Coordinate Representation of the One-Spinon One-Holon Wavefunction and Spinon-Holon Interaction

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By deriving and studying the coordinate representation for the one-spinon one-holon wavefunction we show that spinons and holons in the supersymmetric $t - J$ model with $1/r^2$ interaction attract each other. The interaction causes a probability enhancement in the one-spinon one-holon wavefunction at short separation between the particles. We express the hole spectral function for a finite lattice in terms of the probability enhancement, given by the one-spinon one-holon wavefunction at zero separation. In the thermodynamic limit, the spinon-holon attraction turns into the square-root divergence in the hole spectral function.

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I. INTRODUCTION

Landau’s Fermi liquid theory applies to interacting electron systems that can be adiabatically deformed to a Fermi gas. If the interaction is smoothly switched off, the spectrum of a Fermi liquid reduces to the spectrum of a noninteracting fermionic system. The excitations of a Fermi liquid are given by quasiparticles and quasiholes. Although their lifetime may be short, it always becomes infinite at the Fermi surface [1]. As one is concerned only with energies close to the Fermi surface, Fermi liquid’s picture applies to a wide class of correlated systems. Experimentally, Landau quasiparticles are observed as a resonant peak at the Fermi surface in the spectral density of states measured at fixed momentum.

Nevertheless, there are several low-dimensional strongly correlated systems where Landau’s picture breaks down. In Luttinger liquids the spin and charge degrees of freedom “separate” and the quasiparticles and quasiholes are no longer elementary excitations [2,3]. The same phenomenon was discovered by means of Bethe-ansatz-like techniques in exactly solvable models, like the same phenomenon was discovered by means of Bethe-quasiholes are no longer elementary excitations [2,3]. The degrees of freedom “separate” and the quasiparticles and breaks down. In Luttinger liquids the spin and charge strongly correlated systems where Landau’s picture applies to a wide class of correlated systems. Experimentally, Landau quasiparticles are observed as a resonant peak at the Fermi surface in the spectral density of states measured at fixed momentum.

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a broad spectrum. This was experimentally detected in ARPES experiments on quasi-1D samples [4].

Using Bethe-Ansatz solutions of 1D models, one can work out the energy of a many-spinon many-holon state. In the thermodynamic limit the total energy is the sum of the energies of each isolated particle. However, this does not imply that spinons and holons do not interact. In recent papers [5,6] we carefully studied spinon interaction in an exact solution of the Haldane-Shastry model (HSM) [6,7], a model for a strongly correlated 1D system with no charge degrees of freedom. We showed that spinons interact, although in the thermodynamic limit the energy of a many-spinon solution is additive. We also showed that their interaction is responsible for the lack of integrity of spin waves against decay into spinons, the true low-energy excitations of the system.

In this paper we generalize the formalism introduced in Ref. [3] to study the interaction between spinons and holons in an exact closed-form solution of the supersymmetric extension of the HSM: the supersymmetric $t - J$ model with $1/r^2$-interaction (Kuramoto-Yokoyama (KY) model [12,13]). The KY-model is a system of electrons located at the sites of a circular lattice, where double occupancy of a site is forbidden by strong Coulomb repulsion. Charge hopping, Coulomb interaction and spin-spin antiferromagnetic interaction are all inversely proportional to the square of the chord between the corresponding sites. Charge vacancies at some sites (holes) may be created by filling the system with fewer electrons than the number of sites. In the KYM one is concerned with both spin and charge degrees of freedom.

We investigate the basic features of the KY-model by employing a formalism based on analytic variables on the unit radius circle. As in the case of the HSM, it easier to construct and visualize the spinon and holon excitation...
by using a real-space formalism than by using the Bethe-Ansatz formalism. Within our formalism we derive a “real space” representation of the one-spinon one-holon wavefunction as a solution of an appropriate equation of motion.

By taking the thermodynamic limit of the exact solution of the equation of motion, we find that the probability of finding a spinon and a holon at large separations from each other is independent of the distance between the particles, while it is greatly enhanced when the two particles are on top of each other, phenomenon we refer to as “short-distance spinon-holon probability enhancement”. In the thermodynamic limit, the spinon-holon interaction assumes the same form as the spinon-spinon interaction derived in [8], although the corresponding equations of motion are completely different. The physical interpretation is the same as in the case of the spinon-spinon interaction: a spinon and a holon do not interact when they are widely separated from each other, while they exhibit a short-range attraction at short separations [9].

To show the effects of spinon-holon interaction on the hole spectral density, \( A_h(\omega, q) \), we exactly calculate the contribution to \( A_h(\omega, q) \) from one-spinon one-holon states, \( A_h^{\text{sp,ho}}(\omega, q) \). The spinon-holon interaction has important consequences on the functional form of \( A_h^{\text{sp,ho}}(\omega, q) \). The probability enhancement causes the overlap between the wavefunction for the localized hole and that for a spinon-holon pair to be significant, although not enough to form a spinon-holon bound state. The corresponding matrix element is enhanced - despite the fact that the density of states is uniform at low energy - so as to make the hole excitation fully unstable to decay into a spinon-holon pair. Taking the thermodynamic limit of our result, we show that, as the size of the system increases, spinon-holon interaction turns into a square root singularity at the one-spinon one-holon creation threshold. Correspondingly, \( A_h^{\text{sp,ho}}(\omega, q) \) shows no Landau quasiparticle’s peak, but it rather exhibits a sharp singular threshold, followed by a broad branch cut [10].

The paper is organized as follows: In section II we review the KY Hamiltonian and its supersymmetry; In section III we introduce the ground state of the KY model at half filling and its representation as a function of analytic variables on the unit circle. At half-filling, the ground state is the same as the ground state of the HS-model - a disordered spin singlet. We will briefly review some properties of the ground state, already discussed at length in [9]; In section IV we analyze the one-spinon solution and review its relevant properties; In section V we focus on the one-holon solution and derive its relevant properties. In section VI we derive the action of \( \mathcal{H}_\text{KY} \) on the one-spinon one-holon states, the energy eigenvalues, the corresponding eigenvectors, and their norm; in section VII we write the Schrödinger equation for the one-spinon one-holon wavefunction, whose solutions are simple polynomials. From the behavior of the one-spinon one-holon wavefunction, we infer the nature of the interaction between spinons and holons: a short-range attraction. The physical consequences of such an interaction are discussed at length in Section VIII, where we rederive an exact expression for the contribution of one-spinon one-holon states to the hole spectral function in terms of the spinon-holon wavefunctions and rigorously prove that this contribution is completely determined by the spinon-holon interaction. In the thermodynamic limit spinon-holon interaction turns into the square root divergence in the hole spectral function, obtained by [10]. In Section IX we provide our main conclusions.

II. KURAMOTO-YOKOYAMA HAMILTONIAN

The Kuramoto-Yokoyama model is defined on a lattice with periodic boundary conditions. Sites are parametrized by the \( N \)-th roots of unity, \( z_\alpha (\alpha = 1, \ldots, N) \). Strong electron repulsion forbids double occupancy at each site. Therefore, sites can be occupied by an \( \uparrow \) or a \( \downarrow \)-electron, or they can be empty. The number of empty sites can be tuned by means of an external chemical potential which fixes the total charge of the system. The Kuramoto-Yokoyama Hamiltonian [12] is a generalization of the Haldane-Shastry Hamiltonian [10,11], where also charge dynamics is taken into account. It takes the form:

\[
\mathcal{H}_\text{KY} = J \left( \frac{2\pi}{N} \right)^2 \sum_{\alpha<\beta} \frac{1}{|z_\alpha - z_\beta|^2} P \left\{ \vec{S}_\alpha \cdot \vec{S}_\beta \right\} - \frac{1}{2} \sum_\sigma \left( c_\sigma^\dagger \left( c_\sigma^\dagger c_\sigma^\dagger \right) + \frac{1}{2} \left( n_\sigma + n_\beta \right) - \frac{1}{4} n_\sigma n_\beta - \frac{3}{4} \right) P ,
\]

where the Gutzwiller’s projector

\[
P = \prod_\alpha \left( 1 - c_\sigma^\dagger c_\sigma^\dagger c_\sigma^\dagger c_\sigma^\dagger \right) ,
\]

accounts for the no-double occupancy constraint.

Site occupation and spin operators are given by

\[
n_\sigma = c_\sigma^\dagger c_\sigma^\dagger + c_\sigma^\dagger c_\sigma^\dagger
\]

\[
S_\sigma^a = \frac{1}{2} \sum_{\sigma,\sigma'} \tau^a_{\sigma\sigma'} n_\sigma n_{\sigma'} .
\]

where \( \tau^a, a = x, y, z \), are Pauli matrices. An empty site corresponds to a charge-1 hole that can tunnel to nearby sites by means of the same inverse-square matrix.
element characterizing the spin exchange and the charge-repulsion term in $\mathcal{H}_{KY}$.

Usual bosonic symmetries of a $t - J$-like model are total spin, corresponding to the operator $\vec{S} = \sum_\alpha \vec{S}_\alpha$, and total charge, corresponding to the operator $N = \sum_\alpha n_\alpha$. The equivalence of energy scales for magnetism, charge transport and charge interaction, causes the KY Hamiltonian to be supersymmetric, in the sense that it commutes with the electron or hole injection operators $Q_\alpha = \sum_\sigma P_{c\alpha \sigma}$, $Q_\alpha^\dagger = \sum_\sigma P_{c\alpha \sigma}^\dagger P$.

As in the Haldane-Shastry Hamiltonian, since the complex variable $z$ lays on the unit circle ($z^* = z^{-1}$), the interaction is an analytic function of the coordinates, that is:

$$\frac{1}{|z_\alpha - z_\beta|^2} = \frac{z_\alpha z_\beta}{(z_\alpha - z_\beta)^2} .$$

Throughout the paper we use the representation in terms of the analytic variables $z_\alpha$. This turns out to be very useful for describing the properties of spinons and holons in real space.

**III. GROUND STATE**

In this section and in the following one we discuss the ground-state and the one-spinon eigenstates of the KY-model at half filling. At half-filling the KY-model reduces to the Haldane Shastry (HS) Hamiltonian:

$$\mathcal{H}_{HS} = J \left(\frac{2\pi}{N}\right)^2 2 \sum_{\alpha < \beta} \frac{\vec{S}_\alpha \cdot \vec{S}_\beta}{|\alpha - \beta|^2} . \quad (5)$$

Both the ground state and the one-spinon eigenstates are the same as for the HS-model. Since in we have already applied our formalism to study basic properties of the HS-model, here we will only briefly review the main results in view of their extension to states where holon excitations are present.

