ON SOME DIFFERENTIAL-GEOMETRIC ASPECTS OF THE TORELLI MAP

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ABSTRACT. In this note we survey recent results on the extrinsic geometry of the Jacobian locus inside $A_g$. We describe the second fundamental form of the Torelli map as a multiplication map, recall the relation between totally geodesic subvarieties and Hodge loci and survey various results related to totally geodesic subvarieties and the Jacobian locus.

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1. INTRODUCTION

1.1. Let $M_g$ denote the moduli space of smooth projective curves of genus $g$ and let $A_g$ denote the moduli space of principally polarized abelian varieties of dimension $g$. The Torelli map $j : M_g \to A_g$ associates to the point $[C] \in M_g$ the moduli point of the jacobian of $C$ with the polarization induced from the cup product. Both $M_g$ and $A_g$ have natural structures of quasi-projective varieties and $j$ is a regular map. By Torelli theorem it is injective.

If one works over the complex numbers (as we do systematically), both $M_g$ and $A_g$ can be provided with the structure of complex analytic orbifold. (See [1, XII, 4] for the main definitions.) This allows to work as if $M_g$ and $A_g$ were smooth. (Another possibility, that for our purposes is equivalent, is to fix level structures.) In the following we will sometimes simplify the terminology by omitting the word ”orbifold”.

The map $j$ is an orbifold map, i.e. it lifts to a holomorphic map of the uniformizers. Oort and Steenbrink [30] proved that the restriction of $j$ to the set of non-hyperelliptic curves is an orbifold immersion.

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Next we recall that $A_g$ has a natural metric. Indeed it is the quotient of the Siegel space $S_g$, which is an irreducible Hermitian symmetric space of the non-compact type, by a properly discontinuous group of isometries. We call the induced metric on $A_g$ the *Siegel metric*.

Summing up, if $M_g^* \subset M_g$ denotes the complement of the hyperelliptic locus, then $j(M_g^*)$ is a complex analytic suborbifold of the Riemannian orbifold $A_g$. It is natural to study the extrinsic geometry of $j(M_g^*)$ inside $A_g$. This study is still largely open and the goal of this note is to discuss some of the results obtained so far. The rough idea behind these results is that $j(M_g^*)$ should be “very curved” inside $A_g$. In other words the way in which $M_g$ sits inside $A_g$ should be “complicated”. This statement is extremely vague, but there are at least three ways to make it precise.

1.2. On the one hand the second fundamental form of the embedding $j : M_g^* \hookrightarrow A_g$ should be highly nondegenerate, i.e. it should most of the time be non-zero. This is far from understood. But there are some results on the second fundamental form.

In §2 we explain in some detail how the second fundamental form can be interpreted as a multiplication map. This is based on the fundamental work of Colombo, Pirola and Tortora [9].

1.3. On the other hand one might look at totally geodesic subvarieties of $A_g$ and ask whether $j(M_g^*)$ contains some of them. Here the expectation is that $j(M_g^*)$ should contain very few totally geodesic subvarieties. The analogous statement for a surface in 3-space is that the surface contains no line. In the case of the Jacobian locus this expectation agrees with a rather famous conjecture, the Coleman-Oort conjecture, saying the $j(M_g)$ should contain no Hodge locus of $A_g$.

In §3 we discuss these kind of problems. First of all we prove that Hodge loci of $A_g$ are totally geodesic subvarieties. This is well-known, but it is hard to find an elementary exposition.

Next we recall some non-existence results for totally geodesic subvarieties in $j(M_g)$ based on the second fundamental form. On the other hand we explain that in low genus there are some interesting examples of totally geodesic subvarieties generically contained in $A_g$.

1.4. Using totally geodesic subvarieties in a different way we get to the third way of making precise the fact that $j(M_g)$ is very curved inside $A_g$. Consider a submanifold $M$ of a Riemannian manifold $A$. One can look at the intersection of $M$ with totally geodesic submanifolds $Z \subset A$. The fact that $M \cap Z$ has high codimension in $M$ for any $Z$ is our third way to express the complexity of the embedding $M \hookrightarrow A$. We explain this at the end of §3 and we describe a recent result saying
that in the case of the embedding \( j : \mathcal{M}_g^* \subset A_g \) the intersection of \( j(\mathcal{M}_g^*) \) with any totally geodesic subvariety has codimension at least 2.

1.5. This note is dedicated to the memory of my deeply esteemed teacher and friend Paolo de Bartolomeis. While writing it I thought of Paolo so many times! I was led to recall the glorious times when I was a student and I listened to Paolo’s beautiful lectures. I learned from him so many basic concepts! Lie groups, Lie algebras, symmetric spaces, complex structures, symplectic forms, totally geodesic submanifolds and so on, just to mention the ones that are used continuously in this note.

