A LOCAL PARABOLIC MONOTONICITY FORMULA ON RIELMANNIAN MANIFOLDS

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Abstract. In this article we establish a local parabolic almost monotonicity formula for two phase free boundary problems on Riemannian manifolds, which is an extension of a work of Edquist-Petrosyan.

1. Introduction

In the theory of two-phase free boundary problems, it is well established that regularity of the interface is closely related to asymptotic behavior of solution near the free boundary. In 1984 Alt, Caffarelli and Friedman [2] established a monotonicity formula to describe the interaction of the two pieces of the solution on each side of the free boundary. This formula has been extremely powerful in the regularity theory and it reads as follows: Let $u_1, u_2$ be two non-negative continuous functions in $B_1$ (the unit ball in $\mathbb{R}^n$) such that $\Delta u_i \geq 0$ ($i = 1, 2$) are satisfied in distribution. Suppose $u_1 \cdot u_2 = 0$ and $u_1(0) = u_2(0) = 0$, then

$$\phi(r) = \frac{1}{r^4} \int_{B_r} |\nabla u_1|^2 \frac{dx}{|x|^{n-2}} \int_{B_r} |\nabla u_2|^2 \frac{dx}{|x|^{n-2}}$$

is monotone non-decreasing for $0 < r < 1$.

There have been different extensions of the theorem of Alt-Caffarelli-Friedman under different contexts. For example, Caffarelli [3] established a monotonicity formula for variable coefficient operators, Friedman-Liu [14] have an extension for eigenvalue problems. Another important extension has been achieved by Caffarelli-Jerison-Kenig [10] who replace $\Delta u_i \geq 0$ by $\Delta u_i \geq -1$ ($i = 1, 2$). Under this new assumption they prove that $\phi(r)$ is uniformly bounded for $0 < r < \frac{1}{2}$. This is called an "almost monotonicity formula".

Even though there is no monotonicity in the Caffarelli-Jerison-Kenig formula, it does provide a control of $|\nabla u_1(0)| \cdot |\nabla u_2(0)|$ if $u_1, u_2$ are both smooth at $0$. For many free boundary problems, the control of $|\nabla u_1(0)| \cdot |\nabla u_2(0)|$
usually leads to important regularity results. Moreover, for some real life problems such as the Prandtl-Batchelor problem \([11, 14, 15]\) and some classical problems (e.g. see Shahgholian \([16]\)), the equations may be inhomogeneous and we may not have \(\Delta u_i \geq 0\) \((i = 1, 2)\) on each side of the free boundary. The “almost monotonicity formula” of Caffarelli-Jerison-Kenig is particularly useful in these situations and has provided a theoretical basis for the regularity theory of many new problems (see for example \([10, 16]\)).

For two-phase parabolic free boundary problems, Caffarelli \([9]\) established a monotonicity formula for two sub-caloric functions: Let \(u_1, u_2\) satisfy \(\Delta u_i - \partial_t u_i \geq 0\) in \(\mathbb{R}^n \times (-1, 0)\) \((i = 1, 2)\), \(u_1 \equiv 0, u_1(0, 0) = u_2(0, 0) = 0\). Let

\[
G_0(x, t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2 / 4t}
\]

be the fundamental solution of the heat equation in \(\mathbb{R}^n\), then

\[
\phi(r) = \frac{1}{r^n} \int_{\mathbb{R}^n} \int_{-r^2}^0 |\nabla u_1|^2 G_0(x, -s) dsdx \int_{\mathbb{R}^n} \int_{-r^2}^0 |\nabla u_2|^2 G_0(x, -s) dsdx
\]

is monotone non-decreasing provided that \(u_1, u_2\) have reasonable growth at infinity.

Clearly this monotonicity formula for parabolic free boundary problems is in correspondence with the Alt-Caffarelli-Friedman formula. Later, Edquist-Petrosyan \([13]\) derived the ”almost monotonicity formula” for \(u_1, u_2\) satisfying \(\Delta u_i - \partial_t u_i \geq -1\) instead of being sub-caloric. Similar to the Caffarelli-Jerison-Kenig formula, Edquist-Petrosyan proved the bound of \(\phi(r)\) for \(0 < r < \frac{1}{2}\).

A common feature of all monotonicity and almost monotonicity formulas aforementioned is that they are all designed for problems within Euclidean spaces. From theoretical and application viewpoints it is natural to consider some free boundary problems on Riemannian manifolds. Indeed, it has been pointed out by Caffarelli and Salsa in \([12]\) that the tools developed for free boundary problems on Euclidean spaces should have their counterparts for free boundary problems on manifolds (page ix of the introduction). The analogs of Alt-Caffarelli-Friedman monotonicity formula and Caffarelli-Jerison-Kenig almost monotonicity formula have been developed for the Laplace-Beltrami operator by the authors in \([17]\). The purpose of this article is to derive the analogue of Edquist-Petrosyan formula on Riemannian manifolds. In forthcoming works we shall use these formulas to discuss free boundary problems on Riemannian manifolds.

