Symmetry Analysis in Linear Hydrodynamic Stability Theory:
Classical and New Modes in Linear Shear

Andreas Nold\textsuperscript{1,}\textsuperscript{a} and Martin Oberlack\textsuperscript{2,3,4}

1) Department of Chemical Engineering, Imperial College London, London, SW7 2AZ, United Kingdom
2) Chair of Fluid Dynamics, Technische Universität Darmstadt, 64287 Darmstadt, Germany
3) Center of Smart Interfaces, Technische Universität Darmstadt, 64287 Darmstadt, Germany
4) Graduate School of Computational Engineering, TU Darmstadt, 64293 Darmstadt, Darmstadt, Deutschland

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We present a symmetry classification of the linearised Navier-Stokes equations for a two-dimensional unbounded linear shear flow of an incompressible fluid. The full set of symmetries is employed to systematically derive invariant ansatz functions. The symmetry analysis grasps three approaches. Two of them are existing ones, representing the classical normal modes and the Kelvin modes, while the third is a novel approach and leads to a new closed-form solution of traveling modes, showing qualitatively different behaviour in energetics, shape and kinematics when compared to the classical approaches. The last modes are energy conserving in the inviscid case. They are localized in the cross-stream direction and periodic in the streamwise direction. As for the kinematics, they travel at constant velocity in the cross-stream direction, whilst in the streamwise direction they are accelerated by the base flow. In the viscous case, the modes break down due to damping of high wavenumber contributions.

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\textsuperscript{a)Electronic mail: andreas.nold09@imperial.ac.uk}
I. INTRODUCTION

Flow stability theory deals with the breakdown of an ordered laminar non-uniform flow and the onset of turbulent structures. This transition to turbulence can be found in nature and industrial applications. For example, when observing rivers, winds as well as pipelines and bearings, we notice that accelerating a flow leads to turbulent structures. Apart from its economic relevance, the study of the precise location, time and form of the transition to turbulence has fascinated generations of scientists since the famous experiment of an unstable flow in a pipe by Reynolds in 1883. Due to its complexity, even canonical examples such as the stability of a simple Couette flow between two infinite plates, where one plate is moving steadily in one direction and the other plate is fixed, have been a topic of debate for decades.

The classical approach to stability problems is the so-called normal mode approach as derived by Orr. It consists of periodic modes traveling in the streamwise direction and has been applied to the linear stability problem of a plane Couette flow by Hopf, Wasow, Grohne, Reid and others, yielding a decay of all modes for large times. A second viable approach consists of using the modes introduced by Kelvin, with a time-dependent wavelength in the cross-stream direction. Rosen reviewed this approach and formulated a general solution for perturbations of a plane Couette flow in the linear framework. The three-dimensional Kelvin modes exhibit a period of modest algebraic/transient growth before entering a phase of exponential viscous decay. Decay of all perturbations in shear flows for large times was also shown by Case in a closed-form solution of the bounded inviscid initial-value problem, yielding an algebraic decay of the perturbation with $1/t$ if the initially introduced vorticity is finite. In particular, Romanov showed that all eigenvalues for a small enough perturbation are less than $-C/Re$, where $C$ is a positive constant. Clearly, these results contradict experiments, where transition to turbulence of a Couette flow is observed at Reynolds numbers of around 350.

It was in the 1990’s that a novel approach revealed an explanation for this apparent paradox. It was shown that non-uniform flows are non-normal: they are spectrally stable, but perturbations are able to gain the basic (shear) flow energy transiently and, consequently, exhibit strong growth during a limited time interval. In particular, this is observed in the short-term behaviour of perturbed flows. In the case of large enough initial perturbations,
the strong short-term non-normal growth allows for non-linear effects to take place, which regenerate the transiently growing perturbations. This positive feedback-loop allows for the onset of turbulence and is usually denoted as bypass-transition\textsuperscript{16–18}.

The aim of this work is to perform symmetry analysis of the linear stability problem of an unbounded Couette flow and to show how this mathematical tool can shed new light on long-standing and well-known problems. A symmetry is a transformation which maps the solution manifold of a differential equation onto itself. Special solutions which do not change under a symmetry transformation are denoted as invariant solutions, or, if scaling is part of the transformation, as self-similar solutions. Generally, invariant solutions are a powerful tool for the systematic development of ansatz functions for solutions of partial differential equations\textsuperscript{19–21}. Especially in the area of fluid mechanics, these have been applied successfully in various fields of application\textsuperscript{22–29}.

We search for invariant solutions in the context of the stability of an unbounded Couette flow and perform a symmetry classification of the linearised Navier-Stokes equations for two-dimensional perturbations. We show that the century-old normal mode approach and the Kelvin mode approach both turn out to be among a larger class of invariant solutions. In particular, the normal-mode approach is obtained by a successive symmetry reduction with respect to space- and time-translation symmetries together with a scaling symmetry. The Kelvin mode approach is obtained similarly. It is invariant with respect to a combination of a time-translation symmetry and one symmetry which is due to the linearity of the base flow.

Strikingly, symmetry methods also allow for a third class of invariant solutions so far not known to the authors. In the inviscid case, we obtain a new closed form solution, exhibiting qualitatively new behaviour in kinematics, energetics, and shape. In particular, the modes consist of vortices traveling at constant speed in the cross-stream direction and being accelerated by the linear shear base flow in the streamwise direction. The closed form solution also reveals a particular shape of the modes, which are energy-conserving, decay in the cross-stream direction and are periodic in the streamwise direction. In the viscous case, invariance of these modes is lost. We present a closed-form solution of the initial value problem. In agreement with expectations from Kelvin mode theory, the energy of the modes decays exponentially due to viscous damping effects.

We emphasize that the new invariant function limits its application to two-dimensional
settings, as three-dimensional effects are not taken into account here. Due to the traveling in the cross-stream direction the new invariant function also limits its validity by a finite time interval (during which the solution reaches a boundary). These properties might be given in wind shear or ocean flows. Due to its peculiar behaviour and its analytical simplicity, we believe that despite the limitations, the new approach presented here adds a new perspective to the understanding of shear flow dynamics. The feasibility of the found solution may be confirmed numerically by imposing the invariant function in a plane Couette flow and following the dynamics by direct numerical simulation.