Paolo was really a friend. He had a wonderful sense of humour and I liked that a lot. I was always happy when I was going to meet him at conferences, since talking with him was always very interesting and extremely pleasant. Our last contact was by email. I had just watched for the first time a movie that Paolo liked a lot. I wrote him to tell that I also liked it a lot. His reply was great! Paolo was such a nice guy! I miss him a lot.

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2. The second fundamental form

2.1. If \( C \) is a non-hyperelliptic curve and \( x = [C] \in \mathcal{M}_g \), then \( dj_x : T_x \mathcal{M}_g \to T_{j(x)} A_g \) is injective and we have an exact sequence

\[
0 \to T_x \mathcal{M}_g \xrightarrow{dj_x} T_{j(x)} A_g \xrightarrow{\pi} N_x \to 0,
\]

where \( N \) denotes the normal bundle to \( j(\mathcal{M}_g^*) \subset A_g \). The normal bundle in the Riemannian sense, that is as orthogonal complement, can be identified with the quotient bundle which is holomorphic. Recall that \( T_x \mathcal{M}_g \cong H^1(C, T_C) \) and \( T_x \mathcal{M}_g \cong H^0(C, 2K_C) \). If \( A \) is a principally polarized abelian variety and \( y = [A] \in A_g \), then \( T_y A_g \cong S^2 H^0(A, T_A) \). If \( y = j(x) \), i.e. \( A = H^0(C, K_C)^*/H_1(C, \mathbb{Z}) \), then \( H^0(A, T_A) = H^0(C, K_C) \) and \( T_{j(x)} A_g \cong S^2 H^0(C, K_C)^* \). The transpose of the map \( dj_x \) is the multiplication map

\[
m : T_{j(x)} A_g = S^2 H^0(C, K_C) \to H^0(C, 2K_C) \cong T_x^* \mathcal{M}_g.
\]

Therefore \( N_x^* = \text{Ann}(\text{im} \, j_x) \) is identified with \( \ker m \) and the dual of the above sequence is the following one:

\[
0 \to I_x(K_C) := \ker m \to S^2 H^0(C, K_C) \xrightarrow{m} H^0(C, 2K_C) \to 0.
\]

(See [3] for more details.). Denote by

\[
\Pi_x : S^2 T_x \mathcal{M}_g = S^2 H^1(C, T_C) \to N_x
\]
the second fundamental form of the Torelli embedding with respect to the Siegel metric on \( A_g \). We can identify \( \Pi_x \) with a map

\[
(2.1) \quad \rho_x : N^*_x = I_2(K_C) \to S^2H^0(C, 2K_C).
\]

We will use the two symbols to distinguish the different interpretations, but they are the same object.

Our goal in this section is to interpret the map \( \rho \) in (2.1) as a multiplication map between spaces of sections on \( S := C \times C \).

We start by explaining in some detail how to reinterpret domain and target of \( \rho \) as spaces of sections of appropriate bundles on \( S \).

Call \( p \) and \( q \) the two projections:

\[
\begin{align*}
S & \xrightarrow{q} C \\
& \downarrow p \\
C & \xrightarrow{q} C
\end{align*}
\]

Given line bundles \( L \to C \) and \( M \to C \), set

\[
L \boxtimes M := p^*L \otimes q^*M \to S.
\]

The map

\[
H^0(C, L) \otimes H^0(C, M) \to H^0(S, L \boxtimes M), \quad s \otimes t \mapsto p^*s \otimes q^*t,
\]

is an isomorphism. On \( S \) we have the automorphism

\[
\sigma : S \to S, \quad \sigma(x, y) = (y, x).
\]

For any line bundle \( L \to C \), \( \sigma \) lifts to \( L \boxtimes L \) as follows:

\[
\tilde{\sigma} : (L \boxtimes L)_{(x, y)} = L_x \otimes L_y \to (L \boxtimes L)_{\sigma(x, y)} = L_y \otimes L_x
\]

is simply the map

\[
(2.2) \quad \tilde{\sigma}(u \otimes v) = v \otimes u, \quad u \in L_x, v \in L_y.
\]

Consider the special case \( L = K_C \). We have a canonical isomorphism \( K_S \cong K_C \boxtimes K_C \) given by the map

\[
f : K_C \boxtimes K_C \to K_S, \quad f(\alpha \otimes \beta) = p^*\alpha \wedge q^*\beta, \quad \alpha \in T^*_x C, \quad \beta \in T^*_y C.
\]

On \( K_S \) we have two involutions lifting \( \sigma \). One is simply the pull-back:

\[
\sigma^* : K_S \to K_S. \quad \text{Since } p\sigma = q, \text{ we have}
\]

\[
\sigma^*(p^*\alpha \wedge q^*\beta) = -p^*\beta \wedge q^*\alpha.
\]

The other lift is \( f\tilde{\sigma}f^{-1} : K_S \to K_S \) and satisfies

\[
f\tilde{\sigma}f^{-1}(p^*\alpha \wedge q^*\beta) = f\tilde{\sigma}(\alpha \otimes \beta) = f(\beta \otimes \alpha) = p^*\beta \wedge q^*\alpha.
\]

