MATRICES ELEMENTS OF FOURIER INTEGRAL OPERATORS

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Abstract. This article is concerned with the semi-classical limits of matrix elements
⟨Fφj, ϕj⟩ of eigenfunctions of the Laplacian ∆g of a compact Riemannian manifold (M, g)
with respect to a Fourier integral operator F on L²(M). More generally, we consider matrix
elements of eigensections of quantum maps. Many results exist for the case where F is a
pseudo-differential operator, but matrix elements of Fourier integral operators involve new
considerations. The limits reflect the extent to which the canonical relation of F is invariant
under the geodesic flow of (M, g). When the canonical relation is almost nowhere invariant,
a density one subsequence of the matrix elements tends to zero (related results arose first in
the study of quantum ergodic restriction theorems). The limit states are invariant measures
on the canonical relation of F and their invariance properties are explained. The invariance
properties in the case of Hecke operators answers an old question raised by the author in
[Z1].

One of the main objects of study in quantum ergodicity is the sequence of diagonal matrix
elements
ρj(A) = ⟨Aφj, ϕj⟩
(1)
of zeroth order pseudodifferential operators A ∈ Ψ⁰(M) relative to an orthonormal basis
{φj} of eigenfunctions
∆φj = λ²j φj,  ⟨φj, φk⟩ = 0.
of the Laplacian ∆ of a compact Riemannian manifold (M, g). The diagonal matrix element
(1) define positive linear functionals of mass one,
ρj : Ψ⁰ → R,  ρ(I) = 1,
(2)
on the norm closure of the space Ψ⁰(M) of zeroth order pseudo-differential operators, and
are invariant under the wave group in the sense that
ρj(Uᵗ*AUᵗ) = ρj.
(3)
The well-known consequence of Egorov’s theorem is that any weak* limit µ of the sequence
{ρj} lies in the space ℳlg of invariant probability measure for the geodesic flow Gtg on S*M,
i.e. is a positive linear functional on C(S*M) with µ(1) = 1 and Gᵗ*µ = µ. Moreover, one
has the local Weyl law
lim₇→∞ 1/N(λ) ∑j:λj≤λ ρj = ω₇,
(4)
where ω₇(A) = ∫S*M σₐdµₗ is the Liouville state of integration of the principal symbol of
A with respect to normalized Liouville measure. Also, N(λ) = #{j : λj ≤ λ} is the Weyl
counting function and convergence is in the sense of continuous linear functionals on Ψ⁰. We

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refer to [Z3, Z4, Zw] for background on these statements. The off-diagonal matrix elements are also important and we refer to [Z5] for results on them.

The purpose of the present note is to consider the analogues of these basic results for diagonal (and to a lesser extent, off-diagonal) matrix elements (1) of Fourier integral operators \(F\) associated to a closed canonical relation \(C \subset T^*M \times T^*M\). That is, we consider the functionals

\[
\rho_j : I^r(M \times M, C) \to \mathbb{C}, \quad \rho_j(F) = \langle F\varphi_j, \varphi_j \rangle,
\]

where \(I^r(M \times M, C)\) is the space of Fourier integral operators of order \(r\) with wave front relation along \(C\) (see Vol. 4 of [HoI-IV] for background and notation). The linear functionals \(\rho_j(5)\) are invariant under the wave group in the sense that if \(U_t = e^{it\sqrt{\Delta}}\) is the wave group of \((M, g)\), then

\[
\rho_j(U_t^{-1}FU_t) = \rho_j(F).
\]

(6)

In particular, we are interested in the weak* limits of the sequence \(\{\rho_j\}\).

**Definition:** We define a weak* limit \(\rho_\infty\) of the functionals \(\{\rho_j\}\) to be a functional on \(I^0(M \times M, C)\) so that \(\rho_j(F) \to \rho_\infty(F)\) for all \(F\) in this class.

It is shown in Proposition 1 that \(\rho_\infty(F)\) depends only on the principal symbol of \(F\), i.e. on the half-density symbol on the associated canonical relation and defines a measure on the symbols. We therefore use a convenient abuse of notation and identity the state and the measure, i.e. we put

\[
\rho_\infty(F) = \rho_\infty(\sigma_F).
\]

(7)

The motivation for this problem comes from several sources, for instance:

- (i) The question of weak * limits for matrix elements of Fourier integral operators arose in recent work on quantum ergodic restriction theorems [TZ, TZ2, DZ], and more recently for ray-splitting in [JSS]. For the \(F\) in those articles, the underlying canonical relation is a local canonical graph. A key point was that \(\rho_j(F) \to 0\) along a subsequence of density one when the local canonical graphs are ‘almost nowhere invariant’ under the geodesic flow. We say that \(F\varphi_j\) is almost orthogonal to \(\varphi_j\). One aim of this note is to understand such almost orthogonality more systematically.
- (ii) Hecke operators \(T_p\) are Fourier integral operators associated to local canonical graphs, namely \(C\) is the lift to \(G/\Gamma\) of the Hecke correspondence [E]. All work in arithmetic quantum chaos concerns joint eigenfunctions of \(\Delta\) and of the Hecke operators \(T_p\). An obvious question is the Hecke correspondence invariance properties of the weak * limits of Hecke \(\rho_j\). This question was raised but not settled in [Z1] and an answer will be given in Proposition 6.3 for Hecke operators on spheres and in Proposition 6.6 for arithmetic hyperbolic quotients. Of course, Lindenstrauss [L] has long since proved that \(\rho_j \to \omega_L\), but the result we present appears to be new.
- (iii) A weak* limit problem where the canonical relation is not a local canonical graph arose in [Z10]. To study nodal sets, the \(\Delta\)-eigenfunctions of a real analytic Riemannian manifold \((M, g)\) were analytically continued \(\varphi_j \to \varphi_j^C\) to its Grauert tubes \(M_\tau\) and then restricted to geodesic arcs \(\gamma : \mathbb{R} \to \partial M_\tau\). The pullbacks \(\gamma^*|\varphi_j^C|^2\) can be normalized to form a bounded sequence of measures on compact intervals of \(\mathbb{R}\). All of their weak * limits are constant multiples of Lebesgue measure. The
constants depend on whether the geodesic is closed or not. Related pointwise Weyl laws for $|\phi^C_j(\zeta)|^2$ on all of $\partial M$ have been proved [Z8].

- (iv) Pointwise phase space Weyl laws for matrix elements of coherent state projectors $F_h = \psi_{x,\xi}^j \otimes \psi_{x,\xi}^{j,*}$ were obtained in [PU]. They are somewhat similar to modulus squares $|\phi^C_j(\zeta)|^2$ but involve a different FBI transform. It does not seem that the weak* limit problem was studied explicitly before, but the results are rather similar to the restrictions $\gamma^*|\phi^C_j|^2$.

- (v) Both of the above problems are special cases of weak* limit problems for $\rho_j$ on algebras of Toeplitz operators associated to invariant symplectic cones $\Sigma$ under the geodesic flow. The results in this setting are parallel to the case of $\rho_j$ as states on the algebra $\Psi^0(M)$ in the sense of [Z4]. Different algebras of Fourier integral operators associated to idempotent canonical relations were introduced in [GuSt]. In a special case, the the weak* limit problem was studied in [Z9] (see also [ST]).

