Uncertainty relations in terms of Tsallis entropy

Grzegorz Wilk and Zbigniew Włodarczyk

1The Andrago Soltan Institute for Nuclear Studies, Hoża 69, 00681, Warsaw, Poland
2Institute of Physics, Jan Kochanowski University, Święt okrzyska 15, 25-406 Kielce, Poland

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I. INTRODUCTION

It is known that the usual uncertainty relations, as given by the Heisenberg uncertainty principle, $\Delta x \Delta p \geq \frac{\hbar}{2\pi}$ (which are based on standard deviations $\Delta x$ and $\Delta p$) frequently encounter serious difficulties \cite{1,2}. The best examples are the cases of probability distributions for which these deviations lose their usefulness (being, for example, divergent). It was therefore argued that one should base the formulation of these relations on the information theory approach (see, for example, discussion in \cite{1} and references therein). In this way one avoids the above mentioned problems. The price to be paid is, however, the fact that the information theory approach depends on the type of information measure used, which amounts to dependence on the type of information entropy defining this measure. Examples of Shannon, Rényi and Tsallis information entropies used for this purpose are presented, for example, in \cite{2,3,4,5,6,7} and \cite{2,8}, respectively (for more information see references therein).

Let us notice that the entropic inequality relations involve sums of entropies and are quite different from the standard uncertainty relations. In standard uncertainty relations the product $\Delta x \Delta p$ is strictly determined (i.e., $\Delta p$ is given by $\Delta x$ and vice versa) for a given distribution function and cannot take any values as will be the case further on below. The uncertainty relation such as $\Delta x \Delta p \geq \hbar/4\pi$ is not a statement about the accuracy of our measuring instruments. In contrast, entropic uncertainty relations do depend on the accuracy of the measurement as they explicitly contain the area of the phase space determined by the resolution of the measuring instruments. In this paper we shall revisit, in Section II uncertainty relations emerging from Tsallis entropy \cite{9} and discuss them in detail. Our main result is present in Section III in which we derive the new entropy saturation function. Section IV contains our summary.

II. UNCERTAINTY RELATIONS EMERGING FROM TSALLIS ENTROPY

Let us define probability distributions associated with the measurements of momentum $(p)$ and position $(x)$ of a quantum particle in a pure state as

$$p_k = \int_{k\delta p}^{(k+1)\delta p} \left| \psi(p) \right|^2 dp,$$  

$$x_l = \int_{l\delta x}^{(l+1)\delta x} dx \left| \psi(x) \right|^2,$$  

where indices $k$ and $l$ run from 0 to $\pm \infty$ and the Fourier transform is defined with the physical normalization ($\hbar$ is Planck constant), i.e.,

$$\tilde{\psi}(p) = \frac{1}{\sqrt{\hbar}} \int dx \exp \left( -\frac{2\pi}{\hbar} ipx \right) \psi(x).$$

From the probability distributions $p_k$ and $x_l$ we may construct the corresponding Tsallis entropies \cite{6}, which measure the uncertainties in momentum and position spaces:

$$H^{(p)}_\alpha = \frac{\sum_k p_k^\alpha - 1}{1 - \alpha}; \quad H^{(x)}_\beta = \frac{\sum_l x_l^\beta - 1}{1 - \beta}.$$  

In the respective limits of $(\alpha, \beta) \to 1$ entropies $H^{(r)}$ reduce to the Shannon entropy $(r = p, x)$:

$$S^{(r)} = -\sum_k r_k \ln r_k,$$  

for which the uncertainty relation has been derived long ago and takes the form of a condition imposed on the sum of entropies \cite{6}:

$$S^{(p)} + S^{(x)} \geq -\ln \left( \frac{2\delta x \delta p}{\epsilon \hbar} \right)$$  

