PSL(n|n) Sigma Model as a Conformal Field Theory

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Abstract

We discuss the sigma model on the PSL(n|n) supergroup manifold. We demonstrate that this theory is exactly conformal. The chiral algebra of this model is given by some extension of the Virasoro algebra, similar to the $W$ algebra of Zamolodchikov. We also show that all group invariant correlation functions are coupling constant independent and can be computed in the free theory. The non invariant correlation functions are highly nontrivial and coupling dependent. At the end we compare two and three-point correlation functions of the PSL(1,1|2) sigma model with the correlation functions in the boundary theory of $AdS_3 \times S^3$ and find a qualitative agreement.
1. Introduction

In this paper we discuss two-dimensional nonlinear sigma models on the supergroup manifolds $PSL(n|n)$. The interest in these models is motivated by the recent discoveries in the string theory. It was long suspected by Polyakov and others (see, for example [1] and references therein) that gauge theory can be described by some version of string theory. The first concrete example of this was recently suggested in [2]. The strings propagate in the $AdS$-type supergravity background. The $\mathcal{N} = 4$ supersymmetric gauge theory “lives” on the boundary of the $AdS$ space and provides boundary conditions for the bulk gravity theory. The supergravity describes the large coupling limit of the gauge theory (literally speaking $g^2N = \infty$ limit). The $1/N$ corrections can be identified with string loops. The string theory appears to be critical with constant dilaton which implies that the boundary theory is conformal.

The immediate problem is to understand string theory in the $AdS$-type backgrounds. Unfortunately, the NSR formalism does not seem to be suitable for the description of RR backgrounds. The appearance of the RR vertex operators introduces an arbitrary number of cuts ruining the NSR worldsheet. One can define/compute the scattering amplitudes of several RR vertex operators but it is unclear how to describe a condensate or a background of RR fields. The GS formalism seems to be more appropriate for that. This reminds us of the sigma model with the target space being the supermanifold. Therefore, one can suggest that the $PSU(1,1|2)$, or more generally the $PSL(n|n) = SL(n|n)/U(1)$ sigma models would naturally appear in this description. Indeed, the appearance of the worldsheet scalars/target space fermions is the genuine feature of the GS-type actions. On the other hand the second order fermionic kinetic term indicates the possible relation to RR flux. In fact, there are suggestions for a GS formulation of string theory based on the $PSL(n|n)$ cosets [4–11]. Unfortunately, it is unclear how to quantize this string theory. The above provides more than enough motivation to study these sigma models.

To be a bit more precise, one can say that $PSU(1,1|2)$ sigma model is related to string theory propagating in $AdS_3 \times S^3 \times M^4$ background. The NSR formulation of string theory propagating on this background was given in [3]. The classical formulation of GS superstring was discussed in [4–6]. The $PSU(1,1|2)$ sigma model is not a string theory yet but it is going to be an important ingredient in the construction. Such string theory was recently proposed in [12].

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Similarly, the sigma model on a $PSU(2,2|4)$ is related to the string theory on the $AdS_5 \times S^5$ background. The latter can be regarded as the bosonic part (body) of a quotient superspace of a supergroup by the subgroup

$$\mathcal{T} = \frac{PSU(2,2|4)}{SO(4,1) \times SO(5)} \quad (1.1)$$

The $SO(4,1) \times SO(5)$ action is an isometry without fixed points. Notice, that there is no GKO construction and therefore it is not obvious that the quotient (1.1) leads to the conformal theory. However the subgroup $SO(4,1) \times SO(5)$ turns out to be very special and one can show that the sigma model on a quotient space is one loop finite. Moreover, we expect that sigma model on the quotient space is exactly conformal in all orders in perturbation theory. The classical formulation of GS string theory based on this quotient was discussed in [7], [8–11]. We believe that understanding the $PSU(2,2|4)$ sigma model may shed some light on how the string theory on $AdS_5 \times S^5$ background can be quantized.

It is clear, even without making any calculations, that the $PSL(n|n)$ sigma models are quite remarkable. The dual Coxeter number $C_V$ vanishes for these groups and therefore the one loop beta function, which is proportional to $C_V$, is also zero. Hence, one may suspect that these theories are exactly conformal. Moreover, the supermanifold $PSL(n|n)$ is in a sense a Calabi-Yau supermanifold (see [13]). Namely, the Ricci tensor is identically equal to zero $R_{ij} \sim C_V g_{ij}$. The Ricci flatness implies that the theory is one loop finite. In [13] the finiteness of the theory was ensured by $\mathcal{N} = 2$ supersymmetry, in our case the theory is conformal without worldsheet supersymmetry. The idea of thinking about $PSL(n|n)$ as a Calabi-Yau manifold could be very fruitful for possible future applications. For example, one may try to construct new examples of supermanifolds that lead to exactly conformal field theories by tensoring several $PSL(n|n)$ models and then taking a quotient by certain subgroups. Very little is known about sigma models on supermanifolds and potential applications are enormous.

In this paper we study the sigma model on a supergroup $PSL(n|n)$ in perturbation theory (large volume expansion). Clearly, there is no choice of real structure on the group for which the metric is positive definite. Nevertheless, we assume that the theory can be well defined. For most part of the text our discussion is going to be purely algebraic such that it does not require a choice of the real structure on the group. We specify the signature at the very end when we discuss the relation to $AdS_3 \times S^3$. We prove that the theory is exactly conformal to all orders in perturbation theory for any values of
the coupling constant. We also prove that the correlation functions of the group invariant combinations of operators are given by the gaussian integration and the interaction vertices do not contribute to these calculations. This observation does not make the theory trivial. There are plenty of correlation functions that have a non-trivial dependence on the sigma model coupling. These are the correlation functions that are not invariant under the group action. The spectrum of conformal primary fields is classified by the representations of the left/right multiplication symmetry group – \( V_{(\Lambda_L, \Lambda_R)} \). We will describe a certain class of operators (analogous to vertex operators of string theory) corresponding to representations \((\Lambda_L, \Lambda_R)\) such that all Casimir operators \( \hat{C}^{(N)} \) have the same eigenvalues for both of them \( (C_{\Lambda_L}^{(N)} = C_{\Lambda_R}^{(N)}) \). We conjecture that conformal dimensions of these operators are equal to \( \Delta = \bar{\Delta} = \lambda^2 C^{(2)}/2 \), where \( \lambda \) is the sigma model coupling constant and \( C^{(2)} \) is the eigenvalue of quadratic Casimir. These operators are very similar to the momentum modes of \( c = 1 \) system on a circle. We also find that \( PSL(n|n) \) sigma model has a chiral algebra which is an extension of the Virasoro algebra (similar to W-algebras of Zamolodchikov). Very little is known about such W-algebras and it could be a very promising direction for future studies.

In the next section we discuss the definition of \( PSL(n|n) \) sigma model, as well as some properties of \( PSL(n|n) \) group. In section 3 we study our sigma model in perturbation theory and show that it is exactly conformal using the background method. Then we analyze the possible quantum corrections to the equation \( \partial T_{zz} = 0 \) and show that they vanish. We also present some examples of correlation functions involving currents. In section 4 we present some non-perturbative arguments based on localization. In section 5 we describe the chiral algebra of the \( PSL(n|n) \) sigma model. We believe that the presence of this algebra should be important for solving the theory. Finally, in the last section we discuss physical operators and try to make contact with the string theory on \( AdS_3 \times S^3 \).

While this paper was in a final stage of preparation we received a paper [12], that partially overlaps ours.

2. Principal Chiral Field

The \( PSL(n|n) \) principal chiral field is a two-dimensional non-linear sigma model with the fields \( G(x) \) taking values in a supergroup \( PSL(n|n) \). The action for the principal chiral model is

\[
S[G] = \frac{1}{4\pi\lambda^2} \int \text{Str}(|G^{-1}dG|^2) d^2x .
\]  (2.1)
It is invariant with respect to both left and right multiplications \( G(x) \to U_LG(x)U_R^{-1} \), the corresponding conserved currents are

\[
J_R = G^{-1}dG, \quad J_L = dGG^{-1}.
\]  

(2.2)

In its turn, the left current is invariant under the right multiplication symmetry, while the right current is invariant under the left multiplication. Under the left (right) multiplication the left (right) current transforms by conjugation \( J_{L,R} \to U_{L(R)}J_{L(R)}U_{L(R)}^{-1} \). The presence of this symmetry (especially its fermionic part) makes it difficult to define a theory.

Let us remind the reader what happens in the case of conventional bosonic sigma models. As usual, correlation functions are normalized by the partition function

\[
\langle \prod_i \mathcal{O}_i \rangle = \frac{\int [Dg] \prod \mathcal{O}_i e^{-S}}{\int [Dg] e^{-S}}.
\]

(2.3)

The integral runs over the space of maps from our worldsheet (say, the sphere) into the group manifold. The symmetry groups \( G_L/G_R \) act on this space by left/right multiplications. Therefore one can try to factorize the path integral into the integral over the space of orbits and a finite dimensional integration along the orbit. In the case of correlation functions invariant under either \( G_L \) or \( G_R \) action, integration along the corresponding orbit introduces a multiplicative factor (the volume of the group), which cancels in the numerator and denominator. For non-invariant quantities the group integration projects on the invariant subspace.

One way to compute correlation functions is to use perturbation theory. To build a perturbation theory one is forced to fix the “false” vacuum to expand around. For example, one can impose the condition that at infinity the field \( G(x) \) approaches some fixed element \( G_0 \). This choice clearly breaks the left/right multiplication symmetry. Again, in the case of correlation functions invariant under either \( G_L \) or \( G_R \) action, the remaining integral over \( G_0 \) can be thought of as an integral along the orbit of a symmetry action and hence gives the same volume factor which cancels. The quantum field around \( G_0 \) is a Goldstone boson of this broken symmetry and therefore massless. Consequently, we run into infrared problems. To cure these one can add a small mass term, or the potential around \( G_0 \), or work in the finite volume. However, the above correlation functions are perturbatively well defined, i.e., IR finite \([14]\) \([15]\) and therefore one may trust the perturbative calculations. Still, one has to be careful making conclusions based on them. The perturbation theory
does not feel the global properties of the group as it is built as an expansion around a particular point.

It is instructive to consider a simple example of a free scalar field theory. The theory is invariant with respect to a constant shift \( X \rightarrow X + c \) (zero mode). The correlation function of the vertex operators requires an IR cutoff \( m \)

\[
\langle e^{ikX(z_1)}...e^{ikX(z_n)} \rangle = m^{(\sum k_i)^2/2} \prod |z_i - z_j|^{k_i k_j}.
\]  

(2.4)

This correlation function is invariant under the symmetry \( X \rightarrow X + c \) only if \( \sum k_i = 0 \) (otherwise it gets multiplied by a phase). Invariant correlation functions are independent on the IR cutoff, while non-invariant correlation functions vanish.

