LARGE TIME BEHAVIOR OF SOLUTIONS TO NONLINEAR BEAM EQUATIONS

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ABSTRACT. In this note we analyze the large time behavior of solutions to a class of initial/boundary problems involving a damped nonlinear beam equation. We show that under mild conditions on the damping term of the equation of motions the solutions of the dynamical problem converge to the solution of the stationary problem. We also show that this convergence is exponential.

1. Introduction

In [2] the following initial/boundary value problem is studied

\begin{align}
  u_{tt} + u_{xxxx} + F(u_t) + G(u) &= f(x, t), \quad \text{on } (c, d) \times (0, T] \\
  u(c, t) &= u(d, t), \quad \text{for all } t \in (0, T] \\
  u_{xx}(c, t) &= u_{xx}(d, t), \quad \text{for all } t \in (0, T] \\
  u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x) \quad \text{for all } x \in (c, d).
\end{align}

Problems of this type arise naturally in the study of vibrations in slender beams. In particular, it has been used to model vibrations in suspension bridges and railroad tracks.

The main result of the investigations in [2] is the following,

**Theorem 1.1.** Let \( c \) and \( d \) be two real numbers, with \( c < d \). Let \( T \) be a positive real number. Let \( U \) be the open interval \( (c, d) \). Let \( f \) be a \( C^1 \) map from \([0, T]\) into \( L^2(U) \). Let \( F : \mathbb{R} \to \mathbb{R} \) be a nondecreasing \( C^1 \) function with \( F(0) = 0 \). Let \( G : \mathbb{R} \to \mathbb{R} \) be a continuously differentiable function such that there exists a nonpositive real number \( D \) such that \( \int_0^u G(s)ds \geq D \) for all \( u \in \mathbb{R} \). Furthermore, let \( u_0 \) be an element of \( H^4_0(U) \) and let \( u_1 \) be an element of \( H^2(U) \). Then there exists a unique high regularity weak solution of the initial/boundary value problem (1.1).

(See [2] for a proof of the above theorem.) Here \( H^2(U) \) is the intersection of \( H^1_0(U) \) with \( H^2(U) \); it is equipped with the inner-product

\begin{align}
  (u, v)_{H^2} &= \int_U u_{xx}v_{xx}dx.
\end{align}

\( H^4(U) \) is the elements of \( H^4(U) \) such that \( u \in H^2(U) \) and \( u_{xx} \in H^2(U) \); it is equipped with the inner-product

\begin{align}
  (u, v)_{H^4} &= \int_U u_{xxxx}v_{xxxx}dx.
\end{align}

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In [2] it is shown that the above inner-products make $H^2_x(U)$ and $H^4_x(U)$ Hilbert spaces.

A high regularity weak solution of (1.1) is one such that $u \in L^\infty(0,T;H^4_x(U))$, $u \in W^{1,\infty}(0,T;H^2_x(U))$, $u \in W^{2,\infty}(0,T;L^2(U))$ and such that

$$(1.4) \quad <u''(t),v> + (F_1(u'(t)), v)_{L^2} + (u(t),v)_{H^2_x} + (F_2(u(t)), v)_{L^2} = (f(t), v)_{L^2},$$

for each $v \in H^2_x(U)$ and almost every $t \in [0,T]$. (Here, and for the remainder of the paper, $<\cdot,\cdot>$ is the duality between the topological dual space of $H^2_x(U)$ and $H^2_x(U)$.) Note, here we view $u(t)$ as a Hilbert space valued function of time, and hence $u'(t)$ ($u''(t)$) is the first (second) time derivative of a Hilbert space valued function of time. It does not denote the first (second) derivative with respect to the spatial variable $x$. We will continue to use this notation convention for the remainder of this paper.

The above theorem is definitely fundamental for the study of (1.1), but it leaves open many questions about the evolution of solutions to (1.1). In particular it leaves open the matter of whether or not solutions converge to solutions of the boundary value problem

$$u_{xxxx} + G(u) = f(x) \text{ on } U,$$

$$u(c) = 0 = u(d),$$

$$u_{xx}(c) = 0 = u_{xx}(d).$$

as $t \to \infty$. Since (1.1) is derived from a damped mechanical problem, it’s also natural to ask whether or not the total energy is decreasing or not if $f$ is independent of time. These two questions are the focus of this paper, and the main result is