(where $\epsilon$ is the basis of natural logarithm). The relation \cite{6} reflects the fact that, although probability distributions in Eq. \ref{eq:1} correspond to different observables, nevertheless they describe the same quantum physical state and therefore must be, in general, correlated. Recently Eq. \ref{eq:5} has been generalized to the case of Rényi entropies \cite{4,9,10,11,12},

$$R^{(r)}_\alpha = \frac{1}{1 - \alpha} \ln \left( \sum_k r_k^\alpha \right).$$

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\*Electronic address: \texttt{wilk@fuw.edu.pl}

\textsuperscript{1}Electronic address: \texttt{wlod@pu.kielce.pl}
for which one gets \[4\] that

\[
R^{(p)}_\alpha + R^{(x)}_\beta \geq \frac{1}{2} \left( \frac{\ln \alpha}{1-\alpha} + \frac{\ln \beta}{1-\beta} \right) - \ln \left( \frac{2\delta p x}{h} \right),
\]

where parameters \(\alpha\) and \(\beta\) are assumed to be positive and constrained by the relation

\[
\frac{1}{\alpha} + \frac{1}{\beta} = 2.
\]

Let us now proceed to the case of nonextensive Tsallis entropy and derive for it the corresponding entropic inequality. Our approach differs from that already presented in \[8\] in that we are attempting from the very beginning to provide condition on the sum of the corresponding \(H^{(r)}_\gamma\) entropies (where \(\gamma = (\alpha, \beta)\)), respectively. To do this we shall start from the following Babenko-Beckner inequality relation \[10\].

\[
\left( \sum_k p_k^\alpha \right)^{\frac{1}{\alpha}} \leq \frac{(2\alpha)}{h} \left( \frac{(2\beta)}{h} \right)^{\frac{1}{\beta}} \left( \delta x \right)^{1-\beta} \sum_i x_i^\beta
\]

which has been also used in \[4\] (cf., Eq. (21) there). Parameters \(\alpha\) and \(\beta\) satisfy condition \[3\] and we shall assume at this moment that \(\alpha > \beta\). Notice that \[3\] means that the effects of nonextensivity in \(x\) and \(p\) spaces, as measured by \(\alpha\) and \(\beta\), cannot be identical (\(\alpha = \beta\) only for \(\alpha = 1\) and \(\beta = 1\), i.e., in the case of the Shannon entropy). The more general case of independent indices has been recently discussed in \[8\] but we shall not comment on it here. The inequality \[10\] can be rewritten as

\[
-\left( \sum_k p_k^\alpha \right)^{\frac{1}{\alpha}} \geq -\eta(\alpha, \beta) \left( \sum_i x_i^\beta \right)^{\frac{1}{\beta}},
\]

where

\[
\eta(\alpha, \beta) = \left( \frac{\beta}{\alpha} \right)^{\frac{1}{\beta}} \left( \frac{(2\beta)}{h} \delta x \right)^{\frac{\alpha-1}{\alpha}},
\]

or as

\[
-1 + \frac{\alpha-1}{\alpha} A^{(p)}_\alpha \geq -\eta(\alpha, \beta) + \eta(\alpha, \beta) \frac{\beta-1}{\beta} A^{(x)}_\beta.
\]

where we have used the first order homogenous entropy defined as (as before, \(r = (p, x)\)).

\[
A^{(r)}_\alpha = \frac{\alpha}{\alpha-1} \left[ 1 - \left( \sum r^\alpha \right)^{\frac{1}{\alpha}} \right]
\]

(it has been firstly introduced in \[11\], and then subsequently given a complete characterization in \[12\]). By making use of Eq. \[8\] one can rewrite Eq. \[12\] in the following way:

\[
A^{(p)}_\alpha + \eta(\alpha, \beta) A^{(x)}_\beta \geq \frac{\alpha}{\alpha-1} \left[ 1 - (\eta(\alpha, \beta)) \right] = \frac{\alpha}{\alpha-1} \left[ \left( \frac{\beta}{\alpha} \right)^{\frac{1}{\beta}} \left( \frac{(2\beta h)}{\delta x} \right)^{\frac{\alpha-1}{\alpha}} - 1 \right].
\]

FIG. 1: Example of the dependence (for \(\alpha = 1.3\)) of limitations of the sum of entropies on the size of cell in phase space, \(dx dp/h\). Results for Renyi and Shannon entropies are practically indistinguishable (to expose both the large and small values we used the linear-log scale here, in this case for both entropies one gets straight lines). Horizontal lines indicate the corresponding bounds for limitation imposed on Tsallis entropies for which \(1/(1-\alpha) < \inf\{H^{(x)} + H^{(p)}\} < 1/(\alpha - 1)\).