There is a long-standing question as to the uniqueness of weak* limits (2) when the geodesic flow is sufficiently chaotic. The larger the class of ‘test’ operators one can use, the more control one has over the limits. In general one would like to study the most general possible microlocal defect measures. We refer to [Zw] for general background.

In this article, we concentrate on the case where $C$ is a local canonical graph and the order $r = 0$, and only briefly summarize results on weak * limits of matrix elements in the other cases above. In the canonical graph case, the family $\{\rho_j\}$ of functionals on $I^0(M \times M, C)$ is uniformly bounded and all weak* limits are complex measures on $C$. More precisely, they are linear functionals of the symbol $\sigma_F$ of $F$, which is a $\frac{1}{2}$-density along $C$ (times a Maslov factor, which will be ignored here for simplicity of exposition and because the results do not depend on the Maslov factor). The local Weyl law for Fourier integral operators associated to local canonical graphs was studied in [Z2] (see also [TZ2, JSS]).

Many of the results for local canonical graphs turn out to be negative: the weak * limits of the diagonal functionals $\rho_j$ are very often zero, since most canonical graphs $C$ are almost nowhere invariant under a given geodesic flow. The graph in the Hecke case is invariant [6] and so the question becomes one of determining when the limits are trivial and when they are not. The local canonical (or isotropic) relation in the case of $\gamma^*|\phi^C_j(z)|^2$ or for $|\langle \phi_j, \psi_{x,\xi}^h \rangle|^2$ are not invariant but can be time averaged to become invariant, and the weak * limits (when suitably normalized) are often non-trivial.

The results also suggest that off-diagonal elements

$$\rho_{ij}(F) = \langle F \phi_i, \phi_j \rangle$$

are often more natural when testing against Fourier integral operators. They satisfy

$$\rho^t_{ij}(F) := \rho_{ij}(U^{-t}FU^t) = e^{it(\lambda_i - \lambda_j)} \rho_j(F),$$

and intuitively correspond to canonical transformations which change the energy level. However, the weak* limits are again trivial when the graph is almost nowhere invariant.

0.1. Results for local canonical graphs. To state our results, we need to introduce some further notation. A Fourier integral operator is an operator $F$ whose Schwartz kernel may
be locally represented as a finite sum of oscillatory integrals,

\[ K_F(x, y) \sim \int_{\mathbb{R}^N} e^{i \phi(x, y, \theta)} a(x, y, \theta) d\theta \]  

(10)

for some homogeneous phase \( \varphi \) and amplitude \( a \). It is well known that \( F \) is determined up to compact operators by the canonical relation

\[ C = \{(x, \varphi'_x, y, -\varphi'_y) : \varphi'_\xi(x, y, \xi) = 0\} \subset T^*M \times T^*M, \]

together with the principal \( \sigma_F \) of \( F \), a 1/2-density along \( C \). We denote by \( I_0(M \times M, C) \) the class of Fourier integral operators of order zero and canonical relation \( C \). We refer to [HoI-IV] for the background. Then we may regard (1) as defining continuous linear functionals on \( I_0(M \times M, C) \) with respect to the operator norm. We recall that \( C \) is a local canonical graph when both projections in the diagram

\[ C \subset T^*M \times T^*M \]

\[ \pi_X \not\rightarrow \quad \swarrow \pi_Y \]

(11)

are (possibly branched) covering maps. If we equip \( C \) with the symplectic volume measure pulled back by \( \pi_X \) from \( T^*M \), then we may consider symbols \( \sigma_F \) as functions on \( C \). Some well-known examples are:

- \( F = T_g \) is translation by an isometry of a Riemannian manifold \((M, g)\) possessing an isometry.
- \( F \) is a Hecke operator \( Tf(x) = \sum_{j=1}^k (f(g_jx) + f(g_j^{-1}x)) \) on \( S^n \) or on an arithmetic hyperbolic manifold corresponding to a finite set \( \{g_1, \ldots, g_k\} \) of isometries of the universal cover. In this case \( C \) is the graph of the cotangent lift of the Hecke correspondence and is a local canonical graph [RS, LPS].
- \( F_t = U^t = e^{it\sqrt{\Delta}} \) or its self-adjoint part \( \cos t\sqrt{\Delta} \).
- \( F = W^*W \) where \( Wf = \gamma_H BU^t \) where \( \gamma_H \) is restriction to a hypersurface \( H \subset M \) and \( WF''(B) \) is disjoint from the cotangent directions to \( H \) [La, GS, TZ, TZ2, DZ] among many articles.
- \( F \) is a semi-classical quantum map in the setting of positive Hermitian holomorphic line bundles over Kähler manifolds [Z6].

The first result is:

**Proposition 1.** If \( C \) is a local canonical graph and \( F \in I^0(M \times M, C) \) then the weak limits \( \rho_\infty \) of \( \{F\varphi_j, \varphi_j\} \) are measures on \( SC := C \cap S^*M \times S^*M \), i.e. \( |\rho_\infty(F)| \leq C \sup_{SC} ||\sigma_F||_{c_0} \).

As discussed above (7) we also write the limit functional as functional on the symbol, and thus have \( |\rho_\infty(\sigma_F)| \leq C \sup_{SC} ||\sigma_F||_{c_0} \).

The proof is quite similar to that for \( A \in \Psi^0(M) \). But it can be useful to interpret the quantum limits as living on \( SC \) rather than on \( S^*M \), as will be seen in the case of Hecke operators.

The weak* limit problem for \( \{\rho_j\} \) on \( I^0(M \times M, C) \) is not so different from that of \( \Psi^0(M) \) since there often exists an elliptic element \( F_0 \) of \( I^0(M \times M, C) \), i.e. one with nowhere
vanishing symbol, and then all of the elements have the form $AF_0$ or $F_0B$ where $A, B \in \Psi^0(M)$. But the canonical relation of $U^{-t}FU^t$ equals the image

$$C_t := (G^{-t} \times G^t)(C)$$

of $C$ under the map $G^{-t} \times G^t$ of $T^*M \times T^*M$, and only coincides with the canonical relation $C$ of $F$ if $C$ is $G^t$-invariant. In general,

$$U^{-t}FU^t \in I^0(M \times M, C_t),$$

so that by (6) $\rho_j^t$ induces a functional on $I^0(M \times M, C_t)$. The following initial result shows that the invariance properties of quantum limits depend on whether the canonical relation is invariant under the geodesic flow.

**Proposition 2.** Let $\rho^t_\infty$ be a weak* limit of the functionals $\rho_j^t$ on $I^0(M \times M, C)$. Then there exists a family of measures $\mu_t$ on $C_t$ such that

$$\rho^t_\infty(A) = \int_{C_t} (G^t \times G^{-t})^*\sigma_A \ d\mu_t, \quad A \in I^0(M \times M, C)$$

with $\mu_t = \mu$ on $C_t \cap C$.

Thus,

**Corollary 1.** Let $\rho_\infty$ be a limit of the sequence of functionals $\rho_j(F) = \langle F\varphi_j, \varphi_j \rangle$ on $I^0(M \times M, C)$. Suppose that the canonical relation $C$ is invariant under the geodesic flow $G^{-t} \times G^t$. Then $\rho_\infty$ is a $G^t$-invariant signed measure of mass $\leq 1$ on $C$.

