Research Article

Poincaré Bifurcations of Two Classes of Polynomial Systems

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Using bifurcation methods and the Abelian integral, we investigate the number of the limit cycles that bifurcate from the period annulus of the singular point when we perturb the planar ordinary differential equations of the form

\[
\begin{align*}
\dot{x} &= -yC(x, y), \\
\dot{y} &= xC(x, y),
\end{align*}
\]

with an arbitrary polynomial vector field, where \(C(x, y) = 1 - x^3\) or \(C(x, y) = 1 - x^4\).

1. Introduction and Main Results

In the qualitative theory of real planar differential systems, one of the main problems is to determine the existence and number of the limit cycles of the polynomial differential system. In the general case, this is a very difficult task. Therefore, the researchers consider the weak Hilbert 16th problem. In addition, the existence of invariant algebraic curves in polynomial systems may influence the number of limit cycles. For example, the planar quadratic systems with one invariant line or conic curve or cubic curve can have at most one limit cycle [1–4]. In [3], the authors proved that the cubic systems with four invariant lines have at most one limit cycle. In [5, 6], the authors proved that a real polynomial system of degree \(m\) with irreducible invariant algebraic curves has at most \(1 + (m - 1)(m - 2)/2\) limit cycles if \(m\) is even and \((m - 1)(m - 2)/2\) limit cycles if \(m\) is odd.

In this paper, we consider the weak Hilbert 16th problem that the unperturbed systems have a linear center and an invariant algebraic curve

\[
\begin{align*}
\dot{x} &= -yC(x, y) + \varepsilon P(x, y), \\
\dot{y} &= xC(x, y) + \varepsilon Q(x, y),
\end{align*}
\]

where \(P(x, y)\) and \(Q(x, y)\) are polynomials of degree \(n\) in \(\mathbb{R}^2\), the algebraic curve \(C(x, y)\) satisfies \(C(0, 0) \neq 0\), and \(\varepsilon \in \mathbb{R}\) is a sufficient small parameter.

It is obvious that, on the region \(\Omega = \{(x, y) \mid C(x, y) \neq 0\}\), the system (1) is equivalent to the following form:

\[
\begin{align*}
\dot{x} &= -y + \varepsilon \frac{P(x, y)}{C(x, y)}, \\
\dot{y} &= x + \varepsilon \frac{Q(x, y)}{C(x, y)}.
\end{align*}
\]

When \(\varepsilon = 0\), system (2) is a Hamilton system with a family of ovals

\[
\gamma_h = \{(x, y) \in \mathbb{R}^2 \mid H(x, y) = x^2 + y^2 = h, \ h > 0\}.
\]

Define the Abelian integral

\[
\Phi(h) = \oint_{\gamma_h} \frac{P(x, y) \, dy - Q(x, y) \, dx}{C(x, y)},
\]

which is also called first-order Melnikov function of (2). According to the Poincaré-Pontryagin theorem [7], the number of isolated real zeros of \(\Phi(h)\) controls the number of limit cycles of system (1) that bifurcate from the periodic annulus of the perturbed system (1) with \(\varepsilon = 0\). That is to say, when the \(\Phi(h)\) does not vanish exactly, the maximum number of the isolated real zeros of \(\Phi(h)\) is corresponding to the upper bound of the number of limit cycles which bifurcate from periodic annulus of unperturbed systems.

For the arbitrary polynomials \(P(x, y), Q(x, y)\) of given degree \(n\), the number of limit cycles of (2) depends on
the different choices of $C(x, y)$. At present, several works have figured out this problem for the particular choices of $C(x, y)$. In [8], the authors studied the system (1) with $C(x, y) = 1 + x$ and proved that the number of limit cycles that bifurcate from the period orbits is at most $n$. The authors in [9, 10] studied the number of limit cycles which bifurcate from (1) when $\varepsilon = 0$ with $C(x, y) = 1 + Bx + Ax^2$ and $C(x, y) = 1 + ax + by + cx(x^2 + y^2)$, respectively. The authors in [11] studied the number of limit cycles of system (1) with $C(x, y) = y^2 + Ax^2 + Bx + C$. In [12], the authors studied the number of limit cycles of system (1) with $C(x, y) = (x + a)(x + b)$ and obtained that the system can have at most $3[(n - 1)/2] + 2$ limit cycles if $a \neq b$ and $2[(n - 1)/2] + 1$ if $a = b$, respectively.

In [13], the authors studied the case the curves $C(x, y) = 0$ are three lines, two of them parallel and one perpendicular, and [14, 15] studied the case the curves are $k$ $(k > 3)$ lines, and any two of them are parallel or perpendicular directions. The authors in [16] studied the case the curves are consistent by $k$ nonzero points. The authors in [17] considered system (1) with $C(x, y) = 1 + x^4$ and proved that $3[(n + 1)/2] - 2$ limit cycles can at most bifurcate from the periodic orbits of the unperturbed system. In [18], the authors proved that the system (1) with $C(x, y) = (1 - x)^m$ has at most $n + m - 1$ limit cycles.

The aim of this paper is to investigate the upper bound of the number of limit cycles bifurcate from the periodic annulus of the center of the unperturbed system (1) ($\varepsilon = 0$) with the perturbed polynomials $P(x, y), Q(x, y)$ of given degree $n$, and $C(x, y) = 1 - x^3$ or $C(x, y) = 1 - x^4$.

Consider the planar differential system

$$\begin{align*}
\dot{x} &= -yC(x, y) + \varepsilon \sum_{0 \leq k + j \leq n} a_{k,j} x^k y^j, \\
\dot{y} &= xC(x, y) + \varepsilon \sum_{0 \leq k + j \leq n} b_{k,j} x^k y^j.
\end{align*}$$

(5)

where $\varepsilon \in \mathbb{R}$ is a sufficient small parameter. Applying the Abelian integral, we obtain the following two main theorems.

**Theorem 1.** If $C(x, y) = 1 - x^3$, the lower bound of the maximum number of limit cycles bifurcating from the period orbits of system (5) with $\varepsilon = 0$ is $4[(n + 1)/2] - 2$.

**Theorem 2.** If $C(x, y) = 1 - x^4$, the upper bound of the maximum number of limit cycles bifurcating from the period orbits of system (5) with $\varepsilon = 0$ is $3[(n + 1)/2] - 2$.

Our primary purpose is to calculate the concrete expression of $\Phi(h)$; then we can obtain the number of limit cycles of the perturbed system (5) by determining the isolated real zeros of Abelian integral $\Phi(h)$. In Sections 2 and 3, we prove these two theorems with the different methods, respectively.

