SOME REMARKS ON A (MORE GENERAL) JACOBIAN CONJECTURE FOR DRUZKOWSKI MAPS OF DEGREE 3

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Abstract. For vectors $u, v \in \mathbb{C}^n$, we define $u \ast v := (u_1v_1, \ldots, u_nv_n)$ (coordinate-wise multiplication) and $u^k = u \ast u \ast \ldots \ast u$, and we define by $\Delta[u]$ the diagonal $n \times n$ matrix whose $(i, i)$-th entry is $u_i$. In this paper, we study the following (to be shown to be) more general version of the Jacobian conjecture for Druzkowski maps of degree 3:

Main Conjecture. Let $A$ be an $n \times n$ matrix with coefficients in $\mathbb{C}$ such that $\det(A) = 0$. Let $y, z \in \mathbb{C}^n$ be such that

$$\det(I + \Delta[(tz + sy)^2].A) = 1$$

for all $t, s \in \mathbb{C}$. If moreover,

$$z + A.(z^3 + z \ast (y^3)) = 0,$$

then $z = 0$.

The advantage of this Conjecture, versus the original Jacobian conjecture applied to the Druzkowski maps of degree 3, is that its statement is a lot simpler and it is more amenable to using computer programs to test. We are able to check the validity of this conjecture when either i) $n \leq 3$ or ii) $A$ is of rank 1 and $n \leq 20$, by using a Mathematica program run on a personal computer. For $n = 4$ and $A$ has rank 2, we have checked the Conjecture with many randomly generated matrices. We note that for the matrices $A$ in the Conjecture, the map $F(x) = x + (Ax)^3$ may not be injective. A slightly weaker conjecture is also discussed.

1. Theoretical results

The famous Jacobian conjecture is the following statement:

Jacobian conjecture. Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map such that $JF$ is invertible. Then $F$ has a polynomial inverse.

This conjecture has attracted a lot of works, and many partial results were found. For example, Magnus - Applegate -Onishi -Nagata proved the conjecture for $n = 2$ and $F = (P, Q)$ where the GCD of the degrees of $P, Q$ is either a prime number or $\leq 8$; Moh proved that the conjecture is true for $n = 2$ and $\deg(F) \leq 100$; and Wang proved that the conjecture is true for arbitrary $n$ if $\deg(F) = 2$ (see [4]). For more details the readers can consult the reference list and the references therein. An excellent survey is the book [4]. We note that the $\mathbb{R}$-analog of the Jacobian conjecture is false, by the work of Pinchuk (see Section 10 in [4]).

There have been many reductions of the Jacobian conjecture. One of these reductions is due to Bass, Connell, Wright and Yagzhev, who showed that to prove the Jacobian conjecture for all $n$, it is enough to prove for all $F(x) = x + H(x)$ and all $n$, where $H(x)$ is a homogeneous polynomial of degree 3 (see Section 6.3 in [4]). Druzkowski made a further simplification (see Section 6.3 in [4]).

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Theorem 1.1. The Jacobian conjecture is true for all \( n \) if it is true for all the maps \( F \) of the form \( F(x) = (x_1 + l_1(x)^3, \ldots, x_n + l_n(x)^3) \) and all \( n \) with invertible Jacobian \( JF \), here \( l_1, \ldots, l_n \) are linear forms.

Later (see [3]), Druzkowski simplified even further showing that it is enough to show for the above maps with the additional condition that \( A^2 = 0 \), where \( A \) is the \( n \times n \) matrix whose \( i \)-th row is \( l_i \). For the Druzkowski maps, methods of Linear Algebra may be made use of. The current paper may be seen as one application of this statement.

Notations. We will use the following notations. For vectors \( u, v \in \mathbb{C}^n \), we define \( u * v := (u_1 v_1, \ldots, u_n v_n) \) and \( u^k = u * u * \ldots * u \) (\( k \)-th self-multiplication of \( u \)), and we define by \( \Delta[u] \) the diagonal \( n \times n \) matrix whose \((i, i)\)-th entry is \( u_i \). Thus the Druzkowski maps and their Jacobians can be written as

\[
F(x) = x + (A.x)^3, \\
JH(x) = Id + 3\Delta[(A.x)^2].A.
\]

There is a well-known result that a polynomial self-map of \( \mathbb{C}^n \) is an automorphism if it is injective (see Chapter 3 in [1]). A lot of efforts have been devoted to showing that the Druzkowski maps are injective (see Section 7.1 in [4] and for recent developments see [6], and for a comprehensive reference on this topic see [2]). There are many partial results proved for this class of maps. However, as far as we know the Jacobian conjecture was completely checked only for Druzkowski maps in dimensions \( \leq 8 \) (see [2]).

