Single-atom entropy squeezing for two two-level atoms interacting with a single-mode radiation field

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In this paper we consider a system of two two-level atoms interacting with a single-mode quantized electromagnetic field in a lossless resonant cavity via $l$-photon-transition mechanism. The field and the atoms are initially prepared in the coherent state and the excited atomic states, respectively. For this system we investigate the entropy squeezing, the atomic variances, the von Neumann entropy and the atomic inversions for the single-atom case. We show that the more the number of the parties in the system the less the amounts of the nonclassical effects exhibited in the entropy squeezing. The entropy squeezing can give information on the corresponding von Neumann entropy. Also the nonclassical effects obtained form the asymmetric atoms are greater than those obtained form the symmetric atoms. Finally, the entropy squeezing gives better information than the atomic variances only for the asymmetric atoms.

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I. INTRODUCTION

The squeezed state of light is distinguished by a long-axis variance of noise ellipse for one of its quadratures in the phase space. This property has been used in many optical devices as well as in the quantum information, e.g. a power recycled interferometer [1], a phase-modulated signal recycled interferometer [2], quantum teleportation [3], cryptography [4, 5] and dense coding [6]. It is worth pointing out that experiments for the quantum teleportation have been successfully performed by means of the two-mode squeezed vacuum states [7]. Various methods have been

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proposed for the generation of squeezed states of the electromagnetic field and some of them have been implemented, e.g., [8, 9].

The concept of squeezed states has been extended to atoms [10] and defined in a sense similar to that of the radiation field. In this respect, the atomic squeezing has been obtained a great interest [11] owing to its potential application in the high-resolution spectroscopy [12], the high-precision atomic fountain clock [13], the high-precision spin polarization measurements [14], etc. Spin squeezing is another measure, which has been applied to collected atoms [14]. Additionally, this measure has been used to quantify the entanglement in the multi-atom systems [15]. Furthermore, spin squeezing has been experimentally realized for an ensemble of V -type atoms [16]. In all these cases the atomic squeezing has been treated in the framework of the Heisenberg uncertainty relations (HUR). Nevertheless, the HUR cannot give sufficient information on the atomic squeezing, in particular, when the atomic inversion is zero [17]. This difficulty has been overcome by using the entropic uncertainty relation (EUR) [18]. In this regard one has to use the concept of atomic entropy squeezing. More details about this issue will be given in section 2. So far the entropy squeezing technique has been applied to the single two-level atom interacting with a single mode or two modes, i.e. the Jaynes-Cummings model (JCM) [17, 19]. In this paper we apply this technique to the system of two atoms interacting with the one-mode electromagnetic field being in the \( l \)-th photon resonance with the atomic transition (TJCM). The atoms and the field are initially prepared in the excited atomic states and the coherent state, respectively. Moreover, we do not consider dissipation in the system, which generally leads to the degradation of the nonclassical effects. This means that the system is always being in a pure state. Sometimes this system is called Tavis-Cummings model [20] or Dicke model [21] or two-atom JCM. For this system we make a comparative study to the atomic inversions, the entropy squeezing, the atomic variances and the von Neumann entropy. We treat two cases, namely, symmetric (two identical atoms) and asymmetric (two non-identical atoms) cases. The investigation will be restricted to the single-atom case, where there are difficulties to deal with the entropy squeezing of the compound case, as we argue in section 2. We should stress that the behavior of the individual atoms in the TJCM is generally different from that in the JCM, where the distribution of energy among the parties in the bipartite is completely different from that in the tripartite. Such type of investigation is motivated by the importance of the TJCM in the literatures, e.g. [22, 23, 24, 25, 26, 27]. The atomic inversions of the TJCM have taken much interest [22, 26, 27] since they exhibit different shapes of the revival-collapse phenomenon (RCP). The importance of the TJCM has increased as a result of the progress in the quantum information [28, 29]. In this respect, the entanglement for the TJCM
with the initial coherent state \([24]\), the binomial state \([25]\) and the superposition displaced Fock state \([27]\) has been investigated. Moreover, various schemes have been proposed for the TJCM, e.g., \([30]\). This information indicates that the investigation presented in this paper is important in its own right. Additionally, we obtained many of interesting results. For instance, as is well known for the JCM that the entropy squeezing always gives better information on the atomic system than the atomic variances \([17, 19]\). In this paper we show that this is not always correct, where in some cases they can give identical behaviors. Also when the number of the qubits in the quantum system is increased the amounts of the nonclassical effects occurred in the entropy squeezing decrease. More precisely, we show that the amounts of the nonclassical effects in the entropy squeezing associated with the JCM are greater than those with the TJCM. Most importantly, we show that the entropy squeezing can generally give information on the corresponding von Neumann entropy. Also the amounts of the nonclassical effects obtained form the asymmetric case are greater than those obtained from the symmetric one. These results are valid for both \(l = 1\) and \(l = 2\). Finally, throughout the paper the phrase ”nonclassical effects” means that the entropy squeezing and/or the atomic variance include negative values.