Of course, the quantum invariance (6) implies that

$$\int_{C_t} (G^t \times G^{-t})^*\sigma_A \ d\mu_t = \int_C \sigma_A \ d\mu.$$

But Proposition 2 does not give any non-trivial invariance conditions on the set where canonical relation $C$ is nowhere invariant under the geodesic flow $G^{-t} \times G^t$. The next result identifies the limit measure as zero along a subsequence of density one. We refer to this as the ‘almost-orthogonality’ of $F\varphi_j$ and $\varphi_j$. In the following, $n = \dim M$ so that $2n = \dim C$.

**Proposition 3.** Let $F \in I^0(M \times M, C)$ and assume that for $t > 0$, the set $C_t = C_t \cap C$ has Minkowski $2n$-measure zero. Then there is a density one subsequence of eigenfunctions so that $\langle F\varphi_{\lambda_j}, \varphi_{\lambda_j} \rangle \to 0$.

A special case of this almost orthogonality result was one of the the main ingredients in the proof in [TZ] of the quantum ergodic restriction theorem along hypersurfaces. See Theorem 10 of [TZ] or §8 of [TZ2]. Although this statement is reminiscent of quantum ergodicity, it does not use any dynamical properties such as ergodicity of $G^t$. As Proposition 3 indicates, almost-orthogonality can sometimes be understood in terms of localization on energy surfaces of eigenfunctions. The nowhere commuting condition in that result implies that $F\varphi_\lambda$ localizes on a disjoint set from $\varphi_\lambda$ and thus the two states are almost orthogonal. In this example there are no sparse exceptional subsequences of eigenfunctions. See also Lemma 3.2 of [B1]. But this is not always the case, for instance localization does not seem to play a role in the Kähler analogue of Theorem ??.
0.2. Reality. Additional invariance properties arise when $F$ is self-adjoint or real due to the fact that the eigenfunctions are real valued. We say that $F$ is real if $cF = Fc$ where $c$ denotes complex conjugation. We note that

$$
\langle F^* \varphi_j, \varphi_j \rangle = \langle F \varphi_j, \varphi_j \rangle = \langle Ft \varphi_j, \varphi_j \rangle = \langle F^* \varphi_j, \varphi_j \rangle.
$$

We recall that the transpose of a Lagrangian manifold $\Lambda$ is defined by

$$
\Lambda^t = \{ (y, \eta, x, \xi) : (x, \xi, y, \eta) \in \Lambda \},
$$

i.e. it is the image of $\Lambda$ under the involution,

$$
\iota(x, y, \xi, \eta) = (y, \eta, x, \xi).
$$

The ‘conjugate’ Lagrangian is defined by

$$
\Lambda^* = \{ (y, \eta, x, -\xi) : (x, \xi, y, \eta) \in \Lambda \},
$$

i.e. it is the image under $c \circ \iota$ where $c$ is the conjugation involution $(x, \xi) \rightarrow (x, -\xi)$ in the second variable; the notation is consistent since complex conjugation is the quantization of the map $c$. We say that a canonical relation $C$ is symmetric if $C^t = C$ and self-adjoint if $C^* = C$. A self-adjoint Fourier integral operator always has a self-adjoint canonical relation and a real Fourier integral operator has a canonical relation invariant under $c$. When $F$ is self-adjoint we obtain an additional invariance principle if we consider symbols defined by the functions $\pi^* X, \pi^* Y$ (cf. (11)):

**Proposition 4.** If $F$ is real and self-adjoint then $(\pi^* X)_{\mu} = (\pi^* Y)_{\mu}$ for any quantum limit measure $\mu$ on $C$ of Proposition 2.

This additional principle is useful in obtaining relations between limit measures of Hecke operators. See §6.

0.3. Quantum maps in the Kähler setting. There is a natural analogue of Proposition 3 in the Kähler setting. We let $(M, \omega, L)$ be a compact polarized Kähler manifold. That is, $L \rightarrow M$ is a holomorphic line bundle equipped with a Hermitian metric $h$ whose curvature form $\Theta_h$ equals $i\omega$. Thus, $\omega \in H^{1,1}(M, 2\pi \mathbb{Z})$. We also denote the $k$th tensor power of $L$ by $L^k$ and denote the space of holomorphic sections by $H^0(M, L^k)$; we also put $d_k = \dim H^0(M, L^k)$. We refer to [Z6, Z7] for background.

We further $\chi_1, \chi_2$ be two quantizable symplectic diffeomorphisms of $(M, \omega)$. The definition of quantizable is from [Z6, Z7], which generalizations the implicit standard notion for special cases such as cat maps on a complex one dimensional torus (elliptic curve). Namely, $\chi_j$ are symplectic diffeomorphisms which possess lifts as contact transformations of the unit circle bundle $X_h = \partial D^*_h$ where $D^*_h$ is the unit co-disc bundle in the dual line bundle $L^*$ of $L$ with respect to the dual metric $h$. An exposition of the key notions Kähler quantization can be found [Gu].

We let $\{U_{\chi, k}\}_{k=1}^{\infty}$ denote the semi-classical quantization of $\chi_1$ as a sequence of unitary operators on the Hilbert spaces $H^0(M, L^k)$. Thus, $F = U_{\chi_2, k}$ denotes the quantum map quantizing $\chi_2$. As discussed in [Z6, Z7] the quantizations have the form $U_{\chi, k} := \Pi_h \sigma_{k, \chi} T_\chi \Pi_h$ where $\Pi_h : L^2(M, L^l) \rightarrow H^0(M, L^k)$ is the orthogonal projection (Szegő kernel), where $T_\chi$ is the translation operator by $\chi$ and where $\sigma_{k, \chi}$ is a symbol designed to make $U_{\chi, k}$ unitary. More precisely, $T_\chi$ is the translation operator by the lift of $\chi$ to $X_h$. 
Proposition 5. Let \( \varphi_{k,j} \) denote the eigensections of \( U_{\chi_1,k} \). Suppose that \( \chi_1, \chi_2 \) almost nowhere commute in the sense that the set \( \{ z \in M : \chi_1 \chi_2(z) = \chi_2 \chi_1(z) \} \) has measure zero. Then
\[
\frac{1}{d_k} \sum_{j=1}^{d_k} |\langle U_{\chi_2,k} \varphi_{k,j}, \varphi_{k,j} \rangle|^2 \rightarrow 0.
\]

As a simple example, suppose that \( A_1, A_2 \) are two non-commutating elements of the \( \theta \) subgroup of \( SL(2, \mathbb{Z}) \) (i.e. are congruent to the identity modulo 2). The associated symplectic maps \( \chi_1, \chi_2 \) of \( \mathbb{R}^2/\mathbb{Z}^2 \) are then quantizable and almost nowhere commute. So the eigenfunctions of one ‘quantum cat map’ give rise to zero quantum limits for the other.

An interesting comparison to Proposition 3 is that the semi-classical eigensections \( \varphi_{N,j} \) do not appear to have any localization properties which account for the almost orthogonality of the matrix elements.

