The Dynamic Complexity of Formal Languages

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The paper investigates the power of the dynamic complexity classes DynFO, DynQF and DynPROP over string languages. The latter two classes contain problems that can be maintained using quantifier-free first-order updates, with and without auxiliary functions, respectively. It is shown that the languages maintainable in DynPROP are exactly the regular languages, even when allowing arbitrary precomputation. This enables lower bounds for DynPROP and separates DynPROP from DynQF and DynFO. Further, it is shown that any context-free language can be maintained in DynFO and a number of specific context-free languages, for example all Dyck-languages, are maintainable in DynQF. Furthermore, the dynamic complexity of regular tree languages is investigated and some results concerning arbitrary structures are obtained: There exist first-order definable properties which are not maintainable in DynPROP. On the other hand, any existential first-order property can be maintained in DynQF when allowing precomputation.

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1. INTRODUCTION

Traditional complexity theory asks for the necessary effort to decide whether a given input has a certain property, more precisely, whether a given string is in a certain language. Dynamic complexity, in contrast, asks for the effort to maintain sufficient knowledge to be able to decide whether the input object has the property after a series of small changes of the object. The complexity theoretic investigation of the dynamic complexity of algorithmic problems was initiated by Patnaik and Immerman [2]. They defined the class DynFO of dynamic problems where small changes in the input can be mastered by formulas of (first-order) predicate logic (or, equivalently, poly-size circuits of bounded depth, see [2]). More precisely, the dynamic program makes use of an auxiliary data structure and after each update (say, insertion or deletion) the auxiliary data structure can be adapted by a first-order formula.

Among others, they showed that the dynamic complexity of the following problems is in DynFO: reachability in undirected graphs, minimum spanning forests, multiplication, regular languages, and the Dyck languages $D_n$. Subsequent work has yielded more problems in DynFO [2] some of which are LOGCFL-complete [2] and even PTIME-complete [2; 2] (even though the latter are highly artificial). Other work also considered stronger classes (like Hesse’s result that Reachability in
arbitrary directed graphs is in DYN TC\(^0\) \([\text{?}]\), studied notions of completeness for dynamic problems \([\text{?}]\), and elaborated on the handling of precomputations \([\text{?}]\).

The choice of first-order logic as update language in \([\text{?}]\) was presumably triggered by the hope that, in the light of lower bounds for AC\(^0\), it would be possible to prove that certain problems do not have DYN FO dynamic complexity. As it is easy to show that every DYN FO problem is in PTIME, a non-trivial lower bound result would be that the dynamic complexity of some PTIME problem is not in DYN FO. However, so far there are no results of this kind.

The inability to prove lower bounds has naturally led to the consideration of subclasses of DYN FO. Hesse studied problems with quantifier-free update formulas, yielding DYN PROP if the maintained data structure is purely relational and DYN QF if functions are allowed as well \([\text{?}; \text{?}]\). As further refinements, the subclasses DynOR and DynProjections were studied. In \([\text{?}]\) separation results for subclasses of DYN PROP were shown and the separation between DYN PROP and DYN P was stated as an open problem.

The framework of \([\text{?}]\) allows more general update operations and some of the results we mention depend on the actual choice of operations. Nevertheless, most research has concentrated on insertions and deletions as the only available operations. Furthermore, most work considered underlying structures of the following three kinds.

**Graphs.** Here, edges can be inserted or deleted. One of the main open questions is whether Reachability (aka Transitive Closure) can be maintained in DYN FO for directed, possibly cyclic graphs.

**Strings.** Here, letters can be inserted or deleted. As mentioned above, \([\text{?}]\) showed that regular languages and Dyck languages can be maintained in DYN FO. Later, Hesse proved that the dynamic complexity of regular languages is actually in DYN QF \([\text{?}]\).

**Databases.** The dynamic complexity of database properties were studied in the slightly different framework of First-Order Incremental Evaluation Systems (FOIES) \([\text{?}]\). Many interesting results were shown, including a separation between deterministic and nondeterministic systems \([\text{?}]\) and inexpressibility results for auxiliary relations of small arity \([\text{?}; \text{?}]\). However, general lower bounds have not been shown yet.

Continuing the above lines of research, this paper studies the dynamic complexity of formal languages with a particular focus on dynamic classes between DYN PROP and DYN QF. Our main contributions are as follows (see also Figure ??):

—We give an exact characterization of the dynamic complexity of regular languages: A language can be maintained in DYN PROP if and only if it is regular. This also holds in the presence of arbitrary precomputed (aka built-in) relations. (Section ??)

—We provide new upper bounds for context-free languages: Every context-free language can be maintained in DYN FO, Dyck languages even in DYN QF; Dyck languages with one kind of brackets in a slight extension of DYN PROP, where built-in successor and predecessor functions can be used. (Section ??)

\(^1\)In this figure the dynamic complexity classes are allowed to operate with precomputation. Some of the results also hold without precomputation, for example all results concerning formal languages.
As an immediate consequence, we get a separation between DynPROP and DynQF, thereby also separating DynPROP from DynFO and DynP.

We investigate a slightly different semantic for dynamic string languages, and we show that also regular tree languages can be maintained in DynPROP, when allowing precomputation and the use of built-in functions. (Section ??).

We also study general structures, and show that (bounded-depth) alternating reachability is not maintainable in DynPROP. It follows that not all first-order definable properties are maintainable in DynPROP. On the other hand, we prove that all existential first-order definable properties are maintainable in DynQF when allowing precomputation. (Section ??)

**Related work.** We already discussed most of the related work above. A related research area is the study of incremental computation and the complexity of problems in the cell probe model. Here, the focus is not on structural (parallel) complexity of updates but rather on (sequential) update time [?, ?]. In particular, [?, ?] give efficient incremental algorithms and analyse the complexity of formal language classes based on completely different ideas.

Another area related to dynamic formal languages is the incremental maintenance of schema information (aka regular tree languages) [?, ?] and XPath query evaluation [?] in XML documents. Again, the interest is mainly in fast algorithms, less in structural dynamic complexity. Nevertheless, techniques from dynamic algorithms for string languages also find applications in these settings.

### 2. Definitions

Let $\Sigma = \{\sigma_1, ..., \sigma_k\}$ be a fixed alphabet. We represent words over $\Sigma$ encoded by *word structures*, that is, logical structures $W$ with universe $\{1, ..., n\}$, one unary relation $R_\sigma$ for each symbol $\sigma \in \Sigma$, and the canonical linear order $<$ on $\{1, ..., n\}$. We only consider structures in which, for each $i \leq n$, there is at most one $\sigma \in \Sigma$ such that $R_\sigma(i)$ holds, but there might be none such $\sigma$. We write $W(i) = \sigma$ if $R_\sigma(i)$ holds and $W(i) = \varepsilon$ if no such $\sigma$ exists. The size of $W$ is $n$.

The word $w = \text{word}(W)$ represented by a word structure $W$ is simply the concatenation $W(1) \circ \cdots \circ W(n)$. Notice that, as certain elements in $W$ might not carry a symbol, the actual length of the string can be less than $n$. In particular, every word $w$ can be encoded by infinitely many different word structures. Let $[i, j]$ and $]i, j]$ denote the intervals from $i$ to $j$, respectively, from $i + 1$ to $j - 1$. For a word structure $W$ and positions $i \leq j$ in $[1, n]$, we write $w[i, j]$ for the (sub-)string $W(i) \circ \cdots \circ W(j)$. In particular, $w[i, i - 1]$ denotes the empty substring between positions $i$ and $i - 1$.

By $E_n$ we denote the structure with universe $\{1, ..., n\}$ representing the empty string $\varepsilon$ (thus in $E_n$ all relations $R_\sigma$ are empty).

#### 2.1 Dynamic Languages and Complexity Classes

In this section, we first define dynamic counterparts of formal languages. Informally, a dynamic language consists of all sequences of insertions and deletions of symbols that transform the empty string into a string of a particular (static) language $L$. Here, the empty string is encoded by $E_n$, for some $n$, and inserting a symbol $\sigma$ at position $i$ amounts to adding $i$ to $R_\sigma$. Then we define dynamic programs which keep track of whether the string resulting from a sequence of updates is in $L$. Finally, we define complexity classes of dynamic languages. Most of our definitions are inspired by [?] but, as we consider strings as opposed to arbitrary structures, we try to keep the formalism as simple as possible.

**Dynamic Languages.** We will associate with each string language $L$ a dynamic language $\text{Dyn}(L)$. The idea is that words can be changed by a sequence of insertions and deletions of letters and $\text{Dyn}(L)$ is basically the set of update sequences $\alpha$ which turn the empty string into a string in $L$.

For an alphabet $\Sigma$ we define the set $\Delta := \{\text{ins}_\sigma \mid \sigma \in \Sigma\} \cup \{\text{reset}\}$ of abstract updates. A concrete update is a term of the form $\text{ins}_\sigma(i)$ or $\text{reset}(i)$, where $i$ is a positive integer. A concrete update is applicable in a word structure of size $n$ if $i \leq n$. By $\Delta_n$ we denote the set of applicable
If there is no danger of confusion we will simply write “update” for concrete or abstract updates.

The semantics of applicable updates is defined as expected: \( \delta(W) \) is the structure resulting from

- setting \( R_\sigma(i) \) to true and \( R_{\sigma'}(i) \) to false, for \( \sigma' \neq \sigma \), if \( \delta = \text{ins}_\sigma(i) \); and
- setting all \( R_\sigma(i) \) to false, if \( \delta = \text{reset}(i) \).

For a sequence \( \alpha = \delta_1 \ldots \delta_k \in \Delta_n^+ \) of updates we define \( \alpha(W) \) as \( \delta_k(\ldots(\delta_1(W))\ldots) \).

**Definition 2.1.** Let \( L \) be a language over alphabet \( \Sigma \). The dynamic language \( \text{Dyn}(L) \) is the set of all (non-empty) sequences \( \alpha \) of updates, for which there is an \( n > 0 \) such that \( \alpha \in \Delta_n^+ \) and \( \text{word}(\alpha(E_n)) \in L \). We call \( L \) the underlying language of \( \text{Dyn}(L) \).

**Dynamic Programs.** Informally, a dynamic program is a transition system which reads sequences of concrete updates and stores the current string and some auxiliary relations in its state. It also maintains the information whether the current string is in the (static) language under consideration.

A program state \( S \) is a word structure \( W \) extended by (auxiliary) relations over the universe of \( W \). The schema of \( S \) is the set of names and arities of the auxiliary relations of \( S \). We require that each program has a 0-ary relation ACC.

A dynamic program \( P \) over alphabet \( \Sigma \) and schema \( R \) consists of an update function \( \phi_{\text{op}}^{R}(y; x_1, \ldots, x_k) \), for every \( \text{op} \in \Delta \) and \( R \in R \), where \( k = \text{arity}(R) \). A dynamic program \( P \) operates as follows. Let \( S \) be a program state with word structure \( W \). The application of an applicable update \( \delta = \text{op}(i) \) on \( S \) yields the new state \( S' = \delta(S) \) consisting of \( W' = \delta(W) \) and new relations \( R' = \{ j | S \models \phi_{\text{op}}^{R}(i, j) \} \), for each \( R \in R \). For each \( n \in \mathbb{N} \) and update sequence \( \alpha = \delta_1 \ldots \delta_k \in \Delta_n^+ \) we define \( \alpha(S) \) as \( \delta_k(\ldots(\delta_1(S))\ldots) \). A state \( S \) is accepting if \( S \models \text{ACC} \), that is, if the 0-ary ACC-relation contains the empty tuple.\(^3\)

We say that a dynamic program \( P \) recognizes the dynamic language \( \text{Dyn}(L) \) if for all \( n \in \mathbb{N} \) and all \( \alpha \in \Delta_n^+ \) it holds that \( \alpha(S_n) \) is accepting if and only if \( \text{word}(\alpha(E_n)) \in L \), where \( S_n \) denotes the state with word structure \( E_n \) and otherwise empty relations.

**Dynamic Complexity Classes.** \( \text{DynFO} \) is the class of all dynamic languages that are recognized by dynamic programs whose update functions are definable by first-order formulas. \( \text{DynPROP} \) is the subclass of \( \text{DynFO} \) where all these formulas are quantifier free.

### 2.2 Extended Dynamic Programs

To gain more insight into the subtle mechanics of dynamic computations, we study two orthogonal extensions of dynamic programs: auxiliary functions and precomputations.

**Dynamic programs with functions.** A dynamic program with auxiliary functions \( P \) is a dynamic program over a schema \( R \), possibly containing function symbols, which has, for each \( \sigma \in \Sigma \) and each function symbol \( f \in R \), an update function \( \psi_{\sigma}^{f}(i; x_1, \ldots, x_k) \), where \( k = \text{arity}(f) \).

As we are mainly interested in quantifier free update functions for updating auxiliary functions we restrict ourselves to update functions defined by update terms, such that

- every \( x_i \) is an update term;
- if \( f \in R \) is a function and \( \bar{t} \) contains only update terms then \( f(\bar{t}) \) is an update term; and

\(^3\)There is a danger of confusion as we deal with two kinds of languages: “normal languages” consisting of “normal strings” and dynamic languages consisting of sequences of updates. We use the terms “word” and “string” only for “normal strings” and call the elements of dynamic languages “sequences”.

\(^4\)0-ary relations can be viewed as propositional variables: either they contain the empty tuple (corresponding to TRUE) or not.
—if $\phi$ is a quantifier free formula (possibly using update terms) and $t_1$ and $t_2$ are update terms, then $\text{ite}(\phi, t_1, t_2)$ is an update term.

The semantics of update terms is straightforward for the first two rules. A term $\text{ite}(\phi, t_1, t_2)$ takes the value of $t_1$ if $\phi$ evaluates to true and the value of $t_2$ otherwise.

After an update $\delta$, the auxiliary functions in the new state are defined by the update functions in the straightforward way. Unless stated otherwise, the functions in the initial state $S_n$ map every tuple to its first element.

**Dynamic programs with precomputations.** Sometimes it can be useful for a dynamic algorithm to have a precomputation which prepares some sophisticated data structure. Such precomputations can easily be incorporated into the model of dynamic programs.

In [?], the class $\text{DynFO}^+$ allowed polynomial time precomputations on the auxiliary relations. The structural properties of dynamic algorithms with precomputation were further studied and refined in [?]. In this paper, we do not consider different complexities of precomputations but distinguish only the cases where precomputations are allowed or not.

A dynamic program $P$ with precomputations uses an additional set of initial auxiliary relations (and possibly initial auxiliary functions). For each initial auxiliary relation symbol $R$ and each $n$, $P$ has a relation $R_n^{\text{init}}$ over $\{1, \ldots, n\}$. The semantics of dynamic programs with precomputations is adapted as follows: In the initial state $S_n$ each initial auxiliary relation $R$ is interpreted by $R_n^{\text{init}}$. Similarly, for initial auxiliary function symbol $f$ and each $n$, there is a function $f_n^{\text{init}}$ over $\{1, \ldots, n\}$.

Initial auxiliary relations and functions are never updated, that is, $P$ does not have update functions for them.

The extension of dynamic programs by functions and precomputations can be combined and gives rise to different complexity classes: For $I \in \{\bot, \text{Rel}, \text{Fun}\}$ and $A \in \{\text{Rel}, \text{Fun}\}$, we denote by $\text{DynC}(I, A)$ the class of dynamic languages recognized by dynamic programs

—without precomputations, if $I = \bot$;
—with initial auxiliary relations, if $I = \text{Rel}$;
—with initial auxiliary relations and functions, if $I = \text{Fun}$;
—with (updatable) auxiliary relations only, if $A = \text{Rel}$; and
—with (updatable) auxiliary relations and functions, if $A = \text{Fun}$.

Thus, we have $\text{DynFO} = \text{DynFO}(\bot, \text{Rel})$ and $\text{DynPROP} = \text{DynPROP}(\bot, \text{Rel})$. If the base class $\text{DynC}$ is $\text{DynPROP}$ or $\text{DynFO}$, $\text{DynC}(I, A)$ is clearly monotonic with respect to the order $\bot < \text{Rel} < \text{Fun}$. In particular,

\[ \text{DynPROP}(\text{Rel}, \text{Rel}) \subseteq \text{DynPROP}(\text{Fun}, \text{Rel}) \subseteq \text{DynPROP}(\text{Fun}, \text{Fun}) \]

As we are particularly interested in the class $\text{DynPROP}(\bot, \text{Fun})$, we also denote it more consisely by $\text{DynQF}$.