Trying to repeat the above steps in the case of a supergroup we find that the naive integration along the orbit produces zero – the volume of the supergroup. However we still want to define correlation functions by normalizing them by the partition function as in (2.3). Formally, the numerator and denominator vanish as the consequence of the zero group volume, but we can take a sensible limit to define the ratio. First we need to break the left/right multiplication symmetry, say by the same condition that at infinity the group element \( G(x) \) approaches some fixed element \( G_0 \). The fluctuations around the “false” vacuum \( G_0 \) are massless bosonic and fermionic degrees of freedom. The latter give zero (due to fermionic zero modes) while the massless bosons give rise to the infrared problems. Again, adding a small mass term or the potential around \( G_0 \) fixes both problems and allows one to define the correlation function as the limit of the ratio when the mass/IR cut-off is sent to zero. In fact this is exactly what is computed in perturbation theory for the conventional principal chiral model.

The \( G_L \) or \( G_R \) invariant correlation functions are again IR finite and therefore this procedure provides a consistent way of making sense out of (2.3). The situation with non-invariant quantities is far from being well understood. In the bosonic case the group integration effectively projects on the group invariant subsector. In the case of a supergroup, there is no well defined projection operator and one has to be very careful (some examples of these computations will be discussed later).
2.1. Introduction to $GL(n|n)$, $SL(n|n)$ and $PSL(n|n)$ groups.

The groups $GL(n|n)$, $SL(n|n)$ and $PSL(n|n)$ are closely related to each other. The supergroup $GL(n|n)$ consists of real $(n|n)$ supermatrices with non-zero superdeterminant. $SL(n|n)$ is a subgroup of $GL(n|n)$ – simply matrices with superdeterminant equal to 1. It has a normal $U(1)$ subgroup – matrices that are multiples of the identity. The $PSL(n|n)$ is a factor of $SL(n|n)$ by that subgroup. Unfortunately, $PSL(n|n)$ does not have a representation in $Mat(n|n)$. This $U(1) \subset SL(n|n)$ that one has to factor out to get the $PSL(n|n)$ group appears as an additional gauge symmetry in the $SL(n|n)$ principal chiral field. This suggests a way to think about $PSL(n|n)$ principal chiral field as a gauge invariant subsector of the $SL(n|n)$ sigma model.

We start with properties of the above groups or, rather, their Lie superalgebras. The superalgebra $gl(n|n)$ has a non-degenerate metric given by $g_{ij} = Str(TiTj)$, where $T_i$ are the generators of $gl(n|n)$ in the fundamental representation. We choose the generators of $gl(n|n)$ as follows: first $(2n)^2 - 2$ generators span a subspace of supertraceless and traceless $2n \times 2n$ matrices and we denote them as $T_a$. The remaining generators are the identity matrix $I$, and matrix $J$, $J = \text{diag}(1, \ldots, 1, -1, \ldots, -1)$. Also note, that $T_a$’s together with the identity generate $sl(n|n)$, i.e. their commutators close without $J$.

Generators $T_a$ get projected on generators of $psl(n|n)$ algebra and we keep the same notations for them. The $T_a$’s do not generate $psl(n|n)$, as an identity appears among commutators of $T_a$. However, if we “ignore” the identity part of the commutators

$$[T_a, T_b] = f_{ab}^c T_c + d_{ab}I$$

we would obtain the structure constants of $psl(n|n)$, and the Jacobi identity is satisfied as one can easily check. “Ignoring” means subtracting identity to make the result traceless. Note that the trace and supertrace operations are related to each other $Str(X) = Tr(XJ)$.

In our basis the $gl(n|n)$ metric looks like:

$$
\begin{pmatrix}
g_{ab} & 0 \\
0 & 2n \\
0 & 2n & 0
\end{pmatrix}
$$

The metric $g_{ab}$ is an invariant metric on $psl(n|n)$. 

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2.2. Comments on $GL(n|n)$ principal chiral model

The group $GL(n|n)$ is not semi-simple and it has a family of invariant metrics $g_{ij}^{(\mu)} = Str(T_i T_j) + \mu Str(T_i) Str(T_j)$. This metric is non-degenerate for any value of $\mu$. A simple calculation shows that at one loop a new term is generated

$$S = \frac{1}{\lambda(\Lambda)^2} \int Str(|G^{-1}dG|^2) + \mu(\Lambda) \int |Str(G^{-1}dG)|^2,$$

which effectively introduces the cut-off dependence of the metrics $g_{ab}^{(\mu)}$.

To get the $SL(n|n)$ principal chiral model one just needs to restrict $G$ to lie in $SL(n|n)$ subgroup. There are two related issues (i) the $SL(n|n)$ invariant metric is degenerate and (ii) the $SL(n|n)$ principal chiral action has $U(1)$ gauge symmetry. The appearance of this gauge invariance is quite remarkable and happens only for $SL(n|n)$. It acts as $G \to e^\phi G$. Under this symmetry the current gets shifted $J \to J + d\phi$, but the action is still invariant because $e^\phi$ is proportional to the identity matrix. Restricting oneself to a gauge invariant sector is the equivalent of making a quotient $PSL(n|n) = SL(n|n)/U(1)$. At the same time the problem (i) also disappears, the $PSL(n|n)$ has an invariant non-degenerate metric.

The $gl(n|n)$ Lie algebra has a decomposition $gl(n|n) = F_- \oplus B_0 \oplus F_+$, where $F_\pm$ are the subspaces of the lower and upper triangular odd matrices and $B_0 = gl(n) \times gl(n)$, the even part of the $gl(n|n)$. Every element $Q_{ij} \in F_\pm$ is nilpotent and $[F_-, F_+] \subset B_0$. The $GL(n|n)$ contains many operators that may play a role of the BRST operator. As we already explained, the left/right multiplication symmetry is broken by the choice of the vacuum (for simplicity we choose $G_0 = I$). Still our model is invariant with respect to conjugations. As we will see, the $GL(n|n)$ is essentially a topological sigma model. For example, consider operator\[16\]

$$Q = \sum_{i=1}^{i=n} Q_{n+i,j}.$$

The action of this operator corresponds to conjugation by the matrix

$$\begin{pmatrix} 1 & 0 \\ \epsilon 1 & 1 \end{pmatrix}$$

(2.8)

This operator is clearly nilpotent, but what is more important, the $GL(n|n)$ sigma model action is $Q$-exact! Indeed, if we represent the current $J_\mu$ as $2 \times 2$ block matrix\[16\]

$$J_\mu = \begin{pmatrix} A_\mu & \chi_\mu \\ \eta_\mu & B_\mu \end{pmatrix}$$

\[1\] It does not matter whether we choose the left or the right current. Their transformation properties differ by sign.
then the transformation properties are

\[
\delta \epsilon A_\mu = \epsilon \chi_\mu , \quad \delta \epsilon B_\mu = \epsilon \chi_\mu \\
\delta \epsilon \chi_\mu = 0 , \quad \delta \epsilon \eta_\mu = \epsilon (A_\mu - B_\mu) .
\]

(2.9)

Now it is easy to see that the action (2.1) is

\[
Q\text{-exact}
\]

\[
Tr(A_\mu A^\mu) - Tr(B_\mu B^\mu) + 2Tr(\chi_\mu \eta^\mu) = [Q, Tr(\eta_\mu (A_\mu + B_\mu)]
\]

(2.10)

Similarly, the induced term in (2.6) is also Q-exact. As the result, the computations of group invariant correlation functions reduces to a classic al problem.

The BRST operator introduced in (2.7) turns out to be very useful. For example, using the Q-cohomology technique one can prove the following theorem: if any Casimir operator has a non-zero eigenvalue on an irreducible representation Λ, then the super dimension of this representation is zero.

The proof goes as follows: Consider the universal enveloping algebra \( U(\mathcal{G}) \) (we would be interested in the case when \( \mathcal{G} = gl(n|n), sl(n|n) \) or \( psl(n|n) \)). The action of \( Q \) on \( \mathcal{G} \) can be lifted to an action on \( U(\mathcal{G}) \). Now, using the Poincare-Birkhof-Witt theorem one identifies \( U(\mathcal{G}) \) with \( S(\mathcal{G}) = \bigoplus_{i=0}^{\infty} S^i(\mathcal{G}) \), where \( S^i(\mathcal{G}) \) are symmetric powers of \( \mathcal{G} \). This identification is (i) an isomorphism of \( \mathcal{G} \)-modules and (ii) the space of invariants of \( S(\mathcal{G}) \) is mapped on the center in \( U(\mathcal{G}) \) (Casimirs). Now consider the action of \( Q \) on \( S(\mathcal{G}) \). It is clear that \( Q : S^i(\mathcal{G}) \rightarrow S^i(\mathcal{G}) \) and one can define cohomology \( H^*(Q, S^i(\mathcal{G})) \). Now, it can be shown that

\[
H^*(Q, S^i(\mathcal{G})) = \begin{cases} 
0, & \mathcal{G} = gl(n|n) \quad i > 0 \\
0, & \mathcal{G} = sl(n|n) \quad i > 1 \\
0, & \mathcal{G} = psl(n|n) \quad i > 2 
\end{cases}
\]

(2.11)

For \( sl(n|n) \) the cohomology is non-zero only in degrees \( n = 0,1 \) and is spanned by constants and \( Q \). For \( psl(n|n) \) the cohomology is non-zero in degrees \( n = 0,1,2 \) and the quadratic Casimir spans the cohomology in degree \( n = 2 \).

As a result all Casimir operators are Q-commutators \( \hat{C}^{(N)} = [Q, X^{(N)}] \) for \( N > 2 \). The super dimension of any irreducible representation can be written as

\[
\text{sdim}(\Lambda) = \text{Str}(I) = \frac{1}{C^{(N)}} \text{Str}([Q, X^{(N)}]) = 0 ,
\]

(2.12)

where \( C^{(N)} \) in the eigenvalue of the \( N \)th Casimir. Now, for \( psl(n|n) \) the quadratic Casimir is not \( Q \)-exact, but its square is! Therefore, we can slightly modify our arguments in (2.12) by replacing \( \hat{C}^{(2)} \) by its square.

This BRST-like symmetry can be used to compute some correlation functions for \( PSL(n|n) \) models using the localization technique. We will discuss this issue in one of the next sections.
3. Principal chiral field as a conformal theory

3.1. Background field method and conformal invariance.

The simple way to show conformal invariance of the $PSL(n|n)$ principal chiral model is based on the background field method [17]. In this method the symmetries of the theory impose strong constraints on the form of the renormalized action, which is very useful. We find that our theory is conformal to all orders in perturbation theory.

First of all, the renormalization of the general non-linear sigma model was studied extensively and the renormalization group flow for it was interpreted as the flow in the space of metrics on the target manifold. In other words the shape and size of the manifold changes with the flow [18]. The same topic was also studied using the background field method [19] [20]. The main technical difficulty for the general case is the necessity to expand the action in terms of “linear” fields, e.g. the Riemann normal coordinates. In the case of the group manifold as a target space the expansion simplifies dramatically and terms of any order can be written explicitly. This will allow us to analyze all orders in perturbation theory.

