Generalised space-time and duality

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In this paper we consider the previously proposed generalised space-time and investigate the structure of the field theory upon which it is based. In particular, we derive a $\text{SO}(D,D)$ formulation of the bosonic string as a non-linear realisation at lowest levels of $E_{11} \otimes_s l_1$ where $l_1$ is the first fundamental representation. We give a Hamiltonian formulation of this theory and carry out its quantisation. We argue that the choice of representation of the quantum theory breaks the manifest $\text{SO}(D,D)$ symmetry but that the symmetry is manifest in a non-commutative field theory. We discuss the implications for the conjectured $E_{11}$ symmetry and the role of the $l_1$ representation.
1. Introduction

When it was first conjectured that the maximal supergravities in any dimensions could be extended to possess an $E_{11}$ symmetry [1] it was realised that some modification of space-time would be required rather than the ad hoc introduction of the translation generators as was done in the early papers [1,2]. It was subsequently proposed [3] that one introduce the generators transforming in the fundamental representation $l_1$ of $E_{11}$; more precisely one should take the non-linear realisation of the semi-direct product of $E_{11}$ and the $l_1$ representation, i.e. $E_{11} \otimes_s l_1$ [2]. At lowest levels the $l_1$ multiplet in eleven dimensions begins with the space-time translation generators $P_a$, then a two form $Z^{a_1a_2}$, five form generator $Z^{a_1...a_5}$ and a generator $Z^{a_1...a_7,b}$ together with an infinite number of other generators. In this approach the fields would depend on all the coordinates introduced in this non-linear realisation that is $x^a, x_{a_1a_2}, x_{a_1...a_5}, x_{a_1...a_7,b}, \ldots$ [3].

The simplest application of this idea is to consider the reduction on a circle, that is the IIA theory, and restrict the $E_{11}$ algebra to the $D_{10}$ subalgebra found by deleting node ten. The Dynkin diagram of $E_{11}$ is given in fig 1. In this case, at lowest level, one has the coordinates $x^a, y_a, a = 1, \ldots, 10$, arising as the dimensional reductions of $x^a$ and $x_{a11}$, belonging to the vector representation of $D_{10}$ and the field content $h_{a,b}, B_{a_1a_2}, \phi$ which should now depend on these coordinates which belong to the vector representation of SO(D,D) [4]. Although the non-linear realisation that arises is relatively straightforward to work out, one would be left with the problem of how to recover the usual theory, that is the massless NS-NS sector of the superstring. The introduction of the coordinates $x^a$ and $y_a$ has a long history in the context of $T$ duality in string theory; two of the earliest papers being [5,6]

The content of the $l_1$ representation can be found [7] by considering the algebra whose Dynkin diagram is that for $E_{11}$ but with one node added to the node labeled one of the $E_{11}$ Dynkin diagram by a single line and taking, with respect to the new node, only level one generators in the enlarged algebra. In eleven dimensions we decompose the $l_1$ representation in terms of representations of $A_{10}$ and we find that at lowest levels the generators [7]

\[
P_{\hat{a}}; Z_{\hat{a}_1\hat{a}_2}; Z_{\hat{a}_1...\hat{a}_5}; Z_{\hat{a}_1...\hat{a}_7,b}; Z_{\hat{a}_1...\hat{a}_8}; Z_{\hat{a}_1...\hat{a}_9,(\hat{b}\hat{c})}; Z_{\hat{a}_1...\hat{a}_9,\hat{b}_1\hat{b}_2},
\]

\[
Z_{\hat{a}_1...\hat{a}_{10},\hat{b}}; Z_{\hat{a}_1...\hat{a}_{11},\hat{b}_1...\hat{b}_4,\hat{c}}; Z_{\hat{a}_1...\hat{a}_8,\hat{b}_1...\hat{b}_6}; Z_{\hat{a}_1...\hat{a}_9,\hat{b}_1...\hat{b}_5}, \ldots
\]

where $\hat{a} = 1, \ldots, 11$.

By deleting node labeled $d$ in the Dynkin diagram of the enlarged algebra one can find the content of the $l_1$ representation appropriate to the $d$ dimensional theory, that is decomposed in terms of representations of $E_{11-d} \otimes GL(d)$. The results [4,8,9] are given in table one. Indeed page 13 of the second of these papers contains the point particle multiplet in all dimensions three and above. One can also find this result at low levels by simply carrying out the dimensional reduction by hand on the eleven dimensional generators contained in the above equation. In the non-linear realisation of $E_{11} \otimes_s l_1$ in $d$ dimensions one would introduce coordinates corresponding to the charges in the table.

The first few generators of the $l_1$ representation in eleven dimensions are the charges associated with the point particle, the two brane and the five brane and it has been
proposed that the $l_1$ representations contains all the brane charges \[4,7\] in eleven and lower dimensions. One piece of evidence supporting this conjecture is that for every $A_{10}$ representation in the adjoint representation of $E_{11}$, that is gauge field in the non-linear realisation, we can find an $A_{10}$ representation in the $l_1$ multiplet that has the correct space-time index structure to be interpreted as the charge corresponding to the current of the brane to which the gauge field couples. Put another way the existence of Wess-Zumino term in the dynamics of branes implies a pairing between gauge fields and currents and this implies a correspondence between the representations in the adjoint and $l_1$ representations of $E_{11}$ that holds \[7\].

