Set-theoretical solutions of the Yang-Baxter and pentagon equations on semigroups✩

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Abstract

The Yang-Baxter and pentagon equations are two well-known equations of Mathematical Physics. If \(S\) is a set, a map \(s : S \times S \rightarrow S \times S\) is said to be a set theoretical solution of the Yang-Baxter equation if

\[
s_{23} s_{13} s_{12} = s_{12} s_{13} s_{23},
\]

where \(s_{12} = s \times \text{id}_S\), \(s_{23} = \text{id}_S \times s\), and \(s_{13} = (\text{id}_S \times \tau) s_{12} (\text{id}_S \times \tau)\) and \(\tau\) is the flip map, i.e., the map on \(S \times S\) given by \(\tau(x, y) = (y, x)\). Instead, \(s\) is called a set-theoretical solution of the pentagon equation if

\[
s_{23} s_{13} s_{12} = s_{12} s_{23}.
\]

The main aim of this work is to display how solutions of the pentagon equation turn out to be a useful tool to obtain new solutions of the Yang-Baxter equation. Specifically, we present a new construction of solutions of the Yang-Baxter equation involving two specific solutions of the pentagon equation. To this end, we provide a method to obtain solutions of the pentagon equation on the matched product of two semigroups, that is a semigroup including the classical Zappa product.

Keywords: Quantum Yang-Baxter equation, pentagon equation, set-theoretical solution, semigroup

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1. Introduction

Let \(S\) be a set. A set-theoretical solution of the pentagon equation on \(S\) is a map \(s : S \times S \rightarrow S \times S\) such that

\[
s_{23} s_{13} s_{12} = s_{12} s_{23}.
\]

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where $s_{12} = s \times \text{id}_S$, $s_{23} = \text{id}_S \times s$, and
$s_{13} = (\text{id}_S \times \tau) s_{12} (\text{id}_S \times \tau)$ and $\tau$ is the flip map,
i.e., the map on $S \times S$ given by $\tau(x, y) = (y, x)$.

Set-theoretical solutions of the pentagon equation are used in the pioneering work of
Baaj and Skandalis \cite{1} to obtain multiplicative unitary operators on a Hilbert space.
Later, there have been appeared several papers on this topic: Zakrzewski \cite{38}, Baaj
and Skandalis \cite{2}, Kashaev and Sergeev \cite{23}, Jiang and Liu \cite{22}, Kashaev and Reshetikhin
\cite{24}, Kashaev \cite{23}, and Catino, Mazzotta, and Miccoli \cite{9}.

On the other hand, a map $s : S \times S \to S \times S$ is a
set-theoretical solution of the quantum
Yang-Baxter equation \cite{14} on a set $S$ if the relation

$$s_{23} s_{13} s_{12} = s_{12} s_{13} s_{23}$$

is satisfied, with the same notation adopted for the pentagon equation. Just to give an
example, if $f, g$ are idempotent maps from $S$ into itself such that $fg = gf$, then the map
$s$ defined as $s(x, y) = (f(x), g(y))$ is a solution to the both equations. These examples
are provided by Militaru \cite[Examples 2.4]{31} and in particular they lie in the class of the
well-known Lyubashenko solutions \cite{14}.

Finding set-theoretical solutions of the Yang-Baxter equation is equivalent to deter-
mining set-theoretical solutions of the braid equation, i.e., maps $r : S \times S \to S \times S$ such
that the relation

$$r_{23} r_{12} r_{23} = r_{12} r_{23} r_{13}$$

holds. In particular, a map $s$ is a set-theoretical solution of the quantum Yang-Baxter
equation if and only if $r = \tau s$ is a set-theoretical solution of the braid equation. This map
$r$ is usually written as $r(x, y) = (\lambda_x(y), \rho_y(x))$ with $\lambda_x, \rho_y$ maps from $S$ into itself. Since
the late 1990s a large number of works related to this equation has been produced, in-
cluding the seminal papers of Gateva-Ivanova and Van den Bergh \cite{17}, Etingof, Schedler,
and Soloviev \cite{15}, and Lu, Yan, and Zhu \cite{29}. In particular, the class of involutive non-
degenerate solutions, i.e., $r^2 = \text{id}$ and the maps $\lambda, \rho_x$ are bijective for every $x \in S$, has
been the most studied. Some algebraic structures related to the braid equation have
been introduced and investigated over the years. Just to name a few, we remind those
introduced by Rump, that are cycle sets \cite{33} and a generalization of radical rings, the
braces \cite{34}. The former are in bijective correspondence with involutive non-degenerate
solutions and they are also studied in recent works such as \cite{2, 3, 5, 28}. The latter are a
useful tool to obtain solutions and they are widely investigated in \cite{10, 11, 16, 35}.

Recently, generalizations of braces afford to determine bijective solutions that are not
necessarily involutive, for instance through skew braces \cite{19, 36}, or not bijective solutions
that are left non-degenerate, such as by means of semi-braces, see \cite{7} and \cite{21}. In this
way, Guarnieri and Vendramin \cite{19} determine solutions that arise from finite skew brace
such that $r^n = \text{id}$. Under mild assumptions, Catino, Colazzo, and Stefanelli \cite{8} prove
that the solution $r$ associated to a semi-brace satisfy $r^n = r$ with $n$ natural number
closely linked with the semi-brace structure.

During the last years, Lebed \cite{27} and Matsumoto and Shimizu \cite{30} deal with the
particular class of idempotent solutions, i.e., $r^2 = r$, which are useful tools in the study of
some algebraic structures, such as factorizable monoids. Furthermore, Cvetko-Vah and Verwimp [13] provide solutions $r$ by means of skew lattices that are cubic solutions, i.e., $r^3 = r$. Moreover, Jespers and Van Antwerpen obtain examples of such solutions through specific semi-braces, see [21, Theorem 5.1]. In addition, examples of solutions $r$ with the property $r^4 = \mathrm{id}$ are given by Yang in [37, p. 511].

The aim of this work is to determine new solutions of the Yang-Baxter equation by means of solutions to the pentagon equation. Specifically, we show how to involve two specific solutions of the pentagon equation defined on semigroups $S$ and $T$ in order to construct new solutions of the braid equation on the cartesian product $S \times T$.

In addition to this, among the solutions of the pentagon equation we characterize the special class of those that are also solutions of the quantum Yang-Baxter equation. Our characterization leads to consider specific semigroups belonging to the variety of semigroups $S = [xyz = xwyz]$ that one can deepen in [32]. The solutions defined on such semigroups are different from those known until now. Namely, such solutions $r$ satisfy the property $r^5 = r^4$ and the powers of $r$ are still solutions.

In view of all this, we also introduce a construction of solutions of the pentagon equation on semigroups. Specifically, if $s$ and $t$ are two given solutions on semigroups $S$ and $T$, respectively, we provide suitable conditions to obtain a new solution $s \bowtie t$ on the cartesian product $S \times T$, named the matched solution of $s$ and $t$. In particular, these solutions are defined on the matched semigroup $S \bowtie T$, that includes the classical Zappa product in [26].

Finally, the last section is devoted to some remarks and questions about solutions of the Yang-Baxter equation that arise throughout in this work.

2. New solutions of the pentagon equation

This section is devoted to introducing a new construction of solutions of the pentagon equation defined on a particular semigroup that has the cartesian product of two semigroups as underlying set. In this way, examples of such solutions can be obtained starting from classical Zappa products of semigroups.

