SOME EXTENSIONS OF CLASSES INVOLVING PAIR OF WEIGHTS RELATED TO THE BOUNDEDNESS OF MULTILINEAR COMMUTATORS ASSOCIATED TO GENERALIZED FRACTIONAL INTEGRAL OPERATORS

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Abstract. We deal with the boundedness properties of higher order commutators related to some generalizations of the multilinear fractional integral operator of order $m$, $I^m$, from a product of weighted Lebesgue spaces into adequate weighted Lipschitz spaces, extending some previous estimates for the linear case. Our study includes two different types of commutators and sufficient conditions on the weights in order to guarantee the continuity properties described above. We also exhibit the optimal range of the parameters involved. The optimality is understood in the sense that the parameters defining the corresponding spaces belong to a certain region, being the weights trivial outside of it. We further show examples of weights for the class which cover the mentioned area.

1. Introduction

Many classical operators in Harmonic Analysis whose continuity properties were extensively studied, have shown to behave in a suitable way when their multilinear versions act in the corresponding multilinear spaces. For example, in [6] the authors proved that both, the multilinear Calderón-Zygmund operators and their commutators with BMO symbols, are bounded from a product of weighted Lebesgue spaces to an associated weighted Lebesgue space, with weights belonging to the $A_{\vec{p}}$ multilinear class. This article established the starting point of the weighted theory in this general context. Motivated by the result in [6], similar estimates for multilinear fractional integral operators were obtained in [7], and in [4] for their commutators with BMO symbols. The weights involved in both articles belong to a class that generalizes those given in [6] and in [8] for the linear case.

Regarding the two-weighted theory, also in [7] K. Moen gave a complete discussion showing sufficient bump conditions in order to guarantee the continuity properties of the multilinear fractional integral operator acting between a wider class of Lebesgue spaces than those obtained in the one-weight theory. The obtained results generalize the linear version given, for example, in [17] and [10].

In [2] the authors studied the two-weighted boundedness properties of the multilinear fractional integral operator between a product of weighted Lebesgue spaces into appropriate weighted Lipschitz spaces associated to a parameter $\delta$. They characterized the classes of weights related to this problem, showing also the optimal range of the parameters involved. The optimality is understood in the sense that the parameters defining the corresponding spaces belong to a

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certain region, being the weights trivial outside of it. These results extend the corresponding
proved in [14] for the linear case.

Our main interest in this article is the study of the boundedness properties for higher order
commutators related to some generalizations of the multilinear fractional integral operator of
order \( m \), \( I^m_\alpha \), defined by

\[
I^m_\alpha \vec{f}(x) = \int_{(\mathbb{R}^n)^m} \frac{\prod_{i=1}^{m} f_i(y_i)}{(\sum_{i=1}^{m} |x - y_i|)^{m-\alpha}} \, d\vec{y},
\]

where \( 0 < \alpha < mn \), \( \vec{f} = (f_1, f_2, \ldots, f_m) \) and \( \vec{y} = (y_1, y_2, \ldots, y_m) \). These operators are given by

\[
T^m_\alpha \vec{f}(x) = \int_{(\mathbb{R}^n)^m} K_\alpha(x, \vec{y}) \prod_{i=1}^{m} f_i(y_i) \, d\vec{y},
\]

where \( K_\alpha \) satisfies certain size and regularity conditions (see Section 2). Particularly, the
size condition allows us to conclude that \( |T^m_\alpha \vec{f}| \leq C T^m_\alpha \vec{g} \), where \( \vec{g} = (|f_1|, \ldots, |f_m|) \). This estimation guarantees the boundedness of \( T^m_\alpha \) with different weights, between a product of
Lebesgue spaces into a related Lebesgue space when the class of multilinear weights involved is
an extension of that given in [7] for the continuity properties of \( I^m_\alpha \). Nevertheless, this argument
cannot be applied for other spaces such that \( \text{BMO} \), Lipschitz or Morrey because of the fact that
they have not a growth property such as Lebesgue spaces. So, we shall focus our attention
in studying the boundedness properties of the commutators of \( T^m_\alpha \) acting between a product of
weighted Lebesgue spaces into certain weighted versions of the aforementioned spaces. In
the linear case this problem was studied in [15] for higher order commutators including the
case of the operator \( I_\alpha \), which had been previously given in [14] for the two-weighted setting
in the linear context. For similar problems involving other weighted type of Lipschitz spaces
see [5], [13] and [16]. On the other hand, in the multilinear case and for \( T^m_\alpha = I^m_\alpha \) (that is
\( K_\alpha(x, \vec{y}) = (\sum_{i=1}^{m} |x - y_i|)^{\alpha - mn} \)), generalizations of two-weighted problems can be found in [2]
and [3] (see also [1] for the unweighted problem).

In this paper we study the boundedness of commutators of fractional operators, including \( I^m_\alpha \),
between a product of weighted Lebesgue spaces and weighted generalizations of those introduced
in [9]. For a weight \( w \), the latter are denoted by \( \mathbb{L}_w(\delta) \) and defined as the collection of locally
integrable functions \( f \) for which the inequality

\[
\frac{\|wX_B\|_{\infty}}{|B|^{1+\delta/n}} \int_B |f(x) - f_B| \, dx \leq C
\]

holds for every ball \( B \subset \mathbb{R}^n \), where \( f_B = |B|^{-1} \int_B f \).

We shall consider two different types of higher order commutators of \( T^m_\alpha \). Given an \( m \)-tupel
of functions \( \vec{b} = (b_1, \ldots, b_m) \), where each component belongs to \( L^1_{\text{loc}} \), the first multilinear
commutator we shall be dealing with, \( T^m_{\alpha, \vec{b}} \), is given by the expression

\[
T^m_{\alpha, \vec{b}} \vec{f}(x) = \sum_{j=1}^{m} T^m_{\alpha, b_j} \vec{f}(x),
\]

where \( T^m_{\alpha, b_j} \) is formally defined by

\[
T^m_{\alpha, b_j} \vec{f}(x) = \int_{(\mathbb{R}^n)^m} (b_j(x) - b_j(y_j))K_\alpha(x, \vec{y}) \prod_{i=1}^{m} f_i(y_i) \, d\vec{y}.
\]
On the other hand, the second type of commutator that we shall consider can be expressed as

$$
T_{\alpha,b}^m \tilde{f}(x) = \int_{\mathbb{R}^n} K_\alpha(x,y) \prod_{i=1}^m (b_i(x) - b_i(y)) f_i(y_i) \, dy.
$$

The last two equalities above are a consequence of their definitions given in Section 2, and both commutators were introduced in [12] and [11], respectively.

We shall also be dealing with multilinear symbols with components belonging to the classical Lipschitz spaces $\Lambda(\delta)$ (for more information see Section 2).

Let $\vec{p} = (p_1, p_2, \ldots, p_m)$ be a vector of exponents such that $1 \leq p_i \leq \infty$ for every $i$. Let $\beta$, $\delta$ and $\delta$ be real constants. Given $w$, $\vec{v} = (v_1, v_2, \ldots, v_m)$ and $\vec{p}$, we say that $(w, \vec{v}) \in H_m(\vec{p}, \beta, \delta)$ if there exists a positive constant $C$ such that the inequality

$$
\frac{\|w x_B\|_{L^\infty}}{|B|^{(\beta-\delta)/n}} \prod_{i=1}^m \left( \int_{\mathbb{R}^n} \frac{v_i^{-\beta_i}}{|B|^{1/n} + |x_B - y|^{(n-\beta_i+\delta/m)p_i'}} \, dy \right)^{1/p_i'} \leq C
$$

holds for every ball $B = B(x_B, R)$, with the obvious changes when $p_i = 1$. The numbers $\beta_i$ satisfy $\sum_{i=1}^m \beta_i = \beta$ and also $0 < \beta_i < n$, for every $i$ (see Section 2 for further details related to these classes of weights).

We shall now state our main results. From now on, $1/p = \sum_{i=1}^m 1/p_i$. See Section 2 for details.

**Theorem 1.1.** Let $0 < \alpha < mn$ and $T_{\alpha,b}^m$ be a multilinear fractional operator with kernel $K_\alpha$ satisfying (2.2) and (2.3). Let $0 < \delta < \min\{\gamma, mn - \alpha\}$, $\alpha = \alpha + \delta$ and $\vec{p}$ a vector of exponents that satisfies $p > n/\alpha$. Let $b = (b_1, \ldots, b_m)$ be a vector of symbols such that $b_i \in \Lambda(\delta)$, for $1 \leq i \leq m$. Let $\delta \leq \delta$ and $(w, \vec{v})$ be a pair of weights belonging to the class $H_m(\vec{p}, \alpha, \delta)$ such that $v_i^{-\beta_i} \in RH_m$, for every $i$ such that $1 < p_i \leq \infty$. Then the multilinear commutator $T_{\alpha,b}^m$ is bounded from $\prod_{i=1}^m L^{p_i}(v_i^{\beta_i})$ to $L_w(\delta)$, that is, there exists a positive constant $C$ such that the inequality

$$
\frac{\|w x_B\|_{L^\infty}}{|B|^{1+\delta/n}} \int_B |T_{\alpha,b}^m \tilde{f}(x) - (T_{\alpha,b}^m \tilde{f})_B| \, dx \leq C \prod_{i=1}^m \|f_i v_i\|_{L^{p_i}}
$$

holds for every ball $B$ and every $\tilde{f}$ such that $f_i v_i \in L^{p_i}$, for $1 \leq i \leq m$.

Concerning the commutator $T_{\alpha,b}^m$ we have the following result.

**Theorem 1.2.** Let $0 < \alpha < mn$ and $T_{\alpha,b}^m$ be a multilinear fractional operator with kernel $K_\alpha$ satisfying (2.2) and (2.3). Let $0 < \delta < \min\{\gamma, (mn - \alpha)/m\}$, $\alpha = \alpha + m\delta$ and $\vec{p}$ a vector of exponents that satisfies $p > n/\alpha$. Let $b = (b_1, \ldots, b_m)$ be a vector of symbols such that $b_i \in \Lambda(\delta)$, for $1 \leq i \leq m$. Let $\delta \leq \delta$ and $(w, \vec{v})$ be a pair of weights belonging to the class $H_m(\vec{p}, \alpha, \delta)$ such that $v_i^{-\beta_i} \in RH_m$, for every $1 < p_i \leq \infty$. Then the multilinear commutator $T_{\alpha,b}^m$ is bounded from $\prod_{i=1}^m L^{p_i}(v_i^{\beta_i})$ to $L_w(\delta)$, that is, there exists a positive constant $C$ such that the inequality

$$
\frac{\|w x_B\|_{L^\infty}}{|B|^{1+\delta/n}} \int_B |T_{\alpha,b}^m \tilde{f}(x) - (T_{\alpha,b}^m \tilde{f})_B| \, dx \leq C \prod_{i=1}^m \|f_i v_i\|_{L^{p_i}}
$$

holds for every ball $B$ and every $\tilde{f}$ such that $f_i v_i \in L^{p_i}$, for $1 \leq i \leq m$.

Note that the restriction on the parameter $\delta$ implicated is different in each theorem due to the nature of the considered commutators.
When we deal with $w = \prod_{i=1}^{m} v_i$, which is the natural substitute of the one weight theory in the linear case, we shall say that $\vec{v} \in \mathbb{H}_m(\vec{\beta}, \vec{\delta})$. Then we have the following lemma.

**Lemma 1.3.** Let $0 < \beta < mn$, $\vec{\delta} \in \mathbb{R}$, and $\vec{\beta}$ a vector of exponents. If $\vec{v} \in \mathbb{H}_m(\vec{\beta}, \vec{\delta})$, then $\delta = \beta - n/p$.

The proof of this result follows similar arguments as those of Theorem 1.3 in [2] and we shall omit it. As a consequence of this lemma we can prove that if $\tilde{\delta} < \tau = (\beta - mn)(1 - 1/m) + \delta/m$, then condition (2.5) can be reduced to the class $A_{\vec{\beta}, \infty}$, defined as the collection of multilinear weights $\vec{v} = (v_1, \ldots, v_m)$ for which the inequality

$$[\vec{v}]_{A_{\vec{\beta}, \infty}} = \sup_{B \subset \mathbb{R}^n} \left\| \chi_B \prod_{i=1}^{m} v_i \right\|_{\infty} \prod_{i=1}^{m} \left( \frac{1}{|B|} \int_B v_i^{-p_i'} \right)^{1/p_i'} < \infty$$

holds (see Corollary 6.4).

The paper is organized as follows. In Section 2 we give the main definitions required in the sequel. In Section 3 we state and prove some auxiliary results that will be useful for the proof of the main theorems, which are contained in Section 4 and 5. Finally in Section 6 we prove some properties of the class $\mathbb{H}_m(\vec{\beta}, \vec{\delta})$ and show the optimality of the associated parameters.

### 2. Preliminaries and definitions

Throughout the paper $C$ will denote an absolute constant that may change in every occurrence. By $A \lesssim B$ we mean that there exists a positive constant $c$ such that $A \leq cB$. We say that $A \approx B$ when $A \lesssim B$ and $B \lesssim A$.

By $m \in \mathbb{N}$ we denote the multilinear parameter involved in our estimates. Given a set $E$, with $E^m$ we shall denote the cartesian product of $E$ $m$ times. We shall be dealing with operators given by the expression

$$T_{\alpha}^m f(x) = \int_{(\mathbb{R}^n)^m} K_{\alpha}(x, \vec{y}) \prod_{i=1}^{m} f_i(y_i) d\vec{y},$$

for $0 < \alpha < mn$, where $\vec{f} = (f_1, \ldots, f_m)$, $\vec{y} = (y_1, \ldots, y_m)$ and $K_{\alpha}$ is a kernel satisfying the size condition

$$|K_{\alpha}(x, \vec{y})| \lesssim \frac{1}{\left( \sum_{i=1}^{m} |x - y_i| \right)^{mn-\alpha}}$$

and an additional smoothness condition given by

$$|K_{\alpha}(x, \vec{y}) - K_{\alpha}(x', \vec{y})| \lesssim \frac{|x - x'|^{\gamma}}{\left( \sum_{i=1}^{m} |x - y_i| \right)^{mn-\alpha+\gamma}},$$

for some $0 < \gamma \leq 1$, whenever $\sum_{i=1}^{m} |x - y_i| > 2|x - x'|$. It is easy to check that $T_{\alpha}^m = T_{\alpha}^m$ defined above, when we consider $K_{\alpha}(x, \vec{y}) = (\sum_{i=1}^{m} |x - y_i|)^{\alpha-mn}$.