By taking a particular charge found by dimensional reduction and applying $U$ duality in $d$ dimensions some charge multiplets for point particles and some other branes have been previously found \[10-13\]. For example, for the point particle we can take the charge of one of the the Kaluza Klein particles, that is $\frac{n}{R_i}$ where $R_i$ is one of the radii of the torus used in the dimensional reduction. The results predicted by $E_{11}$ agree with these multiplets, that is the first two columns of table 1 agree with the point particle and string multiplets found earlier. While it is obvious that the decomposition of the $l_1$ multiplet would lead to multiplets of $E_{11-\delta} \otimes GL(d)$ it did not have to lead to the correct representations. Put another way the charge representations in $d$ dimensions did not have to assemble into a single representation of $E_{11}$. Thus there is considerable evidence that the $l_1$ multiplet does contain all the brane charges. The table of figure 1 predicts quite a few other charge multiplets and it would be good to understand their role in string theory.

One can interpret the dependence on the generalised coordinates as encoding the measurement of events by all the different branes using the space-time that they see, that is $x^a$ corresponds to a point particle, $x^a$ and $x_{a_1a_2}$ by a two brane etc. Thus encoding all the coordinates of the $l_1$ multiplet allows one to consider all the possible ways of measuring space-time using all the different probes in a way that reflects the underlying symmetry that is $E_{11}$ and in particular, in lower dimensions, $U$ duality. The approach of introducing a generalised space-time was used to construct the field strengths of all maximal gauged supergravities in five dimensions \[14\]. In particular this paper introduced the coordinates $x^a$, $x^N$, $x_a$, $x_{a_1a_2}$, $x_{a_1a_2a_3}$, $x^{NM}_{a_1a_2a_3}$, $x_{a_1a_2a_3N}$, ... in addition to those of the usual five dimensional space-time, although the dependence on these coordinates was of a rather specific form. However, it is far from clear how to recover our usual supergravity theories in the general situation and it is likely that one is over counting in some way by introducing all the coordinates. It is the purpose of this paper to try to shed some light on this dilemma.

Many of the features suggested in references \[3,7,4\], and discussed above, have appeared in subsequent works on generalised geometry. It would be invidious to reference these papers here.

It has also been proposed \[8,15\] to use a non-linear realisation based on $E_{11} \otimes \mathfrak{l}_1$ to give a description of brane dynamics. Although the algebra is the same as that used for the supergravity theories the dependence on the fields and the choice of local subalgebra is different. We carry out this non-linear realisation in section 2 taking the IIA perspective and only the lowest level as described above, that is the SO(D,D) algebra and the brane coordinates $x^a, y_a, a = 1, \ldots, 10$. We arrive at a formulation of string theory that has
SO(D,D) symmetry and constructed from the coordinates \( x^a, y_a \). It is in fact a formulation found long ago [5].

The quantisation of the dynamics of the usual string based on the coordinate \( x^\mu \), that is the Nambu action, leads to a quantised theory [16] that contains the bosonic string. In this paper we will quantise the SO(D,D) string just mentioned. In section 3 we find its Hamiltonian formulation and in section 4 we quantise this theory to find that the \( x^a \) and \( y_a \) coordinates do not commute. In the quantised theory one can work with just \( x^a \) but then the SO(D,D) symmetry is not manifest. To maintain manifest SO(D,D) symmetry we must work with both \( x^a \) and \( y_a \), but then one is dealing with a non-commutative field theory. In section five we discuss the implications of this work for the \( E_{11} \) conjecture and the role of the \( l_1 \) representation.

2. SO(D,D) symmetric string as a non-linear realisation

Let us begin by briefly summarising the two methods of carrying out a non-linear realisation for an internal group, that is where the space-time coordinates are inert under the group involved [17]. In particular we consider the non-linear realisation of a group \( G \) with local sub-algebra \( H \) with a group element \( g \in G \) which depends on a set of parameters which in turn depend on the space-time coordinates. Thus the parameters in the group element become the fields of the theory. The group element \( g \) is subject to the two transformation \( g \to g_0g \) and \( g \to gh \) where \( g_0 \in G \) and \( h \in H \), but while \( g_0 \) is a rigid transformation and so is independent of the coordinates of space-time, the \( h \) transformation is a local transformation and so does depend on the space-time coordinates. To construct the non-linear realisation we must find some dynamics which is invariant under the above two transformations. Let us assume that \( G \) is a Kac-Moody algebra and that the local sub-algebra \( H \) is the one invariant under the Cartan involution \( I_c \) which acts on the Chevalley generators \( E_a, F_a \) and \( H_a \) as

\[
I_c(E_a) = -F_a, \quad I_c(F_a) = -E_a, \quad I_c(H_a) = -H_a, \tag{2.1}
\]

Another useful operator, denoted by \( I \), acts on group elements as \( I(g) = I_c(g^{-1}) \) and on element \( A \) of the algebra by \( I(A) = I_c(-A) \). We note that this operator also squares to the identity operation and that \( I(AB) = I(B)I(A) \) for any two elements \( A \) and \( B \) of the Lie algebra. Since by definition \( h \) satisfies \( I_c(h) = h \) it follows that \( I(h) = h^{-1} \). Using the local \( H \) symmetry we may set to zero the \( H \) part of \( g \), leaving it to be a member of the Borel sub-algebra of \( G \). Having done this one must then carry out a local \( h \) transformations for a generic \( g_0 \) transformations to preserve the choice of coset representative.