### Abstract and Applied Analysis

## 2. The proof of Theorem 1

Taking the change of variable $x = \sqrt{\varepsilon} \cos \theta, y = \sqrt{\varepsilon} \sin \theta$ $(0 < h < 1)$, we have

$$\Phi(h) = \int_{\gamma_h} \sum_{0 \leq k + j \leq n} a_{k,j} x^k y^j + \sum_{0 \leq k + j \leq n} b_{k,j} x^k y^j \frac{1}{1 - x^3} d\theta$$

$$= \sum_{0 \leq k + j \leq n} h^{(k+j+1)/2} \int_0^{2\pi} a_{k,j} \cos^{k+1} \theta \sin^j \theta + b_{k,j} \cos^k \theta \sin^{j+1} \theta \frac{1}{1 - h^{3/2} \cos^2 \theta} d\theta$$

$$= \sum_{0 \leq k + j \leq n} (a_{k,j} \Phi_{k+1,j}(h) + b_{k,j} \Phi_{k,j+1}(h)),$$

where

$$\Phi_{k,j} = h^{(k+j+1)/2} \int_0^{2\pi} \frac{\cos^k \theta \sin^j \theta}{1 - h^{3/2} \cos^2 \theta} d\theta.$$  

(7)

Firstly, we have the following obvious result.

**Lemma 3.** If $j$ is odd, $\Phi_{k,j} = 0$ $(k \geq 0)$.

According to Lemma 3, we can rewrite the $\Phi(h)$ as follows:

$$\Phi(h) = \sum_{s=0}^n a_{s,0} \Phi_{s+1,0} + \sum_{s=1}^n \sum_{k+j=s} (a_{k,j} + b_{k+1,j-1}) \Phi_{k+1,j}$$

$$+ \sum_{s=0}^n b_{s,0} \Phi_{0,s+1}.$$  

(8)

Denote $l = [(n + 1)/2]$ and $a_{k,j}^* = a_{k,j} + b_{k+1,j-1}$ $(j \geq 1)$; then

$$\sum_{s=0}^n a_{s,0} \Phi_{s+1,0} = \sum_{s=1}^l (a_{2s-1,0} \Phi_{2s,0} + a_{2s-2,0} \Phi_{2s-1,0}),$$

$$\sum_{s=0}^n b_{s,0} \Phi_{0,s+1} = \sum_{s=1}^l b_{s,0} \Phi_{0,2s} + \sum_{s=1}^{l-1} a_{s,2s} \Phi_{2s-2,0} + \sum_{s=1}^{l-1} a_{s,2s-1} \Phi_{2s-1,0}.$$  

(9)
Therefore, we have

\[
\Phi(h) = \sum_{s=1}^{l} \left( a_{2s-1,0} \Phi_{2s,0} + \sum_{m=1}^{s-1} a_{2s-2m-1,2m} \Phi_{2s-2m,2m} \right ) + b_{0,2s-1} \Phi_{0,2s} + a_{2s-2,0} \Phi_{2s-1,0} + \sum_{m=1}^{s-1} a_{2s-2m-2,2m} \Phi_{2s-2m-2,2m} \right ).
\]

Let

\[
\Gamma_0 = 2\pi, \quad \Gamma_2 = \int_{0}^{2\pi} \cos^2 \theta \, d\theta = 2\pi \frac{(2i-1)!!}{2i!!}, \quad (i \geq 1),
\]

\[
A_{k,j} = \sum_{r=0}^{k} (-1)^r C_k^r \Gamma_{2j+2r},
\]

\[
B_{i,j} = a_{2i-1,0} \Gamma_{2i+6j} + \sum_{m=1}^{i-1} a_{2i-2m-1,2m} A_{m,j-m+3j+1} + b_{0,2i-1} A_{i,3j},
\]

\[
C_{i,j} = a_{2i-2,0} \Gamma_{2i+2+6j} + \sum_{m=1}^{i-1} a_{2i-2m-2,2m} A_{m,j-m+3j+1},
\]

where \(a_{i,j}^* = 0\) if \(i < 0\). Then, we have the following lemma.

**Lemma 4.** Abelian integral \(\Phi(h)\) has an expansion in the form

\[
\Phi(h) = h \left( p_0 + p_1 h + p_2 h^2 + \cdots + p_3 h^3 + p_3 h^{3+1} + p_3 h^{3+2} + O(h^{3+2}) \right ),
\]

where \(h \in (0, 1)\),

\[
p_j = \sum_{s=0}^{\left \lfloor (j+1)/3 \right \rfloor} \left( B_{j-3s+1,u} + C_{j-3s,u} \right), \quad \text{if } j < l,
\]

\[
p_j = \sum_{s=0}^{\left \lfloor j/3 \right \rfloor} \left( B_{j-3s,u+s} + C_{j-3s-1,u+s} \right), \quad \text{if } j = l + 3u - 1 \quad (u = 1, 2, 3, \ldots),
\]

\[
p_j = \sum_{s=0}^{\left \lfloor (l-2)/3 \right \rfloor} \left( B_{j-3s-2,u+s} + C_{j-3s,u+s} \right), \quad \text{if } j = l + 3u - 2 \quad (u = 1, 2, 3, \ldots),
\]

\[
p_j = \sum_{s=0}^{\left \lfloor (l-1)/3 \right \rfloor} \left( B_{j-3s-1,u+s} + C_{j-3s-2,u+s} \right),
\]

\[
\text{Proof.} \text{ Firstly, we have}
\]

\[
\Phi_{2s,0} = h^s \int_{0}^{2\pi} \frac{\cos^2 \theta}{1 - h^{3/2} \cos^2 \theta} \, d\theta
\]

\[
= h^s \int_{0}^{2\pi} \left( \cos^2 \theta + h^3 \cos^2 \theta + h^5 \cos^2 \theta + \cdots \right ) \, d\theta
\]

\[
= h^s \left( \Gamma_{2s} + h^3 \Gamma_{2s+6} + \cdots + h^3 \Gamma_{2s+6} \right ),
\]

\[
\Phi_{2s+2m,2m} = h^s \int_{0}^{2\pi} \cos^{2s-2m} \theta \, d\theta
\]

\[
= h^s \left( \Gamma_{2s} + h^3 \Gamma_{2s+6} + \cdots + h^3 \Gamma_{2s+6} \right ),
\]

\[
\Phi_{2s+2,2s} = h^s \int_{0}^{2\pi} \cos^{2s+2} \theta \, d\theta
\]

\[
= h^s \left( \Gamma_{2s} + h^3 \Gamma_{2s+6} + \cdots + h^3 \Gamma_{2s+6} \right ),
\]

\[
\Phi_{0,2s} = h^s \int_{0}^{2\pi} \cos^{2s} \theta \, d\theta
\]

