ON REGULARITY AND IRREGULARITY
OF CERTAIN
HOLOMORPHIC SINGULAR INTEGRAL OPERATORS

LOREDANA LANZANI* AND ELIAS M. STEIN**

Abstract. We survey recent work and announce new results concerning two singular integral operators whose kernels are holomorphic functions of the output variable, specifically the Cauchy-Leray integral and the Cauchy-Szegő projection associated to various classes of bounded domains in $\mathbb{C}^n$ with $n \geq 2$.

1. Introduction

This is a review of recent and forthcoming work concerning a menagerie of singular integral operators in several complex variables whose kernels are holomorphic functions of the output variable. (All proofs have appeared or will appear elsewhere.) Our family of operators consists of the Cauchy-Szegő Projection, namely the orthogonal projection of $L^2(bD, \mu)$ onto the holomorphic Hardy space $\mathcal{H}^2(bD, \mu)$, as well as various higher-dimension analogs of the Cauchy integral for a planar curve that are collectively known as Cauchy-Fantappiè integrals and include the Cauchy-Leray integral as a particularly relevant example. We will henceforth denote such operators $S$ and $C$, respectively. Here $D$ is a bounded domain in complex Euclidean space $\mathbb{C}^n$ with $n \geq 2$; $bD$ is the topological boundary of $D$, while $\mu$ is an appropriate measure supported on $bD$, and we will pay particular attention to two such measures, namely induced Lebesgue measure $\Sigma$, and the Leray-Levi measure $\lambda$.

To be precise, we are interested in the $L^p$-regularity problem for $C$ and for $S$, that is:

* Supported in part by the National Science Foundation, award DMS-1503612.
** Supported in part by the National Science Foundation, award DMS-1700180.

2000 Mathematics Subject Classification: 30E20, 31A10, 32A26, 32A25, 32A50, 32A55, 42B20, 46E22, 47B34, 31B10.
Keywords: Hardy space; Cauchy Integral; Cauchy-Szegő projection; Lebesgue space; pseudoconvex domain; minimal smoothness; measure.

January 14, 2019.
Determine regularity and geometric conditions on the ambient domain $D$ that grant that

- $\mathcal{C} : L^p(bD, \lambda) \to L^p(bD, \lambda)$ is bounded for all $p \in (1, \infty)$.
- $\mathcal{S} : L^p(bD, \Sigma) \to L^p(bD, \Sigma)$ is bounded for $p$ in an interval of maximal size about $p_0 = 2$.

In complex dimension 1 (that is, for $D \Subset \mathbb{C}$) these problems are well understood; here we focus on dimension $n \geq 2$.

We point out that the Cauchy-Leray integral is a Calderón-Zygmund operator, thus $L^p$-regularity for $p = 2$ is equivalent to regularity in $L^p$ for $1 < p < \infty$. On the other hand, the Cauchy-Szegő projection is automatically bounded in $L^2(bD, \mu)$ but $L^2$-regularity does not guarantee $L^p$-regularity for $p \neq 2$: indeed, establishing $L^p$-regularity of $\mathcal{S}$ for $p \neq 2$ is, in general, a very difficult problem. The main difficulty stems from the fact that the Schwartz kernel for $\mathcal{S}$ (that is the Cauchy-Szegő kernel) is almost never explicitly available, even in the favorable setting when $D$ is smooth and strongly pseudoconvex, so direct estimates cannot be performed and one has to rely on other methods, such as asymptotic formulas analogous to those obtained by C. Fefferman \[F\] (for the Bergman kernel) and Boutet de Monvel-Sjöstrand \[BS\], or a paradigm discovered by N. Kerzman and E. M. Stein \[KeSt-1\] that relates $\mathcal{S}$ to a certain Cauchy-Fantappiè integral associated to $D$ (the Kerzman-Stein identity).

About 30 years later, a surge of interest in singular integral operators in a variety of “non-smooth” settings led us to a new examination of these problems from the following point of view:

- to what extent is the $L^p$-boundedness of the aforementioned operators reliant upon the boundary regularity and (natural to this context) upon the amount of convexity of the ambient domain $D$?

Stripping away the smoothness assumptions brings to the fore the geometric interplay between the operators and the domains on which they act: it soon became apparent that new ideas and techniques were needed, even to deal with rather tame singularities such as the class $C^{2,\alpha}$. The following results were proved in \[LaSt-8\] and \[LaSt-5\].

(I) $\mathcal{S} : L^p(bD, \Sigma) \to L^p(bD, \Sigma)$ is bounded for $1 < p < \infty$ if

1. $D \Subset \mathbb{C}^n$ is strongly pseudoconvex, and
2. $bD$ is of class $C^2$.

(II) $\mathcal{C} : L^p(bD, \lambda) \to L^p(bD, \lambda)$ is bounded for $1 < p < \infty$ if
REGULARITY & IRREGULARITY

(i.) $D \Subset \mathbb{C}^n$ is strongly $\mathbb{C}$-linearly convex, and
(ii.) $bD$ is of class $C^{1,1}$.

The purpose of this note is to summarize the main points in the proof of (I) and (II), and to announce new results that will appear in forthcoming papers [LaSt-6] and [LaSt-10] pertaining the optimality of the assumptions made in (I) and (II).