Since there are two distinct symmetries that must be taken into account, that is the above rigid and local symmetries, there two are ways to proceed. One can first find objects which are invariant under \( g_0 \) transformations and then solve the invariance with respect to \( h \) transformations. This is achieved by considering the Cartan forms \( \mathcal{V} = g^{-1}dg \) which are indeed invariant under the former transformations and transform under local \( H \) transformations as \( \mathcal{V} \to h^{-1}\mathcal{V}h + h^{-1}dh \). To construct the dynamics we exploit the properties of the Cartan involution and consider the objects

\[
\mathcal{U} = \mathcal{V} + I(\mathcal{V}) = d\xi^\alpha \mathcal{U}_\alpha, \quad w = \frac{1}{2}(\mathcal{V} - I(\mathcal{V})) = d\xi^\alpha w_\alpha \tag{2.2}
\]
which transform as
\[ \mathcal{U} \to h^{-1} \mathcal{U} h, \quad w \to h^{-1} w h + h^{-1} d h \]  
(2.3)
We note that \( D_\alpha \mathcal{U}_\beta = \partial_\alpha \mathcal{U}_\beta + [w_\alpha, \mathcal{U}_\beta] \) transforms covariantly; \( D_\alpha \mathcal{U}_\beta \to h^{-1} D_\alpha \mathcal{U}_\beta h \). An invariant is given by
\[ Tr(\mathcal{U}_\alpha \mathcal{U}_\beta) \]  
(2.4)
Integrating this expression over space-time and contracting the space-time indices we can take this to be the action.

Alternatively, one can first find objects invariant under the local \( H \) transformations. Such an object is \( M \equiv g I(g) \) which is invariant under local \( h \) as \( I(h) = h^{-1} \) and transforms under the rigid transformations as \( M \to g_0 M I(g_0) \). Clearly,
\[ tr(M^{-1} \partial_\alpha M M^{-1} \partial_\beta M) \]  
(2.5)
is invariant under the local and rigid transformations. In fact the two invariants of equations (2.4) and (2.5) are the same up to a constant of proportionality. Such constructions has proved particularly useful in formulating the symmetries of the scalars in the dimensionally reduced maximal supergravity theories.

We can also consider a non-linear realisation in which the generators of space-time belong to the group \( G \). Such is the case for gravity and supergravity. In particular the group used in such non-linear realisations is extended to include generators that belong to a realisation \( l \) of \( G \) and consider the semi-direct product group formed from \( l \) and \( G \), denoted \( G \otimes_s l \). In terms of the algebra, if the generator \( A \) is in the Lie algebra \( G \) and the generators in the \( l \) representation are denoted by \( Z_s \) then we adopt the commutator
\[ [Z_s, A] = D(A)_s r Z_r \] where \( D(A) \) is the matrix representation of \( G \) of the \( l_1 \) representation. We could take the generators \( Z_s \) to have non-trivial commutators amongst themselves provided this is consistent with the Jacobi identities, but here we will take them to commute. The group element can be written in the form
\[ g = g_I g_B \]  
(2.6)
where \( g_I \) is generated by the \( Z_r \) the coefficients of which are the the space-time coordinates and the \( g_B \) belongs to \( G \) and the coefficients of the generators of \( G \) are the fields of the theory which are taken to depend on the coordinates of the generalised space-time. As above one has the two transformations, one rigid and one local. The local sub-algebra being the Cartan sub-algebra of \( G \). For examples of how this method proceeds see references [2].

Now let us consider the non-linear realisation appropriate to a brane moving in a background, such as gravity and supergravity [2]. The background fields belong to a non-linear realisations as described just above, that is the arise as a non-linear realisation of a Lie algebra \( G \otimes_s l \). The local sub-algebra is chosen to be a sub-algebra \( H_a \) of the Cartan involution invariant sub-algebra \( H \) of \( G \). This corresponds to the breaking of some of the background symmetries by the presence of the brane. Different choices of this local sub-algebra give rise to different branes. The group element is of the form
\[ g = g_I g_B g_a \]  
(2.7)
where \( g_l \) belongs to the Abelian group generated by \( l \) i.e. by the \( Z_r \), and it contains the space-time coordinates; \( g_b \) is a group element of \( G \) which has its \( H \) part removed, so belongs to the Borel sub-algebra of \( G \) and it contains the background fields. Finally \( g_a \) is a group element of \( H \) and it contains more fields associated with the breaking of the symmetries of \( G \) by the brane. We may remove the part of \( g_a \) that belongs to \( H_a \) using the local sub-algebra.

The brane coordinates, which appear in \( g_l \), depend on the parameters that describe the world volume of the brane. The background fields, that appear in \( g_B \), depend on the brane coordinates and \( g_a \) contains further fields, denoted \( \phi \) that also depend on the parameters \( \xi \) of the brane world volume. As \( l \) arises as a representation of \( G \), under a \( g_0 \) transformation \( g_l \to g_0 g_l (g_0)^{-1} \) and so they transform linearly under \( G \).