We shall introduce two versions of commutators of the operators above. For a specified linear operator $T$ and a function $b \in L^1_{\text{loc}}$ we recall that the classical commutator $T_b$ or $[b, T]$ is given by the expression

$$[b, T]f = bf - T(f).$$
When we deal with multilinear functions and symbols, it will be necessary to emphasize how we proceed to perform the commutation. If \( b \in L^1_{\text{loc}}, T \) is a multilinear operator and \( \vec{f} = (f_1, f_2, \ldots, f_m) \) we write
\[
[b, T]_j(\vec{f}) = bT(\vec{f}) - T((f_1, \ldots, b_{fj}, \ldots, f_m)),
\]
that is, \([b, T]_j\) is obtained by commuting \( b \) with the \( j \)-th entry of \( \vec{f} \).

The first version of the commutator is defined as follows. Given an \( m \)-tuple \( b = (b_1, \ldots, b_m) \), with \( b_i \in L^1_{\text{loc}} \) for every \( i \), we define the multilinear commutator of \( T^m_\alpha \) by the expression
\[
T^m_{\alpha, b} \vec{f}(x) = \sum_{j=1}^m T^m_{\alpha, b_j} \vec{f}(x),
\]
where
\[
T^m_{\alpha, b} \vec{f}(x) = [b_j, T^m_\alpha]_j \vec{f}(x).
\]

As a consequence of (2.1) it is not difficult to see that
\[
T^m_{\alpha, b} \vec{f}(x) = \int_{(\mathbb{R}^n)^m} (b_j(x) - b_j(y_j))K_\alpha(x, \vec{y}) \prod_{i=1}^m f_i(y_i) \, d\vec{y}.
\]

We now introduce the second type of commutator of \( T^m_\alpha \). The multilinear product commutator \( T^m_{\alpha, b} \) is defined iteratively as follows
\[
T^m_{\alpha, b} \vec{f}(x) = [b_m, \ldots, [b_2, [b_1, T^m_\alpha]]_2 \ldots]_m \vec{f}.
\]
The expression above does not involve a simple notation, so we shall provide an alternative way to denote this commutator (see Section 3).

By means of (2.1) we can also obtain an integral representation for this operator (see Proposition 3.1 below), namely
\[
T^m_{\alpha, b} \vec{f}(x) = \int_{(\mathbb{R}^n)^m} K_\alpha(x, \vec{y}) \prod_{i=1}^m (b_i(x) - b_i(y_i)) f_i(y_i) \, d\vec{y}.
\]

By a weight we understand any positive and locally integrable function.

Given \( \delta \in \mathbb{R} \) and a weight \( w \) we say that a locally integrable function \( f \in L_w(\delta) \) if there exists a positive constant \( C \) such that the inequality
\[
\frac{\|w\chi_B\|_\infty}{|B|^{1+\delta/n}} \int_B |f(x) - f_B| \, dx \leq C
\]
holds for every ball \( B \), where \( f_B = |B|^{-1} \int_B f \). The smallest constant \( C \) in (2.4) will be denoted by \( \|f\|_{L_w(\delta)} \). If \( \delta = 0 \) the space \( L_w(\delta) \) coincides with a weighted version of the BMO spaces introduced in [8], the classical Lipschitz functions when \( 0 < \delta < 1 \) and the Morrey spaces when \(-n < \delta < 0 \). These classes of functions were also studied in [14].

Regarding the symbols, we shall be dealing with the \( \Lambda(\delta) \) Lipschitz spaces given, for \( 0 < \delta < 1 \), by the collection of functions \( b \) verifying
\[
|b(x) - b(y)| \leq C|x - y|^{\delta}.
\]
The smallest constant for this inequality to hold will be denoted by \( \|b\|_{\Lambda(\delta)} \). For a given \( b = (b_1, \ldots, b_m) \), with \( b_i \in \Lambda(\delta) \) for every \( 1 \leq i \leq m \), we define
\[
\|b\|_{(\Lambda(\delta))^m} = \max_{1 \leq i \leq m} \|b_i\|_{\Lambda(\delta)}.
\]
Let $S_m = \{0, 1\}^m$. Given a set $B$ and $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m) \in S_m$, we define

$$B^\sigma = \begin{cases} B, & \text{if } \sigma_i = 1 \\ \mathbb{R}^n \setminus B, & \text{if } \sigma_i = 0. \end{cases}$$

With the notation $B^\sigma$ we will understand the cartesian product $B^{\sigma_1} \times B^{\sigma_2} \times \cdots \times B^{\sigma_m}$.

For $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m)$ we also define

$$\bar{\sigma}_i = \begin{cases} 1 & \text{if } \sigma_i = 0; \\ 0 & \text{if } \sigma_i = 1, \end{cases}$$

for every $1 \leq i \leq m$ and $|\sigma| = \sum_{i=1}^m \sigma_i$.

We now describe the classes of weights involved in our estimates. Let $\delta$ be a fixed real constant. If $1 \leq p_i \leq \infty$ for every $i$, the $m$-tuple $\vec{p} = (p_1, p_2, \ldots, p_m)$ will be called a vector of exponents. We shall also denote $1/p = \sum_{i=1}^m 1/p_i$. Let $\beta$ and $\tilde{\delta}$ be real constants. Given $w$, $\vec{v} = (v_1, v_2, \ldots, v_m)$ and a vector of exponents $\vec{p}$, we say that $(w, \vec{v}) \in \mathcal{H}(\vec{p}, \beta, \tilde{\delta})$ if there exists a positive constant $C$ such that the inequality

$$\left( \int_{\mathbb{R}^n} \frac{v_i^{-p'_i}}{|B|^{(\delta-\delta)/n} + |x_B - y|^{(n-\beta_i+\delta/m)p'_i}} \, dy \right)^{1/p'_i} \leq C$$

holds for every ball $B = B(x_B, R)$. The numbers $\beta_i$ satisfy $\sum_{i=1}^m \beta_i = \beta$ and also $0 < \beta_i < n$, for every $i$, which leads to $0 < \beta < mn$. We shall also see that the parameters $\tilde{\delta}$ and $\beta$ are related to $\delta$. When $p_i = 1$ for some $i$ the integral above is understood as

$$\left( \frac{v_i^{-1}}{|(B|^{1/n} + |x_B - \cdot|^{n-\beta_i+\delta/m})| \right) \cdot$$

Let $I_1 = \{1 \leq i \leq m : p_i = 1\}$ and $I_2 = \{1 \leq i \leq m : 1 < p_i \leq \infty\}$. Condition (2.5) implies that

$$\left( \frac{v_i^{-1}}{|B|^{(\delta-\delta)/n} + |x_B - \cdot|^{(n-\beta_i+\delta/m)p'_i}} \right)^{1/p'_i} \leq C$$

and

$$\left( \frac{v_i^{-1}}{|x_B - \cdot|^{n-\beta_i+\delta/m}} \right)^{1/p'_i} \leq C.$$

We shall refer to the inequalities above as the local and global conditions, respectively.

Furthermore, given $\sigma \in S_m$ we can estimate the $i$-th factor in (2.5) depending whether $\sigma_i = 0$ or $\sigma_i = 1$. Thus we have that condition (2.5) implies

$$\left( \frac{1}{|B|} \int_B w^s \right)^{1/s} \leq C \int_B w$$
holds for every ball $B$ in $\mathbb{R}^n$. The smallest constant for which the inequality above holds is denoted by $[w]_{\text{RH}_s}$. It is not difficult to see that $\text{RH}_t \subset \text{RH}_s$ whenever $1 < s < t$. We say that $w \in \text{RH}_\infty$ if
\[
\sup_B w \leq \frac{C}{|B|} \int_B w,
\]
for some positive constant $C$.

3. Auxiliary results

We devote this section to state and prove some facts that will be useful in our estimates.

The following proposition establishes an alternative way to denote the product commutator $T_{\alpha,b}^m$.

**Proposition 3.1.** Let $T_{\alpha}^m$ be a multilinear operator as in (2.1) and $b = (b_1, \ldots, b_m)$ where $b_i \in L^{1}_{\text{loc}}$ for $1 \leq i \leq m$. Then we have that
\[
T_{\alpha,b}^m \vec{f}(x) = \sum_{\sigma \in S_m} (-1)^{m-|\sigma|} \left( \prod_{i=1}^{m} b_{\sigma_i}^i(x) \right) T_{\alpha}^m (f_1 b_{\bar{\sigma}_1}^1, \ldots, f_m b_{\bar{\sigma}_m}^m)(x).
\]
Furthermore, we have the integral representation
\[
T_{\alpha,b}^m \vec{f}(x) = \int_{(\mathbb{R}^n)^m} K_{\alpha}(x, \vec{y}) \prod_{i=1}^{m} (b_i(x) - b_i(y_i)) f_i(y_i) \, d\vec{y}.
\]

**Proof.** Let us introduce some notation in order to make our calculations simpler. Fix a symbol $b = (b_1, \ldots, b_m)$ and let $F_k$ be the operator resulting after perform $k$ iterative commutings, $1 \leq k \leq m$, that is
\[
F_k \vec{f} = [b_k, \ldots, [b_2, [b_1, T_{\alpha}^m]_1]]_2 \ldots]_k \vec{f}.
\]
Given $\sigma \in S_m$, let us also denote with
\[
\vec{g}_{b,\sigma}^k = (g_1, g_2, \ldots, g_m)
\]
where
\[
g_i = \begin{cases} 
  f_i b_{\sigma_i}^i & \text{if } 1 \leq i \leq k, \\
  f_i & \text{if } k < i \leq m.
\end{cases}
\]
We shall proceed by induction in order to show that
\[
F_k \vec{f} = \sum_{\sigma \in S_k} (-1)^{k-|\sigma|} \left( \prod_{i=1}^{k} g_{\sigma_i}^i \right) T_{\alpha}^m (\vec{g}_{b,\sigma}^k).
\]

Notice that the case $k = 1$ is immediate after performing one commutation only. Let us assume that the expression above holds for $k$ and we shall prove it for $k + 1$. By the definition and the inductive hypothesis we have
\[
F_{k+1} \vec{f} = [b_{k+1}, F_k]_{k+1} \vec{f} = b_{k+1} \sum_{\sigma \in S_k} (-1)^{k-|\sigma|} \left( \prod_{i=1}^{k} b_{\sigma_i}^i \right) T_{\alpha}^m (\vec{g}_{b,\sigma}^k) - \sum_{\sigma \in S_k} (-1)^{k-|\sigma|} \left( \prod_{i=1}^{k} b_{\sigma_i}^i \right) T_{\alpha}^m (f_1 b_{\sigma_1}^1, \ldots, f_k b_{\sigma_k}^k, f_{k+1} b_{\sigma_{k+1}}, f_{k+2}, \ldots, f_m).\]
Observe that \( \{ \theta \in S_{k+1} \} = \{ \theta \in S_{k+1} : \theta_{k+1} = 1 \} \cup \{ \theta \in S_{k+1} : \theta_{k+1} = 0 \} \). By rewriting the sums above we get

\[
F_{k+1} = \sum_{\theta \in S_{k+1}, \theta_{k+1} = 1} (-1)^{k-|\theta|-1} \left( \prod_{i=1}^{k+1} b_{i}^{\theta_{i}} \right) T_{\alpha}^{m} \left( \tilde{g}_{b, \theta} \right) + \sum_{\theta \in S_{k+1}, \theta_{k+1} = 0} (-1)^{k+1-|\theta|} \left( \prod_{i=1}^{k+1} b_{i}^{\theta_{i}} \right) T_{\alpha}^{m} \left( \tilde{g}_{b, \theta} \right)
\]

Therefore (3.1) holds for every \( 1 \leq k \leq m \), and the case \( k = m \) allows us to conclude the desired estimate.

In order to show the integral representation, we shall prove that

(3.2) \[
F_{k} = \int_{(\mathbb{R}^{n})^{m}} K_{\alpha}(x, \vec{y}) \prod_{i=1}^{k} (b_{i}(x) - b_{i}(y_{i})) \prod_{i=1}^{m} f_{i}(y_{i}) \, dy.
\]

We proceed again by induction on \( k \). If \( k = 1 \) we have that

\[
F_{1} = b_{1}(x)T_{\alpha}^{m} \tilde{f}(x) - T_{\alpha}^{m}((b_{1}f_{1}, f_{2}, \ldots, f_{m}))(x)
\]

Now assume the representation holds for \( k \) and let us prove it for \( k + 1 \). Indeed, by using the definition of \( F_{k} \) and the inductive hypothesis we get

\[
F_{k+1} = b_{k+1}(x)F_{k} - F_{k}(f_{1}, \ldots, f_{k}, f_{k+1}b_{k+1}, f_{k+2}, \ldots, f_{m})(x)
\]

This shows that the representation holds for \( k + 1 \) also. By putting \( k = m \) we get the integral representation for \( T_{\alpha,b}^{m} \).