**A. Ground State wavefunction**

Let $N$ be even. We first give the representation of the ground state $|\Psi_{GS}\rangle$ in terms of the $z$-coordinates and then derive its energy. $|\Psi_{GS}\rangle$ is defined in terms of its projection onto the set of states with $M = N/2$ spins up and the remaining spins down. If $z_1, \ldots, z_M$ are the coordinates of the up spins, one defines the state $|z_1, \ldots, z_M\rangle$ as:

$$|z_1, \ldots, z_M\rangle = \prod_{j=1}^M S_j^+ \prod_{\alpha=1}^N c_{z_\alpha}^\dagger |0\rangle$$

where $|0\rangle$ is the empty state. The projections are given by:

$$\Psi_{GS}(z_1, \ldots, z_M) = \prod_{j<k} (z_j - z_k)^2 \prod_{j=1}^M z_j . \quad (6)$$

$\Psi_{GS}$ is a polynomial in the analytic variables $z_1, \ldots, z_M$. Its norm was first computed by Wilson [10] by using the following identity:

$$C_M = \sum_{z_1, \ldots, z_M} \prod_{i<j} |z_i - z_j|^4$$

$$= \frac{(N/2 \pi i)^M \int \frac{dz_1}{z_1} \ldots \int \frac{dz_M}{z_M} \prod_{i<j} (1 - \frac{z_i}{z_j})^2}{N^M (2M)!} . \quad (7)$$

Basic properties of $\Psi_{GS}$ follow in this section, together with their derivation.

**B. Singlet State**

The ground state is a spin singlet. $|\Psi_{GS}\rangle$ is annihilated by both $S^+$ and $S^-$. $S^+|\Psi_{GS}\rangle = 0$ because $|\Psi_{GS}\rangle$ has an equal number of $\uparrow$ and $\downarrow$ spins, while

$$[S^- |\Psi_{GS}\rangle](z_2, \ldots, z_M) = \sum_{\alpha=1}^N \langle z_2, \ldots, z_M | S^-_\alpha |\Psi_{GS}\rangle$$

$$= \lim_{z_1 \to 0} \sum_{\ell=1}^{N-1} \frac{1}{\ell!} \left( \sum_{\alpha=1}^N z_\alpha^\ell \right) \frac{\partial^\ell |\Psi_{GS}(z_1, \ldots, z_M) = 0, (8)$$

since $\sum_{\alpha=1}^N z_\alpha = N \delta_{\ell 0} \pmod{N}$.

As $\Psi_{GS}$ is a spin singlet, it takes exactly the same form if expressed either in terms of the $\uparrow$-spin coordinates $z_1, \ldots, z_M$, or of the $\downarrow$-spin coordinates, $\eta_1, \ldots, \eta_M$, that is:

$$\Psi_{GS}(z_1, \ldots, z_M) = \Psi_{GS}(\eta_1, \ldots, \eta_M) . \quad (9)$$

Eq.(9) is proved in Appendix A, where we derive the formulas to relate the representation of the states of the system in terms of $\uparrow$-spin coordinates to the representation in terms of $\downarrow$-spin coordinates.

In the thermodynamic limit, half-odd spin chains exhibit a gapless spectrum, although they are not allowed to order [17]. Accordingly, $|\Psi_{GS}\rangle$ is a disordered spin liquid state, and the spin-spin correlation function, $\chi(z_\alpha) = \langle \Psi_{GS} S_\alpha^+ S^-_\alpha |\Psi_{GS}\rangle/\langle\Psi_{GS}|\Psi_{GS}\rangle$, falls off with the distance as $(-1)^{\ell}/\ell$, thus showing absence of spin order [9].

**C. Ground State Energy**

At filling-1/2, $|\Psi_{GS}\rangle$ is the ground state of $\mathcal{H}_{KY}$, with eigenvalue:

$$
\Psi_{GS}$$

is a polynomial in the analytic variables $z_1, \ldots, z_M$. Its norm was first computed by Wilson [10] by using the following identity:

$$C_M = \sum_{z_1, \ldots, z_M} \prod_{i<j} |z_i - z_j|^4$$

$$= \frac{(N/2 \pi i)^M \int \frac{dz_1}{z_1} \ldots \int \frac{dz_M}{z_M} \prod_{i<j} (1 - \frac{z_i}{z_j})^2}{N^M (2M)!} . \quad (7)$$

Basic properties of $\Psi_{GS}$ follow in this section, together with their derivation.
In this subsection we review our technique, in view of its generalization to the case where the filling is \( \neq 1/2 \) and the dynamics of the system is described by the full KY-Hamiltonian. Since \( [S_+^a S^-_\beta \Psi_{GS}] \neq 0 \) for any value on the unit circle. After computing the derivatives, we constrain them again to lattice sites. States with half-odd spin are alleged eigenstates of the singlet sea \([10,11,9]\). Since correlations in the ground state of \( H_{HS} \) makes no difference whether one begins with an odd or even number of sites. Therefore, in the thermodynamic limit, there is no way to distinguish between chains with odd number of sites or chains with even number of sites. Thus, the elementary excitations above the ground state of a correlated 1D electron system are not Landau’s quasiparticles, but rather spinons and holons. Spinons have been identified as the elementary excitations of a spin-1/2 1D antiferromagnet. They can be thought of as localized spin defects carrying total spin-1/2, embedded in an otherwise featureless disordered singlet spin sea. As spinon excitations appear in the KY-chain at any filling, we study spinons at filling-1/2, when the KY-model reduces back to the HS-model. One-spinon states appear as states of the chain with an odd number of sites. In this case, the minimum possible value for the total spin is 1/2, and the state for a localized spinon at \( s \) is created by constraining the spin at \( s \) to be \( \downarrow \), in a surrounding spin singlet state \([11,13]\). Since correlations in the ground state are short-ranged, in the thermodynamic limit it makes no difference whether one begins with an odd or an even number of sites. Therefore, in the thermodynamic limit, there is no way to distinguish between chains with odd number of sites or chains with even number of sites. States with half-odd spin are alleged eigenstates of \( H_{KY} \) at half-filling with an odd number of spinons and no holons. In this section we briefly review the one-spinon wavefunction and discuss its properties. This was first
studied by Haldane and Shastry [10,11], and discussed at length in [9] within the framework of the formalism of analytic variables.

A. One-spinon spin doublet

Let \( N \) be odd and \( M = (N - 1)/2 \). The wavefunction for a localized spinon at \( s \) takes Haldane’s form [9,11]:

\[
\Psi_{sp}^s(z_1, \ldots, z_M) = \prod_{j=1}^{M} (z_j - s) \prod_{i<j} (z_i - z_j)^2 ,
\]

(15)

where \( z_1, \ldots, z_M \) denote again the position of the ↑-spins and \( s \) is the coordinate of a lattice site where the spin is fixed to be ↓.

By definition, \( \Psi_{sp}^s \) is an eigenstate of \( S^z \) with eigenvalue -1/2. In order to prove that it is a spin-1/2 state, indeed, per Eq. (8) we have:

\[
\sum_{z_\beta \neq z} S^\beta_{sp} \Psi_{sp}^s = 0 ,
\]

(16)

which proves that \( \Psi_{sp}^s \) is the ↓-spin component of a spin doublet.

\( \Psi_{sp}^s(z_1, \ldots, z_M) \) is a polynomial of degree less than \( N + 1 \) in each variables \( z_j \). Therefore, we may again apply Taylor’s expansion technique used to calculate the ground state energy. Doing so, we find:

\[
\mathcal{H}_{KY} \Psi_{sp}^s = \frac{J}{2} \left( \frac{2\pi}{N} \right)^2 \left\{ \lambda + \frac{N}{48} (N^2 - 1) \right. + \left. \frac{M}{6} (4M^2 - 1) - \frac{N}{2} M^2 \right\} \Psi_{sp}^s ,
\]

(17)

provided that \( \lambda \) satisfies the following eigenvalue equation for \( \Phi_{sp}^s = \prod_j^M (z_j - s) \):

\[
\left\{ M(M - 1) - s^2 \frac{\partial^2}{\partial s^2} - \frac{N - 3}{2} \left[ M - s \frac{\partial}{\partial s} \right] \right\} \Phi_{sp}^s = \lambda \Phi_{sp}^s .
\]

(18)

One↓-spinon energy eigenstates are given by propagating one-spinon plane waves:

\[
\Psi_{sp}^s(z_1, \ldots, z_M) = \frac{1}{N} \sum_s (s^s)^m \Psi_{sp}^s(z_1, \ldots, z_M) .
\]

(19)

The energy eigenvalue is

\[
\mathcal{H}_{KY} |\Psi_{sp}^m\rangle = \left\{ -J \frac{\pi^2}{24} (N - \frac{1}{N}) \right. + \left. \frac{J}{2} \left( \frac{2\pi}{N} \right)^2 m \frac{N - 1}{2} - m \right\} |\Psi_{sp}^m\rangle ,
\]

(20)

with \( 0 \leq m \leq (N - 1)/2 \) and \( \lambda = m((N - 1)/2 - m) \).

As the total crystal momentum of the state \(|\Psi_{sp}^m\rangle\) is given by

\[
q_m^sp = \frac{\pi}{2} N - \frac{2\pi}{N} (m + \frac{1}{4}) \text{ (mod } 2\pi) ,
\]

(21)

the total energy may be rewritten as:

\[
\mathcal{H}_{KY} |\Psi_{sp}^m\rangle = \left\{ -J \frac{\pi^2}{24} (N + \frac{5}{N}) - \frac{3}{N^2} \right\} + E(q_m^sp) |\Psi_{sp}^m\rangle ,
\]

(22)

that is, the sum of a ground-state contribution, plus the kinetic energy of the propagating spinon. The one-spinon dispersion relation is correspondingly provided by:

\[
E(q_m^sp) = \frac{J}{2} \left( \frac{\pi^2}{2} - (q_m^sp)^2 \right) \text{ (mod } \pi) ,
\]

(23)

As extensively discussed in [9,10,11], the one-spinon dispersion relation shows the typical features of spinon excitations. It spans only the inner or outer half of the Brillouin zone, depending on whether \( N - 1 \) is divisible by 4 or not, which corresponds to the absence of negative energy states, i.e., to the absence of “antispinons”. The spinon dispersion at low energies is linear in \( q \) with a velocity

\[
v_{\text{spinon}} = \frac{\pi}{2} J ,
\]

(24)

The half-band of single elementary excitations for odd \( N \) are the only \( S = 1/2 \) states without extra degeneracies. The ground state of the odd-N spin chain is 4-fold degenerate and is given by \(|\Psi_{sp}^m\rangle\) for \( m = 0 \) and \((N - 1)/2\) and their ↑ counterparts. This corresponds physically to a “left-over” spinon with momentum ±\( \pi \).

The spin density in the state \( \Psi_{sp}^s \) as a function of the spinon position is uniformly zero, as appropriate for the disordered spin singlet, except for an abrupt dip centered at \( z = s \). The dip is identified with a localized spinon at \( s \). Therefore, \( \Psi_{sp}^s \) may be thought of as the wavefunction for a localized spinon at \( s \). Starting from such an interpretation, in Ref. (9) we showed that, although spinons are collective excitations of a strongly correlated system, they can still be treated as real quantum mechanical particles. In this paper we will generalize our formalism to states where both spinons and holons are present.

