

Bₙ-Generalized Pseudo-Kähler Structures

Vicente Cortés¹ · Liana David²

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Abstract
We define the notions of Bₙ-generalized pseudo-Hermitian and Bₙ-generalized pseudo-Kähler structure on an odd exact Courant algebroid E. When E is in the standard form (or of type Bₙ) we express these notions in terms of classical tensor fields on the base of E. This is analogous to the bi-Hermitian viewpoint on generalized Kähler structures on exact Courant algebroids. We describe left-invariant Bₙ-generalized pseudo-Kähler structures on Courant algebroids of type Bₙ over Lie groups of dimension two, three and four.

Keywords Generalized Kähler structures · Odd exact Courant algebroids · Courant algebroids of type Bₙ · Heterotic Courant algebroids

Mathematics Subject Classification 53D18

1 Introduction

Generalized complex geometry is a unification of complex and symplectic geometry and represents an active research area in current mathematics at the interface with mathematical physics. The initial idea was to replace the tangent bundle of a manifold M with the generalized tangent bundle \( T M := TM \oplus T^*M \) and to include complex and symplectic structures in a more general type of structure, a so called generalized complex structure. Later on, other classical structures (like Kähler, quaternionic, hyper-Kähler etc.) were defined and studied in this general setting.
Generalized tangent bundles are the simplest class of Courant algebroids. They are sometimes called exact Courant algebroids, owing to the fact that they fit into an exact sequence

\[ 0 \rightarrow T^*M \xrightarrow{\pi^*} TM \xrightarrow{\pi} TM \rightarrow 0, \]

where \( \pi : TM \rightarrow TM \) is the natural projection (the anchor of \( TM \)) and \( \pi^* : T^*M \rightarrow TM \) is the dual of \( \pi \) (here \( TM \) and \( (TM)^* \) are identified using the natural scalar product of neutral signature of \( TM \)). The notion of a Courant algebroid (see Sect. 2 for its definition) was defined for the first time by Z. J. Liu, A. Weinstein and P. Xu (see [16]) and since then it has been intensively studied. Regular Courant algebroids (i.e. Courant algebroids for which the image of the anchor is a vector bundle) were classified in [3]. This includes a classification of transitive Courant algebroids (i.e. Courant algebroids with surjective anchor). In the context of \( T \)-duality, heterotic Courant algebroids (i.e. Courant algebroids for which the associated quadratic Lie algebra bundle is an adjoint bundle) were considered in [1]. To any regular Courant algebroid \( E \) one can associate a (regular) quadratic Lie algebroid \( E / (\text{Ker } \pi)^\perp \), where \( \perp \) denotes the orthogonal complement with respect to the scalar product of \( E \). Conversely, a regular quadratic Lie algebroid arises in this way from a Courant algebroid if and only if its first Pontryagin class \([2, 15]\) vanishes (see Theorem 1.10 of [3]).

Our setting in this paper are the so called Courant algebroids of type \( B_n \) (and their global analogue, the odd exact Courant algebroids), introduced in [13]. The terminology is justified by the fact that the underlying bundle of a Courant algebroid of type \( B_n \) is of the form \( TM \oplus T^*M \oplus \mathbb{R} \) and its scalar product is the natural scalar product of \( TM \oplus T^*M \oplus \mathbb{R} \) of signature \((n + 1, 1)\), where \( n \) is the dimension of the manifold \( M \). (Along the same lines, exact Courant algebroids were called in [13] Courant algebroids of type \( D_n \), as the natural scalar product of a generalized tangent bundle has neutral signature. One may also define a Courant algebroid of type \( B_n \) as a transitive Courant algebroid in the standard form, with quadratic Lie algebra bundle \( Q = M \times \mathbb{R} \) the trivial line bundle, with positive definite canonical metric, trivial Lie bracket and canonical flat connection. Therefore, from the viewpoint of the classification of transitive Courant algebroids [3], the Courant algebroids of type \( B_n \) represent the next simplest class, after generalized tangent bundles.

Courant algebroids of type \( B_n \) may be seen as the odd analogue of Courant algebroids of type \( D_n \). Following this analogy, generalized complex structures on odd exact Courant algebroids (the so called \( B_n \)-generalized complex structures) were introduced and studied systematically in [12, 13]. It is worth mentioning that there are various approaches to define odd dimensional analogues of \( D_n \)-generalized geometry (see e.g. [11, 14, 17]). In this paper we adopt the viewpoint of [13] and, in analogy with [8, 9], we define generalized pseudo-Kähler structures on odd exact Courant algebroids as an enrichment of \( B_n \)-generalized complex structures.

**Structure of the paper and main results.** In Sect. 2 we recall basic definitions on Courant algebroids of type \( B_n \), odd exact Courant algebroids, generalized metrics and \( B_n \)-generalized complex structures.

In Sect. 3 we define the notion of \( B_n \)-generalized almost pseudo-Hermitian structure on an odd exact Courant algebroid \( E \) as a pair \((\mathcal{G}, \mathcal{F})\) formed by a generalized metric \( \mathcal{G} \) and a \( B_n \)-generalized almost complex structure \( \mathcal{F} \) which satisfy \( \mathcal{G}^{\text{end}} \mathcal{F} = \mathcal{F} \mathcal{G}^{\text{end}} \). The
integrability of \((G, F)\) (and the corresponding notion of \(B_n\)-generalized pseudo-Kähler structure) is defined by imposing integrability on \(F\) and on the second \(B_n\)-generalized almost complex structure \(F_2 := \mathcal{F}\Gamma^\text{end} \).

In analogy with the bi-Hermitian viewpoint on generalized Kähler structures on generalized tangent bundles \([8]\), in Sect. 4 we assume that \(E\) is a Courant algebroid of type \(B_n\) and we describe \(B_n\)-generalized almost pseudo-Hermitian structures on \(E\) in terms of classical tensor fields (called “components”) on the base of \(E\). The case when \(n\) is odd is described in the next proposition (below “\(\perp\)” denotes the orthogonal complement with respect to \(g\), cf. Proposition 13 i).

**Proposition 1** Let \(E\) be a Courant algebroid of type \(B_n\) over an \(n\)-dimensional manifold \(M\), with Dorfman bracket twisted by \((H, F)\). Assume that \(n\) is odd. A \(B_n\)-generalized almost pseudo-Hermitian structure \((G, F)\) on \(E\) is equivalent to the data \((g, J_+, J_-, X_+, X_-)\) where \(g\) is a pseudo-Riemannian metric on \(M\), \(J_\pm \in \Gamma(\text{End} \, TM)\) are \(g\)-skew-symmetric endomorphisms and \(X_\pm \in \mathfrak{X}(M)\) are vector fields of norm one, such that \(J_\pm X_\pm = 0\) and \(J_\pm|_{X_\pm}\) are complex structures.

The case when \(n\) is even is described in the next proposition, cf. Proposition 13 ii).

**Proposition 2** Let \(E\) be a Courant algebroid of type \(B_n\) over an \(n\)-dimensional manifold \(M\), with Dorfman bracket twisted by \((H, F)\). Assume that \(n\) is even. A \(B_n\)-generalized almost pseudo-Hermitian structure \((G, F)\) on \(E\) is equivalent to the data \((g, J_+, J_-, X_+, X_-, c_+)\) where \(g\) is a pseudo-Riemannian metric on \(M\), \(J_\pm \in \Gamma(\text{End} \, TM)\) are \(g\)-skew-symmetric endomorphisms, \(X_\pm \in \mathfrak{X}(M)\) are \(g\)-orthogonal vector fields and \(c_+ \in C^\infty(M)\) is a function, such that \(J_-\) is an almost complex structure on \(M\), \(J_+\) satisfies

\[
J_+ X_+ = -c_+ X_-, \quad J_+ X_- = c_+ X_+,
\]

\[
J_+^2 X = -X + g(X, X_+)X_+ + g(X, X_-)X_- \quad \forall X \in TM
\]

and \(g(X_+, X_+) = g(X_-, X_-) = 1 - c_+^2\).

Our aim in Sect. 5 is to express the integrability of a \(B_n\)-generalized almost pseudo-Hermitian structure \((G, F)\) on a Courant algebroid of type \(B_n\) in terms of its components. When \(n\) is odd we obtain the following summarized description. (Below \(\nabla\) denotes the Levi-Civita connection of \(g\) and \(T_A^{(1,0)} M \subset (TM)_\mathbb{C}\) denotes the \(i\)-eigenbundle of an endomorphism \(A\) of \(TM\)).

**Theorem 3** In the setting of Proposition 1, define connections \(\nabla^\pm\) on \(TM\) by

\[
\nabla^+_X := \nabla_X - \frac{1}{2} H(X) - J_+ F(X) \otimes X_+,
\]

\[
\nabla^-_X := \nabla_X + \frac{1}{2} H(X),
\]

for any \(X \in \mathfrak{X}(M)\), where we are identifying vectors and co-vectors using the metric \(g\), as explained after equation \((18)\), cf. Notation 15, and \(J_+ F(X)\) denotes the
endomorphism $J_+$ applied to the vector field $F(X)$. Then $(\mathcal{G}, \mathcal{F})$ is a $B_n$-generalized pseudo-Kähler structure if and only if $\nabla^\pm$ preserves $T^{(1,0)}_{j_\pm} M$,

$$\nabla_X^+ X_- = 0, \; \nabla_X^- X_+ = -J_+ F(X), \; \forall X \in \mathfrak{X}(M)$$

(1)

and certain algebraic conditions hold (see Theorem 16 for the precise conditions).

The analogous result for $n$ even is stated as follows.

**Theorem 4** In the setting of Proposition 2, assume that $X_+$ and $X_-$ are non-null (at any point), i.e. $c_+(p)^2 \neq 1$ for any $p \in M$. Define connections $D^\pm$ on $TM$ by

$$D^+_X := \nabla_X - \frac{1}{2} H(X) + \frac{c_+}{1 - c_+^2} F(X) \otimes X_+ - \frac{J_+ F(X)}{1 - c_+^2} \otimes X_-,$$

$$D^-_X := \nabla_X + \frac{1}{2} H(X),$$

for any $X \in \mathfrak{X}(M)$. Then $(\mathcal{G}, \mathcal{F})$ is a $B_n$-generalized pseudo-Kähler structure if and only if $D^\pm$ preserves $T^{(1,0)}_{j_\pm} M$, $X_+$ and $X_-$ commute, their covariant derivatives are given by

$$\nabla_X X_+ = \frac{1}{2} H(X, X_+) + c_+ F(X),$$

$$\nabla_X X_- = \frac{1}{2} H(X, X_-) - J_+ F(X),$$

$i_{X_+} F = d c_+$ and certain algebraic conditions hold (see Theorem 20 for the precise conditions).

The proofs of Theorems 3 and 4 are analogues to the proof of Gualtieri on the relation between generalized Kähler structures on exact Courant algebroids and bi-Hermitian data [8]. More precisely, in a first stage we find a criterion for the integrability of a $B_n$-generalized almost pseudo-Hermitian structure $(\mathcal{G}, \mathcal{F})$ in terms of the (Dorfman) integrability of the bundles $L_1, L_1 \cap L_2$ and $L_1 \cap \bar{L}_2$ and their invariance under the Dorfman Lie derivative $L_{u_0}$ (where $L_1$ is the $(1,0)$-eigenbundle of $\mathcal{F}$, $L_2$ is the $(1,0)$-eigenbundle of $\mathcal{G}^{\text{end}} \mathcal{F}$ and $u_0 \in \Gamma(\text{Ker} \mathcal{F})$ is suitably normalized), see Proposition 28. Then the above theorems follow by computing the intersections $L_1 \cap L_2$ and $L_1 \cap \bar{L}_2$ in terms of the components of $(\mathcal{G}, \mathcal{F})$ and applying the above-mentioned criterion. The assumption from Theorem 4 that $X_\pm$ are non-null simplifies the argument considerably. Along the way we determine various further properties of the components of $B_n$-generalized pseudo-Kähler structures as well as a rescaling property for such structures (see Corollary 19 and Corollary 24). Finally, we remark that a generalized Kähler structure on an exact Courant algebroid may be interpreted as a $B_n$-generalized Kähler structure (see Remark 25).

In Sect. 6 we simplify Theorems 3 and 4 under the assumption that $M$ has small dimension. This section is intended as a preparatory material for Sect. 7, which is
devoted to the description of left-invariant $B_n$-generalized pseudo-Kähler structures $(G, \mathcal{F})$ over Lie groups of dimension two, three or four. Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ and $E$ a Courant algebroid of type $B_n$ over $G$, whose Dorfman bracket is twisted by left-invariant forms $(H, F)$. A $B_n$-generalized pseudo-Kähler structure on $E$ is called left-invariant if its components are left-invariant tensor fields on $G$ (in particular, the function $c_+$ is constant). Our main results from Sect. 7 can be roughly summarized as follows (below $c_+$ and $g$ are part of the components of the corresponding $B_n$-generalized pseudo-Kähler structure).

**Theorem 5**

i) Assume that $G$ is 2-dimensional. There is a left-invariant $B_2$-generalized pseudo-Kähler structure on $E$, such that $c_\pm \not\in \{-1, +1\}$, if and only if $\mathfrak{g}$ is abelian.

ii) Assume that $G$ is 3-dimensional, unimodular, with canonical operator $L$ (defined in (75)) diagonalizable. There is a left-invariant $B_3$-generalized pseudo-Kähler structure on $E$ if and only if $\mathfrak{g}$ is abelian, $\mathfrak{g} = \mathfrak{so}(2) \ltimes \mathbb{R}^2$ or $\mathfrak{g} = \mathfrak{so}(1, 1) \ltimes \mathbb{R}^2$.

iii) Assume that $G$ is 3-dimensional and non-unimodular. Let $\mathfrak{g}_0$ be the unimodular kernel of $\mathfrak{g}$. There is a left-invariant $B_3$-generalized pseudo-Kähler structure on $E$ such that $g|_{\mathfrak{g}_0 \times \mathfrak{g}_0}$ is non-degenerate if and only if $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{so}_2$, where $\mathfrak{so}_2$ is the unique non-abelian 2-dimensional Lie algebra.

iv) Assume that $G$ is 4-dimensional, non-unimodular, and its unimodular kernel is non-abelian. There is an adapted (see Definition 47) $B_4$-generalized pseudo-Kähler structure on $E$ such that $c_+ \not\in \{-1, 0, 1\}$ if and only if $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{iso}(2)$, where $\mathfrak{iso}(2)$ denotes the Lie algebra of the isometry group of the standard (positive definite) metric on $\mathbb{R}^2$.

In all cases above $E$ is untwisted (i.e. $H = 0$, $F = 0$).

The description of the left-invariant $B_n$-generalized pseudo-Kähler structures which arise in Theorem 5 can be found in Sect. 7.1, Proposition 42, Proposition 44 and Proposition 48. Cases ii) and iii) of Theorem 5 include a description of all left-invariant $B_3$-generalized Kähler structures on Courant algebroids of type $B_3$ over 3-dimensional Lie groups (see Corollary 45). The various additional assumptions from Theorem 5 were intended to simplify the computations. All examples provided by Theorem 5 live on untwisted Courant algebroids. It would be interesting to find twisted examples.

## 2 Preliminary Material

For completeness of our exposition we recall the definition of a Courant algebroid (the axioms C2 and C3 are redundant and are only included for convenience).

**Definition 6** A Courant algebroid on a manifold $M$ is a vector bundle $E \to M$ equipped with a non-degenerate symmetric bilinear form $(\cdot, \cdot) \in \Gamma(\text{Sym}^2(E^*))$ (called the scalar product), a bilinear operation $[\cdot, \cdot]$ (called the Dorfman bracket) on the space of smooth sections $\Gamma(E)$ of $E$ and a homomorphism of vector bundles $\pi : E \to TM$ (called the anchor) such that the following conditions are satisfied: for all $u, v, w \in \Gamma(E)$ and $f \in C^\infty(M)$,

C1) $[u, [v, w]] = [[u, v], w] + [v, [u, w]],$
C2) \( \pi([u, v]) = \mathcal{L}_{\pi(u)}\pi(v) \),
C3) \([u, fv] = \pi(u)(f)v + f[u, v] \),
C4) \( \pi(u)(v, w) = \{[u, v], w\} + \{v, [u, w]\} \),
C5) \( 2\{[u, u], v\} = \pi(v)\{u, u\} \),

where \( \mathcal{L}_{\pi(u)}\pi(v) \) denotes the Lie bracket of the vector fields \( \pi(u) \) and \( \pi(v) \).

According to [13], a Courant algebroid of type \( B_n \) over a manifold \( M \) of dimension \( n \) is the vector bundle \( E = TM \oplus T^*M \oplus \mathbb{R} \), with scalar product

\[
\langle X + \xi + \lambda, Y + \eta + \mu \rangle = \frac{1}{2} (\eta(X) + \xi(Y)) + \lambda\mu,
\]

for any \( X, Y \in TM, \xi, \eta \in T^*M \) and \( \lambda, \mu \in \mathbb{R} \), anchor given by the natural projection \( \pi : E \to TM \) and Dorfman bracket given by

\[
[X + \xi + \lambda, Y + \eta + \mu] = \mathcal{L}_X Y + \mathcal{L}_X \eta - i_Y d\xi + 2\mu d\lambda + i_X i_Y H - 2(\mu i_X F - \lambda i_Y F) + X(\mu) - Y(\lambda) + F(X, Y),
\]

for any \( X, Y \in \mathfrak{X}(M), \xi, \eta \in \Omega^1(M) \) and \( \lambda, \mu \in C^\infty(M) \), where \( F \in \Omega^2(M) \) is a closed 2-form and \( H \in \Omega^3(M) \) is such that \( dH + F \wedge F = 0 \) (we follow the convention of [13], which differs from the convention from our previous works [4] and [5] by a minus sign in the 3-form \( H \)). We will refer to (3) as the Dorfman bracket twisted by \((H, F)\). The Courant algebroid of type \( B_n \) with Dorfman bracket twisted by \((H, F)\) will be denoted by \( E_{H,F} \) and will be called the Courant algebroid twisted by \((H, F)\). An odd exact Courant algebroid is a Courant algebroid isomorphic to a Courant algebroid of type \( B_n \).

2.1 Generalized Metrics

A generalized metric on an odd exact Courant algebroid \( E \), with scalar product \( \langle \cdot, \cdot \rangle \) and anchor \( \pi : E \to TM \), is a subbundle \( E_- \subset E \) of rank \( n := \dim M \) on which \( \langle \cdot, \cdot \rangle \) is non-degenerate (see e.g. [7]). According to [1], when the restriction \( \pi|_{E_-} : E_- \to TM \) is an isomorphism, the generalized metric is called admissible. Any rank \( n \) subbundle \( E_- \subset E \) with the property that the restriction \( \langle \cdot, \cdot \rangle|_{E_-} \) is negative definite is an admissible generalized metric. Such generalized metrics will be called Riemannian. All generalized metrics considered in this paper are admissible. For this reason, the word ‘admissible’ will be omitted.