Hence

\[
(2.3) \quad \sigma^* = -f\tilde{\sigma}f^{-1}.
\]

Consider now the isomorphism

\[
f : H^0(C, K_C) \otimes H^0(C, K_C) \xrightarrow{\cong} H^0(S, K_S),
\]
induced by \( f \) (and that we still denote by \( f \)). It follows from (2.3) that

\[
H^0(S, K_S)^+ := \{ \alpha \in H^0(S, K_S) : \sigma^*\alpha = \alpha \} = f(\Lambda^2 H^0(C, K_C)),
\]

\[
H^0(S, K_S)^- := \{ \alpha \in H^0(S, K_S) : \sigma^*\alpha = -\alpha \} = f(S^2 H^0(C, K_C)).
\]

Let \( \Delta \subset S \) denote the diagonal. It is a reduced divisor in \( S \). We claim that

\[
f(I_2(K_C)) = \{ \alpha \in H^0(S, K_S)^- : \alpha|_\Delta = 0 \}.
\]

To see this fix a coordinate system \( z : U \subset C \to \mathbb{C} \). From this we get a chart \((z_1, z_2) : U' := U \times U \subset S \to \mathbb{C}^2 \) by setting \( z_1 = zop \) and \( z_2 = zoq \). Further set \( x := (z_1 + z_2)/2 \) and \( y := (z_1 - z_2)/2 \). Then \((x, y)\) is another coordinate system on \( U' \), \( \sigma(x, y) = (x, -y) \) and \( \Delta \cap U' = \{ y = 0 \} \). Any \( \alpha \in H^0(S, K_S) \) has a local expression \( \alpha = \varphi(x, y)dx \wedge dy \) on \( U' \) and

\[
\sigma^*\alpha = -\varphi(x, -y)dx \wedge dy.
\]

Thus \( \alpha \in H^0(S, K_S)^- \) iff \( \varphi(x, -y) = \varphi(x, y) \) i.e. \( \varphi \) is an even function of \( y \). In this case for any odd \( m \) we have

\[
\frac{\partial^m \varphi}{\partial y^m}(x, 0) \equiv 0.
\]

It follows that if \( \alpha \in H^0(S, K_S)^- \) vanishes along \( \Delta \) it vanishes there to second order. Hence

\[
(2.4) \quad I_2(K_C) = H^0(S, K_S(-2\Delta))^-.
\]

We can apply the same analysis to tensor powers of \( K_C \). Using the same notation as above, we see that for any \( n \) there is an isomorphism

\[
f_n : K_C^n \otimes K_C^n \xrightarrow{\sim} K_S^n,
\]

\[
f_n(\alpha^n \otimes \beta^n) = (p^*\alpha \wedge q^*\beta)^n, \quad \alpha \in T_x^*C, \beta \in T_y^*C.
\]

Moreover for any \( n \) there is a lifting \( \sigma^* \) of \( \sigma \) to \( K_S^n \). By the same computation as above we get

\[
\sigma^* = (-1)^n f_n \sigma f_n^{-1}.
\]

Indeed for \( \alpha \in T_x^*C^*, \beta \in T_y^*C^* \), we have

\[
\tilde{\sigma}(\alpha^n \otimes \beta^n) = \beta^n \otimes \alpha^n,
\]

as in (2.2). Thus

\[
f_n \tilde{\sigma} f_n^{-1}(p^*\alpha \wedge q^*\beta)^n = f_n \sigma(\alpha^n \otimes \beta^n) = f_n(\beta^n \otimes \alpha^n) =
\]

\[
= (p^*\beta \wedge q^*\alpha)^n = \sigma^*(q^*\beta \wedge p^*\alpha)^n) = (-1)^n \sigma^*(p^*\alpha \wedge q^*\beta)^n).
\]

Thus

\[
(2.5) \quad H^0(S, K_S^2)^+ := \{ \alpha \in H^0(S, K_S) : \sigma^*\alpha = \alpha \} = f_2(S^2 H^0(C, K_C)).
\]
Finally consider the line bundle $K_S(2\Delta) \to S$. Since $\Delta$ is $\sigma$-invariant, there is a lift of $\sigma$ to $K_S(2\Delta)$, that we still denote by $\sigma^*$. We set
\[ H^0(S, K_S(2\Delta))^\pm = \{ \alpha \in H^0(S, K_S(2\Delta)) : \sigma^*\alpha = \pm \alpha \}. \]

We are finally in the position to state the main theorem about the second fundamental form.