B. The Norm

The squared norm of the one-spinon energy eigenstates is defined as the scalar product:
\begin{equation}
\langle \Psi_{sp}|\Psi_{sp}\rangle = \sum_{z_1,\ldots,z_M} |\Psi_{sp}^m(z_1,\ldots,z_M)|^2 . \tag{25}
\end{equation}

For the Haldane-Shastry model, we derived the formula for Eq.(25) in Ref.\cite{1}. By employing a recursion relation between \(\langle \Psi_{sp}^m|\Psi_{sp}^p\rangle\) and \(\langle \Psi_{sp}^{m-1}|\Psi_{sp}^p\rangle\), we expressed all the norms in terms of \(m\) and of the constant \(C_M\) introduced in Eq.(13). The induction relation is:
\begin{equation}
\frac{\langle \Psi_{sp}^m|\Psi_{sp}^p\rangle}{\langle \Psi_{sp}^{m-1}|\Psi_{sp}^p\rangle} = \frac{(m - \frac{1}{2})(M - m + 1)}{m(M - m + \frac{1}{2})} , \tag{26}
\end{equation}

That recursively gives:
\begin{equation}
\langle \Psi_{sp}^m|\Psi_{sp}^m\rangle = \frac{\Gamma[M + 1]\Gamma[M + \frac{1}{2}]\Gamma[M - m + \frac{1}{2}]}{\Gamma[\frac{1}{2}]\Gamma[M + \frac{1}{2}]\Gamma[m + 1]\Gamma[M - m + 1]} C_M , \tag{27}
\end{equation}

\section{V. ONE-HOLON WAVEFUNCTION.}

Holons are charged, spin-0 elementary excitations of the Kuramoto-Yokoyama Hamiltonian. They are constructed by removing an electron from the center of a spinon. The state for a localized holon at \(h_0\) is given by \(\langle \Psi_{sp}|\Psi_{sp}^{h_0}\rangle\), where \(\langle \Psi_{sp}^p\rangle\) is the state defined in Eq.(13). By construction, in the KY-model, the holon is the supersymmetric partner of the spinon. However, unlike in the spinon case, the Brillouin zone for one-holon states is not halved, as both negative and positive energy holons can be constructed. In this section, we are concerned mainly with holon “kinematics”. We derive the one-holon eigenstates, their norm, their energy and their crystal momentum. In particular, we focus on negative-energy one-holon states, since these states are the ones relevant to the spinon-holon interaction. In the next Sections we analyze holon and spinon dynamics - the interaction between spinons and holons and its relation to the instability of the hole excitations in the KY-model.

\subsection{A. One-Holon Spin Singlet}

Let \(N\) be odd and \(M = (N - 1)/2\). The wavefunction for a propagating, negative energy, holon is given by:
\begin{equation}
\Psi_{hn}^h(z_1,\ldots,z_M|h) = (h^*)^M \prod_j (z_j - h) z_j \prod_{j<k}(z_j - z_k)^2 , \tag{28}
\end{equation}

where \(z_1,\ldots,z_M\) denote the positions of the \(\uparrow\) sites and \(h\) denotes the position of the empty site, all others being \(\downarrow\). Also, \(0 \leq n \leq (N + 1)/2\). Different from the spinon case, in Eq.(28) the holon coordinate \(h\) is not a quantum number but a coordinate variable. Therefore, unlike localized one-spinon eigenstates, \(\Psi_{hn}^h\) takes a well-defined crystal momentum, as we will show later.

\(\Psi_{hn}^h\) is a spin-singlet state. Indeed, by definition its total component of the spin along \(z\) is zero. Following the same steps leading to Eq.(16), we also get:
\begin{equation}
S^z \Psi_{hn}^h = S^− \Psi_{hn}^h = 0 , \tag{29}
\end{equation}

which proves that \(\Psi_{hn}^h\) is a spin singlet.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Spin and charge profiles of the localized holon \(\langle \Psi_{hn}^h\rangle\) defined by Eq.(17). \(\theta\) is defined as \(\theta = -i \ln(z/h_0)\), where \(z\) is the independent variable.}
\end{figure}

\subsection{B. Negative One-Holon Energy Eigenstates}

\(\Psi_{hn}^h\) is an eigenstate of \(\mathcal{H}_{KY}\) with energy eigenvalue given by:
\begin{equation}
\mathcal{H}_{KY}\Psi_{hn}^h = \left\{ -J(\frac{\pi^2}{24})(N - \frac{1}{2}) \right. \\
+ \frac{J}{2}(\frac{2\pi}{N})^2 n(\frac{N + 1}{2}) \left\} \Psi_{hn}^h , \tag{30}
\end{equation}

where \(0 \leq n \leq (N + 1)/2\).

In order to prove Eq.(30), let us first split \(\mathcal{H}_{KY}\) as follows:
\begin{equation}
\mathcal{H}_{KY} = h_S^2 + h_S^h + h_N + h_Q^1 + h_Q^2 . \tag{31}
\end{equation}

We define the various terms when calculating their contributions to the total energy.

- Spin-exchange term:

\[ [h_S^2 \Psi_{hn}^h](z_1,\ldots,z_M|h) \]
\[
\frac{1}{\Delta_N - N} = \frac{1}{\Delta_N - N} \sum_{i<j} \frac{1}{|z_i - z_j|^2} \Psi_n^\text{ho}(z_1, \ldots, z_M|\hbar)
\]
\[
= \left\{ \sum_{\alpha \neq \beta} P S^\alpha_\beta S^\beta_\alpha P \Psi_n^\text{ho}(z_1, \ldots, z_M|\hbar) \right\} (z_1, \ldots, z_M|\hbar)
\]
Their energy is given by the one-spinon eigenstates constructed by supersymmetrically rotating one-spinon bands. The holon dispersion relation around the band minimum is linear in momentum

\[ \Psi_{n}^{ho}(z, \ldots, z_{M}|h) = (\partial_{h}^{2} + \frac{2}{2} h^{2})^{-1} \Psi_{n}^{ho}(z, \ldots, z_{M}|h) . \] (41)

Adding Eqs. (32, 33, 34, 35, 41) all together, we find that:

\[ [H_{KY} \Psi_{n}^{ho}](z, \ldots, z_{M}|h) = \frac{J}{2} (\frac{2\pi}{N})^{2} \left\{ -\frac{N(N^{2} - 1)}{4} + n(n - M - 1) \right\} \Psi_{n}^{ho}(z, \ldots, z_{M}|h) , \] (42)

This is the formula we had to prove.

**C. Crystal Momentum**

The state \( |\Psi_{n}^{ho} \rangle \) is a propagating holon with crystal momentum

\[ q_{n}^{ho} = \frac{\pi}{2} N + 2\pi (n - \frac{1}{4}) \pmod{2\pi} , \] (43)

with the definition

\[ \Psi_{n}^{ho}(z_{1}, \ldots, z_{M}|h) = \exp(iq_{n}^{ho}) \Psi_{n}^{ho}(z_{1}, \ldots, z_{M}|h) . \] (44)

Rewriting the eigenvalue as

\[ H_{KY} |\Psi_{n}^{ho} \rangle = \left\{ -J \left( \frac{\pi^{2}}{24} \right) (N + \frac{5}{N} - \frac{3}{N^{2}}) + E(q_{n}^{ho}) \right\} |\Psi_{n}^{ho} \rangle , \] (45)

we obtain the dispersion relation

\[ E(q_{n}^{ho}) = -\frac{J}{2} \left[ \left( \frac{\pi}{2} \right)^{2} - (q_{n}^{ho})^{2} \right] \pmod{\pi} , \] (46)

We worked out Eq. (46) for the case of negative-energy one-holon eigenstates. Unlike in the spinon case, the holon dispersion relation around the band minimum is quadratic in \( q \), while it is linear in \( q \) near \( \pi/2 \), where the band closes.

Positive-energy one-holon eigenstates may be constructed by supersymmetrically rotating one-spinon eigenstates [13]. Their energy is given by \( |E(q_{n}^{ho})| \) and the momentum spans the remaining half of the Brillouin zone. The corresponding dispersion relation is plotted in Fig. 2. Since for the purpose of studying spinon-holon interaction we only need negative-energy holon states only, we do not discuss here positive-energy holon states. We also need the wavefunction for a localized holon at site \( h_{0} \), \( |\Psi_{h_{0}}^{ho} \rangle \), which is obtained by Fourier transforming the propagating holon wavefunction to real space:

\[ \Psi_{h_{0}}^{ho} = \sum_{n=0}^{(N+1)/2} h_{n}^{-n} \Psi_{n}^{ho} \] (47)

**D. The Norm**

The norm of the state \( |\Psi_{n}^{ho} \rangle \) is defined as:

\[ \langle \Psi_{n}^{ho} | \Psi_{n}^{ho} \rangle = \sum_{z_{1}, \ldots, z_{M}} |\Psi_{n}(z_{1}, \ldots, z_{M}|h)\rangle^{2} \] (48)

From the definition of the norm it immediately follows out that \( \langle \Psi_{n}^{ho} | \Psi_{n}^{ho} \rangle \) does not depend on \( n \). The formula for \( \langle \Psi_{n}^{ho} | \Psi_{n}^{ho} \rangle \) is worked out in Appendix B and is given by:

\[ \langle \Psi_{n}^{ho} | \Psi_{n}^{ho} \rangle = N^{M+1} 4^{M} \Gamma[M] \Gamma[M + 1] \frac{\Gamma[1 + \frac{M}{2} + \frac{1}{2}]}{\Gamma[M + 1 + \frac{1}{2}]} . \] (49)
VI. ONE-SPINON ONE-HOLON WAVEFUNCTION

As for many-spinon configurations, spinons and holons maintain their integrity when many of them are present in the same state. Therefore, we can diagonalize $H_{KY}$ within subspaces with a fixed number of spinons and holons.

In this section we derive the action of $H_{KY}$ on the one-spinon one-holon eigenstates, diagonalize the corresponding matrix and work out the norm of the states.

A. Action of $H_{KY}$ on one-spinon one-holon states

To construct one-spinon one-holon eigenstates of $H_{KY}$ we start with states in a mixed representation, $\Psi_n^s$, where the spinon is localized at site $s$, but the holon is propagating with momentum $q_{\eta}$. For $N$ even and $M = N/2 - 1$, we have

$$\Psi_n^s(z_1, \ldots, z_M|h) = h^n \prod_{j} (z_j - s)(z_j - h) \prod_{j < k} (z_j - z_k)^2,$$

where $1 \leq n \leq M + 2$. To derive the action of the KY-Hamiltonian on $\Psi_n^s$, we split it as in Eq. (31).

