ON HIGHER-DIMENSIONAL COURANT ALGEBROIDS

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Abstract. We define the transgression functor which associates to a (higher-
dimensional) Courant algebroid on a manifold a Lie algebroid on the shifted
tangent bundle of the manifold.

1. Introduction

Courant algebroids ([C], [D], [LWX], [B], [S1], [S2]), as well as higher-
dimensional generalizations thereof ([Z], [V2], [CR]) are objects of finite di-

mensional differential geometry built around Leibniz (rather than Lie) brack-
ets which give rise to field-theoretic Poisson brackets. Leibniz brackets arise as
derived brackets ([KS1], [KS2]) and it is natural to expect Courant algebroids
to arise in this way. Such a construction was given by D. Roytenberg in [R]
and P. Ševera in [S2], Letter 7; (see also [K-S], [U], [Z]).

We describe a construction, which we refer to as transgression, which asso-
ciates to a Courant algebroid \( Q \) on a manifold \( X \) (see Definition 4.1) a Lie
algebroid \( \tau Q \) on the differential graded manifold \( X^\sharp \) (frequently denoted by
\( T[1]X \) in the literature) with the underlying space \( X \) and the structure sheaf
the de Rham complex of \( X \). The Lie algebroid \( \tau Q \) possesses (and, hence, is
uniquely determined by) a universal property with respect to suitably defined
Courant-to-Lie morphisms (see Definition 5.1) of \( Q \) to Lie algebroids on
\( X^\sharp \).

The construction is given in Section 5 which is devoted to the proof of the
main result of the paper (Theorem 5.3 and Corollary 5.4). Furthermore, in
Theorem 6.2 we show that for a special class of Courant algebroids (Courant
extensions of Lie algebroids, see Definition 4.5) transgression restricts to an
equivalence of appropriately defined categories. The Courant algebroid \( Q \) ap-
ppears as a graded component of \( \tau Q \) and is equipped with the structure (the
Leibniz bracket in particular) derived from the Lie algebroid structure of the
latter.

The present work generalizes and expands upon the results of [CR] which
treated the case of exact (higher dimensional) Courant algebroids.

In Section 2 we review some basics of the theory of differential-graded man-
ifolds; (see also [R], [S1], [V1]). In particular, we show that, for \( X \) a manifold,
the differential-graded manifold \( X^\sharp \) is a mapping space, a well-known fact,
[KS].
In Section 3 we define the transgression functors which associate to Lie algebroids and modules over these on $X$ corresponding objects on $X^\#$. In addition we introduce marked Lie algebroids examples of which are the Lie algebroids $\tau Q$ arising from Courant algebroids via transgression.

In Section 4 we give the definition of (higher-dimensional) Courant algebroids and study basic properties thereof, including a review of the classification of exact Courant algebroids. Most of the results are direct generalizations of the corresponding results of [B].

Section 5 contains the key construction of the present paper, namely, of transgression for Courant algebroids, as well as the principal results, Theorem 5.3, Corollary 5.4.

In Section 6 we specialize to the case of Courant extensions and show (Theorem 6.2) that transgression restricts to an equivalence of appropriately defined categories.

For the convenience of the reader those proofs which consist of straightforward but lengthy verification of identities are delegated to Section 7.

1.1. Notation. In order to simplify notations in numerous signs we will write “$a$” instead of “deg($a$)” in expressions appearing in exponents of $-1$. For example, $(-1)^{ab-1}$ stands for $(-1)^{\deg(a)\cdot \deg(b)-1}$.

Throughout the paper “manifold” means a $C^\infty$, real analytic or complex manifold. For a manifold $X$ we denote by $\mathcal{O}_X$ the corresponding structure sheaf of complex valued $C^\infty$, respectively analytic or holomorphic functions. We denote by $\mathcal{T}_X$ (respectively, by $\Omega^k_X$) the sheaf of complex valued vector fields (respectively, differential forms of degree $k$) on $X$.

2. The odd path space

2.1. DG-manifolds. For the purposes of the present note a differential-graded manifold (DG-manifold) is a pair $\mathfrak{X} := (X, \mathcal{O}_X)$, where $X$ is a manifold and $\mathcal{O}_X$ is a sheaf of commutative differential-graded algebras (CDGA) on $X$ locally isomorphic to one of the form $\mathcal{O}_X \otimes S(E)$, where $S(E)$ is the symmetric algebra of a finite-dimensional graded vector space $E$.

Let $\mathfrak{X} = (X, \mathcal{O}_X)$ and $\mathfrak{Y} = (Y, \mathcal{O}_Y)$ be DG-manifolds. A morphism $\phi: \mathfrak{X} \to \mathfrak{Y}$ is a morphism of ringed spaces, which is to say a map $\phi: X \to Y$ of manifolds together with the morphism of graded algebras $\phi^*: \phi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ compatible with the map $\phi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$.

We denote the category of DG-manifolds by $\text{dgMan}$. Let $\text{dgMan}^+$ denote the full subcategory of DG-manifolds $\mathfrak{X} = (X, \mathcal{O}_X)$ such that $\mathcal{O}^i_X = 0$ if $i < N$ for some $N \in \mathbb{Z}$.

Example 2.1. An ordinary manifold is an example of a DG-manifold with the structure sheaf concentrated in degree zero. Each ordinary manifold $X$
determines a DG-manifold $X^\sharp \in \text{dgMan}^+$ defined by $X^\sharp = (X, \Omega^\cdot_X, d)$ and frequently denoted by $T[1]X$ in the literature. There is a canonical morphism $X \to X^\sharp$ of DG-manifolds defined by the canonical map $\Omega^\cdot_X \to O_X$.

**Example 2.2.** Let $\vec{t}$ denote the DG-manifold with the underlying space consisting of one point and the DG-algebra of functions $O_{\vec{t}} = C[\epsilon]$, the free graded commutative algebra with generator $\epsilon$ of degree $-1$ and the differential $\partial \epsilon : \epsilon \mapsto \epsilon + 1$. Note that $\vec{t} \in \text{dgMan}^+$.

**2.2. Products.** The category $\text{dgMan}^+$ has finite products. Suppose that $X = (X, O_X), Y = (Y, O_Y) \in \text{dgMan}^+$.

If $O_X \cong O_X \otimes S(E)$ and $O_Y \cong O_X \otimes S(F)$, where $E$ and $F$ are finite-dimensional graded vector spaces, then

$$O_{X \times Y} \otimes O_X \otimes O_Y \cong O_{X \times Y} \otimes S(E \oplus F)$$

and is bounded below.

Let

$$O_{X \times Y} := O_X \times O_Y \otimes O_X \otimes O_Y.$$

Then, $(X \times Y, O_{X \times Y}) \in \text{dgMan}^+$.

The morphism $\text{pr}_X : X \times Y \to X$ is defined as the projection $\text{pr}_X : X \times Y \to X$ together with the map $\text{pr}^{-1}_X : X \times Y \to O_{X \times Y}$ defined as the composition

$$\text{pr}^{-1}_X O_X \xrightarrow{id \otimes 1} O_X \otimes O_Y \xrightarrow{1 \otimes id} O_{X \times Y}. $$

The morphism $\text{pr}_Y : X \times Y \to Y$ is defined similarly.

**Lemma 2.3.** The DG-manifold $(X \times Y, O_{X \times Y}) \in \text{dgMan}^+$ together with the morphisms $\text{pr}_X$ and $\text{pr}_Y$ represents the product of $X$ and $Y$.

**Proof.** Left to the reader. \qed

**2.3. The odd path space.** We briefly recall the basic facts concerning the internal hom in presheaves. For a category $\mathcal{C}$ we denote by $\hat{\mathcal{C}}$ the category of presheaves on $\mathcal{C}$. For $F, G \in \hat{\mathcal{C}}$ the presheaf $\text{Hom}(F, G)$ is defined by

$$\text{Hom}(F, G)(c) = \text{Hom}_{\hat{\mathcal{C}}}(c, \text{Hom}(F, G)) := \text{Hom}_{\hat{\mathcal{C}}}(c \times F, G)$$

for $c \in \mathcal{C}$. Then, for any $H \in \hat{\mathcal{C}}$,

$$\text{Hom}_{\hat{\mathcal{C}}}(H, \text{Hom}(F, G)) \cong \text{Hom}_{\hat{\mathcal{C}}}(H \times F, G)$$

naturally in $F, G$ and $H$. The case $H = \text{Hom}(F, G)$ and $id_{\text{Hom}(F, G)}$ gives rise to the evaluation map $\text{ev} : \text{Hom}(F, G) \times F \to G$. The case $H = G \times F$ and the projection $\text{pr} : G \times F \to G$ gives rise to the map $i : G \to \text{Hom}(F, G)$. In other words, internal hom is defined as the right adjoint to the product and the maps $\text{ev}$ and $i$ are the counit and the unit of the adjunction.
Theorem 2.4. For a manifold $X$ the DG manifold $X^\sharp$ (see Example [2.1]) represents the presheaf $\text{Hom}(\mathfrak{I}, X)$ on $\text{dgMan}^+$.

Implicit in the statement of the theorem are natural bijections

$(2.3.1) \quad \text{Hom}(\mathfrak{I}, X)(\mathfrak{Y}) = \text{Hom}(\mathfrak{Y} \times \mathfrak{I}, X) \xrightarrow{\phi \mapsto \phi^\sharp} \text{Hom}(\mathfrak{Y}, X^\sharp).$

and

$(2.3.2) \quad \text{Hom}(\mathfrak{Y}, X^\sharp) \xrightarrow{\psi \mapsto \psi^\sharp} \text{Hom}(\mathfrak{Y} \times \mathfrak{I}, X) = \text{Hom}(\mathfrak{I}, X)(\mathfrak{Y}).$

which we are going to define presently.

Let $\phi: \mathfrak{Y} \times \mathfrak{I} \to X$ be a morphism of DG manifolds. The morphism $\phi$ is given by a map of manifolds $\phi: Y \to X$ and a morphism of DGA $\phi^*: \phi^{-1}O_X \to O_{\mathfrak{Y}}[\epsilon]$. The only non-trivial component of the latter is $\phi^*: \phi^{-1}O_X \to O_{\mathfrak{Y}}[\epsilon]$ which we will write as $\phi^*(f) = \phi_0^*(f) + \phi_1^*(f) \cdot \epsilon$ for $f \in O_X$.

Since $O_X$ is concentrated in degree zero it follows that $\partial_{\mathfrak{Y} \times \mathfrak{I}} \circ \phi^* = 0$ which implies that $\partial_{\mathfrak{Y}} \phi_0^*(f) = \phi_1^*(f)$ for $f \in O_X$. Thus, $\phi_1^*$ is a derivation, hence give rise to the $O_X$-linear (in the sense of $\phi_0^*$) map $\phi_1^*: \phi^{-1}\Omega^1_X \to O_{\mathfrak{Y}}$ making the diagram

\[
\begin{array}{ccc}
\phi^{-1}\Omega^1_X & \xrightarrow{\phi_1^*} & O_{\mathfrak{Y}}^1 \\
\downarrow & & \uparrow \partial_{\mathfrak{Y}} \\
\phi^{-1}O_X & \xrightarrow{\phi_0^*} & O_{\mathfrak{Y}}^0 
\end{array}
\]

commutative. Therefore, there exists a unique map of DGA $(\phi^\sharp)^*: \phi^{-1}O_X^\sharp = \phi^{-1}\Omega^\cdot_X \to O_{\mathfrak{Y}}$

whose components in degrees zero and one are equal, respectively, to $\phi_0^*$ and $\phi_1^*$. The pair of maps $\phi: Y \to X$ and $(\phi^\sharp)^*$ define a morphism of DG manifolds $\phi^\sharp: \mathfrak{Y} \to X^\sharp$. The assignment $\phi \mapsto \phi^\sharp$ defines the map (2.3.1).

Suppose that $\psi: \mathfrak{Y} \to X^\sharp$ is a morphism of DG manifolds given by a map of manifolds $\psi: Y \to X$ and a morphism of DGA $\psi^*: \psi^{-1}O_X^\sharp \to O_{\mathfrak{Y}}$. Let

$(2.3.3) \quad (\psi^\sharp)^*: \psi^{-1}O_X \to O_{\mathfrak{Y}}[\epsilon]$

denote the map whose only non-trivial component in degree zero is given by $(\psi^\sharp)^*(f) = \psi^0(f) + \psi^1(df) \cdot \epsilon$.

Lemma 2.5. The map (2.3.3) is a morphism of DGA.

Proof. The map $(\psi^\sharp)^*$ is a map of complexes since

\[
\partial_{\mathfrak{Y} \times \mathfrak{I}}(\psi^0(f) + \psi^1(df) \cdot \epsilon) = (\partial_{\mathfrak{Y}} \pm \partial_{\mathfrak{I}})(\psi^0(f) + \psi^1(df) \cdot \epsilon) = \partial_{\mathfrak{Y}}(\psi^0(f) + \psi^1(df) \cdot \epsilon) - \psi^1(df) \cdot \partial_{\mathfrak{I}}\epsilon = \psi^1(df) \cdot \partial_{\mathfrak{I}}\epsilon
\]

\[
= 0
\]
and a map of algebras since
\[(\psi_b)^*(fg) = \psi^0(fg) + \psi^1(df + gdf) \cdot \epsilon \]
\[= \psi^0(f)\psi^0(g) + \psi^1(fdg + gdf) \cdot \epsilon \]
\[= \psi^0(f)\psi^0(g) + \psi^1(fdg) \cdot \epsilon + \psi^1(gdf) \cdot \epsilon \]
\[= \psi^0(f)\psi^0(g) + \psi^0(f)\psi^1(dg) \cdot \epsilon + \psi^0(g)\psi^1(df) \cdot \epsilon \]
\[= (\psi_b^* f)(\psi_b^* g). \]

It follows from the lemma above that the pair of maps \(\psi: \mathfrak{J} \to X\) and \((\psi_b)^*\) define a morphism of DG manifolds \(\psi_b: \mathfrak{J} \times \mathfrak{I} \to X\). The assignment \(\psi \mapsto \psi_b\) defines the map \((2.3.2)\).

**Proof of Theorem 2.4.** We leave it to the reader to verify that \((2.3.1)\) and \((2.3.2)\) are mutually inverse and natural in the variable \(\mathfrak{J}\). i.e. define an isomorphism of presheaves \(\mathcal{H}om(\mathfrak{I}, X) \cong X^2\).

Let \((2.3.4)\)
\[
ev^*: \mathcal{O}_X \to \mathcal{O}_{X^2}[\epsilon] := \mathcal{O}_{X^2 \times \mathfrak{I}}
\]
denote the map defined by \(f \mapsto f + df \cdot \epsilon\).

**Lemma 2.6.** The map \((2.3.4)\) is a morphism of DG-algebras.

**Proof.** Let \(f, g \in \mathcal{O}_X\). The calculation
\[
ev^*(f) \cdot \ev^*(g) = (f + df \cdot \epsilon) \cdot (g + dg \cdot \epsilon)
\]
\[= fg + fdg \cdot \epsilon + gdf \cdot \epsilon
\]
\[= fg + d(fg) \cdot \epsilon = \ev^*(fg)
\]
shows that \((2.3.4)\) is a morphism of graded algebras. The computation
\[
\partial_{X^2 \times \mathfrak{I}}(\ev^*(f)) = \partial_{X^2 \times \mathfrak{I}}(f + df \cdot \epsilon)
\]
\[= df - df \cdot \partial_\epsilon
\]
\[= df - df = 0.
\]
shows that \((2.3.4)\) is a map of complexes. \(\square\)

Let \((2.3.5)\)
\[
\ev: X^2 \times \mathfrak{I} \to X
\]
denote the map of DG-manifolds with the structure morphism given by \((2.3.4)\).

**Lemma 2.7.** The isomorphism \(X^2 \to \mathcal{H}om(\mathfrak{I}, X)\) given by \((2.3.1)\) corresponds to the map \((2.3.5)\).

**Proof.** Left to the reader. \(\square\)
2.4. Transgression for $\mathcal{O}$-modules. We denote by $\text{pr}: X^\sharp \times \bar{t} \to X^\sharp$ the canonical projection. The diagram

$$\begin{array}{ccc}
X^\sharp \times \bar{t} & \xrightarrow{\text{ev}} & X \\
\text{pr} \downarrow & & \downarrow \\
X^\sharp
\end{array}$$

gives rise to the functor

$$(2.4.1) \quad \text{pr}_* \text{ev}^*: \mathcal{O}_X^{-\text{Mod}} \to \mathcal{O}_{X^\sharp}^{-\text{Mod}}.$$ 

Since the underlying space of both $X^\sharp$ and $X^\sharp \times \bar{t}$ is equal to $X$, the functor $\text{ev}^*$ is given by $\text{ev}^* \mathcal{E} = \mathcal{O}_{X^\sharp}[\varepsilon] \otimes_{\mathcal{O}_X} \mathcal{E}$ and the effect of the functor $\text{pr}_*$ amounts to restriction of scalars along the unit map $\mathcal{O}_{X^\sharp} \to \mathcal{O}_{X^\sharp}[\varepsilon]$.

Lemma 2.8. For $k \geq 0$ the diagram

$$\Omega^k_X \otimes_{\mathcal{O}_X} \Omega^1_X \longrightarrow \Omega^k_X \otimes_{\mathcal{O}_X} (\mathcal{O}_{X^\sharp}[\varepsilon])^0$$

is a push-out square.

Remark 2.9. Suppose that $\mathcal{E}$ is a vector bundle on $X$. The complex $\text{pr}_* \text{ev}^* \mathcal{E}$ with differential denoted $\partial$ may be recognized in more traditional terms as follows.

The differential $\mathcal{E} = (\text{pr}_* \text{ev}^* \mathcal{E})^{-1} \xrightarrow{\partial} (\text{pr}_* \text{ev}^* \mathcal{E})^0$ is a differential operator of order one and therefore factors as $\mathcal{E} \xrightarrow{j^1} \mathcal{J}^1(\mathcal{E}) \to (\text{pr}_* \text{ev}^* \mathcal{E})^0$ the second map being $\mathcal{O}_X$-linear. It is easy to see that it is an isomorphism.

