Modeling of the stability of thin composite plates based on an asymptotic theory

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Abstract. The purpose of this work is the construction of the theory of stability of thin plates based on the asymptotic analysis of the general three-dimensional equations of the theory of elasticity and three dimensional stability theory, without introducing any assumptions regarding the nature of displacements and stresses. According to the proposed theory of stability calculation was made straight plate in uniaxial compression.

Introduction
The purpose of this operation is to construct stability theory on thin plates based common three-dimensional equations of the theory of elasticity and three dimensional stability theory, without introducing any assumptions regarding the nature of displacements and stresses, based only on analysis of basic equations asymptotic decompositions of small geometric parameter representing the ratio of the thickness of the plate to its length.

Problem statements for the base and varied states
According to the three-dimensional stability [1,2] theory is considered primary and varied plate state under the action of loads. The values for the main state of stress tensor \( \sigma^0 \), tensor of small deformations \( \varepsilon^0 \), vector movements \( u^0 \) in base (steady) state have a three-dimensional object of the linear elasticity theory [19]:

\[
\nabla \cdot \sigma^0 = 0, \quad \sigma^0 = C \varepsilon^0, \quad \varepsilon^0 = \frac{1}{2} (\nabla \otimes u + \nabla \otimes u^T),
\]

\[
\n \left|_{\Sigma_0} \right. \quad n \cdot \sigma^0 = S^0, \quad \left|_{\Sigma_0} \right. \quad n \cdot \varepsilon^0 = -\vec{p}_x n, \quad \left|_{\Sigma_0} \right. \quad u = u^0.
\]

For varied state have the following problem of the three-dimensional stability theory [1,2]:

\[
\nabla \cdot (\sigma - \sigma^0 \cdot (B \cdot \tau)) = 0,
\]

\[
\sigma = d \cdot C \cdot \varepsilon(w), \quad \varepsilon(w) = \frac{1}{2} (\nabla \otimes w + \nabla \otimes w^T),
\]

\[
\B = \nabla \otimes \omega, \quad \omega = \frac{1}{2} \tau \cdot \Omega(w), \quad \Omega(w) = \frac{1}{2} (\nabla \otimes w - \nabla \otimes w^T),
\]

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Here are the stress tensor, tensor of small deformations, vector displacements in varied state, normal vector, vector skew, symmetric rotation tensor, tensor Lewy-Chivites [3].

We will assume that the plate is thin and small parameter \( \kappa = h/L \ll 1 \) can be introduced, as the ratio of the total plate thickness \( h \) to the dimension \( L \) of the entire plate (e.g., to its maximum length).

Also, let’s introduce non-dimensional global \( x_k \) and local \( \xi \) coordinates: \( x_k = X^k / L \), \( \xi = X^1 / \kappa, k = 1, 2, 3 \). Coordinate \( \xi \) varies across the thickness of the plate \(-0.5 < \xi < 0.5\).

The basic premise is assumed corresponding to the real conditions of loading of the thin plates, that vector components of forces \( \tilde{S}_j \) and pressures \( \tilde{p}_\pm \) on the outer and inner surfaces of the plates are in the order of small \( O(\kappa^3) \), i.e. \( \tilde{p}_\pm = \kappa^2 p_\pm \), \( \tilde{S}_j = \kappa^3 S_j^0 \). In equations (1) and (2) assume moduli tensor \( C_{ijkl}(\xi) \) components depending on the coordinates \( \xi \) as this tensor is different for different layers of the plate.

Asymptotic expansions

Solution of the problems (1), (2) for the main and varied states, following operations [4-7], we will look for as asymptotic decompositions of the parameter \( \kappa \)

\[
\begin{align*}
    u_k &= u_k^{(0)}(x_j) + \kappa u_k^{(1)}(x_j, \xi) + \kappa^2 u_k^{(2)}(x_j, \xi) + \kappa^3 u_k^{(3)}(x_j, \xi) + \ldots \quad (3) \\
    w_k &= w_k^{(0)}(x_j) + \kappa w_k^{(1)}(x_j, \xi) + \kappa^2 w_k^{(2)}(x_j, \xi) + \kappa^3 w_k^{(3)}(x_j, \xi) + \ldots \quad k = 1, 2, 3; (4)
\end{align*}
\]

The Cauchy equation (3) decomposition will be substituted in the system (1), we obtain asymptotic decomposition for deformations:

\[
\begin{align*}
    \varepsilon_{ij}^{(0)} &= \varepsilon_{ij}^{(0)(0)} + \kappa \varepsilon_{ij}^{(0)(1)} + \kappa^2 \varepsilon_{ij}^{(0)(2)} + \ldots \quad (5)
\end{align*}
\]

