Tidal Deformability Doppelgängers:
I. Differentiability of gravitational waveforms for neutron stars with a low-density phase transition

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Gravitational wave detections of binary neutron star inspirals will be crucial for constraining the dense matter equation of state (EoS). Such constraints rely on the robust association of the inferred tidal deformability to the underlying EoS. Here, we demonstrate the existence of a new family of EoSs that differ by \( \lesssim 30 \) in their tidal deformability curves across the entire neutron star mass range, despite differences of up to \( \sim 0.45 \) km in radius. These “tidal deformability doppelgängers” are indiscernible with current gravitational wave detectors; however, next generation detectors and advances in nuclear theory may be able to constrain or rule out this family of EoSs.

I. INTRODUCTION

Gravitational wave (GW) events offer exciting prospects to constrain the properties of dense matter e.g., [1–4]. In particular, the GW signal emitted during the final orbits of two colliding neutron stars contains imprints of the tidal deformability parameter, \( \Lambda \), that can be related to the properties of dense matter in terms of the equation of state (EoS) e.g., [5–13]. In practice, this inference is limited by the sensitivity to which the tidal deformability can be constrained. For example, for the GW170817 event, \( \Lambda \) was constrained to 300\( ^{+420}_{-230} \) at 90% confidence [13]. Through either full Bayesian inferences or quasi-universal relations that map \( \Lambda \) to the neutron star radius, \( R \), this measurement has been translated to constraints on the radius of \( 10 \lesssim R \lesssim 13 \) km e.g., [5–7, 11].

When advanced LIGO reaches its fifth observing campaign, it is expected that the tidal deformability will be able to be constrained to uncertainties of \( \sigma_\Lambda \approx 46 \) at 68% confidence, for a GW170817-like event. With next-generation (XG) GW detectors, these bounds on \( \Lambda \) will be further improved, leading to anticipated constraints of \( \sigma_\Lambda < 8 \) from the inspiral GW signal for a similar event, and \( \sigma_\Lambda \approx 1 − 4 \) for a population of mergers observed with XG detectors, depending on the merger rate [15]. From the usual quasi-universal relations that map tidal deformabilities to the neutron star radius e.g., [16–18], one would typically assume that small uncertainties in \( \sigma_\Lambda \) directly translate to tight constraints on \( R \), potentially to 50-200 m accuracy [3], assuming that dynamical tides are correctly accounted for in the extraction of \( \Lambda \) [19, 20]. However, as we will show, this tight mapping to the neutron star radius may not be possible in all cases.

In this work, we introduce a new class of EoS models that differ significantly in radii (by up to \( \sim 0.45 \) km in the radius \( R_{1.4} \) of a 1.4 \( M_\odot \) neutron star) and in the pressure at nuclear densities (by a factor of 2.5), but that are extremely similar in tidal deformabilities across the entire range of astrophysically-observed neutron star masses. These models vary by \( \Delta \Lambda \lesssim 30 \) across the full mass range, and by \( \lesssim 10 \) for neutron star masses above 1.25 \( M_\odot \), making them indistinguishable to current gravitational wave detectors, despite their large differences in radius.

In this work, we introduce the key features of these “tidal deformability doppelgängers”, and discuss the prospects for resolving this observational degeneracy in tidal deformability curves with XG detectors, from both inspiral and post-merger GWs. We demonstrate that with additional input from nuclear theory at densities near the nuclear saturation density e.g., [21–24] and radius constraints from X-ray observations [25–34], it may be possible to constrain or even rule out this family of EoS models. At present, however, this near-degeneracy in EoS models might need to be taken into account when performing EoS inferences with current and upcoming GW analyses.

II. TIDAL DEFORMABILITY DOPPELGÄNGERS

We introduce the basic phenomenology of the doppelgänger EoS construction with two examples in Fig. 1. The softer pair of doppelgängers (shown in blue) differ in the radius of a 1.4 \( M_\odot \) neutron star, \( R_{1.4} \), by 0.48 km, while the stiffer pair of doppelgängers (shown in green) differ in \( R_{1.4} \) by 0.40 km. We note that the softer (blue) pair of doppelgängers was previously presented in a different context, as part of a large EoS