A generalized metric \( E_- \) on an odd exact Courant algebroid \( E \) defines an orthogonal isomorphism \( G^\text{end} \in \Gamma(\text{End} E) \) by \( G^\text{end}|_{E_-} = \pm \text{Id} \), where \( E_\pm := E_\perp \) (unless otherwise stated, \( \perp \) will always denote the orthogonal complement with respect to the scalar product of the Courant algebroid). The bilinear form

\[
G(u, v) := \langle G^\text{end} u, v \rangle, \quad u, v \in E
\]

is given by
\[ G = \langle \cdot, \cdot \rangle|_{E^+} - \langle \cdot, \cdot \rangle|_{E^-}. \]  

(5)

It is symmetric and, when \( E_- \) is a generalized Riemannian metric, it is positive definite. When referring to a generalized metric we will freely name either the bundle \( E_- \) or the bilinear form \( G \).

Let \( E_- \) be a generalized metric on a Courant algebroid \( E \) of type \( B_n \) and \( E_+ := E_-^\perp \). Since the restriction of the anchor \( \pi|_{E_-} : E_- \to TM \) is an isomorphism there is an induced pseudo-Riemannian metric on \( M \), defined by

\[ g(X, Y) := -\langle s(X), s(Y) \rangle, \forall X, Y \in TM, \]  

(6)

where \( s : TM \to E_- \) is the inverse of \( \pi|_{E_-} \). When \( E_- \) is a generalized Riemannian metric, \( g \) is positive definite. As for generalized metrics on heterotic Courant algebroids (see [1]), the vector bundles \( E_{\pm} \) are given by

\[
E_- = \{ X - i_X (g - b) - A(X)A + A(X) \mid X \in TM \}, \\
E_+ = \{ X + i_X (g + b) + (A(X) - 2\lambda)A + \lambda \mid X \in TM, \lambda \in \mathbb{R} \},
\]

(7)

where \( A \in \Omega^1(M) \) and \( b \in \Omega^2(M) \).

The next lemma was proved in [1] for heterotic Courant algebroids. The same argument applies to Courant algebroids of type \( B_n \). For completeness of our exposition we include its proof.

**Lemma 7** Let \( E_- \) be a generalized metric on a Courant algebroid \( E = EH, F \), given by (7). There is an isomorphism between \( E \) and another Courant algebroid \( \tilde{E} \) of type \( B_n \) which maps \( E_- \) to a generalized metric on \( \tilde{E} \) of the form (7) with \( A = 0 \) and \( b = 0 \).

**Proof** Let \( \tilde{E} := E_{\tilde{H}, \tilde{F}} \) be the Courant algebroid twisted by \( (\tilde{H}, \tilde{F}) \), where

\[ \tilde{H} := H - db - (2F + dA) \wedge A, \; \tilde{F} := F + dA \]

(note that \( d\tilde{H} + \tilde{F} \wedge \tilde{F} = 0 \) and \( d\tilde{F} = 0 \), since \( dH + F \wedge F = 0 \) and \( dF = 0 \)). The map \( I : E \to \tilde{E} \) defined by

\[ I(X) = X - A(X) - i_X b - A(X)A, \; I(\eta) = \eta, \; I(\lambda) = 2\lambda A + \lambda, \]

for \( X \in TM, \eta \in T^*M \) and \( \lambda \in \mathbb{R} \), is an isomorphism of Courant algebroids which maps \( E_- \) to the generalized metric

\[ \tilde{E}_- = \{ X - i_X g, \; X \in TM \}. \]

\[ \Box \]

Owing to the above lemma we will often assume that a generalized metric on a Courant algebroid of type \( B_n \) is given by a subbundle \( E_- \) like in (7) with \( g \) a pseudo-Riemannian metric, \( b = 0 \) and \( A = 0 \). Such a generalized metric is in the standard form. For more details on generalized metrics on arbitrary Courant algebroids, see e.g. [7].
2.2 $B_n$-Generalized Complex Structures

Following [13], we recall that a $B_n$-generalized almost complex structure on an odd exact Courant algebroid $(E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \pi)$ over a manifold $M$ of dimension $n$ is a complex isotropic rank $n$ subbundle $L \subset E_C$ such that $L \cap \bar{L} = 0$. The orthogonal complement $U_C$ of $L \oplus \bar{L} \subset E_C$ has rank one. It is generated by a section $u_0 \in \Gamma_1(E)$ (unique up to multiplication by ±1) which satisfies $\langle u_0, u_0 \rangle = (-1)^n$. Let $\mathcal{F} \in \Gamma(\text{End} E)$ be the endomorphism with $i$-eigenbundle $L$, $(-i)$-eigenbundle $\bar{L}$ and $\mathcal{F}u_0 = 0$. It is $\langle \cdot, \cdot \rangle$-skew-symmetric and satisfies

$$\mathcal{F}^2 = -\text{Id} + (-1)^n \langle \cdot, u_0 \rangle u_0. \quad (8)$$

We will often call $\mathcal{F}$ (rather than $L$) a $B_n$-generalized almost complex structure. We say that $\mathcal{F}$ is a $B_n$-generalized complex structure (or is integrable) if $L$ is integrable. (A subbundle of $E$ or its complexification $E_C$ is called integrable if its space of sections is closed under the Dorfman bracket). Then $L$ becomes a Lie algebroid with Lie bracket induced by the Dorfman bracket of $E$.

3 Definition of $B_n$-Generalized Pseudo-Kähler Structures

Let $(E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \pi)$ be an odd exact Courant algebroid over a manifold $M$ of dimension $n$.

Definition 8 A $B_n$-generalized almost pseudo-Hermitian structure $(\mathcal{G}, \mathcal{F})$ on $E$ is a generalized metric $\mathcal{G}$ together with a $B_n$-generalized almost complex structure $\mathcal{F}$ such that $\mathcal{G}^\text{end} \mathcal{F} = \mathcal{F} \mathcal{G}^\text{end}$.

The commutativity condition $\mathcal{G}^\text{end} \mathcal{F} = \mathcal{F} \mathcal{G}^\text{end}$ implies that $\mathcal{F}$ preserves the bundles $E_{\pm}$ determined by the generalized metric. The next lemma summarizes some simple properties of $B_n$-generalized almost pseudo-Hermitian structures.

Lemma 9 Let $(\mathcal{G}, \mathcal{F})$ be a $B_n$-generalized almost pseudo-Hermitian structure on $E$, $U := \ker \mathcal{F}$ and $u_0 \in \Gamma(U)$ such that $\langle u_0, u_0 \rangle = (-1)^n$. The following statements hold:

i) $u_0$ is a section of $E_+$ if $n$ is even and a section of $E_-$ if $n$ is odd, i.e.

$$\mathcal{G}^\text{end}(u_0) = (-1)^n u_0. \quad (9)$$

ii) For any $u, v \in E$,

$$\mathcal{G}(\mathcal{F}u, \mathcal{F}v) = \mathcal{G}(u, v) - \langle u, u_0 \rangle \langle v, u_0 \rangle,$$

$$\mathcal{G}(\mathcal{F}u, v) = -\mathcal{G}(u, \mathcal{F}v). \quad (10)$$

Proof i) Since $U$ has rank one and $\mathcal{G}^\text{end} \mathcal{F} = \mathcal{F} \mathcal{G}^\text{end}$, we obtain that $\mathcal{G}^\text{end}(u_0) = \lambda u_0$ for $\lambda \in C^\infty(M)$. From $(\mathcal{G}^\text{end})^2 = \text{Id}$ we obtain that $\lambda = \pm 1$, i.e. $u_0 \in \Gamma(E_+)$ or $u_0 \in \Gamma(E_-)$.
$u_0 \in \Gamma(E_-)$. On the other hand, $F$ is a complex structure on $u_0^\perp$ and preserves $E_\pm$. In particular, $F$ is a complex structure on $u_0^\perp \cap E_\pm$. If $u_0 \in \Gamma(E_+)$, then $E_- = E_- \cap u_0^\perp$ (because $E_\pm$ are orthogonal) and $F|_{E_-}$ is a complex structure on $E_-$. This implies that $n = \text{rank } E_-$ is even. If $u_0 \in \Gamma(E_+)$, then $F|_{E_-}$ is a complex structure on $E_-$. This implies that $\text{dim } E_- = \text{dim } u_0^\perp$. If $u_0 \in \Gamma(E_-)$, then $E_- = (E_- \cap u_0^\perp) \oplus \text{span}\{u_0\}$

and $F$ restricts to a complex structure on $E_- \cap u_0^\perp$. We obtain that $n$ is odd.

ii) Relations (10) are consequences of the properties of $G^{\text{end}}$ and $F$.

In the setting of Lemma 9, let $F_2 := G^{\text{end}}F$. Then $F_2$ is $(\cdot, \cdot)$-skew-symmetric, satisfies relation (8) and $\text{Ker } F_2 = U$. Its $i$-eigenbundle is given by

$$L_2 = L_1 \cap (E_+)_\mathbb{C} \oplus \tilde{L}_1 \cap (E_-)_\mathbb{C},$$

where $L_1$ is the $i$-eigenbundle of $F$. As

$$L_1 = L_1 \cap (E_+)_\mathbb{C} \oplus L_1 \cap (E_-)_\mathbb{C},$$

we obtain that rank $L_2 = \text{rank } L_1 = n$. We deduce that $F_2$ is a $B_n$-generalized almost complex structure. We obtain the following alternative definition of $B_n$-generalized almost pseudo-Hermitian structures on odd exact Courant algebroids.

**Proposition 10** A $B_n$-generalized almost pseudo-Hermitian structure on $E$ is equivalent to a pair $(F_1, F_2)$ of commuting $B_n$-generalized almost complex structures such that the bilinear form $(u, v) \mapsto \langle F_1 F_2(u), v \rangle$ on $U^\perp$ is nondegenerate and

$$\pi : \{u \in U^\perp \mid F_1(u) = -F_2(u)\} \to TM$$

(12)

is an isomorphism when $n$ is even while

$$\pi : \{u \in E \mid F_1(u) = -F_2(u)\} \to TM$$

(13)

is an isomorphism when $n$ is odd. Above $U = \text{Ker } F_1 = \text{Ker } F_2$.

**Proof** Given $(F_1, F_2)$ as in the statement of the proposition, we recover the generalized metric $G$ from $G^{\text{end}} := -F_1 F_2$ on $U^\perp$ and $G^{\text{end}} = (-1)^n \text{Id}$ on $U$. The requirement that (12) and (13) are isomorphisms is equivalent to the fact that $G$ is admissible.

**Remark 11** In the setting of the above proposition, if $-\langle F_1 F_2 u, u \rangle > 0$ for any $u \in U^\perp \setminus \{0\}$ then $G$ is a generalized Riemannian metric.

**Definition 12** A $B_n$-generalized almost pseudo-Hermitian structure $(G, F)$ is called integrable or a $B_n$-generalized pseudo-Kähler structure if $F$ and $G^{\text{end}} F$ are $B_n$-generalized complex structures.
4 Components of $B_n$-Generalized Almost Pseudo-Hermitian Structures

In this section we describe $B_n$-generalized almost pseudo-Hermitian structures $(\mathcal{G}, \mathcal{F})$ on a Courant algebroid $E := E_{H,F}$ of type $B_n$ over a manifold $M$ of dimension $n$ in terms of tensor fields on $M$. We assume that the generalized metric $\mathcal{G}$ is in standard form. Let $E = E_+ \oplus E_-$ be the decomposition of $E$ determined by $\mathcal{G}$. From the description (7) of $E_{\pm}$ (with $b = 0$, $A = 0$) we obtain canonical isomorphisms

$$
E_- \cong TM, \quad X - i_X g \mapsto X, \\
E_+ \cong TM \oplus \mathbb{R}, \quad X + i_X g + \lambda \mapsto X + \lambda, 
$$

(14)

where $g$ is the pseudo-Riemannian metric on $M$ induced by $\mathcal{G}$, see (6). The second isomorphism (14) maps $\langle \cdot, \cdot \rangle|_{E_+}$ to the metric

$$(g + g_{\text{can}})(X + \lambda, Y + \mu) = g(X, Y) + \lambda\mu, \quad \forall X, Y \in TM, \quad \lambda, \mu \in \mathbb{R}$$

on $TM \oplus \mathbb{R}$, where $g_{\text{can}}(\lambda, \mu) := \lambda\mu$. Let $u_0 \in \Gamma(\ker \mathcal{F})$, normalized by $\langle u_0, u_0 \rangle = (-1)^n$. When $n$ is even, $u_0 \in \Gamma(E_+)$ and will be denoted by $u_+$. When $n$ is odd, $u_0 \in \Gamma(E_-)$ and will be denoted by $u_-$. (See Lemma 9.)

In the next proposition ‘$\perp$’ denotes the orthogonal complement in $TM$ with respect to $g$.

**Proposition 13**

i) Assume that $n$ is odd. A $B_n$-generalized almost pseudo-Hermitian structure $(\mathcal{G}, \mathcal{F})$ on $E$ is equivalent to the data $(g, J_+, J_-, X_+, X_-)$ where $g$ is a pseudo-Riemannian metric on $M$, $J_{\pm} \in \Gamma(\text{End} TM)$ are $g$-skew-symmetric endomorphisms and $X_{\pm} \in \mathfrak{X}(M)$ are vector fields, such that $J_{\pm}X_{\pm} = 0$, $J_{\pm}|_{X_{\pm}^\perp}$ are complex structures and $g(X_{\pm}, X_{\pm}) = 1$.

ii) Assume that $n$ is even. A $B_n$-generalized almost pseudo-Hermitian structure $(\mathcal{G}, \mathcal{F})$ on $E$ is equivalent to the data $(g, J_+, J_-, X_+, X_-, c^+)$ where $g$ is a pseudo-Riemannian metric on $M$, $J_{\pm} \in \Gamma(\text{End} TM)$ are $g$-skew-symmetric endomorphisms, $X_{\pm} \in \mathfrak{X}(M)$ are $g$-orthogonal vector fields and $c^+ \in C^\infty(M)$ is a function, such that $J_-$ is an almost complex structure on $M$, $J_+$ satisfies

$$
J_+X_+ = -c^+X_-, \quad J_+X_- = c^+X_+,
$$

$$
J_+^2X = -X + g(X, X_+)X_+ + g(X, X_-)X_- \quad \forall X \in TM
$$

and $g(X_+, X_+) = g(X_-, X_-) = 1 - c^+$. 

iii) In both cases (dim $M$ odd or even), the generalized metric $\mathcal{G}$ is given by

$$
E_- = \{X - i_X g \mid X \in TM\} 
$$

and the $B_n$-generalized almost complex structures $\mathcal{F}_1 = \mathcal{F}$ and $\mathcal{F}_2 = \mathcal{G}^{\text{end}}\mathcal{F}$ are given by
\[ F_1 = \begin{pmatrix} \frac{1}{2}(J_+ + J_-) & \frac{1}{2}(J_+ - J_-) \circ g^{-1} & X \\ \frac{1}{2}g \circ (J_+ - J_-) & -\frac{1}{2}(J_+ + J_-)^* & i_X g \\ \frac{1}{2}i_X g & -\frac{1}{2}X & 0 \end{pmatrix} \] (16)

and

\[ F_2 = \begin{pmatrix} \frac{1}{2}(J_+ - J_-) & \frac{1}{2}(J_+ + J_-) \circ g^{-1} & X \\ \frac{1}{2}g \circ (J_+ + J_-) & -\frac{1}{2}(J_+ - J_-)^* & i_X g \\ \frac{1}{2}i_X g & -\frac{1}{2}X & 0 \end{pmatrix}, \] (17)

where \( X := X_+ \) when \( n \) is odd and \( X := X_- \) when \( n \) is even.

**Proof**

i) The scalar product \(-\langle \cdot, \cdot \rangle|_{E_-}\) corresponds to \( g \) under the first isomorphism (14). Then \( X_- := \pi(u_-) \) satisfies \( g(X_-, X_-) = 1 \), since \( \langle u_-, u_- \rangle = -1 \). By means of the first isomorphism (14), the restriction \( F|_{E_-} \) induces a \( g \)-skew-symmetric endomorphism \( J_- \) of \( TM \), with \( J_- X_- = 0 \) and which is a complex structure on \( X_-^\perp \). Consider now the second isomorphism (14). By means of this isomorphism, \( \langle \cdot, \cdot \rangle|_{E_+} \) corresponds to \( g + g_{\text{can}} \) on \( TM \oplus \mathbb{R} \) and \( F|_{E_+} \) induces a complex structure \( F_+ \) on \( TM \oplus \mathbb{R} \), skew-symmetric with respect to \( g + g_{\text{can}} \). An easy computation shows that \( F_+ \) is of the form

\[ F_+ = \begin{pmatrix} J_+ & X_+ \\ -i_{X_+} g & 0 \end{pmatrix}, \]

where \( X_+ \in \mathcal{X}(M) \) satisfies \( g(X_+, X_+) = 1 \) and \( J_+ \in \Gamma(\text{End} TM) \) is \( g \)-skew-symmetric, \( J_+ X_+ = 0 \) and \( J_+|_{X_+^\perp} \) is a complex structure on \( X_+^\perp \).

ii) The restriction \( F|_{E_-} \) induces, under the first isomorphism (14), a \( g \)-skew-symmetric almost complex structure \( J_- \) on \( M \). Write \( u_+ = X_+ + g(X_+) + c_+ \), where \( X_+ \in \mathcal{X}(M) \) and \( c_+ \in C^\infty(M) \). From \( \langle u_+, u_+ \rangle = 1 \) we obtain \( g(X_+, X_+) = 1 - c_+^2 \). Under the second isomorphism (14), the restriction \( F|_{E_+} \) induces an endomorphism \( F_+ \in \Gamma(\text{End}(TM \oplus \mathbb{R})) \), which satisfies

\[ F_+(X_+ + c_+) = 0, \quad F_+^2 = -\text{Id} + i_{X_+ + c_+} (g + g_{\text{can}}) \otimes (X_+ + c_+) \]

and is skew-symmetric with respect to \( g + g_{\text{can}} \). Writing \( F_+ \) in block form

\[ F_+ = \begin{pmatrix} J_+ & X_- \\ \omega & a \end{pmatrix} \]

where \( J_+ \in \Gamma(\text{End} TM), X_- \in \mathcal{X}(M), \omega \in \Omega^1(M) \) and \( a \in C^\infty(M) \) an easy check shows that \( a = 0, \omega = -i_{X_-} g, g(X_-, X_-) = 1 - c_+^2, g(X_+, X_-) = 0 \) and \( J_+ \) satisfies the required properties.

iii) Relations (16) and (17) follow from i) and ii). For instance, to compute \( F \) on an element \( X \in TM \) it suffices to decompose \( X = \frac{1}{2}(X + i_X g) + \frac{1}{2}(X - i_X g) \in E_+ \oplus E_- \) and to apply the above formulas for \( F|_{E_\pm} \). This yields the first column of (16). The second column is obtained similarly after decomposing an element \( \xi \in T^* M \) as \( \xi = \frac{1}{2}(X + i_X g) - \frac{1}{2}(X - i_X g) \), where \( X = g^{-1}\xi \). \( \square \)
Definition 14 Let \((G, F)\) be a \(B_n\)-generalized almost pseudo-Hermitian structure on a Courant algebroid \(E_{H, F}\) of type \(B_n\) over a manifold \(M\). The tensor fields on \(M\) constructed in Proposition 13, are called the components of \((G, F)\).

5 Components of \(B_n\)-Generalized Pseudo-Kähler Structures

In this section we express the integrability of a \(B_n\)-generalized almost pseudo-Hermitian structure on a Courant algebroid \(E\) of type \(B_n\) in terms of its components. This is done in Theorems 16 and 20 below. The proofs of these theorems will be presented in Sect. 5.2.