Substituting the expression (5) in Hooke’s law in the system (1), we obtain the asymptotic voltage for decomposition:

\[
\begin{align*}
    \sigma_{ij}^{(0)} &= \sigma_{ij}^{(0)(0)} + \kappa \sigma_{ij}^{(0)(1)} + \kappa^2 \sigma_{ij}^{(0)(2)} + \ldots. \quad (6)
\end{align*}
\]

Substituting the decomposition, (3) (6) (5) Cauchy equation in relation with the balance of the system (1) and equating equations in terms \( \kappa^{-1} \) to zero and all other ratios between \( \kappa \) certain values, independent recurrent \( h_{i(0)}, h_{i(1)}, h_{i(2)} \) on \( \xi \) gives local tasks where a task sequence for the zero approximation is:

\[
\begin{align*}
    \sigma_{i3}^{(0)(0)} &= 0, \\
    \sigma_{i3}^{(0)} &= C_{i3kL}^{(0)(0)} + C_{i3k3}^{(0)(0)}, \quad \sigma_{ij}^{(0)} = C_{ijkl}^{(0)(0)} + C_{ijkl}^{(0)(0)}, \\
    \varepsilon_{ij}^{(0)} &= \frac{1}{2} (u_{i,j}^{(0)} + u_{j,i}^{(0)}) + \frac{1}{2} (u_{i,j}^{(0)} + u_{j,i}^{(0)}), \quad \varepsilon_{33}^{(0)} = u_{33}^{(1)}, \\
    \Sigma_{33} : \sigma_{i3}^{(0)(0)} &= 0; \quad \Sigma_{S} : [\sigma_{ij}^{(0)(0)}] = 0, \quad [u_{i}^{(1)}] = 0, \quad <u_{i}^{(1)}> = 0; \quad (7)
\end{align*}
\]
The solution of the problem of local approximation (7)-zero functions \( u_j^{(1)}, \varepsilon_j^{(0)}, \sigma_j^{(0)} \) depend on local coordinates \( \xi_i \) and input task means displacements \( u_j^{(0)}(x_j) \). Similarly, there are solutions for the entire sequence of local tasks. The functions \( h_i^{(0)}, h_i^{(1)}, h_i^{(2)} \) find existence conditions of local decisions tasks [12]:

\[
h_i^{(0)} = < \sigma_{ij,j} >, \quad h_i^{(1)} = < \sigma_{ij,j} > > h_i^{(2)} = < \sigma_{ij,j} > - \Delta p \delta_{i3}, \quad \Delta p = p_+ - p_-
\] (8)

The stability of the system in equation (2) can be written as follows:

\[
\sigma_{ij} - \tau^{ik} B_{kl} \sigma_{kl}^{(0)} = 0
\] (9)

where the components \( B_{\alpha \beta} \) of the gradient tensor from accompanying vector is:

\[
B_{\alpha \beta} = \omega_{\alpha \beta} = \omega_{\beta, \alpha} = \frac{1}{\kappa} B_{\alpha \beta}(-1) + B_{\alpha \beta}^{(0)} + \kappa B_{\alpha \beta}^{(1)} + \kappa^2 B_{\alpha \beta}^{(2)} + \kappa^3 B_{\alpha \beta}^{(3)} + ..
\] (10)

Substituting equations (6), (9), (10) in equation (2), collecting members at similar rates and equating them to constants \( h_i^{(-1)} = 0, h_i^{(0)}, h_i^{(1)} \), we obtain local series resistance problems where local stability problem for approximation is zero:

\[
\sigma_{3j}^{(0)} - \tau^{jk} B_{lm}^{(1)} \sigma_{3k}^{(0)} = 0,
\]

\[
\sigma_{3k}^{(0)} = C_{3jkl} \varepsilon_{kl}^{(0)} + C_{3jk3} \varepsilon_{k3}^{(0)}, \quad \sigma_{ij}^{(0)} = C_{ijk} \varepsilon_{kl}^{(0)} + C_{ik3} \varepsilon_{k3}^{(0)},
\]

\[
\sigma_{3j}^{(0)} = \frac{1}{2} (w_{ij}^{(0)} + w_{jj}^{(0)}), \quad \varepsilon_{ij}^{(0)} = \frac{1}{2} (w_{ij}^{(0)} + w_{ij}^{(1)}), \quad \varepsilon_{13}^{(0)} = w_{13}^{(1)},
\]

\[
\Sigma_{3z} : \sigma_{3j}^{(0)} = 0; \quad \Sigma_S : [\sigma_{3j}^{(0)}] = 0, \quad [w_{ij}^{(1)}] = 0, \quad < w_{ij}^{(1)} > = 0;
\]

The entire sequence of local stability problems similarly dismissed.