- The action of spin-exchange term on $\Psi_n^s$ provides:

$$[h^T S^\eta] \Psi_n^s(z_1, \ldots, z_M|h) =$$

$$- \sum_{j=\uparrow} \sum_{j \neq j} \frac{z_j z_{\beta}}{(z_j - z_{\beta})^2} \Psi_n^s(z_1, \ldots, z_{\beta}, \ldots | h)$$

$$+ \frac{M}{12} (-2N + 5) \Psi_n^s - \sum_{i \neq j = \uparrow} \frac{1}{|z_i - z_j|^2} \Psi_n^s$$

$$+ (M - 1) \left[ s \frac{\partial}{\partial s} \Psi_n^s + h^{n+1} \frac{\partial}{\partial h} \left( \frac{\Psi_n^s}{h^n} \right) \right]$$

$$- s^2 \frac{\partial^2}{\partial s^2} \Psi_n^s - h^{n+2} \frac{\partial^2}{\partial h^2} \left( \frac{\Psi_n^s}{h^n} \right)$$

$$+ \frac{1}{2} \left( \frac{h + s}{h - s} \right) \left[ s \frac{\partial}{\partial s} \Psi_n^s - h^{n+1} \frac{\partial}{\partial h} \left( \frac{\Psi_n^s}{h^n} \right) \right].$$

- The Ising term gives:

$$[h^Y S^\eta] \Psi_n^s(z_1, \ldots, z_M|h) =$$

$$\left\{ \sum_{i \neq j = \uparrow} \frac{1}{|z_i - z_j|^2} - \frac{M N^2 - 1}{2} \right\} \Psi_n^s.$$

- The Coulomb potential term acting on the one-spinon-one-holon wavefunction reads:

$$[h_N S^\eta] \Psi_n^s(z_1, \ldots, z_M|h) = \frac{1 - N^2}{24} \Psi_n^s. \quad (53)$$

- The $\downarrow$ spin contribution to the charge kinetic energy gives:

$$[h_Q S^\eta] \Psi_n^s(z_1, \ldots, z_M|h) =$$

$$- \sum_{n \neq h} \frac{z_n h}{(z_n - h)^2} \Psi_n^s(z_1, \ldots, z_M z_\beta)$$

$$= \left[ \frac{N^2 - 1}{12} + \frac{n(n - N)}{2} \right] \Psi_n^s$$

$$- \left[ \frac{N - 1}{2} - n \right] h^{n+1} \frac{\partial}{\partial h} \left( \frac{\Psi_n^s}{h^n} \right) + \frac{1}{2} h^{n+2} \frac{\partial^2}{\partial h^2} \left( \frac{\Psi_n^s}{h^n} \right).$$

- In order to manipulate the $\uparrow$ spin contribution to the charge kinetic energy, we need to express $\Psi_n^s(z_1, \ldots, z_M|h)$ in terms of the $\downarrow$ spin coordinates $\eta$. As in the one-holon case, we obtain:

$$\Psi_n^s(z_1, \ldots, z_M|h) = \Psi_n^s(\eta_1, \ldots, \eta_M|h).$$

Therefore, we have:

$$[h_Q S^\eta] \Psi_n^s(\eta_1, \ldots, \eta_M|h) =$$

$$- \sum_{n \neq h} \frac{h z_n}{(h - z_n)^2} \Psi_n^s(\eta_1, \ldots, \eta_M z_\beta)$$

$$+ \frac{sh}{(s - h)^2} \Psi_n^s(\eta_1, \ldots, \eta_M s)$$

$$= \left[ \frac{N^2 - 1}{12} + \frac{n(n - N)}{2} \right] \Psi_n^s$$

$$- \left[ \frac{N - 1}{2} - n \right] h^{n+1} \frac{\partial}{\partial h} \left( \frac{\Psi_n^s}{h^n} \right) + \frac{1}{2} h^{n+2} \frac{\partial^2}{\partial h^2} \left( \frac{\Psi_n^s}{h^n} \right)$$

$$+ \frac{sh}{(s - h)^2} \Psi_n^s(\eta_1, \ldots, \eta_M | s). \quad (55)$$
By going back to $\uparrow$-spin coordinates $z$, Eq. (59) provides:

\[
\frac{n^2 - 2n + 1}{2} \Psi_n^s + \left( \frac{1}{2} - n \right) \frac{h + s}{h - s} \Psi_n^s \\
+ \left( \frac{3}{2} - n \right) h^{n+1} \frac{\partial}{\partial h} \left( \frac{\Psi_n^s}{h^n} \right) + \frac{hs}{(h-s)^2} \Psi_n^s \\
+ \frac{1}{2} \left( \frac{h + s}{h - s} \right) h^{n+1} \frac{\partial}{\partial h} \left( \frac{\Psi_n^s}{h^n} \right) + \frac{1}{2} h^{n+2} \frac{\partial^2}{\partial h^2} \left( \frac{\Psi_n^s}{h^n} \right) \\
- \sum_{z_j} \frac{1}{|z_j - h|^2} \Psi_n^s - \frac{sh}{(s-h)^2} \left( \frac{s}{h} \right)^{n-1} \Psi_n^h. \tag{56}
\]

Adding up all the contributions together, we obtain:

\[
h_KY \Psi_n^s(z_1, \ldots, z_M|h) = \\
\left\{ \begin{array}{l}
\left( \frac{-N^3 + 19N}{48} + \left( n - \frac{N}{2} - 1 \right) \right) \Psi_n^s \\
+ (M - 1)s \frac{\partial}{\partial s} \Psi_n^s - s^2 \frac{\partial^2}{\partial s^2} \Psi_n^s \\
+ \frac{1}{2} \frac{h + s}{h - s} \left( \frac{\partial}{\partial h} \Psi_n^s + (1-n)\Psi_n^s \right) \\
+ \frac{hs}{(h-s)^2} \left( \Psi_n^s - \left( \frac{s}{h} \right)^{n-1} \Psi_n^h \right)
\end{array} \right. \tag{57}
\]

In the next subsection we will will solve Eq. (57) by working out the basis of one-spinon one-holon states that diagonalize $h_KY$.

**B. One-spinon one-holon energy eigenstates.**

To diagonalize $h_KY$, we introduce the propagating one-spinon one-holon energy eigenstates

\[
\Psi_{mn}(z_1, \ldots, z_M|h) = \sum_s s^{-m} N \Psi_n^s(z_1, \ldots, z_M|h). \tag{58}
\]

The second and the third rows of Eq. (57) are diagonal in the basis of the states $\Psi_{mn}$, and their contribution is given by:

\[
\epsilon_0 + n(n-1) + m(M-m) |\Psi_{mn} \right. \tag{59}
\]

where:

\[
\epsilon_0 = -\frac{N^3 + 19N}{48}.
\]

On the other hand, the diagonalization of the “interaction” term, given by the fourth and the fifth rows of Eq. (57), needs further work. By using a straightforward, although tedious, application of basic identities proved in the Appendix of Ref. [9], one obtains:

\[
\sum_{s \in S_N} s^{-m} \left\{ \frac{1}{2} \left( \frac{h + s}{h - s} \right) \left( \frac{\partial}{\partial s} \Psi_n^s + (1-n)\Psi_n^s \right) \\
+ \frac{hs}{(h-s)^2} \left( \Psi_n^s - \left( \frac{s}{h} \right)^{n-1} \Psi_n^h \right) \right\}
\]

\[
= \sum_{s \neq h} s^{-m} \left\{ \frac{1}{2} \left( \frac{h + s}{h - s} \right) \left( \frac{\partial}{\partial s} \Psi_n^s + (1-n)\Psi_n^s \right) \\
+ \frac{hs}{(h-s)^2} \left( \Psi_n^s - \left( \frac{s}{h} \right)^{n-1} \Psi_n^h \right) \right\}
\]

\[
= \frac{-N}{2} (m - n + 1) \Psi_{mn} + \sum_{j=0}^{M-m} N(m - n + 1) \Psi_{m-j,n-j}, \tag{60}
\]

if $n - m - 1 > 0$ and

\[
\frac{-N}{2} (n - m - 1) \Psi_{mn} + \sum_{j=0}^{M-m} N(n - m - 1) \Psi_{m+j,n+j}, \tag{61}
\]

if $n - m - 1 \leq 0$. Therefore the action of $h_KY$ on $\Psi_{mn}$ is given by:

\[
[h_KY \Psi_{mn}] = \left[ \epsilon_0 + m(M-m) + (n-1 + \frac{N}{2}) \right] \Psi_{mn}
\]

\[
- \frac{1}{2} (n - m - 1) \Psi_{mn} - (n - m - 1) \sum_{j=1}^{m} \Psi_{m-j,n-j}, \tag{62}
\]

if $m - n + 1 < 0$, and

\[
[h_KY \Psi_{mn}] = \left[ \epsilon_0 + m(M-m) + (n-1 + \frac{N}{2}) \right] \Psi_{mn}
\]
if $m - n + 1 \geq 0$. Since the Hamiltonian matrix is upper (or lower) triangular, complete diagonalization is possible. The energy eigenstates will be linear combinations of $\Psi_{mn}$

$$\Phi_{mn} = \sum_{j=0}^{m} a_j \Psi_{m-j,n-j} ,$$

(64)

if $m - n + 1 < 0$, and:

$$\Phi_{mn} = \sum_{j=0}^{M-m} a_j \Psi_{m-j,n+j} ,$$

(65)

if $m - n + 1 \geq 0$, corresponding to the energy eigenvalues

$$E^+_{mn} = \frac{J}{2} \left( \frac{2\pi}{N} \right)^2 \left[ \epsilon_0 + m(M-m) + n(n - 1 - \frac{N}{2}) - \frac{1}{2}(n - m - 1) \right] ,$$

if $m - n + 1 < 0$, and:

$$E^-_{mn} = \frac{J}{2} \left( \frac{2\pi}{N} \right)^2 \left[ \epsilon_0 + m(M-m) + n(n - 1 - \frac{N}{2}) + \frac{1}{2}(n - m - 1) \right] ,$$

if $m - n + 1 \geq 0$. The coefficients $a_t$ are defined by the recursion relation:

$$a_t = \frac{1}{2t} \sum_{k=0}^{t-1} a_k \quad a_0 = 1 .$$

(66)

In terms of spinon and holon momenta, $E^+_{mn}$ and $E^-_{mn}$ take the same form $E_{mn}$, given by:

$$E_{mn} = E_{GS} + E(q^p_m) + E(q^h_n) - \frac{\pi J}{N} |q^p_m - q^h_n| \frac{1}{2} .$$

(67)

$E_{mn}$ is the sum of the ground-state energy, the energies of an isolated spinon and an isolated holon plus a negative interaction contribution that becomes negligibly small in the thermodynamic limit. Eqs. (64, 65) can be inverted. The result is:

$$\Psi_{mn} = \sum_{j=0}^{M-m} b_j \Phi_{m+j,n+j} ,$$

(69)

if $m - n + 1 \geq 0$. The coefficients are given by:

$$b_j = \frac{\Gamma[j + \frac{1}{2}]}{\Gamma[\frac{1}{2}] \Gamma[j + 1]} .$$

(70)

C. The Norm

The squared norm of the state $\Psi_{mn}$ is defined as:

$$\langle \Phi_{mn} | \Phi_{mn} \rangle = \sum_{z_1,\ldots,z_M} |\Phi_{mn}(z_1,\ldots,z_M)|^2 .$$

(71)

In a similar fashion to the two-spinon case discussed in [9], we compute the norm of the one-spinon one-holon states by means of mathematical induction. The calculation is presented in detail in Appendix C. The basic induction relation in the case $n - k + 1 < 0$ is given by:

$$\frac{\langle \Phi_{kn} | \Phi_{kn} \rangle}{\langle \Phi_{k-1,n} | \Phi_{k-1,n} \rangle} = \frac{(k - \frac{1}{2})(M - k + \frac{3}{2})}{k(M-k+1)} ,$$

(72)

which provides the formula for the norm of the one-spinon one-holon energy eigenstates:

$$\langle \Phi_{kn} | \Phi_{kn} \rangle = N^{M+1} \frac{(2M)!}{2^M} (M + \frac{1}{2}) \Gamma[M-k+1] \Gamma[k+\frac{3}{2}] .$$

(73)

In the complementary case, $n - k + 1 \geq 0$, we obtain:

$$\langle \Phi_{kn} | \Phi_{kn} \rangle = \langle \Phi_{M-k,M-n} | \Phi_{M-k,M-n} \rangle = N^{M+1} \frac{(2M)!}{2^M} (M + \frac{1}{2}) \Gamma[M-k+1] \Gamma[k+\frac{3}{2}] .$$

(74)

FIG. 3. Left panel: $|p_{mn}(e^{i\theta})|^2$ vs. $\theta$ for $m = M, n = 0$; Right panel: the same plot on a log-log scale. The dashed straight line is a plot of $1/\theta$.
VII. SPINON-HOLON ATTRACTION

In this Section we analyze the interaction between a spinon and a holon by constructing the real-space representation of the one-spinon one-holon wavefunction, and by studying the behavior of the corresponding probability as a function of the separation between the two particles. Our exact results show that a spinon and a holon interact through a short-range attraction identical, in the thermodynamic limit, to the attraction between two spinons.

