ON THE EXISTENCE OF THE GREEN FUNCTION FOR ELLIPTIC SYSTEMS IN DIVERGENCE FORM

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Abstract. We study the existence of the Green function for an elliptic system in divergence form \(-\nabla \cdot a \nabla u = 0\) in \(\mathbb{R}^d\), with \(d > 2\). The tensor field \(a = a(x)\) is only assumed to be bounded and \(\lambda\)-coercive. For almost every point \(y \in \mathbb{R}^d\), the existence of a Green’s function \(G(a; \cdot, y)\) centered in \(y\) has been proven in [2]. In this paper we show that the set of points \(y \in \mathbb{R}^d\) for which \(G(a; \cdot, y)\) does not exist has zero \(p\)-capacity, for an exponent \(p > 2\) depending only on the dimension \(d\) and the ellipticity ratio of \(a\).

This paper is an extension of [2] and further investigates the existence of a Green’s function for the second-order elliptic operator \(-\nabla \cdot a \nabla\) in \(\mathbb{R}^d\), with \(d > 2\). We focus on the case of systems, namely when \(a\) is a measurable tensor field \(a : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^{m \times d}; \mathbb{R}^{m \times d})\), with \(m\) being any positive integer. We stress that in this paper we do only assume that \(a\) is bounded and \(\lambda\)-coercive, i.e. that there exists \(\lambda > 0\) such that

\[
\forall \zeta \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^m) \quad \int \nabla \zeta(x) \cdot a(x) \nabla \zeta(x) \, dx \geq \lambda \int |\nabla \zeta(x)|^2 \, dx,
\]

\[
\forall x \in \mathbb{R}^d, \forall \xi \in \mathbb{R}^{m \times d} \quad |a(x)\xi| \leq |\xi|.
\]

In [2], J. Conlon and the authors show that a Green’s function \(G(a; \cdot, y)\) centered in \(y\), exists for every coefficient field \(a\) satisfying (0.1) and for (Lebesgue-)almost every point \(y \in \mathbb{R}^d\). In this paper, we improve this result by showing that the \emph{exceptional set} \(\Sigma\) of points \(y \in \mathbb{R}^d\) for which \(G(a; \cdot, y)\) may not exist has \(p\)-capacity zero, for an exponent \(p > 2\) depending only on the dimension \(d\) and the ellipticity ratio \(\lambda\). This, in particular, implies that for every coefficient field \(a\) that is \(\lambda\)-coercive and bounded, the Hausdorff dimension of \(\Sigma\) is strictly smaller than \(d - 2\) [4][Theorem 4.17].

The result of [2] crucially relies on the idea of studying the Green function as a map \(G(a; \cdot, \cdot)\) in both variables \(x, y \in \mathbb{R}^d\). This yields optimal estimates for the \(L^2\)-norm in \(y\) and \(x\) of \(G(a; \cdot, \cdot), \nabla_x G(a; \cdot, \cdot)\) and \(\nabla_x \nabla_y G(a; \cdot, \cdot)\) both away from the diagonal \(\{x = y\}\) and close to it. By the standard properties of Lebesgue-integrable functions, these estimates allow to give a pointwise meaning in \(y\) to \(G(a; \cdot, y)\), up to a set of Lebesgue-measure zero. The main idea behind the result of this paper is to exploit the integrability of the mixed derivatives \(\nabla_y \nabla_x G(a; \cdot, \cdot)\) and extend the set of Lebesgue points \(y\) where \(G(a; \cdot, y)\) is well-defined up to the set \(\Sigma\) having zero \(p\)-capacity.

We remark that in the case of elliptic systems the set \(\Sigma\) is expected to be non-trivial. There are, indeed, coefficient fields \(a\) satisfying (0.1) for which one may construct unbounded \(a\)-harmonic vector fields. From this, and by means of representation formulas, it follows that the points where such vector fields are unbounded cannot be Lebesgue points for \(G(a; \cdot, y)\). A classical example of a discontinuous \(a\)-harmonic vector field is due to E. De Giorgi [3]: For any dimension \(d > 2\), the vector field \(u : \mathbb{R}^d \to \mathbb{R}^d\)

\[
u(x) = \frac{x}{|x|^\gamma}, \quad \gamma := \frac{d}{2} \left( 1 - \frac{1}{\sqrt{2d - 2}} \right) > 1
\]

solves \(-\nabla \cdot a_0 \nabla u = 0\) in \(\mathbb{R}^d\), with \(a_0\) satisfying (0.1) and being smooth everywhere outside of the origin. We remark that the coefficient \(a_0\) is not only \(\lambda\)-coercive as in (0.1), but also strongly
elliptic: For almost every \( x \in \mathbb{R}^d \) and every matrix \( \xi \in \mathbb{R}^{d \times d} \), it satisfies \( \xi \cdot a_0(x) \xi \geq \lambda |\xi|^2 \), with \( \lambda \) depending on \( d \).

In the case \( d = 3 \), the previous example implies that the exceptional set \( \Sigma \) for \( a_0 \) contains at least the origin. For higher dimensions \( d \geq 3 \), the trivial extension of the vector field \( u \) for \( d = 3 \) is itself \( \tilde{a}_0 \)-harmonic if

\[
\tilde{a}_0 := \begin{pmatrix} a_0 & 0 \\ 0 & I \end{pmatrix}.
\]

This implies, in particular, that \( \Sigma \) for \( \tilde{a}_0 \) has Hausdorff dimension at least \( d - 3 \).

The previous counterexample also implies that for (locally) \( a \)-harmonic vector fields one may only aim at statements on their partial regularity as, for instance, their continuity outside of a singular set. We remark that there exist examples of discontinuous \( a \)-harmonic vector fields with discontinuity much larger than (0.2): We refer, for instance, to the paper by J. Soucek [10], which exhibits an \( a \)-harmonic vector field discontinuous on a dense countable set, and the one by O. John, J. Malý and J. Stará [6], in which, for every countable union of closed sets, an \( a \)-harmonic vector field discontinuous there is constructed.

Without using the equation, the fact that \( a \)-harmonic functions are locally in \( H^1 \) immediately implies that they are 2-quasicontinuous. This means that there exists a set, having 2-capacity which can be chosen arbitrarily small, outside of which the function considered is continuous [4][Definition 4.11]. This argument is oblivious to the difference between scalar and vectorial functions. By using the equation and appealing to Meyers’s [9] or Gehring’s [5] estimates, this notion of continuity may be upgraded from 2-quasicontinuity to \( p \)-quasicontinuity, for an exponent \( p > 2 \). The result of this paper provides an analogous statement for the solution operator for \( -\nabla \cdot a \nabla \). By means of representation formulas, indeed, we prove that for any family \( \mathcal{F} \) of locally \( a \)-harmonic functions that are uniformly bounded in the \( H^1_{loc} \)-norm, there exists a universal set of zero \( p \)-capacity outside of which \( \mathcal{F} \) is equicontinuous (see Corollary 1).

