Fusion Rings Related to Affine Weyl Groups*

P. Furlan* † and V.B. Petkova**

*Dipartimento di Fisica Teorica dell’Università di Trieste and †Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Trieste
Strada Costiera 11
34100 Trieste, Italy
**Institute for Nuclear Research and Nuclear Energy
Tzarigradsko Chaussee 72
1784 Sofia, Bulgaria

Abstract

The construction of the fusion ring of a quasi-rational CFT based on \( \hat{\mathfrak{sl}(3)}_k \) at generic level \( k \notin \mathbb{Q} \) is reviewed. It is a commutative ring generated by formal characters, elements in the group ring \( \mathbb{Z}[\hat{W}] \) of the extended affine Weyl group \( \hat{W} \) of \( \hat{\mathfrak{sl}}(3)_k \). Some partial results towards the \( \hat{\mathfrak{sl}}(4)_k \) generalisation of this character ring are presented.

1 Introduction

Describing the fusion rules of the conformal field theory (CFT) based on the fractional level admissible representations of the affine KM algebras \([1]\) remains an open problem in general. It has been solved in the simplest case of \( \hat{\mathfrak{sl}}(2)_k \) \([2]\) by a direct solution of the singular vector decoupling equations, the result being confirmed by a more abstract analysis in \([3]\). In \([4]\) a general approach was initiated and illustrated

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on the first example with nontrivial fusion multiplicities, the case $\hat{sl}(3)_k$. The problem is treated starting first with a quasi-rational CFT described by an infinite class of “pre-admissible” representations of the affine algebra with generic $k \not\in \mathbb{Q}$ and highest weights labelled by a certain subset of the extended affine Weyl group. The fusion rules of this theory provide the “classical” counterpart of the fusion rules of the fractional level admissible CFT. This is analogous to the relation between the CFT based on the integrable representations and its “classical” counterpart, the “pre-integrable” CFT described by an infinite set of representations with $k \not\in \mathbb{Q}$ and highest weights given by integer dominant weights of the horizontal subalgebra. In that case the fusion rule multiplicities of the generic level CFT are given simply by the tensor product multiplicities of the finite dimensional representations of the horizontal algebra. The standard formal characters of these representations, elements of the group ring of the group of translations, serve as characters of the fusion algebra of the affine algebra representations. These “classical” data are partially incorporated in the integrable representations CFT, namely the formula for the fusion rules derived in [5],[6],[7] is based on the notion of weight diagram (a support of a finite dimensional module), and furthermore the characters of the integrable fusion algebra can be recovered from the standard characters by certain quantisation procedure. Similarly all these structures need to be generalised in order to describe the fractional level admissible CFT. In particular it turns out that the standard formal characters are replaced by some elements in the group ring of the (extended) affine Weyl group.

In the next section we review the main steps in [4] and in the last section we turn to the case of $\hat{sl}(4)_k$.

2 General setting and the case of $\hat{sl}(3)_k$

Let $k \not\in \mathbb{Q}$ and consider the subset $\tilde{\mathcal{W}}^{(+)}$ of the extended Weyl group $\tilde{W}$, defined as

$$\tilde{\mathcal{W}}^{(+)} := \{ y \in \tilde{W} \mid y(\alpha_i) \in \Delta^\text{re}_+ \text{ for } \forall \alpha_i \in \Pi \}, \tag{1}$$

where $\tilde{W} = W \ltimes t_P = W \rtimes A$, $t_P$ is the subgroup of translations in the weight lattice $P$ of $\tilde{g} = sl(n)$, $\tilde{W}$ and $W$ are the Weyl groups of $\tilde{g}$ and $g = \hat{sl}(n)_k$ respectively, and $A$ is the cyclic subgroup of $\tilde{W}$ which keeps invariant the set of simple roots $\Pi = \{ \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \}$ of $g$. In $\tilde{\mathcal{W}}^{(+)} \cap \tilde{W}$ is the set of real positive roots of $g$ and $\Delta^\text{re}_+$ is the set of simple roots of $\tilde{g}$.

