SCALAR CURVATURE RIGIDITY FOR ASYMPTOTICALLY LOCALLY HYPERBOLIC MANIFOLDS

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Abstract. Rigidity results for asymptotically locally hyperbolic manifolds with lower bounds on scalar curvature are proved using spinor methods related to the Witten proof of the positive mass theorem. The argument is based on a study of the Dirac operator defined with respect to the Killing connection. The existence of asymptotic Killing spinors is related to the spin structure on the end. The expression for the mass is calculated and proven to vanish for conformally compact Einstein manifolds with conformal boundary a spherical space form, giving rigidity. In the 4-dimensional case, the signature of the manifold is related to the spin structure on the end and explicit formulas for the relevant invariants are given.

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1. Introduction

Let \((M, g)\) be a complete spin manifold of dimension \(n \geq 3\). We say that \((M, g)\) is asymptotically locally hyperbolic (ALH) if \((M, g)\) has an end \(E\) where the metric is asymptotic to \(\mathbb{H}^n/\Gamma\) where \(\mathbb{H}^n\) is hyperbolic space and \(\Gamma\) is a finite group acting by isometries, see section \(\ref{sec:ALH}\). In particular the sectional curvatures are asymptotic to \(-1\) on \(E\). In this paper we will prove some rigidity results for ALH manifolds whose scalar curvature \(s\) satisfies \(s \geq -n(n-1)\). The notion of an ALH manifold is analogous to the notion of an asymptotically locally Euclidean (ALE) manifold, a class of manifolds which has been extensively studied.

In \cite{Min-Oo} Min-Oo proved a scalar curvature rigidity result for manifolds asymptotic to hyperbolic space. The argument was an adaptation of the Witten proof of the positive mass theorem. The asymptotically hyperbolic case is related to the proof of positivity of the Bondi mass.

From the point of view of Riemannian geometry the positive mass theorem may be viewed as a statement concerning asymptotically Euclidean 3-manifolds with nonnegative scalar curvature. For dimension higher than 3, we have the generalized positive action conjecture, for ALE manifolds with nonnegative scalar curvature, which was shown to be false by LeBrun \cite{LeBrun}. The positive mass argument fails when the spin structure does not allow asymptotically parallel spinors. Even given the existence of asymptotically parallel spinors the conclusion drawn from the positive mass argument in the ALE case is in general just a restriction on holonomy, see however \cite{LeBrun} for a complete discussion of the 4-dimensional case.

In the ALH case, it is natural to consider a modified connection which is such that its curvature vanishes on hyperbolic space. The corresponding Dirac operator and Lichnerowicz identities can then be used in the positive mass argument in much the same way as in the Witten proof, the relevant mass being related to the Bondi mass.

There are different ways to modify the Levi-Civita connection so that it becomes adapted to hyperbolic space. In \cite{Min-Oo} a connection on an extended spinor bundle was studied. This setup models the manifold as an umbilic hypersurface in a Minkowski space of dimension \(n + 1\). Alternatively one may introduce a modified connection \(\tilde{\nabla}\) defined by \(\tilde{\nabla}_X \psi = \nabla_X \psi + \frac{i}{2} X \psi\) on the usual spinor bundle, which has as parallel spinors the imaginary Killing spinors. This is the approach taken in the present paper.

An important difference between Killing spinors and parallel spinors is that the existence of an imaginary Killing spinor allows one to apply the powerful structure theorem of H. Baum, see Theorem \(\ref{thm:Baum}\) and in contrast to the ALE case, instead of restrictions on the holonomy we get rigidity.

If \(M\) is ALH or ALE, then \(M\) has an end \(E\) diffeomorphic to \(\mathbb{R} \times (S^{n-1}/\Gamma)\). The existence of asymptotically parallel or asymptotically Killing spinors is a topological condition on the spin structure on \(E\). When \(M\) is 4-dimensional we use the Atiyah-Patodi-Singer index theorem to relate the existence of asymptotic Killing spinors to the signature of \(M\). From this we derive conditions on
the signature of $M$ from the ALH condition and the lower bound on the scalar curvature.

As a particular case we study conformally compact Einstein manifolds. We calculate the expression for the mass for a conformally compact manifold and in case the conformal boundary is covered by the round sphere we prove that the mass vanishes, which leads to rigidity, see section 5.

**Overview of this paper:** This paper is organized as follows. In section 2 we discuss spin structures on quotients and the problem of relating spinors defined with respect to different metrics. In section 3 we give the necessary background on imaginary Killing spinors. In section 4 we define the notion of ALH manifold and prove the basic rigidity theorem, Theorem 4.8. The main step in the proof is Lemma 4.10 which is an adaptation of the argument in [10] to the present situation. In section 5 we apply these ideas to the particular case of conformally compact Einstein manifolds and prove that when the conformal boundary is a spherical spaceform, the mass vanishes which reduces the problem of proving scalar curvature rigidity to a topological question of whether the spin structure on the end admits an asymptotic Killing spinor. Finally, in section 6 we discuss the relation between the topology of $M$ and the spin structure on $E$ which leads up to Corollary 6.3 and Corollary 6.4.

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2. Preliminaries

Let $(M, g)$ be an oriented Riemannian spin manifold of dimension $n \geq 3$ with spin structure $\text{Spin}(M, g)$ and let $S(M, g)$ be the spinor bundle on $M$ associated to $\text{Spin}(M, g)$.

2.1. Spin structures on quotients. Let $\Gamma$ be a group acting by orientation preserving isometries on $M$. An element $\gamma \in \Gamma$ acts on a frame $f$ by $f \mapsto \gamma_* f$. Assume that this action of $\Gamma$ on the frame bundle lifts to an action on $\text{Spin}(M, g)$, that is we have an action $s \mapsto \tilde{\gamma} s$ which projects to the action $f \mapsto \gamma_* f$. Via the spin representation this defines an action on the spinor bundle where we denote the action of $\gamma$ by $\tilde{\gamma}$.

Assume that $\Gamma$ is a discrete group acting without fixed points. Then $\Gamma$ has a lift if and only if $M/\Gamma$ is spin. In this case the spin bundle on the quotient is given by

$$\text{Spin}(M/\Gamma) = \text{Spin}(M)/\Gamma$$

and the associated spinor bundle is given by

$$S(M/\Gamma) = S(M)/\Gamma.$$ 

This means that given a lift the sections of $S(M/\Gamma)$ are precisely the $\Gamma$-periodic sections of $S(M)$.

Lifts of $\Gamma$ are classified by $\text{Hom}(\Gamma, \mathbb{Z}_2)$ and in case $M$ is simply connected it follows from the isomorphism $H^1(M/\Gamma; \mathbb{Z}_2) = \text{Hom}(\Gamma, \mathbb{Z}_2)$ that this also classifies the spin structures on $M/\Gamma$. 
2.2. Comparing spinors for different metrics. Let \( g, g' \) be Riemannian metrics on a manifold \( M \), and define the positive definite ‘gauge transformation’ \( A \in \text{End}(TM) \) by

\[
\begin{align*}
g(AX, AY) &= g'(X, Y), \\
g(AX, Y) &= g(X, AY).
\end{align*}
\]

Because of the first property \( A \) will map ON-frames for \( g' \) to ON-frames for \( g \), and thus \( A \) induces a map \( \text{SO}(M, g') \overset{A}{\rightarrow} \text{SO}(M, g) \). If \( M \) is spin and we choose equivalent spin-structures for \( g \) and \( g' \) this can be lifted to \( \text{Spin}(M, g') \overset{A}{\rightarrow} \text{Spin}(M, g) \). A spinor field for \( g \) can be viewed as a \( \text{Spin}(n) \)-equivariant map \( \text{Spin}(M, g) \overset{\varphi}{\rightarrow} S \), where \( S \) is the spinor space, so the composition \( \varphi \circ A \) is a map \( \text{Spin}(M, g') \overset{\varphi}{\rightarrow} S \) which also is \( \text{Spin}(n) \)-equivariant. This gives the extension of \( A \) to a map \( \text{Spin}(M, g') \overset{A}{\rightarrow} \text{Spin}(M, g) \) which respects Clifford multiplication;

\[
A(X \cdot \varphi) = (AX) \cdot (A\varphi).
\]

Since the metric on the spinor bundle is given by a fixed Hermitean inner product on \( S \), \( A \) defines a fibrewise isometry. The above can be collected in a diagram.

\[
\begin{CD}
\text{Spin}(M, g') @>{A}>> \text{Spin}(M, g) @>{\varphi}>> S \\
\downarrow{\pi} & & \downarrow{\pi} \\
\text{SO}(M, g') @>{A}>> \text{SO}(M, g)
\end{CD}
\]

We will now look at the relation between the canonical covariant derivatives for \((M, g)\) and \((M, g')\). Let \( \nabla \) and \( \nabla' \) be the Levi-Civita connections for \( g \) and \( g' \), to be able to compare \( \nabla \) and \( \nabla' \) on the frame and spin bundles for \( g \) we define a connection \( \overline{\nabla} \) by

\[
\overline{\nabla}X = A(\nabla' A^{-1}X).
\]

The connection \( \overline{\nabla} \) is metric with respect to \( g \) and has torsion

\[
\begin{align*}
\overline{T}(X, Y) &= \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X, Y] \\
&= -((\nabla'_X A)A^{-1}Y - (\nabla'_Y A)A^{-1}X).
\end{align*}
\]

Expressing the covariant derivative in terms of the Lie bracket and the metric we get

\[
2g(\overline{\nabla}_X Y - \overline{\nabla}_Y X, Z) = g(\overline{T}(X, Y), Z) - g(\overline{T}(X, Z), Y) - g(\overline{T}(Y, Z), X).
\]

