On quantum $K$-groups of partial flag manifolds

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Abstract

We show that the equivariant small quantum $K$-group of a partial flag manifold is a quotient of that of the full flag manifold as a based ring. This yields a variant of the $K$-theoretic analogue of the parabolic version of Peterson’s theorem [Lam-Shimozono, Acta Math. 204 (2010)] that exhibits different shape from the case of quantum cohomology. This note can be seen as an addendum to [K, arXiv:1805.01718 and arXiv:1810.07106].

Introduction

Let $G$ be a simply connected simple algebraic group over $\mathbb{C}$ with a maximal torus $H$ and a Borel subgroup $B$ that contains $H$. For each (standard) parabolic subgroup $B \subset P \subset G$, we have a partial flag variety $G/P$. Let $Gr$ denote the affine Grassmannian of $G$. In this note, we describe the $H$-equivariant small quantum $K$-group $qK_H(G/P)$ of $G/P$ as a quotient of the $H$-equivariant small quantum $K$-group $qK_H(G/B)$ of $G/B$.

The work of Peterson [34] (on quantum cohomology), whose main results appeared as Lam-Shimozono [30], states that we can recover the structure of the $H$-equivariant small quantum cohomology $qH_H(G/P)$ of $G/P$ by using the $H$-equivariant cohomology of $Gr$. In this context, we have a ring surjection $qH_H(G/B) \rightarrow qH_H(G/P)$ as a consequence of detailed study ([32]).

In [21, 20], we shed a light on the $K$-theoretic version of the above relation (for $G/B$) by employing the equivariant $K$-group of a semi-infinite flag manifold ([22]) as a mediator, following an idea by Givental [14]. From this view point, the connection between $qK_H(G/P)$’s for different $P$’s looks simpler as the structure of $K_H(G/P)$ is known to be governed by that of $K_H(G/B)$ through the pullback map $K_H(G/P) \rightarrow K_H(G/B)$.

The goal of this note is to take this advantage to prove the following:

Theorem A ($\doteq$ Theorem 2.18). There exists a surjective morphism

$$qK_H(G/B) \twoheadrightarrow qK_H(G/P)$$

of algebras that sends a Schubert basis to a Schubert basis. Moreover, if $B \subset P' \subset P$ is an intermediate standard parabolic subgroup, then the above algebra map factors through $qK_H(G/P')$.

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The same proof also works for its non-commutative variant (Corollary 2.19).
Here we stress that the existence of this map is purely of quantum nature, and it does not specialize to give an algebra map $K_H(G/B) \rightarrow K_H(G/P)$. In fact, our algebra map specializes some of the Novikov variables to 1, as opposed to 0 employed in the cases of quantum cohomologies [30, 32]. As a consequence, our algebra map exhibits mixed nature of [30] and [32], whose exact meaning is unclear at the moment. By setting $P = G$, we obtain a ring morphism

$$qK_H(G/B) \rightarrow qK_H(pt) = K_H(pt)$$

in Buch-Chung-Li-Mihalcea [7, Corollary 10].

In view of the $K$-theoretic version of the Peterson isomorphism (conjectured in [28] and proved as [21, Corollary C]), we also conclude a surjective morphism

$$K_H(Gr)_{\text{loc}} \rightarrow qK_H(G/P)_{\text{loc}}$$

(0.1)
of suitably localized algebras (Theorem 2.22). This also sends a Schubert basis to a Schubert basis (up to a Novikov monomial), and hence enforces the theme developed in [8, 28, 6] and the references therein.

We remark that the explicit nature of Theorem A and (0.1) allows us to transplant various multiplication formulas of $qK_H(G/B)$ (that can be seen in [29, 22, 33] etc...) to the setting of $qK_H(G/P)$.

The organization of this note is as follows. In §1, we collect preliminary results including those of equivariant quantum $K$-groups and quasi-map spaces. In §2, we cite results from [21, 20] to establish that certain Schubert varieties of parabolic quasi-map spaces have rational singularities (Theorem 2.11). Also, we introduce variants of equivariant $K$-groups $K_H(Q_J)$ of the semi-infinite (partial) flag manifold $Q_J$ different from those in [22] and [21] that are more suited for our purpose (Theorem 2.5 and the proof of Theorem 2.14). Other than these, we mainly follow the arguments of [21] with necessary modifications, though we tried to exhibit them slightly different in flavor. We also provide example calculations for $G = SL(3)$ in §3.

1 Preliminaries

A vector space is always a $C$-vector space, and a graded vector space refers to a $\mathbb{Z}$-graded vector space whose graded pieces are finite-dimensional and its grading is bounded from the above. Tensor products are taken over $C$ unless stated otherwise. We define the graded dimension of a graded vector space as

$$\text{gdim } M := \sum_{i \in \mathbb{Z}} q^i \dim_C M_i \in \mathbb{Q}(q^{-1}).$$

We set $C^0_q := \mathbb{C}[q^{-1}]$, $C_q := \mathbb{C}[q, q^{-1}]$, and $C_q := \mathbb{C}((q^{-1}))$ for the notational convention. As a rule, we suppress $\emptyset$ and associated parenthesis from notation. This particularly applies to $\emptyset = J \subset I$ frequently used to specify parabolic subgroups.

1.1 Groups, root systems, and Weyl groups

We refer to [9, 27] for precise expositions of general material presented in this subsection.
Let $G$ be a connected, simply connected simple algebraic group of rank $r$ over $\mathbb{C}$, and let $B$ and $H$ be a Borel subgroup and a maximal torus of $G$ such that $H \subset B$. We set $N := [B, B]$ to be the unipotent radical of $B$. We denote the Lie algebra of an algebraic group by the corresponding German small letter.

We have a (finite) Weyl group $W := N_G(H)/H$. For an algebraic group $E$, we denote its set of $\mathbb{C}[z]$-valued points by $E[z]$, and its set of $\mathbb{C}[[z]]$-valued points by $E[[z]]$ etc. Let $I \subset G[z]$ be the preimage of $B \subset G$ via the evaluation at $z = 0$ (the Iwahori subgroup of $G[z]$). By abuse of notation, we might consider $I$ and $G[z]$ as group schemes over $\mathbb{C}$ whose $\mathbb{C}$-valued points are given as these.

Let $P := \text{Hom}_{fr}(H, \mathbb{G}_m)$ be the weight lattice of $H$, let $\Delta \subset P$ be the set of roots, let $\Delta^+ \subset \Delta$ be the set of roots that yield root subspaces in $b$, and let $\Pi \subset \Delta^+$ be the set of simple roots. Each $\alpha \in \Delta^+$ defines a reflection $s_\alpha \in W$. Let $Q^\vee$ be the dual lattice of $Q$, be the set of simple coroots, and let $\Delta$ be the set of roots that yield root subspaces in $b$. For $\beta, \gamma \in Q^\vee$, we define $\beta \geq \gamma$ if and only if $\beta - \gamma \in Q^+_\vee$. We set $P_+ := \{ \lambda \in P \mid (\alpha^\vee, \lambda) \geq 0 \text{ for all } \alpha \in \Pi \}$ and $P^+ := \{ \lambda \in P \mid (\alpha^\vee, \lambda) > 0 \text{ for all } \alpha \in \Pi \}$. We fix bijections $\Pi \cong W$ that correspond to $s_\alpha \in W$, its coroot $\alpha^\vee \in \Pi^\vee$, and a simple reflection $s_i = s_\alpha_i \in W$. Let $\{\pi_i\}_{i \in \mathbb{I}} \subset P_+$ be the set of fundamental weights (i.e. $\langle \alpha_i^\vee, \pi_j \rangle = \delta_{ij}$).

For a subset $J \subset \mathbb{I}$, we define $P(J)$ as the standard parabolic subgroup of $G$ corresponding to $J$. I.e. we have $b \subset p(J) \subset g$ and $p(J)$ contains the root subspace corresponding to $-\alpha_i$ ($i \in \mathbb{I}$) if and only if $i \in J$. We set $J^c := \mathbb{I} \setminus J$. Then, the set of characters of $P(J)$ is identified with $P_2 := \sum_{i \in J} \mathbb{Z}\alpha_i$. We also set $P_{2,+} := \sum_{i \in J} \mathbb{Z}_{\geq 0}\alpha_i = P_+ \cap P_2$ and $P_{1,+} := \sum_{i \in J} \mathbb{Z}\alpha_i = P_+ \cap P_2$. We set $Q_2^\vee := \sum_{i \in J} \mathbb{Z}\alpha_i^\vee$ and $Q_{2,+}^\vee := \sum_{i \in J} \mathbb{Z}_{\geq 0}\alpha_i^\vee$. We define $W_J \subset W$ to be the reflection subgroup generated by $\{s_i\}_{i \in J}$. It is the Weyl group of the semisimple quotient of $P(J)$.

Let $\Delta_{af} := \Delta \times \mathbb{Z}di \cup \{ m\delta \}_{m \neq 0}$ be the untwisted affine root system of $\Delta$ with its positive part $\Delta^+ \subset \Delta_{af}$. We set $\alpha_0 := -\vartheta + \delta$, $\Pi_{af} := \Pi \cup \{ \alpha_0 \}$, and $\Gamma_{af} := \mathbb{I} \cup \{ 0 \}$, where $\vartheta$ is the highest root of $\Delta^+$. We set $W_{af} := W \times \mathbb{Q}^\vee$ and call it the affine Weyl group. It is a reflection group generated by $\{s_i \mid i \in \Gamma_{af}\}$, where $s_0$ is the reflection with respect to $\alpha_0$. Let $\ell : W_{af} \to \mathbb{Z}_{\geq 0}$ be the length function and let $w_0 \in W$ be the longest element in $W \subset W_{af}$. Together with the normalization $t_{-\vartheta'} := s_0 s_{\vartheta}$ (for the coroot $\vartheta^\vee$ of $\vartheta$), we introduce the translation element $t_{\beta} \in W_{af}$ for each $\beta \in Q_{af}^\vee$. By abuse of notation, we denote by $W/W_J$ the set of minimal length $W_J$-coset representatives in $W$.

