Entanglement and the $SU(2)$ phase states in atomic systems

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We show that a system of $2^n$ identical two-level atoms interacting with $n$ cavity photons manifests entanglement and that the set of entangled states coincides with the so-called $SU(2)$ phase states. In particular, violation of classical realism in terms of the GHZ and GHSH conditions is proved. We discuss a new property of entanglement expressed in terms of local measurements. We also show that generation of entangled states in the atom-photon systems under consideration strongly depends on the choice of initial conditions and that the parasitic influence of cavity detuning can be compensated through the use of Kerr medium.
I. INTRODUCTION

It has been recognized that entanglement phenomenon touches on the conceptual problems of reality and locality in quantum physics as well as the more technological aspects of quantum communications, cryptography, and computing. In particular, the methods of quantum key distribution in communication channels secured from eavesdropping are based on the use of entangled states (19) (for recent review, see Refs. 6, 7). In turn, the realization of quantum computer 8 is dependent on the fact that its realization seems to be easy available with present experimental technique. The importance of this scheme is caused by the fact that its realization seems to be easy available with present experimental technique.

An interesting approach has been proposed recently 23. Considering the state shared between Alice and Bob as a quantum communication channel, the authors 32 concluded that the information in the case of entanglement is carried mostly by the correlations between the sides of the channel [23]. Following 23, consider a composite system defined in the Hilbert space

\[ \mathcal{H} = \bigotimes_{\ell} \mathcal{H}^{(\ell)}, \quad \ell \geq 2. \]

Let \( G \) be the group of dynamical symmetry of a subsystem in the composite system. Then the Hermitian operators \( g \) associated with representation of \( G \) in \( \mathcal{H}^{(\ell)} \) define the set of local measurement on the corresponding side of the channel provided by a state \( |\psi\rangle \in \mathcal{H} \). For example, in the case of \( \mathcal{H} = C^2 \), corresponding to the EPR spin-\( \frac{1}{2} \) system, \( G = SU(2) \) and the set of local measurements can be specified by the infinitesimal generators of the \( SL(2) \) group with the componentwise action of \( SU(2)^N \). In particular, we show that these states violate the classical realism and discuss their realization.

On the other hand, we will discuss a new condition of entanglement has been proposed recently 23. Let us note in this connection that, in the usual treatment of entanglement, the entangled states of a two-component (in general, multi-component) system are considered as the nonseparable states with respect to the subsystems (e.g., see 24). For example, if the individual components of a two-component system are described by the states \( |\xi_i\rangle \) and \( |\chi_i\rangle \), respectively, the state

\[ |\psi_{ent}\rangle = \sum_i b_i |\xi_i\rangle \otimes |\chi_i\rangle, \]

\[ \langle \xi_i |\xi_k\rangle = \langle \chi_i |\chi_k\rangle = \delta_{ik}, \quad \sum_i |b_i|^2 = 1 \]

is entangled if \( b_i \neq 0 \) for at least two distinct values of the subscript \( i \). From the mathematical point of view, the entanglement is caused by the combination of the superposition principle in quantum mechanics with the tensor product structure of the space of state of the two-or multi-component system 23.

Very often, the existence of entanglement is verified in terms of violation of Bell’s inequalities and their generalizations 25, 26, 27, 28, 29, 30, 31. Another way is based on the use of GHZ theorem 10. A possibility to introduce more general inequalities is also discussed 22.

It should be noted that the use of Bell’s inequalities and their numerous generalizations demonstrate nothing but the nonexistence of hidden variables. Moreover, it is possible to say that the unique, general, and mathematically correct definition of entanglement still does not exist (e.g., see Ref. 22).

If the photon can leak out from the resonator, the absence of entanglement can be represented by the so-called \( SU(2) \) phase states corresponding to the \( SU(2) \) algebra of the odd "spin"

\[ j = \frac{1}{2} \left( \binom{2n}{n} - 1 \right), \quad (1) \]

where \( 2n \) is the even number of atoms and \( n = 1, 2, \cdots \) is the number of cavity photons. In particular, the system considered in Ref. 13 corresponds to the phase states of "spin" \( j = 1/2 \). The \( SU(2) \) phase states were introduced in 19 for an arbitrary spin and then generalized in 21 to the case of the \( SU(2) \) subalgebra in the Weyl-Heisenberg algebra of photon operators (for recent review, see 22). From the mathematical point of view, this is the system of

\[ N = 2j + 1 \]

qubits defined in the Hilbert space

\[ \mathcal{H}_N = (C^2)^\otimes N \]
Then, the local measurement $g$ way by the maximum of reduced entropy (e.g., see Refs. after the maximum entanglement is defined in the usual phase states of spin Pauli matrices $\sigma^{(\ell)}_n$ defined by Eq. (1) in a 2 states.