Let \((M, g)\) be a Riemannian manifold of dimension \(n \geq 2\), let \(B(p, \delta_p)\) be a geodesic ball around \(p\) with radius \(\delta_p = \min\{1, \text{inj}_p\}\) (\(\text{inj}_p\) is the injectivity radius at \(p\)). We shall use the following cut-off function \(\chi\) supported in \(B(p, \delta_p)\):

\[
\chi \equiv 1, \text{ in } B(p, \delta_p/4), \quad \chi \equiv 0, \text{ in } B(p, \delta_p/2).
\]
Let $R_m$ denote the curvature tensor. Assume
\begin{equation}
|R_m| + |\nabla_g R_m| \leq \Lambda.
\end{equation}
In this article, if we do not mention the dependence of a given constant, it is implied that this constant is either universal or depends only on $n$, $\Lambda$ and $\chi$.

Let $Q_r^-(p) = B_p(r) \times (-r^2, 0)$ for $0 < r < \delta_p/2$. If $u_+, u_- \in H^1_{loc}(Q_{\delta_p}^-)$, we define $w_\pm$ as $w_\pm = u_\pm - \chi$ and let
\[
\phi(r) = \frac{1}{r^4} \int_{S_r(p)} |\nabla_g w_+|^2 G(x,-s) dV_g ds \int_{S_r(p)} |\nabla_g w_-|^2 G(x,-s) dV_g ds
\]
where $G(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{d(x,p)^2}{4t}}$, $S_r = B(p, \delta_p) \times (-r^2, 0)$, $dV_g = \sqrt{\det(g)} dx$. Our main result is:

**Theorem 1.1.** Let $u_\pm$ be non-negative continuous functions in $Q_{\delta_p}^-$ that satisfy $u_+ \cdot u_- = 0$ and
\[
(\Delta_g - \partial_t)u_\pm \geq -1, \quad \text{in} \quad Q_p^-(1)
\]
in the weak sense. Then $u_\pm \in H^1_{loc}(Q_{\delta_p}^-)$ and there exists $C > 0$ such that
\[
\phi(r) \leq C(1 + \int_{Q_{\delta_p}^-} u_+^2 dV_g ds + \int_{Q_{\delta_p}^-} u_-^2 dV_g ds)^2
\]
for all $r \in (0, \frac{1}{2} \delta_p)$.

The proof of Theorem 1.1 is along the lines of [10] and [13]. However, since equations, integrals and kernels are defined on a Riemannian manifolds, many perturbation terms have to be properly controlled in different situations. For example, we need to derive a “perturbed” version of Beckner-Kenig-Pipher inequality in Proposition 2.3. As the reader will see, it is crucial in our analysis to have a quantitative estimate of the perturbation of the eigenvalues in this key inequality. In accordance to [10] and [13], assuming Holder continuity of solution, we can infer a more precise control of the functional $\phi$. This is the content of Theorem 3.1 we present at the end of the paper. With our “perturbed” version of Beckner-Kenig-Pipher inequality in hand, the proof of Theorem 3.1 becomes very similar to the corresponding theorem in [13] and is therefore omitted.

2. THE PROOF OF THE MONOTONICITY FORMULA

Let $U(x,t)$ be the heat kernel in the neighborhood of $p$, then for $t$ small we have
\[
U(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{d(x,p)^2}{4t}} \left( \sum_{i=0}^{\infty} \phi_i(x)t^i \right)
\]
(see [15] P109) where $\phi_i$ are smooth functions of $x$, and
\[
\phi_0 = \det(g)^{-\frac{1}{4}} = 1 + O(d(x)^2).
\]
Since we consider a neighborhood of the origin, $G$ and $U$ are clearly comparable in this neighborhood. As a consequence, $G$ can be replaced by $U$ in Theorem 1.1. In the following we will mainly use $U$ in our proof.

Let

$$A_k^\pm = \int_{S_{r-k}} \left| \nabla g w_{\pm} \right|^2 U(x, -s) dV_g ds, \quad b_k^\pm = 4^{4k} A_k^\pm.$$ 

To prove Theorem 1.1 it is enough to prove the bound of $\phi(r)$ for $0 < r < \delta$ where $\delta$ is a small constant depending on $n, \Lambda$ and $\chi$. The bound for $r > \delta$ is obvious. Therefore in the proof we only focus on the estimates of $A_k^\pm$ for $k$ sufficiently large.

We shall prove the following two key propositions:

**Proposition 2.1.** There exist $C_0, C_1 > 0$ such that for $k \geq k_0(n, \Lambda, \chi)$, if $b_k^\pm \geq C_0$,

$$4^4 A_{k+1}^+ A_{k+1}^- \leq A_k^+ A_k^- (1 + \delta_k)$$

where $\delta_k = C_1 \left( \frac{1}{\sqrt{b_k^+}} + \frac{1}{\sqrt{b_k^-}} + 4^{-2k} \right)$.

