Gravitational Collapse of Phantom Fluid in (2 + 1)-Dimensions*

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Abstract
This paper is devoted to the solutions of Einstein’s field equations for a circularly symmetric anisotropic fluid, with kinematic self-similarity of the first kind, in (2 + 1)-dimensional spacetimes. In the case where the radial pressure vanishes, we show that there exists a solution of the equations that represents the gravitational collapse of an anisotropic fluid, and this collapse will eventually form a black hole, even when it is constituted by the phantom energy.

1 Introduction
Self-similar solutions of Einstein field equations have attracted a great deal of recent attentions, not only because they can be studied analytically through

*modified version
simplifications, but also due to their relevance in astrophysics [1] and critical phenomena in gravitational collapses [2, 3].

Lately, we investigated the self-similar solutions in various spacetimes [4]. In particular, we considered a massless scalar field in $(2 + 1)$-dimensional circularly symmetric spacetimes with kinematic self-similarity of the second kind in the context of Einstein’s theory, and acquired all such solutions [5]. We further studied their local and global properties and found that some of them represent gravitational collapses of a massless scalar field, in which black holes are always formed. In another discussion we discussed an anisotropic fluid, with the same self-similarity, and showed that the existing solution is unique when the radial pressure vanishes, and it represents a collapsing dust fluid, where the final output can be either naked singularities or black holes [6].

In this paper, we extend the aforementioned studies to the case of an anisotropic fluid with zero radial pressure while the self-similarity of the first kind is considered.

2 Solutions of the field equations

The general metric can be given by

$$ds^2 = e^{2\phi} dt^2 - e^{2\psi} dr^2 - r^2 S^2 d\theta^2.$$  \hfill (1)

The non-null components of Einstein’s tensor are

\[ G_{tt} = \frac{e^{-2\psi}}{r S} \left\{ e^{2\phi} \psi_r r S, r + e^{2\phi} \psi_t r S, t - e^{2\phi} r S, r - 2 e^{2\phi} S, r + e^{2\phi} \psi_t r S, t \right\}, \] \hfill (2)

\[ G_{tr} = \frac{1}{r S} \left\{ \phi_r r S, t + \psi_t r S, r + \psi_t S - r S, tr - S, t \right\}, \] \hfill (3)

\[ G_{rr} = \frac{e^{-2\phi}}{r S} \left\{ e^{2\phi} \phi_r r S, r + e^{2\phi} \phi_t r S, t + e^{2\phi} \phi_t r S, t - e^{2\psi} r S, tt \right\}, \] \hfill (4)

\[ G_{\theta\theta} = r^2 S^2 \left\{ e^{-2\psi} \phi_r^2 - e^{-2\psi} \phi_r \psi_r + e^{-2\psi} \phi_t \psi_r + e^{-2\psi} \phi_t \psi_t \right. \]
\[ \left. - e^{-2\psi} \psi_t^2 - e^{-2\psi} \psi_t^2 \right\} . \] \hfill (5)
To study properties of these solutions with self-similarity of the first kind, we introduce two new dimensionless variables, \( \chi \) and \( \tau \), through the relations

\[
\chi = \ln \left( \frac{r}{(-t)} \right),
\]

(6)

\[
\tau = -\ln (-t).
\]

(7)

Note that the range of the time coordinate considered is \(-\infty < t \leq 0\).

On the other hand, a self-similar solution is defined by

\[
\phi (t, r) = \phi (\chi), \quad \psi (t, r) = \psi (\chi), \quad S (t, r) = S (\chi).
\]

(8)

Thus, Eqs. (2)-(5) become,

\[
G_{tt} = -\frac{1}{r^2 S e^{2\psi}} \left\{ e^{2\phi} [S_{,\chi \chi} + S_{,\chi} - \psi_{,\chi} (S_{,\chi} + S)] - \frac{r^2}{t^2} S_{,\chi} e^{2\psi} \right\},
\]

(9)

\[
G_{tr} = \frac{1}{tr S} \left\{ S_{,\chi \chi} - \psi_{,\chi} (S_{,\chi} + S) - S_{,\chi} (\phi_{,\chi} - 1) \right\},
\]