Of course the group \( G \otimes s \) is not a Kac-Moody algebra, or in the finite case a semi-simple Lie algebra, nonetheless one can extend the notion of Cartan involution and the involution \( I \) to act on the generators \( Z_r \) to give new generators \( \tilde{Z}_r \) and then use this extended involution to construct invariant dynamics. One can do this by trial and error until one finds an involution of the algebra \( G \otimes s l \) which reduces to the previous involution when restricted to \( G \). However, when the representation \( l \) is one of the fundamental representation of \( G \) we can enlarge the algebra by adding an extra node to the Dynkin diagram of \( G \) attached to the node associated with the fundamental representation. The generators of the \( G \) are the level zero generators of the new algebra and, as explained in [7], the generators of the \( l \) representation are those of level one. The action of the Cartan involution takes these generators into those at level minus one. Thus one can embed the generators of \( G \otimes s l \) into a Kac-Moody algebra whose Dynkin diagram is the enlarged Dynkin diagram just discussed and then we can use the Cartan involutions for any Kac-Moody algebra.

The situation is most readily understood by considering the simplest example; the bosonic \( p \) brane in space-time dimension \( D \) coupled to gravity. For this case \( G = GL(D) \), \( H = SO(1, p) \otimes SO(D - p - 1) \) the \( l \) representation is the fundamental representation associated with first node of the Dynkin diagram of \( A_{D-1} \), which are just the usual translations \( P_a \). The group element has the form

\[
g = e^{x^a(\xi)P_a} e^{K^a_{\;b}h_a \cdot (x^c)} e^{\phi(\xi) J}
\] (2.8)

where the three group elements correspond to the decomposition of equation (2.7) and so \( \phi \cdot J \) contains terms that only belong to \( SO(1, D) \) moded out by \( SO(1, p) \otimes SO(D - p - 1) \) and \( K^a_{\;b} \) is symmetric once one of its indices is lowered. Using the suitable generalisation of the first method discussed above for internal symmetries to construct the non-linear realisation, that is worked with \( V \), one finds covariant constraints that express the fields \( \phi \) in terms of the derivatives of the coordinates \( x^a \), and construct an invariant actions out of these remaining fields. As such one ends up with an invariant action that just contains the the field \( x^a \) and the background metric \( g_{mn} = e_m ^{a} \eta_{ab} e_n ^{b} \) where \( e_m ^{a} = (e^h)_m ^{a} \). It is just the volume swept [2].

The above example is thought to illustrate the general situation, one can find covariant constraints such that the fields \( \phi \) are expressed in terms of derivatives of the brane coordinates. However, there is at present no systematic way of choosing \( H_a \) or of finding
these constraints. As such we will use the analogue of the second of the above methods, namely that based on \( M = gI(g) \). This has a substantial advantage in that the \( g_a \) part of the group element drops out of \( M \) and so one does not need to know the \( H_a \) or need to find the covariant constraints that express \( \phi \) in terms of the derivatives of \( X^a \).

We illustrate this for a bosonic p brane coupled to gravity. The algebra of \( \text{GL}(D) \) is given by

\[
[K^a_b, K^c_d] = \delta^c_d K^a_b - \delta^a_d K^c_b, \tag{2.9}
\]

and their relations with the commuting translations \( P_a \) are

\[
[K^c_b, P_a] = -\delta^c_a P_b \tag{2.10}
\]

The Cartan involution acts as \( I_c(K^a_b) = -K^b_a \) and we identify the Cartan involution invariant sub-algebra of \( \text{SL}(D) \) to be generated by \( I \), \( \text{SO}(D) \). Acting on the translations we find \( I_c(P_a) = -\tilde{P}^a \) where \( \tilde{P}^a \) is a new generator.

We now construct, the ISO(D,D) invariant string from a non-linear realisation. As explained above this is just the non-linear realisation of \( E_{11} \otimes_s l_1 \), as seen from the IIA perspective and at the lowest level. To this end we take \( G \otimes_s l \) to be the group ISO(D,D) whose commutation relations are given by

\[
[K^a_b, K^c_d] = \delta^c_d K^a_b - \delta^a_d K^c_b, \quad [K^a_b, R^{cd}] = \delta^c_d R^{ba} - \delta^d_b R^{ac}, \quad [K^a_b, \tilde{R}^{cd}] = -\delta^c_a \tilde{R}^{bd} + \delta^d_a \tilde{R}^{bc},
\]

\( \text{where} \ \delta^a_b \)
The $l$ representation is the fundamental representation associated with the node of SO(D,D) Dynkin diagram that is on the end of the long tail. It corresponds to the generators $P_a, Q^a$ and ISO(D,D) is indeed the corresponding semi-direct product group. The Cartan involution acts on the generators of SO(D,D) as $\tilde{P}_a, \tilde{Q}_a$ such that $I_c(P_a) = -\tilde{P}_a, I_c(Q^a) = -\tilde{Q}_a$. We take them to commute with $P_a, Q^a$ and their commutation relations with the generators of SO(D,D) are given by

$$[K^a, P_b] = -\delta^c_a P_b, \quad [R^{ab}, P_c] = -\frac{1}{2} (\delta^a_c Q^b - \delta^b_c Q^a), \quad [R^{ab}, P_c] = 0,$$

$$[K^a, Q^c] = \delta^c_b Q^a, \quad [R^{ab}, Q^c] = \frac{1}{2} (\delta^c_b P_a - \delta^c_a P_b), \quad [R^{ab}, Q^c] = 0,$$

We note that the full algebra of equations (2.17) to (2.21) admits another automorphism $J$ which leaves the generators of SO(D,D) invariant but acts as $J(P_a) = \tilde{Q}_a, J(Q^a) = \tilde{P}_a$.