In § II and § III we give a brief overview of symmetry methods and introduce a full symmetry-classification for the linear stability analysis of a Couette flow in stream function formulation. We then show how symmetry methods allow for a systematic derivation of the normal mode approach and the Kelvin mode approach in § IV and V. In the following, we present the new invariant modes in § VI. In the inviscid case, a closed-form solution of the modes is derived in § VI A, whereas the viscous case is studied in § VI B. We conclude in § VII with a summary of our results and discussion. In the Appendix, linearly independent solutions to the viscous problem employing the new invariant approach are presented.

II. SYMMETRY ANALYSIS

We consider an unbounded parallel two-dimensional shear flow \((U(y), 0)^T\) with a perturbation of the form \((u(x, y, t), v(x, y, t))^T\). Applying the curl on the momentum-equations for the perturbations and introducing a stream function \(\psi(x, y, t)\) yields the following linearized fourth order partial differential equation for the stream function:

\[
\frac{\partial}{\partial t} \Delta \psi + U \frac{\partial}{\partial x} \Delta \psi - U'' \frac{\partial \psi}{\partial x} = \nu \Delta \Delta \psi.
\] (1)

where \(\nu\) is the kinematic viscosity and \(\Delta\) is the Laplace operator. A symmetry of this differential equation is given by a point transformation \(T = (\tilde{x}, \tilde{y}, \tilde{t}, \tilde{\psi})\) with

\[
\tilde{x} = \tilde{x}(x, y, t, \psi; \epsilon), \quad \tilde{y} = \tilde{y}(x, y, t, \psi; \epsilon),
\]

\[
\tilde{t} = \tilde{t}(x, y, t, \psi; \epsilon), \quad \tilde{\psi} = \tilde{\psi}(x, y, t, \psi; \epsilon),
\]

for which the transformed quantities satisfy the transformed differential Eq. (1), yielding

\[
\frac{\partial}{\partial \tilde{t}} \tilde{\Delta} \tilde{\psi} + U \frac{\partial}{\partial \tilde{x}} \tilde{\Delta} \tilde{\psi} - U'' \frac{\partial \tilde{\psi}}{\partial \tilde{x}} = \nu \tilde{\Delta} \tilde{\Delta} \tilde{\psi}.
\] (4)
In (2)-(3), $\varepsilon \in \mathbb{R}$ is the group parameter of the transformation and we assume that $T$ is a smooth function of the parameter $\varepsilon$. As an example, in the case of a space translation transformation, $\varepsilon$ is equivalent to the actual translation performed. The rate of change with which the variables are transformed is then given by the tangent vector field $(\xi^x, \xi^y, \xi^t, \eta)$ of the map $T$ at $\varepsilon = 0$, defined by

$$
\xi^{(x,y,t)} = \left. \frac{\partial T}{\partial \varepsilon} \right|_{\varepsilon=0}^{(x,y,t)} \text{ and } \eta = \left. \frac{\partial T}{\partial \varepsilon} \right|_{\varepsilon=0}^{\psi}.
$$

(5)

Sophus Lie first introduced a special kind of transformations, the so-called Lie-point symmetries. In this case, the map $T$ has the special property of being uniquely defined by its tangent vector field, which can also be written as an infinitesimal generator

$$X := \xi^x \frac{\partial}{\partial x} + \xi^y \frac{\partial}{\partial y} + \xi^t \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial \psi},$$

(6)

and which forms a Lie-symmetry group through

$$T = e^{\varepsilon X} x,$$

(7)

where $x = (x, y, t, \psi)^21$.

The first powerful tool of symmetry analysis is the concept of invariant solutions. In a nutshell, invariance means that a solution $\psi(x, y, t)$ is not changed by the application of the transformation $T$. The mathematical condition for this is that the solution $\psi = \psi(x, y, t)$ does not change its functional form after application of the infinitesimal generator:

$$X(\psi - \psi(x, y, t))|_{\psi=\psi(x,y,t)} = \eta - \xi^x \frac{\partial \psi}{\partial x} - \xi^y \frac{\partial \psi}{\partial y} - \xi^t \frac{\partial \psi}{\partial t} = 0.$$ (8)

As a simple example, a solution which is invariant with respect to the translational symmetry in $x$ is represented by $X = \frac{\partial}{\partial x}$. In this case, condition (8) yields

$$\frac{\partial \psi}{\partial x} = 0$$

(9)

and therefore the respective invariant solution does not depend on $x$.

Next to invariance, the second crucial tool of symmetry analysis is the combination of different symmetries. Mathematically, two symmetries $X_1, X_2$ are combined by superposing their infinitesimals $X := a_1 X_1 + a_2 X_2$ with

$$\xi^{(x,y,t)} = a_1 \xi^{(x,y,t),1} + a_2 \xi^{(x,y,t),2} \text{ and } \eta = a_1 \eta^1 + a_2 \eta^2$$

(10)
and coefficients $a_{1,2}$. For example, instead of working with a solution which is invariant with respect to the space translation symmetry alone, we can search for solutions which are invariant to a combined space- and time translation symmetry $X = a_1X_1 + a_2X_2$, such that (8) transforms to
\begin{equation}
-a_1 \frac{\partial \psi}{\partial x} - a_2 \frac{\partial \psi}{\partial t} = 0.
\end{equation}

The corresponding invariant solution represents a traveling wave $\psi (a_2x - a_1t)$ for which the velocity of propagation depends on the ratio of the coefficients $a_1$ and $a_2$.

Together, the concept of invariance and of superposition of symmetries will help to gain new insights for the systematic analysis of ansatz functions for linear stability theory. In this work, we will show how normal modes, Kelvin modes, as well as a new type of base solutions can be systematically derived by searching for invariant solutions with respect to a combination of the full set of available symmetries. For a more detailed introduction to Lie Symmetries, see Bluman and Anco\textsuperscript{19}, Bluman, Cheviakov, and Anco\textsuperscript{21}, Bluman and Kumei\textsuperscript{30}, Cantwell\textsuperscript{31} and Steeb\textsuperscript{20}. The symmetries presented in this work were derived by means of the Lie-Algorithm, using the GeM package of Cheviakov\textsuperscript{32} and the DESOLVE package of Carminati and Vu\textsuperscript{33}.