**Lemma 3.2.** Let \( m \in \mathbb{N} \) and \( a_{i}, b_{i} \) and \( c_{i} \) be real numbers for \( 1 \leq i \leq m \). Then

\[
\prod_{i=1}^{m} (a_{i} - b_{i}) - \prod_{i=1}^{m} (c_{i} - b_{i}) = \sum_{j=1}^{m} (a_{j} - c_{j}) \prod_{i \neq j} (a_{i} - b_{i}) \prod_{i \neq j} (c_{i} - b_{i}).
\]

**Proof.** We proceed by induction on \( m \). If \( m = 1 \) it is immediate, both sides are equal to \( a_{1} - c_{1} \) since the products on the right-hand side are equal to 1.
Assume that the equality holds for \( m = k \) and let us prove it for \( m = k + 1 \). We have that
\[
\prod_{i=1}^{k+1}(a_i - b_i) - \prod_{i=1}^{k+1}(c_i - b_i) = (a_{k+1} - b_{k+1}) \prod_{i=1}^{k}(a_i - b_i) - \prod_{i=1}^{k}(c_i - b_i)
\]
\[
= (a_{k+1} - c_{k+1}) \prod_{i=1}^{k}(a_i - b_i)
\]
\[
+ (c_{k+1} - b_{k+1}) \left( \prod_{i=1}^{k}(a_i - b_i) - \prod_{i=1}^{k}(c_i - b_i) \right)
\]
By using the inductive hypothesis we arrive to
\[
\prod_{i=1}^{k+1}(a_i - b_i) - \prod_{i=1}^{k+1}(c_i - b_i) = (a_{k+1} - c_{k+1}) \prod_{i=1}^{k}(a_i - b_i)
\]
\[
+ (c_{k+1} - b_{k+1}) \sum_{j=1}^{k}(a_j - c_j) \prod_{i<j}^{k}(a_i - b_i) \prod_{i>j}^{k}(c_i - b_i) \]
\[
= \sum_{j=1}^{k+1}(a_j - c_j) \prod_{i<j}^{k+1}(a_i - b_i) \prod_{i>j}^{k+1}(c_i - b_i),
\]
so the result also holds for \( m = k + 1 \). This completes the proof. \( \square \)

The next lemma establishes a useful relation between \( A_{\vec{p}} \) and \( A_{\vec{p},q} \) classes that we shall need in the sequel. Given \( \vec{p} = (p_1, \ldots, p_m) \) with \( 1/p = \sum_{i=1}^{m} 1/p_i \) and \( 1 \leq p_i \leq \infty \) for every \( i \), we say that \( \vec{w} = (w_1, \ldots, w_m) \in A_{\vec{p}} \) if
\[
\sup_{B} \left( \frac{1}{|B|} \int_{B} \prod_{i=1}^{m} w_i^{p_i/p_i} \right)^{1/p} \prod_{i \in I_1} \left\| w_i^{-1}X_B \right\|_{\infty} \prod_{i \in I_2} \left( \frac{1}{|B|} \int_{B} w_i^{1-p_i'} \right)^{1/p_i'} < \infty,
\]
and this supremum is denoted by \([\vec{w}]_{A_{\vec{p}}}\).

On the other hand, given \( 0 < q < \infty \) and \( \vec{p} \) as above, we say that \( \vec{w} = (w_1, \ldots, w_m) \in A_{\vec{p},q} \) if
\[
\sup_{B} \left( \frac{1}{|B|} \int_{B} \prod_{i=1}^{m} w_i^{q} \right)^{1/q} \prod_{i \in I_1} \left\| w_i^{-1}X_B \right\|_{\infty} \prod_{i \in I_2} \left( \frac{1}{|B|} \int_{B} w_i^{-p_i'} \right)^{1/p_i'} < \infty,
\]
and this supremum is denoted by \([\vec{w}]_{A_{\vec{p},q}}\). When \( q = \infty \) this class corresponds to \( A_{\vec{p},\infty} \) given in (1.1).

**Lemma 3.3.** Let \( \vec{p} = (p_1, \ldots, p_m) \) be a vector of exponents, \( 0 < q < \infty \) and \( \vec{w} = (w_1, \ldots, w_m) \). Assume that
\[
\frac{1}{p_i} + \frac{1}{mq} - \frac{1}{mp} > 0
\]
for every \( 1 \leq i \leq m \). We define \( \lambda_m = 1/(mp)' + 1/(mq) \),
\[
\ell_i = \begin{cases} 
1 & \text{if } i \in I_1 \\
(\lambda_m p_i')' & \text{if } i \in I_2,
\end{cases}
\]
\( \ell \) such that \( 1/\ell = \sum_{i=1}^{m} 1/\ell_i \) and \( \vec{z} = (z_1, \ldots, z_m) \) where \( z_i = w_i^{\ell_i}/\ell \), for each \( i \). Then \( \vec{w} \in A_{\vec{p},q} \) if and only if \( \vec{z} \in A_{\vec{\ell}} \).
Proof. We first notice that condition (3.3) guarantees that \( \ell_i > 1 \) for \( i \in \mathcal{I}_2 \). It is immediate from the definition that
\[
\prod_{i=1}^{m} z_i^{\ell_i/\ell} = \prod_{i=1}^{m} w_i^q.
\]

Observe that
\[
\frac{1}{\ell} = \sum_{i=1}^{m} \frac{1}{\ell_i} = \sum_{i \in \mathcal{I}_1} 1 + \sum_{i \in \mathcal{I}_2} \frac{1}{\ell_i} = m - \frac{1}{\ell} \left( \frac{1}{\alpha} - \frac{1}{p} \right) = \frac{1}{q\lambda_m}.
\]

Also notice that for \( i \in \mathcal{I}_2 \)
\[
z_i^{1-\ell_i'} = w_i^{q\ell_i(1-\ell_i')/\ell} = w_i^{-\ell_i'/\lambda_m} = w_i^{-\nu_i'}
\]
and
\[
\frac{1}{\ell_i'} = \frac{1}{\lambda_m \ell_i'} = \frac{q}{\ell' i'}.
\]

These identities imply that
\[
\left( \frac{1}{|B|} \int_B \prod_{i=1}^{m} w_i^q \right)^{1/q} \prod_{i \in \mathcal{I}_1} \|w_i^{-1} X_B\|_\infty \prod_{i \in \mathcal{I}_2} \left( \frac{1}{|B|} \int_B z_i^{1-\ell_i'} \right)^{1/\ell_i'}
\]
can be rewritten as
\[
\left[ \left( \frac{1}{|B|} \int_B \prod_{i=1}^{m} w_i^q \right)^{1/q} \prod_{i \in \mathcal{I}_1} \|w_i^{-1} X_B\|_\infty \prod_{i \in \mathcal{I}_2} \left( \frac{1}{|B|} \int_B w_i^{-\nu_i'} \right)^{1/\ell_i'} \right]^{q/\ell},
\]
from where the equivalence follows. \( \square \)

Remark 1. When \( m = 1 \) condition (3.3) trivially holds, and we get that \( w \in A_{p,q} \) if and only if \( w^q \in A_{1+q/p'} \), a well-known relation between \( A_p \) and \( A_{p,q} \) classes.

4. Proof of Theorem 1.1

We devote this section to prove Theorem 1.1. We shall begin with an auxiliary lemma that will be useful for this purpose.

Lemma 4.1. Let \( 0 < \alpha < mn, 0 < \delta < mn - \alpha, \tilde{\alpha} = \alpha + \delta \) and \( \vec{p} \) a vector of exponents that satisfies \( p > n/\tilde{\alpha} \). Let \( \vec{\alpha} \leq \delta \) and \( (w, \vec{v}) \) be a pair of weights belonging to the class \( \mathbb{H}_m(\vec{p}, \tilde{\alpha}, \vec{\delta}) \) such that \( v_i^{-\nu_i'} \in \text{RH}_m \) for every \( i \in \mathcal{I}_2 \). Then there exists a positive constant \( C \) such that for every ball \( B \) and every \( \vec{f} \) such that \( f_i v_i \in L^{p_i}, 1 \leq i \leq m \), we have that
\[
\int_B |I_{\vec{\alpha}, m}\vec{g}(x)| \, dx \leq C \frac{|B|^{1+\delta/n}}{\|w X_B\|_\infty} \prod_{i=1}^{m} \|f_i v_i\|_{p_i},
\]
where \( \vec{g} = (f_1 X_{2B}, f_2 X_{2B}, \ldots, f_m X_{2B}) \).

Proof. We shall follow similar lines as in the proof of Lemma 3.1 in [2]. We include a sketch for the sake of completeness.

We shall split the set \( \mathcal{I}_2 \) into \( \mathcal{I}_2^1 \) and \( \mathcal{I}_2^2 \) where
\[
\mathcal{I}_2^1 = \{ i \in \mathcal{I}_2 : 1 < p_i < \infty \} \quad \text{and} \quad \mathcal{I}_2^2 = \{ i \in \mathcal{I}_2 : p_i = \infty \}.
\]
Let $m_i = \#\mathcal{I}_i$, for $i = 1, 2$ and $m_j = \#\mathcal{I}_j$, also for $j = 1, 2$. Then $m = m_1 + m_2 = m_1 + m_2^1 + m_2^2$.

Then, by denoting $B = 2B$, for $x \in B$ we have that

$$|I_{\tilde{g}_m}(x)| \leq \int_{B^m} \frac{\prod_{i=1}^m |f_i(y_i)|}{(\sum_{i=1}^m |x - y_i|)^{m - m_2}} \, d\tilde{y}$$

$$\leq \left( \prod_{i \in \mathcal{I}_2^1} \|f_i v_i\|_{\infty} \right) \left( \prod_{i \in \mathcal{I}_2^2} \|f_i \tilde{X}_B\|_1 \right) \int_{B^m_2} \frac{\prod_{i \in \mathcal{I}_1^1} |f_i(y_i)| \prod_{i \in \mathcal{I}_2^2} v_i^{-1}(y_i)}{(\sum_{i \in \mathcal{I}_2^2} |x - y_i|)^{m - \tilde{\alpha}}} \, d\tilde{y}$$

Therefore

$$\tilde{\alpha} > n/p = n \sum_{i=1}^m \frac{1}{p_i} = m_1 n + \frac{n}{p^*},$$

where $1/p^* = \sum_{i \in \mathcal{I}_2} 1/p_i$. Then we can split $\tilde{\alpha} = \tilde{\alpha}^1 + \tilde{\alpha}^2$, where $\tilde{\alpha}^1 > m_1 n$ and $\tilde{\alpha}^2 > n/p^*$. Therefore

$$mn - \tilde{\alpha} = m_2 n - \tilde{\alpha}^2 + m_1 n - \tilde{\alpha}^1.$$

Let us sort the sets $\mathcal{I}_2^1$ and $\mathcal{I}_2^2$ increasingly, so

$$\mathcal{I}_2^1 = \{i_1, i_2, \ldots, i_{m_1^2}\} \quad \text{and} \quad \mathcal{I}_2^2 = \{i_{m_1^2+1}, i_{m_1^2+2}, \ldots, i_{m_2}\}.$$

We now define $\tilde{g} = (g_1, \ldots, g_{m_2})$, where

$$g_j = \begin{cases} |f_{i_j}| & \text{if } 1 \leq j \leq m_1^2, \\ v_i^{-1} & \text{if } m_1^2 + 1 \leq j \leq m_2. \end{cases}$$

Then we can proceed in the following way

$$I(x, B) \leq \int_{B^m_2} \frac{\prod_{i \in \mathcal{I}_1^1} |f_i(y_i)| \prod_{i \in \mathcal{I}_2^2} v_i^{-1}(y_i) (\sum_{i \in \mathcal{I}_2^2} |x - y_i|)^{m - m_2}}{(\sum_{i \in \mathcal{I}_2^2} |x - y_i|)^{m_2 - \tilde{\alpha}^2}} \, d\tilde{y}$$

$$\leq |B|^{\tilde{\alpha}/n - m_1} \int_{B^m_2} \frac{\prod_{i \in \mathcal{I}_1^1} |f_i(y_i)| \prod_{i \in \mathcal{I}_2^2} v_i^{-1}(y_i)}{(\sum_{i \in \mathcal{I}_2^2} |x - y_i|)^{m - \tilde{\alpha}}} \, d\tilde{y}$$

$$= |B|^{\tilde{\alpha}/n - m_1} \int_{B^m_2} \frac{\prod_{j=1}^{m_2} g_j(y_i)}{(\sum_{j=1}^{m_2} |x - y_i|)^{m_2 - \tilde{\alpha}}} \, d\tilde{y}$$

$$\lesssim |B|^{\tilde{\alpha}/n - m_1} I_{\tilde{\alpha}^2, m_2} (\tilde{g} \tilde{X}_{B^m_2})(x).$$

Next we define the vector of exponents $\tilde{r} = (r_1, \ldots, r_{m_2})$ in the following way

$$r_j = \begin{cases} m_2 p_{i_j} / (m_2 - 1 + p_{i_j}) & \text{if } 1 \leq j \leq m_1^2; \\ m_2 & \text{if } m_1^2 + 1 \leq j \leq m_2. \end{cases}$$
This definition yields
\[ \frac{1}{r} = \sum_{j=1}^{m_2} \frac{1}{r_j} = \sum_{j=1}^{m_2^1} \left( \frac{1}{m_2} + \frac{m_2 - 1}{m_2 p_{i_j}} \right) + \sum_{j=m_2^1+1}^{m_2} \frac{1}{m_2} = \frac{m_2^1}{m_2} + \frac{m_2 - 1}{m_2 p^*} + \frac{m_2^2}{m_2} = 1 + \frac{m_2 - 1}{m_2 p^*}. \]

Notice that \( 1/r > 1/p^* \) and also \( n/p^* < \tilde{\alpha}^2 \) by construction. Then there exists an auxiliary number \( \tilde{\alpha}_0 \) such that \( n/p^* < \tilde{\alpha}_0 < n/r \). Indeed, if \( \tilde{\alpha}^2 < n/r \) we can directly pick \( \tilde{\alpha}_0 = \tilde{\alpha}^2 \). Otherwise \( \tilde{\alpha}_0 < \tilde{\alpha}^2 \). We shall first assume that \( m_2 \geq 2 \). We set
\[ \frac{1}{q} = \frac{1}{r} - \frac{\tilde{\alpha}_0}{n}. \]