**Proposition 2.2.** There exists $\epsilon \in (0, 1)$ such that for $k \geq k_0(n, \Lambda)$, if $b_k^\pm \geq C_0$ and $4^4 A_{k+1}^+ > A_k^-$ then $A_{k+1}^- \leq (1 - \epsilon) A_k^-.$

Theorem 1.1 follows from Proposition 2.1 and Proposition 2.2 by standard argument in [10, 17]. Here we note that we shall always assume $u^\pm$ to be smooth, as $u^\pm$ can be mollified to $u^\pm_\epsilon$ such that necessary inequalities for $u^\pm_\epsilon$ can be obtained first. Therefore, the conclusion for $u^\pm$ can be obtained by passing $u^\pm_\epsilon$ to the limit. This part of the argument is standard and is omitted. The interested readers may look into [17] for reference.

The following estimate is important for the proof of both Proposition 2.1 and Proposition 2.2

$$\int_{S_r} \left| \nabla g w \right|^2 U(x, -s) dV_g ds$$

$$\leq C_M r^4 + C_M r^2 \left( \int_{\mathbb{R}^n} w^2_\pm (x, -r^2) U(x, r^2) dV_g \right)^\frac{1}{2}$$

$$+ \frac{1}{2} \int_{\mathbb{R}^n} w^2_\pm (x, -r^2) U(x, r^2) dV_g.$$  \hspace{1cm} (2.1)

where $C_M$ depends on $\int_{S_1} (u^2_+ + u^2_-) dV_g ds$.

**Proof of (2.1):** From $(\Delta_g - \partial_s) u^\pm \geq -1$, we have, by standard computation

$$(\Delta_g - \partial_s)(w^2_\pm) \geq -2w_\pm \chi - 4|\nabla g u_\pm| \cdot |\nabla g \chi| w_\pm - 2u_\pm |\Delta g \chi| w_\pm + 2|\nabla g w| w_\pm^2.$$
From the above we have
\[
2 \int_{S_r} |\nabla g w^\pm|^2 d\nu \leq \int_{S_r} (\Delta_g - \partial_s) (w^\pm_2) d\nu \\
+ 2 \int_{S_r} w^\pm d\nu + 4 \int_{S_r} (|\nabla g u^\pm| |\nabla g \chi| + u^\pm |\Delta g \chi|) w^\pm d\nu \\
= I_1 + I_2 + I_3.
\]
Here we use the notation: \(d\nu = U(x, -s) dV_g ds\). We shall also use \(d\nu^s = U(x, -s) dV_g\).

Using \(w^\pm \geq 0\) and integration by parts we obtain
\[
\tag{2.2} I_1 \leq \int_{\mathbb{R}^n} w^2_\pm (x, -r^2) d\nu - r^2.
\]

Note that we used \(\lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n} w^2_\pm (x, -s) U(x, \epsilon) dV_g = -w^2_\pm (0, 0) \leq 0\).

To estimate \(I_2\) we use the following equation easy to be verified by direct computation:
\[
(\Delta_g - \partial_s) (w^\pm - s) \geq 2 \nabla g u^\pm \nabla_g \chi + u^\pm \Delta_g \chi.
\]
For \(s_2 \leq s_1 \leq 0\) we integrate the above to obtain
\[
\int_{\mathbb{R}^n} (w^\pm - s_1) d\nu^{s_1} \leq \int_{\mathbb{R}^n} (w^\pm - s_2) d\nu^{s_2} \\
+ \int_{s_2}^{s_1} \left( \int_{\mathbb{R}^n} (2|\nabla g u^\pm| |\nabla g \chi| + u^\pm |\Delta g \chi|) d\nu^s \right) ds.
\]

For \(0 \geq s_1 \geq -r^2 \geq s_2 \geq -4r^2\), since the support of \(\nabla \chi\) or \(\Delta \chi\) stays away from the origin, it is elementary to obtain
\[
\int_{\mathbb{R}^n} w^\pm (x, s_1) d\nu^{s_1} \leq \inf_{s \in [-4r^2, -r^2]} \int_{\mathbb{R}^n} w^\pm (x, s) d\nu^s \\
+ C_1(\Lambda, n)r^2 + C_2(M, N)r^N.
\]
where \(M\) is the \(L^2\) norm of \(u^\pm\) on \(Q^{-}\), \(N\) is a large number. Therefore \(I_2\) satisfies
\[
\tag{2.5} I_2 \leq 2r^2 \inf_{s \in [-4r^2, -r^2]} \int_{\mathbb{R}^n} w^\pm (x, s) d\nu^s + C(N, M)r^4.
\]
If we further use Cauchy’s inequality we have
\[
I_2 \leq C(M, \Lambda)r^4 + \inf_{s \in [-4r^2, -r^2]} \int_{\mathbb{R}^n} w^2_\pm (x, s) d\nu^s.
\]
The estimate of \(I_3\) is similar.
\[
\tag{2.6} I_3 \leq C(N)r^N \int_{Q^{-}_{\frac{3}{4}}} (|\nabla g u^\pm|^2 + |u^\pm|^2) \leq C(N, M)r^N \int_{Q^{-}_{\frac{3}{4}}} u^2_\pm.
\]
can be obtained easily from (2.2), (2.5) and (2.6). As a consequence, the following estimates also hold:

\[
\int_{S_r} |\nabla g w^\pm|^2 d\nu \leq C_M r^4 + C_M \inf_{s \in [-4r^2, -r^2]} \int_{\mathbb{R}^n} w^2_s(x,s) d\nu^s. 
\]