The state for a localized spinon at site \( s \) and a localized holon at site \( h_0 \), \( \Psi_{sh_0} \) is defined as the Fourier transform of \( \Psi_s^n \) back to coordinate space:

\[
\Psi_{sh_0} = \sum_{n=1}^{M+2} h_0^{-n} \Psi_s^n .
\]

Following the same steps as for the two-spinon wavefunctions \((67)\), we define the real-space coordinate representation for a spinon-holon pair, \( s^m h_0^{-n} \Phi_{mn}(s/h_0) \), as follows:

\[
\begin{align*}
\Psi_{sh_0} &= \sum_{n=1}^{M+2} \sum_{m=0}^{n-2} s^m h_0^{-n} \sum_{j=0}^{m} b_j \Phi_{m-j,n-j} \\
&+ \sum_{n=1}^{M+2} \sum_{m=n-1}^{M} s^m h_0^{-n} \sum_{j=0}^{M-m} b_j \Phi_{m+j,n+j} \\
&= \sum_{n=1}^{M+2} \sum_{m=0}^{n-2} s^m h_0^{-n} \Phi_{mn}(s/h_0) + \sum_{n=1}^{M+2} \sum_{m=n-1}^{M} s^m h_0^{-n} \Phi_{mn}(s/h_0) ,
\end{align*}
\]

where \( |\Phi_{mn}\rangle \) is an eigenstate of \( H_{K\gamma} \) with eigenvalue \( E_{mn} \), that is:

\[
|\Phi_{mn}\rangle |H_{K\gamma}|\Psi_{sh_0}\rangle = \langle \Phi_{mn}|H_{K\gamma}|\Psi_{sh_0}\langle \rangle .
\]

The matrix element \( \langle \Phi_{mn}|H_{K\gamma}|\Psi_{sh_0}\rangle \) can be written as a differential operator acting on the analytic extension of \( \langle \Phi_{mn}|\Psi_{sh_0}\rangle \), where \( s \) and \( h_0 \) are understood to take any value on the unit circle. Therefore, by equating \( \langle \Phi_{mn}|H_{K\gamma}|\Psi_{sh_0}\rangle \) to \( E_{mn}\langle \Phi_{mn}|\Psi_{sh_0}\rangle \), it is straightforward to write down the equation of motion for the one-spinon one-holon wavefunction, which reads:

\[
(E_{mn} - E_{GS}) |\Phi_{mn}|\Psi_{sh_0}\rangle = \langle \Phi_{mn}|(H_{K\gamma} - E_{GS})|\Psi_{sh_0}\rangle = 0 .
\]

Following the same steps as for the two-spinon wavefunctions \((70)\), \((71)\), \((72)\), we find the following equations for the “relative wavefunctions” \( p_{mn}(z) \) and \( p_{mn}'(z) \):

\[
\begin{align*}
(2 - d/dz) - \frac{1}{(1 - z^2)} &\langle \Phi_{mn}|\Psi_{sh_0}\rangle = \langle \Phi_{mn}|\Psi_{sh_0}\rangle \bigg|_{s/h_0} \frac{\nu}{E_{mn} - E_{GS}} \langle \Phi_{mn}|\Psi_{sh_0}\rangle .
\end{align*}
\]

where \( \nu = M \) if \( m - n + 1 < 0 \), \( \nu = 0 \) otherwise. In the differential operator in Eq.\((75)\), we recognize the sum of the energies of the free spinon and holon, a velocity-dependent interaction, which diverges at small spinon-holon separation, and another term which takes into account the correction for the case when the spinon and the holon are at the same position. By using Eqs.\((67)\), \((68)\), \((72)\), \((73)\), \((74)\), \((75)\), we find the following equations for the “relative wavefunctions” \( p_{mn}(z) \) and \( p_{mn}'(z) \):

\[
\begin{align*}
2 - d/dz - \frac{1}{(1 - z^2)} &\langle \Phi_{mn}|\Psi_{sh_0}\rangle = \langle \Phi_{mn}|\Psi_{sh_0}\rangle ,
\end{align*}
\]

if \( m - n + 1 < 0 \), and

\[
\begin{align*}
2 - d/dz - \frac{1}{(1 - z^2)} &\langle \Phi_{mn}|\Psi_{sh_0}\rangle = \langle \Phi_{mn}|\Psi_{sh_0}\rangle ,
\end{align*}
\]

if \( m - n + 1 \geq 0 \).

Eqs.\((78)\), \((80)\) are first-order “Dirac-like” equations. They are first order because the spinon and the holon energy bands have opposite curvature. In this respect, they differ from the differential equation obtained in the two spinon case, which was second order \((66)\). The corresponding solutions are given by:

\[
p_{mn}(z) = \sum_{k=0}^{M-m-1} \frac{\Gamma[k + \frac{1}{2}]}{\sqrt{\pi} \Gamma[k + 1]} z^k ,
\]

for Eq.\((78)\) and

\[
p_{mn}'(z) = \sum_{k=0}^{m} \frac{\Gamma[k + \frac{3}{2}]}{\sqrt{\pi} \Gamma[k + 1]} \frac{1}{z} k^k ,
\]

for Eq.\((80)\).

The value of the spinon-holon wavefunction at zero separation between the particles is derived in Appendix D. It is given by:

\[
p_{mn}(1) = 2 \frac{\Gamma[M - m + \frac{3}{2}]}{\Gamma[\frac{1}{2}] \Gamma[M - m]} ,
\]

and

\[
p_{mn}'(1) = 2 \frac{\Gamma[m + \frac{3}{2}]}{\Gamma[\frac{1}{2}] \Gamma[m + 1]} ,
\]

Within the framework of our formalism, it is possible to treat spinons and holons, collective excitations of strongly-correlated one-dimensional electron systems, as actual quantum-mechanical particles. We were first
able to associate a two-particle wavefunction to a spinon-holon pair and to write down the corresponding equation of motion (Eqs. (62,64)). We then worked out the exact wavefunctions corresponding to each energy eigenvalue (Eqs. (63,65)).

The squared modulus for the spinon-holon wavefunction, $|p_{mn}(z)|^2$, gives the probability for a spinon and a holon configuration as a function of the separation between the two particles. In Fig. 3, we plot $|p_{mn}(e^{i\theta})|^2$ versus the distance between the spinon and the holon, $\theta$. From Fig. 3, the nature of the interaction between a spinon and a holon may be easily inferred. While at large separations $|p_{mn}(e^{i\theta})|^2$ does not depend on $\theta$, as it is appropriate for noninteracting particles, at small separations it shows a remarkable enhancement. This corresponds to a huge increase of the probability of configurations with the spinon and the holon on top of each other. The enhancement does not depend on the holon momentum. As $N$ gets larger, the probability enhancement peaks up. It survives the thermodynamic limit, even though the interaction energy goes to zero, and the total energy becomes the sum of the energies of the isolated spinon and holon. However, the attraction is not strong enough to create a spinon-holon bound state, even in the thermodynamic limit. This corresponds to the absence of a low-energy stable hole excitation, and it is what causes the quasiparticle peak to disappear.

An intriguing feature of the spinon-holon interaction in the thermodynamic limit is that it has the same power-law form as the spinon-spinon interaction derived in [3]. In the right panel of Fig. 3, we plot $|p_{mn}(e^{i\theta})|^2$ on a log-log scale and compare it with $1/\theta$. The probability falls off as the first power of the separation between the two particles. This shows that, although the equation of motion for a spinon and a holon is quite different from the one for two spinons, the interaction in both cases results in a short-range attraction, and its effects on the corresponding two-particle wavefunction are basically the same.

VIII. HOLE SPECTRAL FUNCTION

In this Section we work out $A_{n}^{sp,ho}(\omega, q)$, the one-spinon one-holon contribution to the hole spectral function $A_{n}(\omega, q)$. We show that this contribution (which provides quite a good approximation to $A_{n}(\omega, q)$ for $q \sim 0$) depends only on the $p_{mn}$’s and the $p_{mn}$’s calculated at $z = 1$ (that is, as the spinon and the holon lay at the same site). This allows us to obtain for any finite $N$ a simple closed-form expression for $A_{n}^{sp,ho}(\omega, q)$, and to relate it to the spinon-holon interaction. In the thermodynamic limit, we obtain the previously known formula for the contribution of the one-spinon one-holon states to $A_{n}^{sp,ho}(\omega, q)$ [14]. The formula in [14] shows that there is no low-energy hole pole in the hole spectral function, but rather a sharp square-root singularity followed by a branch cut. These features have also been experimentally detected by means of ARPES experiments on quasi 1D insulator [15].

The branch cut corresponds to the lack of integrity of the hole excitation, which breaks up into a spinon and a holon. Here we will show that, in the thermodynamic limit, the probability enhancement $p_{mn}(1)$ ($p_{mn}(1)$) turns into the square-root singularity at threshold for a spinon-holon pair. As a consequence, we prove that the square-root singularity in the hole spectral function is a direct consequence of the interaction between spinons and holons. Therefore, it can be directly experimentally measured.

We begin with the calculation of $A_{n}^{sp,ho}(\omega, q)$ for a finite lattice. In Lehman representation we obtain:

$$A_{n}^{sp,ho}(\omega, q) = \frac{3m}{\pi} \sum_{X} \frac{|\langle X | \sum_{h_{0}} \langle h_{0} | - k c_{h_{0}} | \Psi_{GS} \rangle |^{2}}{N(X|X) \langle \Psi_{GS} | \Psi_{GS} \rangle} \times \frac{1}{\omega + i\eta - (E_{X} - E_{GS})}, \quad (85)$$

where $|X \rangle$ is an exact one-spinon one-holon eigenstate of $H_{KY}$, $|\Phi_{mn} \rangle$ with energy $E_{X} = E_{mn}$, and $q = 2\pi k/N$ (below we discuss why only forward propagating states contribute to Eq. (85)).