In theory, we can check, for each given dimension \( n \), whether all Druzkowski maps are injective by using a computer program (for example Mathematica) to find the Groebner basis for the ideal \( \mathcal{I} \) defined by the equations \( x + (A.x)^3 = y + (A.x)^3 \) and

\[
(1.1) \quad \det(Id + \Delta[(A.z)^2].A) = 1
\]

for all \( z \in \mathbb{C}^n \), to see that \( x - y \) belongs to this Groebner basis. However, in practice one faces the difficulty that the number of the polynomials in the ideal defined by the system \( \det(Id + \Delta[(A.z)^2].A) = 1 \) for all \( z \in \mathbb{C}^n \) grows very fast with respect to the dimension \( n \): it is roughly the same as the number of monomials of degrees at most \( n \) in \( n \) variables.

This paper grew out of the author’s curiosity to see whether we can reduce the number of equations defining the ideal \( \mathcal{I} \). It also originates from our trying to ponder on the following question:

Question. If a formal proof of the Jacobian conjecture is to be found for all Druzkowski maps of degree 3 in all dimensions \( n \), how can we make use of the assumption that \( JF \) is invertible?

To be more explicit about this Question, let us first make some simple algebraic reductions. Let \( u, v \in \mathbb{C}^n \) be such that \( F(u) = F(v) \), that is \( u + (A.u)^3 = v + (A.v)^3 \). Then by subtracting and using that \( A \) is a linear map, we find that

\[
(u - v) + (A.u - A.v) * ((A.u)^2 + (A.u) * (A.v) + (A.v)^2) = 0.
\]

If we define \( x = u - v \) then we can write the above equation as

\[
x + (A.x) * ((A.x)^2 + 3(A.x) * (A.v) + 3(A.v)^2) = 0.
\]
Now, by dividing the equation by 2 and substituting
\[ y = \sqrt{3}v + \frac{\sqrt{3}}{2}x, \]
we see that the above equation is reduced to
\[ (1.2) \quad x + (A.x) \ast ((A.x)^2 + (A.y)^2) = 0. \]
Then, the fact that \( F \) is injective is the same as the equation \( x + (A.x) \ast ((A.x)^2 + (A.y)^2) = 0 \) has only the solution \( x = 0 \).

Now, we see that \( y \) appears in the ideal \( I \) only through \( A.y \). Hence it is natural to ask whether the Jacobian conjecture is in fact stronger, that is in Equation (1.2), we can replace \( Ay \) by \( y \) (which of course must satisfy a condition compatible with Equation (1.1)) and still obtain the same conclusion? Hence, we state a weaker version of our main conjecture:

**Conjecture (Weaker version).** Let \( A \) be an \( n \times n \) matrix with constant coefficients in \( \mathbb{C} \). Assume that \( y \in \mathbb{C}^n \) satisfy
\[ \det(\text{Id} + \Delta[(A.z + ty)^2].A) = 1 \]
for all \( t \in \mathbb{C} \) and \( z \in \mathbb{C}^n \). Then, if \( x + (Ax) \ast ((Ax)^2 + y^2) = 0 \), we must have \( x = 0 \).

We may push this investigation further, by asking that in showing that \( x + (A.x) \ast ((A.x)^2 + (A.y)^2) = 0 \) has only the solution \( x = 0 \), do we need the assumption (1.1) somehow on the plane generated by \( A.x \) and \( y \) only? This leads us to state the first formulation of our main conjecture.

**Main Conjecture (First formulation).** Let \( A \) be an \( n \times n \) matrix with constant coefficients in \( \mathbb{C} \) such that \( \det(A) = 0 \). Assume that \( x, y \in \mathbb{C}^n \) satisfy
\[ \det(\text{Id} + \Delta[(sA.x + ty)^2].A) = 1 \]
for all \( s, t \in \mathbb{C} \). Then, if \( x + (Ax) \ast ((Ax)^2 + y^2) = 0 \), we must have \( x = 0 \).

The condition \( \det(A) = 0 \) in the second conjecture is quite natural. It is easy to see that if \( A \) satisfies Equation (1.1), then \( \det(A) = 0 \).

These two conjectures can be seen to be more general than the original Jacobian conjecture. While in the first conjecture, the number of equations in the ideal \( I \) is even bigger that the corresponding one in the case of the original Jacobian, in the second one the number decreases significantly: here we only need \( n^2 + n \) equations.

The appearance of the Main Conjecture can be made simpler. We note that there is an asymmetricity between \( x \) and \( y \) in Equation (1.2): \( y \) appears only through \( A.y \), while both \( x \) and \( A.x \) appear. We can make this asymmetricity become implicit by using the following simple result.

