The no-sticking effect in ultra-cold collisions

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We provide the theoretical basis for understanding the phenomenon in which an ultra cold atom incident on a possibly warm target will not stick, even in the large $n$ limit where $n$ is the number of internal degrees of freedom of the target. Our treatment is non-perturbative in which the full many-body problem is viewed as a scattering event purely within the context of scattering theory. The question of sticking is then simply and naturally identified with the formation of a long lived resonance. One crucial physical insight that emerges is that the many internal degrees of freedom serve to decohere the incident one body wavefunction, thus upsetting the delicate interference process necessary to form a resonance in the first place. This is the physical reason for not sticking.

I. INTRODUCTION

The problem of low energy sticking to surfaces has attracted much attention over the years [1–5]. The controversial question has been the ultralow energy limit of the incoming species, for either warm or cold surfaces. A battle has ensued between two countervailing effects, which we will call classical sticking and quantum reflection. The concept of quantum reflection is intimately tied into threshold laws, and was recognized in the 1930’s by Lennard-Jones [1]. Essentially, flux is reflected from a purely attractive potential with a probability which goes as $1 - \alpha \sqrt{\epsilon}$, as $\epsilon \to 0$, where $\alpha$ is a constant and $\epsilon$ is the translational energy of the particle incident on the surface. Classically the transmission probability is unity. Reflection at long range prevents inelastic processes from occurring, but if the incoming particle should penetrate into the strongly attractive region, the ensuing acceleration and hard collision with the repulsive short range part of the potential leads to a high probability of inelastic processes and sticking.

The blame for the quantum reflection can be laid at the feet of the WKB approximation, which breaks down in the long range attractive part of the potential at low energy. Very far out, the WKB is good even for low energy, because the potential is so nearly flat. Close in, the kinetic energy is high, because of the attractive potential, even if the asymptotic energy is very low, and again WKB is accurate. But in between there is a breakdown, which has been recognized and exploited by several groups [6–11]. We show in the paper following this one that the breakdown occurs in a region around $|V| \approx \epsilon$; i.e. approximately where the kinetic and potential energies are equal.

It would seem that quantum reflection would settle the issues of sticking, since if the particle doesn’t make it in close to the surface there is no sticking. (Fig 1) There is one caveat, however, which must be considered: quantum reflection can be defeated by the existence of a resonance in the internal region, i.e. a threshold resonance. (Fig 2)

The situation is very analogous to a high Q Fabry-Perot cavity, where using nearly 100% reflective, parallel mirrors gives near 100% reflection except at very specific wavelengths.
At these specific energies a resonance buildup occurs in the interior of the cavity, permitting near 100% transmission. Such resonances are rare in a one dimensional world, but the huge number of degrees of freedom in a macroscopic solid particle makes resonance ubiquitous. Indeed, the act of colliding with the surface, creating a phonon and dropping into a local bound state of the attractive potential describes a Feshbach resonance. Thus, the resonances are just the sticking we are investigating, and we must not treat them lightly! Perhaps it is not obvious after all whether sticking occurs.

After the considerable burst of activity surrounding the sticking issue on the surface of liquid Helium \[12,13\], and after a very well executed theoretical study by Clougherty and Kohn \[4\], the controversy has settled down, and the common wisdom has grown that sticking does not occur at sufficiently low energy. While we agree with this conclusion, we believe the theoretical foundation for it is not complete, nor stated in a wide enough domain of physical situations. For example, Ref. \[4\] treats only a harmonic slab with one or two phonon excitation. It is not clear whether the results apply to a warm surface. On the experimental side, even though quantum reflection was observed from a liquid Helium surface, that surface has a very low density of available states (essentially only the ripplons) which could be a special case with respect to sticking. Thus, the need for more rigorous and clear proof of non-sticking in general circumstances is evident. This paper gives such an analysis. In a following paper, application is made to specific atom-surface and slab combinations, and the rollover to the sticking regime as energy is increased (which can be treated essentially analytically) is given.

The strategy we use puts a very general and exact scattering formalism to work, providing a template into which to insert the properties of our target and scatterer. Then very general results emerge, such as the non-sticking theorem at zero energy. The usual procedure of defining model potentials and considering one phonon processes etc. is not necessary. All such model potentials and Hamiltonians wind up as parameters in the R-matrix formalism. The details of a particular potential are of course important for quantitative results, but the range of possible results can be much more easily examined by inserting various parameters into the R-matrix formalism. All the possible choices of R-matrix parameters give the correct threshold laws. Certain trends are built into the R-matrix formalism which are essentially independent of the details of the potentials.

Before commencing with the R matrix treatment, we briefly consider the problem perturbatively in order to better elucidate the role played by quantum reflection. We emphasize that none of the perturbation section is actually necessary for our final conclusions.

In a perturbative treatment for our slab geometry, quantum reflection simply results in the entrance channels’ wave function (at threshold) having its amplitude in the interaction region go to zero as \( k_e \sim \sqrt{\epsilon} \) when normalized to have a fixed incoming flux. \( k_e \) is the magnitude \( |k_e| \) of the incident wavevector of the incoming atom). The inelastic transition probabilities are proportional to the potential weighted overlap of the channel wavefunctions and this immediately leads to the conclusion that the inelastic probability itself vanishes as \( k_e \sim \sqrt{\epsilon} \). As mentioned, this conclusion is shown to rigorously remain true using the R matrix. We show in this paper that in spite of the inherently many-body nature of the problem, in the ultra-cold limit we can correctly obtain the long-range form of the entrance channel’s wavefunction by solving for the one-dimensional motion in the long-range surface-atom attraction (i.e. the diagonal element of the many-channel potential matrix). This
FIG. 1. The stationary state one body wavefunction of the incident atom moving in the $y$-independent mean potential felt by it. The amplitude inside the interaction region is suppressed by $k_e \sim \sqrt{\epsilon}$. This is tantamount to the reflection of the atom.

FIG. 2. A schematic view of a Feshbach resonance wherein the incident atom forms a long lived quasi-bound state with the target. The many body wavefunction in this situation (not shown) has a large amplitude in the ‘interior’ region near the slab.

allows quantitative predictions of the sticking probability, which we do in the following paper. There, we further exploit the perturbative point of view together with an analysis of WKB to predict a ‘post-threshold’ behavior as quantum reflection abates, when the incoming energy is increased.