Then \( 0 < 1/q < 1 \) since
\[ \frac{1}{r} - 1 = \left( 1 - \frac{1}{m_2} \right) \frac{1}{p^*} < \frac{1}{p^*} < \frac{\tilde{\alpha}_0}{n}. \]

By using the well-known continuity property \( I_{\tilde{\alpha}_0,m_2} : \prod_{j=1}^{m_2} L^{r^j} \to L^q \) with respect to the Lebesgue measure (see, for example, [7]) we obtain
\[
\int_B I(x, B) \, dx \lesssim |B|^{\tilde{\alpha}^1/n - m_1 + (\tilde{\alpha}^2 - \tilde{\alpha}_0)/n} \left( \int_B |I_{\tilde{\alpha}_0,m_2}(\bar{g} \chi_B^m)(x)|^q \, dx \right)^{1/q} |B|^{1/q'} \\
\lesssim |B|^{(\tilde{\alpha} - \tilde{\alpha}_0)/n - m_1 + 1/q'} \left( \int_{\mathbb{R}^n} |I_{\tilde{\alpha}_0,m_2}(\bar{g} \chi_B^m)(x)|^q \, dx \right)^{1/q} \\
\lesssim |B|^{(\tilde{\alpha} - \tilde{\alpha}_0)/n - m_1 + 1/q'} \prod_{j=1}^{m_2} \|g_j \chi_B^m\|_{r_j}.
\]

Observe that \( r_j < p_{i_j} \) for every \( 1 \leq j \leq m_2^1 \). Since \( \nu_i^{-p_i'} \in RH_{m} \subseteq RH_m \) for every \( i \in I_2 \), by applying Hölder inequality and then the reverse Hölder condition on these weights we get
\[
\prod_{j=1}^{m_2} \|g_j \chi_B^m\|_{r_j} = \prod_{i \in I_2^1} \left( \int_B |f_i|^{r_i} \nu_i^{-r_i} \right)^{1/r_i} \prod_{i \in I_2^2} \left( \int_B \nu_i^{-m_2 p_i'} \right)^{1/m_2} \\
\leq \prod_{i \in I_2^1} \|f_i \nu_i\|_{p_i} \left( \int_B \nu_i^{-m_2 p_i'} \right)^{1/(m_2 p_i')} \prod_{i \in I_2^2} \left( \int_B \nu_i^{-m_2} \right)^{1/m_2} \\
\leq |B|^{m_2^1/m_2 - 1/(m_2 p^*)} \|f_i \nu_i\|_{p_i} \left( \frac{1}{|B|} \int_B \nu_i^{-p_i} \right)^{1/p_i'} \\
\times \prod_{i \in I_2^2} \left[ \nu_i^{-1} \right]_{RH_m} \left( \frac{1}{|B|} \int_B \nu_i^{-1} \right) \\
\lesssim |B|^{-1/(m_2 p^*)} \prod_{i \in I_2^1} \|f_i \nu_i\|_{p_i} \left( \frac{1}{|B|} \int_B \nu_i^{-p_i} \right)^{1/p_i'} \prod_{i \in I_2^2} \left( \frac{1}{|B|} \int_B \nu_i^{-1} \right).
\]

By combining the estimates above with condition (2.6), we finally arrive to
\[
\int_B |I_{\tilde{\alpha},m\bar{g}}(x)| \, dx \lesssim \left( \prod_{i \in I_2^1} \|f_i \nu_i\|_{\infty} \right) \left( \prod_{i \in I_2^1} \|f_i \chi_B^m\|_1 \right) \int_B I(x, B) \, dx
\]
There are only three possible cases:

1. $m_2 = 0$. In this case we have $m_1 = m_2 = 0$ and this implies $\tilde{p} = (1, 1, \ldots, 1)$. This situation is not possible, because $p > n/\tilde{\alpha}$.

2. $m_1 = 0$ and $m_2 = 1$. In this case $1/p = m - 1$. Condition $p > n/\tilde{\alpha}$ implies $\tilde{\alpha} > (m - 1)n$.

Let $i_0$ be the index such that $p_{i_0} = \infty$. By using Fubini's theorem, we can proceed in the following way

$$
\int_B \int_{B^m} \frac{\prod_{i=1}^m |f_i(y_i)|}{(\sum_{i=1}^m |x - y_i|)^{m\alpha - \alpha}} \, d\tilde{y} \, dx = \int_{B^m} \prod_{i=1}^m |f_i(y_i)| \left( \int_B \left( \sum_{i=1}^m |x - y_i|^{\tilde{\alpha} - mn} \right) \, dx \right) \, d\tilde{y}.
$$

Since

$$
\int_B \left( \sum_{i=1}^m |x - y_i|^{\tilde{\alpha} - mn} \right) \, dx \lesssim \int_0^{4R} \rho^{\tilde{\alpha} - mn} \rho^{n-1} \, d\rho \lesssim |B|^\tilde{\alpha}/n + m + 1,
$$

by (2.6), we get

$$
\int_B |I_{\tilde{\alpha}, m}(\tilde{y})(x)| \, dx \lesssim |B|^{\tilde{\alpha}/n - m + 2} \left( \prod_{i=1}^m \|f_i v_i\|_{p_i} \right) \left( \prod_{i \in I_1} \|v_i^{-1} X_{\tilde{B}^i}\|_{\infty} \right) \left( \frac{1}{|B|} \int_{B_{i_0}} v_i^{-1} \right) \lesssim \left( \prod_{i=1}^m \|f_i v_i\|_{p_i} \right) \frac{|B|^{\tilde{\alpha}/n - m + 2 + \delta/n - \tilde{\alpha}/n + 1/p}}{|w X_{\tilde{B}}|_{\infty}} \lesssim \left( \prod_{i=1}^m \|f_i v_i\|_{p_i} \right) \frac{|B|^{1+\delta/n}}{|w X_{\tilde{B}}|_{\infty}}.
$$

3. $m_1 = 1$ and $m_2 = 0$. If $i_0$ denotes the index for which $1 < p_{i_0} < \infty$, the condition $p > n/\tilde{\alpha}$ implies that

$$
\frac{\tilde{\alpha}}{n} > \frac{1}{p} = m - 1 + \frac{1}{p_{i_0}},
$$
and thus $\tilde{\alpha} > (m - 1)n$. We repeat the estimate given in the previous case. This yields

$$\int_B |I_1 \tilde{f}(x)| \, dx \lesssim |B|^{\tilde{\alpha}/n-m+1/p'_0} \left( \prod_{i=1}^m \| f_i v_i \|_{p_i} \right) \left( \prod_{i \in I_i} \| v_i^{-1} \mathcal{X}_B \|_\infty \right) \left( \frac{1}{|B|} \int_B v_{i_0}^{-p'_0} \right)^{1/p'_0}$$

$$\lesssim \left( \prod_{i=1}^m \| f_i v_i \|_{p_i} \right) \left( \frac{|B|^{\tilde{\alpha}/n-m+1/p'_0} + \delta/n - \tilde{\alpha}/n + 1/p}{\| w \mathcal{X}_B \|_\infty} \right)$$

$$\lesssim \left( \prod_{i=1}^m \| f_i v_i \|_{p_i} \right) \| B \|^{1+\delta/n} \frac{\| w \mathcal{X}_B \|_\infty}{\| w \mathcal{X}_B \|_\infty}.$$ 

We covered all the possible cases for $m_2$ and the proof is complete. \hfill $\square$

**Proof of Theorem 1.1.** It will be enough to prove that

$$\frac{\| w \mathcal{X}_B \|_\infty}{|B|^{1+\delta/n}} \int_B |T_{\alpha,b_j}^m \tilde{f}(x) - c_j| \, dx \leq C \prod_{i=1}^m \| f_i v_i \|_{p_i},$$

for some positive constant $c_j$ and every ball $B$, for each $j$ and with $C$ independent of $B$ and $j$.

Indeed, if (4.1) holds we take $c = \sum_{j=1}^m c_j$ and therefore

$$\frac{\| w \mathcal{X}_B \|_\infty}{|B|^{1+\delta/n}} \int_B |T_{\alpha,b_j}^m \tilde{f}(x) - c| \, dx \leq \sum_{j=1}^m \frac{\| w \mathcal{X}_B \|_\infty}{|B|^{1+\delta/n}} \int_B |T_{\alpha,b_j}^m \tilde{f}(x) - c_j| \, dx$$

$$\leq Cm \prod_{i=1}^m \| f_i v_i \|_{p_i}$$

and the proof would be complete. Then we shall proceed to prove (4.1).

Fix $1 \leq j \leq m$ and a ball $B = B(x_B, R)$. We decompose $\tilde{f} = (f_1, f_2, \ldots, f_m)$ as $\tilde{f} = \tilde{f}_1 + \tilde{f}_2$, where $\tilde{f}_1 = (f_1 \mathcal{X}_2, f_2 \mathcal{X}_2, \ldots, f_m \mathcal{X}_2)$. We take

$$c_j = \left( T_{\alpha,b_j}^m \tilde{f}_2 \right)_B = \frac{1}{|B|} \int_B T_{\alpha,b_j}^m \tilde{f}_2(x) \, dx.$$

We also notice that

$$\frac{1}{|B|} \int_B T_{\alpha,b_j}^m \tilde{f}_2(x) \, dx = \sum_{\sigma \in S_m, \sigma \neq 1} \frac{1}{|B|} \int_B \int_{(2B)^m} (b_j(x) - b_j(y_j)) K_\alpha(x, \tilde{y}) \prod_{i=1}^m f_i(y_i) \, d\tilde{y} \, dx.$$

In order to prove (4.1) we write

$$\frac{\| w \mathcal{X}_B \|_\infty}{|B|^{1+\delta/n}} \int_B |T_{\alpha,b_j}^m \tilde{f}(x) - c_j| \, dx \leq \frac{\| w \mathcal{X}_B \|_\infty}{|B|^{1+\delta/n}} \left( \int_B |T_{\alpha,b_j}^m \tilde{f}_1(x) | \, dx + \frac{1}{|B|} \int_B |T_{\alpha,b_j}^m \tilde{f}_2(x) - c_j| \, dx \right)$$

$$= \frac{\| w \mathcal{X}_B \|_\infty}{|B|^{1+\delta/n}} \left( I + \frac{1}{|B|} II \right).$$

Let us first estimate $I$. By applying Lemma 4.1 we get

$$I = \int_B |T_{\alpha,b_j}^m \tilde{f}_1(x) | \, dx \leq \int_B \int_{(2B)^m} |b_j(x) - b_j(y_j)| \, K_\alpha(x, \tilde{y}) \prod_{i=1}^m |f_i(y_i)| \, d\tilde{y} \, dx$$

$$\lesssim \| b_j \|_{L_\delta} \int_B \int_{(2B)^m} |K_\alpha(x, \tilde{y}) | \prod_{i=1}^m |f_i(y_i)| \, d\tilde{y} \, dx.$$
We now turn our attention to BOUNDEDNESS OF MULTILINEAR COMMUTATORS OF FRACTIONAL INTEGRAL OPERATORS. Let
\[
\sigma \ll \frac{\sum_{i=1}^{m} |f_i(y_i)|}{\sum_{i=1}^{m} |x - y_i|^{mn - \alpha + \gamma}} \ll \frac{1}{B^{\gamma/n}}.
\]
Consequently,
\[
\bigg\| \frac{w \chi_B}{|B|^{1+\delta/n}} \bigg\|_\infty I \lesssim \|b\|_{(\Lambda(\delta))^{m}} \prod_{i=1}^{m} \|f_i v_i\|_{p_i}.
\]
We now turn our attention to II. By (4.2) we can write
\[
II \leq \sum_{\sigma \in S_m, \sigma \neq 1} \int_B \int_B \int (2B)^{\sigma} \left| (b_j(x) - b_j(y_j))K_\alpha(x, y) - (b_j(z) - b_j(y_j))K_\alpha(z, y) \right|
\times \prod_{i=1}^{m} |f_i(y_i)| \, dy \, dx
\leq \sum_{\sigma \in S_m, \sigma \neq 1} \int_B \int_B \int (2B)^{\sigma} \left| (b_j(x) - b_j(y_j))(K_\alpha(x, y) - K_\alpha(z, y)) \right|
\times \prod_{i=1}^{m} |f_i(y_i)| \, dy \, dx
+ \sum_{\sigma \in S_m, \sigma \neq 1} \int_B \int_B \int (2B)^{\sigma} \left| (b_j(x) - b_j(z))K_\alpha(z, y) \right|
\times \prod_{i=1}^{m} |f_i(y_i)| \, dy \, dx
= \sum_{\sigma \in S_m, \sigma \neq 1} (I_1^\sigma + I_2^\sigma).
\]
We shall estimate each sum separately. Fix \(\sigma \in S_m, \sigma \neq 1\). We start with \(I_1^\sigma\). Since we are assuming \(\sigma \neq 1\), condition (2.3) implies that
\[
|K_\alpha(x, y) - K_\alpha(z, y)| \lesssim \frac{|x - z|^{\gamma}}{(\sum_{i=1}^{m} |x - y_i|)^{mn - \alpha + \gamma}} \lesssim \frac{|x - z|^{\gamma}}{B^{\gamma/n}}.
\]
Therefore we have that
\[
I_1^\sigma \lesssim \|b_j\|_{(\Lambda(\delta))^{m}} \|B^{1+\gamma/n} \int_B \int_B (2B)^{\sigma} \frac{|x - y_j|^{\delta} \prod_{i=1}^{m} |f_i(y_i)|}{\sum_{i=1}^{m} |x - y_i|^{mn - \alpha + \gamma}} \, dy \, dx \, dz
\leq \|b_j\|_{(\Lambda(\delta))^{m}} \|B^{1+\gamma/n} \int_B \int_B (2B)^{\sigma} \frac{\prod_{i=1}^{m} |f_i(y_i)|}{(\sum_{i=1}^{m} |x - y_i|)^{mn - \alpha + \gamma - \delta}} \, dy \, dx \, dz
\leq \|b_j\|_{(\Lambda(\delta))^{m}} \|B^{1+\gamma/n} \int_B \int_B (2B)^{\sigma} \frac{\prod_{i=1}^{m} |f_i(y_i)|}{(\sum_{i=1}^{m} |x - y_i|)^{mn - \alpha + \delta}} \, dy \, dx \, dz
= \|b_j\|_{(\Lambda(\delta))^{m}} \|B^{1+\gamma/n} \int_B J_1(x, \sigma) \, dx \, dz.
\]
By separating the factors in \(J_1\) and applying Hölder inequality we arrive to
\[
J_1(x, \sigma) \lesssim \left( \prod_{i: \sigma_i = 1} \int_{2B} \frac{|f_i(y_i)|}{|2B|^{1-\alpha_i/n + \delta/(mn)}} \, dy \right) \left( \prod_{i: \sigma_i = 0} \int_{\mathbb{R} \setminus 2B} \frac{|f_i(y_i)|}{|x - y_i|^{n - \alpha_i + \delta/m}} \, dy \right)
\lesssim \prod_{i=1}^{m} \|f_i v_i\|_{p_i} \left( \prod_{i: \sigma_i = 1} \left\| \frac{\chi_{2B}}{|2B|^{1-\alpha_i/n + \delta/(mn)}} \right\|_{p'_i} \right) \left( \prod_{i: \sigma_i = 0} \left\| \frac{\chi_{\mathbb{R} \setminus 2B}}{|x - y_i|^{n - \alpha_i + \delta/m}} \right\|_{p'_i} \right).
with different parameters. The proof

This completes the proof of (4.1) and we are done.