One deduces easily from Lemma 2.8 and the above isomorphism that for $k \geq 0$ the square

$$\Omega^k_X \otimes_{\mathcal{O}_X} \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E} \longrightarrow \Omega^k_X \otimes_{\mathcal{O}_X} \mathcal{J}^1(\mathcal{E})$$

is cocartesian. The differential $\partial$ on $\text{pr}_* \text{ev}^* \mathcal{E}$ is induced by the canonical flat connection on $\mathcal{J}^\infty(\mathcal{E})$. Namely, the composition

$$\Omega^k_X \otimes_{\mathcal{O}_X} \mathcal{J}^\infty(\mathcal{E}) \xrightarrow{\text{conn}} \Omega^k_{X^\sharp} \otimes_{\mathcal{O}_X} \mathcal{J}^\infty(\mathcal{E}) \to \Omega^k_{X^\sharp} \otimes_{\mathcal{O}_X} \mathcal{J}^1(\mathcal{E}) \to (\text{pr}_* \text{ev}^* \mathcal{E})^{k+1}$$

factors as

$$\Omega^k_X \otimes_{\mathcal{O}_X} \mathcal{J}^\infty(\mathcal{E}) \to \Omega^k_X \otimes_{\mathcal{O}_X} \mathcal{J}^1(\mathcal{E}) \to (\text{pr}_* \text{ev}^* \mathcal{E})^k \xrightarrow{\partial} (\text{pr}_* \text{ev}^* \mathcal{E})^{k+1}.$$
2.5. The “integration” map. The unique map of complexes
\[ \mathcal{O}_X \otimes_{\mathbb{C}} \Omega^k_X \to \mathcal{O}_X \] whose component of degree \(-1\) is the identity map (and, whose component of degree zero equal, therefore, to the de Rham differential) extends by linearity to the map
\[ \tilde{f}: \mathcal{O}_X[\epsilon] \otimes_{\mathbb{C}} \Omega^k_X \to \mathcal{O}_X[1]. \]
The calculation
\[ \tilde{f}(f + df \cdot \epsilon \otimes \omega) = fd\omega + df \wedge \omega = \tilde{f}(1 \otimes f \omega) \]
shows that the map \(\tilde{f}\) factors through the map
\[ (2.5.1) \quad f: \text{pr}_* \text{ev}^* \Omega^k_X = \mathcal{O}_X[\epsilon] \otimes_{\mathcal{O}_X} \Omega^k_X \to \mathcal{O}_X[1]. \]

3. Transgression for Lie algebroids

3.1. Lie algebroids. An \(\mathcal{O}_X\)-Lie algebroid structure on an \(\mathcal{O}_X\)-module \(A\) consists of
(1) a structure of a \(\mathbb{C}\)-Lie algebra \([\cdot , \cdot]_i: A \otimes_{\mathbb{C}} A \to A\);
(2) an \(\mathcal{O}_X\)-linear map \(\sigma: A \to T_X\) of Lie algebras called anchor map.
These data are required to satisfy the compatibility condition (Leibniz rule)
\[ [a, f \cdot b] = \sigma(a)(f) \cdot b + f \cdot [a, b] \]
for \(a, b \in A\) and \(f \in \mathcal{O}_X\).
A morphism of \(\mathcal{O}_X\)-Lie algebroids \(\phi: A_1 \to A_2\) is an \(\mathcal{O}_X\)-linear map of Lie algebras which commutes with respective anchor maps.
With the above definition of morphisms \(\mathcal{O}_X\)-Lie algebroids form a category denoted \(\mathcal{O}_X-\text{LieAlgd}\).
The notion of Lie algebroid generalizes readily to the DG context.

Remark 3.1. Suppose that \(\mathcal{X} = (X, \mathcal{O}_X)\) is a DG-manifold such that \(\mathcal{O}_X^i = 0\) for \(i \leq -1\) and \(A\) is a Lie algebroid on \(\mathcal{X}\). Then, the Leibniz rule implies that for \(i, j \leq -1\) the brackets \([\cdot, \cdot]_i^j: A^i \otimes_{\mathbb{C}} A^j \to A^{i+j}\) are in fact \(\mathcal{O}_X^0\)-bilinear. Indeed, for \(f \in \mathcal{O}_X^0\) and \(a \in A^i\), \(i \leq -1\), \(\sigma(a)(f)\) has strictly negative degree.

3.2. Atiyah algebras. We recall the definition of the Atiyah algebra of an \(\mathcal{O}\)-module.
For an \(\mathcal{O}_X\)-module \(\mathcal{E}\) the Atiyah algebra \(A_{\mathcal{E}}\) is defined as the sheaf whose sections are pairs \((P, \xi)\), where
- \(P \in \text{End}_{\mathbb{C}}(\mathcal{E})\),
- \(\xi \in T_X\).
subject to the condition
\[ \forall f \in \mathcal{O}_X \ [P, f] = \xi(f) \in \text{End}_C(\mathcal{E}). \]

In other words, \( P \) is a differential operator of order one with the principal symbol \( \sigma(P) \) equal to \( \xi \cdot \text{id} \); in particular, \( \xi \) is uniquely determined by \( P \).

The assignment \((P, \xi) \mapsto \sigma(P) = \xi \cdot \text{id}; \) defines the map \( \sigma: \mathcal{A}_E \to \mathcal{T}_X \). With the obvious \( \mathcal{O}_X \)-structure, the bracket given by the commutator of operators and the principal symbol map as anchor, the Atiyah algebra acquires the canonical structure of an \( \mathcal{O}_X \)-Lie algebroid. If \( \mathcal{E} \) is locally free of finite rank, i.e. a vector bundle, the sequence
\[ 0 \to \text{End}_{\mathcal{O}_X}(\mathcal{E}) \to \mathcal{A}_E \xrightarrow{\sigma} \mathcal{T}_X \to 0 \]
is exact.

The above definition of Atiyah algebra generalizes readily to the DG context (cf. [FN]).

3.3. Modules over Lie algebroids. Suppose that \( \mathcal{A} \) is an \( \mathcal{O}_X \)-Lie algebroid.

A structure of an \( \mathcal{A} \)-module on an \( \mathcal{O}_X \)-module \( \mathcal{E} \) is a morphism of \( \mathcal{O}_X \)-Lie algebroids \( \mathcal{A} \to \mathcal{A}_E \), i.e. an action of \( \mathcal{A} \) on \( \mathcal{E} \) by differential operators of order one which satisfies the Leibniz rule.

A morphism \( \phi: \mathcal{E}_1 \to \mathcal{E}_2 \) of \( \mathcal{A} \)-modules is an \( \mathcal{O}_X \)-linear map which commutes with respective actions.

With the above definition of morphisms \( \mathcal{A} \)-modules form a category denoted \( \mathcal{A} - \text{Mod} \).

3.4. Vector fields on the odd path space. The canonical map of Lie algebras \( L: \mathcal{T}_X \to \mathcal{T}_{X^1} \) defined by \( \xi \mapsto L\xi \) which extends to the map of graded Lie algebras \( L: \mathcal{O}_X \otimes \mathbb{C} \mathcal{T}_X \to \mathcal{T}_{X^1} \) by setting \( L_{\epsilon \xi} = \iota_{\xi} \) and, by linearity, to the map of \( \mathcal{O}_{X^1} \)-Lie algebroids
\[ (3.4.1) \quad \tilde{L}: \mathcal{O}_{X^1}[\epsilon] \otimes \mathbb{C} \mathcal{T}_X \to \mathcal{T}_{X^2} \]

Lemma 3.2.

(1) The map \( (3.4.1) \) factors as
\[ \mathcal{O}_{X^1}[\epsilon] \otimes \mathbb{C} \mathcal{T}_X \to \mathcal{O}_{X^1}[\epsilon] \otimes_{\mathcal{O}_X} \mathcal{T}_X = \text{pr}_x \text{ev}^* \mathcal{T}_X \to \mathcal{T}_{X^2}. \]

(2) The map \( \text{pr}_x \text{ev}^* \mathcal{T}_X \to \mathcal{T}_{X^2} \) is an isomorphism of \( \mathcal{O}_{X^1} \)-Lie algebroids.

Proof. The calculation
\[ \tilde{L}((f + df \cdot \epsilon) \otimes \xi) = f \cdot L\xi + df \cdot \iota_{\xi} = L_{f\xi} \]
implies the first claim.
Lemma 3.3. The structure of an $\mathcal{O}_{X^2}$-Lie algebroid on $\mathcal{O}_{X^2}[\epsilon] \otimes \mathcal{A}$ descends to one on $pr_*\ev^*\mathcal{A}$.

Proof. The proof is given in \ref{3.4.1}. \hfill $\Box$

Let $\mathcal{T}_{X^2/X} \subset \mathcal{T}_{X^2}$ denote the normalizer of $\mathcal{O}_X \subset \mathcal{O}_{X^2}$. The map of graded (but not DG) manifolds $p : X^2 \to X$ which corresponds to the inclusion $\mathcal{O}_X \hookrightarrow \mathcal{O}_{X^2}$ gives rise to the short exact sequence of $\mathcal{O}_{X^2}$-modules

$$0 \to \mathcal{T}_{X^2/X} \to \mathcal{T}_{X^2} \to \mathcal{O}_{X^2} \otimes \mathcal{O}_X \mathcal{T}_X \to 0.$$ 

Since the action of $\epsilon \otimes \mathcal{T}_X$ on $\mathcal{O}_{X^2}$ is $\mathcal{O}_X$-linear the restriction of the map $pr_*\ev^*\mathcal{T}_X \to \mathcal{T}_{X^2}$ to $\epsilon \cdot pr_*\ev^*\mathcal{T}_X$ takes values in $\mathcal{T}_{X^2/X}$ and, hence, induces a morphism of short exact sequences

$$0 \longrightarrow \epsilon \cdot pr_*\ev^*\mathcal{T}_X \longrightarrow pr_*\ev^*\mathcal{T}_X \longrightarrow \mathcal{O}_{X^2} \otimes \mathcal{O}_X \mathcal{T}_X \longrightarrow 0$$

It is easy to see that the left vertical map (the $\mathcal{O}_{X^2}$-linear extension of the action by interior multiplication) $\mathcal{O}_{X^2} \otimes \mathcal{O}_X \mathcal{T}_X[1] \to \mathcal{T}_{X^2/X}$ is an isomorphism, and so is the right vertical map. This implies the second claim. \hfill $\Box$

3.5. Transgression for Lie algebroids. Suppose that $\mathcal{A}$ is an $\mathcal{O}_{X^2}$-Lie algebroid as in \ref{3.1}.

The sheaf $\mathcal{O}_{X^2}[\epsilon] \otimes \mathcal{A}$ has a canonical structure of an $\mathcal{O}_{X^2}$-Lie algebroid with

- the anchor map is given by the composition

$$\tilde{\sigma} : \mathcal{O}_{X^2}[\epsilon] \otimes \mathcal{A} \xrightarrow{id \otimes \sigma} \mathcal{O}_{X^2}[\epsilon] \otimes \mathcal{O}_X \mathcal{T}_X \xrightarrow{\epsilon \cdot pr_*\ev^*\mathcal{T}_X} \mathcal{T}_{X^2},$$

- the bracket defined by the composition of the bracket on $\mathcal{A}$ by the Leibniz rule, the anchor above and the action of $\mathcal{T}_{X^2}$ on $\mathcal{O}_{X^2}[\epsilon]$; namely:

$$[\omega_1 \otimes 1 \otimes a_1, \omega_2 \otimes 1 \otimes a_2] = \omega_1 \wedge L_{\sigma(a_1)} \omega_2 \otimes 1 \otimes a_2$$

$$+ (-1)^i j \omega_2 \wedge (\omega_1 \otimes 1 \otimes [a_1, a_2] - L_{\sigma(a_2)} \omega_1 \otimes 1 \otimes a_1),$$

$$[\omega_1 \otimes 1 \otimes a_1, \omega_2 \otimes \epsilon \otimes b_2] = \omega_1 \wedge L_{\sigma(a_1)} \omega_2 \otimes \epsilon \otimes b_2$$

$$+ (-1)^i j \omega_2 \wedge (\omega_1 \otimes \epsilon \otimes [a_1, b_2] - (-1)^i \iota_{\sigma(b_2)} \omega_1 \otimes 1 \otimes a_1),$$

$$[\omega_1 \otimes \epsilon \otimes b_1, \omega_2 \otimes \epsilon \otimes b_2] =$$

$$\omega_1 \wedge \iota_{\sigma(b_1)} \omega_2 \otimes \epsilon \otimes b_2 + (-1)^{ij+j+i} \omega_2 \wedge \iota_{\sigma(b_2)} \omega_1 \otimes \epsilon \otimes b_1,$$

where $\omega_1 \in \mathcal{O}_{X^2}^{i}, \omega_2 \in \mathcal{O}_{X^2}^{j}, a_1, a_2, b_1, b_2 \in \mathcal{A}$.

Lemma 3.3. The structure of an $\mathcal{O}_{X^2}$-Lie algebroid on $\mathcal{O}_{X^2}[\epsilon] \otimes \mathcal{A}$ descends to one on $pr_*\ev^*\mathcal{A}$.

Proof. The proof is given in \ref{3.4.1}. \hfill $\Box$
Notation. In what follows we shall denote the $\mathcal{O}_{X^\sharp}$-Lie algebroid structure on $\text{pr}_* \text{ev}^* A$ given by Lemma 3.3 by $A^\sharp$.

We leave it to the reader to check that the morphism $\text{pr}_* \text{ev}^* \phi: \text{pr}_* \text{ev}^* A_1 \to \text{pr}_* \text{ev}^* A_2$ induced by a morphism $\phi: A_1 \to A_2$ of $\mathcal{O}_X$-Lie algebroids is in fact a morphism of $\mathcal{O}_{X^\sharp}$-Lie algebroids. That is, the assignment $A \mapsto A^\sharp$ extends to a functor

$$\left( \right)^\sharp: \mathcal{O}_X \text{-LieAlg} \to \mathcal{O}_{X^\sharp} \text{-LieAlg}.$$  

3.6. Transgression for modules. Suppose that $E$ is an $\mathcal{O}_X$-module. For $D \in \mathcal{A}_E$ we denote by $\iota_D: (\text{pr}_* \text{ev}^* E)^0 \to E$ the unique $\mathcal{O}_X$-linear map such that $D = \iota_D \circ \partial$, see Remark 2.9. Let

$$\tilde{i}_D: \Omega_X^{k+1} \otimes \mathcal{O}_X (\text{pr}_* \text{ev}^* E)^0 \to (\text{pr}_* \text{ev}^* E)^k$$

denote the map defined by the formula

$$\tilde{i}_D(\alpha \otimes B) = \iota_{\sigma(D)}(\alpha \otimes B) + (-1)^\alpha \alpha \otimes \iota_D B.$$  

Lemma 3.4.

1. The map (3.6.1) factors through the map $\tilde{i}_D: (\text{pr}_* \text{ev}^* E)^{k+1} \to (\text{pr}_* \text{ev}^* E)^k$.
2. The $\mathcal{O}_X$-linear map $\tilde{i}_D: \text{pr}_* \text{ev}^* E \to \text{pr}_* \text{ev}^* E[-1]$ is a differential operator or order one with respect to the $\mathcal{O}_{X^\sharp}$-module structure with principal symbol $\sigma(D) \otimes \epsilon$. In particular, $\tilde{i}_D \in \mathcal{A}_{\text{pr}_* \text{ev}^* E}^{-1}$.

Proof. The proof is given in [7.2].

Let

$$\tilde{D} = [\partial, \tilde{i}_D].$$

Thus, $\tilde{D}: \text{pr}_* \text{ev}^* E \to \text{pr}_* \text{ev}^* E$ is a differential operator or order one with respect to the $\mathcal{O}_{X^\sharp}$-module structure with principal symbol $L_{\sigma(D)} = [\partial, \sigma(D) \epsilon]$. In particular, $\tilde{D} \in \mathcal{A}^0_{\text{pr}_* \text{ev}^* E}$. Thus, the assignment $D \mapsto \tilde{D}$ defines a map

$$\mathcal{A}_E \to \mathcal{A}_{\text{pr}_* \text{ev}^* E}.$$  

Lemma 3.5.

1. For $D \in \mathcal{A}_E$, $A \in \mathcal{O}_{X^\sharp}[\epsilon]$ and $e \in \mathcal{E}$

$$\tilde{D}(A \otimes e) = L_{\sigma(D)} A \otimes e + A \otimes D(e)$$

2. The map $\mathcal{A}_E \xrightarrow{\tilde{D}} \mathcal{A}_{\text{pr}_* \text{ev}^* E}$ is a morphism of Lie algebras.
3. The diagram

$$\begin{array}{ccc}
\mathcal{A}_E & \xrightarrow{\tilde{D}} & \mathcal{A}_{\text{pr}_* \text{ev}^* E} \\
\sigma \downarrow & & \sigma \downarrow \\
\mathcal{T}_X & \xrightarrow{L} & \mathcal{T}_{X^\sharp}
\end{array}$$

(3.6.3)
is commutative.

(4) The induced map \( \text{End}_{O_X}(E) \to \text{End}_{O_X}(\text{pr}_{\ast}\text{ev}^{\ast}E) \) between the kernels of respective anchor maps coincides with the map induced by the functor \( \text{pr}_{\ast}\text{ev}^{\ast} \).

Proof. The proof is given in [7.3] □

Let

\[
(3.6.4) \quad L^\xi : O_{X^1}[\epsilon] \otimes \mathbb{C} A_\xi \to A_{\text{pr}_{\ast}\text{ev}^{\ast}E}
\]

denote the \( O_{X^1} \)-linear extension of the map \( L^\xi : \mathbb{C}[\epsilon] \otimes \mathbb{C} A_\xi \to A_{\text{pr}_{\ast}\text{ev}^{\ast}E} \) defined by \( D \mapsto L^\xi_D = \tilde{D} \) and \( \epsilon \cdot D \mapsto L^\xi_{\epsilon,D} = \tilde{\epsilon}_D \).

Lemma 3.6.

(1) The map (3.6.4) descends to the map

\[
(3.6.5) \quad L^\xi : A_\xi^1 \to A_{\text{pr}_{\ast}\text{ev}^{\ast}E}.
\]

(2) The map (3.6.5) is a morphism of \( O_{X^1} \)-Lie algebroids.

Proof. The proof is given in [7.3] □

Suppose that \( \mathcal{A} \) is an \( O_X \)-Lie algebroid and \( E \) is an \( \mathcal{A} \)-module with the module structure given by the morphism \( \alpha : \mathcal{A} \to \mathcal{A}_\mathcal{E} \). The composition \( \mathcal{A}^2 \xrightarrow{\alpha^1} \mathcal{A}^1 \xrightarrow{(3.6.4)} \mathcal{A}_{\text{pr}_{\ast}\text{ev}^{\ast}E} \) endows \( \text{pr}_{\ast}\text{ev}^{\ast}E \) with a canonical structure of an \( \mathcal{A}^2 \)-module. Thus, the functor (2.4.1) induces the functor

\[
\text{pr}_{\ast}\text{ev}^{\ast} : \mathcal{A} - \text{Mod} \to \mathcal{A}^2 - \text{Mod}.
\]

3.7. Marked Lie algebroids. Suppose that \( \mathfrak{X} = (X, O_X) \) is a DG-manifold.

Lemma 3.7. Suppose that \( \mathcal{A} \) is a \( O_X \)-Lie algebroid. Any (homogeneous) central section \( c \in \Gamma(X;\mathcal{A}) \) (i.e. \( [c,\mathcal{A}] = 0 \)) belongs to the kernel of the anchor map.