(10)

\[
G_{rr} = \frac{1}{r^2 S e^{2\phi}} \left\{ e^{2\phi} [\phi_{,\chi} (S_{,\chi} + S)]
- \frac{r^2}{t^2} e^{2\psi} (S_{,\chi \chi} - S_{,\chi} \phi_{,\chi}) - \frac{r^2}{t^2} e^{2\psi} S_{,\chi} \right\},
\]

(11)

\[
G_{\theta\theta} = S^2 \left\{ e^{-2\psi} [\phi_{,\chi \chi} + \phi_{,\chi} (\phi_{,\chi} - \psi_{,\chi} - 1)] - \frac{r^2 e^{-2\phi}}{t^2} \psi_{,\chi \chi}
- \psi_{,\chi} (\phi_{,\chi} - \psi_{,\chi} - 1) \right\}.
\]

(12)

The momentum-energy tensor is given by

\[
T_{\mu\nu} = \rho u_\mu u_\nu + p_r r_\mu r_\nu + p_\theta \theta_\mu \theta_\nu,
\]

(13)

where \( \rho \) is the energy density, \( p_r \) and \( p_\theta \) are the radial and the tangential pressures, respectively, with \( u_\mu, r_\mu \) and \( \theta_\mu \) being given by

\[
u_\mu = e^{\phi (\chi)} \delta_\mu^t, \quad r_\mu = e^{\psi (\chi)} \delta_\mu^r, \quad \theta_\mu = r S (\chi) \delta_\mu^\theta.
\]

(14)
where a comoving frame is adopted.

Substituting Eqs. (9)-(12) and (13) into the following field equations

\[ G_{\mu \nu} = \kappa T_{\mu \nu}, \]  

we obtain that

\[ \rho = -\frac{1}{\kappa} \left\{ \frac{1}{r^2} e^{-2\psi} [y_\chi + (y + 1)(y - \psi_\chi)] - \frac{1}{l^2} e^{-2\phi} \psi_\chi y \right\}, \]

\[ y_\chi = y\phi_\chi + (y + 1)(\psi_\chi - y), \]

\[ p_r = \frac{1}{l^2 \kappa} \left\{ \frac{1}{r^2} e^{-2\psi} \phi_\chi (y + 1) - \frac{1}{l^2} e^{-2\phi} [y_\chi + y(y - \phi_\chi + 1)] \right\}, \]

\[ p_\theta = \frac{1}{l^2 \kappa} \left\{ \frac{e^{-2\psi}}{r^2} [\phi_{,\chi \chi} + \phi_\chi (\phi_\chi - \psi_\chi - 1)] - \frac{e^{-2\phi}}{l^2} \psi_{,\chi \chi} + \psi_\chi (\phi_\chi - \psi_\chi - 1) \right\}. \]

Note that in above equations we set

\[ y(\chi) = \frac{S_\chi}{S}. \]

Therefore we have 4 equations for the 6 functions to determine, that is, \( \phi, \psi, S, \rho, p_r \) and \( p_\theta \). For this, we introduce two additional equations, \( p_r = 0 \) and \( p_\theta = \omega \rho \) for forming a consistent and solvable system. The two additional state equations furnish

\[ \phi_{,\chi} (y + 1) - e^{2(\chi - \phi + \psi)} [y_\chi + y(y - \phi_\chi + 1)] = 0 \]

and

\[ \phi_{,\chi \chi} + \phi_\chi (\phi_\chi - \psi_\chi - 1) + e^{2(\chi - \phi + \psi)} [-\omega \psi_{,\chi} y - \psi_{,\chi \chi} + \psi_\chi (\phi_\chi - \psi_\chi - 1)] + \omega [y_\chi + (y + 1)(y - \psi_\chi)] = 0 \]

A substitution of equation (17) into equation (21) yields
\[(y + 1) \left[ \phi_{,\chi} - e^{2(\chi - \phi + \psi)} \psi_{,\chi} \right] = 0. \quad (23)\]

Thus, we have two possible solution equations which are

\[(i) \quad y + 1 = 0, \quad (24)\]
\[(ii) \quad \phi_{,\chi} - e^{2(\chi - \phi + \psi)} \psi_{,\chi} = 0. \quad (25)\]

Although we are not able to solve analytically equation (25), for the former case \((i)\), we have the solutions

\[S(\chi) = S_0 e^{-\chi}, \quad (26)\]
\[\psi(\chi) = \psi_0 + \ln \left[ a + e^{(\omega - 1)\chi} \right], \quad (27)\]

and

\[\phi(\chi) = \phi_0, \quad (28)\]

where \(\phi_0, \psi_0, S_0\) and \(a\) are integration constants.