2. The $L^p$-regularity of the Cauchy-Leray integral

Here we take as our model the seminal one-dimensional theory of Calderón [Ca], Coifman-McIntosh-Meyer [CMM], and David [Da] for the Cauchy integral of a planar curve, and in particular its key theorem on the Lipschitz case. As is well known, the initial result was the classical theorem of M. Riesz for the Cauchy integral on the unit disc (i.e. the Hilbert transform on the circle); the standard proofs which developed from this then allowed an extension to a corresponding result where the disc is replaced by a domain $D \subset \mathbb{C}$ whose boundary is relatively smooth, i.e. of class $C^{1,\alpha}$, for $\alpha > 0$. However, going beyond that to the limiting case of regularity, namely $C^1$ and other variants “near $C^{1,\alpha}$”, required further ideas. The techniques introduced in this connection led to significant developments in harmonic analysis such as the “$T(1)$ theorem” and various aspects of multilinear analysis and analytic capacity, [Ch-1], [MeCo], [To-1], [To-2]. The importance of those advances suggests the following fundamental question:\textit{what might be the corresponding results for the Cauchy integral in several variables.} However, in the context of higher dimension geometric obstructions arise (pseudoconvexity or, equivalently, lack of conformal mapping) which in the one-dimensional setting are irrelevant. As a consequence, there is no canonical notion of holomorphic Cauchy kernel: all such kernels must be domain-specific. Indeed, the only kernel that can be deemed “canonical” is the Bochner-Martinelli kernel [Ky], but such kernel is nowhere holomorphic and thus of no use in the applications described below. One is therefore charged with the further task of constructing a holomorphic kernel that is fitted to the specific geometry of the domain and, after that, with supplying proof of regularity of the resulting singular integral operator. As in the one-dimensional setting, this theory was first conceived within the context of smooth ambient domains; if the domain is not sufficiently smooth (of class $C^{2,\alpha}$ or better) the original kernel constructions by Henkin and Ramirez, [He] and [Ram],

*“canonical” in the sense that it is the restriction to $bD$ of a universal kernel defined in $\mathbb{C}^n \times \mathbb{C}^n \setminus \{z = w\}$. 
and the “osculation by the Heisenberg group” technique in Kerzman-Stein [KeSt-1] are no longer applicable. In [LaSt-5] it is shown that the $T(1)$-theorem technique for a space of homogeneous type fitted to the geometry and regularity of the ambient domain can be applied to prove $L^p(bD, \mu)$-regularity, for $1 < p < \infty$ of the Cauchy-Leray integral:

$$\mathcal{C}f(z) = \frac{1}{(2\pi i)^n} \int_{w \in bD} f(w) \frac{\Omega(z, w) \wedge (d_w \Omega(z, w))^{n-1}}{\langle \Omega(z, w), w-z \rangle^n}, \quad z \in D$$

whenever $D \subset \mathbb{C}^n$ is a bounded, strongly $\mathbb{C}$-linearly convex domain whose boundary satisfies the minimal regularity condition given by the class $C^{1,1}$ (that is, the domain admits a defining function $\rho$ of class $C^{1,1}$). Here the generating 1-form $\Omega(z, w)$ is the complex gradient of the domain’s defining function (and we should point out that the definition of $\mathcal{C}$ is independent of the choice of defining function $\rho$). More precisely:

$$\Omega(z, w) = j^* \partial \rho(w),$$

where $j^*$ denotes the pull-back under the inclusion map $j : bD \hookrightarrow \mathbb{C}^n$. The boundary measure $\mu$ belongs to a family that includes induced Lebesgue measure $\Sigma$, as well as the Leray-Levi measure

$$d\lambda(w) := (2\pi i)^{-2n} j^* \left( \partial \rho \wedge (\overline{\partial} \rho)^{n-1} \right)(w), \quad w \in bD.$$

We remark that under our assumptions (class $C^{1,1}$) the factor $\overline{\partial} \rho$ in the definition of the Leray-Levi measure $\lambda$, as well as the factor $d_w \Omega(z, w)$ in the Schwartz kernel for $\mathcal{C}$, are only in $L^\infty(\mathbb{C}^n)$ and therefore may be undefined on $bD$ because the latter is a zero-measure subset of $\mathbb{C}^n$, however it turns out that the tangential component of each of $\overline{\partial} \rho(w)$ and $d_w \Omega(z, w)$, namely $j^* \overline{\partial} \rho(w)$ and $j^* d_w \Omega(z, w)$, are in fact meaningful, leading to a kernel that is well-defined even in our singular context.

3. Counter-examples to the $L^p$-theory for the Cauchy-Leray integral

In [LaSt-6] we construct two examples that establish the optimality of the assumptions made on the ambient domain for the Cauchy-Leray integral $\mathcal{C}$. Both examples are real ellipsoids of the form

$$D_{r,q} := \left\{ (z_1, z_2) \in \mathbb{C}^2 \left| |\text{Re } z_1|^r + |\text{Im } z_1|^q + |z_2|^2 < 1 \right. \right\}.$$

For the first example, $(r, q) = (2, 4)$; in this case the domain is smooth, strongly pseudoconvex and strictly convex, but it is not strongly $\mathbb{C}$-linearly convex. In the second example, $(r, q) = (m, 2)$ for any $1 < m < 2$; this domain is strongly $\mathbb{C}$-linearly convex but is only of class $C^{1,m-1}$ (and no better). In both cases we show that the associated
Cauchy-Leray integral $C$ is well-defined on a dense subset of $L^p(bD_{r,q}, \mu)$ but does not extend to a bounded operator: $L^p(bD_{r,q}, \mu) \to L^p(bD_{r,q}, \mu)$ for any $1 < p < \infty$. Specifically, we prove that there is a function $f \in C^1(bD_{r,q})$ supported in a proper subset of $bD_{r,q}$ such that

(i.) $Cf(z)$ can be defined as an absolutely convergent integral whenever $z \in bD_{r,q}$ is at positive distance from the support of $f$.