Let $W_{af}^-$ denote the set of minimal length representatives of $W_{af}/W$ in $W_{af}$. We set

$$Q_{af}^- := \{ \beta \in Q_{af}^\vee \mid \langle \beta, \alpha_i \rangle < 0, \forall i \in \mathbb{I} \}.$$

For each $\lambda \in P_+$, we denote by $L(\lambda)$ the corresponding irreducible $G$-module with a highest $B$-weight $\lambda$. I.e. $L(\lambda)$ has a $B$-eigenvector with its $H$-weight $\lambda$. For a semi-simple $H$-module $V$, we set

$$\text{ch} V := \sum_{\lambda \in P} e^\lambda \cdot \dim \text{Hom}_H(\mathbb{C}_\lambda, V).$$

If $V$ is a $\mathbb{Z}$-graded $H$-module in addition, then we set

$$\text{gch} V := \sum_{\lambda \in P, n \in \mathbb{Z}} q^n e^\lambda \cdot \dim \text{Hom}_H(\mathbb{C}_\lambda, V_n).$$
Let \( \mathcal{B}_J := G/P(J) \) and call it the (partial) flag manifold of \( G \). We have the Bruhat decomposition
\[
\mathcal{B}_J = \bigsqcup_{u \in W/W_J} Q_J(u)
\]
(1.1)
into \( B \)-orbits such that \( \text{codim}_{\mathcal{B}_J} Q_J(u) = \ell(u) \) for each \( u \in W/W_J \subset W_{af} \). We set \( \mathcal{B}_J(u) := Q_J(u) \subset \mathcal{B} \).

For each \( \lambda \in P_3 \), we have a line bundle \( O_{\mathcal{B}_J}(\lambda) \) such that
\[
H^0(\mathcal{B}_J, O_{\mathcal{B}_J}(\lambda)) \cong L(-w_0\lambda), \quad O_{\mathcal{B}_J}(\lambda) \otimes O_{\mathcal{B}_J}(-\mu) \cong O_{\mathcal{B}_J}(\lambda-\mu) \quad \lambda, \mu \in P_{3,+}.
\]

For each \( u \in W/W_J \), let \( p_u \in Q_J(u) \) be the unique \( H \)-fixed point. We normalize \( p_u \) (and hence \( Q_J(u) \)) so that the restriction of \( H^0(\mathcal{B}_J, O_{\mathcal{B}_J}(\lambda)) \) to \( p_u \) is isomorphic to \( \mathbb{C}_{-u\lambda} \) for every \( \lambda \in P_{3,+} \). (Here we warn that the convention differs from [21] by the twist of \( -w_0 \). This change of convention also applies to \( Q^\gamma \) in §1.2 in order to keep the degree in Theorem 1.2.)

### 1.2 Quasi-map spaces

Here we recall basics of quasi-map spaces from [12, 11, 20].

We have isomorphisms \( H^2(\mathcal{B}_J, \mathbb{Z}) \cong P_3 \) and \( H^2(\mathcal{B}_J, \mathbb{Z}) \cong Q^\gamma_1 \). This identifies the (integral points of the) nef cone of \( \mathcal{B}_J \) with \( P_{3,+} \subset P_3 \) and the effective cone of \( \mathcal{B}_J \) with \( Q^\gamma_{1,+} \). A quasi-map \((f, D)\) is a map \( f : \mathbb{P}^1 \to \mathcal{B}_J \) together with a colored effective divisor
\[
D = \sum_{x \in \mathbb{P}^1(C)} \beta_x \otimes (x) \in Q^\gamma_{1,+} \otimes \text{Div} \mathbb{P}^1 \quad \beta_x \in Q^\gamma_{1,+}
\]
We call \( D \) the defect of the quasi-map \((f, D)\). Here we define the degree of the defect by
\[
\|D\| := \sum_{x \in \mathbb{P}^1(C)} \beta_x \in Q^\gamma_{1,+}.
\]

For each \( \beta \in Q^\gamma_{1,+} \), we set
\[
Q(\mathcal{B}_J, \beta) := \{ f : \mathbb{P}^1 \to X \mid \text{quasi-map s.t. } f_*(\mathbb{P}^1) + |D| = \beta \},
\]
where \( f_*(\mathbb{P}^1) \) is the class of the image of \( \mathbb{P}^1 \) multiplied by the degree of \( \mathbb{P}^1 \to \text{Im } f \).

We denote \( Q(\mathcal{B}_J, \beta) \) by \( Q(\beta) \) in case there is no danger of confusion.

**Definition 1.1** (Drinfeld-Plücker data). Consider a collection \( \mathcal{L} = \{ (\psi_\lambda, L^\lambda) \}_{\lambda \in P_{3,+}} \) of inclusions \( \psi_\lambda : L^\lambda \to L(\lambda) \otimes O_{\mathbb{P}^1} \) of line bundles \( L^\lambda \) over \( \mathbb{P}^1 \). The data \( \mathcal{L} \) is called a Drinfeld-Plücker data (DP-data) if the canonical inclusion of \( G \)-modules
\[
\eta_{\lambda, \mu} : L(\lambda + \mu) \hookrightarrow L(\lambda) \otimes L(\mu)
\]
induces an isomorphism
\[
\eta_{\lambda, \mu} \otimes \text{id} : \psi_{\lambda+\mu}(L^{\lambda+\mu}) \xrightarrow{\cong} \psi_\lambda(L^\lambda) \otimes_{O_{\mathbb{P}^1}} \psi_\mu(L^\mu)
\]
for every \( \lambda, \mu \in P_{3,+} \).

**Theorem 1.2** (Drinfeld, see [12, 2] and [20]). The variety \( Q(\beta) \) is isomorphic to the variety formed by isomorphism classes of the DP-data \( \mathcal{L} = \{ (\psi_\lambda, L^\lambda) \}_{\lambda \in P_{3,+}} \) such that \( \text{deg } L^\lambda = -\langle \beta, \lambda \rangle \).
For each $u \in W/W_1$, let $Q_2(\beta, u) \subset Q_2(\beta)$ be the closure of the set formed by quasi-maps that are defined at $z = 0$, and their values at $z = 0$ are contained in $B_2(u) \subset B_2$. (Hence, we have $Q_2(\beta) = Q_2(\beta, e).$

For each $\lambda \in P_2$ and $w \in W$, we have a $G$-equivariant line bundle $O_{Q_2(\beta, u)}(\lambda)$ obtained by the (tensor product of the) pull-backs $O_{Q_2(\beta, u)}(w_i)$ of the $i$-th $O(1)$ via the embedding

$$Q_2(\beta, u) \hookrightarrow \prod_{i \in P} \mathbb{P}(L(w_i) \otimes \mathbb{C}[z]_{\leq (\beta, w_i)})$$

(1.2)

for each $\beta \in Q^+_2$. Using this, we set

$$\chi(Q_2(\beta, u), O_{Q_2(\beta)}(\lambda)) := \sum_{i \geq 0} (-1)^i \text{gch}(H^i(Q_2(\beta, u), O_{Q_2(\beta, u)}(\lambda))) \in \mathbb{C}_0^P$$

for each $\beta \in Q^+_2$ and $\lambda \in P_2$, where the grading $q$ is understood to count the degree of $z$ detected by the $G_m$-action. Here we understand that

$$\chi(Q_2(\beta, u), O_{Q_2(\beta, u)}(\lambda)) = 0 \quad \beta \notin Q^+_2.$$

### 1.3 Graph and map spaces and their line bundles

For each non-negative integer $n$ and $\beta \in Q^+_2$, we set $\mathcal{S}B_{2,n,\beta}$ to be the space of stable maps of genus zero curves with $n$-parked points to $(\mathbb{P}^1 \times B_2)$ of bidegree $(1, \beta)$, that is also called the graph space of $B_2$. A point of $\mathcal{S}B_{2,n,\beta}$ is a genus zero quasi-stable curve $C$ with $n$-marked points, together with a map to $\mathbb{P}^1$ of degree one. Hence, we have a unique $\mathbb{P}^1$-component of $C$ that maps isomorphically onto $\mathbb{P}^1$. We call this component the main component of $C$ and denote it by $C_0$. The space $\mathcal{S}B_{2,n,\beta}$ is a normal projective variety by [13, Theorem 2] that have at worst quotient singularities arising from the automorphism of curves. The natural $(H \times G_m)$-action on $(\mathbb{P}^1 \times B_2)$ induces a natural $(H \times G_m)$-action on $\mathcal{S}B_{2,n,\beta}$. Moreover, $\mathcal{S}B_{1,0,\beta}$ has only finitely many isolated $(H \times G_m)$-fixed points, and thus we can apply the formalism of Atiyah-Bott-Lefschetz localization (cf. [16, p201L26] and [4, Proof of Lemma 5]).