Thus, without lost of a generality, the corresponding metric (1) can be reformulated in the form,

\[ds^2 = dt^2 - \left[ a + \left( \frac{r}{-t} \right)^{\omega - 1} \right]^2 dr^2 - S_0^2 t^2 d\theta^2, \quad (29)\]

due to the fact that we have set \(\phi_0 = \psi_0 = 0\). The energy density and the tangential pressure can be written, respectively, as

\[\rho(r, t) = \frac{(1 - \omega)r^{\omega - 1}}{l^2 K t^2 \left[ (-t)^{\omega - 1} a + r^{\omega - 1} \right]}, \quad (30)\]

and

\[p_\theta (r, t) = \frac{\omega(1 - \omega)r^{\omega - 1}}{l^2 K t^2 \left[ (-t)^{\omega - 1} a + r^{\omega - 1} \right]}, \quad (31)\]
The Kretschmann’s scalar, for above solutions are given by

\[ K(r,t) = \frac{4(\omega - 1)^2(\omega^2 + 1)}{t^4 e^{4\phi_0} [1 + ar^{(1-\omega)(-t)^(\omega-1)}]^2}. \]  

(32)

So, there is the possibility of the formation of two singularities which are

\[ t = t_{\text{sing}} = 0 \]  

(33)

and

\[ r = r_{\text{sing}} = \left[ \frac{(-t)}{|a|^{1/(1-\omega)}} \right], \quad \text{for} \quad a < 0. \]  

(34)

Further, the geometric radius, \( Rg \), is defined by

\[ Rg = S \, r = -S_0 \, t, \]  

(35)

where \( S_0 \geq 0 \).

We note the fact that a geometric radius decreases with respect to the time indicates that the process represents a collapse.

The expansions of the ingoing and outgoing null geodesics congruence is useful for understanding global properties of the solutions. The latter are given by

\[ \theta_l = \theta_n = -S_0. \]  

(36)

Now, observe that both \( \theta_l \) and \( \theta_n \) are always negative. Therefore, the solutions we have obtained must represent a region inside the event horizon of a black hole. To support this hypothesis, it is necessary to cut the current spacetime and match it with a different spacetime which represents the exterior of a black hole.
3 The energy conditions

For $a > 0$, the signs of the energy density and of the tangential pressure can be determined by the terms $1 - \omega$ and $\omega(1 - \omega)$, respectively. Thus, it is clear to see that for $-1 \leq \omega \leq 1$, all the required energy conditions are satisfied. If we admit the dark energy, we can construct a collapse model of a phantom anisotropic fluid, with $\omega \leq -1$. In the case, all the energy conditions are violated, while the energy density preserves its positivity.

4 The junction conditions

4.1 Junction conditions without thin shells

In order to match the interior fluid solution with the exterior vacuum solution we consider the Darmois junction conditions [7], which requires that the first and the second fundamental forms (that are the metric and the extrinsic curvature, respectively) be continuous across the junction hypersurface.

We divide the current spacetime into following two regions: one of them is called the interior region, constituted by a circularly symmetric anisotropic fluid with kinematic self-similarity of the first kind ($V^-$, for $r \leq r_{\Sigma}$), and the other, called the exterior region, fulfilled by vacuum in the presence of negative cosmological constant ($V^+$, em $r \geq r_{\Sigma}$), where $r_{\Sigma}$ is the radial coordinate of the hypersurface.

The general metric which describes the internal region, $ds_+^2$, has been given by equation (29), with $0 \leq r \leq r_{\Sigma}$, $0 \leq \theta < 2\pi$ and $-\infty < t \leq 0$.