Proof. For \( a \in \mathcal{A} \), \( \gamma \in O_{X^1} \), the calculation

\[
0 = [c, \gamma \cdot a] = \sigma(c)(\gamma) \cdot a
\]

implies that \( \sigma(c)\gamma = 0 \). Applying the anchor map we find that \( \sigma(c)(\gamma) \cdot \sigma(a) = 0 \) for all \( a \in \mathcal{A} \) and \( \gamma \in O_{X^1} \). With \( a = c \) this means that the vector field \( \sigma(c) \) annihilates \( O_{X^1} \).

Definition 3.8. A marked \( O_X \)-Lie algebroid is a pair \( (\mathcal{A}, c) \), where \( \mathcal{A} \) is a \( O_X \)-Lie algebroid and \( c \in \Gamma(X;\mathcal{A}) \) is a homogeneous central section.
A morphism $\phi: (A_1, c_1) \to (A_2, c_2)$ is a morphism of Lie algebroids $\phi: A_1 \to A_2$ such that $\phi(c_1) = c_2$. In particular, $c_1$ and $c_2$ have the same degree.

With the above definitions marked $\mathcal{O}_X$-Lie algebroids and morphisms thereof form a category denoted $\mathcal{O}_X\text{-LieAlgd}$. The full subcategory of marked $\mathcal{O}_X$-Lie algebroids $(A, c)$ with $\deg c = k$ is denoted $\mathcal{O}_X\text{-LieAlgd}^k$.

For a marked Lie algebroid $(A, c)$ with deg $c = n$ let

$$\overline{A} := \ker(\mathcal{O}_X[n] \to A).$$

**Lemma 3.9.** The structure of a Lie algebroid on $A$ descends to one on $\overline{A}$.

**Proof.** Since $c$ is central the submodule $\mathcal{O}_X \cdot c$ satisfies $[\mathcal{O}_X \cdot c, A] \subseteq \mathcal{O}_X \cdot c$. Moreover, by Lemma 3.7 the anchor map factors through $\overline{A}$. \qed

### 3.8. $\mathcal{O}_X[n]$-extensions

Suppose that $B$ is a $\mathcal{O}_X$-Lie algebroid.

**Definition 3.10.** An $\mathcal{O}_X[n]$-extension of $B$ is a marked $\mathcal{O}_X$-Lie algebroid $(A, c)$ with deg $c = n$ together with the identification $A \cong B$ such that the sequence

$$0 \to \mathcal{O}_X[n] \to A \to B \to 0$$

is exact.

A morphism of $\mathcal{O}_X[n]$-extensions of $B$ is a morphism of marked Lie algebroids which induces the identity map on $B$.

We denote the category of $\mathcal{O}_X[n]$-extensions of $B$ by $\mathcal{O}[n] \text{Ext}(B)$.

It follows from the definitions that a morphism of $\mathcal{O}_X[n]$-extensions of $B$ induces a morphism of extensions of $B$ by $\mathcal{O}_X[n]$ (in the abelian category of $\mathcal{O}_X$-modules), hence is an isomorphism of $\mathcal{O}_X$-modules. The inverse map is easily seen to be a morphism of $\mathcal{O}_X[n]$-extensions. It follows that the category $\mathcal{O}[n] \text{Ext}(B)$ is a groupoid.

### 4. Courant algebroids

**Definition 4.1.** A $k$-dimensional Courant algebroid is an $\mathcal{O}_X$-module $Q$ equipped with

1. a structure of a Leibniz $\mathbb{C}$-algebra

$$\{,\} : Q \otimes_\mathbb{C} Q \to Q;$$

2. an $\mathcal{O}_X$-linear map of Leibniz algebras (the anchor map)

$$\pi : Q \to \mathcal{T}_X;$$

3. a symmetric $\mathcal{O}_X$-bilinear pairing

$$\langle , \rangle : Q \otimes_{\mathcal{O}_X} Q \to \Omega^{k-1}_X;$$

4. an $\mathcal{O}_X$-linear map

$$\pi^* : \Omega^k_X \to Q.$$
These data are required to satisfy

\[(4.0.1) \quad \pi \circ \pi^\dagger = 0\]

\[(4.0.2) \quad \{q_1, f q_2\} = f \{q_1, q_2\} + \pi(q_1)(f)q_2\]

\[(4.0.3) \quad \langle \{q, q_1\}, q_2 \rangle + \langle q_1, \{q, q_2\} \rangle = L_{\pi(q)} \langle q_1, q_2 \rangle\]

\[(4.0.4) \quad \{q, \pi^\dagger(\alpha)\} = \pi^\dagger(L_{\pi(q)}(\alpha))\]

\[(4.0.5) \quad \{q, \pi^\dagger(\alpha)\} = \iota_{\pi(q)} \langle q_1, q_2 \rangle - \iota_{\pi(q)} \langle q, q_2 \rangle - \iota_{\pi(q)} \langle q_2, q_1 \rangle\]

\[(4.0.6) \quad \{q_1, q_2\} + \{q_2, q_1\} = \pi^\dagger(d\langle q_1, q_2 \rangle)\]

for \(f \in \mathcal{O}_X\) and \(q, q_1, q_2 \in \mathcal{Q}\).

A morphism \(\phi: \mathcal{Q}_1 \to \mathcal{Q}_2\) of \(k\)-dimensional Courant algebroids on \(X\) is an \(\mathcal{O}_X\)-linear map of Leibniz \(\mathbb{C}\)-algebras such that the diagram

\[
\begin{array}{ccc}
\Omega^k_X & \xrightarrow{\pi^\dagger} & \mathcal{Q}_1 \\
\| & \phi \downarrow & \| \\
\Omega^k_X & \xrightarrow{\pi^\dagger} & \mathcal{Q}_2 \\
\end{array}
\]

is commutative.

With the above definitions \(k\)-dimensional Courant algebroids on \(X\) and morphisms thereof form a category henceforth denoted \(\text{CA}(X)_k\).

**Remark 4.2.** A zero-dimensional Courant algebroid is just a marked \(\mathcal{O}_X\)-Lie algebroid with the marking given by \(\pi^\dagger\).

A one-dimensional Courant algebroid is a Courant algebroid in the sense of [B] (and, in restricted setting, of [LWX], [R] and [S2]). In this case the axiom (4.0.4) is redundant.

**Remark 4.3.** The formula (4.0.4) says that the graded \(\mathcal{O}_X\)-module \(Q[1] \oplus \bigoplus_{i=1}^k \Omega^{-i}_X[1] \oplus \Omega^{-i}_X[2]\) equipped with the bracket whose only non-zero component is \(\langle , \rangle: \mathcal{Q}[1] \otimes \mathcal{Q}[1] \to \Omega^{-i}_X[2]\) is a graded Lie algebra.

**4.1. The associated Lie algebroid.** Suppose that \(\mathcal{Q}\) is a \(k\)-dimensional Courant algebroid on \(X\). Let

\[
\overline{\mathcal{Q}} = \text{coker}(\pi^\dagger).
\]

The formula (4.0.5) implies that \(\{\mathcal{Q}, \text{im}(\pi^\dagger)\} \subseteq \text{im}(\pi^\dagger)\). Formulas (4.0.5), (4.0.7) and (4.0.6) imply that \(\text{im}(\pi^\dagger), \mathcal{Q} \subseteq \text{im}(\pi^\dagger)\). Therefore, the bracket on \(\mathcal{Q}\) descends to a bilinear operation

\[(4.1.1) \quad \{ , \} : \overline{\mathcal{Q}} \otimes \overline{\mathcal{Q}} \to \overline{\mathcal{Q}}.\]
The condition (4.0.1) implies that the anchor map \( \pi : Q \to T_X \) factors through (4.1.2) \[ \pi : \widehat{Q} \to T_X \]

**Lemma 4.4.** The bracket (4.1.1) and the anchor (4.1.2) determine a structure of a Lie algebroid on \( \widehat{Q} \).

**Proof.** According to (4.0.7) the symmetrization of the Leibniz bracket on \( Q \) takes values in the image of \( \pi^\dagger \). Therefore the induced bracket (4.1.1) is skew-symmetric. The Leibniz rule and the Jacobi identity for \( Q \) follow from those for \( \widehat{Q} \). \( \square \)

In what follows we refer to the Lie algebroid \( \widehat{Q} \) as *the Lie algebroid associated to the Courant algebroid \( Q \).*

The assignment \( Q \mapsto \widehat{Q} \) extends to a functor \[ \overline{()}: \mathcal{C}(X)_k \to \mathcal{O}_X - \text{LieAlgd}. \]

### 4.2. Courant extensions.

**Definition 4.5.** For \( A \in \mathcal{O}_X - \text{LieAlgd} \) a *(\( k \)-dimensional) Courant extension of \( A \)* is a \( k \)-dimensional Courant algebroid \( Q \) together with the identification \( Q \cong A \) such that the sequence

(4.2.1) \[
0 \to \Omega^k_X \xrightarrow{\pi^\dagger} Q \to A \to 0
\]

is exact.

A morphism \( \phi : Q_1 \to Q_2 \) of Courant extensions of \( A \) is a morphism of Courant algebroids which is compatible with the identifications \( \overline{Q}_i \cong A \).

**Notation.** We denote the category of \( k \)-dimensional Courant extension of \( A \) by \( \mathcal{C}\text{Ext}(A)_k \).

A morphism in \( \mathcal{C}\text{Ext}(A)_k \) induces a morphism of associated short exact sequences (4.2.1), hence is an isomorphism of underlying \( \mathcal{O}_X \)-modules. It is easy to see that the inverse map is, in fact, a morphism of Courant algebroids. Consequently, \( \mathcal{C}\text{Ext}(A)_k \) is a groupoid.

### 4.3. Transitive Courant algebroids.

A Courant algebroid \( Q \) is *transitive* if the associated Lie algebroid \( \overline{Q} \) is transitive, i.e. if the anchor map is surjective.

**Lemma 4.6.** Suppose that \( Q \) is a transitive Courant algebroid. Then, the map \( \Omega^k_X \xrightarrow{\pi^\dagger} Q \) is injective, hence the sequence

\[
0 \to \Omega^k_X \xrightarrow{\pi^\dagger} Q \to \overline{Q} \to 0
\]

is exact.
Proof. According to (4.0.6) the map $\pi^\dagger$ is adjoint to the map $\pi$. As the latter is surjective, the former is injective. Namely, if $\alpha \in \Omega^k_X$ satisfies $\pi^\dagger(\alpha) = 0$, then $\iota_{\pi(q)} \alpha = 0$ for all $q \in Q$. Since $\pi$ is surjective $\iota_\xi \alpha = 0$ for all $\xi \in T_X$, i.e. $\alpha = 0$. □

4.4. Exact Courant algebroids.

Definition 4.7. A Courant algebroid $Q$ is exact if the anchor map $Q \xrightarrow{\pi} T_X$ is an isomorphism.

Equivalently, by Lemma 4.6 an exact Courant algebroid $Q$ is a Courant extension of $T_X$, i.e. the sequence

\begin{equation}
0 \to \Omega^k_X \xrightarrow{\pi^\dagger} Q \xrightarrow{\pi} T_X \to 0
\end{equation}

is exact.

We denote by $\text{ECA}(X)_k$ the category (groupoid) of exact Courant algebroids.

Example 4.8. Let $Q_0 = \Omega^k_X \oplus T_X$. Let $\pi: Q_0 \to T_X$ (respectively, $\pi^\dagger: \Omega^k_X \to Q_0$) be the projection on the second factor (respectively, the inclusion of the first summand). Let $\langle \, , \, \rangle: Q_0 \otimes_{\mathcal{O}_X} Q_0 \to \Omega^{k-1}$ be the symmetric pairing with $\langle \Omega^k_X, \Omega^k_X \rangle = 0$, $\langle T_X, T_X \rangle = 0$ and $\langle \, , \, \rangle: T_X \otimes_{\mathcal{O}_X} \Omega^k_X \to \Omega^{k-1}$ the interior multiplication. There is a unique structure of a Courant algebroid on $Q_0$ characterized by the property that the canonical inclusion of $T_X$ into $Q_0$ is a map of Leibniz algebras.

Suppose that $Q$ is an exact Courant algebroid. A splitting $\nabla: T_X \to Q$ of (4.4.1) (i.e. a section of the anchor map) is called a connection (on $Q$) if it is isotropic with respect to the symmetric pairing, i.e. $\langle \nabla(\xi), \nabla(\eta) \rangle = 0$ for all $\xi, \eta \in T_X$. We denote by $\text{Conn}(Q)$ the sheaf of locally defined connections on $Q$.

Lemma 4.9. The sheaf $\text{Conn}(Q)$ is locally non-empty.

Proof. Let $s: T_X \to Q$ denote a locally defined splitting of (4.4.1). Let $\phi: T_X \to \Omega^k_X$ denote the map uniquely determined by

$$
\iota_\eta \phi(\xi) = -\frac{1}{2} \langle s(\xi), s(\eta) \rangle .
$$

Let $\nabla = s + \pi^\dagger \circ \phi$. Then,

$$
\langle \nabla(\xi), \nabla(\eta) \rangle = \langle s(\xi) + \pi^\dagger(\phi(\xi)), s(\eta) + \pi^\dagger(\phi(\eta)) \rangle = \\
\langle s(\xi), s(\eta) \rangle + \iota_\eta \phi(\xi) + \iota_\xi \phi(\eta) = \\
\langle s(\xi), s(\eta) \rangle - \frac{1}{2} \langle s(\xi), s(\eta) \rangle - \frac{1}{2} \langle s(\eta), s(\xi) \rangle = 0 .
$$
For a connection $\nabla \in \text{Conn}(Q)$ and $\omega \in \Omega_{X}^{k+1}$ let $\nabla + \omega : T_{X} \to Q$ denote the map defined by

$$\nabla + \omega(\xi) = \nabla(\xi) + \pi(\ell_\xi \omega)$$.

**Lemma 4.10.**

1. For $\nabla \in \text{Conn}(Q)$ and $\omega \in \Omega_{X}^{k+1}$ the map $\nabla + \omega$ is a connection on $Q$.
2. The assignment $\nabla \mapsto \nabla + \omega$ is an action of (the sheaf of groups) $\Omega_{X}^{k+1}$ on $\text{Conn}(Q)$ which endows the latter with a structure of a $\Omega_{X}^{k+1}$-torsor.

**Proof.** For $\xi, \eta \in T_{X}$

$$\langle (\nabla + \omega)(\xi), (\nabla + \omega)(\eta) \rangle = \langle \nabla(\xi) + \pi(\ell_\xi \omega), \nabla(\eta) + \pi(\ell_\eta \omega) \rangle = \langle \nabla(\xi), \nabla(\eta) \rangle + \langle \pi(\ell_\xi \omega), \nabla(\eta) \rangle + \langle \nabla(\xi), \pi(\ell_\eta \omega) \rangle + \langle \pi(\ell_\xi \omega), \pi(\ell_\eta \omega) \rangle$$

with $\langle \nabla(\xi), \nabla(\eta) \rangle = 0$ because $\nabla$ is a connection, $\langle \pi(\ell_\xi \omega), \pi(\ell_\eta \omega) \rangle = 0$ by (4.0.6) and $\langle \nabla(\xi), \pi(\ell_\eta \omega) \rangle = \ell_\xi \ell_\eta \omega = \ell_\eta \ell_\xi \omega = -\ell_\eta \ell_\xi \omega = -\langle \pi(\ell_\xi \omega), \nabla(\eta) \rangle$ by skew-symmetry of $\omega$ and (4.0.6). Hence, $\langle (\nabla + \omega)(\xi), (\nabla + \omega)(\eta) \rangle = 0$ which proves the first claim.

It is clear that the assignment $(\omega, \nabla) \mapsto \nabla + \omega$ defines an action of $\Omega_{X}^{k+1}$ on $\text{Conn}(Q)$. For $\nabla_1, \nabla_2 \in \text{Conn}(Q)$ the difference $\nabla_1 - \nabla_2$ satisfies $\pi \circ \nabla_1 - \nabla_2 = 0$, hence defines a map $\phi : T_{X} \to \Omega_{X}^{k}$ (by $\pi \circ \phi = \nabla_1 - \nabla_2$) or, equivalently, a section of $\Omega_{X}^{k} \otimes_{\mathcal{O}_{X}} \Omega_{X}^{k}$. The calculation above shows that the latter is totally skew-symmetric, i.e. a section of $\Omega_{X}^{k+1}$. By Lemma 4.9 $\text{Conn}(Q)$ is locally non-empty, hence a torsor. □

For a connection $\nabla$ on $Q$ and $\xi, \eta \in T_{X}$ the expression $\{\nabla(\xi), \nabla(\eta)\} - \nabla([\xi, \eta])$ belong to the kernel of the anchor map. Hence there is a unique $c(\nabla)(\xi, \eta) \in \Omega_{X}^{k}$ with

$$\pi(c(\nabla)(\xi, \eta)) = \{\nabla(\xi), \nabla(\eta)\} - \nabla([\xi, \eta])$$.

The assignment $(\xi, \eta) \mapsto c(\nabla)(\xi, \eta)$ defines a map

$$c(\nabla) : T_{X} \otimes_{\mathcal{O}_{X}} T_{X} \to \Omega_{X}^{k}$$

called the curvature of $\nabla$.

**Lemma 4.11.**

1. The curvature $c(\nabla)$ of a connection $\nabla$ satisfies

$$\iota_\eta(c(\nabla)(\xi_1, \xi_2)) = \langle \{\nabla(\xi_1), \nabla(\xi_2)\}, \nabla(\eta) \rangle$$

for $\xi_1, \xi_2, \eta \in T_{X}$.
2. The map

$$\iota_\eta \wedge \ldots \wedge \iota_{\eta_k} (c(\nabla)(\xi_1, \xi_2))$$

is totally skew-symmetric and $\mathcal{O}_{X}$-multilinear.
3. The differential form defined by (4.4.2) is closed.
Proof. The calculation
\[\iota_\eta(c(\nabla)(\xi_1, \xi_2)) = \langle c(\nabla)(\xi_1, \xi_2), \nabla(\eta) \rangle = \langle \{\nabla(\xi_1), \nabla(\xi_2)\}, \nabla(\eta) \rangle = \]
\[L_{\xi_1} \langle \nabla(\xi_2), \nabla(\eta) \rangle - \langle \{\nabla(\xi_1), \nabla(\eta)\}, \nabla(\xi_2) \rangle =
- \langle \{\nabla(\xi_1), \nabla(\eta)\}, \nabla(\xi_2) \rangle = -\iota_{\xi_2}(c(\nabla)(\xi_1, \eta))\]
proves the first claim. Since the map (4.4.2) is skew symmetric in the first two variables as well as in the rest of the variables separately the calculation implies that (4.4.2) is, in fact, totally skew-symmetric. Since it is \(O_X\)-linear in \(\eta_i\) and totally skew-symmetric it follows that it is \(O_X\)-linear in all variables.