Further discussion depends on whether defined by Eq. \[11\] coefficient \(\eta(\alpha, \beta)\) is smaller or greater than unity. In the first case

\[
\eta(\alpha, \beta) \leq 1 \quad \text{or} \quad \delta x \delta p < \frac{1}{2\beta} \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\alpha-1}} h.
\]

We can now write the lhs of Eq. \[12\] as

\[
A^{(p)}_\alpha + A^{(x)}_\beta \geq A^{(p)}_\alpha + \eta(\alpha, \beta) A^{(x)}_\beta,
\]

make use of the fact that for \(\alpha \geq 1\) and \(\beta \leq 1\) one has (see Eq. \[8\]),

\[
\alpha H_\alpha \geq A_\alpha \quad \text{and} \quad \frac{\beta}{2\beta - 1} H_\beta \geq A_\beta,
\]

and get finally that

\[
H^{(p)}_\alpha + H^{(x)}_\beta \geq \frac{1}{\alpha} \left[ A^{(p)}_\alpha + A^{(x)}_\beta \right].
\]

It means that in this case one has

\[
H^{(p)}_\alpha + H^{(x)}_\beta \geq \frac{1}{1-\alpha} \left[ \left( \frac{\beta}{\alpha} \right)^{\frac{1}{\beta}} \left( \frac{(2\beta h)}{\delta x} \right)^{\frac{\alpha-1}{\alpha}} - 1 \right].
\]

In the second case, for \(\eta(\alpha, \beta) > 1\), one gets

\[
H^{(p)}_\alpha + H^{(x)}_\beta \geq \frac{1}{\alpha-1} \left[ \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\beta}} \left( \frac{(2\beta h)}{\delta x} \right)^{\frac{\alpha-1}{\alpha}} - 1 \right].
\]

Both results generalize Eq. \[3\], the result for Shannon entropy, to which they converge when \(\alpha \rightarrow 1\) and \(\beta \rightarrow 1\).
To extend the above results to the case of $\alpha < \beta$ one should use the same Babenko-Beckner inequality \[10\] as in Eq. \[9\] but with the role of $p$ and $x$ interchanged, $p \leftrightarrow x$.

The dependence of the limitations on the sum of entropies on the size of cell in phase space is visualized in Fig. \[1\]

III. THE ENTROPY SATURATION FUNCTION

The inequalities presented above are, so far, purely mathematical in the sense that they allow for negative lower limits for the corresponding sum of entropies. For example, the rhs of equation Eq. \[5\] is positive only for

$$\delta p \delta x \leq \frac{1}{2} \epsilon h.$$ \[21\]

Because the sum of entropies must be non-negative therefore the condition provided by Eq. \[3\] only works together with Eq. \[21\]. The same reasoning can be performed for the remain two entropies leading to the following additional requirements for the products $\delta x \delta p$:

$$\delta x \delta p \leq \frac{1}{2} \hbar \alpha \frac{1}{\sqrt{1 - \beta^2}} \frac{1}{\sqrt{1 - \alpha^2}}.$$ \[22\]