This set is universal in the sense that it depends only on the coefficient \( a \) and on the dimension \( d \), but not on the family \( \mathcal{F} \).

Notation and previous results. For the sake of simplicity, throughout the paper we use a scalar notation by pretending that \( a \) is a matrix field and that the Green function is scalar. For a detailed discussion about this abuse of notation, we refer to [2, Section 2]. Moreover, again for notational convenience, as in [2] we assume that \( a \) is symmetric, i.e. that for almost every \( x \in \mathbb{R}^d \), the the tensor \( a(x) \) is symmetric. If we denote the elements of the product space \( \mathbb{R}^d \times \mathbb{R}^d \) by \((x, y)\), we use the notation \( W^{1,q}_x(a, d) \) to specify in the lower index the differentiation and integration variable. Similarly, we write \( \nabla_x, \nabla_y \) or \( \nabla_{x,y} \) when the gradient is taken with respect to \( x, y \) or both variables \((x, y)\), respectively. We denote by \( W^{1,p}(\mathbb{R}^d, \mathbb{R}^m) \), \( p \geq 1 \) the Sobolev spaces of functions in \( \mathbb{R}^d \) taking values in \( \mathbb{R}^m \); if \( m = 1 \), we use the usual notation \( W^{1,p}(\mathbb{R}^d) \). The same criteria are employed for all the other standard functions spaces used in the paper.

For an open set \( D \subseteq \mathbb{R}^d \), we may define the space

\[
Y^{1,2}(D) := \{ u \in L^2^*(D) : \nabla u \in L^2(D; \mathbb{R}^d) \},
\]

with \( 2^* := \frac{2d}{d-2} \) and equip it with the norm \( \| u \|_{Y^{1,2}(D)} := \| u \|_{L^2(D)} + \| \nabla u \|_{L^2(D)} \). The main theorem of [2, Theorem 1] provides the existence of a map

\[
G(a; \cdot, \cdot) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{m \times m}
\]

such that for all \( 1 \leq q < \frac{d}{d-1} \) and \( r > 0 \)

\[
G(a; \cdot, \cdot) \in W^{1,q}_{loc}(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^{m \times m}) \cap Y^{1,2}(\{|x - y| > r\}; \mathbb{R}^{m \times m})
\]

and for almost every \( y \in \mathbb{R}^d \) it holds (in the weak sense)

\[
-\nabla \cdot a \nabla G(a; \cdot, y) = \delta(\cdot - y) \quad \text{in } \mathbb{R}^d.
\]
Furthermore, the matrix-field $G(a; \cdot, \cdot)$ is unique in the class of fields $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{m \times m}$ solving (0.5) for almost every $y \in \mathbb{R}^d$ and satisfying for some $\frac{d}{2} - 1 < \alpha_0 < \frac{d}{2}$, every $z \in \mathbb{R}^d$ and $R > 0$

\[ \int_{|y-z|<R} \int_{|x-z|<R} |x-y|^{2\alpha_0} |\nabla_{x,y} G(a; x, y)|^2 dx \, dy < +\infty, \tag{0.6} \]
\[ \int_{|y-z|<R} \int_{|x-z|>2R} |\nabla_x G(a; x, y)|^2 dx \, dy < +\infty. \tag{0.7} \]

1. Main result

**Theorem 1.** Let $a$ be symmetric and satisfy assumptions (0.1). Let $G : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{m \times m}$ be the Green function in the sense of [2] and constructed there. Then, there exists a (measurable) set $\Sigma = \Sigma(a) \subseteq \mathbb{R}^d$ with the following properties:

(a) There exists an exponent $p = p(d, \lambda) > 2$ such that

\[ p\text{-cap}(\Sigma) = 0. \]

(b) There exists an exponent $q = q(d, \lambda) > 1$ such that for every $z \in \mathbb{R}^d \setminus \Sigma$ and for all $r > 0$

\[ \int_{|y-z|<\delta} G(a; \cdot, y) dy \to G^*(a; \cdot, z), \quad \delta \downarrow 0^+ \]

in $W^{1,q}_{\text{loc},x}(\mathbb{R}^d; \mathbb{R}^{m \times m}) \cap Y^{1,2}(\{x: |x - z| > r\}; \mathbb{R}^{m \times m})$.

(c) The representative $G^*(a; \cdot, y)$ solves equation (0.5) for every $y \in \mathbb{R}^d \setminus \Sigma$. Furthermore, there exists an exponent $\alpha < \frac{d}{2}$ such that

\[ \int \min\{|x-y|^\alpha, 1\}^2 |\nabla G^*(a; x, y)|^2 dx < +\infty. \]

In addition, as a corollary we have:

**Corollary 1.** Let the coefficient field $a$ and the exponent $p > 2$ be as in Theorem 1. Let

\[ \mathcal{F} := \left\{ a : u \text{ is } a\text{-harmonic in } \{|x| < 4\}, \int_{|x|<4} |\nabla u|^2 \leq 1 \right\}. \]

Then $\mathcal{F}$ is uniformly $p$-quasicontinuous in $\{|x| < 1\}$. More precisely, for every $\varepsilon > 0$ there exists an open set $U^\varepsilon \subseteq \{|x| < 1\}$ having

\[ p\text{-cap}(U^\varepsilon) < \varepsilon \]

such that $\mathcal{F}$ is uniformly equicontinuous in $\{|x| < 1\} \setminus U^\varepsilon$.

2. Proofs

Throughout this whole section we fix the coefficient field $a$ and drop the argument $a$ in the notation for $G$, $\nabla_{x,y} G$ and $\nabla_x \nabla_y G$. We write $\gtrsim$ and $\lesssim$ for $\geq C$ and $\leq C$ with the constant depending only on the dimension $d$, the ellipticity ratio $\lambda$ and the dimension of the target space $m$. Finally, for a function $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, we introduce the notation

\[ \mathcal{M} f(y) := \sup_{\delta > 0} \int_{|y-y|<\delta} |f(\tilde{y})| \, d\tilde{y}, \tag{2.8} \]
\[ \mathcal{M}_0 f(y) := \limsup_{\delta \downarrow 0^+} \int_{|y-y|<\delta} |f(\tilde{y})| \, d\tilde{y}. \tag{2.9} \]

Before giving the proof of Theorem 1, we recall some of the main properties of $G(\cdot, \cdot)$ obtained in [2, Section 2] which will be crucially used in our proofs:
For every $R > 0$, $z \in \mathbb{R}^d$ and $\alpha > \frac{d}{2} - 1$ it holds
\[
\int_{|y - z| < R} \int_{|x - z| < R} |x - y|^{2\alpha} |\nabla_{x,y} G(x, y)|^2 \, dx \, dy \lesssim C(\alpha) R^{2 + 2\alpha},
\] (2.10)
\[
\int_{|y - z| < R} \int_{|x - z| > 2R} |\nabla_x G(x, y)|^2 \, dx \, dy \lesssim R^2,
\] (2.11)
\[
\int_{|y - z| < R} \int_{|x - z| > 2R} |\nabla_y \nabla_x G(x, y)|^2 \, dx \, dy \lesssim 1.
\] (2.12)