5.1 Statement of Results

Let \(E = E_{H, F}\) be a Courant algebroid of type \(B_n\) over a manifold \(M\) of dimension \(n\), with Dorfman bracket twisted by \((F, H)\), and \((G, F)\) a \(B_n\)-generalized almost pseudo-Hermitian structure on \(E\). As in Sect. 4, we assume that the generalized metric \(G\) is in standard form.

Notation 15 i) In analogy with the standard notation for the \((1, 0)\)-bundle of an almost complex structure on a manifold, for any endomorphism \(A \in \Gamma(\text{End} TM)\), we will denote by \(T^{(1,0)}_A M \subset (TM)_C\) its \(i\)-eigenbundle.

ii) Let \(g\) be the pseudo-Riemannian metric which is part of the components of \((G, F)\), see Proposition 13. We identify \(TM\) with \(T^*M\) using \(g\). For \(X, Y \in \mathfrak{X}(M)\) we denote by \(F(X)\) and \(H(X, Y)\) the vector fields identified with the 1-forms \(i_X F\) and \(i_Y i_X H\). A decomposable tensor \(Z \otimes V \in TM \otimes TM\) with \(Z, V \in TM\) is identified with the endomorphism of \(TM\) which assigns to \(X \in TM\) the vector \(g(Z, X)V\). For an endomorphism \(A \in \Gamma(\text{End} TM)\) we denote by \(A^{\text{sym}}\) and \(A^{\text{skew}}\) its \(g\)-symmetric and \(g\)-skew-symmetric parts, respectively.

5.1.1 The Case of Odd \(n\)

Assume that \(n\) is odd and let \((g, J_+, J_-, X_+, X_-)\) be the components of \((G, F)\). Define connections \(\nabla^+\) and \(\nabla^-\) on \(TM\) by

\[
\nabla^+_X := \nabla_X - \frac{1}{2} H(X) - J_+ F(X) \otimes X_+,
\]

\[
\nabla^-_X := \nabla_X + \frac{1}{2} H(X),
\]

where \(X \in \mathfrak{X}(M)\), \(\nabla\) is the Levi-Civita connection of \(g\), \(H(X)\) denotes the skew-symmetric endomorphism \(Y \mapsto H(X, Y)\) and the vector field \(J_+ F(X)\) is metrically identified with a one-form.

Theorem 16 The \(B_n\)-generalized almost pseudo-Hermitian structure \((G, F)\) is \(B_n\)-generalized pseudo-Kähler if and only if the following conditions hold:
i) The connections $\nabla^\pm$ preserve the distributions $T_{\pm}^{(1,0)} M$ (respectively) and
\[ \nabla^- X = -J^+ F(X), \quad \forall X \in \mathcal{X}(M). \tag{19} \]

ii) The forms $H$ and $F$ satisfy the constraints
\[ H|_{\Lambda^3 T_{\pm}^{(1,0)} M} = 0, \quad (iX^+ H)|_{\Lambda^2 T_{\pm}^{(1,0)} M} = iF|_{\Lambda^2 T_{\pm}^{(1,0)} M}, \]
\[ (iX^- H)|_{\Lambda^2 T_{\pm}^{(1,0)} M} = 0, \quad F|_{\Lambda^2 T_{\pm}^{(1,0)} M} = 0, \quad iX^+ F = 0. \tag{20} \]

We make several comments on the above theorem.

**Corollary 17** In terms of the almost contact metric structures $(g, J_{\pm}, \eta_{\pm} := g(X_{\pm}, \cdot))$, the conditions from Theorem 16 i) are equivalent to
\[ \nabla X J_+ = -\frac{1}{2} [H(X), J_-], \quad d\eta_- = iX^+ H, \]
\[ \nabla X J_- = \frac{1}{2} [H(X), J_+] + F(X) \wedge X_+, \quad d\eta_+ = -iX^- H + 2(F \circ J_+)_{\text{skew}}, \tag{21} \]
for any $X \in \mathcal{X}(M)$, together with the facts that $X_-$ is a Killing field and $\mathcal{L}_{X_+} g = -2(F \circ J_+)_{\text{sym}}$. (Here the bivector $F(X) \wedge X_+ = \eta_+(Y) F(X)$.) In particular, if the Courant algebroid is untwisted (i.e. $H = 0$ and $F = 0$), the conditions from Theorem 16 reduce to $\nabla J_\pm = 0$.

**Proof** Assume that the conditions from Theorem 16 hold. In terms of the Levi-Civita connection $\nabla$, relations (19) become
\[ \nabla X J_+ = \frac{1}{2} [H(X), X_+] - J^+ F(X), \tag{22} \]
for any $X \in \mathcal{X}(M)$. From the above relations we obtain the required expressions for $d\eta_\pm$ and $\mathcal{L}_{X_\pm} g = -2(F \circ J_\pm)_{\text{sym}}$. Since $\nabla^\pm$ preserve the distributions $T_{\pm}^{(1,0)} M$, they also preserve the distributions $T_{\pm}^{(0,1)} M$. Together with relations (19), they imply that
\[ \nabla^- X = 0, \quad \nabla^+ X_+ = X_+ \otimes (F(X, X_+)X_+ - F(X)), \tag{23} \]
for any $X \in \mathcal{X}(M)$. Replacing in the above relations $\nabla^\pm$ with their expressions in terms of $\nabla$ provided by relations (18), we obtain the required expressions for $\nabla J_\pm$. Reversing the argument we obtain the first statement of the corollary.

When $E$ is untwisted, relations (21) reduce to $\nabla J_\pm = 0$. Remark that if $\nabla J_\pm = 0$ then $\nabla X_\pm = 0$ as $g(X_\pm, X_\pm) = 1$ and $\text{Ker } J_\pm = \text{span}\{X_\pm\}$. The conditions from Theorem 16 ii) are trivially satisfied.
Corollary 18  In the setting of Theorem 16, if \((G, \mathcal{F})\) is a \(B_n\)-generalized pseudo-Kähler structure with components \((g, J_+, J_-, X_+, X_-)\), then the Killing field \(X_-\) commutes with \(X_+\) and preserves the endomorphisms \(J_\pm\).

Proof  In order to prove that \(X_+\) and \(X_-\) commute, we write
\[
\mathcal{L}_{X_-} X_+ = \nabla_{X_-} X_+ - \nabla_{X_+} X_- = \frac{1}{2} H(X_-, X_+) - \frac{1}{2} H(X_-, X_+) = 0,  \tag{24}
\]
where we used relations (22) and \(i_{X_-} F = 0\). The statements \(\mathcal{L}_{X_-} J_\pm = 0\) follow from
\[
\mathcal{L}_{X_-} J_\pm = \nabla_{X_-} J_\pm - [\nabla X_-, J_\pm],
\]
together with the first relation (22), \((i_{X_-} H)|_{\Lambda^2 T^{(1,0)} M} = 0\) (for \(\mathcal{L}_{X_-} J_-\)) and \(i_{X_-} F = 0\) (for \(\mathcal{L}_{X_-} J_+\)). \(\Box\)

There is a rescaling property of \(B_n\)-generalized pseudo-Kähler structures, which also follows from Theorem 16.

Corollary 19  Let \((G, \mathcal{F})\) be a \(B_n\)-generalized pseudo-Kähler structure on a Courant algebroid \(E_{H, F}\) of type \(B_n\) over an odd dimensional manifold \(M\), with components \((g, J_+, J_-, X_+, X_-)\). For any \(\lambda \in \mathbb{R}\setminus\{0\}\), the data \((\tilde{g} \::= \lambda^2 g, \tilde{J}_+ \::= J_+, \tilde{J}_- \::= J_-, \tilde{X}_+ \::= \frac{1}{\lambda} X_+, \tilde{X}_- \::= \frac{1}{\lambda} X_-)\) defines a \(B_n\)-generalized pseudo-Kähler structure on the Courant algebroid \(E_{\tilde{H}, \tilde{F}}\), where \(\tilde{H} \::= \lambda^2 H\) and \(\tilde{F} \::= \lambda F\).

Proof  Let us denote by \(\tilde{\nabla}^\pm\) the connections associated with the rescaled data. Since
\[
H(X, Y) = g^{-1} H(X, Y, \cdot) = \tilde{g}^{-1} \tilde{H}(X, Y, \cdot) = \tilde{H}(X, Y),
\]
\[
F(X, Y) \otimes X_+ = \tilde{F}(X, Y) \otimes \tilde{X}_+,
\]
for all \(X, Y \in \mathfrak{X}(M)\) and \(\tilde{J}_+ = J_+\), we see that \(\tilde{\nabla}^\pm = \nabla^\pm\). Using this fact, it is easy to check that the rescaled data satisfy the conditions from Theorem 16. For instance, we get
\[
\tilde{\nabla}_X^+ \tilde{X}_+ = \frac{1}{\lambda} \nabla_X^+ X_+ = - \frac{1}{\lambda} J_+ F(X) = - \tilde{J}_+ \tilde{F}(X),
\]
where we have used that \(\tilde{F}(X) = \tilde{g}^{-1} \tilde{F}(X, \cdot) = \frac{1}{\lambda} \tilde{g}^{-1} F(X, \cdot) = \frac{1}{\lambda} F(X)\). \(\Box\)

5.1.2 The Case of Even \(n\)

Assume that \(n\) is even and let \((g, J_+, J_-, X_+, X_-, c_+)\) be the components of \((G, \mathcal{F})\). To simplify the computations we assume that \(X_\pm\) are non-null, i.e. \(g((X_\pm)_p, (X_\pm)_p) \neq 0\) or \(c_+(p) \neq \pm 1\), for any \(p \in M\). We define the connections
\[
D^+_X := \nabla_X - \frac{1}{2} H(X) + \frac{c_+}{1 - c_+^2} F(X) \otimes X_+ - \frac{J_+ F(X)}{1 - c_+^2} \otimes X_-,
\]
\(\square\) Springer
\[ D^-_X := \nabla_X + \frac{1}{2} H(X) \]  

on \( TM \), where \( \nabla \) is the Levi-Civita connection of \( g \).

**Theorem 20** The \( B_n \)-generalized almost pseudo-Hermitian structure \((G, \mathcal{F})\) is \( B_n \)-generalized pseudo-Kähler if and only if the following conditions hold:

i) \( D^\pm \) preserve \( T^{(1,0)}_J \), the vector fields \( \{X^+, X^-\} \) commute and their covariant derivatives \( \nabla X^\pm \) are given by

\[
\nabla_X X^+ = \frac{1}{2} H(X, X^+) + c^+ F(X), \\
\nabla_X X^- = \frac{1}{2} H(X, X^-) - J^+ F(X),
\]

for any \( X \in \mathfrak{X}(M) \).

ii) The forms \( F \) and \( H \) satisfy

\[
F|_{\Lambda^2 T^{(1,0)}_J} = 0, \quad H|_{\Lambda^3 T^{(1,0)}_J} = 0, \quad (i_{X^+} + i_{c^+ X^-} H)|_{\Lambda^2 T^{(1,0)}_J} = 0,
\]

and are related by

\[
F|_{\Lambda^2 T^{(1,0)}_J} = -i (i_{X^-} H)|_{\Lambda^2 T^{(1,0)}_J}.
\]

iii) The function \( c^+ \) is Hamiltonian for the closed 2-form \( F \), with Hamiltonian vector field \( X^+ \):

\[
dc^+ = i_{X^+} F.
\]

We make several comments on the above theorem. When the Courant algebroid is untwisted \((H = 0, F = 0)\), the conditions simplify considerably.

**Corollary 21** In the setting of Theorem 20, assume that \( E \) is untwisted. Then \((G, \mathcal{F})\) is a \( B_n \)-generalized pseudo-Kähler structure if and only if \( \nabla J^\pm = 0 \) and \( \nabla X^\pm = 0 \). If these conditions hold then the function \( c^+ \) is constant.

In computations we shall often replace the condition \( \mathcal{L}_{X^+} X^- = 0 \) from the above theorem with a relation between the forms \( H \) and \( F \), as follows.

**Lemma 22** In the setting of Theorem 20, the commutativity of \( X^+ \) and \( X^- \) can be replaced by the relation

\[
H(X^+, X^-) = c^+ F(X^-) + J^+ F(X^+).
\]

**Proof** The statement follows by writing \( \mathcal{L}_{X^+} X^- = \nabla_{X^+} X^- - \nabla_{X^-} X^+ \) and using relations (26). \( \square \)
**Corollary 23** In the setting of Theorem 20, if \((G, \mathcal{F})\) is a \(B_n\)-generalized pseudo-Kähler structure with components \((g, J_+, J_-, X_+, X_-, c_+)\), then \(X_+\) is a Killing field and the almost complex structure \(J_-\) is integrable.

**Proof** The first relation (26) implies that \(X_+\) is a Killing field. The integrability of \(J_-\) will follow from the proof of Theorem 20 (see Corollary 33 iii) below. □

Another consequence of Theorem 20 is a rescaling property of certain \(B_n\)-generalized pseudo-Kähler structures defined on Courant algebroids \(E_{H,F}\) with trivial 2-form \(F\). On such Courant algebroids the function \(c_+\) is constant for any \(B_n\)-generalized pseudo-Kähler structure (owing to relation (29) above).

**Corollary 24** Let \((G, \mathcal{F})\) be a \(B_n\)-generalized pseudo-Kähler structure on a Courant algebroid \(E_H := E_{H,0}\) of type \(B_n\) over an even dimensional manifold \(M\). Let \((g, J_+, J_-, X_+, X_-, c_+\) be the components of \((G, \mathcal{F})\) and assume that \(c_+ \notin \{-1, 1\}\). Define \(\tilde{g} := \epsilon g\) where \(\epsilon := \text{sign}(1 - c_+^2), \tilde{X}_\pm := |1 - c_+^2|^{-1/2}X_\pm,\) and \(\tilde{J}_\pm \in \Gamma(\text{End} \, TM)\) by \(\tilde{J}_- := J_-, \tilde{J}_+|_{X_+} := J_+|_{X_+}, \tilde{J}_+X_- = \tilde{J}_+X_+ = 0\). Then \((\tilde{g}, \tilde{J}_-, \tilde{J}_+, \tilde{X}_+, \tilde{X}_-, \tilde{c}_+ := 0)\) are the components of a \(B_n\)-generalized pseudo-Kähler structure on the Courant algebroid \(E_{\tilde{H}} := E_{\tilde{H},0}\) where \(\tilde{H} := \epsilon H\).

**Remark 25** The formulation and the proof of Theorem 20 can be adapted to the case when the vector fields \(X_\pm\) are trivial (see Remark 37). Let \((G, \mathcal{F})\) be a \(B_n\)-generalized almost pseudo-Hermitian structure on \(E_{H,F}\) with \(X_+ = X_- = 0\). It turns out that \((G, \mathcal{F})\) is integrable if and only if \(F = 0\) and \((G, \mathcal{F})\) is a generalized pseudo-Kähler structure on the exact Courant algebroid \(TM \oplus T^*M \oplus \mathbb{R}\) with Dorfman bracket twisted by the (closed) 3-form \(H\). Such structures were defined in [8] and are deeply studied in the literature (especially for positive definite signature).

### 5.2 Proofs of Theorems 16 and 20

We start with various general integrability results, which will be used in the proofs of Theorems 16 and 20.

#### 5.2.1 General Integrability Results

Let \((G, \mathcal{F})\) be a \(B_n\)-generalized almost pseudo-Hermitian structure on an odd exact Courant algebroid \((E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \pi)\) over a manifold \(M\) of dimension \(n\) and \(u_0 \in \Gamma(\text{Ker} \, \mathcal{F})\) such that \(\langle u_0, u_0 \rangle = (-1)^n\). For \(u, v \in \Gamma(E)\) and \(A \in \Gamma(\text{End} \, E)\) we define \(L_u v := [u, v]\) and \(L_u A \in \Gamma(\text{End} \, E)\) by

\[(L_u A)(v) := [u, Av] - A[u, v], \quad \forall v \in \Gamma(E)\]

and refer to it as the Dorfman Lie derivative of \(A\) in the direction of \(u\). It is easy to check that \(L_u\) acts as a derivation on the \(\mathbb{R}\)-algebra \(\Gamma(\text{End} \, E)\).
Lemma 26 If $\mathcal{F}$ is integrable, then $L_{u_0}\mathcal{F} = 0$. If $(\mathcal{G}, \mathcal{F})$ is integrable, then $L_{u_0}\mathcal{F} = 0$ and $L_{u_0}G^\text{end} = 0$.

Proof The statement $L_{u_0}\mathcal{F} = 0$ for any $B_n$-generalized complex structure $\mathcal{F}$ and $u_0 \in \Gamma(\text{Ker} \mathcal{F})$ normalized by $\langle u_0, u_0 \rangle = (-1)^n$ was proved in Lemma 4.13 of [13]. Assume that $(\mathcal{G}, \mathcal{F})$ is a $B_n$-generalized pseudo-Kähler structure. Since $\mathcal{F}$ and $\mathcal{F}_2 := G^\text{end} \mathcal{F}$ are integrable, we obtain that $L_{u_0}\mathcal{F} = L_{u_0}\mathcal{F}_2 = 0$. On the other hand, from $L_{u_0}(u_0) = \frac{1}{2}\pi^* d\langle u_0, u_0 \rangle = 0$ we deduce that $L_{u_0}$ preserves $u_0^\perp$. Combined with $G^\text{end} = -\mathcal{F}\mathcal{F}_2$ on $u_0^\perp$ we obtain $L_{u_0}G^\text{end} = 0$. \hfill $\Box$

Let $L_1 \subset E_\mathbb{C}$ be the $i$-eigenbundle of $\mathcal{F}$ and $E_\pm$ the $\pm 1$-eigenbundles of $G^\text{end}$. From (11),

$$L_1 = L_1 \cap (E_+)\mathbb{C} \oplus L_1 \cap (E-)\mathbb{C} = L_1 \cap L_2 \oplus L_1 \cap \bar{L}_2,$$

where $L_2$ is the $i$-eigenbundle of $\mathcal{F}_2 := G^\text{end} \mathcal{F}$. Let $L_1^+ := L_1 \cap L_2$ and $L_1^- := L_1 \cap \bar{L}_2$.

Lemma 27 Assume that $\mathcal{F}$ is integrable. Then the Dorfman Lie derivative $L_{u_0} : \Gamma(E) \to \Gamma(E)$ preserves $\Gamma(L_1^+)$ if and only if it preserves $\Gamma(L_1^-)$.

Proof We remark that

$$L_1^- = \{ v \in L_1 | \langle v, w \rangle = 0, \forall w \in \bar{L}_1^+ \},$$

$$L_1^+ = \{ v \in L_1 | \langle v, w \rangle = 0, \forall w \in L_1^- \}. \tag{31}$$

Assume that $L_{u_0}$ preserves $\Gamma(L_1^+)$. Then $L_{u_0}$ preserves also $\Gamma(\bar{L}_1^+)$. For any $v \in \Gamma(L_1^-)$ and $w \in \Gamma(\bar{L}_1^+)$,

$$\langle [u_0, v], w \rangle = -\langle v, [u_0, w] \rangle = 0. \tag{32}$$

where we used $\langle v, w \rangle = 0$, $[u_0, w] \in \Gamma(\bar{L}_1^+)$ and the first relation (31). Since $\mathcal{F}$ is integrable, $[u_0, v] \in \Gamma(L_1)$ (see Lemma 26). The first relation (31) and relation (32) imply that $[u_0, v] \in \Gamma(\bar{L}_1^-)$. We proved that $L_{u_0}$ preserves $\Gamma(L_1^-)$. The converse statement follows in a similar way, by using the second relation (31). \hfill $\Box$

The next proposition is the analogue of Proposition 6.10 of [8].