The paper is prepared in the following order. In section 2 we give the basic equations and relations for the system under consideration. In section 3 we investigate the atomic inversions, the entropy squeezing, the atomic variances and the von Neumann entropy. The main conclusions are summarized in section 4.

II. BASIC EQUATIONS AND RELATIONS

In this section we give a definition for the atomic squeezing in terms of the entropic information and the atomic variances. We develop the Hamiltonian and the wavefunction for a two two-level atoms multi-photon JCM. Also we deduce the expressions of the entropy squeezing and the von Neumann entropy.

As is well known that for the \(j\)th atom the Pauli spin operators \(\hat{\sigma}_x^{(j)}, \hat{\sigma}_y^{(j)}\) and \(\hat{\sigma}_z^{(j)}\) determine the real, the imaginary parts of the complex dipole moment and the energy of the atom, respectively. This set of operators satisfy the following commutation rule:

\[
\left[\hat{\sigma}_x^{(j)}, \hat{\sigma}_y^{(j)}\right] = 2i\hat{\sigma}_z^{(j)}.
\]  

(1)

The Heisenberg uncertainty relation (HUR) associated with (1) is

\[
\langle \left(\Delta \hat{\sigma}_x^{(j)}\right)^2 \rangle \langle \left(\Delta \hat{\sigma}_y^{(j)}\right)^2 \rangle \geq |\langle \hat{\sigma}_z^{(j)} \rangle|^2.
\]  

(2)
From (2) the atomic system has reduced fluctuations (i.e. squeezing) in the $\hat{\sigma}_x^{(j)}$ or in the $\hat{\sigma}_y^{(j)}$ if
\[
F_k^{(j)} = \left( \langle \Delta \hat{\sigma}_k^{(j)} \rangle^2 - |\langle \hat{\sigma}_z^{(j)} \rangle| \right) < 0, \quad k = x, y.
\] (3)

The inequality (2) is a state dependent and it is trivially satisfied for any atomic state having $\langle \hat{\sigma}_z^{(j)} \rangle = 0$ [17]. In this case (3) fails to provide any useful information on the atomic system. This difficulty has been overcome using the entropic uncertainty relation (EUR) [31, 32]. An optimal EUR for a set of $N + 1$ complementary observable with different eigenvectors in an even $N$-dimensional Hilbert space can be evaluated through the inequality [18]:
\[
\sum_{k=1}^{N+1} H(\hat{\sigma}_k^{(j)}) \geq \frac{N}{2} \ln(\frac{N}{2}) + (\frac{N}{2} + 1)\ln(\frac{N}{2} + 1),
\] (4)

where $H(\hat{\sigma}_k^{(j)})$ is the information entropy associated with the variable $\hat{\sigma}_k^{(j)}$. The quantity $H(\hat{\sigma}_k^{(j)})$ can be described as follows: For an arbitrary atomic system described by the density matrix $\hat{\rho}$, the probability distribution of $N$ possible outcome of measurements of the operator $\hat{\sigma}_k^{(j)}$ is
\[
P_{j'}(\hat{\sigma}_k^{(j)}) = \langle \psi_{kj'} | \hat{\rho} | \psi_{kj'} \rangle, \quad j' = 1, 2, ..., N,
\] (5)

where $|\psi_{kj'}\rangle$ are the eigenstates of $\hat{\sigma}_k^{(j)}$. In this case the associated information entropy is:
\[
H(\hat{\sigma}_k^{(j)}) = -\sum_{j'=1}^{N} P_{j'}(\hat{\sigma}_k^{(j)}) \ln P_{j'}(\hat{\sigma}_k^{(j)}).
\] (6)

For the single-atom JCM we have $N = 2$ and then $0 \leq H(\hat{\sigma}_k^{(j)}) \leq \ln2$, where $k = x, y, z$. For this case the inequality (1) takes the form:
\[
H(\hat{\sigma}_x^{(j)}) + H(\hat{\sigma}_y^{(j)}) \geq \ln4 - H(\hat{\sigma}_z^{(j)}).
\] (7)