5. Proof of Theorem 1.2

We devote this section to prove Theorem 1.2. We shall first establish an auxiliary lemma, which is essentially the boundedness given in Lemma 4.1 with different parameters. The proof can be achieved by following the same steps and we shall omit it.
Lemma 5.1. Let $0 < \alpha < mn$, $0 < \delta < (n - \alpha)/m$, $\tilde{\alpha} = \alpha + m\delta$ and $\tilde{p}$ a vector of exponents that satisfies $p > n/\tilde{\alpha}$. Let $\tilde{\delta} \leq \delta$ and $(w, \tilde{v})$ be a pair of weights belonging to the class $H_m(\tilde{p}, \tilde{\alpha}, \tilde{\delta})$ such that $v_i^{-p_i} \in RH_m$ for every $i \in I_2$. Then there exists a positive constant $C$ such that for every ball $B$ and every $\tilde{f}$ such that $f_i v_i \in L^{p_i}$, $1 \leq i \leq m$, we have that

$$\int_B |I_{\tilde{\alpha}, m} \tilde{f}(x)| \, dx \leq C \frac{|B|^{1 + \delta/n}}{\|w \mathcal{X}_B\|_{\infty}} \prod_{i=1}^m \|f_i v_i\|_{p_i},$$

where $\tilde{g} = (f_1 X_{2B}, f_2 X_{2B}, \ldots, f_m X_{2B})$.

Proof of Theorem 1.2. It will be enough to prove that

$$\left(\frac{\|w \mathcal{X}_B\|_{\infty}}{|B|^{1 + \delta/n}} \int_B |T_{\alpha, b}^m \tilde{f}(x) - c| \, dx \right) \leq C \prod_{i=1}^m \|f_i v_i\|_{p_i},$$

for some constant $c$ and every ball $B$, with $C$ independent of $B$ and $\tilde{f}$.

Fix a ball $B = B(x_B, R)$. By proceeding as in the proof of Theorem 1.1, we split $\tilde{f} = \tilde{f}_1 + \tilde{f}_2$, where $\tilde{f}_1 = (f_1 X_{2B}, f_2 X_{2B}, \ldots, f_m X_{2B})$. We take

$$c = \left(\frac{T_{\alpha, b}^m \tilde{f}_2}{B}\right) = \frac{1}{|B|} \int_B T_{\alpha, b}^m \tilde{f}_2(z) \, dz.$$

By Proposition 3.1, for $z \in B$ we have that

$$T_{\alpha, b}^m \tilde{f}_2(z) = \sum_{\sigma \in S_{m, \sigma \neq 1}} \int_{(2B)^\sigma} K_\alpha(z, \vec{y}) \prod_{i=1}^m (b_i(z) - b_i(y_i)) f_i(y_i) \, d\vec{y}.$$

Thus

$$\left(\frac{\|w \mathcal{X}_B\|_{\infty}}{|B|^{1 + \delta/n}} \int_B |T_{\alpha, b}^m \tilde{f}(x) - c| \, dx \right) \leq \left(\frac{\|w \mathcal{X}_B\|_{\infty}}{|B|^{1 + \delta/n}} \int_B |T_{\alpha, b}^m \tilde{f}_1(x)| \, dx \right) + \left(\frac{\|w \mathcal{X}_B\|_{\infty}}{|B|^{2 + \delta/n}} \int_B \int_B \left|T_{\alpha, b}^m \tilde{f}_2(x) - T_{\alpha, b}^m \tilde{f}_2(z)\right| \, dz \, dx \right)$$

$$= \left(\frac{\|w \mathcal{X}_B\|_{\infty}}{|B|^{1 + \delta/n}} (I + II)\right).$$

Let us first estimate $I$. By applying Proposition 3.1, (2.2) and Lemma 5.1 we get

$$I = \int_B |T_{\alpha, b}^m \tilde{f}_1(x)| \, dx \leq \int_B \int_{(2B)^m} |K_\alpha(x, \vec{y})| \prod_{i=1}^m |b_i(x) - b_i(y_i)| |f_i(y_i)| \, d\vec{y} \, dx$$

$$\leq C \prod_{i=1}^m \|b_i\|_{\Lambda(\delta)} \int_B \int_{(2B)^m} |K_\alpha(x, \vec{y})| \prod_{i=1}^m |f_i(y_i)| \, d\vec{y} \, dx$$

$$\leq C \|b\|_{\Lambda(\delta)}^m \int_B |I_{\tilde{\alpha}, m} \tilde{f}_1(x)| \, dx$$

$$\leq C \|b\|_{\Lambda(\delta)}^m \left(\frac{|B|^{1 + \delta/n}}{\|w \mathcal{X}_B\|_{\infty}} \prod_{i=1}^m \|f_i v_i\|_{p_i}\right).$$

Consequently,

$$\left(\frac{\|w \mathcal{X}_B\|_{\infty}}{|B|^{1 + \delta/n}} I \right) \leq C \|b\|_{\Lambda(\delta)}^m \prod_{i=1}^m \|f_i v_i\|_{p_i}.$$
We continue with the estimate of $II$. We shall see that

$$
|\mathcal{T}_{a,b}\vec{f}_2(x) - \mathcal{T}_{a,b}\vec{f}_2(z)| \leq C\|b\|_{(\lambda(\delta))}^m \frac{|B|^{\delta/n}}{\|w\chi_B\|_\infty} \prod_{i=1}^m \|f_i v_i\|_{p_i},
$$

for every $x, z \in B$. This would imply that $II \leq C\prod_{i=1}^m \|f_i v_i\|_{p_i}$.

By (5.2), for $x \in B$ we can write

$$
|\mathcal{T}_{a,b}\vec{f}_2(x) - \mathcal{T}_{a,b}\vec{f}_2(z)|
\leq \sum_{\sigma \in S_m, \sigma \neq 1} \int_{(2B)^\sigma} |K_\alpha(x, \vec{y})| \prod_{i=1}^m |b_i(x) - b_i(y_i)| \prod_{i=1}^m |f_i(y_i)| d\vec{y}
\leq \sum_{\sigma \in S_m, \sigma \neq 1} \int_{(2B)^\sigma} |K_\alpha(x, \vec{y}) - K_\alpha(z, \vec{y})| \prod_{i=1}^m |b_i(x) - b_i(y_i)| |f_i(y_i)| d\vec{y}
+ \sum_{\sigma \in S_m, \sigma \neq 1} \int_{(2B)^\sigma} |K_\alpha(z, \vec{y})| \prod_{i=1}^m |b_i(x) - b_i(y_i)| - \prod_{i=1}^m |b_i(z) - b_i(y_i)| d\vec{y}
\leq \sum_{\sigma \in S_m, \sigma \neq 1} (I_1^\sigma + I_2^\sigma).
$$

Fix $\sigma \in S_m, \sigma \neq 1$. Let us first estimate $I_1^\sigma$. By applying condition (2.3) we have that

$$
|K_\alpha(x, \vec{y}) - K_\alpha(z, \vec{y})| \lesssim \frac{|x - z|^{\gamma}}{(\sum_{i=1}^m |x - y_i|)^{mn-\alpha+\gamma}}
\lesssim \frac{|B|^{\gamma/n}}{(\sum_{i=1}^m |x - y_i|)^{mn-\alpha+\gamma}}.
$$

Therefore we have that

$$
I_1^\sigma \lesssim |B|^{\gamma/n} \prod_{i=1}^m |b_i|_{\lambda(\delta)} \int_{(2B)^\sigma} \frac{(\sum_{i=1}^m |x - y_i|)^{\sigma_0} \prod_{i=1}^m |f_i(y_i)|}{(\sum_{i=1}^m |x - y_i|)^{mn-\alpha+\gamma}} d\vec{y}
\lesssim \|b\|_{(\lambda(\delta))}^m |B|^{\gamma/n} \int_{(2B)^\sigma} \prod_{i=1}^m |f_i(y_i)| (\sum_{i=1}^m |x - y_i|)^{mn-\alpha+\gamma+\theta} d\vec{y}
\lesssim \|b\|_{(\lambda(\delta))}^m |B|^{\delta/n} \int_{(2B)^\sigma} \prod_{i=1}^m |f_i(y_i)| (\sum_{i=1}^m |x - y_i|)^{mn-\alpha+\gamma+\theta} d\vec{y},
$$

since $\gamma > \delta$ and $\vec{y} \in (2B)^\sigma$ implies that $|x_B - y_j| \geq C|B|^{1/n}$ for at least one index $1 \leq j \leq m$.

From this expression we can continue as in the estimate performed in page 15 in order to obtain

$$
I_1^\sigma \lesssim \|b\|_{(\lambda(\delta))}^m \frac{|B|^{\delta/n}}{\|w\chi_B\|_\infty} \prod_{i=1}^m \|f_i v_i\|_{p_i}.
$$
Next we proceed to estimate $I_2^\sigma$. Fix $\sigma \in S_m$, $\sigma \neq 1$. By applying Lemma 3.2 we have that
\[
\left| \prod_{i=1}^m (b_i(x) - b_i(y_i)) - \prod_{i=1}^m (b_i(z) - b_i(y_i)) \right| \lesssim \sum_{j=1}^m |b_j(x) - b_j(z)| \prod_{i \neq j} |b_i(z) - b_i(y_i)| |b_i(x) - b_i(y_i)|
\]
By using this estimate we can proceed with
\[
I \quad \text{holds for every ball } B \quad \text{and also}
\]
\[
|y| \lesssim |x - y| \lesssim \sum_{j=1}^m |x_B - y_j|
\]
and also
\[
|z - y| \lesssim \sum_{j=1}^m |x_B - y_j|,
\]
for each $i$, regardless $y_i$ belongs to $2B$ or $\mathbb{R}^n \setminus 2B$. Therefore we arrive to
\[
\left| \prod_{i=1}^m (b_i(x) - b_i(y_i)) - \prod_{i=1}^m (b_i(z) - b_i(y_i)) \right| \lesssim \|b\|^{m}_{\Lambda(\delta)} |B|^{\delta/n} \left( \sum_{j=1}^m |x_B - y_j| \right)^{(m-1)\delta},
\]
By using this estimate we can proceed with $I_2^\sigma$ as follows
\[
I_2^\sigma \lesssim \|b\|^{m}_{\Lambda(\delta)} |B|^{\delta/n} \int_{(2B)^\sigma} \frac{\prod_{i=1}^m |f_i(y_i)|}{(\sum_{i=1}^m |x_B - y_i|)^{mn-\alpha+(1-m)\delta}} d\bar{y}
\]
\[
\lesssim \|b\|^{m}_{\Lambda(\delta)} |B|^{\delta/n} \int_{(2B)^\sigma} \frac{\prod_{i=1}^m |f_i(y_i)|}{(\sum_{i=1}^m |x_B - y_i|)^{mn-\alpha+\delta}} d\bar{y}
\]
\[
\lesssim \|b\|^{m}_{\Lambda(\delta)} |B|^{\delta/n} \prod_{i: \sigma_i=1} \int_{2B} |f_i(y_i)| \frac{dy_i}{|x_B - y_i|^{n-\alpha_i+\delta/m}} \prod_{i: \sigma_i=0} \int_{\mathbb{R}^n \setminus 2B} |f_i(y_i)| \frac{dy_i}{|x_B - y_i|^{n-\alpha_i+\delta/m}}.
\]
By applying Hölder inequality and condition (2.8) we get
\[
I_2^\sigma \lesssim \|b\|^{m}_{\Lambda(\delta)} |B|^{\delta/n-\theta(\sigma)} \prod_{i=1}^m ||f_i|_{p_i}} \prod_{i: \sigma_i=1} \|v_i^{-1} \chi_{2B} \|_{p_i'} \prod_{i: \sigma_i=0} \left\| v_i^{-1} \chi_{\mathbb{R}^n \setminus 2B} \right\|_{p_i'}
\]
\[
\lesssim \|b\|^{m}_{\Lambda(\delta)} \frac{|B|^{\delta/n}}{\|w \chi_B\|_{\infty}} \prod_{i=1}^m ||f_i|_{p_i}}
\]
where $\theta(\sigma) = \sum_{i: \sigma_i=1} (1 - \alpha_i/n + \delta/(mn))$. So (5.3) holds and the proof is complete. \[ \square \]

6. The class $\mathbb{H}_m(\tilde{\rho}, \beta, \tilde{\delta})$

In this section we give a complete study of the class $\mathbb{H}_m(\tilde{\rho}, \beta, \tilde{\delta})$ related with the boundedness properties stated in our main results. Recall that $(w, \tilde{v})$ belongs to the class $\mathbb{H}_m(\tilde{\rho}, \beta, \tilde{\delta})$ if there exists a positive constant $C$ for which the inequality
\[
\frac{\|w \chi_B\|_{\infty}}{|B|^{(\delta-\delta)/n}} \prod_{i=1}^m \left( \int_{\mathbb{R}^n} \frac{v_i^{p_i'}}{|B|^{1/n} + |x_B - y|^{(n-\beta_i+\delta/m)p_i'}} dy \right)^{1/p_i'} \leq C
\]
holds for every ball $B = B(x_B, R)$. 