At first, we remind some available results in literature of the pentagon equation and some examples and constructions of solutions of this equation. Hereinafter, we call any set-theoretical solution of the pentagon equation simply a PE solution. According to the notation introduced in [8, Proposition 8], given a set $S$ and a map $s$ from $S \times S$ into itself, we write

$$s(x, y) = (xy, \theta_x(y)),$$

where $\theta_x$ is a map from $S$ into itself, for every $x \in S$. Then, $s$ is a PE solution on $S$ if and only if the following conditions hold

$$(xy)z = x(yz)$$
$$\theta_x(y)\theta_{xy}(z) = \theta_x(yz)$$
$$\theta_{\theta_x(y)}\theta_{xy} = \theta_y$$

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for all $x, y, z \in S$. Moreover, the map $s$ is a PE solution if and only if $t := \tau s \tau$ satisfies

$$t_{12} t_{13} t_{23} = t_{23} t_{12}.$$ 

According to [9, Definition 1], such a map $t$ is called a set-theoretical solution of the reversed pentagon equation, or briefly a reversed solution. Thus, every reversed solution will be written as $t(x, y) = (\theta_y(x), yx)$.

The following are easy examples of PE solutions that we will often use in this work.

Examples 1.

1. Let $S$ be a semigroup and $\gamma$ an idempotent endomorphism of $S$. Then, the map $s : S \times S \to S \times S$ given by

$$s(x, y) = (xy, \gamma(y))$$ 

is a PE solution on $S$. In particular, if $e$ is an idempotent element of $S$, then $s(x, y) = (xy, e)$ is a solution on $S$.

2. If $S$ is a set and $f, g$ are idempotent maps from $S$ into itself such that $fg = gf$, then the map $s : S \times S \to S \times S$ given by

$$s(x, y) = (f(x), g(y))$$ 

is both a PE solution and a reversed solution on $S$ that we call Militaru solution.

In literature, there are few systematic constructions of PE solutions. In [25, Proposition 1] Kashaev and Sergeev provide a construction of PE solutions on a closed under multiplication subset $S$ of a group $G$. Recently, a complete description of PE solutions $s$ on groups $G$ of the form $s(x, y) = (xy, \theta_x(y))$ is presented in [9, Theorem 15].

The property (1) suggests to look for constructions of PE solutions starting from fixed semigroups and then to find maps $\theta_x$ satisfying properties (2) and (3).

In conformity with this idea, we show a new method to obtain PE solutions on inflations of semigroups. In detail, if $X$ is a set, $T$ a semigroup, and $\varphi : X \to T$ a map, consider $S = X \cup T$ and $\bar{\varphi} : S \to T$ the extension map of $\varphi$ such that $\bar{\varphi}|_T = \text{id}_T$. Thus, the set $S$ endowed with the operation given by

$$ab := \bar{\varphi}(a) \bar{\varphi}(b)$$

is a semigroup that is called the inflation of $T$ via $\varphi$. Note that this definition is equivalent to that provided by Clifford and Preston in [12, p. 98].

Given an inflation $S$ of a semigroup $T$ via a map $\varphi$, if $s(u, v) = (uv, \theta_u(v))$ is a PE solution on $T$, then the map $\bar{s} : S \times S \to S \times S$ defined by

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$$\bar{s}(a, b) = (ab, \theta_{\varphi(a)} \varphi(b))$$
is a PE solution on the inflation $S$ of $T$. We name such a solution $\vec{s}$ an inflation of $s$ via $\varphi$.

Now, we provide a more elaborate construction of PE solutions on the cartesian product of two semigroups. At first we need to fix some notations and to give preparatory definitions. For the ease of the reader, given two semigroups $S$ and $T$, we use the letters $a, b, c$ for $S$ and $u, v, w$ for $T$. Moreover, we denote any PE solution $s$ on $S$ and any PE solution $t$ on $T$ by $s(a, b) = (ab, \theta_a(b))$ and $t(u, v) = (uv, \theta_u(v))$, respectively.

**Definition 1.** Let $S$ and $T$ be semigroups, $\alpha : T \to S$ and $\beta : S \to T$ maps, set $\alpha_u := \alpha(u)$, for every $u \in T$, and $\beta_a := \beta(a)$, for every $a \in S$. If $\alpha$ and $\beta$ satisfy the following conditions

\[
\alpha_u (ab) = \alpha_u (a) \alpha_\beta(u)v(b) \quad \text{(S1)}
\]
\[
\beta_a (uv) = \beta_a (u) \beta_\beta(v) \quad \text{(S2)}
\]

for all $a, b \in S$ and $u, v \in T$, then we call $(S, T, \alpha, \beta)$ a matched quadruple of semigroups.

It is a routine computation to verify that $S \times T$ is a semigroup with respect to the operation defined by

\[(a, u) \ (b, v) = (ab, \theta_a(b))\]

if and only if $(S, T, \alpha, \beta)$ is a matched quadruple of semigroups. We call such a semigroup the matched product of $S$ and $T$ and we denote it by $S \triangleleft \triangleright T$.

As a class of examples of matched product of semigroups one can easily find the classical Zappa product \[26\]. Specifically, instead of (S1) and (S2), in this case we require the following conditions

\[
\alpha_u (ab) = \alpha_u (a) \alpha_\beta(u)v(b) \quad \text{(S1')}\]
\[
\beta_a (uv) = \beta_a (u) \beta_\beta(v) \quad \text{(S2')}\]

hold, for all $a, b \in S$ and $u, v \in T$.

**Definition 2.** Let $(S, T, \alpha, \beta)$ be a matched quadruple of semigroups, $s$ and $t$ PE solutions on $S$ and $T$, respectively. If $s, t, \alpha, \text{and } \beta$ satisfy the following conditions

\[
\theta_a \alpha_u = \theta_\alpha(u)\alpha_v \quad \text{(M1)}
\]
\[
\theta_{\alpha uv}(a) = \alpha_{\theta(a)}(v) \quad \text{(M2)}
\]
\[
\beta_{\theta(a)}(u) = \beta_{\theta(a)}(v) \quad \text{(M3)}
\]

for all $a, b, c \in S$ and $u, v \in T$, then we call $(s, t, \alpha, \beta)$ a matched quadruple.

Now, we show how to obtain new PE solutions by means of a matched quadruple. For simplicity, we denote $S \times T \times S \times T$ by $(S \times T)^2$. 

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Theorem 3. Let $S$, $T$ be semigroups and $(s, t, \alpha, \beta)$ a matched quadruple. Then, the map $s \bowtie t : (S \times T)^2 \to (S \times T)^2$ defined by

$$s \bowtie t (a, u; b, v) = (\alpha a u (b), \beta_\delta (u)v; \theta a u (b), \theta_\beta (u)v) ,$$

for all $(a, u), (b, v) \in S \times T$, is a PE solution on the matched product $S \bowtie T$. We call $s \bowtie t$ the matched product of $s$ and $t$.

Proof. First of all, the condition (11) is straightforward since $S \bowtie T$ is a semigroup.