As for the exterior spacetime, we have the BTZ solution described by the metric

$$ds_+^2 = -(\Lambda R^2 + M)dT^2 + \frac{1}{\Lambda R^2 + M}dR^2 - R^2 d\theta^2,$$

where $R_{\Sigma} \leq R < \infty$, $-\infty < T \leq 0$, $0 \leq \theta < 2\pi$, $\Lambda < 0$ and $M$, the mass of the black hole, where event horizon is located on

$$R_{EH} = \sqrt{-\frac{M}{\Lambda}}.$$
On the junction hypersurface, the intrinsic metric reduces to

\[ ds^2 = d\tau^2 - H(\tau)^2 d\theta^2, \tag{38} \]

where \( \tau \) is the proper time.

To apply the junction conditions, we first require the continuity of the metric potential, that is,

\[ (ds^2)_\Sigma = (ds^2)_\Sigma = (ds^2)_\Sigma, \tag{39} \]

which indicates that

\[ d\tau = dt, \tag{40} \]

\[ \frac{dt}{d\tau} = \sqrt{-\left(\Lambda R^2 + M\right)^2 + \left(\frac{dR_\Sigma(T)}{dT}\right)^2}, \tag{41} \]

\[ R^2_\Sigma(T) = S^2_0 \tau^2. \tag{42} \]

Further, the second fundamental form, or the extrinsic curvature, is defined as

\[ K_{ij} = \eta^\pm \frac{\partial^2 x^\alpha}{\partial \xi^i \partial \xi^j} - \eta^\pm \Gamma^\alpha_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^i} \frac{\partial x^\nu}{\partial \xi^j}, \tag{43} \]

where

\[ \eta^\pm = L \frac{\partial f^\pm}{\partial x^\alpha}, \tag{44} \]

are the unit normal vectors and \( f^\pm \) is the function which defines the hypersurface, given by

\[ f^- = r - r_\Sigma = 0, \tag{45} \]

\[ f^+ = R - R_\Sigma(T) = 0. \tag{46} \]
On the other hand, the unit normal vectors, $\eta^\pm_\alpha$, are given respectively by

$$\eta^-_\alpha = \left[ a + \left( \frac{r_\Sigma}{-t} \right)^{(\omega-1)} \right] (0, 1, 0), \quad (47)$$

and

$$\eta^+_\alpha = \frac{\sqrt{\Lambda R^2_\Sigma + M}}{-\Lambda R^2_\Sigma + M + \left( \frac{dR_\Sigma(T)}{dT} \right)^2} \left( -\frac{dR_\Sigma(T)}{dT}, 1, 0 \right). \quad (48)$$

We find that all components of the extrinsic curvature for the interior spacetime are null, while for the exterior spacetime the non-null components are only given by

$$K^\pm_{00} = \frac{\Lambda R^2_\Sigma + M}{\sqrt{-\left( \Lambda R^2_\Sigma + M \right)^2 + \left( \frac{dR_\Sigma(T)}{dT} \right)^2}} 
\times \left\{ \frac{\partial^2 R_\Sigma(T)}{\partial \tau^2} + 3 \left( \frac{\partial R_\Sigma(T)}{\partial \tau} \right)^2 \left( \frac{\Lambda R_\Sigma}{\Lambda R^2_\Sigma + M} \right) 
- \Lambda R_\Sigma (\Lambda R^2_\Sigma + M) \left( \frac{\partial T}{\partial \tau} \right)^2 
- \frac{\partial R_\Sigma(T)}{\partial T} \frac{\partial^2 T}{\partial \tau^2} \right\} \quad (49)$$

as well as

$$K^\pm_{22} = -\frac{\Lambda R^2_\Sigma + M}{\sqrt{-\left( \Lambda R^2_\Sigma + M \right)^2 + \left( \frac{dR_\Sigma(T)}{dT} \right)^2}} R_\Sigma \left( \Lambda R^2_\Sigma + M \right). \quad (50)$$