We begin with a characterization of this class of weights in terms of the global condition (2.7). The proof follows similar lines as Lemma 2.1 in [2] and we shall omit it. We recall the notation $\mathcal{I}_1 = \{ i : p_i = 1 \}$ and $\mathcal{I}_2 = \{ i : 1 < p_i \leq \infty \}$.

**Lemma 6.1.** Let $0 < \beta < mn$, $\tilde{\delta} \in \mathbb{R}$, $\tilde{\mu}$ a vector of exponents and $(w, \tilde{v})$ a pair of weights such that $v_i^{-1} \in \operatorname{RH}_\infty$ for $i \in \mathcal{I}_1$ and $v_i^{-p_i'}$ is doubling for $i \in \mathcal{I}_2$. Then, condition $\mathbb{H}_m(\tilde{\mu}, \beta, \tilde{\delta})$ is equivalent to (2.7).

As an immediate consequence of this lemma we have the following.

**Corollary 6.2.** Under the hypotheses of Lemma 6.1 we have that conditions (2.7) implies (2.6).

The next lemma establishes a useful property in order to give examples of weights in the considered class. We shall assume that $\beta_i = \beta/m$ for every $i$.

**Lemma 6.3.** Let $0 < \beta < mn$, $\tilde{\delta} < \tau = (\beta - mn)(1 - 1/m) + \delta/m$, $\tilde{\mu}$ a vector of exponents and $(w, \tilde{v})$ a pair of weights satisfying condition (2.6). Then $(w, \tilde{v})$ satisfies (2.7).

**Proof.** The proof follows similar lines as in Lemma 2.3 in [2]. We shall give a scheme for the sake of completeness. Let $\theta = n - \beta/m + \delta/m$. Let $B$ be a ball and $B_k = 2^k B$, for $k \in \mathbb{N}$. If $i \in \mathcal{I}_1$ we have that

$$
\left\| \frac{v_i^{-1} \mathcal{X}_{\mathbb{R}^n \setminus B}}{|x_B - \cdot|^\theta} \right\|_\infty \leq \sum_{k=1}^{\infty} \left\| \frac{v_i^{-1} \mathcal{X}_{B_{k+1} \setminus B_k}}{|x_B - \cdot|^\theta} \right\|_\infty
$$

On the other hand, for $i \in \mathcal{I}_2$

$$
\left( \int_{\mathbb{R}^n \setminus B} \frac{v_i^{-p_i'}(y)}{|x_B - y|^\theta |\tilde{v}_i|} \, dy \right)^{1/p_i'} \leq \left( \sum_{k=1}^{\infty} \int_{B_{k+1} \setminus B_k} \frac{v_i^{-p_i'}(y)}{|x_B - y|^\theta |\tilde{v}_i|} \, dy \right)^{1/p_i'}
$$

By taking $\tilde{k} = (k_1, k_2, \ldots, k_m)$, the left-hand side of (2.7) can be bounded by a multiple constant of

$$
\sum_{\tilde{k} \in \mathbb{N}^m} \prod_{i \in \mathcal{I}_1} |B_{k_i}|^{-\theta/n} \left\| v_i^{-1} \mathcal{X}_{B_{k_i+1} \setminus B_k} \right\|_\infty \prod_{i \in \mathcal{I}_2} |B_{k_i}|^{-\theta/n} \left( \int_{B_{k_i+1}} v_i^{-p_i'} \right)^{1/p_i'}
$$

Observe that $\mathbb{N}^m \subset \bigcup_{i=1}^m K_i$, where $K_i = \{ \tilde{k} = (k_1, k_2, \ldots, k_m) : k_i \geq k_j \text{ for every } j \}$. Let us estimate the sum over $K_1$, being similar for the other sets. Therefore

$$
\sum_{\tilde{k} \in K_1} I \left( B, \tilde{k} \right) \leq \sum_{k_1=1}^{\infty} |B_{k_1}|^{-\theta/n} \prod_{i \in \mathcal{I}_1} \left\| v_i^{-1} \mathcal{X}_{B_{k_i+1} \setminus B_{k_i}} \right\|_\infty \prod_{i \in \mathcal{I}_2} \left( \int_{B_{k_i+1}} v_i^{-p_i'} \right)^{1/p_i'} \prod_{i \neq 1} \sum_{k_i=1}^{k_1} |B_{k_i}|^{-\theta/n}.
$$

Notice that

$$
\sum_{k_1=1}^{\infty} |B_{k_1}|^{-\theta/n} = |B|^{-\theta/n} \sum_{k_1=1}^{\infty} 2^{-k_1 \theta} \lesssim |B_{k_1}|^{-\theta/n} \sum_{k_1=1}^{\infty} 2^{(k_1 - k_i) \theta} \lesssim |B_{k_1}|^{-\theta/n} 2^{k_1 \theta}.
$$
Thus, from the estimation above and (2.6) we obtain that
\[
\frac{\|wX^B\|_\infty}{|B|^{(\delta-\delta)/n}} \sum_{k \in K_1} I \left( B, k \right) \leq |B|^{(\delta-\delta)/n} \sum_{k_1=1}^{\infty} 2^{(m-1)k_1} |B_{k_1}|^{-m\theta/n} \|wX_{B_{k_1+1}}\|_\infty \\
\times \prod_{i \in I_1} \|v_i^{-1}X^B_{k_1+1}\|_\infty \prod_{i \in I_2} \left( \int_{B_{k_1+1}} v_i^{-p_i'} \right)^{1/p_i'} \\
\leq \sum_{k_1=1}^{\infty} 2^{-k_1(\theta-mn-\delta+\beta)},
\]
being the last sum finite since \(\tilde{\delta} < \tau\).

The following result establishes that when we restrict the pair of weights \((w, v)\) to satisfy the relation \(w = \prod_{i=1}^{m} v_i\), condition \(\mathbb{H}_m(p, \beta, \tilde{\delta})\) is equivalent, for a suitable range of the parameter \(\tilde{\delta}\), to \(A_{\vec{p},\infty}\).

**Corollary 6.4.** Let \(\tilde{\delta} \in \mathbb{R}, 0 < \beta < mn, \tilde{\delta} < \tau = (\beta-mn)(1-1/m)+\delta/m\). Then \(\vec{v} \in \mathbb{H}_m(p, \beta, \tilde{\delta})\) if and only if \(\vec{v} \in A_{\vec{p},\infty}\).

**Proof.** Let \(\vec{v} \in \mathbb{H}_m(p, \beta, \tilde{\delta})\). Then condition (2.6) holds. On the other hand, by Lemma 1.3, we have that \(\tilde{\delta} = \beta - n/p\). This implies that \(\vec{v} \in A_{\vec{p},\infty}\).

Conversely, let \(\vec{v} \in A_{\vec{p},\infty}\). Since \(\tilde{\delta} < \tau\), by Lemma 6.3 we have that \(\vec{v}\) satisfies (2.7). By Lemma 6.1 it will be enough to check that \(v_i^{-1} \in \text{RH}_{\infty}\) for \(i \in I_1\) and \(v_i^{-p_i'}\) is doubling for \(i \in I_2\). Let us first check that \(v_i^{-1} \in \text{RH}_{\infty}\) for \(i \in I_1\). Let \(r = p/(mp-1)\), fix \(i_0 \in I_1\) and observe that
\[
\frac{1}{|B|} \int_B v_i^{-r} = \frac{1}{|B|} \int_B v_i^{1} \prod_{i \neq i_0} v_i^{-r} \\
\leq \left\|X^B \prod_{i=1}^{m} v_i \right\|_\infty \prod_{i \neq i_0} \left\|v_i^{-1}X^B\right\|_\infty \prod_{i \in I_2} \left( \frac{1}{|B|} \int_B v_i^{-p_i'} \right)^{1/p_i'} \\
\leq \frac{[\vec{v}]_{A_{\vec{p},\infty}}}{\|v_i^{-1}X^B\|_\infty} \\
= [\vec{v}]_{A_{\vec{p},\infty}} \inf_{B} v_i^{-r}.
\]
Therefore, \(v_i^{-r}\) is an \(A_1\) weight. Then we can conclude \(v_i^{-1}\) is an \(\text{RH}_{\infty}\) weight.

On the other hand, observe that \(A_{\vec{p},\infty} \subseteq A_{\vec{p},q}\) for every \(q > 0\). If we pick \(q = p\) we can apply Lemma 3.3 to conclude that \(\vec{z} = (z_1, \ldots, z_m)\) belongs to \(A_{\vec{p}}\), where \(\ell = p, z_i = v_i^{\ell_i}\) and \(\ell_i = p_i\) for every \(i\). This implies (see, for example, Theorem 3.6 in [6]) that \(z_i^{1-\ell_i} \in A_{mp_i}\), that is, \(v_i^{-p_i'} \in A_{mp_i} \subseteq A_{\infty}\), so it is a doubling weight for every \(i \in I_2\). This completes the proof.

The following two theorems allows us to describe the region where we can find nontrivial weights in \(\mathbb{H}_m(p, \beta, \tilde{\delta})\) in terms of the parameters \(p, \beta\) and \(\tilde{\delta}\).

**Theorem 6.5.** Let \(\tilde{\delta} \in \mathbb{R}\) be fixed. Let \(0 < \beta < mn, \tilde{\delta} \in \mathbb{R}\) and \(\vec{p}\) be a vector of exponents. The following statements hold:
(a) If \( \tilde{\delta} > \delta \) or \( \tilde{\delta} > \beta - n/p \) then condition \( \mathbb{H}_m(\bar{\rho}, \beta, \tilde{\delta}) \) is satisfied if and only if \( v_i = \infty \) a.e. for some \( 1 \leq i \leq m \).

(b) The same conclusion holds if \( \tilde{\delta} = \beta - n/p = \delta \).

(c) If \( \tilde{\delta} < \beta - mn \), then condition \( \mathbb{H}_m(\bar{\rho}, \beta, \tilde{\delta}) \) is satisfied if and only if \( v_i = \infty \) a.e. for some \( 1 \leq i \leq m \) or \( w = 0 \) a.e.

Proof. Let \((w, \vec{v}) \in \mathbb{H}_m(\bar{\rho}, \beta, \tilde{\delta})\). We start with the proof of item (a). We shall first assume that \( \tilde{\delta} > \delta \). By picking a ball \( B = B(x_B, R) \) such that \( x_B \) is a Lebesgue point of \( w^{-1} \), from (2.5) we obtain

\[
\prod_{i \in I_1} \left\| \left( \int_{\mathbb{R}^n} \frac{v_i^{-1} \rho_i^{-1}}{(|B|^{1/n} + |x_B - \cdot|^{n-\beta_i+\delta/m})} \right) \frac{1}{\rho_i} \right\|_{B} \lesssim \frac{|B|^{(\delta-\delta)/n}}{\|w X_B\|_\infty} \lesssim \frac{w^{-1}(B)}{|B| R^{\tilde{\delta}-\beta+n/p}}
\]

for every \( R > 0 \). By letting \( R \) approach to zero we can deduce that there exists \( 1 \leq i \leq m \) such that \( v_i = \infty \) almost everywhere.

Let us now consider the case \( \tilde{\delta} > \beta - n/p \). Pick a ball \( B = B(x_B, R) \), being \( x_B \) a Lebesgue point of \( w^{-1} \) and of every \( v_i^{-1} \). Then condition (2.6) implies that

\[
\prod_{i=1}^m \frac{1}{|B|} \left( \int_B v_i^{-1} \right) \prod_{i \in I_2} \left\| v_i^{-1} X_B \right\|_{\infty} \prod_{i \in I_2} \left( \int_B v_i^{-1} \right)^{1/p_i} \lesssim \frac{|B|^{\frac{1}{n} - \frac{n-\beta + \delta}{p} + \frac{1}{p}}}{\|w X_B\|_\infty} \lesssim \frac{w^{-1}(B)}{|B| R^{\tilde{\delta}-\beta+n/p}}
\]

for every \( R > 0 \). If we let again \( R \) approach to zero, we obtain

\[
\prod_{i=1}^m v_i^{-1}(x_B) = 0,
\]

and then \( \prod_{i=1}^m v_i^{-1} \) is zero a.e. This allows us to conclude that the set \( \bigcap_{i=1}^m \{v_i^{-1} > 0\} \) has null measure. Since \( v_i(y) > 0 \) for almost every \( y \) and every \( i \), there exists \( j \) such that \( v_j = \infty \) a.e.