Denote $\mathcal{W}^{(+)} = \mathcal{W}^{(+)} \cap W$, then $\tilde{\mathcal{W}}^{(+)} = \cup_{a \in A} a\mathcal{W}^{(+)}$. The subset $\tilde{\mathcal{W}}^{(+)}(\mathcal{W}^{(+)}$) is a fundamental domain (a “dominant chamber”) in $\tilde{W}(W)$ with respect to the right action of $\tilde{W}$ $[\tilde{g}]$ and the subset $\tilde{\mathcal{W}}^{(+)} \cdot k\Lambda_0$ of weights (or, equivalently, the subset $\tilde{\mathcal{W}}^{(+)} \subset \tilde{W}$ itself) labels the highest weights $\Lambda$ of maximally reducible Verma modules of $g$. Indeed for $\Lambda = y \cdot k\Lambda_0$ and $\beta = y(\alpha)$, s.t. $y \in \tilde{\mathcal{W}}^{(+)}$, the Kac-Kazhdan
singular vector criterion holds true for any positive root $\alpha$ of $\mathfrak{g}$. The Kac-Kazhdan reflections can be identified with the right action of $\tilde{W}$ on $\tilde{W}$, i.e.,

$$w_{y(\alpha)} \cdot \Lambda = yw_{\alpha} \cdot k\Lambda_0.$$ 

Here $\Lambda_0$ is the fundamental weight of $\mathfrak{g}$ dual to the affine root $\alpha_0$, and the shifted action of $\tilde{W}$ is given by $w \cdot \Lambda = w(\Lambda + \rho) - \rho$, $\rho$ being the Weyl vector of $\mathfrak{g}$.

Introducing a map $\iota$ of $\tilde{W}$ into the root lattice $Q$ of $\tilde{g}$

$$\iota : \tilde{W} \ni y = \bar{y} t_{-\lambda} \mapsto n \lambda + \bar{y}^{-1} \cdot 0 \in Q,$$

with the properties

$$\iota(xy) = \bar{y}^{-1}(\iota(x)) + \iota(y),$$

$$\iota(yw) = \bar{w}^{-1} \cdot \iota(y), \quad w \in W,$$

one can express $\tilde{W}^{(+)}$ alternatively as

$$\tilde{W}^{(+)} = \{ y \in \tilde{W} | \iota(y) \in P_+ \},$$

where $P_+ = \oplus_i \mathbb{Z}_{\geq 0} \bar{\Lambda}_i$, $\bar{\Lambda}_i$ being the fundamental weights of $\tilde{g}$. The $\tilde{g}$ Verma modules of highest weight $\iota(y)$ are reducible iff the corresponding $\mathfrak{g}$ Verma modules of highest weight $\Lambda = y \cdot k\Lambda_0$ are reducible.

With any $y \in \tilde{W}^{(+)}$ an element of the group ring $\mathbb{Z}[\tilde{W}]$ of $\tilde{W}$, a formal “character”, is associated

$$\chi_y = \sum_{z \in \tilde{W}, zy^{-1} \in W} m_z^y z,$$

and extended to $\tilde{W}$,

$$\chi_{y\bar{w}} := \det(\bar{w}) \chi_y, \quad y \in \tilde{W}^{(+)}, \quad \bar{w} \in \tilde{W},$$

where the integer coefficients $m_z^y$ are defined in the $n = 3$ case as

$$m_z^y = \overline{m_z^{(y)}},$$

$\overline{m_z^{(y)}}$ being the standard multiplicity of the weight $\mu = \iota(z)$ of the finite dimensional representation of $sl(3)$ of highest weight $\lambda = \iota(y)$. One introduces the notion of a generalised “weight diagram”, $G_y = \{ z \in \tilde{W} | m_z^y \neq 0 \}$, interpreted as the support (i.e., a set of weights with their multiplicities) of a “finite dimensional module” of highest weight $y$. These generalised weight diagrams are explicitly determined by (7) and thus have the structure of the weight diagrams $\Gamma_{\iota(y)}$ of triality zero $sl(3)$ representations, with the weights $\mu \not\in \text{Im}(\iota)$ excluded. In particular we refer to $y$ as a highest weight element in the diagram $G_y$ or in the character $\chi_y$.