For every $R > 0$, $z \in \mathbb{R}^d$ and almost every $y \in \mathbb{R}^d$ such that $|y - z| > 2R$,
\[
-\nabla_x \cdot a \nabla_x \nabla_y G(\cdot, y) = 0 \quad \text{in } \{|x - z| < R\}.
\] (2.13)

For almost every $x, y \in \mathbb{R}^d$
\[
G(x, y) = G(y, x).
\] (2.14)

For every $g \in L^2(\mathbb{R}^d; \mathbb{R}^{m \times d})$ and every $f \in L^2(\mathbb{R}^d; \mathbb{R}^m)$ having compact support, the solution $u \in Y^{1,2}(\mathbb{R}^d; \mathbb{R}^m)$ to
\[
-\nabla \cdot a \nabla u = \nabla \cdot g + f \quad \text{in } \mathbb{R}^d
\]
may be written as the identity (up to a set of Lebesgue measure zero)
\[
u(\cdot) = \int \nabla_y G(\cdot, y) g(y) \, dy + \int G(\cdot, y) f(y) \, dy.
\] (2.15)

**Proof of Theorem 1.** We divide the proof into steps: In Step 1 we give a formulation of the standard Gehring’s estimate tailored for our needs. Roughly speaking, this allows to upgrade estimate (2.12) into an $L^2$-estimate in $x$ and $L^p$ in $y$, for the Gehring exponent $p > 2$. Steps 2-4 contain the main capacitary estimates for the exceptional set $\Sigma$, which is closely related to the set of points $y \in \mathbb{R}^d$ where $G(\cdot, y)$ and $\nabla G(\cdot, y)$ have infinite $W^{1,2}_{1,0} \cap Y^{1,2}(\{ |x - y| > 1 \})$-norms. These estimates on the capacity of $\Sigma$ crucially rely on the upgraded version of (2.12) and are combined with a maximal function estimate for Sobolev functions. Finally, in Step 5 we argue how to construct the representative $G^* (\cdot, y)$ away from the singularity set $\Sigma$.

**Step 1. Gehring’s estimate.** Let $u \in H^1(\{|x| < 2R\}; \mathbb{R}^m)$ be a solution to
\[
-\nabla \cdot a \nabla u = \nabla \cdot g \quad \text{in } \{|x| < 2R\}.
\] (2.16)
Then, there exists an exponent $p = p(d, \lambda) > 2$ such that
\[
\left( \int_{|x| < R} |\nabla u(x)|^p \, dx \right)^{\frac{1}{p}} \lesssim \left( \int_{|x| < 2R} |\nabla u(x)|^2 \, dx \right)^{\frac{1}{2}} + \left( \int_{|x| < 2R} |g(x)|^p \, dx \right)^{\frac{1}{p}}.
\] (2.17)
This is a standard result in elliptic regularity theory and we refer to [5, Chapter V, Theorem 2.1] for its proof.\footnote{In [5] the coefficients are assumed to be very strongly elliptic. However, the argument only relies on Caccioppoli’s and Poincaré-Sobolev’s inequality which hold true also if $a$ is assumed to be only $\lambda$-coercive as in (0.1). Moreover, [5][Inequality (0.2)] corresponds to the standard case of $a$-harmonic functions; our case is an immediate adaptation of the Caccioppoli’s inequality in the case of solutions to (2.16).} We pick a (smooth) cut-off function $\eta$ for $\{|x| < R\}$ in $\{|x| < 2R\}$. Since for any $p \leq \frac{2d}{d-2}$, Poincaré-Sobolev inequality yields
\[
\left( \int_{|x| < 2R} |\eta(x) u(x)|^p \, dx \right)^{\frac{1}{p}} \leq R \left( \int_{|x| < 2R} |\nabla (\eta(x) u(x))|^2 \, dx \right)^{\frac{1}{2}},
\]
it follows that
\[
R^{-1}\left(\int_{|x|<R} |u(x)|^p \, dx\right)^{\frac{1}{p}} + \left(\int_{|x|<R} |\nabla u(x)|^p \, dx\right)^{\frac{1}{p}} \\
\lesssim R^{-1}\left(\int_{|x|<2R} |u(x)|^2 \, dx\right)^{\frac{1}{2}} + \left(\int_{|x|<2R} |\nabla u(x)|^2 \, dx\right)^{\frac{1}{2}} + \left(\int_{|x|<2R} |g(x)|^p \, dx\right)^{\frac{1}{p}}.
\] (2.18)

**Step 2. Capacity estimates: First reduction.** Let \(\mathcal{M}_0\) be as in definition (2.9). We claim that if there exists an exponent \(\alpha(d, \lambda) < \frac{d}{2}\) such that for every \(R > 0\)

\[
p\text{-cap}\left\{ |y - z| < 1 : \mathcal{M}_0 \left(\int_{|x-y| < R} |x-y|^{2\alpha} |\nabla G(x, y)|^2 \, dx\right)^{\frac{1}{2}} = +\infty \right\} = 0,
\] (2.19)

\[
p\text{-cap}\left\{ |y - z| < 1 : \mathcal{M}_0 \left(\int_{|x-y| < R} |x-y|^{2\alpha} |G(x, y)|^2 \, dx\right)^{\frac{1}{2}} = +\infty \right\} = 0,
\] (2.20)

then

\[
p\text{-cap}\left\{ y \in \mathbb{R}^d : \mathcal{M}_0 \left(\int_{|x-y| < 1} |x-y|^{2\alpha} |\nabla G(x, y)|^2 \, dx\right)^{\frac{1}{2}} = +\infty \right\} = 0
\] (2.21)

and we may also find an exponent \(q = q(d, \lambda) > 1\) such that for every \(R > 0\)

\[
p\text{-cap}\left\{ y \in \mathbb{R}^d : \mathcal{M}_0 \|G(\cdot, y)\|_{W^{1,q}_p(\{x-y| < R\}; \mathbb{R}^m)} = +\infty \right\} = 0,
\] (2.22)

\[
p\text{-cap}\left\{ y \in \mathbb{R}^d : \mathcal{M}_0 \|G(\cdot, y)\|_{Y^{1,2}_p(\{x-y| > R\}; \mathbb{R}^m)} = +\infty \right\} = 0.
\] (2.23)

Indeed, since we may cover the whole space \(\mathbb{R}^d\) with a countable number of unit balls, the subadditivity of the capacity and (2.19), (2.20) immediately imply

\[
p\text{-cap}\left\{ y \in \mathbb{R}^d : \mathcal{M}_0 \left(\int_{|x-y| < R} |x-y|^{2\alpha} |\nabla G(x, y)|^2 \, dx\right)^{\frac{1}{2}} = +\infty \right\} = 0,
\] (2.24)

\[
p\text{-cap}\left\{ y \in \mathbb{R}^d : \mathcal{M}_0 \left(\int_{|x-y| < R} |x-y|^{2\alpha} |G(x, y)|^2 \, dx\right)^{\frac{1}{2}} = +\infty \right\} = 0.
\]

Estimate (2.21) is an immediate consequence of the first identity above. Similarly, also (2.23) easily follows from the first identity above and Sobolev’s inequality for the exterior domain \(\{x : |x-y| > R\}\). Since for \(\alpha < \frac{d}{2}\), H"older’s inequality implies that there exists \(1 < q < 2\) such that for any \(u\)

\[
\int_{|x-y| < R} |u(x)|^q \, dx \leq C(R) \left(\int_{|x-y| < R} |x-y|^{2\alpha} |u(x)|^2 \, dx\right)^{\frac{q}{2}}
\]

estimate (2.22) is implied by this inequality together with both identities (2.24).