We now proceed with item (b). Suppose \( \tilde{\delta} = \beta - n/p = \delta \). We define

\[
\xi = \sum_{i=1}^m \frac{1}{\rho_i} = m - \frac{1}{p}.
\]

By applying Hölder inequality we obtain

\[
\left( \int_{\mathbb{R}^n} \left( \frac{(\prod_{i \in I_2} v_i^{-1})^{1/\xi}}{\sum_{i \in I_2} (|x_B - \cdot|^{n-\beta_i+\delta/m})^{\xi}} \right)^{\xi} \right)^{1/\xi} \lesssim \prod_{i \in I_2} \left( \int_{\mathbb{R}^n} \frac{v_i^{-\rho_i}}{\sum_{i \in I_2} (|x_B - \cdot|^{n-\beta_i+\delta/m})^{\xi}} \right)^{1/\rho_i}
\]

and since \((w, \vec{v}) \in \mathbb{H}_m(\bar{\rho}, \beta, \tilde{\delta})\) this implies that

\[
\prod_{i \in I_1} \left\| \left( \int_{\mathbb{R}^n} \frac{v_i^{-1}}{(|B|^{1/n} + |x_B - \cdot|^{n-\beta_i+\delta/m})^{\xi}} \right) \right\|_{\infty} \lesssim \frac{w^{-1}(B)}{|B|},
\]

and so we can deduce that for every ball \( B \)

\[
\left( \int_{\mathbb{R}^n} \frac{(\prod_{i=1}^m v_i^{-1})^{1/\xi}}{\sum_{i=1}^m (|x_B - y|^{mn-\beta+\delta})^{\xi}} \right)^{1/\xi} \lesssim \frac{w^{-1}(B)}{|B|}.
\]
From this inequality we can continue by adapting an argument of [14] in order to conclude that there exists \( i \) such that \( v_i = \infty \) a.e.

We conclude with the proof of item (c). If we assume that \( \tilde{\delta} < \beta - mn \), given a ball \( B = B(x_B, R) \) and \( B_0 \subset B \), condition (2.6) implies that

\[
\| w \chi_{B_0} \|_\infty \prod_{i=1}^{m} \| v_i^{-1} \chi_{B_0} \|_{p_i'} \leq \| w \chi_{B} \|_\infty \prod_{i=1}^{m} \| v_i^{-1} \chi_{B} \|_{p_i'} \lesssim R^{\tilde{\delta} - \beta + mn}.
\]

The right-hand side of the inequality above tends to zero when \( R \) approaches to \( \infty \), which implies that either \( \| w \chi_{B_0} \|_\infty = 0 \) or \( \| v_i^{-1} \chi_{B_0} \|_{p_i'} = 0 \), for some \( i \). By the arbitrariness of \( B_0 \) we obtain either \( w = 0 \) or \( v_i = \infty \) for some \( i \), respectively. \( \square \)

**Theorem 6.6.** Let \( \delta \in \mathbb{R} \) be fixed. Given \( 0 < \beta < mn \), there exist pairs of weights \( (w, \vec{v}) \) satisfying (2.5) for every \( \vec{p} \) and \( \tilde{\delta} \) such that \( \beta - mn \leq \tilde{\delta} \leq \min \{ \delta, \beta - n/p \} \), excluding the case \( \tilde{\delta} = \delta \) when \( \beta - n/p = \delta \).

The next figure depicts the area in which we can find nontrivial pair of weights belonging to \( \mathbb{H}_m(\vec{p}, \beta, \tilde{\delta}) \), for a fixed value \( \delta \) and depending on \( \beta \).

![Diagram](image)

Since the classes \( \mathbb{H}_m(\vec{p}, \beta, \tilde{\delta}) \) have a similar structure as those defined in [2], the proof of the theorem above will follow similar lines as that in Theorem 5.1 of [2], with adequate changes. We include a sketch for the sake of completeness. We shall need the following auxiliary lemma.

**Lemma 6.7.** For a ball \( B = B(x_B, R) \) in \( \mathbb{R}^n \) and \( \alpha > -n \), we have that

\[
\int_B |x|^\alpha \, dx \approx R^n \left( \max \{ R, |x_B| \} \right)^\alpha.
\]

**Proof of Theorem 6.6.** Recall that \( \tau = (\beta - mn)(1 - 1/m) + \delta/m \) is the number appearing in Lemma 6.3, we shall split the proof into the following cases:

(a) \( \beta - mn < \tilde{\delta} < \tau \leq \beta - n/p \);
(b) \( \beta - mn < \tilde{\delta} \leq \beta - n/p < \tau; \)
(c) \( \beta - mn < \tilde{\delta} = \tau < \delta < \beta - n/p; \)
(d) \( \beta - mn < \tilde{\delta} = \tau < \beta - n/p < \delta; \)
(e) \( \tau < \tilde{\delta} \leq \min\{\delta, \beta - n/p\}; \)
(f) \( \tilde{\delta} = \beta - mn. \)

Let us prove (a). Recall that \( I_1 = \{i : p_i = 1\}, I_2 = \{i : p_i > 1\} \) and let \( m_j = \#I_j, \) for \( j = 1, 2. \) Since \( m_1 < m \) by the restrictions on the parameters, we can take

\[ 0 < \varepsilon < \frac{mn - \beta + \tilde{\delta}}{m - m_1}. \]

For \( 1 \leq i \leq m \) we define

\[
\xi_i = \begin{cases} 
0 & \text{if } i \in I_1, \\
\frac{\rho}{p_i} - \varepsilon & \text{if } i \in I_2.
\end{cases}
\]

Let \( \rho = \sum_{i=1}^{m} \xi_i + \tilde{\delta} - \beta + n/p > 0. \) Then we take

\[ w(x) = |x|^\rho \quad \text{and} \quad v_i(x) = |x|^{\xi_i}. \]

By virtue of Lemma 6.3 it will be enough to show that \((w, \bar{v})\) verifies condition (2.6). Let \( B = B(x_B, R) \) and assume that \( |x_B| \leq R. \) If \( i \in I_2, \) by Lemma 6.7 we get

\[
\left( \frac{1}{|B|} \int_B v_i^{p_i'} \right)^{1/p_i'} = \left( \frac{1}{|B|} \int_B |x|^{-\xi_i p_i'} dx \right)^{1/p_i'} \approx R^{-\xi_i},
\]

and \( \|v_i^{-1} \chi_B\|_\infty = 1 \) for \( i \in I_1. \) On the other hand, \( \|w \chi_B\|_\infty \leq R^\rho \) since \( \rho > 0. \) Therefore,

\[
\frac{\|w \chi_B\|_\infty}{|B|^{\delta/n - \beta + n/p}} \prod_{i \in I_1} v_i^{-1} \chi_B \prod_{i \in I_2} \left( \frac{1}{|B|} \int_B |x|^{-\xi_i p_i'} dx \right)^{1/p_i'} \leq R^{\rho - \sum_{i=1}^{m} \xi_i - \tilde{\delta} + \beta - n/p} \leq C.
\]

We now consider the case \( |x_B| > R. \) We have that

\[ \|w \chi_B\|_\infty \lesssim |x_B|^\rho, \]

whilst for \( i \in I_2 \)

\[
\left( \frac{1}{|B|} \int_B |x|^{-\xi_i p_i'} dx \right)^{1/p_i'} \approx |x_B|^{-\xi_i}.
\]

Consequently, since \( \tilde{\delta} < \beta - n/p \) we get

\[
\frac{\|w \chi_B\|_\infty}{|B|^{\delta/n - \beta + n/p}} \prod_{i \in I_1} v_i^{-1} \chi_B \prod_{i \in I_2} \left( \frac{1}{|B|} \int_B |x|^{-\xi_i p_i'} dx \right)^{1/p_i'} \lesssim |x_B|^\rho \lesssim |x_B|^{\rho - \sum_{i=1}^{m} \xi_i - \tilde{\delta} + \beta - n/p} \leq C,
\]

which completes the proof of (a).

We now prove (b). In this case we take \( w = 1 \) and \( v_i = |x|^{\xi_i}, \xi_i = (\beta - \tilde{\delta})/m - n/p_i \) for every \( 1 \leq i \leq m. \) By Lemma 6.3 it will be enough to prove that \((w, \bar{v})\) satisfies condition (2.6). Pick a ball \( B = B(x_B, R) \) and assume that \( |x_B| \leq R. \) If \( i \in I_1 \) we get \( \xi_i < 0, \) since we are assuming \( \tilde{\delta} > \beta - mn. \) In this case we get

\[ \|v_i^{-1} \chi_B\|_\infty \approx R^{-\xi_i}. \]

On the other hand, for \( i \in I_2 \) we have \( \xi_i < n/p_i, \) so Lemma 6.7 yields

\[ \left( \frac{1}{|B|} \int_B |x|^{-\xi_i p_i'} dx \right)^{1/p_i'} \approx R^{-\xi_i}. \]
These two estimates imply that
\[
\frac{\|w \mathcal{X}_B\|_\infty}{|B|^{\delta/n-\beta/n+1/p}} \prod_{i \in \mathcal{I}_1} \|v_i^{-1} \mathcal{X}_B\|_\infty \prod_{i \in \mathcal{I}_2} \left( \frac{1}{|B|} \int_B v_i^{-p_i'} \right)^{1/p_i'} \lesssim \frac{R^{-\sum_{i=1}^m \xi_i}}{R^{\delta-\beta+n/p}} = 1.
\]
If \(|x_B| > R\), we have that \(\|v_i^{-1} \mathcal{X}_B\|_\infty \lesssim |x_B|^{-\xi_i}\) and also
\[
\left( \frac{1}{|B|} \int_B v_i^{-p_i'} \right)^{1/p_i'} \approx |x_B|^{-\xi_i}
\]
by Lemma 6.7. Thus
\[
\frac{\|w \mathcal{X}_B\|_\infty}{|B|^{\delta/n-\beta/n+1/p}} \prod_{i \in \mathcal{I}_1} \|v_i^{-1} \mathcal{X}_B\|_\infty \prod_{i \in \mathcal{I}_2} \left( \frac{1}{|B|} \int_B v_i^{-p_i'} \right)^{1/p_i'} \lesssim \frac{|x_B|^{-\sum_{i=1}^m \xi_i}}{R^{\delta-\beta+n/p}} \leq 1,
\]
since \(\delta \leq \beta - n/p\). This concludes the proof of item (b).

In order to prove (c) we pick \((\beta - \tau)/m - n/p_i < \xi_i < n/p_i'\) for \(i \in \mathcal{I}_2\) and \(\xi_i = 0\) for \(i \in \mathcal{I}_1\). We also take \(\rho = \sum_{i=1}^m \xi_i + \tau - \beta + n/p > 0\) and define \(w(x) = |x|^\rho\) and \(v_i(x) = |x|^{\xi_i}\), for \(1 \leq i \leq m\). We first notice that
\[
\prod_{i \in \mathcal{I}_1} \left\| \frac{v_i^{-1} \mathcal{X}_{B \setminus B}}{|x_B - \cdot|^{n-\beta/m+\delta/m}} \right\|_\infty \leq R^{-\sum_{i \in \mathcal{I}_1} (n-\beta/m+\delta/m)}.
\]

By virtue of Lemma 6.1 we have to prove that condition (2.7) holds. By using the estimate above, it will be enough to show that
\[
(6.1) \quad R^{\delta-\delta-\sum_{i \in \mathcal{I}_1} (n-\beta/m+\delta/m)} \|w \mathcal{X}_B\|_\infty \prod_{i \in \mathcal{I}_2} \left( \int_{\mathbb{R}^n \setminus B} \frac{v_i^{-p_i'}(y)}{|x_B - y|^{(n-\beta/m+\delta/m)p_i}} \ dy \right)^{1/p_i'} \leq C
\]
for every ball \(B\). We shall first assume that \(|x_B| \leq R\). Let \(B_k = B(x_B, 2^k R)\) for \(k \in \mathbb{N}_0\) and \(i \in \mathcal{I}_2\). By Lemma 6.7, we get
\[
\left( \int_{\mathbb{R}^n \setminus B} \frac{v_i^{-p_i'}(y)}{|x_B - y|^{(n-\beta/m+\delta/m)p_i}} \ dy \right)^{1/p_i'} \lesssim \sum_{k=0}^\infty (2^k R)^{-n+\beta/m-\delta/m} \left( \int_{B_{k+1} \setminus B_k} |y|^{-\xi_i} v_i^{p_i'} \ dy \right)^{1/p_i'} \lesssim \sum_{k=0}^\infty (2^k R)^{-n+\beta/m-\delta/m-\xi_i+n/p_i} \lesssim R^{-n/p_i+\beta/m-\delta/m-\xi_i},
\]
since \(-n/p_i+\beta/m-\delta/m-\xi_i < 0\) by the election of \(\xi_i\). Since \(\delta = \tau\), the left-hand side of (6.1) is bounded by a multiple constant of
\[
R^{\delta-\delta-\sum_{i \in \mathcal{I}_1} (n-\beta/m+\delta/m)+\rho-\sum_{i \in \mathcal{I}_2} (n/p_i-\beta/m+\delta/m+\xi_i)} = R^{-\delta-n/p+\beta+\rho-\sum_{i=1}^m \xi_i} = 1.
\]

We now assume \(|x_B| > R\). There exists a number \(N\) such that \(2^N R < |x_B| \leq 2^{N+1} R\). If \(i \in \mathcal{I}_2\) we have that
\[
\left( \int_{\mathbb{R}^n \setminus B} \frac{v_i^{-p_i'}(y)}{|x_B - y|^{(n-\beta/m+\delta/m)p_i}} \ dy \right)^{1/p_i'} \lesssim \sum_{k=0}^\infty (2^k R)^{-n+\beta/m-\delta/m} \left( \int_{B_k} |y|^{-\xi_i} v_i^{p_i'} \ dy \right)^{1/p_i'}
\]
for every ball \(B_k\). By Lemma 6.7, we get
Let \( \theta_i = n/p_i + (\delta - \beta)/m \), for \( 1 \leq i \leq m \). We shall first prove that if \( \theta_i < 0 \), then
\[
(6.2) \quad S_j^i \lesssim |x_B|^{-\xi_i - \theta_i},
\]
for \( j = 1, 2 \). Indeed, by Lemma 6.7 we obtain
\[
S_1^i \lesssim \sum_{k=0}^{N} (2^k R)^{-n+\beta/m-m+n/p_i} |x_B|^{-\xi_i} \lesssim |x_B|^{-\xi_i} \sum_{k=0}^{N} 2^{-k\theta_i} \lesssim |x_B|^{-\xi_i} (2^N R)^{-\theta_i} \lesssim |x_B|^{-\xi_i - \theta_i}.
\]
For \( S_2^i \) we apply again Lemma 6.7 in order to get
\[
S_2^i \lesssim \sum_{k=N+1}^{\infty} (2^k R)^{-n+\beta/m-m+n/p_i} |x_B|^{-\xi_i} \lesssim \sum_{k=N+1}^{\infty} (2^k R)^{-\xi_i - \theta_i} = (2^{N+1} R)^{-\xi_i - \theta_i} \sum_{k=0}^{\infty} 2^{-k(\xi_i + \theta_i)} \lesssim |x_B|^{-\xi_i - \theta_i},
\]
since \( \theta_i + \xi_i = n/p_i + (\delta - \beta)/m + \xi_i > 0 \).