The characters (5) give rise to a commutative ring with identity, interpreted as an extension of the ring of characters of the finite dimensional representations of
The ring contains a subring generated by elements $\chi_y$ labelled by $y \in \mathcal{W}(+) = \tilde{\mathcal{W}}(+) \cap W$, each represented as a polynomial of three “fundamental” characters, $\chi_{w_0}, \chi_{w_{10}}, \chi_{w_{20}}$, subject to one relation. To the elements of $A$ correspond "simple currents", $\chi_a = a, a \in A$,

$$\chi_a \chi_y = \chi_{ay}, \tag{8}$$

so that the fusions (8) with the generator $\gamma = \chi_{\gamma}$ of $A$ recover all $\chi_y, y \in \tilde{\mathcal{W}}(+)$. Having a notion of a generalised weight diagram, a formula for the fusion rule multiplicities, generalising the Weyl-Steinberg (W-S) formula, can be derived,

$$\chi_x \chi_y = \sum_{z \in \mathcal{G}_x} m_z^x \chi_{zy} = \sum_{z \in \mathcal{W}(+)} N^z_{x,y} \chi_z, \tag{9}$$

$$N^z_{x,y} = \sum_{w \in \mathcal{W}} \det(w) m_{zwy}^{-1}. \tag{10}$$

The second equality in (9) is derived as for the usual $\mathfrak{sl}(n)$ characters, using the symmetry in (8) and the fact that $\mathcal{W}(+) (\tilde{\mathcal{W}}(+))$ is a fundamental domain in $W(\tilde{W})$; the summation in $z$ in the last term runs effectively over the shifted weight diagram $\mathcal{G}_x \cap \tilde{\mathcal{W}}(+) \cap \tilde{\mathcal{W}}(+) \cap \mathcal{G}_x$ of ‘shifted highest weight’ $xy$. Using the properties (3) of the map $\iota$ the fusion coefficients (10) can be also expressed in terms of the structure constants $\tilde{N}^z_{\iota(x), \iota(y)}$ of the $\mathfrak{sl}(3)$ character ring

$$N^z_{x,y} = \tilde{N}^z_{\iota(x), \iota(y)} \tag{11}.$$

The generic fusions for the four generators $\chi_{w_0}, \chi_{w_{10}}, \chi_{w_{20}}, \chi_{\gamma}$ were confirmed in [4], [8] to reproduce the CFT fusion rules. Finally a “quantisation” of these formal characters leads to the finite set of characters of the fusion algebra of rational level $k = 3/p$ admissible representations, see [4] for more details.

It turns out that the definition of the generalised weight diagram in [4], based on (7), has to be modified in the $\mathfrak{sl}(4)$ and presumably in all higher rank cases, since it is not consistent with the Weyl-Steinberg formula (9), as can be checked in simple examples, see next section. This in particular implies that the formula (11), alternative to (10) in the $n = 2, 3$ cases, does not hold true in general. Typically the dimension of the “module” prescribed by (7) is higher than what one obtains by fusion, applying the W-S type formula (9) and adopting the definitions (3),(7) for a subset of smaller dimension modules, see below.
3 The case of $\tilde{sl}(4)_k$

We start with recalling some standard notation for the algebra $g = \tilde{sl}(4)_k$ and its Weyl group.

