Geometric counter-vertex for open string scattering on D-branes

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**Abstract**

In arXiv:0801.0218 [hep-th] it was conjectured that quantum effects of open strings moving on D-branes generate the D-brane geometry through a counter-vertex operator. The conjecture has been checked at one-loop in arXiv:0806.3330 [hep-th]. Here we discuss the two-loop extension.
1 Introduction

Before the birth of the D-brane physics [1, 2], an open string was a rather subsidiary object with a closed string at the center of attention, both in theory and phenomenology. The advent of the D-brane physics has brought a shift in importance toward the open string: the open string has become to play much more important roles in string theory. What makes it tempting to consider a further shift is the potential new physics that is associated with the end points of an open string which might have not been fully appreciated. On a corner of the moduli space the end points of an open string may stick together, thereby converting it into a closed string [3, 4, 5, 6]. This seems to suggest unification of an open string and a closed string at the level of degrees of freedom. It may be technically challenging but should be a worthwhile endeavor to try to realize a closed string as some sort of composite or bound state of an open string in the frame-work of purely open string degrees of freedom.

An additional motivation for attempting a further shift comes from the recent works [7, 8, 9] where it was conjectured that quantum effects of open strings moving on D-branes generate the D-brane geometry through a counter-vertex operator. The divergence structures were analyzed for the scattering amplitudes of massless open strings. (Previous works on scattering that involve a D-brane can be found e.g., in [10]. Also the works of [11, 12] are more closely related to [7].) For the analysis, first the massless vertex operators were constructed for the external open strings on the D-branes. One thing note-worthy is that the closure under super symmetry transformation requires the momenta of the scattering states to be solely along the brane directions. (It should be possible to understand this on more physical grounds.) Because of this constraint, one expects that the loop effects and the analysis thereof will go differently from those of a D9-brane since parts of the momentum degrees of freedom are lost. Indeed, that is what we have observed. It was anticipated that it should be possible to remove the divergences by a composite operator. The composite operator is, in essence, the non-linear sigma model action for the corresponding D-brane geometry [13][14][15]. The anticipation was verified at one-loop in the subsequent work [9]. In this work, we set out to establish the two-loop extension of the conjecture.

The task consists of two parts. First one must obtain the two-loop divergence structures. With those available one can carry out a check whether the counter vertex produces suitable amplitudes to cancel the divergences. The two-loop amplitudes were fully calculated only recently. It was first done in the RNS formulation [18][19, 20] and later in the pure spinor formulation [21]. (For the bosonic case see also [22, 23].) The connection with the geometry can, a priori, be such that the counter vertex operator

\footnote{We thank Ehsan Hatefi for bringing the papers in [11] to our attention.}

\footnote{In a broader sense, a connection between the loop effects and geometry goes back to the Fischler-Susskind mechanism [16]. Also the result of [17] can be interpreted in the context of the present paper. For the scattering of massless states the D-branes geometry is that of the extremal case. At least that is what the results so far indicate.}
may depend on the scattering states and the number of loops. However, the results so far indicate that things may work much more remarkable way: the counter vertex operator given in (1) may be the master operator that accounts for any and all the scattering amplitudes, at least for the massless cases.

We would like to make a few remarks before getting into the detailed analysis. Firstly, compared with the previous work, the new ingredient for the two-loop analysis is an introduction of the multiplicative renormalization of the external vertex operators. We also introduce the string tension renormalization for future purposes though it is not necessary for the amplitudes that we consider at the given order, i.e., the two-loop four point vector and scalar amplitude. Once one considers various other amplitudes, which is one of the near-future tasks that we will work on, one may need such renormalization. The second remark is that we minimize the use of T-duality but rather rely on direct computations. That is because we believe that there are subtle issues with T-duality. The issues may have something to do with the fact that the open string states on the D-branes do not have transverse components of their momenta. We postpone the discussion on this until the conclusion. Lastly, when we consider an open string on a Dp brane with $p < 9$, it seems that one doesn’t have to consider the non-planar graphs. For a non-planar graph at least one of the external state must momentarily leave the brane, which would violate the boundary conditions. There will be more on this in the conclusion.

The organization of the paper is as follows. In sec2, we explain the strategy and summarize the results. After a brief review of the one-loop divergence cancellation, we write down and examine the two-loop four point open string amplitude. It can be obtained by inspecting the amplitude for the closed string which has been obtained in the literature in different formulations. It is noted that the (most) divergent part of the amplitude comes with an overall factor of $s + t + u$ where $s, t, u$ are the Mandelstam variables, hence vanishes. Therefore it seems that the two-loop counter vertex operator should not generate any divergence since there is no divergence to cancel against in the first place. In sec3, we confirm this by explicit computations up to the following point. It turns out that some of the vertices in the counter vertex operator do produce divergences. However, the forms of the divergences are such that they have precisely the same forms as the tree level ones. That makes it possible for them to be absorbed by renormalization of the external vertex operators, as shown in sec2. In sec3, we verify by explicit computations that the forms of the divergences are the same as those of the tree level ones. In sec4, we discuss various issues that include some three-loop perspectives and future directions.

3The renormalization of the external vertex seems in the same spirit in the work by S. Weinberg, [24].
2 Strategy and summary of results

We start with a brief review of the one-loop case. As in the previous works [7, 8, 9], we use the light-cone formulation. More details of the conventions and some simpler calculations can be found in the same references. As well-known the formulation is not suited for two- or higher loop computations. For that, one needs to use the RNS formulation [19] or the pure spinor formulation [21]. On the contrary, it is well-suited for our purpose which is to compute the tree level correlators with the counter vertex operator inserted. It is rather awkward that we need to employ more than one formulation: one for computing the tree diagrams with counter vertex inserted and another for computing the loop diagrams. The benefit is that it is presumably the easiest in light-cone formulation to write down the form of the counter vertex operator. It is also straightforward to compute various tree level correlators with counter vertices inserted.

In the one-loop case one can use the light-cone formulation, both for computing the one-loop diagram, \( < VVVV >_{1\text{loop}} \), and the tree diagram with the counter vertex operator, \( V_G \), inserted, \( < VVVV V_G >_{\text{tree}} \). The subscript \( G \) indicates its connection to geometry. The geometry vertex operator [9] before any expansion \(^4\) is given by

\[
V_G = \int (-)^{1/2} \sqrt{h} h^{ij} \left[ \partial_i X^u \partial_j X^v \eta_{uv} (H^{-1/2} - 1) + \partial_i X^m \partial_j X^n \eta_{mn} (H^{1/2} - 1) \right] + \frac{1}{2p^+} \left[ - 2i(\sqrt{h} h^{ij} - \varepsilon^{ij}) \partial_i X^+ H^{-1/4} (S \partial_j S) + \frac{i}{4}(\sqrt{\epsilon} h^{ij} - \varepsilon^{ij}) \partial_i X^+ H^{-7/4} \frac{H'}{r} \partial_j X^u X^m (S \gamma^{um} S) - \frac{i}{4}(\sqrt{h} h^{ij} - \varepsilon^{ij}) \partial_i X^+ H^{-5/4} \frac{H'}{r} \partial_j X^m X^n (S \gamma^{mn} S) \right] + \frac{1}{4(p^+)^2} \sqrt{h} h^{ij} \partial_i X^+ \partial_j X^+ H^{-1/2} \left[ - \frac{17}{1536} \kappa_1 (S \gamma^{uv} S) (S \gamma^{uv} S) + \left( \frac{43}{768} \kappa_1 + \frac{1}{192} \kappa_2 \right) (S \gamma^{su} S) (S \gamma^{su} S) - \left( \frac{1}{192} \kappa_2 + \frac{1}{128} \kappa_1 \right) (S \gamma^{ab} S) (S \gamma^{ab} S) + X^a X^b \frac{1}{r^2} \left( \frac{31}{768} \kappa_1 - \frac{1}{32} \kappa_2 \right) (S \gamma^{au} S) (S \gamma^{bu} S) + X^a X^b \frac{1}{r^2} \left( \frac{1}{32} \kappa_2 + \frac{29}{384} \kappa_1 \right) (S \gamma^{ac} S) (S \gamma^{bc} S) \right] \tag{1}
\]

\(^4\) As commented in some of the previous works, e.g., [9], a correlator with the series inserted is likely to terminate after some orders in an expansion that we call a large-\( r_0 \) expansion [8, 9]. For the scalar scattering, it does terminate due to dimensional regularization. For the vector scattering there is a general tendency that a correlator with many fields inserted vanishes due to one or more reasons presented right above sec3.1.
where $h^{ij}$ is the world-sheet metric $\eta^{ij} = \text{Diag}(-1, 1)$ and

$$H = 1 + \frac{4\pi g^2 \alpha'^2}{r^4}, \quad \kappa_1 = H^{-5/2}(H')^2, \quad \kappa_2 = H^{-3/2}H'\frac{1}{r}$$

In $H$, we have used the open string coupling square, $g^2$, in the place of the more commonly used closed string coupling, $g_c$. About the notations, the ten dimensional bosonic coordinate, $X^M$, consists of

$$X^M = (X^\pm, X^u, X^m)$$

where $u$ runs in the two world-volume directions and $m$ or $a$ runs in the six transverse directions with $r^2 \equiv \sum_m (X^m)^2$. (We do not distinguish the index $a$ from the index $m$.) Below often we use the following abbreviations

$$A = (u, m)$$

The fermionic coordinate, $S$, is the re-scaled coordinate of the standard fermionic coordinate, $\theta$. We refer to [9, 26] for more details on the notations.

The individual counter vertex results from the expansion of $e^{-\frac{1}{2}TV_G}$ where $T$ is the string tension. We define

$$V_G \equiv V_{G, g^2} + V_{G, g^4} + \ldots$$

$$e^{-\frac{1}{2}TV_G} = 1 - \frac{1}{2}TV_G + \frac{1}{8}T^2V_G^2 + \ldots$$

For the two-loop order, we use the same strategy: first we compute a disc with two holes and isolate the divergence from it. The difference is that one cannot use the light-cone formulation in order to compute the diagram. However, the tree amplitudes with the counter vertex operator inserted can still be computed in the light-cone formulation. As a matter of fact, the light-cone Green-Schwarz seems the most convenient for that particular purpose. Fortunately two-loop closed string amplitudes have been computed recently in other formulations [18][20][21]. The measure for the open string amplitudes can be deduced from the closed string ones by noting that an open string surface is related to the corresponding closed string surface by a simple identification procedure. Including the measure, a two-loop four-point vector amplitude, for example, in the hyperelliptic parametrization should take a form of

$$A(1, 2, 3, 4) = K_0 \int \frac{da_1 da_2 da_3}{a_{12}a_{23}a_{31}} \frac{1}{T^6} \prod_{i=1}^4 \frac{dz_i}{y(z_i)} \prod_{i<j} \exp[-k_i \cdot k_j G(z_i, z_j)]$$

$$[s(z_1 z_2 + z_3 z_4) + t(z_1 z_4 + z_2 z_3) + u(z_1 z_3 + z_2 z_4)]$$

5In later sections, we also use the notation,

$$V_G \equiv V_{G, q} + V_{G, q^2} + \ldots$$

In other words, $V_{G, q^n} \equiv V_{G, q^{2n}}$ with $q$ defined in (28).
The result is valid up to an overall numerical constant. $K_0$ is the usual tree-level kinematic factor defined e.g., in (7.4.42) of [26]. The variables, $a_i$, are the moduli parameters of the hyperelliptic representation of the two-loop diagram. We refer to [18]) (and the references therein) for the definitions of $T$ (not to be confused with the string tension $T$) and other quantities. Based on a survey of literatures (e.g.,[18, 19, 22, 25]) we expect that the divergence will presumably occur only when the locations of all four vertex operators coincide. It implies then that the amplitude becomes factorized to contain $s + t + u$ as an overall factor, which vanishes for massless states: the two-loop amplitude seems finite unlike the one-loop amplitude. If there were a divergence we would have tried to cancel it with $< VVVV V_{G,g^4} >_{\text{tree}}$ where $V_{G,g^4}$ is the quartic order term in the coupling constant expansion of $V_G$, which is the order in which the coupling constant appears for the two loop diagrams. Since there is no divergence to cancel against, the counter vertex operator should not generate any divergence or at least any new form of it. If it does not generate any divergence the ”renormalization” procedure will be simpler than otherwise. It turns out that it does generate divergences but fortunately they come with the same kinematic factors as the tree amplitudes. As we discuss now they can be absorbed by re-scaling the external vertex operators. String tension renormalization may be required as well in general.

At two-loop order, there are two types of the counter-vertex terms. Schematically they can be written as

$$(V_{G,g^2})^2 \oplus V_{G,g^4}$$

where $\oplus$ indicates that the precise relative coefficients are not being recorded. They are obtained by the large-$r_0$ expansion of Eq.(1). Various vertices in $(V_{G,g^2})^2$ and $V_{G,g^4}$ contain terms of the types $\int d^2z \,(\cdots)$ and $\int d^2z \,(\cdots) \int dz' (\cdots)$. For the single integral terms, we use a prescription where one freely performs integration by parts, and only at the final moment one replaces $\int d^2z \rightarrow \int dy$. For the terms that come with $\int d^2z \,(\cdots) \int dz' (\cdots)$ one may partially integrate, after inserting the external vertex operators, to get a delta function. One of the two $z$-integrations can be removed by the resulting delta function. (More details are presented in the next section.) Then we follow the same prescription that is used for the single integral terms. We show after lengthy algebra in the next section that it is only the $\partial X \cdot \partial X$-terms (or the terms that produce the $X \cdot \partial X$-terms after a delta function is used) that yield non-vanishing results: together with the external vertex operators, the quartic order counter-vertex

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6To be able to see it explicitly it will be necessary to make change of variables from $z_i$-coordinates to $(q, \nu_i)$-coordinates that are used in e.g., section 8.1 of [26].