0.4. Pointwise squares as matrix elements. Let \( \{ \psi \} \) be a semi-classical Lagrangian state, for instance a coherent state \([CR]\) in the Schrödinger representation, or coherent states induced by a Bergman reproducing kernel, or the sequence of Gaussian beams associated to a closed geodesic \([BB]\). In each case, we consider the norm-squares as matrix elements of semi-classical Fourier integral operators,
\[
|\langle \psi, \varphi \rangle|^2 = \langle F_h \varphi, \varphi \rangle, \quad \text{where} \quad F_h = \psi \otimes \psi^*.
\]

The Schwartz kernel of \( F_h(x, y) = \psi_h(x) \overline{\psi_h(y)} \) inherits an oscillatory integral representation from that of \( \psi_h \). One relevant normalization is to take \( \| \psi_h \|_{L^2} = 1 \).

The underlying Lagrangian or isotropic submanifold of \( \{ \psi_h \} \) may or may not be invariant under the geodesic flow. For instance, they are not invariant for coherent states \( \psi_{x,\xi} \) (where the isotropic submanifold is a point), but they are for the sequence of highest weight spherical harmonics \( Y_k \) on the standard \( S^2 \) or for more general Gaussian beams. However by an averaging argument (see \([4]\)) one can show that the weak* limits must be non-negative measures on the ‘orbit’ of the underlying Lagrangian or isotropic submanifold under the geodesic flow. In the case of a local canonical graph this ‘flowout’ could be dense in \( T^*M \times T^*M \) if \( C \) is almost nowhere invariant under \( G_t \times G_t \) but for an isotropic submanifold the flowout can be a closed submanifold and one can have non-trivial limits.

For the sake of brevity, we only give explain how the recent results in \([Z10]\) fit into the picture of matrix elements of Fourier integral operators, which was not the approach used in that article. There are many related examples that we will not consider here.

0.5. Idempotent canonical relations and algebras of Fourier integral operators. There are other settings where the weak* limit problem is of interest. One is where \( I^0(M \times M, C) \) is a \(*\) algebra of Fourier integral operators. This occurs when the canonical relation is idempotent in the sense that \( C^* = C = C^2 \). One such situation is the algebra of Fourier integral operators associated to a symplectic cone \( \Sigma \subset T^*X \). In fact \( C \) need not be a Lagrangian submanifold. It is sufficient that \( C \) be isotropic (see \([W]\)). An example is when \( \gamma \) is a closed geodesic of a Riemannian manifold and where \( \Sigma = \mathbb{R}_+ \dot{\gamma} \). Then \( C \) is the diagonal of \( \Sigma \times \Sigma \). The averaged coherent state projections
\[
\langle \psi_{x,\xi}^h \otimes (\psi_{x,\xi}^h)^* \rangle_L := \frac{1}{L} \int_0^L U^t (\psi_{x,\xi}^h \otimes (\psi_{x,\xi}^h))^* U^{-t} dt
\]
are Toeplitz operators in the case $\Sigma = \mathbb{R}_+ \dot{\gamma}$.

Idempotent canonical relations also arise as leaf equivalence relations of null foliations of co-isotropic submanifolds $\Sigma \subset T^*M$, also known as flowouts. In the case where the null-foliation is a fiber bundle with compact fiber over a leaf space $S$ (a symplectic manifold), the algebra was denoted $\mathcal{R}_\Sigma$ and was studied in [GuSt]. For the sake of brevity we omit further discussion and refer to [Z9] and to [ST, GU] for the study of quantum ergodicity in this setting.

1. Background

We recall that a Fourier integral operator $A : C^\infty(X) \to C^\infty(Y)$ is an operator whose Schwartz kernel may be represented by an oscillatory integral

$$K_A(x, y) = \int_{\mathbb{R}^N} e^{i\varphi(x,y,\theta)} a(x, y, \theta) d\theta$$

where the phase $\varphi$ is homogeneous of degree one in $\theta$. The critical set of the phase is given by

$$C_\varphi = \{(x, y, \theta) : d_\theta \varphi = 0\}.$$

When the map

$$\iota_\varphi : C_\varphi \to T^*(X,Y), \quad \iota_\varphi(x, y, \theta) = (x, d_x \varphi, y, -d_y \varphi)$$

is an immersion the phase is called non-degenerate. Less restrictive is where the phase is clean, i.e. $\iota_\varphi : C_\varphi \to \Lambda_\varphi$, where $\Lambda_\varphi$ is the image of $\iota_\varphi$, is locally a fibration with fibers of dimension $e$. From [HoI-IV] Definition 21.2.5, the number of linearly independent differentials $d{\partial_\varphi \over \partial \theta}$ at a point of $C_\varphi$ is $N - e$ where $e$ is the excess.

We work in the polyhomogeneous framework of [HoI-IV], and assume that classical polyhomogeneous symbols

$$a(x, y, \theta) \sim \sum_{k=0}^{\infty} a_{-k}(x, y, \theta), \quad (a_{-k} \text{ positive homogeneous of order } -k \text{ in } \theta).$$

All of the results and notions of this note generalize to semi-classical Fourier integral operators with semi-classical symbols $a \in S^{0,0}(T^*H \times (0, h_0]$ of the form

$$a_h(s, \sigma) \sim \sum_{k=0}^{\infty} h^k a_{-k}(s, \sigma), \quad (a_{-k} \in S_{1,0}^{-k}(T^*H)).$$

Since there is no essential difference in the weak* limit results in the two settings, we only consider the poly-homogeneous one.

We recall that the order of $F : L^2(X) \to L^2(Y)$ in the non-degenerate case is given in terms of a local oscillatory integral formula by $m + {N \over 2} - {n \over 4}$, where $n = \dim X + \dim Y$, where $m$ is the order of the amplitude, and $N$ is the number of phase variables in the local Fourier integral representation (see [HoI-IV], Proposition 25.1.5); in the general clean case with excess $e$, the order goes up by $e$ ([HoI-IV], Proposition 25.1.5'). Further, under clean composition of operators of orders $m_1, m_2$, the order of the composition is $m_1 + m_2 - {e \over 2}$ where $e$ is the so-called excess (the fiber dimension of the composition); see [HoI-IV], Theorem 25.2.2.

The symbol $\sigma(\nu)$ of a Lagrangian (Fourier integral) distributions is a section of the bundle $\Omega^{1 \over 2}_2 \otimes \mathcal{M}_3$ of the bundle of half-densities (tensor the Maslov line bundle). In terms of
a Fourier integral representation it is the square root \( \sqrt{d_{C_{\varphi}}} \) of the delta-function on \( C_{\varphi} \) defined by \( \delta(d\theta\varphi) \), transported to its image in \( T^*M \) under \( \iota_{\varphi} \). If \( (\lambda_1, \ldots, \lambda_n) \) are any local coordinates on \( C_{\varphi} \), extended as smooth functions in neighborhood, then

\[
d_{C_{\varphi}} := \frac{|d\lambda|}{|D(\lambda, \varphi_\theta)/D(x, \theta)|},
\]

where \( d\lambda \) is the Lebesgue density.

1.1. Local Weyl law for Fourier integral operators. It was proved in [Z2] (see also [TZ, TZ2, DZ]) that if average the functionals \( \rho_j \), then the limit is a measure on the unit vectors in the intersection \( C \cap \Delta_{T^*M \times T^*M} \) of the canonical relation \( C \) with the diagonal in \( T^*M \times T^*M \). That is, one has the local Weyl law,

\[
\frac{1}{N(\lambda)} \sum_{j: \lambda_j \leq \lambda} \rho_j \rightarrow \rho_{LWL},
\]

where the local Weyl law measure is given by

\[
\rho_{LWL}(F) = \int_{S(C \cap \Delta_{T^*M \times T^*M})} \sigma_F d\nu,
\]

where \( d\nu \) is a ‘half-density measure’ and \( S(C \cap \Delta_{T^*M \times T^*M}) \) is the set of unit covectors in \( C \cap \Delta_{T^*M \times T^*M} \). In the case where \( C \) is a local canonical graph, this intersection is the fixed point set of the correspondence \( \chi \) and we write it as \( S\text{Fix}(\chi) \).