Using $A_{n}^{sp,ho}(\omega, q)$ instead of $A_{n}(\omega, q)$ is equivalent to approximating

$$c_{h_{0}}^{+} |\Psi_{GS} \rangle \approx \langle \Psi_{GS} | |h_{0} \rangle = \sum_{n=1}^{M+2} \sum_{m=0}^{n-2} h_{0}^{m-n} p_{mn}(1) \Phi_{mn} + \sum_{n=1}^{M+2} \sum_{m=n-1}^{M} h_{0}^{m-n} p_{mn}(1) \Phi_{mn}. \quad (86)$$

(Eq. (86) basically amounts to neglecting contributions to $c_{h_{0}}^{+} |\Psi_{GS} \rangle$ coming from multi-spinon one-holon states.)

Since $H_{KY}$ contains the Gutzwiller projector $P$, its matrix elements between states with at least a doubly occupied site are zero. Therefore, at half-filling, $A_{n}(q, \omega)$ takes contributions only from forward-propagating hole states. Hence, using Eqs. (43,45) we obtain:

$$A_{n}^{sp,ho}(\omega, q) = \frac{3m}{\pi} \sum_{l=2}^{M+1} \sum_{m=0}^{l-2} \frac{\delta_{k-l+m+1}}{\omega + i\eta - (E_{mn} - E_{GS})} \langle \Phi_{ml} | \Psi_{ml} \rangle \langle \Psi_{GS} | \Psi_{GS} \rangle \quad (87)$$

Eq. (87) shows that only the $p_{mn}$’s at $z = 1$ determine the spinon-holon contribution to the hole spectral function. Therefore, the contribution is completely determined by the spinon-holon interaction.
Let us now analyze the thermodynamic limit of Eq. (87). In the thermodynamic limit, the gamma functions can be approximated by using Stirling’s formula

$$\Gamma[z] \approx \sqrt{\pi(z - 1)} z^{-\frac{1}{2}} e^{-z(1 - \frac{1}{2} \ln z)} . \quad (88)$$

From Eqs. (85, 88) we get, in the thermodynamic limit:

$$A_{\text{sp,ho}}(q, \omega) \approx \frac{2}{\pi(M + 1)} \times \left( \sum_{l=0}^{M+2} \sum_{m=0}^{l-2} \frac{\delta_{l-m+1}}{\omega + i\eta - (E_{m,n} - E_{\text{GS}})} \sqrt{\frac{M - m + \frac{1}{2}}{m}} + \sum_{l=0}^{M+2} \sum_{m=1}^{M} \frac{\delta_{l-m+1}}{\omega + i\eta - (E_{m,n} - E_{\text{GS}})} \sqrt{\frac{m + \frac{1}{2}}{M - m}} \right) . \quad (89)$$

(Notice that, in order to stabilize the hole occupation at 1, we had to introduce a chemical potential $J\pi^2/4$, which is added to the energies $E_{m,n}$.)

By defining the auxiliary variables:

$$q_{\text{sp}} = \frac{2\pi}{N} m , \quad q_{\text{ho}} = \frac{2\pi}{N} l ,$$

Eq. (89), in the thermodynamic limit, may be written in the form already obtained in [14]:

$$A_{\text{sp,ho}}(\omega, q) = 23m \int_0^\pi \frac{dq_{\text{ho}}}{\pi} \left( \int_0^{q_{\text{ho}}} \frac{dq_{\text{sp}}}{\pi} \sqrt{\frac{\pi - q_{\text{sp}}}{q_{\text{sp}}}} + \right) \delta(q - q_{\text{sp}} + q_{\text{ho}}) \left( \omega - \mu + i\eta - E(q_{\text{sp}}, q_{\text{ho}}) \right) \frac{dq_{\text{sp}}}{\pi} . \quad (90)$$

In the region $0 \leq q \leq \pi$ the integration of Eq. (90) gives:

$$A_{\text{sp,ho}}(\omega, q) = \frac{1}{\pi^2 J} \sqrt{J(q + \frac{\pi}{2})^2 - \omega} \Theta \left[ J\frac{\pi^2}{4} + q(\pi - q) - \omega \right] . \quad (91)$$

This formula shows that the spinon-holon probability enhancement $p_{\text{sp,ho}}^{\text{enh}}(1)$ turned into a square-root singularity in $A_{\text{sp,ho}}(\omega, q)$ at the threshold energy for creation of a spinon-holon pair. Because the spinon-holon joint density of states is uniform, the main conclusion we trace from our calculation is that the sharp nonanalytic threshold in $A_{\text{sp,ho}}(\omega, q)$ is the direct consequence of spinon-holon interaction.

**IX. CONCLUSIONS**

In this paper, we have extended the formalism introduced in [3] to analyze spinon interaction in the Haldane-Shastry model, to the case where also charge degrees of freedom are involved. Our formalism allows us to define a quantum-mechanical real-space representation of the one-spinon one-holon wavefunction. We construct a Dirac-like equation, whose solution is the spinon-holon wavefunction in real space coordinates. By means of a careful study of the real-space one spinon one holon wavefunction, we show the existence of the spinon-holon interaction and its survival in the thermodynamic limit. Spinon-holon interaction generates a short-range enhancement in the probability for a spinon and a holon to be on the same site. The attraction, however, is not strong enough to form a spinon-holon bound state, which would correspond to a Landau’s quasihole resonance. This makes, in the thermodynamic limit, the hole excitation fully unstable against decay in one-holon multi-spinon states and the quasiparticle peak disappear. Correspondingly, in the thermodynamic limit, the probability enhancement develops into a square root singularity followed by a branch cut, which reflects the full instability of the hole excitation. Hence, by means of a sequence of exact, straightforward steps, we prove that spinon-holon attraction is what makes Landau’s Fermi liquid theory break down in 1-d strongly correlated electron systems.

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**APPENDIX A: FUNCTIONS EXPRESSED IN TERMS OF ↓-SPIN COORDINATES.**

In this Section we prove the formulas that express the states of the KY-model in terms of the ↓-spin coordinates, once the expression in terms of the ↑-spin coordinates is known. The starting point is the following identity $(z_\alpha, z_\beta$ are $N - th$ roots of the identity):

$$\prod_{\alpha: z_\alpha \neq z_\beta} (z_\alpha - z_\beta) = \lim_{z \to z_\beta} \frac{z^N - 1}{z - z_\beta} = \frac{N}{z_\beta} . \quad (A1)$$
The ground state wavefunction at filling-1/2 expressed in terms of the \( \uparrow \)-spin coordinates is given by:

\[
\Psi_{GS}(z_1, \ldots, z_M) = \prod_{i<j}^M (z_i - z_j)^2 \prod_{t}^M z_t , 
\]  
(A2)

where \( N \) is even and \( M = N/2 \). Let \( \eta_1, \ldots, \eta_M \) be the \( \downarrow \)-spin coordinates. Upon applying Eq. (A1), we get:

\[
\prod_{i<j}^M (z_i - z_j)^2 \prod_{t}^M z_t = (-1)^{i<j} \prod_{i\neq j}^M (z_i - z_j) \prod_{t}^M z_t =
\]

\[
(-1)^{i<j} \prod_{i\neq j}^M (z_i - z_j) \prod_{t}^M z_t = \prod_{i<j}^M (\eta_i - \eta_j)^2 \prod_{t}^M \eta_t , 
\]  
(A3)

which proves that \( \Psi_{GS}(z_1, \ldots, z_M) = \Psi_{GS}(\eta_1, \ldots, \eta_M) \).

The one-holon wavefunction is given by:

\[
\Psi^h_n(z_1, \ldots, z_M) = h^n \prod_{i<j}^M (\eta_i - \eta_j)^2 \prod_{t}^M \eta_t , 
\]  
(A4)

where now \( N \) is odd, \( M = (N - 1)/2 \) and \( h \) is the coordinate of the empty site.

The same steps as for \( \Psi_{GS} \) apply to the one-holon wavefunction. We have:

\[
h^n \prod_{j}^M (\eta_i - h)^2 \prod_{t}^M \eta_t =
\]

\[
h^n (-1)^{i<j} \prod_{i\neq j}^M (\eta_i - h) \prod_{t}^M \eta_t =
\]

\[
h^n \prod_{i<j}^M (\eta_i - h)^2 \prod_{t}^M \eta_t ; 
\]  
(A5)

this proves that \( \Psi^h_n(z_1, \ldots, z_M|h) = \Psi^h_n(\eta_1, \ldots, \eta_M|h) \).

The one-spinon one-holon state \( \Psi^s_n(z_1, \ldots, z_M|h) \) is given by:

\[
\Psi^s_n(z_1, \ldots, z_M|h) =
\]

\[
h^n \prod_{j}^M (z_j - h)(z_j - h)^2 \prod_{t}^M z_t , 
\]  
(A6)

where \( N \) is even, \( M = N/2 - 1 \), \( s \) is the coordinate of the \( \downarrow \)-spin and \( h \) is the location of the empty site. As in the previous cases, we have:

\[
h^n \prod_{j}^M (z_j - s)(z_j - h) \prod_{t}^M (z_t - j)^2 \prod_{t}^M z_t
\]

\[
= h^n (-1)^{i<j} \prod_{i\neq j}^M (\eta_i - j) \prod_{t}^M (\eta_i - z_t)
\]

\[
= h^n \prod_{i<j}^M (\eta_i - s)(\eta_i - h) \prod_{t}^M (\eta_i - \eta_j)^2 \prod_{t}^M \eta_t . 
\]  
(A7)

Eq. (A7) provides the proof that \( \Psi^s_n(z_1, \ldots, z_M|h) = \Psi^s_n(\eta_1, \ldots, \eta_M|h) \).

The last identity we need refers to the case where the spinon and holon are at the same site, which we had to consider in deriving Eq. (B6). We have:

\[
\Psi^s_n(z_1, \ldots, z_M|s) = s^n \prod_{j}^M (z_j - s)^2 \prod_{i<j}^M (z_i - z_j)^2 \prod_{t}^M z_t
\]

\[
= s^n (-1)^{i<j} \prod_{i\neq j}^M (z_i - s)(\eta_i - j) \prod_{t}^M (\eta_i - \eta_t)
\]

\[
= s^n \prod_{i<j}^M (z_i - s)(\eta_i - j) \prod_{t}^M (\eta_i - \eta_t) \prod_{t}^M (\eta_i - \eta_j)
\]

\[
\times \prod_{i<j}^M (\eta_i - \eta_j)^2 \prod_{t}^M (\eta_i - s)(\eta_i - h)
\]

\[
= -(s/h)^{n-1} h^n \prod_{i}^M (\eta_i - h)^2 \prod_{i<j}^M (\eta_i - \eta_j)^2 \prod_{t}^M \eta_t . 
\]  
(A8)

Eq. (A8) proves the identity

\[
\Psi^s_n(z_1, \ldots, z_M|s) = -(s/h)^{n-1} \Psi^s_n(\eta_1, \ldots, \eta_M|h) . 
\]

**APPENDIX B: THE NORM OF ONE-HOLON WAVEFUNCTION.**

In this Appendix we discuss in detail the calculation of the norm of the negative-energy one-holon states, \( \langle \Psi^h_n | \Psi^h_n \rangle \).

\[
\langle \Psi^h_n | \Psi^h_n \rangle = \sum_{z_1, \ldots, z_M} \langle \Psi^h_n(z_1, \ldots, z_M|h) \rangle^2 = P_z^m . 
\]  
(B1)