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deformability, \( \Lambda_{1.4} \), of a 1.4\( M_\odot \) star agrees to within \( \Delta \Lambda_{1.4} < 1 \). This small difference in \( \Lambda_{1.4} \) is already surprising, if we consider the quasi-universal relation between \( \Lambda \) and the stellar compactness, \( C \), of [17], which predicts that for a pair of neutron stars that differ in \( R_{1.4} \) from 10.8 to 11.2 km (as in Fig. 1), \( \Lambda_{1.4} \) should vary by 56, while for the 12.8 and 13.2 km stars, the same relation predicts \( \Lambda_{1.4} \) would vary by 115. More generally, it has been shown that \( \Lambda \propto R^\alpha \) with \( \alpha \approx 6 \) [e.g., 10, 11, 18], thus one would traditionally expect that a 0.4 km difference in radius should propagate to a much larger difference in \( \Lambda \). For the doppelgängers, we instead find that \( \alpha \sim 1 \) over a narrow range of radii, and that the differences in \( \Lambda_{1.4} \) for these models are thus much smaller than expected. We note, however, that these differences still fall within the broad error band of the standard Love-C relation [17], and also within the errors of the quasi-universal relations with \( \Lambda \), which are commonly used in EoS inferences from GW data [10,12,16,36] (see Appendix C).

Even more surprising than this small difference in \( \Lambda_{1.4} \), is the fact that the difference in the tidal deformability for a given pair of doppelgängers remains very small \(( \lesssim 30 \) at all astrophysically-observed masses, which we consider to span from the lightest observed pulsar at 1.17\( M_\odot \) [37] to 2.01\( M_\odot \) [38, 39]. In other words, the entire mass-tidal deformability curve is nearly identical between the two EoSs, despite their large differences in radii.

It has previously been shown that changing the crust EoS (at densities below \( 10^{14} \) g/cm\(^3 \)) can change the radius without significantly affecting the tidal deformabilities [10], here, however, we find that large differences in the EoS at supranuclear densities can potentially leave no discernible imprint on the tidal deformability curve. This near-degeneracy in tidal deformability curves may pose a challenge for gravitational wave inferences of the EoS. For example, in Fig. 1 we include mock measurements of the tidal deformability from a series of GW170817-strength events observed with the proposed XG detector Cosmic Explorer (CE) [41], for which it is estimated that the 68% measurement uncertainty in \( \Lambda \) will be 8 [15]. Even with such a series of individually-precise measurements across the entire range of masses, a given doppelganger cannot be differentiated from its companion. This is in spite of \( \sim 0.5 \) km differences in radii and factors of \( \sim 2.5 \times \) difference in the pressures at the nuclear saturation density. For such doppelgängers, it may be necessary to perform a population-level analysis of \( \Lambda \) with CE-level sensitivity (in which case \( \sigma_\Lambda = 1 - 4 \), depending on the merger rate [15]), or to adopt more informative nuclear priors, in order to tell these models apart.

These EoSs were constructed using a piecewise polytropic framework [12] starting from \( 0.5\rho_{\text{sat}} \), where \( \rho_{\text{sat}} = 2.7 \times 10^{14} \) g/cm\(^3 \) is the nuclear saturation density, and using the cold, \( \beta \)-equilibrium slice of the SFHo EoS table [43] to describe the crust EoS at lower densities \(( \rho < 0.5\rho_{\text{sat}} )\). The parameters of the high-density EoS were specifically chosen to maximize differences in radius, while minimizing differences in the tidal deformability. We provide more details and the parameters of these four representative EoSs in Appendix A.
and tidal deformability are averaged over the astrophysically-observed mass range of $1.17 - 2.01 M_\odot$. These doppelgängers have been selected to maximize $\Delta R$ and minimize $\Delta \Lambda$. As more restrictions are added – e.g., by trusting the crust EoS to nuclear densities – the allowed doppelgängers become less extreme.

