How to split the electron in one dimension

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Using the example of the Davydov soliton - a large acoustic polaron in one dimension - we demonstrate that the electron wave function can be fissioned in two or more long-lived, well-localized and spatially arbitrarily far separated fragments. The phenomenon of wave function splitting is a result of the electron-medium interaction, and takes place under a variety of conditions provided the initial wave function of the electron is localized and has at least one node.

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The concept of the wave function and the invention of quantum mechanics are among the most fascinating and powerful achievements of physics. The purpose of this Letter is to show yet another intriguing manifestation of quantum mechanics, namely that the wave function of an elementary particle can be fissioned into well-localized, spatially well-separated, and long-lived pieces.

The idea of fissioning has been put forward by Maris in his analysis of the physics of electron bubbles in superfluid helium [1]. It has been known for years that an electron injected into liquid helium can lower its energy by self-localizing in a 1s state inside a spherical cavity from which helium atoms are expelled. Maris argues that if the electron is optically excited from the 1s state to the 1p state, the pressure exerted by the electron in the excited state will no longer support the spherical cavity – the bubble walls will be set into motion, and the inertia of the liquid around the bubble will suffice to break the bubble into two smaller bubbles each carrying half of the original wave function. Maris further argues that the daughter bubbles will act in such a way as if they were fractions of the original particle, and that the existence of “fractional” bubbles explains a substantial amount of otherwise unexplained experimental data.

Elser [2] disagrees, arguing that the failure of the adiabatic approximation used in [1] makes it impossible to fission the wave function.

Jackiw, Rebbi and Schrieffer [3] do not contest the existence of a bubble carrying “half of the electron’s wave function” but point out that an electron in a helium bubble is entangled rather than fractional: before a measurement one cannot state in which bubble the electron resides; after the measurement a full electron will be found in one bubble and nothing in the other.

The dispute initiated by the proposal [1] raises a fundamental question: Is it in principle possible to fission the wave function of an elementary particle in long-lived localized well-separated fragments?

We found an affirmative answer to this question in a related physical context. Part of the difficulty in making definite statements about electron bubbles in helium is that the treatments involve too many approximations [1]; some of which are hardly justifiable [3]. At the same time it seems difficult to improve the approximations without making the problem unmanageable.

Electrons form bubbles in helium as a consequence of a kind of polaronic effect [4]: they become self-localized as a result of the response of the medium. Instead we will look at a quantum particle in the presence of a deformable classical medium, a one-dimensional polaron problem. Among the attractive features of this system is that self-localization takes place regardless of the magnitude of the coupling between the electron and the medium [1]. When the coupling is small, the scale of the self-localized state can be much bigger than the atomic spacing (“large” polaron limit), and thus a macroscopic approach that ignores the discreteness of the medium can be adopted. From a practical viewpoint, polaronic effects play an important role in the physics of quasi-one-dimensional conductors. Polarons (also called Davydov solitons) are known to exist in α-helical protein molecules of living organisms, and play a crucial role in the transport of vibrational energy and charge [4]. These systems represent experimental candidates where the effects described below can be tested.

The dynamics of the system is described by the set of Davydov equations [4]:

\[ i\hbar \partial_t \Psi = -\frac{\hbar^2}{2m} \partial_x^2 \Psi + \lambda \partial_x u \Psi, \]  
\[ \rho \partial_t^2 u = B \partial_x^2 u + \lambda \partial_x |\Psi|^2, \]  

where \( m \) is the electron mass, \( \Psi(x,t) \) is the electron wave function normalized to unity, \( u(x,t) \) is the classical displacement field of the medium, \( B \) is the bulk constant of the medium, \( \rho \) is the linear mass density of the medium, and \( \lambda \) is the constant of electron-medium interaction. The first of the equations (1) is the Schrödinger equation for the electron moving in a self-induced potential \( \lambda \partial_x u \) due to the strain in the medium, while the second equation (2) is the inhomogeneous wave equation with an extra term accounting for the force exerted by the electron on the particles of the medium. A classical description of the medium degrees of freedom is justified as
the medium is composed of a large number of particles which are much heavier than the electron.