As auxiliary functions can be simulated by auxiliary relations when the update functions are first-order formulas, we also have $\text{DynFO}(\text{Rel}, \text{Rel}) = \text{DynFO}(\text{Fun}, \text{Fun})$ and $\text{DynFO} = \text{DynFO}(\bot, \text{Fun})$. Thus, in our setting there are only two classes with base class $\text{DynFO}$: the one with and the one without precomputations.

We also examine the setting where we only allow a specific set of initial auxiliary (numerical) functions, namely built-in successor and predecessor functions. For each universe size $n$ let $\text{succ}$ be the function that maps every universe element to its successor (induced by the ordering) and the element $n$ to itself, let $\text{pre}$ be the function mapping each element to its predecessor and the element 1 to itself, and let $\text{min}$ be the constant (that is, nullary function) mapping to the minimal element 1 in the universe. Then $\text{DynPROP}(\text{Succ, Rel})$ is the class of dynamic languages recognized by dynamic
programs using quantifier free formulas with initial (precomputed) auxiliary relations, the auxiliary functions succ, pre and min and updatable auxiliary relations.

Dynamic Programs with initialisation. In some cases, dynamic programs need some weak kind of precomputation. In these cases, it will be useful to be able to suitably initialize the auxiliary relations, in particular in settings where no precomputation is allowed. The following lemma shows that this is indeed possible, if the initialization functions can be defined in the same logic as the update functions. A dynamic program with initialization is a dynamic program with an additional quantifier free formula $\beta_R(\vec{x})$, for each auxiliary relation $R$. The value of each relation $R$ in the initial state $S_n$ is then determined by $\beta_R$.

Lemma 2.2. For each dynamic DynPROP- or DynFO-program $P$ with initialization there is an equivalent program $P'$ that does not use initialization.

Proof. The simulating program $P'$ uses an additional 0-ary relation $I_0$ which contains the empty tuple if some update has already occurred. The update formulas of $P'$ are obtained from those of $P$ by replacing each atom of the form $R(\vec{x})$ by $(I_0 \land \beta_R(\vec{x})) \lor (\neg I_0 \land R(\vec{x}))$. The update formulas for $I_0$ are constantly true.

Finally, to simplify presentation, we restrict the update sequences under consideration. An update sequence is useful if (1) whenever an update reset($z$) occurs, the position $z$ carried a symbol before the update, and (2) whenever an update ins$_{\sigma}(z)$ occurs, the position $z$ was empty (that is, did not carry a symbol). Restricting the update sequences to useful ones can be done without loss of generality, as shown in the following lemma. Therefore, when constructing dynamic programs in the remainder of the paper, we always assume that it only has to deal with useful update sequences.

Lemma 2.3. For every dynamic program $P$ working only on useful update sequences, there exists an equivalent one $P'$ working on all update sequences.

Proof. The program $P'$ can simulate $P$. Indeed, for the reset operation, $P'$ can test whether $z$ was empty before the update, in which case it returns the original value of the updated relation or function; or, if $z$ carried a symbol, it uses the update functions of $P$. In the case of an insertion at a position $z$ for which $z$ already carried a symbol, $P'$ can simulate what would happen if in $P$ consecutively the updates reset($z$) and ins$_{\sigma}(z)$ would occur. Technically, this can be achieved by replacing in all formulas $\phi^{R}_{\sigma}$ any occurrence of a relation name $R'$ by $\phi^{R'}_{\text{reset}}$. These modified update formulas then compute exactly the relations and functions $P$ would compute after handling the updates reset($z$) and ins$_{\sigma}(z)$.

2.3 Types

We finally introduce the concept of the type of a tuple of elements. Informally, the type of a tuple captures all information a quantifier free formula can express about a tuple. Let $\vec{i} = (i_1, \ldots, i_l)$ be an $l$-tuple of elements of a state $S$ and let $\varphi$ be a quantifier free formula using variables from $x_1, \ldots, x_l$. We write $\varphi[\vec{i}]$ for the formula resulting from $\varphi$ by replacing each $x_j$ with $i_j$. For instance, for $\vec{i} = (2, 5, 4)$ and the atom $\varphi = R(x_3, x_1)$, we have $\varphi[\vec{i}] = R(4, 2)$.

Let the type $\langle S, \vec{i} \rangle$ of an $l$-tuple $\vec{i} = (i_1, \ldots, i_l)$ in state $S$ be the set of those atomic formulas $\varphi$ over $x_1, \ldots, x_l$ for which $\varphi[\vec{i}]$ holds in $S$. We also sometimes call $\langle S, \vec{i} \rangle$ an $l$-type. A tuple $\vec{i} = (i_1, \ldots, i_l)$ is ordered if $i_1 < i_2 < \cdots < i_l$. An ordered type is the type of an ordered tuple.

3. Dynamic Complexity of Regular Languages

As already mentioned in the introduction, every regular language can be recognized by a DynFO program [?]; and the full power of DynFO is actually not needed: Every regular language is recognized by some DynQF program [?].

Our first result is a precise characterization of the dynamic languages Dyn($L$) with an underlying regular language $L$: They exactly constitute the class DynPROP. Before stating the result
formally and sketch its proof, we will give a small example to illustrate how regular languages can be maintained in DYNPROP.

Example 3.1. We consider the regular language \((a + b)^*a(a + b)^+\) over the alphabet \(\{a, b\}\). The dynamic program maintains one binary relation \(A\) containing all pairs \((i, j)\) for which \(i < j\) and there exists \(k \in ]i, j[\) such that \(w[k, k] = a\). It further uses a unary relation \(I\) containing all \(j\) for which there exists a \(k < j\) such that \(w[k, k] = a\) and, dually, a unary relation \(F\) containing all \(i\) for which there exists a \(k > i\) such that \(w[k, k] = a\).

We will state here the update formulas for the three kinds of operations: \(\text{ins}_a\), \(\text{ins}_b\), and \(\text{reset}\). The formulas for the insertion of the symbol \(b\) into the string or the deletion of a string symbol are the same, since the language only distinguishes whether there is an \(a\) in the string or not.

After the operation \(\text{ins}_a(y)\), the relations can be updated as follows

\[
\begin{align*}
\phi^A_{\text{ins}_a}(y; x_1, x_2) & \equiv [(y \leq x_1 \lor y \geq x_2) \land A(x_1, x_2)] \lor [x_1 < y < x_2] \\
\phi^I_{\text{ins}_a}(y; x) & \equiv [y \geq x \land I(y)] \lor [y < x] \\
\phi^F_{\text{ins}_a}(y; x) & \equiv [y \leq x \land F(y)] \lor [y > x] \\
\phi^{\text{ACC}}_{\text{ins}_a}(y) & \equiv \text{true},
\end{align*}
\]

and after the operations \(\text{ins}_b(y)\) and \(\text{reset}(y)\), the relations can be updated as follows

\[
\begin{align*}
\phi^A_{\text{reset/ins}_b}(y; x_1, x_2) & \equiv [(y \leq x_1 \lor y \geq x_2) \land A(x_1, x_2)] \lor [x_1 < y < x_2 \land A(x_1, y) \lor A(y, x_2)] \\
\phi^I_{\text{reset/ins}_b}(y; x) & \equiv [y \geq x \land I(y)] \lor [y < x \land I(y) \lor A(y, x)] \\
\phi^F_{\text{reset/ins}_b}(y; x) & \equiv [y \leq x \land F(y)] \lor [y > x \land F(y) \lor A(y, x)] \\
\phi^{\text{ACC}}_{\text{reset/ins}_b}(y) & \equiv I(y) \lor F(y).
\end{align*}
\]

It is crucial here that \(A(i, j)\) refers to the substring from \(i+1\) up to position \(j-1\) (as opposed to \(i\) and \(j\)). Otherwise, it would not be possible to maintain these auxiliary relations. In the update formula \(\phi^A_{\text{ins}_a}(y; x_1, x_2)\) for example, one can only use the three variables \(y, x_1\) and \(x_2\) to compute the new value of \(A(x_1, x_2)\) but needs to know the length of the substring on the intervals \([x_1, y]\) and \([y, x_2]\).

As said before, the dynamic languages Dyn(L) with an underlying regular language \(L\) exactly constitute the class DYNPROP.

Theorem 3.2. Let \(L\) be a language. Then, the following are equivalent:

(a) \(L\) is regular.
(b) \(\text{Dyn}(L) \in \text{DYNPROP}\).
(c) \(\text{Dyn}(L) \in \text{DYNPROP}(\text{Rel}, \text{Rel})\).

The rest of this section is devoted to proving this theorem. First, in Proposition ?? we show that, for every regular language \(L\), \(\text{Dyn}(L) \in \text{DYNPROP}\) (\(a \Rightarrow b\)). Its converse is proven in Proposition ?? (\(b \Rightarrow a\)). As any dynamic language in DYNPROP is, by definition, also in DYNPROP(Rel, Rel) (\(b \Rightarrow c\)); it then suffices to extend Proposition ?? to Proposition ??, which states that for every dynamic language Dyn(L) in DYNPROP(Rel, Rel), \(L\) is regular (\(c \Rightarrow a\)). Together, these imply Theorem ??.

Proposition 3.3. For every regular language \(L\), \(\text{Dyn}(L) \in \text{DYNPROP}\).

Proof. Let \(A = (Q, \delta, s, F)\) be a DFA accepting \(L\). Here, \(Q\) is the set of states, \(\delta : Q \times \Sigma \rightarrow Q\) is the transition function, \(s\) is the initial state and \(F\) is the set of accepting states. As usual, we denote by \(\delta^* : Q \times \Sigma^* \rightarrow Q\) the reflexive, transitive closure of \(\delta\). Then, \(w \in L(A)\) if and only if \(\delta^*(s, w) \in F\).

The program \(P\) recognizing \(\text{Dyn}(L)\) uses the following relations.
For any pair of states \( p, q \in Q \), a relation
\[
R_{p,q} = \{(i,j) \mid i < j \land \delta^*(p, w[i+1, j-1]) = q\};
\]

- For each state \( q \), a relation \( I_q = \{ j \mid \delta^*(s, w[1, j-1]) = q \}; \) and
- For each state \( p \), a relation \( F_p = \{ i \mid \delta^*(p, w[i+1, n]) \in F \}\),
where \( n \) is the size of the word structure.

As already mentioned in example ??, it is crucial here that \( R_{p,q}(i, j) \) refers to the substring from position \( i+1 \) up to position \( j-1 \) (as opposed to \( j \)), as will become clear in the following.

Thanks to Lemma ?? we can assume that these relations are initialized as follows.

- \( R_{p,p} = \emptyset \) and \( R_{p,q} = \emptyset \), for \( p \neq q \);
- \( I_s = \{1, \ldots, n\} \) and \( I_q = \emptyset \), for \( q \neq s \);
- \( F_p = \{1, \ldots, n\} \) if \( p \in F \) and \( F_p = \emptyset \), otherwise.

We now show how these relations can be maintained. First, for each \( \sigma \in \Sigma \) and \( p, q \in Q \), we have the following update formulas for relations \( R_{p,q} \):
\[
\phi_{\text{ins}}^{R_{p,q}}(y; x_1, x_2) \equiv \left( y \notin x_1, x_2 \land R_{p,q}(x_1, x_2) \right)
\lor \left( y \in x_1, x_2 \land \bigvee_{p', q' \in Q} R_{p,p'}(x_1, y) \land R_{q', q}(y, x_2) \right);
\]

\[
\phi_{\text{reset}}^{R_{p,q}}(y; x_1, x_2) \equiv \left( y \notin x_1, x_2 \land R_{p,q}(x_1, x_2) \right)
\lor \left( y \in x_1, x_2 \land \bigvee_{p' \in Q} R_{p,p'}(x_1, y) \land R_{p', q}(y, x_2) \right).
\]

The formulas for the other relations are along the same lines, e.g., for each \( \sigma \in \Sigma \) and \( q \in Q \), and the relation \( I \) we have the following update formula:
\[
\phi_{\text{ins}}^{I_q}(y; x) \equiv \left( y \geq x \land I_q(x) \right)
\lor \left( y < x \land \bigvee_{p', q' \in Q} I_{p'}(y) \land R_{q', q}(y, x) \right).
\]

Finally, \( \text{ACC} \) can be updated by the formulas
\[
\phi_{\text{ins}}^{\text{ACC}}(y) \equiv \bigvee_{p', q' \in Q} I_{p'}(y) \land F_{q'}(y) \quad \text{and} \quad \phi_{\text{reset}}^{\text{ACC}}(y) \equiv \bigvee_{p' \in Q} I_{p'}(y) \land F_{p'}(y).
\]

We next show that the converse of Proposition ?? is also true.

**Proposition 3.4.** Let \( \text{Dyn}(L) \) be a dynamic language in \( \text{DYNPROP} \). Then \( L \) is regular.

**Proof.** The idea of the proof is as follows. We consider a dynamic program \( P \) for \( \text{Dyn}(L) \) and the situation where, starting from the empty word, the positions of a word are set in a left-to-right fashion. Since the acceptance of the word by \( P \) does not depend on the order of updates used to produce the word, it suffices to consider this particular sequence of updates. We then show that, for such update sequences, we can construct a finite automaton which simulates \( P \) and thus recognizes \( L \). It then follows that \( L \) is regular.

The three following observations enable us to simulate \( P \) by an automaton.

1. After each update, all tuples of positions that have not been set yet have the same type (cf. Subsection ??).
There is only a bounded number (depending only on the number and the maximal arity of the auxiliary relations of \( P \)) of different types of such tuples.

(3) The new type of the tuples after one update is uniquely determined by the inserted symbol and the previous type.

Together these observations will enable us to define a finite automaton for \( L \).

To state these observations more formally, we introduce some notation. Let \( P \) be a \textsc{DynPROP} program recognizing a dynamic language \( \text{Dyn}(L) \) and let \( k \geq 1 \) be the highest arity of any auxiliary relation of \( P \). We make the observations above concrete. Recall that \( S_n \) is the state over universe \( \{1, \ldots, n\} \) containing both the auxiliary relations and the relations encoding a word, in which all relations are empty.

**Observation 1.** Let \( S \) be a state that is reached from \( S_n \) by insertions and deletions at positions \( \leq m \), for some \( m \). Then, the set \( \{m+1, \ldots, n\} \) is \( k \)-indiscernible.

**Proof.** Consider two ordered \( k \)-tuples \( \vec{j} = (j_1, \ldots, j_k) \) and \( \vec{j}' = (j'_1, \ldots, j'_k) \) of elements from \( \{m+1, \ldots, n\} \). Let \( \vec{i} \) and \( \vec{i}' \) be the tuples \( (1, \ldots, m, j_1, \ldots, j_k) \) and \( (1, \ldots, m, j'_1, \ldots, j'_k) \). We show by induction on the number of update steps that after every sequence of updates starting in state \( S_n \) and resulting in state \( S \) it holds that

\[
\langle S, \vec{i} \rangle = \langle S, \vec{i}' \rangle.
\]

As this holds for every pair of ordered \( k \)-tuples in \( \{m+1, \ldots, n\} \) we can conclude that \( \{m+1, \ldots, n\} \) is indeed \( k \)-indiscernible. Obviously, in the state \( S_n \) the equation holds. Assume now that in some state \( S \) the equation holds and consider one update operation on an element \( m' \in \{1, \ldots, m\} \) resulting in state \( S' \). Then, \( \langle S', \vec{i} \rangle = \langle S', \vec{i}' \rangle \) if, for every atom \( \varphi \) over the set of variables \( \{x_1, \ldots, x_{m+k}\} \), it holds that \( S' \models \varphi[\vec{i}] \) if and only if \( S' \models \varphi[\vec{i}'] \). However, the truth values of \( \varphi[\vec{i}] \) and \( \varphi[\vec{i}'] \) after the update are determined by the evaluation in \( S \) of a quantifier free formula \( \psi \) in which the free variables are replaced by elements of \( \vec{i} \) and \( \vec{i}' \), respectively. As \( \langle S, \vec{i} \rangle = \langle S, \vec{i}' \rangle \), \( S \models \psi[m'; \vec{i}] \) if and only if \( S \models \psi[m'; \vec{i}'] \) and, hence, \( S' \models \varphi[\vec{i}] \) if and only if \( S' \models \varphi[\vec{i}'] \).

**Observation 2.** Let \( S \) be a state and let \( l > k \). If a set \( I \) of at least \( l \) elements from \( S \) is \( k \)-indiscernible then it is also \( l \)-indiscernible. Furthermore, the type of any ordered \( l \)-tuple over \( I \) is uniquely determined by the type of its first \( k \) elements.