In the background field method we parameterize the quantum field $G(x)$ as $g(x)G_0(x)$ where $G_0(x)$ is the classical background and the field $g(x)$ describes quantum fluctuations. The “linear” field for the quantum fluctuation is defined by $g(x) = e^{\lambda A(x)}$ where $\lambda$ is inserted just to have a convenient normalization later. $A(x)$ is an element of the Lie algebra and transforms as a tangent vector at the point $G_0(x)$. In terms of these fields the (right) current of the model becomes

$$J_G^\mu \equiv G^{-1}\partial_\mu G = G_0^{-1}\partial_\mu G_0 + G_0^{-1}j^A_\mu G_0 = J^0_\mu + G_0^{-1}j^A_\mu G_0 , \quad (3.1)$$

where we denoted the background current by $J^0$ and the current corresponding to quantum fluctuations as $j^A_\mu = e^{-\lambda A}\partial_\mu e^{\lambda A}$. The action is then

$$S[G] = S[e^{\lambda A}] + S[G_0] + \frac{1}{2\pi\lambda^2} \int Str (j^A_\mu (\partial^\mu G_0)G_0^{-1}) \quad (3.2)$$

The current $j^A$ written in terms of $A$ has the following expansion:

$$j^A_\mu \equiv e^{-\lambda A}\partial_\mu e^{\lambda A} = \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} [[[\partial_\mu A, A], A] \ldots A] = \lambda \partial_\mu A + \frac{\lambda^2}{2} [\partial_\mu A, A] + \frac{\lambda^3}{3!} [[[\partial_\mu A, A], A] + \ldots \quad (3.3)$$
We will need the polynomial expansion of (3.2) in terms of $A$:

$$L(e^{\lambda A}) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\lambda^{2n-2}}{(2n)!} \text{Str}(\ldots[[\partial_\mu A, A], A] \ldots A) \partial^\mu A) =$$

$$= \frac{1}{4\pi} \left( \text{Str}(\partial_\mu A \partial^\mu A) + \frac{\lambda^2}{12} \text{Str}([[\partial_\mu A, A] \partial^\mu A) + \ldots \right)$$

(3.4)

Putting it all together we find that our Lagrangian for the quantum field $A(x)$ in the background $G_0(x)$ contains the following terms: an $A$-independent part which is the Lagrangian for the background field itself, free-field kinetic energy $\text{Str}(\partial_\mu A \partial^\mu A)$ and the interaction terms of two kinds. First, the interaction terms which involve the background current and single derivative of $A$ and second, the terms that appear in the expansion (3.4), which contain two derivatives of $A$. An important point to notice is that all interaction vertices are built from structure constants of $\text{psl}(n|n)$. Schematically, each interaction vertex can be represented as a tree diagram, shown in Fig. 1. These diagrams describe the “color structure” of the interaction vertices. Each three-vertex in the picture represents a $\text{psl}(n|n)$ structure constant $f_{abc}$ and the dashed lines correspond to the contraction of indices with the $\text{psl}(n|n)$ invariant metric $g_{ab}$ but no propagator insertion.

![Fig.1 Typical interaction vertices](image)

To compute effective action for the background field $G_0(x)$, one has to evaluate all 1PI diagrams with external background lines only. The typical diagram is represented in Fig. 2. To find the beta function we have to renormalize UV divergent diagrams. There are of course IR divergent diagrams, but we assume that they have been regularized as discussed before, say, by including a small mass term. Notice that the structure of interaction vertices is such that the background current appears in vertices with a single derivative of $A$. Therefore, by power counting, all primitively divergent diagrams contain no more than two external lines of the background field. Those diagrams with no background external lines at all will lead to renormalization of the action for the quantum field $A$. The resulting wave function/vertex renormalization for it would not affect our conclusions because there are no external $A$-lines. By choosing the renormalization scheme that preserves the symmetry
of the problem, one also ensures that the group structure of renormalized vertices does not change.

Fig. 2: Example of a diagram with two external background lines

Now we will demonstrate that the diagrams with one or two background external lines are identically equal to zero, even before one computes the momentum integral. All these diagrams vanish because of the group factors. Consider first the diagrams with a single external background line. As we mentioned, we represented the interaction vertices as tree diagrams constructed from three-vertices (structure constants). Let us take the 3-vertex where the external line enters and pull it out of the diagram (see Fig. 3a). The rest of the diagram can be represented as a blob with two external lines. Its group structure is given by a second rank invariant tensor. There is only one such tensor – the metric and its contraction with the structure constants vanishes.

Fig. 3

Next let us consider the diagrams with two external background lines and do the same trick. Let us pull out the 3-vertex where the external line enters. The rest of the diagram is now a blob with three external lines, one of them is the background field line and the other two are contracted with the structure constants (see Fig. 3b). This blob is also an invariant tensor. Luckily enough, \( \text{psl}(n|n) \) has only one invariant rank 3 tensor – structure constants. This can be proved as follows: all rank 3 invariant tensors come from the \( \text{gl}(n|n) \). For \( \text{gl}(n|n) \) there are 6 invariant rank 3 tensors but only one survives the reduction to \( \text{psl}(n|n) \). As a result, the whole contraction is proportional to the metric times the dual Coxeter number \( f_{abc} f_d^{bc} = C_V g_{ad} \). However, that number vanishes for the \( \text{psl}(n|n) \) groups.\(^2\)

\(^2\) There is another group for which dual Coxeter number vanishes – \( \text{osp}(2n + 2|2n) \). It is possible that the \( \text{osp}(2n + 2|2) \) sigma model is also conformal.
What we just showed is that there are no divergent diagrams for the effective action and therefore the coupling $\lambda$ is not renormalized to all orders in perturbation theory. Hence the $\text{psl}(n|n)$ principal chiral model is perturbatively conformal.

3.2. Perturbation theory.

In this section we present an analysis of the perturbation theory for the $\text{PSL}(n|n)$ principal chiral model. We find that some correlation functions can be computed exactly to all orders in perturbation theory, in an analogy to the calculations in the background field method. We will also prove conformal invariance in a different way.

The very definition of the model requires breaking of the $G_L \times G_R$ invariance in order to get rid of the fermionic zero modes. We choose a “false” vacuum $G(x) = G_0$ and, if working in an infinite volume, add a small potential term making $G_0$ a true ground state. This is exactly what is necessary to build a perturbation theory. The action in the “background” of the $G_0$ vacuum is given by (3.2) plus the potential term that we choose as $m^2 \text{Str}(A^2)$

$$S_{\text{pert}}[A; G_0] = S[e^{\lambda A}] - \frac{1}{4\pi} \int d^2 \sigma \; m^2 \text{Str}(A^2) =$$

$$= \frac{1}{4\pi} \int d^2 \sigma \; \text{Str}(\partial_\mu A \partial^\mu A - m^2 A^2) + S_{\text{int}} \quad (3.5)$$

Although this action explicitly breaks $G_L \times G_R$ symmetry it is still invariant under the subgroup which leaves $G_0$ invariant. This remaining symmetry acts on $A$ simply by conjugation $A \rightarrow h^{-1} Ah$ and on $G(x)$ by $G(x) \rightarrow hG(x)G_0^{-1}h^{-1}G_0$. We will explore consequences of this symmetry later on.

Once again, the most important thing to notice about (3.5) is that all the interaction vertices in $S_{\text{int}}$ are built from the structure constants of $\text{psl}(n|n)$ in exactly the same fashion as we found in the background field method (See Fig. 1). In particular, all vertices have the group structure of tensors invariant under the remaining symmetry, i.e., tensors invariant under the conjugation by the elements of the Lie algebra of $\text{psl}(n|n)$.

Now suppose we want to compute a correlation function which is manifestly invariant under the left and right multiplication symmetry $G_L \times G_R$. As explained, we expect to trust the answer computed in the broken symmetry “phase” and we also know that in perturbation theory the result is IR-finite. Invariance under $G_L \times G_R$ symmetry implies that the correlation functions are invariant under the symmetry remaining in the perturbation theory – the conjugation of $A$. It means that in every order in $A$ the expression inside the
correlation function can be written as a product of $A$'s and their derivatives with all their group indices contracted with an invariant tensor:

$$\langle (\partial) A^{a_1}(x_1)(\partial) A^{a_2}(x_2) \ldots (\partial) A^{a_n}(x_n) d_{a_1a_2\ldots a_n} \rangle$$

Here $d_{a_1a_2\ldots a_n}$ is an invariant $n$-tensor. Then the group structure of any Feynman diagram contributing to such a correlation function, even before the momentum integration, can be represented as follows. It is a collection of invariant tensors of $psl(n\mid n)$, one coming from the above expression and others from $S_{int}$, and all their indices are contracted with the help of the inverse metric on $psl(n\mid n)$. Most importantly, any subdiagram is then also an invariant tensor. The rank of this tensor is given by the number of legs connecting it to the rest of the diagram.

This allows us to do the same trick as before. Consider any interaction vertex inside a given Feynman graph. Take any three-vertex in it and pull it out. (see Fig. 4) The rest of the diagram is an invariant tensor with three indices. There is only one such tensor for $psl(n\mid n)$ - structure constants. The pairwise contraction of all indices of two three-tensors is proportional to the dual Coxeter number $f_{abc}f_{abc} = C_V \delta^a_a = 0$. Thus all the diagrams without external lines and with at least one interaction vertex vanish. The remaining diagrams represent the calculation made in the free theory, $S_{int} = 0$. Moreover, the $A$’s are effectively commuting variables in such calculations since commutators introduce more of the structure constants in the graph and their contributions vanish.

![Fig. 4](image-url)

It may also happen that the invariant tensor $d_{a_1a_2\ldots a_n}$ entering the correlation function would itself contain $f_{abc}$. Such correlation functions all vanish, by the same reasoning.

Let us illustrate the preceding discussion with some examples. Consider $G_L$ and $G_R$ invariant bilinears built from the (right) currents: $Str(J_z J_z)$ and $Str(J_z J_{\bar{z}})$. We can compute expectation values of strings of such operators. The currents themselves are complicated expressions in terms of $A$, given in (3.3). Fortunately, all the terms but one contain commutators. Therefore the invariant tensors appearing in the correlation functions with those terms all contain structure constants. Thus the only contributing
terms are coming from $\partial A$ and $\bar{\partial} A$. The calculation reduces to computing expectation values of strings of $\text{Str}(\partial A \partial A)$ and $\text{Str}(\partial A \bar{\partial} A)$ in the free theory with commuting $A$’s. For example:

$$\langle \text{Str}(J_z(z, \bar{z}) J_z(z, \bar{z})) \text{Str}(J_z(0,0) J_z(0,0)) \rangle = -\frac{4\lambda^4}{z^4}$$

(3.6)

Analogously we find

$$\langle \text{Str}(J_z(z, \bar{z}) J_{\bar{z}}(z, \bar{z})) \text{Str}(J_z(0,0) J_{\bar{z}}(0,0)) \rangle = 0$$

$$\langle \text{Str}(J_z(z, \bar{z}) J_{\bar{z}}(z, \bar{z})) \text{Str}(J_z(0,0) J_{\bar{z}}(0,0)) \rangle = \frac{2\lambda^4}{|z|^4}$$

Given these correlation functions we can present another proof that our theory is conformal at the quantum level. The classical theory is conformal and therefore the $z - \bar{z}$ component of the stress-energy tensor is equal to zero. Quantum mechanically, there could be an anomaly. However the symmetries of the problem restrict its possible form [21]. It can only be that $T_{zz} = \alpha \text{Str}(J_zJ_z)$, where $\alpha$ is a constant, possibly dependent on the scale. Now we want to show that $\alpha = 0$. Following Zamolodchikov [22] we introduce functions $G(z\bar{z})$ and $H(z\bar{z})$ (see also [23])

$$G(z\bar{z}) = 4z^3\bar{z}\langle T_{zz}(z, \bar{z})T_{zz}(0) \rangle$$

$$H(z\bar{z}) = 16z^2\bar{z}^2\langle T_{\bar{z}\bar{z}}(z, \bar{z})T_{\bar{z}\bar{z}}(0) \rangle$$

(3.7)

The conservation of stress-energy tensor and rotational invariance imply that

$$4\dot{G} - 4G + \dot{H} - 2H = 0 ,$$

where dot denotes the derivative with respect to $r^2 = z\bar{z}$. Now, taking into account that $T_{zz} = -\frac{1}{2\lambda^2} \text{Str}(J_zJ_z)$ we conclude that $\dot{H} - 2H = -32\alpha^2\lambda^4 = 0$. This means that $T_{zz}$ is actually zero quantum mechanically in perturbation theory. Notice, that we never used any information about whether the theory is unitary or not.