We have a morphism $\pi_{2,n,\beta} : \mathcal{S}B_{2,n,\beta} \to Q_2(\beta)$ that factors through $\mathcal{S}B_{2,0,\beta}$ (Givental’s main lemma [17]; see [11, §8] and [13, §1.3]). Let $\mathbf{ev}_j : \mathcal{S}B_{1,n,\beta} \to \mathbb{P}^1 \times B_2$ ($1 \leq j \leq n$) be the evaluation at the $j$-th marked point, and let $e\mathbf{v}_j : \mathcal{S}B_{1,n,\beta} \to B_2$ be its composition with the second projection.

The following result is responsible for the basic case (the case of $J = \emptyset$) of our computation:

**Theorem 1.3** (Braverman-Finkelberg [3, 4, 5]). The morphism $\pi_{0,\beta}$ is a rational resolution of singularities (in an orbifold sense). □

We note that $\mathcal{S}B_{2,n,\beta}$ is irreducible ([23]).

For each $\lambda \in P_2$, we have a line bundle $O_{\mathcal{S}B_{2,n,\beta}}(\lambda) := \pi_{2,n,\beta}^* O_{Q_2(\beta)}(\lambda)$. For a $(H \times G_m)$-equivariant coherent sheaf on a projective $(H \times G_m)$-variety $X$, let $\chi(X, F) \in \mathbb{C}_0^P$ denote its Euler-Poincaré characteristic (that enhances the element $\chi(Q_2(\beta, w), O_{Q_2(\beta, w)}(\lambda))$ defined in §1.2).
1.4 Equivariant quantum $K$-group of $B_2$

We introduce a polynomial ring $\mathbb{C}Q_{1,+}^\prime$ with its variables $Q_i = Q_i^\prime$ $(i \in J^c)$. We set $Q_{\beta} := \prod_{i \in J^c} Q_i^{[\beta,0_i]}$ for each $\beta \in Q_{1,+}^\prime$. We define the $H$-equivariant (small) quantum $K$-group of $B_2$ as:

$$qK_H(B_2) := K_H(B_2) \otimes \mathbb{C}Q_{1,+}^\prime,$$

(1.3)

where $K_H(B_2)$ is the complexified $H$-equivariant $K$-group of $B_2$.

Thanks to (the $H$-equivariant versions of) [15, 31] and the finiteness of the quantum multiplication [1], $qK_H(B_2)$ is equipped with the commutative and associative product $\ast$ (called the quantum multiplication) such that:

1. the element $[O_{B_i}] \otimes 1 \in qK_H(B_2)$ is the identity (with respect to $\cdot$ and $\ast$);
2. the map $Q_{\beta} \ast (\beta \in Q_{1,+}^\prime)$ is the multiplication of $Q_{\beta}$ in the RHS of (1.3);
3. we have $\xi \ast \eta \equiv \xi \cdot \eta \mod (Q_i; i \in J^c)$ for every $\xi, \eta \in K_H(B_2) \otimes 1$.

We set

$qK_{H \times G_m}(B_2) := K_H(B_2) \otimes \mathbb{C}qQ_{1,+}^\prime$ and $qK_{H \times G_m}(B_2)^\wedge := K_H(B_2) \otimes \mathbb{C}[Q_{1,+}^\prime]$.

We can localize $qK_H(B_2)$ (resp. $qK_{H \times G_m}(B_2)$ and $qK_{H \times G_m}(B_2)^\wedge$) in terms of $(Q_{\beta})_{\beta \in Q_{1,+}^\prime}$ to obtain a ring $qK_H(B_2)_{\text{loc}}$ (resp. vector spaces $qK_{H \times G_m}(B_2)_{\text{loc}}$ and $qK_{H \times G_m}(B_2)^{\wedge}_{\text{loc}}$).

We sometimes identify $K_H(B_2)$ with the submodule $K_H(B_2) \otimes 1$ of $qK_H(B_2)$ or $qK_{H \times G_m}(B_2)$. We set $p_i := [O_{B_i}(\pi_i)]$ for $i \in J^c$, and we sometimes consider it as an endomorphism of $qK_{H \times G_m}(B_2)$ through the scalar extension of the product of $K_H(B_2)$ (i.e. the classical product). For each $i \in J^c$, let $qQ_{i,0_i}$ denote the $\mathbb{C}q,P$-endomorphism of $qK_{H \times G_m}(B_2)$ such that

$qQ_{i,0_i} : (\xi \otimes Q_{\beta}) = q^{(\beta,0_i)} \xi \otimes Q_{\beta} \quad \xi \in K_H(B_2), \beta \in Q_{1,+}^\prime.$

Following [18, §2.4], we consider the operator $T \in \text{End}_{\mathbb{C}q,P} qK_{H \times G_m}(B_2)^{\wedge}$ (obtained from the same named operator in [18] by setting $0 = t \in K(B_2)$).

Then, we have the shift operator (also obtained from an operator $A_i(q,t)$ in [18] by setting $t = 0$) defined by

$$A_i(q) = T^{-1} \circ p_i^{-1} qQ_{i,0_i} \circ T \in \text{End} qK_{H \times G_m}(B_2)^{\wedge} \quad i \in J^c.$$

(1.4)

Theorem 1.4 ([18] and [1]). For $i \in J^c$, the operator $A_i(1)$ is well-defined and defines the $\ast$-multiplication by $[O_{B_i}(-\varpi_i)]$ in $qK_H(B_2)$.

Proof. The well-definedness of the substitution $q = 1$ is by [18, Remark 2.14]. By [18, Corollary 2.9] and [1, Theorem 8], the set $\{A_i(1)\}_{i \in J^c}$ defines mutually commutative endomorphisms of $qK_H(B_2)$ that commutes with the $\ast$-multiplication. Since $\text{End}_R R \cong R$ for every ring $R$, we conclude the assertion by $A_i(1)([O_{B_i}]) = [O_{B_i}(-\varpi_i)]$ ([1, Lemma 6]).

2 A description of the quantum $K$-groups

We continue to work in the setting of the previous section.
2.1 $K$-groups of semi-infinite flag manifolds

Let $J \subset \mathfrak{I}$ be a subset. The semi-infinite partial flag manifold $Q^\text{rat}_J$ is the reduced closed ind-subscheme of $\prod_{i \in J} P(L(\pi_i)((z)))$ whose set of $\mathbb{C}$-valued points is

$$G((z))/H(\mathbb{C}) \cdot ([P(J), P(J)]((z))).$$

This is a pure ind-scheme of ind-infinite type [22, 20]. Note that the group $Q^\vee \subset H((z))/H$ acts on $Q^\text{rat}_J$ from the right, whose action factors through $Q^\vee_J$ via the projection described in the below. The indscheme $Q^\text{rat}_J$ is equipped with a $G[z]$-equivariant line bundle $\mathcal{O}_{Q^\text{rat}}(\lambda)$ for each $\lambda \in P_J$. Here we normalized so that $\Gamma(Q^\text{rat}, \mathcal{O}_{Q^\text{rat}}(\lambda))$ is co-generated by its $H$-weight $(-u_0\lambda)$-part as a $B^{-}[z]$-module.

The following two results are not recorded in the literature in a strict sense, but they are straightforward consequences of the set-theoretic consideration that is allowed in view of [20, Theorem A].

**Theorem 2.1.** We have an $I$-orbit decomposition

$$Q^\text{rat}_J = \bigsqcup_{\beta' \in Q^\vee_J} O_J(ut_{\beta'}).$$

**Corollary 2.2.** The natural quotient map $Q^\text{rat} \rightarrow Q^\text{rat}_J$ sends the $I$-orbit $O(ut_{\beta})$ to $O_J(ut_{\beta'})$, where $u' \in W_J$ is characterized by $u' \in W/W_J$ and $\beta' \in Q^\vee_J$ is defined as the projection:

$$\beta' := \beta - \sum_{\gamma \in J} \langle \beta, \pi\gamma \rangle \alpha_i^\gamma.$$

For $u \in W$ and $\beta \in Q^\vee$, we denote the element $u't_{\beta'} \in W_{af}$ obtained in Corollary 2.2 by $[ut_{\beta}]$. By abuse of notation, we also write $\beta'$ by $[\beta]_J$.

We have embeddings $B_J(u) \subset Q_J(\beta, u) \subset Q_J(u) (u \in W/W_J)$ so that the line bundles $\mathcal{O}(\lambda)$ ($\lambda \in P_J$) corresponds to each other by restrictions ([4, 19, 22]).

**Theorem 2.3 ([20] Corollary C and Appendix A).** For each $u \in W/W_J$, and $\lambda \in P_{J, +}$, we have

$$\lim_{\beta \rightarrow \infty} \chi(Q_J(\beta, u), \mathcal{O}_{Q_J(\beta, u)}(\lambda)) = gch H^0(Q_J(u), \mathcal{O}_{Q_J(u)}(\lambda)) \in C_q P. \quad (2.1)$$

Moreover, we have $H^{>0}(Q_J(u), \mathcal{O}_{Q_J(u)}(\lambda)) = \{0\}$.