**Theorem 2.2.** If $C$ is not hyperelliptic of genus at least 4, then there exists a section $\eta \in H^0(S, K_S(2\Delta))^+$ such that, using (2.4) and (2.5), the second fundamental form (2.1) for $x = [C]$ gets identified with the multiplication map by $\eta$:
\[ \rho_x : H^0(S, K_S(-2\Delta))^+ \to H^0(S, 2K_S)^+, \quad \rho_x(\alpha) = \eta \cdot \alpha. \]

2.3. The proof of the Theorem is rather complicated and deep. The most important part is in [9]. The reduction to a multiplication was achieved in [11, 14]. Here we will only give some idea about the construction of $\eta$.

Fix a curve $C$ of genus $g$ and a point $x \in C$. The space $H^0(C, K_C(2x))$ is contained in the space of closed 1-forms on $C \setminus \{x\}$. The induced map $H^0(C, K_C(2x)) \to H^1(C \setminus \{x\}, \mathbb{C})$ is injective as soon as $g > 0$. By Mayer-Vietoris $H^1(C, \mathbb{C}) \cong H^1(C \setminus \{x\}, \mathbb{C})$. Hence we get an injection $H^0(C, K_C(2x)) \hookrightarrow H^1(C, \mathbb{C})$. We identify $H^0(C, K_C(2x))$ with its image inside $H^1(C, \mathbb{C})$. The space $H^{1,0}(C)$ is contained in $H^0(C, K_C(2x))$. Since $h^0(C, K_C(2x)) = g + 1$, the intersection $H^0(C, K_C(2x)) \cap H^{0,1}(C)$ is a 1-dimensional. If $u \in T_xC$ we choose a local coordinate at $x$ such that $u = \partial/\partial z$. Then there is a unique $\varphi_u \in H^0(C, K_C(2x)) \cap H^{0,1}$ such that locally $\varphi_u = f(z)dz$ with $f(z) = 1/z^2 + h(z)$ and $h$ is holomorphic. Note that $\varphi_u$ only depends on $u$, not on the coordinate $z$. (If $u = 0$, set $\varphi_u = 0$.) Moreover the map $u \mapsto \varphi_u$ is linear. Consider the vector bundle $V := p_*(q^*K_C(2\Delta))$ over $C$. We have $V_x = H^0(C, K_C(2x))$. The maps $T_xC \to V_x$, $u \mapsto \varphi_u$ give a holomorphic section of $K_C \otimes V = p_*(p^*K_C \otimes q^*K_C(2\Delta)) = p_*(K_S(2\Delta))$. [6] Prop. 3.4. But global sections of $p_*(K_S(2\Delta))$ are the same as global sections of $K_S(2\Delta)$. Thus we get a global section of $K_S(2\Delta)$ and this is the form $\eta$ in the theorem!

The definition of $\eta$ clearly involves Hodge theory. It is not clear how to control the behaviour of the form $\eta$ at points in $S - \Delta$. On the other hand the behaviour of $\eta$ along the diagonal is closely related to the second Wahl map, which is an algebraic object.

3. Totally geodesic subvarieties and Hodge loci in $A_g$

In this § we are interested in totally geodesic submanifolds of Siegel space. The starting point is the following characterization of totally geodesic submanifolds in symmetric spaces. (See [12] p. 19 for a proof.)
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Theorem 3.1. Let $X$ be a symmetric space and let $X' \subset X$ be a closed connected submanifold. Set $G := \text{Isom}(X)^0$. Fix a point $o \in X'$, set $K := G_o$ and let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ be the Cartan decomposition. The following conditions are equivalent.

1. $X'$ is totally geodesic.
2. For any $x \in X'$ we have $s_x(X') = X'$.
3. $X' = \exp_o \mathfrak{m}$, where $\mathfrak{m} \subset \mathfrak{p}$ is a Lie triple system i.e. $[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}] \subset \mathfrak{m}$.

3.2. We recall the definition and the main properties of Siegel space. Fix a real symplectic vector space $(V, \omega)$ of dimension $2g$. If $J$ is a complex structure on $V$, we let $V_J$ denote the complex vector space obtained using $V$ as underlying real vector space and letting multiplication by $i$ act as $J$. We have $J^*\omega = \omega$ if and only if the bilinear form $g_J := \omega(\cdot, J\cdot)$ is symmetric. Set

\[ \mathcal{G}(V, \omega) := \{ J \in \text{End} V : J^2 = -\text{id}_V, J^*\omega = \omega, \quad g_J \text{ is positive definite} \}. \]

If $J \in \mathcal{G}(V, \omega)$, then $H_J(x, y) := g_J(x, y) - i\omega(x, y)$ is a Hermitian product on $V_J$. The group $\text{Sp}(V, \omega)$ acts on $\mathcal{G}(V, \omega)$ by conjugation: $a \cdot J := aJa^{-1}$. We claim that this action is transitive. Indeed let $J, J' \in \mathcal{G}(V, \omega)$. Fix an $H_J$-unitary basis $\{u_i\}$ of $V_J$. Similarly let $\{u'_i\}$ be an $H_{J'}$-unitary basis of $V_{J'}$. Setting $b(u_i) := u'_i$ we get a complex isometry $b : V_J \to V_{J'}$, so $b^*H_{J'} = H_J$. Let $a \in \text{End} V$ denote the underlying real isomorphism. It satisfies $a^*\omega = -a^*(\text{Im} H_J) = -\text{Im} H_J = \omega$ and $aJ = J'a$. Thus $a \in \text{Sp}(V, \omega)$ and $a \cdot J = J'$. This proves the claim.