For any \(\xi, \eta, \gamma \in T_X\) we compute the left hand side and the right hand side of the Jacobi identity,
\[\{\nabla \xi, \{\nabla \eta, \nabla \gamma\}\} = \{\{\nabla \xi, \nabla \eta\}, \nabla \gamma\} + \{\nabla \eta, \{\nabla \xi, \nabla \gamma\}\}.
\]
The left hand side is equal to:
\[\{\nabla \xi, \{\nabla \eta, \nabla \gamma\}\} = \{\nabla \xi, \pi^\dagger(c(\nabla)(\eta, \gamma)) + \nabla[\eta, \gamma]\}
= \pi^\dagger(L_\xi c(\nabla)(\eta, \gamma)) + \pi^\dagger(c(\nabla)(\xi, [\eta, \gamma])) + \nabla[\xi, [\eta, \gamma]]
= \pi^\dagger(L_{\xi_1} c(\nabla) + \iota_\xi L_{\eta \eta} c(\nabla) - \iota_\xi L_{\eta_\gamma} L_\eta c(\nabla)) + \nabla([\xi, [\eta, \gamma]]).
\]
Similarly, the right hand side is equal to:
\[\{\nabla \xi, \{\nabla \eta, \nabla \gamma\}\} + \{\nabla \eta, \{\nabla \xi, \nabla \gamma\}\} =
\{\pi^\dagger(c(\nabla)(\xi, \eta)) + \nabla[\xi, \eta], \nabla \gamma\} + \{\nabla \eta, \pi^\dagger(c(\nabla)(\xi, \gamma)) + \nabla[\xi, \gamma]\}
= \pi^\dagger((L_{\eta_\xi} c(\nabla) - \iota_\eta L_{\xi_\eta} - \iota_\gamma d_{\xi_\eta} - L_{\eta_\xi} L_{\eta_\gamma} - \iota_{\eta_\xi} L_{\eta_\gamma} c(\nabla))
+ \nabla([\xi, \eta], [\gamma] + [\eta, [\xi, \gamma]])
\]
Since (4.4.3) is equal to (4.4.4), after cancellation of terms, the equality is written as,
\[0 = \iota_{[\eta, \xi]} L_{\eta_\gamma} - \iota_\eta L_{\eta_\xi} L_{\xi_\eta} + \iota_\gamma L_{\xi_\eta} L_{\eta_\gamma} c(\nabla)
= \iota_\gamma L_{\xi_\eta} L_{\eta_\gamma} c(\nabla) + \iota_{[\eta, \xi]} L_{\eta_\gamma} + \iota_{\eta_\xi} L_{\eta_\gamma} d(\nabla)
= \iota_{\eta_\xi} L_{\eta_\gamma} d(\nabla),\]
thus \(c(\nabla)\) is a closed differential form. \(\square\)

In view of Lemma 4.11 we may and will regard the curvature \(c(\nabla)\) as a closed differential \((k + 2)\)-form. Thus, the assignment \(\nabla \mapsto c(\nabla)\) defines a map
\[c: \text{Conn}(Q) \rightarrow \Omega^{k+2,cl}_X.\]
Lemma 4.12. The map (4.4.5) satisfies $c(\nabla + \omega) = c(\nabla) + d\omega$. Thus, the pair $(\text{Conn}(\mathcal{Q}), c)$ is a $(\Omega^{k+1}_X \xrightarrow{d} \Omega^{k+2,cl}_X)$-torsor.

The assignment $\mathcal{Q} \mapsto (\text{Conn}(\mathcal{Q}), c)$ extends to a functor

(4.4.6) \[ \text{ECA}(X)_k \rightarrow (\Omega^{k+1}_X \xrightarrow{d} \Omega^{k+2,cl}_X)\text{-torsors} \]

Lemma 4.13. The functor (4.4.6) is an equivalence.

Proof. We sketch a construction of a quasi-inverse. Suppose that $(\mathcal{C}, c)$ is a $(\Omega^{k+1}_X \xrightarrow{d} \Omega^{k+2,cl}_X)$-torsor, i.e. $\mathcal{C}$ is a $\Omega^{k+1}$-torsor and the map $c: \mathcal{C} \rightarrow \Omega^{k+2,cl}_X$ satisfies $c(\nabla + \omega) = c(\nabla) + d\omega$. We associate to it the exact Courant algebroid which is the $\mathcal{C}$ twist of the Courant algebroid $\mathcal{Q}_0$ of Example 4.8 constructed as follows.

The underlying extension of $\mathcal{T}_X$ by $\Omega^k$ is the $\mathcal{C}$ twist $\mathcal{Q}_0^\mathcal{C}$ of the trivial extension $\mathcal{Q}_0 = \Omega^k \oplus \mathcal{T}_X$, i.e. $\mathcal{Q}_0^\mathcal{C} = \mathcal{C} \times_{\Omega^{k+1}_X} \mathcal{Q}_0$. Since the action of $\Omega^{k+1}_X$ on $\mathcal{Q}_0$ preserves the symmetric pairing, it follows that $\mathcal{Q}_0^\mathcal{C}$ is equipped with the induced symmetric pairing. The Leibniz bracket on $\mathcal{Q}_0^\mathcal{C}$ is given by the formula

\[
[(s_1, q_1), (s_2, q_2)] = (s_1, [q_1, q_2 + \tau_{\pi(q_1)}(s_1 - s_2)]_0 + \tau_{\pi(q_1)}\tau_{\pi(q_2)}(s_1))
\]

where $s_i \in \mathcal{C}$, $q_i \in \mathcal{Q}_0$, $s_1 - s_2 \in \Omega^{k+1}_X$ is the unique form such that $s_1 = s_2 + (s_1 - s_2)$ and $[\ , \ ]_0$ is the bracket on $\mathcal{Q}_0$. \hfill \Box

The equivalence (4.4.6) induces a bijection between the respective sets of connected components $\pi_0\text{ECA}(X)_k \cong H^1(X; \Omega^{k+1}_X \xrightarrow{d} \Omega^{k+2,cl}_X)$. The category $\text{ECA}(X)_k$ has a canonical structure of a Picard groupoid with the monoidal structure (induced by the Baer sum of extensions), as does the category $(\Omega^{k+1}_X \xrightarrow{d} \Omega^{k+2,cl}_X)$-torsors; the functor (4.4.6) is monoidal.

5. Transgression for Courant Algebroids

5.1. The category of Courant-to-Lie morphisms. Suppose that $\mathcal{Q}$ is a $(k - 1)$-dimensional Courant algebroid on $X$ (see Definition 3.1) and $(\mathcal{A}, c)$ is marked Lie algebroid on $X^\sharp$ with deg $c = k$ (see Definition 3.3).

Definition 5.1. A Courant-to-Lie (CtL) morphism $\mathcal{Q} \xrightarrow{\phi} (\mathcal{A}, c)$ is a morphism of $\mathcal{O}_X$-modules $\phi: \mathcal{Q}[1] \rightarrow \mathcal{A}$ such that the diagrams

\[
\begin{align*}
\text{(A)} & \quad \mathcal{Q} \xrightarrow{\phi} \mathcal{A}^{-1} \\
& \downarrow \pi \quad \downarrow \sigma \\
\mathcal{T}_X \xrightarrow{\cong} \mathcal{T}_X^{-1}
\end{align*}
\quad \begin{align*}
\text{(B)} & \quad \Omega^{k-1}_X \xrightarrow{\cong} (\mathcal{O}_{X^\sharp}[k])^{-1} \\
& \downarrow \pi^t \quad \downarrow \tau \\
\mathcal{Q} \xrightarrow{\phi} \mathcal{A}^{-1}
\end{align*}
\]
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\[
\begin{array}{ccc}
\mathcal{Q} \otimes_{\mathcal{O}_X} \mathcal{Q} & \xrightarrow{\phi \otimes \phi} & \mathcal{A}^{-1} \otimes_{\mathcal{O}_X} \mathcal{A}^{-1} \\
\langle \cdot, \cdot \rangle & \downarrow & \langle [\cdot, \cdot]^{-1,-1}, \cdot \rangle \\
\mathcal{O}_X^{k-2} & \xrightarrow{\epsilon} & \mathcal{A}^{-2} \\
\Phi \quad \downarrow & & \downarrow \Phi_L \\
\mathcal{Q} & \xrightarrow{\phi} & \mathcal{A}^{-1}
\end{array}
\]

are commutative. Note that since \( \mathcal{O}_X^i = 0 \) for \( i \leq -1 \), the bracket \( \langle [\cdot, \cdot], \cdot \rangle \) is \( \mathcal{O}_X \)-bilinear (see Remark 3.1).

A morphism \( (\mathcal{Q}_1 \xrightarrow{\phi_1} (\mathcal{A}_1, \mathcal{c}_1)) \xrightarrow{\Phi} (\mathcal{Q}_2 \xrightarrow{\phi_2} (\mathcal{A}_2, \mathcal{c}_2)) \) of \( \text{CtL} \)-morphisms is a pair \( \Phi = (\Phi_C, \Phi_L) \), where \( \Phi_C : \mathcal{Q}_1 \to \mathcal{Q}_2 \) is a morphism of Courant algebroids and \( \Phi_L : (\mathcal{A}_1, \mathcal{c}_1) \to (\mathcal{A}_2, \mathcal{c}_2) \) is a morphism of marked Lie algebroids such that the diagram

\[
\begin{array}{ccc}
\mathcal{Q}_1[1] & \xrightarrow{\Phi_C[1]} & \mathcal{Q}_2[1] \\
\phi_1 & \downarrow & \phi_2 \\
\mathcal{A}_1 & \xrightarrow{\Phi_L} & \mathcal{A}_2
\end{array}
\]

is commutative. We leave it to the reader to check that with the above definitions \( \text{CtL} \) morphisms form a category.

**Notation.** We denote the category of Courant-to-Lie morphisms as above by \( \text{CtL}(X)_k \).

The assignments \( (\mathcal{Q} \xrightarrow{\phi} (\mathcal{A}, \mathcal{c})) \mapsto \mathcal{Q}, \Phi = (\Phi_C, \Phi_L) \mapsto \Phi_C \) define a functor

\[
q : \text{CtL}(X)_k \to \text{CA}(X)_{k-1}.
\]

Suppose that \( \psi_1 : \mathcal{Q}_1 \to \mathcal{Q}_2 \) is a morphism in \( \text{CA}(X)_{k-1} \) and \( (\mathcal{Q}_2 \xrightarrow{\phi_2} (\mathcal{A}_2, \mathcal{c}_2)) \in \text{CtL}(X)_k \). We leave it to the reader to verify that the composition \( \mathcal{Q}_1[1] \xrightarrow{\psi_1[1]} \mathcal{Q}_2[1] \xrightarrow{\phi_2[1]} \mathcal{A}_2 \) satisfies the conditions of Definition 5.1, i.e. defines an object of \( \text{CtL}(X)_k \).

**Notation.** The object \( (\mathcal{Q}_1 \xrightarrow{\phi_2 \circ \psi_1} (\mathcal{A}_2, \mathcal{c}_2)) \) will be denoted \( \psi_1^*(\mathcal{Q}_2 \xrightarrow{\phi_2} (\mathcal{A}_2, \mathcal{c}_2)) \).

There is a canonical morphism

\[
\tilde{\psi}_1 : \psi_1^*(\mathcal{Q}_2 \xrightarrow{\phi_2} (\mathcal{A}_2, \mathcal{c}_2)) \to (\mathcal{Q}_2 \xrightarrow{\phi_2} (\mathcal{A}, \mathcal{c}))
\]

in \( \text{CtL}(X) \) given by \( \tilde{\psi}_1 = (\psi_1, \id_{\mathcal{A}}) \).

**Proposition 5.2.**

1. The morphism \( (5.1.2) \) is cartesian.
2. The functor \( (5.1.1) \) is a Grothendieck fibration.
Proof. Suppose that $\Phi = (\Phi_C, \Phi_L): (\mathcal{Q} \xrightarrow{\phi} (\mathcal{A}, c)) \to (\mathcal{Q}_2 \xrightarrow{\phi_2} (\mathcal{A}_2, c_2))$ with $q(\Phi) = \Phi_C = \psi_1 \circ \psi$. Then $\Psi = (\psi, \Phi_L): (\mathcal{Q} \xrightarrow{\phi} (\mathcal{A}, c)) \to \psi_1(\mathcal{Q}_2 \xrightarrow{\phi_2} (\mathcal{A}_2, c_2))$ is the unique morphism such that $\Phi = \psi_1 \circ \Psi$.

Since for any morphism $\psi_1: \mathcal{Q}_1 \to \mathcal{Q}_2$ in $\text{CA}(X)_{k-1}$ and any object $(\mathcal{Q}_2 \xrightarrow{\phi_2} (\mathcal{A}_2, c_2)) \in \text{CtL}(X)_k$ above $\mathcal{Q}_2$ there is a cartesian morphism $\tilde{\psi}_1$ with $q(\tilde{\psi}_1) = \psi_1$, it follows that the functor $(5.1.1)$ is a fibration. □

For a $(k-1)$-dimensional Courant algebroid $\mathcal{Q}$ we consider the fiber $\text{CtL}(X)_k/\mathcal{Q}$ of $(5.1.1)$ over $\mathcal{Q}$. Explicitly, the objects of $\text{CtL}(X)_k/\mathcal{Q}$ are CtL morphisms $\mathcal{Q} \xrightarrow{\phi} (\mathcal{A}, c)$. A morphism $\Phi = (\Phi_C, \Phi_L)$ is in $\text{CtL}(X)_k/\mathcal{Q}$ if $\Phi_C = \text{id}_\mathcal{Q}$.

The rest of this section is devoted to the proof of the following theorem.

**Theorem 5.3.** For any Courant algebroid $\mathcal{Q}$ the category $\text{CtL}(X)_k/\mathcal{Q}$ has an initial object.

**Notation.** For a Courant algebroid $\mathcal{Q}$ we will denote the initial object of $\text{CtL}(X)_k/\mathcal{Q}$ by $\tau \mathcal{Q}$. By abuse of notation we will refer to the marked Lie algebroid constituent thereof by $\tau \mathcal{Q}$ as well.

**Corollary 5.4.** The assignment $\mathcal{Q} \mapsto \tau \mathcal{Q}$ defines a section

$$\tau: \text{CA}(X)_{k-1} \longrightarrow \text{CtL}(X)_k$$

of $(5.1.1)$ (i.e. a functor which satisfies $q \circ \tau = \text{id}$) which is left adjoint to $q$.

**Proof.** Suppose that $\psi: \mathcal{Q}_1 \to \mathcal{Q}_2$ is a morphism in $\text{CA}(X)_{k-1}$. Let $\tau(\psi): \tau \mathcal{Q}_1 \to \tau \mathcal{Q}_2$ denote the composition $\tau \mathcal{Q}_1 \to \psi^* \tau \mathcal{Q}_2 \xrightarrow{\tilde{\psi}} \tau \mathcal{Q}_2$. We leave it to the reader to verify that the assignments $\mathcal{Q} \mapsto \tau \mathcal{Q}$, $\psi \mapsto \tau(\psi)$ define a functor. It is clear from the definition that $q \circ \tau = \text{id}$.

For $\mathcal{Q}_1 \in \text{CA}(X)_{k-1}$ and $\phi = (\mathcal{Q}_2 \xrightarrow{\phi} (\mathcal{A}_2, c_2)) \in \text{CtL}(X)_k$ with $q(\phi) = \mathcal{Q}_2$ the map

$$\text{Hom}_{\text{CtL}(X)_k}(\tau \mathcal{Q}_1, \phi) \to \text{Hom}_{\text{CA}(X)_{k-1}}(\mathcal{Q}_1, q(\phi))$$

is defined by $\Phi = (\Phi_C, \Phi_L) \mapsto \Phi_C$. This map is clearly natural (i.e. a morphism of bi-functors) and is, in fact an isomorphism of such with the inverse given by $\psi \mapsto (\psi, \Psi)$ where $\Psi$ is the composition $\tau \mathcal{Q}_1 \to \psi^* \tau \mathcal{Q}_2 \xrightarrow{\tilde{\psi}} \tau \mathcal{Q}_2 \to \mathcal{A}_2$. □

5.2. **Construction of a marked Lie algebroid.** In this section we are going to construct a marked Lie algebroid that we will denote by $(\tau \mathcal{Q}, c)$. Let $\tau \mathcal{Q}$ denote the cokernel of the map

$$\tau \mathcal{Q} := \text{coker}(\text{pr}_* \text{ev}^* \Omega_{X_{\mathcal{Q}}}^{k-1} \xrightarrow{(f, -\text{pr}_* \text{ev}^*(\pi^*))} \mathcal{O}_{X_{\mathcal{Q}}}^3[k] \oplus \text{pr}_* \text{ev}^* \mathcal{Q})$$.
where $f$ is the map (2.5.1). In other words, the square

$$\begin{array}{ccc}
\text{pr}_* \text{ev}^* \Omega^{k-1}_X & \xrightarrow{\text{pr}_* \text{ev}^*(\pi_1)} & \text{pr}_* \text{ev}^* Q \\
\downarrow & & \downarrow \\
\mathcal{O}_{X^1}[k] & \longrightarrow & \tau Q
\end{array}$$

(5.2.1)

is cocartesian.

5.2.1. The bracket on $\tau Q$. Let

$$\left[ , \right]: (\mathcal{O}_{X^1}[k] \oplus \mathcal{O}_{X^1}[\epsilon] \otimes_C \mathcal{Q}) \otimes_C (\mathcal{O}_{X^1}[k] \oplus \mathcal{O}_{X^1}[\epsilon] \otimes_C \mathcal{Q}) \to \mathcal{O}_{X^1}[k] \oplus \mathcal{O}_{X^1}[\epsilon] \otimes_C \mathcal{Q}$$

denote the extension by Leibniz rule of the binary operation on $\mathcal{O}_{X^1}[k] \oplus \mathcal{O}_{X^1}[\epsilon] \otimes_C \mathcal{Q}$ determined by

1. $[\mathcal{O}_{X^1}[k], \mathcal{O}_{X^1}[k]] = 0$
2. $[\epsilon \otimes q, \beta] = -(\mathcal{L}_{\pi(q)} \beta) = \tau_{\pi(q)} \beta$
3. $[1 \otimes q, \beta] = -[\beta, 1 \otimes q] = L_{\pi(q)} \beta$
4. $[\epsilon \otimes q_1, \epsilon \otimes q_2] = \langle q_1, q_2 \rangle \in (\mathcal{O}_{X^1}[k])^{-2}$
5. $[1 \otimes q_1, \epsilon \otimes q_2] = \epsilon \otimes \{q_1, q_2\}$
6. $[\epsilon \otimes q_1, 1 \otimes q_2] = -d\langle q_1, q_2 \rangle + \epsilon \otimes \{q_1, q_2\}$

where $q, q_1, q_2 \in \mathcal{Q}$ and $\beta \in \mathcal{O}_{X^1}$.