Now, if $(a, u), (b, v), (c, w) \in S \times T$, we have

$$\theta_{(a, u)} (b, u) \theta_{(a, u)} (b, v) (c, w)$$

and

$$\begin{align*}
\theta_{(a, u)} ((b, u) (c, w)) &= \theta_{(a, u)} (b a u (c), \beta_c (v) w) \\
&= \left( \theta_{(a, u)} (b a u (c), \beta_b (u) v) \right) \left( \theta_{(a, u)} (b a u (c), \beta_b (u) v) \right) \\
&= \theta_{(a, u)} (b a u (c), \beta_b (u) v) \theta_{(a, u)} (b a u (c), \beta_b (u) v)
\end{align*}$$

In addition,

$$\theta a u b a u (c)$$

and

$$\theta_{(a, u)} \beta_{(a, u)} (u) \beta_c (v) w$$

hence the condition (12) is satisfied. Furthermore, we have

$$\begin{align*}
\theta_{(a, u)} ((b, u) (c, w)) &= \theta_{(a, u)} (b a u (c), \beta_c (v) w) \\
&= \theta_{(a, u)} (b a u (c), \beta_b (u) v) \theta_{(a, u)} (b a u (c), \beta_b (u) v)
\end{align*}$$
thus the condition (3) holds. Therefore, the map $s \triangleleft \triangleright t$

Example 2. Let $S, T$ be semigroups, $s$ a PE solution on $S$, and $t$ a PE solution on $T$
both in the class of solutions in Examples 1 given by $s(a, b) = (ab, \gamma(b))$ and $t(u, v) = (uv, \delta(v))$, respectively. Assuming $\alpha_u = \gamma$ and $\beta_u = \delta$, for all $u \in T$ and $a \in S$, we obtain that $(s, t, \alpha, \beta)$ is a matched quadruple. Then, by Theorem 3 the map

$$s \triangleright t (a, u; b, v) = (a \gamma(b), \delta(u); \gamma(b), \delta(v))$$

is the matched product of $s$ and $t$.

The following PE solution is defined on a matched product $S \bowtie T$ that do not lie in the class of Zappa product of semigroups.

Example 3. Let $f$ be an idempotent map from $\mathbb{N}_0$ into itself, $S$ the semigroup on the positive integers $\mathbb{N}_0$ with multiplication defined by $ab = f(a)$, and $T$ the monoid $(\mathbb{N}_0, +)$. If $\alpha_u = f$, for every $u \in T$, and $\beta_u : T \rightarrow T$ is the map defined by $\beta_u(a) = f(a) + u$, then $(S, T, \alpha, \beta)$ is a matched quadruple of semigroups. Indeed, if $a, b, u, v \in \mathbb{N}_0$, we have that $\alpha_u (aa\alpha_u(b)) = f^2(a) = \alpha_u (a) \alpha_{\beta_u(a)}(b)$, hence holds, and

$$\beta_u(\beta_u(u) + v) = f(u) + f(b) + u + v = f^2(b) + u + f(a) + v$$

Moreover, if $s$ is the PE solution on $S$ defined by $s(a, b) = (f(a), f(b))$ and $t$ the PE solution on $T$ given by $t(u, v) = (uv, v)$ it follows that $(s, t, \alpha, \beta)$ is a matched quadruple. Indeed, (M1) and (M2) are trivially satisfied. Furthermore, it holds

$$\beta_{\alpha_u(a)}(a) \beta_{\beta_u(c)}(v) = \beta f^2(c)(v) = \beta f(c)(v) = f^2(c) + v = \beta c(v) = \theta_{\alpha u(c)}(v) \beta c(v),$$
i.e., the condition (M3) is satisfied. Therefore, by Theorem 3 the map defined by
\[ s \bowtie t (a, u; b, v) = (f(a), f(b) + u + v; f(b), v) \]
is the matched product of \( s \) and \( t \).

Given a matched product of solutions \( s \bowtie t \), in order to extrapolate the solutions \( s \) and \( t \), we need to find isomorphic copies of \( S \) and \( T \) inside \( S \bowtie T \). For this purpose, we have to require additional properties on semigroups \( S \) and \( T \).

**Proposition 4.** [cf. Theorem 5] Let \( S \bowtie T \) be a matched product of semigroups via \( \alpha \) and \( \beta \), \( e_S \) a right identity for \( S \), and \( e_T \) a left identity for \( T \). Then, if the following conditions
\[ \alpha_{e_T} = \text{id}_S \quad \forall a \in S \quad \beta_{e_T} = e_T \] hold, then \( S^* = \{ (a, e_T) \mid a \in S \} \) is a subsemigroup of \( S \bowtie T \) isomorphic to \( S \).
Similarly, if
\[ \beta_{e_S} = \text{id}_T \quad \forall u \in T \quad \alpha_{e_S} = e_S \] hold, then \( T^* = \{ (e_S, u) \mid u \in T \} \) is a semigroup of \( S \bowtie T \) isomorphic to \( T \).

**Example 4.** Let \( X = \{0, 1, 2\} \), \( T \) the right zero semigroup on \( X \), and \( S \) the semigroup on \( X \) with the multiplication given by
\[ \forall a \in X \quad a \cdot 0 = 0 \cdot a = 0, \]
\[ \forall a, b \in X \setminus \{0\} \quad a \cdot b = a. \]
Then, \( e_S = 1 \) is a right identity for \( S \) and \( e_T = 0 \) is a left identity for \( T \). It is a routine computation to check that the map \( \gamma : X \to X \) defined by \( \gamma(x) = 1 \cdot x \) is an idempotent endomorphism of \( S \). Moreover, let \( \alpha : T \to S^0 \) and \( \beta : S \to T^0 \) be the maps such that
\[ \alpha_a = \begin{cases} \text{id}_S & \text{if } u = 0 \\ \gamma & \text{if } u \neq 0 \end{cases} \quad \beta_a = \begin{cases} \text{id}_T & \text{if } a = 1 \\ \gamma & \text{if } a \neq 1 \end{cases} \]
Then, \( S \bowtie T \) is a matched product of semigroups via the maps \( \alpha \) and \( \beta \) (that do not lie in the class of Zappa product). Furthermore, conditions (S3), (S4), (S5), and (S6) are trivially satisfied and thus, by Proposition 4, one can find isomorphic copies of \( S \) and \( T \) inside \( S \bowtie T \).

Let \( s \) be the PE solution on \( S \) given by \( s(a, b) = (a \cdot b, \gamma(b)) \) and \( t \) the PE solution on \( T \) given by \( t(u, v) = (v, v) \). Then, it is a routine computation to check that (M1), (M2), and (M3) and so by Theorem 3 the map
\[ s \bowtie t (a, u; b, v) = (a \cdot \alpha_a(b), v; \gamma(b), v) \]
is the matched product of \( s \) and \( t \).
Remark 5. Let \((S, T, \alpha, \beta)\) be a matched quadruple of semigroups and \(e_S\) a right identity for \(S\). If conditions (S5) and (S6) hold, then the condition (M2) becomes easier and it is equivalent to
\[
\theta_a = \alpha \theta_a \theta_a = \alpha \theta_a \theta_a = \alpha \theta_a \theta_a \tag{M2'}
\]
for all \(a \in S\) and \(u, v \in T\). Indeed, if (M2) holds, with \(b = e_S\), we get
\[
\theta_a = \theta_{ab} = \alpha \theta_{ab} \theta_a \theta_{ab} \theta_a \theta_{ab} \theta_a = \alpha \theta_a \theta_a = \alpha \theta_a \theta_a,
\]
i.e., (M2') holds. Conversely, if (M2') holds, with \(a = \alpha \theta_a \theta_a\) and \(u = \beta b(u)\), we clearly obtain that condition (M2) is satisfied.