7As a matter of fact, one should also consider the contribution form $< VVVV V_{G,g^2} >_{1 \text{ loop}}$ to be complete as would be expected in analogy with a normal quantum field theory analysis. This point has been brought to our attention by a few people. The outcome of such computation will constitute another crucial test of our conjecture. Since it is a five-point or more in one-loop order, we will take the task in some place else in the near future.
operator yields
\[ < VVVV [(V_{G,g^2})^2 \oplus V_{G,g^4}] > \propto < VVVV > \int \partial X \cdot \partial X > \]  

(9)

More precisely, we show in the next section the following results for the vector multiplet correlators and the scalar multiplet correlators

\[ < V_v V_v V_v V_v G,g > = -q^2 \frac{3}{16} < V_v V_v V_v V_v \int_{x_1}^{\infty} (\partial_i X^u \partial_i X^u) > \]

\[ < V_v V_v V_v (V_{G,g^2})^2 > = q^2 \frac{8}{16} < V_v V_v V_v \int_{x_1}^{\infty} (\partial_i X^u \partial_i X^u) > \]

\[ < V_s V_s V_s V_s G,g^4 > = q^2 \frac{16}{16} < V_s V_s V_s V_s \int_{x_1}^{\infty} (\partial_i X^m \partial_i X^m) > \]

\[ < V_s V_s V_s (V_{G,g^2})^2 > = q^2 \frac{8}{16} < V_s V_s V_s \int_{x_1}^{\infty} (\partial_i X^m \partial_i X^m) > \]  

(10)

The definition of the parameter \( q \) can be found in (28). The explanation of the particular form of the integration can be found in, e.g., sec3 of [9]. The RHS’s yield the tree diagram kinematic factors with additional divergence factors. These divergences can be absorbed in the following way. In our convention the free part of the action is given by

\[ - \frac{T}{2} \int d^2 \sigma \partial X \partial X \]  

(11)

The free fermionic pieces have been omitted: as in the one-loop case [9], they only lead to vanishing results. (See also the discussion below (32).) For the amplitudes that we consider at the given order, it is not necessary to perform string tension renormalization. However, we still consider it just in case we need when computing other and/or higher order amplitudes in the near future: we write

\[ T = T_0 + T_1 g^2 + T_2 g^4 + ... \]  

(12)

We also introduce the vertex operator renormalization

\[ V \Rightarrow \mathcal{N}_V V \quad \text{where} \quad \mathcal{N}_V = 1 + \mathcal{N}_V^{(1)} g^2 + \mathcal{N}_V^{(2)} g^4 + ... \]  

(13)

where \( \mathcal{N}_V \equiv \mathcal{N}_v \) for the vector operator and \( \mathcal{N}_V \equiv \mathcal{N}_s \) for the scalar operator. The counter correlator is given by

\[ < (\mathcal{N}_V^4)^4 VVVV e^{-\frac{T_0}{2}} \int d^2 \sigma \partial X \partial X e^{-\frac{T}{2} V_G} > \]  

(14)

\[ \text{8The the vector multiplet, } V_v, \text{ and the scalar multiplet, } V_s, \text{ were obtained in [7] and are quoted for convenience in one of the appendices.} \]
Writing
\[ V_G = V_{G,g^2} + V_{G,g^4} + \ldots \] (15)
and focusing on the exponential terms one gets
\[
(N_V)^4 e^{-\frac{T-T_0}{2}} \int d^2\sigma \partial X \partial X e^{-\frac{T}{2}V_G} - 1
= -\frac{1}{2} \left( T_1 g^2 \int \partial X \partial X + T_0 V_{G,g^2} \right) + 4N_V^{(1)} g^2
-\frac{1}{2} \left( T_2 g^4 \int \partial X \partial X + T_1 g^2 V_{G,g^2} + T_0 V_{G,g^4} \right) + \frac{1}{8} \left( T_1 g^2 \int \partial X \partial X + T_0 V_{G,g^2} \right)^2 + \ldots
+ 4N_V^{(1)} g^2 \left( -\frac{1}{2} \right) \left( T_1 g^2 \int \partial X \partial X + T_0 V_{G,g^2} \right) + (6[N_V^{(1)}]^2 + 4N_V^{(2)}) g^4
\] (16)
The first line should cancel the one-loop divergence. It implies that
\[ \frac{g^2}{4} \frac{1}{\epsilon'} + \frac{1}{2} \frac{1}{\epsilon} \left( T_1 g^2 - \frac{qT_0}{2} \right) + 4N_V^{(1)} g^2 = 0 \] (17)
where the first term is the one-loop divergence. The symbol, \( \epsilon' \), is the cutoff that was used in the one-loop computation [9]. (It is called \( \epsilon_y \) in Eq.(25) of [9].) Similarly \( \epsilon \) is associated with the two-loop terms.\(^9\)
The fact that we have assigned a different regulator, \( \epsilon \), for the two loop is matter of convenience: it is to keep track of the one loop and the two loop computations separately. At any point, \( \epsilon' \) can be set \( \epsilon' = \epsilon \). The loop expansion parameter \( q \) is defined as
\[ q = \frac{4\pi g^2 \alpha'^2}{r_0^4} \] (18)
in (28), below which the definition of \( r_0 \) can be found as well. We also have used
\[ < VVVV \partial X \partial X > = -\frac{1}{\epsilon} < VVVV > \] (19)
Since eq(17) is valid both for the vector and the multiplets, it implies
\[ N_V^{(1)} = N_s^{(1)} \equiv N^{(1)} \] (20)
After substituting (18) into (17) one gets
\[ N^{(1)} = -\frac{1}{16} \frac{1}{\epsilon'} - \frac{1}{8} \frac{1}{\epsilon} T_1 + \frac{1}{16} \frac{\pi}{\epsilon} \frac{T_0}{r_0^4} \] (21)
\(^9\)As with our previous work [9], we use dimensional regularization and world-sheet integration cutoff together. It seems that it is not unusual in string theory to use two different regularization methods at the same time. For example, a combined use of normal ordering and string tension renormalization is made in [26]. Although it is not certain at this point, it might be that dimensional regularization should be viewed as a world-sheet regularization whereas the cut-off together with the deformed geometry should be viewed as a space-time renormalization.
The second and third lines of (16) together should remove the two-loop divergence. As we have discussed above, the two-loop amplitudes seem finite. Therefore we require the correlator that involves the second and the third line to vanish when computed with the four external vertex operators inserted. For the vector states it yields

\[
\begin{align*}
& < \left\{ \frac{1}{2} \left( T_2 g^4 \int \partial X \partial X + T_1 g^2 V_{G,g^2} + T_0 V_{G,g^4} \right) \\
& - \frac{1}{8} \left( T_1^2 g^4 \left( \int \partial X \partial X \right)^2 + 2 T_1 g^2 \left( \int \partial X \partial X \right) T_0 V_{G,g^2} + T_0^2 (V_{G,g^2})^2 \right) \\
& - \left[ -2 \mathcal{N}_v^{(1)} g^2 \left( T_1 g^2 \int \partial X \partial X + T_0 V_{G,g^2} \right) + (6 \mathcal{N}_v^{(1)})^2 + 4 \mathcal{N}_v^{(2)} g^4 \right] \right\} V_v V_v V_v V_v > \\
& = 0 \tag{22}
\end{align*}
\]

One gets after using (10) and the results of [8, 9]

\[
\kappa \left[ g^4 T_2 - \frac{q}{2} g^2 T_1 - \frac{q^2}{16} T_0 - \frac{1}{4} \left( -2 c g^4 T_1^2 + 2 c g^2 T_1 T_0 - \frac{g^2}{8} T_0^2 \right) \\
+ 4 \mathcal{N}_v^{(1)} g^2 \left( T_1 g^2 - \frac{q}{2} T_0 \right) \right] + 2 (6 \mathcal{N}_v^{(1)})^2 + 4 \mathcal{N}_v^{(2)} g^4 = 0 \tag{23}
\]

Substituting (18) and dividing by \( g^4 \) it simplifies to

\[
- \frac{1}{\epsilon} \left[ T_2 - \frac{\pi^2}{2 r_0^2} T_1 - \frac{3 \pi^2}{16 r_0^2} T_0 - \frac{1}{2} T_1^2 + \frac{1}{2} \frac{\pi}{r_0^2} T_1 T_0 - \frac{\pi^2}{32 r_0^2} T_0^2 \\
+ 4 \mathcal{N}_v^{(1)} \left( T_1 - \frac{\pi}{2 r_0^2} T_0 \right) \right] + 12 \mathcal{N}_v^{(1)}^2 + 8 \mathcal{N}_v^{(2)} = 0 \tag{24}
\]

Similarly, requiring

\[
< \left\{ \frac{1}{2} \left( T_2 g^4 \int \partial X \partial X + T_1 g^2 V_{G,g^2} + T_0 V_{G,g^4} \right) \\
- \frac{1}{8} \left( T_1^2 g^4 \left( \int \partial X \partial X \right)^2 + 2 T_1 g^2 \left( \int \partial X \partial X \right) T_0 V_{G,g^2} + T_0^2 (V_{G,g^2})^2 \right) \\
- \left[ -2 \mathcal{N}_s^{(1)} g^2 \left( T_1 g^2 \int \partial X \partial X + T_0 V_{G,g^2} \right) + (6 \mathcal{N}_s^{(1)})^2 + 4 \mathcal{N}_s^{(2)} g^4 \right] \right\} V_s V_s V_s V_s > \\
= 0 \tag{25}
\]

for the scalar states leads to

\[
- \frac{1}{\epsilon} \left[ T_2 - \frac{\pi^2}{2 r_0^2} T_1 + \frac{\pi^2}{16 r_0^2} T_0 - \frac{1}{2} T_1^2 + \frac{1}{2} \frac{\pi}{r_0^2} T_1 T_0 - \frac{\pi^2}{32 r_0^2} T_0^2 \\
+ 4 \mathcal{N}_s^{(1)} \left( T_1 - \frac{\pi}{2 r_0^2} T_0 \right) \right] + 12 \mathcal{N}_s^{(1)}^2 + 8 \mathcal{N}_s^{(2)} = 0 \tag{26}
\]
By subtracting (26) from (24) one gets
\[ -\mathcal{N}_s^{(2)} + \mathcal{N}_v^{(2)} - \frac{1}{32} \pi^2 \varepsilon r_0^8 T_0 = 0 \]  
(27)
Therefore it is verified that the one- and two-loop divergences can be absorbed by the operator renormalization relations (20), (21), (24), (26) and (27).

3 Two-loop counter terms

In this section, we prove the results given in (10). The geometry vertex operator, \( V_G \), given in (1) is the unique counter vertex operator in the sense that the form does not depend on the external scattering states and the number of loops. This is true at least for the correlators that we have computed so far, which is rather remarkable. Below we expand \( V_G \) in terms of two parameters, \( g, r_0 \). The parameter \( q \) is defined as
\[ q = \frac{4\pi g^2 \alpha^2}{r_0^4} \]  
(28)
where \( r_0^2 = \sum_m (X_0^m)^2 \) with \( X_0 \) appearing in the shift, \( X^m \to X^m + X_0^m \) [9]. It counts the number of loops among other things. Since we are interested in the two-loop amplitudes, we select the terms with \( q^2 \). As for the \( r_0 \), we keep the terms up to (and including) \( \frac{1}{r_0^4} \) order. With the expanded vertex operator inserted we consider in the next section two scattering amplitudes: the four vector scattering amplitude and the four scalar scattering amplitude. As emphasized in the previous works (e.g., [9]), the computations are at the tree level.

There are two terms that appear in (15) at this order: the q-order vertex operator, \( V_{G,q} \), and the \( q^2 \)-order vertex operator, \( V_{G,q^2} \). One can show by straightforward algebra that they are respectively given by\textsuperscript{10}
\[ V_{G,q} = q \int \left( -\frac{1}{2} \sqrt{h} h^{ij} \left[ \partial_i X^u \partial_j X^v \eta_{uv} \left( -\frac{1}{2} + \frac{2 X_0 \cdot X}{r_0^2} + \frac{r^2}{r_0^2} - \frac{6 (X_0 \cdot X)^2}{r_0^4} \right) \right] \right. \]
\[ \left. + \partial_i X^m \partial_j X^n \eta_{mn} \left( \frac{1}{2} - \frac{2 X_0 \cdot X}{r_0^2} + \frac{6 (X_0 \cdot X)^2}{r_0^4} - \frac{r^2}{r_0^2} \right) \right) \]
\[ + \frac{1}{2p^+} (\sqrt{h} h^{ij} - \varepsilon^{ij}) \partial_i X^{+} \left[ -2i(S \partial_j S) \left( -\frac{1}{4} + \frac{X_0 \cdot X}{r_0^2} + \frac{r^2}{2r_0^2} - \frac{3 (X_0 \cdot X)^2}{r_0^4} \right) \right] \]
\[ - i \partial_j X^u (X + X_0)^m (S \gamma^{um} S) + i \partial_j X^m (X + X_0)^n (S \gamma^{mn} S) \]
\[ + \frac{1}{4(p^+)^2} \sqrt{h} h^{ij} \partial_i X^{+} \partial_j X^{+} \left[ 1 \right. \]
\[ \left. + \frac{1}{48} (S \gamma^{au} S)(S \gamma^{au} S) + \frac{1}{48} (S \gamma^{ab} S)(S \gamma^{ab} S) \right. \]
\[ + \frac{1}{8r_0^2} (X + X_0)^a (X + X_0)^b \left. (S \gamma^{ab} S)(S \gamma^{ab} S) - \frac{1}{8} (S \gamma^{ac} S)(S \gamma^{bc} S) \right) \]  
(29)
\textsuperscript{10}As mentioned in one of the previous footnotes, our notation is such that \( V_{G,q^n} \equiv V_{G,q^{2n}} \).
\[ V_{G,q^2} = q^2 \int (-\frac{1}{2}) \sqrt{h} h^{ij} \left[ \partial_i X^u \partial_j X^v \eta_{uv} \left( \frac{3}{8} - \frac{3 X_0 \cdot X}{r_0^2} - \frac{3 r^2}{2 r_0^2} + \frac{15 (X_0 \cdot X)^2}{r_0^4} \right) \\
+ \partial_i X^m \partial_j X^n \eta_{mn} \left( -\frac{1}{8} + \frac{X_0 \cdot X}{r_0^2} + \frac{r^2}{2 r_0^2} - \frac{5 (X_0 \cdot X)^2}{r_0^4} \right) \right] \\
+ \frac{1}{2p^+} (\sqrt{h} h^{ij} - \delta^{ij}) \partial_i X^+ \left[ -2i(S \partial_j S) \left( \frac{5}{32} - \frac{5 X_0 \cdot X}{4 r_0^2} + \frac{25 (X_0 \cdot X)^2}{8 r_0^4} - \frac{5 r^2}{8 r_0^2} \right) \\
+ \frac{7i}{4} \partial_j X^u \frac{(X + X_0)^m}{r_0^2} (S \gamma^{um} S) - \frac{5i}{4} \partial_j X^m \frac{(X + X_0)^n}{r_0^2} (S \gamma^{mn} S) \right] \\
+ \frac{1}{4(p^+)^2} \sqrt{h} h^{ij} \partial_i X^+ \partial_j X^+ \frac{1}{r_0^2} \left[ - \frac{17}{96} (S \gamma^{uv} S) (S \gamma^{uv} S) + \frac{45}{48} (S \gamma^{au} S) (S \gamma^{au} S) \\
- \frac{1}{6} (S \gamma^{ab} S) (S \gamma^{ab} S) \right] \\
+ \frac{(X + X_0)^a}{r_0^2} (X + X_0)^b \left( \frac{19}{48} (S \gamma^{au} S) (S \gamma^{ba} S) + \frac{35}{24} (S \gamma^{ac} S) (S \gamma^{bc} S) \right) \right] \]  
(30)