We can now construct the non-linear realisation for the string, as discussed above. We take $G = SO(D,D)$ with $H = SO(D) \otimes SO(D)$ and $H_a = SO(D-2) \otimes SO(D-2) \otimes SO(1,1) \otimes SO(1,1)$. The corresponding group element is given by

$$g = e^{x^a(\xi) P_a + y_a(\xi) Q^a} e^{K^a \phi(x^c) e B_{bc}(x^c) R^{bc} e^{\phi(\xi)} J}$$

In fact this is not quite the complete lowest level $E_{11} \otimes s l_1$ realisation as we have discarded the dilaton and its associated Abelian generator. Its inclusion would not significantly alter the conclusions.

It is straightforward to calculate, in a constant background, the locally $H$ invariant object

$$M^{-1} \partial_\alpha M = \partial_\alpha y_n \tilde{Q}^n + \partial_\alpha x^n \tilde{P}^n + \nabla_\alpha x_m P_m + \nabla_\alpha y^m Q^m$$

where

$$\nabla_\alpha x_m = \partial_\alpha x^n g_{pm} - \partial_\alpha y_n B^n m - \partial_\alpha x^n B_{pq} B^q m, \quad \nabla_\alpha y^m = \partial_\alpha y_p g^{pm} + \partial_\alpha x^n B^n m$$

Under a rigid $g_0$ transformation $M^{-1} \partial_\alpha M \to (I(g_0))^{-1} M^{-1} \partial_\alpha M I(g_0)$. We may write $g_0$ as $g_0 = kl_0$ where $k \in SO(D,D)$ and $l_0$ is generated by $P_a, Q^a$. As the latter commute with themselves and with $\tilde{P}_a, \tilde{Q}_a$ we find that $M^{-1} \partial_\alpha M \to (I(k))^{-1} M^{-1} \partial_\alpha M I(k)$. Now consider the above automorphism $J$, as it leaves elements of SO(D,D) inert we find that
$J(M^{-1}\partial_\alpha M)$ transforms just like $M^{-1}\partial_\alpha M$. Consequently, the first order equations of motion

$$\epsilon^{\alpha\beta} J(M^{-1}\partial_\beta M) = \sqrt{-\gamma}\gamma^{\alpha\beta} M^{-1}\partial_\beta M$$  \hspace{1cm} (2.25)

are invariant under local and rigid transformations provided we can find a $\gamma_{\alpha\beta}$ which is also invariant and transforms like a metric under two dimensional reparameterisations. Leaving aside this one point for the moment the above equations of motion are given in terms of the fields by

$$\epsilon^{\alpha\beta}\partial_\beta y_n = \sqrt{-\gamma}\gamma^{\alpha\beta}\nabla_\beta x_n, \hspace{1cm} (2.26)$$

$$\epsilon^{\alpha\beta}\partial_\beta x_n = \sqrt{-\gamma}\gamma^{\alpha\beta}\nabla_\beta y_n. \hspace{1cm} (2.27)$$

The other two equations contained in equation (2.25) are equivalent to these two equations as is to be expected as the equation is invariant under the action of $J$. In fact one requires only one of the above equations as the first implies the second. These equations were first given in reference [5]. We note that the doubling of coordinates is similar to the introduction of dual fields, such as for electromagnetism; they both allow the equation of motion to be written in two ways and these together form a multiplet under the duality group.

At first sight equations (2.26) and (2.27) provide a manifestly SO(D,D) set of equations of motion for the string, however, this assumes one can find an expression for $\gamma_{\alpha\beta}$ that is manifestly SO(D,D) invariant. We have at our disposal the invariant $\text{tr}(M^{-1}\partial_\alpha MM^{-1}\partial_\beta M)$ which, using equation (2.23) evaluates to

$$I^1_{\alpha\beta} \equiv \text{tr}(M^{-1}\partial_\alpha MM^{-1}\partial_\beta M) = \partial_\alpha x^n \nabla_\beta x_n + \partial_\alpha y_n \nabla_\beta y_n + (\alpha \leftrightarrow \beta) \hspace{1cm} (2.28)$$

This would seem, at first sight, a good choice for $\gamma_{\alpha\beta}$. However, multiply equation (2.26) by $\partial_\gamma x^n$ and equation (2.27) by $\partial_\gamma y_n$ and adding we find, multiplying by $\epsilon_\delta\alpha$ that the left hand side is symmetric in $\delta\gamma$, but the right hand side is anti-symmetric. Hence the Invariant $I^1_{\alpha\beta}$ vanishes if we use the equations of motion.

There exists another invariant, namely $\text{tr}(M^{-1}\partial_\alpha MJ(M^{-1}\partial_\beta M))$ which can be evaluated to

$$I^2_{\alpha\beta} \equiv \frac{1}{2}\text{tr}(M^{-1}\partial_\alpha MJ(M^{-1}\partial_\beta M)) = \partial_\alpha x^n \partial_\beta y_n + \nabla_\alpha x_n \nabla_\beta y_n + (\alpha \leftrightarrow \beta)$$

$$= \partial_\alpha x^n \partial_\beta y_n + \partial_\beta x^n \partial_\alpha y_n \hspace{1cm} (2.29)$$

However, setting $\gamma_{\alpha\beta}$ equal to $I^2_{\alpha\beta}$ again leads to a contradiction. In fact there is no manifestly ISO(D,D) expression for $\gamma_{\alpha\beta}$ that leads to a consistent set of equations of motion.