The paper is organized as follows. In Sec. II, we consider the representation of the SU(2) phase states. As a particular example, we examine the system of two identical two-level atoms, interacting with a single cavity photon and show that the maximum entangled atomic states of the Ref. [18] belong to the class of the SU(2) phase states (2). Let $|e\rangle$ and $|g\rangle$ denote the excited and ground atomic states of the $\ell^{th}$ atom, respectively. Then, the entangled, maximum excited atomic states in the system ”2 atoms plus 1 photon” considered in [18] are

$$|\psi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|e_1g_2\rangle \pm |g_1e_2\rangle).$$

Then, the local measurement $g$ can be described by the Pauli matrices

$$\sigma^{(\ell)}_1 = |e\rangle\langle g| + |g\rangle\langle e|,$$

$$\sigma^{(\ell)}_2 = -i|e\rangle\langle g| + i|g\rangle\langle e|,$$

$$\sigma^{(\ell)}_3 = |e\rangle\langle e| - |g\rangle\langle g|,$$

i.e., by the infinitesimal generators of the algebra $SL(2)$. It is now a straightforward matter to check that

$$\forall i, \ell \langle \psi_{\pm}|\sigma^{(\ell)}_i|\psi_{\pm}\rangle = 0,$$

where averaging is taken over the states (4). Another example is provided by the GHZ states [10]

$$|\psi_{\pm}^{(GHZ)}\rangle = \frac{1}{\sqrt{2}}(|e_1e_2e_3\rangle \pm |g_1g_2g_3\rangle),$$

corresponding to the maximum atomic excitation in the 3 + 3-system. It is easily seen that the averaging of the local operators (3) over (5) gives the same result as (4).

This property (4) can be used to define the entangled states.

We will show that the SU(2) phase states of spin $j$ defined by Eq. (1) in a 2$n + n$-type atom-photon system obey the nonseparability conditions, have the property (4), and manifest the violation of classical realism expressed in terms of the GHZ [10] and CHSH (Clauser-Horne-Shimoni-Holt) [33] conditions.

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II. REPRESENTATION OF THE SU(2) PHASE STATES

An arbitrary spin $j$ can be described by the generators $J_+, J_-, J_z$ of the SU(2) algebra such that

$$[J_+, J_-] = 2J_z, \quad [J_z, J_\pm] = \pm J_\pm,$$

$$J^2 = J_z^2 + \frac{1}{2}(J_+J_- + J_-J_+) = j(j + 1) \times 1,$$

where 1 is the unit operator in the 2$j + 1$ dimensional Hilbert space. Since

$$J_\pm = J_x \pm iJ_y,$$

it is possible to say that the generators $J_+, J_-, J_z$ in (6) correspond to the Cartesian representation of the SU(2) algebra. Following [14], one can introduce the representation in spherical coordinates via the polar decomposition of (6) of the form

$$J_+ = J_r \epsilon, \quad J_r = J_r^+, \quad \epsilon \epsilon^* = 1,$$

where the Hermitian operator $J_r$ corresponds to the radial contribution, while $\epsilon$ gives the exponential of the azimuthal phase operator. It is a straightforward matter to show that $\epsilon$ can be represented by the following $(2j + 1) \times (2j + 1)$ matrix

$$\epsilon = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 1 \\
e^{i\psi} & 0 & 0 & 0 & 0
\end{pmatrix}$$

in the 2$j + 1$-dimensional Hilbert space. Here $\psi$ is an arbitrary real parameter (reference phase). The eigenstates of the operator (8)

$$\epsilon|\phi_{n}^{(j)}\rangle = e^{i\phi_{n}^{(j)}}|\phi_{n}^{(j)}\rangle, \quad n = 1, \cdots, (2j + 1),$$
form the basis of the so-called phase states

\[ |\psi_n^{(j)}\rangle = \frac{1}{\sqrt{2j+1}} \sum_{k=0}^{2j} e^{ik\phi_n^{(j)}} |\psi_k\rangle \]  

(10)
dual with respect to the basis of individual states |\psi_k\rangle of the Hilbert space.

As a physical example of some considerable interest, consider now the system of the two identical two-level atom interacting with the single cavity photon (see [13]). If the cavity photon is absorbed by either atom, the atomic subsystem can be observed in the following states

\[ |\psi_1\rangle = |e_1 g_2\rangle, \quad |\psi_2\rangle = |g_1 e_2\rangle, \]

(11)

where |e_1 g_2\rangle = |e_1\rangle \otimes |g_2\rangle and |e\rangle and |g\rangle denote the excited and ground atomic states, respectively. The subscript marks the atom. Using the atomic basis (11), we can construct the following representation of the SU(2) algebra:

\[ J_+ = |e_1 g_2\rangle \langle g_1 e_2|, \quad J_- = |g_1 e_2\rangle \langle e_1 g_2|, \]

\[ J_3 = \frac{1}{2}(|e_1 g_2\rangle \langle e_1 g_2| - |g_1 e_2\rangle \langle g_1 e_2|). \]

(12)