In order to explore the ubiquity of doppelgänger models more generally, we perform a large-scale parameter survey of EoSs in a companion paper. In particular, we generate four samples of $\mathcal{O}(10^5)$ piecewise polytropic EoSs that are constructed to obey minimal physical constraints (namely, obeying causality and hydrostatic stability, and being able to support massive neutron stars). Each sample is subject to a different set of nuclear input, in terms of the density to which the crust EoS is utilized (either $0.5 \rho_{\text{sat}}$ or $\rho_{\text{sat}}$; above which we adopt a polytropic representation of the EoS) and in terms of a prior on the second density-derivative of the pressure. We search within these EoS samples for pairs of EoSs that have a small maximum difference in their tidal deformabilities, but a large minimum difference in their mass-radius curves, across a range of masses from 1.17-2.01 $M_\odot$. We define a “doppelgänger score” based off these criteria, according to

$$D = \exp \left\{ -\frac{(\Delta \Lambda_{\text{max}})^2}{2\sigma_\Lambda^2} \right\} \left\{ 1 - \exp \left[ -\frac{(\Delta R_{\text{min}})^2}{2\sigma_R^2} \right] \right\} \quad (1)$$

where $\Delta \Lambda_{\text{max}}$ is defined as the maximum difference in $\Lambda$ between the two EoSs at any mass across the range 1.17-2.01 $M_\odot$, $\Delta R_{\text{min}}$ is the minimum difference in radius across the same mass range, and we set the scale to $\sigma_\Lambda = 10$ and $\sigma_R = 0.3$ km based off of the characteristic differences found in Fig. 1. We use this scoring criteria to select the most extreme examples of doppelgängers, and we find that doppelgänger EoSs occur naturally and generically within these samples of randomly-generated EoSs. For each sample, we find $\mathcal{O}(10^5)$ pairs of doppelgänger EoSs that have significant differences in radii and only very small differences in $\Lambda$ across the entire mass range considered.

Figure 2 shows contours of the mass-averaged difference in radius and tidal deformability, calculated for the highest-scoring pairs of doppelgängers identified with eq. (1). These results are for the case of a weak (less restrictive) prior on the second derivative of the pressure; we will explore the impact of adopting a stronger prior in [49]. The green contours show the doppelgängers from the EoS sample for which the crust EoS is adopted only up to $0.5 \rho_{\text{sat}}$, while the orange contours show the doppelgängers from the sample in which the crust EoS is used to $\rho_{\text{sat}}$. In both EoS samples, we find that the mass-averaged difference in radius is at least 0.1 km, and can reach up to $\sim 0.5$ km in the most extreme cases, while the average difference in $\Lambda$ is $\lesssim 10$.

We note that the values for $\Delta R$ and $\Delta \Lambda$ in Fig. 2 are comparable to the examples constructed by hand in Fig. 1 but that these $\mathcal{O}(10^5)$ doppelgängers were found generically in samples of randomly-generated EoSs. This has important implications for EoS inferences with the current generation of tidal deformability measurements. Namely, in randomly-generated EoS samples, there will
FIG. 4: Pressure-density histograms for the set of doppelgängers identified from the randomly-generated sample of PWP EoSs, for which the parametrization starts at $0.5\rho_{\text{sat}}$. We classify each EoS in a given pair of doppelgängers as “stiff” or “soft” based on the pressure at the first fiducial density, and we plot the 2D histograms for each subclass in red and blue respectively. The general doppelgänger behavior is caused by allowing for a phase transition at densities near the nuclear saturation density $\rho_{\text{sat}}$. The onset of the phase transition can be pushed to higher densities by adopting more restrictive nuclear input [45].

likely exists pairs of EoSs that differ substantially in radii (and, accordingly, in the pressure near nuclear densities), but that are not distinguishable by the current level of GW detector sensitivity. At present, these models will be differentiated purely by the choice of priors that are adopted in the EoS inference.

For example, Fig. 2 already demonstrates that by adopting a crust EoS to higher densities, the extremity of doppelgänger models can be reduced. That is, there still exist models with comparable tidal deformability curves, but they differ by a smaller degree in radius (and, accordingly, the pressure). We will explore in a companion paper [45] the impact of adopting stronger priors on the second derivative of the pressure.

Already, the doppelgänger models exist in a particular region of EoS parameter space, corresponding to relatively compact stars. We demonstrate this in Fig. 3, which shows the mass-radius parameter space spanned by the doppelgänger models from Fig. 2. For reference, Fig. 3 also includes the bounds spanned by the full, randomly-generated EoS sample for each case (in the lighter shaded regions). Even though the full sample of EoSs extends to very large radii, the models that we identify as doppelgängers are found in a narrow and compact region of the parameter space.