If \(S\) and \(T\) are monoids and the identities \(1_S\) and \(1_T\) satisfy the conditions (S3), (S4), (S5), (S6), we call any matched quadruple \((S, T, \alpha, \beta)\) a matched quadruple of monoids.

Moreover, \(S \bowtie T\) is a monoid with identity \((1_S, 1_T)\) if and only if \((S, T, \alpha, \beta)\) is a matched quadruple of monoids. Note that in such a case this construction is equivalent to the classical Zappa product of two monoids.

In the following we show that under the assumption of \((S, T, \alpha, \beta)\) is a matched quadruple of monoids, then the conditions in Definition 2 become easier.

Proposition 6. Let \((S, T, \alpha, \beta)\) be a matched quadruple of monoids. Then, \((s, t, \alpha, \beta)\) is a matched quadruple if and only if the following conditions

1. \(\theta_a = \alpha \theta_a \theta_a\),
   \[
   \theta_a = \theta_{a_1} = \alpha \theta_{a_1} \theta_{a_1} = \alpha \theta_{a_1} \theta_{a_1} = \alpha \theta_{a_1} \theta_{a_1},
   \]
   hold, for all \(a, b \in S\) and \(u, v \in T\).

Proof. Initially, note that by Remark 5 the first equality of condition 1 is equivalent to the condition (M2).

Now, suppose that \((s, t, \alpha, \beta)\) is a matched quadruple. If \(a \in S\) and \(v \in T\), then
\[
\theta_a = \theta_{a_1} = \alpha \theta_{a_1} \theta_{a_1} = \alpha \theta_{a_1} \theta_{a_1} = \alpha \theta_{a_1} \theta_{a_1},
\]
by (S3)

Moreover, we have
\[
\beta \theta_{a_1} = \beta \theta_{a_1} = \beta \theta_{a_1} = \beta \theta_{a_1} = \beta \theta_{a_1} = \beta \theta_{a_1} = \beta \theta_{a_1},
\]
by (S5)-(S6)
Conversely, suppose that conditions 1. and 2. hold. Then, if \( a, b \in S \) and \( u, v \in T \), since \( \alpha \) is a homomorphism, we get

\[
\theta \alpha u = \theta \alpha_x(a) \alpha \beta_y(u) \alpha u = \theta \alpha_x(a) \alpha \beta_y(u) u,
\]

i.e., the condition (M1) holds. Furthermore, set \( a = a \alpha (b), u = \beta_y(u), c = b \), we trivially obtain the condition (M3).

**Example 5.** Let \( S = \{1_S, x, y\} \) be the monoid such that \( x^2 = x, y^2 = y, xy = y = yx \) and \( T = \{1_T, z\} \) the monoid such that \( z^2 = z \). Consider the maps \( \alpha : T \to S \) such that \( \alpha_1 = \text{id}_S \) and \( \alpha_z = \gamma \), where \( \gamma \) is an idempotent endomorphism of \( S \), and \( \beta : S \to T \) such that \( \beta_1 = \text{id}_T \) and \( \beta_x(u) = \beta_y(u) = 1_T \), for every \( u \in T \). Then, it is easy to check that \((S, T, \alpha, \beta)\) is a matched quadruple of monoids.

Moreover, let \( s \) be the PE solution on \( S \) given by \( s(a, b) = (ab, \theta \alpha (b)) \) and \( t \) the PE solution on \( T \) given by \( t(u, v) = (uv, v) \). Then, by Proposition 6 \((s, t, \alpha, \beta)\) is a matched quadruple and so the map

\[
s \bowtie t(a, u; b, v) = \begin{cases} 
  (a, v; 1_S, v) & \text{if } b = 1_S, u = 1_T \\
  (a, uv; 1_S, v) & \text{if } b = 1_S, u \neq 1_T \\
  (ab, v; \gamma(b), v) & \text{if } b \neq 1_S, u = 1_T \\
  (a\gamma(b), v; \gamma(b), v) & \text{if } b \neq 1_S, u \neq 1_T 
\end{cases}
\]

is the matched product of \( s \) and \( t \).

### 3. QYBE solutions of pentagonal type

This section is devoted to the special class of PE solutions on semigroups of the form \( s(a, b) = (ab, \theta \alpha (b)) \) that are also solutions of the quantum Yang-Baxter equation. In particular, we focus on semigroups belonging to the variety \( S = [abc = adbc] \) (cf. [32]) on which we are able to describe all such solutions.

Hereinafter, we call any set-theoretical solution of the quantum Yang-Baxter equation briefly a *QYBE solution*. Initially, we provide a characterization of PE solutions of the form \( s(a, b) = (ab, \theta \alpha (b)) \) that are also QYBE solutions.

**Proposition 7.** Let \( S \) be a semigroup and \( s \) a PE solution on \( S \) defined by \( s(a, b) = (ab, \theta \alpha (b)) \). Then, the map \( s \) is a QYBE solution if and only if the following conditions

\[
\begin{align*}
  abc &= a \theta \alpha (c) bc \\
  \theta \theta \theta_0 &= \theta \theta_0 \\
  \theta_\alpha (bc) &= \theta \theta \alpha (c) (bc)
\end{align*}
\]

are satisfied, for all \( a, b, c \in S \).
Proof. If \( a, b, c \in S \), we have
\[
s_{12}s_{13}s_{23}(a, b, c) = (a\theta_b(c)bc, \theta_a\theta_b(c)(bc), \theta_a\theta_b)
\]
and by (2) and (3)
\[
s_{23}s_{13}s_{12}(a, b, c) = (abc, \theta_a(b)\theta_a(b)(c), \theta_a(b)(c)) = (abc, \theta_a(b)(bc), \theta_a(b(c))).
\]
Assuming \( s \) is a QYBE solution, we note that comparing the first and the third components, the conditions (Y1) and (Y2) hold. Moreover, by (Y2) and (3), we obtain
\[
\theta_a(b)\theta_a(b) = \theta_a(b)(bc) = \theta_a(b(c)).
\]
Conversely, if (Y1), (Y2), and (Y3) hold, then by using (4) we get the claim.

Recalling that a map \( s \) is a PE solution if and only if the map \( t = \tau s \tau \) is a reversed solution, we have the following result.

**Corollary 8.** Let \( S \) be a semigroup and \( t \) a reversed solution on \( S \) defined by \( t(a, b) = (\theta_b(a), ba) \). Then, the map \( t \) is a QYBE solution on \( S \) if and only if the conditions (Y1), (Y2), and (Y3) are satisfied.

**Definition 9.** A PE solution \( s \) satisfying (Y1), (Y2), and (Y3) is said to be a QYBE solution of pentagonal type, or briefly a P-QYBE solution. Similarly, a reversed solution satisfying (Y1), (Y2), and (Y3) is called a QYBE solution of reversed pentagonal type, or briefly a R-QYBE solution.

From now on, we will show some results for P-QYBE solutions that can be equivalently obtained for R-QYBE solutions.

**Proposition 10.** Let \( s(a, b) = (ab, \theta_a(b)) \) be a P-QYBE solution on a semigroup \( S \). Then, the following hold:

1. the map \( \theta_a \) is idempotent, for every \( a \in S \);
2. \( \theta_{a|a^2} = \theta_{b|a^2} \), for all \( a, b \in S \);
3. if \( S^2 = S \), then \( s(a, b) = (ab, \bar{\theta}(b)) \), where \( \bar{\theta} \) is an idempotent endomorphism of \( S \).