\[
= h^s \left( \Gamma_{2s} + h^3 \Gamma_{2s+6} + \cdots + h^3 \Gamma_{2s+6} \right ),
\]

\[
\Phi_{2s+1,0} = h^s \int_{0}^{2\pi} \frac{\cos^{2s+1} \theta}{1 - h^{3/2} \cos^2 \theta} \, d\theta
\]

\[
= h^s \int_{0}^{2\pi} \left( \cos^{2s+1} \theta + h^9 \cos^{2s+8} \theta + \cdots \right ) \, d\theta
\]

\[
= h^s \left( \Gamma_{2s+1} + h^{9/2} \Gamma_{2s+8} + \cdots + h^9 \Gamma_{2s+8} \right ),
\]

\[
\Phi_{2s+1,1,0} = h^s \int_{0}^{2\pi} \frac{\cos^{2s+1} \theta}{1 - h^{3/2} \cos^2 \theta} \, d\theta
\]

\[
= h^s \int_{0}^{2\pi} \left( \cos^{2s+1} \theta + h^9 \cos^{2s+8} \theta + \cdots \right ) \, d\theta
\]

\[
= h^s \left( \Gamma_{2s+1} + h^{9/2} \Gamma_{2s+8} + \cdots + h^9 \Gamma_{2s+8} \right ),
\]

\[
\Phi_{2s+1,1,1,0} = h^s \int_{0}^{2\pi} \frac{\cos^{2s+1} \theta}{1 - h^{3/2} \cos^2 \theta} \, d\theta
\]

\[
= h^s \int_{0}^{2\pi} \left( \cos^{2s+1} \theta + h^9 \cos^{2s+8} \theta + \cdots \right ) \, d\theta
\]

\[
= h^s \left( \Gamma_{2s+1} + h^{9/2} \Gamma_{2s+8} + \cdots + h^9 \Gamma_{2s+8} \right ),
\]

\[
\Phi_{2s+1,1,1,1,0} = h^s \int_{0}^{2\pi} \frac{\cos^{2s+1} \theta}{1 - h^{3/2} \cos^2 \theta} \, d\theta
\]

\[
= h^s \int_{0}^{2\pi} \left( \cos^{2s+1} \theta + h^9 \cos^{2s+8} \theta + \cdots \right ) \, d\theta
\]

\[
= h^s \left( \Gamma_{2s+1} + h^{9/2} \Gamma_{2s+8} + \cdots + h^9 \Gamma_{2s+8} \right ),
\]
\[
\Phi_{2s-2m-1,2m} = h^{-1/2} \int_{0}^{2\pi} \cos^{2s-2m-1} \theta \sin^{2m} \theta \, d\theta \\
= h^{-1/2} \times \sum_{r=0}^{m} (-1)^r C_r \int_{0}^{2\pi} (h^{3/2} \cos^{2s-2m+2r} \theta \\
+ h^{9/2} \cos^{2s-2m+8+2r} \theta + \ldots \\
+ h^{3/2+3i} \cos^{2s-2m+2r+6i} + \ldots) \, d\theta \\
= h^r \left( h A_{m,s-m+1} + h^4 A_{m,s-m+4} \\
+ \ldots + h^{1+3i} A_{m,s-m+1+3i} + \ldots \right).
\] (16)

Substituting the previous formulas into (10), we have
\[
\Phi (h) = \sum_{j=1}^{l} h^j \\
\times \left[ \left( a_{2s-1,0} \Gamma_{2s} + \sum_{m=1}^{s-1} a_{2s-2m-1,2m} A_{m,s-m} \\
+ b_{0,2s-1} A_{s,0} \right) \\
+ h^3 \left( a_{2s-1,0} \Gamma_{2s+6} + \sum_{m=1}^{s-1} a_{2s-2m-1,2m} A_{m,s-m+3} \\
+ b_{0,2s-1} A_{s,3} \right) + \ldots \\
+ h^{3i} \left( a_{2s-1,0} \Gamma_{2s+6i} + \sum_{m=1}^{s-1} a_{2s-2m-1,2m} A_{m,s-m+3i} \\
+ b_{0,2s-1} A_{s,3i} \right) + \ldots \\
+ h \left( a_{2s-2,0} \Gamma_{2s+2} + \sum_{m=1}^{s-1} a_{2s-2m-2,2m} A_{m,s-m+1} \right) \\
+ h^{3i} \left( a_{2s-2,0} \Gamma_{2s+2+6i} + \sum_{m=1}^{s-1} a_{2s-2m-2,2m} A_{m,s-m+4} \right) + \ldots \\
+ h^{1+3i} \left( a_{2s-2,0} \Gamma_{2s+6+2i} + \sum_{m=1}^{s-1} a_{2s-2m-2,2m} A_{m,s-m+3+1} \right) + \ldots \right].
\] (17)

Using (12) and (13), (17) can be written as follows:
\[
\Phi (h) = \sum_{j=1}^{l} h^j \left( B_{s,0} + h B_{s,1} + \ldots + h^3 B_{s,i} \right) \\
+ \ldots + h^{1+3i} C_{s,i+1} + \ldots + h^{1+3i} C_{s,i+3} + \ldots \\
= h \left[ (B_{s,0} + h^3 B_{s,1} + \ldots + h^3 B_{s,i+1} + \ldots + h^3 B_{s,i+3}) \\
+ h^4 C_{s,1} + \ldots + h^{1+3i} C_{s,i+1} + \ldots + h^{1+3i} C_{s,i+3}) + \ldots \\
+ h^{1-i} (B_{s,0} + h^3 B_{s,1} + \ldots + h^3 B_{s,i+1} + \ldots + h^3 B_{s,i+3}) \\
+ \ldots + h^{1+3i} C_{s,i+1} + \ldots + h^{1+3i} C_{s,i+3}) \right].
\] (18)