Proposition 28 Let $(\mathcal{G}, \mathcal{F})$ be a $B_n$-generalized almost pseudo-Hermitian structure. Then $(\mathcal{G}, \mathcal{F})$ is integrable if and only if $L_1$, $L_1^\pm$ are integrable and any of the equivalent conditions from Lemma 27 holds.

Proof If $(\mathcal{G}, \mathcal{F})$ is integrable then obviously $L_1^\pm$ and $L_1$ are integrable. From Lemma 4.13 of [13], $L_{u_0}$ preserves $\Gamma(L_1^\pm)$.

For the converse, we need to show that $L_2 = L_1^+ \oplus L_1^-$ is integrable, i.e.

$$[\Gamma(L_1^+), \Gamma(L_1^-)] \subset \Gamma(L_1^+ \oplus L_1^-). \tag{33}$$
The map which assigns to $X \in \mathcal{L}_1$ the covector $\xi \in \mathcal{L}_1^*$ defined by $\xi(Y) := \langle X, Y \rangle$, for any $Y \in \mathcal{L}_1$, is an isomorphism, which maps $\mathcal{L}_1^* \subset \mathcal{L}_1$ onto $\text{Ann} \left( \mathcal{L}_1^* \right) \subset \mathcal{L}_1^*$. We identify $\mathcal{L}_1$ with $\mathcal{L}_1^*$ by means of this isomorphism. In particular, $\mathcal{L}_1^*$ inherits a Lie algebroid structure from the Lie algebroid structure of $\mathcal{L}_1$. As proved in [13] (Section 4.4, page 66), for any $X \in \Gamma(\mathcal{L}_1)$ and $\xi \in \Gamma(\mathcal{L}_1^*)$,

$$\left[ X, \xi \right] = \mathcal{L}_X \xi - i_\xi d_{\mathcal{L}_1} X + (-1)^n (\left[ u_0, X \right], \xi) u_0, \quad (34)$$

where $\mathcal{L}_X$ denotes the Lie derivative of the Lie algebroid $\mathcal{L}_1$, defined by the Cartan formula $\mathcal{L}_X := i_X d_{\mathcal{L}_1} + d_{\mathcal{L}_1} i_X$, and

$$d_{\mathcal{L}_1} : \Gamma(\Lambda^k \mathcal{L}_1) \rightarrow \Gamma(\Lambda^{k+1} \mathcal{L}_1), \quad d_{\mathcal{L}_1}^* : \Gamma(\Lambda^k \mathcal{L}_1^*) \rightarrow \Gamma(\Lambda^{k+1} \mathcal{L}_1^*)$$

are the exterior derivatives of the Lie algebroids $\mathcal{L}_1$ and $\mathcal{L}_1^*$. Assume now that $X \in \Gamma(\mathcal{L}_1^*)$ and $\xi \in \Gamma(\mathcal{L}_1^*)$. Since $u_{a_0}$ preserves $\Gamma(\mathcal{L}_1^*)$, the last term from (34) vanishes and we obtain

$$\left[ X, \xi \right] = \mathcal{L}_X \xi - i_\xi d_{\mathcal{L}_1} X = i_X d_{\mathcal{L}_1} \xi - i_\xi d_{\mathcal{L}_1}^* X, \quad (35)$$

where we used $\xi(X) = 0$. On the other hand, if $\mathcal{A}$ is an arbitrary Lie algebroid with a decomposition $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ where $\mathcal{A}_i$ are integrable subbundles of $\mathcal{A}$, then $i_Y d_{\mathcal{A}} \eta \in \Gamma(\text{Ann}(\mathcal{A}_i))$, for any $Y \in \Gamma(\mathcal{A}_i)$ and $\eta \in \Gamma(\text{Ann}(\mathcal{A}_i))$. Applying this result to the Lie algebroids $\mathcal{L}_1 = \mathcal{L}_1^+ \oplus \mathcal{L}_1^-$ and

$$\mathcal{L}_1^* = (\mathcal{L}_1^+)^* \oplus (\mathcal{L}_1^-)^* = \text{Ann}(\mathcal{L}_1^+) \oplus \text{Ann}(\mathcal{L}_1^+) = \mathcal{L}_1^- \oplus \mathcal{L}_1^-$$

we obtain that $i_X d_{\mathcal{L}_1} \xi$ is a section of $\text{Ann}(\mathcal{L}_1^+) = \mathcal{L}_1^-$ and $i_\xi d_{\mathcal{L}_1}^* X$ is a section of $\mathcal{L}_1^+$ (in the last statement we identified $\mathcal{L}_1^+$ with $\text{Ann}(\mathcal{L}_1^+)$ in the natural isomorphism $\mathcal{L}_1 = (\mathcal{L}_1^*)^*$). Relation (33) follows.

Let $E = E_{H, F}$ be a Courant algebroid of type $B_\nu$ over a manifold $M$ with Dorfman bracket twisted by $(H, F)$. On $M$ we consider a pseudo-Riemannian metric $h$, a complex distribution $\mathcal{D} \subset (TM)_\mathbb{C}$ and a vector field $X_0$ (real or complex), $h$-orthogonal to $\mathcal{D}$ and such that $h(X_0, X_0) = 1$. We define the subbundles of $E_C$ by

$$L^{h, \mathcal{D}} := \{ X + h(X) \mid X \in \mathcal{D} \},$$
$$L^{h, \mathcal{D}, X_0} := L^{h, \mathcal{D}} \oplus \text{span}_\mathbb{C} \{ X_0 + h(X_0) + i \}. $$

Let $Y_0 \in \mathcal{X}(M)$, $f \in C^\infty(M, \mathbb{C})$ and

$$u^{Y_0} := Y_0 + h(Y_0), \quad u^{Y_0, f} := Y_0 - h(Y_0) + f,$$

which are sections of $E_C$. Let $\nabla$ be the Levi-Civita connection of $h$. 

\[\square\] Springer
Lemma 29  

i) The bundle $L^{h, D}$ is integrable if and only if $D$ is involutive, $F|\Lambda^2 D = 0$ and, for any $X, Y \in \Gamma(D)$ and $Z \in \mathfrak{X}(M)$, 

$$h(\nabla_Z X, Y) = \frac{1}{2} H(X, Y, Z).$$  

(36)

ii) The Dorfman Lie derivative $L_{uY_0}$ preserves $\Gamma(L^{h, D})$ if and only if 

$$\mathcal{L}_{Y_0} \Gamma(D) \subset \Gamma(D), \ (i_{Y_0} F)|_D = 0$$

and, for any $X \in \Gamma(D)$ and $Z \in \mathfrak{X}(M)$,

$$h(\nabla_Z Y_0, X) = \frac{1}{2} H(Y_0, X, Z).$$

(37)

iii) The Dorfman Lie derivative $L_{uY_0 + }$ preserves $\Gamma(L^{h, D})$ if and only if

$$\mathcal{L}_{Y_0} \Gamma(D) \subset \Gamma(D), \ (i_{Y_0} F)|_D = (df)|_D$$

and, for any $X \in \Gamma(D)$ and $Z \in \mathfrak{X}(M)$,

$$h(\nabla_X Y_0, Z) = \frac{1}{2} H(Y_0, X, Z) - f F(X, Z).$$

(38)

iv) The bundle $L^{h, D, X_0}$ is integrable if and only if the following conditions hold:

1. for any $Y, Z \in \Gamma(D)$,

$$\mathcal{L}_Y Z + i F(Y, Z) X_0 \in \Gamma(D),$$

$$\mathcal{L}_{X_0} Y + i F(X_0, Y) X_0 \in \Gamma(D);$$

(39)

2. for any $Y, Z \in \Gamma(D)$ and $X \in \mathfrak{X}(M)$,

$$h(\nabla_X Y, Z) = \frac{1}{2} H(X, Y, Z),$$

$$h(\nabla_X X_0, Y) = -\frac{1}{2} H(X_0, X, Y) + i F(X, Y).$$

(40)

Proof The proof is a straightforward computation, which uses the expression (3) of the Dorfman bracket. For example, to prove claim i) let $X, Y \in \Gamma(D)$. Then

$$[X + h(X), Y + h(Y)] = \mathcal{L}_X Y + \mathcal{L}_X (h(Y)) - i_Y dh(X) - H(X, Y, \cdot) + F(X, Y)$$

(41)

is a section of $L^{h, D}$ if and only if $\mathcal{L}_X Y \in \Gamma(D)$, $F(X, Y) = 0$ and

$$h(\mathcal{L}_X Y) = \mathcal{L}_X (h(Y)) - i_Y dh(X) - H(X, Y, \cdot).$$
Applying the above relation to $Z \in \mathfrak{X}(M)$ and writing it in terms of $\nabla$ we obtain (36).

The other claims can be obtained similarly.

**Lemma 30** Let $(E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \pi)$ be a Courant algebroid of type $B_n$ over a manifold $M$, with Dorfman bracket twisted by $(H, F)$, $h$ be a pseudo-Riemannian metric on $M$ and $\mathcal{D}_\pm \subset (TM)_C$ two complex distributions, such that $\mathcal{D}_+$ is isotropic. Let $X_0 \in \mathfrak{X}(M)_C$ be a complex vector field orthogonal to $\mathcal{D}_+$, such that $h(X_0, X_0) = 1$. Assume that the following relations hold:

i) for any $X \in \Gamma(\mathcal{D}_-)$ and $Y \in \Gamma(\mathcal{D}_+)$,
\[ \nabla_Y X - \frac{1}{2} H(X, Y) \in \Gamma(\mathcal{D}_-); \]  
(42)

ii) for any $X \in \Gamma(\mathcal{D}_-)$ and $Y \in \Gamma(\mathcal{D}_+)$,
\[ \nabla_X Y - \frac{1}{2} H(X, Y) \in \Gamma(\mathcal{D}_+ \oplus \text{span}_C\{X_0\}); \]  
(43)

iii) for any $X \in \Gamma(\mathcal{D}_-)$,
\[ \nabla_{X_0} X + \frac{1}{2} H(X_0, X) \in \Gamma(\mathcal{D}_-); \]  
(44)

iv) for any $X \in \Gamma(\mathcal{D}_-)$,
\[ \nabla_X X_0 + \frac{1}{2} H(X_0, X) - iF(X) \in \Gamma(\mathcal{D}_+ \oplus \text{span}_C\{X_0\}); \]  
(45)

v) for $X \in \Gamma(\mathcal{D}_-)$,
\[ F(X) \in \Gamma(\mathcal{D}_-). \]  
(46)

Then
\[ [\Gamma(L^{-h,\mathcal{D}_-}), \Gamma(L^h,\mathcal{D}_+,X_0)] \subset \Gamma(L^{-h,\mathcal{D}_-} \oplus L^h,\mathcal{D}_+,X_0). \]  
(47)

**Proof** In order to prove (47) we need to show that $[X - h(X), Y + h(Y)]$ and $[X - h(X), X_0 + h(X_0) + i]$ are sections of $L^{-h,\mathcal{D}_-} \oplus L^h,\mathcal{D}_+,X_0$, for any $X \in \Gamma(\mathcal{D}_-)$ and $Y \in \Gamma(\mathcal{D}_+)$. Remark that
\[ L^{-h,\mathcal{D}_-} \oplus L^h,\mathcal{D}_+,X_0 \]
\[ = \{ X - h(X) \mid X \in \mathcal{D}_- \} \oplus \{ Y + h(Y) + ih(Y, X_0) \mid Y \in \mathcal{D}_+ \oplus \text{span}_C\{X_0\} \}, \]
where we used that $X_0$ is of norm one and orthogonal to $\mathcal{D}_+$. Writing $\mathcal{L}_X Y = \mathcal{F}_+(X, Y) - \mathcal{F}_-(X, Y)$, where
\[ \mathcal{F}_+(X, Y) := \nabla_X Y - \frac{1}{2} H(X, Y) \in \Gamma(\mathcal{D}_+ \oplus \text{span}_C\{X_0\}), \]
\[ \mathcal{F}'_-(X, Y) := \nabla_Y X - \frac{1}{2} H(X, Y) \in \Gamma(D_-), \]

(compare relations (42) and (43)), we obtain

\[
[X - h(X), Y + h(Y)] = \mathcal{F}'_+(X, Y) - \mathcal{F}'_-(X, Y) + \mathcal{L}_X (h(Y)) + iY dh(X) - H(X, Y, \cdot) + F(X, Y).
\]

On the other hand, taking the \( h \)-inner product with \( Y \) of the left hand side of (45) and using that \( D_+ \) is isotropic and \( X_0 \) is orthogonal to \( D_+ \), we obtain

\[
F(X, Y) = ih(\mathcal{F}'_+(X, Y), X_0).
\]  

Using the definition of \( \mathcal{F}'_\pm \) it is easy to see that

\[
\mathcal{L}_X (h(Y)) + iY dh(X) - H(X, Y, \cdot) = h(\mathcal{F}'_+(X, Y)) + h(\mathcal{F}'_-(X, Y)).
\]  

We deduce that

\[
[X - h(X), Y + h(Y)] = \mathcal{F}'_+(X, Y) + h(\mathcal{F}'_+(X, Y)) + ih(\mathcal{F}'_+(X, Y), X_0)
- \mathcal{F}'_-(X, Y) + h(\mathcal{F}'_-(X, Y))
\]

is a section of \( L^{-h,D_-} \oplus L^{h,D_+,X_0} \). Similar computations show that

\[
[X - h(X), X_0 + h(X_0) + i] = \mathcal{F}_+(X) + h(\mathcal{F}_+(X)) + ih(\mathcal{F}_+(X), X_0)
- \mathcal{F}_-(X) + h(\mathcal{F}_-(X))
\]

is also a section of \( L^{-h,D_-} \oplus L^{h,D_+,X_0} \), where

\[
\mathcal{F}_+(X) := \nabla_X X_0 + \frac{1}{2} H(X_0, X) - iF(X) \in \Gamma(D_+ \oplus \text{span}_\mathbb{C}\{X_0\}),
\]

\[
\mathcal{F}_-(X) := \nabla_X X_0 + \frac{1}{2} H(X_0, X) - iF(X) \in \Gamma(D_-)
\]

(from relations (44), (45) and (46)).

\[\square\]

### 5.2.2 Application of General Integrability Results

We now turn to the setting of Sect. 5.1, and, using the results from the previous section, we prove Theorems 16 and 20. Consider the setting of these theorems. We start by computing the bundles \( L^{\pm}_1 \) associated to the \( B_n \)-generalized pseudo-Hermitian structure \( (\mathcal{G}, \mathcal{F}) \), as in Proposition 28. They turn out to be of the form \( L^{h,D} \) or \( L^{h,D,X_0} \) for suitably chosen \( h, D \) and \( X_0 \).
Lemma 31  
i) If \( n \) is odd then
\[
L^+_1 = \{ X + g(X) \mid X \in T^{(1,0)}_{J_+}(M) \} \oplus \text{span}_\mathbb{C}\{X_+ + g(X_+) + i\},
\]
\[
L^-_1 = \{ X - g(X) \mid X \in T^{(1,0)}_{J_-}(M) \}.
\] (52)

ii) If \( n \) is even then
\[
L^+_1 = \{ X + g(X) \mid X \in T^{(1,0)}_{J_+}(M) \} \oplus \text{span}_\mathbb{C}\{V_- + g(V_-) + i\},
\]
\[
L^-_1 = \{ X - g(X) \mid X \in T^{(1,0)}_{J_-}(M) \},
\] (53)

where \( V_- := \frac{1}{1-c_+}(X_- - i c_+ X_+) \) is of norm one and orthogonal to \( T^{(1,0)}_{J_+}(M) \).

**Proof**  
The proof is straightforward from Proposition 13. In particular, one can easily check from the formulas for \( J_+ \) and \( F_\pm \) in Proposition 13 that \( J_+ (X_+ + i) = i (X_+ + i) \) and \( F_+ (V_- + i) = i (V_- + i) \), which are equivalent to \( F (X_+ + g(X_+) + i) = i (X_+ + g(X_+) + i) \) and \( F (V_- + g(V_-) + i) = i (V_- + g(V_-) + i) \), respectively. \( \square \)

**Corollary 32**  
Assume that \( n \) is odd. The following are equivalent:

i) \( (G, F) \) is a \( B_n \)-generalized pseudo-Kähler structure;

ii) \( L^+_1 \) are integrable and the Dorfman Lie derivative \( L_u_- \) preserves \( \Gamma (L^-_1) \), where \( u_- := X_+ - g(X_-) \);

iii) \( T^{(1,0)}_{J_+}(M) \) is involutive, \( F \mid_{\Lambda^2 T^{(1,0)}_{J_+}(M)} = 0, i_{X_-} F = 0 \), the Lie derivative \( \mathcal{L}_{X_-} \) preserves \( \Gamma (T^{(1,0)}_{J_+}(M)) \) and the following relations hold:

1. for any \( X, Y \in \Gamma(T^{(1,0)}_{J_+}(M)) \) and \( Z, V \in \mathcal{X}(M) \),
\[
g (\nabla_Z X, Y) = -\frac{1}{2} H(X, Y, Z),
\]
\[
g (\nabla_Z X_-, V) = -\frac{1}{2} H(X_-, V, Z);
\] (54)

2. for any \( Y, Z \in \Gamma(T^{(1,0)}_{J_+}(M)) \),
\[
\mathcal{L}_Y Z + i F(Y, Z) X_+ \in \Gamma(T^{(1,0)}_{J_+}(M)),
\]
\[
\mathcal{L}_{X_+} Z + i F(X_+, Z) X_+ \in \Gamma(T^{(1,0)}_{J_+}(M));
\] (55)

3. for any \( Y, Z \in \Gamma(T^{(1,0)}_{J_+}(M)) \) and \( X \in \mathcal{X}(M) \),
\[
g (\nabla_X Y, Z) = \frac{1}{2} H(X, Y, Z),
\]
\[
g (\nabla_X X_+, Y) = -\frac{1}{2} H(X_+, X, Y) + i F(X, Y).
\] (56)
Proof The implication i) $\implies$ ii) follows from Proposition 28, while the equivalence between ii) and iii) follows from Lemma 31 i), Lemma 29 ii) with $h := -g$ and $D := T_{J_+}^{(1,0)} M$, Lemma 29 iv) and Lemma 29 iv) with $h := g$, $D := T_{J_-}^{(1,0)} M$ and $Y_0 := X_-$ in order to prove that iii) implies i) we apply again Proposition 28. We need to show that the relations from iii) imply that $L_1$ is integrable, or

$$\left[ \Gamma(L_1^+), \Gamma(L_1^-) \right] \subset \Gamma(L_1) = \Gamma(L_1^+ \oplus L_1^-).$$  \ (57)

The above relation follows from Lemma 30, by noticing that the assumptions from this lemma are implied by the conditions from iii). (The lemma is specialized to $D_\pm = T_{J_\pm}^{(1,0)} M$, $h = g$ and $X_0 = X_+$). For example, the first relation (54) implies that $\nabla X - \frac{1}{2} H(X, Y)$ is orthogonal to $T_{J_-}^{(1,0)} M$, i.e. is a section of $T_{J_-}^{(1,0)} M \oplus \text{span}_\mathbb{C} \{X_\pm\}$, for any $X \in \Gamma(T_{J_-}^{(1,0)} M)$ and $Y \in \Gamma(T_{J_-}^{(1,0)} M)$. From the second relation (54), it is also orthogonal to $X_-$. Relation (42) follows. The other relations can be checked similarly.

