Using rigorous ray tracing to incorporate reflection into the parabolic approximation

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Abstract

We present a parabolic approximation that incorporates reflection. With this approximation, there is no need to solve the parabolic equation for a coupled pair of solutions consisting of the incident and reflected waves. Rather, this approximation uses a synthetic wave whose spectral components manifest the incident and reflected waves.

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The (Leontovich-Fock) parabolic approximation, which approximates the elliptic Helmholtz equation by a parabolic partial differential equation, was originally applied to electromagnetic propagation. Tappert and Hardin introduced the parabolic approximation to acoustic propagation in the ocean to account for inseparable range effects in the sound speed profile. In ocean acoustics, the parabolic equation is a useful computational tool for tackling inseparable indices of refraction for which the sound speed profile changes slowly with respect to range.

One of the deficiencies of the standard parabolic approximation is that it neglects backscatter. Heretofore, to account for backscatter, one solved the parabolic equation for a coupled pair of solutions (the incident and reflected solutions). Attempts to account for backscatter include among others the works of Collins et al, which uses a two-way parabolic approximation. Herein, we present a different approach to include backscatter. Based on rigorous ray tracing, we combine the incident and reflected waves into a modulated synthetic wave that progresses in the incident direction.

Rigorous ray tracing has been developed in a generalized Hamilton-Jacobi representation that accounts for terms ignored by classical ray tracing and other asymptotic methods. It has provided insight into propagation phenomena. Rigorous ray tracing has shown that the existence of a sound-speed gradient is sufficient to induce linear (material) dispersion and angular (geometric) dispersion even for isotropic frequency-independent sound-speed profiles, that rays are not generally orthogonal to wave fronts, that classical ray tracing does not predict all caustics, and that rigorous ray tracing may be solved in closed form whenever the corresponding wave equation may be solved in closed form. Its quantum mechanical analogy, the trajectory representation, has shown how to derive the Helmholtz equation (the stationary Schrödinger equation) from the generalized Hamilton-Jacobi equation. This allows us to construct the wave function or normal mode from Hamilton’s characteristic function (a generator of the motion for the trajectory or ray path). These normal modes can be synthetic normal modes that contain the incident and reflected waves as spectral components. We shall use such a normal mode to develop the parabolic equation that accounts for reflection.

Our objective in this letter is to present a parabolic equation that accounts for reflection. It is beyond the scope of this letter to solve the resultant parabolic equation. The acoustical community is free to solve
this equation by the methods of their choice. This work is presented in two dimensions, which is sufficient to illustrate how to incorporate reflection into the parabolic equation.

We assume that the ocean to first order is a stratified medium whose index of refraction varies with depth due to temperature and pressure changes. The range dependence of the index of refraction is second order. This index of refraction is dependent upon two cartesian coordinates: \((x, z)\) for range and depth respectively. The index of refraction varies much more rapidly in the \(z\)-direction than in the \(x\)-direction. We also assume that, for propagation of a wave train through the ocean medium, the reflected wave is much smaller than the incident wave consistent with the concept of backscatter.

Recently, the trajectory representation of quantum mechanics (the quantum analogue to rigorous ray tracing) showed how the reflected and incident waves can be combined to synthesize a wave whose front monotonically advances in the direction of incidence. The synthetic wave is given by

\[
\alpha \exp[i(kx - \omega t)] + \beta \exp[-i(kx + \omega t)] = \left[\alpha^2 + \beta^2 + 2\alpha\beta \cos(2kx)\right]^{1/2}
\]

where \(\alpha\) is the amplitude of incident wave and \(\beta\) is the amplitude of reflected wave, where \(|\beta| < |\alpha|\), and where \(k\) is the wavenumber and \(\omega\) is the angular frequency. This synthetic wave is a normal mode (follows from the superposition principle).

The synthetic wave has spatially modulation in phase and amplitude as shown by right side of Eq. (1). For completeness, the right side of Eq. (1) was derived from the generator of the motion for the trajectory in Ref. 6, and the left side was subsequently developed by analysis. While the right side of Eq. (1) was first developed from Hamilton’s characteristic function by the quantum analogy to rigorous ray tracing, we subsequently learned how to do it in a wave representation. This is the contribution of rigorous ray tracing that we use here.

The wave equation in two dimensions is given by

\[
\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial z^2} = C^{-2}(x, z)\frac{\partial^2 \Psi}{\partial t^2}
\]

The speed of sound, \(C\), is isotropic and only spatially dependent. The wave equation is separable in time so that \(\Psi(x, z, t) = \psi(x, z) \exp(i\omega t)\) Hence, the wave equation is reduced to the two-dimensional Helmholtz equation

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} + \kappa^2(x, z)\psi = 0
\]

where \(\kappa(x, z) = \omega/C(x, z)\).

For reference, the standard parabolic approximation substitutes

\[
\psi(x, z) = \theta_{\text{standard}}(x)\phi_{\text{standard}}(x, z) = \exp(i k x)\phi_{\text{standard}}(x, z)
\]

into Eq. (2) to produce, after a standard simplification, the standard parabolic equation given by

\[
\frac{\partial^2 \phi_{\text{standard}}}{\partial z^2} + i2k\frac{\partial \phi_{\text{standard}}}{\partial x} + (\kappa^2 - k^2)\phi_{\text{standard}} = 0,
\]

which does not incorporate reflection.