This representation formally corresponds to (6) at the spin \( j = 1/2 \). Then, the corresponding exponential of the phase operator (8) takes the form

\[ \epsilon = |e_1 g_2\rangle \langle g_1 e_2| + e^{i\psi}|g_1 e_2\rangle \langle e_1 g_2|. \]

(13)

In turn, the phase states (9) and (10) are

\[ |\phi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|e_1 g_2\rangle + e^{i\phi_{\pm}} |g_1 e_2\rangle), \]

\[ \phi_{\pm} = \psi/2 + (1 \mp 1)\pi/2. \]

(14)

It is easily seen that the phase states (14) form the set of entangled atomic states in the two-atom system under consideration. Definitely, these states obey the nonseparability condition. It is also seen that (14) coincides with the maximally entangled states (2) of Ref. [18] when the reference phase \( \psi = 0 \).

Consider now a general 2n + n system at \( n \geq 1 \). Then, the maximum excited atomic states

\[ |\psi_i\rangle = |\{e\}_n, \{g\}_n\rangle, \]

(15)
can be used to construct a representation of the SU(2) algebra (6) of spin \( j \) defined in (1). Here \( i = 1, 2, \cdots, N \) and

\[ N = 2j + 1 = \binom{2n}{n} \]

is the total number of such a states. In the basis (15), we can construct the polar decomposition of the SU(2) algebra of spin (1) and the corresponding exponential of the phase operator (8) and the phase states (10). Let us rename the states (15) as follows

\[ |\psi_k\rangle \rightarrow |\psi_{k'}\rangle, \quad k' = k - 1 = 0, \cdots, N - 1. \]

Then, the SU(2) phase states (10) take the form

\[ |\phi_k\rangle = \frac{1}{\sqrt{N}} \sum_{k'=0}^{N-1} e^{ik'\phi_k} |\psi_{k'}\rangle, \]

(16)

where

\[ \phi_k = (\psi + 2k\pi)/N. \]

These states (16) form a basis dual with respect to (15) and spanning the Hilbert space of the maximum excited atomic states in the 2n + n system under consideration. By construction, the phase states (16) are nonseparable with respect to contributions of individual atoms and thus entangled [24]. Let us stress that the choice of the phase factors in (16) is irrelevant to entanglement, which holds for arbitrary phase factors. This choice is caused by the aspiration for getting the dual with respect to (15) basis of entangled states.

It is easily seen that the states (16) obey the condition (4). In fact, the action of the flip-operators \( \sigma_{1,2}^{(i)} \) in (3) on the states (16) leads to the change of the number of either excited or de-excited atoms:

\[ \sigma_{1,2}^{(i)} |\psi_k\rangle \rightarrow \begin{cases} |\{e\}_{n-1}, \{g\}_{n+1}\rangle & \ell \in \{g\} \\ |\{e\}_{n+1}, \{g\}_{n-1}\rangle & \ell \in \{e\} \end{cases} \]

and therefore \( \langle \sigma_{1,2}^{(i)} \rangle = 0 \) in the case of averaging over the states (16). Since each state (15) contains equal number of excited and de-excited atoms, the action of the parity operator in (3) on the phase states (16) should lead to the state which differ from (16) by the multiplication of a certain n terms by the factor of \(-1\). Hence

\[ \langle \sigma_{3}^{(i)} \rangle = \frac{1}{N} \left( \sum_{i=1}^{N/2} 1 - \sum_{i=N/2+1}^{N} 1 \right) = 0. \]

By construction, \( N \) is always an even number.

Thus, the SU(2) phase states (16), corresponding to the maximum excited atomic states in the 2n + n system, are entangled because they are nonseparable and, at the same time, obey the condition (4) for the local measurements. In the next Section, we show that the states (16) manifest violation of classical realism as well.

Before we begin to discuss this subject, let us note that the SU(2) phase states of the atomic system under consideration with integer spin do not provide the entanglement. Consider as an example the system of three identical two-level atoms, interacting with a single cavity photon. There are the three excited atomic states

\[ |e_1 g_2 g_3\rangle, \quad |g_1 e_2 g_3\rangle, \quad |g_1 g_2 e_3\rangle \]

(17)

and the three dual phase states of the type of (16)

\[ |\psi_k\rangle = \frac{1}{\sqrt{3}}(|e_1 g_2 g_3\rangle + e^{ik\phi} |g_1 e_2 g_3\rangle + e^{2ik\phi} |g_1 g_2 e_3\rangle). \]

(18)
It is clear that the states (18) are the phase states of spin \( j = 1 \). Here
\[
\phi_k = (\psi + 2k\pi)/3, \quad k = 0, 1, 2.
\]

By construction, they describe the spin \( j = 5/2 \) system. In turn, the exponential of the phase operator (8) takes the form
\[
\epsilon = |e_1e_2g_3g_4\rangle\langle e_1e_2g_3g_4| + |e_1g_2e_3g_4\rangle\langle e_1g_2e_3g_4| \\
+ |e_1g_2e_3g_4\rangle\langle g_1e_2g_3e_4| + |g_1e_2g_3e_4\rangle\langle g_1e_2g_3e_4| + e^{i\phi_k}|g_1e_2g_3e_4\rangle\langle e_1e_2g_3g_4|.
\]