In view of the fact, that local stability problems are one-dimensional in a local variable \( \xi \), their solution can be found analytically [15]:

\[
u_j^{(1)} = -\xi \varepsilon_j^{(0)} + 2 S_{ijkl} \varepsilon_{kl}^{(0)}, \quad u_j^{(1)} = S_{ikl} \varepsilon_{kl}^{(0)}, \quad \sigma_{ij}^{(0)} = C_{ijkl} \varepsilon_{kl}^{(0)}, \quad \sigma_{ij}^{(0)} = 0
\]

where the designated functions depend on \( \xi \)

\[
S_{ijkl} = \int_{-\xi}^{\xi} C_{3j3l} C_{3jkl} d\xi = \int_{-\xi}^{\xi} C_{3j3l} C_{3jkl} d\xi
\]

\[
S_{kl} = \int_{-0.5}^{0.5} C_{3k3l} C_{3k3l} d\xi = \int_{-0.5}^{0.5} C_{3k3l} C_{3k3l} d\xi
\]

\[
C_{ijkl} = C_{ijkl} - C_{ik3} C_{3jkl} - C_{ik3} C_{3jkl}
\]

Substituting equation (8) into equation of equilibrium (1) multiplying functions \( h_i^{(0)}, h_i^{(0)}, h_i^{(2)} \) after administration at all \( \xi \kappa \), and then integrating over the thickness, we obtain the following equation auxiliary:

\[
\kappa(\xi \varepsilon_j^{(0)} > < \sigma_{ij}^{(0)} > + \kappa^2 (\xi \sigma_j^{(0)} > < \sigma_{ij}^{(0)} > > < \sigma_{ij}^{(0)} > ) + ... = 0
\] (11)
Let’s introduce notation for forces \( T^0_{ij} \) and moments \( M^0_{ij} \) and shear forces \( Q^0_i \) in main and varied state:

\[
T^0_{ij} = \langle \sigma^0_{ij}^0 \rangle > + \kappa < \sigma_{ij}^{(1)} > + \kappa^2 < \sigma_{ij}^{(2)} > + \ldots ,
\]

\[
Q^0_i = \kappa < \sigma_{13}^{(1)} > + \kappa^2 < \sigma_{13}^{(2)} > + \ldots 
\]

\[
M^0_{ij} = \kappa < \xi \sigma_{ij}^{(0)} > + \kappa^2 < \xi \sigma_{ij}^{(1)} > + \ldots .
\]

Then, the equation (11) can be written as a classical equilibrium equations plate where \( \Delta \tilde{p} = \kappa^2 \Delta p \)

\[
T^0_{ij,j} = 0, \quad Q^0_{i,j} = \Delta \tilde{p}, \quad M^0_{ij,j} - Q^0_i = 0 
\]

The forces \( T^0_{ij} \), moments \( M^0_{ij} \), shear forces \( Q_i \) in varied state have a similar decomposition while maintaining only main members. Thus, the resulting equations will assume a resistance:

\[
\sum_{\gamma=1}^{2} T_{\alpha,\gamma,j} + (-1)^{\gamma} (B_{ij}^{(0)} \sigma_{ij}^{(0)} > + < B_{ij}^{(0)} \sigma_{ij}^{(0)} >) = 0, \quad \alpha, \beta = 1, 2, \quad \alpha \neq \beta \quad (12)
\]

\[
\sum_{\gamma=1}^{2} M_{\alpha,\gamma,j} - Q_{\alpha} + (-1)^{\gamma} (\xi B_{ij}^{(0)} \sigma_{ij}^{(0)} > + < \xi B_{ij}^{(0)} \sigma_{ij}^{(0)} >) = 0 \quad (13)
\]

The averaged ratio for defining plates in main state are:

\[
T^0_{ij} = \bar{C}_{ijkl} \varepsilon_{kl}^{0(0)} + B_{ijkl} \eta_{kl}^{0} + K_{ijklm} \varepsilon_{klm}^{0(0)} ,
\]

\[
M^0_{ij} = B_{ijkl} \varepsilon_{kl}^{0(0)} + D_{ijkl} \eta_{kl}^{0} + \bar{K}_{ijklm} \varepsilon_{klm}^{0(0)} ,
\]

\[
Q^0_i = K_{ijkl} \varepsilon_{kl}^{0(0)} + \kappa^2 < \sigma_{ij}^{(2)} > 
\]