Then the continuity of the second fundamental forms, which are given by

$$\left( K^-_{00} \right)_\Sigma = \left( K^0_{00} \right)_\Sigma = \left( K^+_{00} \right)_\Sigma \quad (51)$$

and
furnishes

\[
\left( K_{22}^{-}\right)_{\Sigma} = (K_{22})_{\Sigma} = (K_{22}^{+})_{\Sigma}, \tag{52}
\]

\[
\begin{align*}
\sqrt{-\left( \Lambda R_{\Sigma}^{2} + M \right)} & + \left( \frac{\partial R_{\Sigma}(T)}{\partial \tau} \right)^{2} \\
& \times \left\{ \frac{\partial^{2} R_{\Sigma}(T)}{\partial \tau^{2}} + 3 \left( \frac{\partial R_{\Sigma}(T)}{\partial \tau} \right)^{2} \left( \frac{\Lambda R_{\Sigma}}{\Lambda R_{\Sigma}^{2} + M} \right) \right. \\
& \left. - \Lambda R_{\Sigma}(\Lambda R_{\Sigma}^{2} + M) \left( \frac{\partial T}{\partial \tau} \right)^{2} - \frac{\partial R_{\Sigma}(T)}{\partial T} \frac{\partial^{2} T}{\partial \tau^{2}} \right\} = 0 \tag{53}
\end{align*}
\]

and

\[
- \sqrt{-\left( \Lambda R_{\Sigma}^{2} + M \right)} \left( \frac{\partial R_{\Sigma}(T)}{\partial \tau} \right)^{2} R_{\Sigma} \left( \Lambda R_{\Sigma}^{2} + M \right) = 0. \tag{54}
\]

It follows that there are two solutions, which satisfy the constraints given by equations (53) and (54) simultaneously, and they are

\[ R_{\Sigma}(T) = 0 \tag{55} \]

and

\[ R_{\Sigma}(T) = \sqrt{-\frac{M}{\Lambda}} = R_{EH}. \tag{56} \]

Equations (55) and (56) imply that it is not possible to matching the space-times without the introduction of a thin shell.
4.2 Junction conditions with thin shells

It is observed that conditions (40) and (41) furnish

\[
\frac{dT}{d\tau} = \sqrt{\frac{S_0^2 - (\Lambda R^2_\Sigma + M)}{(\Lambda R^2_\Sigma + M)^2}},
\]

(57)

and

\[
\frac{d^2T}{d\tau^2} = \frac{S_0\Lambda R_\Sigma}{(\Lambda R^2_\Sigma + M)^2\sqrt{S_0^2 - (\Lambda R^2_\Sigma + M)}} \left[2S_0^2 - (\Lambda R^2_\Sigma + M)\right].
\]

(58)

Substituting the equations (57), (58) into (49) and (50), we may rewrite the components of the \(K^+_{ij}\) tensor as

\[
K^+_{00} = \frac{\Lambda R_\Sigma \{4S_0^4 - (\Lambda R^2_\Sigma + M)(2S_0^2 + (\Lambda R^2_\Sigma + M))\}}{(\Lambda R^2_\Sigma + M)^2 \sqrt{S_0^2 - (\Lambda R^2_\Sigma + M)}},
\]

(59)

and

\[
K^+_{22} = -R\sqrt{S_0^2 - (\Lambda R^2_\Sigma + M)}.
\]

(60)

The momentum-energy tensor of the thin shell can be rewritten in an alternative form as

\[
\Pi_{\mu\nu} = \frac{1}{\kappa} \left\{ K^\mu_\nu - K^+_{\mu\nu} - g_{\mu\nu} g^{\alpha\beta} \left[ K^{-}_\alpha\beta - K^+_{\alpha\beta} \right] \right\},
\]

(61)

where \(\kappa = 8\pi\). The non-null components are thus given by

\[
\Pi_{00} = -\frac{1}{R_{\Sigma}\kappa} \left\{ (\Lambda R^2_\Sigma + M)\sqrt{S_0^2 - (\Lambda R^2_\Sigma + M)} \right\}
\]

(62)

and

\[
\Pi_{22} = \frac{\Lambda R^3_\Sigma}{\kappa} \left\{ \frac{4S_0^4 - (\Lambda R^2_\Sigma + M)(2S_0^2 + (\Lambda R^2_\Sigma + M))}{(\Lambda R^2_\Sigma + M)^3 \sqrt{S_0^2 - (\Lambda R^2_\Sigma + M)}} \right\}.
\]