As a result we find that our theory is conformal. It also follows from (3.6) that the central charge is equal to $(-2)$ for both left and right Virasoro algebras, indeed

$$\langle T(z)T(w) \rangle = \frac{-1}{(z-w)^4}.$$ 

(3.8)

Finally, we can extend our results on correlation functions that can be computed exactly. Suppose that we want to compute a correlation function which is invariant only
under one of the $G_L$ or $G_R$ and is an invariant tensor under the action of the other. It turns out that if this tensor is of the rank two, we can still do computations as in the free theory. There are no invariant tensors of rank one. The only rank-two tensor is the metric, so the correlation function is proportional to it. To find the coefficient of proportionality we can contract those two indices with the inverse metric, which gives $c$ times $g_{ab}g^{ab} = -2$. The correlation function thus becomes invariant and can be computed in the free theory just as before.

3.3. Some examples of perturbative calculations

The $psl(n|n)$ is a conformal theory and therefore all fields can be decomposed into represenations of the Virasoro algebra. The current $J_\mu$ is an interesting example. It satisfies the equation of motion $\partial_\mu J^\mu = 0$ and is not holomorphic. On the other hand one can suggest that the current component $J_z$ (or $J_{\bar{z}}$) is a $(1,0)$ (or $(0,1)$) conformal primary field. Notice that the theory in question is not unitary and therefore the zero norm states (such as $\bar{\partial}J_z$) do not decouple. The currents are invariant with respect to either left or right multiplication symmetry and therefore one can analyze multipoint correlation functions of left (or right) currents using perturbation theory. For example, the two point correlation function of currents is known exactly

$$\langle J_a(x)J_b(y) \rangle = \frac{\lambda^2 g_{ab}}{(x-y)^2}$$

(3.9)

As was explained before, all loop diagrams (in other words, diagrams with interaction vertices) that may contribute to this calculation have two external lines and therefore vanish identically. The 3-point correlation function is almost uniquely fixed by conformal invariance

$$\langle J_a(x)J_b(y)J_c(z) \rangle = \frac{3}{2} \frac{\lambda^4 f_{abc}}{(x-y)(x-z)(y-z)},$$

(3.10)

modulo the scale coefficient that we claim is exactly equal to $3\lambda^4/2$. Let us present the arguments that this is an exact answer. Consider the Feynman diagram that contributes to (3.10). It contains three-vertices coming both from the expansion of the currents (3.3) and from the interaction vertices. Diagrams containing only one three-vertex come from the second term in the current and give (3.10). We claim that any Feynman diagram containing two or more three-vertices vanishes. To show this let us pull out the three-vertex where the external line enters. The rest of the diagram is a blob with four external lines (see Fig. 5). The group structure of this blob is a rank 4 invariant tensor. Let us contract
any of its two indices. The resulting graph contains a three-vertex inside and vanishes by our previous arguments. Therefore the blob is a traceless rank 4 tensor. There are only five independent rank 4 invariant tensors: $g_{ab}g_{cd}$ and two other tensors with indices being permuted (three in total) and $f_{ab}^r f_{cdr}$ and $f_{ac}^r f_{bdr}$. Only three combinations of those are traceless: $f_{ab}^r f_{cdr}$, $f_{ac}^r f_{bdr}$ and $g_{ab}g_{cd} + g_{ac}g_{bd} + g_{ad}g_{bc}$. Luckily, for all of the traceless tensors the contraction of any two indices with the structure constant $f_{abc}$ produces zero. This explains why (3.10) is exact.

![Fig. 5](image)

The higher point correlation functions are nontrivial and have complicated $\lambda$ dependence. For example on the general grounds one can conclude that

$$
\langle J_{a_1}(z_1)J_{a_2}(z_2)J_{a_3}(z_3)J_{a_4}(z_4) \rangle = \lambda^4 \prod_{i<j}(z_i - z_j)^{-2/3} \sum_k t_{a_1...a_4}^{(k)} f_k(x) g_k(\bar{x}),
$$

where $t^{(k)}$ are possible tensor structures and $x$ is an anharmonic ratio. This correlation function is clearly not holomorphic. Notice, that $\lambda^4$ is just a normalization factor. The functions $f_k(x), g_k(\bar{x})$ also have a non-trivial dependence on $\lambda$.

3.4. Comments on the WZW term

In the previous sections we proved a remarkable statement – the principal chiral field for $psl(n|n)$ is a conformal theory. There is another well known way to associate a conformal field theory with any group (allowing a non-degenerate invariant second rank tensor) – WZW theory [24] [25]. The action for WZW theory is just a combination of principal chiral action and a WZW term. The theory is conformal (and possesses the affine symmetry algebra) when the coefficient in front of the principal chiral action $\frac{1}{\lambda^2}$ coincides with the coefficient $k$ in front of the WZW term. If these coefficients are different the theory flows, or in other words it is not conformal.

For $psl(n|n)$ group the story is different. The theory is conformal for any values of $k$ and $\lambda$

$$
S_{wzw}[G] = \frac{1}{4\pi\lambda^2} \int \text{Str}(|G^{-1}dG|^2) + k\Gamma_{wzw}[G].
$$

(3.11)
The proof of this statement is almost identical to the one we just gave for $\text{psl}(n|n)$ principal chiral field. It is enough to say that all vertices introduced by WZW term have the group structure which can be described by the similar tree-like diagrams (see Fig. 1).

As a result we have a two parameter family of conformal field theories, parameterized by two charges $k$ and $\lambda$. For compact groups (or for groups having non-trivial $H^3$) the coefficient $k$ is quantized, then it is natural to think about parameter $\lambda$ as an exactly marginal perturbation. At point $k = 1/\lambda^2$ the global left/right multiplication symmetry is enhanced and the theory is invariant under the left/right current algebra. As we see, the theory still possesses a huge chiral algebra for any value of $\lambda$, which is an extension of Virasoro algebra.

Let us see what is the physical meaning of the coefficient $k$ in front of the WZW term and the radius of sigma model $R = \frac{1}{\lambda}$ in the $\text{AdS}_3 \times S^3$ applications. The simple way to realize $\text{AdS}_3 \times S^3$ geometry is to consider configuration of $Q$ NS 5-branes and $p$ fundamental strings [26]. This configuration can be described by WZW model (see for example [3]) with $\frac{1}{\lambda^2} = k_{\text{WZW}} = Q$. Now, performing the S-duality transformation, one can map this configuration on $(Qa, Qc)$ 5-branes and $(pa, pc)$ strings (the second integer indicates the number of D-objects). Then the radius of $\text{AdS}_3$, measured in Einstein frame, remains invariant under the S-duality transformation. As a result we end up with $1/\lambda^2 = R^2 = \sqrt{(Qa)^2 + (Qc)^2 g^2}$, where $g$ is the string coupling constant and $Qc$ counts the number of $D$ 5-branes. The coefficient in front of the WZW term counts the NS 5-brane charge and is equal to $Qa$. It was noticed in [12] that for $1/\lambda^2 < Qa$ the theory becomes ill defined.

4. Comments on Localization

In this section we give some non-perturbative arguments about $\text{PSL}(n|n)$ principal chiral model. It will add strength to the perturbative calculations of the previous section.

Recall that the symmetry groups of our model, $G_L$ and $G_R$, are actually supergroups and as such possess a number of fermionic generators. Those generators act without fixed points and because of that we have to break the $G_L \times G_R$ symmetry and define the model as a limit of the theory with a unique vacuum, $G_0$. The symmetry that remains in that latter theory is the diagonal subgroup of $G_L \times G_R$. It acts by conjugation (we chose $G_0 = 1$ for simplicity)

$$G(x) \to UG(x)U^{-1}, \quad U \in \text{PSL}(n|n)$$  \hspace{1cm} (4.1)
Now, $PSL(n|n)$ has $2n^2$ fermionic generators which can be considered as BRST charges. We split them into two subgroups mentioned earlier, $F_+$ and $F_-$

$$F_+ = \begin{pmatrix} 1 & \chi^+ \\ 0 & 1 \end{pmatrix} \quad F_- = \begin{pmatrix} 1 & 0 \\ \chi^- & 1 \end{pmatrix}$$

where $\chi^\pm$ are fermionic $n \times n$ matrices. Both $F_+$ and $F_-$ are abelian subgroups of $PSL(n|n)$.

The $G_L \times G_R$ invariant correlation functions are at the same time BRST invariant and one can localize the path integral to the arbitrarily small neighborhood of a set fixed by the BRST action [27]. In our case the generators of the $F_\pm$ can all be considered as BRST charges and the only point fixed by them is the vacuum state $G(x) = 1$. Thus to evaluate the path integral in the broken symmetry “phase” with $G_L \times G_R$ invariant observables it is sufficient to restrict ourselves to the infinitesimal neighborhood of the vacuum, i.e. the Gaussian approximation is exact. This confirms what we found earlier by analyzing perturbation theory. Namely, in computing $G_L$ and $G_R$ invariant correlation functions we can set $S_{int}$ to zero. In the case of correlation functions invariant only under a subgroup of $G_L \times G_R$ it is possible to localize the path integral on a bigger set.

The above arguments can also be made in the case of $GL(n|m)$ principal chiral models. However, it doesn’t make those theories conformal. An important ingredient for doing exact computations in perturbation theory is missing. In the $PSL(n|n)$ case any invariant diagram containing the three-vertex vanishes, even if the vertex comes not from the action but from the expansion of the operators. In the case of $GL(n|m)$ this is no longer true and although the calculations can be made in the free theory, the infinite number of terms arising, for instance, from the expansion of $J$ would all contribute.

5. Chiral algebra

As we just demonstrated, the $PSL(n|n)$ principal chiral model is a conformal field theory. The central charge is equal to $c = -2$ and is independent on $\lambda$. As we will see, the parameter $\lambda$ is very similar to the radius parameter in the free theory on a circle.