**Step 3. Capacity estimates: Second reduction.** We now further argue that for (2.19)-(2.20) it suffices to prove that for every \(r > 0\) and all \(\lambda > 0\)

\[
p\text{-cap}\left\{ |y - z| < \frac{r}{2} : \mathcal{M}_0 \left(\int_{|x-y| > 8r} |\nabla G(x, y)|^2 \, dx\right)^{\frac{1}{2}} > \lambda \right\} \lesssim \lambda^{-p} r^{-\frac{d}{2} + \frac{d}{q} + \frac{1}{2}}.
\] (2.25)

\[
p\text{-cap}\left\{ |y - z| < \frac{r}{2} : \mathcal{M}_0 \left(\int_{8r < |x-y| < 16r} |G(x, y)|^2 \, dx\right)^{\frac{1}{2}} > \lambda \right\} \lesssim \lambda^{-p} r^{-\frac{d}{2} + \frac{d}{q} + 1}.
\]

We show that the first inequality in (2.25) implies (2.19); the argument for (2.20) relying on the second inequality in (2.25) is analogous.
Without loss of generality, we prove (2.19) in the case $z = 0$: For any $0 < r \leq 1$ fixed, we may cover the set $\{|y| < 1\}$ by $N \lesssim r^{-d}$ balls of radius $r$ having centres $\{z_i\}_{i=1}^N$; the subadditivity of the capacity and the first estimate in (2.25) with $z = z_i, i \in \{1, \cdots, N\}$ imply that for all $\lambda > 0$

$$
p-cap \left\{|y| < 1 : M_0 \left( \int_{|x-y|>8r} |\nabla_x G(x,y)|^2 \, dx \right)^{\frac{1}{p}} > \lambda \right\} \lesssim \lambda^{-p_r^{-\frac{2d}{p} + \frac{2}{p}}}. \tag{2.26}
$$

If we now choose $r = \frac{1}{8}$ and send $\lambda \uparrow +\infty$, we obtain

$$
p-cap \left\{|y| < 1 : M_0 \left( \int_{|x-y|>1} |\nabla G(x,y)|^2 \, dx \right)^{\frac{1}{p}} = +\infty \right\} = 0. \tag{2.27}
$$

Since the exponent $\alpha$ in (2.19) is positive, to conclude (2.19) it only remains to show that

$$
p-cap \left\{|y| < 1 : M_0 \left( \int_{|x-y|<1} |x-y|^{2\alpha} |\nabla G(x,y)|^2 \, dx \right)^{\frac{1}{p}} = +\infty \right\} = 0. \tag{2.28}
$$

To do this, we smuggle the weight $r^{\alpha}$ in the l. h. s. of inequality (2.26) to get

$$
p-cap \left\{|y| < 1 : M_0 \left( r^{2\alpha} \int_{|x-y|>8r} |\nabla G(x,y)|^2 \, dx \right)^{\frac{1}{p}} > \lambda r^{\alpha} \right\} \lesssim \lambda^{-p_r^{-\frac{2d}{p} + \frac{2}{p}}}. \tag{2.29}
$$

By redefining $r^{\alpha} \lambda$ as $\lambda$ and reducing the domain of integration from $\{|x-y| > 8r\}$ to $\{8r < |x-y| < 16r\}$, we further obtain

$$
p-cap \left\{|y| < 1 : M_0 \left( r^{2\alpha} \int_{8r<|x-y|<16r} |\nabla G(x,y)|^2 \, dx \right)^{\frac{1}{p}} > \lambda \right\} \lesssim \lambda^{-p_r^{-\frac{2d}{p} + \frac{2}{p} + \alpha p}}. \tag{2.30}
$$

Since

$$\int_{8r<|x-y|<16r} |x-y|^{2\alpha} |\nabla G(x,y)|^2 \, dx \lesssim r^{2\alpha} \int_{8r<|x-y|<16r} |\nabla G(x,y)|^2 \, dx,$$

we conclude

$$
p-cap \left\{|y| < 1 : M_0 \left( \int_{8r<|x-y|<16r} |x-y|^{2\alpha} |\nabla G(x,y)|^2 \, dx \right)^{\frac{1}{p}} > \lambda \right\} \lesssim \lambda^{-p_r^{-\frac{2d}{p} + \frac{2}{p} + \alpha p}}. \tag{2.31}
$$

We now define

$$A := \left\{|y| < 1 : M_0 \left( \int_{|x-y|<1} |x-y|^{2\alpha} |\nabla G(x,y)|^2 \, dx \right)^{\frac{1}{p}} > \lambda \right\} \tag{2.32}
$$

and, given a sequence of weights

$$\{\omega_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}_+ \text{ such that } \sum_n \omega_n \leq 1, \tag{2.33}
$$

the sets

$$B_n := \left\{|y| < 1 : M_0 \left( \int_{2^{-n}<|x-y|<2^{-n+1}} |x-y|^{2\alpha} |\nabla G(x,y)|^2 \, dx \right)^{\frac{1}{p}} > \omega_n \lambda \right\}, \tag{2.34}
$$

for $n \in \mathbb{N}$. We claim that

$$A \subseteq \bigcup_n B_n. \tag{2.35}
$$

This can be easily seen by proving the complementary statement $\cap_n B_n^c \subseteq A^c$: Indeed, if for all $n \in \mathbb{N}$

$$M_0 \left( \int_{2^{-n}<|x-y|<2^{-n+1}} |x-y|^{2\alpha} |\nabla G(x,y)|^2 \, dx \right)^{\frac{1}{p}} \leq \omega_n \lambda,$$
then the inclusion of the sequence spaces $\ell^1 \subseteq \ell^2$, the sublinearity of the operator $M_0$ and assumption (2.31) yield
\[
M_0 \left( \int_{|x-y|<2^{-n}} |x-y|^{2\alpha} |\nabla G(x,y)|^2 \, dx \right)^{\frac{1}{2}} 
\leq \sum_{n\in\mathbb{N}} M_0 \left( \int_{2^{-n}<|x-y|<2^{-n+1}} |x-y|^{2\alpha} |\nabla G(x,y)|^2 \, dx \right)^{\frac{1}{2}} \leq \lambda \sum_{n\in\mathbb{N}} \omega_n^{(2.31)} \leq \lambda.
\]
We thus established (2.33).