### III. SYMMETRIES OF A PLANAR COUETTE FLOW

In this section, we present a full symmetry classification of the problem in stream function formulation. We have also performed a symmetry classification of the momentum equations and the continuity equation. However, no additional symmetries exist for this formulation of the problem. Consequently, we will present all results in the stream function formulation.

We have performed a symmetry analysis for two separate cases: First, a general shear flow $U(y)$ was considered. In this case, the stream function formulation of the problem (I) allows for four different symmetry transformations (see also table II). Second, a linear shear flow $U(y) = Ay$ was considered. This restriction allows for an additional symmetry which reflects the effects of a base flow with a constant shear rate $A$ (see table II). As mentioned in section II, a linear combination of the infinitesimals allows for the construction of insightful invariant solutions. A general symmetry can be
\[ X_0 = \psi_0 \frac{\partial}{\partial \psi_0} \]
\[ X_1 = \frac{\partial}{\partial x} \]
\[ X_2 = \frac{\partial}{\partial t} \]
\[ X_3 = \psi \frac{\partial}{\partial \psi} \]
\[ X_4 = A \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \]

\[ \tilde{x} = x + x_0 \]
\[ \tilde{t} = t + t_0 \]
\[ \tilde{\psi} = C \psi \]
\[ \tilde{y} = y - y_0 \]

\[ X = a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4. \]  

TABLE I. Complete set of symmetries of Eq. (1) for two-dimensional perturbations of a viscous fluid for general shear flows \( U(y) \) and for linear shear flows \( U(y) = Ay \). \( X_{0,1,2,3} \): Symmetries for an arbitrary parallel shear flow \( U(y) \). \( X_4 \): Additional symmetry if the base flow is restricted to \( U(y) = Ay \). \( X_1 \) and \( X_2 \) are the space and time translation symmetries. \( X_0 \) is the superposition symmetry and \( X_3 \) is the scaling symmetry. \( X_0 \) and \( X_3 \) are both due to the linearisation of the Navier-Stokes equations for small perturbations.

defined by

\[ X = a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4. \]  

We conclude that finding the invariant solutions amounts to solving (8) with

\[ \xi^{x,y,t} = \sum_{i=1}^{4} a_i \xi^{x,y,t}, \]
\[ \eta^{i} = \sum_{i=1}^{4} a_i \eta^{i}, \]  

where \( \xi^{x,y,t} \) and \( \eta^{i} \) are defined in table I. A brief examination of Eq. (8) shows that the resulting invariant solutions remain unchanged up to an equal scaling of all parameters by some factor \( C \in \mathbb{C} \). This allows us to set one parameter \( a_i \) which is unequal zero to one, without loss of any information. We are also not interested in solutions which are invariant with respect to only one symmetry, because this leads to trivial simplifications. Furthermore, we will exclude the superposition symmetry \( X_0 \) from our further considerations, as the symmetry itself already includes a solution of the equation under consideration.
\[
| a_1 | a_2 | a_3 | a_4 |
|-----|-----|-----|-----|
| \mathbb{C} | \mathbb{C} | \mathbb{C} | 0   |
| \mathbb{C} | 0   | \mathbb{C} | 1   |
| \mathbb{C} | 1   | \mathbb{C} | \mathbb{C}\setminus\{0\} |
\]

Normal Modes
Kelvin Modes
New invariant Modes

TABLE II. Classes of invariant solutions depending on the choice of parameters \( a_i \). \( a_4 \) can only be nonzero in the case of a linear shear flow \( U(y) = Ay \). We note that the invariant solutions remain unchanged if the parameters \( a_i \) are equally scaled by some coefficient \( C \in \mathbb{C} \) (see Eq. 8).

FIG. 1. Schematic view of selected invariant solutions. The symmetry \( X_4 \) is only valid for the case of a linear shear flow. If \( X_4 \) is not used, then successive symmetry reductions lead to the normal mode approach. Employing \( X_4 \), but excluding the time-translation symmetry \( X_2 \) leads to the Kelvin modes, whereas using the full set of symmetries leads to the new invariant solutions presented in this paper.

All other invariant solutions can be divided into three classes, such as shown in table II and in Figure I. In the following chapters we will systematically derive the invariant solutions for these three classes. For simplicity, we will present all solutions in complex-valued form. As Eq. (II) is real-valued, the complex conjugate \( \bar{\psi} \) of any solution \( \psi \) will equally solve the problem. This allows us to construct a real-valued solution from any complex-valued \( \psi \) by the simple superposition \( \psi + \bar{\psi} \).
IV. NORMAL MODES

The classical normal mode approach turns out to be an invariant solution with respect to the combination of the three symmetries $X_1$-$X_3$:

$$X^{(N)} := a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial t} + a_3 \psi \frac{\partial}{\partial \psi},$$

with the complex prefactors $a_{1,2,3} \in \mathbb{C}$ and $a_4 = 0$ in (12). Condition (8) then reduces to

$$a_3 \psi - a_1 \frac{\partial \psi}{\partial x} - a_2 \frac{\partial \psi}{\partial t} = 0.$$  \hspace{1cm} (15)

If $a_2 \neq 0$, then the method of characteristics provides us with the solution

$$\psi^{(N)}(x, y, t) = f^{(N)}(\xi, y) e^{\frac{a_3}{a_2} t},$$

with the new variable

$$\xi = x - a_1 t,$$

and where $f^{(N)}$ solves the fourth order differential equation

$$\left( U - \frac{a_1}{a_2} \right) \frac{\partial}{\partial \xi} \Delta f^{(N)} + \frac{a_3}{a_2} \Delta f^{(N)} - U'' \frac{\partial}{\partial \xi} f^{(N)} = \nu \Delta \Delta f^{(N)},$$

obtained by inserting ansatz (16) into Eq. (1). We note that this first symmetry reduction has not simplified the PDE considerably. While the number of variables has been reduced by one, the PDE is still of fourth order and the number of parameters even increased by two: Additional to the viscosity $\nu$, we now also have $a_1/a_2$ and $a_3/a_2$. Fortunately, by performing a symmetry analysis we find that Eq. (18) admits two symmetries: the scaling symmetry $f \frac{\partial}{\partial f}$ as well as the translational symmetry $\frac{\partial}{\partial \xi}$. Consequently, we repeat the procedure leading to (16) by choosing $f$ to be invariant under a combination of both infinitesimal generators