(ii.) The inequality: $\|C(f)\|_{L^p(S, d\mu)} \leq A_p \|f\|_{L^p(bD_{r,q}, d\mu)}$ (with $A_p$ independent of $f$ and $S$) fails whenever $S \subset bD_{r,q}$ is disjoint from the support of $f$.

Here $\mu$ is a boundary measure that belongs to a family that includes induced-Lebesgue measure $\Sigma$ and the Leray-Levi measure $\lambda$, as well as Fefferman’s measure $\mu^F$, see [E]; for the first example (the smooth domain $D_{2,4}$) all such measures are mutually absolutely continuous. For the second example (the non-smooth domain $D_{m,2}$) these measures are essentially different, yet the counter-example holds in all cases.

The main tool for proving (i.) and (ii.) is a scaling and limiting process that transfers the problem to specific, unbounded smooth domains, namely $\{2 \text{Im } z_2 > (\text{Re } z_1)^2\}$ in the first case, and $\{2 \text{Im } z_2 > |\text{Re } z_1|^m\}$ in the second. On the unbounded domains, explicit computations are carried out to prove failure of the $L^p$-boundedness of the transported operator. There is also the matter of showing that the Cauchy-Leray integral for $D_{r,q}$ maps $L^p$ into the holomorphic Hardy space $H^p$: this question is addressed in [LaSt-7] where it is shown that

(iii.) $Cf(z)$, for $z \in bD_{r,q}$ as in item (i.) above, arises as “boundary value” of a function $F$ holomorphic in $D_{r,q}$.

The proof of (iii.) requires three different approaches, each tailored to the particular type of singularity displayed by the example under consideration: in dealing with the non-smooth domain $D_{m,2}$ one has to distinguish the case when $1 < m \leq 3/2$ from the case $3/2 < m < 2$: in the second case, a global integration by parts gives that $Cf(z)$ is the restriction to $bD_{m,2}$ of a holomorphic $F \in C^1(\overline{D}_{m,2})$. On the other hand, when $1 < m \leq 3/2$ such method is no longer viable but we show nonetheless, that $Cf$ extends to a holomorphic $F$ that is continuous everywhere on $\overline{D}_{m,2}$ except for a 0-measure subset of the boundary (namely the sphere $\{|\text{Re } z_1|^2 + |z_2|^2 = 1\}$).

Finally, the lack of strong $C$-linear convexity in the first example (the domain $D_{2,4}$) prevents us from carrying a global integration by parts: instead, one shows that $Cf$ extends to a holomorphic $F \in C(\overline{D}_{2,4})$ by using a local integration by parts which depends on the location of the coordinate patch with respect to the “flat” part of the boundary. It
should be noted that an earlier result in Barrett-Lanzani \cite{BaLa} already gave an example with irregularity in $L^2(bD,\mu)$, however the less explicit and more complex nature of the construction did not provide insight for $L^p(bD,\mu)$ when $p \neq 2$.

4. The $L^p$-regularity of the Cauchy-Szegő projection

4.1. Discussion of the problem. We recall that the Cauchy-Szegő projection $S$ is the unique, orthogonal (equivalently, selfadjoint) projection operator of $L^2(bD,\Sigma)$ onto the Hardy space of holomorphic functions; here $\Sigma$ is the induced Lebesgue measure on $bD$. As mentioned earlier, one must come to terms with the fact that, in general, orthogonal projections are not Calderon-Zygmund operators, thus $L^p$-regularity for $p \neq 2$ does not follow from $L^2$-regularity; also, one may have $L^p$-regularity only for $p$ in a proper sub-interval of $(1,\infty)$, see e.g. \cite{LaSt-1}. (By contrast, for a Calderon-Zygmund operator boundedness in $L^2$ implies boundedness in $L^p$ for $1 < p < \infty$.) Regularity properties of the Cauchy-Szegő projection, in particular $L^p$-regularity, have been the object of considerable interest for more that 40 years. When the boundary of the domain $D$ is sufficiently smooth, decisive results were obtained in the following settings: (a), when $D$ is strongly pseudoconvex \cite{BS,PS}; (b), when $D \subset \mathbb{C}^2$ and its boundary is of finite type \cite{Mc-1,NRSW}; (c), when $D \subset \mathbb{C}^n$ is convex and its boundary is of finite type \cite{Mc-1,McSt}; (d), when $D \subset \mathbb{C}^n$ is of finite type and its Levi form is diagonalizable \cite{CD}. Related results include \cite{AS-1,AS-3,Ba,BoLo,KrPe,NaPr,PoSt,Ro-1,Ro-2,Z}. The main difference when dealing with the situation when $D$ has lower (in fact minimal) regularity than the setting of the more regular domains treated in (a) – (d), is that in each of those cases known formulas for the Cauchy-Szegő kernel, or at least size estimates, played a decisive role. In our general situation such estimates are unavailable and one must proceed by a different analysis that relies upon (i.), the $T(1)$-theorem technique of \cite{LaSt-5} and (ii.), a new, tricky variant of the original Kerzman-Stein paradigm \cite{KeSt-1} described below.