By (2.33) and the subadditivity of the capacity we get
\[
p\text{-}\cap(A) \leq \sum_n p\text{-}\cap(B_n),
\]
and, recalling definition (2.32), we use estimate (2.29) with $\lambda$ and $r$ substituted by $\omega_n \lambda$ and $2^{-n+3}$ to bound
\[
p\text{-}\cap(A) \leq \lambda^{-p} \sum_n \omega_n^{-p} 2^{-n(-\frac{d}{2} + \frac{d}{p} + \alpha p)}.
\]
By choosing in (2.31) $\omega_n = \frac{6}{(M_n)^p}$, the sum on the right-hand side converges provided that the exponent $\alpha$ satisfies $\alpha > (\frac{3}{2} - \frac{1}{p}) \frac{d}{p}$. Since $p > 2$ by Step 1, there exists $\alpha < \frac{d}{2}$ such that
\[
p\text{-}\cap(A) \leq \lambda^{-p}.
\]
By definition (2.30), if we send $\lambda \uparrow +\infty$ we recover (2.28). Together with (2.27), this allows us to conclude the proof for (2.19) and of Step 3.

**Step 4. Maximal function estimate.** We now prove (2.25) and begin with the first estimate. Without loss of generality, we focus on the case $z = 0$. For any $0 < r \leq 1$, let
\[
F_r(y) := \left( \int_{|x|>4r} |\nabla_x G(x,y)|^2 \, dx \right)^{\frac{1}{2}}.
\]
We first claim that it suffices to show that for every $r > 0$
\[
\left( \int_{|y|<r} \left( r^{-p} |F_r(y)|^p + |\nabla F_r(y)|^p \right) \, dy \right)^{\frac{1}{p}} \lesssim r^{-\frac{d}{2} + \frac{d}{p}}.
\]
Indeed, if $\eta_r$ is a smooth cut-off function for $\{|y| < \frac{r}{2}\}$ in $\{|y| < r\}$, then by (2.35) the function $\eta_r F_r$ satisfies
\[
\|\eta_r F_r\|_{W^{1,p}(\mathbb{R}^d)} \overset{(2.35)}{\lesssim} r^{-\frac{d}{2} + \frac{d}{p}}.
\]
We thus apply the maximal function estimate[7, Inequality (3.1)] to $\eta_r F_r$ and infer that for every $\lambda > 0$
\[
p\text{-}\cap\left\{|y| < r : M(\eta_r F_r)(y) > \lambda\right\} \lesssim \lambda^{-p} r^{-\frac{d}{2} + \frac{d}{p}},
\]
where $M$ is defined in (2.8). Since by the assumption on $\eta_r$ and definitions (2.8)-(2.9) we have
\[
M_0 F_r \leq M(\eta_r F_r)
\]
on $\{|y| < \frac{r}{2}\}$, the monotonicity of the capacity also yields
\[
p\text{-}\cap\left\{|y| < \frac{r}{2} : M_0 F_r(y) > \lambda\right\} \lesssim \lambda^{-p} r^{-\frac{d}{2} + \frac{d}{p}}.
\]
Furthermore, since when $|y| < \frac{x}{2}$ we have $\{|x-y| > 8r\} \subseteq \{|x| > 4r\}$ and thus
\[
\left( \int_{|x-y|>8r} |\nabla_x G(x,y)|^2 \, dx \right)^{\frac{1}{2}} \leq F_r(y),
\]
again by monotonicity, we conclude (2.25) for $\nabla_x G$ from (2.36).

To complete the argument for the first line in (2.25) it remains to prove (2.35): The main ingredient for this are inequalities (2.11) and (2.12) which, by setting $R = 2r$ and $z = 0$, we rewrite as
\[
\int_{|x|>4r} \int_{|y|<2r} \left( |\nabla_y \nabla_x G(x,y)|^2 + r^{-2} |\nabla_x G(x,y)|^2 \right) \, dx \, dy \lesssim 1.
\]
Since by (2.14) and (2.13) the vector field $\nabla_x G(x, \cdot)$ is $a$-harmonic in $\{|y| < 2r\}$ for almost every $x$ such that $|x| > 4r$, we apply (2.18) of Step 1 and upgrade the previous estimate to
\[
\int_{|x|>4r} \left( \int_{|y|<r} |\nabla_y \nabla_x G(x,y)|^p \, dy \right)^{\frac{1}{p}} \, dx \lesssim r^{-d+\frac{2d}{p}}, \tag{2.37}
\]
\[
\int_{|x|>4r} \left( \int_{|y|<r} |\nabla_x G(x,y)|^p \, dy \right)^{\frac{1}{p}} \, dx \lesssim r^{-d+\frac{2d}{p}+2}, \tag{2.38}
\]
with $p > 2$ from Step 1.

If we differentiate the right-hand side of (2.34) in $y$, the chain rule and an application of Cauchy-Schwarz’s inequality yield
\[
|\nabla F_r(y)|^2 \leq \int_{|x|>4r} |\nabla_y \nabla_x G(x,y)|^2 \, dx, \tag{2.39}
\]
and thus
\[
\int_{|y|<r} |\nabla F_r(y)|^p \, dy \leq \int_{|y|<r} \left( \int_{|x|>4r} |\nabla_y \nabla_x G(x,y)|^2 \, dx \right)^{\frac{1}{p}} \, dy.
\]
We now apply Minkowski’s inequality to the r. h. s. and get
\[
\left( \int_{|y|<r} |\nabla F_r(y)|^p \, dy \right)^{\frac{1}{p}} \leq \left( \int_{|x|>4r} \left( \int_{|y|<r} |\nabla_y \nabla_x G(x,y)|^p \, dy \right)^{\frac{2}{p}} \, dx \right)^{\frac{1}{2}}.
\]
Thanks to (2.37), this implies (2.35) for $\nabla F_r$. We argue in favour of (2.35) for $F_r$ itself in a similar way: Using again Minkowski’s inequality and estimate (2.38) we indeed get
\[
\left( \int_{|y|<r} |F_r(y)|^p \, dy \right)^{\frac{1}{p}} \lesssim r^{-\frac{d}{2}+\frac{d}{p}+1}.
\]
This concludes the proof of (2.35) and of (2.25) for $\nabla G$.