It might seem that there is no manifestly ISO(D,D) covariant set of equations of motion as equations (2.26) and (2.27) do not allow one to solve for $\gamma_{\alpha\beta}$. The way out is to set $I^2_{\alpha\beta} = 0$ that is adopt in addition to equations (2.26) and (2.27) the condition

$$\partial_\alpha x^n \partial_\beta y_n + \partial_\beta x^n \partial_\alpha y_n = 0 \hspace{1cm} (2.30)$$
Using equation (2.26) we can eliminate either \(y_n\), or \(x^n\), in equation (2.30) and then find an equation form which we can solve for \(\gamma_{\alpha\beta}\). Eliminating \(y_n\) one finds that

\[
(\sqrt{-\gamma})^{-1}\gamma_{\alpha\beta} = \frac{\partial_\alpha x^n G_{nm} \partial_\beta x^m}{\sqrt{\det \partial_\alpha x^n G_{nm} \partial_\beta x^m}}
\]  

(2.31)

while eliminating \(x^n\) we find that

\[
(\sqrt{-\gamma})^{-1}\gamma_{\alpha\beta} = \frac{\partial_\alpha y_n G_{nm} \partial_\beta y_m}{\sqrt{\det \partial_\alpha y_n G_{nm} \partial_\beta y_m}}
\]  

(2.32)

While these are not manifestly ISO(D,D) invariant expressions, they arise from equations that are manifestly covariant and one can verify that they are inert of one uses the equation of motion of equation (2.26). Substituting these expression for \((\sqrt{-\gamma})^{-1}\gamma_{\alpha\beta}\) of equation (2.31) into equation (2.26) and differentiating with respect to \(\partial_\alpha\) we find the standard equation of motion of the bosonic string. On the otherhand substituting these expression for \((\sqrt{-\gamma})^{-1}\gamma_{\alpha\beta}\) of equation (2.32) into equation (2.27) and differentiating with respect to \(\partial_\alpha\) we find the equations of motion in terms of the dual variable \(y_n\).

Hence we have derived from the non-linear realisation a set of manifestly ISO(D,D) covariant equations of motion; for completeness we summarize them

\[
\epsilon^{\alpha\beta} J(M^{-1}\partial_\beta M) = \sqrt{-\gamma}\gamma^{\alpha\beta} M^{-1}\partial_\beta M \quad \text{or} \quad \epsilon^{\alpha\beta} \partial_\beta y_n = \sqrt{-\gamma}\gamma^{\alpha\beta} \nabla_\beta x_n,
\]

\[
tr(M^{-1}\partial_\alpha M J(M^{-1}\partial_\beta M)) = 0 \quad \text{or} \quad \partial_\alpha x^n \partial_\beta y_n + \partial_\beta x^n \partial_\alpha y_n = 0
\]  

(2.33)

3. SO(D,D) invariant Hamiltonian formulation of the string

To investigate the quantisation of the SO(D,D) string we require a Hamiltonian formulation. One might deduce such a formulation from a Lagrangian that can lead to the string motion described in term of either \(x^\mu\), or the \(y_\mu\), depending which equation of motion one chooses to implement first. However, this turns out to be complicated involving first and second class constraints that must be separated. Here we will content ourselves with producing a Hamiltonian and Poisson brackets that do lead to the SO(D,D) invariant description of the string motion given in the previous section.

We first introduce the fields \(X^N = (x^\mu, y_\mu)\) which transform according to the vector representation of SO(D,D). We take as our Hamiltonian

\[
H = \frac{1}{2} \int d\sigma (\delta \partial_1 X^M G_{MN} \partial_1 X^N + \epsilon \partial_1 X^M \Omega_{MN} \partial_1 X^N)
\]  

(3.1)

where \(\epsilon\) and \(\delta\) are new fields, \(G_{MN}\) and \(\Omega^{MN} = \Omega^{NM}\) are given by

\[
G_{MN} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & (g^{-1})_{\mu\nu} \end{pmatrix}, \quad \Omega^{MN} = \begin{pmatrix} 0 & \delta^\nu_{\mu} \\ \delta^\mu_{\nu} & 0 \end{pmatrix}
\]  

(3.2)

and \(\Omega_{MN} = (\Omega^{-1})_{MN}\). Here \(g_{\mu\nu}\) is the background metric. For simplicity we have set the background two form to zero. The tensor \(\Omega^{MN}\) is an SO(D,D) invariant tensor and one
can verify that $G \Omega G = \Omega^{-1}$. The latter states that $G$ viewed as a matrix is an element of $\text{SO}(D,D)$.

We adopt as our Poisson bracket the relation

$$\{X^N(\sigma), X^M(\sigma')\} = \theta(\sigma - \sigma')\Omega^{MN} \tag{3.3}$$

where $\theta$ is the step function which obeys $\frac{\partial \theta}{\partial \sigma} \theta(\sigma - \sigma') = \delta(\sigma - \sigma')$.