The ground state of the coupled electron-medium system is described by the soliton solution to (1) and (2) of the form \( \Phi_0 = \exp(-iE_0t/\hbar)\text{sech}(x/\xi)/(2\xi)^{1/2} \) and \( \partial_x u_0 = -(\lambda/B)|\Psi_0|^2 \), where \( E_0 = -\hbar^2/(2m\xi^2) \), and \( \xi = 2B\hbar/(m\lambda^2) \). The physical meaning of this solution is that the electron deforms the medium, which acts as a potential well.

Including the deformation energy, the total energy change due to the formation of the soliton is \( E_0/3 \), which is still negative. Thus the soliton forms spontaneously. The location of the soliton center is arbitrary, and the localization length \( \xi \) will be assumed to be much larger than the atomic scale. The Davydov soliton is remotely analogous to the 1s electron bubble in helium \( \Psi_0 \). The potential well \( \lambda\partial_x u_0 = -(\lambda^2/B)|\Psi_0|^2 = -(\hbar^2/m\xi^2)\text{sech}^2(x/\xi) \) created and sensed by the electron is analogous to the cavity of expelled helium atoms confining the electron.

Using the dimensionless variables for length \( y = x/\xi \), time \( \tau = E_0t/\hbar \), the wave function \( \Phi = \xi^{1/2}\Psi \), and the potential felt by the electron \( w = \lambda\partial_x u/|E_0| \), the system of equations (3) and (4) simplifies to:

\[
i\partial_x \Phi = -\partial_y^2 \Phi + w\Phi, \tag{3}
\]

\[
s\partial_y^2 w = \partial_x^2 [w + 4|\Phi|^2]. \tag{4}
\]

It is interesting that the parameter \( s = \lambda^4\rho/(16B^3\hbar^2) \) controlling the strength of electron-medium coupling does not depend on the electron mass. The dimensionless description (3) and (4) will be used hereafter.

We are now ready to describe how to split the wave function into fragments. In the limit that a fragment is infinitely far away from other features of the wave function it is necessarily of the sech form and the supporting medium well is of the sech\(^2\) form. For example, half of the Davydov soliton is described by \( \Phi_{1/2} = \exp(i\tau/4)\text{sech}(y/2)/\sqrt{2} \) (normalized to one half) and \( w_{1/2} = -\text{sech}^2(y/2)/2 \). It is straightforward to calculate that the total energy gain from having in the system two infinitely far separated halves of the Davydov soliton is four times smaller than that from having one full soliton, showing that the Davydov soliton is a stable object which does not spontaneously break in half. However the soliton can be split if extra energy is invested into the system.

To prepare the initial state which will split into fragments one can try, following Maris [1], to optically excite the electron into the first excited state of the self-induced potential [8]. However this idea will not work in one dimension: the self-induced potential \( w_0 = -4|\Phi_0|^2 = -2\text{sech}^2 y \) has only one bound state [3, 3]. Nonetheless, it is possible to excite the electron into an initial state that is well-localized, even though it is a superposition of the unbound states, such that the wave function separates into distinct and well-localized components. For example, for the potential \( w_0 = -2\text{sech}^2 y \) the zero-energy state is on the verge of becoming discrete [8]. Then imposing arbitrarily weak even external confining potential will turn the marginal state into an odd localized excited state. Exciting the electron into this newly formed state and turning off the external potential, we arrive at the initial condition resembling the 1p electron inside the spherical helium bubble [3].

More generally, if initially the wave function is odd and well-localized, and the strain field is even and well-localized, the zero of the wave function persists through the evolution. This node plays the role of an impenetrable wall that repels the wings of the wave function. Without the coupling with the medium there would be a long-range repulsion between the wings and the node and the wave function will just spread out away from the origin. The presence of the medium prevents spreading by localizing the wave function and making its interaction with the node short ranged. After the initial breakup the fragments of the wave function will start aggregating into the appropriate sech form, and will virtually stop feeling the presence of the node as soon as they are several localization lengths apart from the origin. At the same time the material degrees of freedom will keep adjusting into the appropriate sech\(^2\) form. This process of ”settling down” into the Davydov form will be accompanied by the fragments emitting sound waves which establishes a ”communication” between the pieces even when they are far apart. The exchange of sound waves leads to a very long-range repulsion between the fragments, since the amplitude of sound waves is a constant independent of the distance to the source of sound (in one dimension). Thus the combination of inertia acquired as a result of the initial breakup, and the long-range repulsion through exchange by sound will separate the fragments arbitrarily far from each other.