Corollary 33 Assume that $n$ is even. The following are equivalent:

i) $(\mathcal{G}, \mathcal{F})$ is a $B_n$-generalized pseudo-Kähler structure;

ii) $L_\pm$ are integrable and the Dorfman Lie derivative $\mathbf{L}_{u_\pm}$ preserves $\Gamma(L_1^-)$, where $u_+ := X_+ + g(X_+) + c_+$;

iii) $J_-$ is integrable, $F \mid_{\Lambda^2 T_{J_-}^{(1,0)} M} = 0$ and the following relations hold:

1. for any $X, Y \in \Gamma(T_{J_-}^{(1,0)} M)$ and $Z \in \mathfrak{X}(M)$,

$$g(\nabla_Z X, Y) = -\frac{1}{2} H(X, Y, Z),$$

$$g(\nabla_X X_+, Z) = -\frac{1}{2} H(X_+, X, Z) + c_+ F(X, Z);$$  \ (58)

2. for any $X, Y \in \Gamma(T_{J_+}^{(1,0)} M)$,

$$\mathbf{L}_X Y + i F(X, Y) V_- \in \Gamma(T_{J_+}^{(1,0)} M),$$

$$\mathbf{L}_V_- X + i F(V_-, X) V_- \in \Gamma(T_{J_+}^{(1,0)} M),$$  \ (59)

where, we recall, $V_- = \frac{1}{1-c_+}[X_--i c_+ X_+]$;

3. for any $X, Y \in \Gamma(T_{J_+}^{(1,0)} M)$ and $Z \in \mathfrak{X}(M)$,

$$g(\nabla_Z X, Y) = \frac{1}{2} H(X, Y, Z),$$

$$g(\nabla_Z V_-, X) = \frac{1}{2} H(V_-, X, Z) - i F(X, Z);$$  \ (60)
4.

\[ i_{X^+}F = dc_+. \tag{61} \]

**Proof** The claims follow with a similar argument, as in Corollary 32. We only explain the implication (ii) \( \implies \) (iii). We use Lemma 31 (ii), Lemma 29 (i) with \( h := -g \) and \( D := T_{J_+}^{(1,0)}M \), Lemma 29 (iii) with \( h := -g \), \( D := T_{J_-}^{(1,0)}M \), \( Y_0 := X_+ \), \( f := c_+ \) and Lemma 29 (iv) with \( h := g \), \( D := T_{J_+}^{(1,0)}M \), \( X_0 := V_- \). We obtain that \( L_1^\pm \) are integrable and the Dorfman Lie derivative \( L_{u^+} \) preserves \( \Gamma(L_1^-) \) if and only if the relations 1., 2. and 3. from (iii) hold, together with \( i_{X^+}F = dc_+ \) on \( T_{J_-}^{(1,0)}M \) and \( L_{X^+} \Gamma(T_{J_+}^{(1,0)}M) \subset \Gamma(T_{J_-}^{(1,0)}M) \). However, the real one-form \( i_{X^+}F - dc_+ \) vanishes on \( T_{J_-}^{(1,0)}M \) if and only if it is zero, since \( J_- \) is an almost complex structure. Finally, \( L_{X^+} \Gamma(T_{J_+}^{(1,0)}M) \subset \Gamma(T_{J_-}^{(1,0)}M) \) follows by combining the first relation (58) with \( Z := X_+ \) with the second relation (58) with \( Z := Y \) and using that \( F(X, Y) = 0 \) for any \( X, Y \in \Gamma(T_{J_-}^{(1,0)}M) \), since \( F \) is of type \((1, 1)\) with respect to \( J_- \). \( \square \)

**Remark 34** We remark that on the support of the function \( c_+ \) the relation (61) is automatically satisfied if the other conditions listed under (iii) in Corollary 33 hold. In fact,

\[ (i_{X^+}F - dc_+)_{|T_{J_-}^{(1,0)}M} = 0, \]

and hence (61), follows from the second relation (58), by letting \( Z := X_+ \) and using \( g(X_+, X_+) = 1 - c_+^2 \).

We conclude the proofs of Theorems 16 and 20 by noticing that the conditions from Corollary 32 (iii) and 33 (iii) are equivalent to the conditions from these theorems. This is done in the next two lemmas.

**Lemma 35** Assume that \((g, J_+, J_-, X_+, X_-)\) are the components of a \(B_n\)-generalized almost pseudo-Hermitian structure on a Courant algebroid \( E = EH_1F\) of type \(B_n\) over an odd dimensional manifold \(M\). The conditions from Corollary 32 (iii) are equivalent to the conditions from Theorem 16.

**Proof** Assume that the conditions from Corollary 32 (iii) are satisfied. Relations (54) imply that \( \nabla^- \) preserves \( T_{J_-}^{(1,0)}M \) and \( \nabla^- X_- = 0 \). The second relation (56) implies that

\[ \nabla_X X_+ + \frac{1}{2} H(X_+, X) - iF(X) \in \Gamma(T_{J_+}^{(1,0)}M \oplus \text{span}_\mathbb{C}\{X_+\}), \tag{62} \]

for any \( X \in \mathfrak{X}(M) \). Using that \( g(X_+, X_+) = 1 \) we obtain

\[ \nabla_X X_+ + \frac{1}{2} H(X_+, X) - i(F(X) + F(X_+, X)X_+) \in \Gamma(T_{J_+}^{(1,0)}M). \tag{63} \]
Since both, the real and the imaginary parts of the left-hand side of (63) are orthogonal to \( X_+ \), \( J_+ \mid_{\text{span}(X_+)} \) is a complex structure and \( J_+ X_+ = 0 \), we deduce from (63) that

\[
\nabla_{X_+} X_+ = \frac{1}{2} H(X_+, X_+) - J_+ F(X), \quad \forall X \in \mathfrak{X}(M), \tag{64}
\]

which is equivalent to the second relation (19). We now prove that \( \nabla^+ \) preserves \( T_{J_+}^{(1,0)} M \). The first relation (56) implies that for any \( X \in \mathfrak{X}(M) \) and \( Y \in \Gamma(T_{J_+}^{(1,0)} M) \),

\[
\nabla_{X_+} Y - \frac{1}{2} H(X, Y) \in \Gamma(T_{J_+}^{(1,0)} M \oplus \text{span}_C\{X_+\}). \tag{65}
\]

It is easy to see that (64) combined with (65) imply that \( \nabla^+ \) preserves \( T_{J_+}^{(1,0)} M \). The relations from Theorem 16 i) follow.

The relations from Theorem 16 ii) can be obtained by similar computations. To prove that \( H \mid_{\Lambda^3 T_{J_+}^{(1,0)} M} = 0 \) let \( X, Y \in \Gamma(T_{J_+}^{(1,0)} M) \) and write

\[
\mathcal{L}_X Y = \nabla^+_X Y - \nabla^+_Y X - H(X, Y). \tag{66}
\]

Since \( T_{J_+}^{(1,0)} M \) is involutive (from the first condition of Corollary 32 iii)) and \( \nabla^- \) preserves \( T_{J_-}^{(1,0)} M \), we obtain that \( H(X, Y) \) is a section of \( T_{J_-}^{(1,0)} M \). In particular, \( H \mid_{\Lambda^3 T_{J_-}^{(1,0)} M} = 0 \). To prove that \( H \mid_{\Lambda^3 T_{J_+}^{(1,0)} M} = 0 \) and \( (i X_+ H) = i F \) on \( \Lambda^2 T_{J_+}^{(1,0)} M \) we use the first relation (55), together with

\[
\mathcal{L}_Y Z = \nabla^+_Y Z - \nabla^+_Z Y + H(Y, Z) - (F(Y, J_+ Z) - F(Z, J_+ Y)) X_+ \\
= \nabla^+_Y Z - \nabla^+_Z Y + H(Y, Z) - 2i F(Y, Z) X_+ ,
\]

for any \( Y, Z \in \Gamma(T_{J_+}^{(1,0)} M) \), and the fact that the distribution \( T_{J_+}^{(1,0)} M \) is preserved by the connection \( \nabla^+ \). (Recall that \( \nabla^+ \) is related to \( \nabla \) by (18).) Finally, relation \( (i X_+ H) \mid_{\Lambda^2 T_{J_-}^{(1,0)} M} = 0 \) follows by writing

\[
\mathcal{L}_{X_-} X = \nabla^-_{X_-} X - H(X_- , X)
\]

and using that \( \mathcal{L}_{X_-} \) and \( \nabla^- \) preserve \( \Gamma(T_{J_-}^{(1,0)} M) \).

In the same vein one proves that conditions from Theorem 16 imply those from Corollary 32 iii).

It remains to explain how Theorem 20 follows from Corollary 33.

**Lemma 36** Assume that \((g, J_+, J_-, X_+, X_-, c_+)\) are the components of a \( B_n\)-generalized almost pseudo-Hermitian structure on a Courant algebroid \( E = E_{H,F} \) of type \( B_n \) over an even dimensional manifold \( M \), such that \( X_+ \) and \( X_- \) are non-null. Then the conditions from Corollary 33 iii) are equivalent to the conditions from Theorem 20.
Proof} Straightforward computations as in the previous lemma show that all conditions from Corollary 33 iii), except the second relation (59), are equivalent to all conditions from Theorem 20, except $L_{X_+}X_- = 0$ (or relation (30), see Lemma 22). Assume that these equivalent conditions hold. Under this hypothesis, we show that the second equivalent to the statement that the expression

$$1 - c_+X_+.$$ 

Writing

$$L_{V_-}X = \nabla_{V_-}X - \nabla_X V_- = D_{V_-}^+X + \frac{1}{2}H(V_-, X) - iF(V_-, X)V_-$$

$$- X(\frac{1}{1 - c_+^2})(X_+ - ic_+X_+) - \frac{1}{1 - c_+^2}(\nabla_X X_- - iX(c_+)X_+ - ic_+\nabla_X X_+)$$

and using the expressions of $\nabla_X X_+$ and $\nabla_X X_-$ provided by relations (26), together with the fact that $D_+$ preserves $T_{J_+}^{(1,0)}M$, we obtain that the second relation (59) is equivalent to the statement that the expression

$$\text{Expr}(X) := H(V_-, X) + \frac{X(c_+)}{1 - c_+^2}(iX_+ - 2c_+V_-) + \frac{1}{1 - c_+^2}(J_+F(X) + ic_+^2F(X))$$

is a section of $T_{J_+}^{(1,0)}M$, for any $X \in \Gamma(T_{J_+}^{(1,0)}M)$. Equivalently, $\text{Expr}(X)$ is orthogonal to $T_{J_+}^{(1,0)}M$, $X_+$ and $X_-$. The condition

$$g(\text{Expr}(X), Y) = 0, \forall X, Y \in \Gamma(T_{J_+}^{(1,0)}M)$$

is equivalent to $(i_{V_-}H)|_{\Lambda^2T_{J_+}^{(1,0)}M} = iF|_{\Lambda^2T_{J_+}^{(1,0)}M}$, which follows from the third relation (27) and the first relation (28). Similarily, one can show that the relations

$$g(\text{Expr}(X), X_\pm) = 0, \forall X \in \Gamma(T_{J_+}^{(1,0)}M)$$

are equivalent to

$$H(X_+, X_-, X) = c_+F(X_-, X) - (J_+X)(c_+), \quad (67)$$

for all $X \in \text{span}\{X_+, X_-\}$. Using $dc_+ = ic_+F$ we obtain that relation (67) is equivalent to relation (30), as needed. \hfill \square

We end this section with a comment which shows that the notion of $B_n$-generalized Kähler structure includes the notion of generalized Kähler structure on exact Courant algebroids, as defined in [8].

Remark 37 Let $(G, \mathcal{F})$ be a $B_n$-generalized almost pseudo-Hermitian structure with components $(g, J_+, J_-, X_+, X_-, c_+)$ on a Courant algebroid $E_{H, F}$ of type $B_n$ over a manifold $M$ of even dimension. Assume that $X_+ = X_- = 0$ (in particular $c_+ \in \{\pm 1\}$).
Then $J_\pm$ are $g$-skew-symmetric almost complex structures on $M$ and (in the notation introduced in Sect. 4) $u_+ = c_+$. The bundles $L_1^\pm$ from Proposition 28 are given by

$$L_1^{\pm} = \{ X \pm g(X) \mid X \in T_{J_\pm}^{(1,0)}M \}. \quad (68)$$

Adapting the above arguments one can show that $L_1^{\pm}$ are integrable and $L_{u_+}$ preserves $\Gamma(L_1^{\pm})$ if and only if $F = 0$, $J_\pm$ are (integrable) complex structures and the connections $\nabla_X \pm \frac{1}{2} H(X)$ preserve $J_\mp$. These conditions hold if and only if $(G, \mathcal{F})$ is a $B_n$-generalized pseudo-Kähler structure. When these conditions hold and $G$ is positive definite, $(g, J_+, J_-, b := 0)$ is the bi-Hermitian structure of a generalized Kähler structure on the exact Courant algebroid $TM \oplus T^*M$ with Dorfman bracket twisted by the closed 3-form $H$ (see Theorem 6.28 of [8]).

### 6 $B_2^\pm, B_3^\pm, B_4$-Generalized Pseudo-Kähler Structures

In this section we show that Theorems 16 and 20 simplify considerably when $M$ has dimension two, three or four. Below we denote by $X^g$ the 1-form $g$-dual to a vector field $X \in \mathcal{X}(M)$.

**Corollary 38**  

i) Let $(g, J_+, J_-, X_+, X_-, c_+)$ be the components of a $B_2$-generalized pseudo-Kähler structure $(G, \mathcal{F})$ on a Courant algebroid $E_F := E_{0,F}$ over a 2-dimensional manifold $M$. The metric $g$ is either positive or negative definite and there is $\epsilon_0 \in \{ \pm 1 \}$ such that

$$J_- X_+ = -\epsilon_0 X_-, \quad J_- X_- = \epsilon_0 X_+, \quad J_+ = \epsilon_0 c_+ J_-.$$  \quad (69)

If $c_+(p) \neq 0$ for any $p \in M$ then $X_+$ is a Killing field with $g(X_+, X_+) < 1$ and $F = \frac{1}{2c_+} dX_+^g$. If $c_+(p) = 0$ for any $p \in M$ then $X_+$ is a parallel unit field and $F = 0$.

ii) Conversely, any pair $(g, X_+)$ formed by a pseudo-Riemannian metric $g$ of definite signature and a vector field $X_+$ such that either $X_+$ is a Killing field with $g(X_+, X_+) < 1$ or $X_+$ is a parallel unit field defines a $B_2$-generalized pseudo-Kähler structure on $E_F = E_{0,F}$ with $F$ defined as in i). The vector field $X_-$ is arbitrarily chosen, orthogonal to $X_+$ and of the same norm as $X_+$, the endomorphisms $J_\pm$ are defined by (69) and $c_+ := \epsilon_+(1 - g(X_+, X_+))^{1/2}$, where $\epsilon_+ \in \{ \pm 1 \}$.

**Proof**  

i) Since $J_-$ is a $g$-skew-symmetric complex structure and $M$ is two-dimensional, $g$ has signature $(2, 0)$ or $(0, 2)$. The first two relations (69) follow from the fact that $X_\pm$ are orthogonal of the same norm, $M$ is two-dimensional and $J_-$ is a $g$-skew-symmetric complex structure. The last relation (69) follows from $J_+ X_+ = -c_+ X_-$ and $J_+ X_- = c_+ X_+$. Since $M$ is 2-dimensional, $H = 0$, $F$ is of type $(1, 1)$ with respect to $J_-$ and the conditions from Theorem 20 (see also Lemma 22) reduce to

$$\nabla_X X_+ = c_+ F(X), \quad \nabla_X X_- = -J_+ F(X), \quad \nabla_X J_- = 0, \quad i_{X_+} F = d c_+, \quad (70)$$
for any $X \in \mathcal{X}(M)$. However, using (69) one can show that the second and third relation (70) are implied by the first and fourth relation. When $c_+(p) \neq 0$, for any $p \in M$, the first relation (70) implies that $F = \frac{1}{2c_+} dX_+^a$ and the last relation (70) is satisfied. When $c_+(p) = 0$, for any $p \in M$, we obtain from the first and last relation (70) that $\nabla X_+ = 0$ and $F = 0$.

ii) Reversing the argument from i) we obtain claim ii).

Corollary 39 In the setting of Theorem 16 assume that $M$ is 3-dimensional. Then $(\mathcal{G}, \mathcal{F})$ is a $B_3$-generalized pseudo-Kähler structure if and only if $i_X F = 0$ and

\[
\begin{align*}
\nabla X X_- &= -\frac{1}{2} H(X, X_-), \\
\nabla X X_+ &= \frac{1}{2} H(X, X_+) - J_+ F(X),
\end{align*}
\]

(71)

for any $X \in \mathcal{X}(M)$.

Proof Assume that $(\mathcal{G}, \mathcal{F})$ is a $B_3$-generalized pseudo-Kähler structure. Relations (71) were already obtained in the proof of Corollary 17 (see relations (22)). Since the bundles $T_{J_\pm}^{(1,0)} M$ have rank one, relations (20) reduce to $i_{X_-} F = 0$. We now prove that relations (71) imply that $\nabla^\pm$ preserve $T_{J_\pm}^{(1,0)} M$. For this, let $v_-$ be a non-zero local section of $T_{J_-}^{(1,0)} M$. Since $\nabla_- g = 0$ and $v_-$ is isotropic, $g(\nabla^- X v_-, v_-) = 0$. Moreover,

\[
g(\nabla^- X v_, X_-) = -g(v_-, \nabla^- X X_-) = 0
\]

and we obtain that $\nabla^- X v_-$ is a section of $T_{J_-}^{(1,0)} M$, i.e. $\nabla^-$ preserves $T_{J_-}^{(1,0)} M$. A similar argument which uses the second relation (71) and the definition (18) of $\nabla^+$ shows that $\nabla^+$ preserves $T_{J_+}^{(1,0)} M$. Reversing the argument we obtain our claim. □

Corollary 40 In the setting of Theorem 20, assume that $M$ is four dimensional. Then $(\mathcal{G}, \mathcal{F})$ is $B_4$-generalized pseudo-Kähler structure if and only if the following conditions hold:

i) the covariant derivatives of $X_\pm$ with respect to the Levi-Civita connection $\nabla$ of $g$ are given by

\[
\begin{align*}
\nabla X X_+ &= \frac{1}{2} H(X, X_+) + c_+ F(X), \\
\nabla X X_- &= \frac{1}{2} H(X, X_-) - J_+ F(X),
\end{align*}
\]

(72)

for any $X \in \mathcal{X}(M)$.

ii) the connection $D_X := \nabla_X + \frac{1}{2} H(X)$ preserves $J_-;

iii) $F$ is of type $(1, 1)$ with respect to $J_-$, $i_X F = dc_+$ and

\[
H(X_+, X_-) = c_+ F(X_+) + J_+ F(X_+).
\]

(73)
Proof Since \(M\) has dimension four, rank \(T_{J^+}^{(1,0)}M = 1\) and rank \(T_{J^-}^{(1,0)}M = 2\). Like in the proof of Corollary 39, one can show that relations (26) and rank \(T_{J^+}^{(1,0)}M = 1\) imply that \(D^+\) preserves \(T_{J^+}^{(1,0)}M\). We deduce that the conditions from Theorem 20 (see also Lemma 22) reduce to the conditions from Corollary 40.

\[ \square \]

7 Examples Over Lie Groups

In order to illustrate our theory, in this section we construct examples of \(B_n\)-generalized pseudo-Kähler structures on Courant algebroids of type \(B_n\) over Lie groups of dimension two, three and four.