The task now is to compute the four correlators: \( < V_\nu V_\nu V_\nu (V_{G,q^2}) >, \ < V_\nu V_\nu V_\nu (V_{G,q^2})^2 > \), \( < V_\nu V_\nu V_\nu V_\nu (V_{G,q^2}) > \) and \( < V_\nu V_\nu V_\nu V_\nu (V_{G,q^2})^2 > \). There are some terms in (29) and (30) that do not contribute to any of these correlators for obvious reasons. Below we will explain them with specific examples. In the actual computations with the individual vertices in (29) and (30), many of them turn out to vanish for certain reasons. Some of the contractions need not be executed as dimensional regularization is being used. Some contractions produce delta functions due to the relation,

\[ < \partial^2 X(x) X(x') > \sim \delta(x - x') \]  
(31)

If both of the arguments are, for example, those of the external states, the contraction will vanish since the locations of the external vertex operators are such that \( x_1 > 1 > x > 0 \). The only potentially non-zero contractions come from the following sets,

\[ (x, x') = (x_1, y) , \ (y, x_1) , \ (y, y') \]  
(32)

where the \( y \)'s are the arguments for the geometry vertices. A delta function also appears when \( (\sqrt{h} h^{ij} - \delta^{ij}) \partial_j S \) contracts with another \( S \). It is why, as in the one-loop case, the \( S \partial_j S \)-terms in (29) and (30) may be dropped: to avoid a vanishing delta both factors of \( S \) need to contract with the fermionic part of the external vertex operator, \( S(x_1) \gamma^{ij} S(x_1) \). It yields \( \text{Tr} \gamma^{A_1 B_1} = 0 \): henceforth the \( S \partial S \)-term will be omitted. Some of the terms appear with \( \frac{1}{x_{1 i}^2} \) or higher power. They vanish since the measure contains only \( x_{1 i}^2 \) and it is taken \( x_1 \to \infty \) at the end. There are other terms that have an overall factor of either \( k_1^2, k_1 \cdot \xi_1 \), or \( s + t + u \). They vanish due to the masslessness of the external states, the polarization constraint, or momentum conservation. Below we examine each of the terms in (29) and (30) together with the four external vertex operators. As just described, the majority of terms vanish due to one or more of the following reasons:
3.1 Four vector scattering

For convenience, we record the explicit form of the product of four vector vertex operators,

\[
V_v^{u_1}(x_1)V_v^{u_2}(x_2)V_v^{u_3}(x_3)V_v^{u_4}(x_4)
= \hat{X}^{u_1} X^{u_2} X^{u_3} X^{u_4} + i^8 R^{u_1 u_2} k_1^u R^{u_2 u_3} k_2^v R^{u_3 u_4} k_3^u R^{u_4 u_1} k_4^v
\]

\[
- l^2 \left[ \hat{X}^{u_1} X^{u_2} X^{u_3} R^{u_4 u_4} k_4^v + \hat{X}^{u_1} X^{u_2} X^{u_4} R^{u_4 u_3} k_3^v + \hat{X}^{u_2} X^{u_3} X^{u_4} R^{u_4 u_1} k_1^v \right]
+ \hat{X}^{u_1} X^{u_2} R^{u_3 u_3} k_2^v R^{u_4 u_3} k_4^v + \hat{X}^{u_3} X^{u_4} R^{u_1 u_1} k_1^v R^{u_2 u_2} k_2^v
\]

\[
+ \hat{X}^{u_2} X^{u_3} R^{u_1 u_1} k_1^v R^{u_4 u_4} k_4^v + \hat{X}^{u_3} X^{u_4} R^{u_1 u_1} k_1^v R^{u_3 u_3} k_3^v
+ \hat{X}^{u_1} X^{u_2} R^{u_3 u_3} k_2^v R^{u_4 u_4} k_4^v + \hat{X}^{u_2} X^{u_3} R^{u_1 u_1} k_1^v R^{u_4 u_4} k_4^v
\]

\[
- l^2 \left[ \hat{X}^{u_1} X^{u_2} X^{u_3} R^{u_4 u_4} k_4^v + \hat{X}^{u_1} X^{u_2} X^{u_4} R^{u_4 u_3} k_3^v + \hat{X}^{u_2} X^{u_3} X^{u_4} R^{u_4 u_1} k_1^v \right]
+ \hat{X}^{u_1} X^{u_2} R^{u_3 u_3} k_2^v R^{u_4 u_3} k_4^v + \hat{X}^{u_3} X^{u_4} R^{u_1 u_1} k_1^v R^{u_2 u_2} k_2^v
\]

\[
+ \hat{X}^{u_2} X^{u_3} R^{u_1 u_1} k_1^v R^{u_4 u_4} k_4^v + \hat{X}^{u_3} X^{u_4} R^{u_1 u_1} k_1^v R^{u_3 u_3} k_3^v
+ \hat{X}^{u_1} X^{u_2} R^{u_3 u_3} k_2^v R^{u_4 u_4} k_4^v + \hat{X}^{u_2} X^{u_3} R^{u_1 u_1} k_1^v R^{u_4 u_4} k_4^v
\]

\[
(33)
\]

We loosely refer to each type of these terms as XXXX, RRRR, XXXR, XXRR, and XRRR respectively. The parameter \( l \) is the standard one given by \( l = \sqrt{2\alpha'} \). In some places we set \( l = 1 \).

3.1.1 \( V_{G,q^2} \) contribution

For the four vector scattering the vertex \( V_{G,q^2} \) given in (30) simplifies to

\[
V_{G,q^2} = q^2 \int \left[ - \frac{3}{16} \sqrt{h} h^{ij} \partial_i X^u \partial_j X^v \eta_{uv} - \frac{5}{32} p^+ (\sqrt{h} h^{ij} - \varepsilon^{ij}) \partial_i X^+ (S \partial_j S) \right.
\]

\[
+ \frac{1}{4(p^+)^2 r_0^2} \sqrt{h} h^{ij} \partial_i X^+ \partial_j X^+ \left[ \right.
- \frac{17}{96} (S_{\gamma^{au}} S) (S_{\gamma^{uv}} S) + \frac{15}{16} (S_{\gamma^{au}} S) (S_{\gamma^{au}} S) - \frac{1}{6} (S_{\gamma^{ab}} S) (S_{\gamma^{ab}} S)
\]

\[
+ \frac{19}{48} \frac{X_0^a X_0^b}{r_0^2} (S_{\gamma^{au}} S) (S_{\gamma^{bu}} S) + \frac{35}{24} \frac{X_0^a X_0^b}{r_0^2} (S_{\gamma^{ac}} S) (S_{\gamma^{bc}} S) \left. \right] \]

(34)
This is due to the fact that all the terms that contain $X^m$ drop due to dimensional regularization. Also some of the terms were shown to vanish in the one-loop analysis [9]. Consider

$$< V_v^{u_1}(x_1)V_v^{u_2}(x_2)V_v^{u_3}(x_3)V_v^{u_4}(x_4); V_{G,q^2} > $$ (35)

The $\partial X \partial X$-term gives the tree level kinematic factor. We now show that none of the other terms makes a contribution. Let’s focus on the terms with four $S$’s:

$$< V_v^{u_1}(x_1)V_v^{u_2}(x_2)V_v^{u_3}(x_3)V_v^{u_4}(x_4); S_{\gamma}^{AB}SS\gamma^{CD}S >$$

$$= \left[ t^4 \left( X^{u_1} X^{u_2} R^{u_3} v_2 k_2 k_3 k_4 + X^{u_3} R^{u_4} v_4 k_1 k_2 k_3 \right) + R^{u_4} v_4 k_1 k_2 k_3 \right]$$

$$= - l^6 \left( X^{u_1} R^{u_2} v_2 k_2 k_3 k_4 + X^{u_2} R^{u_1} v_1 k_1 k_2 k_3 \right)$$

$$+ l^8 R^{u_1} v_1 k_1 k_2 k_3 k_4$$

$$S_{\gamma}^{AB}SS\gamma^{CD}S >$$ (36)

One can show that these $SSSS$-terms yield vanishing expressions. For example, by explicit calculation one can verify that

$$< V_v^{u_1}(x_1)V_v^{u_2}(x_2)V_v^{u_3}(x_3)V_v^{u_4}(x_4); S_{\gamma}^{au}SS\gamma^{bu}S >= 0 $$ (37)

All the other remaining terms turn out to give vanishing expressions. For an illustration, let us take the terms that contain a factor, $(\zeta_2 \cdot \zeta_3)$. One can show after the correlators in (36) are explicitly calculated that the coefficient of $(\zeta_2 \cdot \zeta_3)$ is proportional to

$$\left\{ \left( \frac{q^2}{r_0} \right) \left[ - \frac{17}{96} (S_{\gamma}^{uv})(S_{\gamma}^{uv}) - \frac{1}{6} (S_{\gamma}^{ab})(S_{\gamma}^{ab}) \right] + \left( \frac{q^2}{r_0} \right) X_3 X_0 \left[ \frac{35}{24} (S_{\gamma}^{ac})(S_{\gamma}^{bc}) \right] \right\}$$

$$\Rightarrow - \frac{1}{(1-x)^2} \zeta_1 \cdot k_4 \zeta_4 \cdot k_1 + \left( \zeta_4 \cdot k_2 + \frac{\zeta_4 \cdot k_3}{x} \right) \frac{1}{1-x}(\zeta_1 \cdot k_2 u - \zeta_1 \cdot k_3 s)$$

$$- \frac{1}{(1-x)^2} t \zeta_1 \cdot k_4 \zeta_4 \cdot k_1$$

$$- \frac{1}{2x} (s \zeta_1 \cdot k_2 \zeta_4 \cdot k_1 - t \zeta_1 \cdot k_2 \zeta_4 \cdot k_3 + s \zeta_1 \cdot k_4 \zeta_4 \cdot k_3 - t \zeta_1 \cdot k_4 \zeta_4 \cdot k_1$$

$$+ u \zeta_1 \cdot k_4 \zeta_4 \cdot k_2 + u \zeta_1 \cdot k_3 \zeta_4 \cdot k_1 - t \zeta_1 \cdot k_3 \zeta_4 \cdot k_2$$

$$+ \frac{1}{2x (1-x)} (-s \zeta_1 \cdot k_2 \zeta_4 \cdot k_1 + t \zeta_1 \cdot k_2 \zeta_4 \cdot k_3 + u \zeta_1 \cdot k_3 \zeta_4 \cdot k_1$$

\[1\] We refer to [9] for simpler sample calculations.
\[-t\zeta_1 \cdot k_3 \cdot \zeta_4 \cdot k_2 - s\zeta_1 \cdot k_4 \cdot \zeta_4 \cdot k_3 + u\zeta_1 \cdot k_4 \cdot \zeta_4 \cdot k_3 - t\zeta_1 \cdot k_4 \cdot \zeta_4 \cdot k_1 \]
\[-\frac{1}{2(1-x)}(-u\zeta_1 \cdot k_3 \cdot \zeta_4 \cdot k_1 + t\zeta_1 \cdot k_3 \cdot \zeta_4 \cdot k_2 + s\zeta_1 \cdot k_2 \cdot \zeta_4 \cdot k_1 - t\zeta_1 \cdot k_2 \cdot \zeta_4 \cdot k_3 - u\zeta_1 \cdot k_4 \cdot \zeta_4 \cdot k_2 + s\zeta_1 \cdot k_4 \cdot \zeta_4 \cdot k_3 - t\zeta_1 \cdot k_4 \cdot \zeta_4 \cdot k_1)\]

where the Wick rotation has been taken into account. It vanishes after the \(x\)-integration. All the other type of terms such as \((\zeta \cdot \zeta_1)\) etc and \((\zeta \cdot k)(\zeta \cdot k)(\zeta \cdot k)\) vanish. Additional examples are presented in one of the appendices. Therefore one gets

\[V_{G,q^2} \Rightarrow (-q^2) \int \frac{3}{16} \sqrt{h} \delta^{ij} \partial_i X^\mu \partial_j X^\nu \eta_{uv}\]

(39)

### 3.1.2 \((V_{G,q})^2\) contribution

Here again, the terms with the factor, \(S\partial S\), drop as one can see as follows. The fermionic parts of the correlators that we need to consider contain the contraction of the type

\[S(y)\partial S(y) \ S(y')\partial S(y') \ S(x_i)\gamma^{uv\nu_1} S(x_i)\]

(40)

or

\[S(y)\partial S(y) \ S(y')\gamma^{AB} S(y') \ S(x_i)\gamma^{uv\nu_1} S(x_i)\]

(41)

Let’s consider the first one. When \(\partial S(y)\) and \(S(y')\) contract the remaining contractions yield a vanishing result. Similarly the contraction between \(\partial S(y)\) and \(S(x_i)\) leads to a null result. Consider the second equation. Eventually it is necessary to have \((A, B) = (u, v)\) for a non-vanishing trace of product of \(\gamma\)-matrices. However, such a term is absent in \(V_{G,q}\).

The term with \(\partial_j X^u X^m (S\gamma^{um} S)\) vanishes since it can only be multiplied with \(-\partial_i X^u \partial_j X^\nu \eta_{uv} + \partial_i X^m \partial_j X^\nu \eta_{mn}\), and the number of \(X^m\)’s is odd. For the \(\partial_j X^u X^m_0 (S\gamma^{um} S)\)-term, it is necessary to consider XXXR-, XXRR-, XRRR- and RRRR-terms when it gets multiplied with \(-\partial_i X^u \partial_j X^\nu \eta_{uv} + \partial_i X^m \partial_j X^\nu \eta_{mn}\). The index structures remove all of them. When it gets squared, i.e., \(\partial_j X^u X^m_0 (S\gamma^{um} S) \partial_j^' X^{u'} X^{m'}_0 (S\gamma^{u'm'} S)\), it seems more convenient to partially integrate to put it into \(X^u X^m_0 (\partial_j S\gamma^{um} S) X^{u'} X^{m'}_0 (\partial_j^' S\gamma^{u'm'} S)\).