**Proof.**
1. The claim follows by (Y2).
2. The claim follows by (Y3).
3. By 2., the maps $\theta_x$ are all equal. Let $x \in S$, set $\theta_x := \bar{\theta}$, we have that $\bar{\theta}$ is an idempotent endomorphism of $S$. Indeed, by (3) it holds $\bar{\theta}^2 = \bar{\theta}$ and we obtain
\[ \bar{\theta}(ab) = \theta_x(ab) = \theta_x(a)\theta_x(b) = \bar{\theta}(a)\bar{\theta}(b), \]
for all $a, b \in S$.

The following are simple examples of PE solutions of the form $s(a, b) = (ab, \gamma(b))$ in Examples 1.1 that are of P-QYBE type.

**Examples 6.**

1. The solution $s(a, b) = (ab, e_S)$, with $e_S$ a left identity (or a right identity) for $S$.

   Note that in the particular case of $S$ a group, by (Y1) the unique P-QYBE solution $s$ on $S$ is given by $s(a, b) = (ab, 1)$.

2. The map $s(a, b) = (ab, b)$, with $S$ is a left quasi-normal semigroup, i.e., $abc = acbc$, for all $a, b, c \in S$, (for more details see [18]).

In order to find more examples, we note that the condition (Y1) leads to consider special classes of semigroups. Thereby, we focus on semigroups $S$ belonging to the variety
\[ S : = \{ abc = adbc \} \] (5)
in [32] which immediately ensures (Y1). In this way, one has to find maps $\theta_a$ from $S$ into itself satisfying just (Y2) and (Y3).

**Example 7.** Let $S \in S$ and $s(a, b) = (ab, \gamma(b))$ the PE solution on $S$ in Examples 1.1. Then, $s$ is a P-QYBE solution.

As a direct consequence of Proposition 10 and Example 7, we provide a complete description of P-QYBE solutions in the particular case of semigroups $S \in S$ such that $S^2 = S$.

**Proposition 11.** Let $S \in S$ such that $S^2 = S$. Then, the unique P-QYBE solutions on $S$ are of the form
\[ s(a, b) = (ab, \bar{\theta}(b)), \] (6)
with $\bar{\theta}$ an idempotent endomorphism of $S$.

The following are examples of P-QYBE solutions defined on a semigroup $S \in S$ for which in general $S^2 \neq S$.

**Examples 8.**

1. The map $s(a, b) = (ab, bab)$.

2. The Militaru solution $s(a, b) = (f(a), g(b))$, where $ab = f(a)$, for all $a, b \in S$. 


The following proposition shows some properties related to the powers of P-QYBE solutions.

**Proposition 12.** Let $s$ be a P-QYBE solution on $S$. Then, for every $n \in \mathbb{N}$, $n \geq 2$,

$$s^n(a, b) = \left( ab \theta_a (b)^{n-1}, \theta_a (b) \right),$$

for all $a, b \in S$. In particular, if $S$ is an idempotent semigroup, it holds that $s^3 = s^2$.

**Proof.** We prove the claim by induction on $n$. The case $n = 2$ follows from the definition of $s$. Suppose that the thesis holds for $n > 2$ and so by induction hypothesis, we have

$$s^{n+1}(a, b) = s \left( ab \theta_a (b)^{n-1}, \theta_a (b) \right),$$

by (Y2)

$$= \left( ab \theta_a (b)^n, \theta_{ab\theta_a(b)}^{n-1} \theta_a (b) \right).$$

Therefore, the statement follows.

By using Proposition 12, we are able to provide P-QYBE solutions whose powers are still P-QYBE solutions.

**Examples 9.**

1. The map $s(a, b) = (ab, bab)$ that satisfies $s^3 = s^2$.
2. The Militaru solution $s(a, b) = (f(a), g(b))$ for which $s^2 = s$ holds.
3. The map $s(a, b) = (ab, e_S)$ in Examples 6.1. In particular, it holds
   - $s^2 = s$, if $e_S$ is a right identity for $S$;
   - $s^3 = s^2$, if $e_S$ is a left identity for $S$.
4. The map $s(a, b) = (ab, b)$ in Examples 6.2 for which it holds $s^3 = s^2$.

Instead, the following is an example of P-QYBE solution whose powers are not all P-QYBE solutions.

**Example 10.** Let $S \in S$ and $s$ the P-QYBE solution on $S$ given by $s(a, b) = (ab, \gamma(b))$ in Example 7 with $\gamma \neq \text{id}_S$. Then, $s$ satisfies $s^4 = s^3$ and in this case $s^3$ is a solution P-QYBE, while $s^2$ is neither a PE solution nor a QYBE solution.
4. Particular classes of YBE solutions

This section is devoted to studying the powers of the braid version \( r(a, b) = (\theta_a(b), ab) \) of specific P-QYBE solutions. As observed by Yang [37, p. 16], if \( r \) is a YBE solution, its \( n \)th power \( r^n \) is not necessarily a YBE solution. In contrast to this fact, we show that the powers of solutions to the braid equation defined on semigroups \( S \in \mathcal{S} \) in (5) are still solutions. We underline that these maps \( r \) lie in the class of degenerate solutions, unless trivial cases.

From now on, we call any set-theoretical solution of the braid equation briefly a YBE solution and the braid version of any P-QYBE solution simply a P-YBE solution.

In the next theorem we provide sufficient conditions so that the powers of any P-YBE solution are still solutions.

**Theorem 13.** Let \( S \in \mathcal{S} \) and \( r(a, b) = (\theta_a(b), ab) \) a P-YBE solution on \( S \). Then, it holds
\[
r^5 = r^3
\]
and the powers \( r^2, r^3, r^4 \) of the map \( r \) are still YBE solutions.

In addition, if \( S \) is idempotent, it holds \( r^4 = r^2 \).

**Proof.** First, denote the map \( \theta_{\alpha|_{\mathcal{S}}} \) as \( \bar{\theta} \), for every \( a \in S \). Note that, by Proposition 7 conditions (Y2) and (Y3) hold. If \( a, b \in S \), we have that
\[
r^2(a, b) = (\theta_{\alpha_2}(ab), \theta_a(b) ab) = (\bar{\theta}(a) \theta_a(b) ab)
\]
and
\[
r^3(a, b) = (\bar{\theta}(\theta_a(b) ab), \bar{\theta}(ab) \theta_a(b) ab)
\]
\[
= (\bar{\theta}(\theta_a(b) ab), \bar{\theta}(a) ab) \quad \text{by (2)-(5)}
\]
\[
= (\theta_a(b) \bar{\theta}(ab), \bar{\theta}(a) ab) \quad \text{by (2)-(Y2)}
\]

By proceeding in this way, one obtains
\[
r^4(a, b) = (\bar{\theta}(a) \bar{\theta}(ab), \theta_a(b) ab), \quad r^5(a, b) = (\theta_a(b) \bar{\theta}(ab), \bar{\theta}(a) ab).
\]
Consequently, it holds \( r^5 = r^3 \).

In particular, if \( S \) is idempotent, comparing the first components of \( r^2 \) and \( r^4 \), we get
\[
\bar{\theta}(ab) = \bar{\theta}(aab)
\]
\[
= \bar{\theta}(a) \theta_a(ab) \quad \text{by (4)}
\]
\[
= \bar{\theta}(a) \bar{\theta}(ab).
\]
Therefore, it holds \( r^4 = r^2 \).