Thus, we have
\[
\text{if } j < l, \\
p_j = B_{j+1,0} + B_{j-2,1} + \ldots + B_{j-3i+1,1} + \ldots + C_{j,0} \\
+ C_{j-3,1} + \ldots + C_{j-3i,3} + \ldots, \\
\] (19)
\[
\text{if } j = l + 3u - 1 \ (u = 1, 2, 3, \ldots), \\
p_j = B_{l,u} + B_{l-3,u+1} + \ldots + B_{l-3i,u+i} \\
+ \ldots + C_{l-1,u} + C_{l-4,u+1} + \ldots + C_{l-3i-1,u+i} + \ldots, \\
\] (20)
\[
\text{if } j = l + 3u \ (u = 0, 1, 2, \ldots), \\
p_j = B_{l-2,u+1} + B_{l-5,u+2} + \ldots + B_{l-3i-2,u+i+1} \\
+ \ldots + C_{l-1,u} + C_{l-3,u+1} + \ldots + C_{l-3i,u+i} + \ldots, \\
\] (21)
\[
\text{if } j = l + 3u - 2 \ (u = 1, 2, 3, \ldots), \\
p_j = B_{l-1,u} + B_{l-4,u+2} + \ldots + B_{l-3i-1,u+i+1} + \ldots \\
+ C_{l-2,u} + C_{l-5,u+1} + \ldots + C_{l-3i-2,u+i} + \ldots, \\
\] (22)

where \( i = 0, 1, 2, \ldots \) and \( B_{i,j} = C_{i,j} = 0 \) if \( i < 0 \). The proof is completed. 
\]
By Lemma 4, we regard $a_{j-1,0}, a_{j-3,2}, \ldots, a_{j-2j-1, l_{0, 2j-1}}, a_{j-2,0}, a_{j-4,2}, \ldots, a_{j-2j-4, a_{0, 2j-2}}$ (as free parameters, $j \geq 1$) and denote vectors

$$\alpha_j = \left( a_{j-1,0}, a_{j-3,2}, \ldots, a_{j-2j-1, l_{0, 2j-1}} \right),$$

and Jacobian matrices

$$J_j = \frac{\partial (p_{j-1}, p_{j+2}, p_{j+5}, \ldots, p_{4j-1})}{\partial \alpha_j},$$

$$H_j = \frac{\partial (p_j, p_{j+3}, p_{j+6}, \ldots, p_{4j-3})}{\partial \beta_j},$$

and $J_j$ is $(j+2) \times (j+2)$ matrix, $H_j$ is $j \times j$ matrix, and $I_l$ is $(4l-1) \times (l^2+2)$ matrix.

For matrices $J_j, H_j$ and $I_l$, we have Lemmas 5 and 6, respectively.

Lemma 5. For $j \geq 1$, $\det(J_j) \neq 0$ and $\det(H_j) \neq 0$. That is, $\text{rank}(J_j) = j+1$, $\text{rank}(H_j) = j$.

**Proof.** According to Lemma 4 and (24), we can obtain that

$$J_j = \left( \begin{array}{cccccc}
\Gamma_{3j} & A_{1,j-1} & A_{2,j-2} & \cdots & A_{j-2,2} & A_{j-1,1} & A_{j,0} \\
\Gamma_{3j-6} & A_{1,j+1} & A_{2,j+2} & \cdots & A_{j,2} & A_{j+1} & A_{j+2} \\
\Gamma_{2j+12} & A_{1,j+3} & A_{2,j+4} & \cdots & A_{j-2,3} & A_{j-1,2} & A_{j,1} \\
\Gamma_{2j+15} & A_{1,j+4} & A_{2,j+5} & \cdots & A_{j-2,4} & A_{j-1,3} & A_{j,1} \\
\Gamma_{2j+18} & A_{1,j+5} & A_{2,j+6} & \cdots & A_{j-2,5} & A_{j-1,4} & A_{j,1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\Gamma_{8j-6} & A_{1,j+9} & A_{2,j+10} & \cdots & A_{j-2,10} & A_{j-1,9} & A_{j,1} \\
\Gamma_{8j} & A_{1,j+10} & A_{2,j+10} & \cdots & A_{j-2,11} & A_{j-1,10} & A_{j,1} \\
\end{array} \right),$$

Define $(j+1) \times (j+1)$ matrix

$$D_i = \begin{pmatrix}
1 & (-1)^{i-2} C_{j-i}^2 \\
1 & (-1)^{i-3} C_{j-i}^2 \\
\vdots & \vdots \\
1 & (-1)^i C_{j-i}^1 \\
1 & 1 \\
\end{pmatrix},$$

where $j, l \geq 1$. Then, $J_j$ is $(j+2) \times (j+2)$ matrix, $H_j$ is $j \times j$ matrix, and $I_l$ is $(4l-1) \times (l^2+2)$ matrix.

For matrices $J_j, H_j$ and $I_l$, we have Lemmas 5 and 6, respectively.

Lemma 5. For $j \geq 1$, $\det(J_j) \neq 0$ and $\det(H_j) \neq 0$. That is, $\text{rank}(J_j) = j+1$, $\text{rank}(H_j) = j$.

**Proof.** According to Lemma 4 and (24), we can obtain that

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\Gamma_{2j+6} & A_{1,j+1} & A_{2,j+2} & \cdots & A_{j,2} & A_{j+1} & A_{j+2} \\
\Gamma_{2j+12} & A_{1,j+3} & A_{2,j+4} & \cdots & A_{j-2,3} & A_{j-1,2} & A_{j,1} \\
\Gamma_{2j+15} & A_{1,j+4} & A_{2,j+5} & \cdots & A_{j-2,4} & A_{j-1,3} & A_{j,1} \\
\Gamma_{2j+18} & A_{1,j+5} & A_{2,j+6} & \cdots & A_{j-2,5} & A_{j-1,4} & A_{j,1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\Gamma_{8j-6} & A_{1,j+9} & A_{2,j+10} & \cdots & A_{j-2,10} & A_{j-1,9} & A_{j,1} \\
\Gamma_{8j} & A_{1,j+10} & A_{2,j+10} & \cdots & A_{j-2,11} & A_{j-1,10} & A_{j,1} \\
\end{array} \right),$$

Then, we have

$$J_j' = J_j D_2 D_3 \cdots D_{j+1},$$

$$J_j' = \left( \begin{array}{cccccc}
\Gamma_{2j} & \Gamma_{2j-2} & \Gamma_{2j-4} & \cdots & \Gamma_{4} & \Gamma_{2} & \Gamma_{0} \\
\Gamma_{2j+6} & \Gamma_{2j+4} & \cdots & \Gamma_{4} & \Gamma_{2} & \Gamma_{0} \\
\Gamma_{2j+12} & \Gamma_{2j+10} & \cdots & \Gamma_{4} & \Gamma_{2} & \Gamma_{0} \\
\Gamma_{2j+18} & \Gamma_{2j+16} & \cdots & \Gamma_{4} & \Gamma_{2} & \Gamma_{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\Gamma_{8j-6} & \Gamma_{8j-8} & \cdots & \Gamma_{6j-2} & \Gamma_{6j-4} & \Gamma_{6j-6} \\
\Gamma_{8j} & \Gamma_{8j-2} & \cdots & \Gamma_{6j-4} & \Gamma_{6j-2} & \Gamma_{6j} \\
\end{array} \right) \cdot D_i,$$