In Fig. 4, we show histograms of the EoSs for the population of doppelgängers in Fig. 2 that start their PWP parametrization at $0.5\rho_{\text{sat}}$. For a given pair of doppelgängers, we identify one EoS as “softer” and one as “stiffer”, based on which has the larger pressure at the first fiducial density in our parametrization, and show the corresponding histograms in blue and red, respectively. We find that the doppelgänger EoSs are generically characterized by a phase transition at relatively low densities near $\rho_{\text{sat}}$, where the exact transition density can be pushed to higher densities depending on the choice of parametrization and crust EoS [45]. As a result of this phase transition, these models tend to predict more compact (i.e., small radius) stars.

Thus, by combining future neutron star radius constraints together with improvements from nuclear theory in the EoS near nuclear densities, it may be possible to constrain or even rule out some classes of doppelgänger models. For example, if one trusts the specific crust EoS used in Fig. 3 to $\rho_{\text{sat}}$ and the neutron star radius were robustly measured to be $\gtrsim 12$ km, then the most extreme doppelgängers would be ruled out. However, current NICER constraints (shown in gray dashed lines in Fig. 3) are not yet able to exclude these models [28, 29, 32, 33].

IV. PROSPECTS FOR NEXT-GENERATION DETECTORS

From the gravitational wave signal alone, the small differences in $\Lambda$ shown in Figs. 1 and 2 will be challenging to discern with the current generation of GW detectors. This is true not just for the individual tidal deformabilities ($\Lambda_1$ and $\Lambda_2$), but also for the binary tidal deformability, $\tilde{\Lambda}$, which is a mass-weighted average of the two.

As expected, given the similarity between the individual tidal deformabilities for a pair of doppelgängers, we find that the differences in their binary tidal deformabilities are also very small, for a wide range of binary masses ($\Delta \tilde{\Lambda} \sim 10$; see Appendix B).

For comparison, the measurement uncertainty for the flagship event GW170817 was $\sigma_{\tilde{\Lambda}} \approx 198$ (at the 1$\sigma$-level) [14]. It has been estimated that a comparable event measured with aLIGO would have $\sigma_{\tilde{\Lambda}} \approx 46$. With a population of 79 mergers (corresponding to an intermediate estimate of the astrophysical merger rate), $\sigma_{\tilde{\Lambda}}$ would improve to 25, which would still be insufficient to differentiate the models shown in Fig. 1 for all but an extremely low-mass system [15]. On the other hand, with proposed XG facilities such as Einstein Telescope (ET) [50], Cosmic Explorer (CE) [11] or NEMO [51], the expected measurement uncertainty for a GW170817-like event improves to $\sigma_{\tilde{\Lambda}} = 7$ or 8 for ET or CE respectively, while with one year of observations, the measurement uncertainty will reduce to $\sim 1 - 4$, depending on the merger rate [15]. These XG detectors will thus likely be necessary to fully distinguish between these doppelgängers, based on their tidal deformabilities alone.

Another exciting prospect of XG detectors is the possibility of capturing the post-merger GW emission [see, e.g., 52, 54]. Much work has been devoted to understanding the connection of EoS models to the post-merger frequency spectrum. These quasi-universal relations rely to
a large extent on correlations between the dominant frequency $f_2$ and the tidal deformabilities or radii of cold neutron stars [e.g., 46, 47, 55–63]. One might naturally expect doppelgängers to challenge these relations too.

To test this, we perform binary neutron star merger simulations for the two pairs of doppelganger EoSs shown in Fig. 1. We extend the zero-temperature EoSs of Fig. 1 to finite-temperatures and arbitrary compositions using the framework of [64], and perform merger simulations for each EoS using GW170817-like binary parameters. Two of these fully-finite temperature doppelganger models have been simulated previously [35, 47], and the numerical set-up [65, 66] of our simulations here is identical to that work [35]; for key details, see Appendix D.