It turns out that the chiral algebra of the theory is not just Virasoro algebra, but much bigger. Classically, any chiral model contains an infinite set of holomorphic currents. To be more precise, for every invariant symmetric tensor $t_{a_1...a_n}$ one can construct a holomorphic current

$$W[t] = \frac{1}{n!} t_{a_1...a_n} J^{a_1} J^{a_2} ... J^{a_n}$$  (5.1)
It is irrelevant whether we use the left or the right current in the definition of $W_t$. The equation of motion

$$\partial_\mu J^\mu = \partial \bar{z} J_z + \partial z J_{\bar{z}} = 0 ,$$  

(5.2)

being combined with the flatness condition

$$\partial \bar{z} J_z - \partial z J_{\bar{z}} + [J_{\bar{z}}, J_z] = 0$$

(5.3)
yields the relation

$$\partial \bar{z} J_z + \frac{1}{2} [J_{\bar{z}}, J_z] = 0$$

(5.4)

It is easy to see that the commutator in the last equation does not contribute to $\partial \bar{z} W_t$ and as a result, $W_t$ turns out to be holomorphic at the classical level (see for example [21]). For an arbitrary group $G$ the holomorphicity of the currents $W_t$ is destroyed by quantum corrections. However, the case of $PSL(n|n)$ group is very remarkable and we believe that the currents $W_n$ remain holomorphic.

The construction of totally symmetric invariant tensors is given in the Appendix. Here we just show one example. Consider $SL(n|n)$ invariant tensor corresponding to normalized trace $d_n(X_1, X_2, ... X_n) = (1/n!) Str(X_1 X_2 ... X_n)$. Under $U(1)$ transformation $G \to e^{i\phi} G$ it transforms as follows: $\delta d_n(...) = (1/n) \partial \phi \sum_i d_{n-1}(...)$. Given this, it is easy to see that for every $n$ one can construct an invariant tensor of degree $2n$

$$t_{2n} \sim \frac{1}{2} d_n^2 + \sum (-1)^i d_{n-i}^i$$

(5.5)

Each $psl(n|n)$ invariant tensor $t_n$ gives rise to a holomorphic current $W_t$ that remains holomorphic even at the quantum level. One can check this by analyzing the perturbation theory (we checked the first few non-trivial orders in perturbation theory). The field $W_t$ is a conformal primary field of dimension $\Delta = \text{rank}(t)$ if tensor $t$ is traceless. This can be always achieved by subtracting the traces.

The fields $\{W_t\}$ generate the chiral algebra of the theory. The first field is the stress energy tensor $T = W_2$, the next ones are $W_{t_6}$ and $W_{t_8} = T^4/480$. For $psl(2|2)$, all other fields $W_t$ are in a sense “dependent”. This dependence is non-polynomial. For example

$$W_{[t_6]} T^2 - \frac{2}{5} W_{[t_6]} W_{[t_6]} - \frac{1}{480} T^6 + ... = 0 ,$$
where the dots represent possible quantum corrections. This chiral algebra is an example of a $W$-algebra. The OPE of $W_{[t_6]}$ with itself has the following structure

$$W_{[t_6]}(z)W_{[t_6]}(w) = \frac{C_{66}^8}{(z-w)^4}[W_{[t_8]}] + \frac{C_{66}}{(z-w)^8}[U].$$

The brackets $[X]$ denote the conformal block of operator $X$. The field $U$ is given by a combination of square of stress energy tensor and its second derivative. In general, for $psl(n|n)$ theory there are $(n-2)$ independent operators. The algebra generated by these currents is quite complicated and it would be nice to have some explicit description of this algebra. For example, it is plausible that this chiral algebra can be obtained by Hamiltonian reduction from $psl(n|n)$ affine algebra. This representation, being constructed, should be very helpful for representation theory.

6. Physical operators and Kac representations

6.1. Sigma model operators

Let us describe operators that appear in the quantization of the $PSL(n|n)$ sigma model. All fields can be classified by representations of the symmetry group $G_L \times G_R$, or in other words, by a pair of weights $(\Lambda_L, \Lambda_R)$. Let us denote by $V_{(\Lambda_L, \Lambda_R)}$ a primary field of the chiral algebra. Observe that all the $W_{[t]}$ can be constructed either from the left current $J_L$, or from the right current $J_R$. Therefore, in order to avoid the contradiction, we have to assume that the weights $(\Lambda_L, \Lambda_R)$ are not arbitrary but such that all the Casimir operators have the same eigenvalues for both $\Lambda_L$ and $\Lambda_R$ representations.

In this paper we discuss only highest/lowest weight representations. For generic weight $\Lambda$ the Kac representation (see definition in the Appendix) is irreducible, but for certain weights $\Lambda$ it turns out to be reducible (atypical). The resulting irreducible representation turns out to be smaller and we call it a short representation. These representations will play an important role in the AdS/CFT duality. They correspond to chiral multiplets in the boundary theory.

One can easily construct operators corresponding to the weights $(\Lambda, \Lambda' = \Lambda)$ (both left and right representations are the same) of $PSL(n|n)$. For our group every Lie algebra representation

\[ 3 \] Although not all the representations are of the highest weight type, we would not have the need for those in our discussion.
representation gives rise to the group representation. Let $T^a$ be the generators of the Lie algebra in the representation $\Lambda$ and let $A^a$ be the corresponding coordinates on the Lie algebra. Then we propose $V_{(\Lambda,\Lambda)} = e^{\lambda A^a T^a}$ as a candidate for the conformal primary field. Classically, this field has zero dimension, but it acquires an anomalous dimension. In the leading order it is equal to $\Delta = \bar{\Delta} = \lambda^2 C_\Lambda/2$, where $C_\Lambda$ is the eigenvalue of the quadratic Casimir. We conjecture that this expression for the dimension is exact to all orders in $\lambda$.

Let us discuss the correlation functions of these operators. We first consider two-point correlation function of a group invariant combination. In order to construct a group invariant combination of two representations $\Lambda_1, \Lambda_2$, one needs to have a way of pairing two representations ("contracting indices") \footnote{In other words, one has to choose elements in $Hom(\Lambda_1, \Lambda_2^*)$ and $Hom(\Lambda_1^*, \Lambda_2)$.}. For two irreducible representations this pairing exists only if $\Lambda_1 = \Lambda_2^*$ and in this case $(V_{(\Lambda_1^*, \Lambda_2^*)})^k_j = (V_{(\Lambda_1, \Lambda)}^{-1})^k_j$. Then the pairing is just the supertrace of the product of two matrices. As we know, an invariant two point correlation function can be easily computed to all orders in perturbation theory

$$\langle Str(V_{(\Lambda,\Lambda)}(z)V_{(\Lambda^*,\Lambda^*)}(w)) \rangle = \frac{sdim(\Lambda)}{|z-w|^{2\lambda^2 C_\Lambda}}$$

(6.1)

Unfortunately, this result is not too exciting. The superdimension is equal to zero unless $C_\Lambda = 0$. Therefore, this correlation function differs from zero only for short representations. One may try to consider correlation functions of non-invariant operators, for example $V_{(\Lambda,\Lambda)}(z) \cdot V_{(\Lambda^*,\Lambda^*)}(w)$ (no supertrace). This correlation function is invariant under the right multiplication and so can be computed around any $G_0$ vacuum. On general grounds one expects the answer to be of the form

$$\langle (V_{(\Lambda,\Lambda)})^i_k(z)(V_{(\Lambda^*,\Lambda^*)})^l_j(w) \rangle = \frac{N \delta^i_j}{|z-w|^{2\lambda^2 C_\Lambda}} \cdot$$

(6.2)

To compute the normalization factor $N$ one may project on invariant subsector by taking a trace. As a result one gets $N sdim(\Lambda) = sdim(\Lambda)$. Therefore it implies that $N = 1$, if $sdim(\Lambda) \neq 0$. In the case when $sdim(\Lambda) = 0$ we can not justify that $N = 1$ using this kind of argument. Still, perturbation theory predicts that $N = 1$.

The correlation function of matrix elements $\langle (V_{(\Lambda,\Lambda)})^i_k(z)(V_{(\Lambda^*,\Lambda^*)})^l_j(w) \rangle$ can not be computed in perturbation theory. However, from the transformation properties of the correlation function we expect that

$$\langle (V_{(\Lambda,\Lambda)})^i_k(z)(V_{(\Lambda^*,\Lambda^*)})^l_j(w) \rangle = \delta^i_j \delta^l_k (-1)^{k^2} f(z, w)$$

(6.3)
where $\tilde{k}$ is the parity of the vector $k$ in the representation $\Lambda$. To compute $f(z, w)$, one can just evaluate the trace over indices $k, l$ producing a simple relation $\text{sdim}(\Lambda) f(z, w) = 1/|z - w|^{\lambda^2 C_\Lambda}$. For short representations $\text{sdim}(\Lambda) \neq 0$ and at the same time $C_\Lambda = 0$ (see section 2.2) and we obtain $f(z, w) = 1/\text{sdim}(\Lambda)$. For generic representations these arguments fail and we do not know how to compute correlation functions of matrix elements.

In fact, we can also discuss the multipoint correlation functions of group invariant combination of operators. These correlation functions can be computed to all orders in perturbation theory. Again, to define the three point correlation functions, one needs Clebsch-Gordan coefficients in order to contract indices. Saying it differently, one has to fix two invariant tensors $\phi$ and $\chi$. When the choice of $\phi$ and $\chi$ is unique, they are necessarily proportional to Clebsch-Gordan coefficients (as in the case of the classical groups). This choice fixes the gauge invariant combination. Now, going through the similar computations one finds that

$$\langle \phi \otimes \chi (V_{(\Lambda_1, \Lambda_2)}(z)V_{(\Lambda_2, \Lambda_3)}(w)V_{(\Lambda_3, \Lambda_3)}(x)) \rangle = \frac{\langle \phi, \eta \rangle}{|z - w|^{\gamma_{12}}|z - x|^{\gamma_{13}}|w - x|^{\gamma_{23}}},$$

where $\gamma_{12} = 2\lambda^2(C_1 + C_2 - C_3), \gamma_{13} = 2\lambda^2(C_1 + C_3 - C_2), \gamma_{23} = 2\lambda^2(C_2 + C_3 - C_1), C_j$ is the eigenvalue of the Casimir operator on representation $\Lambda_j$ and $\langle \phi, \eta \rangle = \sum_{ijk} \phi_{ijk} \chi_{ijk}$. Unfortunately, $\langle \phi, \eta \rangle$ vanishes unless all $C_i = 0$. The group invariant three point correlation function is not zero only for short representations and in this case it computes pairing $\langle \phi, \eta \rangle$. This pairing is the analog of the square of Clebsch-Gordan coefficients $\sum_{ijk} |C_{\mu \nu \lambda}^{ijk}|^2$ for classical groups.