Then, the six phase states (9) have the form (16) with \( N = 6 \) and
\[
\phi_k = \frac{\psi}{6} + \frac{k\pi}{3}, \quad k = 0, 1, \ldots, 5.
\]

As well as (16), these states are nonseparable and hence entangled and obey the condition (4) for local variables. To show that these phase states violate the classical realism, let us first represent the states (16) at \( N = 6 \) in the following way
\[
|\phi_k\rangle = \frac{1}{\sqrt{3}} (|\chi_{1k}\rangle + e^{i\phi_k}|\chi_{2k}\rangle + e^{i\phi_k}|\chi_{3k}\rangle),
\]
where
\[
|\chi_{1k}\rangle = \frac{1}{\sqrt{2}} (|e_1e_2g_3g_4\rangle + e^{5i\phi_k}|g_1e_2g_3e_4\rangle), \\
|\chi_{2k}\rangle = \frac{1}{\sqrt{2}} (|g_1e_2g_3e_4\rangle + e^{3i\phi_k}|e_1g_2e_3g_4\rangle), \\
|\chi_{3k}\rangle = \frac{1}{\sqrt{2}} (|g_1e_2g_3e_4\rangle + e^{i\phi_k}|e_1g_2e_3g_4\rangle).
\]

It is easily seen that each set of six states \(|\chi_{jk}\rangle\) with \( p = 1, 2, 3 \) and \( k = 0, \ldots, 5 \) consists of the nonseparable and hence entangled states. Consider, for example, the states \(|\chi_{1k}\rangle\) in (22). Because of the definition of the phase angle \( \phi_k \) at \( N = 6 \), they consist of the three sets of the pairwise orthogonal states
\[
\{|\chi_{10}\rangle, |\chi_{11}\rangle\}, \quad \{|\chi_{11}\rangle, |\chi_{14}\rangle\}, \quad \{|\chi_{12}\rangle, |\chi_{15}\rangle\}.
\]

It is also seen that the second and third sets here are obtained from the first set by the successive rotations of the reference frame.

Now the violation of classical realism can be proved through the use of the GHZ theorem [10]. Consider first the state \(|\chi_{10}\rangle\) in (22). It is easy to verify that this state obey the conditions
\[
\forall i, \ell \quad \bigotimes_{\ell=1}^{4} \sigma_{i_\ell}^{(\ell)} |\chi_{10}\rangle = |\chi_{10}\rangle
\]
and
\[
\sigma_1^{(1)} \sigma_2^{(2)} \sigma_3^{(3)} \sigma_4^{(4)} |\chi_{10}\rangle = -|\chi_{10}\rangle, \\
\sigma_2^{(1)} \sigma_1^{(2)} \sigma_3^{(3)} \sigma_4^{(4)} |\chi_{10}\rangle = -|\chi_{10}\rangle, \\
\sigma_1^{(1)} \sigma_2^{(2)} \sigma_3^{(3)} \sigma_4^{(4)} |\chi_{10}\rangle = |\chi_{10}\rangle, \\
\sigma_1^{(1)} \sigma_2^{(2)} \sigma_3^{(3)} \sigma_4^{(4)} |\chi_{10}\rangle = |\chi_{10}\rangle, \\
\sigma_1^{(1)} \sigma_2^{(2)} \sigma_3^{(3)} \sigma_4^{(4)} |\chi_{10}\rangle = |\chi_{10}\rangle.
\]

III. THE 4 + 2-SYSTEM

To show that the phase states (16) of a \( 2n + n \) system violate the classical realism, consider the system of four identical two-level atoms interacting with two cavity photons. The maximum excited atomic states at \( n = 2 \) are
\[
|e_1e_2g_3g_4\rangle, \quad |e_1g_2e_3g_4\rangle, \quad |e_1g_2g_3e_4\rangle, \\
|g_1e_2g_3g_4\rangle, \quad |g_1e_2e_3g_4\rangle, \quad |g_1g_2e_3e_4\rangle.
\]