$$\tilde{X}^{(N)} = b_1 \frac{\partial}{\partial \xi} + b_2 f \frac{\partial}{\partial f},$$

with the complex prefactors $b_{1,2} \in \mathbb{C}$. Applying condition (8) for an invariant solution with respect to the general symmetry (19), we obtain ansatz

$$f^{(N)}(\xi, y) = g^{(N)}(y) e^{\frac{b_2}{b_1} \xi}.$$  \hspace{1cm} (20)
Note that here we have assumed that $b_1 \neq 0$. If $b_1 = 0$, then (8) leads to the trivial solution $\psi = 0$. Inserting this approach into (16) leads to

$$\psi^{(N)}(x, y, t) = g^{(N)}(y) \exp \left( \frac{b_2}{b_1} x + \frac{a_3 b_1 - a_1 b_2}{a_2 b_1} t \right).$$

Let us now substitute the coefficients and assume that the solution is bounded for $x \to \pm \infty$ (i.e. $\Re (b_2 / b_1) = 0$). This yields the classical normal mode approach

$$\psi^{(N)}(x, y, t) = g^{(N)}(y) e^{i\alpha (x - ct)},$$

with wavelength $\alpha = \Im (b_2 / b_1)$ and wave speed $c = a_1 / a_2 - (a_3 b_1) / (a_2 b_2)$. Insertion into (1) leads to the Orr-Sommerfeld equation

$$(U - c) \left( \frac{d^2}{dy^2} - \alpha^2 \right) g^{(N)} - U'' g^{(N)} = \nu \frac{i\alpha}{\alpha^2} \left( \frac{d^2}{dy^2} - \alpha^2 \right)^2 g^{(N)}.$$

We note that the analysis can be repeated analogously if $a_2 = 0$ and $a_1 \neq 0$. Summarizing this section, the Orr-Sommerfeld equation is derived through a successive symmetry reduction of the linearised Navier-Stokes equations each time using the full set of admitted symmetries for an arbitrary $U(y)$ in Eq. (11). This holds true for both the viscous and the inviscid case and is usually referred to as normal mode or modal approach. In this work, we will repeatedly apply the method of successive symmetry reductions in order to reproduce existing and find new approaches.

V. KELVIN MODES

In the following, we restrict ourselves to the analysis of a linear shear flow

$$U(y) = Ay.$$ (24)

Analogously to the derivation leading to the Orr-Sommerfeld equation, we search for a solution which is invariant with respect to the following combination of symmetries $X_1, X_3$ and $X_4$, excluding the time-translation symmetry by setting $a_2 = 0$ in (12):

$$X^{(K)} = a_1 \frac{\partial}{\partial x} + a_3 \psi \frac{\partial}{\partial \psi} + \left( At \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right).$$

In this case, the defining Eq. (8) for the invariant solution becomes

$$a_3 \psi - (a_1 + At) \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial y} = 0.$$
The respective invariant solution is given by
\[
\psi^{(K)}(x, y, t) = f^{(K)}(\zeta, t) e^{a_3 y}. \tag{27}
\]
with the new variable
\[
\zeta = x - y (At + a_1). \tag{28}
\]

\(f^{(K)}\) solves the equation obtained by inserting ansatz (27) into Eq. (1):
\[
\frac{\partial}{\partial t} \left( \left( a_3 - (At + a_1) \frac{\partial}{\partial \zeta} \right)^2 + \frac{\partial^2}{\partial \zeta^2} \right) f^{(K)} = \nu \left( \left( a_3 - (At + a_1) \frac{\partial}{\partial \zeta} \right)^2 + \frac{\partial^2}{\partial \zeta^2} \right)^2 f^{(K)}. \tag{29}
\]

Similar to the previous section, this equation admits the scaling symmetry \(f \frac{\partial}{\partial f}\) and the translational symmetry \(\frac{\partial}{\partial \zeta}\). Superposing the two infinitesimal generators
\[
\tilde{X}^{(K)} = b_1 \frac{\partial}{\partial \zeta} + b_2 f \frac{\partial}{\partial f}, \tag{30}
\]
with the complex prefactors \(b_{1,2} \in \mathbb{C}\) and applying condition (8) for an invariant solution with respect to the general symmetry (30), we obtain ansatz
\[
f^{(K)}(\zeta, t) = g^{(K)}(t) e^{b_2 \zeta}. \tag{31}
\]

Note that here we have assumed that \(b_1 \neq 0\). \((b_1 = 0\) only leads to the trivial solution \(\psi = 0\). Inserting (31) into (27) leads to
\[
\psi^{(N)}(x, y, t) = g^{(N)}(t) \exp \left( \frac{b_2}{b_1} (x - y (At + a_1)) + a_3 y \right). \tag{32}
\]

Let us now substitute the coefficients \(a_1, a_3, b_1, b_2\), assuming that that the solution is bounded for \(x \to \pm \infty\) (i.e. \(\Re(b_2/b_1) = 0\)) and for \(y \to \pm \infty\) (i.e. \(\Re(a_3) = 0\)). We also set \(a_1\) to zero, as \(a_1\) only leads to a time-translation of the solutions. This yields the Kelvin mode approach
\[
\psi^{(K)}(x, y, t) = g^{(K)}(t) e^{i \kappa_x x (x - Ayt) + i \kappa_y y}, \tag{33}
\]
with wavelength \(\kappa_x \in \mathbb{R}\) in the streamwise direction and a time-dependent wavelength in the cross-stream direction \(\kappa_y - \kappa_x At\), where \(\kappa_y \in \mathbb{R}\). Finally, inserting the Kelvin mode approach into the stream function form of the Navier-Stokes equation (11) gives the following ODE for \(g^{(K)}:\)
\[
-\frac{d}{dt} \left( (\kappa_x^2 + (A \kappa_x t - \kappa_y)^2) g^{(K)} \right) = \nu \left( \kappa_x^2 + (A \kappa_x t - \kappa_y)^2 \right)^2 g^{(K)}. \tag{34}
\]
The solution of this first order ODE yields for the stream function is

$$\psi^{(K)}(x, y, t) = \frac{\kappa_x^2 + \kappa_y^2}{\kappa_x^2 + (\kappa_x At - \kappa_y)^2} \exp(i\kappa_x(x - Ayt) + i\kappa_y y) \times$$

$$\times \exp \left(-\nu t \left( \frac{1}{3} \kappa_x^2 A^2 t^2 - \kappa_y \kappa_x At + \kappa_y^2 + \kappa_x^2 \right) \right), \quad (35)$$

which corresponds to the solution derived by Rosen, for two dimensional perturbations.