Estimate (2.25) for $G$ follows by a similar argument. This time, we define the family of functions
\[
F_r(y) = \left( \int_{8r<|x|<16r} |G(x,y)|^2 \, dx \right)^{\frac{1}{2}}.
\]
Also this definition of $F_r$ satisfies inequality (2.35), this time with the r.h.s. being $r^{-\frac{d}{2}+\frac{d}{p}+1}$. Similarly to (2.39), it holds indeed that
\[
|\nabla F_r(y)|^2 \leq \left( \int_{8r<|x|<16r} |\nabla_y G(x,y)|^2 \, dx \right)^{\frac{1}{2}}. \tag{2.40}
\]
In addition, by Poincaré’s inequality in \{8r < |x| < 16r\} and Sobolev’s inequality in the exterior domain \{|x| > 8r\} we bound
\[
\int_{|y|<r} |F_r(y)|^p \, dy \leq r^p \int_{|y|<r} \left( \int_{8r<|x|<16r} |G(x,y)|^{2p} \, dx \right)^{\frac{p-2}{2}} \, dy
\]
\[
\leq r^p \int_{|y|<r} \left( \int_{8r<|x|<16r} |\nabla G(x,y)|^2 \, dx \right)^{\frac{p}{2}} \, dy,
\]
and also
\[
\int_{|y|<r} |\nabla F_r(y)|^p \, dy \overset{(2.40)}{\leq} \int_{|y|<r} \left( \int_{8r<|x|<16r} |\nabla_y G(x,y)|^2 \, dx \right)^{\frac{p}{2}} \, dy
\]
\[
\leq r^p \int_{|y|<r} \left( \int_{8r<|x|<16r} |\nabla_y G(x,y)|^{2p} \, dx \right)^{\frac{2-2p}{2}} \, dy
\]
\[
\leq r^p \int_{|y|<r} \left( \int_{|x|>8r} |\nabla \nabla G(x,y)|^2 \, dx \right)^{\frac{p}{2}} \, dy.
\]
By appealing to (2.38) and (2.37), from these two inequalities we obtain the analogue (2.35) for this definition of \(F_r\). We may now pass from (2.35) to (2.25) as done in the case of \(\nabla G\). This concludes the proof of Step 4.

**Step 5. Construction of** \(G^*(a;\cdot;\cdot)\). By wrapping up Steps 2-4, we have that \(G\) and \(\nabla G\) satisfy (2.25) and therefore also \((2.19)-(2.20)\) and \((2.22)-(2.23)\). Equipped with these identities, we now proceed to prove the existence of \(G^*(\cdot;y)\) for \(y\) outside an exceptional set \(\Sigma\) satisfying (a) in the statement of Theorem 1.

For a test function \(\zeta \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^m)\), we consider the vector field
\[
u(y) = \int G(x,y)\zeta(x) \, dx.
\]
By symmetry (2.14) and the representation formula (2.15), \(u \in Y^{1,2}(\mathbb{R}^d; \mathbb{R}^m)\) and (weakly) solves
\[
-\nabla \cdot \mathbf{a} \nabla u = \zeta \quad \text{in } \mathbb{R}^d.
\]
Since \(\zeta \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^m)\), Gehring’s estimate (2.18) imply that \(u \in W_{loc}^{1,p}(\mathbb{R}^d; \mathbb{R}^m)\) for \(p > 2\). By Lebesgue’s theorem for Sobolev functions [8, Theorem 2.55], we infer that
\[
\lim_{\delta \downarrow 0} \int_{|\tilde{y} - y| < \delta} u(\tilde{y}) \, d\tilde{y} = \lim_{\delta \downarrow 0} \int_{|\tilde{y} - y| < \delta} \zeta(x) G(x,\tilde{y}) \, dx \, d\tilde{y}
\]
exists as an element of \(\mathbb{R}^m\) for all \(y \in \mathbb{R}^d\) outside a set of zero \(p\)-capacity. Since \(C_0^\infty(\mathbb{R}^d; \mathbb{R}^m)\) is separable, we may restrict ourselves to a countable subset \(\{\zeta_n\}_{n \in \mathbb{N}} \subseteq C_0^\infty(\mathbb{R}^d; \mathbb{R}^m)\) dense w. r. t. the \(C^1\) topology. Hence, there exists a set \(\tilde{\Sigma}\), with \(p\text{-cap}(\tilde{\Sigma}) = 0\) such that
\[
\forall y \in \mathbb{R}^d \setminus \tilde{\Sigma}, \forall n \in \mathbb{N} \lim_{\delta \downarrow 0} \int_{|\tilde{y} - y| < \delta} \zeta_n(x) G(x,\tilde{y}) \, dx \, d\tilde{y} \text{ exists.}
\]
Let \(\Sigma_1\) and \(\Sigma_2\) be the \(p\)-capacity zero sets of (2.22), (2.23) in Step 2 and define
\[
\Sigma := \tilde{\Sigma} \cup \Sigma_1 \cup \Sigma_2.
\]
With this definition, \(\Sigma\) satisfies (a) of Theorem 1. By (2.19) of Step 2, we remark that may chose the previous sets to be such that for every \(y \notin \Sigma\), it also holds
\[
\mathcal{M}_0(\int (|x - y|^{2\alpha} \wedge 1)|\nabla G(x,y)|^2 \, dx) < +\infty, \quad (2.43)
\]
for the exponent \( \alpha < \frac{d}{2} \) of Step 2. By Jensen’s inequality, for every ball \( \{|x - y| < r\} \) with \( r > 0 \) and \( y \in \mathbb{R}^d \) we have

\[
\| \int_{|y - \tilde{y}| < \delta} G(\cdot, \tilde{y}) \, d\tilde{y} \|_{W^{1, q}_x((|x - y| < r); \mathbb{R}^{m \times m})} \leq \int_{|y - \tilde{y}| < \delta} \|G(\cdot, \tilde{y})\|_{W^{1, q}_x((|x - y| < r); \mathbb{R}^{m \times m})} \, d\tilde{y},
\]

so that by (2.22) and weak compactness of \( W^{1,q}_{x,loc} \), with \( q > 1 \), we infer that for every \( y \in \mathbb{R}^d \setminus \Sigma \) there exists a subsequence \( \delta_k \downarrow 0 \) (a priori depending on \( y \)) and a limit \( G^*(a; \cdot, y) \) for which

\[
\int_{|y - \tilde{y}| < \delta_k} G(\cdot, \tilde{y}) \, d\tilde{y} \rightharpoonup G^*(\cdot, y) \quad \text{in} \quad W^{1,q}_{x,loc}(\mathbb{R}^d; \mathbb{R}^{m \times m}),
\]

\[
\int_{|y - \tilde{y}| < \delta_k} \nabla G(\cdot, \tilde{y}) \, d\tilde{y} \rightharpoonup \nabla G^*(\cdot, y) \quad \text{in} \quad L^2(\{{x: |x - y| > r}\}; \mathbb{R}^{m \times m}), \quad \text{for all} \quad r > 0.
\]

Moreover, inequality (2.43) and weak lower-semicontinuity also yield

\[
\int (|x - y|^{2\alpha} \wedge 1)|\nabla G^*(x, y)|^2 \, dx < +\infty.
\]

We now show that (2.44), (2.45) hold for the entire family \( \delta \downarrow 0 \): Let us assume, indeed, that this were not the case, i.e. that there exist two sequences \( \{\delta_k^{(1)}\}_k, \{\delta_k^{(2)}\}_k \) along which we obtain in (2.44), (2.45) two different limits \( G^{(1)}(\cdot, y), G^{(2)}(\cdot, y) \). Appealing to (2.42), to Fubini’s theorem to exchange the order of the integrals, and to (2.44) we infer that for every \( n \in \mathbb{N} \)

\[
\int \zeta_n(x) G^{(1)}(x, y) \, dx = \int \zeta_n(x) G^{(2)}(x, y) \, dx.
\]

Since the subset \( \{\zeta_n\}_{n \in \mathbb{N}} \) is chosen to be dense, we conclude that \( G^{(1)}(x, y) = G^{(2)}(x, y) \) for almost every \( x \in \mathbb{R}^d \).

