot E) \right. \\
+ \frac{1}{\hbar^2} \sum_{a'=0, x, y, z} \sigma_{a'} \int dt''' \int dt'''' \int \frac{d^3p_2}{(2\pi)^3} \left. \times \left[ \text{Tr} [\tau_a G(p_1 + q; t, t''') \tau_a G(p_2 + q; t''', t') \tau_b G(p_2; t', t'') \tau_b G(p_2; t', t'') \tau_a G(p_2; t'', t) (p_1 \cdot E) \\
+ \tau_a G(p_1 + q; t, t''') \tau_a G(p_2 + q; t''', t') \tau_b G(p_2; t', t'') \tau_a G(p_2; t', t'') \tau_a G(p_2; t'', t) (p_2 \cdot E)] \right] \right\}, \quad (43)
\]

where the trace is again over spin space. In this expression the first, second and third terms correspond to the Feynman diagrams in Fig. 3 a), b) and c), respectively. Determining the low-frequency long-wavelength behavior of the retarded, advanced, and Keldysh components from Eq. (43) is straightforward but rather tedious. Typical steps in the calculations are illustrated in the appendix for the spin-density spin-density response function. Here, we directly present the results. Note that to obtain the spin transfer torques, and in particular the $\beta$-coefficient that characterizes the non-adiabatic spin transfer torque, it is sufficient to perform a zero-frequency long-wavelength expansion.

The results for the various parts of the function $K_{ab}(q; t, t')$, that ultimately determine the adiabatic spin transfer torque, are given by

\[
K_{x\gamma}^{K}(q, \omega) = K_{y\gamma}^{K}(q, \omega) = 0; \\
K_{x\gamma}^{(\pm)}(q, \omega) = -K_{y\gamma}^{(\pm)}(q, \omega) \\
= \frac{i\Delta^2 |e| (q \cdot E)}{16mM^2} \left( n_{\uparrow} \gamma_{\uparrow} \gamma_{\downarrow} - n_{\downarrow} \gamma_{\downarrow} \gamma_{\uparrow} \right) . \quad (44)
\]

We note that these off-diagonal parts correspond to the reactive part of the photon two-magnon interaction vertex. The dissipative part that gets contributions from decay processes and determines the non-adiabatic spin transfer torque is given by

\[
K_{xx}^{(\pm)}(q, \omega) = K_{yy}^{(\pm)}(q, \omega) \\
= \frac{\pm i\Delta^2 \hbar^2 |e| (q \cdot E)}{16mM^3} \left( n_{\uparrow} \gamma_{\uparrow} \gamma_{\downarrow} - n_{\downarrow} \gamma_{\downarrow} \gamma_{\uparrow} \right)
\]
\[-\frac{\hbar}{\tilde{U}} \left( n_{1\downarrow} u_1 - n_{2\downarrow} u_2 \right) \left( v_{\uparrow} \tau_+^{\uparrow \downarrow} - v_{\downarrow} \tau_+^{\downarrow \uparrow} \right) \]

\[
K_{xx}^{\alpha}(q, \omega) = K_{yy}^{\alpha}(q, \omega) = 0 .
\] (45)

From the above expression we note that, to first order in the electric field, the Keldysh part of the photon two-magnon interaction vertex is zero. Ultimately, this implies that the current does not alter the thermal fluctuations, at least to first order perturbation theory in the electric field. Finally, we remark that inserting the above results from Eqs. (44) and (45) into the general equation of motion in Eq. (33) leads in a straightforward manner to the results for \(v_s\) and \(\beta\) presented in Sec. IV.

IV. CONCLUSIONS

In conclusion, we have presented a general framework for the derivation of the effective equations of motion for the magnetization direction of a metallic ferromagnet, including nonzero-temperature effects and current. An important aspect of our approach is that the functional Keldysh methods we employ enable us to incorporate thermal fluctuations via stochastic forces in a unifying manner, without explicitly invoking the fluctuation-dissipation theorem. As a specific example, we have carried out detailed calculations for the model of a disordered itinerant ferromagnet used by Kohno et al. Our results for the Gilbert damping parameter and the \(\beta\)-parameter that characterizes the non-adiabatic torque are identical to the results found by these authors using imaginary-time methods. We have in addition determined the thermal fluctuations and found that, although the non-adiabatic torque corresponds to a dissipative process and the current-carrying situation makes application of the fluctuation-dissipation theorem questionable, to first order in the electric field the usual fluctuation-dissipation relation to the Gilbert damping holds.