Note that the trace operation concentrated the average of the \( \rho_j \) on the diagonal part of \( C \). The individual matrix elements do not have this property, even though \( \langle F\varphi, \varphi \rangle = TrF\varphi \otimes \varphi^* \) is a trace.

2. Invariant states on \( I^0(M \times M, C) \): Proof of Propositions \textit{1} and \textit{2}

2.1. Fourier integral operators associated to local canonical graphs. The proof of Proposition \textit{2} is similar to the case of pseudo-differential operators proved in [W].

We first recall

**Theorem 2.1.** ([HoI-IV, Theorem 25.3.1]) If \( C \) is a local canonical graph and \( A \in I^0(M \times M, C) \), then \( A : L^2(M) \rightarrow L^1(M) \) is bounded, and it is compact if the symbol of \( A \) tends to 0 as \( |\xi| \rightarrow \infty \). 

We then prove

**Lemma 2.2.** If \( C \) is a local canonical graph and \( A \in I^0(M \times M, C) \) then

\[
\sup_{T^*M} |\sigma_A| = \inf_K ||A + K||.
\]

**Proof.** The equality (15) is well known for \( A \in \Psi^0(M) \). To generalize it to Fourier integral operators associated to local canonical graphs it suffices to use that, in a sufficiently small cone, \( C \) is the graph of a canonical transformation. Then as in the proof of [HoI-IV, Theorem 25.3.1], \( A^*A \in \Psi^0(M) \) and \( \sigma_{A^*A} = |\sigma_A|^2 \). It follows that

\[
\sup_{T^*M} |\sigma_A|^2 = \inf_{K \text{ compact}} ||A^*A + K|| = \inf_{K \text{ compact}} ||A + K||^2.
\]
Here we use that for any \( u \in L^2 \) with \( ||u|| = 1 \), \( ||(A + K)u||^2 = \langle (A^*A + K_1)u, u \rangle \) for another compact operator \( K_1 \) and that \( ||A^*A + K|| = \sup_{||u||=1} |\langle (A^*A + K_1)u, u \rangle| \) when \( K \) is self-adjoint.

If \( K \) is any compact operator on \( L^2(M) \) then \( \langle K\varphi_j, \varphi_j \rangle \to 0 \). Indeed, \( \varphi_j \to 0 \) weakly in \( L^2 \) and so \( K\varphi_j \to 0 \) in norm. The principal symbol of \( F \) determines \( F \) up to an element of \( I^{-1}(M \times M, C) \) and the operators in this class are compact. This proves Proposition \( \boxed{\text{II}} \).

We now complete the proof of Propositions \( \text{I} - \text{II} \).

Proof. For any compact operator \( K \), \( \langle K\varphi_j, \varphi_j \rangle \to 0 \). Hence, any limit of \( \langle A\varphi_k, \varphi_k \rangle \) is equally a limit of \( \langle (A+K)\varphi_k, \varphi_k \rangle \). By the norm estimate, the limit is bounded by \( \inf_K ||A+K|| \) (the infimum taken over compact operators). Hence any weak limit is bounded by a constant times \( ||\sigma_A||_{L^\infty} \) and is therefore continuous on \( C(S^*M) \). It is a positive functional since each \( \rho_j \) is and hence any limit is a probability measure.

To prove the invariance of the limit measure, we apply an Egorov type theorem to \( U^{-t}FU^t \) for \( F \in I^0(M \times M, C) \) and for fixed \( t \). The canonical relation of the composition is the composition

\[ \Gamma_t^* \circ C \circ \Gamma_t = C, \]

where

\[ \Gamma_t = \{(x, \xi, G^t(x, \xi)) : |\tau| + |\xi| = 0\}. \]

Hence only the symbol \( \sigma_F \) is changed. If we choose a nowhere vanishing half-density on \( C \) (e.g. the graph half-density corresponding to the symplectic volume density on \( T^*M \)), then \( \sigma_F \) may be identified with a scalar function and the composite symbol is its pull-back under \( G^t \).

By invariance of the \( \rho_k \), any limit of \( \rho_k(A) \) is a limit of \( \rho_k(Op(\sigma_A \circ \Phi^t)) \) and hence the limit measure is invariant. It is also time-reversal since the eigenfunctions are real-valued, i.e. complex conjugation invariant.\( \square \)

2.2. Off-diagonal matrix elements. A similar argument applies to off-diagonal matrix elements

\[ \rho_{jk}(A) = \langle A\varphi_j, \varphi_k \rangle. \] (16)

The discussion is very similar to that in [Z5] in the pseudo-differential case.

**Proposition 2.3.** Let \( \rho_{\infty} \) be a limit of the sequence of functionals \( \{16\} \) with the gap \( \lambda_j - \lambda_k \to \tau \) on \( I^0(M \times M, C) \). Suppose that the canonical relation \( C \) is invariant under the geodesic flow \( G^{-t} \times G^t \). Then \( \rho_{\infty} \) is a signed \( G^t \)-eigenmeasure of mass \( \leq 1 \) on \( C \).

The only change to the proof of Proposition \( \boxed{\text{II}} \) is in the last step. The functionals \( \rho_{jk} \) are no longer invariant but rather satisfy \( \{9\} \). It follows that if \( \lambda_j - \lambda_k \to \tau \) then any limit measure is a \( G^t \) eigenmeasure on \( C \) with eigenvalue \( e^{it\tau} \).

3. **Almost orthogonality: Proof of Proposition \(

This section is motivated by the proof of the quantum ergodic restriction theorem in [TZ, TZ2] (see also [DZ]). The criterion for QER in those papers is an almost nowhere commutativity condition between two canonical transformations, or equivalently an almost nowhere invariance problem. It is not clear that the condition for QER in those papers is sharp.
We now prove Proposition 3.

Proof. We are assuming that for $t > 0$, the set $C_t = \{ (x, \xi) \in S^*M : G^t \chi = \chi G^t \}$ has Liouville measure zero.

It suffices to show that

$$\frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle F \varphi_{\lambda_j}, \varphi_{\lambda_j} \rangle|^2 = o(1) \text{ as } \lambda \to \infty.$$  \hfill (17)

We put

$$F(t) = U^{*F} T, \quad \langle F \rangle_T := \frac{1}{T} \int_{-T}^{T} F(s) ds, \text{ where } U^t = e^{it\sqrt{\Delta}}.$$

Then $\langle F(t) \varphi_{\lambda_j}, \varphi_{\lambda_j} \rangle = \langle F \varphi_{\lambda_j}, \varphi_{\lambda_j} \rangle$.