For the initial conditions described above the exchange of probabilities between the fragments is forbidden by symmetry. In reality there can be deviations from the ”ideal” conditions. Then the initial zero of the wave function will not be preserved by the evolution – fragments of unequal sizes can start exchanging by a flow of probability density. This might lead to the larger fragment absorbing the smaller one, thus restoring the full Davydov soliton in a microscopic time. To understand this case we resort to numerical experiments which are described below.

We tested these ideas by numerically solving the system of equations (3) and (4) under a variety of initial conditions. We use a Fourier transform algorithm to integrate forward the Schrödinger equation, thus maintaining unitary evolution perfectly. The wave equation for the stress field is integrated forward using the Ver-
let algorithm \[1\]. Figures 1 and 2 show “carpets” of the electron density \(\Phi^2\) and the well function \(w\) as functions of position and time. The horizontal direction corresponds to space while the vertical to time; the time arrow runs from the bottom to the top, and the bottom of the Figures represents the initial conditions. The electron density is strictly non-negative and shown using varying shades of red – the higher the density, the deeper the color. The well function is mainly negative and shown using varying shades of blue – the intensity of the color is proportional to the well depth. The places where the well function becomes positive are color-coded in red with the same intensity-magnitude correspondence. The Figures are constructed for \(s = 1\) and \(\Phi(y, 0) \sim \exp[-0.5(y - y_c)^2/d^2]\tanhy - \) for \(y_c = 0\) this is an odd function localized near the origin (\(d = 30\) was selected), while the parameter \(y_c\) controls the asymmetry of the initial electron density. The Figures are produced for \(w(y, 0) = -4|\Phi(y, 0)|^2\) which corresponds to the local equilibrium of the wave function and the medium - this has an advantage of minimizing transient effects which are unavoidable in the system very far from the ground state. We also tried other functional forms and values of parameters, and always found similar results as long as the initial functions were spatially well-localized, and the wave function had a node.

We show the results of a study using a ring geometry, consisting of \(2L = 512\) units (larger rings were also looked at and the same results were found). In order to have a fair representation of an infinite system we had to make sure that the excitations of the \(w\) field do not travel around the ring, and thus do not interfere with the dynamics which would be present in truly infinite system. This was accomplished by adding a localized dissipation source at \(y = \pm L\) – we have included a term of the form \(A \exp[-(y \pm L)^2/l^2]\partial_t w\) on the left hand side of \[1\] with adjustable amplitude \(A\) and width \(l\) \((A = 5\) and \(l = 30\) corresponds to the Figures). Similarly, to eliminate communication between parts of the wave function which are solely due to the ring geometry, we imposed a very large potential barrier at the distances \(l\) away from the center of the dissipation source – this mimics the vanishing of the wave function at infinity. These measures also turned out to be an effective way to asymptotically slow down and confine the soliton fragments in the vicinity of the dissipation region.

In Figure 1 the initial conditions for the electron density and the well function were selected to be even \((y_c = 0)\). This symmetry is preserved by the equations of motion \([3]\) and \([4]\) which allows us to show the \(y < 0\) part of the electron density carpet (left) and the \(y > 0\) part of the well function (right) on the same plot. Ignoring the fine structure, the left and right sides of Figure 1 are almost perfect mirror images of each other. Most of the fine structure consists of ripples of the \(w\) field which are sound waves propagating with constant velocity \(s^{-1/2}\). It is obvious that very quickly after the breakup two well-localized objects form which travel away from each other, with the electron density and the well function peaks coinciding. As time progresses, the magnitude of the fragment oscillations decreases as they approach the Davydov functional form. This is because the extra kinetic energy emitted by the fragments outward in the form of sound waves is carried away to infinity – in our finite system it is absorbed by the dissipative segment of the ring opposite to the origin. This segment can be recognized on the \(w\) carpet as sound-absorbing ripple-free stripe.