Notation. From now on we will denote $\alpha \otimes 1 \otimes q$ (respectively $\alpha \otimes \epsilon \otimes q$) by $\alpha \otimes q$ (respectively $\alpha \otimes \epsilon \otimes q$).

A general element of $\mathcal{O}_{X^1}[k] \oplus \mathcal{O}_{X^1}[\epsilon] \otimes_C \mathcal{Q}$ is sum of ones of the form $\omega$, $\alpha \epsilon \otimes q$ and $\beta \otimes r$, where $\omega, \alpha, \beta \in \mathcal{O}_{X^1}$ and $q, r \in \mathcal{Q}$. The bracket (5.2.2) of two such is given explicitly by formulas formulas (5.2.3)–(5.2.8) below, where $\alpha, \alpha_1, \alpha_2, \beta \in \mathcal{O}_{X^1}$ are homogeneous elements and $q_1, q_2 \in \mathcal{Q}$.

$$\begin{align*}
[\alpha_1 \epsilon \otimes q_1, \alpha_2 \epsilon \otimes q_2] &= \alpha_1 \wedge (\mathcal{L}_{\pi(q_1)} \alpha_2) \epsilon \otimes q_2 + (\tau_{\pi(q_1)} \alpha_2) \wedge \alpha_1 \wedge \langle q_1, q_2 \rangle \\
&\quad + (\mathcal{L}_{\pi(q_2)} \alpha_1) \epsilon \otimes q_1 \\
&\quad + (\mathcal{L}_{\pi(q_1)} \alpha_2) \wedge \alpha_1 \wedge \langle q_1, q_2 \rangle
\end{align*}$$

(5.2.3)

$$\begin{align*}
[\alpha_1 \otimes q_1, \alpha_2 \epsilon \otimes q_2] &= \alpha_1 \wedge (\mathcal{L}_{\pi(q_1)} \alpha_2) \epsilon \otimes q_2 + (\mathcal{L}_{\pi(q_1)} \alpha_2) \wedge \alpha_2 \epsilon \otimes \{q_1, q_2\} \\
&\quad - (\mathcal{L}_{\pi(q_2)} \alpha_1) \otimes q_1
\end{align*}$$

(5.2.4)
\[ (5.2.5) \quad [\alpha_1 \epsilon \otimes q_1, \alpha_2 \otimes q_2] = \\
\alpha_1 \land (l_{\pi(q_1)} \alpha_2) \otimes q_2 - (-1)^{(\alpha_1 - 1)\alpha_2} \alpha_1 \land d(q_1, q_2) \\
+ (-1)^{(\alpha_1 - 1)\alpha_2} \alpha_1 \land \alpha_1 \epsilon \otimes \{q_1, q_2\} - (-1)^{(\alpha_1 - 1)\alpha_2} \alpha_2 \land (L_{\pi(q_2)} \alpha_1) \epsilon \otimes q_1. \]

\[ (5.2.6) \quad [\alpha_1 \otimes q_1, \alpha_2 \otimes q_2] = \\
\alpha_1 \land (L_{\pi(q_1)} \alpha_2) \otimes q_2 + (-1)^{(\alpha_1 - 1)\alpha_2} \alpha_1 \land \alpha_1 \epsilon \otimes \{q_1, q_2\} - (-1)^{(\alpha_1 - 1)\alpha_2} \alpha_2 \land (L_{\pi(q_2)} \alpha_1) \epsilon \otimes q_1. \]

\[ (5.2.7) \quad [\alpha \epsilon \otimes q, \beta] = -(-1)^{\alpha \beta}[\beta, \alpha \epsilon \otimes q] = \alpha \land L_{\pi(q)} \beta \]

\[ (5.2.8) \quad [\alpha \otimes q, \beta] = -(-1)^{\alpha \beta}[\beta, \alpha \otimes q] = \alpha \land L_{\pi(q)} \beta. \]

**Lemma 5.5.** The operation \((5.2.2)\) satisfies
\[ \partial([a, b]) = [\partial(a), b] + (-1)^{a}[a, \partial(b)], \]
for any pair of homogeneous elements \(a, b \in \mathcal{O}_{X^1}[k] \oplus \mathcal{O}_{X^1}[\epsilon] \otimes_{\mathbb{C}} \mathcal{Q}. \)

**Proof.** The proof is given in 7.5 \(\Box\)

Let
\[ K := \ker(\mathcal{O}_{X^1}[k] \oplus \mathcal{O}_{X^1}[\epsilon] \otimes_{\mathbb{C}} \mathcal{Q} \rightarrow \tau \mathcal{Q}). \]
It is equal to the \(\mathcal{O}_{X^2}\)-submodule of \(\mathcal{O}_{X^1}[k] \oplus \mathcal{O}_{X^1}[\epsilon] \otimes_{\mathbb{C}} \mathcal{Q} \) generated by
1. \(\alpha - \epsilon \otimes \pi^1(\alpha)\) for \(\alpha \in \Omega_{X}^{k-1}\)
2. \(d\alpha - 1 \otimes \pi^1(\alpha)\) for \(\alpha \in \Omega_{X}^{k-1}\)
3. \(\beta \land \alpha - \beta \epsilon \otimes \pi^1(\alpha)\) for \(\beta \in \mathcal{O}_{X^2}\) and \(\alpha \in \Omega_{X}^{k-1}\)
4. \(\epsilon \otimes f q - f \epsilon \otimes q\) for \(f \in \mathcal{O}_X\) and \(q \in \mathcal{Q}\)
5. \(1 \otimes f q - f \otimes q - df \epsilon \otimes q\) for \(f \in \mathcal{O}_X\) and \(q \in \mathcal{Q}\)

Note that \(\partial(K) \subset K\).

**Lemma 5.6.** \(K\) is a two-sided ideal with respect to the operation \((5.2.2)\), i.e.
\[ [\mathcal{O}_{X^1}[k] \oplus \mathcal{O}_{X^1}[\epsilon] \otimes_{\mathbb{C}} \mathcal{Q}, K] \subset K, \quad [K, \mathcal{O}_{X^1}[k] \oplus \mathcal{O}_{X^1}[\epsilon] \otimes_{\mathbb{C}} \mathcal{Q}] \subset K. \]

**Proof.** The proof is given in 7.6 \(\Box\)

Lemma 5.6 implies that the operation \((5.2.2)\) descends to the operation
\[ (5.2.10) \quad [, ] : \tau \mathcal{Q} \otimes_{\mathbb{C}} \tau \mathcal{Q} \rightarrow \tau \mathcal{Q}. \]
5.2.2. The anchor map and the marking of \( \tau \mathcal{Q} \). The composition

\[
pr^*_ev^* \Omega^{k-1} \stackrel{(f,-pr^*ev^*(\pi^1))}{\longrightarrow} \mathcal{O}_{X^1}[k] \oplus pr^*_ev^* \mathcal{Q} \rightarrow \\
pr^*_ev^* \mathcal{Q} \stackrel{pr^*ev^*(\pi)}{\longrightarrow} pr^*_ev^* T_X = T_{X^2}
\]

is equal to zero, hence factors through \( \tau \mathcal{Q} \). Let

\[(5.2.11) \quad \sigma : \tau \mathcal{Q} \to T_{X^2}\]

denote the induced map of \( \mathcal{O}_{X^2} \)-modules.

**Lemma 5.7.** The map \((5.2.11)\) preserves the brackets, i.e.

\[
\sigma([a,b]) = [\sigma(a),\sigma(b)],
\]

for any \( a,b \in \tau \mathcal{Q} \).

**Proof.** Left to the reader. \( \square \)

Let \( c \in \tau \mathcal{Q}^{-k} \) denote the image of \( 1 \in \Gamma(X; (\mathcal{O}_{X^1}[k])^{-k}) \) under the composition

\[
\mathcal{O}_{X^1}[k] \stackrel{(id,0)}{\longrightarrow} \mathcal{O}_{X^1}[k] \oplus pr^*_ev^* \mathcal{Q} \rightarrow \tau \mathcal{Q}
\]

(equal to the bottom horizontal map in \((5.2.11)\)).

**Proposition 5.8.** The operation \((5.2.10)\), the anchor map \((5.2.11)\) and the marking \( c \) endow the \( \mathcal{O}_{X^2} \)-module \( \tau \mathcal{Q} \) with a structure of a marked \( \mathcal{O}_{X^2} \)-Lie algebroid.

**Proof.** The proof is given in \(7.7\). \( \square \)

Since the diagram \((5.2.1)\) is commutative, the (right vertical) map \( pr^*_ev^* \mathcal{Q} \to \tau \mathcal{Q} \) induces the map

\[(5.2.12) \quad \text{coker}(pr^*_ev^*(\pi^1)) \to \text{coker}(\mathcal{O}_{X^1}[k] \xrightarrow{\iota} \tau \mathcal{Q}) = \overline{\tau \mathcal{Q}}.
\]

**Proposition 5.9.** The marked \( \mathcal{O}_{X^2} \)-Lie algebroid \( \tau \mathcal{Q} \) enjoys the following properties:

1. \( \text{coker}(pr^*_ev^*(\pi^1)) = \overline{\mathcal{Q}}^i \)
2. the map \((5.2.12)\) is an isomorphism, in particular \( \overline{\mathcal{Q}}^i \cong \tau \mathcal{Q} \) and \( \overline{\tau \mathcal{Q}}^i = 0 \) for \( i \leq -2 \);
3. the map \( \Omega_{X^2}^{i+k} \to \tau \mathcal{Q}^i \) induced by the map \( \mathcal{O}_{X^1}[k] \xrightarrow{\iota} \tau \mathcal{Q} \) is an isomorphism for \( i \leq -2 \);
4. if \( \mathcal{Q} \) is a \( \Omega_{X^1}^{k-1} \)-extension of \( \mathcal{A} \) then \( \tau \mathcal{Q} \) is a \( \mathcal{O}_{X^1}[k] \)-extension of \( \mathcal{A}^i \).

**Proof.** The map \((5.2.12)\) is an isomorphism since the square \((5.2.1)\) is cocartesian. Since the functor \( pr^*_ev^* \) is exact, it follows that \( \text{coker}(pr^*_ev^*(\pi^1)) = \overline{\mathcal{Q}}^i \).

We leave it to the reader to check that the map \( \overline{\mathcal{Q}}^i \to \overline{\tau \mathcal{Q}} \) is a morphism of \( \mathcal{O}_{X^2} \)-Lie algebroids.
The second claim is a direct consequence of the construction of $\tau Q$.

If the sequence (4.2.1) is exact then so is the sequence obtained from (4.2.1) by applying the exact functor $pr_* ev^*$. The latter short exact sequence gives rise, by push-out along the map $\int : pr_* ev^* \Omega_{X}^{k-1} \to O_{X^i}[k]$ (see (5.2.1)), to the short exact sequence

$$0 \to O_{X^i}[k] \overset{\tau Q}{\to} A^i \to 0.$$ \hfill \(\square\)

5.3. The initial CtL morphism.

**Lemma 5.10.** The composition

(5.3.1) \hspace{1cm} $Q[1] \to pr_* ev^* Q \to O_{X^i}[k] \oplus pr_* ev^* Q \to \tau Q$

defines a CtL morphism.

**Proof.** We need to show that the squares $\square A$, $\square B$, $\square C$ and $\square D$ in Definition 5.1 commute.

Square $\square A$ commutes since the two compositions $q \mapsto \epsilon \otimes q \mapsto \iota_{\pi(q)}$ and $q \mapsto \pi(q) \mapsto \iota_{\pi(q)}$ are equal.

Square $\square B$ commutes since the two compositions are $\alpha \mapsto \alpha \cdot c$ and $\alpha \mapsto \epsilon \otimes \pi^\dagger(\alpha)$ and $\alpha \cdot c = \epsilon \otimes \pi^\dagger(\alpha)$ in $\tau Q$.

Square $\square C$ commutes since $[\epsilon \otimes q_1, \epsilon \otimes q_2] = \langle q_1, q_2 \rangle \cdot c$ by definition of the bracket on $\tau Q$ (see (5.2.1)).

Square $\square D$ commutes since $\delta(\epsilon \otimes q_1) = 1 \otimes q_1$ and $[1 \otimes q_1, \epsilon \otimes q_2] = \epsilon \otimes \{q_1, q_2\}$ by definition of the bracket on $\tau Q$ (see (5.2.1)). \hfill \(\square\)

**Proposition 5.11.** The object $(Q \to (\tau Q, c)) \in Q/\text{CtL}$ corresponding to the CtL morphism (5.3.1) is initial.

**Proof.** Suppose that $Q \xrightarrow{\phi} (A, c_A)$ is a CtL morphism. The morphism $\phi: Q[1] \to A$ admits a unique extension to a morphism of $O_{X^i}$-modules $\phi: pr_* ev^* Q \to A$ given by the formula

$$\phi(\omega \otimes q_1 + \beta \epsilon \otimes q_2) = \omega \cdot \delta(\phi(q_1)) + \beta \cdot \phi(q_2),$$

where $\omega, \beta \in O_{X^i}$ and $q_1, q_2 \in Q$. Since the diagram

$$\begin{array}{ccc}
pr_* ev^* \Omega_{X}^{k-1} & \xrightarrow{pr_* ev^* (\pi^\dagger)} & pr_* ev^* Q \\
\downarrow f & & \downarrow \phi \\
O_{X^i}[k] & \xrightarrow{c} & A
\end{array}$$

is commutative, there exists a unique morphism of $O_{X^i}$-modules

(5.3.2) \hspace{1cm} $\tilde{\phi}: \tau Q \to A$
It is given by
\[ \tilde{\varphi}(\theta + \omega \epsilon \otimes q_1 + \beta \otimes q_2) = \theta \cdot c_A + \omega \cdot \phi(q_1) + \beta \cdot \delta(\phi(q_2)) \].

We leave it to the reader to verify that (5.3.2) is a morphism of marked Lie algebroids. Since, clearly, the diagram
\[ \begin{array}{ccc}
Q[1] & \xrightarrow{\phi} & Q[1] \\
\downarrow^{\delta} & & \downarrow^{\phi} \\
\tau Q & \xrightarrow{\tilde{\varphi}} & A
\end{array} \]
is commutative, (5.3.2) is a morphism in \( Q/\text{CtL} \). □

6. Trasgression for extensions

6.1. From Lie to Courant. Suppose that \((B, c) \in \mathcal{O}_{X^2} - \text{LieAlgd}^*_n\) satisfies

1. the sequence
   \[ 0 \to \mathcal{O}[n] \xrightarrow{\delta} B \to \mathcal{B} \to 0 \]
is exact, i.e. \((B, c)\) is an \( \mathcal{O}[n] \)-extension of \( \mathcal{B} \);

2. \( \mathcal{B} = 0 \) for \( i \leq -2 \), i.e. the map \( \mathcal{O}[n] \xrightarrow{\delta} B \) induces isomorphisms \( \Omega^{i+n}_X \xrightarrow{\cong} \mathcal{B}^i \) for \( i \leq -2 \).

Let \( Q(B, c) \) denote the \( \mathcal{O}_X \)-module \( B^{-1} \) equipped with the following structures:

1. the map \( \pi: B^{-1} \to T_{\mathcal{O}_X^{-1}} = T_X \) is the restriction of the anchor map \( B \to T_{\mathcal{O}_X} \);
2. the map \( \pi^\dag: \Omega^{n-1} \to B^{-1} \) is the restriction of the map \( \mathcal{O}[n] \xrightarrow{\delta} B \);
3. the pairing \( \langle \ , \ \rangle: B^{-1} \otimes B^{-1} \to \Omega^{n-2} \) is determined by the equation \( \langle a, b \rangle \cdot c = [a, b]^{-1, -1} \);
4. the binary operation \( \{ \ , \ \}: B^{-1} \otimes B^{-1} \to B^{-1} \) is the derived bracket, i.e. is given by the formula \( \{ a, b \} = [\delta a, b]^{0, -1} \).

Lemma 6.1.

1. \( Q(B, c) \) is an \((n - 1)\)-dimensional Courant algebroid.
2. \( Q(B, c) = \mathcal{B}^{-1} \); in particular, the derived bracket defines a structure of a Lie algebroid on \( \mathcal{B}^{-1} \) and \( Q(B, c) \) is a Courant extension of \( \mathcal{B}^{-1} \).
3. The inclusion \( B^{-1}[1] \to B \) defines a CtL morphism \( Q(B, c) \to (B, c) \).

Proof. We leave it to the reader to verify that \( Q(B, c) \) is a Courant algebroid. Note that the sequence
\[ 0 \to \Omega^{n-1}_X \to B^{-1} \to \mathcal{B}^{-1} \to 0 \]
is exact by assumption.

The last claim is a direct consequence of the definition of Courant algebroid structure on $Q(B, c)$. □

The assignment $(B, c) \mapsto (Q(B, c) \to (B, c))$ extends to a functor

$$\mathcal{O}[n] \text{ Ext}(\mathcal{B}) \to Q(B, c) / \text{CtL}.$$ (6.1.1)

Composing with the forgetful functor $Q(B, c) / \text{CtL} \to \mathcal{C} \text{Ext}(B^{-1})_{n-1}$ we obtain

the functor

$$Q: \mathcal{O}[n] \text{ Ext}(\mathcal{B}) \to \mathcal{C} \text{Ext}(\mathcal{A})_{n-1}.$$ (6.2.1)

6.2. From Courant to Lie. Suppose that $\mathcal{A}$ is an $\mathcal{O}_X$-Lie algebroid. According to 3.5 $\mathcal{A}$ gives rise to the $\mathcal{O}_X^\#$-Lie algebroid $\mathcal{A}^\#$ with $(\mathcal{A}^\#)^{-1} = \mathcal{A}$. Then, any $B \in \mathcal{O}[n] \text{ Ext}(\mathcal{A}^\#)$ satisfies the assumptions of 6.1, whence the functor $Q: \mathcal{O}[n] \text{ Ext}(\mathcal{A}^\#) \to \mathcal{C} \text{Ext}(\mathcal{A})_{n-1}$.