The method presented here is quite general, and in the near future we intend to apply it also to other models of ferromagnets. We briefly comment on the generality of our results and what to expect for other models. In practice ferromagnetism in metals is usually described in terms of some combination of ground state and time-dependent spin-density-functional (SDF) theory. The structure of the ground state theory is then the same as that of the saddle-point mean-field equations that arise in our theory, with the spin-dependent interaction in our theory replaced by the spin-dependence of the exchange-correlation potential in SDF theory and our parabolic bands replaced by more complex bands specific to a particular material. For transition metals the more realistic bands of SDF theory are hybridized \(s\) and \(d\) bands with \(k\) dependent spin-splitting which tends to be larger in bands with dominant \(d\)-character. Transition metal ferromagnets are sometimes described by a crude model in which hybridization is not explicitly accounted for and the \(d\)-orbitals are assumed to be fully spin-polarized. In this \(s - d\) model the \(d\)-orbitals do not contribute to the density of states at the Fermi level since the majority spins are fully occupied and the minority spins are empty. It follows that the \(d\)-orbitals do not contribute to transport or to any other property that involves only orbitals at the Fermi energy. When the formalism of our paper is applied to an \(s - d\) model rather than to the single-band model we discuss, the \(d\) orbitals can contribute to properties associated with the reactive pieces of the response functions we evaluate, but not to the properties that come from the low-energy limits of the dissipative response function pieces. The \(d\)-orbitals do contribute to the coefficient of \(\omega\) which translates into the dissipative precessional dynamics time derivative for example. [See the last line of Eq. (11).] This \(q = 0\) time-derivative can be interpreted as capturing the Berry phase associated with adiabatic spin-dynamics. It follows that the \(d\) and \(s\) orbital contributions to this coefficient are proportional to their respective contributions to the total spin-density. The \(d\)-orbitals of an \(s - d\) model do not, however, contribute to the reactive adiabatic spin-torque \((v_s)\) term because the \(d\)-bands are either full or empty and therefore do not respond to an electric field.

The \(\alpha\) and \(\beta\) dissipative parameters are both defined as dimensionless ratios of coefficient contributions from dissipative and non-dissipative terms in the equation of motion Eq. (11). Because \(\alpha\) parameterizes the ratio of the two time-derivative terms it is indirectly altered by the \(d\)-bands. In contrast, \(\beta\) parameterizes the ratio of the two space-derivative terms neither of which has a \(d\)-orbital contribution. This is the reason, as noted in previous studies, why \(\beta\) tends to be larger than \(\alpha\) in \(s - d\) models, especially when the \(d\) orbitals make the dominant contribution to the spin-density. As we have mentioned previously, and originally shown by Tserkovnyak et. al., for spin-dependent scattering models with parabolic dispersion \(\alpha \approx \beta\) in a Stoner band model when the exchange splitting is much smaller than the Fermi energy, but not when the \(d\)-orbital Berry phase is added to the reactive time derivative of an \(s - d\) model. It is perhaps expected that \(\alpha\) should approximately equal to \(\beta\) in this limit since all the ingredients necessary for macroscopic Galilean invariance seem to be present. When the spin-polarization is small there is little to distinguish one direction of spin-polarization from another and therefore one position in a spin texture from another. When the bands are parabolic in addition, an external electric field simply accelerates the system’s center of mass. Explicit calculations for the present toy model demonstrate conclusively that the \(\alpha = \beta\) condition which corresponds to macroscopic Galilean invariance is not satisfied generally. For more general spin-dependent disorder models or more realistic exchange splitting values \(\alpha\) and \(\beta\) are never equal.

These considerations to not apply directly to transition metal ferromagnets because of \(s - d\) hybridization and because of the large \(d\)-orbital contribution to the minority-spin density of states. It is nevertheless true that the two reactive term coefficients can be expressed approximately
as the sum of $s$ and $d$ orbital contributions. In the absence of spin-orbit coupling, the coefficient of $\frac{d\Omega}{dt}$ in Eq. (1) is rigorously equal to one because the Berry phase is proportional to the total spin-density, including the $s$ and the dominant $d$ contribution. Similarly the reactive coefficient of $\nabla \Omega$ can be understood in terms of the cancellation between convective and precessional contributions to spin-dynamics in the static limit. It follows that the $d$-orbital weight in this reactive coefficient is not zero, as in the $s$--$d$ model, but still relatively smaller than the $d$ contribution to the reactive time-derivative coefficient. These considerations suggest that $\beta / \alpha$ will tend to be larger than one in most transition metal ferromagnets. The main challenges to addressing this issue more quantitatively for a specific material are achieving an understanding of the nature of its spin-independent and spin-dependent disorder, accounting for the spin-orbit coupling present in the bands of the perfect crystal, and evaluating the vertex corrections (whose essential role is established by these toy model calculations) in systems with complex band structures.