Next we compare \( \nabla, \overline{\nabla} \) when lifted to the spinor bundle \( S(M, g) \). Let \( \{e_i\} \) be a local orthonormal frame for \( g \), and let \( \{\sigma_\alpha\} \) be the corresponding local orthonormal frame of the spinor bundle. Denote by \( \omega_{ij}, \overline{\omega}_{ij} \) the connection one-forms for \( \nabla, \overline{\nabla} \) defined with respect to \( \{e_i\} \),

\[
\omega_{ij} = g(\nabla e_i, e_j),
\]

\[
\overline{\omega}_{ij} = g(\overline{\nabla} e_i, e_j),
\]
then the covariant derivatives of \( \varphi = \varphi^\alpha \sigma_\alpha \) are given by [8, Thm 4.14]
\[
\nabla \varphi = d\varphi^\alpha \otimes \sigma_\alpha + \frac{1}{2} \sum_{i<j} \omega_{ij} \otimes e_i e_j \varphi,
\]
and hence the difference between \( \nabla \) and \( \nabla \) acting on \( \varphi \) is
\[
\nabla \varphi - \nabla \varphi = \frac{1}{2} \sum_{i<j} (\omega_{ij} - \omega_{ij}) \otimes e_i e_j \varphi.
\] (4)

Using (2) and (3) we can estimate
\[
|| (\omega_{ij} - \omega_{ij})(e_k) || \leq C|A^{-1}|||\nabla' A||.
\]

We have proved the following lemma

**Lemma 2.1.** Let \( Y \) be a vectorfield and let \( \varphi \) be a spinor (w.r.t the \( g \) spin bundle), then
\[
|\nabla Y - \nabla Y| \leq C|A^{-1}|||\nabla' A|||Y||,
\] (5)
\[
|\nabla \varphi - \nabla \varphi| \leq C|A^{-1}|||\nabla' A|||\varphi||
\] (6)

and
\[
|D\varphi - D\varphi| \leq C|A^{-1}|||\nabla' A|||\varphi||,
\] (7)

where \( D, \bar{D} \) are the Dirac operators associated to the connections \( \nabla, \nabla \). □

### 3. Killing spinors

#### 3.1. The Killing connection.** Define a modified connection on the spinor bundle by
\[
\hat{\nabla}_X = \nabla_X + \frac{i}{2} X,
\]
we call this the Killing connection, spinors parallel with respect to this connection are called imaginary Killing spinors. There is an analogous concept of real Killing spinors. In this paper we will discuss only the imaginary Killing spinors and write simply Killing spinors. The connection \( \hat{\nabla} \) will respect the Clifford module structure of \( \mathcal{S}(M, g) \) if we define its action on the Clifford algebra bundle on \( (M, g) \) so that
\[
\hat{\nabla}_X Y = \nabla_X Y + \frac{i}{2} [X, Y],
\]
where \([,] \) denotes the commutator in the Clifford algebra.

The Dirac operator corresponding to \( \hat{\nabla} \) is
\[
\hat{D} = D - \frac{i}{2} n.
\]
The curvature of $\hat{\nabla}$ is
\[
\hat{R}(X,Y)\varphi = (\hat{\nabla}_X\hat{\nabla}_Y - \hat{\nabla}_Y\hat{\nabla}_X - \hat{\nabla}_{[X,Y]}\varphi)
\]
\[
= (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\varphi)
\]
\[
+ \frac{1}{2}((\nabla_XY) - (\nabla_YX) - [X,Y])\varphi - \frac{1}{4}(X \cdot Y - Y \cdot X) \cdot \varphi
\]
\[
= R(X,Y)\varphi - \frac{1}{4}(X \cdot Y - Y \cdot X) \cdot \varphi
\]
Through the natural isomorphism of Lie algebras $\text{spin}(n) \leftrightarrow \mathfrak{so}(n)$ ([8, §I.6]) the second term corresponds to the skew-adjoint endomorphism $X \wedge Y$ of $TM$ defined by
\[
(X \wedge Y)(Z) = \langle X,Z \rangle Y - \langle Y,Z \rangle X.
\]
This is also the curvature tensor $R_{-1}$ on $TM$ of constant sectional curvature $-1$, so we see that the curvature of the Killing connection viewed as an $\mathfrak{so}(n)$-valued two form is
\[
\hat{R}(X,Y) = R(X,Y) - R_{-1}(X,Y).
\]

3.2. Manifolds with Killing spinors. This expression for the curvature of $\hat{\nabla}$ tells us that if there is a local basis of Killing spinors then $\hat{R}$ vanishes and $(M,g)$ is locally isometric to hyperbolic space.
If a manifold has one spinor parallel with respect to $\nabla$ it is Ricci-flat, this follows from the formula
\[
\sum e_i R(X,e_i)\varphi = -\frac{1}{2} \text{Ric}(X) \cdot \varphi.
\]
We calculate using (8) for a Killing spinor $\varphi$
\[
0 = \sum e_i \hat{R}(X,e_i)\varphi = -\frac{1}{2}(\text{Ric}(X) + (n-1)X) \cdot \varphi,
\]
from which it follows that a manifold with a Killing spinor must be Einstein. But for a complete manifold with a Killing spinor there is the following result of H. Baum.

Theorem 3.1 (H. Baum [3] [4, Chapter 7]). A complete manifold $M$ has a Killing spinor $\varphi$ if and only if it is isometric to $P \times \mathbb{R}$ with the metric $e^{2t}h + dt^2$ where $(P,h)$ is a complete manifold with a parallel spinor.

On hyperbolic space $\mathbb{H}^n$ there is a full set of Killing spinors. We describe them using the ball model of $\mathbb{H}^n$, which is $B^n = \{|x| < 1\}$ with the metric $g = \rho(x)^{-2}g_0$ where $\rho(x) = \frac{1}{2}(1 - |x|^2)$ and $g_0$ is the standard flat metric. Since this is conformal to the flat metric on the ball we get an identification $\text{SO}(\mathbb{H}^n) = B^n \times \text{SO}(n)$ which gives trivializations $\text{Spin}(\mathbb{H}^n) = B^n \times \text{Spin}(n)$ and $\mathcal{S}(\mathbb{H}^n) = B^n \times \mathcal{S}$.

Proposition 3.2 ([3] [4]). In the above trivialization, the Killing spinors on $\mathbb{H}^n$ are
\[
\varphi_u(x) = \rho(x)^{-\frac{1}{2}}(1 + ix \cdot)u,
\]
where $u \in \mathcal{S}$.
3.3. The Lichnerowicz formula. Fix a spinorfield $\varphi$ and define a one-form

$$\hat{\alpha}(X) = \langle (\hat{\nabla}_X + X \cdot \hat{D})\varphi, \varphi \rangle.$$ 

A computation gives

$$\text{div} \, \hat{\alpha} = \frac{\hat{s}}{4} |\varphi|^2 + |\hat{\nabla}\varphi|^2 - |\hat{D}\varphi|^2.$$ 

where $\hat{s} = s + n(n - 1)$ and $s$ is the scalar curvature of $(M, g)$. Integrating over a manifold $M$ with boundary $\partial M$ we get the Lichnerowicz formula

$$\int_M \left( \frac{\hat{s}}{4} |\varphi|^2 + |\hat{\nabla}\varphi|^2 - |\hat{D}\varphi|^2 \right) = \int_{\partial M} \langle (\hat{\nabla}_\nu + \nu \hat{D})\varphi, \varphi \rangle,$$

(9)

where $\nu$ is the outward normal of the boundary. We also note the integration by parts formula for the Dirac operator $\hat{D}$.

$$\int_M \langle (\hat{D} + in)\varphi, \psi \rangle = \int_{\partial M} \langle \nu \varphi, \psi \rangle + \int_M \langle \varphi, \hat{D}\psi \rangle.$$

(10)

3.4. A little analysis. We end this section by proving some analytical results using the formulas derived above. Let $M$ be a complete spin manifold with $\hat{s}$ non-negative and bounded. Let $C^\infty_0 S(M)$ be the space of smooth sections with compact support and define a sesquilinear form

$$B(\varphi, \psi) = \int_M \langle \hat{D}\varphi, \hat{D}\psi \rangle.$$ 

which is obviously bounded. $B$ is Hermitean and using (9) we get

$$B(\varphi, \varphi) = \int_M \frac{\hat{s}}{4} |\varphi|^2 + |\hat{\nabla}\varphi|^2$$

$$= \int_M \frac{\hat{s} + n}{4} |\varphi|^2 + |\nabla\varphi|^2$$

Define a scalar product $(\cdot, \cdot)_1$ by

$$(\varphi, \psi)_1 = \int_M \langle \nabla\varphi, \nabla\psi \rangle + \frac{n}{4} \langle \varphi, \psi \rangle$$

and let $H^1 S(M)$ be the closure of $C^\infty_0 S(M)$ with respect to the norm $||\varphi||_1 = (\varphi, \varphi)_1^{1/2}$. Then $(H^1 S(M), (\cdot, \cdot)_1)$ is a Hilbert space. If $\hat{s}$ is bounded, $B$ extends to $H^1 S(M)$ as a bounded sesquilinear form and if $\hat{s}$ is non-negative we see that $B$ is coercive on $H^1 S(M)$.

We have shown that $B$ satisfies the conditions of the Theorem of Lax-Milgram, the conclusion is

**Proposition 3.3.** Let $(M, g)$ be as above and assume that $\hat{s}$ is nonnegative and bounded. Then for every bounded linear functional $l$ on $H^1 S(M)$ there is a unique $\varphi \in H^1 S(M)$ so that for all $\psi \in H^1 S(M)$

$$B(\varphi, \psi) = l(\psi).$$

□
Remark 3.4. In the rest of the paper we will assume that $\hat{s}$ is bounded. This assumption is technical and can be removed by using appropriately weighted spaces of sections. □

4. Rigidity for ALH manifolds

In this section we consider manifolds satisfying a condition which is analogous to the vanishing of the Bondi mass. By the positive mass argument this leads to rigidity theorems.