**Theorem 1.2.** Let $c$ and $d$ be two real numbers, with $c < d$. Let $m$ be a positive real number such that $m \geq 2$. Let $T$, $a_1$, and $a_2$ be three positive real numbers with $a_1 \leq a_2$. Let $U$ be the open interval $(c,d)$. Let $f$ be a $C^1$ map from $[0,T]$ into $L^2(U)$ such that $f \equiv 0$. Let $F : \mathbb{R} \to \mathbb{R}$ be a non decreasing function such that

$$|a_1(x + |x|^{m-2})| \leq |F(x)| \leq |a_2(x + |x|^{m-2})|.$$  

Suppose $G : \mathbb{R} \to \mathbb{R}$ has the property $G \equiv 0$. Furthermore, let $u_0$ be an element of $H^4_x(U)$ and let $u_1$ be an element of $H^2_x(U)$. Then there exists a unique high regularity weak solution, $u(t)$, of the initial/boundary value problem (1.1). Furthermore, $u(t) \to \bar{u}$ exponentially with respect to the $H^2_x(U)$ norm and $u'(t) \to 0$ in exponentially with respect to the $L^2(U)$ norm.

An immediate consequence of the above is the following

**Corollary 1.3.** Let $c$ and $d$ be two real numbers, with $c < d$. Let $m$ be a positive real number such that $m \geq 2$. Let $T$, $a_1$, and $a_2$ be three positive real numbers with $a_1 \leq a_2$. Let $U$ be the open interval $(c,d)$. Let $f$ be a $C^1$ map from $[0,T]$ into $L^2(U)$ that is independent of $t \in [0,T]$. Let $F : \mathbb{R} \to \mathbb{R}$ be a non decreasing function such that

$$|a_1(x + |x|^{m-2})| \leq |F(x)| \leq |a_2(x + |x|^{m-2})|.$$  

Suppose $G : \mathbb{R} \to \mathbb{R}$ has the property $G \equiv 0$. Furthermore, let $u_0$ be an element of $H^4_x(U)$ and let $u_1$ be an element of $H^2_x(U)$. Then there exists a unique high regularity weak solution, $u(t)$, of the initial/boundary value problem (1.1). Furthermore, $u(t) \to \bar{u}$ exponentially with respect to the $H^2_x(U)$ norm and $u'(t) \to 0$ in
exponentially with respect to the $L^2(U)$ norm. Here $\hat{u}$ is the unique solution of the boundary value problem (1.5).

In section two of this paper we make some observations about the energy of solutions to (1.1). In particular, we will see that it is nonincreasing. A consequence of this is that the $C(U)$ norm of the solution is bounded independent of $t \in [0, \infty)$. We will also prove an inequality that will become very useful when combined with the boundedness of solutions.

In section three of this paper we will prove Theorem 1.2. The idea behind the proof was borrowed from [1]. One perturbs the energy by a small quantity and then shows that this modified energy decays exponentially. The only results needed are the standard inequalities one uses while studying partial differential equations along with the inequality that is proved in section two. The main difference between the proof presented below and the one found in [1] is that we do not have to consider different cases of the size of the $L^m$ norm of the solutions as $t$ increases. This is because we do not consider equations with source terms.

2. Preliminaries

2.1. The Solution Energy. Let $u(t)$ be a high regularity weak solution to (1.1), where $F$ satisfies the hypotheses contained in the statement of Theorem 1.2; where $G \equiv 0$; and where $f \equiv 0$. We will call

\begin{equation}
E(t) = \frac{1}{2} \int_U (u_t(t))^2 \, dx + \frac{1}{2} \int_U (u_{xx}(t))^2 \, dx,
\end{equation}

the solution energy. An important property of the solution energy is that

\begin{equation}
E'(t) = -\int_U F(u_t(t))u_t(t) \, dx \leq 0.
\end{equation}

An important consequence of this property is that there exists a positive real number $C$ such that $||u(t)||_{C(U)} \leq C$ for all $t \in [0, \infty)$.

2.2. Inequalities. An inequality that will be used multiple times in the proof of Theorem 1.2 is as follows. There exists a positive real number $B$ such that

\begin{equation}
||u||_2 \leq B||u||_{H^2},
\end{equation}

for all $u \in H^2_2(U)$. This inequality is proven in Section Two of [2].