On the other hand, according to Proposition 5.9, the transgression functor restricts to the functor $\tau: \mathcal{C} \text{Ext}(\mathcal{A})_{n-1} \to \mathcal{O}[n] \text{ Ext}(\mathcal{A}^\#)$. (6.2.2)

Theorem 6.2. The functors (6.2.1) and (6.2.2) are mutually quasi-inverse equivalences of categories.

Proof. It is clear that $Q \circ \tau = \text{id}$.

For $(B, c) \in \mathcal{O}[n] \text{ Ext}(\mathcal{A}^\#)$ the CtL morphism $Q(B, c) \to (B, c)$ gives rise to the morphism $\tau Q(B, c) \to B$ of $\mathcal{O}_X$-Lie algebroids and, in fact, of $\mathcal{O}_X[n]$-extensions of $\mathcal{A}^\#$. Therefore, it is an isomorphism. It is clearly natural in $(B, c)$, hence $\tau \circ Q \cong \text{id}$. □

6.3. Transgression and symplectic NQ-manifolds of degree 2. Below we sketch the relationship between the transgression functor and the construction of a NQ-manifold of degree 2 associated to a (1-dimensional) Courant algebroid of $[R]$ and attributed to A. Weinstein in [S2].

Suppose that $\mathcal{E}$ is a vector bundle on $X$ and $\langle \ , \ \rangle: \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{O}_X$ is a non-degenerate symmetric pairing. Let $\mathfrak{so}(\mathcal{E}, \langle \ , \ \rangle)$ denote the subshaf of endomorphisms $\phi \in \text{End}_{\mathcal{O}_X}(\mathcal{E})$ such that $\langle \phi(e), f \rangle + \langle e, \phi(f) \rangle = 0$ for all $e, f \in \mathcal{E}$. Let $\theta: \bigwedge^2 \mathcal{E} \to \mathfrak{so}(\mathcal{E}, \langle \ , \ \rangle)$ denote the map defined by $\theta(e_1 \wedge e_2)(f) = 1/2 \langle \langle e_2, f \rangle e_1 - \langle e_1, f \rangle e_2 \rangle$. The map $\theta$ is an isomorphism since the pairing $\langle \ , \ \rangle$ is non-degenerate.

Let $\mathcal{A}_{\mathcal{E},\langle \ , \ \rangle}$ denote the subsheaf of operators $D \in \mathcal{A}_{\mathcal{E}}$ which satisfy $\langle D(e), f \rangle + \langle e, D(f) \rangle = \sigma(D)(\langle e, f \rangle)$. The sheaf $\mathcal{A}_{\mathcal{E},\langle \ , \ \rangle}$ is a $\mathcal{O}_X$-algebroid. The anchor map $\sigma: \mathcal{A}_{\mathcal{E},\langle \ , \ \rangle} \to \mathcal{T}_X$ is surjective with kernel $\mathfrak{so}(\mathcal{E}, \langle \ , \ \rangle)$. 
Let \( I \subset \mathcal{S}_{\mathcal{O}_X}(\mathcal{E}[-1] \oplus \mathcal{A}_{\mathcal{E},(\cdot,\cdot)}[-2]) \) denote the ideal generated by the sections of the form \( \theta(g) - g \in \mathcal{S}_{\mathcal{O}_X}(\mathcal{E}[-1] \oplus \mathcal{A}_{\mathcal{E},(\cdot,\cdot)}[-2])^2, \ g \in \wedge^2 \mathcal{E} \). Let
\[
e_3(\mathcal{E},(\cdot,\cdot)) := \mathcal{S}_{\mathcal{O}_X}(\mathcal{E}[-1] \oplus \mathcal{A}_{\mathcal{E},(\cdot,\cdot)}[-2])/I.\
\]
Since the relations are homogeneous \( e_3(\mathcal{E},(\cdot,\cdot)) \) inherits a structure of a graded commutative \( \mathcal{O}_X \)-algebra. Moreover, \( e_3(\mathcal{E},(\cdot,\cdot))^0 = \mathcal{O}_X, \ e_3(\mathcal{E},(\cdot,\cdot))^1 = \mathcal{E}, \ e_3(\mathcal{E},(\cdot,\cdot))^2 = \mathcal{A}_{\mathcal{E},(\cdot,\cdot)} \).

The bracket on \( \mathcal{A}_{\mathcal{E},(\cdot,\cdot)} \), the action of the latter on \( \mathcal{E} \) and the pairing \( (\cdot,\cdot) \) extend canonically to a structure of a Lie algebra on \( \mathcal{S}_{\mathcal{O}_X}(\mathcal{E}[-1] \oplus \mathcal{A}_{\mathcal{E},(\cdot,\cdot)}[-2])[2] \) so that the bracket is a bi-derivation of the product. Since \( \{\mathcal{S}_{\mathcal{O}_X}(\mathcal{E}[-1] \oplus \mathcal{A}_{\mathcal{E},(\cdot,\cdot)}[-2])[2], I\} \subset I \), the bracket descends to \( e_3(\mathcal{E},(\cdot,\cdot))[2] \). The commutative algebra \( e_3(\mathcal{E},(\cdot,\cdot)) \) equipped with the bracket \( (\cdot,\cdot) \) is a \( e_3 \)-algebra.

A structure on a Courant algebroid on \( \mathcal{E} \) with the symmetric pairing given by \( (\cdot,\cdot) \) gives rise to a derivation of the \( e_3 \)-algebra \( e_3(\mathcal{E},(\cdot,\cdot)) \) of degree one and square zero. As a consequence, there is a canonical map of DGA \( \mathcal{O}_{X^2} \to e_3(\mathcal{E},(\cdot,\cdot)) \) which extends the identity map of \( \mathcal{O}_X \) in degree zero.

Suppose that \( \mathcal{Q} \) is a Courant algebroid. Thus, \( e_3(\mathcal{Q},(\cdot,\cdot)) \) is a differential \( e_3 \)-algebra equipped with the morphism of DGA \( \pi^1: \mathcal{O}_{X^2} \to (e_3(\mathcal{Q},(\cdot,\cdot)) which we assume to be a monomorphism. Let \( F_1 e_3(\mathcal{Q},(\cdot,\cdot))[2] \) denote the normalizer of \( \text{im}(\pi^1) \). The sheaf \( F_1 e_3(\mathcal{Q},(\cdot,\cdot))[2] \) has a canonical structure of a \( \mathcal{O}_{X^2} \)-Lie algebroid marked by \( \pi^1(1) \). Note that \( \mathcal{Q} = F_1 e_3(\mathcal{Q},(\cdot,\cdot))^1 = e_3(\mathcal{Q},(\cdot,\cdot))^1 \) and \( \mathcal{A}_{\mathcal{Q},(\cdot,\cdot)} \subset F_1 e_3(\mathcal{Q},(\cdot,\cdot))^2 \).

The map \( \mathcal{Q}[1] \to F_1 e_3(\mathcal{Q},(\cdot,\cdot))[2] \) is a CtL morphism \( \mathcal{Q} \to (F_1 e_3(\mathcal{Q},(\cdot,\cdot))[2], \pi^1(1)) \) hence extends to a morphism of marked \( \mathcal{O}_{X^2} \)-Lie algebroids
\[
(\tau \mathcal{Q}, \mathcal{Q}) \to (F_1 e_3(\mathcal{Q},(\cdot,\cdot))[2], \pi^1(1)).
\]

### 6.4. Some examples.

**Example 6.3. (Exact Courant algebroids)** Applying Theorem 6.2 to \( \mathcal{T}_X \) we obtain the equivalence of [CR], namely the equivalence of categories \( \text{ECA}(X)_{n-1} \cong \mathcal{O}[n] \text{Ext}(\mathcal{T}_X) \) (denoted \( \mathcal{O}_{X^1} = \text{LA}(X^2) \) in loc. cit.).

**Example 6.4. (Commutative Courant algebroids)** The sheaf of differential forms of degree \( n-1 \) has a canonical structure of a commutative \( (n-1) \)-dimensional Courant algebroid. The \( \mathcal{O}_{X^2} \)-Lie algebroid \( \tau \Omega^n_X \) is \( \mathcal{O}_{X^2}[n] \) with trivial bracket and anchor.

**Example 6.5. (Quadratic Lie algebras)** Suppose that \( \mathcal{A} \) is a transitive \( \mathcal{O}_X \)-Lie algebroid. Let \( \mathfrak{g} \) denote the kernel of the anchor map. Let \( (\cdot,\cdot) \) be a \( \mathcal{A} \)-invariant symmetric \( \mathcal{O}_X \)-bilinear pairing on \( \mathfrak{g} \). This data determines a Courant algebroid with trivial anchor \( \mathfrak{g} \) which is a Courant extension of \( \mathfrak{g} \) (see [3], 3.2 for details). The Courant algebroid \( \mathfrak{g} \) transgresses to an abelian extension of the DGLA \( \mathfrak{g}^\mathbb{C} \) by \( \mathcal{O}_{X^2}[2] \).
The construction applies when \( X \) is a connected manifold of dimension zero, in which case \( g \) is a \( \mathbb{C} \)-Lie algebra equipped with an invariant symmetric bilinear form. In this case the DGLA \( \tau g \) is easily seen to be \( \mathbb{C}[2] \oplus g[1] \oplus g \) with the bracket given by the symmetric form, the adjoint action and the bracket on \( g \) and the only non-trivial component of the differential the identity map of \( g \).

7. Proofs

7.1. Lemma 3.3. Suppose that \( f \in \mathcal{O}_X, \omega_1, \omega_2 \in \mathcal{O}_{X^2}, a_1, a_2 \in \mathcal{A} \). The identities

\[
\begin{align*}
(7.1.1) \quad \{ \omega_1 \otimes 1 \otimes f a_1, \omega_2 \otimes 1 \otimes a_2 \} & = [f \omega_1 \otimes 1 \otimes a_1, \omega_2 \otimes 1 \otimes a_2] + [\omega_1 \wedge df \otimes \epsilon \otimes a_1, \omega_2 \otimes 1 \otimes a_2]; \\
(7.1.2) \quad \{ \omega_1 \otimes 1 \otimes f a_1, \omega_2 \otimes \epsilon \otimes a_2 \} & = [f \omega_1 \otimes 1 \otimes a_1, \omega_2 \otimes \epsilon \otimes a_2] + [\omega_1 \wedge df \otimes \epsilon \otimes a_1, \omega_2 \otimes \epsilon \otimes a_2]; \\
(7.1.3) \quad \{ f \omega_1 \otimes \epsilon \otimes a_1, \omega_2 \otimes \epsilon \otimes a_2 \} & = [\omega_1 \otimes \epsilon \otimes f a_1, \omega_2 \otimes \epsilon \otimes a_2]; \\
(7.1.4) \quad \{ f \omega_1 \otimes \epsilon \otimes a_1, \omega_2 \otimes 1 \otimes a_2 \} & = [\omega_1 \otimes \epsilon \otimes f a_1, \omega_2 \otimes 1 \otimes a_2];
\end{align*}
\]

show that the bracket on \( \mathcal{O}_{X^2}[\epsilon] \otimes \mathcal{A} \) descends to \( \text{pr}_* \text{ev}^* \mathcal{A} \). We will verify equations (7.1.1) and (7.1.3) leaving (7.1.2) and (7.1.4) to the reader.

\[
\begin{align*}
\{ \omega_1 \otimes 1 \otimes f a_1, \omega_2 \otimes 1 \otimes a_2 \} & = \omega_1 \wedge (f L_{\sigma(a_1)} \omega_2 + df \wedge (\iota_{\sigma(a_1)} \omega_2) \otimes 1 \otimes a_2 + \\
& \quad + (-1)^{\omega_1 \omega_2} \omega_2 \wedge \omega_1 \otimes 1 \otimes f[a_1, a_2] - (1)^{\omega_1 \omega_2} \omega_2 \wedge \omega_1 \otimes 1 \otimes L_{\sigma(a_2)} f \cdot a_1 + \\
& \quad - (1)^{\omega_1 \omega_2} \omega_2 \wedge L_{\sigma(a_2)} \omega_1 \otimes 1 \otimes f \cdot a_1 \\
& = [f \omega_1 \otimes 1 \otimes a_1, \omega_2 \otimes 1 \otimes a_2] + [\omega_1 \wedge df \otimes \epsilon \otimes a_1, \omega_2 \otimes 1 \otimes a_2];
\end{align*}
\]

\[
\begin{align*}
\{ f \omega_1 \otimes \epsilon \otimes a_1, \omega_2 \otimes 1 \otimes a_2 \} & = f \omega_1 \wedge (\iota_{\sigma(a_1)} \omega_2) \otimes 1 \otimes a_2 + \\
& \quad + (-1)^{(\omega_1 - 1) \omega_2} \omega_2 \wedge f \omega_1 \otimes \epsilon \otimes [a_1, a_2] - (1)^{(\omega_1 - 1) \omega_2} \omega_2 \wedge L_{\sigma(a_2)} (f \omega_1) \otimes \epsilon \otimes a_1 = \\
& = [\omega_1 \otimes \epsilon \otimes f a_1, \omega_2 \otimes 1 \otimes a_2].
\end{align*}
\]

The calculations

\[
\begin{align*}
\sigma(1 \otimes \epsilon \otimes f a) & = \iota_{\sigma(fa)} = f \iota_{\sigma(a)} = \sigma(f \otimes \epsilon \otimes a), \\
\sigma(1 \otimes 1 \otimes f a) & = L_{\sigma(fa)} = f L_{\sigma(a)} + df \wedge \iota_{\sigma(a)} \\
& = \sigma(f \otimes 1 \otimes a) + \sigma(df \otimes \epsilon \otimes a)
\end{align*}
\]

show that the anchor map descends to \( \text{pr}_* \text{ev}^* \mathcal{A} \).
7.2. Lemma 3.4
(1) Suppose that $f \in \mathcal{O}_X$, $\alpha \in \Omega^{k+1}_X$ and $B \in (\text{pr}_* \text{ev}^* \mathcal{E})^0$. From the computations

$$
\tilde{i}_D(f \alpha \otimes B) = \iota_{\sigma(D)}(f \alpha) \otimes B + (-1)^\alpha f \alpha \otimes i DB \\
= f \iota_D(\alpha) \otimes B + (-1)^\alpha f \alpha \otimes i DB \\
= f(\tilde{i}_D \alpha \otimes B)
$$

and

$$
\tilde{i}_D(\alpha \otimes fB) = \iota_{\sigma(D)}(\alpha) \otimes fB + (-1)^\alpha \alpha \otimes i DB \\
= f \tilde{i}_D(\alpha \otimes B),
$$

it follows that $\tilde{i}_D(\alpha \otimes fB) = \tilde{i}_D(f \alpha \otimes B)$.

(2) Suppose that $\gamma \in \Omega^k_X$ and $B \in (\text{pr}_* \text{ev}^* \mathcal{E})$. The principal symbol of $\tilde{i}_D$ is equal to

$$
[i_D, \gamma](B) = (i_D \circ \gamma - (-1)^\gamma \gamma \circ i_D)B \\
= \iota_{\sigma(D)} \gamma \otimes (B) + (-1)^\gamma \gamma i_D(B) - (-1)^\gamma \gamma i_D(B) \\
= \iota_{\sigma(D)} \gamma \otimes (B) \\
= (\sigma(D) \otimes e) \gamma \otimes B.
$$

7.3. Lemma 3.5
(1)

$$
\tilde{D}(A \otimes e) = [\partial, \tilde{i}_D](A \otimes e) \\
= (\partial \circ \tilde{i}_D + \tilde{i}_D \circ \partial)(A \otimes e) \\
= d(\iota_{\sigma(D)} A \otimes e + (-1)^A A e \otimes De) + \iota_D(\partial A \otimes e) \\
= d\iota_{\sigma(D)} A \otimes e + A \otimes De + \iota_{\sigma(D)} dA \otimes e \\
= L_{\sigma(D)} A \otimes e + A \otimes De
$$
(2) Suppose that $D_1, D_2 \in A_E$. The calculation

$$[\widetilde{D}_1, \widetilde{D}_2](A \otimes e) = (\widetilde{D}_1 \circ \widetilde{D}_2 - \widetilde{D}_2 \circ \widetilde{D}_1)(A \otimes e)$$

$$= \widetilde{D}_1(L_{\sigma(D_2)}A \otimes e + A \otimes D_2 e) - \widetilde{D}_2(L_{\sigma(D_1)}A \otimes e + A \otimes D_1 e)$$

$$= L_{\sigma(D_1)}L_{\sigma(D_2)}A \otimes e + L_{\sigma(D_2)}A \otimes D_1 e$$

$$+ L_{\sigma(D_1)}A \otimes D_2 e + A \otimes D_1 D_2 e - L_{\sigma(D_2)}L_{\sigma(D_1)}A \otimes e$$

$$- L_{\sigma(D_1)}A \otimes D_2 e - L_{\sigma(D_2)}A \otimes D_1 e - A \otimes D_2 D_1 e$$

$$= [L_{\sigma(D_1)}, L_{\sigma(D_2)}]A \otimes e + A \otimes [D_1, D_2] e$$

$$= [\widetilde{D}_1, \widetilde{D}_2](A \otimes e)$$

shows that the map $D \mapsto \tilde{D}$ is compatible with brackets.

(3) The commutativity of the diagram follows from the first item.

(4) Left to the reader.