Now, in order to show that \( r^2 \) is a YBE solution, set
\[
\lambda_a(b) := \bar{\theta}(ab) \quad \rho_b(a) := \theta_a(b) ab
\]
and we check that

\[
\lambda_a \lambda_b (c) = \lambda_{\lambda_a (b) \lambda_{\rho_b (a)} (c)} \\
\rho_{\lambda a} (a) = \rho_{\lambda a (b) \lambda_{\rho_b (a)} (c)} \\
\lambda_{\rho_{\lambda a (b)} (a) \rho_{\lambda c (c)}} (b) = \rho_{\lambda_{\rho_{\lambda a (b)}} (a) \lambda_{\rho_b (a)} (c)}
\]

for all \(a, b, c \in S\). Then, we have

\[
\lambda_a \lambda_b (c) = \lambda_a \theta (bc) = \theta (\lambda \theta (bc)) \\
= \theta (a \theta (bc)) \quad \text{by (2)-(Y2)}
\]

and

\[
\lambda_{\lambda_a (b) \lambda_{\rho_b (a)}} (c) = \lambda_{\theta (a) \theta (bc)} \theta (\theta_a (b) bc) \\
= \theta (\theta_a (b) \theta (bc)) \\
= \theta (a \theta (bc)) \quad \text{by (2)-(Y2)}
\]

Moreover, we compute

\[
\rho_{\lambda c} (a) = \rho_c (\theta_a (b) ab) = \theta_{\theta_a (b) ab} (c) \theta_a (b) abc \\
= \theta_b (c) bc \quad \text{by (1)-(3)}
\]

and

\[
\rho_{\lambda c} (a) = \rho_{\theta_a (b) ab} (c) \theta_{\theta_a (b) ab} (c) \theta_a (b) abc \\
= \theta_a (b) \theta (bc) \theta (bc) \theta_b (c) bc \\
= \theta_b (c) bc \quad \text{by (Y2)-(5)}
\]

Finally, we have that

\[
\lambda_{\rho_{\lambda a (b)}} (a) \rho_{\lambda c} (b) = \lambda_{\theta_a (b) \theta (bc)} (\theta_b (c) bc) = \lambda_{\theta_a (b) \theta (bc) \theta_b (c) bc} \theta_a (b) (c) bc \\
= \lambda_{\theta_a (b) \theta (bc)} (\theta_b (c) bc) \\
= \theta (\theta (b) \theta (bc) \theta_b (c) bc) \\
= \theta (\theta (b) \theta (bc)) \theta_b (c) bc \quad \text{by (3)}
\]

by (Y2)-(2)
\[\rho_{\lambda \mu(a)(c)} \lambda_a(b) = \rho_{\lambda \mu(b)(a)(c)} \tilde{\mu}(ab) = \rho_{\theta(\lambda \mu(b)(a)(c))} \tilde{\mu}(ab)\]
\[= \rho_{\theta(\lambda \mu(b)(a)(c))} \tilde{\mu}(ab)\]
\[= \theta_{\tilde{\mu}(ab)}(\theta_a(b) \tilde{\mu}(bc)) \tilde{\mu}(ab) \theta_a(b) \tilde{\mu}(bc)\]
\[= \theta_{\tilde{\mu}(ab)}(\theta_a(b) \tilde{\mu}(bc)) \tilde{\mu}(bc)\]
\[= \theta_a(b) \tilde{\mu}(bc)\]
\[= \theta_a(b) \tilde{\mu}(bc)\] by (Y2)–(Y3)
\[= \theta_a(b) \tilde{\mu}(bc)\]

By (Y3) we obtain that
\[\theta_a(b) \tilde{\mu}(bc) = \theta_a(b) \tilde{\mu}(bc)\]
\[= \theta_a(b) \tilde{\mu}(bc)\]
\[= \theta_a(b) \tilde{\mu}(bc)\] by (Y2)–(Y3)
\[= \theta_a(b) \tilde{\mu}(bc)\]

Therefore, \(r^2\) is a YBE solution.

With similar computations one can check that \(r^3, r^4\) are YBE solutions.

**Remark 14.** There exist P-YBE solutions \(r\) for which \(r^5 = r^3\), but the powers of \(r\) are not solutions. The P-YBE solution \(r(a, b) = (b, ab)\) defined on a left quasi-normal semigroup \(S\) is such an example. We highlight that this class of semigroups strictly contains \(S\) in (5).

The following example shows that the conditions in Theorem 13 are not necessary.

**Example 11.** Let \(S\) be a semigroup, \(e_S\) a left identity for \(S\), and \(r\) the P-YBE solution on \(S\) defined by \(r(a, b) = (e_S, ab)\). Then, it holds \(r^2 = r\) and so clearly \(r^3 = r^5\).

Since solutions in Example 11 and Examples 8 satisfy the hypotheses of Theorem 13 we list the powers of the braid version of the solutions therein that are still solutions.

**Examples 12.**

1. The map \(r(a, b) = (\gamma(b), ab)\) satisfies \(r^5 = r^3\) and the solutions \(r^2, r^3, r^4\) are
   - \(r^2(a, b) = (\gamma(ab), \gamma(b)ab)\)
   - \(r^3(a, b) = (\gamma(bab), \gamma(a)ab)\)
   - \(r^4(a, b) = (\gamma(a^2b), \gamma(b)ab)\).

2. The map \(r(a, b) = (bab, ab)\) satisfies \(r^4 = r^2\) and the solutions \(r^2, r^3\) are
   - \(r^2(a, b) = (aab, bab)\)
   - \(r^3(a, b) = (bab, aab)\).

3. The solution \(r(a, b) = (g(b), f(a))\) satisfies \(r^4 = r^2\) and \(r^2, r^3\) are respectively
   - \(r^2(a, b) = (fg(a), fg(b))\)
   - \(r^3(a, b) = (fg(b), fg(a))\).
Since $ab = f(a)$, note that in general $S$ is not idempotent and so the vice versa of the last claim of Theorem 13 does not hold.

To conclude this section, we give an application of Theorem 13 in the case of a rectangular band $S$, i.e., an idempotent semigroup such that $abc = ac$, for all $a, b, c \in S$ (cf. [20]).

**Example 13.** Let $S$ be a rectangular band. Then, by Proposition 11 the unique P-YBE solutions on $S$ are given by

$$r(a, b) = (\bar{\theta}(b), ab),$$

where $\bar{\theta}$ is an idempotent endomorphism of $S$.

Since $S \in S$ and $S$ is idempotent, by Theorem 13 we obtain that $r^4 = r^2$ and the solutions $r^2$ and $r^3$ are respectively

- $r^2(a, b) = (\bar{\theta}(a), \bar{\theta}(b)ab),$
- $r^3(a, b) = (\bar{\theta}(b), \bar{\theta}(a)ab)$.

5. **YBE solutions derived from PE solutions**

The aim of this section is to introduce a new method to construct solutions of the Yang-Baxter equation on the cartesian product of two sets through solutions of the pentagon equation.

For the ease of the reader, before proving the main theorem we provide the conditions to obtain a YBE solution in an easier case involving a PE solution and an R-QYBE solution.