For any operator $A$ we have

$$|\langle A \varphi_{\lambda}, \varphi_{\lambda} \rangle|^2 \leq \langle A^* A \varphi_{\lambda}, \varphi_{\lambda} \rangle.$$

It follows that

$$|\langle F \varphi_{\lambda_j}, \varphi_{\lambda_j} \rangle|^2 \leq \langle \langle F \rangle_T^* \langle F \rangle_T \varphi_{\lambda}, \varphi_{\lambda} \rangle.$$  \hfill (18)

We now let $\lambda \to \infty$ and use the local Weyl law (14) for Fourier integral operators. We have,

$$\langle F \rangle_T^* \langle F \rangle_T = \frac{1}{T^2} \int_{-T}^{T} \int_{-T}^{T} F(s)^* F(t) ds dt.$$

Since we are taking a trace, we can cycle the $U^t$ to get

$$\langle F \rangle_T^* \langle F \rangle_T = \frac{1}{T^2} \int_{-T}^{T} \int_{-T}^{T} U(t-s)^* F^* U(t-s) F ds dt.$$  \hfill (19)

We change variables to $u = \frac{t-s}{2}, v = \frac{t+s}{2}$ and simplify to get

$$\langle F \rangle_T^* \langle F \rangle_T = \frac{1}{T} \int_{-T}^{T} U(t)^* F^* U(t) F \rho_T(t) dt.$$

Here, $\rho_T(t)$ is the measure in $[-T, T] \times [-T, T]$ of $\{(s, s') : s - s' = t\}$. For each $t$ the local Weyl law gives

$$\frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \langle U(t)^* F^* U(t) F \varphi_{\lambda}, \varphi_{\lambda} \rangle \to \int_{S_{\text{Fix}} G^{-1} \chi} \sigma_{t,F},$$  \hfill (19)

where $\sigma_{t,F}$ is a composite density on the fixed point set. The fixed point set is exactly the set where $\chi G_t = G^t \chi$ and has measure zero for all $t \neq 0$. Hence the integral tends to zero for all $T > 0$.

\[\square\]

This arguments works too well because the assumption is so strong. A related argument is to just assume that there exists $t_0$ so that $\chi G^{nt_0} = G^{nt_0} \chi$ only holds on a set of measure zero. It is this argument which was in effect used in [TZ, TZ2].
Proposition 3.1. Let $F$ be a Fourier integral operator associated to a symplectic correspondence $\chi$. Assume that there exists $t_0 \neq 0$ so that for $n = 1, 2, 3, \ldots$, the set $\mathcal{C}_n = \{(x, \xi) \in S^*M : G^{nt_0} \chi = \chi G^{nt_0}\}$ has Liouville measure zero. Then there is a density one subsequence of eigenfunctions so that $\langle F \varphi_{\lambda_j}, \varphi_{\lambda_j} \rangle \to 0$.

Proof. We define

$$\langle F \rangle_M := \frac{1}{2M} \sum_{m=-M}^{M} U^{-mt_0} F U^{mt_0}.$$ 

Going through the same argument gives the upper bound

$$\frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle F \varphi_{\lambda_j}, \varphi_{\lambda_j} \rangle|^2 \leq \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \langle \langle F \rangle_M^{*}, \langle F \rangle_M \varphi_{\lambda}, \varphi_{\lambda} \rangle. \quad (20)$$

We then have

$$\langle F \rangle_M^{*} \langle F \rangle_M = \frac{1}{M^2} U^{t_0(m-n)} F^{*} U^{t_0(m-n)} F,$$

where $\#p = \#\{(m, n) \in [-M, M] \times [-M, M] : m - n = p\}$. We then apply the local Weyl law and find that the only term which makes a non-vanishing contribution is $p = 0$. So it is $O\left(\frac{1}{M}\right)$.

3.1. Almost disjoint energy surfaces. The commutator $[\sqrt{\Delta}, F]$ is always of order one if $F \in \mathcal{I}(M \times M, C)$. The symbol of $\sqrt{\Delta} F$ at $(x, \xi, y, \eta)$ is the product $|\xi|_{x} \sigma_F$ while in the other order we have $|\eta|_{y} \sigma_F$. So they do not cancel unless $\chi$ preserves $S^*_y X$, which is not the case when they almost never commute.

Lemma 3.2. Suppose that $\chi$ and $G^t$ almost never commute. Then $\chi(S^*_y M) \cap S^*_y M$ has Liouville measure zero.

Let $H(x, \xi) = |\xi|_y$ be the metric norm function. The Hamiltonian flow of $\chi^* H$ is $\chi G^t \chi^{-1}$. The orbit of $(x, \xi)$ under this flow is almost certainly disjoint from the $G^t$ orbit of $(x, \xi)$. Also the Hamiltonian vector field $\xi_H$ of $H$ almost certainly satisfies $\chi_* \xi_H \neq \xi_H$. If $\chi^* H = H$ on an open set, then we take the Hamiltonian vector field of both sides to get $\chi_* \xi_H = \xi_H$. Similarly for any set of positive measure.

We can then give a second proof of Proposition 3.1.

Proof. It is well-known that $\Delta$-eigenfunctions concentrate microlocally on the energy surfaces $|\xi| = E$ (see [Zw], Theorem 6.4). In the homogeneous setting we may identify all the energy surfaces with $S^*M$. It follows that $F \varphi_j$ microlocally concentrates on $\chi(S^*M)$. In other words we may construct semi-classical cutoffs $Op_h(b)$ to $S^*M$ and $Op_h(\chi^* b)$ so that $Op_h \varphi_j = \varphi_j + \mathcal{O}(h), Op_h(\chi^* b) F \varphi_j = F \varphi_j + \mathcal{O}(h)$. By Lemma 3.2 the intersection has Liouville measure zero. It follows that

$$\langle F \varphi_j, \varphi_j \rangle = \langle Op_h(\chi^* b) F \varphi_j, Op_h(\chi^* b) \varphi_j \rangle + \mathcal{O}(h) \leq ||Op_h(\chi^* b) Op_h(\chi^* b) \varphi_j|| + \mathcal{O}(h).$$
But $Op_h(\chi^* b) Op_h(b) = Op_h(b \chi^* b) + \mathcal{O}(\hbar)$. Then
\[
\frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle F \varphi_{\lambda_j}, \varphi_{\lambda_j} \rangle|^2 \leq \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \langle (Op_h(\chi^* b) \varphi_{\lambda_j}, \varphi_{\lambda_j}) \rangle^2 + \mathcal{O}(\lambda^{-1})
\]
\[
\leq \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \langle (Op_h(\chi^* b)^* p_h(\chi^* b) \varphi_{\lambda_j}, \varphi_{\lambda_j}) \rangle^2 + \mathcal{O}(\lambda^{-1})
\]
\[
\to \int_{S^* M} b^2 \chi^* b^2 \text{ as } \lambda \to \infty.
\]
But by assumption, for any $\varepsilon > 0$ one may construct $b$ so that the support of $b \times \chi^* b$ has volume $\leq \varepsilon$. It follows that $\int_{S^* M} b^2 \chi^* b^2 \leq \varepsilon$, and therefore $\frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle F \varphi_{\lambda_j}, \varphi_{\lambda_j} \rangle|^2 \to 0$. □

The proof indicates that although a subsequence of density one of the matrix elements tends to zero, not all them need to. It could be the case that a sub sequence of eigenfunctions $\varphi_{j_k}$ concentrates microlocally on a closed geodesic $\gamma$ and that $\chi(\gamma) = \gamma$. Then even though $\chi(S^* M) \cap S^* M$ has Liouville measure zero, the full sequence of matrix elements $\langle F \varphi_j, \varphi_j \rangle$ need not tend to zero.