The spin-density spin-density response function in Eq. (30) is given by

$$\Pi_{ab}(\mathbf{q}; t, t') = \Pi_{ab}^0(\mathbf{q}; t, t') + \Pi_{ab}^1(\mathbf{q}; t, t') ,$$

where the first term on the right-hand side is the lowest-order diagram in Fig. (a) and the second term is the vertex correction in Fig. (b). In the first part of this appendix we evaluate the lowest-order diagram. In the second part the vertex correction is calculated.

1. No vertex corrections

Without vertex corrections the response function

$$\Pi_{ab}^0(\mathbf{q}; t, t') = \frac{i \Delta^2}{8 \hbar} \int \frac{d\mathbf{p}}{(2\pi)^3} \text{Tr} \left[ \tau_a G(\mathbf{p} + \mathbf{q}; t, t') \tau_b G(\mathbf{p}; t', t) \right] ,$$

is shown diagrammatically in Fig. (a). The analytic pieces are now easily determined from the above result and given by

$$\Pi_{ab}^{0, >}(\mathbf{q}; t, t') = \frac{i \Delta^2}{8 \hbar} \int \frac{d\mathbf{p}}{(2\pi)^3} \text{Tr} \left[ \tau_a G^>(\mathbf{p} + \mathbf{q}; t, t') \tau_b G^< (\mathbf{p}; t', t) \right] ;$$

$$\Pi_{ab}^{0, <}(\mathbf{q}; t, t') = \frac{i \Delta^2}{8 \hbar} \int \frac{d\mathbf{p}}{(2\pi)^3} \text{Tr} \left[ \tau_a G^<(\mathbf{p} + \mathbf{q}; t, t') \tau_b G^>( \mathbf{p}; t', t) \right] .$$

Using the results in Eqs. (37)-(40) we have for the retarded and advanced components of the zero-momentum Fourier-transformed response function that

$$\Pi_{ab}^{0, (\pm)}(\mathbf{q}, \omega) = -\frac{\Delta^2}{8} \int \frac{d\epsilon}{(2\pi)} \int \frac{d\epsilon'}{(2\pi)} \int \frac{d\mathbf{p}}{(2\pi)^3} \left[ N_F(\epsilon - \mu) - N_F(\epsilon' - \mu) \right] \frac{\epsilon - \epsilon' - \hbar \omega^{\pm}}{\epsilon - \epsilon' - \hbar \omega^{\pm}} \text{Tr} \left[ \tau_a A(\mathbf{p}, \epsilon) \tau_b A(\mathbf{p}, \epsilon') \right] .$$

Expanding for small energies we find that

$$\Pi_{ab}^{0, (\pm)}(\mathbf{q}, \omega) \simeq -\frac{\Delta^2}{8} \int \frac{d\epsilon}{(2\pi)} \int \frac{d\epsilon'}{(2\pi)} \int \frac{d\mathbf{p}}{(2\pi)^3} \left[ N_F(\epsilon - \mu) - N_F(\epsilon' - \mu) \right] \frac{\mathcal{P}_{\epsilon - \epsilon'}}{(\epsilon - \epsilon')} \text{Tr} \left[ \tau_a A(\mathbf{p}, \epsilon) \tau_b A(\mathbf{p}, \epsilon') \right] .$$

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APPENDIX: CALCULATION OF THE RETARDED, IMAGINARY AND KELDYSH PARTS OF THE SPIN-DENSITY SPIN-DENSITY RESPONSE FUNCTION
\[
\begin{align*}
&\frac{\Delta^2 \hbar \omega}{8} \int \frac{d\epsilon}{(2\pi)^2} \int \frac{d\epsilon'}{(2\pi)^2} \int \frac{d\mathbf{p}}{(2\pi)^3} \left[ N_F(\epsilon - \mu) - N_F(\epsilon' - \mu) \right] \frac{\mathcal{P}}{(\epsilon - \epsilon')^2} \text{Tr} \left[ \tau_0 A(p, \epsilon) \tau_b A(p, \epsilon') \right] \\
&\pm \frac{i \Delta^2 \hbar \omega}{32\pi} \int \frac{d\mathbf{p}}{(2\pi)^3} \text{Tr} \left[ \tau_0 A(p, \mu) \tau_b A(p, \mu) \right].
\end{align*}
\] (A.5)