These orthonormal states form the six-dimensional basis of the Hilbert space in which the representation of the generators (6) has the form
\[
J_+ = \sqrt{5}|e_1e_2g_3g_4\rangle\langle e_1e_2g_3g_4| + \sqrt{5}|e_1g_2e_3g_4\rangle\langle e_1g_2e_3g_4| \\
+ 3|e_1g_2g_3e_4\rangle\langle e_1g_2g_3e_4| + \sqrt{5}|g_1e_2g_3e_4\rangle\langle g_1e_2g_3e_4| \\
+ \sqrt{5}|g_1e_2e_3g_4\rangle\langle g_1e_2e_3g_4| \\
J_3 = \frac{5}{2}|e_1e_2g_3g_4\rangle\langle e_1e_2g_3g_4| + \frac{3}{2}|e_1g_2e_3g_4\rangle\langle e_1g_2e_3g_4| \\
+ \frac{1}{2}|e_1g_2g_3e_4\rangle\langle e_1g_2g_3e_4| - \frac{1}{2}|g_1e_2g_3e_4\rangle\langle g_1e_2g_3e_4| \\
- \frac{3}{2}|g_1e_2e_3g_4\rangle\langle g_1e_2e_3g_4| - \frac{1}{2}|g_1g_2g_3e_4\rangle\langle g_1g_2g_3e_4| \\
- \frac{3}{2}|g_1g_2e_3e_4\rangle\langle g_1g_2e_3e_4| - \frac{1}{2}|g_1g_2e_3e_4|\langle g_1g_2e_3e_4|.
\]
It is possible to say that these equalities (23) and (24) express a kind of EPR “action at distance” in the maximum excited states of the system of four atoms interacting with two photons. In other words, the correlations represented by (23) and (24) permit us determine in a unique way the state of the fourth atom via measurement of the states of other three atoms.

The operator equalities (23) and (24) can be used to obtain the relations similar to those in the GHZ theorem. Following [3], we have to assign the classical quantities \( m_i^{(t)} \) to the local operators. Here

\[
m_1^{(t)}, m_2^{(t)} = \pm 1.
\]

Then, it follows from (23) that

\[
\prod_{\ell=1}^{4} m_{1}^{(\ell)} = 1. \tag{25}
\]

At the same time, it follows from (24) that

\[
[
|\sigma_1^{(1)}|, |\sigma_2^{(1)}|, |\sigma_1^{(2)}|, |\sigma_2^{(2)}|, |\sigma_1^{(3)}|, |\sigma_2^{(3)}|, |\sigma_1^{(4)}|, |\sigma_2^{(4)}|
\]

\[
\chi_{10}
\]

Employing the classical variables instead of the local operators allows this to be cast into the form

\[
(m_1^{(1)})^3 m_1^{(2)} (m_2^{(2)})^2 m_1^{(3)} (m_2^{(3)})^2 m_1^{(4)} (m_2^{(4)})^2 = -1.
\]

Since \((m_1^{(t)})^2 = (m_2^{(t)})^2 = 1\), we get an equivalent equality

\[
m_1^{(1)} m_1^{(2)} m_1^{(3)} m_1^{(4)} = -1,
\]

which contradicts (25). Hence, the state \( |\chi_{10}\rangle \) in (22) obey the GHZ theorem. Similar result can be obtained for all other states in (22) and hence, for the phase states (21).

Our consideration so far have applied to the local measurements touching on a single atom. We now note that the phase states (21) allow another kind of entanglement in the case of pairwise measurement. Consider again the state \( |\chi_{10}\rangle \) in (22) and assume that the measurements \( a \) and \( b \) corresponds to a pair of atoms:

\[
a = \cos \theta_a |e_1 e_2\rangle + \sin \theta_a |e_1 e_2\rangle |g_1 g_2\rangle + |g_1 g_2\rangle |e_1 e_2\rangle - \cos \theta_a |g_1 g_2\rangle |e_1 e_2\rangle,
\]

\[
b = \cos \theta_b |e_3 e_4\rangle + \sin \theta_b |e_3 e_4\rangle |g_3 g_4\rangle + |g_3 g_4\rangle |e_3 e_4\rangle - \cos \theta_b |g_3 g_4\rangle |e_3 e_4\rangle. \tag{26}
\]

Assume now that we make the two measurements \( a \) and \( a' \) with the angles \( \theta_1 = \pi \) and \( \theta'_a = \pi/2 \) and the two more measurements \( b \) and \( b' \) with the angles \( \theta'_b = -\theta_b \), respectively. Then, the averaging over the state \( |\chi_{10}\rangle \) gives

\[
\langle ab \rangle = \langle a'b' \rangle = \cos \theta_b, \quad \langle a'b \rangle = \sin \theta_b = -\langle a'b' \rangle.
\]

Employing the CHSH inequality [3]

\[
|\langle ab \rangle + \langle a'b \rangle - \langle a'b' \rangle - \langle ab' \rangle| \leq 2 \tag{27}
\]

then gives

\[
|\cos \theta_b - \sin \theta_b| \leq 1.
\]

Violation of this inequality and hence, of the classical realism occurs at small negative \( \theta_n \), when we can put

\[
|\cos \theta_b - \sin \theta_b| \sim 1 + |\theta_b| > 1.
\]

Similar consideration can be done for all over states in (22) through the use of proper pairwise measurements. At the same time, the phase states (21) do not manifest entanglement with respect to the pairwise measurements.

The phase states (16) for the \( 6 + 3, 8 + 4, \cdots \) systems, corresponding to the spin (1) equal to 19/2, 69/2, \cdots, respectively, can be considered as above.