Another inequality that will prove to be useful below is the following

**Lemma 2.1.** Let $n$ be a positive integer, and let $m$ be a positive real number such that $m \geq 2$. Let $U$ be a bounded, open subset of $\mathbb{R}^n$ with $C^1$ boundary. Let $u$ be an element of $C([0, \infty); C(U))$ such that there exists a positive real number $M_1$ such that $||u(t)||_{C(U)} \leq M_1$ for all $t \in [0, \infty)$. Then we have the existence of a positive real number $C$ such that

\begin{equation}
\int_U |u(t)|^m \, dx \leq C \int_U (u(t))^2 \, dx,
\end{equation}

for all $t \in [0, \infty)$.

**Proof.** Let $v$ be an element of $C([0, \infty); C(U))$ such that there exists a positive real number $M_2$ such that $||v(t)||_{C(U)} \leq M_2$ for all $t \in [0, \infty)$. Let $M := \max\{M_1, M_2\}$. 
Let \( h : \mathbb{R} \to \mathbb{R} \) be a function such that \( h(x) = |x|^{m/2} \). Since \( h \) is locally Lipschitz we have the existence of a positive real number \( L_h \) such that
\[
|h(s_1) - h(s_2)| \leq L_h |s_1 - s_2| \text{ for any } s_1, s_2 \in [-M, M].
\]
This, in turn, allows us to write
\[
\|h(u(t)) - h(v(t))\|^2 \leq (L_h)^2 \|u(t) - v(t)\|^2
\]
for all \( t \in [0, \infty) \). Setting \( v(t) \equiv 0 \) for all \( t \in [0, \infty) \), we obtain the lemma. \( \Box \)