**Proof.** Suppose \( I \) is \( k \)-indiscernible. Let \( \vec{i} = (i_1, \ldots, i_l) \) and \( \vec{i}' = (i'_1, \ldots, i'_l) \) be two ordered \( l \)-tuples over \( I \). We show that \( \langle S, \vec{i} \rangle = \langle S, \vec{i}' \rangle \), from which it then follows that \( I \) is \( l \)-indiscernible. To do this, let \( R \) be any relation from \( S \) and let \( k' \) be its arity, and let \( j_1, \ldots, j_{k'} \in [1, l] \). Because \( k' \leq k \) and \( I \) is \( k \)-indiscernible \( S \models R(i_1, \ldots, i_{k'}) \) if and only if \( S \models R(i'_1, \ldots, i'_{k'}) \). As this holds for every \( R \) and every sequence \( j_1, \ldots, j_{k'} \) we conclude that indeed \( \langle S, \vec{i} \rangle = \langle S, \vec{i}' \rangle \) and thus \( I \) is \( l \)-indiscernible.

Next we show that the type of an ordered \( l \)-tuple \( \vec{i} = (i_1, \ldots, i_l) \) is determined by the type of its first \( k \) elements \( i_1 \) to \( i_k \). Indeed, the type of \( \vec{i} \) only depends on whether \( S \models R(i_1, \ldots, i_{j_k}) \), for every relation \( R \) with \( \text{arity}(R) = k' \), and \( j_1, \ldots, j_{k'} \in [1, l] \). However, as \( k' \leq k \), the set \( \{i_1, \ldots, i_{k'}\} \) contains at most \( k \) different elements and hence as \( I \) is \( k \)-indiscernible, there is a tuple \( \vec{i}' \) containing only elements in \( \{1, \ldots, k\} \) such that \( S \models R(i_1, \ldots, i_{j_k}) \) if and only if \( S \models R(i'_1, \ldots, i'_{j_k}) \). Hence, the type of \( \vec{i} \) is determined by that of its first \( k \) elements.

**Observation 3.** Let \( S_1, S'_1 \) be states with universes of size \( n \) and \( n' \), respectively and \( m, m', l \in \mathbb{N} \) such that \( \langle S_1, m, \ldots, m+l \rangle = \langle S'_1, m', \ldots, m'+l \rangle \). Let \( S_2 \) and \( S'_2 \) be the states resulting from \( S_1 \) and \( S'_1 \) by inserting the same symbol \( \sigma \) at positions \( m \) and \( m' \), respectively. Then \( \langle S_2, m+1, \ldots, m+l \rangle = \langle S'_2, m'+1, \ldots, m'+l \rangle \).
This observation can be proved along the same lines as Observation 1.

The automaton for $L$ is defined as follows. A type $\tau$ of ordered $k$-tuples is allowed if there is a (not necessarily reachable) state $S$ with elements $1, \ldots, k + 1$ for which every ordered $k$-tuple is of type $\tau$. Let $Q$ be the set of allowed types of ordered $k$-tuples. For each such type $\tau$ and each symbol $\sigma$ let $\delta(\tau, \sigma)$ be determined as follows: Let $S$ be a state in which every ordered $k$-tuple is of type $\tau$. Let $S'$ be the state reached from $S$ after the update $\text{ins}_\sigma(1)$. Then $\delta(\tau, \sigma)$ is $(S', 2, \ldots, k + 1)$. This new type is also allowed, which can be seen as follows. Because $\tau$ is an allowed type, the set $\{1, \ldots, k + 1\}$ is $k$-indiscernible before the update, and hence $k'$-indiscernible for any $k' \leq k$. Therefore also the set $\{2, \ldots, k + 1\}$ has to be $k'$-indiscernible, for any $k' \leq k$ after the update operation. Now we can add one more element $k + 2$ and define the auxiliary relations of all tuples containing $k + 2$ just like any arbitrary other tuple (not containing $k + 2$) with the same ordering on the elements. Let $F$ be the set of types for which ACC holds. Then $A = (Q, \delta, \tau_0, F)$, where $\tau_0 = \langle E'_k, 1, \ldots, k \rangle$. Notice that as the number of $k$-types is bounded, $A$ is indeed a finite automaton.

We now argue that $L(A) = L$. To this end, consider any word $w = \sigma_1 \cdots \sigma_n$, and the associated update sequence $\alpha_w = \text{ins}_{\sigma_1}(1) \cdots \text{ins}_{\sigma_n}(n)$. Now, we consider an execution of $P$ on this update sequence in a universe of size $n + k$. Then, $\text{word}(\alpha_w(E'_{n+k})) = w$, and hence $\alpha_w(E'_{n+k}) \models \text{ACC}$ iff $w \in L$. Using the observations above it can now be shown that, for any $i \in [0, n]$, it holds in state $\alpha_w[1,i](E'_{n+k})$ that $1$ the set $\{i + 1, \ldots, n + k\}$ is $l$-indiscernible, for any $l$; and $2$ $\delta(w[1, i], \tau_0)$ is exactly the $k'$-type of the set $\{i + 1, \ldots, i + k\}$, determining the type of the entire set $\{i + 1, \ldots, n + k\}$.

As $\tau \in F$ iff ACC holds in $\tau$, it follows that $w \in L(A)$ iff $\alpha_w(E'_{n+k}) \models \text{ACC}$.

**Remark 3.5.** Proposition ?? can be considered a lower bound result as, of course, for every non-regular language $L$, $\text{Dyn}(L) \not\in \text{DYNPROP}$.

The proof of Proposition ?? intuitively relies on the fact that, if a string is constructed by successive insertions in a left-to-right fashion, all remaining string positions cannot be distinguished before they are updated. Using a Ramsey argument, this idea can be generalized to the setting with precomputations, thus showing that (relational) precomputations do not increase the expressive power of DYNPROP-programs.

**Proposition 3.6.** Let $\text{Dyn}(L)$ be a dynamic language in $\text{DYNPROP}(\text{Rel, Rel})$. Then $L$ is regular.

**Proof.** The only thing left to prove is that for any language $L$ such that $\text{Dyn}(L)$ is recognized by a DYNPROP(\text{Rel, Rel}) program, it holds that $L$ is regular. Thus, we extend the technique of the proof of Proposition ?? to also handle DYNPROP programs with precomputations. The proof is a generalization of that proof by a Ramsey argument.

To this end, let $P$ be a DYNPROP(\text{Rel, Rel}) program recognizing a dynamic language $\text{Dyn}(L)$ and let $k \geq 1$ be the highest arity of any auxiliary or initial auxiliary relation of $P$. Again, our goal is to construct a finite automaton for $L$ thus showing that $L$ is regular.

The key to the proof is the following observation.

**Observation 1'.** For each $n$ there is some $m$ such that for every state $S$ over a universe of size $m$ there is a $k$-indiscernible set $I$ of size $n$.

**Proof.** of Observation 1': The proof uses a version of Ramsey’s theorem for hypergraphs [?]:

Given a number $c$ of colors and a natural number $n$ there exists a number $R_c(n)$ such that if the edges of a complete $k$-hypergraph (all edges are of size $k$) with $R_c(n)$ vertices are colored with $c$ colors, then it must contain a complete sub-$k$-hypergraph with $n$ vertices whose edges are all colored with the same color.

---

4The states of $P$ should not be confused with the states of $A$. We reserve the word "state" for the former and refer to the latter as $A$-states.
Let $c$ be the number of different ordered $k$-types (which only depends on the number and arity of the initial auxiliary relations). Then $m$ can be chosen as $R_c(n)$. Consider a state $S$ over a universe of size $m$. Construct a hypergraph $G$ as follows. As the vertex set use the set of universe elements and add for every set of elements of size $k$ a $k$-hyperedge colored with its $k$-type. This leads to a complete $k$-hypergraph for which the vertex set of each complete monocolored sub-$k$-hypergraph corresponds to a $k$-indiscernible set. By Ramsey’s theorem, $G$ must contain a monocolored sub-$k$-hypergraph of size at least $n$ and hence $S$ contains a $k$-indiscernible set $I$ of size $n$.

To complete the proof of Proposition ?? we only need to consider computations of $P$ which set the elements of some $k$-indiscernible set in a left-to-right fashion. The automaton $A$ is constructed similarly as in the proof of Proposition ?? . Now for every string $w$ of some length $n$ there is, by Observation 1′, an $m$ such that every state over $m$ elements has a $k$-indiscernible set $I = \{i_1, \ldots , i_{n+k}\}$ of size $n + k$. By considering the left-to-right update sequence $\sigma_w = \text{ins}_{\sigma_1}(i_1) \cdots \text{ins}_{\sigma_n}(i_n)$ which sets the word $w = \sigma_1 \cdots \sigma_n$ on the elements of $I$, in a universe of size $m$, it is easy to show that $w \in L$ if and only if $w$ is accepted by $A$.

4. DYNAMIC COMPLEXITY OF CONTEXT-FREE LANGUAGES

In the previous section, we have seen that the regular languages are exactly those languages that can be recognized by a DynPROP program. In this section, we will study the dynamic complexity of context-free languages. We first show that any context-free language can be maintained in DynFO. Later, we exhibit languages that can be maintained in DynQF or a weak extension of DynPROP.

Theorem 4.1. Let $L$ be a context-free language. Then, Dyn($L$) is in DynFO.

Proof. Let $L$ be a context-free language defined by grammar $G = (V, S, D)$ over an alphabet $\Sigma$. Here, $V$ is the set of non-terminals, $S \in V$ is the initial non-terminal, and $D$ is the set of derivation rules. W.l.o.g. we assume that $G$ is in chomsky normal form, that is, every rule in $D$ is either of the form $U \rightarrow XY$, with $X, Y \in V$; $U \rightarrow a$, with $a \in \Sigma$; or $U \rightarrow \varepsilon$. Further, w.l.o.g., we assume that there is a distinguished non-terminal $E \in V$ such that $E \rightarrow \varepsilon$ and for all $U \in V$, $U \rightarrow UE$ and $U \rightarrow EU$. For $U \in V$, and $w \in (V \cup \Sigma)^*$, we write $U \rightarrow^* w$ when $w$ can be derived from $U$. Then, $L(G) = \{w \mid w \in \Sigma^* \land S \rightarrow^* w\}$.

Our dynamic program $P$ recognizing $L$ maintains, for all $X, Y \in V$, the following relation:

$$R_{X,Y} = \{(i_1, i_2, j_1, j_2) \mid [j_1, j_2] \subseteq [i_1, i_2] \land X \rightarrow^* w[i_1, j_1 - 1]w[j_2 + 1, i_2]\}$$

Intuitively, $(i_1, i_2, j_1, j_2) \in R_{X,Y}$ implies that, assuming $Y \rightarrow^* w[j_1, j_2]$, it follows that $X \rightarrow^* w[i_1, i_2]$. Notice also that, by our assumptions above, we have $X \rightarrow w[i_1, j_1 - 1]w[j_2 + 1, i_2]$ if and only if $R_{X,E}(i_1, i_2, j_1, j_2)$.

We now state the update formulae. For every $\sigma \in \Sigma$ and $X, Y \in V$, the update formula for $\phi^{R_{X,Y}}_{\text{ins}_{\sigma}}(z; x_1, x_2, y_1, y_2)$ is

$$[y_1, y_2] \subseteq [x_1, x_2] \land \phi_1 \land \phi_2 \land \phi_3,$$

where $\phi_1$, $\phi_2$, and $\phi_3$ are defined according to the position of $z$ with respect to the other variables.

---

When $z$ lies in the interval $[y_1, y_2]$ or outside $[x_1, x_2]$, nothing changes:

$$\phi_1 \equiv (z \notin [x_1, x_2] \lor z \in [y_1, y_2]) \land R_{X,Y}(x_1, x_2, y_1, y_2).$$

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Fig. 2. Update of $R_{X,Y}$ after operation $\text{ins}_\sigma(z)$

—When $z$ lies in $[x_1, y_1[$, the truth value of $R_{X,Y}(x_1, x_2, y_1, y_2)$ can be modified. Figure 2 illustrates this situation.

$$\phi_2 \equiv \exists u_1, u_2, u_3 : u_1 \leq u_2 < u_3 \land u_1, u_2 \in [x_1, y_1] \land u_3 \in [y_2, x_2] \land R_{X,U}(x_1, x_2, u_1, u_3) \land R_{U_1,Z}(u_1, u_2, z, z) \land R_{U_2,Y}(u_2 + 1, u_3, y_1, y_2).$$

—When $z$ lies in the interval $]y_2, x_2[$, the situation is very similar to the previous one; and, hence, $\phi_3$ is almost identical to $\phi_2$.

For all $X, Y \in V$, the update formula $\phi^{R_{X,Y}}_{\text{ins}_\sigma}$ is defined very similar as the formula for $\phi^{R_{X,Y}}_{\text{ins}_\sigma}$ above. The only difference is that $Z$ (for which $Z \to \sigma \in D$) is replaced by $E$ (for which $E \to \varepsilon \in D$).

We finally give the update formulae for the acceptance relation $\text{ACC}$:

$$\text{ACC}_{\text{ins}_\sigma}(z) \equiv \bigvee_{Z \in V, Z \to \sigma \to U \in U \in D} R_{S,Z}(\min, \max, z, z)$$

and

$$\text{ACC}_{\text{reset}}(z) \equiv R_{S,E}(\min, \max, z, z).$$

Notice that we have used many abbreviations in the above formulae. However, these can all easily seen to be definable in first-order logic using the built-in order. In particular, the constants min and max and the successor function are definable and are hence not precomputed functions as in other settings considered in this paper.

However, we cannot hope for an equivalence between $\text{DynFO}$ and the context-free languages, as for $\text{DynPROP}$ and the regular languages before. This is immediate as, opposed to the class of context-free languages, $\text{DynFO}$ is closed under intersection and complement. Furthermore, non-context-free languages can be maintained in $\text{DynQF}$ and $\text{DynPROP}(\text{Succ}, \text{Rel})$. This is because, in some sense, dynamic programs in the latter classes can count. For any $n$, let $\text{EQUAL}_n$ be the language over the alphabet $\Sigma = \{a_1, \ldots, a_n\}$ containing all strings with an equal number of occurrences of each symbol $a_i$. Note that $\text{EQUAL}_n$, for $n \geq 3$, is not context-free.

**Proposition 4.2.** For any $n$,

1. $\text{Dyn}(\text{EQUAL}_n) \in \text{DynPROP}(\text{Succ}, \text{Rel})$; and
2. $\text{Dyn}(\text{EQUAL}_n) \in \text{DynQF}$. 

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Proof. In both cases, we prove the proposition for the language \textsc{Equal$_2$}. The general case is an easy generalization of this proof.

(1) We maintain the language \textsc{Equal$_2$} by implementing a unary counter, which can be done in \textsc{DynPROP}(\text{Succ, Rel}). This counter will count the difference of the number of occurrences of the symbols $a_1$ and $a_2$ in the string. For $i \in [1, 2]$, let $\sharp a_i$ denote the number of $a_i$'s in the current string. We maintain the following relations:

--- Nullary relations (flags) $A_1$ and $A_2$ such that $A_1$ is true if and only if $\sharp a_1 > \sharp a_2$ and $A_2$ is true if and only if $\sharp a_2 > \sharp a_1$.

--- A unary relation $C$ such that $C(i)$ is true if and only if $|\sharp a_1 - \sharp a_2| = i$. Hence, as the universe consists of the elements $\{1, \ldots, n\}$, at each time $C$ is true for one value $i$ if $\sharp a_1 \neq \sharp a_2$ and is false for all $i$ if and only if $\sharp a_1 = \sharp a_2$.

We give the update functions for these relations only for the case of the insertion of a symbol $a_1$. The deletion and the insertion of $a_2$ work similarly.

For the update $\text{ins}_{a_1}(x)$, the flags $A_1$ and $A_2$ can be updated as follows

$\phi^{A_1}_{\text{ins}_{a_1}} \equiv \neg A_2$ and $\phi^{A_2}_{\text{ins}_{a_1}} = A_2 \land \neg C(\text{min})$.

For the update of $C$ we distinguish three cases:

$\phi^C_{\text{ins}_{a_1}} (x) \equiv (\neg (A_1 \lor A_2) \land x = \text{min}) \lor$

$(A_1 \land C(\text{pre}(x)) \land x \neq \text{min}) \lor$

$(A_2 \land C(\text{succ}(x)))$.

The acceptance query tests whether both $A_1$ and $A_2$ are false after the update. That is,

$\phi^{\text{ACC}}_{\text{ins}_{a_1}} (x) \equiv \neg \phi^{A_1}_{\text{ins}_{a_1}} (x) \land \neg \phi^{A_2}_{\text{ins}_{a_1}} (x)$.

(2) To prove that $\text{Dyn} (\text{EQUAL$_2$}) \in \text{DYNQF}$, we will use the same algorithm as before. But, of course, the previous algorithm makes extensive use of the functions of Succ, which are not available in DYNQF. Instead, we incrementally construct the min, succ, and pre functions, using the power of DYNQF.