(2.8)

\[
\int_{S_r} |\nabla g w^\pm|^2 d\nu \leq C_M r^4 + \frac{C_M}{r^2} \int_{S_{2r}\setminus S_r} w^2_s(x,s) d\nu. 
\]

In the following, we shall always re-scale \(w^\pm\) as follows:

\[
\bar{w}^\pm(y,s) = \frac{1}{r^2} w^\pm(r y, r^2 s). 
\]

\(\bar{w}^\pm\) is understood similarly. Correspondingly we let

\[
\bar{g}_{ij}(\cdot) = g_{ij}(r \cdot) = \delta_{ij} + O(r^2) |\cdot|^2 
\]

be the re-scaled metric, \(d\bar{\nu}\) and \(d\bar{\nu}^s\) are defined as

\[
d\bar{\nu} = r^n U(r y, -r^2 s) dV_g ds, \quad d\bar{\nu}^s = r^n U(r y, -r^2 s) dV_g, 
\]

then \(\bar{u}^\pm\) satisfy

\[
(\Delta_g - \partial_s)\bar{u}^\pm \geq -1. 
\]

We use \(\Omega^\pm\) to represent the set where \(w^\pm\) is positive. The corresponding set for \(\bar{w}^\pm\) is \(\bar{\Omega}^\pm\).

**Lemma 2.1.** For \(r < \delta_p\) and \(s = -\frac{1}{2}\)

\[
(2.9) \quad \log \frac{1 + O(r^2)}{|\bar{w}^\pm|_{\bar{\nu}^s}} \int_{\mathbb{R}^n} \bar{w}^2_s d\bar{\nu}^s \leq 2(1 + O(r^2)) \int_{\mathbb{R}^n} |\nabla g \bar{w}^\pm|^2 d\bar{\nu}^s. 
\]

**Proof of Lemma 2.1:** We use \(d\nu_0\) to represent the Gauss measure in Euclidean spaces:

\[
d\nu_0 = \frac{1}{(2\pi)^{n/2}} e^{-\frac{|x|^2}{2}} dx. 
\]

We perform two transformations on the three integral terms in (2.9):

\[
|\bar{w}^\pm|_\nu^s, \quad \int_{\mathbb{R}^n} \bar{w}^2_s d\bar{\nu}^s, \quad \int_{\mathbb{R}^n} |\nabla g \bar{w}^\pm|^2 d\bar{\nu}^s 
\]

to reduce them to the Euclidean case. First, using \(y = \phi_1(x)\) (\(\phi_1\) to be determined) we have

\[
|\nabla g \bar{w}^\pm(x)|^2 = \bar{g}^{ij}(x) \frac{\partial \bar{w}^\pm}{\partial x_i} \frac{\partial \bar{w}^\pm}{\partial x_j} 
\]

(2.10)

\[
= \bar{g}^{ij}(\phi^{-1}_1(y)) \frac{\partial y^m}{\partial x_i} \frac{\partial y^l}{\partial x_j} \frac{\partial \bar{w}^\pm}{\partial y_m} \frac{\partial \bar{w}^\pm}{\partial y_l} 
\]

Here repeated indices imply summation. Since \(\bar{g}_{ij}(x)\) is symmetric and

\[
\bar{g}_{ij}(x) = \delta_{ij} + O(r^2 |x|^2) \text{ for } |x| \leq r^{-1} \delta_2 \quad (\delta_2 \text{ small}), 
\]

we can choose \(\phi_1\) so that

\[
\frac{dy}{dx} = \left(\bar{g}_{ij}(x)\right)^{-\frac{1}{2}} = \left(\delta_{ij} + O(r^2 |x|^2)\right) 
\]
Consequently
\[ g^{ij}(\phi_1^{-1}(y)) \frac{\partial y^m}{\partial x_i} \frac{\partial y^l}{\partial x_j} = \delta^{ml} \]
and (2.10) becomes
\[ |\nabla g \bar{w}(x)|^2 = \sum_{i=1}^{n-1} |\frac{\partial \bar{w}}{\partial y_i}|^2. \]
Moreover the Jacobian of the mapping is of the order \( 1 + O(|\nabla \phi_1|) \) for \(|y| \leq \delta_3 r\) with \( \delta_3 \) small. With this \( \phi_1 \) we combine the Jacobian with the heat kernel:

\[ d\bar{v}_y := d\bar{v} J_{\phi_1} = r^n U(r\phi^{-1}(y), -r^2 s) J_{\phi_1} dV_y ds. \]

Using the definition of \( \phi_1 \) we now have (recall that \( s = -\frac{1}{2} \))

\[ r^n U(r\phi^{-1}(y), -r^2 s) J_{\phi_1} dV_y = \frac{1}{(2\pi)^{n/2}} e^{-\nu|y|^2} (1 + O(r^2|y|^2)) dy. \]