From (7) the components $\hat{\sigma}_k^{(j)}(k \equiv x, y)$ are said to be squeezed with respect to the information entropy if one or both of them satisfy the condition 17:
\[
E_k^{(j)} = \delta H(\hat{\sigma}_k^{(j)}) - \frac{2}{\sqrt{\delta H(\hat{\sigma}_z^{(j)})}} < 0, \quad k = x, y,
\] (8)

where $\delta H(\hat{\sigma}_k^{(j)}) = \exp[H(\hat{\sigma}_k^{(j)})]$. It is worth mentioning that $\delta H(\hat{\sigma}_k^{(j)}) = 1$ ($\delta H(\hat{\sigma}_k^{(j)}) = 2$) corresponds to the atom being in a pure (mixed) state. As we mentioned in the Introduction that throughout the paper the phrase "nonclassical effects" means that $E_k^{(j)} < 0$ or $F_k^{(j)} < 0$. Furthermore, the optimal nonclassical entropy squeezing $E_k^{(j)} \simeq -0.4140$ is associated with the eigenstates
\[ |\phi^{(x)}_\pm \rangle = \frac{1}{\sqrt{2}} (|+\rangle \pm |-\rangle), \quad |\phi^{(y)}_\pm \rangle = \frac{1}{\sqrt{2}} (|+\rangle \pm i |-\rangle), \tag{9} \]

where \(|+\rangle\) and \(|-\rangle\) denote the excited and the ground atomic states; and the superscripts \((x)\) and \((y)\) mean that the nonclassical effects only occur in \(E_x(.)\) and \(E_y(.)\), respectively. Throughout the paper we only study the entropy squeezing for the single-atom case since the EUR (4) is not relevant for \(N(\geq 2)\) atomic system, where, e.g., for the TJCM there are degenerate eigenvalues for the compound spin operators \(\hat{\sigma}_x^{(1)} + \hat{\sigma}_x^{(2)}\) and \(\hat{\sigma}_y^{(1)} + \hat{\sigma}_y^{(2)}\).

The Hamiltonian describing the two two-level atoms interacting with the single-mode electromagnetic field through multi-photon transition, namely, two atoms Jaynes-Cummings model (TJCM), in the rotating wave approximation takes the form, e.g., [22, 23, 24, 25, 26, 27]:

\[ \hat{H} = \hat{H}_0 + \hat{H}_I, \tag{10} \]

\[ \hat{H}_0 = \omega \hat{a}^\dagger \hat{a} + \frac{1}{2} \omega_a (\hat{\sigma}_z^{(1)} + \hat{\sigma}_z^{(2)}), \quad \hat{H}_I = \sum_{j=1}^2 \lambda_j (\hat{a}^\dagger \hat{\sigma}_+^{(j)} + \hat{a} \hat{\sigma}_-^{(j)}), \]

where \(\hat{H}_0\) and \(\hat{H}_I\) are the free and the interaction parts of the Hamiltonian, \(\hat{\sigma}_\pm^{(j)}\) and \(\hat{\sigma}_z^{(j)}\) are the Pauli spin operators of the \(j\)th atom; \(\hat{a} \quad (\hat{a}^\dagger)\) is the annihilation (creation) operator denoting the cavity mode, \(\omega\) and \(\omega_a\) are the frequencies of the cavity mode and the atomic systems (we assume that the two atoms have the same frequency), \(\lambda_j\) is the atom-field coupling constant of the \(j\)th atom and \(l\) is the transition parameter. From the Hamiltonian (10) it is evident that the atoms do not interact directly, but only through the common radiation field. Throughout the investigation we mainly deal with the ratio \(g = \lambda_2 / \lambda_1\), when \(g = 1\) \((g \neq 1)\) it is called symmetric (asymmetric) case. Additionally, we assume that \(\omega_a = 2l\omega\) (i.e. the exact resonance case) and the two atoms and the field are initially prepared in the excited atomic states and coherent state \(|\alpha\rangle\), respectively. Under these initial conditions, the dynamical state of the system can be easily evaluated as [26]:

\[ |\Psi(T)\rangle = \sum_{n=0}^{\infty} C_n \left[ X_1(T,n)|+,+,n\rangle + iX_2(T,n)|+,-,n+l\rangle + iX_3(T,n)|-,+,n+l\rangle + X_4(T,n)|-,-,n+2l\rangle \right], \tag{11} \]

where \(T = \lambda_1 t\) is the scaled time and \(C_n = \frac{\alpha^n}{\sqrt{n!}} \exp(-\frac{\alpha^2}{2})\) with the real amplitude \(\alpha\). The explicit general forms for the real dynamical coefficients \(X_j(T,n)\) are given, e.g., in [26]. Nevertheless, for
reasons will be made clear shortly we give the forms of these coefficients for the case \((l, g) = (1, 1)\) as:

\[
X_1(T, n) = \frac{1}{2n+3}[(n + 1) \cos(T \theta_n) + (n + 2)],
\]

\[
X_2(T, n) = X_3(T, n) = -\frac{\sqrt{n+1}}{\theta_n} \sin(T \theta_n),
\]

\[
X_4(T, n) = \frac{\sqrt{(n+1)(n+2)}}{2n+3} \cos(T \theta_n) - 1, \quad \theta_n = \sqrt{4n+6}.
\]  