4.2. $L^p$-regularity of the Cauchy-Szegő projection. Strong $\mathbb{C}$-linear convexity implies strong pseudoconvexity whenever the domain enjoys enough regularity for the latter to be meaningful. In \cite{LaSt-8} some of the techniques from \cite{LaSt-5} are adapted to study the $L^p$-regularity problem for the Cauchy-Szegő projection of strongly Levi-pseudoconvex domains $D \subset \mathbb{C}^n$ with minimal boundary regularity, namely the class $C^2$ (which is the minimal regularity for strong Levi-pseudoconvexity to hold), leading to the conclusion that $L^p$-boundedness
of $S$ holds in the full range $1 < p < \infty$. As mentioned above, in this general setting a direct analysis of the Cauchy-Szegő kernel does not lead to the desired result. Instead, our starting point is the original Kerzman-Stein paradigm [KeSt-1] for domains that are sufficiently smooth: this proceeded by constructing a holomorphic Cauchy-Fantappiè integral $C$ in the same spirit of (2.1) but for a different choice of generating form $\Omega$. The analysis of $S$ begins with the representation:

$$C = S(I - A)$$

on $L^2(bD, \Sigma)$, where $I$ is the identity and $A$ denotes the difference of $C$ and its formal $L^2$-adjoint, that is: $A = C^* - C$. This identity follows from the fact that, just like the Cauchy-Szegő projection, the Cauchy-Fantappiè integral $C$ is also a projection of $L^2(bD, \Sigma)$ onto the holomorphic Hardy space† (albeit not the orthogonal projection!). In particular, since $A^* = -A$ it follows that the operator $(I - A)$ is invertible in $L^2(bD, \Sigma)$ with bounded inverse, and we obtain:

$$S = C(I - A)^{-1}$$
on $L^2(bD, \Sigma)$.

Kerzman and Stein [KeSt-1] proved that if the (strongly pseudoconvex) domain is sufficiently smooth (e.g. of class $C^3$) the singularities of $C$ and $C^*$ cancel out and as a result $A$ is “small” in the sense that it is compact in $L^2(bD, \Sigma)$ (indeed smoothing); from this it follows that the righthand side of the above identity is bounded in $L^p(bD, \Sigma)$ for all $1 < p < \infty$ and therefore so is $S$, giving the solution to the $L^p$-regularity problem for $S$ in the full range $1 < p < \infty$.

If the domain is only of class $C^2$ this argument is no longer applicable because $A$ in general fails to be compact on $L^2(bD, \Sigma)$, see [BaLa]. Instead, in [LaSt-8] we work with a family of holomorphic Cauchy-Fantappiè integrals $\{C_\epsilon\}$, whose kernels are constructed via a first-order perturbation of the Cauchy-Leray kernel (2.1) that makes use of a smooth approximation $\{\tau_\epsilon\}$ of certain second-order derivatives of the defining function of the domain. As in the case of the Cauchy-Leray integral $C$, here there are two boundary measures at play: the induced Lebesgue measure $\Sigma$, and the Leray-Levi measure $\lambda$, see (2.2), which in this new context is absolutely continuous with respect to $\Sigma$ because of the relation

$$d\lambda(w) \approx |\varphi(w)|d\Sigma(w), \ w \in bD$$

where $\varphi(w)$ is the determinant of the Levi matrix. The operators $\{C_\epsilon\}$ are then seen to be bounded in $L^p(bD, \lambda)$ and $L^p(bD, \Sigma)$ for all $1 < p < \infty$ by an application of the $T(1)$-theorem. On the other hand, in defining the Cauchy-Szegő projection it is imperative to specify the

†It is failure of this property that renders the Bochner-Martinelli integral unsuitable for the analysis of $S$. 
underlying measure for $bD$ that arises in the notion of orthogonality that is being used. Correspondingly, we now have two distinct Cauchy-Szegő projections $S_\Sigma$ and $S_\lambda$ but these, in our general setting, are not directly related to one another. It turns out that the Leray-Levi measure $\lambda$ has a “mitigating” effect that leads to a new smallness argument for the difference $C^*_\epsilon - C_\epsilon$ that occurs when the adjoint $C^*_\epsilon$ is computed with respect to $\lambda$. While the $\{C_\epsilon\}_\epsilon$ do not approximate $S_\lambda$ (in fact the norms of the $C_\epsilon$ are in general unbounded as $\epsilon \to 0$), we show that for each fixed $1 < p < \infty$ (in fact for $p < 2$) there is $\epsilon = \epsilon(p)$ such that $C^*_\epsilon - C_\epsilon$ splits as the sum $B_\epsilon + A_\epsilon$, where $B_\epsilon : L^p(bD, \lambda) \to C(bD)$, and $\|A_\epsilon\|_{L^p \to L^p} \leq \epsilon$: this is the new, “tricky” variant of the original Kerzman-Stein paradigm that was alluded to earlier, and it gives us the identity

$$S_\lambda = (S_\lambda B_\epsilon + C_\epsilon)(I - A_\epsilon)^{-1} \text{ in } L^2(bD, \lambda).$$

Then one proves that the righthand side is bounded on $L^p(bD, \lambda)$ (here we also use that $p < 2$ and that $D$ is bounded) and we conclude that $S_\lambda$ is bounded in $L^p(bD, \lambda)$ whenever $1 < p < 2$; the result for $p > 2$ follows by duality. A similar argument is needed to treat $S_\Sigma$, but there is no direct way to show smallness for $C^*_\epsilon - C_\epsilon$ when the adjoint $C^*_\epsilon$ is computed with respect to the induced Lebesgue surface measure $\Sigma$. Instead, one recovers such smallness from the corresponding result for $C^*_\epsilon - C_\epsilon$, by observing that $C_\epsilon - C^*_\epsilon = C_\epsilon - C^*_\epsilon + |\varphi|^{-1}[|\varphi|, C^*_\epsilon]$, where $\varphi$ is as in (4.2), and by controlling the size of the operator norm of the commutator $[|\varphi|, C^*_\epsilon]$.