The fermionic part of the correlators will be of the form

\[(\partial_j S\gamma^{um} S)(\partial_j^' S\gamma^{u'm'} S) S\gamma^{uv\nu_1} S\]

(42)

When \(\partial_j S\) contracts with the next term, the remaining contractions vanish due to dimensional regularization. When it contracts with the third factor, it is equivalent to the previous case unless \((\partial_j S\gamma^{um} S)\) entirely contracts with \(S\gamma^{uv\nu_1} S\), which yields zero.

Consider the terms with \(\partial_j X^m (X + X_0)^n (S\gamma^{mn} S)\). Within the order that we are considering, \(\partial_j X^m X^n\) can only be multiplied with \(\partial_i X^m \partial_j X^\nu \eta_{mn}\). The correlator vanishes due to the index structure: \(m\) and \(n\) must be the same while \(m'\) and \(n'\) must be
different. Consider \( \partial_j X^m X_0^n (S\gamma^{mn} S) \)-term. The only cross term that is relevant is

\[
\partial_j X^{m'} X'_0 (S\gamma^{m'n'} S) \partial_i X^u \partial_j X^v \eta_{uv} \frac{2 X_0 \cdot X}{r_0^2}
\]

(43)

Potentially non-vanishing terms in (33) are of the types, XXXR, XXRR, XRRR, and RRRR. Fermionic index structures kill them all. Consider the diagonal term

\[
\partial_j X^{m'} X'_0 (S\gamma^{m'n'} S) \partial_j X^m X_0^n (S\gamma^{mn} S)
\]

(44)

By partial integration one gets e.g., \( \partial((S\gamma^{m'n'} S)) \). Due to the fermionic index structures, \( \partial((S\gamma^{m'n'} S)) \) can only contract with \( (S\gamma^{mn} S) \) but not with external states. It will produce a delta function. Then the X-correlator vanishes due to dimensional regularization upon using the delta function.

With these results, the vertex operator can be tailored to

\[
V_{G,q} = q \int \left( -\frac{1}{2} \sqrt{h} h^{ij} \left( \partial_i X^u \partial_j X^v \eta_{uv} \left[ -\frac{1}{2} + \frac{2 X_0 \cdot X}{r_0^2} + \frac{r^2}{r_0^2} - \frac{6 (X_0 \cdot X)^2}{r_0^4} \right] \\
+ \partial_i X^m \partial_j X^n \eta_{mn} \left[ \frac{1}{2} - \frac{2 X_0 \cdot X}{r_0^2} + \frac{6 (X_0 \cdot X)^2}{r_0^4} - \frac{r^2}{r_0^2} \right] \right) \\
- \frac{1}{(p^+)^2} \sqrt{h} h^{ij} \partial_i X^+ \partial_j X^+ \frac{1}{r_0^2} \left[ \frac{1}{192} (S\gamma^{au} S)(S\gamma^{au} S) - \frac{1}{192} (S\gamma^{ab} S)(S\gamma^{ab} S) \\
- \frac{1}{32} (S\gamma^{au} S)(S\gamma^{ba} S) \frac{X_0^a X_0^b}{r_0^2} + \frac{1}{32} (S\gamma^{ac} S)(S\gamma^{bc} S) \frac{X_0^a X_0^b}{r_0^2} \right] \right)
\]

(45)
where in the second equality we have collected the same order terms in $\frac{1}{r_0^4}$. Squaring this, one gets, upon dropping some terms that contain the odd number of $X^m$'s,

$$
(V_{G,q})^2 = q^2 \int \int \left[ \frac{1}{4} \sqrt{h} h^{ij'} \sqrt{h} h^{ij} + \frac{1}{4} \left( \partial_\nu X^{\nu'} \partial_{\nu'} X^{\nu'} - \partial_\nu X^{m'\nu'} \partial_{\nu'} X^{m'} \right) \left( \partial_\xi X^{u} \partial_{\xi} X^{u} - \partial_\xi X^{m} \partial_{\xi} X^{m} \right) + \frac{4}{r_0^2} \left( \partial_\nu X^{u} \partial_{\nu} X^{u} - \partial_\nu X^{m} \partial_{\nu} X^{m} \right) \right] - \frac{1}{2} \sqrt{h} h^{ij'} \left( \partial_\nu X^{u} \partial_{\nu} X^{u} - \partial_\nu X^{m} \partial_{\nu} X^{m} \right) \left( \frac{1}{r_0^2} \left( \partial_\xi X^{u} \partial_{\xi} X^{u} - \partial_\xi X^{m} \partial_{\xi} X^{m} \right) \right)
$$

Let’s focus on the third term

$$
-\frac{1}{2} \sqrt{h} h^{ij'} \left( \partial_\nu X^{u} \partial_{\nu'} X^{u} \eta_{\nu' \nu'} - \partial_\nu X^{m} \partial_{\nu'} X^{m} \eta_{\nu' \nu'} \right) + \frac{1}{2} \sqrt{h} h^{ij} \left( \frac{r_0^2}{r_0^4} - \frac{6 (X_0 \cdot X)^2}{r_0^4} \right) \left( \partial_\xi X^{u} \partial_{\xi} X^{u} \eta_{\xi \xi} - \partial_\xi X^{m} \partial_{\xi} X^{m} \eta_{\xi \xi} \right)
$$

$$
-\frac{1}{2} \sqrt{h} h^{ij'} \left( \partial_\nu X^{u} \partial_{\nu} X^{u} - \partial_\nu X^{m} \partial_{\nu} X^{m} \right) \left( \frac{1}{r_0^2} \left( \partial_\xi X^{u} \partial_{\xi} X^{u} - \partial_\xi X^{m} \partial_{\xi} X^{m} \right) \right)
$$

$$
= \frac{1}{2} \sqrt{h} h^{ij'} \partial_\nu X^{u} \partial_{\nu'} X^{u} \eta_{\nu' \nu'} - \frac{1}{2} \sqrt{h} h^{ij} \left( \frac{r_0^2}{r_0^4} - \frac{6 (X_0 \cdot X)^2}{r_0^4} \right) \left( \partial_\xi X^{u} \partial_{\xi} X^{u} \eta_{\xi \xi} - \partial_\xi X^{m} \partial_{\xi} X^{m} \eta_{\xi \xi} \right)
$$

All the other terms vanish due to dimensional regularization. Together with the four external vector vertex operators the correlator has the form of

$$
< V^{u_1}_{v_1}(x_1) V^{u_2}_{v_2}(x_2) V^{u_3}_{v_3}(x_3) V^{u_4}_{v_4}(x_4) \partial_\nu X^{u'} \partial_{\nu'} X^{u'} \eta_{\nu' \nu'} \eta u v' (S_{\gamma^{AB}})(S_{\gamma^{CD}}) >
$$
\[ \begin{align*}
&= +l^8 R^{u_1 v_1 k_1} R^{u_2 v_2 k_2} R^{u_3 v_3 k_3} R^{u_4 v_4 k_4} \\
&+ l^4 \left[ \dot{X}^{u_1} \dot{X}^{u_2} R^{u_3 v_3 k_3} R^{u_4 v_4 k_4} + \dot{X}^{u_3} \dot{X}^{u_4} R^{u_1 v_1 k_1} R^{u_2 v_2 k_2} + \dot{X}^{u_1} \dot{X}^{u_3} R^{u_2 v_2 k_2} R^{u_4 v_4 k_4} + \dot{X}^{u_2} \dot{X}^{u_4} R^{u_1 v_1 k_1} R^{u_3 v_3 k_3} \right] \\
&- l^6 \left[ \dot{X}^{u_1} R^{u_2 v_2 k_2} R^{u_3 v_3 k_3} R^{u_4 v_4 k_4} + \dot{X}^{u_2} R^{u_1 v_1 k_1} R^{u_3 v_3 k_3} R^{u_4 v_4 k_4} + \dot{X}^{u_3} R^{u_2 v_2 k_2} R^{u_4 v_4 k_4} + \dot{X}^{u_4} R^{u_1 v_1 k_1} R^{u_2 v_2 k_2} R^{u_3 v_3 k_3} \right] \\
&\partial_{\nu'} X^{u'} \partial_{\nu} X^{\nu'} \eta_{u'\nu'} (S^{\gamma AB} S)(S^{\gamma CD} S) \tag{48}
\end{align*} \]

Consider the 8S terms,
\[ 
+ l^4 \left[ \dot{X}^{u_1} \dot{X}^{u_2} R^{u_3 v_3 k_3} R^{u_4 v_4 k_4} + \dot{X}^{u_3} \dot{X}^{u_4} R^{u_1 v_1 k_1} R^{u_2 v_2 k_2} + \dot{X}^{u_1} \dot{X}^{u_3} R^{u_2 v_2 k_2} R^{u_4 v_4 k_4} + \dot{X}^{u_2} \dot{X}^{u_4} R^{u_1 v_1 k_1} R^{u_3 v_3 k_3} \right] \\
\partial_{\nu'} X^{u'} \partial_{\nu} X^{\nu'} \eta_{u'\nu'} (S^{\gamma AB} S)(S^{\gamma CD} S) \tag{49}
\]

Since \( < RRRSSS > \)-terms start with \( \frac{1}{x^4} \) order, the X-correlators must have \( O(1) \)-order terms. The leading order comes when each \( \partial_{\nu'} X^{u'} \) contracts with \( e^{ik \cdot X} \). This means that the above correlators are proportional to the corresponding part of \( < VVVVSSSS > \), which vanishes. As a matter of fact this argument applies to the other cases i.e., \( < RRRSSS > \) and \( < RRRRSSSS > \). With these one gets
\[ < V^{u_1 v_1} V^{u_2 v_2} V^{u_3 v_3} V^{u_4 v_4} \partial_{\nu'} X^{u'} \partial_{\nu} X^{\nu'} S^{\gamma AB} S S^{\gamma CD} S > = 0 \tag{50} \]

Therefore the third term entirely vanishes: the equation, (46), simplifies to
\[ (V_{G,q})^2 = \frac{q^2}{4} \int \sqrt{h} \ h^{ij} \sqrt{h} \ h^{ij} \left[ + \frac{1}{4} \left( \partial_{\nu'} X^{u'} \partial_{\nu} X^{\nu'} \left( \partial_{\nu} X^{u'} \partial_{\nu} X^{\nu'} \right) \right) \right] \]
\[ + 4 \frac{X^{i'} X^{i}}{r_0^2} \frac{X^{i'} X^{i}}{r_0^2} \left( \partial_{\nu'} X^{u'} \partial_{\nu} X^{\nu'} \partial_{\nu} X^{u} \partial_{\nu} X^{\nu} + \partial_{\nu'} X^{m'} \partial_{\nu} X^{m'} \partial_{\nu} X^{m} \partial_{\nu} X^{m} \right) \tag{51} \]

where the terms that vanish due to dimensional regularization have been dropped. The second term in (51) drops as one can see as follows. The second term in parentheses can be re-written,
\[ < X^{i'} \partial_{\nu'} X^{u'} \partial_{\nu} X^{u'} X^{i} \partial_{\nu} X^{m} \partial_{\nu} X^{m} >= - < \partial_{\nu'} X^{i'} X^{u'} \partial_{\nu} X^{u'} X^{i} \partial_{\nu} X^{m} \partial_{\nu} X^{m} > \]
\[ = \frac{1}{2} < \partial^{2} X^{i'} X^{u'} X^{u'} X^{i} \partial_{\nu} X^{m} \partial_{\nu} X^{m} > \tag{52} \]

which vanishes due to dimensional regularization. Consider the second term
\[ < X^{i'} \partial_{\nu} X^{u} \partial_{\nu} X^{u} \partial_{\nu'} X^{u'} \partial_{\nu} X^{u'} > \]
Finally consider the correlators with \( XXXX \). The correlator comes with \( \zeta \) and \( \frac{q^2}{16} \) in the leading order contributions. Consider this together with \( VVVV \).

\[
\begin{align*}
= & \, - \langle \partial_i X^t X^u \partial_i X^u \partial_j X^u \partial_j X^u \rangle - \langle X^t X^u \partial^2 X^u \partial_j X^u \partial_j X^u \rangle \\
= & \, - \langle \partial_i X^t X^u \partial_i X^u \partial_j X^u \partial_j X^u \rangle + \langle \partial_j X^t X^l X^u \partial^2 X^u X^u \partial_j X^u \rangle \\
& + \langle X^t X^l X^u \partial^2 X^u X^u \partial_j X^u \rangle 
\end{align*}
\]

(53)

The third term generates \( \delta(y - x_1)\delta(y' - x_1) \) so it vanishes due to dimensional regularization:

\[
\begin{align*}
= & \, - \partial_i X^t X^u \partial_i X^u \partial_j X^u \partial_j X^u + \partial_j X^t X^l X^u \partial^2 X^u X^u \partial_j X^u \\
= & \, \frac{1}{2} \partial^2 X^t X^u X^u \partial_j X^u \partial_j X^u - \frac{1}{2} \partial^2 X^t X^l X^u \partial^2 X^u X^u \partial_j X^u \\
= & \, \frac{1}{2} X^u X^u \partial_j X^u \partial_j X^u - \frac{1}{2} X^u \partial^2 X^u X^u X^u \\
= & \, X^u X^u \partial_j X^u \partial_j X^u + X^u \partial_i X^u \partial_i X^u X^u 
\end{align*}
\]

(54)

Consider this together with \( VVVV \). The RRRR-type terms do not contribute due to momentum conservation. The computations with \( XXXX \), XRRR, XXRR have been carried out using Mathematica. The summary of the results is as follows. The XXRR terms vanish either because they are subleading in \( \frac{1}{x_1} \) or they contain overall \( k_1^2 \). Next, the correlators with the XRRR-type terms,

\[
\begin{align*}
& \langle -t^6 \left[ \hat{X}^{u_1} R^{u_2 v_2} k_2 R^{u_3 v_3} k_3 R^{u_4 v_4} k_4 + \hat{X}^{u_2} R^{u_1 v_1} k_1 R^{u_3 v_3} k_3 R^{u_4 v_4} k_4 \\
& + \hat{X}^{u_3} R^{u_1 v_1} k_1 R^{u_2 v_2} R^{u_4 v_4} k_4 + \hat{X}^{u_4} R^{u_1 v_1} k_1 R^{u_2 v_2} R^{u_3 v_3} k_3 \right] (X^u X^u \partial_j X^v \partial_j X^v + X^u \partial_i X^u \partial_i X^u X^v) \rangle 
\end{align*}
\]