![Figure 1](image)

**FIG. 1.** Symmetric splitting of the soliton in half. For clarity the vertical line is drawn to separate the electron density (left) and the well function (right) carpets.

It is clear from Figure 1 that the fragments are slightly accelerating away from each other even when they are very far apart. This is because they keep exchanging by sound waves, as can be seen from the \(w\) part of the carpet, which produces an effective repulsion between the fragments. We tested this idea by putting at the origin a well-localized source of dissipation of the same kind as we have at the system edges. The role of this source is to prevent the fragments from “talking” to each other via sound waves while allowing them to communicate via the wave function. Then we re-ran the program with exactly the same parameters, and turned on the dissipation source half-way through the evolution. As a result we observed that after the dissipation source was turned on, the fragment trajectories became ballistic thus confirming that the acceleration was lattice-mediated. As time progresses the efficiency of this effective repulsion decreases as sound waves partially transmit through the fragments and leave the system.

For asymmetric initial conditions we observed that the zero of the wave function is not preserved during the evolution but the phase discontinuity at \(y = 0\) persists. The weak asymmetry cases are fairly similar to the symmetric ones and not shown. Figure 2 shows the \(w\) carpet for a strongly asymmetric breakup of the soliton with
the asymmetry parameter $y_c = 4$ (there is initially substantially more probability on the right than on the left). The interesting feature of this evolution is that a smaller third fragment persists for quite a long time until it coalesces with one of the bigger fragments, and we end up with two unequal pieces moving away from each other. The density carpet (free of the sound wave ripples) looks almost identical to Figure 2 and is not shown.

FIG. 2. Carpet of the well function for asymmetric splitting of the soliton.

We note that in Figures 1 and 2 the velocities of the fragments are substantially smaller than the sound velocity which justifies the use of linear elasticity theory in describing the medium degrees of freedom. We also observed breakups into three and four fragments - this happens for smoother initial conditions than those used to construct Figure 2.

Our results indicate that the wave function fragments have infinite life time. However a finite large life time cannot be ruled out if the medium degrees of freedom would be treated quantum-mechanically.

In interpreting our results it is important to remember that the fissioning of the wave function is a statement about the motion of the electron, and not about its parameters such as charge and mass, that characterize the electron as a particle [10]. The process which leads to the splitting is an example of a measurement in which the quantum system (the electron) interacts with a classical apparatus (the medium) [10]. This measurement process changes the state of the apparatus and affects the electron by splitting its wave function. The field $w(x,t)$ is the apparatus “readings” from which we can draw conclusions about the state of the electron, namely the electron probability density $|\Phi(x,t)|^2$, which can be verified in a subsequent measurement. If this is a position measurement, a full electron will be found in one of the fragments, and the wave function will instantly vanish elsewhere. In response to that the classical field $w$ in the “empty” fragment(s) will start relaxing away which will be the experimental sign that before the position measurement the electron was in several arbitrarily far separated fragments at once.

We also conducted a similar numerical investigation of the two-dimensional version of the problem, and did not find the effect of fissioning which is so apparent in one dimension. In two dimensions the electron self-localization takes place if the coupling with the medium exceeds some critical value, and the polaron is always “small” – its size is of order the lattice spacing [4]. However as long as the observed size of the density (or the well) peaks is much larger than the lattice spacing, the continuum theory is still valid, and the subsequent conclusions refer to this macroscopic regime. We found that deep in the self-localizing regime the two initial density peaks, symmetric with respect to a line where the wave function vanishes, do not separate – they just become sharper and remain independent. The same was also true for an asymmetry in the initial conditions. These observations can be understood by noticing that above one dimension the amplitude of the sound waves is a decreasing function of the distance from the source - the lattice-mediated interaction is a much weaker effect than in one dimension. We also observed that when the coupling with the medium is not too large, slightly asymmetric peaks have a very different fate – the higher one becomes self-localized, while the lower one diffuses away. This is reminiscent of the scenario put forward by Elser [2] in the context of electron bubbles in helium.

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