To complete the proof one also needs the requisite representation formulae and density results for the holomorphic Hardy spaces of the domains that satisfy the minimal boundary regularity conditions stated in (I) and (II): these are obtained in [LaSt-9].

5. A COUNTER-EXAMPLE TO THE $L^p$-THEORY FOR THE CAUCHY-SZEGŐ PROJECTION

The forthcoming work [LaSt-10] investigates a long-standing open question concerning $L^p$-irregularity of the Cauchy-Szegő projection for the Diederich-Fornaess worm domains:

$$W_{k,h} := \left\{ (z_1, z_2) \in \mathbb{C}^2, \ |z_2 - ie^{ih(\|z_1\|)}|^2 < 1 - k(\|z_1\|) \right\}.$$  

Appropriate choices of the functions $h$ and $k$ produce domains that are smooth and pseudoconvex but only weakly pseudoconvex along a 2-dim subset of their topological boundary. (The nick-name “worm” is meant to illustrate winding caused by the argument $h(\|z_1\|)$.)
Developed by Diederich and Fornæss in 1977 as examples of smooth, weakly pseudoconvex domains with non-trivial Nebenhülle\(^\dagger\), the class (5.1) has since proved to be a reliable source of counter-examples to a variety of phenomena in complex function theory. Of special relevance here are the seminal paper \cite{Ba} and the related work \cite{KrPe} that prove \textit{irregularity} of the Bergman projection\(^\S\) for the worm domain in the Sobolev- and Lebesgue-space scales, respectively, when the following choices are made for \(h\) and \(k\):

\[
(5.2) \quad h(|z_1|) := \log|z_1|^2; \quad k(z_1) := \phi(h(|z_1|))
\]

with \(\phi\) a smooth, non-negative even function chosen so that \(W_{h,k}\) is smooth, bounded, connected and pseudoconvex, and moreover \(\phi^{-1}(0) = \{ |t| \leq \beta - \pi/2 \} \) for fixed, given \(\beta > \pi/2\).

In contrast with the situation for the Cauchy-Leray integral, the Cauchy-Szegő and Bergman projections are always bounded in \(L^2\) (that is for \(p = 2\)) so in this context “\(L^p\)-irregularity” should be interpreted as “failure of \(L^p\)-regularity in the full range \(1 < p < \infty\)”.

The results described henceforth will appear in \cite{LaSt-10}.

\textbf{Theorem 1} (Main result). \textit{For any} \(p \neq 2\) \textit{there is} \(\beta = \beta(p) > \pi/2\) \textit{such that for} \(W = W_{h,k}\) \textit{with} \(h, k\) \textit{as in (5.2)}, \textit{the Cauchy-Szegő projection associated to} \(W\) \textit{is not bounded}: \(L^p(bW, \Sigma) \not\rightarrow L^p(bW, \Sigma)\).

Here \(\Sigma\) is induced Lebesgue measure for \(bW\). The strategy of proof is similar in spirit to the original arguments \cite{Ba} and \cite{KrPe} for the Bergman projection (which also inspired the strategy of proof for the examples for the Cauchy-Leray integral described in the previous section): one starts with a (biholomorphic) scaling of the original domain \(W\) leading to a family of smooth domains \(\{W_\lambda\}_\lambda\); then a limiting process transfers the \(L^p\)-regularity problem to a specific, unbounded limiting domain \(W_\infty\). On the latter, explicit computations are carried out that prove failure of \(L^p\)-regularity of the relevant operator for \(W_\infty\). The scaling and limiting arguments then allow to percolate failure of \(L^p\)-regularity back to \(W\) via a suitable transformation law under the scaling map.

When carrying out this scheme for the Cauchy-Szegő projection several new obstacles arise that were non-existent in the analysis of the Bergman projection and of the Cauchy-Leray integral: here we focus on just one, namely the fact that the limiting domain \(W_\infty\) is unbounded

\(^\dagger\) the domain is pseudoconvex but cannot be “exhausted” by smooth pseudoconvex “super-domains”.

\(^\S\) that is, the orthogonal projection of \(L^2(D, dV)\) onto the Bergman space \(A^2(D) := \mathcal{D}(D) \cap L^2(D, dV)\).
and non-smooth (it is a Lipschitz domain), thus for $W_\infty$ there is no canonical notion of holomorphic Hardy space nor of Cauchy-Szegő projection (by contrast, the definition of the Bergman space $A^2(W_\infty, dV)$ is standard, and so is the associated Bergman projection). It is not hard to see that the topological boundary of $W_\infty$ splits into three distinct parts: two of these, denoted $\tilde{W}_\infty$ and $\bar{W}_\infty$ have full induced-Lebesgue measure, while the third part is the distinguished boundary $d_{b}W_\infty$. In [MoPe-1] the authors prove irregularity of the Cauchy-Szegő projection associated to $d_{b}W_\infty$ (defined with respect to induced Lebesgue measure for $d_{b}W_\infty$). However the small size of the distinguished boundary (it is a codimension-1 subset of the topological boundary) makes it impossible to percolate the result for $d_{b}W_\infty$ back to the Cauchy-Szegő projection for the full boundary of the original worm $W$. Here we focus instead on the full-measured part of the boundary denoted $\check{W}_\infty$ because this particular piece of the boundary supports a natural notion of “quasi-product measure” $\mu_\infty$ that captures the main features of the full boundary of $W_\infty$, as indicated by the following key observation:

**Proposition 2.** Suppose that $F \in C_0 \left( \bigcup_{\lambda > 0} W_\lambda \right)$. Then

$$\lim_{\lambda \to \infty} \int_{bW_\lambda} F \, d\mu_\lambda = \int_{\check{W}_\infty} F \, d\mu_\infty.$$  

Here $\mu_\lambda$ is the transported induced Lebesgue measure for $W$ via the scaling map. (In fact a more sophisticated version of the above proposition is needed, one that is valid for $F$ in a larger function space that is dense in the Hardy space for $W$, but the above already provides the required “supporting evidence”.)