(55)

When \( R^{u_1 v_1}(x_1) \) it is already \( \frac{1}{x_1^2} \)-order. The remaining \( X \) correlators produce terms with \( O(\frac{1}{x_1}) \). As for the terms with RRR that do not include \( R^{u_1 v_1}(x_1) \), the leading order contributions come from the following \( X \)-contraction

\[
\langle \hat{X}^{u_1} (X^u X^u \partial_j X^v \partial_j X^v + X^u \partial_i X^u \partial_i X^v X^v) \rangle 
\]

(56)

All the terms of order \( \frac{1}{x_1^2} \) contain at least one overall factor of \( k_1^2 \), therefore they vanish. Finally consider the correlators with \( XXXX \). The \( \langle XXXX \rangle \) correlator comes with \( k_1^2 = 0 \). The \( \langle XXXX \rangle \) correlator turns out to contain overall \( \zeta_1 \cdot k_1 \), therefore vanishes: one gets

\[
(V_{G,b})^2 \Rightarrow \frac{q^2}{16} \int \int \sqrt{h} h^{i j} \sqrt{h} h^{i j} (\partial_i X^u \partial_j X^u) (\partial_i X^u \partial_j X^u) \\
\Rightarrow \frac{q^2}{8} \int \sqrt{h} h^{i j} (\partial_i X^u \partial_j X^u) 
\]

(57)

3.2 Four scalar scattering

Similar steps are involved for the four scalar correlators. The explicit form of product of the four scalar vertex operators is

\[
V_{s}^{m_1}(x_1)V_{s}^{m_2}(x_2)V_{s}^{m_3}(x_3)V_{s}^{m_4}(x_4)
\]

18
\[= X^{m_1} X^{m_2} X^{m_3} X^{m_4} + l^8 R^{m_1 v_1} k_1 R^{m_2 v_2} k_2 R^{m_3 v_3} k_3 R^{m_4 v_4} k_4
- l^2 [X^{m_1} X^{m_2} X^{m_3} R^{m_4 v_1} k_1^4 + X^{m_1} X^{m_2} X^{m_4} R^{m_3 v_3} k_3^4]
+ X^{m_1} X^{m_3} X^{m_4} R^{m_2 v_2} k_2^4 + X^{m_2} X^{m_3} X^{m_4} R^{m_1 v_1} k_1^4]
+ l^4 [X^{m_1} X^{m_2} R^{m_3 v_3} k_3^4 R^{m_4 v_4} k_4^4 + X^{m_3} X^{m_4} R^{m_1 v_1} k_1^4 R^{m_2 v_2} k_2^4
+ X^{m_2} X^{m_4} R^{m_1 v_1} k_1^4 R^{m_3 v_3} k_3^4 + X^{m_1} X^{m_3} R^{m_2 v_2} k_2^4 R^{m_4 v_4} k_4^4]
+ l^6 [X^{m_1} R^{m_2 v_2} k_2^4 R^{m_3 v_3} k_3^4 R^{m_4 v_4} k_4^4 + X^{m_2} R^{m_1 v_1} k_1^4 R^{m_3 v_3} k_3^4 R^{m_4 v_4} k_4^4]
+ X^{m_3} R^{m_1 v_1} k_1^4 R^{m_2 v_2} k_2^4 R^{m_4 v_4} k_4^4 + X^{m_4} R^{m_1 v_1} k_1^4 R^{m_2 v_2} k_2^4 R^{m_3 v_3} k_3^4]\] (58)

3.2.1 \(V_{G,q^2}\) contribution

The operator takes the form of

\[
V_{G,q^2} = q^2 \int \left( -\frac{1}{2} \sqrt{h} k^{ij} \left( \partial_i X^u \partial_j X^v \eta_{uv} \left[ \frac{3}{8} - \frac{3 X_0 \cdot X}{r_0^2} + \frac{-3 r^2}{2 r_0^2} + \frac{15 (X_0 \cdot X)^2}{r_0^4} \right] + \partial_i X^m \partial_j X^n \eta_{mn} \left[ \frac{1}{8} + \frac{X_0 \cdot X}{r_0^2} + \frac{r^2}{2 r_0^2} - \frac{5 (X_0 \cdot X)^2}{r_0^4} \right] \right) + \frac{i}{8 p^+ r_0^2} (\sqrt{h} k^{ij} \epsilon^{ij}) \partial_i X^+ \right) \left[ 7 \partial_j X^u (X + X_0)^m (S_{\gamma^m S}) - 5 \partial_j X^m (X + X_0)^n (S_{\gamma^n S}) \right] + \frac{1}{4 (p^+)^2} \sqrt{h} \epsilon^{ij} \partial_i X^+ \partial_j X^+ \frac{1}{r_0^2} \left[ -\frac{17}{96} (S_{\gamma^u S}) (S_{\gamma^v S}) + \frac{45}{48} (S_{\gamma^a S}) (S_{\gamma^b S}) \right] \left( \frac{X_0^a X_0^b}{r_0^2} + \frac{35}{24} (S_{\gamma^a S}) (S_{\gamma^c S}) (S_{\gamma^b S}) \frac{X_0^a X_0^b}{r_0^2} \right) \right] (59)
\]

The reason for dropping \(\frac{1}{p+}\)-terms is that these correlators already appeared in the one-loop analysis [9]. Let’s briefly review the discussion. The vanishing of \(< VVVV \partial X^m X_0^m S_{\gamma^m S} >\) was explicitly discussed in [9]. Consider \(\partial X^u X_0^m S_{\gamma^m S}\)

\[
< V_s^{m_1} (x_1) V_s^{m_2} (x_2) V_s^{m_3} (x_3) V_s^{m_4} (x_4) \partial X^u X_0^m S_{\gamma^m S} >
= < l^6 R^{m_1 v_1} k_1 R^{m_2 v_2} k_2 R^{m_3 v_3} k_3 R^{m_4 v_4} k_4
+ l^4 [X^{m_1} X^{m_2} R^{m_3 v_3} k_3^4 R^{m_4 v_4} k_4^4 + X^{m_3} X^{m_4} R^{m_1 v_1} k_1^4 R^{m_2 v_2} k_2^4
+ X^{m_1} X^{m_4} R^{m_2 v_2} k_2^4 R^{m_3 v_3} k_3^4 + X^{m_1} X^{m_3} R^{m_2 v_2} k_2^4 R^{m_4 v_4} k_4^4]
+ l^6 [X^{m_1} R^{m_2 v_2} k_2^4 R^{m_3 v_3} k_3^4 R^{m_4 v_4} k_4^4 + X^{m_2} R^{m_1 v_1} k_1^4 R^{m_3 v_3} k_3^4 R^{m_4 v_4} k_4^4]
+ X^{m_3} R^{m_1 v_1} k_1^4 R^{m_2 v_2} k_2^4 R^{m_4 v_4} k_4^4 + X^{m_4} R^{m_1 v_1} k_1^4 R^{m_2 v_2} k_2^4 R^{m_3 v_3} k_3^4]\)
\] (60)

The \(RR\) terms vanish due to the index structure. One can easily see this just by explicitly considering the index structure of \(< RRSS >\). Consider the \(RRRR\) term.
All the terms in the resulting computation contain traces of either three or five $\gamma^{mu}$-matrices. When the trace is expressed in terms of the product of deltas each term is bound to contain at least one $\delta_{mu}$, therefore vanishes. Similarly XXRR terms vanish. Consider $\partial X^m X^n S \gamma^{mn} S$. Among (58) terms only XXRR can give a potentially non-zero result

$$\partial X^m X^n S \gamma^{mn} S >$$

Explicit computation shows that these terms are subleading in $x_1$, i.e., $O \left( \frac{1}{x_1} \right)$. One can similarly show that the $\partial X^u X^m S \gamma^{am} S$-correlator vanishes. As in the vector scattering case one can show that $< V_a V_b V_c V_d S S S S S S > = 0$ and from the one-loop analysis [9] we know that $< V_a V_b V_c V_d \partial_j X^u \partial_j X^v \eta_{uv} >= 0$. With these results, (59) can be re-written as

$$V_{G,q^2} = q^2 \int \left\{ \frac{1}{2} \sqrt{h} h^{ij} \right. \left[ -\frac{3 X_0 \cdot X}{r_0^2} + \frac{3}{2} \frac{r^2}{r_0^2} + \frac{15}{r_0^4} (X_0 \cdot X)^2 \right] 
+ \partial_i X^m \partial_j X^m \left[ -\frac{1}{8} + \frac{X_0 \cdot X}{r_0^2} + \frac{r^2}{2 r_0^2} - \frac{5}{r_0^4} (X_0 \cdot X)^2 \right] \right\}$$

The first term goes as follows. Each $\partial_i X^u$ must contract with $e^{ik \cdot X}$. The leading term comes with $k_1^2$, therefore vanishes. The second term vanishes as follows. The $\partial_i X^m \partial_j X^m (X_0 \cdot X)$-term appeared in the one-loop as well. It vanishes due to the index structure. Finally the only non-trivial piece of the quartic term has the form of

$$< X^{m1} X^{m2} X^{m3} X^{m4} (\partial_i X^m \partial_j X^m X^p X^q) >$$

One can check that it only produces terms of $\frac{1}{x_1}$ or higher, hence no contribution. Therefore we prove the third equation of (10),

$$V_{G,q^2} = \frac{q^2}{16} \int \sqrt{h} h^{ij} \partial_i X^m \partial_j X^m \quad (64)$$

### 3.2.2 $V_{G,q^2}^2$ contribution

Tailoring (29) to the given order, one gets

$$V_{G,q^2}^2 = q^2 \int \int \frac{1}{4} \sqrt{h} h^{ij} \sqrt{h} h^{i'j'}$$
\[
\left[ -\frac{1}{2} \partial_i X^u \partial_j X^v \eta_{uv} + \frac{1}{2} \partial_i X^m \partial_j X^n \eta_{mn} \\
+ \partial_i X^u \partial_j X^v \eta_{uv} \frac{2 X_0 \cdot X}{r_0^2} - \partial_i X^m \partial_j X^n \eta_{mn} \frac{2 X_0 \cdot X}{r_0^2}
\right]
+ \partial_i X^u \partial_j X^v \eta_{uv} \left( \frac{r^2}{r_0^2} - \frac{6 (X_0 \cdot X)^2}{r_0^4} \right)
+ \partial_i X^m \partial_j X^n \eta_{mn} \left( \frac{6 (X_0 \cdot X)^2}{r_0^4} - \frac{r^2}{r_0^2} \right)
\right]^2
+ \frac{1}{4 (p^+)^2} \left[ - i(\sqrt{h} \delta^{ij} - \epsilon^{ij}) \partial_i X^+ \partial_j X^m (S \gamma^{um} S) \frac{1}{r_0^2} \right.
\left. + i(\sqrt{h} \delta^{ij} - \epsilon^{ij}) \partial_i X^+ \partial_j X^m (S \gamma^{mn} S) \frac{1}{r_0^2} \right]^2
+ \frac{1}{p^+} \left[ \frac{i}{4} (\sqrt{h} \delta^{ij} - \epsilon^{ij}) \partial_i X^+ \partial_j X^m (X + X_0)^m (S \gamma^{um} S) \left( \frac{4}{r_0^2} + \frac{24 X_0 \cdot X}{r_0^4} \right) \right.
\left. - \frac{i}{4} (\sqrt{h} \delta^{ij} - \epsilon^{ij}) \partial_i X^+ \partial_j X^m (X + X_0)^n (S \gamma^{mn} S) \left( \frac{4}{r_0^2} + \frac{24 X_0 \cdot X}{r_0^4} \right) \right]
\sqrt{h} \delta^{ij} \left[ \partial_i X^u \partial_j X^v \eta_{uv} \left( -\frac{1}{2} + \frac{2 X_0 \cdot X}{r_0^2} \right) + \partial_i X^m \partial_j X^n \eta_{mn} \left( \frac{1}{2} - \frac{2 X_0 \cdot X}{r_0^2} \right) \right]
\]

where we have used

\[
<V_s V_s V_s V_s \partial_i X^m \partial_j X^m (S \gamma^{AB} S) (S \gamma^{CD} S) >= 0 \tag{66}
\]

and

\[
<V_s V_s V_s V_s \partial_i X^u \partial_j X^v \eta_{uv} (S \gamma^{AB} S) (S \gamma^{CD} S) >= 0 \tag{67}
\]

Let’s work out one of the last terms with the factor,

\[
\partial_j X^u (X + X_0)^m (S \gamma^{um} S) \left( \frac{4}{r_0^2} + \frac{24 X_0^p X^p}{r_0^4} \right)
\left[ \partial_i X^u \partial_j X^v \eta_{uv} \left( -\frac{1}{2} + \frac{2 X_0 \cdot X}{r_0^2} \right) + \partial_i X^m \partial_j X^n \eta_{mn} \left( \frac{1}{2} - \frac{2 X_0 \cdot X}{r_0^2} \right) \right]
\]

\[
= \partial_j X^u X^m (S \gamma^{um} S) \left( \frac{4}{r_0^2} \right) \left[ \partial_i X^u \partial_j X^v \eta_{uv} \left( -\frac{1}{2} \right) + \partial_i X^m \partial_j X^n \eta_{mn} \left( \frac{1}{2} \right) \right]
+ \partial_j X^u X^m (S \gamma^{um} S) \left( \frac{4}{r_0^2} + \frac{24 X_0^p X^p}{r_0^4} \right)
\left[ \partial_i X^u \partial_j X^v \eta_{uv} \left( -\frac{1}{2} + \frac{2 X_0 \cdot X}{r_0^2} \right) + \partial_i X^m \partial_j X^n \eta_{mn} \left( \frac{1}{2} - \frac{2 X_0 \cdot X}{r_0^2} \right) \right] \tag{68}
\]
Consider the first line. With rather lengthy algebra, one can show, after the Wick rotation, that

\[ < V_{s_1}^{m_1}(x_1)V_{s_2}^{m_2}(x_2)V_{s_3}^{m_3}(x_3)V_{s_4}^{m_4}(x_4) \partial_j X^{',m'} \partial_{j'} X^{m'} \partial_j X^u X^m (S \gamma^{um} S) > = 2stu \left[ \frac{\xi_1 \cdot \xi_4 \xi_2 \cdot \xi_3}{1 + \alpha' t} + \frac{\xi_1 \cdot \xi_3 \xi_2 \cdot \xi_4}{1 + \alpha' u} + \frac{\xi_1 \cdot \xi_2 \xi_3 \cdot \xi_4}{1 + \alpha' s} \right] \]