The occurrence of negative values in the limitations of the sum of entropies, $H_\alpha(p) + H_\beta(x)$, is the consequence of the fact that for large values of $\delta x \delta p / h$ we have $\eta(\alpha, \beta) > 1$. We shall now look at this problem more closely. Evaluating $\eta(\alpha, \beta)$ we use the integral form of Jensen's inequalities (which state that for convex functions the values of the function at the average point does not exceeds the average value of the function, the opposite being true for concave functions \[13\]):

$$\left[ \frac{1}{\delta p} \int_{k \delta p}^{(k+1) \delta p} dp \tilde{\rho}(p) \right]^\alpha \leq \frac{1}{\delta p} \int_{k \delta p}^{(k+1) \delta p} dp \tilde{\rho}(p)$$ \[23\]

and

$$\left[ \frac{1}{\delta x} \int_{l \delta x}^{(l+1) \delta x} dx \rho(x) \right]^{\beta} \geq \frac{1}{\delta x} \int_{l \delta x}^{(l+1) \delta x} dx \rho(x)$$ \[24\]

where the probability densities are \(\tilde{\rho}(p) = |\tilde{\psi}(p)|^2\) and \(\rho(x) = |\psi(x)|^2\) (cf. \[1\] for more details). It turns out that differences between the left ($L$) and the right ($R$) hand sides of inequalities \[23\] and \[24\] can be rather substantial and can introduces serious bias to the results. Its magnitude can be estimated using Taylor expansion: $E[f(p_k)] \sim E[f(E[p_k]) + \frac{1}{2} f''(E[p_k]) \text{Var}(p_k)]$ where $f(z) = z^\alpha$. However, this is possible only when the functional form of probability $p_k$ is known. In Fig. \[2\] we show an example of the ration $R/L$ for inequality \[23\] calculated for a Gaussian shape of $\tilde{\rho}(p) = |\tilde{\psi}(p)|^2$. The increase in discrepancy is clearly visible. Instead of this, we shall now demonstrate that the accuracy of Jensen's inequality can be dramatically improved by a suitable change of variables. Namely, we consider the following maps, which transform an infinite interval to some finite interval, $r = (p, x) \in (-\infty, \infty) \implies \tau \in (-1, 1)$:

$$t_r = \frac{r}{|r| + s_r},$$ \[25\]

where $s_r$ is scale parameter such that $s_x s_p = h$. In new variables the probability densities are given by

$$\rho(t_x) = \rho[x(t_x)] \left| \frac{dx}{dt_x} \right| = \rho(x) \frac{s_x}{(|t_x| - 1)^2},$$ \[26\]

and

$$\tilde{\rho}(t_p) = \tilde{\rho}[p(t_p)] \left| \frac{dp}{dt_p} \right| = \tilde{\rho}(p) \frac{s_p}{(|t_p| - 1)^2}.$$ \[27\]

Using these new variables in analogous way as in Eqs. \[23\] and \[24\], one can now write the following inequalities:

$$\left[ \frac{1}{\delta t_p} \int_{k \delta t_p}^{(k+1) \delta t_p} \tilde{\rho}(t_p) dt_p \right]^\alpha \leq \frac{1}{\delta t_p} \int_{k \delta t_p}^{(k+1) \delta t_p} \tilde{\rho}(t_p)$$ \[28\]

and

$$\left[ \frac{1}{\delta t_x} \int_{l \delta t_x}^{(l+1) \delta t_x} \rho(t_x) dt_x \right]^{\beta} \geq \frac{1}{\delta t_x} \int_{l \delta t_x}^{(l+1) \delta t_x} \rho(t_x)$$ \[29\]

The ratio $R/L$ for inequality \[28\] calculated for a Gaussian shape of $\tilde{\rho}(p) = |\tilde{\psi}(p)|^2$ is shown in Fig. \[2\] and, as one can see, grows very weakly with the bin size $\delta p$.

Establishing this finding, let us now proceed to a calculation of the corresponding entropic inequalities using new variables. The probabilities corresponding to \[1\] are now:

$$p'_k = \int_{k \delta t_p}^{(k+1) \delta t_p} \tilde{\rho}(t_p) dt_p, \quad x'_l = \int_{l \delta t_x}^{(l+1) \delta t_x} \rho(t_x) dt_x.$$