We define a(n uncompleted version of the) $C^0_q P$-module $K_{H \times G_m}(Q^\text{rat}_J)$ as:

$$K''_{H \times G_m}(Q^\text{rat}_J) := \{ \sum_{u \in W/W_J, \beta' \in Q^\vee_J} a_{u, \beta} [\mathcal{O}_{Q_J(ut_{\beta'})}] \mid a_{u, \beta} \in C^0_q P \}.$$

We set $K'_{H \times G_m}(Q^\text{rat}_J) := C_q \otimes_{C_q} K''_{H \times G_m}(Q^\text{rat}_J)$. For each $\gamma \in Q^\vee_J$, we also define

$$K''_{H \times G_m}(Q_J(t_\gamma)) := \{ \sum_{u \in W/W_J, \beta-\gamma \in Q^\vee_J} a_{u, \beta} [\mathcal{O}_{Q_J(ut_{\beta})}] \in K''_{H \times G_m}(Q^\text{rat}_J) \}.$$

We sometimes also consider its completion

$$K''_{H \times G_m}(Q^\text{rat}_J)^\wedge := C_q \otimes_{C_q} \lim_{\gamma} K''_{H \times G_m}(Q^\text{rat}_J)/K''_{H \times G_m}(Q^\text{rat}_J(t_\gamma)).$$
and its subset
\[ K'_{H \times G_m}(Q_{j}^{\text{rat}}) := \{ \sum_{u \in W/W_j, \beta \in Q_{j}^\vee} a_{u, \beta} [O_{Q_j(u \beta)}] \in K_{H \times G_m}(Q_{j}^{\text{rat}}) \mid \sum_{u, \beta} |a_{u, \beta}| \in C_q P \}, \]

where the absolute value is taken for each coefficient of monomials.

We have a \( C_q P \)-linear surjective morphism
\[ \phi_j : K'_{H \times G_m}(Q^{\text{rat}}) \ni [O_{Q_j(w)}] \mapsto [O_{Q_j([w]_J)}] \in K'_{H \times G_m}(Q_{j}^{\text{rat}}) \quad w \in W_{af}. \]

The \( q = 1 \) specializations of \( K'_{H \times G_m}(Q_{j}^{\text{rat}}) \) and \( K_{H \times G_m}(Q_{j}^{\text{rat}}) \) are denoted by \( K'_{H}(Q_{j}^{\text{rat}}) \) and \( K_{H}(Q_{j}^{\text{rat}}) \), respectively.

**Theorem 2.4** ([22] Corollary 4.29 and [20] Appendix A). For \( w \in W_{af} \) and \( \lambda \in P_1 \), we have
\[ \text{gch} H^0(Q_j([w]_J), O_{Q_j([w]_J)}(\lambda)) = \text{gch} H^0(Q(w), O_{Q(w)}(\lambda)) \in \mathbb{Z}_{\geq 0}[q^{-1}] P. \]

They yields zero if \( \lambda \notin P_{j,+} \). Moreover, their higher cohomologies vanish. \( \square \)

Let \( \text{Fun}_{P_1}(C_q P) \) denote the set of functionals on \( P_1 \) whose value is in \( C_q P \).

We set
\[ \text{Fun}_{P_1}^{\text{neg}}(C_q P) := \{ f \in \text{Fun}_{P_1}(C_q P) \mid \exists \gamma \in P_1 \text{ s.t. } f(\lambda) = 0 \text{ for each } \lambda \in \gamma + P_{j,+} \} \]

and \( \text{Fun}_{\text{ess}}(C_q P) := \text{Fun}_{P_1}(C_q P)/\text{Fun}_{P_1}^{\text{neg}}(C_q P) \).

**Theorem 2.5.** The assignment
\[ K'_{H \times G_m}(Q_{j}^{\text{rat}}) \ni \sum_{u \subseteq W/W_j, \beta \in Q_{j}^\vee} a_{u, \beta} [O_{Q_j(u \beta)}] \mapsto \left( \lambda \mapsto \sum_{u, \beta} a_{u, \beta} \text{gch} H^0(Q_{j}^{\text{rat}}, O_{Q_j(u \beta)}(\lambda)) \right) \in \text{Fun}_{P_1}^{\text{ess}}(C_q P) \]
is an injective \( C_q P \)-linear map. This prolongs to a \( C_q P \)-linear map
\[ K'_{H \times G_m}(Q_{j}^{\text{rat}}) \longrightarrow \text{Fun}_{P_1}^{\text{ess}}(C_q P). \]

**Proof.** The first assertion reduces to the \( C_q P \)-linear independence of the functionals
\[ P_{j,+} \ni \lambda \mapsto \text{gch} H^0(Q_j(u \beta), O_{Q_j(u \beta)}(\lambda)) \quad u \in W/W_j, \beta \in Q_{j}^\vee. \]

In view of Theorem 2.4, this follows as in [22, Proof of Proposition 5.8].

We prove the second assertion. The \( C_q P \)-coefficients \( \{ a_{u, \beta} \} \) of an element of \( K'_{H \times G_m}(Q_{j}^{\text{rat}}) \) satisfies \( a_{u, \beta} = 0 \) for \( \beta \geq \beta_0 \) for some \( \beta_0 \in Q_{j,+}^\vee \), each of them are Laurant polynomials with a uniform upper bound on its \( q \)-degree, and \( \sum_{u, \beta} |a_{u, \beta}| \in C_q P \).

In view of [20, Theorem 2.31] and Theorem 2.4, we have
\[ \text{gch} H^0(Q_j(u \beta), O_{Q_j(u \beta)}(\lambda)) \leq \text{gch} H^0(Q_j(t \beta_0), O_{Q_j(t \beta_0)}(\lambda)) \quad (2.2) \]
for each \( \lambda \in P_1, u \in W_j, \beta_0 \leq \beta \in Q_{j,+}^\vee \), where the inequality is understood to be coefficient-wise (in \( \mathbb{Z}_{\geq 0} \)). The RHS of (2.2) belongs to \( C_q P \) (cf. [19]).

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We set \( a := \sum_{u,\beta} |a_{u,\beta}| \in C_q P \). From the above, we deduce
\[
\sum_{u,\beta} |a_{u,\beta}| gch H^0 (\mathcal{Q}_J (ut_\beta), \mathcal{O}_{\mathcal{Q}_J (ut_\beta)} (\lambda)) \leq a \cdot gch H^0 (\mathcal{Q}_J (t_{\beta_0}), \mathcal{O}_{\mathcal{Q}_J (t_{\beta_0})} (\lambda)),
\]
that implies the convergence of our functional for each \( \lambda \in P_\ast \).

We define \( K'_{H \times G_m} (\mathcal{Q}_J^{\text{rat}}) \) as the image of \( K_{H \times G_m}^{\ast} (\mathcal{Q}_J^{\text{rat}}) \) in \( \text{Fun}_{\text{ess}}^P (C_q P) \).

**Theorem 2.6** (22 Theorem 5.10 for the case \( J = \emptyset \)). For each \( \lambda \in P_\ast \), there exists a \( C_q P \)-linear endomorphism
\[
[\mathcal{O}_{\mathcal{Q}_J (ut_\beta)}] \mapsto [\mathcal{O}_{\mathcal{Q}_J (ut_\beta)} (\lambda)] \in K'_{H \times G_m} (\mathcal{Q}_J^{\text{rat}}) \quad u \in W/W_\ast, \beta \in \mathcal{Q}_J^{\text{rat}}
\]
which is an automorphism of \( K'_{H \times G_m} (\mathcal{Q}_J^{\text{rat}}) \) (that we call \( \Xi (\lambda) \) in the below).

**Proof.** The reasoning we need is the same as those provided in [22, Proof of Theorem 5.10] and [21, Proof of Theorem 1.13] in view of the definition of \( K'_{H \times G_m} (\mathcal{Q}_J^{\text{rat}}) \) and Theorem 2.4.

**Remark 2.7.** In view of [21, Lemma 2.14] and [33, Theorem 1] (cf. [21, Theorem 2.1] for the \( q = 1 \) case), we deduce that \( \Xi (\omega_i) (i \in I) \) defines an automorphism of \( K'_{H \times G_m} (\mathcal{Q}^{\text{rat}}) \). However, an explicit formula [22, Theorem 5.10] tells that \( \Xi (\omega_i) (i \in I) \) never defines an automorphism of \( K'_{H \times G_m} (\mathcal{Q}_J^{\text{rat}}) \).

**Theorem 2.8.** For each \( i \in I^c \), the endomorphism \( \Xi (\omega_i) \) descends to an endomorphism \( \Xi (\omega_i) \) of \( K'_{H \times G_m} (\mathcal{Q}_J^{\text{rat}}) \) through \( \phi_J \). In addition, the map \( \phi_J \) induces a surjective \( C \)-module map \( K'_{H} (\mathcal{Q}_J^{\text{rat}}) \rightarrow K'_{H} (\mathcal{Q}_J^{\text{rat}}) \) such that \( \Xi (\omega_i) \) induces an endomorphism \( \Xi (\omega_i) \) of \( K'_{H} (\mathcal{Q}_J^{\text{rat}}) \).