(69)

What happens is that this contribution is precisely canceled by the other term in the first line. In other words,

\[ < V_{s_1}^{m_1}(x_1)V_{s_2}^{m_2}(x_2)V_{s_3}^{m_3}(x_3)V_{s_4}^{m_4}(x_4) \partial_j X^{',m'} \gamma^{ul} \left( S \gamma^{uml} S \right) [ - \partial_j X^u \partial_j X^v \partial_j \eta_{uv} + \partial_j X^m \partial_j X^n \partial_j \eta_{mn} ] = 0 \]

(70)

This result has some implication for the three-loop computation as we will discuss in the conclusion. The second term of (68) is

\[ \partial_j X^{',m'} \gamma^{ul} \left( S \gamma^{uml} S \right) \left( - \frac{4}{r_0^2} + \frac{24 X_0^' \cdot X^' t}{r_0^4} \right) \]

\[ \left[ \partial_j X^u \partial_j X^u \partial_j \eta_{uu} \left( - \frac{1}{2} t X_0^' \cdot X \right) + \partial_j X^m \partial_j X^n \partial_j \eta_{mn} \left( \frac{1}{2} - \frac{2X_0^' \cdot X t}{r_0^2} \right) \right] \]

\[ = - \frac{2}{r_0^2} \partial_j X^{',m'} \gamma^{ul} \left( S \gamma^{uml} S \right) [ - \partial_j X^u \partial_j X^u \partial_j \eta_{uv} + \partial_j X^m \partial_j X^n \partial_j \eta_{mn} ] \]

\[ + \frac{8}{r_0^2} \partial_j X^{',m'} \gamma^{ul} \left( S \gamma^{uml} S \right) \frac{X_0^' \cdot X}{r_0^4} \left[ \partial_j X^u \partial_j X^v \partial_j \eta_{uv} - \partial_j X^m \partial_j X^n \partial_j \eta_{mn} \right] \]

\[ + 12 \partial_j X^{',m'} \gamma^{ul} \left( S \gamma^{uml} S \right) \left( \frac{X_0^' \cdot X_0^'}{r_0^4} \right) \left[ - \partial_j X^u \partial_j X^v \partial_j \eta_{uv} + \partial_j X^m \partial_j X^n \partial_j \eta_{mn} \right] \]

(71)

where the last equality follows from the fact that the second term vanishes due to the index structure. The vanishing of the terms in the last line goes almost identical to one of the previous calculations that leads to (70). The above equation becomes

\[ = - \frac{8}{r_0^2} \partial_j X^{',m'} \gamma^{ul} \left( S \gamma^{uml} S \right) \frac{X_0^' \cdot X}{r_0^4} \left[ \partial_j X^u \partial_j X^v \partial_j \eta_{uv} - \partial_j X^m \partial_j X^n \partial_j \eta_{mn} \right] \]

(72)

Correlators with XXRR and RRRR types of terms vanish since there are only odd number of \( X^m \)'s. First the correlator involving the XXXR terms,

\[ XXXR \partial_j X^{',m'} \gamma^u \left[ \partial_j X^u \partial_j X^v \partial_j \eta_{uv} - \partial_j X^m \partial_j X^n \partial_j \eta_{mn} \right] (S \gamma^{uml} S) \]

(73)
Partially integrate $\partial_j X^u$ so that the derivative now acts on $(S\gamma^{u'm'}S)$. It should contract only with $R_{m'1}v_1$. The $\partial' S'$ contraction gives a delta function. The other $S$ will vanish due to dimensional regularization. Next the XRRR-correlator,

$$XRRR \partial_j X^u X^v [\partial_i X^u \partial_j X^v \eta_{uv} - \partial_i X^m \partial_j X^n \eta_{mn}] (S\gamma^{u'm'}S)$$

Again partially integrate $\partial_j X^u$ so that the derivative now acts on $(S\gamma^{u'm'}S)$. It is easy to check that both of these terms produce sub-leading terms in $\frac{1}{x_1}$, i.e., $O \left( \frac{1}{x_1^3} \right)$.

The third term in (68) can also be shown to vanish,

$$< V_s V_s V_s V_s \left[ \partial_i X^u \partial_j X^u \left( -\frac{1}{2} + \frac{2X_0 \cdot X}{r_0^2} \right) + \partial_i X^m \partial_j X^m \left( \frac{1}{2} - \frac{2X_0 \cdot X}{r_0^2} \right) \right] >= 0$$

(75)

To prevent this section from becoming too lengthy we do not present the discussion for the rest of the terms in (65). One can show by following the similar steps that all the remaining terms in (65) vanish except one term:

$$V^2_{G,q} \Rightarrow \frac{q^2}{16} \int \int \sqrt{h} h^{ij} \sqrt{h} h^{i'j'} (\partial_i X^{m'} \partial_j X^{m'})(\partial_i X^m \partial_j X^m)$$

$$\Rightarrow \frac{q^2}{8} \int \sqrt{h} h^{ij} (\partial_i X^m \partial_j X^m)$$

(76)

4 Conclusion

In this paper, we have discussed the two-loop divergence cancellation of the four vector and the four scalar scattering amplitudes. At two-loop, the divergence vanishes.\textsuperscript{12} Compared with the one-loop analysis, a new ingredient is the introduction of renormalization of the external vertex operators. The divergences could be absorbed as they have precisely the same forms as the tree-level ones. It is necessary to consider some other scattering amplitudes in order to complete the renormalization program in this paper. By doing so, it should be possible to fix the various constants that appear in (20), (21), (24), (26) and (27). Some of these equations may get modified by the consideration of $< VVVVV_{G,g^2} >_{1\text{ loop}}$, which was noted in the footnote 7. We will check in the near future whether the divergences from $< VVVVV_{G,g^2} >_{1\text{ loop}}$ indeed preserve the consistency of the proposal other than possibly modifying the constraint equations. It will be interesting to compare those results with the wave function renormalization of N=4 SYM. This will set the ground for the computations of the open string corrections for the anomalous dimensions of the SYM operators. The results may also be compared with the results based on [27].

\textsuperscript{12}At least the one that we have considered vanishes. The three-loop result will most likely generate new forms of divergences. Any new form of divergence cannot be cancelled by the counter-terms based on the flat action. In this light, the three-loop result is expected to completely rule out the flat action.
At the two-loop order the counter vertex operator contains a large number of terms. As we have discussed in the main body, the majority of these terms have turned out to vanish. All the non-vanishing terms yield the same kinematic structure as that of the tree diagrams. Because of this one may wonder whether it would be possible at three-loop to have non-vanishing contributions that do not have the same structures as the tree diagrams. As a matter of fact, there are a few terms that have appeared at two-loop which will re-appear at three-loop. For example, one of them is

$$\partial_j X^{u'} \bar{X}^{m'} (S \gamma^{u'm'} S) \partial_i X^u \partial_j X^v \eta_{uv}$$

(77)

At two-loop the contribution of this term gets cancelled by a similar term

$$\partial_j X^{u'} \bar{X}^{m'} (S \gamma^{u'm'} S) \partial_i X^m \partial_j X^n \eta_{mn}$$

(78)

as seen in (70). At three-loop, however, these two terms emerge with coefficients of different magnitudes. Therefore we expect that there will be a non-tree like divergence that is provided by the counter vertex operator. This may be viewed as a prediction that the three-loop diagrams will be divergent but not finite. It will be a very interesting and crucial test of the conjecture in [8, 9] to check whether the picture holds at three-loop.\(^\text{13}\) The three-loop computation seems to be more within the range with the development of the pure spinor formulation [21][28]. (See also [29][30][31] for related works.) Once one has a handle on the three loop computations, one may be in a good position to concretely test the idea put forward in [32] (and subsequent works) concerning the potential roles of an open string in gauge/gravity type conjectures in general. (Related ideas can be found, e.g., in [33][34][35].)

In the introduction, it was mentioned that with our set-up one may not have to consider the non-planar graphs when one considers an open string on a Dp-brane with \(p < 9\). The reason is that, for a non-planar graph, at least one of the external states must leave the brane at some point. This would be a violation of the boundary conditions. This is a different situation from the space-time filling brane where the end points will still move within the brane since there is no leaving the brane. It will be interesting and worthwhile to check explicitly what would go wrong mathematically in the construction of such a process. (It might be that all such amplitudes vanish.) If true (we believe that it is), one may say that the planarity is built in the present frame-work but not something that one imposes as an extra condition. It also has an implication for T-duality. Naively one would expect to include non-planar graphs since they are included in the D9-brane case. Therefore this is an indication that care is needed when applying T-duality. We will report on this issue and the ones given above in the near future.

Finally we elaborate on the subtle issues with T-duality in addition to the one just described. The issue will force us to face a non-trivial question concerning how, with

\(^\text{13}\)The two terms above are absent in the flat space action. If they play a role in the three-loop divergence cancellation as expected, it will be another argument in addition to the one given in [9] that the flat space action is simply not suitable.
regard to the degrees of freedom, one should formulate the string theory, open and/or closed. Here we examine the problem from the angle of the current context and point out a few things that may be useful to precisely pin-down the problem. We also list future tasks that may lead to a satisfying resolution on how to formulate the string theory.

It seems that we are dealing with two T-dualities depending on where we apply it. The first is a T-duality that relates a D9 to a Dp with $p < 9$. The other is a "T-duality within the same Dp-brane theory". (Below we explain what is meant by this.) To some extent, the subtlety hinges on the set-up that we are using. Let’s start with the T-duality in the conventional set-up where both an open string and a closed string are put on an equal footing. Although it may sound straightforward, it is not the case once one considers loops of the open strings that are moving on Dp-branes. We will name a few. In this set-up one would consider the momentum modes along the compactified directions and winding modes. Therefore there is no loss of degrees of freedom. However, an explicit construction of the open string scattering states requires them to have momentum components solely along the longitudinal directions as shown in [7]. It is not clear to us how to reconcile or whether it is even possible to do so. (Also in light of the open string based framework which we discuss below, it is not clear whether including those modes is justified. See the discussion below.) We believe that a more serious issue exists within the set-up itself. It is the redundancy issue. The issue is not limited to the current context but much more general. In a certain circumstance, the end points of an open string can stick together, thereby converting it to a closed string. That seems to suggest that, in principle, the resulting closed string should have a representation in terms of the open string that one started with: putting independent closed string coordinates seems redundant. Furthermore, independent closed string coordinates do not seem to help when it comes to cancelling the open string divergences. A closed string amplitude comes with the kinematic factor that is distinct from that of an open string amplitude. As well known, one has to "square" the latter to go to the former. We find obscure (and could not find an explicit proof of) the statements in the literatures that those divergence can be cancelled by closed string amplitudes. If successful at the three-loop, one of the things that the current program will bring as a by-product is a firmer establishment of the open string based setup.

We turn to the open string based set-up that has served as a basis for the rationale of our current and previous works. In this frame-work, one tries to cancel the given loop divergence by inserting a composite vertex operator. Suppose one starts with D9 and goes to a Dp-brane by T-duality. One then tries to relate the loop amplitude of the D9 to that of the Dp. As stated above and in the previous works, there is a loss of degrees of freedom since the momentum can only be along the longitudinal directions. If one deals with a corresponding situation in an ordinary quantum field theory, it is obvious that one’s attempt to relate the two results is not justified, since the lower dimensional theory would be a dimensionally reduced theory. In a string theory the
the fact that it is not justified seems less obvious. First of all, the Dp-brane theory is still a ten-dimensional theory. Secondly, the simple-minded application of T-duality to the final form of the D9-brane amplitude leads to the same Dp-brane amplitude that is obtained by direct computation. We believe that it should be a coincidence because the loss of degrees of freedom still remains true. The reason behind the coincidence must be that the form of the amplitude is highly constrained by various symmetries and the dimensional analysis. The other T-duality i.e., the ”T-duality within the Dp-brane theory” seems safer. For example, one may be able to deduce the scalar amplitude from the vector amplitude and vice versa. Here those two multiplets are on an equal footing. It might not be completely subtlety-free, however, because one would have to make sure, e.g., that T-duality commutes with dimensional regularization. (In all of the examples that we have considered so far T-duality gives the same results as those obtained by direct computations.) In this work, therefore, we have explicitly and directly computed all the correlators.

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\(^{14}\text{The term, "T-duality within the same Dp", is used in this sense: as in a usual quantum field theory, amplitudes of different set of fields may sometimes be connected by a certain transformation. If one traces its stringy origin, it is likely to originate from the usual T-dualities combined in some way.}\)
Appendices

A Definitions of $V_v$ and $V_s$

For convenience we quote from [7] the expressions for the vector multiplet, $V_v$, (which was called $V_v$ in [7]) and the scalar multiplet, $V_s$.\textsuperscript{15} Define $k^i = (k^u, 0)$, $\zeta^i = (\zeta^u, 0)$ with $i = (u, m)$. The index $i$ here is not to be confused with the world-sheet label used in the main body of the text. The bosonic vector vertex operator is

$$V_v(\zeta, k) = (\zeta^u B_v^u - \zeta^v B_v^v) e^{ik \cdot X} \quad (A.1)$$

where

$$B_v^+ = p^+$$
$$B_v^u = (X^u - R_v^{uj} k^j)$$
$$F_{\dot{a}} = \frac{1}{\sqrt{2p^+}} [((\gamma^u)^T \dot{X}^u S) \dot{a} - ((\gamma^m)^T X^m S) \dot{a} + \frac{1}{3} : ((\gamma^i)^T S) \dot{a} R_v^{ij} : k^j]$$
$$F_{\dot{a}} = \sqrt{\frac{p^+}{2}} S^a \quad (A.2)$$

where $R_v^{ij} = \frac{1}{4} S \gamma^{ij} S$. For the scalar vertex operator, we define $k^i = (k^u, 0)$, $\xi^i = (0, \xi^m)$:

$$V_s(\xi, k) = \xi \cdot B_s e^{ik \cdot X} = (\xi^m B_s^m) e^{ik \cdot X} \quad (A.3)$$

where

$$B_s^m = (X^m + R_s^{mj} k^j)$$
$$F_{\dot{a}} = \frac{1}{\sqrt{2p^+}} [((\gamma^u)^T \dot{X}^u S) \dot{a} - ((\gamma^m)^T X^m S) \dot{a} - \frac{1}{3} : ((\gamma^i)^T S) \dot{a} R_s^{ij} : k^j]$$
$$F_{\dot{a}} = -\sqrt{\frac{p^+}{2}} S^a \quad (A.4)$$

where $R_s^{ij} = \frac{1}{4} S \gamma^{ij} S = R_v^{ij}$.