Carrying out the Hamiltonian analysis we realise that the momenta $\rho$ and $\tau$ conjugate to $\delta$ and $\epsilon$ are absent and so we have the constraints $\rho = 0 = \tau$. Following the Dirac procedure we must insist that the time evolution of these constraints vanish that is

$$\dot{\rho} = \{\rho, H\} = C_1 \equiv \partial_1 X^M G_{MN} \partial_1 X^N = 0 \tag{3.4}$$

and

$$\dot{\tau} = \{\tau, H\} = C_2 \equiv \partial_1 X^M \Omega_{MN} \partial_1 X^N = 0 \tag{3.5}$$

Thus we find two new constraints which obey the Poisson brackets

$$\{C_1(\sigma), C_1(\sigma')\} = -4C_1(\sigma')\partial_1 \delta(\sigma - \sigma') + 2\partial_1 C_1(\sigma)\delta(\sigma - \sigma'),$$

$$\{C_2(\sigma), C_2(\sigma')\} = -4C_2(\sigma')\partial_1 \delta(\sigma - \sigma') + 2\partial_1 C_2(\sigma)\delta(\sigma - \sigma'),$$

$$\{C_1(\sigma), C_2(\sigma')\} = -4C_1(\sigma')\partial_1 \delta(\sigma - \sigma') + 2\partial_1 C_1(\sigma)\delta(\sigma - \sigma') \tag{3.6}$$

These constraints are first class and generate the Virasoro algebra as expected. We note that this also ensures that taking the time evolution of the constraints of equation (3.4) and (3.5) generates no new constraints.

The equation of motion of $X^M$ is given by

$$\partial_0 X^M = \{X^M, H\} = \delta \Omega^{MN} G_{NP} \partial_1 X^P + \epsilon \partial_1 X^M \tag{3.7}$$

which we may write in matrix form as

$$\partial_0 X = (\epsilon + \delta \Omega G) \partial_1 X \tag{3.8}$$

This in turn implies that

$$\partial_1 X = \frac{1}{\delta^2 - \epsilon^2} (-\epsilon + \delta \Omega G) \partial_0 X \tag{3.9}$$

Introducing the two new variable $\tilde{\gamma}^{\alpha\beta} = \sqrt{-\gamma} \gamma^{\alpha\beta}$, where $\gamma = \det \gamma_{\alpha\beta}$ by setting

$$\delta = \tilde{\gamma}_{00}, \quad \epsilon = -\frac{\tilde{\gamma}^{01}}{\tilde{\gamma}^{00}} \tag{3.10}$$

and substituting into equations (3.8) and (3.9) we find that they are the same as

$$\Omega_{MN} \epsilon^{\alpha\beta} \partial_\beta X^N = \sqrt{-\gamma} \gamma^{\alpha\beta} G_{MN} \partial_\beta X^N \tag{3.11}$$
For example, taking $\alpha = 0$ in this latter equation, bring all the $\partial_1 X$ terms to one side and dividing to by $\tilde{\gamma}^{00}$ we find equation (3.8).

We next show that the constraints of equations (3.4) and (3.5) are equivalent to the condition
\[
\partial_\alpha X^M \Omega_{MN} \partial_\beta X^N = 0
\]
(3.12)
provided one uses the equation of motion of $X^M$. Taking $\alpha = 1 = \beta$ we find the constraint of equation (3.5). Substituting the equation of motion of equation (3.11) with $\alpha = 0$ that is the equation
\[
\partial_1 X = \delta^{-1} \Omega G \partial_0 X - \epsilon \Omega G \partial_1 X
\]
(3.13)
we find, using equation (3.5), that
\[
\partial_1 X^M G_{MN} \partial_1 X^N = 0
\]
(3.14)
Proceeding in this way we find the constraint of equation (2.30), or equivalently equation (2.33), as well as the constraint
\[
\partial_\alpha X^M G_{MN} \partial_\beta X^N = 0
\]
(3.15)
In fact this is not an independent constraint as it follows from that of equation (3.5) using the equation of motion of equation (3.11).

Thus the Hamiltonian system introduced above is equivalent to the motion of equation (2.33) and so we can be confident that it is the correct Hamiltonian system.

4. Quantisation of the SO(D,D) symmetric string

To quantise the SO(D,D) formulation of the string is straightforward. The Poisson brackets of equation (3.3) become the commutators
\[
[X^M(\sigma), \hat{X}^N(\sigma')] = i \Omega^{MN}_\sigma (\sigma - \sigma') \quad \text{or equivalently} \quad [\hat{x}^\mu(\sigma), \hat{y}_\nu(\sigma')] = i \delta^\mu_\nu \theta(\sigma - \sigma')
\]
(4.1)
The Hamiltonian of equation (3.1) and constraints of equations (3.4) and (3.5) now contain operator valued $X^M$’s. We impose the constraints on the wavefunction
\[
\hat{C}_1 \Psi = 0 = \hat{C}_2 \Psi
\]
(4.2)
The Schrödinger equation then states that the wavefunction is independent time as the Hamiltonian on it now vanishes. In fact such a commutator was suggested in [6] on the grounds that the $y_\mu$ coordinates are related to the momenta of the theory with just $x^\mu$ in the linearised theory.