II. GEOMETRY AND NOTATION

The incident atom is treated as a point particle at position $(x, y)$. To keep the notation simple we leave out the $z$-coordinate and confine our discussion to two spatial dimensions. Thus a cross-section will have dimensions of length etc. It will be quite obvious how and where $z$ may be inserted in all that follows. Let $u$ represent all the bound degrees of freedom of the scattering target, which we take to be a slab of crystalline or amorphous material. Let $\Omega_c(u)$, $c = 1, 2, \cdots$, be the manybody target wave functions in the absence of interactions with the incident particle, and having energy $E_{c,\text{target}}$. These are normalized as $\int_{u} \lvert \Omega_c(u) \rvert^2 = 1$. $x$ is the distance of the scatterer (atom) from the face of the slab which
is approximately (because the wall is rough) along the line $x = 0$. The internal constituents of the slab lie to the left of $x = 0$ and the scatterer is incident from the right with kinetic energy $\epsilon = \hbar^2k_e^2/2m$. The total energy $E$ of the system is

$$E = \epsilon + E_e^{\text{target}}$$

where $c = e$ is the index of the ‘entrance channel’ i.e. the initial internal state of the slab before the collision is $\Omega_e(u)$. Notice that we say nothing about the value of $E_e^{\text{target}}$ itself. In particular the slab need not be cold. $k_c$ is the magnitude of the wave vector $\vec{k}_c$ of the particle when it leaves the target in the state $\Omega_c(u)$ after the collision. Our interest focuses on $k_e \to 0$. $k_e$ is the magnitude of the wavevector of the incoming particle. For the open channels $c = 1, \cdots n$ (this defines $n$) for which $E > E_c^{\text{target}}$

$$k_c \equiv \sqrt{\frac{2m(E - E_c^{\text{target}})}{\hbar^2}} \quad (c \leq n) ; \quad (2)$$

whereas for the closed channels $(c > n)$, $E < E_c^{\text{target}}$ and

$$k_c \equiv i\sqrt{\frac{2m(E_c^{\text{target}} - E)}{\hbar^2}} \equiv i\kappa_c \quad (c > n) . \quad (3)$$

$\kappa_c > 0$. We will use $(k_{cx}, k_{cy})$ as the $x, y$ components of $\vec{k}_c$. Let $U_{\text{int}}(x, y, u) = (2m/\hbar^2)V_{\text{int}}(x, y, u)$, where $V_{\text{int}}(x, y, u)$ describes quite generally the interaction potential between the incident atom and all the internal degrees of freedom of the slab. For simplicity we assume for the moment that there is no interaction between slab and atom for $x > a$.

III. PRELIMINARIES: PERTURBATION

As stated above, we excercise the perturbative treatment for insight only; our final conclusions are based on nonperturbative arguments.

We treat the interaction $U_{\text{int}}(x, y, u)$ between slab and atom by separating out a ‘mean’ potential felt by the atom that is independent of $y$ and $u$; call it $U^{(0)}(x)$. The remainder $U^{(1)}(x, y, u) \equiv U_{\text{int}}(x, y, u) - U^{(0)}(x)$ is treated as a perturbation.

Now the incident beam is scattered by the entire length (say from $y = -L$ to $L = 2L$) of wall which it illuminates. If all measurements are made close to the wall so that its length $2L$ is the largest scale in the problem, then it is appropriate to speak of a cross-section per unit length of wall, a dimensionless probability. More specifically, we will assume that the matrix elements $U_{cc'}^{(1)}(x, y) \equiv \int du \Omega_c^*(u)U^{(1)}(x, y, u)\Omega_{c'}(u)$ of the perturbation $U^{(1)}(x, y, u)$ in the $\Omega_c(u)$ basis are given by the simple form $U_{cc'}^{(1)}(x, y) = U_{cc'}^{(1)}(x)f(y)$ for $y \in [-L, L]$ and 0 elsewhere. $f(y)$ is a random persistent (does not die to 0 as $|L| \to \infty$) function that models the random roughness of the slab and is characterized by its so-called spectral density function $S$, a smooth positive-valued non-random function, such that

$$\left| \int_{-L}^{L} dy e^{iky}f(y) \right|^2 \equiv 2LS(k) \quad \forall k \quad (4)$$
as $L \to \infty$.

Now, applying either time-independent perturbation (equivalently the Born approximation for this geometry) or time-dependent perturbation theory via the Golden Rule, gives that the cross-section per unit length of wall for inelastic scattering to a final channel $c$ is

$$P_{c\leftarrow e}^\text{in}(\theta) = \frac{2\pi}{k_e} \left( \int_{-\infty}^{0} dx' \phi(x'; k_{ex}) U_{ce}^{(1)}(x') \phi(x'; k_{ex}) \right)^2 S(k_{cy} - k_{ey})$$  \hspace{1cm} (5)

where $\phi(x; k_x)$ is the solution of the o.d.e.

$$\left( \frac{d^2}{dx^2} - U^{(0)}(x) + k^2_x \right) \phi(x; k_x) = 0$$  \hspace{1cm} (6)

which is regular or goes to zero as $x \to -\infty$ inside the slab and is normalized as

$$\phi(x; k_x) \sim \sin(k_x x + \delta) \quad \text{as} \quad x \to \infty$$  \hspace{1cm} (7)

Accepting for the moment that as $k_e \to 0$ the amplitude of $\phi(x; k_{ex})$ in the internal region $x < a$ goes to zero as $k_e \sim \sqrt{\epsilon}$, then the square of the overlap integral in Eq. (5) behaves as $k^2_e$, because by our proposition the amplitude of $\phi(x'; k_{ex}) \sim k_{ex} \sim k_e$. Together with the $1/k_e$ prefactor we get an overall behavior of $k_e$ for the inelastic probability as claimed.

To show that indeed as $k_e \to 0$ the amplitude of $\phi(x; k_{ex})$ in the internal region $x < a$ goes to zero as $k_e \sim \sqrt{\epsilon}$, we temporarily disregard the required normalization of $\phi(x; k_x)$ of Eq. (7) and fix its initial conditions (slope and value) at some point inside the interaction region $x < a$ such that the regularity condition is ensured. We then integrate out to $x = a$. Let us denote this unnormalized solution with a prime, as $\phi'(x; k_x)$. The point is for $k_x$ varying near 0, both $v$(the value) and $s$(the slope) that the solution emerges with at $x = a$, are independent of $k_x$ and in fact the interior solution thus obtained is itself independent of $k_x$.

This is because the local wave vector $k(x) = \sqrt{2m(\epsilon - U(x))}/\hbar$ essentially stays the same function of $x$ for all $\epsilon$ near 0. Therefore for $x > a$ $\phi(x; k_x)$ continues onto

$$v \cos[k_x(x - a)] + \frac{s}{k_x} \sin[k_x(x - a)] \quad x > a$$  \hspace{1cm} (8)

This is a phase-shifted sine wave of amplitude $\sim 1/k_x$. We must enforce the normalization of Eq. (7) and get $\phi(x; k_x) \sim k_x \phi'(x; k_x)$. As a result, the interior solution gets multiplied by $k_x$ and we thereby have our result. $\phi(x; k_x)$ is the solution of a one-dimensional Schrödinger equation for the incoming particle in the one-dimensional long-range potential created by the slab. The suppression of its amplitude by $\sqrt{\epsilon}$ near the slab is due to the reflection it suffers where the interaction turns on. Within the perturbative set-up the non-sticking conclusion is then already foregone [1].