4.1. Asymptotic Killing spinors. Let $(M,g)$ be a complete Riemannian manifold with one end $E$ and a diffeomorphism $N \times (0, \infty) \stackrel{\varphi}{\rightarrow} E$ such that $\varphi^* g = L^2 dR^2 + h$ where $h$ is the induced metric on $N_R = \varphi(N \times \{R\})$ and $L > 0$. Let $L_{\min}(t) = \min_{N_R} L$. We say that $\varphi$ gives a nondegenerate foliation at infinity if $\int_0^\infty L_{\min}(t) dt = \infty$. Let $\{M_R\}$ be a family of compact manifolds with boundary exhausting $M$. We say that that $\{M_R\}$ is nondegenerate if there is a nondegenerate foliation at infinity such that $\partial M_R = N_R$.

We have

Proposition 4.1. Let $(M,g)$ be a complete Riemannian manifold with a nondegenerate exhausting family $M_R$. Let $f$ be a nonnegative integrable function on $M$. Then $\lim \inf_{R \to \infty} \int_{\partial M_R} f = 0$. □

The generalization to manifolds with more than one end is trivial.

Definition 4.2. Let $(M,g)$ be a complete Riemannian spin manifold.

1. We say that a spinor $\varphi_0$ is an asymptotic Killing spinor on $M$ if $\hat{\nabla} \varphi_0 \in L^2(T^* \otimes S)(M)$, $\varphi_0 \notin H^1S(M)$.
2. If $\varphi_0$ is an asymptotic Killing spinor we say that the mass of $M$ (w.r.t. $\varphi_0$) is zero if there is a nondegenerate family $\{M_R\}$ exhausting $M$ so that either

$$\lim_{R \to \infty} \int_{\partial M_R} \langle (\hat{\nabla}_\nu + \nu \hat{D}) \varphi_0, \varphi_0 \rangle = 0$$

(11)

or

$$\langle (\hat{\nabla}_\nu + \nu \hat{D}) \varphi_0, \varphi_0 \rangle \in L^1(M).$$

(12)

□

Remark 4.3. In the setting of Definition 4.2, if the limit

$$\lim_{R \to \infty} \int_{\partial M_R} \langle (\hat{\nabla}_\nu + \nu \hat{D}) \varphi_0, \varphi_0 \rangle$$

exists, then it is a natural to think of this as an analog of a component of the Bondi momentum in general relativity. □

The following Lemma is the essential step in the positive mass argument.

Lemma 4.4. If $M$ has an asymptotic Killing spinor, mass zero in the sense of Definition 4.2 and $\hat{s} \geq 0$, then $M$ has a Killing spinor.
Proof: Since $\hat{D}\varphi_0 \in L^2 S(M)$ the linear functional
\[ l(\psi) = \int_M \langle \hat{D}\varphi_0, \hat{D}\psi \rangle \]
is bounded on $H^1 S(M)$, so by Proposition 3.3 there is a unique $\varphi_1 \in H^1 S(M)$ so that for all $\psi \in H^1 S(M)$
\[ \int_M \langle \hat{D}\varphi_1, \hat{D}\psi \rangle = B(\varphi, \psi) = l(\psi) = \int_M \langle \hat{D}\varphi_0, \hat{D}\psi \rangle. \]
Set $\varphi = \varphi_0 - \varphi_1$, then
\[ \int_M \langle \hat{D}\varphi, \hat{D}\psi \rangle = 0, \]
we will show that this implies $\hat{D}\varphi = 0$. Set $a = \hat{D}\varphi$. Integrating by parts using (10) we get for any $\psi \in C^\infty_0 S(M)$
\[ 0 = \int_M \langle a, \hat{D}\psi \rangle = \int_M \langle (\hat{D} + in) a, \psi \rangle, \]
so
\[ (\hat{D} + in) a = 0. \tag{13} \]
By elliptic regularity we get that $a$ is smooth and by (13) $\hat{D}^k a \in L^2 S(M)$ for any integer $k \geq 0$. Therefore the usual cutoff argument shows that
\[ \int_M \langle \hat{D} a, \hat{D} a \rangle = \int_M \langle (\hat{D} + in) \hat{D} a, a \rangle \]
\[ = \int_M \langle \hat{D}(\hat{D} + in) a, a \rangle \]
\[ = 0 \]
and thus $\hat{D} a = 0$ which together with $(\hat{D} + in) a = 0$ implies $a = \hat{D}\varphi = 0$. Let $M_R$ be as in definition 4.2. We use (8) to get
\[ \int_{M_R} |\hat{\nabla}\varphi|^2 \leq \int_{M_R} \left( \frac{8}{4} |\varphi|^2 + |\hat{\nabla}\varphi|^2 \right) \]
\[ = \int_{\partial M_R} \langle (\hat{\nabla}_\nu + \nu \hat{D}) \varphi, \varphi \rangle. \tag{14} \]
Write $\hat{B} = \hat{\nabla}_\nu + \nu \hat{D}$. Since $\hat{B}$ is self adjoint on $\partial M_R$ we may write (14) in the form
\[ \int_{\partial M_R} \langle \hat{B}\varphi_0, \varphi_0 \rangle + \int_{\partial M_R} \langle \hat{B}\varphi_1, \varphi_1 \rangle + \int_{\partial M_R} \langle \hat{B}\varphi_0, \varphi_1 \rangle + \int_{\partial M_R} \langle \varphi_1, \hat{B}\varphi_0 \rangle. \tag{15} \]
In case (11) holds, the first term tends to zero as $R \to \infty$ from the vanishing of the mass w.r.t. $\varphi_0$. By assumption, $\hat{\nabla}\varphi_0 \in L^2(T^* \otimes S)(M)$ and by construction $\varphi_1 \in H^1 S(M)$, therefore we have $|\langle \hat{B}\varphi_0, \varphi_1 \rangle| \in L^1(M)$ and $|\langle \hat{B}\varphi_1, \varphi_1 \rangle| \in L^1(M)$ and by Proposition 4.1 there is a sequence $\{R_j\}$ such that the last three terms tend to zero as $j \to \infty$. This tells us that $\hat{\nabla}\varphi = 0$ and since $\varphi_0 \notin H^1 S(M)$ by assumption we have $\varphi \neq 0$. Finally in case (12) holds, we may apply Proposition 4.1 as above to all the terms in (15). □
4.2. Asymptotically locally hyperbolic manifolds.

**Definition 4.5.** Two metrics $g$ and $g'$ are strongly asymptotic if the gauge-transformation $A$ of Section 2.2 satisfies

1. There is a $k$ so that for all $X \in TM, |X| = 1$ we have $k^{-1} \leq |AX| \leq k$,
2. $(|\nabla' A|^2 + |A - Id|^2)^{\frac{r}{2}} \in L^2(M) \cap L^1(M)$ w.r.t. the measure $e^r d\text{vol}(g)$,
3. $|\nabla^2 A| \in L^1(M)$

where $r$ is the $g'$-distance from a fixed point.

In the case when the model metric $g'$ is hyperbolic, this leads to the concept of asymptotically locally hyperbolic manifolds, analogous to the notion of asymptotically locally Euclidean spaces.

**Definition 4.6.** Let $\Gamma$ be a finite group acting freely and linearly on the sphere and let $\mathbb{H}^n_\ast$ be $\mathbb{H}^n \setminus B$ where $B$ is a closed ball centered at the origin. Suppose that $(M, g)$ has an end $E$ for which there is an identification of with $\mathbb{H}^n_\ast/\Gamma$ where $\Gamma$ acts in the natural way on the ball model of hyperbolic space. Let $g'$ be a metric on $M$ which on the end $E$ is isometric to $\mathbb{H}^n_\ast/\Gamma$ with the hyperbolic metric. Then we say that $(M, g)$ is asymptotically locally hyperbolic with the asymptotically locally hyperbolic end $E$ with group $\Gamma$ if $g$ and $g'$ are strongly asymptotic on $E$. In case $\Gamma = \{1\}$ we call $M$ asymptotically hyperbolic.

As in the above definition let $\Gamma$ be a finite subgroup of $\text{SO}(n)$ acting freely on the sphere. We will now consider the conditions for existence of Killing spinors on the quotient $\mathbb{H}^n_\ast/\Gamma$. In the trivialization used in Proposition 3.2 $\gamma \in \Gamma$ acts on $\text{SO}(\mathbb{H}^n_\ast)$ as

$$(x, f) \xrightarrow{\gamma} (\gamma(x), \gamma f).$$

Assume that $\mathbb{H}^n_\ast/\Gamma$ is spin. Then by the discussion in Section 2.2 there is a bijective lift of $\Gamma$ to a subgroup $\tilde{\Gamma} \subset \text{Spin}(n)$, which specifies the action of $\Gamma$ on the spin and spinor bundles of $\mathbb{H}^n_\ast$. The sections of $\mathcal{S}(\mathbb{H}^n_\ast/\Gamma)$ are naturally identified with the $\Gamma$-periodic sections of $\mathcal{S}(\mathbb{H}^n_\ast)$. Since the Killing condition is local the Killing spinors on $\mathbb{H}^n_\ast/\Gamma$ are precisely given by the spinors $\varphi_u$ in Proposition 3.2 which are also $\Gamma$-periodic. The $\varphi_u$ are acted on as follows

$$(x, \varphi_u(x)) \xrightarrow{\gamma} (\gamma(x), \rho(\gamma(x))^{-\frac{1}{2}} \tilde{\gamma}(1 + i\gamma \cdot \cdot) u)$$

and

$$\tilde{\gamma}(1 + i\gamma \cdot \cdot) u = (1 + i\tilde{\gamma} \cdot x \cdot \tilde{\gamma}^{-1} \cdot) \tilde{\gamma} \cdot u.$$ 

If we view $\text{Spin}(n)$ as a subgroup of the Clifford algebra the conjugation by $\tilde{\gamma}$ is just the projection to $\gamma$ in $\text{SO}(n)$. Therefore, if we assume that the fixed element $u$ in $\mathcal{S}$ satisfies

$$\tilde{\gamma} \cdot u = u$$

for all $\tilde{\gamma}$ the Killing spinor will be $\Gamma$-periodic

$$(x, \varphi_u(x)) \xrightarrow{\gamma} (\gamma(x), \rho(\gamma(x))^{-\frac{1}{2}} (1 + i\gamma \cdot \cdot) u = (\gamma(x), \varphi_u(\gamma(x))).$$

Conversely if $\varphi_u$ is $\Gamma$-periodic we must have $\tilde{\gamma} \cdot u = u$ for all $\tilde{\gamma}$. The Killing spinors on the quotient thus correspond precisely to the spinors $u \in \mathcal{S}$ which...
are fixed by the spin-representation of the lifted group $\tilde{\Gamma}$, they depend both on $\Gamma$ and via the choice of lift the spin structure on the quotient. We have proved the following Proposition.