The Weyl group of $\mathfrak{g} = sl(4)$ consists of 24 elements and its Cayley graph is drawn in Fig. 1. It is an octahedron with truncated tips, so it has 8 elementary hexagons and 6 squares. They correspond to the Artin relations $w_i w_{i+1} w_i = w_{i+1} w_i w_{i+1}$, $i = 1, 2$ and $w_1 w_3 = w_3 w_1$ respectively. A finite part of the Cayley graph of the affine Weyl group $W$ is depicted in Fig. 2. with the unit element 1 denoted by $\star$. It is made of elementary truncated octahedrons isomorphic to the graph representing $W$ with the generators $w_i$, $i = 1, 2, 3$ replaced by $aw_i a^{-1}$, $a \in A$ (i.e., by $(w_2, w_3, w_0)$, etc). Note that unlike the $sl(3)$ Cayley graphs exploited in [4] here the graphs are not drawn as geometric graphs with angles and lengths consistent with the projection to the root diagram of $sl(4)$ defined by the map $\iota$ in (2). Rather each elementary truncated octahedron isomorphic to the finite Cayley graph of $W$ is “squashed” so that two adjacent hexagons originally meeting at an angle $2 \arctg \sqrt{2}$ now lie on one and the same plane, cf. Fig. 1. Thus the whole $W$ graph is organized into layers (part of a layer is clearly seen in Fig. 2), and if we single out a positive direction along anyone of the edges belonging to the basis of the two opposite pyramids forming an octahedron, we see that within any layer we can distinguish three kinds of vertices: those ones only connected to vertices of the preceding layer (denoted by a white circle), those ones only connected to vertices of the following layer (denoted by a black circle) and those ones only connected to vertices of the same layer (denoted by no circle).

The cyclic subgroup $A$ of $\tilde{W}$ is generated by $\gamma = t \bar{\gamma}$, where $\bar{\gamma} = w_{123} = w_1 w_2 w_3$ is a Coxeter element in $\overline{W}$ generating the cyclic subgroup $\tilde{A}$ of $\overline{W}$. One has $\gamma(\alpha_j) = \alpha_{j+1} = \gamma^{j+1}(\alpha_0)$ for $j = 0, 1, 2, 3$, identifying $\alpha_4 \equiv \alpha_0$, and accordingly $A$ gives an automorphism of $W$, $w_\alpha \rightarrow \gamma w_\alpha \gamma^{-1} = w_{\gamma(\alpha)}$, $\alpha \in \Pi$. The Cayley graph of $\tilde{W}$ can be thought of as a 4-sheeted covering of the graph of $W$ with, for example, the “fiber” over the edge ‘0’ connecting the vertices 1 and $w_0$ being the set $\{A, A w_0\}$. In Fig. 2 we have drawn schematically part of the first sheet $\tilde{W}(+)\overline{W}(+)\overline{W}(+)$ with the first layer and part of the interior structure indicated; e.g., the second layer starts with the point $w_{210}$ as an “origin” and part of its edges are indicated with dashed lines. We have also indicated the position of the $\iota$ - images of some of the elements in $\tilde{W}(+)$, but we have to caution the reader that among the black and white points, visually appearing on the second fundamental axis $\overline{\Lambda}_2$, only the white circled elements have in reality their $\iota$-images lying on this axis with coordinates $(0, 4n, 0)$, $n = 0, 1, \ldots$.

Define the subset of $\overline{W}$

$$U = \{A, A w_0, A w_{10}, A w_{30}, A w_{310}, A w_{2310}\}. \quad (12)$$
Its projection $\bar{U}$ onto the subgroup $\bar{W}$ gives the right cosets of $\bar{A}$. It is easy to show that the chamber $\tilde{\mathcal{W}}^{(+)}$ is represented as $\tilde{\mathcal{W}}^{(+)} = \mathcal{U} \cdot t_{-p_+}$. We define 4 term “class” elements in the group ring $\mathbb{Z}[W]$

$$F_{rst\ldots} \equiv F_{w_{rst\ldots}} := \sum_{a \in A} a w_{rst\ldots} a^{-1}, \quad w_{rst\ldots} \in W.$$ (13)

We shall furthermore denote $"F^m = F/2$ in the cases when $F$ contains two terms with multiplicity $2$, an example is provided by $"F_{13} = w_{13} + w_{02} = "F_{20}$. Note that since $\iota(a) = 0$ for any $a \in A$, each of the terms in a given $F_y$ is mapped to one and the same $\bar{W}$ orbit in the $sl(4)$ weight diagram and their images form a $\bar{A}$ orbit. Apparently $F_y = F_{aya^{-1}}$ for $\forall a \in A$ while for the product of two such elements we have, with the multiplication inherited from the multiplication in $\mathbb{Z}[W],$

$$F_x F_y = \sum_{a \in A} F_{xa^{-1}}y = \sum_{a \in A} F_{ax^{-1}}y.$$ (14)

In general $F_x F_y \neq F_y F_x$, but e.g., $F_0 \cdot F_{30} + "F_{13}" \cdot F_{10} + "F_{13}"$ commute between themselves.