(Notation is such that primed quantities correspond to using the new variable $t_r$ and non-prime ones to the standard variable $r = (x, p)$). Whereas before, in variables $(x, p)$, $k$ and $l$ were varying from $0$ to $\pm \infty$, now
\( k \in (0, \pm k_{\text{max}}) \) and \( l \in (0, \pm l_{\text{max}}) \) where \( k_{\text{max}} \delta t_p = 1 \) and \( l_{\text{max}} \delta t_x = 1 \). For these probabilities we get the following equivalent of Eq. (10),

\[
- \left( \sum_k p_k^\alpha \right)^{1/\alpha} \geq -\eta(\alpha, \beta) \left( \sum_l x_l^\beta \right)^{1/\beta},
\]

(31)

where now

\[
\eta(\alpha, \beta) = \left( \frac{\beta}{\alpha} \right)^{1/\beta} \frac{1}{(2\beta \delta t_x \delta t_p)^{\frac{\alpha-1}{\alpha}}}. \]

(32)

Notice that now \( \eta(\alpha, \beta) \leq 1 \) always, this means that we shall no more encounter problems with negative values for the limits of the sum of entropies.

To be more specific, notice that for entropies \( H^{(p)}_\alpha = \left[ 1 - \sum (p_k^\alpha) \right] / (\alpha - 1) \) and \( H^{(x)}_\beta = \left[ 1 - \sum (x_l^\beta) \right] / (\beta - 1) \) we have that

\[
H^{(p)}_\alpha + H^{(x)}_\beta \geq \frac{1}{\alpha - 1} \left[ 1 - \left( \frac{\beta}{\alpha} \right)^{1/\beta} \left( 2\beta \delta t_x \delta t_p \right)^{\frac{\alpha-1}{\alpha}} \right].
\]

(33)

Putting \( \alpha \rightarrow 1 \) and proceeding to Shannon entropy one gets that (in bits)

\[
S^{(p)} + S^{(x)} \geq \left[ \ln \left( \frac{1}{\delta t_x \delta t_p} \right) + 1 \right] \frac{1}{\ln 2} - 1.
\]

(34)

It is interesting to note that for the uniform distribution in the variable \( t_r \in (-1, 1) \) one has

\[
\ln \left( \frac{1}{\delta t_x \delta t_p} \right) \left( \frac{1}{\ln 2} + 2 \right)
\]

bits of information (the number of bins are \( 2/\delta t_x \) and \( 2/\delta t_p \)). The interval of variability of \( S^{(p)} + S^{(x)} \) is narrow and equals \( 3 - 1/\ln 2 \approx 1.557 \) bits (this is the difference between the maximal and minimal limitations).

Let us notice at this point that, whereas inequalities (35), (7), (19) and (20) are for the fixed values of intervals \( \delta x \) and \( \delta p \), the inequality (33) is for the fixed values of intervals \( \delta t_x \) and \( \delta t_p \). Formal recalculation of these intervals results in their dependence on \( k \) and \( l \), they are not fixed anymore but their values change in the following way: for \( \delta t_r = \text{const} \) one has

\[
\delta r = \int_{k \delta t_r}^{(k+1)\delta t_r} \frac{s_r \delta t_r}{(|t_r| - 1)^2} dt_r = \frac{s_r \delta t_r}{(1 - |k|\delta t_r) |1 - |k|\delta t_r|},
\]

(35)

whereas for \( \delta t = \text{const} \) one has

\[
\delta t_r = \int_{k \delta t_r}^{(k+1)\delta t_r} \frac{r s_r}{(s_r + |r|)^2} dr = \frac{s_r \delta r}{(s_r + |k|\delta r) |s_r + |k|\delta r|}.
\]

(36)

Notice that because Eq. (20) is an odd function of \( r \) and has rotational symmetry with respect to the origin, one has exactly the same intervals \( \delta r \) and \( \delta t_r \) for the negative values of \( k, k = -\kappa \), and for its positive values, \( k = \kappa - 1 \), where \( \kappa = 1, 2, 3, \ldots \).