**Proposition 4.7.** Let $\Gamma$ be a subgroup of $\text{SO}(n)$ acting freely on the sphere and let $\tilde{\Gamma}$ be a bijective lift to $\text{Spin}(n)$, then the Killing spinors on $\mathbb{H}^n_\ast/\Gamma$ with the spin-structure defined by $\tilde{\Gamma}$ correspond to the $\varphi_u$ given by Proposition 3.2 for spinors $u \in S$ which are fixed by the spinor representation of $\tilde{\Gamma}$. \(\square\)

Now let $M$ be an asymptotically locally hyperbolic spin manifold with group $\Gamma$ and fix a spin structure on $M$. The restriction of the spin structure to the locally hyperbolic end will be equivalent to the spin structure on $\mathbb{H}^n_\ast/\Gamma$ defined by some lift $\tilde{\Gamma}$ of $\Gamma$. Thus the number of asymptotic Killing spinors on $M$ is via the lift $\tilde{\Gamma}$ controlled by spin structure, a purely topological condition.

**4.3. A rigidity theorem.** We have the following rigidity theorem.

**Theorem 4.8.** Let $M$ complete spin manifold with an asymptotically locally hyperbolic end $E$ with group $\Gamma$. Suppose that the spin structure on $E$ is equivalent to the spin structure on $\mathbb{H}^n_\ast/\Gamma$ defined by the lift $\tilde{\Gamma}$ of $\Gamma$. If $\tilde{\Gamma}$ fixes some non–zero spinor $u \in S$ and if $s \geq 0$ then $\Gamma = \{1\}$ and $M$ is isometric to $\mathbb{H}^n$.

**Remark 4.9.** The condition on the spin structure can be formulated as the vanishing of a relative Stiefel–Whitney class, see [1]. \(\square\)

We begin by proving the existence of an asymptotic Killing spinor with mass zero.

**Lemma 4.10.** Let $M$ complete spin manifold with an asymptotically locally hyperbolic end $E$ with group $\Gamma$. Suppose that the spin structure on $E$ is equivalent to the spin structure on $\mathbb{H}^n_\ast/\Gamma$ defined by the lift $\tilde{\Gamma}$ of $\Gamma$ and $\tilde{\Gamma}$ fixes some non–zero spinor $u \in S$. Then $M$ has an asymptotic Killing spinor with mass zero.

**Proof:** Let $E$ be the locally hyperbolic end and let $g'$ be the hyperbolic metric. On the end $\mathbb{H}^n_\ast/\Gamma$ with the hyperbolic metric there is by Proposition 4.7 a Killing spinor $\varphi'$. Since the spin-structures match up $\varphi'$ can be moved to a spinor $A\varphi'$ on $E$ where $A$ is the gauge transformation of section 2.2.

Let $f$ be a smooth function with supp$(df)$ compact, $f = 0$ outside $E$ and $f = 1$ at infinity. Define the spinor $\varphi_0 = f A \varphi'$ on $(M, g)$. We are going to show that $\varphi_0$ is an asymptotic Killing spinor.

First we show that $\nabla \varphi_0 \in L^2 S(M)$.

\[
\hat{\nabla}_X \varphi_0 = (\nabla_X + \frac{i}{2} X)f A \varphi' = (X f) A \varphi' + f(\nabla_X A \varphi' + \frac{i}{2} X A \varphi') = (X f) A \varphi' + f(\nabla_X - \frac{i}{2} X)f A \varphi'
\]
where the first term is compactly supported and the second can be estimated using (7), since
\( \hat{\nabla}\phi' = 0 \) the third term is \( f \) times
\[
(\nabla_X + \frac{i}{2}X)A\phi' = A\nabla_X^i\phi' + \frac{i}{2}XA\phi' \\
= -\frac{i}{2}(AX - X)A\phi' \\
= -\frac{i}{2}(A - Id)XA\phi'.
\]
Thus we get for large \( r \)
\[
|\hat{\nabla}\phi_0| \leq C(|\nabla'A|^2 + |A - Id|^2)|\phi_0|^2.
\]
Next we estimate \( |\phi_0|^2 \), since \( \hat{\nabla}\phi' = 0 \) we have for \( X \) with \( |X|^' = 1 \)
\[
|X(|\phi|^2)| \leq 2|\langle \nabla_X\phi', \phi' \rangle| = |\langle X \cdot \phi', \phi' \rangle| \leq |\phi'|^2,
\]
if we take \( X = \frac{\partial}{\partial r} \) and integrate along a radial geodesic this gives
\[
C^{-1}e^{-r} \leq |\phi'|^2 \leq Ce^r
\]
and since \( A \) is an isometry on spinors, we have asymptotically that
\[
C^{-1}e^{-r} \leq |\phi_0|^2 \leq Ce^r.
\]
Thus we have proved
\[
|\hat{\nabla}_X\phi_0|^2 \leq C(|\nabla'A|^2 + |A - Id|^2)e^r|X|^2,
\]
\[
|\hat{D}\phi_0|^2 \leq C(|\nabla'A|^2 + |A - Id|^2)e^r,
\]
which together with point 2 of Definition 4.5 tells us that \( \hat{\nabla}\phi_0 \) is in \( L^2(T^* \otimes S)(M) \). Next we have to show that the mass is zero. Let \( R \) be large and let \( S_R = \{ x \in E; r(x) = R \} \) denote the distance sphere on \( E \) with outward normal \( \nu \).

From (15), (19) we get
\[
|\langle (\hat{\nabla}_\nu + \nu\hat{D})\phi_0, \phi_0 \rangle| \leq C(|\nabla'A|^2 + |A - Id|^2)\frac{1}{2}|A\phi'|^2 \\
\leq C(|\nabla'A|^2 + |A - Id|^2)\frac{1}{2}e^r
\]
and by point 2 of Definition 4.5 this is in \( L^1(M) \).

It remains to check that \( \phi_0 \notin H^1S(M) \). From (17) it follows that \( |\phi_0|^2 \geq Ce^{-r} \) for large \( r \) and hence \( \phi_0 \notin L^2S(M) \), since the sphere \( S_R \) has area of the order \( e^{(n-1)R} \).

This completes the proof of Lemma 4.10.

We have now showed that \( \phi_0 \) is an asymptotic Killing spinor with mass zero and by Lemma 4.4 we conclude that \( M \) has a Killing spinor \( \phi \) and therefore \( (M, g) \) is of the form described in Theorem 3.1. (\( M, g \)).

The proof of Theorem 4.8 is concluded by the following Lemma.

**Lemma 4.11.** Let \( (M, g) \) be a warped product \( (P \times \mathbb{R}, e^{2t}h + dt^2) \) where \( (P, h) \) is a complete Riemannian manifold. Suppose that \( M \) has an asymptotically locally hyperbolic end. Then \( M \) is isometric to \( \mathbb{H}^n \).
Proof: Let $\xi, \eta$ be orthonormal vectors orthogonal to $\frac{\partial}{\partial t}$, then $e^t \xi, e^t \eta$ are orthonormal w.r.t. $h$. The sectional curvature on the plane spanned by $\xi, \eta$ is ([11, Chapter 7, Prop. 42])

$$K(\xi, \eta) = e^{-2t}K_0(\xi, \eta) - 1,$$

where $K_0$ is the sectional curvature of $h$. On the other hand we know that there is an end on which $g(AX, AY)$ equals the hyperbolic metric $g'(X, Y)$ of constant sectional curvature $-1$ where $A$ is as in Definition 4.5. The curvature tensor of the hyperbolic metric $g'$ satisfies

$$R'(X,Y)Z = g'(X,Z)Y - g'(Y,Z)X.$$

From (1) we see that $R'$ is related to the curvature $\nabla$ by

$$R(X,Y)Z = A \circ R'(X,Y) \circ A^{-1}Z.$$

It follows that

$$K(\xi, \eta) = g(\xi, A\eta)^2 - g(\xi, A\xi)g(\eta, A\eta).$$

and

$$|K(\xi, \eta) + 1| \leq C|A - Id|.$$

Define the tensor $\delta$ by

$$\nabla_X = \nabla_X + \delta_X.$$

The curvature tensors for $\nabla'$ and $\nabla$ are related by

$$\nabla(X,Y)Z = R(X,Y)Z + (\nabla_X \delta_Y - \nabla_Y \delta_X - [\delta_X, \delta_Y] + \delta_{[X,Y]})Z.$$

Using (2), (3) and point 1 of Definition 4.5 we get

$$|K'(\xi, \eta) - K(\xi, \eta)| \leq C(|\nabla' A|^2 + |\nabla' A| + |\nabla'^2 A|)$$

and

$$e^{-2t}|K_0| = |K + 1 - \overline{K} + \overline{K}|$$

$$\leq |K - \overline{K}| + |\overline{K} + 1|$$

$$\leq C(|\nabla'^2 A| + |\nabla' A|^2 + |\nabla' A| + |A - Id|).$$

For $x \in P$, let $K_{max}(x)$ be the maximum of $|K_0|$ over the two–planes at $x$. Then we get by integrating over the slice $t = T$

$$e^{-2T} \int_P K_{max}|d\text{vol}(h)|$$

$$\leq C \int_{t=T} \left(|\nabla'^2 A| + |\nabla' A|^2 + |\nabla' A| + |A - Id| \right) \frac{\partial}{\partial t} d\text{vol}(g).$$

By points 2 and 3 of Definition 4.5 and Proposition 4.1, there is a sequence $\{T_j\}$ tending to $\infty$ so that the right hand side vanishes as $j \to \infty$. Therefore since the integral on the left hand side is independent of $T$ and $n \geq 3$ it must be equal to zero.