(63)
We can also rewrite the $\Pi_{\mu\nu}$ tensor in the form

$$\Pi_{\mu\nu} = \sigma u_{\mu} u_{\nu} + \xi_{\theta} \Theta_{\mu} \Theta_{\nu}, \quad (64)$$

where $\sigma$ and $\xi$ are the energy density and the tangential pressure of the shell, respectively, and

$$u_{\mu} = \sqrt{g_{\mu\nu}} \delta_{\mu}^{t} \text{ and } \Theta_{\mu} = \sqrt{-g_{\nu}^{2}} \delta_{\mu}^{\theta}.$$

Utilizing the components of the metric, momentum-energy tensor and equations (62) and (63) for equation (64), we observe immediately that the shell’s energy density and the tangential pressures can be given by

$$\sigma = \frac{1}{R_{\Sigma} \kappa} \sqrt{S_{0}^{2} - (\Lambda R_{\Sigma}^{2} + M)} \quad (65)$$

and

$$\xi_{\theta} = \frac{\Lambda R_{\Sigma}}{\kappa} \left\{ \frac{4 S_{0}^{4} - (\Lambda R_{\Sigma}^{2} + M) [2 S_{0}^{2} + (\Lambda R_{\Sigma}^{2} + M)]}{(\Lambda R_{\Sigma}^{2} + M)^{3} \sqrt{S_{0}^{2} - (\Lambda R_{\Sigma}^{2} + M)}} \right\}. \quad (66)$$

### 4.3 The energy conditions of thin shell

A weak energy condition requires

$$\sigma \geq 0 \quad (67)$$

and

$$\sigma + \xi_{\theta} \geq 0, \quad (68)$$

while the strong energy condition requires only the condition (68) and the dominant energy condition requires equations (67), (68) and further,

$$\sigma - \xi_{\theta} \geq 0. \quad (69)$$

In order to secure real $\sigma$, we must have
\[ S_0^2 - (\Lambda R^2_\Sigma + M) \geq 0 \quad \rightarrow \quad S_0^2 \geq \Lambda R^2_\Sigma + M, \quad (70) \]

which implies that the value of \( \sigma \) is always positive, while equations (68) and (69) furnish, respectively,

\[ S_0^2 - (\Lambda R^2_\Sigma + M) \geq \frac{|\Lambda| R_\Sigma F}{(\Lambda R^2_\Sigma + M)^3}, \quad (71) \]
\[ S_0^2 - (\Lambda R^2_\Sigma + M) \geq -\frac{|\Lambda| R_\Sigma F}{(\Lambda R^2_\Sigma + M)^3}, \quad (72) \]

where \( F = -\Lambda^2 R^4_\Sigma - 2\Lambda(M + S^2_0)R^2_\Sigma + G(S_0, M) \).

The metric (37), which describes the BTZ solution, is defined only for \( (\Lambda R^2_\Sigma + M) < 0 \), representing the exterior spacetime of a black hole, which indicates that \( S_0^2 - (\Lambda R^2_\Sigma + M) \geq 0 \) is always true. Thus, to summarize, we may rewrite conditions (71) and (72) for the cases when \( F < 0 \) and \( F \geq 0 \) as

\( i \) \( F < 0 \):
\[ S_0^2 - (\Lambda R^2_\Sigma + M) \geq \frac{|\Lambda| R_\Sigma F}{(\Lambda R^2_\Sigma + M)^3}, \quad (73) \]

for \( S_0^2 - (\Lambda R^2_\Sigma + M) < -\left[ S^2_0 + S_0 \sqrt{S^2_0 + 4} \right] \),

\( ii \) \( F \geq 0 \):
\[ S_0^2 - (\Lambda R^2_\Sigma + M) \geq \frac{|\Lambda| R_\Sigma F}{(\Lambda R^2_\Sigma + M)^3}, \quad (74) \]

for \( -\left[ S^2_0 + S_0 \sqrt{S^2_0 + 4} \right] \leq S_0^2 - (\Lambda R^2_\Sigma + M) < 0 \).

5 Conclusion

We obtain a solution of the Einstein equations for an anisotropic and circularly symmetric, self-similar fluid of the first kind in a (2+1)-dimensional spacetime. We introduce the state equations, \( p_r = 0 \) as well as \( p_\theta = \omega \rho \), in
order to solve the anticipated problem. It is shown that there is such a solution which represents gravitational collapse of an anisotropic fluid. The final output can be either a normal black hole or a black hole made of phantom.

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