The natural question is then in what way, for some given fixed intervals \( \delta r = (\delta x, \delta p) \), one should choose intervals \( \delta t_r = (\delta t_x, \delta t_p) \) in inequality (33). If we take the maximal values of intervals \( \delta t_r \) (corresponding to \( k = 0 \) or \( k = -1 \)) and make use of the fact that now

\[
\delta t_x \delta t_p = \frac{\delta x \delta p}{s_r + \delta x s_p + \delta p} \leq \frac{\delta x \delta p}{h + \delta x \delta p},
\]

(37)

then we obtain that the right-hand-side of inequality (33) will be limited by

\[
\frac{1}{\alpha - 1} \left[ 1 - \left( \frac{\beta}{\alpha} \right)^{1/\beta} \left( 2\beta \delta t_x \delta t_p \right)^{\frac{\alpha-1}{\alpha}} \right] \geq \frac{1}{\alpha - 1} \left[ 1 - \left( \frac{\beta}{\alpha} \right)^{1/\beta} \left( 2\beta \frac{\delta x \delta p}{h + \delta x \delta p} \right)^{\frac{\alpha-1}{\alpha}} \right].
\]

(38)

Actually, taking exactly the results of (35) and (36) we would obtain equality, not inequality in Eq. (33). However, in such case one would not have at the same time \( \delta x = \text{const} \) and \( \delta t_x = \text{const} \) (or \( \delta p = \text{const} \) and \( \delta t_p = \text{const} \)). Choosing intervals corresponding to \( k = 0 \) or \( k = -1 \) (for which we have maximal interval \( \delta t_p \) equal to \( \delta t_p = \delta p / (s_p + \delta p) \) or, equivalently, minimal interval \( \delta p \) equal to \( \delta t_p = s_p \delta t_p / (1 - \delta t_p) \) we can see that for each \( p_k \) (given by Eq. (30)) we have \( p_k \) (given by Eq. (11)), which satisfies the inequality \( p_k \leq p_{k'} \) for \( \alpha > 1 \) we have \( p_k \leq p_{k'} \) (see (19)). However, because the number of bins in both cases is different, there will be some \( p_k \) left for which there will be no \( p_{k'} \) assigned. Nevertheless, one can construct some new \( p_k \)'s by performing division of \( p_{k'} \). Preserving always the normalization, i.e., assuming that \( \sum_k p_k = \sum_{k'} p_{k'} = 1 \) one has that (20)

\[
\sum (p_k)^{\alpha} \leq \sum (p_{k'})^{\alpha}.
\]

(39)

We have then for entropies \( H_\alpha = \left[ 1 - \sum (p_k)^{\alpha} \right] / (\alpha - 1) \) and \( H_\beta' = \left[ 1 - \sum (p_{k'})^{\alpha} \right] / (\alpha - 1) \) the inequality that \( H_\alpha \geq H_\beta' \). Analogously, repeating the above procedure for probabilities \( x_l \) and \( x_{l'} \) one gets that \( H_\beta \geq H_\beta' \) (where now, according to (33), \( \beta < 1 \)). The limitation for the left-hand-side of inequality (33) is then

\[
H^{(p)}_\alpha + H^{(x)}_\beta \geq H^{(p)}_\alpha + H^{(x)}_\beta \cdot
\]

(40)

Finally, for the Tsallis entropy we can write:

\[
H^{(p)}_\alpha + H^{(x)}_\beta \geq \frac{1}{\alpha - 1} \left[ 1 - \left( \frac{\beta}{\alpha} \right)^{1/\beta} \left( 2\beta \frac{\delta x \delta p}{h + \delta x \delta p} \right)^{\frac{\alpha-1}{\alpha}} \right].
\]

(41)

For \( \delta x \delta p / h \ll 1 \) we recover the previous result given by Eq. (19) whereas in the limit of \( \delta x \delta p / h \rightarrow \infty \) we have

\[
H^{(p)}_\alpha + H^{(x)}_\beta \rightarrow \frac{1}{\alpha - 1} \left[ 1 - (2\alpha)^{1/\beta} (2\beta)^{1/\beta} \right] > 0.
\]