This shows that $h$ is a complete flat metric and since the fundamental group of the end is finite we must have that $(P, h)$ is $\mathbb{R}^n$ with the flat metric and hence $M = \mathbb{H}^n$. \qed
5. Rigidity for conformally compact Einstein manifolds

In this section we prove that the mass for conformally compact Einstein manifolds with conformal boundary a spherical space form vanishes. This means we can apply the results of section 4.

5.1. Conformally compact Einstein manifolds.

Definition 5.1. A Riemannian manifold \((M, g)\) is called conformally compact if \(M\) is the interior of a compact manifold \(\tilde{M}\) with boundary \(\partial M\) and \(g\) is conformal to a smooth metric \(\tilde{g}\) on \(\tilde{M}\),

\[
g = \rho^{-2}\tilde{g},
\]

where \(\rho\) is a defining function for the boundary, that is \(\partial M = \{x : \rho(x) = 0\}\) and \(d\rho \neq 0\) on \(\partial M\).

In the work of Fefferman and Graham [5] conformally compact Einstein metrics which are such that the conformal background metric \(\tilde{g}\) is an even function of the defining function are studied near \(\partial M\). They prove that in case \(M\) is of even dimension, the formal power series expansion of \(\tilde{g}\) in terms of \(\rho\) is uniquely determined, while in the odd dimensional case, there is an undetermined term at the \(n-1\):st order. Here we study the case when the conformal boundary is locally conformally flat and covered by a sphere. Using a combination of the spinor arguments above and an analysis of the power series expansion of \(\tilde{g}\) we prove the following rigidity theorem.

Theorem 5.2. Let \((M, g)\) be a conformally compact Einstein spin manifold of dimension \(n\) for which the boundary of the conformal background metric is isometric to \(S^{n-1}\), then \((M, g)\) is isometric to \(H^n\).

Let \((S^{n-1}, h_0)\) be the standard sphere of dimension \(n - 1\). Hyperbolic space \(H^n\) can be represented as \(\mathbb{R}_+ \times S^{n-1}\) with the metric

\[
\sinh^{-2}(x)(dx^2 + h_0).
\]

We will use a conformal gauge change \(\tilde{g} \to \theta^2\tilde{g}\) and \(\rho \to \theta\rho\) where \(\theta\) is a positive function to put a general conformally compact Einstein metric on a form similar to this.

Lemma 5.3. Let \((M, g)\) be as in Theorem 5.2. There is a conformal gauge change on a collar neighborhood \(U\) of \(\partial M\) so that \(\rho = \sinh(x)\) where \(x\) is the distance to \(\partial M\) in \(\tilde{g}\).

Proof: Given are \(M, \rho, \tilde{g}\) such that \(g = \rho^{-2}\tilde{g}\) is Einstein. The relation between the Ricci-tensors of \(g\) and \(\tilde{g}\) is

\[
\text{Ric} = \tilde{\text{Ric}} + \rho^{-1}((n - 2)\tilde{\nabla}d\rho + \text{tr}\tilde{g}(\tilde{\nabla}d\rho)\tilde{g}) - (n - 1)\rho^{-2}\tilde{g}(d\rho, d\rho)\tilde{g},
\]

(20)

Since \(\text{Ric} = -(n - 1)\tilde{g}\) this gives

\[-(n - 1)\tilde{g} = \rho^2\tilde{\text{Ric}} + \rho((n - 2)\tilde{\nabla}d\rho + \text{tr}\tilde{g}(\tilde{\nabla}d\rho)\tilde{g}) - (n - 1)\tilde{g}(d\rho, d\rho)\tilde{g},
\]

which implies that \(\tilde{g}(d\rho, d\rho) = 1\) on \(\partial M\) so \(1 - \tilde{g}(d\rho, d\rho) = \rho a\) where \(a\) is some smooth function. We are going to find \(\theta\) so that \(\rho := \theta\rho = \sinh(\hat{x})\) where \(\hat{x}\).
is the distance to the boundary in the metric \( \tilde{g} := \theta^2 \hat{g} \), or equivalently so that \( \hat{f} := \sinh^{-1}(\hat{\rho}) = \hat{x} \). We begin by solving for a \( \theta \) such that

\[
|d\hat{f}|_{\tilde{g}} = 1.
\]

Written in terms of the given data this means that \( \theta \) must solve the equation

\[
\rho^2 \tilde{g}(d\theta, d\theta) + 2\rho \theta \tilde{g}(d\theta, d\rho) = \theta^4 \rho^2 + \theta^2 (1 - \tilde{g}(d\rho, d\rho)),
\]

or

\[
\rho \tilde{g}(d\theta, d\theta) + 2\theta \tilde{g}(d\theta, d\rho) = \theta^4 \rho + \theta^2 a.
\]

One verifies easily that this equation with the initial data \( \theta = 1 \) on \( \partial M \) satisfies the conditions for existence of a solution in a neighbourhood of \( \partial M \) \( [12, \text{ volume } 5, \text{ pp. } 39-40] \). In a small enough neighbourhood \( U \) of \( \partial M \) the solution is positive and we continue this to a positive function \( \theta \) on all of \( M \) for which \( |d\hat{f}|_{\tilde{g}} = 1 \) on \( U \). This implies that the gradient curves of \( \hat{f} \) on \( U \) are unit-speed geodesics with respect to \( \tilde{g} \) and since \( \hat{f} = 0 \) precisely on \( \partial M \) we have \( \hat{f} = \hat{x} \) on \( U \).

Let \( \rho, x, \tilde{g} \) be as in Lemma 5.3 and let \( \partial M_x \) be the level surfaces of \( x \). Then \( \tilde{g} \) is of the form \( \tilde{g} = dx^2 + h \) where \( h \) is the induced metric on \( \partial M_x \). The normal vector field to the foliation \( \partial M_x \) w.r.t. \( \tilde{g} \) is \( \eta = \frac{\partial}{\partial x} = \text{grad}(x) \) and the second fundamental form of \( \partial M_x \) is

\[
\lambda(X,Y) = \tilde{g}(\eta, \nabla_X Y) = -\frac{1}{2}(\mathcal{L}_\eta \tilde{g})(X,Y)
\]

Observe that this identity makes sense for \( X,Y \) not tangential to \( \partial M_x \) by defining \( \lambda(\eta, \cdot) = 0 \).

We define a hyperbolic background metric \( g' \) on \( U \) by setting

\[
g' = \sinh^{-2}(x) \tilde{g}'
\]

where \( \tilde{g}' \) is the cylindrical metric \( dx^2 + h_0 \). There is a Killing spinor \( \varphi' \) with respect to \( g' \) defined on \( U \). Let \( A \) be the gauge transformation between \( g \) and \( g' \) and let \( \varphi_0 = fA\varphi' \) where \( f \) is a cut-off function. The proof of Theorem 5.2 given in section 5.3 will consist of showing that \( \varphi_0 \) is an asymptotic Killing spinor and that the mass of \( (M, g) \) with respect to \( \varphi_0 \) is zero. We begin by computing the asymptotics of the metric \( \tilde{g} \).

**Lemma 5.4.** Let \( n > 3 \). The gauge transformation \( A \) satisfies

\[
(A - Id)\eta = 0, \\
\text{tr}_{h_0}(A - Id) = O(x^{2n-2}), \\
A - Id = O(x^{n-1}), \\
|\nabla A|_g = O(x^{n-1}), \\
|\nabla^2 A|_g = O(x^{n-1}).
\]

Further, \( x^{-(n-1)}(A - Id) \) is smooth up to \( \partial M \).

**Proof:** By Taylor’s theorem we may expand \( h \) in a asymptotic series

\[
h = \sum_{k=0}^{\infty} \frac{x^k}{k!} h^{(k)}.
\]
where \( h^{(0)} = h_0 \) and \( h^{(k)} \) are symmetric tensors on \( \partial M \). Then

\[
\lambda(X,Y) = -\frac{1}{2} \sum_{k=0}^{\infty} \frac{x^k}{k!} h^{(k+1)}(X,Y).
\] (22)

The Hessian of \( x \) equals \(-\lambda\)

\[
\nabla^2 \rho = \rho''(x) dx^2 - \rho' \lambda
\]

and, by assumption, the metric \( g \) satisfies \( \text{Ric} = -(n-1)g \). Using this and \( \rho = \sinh(x) \) in (20) gives us the identity

\[
\rho(\text{Ric} + (n-2) dx^2 - (n-2)\tilde{g}) = \rho'((n-2)\lambda + \text{tr}_g \lambda \tilde{g}).
\] (23)

Inserting (22) in (23) and setting \( x = 0 \) we get \( h^{(1)} = 0 \).