Here, we do not require that the constructed min, succ and pre functions are consistent with the order relation. Instead, min will be the first position where a symbol is inserted, its successor the second such position etc. At each point in time, succ and pre therefore define a successor function on those positions that carry a symbol or carried a symbol earlier, which we call the active elements. We will not give the precise update functions which are necessary to construct these auxiliary functions, but simply mention the ideas necessary to construct them.

We additionally maintain a unary relation Act, containing all active elements currently included in the successor function, and a constant (that is, nullary function) Max denoting the last element of the successor ordering. Recall that $\text{succ}(\text{Max}) = \text{Max}$ and $\text{pre}(\text{min}) = \text{min}$ should hold by definition of our successor and predecessor functions.

Then, when an update on an element $x$ occurs there are two possibilities. Either $\text{Act}(x)$ already holds in which case nothing has to be changed, or $\text{Act}(x)$ does not hold and hence $x$ has to be added to the successor structure. This is done by setting $\text{Act}(x)$, making $x$ the maximal element and setting the predecessor and successor functions of $x$, min, and (the old) Max corresponding to the new situation.

We finally argue that the program constructed above still works properly when using these on-the-fly constructed functions instead of the precomputed ones in Succ. To this end, notice that there are only two differences. First, the constructed successor functions are not consistent with the built-in order relation. As the original program does not make use of this order relation, this does
not make a difference. Second, at any time the constructed successor functions are only defined on \( k \) elements, where \( k \) is the number of active elements. However, observe that whenever only \( k \) elements are active, the current string cannot contain more than \( k \) symbols, and hence \( C(i) \) does not hold for \( i > k \). It should be noted, however, that \( C(k) \) can hold. Therefore, for every update, we first compute the new successor functions and use these newly computed functions in the updates of the other relations. This can also done without any problems, and hence we can conclude that the original program still works correctly.

From Proposition ?? and Theorem ?? one can conclude the following:

**Corollary 4.3.**

(1) \( \text{DynPROP} \subseteq \text{DynPROP}(\text{Succ}, \text{Rel}) \)

(2) \( \text{DynPROP} \subseteq \text{DynQF} \)

One can also get better upper bounds for the Dyck-languages, the languages of properly balanced parentheses. For a set of opening brackets \( \{1, \ldots, n\} \) and the set of its closing brackets \( \{1, \ldots, n\} \), the language \( D_n \) is the language produced by the context free grammar

\[
S \rightarrow SS \mid (S)_1 \mid \ldots \mid (nS)_n \mid \varepsilon.
\]

**Proposition 4.4.** For every \( n > 0, D_n \in \text{DynQF} \).

**Proof.** The basic idea is similar to the proof of Theorem ???. We maintain relations \( R_1 \) and \( R_2 \) corresponding to \( R_{SE} \) and \( R_{SS} \) in the terminology of Theorem ???. More precisely, \( R_1(i_1, i_2) \) holds if the current substring \( w[i_1, i_2] \) is well-bracketed. Likewise, \( R_2(i_1, i_2, j_1, j_2) \) holds if the string \( w[i_1, i_2] \) without the symbols at positions \( j_1, \ldots, j_2 \) is well-bracketed. More formally, \( R_2(i_1, i_2, j_1, j_2) \) holds if and only if \( S \rightarrow^* w[i_1, j_1 - 1]w[j_2 + 1, i_2] \).

Nevertheless, the update formulas in the proof of Theorem ?? make use of existential quantifiers which are not available in \( \text{DynQF} \). In the current proof we therefore release these existential quantifiers by functions. To this end, we maintain several additional functions described below.

As in the proof of Proposition ??(2), we make use of on-the-fly constructed functions \( \text{min}, \text{succ}, \) and \( \text{pre} \), defined at any time on the elements on which an update already occurred in the update sequence. Then, we associate numbers with elements in this successor function, and let \( \text{min} \) denote and \( \text{pre} \), defined at any time on the elements on which an update already occurred in the update.

Now, we define the four auxiliary functions needed to maintain \( R_1 \) and \( R_2 \). In the following, for two positions \( i_1 < i_2 \), we write \( d(i_1, i_2) \) for the number of closing brackets in \( [i_1, i_2] \) minus the number of opening brackets in \( [i_1, i_2] \). We write \( \text{Cl}(v) \) if position \( v \) carries a closing bracket and \( \text{Op}(v) \) if it carries an opening bracket.

\[
f^-(u, v) := \min \{u' \mid \langle v \rangle \geq 1 \land \text{Cl}(u') \land u' > u \land d(u + 1, u') = \langle v \rangle\}.
\]

Intuitively, \( f^-(u, v) \) is the position to the right of \( u \) where, for the first time, \( \langle v \rangle \) many brackets pending at \( u \) could be closed.

\[
f^+(u, v) := \max \{u' \mid \langle v \rangle \geq 1 \land \text{Op}(u') \land u' < u \land d(u', u - 1) = \langle v \rangle\}.
\]

Thus, \( g^-(u, v) \) gives the maximum surplus of closing brackets in a prefix of \( w[u + 1, v] \). Intuitively, this is the maximum number of pending open brackets at \( u \) that can be “digested” by \( w[u + 1, v] \). Note that the value of \( g^-(u, v) \) might well be 0.

\[
g^-(u, v) := \max \{d(u + 1, u') \mid u < u' \leq v\}.
\]

The attentive reader might have noticed that these functions are not always defined for all combinations of arguments \( u \) and \( v \). To this end, for each of them there is an accompanying
relation, telling which function values are valid. For instance, \( R_f^-(u, v) \) holds if and only if \( f^-(u, v) \) is defined.

As some of the update terms in the dynamic program for \( D_n \) are slightly involved we present the formulas by means of update programs in a pseudocode. These update programs (which should not be confused with the overall dynamic program) get the parameters of the relation or function as input, can assign (position) values to local variables, use conditional branching and return a function value (or TRUE or FALSE for relations). We abstain from a formal definition of update programs but it is straightforward to transform them into update terms by successively replacing each local variable with its definition.

We now give the update formulas for the different relations and functions. In the update programs the following subroutine \( P_0 \) will appear three times in update programs for \( D_n \). Its meaning will become clear when it is first used.

Subroutine \( P_0 \)
1: \{Parameters: \( x_1, x_2, y_1, y_2, i_0, j_0, z \}\n2: \( m := g^-(j_0, y_1 - 1) \)
3: \( j_1 := f^-(j_0, m) \)
4: \( i_1 := f^-(i_0, m) \)
5: \( m' := g^-(y_1, j_1 + 1) \)
6: \( j_2 := f^-(y_2, m') \)
7: \( \text{if } R_1(i_0 + 1, j_0 - 1) \text{ AND } R_2(i_1, j_1, i_0, j_0) \text{ AND } R_2(j_1 + 1, j_2, y_1, y_2) \text{ AND } R_2(x_1, x_2, i_1, j_2) \text{ then} \)
8: \( \text{Return TRUE} \)
9: \( \text{else} \)
10: \( \text{Return FALSE} \)

We first give the update program for \( R_2(x_1, x_2, y_1, y_2) \) for insertions of a symbol (\( i \) at a position \( z \). Only the case where \( z \) is in the left interval (that is, in \([x_1, y_1 - 1]\)) is considered. The other case is symmetric to the insertion of \( j \) into the left interval which will be handled below.

Intuitively, the string is split into four parts each of which has to be well-bracketed:

—The string between \( i_0 = z \) and the corresponding bracket to the right at \( j_0 \) (assuming that this is before \( y_1 \));
—The maximally bracketed string (from \( i_1 \) to \( j_1 \)) around \( z \) inside \([x_1, y_1 - 1]\) without \([i_0, j_0]\);
—The substring starting to the right of \( j_1 \) and ending at the corresponding (= matching) position \( j_2 \) in \([y_2 + 1, x_2]\); and
—The remaining string before \( i_1 \) and after \( j_2 \).

An illustration can be found in Figure ??(a). If the matching bracket for \( z \) is not before \( y_1 \) the construction is slightly different (Figure ??(b)):

—The string between \( z \) and its matching bracket at \( j_0 \) in \([y_2 + 1, x_2]\) has to be well-bracketed; and
—The remaining string consisting of \( w[x_1, z - 1] \) and \( w[j_0 + 1, x_2] \) has to be well-bracketed.
Fig. 3. Illustration of the update programs for (a) insertion of ( if the matching bracket is in the left string, (b) if it is in the right string.

Update $R_2(x_1, x_2, y_1, y_2)$: insert ( $\mathopen{\text{at}}$ $z$

1: if $z \in [x_1, y_1 - 1]$ then
2: $i_0 := z$
3: $j_0 := f^{-1}(z, 1)$ \{find the matching closing bracket\}
4: if $j_0 < y_1$ then
5: if $R_{i_0}(j_0)$ then
6: $P_0$
7: else
8: Return FALSE
9: else
10: $m := g^{-1}(y_1, z)$
11: $j_0 := f^{-1}(y_2, m + 1)$
12: if $R_{i_0}(j_0)$ AND $R_2(z + 1, j_0 - 1, y_1, y_2)$ AND $R_2(x_1, x_2, z, j_0)$ then
13: Return TRUE
14: else
15: Return FALSE
16: else
17: \{Symmetric case $z \in [y_2 + 1, x_2]$\}

Note that the internal variable $m$ is used for a position that is interpreted as a number (encoded as explained before). Thus, $m + 1$ is an abbreviation for $\text{succ}(m)$. Likewise, 0 is an abbreviation for min.

It could be the case that in line 3 no matching bracket is found. In this case the update program fails and returns FALSE. In the actual function terms this can be handled by using the relation $R_2^{-}$. We will stick to this convention in the following as well: Whenever a function value is not defined the value of the update program becomes FALSE (corresponding to undefined values for the function update programs below).

Next, we describe the update program for the insertion of ) $\mathclose{$. This case is very similar to the insertion of (): The only difference is that $j_0$ is now the position $z$ and $i_0$ is the matching position to the left. Furthermore, there is no case distinction as $j_0$ is always in the left string.

Update $R_2(x_1, x_2, y_1, y_2)$: insert ) $\mathclose{$ $\mathopen{\text{at}}$ $z$

1: if $z \in [x_1, y_1 - 1]$ then
2: $i_0 := f^{-1}(z, 1)$
3: $j_0 := z$
4: $P_0$
5: else
6: \{Symmetric case $z \in [y_2 + 1, x_2]$\}

Finally, the following update program handles reset operations. This can be handled just as an insertion with the difference that there is no string between $i_0$ and $j_0$. 
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Fig. 4. Illustration of the update program for \( f^\rightarrow \) under insertion of a closing bracket.

Update \( R_2(x_1, x_2, y_1, y_2) \): reset \( z \)

1: if \( z \in [x_1, y_1 - 1] \) then
2: \( i_0 := z \)
3: \( j_0 := z \) \{The empty string \( w[z + 1, z - 1] \)
   is well-bracketed...\}
4: \( P_0 \)
5: else
6: \{Symmetric case \( z \in [y_2 + 1, x_2] \}\}

The update programs for \( R_1 \) are similar but easier. We now describe the update programs for the functions \( f^\leftarrow, f^\rightarrow, g^\leftarrow, \) and \( g^\rightarrow \). We only give the update programs for \( f^\rightarrow \) and \( g^\rightarrow \) as \( f^\leftarrow \) and \( g^\leftarrow \) are again symmetric. We do not explicitly state the update programs for \( R_f^\leftarrow \) and \( R_g^\leftarrow \) as they are completely analogous to the programs for the functions.

For \( f^\rightarrow(x, m) \) we only need to consider the case where \( m \) has a corresponding number and is different from \( \text{min} \). The insertion of \( ( \) at position \( z \) only affects \( f^\rightarrow(x, m) \) if \( x < z < f^\rightarrow(x, m) \). In that case, the insertion of \( z \) increases \( d(x, w) \) by one for all \( w > z \) and therefore the previous value of \( f^\rightarrow(x, m + 1) \) is the new value for \( f^\rightarrow(x, m) \).

Update \( f^\rightarrow(x, m) \): insert \( ( \) at \( z \)

1: if \( z \leq x \) then
2: return \( f^\rightarrow(x, m) \)
3: else
4: \( y := f^\rightarrow(x, m) \)
5: if \( y < z \) then
6: \( y := f^\rightarrow(x, m) \)
7: else
8: \( y := f^\rightarrow(x, m + 1) \)

Notice that in this program we are using the assumption that \( z \) was empty before the insertion (see Lemma ??). The update of \( f^\rightarrow(x, m) \) after the insertion of a closing bracket is slightly more involved. If \( x < z < f^\rightarrow(x, m - 1) \) then the new value is just \( f^\rightarrow(x, m - 1) \). Otherwise, we have to identify the maximal pair of matching brackets around \( z \) where the left bracket is to the right of \( f^\rightarrow(x, m - 1) \) (= \( y \)). Due to the additional closing bracket at \( z \) the right bracket of this pair (\( y' \)) is then the new value for \( f^\rightarrow(x, m) \). In case \( m = 1 \) we simply replace the role of \( f^\rightarrow(x, m - 1) \) by \( x \). The main case is illustrated by Figure ??.
Update $f^-(x,m)$: insert $)$ at $z$

1: if $z \leq x$ then
2: Return $f^-(x,m)$
3: else
4: if $m = 1$ then
5: $y := x$
6: else
7: $y := f^-(x,m-1)$
8: if $y > z$ then
9: Return $y$
10: else
11: $m' := g^-(z,y+1)$
12: if $m' = 0$ then
13: Return $z$
14: else
15: Return $f^-(z,m')$

The update program for a reset operation is similar to the insertion of $)$ in case $z$ carries an opening bracket and simpler if $z$ carries a closing bracket.

Update $f^-(x,m)$: reset $z$

1: if $z$ carries a closing bracket then
2: $y := f^-(x,m)$
3: if $y < z$ then
4: Return $y$
5: else
6: Return $f^-(x,m+1)$
7: else
8: if $z \leq x$ then
9: return $f^-(x,m)$
10: else
11: if $m = 1$ then
12: $y := x$
13: else
14: $y := f^-(x,m-1)$
15: if $y > z$ then
16: Return $y$
17: else
18: $m' := g^-(z,y+1)$
19: Return $f^-(z,m'+1)$

Next, we give the update programs for $g^-(x,y)$. The first one handles the insertion of an opening bracket and also the reset for closing brackets.

Update $g^-(x,y)$: insert $($ at $z$

1: if $z \leq x$ OR $z > y$ then
2: Return $g^-(x,y)$
3: $m := g^-(x,y)$
4: $v := f^-(x,m)$
5: if $v < z$ then
6: Return $m$
7: else
8: Return $m - 1$

The next one handles insertion of closing brackets.
Update $g^{-}(x, y)$: insert $)$ at $z$
1: if $z \leq x$ OR $z > y$ then
2: Return $g^{-}(x, y)$
3: $m := g^{-}(x, y)$
4: $v := f^{-}(x, m)$
5: if $v > z$ then
6: Return $m + 1$
7: $m' := g^{-}(z, v)$
8: if $m' = 0$ then
9: Return $m + 1$
10: if $f^{-}(z, m' + 1) \leq y$ then
11: Return $m + 1$
12: else
13: Return $m$

The last update program takes care of reset of opening brackets.

Update $g^{-}(x, y)$: reset $($ at $z$
1: if $z \leq x$ OR $z > y$ then
2: Return $g^{-}(x, y)$
3: $m := g^{-}(x, y)$
4: $v := f^{-}(x, m)$
5: if $v > z$ then
6: Return $m + 1$
7: $m' := g^{-}(z, v)$
8: if $f^{-}(z, m' + 1) \leq y$ then
9: Return $m + 1$
10: else
11: Return $m$

Finally, we give the update formulas for the acceptance relation $ACC$. To this end, we maintain two additional constants (0-ary functions) first and last. Here, first will denote the first element (first according to the given order, not the constructed successor functions) which has been updated, and, similarly, last denotes the last such element. Hence, at any time $w[1, \text{first} - 1] = w[\text{last} + 1, n] = \varepsilon$. These functions can easily be maintained. We give the update formulas for the acceptance relation again in our usual formalism:

$$
\phi_{\text{ins}}^{\text{ACC}}(z) \equiv \phi_{\text{ins}}^{\text{R}_1}(z; \phi_{\text{first}}^{\text{ins}}(z), \phi_{\text{last}}^{\text{ins}}(z))
$$

and

$$
\phi_{\text{reset}}^{\text{ACC}}(z) \equiv \phi_{\text{reset}}^{\text{R}_1}(z; \phi_{\text{first}}^{\text{reset}}(z), \phi_{\text{last}}^{\text{reset}}(z)).
$$

That is, the string is valid if and only if $R_1(\text{first}, \text{last})$ holds after the update has occurred. This completes the description of the update programs. The correctness proof is tedious but straightforward.