We use \( d\bar{v}_y^s \) to denote \( r^n U(rx, -r^2 s) J_{\phi_1} dV_y \). With these notations, the integral forms in (2.9) become

\[ \int_{\mathbb{R}^n} \bar{w}_\pm (\phi_1^{-1}(y)) d\bar{v}_y^s \]
\[ \int_{\mathbb{R}^n} \bar{w}_\pm^2 (x) d\bar{v}^s = \int_{\mathbb{R}^n} \bar{w}_\pm^2 (\phi_1^{-1}(y)) d\bar{v}_y^s \]

\[ \int_{\mathbb{R}^n} |\nabla \bar{g} \bar{w}_\pm|^2 d\bar{v}^s = \int_{\mathbb{R}^n} |\nabla \bar{g} (\phi_1^{-1}(y))|^2 d\bar{v}_y^s. \]

The purpose of the second transformation is to make \( d\bar{v}_y^s \) as close to the Gauss measure on Euclidean spaces as possible. To this end we write \( d\bar{v}_y^s \) as

\[ d\bar{v}_y^s = \frac{1}{(2\pi)^{n/2}} e^{-\nu|y|^2} (1 + A(y)) dy \]

where

\[ |A(y)| \leq Cr^2(1 + |y|)^2, \quad |DA(y)| \leq Cr^2(1 + |y|), \quad |y| \leq \delta_3 r^{-1}. \]

where \( \delta_3 \) is a small number. The second transformation is defined as follows:

\[ y = z + \psi(z) \]

where \( \psi \) satisfies \( z \cdot \psi(z) = \ln(1 + A(z)) \) for \(|z| > 1\). It is easy to obtain from the estimate of \( A \) that

\[ |\psi(z)| \leq Cr^2 |z|, \quad |D\psi(z)| \leq Cr^2, \quad 1 < |z| < \delta_4 r^{-1} \]

where \( \delta_4 \) is a small positive number. Then extend the definition of \( \psi \) to \( B_1 \) in such a way that both \( |\psi| \) and \( |D\psi| \) are of the order \( O(r^2) \) in \( B_1 \).

Using (2.13) and (2.14) we verify by direct computation that

\[ \frac{1}{(2\pi)^{n/2}} e^{-\nu|y|^2} (1 + A(y)) J_{\phi_1} d\bar{v}_y = \frac{1}{(2\pi)^{n/2}} e^{-\nu|z|^2} (1 + O(r^2)). \]
Let \( f_\pm(z) = \bar{w}_\pm(\phi^{-1}(y(z))) \), since the Jacobian \( J_{\frac{dy}{dz}} = 1 + O(r^2) \), the three integral terms in (2.9) are of the form (see (2.12)):

\[
|\bar{w}_\pm|_\nu^s = \int_{\mathbb{R}^n} f_\pm(z) d\nu_0^s(1 + O(r^2))
\]

\[
\int_{\mathbb{R}^n} \bar{w}_\pm^2(x) d\bar{\nu}^s = \int_{\mathbb{R}^n} f_\pm^2(z) d\nu_0^s(1 + O(r^2))
\]

(2.16)

\[
\int_{\mathbb{R}^n} \bar{w}_\pm|\nabla \bar{g}| \, d\bar{\nu}^s = \int_{\mathbb{R}^n} |\nabla f_\pm(z)|^2 d\nu_0^s(1 + O(r^2)).
\]

Note that in the last equality, we used \( \frac{dy}{dz} = id + O(r^2) \) where \( id \) is the identity matrix. For \( f_\pm \) we use the Poincare’s inequality on Euclidean spaces (see [13]):

\[
\log \frac{1}{|f_\pm|_{d\nu_0^s}} \int_{\mathbb{R}^n} f_\pm^2 d\nu_0^s \leq 2 \int_{\mathbb{R}^n} |\nabla f_\pm|^2 d\nu_0^s.
\]

Lemma 2.1 follows from the equation above and (2.16).

The following two lemmas have analogues in [10, 13] and their proofs are similar to their counterparts in [13], we include the proofs here for the convenience of the readers.

Lemma 2.2. Let \( w \) be \( w_+ \) or \( w_- \), suppose

\[
\int_{\Omega \cap S_4} |\nabla g w_\pm|^2 d\nu = \alpha r^4 < \infty, \quad \int_{\Omega \cap S_1} |\nabla g w|^2 d\nu \geq \frac{\alpha r^4}{256}.
\]

Then there exists \( C > 0 \) and \( C_1(n, \Lambda, M) > 0 \) such that if \( \alpha > C, \ |\Omega \cap S_4 \setminus S_1|_{d\nu} \geq C_1 r^2 > 0 \), where \( \Omega \) is the set on which \( w \) is positive.