The problem is whether we can really accept this verdict of the one-dimensional unperturbed solution, when in fact we know that the turning on of the perturbation (many body interactions) causes a multitude of resonances to be created, internal resonances being exactly the situation in which the Proposition above is known to badly fail. It appears that the perturbation is in no sense a small physical effect. Therefore a nonperturbative approach is needed. Here we use R-matrix theory in its general form to accomplish the task.
IV. S-MATRIX AND R-MATRIX

One point that the preceding section has made clear is that it is the energies (both initial and final) in the $x$-direction, perpendicular to the slab that are most relevant. In fact as regards the final form of our answers the motion of the $y$ degree of freedom may as well have been the motion of another internal degree of freedom of the slab. In other words, mathematically speaking, the $y$ degree of freedom may be subsumed by incorporating it as just another $u$. For example, we may imagine the incident atom being confined in the $y$-direction by the walls of a wave-guide at $y = -L$ and $L$ that is large enough so that it could not possibly change the physics of sticking. Then we quite rigorously have a bound internal state of the form

$$\Omega_{c,n}(y,u) = \Omega_c(u) \sin \frac{n\pi y}{L}$$

$x$ is now the only scattering degree of freedom. There will be no necessity in carrying along the extra index $n$ and variable $y$ as in Eq. (9), and we will simply continue to write $\Omega_c(u)$ instead. Thus with this understanding, the problem is essentially one-dimensional in the scattering degree of freedom.

We proceed to derive the expression for the $S$ matrix in terms of the so-called $R$ matrix, and derive the structure of the $R$ matrix. For simplicity we continue to assume for the moment that there is no interaction for $x > a$. Then for $x > a$, the scattering wavefunction of the interacting system corresponding to the scattering particle coming in on one entrance channel, say $c = e$, with energy $\epsilon = \hbar^2 k^2_e / (2m)$ is

$$\psi(x,u) = \sum_{c=1}^{\infty} \left( \frac{e^{-ik_c x}}{\sqrt{k_c}} \delta_{ce} - \frac{e^{ik_c x}}{\sqrt{k_c}} S_{ce} \right) \Omega_c(u) \quad x > a$$

where the sum must include all channels, even though the open channels are finite in number. The factors of $k_c^{-1/2}$ in Eq. (10) mean that the flux in each channel is proportional only to the square of the coefficient and hence ensure the unitarity of $S$. With this convention, the open-open part of the $S$-matrix—the $n \times n$ submatrix $S_{cc'}$ with $c, c' = 1, 2, \ldots, n$—is unitary.

$\sqrt{k_c} \equiv e^{i\pi/4} \sqrt{k_c}$ may be arbitrarily chosen since it cannot affect the open-open part of $S$.

$S$ is found in analogy to the one-dimensional case by introducing the matrix version of the inverse logarithmic derivative at $x = a$ called $R(E)$ the Wigner $R$-matrix defined by

$$\vec{v} = R(E) \vec{s}$$

where the components of $\vec{v}$ and $\vec{s}$ are the expansion coefficients of $\psi(x = a, u)$ and $\frac{\partial \psi(x = a, u)}{\partial x}$ respectively in the $\Omega_c(u)$ basis. Supposing $\frac{\partial \psi(x = a, u)}{\partial x}$ to be known, we will (like in electrostatics) use the Neumann Green’s function $G_N(x, u; x', u')$ to construct $\psi(x, u)$ everywhere in the interior $x < a$. $\psi(x, u)$ satisfies the full Schrödinger equation with energy $E$. We need $\chi_\lambda(x, u) \lambda = 1, 2, \ldots$, the normalized eigenfunctions of the full Schrödinger equation in the interior $x < a$ with energies $E_\lambda$, satisfying Neumann boundary conditions $\frac{\partial \chi_\lambda(x = a, u)}{\partial x} = 0$. So

$$\left( -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{int}}(x, u) - E \right) \psi(x, u) = 0$$

(12)
\[
\left(\frac{-\hbar^2}{2m} \nabla^2 + V_{\text{int}}(x, u) - E\lambda\right) \chi_\lambda(x, u) = 0 \tag{13}
\]

\[
\left(\frac{-\hbar^2}{2m} \nabla^2 + V_{\text{int}}(x, u) - E\right) G_N(x, u; x', u') = \delta(x - x')\delta(u - u') \tag{14}
\]

where \( \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial u^2} \) and

\[
\frac{\partial G_N(x = a, u; x', u')}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \chi(x = a, u)}{\partial x} = 0 \tag{15}
\]

\[
G_N(x, u; x', u') = \sum_{\lambda=1}^{\infty} \frac{\chi_\lambda(x, u)\chi_\lambda(x', u')}{E_\lambda - E} \tag{16}
\]

\( G_N \) is symmetric in the primed and unprimed variables. By Stokes’ Theorem,

\[
(-\hbar^2/2m) \int_{x'<a} \int_{u'} \phi_1 \nabla'^2 \phi_2 - \phi_2 \nabla'^2 \phi_1 = (-\hbar^2/2m) \int_{x'=a} \int_{u'} \phi_1 \nabla' \phi_2 - \phi_2 \nabla' \phi_1 \tag{17}
\]

where \( \nabla'_\parallel (\cdot) \equiv \hat{x}'(\cdot) \cdot \nabla' \) with \( \phi_1 = \psi(x', u') \) and \( \phi_2 = G_N(x, u; x', u') \) gives

\[
\psi(x, u) = \frac{\hbar^2}{2m} \int_{u'} G_N(x, u; x', u') \frac{\partial \psi(x' = a, u')}{\partial x'} \quad x < a \tag{18}
\]

Put \( x = a \) and it is deduced using Eqs. (11) and (18) together that

\[
R_{cc'}(E) = \sum_{\lambda=1}^{\infty} \frac{\gamma_{\lambda c} \gamma_{\lambda c'}}{E_\lambda - E} \tag{19}
\]

where \( \gamma_{\lambda c} = \sqrt{\frac{\hbar^2}{2m}} \int_{u} \chi_\lambda(a, u)\Omega_c(u) \).

### A. The S matrix

Now shifting attention to the outside \((x > a)\), we see that we can compute both \( \nabla^\parallel \psi(a, u) \) and \( \psi(a, u) \) on the surface \( x = a \) using the asymptotic form of Eq. (11) which automatically gives these expanded in the \( \Omega_c(u) \) basis. Writing the matrix Eq. (11) is now simple. It is best to do it all in matrix notation, and thus be able to treat all possible independent asymptotic boundary conditions simultaneously.