**Proof.** Consider the \( C_q P \)-linear map generated by
\[
K'_{H \times G_m} (\mathcal{Q}_J^{\text{rat}}) \ni \sum_{w \in W_\ast} a_w [\mathcal{O}_{\mathcal{Q}_J (w)}] \rightarrow (\lambda \mapsto \sum_{w} a_w gch H^0 (\mathcal{Q}_J^{\text{rat}}, \mathcal{O}_{\mathcal{Q}_J (w)} (\lambda))) \in \text{Fun}_{\text{ess}}^P (C_q P).
\]

By Theorem 2.4, this map factors through \( K'_{H \times G_m} (\mathcal{Q}_J^{\text{rat}}) \) as \( [\mathcal{O}_{\mathcal{Q}_J (w)}] \mapsto [\mathcal{O}_{\mathcal{Q}_J (w^J)}] \) for \( w \in W_\ast \). By Remark 2.7, we know that \( \Xi (\omega_i) (i \in I) \) is an endomorphism of \( K'_{H \times G_m} (\mathcal{Q}_J^{\text{rat}}) \). In view of Theorem 2.5, the endomorphism \( \Xi (\omega_i) \) on \( K'_{H \times G_m} (\mathcal{Q}_J^{\text{rat}}) \) descends to an endomorphism of \( K'_{H \times G_m} (\mathcal{Q}_J^{\text{rat}}) \) for each \( i \in I^c \) via the map \( \phi_J \). By specializing \( q = 1 \), we conclude that \( \phi_J \) induces a \( C \)-module surjection \( K'_{H} (\mathcal{Q}_J^{\text{rat}}) \rightarrow K'_{H} (\mathcal{Q}_J^{\text{rat}}) \) on which \( \Xi (\omega_i) \) descends to an endomorphism.

By abuse of notation, we denote the surjective map \( K'_{H} (\mathcal{Q}_J^{\text{rat}}) \rightarrow K'_{H} (\mathcal{Q}_J^{\text{rat}}) \) in Theorem 2.8 by \( \phi_J \). We also denote the \( q = 1 \) specializations of the automorphisms \( \Xi (\omega_i) \) and \( \Xi (\omega_i) \) in Theorem 2.8 by the same symbols.
2.2 $Q_1(\beta, w)$ has at worst rational singularities

Let $X_1(\beta)$ denote the subvariety of $\mathcal{SB}_{1,2,\beta}$ such that the first marked point projects to $0 \in \mathbb{P}^1$, and the second marked point projects to $\infty \in \mathbb{P}^1$ through the projection of quasi-stable curves $C$ to the main component $C_0 \cong \mathbb{P}^1$. Let us denote the restriction of $ev_i$ ($i = 1, 2$) to $X_1(\beta)$ by the same letter. Since $X_1(\beta)$ is a normal scheme at worst quotient singularity, we might regard it as a smooth stack ([13]). As we know that $Q_1(\beta)$ is normal ([20]), we conclude that $\pi_{1,2,\beta}$ restricted to $X_1(\beta)$ also gives a resolution of singularities of $Q_1(\beta)$.

For each $\beta \in Q^\vee_{1,+}$ and $u \in W/W_2$, we set $X_2(\beta, u) := ev^{-1}_1(B_2(u))$.

**Lemma 2.9.** For each $\beta \in Q^\vee_{1,+}$ and $u \in W/W_2$, the variety $X_2(\beta, u)$ is projective, normal, and has at worst rational singularities.

**Proof.** Being a closed subset of a projective variety $\mathcal{SB}_{1,2,\beta}$, we find that $X_2(\beta, u)$ is projective. The evaluation map $ev_1 : X_2(\beta) \to B_2$ is homogeneous with respect to the $G$-action. Let $N_2 \subset N$ be the unipotent radical of $P(3)$. By restricting to the open $N$-orbit $N_2 \times \{p_e\} \cong Q_1(e) \subset B_2$, we deduce that $ev^{-1}_1(Q_1(e)) \cong N_2 \times ev^{-1}_1(p_e)$. By translating using the $G$-action, we conclude that $ev_1$ is a locally trivial fibration. We know that $B_2(u) \ (u \in W/W_2)$ is normal and has at worst rational singularities (see [24]). Thus, the singularity of $X_2(\beta, u)$ is locally a product of two rational singularities. From basic properties of rational singularities [26, §5.1], we deduce that being rational singularity is a local condition and it is preserved by taking products. Therefore, we conclude that $X_2(\beta, u)$ has at worst rational singularities (and the normality is its consequence).

We have $X_1(\beta) = X_1(\beta, e)$. The map $\pi_{1,2,\beta}$ restricts to a $(B \times \mathbb{G}_m)$-equivariant proper map

$$\pi_{1,\beta,u} : X_2(\beta, u) \to Q_1(\beta, u)$$

by inspection. Let $O_{X_2(\beta, u)}(\lambda)$ denote the restriction of $O_{X_1(\beta)}(\lambda)$ to $X_2(\beta, u)$ for each $\lambda \in P_2$ and $u \in W/W_2$.

**Theorem 2.10** (Kollár [25] Theorem 7.1). Let $f : X \to Z$ be a surjective map between projective varieties, $X$ smooth, and $Z$ normal. Let $F$ be the geometric generic fiber of $f$ and assume that $F$ is connected. The following two statements are equivalent:

1. $\mathbb{R}^i f_* O_X = 0$ for all $i > 0$;
2. $Z$ has rational singularities and $H^i(F, O_F) = 0$ for all $i > 0$.

**Theorem 2.11.** For each $\beta \in Q^\vee_{1,+}$ and $u \in W/W_2$, the variety $Q_1(\beta, u)$ has at worst rational singularities. In addition, we have

$$(\pi_{1,\beta,u})_* O_{X_2(\beta, u)} \cong O_{Q_1(\beta, u)}, \quad \mathbb{R}^>0(\pi_{1,\beta,u})_* O_{X_2(\beta, u)} \cong \{0\}.$$

**Proof.** By [20, Corollary 4.20], the variety $Q_1(\beta, u)$ is normal. By Lemma 2.9, we know that $X_1(\beta, u)$ has at worst rational singularities. The same is true for $J = 0$ by [3, 13]. The coarse moduli property of $X(\beta)$ yields a morphism $X(\beta^+) \to X_2(\beta)$ for every $\beta^+ \in Q^\vee_{1,+}$ such that $\beta = [\beta^+]_J$. In view of [20,
Remark 3.31] (cf. Woodward [35]), we can choose $\beta^+$ such that $\Omega(\beta^+, u) \to \Omega_J(\beta, u)$ is surjective.

We have the following commutative diagram:
\[
\begin{array}{ccc}
\mathcal{X}(\beta^+, u) & \xrightarrow{\eta} & \mathcal{X}_J(\beta, u) \\
\pi_{\beta^+, u} & & \pi_{J, \beta, u} \\
\Omega(\beta^+, u) & \xrightarrow{\eta} & \Omega_J(\beta, u).
\end{array}
\]
Here the maps $\pi_{\beta^+, u}$ and $\pi_{J, \beta, u}$ are birational. Thus, the map $\tilde{\eta}$ is also surjective. Moreover, we have $R^*\eta_*O_{\mathcal{X}(\beta^+, u)} = O_{\mathcal{X}_J(\beta, u)}$ by [20, Corollary 3.30]. We find $R^*(\pi_{\beta^+, u})_*O_{\mathcal{X}(\beta^+, u)} = O_{\mathcal{X}_J(\beta, u)}$ by [21, Corollary 4.5]. By the Leray spectral sequence applied to the composition map $\eta \circ \pi_{\beta^+, u}$, we find that
\[
R^*(\eta \circ \pi_{\beta^+, u})_*O_{\mathcal{X}(\beta^+, u)} = O_{\mathcal{X}_J(\beta, u)}.
\]

This implies that the geometric generic fiber of the composition map $(\eta \circ \pi_{\beta^+, u})$ has trivial higher cohomology. Since $\pi_{J, \beta, u}$ is birational, the geometric generic fiber of $(\eta \circ \pi_{\beta^+, u})$ is the same as $\tilde{\eta}$. Therefore, we conclude
\[
R^*(\eta \circ \pi_{\beta^+, u})_*O_{\mathcal{X}(\beta^+, u)} = O_{\mathcal{X}_J(\beta, u)}
\]
by Theorem 2.10 (by replacing $\mathcal{X}(\beta^+, u)$ with its resolution of singularity if necessary, cf. [26, Theorem 5.10]). By the above commutative diagram, the Leray spectral sequence applied to the composition map $\pi_{J, \beta, u} \circ \tilde{\eta} = \eta \circ \pi_{\beta^+, u}$ implies
\[
R^*(\pi_{J, \beta, u})_*O_{\mathcal{X}_J(\beta, u)} \cong O_{\mathcal{X}_J(\beta, u)}
\]
from (2.3). This shows that $\mathcal{X}_J(\beta, u)$ has at worst rational singularities by [26, Theorem 5.10].

**Corollary 2.12.** For each $\beta \in Q^\vee_{1, +}$, $u \in W/W'_1$, and $\lambda \in P$, we have
\[
\chi(\mathcal{X}_J(\beta, u), O_{\mathcal{X}_J(\beta, u)}(\lambda)) = \chi(\mathcal{Q}_J(\beta, u), O_{\mathcal{Q}_J(\beta, u)}(\lambda)) \in C_0^q P.
\]

**Proof.** Apply the projection formula to Theorem 2.11.

For $\vec{n} = \{n_i\}_{i \in J^e} \subset \mathbb{Z}_{\geq 0}^e$, we set $x^{\vec{n}} := \prod_{i \in J^e} x_i^{n_i}$. For $\lambda \in P$, we set $\lambda(\vec{n}) := \lambda - \sum_{i \in J^e} n_i \omega_i$.