Proof of Lemma 2.2. Let \( \bar{w} \) be \( \bar{w}_+ \) or \( \bar{w}_- \). Let \( \Omega \) be the set on which \( \bar{w} \) is positive. Then the assumptions become

\[
\int_{\Omega \cap S_1} |\nabla g \bar{w}_\pm|^2 d\bar{\nu} = \alpha, \quad \int_{\Omega \cap S_4} |\nabla g \bar{w}_\pm|^2 d\bar{\nu} \geq \frac{\alpha}{256}.
\]

We want to show that if \( \alpha \) is large, \( |\Omega \cap S_1 \setminus S_4|_{d\nu} > c_1(n) \). As a result of (2.7) we have

\[
\frac{\alpha}{256} \leq \int_{\Omega \cap S_1 \setminus S_4} |\nabla g \bar{w}_\pm|^2 d\bar{\nu} \leq C + C \inf_{s \in \left[ -\frac{1}{16}, -\frac{1}{8} \right]} \int_{\mathbb{R}^n} \bar{w}_\pm^2(x, s) d\bar{\nu}^s.
\]

Therefore for \( \alpha \) large

(2.17) \[
\inf_{s \in \left[ -\frac{1}{16}, -\frac{1}{8} \right]} \int_{\mathbb{R}^n} \bar{w}_\pm^2(x, s) d\bar{\nu}^s \geq \frac{\alpha}{512C}.
\]

From \( \int_{S_1} |\nabla g \bar{w}_\pm|^2 d\bar{\nu} = \alpha \), we see that

\[
\int_{\mathbb{R}^n} |\nabla g \bar{w}_\pm|^2 d\bar{\nu}^s \leq 16 \alpha
\]
except on a set of line measure no more than \( \frac{1}{16} \). So there exists a set \( E \) of line measure at least \( \frac{1}{8} \) on \([ -\frac{1}{4}, -\frac{1}{16}]\) such that
\[
\int_{\mathbb{R}^n} |\nabla \bar{w}|^2 d\nu^s \leq 16\alpha, \quad s \in E.
\]

For each \( s \in E \), either \( |\bar{w}|_{d\nu^s} > \frac{1}{2} \), or \( |\bar{w}|_{d\nu^s} \leq \frac{1}{2} \). In this latter case we apply Lemma 2.1 and (2.17) to get \( |\bar{w}|_{d\nu^s} \geq c(n) \). Recall that we always assume \( r \) to be small. So in either case there exists \( c(n) > 0 \) such that
\[
|\bar{w}|_{d\nu^s} > c(n)
\]
for all \( s \in E \). Therefore Lemma 2.2 is established by scaling. \( \square \)

**Lemma 2.3.** Let \( \bar{w}, \Omega \) be the same as those in Lemma 2.2, assume \( \mu \in (0, 1) \) such that
\[
|\Omega \cap S_{1/2} \setminus S_{1/4}|_{d\nu} \leq (1 - \mu)|S_{1/2} \setminus S_{1/4}|_{d\nu}.
\]
Then there exists \( \lambda(\mu) \in (0, 1) \) such that
\[
\int \int_{S_{1/4}} |\nabla \bar{g}\bar{w}|^2 d\nu \leq \lambda \int \int_{S_{1/2}} |\nabla \bar{g}\bar{w}|^2 d\nu.
\]

**Proof of Lemma 2.3:**
\[
|\Omega \cap S_{1/2} \setminus S_{1/4}|_{d\nu} = \int_{-(\frac{1}{2})^2}^{-(\frac{1}{4})^2} |F(s)|ds
\]
where \( |F(s)| \) is the measure of the positive set with respect to the \( d\nu^s \). We know that \( |F(s)| \leq 1 - \frac{2}{3} \) in a set \( E \subset \left[ -\frac{1}{2}, -\frac{1}{4} \right] \) with the line measure greater than or equal to \( \frac{1}{2}|S_{1/2} \setminus S_{1/4}|_{d\nu} \).

Using (2.3) we have, for small \( r \) and \( s \in E \), that
\[
\int_{\mathbb{R}^n} \bar{w}^2(\cdot, s)^2 d\nu^s \leq C \int_{\mathbb{R}^n} |\nabla \bar{g}\bar{w}(\cdot, s)|^2 d\nu^s
\]
If \( \int_{\Omega \cap S_{1/4}} |\nabla \bar{g}\bar{w}|^2 d\nu \leq \frac{\alpha}{2} \), there is nothing to prove. Suppose this is not the case, then by using (2.7) and the largeness of \( \alpha \), we have
\[
\frac{\alpha}{4C} \leq \int_{\mathbb{R}^n} \bar{w}^2(\cdot, s) d\nu \quad \forall s \in \left[ -\frac{1}{4}, -\frac{1}{16} \right].
\]
Specifically for \( s \in E \) we have
\[
\frac{\alpha}{4C} \leq \int_{\mathbb{R}^n} \bar{w}^2(\cdot, s) d\nu \leq C \int_{\mathbb{R}^n} |\nabla \bar{g}\bar{w}(\cdot, s)|^2 d\nu^s, \quad \forall s \in E.
\]
This implies that
\[
\int_{\Omega \cap S_{1/2} \setminus S_{1/4}} |\nabla \bar{g}\bar{w}|^2 d\nu \geq \frac{\alpha|E|}{4C^2}.
\]
Lemma 2.3 follows easily from the above. \( \square \)