**Remark:** It is natural to ask how sharp the almost nowhere invariance or commutation conditions are in the proof of almost orthogonality. That is, we ask whether the following converse to Proposition 3 is true:

**Question:** Denote the canonical relation of $F$ by $C$. Then in the notation of this section, do we have,
\[
\frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle F \varphi_{\lambda_j}, \varphi_{\lambda_j} \rangle|^2 \to 0 \implies C \text{ is almost nowhere invariant under } g^t?
\]

### 3.2. Almost nowhere commuting quantum maps on Kähler manifolds.

We now prove the analogue, Proposition 5, in the Kähler setting. Let $\chi_1, \chi_2$ be two quantizable symplectic diffeomorphisms of a compact Kähler manifold $(M, \omega)$ which almost nowhere commute in the sense that the set where $\chi_1 \chi_2 = \chi_2 \chi_1$ has measure zero. We then let $U_{\chi_1}$ denote the quantization of $\chi_1$ as a unitary operator on $H^0(M, L^k)$ and we let $F$ denote any quantum map quantizing $\chi_2$.

An interesting comparison to the previous case is that the semi-classical eigensections $\varphi_{k,j}$ do not appear to have any localization properties which account for the almost orthogonality of the matrix elements.

**Proof of Proposition 5**

We again consider the partial time average
\[
\langle F \rangle_M := \frac{1}{2M} \sum_{m=-M}^M U^{-m} F U^m.
\]

Going through the same argument gives the upper bound
\[
\frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle F \varphi_{\lambda_j}, \varphi_{\lambda_j} \rangle|^2 \leq \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle F \rangle_M^* \langle F \rangle_M \varphi_{\lambda_j}, \varphi_{\lambda_j} \rangle|.
\]
We then have
\[ \langle F \rangle_M^* \langle F \rangle_M = \frac{1}{M^2} U^t_0 (m-n)^* F^* U^t_0 (m-n) F \]
\[ = \frac{1}{M} \sum_{p=-M}^{M} \# p \, \frac{1}{M} U^t_0 F^* U^t_0 F, \]
where \( \# p = \{ (m, n) \in [-M, M] \times [-M, M] : m - n = p \} \). We then apply the local Weyl law and find that the only term which makes a non-vanishing contribution is \( p = 0 \). So it is \( O\left(\frac{1}{M}\right) \) and therefore the limit equals zero. QED

4. Modulus squares as matrix elements

In this section, we tie together the weak * limit problem for Fourier integral operators with some recent results on modulus squares \( |\varphi_j^C(z)|^2 \) of analytic continuations of eigenfunctions to Grauert tubes and their restrictions to geodesic arcs (cf. [Z8, Z10]), and to some related pointwise Weyl laws for coherent state projections in [PU]. The relevant matrix elements differ from the preceding ones in that the underlying canonical or wave front relations are not local canonical graphs. The results are correspondingly different: the canonical relations are not invariant under the geodesic flow, but Proposition 3 is false for them. When properly normalized the weak* limits can be non-zero. In fact, we will use an averaging argument or a flowout construction to make the canonical relation geodesic flow invariant when \((x, \xi)\) is a periodic point. This is impossible for a local canonical graph which is almost nowhere invariant.

4.1. \(|\varphi_j^C(z)|^2\) on a Grauert tube. We first consider pointwise modulus squares of \( |\varphi_j^C(z)|^2 \) of analytic continuations of eigenfunctions to Grauert tubes in the complexification of \( M \). We refer to [Z8, Z10] and their references for background on the analytic continuation and the geometry of Grauert tubes.

As discussed at length in [GS1, Z8], a Grauert tube \( M_\varepsilon \) is a strictly pseudo-convex domain in the complexification \( M_C \) of a real analytic Riemannian manifold \( (M, g) \). Its defining function \( \rho(z) \) is the analytic continuation of the squared distance function \( r^2(x, y) \) to \( x = z, y = \bar{z} \). The Grauert tube function is its square root \( \sqrt{\rho}(z) \) (we are ignoring here some constants). A key point is that is the image of a ball bundle \( B_\varepsilon^* M \) under the imaginary time exponential map \( E : B_\varepsilon^* M \rightarrow M_\varepsilon, E(x, \xi) = \exp_x i\xi \). The Grauert tube function \( \sqrt{\rho} \) corresponds to the norm function \( |\xi|_g \) of the metric under \( E \). Moreover \( E \) conjugates the geodesic flow to a Hamiltonian flow \( g^t \) on \( M_\varepsilon \) with respect to its adapted Kähler form.

The principal Fourier integral operator in this context is the Poisson kernel \( P^\varepsilon(z, y) \) on \( \partial M_\varepsilon \times M \), defined as follows: The wave group of \((M, g)\) is the unitary group \( U(t) = e^{it\sqrt{\Delta}} \). Its kernel \( U(t, x, y) \) solves the ‘half-wave equation’,
\[
\left( \frac{1}{i} \frac{\partial}{\partial t} - \sqrt{\Delta} \right) U(t, x, y) = 0, \quad U(0, x, y) = \delta_y(x).
\] (22)

The Poisson-wave kernel \( P^\tau(z, y) \) is the analytic continuation \( U(i\tau, x, y) \) of the wave kernel with respect to time, \( t \rightarrow i\tau \in \mathbb{R} \times \mathbb{R}_+ \) and then in \( x \), i.e.
\[
P^\tau(z, y) = U(i\tau, z, y).
\] (23)
Thus, the Poisson-kernel has the eigenfunction expansion for $\tau > 0$,

$$U(i\tau, x, y) = \sum_j e^{-\tau \lambda_j} \varphi_{\lambda_j}^C(z) \varphi_{\lambda_j}(y), \quad z \in \partial M_\tau, y \in M.$$  \hspace{1cm} (24)

Thus, $P^\tau \varphi_j = e^{-\tau \lambda_j} \varphi_j^C$.

The key fact is that $P^\tau(z, y)$ is a Fourier integral operator with complex phase. It is adapted to the symplectic isomorphism of $T^*M$ with the symplectic cone $\Sigma_\tau \subset T^*\partial M_\tau$ generated by the contact form $\alpha_z$. This result was stated by Boutet de Monvel in [Bou] but only recently have detailed proofs been published (cf. [Z8, Sl]). The main ingredient is the analytic continuation of the wave kernel due to J. Hadamard.

One of the main problems with the ‘states’ (27) is that they are not normalized. It is proved in [Z8] (Corollary 2) that $e^{-2\tau \lambda_j} |\varphi_j^C(z)|^2 \leq C\lambda^{m-1}$. The upper bound is sharp (it is attained by highest weight spherical harmonics on the standard $S^m$) but it is generally not attained on a generic analytic Riemannian manifold, nor is it attained at general points $z$ even when it is attained at some point; and even when it is attained at some $z$, it is only attained by a sparse subsequence.