7.4. Lemma 3.6

(1) Suppose that $D \in A_E$ and $\omega_1, \omega_2 \in \mathcal{O}_{X^1}$. By definition of $L^E$

$$L^E((\omega_1 + \omega_2 \otimes \epsilon) \otimes D) = \omega_1 \otimes \tilde{D} + \omega_2 \otimes i_{\tilde{D}},$$

For each $f \in \mathcal{O}_X$ the formula $\tilde{fD} = df \otimes \epsilon + f \tilde{D}$ holds. The calculation

$$L^E((\omega_1 + \omega_2 \otimes \epsilon) \otimes fD) = \omega_1 \otimes \tilde{fD} + \omega_2 \otimes fi_{\tilde{D}}$$

$$= f\omega_1 \otimes \tilde{D} + (\omega_1 + f\omega_2) \otimes i_{\tilde{D}}$$

$$= L^E((f\omega_1 + (f\omega_2 + \omega_1 \wedge df)\epsilon) \otimes D)$$

$$= L^E((\omega_1 + \omega_2 \otimes \epsilon)(f + df \otimes \epsilon) \otimes D),$$

shows that the map $L^E$ descends to the map $A^*_E \to A_{pr \cdot ev^*E}$. 
(2) Let \( \omega_1, \omega_2, \gamma_1, \gamma_2 \in \mathcal{O}_{X^2} \) and \( D_1, D_2 \in \mathcal{A}_\mathcal{E} \). We leave verification of the identities \([\tilde{D}_1, i_{\tilde{D}_2}] = i_{[\tilde{D}_1, \tilde{D}_2]} \) and \([i_{\tilde{D}_1}, i_{\tilde{D}_2}] = 0 \) to the reader. The calculation

\[
L^E(\omega_1 \otimes D_1 + \gamma_1 \otimes \epsilon \otimes D_1, \omega_2 \otimes D_2 + \gamma_2 \otimes \epsilon \otimes D_2)) = \\
L^E(\omega_1 \wedge L_{\sigma(D_1)} \omega_1 \otimes D_2) + (-1)^{\omega_1 \omega_2} \omega_2 \wedge \omega_1 \otimes \sigma(D_1, D_2) + \\
(-1)^{\omega_1 \omega_2} \omega_2 \wedge L_{\sigma(D_2)} \omega_1 \otimes D_1) + L^E(\omega_1 \wedge \sigma(D_1) \gamma_2 \otimes \epsilon \otimes D_2) + \\
(-1)^{\omega_1 \omega_2} \omega_2 \wedge \omega_1 \otimes \epsilon \otimes \sigma(D_1, D_2) + L^E(\gamma_1 \wedge \sigma(D_1) \gamma_2 \otimes \epsilon \otimes D_2) + \\
(-1)^{\omega_1 \omega_2} \omega_2 \wedge \omega_1 \otimes \epsilon \otimes \sigma(D_2) \gamma_1 \otimes \epsilon \otimes D_1) = \\
\omega_1 \wedge L_{\sigma(D_1)} \omega_2 \wedge \tilde{D}_2 + (-1)^{\omega_1 \omega_2} \omega_2 \wedge \omega_1 \otimes \tilde{D}_1 + \\
(-1)^{\omega_1 \omega_2} \omega_2 \wedge \omega_1 \wedge L_{\sigma(D_1)} \gamma_2 \otimes \tilde{D}_2 + \\
(-1)^{\omega_1 \omega_2} \omega_2 \wedge \omega_1 \otimes \tilde{D}_1 + (-1)^{\omega_1 \omega_2} \omega_2 \wedge \omega_1 \otimes \tilde{D}_1 + \\
\gamma_1 \wedge \sigma(D_2) \omega_2 \wedge \tilde{D}_2 + (-1)^{\omega_1 \omega_2} \omega_2 \wedge \gamma_1 \otimes \tilde{D}_2 + \\
(-1)^{\omega_1 \omega_2} \omega_2 \wedge \omega_1 \wedge \sigma(D_2) \gamma_1 \wedge \tilde{D}_2 + \\
(-1)^{\omega_1 \omega_2} \omega_2 \wedge \omega_1 \wedge \sigma(D_2) \gamma_1 \otimes \tilde{D}_2 = \\
[L^E(\omega_1 \otimes D_1 + \gamma_1 \otimes \epsilon \otimes D_1), L^E(\omega_2 \otimes D_2 + \gamma_2 \otimes \epsilon \otimes D_2)]]
\]

shows that the map \( L^E \) is a map of Lie algebras. We leave to the reader to check that \((3.6.5)\) commutes with respective anchor maps. Hence it is a map of \( \mathcal{O}_{X^1} \)-Lie algebroids.

7.5. Lemma 5.5. Suppose that \( a = \omega_1 + \alpha_1 \epsilon \otimes q_1 + \beta_1 \otimes r_1 \) and \( b = \omega_2 + \alpha_2 \epsilon \otimes q_2 + \beta_2 \otimes r_2 \) are homogeneous elements, where \( \omega_i, \alpha_i, \beta_i \in \mathcal{O}_{X^2} \) and \( q_i, r_i \in \mathcal{Q} \) for \( i = 1, 2 \). The calculation

\[
\partial(\alpha_1 \epsilon \otimes q_2, \alpha_2 \epsilon \otimes q_2) = \partial(\alpha_1 \wedge (\iota_{q_1}) \alpha_2) \epsilon \otimes q_2) \\
+ (\alpha_1)_{\alpha_2} + \alpha_2 \partial(\alpha_2 \wedge (\iota_{q_2}) \alpha_1) \epsilon \otimes q_1 + (\alpha_1)_{\alpha_2} \delta(\alpha_2 \wedge \alpha_1 \wedge q_1, q_2) \\
= d(\alpha_1 \wedge (\iota_{q_1}) \alpha_2) \epsilon \otimes q_2 + (\alpha_1)_{\alpha_2} \delta(\alpha_2 \wedge \alpha_1 \wedge q_1, q_2) \\
+ (\alpha_1)_{\alpha_2} \alpha_2 \delta(\alpha_2 \wedge \alpha_1 \wedge q_1, q_2) + (\alpha_1)_{\alpha_2} \alpha_2 \delta(\alpha_2 \wedge \alpha_1 \wedge q_1, q_2) \\
= [\partial(\alpha_1 \epsilon \otimes q_1), \alpha_2 \epsilon \otimes q_2] + (\alpha_1)_{\alpha_2} \delta(\alpha_2 \epsilon \otimes q_2) \\
\]

shows that \((5.2.9)\) holds in the case \((5.2.3)\) of the operation \((5.2.2)\). We leave the cases \((5.2.4)-(5.2.6)\) to the reader.
7.6. Lemma [5.6] Suppose that \( a = \omega + \alpha \epsilon \otimes q + \beta \otimes r \in \mathcal{O}_{X^1}[k] \oplus \mathcal{O}_{X^1}[\epsilon] \otimes \mathcal{C} \mathcal{Q} \) is a homogeneous element and \( b = \gamma \otimes f s - f \gamma \otimes s - \gamma \wedge d f \epsilon \otimes s \in K \), where \( \omega, \alpha, \beta, \gamma \in \mathcal{O}_{X^1}, f \in \mathcal{O}_X \) and \( q, r, s \in \mathcal{Q} \). Then,
\[
[a, b] = [\omega, b] + [\alpha \epsilon \otimes q, b] + [\beta \otimes r, b].
\]
The formulas (5.2.7) and (5.2.8) imply that \( [\omega, b] \in K \). The calculations,
\[
[\alpha \epsilon \otimes q, b] = [\alpha \epsilon \otimes q, \gamma \otimes f s - f \gamma \otimes s - \gamma \wedge d f \epsilon \otimes s]
= \alpha \wedge \iota_{\pi(q)} \gamma (1 \otimes f s - f \otimes s - d f \epsilon \otimes s) + (-1)^{(a-1)} \gamma \otimes (1 \epsilon f \{q, s\})
- f \epsilon \{q, s\} \) + (-1)^{\gamma} \alpha \wedge (1 \otimes (L_{\pi(q)}) f s - \iota_{\pi(q)} (d f \epsilon \otimes s)
\]
and
\[
[\beta \otimes r, b] = [\beta \otimes r, \gamma \otimes f s - f \gamma \otimes s - \gamma \wedge d f \epsilon \otimes s]
= \beta \wedge (L_{\pi(r)} \gamma)(1 \otimes f s - f \otimes s - d f \epsilon \otimes s) + (-1)^{\beta} \gamma \wedge (1 \otimes (L_{\pi(r)}) f s)
- (L_{\pi(r)}) f \otimes s - d (L_{\pi(r)} f \epsilon \otimes s) + (-1)^{\beta} \gamma \wedge (1 \otimes f \{r, s\}) - f \otimes \{r, s\}
- d f \epsilon \otimes \{r, s\})
\]
show that \( [\alpha \epsilon \otimes q, b], [\beta \otimes r, b] \in K \). We leave it to the reader to verify that \( [b, a] \in K \).

7.7. Proposition [5.8] The proof of the proposition proceeds in several steps. All elements are assumed to be homogeneous. Suppose that \( \omega, \alpha_i \in \mathcal{O}_{X^1}, q, q_i \in \mathcal{Q} \) for \( i = 1, 2 \).

7.7.1. Skew-symmetry. Calculations
\[
[\alpha_1 \epsilon \otimes q_1, \alpha_2 \epsilon \otimes q_2] + (-1)^{(\alpha_1 - 1)(\alpha_2 - 1)} [\alpha_2 \epsilon \otimes q_2, \alpha_1 \epsilon \otimes q_1] = 0,
\]
\[
[\alpha_1 \otimes q_1, \alpha_2 \epsilon \otimes q_2] + (-1)^{\alpha_1 (\alpha_2 - 1)} [\alpha_2 \epsilon \otimes q_2, \alpha_1 \otimes q_1] =
(-1)^{\alpha_1 \alpha_2} \alpha_1 \wedge \alpha_2 (\epsilon \otimes \pi^\dagger (d \langle q_1, q_2 \rangle) - d \langle q_1, q_2 \rangle) = 0,
\]
\[
[\alpha_1 \otimes q_1, \alpha_2 \otimes q_2] + (-1)^{\alpha_1 \alpha_2} [\alpha_2 \otimes q_2, \alpha_1 \otimes q_1] =
(-1)^{\alpha_1 \alpha_2} \alpha_2 \wedge \alpha_1 \otimes \pi^\dagger (d \langle q_1, q_2 \rangle) = 0,
\]
\[
[\alpha \epsilon \otimes q, \omega] + (-1)^{(\alpha - 1)\omega} [\omega, \alpha \epsilon \otimes q] = 0,
\]
\[
[\alpha \otimes q, \omega] + (-1)^{\alpha \omega} [\omega, \alpha \otimes q] = 0
\]
show that symmetrization of the brackets given by formulas (5.2.3) – (5.2.8) is trivial. Thus the operation (5.2.2) is skew-symmetric on \( \tau \mathcal{Q} \).
7.7.2. *Jacobi identity.* We outline the proof of the Jacobi identity leaving details to the reader. It is easy to see that Jacobi identity holds for the rules (1)-(6) that define the operation \((5.2.2)\) on \(\tau Q\). By the linearity of the operation \((5.2.2)\), it is enough to check on it the Jacobi identity for the combination of elements:

\[(7.7.1) \quad [\alpha_1 \epsilon \otimes q_1, [\alpha_2 \epsilon \otimes q_2, \alpha_3 \epsilon \otimes q_3]] =
\left[[\alpha_1 \epsilon \otimes q_1, \alpha_2 \epsilon \otimes q_2], \alpha_3 \epsilon \otimes q_3\right] \pm [\alpha_2 \epsilon \otimes q_2, [\alpha_1 \epsilon \otimes q_1, \alpha_3 \epsilon \otimes q_3]]
\]

\[(7.7.2) \quad [\alpha_1 \otimes q_1, [\alpha_2 \otimes q_2, \alpha_3 \otimes q_3]] =
\left[[\alpha_1 \otimes q_1, \alpha_2 \otimes q_2], \alpha_3 \otimes q_3\right] \pm [\alpha_2 \otimes q_2, [\alpha_1 \otimes q_1, \alpha_3 \otimes q_3]]
\]

\[(7.7.3) \quad [\alpha_1 \epsilon \otimes q_1, [\alpha_2 \otimes q_2, \alpha_3 \otimes q_3]] =
\left[[\alpha_1 \epsilon \otimes q_1, \alpha_2 \otimes q_2], \alpha_3 \otimes q_3\right] \pm [\alpha_2 \otimes q_2, [\alpha_1 \epsilon \otimes q_1, \alpha_3 \otimes q_3]]
\]

\[(7.7.4) \quad [\alpha_1 \epsilon \otimes q_1, [\alpha_2 \epsilon \otimes q_2, \alpha_3 \otimes q_3]] =
\left[[\alpha_1 \epsilon \otimes q_1, \alpha_2 \epsilon \otimes q_2], \alpha_3 \otimes q_3\right] \pm [\alpha_2 \epsilon \otimes q_2, [\alpha_1 \epsilon \otimes q_1, \alpha_3 \otimes q_3]]
\]

\[(7.7.5) \quad [\alpha_1 \epsilon \otimes q_1, [\alpha_2 \otimes q_2, \alpha_3 \epsilon \otimes q_3]] =
\left[[\alpha_1 \epsilon \otimes q_1, \alpha_2 \otimes q_2], \alpha_3 \epsilon \otimes q_3\right] \pm [\alpha_2 \otimes q_2, [\alpha_1 \epsilon \otimes q_1, \alpha_3 \epsilon \otimes q_3]]
\]

\[(7.7.6) \quad [\alpha_1 \otimes q_1, [\alpha_2 \epsilon \otimes q_2, \alpha_3 \epsilon \otimes q_3]] =
\left[[\alpha_1 \otimes q_1, \alpha_2 \epsilon \otimes q_2], \alpha_3 \epsilon \otimes q_3\right] \pm [\alpha_2 \epsilon \otimes q_2, [\alpha_1 \otimes q_1, \alpha_3 \epsilon \otimes q_3]]
\]

\[(7.7.7) \quad [\alpha_1 \otimes q_1, [\alpha_2 \epsilon \otimes q_2, \alpha_3 \otimes q_3]] =
\left[[\alpha_1 \otimes q_1, \alpha_2 \epsilon \otimes q_2], \alpha_3 \otimes q_3\right] \pm [\alpha_2 \epsilon \otimes q_2, [\alpha_1 \otimes q_1, \alpha_3 \otimes q_3]]
\]

\[(7.7.8) \quad [\alpha_1 \otimes q_1, [\alpha_2 \epsilon \otimes q_2, \alpha_3 \epsilon \otimes q_3]] =
\left[[\alpha_1 \otimes q_1, \alpha_2 \epsilon \otimes q_2], \alpha_3 \epsilon \otimes q_3\right] \pm [\alpha_2 \epsilon \otimes q_2, [\alpha_1 \otimes q_1, \alpha_3 \epsilon \otimes q_3]]
\]

Since \((4.0.3)\) holds in \(Q\), \((7.7.1)\) and \((7.7.2)\) are deduced using formulas \((5.2.3)-(5.2.8)\) and the Leibniz rule. The verification of \((7.7.3)-(7.7.7)\) is reported in the next calculations.
The left hand side of equation (7.7.3) is equal to:

$$[\alpha_1 \epsilon \otimes q_1, [\alpha_2 \otimes q_2, \alpha_3 \otimes q_3]] = \alpha_1 \land (\iota_{\pi(q_1)}(\alpha_2 L_{\pi(q_2)}\alpha_3) \otimes q_3) \pm$$

$$\alpha_2 \land (L_{\pi(q_2)}\alpha_3) \land \alpha_1 d(q_1, q_3) \pm \alpha_2 \land (L_{\pi(q_3)}\alpha_3) \land \alpha_1 \epsilon \otimes \{q_1, q_3\} \pm$$

$$\alpha_2 \land (L_{\pi(q_2)}\alpha_3) \land (L_{\pi(q_3)}\alpha_1) \epsilon \otimes q_1 \alpha_1 \land \iota_{\pi(q_1)}(\alpha_3 \alpha_2)\{q_2, q_3\} \pm \alpha_3 \land \alpha_2 \land \alpha_1 \epsilon \otimes \{q_1, q_2, q_3\} \pm$$

$$\alpha_3 \land \alpha_2 \land \alpha_1 \epsilon \otimes \{q_1, q_2\} \pm \alpha_3 \land \alpha_2 \land (L_{[\pi(q_2), \pi(q_3)]}\alpha_1) \epsilon \otimes q_1 \pm$$

In the right hand side of (7.7.3) the first summand is equal to,

$$[[\alpha_1 \epsilon \otimes q_1, \alpha_2 \otimes q_2], \alpha_3 \otimes q_3] = \alpha_1 \land (\iota_{\pi(q_1)}(\alpha_2 L_{\pi(q_2)}\alpha_3) \otimes q_3) \pm$$

$$\alpha_3 \land \alpha_1 \land (\iota_{\pi(q_1)}\alpha_2) \otimes \{q_2, q_3\} \pm \alpha_3 \land L_{\pi(q_3)}(\alpha_1 \land \iota_{\pi(q_1)}\alpha_2) \otimes q_2 \pm$$

$$\alpha_3 \land \alpha_3 L_{\pi(q_3)}(\alpha_2 \land \alpha_1 \land d(q_1, q_2)) \pm \alpha_2 \land \alpha_1 \land (\iota_{\{q_1, q_2\}}\alpha_3) \otimes q_3 \pm$$

The second summand is equal to:

$$[[\alpha_2 \otimes q_2, [\alpha_1 \epsilon \otimes q_1, \alpha_3 \otimes q_3]] = \alpha_2 \land L_{\pi(q_2)}(\alpha_2 \land \iota_{\pi(q_1)}\alpha_3) \otimes q_3 \pm$$

$$\alpha_1 \land (\iota_{\pi(q_1)}\alpha_3) \land \alpha_2 \otimes \{q_2, q_3\} \pm \alpha_1 \land (\iota_{q_1}\alpha_3) \land (L_{\pi(q_3)}\alpha_2) \otimes q_2 \pm$$

$$\alpha_2 \land L_{\pi(q_3)}(\alpha_3 \land \alpha_1 \land d(q_1, q_3)) \pm \alpha_2 \land L_{\pi(q_3)}(\alpha_3 \land \alpha_1) \epsilon \otimes \{q_1, q_3\} \pm$$

Using Leibniz rule the above calculations reduce (7.7.3) to

$$\alpha_3 \land (L_{\pi(q_3)}\alpha_2) \land \alpha_1 \land d(q_1, q_2) \pm \alpha_3 \land (L_{\pi(q_3)}\alpha_2) \land \alpha_1 \epsilon \otimes \{q_1, q_2\} =$$

$$\alpha_3 \land L_{\pi(q_3)}(\alpha_2 \land \alpha_1) \epsilon \otimes \{q_1, q_2\} \pm \alpha_3 \land (L_{\pi(q_3)}\alpha_1) \land \alpha_2 \epsilon \otimes \{q_2, q_1\}$$
The latter equation follows from $\epsilon \otimes \{q_2, q_1\} = \epsilon \otimes (-\{q_1, q_2\} + \pi^d\langle q_1, q_2 \rangle)$. In the case of equation (7.7.3), the left hand side is equal to

$$[\alpha_1 \epsilon \otimes q_1, [\alpha_2 \epsilon \otimes q_2, \alpha_3 \otimes q_3]] =$$

$$\alpha_1 \wedge \epsilon_\pi(q_1)(\alpha_2 \wedge \epsilon_\pi(q_2)\alpha_3) \otimes q_3 = \alpha_2 \wedge \epsilon_\pi(q_2)(\alpha_3 \wedge \alpha_1 \wedge \pi^d\langle q_1, q_2 \rangle)\alpha_1 \wedge \epsilon_\pi(q_1)\alpha_3 \wedge \epsilon_\pi(q_2)\alpha_2 \wedge \epsilon_\pi(q_3)\alpha_1 \wedge \epsilon_\pi(q_2)\alpha_3 \wedge \epsilon_\pi(q_3)\alpha_2$$

$$\alpha_2 \wedge \alpha_3 \wedge \epsilon_\pi(q_2)\alpha_1 \wedge \epsilon_\pi(q_3)\alpha_2 \wedge \epsilon_\pi(q_3)\alpha_1$$

$$\alpha_1 \wedge \epsilon_\pi(q_1)(\alpha_3 \wedge \alpha_2 \wedge \epsilon_\pi(q_2)\alpha_3) \otimes q_3$$

$$\alpha_3 \wedge \alpha_2 \wedge \alpha_1 \wedge \epsilon_\pi(q_1)\alpha_2 \wedge \epsilon_\pi(q_3)\alpha_1 \wedge \epsilon_\pi(q_2)\alpha_3 \wedge \epsilon_\pi(q_3)\alpha_2$$

$$\alpha_3 \wedge \epsilon_\pi(q_3)\alpha_2 \wedge \epsilon_\pi(q_2)\alpha_1 \wedge \epsilon_\pi(q_2)\alpha_3 \wedge \epsilon_\pi(q_3)\alpha_2 \wedge \epsilon_\pi(q_3)\alpha_1$$