We summarize the peak frequencies of the post-merger GW emission for each of the four simulations in Fig. 5, where we compare these results against two existing quasi-universal relations, as well as one with a proposed correction that depends on the mass-radius slope, appropriate for the EoS models considered here [47]. We conservatively estimate the numerical uncertainty of $f_2$ to be at the 10% level [49]. Starting with the correlation between $f_2$ and the radius for a 1.8$M_\odot$ star, $R_{1.8}$, we find that (to within current numerical and systematic uncertainties) the doppelganger models considered here do not violate the existing relations of [46, 47] shown in the top panel of Fig. 5. Concerning the tidal coupling constant $\kappa_2^T$ (see eq. 6 of [48]), the relation with $f_2$ is also approximately consistent with the existing relations proposed by [48], to within the numerical errors of the calculations. We note that the disagreement between the computed frequencies at small $\kappa_2^T$ and the universal relations of [48] is likely a result of numerical uncertainty and limited calibration in this regime in terms of EoS coverage [48, 49, 64–69], and may also be affected by uncertainties in finite-temperature [70, 72] and neutrino physics [73–76]. We confirm that these findings are not sensitive to the choice of EoS parametrization by repeating these merger simulations for the $R_{1.4} \sim 13$ km pair of doppelgängers. Specifically, we re-parameterize this pair of doppelgängers with the generalized PWP framework to ensure a smooth speed of sound [77, 78], and find errors and indistinguishability of doppelgängers compatible with those reported in Fig. 5. In summary, to within the current uncertainties of numerical simulations, we find that the post-merger peak frequencies may not be able to tell doppelgängers apart, although the field is likely to progress significantly by the XG era.

V. SUMMARY

In this work, we have investigated a new family of EoSs, which have been constructed to feature EoS companions, so-called doppelgängers, that have almost identical tidal deformability curves ($\Delta \Lambda \lesssim 30$), but differ maximally in the neutron star radius by $\sim 0.45$ km. These doppelgängers are the result of low density phase transitions, which we have investigated for a sample of $O(10^3)$ example doppelganger EoSs (see Fig. 2), finding that they are generically present when randomly sampling EoSs in a parametric fashion. While they remain indistinguishable to current GW detectors, XG detectors might be able to constrain the tidal deformability to within $\sigma_\Lambda \approx 8$ for a GW170817-like event [15], and therefore rule out or constrain the extremity of the doppelganger models. Since these detectors will potentially also be able to detect the post-merger part of the GW emission [e.g., 52, 54], we have performed numerical relativity simulations of a GW170817-like event for two example pairs of doppelganger EoSs. We have found that, to within current uncertainties in the numerical accuracy of and included physics in these simulations, doppelgängers may remain indistinguishable in the peak frequency of the post-merger GWs. Finally, advances in nuclear theory might be able to further constrain the EoS around nuclear saturation [79, 80], in turn putting potentially strong constraints on the existence of tidal deformability doppelgängers.
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Appendix A: Equation of state construction for the simulated pairs of doppelgängers

We construct the zero-temperature, $\beta$-equilibrium equation of state (EoS) using the piecewise polytropic (PWP) framework of [42, 43, 82]. In particular, we use five piecewise polytropic segments which are spaced uniformly in the logarithm of the density between $\rho_0$ and $7.4\rho_{\text{sat}}$, and use a tabulated crust EoS at densities below $\rho_0$. In the PWP formulation, the pressure $P$ between two
fiducial densities $\rho_{i-1}$ and $\rho_i$ is given by

$$P(\rho) = K_i \rho^\Gamma_i, \quad \rho_{i-1} < \rho < \rho_i$$

(A1)

where $\rho$ is the density, the polytropic constant, $K_i$, is determined by requiring continuity between adjacent polytropic segments, according to

$$K_i = \frac{P_{i-1}}{\rho_{i-1}^{\Gamma_i}} = \frac{P_i}{\rho_i^{\Gamma_i}}.$$  

(A2)

and the polytropic index, $\Gamma_i$, is given by

$$\Gamma_i = \frac{\partial \ln P}{\partial \ln \rho} = \frac{\ln (P_i/P_{i-1})}{\ln (\rho_i/\rho_{i-1})}.$$  

(A3)

For the two pairs of doppelgänger EoSs shown in Fig. 1 of the main paper, we start our parametrization at $\rho_0 = 0.5 \rho_{\text{sat}}$. For the crust EoS at $\rho < \rho_0$, we use the SFHo EoS [44]. In order to ensure continuity in the EoS, we fix the pressure at $\rho_0$ to that of SFHo. We then vary the pressures at higher densities to construct examples of tidal deformability degeneracy, while enforcing a set of minimal physical constraints, namely that the sound speed remain sub-luminal, that the star remain hydrostatically stable, and that the EoS be able to support massive ($2 M_\odot$) neutron stars.