Here, we see that the linear shearing and with that the time-dependent wavelength in cross-stream direction in fact reflects the effect of the symmetry \(X_4\) of the base flow. However, we emphasise that in the derivation of the Kelvin modes, not the full set of symmetries was used. Instead, the time translation symmetry \(X_2\), i.e. \(\tilde{t} = t + t_0\) in Table I, was left out by setting the corresponding group parameter \(a_2\) artificially to zero.

VI. NEW INVARIANT MODES

The symmetry analysis leading to the Kelvin modes excluded the time translation symmetry \(X_2\), i.e. \(\tilde{t} = t + t_0\) (see also table I). We will now show how a new class of ansatz functions can be obtained by also including this symmetry. The general infinitesimal in this case is (12) with \(a_2\) and \(a_4\) nonzero. As described in section II, we can rescale the infinitesimal generator by an arbitrary constant. For means of simplicity, we rescale the infinitesimal generator such that \(a_2 = 1:\)

$$X^{(I)} = a_1 \frac{\partial}{\partial x} + \frac{\partial}{\partial \tilde{t}} + a_3 \psi \frac{\partial}{\partial \psi} + a_4 \left(At \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right). \quad (36)$$

Condition (8) then becomes

$$a_3 \psi - (a_1 + a_4 At) \frac{\partial \psi}{\partial x} - a_4 \frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial \tilde{t}} = 0, \quad (37)$$

which is solved by

$$\psi^{(I)}(x, y, t) = f^{(I)}(\bar{x}, \bar{y}) e^{a_3(t + \frac{a_1}{Aa_4})}, \quad (38)$$

with the new variables

$$\bar{x} = x - \frac{Aa_4}{2} \left(t + \frac{a_1}{a_4} \right)^2 \quad \text{and} \quad \bar{y} = y - a_4 \left(t + \frac{a_1}{Aa_4} \right). \quad (39)$$

Insertion into (1) yields

$$\left(A\bar{y} \frac{\partial}{\partial \bar{x}} - a_4 \frac{\partial}{\partial \bar{y}} + a_3 \right) \Delta f^{(I)} = \nu \Delta \Delta f^{(I)}. \quad (40)$$
Analogously to the derivation of the normal and the Kelvin modes, we apply a successive symmetry reduction. Equation (40) admits a scaling symmetry and a translational symmetry in $\bar{x}$, yielding the general infinitesimal generator
\[ \tilde{X}^{(i)} = b_1 \frac{\partial}{\partial \bar{x}} + b_2 f \frac{\partial}{\partial f}. \] (41)

The invariant solution to this symmetry is given by
\[ f^{(i)}(\bar{x}, \bar{y}) = g^{(i)}(\bar{y}) e^{b_2 \bar{x}}, \] (42)
which - after insertion into Eq. (38) - yields the new ansatz function
\[ \psi^{(i)}(x, y, t) = g^{(i)} \left( y - a_4 \left( t + \frac{a_1}{Aa_4} \right) \right) \times \exp \left( b_2 \left( x - \frac{Aa_4}{2} \left( t + \frac{a_1}{Aa_4} \right)^2 \right) + a_3 \left( t + \frac{a_1}{Aa_4} \right) \right). \] (43)

Let us now rename the coefficients, and assume that $\Re \left( \frac{b_2}{b_1} \right) = 0$ in order to assure that the solution remains bounded for $x \to \pm \infty$. We also set $a_1$ to zero, as this parameter only leads to a time offset. Furthermore, we scale the argument of $g^{(i)}$ with $\kappa$ and obtain
\[ \psi^{(i)}(x, y, t) = g^{(i)} \left( \kappa y - \frac{t}{T} \right) \exp \left( i\kappa \left( x - \frac{A t^2}{2 \kappa T} \right) + c \kappa t \right). \] (44)

Based on the coefficients $a_i$ and $b_i$, we have introduced the wavelength $\kappa = \Im \left( \frac{b_2}{b_1} \right)$, the time-scale $T$, such that $\kappa T = \frac{1}{a_4}$. We also introduce a parameter $c = a_3 \kappa^{-1}$. As explained above, $\kappa$ is real. In the next section, we will show that in the inviscid case, physical consistency requires $T$ to be real and $c$ to be imaginary.

The novel ansatz function (44) describes modes traveling at a constant speed $(\kappa T)^{-1}$ in the cross-stream direction. In the streamwise direction, the modes are periodic and are accelerated by the base flow, such that for an outer observer, the modes travel in parabola-shaped curves described by
\[ \left( \frac{x(t) - x_0}{y(t) - y_0} \right) = \frac{1}{\kappa T} \left( \frac{\frac{1}{2} A t^2}{t} \right). \] (45)

Here, $(x_0, y_0) \in \mathbb{R}^2$ defines the initial position of the mode (see Figure 2) and $c = 0$ for simplicity. The shape of the modes is defined by $g^I$, which has to satisfy the equation
\[ -\frac{d}{dy} + i S \bar{y} + \bar{c} \left( \frac{d^2}{dy^2} - 1 \right) g^I = \frac{1}{Re} \left( \frac{d^2}{dy^2} - 1 \right)^2 g^I, \] (46)
FIG. 2. Perturbation streamlines of the new inviscid invariant modes for a shear rate $A = 1/T$ at different points in time. Note that at every instant in time, the solution is periodic in $x$. In time, the streamlines are translated along the dashed lines, defined by Eq. (45).

with

$$\bar{y} = \kappa y - \frac{t}{T},$$

(47)

obtained by insertion of (44) into (1). Note that we have formulated the problem in dimensionless form, introducing the dimensionless shear rate $S$, the growth rate $\tilde{c}$ and a Reynolds-number:

$$S := AT, \quad \tilde{c} := \kappa Te \quad \text{and} \quad Re := \frac{1}{\nu \kappa^2 T}.$$  

(48)

In the sequel, we present an invariant solution of (46) in the inviscid case. We note that in the viscous case, it is not possible to obtain physically consistent invariant solutions (see Appendix [A] for details).