The second summand is equal to

$$[[\alpha_1 \epsilon \otimes q_1, [\alpha_2 \epsilon \otimes q_2, \alpha_3 \otimes q_3]] =$$

$$\alpha_2 \wedge \epsilon_\pi(q_2)(\alpha_3 \wedge \alpha_1 \wedge \epsilon_\pi(q_1)\alpha_2 \wedge \epsilon_\pi(q_3)\alpha_2 \wedge \epsilon_\pi(q_3)\alpha_2 \wedge \epsilon_\pi(q_3)\alpha_2 \wedge \epsilon_\pi(q_3)\alpha_2 \wedge \epsilon_\pi(q_3)\alpha_2 \wedge \epsilon_\pi(q_3)\alpha_2 \wedge \epsilon_\pi(q_3)\alpha_2 \wedge \epsilon_\pi(q_3)\alpha_2.$$
The first summand on the right hand side is equal to

\[
[[\alpha_1 \epsilon \otimes q_1, \alpha_2 \otimes q_2], \alpha_3 \epsilon \otimes q_3] = \\
\alpha_1 \wedge (\iota_{\pi(q_1)} \alpha_2) \wedge (L_{\pi(q_2)} \alpha_3) \epsilon \otimes q_3 \pm \alpha_3 \wedge \alpha_1 \wedge (\iota_{\pi(q_1)} \alpha_2) \epsilon \otimes \{q_2, q_3\} \pm \\
\alpha_3 \wedge \iota_{\pi(q_3)} (\alpha_1 \wedge \iota_{\pi(q_1)} \alpha_2) \otimes q_2 \pm \alpha_3 \wedge \iota_{\pi(q_3)} (\alpha_2 \wedge \alpha_1 \wedge d(q_1, q_2)) \pm \\
\alpha_2 \wedge \alpha_1 \wedge (\iota_{\pi(q_1), \pi(q_2)} \alpha_3) \epsilon \otimes q_3 \pm \alpha_3 \wedge \alpha_2 \wedge \alpha_1 \wedge \{q_1, \{q_2, q_3\}\} \pm \\
\alpha_3 \wedge \iota_{\pi(q_3)} (\alpha_2 \wedge \alpha_1) \epsilon \otimes \{q_1, q_2\} \pm \alpha_2 \wedge (L_{\pi(q_2)} \alpha_1) \wedge (\iota_{\pi(q_1)} \alpha_3) \epsilon q_3 \pm \\
\alpha_3 \wedge \alpha_2 \wedge (L_{\pi(q_2)} \alpha_1) \wedge \{q_1, q_3\} \pm \alpha_3 \wedge \iota_{\pi(q_3)} (\alpha_2 \wedge L_{\pi(q_2)} \alpha_1) \epsilon \otimes q_1
\]

The second summand on the right hand side is equal to

\[
[\alpha_2 \otimes q_2, [\alpha_1 \epsilon \otimes q_1, \alpha_3 \epsilon \otimes q_3]] = \\
\alpha_2 \wedge L_{\pi(q_2)} (\alpha_1 \wedge \iota_{\pi(q_1)} \alpha_3) \epsilon \otimes q_3 \pm \alpha_1 \wedge (\iota_{\pi(q_1)} \alpha_3) \wedge \alpha_2 \epsilon \otimes \{q_2, q_3\} \pm \\
\alpha_1 \wedge (\iota_{\pi(q_1)} \alpha_3) \wedge (\iota_{\pi(q_3)} \alpha_2) \otimes q_2 \pm \alpha_2 \wedge L_{\pi(q_2)} (\alpha_3 \wedge \alpha_1 \wedge \{q_1, q_3\}) \pm \\
\alpha_2 \wedge L_{\pi(q_2)} (\alpha_3 \wedge \iota_{\pi(q_3)} \alpha_1) \epsilon \otimes q_1 \pm \alpha_3 \wedge (\iota_{\pi(q_3)} \alpha_1) \wedge \alpha_2 \epsilon \otimes \{q_1, q_2\} \pm \\
\alpha_3 \wedge (\iota_{\pi(q_3)} \alpha_1) \wedge \alpha_2 \epsilon \otimes \pi^\dagger (d(q_1, q_2)) \pm \alpha_3 \wedge (\iota_{\pi(q_3)} \alpha_1) \wedge (\iota_{\pi(q_1)} \alpha_2) \otimes q_2
\]

Using Leibniz rule the above calculations reduce \((7.7.6)\) to

\[
\epsilon = \alpha_3 \wedge (\iota_{\pi(q_3)} \alpha_1) \wedge \alpha_2 \wedge d(q_1, q_2) - \alpha_3 \wedge (\iota_{\pi(q_3)} \alpha_1) \wedge \alpha_2 \epsilon \otimes \pi^\dagger (d(q_1, q_2))
\]

which is one of the defining relations in \(\tau Q\).

In the case of equation \((7.7.6)\) the left hand side is equal to

\[
[\alpha_1 \otimes q_1, [\alpha_2 \otimes q_2, \alpha_3 \epsilon \otimes q_3]] = \\
\alpha_1 \wedge L_{\pi(q_1)} (\alpha_2 \wedge L_{\pi(q_2)} \alpha_3) \epsilon \otimes q_3 \pm \alpha_2 \wedge (L_{\pi(q_2)} \alpha_3) \wedge \alpha_1 \epsilon \otimes \{q_1, q_3\} \pm \\
\alpha_2 \wedge (L_{\pi(q_2)} \alpha_3) \wedge (\iota_{\pi(q_3)} \alpha_1) \otimes q_1 \pm \alpha_1 \wedge L_{\pi(q_1)} (\alpha_3 \wedge \alpha_2) \epsilon \otimes \{q_2, q_3\} \pm \\
\alpha_3 \wedge \alpha_2 \wedge \alpha_1 \epsilon \otimes \{q_1, \{q_2, q_3\}\} \pm \alpha_3 \wedge \alpha_2 \wedge (\iota_{\pi(q_2), \pi(q_3)} \alpha_1) \otimes q_1 \pm \\
\alpha_1 \wedge L_{\pi(q_1)} (\alpha_3 \wedge \iota_{\pi(q_3)} \alpha_2) \otimes q_2 \pm \alpha_3 \wedge (\iota_{\pi(q_3)} \alpha_1) \wedge \alpha_1 \otimes \{q_1, q_2\} \pm \\
\alpha_3 \wedge (\iota_{\pi(q_3)} \alpha_2) \wedge (L_{\pi(q_2)} \alpha_1) \otimes q_1
\]

The first summand of the right hand side is equal to

\[
[[\alpha_1 \otimes q_1, \alpha_2 \otimes q_2], \alpha_3 \epsilon \otimes q_3] = \\
\alpha_1 \wedge (L_{\pi(q_1)} \alpha_2) \wedge (L_{\pi(q_2)} \alpha_3) \epsilon \otimes q_3 \pm \alpha_3 \wedge \alpha_1 \wedge (L_{\pi(q_1)} \alpha_2) \epsilon \otimes \{q_2, q_3\} \pm \\
\alpha_3 \wedge \iota_{\pi(q_3)} (\alpha_1 \wedge L_{\pi(q_1)} \alpha_2) \otimes q_2 \pm \alpha_2 \wedge (L_{\pi(q_1)} \alpha_2) \wedge \alpha_1 \wedge (\iota_{\pi(q_1), \pi(q_2)} \alpha_3) \epsilon \otimes q_3 \pm \\
\alpha_3 \wedge \alpha_2 \wedge \alpha_1 \epsilon \otimes \{q_1, q_2\} \pm \alpha_3 \wedge \iota_{\pi(q_3)} (\alpha_2 \wedge \alpha_1) \otimes \{q_1, q_2\} \pm \\
\alpha_2 \wedge (L_{\pi(q_2)} \alpha_1) \wedge (L_{\pi(q_1)} \alpha_3) \epsilon \otimes q_3 \pm \alpha_3 \wedge \alpha_2 \wedge (L_{\pi(q_2)} \alpha_1) \otimes \{q_1, q_3\} \pm \\
\alpha_3 \wedge \iota_{\pi(q_3)} (\alpha_2 \wedge L_{\pi(q_2)} \alpha_1) \otimes q_1
\]
The second summand is equal to

\[ [\alpha_2 \otimes q_2, [\alpha_1 \otimes q_1, \alpha_3 \epsilon \otimes q_3]] = \]

\[ \alpha_2 \wedge L_{\pi(q_2)}(\alpha_1 \wedge L_{\pi(q_1)}\alpha_3) \epsilon \otimes q_3 \pm \alpha_1 \wedge (L_{\pi(q_1)}\alpha_2) \wedge \alpha_2 \epsilon \otimes \{q_2, q_3\} \pm \]

\[ \alpha_1 \wedge (L_{\pi(q_1)}\alpha_3) \wedge (t_{\pi(q_2)}\alpha_2) \otimes q_2 \pm \alpha_2 \wedge L_{\pi(q_2)}(\alpha_3 \wedge \alpha_1) \epsilon \otimes \{q_1, q_3\} \pm \]

\[ \alpha_3 \wedge \alpha_1 \wedge \alpha_2 \epsilon \otimes \{q_2, \{q_1, q_3\}\} \pm \alpha_3 \wedge \alpha_1 \wedge (t[\pi(q_1), \pi(q_3)]\alpha_2) \otimes q_2 \pm \]

\[ \alpha_2 \wedge (t_{\pi(q_2)}(\alpha_3 \wedge t_{\pi(q_3)}\alpha_1) \otimes q_1 \pm \alpha_3 \wedge (t_{\pi(q_3)}\alpha_1) \wedge \alpha_2 \otimes \{q_1, q_2\} \pm \]

\[ \alpha_3 \wedge (t_{\pi(q_3)}\alpha_1) \wedge \alpha_2 \otimes \pi^\dagger(d\langle q_1, q_2 \rangle) \pm \alpha_3 \wedge (t_{\pi(q_3)}\alpha_1) \wedge (L_{\pi(q_1)}\alpha_2) \otimes q_2 \]

Using the Leibniz rule the above calculations reduce (7.7.6) to

\[ 0 = \alpha_3 \wedge (t_{\pi(q_3)}\alpha_1) \wedge \alpha_2 \otimes \pi^\dagger(d\langle q_1, q_2 \rangle) \]

which holds since \( \alpha_3 \wedge (t_{\pi(q_3)}\alpha_1) \wedge \alpha_2 \otimes \pi^\dagger(d\langle q_1, q_2 \rangle) = \alpha_3 \wedge (t_{\pi(q_3)}\alpha_1) \wedge \alpha_2 (dd\langle q_1, q_2 \rangle) \).

The left hand side of equation (7.7.7) is equal to:

\[ [\alpha_1 \otimes q_1, [\alpha_2 \epsilon \otimes q_2, \alpha_3 \otimes q_3]] = \]

\[ \alpha_1 \wedge L_{\pi(q_1)}((\alpha_2 \wedge t_{\pi(q_2)}\alpha_3) \otimes q_3 \pm \alpha_2 \wedge (t_{\pi(q_2)}\alpha_3) \wedge \alpha_1 \otimes \{q_1, q_3\} \pm \]

\[ \alpha_2 \wedge (t_{\pi(q_2)}\alpha_3) \wedge (L_{\pi(q_3)}\alpha_1) \otimes q_1 \pm \alpha_1 \wedge L_{\pi(q_1)}((\alpha_3 \wedge \alpha_2 \wedge d\langle q_2, q_3 \rangle) \pm \]

\[ \alpha_1 \wedge L_{\pi(q_1)}(\alpha_3 \wedge \alpha_2) \epsilon \otimes \{q_2, q_3\} \pm \alpha_3 \wedge \alpha_2 \wedge \alpha_1 \epsilon \otimes \{q_1, \{q_2, q_3\}\} \pm \]

\[ \alpha_3 \wedge \alpha_2 \wedge (t[\pi(q_2), \pi(q_3)]\alpha_1) \otimes q_1 \pm \alpha_1 \wedge L_{\pi(q_1)}((\alpha_3 \wedge L_{\pi(q_3)}\alpha_2) \epsilon \otimes q_2 \pm \]

\[ \alpha_3 \wedge (L_{\pi(q_3)}\alpha_2) \wedge \alpha_1 \epsilon \otimes \{q_1, q_2\} \pm \alpha_3 \wedge (L_{\pi(q_3)}\alpha_2) \wedge (t_{\pi(q_2)}\alpha_1) \otimes q_1 \]

The first summand of the right hand side is equal to:

\[ [[\alpha_1 \otimes q_1, \alpha_2 \epsilon \otimes q_2], \alpha_3 \otimes q_3] = \]

\[ \alpha_1 \wedge (L_{\pi(q_1)}\alpha_2) \wedge (t_{\pi(q_2)}\alpha_3) \otimes q_3 \pm \alpha_3 \wedge \alpha_1 \wedge (L_{\pi(q_1)}\alpha_2) \wedge d\langle q_2, q_3 \rangle \pm \]

\[ \alpha_3 \wedge \alpha_1 \wedge (L_{\pi(q_1)}\alpha_2) \epsilon \otimes [q_2, q_3] \pm \alpha_3 \wedge L_{\pi(q_3)}((\alpha_1 \wedge L_{\pi(q_1)}\alpha_2) \epsilon \otimes q_2 \pm \]

\[ \alpha_2 \wedge \alpha_1 \wedge (t[\pi(q_1), \pi(q_2)]\alpha_3) \otimes q_3 \pm \alpha_3 \wedge \alpha_2 \wedge \alpha_1 \wedge d\langle q_1, q_2, q_3 \rangle \pm \]

\[ \alpha_3 \wedge \alpha_2 \wedge \alpha_1 \epsilon \otimes \{\{q_1, q_2\}, q_3\} \pm \alpha_3 \wedge L_{\pi(q_3)}((\alpha_2 \wedge \alpha_1) \epsilon \otimes \{q_1, q_2\} \pm \]

\[ \alpha_2 \wedge (t_{\pi(q_2)}\alpha_1) \wedge L_{\pi(q_1)}\alpha_3 \otimes q_3 \pm \alpha_3 \wedge \alpha_2 \wedge (t_{\pi(q_2)}\alpha_1) \otimes \{q_1, q_3\} \pm \]

\[ \alpha_3 \wedge L_{\pi(q_3)}((\alpha_2 \wedge t_{\pi(2)}\alpha_1) \otimes q_1 \pm \]

\[ \]
The second summand is equal to
\[ [α_2 ε ⊗ q_2, [α_1 ⊗ q_1, α_3 ⊗ q_3]] = \]
\[ α_2 ∧ t_π(q_2)(α_1 ∧ L_π(q_1)α_3) ⊗ q_3 \pm α_1 ∧ (L_π(q_1)α_3) ∧ α_2 ∧ d(q_2, q_3)\pm \]
\[ α_1 ∧ (L_π(q_1)α_3) ∧ α_2 ε ⊗ \{q_2, q_3\} \pm α_1 ∧ (L_π(q_1)α_3) ∧ L_π(q_3)α_2 ε ⊗ q_2 \]
\[ α_2 ∧ t_π(q_2)(α_3 α_1) ⊗ \{q_1, q_3\} \pm α_3 ∧ α_1 ∧ α_2 ∧ d(q_2, \{q_1, q_3\}) \pm \]
\[ α_3 ∧ α_1 ∧ α_2 ε ⊗ \{q_2, \{q_1, q_3\}\} \pm α_3 ∧ α_1 ∧ (L_π(q_1))α_2 ε ⊗ q_2 \pm \]
\[ α_2 ∧ t_π(q_2)(α_3 L_π(q_3)α_1) ⊗ q_1 \pm α_3 ∧ (L_π(q_1)α_1) ∧ α_2 ∧ d(q_1, q_2) \pm \]
\[ α_3 ∧ (L_π(q_2)α_1) ∧ α_2 ε ⊗ \{q_1, q_2\} \pm α_3 ∧ (L_π(q_3)α_1) ∧ α_2 ∧ d(q_1, q_2) \pm \]
\[ α_3 ∧ (L_π(q_3)α_1) ∧ (L_π(q_1)α_2 ε ⊗ q_2 \]

Now, equation (7.7.7) follows from the Leibniz rule. It is left to the reader verification of equation (7.7.8) using Leibniz rule.

7.7.3. Leibniz rule. Suppose that \( a = \omega_1 + α_1 ε ⊗ q_1 + β_1 ⊗ r_1 \) and \( b = \omega_2 + α_2 ε ⊗ q_2 + β_2 ⊗ r_2 \) are homogeneous elements, where \( ω_i, α_i, β_i ∈ O_{X^i} \) and \( q_i, r_i ∈ Q \) for \( i = 1, 2 \). For a homogeneous \( ψ ∈ O_{X^i} \), the Leibniz rule says
\[ (7.7.9) [a, ψ · b] = σ(a)(ψ) · b + (-1)^{α_i ω_i} ψ · [a, b]. \]

We verify (7.7.9) in the case (5.2.3) leaving the cases (5.2.4)-(5.2.6) to the reader.

\[ [α_1 ε ⊗ q_1, γ · (α_2 ε ⊗ q_2)] = [α_1 ε ⊗ q_1, (γ ∧ α_2) ε ⊗ q_2] = \]
\[ = α_1 ∧ t_π(q_1)γ ∧ α_2) ε ⊗ q_2 \pm γ ∧ α_2 ∧ (t_π(q_2)α_1 ε ⊗ q_1) \pm \]
\[ γ · [α_1 ε ⊗ q_1, α_2 ε ⊗ q_2] \pm α_1 ∧ (t_π(q_1))γ) ∧ (α_2 ε ⊗ q_2) \pm \]
\[ γ · [α_1 ε ⊗ q_1, α_2 ε ⊗ q_2] ± (π(α_1 ε ⊗ q_1)γ) · (α_2 ε ⊗ q_2). \]

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