Definition 41 A Courant algebroid \(E_{H,F}\) of type \(B_n\) over a Lie group \(G\) is called left-invariant if the forms \(H \in \Omega^3(G)\) and \(F \in \Omega^2(G)\) are left-invariant. A \(B_n\)-generalized pseudo-Kähler structure is left-invariant if it is defined on a left-invariant Courant algebroid and its components are left-invariant tensor fields.

We identify as usual left-invariant tensor fields on a Lie group \(G\) with tensors on the Lie algebra \(g\) of \(G\). In particular, the forms \(H\) and \(F\) which define a left-invariant Courant algebroid \(E_{H,F}\) over \(G\), as well as the components of a left-invariant \(B_n\)-generalized pseudo-Kähler structure on \(E_{H,F}\), will be considered as tensors on the Lie algebra \(g\).

7.1 The Case \(\dim G = 2\)

Let \(G\) be a 2-dimensional Lie group with Lie algebra \(g\) and \((G, F)\) a left-invariant \(B_2\)-generalized pseudo-Kähler structure on a (left-invariant) Courant algebroid \(E_F := E_{0,F}\) of type \(B_2\) over \(G\), with components \((g, J^+, J^-, X^+, X^-, c^+).\) Recall that a 2-dimensional Lie group with a left-invariant metric admits a (non-zero) left-invariant Killing field if and only if it is abelian. Since (under our overall assumption \(c^+ \neq 1\)) \(X^+\) is a Killing field (see Corollary 23) we deduce that \(G\) is abelian. From Corollary 38, there is a \(g\)-orthonormal basis \(\{v_1, v_2\}\) of \(g\) such that

\[
\begin{align*}
X^+ &= yv_2, & X^- &= yv_1, & J^+v_1 &= \epsilon_0 v_2, \\
J^-v_2 &= -\epsilon_0 v_1, & J^+ &= \epsilon_0 \epsilon_+ (1 - \epsilon y^2)^{1/2} J^-, \\
\end{align*}
\]

where \(\epsilon_0, \epsilon_+ \in \{\pm 1\}\) and \(y \in \mathbb{R}\backslash\{0\}\) is such that \(\epsilon y^2 \leq 1\), where \(\epsilon := g(v_1, v_1) \in \{\pm 1\}.\) The condition \(i_{X^+}F = dc^+\) together with \(c^+\) constant imply \(F = 0.\) Note that examples with \(c^2_+ = 1\) do also exist. These are two-dimensional solvable Kähler Lie groups \((G, J, g)\) considered as generalized Kähler structures on the untwisted generalized tangent bundle of \(G\), which in turn interpreted as odd exact examples with \(X^+_+ = X^- = 0\) as in Remark 37. The Lie algebra \(g\) admits an orthonormal basis \(\{e_1, e_2\}\) such that \(J e_1 = e_2\) and \([e_1, e_2] = \mu e_2, \mu \geq 0.\)
7.2 The Case $\dim G = 3$

Let $G$ be a 3-dimensional Lie group with Lie algebra $\mathfrak{g}$. A (non-degenerate) metric $g$ on $\mathfrak{g}$ defines a canonical operator $L \in \text{End} (\mathfrak{g})$, unique up to multiplication by $\pm 1$. By choosing an orientation on $\mathfrak{g}$, the operator $L$ relates the Lie bracket of $\mathfrak{g}$ with the cross product determined by $g$ and the orientation, by

$$[u, v] = L(u \times v), \; \forall u, v \in \mathfrak{g}. \quad (75)$$

Reversing the orientation, the operator $L$ multiplies by $-1$. It is well-known (see [10] for $g$ positive definite and [6] for $g$ of arbitrary signature) that $G$ is unimodular (i.e. $\text{tr} (\text{ad} \chi) = 0$ for any $X \in \mathfrak{g}$) if and only if $L$ is self-adjoint. In the next proposition we assume for simplicity that $L$ is diagonalizable. This is always the case when $g$ is positive definite and $\mathfrak{g}$ is unimodular.

**Proposition 42** Let $G$ be a 3-dimensional unimodular Lie group, with Lie algebra $\mathfrak{g}$. There is a left-invariant $B_3$-generalized pseudo-Kähler structure $(\mathcal{G}, \mathcal{F})$ on a Courant algebroid $E = E_{H,F}$ over $G$, with components $(g, J_+, J_-, X_+, X_-)$, such that the operator $L \in \text{End} (\mathfrak{g})$ associated to $g$ is diagonalizable, if and only if one of the following two cases holds:

i) there is a $g$-orthonormal basis $\{v_1, v_2, v_3\}$ of $\mathfrak{g}$, such that $g(v_2, v_2) = 1$, in which the Lie bracket of $\mathfrak{g}$ is given by

$$[v_1, v_2] = \epsilon_3 \lambda v_3, \; [v_3, v_1] = 0, \; [v_2, v_3] = \epsilon_1 \lambda v_1, \quad (76)$$

where $\epsilon_i := g(v_i, v_i) \in \{\pm 1\} (i \in \{1, 3\}), \lambda \in \mathbb{R} \setminus \{0\}, X_- = v_2$ and $X_+ = \pm v_2$.

In particular, $\mathfrak{g}$ is isomorphic to the Lie algebra of Killing fields of Euclidean or Minkowskian 2-space, $\mathfrak{g} \cong \text{iso}(2) = \mathfrak{so}(2) \ltimes \mathbb{R}^2$ or $\mathfrak{g} \cong \text{iso}(1, 1) = \mathfrak{so}(1, 1) \ltimes \mathbb{R}^2$, depending on whether $\epsilon_1 \epsilon_3 = 1$ or $\epsilon_1 \epsilon_3 = -1$, cf. Remark 43.

ii) $\mathfrak{g}$ is abelian, $g$ is an arbitrary (non-degenerate) metric on $\mathfrak{g}$ and $X_{\pm} \in \mathfrak{g}$ are arbitrary space-like unit vectors, i.e. $g(X_{\pm}, X_{\pm}) = 1$.

In both cases $H = 0$, $F = 0$ and $J_{\pm} \in \text{End}(\mathfrak{g})$ are arbitrary $g$-skew-symmetric endomorphisms, which satisfy $J_{\pm} X_{\pm} = 0$ and are complex structures on $X_{\pm}$.

**Proof** Assume that $\mathfrak{g}$ is non-abelian. Since $L$ is diagonalizable, there is a $g$-orthonormal basis $\{v_1, v_2, v_3\}$ in which the Lie bracket of $\mathfrak{g}$ is given by

$$[v_1, v_2] = \epsilon_3 \lambda_3 v_3, \; [v_3, v_1] = \epsilon_2 \lambda_2 v_2, \; [v_2, v_3] = \epsilon_1 \lambda_1 v_1, \quad (77)$$

where $\lambda_i \in \mathbb{R}$ (not all of them zero) and $\epsilon_i := g(v_i, v_i) \in \{\pm 1\}$. The Levi-Civita connection of $g$ is given by

$$\nabla_{v_1} v_2 = \frac{\epsilon_3}{2} (-\lambda_1 + \lambda_2 + \lambda_3) v_3, \quad \nabla_{v_2} v_1 = \frac{\epsilon_3}{2} (-\lambda_1 + \lambda_2 - \lambda_3) v_3,$$

$$\nabla_{v_1} v_3 = \frac{\epsilon_2}{2} (\lambda_1 - \lambda_2 - \lambda_3) v_2, \quad \nabla_{v_2} v_1 = \frac{\epsilon_2}{2} (\lambda_1 + \lambda_2 - \lambda_3) v_2.$$
\[
\n\nabla_{v_2} v_3 = \frac{\epsilon_1}{2} (\lambda_1 - \lambda_2 + \lambda_3) v_1, \quad \nabla_{v_3} v_2 = \frac{\epsilon_1}{2} (-\lambda_1 - \lambda_2 + \lambda_3) v_1,
\]

and

\[
\nabla_{v_i} v_i = 0, \quad \forall 1 \leq i \leq 3.
\]

Recall, from Corollary 17, that \(X_-\) is a Killing field of norm one. It turns out that \(g\) admits such a left-invariant Killing field in the following cases:

1) \(\lambda_1 \neq \lambda_2, \lambda_3 = \lambda_1\) and \(\epsilon_2 = 1\) (up to permutation). Then any space-like left-invariant unit Killing field (in particular, \(X_-\)) is of the form \(X_- = \pm v_2\). Replacing \(v_2\) by \(-v_2\) (and leaving \(v_1\) and \(v_3\) unchanged), we may (and will) assume that \(X_- = v_2\).

2) \(\lambda_1 = \lambda_2 = \lambda_3 \neq 0\). Any left-invariant vector field is Killing.

Consider case 1) and let \(\{v_1^*, v_2^*, v_3^*\}\) be the basis of \(\{v_1, v_2, v_3\}\), i.e. \(v_i^*(v_j) = \delta_{ij}\). The covectors \(v_i^*\) correspond to \(\epsilon_i v_i\) in the duality defined by \(g\). From \(i_X F = 0\), \(X_- = v_2\) we deduce that

\[
F = F_{13} v_1^* \wedge v_3^*.
\]

where \(F_{13} \in \mathbb{R}\). Using relations (78) (with \(\lambda_3 = \lambda_1\)) and relations (79) we obtain from the first relation (71) with \(X_- = v_2\) that

\[
H = -\lambda_2 v_1^* \wedge v_2^* \wedge v_3^*.
\]

Recall now, from Corollary 18, that \(X_+\) and \(X_-\) commute.

When \(\lambda_3 = \lambda_1 \neq 0\) the conditions that \(X_\pm\) have the same norm and \(\mathcal{L}_{X_+} X_- = 0\) imply that \(X_+ = \epsilon X_-\) with \(\epsilon \in \{\pm 1\}\). The second relation (71) reduces to

\[
\epsilon H(X_-, X) = -J_+ F(X), \quad \forall X \in \mathfrak{g}.
\]

Using that \(J_+\) is \(g\)-skew-symmetric, \(J_+ X_+ = 0\) and \(\text{rank}(\text{Ker} J_+) = 1\), together with (80) and (81) we obtain from (82) that \(H = 0, F = 0\) and \(\lambda_2 = 0\). This leads to the generalized pseudo-Kähler structure from claim i).

When \(\lambda_3 = \lambda_1 = 0\), \(X_- = v_2\) belongs to the center of \(\mathfrak{g}\) and \(\mathcal{L}_{X_+} X_- = 0\) does not impose any restrictions on \(X_+\) (as it happened when \(\lambda_3 = \lambda_1 \neq 0\)). Since \(g(X_+, X_+) = 1\), the vector field \(X_+\) is of the form \(X_+ = \sum a_i v_i\), where \(a_1, a_2, a_3 \in \mathbb{R}\), such that \(\epsilon_1 a_1^2 + a_2^2 + \epsilon_3 a_3^2 = 1\). The second relation (71) becomes

\[
\sum a_i \nabla_X v_i - \frac{\lambda_2}{2} (a_1 v_2^* \wedge v_3^* - a_2 v_1^* \wedge v_3^* + a_3 v_1^* \wedge v_2^*) (X)
= -F_{13} J_+ (v_1^* \wedge v_3^*) (X),
\]

for any \(X \in \mathfrak{g}\). Since \(\mathfrak{g}\) is not abelian, \(\lambda_2 \neq 0\). Relation (83) with \(X := v_1\) and \(X := v_2\) implies \(a_2 = a_3 = 0\), i.e. \(X_+ = a_1 v_1\). In particular, as \(J_+ X_+ = 0\), we
obtain that $J^+v_1 = 0$. Relation (83) with $X := v_3$ implies that $F_{13}J^+v_1 = \epsilon_1 a_1 \lambda_2 v_2$. Combined with $J^+v_1 = 0$, this implies that $a_1 = 0$, which is a contradiction. Similar computations show that case 2) leads to a contradiction as well.

In the remaining case, i.e. when $\mathfrak{g}$ is abelian, Corollary 39 implies immediately that $H$ and $F$ are zero and the remaining components of $(\mathcal{G}, \mathcal{H})$ are unconstrained. □

**Remark 43** Consider the $B_3$-generalized pseudo-Kähler structure $(\mathcal{G}, \mathcal{F})$ from Proposition 42 i) and the new basis $\{w_1 := \frac{1}{\lambda} v_1, w_2 := \frac{1}{\lambda} v_2, w_3 := \frac{1}{\lambda} v_3\}$ of $\mathfrak{g}$. Rescaling $(\mathcal{G}, \mathcal{F})$ by $\lambda^2$ (according to Corollary 19) we obtain a $B_3$-generalized pseudo-Kähler structure on the untwisted Courant algebroid of type $B_3$ over $G$, with components $(\tilde{g}, \tilde{J}^+, \tilde{J}^-, \tilde{X}^+, \tilde{X}^-)$, such that the basis $\{w_1, w_2, w_3\}$ is $\tilde{g}$-orthogonal, and

$$\tilde{g}(w_i, w_i) = \epsilon_i (i \in \{1, 3\}), \tilde{g}(w_2, w_2) = 1, \tilde{X}^- = w_2, \tilde{X}^+ = \pm w_2,$$ \hspace{1cm} (84)

where $\epsilon_i \in \{\pm 1\}$ for $i \in \{1, 3\}$. As above, $\tilde{J}_\pm \in \text{End} (\mathfrak{g})$ are arbitrary $g$-skew-symmetric endomorphisms, such that $\tilde{J}_\pm w_2 = 0$ and $\tilde{J}_\pm$ are complex structures on $\text{span}_\mathbb{R}\{w_1, w_3\}$. In the new basis the structure constants of $\mathfrak{g}$ take the standard form (as in [10]):

$$[w_1, w_2] = \epsilon_3 w_3, [w_2, w_3] = \epsilon_1 w_1, [w_3, w_1] = 0.$$ \hspace{1cm} (85)

Let $G$ be a 3-dimensional non-unimodular Lie group, with Lie algebra $\mathfrak{g}$. Since $\mathfrak{g}$ is 3-dimensional, its unimodular kernel $\mathfrak{g}_0 := \{X \in \mathfrak{g} : \text{tr} (\text{ad}_X) = 0\}$

is 2-dimensional (and unimodular), hence abelian. Choose a basis $\{v_2, v_3\}$ of $\mathfrak{g}_0$ and a vector $v_1 \notin \mathfrak{g}_0$. In the basis $\{v_1, v_2, v_3\}$ the Lie bracket of $\mathfrak{g}$ is given by

$$[v_1, v_2] = \alpha v_2 + \beta v_3, [v_1, v_3] = \gamma v_2 + \delta v_3, [v_2, v_3] = 0,$$ \hspace{1cm} (87)

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\alpha + \delta \neq 0$. Up to a multiplicative factor the constants $\alpha, \beta, \gamma, \delta$ are independent of the choice of $v_1$.

**Proposition 44** Let $G$ be a 3-dimensional non-unimodular Lie group, with Lie algebra $\mathfrak{g}$ and unimodular kernel $\mathfrak{g}_0$. There is a left-invariant $B_3$-generalized pseudo-Kähler structure $(\mathcal{G}, \mathcal{F})$ on a Courant algebroid $E = E_{H,F}$ of type $B_3$ over $G$, with components $(g, J^+, J^-, X^+, X^-)$, such that $\mathfrak{g}_0^g \cap \mathfrak{g}_0 = 0$, if and only if $\mathfrak{g}$ is isomorphic to $\mathbb{R} \oplus \mathfrak{so}(2)$, where $\mathfrak{so}(2)$ is the unique non-abelian Lie algebra of dimension 2. If $\mathfrak{g}$ is isomorphic to $\mathbb{R} \oplus \mathfrak{so}(2)$, then there is a basis $\{w_1, w_2, w_3\}$ of $\mathfrak{g}$ in which the Lie brackets take the form

$$[w_1, w_2] = w_2, [w_1, w_3] = [w_2, w_3] = 0,$$ \hspace{1cm} (88)

the metric $g$ is given by

$$g(w_1, w_1) = \frac{\epsilon}{\delta^2}, g(w_2, w_2) = \epsilon', g(w_3, w_3) = 1, g(w_i, w_j) = 0 \forall i \neq j.$$ \hspace{1cm} (89)
where $\delta \in \mathbb{R}\setminus\{0\}$, $\epsilon, \epsilon' \in \{\pm 1\}$ are arbitrary, $X_- = w_3$, $X_+ = \pm w_3$ and $J_\pm \in \text{End}(\mathfrak{g})$ are arbitrary $g$-skew-symmetric endomorphisms, which satisfy $J_\pm w_3 = 0$ and are complex structures on $\text{span}_{\mathbb{R}}\{w_1, w_2\}$. Moreover, $H = 0$ and $F = 0$.

**Proof** Let $g$ be a left-invariant metric on $G$ such that $\mathfrak{g}^0_0 \cap \mathfrak{g}_0 = 0$. Choose a $g$-orthonormal basis $\{v_1, v_2, v_3\}$ of $\mathfrak{g}$ such that $v_2, v_3 \in \mathfrak{g}_0$. The Lie bracket of $\mathfrak{g}$ takes the form (87) in the basis $\{v_1, v_2, v_3\}$ and the Levi-Civita connection $\nabla$ of $g$ is given by

\[
\begin{align*}
\nabla_{v_1} v_1 &= 0, \quad \nabla_{v_2} v_2 = \epsilon_1 \epsilon_2 \alpha v_1, \quad \nabla_{v_3} v_3 = \epsilon_1 \epsilon_3 \delta v_1, \\
\nabla_{v_1} v_2 &= \frac{1}{2} (\beta - \epsilon_2 \epsilon_3 \gamma) v_3, \quad \nabla_{v_2} v_1 = -\frac{1}{2} (\epsilon_2 \epsilon_3 \gamma + \beta) v_3 - \alpha v_2, \\
\nabla_{v_3} v_1 &= -\frac{1}{2} (\epsilon_2 \epsilon_3 \beta + \gamma) v_2 - \delta v_3, \quad \nabla_{v_1} v_3 = \frac{1}{2} (\gamma - \epsilon_2 \epsilon_3 \beta) v_2, \\
\n\nabla_{v_3} v_2 &= \nabla_{v_2} v_3 = \frac{\epsilon_1}{2} (\epsilon_2 \gamma + \epsilon_3 \beta) v_1,
\end{align*}
\]

(90)

where $\epsilon_i := g(v_i, v_i) \in \{\pm 1\}$.