It is easy to check that if $a \in \text{Sp}(V, \omega)$, then $a \cdot J = J$ if and only if $a^*H_J = H_J$. It follows that the stabilizer of $J$ in $\text{Sp}(V, \omega)$ is the unitary group $U(V_J, H_J)$. For $L \in \text{End} V$, let $LT_J$ be the transposed operator with respect to the scalar product $g_J$. We claim that for $a \in \text{Sp}(V, \omega)$

\[ a^{T_J} = -Ja^{-1}J = Ja^{-1}J^{-1}. \]

This follows easily using $\omega = g_J(J\cdot, \cdot)$ and $a^*\omega = \omega$. Thus

\[ \theta_J(a) := (a^{-1})^{T_J} = JaJ^{-1}, \]

is a Cartan involution on $\text{Sp}(V, \omega)$. At the Lie algebra level $\theta_J = \text{Ad} J$.

The 1-eigenspace of $\theta_J$ on $\mathfrak{sp}(V, \omega)$ is $\mathfrak{u}(V_J, H_J)$. This show that $\mathcal{G}(V, \omega)$ is a symmetric space, see [20, p. 209].

3.3. Up to now we only used the symplectic vector space $(V, \omega)$. Now we add an extra structure, namely we fix a lattice $\Lambda \subset V$ and we assume that $\omega(\Lambda \times \Lambda) \subset \mathbb{Z}$. In appropriate bases of $\Lambda$ the matrix of $\omega$ has the form

\[ \begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix} \quad \text{with} \quad T = \text{diag}(d_1, \ldots, d_g) \]
and $0 < d_1 |d_2| \cdots |d_g$, see e.g. [16, p. 391]. The vector $(d_1, \ldots, d_g)$ is called the \textit{type} of the form $\omega$. Our final assumption is that $\omega$ has type $(1, \ldots, 1)$. The group $\Gamma := \text{Sp}(\Lambda, \omega)$ is a discrete subgroup of $\text{Sp}(V, \omega)$ and acts properly discontinuously on $\mathfrak{G}(V, \omega)$. Set

$$A_g := \Gamma \backslash \mathfrak{G}(V, \omega).$$

By a theorem of H. Cartan this quotient is a normal complex analytic space. It has also a natural structure of complex analytic orbifold. It can be shown that it is a quasiprojective variety. Since $\Gamma$ is a group of isometries, the symmetric Riemannian structure on $\mathfrak{G}(V, \omega)$ induces a locally symmetric orbifold metric on $A_g$, which we call \textit{Siegel} metric.

3.4. Once the lattice $\Lambda \subset V$ has been fixed there is a natural (and tautological) $\mathbb{Z}$-variation of Hodge structure on $\mathfrak{G}(V, \omega)$: take the constant lattice $\Lambda$ and for $J \in \mathfrak{G}(V, \omega)$ consider the Hodge structure $(\Lambda, V^{1,0}_J)$. Since this Hodge structure only depends on $J$, we denote it simply by $J$. This variation of Hodge structure descends to an (orbifold) variation over $A_g$. We are interested in the Hodge loci of this variation on $A_g$. For the main definitions and facts regarding Hodge loci and Mumford-Tate groups we refer to [28, 33, 36]. Here we recall what we need only in the case of weight 1.

Given $J \in \mathfrak{G}(V, \omega)$ we define a representation $\rho_J : \mathbb{C}^* \to \text{GL}(V)$, setting $\rho_J(z)v = z \cdot v$ for $v \in H^{1,0}$ and $\rho_J(z)v = \bar{z} \cdot v$ for $v \in H^{0,1}$. The \textit{Mumford-Tate group} of $J$, denoted $\text{MT}(J)$, is the smallest algebraic subgroup of $\text{GL}(V)$ defined over $\mathbb{Q}$, whose real points contain $\text{im} \rho_J$. The main property of the Mumford-Tate group is the following: given multi-indices $d, e \in \mathbb{N}^m$ set

$$T_{d,e}^d (\Lambda) := \bigoplus_{j=1}^m \Lambda_\mathbb{Q}^{d_j} \otimes (\Lambda_\mathbb{Q}^*)^{e_j}.$$

This space is a pure Hodge structure. A vector $v \in T_{d,e}^d (\Lambda)$ is invariant by $\text{MT}(H)$ if and only if it is a Hodge class of type $(0,0)$, i.e a rational vector of type $(0,0)$. Moreover the Mumford-Tate group is characterized by this property in the following sense: if $G \subset \text{GL}(H)$ is the subgroup containing the elements that fix the Hodge classes of $T_{d,e}^d$ for any $d$ and $e$, then $G = \text{MT}(H)$, see e.g. [33]. The \textit{Hodge group} or \textit{special Mumford-Tate group}, denoted $\text{Hg}(J)$ is the smallest algebraic subgroup of $\text{GL}(V)$ defined over $\mathbb{Q}$, whose real points contain $\rho_J(S^1)$. If $D$ denotes the subgroup of diagonal matrices in $\text{GL}(V)$, then $\text{MT}(J) = D \cdot \text{Hg}(J)$.