Now we are in a position to calculate the single-atom entropy squeezing, the atomic variances and the von Neumann entropy for the first atom. In doing so we assume that \(A_1, A_2\) and \(f\) denote the first atom, the second atom and the radiation field, respectively. The density matrix of the whole system is \(\hat{\rho}_{A_1A_2f}(T) = |\Psi(T)\rangle \langle \Psi(T)|\), where \(|\Psi(T)\rangle\) is given by (11). As we treat the evolution of the single-atom case we have to trace out the remaining part of the density matrix. For instance, the density matrix of the first atom can be obtained as:

\[
\hat{\rho}_{A_1}(T) = \text{Tr}_{A_2f} \hat{\rho}_{A_1A_2f}(T),
\]

\[
= \sum_{n=0}^{\infty} [Q_1(n, n)|+\rangle \langle +| + Q_2(n, n)|-\rangle \langle -| + Q_3(n, n + l)|+\rangle \langle -| + Q_3^*(n, n + l)|-\rangle \langle +|],
\]

where

\[
Q_1(n, n) = C_{n,n}[X_1^2(T, n) + X_2^2(T, n)],
\]

\[
Q_2(n, n) = C_{n,n}[X_3^2(T, n) + X_4^2(T, n)],
\]

\[
Q_3(n, n + l) = iC_{n+l,n}[X_2(T, n + l)X_4(T, n) - X_3(T, n)X_1(T, n + l)]
\]

and \(C_{n,m} = C_n C_m\). By means of (13) one can deduce the followings:

\[
\langle \sigma_z^{(1)}(T) \rangle = \sum_{n=0}^{\infty} [Q_1(n, n) - Q_2(n, n)],
\]

\[
\langle \sigma_x^{(1)}(T) \rangle = 2 \text{Re} \sum_{n=0}^{\infty} Q_3(n, n + l) = 0,
\]

\[
\langle \sigma_y^{(1)}(T) \rangle = 2 \text{Im} \sum_{n=0}^{\infty} Q_3(n, n + l),
\]
where Re and Im stand for the real and imaginary parts of the complex quantity. From (5), (6) and (13) one can obtain the information entropies of the atomic operators $\sigma_x^{(1)}, \sigma_y^{(1)}, \sigma_z^{(1)}$ as

$$H(\hat{\sigma}_x^{(1)}) = -\frac{1}{2} \left[ 1 + \langle \sigma_x^{(1)}(T) \rangle \right] \ln \left[ \frac{1}{2} + \frac{1}{2} \langle \sigma_x^{(1)}(T) \rangle \right] - \frac{1}{2} \left[ 1 - \langle \sigma_x^{(1)}(T) \rangle \right] \ln \left[ \frac{1}{2} - \frac{1}{2} \langle \sigma_x^{(1)}(T) \rangle \right] = \ln 2,$$

$$H(\hat{\sigma}_y^{(1)}) = -\frac{1}{2} \left[ 1 + \langle \sigma_y^{(1)}(T) \rangle \right] \ln \left[ \frac{1}{2} + \frac{1}{2} \langle \sigma_y^{(1)}(T) \rangle \right] - \frac{1}{2} \left[ 1 - \langle \sigma_y^{(1)}(T) \rangle \right] \ln \left[ \frac{1}{2} - \frac{1}{2} \langle \sigma_y^{(1)}(T) \rangle \right], \quad (16)$$

$$H(\hat{\sigma}_z^{(1)}) = -\frac{1}{2} \left[ 1 + \langle \sigma_z^{(1)}(T) \rangle \right] \ln \left[ \frac{1}{2} + \frac{1}{2} \langle \sigma_z^{(1)}(T) \rangle \right] - \frac{1}{2} \left[ 1 - \langle \sigma_z^{(1)}(T) \rangle \right] \ln \left[ \frac{1}{2} - \frac{1}{2} \langle \sigma_z^{(1)}(T) \rangle \right].$$