**IV. INITIAL CONDITIONS AND ATOMIC ENTANGLEMENT**

It is clear that the evolution of the \( 2n + n \) system strongly depends on the choice of initial conditions. To trace the proper choice leading to the atomic entanglement, let us ignore the relaxation processes. Then, the steady-state evolution of the \( 2n + n \) system under consideration is governed by the Hamiltonian

\[
H = \Delta a^+ a + \omega_0 N + \gamma \sum_{\ell} (R_\ell^+ a + a^+ R_\ell).
\]

Here \( \Delta \) is the cavity detuning, \( \omega_0 \) is the atomic transition frequency, \( \gamma \) is the atom-field coupling constant, operators \( a \) and \( a^+ \) describe the cavity photons, \( N = a^+ a \), and the atomic operators are defined as follows

\[
R_\ell^+ = |e_\ell\rangle \langle g_\ell| \bigotimes_{\ell' \neq \ell} 1^{(\ell')},
\]

and the atomic operators are defined as follows

\[
R_\ell^+ = |e_\ell\rangle \langle g_\ell| \bigotimes_{\ell' \neq \ell} 1^{(\ell')}.
\]

Here \( 1^{(t)} \) denotes the unit operator in the two-dimensional Hilbert space of the \( \ell^{th} \) atom. It is seen that \( [N, H] = 0 \). It is also seen that the atomic operators are similar, in a certain sense, to the local operators (3). In fact

\[
R_\ell^+ = \sigma_\ell^{(t)} \pm i \sigma_\ell^{(t)}.
\]

Consider first the case of two atoms and single cavity photon when \( \ell = 1, 2 \) and the Hamiltonian (28) coincides with that of Ref. [18]. For simplicity, we use here the
same coupling constant $\gamma$ for both atoms. Our consideration can easily be generalized on the case of coupling constant depending on the atomic position. Let us note that, in the case of only two atoms, the Hamiltonian (28) can be represented as follows
\[
H \rightarrow H_\phi = \Delta a^+ a + \omega_0 N_\phi + \gamma \sqrt{R}(R^+ a + a^+ R), \tag{29}
\]
where
\[
N_\phi = a^+ a + \sum_{k=\pm 1} |\phi_k\rangle \langle \phi_k|
\]
and
\[
R^+ = |\phi_+\rangle \langle g_1 g_2|.
\]
Here $|\phi_\pm\rangle$ denote the phase states (14).

Using the Hamiltonian (29) as the generator of evolution, for the time dependent wave function we get
\[
|\Psi(t)\rangle = e^{-iH_\phi t}|\Psi(0)\rangle = |C_- (t)|\phi_-angle + C_+(t)|\phi_+\rangle \otimes |0\rangle_{ph} + C(t)|g_1 g_2\rangle \otimes |1\rangle_{ph}, \tag{30}
\]
where $|\cdots\rangle_{ph}$ denotes the states of the cavity field. The coefficients $C_- (t)$ and $C(t)$ in (30) are completely determined by the initial conditions and normalization condition.

It is easily seen that the state $|\phi_+\rangle \otimes |0\rangle_{ph}$ is the eigenstate of the Hamiltonian (29). Hence, at
\[
C_- (0) = 1, \quad C_+(0) = C(0) = 0,
\]
the atomic phase state $|\phi_+\rangle$ in (14) provides the stationary, maximum entangled atomic state in the system under consideration. At the same time, it is not very clear how to prepare such a state.

Therefore we consider a more realistic initial state provided by excitation of either atom, while the cavity field is in the vacuum state. To realize such a state, we can assume, for example, that one of the atoms (initially de-excited) is trapped in the cavity, while the second atom (initially excited) slowly passes through the cavity like in the experiments discussed in Refs. [8, 9]. Assume for definiteness that
\[
|\Psi(0)\rangle = |e_1 g_2\rangle \otimes |0\rangle_{ph}. \tag{31}
\]
Then, the coefficients of the wave function (30) take the form
\[
C_- (t) = \frac{1}{\sqrt{2}} e^{-i\omega_0 t},
\]
\[
C_+(t) = \frac{1}{\sqrt{2}} \left( \cos \Omega t + \frac{i \Delta}{2 \Omega} \sin \Omega t \right) e^{-i(\omega_0 + \Delta/2) t},
\]
\[
C(t) = -\frac{i \gamma}{\Omega} e^{-i(\omega_0 + \Delta/2) t} \sin \Omega t,
\]
where $\Omega = [2\gamma^2 + (\Delta/2)^2]^{1/2}$. At first site, the probabilities
\[
P_\pm (t) = |\langle 0|_{ph} \otimes \langle \phi_{pm}\rangle |\Psi(t)\rangle|^2 = |C_\pm (t)|^2
\]
to observe the states (14) corresponding to the maximum atomic entanglement, are
\[
P_- (t) = \frac{1}{2},
\]
\[
P_+ (t) = \frac{\Delta^2}{8\Omega^2} + \frac{\gamma^2}{4\Omega^2} \cos^2 \Omega t \leq \frac{1}{2},
\]
respectively. At the same time, the absence of photon counts, which is considered in Ref. [18] as a sign of the atomic entanglement, corresponds here to the case when both probabilities $P_+ (t_k) = 1/2$ at a certain time $t_k$. In other words, the mutually orthogonal entangled states (14) have the same probability to be observed at $t = t_k$. This means that there is no atomic entanglement at all but we definitely know which atom is in the excited state.