**Theorem 2.13** (Iritani-Milanov-Tonita [18], cf. Givental-Lee [16]). For each
\[
\sum_{\beta \in Q^\vee_J, u \in W/W'_1, \vec{n} \in \mathbb{Z}_{\geq 0}^e} f_{\beta, u, \vec{n}}(q) x^{\vec{n}} Q^\beta \in \mathbb{C}_0^q P[[x_i]_{i \in J^e}][Q^\vee_J]
\]
such that
\[
\sum_{\beta \in Q^\vee_J, u \in W/W'_1, \vec{n} \in \mathbb{Z}_{\geq 0}^e} f_{\beta, u, \vec{n}}(q) \left( \prod_{i \in J^e} A_i^{n_i} \right) Q^\beta [O_{\mathcal{Q}_J(u)}] = 0 \in qK_{G \times E_u}(\mathcal{B}_J)^{\lambda},
\]
we have the following equalities:
\[
\sum_{\beta \in Q^\vee_J, u \in W/W'_1, \vec{n} \in \mathbb{Z}_{\geq 0}^e} f_{\beta, u, \vec{n}}(q) q^{-(\beta, \lambda(\vec{n}))} \chi(\mathcal{X}_J(\gamma - \beta, u), O_{\mathcal{X}_J(\gamma - \beta, u)}(\lambda(\vec{n}))) = 0
\]
for each $\lambda \in P_{1, +}$ and $\gamma \in Q^\vee_{1, +}$.
Proof. The assertion follows by plugging (2.4) into [18, Proposition 2.20] and observe that $A_i$ becomes the line bundle twist by $O(-\varpi_i)$ up to $q^{(\beta,\varpi_i)}$. $Q_i$ twists the Novikov variable (and hence the degree of the stable map spaces), and the effect of $O_{B_j(u)}$ is to restrict the whole variety to $X_j(\bullet,u)$ via $ev^*_j$. It can be also seen as a variant of [21, Theorem 3.5 and Theorem 3.6].

2.3 Comparison of equivariant $K$-groups

Theorem 2.14. We have a $\mathbb{C}_q P$-module isomorphism

$$\Psi_{1,q} : qK_{H\times G_m}(B_j)_{\text{loc}} \cong K'_{H\times G_m}(Q^\text{rat}_j)$$

such that

1. $\Psi_{1,q}(\mathcal{O}_{B_j(u)}[Q^\beta]) = \mathcal{O}_{Q^\beta(ut_\beta)}$ for each $u \in W/W_j$ and $\beta \in Q^\beta_j$

2. $\Psi_{1,q}(A_i(\bullet)) = \Xi_j(-\varpi_i)(\Psi_{1,q}(\bullet))$ for each $i \in J^c$.

Corollary 2.15. By specializing $q = 1$, we obtain a $\mathbb{C}P$-module isomorphism

$$\Psi_2 : qK_H(B_j)_{\text{loc}} \cong K'_H(Q^\text{rat}_j)$$

such that

1. $\Psi_2(\mathcal{O}_{B_j(u)}[Q^\beta]) = \mathcal{O}_{Q^\beta(ut_\beta)}$ for each $u \in W/W_j$ and $\beta \in Q^\beta_j$

2. $\Psi_2(A_i(\bullet)) = \Xi_j(-\varpi_i)(\Psi_2(\bullet))$ for each $i \in J^c$.

Proof of Corollary 2.15. Taking Theorem 2.14 into account, it remains to observe that $A_i(\bullet)$ specializes to $[O_{B_j(-\varpi_i)}]_{\bullet}$ by Corollary 1.4.

Proof of Theorem 2.14. By the definitions of $qK_{H\times G_m}(B_j)_{\text{loc}}$ and $K'_{H\times G_m}(Q^\text{rat}_j)$, we find that $\Psi_{1,q}$ is a $\mathbb{C}_q P$-linear isomorphism. The map $\Psi_{1,q}$ prolongs to an isomorphism

$$qK'_{H\times G_m}(B_j)_{\text{loc}} \cong K_{H\times G_m}(Q^\text{rat}_j),$$

where $qK'_{H\times G_m}(B_j)_{\text{loc}}$ is the quotient of some subset of $qK_{H\times G_m}(B_j)_{\text{loc}}$ subject to the analogous convergence condition as in $K'_{H\times G_m}(Q^\text{rat}_j)$ (so that we have $qK_{H\times G_m}(B_j)_{\text{loc}} \subset qK'_{H\times G_m}(B_j)_{\text{loc}}$).

For each $u \in W/W_j$, we expand $A_i([\mathcal{O}_{B_j(u)}])$ as a formal linear combination

$$A_i([\mathcal{O}_{B_j(u)}]) = \sum_{v \in W/W_j, \gamma \in Q^\gamma_j} a_{i,u}^{v,\gamma}Q^\gamma_v[\mathcal{O}_{B_j(v)}] \quad a_{i,u}^{v,\gamma} \in \mathbb{C}_q P$$

by [18, Remark 2.14].

Applying Theorem 2.13 and Corollary 2.12, we have

$$\chi(Q_j(\beta,u),\mathcal{O}_{Q_j(\beta,u)}(\lambda-\varpi_i)) = \sum_{v \in W/W_j, \gamma \in Q^\gamma_j} a_{i,u}^{v,\gamma}q^{-\gamma}\chi(Q_j(\beta,\gamma,v),\mathcal{O}_{Q_j(\beta,\gamma,v)}(\lambda))$$

for each $\beta \in Q^\beta_j$ and $\lambda \in P_j$. We have $\chi(Q_j(\beta,v),\mathcal{O}_{Q_j(\beta,v)}(\lambda)) \in \mathbb{C}_q P$ for every $u \in W/W_j$, $\beta \in Q^\beta_j$, and $\lambda \in P_j$. By [20, Theorem 3.28], the $\mathbb{C}$-coefficients of the series $\{\chi(Q_j(\beta,u),\mathcal{O}_{Q_j(\beta,u)}(\lambda))\}_{\beta} \subset \mathbb{C}_q P$ belongs to $\mathbb{Z}_{\geq0}[q^{-1}] P$ and is
monotonically non-decreasing with respect to $\beta$. By examining the cases $\beta = \gamma$, we deduce $a_{i,u}^{\gamma,\gamma} \in \mathbb{Z}[q^{-1}]P$ by induction (from the case $\beta = 0$). Moreover, the limit $\beta \to \infty$ of the LHS of (2.5) is convergent ([20, Theorem 3.28]). In order that the RHS of (2.5) to be equal to the LHS, we further need $\sum_{v,\gamma} |a_{i,u}^{v,\gamma}| \in C_qP$. Therefore, we conclude $A_i([\mathcal{O}_B(u)]) \in qK'_H \times G_m(B)\gamma.$

By taking the limit $\beta \to \infty$ (cf. [22, Proposition D.1]), we obtain

$$\chi(\mathcal{Q}_J(u\beta), \mathcal{O}_{\mathcal{Q}_J(u\beta)}(\lambda - \omega_i)) = \sum_{v \in W/W_1, \gamma \in \bar{Q}_J^+} a_{i,u}^{v,\gamma} \chi(\mathcal{Q}_J(v\gamma), \mathcal{O}_{\mathcal{Q}_J(v\gamma)}(\lambda))$$

for each $\lambda \in P_{J,+}$ by Theorem 2.1. This implies

$$[\mathcal{O}_{\mathcal{Q}_J(u\beta)}(-\omega_i)] = \sum_{v \in W/W_1, \gamma \in \bar{Q}_J^+} a_{i,u}^{v,\gamma} [\mathcal{O}_{\mathcal{Q}_J(v\gamma)}]$$

in view of Theorem 2.5. Hence, we conclude

$$\Psi_{J,q}(A_i([\mathcal{O}_B(u)]) = \Xi_0(-\omega_i)(\Psi_{J,q}(\mathcal{O}_{\mathcal{B}_1(u)}))) \quad u \in W/W_1,$$

where the equality is in $K_{H \times G_m}(Q_1)$. In view of [33, Theorem 1] (or [1]) and Theorem 2.4, this is in fact an equality in $K_{H \times G_m}(Q_1^\ast)$. Since $\Psi_{J,q}, A_i$, and $\Xi_0(-\omega_i)$ ($i \in J^\ast$) are $C_qP$-linear, we conclude the result. \qed

We consider the subring of $qK_H(B)_{\geq 0} \subset qK_H(B)$ generated by $CP$, $C_{Q_1^\ast}$, and $\{[\mathcal{O}_B(-\omega_i)] \ast \}_{i \in J^\ast}$.

**Lemma 2.16.** For each $i \in I$, the $C_qP$-subspace $K_i^q \subset qK_{H \times G_m}(B)$ spanned by the set

$$\{[\mathcal{O}_B(u)]Q^\beta - [\mathcal{O}_B(u_{\alpha_i})]Q^\beta' \mid u \in W, \beta, \beta' \in Q_1^\ast, \text{ s.t. } \beta - \beta' \in \mathbb{Z}a_i^\ast\}$$

is stable by the action of $A_i(q)$ ($i \in I$). In particular, its specialization $q = 1$ yields a $CP$-subspace $K_i \subset qK_H(B)$ that is stable by the $qK_H(B)_{\geq 0}$-action.

**Remark 2.17.** Lemma 2.16 does not hold if we replace $qK_H(B)$ with $K_H(B)$.

We set $G = SL(2)$ (and hence $B = P^1$ and $I = \{1\}$). We have an equality $[\mathcal{O}_B(-\omega_1)] = e^{-\varpi_1}[\mathcal{O}_B(\alpha_1)] \in K_H(B)$, that implies

$$[\mathcal{O}_B(-\omega_1)] - [\mathcal{O}_B(s_1)(-\omega_1)] = e^{-\varpi_1}[\mathcal{O}_B] - (e^{-\varpi_1} + e^{-\varpi_1})[\mathcal{O}_B(s_1)] \notin CP([\mathcal{O}_B] - [\mathcal{O}_B(s_1)])$$

In other words, the vanishing part of Theorem 2.4 is crucial in our consideration.