Let $\Gamma_j^l = l! / l!!$; then $\Gamma_j = 2\pi \Gamma_j^{-1}$ ($j \geq 1$). Denote

$$T_{j-k} = \left( \begin{array}{cccccc}
\Gamma_{2j+4k-1} & \Gamma_{2j+4k-3} & \cdots & \Gamma_{4k+1} & \Gamma_{4k+2} & \Gamma_{4k+6} \\
\Gamma_{2j+4k+5} & \Gamma_{2j+4k+3} & \cdots & \Gamma_{4k+1} & \Gamma_{4k+2} & \Gamma_{4k+6} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\Gamma_{8j-2k-7} & \Gamma_{8j-2k-9} & \cdots & \Gamma_{6j-5} & \Gamma_{6j-2} & \Gamma_{6j-2} \\
\Gamma_{8j} & \Gamma_{8j-2} & \cdots & \Gamma_{6j-5} & \Gamma_{6j-2} & \Gamma_{6j-2} \\
\end{array} \right) \cdot D_i,$$

where $T_{j-k}$ is $(j-k+1) \times (j-k+1)$ matrix. Then, $\det(J_j) = \det(J_j') = (2\pi)^{j+1} \det(T_j)$. 

We add entries of the $i + 1$th ($0 < i < j - k + 1$) column which times $-(2(j + 2 - i) - 1)/2(j + 2 - i)$ to the $i$th column and obtain

$$T_{j-k} \rightarrow T^1_{j-k}$$

$$= \left( \begin{array}{ccccccc}
0 & 0 & \ldots & 0 \\
6k + 3 & 6k + 3 & \ldots & 6k + 3 \\
\frac{12k + 6}{j + 3k} & \frac{12k + 6}{j + 3k} & \ldots & \frac{12k + 6}{j + 3k} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{(6k + 3) (j - k - 1)}{j + 3k} & \frac{(6k + 3) (j - k - 1)}{j + 3k} & \ldots & \frac{(6k + 3) (j - k - 1)}{j + 3k} \\
\end{array} \right) \left( \begin{array}{c}
\frac{1}{j + 3k} \\
\frac{1}{j + 3k - 1} \\
\vdots \\
\frac{1}{4k + 1} \\
\end{array} \right);$$

where $a_k = (\Gamma_{8k+5}, \Gamma_{8k+12}, \ldots, \Gamma_{6j+2k-6}, \Gamma_{6j+2k})^T$. We can write $T^0_{j-k-1}$ as follows:

$$T^0_{j-k-1} = \left( \begin{array}{c}
6k + 3 \\
12k + 6 \\
\vdots \\
(6k + 3) (j - k) \\
\end{array} \right) \times \left( \begin{array}{c}
\frac{1}{j + 3k} \\
\frac{1}{j + 3k - 1} \\
\vdots \\
\frac{1}{4k + 1} \\
\end{array} \right);$$

then $\det(T_{j-k}) = (-1)^{j-k+2} \frac{1}{6k} (6k + 3)^{j-k} (j - k)!((4k)!/(j + 3k))! \det(T^1_{j-k-1})$.

Summarizing above results, we have

$$\det(J) = (2\pi)^{j+1}(-1)^{(j+2)+(j+1)\ldots+3}j^j 15^{j+2} \cdots (6j - 9)^2 (6j - 3) j! (j - 1)! \cdots 2! \times \Gamma_{8j+11}^{5j+1j+17} \cdots \Gamma_{8j}^{5j-1} \frac{4!}{j! (j + 3)! (j + 6)!} \Gamma_8^{5j+1j+17} \cdots \Gamma_8^{5j-1} \frac{8!}{j! (j + 3)! (j + 6)!}$$

$$\cdots \frac{(4j - 4)!}{(4j - 3)!} \neq 0;$$

therefore, $\text{rank}(J) = j + 1$.

According to Lemma 4 and (25), we have

$$H_{j-1}$$

By the above proof procedure, we can obtain $\text{rank}(H_{j-1}) = j - 1$ in a similar way. That is $\text{rank}(H_j) = j$. The proof is completed.

**Lemma 6.** For $l \geq 1$, $\text{rank}(L_l) = 4l - 1$.

**Proof.** Firstly, if $l = 1$,

$$I_1 = \frac{\partial (p_0, p_1, p_3)}{\partial (\alpha_1, \beta_1)}$$

$$= \left( \begin{array}{c}
\Gamma_2 \\
\Gamma_2 \\
\Gamma_8 \\
\Gamma_8 \\
\end{array} \right) \left( \begin{array}{c}
A_{1,0} \\
A_{1,0} \\
A_{1,3} \\
A_{1,3} \\
\end{array} \right)$$

it is easy to know that $\text{rank}(I_1) = 3$. 
If \( l = 2 \), we have
\[
I_2 = \frac{\partial \left( p_0, p_1, p_2, p_3, p_4, p_5, p_7 \right)}{\partial (\alpha_1, \beta_1, \alpha_2, \beta_2)}
\]
\[
= \begin{pmatrix}
\Gamma_2 A_{1,0} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \Gamma_4 A_{1,1} & A_{2,0} & \Gamma_4 & 0 & 0 & 0 \\
0 & 0 & \Gamma_6 A_{1,2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Gamma_8 A_{1,3} & 0 & 0 & 0 \\
0 & 0 & \Gamma_{10} A_{1,4} & A_{2,3} & \Gamma_{10} & 0 & 0 \\
0 & 0 & 0 & 0 & \Gamma_{12} A_{1,5} & 0 & 0 \\
0 & 0 & \Gamma_{16} A_{1,7} & A_{2,6} & \Gamma_{16} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]\]

where \((x_1, x_2, \ldots, x_{l+1}), (y_1, y_2, \ldots, y_{l})\) and \((z_1, z_2, \ldots, z_{l})\) are nonzero vectors.

**Proof.** According to Lemma 4, if \( j = l+3u-1 \ (u = 1, 2, 3, \ldots) \),
\[
p_j = \sum_{s=0}^{\lfloor l/3 \rfloor} (B_{l-3s+u} + C_{l-3s-1+u})
\]
Substituting \( p_{l-1}, p_{l+2}, \ldots, p_{q+1} \) into the first equation, we obtain a linear equation
\[
d_{l+3u-1} = y_j^T X \ (u > 1),
\]
where \( X = (x_1, x_2, \ldots, x_{l+1})^T, d_{l+3u-1} = (\Gamma_{2l+6u}, A_{1,1+3u}, A_{2,1+3u-2}, \ldots, A_{l-1,3u-1}, A_{l,3u})^T \). According to Lemma 5, \( \det(I_j^T) \neq 0 \), (41) has unique solution \( X = (x_1', x_2', \ldots, x_{l+1}')^T \neq 0 \). That is, the first formula holds.