Consider one more realistic initial state when both atoms are trapped in the cavity in de-excited state, while the cavity field contains a photon:
\[
|\Psi(0)\rangle = |g_1 g_2\rangle \otimes |1\rangle_{ph}. \tag{32}
\]
Then, for all times we get $C_- (t) = 0$ and
\[
C_+(t) = \frac{i \gamma}{\Omega} e^{-i(\omega_0 + \Delta/2) t} \sin \Omega t,
\]
\[
C(t) = \left( \cos \Omega t - \frac{i \Delta}{2 \Omega} \sin \Omega t \right) e^{-i(\omega_0 + \Delta/2) t}.
\]

Hence, under this initial condition, the entangled state $|\phi_-\rangle$ cannot be achieved at all, while the second entangled state $|\phi_+\rangle$ in (14) can be achieved. It is seen that, in the case of initial state (32), the probability to detect the photon is
\[
P_{ph} (t) = |C(t)|^2 = \cos^2 \Omega t + \frac{\Delta^2}{4\Omega^2} \sin^2 \Omega t.
\]
This expression takes the minimum value
\[
\min P_{ph} = P_{ph} (t_m) = \frac{\Delta^2}{4\Omega^2}
\]
at $t = t_m = \pi (2m + 1)/2 \Omega$, $m = 0, 1, \ldots$. At the same time $t_m$, the probability to have the entangled atomic state $|\phi_+\rangle$ takes the maximum value
\[
P_+ (t_m) = |C_+ (t_m)|^2 = \frac{2\gamma^2}{2\gamma^2 + (\Delta/2)^2}.
\]
It is seen that the pure atomic entanglement with $P_+ (t_m) = 1$ is realized at $t = t_m$ only in the absence of the cavity detuning when $\Delta \rightarrow 0$.

The parasitic influence of the cavity detuning can be compensated through the use of Kerr medium filling the cavity. In this case, the Hamiltonian (28) should be supplemented by the term
\[
H_\kappa = \kappa (a^+ a)^2,
\]
which leads to the following renormalization of the Rabi frequency
\[ \Omega \rightarrow \Omega_\kappa = \sqrt{2\gamma^2 + (\Delta + \kappa)^2/4}. \]

Then, the proper choice of the Kerr parameter \( \kappa = -\Delta \) should lead to the pure entangled atomic state \( |\phi_+\rangle \) at a certain times.

Consider now the case of four atoms and two photons. In contrast to the previous case, neither phase state in (21) is an eigenstate of the Hamiltonian (28). Then, the choice of the initial state either as a state with two excited atoms or as a state with one excited atom plus cavity photon does not lead to a pure atomic entanglement. As in the case of two atoms, the pure atomic entanglement can be achieved under the choice of the state with the absence of the atomic excitations in the initial state. The influence of the cavity medium detuning can be compensated by the presence of Kerr medium as well as in the case of two atoms.

\[
C = \frac{1}{2} (\epsilon + \epsilon^+),
\]

where \( \epsilon \) is defined by Eq. (8). This operator \( C \) can be considered as a "Hamiltonian", describing the correlations between the different atoms. For example, in the case of the two atoms interacting with the single photon, the operator \( C \) takes the form

\[ C = \sigma_+^{(1)} \sigma_-^{(2)} + \sigma_-^{(2)} \sigma_+^{(1)}, \]

where
\[ \sigma_\ell^{(\pm)} = \frac{\sigma_\ell^{(1)} \pm i \sigma_\ell^{(2)}}{2}. \]

The operator structure of (33) coincides with that of the so-called model of plane rotator, which is a particular case of the Heisenberg model of ferromagnetism widely used in statistical physics and in quantum information theory.

Let us also stress that the \( SU(2) \) phase states similar to those considered in sections II and III, have been discussed recently in the context of quantum coding.