The following proposition makes use of the two transformations used in the proof of Lemma 2.1.
Proposition 2.3. Let $\tilde{\Omega}_+^1 \subset \mathbb{R}^n$ be the set where $\tilde{w}_+\cdot (-1)$ is positive. $\tilde{\Omega}_-^1$ is understood similarly. There exists $C > 0$ such that
\[
\frac{\int_{\tilde{\Omega}_+^1} |\nabla \tilde{g}\tilde{w}_+\cdot (-1)|^2 d\tilde{v}^{-1}}{\int_{\tilde{\Omega}_+^1} \tilde{w}_+\cdot (-1)^2 d\tilde{v}^{-1}} + \frac{\int_{\tilde{\Omega}_-^1} |\nabla \tilde{g}\tilde{w}_-\cdot (-1)|^2 d\tilde{v}^{-1}}{\int_{\tilde{\Omega}_-^1} \tilde{w}_-\cdot (-1)^2 d\tilde{v}^{-1}} \geq 1 - Cr^2.
\]

Proof of Proposition 2.3:

We make the two transformations as used in the proof of Lemma 2.1. After the transformations, $\tilde{\lambda}_\pm(\cdot, -1)$ become $\tilde{w}_\pm$, $\tilde{\Omega}_\pm^1$ become $\tilde{\Omega}_\pm^1$. We still have $\tilde{\Omega}_+^1 \cap \tilde{\Omega}_-^1 = \emptyset$ and $\tilde{\Omega}_+^1 \cup \tilde{\Omega}_-^1 = B(0, \delta r^{-1})$ for some $\delta > 0$ small. $\tilde{w}_\pm$ are supported in $\tilde{\Omega}_\pm^1$, respectively. Moreover, by the same estimates as in the proof of Lemma 2.1 we have
\[
(2.18) \quad \int_{\tilde{\Omega}_\pm^1} \tilde{w}_\pm\cdot (-1)^2 d\tilde{v}^{-1} \leq (1 + Cr^2) \int_{\tilde{\Omega}_\pm^1} \tilde{w}_\pm^2 d\nu_0^{-1}
\]
and
\[
(2.19) \quad \int_{\tilde{\Omega}_\pm^1} |\nabla \tilde{g}\tilde{w}_\pm\cdot (-1)|^2 d\tilde{v}^{-1} \geq (1 - Cr^2) \int_{\tilde{\Omega}_\pm^1} |\nabla \tilde{w}_\pm|^2 d\nu_0^{-1}.
\]

Recall that $d\nu_0^{-1} = \frac{1}{(4\pi)^{n/2}} e^{-\frac{|x|^2}{4}} dx$.

Beckner-Kenig-Pipher inequality (a proof of which can be found in [11]) gives
\[
(2.20) \quad \frac{\int_{\tilde{\Omega}_+^1} |\nabla \tilde{w}_+|^2 d\nu_0^{-1}}{\int_{\tilde{\Omega}_+^1} \tilde{w}_+^2 d\nu_0^{-1}} + \frac{\int_{\tilde{\Omega}_-^1} |\nabla \tilde{w}_-|^2 d\nu_0^{-1}}{\int_{\tilde{\Omega}_-^1} \tilde{w}_-^2 d\nu_0^{-1}} \geq 1.
\]

Therefore Proposition 2.3 follows immediately from (2.18), (2.19) and (2.20). $\square$

3. Proof of Proposition 2.1 and Proposition 2.2

First we observe that Proposition 2.2 is a direct consequence of Lemma 2.2 and Lemma 2.3 as in [10, 13]. To prove Proposition 2.1 we let
\[
\tilde{v}_\pm(y, s_1) = \frac{1}{4^{2k}} w_\pm(4^k y, 4^{2k} s_1), \quad (y, s_1) \in \tilde{\Omega}_\pm.
\]

It is easy to see that $\tilde{\Omega}_+$ and $\tilde{\Omega}_-$ are disjoint subsets of $B(0, \delta 4^k)$ where $\delta > 0$ is small. Let $\tilde{g}_{ij}$ be the scaled metric, $d\tilde{v}$ and $d\tilde{v}^s$ be the new measures. Let \[
\tilde{\phi}(r) = \frac{1}{r^4} \int_{S_r} |\nabla \tilde{g}\tilde{v}_+|^2 d\tilde{v} \int_{S_r} |\nabla \tilde{g}\tilde{v}_-|^2 d\tilde{v}.
\]

Also we set
\[
\tilde{A}_\pm(r) = \int_{S_r} |\nabla \tilde{g}\tilde{v}_\pm|^2 d\tilde{v}, \quad \tilde{B}_\pm(r) = \int_{\mathbb{R}^n} |\nabla \tilde{g}\tilde{v}_\pm|^2 \tilde{U}(y, -r^2) d\tilde{V}_\tilde{g}.
\]
We want to show that for $\frac{1}{4} \leq r \leq 1$, if $\tilde{A}_\pm = \tilde{A}_\pm(1)$ are both large, then
\begin{equation}
\tilde{\varphi}'(r) \geq -C\tilde{\varphi}(r)\left(\frac{1}{\sqrt{\tilde{A}_+}} + \frac{1}{\sqrt{\tilde{A}_-}} + 4^{-2k}\right)
\end{equation}
where $C$ is independent of $k$. Once (3.1) is established, the integration of (3.1) gives
\begin{equation}
\tilde{\varphi}\left(\frac{1}{4}\right) \leq \tilde{\varphi}(1)(1 + C\delta_k), \quad \delta_k = \frac{1}{\sqrt{\tilde{A}_+}} + \frac{1}{\sqrt{\tilde{A}_-}} + 4^{-2k}.
\end{equation}
Then by scaling, (3.2) is equivalent to Proposition 2.1.