Let \( e^{ikx}, \sqrt{k} \) and \( 1/\sqrt{k} \) be diagonal matrices with diagonal elements \( e^{ikc}x, \sqrt{k_c} \) and \( 1/\sqrt{k_c} \). Then Eq. (11) reads

\[
e^{-ika} \frac{1}{\sqrt{k}} - e^{ika} \frac{1}{\sqrt{k}} S = iRk \left( -e^{-ika} \frac{1}{\sqrt{k}} - e^{ika} \frac{1}{\sqrt{k}} S \right) \tag{20}
\]

Each column \( c = 1, \ldots, n \) of the matrix equation above is just Eq. (11) for the solution corresponding to an incoming wave only in channel \( c \) (For \( c > n \) the wavefunctions blow up as \( x \to \infty \)). Remembering that non-diagonal matrices don’t commute, we solve for \( S \) to get
\[ S = e^{-ika} \sqrt{k} \frac{1}{1 - i Rk} (1 + i Rk) \frac{1}{\sqrt{k}} e^{-ika} \]  

(21)

or, with some simple matrix manipulation,

\[ S = e^{-ika} \frac{1}{1 - i \sqrt{k} R \sqrt{k}} (1 + i \sqrt{k} R \sqrt{k}) e^{-ika}. \]  

(22)

V. S MATRIX NEAR A RESONANCE

As discussed in the introduction, the resonances are a key to the sticking issue. Sticking is essentially a long lived Feshbach resonance in which energy has been supplied to surface and bulk degrees of freedom, temporarily dropping the scattering particle into a bound state of the attractive potential. Thus we must study resonances in various circumstances in the low incident translational energy regime. We derive the approximation for \( S(E) \) near \( E = E_0 \), a resonant energy of the compound system. \( E_0 \) is the total energy of the joined (resonant) system. Within the R-matrix approach, the \( \chi_{\lambda}(x, u) \) of section IV are bound, compound states with Neuman boundary conditions at \( x = a \). R-matrix theory properly couples these bound state to the continuum, but some of the eigenstates are nonetheless weakly coupled to the continuum, as evidenced by small values of the \( \gamma_{\lambda c} \)'s of section IV; these are the measure of the strength of the continuum couplings. While every one of the \( R \)-matrix bound states will result in a pole \( E_\lambda \) in the \( R \) matrix expansion, only the weakly coupled ones are the true long lived Feshbach resonances of physical interest. It is also helpful to know that the values of these ‘truly’ resonant poles at \( E_\lambda \) are the most stable to changes in the position \( x = a \) of the box. This in fact provides one unambiguous way to identify them. Our purpose here is to derive the resonant approximation to the S matrix in the vicinity of one of these Feshbach resonances. We do so using the form of the \( R \)-matrix in Eq. (19). Note that the energy density \( \rho(E) = 1/D(E) \) of these Feshbach resonances will be large because of the large number of degrees of freedom of the target. \( D(E) \) is the level spacing of the quasibound, resonant states.

A. Isolated Resonance

As mentioned, the point of view we will take is to identify a resonant energy with a particular pole \( E_\lambda \) in the \( R \) matrix expansion of Eq. (19). Those \( E_\lambda \) corresponding to resonances are a subsequence of the \( E_\lambda \) appearing in the expansion in Eq. (19). For \( E \) near a well isolated resonance at \( E_\lambda \) we separate the sum-over-poles expansion of the R-matrix into a single matrix term having elements \( \frac{\gamma_{\lambda c} \gamma_{\lambda c}'}{E_\lambda - E} \), plus a sum over all the remaining terms, call it \( N \). If the energy interval between \( E_\lambda \) and all the other poles is large compared to the open-open residue at \( E_\lambda \) then we may expect that the \( n \times n \) open-open block of \( N \) will have all its elements to be small. Then rewriting the inverse in Eq. (22)

\[ \frac{1}{1 - i \sqrt{k} R \sqrt{k}} \equiv \frac{1}{1 - i \left(M + \frac{V}{E_\lambda - E}\right)} \]  

(23)

8
where $M \equiv \sqrt{kN} \sqrt{k}$ and $V_{cc'} \equiv (\sqrt{k_c\gamma_{\lambda c}})(\sqrt{k_{c'}\gamma_{\lambda c'}})$, and setting $M = 0$ allows us to simplify the central term in Eq. (22) exactly. (We will return to the case $M \neq 0$.)

\[
\frac{1}{1 - i\sqrt{kR} \sqrt{k}} (1 + i\sqrt{kR} \sqrt{k})
\]

(24)

\[
= 1 + \frac{1}{1 - i\sqrt{kR} \sqrt{k}} 2i\sqrt{kR} \sqrt{k}
\]

(25)

\[
= 1 + \frac{1}{1 - \frac{V}{E_{\lambda} - E}} 2i\frac{V}{E_{\lambda} - E} \quad \text{(with } M = 0)
\]

(26)

\[
= 1 + \frac{1}{E_{\lambda} - E - iV} 2iV
\]

(27)

\[
= 1 + \frac{1}{E_{\lambda} - E - iV} 2iV k
\]

(28)

where we used

\[
V^2 = \left( (\gamma_{\lambda 1}^2 k_1 + \cdots + \gamma_{\lambda n}^2 k_n) + (\gamma_{\lambda (n+1)}^2 \kappa_{n+1} + \cdots) \right) V
\]

(29)

\[
\equiv \left( \left( \frac{\Gamma_{\lambda 1}}{2} + \cdots + \frac{\Gamma_{\lambda n}}{2} \right) + i(\gamma_{\lambda (n+1)}^2 \kappa_{n+1} + \cdots) \right) V
\]

(30)

\[
\equiv \left( \frac{\Gamma_{\lambda}}{2} + i\Delta E_{\lambda} \right) V
\]

(31)

to get the identities

\[
[E_{\lambda} - E - iV]V = [E_{\lambda} - E - i(\Gamma_{\lambda}/2 + i\Delta E)]V
\]

(32)

\[
\Rightarrow \quad \frac{1}{E_{\lambda} - E - i(\Gamma_{\lambda}/2 + i\Delta E)} V = \frac{1}{E_{\lambda} - E - i\nu} V
\]

(33)

Also define $(\Gamma_{\lambda c}/2)^{1/2} \equiv \gamma_{\lambda c} \sqrt{k_c}$, $c = 1, 2, \cdots, n$. This defines the sign of the square-root on the lhs. to be the sign of $\gamma_{\lambda c}$ and allows the convenience of expressing things in terms of the $\Gamma_{\lambda c}$’s and their square-roots, and not having to use the $\gamma_{\lambda c}$’s themselves. Thus we arrive at

\[
S_{cc'} = e^{-ik_{c'a}} \left( \delta_{cc'} + \frac{i\Gamma_{\lambda c}^{1/2} \Gamma_{\lambda c'}^{1/2}}{E_{\lambda}^{(r)} - E - i\Gamma_{\lambda}/2} e^{-ik_{c'a}} \right)
\]

(34)

where $E_{\lambda}^{(r)} \equiv E_{\lambda} + \Delta E_{\lambda}$, for the $n \times n$ open-open unitary block of $S$ in the neighbourhood of a single isolated resonance after neglecting the contribution of the background matrix $M$. For us the essential point is that

\[
\Gamma_{\lambda c} = 2k_c(E)\gamma_{\lambda c}^2
\]