From this find \( \Pi_{xx}^{0,\pm}(\mathbf{q}, \omega) = \Pi_{yy}^{0,\pm}(\mathbf{q}, \omega) \), and \( \Pi_{xy}^{0,\pm}(\mathbf{q}, \omega) = -\Pi_{yx}^{0,\pm}(\mathbf{q}, \omega) \). Carrying out the remaining integrations we have, in the limit \( \gamma_0 / M \to 0 \), that

\[
\begin{align*}
\Pi_{xx}^{0,\pm}(\mathbf{q}, \omega) &= \frac{\Delta^2 \rho_s}{8M} \pm \frac{i \Delta^2 \hbar \omega}{8} \left\{ n_s u_1^2 \nu_1 \nu_\perp + n_s u_2^2 \left( \frac{\nu_1^2 + \nu_\perp^2}{M^2} + \frac{\nu_2^2 \nu_\perp \nu_\parallel}{M^2} \right) \right\} , \\
\Pi_{xx}^{0,\pm}(0, \omega) &= \frac{\Delta^2 \rho_s}{16M^2} \hbar \omega .
\end{align*}
\] (A.6)

The Keldysh component of the response function is in first instance given by

\[
\begin{align*}
\Pi_{xx}^{0,K}(\mathbf{q}, \omega) &= \pi i \frac{\Delta^2}{4} \int \frac{d\epsilon}{(2\pi)^2} \int \frac{d\epsilon'}{(2\pi)^2} \int \frac{d\mathbf{p}}{(2\pi)^3} \delta(\hbar \omega - \epsilon + \epsilon') \\
&\times \left\{ [1 - N_F(\epsilon - \mu)] N_F(\epsilon' - \mu) + N_F(\epsilon' - \mu) [1 - N_F(\epsilon' - \mu)] \right\} \text{Tr} \left[ \tau_0 A(p, \epsilon) \tau_b A(p, \epsilon') \right].
\end{align*}
\] (A.7)

From this we see that \( \Pi_{xx}^{0,K}(\mathbf{q}, \omega) = \Pi_{yy}^{0,K}(\mathbf{q}, \omega) \), and \( \Pi_{xy}^{0,K}(\mathbf{q}, \omega) = \Pi_{yx}^{0,K}(\mathbf{q}, \omega) = 0 \). We find that

\[
\Pi_{xx}^{0,K}(\mathbf{q}, \omega) = -\frac{i \pi \Delta^2 k_B T}{2} \left\{ n_2 u_2^2 \nu_2 \nu_\perp + n_1 u_1^2 \left( \frac{\nu_2^2 + \nu_\perp^2}{M^2} + \frac{\nu_1^2 \nu_\perp \nu_\parallel}{M^2} \right) \right\} .
\] (A.8)

Moreover, we have that

\[
\Pi_{xx}^{0,K}(\mathbf{q}, \omega) = \pm 2i \left[ 2N_B(\hbar \omega) + 1 \right] \text{Im} \Pi_{xx}^{0,\pm}(\mathbf{q}, \omega) ,
\] (A.9)

where \( N_B(x) \) is the Bose distribution function. This is the fluctuation-dissipation theorem, which emerges naturally from the formalism.

2. Vertex correction

The first-order vertex correction is shown in Fig. 2 (b), and given by

\[
\begin{align*}
\Pi_{ab}^{1}(\mathbf{q}; t, t') &= \frac{i \Delta^2}{8\hbar^3} \sum_{\alpha'} \sigma_{\alpha'} \int \frac{d\mathbf{p}_1}{(2\pi)^3} \int \frac{d\mathbf{p}_2}{(2\pi)^3} \int_{C^-} dt'' \int_{C^+} dt''' \\
&\times \text{Tr} \left[ \tau_0 G(\mathbf{q} + \mathbf{p}_1; t, t'') \tau_{\alpha'} G(\mathbf{q} + \mathbf{p}_2; t'''', t') \tau_0 G(\mathbf{p}_1; t'''', t) \right].
\end{align*}
\] (A.10)