We shall now introduce a finite set of formal characters $\chi_y, y \in \mathcal{W}^{(+)}$, as in (3), for all of which we will adopt the definition (7). In employing the map (2)
and comparing with the standard $sl(4)$ weight diagrams one can use the recursive formula for the multiplicity of a weight $\mu$ (see, e.g. [9])

$$m_\mu = - \sum_{\omega \in W; \omega \neq 1} \det(\omega) m_{\mu + \rho - \omega(\rho)}, \quad (15)$$

with the weights in the r.h.s. strictly greater than $\mu$.

We have for $y \in W^{(0)}$ and of length $l(y) \leq 3$

$$\chi_{w_0} = 3 + w_0 + w_1 + w_2 + w_3 \equiv 3 + F_0, \quad \iota(w_0) = (1, 0, 1),$$
$$\chi_{w_{10}} = 3 + 2F_0 + "F_{13}" + F_{10}, \quad \iota(w_{10}) = (0, 1, 2),$$
$$\chi_{w_{30}} = 3 + 2F_0 + "F_{13}" + F_{30}, \quad \iota(w_{30}) = (2, 1, 0),$$
$$\chi_{w_{230}} = \chi_{\gamma_{\overrightarrow{13}}} = 1 + F_0 + "F_{13}" + F_{30} + F_{230}$$

Fig. 2
where $\iota(w_{130}) = (1, 2, 1)$ of dimension 7, 17, 17, 15, 15, 63 respectively. Here $\overline{X}_{\Lambda}$ are standard formal $\mathfrak{sl}(4)$ characters, i.e., elements in the group ring $\mathbb{Z}[t_p]$. The terms combined in parentheses have $\iota$ images belonging to one and the same orbit of $\overline{W}$. To each of these characters we associate a weight diagram, which can be identified with a finite subset of the Cayley graph of $W$, the identity term being identified with the identity element in $W$, with multiplicities assigned to each vertex. E.g., the diagram $G_{w_0}$ for $\chi_{w_0}$ consists of the identity vertex in the center, with assigned multiplicity 3 and the four vertices connected to it by the generators $w_i$.

Exploiting (14) one obtains by a direct computation the relations

$$\chi_{w_0} \chi_{w_0} = 1 + 2 \chi_{w_0} + \chi_{w_{10}} + \chi_{w_{30}},$$

$$\chi_{w_0} \chi_{w_{10}} = \chi_{w_0} + 2 \chi_{w_{10}} + \chi_{w_{130}} + \chi_{w_{210}},$$

$$\chi_{w_0} \chi_{w_{30}} = \chi_{w_0} + 2 \chi_{w_{30}} + \chi_{w_{130}} + \chi_{w_{230}}.$$  \hfill (17)

These relations are consistent with the W-S formula (9), (10). E.g., for the first fusion we get following (9)

$$\chi_{w_0} \chi_{w_0} = \sum_{z \in G_{w_0}} m_{w_0}^w \chi_{zw_0} = 1 + 3 \chi_{w_0} + \chi_{w_{10}} + \chi_{w_{30}} + \chi_{w_{20}}.$$  \hfill (18)

The last term $\chi_{w_{20}} = \chi_{w_{02}}$ has a highest weight $w_{20}$ which does not belong to the dominant chamber $\overline{W}^\mathbb{R}$ and according to (4)

$$\chi_{w_{20}} = \chi_{w_0 w_2} = \det(w_2) \chi_{w_0} = -\chi_{w_0},$$

thus $\chi_{w_{20}} + 3 \chi_{w_0} = 2 \chi_{w_0},$ recovering the first equality in (17) with fusion multiplicity $N_{w_0 w_0} = 2$.