**Lemma 3.5.** For any $J \in \mathfrak{G}(V, \omega)$, the following properties hold.

1. $J \in \text{Hg}(J) \cap \text{Lie Hg}(J)$.
2. $\text{Hg}(J)$ is invariant by $\theta_J$.
3. The stabilizer of $J$ in $\text{Hg}(J)$ coincides with the centralizer of $J$ in $\text{Hg}(J)$ and it is a maximal compact subgroup of $\text{Hg}(J)$.
((4) The orbit $Hg(J) \cdot J \subset \mathcal{G}(V, \omega)$ is a complex totally geodesic submanifold of $\mathcal{G}(V, \omega)$. With the induced metric it is a Hermitian symmetric space of the non-compact type.

Proof. $J \in \rho_J(S^1) \subset Hg(J)$. Moreover $\rho_J(e^{it}) = \cos t \cdot I + \sin t \cdot J$. Thus $J = d\rho(0)(i) \in \text{Lie } Hg(J)$. This proves (1). To prove (2) use (3.1) and the fact that $J \in Hg(J)$: for $a \in Hg(J)$, $\theta_J(a) = Ja^{-1}J^{-1} \in Hg(J)$. The restriction of $\theta_J$ to $Hg(J)$ is a Cartan involution on $Hg(J)$. If $a \in \text{Sp}(V, \omega)$, then $\theta_J(a) = a$ iff $aJ = Ja$ iff $a \cdot J = J$. Thus the stabilizer of $J$ in $Hg(J)$ is the fixed set of $\theta_J$ in $Hg(J)$, which is a maximal compact subgroup. This proves (3). Set $g := \text{Lie } Hg(J)$ and let $g = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition corresponding to $\theta_J$. Then $\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. Thus $\mathfrak{p}$ is a Lie triple system and $Hg(J) \cdot J = \exp_J p$ is a totally geodesic submanifold by Theorem 3.1. It is complex since ad$J$ preserves $\mathfrak{p}$. □

3.6. The Hodge loci of the natural variation of Hodge structure on $A_g$ are defined as follows. Given $d, e$ and $t \in T^{d,e}(\Lambda_Q)$, set

$$Y(t) := \{J \in \mathcal{G}(V, \omega) : t \in T^{d,e}(V_J)^{0,0}\}.$$

$Y(t)$ is an analytic subset of $\mathcal{G}(V, \omega)$, see [35, p. 404]. If $t_1, \ldots, t_r$ are rational vectors in various tensor constructions, set $Y(t_1, \ldots, t_r) := Y(t_1) \cap \cdots \cap Y(t_r)$. We call $Y(t_1, \ldots, t_r)$ proper if $Y(t_1, \ldots, t_r) \neq \mathcal{G}(V, \omega)$. Let

$$\pi : \mathcal{G}(V, \omega) \longrightarrow A_g,$$

denote the canonical projection. A Hodge locus of $A_g$ is an irreducible component of a proper $\pi(Y(t_1, \ldots, t_r)) \subset A_g$. It is easy to check that Hodge loci are exactly the subsets of the form $\pi(Z)$, where $Z$ is an irreducible component of some proper $Y(t_1, \ldots, t_r)$. The irreducible components of proper subsets $Y(t_1, \ldots, t_r)$ form a countable family $\{Z_i\}_{i \in \mathbb{N}}$ of proper subsets of $\mathcal{G}(V, \omega)$. Set further

$$Z_i^0 := Z_i \setminus \bigcup_{j : Z_j \subset Z_i} Z_j.$$

Theorem 3.7. For $J \in Z_i^0$ we have $Z_i = Hg(J) \cdot J$. In particular $Z_i$ is a totally geodesic submanifold of $\mathcal{G}(V, \omega)$.