Recall the second variation formula

\[
(\mathcal{L}_\eta \lambda)(X,Y) = -\lambda \times \lambda(X,Y) + \mathcal{R}_\eta(X,Y),
\] (24)

where \( \mathcal{R}_\eta(X,Y) = \tilde{g}(\tilde{R}(X,\eta),\eta,Y) \) and \( \times \) denotes the product on 2-tensors defined by \( a \times b(X,Y) = \sum_i a(X,e_i)b(\tilde{e}_i,Y) \) where \( \{e_i\} \) is an ON frame w.r.t. \( \tilde{g} \). Now let \( \text{ric} \) denote the Ricci tensor of the induced metric \( h \) on \( \partial M_x \), then contracting the Gauss equation gives

\[
\tilde{\text{Ric}} = \text{Ric} + \mathcal{R}_\eta + \lambda \times \lambda - \text{tr}_g(\lambda)\lambda
\] (25)

on \( \partial M_x \). Finally we plug (24) and (25) into equation (23) to get

\[
\rho(\text{Ric} - (n-2)h + \mathcal{L}_\eta \lambda + 2\lambda \times \lambda - \text{tr}_g(\lambda)\lambda) = \rho'((n-2)\lambda + \text{tr}_g(\lambda)h).
\] (26)

Further, the normal part of equation (23) is

\[
\rho \tilde{\text{Ric}}(\eta,\eta) = \rho'((n-2)\lambda(\eta,\eta) + \text{tr}_g(\lambda)\tilde{g}(\eta,\eta)) = \rho' \text{tr}_g(\lambda).
\]

Using \( \tilde{\text{Ric}}(\eta,\eta) = \text{tr} \mathcal{R}_\eta \) and the trace of equation (24) we get

\[
\rho' \text{tr}_g(\lambda) = \rho(\text{tr}_g(\mathcal{L}_\eta \lambda) + \text{tr}_g(\lambda \times \lambda)).
\] (27)

We now start an induction by assuming that

\[
h = h^{(0)} + \sum_{k=0}^{\infty} \frac{x^k}{k!} h^{(k)} = h^{(0)} + h^{\text{rem}},
\] (28)

we have already seen that this assumption holds with \( k_0 = 2 \). The Ricci tensor depends smoothly on the metric. Therefore \( \text{Ric} - (n-2)h_0 \) vanishes in \( x \) to the same order as \( h^{\text{rem}} \), that is \( \text{Ric} - (n-2)h = O(x^{k_0}) \). If (28) holds then \( \lambda = O(x^{k_0-1}) \) and we conclude using \( \rho = x + O(x^3) \) that the lowest order terms in (24) are the terms of order \( x^{k_0-1} \), they are the Lie-derivative term on the left side and both terms on the right side. Keeping only the lowest order terms in (26) and simplifying gives us the equation

\[
(n - 1 - k_0)h^{(k_0)} + \text{tr}_{h_0}(h^{(k_0)})h^{(0)} = 0.
\] (29)

Next we look at the trace \( \text{tr}_{h_0}(h^{(k_0)}) \). First we assume that \( k_0 > 2 \). Then if we look at the lowest order terms the \( \lambda \times \lambda \) term vanishes and after simplification

\[
(k_0 - 2) \text{tr}_{h_0} h^{(k_0)} = 0,
\] (30)
which gives $\text{tr}_{h_0} h^{(k_0)} = 0$. If $k_0 = 2$ we take the $h_0$-trace of equation (23) and get $2(n - 2) \text{tr}_{h_0} h^{(2)} = 0$, so $\text{tr}_{h_0} h^{(k_0)}$ vanishes and (29) becomes

$$ (n - 1 - k_0) h^{(k_0)} = 0. \quad (31) $$

This means that as long as $k_0 < n - 1$, (28) implies $h^{(k_0)} = 0$, and we have showed that

$$ h = h^{(0)} + \sum_{k=n-1}^{\infty} \frac{x^k}{k!} h^{(k)}. \quad (32) $$

In order to show that the trace of $A - I_d$ vanishes to higher order, we do an induction on $\text{tr}_{h_0} h^{(k)}$. Assume $\text{tr}_{h_0} h^{(k)} = 0$ for $1 \leq k \leq j$. By the above this holds for $j = n - 2$ which is the base case for the induction. From (32) and (22) we have for $j < 2n - 3$

$$ \text{tr}_{\tilde{g}} \lambda = \text{tr}_{h_0} \lambda + O(x^{2n-3}) = -\frac{1}{2} \frac{x^j}{j!} \text{tr}_{h_0} h^{(j+1)} + O(x^{j+1}), $$

$$ \text{tr}_{\tilde{g}}(L_{\eta} \lambda) = \text{tr}_{h_0}(L_{\eta} \lambda) + O(x^{2n-4}) = -\frac{1}{2} \frac{x^{j-1}}{(j-1)!} \text{tr}_{h_0} h^{(j+1)} + O(x^j). $$

Now use $n > 3$, $\text{tr}_{\tilde{g}}(\lambda \times \lambda) = O(x^{2n-4})$ and (17) to get, after simplifying and discarding higher order terms,

$$ \text{tr}_{h_0} h^{(j+1)} = 0, \quad \text{for } j < 2n - 3. $$

From the definitions,

$$ \tilde{g}'(A^{-2}X,Y) = \tilde{g}(X,Y) = \tilde{g}'((I_d + H)X,Y), $$

where $H$ is defined by $\tilde{g}'(H,\eta X,Y) = h^{\text{rem}}(X,Y)$. Thus from the above $|H| = O(x^{n-1})$ and $\text{tr} H = O(x^{2n-2})$. By definition $A$ and $H$ are self adjoint linear maps w.r.t. the inner product $\tilde{g}'$. From the equation $A^2 = (I_d + H)^{-1}$ we find

$$ A = \sum_{k=0}^{\infty} \left( -\frac{1}{2k} \right) H^k = I_d - \frac{1}{2} H + O(x^{2n-2}) $$

This completes the proof of Lemma 5.4. \hfill \Box

**Remark 5.5.** Under the assumptions of Fefferman and Graham [5], in case $n$ is even, $h^{(n-1)} = 0$, and therefore instead of (32) we get that $h$ agrees with $h_0$ to any order in $x$. \hfill \Box

### 5.2. Proof of Theorem 5.2.

In case $n = 3$ $(M, g)$ Einstein implies constant curvature and the theorem follows from [2, Theorem 6.9], so we assume $n \geq 4$ in the following.

From equation (16) we have

$$ \hat{\nabla}_X \varphi_0 = (\nabla_X - \nabla_X) \varphi_0 - \frac{1}{2} (AX - X) \varphi_0, $$

outside the compact set where $f \neq 1$. Lemma 5.4 together with (8), (17) and the fact that $xe^x$ is bounded on $U$ gives

$$ |\hat{\nabla} \varphi_0|^2 \leq C(|A^{-1}|^2 |\nabla^2 A|^2 + |A - I_d|^2) x^{-1} = O(x^{2n-3}), $$
which using $\mu = \rho^{-n^-}\bar{\mu}$ shows that $\nabla \varphi_0 \in L^2(T^* \otimes S)(M)$. Clearly $\varphi_0 \notin H^1 S(M)$ and therefore $\varphi_0$ is an asymptotic Killing spinor.

Next we are going to show that the boundary integral (where the normal $\nu = \rho \eta$)

$$\lim_{x \to 0} \int_{\partial M_x} g((\nabla_\nu + \nu D)\varphi_0, \varphi_0) d\mu_x$$

vanishes, where $d\mu_x$ is the volume element of the induced metric on $\partial M_x$. Introduce an ON frame $\{e_i'\}$ w.r.t. $g'$ adapted to the foliation $\partial M_x$ so that $e_i' = \nu$ and $e_i' \perp \partial M_x$ for $i \geq 2$. Set $e_i = Ae_i'$. Then $\{e_i\}$ is an ON frame for $g$ adapted to the foliation $\partial M_x$. In the integrand we have

$$(\nabla_\nu + \nu D)\varphi_0 = \sum_i (\delta_{1i} + e_1 e_i) \nabla_{e_i} \varphi_0 = \sum_i \frac{1}{2} \sigma_{1i} \nabla_{e_i} \varphi_0$$

where $\sigma_{ij} = [e_i, e_j]$ is the commutator in the Clifford algebra. We compute

$$(\nabla X - \nabla X)\varphi_0 = \frac{1}{4} \sum_{i,j} (\omega_{ij}(X) - \overline{\omega}_{ij}(X)) e_i e_j \varphi_0 = \frac{1}{4} \sum_{i,j} (\omega_{ij}(X) - \overline{\omega}_{ij}(X)) \sigma_{ij} \varphi_0,$$

and

$$g((\nabla_\nu + \nu D)\varphi_0, \varphi_0) = \frac{1}{8} \sum_{i,j,k} (\omega_{jk}(e_i) - \overline{\omega}_{jk}(e_i)) g(\sigma_{1i}\sigma_{jk} \varphi_0, \varphi_0)$$

$$- \frac{1}{4} \sum_i g(\sigma_{1i}(Ae_i - e_i) \varphi_0, \varphi_0).$$

Since the integral in (33) is real, we need not bother about the imaginary terms in (34). What is left of the first sum is then

$$\sum_i (\omega_{1i}(e_i) - \overline{\omega}_{1i}(e_i)) g(\varphi_0, \varphi_0) + \frac{1}{2} \sum_{i,j,k} (\omega_{jk}(e_i) - \overline{\omega}_{jk}(e_i)) g(\sigma_{1i}\sigma_{jk} \varphi_0, \varphi_0),$$