Similar to the two point correlation functions, one can try to define a three point correlation function for the individual matrix elements. We can compute these correlation functions (i) only for short representations and (ii) under the assumption that invariant tensors $\phi$ and $\chi$ are unique (modulo scaling), and in this case they generalize the notion of Clebsh-Gordan coefficients. Observe that two and three point correlation functions turn out to be coupling independent for short representations.

\[5\] In other words, $\phi \in \text{Hom}(\Lambda_1 \times \Lambda_2, \Lambda_3^*)$ and $\chi \in \text{Hom}(\Lambda_1^* \times \Lambda_2^*, \Lambda_3)$ are non-zero if representation $\Lambda_3$ appears in decomposition $\Lambda_1 \times \Lambda_2$. 

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6.2. Strings on $AdS_3 \times S^3$ and $PSU(1,1|2)$

In this section we want to make contact with some physical models, in particular with strings propagating on $AdS$-type backgrounds. Our arguments that $PSL(n|n)$ principal chiral model is conformal were purely algebraic and independent of the choice of the real structure of the group. Now, to relate our discussion to strings on $AdS_3 \times S^3$ we have to consider $PSU(1,1|2)$ principal chiral field. Similarly, strings propagating on $AdS_5 \times S^5$ are related to the coset \{1|1\} of $PSU(2,2|4)$ principal chiral model.

Let us concentrate on the $PSU(1,1|2)$ case. Our sigma model is non-unitary and therefore there is no state operator correspondence. Still both states and operators can be classified by the representations of the symmetry group $PSU(1,1|2)_L \times PSU(1,1|2)_R$. Each individual weight $\Lambda_L$, $\Lambda_R$ is nothing else but a pair of $su(2)$ and $su(1,1)$ weights $\Lambda_L = (\lambda, \mu)_L$ and $\Lambda_R = (\lambda', \mu')_R$ (the weight of $su(2)$ is twice the spin and is always positive integer $\lambda \in \mathbb{Z}_+$). To conform with the physics literature, $\lambda$ denotes the highest weight of the $su(2)$ while $\mu$ is the lowest weight, in the sense that it has the lowest eigenvalue of $L_0$.

We have to consider both finite and infinite dimensional representations of the symmetry group. See for example the discussion of the spectrum of the string theory on $AdS_3 \times S^3$ in [28]. Not all of the infinite-dimensional representations are highest weight representations, but we will not have the need for those. The unitary infinite-dimensional representations correspond to normalizable modes in $PSU(1,1|2)$ and non-unitary ones to non-normalizable. Representation $((\lambda, \mu), (\lambda', \mu'))$ is unitary if both $\mu$’s are positive. Also note that for a given representation the sum of the left and right quadratic Casimirs is the eigenvalue of the Laplacian on the corresponding modes in $PSU(1,1|2)$.

The generic representation $(\lambda, \mu)$ of $psu(1,1|2)$ is irreducible. But when $\mu = -\lambda$ or $\mu = (\lambda + 2)$ the representation turns out to be reducible\footnote{The Kac representation is constructed as a product $V_{\lambda}^{su(2)} \times V_{\mu}^{su(1,1)} \times C(\theta_1, \theta_2, \theta_3, \theta_4)$, where $V_{\mu}^{su(1,1)}$ is a finite or infinite dimensional l.w.s module corresponding to weight $\mu$.}. In this case the irreducible representations are smaller (one can find the discussion of reducible finite dimensional representations for $psl(2,2)$ in the appendix). In the physical language these are the short representations, similar to those that appear in the description of BPS states. For the future let us denote the unitary short representations with the following weights as $[\lambda, \lambda'] \equiv ((\lambda, \lambda + 2)_L, (\lambda', \lambda' + 2)_R)$.
The sigma model operators $V_{(Λ, Λ')}$ would enter into the construction of string vertex operators. For both $Λ$ and $Λ'$ short representations their quadratic Casimirs vanish. We can write it as

$$\left(\Delta - \lambda(\lambda + 2) - \lambda'(\lambda' + 2)\right)V_{(Λ, Λ')} = 0,$$

(6.5)

where $Δ$ is the Laplacian operator on $AdS_3$. This is nothing else but the supergravity mass shell condition. In other words, operators $V_{(Λ, Λ')}$ correspond to the massless excitations around the supergravity background. As it was explained in [29], the only representations arising in supergravity have left and right weights corresponding to the short representations and their $\{su(2)_L, su(2)_R\}$ weights are $\{λ, λ\}$, $\{λ ± 1, λ\}$ or $\{λ ± 2, λ\}$, where the smaller weight is positive. This restriction comes from the fact that the supergravity sector contains fields of spin two at most. Larger differences would correspond to the higher spin fields. In particular, the short representations with equal $su(2)$ weights correspond to the modes of massless scalars in supergravity. The weights $((λ, λ + 2), (λ, λ + 2))$ labels normalizable modes, while $((λ, −λ), (λ, −λ))$ labels non-normalizable modes which lie in the highest/lowest weight representations. Clearly, the spectrum of the stringy modes is much more complicated.

The conformal field theory on the boundary of $AdS_3$ possesses left and right $\mathcal{N} = 4$ superconformal algebras. Those have left and right $PSU(1, 1|2)$’s as finite subalgebras. Indeed, one can easily identify them: $J^3, J^\pm$ and $L_0, L_{±1}$ generate $su(2) \times su(1, 1)$ subalgebra and $G_{±1/2}^a$ and $\bar{G}_{±1/2}^a$ are the fermionic generators. The chiral multiplets of the boundary CFT then correspond to the short representations of $PSU(1, 1|2)_L \times PSU(1, 1|2)_R$. The two and three-point correlation functions of chiral operators in the boundary theory were computed in [30] [31] and they are given by the coupling-independent (in the appropriate normalization) expressions proportional to Clebsch-Gordan coefficients. As we found, the two and three-point worldsheet correlation functions of the vertex operators corresponding to short representations $(Λ, Λ)$ are also independent of the coupling and uniquely determined by the group structure.

To compute correlation functions of boundary CFT in $AdS$ supergravity, one has to find the solution of SUGRA equations of motion which approaches given boundary values at infinity [32]. Consider a massless scalar field in supergravity. For its Kaluza-Klein harmonic lying in the representation with $su(2)$ weight $λ$, the apparent mass in $AdS_3$ is $m^2 = λ(λ + 2)$. For such a scalar the solution approaching a $δ(x)$ on the boundary was found in [32]. Remarkably, this solution is exactly the lowest weight state in the
representation of the \( \text{AdS}_3 \) isometry group \( SU(1,1)_L \times SU(1,1)_R \) with both weights equal to \( \mu = 1 + \sqrt{1 + m^2} = \lambda + 2 \). Thus it is actually a state of the lowest \( SU(1,1) \) weights in the short representation of the \( PSU(1,1|2)_L \times PSU(1,1|2)_R \) which we earlier denoted \([\lambda, \lambda]\). Therefore we can identify the vertex operator representing the same state in \( V_{[\lambda,\lambda]} \) as the one which corresponds to the insertion of the boundary operator at the origin. Let us call this operator \( V_{[\lambda,\lambda]}(z, \bar{z}|0) \) where \( z \) is the world-sheet coordinate and \( 0 \) is, in a sense, a boundary coordinate. We suppress the indices labelling the states inside the \( V_{[\lambda,\lambda]} \) module.

To shift the above operator from the origin, we can, of course, act on it with \( su(1,1) \) lowering operators \( L_{-1} \) and \( \bar{L}_{-1} \) and thus define

\[
V_{[\lambda,\lambda]}(z, \bar{z}|x, \bar{x}) = \sum_{x} \frac{x^n \bar{x}^m}{n! m!} [L_{-1}, [\bar{L}_{-1}, V_{[\lambda,\lambda]}(z, \bar{z})]] ,
\]

where \((z, \bar{z})\) are worldsheet coordinates and \((x, \bar{x})\) are now boundary coordinates. Notice, that one can equally well generalize these formulas in the case of different left and right short representations.

Clearly the general string amplitudes are given by complicated expressions that involve ghost fields and worldsheet integrations over positions of vertex operators, but two- and three- scattering amplitudes are given by simple minded expressions – no ghost fields and, thanks to worldsheet \( sl(2,\mathbb{C}) \), no integrations. Therefore we can completely neglect the worldsheet dependence of the correlation functions assuming that our operators are inserted say at 0 and 1 for propagator and at 0, 1 and \( \infty \) for the three point scattering.

Now we can write two- and three-point correlation functions for our vertex operators. We find that the dependence on the boundary coordinates produces exactly right scaling behavior

\[
\langle V_{[\lambda,\lambda]}(0|x, \bar{x})V_{[\mu,\mu]}(\infty|y, \bar{y}) \rangle = \frac{\delta_{\lambda\mu}}{|x - y|^{2(\lambda + 2)}},
\]

\[
\langle V_{[\lambda,\lambda]}(0|x, \bar{x})V_{[\mu,\mu]}(1|y, \bar{y})V_{[\nu,\nu]}(\infty|u, \bar{u}) \rangle = \frac{T_{\lambda\mu\nu}}{|x - y|^{\gamma_{xy}} |x - u|^{\gamma_{xu}} |y - u|^{\gamma_{yu}}},
\]

where \( \gamma_{xy} = (\lambda + \mu + 2 - \nu) \), \( \gamma_{xu} = (\lambda + \nu + 2 - \mu) \), \( \gamma_{yu} = (\nu + \mu + 2 - \lambda) \). The coefficients \( T_{\lambda\mu\nu} \) are coupling independent (!) and can be computed in terms of Clebsh-Gordan coefficients.

Strictly, we can not make computations in the case of infinite dimensional representation. Still, morally speaking this answer is very similar to that presented at the end of the previous section. Both answers are coupling constant independent and are expressed in terms of Clebsh-Gordan coefficients (compare (6.7) and [31]).

\[\text{7} \] Tensors \( \delta_{\lambda\mu} \) and \( T_{\lambda\mu\nu} \) have suppressed indices that label the states inside representations.
7. Discussion

We believe that by now we have convinced the reader that $PSL(n|n)$ sigma model is quite remarkable. It gives rise to a conformal field theory. This conformal field theory is non-trivial, but it contains a certain subsector corresponding to short representations that is easy to analyze. These are short (atypical) representations that correspond to chiral primary fields in the boundary theory. The long representations are difficult to analyze and they give rise to stringy modes (in the corresponding string theory). In a sense, the subsector of short representations is reminiscent to the ground ring of $c = 1$ model. The chiral algebra of this sigma model is not just a Virasoro algebra but its extension similar to $W$ algebras. We believe that the study of the representations of this chiral algebra will be an important ingredient in the solution of the theory.

It follows from our presentation that all group invariant correlation functions are coupling constant independent, including the higher genus calculations. For example, the one loop partition function is coupling independent and after eliminating the contribution of zero modes is equal to $Z' = \eta^2(q)$ (this is just a contribution of fermionic $(b,c)$ system). This partition function does not say much about the spectrum of the theory. It is clear that various twisted version of one loop partition function would contain more information about the spectrum. This question is currently under investigation.