Let us incorporate reflection by considering

\[
\psi(x, z) = \theta(x)\phi(x, z)
\]

where
\[ \theta = [\alpha^2 + \beta^2 + 2\alpha\beta \cos(2kx)]^{1/2} \exp \left[ i \arctan \left( \frac{\alpha - \beta}{\alpha + \beta} \tan(kx) \right) \right]. \]

There is flexibility in choosing the form of \( \theta \). Different choices of \( \theta \) lead to different parabolic equations. We have chosen a \( \theta \) that is the spatial component of the synthetic wave, Eq. (1). This \( \theta \) includes reflection while progressing in the incident direction. In the standard parabolic equation, the corresponding \( \theta_{\text{standard}} \) in Eq. (4) is given by \( \theta_{\text{standard}} = \exp(ikx) \), which only includes the incident wave. Substituting \( \psi = \theta \phi \) into the Helmholtz equation leads to

\[ \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \phi}{\partial x^2} + 2 \frac{\partial \theta}{\partial x} \frac{\partial \phi}{\partial x} + \left( k^2 - \frac{\beta}{\alpha} \right) \phi = 0 \]

where \( \partial^2 \theta/\partial x^2 = k^2 \theta \) by the superposition principle or by direct substitution.

We now examine \( (\partial \theta/\partial x)/\theta \), which is given by

\[ \frac{\partial \theta}{\partial x} = ik \left( \frac{\alpha^2 + \beta^2 - 2\alpha\beta \cos(2kx)}{\alpha^2 + \beta^2 + 2\alpha\beta \cos(2kx)} \right)^{1/2} \times \exp \left[ i \arctan \left( \frac{\alpha + \beta}{\alpha - \beta} \tan(kx) \right) - i \arctan \left( \frac{\alpha - \beta}{\alpha + \beta} \tan(kx) \right) \right]. \]

For small reflections, \( \beta \ll \alpha \), Eq. (5) may be simplified to

\[ \frac{\partial \theta}{\partial x} = ik[1 - (2\beta/\alpha) \cos(2kx)] \exp[i(2\beta/\alpha) \sin(2kx)] + O[(\beta/\alpha)^2]. \]

Now the transformed Helmholtz equation for small reflection becomes

\[ \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \phi}{\partial x^2} + i2k[1 - (2\beta/\alpha) \cos(2kx)] \exp[i(2\beta/\alpha) \sin(2kx)] \frac{\partial \phi}{\partial x} + \frac{\partial^2 \phi}{\partial x^2} + \left( k^2 - \frac{\beta}{\alpha} \right) \phi = O[(\beta/\alpha)^2]. \]

The critical assumption for the validity of the parabolic assumptions is that \( \phi \) is well behaved (smooth) in range so that

\[ \left| \frac{\partial^2 \phi}{\partial x^2} \right| \ll 2k \frac{\partial \phi}{\partial x}. \]

This assumption, Eq. (6) is standard for simplifying the elliptic Helmhotz equation to an approximating parabolic equation. The resulting parabolic wave equation with reflection to first order in \( (\beta/\alpha) \) is given by

\[ \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \phi}{\partial x^2} + 2ik[1 - (2\beta/\alpha) \cos(2kx)] \exp[i(2\beta/\alpha) \sin(2kx)] \frac{\partial \phi}{\partial x} + \left( k^2 - \frac{\beta}{\alpha} \right) \phi = 0 \]

or

\[ \frac{\partial^2 \phi}{\partial z^2} + 2ik \exp[i(2\beta/\alpha) \sin(2kx)] \frac{\partial \phi}{\partial x} + \left( k^2 - \frac{\beta}{\alpha} \right) \phi = 0. \]

Equation (7) is the parabolic equation that incorporates reflection. The difference between Eq. (7) and the standard parabolic equation, Eq. (4), is the additional factor \( \exp[i(2\beta/\alpha) \sin(2kx)] \) in the \( \partial \phi/\partial x \) term in Eq. (7). Relative reflection as a function of the fraction \( \beta/\alpha \) is thereby incorporated to first order into \( \phi(x, z) \).

In order to account for the effect of reflection, contemporary solutions to the parabolic approximation solve the parabolic equation for an interacting pair of solutions (incident and reflected) or decouple the pair by simplification. Herein, we avoid the problem of coupled solutions. Our solution to Eq. (7) is a single synthetic wave that manifests both the incident and reflected wave throughout the domain.

The initialization of \( \phi \) at some initial range, \( x_i \), over the depth column, \( z \), renders the value, \( \phi(x_i, z) \), over an open boundary thereby establishing the Dirichlet boundary conditions for a unique, stable solution.
for $\phi$. This initialization process is similar to that for the standard parabolic equation, but here we must also specify the fraction $\beta/\alpha$. (As a starter, one could use Urick and the references therein to predict $\beta/\alpha$ from reverberation and backscatter.) Solving Eq. (3) (which is beyond the scope of this letter) in practice, one must not only take the usual precautions associated with the standard parabolic approximation but also take into account that Eq. (3) is an approximation that ignores some second-order terms of $(\beta/\alpha)$.

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