The relations (17) imply that given the characters labelled by $w_0, w_{10}, w_{210}$ (or by $w_0, w_{30}, w_{230}$), one can generate the remaining characters with highest weight $y$ of length $\leq 3$. In the next step there are 4 characters $\chi_y$ with $y \in \overline{W}^\mathbb{R}$ of length 4, and all of them are recovered recursively fusing according to (9) and comparing with the result of the explicit multiplication, e.g.,

$$\chi_{w_0} \chi_{w_{230}} = \chi_{w_{30}} + \chi_{w_{230}} + \chi_{w_{1230}}; \quad \iota(w_{1230}) = (5, 0, 1),$$

$$\chi_{w_0} \chi_{w_{210}} = \chi_{w_{10}} + \chi_{w_{210}} + \chi_{w_{3210}}; \quad \iota(w_{3210}) = (1, 0, 5),$$

$$\chi_{w_{10}} \chi_{w_{30}} = 1 + 2 \chi_{w_0} + \chi_{w_{10}} + \chi_{w_{30}} + \chi_{w_{130}} + \chi_{w_{2130}}; \quad \iota(w_{2130}) = (2, 2, 2),$$

$$\chi_{w_{10}} \chi_{w_{10}} = \chi_{w_{10}} + \chi_{w_{30}} + \chi_{w_{210}} + 2 \chi_{w_{130}} + \chi_{w_{3210}} + \chi_{w_{0130}},$$

$$\chi_{w_{30}} \chi_{w_{30}} = \chi_{w_{10}} + \chi_{w_{30}} + \chi_{w_{230}} + 2 \chi_{w_{130}} + \chi_{w_{1230}} + \chi_{w_{0130}}.$$
The last character, $\chi_{w_{0130}} = \chi_{\gamma,2} \iota$, with $\iota(w_{0130}) = (0, 4, 0)$, contains a term $F_{0130} + "F_{0213}" = \gamma^2 \chi_{w_{0130}}$.

While the characters $\chi_{w_{1230}}, \chi_{w_{2310}}, \chi_{w_{0130}}$, obtained by the fusions in (18) coincide with the expressions prescribed by (7), the character $\chi_{w_{2130}}$, with $\iota(w_{2130}) = (2, 2, 2)$, provides the first example in which formula (7) fails. Indeed the expression obtained by the fusion is “smaller”, with some terms missing, or multiplicities decreased, effectively implying that the corresponding weight diagram is a subset of the one determined by (7). Using the above fusion the character (corresponding to the “true” weight diagram) is given by

$$\chi_{w_{2130}} = 11 + 9F_0 + 8"F_{13}" + 5F_{12} + 5F_{21} + 3F_{121} + 3(F_{213} + F_{132}) + 2F_{123} + 2F_{321} + (F_{1213} + F_{1232}) + (F_{1321} + F_{2321}) + "F_{0213}" + F_{2130}.$$
From the Table we see one of the reasons why (4) comes into contradiction with the fusion based on (3): there are elements of higher length than the length of the highest weight element \( y = w_{2310} \) which have images in the weight diagram of the \( sl(4) \) representation \( \iota(y) \). Indeed, according to (7), the longest element \( u_{121321} \) of \( \mathcal{W} \) contributes to the character along with the highest weight element \( w_{2130} \), since both \( \iota(u_{121321}) \) and \( (2,2,2) = \iota(w_{2130}) \) appear in the outer Weyl orbit in the \( sl(4) \) weight diagram. On the other hand \( F_{121321} \), looked as a constituent of \( u_{121321} \), cannot be produced multiplying two elements of \( \mathcal{W} \) with a sum of lengths giving 4.