So we are left with the proof of (3.1). By scaling, the case for $\frac{1}{4} \leq r \leq 1$ can be treated as $r = 1$. Then the proof is very similar to the standard one:
\begin{equation}
\tilde{\varphi}'(1) = -4\tilde{A}_+\tilde{A}_- + 2\hat{B}_+\tilde{A}_- + 2\tilde{A}_+\hat{B}_-.
\end{equation}
If $\hat{B}_+ \geq 2\tilde{A}_+$ or $\hat{B}_- \geq 2\tilde{A}_-$, $\tilde{\varphi}'(1) \geq 0$. So we only assume $\hat{B}_\pm \leq 4\tilde{A}_\pm$.

Now we apply (2.1) to $\tilde{v}_\pm$ to get (using $d\tilde{\nu}^{-1} = \tilde{U}(\cdot, 1)dV_{\tilde{g}}$)
\begin{equation}
\tilde{A}_\pm \leq C + C\left(\int_{\mathbb{R}^n} \tilde{v}_\pm^2 d\tilde{\nu}^{-1}\right)^{\frac{1}{2}} + \frac{1}{2} \int_{\mathbb{R}^n} \tilde{v}_\pm^2 d\tilde{\nu}^{-1}
\end{equation}
\begin{equation}
\leq C + \frac{C}{\sqrt{\lambda_+}}\sqrt{\hat{B}_+} + \frac{1}{2\lambda_+}\hat{B}_+.
\end{equation}

Note that we can assume $\lambda_+$ and $\lambda_-$ are both positive, because if, say $\tilde{v}_+ \equiv 0$, we obtain from the first line of (3.3) that
\begin{equation}
\tilde{A}_+ \leq C,
\end{equation}
which is a contradiction to the largeness of $\tilde{A}_+$.

From (3.3) we see that if $\lambda_+ \geq 2$ or $\lambda_- \geq 2$, (3.1) is established easily. Therefore we assume $\lambda_\pm \leq 2$. In this case we obtain from (3.3) that
\begin{equation}
2\lambda_+\tilde{A}_+ \leq C + \frac{C}{\sqrt{\lambda_+}}\sqrt{\hat{B}_+} + \hat{B}_+.
\end{equation}

There is a similar equation for $2\lambda_-\tilde{A}_-$. Multiplying $\tilde{A}_+$ to (3.4), $\tilde{A}_+$ to the corresponding equation, we have, by adding these two equations
\begin{equation}
2(\lambda_+ + \lambda_-)\tilde{A}_+\tilde{A}_- \leq C(1 + \tilde{A}_-\sqrt{\frac{\hat{B}_+}{\lambda_+}} + \tilde{A}_+\sqrt{\frac{\hat{B}_-}{\lambda_-}}) + \hat{B}_+\tilde{A}_- + \hat{B}_-\tilde{A}_+.
\end{equation}

Proposition 2.3 gives $\lambda_+ + \lambda_- \geq 1 - C4^{-2k}$. Using this in (3.5) we obtain (3.1). Proposition 2.1 is established. □

Theorem 1.1 follows from Proposition 2.1 and Proposition 2.2 as in [10] and [13].
If further information on the growth of $u_\pm$ is known near the origin, then the behavior of $\phi(r)$ can be made more precise. This is the observation in [10, 13].

**Theorem 3.1.** Let $u_\pm, w_\pm, \chi$ be the same as in Theorem 1.1, suppose in addition that

$$|u_\pm(x, s)| \leq C_\epsilon(|x|^2 + |s|)^{\frac{\epsilon}{2}}$$

for $(x, s) \in Q_{\delta_p}$ and $\epsilon \in (0, 1)$. Then

$$\phi(r) \leq (1 + \rho')\phi(\rho) + C_M \rho', \quad 0 < \rho \leq \delta_p/4$$

where $C_M$ depends on $n, \Lambda, M = \|u_+\|_{L^2(Q_{\delta_p})} + \|u_-\|_{L^2(Q_{\delta_p})}, \chi, \epsilon$.

Since the proof of Theorem 3.1 is similar to its analogue in [13], we leave the detail to the interested readers. It is worthwhile to point out here that the only difference in the proof comes from the correction term in the Beckner-Kenig-Pipher inequality, [6]. The readers can easily see that at this point the extra term does not cause further difficulties in the argument.

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