Let \( P_m(z_1, \ldots, z_M) \) be the \( m-th \) degree symmetric polynomial in \( z_1, \ldots, z_M \). One obtains [3]:

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\( \Psi_{n}^{\text{ho}}(z_1, \ldots, z_M | h) \)

\[
= \sum_{m=0}^{M} h^{m+n} P_{m}(z_1, \ldots, z_M) \prod_{i<j} (z_i - z_j)^{2}. \tag{B2}
\]

From Eqs. (B1, B2), we derive:

\[
\langle \Psi_n^{\text{ho}} | \Psi_n^{\text{ho}} \rangle = N \sum_{m=0}^{M} \sum_{z_1, \ldots, z_M} |P_{m}(z_1, \ldots, z_M)|^{2} \prod_{i<j} (z_i - z_j)^{4}
= N \sum_{m=0}^{M} \langle \Psi_m^{\text{sp}} | \Psi_m^{\text{sp}} \rangle. \tag{B3}
\]

The norm of one-spinon wavefunctions, \( \langle \Psi_m^{\text{sp}} | \Psi_m^{\text{sp}} \rangle \), has been derived in Ref. [9], where it has been shown that:

\[
\langle \Psi_m^{\text{sp}} | \Psi_m^{\text{sp}} \rangle = \frac{\Gamma[M + 1]}{\Gamma[M + \frac{1}{2}]} \prod_{m=0}^{M} \Gamma[m + \frac{1}{2}] \prod_{i<j} (z_i - z_j)^{4}. \tag{B4}
\]

Therefore, we can re-write Eq. (B3) as:

\[
\langle \Psi_n^{\text{ho}} | \Psi_n^{\text{ho}} \rangle = \Gamma[M + 1] \prod_{m=0}^{M} \prod_{i<j} (z_i - z_j)^{4}. \tag{B5}
\]

where the polynomial \( g_{mn}(z) \) is the two-spinon relative wavefunction, denoted by \( p_{mn}(z) \) in Ref. [9]:

\[
g_{mn}(z) = \frac{\prod_{m=0}^{M} \prod_{i<j} (z_i - z_j)^{4}}{\Gamma[m + \frac{1}{2}] \Gamma[m - n + 1/2]}. \tag{B6}
\]

By means of generic properties of hypergeometric functions \([10]\), one gets:

\[
g_{mn}(1) = \frac{\prod_{m=0}^{M} \prod_{i<j} (z_i - z_j)^{4}}{\Gamma[m + \frac{1}{2}] \Gamma[m - n + 1/2]}. \tag{B7}
\]

Therefore, we obtain:

\[
\langle \Psi_n^{\text{ho}} | \Psi_n^{\text{ho}} \rangle = N^{M+1} \frac{(2M)! \prod_{m=0}^{M} \prod_{i<j} (z_i - z_j)^{4}}{\prod_{m=0}^{M} \prod_{i<j} (z_i - z_j)^{4}}; \forall n. \tag{B8}
\]

**APPENDIX C: THE NORM OF ONE-SPINON ONE-HOLON ENERGY EIGENSTATES.**

In this section we generalize the recursion procedure introduced in Ref. [8,9] to calculate the norm of one-spinon and two-spinon wavefunction to the calculation of the norm of one-spinon one-holon energy eigenstates. As in the calculation in Refs. [8,9], the key operator is given by \( e_1 \) (z_1, \ldots, z_M), defined as:

\[
e_1(z_1, \ldots, z_M) = z_1 + \ldots + z_M. \tag{C1}
\]

The state for one holon and one spinon localized at \( s \) is given by:

\[
\Psi_{s}^{n}(z_1, \ldots, z_M) = \Phi_{s}^{n}(z_1, \ldots, z_M | h) \Psi_{GS}(z_1, \ldots, z_M), \tag{C2}
\]

where:

\[
\Phi_{s}^{n}(z_1, \ldots, z_M | h) = h^{n} \prod_{j} (z_j - s)(z_j - h). \tag{C3}
\]

On \( \Psi_{s}^{n}(z_1, \ldots, z_M) \), \( e_1 \) acts as a ladder operator, as we are going to show next.

In order to work out the action of \( H_{KY} \) on holon eigenstate, we splitted it into five terms: \( H_{KY} = \frac{1}{2} \left( \frac{\Psi_{s}^{n}}{h^{n}} \right)^{2} = h_{S}^{2} + h_{N} + h_{Q} + h_{Q}^{2}. \) Among those five terms, the only ones that do not commute with \( e_1 \) are the spin exchange operator \( h_{S}^{2} \) and the \( \uparrow \)-spin charge propagation operator \( h_{Q}^{2}. \) On the state \( \Psi_{s}^{n} \), \( h_{S}^{2} \) is realized as:

\[
h_{S}^{2} = \frac{1}{2} \sum_{i} z_{i}^{2} \frac{\partial^{2}}{\partial z_{i}^{2}} + 2 \sum_{i \neq j} \frac{z_{i}^{2} \partial}{\partial z_{i}} - N \frac{3}{2} \sum_{i} z_{i}^{2} \frac{\partial}{\partial z_{i}} \Phi_{s}^{n}. \tag{C4}
\]

From Eq. (C3), we derive that:

\[
[h_{S}^{2} e_1 | \Psi_{GS}] = e_1 \Phi_{s}^{n} h_{S}^{2} | \Psi_{GS}]
= \Psi_{GS} \left\{ \left( M - \frac{3}{2} \right) e_1 \Phi_{s}^{n} + \sum_{i} z_{i}^{2} \frac{\partial}{\partial z_{i}} \Phi_{s}^{n} \right\} \tag{C5}
= \Psi_{GS} \left\{ (M + \frac{1}{2}) e_1 \Phi_{s}^{n} + M(s + h) \Phi_{s}^{n} - s^{2} \frac{\partial}{\partial s} \Phi_{s}^{n} - h^{n+2} \frac{\partial}{\partial h} \left( \Phi_{s}^{n} \right) h^{n} \right\}. \tag{C6}
\]

In the sense clarified by Eq. (C4), \( e_1 \) commutes with \( h_{S}^{2}, h_{N} \) and \( h_{Q}^{2}. \) On the other hand, it does not commute with \( h_{Q}^{2}. \) Indeed, since, in order to derive the action of \( h_{Q}^{2} \) on \( \Psi_{s}^{n} \) we have to express the state in terms of the \( \downarrow \)-spin coordinates, we must do the same with \( e_1. \) In order to do so, we notice that \( \{ z_j \}, \{ \eta_j \}, h \) and \( s \) taken all together.
are the set of the $N$ $n$-th roots of 1. Therefore, we obtain:

$$z_1 + \ldots + z_M + \eta_1 + \ldots + \eta_M + h + s = 0,$$

which implies:

$$e_1(z_1, \ldots, z_M) = -e_1(\eta_1, \ldots, \eta_M) - h - s. \quad (C5)$$

The action of $h_Q^\dagger$ on $e_1 \Psi^n_s$ gives:

$$[h_Q^\dagger e_1] \Psi^n_s = \Psi_{GS} \left\{ \left[ \frac{N^2 - 1}{12} + \frac{n(n - N)}{2} \right] e_1 \Phi_s^n(\{\eta\}|h) - \left[ \frac{N - 1}{2} - n \right] h^{n+1} \frac{\partial}{\partial h} \left[ (-h - s - \sum_j \eta_j) \frac{\Phi_s^n(\{\eta\}|h)}{h^n} \right] \right\}$$

$$+ \frac{1}{2} h^{n+2} \frac{\partial^2}{\partial h^2} \left[ (-h - s - \sum_j \eta_j) \frac{\Phi_s^n(\{\eta\}|h)}{h^n} \right]$$

$$+ \frac{sh}{(s - h)^2} \left\{ - \sum_j \eta_j + 2s \right\} \Phi_s^n(\{\eta\}|s) \right\} = \Psi_{GS} e_1 \left[ h_Q^\dagger \Phi_s^n(\{\eta\}|h) \right]$$

$$+ \left[ \frac{N - 1}{2} - n \right] h \Phi_s^n(\eta_1, \ldots, \eta_M|h)$$

$$- h^{n+2} \frac{\partial}{\partial h} \left( \frac{\Phi_s^n(\eta_1, \ldots, \eta_M|h)}{h^n} \right)$$

$$+ \frac{h}{h - s} \Phi_s^{n+1}(\eta_1, \ldots, \eta_M|s). \quad (C6)$$

Eq.\,(C6) yields the result:

$$[h_Q^\dagger, e_1] \Psi^n_s =$$

$$\Psi_{GS} \left\{ h \frac{h}{h - s} \left[ \Phi_s^n(z_1, \ldots, z_M|h) - \left( \frac{s}{h} \right)^n \Phi_s^n(z_1, \ldots, z_M|h) \right] \right\}$$

$$+ h^{n+2} \frac{\partial}{\partial h} \left( \frac{\Phi_s^n(z_1, \ldots, z_M|h)}{h^n} \right) - nh \Phi_s^n(z_1, \ldots, z_M|h). \quad (C7)$$

Upon summing Eq.\,(C6) and Eq.\,(C7), we derive the basic relation we need in order to work out the recursion relations:

$$\Psi_{NS} \left\{ \left[ \frac{H_{KY}}{2 \left( \frac{2\pi}{N} \right)^2} \right] e_1 \right\} \Psi^n_s = (M + \frac{1}{2}) e_1 \Psi^n_s + M(s + h) \Psi^n_s$$

$$- s^2 \frac{\partial}{\partial s} \Psi^n_s + h \frac{h}{h - s} \sum_{m=0}^M \left[ s^m - \left( \frac{s}{h} \right)^n h^m \right] \Psi_{mn}$$

$$- nh \Psi^n_s, \quad (C8)$$

where we have introduced the one-spinon one-holon plane waves $\Psi_{mn}$ defined in Section VI-B.

In order to further manipulate term in Eq.\,(C8), let us consider, now, the identity:

$$h \frac{h}{h - s} \sum_s \frac{s^{k}}{N} \sum_{m=0}^M \left[ s^m - \left( \frac{s}{h} \right)^n h^m \right] \Psi_{mn}$$

$$= \sum_{m=0}^M \Psi_{mn} h^{1+m-k} \sum_{\{s/h\}} \left[ \left( \frac{s}{h} \right)^{n-k} - \left( \frac{s}{h} \right)^{m-k} \right]$$

$$= \frac{1}{N} \sum_{m=0}^N \Psi_{mn} h^{1+m-k} \sum_{\{s/h\}} \sum_{r=0}^{n-m-1} \left( \frac{s}{h} \right)^{n+r-k}$$

$$- \frac{1}{N} \sum_{m=n+1}^M \Psi_{mn} h^{1+m-k} \sum_{\{s/h\}} \sum_{r=0}^{n-m-1} \left( \frac{s}{h} \right)^{n+r-k}. \quad (C9)$$

(The symbol $\sum_{\{s/h\}}$ means that we have to sum over $s/h \in S^N$).