Let $S, T$ be semigroups, $s$ a PE solution on $S$ given by $s(a, b) = (ab, \theta_a(b))$, and $t$ an R-QYBE solution on $T$ given by $t(u, v) = (\theta_u(v), vu)$. If $\alpha : T \to S^S$ is a map, set $\alpha_u := \alpha(u)$, for every $u \in T$, and

$$a_u b_v := \alpha_u(a) \alpha_{\theta_v(u)}(b),$$

for all $a, b \in S$ and $u, v \in T$. If the following conditions hold

1. $a b_u c_v = a \theta_u \alpha_v(c) b_u c_v$ (p1)
2. $\theta_a \theta_b \alpha_u = \theta_{ab} \alpha_{\theta_u(b)}\alpha_v$ (p2)
3. $\theta_{ab} = \theta_{ab} \alpha_u \alpha_v$ (p3)
4. $a_{ab} = \alpha_a \alpha_v (a \alpha_v(b))$ (p4)
5. $\theta_a = \alpha_a \theta_a$ (p5)
for all \(a, b, c \in S\) and \(u, v, w \in T\), then the map
\[
\begin{align*}
    r(a, u; b, v) &= (\theta_a \alpha_u(b), vu; a\alpha_u(b), \theta_v(u)),
\end{align*}
\]
is a YBE solution on \(S \times T\).

The proof of this statement is omitted since it will be contained in Theorem 16. Let us see the efficacy of this method by using easy PE solutions.

**Example 14.** Let \(S \in W = [abc = abdc, a^3 = a^2]\) (see [32, p. 370]), \(k\) a given element of \(S\), and \(s\) the PE solution on \(S\) defined by \(s(a, b) = (ab, k^2)
\). Note that \(s\) does not satisfy (Y1) and so it is not a QYBE solution.

Moreover, let \(T \in S\) in (5), and \(t\) the R-QYBE solution on \(T\) given by \(t(u, v) = (u, vu)\). If we consider \(\alpha_u(a) = k^2\), for every \(a \in S\) and \(u \in T\), then conditions from (p1) to (p5) are trivially satisfied, hence the map given by
\[
    r(a, u; b, v) = (k^2, vu; ak^2, u)
\]
is a YBE solution on \(S \times T\). One can check that \(r^5 = r^3\) and \(r^2, r^3, r^4\) are still YBE solutions and they respectively are
\[
\begin{align*}
    - r^2(a, u; b, v) &= (k^2, uvu; k^2, vu), \\
    - r^3(a, u; b, v) &= (k^2, vvu; k^2, uvu), \\
    - r^4(a, u; b, v) &= (k^2, uvu; k^2, vvu).
\end{align*}
\]

Note that if \(S\) is a right zero semigroup, then by (2) and (3) one can see that the unique PE solutions \(s\) on \(S\) are of the form \(s(a, b) = (b, \varphi(b))\), where \(\varphi\) is an idempotent map from \(S\) into itself. Clearly, such a map \(s\) is a P-QYBE solution. Thus, we provide the following example.

**Example 15.** Let \(S\) be a right zero semigroup, \(\varphi\) an idempotent map from \(S\) into itself, and \(s\) the P-QYBE solution on \(S\) given by \(s(a, b) = (b, \varphi(b))\). Moreover, let \(T\) be a rectangular band and \(t\) the R-QYBE solution on \(T\) defined by \(t(u, v) = (u, vu)\). Set \(\alpha_u := \varphi\) from \(S\) into itself, for every \(u \in T\). Then, conditions from (p1) to (p5) trivially hold and so the map
\[
    r(a, u; b, v) = (\varphi(b), vu; \varphi(b), u)
\]
is a YBE solution. Moreover, it holds \(r^3 = r\) and \(r^2\) is still a YBE solution and it is given by
\[
    r^2(a, u; b, v) = (\varphi(b), u; \varphi(b), vu),
\]
for all \(a, b \in S\) and \(u, v \in T\).

In order to present the main theorem, we introduce the following definition.
Definition 15. Let $S, T$ be two semigroups, $s(a, b) = (ab, b)$ a PE solution on $S$ given by $s(a, b) = (ab, 	heta_u(b))$, and $t$ a reversed solution on $T$ given by $t(u, v) = (\theta_v(u), vu)$. If $\alpha : T \to S^S$ and $\beta : S \to T^T$ are maps, set $\alpha_u := \alpha(u)$, for every $u \in T$, and $\beta_u := \beta(u)$, for every $a \in S$ and

$$a_u b_v := \alpha_u(a) \alpha_{u \beta_v(\alpha_u)}(b) \quad \theta_{ab_v} (c_v) := \theta_{\alpha_{u \beta_v(\alpha_u)}(b) \alpha_{u \beta_v(\alpha_u)}(c)}$$

for all $a, b \in S$ and $u, v, w \in T$. If the following conditions hold

$$a b c = a \theta_{b \alpha_v(c)} b c \quad (p1) \quad u v w = u \theta_v \beta_b(w) v_w (r1)$$

$$\theta_a \theta_{b \alpha_v} = \theta_{a \alpha_v(b) \alpha_{u \beta_v (\alpha_u)}} \quad (p2) \quad \theta_a \theta_{b \beta_u} = \theta_{\beta_u(b) \beta_{u \beta_v (\alpha_u)}} \quad (r2)$$

$$\theta_a (bc) = \theta_{a \alpha_{u \beta_v (\alpha_u)}} (bc) \quad (p3) \quad \theta_a (uv) = \theta_{u \theta_v \beta_b (w)} \quad (r3)$$

$$a_u b_v = \alpha_{u \beta_v (\alpha_u)}(a \alpha_v(b)) \quad (p4) \quad u_v w = \beta_{u \beta_v (\alpha_u)}(u \beta_v(w)) \quad (r4)$$

$$\theta_a = \alpha_u \theta_{\alpha_u} \quad (p5) \quad \theta_a = \beta_u \theta_{\beta_u} \quad (r5)$$

for all $a, b, c \in S$ and $u, v, w \in T$, then $(s, t, \alpha, \beta)$ is called a pentagon quadruple.

Theorem 16. Let $S, T$ be semigroups and $(s, t, \alpha, \beta)$ a pentagon quadruple. Then, the map $r : (S \times T)^2 \to (S \times T)^2$ defined by

$$r(a, u; b, v) := (\theta_a \alpha_u (b), v \beta_b (u) ; a \alpha_u (b), \theta_v \beta_b (u)),$$

for all $a, b \in S$ and $u, v \in T$, is a YBE solution on $S \times T$.