Let us denote by \( \mathcal{F} \) the set of the first five characters in (16) with the identity \( \chi_1 = 1 \) added,
\[
\mathcal{F} = \{1, \chi_{w_0}, \chi_{w_{10}}, \chi_{w_{20}}, \chi_{w_{210}}, \chi_{w_{230}}\}. \tag{19}
\]
The elements in (15) commute between themselves and are restricted by two algebraic relations implied by (17). Then we have

**Proposition** For any \( y \in \mathcal{W}^{(+)} \) there is a formal character \( \chi_y \) of the form of (4), which is obtained recursively, using (9), as a polynomial of the elements in (17).

Let us sketch the proof of the proposition.

We have to prove that by a proper fusion, identifying \( G_x \) with the weight diagram of some of the elements in \( \mathcal{F} \), the r.h.s. of (9) always produces one and only one “new” character with highest weight in \( \mathcal{W} \). It is useful to draw the weight diagrams of the basic elements in \( \mathcal{F} \). The shifted diagram \( G_x y \) is visualised identifying the center of the weight diagram \( G_x \), i.e., the identity element in any of the first five characters in (16), with the point \( y \in \mathcal{W}^{(+)} \), which locates \( G_x \) generically in the chamber \( \mathcal{W}^{(+)} \), each vertex counted with the corresponding multiplicity. Whenever some points of the shifted weight diagram appear outside of \( \mathcal{W}^{(+)} \) their contribution is cancelled, using the right action of \( \mathcal{W} \) in (9), i.e., the weight multiplicities are replaced by the fusion multiplicities (4) as in the last equality in (9).

We first consider the first layer in Fig. 2. Its vertices form the subset \( U t_{-n_1 \Lambda_1 - n_2 \Lambda_2} \cap \mathcal{W} \) of \( \mathcal{W}^{(+)} \).

We can split this layer into a disjoint union of even and odd horizontal “floors”, to be denoted \( T_{2j} \) and \( T_{2j+1} \), \( j = 0,1,2 \ldots \) containing \( 2j + 1 \) and \( j + 1 \) elements respectively. E.g., \( T_0 = \{1\}, T_1 = \{w_0\}, T_2 = \{w_{230}, w_{230}, w_{110}\}, T_3 = \{w_{1230}, w_{1230}, w_{1230}, w_{230}\} \), etc. Their \( \iota \) - images are given by
\[
\iota(T_{2j}) = \{4j \Lambda_1, 4j \Lambda_1 - \alpha_1, 4j \Lambda_1 - 2\alpha_1 + \alpha_3, 4j \Lambda_1 - 3\alpha_1 + \alpha_3, 4j \Lambda_1 - 4\alpha_1 + \alpha_3, 4j \Lambda_1 - 5\alpha_1 + \alpha_3, \ldots, 4j \Lambda_1 - 2j \alpha_1, \text{ for } j\text{-even, (or, } 4j \Lambda_1 - 2j \alpha_1 + \alpha_3 \text{ for } j\text{-odd}) \},
\]
\[
\iota(T_{2j+1}) = \{4j \Lambda_1 + \theta, 4j \Lambda_1 + \theta - 2\alpha_1, 4j \Lambda_1 + \theta - 4\alpha_1, \ldots, 4j \Lambda_1 + \theta - 2j \alpha_1 \}.
\]

The first three floors contain only elements labelling the highest weights of the fundamental set \( \mathcal{F} \). We shall identify them with the corresponding characters. Assume that for a given \( j \geq 1 \) all characters in \( T_i \) with \( l \leq 2j \) are generated. Then starting from the “white” circles in \( T_{2j} \) we recover fusing by \( w_0 \) all the elements
of the next floor $T_{2j+1}$, by $w_{10}$ — all “black” circles in $T_{2j+2}$, and by $w_{30}$ — all white circles in $T_{2j+2}$ with the exception of the utmost left one, with an $\iota$ image $(4j + 4)\bar{\Lambda}_1$; the latter is recovered by $\chi_{w_{230}}$ starting from the utmost left white circle in $T_{2j}$ mapped to $4j\bar{\Lambda}_1$. All of these fusions contain only one new element, the remaining terms corresponding to elements in $T_I$ with smaller $l$. In this way one generates all the characters corresponding to the points on the first layer.