If \( j = l + 3u - 2 \ (u = 1, 2, 3, \ldots) \), in a similar way, we can prove that the second formula holds.

If \( j = l + 3u \ (u = 0, 1, 2, \ldots) \), we have
\[
p_j = \sum_{s=0}^{\lfloor l/3 \rfloor} B_{l-3s-2u+1} + \sum_{s=0}^{\lfloor l/3 \rfloor} C_{l-3s-2u+1}
\]
substituting \( p_{l+1}, p_{l+4}, \ldots, p_{q+3} \) into the third equation, we obtain a linear equation
\[
c_{l+3u} = h_j^T Z \ (u > 1 - 1),
\]
where \( Z = (z_1, z_2, \ldots, z_{l+1})^T, c_{l+3u} = (\Gamma_{2l+6u+2}, A_{1,1+3u}, A_{2,1+3u-1}, \ldots, A_{l-2,3u+1}, A_{l-1,3u+1})^T \). According to Lemma 5, \( \det(H_j^T) \neq 0 \), (43) has unique solution \( Z = (z_1', z_2', \ldots, z_{l+1}')^T \neq 0 \). That is, the third formula also holds. The proof is completed.

Now, we prove Theorem 1.

**Proof.** For \( h \in (0, 1) \), Abelian integral \( \Phi(h) \) has an expansion of the following form:
\[
\Phi(h) = h \left( p_0 + p_1 h + p_2 h^2 + \cdots + p_{l+3u-3} h^{l+3u-3} \right) + p_{q+1} h^{l+1} + o(h^{l+1})
\]
According to Lemma 6, for \( l \geq 1, p_0, p_1, p_2, \ldots, p_{l-1}, p_{l+1}, p_{l+4}, \ldots, p_{q-2} \), \( p_{q-1} \) are independent. Now, we have the following lemma.

**Lemma 7.** For \( l \geq 1, we can write \( p_j \) as follows:
\[
\begin{align*}
p_{l+3u-1} &= x_1 p_{l-1} + x_2 p_{l+2} + \cdots + x_{l+1} p_{4l-1}, \\
p_{l+3u-2} &= y_1 p_{l-2} + y_2 p_{l+2} + \cdots + y_{l+1} p_{4l-5}, \\
p_{l+3u} &= z_1 p_{l+1} + z_2 p_{l+4} + \cdots + z_{l+1} p_{4l-3},
\end{align*}
\]
where \((x_1, x_2, \ldots, x_{l+1}), (y_1, y_2, \ldots, y_{l})\) and \((z_1, z_2, \ldots, z_{l})\) are nonzero vectors.

By Lemma 6, it is obvious that \( p_0, p_1, p_2, \ldots, p_{l-1}, p_{l+1}, p_{l+4}, \ldots, p_{q-2} \) are independent. Let \( p_0 = p_1 = \cdots = p_{q-2} = 0 \) and \( p_{q-1} = 1 \); then, by Lemma 7, \( p_{q-2} = 0 \). Thus (44) becomes \( \Phi(h) = h^{l+1} + o(h^{l+1}) \), and \( \Phi(h) > 0 \) if \( h \in (0, 1) \). Furthermore, we take \( p_0 = p_1 = p_2 = \cdots = p_{q-2} = 0 \) and \( p_{q-1} = 1 \); then, \( p_{q-2} = 0 \) still holds. Choosing proper \( p_{q-3} \in (-1, 0) \) such that \( \Phi(h) = p_{q-3} h^{l+3u-2} + h^{l+1} + o(h^{l+1}) \) < 0, by Descartes' rule of signs, (44) has a root \( h_1 \) on interval \((0, 1)\).

Let \( p_0 = p_1 = p_2 = \cdots = p_{q-6} = 0 \) and choose proper \( p_{q-4} \neq 0 \) (so that \( \Phi(h) = p_{q-4} h^{l+3u-2} + p_{q-3} h^{l+3u-3} + h^{l+1} + o(h^{l+1}) > 0 \); then (44) has the second root \( h_2 \) on interval \((0, 1)\). In a similar way, we take proper \( p_i \ (i = 4l - 6, 4l - 7, \ldots, 2, 1, 0) \) in turn such that
primary computations to express the Abelian integral and some concerned lemmas are presented.

In this section, we will prove Theorem 2. At first, a little by (4) we have

\[\phi(h) = \int \sum \frac{\phi_{k,j+1} y_j^i + \sum b_{k,j} x_k y_j^{i+1}}{1-x^2} \, d\theta,\]

where

\[\phi^1 = \int \sum \frac{\phi_{k,j} x_k^i y_j^i}{1-x^2} \, d\theta,\]

\[\phi^2 = \int \sum \frac{\phi_{k,j} x_k^i y_j^i}{1-x^2} \, d\theta,\]

Denote

\[a_{k,j} = 1 - x^2 \quad \text{and} \quad b_{k,j} = 1 - x^2 \]

where \(i^2 = -1\); then we have

\[\phi^1 = \sum \frac{a_{k,j} \phi_{k+1,j} + b_{k,j} \phi_{k,j+1}}{1-x^2},\]

\[\phi^2 = \sum \frac{a_{k,j} \phi_{k+1,j} + b_{k,j} \phi_{k,j+1}}{1-x^2}.\]

Lemma 8. Let \(\omega_1 = 1, \omega_2 = -1, \omega_3 = i, \omega_4 = -i\); then

\[\psi_s = \int \frac{d\theta}{y_0 - \omega_s x} = \frac{2\pi}{\sqrt{1 - \omega_s^2 h}}.\]

where \(s = 1, 2, 3, 4\). And

\[\phi_0^1 = \int \sum \frac{d\theta}{y_0 - (1-x)(1+x)} = \frac{2\pi}{\sqrt{1-h}},\]

\[\phi_{1,0}^1 = \int \sum \frac{xd\theta}{y_0 - (1-x)(1+x)} = 0,\]

\[\phi_{0,0}^2 = \int \sum \frac{d\theta}{y_0 - (1-x)(1+x)} = \frac{2\pi}{\sqrt{1+h}},\]

\[\phi_{1,0}^2 = \int \sum \frac{xd\theta}{y_0 - (1-x)(1+x)} = 0.\]

Proof. We use the residue theorem to compute the \(\psi_s\) (\(s = 1, 2, 3, 4\)).