### 3. Proof of Theorem 1

Proceeding as in \([1]\), we start with the following observation. Notice that
\[
E'(t) = -\int_U (F(u_t))_x(x, t) dx
\]
\[
\leq -a_1(\int_U (u_t(x, t))^2 dx + \int_U |u_t(x, t)|^m dx).
\]
Define
\[
H(t) = E(t) + \epsilon \int_U u u_t(x, t) dx
\]
for an \( \epsilon \) to be specified later. Notice that we can assume without loss of generality that \( B \geq 1 \), where \( B \) is the constant appearing in \((2.3)\), so let us do so. Then the Schwarz inequality gives us
\[
|\epsilon \int_U u u_t(x, t) dx| \leq \frac{\epsilon}{2} \int_U u(x, t)^2 dx + \frac{\epsilon}{2} \int_U (u_t(x, t))^2 dx
\]
\[
\leq \frac{\epsilon^2}{2} \int_U (u_{xx}(x, t))^2 dx + \frac{\epsilon^2 B^2}{2} \int_U (u_t(x, t))^2 dx + \frac{\epsilon}{2} \int_U (u_t(x, t))^2 dx
\]
\[
\leq \epsilon B^2 E(t).
\]
It follows that
\[
|H(t) - E(t)| \leq \epsilon B^2 E(t).
\]
Differentiating \((3.2)\) we see that
\[
H'(t) \leq -a_1 \int_U (u_t(x, t))^2 dx - a_1 \int_U (u_t(x, t))^m dx + \epsilon \int_U (u_t(x, t))^2 dx
\]
\[
- \epsilon \int_U (u_{xx}(x, t))^2 dx - \epsilon \int_U F(u_t) u(x, t) dx
\]
\[
\leq -a_1 \int_U (u_t(x, t))^2 dx - a_1 \int_U |u_t(x, t)|^m dx + \frac{3}{2} \epsilon \int_U (u_t(x, t))^2 dx
\]
\[
- \frac{1}{2} \int_U (u_{xx}(x, t))^2 dx - \epsilon \int_U F(u_t) u(x, t) dx - \epsilon E(t)
\]
\[
\leq -a_1 \int_U (u_t(x, t))^2 dx - a_1 \int_U |u_t(x, t)|^m dx + \frac{3}{2} \epsilon \int_U (u_t(x, t))^2 dx
\]
\[
- \frac{1}{2} \int_U (u_{xx}(x, t))^2 dx + \epsilon a_2 \int_U |u_t||u|(x, t) dx + \epsilon a_2 \int_U |u_t|^{m-1} |u|(x, t) dx
\]
\[
- \epsilon E(t).
\]
By using
\[ a_2 \int_{U} |u_t||u|(x,t) \, dx \leq \frac{1}{4} \int_{U} (u_{xx}(x,t))^2 + a_2^2 B \int_{U} |u_t(x,t)|^2 \, dx, \]
inequality (3.5) takes then the form
\[ H'(t) \leq -a_1 \int_{U} |u_t(x,t)|^2 \, dx - a_1 \int_{U} |u_t(x,t)|^m \, dx \]
\[ + \left( \frac{3}{2} + a_2 B \right) \epsilon \int_{U} (u_t(x,t))^2 \, dx - \frac{1}{4} \epsilon \int_{U} (u_{xx}(x,t))^2 \, dx \]
\[ - a_2 \epsilon \int_{U} |u_t|^{m-1}|u|(x,t) \, dx - \epsilon E(t). \]
We then exploit Young's inequality
\[ XY \leq \delta X^r + c(\delta)Y^s, \]
where \( X, Y, \delta, c(\delta) \geq 0 \) and \( \frac{1}{r} + \frac{1}{s} = 1 \), with \( r = m \) and \( s = \frac{m}{m-1} \) to get
\[ \int_{U} |u_t|^{m-1}|u|(x,t) \, dx \leq \delta \|u\|_m^m + c(\delta)\|u_t\|_m^m, \]
for all \( \delta > 0 \). We can now combine the fact that there exists a positive real number \( C \) such that \( \|u(t)\|_{C(\bar{T})} \leq C \) for all \( t \in [0, \infty) \) with Lemma 2.1 to see that there exists a positive real number \( \gamma \) such that
\[ \int_{U} |u_t|^{m-1}|u|(x,t) \, dx \leq \gamma \|u\|_2^2 + c(\delta)\|u_t\|_m^m. \]
Therefore (3.7) becomes
\[ H'(t) \leq -a_1 \int_{U} |u_t(x,t)|^2 \, dx - a_1 \int_{U} |u_t(x,t)|^m \, dx \]
\[ + \left( \frac{3}{2} + a_2 B \right) \epsilon \int_{U} (u_t(x,t))^2 \, dx - \frac{1}{4} \epsilon \int_{U} (u_{xx}(x,t))^2 \, dx \]
\[ + a_2 \epsilon \gamma \|u_t\|_2^2 + c(\delta)\|u_t\|_m^m - \epsilon E(t), \]
for all \( \delta > 0 \). This, in turn, allows us to write
\[ H'(t) \leq -\epsilon E(t) - \frac{1}{4} \epsilon \|u_{xx}\|_2^2 + a_2 \epsilon \gamma \|u_t\|_2^2 \]
\[ - \left[ a_1 - \left( \frac{3}{2} + a_2 B \right) \epsilon \right]\|u_t\|_2^2 - a_1 \left[ 1 - \frac{a_2}{a_1} \epsilon c(\delta) \right] \|u_t\|_m^m, \]
for all \( \delta > 0 \). Now, picking \( \delta = \frac{B^2}{4a_2 \gamma} \), we see that
\[ H'(t) \leq -\epsilon E(t) - a_1 \left[ 1 - \frac{a_2}{a_1} \epsilon c(\delta) \right] \|u_t\|_m^m - \left[ a_1 - \left( \frac{3}{2} + a_2 B \right) \epsilon \right]\|u_t\|_2^2. \]
Now, if we pick \( \epsilon \leq \min \left\{ \frac{a_1}{a_2 c(\delta)}, \frac{a_1}{3/2 + a_2 B} \right\} \), we see that \( H'(t) \leq -\epsilon E(t) \). Now recall that
\[ |H(t) - E(t)| \leq \epsilon B^2 E. \]
It follows that \( E(t) \geq \frac{1}{1 + \epsilon B^2} H(t) \), and hence
\[ H'(t) \leq \left( -\epsilon \frac{1}{1 + \epsilon B^2} \right) H(t). \]
Calculus thus gives us that there exists a positive real number $r$ such that
\begin{equation}
H(t) \leq H(0)e^{-rt}.
\end{equation}
Now, recalling (3.14), we see that
\begin{equation}
E(t) \leq \frac{H(t)}{(-\epsilon B^2 + 1)},
\end{equation}
provided that $\epsilon$ is chosen small enough so that $-\epsilon B^2 + 1 > 0$. It follows that if we pick $\epsilon$ small enough, we have
\begin{equation}
E(t) \leq \frac{H(t)}{-\epsilon B^2 + 1} \leq \frac{H(0)}{-\epsilon B^2 + 1} e^{-rt},
\end{equation}
with $-\epsilon B^2 + 1 > 0$. The theorem follows.

References

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