A. The Inviscid Case

In this section, we give a detailed analysis of the invariant modes (44) in the inviscid case, including a proof of their energy-conservation and a study of the vorticity- and the velocity field. We will also present a link to the Kelvin modes.
Integrating the first differential operator of Eq. (46) yields up to a constant pre factor

$$
\left(\frac{d^2}{dy^2} - 1\right) g^{(l)}_\infty = \exp\left(\frac{iS\tilde{y}^2}{2} + \tilde{c}\tilde{y}\right).
$$

(49)

Note that according to ansatz (44), up to a pre factor $\kappa^2$ the expression above corresponds with the negative vorticity for $x, t = 0$. We require that the initial vorticity remains finite for $y \to \pm\infty$, such that necessarily we have to require that $S$ is real (which implies that $T$ is real) and that $\Re(\tilde{c}) = 0$. The imaginary part of $\tilde{c}$ will only lead to a translation in the cross-stream direction, such that for simplicity we set $\tilde{c} = 0$. Requiring $\Re(\tilde{c}) = 0$ enforces zero exponential growth or decay of the amplitude of the modes over time. Solving Eq. (49) for $g^{(l)}_\infty$ then yields the closed-form analytical solution

$$
g^{(l)}_\infty(\tilde{y}) = e^{\tilde{y}}\text{erfc}\left(\frac{1 - i}{2\sqrt{S}}\left(\tilde{y} + \frac{i}{S}\right)\right) + e^{-\tilde{y}}\text{erfc}\left(\frac{-1 + i}{2\sqrt{S}}\left(\tilde{y} - \frac{i}{S}\right)\right).
$$

(50)

In the following, we will scrutinize this result and show some of its physical implications.

The peculiar form of the invariant solution can be obtained by a special combination of Kelvin modes (see Eq. (35)). This combination can be obtained by decomposing the initial condition into Fourier modes

$$
\psi^{(l)}(x, y, 0) = \iint W_T(\kappa_x, \kappa_y) e^{i\kappa_x x + i\kappa_y y} d\kappa_x d\kappa_y,
$$

weighted with

$$
W_T(\kappa_x, \kappa_y) = \frac{\exp\left(-\frac{i}{2AT} \frac{\kappa_x^2}{\kappa_x^2 + \kappa_y^2}\right)}{\kappa_x^2 + \kappa_y^2} \delta(\kappa_x - \kappa_x) \delta(\kappa_y - \kappa_y).
$$

(51)

For a comparison of the new invariant solution with the Kelvin mode solution in phase space, see Figure 4. A quick computation confirms that this is consistent with modes traveling in parabola-shaped curves:

$$
\psi^{(l)}(x, y, t) = \iint W_T(\kappa_x, \kappa_y) \frac{\kappa_x^2 + \kappa_y^2}{\kappa_x^2 + (\kappa_y - \kappa_x At)^2} e^{i\kappa_x(x - y At) + i\kappa_y y} d\kappa_x d\kappa_y
$$

(52)

$$
= \iint e^{-\frac{i}{2AT} \frac{(\kappa_y + \kappa_y At)^2}{\kappa_x^2 + \kappa_y^2}} e^{i\kappa_x x + i\kappa_y y} d\kappa_y
$$

(53)

$$
= \iint e^{-\frac{i}{2AT} \frac{\kappa_y^2}{\kappa_x^2 + \kappa_y^2}} e^{i\kappa(x - \frac{At^2}{2\kappa T} + i\kappa_y(y - \frac{t}{\kappa T}))} d\kappa_y
$$

(54)

$$
= \psi^{(l)}\left(x - \frac{At^2}{2\kappa T}, y - \frac{t}{\kappa T}, 0\right).
$$

(55)
FIG. 3. Time-Evolution of the stream function of a single Kelvin mode $\psi^{(35)}$ in phase space. The stream functions was normalized s.t. the maximal value corresponds to unity. The inset depicts the stream function in wave-space in a $\kappa_x, \kappa_y$ plane.

Let us now examine the vorticity of the solution. According to (44) and (49), we obtain

$$\omega(x, y, t) = -\Delta \psi = -\kappa^2 \exp \left( i\kappa \left( x - \frac{At^2}{2\kappa T} \right) + i \frac{AT}{2} \left( \kappa y - \frac{t}{T} \right)^2 \right)$$

$$= -\kappa^2 \exp \left( i\kappa (x - Ayt) + i \frac{AT}{2} \kappa^2 y^2 \right)$$

$$= \omega(x - Ayt, y, 0).$$

Interestingly, the parabola-shaped trajectory of the perturbations, given in Eq. (45), is consistent with a shearing of the vorticity over time. This is due to the parabola-shaped isolines of the vorticity (see Figure 5). In particular, the mapping

$$h(x, y) \rightarrow h(x - Ayt, y),$$

when applied on a simple parabola $h(x, y) = x + by^2$, yields

$$x + by^2 \rightarrow x - Ayt + by^2 = x - \left( \frac{(At)^2}{4b} \right) + b \left( y - \frac{At}{2b} \right)^2,$$

where in this case $b = AT\kappa/2$. Consequently, shearing the parabola is equivalent to translating it in a parabolic trajectory. This property was automatically made use of by the application of symmetry methods.
Another interesting property of the new invariant modes is that they are energy-conserving. This can be shown by integration in wave space,

$$\int \int |\hat{\psi}^{(I)}|^2 \, d\bar{\kappa}_y \, d\kappa_x = \int \int \frac{1}{(\kappa^2 + \bar{\kappa}_y^2)^2} \delta (\kappa_x - \kappa) \, d\bar{\kappa}_y \, d\kappa_x = \frac{\pi}{2\kappa^3},$$  \quad (59)$$

where $\hat{\psi}^{(I)}$ is the representation of $\psi^{(I)}$ in wave space, obtained from Eq. (53). Interestingly, the modes presented here are not the only energy-conserving solutions of (1). $W_T(\kappa_x, \kappa_y)$ can be manipulated by changing its phase in order to obtain further energy-conserving modes, e.g. by setting

$$\tilde{W}_T(\kappa_x, \kappa_y) = \exp \left( -\frac{i \kappa^4}{2\Delta T \kappa_x^2} \right) \delta (\kappa_x - \kappa).$$  \quad (60)$$