For every point \( y \) outside \( \Sigma \), we thus constructed a tensor field \( G^*(\cdot, y) \in W^{1,q}_{x,loc}(\mathbb{R}^d; \mathbb{R}^{m \times m}) \cap Y^{1,2}(\{{x: |x - y| > r}\}; \mathbb{R}^{m \times m}) \) for all \( r > 0 \), which is the weak limit of \( \int_{|y - \tilde{y}| < \delta} G(\cdot, \tilde{y}) \, d\tilde{y} \) and satisfies (2.46). Furthermore, since \( G(\cdot, \tilde{y}) \) solves equation (0.5) for almost every \( \tilde{y} \in \mathbb{R}^d \), for every \( \zeta \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^m) \), every \( y \in \mathbb{R}^d \) and \( \delta > 0 \) we have

\[
\int_{|y - \tilde{y}| < \delta} \nabla \zeta(x) \cdot a(x) \nabla G(x, \tilde{y}) \, dx \, \tilde{y} = \int_{|y - \tilde{y}| < \delta} \zeta(\tilde{y}) \, d\tilde{y},
\]

and by Fubini’s theorem that

\[
\int \nabla \zeta(x) \cdot a(x) \int_{|y - \tilde{y}| < \delta} \nabla G(x, \tilde{y}) \, dx \, \tilde{y} = \int \zeta(\tilde{y}) \, d\tilde{y}.
\]

By taking the limit \( \delta \downarrow 0^+ \), the assumption on \( \zeta \), the boundedness (0.1) of \( a \) and (2.44) yield that for all \( y \in \mathbb{R}^d \setminus \Sigma \) it holds

\[
\int \nabla \zeta(x) \cdot a(x) \nabla G^*(x, y) \, dx = \zeta(y).
\]

Since \( \zeta \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^m) \) is arbitrary, we conclude that \( G^*(\cdot, y) \) solves equation (0.5) and (b)-(c) of Theorem 1 by (2.44) and (2.46), respectively. The proof of Theorem 1 is therefore complete.

\[\square\]

**Proof of Corollary 1.** Let \( \Sigma \) be as in the statement of Theorem 1 and let \( \eta \) be any smooth cut-off function for \( \{|x| < 4\} \) in \( \{|x| < 2\} \). With no loss of generality we may assume that each \( u \) satisfies \( \int_{|x| < 4} u = 0 \). The vector field \( \eta u \in H^1_0(\{|x| < 4\}; \mathbb{R}^m) \subseteq Y^{1,2}(\mathbb{R}^d; \mathbb{R}^m) \) solves

\[
-\nabla \cdot a(\nabla (\eta u)) = \nabla \cdot g + f \quad \text{in} \quad \mathbb{R}^d
\]
with

\[ g := -ua \nabla \eta, \quad f := -\nabla \eta \cdot a \nabla u. \]

Both \( g \) and \( f \) are supported in \( \{2 < |x| < 4\} \) and by the definition of \( \eta \), the second inequality in (0.1), the bound on the Dirichlet energy of \( u \) and Poincaré’s inequality satisfy

\[
\int |g(x)|^2 \, dx + \int |f(x)|^2 \, dx \lesssim 1. \tag{2.51}
\]

Furthermore, the representation formula (2.15) and Theorem 1 imply that for every \( y \notin \Sigma \) with \( |y| < \frac{3}{2} \) we may define as representative

\[
u(y) = \int_{2<|x|<4} \nabla_x G^*(x,y)g(x) \, dx + \int_{2<|x|<4} G^*(x,y)f(x) \, dy.
\]

To make our notation leaner, we define

\[
v(y) := \int g(x) \cdot \nabla_x G^*(x,y) \, dx, \quad w(y) := \int f(x)G^*(x,y) \, dx. \tag{2.52}
\]

and prove the statement of the corollary for \( v \). The vector field \( w \) may be treated analogously. We adapt the proof of [4, Theorem 4.19] to show that there exist a sequence of sets \( \{B_j\}_{j\in\mathbb{N}} \subseteq \{|y| < 1\} \) having

\[
p\text{-cap}\{B_j\} < \frac{1}{2^j} \tag{2.53}
\]

and moduli of continuity \( \omega_j : \mathbb{R}_+ \to \mathbb{R}_+ \) such that the following holds: For every \( v \) constructed as above, there exists a sequence \( \{v_j\}_{j \in \mathbb{N}} \) satisfying for all \( j \in \mathbb{N} \)

\[
\sup_{\{|y| < 1\} \setminus B_j} |v(y) - v_j(y)| < \frac{1}{2^j} \tag{2.54}
\]

and

\[
|v_j(y) - v_j(\tilde{y})| \leq \omega_j(|y - \tilde{y}|) \quad \forall y, \tilde{y} \text{ s.t. } |y| < 1, |\tilde{y}| < 1. \tag{2.55}
\]

From this, the statement of the corollary follows easily: For each \( \varepsilon > 0 \) fixed, let \( j_0 \in \mathbb{N} \) such that

\[
p\text{-cap}\left(\bigcup_{j \geq j_0} B_j\right) \lesssim \frac{\varepsilon}{2}. \tag{2.53}
\]

By definition of capacity (see, for instance, [4, Theorem 4.15 (i)]) we may find an open set \( U^\varepsilon \supset \bigcup_{j \geq j_0} B_j \) having \( p\text{-cap}(U^\varepsilon) < \varepsilon \). We prove that on \( \{|y| < 1\} \setminus U^\varepsilon \), the vector fields \( v \) in (2.52) are uniformly equicontinuous. This means proving that for each \( \kappa > 0 \), there exists \( \delta = \delta(\kappa) > 0 \) such that for all \( v \) as in (2.52) and all \( y, \tilde{y} \notin U^\varepsilon \) and such that \( |y| < 1, |\tilde{y}| < 1 \) and \( |y - \tilde{y}| < \delta \) we may bound

\[
|v(y) - v(\tilde{y})| < \kappa.
\]

By the triangle inequality, (2.54) and the definition of \( U^\varepsilon \), we know indeed that if we fix \( j \geq j_0 \) such that \( 2^{-j} < \frac{\kappa}{3} \), then

\[
|v(y) - v(\tilde{y})| \leq |v(y) - v_j(y)| + |v(\tilde{y}) - v_j(\tilde{y})| + |v^j(\tilde{y}) - v_j(y)| \lesssim \frac{3}{2} \kappa + |v_j(\tilde{y}) - v_j(y)| \lesssim \frac{3}{2} \kappa + \omega_j(|y - \tilde{y}|). \tag{2.55}
\]