(35)

that the partial widths $\Gamma_{\lambda c}$ depend on the energy $E$, through the kinematic factor $k_c(E)$. Mostly this energy dependence is small and irrelevant except where the $k_c$’s and hence $\Gamma_{\lambda c}$’s are varying near 0. These are the partial widths of the open channels near threshold. Hence $|S_{cc'}|^2$ ($c \neq e$) an inelastic probability behaves like $k_c \sim \sqrt{\epsilon}$ when the entrance channel is at threshold. Including the background term ($M \neq 0$) does not change this. To see this we may
perform the inverse in Eq. (22) to first order in $M$ and then get an additional contribution of the terms

$$e^{-ika} \left( \frac{2i}{1 - \frac{iV}{E_\lambda - E}} M + \frac{1}{1 - \frac{iV}{E_\lambda - E}} + \frac{1}{1 - \frac{iV}{E_\lambda - E}} \frac{iM}{1 - \frac{iV}{E_\lambda - E}} 2iV \right) e^{-ika} \quad (36)$$

to the $S$-matrix. Now, both $M$ and $V$ have a factor of $\sqrt{k_e}$ multiplying their $c$th columns (and rows) from their definitions and so a matrix element $b_{cc'}$ of the matrix in parentheses in Eq. (36) will have a $\sqrt{k_e}$ and $\sqrt{k_{c'}}$ dependence. An inelastic element of $S(c \neq c')$ would now take the form

$$S_{cc'} = e^{-ika} \left( b_{cc'} + \frac{i\Gamma^{1/2}_{\lambda c} \Gamma^{1/2}_{\lambda c'}}{E_\lambda^{(r)} - E - i\Gamma_\lambda/2} \right) e^{-ik_{c'}a}, \quad (37)$$

As mentioned our interest is in the case when the entrance channel is at threshold so that this dependence is $\sqrt{k_e}$, making the inelastic probability $|S_{cc}|^2$ still continue to behave as $k_e \sim \sqrt{\epsilon}$.

### B. Overlapping Resonances

Here we require the form of the $S$ matrix near an energy $E$ where many of the quasi-bound states may be simultaneously excited, i.e. the resonances overlap. Again, neglecting background for the moment, the $S$ matrix is simply taken to be a sum over the various resonances.

$$S = 1 - \sum_{\lambda} \frac{iA_\lambda}{E - E_\lambda^{(r)} + i\Gamma_\lambda/2} \quad (38)$$

where $A_\lambda$ is a $n \times n$ rank 1 matrix with the $cc'$th component as $\Gamma^{1/2}_{\lambda c} \Gamma^{1/2}_{\lambda c'}$. There is no entirely direct justification of this form, but one can see that there is much which it gets correct.

The $A_\lambda$ are symmetric, hence $S$ is symmetric. Obviously it has the poles in the right places allowing the existence of decaying states with a purely outgoing wave at the resonant energies. A crucial additional assumption that also makes $S$ approximately unitary is that the signs of the $\Gamma^{1/2}_{\lambda c}$ are random and uncorrelated both in the index $\lambda$ as well as $c$, regardless of how close the energy intervals involved may be. One simple consequence is that we approximately have that

$$A_\lambda A_{\lambda'} = \delta_{\lambda\lambda'} \Gamma_\lambda A_\lambda \quad (39)$$

in the sense that the l.h.s. is negligible for $\lambda \neq \lambda'$ in comparison to the value for $\lambda = \lambda'$. With Eq. (39) it is easy to verify the approximate unitarity of $S$.

We investigate now the onset of the overlapping regime as $E$ increases. $D(E)$, the level spacing of the resonant $E_\lambda^{(r)}$, is a rapidly decreasing function of its argument. On the other hand, $\Gamma_\lambda = \Gamma_{\lambda 1} + \Gamma_{\lambda 2} + \cdots + \Gamma_{\lambda n}$, and since more channels are open at higher energy, $\Gamma_\lambda$ is increasing with the energy of the resonance. The widths must therefore eventually overlap,
and $\Gamma_\lambda \gg D \left( E^{(r)}_\lambda \right)$ for the larger members of the sequence of $E^{(r)}_\lambda$'s. In this regard there is a useful estimate due to Bohr and Wheeler \[15\], that for $n$ large

$$\frac{\Gamma_\lambda}{D(E^{(r)}_\lambda)} \simeq n . \quad (40)$$

Appendix A derives this using a phase space argument. Here we point out that this is entirely consistent with the assumption of the random signs, indeed requiring it to be true. Take for example a typical inelastic amplitude

$$S_{cc'} = -i \sum_\lambda \frac{\Gamma^{1/2}_\lambda \Gamma^{1/2}_{\lambda c}}{E^{(r)}_\lambda - E - i\Gamma_\lambda/2} \quad (c \neq c') \quad (41)$$

First let us note that the $\Gamma_\lambda$ being the sum of many random variables (the partial widths $\Gamma_{\lambda c}$) do not fluctuate much. Let $\Gamma$ denote their typical value over the $n$ overlapping resonances. Also since $\Gamma = nD$ it follows that the typical size of a partial width $\Gamma_{\lambda c}$ is $D$. Therefore the typical size of the product $\Gamma^{1/2}_\lambda \Gamma^{1/2}_{\lambda c}$ is $D$ but these random variables fluctuate randomly over the index $\lambda$, and moreover the sign is random. Thus for energies in the overlapping domain $S_{cc'}$ is a sum of $n$ complex numbers each of typical size $D/\Gamma = 1/n$, but random in sign. This makes for a sum of order $1/\sqrt{n}$. Clearly this is as required to make the $n \times n$ matrix $S$ unitary. Note that the above argument fails (as is should) if $c \neq c'$ because then the signs of $\Gamma^{1/2}_\lambda \Gamma^{1/2}_{\lambda c} = \Gamma_\lambda > 0$ are of course not random.

Unlike the case of the isolated resonance, the $S$-matrix elements here are smoothly varying in $E$. Addition of a background term $B_{cc'}$

$$S_{cc'} = B_{cc'} - i \sum_\lambda \frac{\Gamma^{1/2}_\lambda \Gamma^{1/2}_{\lambda c}}{E^{(r)}_\lambda - E - i\Gamma_\lambda/2} . \quad (42)$$

just shifts this smooth variation by a constant. If $B_{cc'}$ is also thought of as arising from a sum over the individual backgrounds then for the same reasons as discussed at the end of the preceding section $|B_{cc}|^2 \sim k_c \sim \sqrt{\epsilon}$ for an entrance channel near threshold. For simplicity we will continue to take $B_{cc'}$ to be 0 and look at the case with background in the appendix.

VI. Q-MATRIX AND STICKING

From the viewpoint of scattering theory, the sticking of the incident particle to the target is just a long-lived resonance. It is natural then to investigate the time-delay for the collision. Smith \[14\] introduced the collision lifetime or $Q$-matrix

$$Q \equiv \hbar S \frac{\partial S^\dagger}{\partial E} \quad (43)$$

which encapsulates such information. We review some of the relevant properties of $Q$. The rhs of Eq. (43) involves the ‘open-open’ upper left block of $S$ so that $Q$ is also an $n \times n$ energy-dependent matrix, having dimensions of time. For 1-dimensional elastic potential scattering $S = e^{i\vartheta(c)}$ and $Q$ reduces to the familiar time delay $\hbar \frac{\partial \vartheta(E)}{\partial E}$. If $\vec{v}$ is a vector
whose entries are the coefficients of the incoming wave in each channel then \( \vec{v}^{tr} Q(E) \vec{v} \) is the average delay time experienced by such an incoming wave. Because physically the particle is incident on only one channel, \( \vec{v} \) consists of all 0’s except for a 1 in the \( e \)th slot so that the relevant quantity is just the matrix element \( Q_{ee}(E) \). Smith shows that this delay time is the surplus probability of being in a neighborhood of the target (measured relative to the probability if no target were present) divided by the flux arriving in channel \( e \). This matches our intuition that when the delay time is long, there is a higher probability that the particle will be found near the target.