We close this section by writing the expression of the single-atom von Neumann entropy as:

$$\gamma(T) = -\Tr[\hat{\rho}_{A_1}(T) \ln \hat{\rho}_{A_1}(T)] = -\mu_-(T) \ln \mu_-(T) - \mu_+(T) \ln \mu_+(T), \quad (17)$$

where $\mu_\pm$ are the eigenvalues of the density matrix $\hat{\rho}_{A_1}(T)$, which can be easily evaluated from (13) as:

$$\mu_\pm = \frac{1}{2} \sum_{n=0}^{\infty} (Q_1(n, n) + Q_2(n, n)) \pm \frac{1}{2} \sqrt{\left\{ \sum_{n=0}^{\infty} (Q_1(n, n) - Q_2(n, n))^2 + 4 \sum_{n=0}^{\infty} Q_3(n, n+l)^2 \right\}^2}. \quad (18)$$

For the future purpose, by means of (15) the eigenvalues (18) can be re-expressed as:

$$\mu_\pm = \frac{1}{2} \pm \frac{1}{2} \sqrt{\langle \sigma_x^{(1)}(T) \rangle^2 + \langle \sigma_y^{(1)}(T) \rangle^2 + \langle \sigma_z^{(1)}(T) \rangle^2}. \quad (19)$$

It is worth mentioning that the relations related to the second atom can be obtained from those of the first one by using the interchange $X_2(T, n) \leftrightarrow X_3(T, n)$. In the following section we use the relations obtained above to investigate the single-atom entropy squeezing, the atomic variances, the von Neumann entropy and the atomic inversions for the system under consideration.

### III. DISCUSSION OF THE RESULTS

In this section we investigate the single-atom atomic inversions, the entropy squeezing, the atomic variances and the von Neumann entropy for the system under consideration. From (8), (15) and (16) one can prove for the $x$-component entropy squeezing that

$$E_x^{(j)}(T) = 2 \left[ 1 - \frac{1}{\sqrt{\delta H(\hat{\sigma}_x^{(1)})}} \right] \geq 0, \quad (20)$$

where $\frac{1}{\sqrt{\delta H(\hat{\sigma}_x^{(1)})}} \leq 1$. This means that $E_x^{(j)}$ cannot exhibit nonclassical effects. Similar arguments show that $F_x^{(j)} \geq 0$. Thus throughout the discussion we focus the attention on the $y$-component of...
FIG. 1: The evolution of the atomic inversions, the entropy squeezing and the atomic variance as indicated for $(\alpha, g, l) = (5, 0.5, 1)$. In (a) the curves A and B are given for $\langle \hat{\sigma}_z^{(1)}(T) \rangle$ and $\langle \hat{\sigma}_z^{(2)}(T) \rangle + 2$, respectively. The straight lines in (b)–(d) are given to show the nonclassical effects bounds.

both the entropy squeezing and the atomic variances. Furthermore, for a weak initial field intensity, i.e. $\alpha \leq 1$, we have found that $E_y^{(j)}$ and $F_y^{(j)}$ cannot exhibit nonclassical effects. This is because the coherent state $|\alpha\rangle$ tends to the vacuum state $|0\rangle$ and/or the Fock state, which manifest themselves as a periodic behavior in these quantities. Moreover, we have noted that the entropy squeezing cannot exhibit nonclassical effects for $l > 2$. When the initial field intensity is strong the atomic inversion of the system exhibits the RCP, which is connected with the occurrence of the nonclassical effects in the system. For instance, the JCM generates the Schrödinger-cat states at one-half of the revival time \[33\], however, the symmetric (asymmetric) TJCM generates asymmetric (symmetric) cat states at the quarter of the revival time \[27\]. Additionally, for the JCM the entropy squeezing exhibits nonclassical effects only in the course of the collapse regions of the corresponding atomic inversion, however, the atomic variance fails to give any useful information \[17\]. As a result of these facts we'll study the evolution of the atomic inversions for the TJCM, too. As we mentioned in the Introduction that the behavior of the individual atoms in the TJCM is generally different.
from that in the JCM, where the latter (the former) includes one (two) interaction mechanism(s).