It turns out that the quasi-product measure $\mu_\infty$ leads to a meaningful notion of Hardy space for $\check{W}_\infty$ and furthermore, that the topological boundary of the original (smooth) worm $W$ also supports a “quasi-product” measure $\mu_0$ that is mutually absolutely continuous with respect to induced Lebesgue measure $\Sigma$ and enjoys a certain stability under the scaling maps, leading us to the following result:

**Theorem 3.** Let $S_\infty$ denote the Cauchy-Szegő projection for $H^2(\check{W}_\infty, \mu_\infty)$, and let $S_{bW}$ denote the Cauchy-Szegő projection for $H^2(bW, \Sigma)$.

If $S_{bW} : L^p(bW, \Sigma) \to L^p(bW, \Sigma)$ is bounded, then $S_\infty : L^p(\check{W}_\infty, \mu_\infty) \to L^p(\check{W}_\infty, \mu_\infty)$ is bounded and

$$\|S_\infty\|_{L^p(\check{W}_\infty, \mu_\infty)} \leq \|S_{bW}\|_{L^p(bW, \Sigma)}.$$
Finally, a direct examination shows that $S_\infty$ is unbounded on $L^p(\dot{W}_\infty, \mu_\infty)$, giving us the proof of Theorem 1.

Bibliography

[AS-1] Ahern P. and Schneider R., *A smoothing property of the Henkin and Cauchy-Szegő projections*, Duke Math. J. 47 (1980), 135 - 143.

[AS-3] Ahern P. and Schneider R., *The boundary behavior of Henkin’s kernel*, Pacific J. Math. 66 (1976), 9 - 14.

[APS] M. Andersson, M. Passare and R. Sigurdsson *Complex Convexity and Analytic Functionals*, Birkhäuser, Boston (2004).

[Barrett] Barrett D., *Behavior of the Bergman projection on the Diederich-Fornæss worm*, Acta Math. 168 (1992), 1 - 10.

[BEP] Barrett D. E., Ehsani D. and Peloso M., *Regularity of projection operators attached to worm domains*, Doc. Math. 20 (2015), 1207 - 1225.

[Barrett, Lanzani] Barrett, D. and Lanzani, L. *The Spectrum of the Leray Transform for Convex Reinhardt Domains in $\mathbb{C}^2$*. J. Funct. Analysis, 257 (2009) 2780-2819.

[Barrett, Vassiliadou] Barrett, D. E. and Vassiliadou, S., *the Bergman kernel on the intersection of two balls in $\mathbb{C}^2$*, Duke Math. J. 120 (2003), 441-467.

[BoCh] Bonami, A. and Charpentier, P. *Comparing the Bergman and Cauchy-Szegő projections*, Math. Z. 204 (1990), 225-233. Int. J. of Math. and Math. Sc., 29 (2002), 613-627.

[BoLo] Bonami, A. and Lohoué, N. *Projecteurs de Bergman et Cauchy-Szegô pour une classe de domaines faiblement pseudo-convexes et estimations $L^p$* Compositio Math. 46 (1982), no. 2, 159–226.

[BS] Boutet de Monvel, L. and Sjöstrand, J. *Sur la singularité des noyaux de Bergman et le Szegô*, Journées: équations aux dérivées partielles de Rennes (1975), pp. 123 - 164. Astérisque, no. 34-35, Soc. Math. France, Paris 1976.

[Ca] Calderón, A., *Cauchy integrals on Lipschitz curves and related operators*, Proc. Nat. Acad. Sci. U.S.A. 74 (1977), 1324-1327.

[ChZe] Chakrabarti D. and Zeytuncu Y., *$L^p$ mapping properties of the Bergman projection on the Hartogs triangle*, Proc. AMS 144 no. 4 (2016), 1643 - 1653.

[CD] Charpentier P. and Dupain Y., *Estimates for the Bergman and Cauchy-Szegö projections for pseudoconvex domains of finite type with locally diagonalizable Levi forms*, Publ. Mat. 50 (2006), 413 - 446.

[CheZe] Chen L.W. and Zeytuncu Y., 1271 - 1282. *Weighted Bergman projections on the Hartogs triangle: exponential decay*, New York J. Math. 22 no 16. (2016),
[Ch-1] Christ, M. A T(b) theorem with remarks on analytic capacity and the Cauchy integral, Colloq. Math. 60/61 (1990) no. 2, 601-628.

[Ch-2] Christ, M. Lectures on singular integral operators, AMS-CBMS 77 (1990).

[CMM] Coifman, R. R., McIntosh, A. and Meyer, Y., L’intégrale de Cauchy définit un opérateur borné sur $L^2$ pour les courbes Lipschitziennes, Ann. Math. 116 (1982), 361-387.