Furthermore, as a Hermitian matrix, \( Q(E) \), can be resolved into its eigenstates \( \vec{v}^{(1)} \cdots \vec{v}^{(n)} \) with eigenvalues \( q_1 \cdots q_n \). The components of \( \vec{v}^{(1)} \) are the incoming coefficients of a quasi-bound state with lifetime \( q_1 \) and so on. Then

\[
\vec{v}^{tr} Q(E) \vec{v} = \sum_{j=1}^{n} q_j |\vec{v}^{(j)} \cdot \vec{v}|^2. \tag{44}
\]

As can be seen from this expression, the average time delay results, in general, from the excitation of multiple quasi-stuck states each with its lifetime \( q_j \) and probability of formation \( |\vec{v}^{(j)} \cdot \vec{v}|^2 \). However, we will find that using our resonant approximation to the \( S \) matrix near a resonant energy \( E^{(r)}_\lambda \) the time delay will consist of only one term from the sum on the rhs of Eq. (44), all the other eigenvalues being identically 0.

Using equation Eq. (43),

\[
Q(E) = i\hbar \left( \sum \frac{-iA_{\lambda'}}{[E - E^{(r)}_{\lambda'} - i\Gamma_{\lambda'}/2]^2} - \sum_{\lambda,\lambda'} \frac{A_{\lambda}A_{\lambda'}}{[E - E^{(r)}_{\lambda} + i\Gamma_{\lambda}/2][E - E^{(r)}_{\lambda'} - i\Gamma_{\lambda'}/2]^2} \right), \tag{45}
\]

which using Eq. (39) simplifies to

\[
= \sum_{\lambda} \frac{\hbar}{(E - E^{(r)}_{\lambda})^2 + (\Gamma_{\lambda}/2)^2} A_{\lambda}, \tag{46}
\]

a remarkably simple answer. We need \( Q_{ee}(E) \), where \( e \) is the entrance channel.

\[
Q_{ee}(E) = \sum_{\lambda} \frac{\hbar\Gamma_{\lambda e}}{(E - E^{(r)}_{\lambda})^2 + (\Gamma_{\lambda}/2)^2} \tag{47}
\]

\[
= \sum_{\lambda} \left( \frac{\hbar\Gamma_{\lambda}}{(E - E^{(r)}_{\lambda})^2 + (\Gamma_{\lambda}/2)^2} \times \frac{\Gamma_{\lambda e}}{\Gamma_{\lambda}} \right) \tag{48}
\]

where the second equation has the interpretation (for each term) as the life-time of the mode, multiplied by the probability of its formation. Note how for each resonance \( E^{(r)}_{\lambda} \) there is only one term corresponding to the decomposition of Eq. (44). The actual measured lifetime is the average of \( Q_{ee}(E) \) averaged over the energy spectrum \( |g(E)|^2 \) of the collision process.

\[\text{A. Energy averaging over spectrum}\]

With the target in state \( \Omega_{c}(u) \) where \( c = e \) is the entrance channel, the energy of the target is fixed, and the time-dependent solution will look like
\[ \psi(x, u, t) = \int dE \left( g(E) \sum_{c=1}^{\infty} \left( \frac{e^{-i k_c(E)x}}{\sqrt{k_c(E)}} - \frac{e^{i k_c(E)x}}{\sqrt{k_c(E)}} S(E)_{ce} \right) \Omega_c(u) \right). \]  

(49)

Recall, \( E \) is the total energy of the system. We are interested in the threshold situation where the incident kinetic energy of the incoming particle \( \epsilon \to 0 \). This can be arranged if \( g(E) \) is peaked at \( E_0 \) with a spread \( \Delta E \) such that i) \( E_0 \) is barely above \( \epsilon_{\text{target}} \) and ii) \( \Delta E = \delta \epsilon \) is some small fraction of \( \epsilon \), the mean energy of the incoming particle. The second condition ensures that we may speak unambiguously of the incoming particle’s mean energy. So,

\[ \langle Q_{ee}(E) \rangle \equiv \int dE |g(E)|^2 Q_{ee}(E) \]

(50)

\[ \simeq \frac{1}{\Delta E} \int dE Q_{ee}(E) \]

(51)

\( \langle \rangle \) denotes the average over the \( \Delta E \) interval. Now \( Q_{ee}(E) \) is just a sum of Lorentzians centred at the \( E_\lambda^{(r)} \)'s with width \( \Gamma \) and Eq. (51) is just a measure of their mean value over the \( \Delta E \) interval.

So long as the \( \Delta E \) interval around which we are averaging, is broad enough to straddle many of these Lorentzians, the mean height is just

\[ \frac{1}{\Delta E} \times \rho(E) \Delta E \times \frac{\hbar \pi \Gamma_{\lambda e}}{\Gamma_{\lambda}} \]

(52)

where the second factor is the number of Lorentzians in the \( \Delta E \) interval and the third factor is the area under the ‘\( \lambda \)'th Lorentzian. This is true regardless of whether or not they are overlapping. It will be convenient to write \( \Gamma_{\lambda} \) as

\[ \Gamma_{\lambda} = n \times 2\bar{k}_{\lambda} \text{var}(\gamma_{\lambda}) \]

(53)

where \( \text{var}(\gamma_{\lambda}) \) is the variance of the set of \( \gamma_{\lambda e} \)'s over the \( n \) open channels and \( \bar{k}_{\lambda} \) is a mean or effective wavenumber \( k_e \) over the open channels, which for a particular realization \( \lambda \) we take to be defined by Eq. (53) itself. Let \( \langle \rangle \) denote the average over the occurrences of the quantity in the \( \Delta E \) interval. \( \Gamma_{\lambda} \equiv \langle \Gamma_{\lambda} \rangle \), \( \bar{k}_{\lambda} \equiv \langle \bar{k}_{\lambda} \rangle \). Then Eq. (52) simplifies

\[ \langle Q_{ee}(E) \rangle \simeq \frac{\hbar}{D} \frac{k_e \langle \gamma_{\lambda e}^2 \rangle}{nk_{\lambda} \text{var}(\gamma_{\lambda})} \]

(54)

\[ \simeq \frac{\hbar k_e}{\Gamma_{\lambda}} \]

(55)

which tends to 0 as \( k_e \sim \sqrt{\epsilon} \). The form of Eq. (55) and all the steps leading up to it remain valid whether the Lorentzians are overlapping or not, as long as the \( \Delta E = \Delta \epsilon \) interval which we are averaging over includes many of them.