Now we start the investigation with the asymmetric case for the single-photon transition mechanism. For this case we plot the atomic inversions, the entropy squeezing and the atomic variance in Figs. 1 as indicated for the given values of the interaction parameters. In these figures we take $\lambda_1 = 2\lambda_2$, i.e. the interaction between the field and the first atom is two times stronger than that with the second atom. This is manifested in the evolution of the atomic inversions, where the collapse regions in $\langle \hat{\sigma}_z^{(2)}(T) \rangle$ are two times greater than those in $\langle \hat{\sigma}_z^{(1)}(T) \rangle$ (see Fig. 1(a)). This means that the rate of energy interchange between the radiation field and the first atom is two times faster than that with the second atom. The behaviors of the atomic inversions are roughly connected with the evolution of the entropy squeezing and the atomic variance (see Fig. 1(b)–(d)). For instance, $E_y^{(1)}(T)$ and $E_y^{(2)}(T)$ exhibit nonclassical effects immediately after switching on the interaction. At this stage the atomic inversions provide their zero revival patterns. As the interaction proceeds the nonclassicality completely disappears in $E_y^{(1)}(T)$, however, $E_y^{(2)}(T)$ provides its maximum value approximately at one-half of the revival time in the $\langle \hat{\sigma}_z^{(2)}(T) \rangle$. The comparison between Fig. 1(b) and Fig. 1(c) shows that the amounts of the nonclassical effects exhibited in $E_y^{(2)}(T)$ are much greater than those in $E_y^{(1)}(T)$. These amounts can be increased by increasing the value of $\alpha$. We have checked this fact. Additionally, the behaviors of $E_y^{(1)}(T)$ and $E_y^{(2)}(T)$ in the Fig. 1(b) and (c) can be reversed if one takes $\lambda_2 = 2\lambda_1$ and $T = t\lambda_2$. On the other hand, we have noted that $F_y^{(1)}(T)$ and $F_y^{(2)}(T)$ provide similar behaviors. This indicates that the entropy squeezing is more sensitive to the interaction mechanisms in the compound system than the atomic variance. We have plotted only $F_y^{(2)}(T)$ in Fig. 1(d). From this figure we can see that $F_y^{(2)}(T)$ exhibits nonclassical effects only after switching on the interaction. Comparison between Fig. 1(c) and Fig. 1(d) confirms that the entropy squeezing gives better information on the atomic system than the atomic variance [17]. We’ll show below that this statement is not always correct.

Now we draw the attention to the two-photon transition case, i.e. $l = 2$, which is plotted in Figs. 2 for the given values of the interaction parameters. From Fig. 2(a) $\langle \hat{\sigma}_z^{(1)}(T) \rangle$ and $\langle \hat{\sigma}_z^{(2)}(T) \rangle$ exhibit periodic, systematic, and compact RCP with periods $\pi$ and $2\pi$, respectively. This behavior is connecting with the two-photon nature of the system. Also one can observe the occurrence of the subsidiary revivals in the evolution of $\langle \hat{\sigma}_z^{(1)}(T) \rangle$, i.e. each revival pattern is followed by a subsidiary one (see the curve A in Fig. 2(a)). It is worth reminding that the subsidiary revivals have been observed also for the two-mode single-photon JCM [34]. On the other hand, from the solid curve in the Fig. 2(b) one can observe that $E_y^{(1)}(T)$ periodically (with period $\pi$) exhibits nonclassical effects in the course of the revival patterns in $\langle \hat{\sigma}_z^{(1)}(T) \rangle$. Moreover, the amounts of the nonclassical
FIG. 2: The evolution of the atomic inversions, the entropy squeezing and the atomic variance as indicated in the figures for \((\alpha, g, l) = (5, 0.5, 2)\). In (a) the curves A and B are given for \(\langle \hat{\sigma}_z^{(1)}(T) \rangle\) and \(\langle \hat{\sigma}_z^{(2)}(T) \rangle + 2\), respectively. In (b) and (c) the solid and dashed curves are given for the first and second atom, respectively. The inset in (b) is given to show the behavior of the entropy squeezing through a very short period around \(T = 2\pi\). The short-dashed curve in this inset is given for \(E_y^1\) of the case \(g = 1\). The straight lines in (b)–(c) are given to show the nonclassical bounds.

effects occurred in \(E_y^{(1)}(T)\) around \(T = s\pi\) are more pronounced than those around \(T = s'\pi\), where \(s\) and \(s'\) are odd and even integers, respectively. Also we have found \(E_y^{(1)}(T = \bar{s}\pi) \simeq 0\), where \(\bar{s}\) is integer. For instance, this is obvious from the inset in Fig. 2(b) for a short period around \(T = 2\pi\). Next, the evolution of \(E_y^{(2)}(T)\) exhibits periodical compound behavior with period \(2\pi\) (see the dashed curve in Fig. 2(b)). More illustratively, \(E_y^{(2)}(T)\) exhibits long-lived nonclassical effect at \(T \simeq \pi/2, 3\pi/2\) and instantaneous nonclassical effect around \(T = 2\pi\). The comparison between the curve B in the Fig. 2(a) and the dashed curve in the Fig. 2(b) shows that \(E_y^{(2)}(T)\) provides nonclassical effects in the course of both of the collapse regions and the revival patterns of \(\langle \hat{\sigma}_z^{(2)}(T) \rangle\).