Apparently there is another infinite series of supergroups that have a chance of being conformal – the $Osp(2n + 2|2n)$ groups. This group contains $SO(2n + 2) \times Sp(2n)$ as a bosonic subgroup. The dual Coxeter number vanishes for these groups and therefore the one loop beta function is zero. We believe that this group has unique rank 3 totally antisymmetric tensor, which would imply that the theory is exactly conformal. It will be very interesting to analyze this series.

One can try to go even further. The $PSL(n|n)$ principal chiral model is a generalization of the $G \times G$ sigma model to a case of a very special supergroup. It may be promising to generalize $O(N)$ sigma models in a similar way. We only note that the action for such model looks similar to the action found in [12].

As we already mentioned in the introduction, the $PSL(n|n)$ group manifold is in a sense a Calabi-Yau manifold. One of the directions for future work could be the construction of other supermanifolds that give rise to conformal field theories. For example, various quotients over special subgroups also yield the conformal field theories [14].

There is another observation which is somewhat beyond the scope of this paper but nevertheless is quite interesting. Consider a four dimensional gauge theory based on a
supergroup $PSL(n|n)$. In other words, there is no supersymmetry in four dimensions, but there are two kinds of vector particles – the usual bosonic as well as fermionic. It is a four dimensional non-unitary theory. It is possible that certain version of this theory might appear in the description of the system that contains both D-branes and anti D-branes. Now, according to the arguments presented in this paper, this theory is going to be exactly conformal! There is not much difference in group structure between two dimensional and four dimensional Feynman diagrams. Therefore, we can repeat all the steps of our proof in the case of this four dimensional theory. Still, it is unclear whether one can make sense out of this theory.

It is a challenge to understand the $PSL(n|n)$ sigma models and we hope to return to this subject in the future.

**Acknowledgments:** We would like to thank A. Bernstein, S. Coleman, P. Etingoff, D. Kazhdan, D. Kutasov, N. Nekrasov, H. Ooguri, V. Serganova, A. Strominger, S. Gubser, C. Vafa and J. Zinn-Justin for many valuable discusions. This research was supported by NSF grant PHY-92-18167. In addition, the research of M.B is supported in addition by NSF 1994 NYI award and the DOE 1994 OJI award.
8. Appendix. Representations of \( psl(n|n) \)

The simple Lie superalgebra \( psl(n|n) \) (or \( A(n-1|n-1) \) in Kac’s notation \([33]\)) stands out among other superalgebras of the \( A(m|n) \) series in many ways and is relatively less studied. Here we will discuss some of its features.

First of all, \( psl(n|n) \) it is not a subalgebra of the matrix superalgebra \( gl(n|n) \) unlike all other algebras of the \( A \) series. Indeed, the traceless subalgebra \( sl(n|n) \) of \( gl(n|n) \) is not simple since it has a non-trivial center \( Z = C \cdot I \) generated by the identity matrix \( I \in sl(n|n) \). The quotient superalgebra \( psl(n|n) = sl(n|n)/Z \) is simple for \( n > 1 \).

The projection
\[
p : sl(n|n) \rightarrow psl(n|n)
\] (8.1)
cannot be split (i.e. \( psl(n|n) \) cannot be embedded into \( sl(n|n) \) as a subalgebra) and therefore \( sl(n|n) \) is a non-trivial central extension of the algebra \( psl(n|n) \). This is the only series of basic classical Lie superalgebras admitting a non-trivial central extension (the only other examples being the series of “strange” superalgebras \( Q(n) \) and Hamiltonian superalgebras \( H(n) \)). This phenomenon places \( psl(n|n) \) in one class with the Virasoro or current Lie algebras and shows that not every irreducible representation of the matrix superalgebra \( sl(n|n) \) factors through the projection (8.1) and gives a representation of \( psl(n|n) \), but only those with vanishing “central charge” (the eigenvalue of \( I \)).

Another peculiarity of \( psl(n|n) \) is that it has rank \( 2n - 2 \) (the dimension of the Cartan subalgebra) and \( 2n - 1 \) simple roots (i.e. its simple roots are linearly dependent).

Before discussing representations of \( psl(n|n) \) let us fix some notation. Since \( psl(n|n) \) does not have a natural matrix representation we will be working with it in terms of \( sl(n|n) \) or \( gl(n|n) \) generators keeping in mind that two matrices from \( sl(n|n) \) whose difference is a scalar multiple of \( I \) represent the same element of \( psl(n|n) \). The Cartan subalgebra \( H_s \) of \( sl(n|n) \) has dimension \( 2n - 1 \) and is spanned by elements
\[
h_i = E_{ii} - E_{i+1,i+1}, \quad 1 \leq i \leq n - 1, \quad \text{and} \quad n + 1 \leq i \leq 2n - 1,
\]
\[
h_n = E_{nn} + E_{n+1,n+1},
\] (8.2)

where \( E_{ij} \) is the elementary matrix whose only non-zero entry is 1 on the intersection of the \( i \)-th row and the \( j \)-th column. Elements \( E_{ii} \quad 1 \leq i \leq 2n \) generate the Cartan subalgebra \( H_g \) of \( gl(n|n) \).
The remaining part of the root decomposition is the same for both \( sl(n|n) \) and \( gl(n|n) \) and is given by the following root vectors and the corresponding roots:

\[
E_{ij} , \quad \epsilon_i - \epsilon_j , \\
E_{n+i,n+j} , \quad \delta_i - \delta_j , \\
E_{i,m+j} , \quad \epsilon_i - \delta_j , \\
E_{m+i,j} , \quad \delta_i - \epsilon_j , \\
\]

(8.3)
i, j = 1, \ldots, n, where \( \epsilon_i, \delta_i \in H_g^* \) are functionals on \( H_g \) such that \( \epsilon_i(x) = x_i \) and \( \delta_i(x) = x_{n+i} \) for \( x = \text{diag}(x_1, \ldots, x_{2n}) \in H_g \). The roots \( \epsilon_i - \delta_j \) and \( \delta_i - \epsilon_j \) are odd (which means that the corresponding root vector is odd), the remaining ones are even. The so-called distinguished system of simple roots is chosen as follows:

\[
\alpha_i = \epsilon_i - \epsilon_{i+1}, \quad 1 \leq i \leq n - 1 , \\
\alpha_n = \epsilon_n - \delta_1 , \\
\alpha_{n+i} = \delta_i - \delta_{i+1}, \quad 1 \leq i \leq n - 1 .
\]

(8.4)

With this choice of simple roots, the \( 2n^2 - n \) positive roots of \( sl(n|n) \) are \( E_{ij} \) for \( i < j \), \( E_{n+i,n+j} \) for \( i < j \), and \( E_{i,m+j} \). In the distinguished system there is only one odd simple root \( \alpha_n \).

There are two different ways to represent \( sl(n|n) \) weights (i.e., elements of \( H_s^* \)) in coordinates both of which have some advantages. First, we can express \( \Lambda \in H_s^* \) in terms of the basis (8.4)

\[
\Lambda = [a_1, a_2, \ldots, a_{2n-1}] = \sum_{i=1}^{2n-1} a_i \alpha_i , \quad \text{where} \quad a_i = \Lambda(h_i) .
\]

(8.5)

On the other hand, since \( H_s^* \) is a quotient of \( H_g^* \) by the element

\[
\omega_0 = \sum_{i=1}^{n} \epsilon_i - \sum_{i=1}^{n} \delta_i ,
\]

we can represent \( \Lambda \) in terms of \( \epsilon_i \) and \( \delta_i \) as

\[
\Lambda = (\lambda_1, \ldots, \lambda_n; \mu_1, \ldots, \mu_n) = \sum_{i=1}^{n} \lambda_i \epsilon_i + \sum_{i=1}^{n} \mu_i \delta_i ,
\]

(8.6)
keeping in mind that the strings \( \Lambda = (\lambda_1, \ldots, \lambda_n; \mu_1, \ldots, \mu_n) \) and \( \Lambda + \mu = (\lambda_1 + 1, \ldots, \lambda_n + 1; \mu_1 - 1, \ldots, \mu_n - 1) \) represent the same element in \( H_s^* \). The relation between the two coordinate systems is given by

\[
(\lambda_1, \ldots, \lambda_n; \mu_1, \ldots, \mu_n) = [\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n, \lambda_n + \mu_1, \mu_1 - \mu_2, \ldots, \mu_{n-1} - \mu_n]. \quad (8.7)
\]

The dual space \( H^* \) of the Cartan subalgebra \( H \) of \( psln(n|n) \) is a codimension one subspace of \( H_s^* \) that consists of linear functionals on \( H_s \) vanishing on the vector \( I \in sl(n|n) \). The equation

\[
I = h_1 + 2h_2 + 3h_3 + \ldots nh_n - (n - 1)h_{n-1} - \ldots - 2h_{2n-2} - h_{2n-1},
\]

shows that \([a_1, a_2, \ldots, a_{2n-1}]\) belongs to \( H^* \) if and only if

\[
\alpha_1 + 2\alpha_2 + \ldots + n\alpha_n + (n - 1)\alpha_{n-1} + \ldots + 2\alpha_{2n-2} + \alpha_{2n-1} = 0 . \quad (8.8)
\]

and, therefore,

\[
a_n = -\frac{1}{n} \left( \sum_{i=1}^{n-1} ia_i - \sum_{i=1}^{n-1} (n - i)a_{n+i} \right).
\]

In the \((\lambda, \mu)\) form, the equation \((8.8)\) looks simpler

\[
\sum_i (\lambda_i + \mu_i) = 0.
\]

The equation \((8.8)\) implies in particular that the simple roots \((8.4)\) of \( psln(n|n) \) are not linearly independent.

Since \( psln(n|n) \) is a quotient of \( sl(n|n) \), every representation of \( psln(n|n) \) is automatically a representation of \( sl(n|n) \). The relationship between representations of \( sl(n|n) \) and \( gl(n|n) \) is slightly different because \( sl(n|n) \) is only a subalgebra and not a quotient of \( gl(n|n) \) (which is the case with \( gl(m|n) \) for \( m \neq n \)). Therefore, every representation of \( gl(n|n) \) is a representation of \( sl(n|n) \), but the converse is not true in general. However, since every finite-dimensional irreducible representation of \( sl(n|n) \) is a highest weight representation \( V_\Lambda \), it is determined by a weight \( \Lambda \in H_s^* \) (which can be considered as a one-dimensional representation of the Cartan subalgebra \( H_s \)). But every weight \( \Lambda \in H_s^* \) can be extended (not uniquely) to a weight \( \tilde{\Lambda} \in H_g^* \) and thus the action of \( sl(n|n) \) on \( V_\Lambda \) can be extended to a \( gl(n|n) \)-action. This allows us to work with irreducible representations of \( psln(n|n) \) in
terms of $sl(n|n)$ or $gl(n|n)$ representations. In particular, every irreducible representation
of $psl(n|n)$ lifts to an irreducible representation of $gl(n|n)$.