From Corollary 17, $X_-$ is a Killing field. It turns out that $g$ admits a non-zero left-invariant Killing field only in the following cases:

1) $\alpha = 0$, $\delta \neq 0$ and $\beta \gamma = 0$. Any non-zero left-invariant Killing field (in particular $X_-$) is of the form $X_- = b(v_2 - \frac{\beta}{\delta} v_3)$, where $b \in \mathbb{R}\setminus\{0\}$; or

2) $\alpha \neq 0$, $\alpha + \delta \neq 0$ and $\gamma \beta = \delta \alpha$. Any non-zero left-invariant Killing field (in particular $X_-$) is of the form $X_- = c(\frac{\alpha}{\delta} v_2 + v_3)$, where $c \in \mathbb{R}\setminus\{0\}$.

Consider case 1) and let $\{v_1^*, v_2^*, v_3^*\}$ be the dual basis. Relation $i_{X_-} F = 0$ implies that

\[ F = F_{13}(\frac{\beta}{\delta} v_1^* \wedge v_2^* + v_1^* \wedge v_3^*), \]

(91)

where $F_{13} \in \mathbb{R}$. We write $H = H_{123} v_1^* \wedge v_2^* \wedge v_3^*$, where $H_{123} \in \mathbb{R}$. Using relations (90) (with $\alpha = 0$) we obtain that the first relation (71) is equivalent to $H_{123} = \epsilon_2 \gamma - \epsilon_3 \beta$, i.e.

\[ H = (\epsilon_2 \gamma - \epsilon_3 \beta) v_1^* \wedge v_2^* \wedge v_3^*. \]

(92)

For the second relation (71), let $X_+ := \sum a_i v_i$, where $a_1, a_2, a_3 \in \mathbb{R}$. With this notation, the second relation (71) is equivalent to

\[
\begin{align*}
J_+ F(v_1) &= (\beta - \epsilon_2 \epsilon_3 \gamma)(\epsilon_2 \epsilon_3 a_3 v_2 - a_2 v_3), \\
J_+ F(v_2) &= \beta(\epsilon_1 \epsilon_3 a_3 v_1 + a_1 v_3), \\
J_+ F(v_3) &= -\epsilon_1 (\epsilon_2 a_2 \gamma + \epsilon_3 a_3 \delta) v_1 + a_1 \gamma v_2 + a_1 \delta v_3,
\end{align*}
\]

or, using (91), to

\[
F_{13} J_+ (\frac{\beta}{\delta} v_2 + \epsilon_3 v_3) = (\beta - \epsilon_2 \epsilon_3 \gamma)(\epsilon_2 \epsilon_3 a_3 v_2 - a_2 v_3),
\]
\[
\frac{\beta}{\delta} F_{13} J_+ (v_1) = \beta (\epsilon_3 a_3 v_1 - \epsilon_1 a_1 v_3),
\]
\[
F_{13} J_+ (v_1) = (\epsilon_2 a_2 \gamma + \epsilon_3 a_3 \delta) v_1 - \epsilon_1 a_1 \gamma v_2 - \epsilon_1 a_1 \delta v_3. \tag{93}
\]

Recall now that \( J_+ \in \text{End}(g) \) is \( g \)-skew-symmetric, \( J_+ X_+ = 0 \) and \( J_+ \) is a complex structure on the orthogonal complement \( X_\perp \). These conditions combined with (93) imply that \( F_{13} = H_{123} = \beta = \gamma = a_1 = a_3 = 0 \). To summarize: case 1) provides a basis \( \{v_1, v_2, v_3\} \) of \( g \) and a one parameter family (indexed by \( \delta \in \mathbb{R}\setminus\{0\} \), see below) of \( B_3 \)-generalized pseudo-Kähler structures on the untwisted Courant algebroid \( (H = 0, F = 0) \), with components \( (g, J_+, J_-, X_+, X_-) \), such that the basis \( \{v_1, v_2, v_3\} \) is \( g \)-orthonormal, the Lie bracket of \( g \) is given by

\[
[v_1, v_2] = 0, \ [v_1, v_3] = \delta v_3, \ [v_2, v_3] = 0, \tag{94}
\]

where \( \delta \in \mathbb{R}\setminus\{0\} \), \( X_- = v_2 \) (replacing, if necessary, \( v_2 \) by \( -v_2 \)) and \( X_+ = \pm v_2 \). From \( g(X_\pm, X_\pm) = 1 \) we obtain that \( g(v_2, v_2) = 1 \). In the new basis \( \{w_1 := \frac{1}{\delta} v_1, w_2 := v_3, w_3 := v_2\} \) this family takes the form described in the statement of the proposition, where \( \epsilon = \epsilon_1 \) and \( \epsilon’ = \epsilon_3 \).

Similar arguments show that case 2) with \( \gamma \neq 0 \) leads to a basis \( \{v_1, v_2, v_3\} \) of \( g \) and a family of \( B_3 \)-generalized pseudo-Kähler structures on the untwisted Courant algebroid \( (H = 0, F = 0) \), with components \( (g, J_+, J_-, X_+, X_-) \), such that \( \{v_1, v_2, v_3\} \) is \( g \)-orthonormal, the Lie bracket takes the form

\[
[v_1, v_2] = \alpha v_2 + \gamma \epsilon_2 \epsilon_3 v_3, \ [v_1, v_3] = \gamma v_2 + \epsilon_2 \epsilon_3 \frac{\gamma^2}{\alpha} v_3, \ [v_2, v_3] = 0, \tag{95}
\]

where \( \alpha, \gamma \in \mathbb{R}\setminus\{0\}, \epsilon_i := g(v_i, v_i) \in \{\pm 1\}, X_- = c(-\frac{\gamma}{\alpha} v_2 + v_3) \) and \( X_+ = \pm X_- \), where \( c \in \mathbb{R} \) satisfies \( c^2(\alpha^2 \epsilon_3 + \gamma^2 \epsilon_2) = \alpha^2 \). In the new basis

\[
\begin{align*}
\{w_1 := \frac{\epsilon_3 c^2}{\alpha} v_1, w_2 := \frac{\epsilon_3 c}{\alpha} (\alpha v_2 + \epsilon_2 \epsilon_3 \gamma v_3), w_3 := -\frac{c}{\alpha} (\gamma v_2 - \alpha v_3)\}
\end{align*}
\]

the Lie brackets take the form (88) and the metric \( g \) and vector fields \( X_\pm \) are given by

\[
g(w_1, w_1) = \frac{c^4}{\alpha^4} \epsilon_1, \ g(w_2, w_2) = \epsilon_2 \epsilon_3, \ g(w_3, w_3) = 1,
\]
\[
g(w_1, w_2) = g(w_1, w_3) = g(w_2, w_3) = 0,
\]
\[
X_- = w_3, \ X_+ = \pm w_3. \tag{96}
\]

Letting \( \delta \) such that \( \delta^2 = \alpha^2 / c^4 \) we arrive again at the \( B_3 \)-generalized pseudo-Kähler structures described in the statement of the proposition, where now \( \epsilon = \epsilon_1 \) and \( \epsilon’ = \epsilon_2 \epsilon_3 \).

Case 2) with \( \gamma = 0 \) leads to the family of \( B_3 \)-generalized pseudo-Kähler structures obtained in case 1), but with \( v_2 \) and \( v_3 \) interchanged. Therefore, they provide no further \( B_3 \)-generalized pseudo-Kähler structures besides those described above.

\( \square \)
The next corollary summarizes our results from this section in the positive definite case.

**Corollary 45** Let \((G, \mathcal{F})\) be a left-invariant \(B_3\)-generalized Kähler structure on a Courant algebroid \(E = \mathcal{E}_{H,F}\) of type \(B_3\) over a 3-dimensional Lie group \(G\) with Lie algebra \(\mathfrak{g}\). Let \((g, J_+, J_-, X_+, X_-)\) be the components of \((G, \mathcal{F})\). Up to rescaling of \((G, \mathcal{F})\) one of the following situations holds:

i) there is a \(g\)-orthonormal basis \(\{w_1, w_2, w_3\}\) of \(g\) in which the Lie brackets take the form

\[
[w_1, w_2] = w_3, \quad [w_2, w_3] = w_1, \quad [w_3, w_1] = 0
\]  

(97)

and \(X_- = w_2, X_+ = \pm w_2\).

ii) there is a \(g\)-orthogonal basis \(\{w_1, w_2, w_3\}\) of \(g\) in which the Lie brackets take the form

\[
[w_1, w_2] = w_2, \quad [w_1, w_3] = [w_2, w_3] = 0,
\]  

and

\[
g(w_1, w_1) = \frac{1}{\delta^2}, \quad g(w_2, w_2) = 1, \quad g(w_3, w_3) = 1,
\]  

where \(\delta \in \mathbb{R}\{0\}\), \(X_- = w_3, X_+ = \pm w_3\).

iii) the Lie algebra \(g\) is abelian, \(g\) is any Riemannian metric on \(g\) and \(X_\pm\) are arbitrary vectors from \(g\), of norm one.

In all cases above \(H = 0\), \(F = 0\) and \(J_\pm \in \text{End} (g)\) are \(g\)-skew-symmetric endomorphisms, with the properties that \(J_\pm |_{X_\perp} = 0\) and \(J_\pm |_{X_\perp} \) are complex structures.

**Proof** The claim follows from Propositions 42 and 44 (see also Remark 43), together with the observation that if \(g\) is positive definite then the operator \(L\) from Proposition 42 is diagonalizable and the assumption \(g_0^\perp \cap g_0 = 0\) from Proposition 44 is satisfied.

\[\square\]

### 7.3 The Case \(\dim G = 4\)

Let \(G\) be a 4-dimensional Lie group with Lie algebra \(\mathfrak{g}\). We assume that \(\mathfrak{g}\) is of the form

\[
\mathfrak{g} = u + \mathfrak{g}_0,
\]  

(100)

where \(\mathfrak{g}_0\) is a 3-dimensional unimodular non-abelian Lie algebra and \(u\) is 1-dimensional and acts on \(\mathfrak{g}_0\) as a (1-dimensional) family of derivations. Note that such a Lie algebra \(\mathfrak{g}\) is unimodular if and only if \(\text{tr} (\text{ad}_X) = 0\) for a non-zero element \(X \in u\).
**Example 46** Every non-unimodular Lie algebra $\mathfrak{g}$ admits a (unique) codimension one unimodular ideal $\mathfrak{g}_0 = \text{Ker} (\text{tr} \circ \text{ad})$, called the unimodular kernel of $\mathfrak{g}$. Choosing a complementary line $u$ we arrive at a decomposition $\mathfrak{g} = u + \mathfrak{g}_0$. So the assumptions of this section are satisfied by any 4-dimensional non-unimodular Lie algebra with non-abelian unimodular kernel.

As an application of Corollary 40, we now describe a class (called adapted) of left-invariant $B_4$-generalized pseudo-Kähler structures over $G$.

**Definition 47** A left-invariant $B_4$-generalized pseudo-Hermitian structure $(\mathcal{G}, \mathcal{F})$ on a Courant algebroid over $G$, with components $(g, J_+, J_-, X_+, X_-, c_+)$, is called adapted if the decomposition $\mathfrak{g} = u + \mathfrak{g}_0$ is orthogonal with respect to $g$, the operator $L$ associated to $(\mathfrak{g}_0, g|_{\mathfrak{g}_0 \times \mathfrak{g}_0})$ is diagonalizable, $J_-(u)$ is included in an eigenspace of $L$ and $u$ and $X_{\pm}$ are non-null (i.e. $c_+ \in \mathbb{R} \setminus \{ \pm 1 \}$).

If $(g, J_+, J_-, X_+, X_-, c_+)$ are the components of an adapted $B_4$-generalized pseudo-Hermitian structure, then there is a $g$-orthonormal basis of $\mathfrak{g}$ (called adapted) of the form $\{u, e_1, e_2, e_3\}$, where $u \in u$ and $e_i \in \mathfrak{g}_0$,

$$J_+ u = e_1, \ J_- e_2 = e_3,$$  \hspace{1cm} (101)

in which the Lie bracket takes the form

$$[e_1, e_2] = e_3 \lambda_3 e_3, \ [e_2, e_3] = e_1 \lambda_1 e_1, \ [e_3, e_1] = e_2 \lambda_2 e_2, \ [u, e_i] = \sum_{j=1}^{3} a_{ij} e_j,$$  \hspace{1cm} (102)

where $a_{ij} \in \mathbb{R}$ and $\lambda_i \in \mathbb{R}$ (at least one non-zero). Since $\text{ad}_u$ is a derivation of $\mathfrak{g}_0$,

$$\begin{align*}
\lambda_1(-a_{11} + a_{22} + a_{33}) &= 0, \\
\lambda_2(a_{11} - a_{22} + a_{33}) &= 0, \\
\lambda_3(a_{11} + a_{22} - a_{33}) &= 0, \\
\epsilon_i \lambda_i a_{ij} &= -\epsilon_j \lambda_j a_{ji}, \ \forall i \neq j,
\end{align*}$$  \hspace{1cm} (103)

where $\epsilon_i := g(v_i, u_i) \in \{ \pm 1 \}$ for any $i \in \{1, 2, 3\}$. Remark that $\epsilon_0 := g(u, u) = \epsilon_1$ and $\epsilon_2 = \epsilon_3$ since $J_-$ is $g$-skew-symmetric.

**Proposition 48** i) There is an adapted $B_4$-generalized pseudo-Kähler structure $(\mathcal{G}, \mathcal{F})$ on a Courant algebroid $E = EH_{1, F}$ over $G$, with components $(g, J_+, J_-, X_+, X_-, c_+)$, such that $c_+ \neq 0$, if and only if there is an adapted $g$-orthonormal basis of $\{u, e_1, e_2, e_3\}$ of $\mathfrak{g}$ such that

$$\begin{align*}
[e_1, e_2] &= e_3 \lambda_3 e_3, \ [e_2, e_3] = 0, \ [e_3, e_1] = e_2 \lambda e_2, \\
[u, e_1] &= 0, \ [u, e_2] = \beta e_3, \ [u, e_3] = -\beta e_2,
\end{align*}$$  \hspace{1cm} (104)
where \( \lambda \in \mathbb{R}\setminus\{0\}, \beta \in \mathbb{R}, \epsilon_i := g(e_i, e_i) \in \{\pm 1\}, g(u, u) = \epsilon_1 \) and \( \epsilon_2 = \epsilon_3 \). The remaining data are given by: \( X_+ = au + b e_1, X_- = \bar{a} u + \bar{b} e_1, J_+ X_+ = -c_+ X_-, J_+ X_- = c_+ X_+ \), where \( a, b, \bar{a}, \bar{b} \in \mathbb{R} \) are such that

\[
\epsilon_1(a^2 + b^2) = \epsilon_1(\bar{a}^2 + \bar{b}^2) = 1 - c_+^2, \quad a \bar{a} + b \bar{b} = 0, \tag{105}
\]

\( c_+ \in \mathbb{R}\setminus\{-1, 0, 1\} \), the complex structure \( J_- \) is given by (101) and \( J_+|_{\text{span}_g\{e_2, e_3\}} \) is any \( g \)-skew-symmetric complex structure. Moreover, \( H = 0 \) and \( F = 0 \).

ii) Changing the decomposition \( g = u + g_0 \) if necessary, the orthonormal vectors \( u, e_1 \) can be always chosen such that \( \beta = 0 \). In particular, \( g \cong \mathbb{R} \oplus \text{iso}(2) \).

We divide the proof of Proposition 48 into several lemmas. Let \((G, F)\) be an adapted \( B_4 \)-generalized pseudo-Hermitian structure on \( G \), with components \((g, J_+, J_-, X_+, X_-, c_+)\), where \( c_+ \in \mathbb{R}\setminus\{\pm 1\} \) is arbitrary. As above, let \( \{u, e_1, e_2, e_3\} \) be an adapted basis, \( \epsilon_i := g(e_i, e_i) \) and \( \epsilon_0 := g(u, u) \).

Lemma 49 The Levi-Civita connection of \( g \) is given by

\[
\nabla_{e_1} e_2 = \frac{\epsilon_3}{2}(-\lambda_1 + \lambda_2 + \lambda_3)e_3 + \frac{\epsilon_0}{2}(\epsilon_2 a_{12} + \epsilon_1 a_{21})u, \\
\nabla_{e_2} e_1 = \frac{\epsilon_3}{2}(-\lambda_1 + \lambda_2 - \lambda_3)e_3 + \frac{\epsilon_0}{2}(\epsilon_2 a_{12} + \epsilon_1 a_{21})u, \\
\nabla_{e_1} e_3 = \frac{\epsilon_2}{2}(\lambda_1 - \lambda_2 - \lambda_3)e_2 + \frac{\epsilon_0}{2}(\epsilon_3 a_{13} + \epsilon_1 a_{31})u, \\
\nabla_{e_3} e_1 = \frac{\epsilon_2}{2}(\lambda_1 + \lambda_2 - \lambda_3)e_2 + \frac{\epsilon_0}{2}(\epsilon_3 a_{13} + \epsilon_1 a_{31})u, \\
\nabla_{e_2} e_3 = \frac{\epsilon_1}{2}(\lambda_1 - \lambda_2 + \lambda_3)e_1 + \frac{\epsilon_0}{2}(\epsilon_3 a_{23} + \epsilon_2 a_{32})u, \\
\nabla_{e_3} e_2 = \frac{\epsilon_1}{2}(\lambda_1 - \lambda_2 + \lambda_3)e_1 + \frac{\epsilon_0}{2}(\epsilon_3 a_{23} + \epsilon_2 a_{32})u, \\
\n\nabla_u e_i = \frac{1}{2} \sum_j (a_{ij} - \epsilon_i \epsilon_j a_{ji})e_j, \quad \nabla_{e_i} e_j = \epsilon_0 \epsilon_i a_{ij} u, \\
\n\nabla_{e_i} u = -\frac{1}{2} \sum_j (a_{ij} + \epsilon_i \epsilon_j a_{ji})e_j, \quad \nabla_u u = 0.
\]

Let \( \{u^*, e_1^*, e_2^*, e_3^*\} \) be the dual basis of \( \{u, e_1, e_2, e_3\} \), i.e.

\[
e_i^*(e_j) = \delta_{ij}, \quad e_i^*(u) = 0, \quad u^*(u) = 1, \quad u^*(e_i) = 0, \quad \forall i, j.
\]

We write the left-invariant forms \( H \) and \( F \) as

\[
H = H_{123} e_1^* \wedge e_2^* \wedge e_3^* + \frac{1}{2} H_{ij} u^* \wedge e_i^* \wedge e_j^*, \\
F = \frac{1}{2} F_{ij} e_i^* \wedge e_j^* + F_i u^* \wedge e_i^*, \tag{106}
\]
where $H_{123}, H_{ij}, F_{ij}, F_i \in \mathbb{R}, H_{ij} = -H_{ji}, F_{ij} = -F_{ji}$ for any $i, j$, and to simplify notation we omitted the summation signs. As the pair $(H, F)$ defines a Courant algebroid of type $B_4$, the coefficients of $H$ and $F$ are subject to various constraints which come from $dF = 0$ and $dH + F \wedge F = 0$.