In Fig. 6, we compare the time evolution of streamlines of the invariant modes (51) and (60). We note that the modes described by (60) are not invariant, do not conserve their shape and hence are of a different nature than the modes obtained by symmetry analysis.
Following (56), the isolines at time \( t \) are described by
\[
(\kappa x - \frac{At^2}{T}) + \frac{AT}{2} \left( \kappa y - \frac{t}{T} \right)^2 = c
\]
for some \( c \in \mathbb{R} \). With time, shearing of the parabola-shaped isolines leads to a displacement following a parabola-shaped trajectory, described by \( \kappa x = \kappa x_0 + \frac{At^2}{T} \) and \( \kappa y = \kappa y_0 + \frac{t}{T} \) (dashed line).

Let us now study the implications for the far-field behavior of the velocity field of the perturbations. These can be written as
\[
\begin{align*}
   u &= \kappa \left( g^{(1)}_{\infty} \right)' \left( \kappa y - \frac{t}{T} \right) e^{i \left( \kappa x - \frac{At^2}{T} \right)} \quad (61) \\
   v &= -i \kappa g^{(1)}_{\infty} \left( \kappa y - \frac{t}{T} \right) e^{i \left( \kappa x - \frac{At^2}{T} \right)} \quad (62)
\end{align*}
\]

Hence, in a moving frame of reference defined by \( \tilde{y} = \kappa y - \frac{t}{T} \), the velocities decay algebraically as
\[
\begin{align*}
   |u| &\sim (AT)^{-1/2} \tilde{y}^{-1} + O \left( \tilde{y}^{-3} \right) \quad (63) \\
   |v| &\sim (AT)^{-3/2} \tilde{y}^{-2} + O \left( \tilde{y}^{-4} \right) \quad (64)
\end{align*}
\]
for \( \tilde{y} \to \infty \). In Figure 7, we compare the algebraic decay of the velocities in the cross-stream and the streamwise direction.

This leads to another major difference of the new invariant modes if compared to the Kelvin modes: Whereas the Kelvin modes represent a complete set of solutions to (1), the
spectrum $W_T(\kappa_x, \kappa_y)$ of the new invariant solutions is always even. A superposition of even functions will always be even, too. Hence, the set of solutions obtained by using the new invariant functions as basis functions is more restricted in that it only accepts solutions with an even spectrum in cross-stream direction. Following, we show how one can superimpose the new invariant solutions with a weight function $V_{\kappa_y}(T) = -\frac{\kappa_y}{2\pi AT - \kappa_x^2} e^{\frac{i \kappa_y T}{2\pi A T}}$ to obtain the subset of even functions spanned by the Kelvin mode solutions. In particular, we show how to obtain the sum of two Kelvin mode solutions with $\kappa_y$ of different sign:
FIG. 7. Plot of the absolute value of the velocity components $u$ and $v$ in the streamwise and the cross-section direction as functions of $\tilde{y} = \kappa y - \frac{At}{T}$ for shear rates $A = 1/T$. The dashed-dotted and dashed lines correspond with the asymptotic behaviour given in Eqs. (63) and (64). The insets a) and b) show isolines of the velocity components $u$ and $v$, respectively. Both are plotted in a moving frame of reference, given by $\tilde{x} = \kappa x - \frac{At}{2T}$ and $\tilde{y}$ as above.

\[
\int \psi^{(1)} (x, y, t; \kappa, \kappa_Y) V_{\kappa Y} (T) \, dT = \int \int \left( \int W_T (\kappa_x, \kappa_y) V_{\kappa Y} (T) \, dT \right) \times \\
\frac{\kappa^2 + \kappa_y^2}{\kappa_x^2 + (\kappa_y - \kappa x A T)^2} e^{i \kappa_x (x - x A T) + i \kappa_y y} d\kappa_x d\kappa_y
\]

(65)

\[
= e^{i \kappa (x - y A T)} \left( \frac{\kappa^2 + (\kappa_Y - \kappa A T)^2}{\kappa^2 + (\kappa_Y + \kappa A T)^2} \right),
\]

(66)

\[
= \psi^{(K)} (x, y, t; \kappa, \kappa_Y) + \psi^{(K)} (x, y, t; \kappa, -\kappa_Y)
\]

(67)

where we have used (52) and that

\[
\int W_T (\kappa_x, \kappa_y) V_{\kappa Y} (T) \, dT = \frac{\kappa_Y}{2 \pi A \kappa_x^2} \frac{\delta (\kappa_x - \kappa)}{\kappa_x^2 - \kappa_y^2} \int \exp \left( \frac{i}{2 A T} \frac{\kappa_Y^2 - \kappa_y^2}{\kappa_x^2} \right) \, d \left( \frac{1}{T} \right)
\]

(68)

\[
= 2 \frac{\kappa_Y}{\kappa_x^2 + \kappa_y^2} \delta (\kappa_x - \kappa) \delta (\kappa_y^2 - \kappa_Y^2)
\]

(69)

\[
= \frac{\delta (\kappa_x - \kappa)}{\kappa_x^2 + \kappa_y^2} \left( \delta (\kappa_y - \kappa_Y) + \delta (\kappa_y + \kappa_Y) \right).
\]

(70)
Contrary to the Kelvin modes, the new modes are also not orthogonal, which is verified in phase space by:

$$\int W_{T_1}(\kappa_y) W_{T_2}(\kappa_y) \left( \frac{\kappa_x^2 + \kappa_y^2}{\kappa_x^2 + (\kappa_y - \kappa_x At)^2} \right)^2 d\kappa_y = \int \frac{e^{-\frac{1}{2\nu}(\frac{1}{T_1} - \frac{1}{T_2})\kappa_y^2}}{(\kappa_x^2 + (\kappa_y - \kappa_x At)^2)^2} d\kappa_y. \quad (71)$$

We conclude that the new invariant modes form a non-orthogonal set of energy-conserving solutions which are even in cross-stream direction. Each invariant solution travels on a parabola-shaped curve with a constant velocity in cross-stream direction.