[Cu] Cumenge, A. Comparaison des projecteurs de Bergman et Cauchy-Szegö et applications, Ark. Mat. 28 (1990), 23-47.

[Da] David, G. Opérateurs intégraux singuliers sur certain courbes du plan complexe, Ann. Sci. École Norm. Sup. 17 (1984), 157-189.

[DLWW] Duong X.-T., Lacey M., Li J., Wick B. and Wu Q., Commutators of Cauchy-type integrals for domains in $\mathbb{C}^n$ with minimal smoothness, preprint (2018) (ArXiv: 1809.08335).

[Du] Duren, P. L., Theory of $H^p$ spaces, Dover (2000).

[F] Fefferman, C. The Bergman kernel and biholomorphic mappings of pseudoconvex domains, Invent. Math. (1974), 1-65.

[F1] Fefferman, C. Parabolic invariant theory in complex analysis, Adv. in Math. 31 (1979), 131-162.

[Gu-2] Gupta, P. Lower-dimensional Fefferman measures via the Bergman kernel, Contemp. Math. 681 (2017), 137 - 151.

[Han] T. Hansson, On Hardy spaces in complex ellipsoids, Ann. Inst. Fourier (Grenoble) 49 (1999), 1477–1501.

[He] G. M. Henkin, Integral representation of functions which are holomorphic in strictly pseudoconvex regions, and some applications, (Russian) Mat. Sb. (N.S.) 78 (1969), 611–632.

[Ho] Hörmander, L., Notions of convexity, Progress in Mathematics 127 (1994), Birkhäuser, Boston.

[Ke-1] Kenig, C., Weighted $H^p$ spaces on Lipschitz domains, Amer. J. Math., 102 (1980), 129-163.

[KeSt-1] Kerzman, N. and Stein, E.M., The Cauchy-Szegő kernel in terms of Cauchy-Fantappié kernels Duke Math. J. 45 (1978), 197-224.

[KeSt-2] Kerzman, N. and Stein, E.M., The Cauchy kernel, the Cauchy-Szegő kernel and the Riemann mapping function, Math. Ann. 236 (1978), 85-93.

[Ki] Kiselman, C., A study of the Bergman projection in certain Hartogs domains, in Several Complex Variables and Complex Geometry, Part 3 Proc. Sympos. Pure Math 52, Part 3, Amer. Math. Soc. Providence RI, 1991.

[Ko-1] Koenig, K. D. Comparing the Bergman and Cauchy-Szegö projections on domains with subelliptic boundary Laplacian, Math. Annalen 339 (2007), 667-693.
[Ko2] Koenig, K. D. An analogue of the Kerzman-Stein formula for the Bergman and Cauchy-Szegö projections, J. Geom. Analysis 14 (2004), 63-86.

[KoLa] Koenig, K. D. and Lanzani, L. Bergman vs. Szegö via Conformal Mapping Indiana U. Math. J. 58 (2009), 969 – 997.

[Kr1] Krantz, S. G. Integral formulas in complex analysis, in Beijing Lectures in Harmonic Analysis, Ann. of Math. Stud. 112, 185-240.

[Kr2] Krantz S. G. Function theory of several complex variables, John Wiley & Sons (1982).

[KrPe] Krantz S. and Peloso M., The Bergman kernel and projection on non-smooth worm domains, Houston J. Math. 34 (2008), 873 - 950.

[Ky] Kytmanov A. M., The Bochner-Martinelli integral and its applications, Birkhäuser, Basel (1992), ISBN: 3-7643-5240.

[La-1] Lanzani, L. Cauchy-Szegö Projection Versus Potential Theory For Non-Smooth Planar Domains. Indiana Univ. Math. J. 48 (1999), 537-556.

[La-2] Lanzani, L. Cauchy transform and Hardy spaces for rough planar domains, Contemp. Math., 251 (2000), 409-428.

[La-6] Lanzani L., Harmonic Analysis Techniques in Several Complex Variables, Bruno Pini Math. Analysis Seminar, Series 1, (2014), 83-110. ISSN 2240-2829.

[LaSt-1] Lanzani, L. and Stein, E. M., Cauchy-Szegö and Bergman projections on non-smooth planar domains, J. Geom. An. 14 (2004), 63 – 86.

[LaSt-3] Lanzani L. and Stein E. M., The Bergman projection in $L^p$ for domains with minimal smoothness, Illinois J. Math. 56 (1) (2013) 127 – 154.

[LaSt-4] Lanzani L. and Stein E. M., Cauchy-type integrals in several complex variables, Bull. Math. Sci. 3 (2) (2013), 241 – 285.

[LaSt-5] Lanzani, L. and Stein E. M., The Cauchy integral in $\mathbb{C}^n$ for domains with minimal smoothness, Adv. Math. 264 (2014), 776 – 830.

[LaSt-6] Lanzani L. and Stein E. M., The Cauchy-Leray integral: counterexamples to the $L^p$-theory, to appear in Indiana U. Math. J. (ArXiv: 1701.03812).

[LaSt-7] Lanzani L. and Stein E. M., The role of an integration identity in the analysis of the Cauchy-Leray transform, Science China Mathematics, 60 (2017) 1923 - 1936.

[LaSt-8] Lanzani L. and Stein E. M., The Cauchy-Szegö projection for domains with minimal smoothness, Duke Math. J. 166 no. 1 (2017), 125-176.