B Identities

For the computations in sec3, it is useful to note the several identities. For example, one has

$$< R_{u_1 v_1} R_{u_2 v_2} > = -\left( \frac{\delta_{u_1 u_2} \delta_{v_1 v_2} - \delta_{u_1 v_2} \delta_{u_2 v_1}}{(x_1 - x_2)^2} \right) \quad (B.1)$$

\textsuperscript{15}Here we only quote the bosonic vertex operators. The fermionic vertex operators can be found in [7]. The vertex operators were obtained in the world-sheet strip. To compute correlators using the Wick contraction technique as done in this work and the previous works, one should go to the upper half-plane by the standard conformal transformation.
and
\[
< R_{uv}^{u_1 v_1}(x_1) R_{uv}^{u_2 v_2}(x_2) R_{uv}^{u_3 v_3}(x_3) >
\]
\[
= - \frac{1}{x_1 x_2 x_3 x_4} (\delta_{u_2 u_3} \delta_{u_1 v_2} \delta_{v_1 v_3} - \delta_{u_2 u_3} \delta_{u_1 v_4} \delta_{v_3 v_2} - \delta_{u_2 v_3} \delta_{u_1 v_2} \delta_{v_1 v_4} \\
+ \delta_{u_2 v_3} \delta_{u_1 v_4} \delta_{v_3 v_2} - \delta_{u_3 v_2} \delta_{u_1 v_2} \delta_{v_1 v_3} + \delta_{u_3 v_2} \delta_{u_4 v_3} \delta_{v_2 v_1} \\
+ \delta_{v_2 v_3} \delta_{u_1 v_2} \delta_{v_1 v_4} - \delta_{v_2 v_3} \delta_{u_1 v_4} \delta_{v_3 v_2} )
\]  
(B.2)

From these one can easily deduce
\[
\text{Tr} \, \gamma_{u_1 v_1} \gamma_{u_2 v_2} = -8(\delta_{u_1 u_2} \delta_{v_1 v_2} - \delta_{u_1 v_2} \delta_{u_2 v_1})
\]  
(B.3)

and
\[
\text{Tr} \, (\gamma_{u_1 v_1} \gamma_{u_2 v_2} \gamma_{u_3 v_3})
= 8(\delta_{u_2 u_3} \delta_{u_1 v_2} \delta_{v_1 v_3} - \delta_{u_2 u_3} \delta_{u_1 v_4} \delta_{v_3 v_2} - \delta_{u_2 v_3} \delta_{u_1 v_2} \delta_{v_1 v_4} + \delta_{u_2 v_3} \delta_{u_1 v_4} \delta_{v_3 v_2} \\
- \delta_{u_3 v_2} \delta_{u_2 v_3} \delta_{v_1 v_2} + \delta_{u_3 v_2} \delta_{u_1 v_3} \delta_{v_2 v_1} + \delta_{v_2 v_3} \delta_{u_1 u_2} \delta_{v_1 v_3} - \delta_{v_2 v_3} \delta_{u_1 u_3} \delta_{v_2 v_1} )
\]  
(B.4)

The proof of the trace that involves four $\gamma$'s is more lengthy: it yields\textsuperscript{16}
\[
\frac{1}{24} \text{Tr} \gamma_{u_1 v_1} \gamma_{u_2 v_2} \gamma_{u_3 v_3} \gamma_{u_4 v_4} 
= \delta_{u_1 u_2} \delta_{u_3 u_4} (\delta_{v_1 v_2} \delta_{v_3 v_4} - \delta_{v_1 v_4} \delta_{v_2 v_3}) \\
+ \delta_{u_1 u_3} \delta_{u_2 u_4} (-\delta_{v_1 v_2} \delta_{v_3 v_4} + \delta_{v_1 v_4} \delta_{v_2 v_3}) \\
+ \delta_{u_1 u_4} \delta_{u_2 u_3} \delta_{v_1 v_2} \delta_{v_3 v_4} - \delta_{u_1 u_3} \delta_{v_1 v_2} \delta_{v_3 v_4} - \delta_{u_1 u_4} \delta_{v_2 v_3} \delta_{v_1 v_4} + \delta_{v_1 v_3} \delta_{v_2 v_4} \delta_{v_3 v_4} \\
+ \delta_{u_2 u_3} (\delta_{u_1 u_2} \delta_{u_3 u_3} \delta_{u_4 v_4} - \delta_{v_1 v_2} \delta_{u_3 u_4} \delta_{v_3 v_4}) \\
+ \delta_{u_2 u_4} (\delta_{u_1 u_2} \delta_{u_3 u_3} \delta_{u_4 v_4} - \delta_{v_1 v_2} \delta_{u_3 u_4} \delta_{v_3 v_4}) \\
+ \delta_{u_3 u_4} (\delta_{u_1 u_2} \delta_{u_3 u_3} \delta_{u_4 v_4} - \delta_{v_1 v_2} \delta_{u_3 u_4} \delta_{v_3 v_4}) \\
+ \delta_{u_2 u_3} (\delta_{u_1 u_2} \delta_{u_3 u_3} \delta_{u_4 v_4} - \delta_{v_1 v_2} \delta_{u_3 u_4} \delta_{v_3 v_4}) \\
+ \delta_{u_2 u_4} (\delta_{u_1 u_2} \delta_{u_3 u_3} \delta_{u_4 v_4} - \delta_{v_1 v_2} \delta_{u_3 u_4} \delta_{v_3 v_4}) \\
+ \delta_{u_3 u_4} (\delta_{u_1 u_2} \delta_{u_3 u_3} \delta_{u_4 v_4} - \delta_{v_1 v_2} \delta_{u_3 u_4} \delta_{v_3 v_4}) \\
+ \delta_{u_2 u_3} (\delta_{u_1 u_2} \delta_{u_3 u_3} \delta_{u_4 v_4} - \delta_{v_1 v_2} \delta_{u_3 u_4} \delta_{v_3 v_4}) \\
+ \delta_{u_2 u_4} (\delta_{u_1 u_2} \delta_{u_3 u_3} \delta_{u_4 v_4} - \delta_{v_1 v_2} \delta_{u_3 u_4} \delta_{v_3 v_4}) \\
+ \delta_{u_3 u_4} (\delta_{u_1 u_2} \delta_{u_3 u_3} \delta_{u_4 v_4} - \delta_{v_1 v_2} \delta_{u_3 u_4} \delta_{v_3 v_4}) \\
+ \delta_{u_2 u_3} (\delta_{u_1 u_2} \delta_{u_3 u_3} \delta_{u_4 v_4} - \delta_{v_1 v_2} \delta_{u_3 u_4} \delta_{v_3 v_4}) \\
+ \delta_{u_2 u_4} (\delta_{u_1 u_2} \delta_{u_3 u_3} \delta_{u_4 v_4} - \delta_{v_1 v_2} \delta_{u_3 u_4} \delta_{v_3 v_4}) \\
+ \delta_{u_3 u_4} (\delta_{u_1 u_2} \delta_{u_3 u_3} \delta_{u_4 v_4} - \delta_{v_1 v_2} \delta_{u_3 u_4} \delta_{v_3 v_4}) \\
+ \delta_{u_2 u_3} (\delta_{u_1 u_2} \delta_{u_3 u_3} \delta_{u_4 v_4} - \delta_{v_1 v_2} \delta_{u_3 u_4} \delta_{v_3 v_4}) \\
+ \delta_{u_2 u_4} (\delta_{u_1 u_2} \delta_{u_3 u_3} \delta_{u_4 v_4} - \delta_{v_1 v_2} \delta_{u_3 u_4} \delta_{v_3 v_4}) \\
+ \delta_{u_3 u_4} (\delta_{u_1 u_2} \delta_{u_3 u_3} \delta_{u_4 v_4} - \delta_{v_1 v_2} \delta_{u_3 u_4} \delta_{v_3 v_4}) \\
+ \delta_{u_2 u_3} (\delta_{u_1 u_2} \delta_{u_3 u_3} \delta_{u_4 v_4} - \delta_{v_1 v_2} \delta_{u_3 u_4} \delta_{v_3 v_4}) \\
+ \delta_{u_2 u_4} (\delta_{u_1 u_2} \delta_{u_3 u_3} \delta_{u_4 v_4} - \delta_{v_1 v_2} \delta_{u_3 u_4} \delta_{v_3 v_4}) \\
+ \delta_{u_3 u_4} (\delta_{u_1 u_2} \delta_{u_3 u_3} \delta_{u_4 v_4} - \delta_{v_1 v_2} \delta_{u_3 u_4} \delta_{v_3 v_4}) \\
+ \delta_{u_2 u_3} (\delta_{u_1 u_2} \delta_{u_3 u_3} \delta_{u_4 v_4} - \delta_{v_1 v_2} \delta_{u_3 u_4} \delta_{v_3 v_4}) \\
+ \delta_{u_2 u_4} (\delta_{u_1 u_2} \delta_{u_3 u_3} \delta_{u_4 v_4} - \delta_{v_1 v_2} \delta_{u_3 u_4} \delta_{v_3 v_4})
\]

\textsuperscript{16}Note that the following identity is for 16 by 16 gamma matrices. For 8 by 8 matrices, a factor $\frac{1}{2^4}$ should appear instead of $\frac{1}{2^4}$ in the first line.  

28
\[
\begin{align*}
\delta_{u_{1}v_{2}} \delta_{v_{1}v_{3}} \delta_{u_{3}v_{4}} &+ \delta_{u_{1}v_{2}} \delta_{v_{1}v_{3}} \delta_{u_{3}v_{4}} + \delta_{u_{1}v_{2}} \delta_{v_{1}v_{3}} \delta_{u_{3}v_{4}} - \delta_{u_{1}v_{2}} \delta_{v_{1}v_{3}} \delta_{u_{3}v_{4}} \\
+ \delta_{u_{1}v_{3}} \delta_{v_{1}v_{4}} \delta_{u_{2}v_{3}} &- \delta_{u_{1}v_{4}} \delta_{v_{1}v_{3}} \delta_{u_{2}v_{3}} + \delta_{u_{1}v_{4}} \delta_{v_{1}v_{3}} \delta_{u_{2}v_{3}} - \delta_{u_{1}v_{3}} \delta_{v_{1}v_{4}} \delta_{u_{2}v_{3}} \\
+ \delta_{u_{2}v_{4}} \delta_{v_{1}v_{3}} \delta_{u_{2}v_{3}} &- \delta_{u_{2}v_{4}} \delta_{v_{1}v_{3}} \delta_{u_{2}v_{3}} + \delta_{u_{2}v_{4}} \delta_{v_{1}v_{3}} \delta_{u_{2}v_{3}} - \delta_{u_{2}v_{4}} \delta_{v_{1}v_{3}} \delta_{u_{2}v_{3}} \\
+ \delta_{u_{2}v_{3}} \delta_{v_{1}v_{4}} \delta_{u_{2}v_{3}} &- \delta_{u_{2}v_{3}} \delta_{v_{1}v_{4}} \delta_{u_{2}v_{3}} + \delta_{u_{2}v_{3}} \delta_{v_{1}v_{4}} \delta_{u_{2}v_{3}} - \delta_{u_{2}v_{3}} \delta_{v_{1}v_{4}} \delta_{u_{2}v_{3}} \\
+ \delta_{u_{3}v_{4}} \delta_{v_{1}v_{3}} \delta_{u_{2}v_{3}} &- \delta_{u_{3}v_{4}} \delta_{v_{1}v_{3}} \delta_{u_{2}v_{3}} + \delta_{u_{3}v_{4}} \delta_{v_{1}v_{3}} \delta_{u_{2}v_{3}} - \delta_{u_{3}v_{4}} \delta_{v_{1}v_{3}} \delta_{u_{2}v_{3}} \\
\vdots &
\end{align*}
\]

(B.5)

The following three relations are also used in sec3.

\[
\begin{align*}
\langle S(x_1) \gamma^{A_1 B_1} S(x_2) \rangle &+ \langle S(x_1) \gamma^{A_2 B_2} S(x_2) \rangle + \langle S(x_1) \gamma^{A_3 B_3} S(x_2) \rangle \\
\rangle &+ \langle S(x_1) \gamma^{A_4 B_4} S(x_2) \rangle \\
\rangle &+ \langle S(x_1) \gamma^{A_5 B_5} S(x_2) \rangle \\
\rangle &+ \langle S(x_1) \gamma^{A_6 B_6} S(x_2) \rangle
\end{align*}
\]

(B.6)
and

\[
\begin{align*}
S(x_1) & \gamma^{A_1 B_1} S(x_1) \ S(x_2) \gamma^{A_2 B_2} S(x_2) \ S(x_3) \gamma^{A_3 B_3} S(x_3) \ S(x_4) \gamma^{A_4 B_4} S(x_4) \\
S(y) & \gamma^{A_5 B_5} S(y) \ S(y) \gamma^{A_6 B_6} S(y)
\end{align*}
\]

\[
= \frac{8 \text{Tr}[\gamma^{A_3 B_3} \gamma^{A_4 B_4}]}{(x_3 - x_4)^2 (x_2 - y)^2} \left( \text{Tr}[\gamma^{A_1 B_1 \gamma^{A_6 B_6}}] \text{Tr}[\gamma^{A_2 B_2 \gamma^{A_5 B_5}}] + \text{Tr}[\gamma^{A_1 B_1 \gamma^{A_5 B_5}}] \text{Tr}[\gamma^{A_2 B_2 \gamma^{A_6 B_6}}] \right) \\
+ \frac{8 \text{Tr}[\gamma^{A_2 B_2 \gamma^{A_4 B_4}}]}{(x_2 - y)^2 (x_3 - y)^2} \left( \text{Tr}[\gamma^{A_1 B_1 \gamma^{A_6 B_6}}] \text{Tr}[\gamma^{A_3 B_3 \gamma^{A_5 B_5}}] + \text{Tr}[\gamma^{A_1 B_1 \gamma^{A_5 B_5}}] \text{Tr}[\gamma^{A_3 B_3 \gamma^{A_6 B_6}}] \right) \\
+ \frac{32}{(x_2 - y)^2 (x_3 - y)^2} \left( \text{Tr}[\gamma^{A_1 B_1 \gamma^{A_6 B_6}}] \text{Tr}[\gamma^{A_2 B_2 \gamma^{A_4 B_4 \gamma^{A_3 B_3 \gamma^{A_5 B_5}}}}] + \text{Tr}[\gamma^{A_1 B_1 \gamma^{A_5 B_5}}] \text{Tr}[\gamma^{A_2 B_2 \gamma^{A_4 \gamma^{A_3 B_3 \gamma^{A_6 B_6}}}}] \right)
\]