It thus remains to pick \( \delta \) such that the last term on the right-hand-side is smaller than \( \frac{\varepsilon}{3} \). This concludes the statement of the corollary.
We now show (2.53), (2.54) and (2.55). To do so, we begin by observing that, if \( p > 2 \) is as in Theorem 1, then the triangle inequality allows us to bound
\[
\int_{|y| < \frac{3}{2}} \left( \int_{2 < |x| < 4} |\nabla \nabla^* (x, y)|^2 \, dx \right)^{\frac{p}{2}} \, dy \\
\leq \left( \int_{2 < |x| < 4} \left( \int_{|y| < \frac{3}{2}} |\nabla \nabla^* (x, y)|^p \, dy \right)^{\frac{2}{p}} \, dx \right)^{\frac{p}{2}}.
\]
By (2.37) in the proof of Theorem 1, we further infer that
\[
\int_{|y| < \frac{3}{2}} \left( \int_{2 < |x| < 4} |\nabla \nabla^* (x, y)|^2 \, dx \right)^{\frac{p}{2}} \, dy \lesssim 1.
\]
Similarly, this time by using (2.38), we have that
\[
\int_{|y| < \frac{3}{2}} \left( \int_{2 < |x| < 4} |\nabla_x \nabla^* (x, y)|^2 \, dx \right)^{\frac{p}{2}} \, dy \\
\leq \left( \int_{2 < |x| < 4} \left( \int_{|y| < \frac{3}{2}} |\nabla_x \nabla^* (x, y)|^p \, dy \right)^{\frac{2}{p}} \, dx \right)^{\frac{p}{2}} \lesssim 1.
\]
By standard approximation arguments adapted to Banach-valued functions (see e.g. [1, Corollary 1.4.37]), we may find a sequence \( \{F_j\}_{j \in \mathbb{N}} \) of continuous maps
\[
F_j : \{|y| < \frac{3}{2}\} \to L^2(\{2 < |x| < 4\}; \mathbb{R}^{d \times m \times m})
\]
such that for each \( j \in \mathbb{N} \) we have
\[
\int_{|y| < \frac{3}{2}} \left( \int_{2 < |x| < 4} |\nabla_x \nabla^* (x, y) - F_j (x, y)|^2 \, dx \right)^{\frac{p}{2}} \, dy \\
+ \int_{|y| < \frac{3}{2}} \left( \int_{2 < |x| < 4} |\nabla_y \nabla^* (x, y) - \nabla_y F_j (x, y)|^2 \, dx \right)^{\frac{p}{2}} \, dy \leq \frac{1}{2^{(p+1)j}}. \tag{2.56}
\]
Let \( \Sigma \) be the exceptional set of Theorem 1 and let \( M_0 \) be the maximal function operator (see (2.9)). We claim that
\[
B_j := \{ y : |y| < 1, \ M_0 \left( \int_{2 < |x| < 4} |\nabla_x \nabla^* (a; x, \cdot) - F_j (x, \cdot)|^2 \, dx \right)^{\frac{p}{2}} (y) > \frac{1}{2^j} \} \cup \Sigma \tag{2.57}
\]
and \( \omega_j \) such that for all \( R > 0 \)
\[
\omega_j (R) := 4 \sup \left\{ \left( \int_{2 < |x| < 4} |F_j (x, y) - F_j (x, \tilde{y})|^2 \, dx \right)^{\frac{1}{2}} : |y| \leq 1, |\tilde{y}| \leq 1, |y - \tilde{y}| < R \right\}, \tag{2.58}
\]
satisfy (2.53), (2.54) and (2.55) provided that for every \( v \) we choose as approximating sequence
\[
v_j (y) := \int g(x) \cdot F_j (x, y) \, dx. \tag{2.59}
\]
Here, \( g \) is the vector field in the definition (2.52) of \( v \). We stress that, since each \( F_j \) is continuous in \( \{|y| \leq 1\} \) with values in \( L^2(\{2 < |x| < 4\}; \mathbb{R}^{d \times m \times m}) \), the above function is a well-defined modulus of continuity by Heine-Cantor theorem. Furthermore, by appealing to Cauchy-Schwarz’s inequality, (2.51) and (2.58), definition (2.59) immediately imply that \( \{v_j\}_{j \in \mathbb{N}} \) satisfy (2.55).
It remains to show (2.53) and (2.54): Since by Theorem (1) we have p-cap(Σ) = 0, we may argue for B_j as done for (2.25) and obtain that
\[
p-cap(B_j) \lesssim 2^{pj} \int_{|y| < \frac{1}{2}} \left( \int_{2 < |x| < 4} |\nabla x G^*(x, y) - F_j(x, y)|^2 \, dx \right)^{\frac{p}{2}} \, dy \\
+ 2^{pj} \int_{|y| < \frac{1}{2}} \left( \int_{2 < |x| < 4} |\nabla y \nabla x G^*(x, y) - \nabla y F_j(x, y)|^2 \, dx \right)^{\frac{p}{2}} \, dy \lesssim 2^{-j},
\]
i.e. inequality (2.53). It remains to show that \( \{v_j\}_{j \in \mathbb{N}} \) defined in (2.59) satisfies (2.54): For every \( y \notin B_j \cup \Sigma \) with \( |y| < 1 \), we use the definition (2.52) of \( v \) and (b) of Theorem 1 to rewrite
\[
|v_j(y) - v(y)| = \limsup_{r \downarrow 0} \left| \int_{|\tilde{y} - y| < r} v_j(\tilde{y}) \, d\tilde{y} \right|.
\]
By the triangle inequality, we bound
\[
|v_j(y) - v(y)| \leq \limsup_{r \downarrow 0} \left| \int_{|\tilde{y} - y| < r} v_j(\tilde{y}) \, d\tilde{y} \right| + \limsup_{r \downarrow 0} \left| \int_{|\tilde{y} - y| < r} (v(\tilde{y}) - v_j(\tilde{y})) \, d\tilde{y} \right|.
\]
By (2.55), the first limit supremum on the right-hand side is zero. Hence, we have that
\[
|v_j(y) - v(y)| \leq \limsup_{r \downarrow 0} \int_{|\tilde{y} - y| < r} |v_j(\tilde{y}) - v(\tilde{y})| \, d\tilde{y}.
\]
Furthermore, the definitions of \( v \) and \( v_j \), Cauchy-Schwarz’s inequality and the definition of \( B_j \) together with (2.51) allow us to bound
\[
\limsup_{r \downarrow 0} \int_{|\tilde{y} - y| < r} |v_j(\tilde{y}) - v(\tilde{y})| \, d\tilde{y} \lesssim \limsup_{r \downarrow 0} \int_{|\tilde{y} - y| < r} \left( \int_{2 < |x| < 4} |\nabla x G^*(x, \tilde{y}) - F_j(x, \tilde{y})|^2 \, dx \right)^{\frac{p}{2}} \, d\tilde{y} \lesssim 2^{-j}.
\]
By inserting this into (2.61) we conclude (2.54). The proof of Corollary 1 is complete.

\[\square\]

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