It is also known that the \( SU(2) \) phase states have direct connection with the quantum description of polarization of spherical photons emitted by the multipole transitions in atoms and molecules. Therefore, the polarization entanglement of photons can be examined in direct analogy to the above discussed atomic entanglement. At the same time, the consideration of spherical photons requires the use of more quantum degrees of freedom. Consider as an example the cascade decay of a two-level atom specified by the transition
\[
|J = 2, m = 0 \rangle \rightarrow |J' = 0, m' = 0 \rangle.
\]
Here \( J, J' \) and \( m, m' \) denote the angular momentum and projection of the angular momentum of the excited and ground atomic states, respectively. This transition gives rise to an entangled photon twins. Each photon carries spin 1, but because of the conservation of the angular momentum in the process of radiation, the sum of projections of the angular momenta of the two photons should be equal to zero. Denoting the state of a photon with given \( m \) by \(| m \rangle\), we get the three possible states of the photon subsystem

\[
| +1 \rangle \otimes | -1 \rangle, \quad | 0 \rangle \otimes | 0 \rangle, \quad | -1 \rangle \otimes | +1 \rangle.
\]

These three "individual" states can be used to construct the dual basis of the \( SU(2) \) phase states

\[
| \phi_k \rangle = \frac{1}{\sqrt{3}} (| +1 \rangle \otimes | -1 \rangle + e^{i\phi_k} | 0 \rangle \otimes | 0 \rangle + e^{2i\phi_k} | -1 \rangle \otimes | +1 \rangle),
\]

\[
\phi_k = \frac{\psi + 2k\pi}{3}, \quad k = 0, 1, 2 \quad (34)
\]

similar to (18). It can be easily seen that these states manifest the maximum entanglement.

Similar entangled states have been discussed in the context of the so-called biphoton excitations \[11\] (photon pairs in symmetric Fock states). They can also be used in quantum cryptography \[12\].

Let us stress that the general condition of the type of (4) is also valid in the case of states (34). However, the definition of local measurement should be changed in this case. Because of the number of degrees of freedom per photon is equal to three, the Hermitian operators associated with the \( SU(3) \) group should be considered instead of the infinitesimal generators of the \( SL(2) \) group. For example, the set of the Stokes operators of the Ref. \[21\], corresponding to the representation of the \( SU(3) \) subalgebra in the Weyl-Heisenberg algebra of spherical photons, can be used to define the complete set of local measurements in this case.

It is shown in section IV that the realization of a pure atomic entanglement in the \( 2n + n \)-type atom + photon systems strongly depends on the choice of initial state. Viz, the entangled states can be reached in the process of steady-state evolution only if all \( 2n \) atoms are initially in the de-excited states, while the cavity contains just \( n \) photons. This condition has an intuitively clear explanation: the excitations of different atoms have the same probability and therefore each photon in the \( 2n + n \)-system is shared with a couple of atoms.

It is also shown in section IV that the presence of the cavity detuning lampers the creation of a pure entangled atomic state. This negative effect can be compensated through the use of Kerr medium in the cavity.

We now note that the practical realization of a long-lived, maximum entanglement in a quantum mechanical system strongly depends on the interaction between this system and environment. The point is that the state of a closed quantum mechanical system changes periodically, providing the maximum entanglement as an instant event only at a certain times (see section IV). Such a periodicity is caused by a finite number of degrees of freedom in the system. To destruct such a periodicity, it is necessary to connect the system to a "heat bath", which would tune in the system to a required state. In Ref. \[3\], it has been proposed to support the atomic entanglement by the cavity losses. In this case, the absence of the photon counting outside the cavity can be associated with the existence of the entangled atomic state in the cavity.

Let us stress that an advantage of the use of the \( SU(2) \) phase states as the maximum entangled atomic states consists in the simple preparation of the initial states discussed in section IV.

In view of realization of atomic entanglement with present experimental technique, it seems to be more convenient if the existence of entangled state in a cavity would manifest itself via a signal photon rather than the absence of photon leakage from the cavity. In this case, there should be at least the two modes such that one of them (the cavity mode) provides the correlation between the atoms, while the second can freely leave the resonator to signalize the existence of the entanglement. Such a process can be realized through the use of the Raman process in atoms shown in Fig. 1 (e.g., see \[3\]). Here the dipole transitions are allowed between the levels 1 and 2 and 2 and 3, while forbidden between 1 and 3 because of the parity conservation. In the simplest case, we should assume that the two identical atoms of this type are located in a cavity, which has a very high quality with respect to the pumping mode \( \omega_p \), while the Stokes photons with frequency \( \omega_{Sh} \) can leak away freely.

Assume that the atoms are initially in the ground state 1, the Stokes field is in the vacuum state, and the pump field consists of a single photon. The evolution of the system can lead to the absorption of the cavity photon by either atom with further emission of the Stokes photon, which leaves the cavity. After that, the atoms are in entangled state, corresponding to the excitation of the atomic level 3 shared between the atoms. Since the inverse process cannot be realized without assistance of the Stokes photon, such a state represents a durable atomic entangled state.

It is clear that the above consideration of the atomic entanglement in the multi-atom system can be generalized with easy on the case of Raman process in atoms. In other words, the \( SU(2) \) phase states similar to (16) form the class of the maximum entangled atomic states in the case of Raman-type processes in the three-level atoms as well. An evident advantage of the use of the Raman process is the long-lived maximum entanglement in atomic subsystems.

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FIG. 1. Atomic Raman-type interaction with pump (P) and Stokes (S) photons.