The averaged ratio defining plate to have the appearance of a state varied:

\[
T^0_{ij} = \bar{C}_{ijkl} \varepsilon_{kl}^{0} + \bar{B}_{ijkl} \eta_{kl} + \bar{K}_{ijklm} \varepsilon_{klm}^{0} + \bar{\nu}_{ijklm} B_{ijklm}^{0} \varepsilon_{klm}^{0(0)}
\]

\[
M^0_{ij} = B_{ijkl} \varepsilon_{kl}^{0} + D_{ijkl} \eta_{kl} + \bar{K}_{ijklm} \varepsilon_{klm}^{0} + \bar{\nu}_{ijklm} B_{ijklm}^{0} \varepsilon_{klm}^{0(0)}
\]

where the plate tensors are averaged elastic constants

\[
\bar{C}_{ijkl} = \langle C_{ijkl}^{0} \rangle , \quad \bar{B}_{ijkl} = \kappa < \xi C_{ijkl}^{0} > , \quad \bar{D}_{ijkl} = \kappa^2 < \xi^2 C_{ijkl}^{0} >
\]

\[
\bar{K}_{ijklm} = \kappa < \tilde{N}_{ijklm}^{(0)} > , \quad \bar{K}_{ijklm} = - \kappa < \{ C_{ijklm}^{0} \} > , \quad \bar{\nu}_{ijklm} = \kappa^2 < \xi \tilde{N}_{ijklm}^{(0)} > 
\]

**Example for plate stability**

Consider a classic problem of stability on the plate under the action of longitudinal compressive load \( T_{ij}^{(0)} \equiv - T^0 < 0 \). The axis \( OX^1 \) is oriented in the direction of the longitudinal axis of the plate. In the plate in the main state, a state of uniaxial stretching (compression), which corresponds to the following solution

\[
u_{1}^{(0)} = \Pi_{11} T_{11}^{(0)} X^1 + u_{10}, \quad u_{3}^{(0)} = 0, \quad u_{2}^{(0)} = 0, \quad \eta_{kl} = \eta_{3}^{(0)} = 0
\]

\[
\sigma_{11}^{(0)} = T_{11}^{0}, \quad \epsilon_{11}^{(0)} = \Pi_{11} \sigma_{11}^{(0)}, \quad \epsilon_{22}^{(0)} = \Pi_{12} \sigma_{12}^{(0)}
\]
where $\Pi_{IJKL}$ components of the tensor compliances, back to $C_{IJKL}$ the $u_{I0}$ constant of integration, determined from the boundary condition at the end of the plate.

The system of equations (12)-(13) stability theory has only two non-zero equations with two unknown functions $w_{3}^{(0)}$ and $w_{3}^{(1)}$

$$M_{I1J1} - Q_{I} = 0, \quad Q_{I1} + F_{I3}^{(0)} = 0$$

Excluding from two equations, we obtain equation shear force $Q_{I}$

$$M_{I1J1} + F_{I3}^{(0)} = 0$$

where

$$M_{I1J1} = -D_{1111} w_{3,1111}^{(0)}, \quad F_{I3}^{(0)} = -\sigma_{11}^{(0)} \int_{-h/2}^{h/2} (w_{Ii,3}^{(1)}) dX^3 = -T_{11}^{0} \int_{-h/2}^{h/2} (w_{Ii,3}^{(1)}) dX^3$$

Equation (15) will be substituted into equation (14) and equation gives overall shape plate stability theory

$$w_{3,1111}^{(0)} + k^2 w_{3,11}^{(0)} = 0$$

where

$$k^2 = \frac{T_{11}^{0}}{D_{1111}}$$

Equation (16) with the boundary conditions of the hinge fastening:

$X^1 = 0: \quad U_{1}^{0} = 0, \quad w_{3}^{(0)} = 0, \quad M_{I1} = 0$;

$X^1 = l: \quad w_{3}^{(0)} = 0, \quad M_{I1} = 0$

It has a minimum eigenvalue $k = \pi / l$ corresponds to the critical compressive load $T_{11}^{0}$ at which the loss of stability of the plate:

$$T_{11}^{0} = \frac{\pi^2 D_{1111}}{l^2}$$

We classical Euler formula for the critical force.

**Conclusions**

Thus, herein were derived equations for stresses of deformation forces, shear moments and forces for the plate on the basis of three-dimensional equations of the mechanics of the media with the final deformations using the asymptotic theory and theory of Timoshenko shells and are generally local tasks to third order varied inclusive for main and local object plate condition and is approaching zero, manufactured by calculation of forward resistance plate proposed theory.

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