When \(x = \sqrt{h} \cos \theta\), we have

\[\psi_s = \int \frac{d\theta}{1 - \omega_s x - 2z + \omega_s \sqrt{h} \cos \theta}.\]

Let \(e^{i\theta} = z\); then \(\cos \theta = (z^2 + 1)/2z, d\theta = dz/iz\). The previous formula becomes

\[\psi_s = \int \frac{dz}{1 - \omega_s \sqrt{h} \cos \theta} \left( 1 - \frac{1 - \omega_s^2 h}{\omega_s \sqrt{h}} \right)\]

\[= \frac{2\pi}{\sqrt{1 - \omega_s^2 h}}.\]

hence (49) holds. For the first formula of (50),

\[\phi_0^1 = \frac{1}{2} \left( \int \frac{d\theta}{y_0 - (1-x)(1+x)} + \int \frac{d\theta}{y_0 - \omega_1 x} \right)\]

\[= \frac{1}{2} \left( \frac{2\pi}{\sqrt{1-h}} + \frac{2\pi}{\sqrt{1+h}} \right) = \frac{2\pi}{\sqrt{1-h}},\]

and the others can be proved in a similar way. The proof is completed. \(\square\)

Lemma 9. If \(j\) is odd, the integrands in \(\phi_{k,j}^1\) and \(\phi_{k,j}^2\) are odd functions with respect to the variable \(\theta\); therefore, \(\phi_{k,j}^1 = 0\) and \(\phi_{k,j}^2 = 0\).

Define

\[\phi_{k,j} = \int \frac{a_{k,j} x_k^{j+1} y_j^i + b_{k,j} x_k^{j+1} y_j^i}{1 - x^2} \, d\theta,\]

then, according to Lemmas 8 and 9 and the definition of \(\phi(h)\), one knows that \(\phi_{0,0} = 0\).
Lemma 10. If \( j \) is even, then
\[
\Phi^1_{k,j} = \sum_{s=0}^{j/2} (-1)^s h^{j/2-s} C_{j/2}^s \Phi^1_{k+2s,0},
\]
(55)
\[
\Phi^2_{k,j} = \sum_{s=0}^{j/2} (-1)^s h^{j/2-s} C_{j/2}^s \Phi^2_{k+2s,0}.
\]
(56)

Proof. If \( j \) is even, then
\[
\Phi^1_{k,j} = \frac{2\pi}{\sqrt{1-h}} \left[ 1 - \sum_{l=1}^{k} 2^{-k-l} h^{(k-l)/2} \sqrt{1-h} C_{k-l}^{(k-l)/2} \right].
\]
(58)
\[
\Phi^2_{k,j} = \frac{2\pi}{\sqrt{1-h}} \left[ 1 - \sum_{l=1}^{k} 2^{-k-l} h^{(k-l)/2} \sqrt{1-h} C_{k-l}^{(k-l)/2} \right].
\]
(59)

Lemma 11. (i) If \( k \) is odd, \( \Phi^1_{k,0} = 0 \).
(ii) If \( k \) is even, then
\[
\Phi^1_{k,0} = \frac{2\pi}{\sqrt{1-h}} \left[ 1 - \sum_{l=1}^{k} 2^{-k-l} h^{(k-l)/2} \sqrt{1-h} C_{k-l}^{(k-l)/2} \right].
\]
(60)

Proof. When \( k > 1 \),
\[
\Phi^1_{k,0} = \frac{2\pi}{\sqrt{1-h}} \left[ 1 - \sum_{l=1}^{k} 2^{-k-l} h^{(k-l)/2} \sqrt{1-h} C_{k-l}^{(k-l)/2} \right].
\]
(61)

We use the residue theorem to compute the integrals \( M_1 \) and \( M_2 \). Denote \( e^{i\theta} = z \); thus, \( \cos \theta = (z^2 + 1)/2z, \theta = d\theta/dz \). We have
\[
M_1 = 2\pi \sqrt{1-h} \left[ 1 - \sum_{l=1}^{k} 2^{-k-l} h^{(k-l)/2} \sqrt{1-h} C_{k-l}^{(k-l)/2} \right].
\]
(62)

Since \( z_1 = (1 - \sqrt{1-h})/\sqrt{h} \) is the first-order zero of the equation \( \sqrt{h}z^2 - 2z + \sqrt{h} = 0 \), the residue of \( M_1 \) at \( z_1 \) is
\[
\text{Res}(M_1, z_1) = \frac{1}{2} \frac{(z^2 + 1)/z}{\sqrt{h} - 1}.
\]
(63)

For the residue at \( z = 0 \), we have the expansion of \( M_1 \) in the form
\[
M_1 = \frac{1}{2} \frac{(z^2 + 1)/z}{\sqrt{h} - 1}.
\]
(64)

Substituting them into (60), we have
\[
M_1 = -2^{-k} \cdot h^{k/2} \left[ 1 - \sum_{l=1}^{k} 2^{-k-l} h^{(k-l)/2} \sqrt{1-h} C_{k-l}^{(k-l)/2} \right].
\]
(65)

We can compute \( M_2 \) in a similar way and obtain
\[
M_2 = \frac{2\pi}{\sqrt{1-h}} \left[ 1 - \sum_{l=1}^{k} 2^{-k-l} h^{(k-l)/2} \sqrt{1-h} C_{k-l}^{(k-l)/2} \right].
\]
By the formulas (64) and (65), $\Phi_{k,0}^1$ becomes

$$\Phi_{k,0}^1 = \frac{\pi}{\sqrt{1 - h}}$$

$$\times \left[ 1 - \sum_{l=k}^{k} 2^{l-k} h^{(k-l)/2} \sqrt{1 - hC_{k-l}^{(k-l)/2}} + (-1)^k \right]$$

$$+ \sum_{l=k+1}^{k} (-1)^{l-k} 2^{l-k} h^{(k-l)/2} \sqrt{1 + hC_{k-l}^{(k-l)/2}}$$

(66)

from the previous formula, it is easy to know that, if $k$ is odd, $\Phi_{k,0}^1 = 0$, and if $k$ is even, formula (58) is obtained. The proof is completed. \qed

In a similar way, we can prove the following lemma.

**Lemma 12.** (i) If $k$ is odd, $\Phi_{k,0}^2 = 0$.

(ii) If $k$ is even, then

$$\Phi_{k,0}^2 = \frac{2\pi i}{\sqrt{1 + h}}$$

$$\times \left[ 1 - \sum_{l=k}^{k} 2^{l-k} h^{(k-l)/2} i^{-l-k} \sqrt{1 + hC_{k-l}^{(k-l)/2}} \right].$$

(67)

Using Lemmas 11 and 12, it is easy to see that, if $k$ is even, $\Phi_{k,0} = 0$. Therefore, by Lemma 10 and the definition of $\Phi_{k,j}$, if $n$ is even, $\Sigma_{k+j=n} \Phi_{kj} = 0$.