Suppose, now, $n - k + 1 < 0$. In this case, the sum in Eq.\,(C9) over $m$ from $n + 1$ to $M$ will give 0, while the second sum will be reduced to:

$$\sum_{m=0}^k \Psi_{mn} h^{1+m-k} \sum_{\{s/h\}} \sum_{r=0}^{n-m-1} \left( \frac{s}{h} \right)^{n+r-k}$$

On the other hand, as $k - n + 1 \geq 0$, the only nonzero term in the sum will be given by:

$$\sum_{m=0}^M \Psi_{mn} h^{1+m-k} \sum_{\{s/h\}} \sum_{r=0}^{n-m-1} \left( \frac{s}{h} \right)^{n-k+r}.$$
\[ + (M - k + 1) \Psi_{k-1,n} + (M - n + 1) h \Psi_{kn} \]
\[ + \sum_{m=0}^{k-1} \Psi_{mn} \frac{\hbar^{1+m-k}}{N} \sum_{\{s/h\}} \sum_{r=0}^{n-m-1} \left( \frac{s}{\hbar} \right)^{m-k+r} . \]  
(C10)

- If \( k - n + 1 \geq 0 \):

\[ \left[ \frac{\hbar}{2(\frac{2\pi}{\hbar})^2} \right]_{1} \Psi_{kn} = (M + \frac{1}{2}) e_{1} \Psi_{kn} \]
\[ + (M - k + 1) \Psi_{k-1,n} + (M - n) h \Psi_{kn} \]
\[ - \sum_{m=k+1}^{M} \Psi_{mn} \frac{\hbar^{1+m-k}}{N} \sum_{\{s/h\}} \sum_{r=0}^{n-m-1} \left( \frac{s}{\hbar} \right)^{n-k+r} . \]  
(C11)

Let us now work out the basic recursion relation in both cases.

- Case \( k - n + 1 < 0 \).

In this case energy eigenstates are given by:

\[ \Phi_{kn} = \sum_{\ell=0}^{k} a_{\ell} \Phi_{k-\ell,n-\ell} . \]

(see Section VI-B for the definition of the coefficients \( a_{\ell} \)).

Therefore, we have:

\[ \left[ \frac{\hbar}{2(\frac{2\pi}{\hbar})^2} \right]_{1} \Phi_{kn} = \sum_{\ell=0}^{k} a_{\ell} \left[ \frac{\hbar}{2(\frac{2\pi}{\hbar})^2} \right]_{1} \Phi_{k-\ell,n-\ell} \]
\[ = \sum_{\ell=0}^{k} a_{\ell} (M + \frac{1}{2}) e_{1} \Phi_{k-\ell,n-\ell} \]
\[ + \sum_{\ell=0}^{k-1} a_{\ell} (M - k + \ell + 1) \Phi_{k-1,n-\ell} \]
\[ + \sum_{\ell=0}^{k} a_{\ell} (M - n + \ell + 1) \Phi_{k,n-\ell+1} \]
\[ + \sum_{\ell=0}^{k-1} \sum_{m=0}^{k-\ell-1} \Psi_{m,n+1+m-k} . \]

From Eq.(C12), we get:

\[ (E_{k-1,n} - E_{kn} - M - \frac{1}{2}) \langle \Phi_{k-1,n} | e_{1} | \Phi_{kn} \rangle \]
\[ = \sum_{\ell=0}^{k-1} a_{\ell} (M - k + \ell + 1) \langle \Phi_{k-1,n} | \Phi_{k-\ell,n-\ell} \rangle + \sum_{\ell=0}^{k} a_{\ell} (M - n + \ell + 1) \langle \Phi_{k-1,n} | \Phi_{k-n+\ell+1} \rangle \]
\[ = [M - k + 2 + b_{1}(M - n + 1) + a_{1}(M - n + 2)] \langle \Phi_{k-1,n} | \Phi_{k-1,n} \rangle \]
\[ = \langle M - k + \frac{3}{2} \rangle \langle \Phi_{k-1,n} | \Phi_{k-1,n} \rangle . \]  
(C13)

(notice that, from their definition, we have \( a_{1} = -1/2 \), \( b_{1} = 1/2 \)). Eq.(C13) may be recast in the following compact form:

\[ \frac{\langle \Phi_{k-1,n} | e_{1} | \Phi_{kn} \rangle}{\langle \Phi_{k-1,n} | \Phi_{k-1,n} \rangle} = \frac{M - k + \frac{3}{2}}{2(M - k + 1)} \]  
(C14)

- Case \( k - n + 1 \geq 0 \).

In this case we have:

\[ \Phi_{kn} = \sum_{\ell=0}^{M-k} a_{\ell} \Phi_{k+\ell,n+\ell} . \]

Eq.(C12) now takes the form:

\[ \left[ \frac{\hbar}{2(\frac{2\pi}{\hbar})^2} \right]_{1} \Phi_{kn} \]
\[ = \sum_{\ell=0}^{M-k} a_{\ell} (M + \frac{1}{2}) e_{1} \Phi_{k+\ell,n+\ell} \]
\[ + \sum_{\ell=0}^{M-k} a_{\ell} (M - k - \ell + 1) \Phi_{k+\ell-1,n+\ell} \]
\[ + \sum_{\ell=0}^{M-k} \sum_{m=0}^{M-k-\ell} \Psi_{m,n+1+m-k} . \]

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Eq. (C17), we obtain:

\[ -\sum_{\ell=0}^{M-k} a_\ell \sum_{m=k+\ell+1}^{M} \Psi_{m,n+1+m-k} \]

\[ \times \frac{1}{N} \sum_{\{s,h\}} \left[ \frac{m-n-1}{s!h!} \left( \frac{s}{N} \right)^{n-k+r} \right] . \quad (C15) \]

From Eq. (C15) and by working exactly as in the previous case, we get the identity:

\[ (E_{k-1,n} - E_{kn} - M - \frac{1}{2}) (\Phi_{k-1,n}|e_1|\Phi_{kn}) \]

\[ = (M - k + 1) (\Phi_{k-1,n}|e_1|\Phi_{kn}) , \quad (C16) \]

which yields the relation:

\[ \frac{\langle \Phi_{k-1,n}|e_1|\Phi_{kn} \rangle}{\langle \Phi_{k-1,n}|\Phi_{kn} \rangle} = -\frac{M - k + 1}{2(M - k + \frac{1}{2})} . \quad (C17) \]

In order to complete the recursion procedure, we need one more induction relation that is derived by inserting in the product \( \langle \Phi_{ab}|e_1|\Phi_{cd} \rangle \) the product \( e_M e_M^* \) = 1, where:

\[ e_M(z_1, \ldots, z_M) = z_1 \cdot \ldots \cdot z_M . \]

From the definition of scalar product, Eq. (25), it is straightforward to show that:

\[ \langle \Phi_{ab}|e_1|\Phi_{cd} \rangle = \langle (e_M)^2 \Phi_{ab}|e_1|(e_M)^2 \Phi_{cd} \rangle \]

\[ = \langle \Phi_{M-c,M-d}|e_1|\Phi_{M-a,M-b} \rangle . \quad (C18) \]

By applying Eq. (C18) to Eq. (C13) and by using Eq. (C17), we obtain:

\[ \langle \Phi_{k-1,n}|e_1|\Phi_{kn} \rangle = \langle \Phi_{M-k,M-n}|e_1|\Phi_{M-k+1,M-n} \rangle \]

\[ = -\frac{k}{2(k - \frac{1}{2})} \langle \Phi_{kn}|\Phi_{kn} \rangle . \quad (C19) \]

By putting together Eq. (C15) and Eq. (C19) one finally obtains:

\[ \frac{\langle \Phi_{kn}|\Phi_{kn} \rangle}{\langle \Phi_{k-1,n}|\Phi_{kn} \rangle} = \frac{(k - \frac{1}{2})(M - k + \frac{3}{2})}{k(M - k + 1)} , \quad (C20) \]

that provides the formula for the norm of the one-spinon one-holon energy eigenstates in the case \( k - n + 1 < 0 \):

\[ \langle \Phi_{kn}|\Phi_{kn} \rangle = N^{M+1} \frac{(2M)!}{2^M} (M + \frac{1}{2})^{\Gamma[M - k + \frac{1}{2}] / \Gamma[M - k + 1] \Gamma[k + \frac{1}{2}]} \] . \quad (C21)

In the complementary case, \( k - n + 1 \geq 0 \), we may follow the same step to prove that:

\[ \langle \Phi_{kn}|\Phi_{kn} \rangle = \langle \Phi_{M-k,M-n}|\Phi_{M-k,M-n} \rangle \]

\[ = N^{M+1} \frac{(2M)!}{2^M} (M + \frac{1}{2})^{\Gamma[M - k + \frac{1}{2}] / \Gamma[M - k + 1] \Gamma[k + \frac{1}{2}]} . \quad (C22) \]

**APPENDIX D: SOLUTION OF THE EQUATION OF MOTION FOR THE ONE-SPINON ONE-HOLON WAVEFUNCTION**

In this Appendix we will derive the solution to the equation of motion for the relative coordinate part of the spinon-holon wavefunctions, \( p_{mn}(z) \). In order to do so, let us consider first the case \( m - n + 1 < 0 \), in which we express the solution to Eq. (79) as a power series of \( z \):

\[ p_{mn}(z) = \sum_{k} a_k z^k . \quad (D1) \]

From Eq. (D1), we get the following equation for the coefficients \( a_k \):

\[ -2 \sum_{k} k a_{k+1} z^k + \sum_{k} (2k - 1) a_k z^k - z^{M-m} \sum_{k} a_k = 0 . \quad (D2) \]

As \( k \leq M - m \), the following recursion relation between the \( a_k \)’s holds:

\[ a_{k+1} = \frac{k + \frac{1}{2}}{k + 1} . \quad (D3) \]

Eq. (D2) is satisfied by:

\[ a_k = \frac{\Gamma[k + \frac{1}{2}]}{\Gamma[k + 1]} . \quad (D4) \]

To calculate \( a_{M-m+1} \), we need the following identity, valid for any positive integer \( R \) (\( C_0 \) is a closed path centered at \( z = 0 \)):

\[ \sum_{k=0}^{R} \frac{\Gamma[k + \frac{1}{2}]}{\Gamma[k + 1]} = \sum_{k=0}^{R} \oint_{C_0} \frac{dz}{2\pi i z^{k+1}} \frac{1}{\sqrt{1-z}} = \]

\[ \oint_{C_0} \frac{dz}{2\pi i z^{R+1}} \frac{1}{\sqrt{1-z}} = 2 \frac{\Gamma[R + \frac{1}{2}]}{\Gamma[R + 1]} (R + \frac{1}{2}) . \quad (D5) \]

Since the recursion relation for \( a_{M-m} \) is:

\[ -2(M - m) a_{M-m} + 2(M - m - \frac{1}{2}) a_{M-m-1} - \sum_{k} a_k = 0 , \quad (D6) \]

Eq. (D2) implies \( a_{M-m} = 0 \) and:

\[ p_{mn}(z) = \sum_{k=0}^{M-m-1} \frac{\Gamma[k + \frac{1}{2}]}{\Gamma[k + 1]} z^k . \quad (D7) \]
Eq. (D7) provides the formula for $p_{mn}(z)$ used throughout the paper.

The same steps followed in the case $m - n + 1 < 0$ allow us to find the spinon-holon wavefunction in the case $m - n + 1 \geq 0$, provided the coordinate $z$ is substituted with $\xi = 1/z$.

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