**B. On an isolated resonance**

If the target is cold enough that the resonances are isolated, then as the incident particle’s energy \( \epsilon \to 0 \), adhering to the condition \( \Delta \epsilon < \epsilon \) will eventually result in \( \Delta \epsilon \) becoming
narrower than the resonance widths. It becomes possible then for $\Delta \epsilon$ to be centered right around a single isolated resonance at $E_{\lambda}^{(r)}$. In this case $\langle Q_{ee}(E) \rangle$ is found simply by putting $E = E_{\lambda}^{(r)}$, because the spectrum $|g(E)|^2$ is well approximated by $\delta(E - E_{\lambda}^{(r)})$. So

$$\langle Q_{ee}(E) \rangle = \frac{\hbar \Gamma_{\lambda}}{\Gamma_{\lambda}^2} = \frac{\hbar \Gamma_{\lambda}}{\Gamma_{\lambda}^2} \frac{\hbar}{\Gamma_{\lambda}} \frac{k_e}{\bar{k}}.$$  

(56)

Even in this case there is the $\sqrt{\epsilon}$ behavior as $\epsilon \to 0$ and there is no sticking.

In the extreme case that there are no other open channels at all ($n = 1$), $\langle Q_{ee}(E) \rangle \simeq \frac{\hbar \Gamma_{\lambda}}{\Gamma_{\lambda}^2}$ because $\Gamma_{\lambda} = \Gamma_{\lambda e}$. In fact, $e = 1$, and $\langle Q_{ee}(E) \rangle$ diverges, implying in this case that it is possible to have the particle stick. This is an exception to all the cases above but is experimentally not so relevant because we may always expect to find some exothermic channels open for a target with many degrees of freedom.

VII. INELASTIC CROSS SECTIONS AND STICKING

Another physically motivated measure of the sticking probability may be obtained by studying the total inelastic cross-section of the collision. The idea is that any long lived "sticking" is overwhelmingly likely to result in an inelastic collision process; i.e. that the scattering particle will leave in a different channel than it entered with. Using the original Wigner approach it is possible to show that for our case where we have only one scattering degree of freedom, the inelastic probability for an exothermic and endothermic collision vanishes like $k_e$. The only possible exception to this is a measure zero chance of a resonance exactly at the threshold energy, $E_{\epsilon}^{\text{target}}$. In the event that there is a resonance $E_{\lambda}^{(r)}$ close to but above this threshold energy, it is only necessary that $E$ is below $E_{\lambda}^{(r)}$ (by an energy of at least $\Delta E$, the spread in energy) in order to observe the usual Wigner threshold behavior:

$$P_{\text{inelastic}} \to 0 \text{ like } k_e \propto \sqrt{\epsilon}$$  

(57)

for the inelastic probability. However our problem is unusual in the sense that because of the large number of degrees of freedom of the target, we will always find resonances between $E_{\epsilon}^{\text{target}}$ and $E$ no matter how small $E - E_{\epsilon}^{\text{target}} = \epsilon$ is. Thus the Wigner regime is not accessible. Still the surprise is that a simple computation reveals the same behavior holds for large $n$:

$$P_{\text{inelastic}}(E) = \sum_{c \neq e} P_{c\epsilon\epsilon}(E)$$  

(58)

$$= \sum_{c \neq e} |S_{c\epsilon\epsilon}(E)|^2$$  

(59)

$$= \sum_{c \neq e} \sum_{\lambda} \sum_{\lambda'} \frac{\Gamma_{\lambda e}^{1/2} \Gamma_{\lambda e}^{1/2}}{E - E_{\lambda}^{(r)} - i \Gamma_{\lambda}/2} \frac{\Gamma_{\lambda e}^{1/2} \Gamma_{\lambda e}^{1/2}}{E - E_{\lambda}^{(r)} + i \Gamma_{\lambda}/2}$$  

(60)

$$\Rightarrow P_{\text{inelastic}}(E) = \sum_{\lambda} \frac{\Gamma_{\lambda}}{(E - E_{\lambda}^{(r)})^2 + (\Gamma_{\lambda}/2)^2} \frac{\Gamma_{\lambda e}}{\bar{k}}$$  

(61)

where we used the random sign property of the $\Gamma_{\lambda e}^{1/2}$'s and the understanding that $\sum_{c \neq e} \Gamma_{\lambda e} \simeq \sum_{\lambda} \Gamma_{\lambda e} = \Gamma_{\lambda}$. Since the sum $\sum_{c \neq e}$ is over the $n \gg 1$ open channels, omission of a single term
can hardly matter. Apart from the factor $\hbar/\Gamma_\lambda$, the rhs of the above equation is identical to the expression for $Q_{ee}(E)$ in Eq. (48). Averaging $P_{\text{inelastic}}(E)$ over many resonances $E^{(r)}_\lambda$ (overlapping or not) we may use the same algebraic simplifications as before to show

$$\langle P_{\text{inelastic}} \rangle = \frac{k_e}{k} \quad (62)$$

As $k_e$ tends to 0, this gives the $\sqrt{\epsilon}$ Wigner behavior showing that there is no sticking.

The above argument fails when there is only one open channel. There are no inelastic channels to speak of. In this case, if the energy $E$ coincides with a resonant energy $E^{(r)}_\lambda$ we will have the exceptional case of sticking, as discussed at the end of the previous section. But as pointed out there, this is primarily of theoretical interest only.

VIII. CHANNEL DECOHERENCE

The only case for which we stick is seen to be the case of when we are sitting right on top of a resonance with the incoming energy so well resolved that we are completely within the resonance width, AND there are no exothermic channels open. Having no such channels open amounts to an infinitesimally low energy for a large target. Otherwise, the sticking probability tends to 0 as $\sqrt{\epsilon}$ in every case.

A. Time dependent picture

From the time independent point of view, the physical reason for the absence of low energy sticking is contained in the factor $\frac{\Gamma_{ee}}{\Gamma_\lambda}$ of Eq. (48). This is the formation probability for the compound state. We will explain physically why it is small for $n \gg 1$. The resonance state is a many-body entangled state. If we imagine the decay of this compound state (already prepared by some other means say) each open channel carries away some fraction of the outgoing flux, with no preference for any one particular channel. Running this whole process in reverse it becomes evident that the optimum way to form the compound state is to have each channel carry an incoming flux with exactly the right amplitude and phase. This corresponds to however an entangled initial state. With all the incoming flux instead constrained to be in only one channel it becomes clear that we are not exciting the resonance in the optimal way and the buildup of amplitude inside is not so large; i.e., the compound state has a small probability of forming.