[LaSt-9] Lanzani L. and Stein E. M., Hardy Spaces of Holomorphic functions for domains in $\mathbb{C}^n$ with minimal smoothness in Harmonic
Analysis, Partial Differential Equations, Complex Analysis, and Operator Theory: Celebrating Cora Sadosky's life, AWM-Springer vol. 1 (2016), 179 - 200. ISBN-10: 3319309595.

[LaSt-10] Lanzani L. and Stein E. M., *On irregularity of the Cauchy-Szegő projection for the Diederich-Fornaess worm domain*, manuscript in preparation.

[Mc-1] McNeal J., *Boundary behavior of the Bergman kernel function in $\mathbb{C}^2$*, Duke Math. J. **58** no. 2 (1989), 499 - 512.

[Mc-2] McNeal, J., *Estimates on the Bergman kernel of convex domains* Adv. Math. **109** (1994) 108 –139.

[McSt] McNeal J. and Stein E. M., *Mapping properties of the Bergman projection on convex domains of finite type*, Duke Math. J. **73** no. 1 (1994), 177 - 199.

[MeCo] Meyer Y. and Coifman R., *Ondelettes et Opérateurs III Opérateurs multilinéaires* Actualités Mathématiques, Hermann (Paris), 1991, pp. i-xii and 383-538. ISBN: 2-7056-6127-1.

[Mo-1] Monguzzi, A. *Hardy spaces and the Cauchy-Szegő projection of the non-smooth worm domain $D_\beta'$*, J. Math. Anal. Appl. **436** (2016), 439 - 466.

[Mo-2] Monguzzi, A. *On Hardy spaces on worm domains*, Concr. Oper. **3** (2016), 29 - 42.

[MoPe-1] Monguzzi A. and Peloso M. *Sharp estimates for the Cauchy-Szegő projection on the distinguished boundary of model worm domains*, Integral Eqns Op. Th. **89** (2017), 315 - 344.

[MoPe-2] Monguzzi A. and Peloso M. *Regularity of the Cauchy-Szegő projection on model worm domains*, Complex Var. Elliptic Equ. **62** no. 9 (2017), 1287 - 1313.

[MuZe-1] Munasinghe, S. and Zeytuncu Y. *Irregularity of the Cauchy-Szegő projection on bounded pseudoconvex domains in $\mathbb{C}^2$*, Integral Eqns Op. Th. **82** (2015), 417 - 422.

[MuZe-2] Munasinghe, S. and Zeytuncu Y. *$L^p$-regularity of weighted Cauchy-Szegő projections on the unit disc*, Pacific J. Math. **276** (2015), 449 - 458.

[NaPr] Nagel A. and Pramanik M., *Diagonal estimates for the Bergman kernel on certain domains in $\mathbb{C}^n$*, preprint.

[NRSW] Nagel A., Rosay J.-P., Stein E. M. and Wainger S., *Estimates for the Bergman and Cauchy-Szegő kernels in $\mathbb{C}^2$*, Ann. of Math. **129** no. 2 (1989), 113 - 149.

[NRSW] Nagel A., Rosay J.-P., Stein E. M. and Wainger S., *Estimates for the Bergman and Cauchy-Szegő kernels in $\mathbb{C}^2$*, Ann. of Math. **129** no. 2 (1989), 113 - 149.

[NTV-2] Nazarov F., Treil S. and Volberg A., *Cauchy integral and Calderón-Zygmund operators on nonhomogeneous spaces* Int. Math. Res. Not. **15** (1997), 703 - 726.
Phong D. and Stein E. M., Estimates for the Bergman and Cauchy-Szegő projections on strongly pseudoconvex domains, Duke Math. J. 44 no.3 (1977), 695 - 704.

Poletsky E. and Stessin M., Hardy and Bergman spaces on hyperconvex domains and their composition operators Indiana Univ. Math. J. 57 (2008), 2153-2201.

Range, R. M. Holomorphic functions and integral representations in several complex variables, Graduate texts in Mathematics, 108, Springer Verlag, 1986.

E. Ramírez de Arellano, Ein Divisionsproblem und Randintegraldarstellungen in der komplexen Analysis, Math. Ann. 184 (1969/1970), 172–187.

Rotkevich A. S., Cauchy-Leray-Fantappiè integral in linearly convex domains, J. of Math. Sci., 194 (2013), 693 - 702.

Rotkevich A. S., The Aizenberg formula in non convex domains and some of its applications, Zap. Nauchn. Semin. POMI 389 (2011), 206 - 231.

Semmes, S., The Cauchy integral and related operators on smooth curves, Thesis, Washington University (1983).

Stein, E. M. Boundary behavior of holomorphic functions of several complex variables, Princeton University press, Princeton, NJ, 1972.

Stein, E. M. Harmonic Analysis Princeton Univ. Press (1993).

Stout, E. L. $H^p$ functions on strictly pseudoconvex domains, Amer. J. Math., 98 (1976), 821-852.

Tolsa X., Analytic capacity, rectifiability, and the Cauchy integral, International Congress of Mathematicians, Vol. II, 1505 - 1527, Eur. Math. Soc. Züri 2006.

Tolsa, X., The semiadditivity of continuous analytic capacity and the inner boundary conjecture, Amer. J. Math. 126 (2004), 523-567.

Verdera, J. $L^2$ boundedness of the Cauchy integral and Menger curvature, in “Harmonic analysis and boundary value problems”, 139–158, Contemp. Math., 277, Amer. Math. Soc., Providence, RI, 2001.

Zeytuncu Y., $L^p$-regularity of weighted Bergman projections, Trans. AMS 365 (2013), 2959 - 2976.