It remains to repeat the steps for any consecutive layer since any layer repeats the structure of the first one with the only difference that there are edges going backwards, i.e., producing characters already generated in the previous steps. What we need is a starting point replacing the identity element, e.g., for the second layer this point is $w_{210}$, the corresponding character being in the fundamental set (19). Then the first three floors are given by $T_0^{(2)} = \{w_{210}\}$, $T_1^{(2)} = \{w_{30}w_{210}\}$, generated by $\chi_{w_0}$ (recall that $F_0 = F_3$), and $T_2^{(2)} = \{w_{123}w_{210}, w_{23}w_{210}, w_{03}w_{210}\}$, generated by $\chi_{w_{230}}, \chi_{w_{30}}, \chi_{w_{10}}$ resp., acting on $\chi_{w_{210}}$; recall that $F_{230} = F_{123}$, etc.). The starting points of the next layers are depicted as white circles with images on the $\bar{\Lambda}_3$ axis. Once the preceding layer is recovered, each of the characters labelled by these points is produced starting from the preceding one and applying $\chi_{w_{210}}$. □

The Proposition implies that comparing the result of the direct multiplication in the l.h.s of (10) with the r.h.s., which contains at every recursive step only one “new” character, the latter is obtained explicitly, i.e., the multiplicities in (5) are determined. It is also clear from the construction that the element in $\mathbb{Z}[W]$ corresponding to the highest weight $y$ of $\chi_y$, as well as the full term $F_y$, appears with multiplicity one. However the nonnegativity of the general multiplicities in (5) remains to be proved. We have checked this for up to length 6 highest weight words. All the examples confirm the result for the character $\chi_{w_{2130}}$ described above and suggest that the relevant definition of the weight diagrams generalising the one in the $\hat{sl}(3)_k$ case has to be based again on the correspondence via the map $\iota$ to the standard $sl(4)$ weight diagrams. Namely a term $F_z$ is present in $\chi_y$ only if $\iota(z) \in \Gamma_{\iota(y)}$, however in general $m_{z}^{y} \leq m_{\iota(z)}^{\iota(y)}$. We recall that in the $\hat{sl}(3)_k$ case the multiplicities $m_{z}^{y}$ were derived starting from supports of generalised “Verma modules” and prescribing their multiplicities by comparing, via the map $\iota$, with the multiplicities of the $sl(3)$ Verma modules determined by the Kostant partition function; this led to a generalised Weyl formula for the characters. Thus the problem can be reformulated as the problem of finding proper definition for the generalised Verma module supports.

We conclude with the remark that there is a sufficient evidence that as in the $\hat{sl}(3)$ case, the $\hat{sl}(4)_k$ fusion ring is an extension of the $sl(4)$ character ring $\overline{W}$. Indeed denote the commuting combinations $Y_0 = F_0, Y_1 = F_{30} + "F_{13}"$, $Y_2 = F_{10} + "F_{13}"$; $\{Y_j\}$ also commute with the elements of the cyclic group $\bar{A}$ as well as with the standard $sl(4)$ characters $\{\chi_{\bar{A}}\}$. The latter commute with any $w \in \overline{W}$. The fusions (17), (18), imply, postulating the expression for the character $\chi_{0130}$ as being given
by the prescription in (7), the following relations with coefficients $C_l, B^i_j, D$ in the group ring $\mathbb{W}[A],$

\[
Y_0^2 = 4 + Y_1 + Y_2,
\]

\[
Y_j^2 = C_j + \sum_{i=0}^{2} B^i_j Y_i, \quad j = 1, 2,
\]

\[
Y_0(Y_1 - Y_2) = D = \gamma^2 \chi_{\Lambda_1} - \gamma^3 \chi_{\Lambda_3}.
\]

This suggests to consider linear combinations of the remaining five independent variables (e.g., $Y_0, Y_1, Y_2, Y_0Y_1, Y_1Y_2$), with coefficients in $\mathbb{W}[A]$. Requiring the validity of the multiplication rule in (9) for any $\chi_x \in \mathcal{F}$ will impose restrictions on these coefficients which might provide another route to the explicit construction of the characters.

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