The time dependent view is even more revealing. Imagine a wave packet incident on the system. For a single open channel Feshbach resonance, the build-up of amplitude in the interior region can be decomposed as follows. As the leading edge of the wavepacket approaches the region of attraction, most is turned away due to the quantum reflection phenomena. (It is a useful model to think of the quantum reflection as due to a barrier located some distance away from the interaction region.) The wavefunction in the interaction region constructively interferes with new amplitude entering the region. At the same time, the amplitude leaving the region is out of phase with the reflected wave, cancelling it and assisting more amplitude to enter.
Now suppose many channels are open. All the flux entering the interior must of course return, but it does so fragmented into all the other open channels. Only the fraction that makes it back into the entrance channel has the opportunity to interfere (constructively) with the rest of the entering wavepacket. The constructive interference is no longer efficient and is in fact almost negligible for \( n \gg 1 \), thereby ruining the delicate process that was responsible for the buildup of the wave function inside. The orthogonality of the other channels prevents interference in the scattering dimension. If we trace over the target coordinates, leaving only the scattering coordinate, most of the coherence and the constructive interference is lost, and no resonant buildup occurs. Therefore, one way to understand the non-sticking is to say that decoherence is to blame.

**B. Fabry-Perot and Measurement Analogy**

Suppose we have a resonant quantum mechanical Fabry-Perot cavity, where the particle has a high probability of being found in between the two reflecting barriers. Now, during the time it takes for the probability to build up in the interior, suppose we continually measure the position of the particle inside. In doing so we decohere the wave function and in fact never find it there at all. Alternatively, imagine simply tilting one barrier (mirror) to make it non-parallel to the first and redirecting the flux into an orthogonal direction, again spoiling the resonance. Measurement entangles other (orthogonal) degrees of freedom with the one of interest, resulting in flux being effectively re-directed into orthogonal states. Thus the states of the target (if potentially excitable) are in effect continually monitoring (measuring) to see if the incoming particle has made it in inside, ironically then preventing it from ever doing so. The buildup process of constructive interference in the interaction region, described in the preceding paragraph, is slower than linear in \( t \). Therefore, the constant measurement of the particle’s presence (and resultant prevention of sticking) is an example of the Zeno “paradox” in measurement theory.

**IX. CONCLUSION**

We have presented a general approach to the low energy sticking problem, in the form of \( R \)-matrix theory. This theory is well suited for the task, since it highlights the essential features of multichannel scattering at low incident translational energy. We did not need to make a harmonic or other approximate assumptions about the solid target, which is characterized by its long range interaction with the incoming particle and its density of states. “Warm” surfaces are included in the formalism, and do not change the non-sticking conclusion.

Several supporting arguments for the non-sticking conclusion were given. Perhaps most valuable is the physical decoherence picture associated with the conclusion that there is no sticking in the zero translational energy limit.

Reviewing the observations leading up to the non-sticking conclusion, we start with the near 100% sticking in the zero translational energy limit classically (sticking probability 1). We then invoke the phenomenon of quantum reflection (Fig. 1), which keeps the incident particle far from the surface (sticking probability 0). Third, we note that quantum reflection
can be overcome by resonances (Fig. 2), and since resonances are ubiquitous in a many body target, being the Feshbach states by which a particule could stick to the surface, perhaps sticking approaches 1 after all. Fourth, we suggest that decoherence (from the perspective of the incoming channel, with elastic scattering defined as coherent) ruins the resonance effect, reinstating the quantum reflection as the determining effect. Finally, then, there is no sticking, and the short answer as to why is: quantum reflection and many channel decoherence. The ultrashort explanation is simply quantum reflection, but this is dangerous and non-rigorous, as we have tried to show.

All this does not tell us much about how sticking turns on as incident translational energy is raised. This is the subject of the following paper, where a WKB analysis proves very useful. Quantum reflection is a physical phenomenon liked directly to the failure of the WKB approximation.

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APPENDIX A: \( \Gamma \approx ND \)

With the large number of degrees of freedom involved and assuming thorough phase space mixing associated with the resonance we may reasonably describe the compound state wavefunction by a classical ensemble of points \((x, p_x, u, p_u)\) in the combined phase space of the joint system given by the normalized distribution

\[
\frac{1}{\rho_C(E)} \delta(E - H(x, p_x, u, p_u)).
\]  

(A1)

It is understood in the above that the system is restricted to be in the region \(x < a\). This makes all accessible states of energy \(E\) with \(x < a\) equally likely. Then the rate of escape \(\Gamma/\hbar\) through the hypersurface \(x = a\) of the members of this ensemble is

\[
\frac{\Gamma}{\hbar} = \frac{1}{\rho_C(E)} \int_{x=a} dudp_u \int_{p_x \in [0,\infty]} dp_x \frac{p_x}{m} \delta(E - H(x, p_x, u, p_u)).
\]  

(A2)

\(p_x/m\) is just the velocity in phase space of a point at \(x = a\) in the \(\hat{x}\) direction. At \(x = a\) we have supposed no interaction. Hence the Hamiltonian separates in Eq. (A2). Therefore

\[
\frac{\Gamma}{\hbar} = \frac{1}{\rho_C(E)} \int dudp_u \int_0^\infty d \left(\frac{p_x^2}{2m}\right) \delta \left(E - \left(\frac{p_x^2}{2m} + H_{\text{target}}(u, p_u)\right)\right)
\]  

(A3)

\[
= \frac{1}{\rho_C} \int_{H_{\text{target}}(u, p_u) < E} dudp_u
\]  

(A4)

\[
= \frac{1}{\rho_C} \Omega_C \simeq \frac{1}{2\pi \hbar \rho_Q} \Omega_Q = \frac{1}{2\pi \hbar} nD.
\]  

(A5)
Therefore $\rho \simeq n$. $\rho_Q (\rho_C)$ is the quantum (classical) density of states (phase space volume) of the joint system at energy $E$. $\Omega_Q (\Omega_C)$ is the quantum (classical) total number of states (total phase space volume) of only the target below energy $E$. We have used the correspondence between the Classical and Quantum density of states. $1/\rho_Q$ is identified with $D$, and the number of states of the target having energy less that $E$ is just $n$, the number of open channels.

**APPENDIX B: INELASTIC PROBABILITY WITH BACKGROUND**

We show here that the inelastic probabilities remain essentially unaffected in magnitude with the presence of a background term in the S-matrix. In the isolated case the addition of $b_{cc'}$ to an inelastic element $S_{cc'}$ simply changes the Lorentzian profile of $|S_{cc'}|^2$. In the more important overlapping case, the energy variation of $S_{cc'}$ is smooth in any case without background and

$$
|S_{cc'}|^2 = \left| B_{cc'} - i \sum_{\lambda} \frac{\Gamma_{1/2} \Gamma_{\lambda}^{1/2}}{E_{\lambda} - E - i\Gamma_{\lambda}/2} \right|^2 \tag{B1}
$$

$$
= |B_{cc'}|^2 + \sum_{\lambda} \frac{\Gamma_{\lambda} \Gamma_{\lambda'}^{1/2}}{(E_{\lambda} - E)^2 + \Gamma_{\lambda}^2/4} \tag{B2}
$$

where we have used the random sign property of the products $\Gamma_{1/2}^{1/2} \Gamma_{\lambda}^{1/2}$ to neglect the 2nd cross-term in comparison to the last one where again the same property is used to simplify the double sum to a single one. Summing over all the inelastic channels then leads to the same result of Eq. (61) with an added term of $\sum_{\lambda \neq \lambda'} |B_{cc'}|^2$ which itself is proportional to $k_e$ as discussed at the end of Section VB.

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