**Lemma 1.2.** The following two statements are equivalent:

1) There is a non-zero solution \( x \) to \( x + (A.x)^3 + (A.x) \ast y^2 = 0 \),

and

2) There is a non-zero solution \( z \) to \( z + A.(z^3 + z \ast y^2) = 0 \).
Proof. \((\Rightarrow)\) If \(x\) is a non-zero solution to \(x + (A.x)^3 + (A.x) * y^2 = 0\) then \(z = A.x\) is non-zero. Moreover, we have
\[
0 = A(x + (A.x)^3 + (A.x) * y^2) = A(x + z^3 + z * y^2) = A(x) + A(z^3 + z * y^2) = z + A(z^3 + z * y^2).
\]

\((\Leftarrow)\) If \(z\) is a non-zero solution to \(z + A(z^3 + z * y^2) = 0\), by defining \(x = -(z^3 + z * y^2)\) we see that \(Ax = z\). In particular, \(x\) is also non-zero. Moreover,
\[
0 = x + z^3 + z * y^2 = x + (A.x)^3 + (A.x) * y^2.
\]

We are now ready to state our main conjecture.

**Main Conjecture (Second formulation).** Let \(A\) be an \(n \times n\) matrix with coefficients in \(\mathbb{C}\) such that \(\det(A) = 0\). Let \(y, z \in \mathbb{C}^n\) be such that
\[
(1.3) \quad \det(Id + \Delta[(sz + ty)^2].A) = 1
\]
for all \(s, t \in \mathbb{C}\). If moreover,
\[
(1.4) \quad z + A.(z^3 + z * y^2) = 0,
\]
then \(z\) must be 0.

If in Equations \((1.3)\) and \((1.4)\) we decrease the exponents by 1, that is to require instead
\[
(1.3) \quad \det(Id + \Delta[(sz + ty)].A) = 1,
\]
\[
(1.4) \quad z + A.(z^2 + z * y) = 0,
\]
for all \(s, t \in \mathbb{C}\), then it is easy to see that \(z = 0\). This gives some evidence to that the Main Conjecture may be true.

We note that Equations \((1.3)\) and \((1.4)\) of the Main Conjecture are much simpler than the corresponding ones \((1.1)\) and \((1.2)\) of the original Jacobian conjecture. Moreover, the ideal defining the Equations \((1.3)\) and \((1.4)\) consists of only \(n^2 + n\) polynomials, a lot smaller than the corresponding one for the original equations \((1.1)\) and \((1.2)\). Hence, from a computational aspect, if the Main Conjecture is true, then it probably needs less effort to be proven than the original statement. Also, the simpler statement of the Main Conjecture may suggest how to prove formally the Jacobian conjecture.

In the next section we will present computational results concerning the Main Conjecture.

2. Computational results and final remarks

We have used a Mathematica program, run on a personal computer, to check the validity of the Main Conjecture in two cases: i) \(n \leq 3\) and ii) \(A\) has rank 1 and \(n \leq 20\). For \(n = 4\) and larger, the computer is not strong enough to finish the task. For \(n = 4\) and \(A\) has rank 2, we have checked the Main Conjecture with many randomly generated imatrices. A Mathematica source file is available upon request.

In the first subsections below we will present the computational results. In the final subsection, we state some final remarks.
2.1. The case $n = 2$. The system (1.3) consists of 4 polynomials of total degree 4 in $z$, $y$ and the entries of the matrix $A$. The Groebner basis for the union of the two systems (1.3) and (1.4) has 4 elements, including $z$, $\det(A)$, and the other one is the polynomial defined by the system $\det(Id + t\Delta[y^2]A) = 1$ for all $t \in \mathbb{C}$.

The system (1.1) has 4 polynomials in the entries of $A$. The Groebner basis for the union of the two systems (1.1) and (1.2) has 7 elements.

2.2. The case $n = 3$. The system (1.3) consists of 9 polynomials in $x$, $y$ and the entries of the matrix $A$. The Groebner basis for the union of the two systems (1.3) and (1.4) has 13 elements, including $z$, $\det(A)$, and the other ones are defined by the system $\det(Id + t\Delta[y^2]A) = 1$ for all $t \in \mathbb{C}$.

In contrast, the system (1.1) has 22 polynomials in the entries of $A$. It takes longer to compute the Groebner basis. The Groebner basis for the union of the two systems (1.1) and (1.2) grows significantly to 261 elements.

2.3. The case $A$ has rank 1. We parametrize the matrix $A$ by two vectors $u$ and $v$, with the first entry of $u$ is normalized to be 1. Hence, in this case we consider $4n - 1$ variables.