We report the pressures that uniquely characterize the PWP parametrization for the four doppelgänger EoSs from Fig. 1 in Table I.

| $R_{1.4}$ [km] | $\Lambda_{1.4}$ | $P(0.86\rho_{\text{sat}})$ | $P(1.47\rho_{\text{sat}})$ | $P(2.52\rho_{\text{sat}})$ | $P(4.32\rho_{\text{sat}})$ | $P(7.40\rho_{\text{sat}})$ |
|---------------|----------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| 10.77         | 193            | 2.22e+33                    | 5.46e+33                    | 3.90e+34                    | 3.76e+35                    | 1.30e+36                    |
| 11.23         | 193            | 5.44e+33                    | 1.36e+34                    | 3.12e+34                    | 4.25e+35                    | 1.09e+36                    |
| 12.64         | 522            | 1.82e+33                    | 1.97e+34                    | 7.75e+34                    | 5.21e+35                    | 1.50e+36                    |
| 13.01         | 522            | 6.81e+33                    | 1.56e+34                    | 8.15e+34                    | 5.22e+35                    | 1.50e+36                    |

TABLE I: Model parameters for four example doppelgänger EoSs. The first column reports the radius of a 1.4 $M_\odot$ neutron star; the second column reports the tidal deformability of a 1.4 $M_\odot$ star; and the remaining columns report the pressures at each fiducial density in our parametrization. All pressures are in units of dyn/cm².

and 13 km models. At densities below $\rho_{\text{sat}}$, we smoothly connect to the full SFHo table to describe the low-density EoS, using the free-energy matching scheme of [34]. The implementation details for constructing the full, finite-temperature versions of these EoS tables are otherwise identical to the procedure described in [35].

Appendix B: Binary tidal deformability

The inspiral gravitational waves are most sensitive not to the tidal deformabilities of the individual neutron stars ($\Lambda_1$ and $\Lambda_2$), but rather to the binary tidal deformability, $\tilde{\Lambda}$, which is a mass-weighted average of the two, defined according to

$$\tilde{\Lambda} = \frac{16}{13} \frac{(m_1 + 12m_1)m_1^2\Lambda_1 + (m_2 + 12m_2)m_2^2\Lambda_2}{(m_1 + m_2)^3}.$$  

(B1)

where $m_{1,2}$ are the component masses. We calculate the binary tidal deformability for each of the simulated pairs of doppelgängers from Fig. 1 and show the differences between each pair of doppelgängers for a range of masses and mass ratios in Fig. 6. As expected given the similarity between the tidal deformabilities of these models, the differences in $\tilde{\Lambda}$ are also very small, and are $\lesssim 10$ for most binary masses.

Appendix C: Quasi-universal relations

In this paper, we have found that the differences in tidal deformability for the doppelgänger models are much smaller than would be expected based on the Love-C quasi-universal relations, but these differences still fall within the broad error band of the existing Love-C relations [17] [85] (for further analysis, see [44]). There are additionally other quasi-universal relations for the binary tidal deformability parameters, which are commonly used in EoS inferences from GW data [16] [12] [16] [36]. In this appendix, we demonstrate that the doppelgängers summarized in Fig. 2 also fall within the existing error bands of, e.g., the binary-Love relation of [16] as well as the $\tilde{\Lambda} - R_{1.4}$ relation of [10] [11].

We start with the binary-Love relation of [17], which relates the symmetric and antisymmetric combinations
FIG. 6: Contours showing the difference in binary tidal deformability, $|\Delta \tilde{\Lambda}|$, as a function of the primary mass, $M_1$, and the mass ratio, $q$, for the softer pairs of doppelgänger EoSs ($R_{1.4} \simeq 11$ km) from Fig. 1 in the top panel, and the stiffer pair ($R_{1.4} \simeq 13$ km) in the bottom panel.

of the tidal deformability, according to

$$\Lambda_{a,s} = \frac{\Lambda_1 \pm \Lambda_2}{2}$$  \hspace{1cm} (C1)

where $\Lambda_{1,2}$ are tidal deformabilities of the component stars. We show $\Lambda_{a,s}$ in the left column of Fig. 7 for the two sets of doppelgängers identified in Fig. 2. The dashed line shows the empirical binary-Love relation from [17], and we show the fractional residuals between the doppelgänger models and this fit in the bottom panel. We find that the large majority (~95%) of the doppelgängers obey this quasi-universal relation to within $\lesssim 15\%$, which is consistent with the degree of scatter reported in [17]. The largest degree of scatter occurs for small values of the tidal deformability (corresponding to large neutron star masses), which is also consistent with the findings of [17].