Proof. Let $(a, u), (b, v), (c, w) \in S \times T$. In order to show that $r$ is a YBE solution, set

$$\lambda_{(a, u)}(b, v) := (\theta_a \alpha_u (b), v \beta_b (u)) \quad \rho_{(b, v)}(a, u) := (a \alpha_u (b), \theta_v \beta_b (u)),$$

we have to check that

$$\lambda_{(a, u)} \lambda_{(b, v)} (c, w) = \lambda_{(a, u)(b, v)} \lambda_{(b, v)(a, u)} (c, w)$$

$$\rho_{(c, w)} \rho_{(b, v)}(a, u) = \rho_{(c, w)(b, v)} \rho_{(b, v)(c, w)} (a, u)$$

$$\lambda_{(b, v)(c, w)}(a, u) \rho_{(c, w)}(b, v) = \rho_{(b, v)(c, w)}(a, u) \lambda_{(b, v)}(b, v).$$

Initially, note that

$$\lambda_{(a, u)} \lambda_{(b, v)} (c, w) = (\theta_a \alpha_u \theta_b \alpha_v (c), w \beta_v (c) \beta_{u \beta_v (\alpha_u)} (w)),$$

$$\lambda_{(a, u)} \lambda_{(b, v)} (c, w) = (\theta_a \alpha_u (b) \alpha_{u \beta_v (\alpha_u)} \theta_{u \beta_v (\alpha_u)} \alpha_{u \beta_v (\alpha_u)} (c), w \beta_v \beta_b (u) \beta_{u \beta_v (\alpha_u)} \alpha_{u \beta_v (\alpha_u)} (w)),$$

$$= \left( \theta_a \alpha_u (b) \alpha_{u \beta_v (\alpha_u)} \theta_{u \beta_v (\alpha_u)} \alpha_{u \beta_v (\alpha_u)} (c), w \beta_v \beta_b (u) \beta_{u \beta_v (\alpha_u)} \alpha_{u \beta_v (\alpha_u)} (w) \right).$$
Therefore, \( r \) is a YBE solution.
Example 16. Let $S$ be a rectangular band and $s$ the P-QYBE solution on $S$ given by 
$s(a, b) = (ab, \gamma(b))$. Moreover, let $t$ be the Militaru solution on a semigroup $T$ given by 
$t(u, v) = (f(u), f(v))$.
Set $\alpha_u = \gamma$, for every $u \in T$, and $\beta_a = f$, for every $a \in S$. Then, $(s, t, \alpha, \beta)$ is a pentagon quadruple and, by Theorem 16, the map 
$$r(a, u; b, v) = (\gamma(b), f(v); a\gamma(b), f(u))$$
is a YBE solution on $S \times T$. Moreover, it holds $r^4 = r^2$ and the powers of $r$ are still YBE solutions and they are 
- $r^2(a, u; b, v) = (\gamma(ab), f(u); \gamma(b), f(v))$
- $r^3(a, u; b, v) = (\gamma(b), f(v); \gamma(ab), f(u))$.

Example 17. Assume that $S$ and $T$ are groups and $(s, t, \alpha, \beta)$ is a pentagon quadruple. Conditions (p1) and (r1) become 
$$\theta_u \alpha_u(b) = 1_S$$
$$\theta_u \beta_a(v) = 1_T.$$ (7) (8)
By (7), (8), and [9, Lemma 11], it follows that (p2), (r2), (p3), and (r3) are trivially satisfied. Moreover, (p4) and (r4) become 
$$\alpha_u(a) = \alpha_1_T(a),$$
$$\beta_a(u) = \beta_1_S(u).$$ (9) (10)
By (p5), we get $\alpha_u(1_S) = 1_S$, for every $u \in T$ and similarly, by (p5), $\beta_a(1_T) = 1_T$, for every $a \in S$. Moreover, by (9), with $b = 1_S$ and $v = 1_T$ we obtain 
$$\alpha_u(a) = \alpha_1_T(a),$$
for all $u \in T$ and $a \in S$. Thus, all the maps $\alpha_u$ are all equal to a map $\bar{\alpha}$, for every $u \in T$.
Similarly, the maps $\beta_a$ are all equal to $\bar{\beta}$, for all $a \in S$. Therefore, by Theorem 16 the map 
$$r(a, u; b, v) = (1_S, v\bar{\beta}(u); a\bar{\alpha}(b), 1_T),$$
is a YBE solution on $S \times T$. Note that 
$$r^n(a, u; b, v) = (1_S, \bar{\beta}^{n-1}(v)\bar{\beta}(u); \bar{\alpha}^{n-1}(a)\bar{\alpha}(b), 1_T),$$
for every $n \in \mathbb{N}$. In this way, by choosing $\bar{\alpha}$ an idempotent endomorphism of $S$ and $\bar{\beta}$ an idempotent endomorphism of $T$, then we obtain examples of solutions $r$ with the property $r^3 = r^2$.
To find other examples on groups, by conditions (7) and (8) one has to look at maps
and \( \bar{\beta} \) whose images \( \text{Im} \, \bar{\alpha} \) and \( \text{Im} \, \bar{\beta} \) are contained in the kernel \( K_s \) of \( s \) and \( K_t \) of \( t \), respectively. Given a PE solution \( s(a, b) = (ab, \theta_{a}(b)) \) on a group \( S \), we remind that the subset of \( S \) defined by
\[
K_s := \{ a \mid a \in S, \theta_{1S}(a) = 1_S \}
\]
is a normal subgroup of \( S \) called the kernel of \( s \) (cf. [9, Lemma 13]).

6. Some comments and questions

We conclude with some remarks and questions that arise throughout this work. Specifically, our purpose is to show that if some properties hold for a YBE solution \( r \), then it does not necessarily hold for the QYBE solution \( s := \tau r \) and vice versa.

First, we give the definitions of the index and the period of a map \( f \) as
\[
i(f) := \min \{ j \mid j \in \mathbb{N}_0, \exists l \in \mathbb{N}, f^j = f^l, j \neq l \}
\]
\[
p(f) := \min \{ k \mid k \in \mathbb{N}, f^{i(r) + k} = f^{i(r)} \},
\]
respectively. As observed in [8], these definitions are slightly different from the classical ones (cf. [20, p. 10]).

In general, we note that the index or the period of a QYBE solution \( s \) are independent from the index or the period of its YBE solution \( r \). The following are examples of QYBE solutions \( s \) for which it holds \( i(s) = 1 \) and \( p(s) = 1 \), but the corresponding YBE solutions \( r = \tau s \) have different periods.

Examples 18.

1. Let \( S \) be a rectangular band, i.e., an idempotent semigroup with the property \( abc = ac \), for all \( a, b, c \in S \), and \( s \) the P-QYBE solution on \( S \) given by \( s(a, b) = (ab, b) \).

Then, they hold \( i(r) = 1 \) and \( p(r) = 2 \).

2. Let \( S \) be a monoid and \( s \) the solution P-QYBE on \( S \) given by \( s(a, b) = (ab, 1) \).

Then, in this case we have \( i(r) = 1 \) and \( p(r) = 1 \).

Another aspect is that if \( r \) is a YBE solution for which its powers \( r^n \) are still solutions, then the maps \( s^n := (\tau r)^n \) are not necessarily QYBE solutions. The following is an example of YBE solution \( r \) for which the powers are all solutions while the powers of \( s = \tau r \) are not all QYBE solutions.

Example 19. Let \( r \) be the P-YBE solution given by \( r(a, b) = (\gamma(b), ab) \) on a semigroup \( S \). Then, by Theorem 13, we have that \( r^5 = r^3 \) and the powers of \( r \) are still YBE solutions. Now, if we consider \( s = \tau r \), it holds that \( s^4 = s^3 \) and one can check that \( s^3 \) is still a QYBE solution, while \( s^2 \) is not.
As observed in Examples 9.2, the powers of the YBE solution \( r(a, b) = (bab, ab) \) are all YBE solutions and as seen in Examples 12.1, the map \( s = \tau r \) is such that its powers are all Q-YBE solutions. Hence, a question arises.

**Question.** Classify the YBE solutions \( r \) such that their powers are still YBE solutions and for which also the powers of the QYBE solution \( s = \tau r \) are still QYBE solutions.

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