**Proof of Lemma 2.16.** By Theorem 2.4, elements in $\Psi_{q^{-1}}(K_i^q)$ are precisely the elements in $\Psi_{q^{-1}}(qK_{H \times G_m}(B))$ that vanishes via the functional in Theorem 2.5 restricted to $\lambda \in P_{1,i}$. Hence, $\Psi_{q^{-1}}(K_i^q)$ is stable under the action of $\{\Xi(-\omega_i)\}_{i \in I}$. It follows that the set $\Psi_{q^{-1}}(K_i)$ is stable by the multiplication by $qK_H(B)_{\geq 0}$.

\section*{2.4 Comparison between equivariant quantum $K$-groups}

The following crucial observation is due to Buch-Chaput-Mihalcea-Perrin [6, §5] (see also [1, §1.2], cf. [10, Lemma 4.1.3]):
Theorem 2.18. We have a diagram (represented by real arrows) of

\[ \text{Proof.}\]

In view of the equality (cf. [21, Theorem 1.1])

\[ [O_{B,j}(-\varpi_i)] = e^{-w_0\varpi_i}([O_{B,j}] - [O_{B,j}(\varpi_i)]) \in K_H(B_j) \quad i \in J^c, \]

we can rephrase this as:

- The multiplication rule of \( qK_H(B_j) \) as a \( CP \otimes CQ_+ \)-algebra is completely
determined by the \( \ast \)-multiplication table of \( O_{B,j}(\varpi_i) \) for \( i \in J^c \).

These fact holds as \( qK_H(B_j) \) is generated by \( \{ [O_{B,j}(-\varpi_i)] \}_{i \in J^c} \) after localization to \( C(P \oplus Q_+) \) [6, Remark 5.10]. In other words, we have

\[ \mathbb{C}(P \oplus Q_+^J) \otimes (CP)Q_+^J qK_H(B_j)_{\geq 0} = \mathbb{C}(P \oplus Q_+) \otimes (CP)Q_+^J qK_H(B_j) \]

and the multiplication rule of \( \{ [O_{B,j}(-\varpi_i)] \}_{i \in J^c} \) on some \( \mathbb{C}(P \oplus Q_+^J) \)-basis of
\( \mathbb{C}(P \oplus Q_+) \otimes (CP)Q_+^J qK_H(B_j) \) determines the product structure of \( qK_H(B_j) \).

**Theorem 2.18.** We have a surjective morphism

\[ qK_H(B) \longrightarrow qK_H(B) \]

of commutative algebras such that the image of \( [O_{B(w)}] \) is \( [O_{B,i}(w_i)] \) for each \( w \in W \), and the image of \( Q^b \) is \( Q_+^b \) for each \( \beta \in Q_+^b \).

**Proof.** We have a diagram (represented by real arrows) of \( CP \otimes CQ^J \)-modules

\[
\begin{array}{ccc}
K_H'(\mathbb{Q}_G(e)) & \xrightarrow{\Psi} & qK_H(B) \\
\phi_3 \downarrow & & \downarrow \\
K'_H(\mathbb{Q}_G(e)) & \xrightarrow{\Psi_3} & qK_H(B) \\
\end{array}
\]

such that their bases correspond as \( \phi_1([O_{\mathbb{Q}_G(w)}]) = [O_{\mathbb{Q}_G(w_i)}] \) \( (w \in W \times Q_+^J \subseteq W_J) \). The kernel of the map \( \phi_3 \) is the preimage of the sum of \( K_3 \) (borrowed from Lemma 2.16) for \( i \in J \). This defines an ideal of \( \Psi^{-1}(qK_H(B)_{\geq 0}) \). Therefore, the map \( \phi_3 \) induces some \( CP \)-algebra structure on

\[ \phi_3(\Psi^{-1}(qK_H(B)_{\geq 0})) \subset K'_H(\mathbb{Q}_G). \]

(If \( J = I \), then we have \( \phi_3([O_{\mathbb{Q}_G(w)}]) = 1 \) and \( \text{Im} \ \phi_3 = K_H(\text{pt}) = CP \). Hence this algebra structure must be the correct one and the result follows in this case.)

In view of Theorem 2.5 and Theorem 2.4, we find that

\[ \Xi_{J}(-\varpi_i) \circ \phi_3 = \phi_3 \circ \Xi(-\varpi_i) \quad i \in J^c. \]

Thus, the above observation and Corollary 2.15 imply that the above module map induces an algebra map

\[ qK_H(B) \longrightarrow qK_H(B) \]

with the desired properties (here we used that the both sides are algebras also by the \( \ast \)-products). \( \square \)
The same reasoning yields the following:

**Corollary 2.19.** We have a surjective \( \mathbb{C}_q \mathcal{P} \)-module morphism
\[
qK_H \times \mathbb{C}_q \mathcal{P} \longrightarrow qK_H \times \mathbb{C}_q \mathcal{P}
\]
that intertwines the actions of \( \mathbb{A}_i(q) \) \((i \in I)\), and the image of \([\mathcal{O}_{\mathcal{B}(u)}]Q^\beta\) is \([\mathcal{O}_{\mathcal{B}_i([u])}]Q^{[\beta]}_j\) for each \( w \in W \) and \( \beta \in Q^+_\mathbb{Q} \). \( \square \)

### 2.5 Comparison with affine Grassmanians

In this subsection, we deal with an algebra \( K_H(\text{Gr}) \) that can be seen as the \( H \)-equivariant \( K \)-group of the affine Grassmannian of \( G \) whose product structure is given by the Pontryagin product. For background materials, see [29, 21].

For \( w \in W^- \), we consider a formal symbol \( \text{Gr}_w \) and set
\[
K_H(\text{Gr}) := \bigoplus_{w \in W^-} \mathbb{C}_q [\mathcal{O}_{\text{Gr}_w}].
\]

**Theorem 2.20** (Lam-Schilling-Shimozono see [21] §1.3). There exists a commutative algebra structure (whose multiplication is denoted by \( \odot \)) on \( K_H(\text{Gr}) \) such that
\[
[\mathcal{O}_{\text{Gr}_w}] \odot [\mathcal{O}_{\text{Gr}_\beta}] = [\mathcal{O}_{\text{Gr}_{w\beta}}]
\]
for each \( w \in W^- \) and let \( \beta \in Q^+_\mathbb{Q} \).

We call the multiplication \( \odot \) of \( K_H(\text{Gr}) \) the Pontryagin product. Theorem 2.20 implies that the set
\[
\{[\mathcal{O}_{\text{Gr}_\beta}] | \beta \in Q^+_\mathbb{Q} \} \subset (K_H(\text{Gr})_{\text{loc}}, \odot)
\]
forms a multiplicative system. We denote by \( K_H(\text{Gr})_{\text{loc}} \) its localization. The action of an element \([\mathcal{O}_{\text{Gr}_\beta}]\) on \( K_H(\text{Gr}) \) in Theorem 2.20 is torsion-free, and hence we have an embedding \( K_H(\text{Gr}) \hookrightarrow K_H(\text{Gr})_{\text{loc}} \).

**Theorem 2.21** ([21] Corollary C). There exists an isomorphism
\[
\Phi : (K_H(\text{Gr})_{\text{loc}}, \odot) \longrightarrow (qK_H(\mathcal{B})_{\text{loc}}, \star)
\]
of algebras such that
\[
\Phi([\mathcal{O}_{\text{Gr}_{w\beta_1}}] \odot [\mathcal{O}_{\text{Gr}_{w\beta_2}}])^{-1} = [\mathcal{O}_{\mathcal{B}(u)}]Q^{\beta_1 - \beta_2} \quad u \in W, \beta_1, \beta_2 \in Q^+_\mathbb{Q}.
\]

**Theorem 2.22.** There exist a surjective algebra map
\[
\eta_J : (K_H(\text{Gr})_{\text{loc}}, \odot) \longrightarrow (qK_H(\mathcal{B}_J)_{\text{loc}}, \star)
\]
such that
\[
\eta_J([\mathcal{O}_{\text{Gr}_{w\beta_1}}] \odot [\mathcal{O}_{\text{Gr}_{w\beta_2}}])^{-1} = [\mathcal{O}_{\mathcal{B}_J([u])}]Q^{[\beta_1 - \beta_2]}_J \quad u \in W, \beta_1, \beta_2 \in Q^+_\mathbb{Q}.
\]

**Proof.** Combine Theorem 2.21 with Theorem 2.18. \( \square \)
3 Examples: $G = SL(3)$

Keep the setting of the previous section with $G = SL(3)$. We have $W = \langle s_1, s_2 \rangle \cong S_3$, $P = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$, and $Q^\vee = \mathbb{Z}^\alpha \oplus \mathbb{Z}^\gamma$. Recall that $\vartheta := \alpha_1 + \alpha_2$ and $\vartheta^\vee := \alpha^\vee_1 + \alpha^\vee_2$. We have $w_0 = s_1s_2s_1 = s_2s_1s_2$. In our case, we have three possible choices of $\emptyset \neq J \subset I = \{1, 2\}$. In view of [21, Corollary 3.2 or Proposition 2.12], we may consult [28, §4.2] (with the convention of $H$-characters twisted by $w_0$) or [33, Appendix A] (with the convention of $H$-characters twisted by $-w_0$) to justify the first equality in each item. The other equalities are consistent with [8, §5.5].