We have computed up to $n = 20$. The system (1.3) always consists of 3 homogeneous polynomials of total degree 2 in $z$ and $y$. The Groebner basis of the system consists of the two systems (1.3) and (1.4) always consists of $n + 1$ elements, including $z$, and the other one is the polynomial of degree 2 in $y$ given by the system $\det(Id + t\Delta[y^2]A) = 1$ for all $t \in \mathbb{C}$. This is also a polynomial of total degree 2 in the entries of the matrix $A$.

In contrast, for $n = 20$, the system (1.1) consists of 211 equations, and it takes longer to compute the Groebner basis. The Groebner basis for the union of the two systems (1.1) and (1.2) in this case also has 21 elements, however these there is a complicated polynomial of degree 12 in the entries of the matrix $A$.

2.4. $n = 4$ and $A$ is a random matrix of rank 2. We parametrize a $4 \times 4$ matrix $A$ by a $2 \times 4$ matrix denoted by FirstRows (representing the first two rows of $A$, which are assumed to be linearly independent) and a $2 \times 2$ matrix Coefficients (representing the coefficients of the last two rows with respect to the first two rows). We use the RandomInteger command of Mathematica to generate randomly the two matrices FirstRows and Coefficients. Here we restrict the range of the random numbers to $[-5, 5]$. We then use the Rank command to detect that the matrix $A$ constructed from these two random matrices is of rank 2. Then we also use the GroebnerBasis command to check that the system (1.3) has non-trivial solutions $z, y$. Finally, we use the GroebnerBasis command to check that the union of the two systems (1.3) and (1.4) has only the solution $z = 0$ as wanted.

For example, one such randomly generated matrix $A$ is the following

$$A = [[-3, -3, 0, 1], [2, -2, 1, 1], [-2, -10, 2, 4], [-5, -13, 2, 5]]$$

For this example, the Groebner basis of the system (1.3) has 90 polynomials, while the Groebner basis for the union of the two systems (1.3) and (1.4) has only 6 polynomials, including $z$. 
Another example of randomly generated $4 \times 4$ matrices of rank 2 is the following

$$A = \begin{bmatrix}
4 & 3 & 1 & 5 \\
-1 & 2 & 4 & -5 \\
4 & -8 & -16 & 20 \\
-5 & -1 & 3 & -10
\end{bmatrix}$$

For this example, the Groebner basis of the system (1.3) has 207 polynomials, while the Groebner basis for the union of the two systems (1.3) and (1.4) has only 6 polynomials, including $z$.

It seems from our experiment that for $4 \times 4$ random matrices of rank 2, the computer works quite fast and always give that the Groebner basis for the union of the two systems (1.3) and (1.4) has only 6 polynomials, including $z$. However, when we tried to increase either $r$ to 3 or $n$ to 5, the computer could not be able to finish the task.

Remarks.

1) We do not know how to employ a random check for the union of the two systems (1.1) and (1.2).

2) It is natural to wonder since we have generated randomly the matrices $A$, whether the system (1.4) trivially follow from the system (1.3), in the sense that the system (1.3) alone has no nonzero solution $z$ such that $z = A.x$ for some nonzero $x$. We have checked for each of the matrix $A$ we experimented, and to our surprise this is not the case (by checking that there is no polynomial of degree 1 appearing in the Groebner basis for the system $\det(Id + \Delta[(tA.x + sy)^2].A) = 1$ for all $t, s \in \mathbb{C}$). We do not know whether this is just a coincidence or whether the following should be true.

Conjecture. Let $A$ be an $n \times n$ matrix with coefficients in $\mathbb{C}$ such that $\det(A) = 0$. Then there are $x, y \in \mathbb{C}^n$ such that $A.x \neq 0$ and $\det(Id + \Delta[(sA.x + ty)^2].A) = 1$ for all $s, t \in \mathbb{C}$.

2.5. Final remarks. The previous computational results give support to that the Main Conjecture may be true. These experiments also show that the union of the two systems (1.3) and (1.4) is more amenable to be checked by computer programs than the union of the two systems (1.1) and (1.2). We have used a very primitive Mathematica program, and it may be improved further, for example by taking into accounts the symmetric structures of the systems we are considering (invariant under the permutations of the set $\{1, \ldots, n\}$), or by using a more suitable method rather than just the GroebnerBasis command. Also, we used only a personal computer, and stronger computers may help to check the conjecture in higher dimensions.

Finally, from the computational experiments, we feel that the following is true, and may be helpful in finding a formal proof for the Main Conjecture:

Conjecture. The Groebner basis for the union of the two systems (1.3) and (1.4), with variables the entries of $z$, $y$ and $A$, where $A$ is constrained to have $\det(A) = 0$, consists exactly of the following: $\det(A)$; $z$; and the coefficients in the polynomial $\det(Id + t\Delta[y^2].A) - 1$ (viewed as a polynomial of $t$).

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