(42)
Notice that now the limit is always positive. For large intervals, i.e., for $\alpha \to 1$ we get a limitation for Shannon entropy, which now reads

$$S^{(p)} + S^{(x)} \geq -\ln \left( \frac{2 \delta x \delta p}{\epsilon h + \delta x \delta p} \right).$$

(43)

For large intervals, i.e., for $\delta x \delta p/h \to \infty$, one gets $S^{(p)} + S^{(x)} \to 1 - \ln 2$ [24]. It should be noticed that this new inequality (43) for Shannon entropy is stronger (for all values of interval $\delta p \delta x$) than the previous limitation [5] derived in [8]. The new dependencies of limitations on different entropies on the size of the phase space cell $\delta x \delta p/h$ are displayed in Fig. 3 [22].

IV. SUMMARY

We have derived uncertainty relations based on Tsallis entropy. We have also found a positively defined function that saturates the so called entropic inequalities for entropies characterizing physical states under consideration, cf. Eq. (41). In case of Shannon entropy (Eq. (43)) the limit provided is more stringent than the previously derived. Formally, our results show that changing $\delta p \delta x/h$ to $\delta p \delta x/(h + \delta p \delta x)$ one avoids (in all cases: Shannon, Renyi and Tsallis entropies) the appearance of unphysical negative values in the entropy bounds.

Let us close with the remark that in some applications of the nonextensive statistics the nonextensivity parameter $q$ (corresponding to $\alpha$ and $\beta$ here) describes intrinsic fluctuations existing in the physical system under consideration [18]. This raises an interesting question of the possible existence of such relations also in the applications mentioned above. In particular there still remains the question of whether our results will survive the other choices of inequality used in [19] and/or in the case of independent indices $(\alpha, \beta)$ as discussed in [6]. We plan to address this point elsewhere.

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\( p_k(\bar{r}) \leq p_k'(\bar{t}_r) \). Notice now that the number of bins in both cases is different and that probabilities in this inequality are not for the same bin number \( k \) but for the corresponding position in variables \( r \) and \( t_r \).

[20] For the condition \( \sum k p_k = \sum k p'_k = 1 \) increasing the number of divisions leads, for \( \alpha > 1 \), to decreasing of \( P_k = p_k' \) and to its increasing for \( \alpha < 1 \). Let us notice that \( P_n |k| p_{\alpha}^n = P_{n+1} |k| p_{\alpha}^{n+1} \) one gets, for \( \alpha > 1 \), that \( p_{\alpha}^n = (p_n + p_{n+1})^\alpha \geq p_n^\alpha + p_{n+1}^\alpha \), which leads to \( p_{\alpha}^n \geq p_n^\alpha \). Repeating this procedure of dividing the \( p_k' \) (and possibly also dividing again \( p_k \)) one gets that \( P_n |k| p_{\alpha}^n \geq P_{m+1} |k| p_{\alpha}^{m+1} \), where \( m > n \). Actually, this inequality is true also for \( m \to \infty \).

[21] Actually this bound was first conjectured by Hirschman [15] and proven by Beckner [16]. For Renyi entropy it was derived in [17] and reads: \( R_\alpha^p + R_\beta^x \geq \frac{1}{2} \ln \frac{1}{\alpha/(\alpha - 1) + \ln \beta/(\beta - 1)} - \ln 2 \). Our Eq. (42) extends therefore Hirschman uncertainty to Tsallis entropy.

[22] To get a feeling of difference between limitations represented by Eqs. (5) and (43) let us consider, as example, gaussian probability densities (corresponding to the gaussian wave-function) with dispersion equal unity, for which we can evaluate \( S(p) \) and \( S(x) \) using definition (4).

For \( \delta x/s_x = \delta p/s_p = \sqrt{2} \) the sum of entropies is equal to \( S(p) + S(x) \approx 1.76 \). The lower limit provided in this case by Eq. (43) is \( S(p) + S(x) \geq 0.712 \), which is much stringent than the corresponding limit \( S(p) + S(x) \geq -0.386 \) provided by Eq. (5) (notice that it leads to negative value of the sum of entropies).