where $\sigma_{1i,j,k} = e_1 e_i e_j e_k$ if $1, i, j, k$ are all different or else $\sigma_{1i,j,k} = 0$. From (2) and (3) we get the identities

$$\omega_{1i}(e_i) - \overline{\omega}_{1i}(e_i) = g(\overline{T}(e_i, e_1), e_i)$$

$$= g((\nabla_{e_1} A^{-1} e_i, e_i) - g((\nabla_{e_1} A) A^{-1} e_1, e_i))$$

and

$$2(\omega_{jk}(e_i) - \overline{\omega}_{jk}(e_i)) = -g(\overline{T}(e_i, e_j), e_k) + g(\overline{T}(e_i, e_k), e_j) + g(\overline{T}(e_j, e_k), e_i),$$

where the last two terms taken together are symmetric in $ij$ and vanish when summed against $\sigma_{1i,j,k}$. This leaves

$$\sum_i (g((\nabla_{e_1} A^{-1} e_i, e_i) - g((\nabla_{e_1} A) A^{-1} e_1, e_i)) g(\varphi_0, \varphi_0)$$

$$+ \frac{1}{2} \sum_{i,j,k} g((\nabla_{e_1} A) A^{-1} e_j, e_k) g(\sigma_{1i,j,k} \varphi_0, \varphi_0).$$
Next we look at the second sum in (34). We have
\[
\sum_i g(\sigma_{1i}(Ae_i - e_i)\varphi_0, \varphi_0) = 2 \sum_i g((\delta_{1i} + e_1 e_i)(Ae_i - e_i)\varphi_0, \varphi_0)
= 2 \sum_i g(e_1 e_i (Ae_i - e_i)\varphi_0, \varphi_0)
= 2 \sum_{i,j} (A^i_j - \delta^i_j) g(e_1 e_i e_j \varphi_0, \varphi_0)
\]
where \(A^i_j\) is defined by \(Ae_i = A^i_j e_j\) so that \(Ae_i = A^i_i e_i\). Since \(A^i_i\) is symmetric this simplifies to
\[
-2 \sum_i (A^i_i - \delta^i_i) g(e_1 \varphi_0, \varphi_0) = -2(\text{tr}_g(A) - n) g(e_1 \varphi_0, \varphi_0)
\]
and (the real part of) the integrand in (33) is
\[
\sum_i (g((\nabla'_{e_i} A) A^{-1} e_i, e_i) - g((\nabla'_{e_i} A) A^{-1} e_1, e_i)) g(\varphi_0, \varphi_0)
+ \frac{1}{2} \sum_{i,j,k} g((\nabla'_{e_i} A) A^{-1} e_j, e_k) g(\sigma_{1ijk} \varphi_0, \varphi_0) + \frac{1}{2} (\text{tr}_g(A) - n) g(e_1 \varphi_0, \varphi_0).
\]

We consider each term separately. To simplify the notation we will write \(f_1 \simeq f_2\) if \(f_1\) and \(f_2\) are both \(O(x^k)\) for some \(k\) and \(f_1 - f_2 = O(x^{k+n-1})\). Let \(B = A - Id\), note that \(B\) is symmetric and \(A^{-1} - Id \simeq -B\). In the following computations we use the summation convention. We have
\[
g((\nabla'_{e_i} A) A^{-1} e_i, e_i) = g'(A^{-1}(\nabla'_{e_i} A) e_i', e_i')
= g'(A^{-1}(\nabla'_{e_i} (Ae_i) - A \nabla'_{e_i} e_i'), e_i')
= (e_1 A^i_i) g'(A^{-1} e_i', e_i') + A^i_i g'(A^{-1} \nabla'_{e_i} e_i', e_i') - g'(\nabla'_{e_i} e_i', e_i')
\approx (e_1 A^i_i) g'(e_i', e_i') + B^i_i g'(\nabla'_{e_i} e_i', e_i') - g'(B \nabla'_{e_i} e_i', e_i')
= (e_1 A^i_i) + B^i_i g'(\nabla'_{e_i} e_i', e_i') - B^i_i g'(\nabla'_{e_i} e_i', e_i')
= (e_1 A^i_i),
\]
where in the last row we used the fact that \(B\) is symmetric. Similarly we have
\[
g((\nabla'_{e_i} A) A^{-1} e_1, e_i) = g'(A^{-1}(\nabla'_{e_i} A) e_1', e_i')
= g'(A^{-1}(Id - A) \nabla'_{e_i} e_1', e_i')
\approx -g'(B \nabla'_{e_i} e_1', e_i')
= -B^i_i g'(\nabla'_{e_i} e_1', e_i')
\approx -B^i_i g'(\nabla'_{e_i} e_1', e_i')
= -B^i_i (\rho \delta_{i1} \delta_{ii} + \rho g'(\nabla'_{e_i} e_1', e_i')),\]
where \(e_i' = \rho^{-1} e_i\) form an ON frame for \(\tilde{g}' = \rho^2 g'\). We now use the formula
\[
\tilde{g}'(\nabla_X e_i', e_j') = \omega'_{ij}(X) = \omega_{ij}(X) - \rho^{-1} (e_j' \rho) \tilde{g}'(e_i', X) - (e_i' \rho) \tilde{g}'(e_j', X))
\]
for the connection coefficients for $\nabla'$ w.r.t. the frame $\{\tilde{e}'_i\}$. Note that $\tilde{\omega}'_{1l}(\tilde{e}'_i) = 0$ for the cylindrical metric. Therefore
\[\rho g'((\nabla' e'_i)A^{-1} e_j, e_k) = -\rho' \delta_{1l} \delta_{1l} + \rho' \delta_{il}\]
and hence,
\[g((\nabla' e_i)A^{-1} e_j, e_k) \simeq -\rho' B^i_k.\]
Next we consider the terms in (37) containing $\sigma_{ijl}$. We have
\[g((\nabla' e_i)A^{-1} e_j, e_k)\sigma_{ijl} = g'(A^{-1}(\nabla' e_i)A^{-1} e_j, e_k)\sigma_{ijl} - g'(A^{-1}(\nabla' e_i)A^{-1} e_j, e_k)\sigma_{ijl} + \rho' g((\nabla' e_i)A^{-1} e_j, e_k)\sigma_{ijl}\]
\[\simeq \left(e_i A^k_j + B^k_j g((\nabla' e_i)A^{-1} e_j, e_k) - B^k_j g((\nabla' e_i)A^{-1} e_j, e_k)\right)\sigma_{ijl} + \rho' g((\nabla' e_i)A^{-1} e_j, e_k)\sigma_{ijl} = 0.\]
This means that $Q = g((\nabla' e_i)A^{-1} e_j, e_k)\sigma_{ijl}$ satisfies $Q = O(x^{2n-2})$. Finally, note that $\text{tr}_g A - n \simeq B^i_k$ where the components are defined w.r.t. the frame $\{e'_i\}$. This gives the following expression for the mass
\[\lim_{x \to 0} \int_{\partial M_x} g((\tilde{\nabla}_\nu + \nu D)\varphi_0, \varphi_0) d\mu_x = \lim_{x \to 0} \int_{\partial M_x} \left(\left((e_i A^i_j) + \rho' B^i_j\right)g(\varphi_0, \varphi_0) + \frac{1}{2} B^i_j g(e_i \varphi_0, \varphi_0)\right) d\mu_x.\]
(38)
From Lemma 5.4 we have $e_i A^i_j = O(x^{2n-2}), B^i_k = O(x^{2n-2})$ and as we have seen, $Q = O(x^{2n-2})$. Further, using (17) and $xe^r$ bounded on $U$ together with $d\mu_x = \rho^{-(n-1)}d\mu(h)$ we find that the integrand in (38) is $O(x^{n-2})$ which vanishes as $x \to 0$ and hence the mass is zero.

It follows from Lemma 4.4 that there is a Killing spinor $\varphi$ for $g$ associated to any Killing spinor $\varphi_0$ for the hyperbolic background metric $g'$ and hence there is a full set of linearly independent Killing spinors for $g$. This proves that $\tilde{R} = 0$ and hence $(M, g)$ has sectional curvature $-1$ and is isometric to hyperbolic space $\mathbb{H}^n$. □

5.3. Spherical space form boundary.

**Theorem 5.6.** Let $M$ conformally compact Einstein spin manifold with conformal boundary $\partial M = S^{n-1}/\Gamma$. Suppose that the spin structure on a collar neighborhood $U$ of $\partial M$ is equivalent to the spin structure on $\mathbb{H}^n/\Gamma$ defined by the lift $\tilde{\Gamma}$ of $\Gamma$. If $\tilde{\Gamma}$ fixes some non–zero spinor $u \in S$ then $\Gamma = \{1\}$ and $M$ is isometric to $\mathbb{H}^n$.

**Proof:** In case $n = 3$ $(M, g)$ Einstein implies constant curvature and the theorem follows from [3, Theorem 6.9], so we assume $n \geq 4$ in the following.
By passing to a cover of \( U \), the estimates of Lemma 5.4 apply without change. From the assumptions it follows that \( U \) with the hyperbolic background metric \( g' \) and its induced spin structure has a Killing spinor. Therefore by the same argument as in the proof of Theorem 5.2 there is an asymptotic Killing spinor with mass zero.

By Lemma 4.4 there is a Killing spinor on \((M, g)\) and therefore \((M, g)\) is a warped product as in Theorem 3.1.

It follows from the estimates of Lemma 5.4 that an integral of the form
\[
\int_{\partial M} (|\nabla'^2 A| + |\nabla' A|^2 + |A - Id|) \, d\mu_x
\]
is bounded. Therefore since we are considering only \( n \geq 4 \) the proof of Lemma 4.11 applies to the present case.