We expect the result to hold for a broader class of context-free languages which has yet to be pinned down exactly. It is even conceivable that all deterministic or unambiguous context-free languages are in DynQF.

We next strengthen the above result for the Dyck language with only one kind of bracket, that is, $D_1$. Here, auxiliary functions are not needed if built-in successor and predecessor functions are given.

**Proposition 4.5.** $D_1 \in \text{DynPROP}(\text{Succ}, \text{Rel})$

**Proof.** In [?] it was shown that $D_1$ is maintainable in DynFO using the so-called "level trick". To each position $i$ of the string, a number $L(i)$ (the level) is assigned such that $L(i)$ is equal to
the number of opening brackets minus the number of closing brackets in the substring $w[1..i]$. The string is in $D_1$ if and only if there every position has a positive level and the level of the last position in the string equals 0.

In the following program we maintain a data structure, called a ringlist, capable of storing a set of elements. Technically, a ringlist is the edge relation of a directed graph that is a circle. For instance, the set $\{a, b, c\}$ can be stored by the edge relation $\{(a, b), (b, c), (c, a)\}$.

The DynPROP(Succ, Rel)-program for $D_1$ will maintain for all pairs $(i, j)$ of positions in the string and for each number $l \in \{-n, \ldots, -2, -1, 0, 1, 2, \ldots, n\}$ a ringlist of all positions $k \in \{i, \ldots, j\}$ of level $l$. For this purpose, we use the following relations:

- $L_0(i, j, \cdot, \cdot)$ is a 4-ary relations containing the ringlist of all string positions of level 0. That is, $L_0(i, j, a, b)$ holds when $(a, b)$ is an edge in the ringlist for positions $i$ and $j$ and level 0. We also denote this ringlist by $L_0(i, j)$.
- $L_+(i, j, l, \cdot, \cdot)$ and $L_-(i, j, l, \cdot, \cdot)$ are 5-ary relations containing ringlists for the positive and negative level $l$ and $-l$. Similarly, we denote these ringlists by $L_+(i, j, l)$ and $L_-(i, j, l)$.
- $F_0(i, j)$ holds if $L_0(i, j)$ is not empty.
- $F_+(i, j, l)$ and $F_-(i, j, l)$ hold when $L_+(i, j, l)$ and $L_-(i, j, l)$ are not empty.
- $F_{\text{max}0}(i)$ holds when $F_0(i, n)$ holds, where $n$ is the universe size (remember that we do not have a constant for the maximal element).
- $F_{\text{max}+}(i, l)$ and $F_{\text{max}+}(i, l)$ hold when $F_+(i, n, l)$ and $F_+(i, n, l)$ hold.
- $\text{Min}_0(i, j, k)$ (respectively, $\text{Max}_0(i, j, k)$) holds if and only if $k$ is the minimal (respectively, maximal) element of the ringlist $L_0(i, j)$.
- $\text{Min}_+(i, j, l, k)$, $\text{Min}_-(i, j, l, k)$, $\text{Max}_+(i, j, l, k)$ and $\text{Max}_-(i, j, l, k)$ are the corresponding relations for the ringlist of the other levels beside 0.
- $\text{Last}_0$ is a nullary relation which holds when the level of the last position is 0.
- $\text{Last}_0$ and $\text{Last}_+(l)$ store the level of the last position.

Initially, for all $i$ and $j$, $F_0(i, j)$, $F_{\text{max}0}(i)$, $\text{Last}_0$, $\text{Min}_0(i, j, i)$ and $\text{Max}_0(i, j, j)$ are true and

$$L_0(i, j, a, b) = (a, b \in \{i, \ldots, j\} \land b = \text{succ}(a)) \lor (a = j \land b = i).$$

Thanks to Lemma 10, we can assume these initializations to take place before the computation of the program.

We can maintain these relations because of the following observation: After an update operation on some position $x$ in the string, the level of all succeeding positions increases or decreases simultaneously by 1.

Thus, when we for instance insert a ( at position $x$, then a position is of level $l$ after the update if it either comes before $x$ and had level $l$ before the update, or if it is $x$ or comes after $x$ and had level $l - 1$ before the update.

So, to construct the new ringlist for a level $l$ after an update at a position $x$ one has to merge the ringlist for the position between $i$ and $\text{pre}(x)$ of level $l$ and the one for position between $x$ and $j$ of level $l + 1$ or $l - 1$. In order to do this, only the relations around the update position $x$, its two borders $i$ and $j$ and the minimal and maximal element (relative to the ordering) of the considered ringlists have to be changed. We will show that it is possible to express these updates using quantifier free formulas.

Let us first consider the update function for $L_0(i, j)$ and the operation ins$(x)$. Here, the levels of all positions from $x$ to $n$ have to increase by one. The update formulas for the relations $L_-$ and $L_+$ are along the same line, and the ones for the update operations ins$(x)$ and reset$(x)$ are also very similar. For readability, we use case distinctions and state the formulae for each case separately.

They can easily be put together in one (quantifier free) formula.
—If $x$ does not lie in the interval $[i,j]$ then nothing happens: $L_0(i, j)$ remains the same.
—If $x = i$ then the whole list has to be increased by one, so
\[ \phi_{\text{ins}}^{L_0}(x; i, j, a, b) \equiv L_-(i, j, 1, a, b). \]

Here, the constant 1 is not included as a nullary function but can be accessed via $\text{succ}(\text{min})$.
—Else, if $x \in \{\text{succ}(i), j\}$, then he lists $L_0(i, \text{pre}(x), \cdot, \cdot)$ and $L_-(x, j, 1, \cdot, \cdot)$ have to be merged. Here, the emptiness-relations $F_0(i, \text{pre}(x))$ and $F_-(x, j, 1)$ come into play. Indeed, if one of the corresponding ringlists is empty, the other ringlist just has to be copied. If both are empty, then $L_0(i, j, \cdot, \cdot)$ has to be empty after the update. Only if both $F_0(i, \text{pre}(x))$ and $F_-(x, j, 1)$ are false, the following formula applies:
\[ \phi_{\text{ins}}^{L_0}(x; i, j, a, b) \equiv a < b < x \land \text{pre}(x) \lor \begin{cases} a < x \leq b \land (\text{Max}_0(i, \text{pre}(x), a) \land \text{Min}_-(x, j, 1, b)) \lor \hfill \\
 x < a < b \land \text{Max}_-(x, j, 1, a, b) \lor \hfill \\
 b < a \land (\text{Min}_0(i, \text{pre}(x), b) \land \text{Max}_-(x, j, 1, a)) \end{cases} \]

Similar to the level-relations $L_0$, $L_-$ and $L_+$ we only give the update formulae for $F_0(i, j)$ after the update operation $\text{ins}_1(x)$. The formulas for the other emptiness-relations $F_-$ and $F_+$ and for the other kind of update operations are similar.
—If $x$ does not belong to $[i,j]$, then $F_0(i, j)$ stays the same.
—If $x = i$ then
\[ \phi_{\text{ins}}^{F_0}(x; i, j) \equiv F_-(i, j, 1) \]

because the whole ringlist $L_-(i, j, 1, \cdot, \cdot)$ was shifted to $L_0$. Hence, if $L_-$ was empty before the update then, after the update, $L_0$ is empty.
—In the third case, if $x \in \{\text{succ}(i), j\}$ then $F_0(i, j)$ is non-empty if either $L_0(i, \text{pre}(x), \cdot, \cdot)$ or $L_-(x, j, 1, \cdot, \cdot)$ was non-empty before the update operation. So
\[ \phi_{\text{ins}}^{F_0}(x; i, j) \equiv F_0(i, \text{pre}(x)) \lor F_-(x, j, 1). \]

The relations $F\text{max}_0$, $F\text{max}_-$ and $F\text{max}_+$ can be maintained in a similar way.

Next, we show how to maintain the relation $\text{Min}_0(i, j, k)$ after the update operation $\text{ins}_1(x)$. Again, three cases have to be distinguished.
—If $x \notin [i,j]$ then nothing changes.
—If $x = i$, then
\[ \phi_{\text{ins}}^{\text{Min}_0}(x; i, j, k) \equiv \text{Min}_-(i, j, 1, k) \]
—Else, we have to check whether the list $L_0(i, \text{pre}(x), \cdot, \cdot)$ is empty. If it is empty, then the minimum has to be taken from the list $L_-(x, j, 1)$. Otherwise, its minimum remains the same. So, we have the following formula for the third case:
\[ \phi_{\text{ins}}^{\text{Min}_0}(x; i, j, k) \equiv (F_0(i, \text{pre}(x)) \land \text{Min}_0(i, \text{pre}(x), k)) \lor (\neg F_0(i, \text{pre}(x)) \land \text{Min}_-(x, j, 1, k)) \]

Again, the relations $\text{Max}_0$, $\text{Min}_-$, $\text{Min}_+$, $\text{Max}_+$ and $\text{Max}_-$ can be updated similarly. The last relations which have to be updated are $\text{Last}_0$, $\text{Last}_-$ and $\text{Last}_+$. However, their change does not depend on the position of the actual update operation, but only on the type of the inserted or deleted symbol. In fact they only have to count the difference between the number of opening and closing brackets in the string. The maintenance of these relations is straightforward. For example after the insertion of an opening bracket we have
\[ \phi_{\text{ins}_1}^{\text{Last}_0}(x) \equiv \text{Last}_-(1). \]
Now only the acceptance of a string has to be detected. The string is accepted if and only if, after the update, the level of the last position equals 0 and the ringlist of level -1 is empty. We only have to check the level -1, and not all negative levels. Indeed, if there is a position with level less then -1 there also has to be a position which has level -1. So, for instance for the update ins_i, the update formula for ACC is
\[
\phi_{\text{ins}_i}^{\text{ACC}}(x) \equiv \neg \phi_{\text{ins}_i}^{\text{Fmax} -}(x; \min, 1) \wedge \phi_{\text{ins}_i}^{\text{Last} -1}(x).
\]

So, whereas built-in relations do not increase the expressive power of DYNPROP, already the three simple functions succ, pre and min allow the maintenance of non-regular languages.

5. VARIATIONS

Alternative Semantics. Following [7], we have introduced in Section ?? dynamic languages in which it is both allowed to insert or change labels at positions in the string and to delete elements at positions. In a universe of size \( n \), one can thus create all strings of length smaller or equal than \( n \).

However, one can also consider the setting in which each position in the string must at any time be assigned a symbol. Although this setting is less “dynamic”, it has the advantage that a word is always associated with its canonical logical structure. This can be achieved by starting with an initial structure in which each symbol is already assigned a symbol, and subsequently only allowing labels to be changed (and not deleted).

More formally, we assign to every language \( L \) a dynamic language Dyn-alt(\( L \)) as follows. For a distinguished initial symbol \( a \in \Sigma \) and \( n \in \mathbb{N} \), let \( E_n^a \) be the word structure in which \( R_a(i) \) is true, for all \( i \), and \( R_{\sigma} \) is empty, for all \( \sigma \neq a \). Further, \( \Delta_n = \{ \text{ins}_\sigma \mid \sigma \in \Sigma \} \). Then, Dyn-alt(\( L \)) = \{ \( (n, \delta) \mid \delta \in \Delta_n^a \wedge \text{word}(\delta(E_n^a)) \in L \}^5.

Proposition ?? shows that the situation is less appealing than in the original semantics. In particular, there are regular languages which cannot be maintained without precomputation; and with precomputation all regular, but also non-regular, languages can be maintained. Here, MIDDLE = \{ \text{wba'} \mid |w| = |w'| \} is the language over the alphabet \( \Sigma = \{a, b\} \) which contains all strings whose middle element is \( b \), which is clearly not regular.

**Proposition 5.1.**

1. Dyn-alt(\( L((aa)^*) \)) \( \notin \) DYNPROP
2. For any regular language \( L \), Dyn-alt(\( L \)) \( \in \) DYNPROP(\( \text{Rel}, \text{Rel} \))
3. Dyn-alt(MIDDLE) \( \in \) DYNPROP(\( \text{Rel}, \text{Rel} \))

**Proof.** (1) Let \( L = L((aa)^*) \). Let \( n \) be any positive even integer, and \( \delta = \text{ins}_a(1) \). Then, \( \text{word}(\delta(E_n^a)) \in L \), and \( \text{word}(\delta(E_{n+1}^a)) \notin L \). Hence, \( (n, \delta) \in \text{Dyn-alt}(L) \) and \( (n + 1, \delta) \notin \text{Dyn-alt}(L) \). We show that for any program \( P \in \text{DYNPROP} \), \( (n, \delta) \in L(P) \) if and only if \( (n + 1, \delta) \notin L(P) \), which implies the proposition.

First, notice that \( (n, \delta) \in L(P) \) if and only if \( S_n^a \models \phi_{\text{ins}_a}^{\text{ACC}}(1) \), and, correspondingly, \( (n + 1, \delta) \in L(P) \) if and only if \( S_{n+1}^a \models \phi_{\text{ins}_a}^{\text{ACC}}(1) \). However, these two questions can be decided in an identical manner: Take \( \phi_{\text{ins}_a}^{\text{ACC}} \), replace any occurrence of \( R_a \) by true and any occurrence of a relation symbol different from \( R_a \) by false, and evaluate the obtained boolean formula. Hence, \( S_n^a \models \phi_{\text{ins}_a}^{\text{ACC}}(1) \) if and only if \( S_{n+1}^a \models \phi_{\text{ins}_a}^{\text{ACC}}(1) \), which concludes the proof.

\(^5\)Notice that Dyn(\( L \)) consists only of update sequences \( \delta \), whereas Dyn-alt(\( L \)) contains tuples \( (n, \delta) \). This change is necessary as the membership of a word of a language under the current semantics can depend both on the size of the initial structure \( n \), and the update sequence \( \delta \).
(2) As seen in the previous proof, DynPROP program without precomputation are not capable of maintaining all regular languages. The reason for this is that the initial string is $a^n$, for some $n$, whereas the initial string was empty in the original semantics. Then, when the computation starts, the DynPROP program did not have the chance to initialize its data structures according to $a^n$ and is immediately lost.

However, when allowing precomputation, we can simply reuse the program $P$ defined in the proof of Proposition ??, Indeed, the only difference is in the initialization of the relations. Whereas they could be initialized by quantifier free formulas when the initial string was empty, we now have to use the power of precomputations to initialize them. In particular, for a language $L$ accepted by automaton $A = (Q, \delta, s, F)$ they should be initialized as follows:

$$-R_{p,q} = \{(i, j) \mid i < j \land (p, a^{j-i-1}, q) \in \delta\};$$
$$-I_q = \{i \mid (s, a^{i-1}, q) \in \delta\};$$ and
$$-F_p = \{i \mid (p, a^{n-1}, q_f) \in \delta, \text{ for some } q_f \in F\}.$$

From the correctness of the program of Proposition ?? and this precomputation, the correctness of this modified program immediately follows.

(3) The dynamic program $P$ maintaining Dyn-alt(MIDDLE) will make use of the precomputed unary relation $M$ containing the middle element of the structure, if the universe size is odd. Formally, for $n \in \mathbb{N}$, $M_{n}^{\text{init}} = \{\lceil n/2 \rceil \mid n \text{ is odd}\}$. The program $P$ only needs to maintain the acceptance relation, which can be done as follows:

$$\phi_{\text{ins}}^{\text{ACC}}(x) \equiv \text{ACC} \land \neg M(x)$$

and

$$\phi_{\text{ins}}^{\text{ACC}}(x) \equiv \text{ACC} \lor M(x).$$

Notice that, contrary to Theorem ??, Proposition ?? does not allow to infer lower bounds for DynPROP(\text{Rel, Rel}) under the current semantics. However, if we consider the class of languages with neutral elements, this becomes possible again. We say that a language $L$ has a neutral element $a$ if for all $w, w' \in \Sigma^*$ it holds that $ww' \in L$ if and only if $waw' \in L$. If a language has at least one neutral element, we assume that the initial symbol for its dynamic algorithm is one of these neutral elements.

A straightforward generalization of Theorem ?? yields the following proposition which implies, for instance, that Dyn-alt($L$) $\notin$ DynPROP(\text{Rel, Rel}) for all non-regular languages $L$ which have a neutral element.

PROPOSITION 5.2. Let $L$ be a language which has a neutral element. Then, the following are equivalent:

1. $L$ is regular;
2. Dyn-alt($L$) $\in$ DynPROP; and
3. Dyn-alt($L$) $\in$ DynPROP(\text{Rel, Rel}).