Proof. We claim that the set of Hodge classes is constant on $Z_i^0$. Indeed let $x, y \in Z_i^0$ be distinct points. Assume that $Z_i$ is a component of $Y(t_1, \ldots, t_r)$ and assume by contradiction that there is $t_{r+1}$ that is a Hodge class at $x$, but not at $y$. Let $Z'$ be the irreducible component of $Y(t_1, \ldots, t_r, t_{r+1})$ containing $x$. Then $y \notin Z'$, so $Z'$ is among the $Z_j$'s with $Z_i \nsubseteq Z_j$. But then $x \notin Z_i^0$. This proves the claim. By the property mentioned in (3.3) we conclude that the Mumford-Tate and
Hodge groups are constant on $Z_i^0$. Fix $J_0 \in Z_i^0$ and set $G := Hg(J_0)$. We just proved that

$$Z_i^0 \subset Y := \{ J \in \mathcal{G}(V, \omega) : Hg(J) = G \}.$$  

Set $K := G_{J_0}$ and $A := Z(K)$. We claim that $A \cap \mathcal{G}(V, \omega)$ is a finite set. Indeed $A$ is abelian and acts unitarily and faithfully on $(V_{J_0}, H_{J_0})$. Hence $V_{J_0}$ splits in one-dimensional subrepresentations $V_i$ on which $A$ acts by a character $\chi_i$. But if $J \in A \cap \mathcal{G}(V, \omega)$, then $J^2 = -I$, so $\chi_i(J) = \pm i$ for any $i$. Since the representation is faithful this shows that $A \cap \mathcal{G}(V, \omega)$ is finite.

Since $Z_i$ is irreducible, $Z_i^0$ is connected. Let $Y'$ be the connected component of $Y$ that contains $Z_i^0$. We claim that

$$Y' \subset G \cdot J_0.$$  

Indeed let $J$ be a point in $Y'$. Then $Hg(J) = G$. So $G$ is $\theta_J$-invariant and the stabilizer $G_J$ is a maximal compact subgroup of $G$. Since $G$ is connected there is $a \in G$ such that $a^{-1}G_J a = K$. Moreover $J$ belongs to the center of $G_J$, hence $a^{-1} J a$ belongs to the center of $K$. This shows that $a^{-1}J a \in A \cap \mathcal{G}(V, \omega)$, which is a finite set. Since $Y'$ is connected we have necessarily $a^{-1} J a = J_0$, i.e. $J = a \cdot J_0$. This proves (3.2).

Finally we claim that $G \cdot J_0 \subset Z_i$. Assume that $J = a \cdot J_0$ with $a \in G$. Then $J = Ad(a)(J_0) \in g$, so $\rho_J(e^{it}) \in G$ for any $t$. Hence $Hg(J) \subset G$. By assumption $Z_i$ is an irreducible component of some proper subset $Y(t_1, \ldots, t_r)$. Then by (3.4) we have

$$MT(J_0) = \{ a \in GL(A_\mathbb{Q}) : a \cdot t_j = t_j \text{ for } j = 1, \ldots, r \}.$$  

Since $Hg(J) \subset G$, $MT(J) \subset MT(J_0)$, so $J \in Y(t_1, \ldots, t_r)$. We have proved that $G \cdot J_0 \subset Y(t_1, \ldots, t_r)$. Since $G \cdot J_0$ is irreducible, we have in fact $G \cdot J_0 \subset Z_i$.

We have proved the inclusions

$$Z_i^0 \subset Y' \subset G \cdot J_0 \subset Z_i.$$  

Since $G \cdot J_0$ is a closed subset of $\mathcal{G}(V, \omega)$ and $\overline{Z_i} = Z_i$, we conclude that $G \cdot J_0 = Z_i$ as desired. The last statement follows from Lemma 3.5 (4).

Definition 3.8. A totally geodesic subvariety of $A_g$ is a closed algebraic subvariety $Z \subset A_g$, such that $Z = \pi(X)$ for some totally geodesic submanifold $X \subset \mathcal{G}(V, \omega)$.

Corollary 3.9. Hodge loci of $A_g$ are totally geodesic subvarieties.

Proof. Let $W \subset A_g$ be a Hodge locus. By the theorem of Cattani-Deligne-Kaplan [2] $W$ is a closed algebraic subset of $A_g$. As noticed in 3.6 $W = \pi(Z_i)$ for some $Z_i \subset \mathcal{G}(V, \omega)$ for some irreducible component $Z_i$. By Theorem 3.7 $Z_i$ is a totally geodesic submanifold of $\mathcal{G}(V, \omega)$. □
Hodge loci of $A_g$ are also called special subvarieties or Shimura subvarieties. A CM point of $A_g$ is by definition a moduli point $[A]$, where $A$ is an abelian variety with complex multiplication, see e.g. [29]. This condition is of arithmetic nature. Shimura varieties always contain CM points. This condition in fact characterises them among totally geodesic subvariety, see [29, 26]. For this reason Shimura varieties play a prominent role in arithmetic algebraic geometry. We say that a subvariety $Z \subset A_g$ is generically contained in $j(M_g)$, if $Z \subset j(M_g)$ and $Z \cap j(M_g) \neq \emptyset$. The following conjecture is rather important in arithmetic algebraic geometry.

**Conjecture 3.10** (Coleman-Oort). For large $g$ there is no Shimura variety $Z \subset A_g$ generically contained in $j(M_g)$.