We now assume that \( \theta_i = 0 \). By proceeding similarly as in the previous case, we have
\[
S_1^i \lesssim |x_B|^{-\xi_i} \lesssim |x_B|^{-\xi_i} \log_2 \left( \frac{|x_B|}{R} \right),
\]
and
\[
S_2^i \lesssim |x_B|^{-\xi_i}
\]
since \( \xi_i > 0 \) when \( \theta_i = 0 \). Consequently,
\[
(6.3) \quad S_1^i + S_2^i \lesssim |x_B|^{-\xi_i} \left( 1 + \log_2 \left( \frac{|x_B|}{R} \right) \right) \lesssim |x_B|^{-\xi_i} \log_2 \left( \frac{|x_B|}{R} \right).
\]
We finally consider the case \( \theta_i > 0 \). For \( S_2^i \) we can proceed exactly as in the case \( \theta_i < 0 \) and get the same bound. On the other hand, for \( S_1^i \) we have that
\[
S_1^i \lesssim \sum_{k=0}^{N} (2^k R)^{-n+\beta/m-m+n/p_i} |x_B|^{-\xi_i} \lesssim |x_B|^{-\xi_i} R^{-\theta_i} \sum_{k=0}^{N} 2^{-k\theta_i} \lesssim |x_B|^{-\xi_i - \theta_i} 2^{N\theta_i}.
\]
Therefore, if \( i \in I_2 \) and \( \theta_i > 0 \) we get
\[
(6.4) \quad S_1^i + S_2^i \lesssim |x_B|^{-\xi_i - \theta_i} \left( 1 + 2^{N\theta_i} \right) \lesssim 2^{N\theta_i} |x_B|^{-\xi_i - \theta_i}.
\]
By combining (6.2), (6.3) and (6.4) we obtain
\[
\prod_{i \in I_2} \left( \int_{\mathbb{R}^n \setminus B} \frac{v_i^{-p_i'(y)}}{|x_B - y|^{n-\beta/m+\delta/m}\mu_i} dy \right)^{1/p_i'} \lesssim \prod_{i \in I_2, \theta_i < 0} |x_B|^{-\xi_i - \theta_i} \prod_{i \in I_2, \theta_i = 0} |x_B|^{-\xi_i} \log_2 \left( \frac{|x_B|}{R} \right) \times \prod_{i \in I_2, \theta_i > 0} |x_B|^{-\xi_i - \theta_i} 2^{N\theta_i}
\]
so the left-hand side of (6.1) can be bounded by a multiple constant of
\[ R^{\tilde{\delta}-(n-\beta/m+\delta/m)m_1} |x_B|^{\rho-\sum_{i \in I_2}(\xi_{i}+\theta_{i})} 2^{N \sum_{i \in I_2, \theta_i > 0} \theta_i} \left( \log_2 \left( \frac{|x_B|}{R} \right) \right)^{\# \{i \in I_2, \theta_i = 0\}}, \]
which is equivalent to
\[ (\frac{|x_B|}{R})^{\tau-\delta+(n-\beta/m+\delta/m)m_1+\sum_{i \in I_2, \theta_i > 0} \theta_i} \left( \log_2 \left( \frac{|x_B|}{R} \right) \right)^{\# \{i \in I_2, \theta_i = 0\}}. \]
(6.5)
Since \( \theta_i < n + (\delta - \beta)/m \) for \( i \in I_2 \), there exists \( \varepsilon > 0 \) that verifies
\[ \sum_{i \in I_2, \theta_i > 0} \theta_i + \varepsilon \# \{i \in I_2, \theta_i = 0\} \leq (n + \frac{\delta - \beta}{m}) \# \{i \in I_2, \theta_i > 0\}. \]
By using the fact that \( \log_2 t \lesssim t^{\varepsilon} \) for every \( t \geq 1 \), we can majorize (6.5) by a constant factor provided that
\[ \tau - \delta + \left( n + \frac{\delta - \beta}{m} \right) \left( m_1 + \# \{i \in I_2 : \theta_i > 0\} \right) \leq \tau - \delta + \left( n + \frac{\delta - \beta}{m} \right) (m - 1) = 0. \]
Indeed, if this last inequality did not hold, then we would have that \( \theta_i > 0 \) for every \( i \in I_2 \). We also observe that \( \theta_i > 0 \) for \( i \in I_1 \). This would lead to \( n/p > \beta - \delta \), a contradiction.

In order to prove (d) we only consider two cases. If there exists some \( i \in I_2 \) such that \( \theta_i \leq 0 \), the proof follows exactly as in (c). If not, that is \( \theta_i > 0 \) for every \( i \in I_2 \), we observe that
\[ \tau - \delta + \left( n + \frac{\delta - \beta}{m} \right) m_1 + \sum_{i \in I_2, \theta_i > 0} \theta_i = \tau - \delta + \sum_{i=1}^{m} \left( \frac{n}{p_i} + \frac{\delta - \beta}{m} \right) \]
\[ = \tau + \frac{n}{p} - \beta \]
\[ < 0, \]
then we can choose \( \varepsilon > 0 \) small enough so that the resulting exponent for \( |x_B|/R \) in (6.5) becomes negative.

We now proceed with the proof of (e). Let us first suppose that \( \tilde{\delta} < \min\{\delta, \beta - n/p\} \). We take \( \rho = \tilde{\delta} - \tau > 0 \) and \( \tilde{\xi}_i = (\delta - \tau)/m - \theta_i \), for every \( i \). Then we define \( w(x) = |x|^\rho \) and \( v_i = |x|^\tilde{\xi}_i \), \( 1 \leq i \leq m \). These functions are locally integrable since \( \rho > 0 \) and \( \tilde{\xi}_i < n/p_i \). Furthermore, \( \tilde{\xi}_i < 0 \) for \( i \in I_1 \), so \( v_i^{-1} \in \text{RH}_{\infty} \) for these index. Then, by Lemma 6.1, it will be enough to show that condition (2.7) holds. Fix a ball \( B = B(x_B, R) \) and assume that \( |x_B| < R \). Then we get
\[ \left\| \frac{wX_B}{|B|^{(\delta - \delta)/n}} \right\|_\infty \leq R^{\delta - \delta + \rho} = R^{\delta - \tau}. \]
(6.6)
On the other hand, if \( i \in I_1 \) we have
\[ \left\| \frac{v_i^{-1}X_{B^n \setminus B}}{|x_B - |n-\beta/m+\delta/m|}_\infty \right\| \lesssim \sum_{k=0}^{\infty} \left\| \frac{v_i^{-1}X_{B_{k+1} \setminus B_k}}{|x_B - |n-\beta/m+\delta/m|}_\infty \right\|.
Finally, if $\xi_i + \theta_i = (\delta - \tau)/m > 0$ for $i \in I_2$, we can proceed as in page 25 to obtain

\[(6.8) \quad \prod_{i \in I_2} \left( \int_{\mathbb{R}^n \setminus B} \frac{v_i^{-\lambda_i'(y)}}{|x_B - y|^{(n-\beta/m+\delta/m)p_i'}} \, dy \right)^{1/p_i'} \lesssim R^{m_2(\delta-\delta)/m}.\]

By combining (6.6), (6.7) and (6.8), the left-hand side of (2.7) is bounded by a constant $C$.

We now consider the case $|x_B| > R$. We have that

\[(6.9) \quad \frac{\|w \chi_B\|_\infty}{|B|^{(\delta-\delta)/n}} \lesssim R^{\delta-\delta} |x_B|^\rho.\]

Since $|x_B| > R$, there exists a number $N \in \mathbb{N}$ such that $2^N R < |x_B| \leq 2^{N+1} R$. For $i \in I_1$ we write

\[
\left\| \frac{v_i^{-1} \chi_{R^n \setminus B}}{|x_B - .|^{n-\beta/m+\delta/m}} \right\|_\infty \lesssim \sum_{k=0}^{N} \left\| \frac{v_i^{-1} \chi_{B_{k+1} \setminus B_k}}{|x_B - .|^{n-\beta/m+\delta/m}} \right\|_\infty + \sum_{k=N+1}^{\infty} \left\| \frac{v_i^{-1} \chi_{B_{k+1} \setminus B_k}}{|x_B - .|^{n-\beta/m+\delta/m}} \right\|_\infty
= S_i^1 + S_i^2.
\]

By proceeding as in page 26 with $p_i = 1$ we have that

\[(6.10) \quad \prod_{i \in I_1} \left\| \frac{v_i^{-1} \chi_{R^n \setminus B}}{|x_B - .|^{n-\beta/m+\delta/m}} \right\|_\infty \lesssim |x_B|^{-\sum_{i \in I_1} (\xi_i + \theta_i)} 2^N \sum_{i \in I_1} \theta_i.
\]

Finally, if $i \in I_2$ our choice of $\xi_i$ allows us to follow the argument given in page 25 to conclude that

\[
\prod_{i \in I_2} \left( \int_{\mathbb{R}^n \setminus B} \frac{v_i^{-\lambda_i'(y)}}{|x_B - y|^{(n-\beta/m+\delta/m)p_i'}} \, dy \right)^{1/p_i'} \lesssim |x_B|^{-\sum_{i \in I_2} (\xi_i + \theta_i)} 2^N \sum_{i \in I_2} \theta_i > \theta_i
\]

\[
\times \left( \log_2 \left( \frac{|x_B|}{R} \right) \right)^{\#\{i \in I_2, \theta_i = 0\}}.
\]

By combining the inequality above with (6.9) and (6.10), the left-hand side of (2.7) can be bounded by a multiple constant of

\[
R^{\delta-\delta} |x_B|^{\rho - \sum_{i=i}^{m} (\theta_i + \xi_i)} 2^N \sum_{i: \theta_i > 0} \theta_i \left( \log_2 \left( \frac{|x_B|}{R} \right) \right)^{\#\{i \in I_2, \theta_i = 0\}}
\]

which is equal to

\[
\left( \frac{R}{|x_B|} \right)^{\delta-\delta - \sum_{i: \theta_i > 0} \theta_i} \left( \log_2 \left( \frac{|x_B|}{R} \right) \right)^{\#\{i \in I_2, \theta_i = 0\}}.
\]
If \( \theta_i < 0 \) for every \( i \) then the exponent of \( R/|x_B| \) is positive. On the other hand, if \( \theta_i \geq 0 \) for every \( i \) then
\[
\delta - \tilde{\delta} - \sum_{i: \theta_i > 0} \theta_i = \delta - \tilde{\delta} - \sum_{i=1}^m \theta_i = \delta - \tilde{\delta} - \frac{n}{p} + \beta - \delta > 0,
\]
since \( \tilde{\delta} < \beta - n/p \). In both cases we can repeat the argument given in page 27 to conclude that \( (w, \tilde{\theta}) \) belongs to \( \mathcal{M}_{n/p}(\tilde{\beta}, \delta, \tilde{\delta}) \). Let us observe that, for example, if \( \beta \leq \delta \) then every \( \theta_i \) is nonnegative.

If \( \tilde{\delta} = \delta < \beta - n/p \) or \( \tilde{\delta} = \beta - n/p < \delta \) the same estimation as above works when we take \( \theta_i < 0 \) for every \( i \). The second case also works when \( \theta_i > 0 \) for every \( i \).

We finish with the proof of item (i). In this case we fix \( \rho > 0 \) and take \( w(x) = (1 + |x|^{\rho})^{-m_1} \). If \( g_i \) are nonnegative fixed functions in \( L^{\bar{p}_i}(\mathbb{R}^n) \) for \( i \in I_2 \), we define
\[
v_i(x) = \begin{cases} 
eq |x| & \text{if } i \in I_1, \\ g_i^{-1} & \text{if } i \in I_2.
\end{cases}
\]

Fix a ball \( B = B(x_B, R) \). It is enough to check condition (2.6), since \( \tilde{\delta} = \beta - mn < \tau \). Notice that
\[
\|w\mathcal{X}_B\|_{\infty} \prod_{i \in I_1} \|v_i^{-1}\mathcal{X}_B\|_{\infty} \leq \prod_{i \in I_1} \|(1 + | \cdot |^{\rho})^{-1}\mathcal{X}_B\|_{\infty} \|e^{-\delta | \cdot |}\mathcal{X}_B\|_{\infty} \leq 1.
\]
Therefore,
\[
\frac{\|w\mathcal{X}_B\|_{\infty}}{|B|^{\delta/\tau - \beta - n + 1/p}} \prod_{i \in I_1} \|v_i^{-1}\mathcal{X}_B\|_{\infty} \prod_{i \in I_2} \left( \frac{1}{|B|} \int_B v_i^{\tau} \right)^{1/p_i} \lesssim \prod_{i \in I_2} \|g_i\|_{p_i}^p,
\]
for every ball \( B \). This concludes the proof of (i).

\[ \square \]

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