(C) More details on proof of (36)

In sec3, it was stated that the four S-correlator eq(36) vanishes. As an illustration, the computation for the coefficient of \((\zeta_2 \cdot \zeta_3)\) was presented. Below we give two additional coefficients, the coefficient of \((\zeta_2 \cdot \zeta_4)\) and the coefficient of \(\zeta \cdot k \cdot \zeta \cdot k \cdot \zeta \cdot k\).
C.1 coeff. of $\left(\zeta_2 \cdot \zeta_4\right)$

In sec3, it was stated

$$< V^u(x_1)V^u(x_2)V^u(x_3)V^u(x_4); \gamma^{AB}SS\gamma^{CD}S > = 0$$  \hfill (C.1)

We showed that terms with $\left(\zeta_2 \cdot \zeta_3\right)$ add to zero. Here we present more details taking terms with a factor, $\zeta_2 \cdot \zeta_4$, and terms with a factor, $\zeta \cdot k \cdot \zeta \cdot k \cdot \zeta \cdot k$ for illustrations. One can show that

$$\frac{17}{96} < VVVV (S\gamma^{uu})(S\gamma^{uu}) > = \frac{17}{96} l^4 \epsilon_{x_1}^2 \zeta_2 \cdot \zeta_4 \left\{ 4 \left[ \frac{\alpha'}{(x_2 - x_4)^2} \right] \zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_1 \\
- l^2 \left[ \frac{(-2\alpha')}{(x_2 - x_4)} \left( \frac{i\zeta_3 \cdot k_2}{x_3 - x_2} + \frac{i\zeta_3 \cdot k_4}{x_3 - x_4} \right) (\zeta_1 \cdot k_2 t - \zeta_1 \cdot k_4 s) \right] \\
- l^4 \left[ \frac{1}{(x_2 - x_4)^2} (\zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_1) u \right] \\
+ \frac{1}{2(x_3 - x_4)} \frac{1}{(x_2 - x_4)} (-s\zeta_1 \cdot k_2 \cdot \zeta_3 \cdot k_1 + u\zeta_1 \cdot k_2 \cdot \zeta_3 \cdot k_1 + t\zeta_1 \cdot k_4 \cdot \zeta_3 \cdot k_1 \\
- u\zeta_1 \cdot k_4 \cdot \zeta_3 \cdot k_2 - s\zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_4 + t\zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_2 - u\zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_1) \\
- \frac{1}{2(x_3 - x_4)} \frac{1}{(x_2 - x_4)} (s\zeta_1 \cdot k_2 \cdot \zeta_3 \cdot k_1 - u\zeta_1 \cdot k_2 \cdot \zeta_3 \cdot k_4 + s\zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_4 \\
- u\zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_1 + t\zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_2 + t\zeta_1 \cdot k_4 \cdot \zeta_3 \cdot k_1 - u\zeta_1 \cdot k_4 \cdot \zeta_3 \cdot k_2) \\
+ \frac{1}{2(x_3 - x_4)} \frac{1}{(x_2 - x_4)} (-u\zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_1 + s\zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_4 - t\zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_2 \\
+ s\zeta_1 \cdot k_2 \cdot \zeta_3 \cdot k_1 - u\zeta_1 \cdot k_2 \cdot \zeta_3 \cdot k_4 - t\zeta_1 \cdot k_4 \cdot \zeta_3 \cdot k_1 + u\zeta_1 \cdot k_4 \cdot \zeta_3 \cdot k_2) \right\} \hfill (C.2)$$

Similarly,

$$\frac{1}{6} < VVVV (S\gamma^{ab})(S\gamma^{ab}) > = \frac{112}{3} l^4 \epsilon_{x_1}^2 \zeta_2 \cdot \zeta_4 \left\{ 4 \left[ \frac{\alpha'}{(x_2 - x_4)^2} \right] \zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_1 \\
- l^2 \left[ \frac{(-2\alpha')}{(x_2 - x_4)} \left( \frac{i\zeta_3 \cdot k_2}{x_3 - x_2} + \frac{i\zeta_3 \cdot k_4}{x_3 - x_4} \right) \frac{1}{(x_2 - x_4)} (\zeta_1 \cdot k_2 t - \zeta_1 \cdot k_4 s) \right] \\
- l^4 \left[ \frac{1}{(x_2 - x_4)^2} (\zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_1) u \right] \\
+ \frac{1}{2(x_3 - x_4)} \frac{1}{(x_2 - x_4)} (-s\zeta_1 \cdot k_2 \cdot \zeta_3 \cdot k_1 + u\zeta_1 \cdot k_2 \cdot \zeta_3 \cdot k_4 + t\zeta_1 \cdot k_4 \cdot \zeta_3 \cdot k_1 \\
- u\zeta_1 \cdot k_4 \cdot \zeta_3 \cdot k_2 - s\zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_4 + t\zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_2 - u\zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_1) \right\} \hfill (C.2)$$
\[
-u_{1} \cdot k_{4} \cdot \zeta_{3} \cdot k_{2} - s_{1} \cdot k_{3} \cdot \zeta_{3} \cdot k_{4} + t_{1} \cdot k_{3} \cdot \zeta_{3} \cdot k_{2} - u_{1} \cdot k_{3} \cdot \zeta_{3} \cdot k_{1} \\
-\frac{1}{2(x_{3} - x_{1})(x_{2} - x_{3})}(s_{1} \cdot k_{2} \cdot \zeta_{3} \cdot k_{1} - u_{1} \cdot k_{2} \cdot \zeta_{3} \cdot k_{4} + s_{1} \cdot k_{3} \cdot \zeta_{3} \cdot k_{4}) \\
-\frac{1}{2(x_{2} - x_{3})(x_{2} - x_{4})}(-u_{1} \cdot k_{3} \cdot \zeta_{3} \cdot k_{1} + s_{1} \cdot k_{3} \cdot \zeta_{3} \cdot k_{4} - t_{1} \cdot k_{3} \cdot \zeta_{3} \cdot k_{2}) \\
+ \frac{1}{2(x_{2} - x_{3})(x_{2} - x_{4})}(s_{1} \cdot k_{1} \cdot \zeta_{3} \cdot k_{1} - s_{1} \cdot k_{2} \cdot \zeta_{3} \cdot k_{4} - t_{1} \cdot k_{1} \cdot \zeta_{3} \cdot k_{2} + s_{1} \cdot k_{1} \cdot \zeta_{3} \cdot k_{2})
\]
\]

Now consider the terms that come with \( \frac{2}{r_0^2} X_0^a X_0^b \):

\[
\frac{35}{24} < VVVV (S_{\gamma}^{ac} S)(S_{\gamma}^{bc} S) >
\]

\[
= -\frac{35 \cdot 7 \cdot 4}{24} \left\{ \frac{l^4}{\epsilon x_1^2} \zeta_2 \cdot \zeta_4 \left[ \frac{\alpha'}{\epsilon x_1^2 (x_2 - x_4)^2} \right] \zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_1 \\
- \frac{l^2}{\epsilon x_1^2 (x_2 - x_4)^2} \left( -2\alpha' \right) \left( \frac{i \zeta_3 \cdot k_2}{x_3 - x_2} + \frac{i \zeta_3 \cdot k_4}{x_3 - x_4} \right) \frac{1}{\epsilon x_1^2 (x_2 - x_4)} (\zeta_1 \cdot k_2 \cdot t - \zeta_1 \cdot k_4 \cdot s) \right\} \\
- \frac{l^4}{\epsilon x_1^2 (x_2 - x_4)^2} \left( \frac{1}{(x_2 - x_4)^2} (\zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_1) u \right)
\]

\[
+ \frac{1}{2(x_3 - x_1)(x_2 - x_4)} \left( -s_{1} \cdot k_{2} \cdot \zeta_{3} \cdot k_{1} + u_{1} \cdot k_{2} \cdot \zeta_{3} \cdot k_{4} + t_{1} \cdot k_{3} \cdot \zeta_{3} \cdot k_{1} - u_{1} \cdot k_{3} \cdot \zeta_{3} \cdot k_{4} + s_{1} \cdot k_{3} \cdot \zeta_{3} \cdot k_{1} - u_{1} \cdot k_{1} \cdot \zeta_{3} \cdot k_{4} + t_{1} \cdot k_{1} \cdot \zeta_{3} \cdot k_{2} \\
- \frac{1}{2(x_3 - x_1)(x_2 - x_3)}(s_{1} \cdot k_{2} \cdot \zeta_{3} \cdot k_{1} - u_{1} \cdot k_{2} \cdot \zeta_{3} \cdot k_{4} + s_{1} \cdot k_{3} \cdot \zeta_{3} \cdot k_{4}) \\
- \frac{1}{2(x_3 - x_2)(x_2 - x_4)}(-u_{1} \cdot k_{3} \cdot \zeta_{3} \cdot k_{1} + t_{1} \cdot k_{3} \cdot \zeta_{3} \cdot k_{2} + t_{1} \cdot k_{4} \cdot \zeta_{3} \cdot k_{1} - u_{1} \cdot k_{4} \cdot \zeta_{3} \cdot k_{2}) \\
+ \frac{1}{2(x_3 - x_2)(x_2 - x_4)}(-u_{1} \cdot k_{1} \cdot \zeta_{3} \cdot k_{1} + u_{1} \cdot k_{1} \cdot \zeta_{3} \cdot k_{2} - t_{1} \cdot k_{1} \cdot \zeta_{3} \cdot k_{2})
\right\}
\]

Therefore apart from some common factors one gets after taking Wick rotation into account

\[
\left\{ \left( \frac{q^2}{r_0^2} \right) \left[ -\frac{17}{96} (S_{\gamma}^{uv} S)(S_{\gamma}^{uv} S) - \frac{1}{6} (S_{\gamma}^{ab} S)(S_{\gamma}^{ab} S) \right] \\
+ \left( \frac{q^2}{r_0^2} \right) X_0^a X_0^b \left[ \frac{35}{24} (S_{\gamma}^{ac} S)(S_{\gamma}^{bc} S) \right] \right\}
\]

\[
\Rightarrow -\frac{2}{(x_2 - x_4)^2} \zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_1
\]
\(-\left(\frac{\zeta_3 \cdot k_2}{x_3 - x_2} + \frac{\zeta_3 \cdot k_4}{x_3 - x_4}\right) \cdot \frac{1}{(x_2 - x_4)}(\zeta_1 \cdot k_2 \cdot t - \zeta_1 \cdot k_4 s)\) \\
- \left[ \frac{1}{(x_2 - x_4)^2}(\zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_1)u \right] \\
+ \frac{1}{2(x_3 - x_4)(x_2 - x_4)} \left[ -s\zeta_1 \cdot k_2 \cdot \zeta_3 \cdot k_1 + u\zeta_1 \cdot k_2 \cdot \zeta_3 \cdot k_4 + t\zeta_1 \cdot k_4 \cdot \zeta_3 \cdot k_1 \\
- u\zeta_1 \cdot k_2 \cdot \zeta_3 \cdot k_4 \cdot s\zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_4 \cdot t\zeta_1 \cdot k_4 \cdot \zeta_3 \cdot k_1 \\
- u\zeta_1 \cdot k_3 \cdot \zeta_1 \cdot k_4 \cdot t\zeta_1 \cdot k_4 \cdot \zeta_3 \cdot k_2 + u\zeta_1 \cdot k_4 \cdot \zeta_3 \cdot k_2 \right] \\
\leq -2\zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_1 \\
- \left(\frac{\zeta_3 \cdot k_2}{x - 1} + \frac{\zeta_3 \cdot k_4}{x}\right) \cdot (\zeta_1 \cdot k_2 \cdot t - \zeta_1 \cdot k_4 s) \\
- (\zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_1)u \\
- \frac{1}{2x} \left[ -s\zeta_1 \cdot k_2 \cdot \zeta_3 \cdot k_1 + u\zeta_1 \cdot k_2 \cdot \zeta_3 \cdot k_4 + t\zeta_1 \cdot k_4 \cdot \zeta_3 \cdot k_1 \\
- u\zeta_1 \cdot k_2 \cdot \zeta_3 \cdot k_4 \cdot s\zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_4 \cdot t\zeta_1 \cdot k_4 \cdot \zeta_3 \cdot k_1 \\
- u\zeta_1 \cdot k_3 \cdot \zeta_1 \cdot k_4 \cdot t\zeta_1 \cdot k_4 \cdot \zeta_3 \cdot k_2 + u\zeta_1 \cdot k_4 \cdot \zeta_3 \cdot k_2 \right] \\
\leq -\frac{1}{2(1 - x)} \left[ -u\zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_1 + s\zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_4 - t\zeta_1 \cdot k_3 \cdot \zeta_3 \cdot k_2 \\
+ s\zeta_1 \cdot k_2 \cdot \zeta_3 \cdot k_2 - u\zeta_1 \cdot k_2 \cdot \zeta_3 \cdot k_4 - t\zeta_1 \cdot k_4 \cdot \zeta_3 \cdot k_1 + u\zeta_1 \cdot k_4 \cdot \zeta_3 \cdot k_2 \right] (C.5) \\
which leads to a vanishing result after the \(x\)-integration. 

C.2 coeff. of \(\zeta \cdot k \cdot \zeta \cdot k \cdot \zeta \cdot k \cdot \zeta \cdot k\)

Similarly one finds after some algebra

\[
\left\{ \left( \frac{q^2}{r_0^2} \right) \left[ -\frac{17}{96} (S_{\gamma}^{uu} S) (S_{\gamma}^{uu} S) - \frac{1}{6} (S_{\gamma}^{ab} S) (S_{\gamma}^{ab} S) \right] + \left( \frac{q^2}{r_0^2} \right) X_0^a X_0^b \left[ \frac{35}{24} (S_{\gamma}^{ac} S) (S_{\gamma}^{bc} S) \right] \right\} \\
= - \left( -\zeta_4 \cdot k_2 - \frac{\zeta_4 \cdot k_3}{x} \right) \left( \frac{\zeta_3 \cdot k_2}{x - 1} + \frac{\zeta_3 \cdot k_4}{x} \right) \zeta_1 \cdot k_2 \cdot \zeta_2 \cdot k_1 \right)
\]
where Wick rotation has been taken into account. Once again the $x$-integration yields zero.
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