From the dashed-curve in the inset in the Fig. 2(b), one can realize that \(E_y^{(2)}(T = 2\bar{s}\pi) \simeq 0\). We’ll return to this point shortly. From the above investigation it is obvious for the case \(l = 2\) that there is no clear relationship between the occurrence of the RCP in the atomic inversions and the nonclassical effects in the entropy squeezing. Now we draw the attention to \(F_y^{(1,2)}(T)\). Generally, the behaviors of \(F_y^{(1,2)}(T)\) are similar to those of \(E_y^{(1,2)}(T)\) (compare Fig. 2(b) and Fig. 2(c)). Nevertheless, \(F_y^{(1,2)}(T)\) provide only instantaneous nonclassical effects with amounts smaller than those exhibited in \(E_y^{(1,2)}(T)\).

We conclude this part by showing an important fact: the entropy squeezing can give information on the corresponding von Neumann entropy. To clarify this point we plot—as an example—\(\gamma(T)\) for the second atom in the Fig. 3(a) and (b) when \(l = 1\) and 2, respectively. The comparison between these figures and the corresponding \(E_y^{(2)}\) in Figs. 1 and 2 is instructive. Of course these quantities
The evolution of the von Neumann entropy for the second atom for \((\alpha, g) = (5, 0.5)\) when \(l = 1\) (a) and \(l = 2\) (b).

Include different scales, e.g. \(0 \leq \gamma(T) \leq \ln 2\) and \(-0.4 \leq E_y^{(j)}(T) \leq 0.6\) (see Figs. 1-2). The limitations of the \(E_y^{(j)}(T)\) have been numerically obtained. From Figs. 1–3 one can realize when \((E_y^{(2)}, \gamma) \simeq (-0.4, 0)\) or \((0.6, 0.639)\) the bipartite (i.e. \(A_2, fA_1\)) is disentangled or maximally entangled. In this respect, the entropy squeezing can play two roles, one for squeezing and the other for entanglement. For the latter, it can be interpreted as follows: Apart from \(T > 0\) for \(E_j < 0\) (\(> 0\)) the bipartite will be close to the disentangled (entangled) form till \(E_j = -0.41\) (0.6) the bipartite will be in a complete disentangled (maximally entangled) states. Additionally, the bipartite is disentangled when \(E_j(T)\) has an inclined point at \(E_j(T) = 0\). The inclined point means that \(E_j(T)\) changes its behavior around this point, i.e. the function changes its behavior from an increasing function to a decreasing one or vice versa around \(E_j(T) = 0\). This is quite obvious from the inset in Fig. 2(b). Also from this inset one can realize that the tripartite is periodically (with period \(2\pi\)) in an approximate disentangled form, e.g. \(\hat{\rho}(T = 2\pi) \simeq \hat{\rho}_{A_1} \otimes \hat{\rho}_{A_2} \otimes \hat{\rho}_f\), however, it is difficult to find the asymptotic forms for these density matrices. Nevertheless, for the symmetric case these forms have been already derived in [27] and confirmed by the short-dashed curve in the inset in Fig. 2(b), which involves an explicit inclined point at \(T = 2\pi\). Now, the reason why the entropy squeezing \(E_y^{(j)}(T)\) and the von Neumann entropy have similar behaviors is that both of them are functions in the \(\langle \sigma_z^{(2)}(T) \rangle\) and \(\langle \sigma_y^{(2)}(T) \rangle\) (c.f. [8], [10]-[12]). Also from these equations, for \(\langle \sigma_z^{(2)}(T) \rangle, \langle \sigma_y^{(2)}(T) \rangle = (0, \pm 1)\) and \((\pm 1, 0)\) one can prove \((E_y^{(2)}, \gamma) \simeq (-0.4, 0)\) and \((0, 0)\), respectively. These results are in a good agreement with the above discussion. We have to stress that this relationship between the von Neumann and the entropy squeezing is universal (for the JCM and TJCM) provided that the atoms are initially prepared in the excited or ground
FIG. 4: The evolution of the $E_y(T)$ for the TJCM (with $g = 1$) (a) and the JCM (b) when $(\alpha, l) = (5, 1)$. The straight line in (b) is given to show the nonclassical effects bound.

states. We have checked this fact. The final remark: the inverse situation is not correct. More illustratively, the von Neumann entropy cannot be used to give information on the nonclassicality in the entropy squeezing. For instance, when $\gamma(T) \simeq 0$ this may be corresponding to $E_j = -0.41$ or $E_j = 0$, as we have shown above.