**Lemma 13.** Consider a function of the form

$$F(x) = P_0(x) + \frac{P_1(x)}{\sqrt{1+x}} + \frac{P_2(x)}{\sqrt{1-x}},$$

(68)

where $P_j(x)$ $(j = 1, 2)$ are real polynomials of degree $n$ and the degree of $P_0$ is $n_0$. Then, the number $\mathcal{X}(F)$ of real zeros of $F(x)$ in $U = [0, 1)$, taking into account their multiplicities, satisfies $\mathcal{X}(F) \leq 2n + n_0 + 2$; here $\deg(0) = -1$.

To prove Lemma 13, we need the following lemma and a known principle, the Derivation-division algorithm.

**Lemma 14.** For any $n \geq 0$, $m \geq 1$ and the real constants $\alpha$,

$$\mathcal{D} (p_n(x)(x+1)^\alpha) = q_n(x)(x+1)^{\alpha-1},$$

$$\mathcal{D} \left( \frac{p_n(x)}{1-x} \right)^\alpha = q_{n+1}(x) \frac{(1+x)^{\alpha-1}}{(1-x)^{\alpha+1}},$$

(69)

$$\mathcal{D}^m \left( \frac{p_n(x)}{1-x} \right)^\alpha = q_{m+n}(x) \frac{(1+x)^{\alpha-m}}{(1-x)^{\alpha+m}}.$$
By Lemmas 11 and 12, the previous formula becomes

\[
\Phi(h) = \frac{1}{2} \left[ \sum_{0 \leq k + j \leq n \atop k \text{ odd}, j \text{ even}} a_{k,j} \left( \Phi^1_{k+1,j} + \Phi^2_{k+1,j} \right) + \sum_{0 \leq k + j \leq n \atop k \text{ even}, j \text{ odd}} b_{k,j} \left( \Phi^1_{k,j+1} + \Phi^2_{k,j+1} \right) \right].
\] (74)

According to Lemma 10, we have that, if \( j \) is even,

\[
\Phi^\sigma_{k+1,j} = \sum_{s=0}^{j/2} (-1)^s h^{j/2-s} C^\sigma_{j/2} \Phi^\sigma_{k+1,2s,0^s}.
\] (75)

if \( j \) is odd,

\[
\Phi^\sigma_{k,j+1} = \sum_{s=0}^{(j+1)/2} (-1)^s h^{(j+1)/2-s} C^\sigma_{(j+1)/2} \Phi^\sigma_{k,2s,0^s}.
\] (76)

where \( \sigma = 1, 2 \).

According to (75) and (76), we can obtain that, if \( k \) is odd, \( j \) is even,

\[
\frac{1}{2} \left( \Phi^1_{k+1,j} + \Phi^2_{k+1,j} \right)
= \sum_{s=0}^{j/2} (-1)^s h^{j/2-s} C^s_{j/2}
\times \left[ \frac{\pi}{\sqrt{1-h}} + \frac{\pi^{k+1+2s}}{\sqrt{1+h}} - \sum_{l=1}^{k+1+2s} 2^{l-k-1-2s} \pi \right].
\] (77)

and if \( k \) is even, \( j \) is odd,

\[
\frac{1}{2} \left( \Phi^1_{k,j+1} + \Phi^2_{k,j+1} \right)
= \sum_{s=0}^{(j+1)/2} (-1)^s h^{(j+1)/2-s} C^s_{(j+1)/2}
\times \left[ \frac{\pi}{\sqrt{1-h}} + \frac{\pi^{k+1+2s}}{\sqrt{1+h}} - \sum_{l=1}^{k+1+2s} 2^{l-k-1-2s} \pi \right].
\] (78)

From (77) and (78), we have, if \( k \) is odd, \( j \) is even,

\[
\frac{1}{2} \left[ a_{k,j} \left( \Phi^1_{k+1,j} + \Phi^2_{k+1,j} \right) + b_{k+1,j-1} \left( \Phi^1_{k+1,j} + \Phi^2_{k+1,j} \right) \right]
= \left( a_{k,j} + b_{k+1,j-1} \right)
\times \sum_{s=0}^{j/2} (-1)^s h^{j/2-s} C^s_{j/2}
\times \left[ \frac{\pi}{\sqrt{1-h}} + \frac{\pi^{k+1+2s}}{\sqrt{1+h}} - \sum_{l=1}^{k+1+2s} 2^{l-k-1-2s} \pi \right].
\] (79)

Let \( m = [(n+1)/2] \) and

\[
\Psi_{k,j} = \frac{1}{2} \left[ a_{k,j} \left( \Phi^1_{k+1,j} + \Phi^2_{k+1,j} \right) + b_{k+1,j-1} \left( \Phi^1_{k+1,j} + \Phi^2_{k+1,j} \right) \right],
\] (80)

where \( k \) is odd, \( j \) is even.
According to (79), the Abelian integral $\Phi(h)$ of system (5) with $C(x, y) = 1 - x^4$ has the following form:

$$\Phi(h) = \sum_{0 \leq k + j \leq n} \Phi_{k,j} = \sum_{u=1}^{m} \sum_{k \text{ odd}} \Psi_{k,j} + \frac{1}{2} \sum_{j=1}^{n} b_{0,j} \left( \Phi_{0,j+1} + \Phi_{0,j+1}^2 \right)$$

$$= \sum_{u=0}^{m} \left( P_{u-1}(h) + \frac{Q_{u-1}(h)}{\sqrt{1-h}} + \frac{R_{u-1}(h)}{\sqrt{1+h}} \right)$$

$$\equiv P_{m-1}(h) + \frac{Q_{m-1}(h)}{\sqrt{1-h}} + \frac{R_{m-1}(h)}{\sqrt{1+h}},$$

where $P_m(h), Q_m(h),$ and $R_m(h)$ denote a polynomial of variable $h$ of degree $m,$ whose coefficients are linear combinations of $a_{k,j}, b_{0,j}$ $(0 \leq k + j \leq n)$.

According to Lemma 13, taking into account their multiplicities, the maximum number of real zeros of (81) in interval $U = [0, 1)$ is $3m - 1.$ From (73), (75), and (76), we know that $\Phi(0) = 0.$ Hence, $\Phi(h)$ has at least $3m - 2$ real zeros in the open interval $(0, 1).$

Applying the Poincaré-Pontryagin theorem, the upper bound of number of limit cycles for the system (5) with $C(x, y) = 1 - x^4$ is $3m - 2.$ That is, the maximum number of limit cycles bifurcating from the period orbits of system (5) with $z = 0$ is $3[(n + 1)/2] - 2.$ The proof of Theorem 2 is completed.

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