- We have
  
  \[ \alpha_2 \] $\langle \wedge \rangle \sim \frac{1}{2} \langle \vee \rangle$.

Applying Theorem 2.18, we deduce

\[ [O_B(s_1)] \ast [O_B(s_2)] = (1 - e^{\alpha_2})[O_B(s_1)] + e^{\alpha_2}[O_B]Q^\alpha + e^{\alpha_2}[O_B(s_2)] - e^{\alpha_2}[O_B(s_2)]Q^\alpha_2. \]

- We have $[O_B(s_1)] \ast [O_B(s_2)] = [O_B(s_1, s_2)] + [O_B(s_1)] - [O_B(s_1)].$ From this, we deduce
  
  \[ [O_B(s_1)] \ast [O_B(s_2)] = [O_B(s_1, s_2)]. \]

- We have $[O_B(s_2)] \ast [O_B(s_1, s_2)] = (1 - e^{\alpha_2})[O_B(s_1, s_2)] + e^{\alpha_2}[O_B(s_1)].$ From this, we deduce
  
  \[ [O_B(s_2)] \ast [O_B(s_1, s_2)] = [O_B(s_1, s_2)]. \]

- We have $[O_B(s_1)] \ast [O_B(s_2, s_1)] = (1 - e^{\alpha_2})[O_B(s_2, s_1)] + e^{\alpha_2}[O_B(s_1)].$ From this, we deduce
  
  \[ [O_B(s_1)] \ast [O_B(s_2, s_1)] = [O_B(s_2, s_1)]. \]

- We have $[O_B(s_1)] \ast [O_B(s_2)] = (1 - e^{\alpha_2})[O_B(s_2)] + e^{\alpha_2}[O_B(s_1)].$ From this, we deduce
  
  \[ [O_B(s_1)] \ast [O_B(s_2)] = [O_B(s_2)]. \]

- We have $[O_B(s_1)] \ast [O_B(s_2)] = (1 - e^{\alpha_2})[O_B(s_2)] + e^{\alpha_2}[O_B(s_1)].$ From this, we deduce
  
  \[ [O_B(s_1)] \ast [O_B(s_2)] = [O_B(s_2)]. \]

- We have $[O_B(s_1)] \ast [O_B(s_2)] = (1 - e^{\alpha_2})[O_B(s_2)] + e^{\alpha_2}[O_B(s_1)].$ From this, we deduce
  
  \[ [O_B(s_1)] \ast [O_B(s_2)] = [O_B(s_2)]. \]
In all cases, the above calculations recover [7, Corollary 10] as:

\[ 1 \star 1 = 1 \in qK_H(\mathcal{B}_{(1,2)}) \equiv qK_H(G/G) = K_H(G/G) = \mathbb{C} \mathbb{P} \]

by setting \([\mathcal{O}_{\mathcal{B}(w)}] \equiv 1 \equiv Q_{\alpha_i}^w (w \in W, i = 1, 2)\).

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References

[1] David Anderson, Linda Chen, and Hsian-Hua Tseng. The quantum K-theory of a homogeneous space is finite. arXiv:1804.04579v2, 2018.
[2] A. Braverman, M. Finkelberg, D. Gaitsgory, and I. Mirković. Intersection cohomology of Drinfeld’s compactifications. Selecta Math. (N.S.), 8(3):381–418, 2002.
[3] Alexander Braverman and Michael Finkelberg. Semi-infinite Schubert varieties and quantum K-theory of flag manifolds. J. Amer. Math. Soc., 27(4):1147–1168, 2014.
[4] Alexander Braverman and Michael Finkelberg. Weyl modules and q-Whittaker functions. Math. Ann., 359(1-2):45–59, 2014.
[5] Alexander Braverman and Michael Finkelberg. Twisted zastava and q-Whittaker functions. J. London Math. Soc., 96(2):309–325, 2017, arXiv:1410.2365.
[6] Anders S. Buch, Pierre-Emmanuel Chaput, Leonardo C. Mihalcea, and Nicolas Perrin. A Chevalley formula for the equivariant quantum K-theory of cominuscule varieties. Algebrinc Geometry, 5(5):568–595, 2018.
[7] Anders S. Buch, Sjuvon Chung, Changzheng Li, and Leonardo C. Mihalcea. Euler characteristics in the quantum K-theory of flag varieties. arXiv:1903.02215, 2019.
[8] Anders S. Buch and Leonardo C. Mihalcea. Quantum K-theory of Grassmannians. Duke Math. J., 156(3):501–538, 2011.
[9] Neil Chriss and Victor Ginzburg. Representation theory and complex geometry. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2010. Reprint of the 1997 edition.
[10] Ionut Ciocan-Fontanine, Bumsig Kim, and Claude Sabbah. The Abelian/Nonabelian correspondence and Frobenius manifolds. Invent. Math., 171(2):301–343, 2008.
[11] Boris Feigin, Michael Finkelberg, Alexander Kuznetsov, and Ivan Mirković. Semi-infinite flags. II. Local and global intersection cohomology of quasimaps’ spaces. In Differential topology, infinite-dimensional Lie algebras, and applications, volume 194 of Amer. Math. Soc. Transl. Ser. 2, pages 113–148. Amer. Math. Soc., Providence, RI, 1999.
[12] Michael Finkelberg and Ivan Mirković. Semi-infinite flags. I. Case of global curve P1. In Differential topology, infinite-dimensional Lie algebras, and applications, volume 194 of Amer. Math. Soc. Transl. Ser. 2, pages 81–112. Amer. Math. Soc., Providence, RI, 1999.
[13] William Fulton and Rahul Pandharipande. Notes on stable maps and quantum cohomology. In Robert Lazarsfeld János Kollár and David Morrison, editors, Algebraic Geometry: Santa Cruz 1995. Amer Mathematical Society, 1995.
[14] Alexander Givental. Homological geometry and mirror symmetry. Proceedings of the International Congress of Mathematicians 1994, pages 472–480, 1995.
[15] Alexander Givental. On the WDVV equation in quantum K-theory. Michigan Math. J., 48:295–304, 2000.
[16] Alexander Givental and Yuan-Pin Lee. Quantum K-theory on flag manifolds, finite-difference Toda lattices and quantum groups. Invent. Math., 151(1):193–219, 2003.
[17] Alexander B. Givental. Equivariant Gromov-Witten invariants. Internat. Math. Res. Notices, 13:613–663, 1996.
[18] Hiroshi Iritani, Todor Milanov, and Valentina Tonita. Reconstruction and convergence in quantum K-Theory via Difference Equations. *International Mathematics Research Notices*, 2015(11):2887–2937, 2015, 1309.3750.

[19] Syu Kato. Demazure character formula for semi-infinite flag varieties. *Math. Ann.*, 371(3):1769–1801, 2018, arXiv:1605.0279.

[20] Syu Kato. Frobenius splitting of Schubert varieties of semi-infinite flag manifolds. arXiv:1810.07106, 2018.

[21] Syu Kato. Loop structure on equivariant K-theory of semi-infinite flag manifolds. arXiv:1805.01718, 2018.

[22] Syu Kato, Satoshi Naito, and Daisuke Sagaki. Equivariant K-theory of semi-infinite flag manifolds and Pieri-Chevalley formula. arXiv:1702.02408, 2017.

[23] B. Kim and R. Pandharipande. The connectedness of the moduli space of maps to homogeneous spaces. In *Symplectic geometry and mirror symmetry (Seoul, 2000)*, pages 187–201. World Sci. Publ., River Edge, NJ, 2001.

[24] Allen Knutson, Thomas Lam, and David E Speyer. Projections of Richardson varieties. *J. reine angew. Math.*, 687:133–157, 2014.

[25] János Kollár. Higher direct images of dualizing sheaves, I. *Ann. of Math. (2)*, 123(1):11–42, 1986.

[26] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[27] Shrawan Kumar. *Kac-Moody groups, their flag varieties and representation theory*, volume 204 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2002.

[28] Thomas Lam, Changzheng Li, Leonardo C. Mihalcea, and Mark Shimozono. A conjectural Peterson isomorphism in K-theory. *J. Algebra*, 513:326–343, 2018.

[29] Thomas Lam, Anne Schilling, and Mark Shimozono. K-theory Schubert calculus of the affine Grassmannian. *Compositio Mathematica*, 146(4):811–852, 2010, 0901.1506.

[30] Thomas Lam and Mark Shimozono. Quantum cohomology of G/P and homology of affine Grassmannian. *Acta Mathematica*, 204(1):49–90, 2010, arXiv:0705.1386v1.

[31] Yuan-Pin Lee. Quantum K-theory, I. Foundations. *Duke Math. J.*, 121(3):389–424, 2004.

[32] Naichung Conan Leung and Changzheng Li. Functorial relationships between $QH^*(G/B)$ and $QH^*(G/P)$. *Journal of Differential Geometry*, 86:303–354, 2010.

[33] Satoshi Naito, Daniel Orr, and Daisuke Sagaki. Pieri-Chevalley formula for anti-dominant weights in the equivariant K-theory of semi-infinite flag manifolds. arXiv:1808.01468.

[34] Dale Peterson. Quantum cohomology of G/P, 1997.

[35] Christopher T. Woodward. On D. Peterson’s comparison formula for Gromov-Witten invariants of G/P. *Proc. Amer. Math. Soc.*, 133(6):1601–1609, 2005.