We close this section by discussing the symmetric case, in particular, $(g, l) = (1, 1)$. In this case the energy interchanged between the field and the atoms is equally distributed between the two atoms. Therefore, the evolution of the different quantities related to the individual atoms are identical. For this case the atomic inversion has an identical behavior with that of the JCM (see the curve A Fig. 1(a)). In spite of this fact the evolution of $E_y(T)$ of the JCM and of the TJCM are completely different (see Figs. 4). From Fig. 4(a) one can observe that $E_y(T)$ exhibits nonclassical effects only after switching on the interaction, however, this is not the case for the JCM and/or Fig. 4(b). Moreover, the comparison between these figures shows that the amounts of the nonclassical effects associated with $E_y(T)$ of the JCM are much greater than those exhibited for the TJCM. This conclusion is also valid for the asymmetric case (compare Fig. 1(c) to Fig. 4(b)). This means when the number of the qubits in the quantum system is increased, e.g. Ising model [33], the amounts of the nonclassical effects in $E_k(T)$ of the individual qubits decrease. We can analytically explain this fact for the TJCM as follows: $E_y$ depends on $\langle \sigma_z(T) \rangle$ and $\langle \sigma_y(T) \rangle$, however, for the JCM and the symmetric TJCM the quantity $\langle \sigma_z(T) \rangle$ has a quite similar behavior for both of them, as we mentioned above. Thus $\langle \sigma_y(T) \rangle$ plays the essential role in the evolution of the entropy squeezing. When $\alpha >> 1$ the probability distribution $P(n) = C_\alpha^2$ is Piossonian and has the main contribution around $n \simeq \alpha^2$. In this case the harmonic approximation can be applied
to (12), i.e. $\epsilon/n \to 1$, where $\epsilon$ is an arbitrary c-number) and hence the quantity $\langle \sigma_y(T) \rangle$ in (15) can be simplified as:

$$\langle \sigma_y(T) \rangle \simeq \sum_{n=0}^{\infty} C_{n,n+1} \left\{ \frac{1}{2} \sin[T(\theta_n - \theta_{n+1})] + \sin[T(\theta_n + \theta_{n+1})] \cos[T(\theta_n - \theta_{n+1})] \right\}. \quad (21)$$

Nevertheless, for the JCM we have

$$\langle \sigma_y(T) \rangle = 2 \sum_{n=0}^{\infty} C_{n,n+1} \cos(T\sqrt{n+2}) \sin(T\sqrt{n+1}). \quad (22)$$

The comparison between the two above expressions leads to that generally the amplitudes of the $\langle \sigma_y(T) \rangle$ of the TJCM is one-half of that of the JCM. We have numerically checked this fact. This completes the explanation. Similar conclusions have been also realized for the case $l = 2$.

Finally, we have found for the symmetric case that $F_y(T)$ and $E_y(T)$ provide similar behaviors. This indicates that the entropy squeezing does not always give better information than the atomic variances.

**IV. CONCLUSION**

In this paper we have investigated a system of two two-level atoms interacting with a single-mode quantized electromagnetic field in a lossless resonant cavity via $l$-photon-transition mechanism. The atoms and the field are initially prepared in the excited atomic states and the coherent state, respectively. We have found the partial density matrix for an individual atom by tracing the state over the variables of the other atom and the field. Two cases have been treated, namely, the symmetric and the asymmetric cases. The investigations have been focused on the entropy squeezing, the atomic variances, the von Neumann entropy and the atomic inversions for the pseudo-spin components of the individual atom. We have shown that the values of the transition parameter $l$, the ratio $g$ and the intensity $\alpha$ are important for the occurrence of the nonclassical effects in the entropy squeezing. We have numerically shown that the system cannot provide nonclassical effects in $E_k^{(j)}$ for $l > 2$. The amounts of the nonclassical effects in $E_k^{(j)}$ obtained from the TJCM are smaller than those obtained from the JCM. In other words, when the number of the parties in the quantum system is increased the amounts of the nonclassical effects involved in the entropy squeezing decrease. The nonclassical effects in $E_k^{(j)}$ obtained from the asymmetric case are greater than those obtained from the symmetric one. Also we have shown that the entropy squeezing does not always give better information than the atomic variances. Most importantly, we have shown that the entropy squeezing can generally give information on the corresponding von
Neumann entropy. These results are verified for both of the cases $l = 1$ and $l = 2$. Additionally, there is no clear relationship between the occurrence of the RCP in the atomic inversions and the nonclassical effects in the entropy squeezing for the case $l = 2$.

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