B. The Viscous Case

In the viscous case, it is not possible to obtain physically consistent invariant solutions (see Appendix A). Hence, we compute here the time-evolution of a configuration where the inviscid modes are imposed as initial conditions, but in a viscous setting. For simplicity, we employ a superposition of Kelvin modes, weighted by $W_T$, to obtain

$$\psi(V)(x, y, t) = \int \int W_T(\kappa_x, \kappa_y) \frac{\kappa_x^2 + \kappa_y^2}{\kappa_x^2 + (\kappa_y - \kappa_x At)^2} e^{ik_x(x-yAt)+ik_yy} \times$$

$$\times \exp \left( -\nu t \left( \frac{1}{3} \kappa_x^2 A^2 t^2 - \kappa_y \kappa_x At + \kappa_y^2 + \kappa_x^2 \right) \right) d\kappa_x d\kappa_y. \quad (72)$$

It can be readily seen that for finite times, contributions of high wave numbers $\kappa_y \to \infty$ will be quickly damped by viscous effects. As $W_T$ includes contributions from the complete range of wave numbers, we expect the shape and intensity of the modes to break down. In Figure 8, we contrast the longevity of the modes for a low-viscosity regime with the quite fast breakdown of the intensity and shape of the modes for higher viscosities.

VII. CONCLUSION

We have presented a symmetry classification of the stream function form of the linearized Navier-Stokes equation for two-dimensional perturbations. In particular, we have applied symmetry analysis to generate a general set of ansatz functions which goes beyond the approaches known up-to-date, namely the normal mode approach and the Kelvin mode approach. We found that for a general base flow, the equation allows for a time- and space translation symmetry together with a scaling symmetry. If the base flow is restricted to a linear shear flow, we obtain an additional symmetry revealing specific properties of the flow.
FIG. 8. The graphs depict the evolution of the kinetic energy of the viscous initial-value solution (72) at different Reynolds numbers and for shear rates $\lambda = 1/T$. The dashed-dotted and the solid lines depict the time-evolution of the energy for $Re = 10$ and $Re = 1000$, respectively. The insets show isolines of the real part of the stream function defined by Eq. (72) at different points in time in a moving frame of reference $(\tilde{x}, \tilde{y}) \in [-1, 3] \times [-4, 4]$ for $\tilde{x} = \kappa x - \frac{4t^2}{2T}$ and $\tilde{y} = \kappa y - \frac{t}{T}$.

We have shown that the normal mode approach as well as the Kelvin mode approach can be systematically derived using successive symmetry reductions. The classical normal mode approach leading to the Orr-Sommerfeld equation is based on the three symmetries of the equation for a general base flow. Meanwhile, the Kelvin mode approach is based on an additional symmetry obtained through the restriction of the base flow to a linear shear flow. We note that here, the complete set of symmetries is not used, rather the time translation symmetry is excluded.

Including all relevant symmetries of the system leads to a new invariant ansatz function exhibiting qualitatively different behaviour. Kinematically, the new approach describes modes traveling at a constant speed in the cross-stream direction and being accelerated in the streamwise direction by the base flow. In the inviscid case, we have presented an analytical closed-form solution of these modes. The modes are energy-conserving in time and are non-periodic/decay in the cross-stream direction. In the viscous case, the modes break down because of the contributions of high wave numbers $\kappa y \to \infty$, which are quickly
damped, in agreement with the expected behavior from the Kelvin modes.

We emphasise that the invariant approach presented in this work is restricted to two-dimensional perturbations. Due to the constant translation of the modes in the cross-stream direction, it is also only applicable for finite times until the perturbations reach a boundary of the system.

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Appendix A: Viscous invariant Ansatz

We solve the linearized Navier-Stokes equation for a perturbation of a linear shear flow in stream function formulation (see Eq. (1)) using the invariant ansatz function (44). This approach leads to Eq. (46). Two naive solutions of this equation are

\[ g_1^{(I)}(\tilde{y}) = e^{\tilde{y}} \quad \text{and} \quad g_2^{(I)}(\tilde{y}) = e^{-\tilde{y}}. \]  

(A1)

The other two independent solutions of Eq. (46) are obtained by substituting

\[ u(\tilde{y}) = e^{\frac{Re}{2} \tilde{y}} \left( \frac{d^2}{d\tilde{y}^2} - 1 \right) g(\tilde{y}). \]

(A2)

We then obtain the following second order ODE

\[ u \left( 1 + \tilde{c}Re + \left( \frac{Re}{2} \right)^2 + i\tilde{y}SRe \right) - u'' = 0, \]  

(A3)

which is solved by the Airy-Functions

\[ u_1(\tilde{y}) = Ai(d_1\tilde{y} + d_2) \]  

(A4)

and \[ u_2(\tilde{y}) = Bi(d_1\tilde{y} + d_2) \]  

(A5)
with the parameters
\[
d_1 := (i S \text{Re})^{1/3}
\]
and
\[
d_2 := (i S \text{Re})^{-2/3} \left( 1 + c \text{Re} + \left( \frac{\text{Re}}{2} \right)^2 \right).
\]

Inverting (A2) yields
\[
g(\tilde{y}) = \int_{0}^{\tilde{y}} \sinh (\tilde{y} - \tilde{y}') e^{-\frac{Rs}{2} \tilde{y}'} u(\tilde{y}') d\tilde{y}',
\]
such that we obtain the independent solutions
\[
g_3^{(I)}(\tilde{y}) = \int_{0}^{\tilde{y}} \sinh (\tilde{y} - \tilde{y}') e^{-\frac{Rs}{2} \tilde{y}'} \text{Ai}(d_1 \tilde{y}' + d_2) d\tilde{y}',
\]
\[
g_4^{(I)}(\tilde{y}) = \int_{0}^{\tilde{y}} \sinh (\tilde{y} - \tilde{y}') e^{-\frac{Rs}{2} \tilde{y}'} \text{Bi} (d_1 \tilde{y}' + d_2) d\tilde{y}'.
\]

We note that all independent solutions \(g_{1-4}\) diverge for \(\tilde{y} \to \pm\infty\), violating the consistency of the initial condition.

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