3.11. Using the results on the second fundamental form one can get some constraints on the existence of totally geodesic subvarieties of $A_g$ contained in $M_g$. Since these methods are of local nature, the results apply to analytic germs of such subvarieties.

**Theorem 3.12.** Assume that $C$ is a $k$-gonal curve of genus $g$ with $g \geq 4$ and $k \geq 3$. Let $Y$ be a germ of a totally geodesic subvariety of $A_g$ which is contained in the Jacobian locus and passes through $j([C])$. Then $\dim(Y) \leq 2g + k - 4$.

This immediately yields a bound which only depends on $g$.

**Theorem 3.13.** If $g \geq 4$ and $Y$ is a germ of a totally geodesic subvariety of $A_g$ contained in the Jacobian locus, then $\dim Y \leq \frac{5}{2}(g - 1)$.

Thus the existence of totally geodesic subvarieties (and in particular of Shimura varieties) of very large dimension is excluded. This agrees with the Coleman-Oort conjecture. One in fact expects a much better bound to hold than the one in the previous theorem. But up to now that is the best known.

3.14. The Coleman-Oort conjecture precludes the existence of Shimura varieties generically contained in the Jacobian locus for large genus. But for low genus, namely for $g \leq 7$, one can construct examples of totally geodesic subvarieties contained in $M_g$. Most of these examples are constructed using families of Galois covers of $\mathbb{P}^1$ and are in fact Shimura varieties. The first examples obtained in this way were cyclic covers, see e.g. [10, 27, 32]. A complete list of the Shimura varieties that can be obtained using cyclic covers has been given by Moonen [27] using deep results in positive characteristic. It would be interesting to have a simple differential-geometric proof of this result. Some results in that direction are contained in [6], but at the moment one there is no proof of Moonen’s result using differential geometry. Other examples of Shimura varieties generically contained in the Jacobian locus were
later constructed using non-cyclic Galois covers of $\mathbb{P}^1$, see [28] and [13]. Finally some examples were gotten using Galois cover of elliptic curves, see [14]. This paper studies in particular a 3-dimensional Shimura variety generically contained in $\mathcal{M}_4$. This variety was first constructed by Pirola [31] to disprove a conjecture of Xiao. The same variety has been studied in [17], where it is shown that it is fibered in totally geodesic curves. Since CM points are countable, only countably many fibres are Shimura varieties. Therefore most of these fibres are totally geodesic curves that are not Shimura. Obviously they are generically contained in the Jacobian locus.

3.15. Other works studying totally geodesic subvarieties in the Jacobian locus include [34, 19, 11, 21, 3, 22, 23, 18, 25, 15]. The papers [8] and [5] consider the corresponding problem for the Prym locus instead of the Jacobian locus.

3.16. As mentioned in the introduction, the idea behind all the results we have recalled is that the way in which $\mathcal{M}_g$ sits inside $\mathbb{A}_g$ should be rather “complicated”. Here we wish to make this statement more precise in the third way sketched in 1.4. Consider a Riemannian manifold $A$ and a submanifold $M \subset A$. If the manifold $A$ has a lot of totally geodesic submanifolds (and this is the case for symmetric spaces) the study of the intersections $M \cap Z$, where $Z$ is a totally geodesic submanifold of $A$, gives information on $M$. The first question one asks in this setting is whether there is a totally geodesic $Z$ such that $M \cap Z = M$, i.e. $M \subset Z$. If this does not happen one says that $M$ is full. In Euclidean space totally geodesic submanifolds are affine subspaces, so being full means that $M \subset \mathbb{R}^n$ is not contained in a hyperplane. If $M$ is full, then for any totally geodesic $Z$ the intersection $M \cap Z$ is a possibly singular proper submanifold of $M$. One might consider the totally geodesic submanifolds of $A$ as analogues of affine linear subspaces. If the codimension of $M \cap Z$ is always $\geq k$, not only $M$, but also all its submanifolds of codimension $> k$, are full. This is a measure of the complexity of $M$ in $A$, at least when $A$ has a lot of totally geodesic submanifolds.

The following result was obtained recently in [7].

**Theorem 3.17.** If $g \geq 3$ and $Y \subset \mathcal{M}_g$ is a divisor, then $j(Y)$ is full in $\mathbb{A}_g$.

This shows that in the case of the embedding $j : \mathcal{M}_g^* \hookrightarrow \mathbb{A}_g$ $k \geq 2$, i.e. for every totally geodesic subvariety $\mathcal{M}_g \cap Z$ has codimension at least 2 in $\mathcal{M}_g$. The proof is based on induction on the genus. The case $g = 3$ follows from the fact that $\mathbb{G}_3$ contains no totally geodesic divisor. The inductive step depends on algebro-geometric techniques developed in
and on simple Lie theoretic arguments. It is an interesting problem to get better estimates for $k$.

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