6. Four dimensions

We now consider the four dimensional case. Here we have isomorphisms
\[
\text{Spin}(4) = \text{SU}(2) \times \text{SU}(2) = \text{Sp}(1) \times \text{Sp}(1).
\]
The projection \( \text{Spin}(4) \to \text{SO}(4) \) takes \((p, q)\) to the map \( v \to pq^* \) (quaternion multiplication). The spinor representation is \( \text{SU}(2) \times \text{SU}(2) \) acting on \( S = S^+ \oplus S^- = \mathbb{C}^2 \oplus \mathbb{C}^{2\ast} \).

6.1. ALH ends in four dimensions. Suppose that \( \Gamma \subset \text{SO}(4) \) has a lift to \( \tilde{\Gamma} \subset \text{Spin}(4) \) such that for all \( \tilde{\gamma} \in \tilde{\Gamma} \), \( \tilde{\gamma} \cdot u = u \) where \( 0 \neq u \in S \). Then the same holds for the parts \( u_+, u_- \), by a choice of orientation we may assume \( u_- \neq 0 \). This means that for all \( \tilde{\gamma} \) the part in the second \( \text{SU}(2) \) factor, \( \tilde{\gamma}_- \), has one eigenvalue equal to one and since the determinant is one we must have \( \tilde{\gamma}_- = Id \). So \( S^- \) is fixed by \( \tilde{\Gamma} \) and \( \tilde{\Gamma} \) is a subgroup of the first \( \text{SU}(2) \) factor, which also means that \( \Gamma \subset \text{Sp}(1) = \text{SU}(2) \subset \text{SO}(4) \). The same reasoning gives that every finite subgroup of \( \text{SU}(2) \) acts freely on the sphere. We conclude;

**Proposition 6.1.** If \( \Gamma \) is a finite subgroup of \( \text{SU}(2) \) then with the above choice of orientation, the spinors \( \varphi_u \) with \( u \in S^- \) give Killing spinors on the quotient \( (\mathbb{H}^4_\ast) / \Gamma \) provided we choose the spin structure
\[
\text{Spin}( (\mathbb{H}^4_\ast) / \Gamma ) = \text{Spin}(\mathbb{H}^4_\ast) / \tilde{\Gamma}
\]
defined by the lift \( \gamma \to \tilde{\gamma} = (\gamma, Id) \). Except for reversing orientation these are the only cases allowing Killing spinors.

So we can only find asymptotic Killing spinors on an asymptotically locally hyperbolic four manifold if the fundamental group of the locally hyperbolic end is a finite subgroup of \( \text{SU}(2) \). The following groups are up to conjugation all finite subgroups of \( \text{SU}(2) \) (\cite[Thm. 2.6.7]{[Ref]}).

- **A\(_n\):** The cyclic group of order \( n \) generated by \( z = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \) where \( \zeta = e^{2\pi i/n} \).
- **D\(_n\):** The binary dihedral group of order \( 4n \), this consists of \( \{z^a, jz^a\}_{a=0}^{2n-1} \) where \( z = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \) with \( \zeta = e^{2\pi i/2n} \) and \( j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).
\(T^*\): The binary tetrahedral group.
\(O^*\): The binary octahedral group.
\(I^*\): The binary icosahedral group.

For these groups there is a canonical lift to Spin(4) and an associated canonical spin-structure on the end which we call the trivial spin structure. Any other lift of \(\Gamma\) to Spin(4) is given up to conjugation by an element \(\kappa \in \text{Hom}(\Gamma, \mathbb{Z}_2)\) as follows

\[\Gamma \ni \gamma \rightarrow \tilde{\gamma} = \kappa(\gamma)(\gamma, \text{Id}) \in \text{Spin}(4).\]

The elements of \(\text{Hom}(\Gamma, \mathbb{Z}_2)\) are

- \(A^n\): \(\kappa_0 = 1\) and if \(n\) is even \(\kappa_1\) defined by \(\kappa_1(z) = -1\).
- \(D^n\): \(\kappa_{pq}, p, q = 0, 1\) defined by \(\kappa_{pq}(z) = (-1)^p\) and \(\kappa_{pq}(j) = (-1)^q\).
- \(T^*\): 1.
- \(O^*\): \(\kappa_0 = 1\) and \(\kappa_1\) which is nontrivial.
- \(I^*\): 1.

These are thus also the possible spin structures on an asymptotically locally hyperbolic end.

6.2. Detecting the spin structure on the end. We will now see that the spin structure on an asymptotically locally hyperbolic end can almost be detected by the signature of the manifold. For simplicity we assume for the rest of this section that \(M\) is an asymptotically locally hyperbolic manifold with only one end. The problem of detecting the spin-structure is a purely topological one so we can forget the original metric on \(M\) and instead put on \(M\) a metric which is a product \(dt^2 + g_1\) on the end \((0, 1] \times (S^3/\Gamma)\), where \(g_1\) is the spherical metric. Note that we have added a boundary isometric to the spherical space form \((S^3/\Gamma)\). We can then apply the Atiyah–Patodi–Singer index theorem to compute the index of the Dirac operator and the signature of the manifold

\[\text{ind}(D) = -\frac{1}{24} \int_M p_1 - \eta_D(S^3/\Gamma),\]  
\[\sigma(M) = \frac{1}{3} \int_M p_1 - \eta_\sigma(S^3/\Gamma).\]

where \(\eta_\sigma(S^3/\Gamma)\) and \(\eta_D(S^3/\Gamma)\) are the eta-invariants of the Signature- and the Dirac-operator on the boundary. These formulas combine to

\[\sigma(M) + 8 \text{ind}(D) = -\eta_\sigma(S^3/\Gamma) - 8\eta_D(S^3/\Gamma).\]

On a four dimensional manifold the index of the Dirac operator is even since the spin representation is quaternionic, this means that the \(8 \text{ind}(D)\) term vanishes modulo 16 and

\[\sigma(M) \equiv -\eta_\sigma(S^3/\Gamma) - 8\eta_D(S^3/\Gamma) \pmod{16}.\]

The right hand side of (11) is known as the Rochlin invariant. This expression is useful since the eta invariants are explicitly computable and \(\eta_D\) involves the spin structure via \(\kappa\). The details of the computation of the eta invariants will be described in [1]. We summarize the results for the case of \(\Gamma \subset SU(2)\) in Table 2. From (11) we now have
Theorem 6.2. Let $M$ be a spin manifold of dimension 4 with boundary $\partial M = S^3/\Gamma$ where $\Gamma \subset SU(2)$. Fix the orientation of $M$ so that the induced orientation on $\partial M$ is standard. Then if $\sigma(M) \pmod{16}$ does not take the value corresponding to one of the nontrivial spin structures in Table 3, then the induced spin structure on $\partial M$ is trivial.

In the following we assume the orientation of $M$ is chosen as in Theorem 6.2. Combining Theorem 6.2 with Theorem 4.8 proves

Corollary 6.3. Let $(M,g)$ be a 4 dimensional complete spin manifold with one ALH end with group $\Gamma \subset SU(2)$ and $s \geq 0$. Then $\sigma(M) \pmod{16}$ does not take the value of the trivial spin structure corresponding to $\Gamma$ in Table 3 unless $(M,g)$ is isometric to $\mathbb{H}^4$. Therefore the allowed groups and signatures are those listed in Table 4.

Similarly, Theorem 6.2 combined with Theorem 5.6 proves

Corollary 6.4. Let $(M,g)$ be a 4 dimensional Einstein spin manifold which is conformally compact with conformal boundary $\partial M = S^3/\Gamma$ with $\Gamma \subset SU(2)$. Then $\sigma(M) \pmod{16}$ does not take the value of the trivial spin structure corresponding to $\Gamma$ in Table 3 unless $(M,g)$ is isometric to $\mathbb{H}^4$. Therefore the allowed groups and signatures are those listed in Table 4.

Table 1. Allowed groups and signature

| Group | $\sigma(M) \pmod{16}$ |
|-------|------------------|
| $\{id\}$ | 0 |
| $A_n$ (n even) | 1 |
| $D_n^+$ | $-n,-2,0$ |
| $O^*$ | $-1$ |

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Table 2. Eta invariants and signature

| Group       | Spin structure | $\eta_0$ | $\eta_D$ | $\sigma(M)(\text{mod }16)$ |
|-------------|----------------|----------|----------|---------------------------|
| $A_n$ (n even) | $\kappa_0$ | $\frac{(n-1)(n-2)}{3n}$ | $\frac{n^2-1}{12n}$ | $-n + 1$ |
|             | $\kappa_1$ | $-\frac{n^2+2}{12n}$ | $1$ |                     |
| $A_n$ (n odd) | $\kappa_0$ | $\frac{(n-1)(n-2)}{3n}$ | $\frac{n^2-1}{12n}$ | $-n + 1$ |
| $D^*_n$     | $\kappa_{00}$ | $\frac{2n^2+1}{6n}$ | $\frac{4n^2+12n-1}{48n}$ | $-n - 2$ |
|             | $\kappa_{01}$ | $\frac{4n^2-1}{48n}$ | $-n$ |                     |
|             | $\kappa_{10}$ | $-\frac{2n^2-12n+1}{48n}$ | $-2$ |                     |
|             | $\kappa_{11}$ | $-\frac{2n^2+1}{48n}$ | $0$ |                     |
| $T^*$       |               | $49$ | $167$ | $-6$ |
| $O^*$       | $\kappa_0$ | $\frac{121}{72}$ | $\frac{383}{576}$ | $-7$ |
|             | $\kappa_1$ | $\frac{49}{576}$ | $-1$ |                     |
| $I^*$       |               | $\frac{361}{180}$ | $\frac{1079}{1440}$ | $-8$ |

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