PROOF. As (2) $\Rightarrow$ (3) follows by definition, it suffices to show (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): Let $L$ be a regular language with neutral element and $A$ be the minimal DFA accepting $L$. Then, the DynPROP program $P$, accepting Dyn($L$), constructed in the proof of Proposition ?? accepts exactly Dyn-alt($L$).

It should be clear that the correctness of the update functions of $P$ carries over immediately to the current setting. To see that also the initialization of the different relations is correct, notice that, as $A$ is minimal and $a$ is a neutral element, $(q, a, p) \in \delta$ if and only if $q = p$, for all states $p$ and $q$ of $A$. Since word($E^n_{\text{ins}}$) = $a^n$ it follows that the different relations are properly initialized.
(3) ⇒ (1): Let \( L \) be a language such that Dyn-alt\((L)\) is accepted by a DynPROP\((\text{Rel}, \text{Rel})\) program \( P \). We show that \( L \) is regular by constructing a finite automaton accepting \( L \). Again, this can be done almost identically as in the proof of implication (3) ⇒ (1) in Theorem ???. The key point to notice is that a position which is labeled \( a \) in the current semantics can intuitively be seen as an empty, that is, not-labeled, position in the original semantics because \( a \) is a neutral element.

Therefore, we proceed in two steps. First, completely ignoring the symbol \( a \), we create the automaton \( A' \) exactly as in the proof of Theorem ???. Denote \( \Sigma \setminus \{a\} \) by \( \Sigma_a \). Then, as before, it can be shown that \( L(A) = L \cap \Sigma_a \), that is, \( A \) accepts all strings in \( L \) that do not contain an \( a \). As \( a \) is a neutral element of \( L \), it holds that \( L = \bigcup_{\sigma = \sigma_1 \cdots \sigma_n \in L(A)} L(a^* \sigma_1 a^* \cdots a^* \sigma_n a^*) \). Hence, the desired automaton \( A' \), with \( L(A') = L \), can be obtained from \( A \) by adding the transition \((q, a, q)\) to \( A \), for all states \( q \) of \( A \).

**Regular Tree Languages.** We now investigate the dynamic complexity of the regular tree languages. To this end, we define dynamic tree languages. A tree \( t \) over an alphabet \( \Sigma \) is encoded by a logical structure \( T \) with as universe the first \( n \) elements of the list \((1, 11, 12, 111, 112, 121, 122, \ldots)\), for some \( n \in \mathbb{N} \), and consisting of (1) one unary relation \( R_\sigma \), for each symbol \( \sigma \in \Sigma \); (2) a constant root, denoting the element 1; and (3) binary relations \( \text{L-child} \) and \( \text{R-child} \), containing all tuples \((u,u)\) and \((u,u2)\), respectively.

The updates are terms \( \text{ins}_\sigma(u) \) and \( \text{reset}(u) \), setting and resetting the label of node \( u \) in \( T \), exactly as in the string case. So, the logical structure \( T \) is a fixed balanced binary tree in which the labels can change. Then, the tree \( t \) encoded by \( T \) is the largest subtree of \( T \) whose root is the element 1 and in which all nodes are labelled with an alphabet symbol. Hence, a node of \( T \) is included in \( t \) if it, and all its ancestors, carry an alphabet symbol.

Exactly as for the word languages, for a tree language \( L \), we let \( \text{Dyn}(L) \) be the set of update sequences leading to a tree \( t \in L \). A dynamic program works on a dynamic tree language as it does on a dynamic language. We then obtain the following result.

**Proposition 5.3.** Let \( L \) be a regular tree language. Then, \( \text{Dyn}(L) \in \text{DynPROP}(\text{Fun}, \text{Rel}) \).

**Proof.** We first introduce some notation. For a node \( u \) of \( T \), let \( \text{subtree}_T(u) \) be the largest subtree of \( T \) whose root is \( u \) and in which all nodes are labelled with an alphabet symbol. Hence, \( T \) encodes the tree \( \text{subtree}_T^{\text{root}}(u) \). Further, for a tree \( t \), we denote its set of nodes by \( \text{nodes}(t) \), and for \( u \in \text{nodes}(t) \), \( \text{lab}_t(u) \) denotes the label of \( u \) in \( t \).

The program will make use of the following precomputed relations and functions on \( T \):

- a binary relation \( \text{Anc} \), such that \( \text{Anc}(x, y) \) holds if \( x \) is an ancestor of \( y \);
- a binary function \( \text{lca} \), such that \( \text{lca}(x, y) = z \) if \( z \) is the least common ancestor of \( x \) and \( y \);
- a unary function \( \text{parent} \) such that \( \text{parent}(u) = v \) if \( \text{L-child}(v, u) \) or \( \text{R-child}(v, u) \); and \( \text{parent}(u) = u \), if \( u \) is root;
- unary functions \( \text{l-child} \) and \( \text{r-child} \) such that \( \text{l-child}(u) = v \) if \( \text{L-child}(u, v) \) and \( \text{l-child}(u) = u \), otherwise; and \( \text{r-child}(u) = v \) if \( \text{R-child}(u, v) \) and \( \text{r-child}(u) = u \), otherwise.

Let \( L \) be a regular (binary) tree language, and \( A = (Q, \delta, (q^L_\sigma)_{\sigma \in \Sigma}, F) \) be a bottom-up deterministic tree automaton accepting \( L \), with \( \delta : Q \times Q \times \Sigma \rightarrow Q \) the (complete) transition function. A run of \( A \) on a tree \( t \) is a mapping \( \rho : \text{nodes}(t) \rightarrow Q \) such that (1) for all leaf nodes \( u \) of \( t \), \( \rho(u) = q^L_\sigma \), where \( \text{lab}_t(u) = \sigma \); and (2) for all non-leaf nodes \( u \), with children \( u_1, u_2 \), we have \( \delta(\rho(u_1), \rho(u_2), \text{lab}(u)) = \rho(u) \). If \( \rho(\text{root}(t)) = q \), we say that \( \rho \) is a run of \( A \) on \( t \) to \( q \). A tree \( t \) is accepted if there is a run of \( A \) on \( t \) to \( q_f \), for some \( q_f \in F \).

We denote by \( \text{subtree}_T(u) \) the subtree of \( t \) rooted at \( u \) and by \( \text{subtree}^{u,v}_T \) the subtree of \( t \) with root \( u \) which contains all descendants of \( u \) but no descendants of \( v \). When \( t \) is clear from the context, we
omit it as a subscript. For such a tree subtree \( u,v \), we are also interested in runs which assign a state \( p \) to the new leaf node \( v \), not necessarily consistent with the label of \( v \), and are valid runs otherwise. Thus, a function \( \rho : \text{nodes(subtree}_u^v) \rightarrow Q \) is a run of \( A \) on subtree \( u,v \) to \( q \) if and only if \( \rho(u) = q, \rho(v) = p, \) and \( \rho \) is a valid run of \( A \) on subtree \( u,v \), except that \( p = q_{\text{lab}(v)} \) does not have to hold.

Before giving the relations we will maintain, we define a few subformulas which will be used several times in the subsequent definitions and formulas.

\[
\text{Anc-self}(x,y) \equiv \text{Anc}(x,y) \lor x = y;
\]

\[
\text{Epsilon}(x) \equiv \bigwedge_{\sigma \in \Sigma} \neg R_{\sigma}(x); \text{ and}
\]

\[
\text{Leaf}(x) \equiv (\text{l-child}(x) = x \lor \text{r-child}(x) = x \lor (\text{Epsilon}(\text{l-child}(x)) \land \text{Epsilon}(\text{r-child}(x)))).
\]

Our dynamic program will maintain the following relations:

\[
- \text{Con} = \{ (x,y) \mid \text{Anc-self}(x,y) \land \forall z \text{ with Anc-self}(x,z) \land \text{Anc}(z,y), R_{\sigma}(z) \text{ is true, for some } \sigma \in \Sigma \};
\]

\[
- R_q = \{ x \mid \text{there is a run of } A \text{ on subtree}^x \text{ to } q \}; \text{ and }
\]

\[
- R_{q_1,q_2} = \{ (x_1,x_2) \mid \text{there is a run of } A \text{ on subtree}^{x_1,x_2} \text{ on } [x_2 \rightarrow q_2] \text{ to } q_1 \}.
\]

That is, the relation Con expresses that elements \( x \) and \( y \) are connected in \( T \), that is, that all nodes on the path from \( x \) to \( y \), except possibly \( y \) itself, carry an alphabet symbol. The relation \( R_q \) contains all nodes \( x \) for which there is a run on subtree \( x \) to \( q \); and \( (x_1,x_2) \in R_{q_1,q_2} \) intuitively holds if, assuming there is a run on subtree \( x_2 \) to \( q_2 \), then there is a run on subtree \( x_1 \) to \( q_1 \).

First of all, by Lemma ??, we can assume that these relations are initialized correctly as follows:

\[
- \text{Con} = \{ (x,x) \};
\]

\[
- \text{for all } q \in Q, R_q = \emptyset; \text{ and}
\]

\[
- \text{for all } q_1,q_2 \in Q, R_{q_1,q_2} = \emptyset \text{ if } q_1 \neq q_2, \text{ and } R_{q_1,q_2} = \{ (x,x) \}, \text{ otherwise.}
\]

We now give the update formulæ for the different relations. First, the relation Con can easily be maintained. For all \( \sigma \in \Sigma \),

\[
\phi^\text{Con}_{\text{ins}}(y;x_1,x_2) \equiv \neg (\text{Anc-self}(x_1,y) \land \text{Anc}(y,x_2)) \land \text{Con}(x_1,x_2) \lor \left[ \text{Con}(x_1,y) \land (\text{Con}(\text{l-child}(y),x_2) \lor \text{Con}(\text{r-child}(y),x_2)) \right]
\]

\[
\phi^\text{Con}_{\text{reset}}(y;x_1,x_2) \equiv \neg (\text{Anc-self}(x_1,y) \land \text{Anc}(y,x_2)) \land \text{Con}(x_1,x_2).
\]

Before giving the update formulæ for \( R_q \) and \( R_{q_1,q_2} \) we define a formula which will be used several times. For \( p \in Q \) and \( \sigma \in \Sigma \), the following formula intuitively says that “if node \( x \) is labeled \( \sigma \), then there is a run on subtree \( x \) to \( p \) :

\[
\phi^\sigma_q(x) \equiv [(\text{Leaf}(x) \land q^l_{\sigma} = p) \lor \neg \text{Leaf}(x) \lor \bigvee_{p_1,p_2 \in Q, \delta(p_1,p_2,\sigma)=p} (R_{p_1}(\text{l-child}(x)) \land R_{p_2}(\text{r-child}(x))).]
\]

We can now give the different update formulæ for the insert operation. For each \( \sigma \in \Sigma \) and \( q \in Q \), the relation \( R_q \) can be updated as follows:

\[
\phi^\text{ins}_{R_q}(y;x) \equiv \neg (\text{Anc-self}(x,y) \land \text{Con}(x,y)) \land R_q(x) \lor \left[ \text{Anc-self}(x,y) \land \text{Con}(x,y) \land \bigvee_{p \in Q} (\phi^p_q(y) \land R_{q,p}(x,y)) \right].
\]
The update formula for $R_{q_1,q_2}$ is similar but more involved. It is defined as follows:

$$
\phi_{\text{reset}}^{R_{q_1,q_2}}(y; x_1, x_2) \equiv \text{Anc-self}(x_1, x_2) \land \phi_{\text{ins}}^\Sigma(y; x_1, x_2) \land (\phi_1 \land \phi_2 \land \phi_3 \land \phi_4 \land \phi_5),
$$

where $\phi_1$ to $\phi_5$ are formulas defined according to the position of $y$ with respect to $x_1$ and $x_2$:

—If $y$ does not occur in subtree $x_1,x_2$ after ins$_\Sigma(y)$, or $y = x_2$, then the truth value of $R_{q_1,q_2}(x_1, x_2)$ is not changed:

$$\phi_1 \equiv (-\text{Con}(x_1, y) \lor \text{Anc-self}(x_2, y)) \land R_{q_1,q_2}(x_1, x_2).$$

—Let lca$(x_2, y) = z$. If $y = z$ and $x_2$ is a left descendant of $y$, that is, Anc-self(l-child$(y), x_2)$, we can determine the state $p$ of $z$ and use this information to decide whether $R_{q_1,q_2}(x_1, x_2)$:

$$\phi_2 \equiv y = z \land \text{Anc-self}(l\text{-child}(y), x_2) \land \bigvee_{p,p_1,p_2 \in Q \leftarrow \delta(p_1,p_2,\sigma) = p} (R_{p_1,q_2}(l\text{-child}(y), x_2) \land R_{p_2,r\text{-child}(y)} \land R_{q_1,p}(x_1, y)).$$

—Else if $y = z$, and $x_2$ is a right descendant of $y$, then $\phi_3$ is almost identical to $\phi_2$.

—Else if $y \neq z$ and $y$ is a left descendant of $z$, then:

$$\phi_4 \equiv y \neq z \land \text{Anc-self}(l\text{-child}(z), y) \land \bigvee_{p \in Q} \left(\phi_p^\Sigma(y) \land \bigvee_{r,r_1,r_2 \in Q, \sigma \in \Sigma \leftarrow \delta(r_1,r_2,\sigma) = r} [R_{q_1'}(r) \land R_{r_1,p}(l\text{-child}(z), y) \land R_{r_2,q_2}(r\text{-child}(z), x_2) \land R_{q_1,r}(x_1, z)]\right).$$

—Else if $y \neq z$ and $y$ is a right descendant of $z$, then $\phi_5$ is almost identical to $\phi_4$.

We now give the different formulae for the reset operation. Again, we first define a subformula which will be used several times. The following formula intuitively says that “if node $y$ is reset, and $y'$ is its parent, then there is a run on subtree $y'$ to $p$”:

$$\psi_p(y, y') \equiv \bigvee_{\sigma \in \Sigma \leftarrow p = q^p_\sigma} [\text{l-child}(y') = y \land \text{Epsilon}(r\text{-child}(y')) \lor (\text{r-child}(y') = y \land \text{Epsilon}(r\text{-child}(y'))].$$

We can now define the different formulae for the reset operation. For all $q \in Q$,

$$\phi_{\text{reset}}^{R_q}(y; x) \equiv [\neg(\text{Anc-self}(x, y) \land \text{Con}(x, y)) \land R_q(x)] \lor [\text{Anc}(x, y) \land \bigvee_{p \in Q} \psi_p(y, \text{parent}(y)) \land R_{q,p}(x, \text{parent}(y))].$$

Again, the formula $\phi_{\text{reset}}^{R_{q_1,q_2}}$ is similar but more involved:

$$\phi_{\text{reset}}^{R_{q_1,q_2}}(y; x_1, x_2) \equiv \text{Anc-self}(x_1, x_2) \land \neg(\text{Anc-self}(x_1, y) \land \text{Anc}(x_2, y)) \land \text{Con}(x_1, x_2) \land (\phi_1 \lor \phi_2 \lor \phi_3).$$

Notice that if any of these conditions is not satisfied then $R_{q_1,q_2}(x_1, x_2)$ cannot hold after reset$(y)$. The formulas $\phi_1$, $\phi_2$ and $\phi_3$ depend on the possible remaining positions of $y$ w.r.t. $x_1$ and $x_2$. We only have to distinguish three cases here, as opposed to five before, because we do not have to consider the case lca$(y, x_2) = y$. Indeed, if lca$(y, x_2) = y$, resetting $y$ disconnects $x_1$ from $x_2$.

—If $y$ does not occur in subtree $x_1,x_2$, then the truth value of $R_{q_1,q_2}(x_1, x_2)$ is not changed:

$$\phi_1 \equiv (-\text{Con}(x_1, y) \lor \text{Anc-self}(x_2, y)) \land R_{q_1,q_2}(x_1, x_2).$$
—Let \( \text{lca}(x_2, y) = z \) and \( \text{parent}(y) = y' \). If \( y \neq z \) and \( y \) is a left descendant of \( z \), then

\[
\phi_4 \equiv y \neq z \land \text{Anc-self}(l\text{-child}(z), y) \land \bigvee_{p \in Q} (\psi^p(y, y') \land \left[ R_{\sigma}(z) \land R_{r_1, p}(l\text{-child}(z), y') \land R_{r_2, q_2}(r\text{-child}(z), x_2) \land R_{q_1, r}(x_1, z) \right])
\]

—If \( y \neq z \) and \( y \) is a right descendant of \( z \), the formula \( \phi_3 \) is almost identical to \( \phi_2 \).