Before we proceed, we state some rules for calculus involving the Keldysh contour. From an equation like

\[
A(t, t') = \int_{C^-} dt'' B(t, t'') C(t'', t') ,
\] (A.11)

we find the analytic pieces as

\[
A^\pm(t, t') = \int_{-\infty}^{\infty} dt'' B^{(\pm)}(t, t'') C^\pm(t'', t') + \int_{-\infty}^{\infty} dt'' B^\pm(t, t'') C^{(\mp)}(t'', t') ,
\] (A.12)

where the retarded and advanced components are defined in Eq. 20. It is then also straightforward to show that

\[
A^{(\pm)}(t, t') = \int_{-\infty}^{\infty} dt'' B^{(\pm)}(t, t'') C^{(\pm)}(t'', t') .
\] (A.13)
Using these rules we find from Eq. (A.10)

\[
\Pi_{ab}^{1(\pm)}(q, \omega) = -\frac{\Delta^2}{8} \sum_{a' \in \{0, x, y, z\}} \sigma_{a'} \int \frac{d\epsilon}{(2\pi)} \int \frac{d\epsilon'}{(2\pi)} \int \frac{dp_1}{(2\pi)^3} \int \frac{dp_2}{(2\pi)^3} \frac{1}{\hbar \omega - \epsilon + \epsilon'} 
\times \text{Tr} \left\{ \left[ \tau_a G^{(+)}(q + p_1; \epsilon) \tau_{a'} G^{>} (q + p_2; \epsilon) + \tau_a G^{>}(q + p_1; \epsilon) \tau_{a'} G^{(-)}(q + p_2; \epsilon) \right] \right.
\left. \left[ \tau_a G^{(+)}(p_2; \epsilon') \tau_{a'} G^{<}(p_1; \epsilon') + \tau_a G^{<}(p_2; \epsilon') \tau_{a'} G^{(-)}(p_1; \epsilon') \right] \right\}.
\]

(A.14)

The Keldysh component reads

\[
\Pi_{ab}^{1K}(q, \omega) = \frac{i\pi \Delta^2}{4} \sum_{a' \in \{0, x, y, z\}} \sigma_{a'} \int \frac{d\epsilon}{(2\pi)} \int \frac{d\epsilon'}{(2\pi)} \int \frac{dp_1}{(2\pi)^3} \int \frac{dp_2}{(2\pi)^3} \delta(\hbar \omega - \epsilon + \epsilon') 
\times \text{Tr} \left\{ \left[ \tau_a G^{(+)}(q + p_1; \epsilon) \tau_{a'} G^{>} (q + p_2; \epsilon) + \tau_a G^{>}(q + p_1; \epsilon) \tau_{a'} G^{(-)}(q + p_2; \epsilon) \right] \right.
\left. \left[ \tau_a G^{(+)}(p_2; \epsilon') \tau_{a'} G^{<}(p_1; \epsilon') + \tau_a G^{<}(p_2; \epsilon') \tau_{a'} G^{(-)}(p_1; \epsilon') \right] \right\} + \left[ \tau_a G^{(+)}(q + p_1; \epsilon) \tau_{a'} G^{<} (q + p_2; \epsilon) + \tau_a G^{<}(q + p_1; \epsilon) \tau_{a'} G^{(-)}(q + p_2; \epsilon) \right] \left[ \tau_a G^{(+)}(p_2; \epsilon') \tau_{a'} G^{<}(p_1; \epsilon') + \tau_a G^{<}(p_2; \epsilon') \tau_{a'} G^{(-)}(p_1; \epsilon') \right].
\]

(A.15)

We focus now on the imaginary part of \(\Pi_{xx}^{1(\pm)}(q, \omega) = \Pi_{yy}^{1(\pm)}(q, \omega)\) since these determine the Gilbert damping constant. Carrying out the momentum and energy integrals results in

\[
\Pi_{xx}^{1(\pm)}(q, \omega) = \mp \frac{i\pi \Delta^2 \hbar \omega}{8} \left[ n_i \nu_i^\alpha \nu_i^\beta - n_s \nu_s^\alpha \nu_s^\beta \right] \frac{\hbar^2}{M^2}
\]

(A.16)

The corresponding Keldysh part is given by

\[
\Pi_{xx}^{1K}(q, \omega) = \frac{\pi i \Delta^2 k_B T}{2} \left[ n_i \nu_i^\alpha \nu_i^\beta - n_s \nu_s^\alpha \nu_s^\beta \right] \frac{\hbar^2}{M^2}
\]

(A.17)

Adding the results from Eqs. (A.6) and (A.16) we get the result presented in Eq. (11). Similarly, adding the result in Eq. (A.8) to Eq. (A.17) reproduces Eq. (12).
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