Let us recall Kac’s construction of representations of $sl(n|n)$ (which works with some
modifications also for other basic classical Lie superalgebras). Denote $L = sl(n|n)$ and let
$L = L_{-1} \oplus L_0 \oplus L_1$, where

$$L_0 = sl(n) \oplus \mathbb{C} \oplus sl(n)$$

(8.9)
is the even subalgebra of $L$ and $L_{\pm 1}$ the subalgebras corresponding to upper- and lower-
triangular odd matrices in $sl(n|n)$. Pick a representation of the subalgebra $L_0$ and extend
it to a representation of the subalgebra $K = L_0 \oplus L_1$ by setting $L_1 V = 0$. The Kac representation

$$\tilde{V} = \text{Ind}_K^L V = U(L) \otimes U(K) V$$

(8.10)
corresponding to $V$ (also called the Kac or induced module) as a vector space is isomorphic
to the tensor product

$$V \otimes \wedge L_{-1},$$

where $\wedge L_{-1}$ is the Grassman algebra of the vector space $L_{-1}$.

If we start with an irreducible representation $V = V_\Lambda$ of the even subalgebra (8.9)
corresponding to a a weight vector

$$\Lambda = [a_1, a_2, \ldots, a_{n-1}, c, b_1, b_2, \ldots, b_{n-1}] \in H^*_s,$$

(8.11)

where $[a_1, a_2, \ldots, a_{n-1}]$ and $[b_1, b_2, \ldots, b_{n-1}]$ are dominant $sl(n)$ weights (i.e. all $a_i$ and $b_j$
are non-negative integers), then the Kac module $V(\Lambda) = \tilde{V}_\Lambda$ has dimension

$$2^{n^2} \prod_{1 \leq i \leq j \leq n-1} \frac{((a_i + 1) + \ldots + (a_j + 1))((b_i + 1) + \ldots + (b_j + 1))}{(j - i + 1)^2}.$$ 

For a generic (typical) weight $\Lambda$ the representation $V(\Lambda)$ is irreducible, otherwise it has
a unique maximal invariant subspace $U$ and the quotient representation $M(\Lambda) = V(\Lambda)/U$ is
irreducible. All irreducible representations of $sl(n|n)$ can be obtained by this construction. If the Kac representation (8.10) is reducible, the weight $\Lambda$ is called atypical. This happens exactly when

$$a_i + a_{i+1} + \ldots + a_n - a_{n+1} - \ldots - a_{n+j-1} + i - j - n + 1 = 0$$

(8.12)
for some $1 \leq i, j \leq n$. This condition means that at least one of the inner products
\[
\langle \Lambda + \rho | \sigma_{ij} \rangle
\]
vanishes, where $\sigma_{ij} = \epsilon_i - \delta_j$ a positive odd root and
\[
\rho = [1, 1, \ldots, 1, 0, 1, \ldots, 1] = \frac{1}{2}(2n - 1, 2n - 3, \ldots, -2n + 1)
\]
is the half-sum of the positive roots.

In terms of the $gl(n|n)$ coordinates (8.6) the condition (8.12) can be rewritten as
\[
\lambda_i + \mu_j + n - i - j + 1 = 0.
\]
(8.14)

For example, the Kac representation $V(\Lambda)$ of $sl(2|2)$ with $\Lambda = [a, c, b], \ a, b \in Z_+$ is atypical if $c$ is equal to one of the following
\[
0, -a - 1, b + 1, b - a.
\]
(8.15)

Now when does a highest weight representation of $sl(n|n)$ descend to a representation of $psl(n|n)$ via projection (8.1)? This happens when the weight $\Lambda$ belongs to the subspace of $H^*$ given by the constraint (8.8). This gives the following equation for the component $c$ of the weight (8.11)
\[
c = -\frac{1}{n} \left( \sum_{k=1}^{n-1} ka_k - \sum_{k=1}^{n-1} kb_{n-k+1} \right).
\]
(8.16)

Therefore, irreducible representations of $psl(n|n)$ are determined by a pair of $sl(n)$ weights $\alpha = [a_1, \ldots, a_{n-1}]$ and $\beta = [b_1, \ldots, b_{n-1}]$.

This shows another feature of $psl(n|n)$ that distinguishes it from other algebras of the $A$ type (and from most of other simple Lie superalgebras) — its irreducible representations depend on $2n - 2$ integer parameters and do not have continuous parameters. (Irreducible representations of $sl(m|n)$ with $m \neq n$ can be deformed, because there are no restrictions on the coordinate $c$ of the highest weight.)

In the $psl(2|2)$ case for example, we have from (8.16) that $c = \frac{(b-a)}{2}$ and the atypicality condition (8.15) now becomes just $a = b$. The Kac module $V_\Lambda$ is reducible when $\Lambda = [a, 0, a]$. Let us denote the corresponding irreducible representation by $M_a$. If $a \neq 0, 1$ the structure of the module $V_\Lambda$ is very simple, namely
\[
M_a \subset K_a \subset V_\Lambda,
\]
(8.17)
where $V_{\Lambda}/K_a = M_a$ and $K_a/M_a = M_{a+1} \oplus M_{a-1}$.

For $\mathfrak{psl}(3|3)$ there are six different types of atypicality some of which can happen simultaneously:

\[
\begin{align*}
  a_1 + 2a_2 - 2a_4 - a_5 &= 0, \\
  -a_1 + a_2 - a_4 + a_5 &= 0, \\
  2a_1 + a_2 + 2a_4 + a_5 + 2 &= 0, \\
  2a_1 + a_2 - a_4 + a_5 + 1 &= 0, \\
  -a_1 + a_2 + 2a_4 + a_5 + 1 &= 0, \\
  -a_1 + a_2 - a_4 - 2a_5 - 1 &= 0.
\end{align*}
\] (8.18)

9. Appendix. Casimir operators of $\mathfrak{psl}(n|n)$

The algebra of casimirs — the center of the universal enveloping algebra — for any Lie superalgebra $L$ with an invariant inner product is isomorphic to the algebra of invariant polynomial functions on $L$. For classical matrix Lie superalgebras this algebra $I(L)$ of invariants is well studied (see, e.g. [36]). An analog of the classical Chevalley’s theorem describes $I(L)$ in terms of restrictions of the invariant polynomials to the Cartan subalgebra $H$ of $L$. For $L = \mathfrak{sl}(n|n)$ for example, $I(L)$ is generated by polynomials $t_n$ where $t_k(X) = \text{str}(X^k)$ for $X \in \mathfrak{sl}(n|n)$ $k = 2, 3, \ldots$. Only the first $2n-1$ functions $t_k$ are algebraically independent, the rest being rational functions of the first $2n-1$ ones. However, when $n \to \infty$, all $t_k$ become algebraically independent.

The case of $\mathfrak{psl}(n|n)$ is more interesting and difficult. Since every invariant polynomial on $\mathfrak{psl}(n|n)$ is also an invariant polynomial on $\mathfrak{sl}(n|n)$, the algebra of Casimirs for $\mathfrak{psl}(n|n)$ is a subalgebra $A$ of $B = \mathbb{C}[t_1, t_2, \ldots, t_k, \ldots]$ that consists of all polynomials in $t_1, t_2, \ldots$ that can be pushed to a well-defined function on $\mathfrak{psl}(n|n)$, i.e.

\[
A = \{ f \in B | f(X + kI) = f(X), \text{ for any } X \in \mathfrak{sl}(n|n) , k \in \mathbb{C} \}. \] (9.1)

This is a rather strong condition and a priori it is not even clear whether any non-constant function with this properties should exist.

It turns out, however, that the algebra $A$ of Casimirs for $\mathfrak{psl}(n|n)$ is quite large and has a rich and interesting structure.

First, it is easy to check that the quadratic polynomial $t_2$ belongs to $A$. It corresponds to the invariant inner product on $\mathfrak{psl}(n|n)$. 

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To construct new Casimirs of higher order let us consider the following formal power series in infinitely many variables $s_1, s_2, \ldots$:

$$f(X; s_1, s_2, \ldots) = \text{str}(e^{s_1 X}) \text{str}(e^{s_2 X}) \ldots$$  \hspace{1cm} (9.2)

If only finitely many, say the first $p$, of the variables $s_1, s_2, \ldots$ are non-zero and $\sum_{k=1}^{p} s_k = 0$, then, obviously,

$$f(X; s_1, s_2, \ldots) = f(X + kI; s_1, s_2, \ldots).$$

Therefore, the coefficient $c(a_1, a_2, \ldots, a_{p-1})$ of $f$ at $s_1^{a_1} s_2^{a_2} \ldots s_{p-1}^{a_{p-1}}$ after expanding $f$ in $s_1, s_2, \ldots, s_{p-1}$ belongs to the algebra of Casimirs $A$.

For example,

$$c(2p) = 2d_2 d_{2p-2} - 2d_3 d_{2p-3} + \ldots + (-1)^p d_p^2,$$  \hspace{1cm} (9.3)

where we set $d_k = t_k/k!$ so that $\text{str}(e^{s_i X}) = \sum_{k \geq 2} s_i^k d_k$, since $t_0 = \text{str} I = 0$ and $t_1 = \text{str} X = 0$. Analogously we have invariants with more terms in the products, for example

$$c(p, 2q) = \sum_{0 \leq i \leq p} \sum_{0 \leq j \leq 2q} (-1)^{p+2q-i-j} C_{p-i}^{p+2q-i-j} d_i d_j d_{p+2q-i-j}$$

and, in particular,

$$c(3, 6) = -3d_2^2 d_5 + 3d_2 d_3 d_4 - d_3^3.$$  

Here is the simplest example of an invariant involving products of four $t_i$'s that does not belong to the subalgebra generated by $c(2p)$ and $c(p, 2q)$:

$$c(3, 4, 8) = -252d_2^3 d_9 + 252d_2^2 d_3 d_8 - 98d_2^2 d_4 d_7 + 30d_2^2 d_5 d_6 - 77d_2 d_3^2 d_7$$
$$+ 68d_2 d_3 d_4 d_6 - 30d_2 d_3 d_5^2 - d_2^2 d_4^2 d_5 + 3d_3 d_4 d_5 - 3d_3^2 d_4 d_5 + d_3^3.$$  \hspace{1cm} (9.4)

It can be proved that the whole algebra of Casimirs is generated by $t_2$ and the invariants

$$c(a_1, a_2, \ldots, a_{m-1}, 2a_m) \text{ with } 3 \leq a_1 < a_2 < \ldots < a_{m-1} \leq a_m.$$  \hspace{1cm} (9.5)

The leading in the lexicographical order of this invariant is $d_{a_1} d_{a_2} \ldots d_{a_{m-1}} a_m^2$.

For $\text{psl}(2|2)$ we have $t_{2k+1} = 0$, and therefore, all invariant polynomials have even degrees. In this case, the invariants $t_2$ and $c(6)$ are algebraically independent, while all the other are rational functions of these two. For example,

$$t_2^2 c(8) = \frac{2}{5} c(6)^2 + \frac{1}{720} t_5^6,$$
$$t_2^4 c(10) = \frac{3}{35} c(6)^3 + \frac{1}{1008} c(6) t_5^6.$$  \hspace{1cm} (9.6)

For $n \geq 2$ only $2n - 2$ of the invariants (5.5) are algebraically independent, but for $n \to \infty$ they become algebraically independent, and $A$ a free algebra with infinitely many generators.
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