Finally, the update formulae for the acceptance relation depend only on the new value of the relations \( R_q \), for \( q \in Q \). That is, for all \( \sigma \in \Sigma \),

\[
\phi^\text{ACC}_{\text{ins}}(x) = \bigvee_{q \in F} \phi^R_{\text{ins}, q}(x, \text{root})
\]

and

\[
\phi^\text{ACC}_{\text{reset}}(x) = \bigvee_{q \in F} \phi^R_{\text{reset}, q}(x, \text{root}).
\]

\section{Beyond Formal Languages}

The definitions given in Section 5 only concern dynamic problems for word structures. Following \cite{??}, we now extend these definitions to arbitrary structures. To this end, let \( \gamma \) be a vocabulary containing relation symbols of arbitrary arities. We assume that a structure over \( \gamma \) of size \( n \) as has universe \( \{1, \ldots, n\} \). The empty structure over vocabulary \( \gamma \) of size \( n \) and only empty relations is denoted \( E_n(\gamma) \).

The set of abstract updates \( \Delta(\gamma) \) is defined as \( \{ \text{ins}_R, \text{del}_R \mid R \in \gamma \} \). A concrete update is a term of the form \( \text{ins}_R(i_1, \ldots, i_k) \text{ or } \text{del}_R(i_1, \ldots, i_k) \), where \( k = \text{arity}(R) \). A concrete update is applicable in a structure of size \( n \) if \( i_j \leq n \), for all \( j \in [1, k] \). By \( \Delta_n(\gamma) \) we denote the set of applicable concrete updates for structures over \( \gamma \) of size \( n \). For a sequence \( \alpha = \delta_1 \ldots \delta_k \in (\Delta_n(\gamma))^+ \) of updates we define \( \alpha(A) = \delta_k(\ldots(\delta_1(A))\ldots) \), where \( \delta(A) \) is the structure obtained from \( A \) by setting \( R(i_1, \ldots, i_k) \) to true if \( \delta = \text{ins}_R(i_1, \ldots, i_k) \), and setting \( R(i_1, \ldots, i_k) \) to false if \( \delta = \text{del}_R(i_1, \ldots, i_k) \).

\begin{definition}
Let \( \gamma \) be a vocabulary and \( F \) be a set of \( \gamma \)-structures. The dynamic problem \( \text{Dyn}(F) \) is the set of all pairs \( (n, \alpha) \), with \( n > 0 \) and \( \alpha \in (\Delta_n(\gamma))^+ \), such that \( \alpha(E_n(\gamma)) \in F \). We call \( F \) the underlying static problem of \( \text{Dyn}(F) \).
\end{definition}

We now explain how a dynamic program operates on a dynamic problem. For a program \( P \), there again is a program state \( S \) containing the current structure and auxiliary relations, one of which is ACC, which are updated according to the updates which occur and the update functions of \( P \). The state \( S \) is accepting if \( S \models \text{ACC} \). Then, let \( L(P) = \{ (n, \alpha) \mid \alpha \in (\Delta_n(\gamma))^+ \text{ and } \| S_n(\gamma) \| \text{ is accepting} \} \), where \( S_n(\gamma) \) denotes the structure \( E_n(\gamma) \) extended with empty auxiliary relations.

A program \( P \) accepts a problem \( F \) if \( L(P) = \text{Dyn}(F) \). If \( P \in C \), for some dynamic complexity class \( C \), we also write \( \text{Dyn}(F) \in C \).

\textbf{Incomparability of FO and DynPROP}. As we have seen in the previous sections, when restricted to monadic input schemas, DynPROP in a sense has the power of MSO. However, we show that if one binary relation is added, DynPROP cannot even capture first-order logic. This is also true if we allow the program to use precomputed functions from the set Succ.

Thus, we consider alternating graphs, encoded by the binary edge relation \( E \) and two unary relations \( A \) and \( B \) that form a decomposition of the set of nodes \( V \) into the set of existential and universal nodes, respectively. Given a node \( s \in V \), the set of all reachable nodes \( \text{Reach}(s) \) is defined as the smallest set satisfying

\[ A \sqsubseteq \text{Reach}(s) \]
— $s \in \text{Reach}(s)$;
— if $u \in A$ and there is a $v \in \text{Reach}(s)$ such that $(u, v) \in E$, then $u \in \text{Reach}(s)$; and
— if $u \in B$ and for all $v \in V$ with $(u, v) \in E$, we have $v \in \text{Reach}(s)$, then $u \in \text{Reach}(s)$.

We define ALT-REACH as the problem, given an alternating graph $G = (A, B, E)$ and two nodes $s$ and $t$, is $t \in \text{Reach}(s)$. Note that ALT-REACH is P-complete (see, for example, [7]).

**Proposition 6.2.** $\text{DYN}(\text{ALT-REACH}) \notin \text{DYNPROP}(\text{Rel}, \text{Rel})$

Before proving the proposition, we state a lemma that describes an important property of DYNPROP programs. An update sequence working on $k$-tuples $\alpha$ is a sequence of updates over the (abstract) universe $\{1, \ldots, k\}$. Given a $k$-tuple $\vec{i} = (i_1, \ldots, i_k)$, $\alpha(\vec{i})$ denotes the sequence of updates one obtains when applying the updates on the elements of the $k$-tuple $\vec{i}$. So, instead of using the (abstract) universe element $x$ the element $i_x$ should be used. For example, the update $\text{ins}_{R}(1, 4, 2)$ results in an update $\text{ins}_{R}(i_1, i_4, i_2)$.

**Lemma 1.** Let $\alpha$ be a sequence of updates working on $k$-tuples. Let $P$ be a DYNPROP($\text{Rel}, \text{Rel}$) program, $S$ a state of $P$ and consider two tuples of elements $i = (i_1, \ldots, i_k)$ and $j = (j_1, \ldots, j_k)$ of $S$ such that $(S, \vec{i}) = (S, \vec{j})$. Then, $(\alpha(i))(S) = (\alpha(j))(S)$, that is, the type of $i$ after applying $\alpha(i)$ and the type of $j$ after applying $\alpha(j)$ are the same. In particular, the value of the ACC-relation is the same in $\alpha(i)(S)$ and $\alpha(j)(S)$.

**Proof.** It suffices to consider one update operation $\delta$ working on $k$-tuples. Then the lemma follows by induction on the length of the update sequence $\alpha$. Let $i$ be the tuple of elements in $\{1, \ldots, k\}$ which form the parameters of $\delta$, and consider any (auxiliary) relation $R$ updated by the program $P$ on a tuple $\kappa$ of elements also from $\{1, \ldots, k\}$. Let $i(\vec{i}), j(\vec{j}), \kappa(\vec{i})$ and $\kappa(\vec{j})$ denote the corresponding tuples in state $S$. Then, the evaluation of the update formula for $R$ on $\kappa(\vec{i})$ after the operation $\delta$ with parameters $\iota(\vec{i})$ depends only on the type of the elements in $\kappa(\vec{i}) \cup \iota(\vec{i})$. The same holds for the tuples corresponding to $\vec{j}$. Since the types $(S, \vec{i})$ and $(S, \vec{j})$ are equal, the update formula evaluates to the same value.

**Proof of Proposition ??**. We first define a family of alternating graphs $G = \{G_m \mid m \in \mathbb{N}\}$. Every graph $G_m$ consists of the following set $V_m$ of nodes:

— two nodes $s$ and $t$;
— a set of $2m$ nodes $P = \{p_1, \ldots, p_{2m}\}$;
— for each subset $I$ of $P$ of size $m$ a node $q_I$, forming the set $Q$ (of size $2^m$); and
— for each subset $J$ of $Q$ a node $r_J$, forming the set $R$ (of size $2^{|Q|}$).

All nodes are existential nodes except the nodes in set $Q$, which are universal. Thus, $A_m = V_m \setminus Q$ and $B_m = Q$. Further, the following set of edges $E_m$ is already present in the graph $G_m$:

— for each subset $I$ of $P$ of size $m$, the set of edges $\{(q_I, p) \mid p \in I\}$ and
— for each subset $J$ of $Q$ the set of edges $\{(r_J, q) \mid q \in J\}$.

As updates we will only consider insertions of edges from $s$ to nodes in the set $R$ and from nodes in the set $P$ to $t$.

We show that no dynamic program can maintain auxiliary relations such that it can incrementally answer, for every $n \in \mathbb{N}$, the question whether $t$ is reachable from $s$ in the alternating graph $G_m$, starting from $G_m$ and arbitrary precomputation on the auxiliary relations. The proposition then follows.

We make use of the following two lemmas:

**Lemma 2.** For every $m$ and every pair of distinct nodes $r, r' \in R$ of $G_m$, there exists a set $I \subseteq P$ of size $m$ such that in the graph $G'_m := (A_m, B_m, E_m \cup \bigcup_{p \in I} \{p, t\})$, $t \in \text{Reach}(r)$ and $t \notin \text{Reach}(r')$. 

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Fig. 5. An abstract illustration of the graph $G_m$. Full edges represent edges already present in $G_m$; dashed ones will be inserted later.

**Proof.** Each of the nodes in $R$ corresponds to a (unique) subset of $Q$. Hence, by definition of $G_m$, there is a set $I$ and a node $q_I \in Q$ which in $G_m$ is reachable from $r$ but not from $r'$. We show that the set $I \subset P$ is the desired set, that is, for $G_m' := \langle A_m, B_m, E_m \cup \bigcup_{p \in I} (p, t) \rangle$, it holds that $t \in \text{Reach}(r)$ and $t \notin \text{Reach}(r')$. We note that in $G_m'$, $q_I$ is the only node in the set $Q$ such that $t \in \text{Reach}(q_I)$ because all nodes in $Q$ are universal nodes. But now, as $r$ and $r'$ are existential nodes, and $r$ is connected to $q_I$ but $r'$ is not, $t \in \text{Reach}(r)$ and $t \notin \text{Reach}(r')$. This concludes the proof. \hfill \Box

**Lemma 3.** The number of possible $k$-types of a structure with $x$ auxiliary relations of maximal arity $y$ is bounded by $2^{x \cdot ky}$.

**Proof.** A $k$-type is constructed from a set of atoms $R(\vec{j})$ (where each element in $\vec{j}$ is in $[1, k]$) by adding either $R(\vec{j})$ or $\neg R(\vec{j})$ to the $k$-type. Hence, there exist at most $2^{|\text{atoms}|}$ different $k$-types where $|\text{atoms}|$ denotes the number of different atoms. For one $y$-ary relation $R$ all atoms of $R$ can be seen as the set of all $y$-tuples of elements in $\vec{i}$. So, one relation of arity $y$ produces $k^y$ different atoms. As there are $x$ different relations, there are at most $x \cdot k^y$ atoms, and thus at most $2^{x \cdot k^y}$ different $k$-types. \hfill \Box

Now, assume, towards a contradiction, that there exists a dynamic program $P$ for $\text{Dyn}(\text{ALT-REACH})$ in $\text{DYNPROP}(\text{Rel}, \text{Rel})$ that makes use of $a$ auxiliary relations of maximal arity $b$. For a graph $G_m$ and node $r \in R$, we consider the tuple $V_r := (s, t, r, p_1, \ldots, p_{2m})$.

Since

$$|Q| = \binom{2m}{m} \geq \prod_{i=0}^{m-1} \frac{2m-i}{m-i} = 2^2m^m$$

there exists a number $m$ such that $|R|$ is bigger than the number of $(2m+3)$-types in any state $S$ of $P$. Indeed, by Lemma ?? and since the program can use $a + 6$ relations (the auxiliary relations, the input relations $E$, $A$, and $B$, and the equality, order and ACC relations) of maximal arity $b$, the number of $(2m+3)$-types in $S$ is bounded by $2^{(a+6)(2m+3)^b}$. For a large enough value of $m$, this is clearly dominated by $2^{2^m}$. Fix such an $m$ and corresponding graph $G_m$, and let $S$ be a state $P$ is in when the current graph is $G_m$. Then, by the above reasoning, in the set $R$ (of $G_m$) there must exist two distinct elements $r$ and $r'$ such that $\langle S, V_r \rangle = \langle S, V_{r'} \rangle$.

According to Lemma ??, there is a set $I$ of $m$ elements in $P$ such that after the insertion of all edges $\{(p, t) \mid p \in I\}$ in $G_m$, $t \in \text{Reach}(r)$ and $t \notin \text{Reach}(r')$. Let $I = \{p_{i_1}, \ldots, p_{i_m}\}$ and consider the two sequences of update operations

$$\alpha = (\text{ins}_E(p_{i_1}, t), \ldots, \text{ins}_E(p_{i_m}, t), \text{ins}_E(s, r))$$

$$\alpha' = (\text{ins}_E(p_{i_1}, t), \ldots, \text{ins}_E(p_{i_m}, t), \text{ins}_E(s, r')).$$

Notice that $\alpha(G_m)$ yields a graph in which $t \in \text{Reach}(s)$, whereas $t \notin \text{Reach}(s)$ in $\alpha'(G_m)$. However, as $\langle S, V_r \rangle = \langle S, V_{r'} \rangle$, it follows from Lemma ?? that also $\langle \alpha(S), V_r \rangle = \langle \alpha'(S), V_{r'} \rangle$. Hence, $P$ will either in both cases claim that $t \in \text{Reach}(s)$ (if ACC holds in $\langle \alpha(S), V_r \rangle$) or claim in both cases that $t \notin \text{Reach}(s)$. We can conclude that there does not exist a $\text{DYNPROP}(\text{Rel}, \text{Rel})$ program for $\text{ALT-REACH}$. \hfill \Box
This proof can be adapted to show that even with a precomputed successor-function, one cannot maintain the reachability problem for alternating graphs.

**Proposition 6.3.** \( \text{Dyn(ALT-REACH)} \notin \text{DYNPROP}(\text{Succ, Rel}) \)

In order to prove this we need an observation similar to Lemma 2 for DYNPROP(\text{Succ, Rel}). For an element \( i \) and a number \( l \), let the \( l \)-neighborhood of \( i \), denoted \( N_{l}(i) \), be the following tuple of elements:

\[
(\text{pre}^l(i), \text{pre}^{l-1}(i), ..., \text{pre}(i), i, \text{succ}(i), ..., \text{succ}^{l-1}i, \text{succ}^l(i)).
\]

For a tuple of elements \( \vec{i} \), we denote by \( N_{l}(\vec{i}) \) the tuple \((\mathcal{N}_{l}(\text{min}), \mathcal{N}_{l}(i_1), ..., \mathcal{N}_{l}(i_k))\).

**Lemma 4.** Let \( \alpha \) be a sequence of updates working on \( k \)-tuples which consists of \( l \) updates. For each DYNPROP(\text{Succ, Rel}) program \( P \), there exists a number \( c \), depending only on \( P \), such that the following holds: Let \( S \) be a state of \( P \), and \( \vec{i} = (i_1, ..., i_k) \) and \( \vec{j} = (j_1, ..., j_k) \) tuples of elements of \( S \) such that \( \langle S, N_{c,l}(\vec{i}) \rangle = \langle S, N_{c,l}(\vec{j}) \rangle \). Then, \( \langle \alpha(\vec{i})(S), \vec{i} \rangle = \langle \alpha(\vec{j})(S), \vec{j} \rangle \).