There are also quasi-universal relations for the binary tidal deformability (eq. 31). It has been shown that there is an approximately one-to-one mapping between $\tilde{\Lambda}$ and the characteristic radius of a neutron star [10, 11]. We show contours of $\tilde{\Lambda}$ and the $R_{1.4}$ for the doppelgänger EoSs in the right column of Fig. 7. Figure 7 also includes two quasi-universal relations for reference: from [10] which derives an analytic mapping based on a quasi-Newtonian description of the tidal deformability for an $n = 1$ polytrope, and from [11] which fits for an empirical relation to a large number of candidate EoSs and thus includes an estimate of the uncertainty to the relationship. The fractional residuals in $R_{1.4}$ compared to the relationship of [11] are shown in the bottom panel. We find that the doppelgänger EoSs approximately obey this quasi-universal relation, with scatter at the 4% level for the doppelgänger models that start their PWP parametrization at 0.5 $\rho_{\text{sat}}$. For the doppelgängers that start their parametrization at $\rho_{\text{sat}}$, the scatter is $\lesssim 2\%$ for 95% of the sample. This is consistent with the 2% scatter reported by [11]. We note that the doppelgängers deviate more significantly from the relationship predicted by [10], due to the more restrictive assumption of an $n = 1$ polytrope that was adopted in that work, which the doppelgänger EoSs strongly violate.

In summary, we find that the doppelgängers that begin their parametrizations at $\rho_{\text{sat}}$ generally obey the existing quasi-universal relations for the binary tidal deformability, to within the degree of scatter that has been previously reported. For the sample of doppelgängers that begin their parametrizations at 0.5 $\rho_{\text{sat}}$, there is more scatter, but the general trends remain the same. The fact that it is possible to construct tidal deformability doppelgängers within the scatter of these existing relationships may limit the ultimate accuracy to which such relations can constrain the EoS.

Appendix D: Numerical relativity simulations of doppelgängers

For our numerical relativity simulation of the doppelgänger models, we use the same setup as in [35]. In fact the simulation results for the $R_{1.4} \simeq 11$ km models have already been presented there. In this appendix, we summarize the main features of the simulations. We solve the coupled Einstein-hydrodynamics system using the Frankfurt-/IllinoisGRMHD code (FIL) code [65, 66, 86]. This solves the general-relativistic (magneto-)hydrodynamics system [87, 88] together with the Z4c formulation of Einsteins equations [89, 90]. In addition, we include a neutrino leakage scheme [91, 92], and evolve the system for vanishing magnetic fields. The computational infrastructure is provided by the Einstein Toolkit [93] infrastructure. We impose reflection symmetry across the orbital plane, and adopt a finest-level resolution of 262 m, with 8 levels of fixed mesh refinement [94]. The simulations are performed for about 10$-15$ ms after the merger, sufficient to extract the $f_2$ peak frequency of the post-merger spectrum.
FIG. 7: **Left**: Binary-Love relation between the symmetric and anti-symmetric tidal deformabilities, calculated assuming the mass-parameters of GW170817 (chirp mass of $1.186 M_\odot$ and mass ratio of 0.85). The green and orange contours correspond to the sample of $O(10^3)$ doppelgängers from Fig. 2. Contours are shown for 68%, 95%, and 99% confidence intervals. The dashed gray line corresponds to the quasi-universal relation introduced by [17], while the bottom panel shows the residuals between this fit-relation and the doppelgänger EoSs. **Right**: Correlation between the binary tidal deformability and the radius of a $1.4 M_\odot$ neutron star, assuming the mass parameters of GW170817, for the same sets of doppelgängers. The gray dashed lines represent the quasi-universal relation (and 2% estimated error) from [11], while the solid gray line is the quasi-Newtonian relation derived in [10] for an $n = 1$ polytrope. The bottom panel shows the residuals with respect to the median [11] relation.