**Lemma 50**

i) The equality $dH + F \wedge F = 0$ holds if and only if

\[ H_{123}(a_{11} + a_{22} + a_{33}) = 2(F_1F_{23} + F_3F_{12} + F_2F_{31}). \]  

(107)

ii) The 2-form $F$ is closed if and only if

\[
\begin{align*}
\epsilon_1 \lambda_1 F_1 &= F_{23}(a_{22} + a_{33}) + F_{21}a_{31} + F_{13}a_{21}, \\
\epsilon_2 \lambda_2 F_2 &= F_{31}(a_{11} + a_{33}) + F_{21}a_{32} + F_{32}a_{12}, \\
\epsilon_3 \lambda_3 F_3 &= F_{12}(a_{11} + a_{22}) + F_{32}a_{13} + F_{13}a_{23}.
\end{align*}
\]

(108)

**Proof** The claims follow from a straightforward computation, which uses that the 1-form $u^*$ is closed and the exterior derivatives of the 1-forms $e_i^*$ are given by

\[
\begin{align*}
d(e_1^*) &= -\epsilon_1 \lambda_1 e_2^* \wedge e_3^* - a_{11} u^* \wedge e_j^*, \\
d(e_2^*) &= -\epsilon_2 \lambda_2 e_3^* \wedge e_1^* - a_{12} u^* \wedge e_j^*, \\
d(e_3^*) &= -\epsilon_3 \lambda_3 e_1^* \wedge e_2^* - a_{13} u^* \wedge e_j^*.
\end{align*}
\]

For instance, these equations imply

\[ dH = -\text{tr}(A) H_{123} u^* \wedge e_1^* \wedge e_2^* \wedge e_3^*, \quad A := (a_{ij}), \]

and comparing with

\[ F \wedge F = 2 \left( \sum_{\mathcal{S}} F_{ij} F_k \right) u^* \wedge e_1^* \wedge e_2^* \wedge e_3^* \]

yields (107), where $\mathcal{S}$ indicates the sum over cyclic permutations.

We now apply Corollary 40 with $c_+$ a non-zero constant. We consider each condition from this corollary separately.

**Lemma 51** The connection $D^- = \nabla + \frac{1}{2} H$ preserves $J_-$ if and only if

\[
\begin{align*}
a_{21} &= a_{31} = 0, \quad a_{23} + a_{32} = 0, \\
a_{22} - a_{33} &= \epsilon_2(\lambda_3 - \lambda_2), \\
H_{23} &= 0, \quad H_{12} = -\epsilon_2 a_{12}, \quad H_{13} = -\epsilon_3 a_{13}, \\
H_{123} &= 2a_{22} \epsilon_2 - \lambda_1 + \lambda_2 - \lambda_3.
\end{align*}
\]

(109)
Proof Recall the definition (101) of $J_-$. Using Lemma 49, one can check that $(D_X J_-)(u) = 0$ for any $X \in g$ is equivalent to relations (109). Moreover, these relations imply $(D_X J_-)(e_i) = 0$ for any $i$.

Since $c_+ \neq 0$, from the first relation (72) we obtain that

$$F = \frac{1}{2c_+} (dX^b + i_X H),$$

(110)

where $X^b_+$ is the 1-form $g$-dual to $X_+$. The first relation (72) also implies that $X_+$ is a Killing field. On the other hand, remark that if $X$ is a Killing field of constant norm for a pseudo-Riemannian metric $h$, then $i_X dX^b = 0$ where $X^b$ is the $h$-dual to $X$. We deduce that the condition $i_X F = 0$ from Corollary 40 is satisfied, once we know that $X_+$ is a Killing field. The next lemma determines the conditions satisfied by the coefficients of Killing fields for $g$.

Lemma 52 Assume that $a_{21} = a_{31} = 0$ and $a_{23} + a_{32} = 0$. A vector field $X_+ = au + be_1 + ce_2 + de_3$ is Killing for $g$ if and only if

$$ba_{11} = aa_{ii} = ba_{12} + ca_{22} + da_{32} = 0, \forall i,$$

$$ba_{13} + ca_{23} + da_{33} = 0,$$

$$\epsilon_2 aa_{12} + d(\lambda_2 - \lambda_1) = 0,$$

$$\epsilon_3 aa_{13} + c(\lambda_1 - \lambda_3) = 0,$$

$$b(\lambda_3 - \lambda_2) = 0.$$  

(111)

Lemma 53 Assume that $a_{21} = a_{31} = 0$, $a_{23} + a_{32} = 0$, $H_{23} = 0$, $H_{12} = -\epsilon_2 a_{12}$ and $H_{13} = -\epsilon_3 a_{13}$. The form $F$ defined by (110) is of type $(1, 1)$ with respect to $J_-$ if and only if the following relations hold:

$$\epsilon_2 aa_{12} - d(H_{123} - \lambda_3) = \epsilon_2 (ca_{23} - da_{33} + ba_{13}),$$

$$\epsilon_3 aa_{13} + c(H_{123} - \lambda_2) = \epsilon_2 (ca_{22} + da_{23} - ba_{12}).$$

(112)

Proof From the definition of $J_-$, a 2-form as in (106) is of type $(1, 1)$ with respect to $J_-$ if and only of

$$F_{12} = -F_3, \ F_{13} = F_2.$$  

(113)

Using relation (110), a simple computation shows that

$$F_{12} = \frac{1}{2c_+} (-\epsilon_2 aa_{12} + d(H_{123} - \lambda_3)),$$

$$F_{13} = -\frac{1}{2c_+} (\epsilon_3 aa_{13} + c(H_{123} - \lambda_2)),$$

$$F_1 = -\frac{1}{2c_+} (\epsilon_1 ba_{11} + 2\epsilon_2 ca_{12} + 2\epsilon_2 da_{13}).$$
\[ F_2 = -\frac{\epsilon_2}{2c^+} (ca_{22} + da_{23} - ba_{12}), \]
\[ F_3 = \frac{\epsilon_2}{2c^+} (ca_{23} - da_{33} + ba_{13}). \]  
(114)

The claim follows from (114) and (113).

\[ \square \]

**Remark 54** For later use, remark that

\[ F_{23} = \frac{1}{2c^+} b(H_{123} - \lambda_1). \]  
(115)

The next lemma describes the Lie bracket of the Lie algebra \( g \), the Killing field \( X_+ \), the forms \( H \) and \( F \), such that all conditions from Corollary 40 are satisfied, except the normalization \( g(X_+, X_+) = 1 - c_+^2 \) and the conditions which involve \( X_- \) and \( J_+ \).

**Lemma 55** Let \( \epsilon_1, \epsilon_2 = \epsilon_3 \in \{ \pm 1 \} \) and \( c_+ \in \mathbb{R}\setminus\{0\} \). There are eight classes

\[ (a_{ij}, \lambda_i, H_{i123}, H_{ij}, F_{ij}, F_i, a, b, c, d) \]  
of solutions of the systems (103), (107), (108), (109), (111), (112), (114), (115) with at least one of the constants \( a, b, c, d \) non-zero and excluding the case \( \lambda_1 = \lambda_2 = \lambda_3 = 0 \), i.e. the case when the ideal \( g_0 \subset g \) is abelian. They are given as follows (below the constants \( H_{123}, H_{ij}, F_{ij}, F_i, a, b, c, d \) are incorporated in the corresponding Killing field \( X_+ \), and forms \( H \) and \( F \)):

1) \( a_{12} = a_{13} = a_{ii} = 0 \) for any \( i, a_{23} \in \mathbb{R}, \lambda_1 \in \mathbb{R}\setminus\{0\}, \lambda_2 = \lambda_3 \in \mathbb{R}\setminus\{\lambda_1\} \), \( X_+ = au + be_1 \) (with \( a \in \mathbb{R}\setminus\{0\} \) and \( b \in \mathbb{R} \)), and

\[ H = -\lambda_1 e_1^* \wedge e_2^* \wedge e_3^*, \ F = -\frac{b\lambda_1}{c_+} e_2^* \wedge e_3^*. \]  
(116)

2) \( a_{12} = a_{13} = a_{ii} = 0 \) for any \( i, a_{23} \in \mathbb{R}, \lambda_1 = \lambda_2 = \lambda_3 \in \mathbb{R}\setminus\{0\} \), \( X_+ = au + be_1 \) (with \( a \in \mathbb{R}\setminus\{0\} \) and \( b \in \mathbb{R} \)). The forms \( H \) and \( F \) are given by (116).

3) \( a_{12} = -\frac{\lambda_2 \epsilon_2}{a}, a_{13} = \frac{\lambda_3 \epsilon_3}{a}, a_{ii} = 0 \) for any \( i, a_{23} \in \mathbb{R}, \lambda_1 = 0, \lambda_2 = \lambda_3 \in \mathbb{R}\setminus\{0\}, X_+ = au - \frac{\lambda_3 \epsilon_3}{\lambda_2} e_1 + ce_2 + de_3 \) (with \( a \in \mathbb{R}\setminus\{0\} \) and \( c, d \in \mathbb{R} \)) and

\[ H = \frac{\lambda_2}{a} u^* \wedge (de_1^* \wedge e_2^* - ce_1^* \wedge e_3^*), \ F = 0. \]  
(117)

4) \( a_{12} = a_{13} = a_{ii} = 0 \) for any \( i, a_{23} \in \mathbb{R}, \lambda_1 = 0, \lambda_2 = \lambda_3 \in \mathbb{R}\setminus\{0\}, X_+ = au + be_1 \) (with \( a \in \mathbb{R}\setminus\{0\} \) and \( b \in \mathbb{R}\setminus\{-\frac{\epsilon_3 a \epsilon_2}{\lambda_2}\} \)). The forms \( H \) and \( F \) are trivial.

5) \( a_{12} = a_{13} = a_{ii} = 0 \) for any \( i, a_{23} \in \mathbb{R}, \lambda_1 = \lambda_2 = \lambda_3 \in \mathbb{R}\setminus\{0\}, X_+ = be_1 \) (with \( b \in \mathbb{R}\setminus\{0\} \)) and

\[ H = -\lambda_1 e_1^* \wedge e_2^* \wedge e_3^*, \ F = -\frac{b\lambda_1}{c_+} e_2^* \wedge e_3^*. \]  
(118)

6) \( a_{23} = a_{33} = 0, a_{11} = -\epsilon_2 \lambda_3, a_{22} = \epsilon_2 \lambda_3, a_{12}, a_{13} \in \mathbb{R}, \lambda_1 = \lambda_2 = 0, \lambda_3 \in \mathbb{R}\setminus\{0\}, X_+ = de_3 \) (with \( d \in \mathbb{R}\setminus\{0\} \) and

\[ H = \lambda_3 e_1^* \wedge e_2^* \wedge e_3^* - \epsilon_2 u^* \wedge (a_{12} e_1^* \wedge e_2^* + a_{13} e_1^* \wedge e_3^*), \]  
(119)
\[ F = -\frac{\epsilon_3 da_{13}}{c_+} u^* \wedge e_1^*. \]

7) \( a_{22} = a_{23} = 0, a_{11} = -\epsilon_3 \lambda_2, a_{33} = \epsilon_3 \lambda_2, a_{12}, a_{13} \in \mathbb{R}, \lambda_1 = \lambda_3 = 0, \lambda_2 \in \mathbb{R} \setminus \{0\}, X_+ = ce_2 \) (with \( c \in \mathbb{R} \setminus \{0\} \)) and

\[
H = \lambda_2 e_1^* \wedge e_2^* \wedge e_3^* - 2 e_2 u^* \wedge (a_{12} e_1^* \wedge e_2^* + a_{13} e_1^* \wedge e_3^*),
\]

\[
F = -\frac{\epsilon_2 c a_{12}}{c_+} u^* \wedge e_1^*.
\]

8) \( a_{12} = a_{13} = a_{ij} = 0 \) for any \( i, a_{23} \in \mathbb{R}, \lambda_1 \in \mathbb{R}, \lambda_2 = \lambda_3 \in \mathbb{R} \setminus \{\lambda_1\}, X_+ = be_1 \) (with \( b \in \mathbb{R} \setminus \{0\} \)) and

\[
H = -\lambda_1 e_1^* \wedge e_2^* \wedge e_3^*, \quad F = -\frac{b \lambda_1}{c_+} e_2^* \wedge e_3^*.
\]  \hfill (119)

In all of the above cases, \( a_{21} = a_{31} = 0 \) and \( a_{32} = -a_{23}. \)

**Proof** The claim follows from elementary algebraic computations. When \( a \neq 0 \), the second relation (111) implies that \( a_{ii} = 0 \) for any \( i \) and the third and last relation (109) imply that \( \lambda_2 = \lambda_3 \) and \( H_{123} = -\lambda_1. \) The case \( a \neq 0 \) leads to the first four classes of the statement. When \( a = 0, \) relations (111) imply

\[
d(\lambda_2 - \lambda_1) = c(\lambda_1 - \lambda_3) = b(\lambda_2 - \lambda_3) = 0.
\]  \hfill (120)

In particular, at least two from the \( \lambda_i \)'s coincide (otherwise \( X_+ = 0 \)). Considering all possibilities \( (\lambda_1 = \lambda_2 = \lambda_3, \lambda_1 = \lambda_2 \neq \lambda_3, \lambda_1 = \lambda_3 \neq \lambda_2 \) and \( \lambda_2 = \lambda_3 \neq \lambda_1) \) we obtain the remaining four cases from the statement. The expressions of \( H \) and \( F \) follow from (106), (109), (114) and (115).

The next lemma concludes the proof of Proposition 48 i).

**Lemma 56** Consider the classes of solutions from Lemma 55 with \( X_+ \) such that \( 0 \neq g(X_+, X_+) < 1 \) and \( c_+ \in \mathbb{R} \setminus \{-1, 0, 1\} \) such that \( c_+^2 = 1 - g(X_+, X_+) \). There is a left-invariant vector field \( X_- \) on \( G \) and a left-invariant endomorphism \( J_+ \in \Gamma(\text{End} \, TG) \), such that \( (g, J_+, J_-, X_+, X_-, c_+) \) are the components of an adapted \( B_4 \)-generalized pseudo-Kähler structure, only in cases 3) (if \( c = d = 0 \), 4) and 8) of Lemma 55. The resulting \( B_4 \)-generalized pseudo-Kähler structures from cases 3), 4) and 8) (with \( a_{23} \) replaced by \( \beta \)) are described in Proposition 48.

**Proof** According to Corollary 40 we need to determine \( X_- \in \mathfrak{g} \) and \( J_+ \in \text{End} \, (\mathfrak{g}) \) such that \( X_- \) is orthogonal to \( X_+ \) and has the same norm as \( X_+ \), \( J_+ \) is \( g \)-skew-symmetric, \( J_+ X_+ = -c_+ X_- \), \( J_+ X_- = c_+ X_+ \), \( J_+ |_{(X_+, X_-)} \) is a complex structure and

\[
\nabla_X X_- - \frac{1}{2} H(X, X_-) = -J_+ F(X), \quad \forall X \in \mathfrak{g},
\]

\[
H(X_+, X_-) = c_+ F(X_-),
\]  \hfill (121)
where we used that $F(X_+) = 0$. Consider case 1) from Lemma 55. Then $X_+ = au + be_1$ with $a \not= 0$ and

$$F(u) = F(e_1) = 0, \quad F(e_2) = -\frac{\epsilon_3 \lambda_1 b}{c_+} e_3, \quad F(e_3) = \frac{\epsilon_2 \lambda_1 b}{c_+} e_2.$$

(122)

Let $X_- = \tilde{a}u + \tilde{b}e_1 + \tilde{c}e_2 + \tilde{d}e_3$, where $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathbb{R}$. We compute

$$\nabla_{e_1} X_- + \frac{1}{2} H(X_-, e_1) = \epsilon_3 \lambda_2 (-\tilde{d}e_2 + \tilde{c}e_3),$$

$$\nabla_{e_2} X_- + \frac{1}{2} H(X_-, e_2) = \lambda_1 (\epsilon_1 \tilde{d}e_1 - \epsilon_3 \tilde{b}e_3),$$

$$\nabla_{e_3} X_- + \frac{1}{2} H(X_-, e_3) = \lambda_1 (-\epsilon_1 \tilde{c}e_1 + \epsilon_2 \tilde{b}e_2),$$

$$\nabla_u X_- + \frac{1}{2} H(X_-, u) = a_{23} (-\tilde{d}e_2 + \tilde{c}e_3).$$

(123)

Combining the first relation (121) with (122) and (123), we obtain

$$a_{23} (-\tilde{d}e_2 + \tilde{c}e_3) = 0, \quad \lambda_2 (-\tilde{d}e_2 + \tilde{c}e_3) = 0,$$

$$\frac{\epsilon_3 b}{c_+} J_+ e_3 = \epsilon_1 \tilde{d}e_1 - \epsilon_3 \tilde{b}e_3, \quad \frac{\epsilon_2 b}{c_+} J_+ e_2 = \epsilon_1 \tilde{c}e_1 - \epsilon_2 \tilde{b}e_2.$$

Since $J_+$ is $g$-skew and the basis $\{u, e_1, e_2, e_3\}$ is orthonormal, $\tilde{b} = 0$ and the second line above becomes

$$bJ_+ e_3 = \epsilon_1 \epsilon_3 c_+ \tilde{d}e_1, \quad bJ_+ e_2 = \epsilon_1 \epsilon_2 c_+ \tilde{c}e_1.$$

(124)

On the other hand $\tilde{b} = 0$ and $g(X_+, X_-) = 0$ imply that $\tilde{a} = 0$. Thus $X_- = \tilde{c}e_2 + \tilde{d}e_3$. Relation (124) also implies that $b \not= 0$ (otherwise $X_- = 0$). It is now straightforward to check that relations (124) combined with $J_+ X_- = c_+ X_+$ and $c_+ a \not= 0$ lead to a contradiction. The remaining cases from Lemma 55 can be treated similarly. □

It remains to prove Proposition 48 ii). If $\beta = 0$ there is nothing to show. Otherwise, the action of $\text{ad}_u$ on the ideal spanned by $e_2$ and $e_3$ is always a non-zero multiple of the action of $\text{ad}_e_1$ and we can find a linear combination of $u$ and $e_1$ the adjoint action of which is trivial on the ideal. Rescaling that linear combination we obtain a new unit vector $u' \in g \setminus g_0$, a new line $u' := \mathbb{R} u'$, a new ideal $g'_0 := (u')^\perp$ and a new orthonormal basis $\{u', e'_1, e_2, e_3\}$ by defining $e'_1 := J_- u'$. The decomposition $u' + g'_0$ and the basis $\{u', e'_1, e_2, e_3\}$ have then the claimed properties. The proof of Proposition 48 is now completed.

While this section was concerned with $B_4$-generalized pseudo-Kähler structures with $c_+ \not= 0$, it remains to find examples of such structures with $c_+ = 0$. Several hints in this direction are given below.
Example 57  
i) By rescaling (see Corollary 24), we obtain from Proposition 48 adapted $B_4$-generalized pseudo-Kähler structures with corresponding vector fields $X_\pm$ of norm one (i.e. $c_+ = 0$).

ii) Using similar computations as above, one can construct further left-invariant $B_4$-generalized pseudo-Kähler structures with $X_\pm$ of norm one. The 2-form $F$ satisfies relations (107), (108) and (113) also in this case, but it is no longer related to $X_+$ by relation (110).

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