**Proof.** The \( c \) of the lemma is the maximal nesting depth of the functions succ and pre used in \( P \). For example the term succ(succ(pre(x))) has nesting depth 3. Let \( \alpha_n \) be the prefix of length \( n \) of the update sequence \( \alpha \). We will here prove the slightly stronger statement that, assuming the conditions of the lemma, \( \langle \alpha_n(\vec{i})(S), N_{c,(l-n)}(\vec{i}) \rangle = \langle \alpha_n(\vec{j})(S), N_{c,(l-n)}(\vec{j}) \rangle \). Then the lemma follows because \( N_{0}(\vec{i}) = (\text{min}, \vec{i}) \). The proof works by induction on \( n \) (assuming \( n < l \)). For \( n = 0 \) the statement is contained in the condition of the lemma. So, assume the statement holds for \( n < l \), we show that it still holds for \( n + 1 \). Let \( \delta \) be the update such that \( \alpha_{n+1} = \alpha_n \cdot \delta \). Just as in the proof of Lemma 2 consider any (auxiliary) relation \( R \) updated by the program \( P \) on elements in \( N_{c,(l-(n+1))}(\vec{i}) \). The evaluation of the update formula for \( R \) depends only on the type of \( N_{c,(l-(n+1))}(\vec{i}) \). This is true because one can reach other elements in the universe only by using the functions. Since these are nested at most \( c \) times, from any element in \( N_{c,(l-(n+1))}(\vec{i}) \) only elements in \( N_{c,(l-n)}(\vec{i}) \) can be reached. The same holds for the tuples corresponding to \( \vec{j} \). As \( \langle \alpha_n(\vec{i})(S), N_{c,(l-n)}(\vec{i}) \rangle = \langle \alpha_n(\vec{j})(S), N_{c,(l-n)}(\vec{j}) \rangle \), and \( \alpha_{n+1} = \alpha_n \cdot \delta \), it follows that \( \langle \alpha_{n+1}(\vec{i})(S), N_{c,(l-(n+1))}(\vec{i}) \rangle = \langle \alpha_{n+1}(\vec{j})(S), N_{c,(l-(n+1))}(\vec{j}) \rangle \). □

**Proof of Proposition 6.3.** The proof is along the same lines as that of Theorem 6.2. Assume that there exists a DYNPROP(\text{Succ, Rel}) program \( P \) for Dyn(ALT-REACH) making use of \( a \) auxiliary relations of maximal arity \( b \). We will again consider (in a graph \( G_m \)) the tuples \( V_r \) and \( V_r' \), but now their corresponding \( (m+1) \)-neighborhoods \( N_{c(m+1)}(V_r) \) and \( N_{c(m+1)}(V_r') \), where \( c \) is the constant of Lemma 2 which only depends on \( P \). Using Lemma 2, we know that in any state \( S \) of \( P \) the number of types of these neighborhoods is bounded by \( 2^{(a+6) \cdot ((2(m+1)+1)(2(m+3)+1))^{k}} \). Hence we can again find a number \( m \) big enough such that there are distinct \( r, r' \) in \( G_m \) such that \( \langle S, N_{c(m+1)}(V_r) \rangle \) and \( \langle S, N_{c(m+1)}(V_{r'}) \rangle \). Using the same argument as above and Lemma 2, we get the desired contradiction. □

**Remark 6.4.** The proofs of the foregoing lemma and proposition depend heavily on the fact that, with each update operation, the neighborhood of a tuple increases only by a constant term. This is because the functions pre and succ are complementary in the sense that \( \text{pre}(\text{succ}) = \text{succ}(\text{pre}) \). So, the order of their usage is not important. If one allows two independent functions (for example, two different successor-functions on the universe) the size of the neighborhood possibly doubles after each operation so the proof of the proposition (based on a counting argument) does not work.

From the proof of the above proposition one can conclude an even stronger statement. The graphs used in the proof are very restricted in the sense that the length of the longest path is bounded by a constant. Let ALT-REACH\(_{\text{depth} \leq d} \) be the alternating reachability problem on graphs of depth at most \( d \). It is easily seen that ALT-REACH\(_{\text{depth} \leq d} \) is expressible by a FO-formula, so we get the following

**Theorem 6.5.** There exists a problem \( F \in \text{FO} \) such that \( \text{Dyn}(F) \notin \text{DYNPROP}(\text{Succ, Rel}) \). □
On the other hand, the reachability problem on acyclic deterministic directed graphs can be maintained in DynPROP [7] but cannot be expressed in FO. So these classes are incomparable.

**Using functions to maintain EFO.** Next, we exhibit a class of properties which can be maintained in DynQF with precomputation. An existential first-order (EFO) sentence is a first-order sentence of the form $\exists x_1 \cdots \exists x_k \phi(\vec{x})$, where $\phi(\vec{x})$ is a quantifier free formula.

**Theorem 6.6.** For any EFO-definable problem $F$, $\text{Dyn}(F) \in \text{DynPROP}(\text{Fun, Fun})$

**Proof.** Let $\psi = \exists x_1 \cdots \exists x_k \phi(\vec{x})$ be an EFO-sentence over vocabulary $\gamma$. We show that there exists a DynPROP(\text{Fun, Fun}) program $P$ which maintains whether $A \models \psi$, for any $\gamma$-structure $A$.

We first introduce some notation. A tuple $\vec{t} = (i_1, \ldots, i_l)$ is disjoint if $i_j \neq i_k$, for all $j, k \in [1, \ell]$, with $j \neq k$. A disjoint type is the type of a disjoint tuple. For a type $\tau$, let $\phi_{\tau}$ be an EFO sentence which is satisfied in a structure $A$ if and only if $A$ contains a tuple $\vec{x}$ such that $\langle A, \vec{x} \rangle = \tau$.

It is well known and easy to see that for any EFO sentence $\phi = \exists x_1 \cdots \exists x_k \psi(\vec{x})$, there exists a set $\theta_\psi$ of disjoint $\ell$-types, with $\ell$ ranging from 1 to $k$, such that $\psi$ is equivalent to $\bigvee_{\tau \in \theta} \phi_{\tau}$. Note that if we would not require the types to be disjoint, we would only need to consider $k$-types, and not $\ell$-types, for all $\ell \leq k$. However, the latter restriction, and corresponding extension, will prove technically more convenient.

Using the information that $A \models \phi$ is completely determined by the set of types $\theta_\phi$, realized in $A$, we now present our dynamic algorithm. It will maintain the following functions. For every disjoint $\ell$-type $\tau$, with $\ell \leq k$, and set $I = \{i_1, \ldots, i_{|I|}\} \subseteq \{1, \ldots, \ell\}$, let

$$f^I_{\tau}(x_1, \ldots, x_{|I|}) = |\{(a_1, \ldots, a_{|I|}) \mid \langle A, \vec{a}\rangle = \tau \land \forall j \in [1, |I|] : a_j = x_j\}|$$

Here, we write $I = \{i_1, \ldots, i_{|I|}\}$ such that $i_j < i_{j+1}$, for all $j \in [1, |I| - 1]$. Then, for $I = \emptyset$, $f^\emptyset$ defines the number of disjoint tuples in $A$ which have type $\tau$. When $I = \{i_1, \ldots, i_{|I|}\} \neq \emptyset$, and given $\vec{x} = (x_1, \ldots, x_{|I|})$, $f^I_{\tau}(\vec{x})$ defines the number of tuples in $A$ which (1) have type $\tau$ and (2) have at position $i_j$ exactly element $x_j$, for all $j \in [1, |I|]$.

Notice that the numbers defined by the above functions can become bigger than $n$, the number of universe elements, but are always smaller than $n^k$. Hence, every such number can be encoded as a number with $k$ digits in base $n$, which is exactly how our functions will encode these numbers. Thus, for every function $f^I_{\tau}$ mentioned above, there are actually $k$ functions $f^{\ell,1}_{\tau}, \ldots, f^{\ell,k}_{\tau}$, each defining one digit of the desired number defined by $f^I_{\tau}$. For clarity, we use the functions $f^I_{\tau}$ instead of the actual ones encoding their digits.

As we are in the setting where precomputation is allowed, we can assume that the functions are properly initialized. For any $l \in [1, k]$, let $\tau_{\vec{x}}$ be the unique $l$-type containing only negated atoms, that is, atoms of the form $\neg R(i)$. Then, for all $l$-types $\tau \neq \tau_{\vec{x}}$, set $I$, and tuple $\vec{x}$, initially

$$f^I_{\tau}(\vec{x}) = 0,$$

and for $\vec{x} = (x_1, \ldots, x_{|I|})$ it holds that

$$f^I_{\tau}(\vec{x}) = 0 \text{ if } x_i = x_j, \text{ for some } i \neq j,$$

and

$$f^I_{\tau}(\vec{x}) = (n - |I|) \cdot (n - (|I| + 1)) \cdots (n - l), \text{ otherwise.}$$

We now show how to incrementally maintain these functions. To this end, we give the precomputed functions and relations which will be used for the updates. For simplicity, we assume the universe of size $n$ consists of the elements $\{0, \ldots, n - 1\}$. Then, there is a constant (0-ary function) $\text{min}$ denoting 0, functions plus and minus such that $\text{plus}(x, y) = x + y \pmod n$ and $\text{minus}(x, y) = x - y \pmod n$, and accompanying relations $R_{\text{plus}}$ and $R_{\text{minus}}$ such that $R_{\text{plus}}(x, y)$ holds if and only if $x + y \geq n$, and $R_{\text{minus}}(x, y)$ holds if and only if $x - y < 0$. That is, the functions plus and minus are defined on all parameters and count modulo $n$. The accompanying relations $R_{\text{plus}}$ and $R_{\text{minus}}$ contain the additional information saying whether the addition or subtraction indeed
went above \( n - 1 \) or below \( 0 \). These functions allow to define addition and subtraction on the \( k \)-digit base-\( n \) numbers used in the functions. Therefore, we simply perform addition and subtraction on these numbers in the sequel.

Second, we introduce some additional notation. As before, we write \( \vec{x} \) for a tuple of elements, but abuse notation and also denote the set of elements in \( \vec{x} \) by \( \vec{x} \). and, correspondingly, apply set-theoretic operations on them, for instance, \( \vec{x} \cup \vec{y} \).

Further, for an integer \( \ell \), set \( I = \{i_1, \ldots, i_{|I|}\} \subseteq \{1, \ldots, \ell\} \), and tuples \( \vec{x} = (x_1, \ldots, x_{|I|}) \) and \( \vec{y} \), we let an indexing for \( \ell, I, \vec{x}, \vec{y} \) be a function \( \text{ind} : \vec{x} \cup \vec{y} \rightarrow \{1, \ldots, \ell\} \) such that for all \( j \in [1, |I|] \), \( \text{ind}(x_j) = i_j \). The indexing \( \text{ind} \) is proper if for all \( z, z' \in \vec{x} \cup \vec{y} \), \( \text{ind}(z) = \text{ind}(z') \) if and only if \( z = z' \). Hence, a proper indexing \( \text{ind} \) associates elements of \( \vec{x} \) to their corresponding elements in \( I \), and associates elements of \( \vec{y} \) to elements of \( \{1, \ldots, \ell\} \) such that elements have the same index if and only if they are equal. Notice that while the fact whether a function \( \text{ind} \) is an indexing only depends on \( I \) and \( \ell \), whether it is proper depends on the actual values of \( \vec{x} \) and \( \vec{y} \). However, this can easily be tested by the following formula:

\[
\phi_{\text{ind}}(\vec{y}, \vec{x}) = \bigwedge_{z, z' \in \vec{x} \cup \vec{y}} z = z' \land \bigwedge_{z, z' \in \vec{x} \cup \vec{y}} z \neq z' \land \text{ind}(z) = \text{ind}(z')
\]

Given \( \vec{x} \) and \( \vec{y} \) and a proper indexing \( \text{ind} \), we write \( (\vec{x}, \vec{y})_{\text{ind}} \) for the sequence \((u_1, \ldots, u_m)\), for some \( m \), such that (1) \( u \) contains every element in \( \vec{x} \cup \vec{y} \) exactly once and (2) \( \text{ind}(u_i) < \text{ind}(u_{i+1}) \), for all \( i \in [1, m - 1] \). Hence, \( u \) is obtained from \( \vec{x} \cup \vec{y} \) by eliminating elements which are equal (and thus have the same index), and ordering the elements by their index. Further, we write \( \text{ind}(\vec{y}) \) to denote the type \( \text{ind}(y_1), \ldots, \text{ind}(y_m) \). Finally, for a type \( \tau \), and \( \text{R}(\vec{i}) \notin \tau \), let \( \tau + \text{R}(\vec{i}) \) denote the type obtained from \( \tau \) by removing \( \text{R}(\vec{i}) \) and adding \( \text{R}(\vec{i}) \). When \( \text{R}(\vec{i}) \notin \tau \), \( \tau + \text{R}(\vec{i}) \) is defined similarly by removing \( \text{R}(\vec{i}) \) and adding \( \text{R}(\vec{i}) \).

We are now ready to give the update functions. For clarity, we write the \( \text{tuple}(\phi, t_1, t_2) \) construct as “if \( \phi \) then \( t_1 \) else \( t_2 \)”. Then, for relation symbol \( R \), \( \ell \)-type \( \tau \), with \( \ell \leq k \), and \( I \subseteq \{1, \ldots, \ell\} \), let

\[
\phi_{\text{ins}_R}(\vec{y}, \vec{x}) = f_{\tau}(\vec{x}) + \sum_{\text{ind} \in \{1, \ldots, \ell\}, \vec{x}, \vec{y}} \text{if } \phi_{\text{ind}}(\vec{y}, \vec{x}) \text{ then } f_{\tau \leftarrow \text{R}(\text{ind}(\vec{y}))}(\vec{x}, \vec{y})_{\text{ind}} \text{ else } 0
\]

and, similarly,

\[
\phi_{\text{del}_R}(\vec{y}, \vec{x}) = f_{\tau}(\vec{x}) + \sum_{\text{ind} \in \{1, \ldots, \ell\}, \vec{x}, \vec{y}} \text{if } \phi_{\text{ind}}(\vec{y}, \vec{x}) \text{ then } f_{\tau \leftarrow \text{R}(\text{ind}(\vec{y}))}(\vec{x}, \vec{y})_{\text{ind}} \text{ else } 0
\]

Intuitively, both formulas compute the number of tuples with the given type \( \tau \) in the same manner: Take the number of tuples which used to have type \( \tau \), add those which obtained type \( \tau \) by the update, and remove the ones which had type \( \tau \), but do not anymore.
We briefly explain the correctness of these formulas by arguing that after an update \( \text{ins}_R(\vec{y}) \) for a tuple \( \vec{x} \) the number of tuples which did not have type \( \tau \) but do after the update is indeed equal to the number computed on the second line of the update formula \( \phi^f_{\text{ins}_R}(\vec{y}; \vec{x}) \).

Let \( \vec{a} = (a_1, \ldots, a_l) \) be a disjoint tuple consistent with \( \vec{x} \) and \( I \), that is, for all \( j \in [1, |I|] \), \( x_j = a_{i_j} \). We denote the structure obtained from \( A \) after the update \( \text{ins}_R(\vec{y}) \) by \( A' \). Now, suppose \( (A, \vec{a}) \neq \tau \), but \( (A', \vec{a}) = \tau \). This can only hold if \( \vec{y} \subseteq \vec{a} \) and thereby the insertion of \( R(\vec{y}) \) has changed the type of \( \vec{a} \) in \( A \). More precisely, if we define \( \vec{k} = k_1, \ldots, k_m \) such that for all \( j \in [1, m] \), \( y_j = a_{k_j} \), then \( (A, \vec{a}) = \tau = R(\vec{k}) \) must hold. Notice also that \( \vec{k} \) is uniquely defined because \( \vec{a} \) is disjoint. Now, \( \vec{k} \), in turn, defines a proper indexing \( \text{ind} \) on \( \vec{x} \) and \( \vec{y} \) as follows: for all \( j \in [1, |I|] \), \( \text{ind}(x_j) = i_j \) (by definition) and for all \( j \in [1, m] \), \( \text{ind}(y_j) = k_j \). In this manner we can thus associate a unique proper indexing to all tuples \( \vec{a} \) which did not have type \( \tau \), but do now. Then, for any indexing \( \text{ind} \), the expression \( \bigvee_{\tau \in \theta_\phi} f^f_{\text{ins}_R}(\vec{y}) \) defines exactly all such tuples with which \( \text{ind} \) is associated. By iterating over all proper indexings we hence count exactly all desired tuples.

Finally, for the acceptance relation we have to check whether there is a tuple in the new structure which has a type contained in \( \theta_\phi \):

\[
\phi^{\text{ACC}}_{\text{ins}_R}(\vec{y}) \equiv \bigvee_{\tau \in \theta_\phi} \phi^f_{\text{ins}_R}(\vec{y}) \neq 0 \quad \text{and} \quad \phi^{\text{ACC}}_{\text{del}_R}(\vec{y}) \equiv \bigvee_{\tau \in \theta_\phi} \phi^f_{\text{del}_R}(\vec{y}) \neq 0.
\]

7. CONCLUSION

We have studied the dynamic complexity of formal languages and, by characterizing the languages maintainable in DynPROP as exactly the regular languages, obtained the first lower bounds for DynPROP. This yields a separation of DynPROP from DynQF and DynFO. We proved that every context-free language can be maintained in DynFO and investigated the power of functions for dynamic programs in maintaining specific context-free and non context-free languages.

As a modest extension we also proved a lower bound for DynPROP with built-in successor functions. Hence, we are now one step closer to proving lower bounds for DynFO, but, of course, a number of questions arise:

—Can the results on the Dyck languages be extended to show that an entire subclass of the context-free languages, such as the deterministic or unambiguous context-free languages, can be maintained in DynQF?

—We showed that \( D_1 \in \text{DynPROP}(\text{Succ, Rel}) \). Can it be shown that \( D_2 \notin \text{DynPROP}(\text{Succ, Rel}) \)?

—Can some of the lower bound techniques for DynPROP be extended to DynQF, in order to separate DynQF from DynFO, or at least from DynP? Is there a context-free language that is not maintainable in DynQF?

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