Following the same treatment used to derive the standard uncertainty principle we find that
\[
(\Delta x^\mu)(\Delta y_\nu)^2 = <x|(x^\mu - <x^\mu>)^2|x><y|(y_\nu - <y_\nu>)^2|y>
\]
\[
\geq |<x|(x^\mu - <x^\mu>)(y_\nu - <y_\nu>)|y>|^2 \geq \delta^\mu_\nu \theta(\sigma - \sigma')|<x|y>|^2
\]
(4.3)
Hence, one can not measure both $x^\mu$ and $y^\nu$. This is consistent with the well known observation that when a string is wrapped on a circle only distances down to a minimum radius are observable.

The simplest way to proceed is to choose the operators to be given by

$$\hat{x}^\mu(\sigma) = x^\mu(\sigma) \quad \text{and} \quad \hat{y}_\mu(\sigma) = -i \int_\sigma^\sigma' \frac{d\sigma'}{\delta x^\mu(\sigma')}$$

In this case the wavefunction depends on $x^\mu(\sigma)$ and we arrive at the standard picture of the second quantised bosonic string as studied in [17].

However, we could equally well take the representation

$$\hat{x}^\mu(\sigma) = i \int_\sigma^\sigma' \frac{d\sigma'}{\delta y_\mu(\sigma')} \quad \text{and} \quad \hat{y}_\mu(\sigma) = y_\mu(\sigma)$$

The relation between the two representations is given by

$$\psi[x^\mu(\sigma)] = \langle x^\mu(\sigma) | \psi \rangle = \int \mathcal{D}y_\nu(\sigma') e^{i \int d\sigma'' x^\nu(\sigma'') \partial_1 y_\nu(\sigma'')} \psi[y(\sigma)]$$

Thus although one can use either representation, or any representation that is related by a SO(D,D) rotations, by choosing a representation one makes a choice and breaks the manifest SO(D,D) symmetry. However, this is not an actually breaking of this symmetry as considering all representations on an equal footing preserves SO(D,D). However, to work in a way that manifestly preserves the SO(D,D) symmetry one must keep both $x^\mu$ and $y_\mu$ and use the techniques of non-commutative field theory.

The $E_{11} \otimes s_1$ non-linear realisation, viewed from the IIA perspective, and at lowest level is just the SO(D,D) string. Hence quantising the string dynamics that follows from this non-linear realisation leads to the same results at lowest order. In general we expect the higher level effects to follow the same pattern; the coordinates will obey non-trivial commutation relations and in order to keep the symmetry manifest one must work with a non-commutative field theory. It is likely that in general the set of commuting coordinates is larger than the set of space-time generators; for example in the dimensions lower than ten we would expect the generators of spacetime translations associated with a torus dimensional reduction to commute.

The adoption of just the space-time translations in the non-linear realisation of $E_{11}$ has, in a number of circumstances, worked better than one might expect given that the next coordinate is only one level more than the usual translations. The analysis given here suggests that keeping only the space-time translations preserves more of the $E_{11}$ symmetry than expected. It would certainly be good to understand how much of the symmetry is hidden in this way and how much is automatically encoded by the existence of different representations.

The string resulting from the choice of coordinates of the representation given in equation (4.4) contains as massless fields the graviton, antisymmetric tensor field and tachyon which depend only on $x^\mu$. Apart from the tachyon these are the fields that are contained in a non-linear realisation of ISO(D,D) with local subgroup $SO(D) \otimes SO(D)$. 

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However, for the choice of equation (4.5) we find the same fields but they now depend on $y_\mu$. These two formulations are related by an SO(D,D) rotation. However, a manifestly SO(D,D) invariant formulation requires both $x^\mu$ and $y_\mu$ but, as we have just pointed out, it is not a usual quantum fields theory but a non-commutative field theory.

It would be interesting to repeat this calculations given in this paper for the membrane in eleven dimensions where the lowest level coordinates are $x^a$ and $x^{ab}$. These are likely to obey non-trivial commutation relations whose right hand sides are field dependent.

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Fig 1. The $E_{11}$ Dynkin diagram

| D | G | $Z$ | $Z^a$ | $Z^{a_1\ldots a_2}$ | $Z^{a_1\ldots a_3}$ | $Z^{a_1\ldots a_4}$ | $Z^{a_1\ldots a_5}$ | $Z^{a_1\ldots a_6}$ | $Z^{a_1\ldots a_7}$ |
|---|---|---|---|---|---|---|---|---|---|
| 8 | $SL(3) \otimes SL(2)$ | $(3, 2)$ | $(3, 1)$ | $(1, 2)$ | $(3, 1)$ | $(3, 2)$ | $(1, 3)$ | $(8, 1)$ | $(6, 2)$ |
| 7 | $SL(5)$ | 10 | 5 | 5 | 10 | 24 | 1 | 40 | 15 |
| 6 | $SO(5, 5)$ | 16 | 10 | 16 | 45 | 1 | 144 | 16 | 320 | 126 |
| 5 | $E_6$ | 27 | 27 | 78 | 1 | 351 | 27 | 1728 | 351 |
| 4 | $E_7$ | 56 | 133 | 912 | 56 | 8645 | 1539 | 133 | 1 |
| 3 | $E_8$ | 248 | 3875 | 1 | 147250 | 30380 | 3875 | 248 | 1 |