To deal with these issues, we first observe that the diagonal matrix elements of $\Phi_z^\tau \otimes (\Phi_z^\tau)^*$ are the same as for the partial time averages

$$\langle \Phi_z^\tau \otimes (\Phi_z^\tau)^* \rangle_T := \frac{1}{2T} \int_{-T}^T U^t (\Phi_z^\tau \otimes (\Phi_z^\tau)^*) U^{-t} dt.$$  \hspace{1cm} (29)

The full time average is the limit $T \to \infty$ in (29) in the weak operator topology. In general, the Fourier integral properties of $\Phi_z^\tau \otimes (\Phi_z^\tau)^*$ are destroyed by infinite time averaging, or at best they are unclear. But if $z$ is a periodic point for $g^t$ of period $L$, then $\langle \Phi_z^\tau \otimes (\Phi_z^\tau)^* \rangle_L$ is a Fourier integral operator with complex phase on $M$ which commutes to leading order with
\[ \sqrt{\Delta}, \text{ i.e.} \]

\[
\left[ \sqrt{\Delta}, \langle \Phi_x^\tau \otimes (\Phi_x^\tau)^* \rangle_L \right] = \frac{1}{2L} \int_{-L}^{L} \frac{d}{dt} U^t (\Phi_x^\tau \otimes (\Phi_x^\tau))^* U^{-t} dt
\]

\[ = \frac{1}{2L} \left( U^L (\Phi_x^\tau \otimes (\Phi_x^\tau))^* U^{-L} - (\Phi_x^\tau \otimes (\Phi_x^\tau))^* \right). \]

So for a periodic point, we set \( T \) equal to an integer multiple of the periodic and substitute the time average into (26). Geometrically, this corresponds to replacing (28) by its flowout

\[ \bigcup_{t \in [0,L]} \Lambda_{g^t(z)} \times \Lambda_{g^t(z)} = \gamma \times \gamma, \]

(30)

where \( \gamma \) is the periodic orbit of \( z \). Obviously it is invariant under \( g^t \times g^t \).

Secondly, we pull back \( \varphi_j^* \) under a parametrization \( \gamma^*(t) \) of the \( g^t \) orbit of \( z \) on \( \partial M_\tau \). That is, we restrict the complexified eigenfunction to the orbit of \( z \). We then re-normalize it with \( e^{-2\pi \lambda_j} |\varphi_j^*(z)|^2 \) to have mass one on \( \gamma \). Here we use the orbit of \( g^t \) on \( \partial M_\tau \) and define

\[ U_j^\tau(t) = \frac{\varphi_j^* (\gamma^*(t))}{\| \gamma^* \varphi_j^* \|_{L^2([0,L])}}. \]

(31)

We then have

**Proposition 4.1.** [**Z10**] (Proposition 2) If \( z \) is a periodic point, and if \( U_j \) does not vanish identically on the orbit of \( z \), then the unique weak * limit of the positive unit mass measures \( \{|U_j|^2\} \) on the orbit \( \gamma \) of \( z \) in \( \partial M_\tau \) is \( \frac{1}{T} dt \), the normalized periodic orbit measure on \( \gamma \). Equivalently, \( \{|U_j|^2\} \rightarrow dt \) weakly as \( j \rightarrow \infty \).

The proof is given in [**Z10**]. One proves an Egorov type theorem in the class of Toeplitz operators. It shows that the weak* limits must be invariant under \( g^t \). Since they are invariant probability measures on \( \gamma \) the only possible limit is \( \frac{1}{T} dt \).

In [**Z8**], an asymptotic formula for the average over the spectrum of the matrix elements (27) is given. It involves the stability matrix of the geodesic flow along the orbit of \( z \). However, this data cancels when we take the quotient with \( \| \gamma^* \varphi_j^* \|_{L^2([0,L])} \).

One may understand Proposition 4.1 in terms of time averages of the Fourier integral operator \( \Phi_x^\tau \otimes (\Phi_x^\tau)^* \). The leading order evolution \( U^t \Phi_x^\tau \otimes (\Phi_x^\tau)^* U^{-t} \) is somewhat analogous to the evolution of the standard coherent states

\[ \psi_{x,\xi}(y) = 2^{-n/4}(2\pi\hbar)^{-n/2} e^{-i\frac{x \cdot \xi}{2\hbar}} e^{i\frac{x \cdot y}{\hbar}} e^{-\frac{(x-y)^2}{2\hbar}} \]

on \( \mathbb{R}^n \), although it is closer to that of coherent states in the Bargmann-Fock representation. The evolution of coherent states has been studied extensively in various settings. The model case of evolution of standard coherent states in the Schrödinger or Bargmann-Fock representations under linear Hamilton flows is discussed in detail in [**CR**], and the evolution of coherent states on manifolds are discussed in [**PU**]. The case relevant here is that of the Poisson FBI transform to Grauert tubes and the the Poisson coherent states are discussed in [**Z8**] [**Z10**]. In each case, to leading order the projection evolves as that of a distorted coherent state projection \( \Phi_{g^t(z)}^* \otimes (\Phi_{g^t(z)}^*)^* \) ‘centered’ on the orbit of \( z \). The shape distortion is due to the Jacobi stability matrix (i.e. \( Dg^t \) along the orbit). Thus, the time average (29) with \( T = L_\gamma \) (the period of the orbit) is a Fourier integral operator with complex phase space with wave front set along \( \gamma \times \gamma \) and with symbol constant along the orbit.
It would be interested to find the optimal analogue of Proposition 4.1 when $z$ is not a periodic point, e.g. when it is a regular point for $g^t$ and when $g^t$ is ergodic on $\partial M_r$. This case is also studied in [Z10]. The problem is that one cannot $L^2$ normalize on the entire orbit unless one uses a weight, e.g. the characteristic function of an interval. But it is apriori possible that the local $L^2$ norms are incommensurable along the orbit. This cannot happen over fixed compact sets but the norms of the restrictions could have different orders of magnitudes on parameter intervals for $\gamma : \mathbb{R} \to \partial M_r$ which are separated by an amount greater than the Ehrenfest time $C \log \lambda$. 

4.2. Coherent states. A variation on the preceding example is to study matrix elements for coherent state projectors, where the coherent states are defined by (32), or more generally (as in [PU]),

$$\psi_{x,\xi}^a(y) = \rho(x - y)2^{-n/4}(2\pi \hbar)^{-n/2} e^{-i\epsilon x^\xi} e^{i\epsilon x^\xi} a((x - y) / \sqrt{\hbar})$$

for $a \in \mathcal{S}(\mathbb{R})$. The coherent states (or wave packet) transform is defined by $f \to \langle f, \psi_{x,\xi}^h \rangle$. Like (26), it is an FBI transform but it is adapted to the heat kernel rather than the Poisson kernel and has different inversion properties.

As above, $\psi_{x,\xi}^a \otimes \psi_{x,\xi}^{a*}$ is a semi-classical Fourier integral operator with complex phase. If $(x, \xi)$ is a periodic point of period $L$ we again consider,

$$\langle \psi_{x,\xi}^a \otimes \psi_{x,\xi}^{a*} \rangle_L := \frac{1}{2L} \int_{-L}^L U^t(\psi_{x,\xi}^a \otimes \psi_{x,\xi}^{a*})U^{-t} dt.$$ 

As above, $\langle \langle \psi_{x,\xi}^a \otimes \psi_{x,\xi}^{a*} \rangle_L \psi_j, \psi_j \rangle = \langle \psi_{x,\xi}^a \otimes \psi_{x,\xi}^{a*} \psi_j, \psi_j \rangle$. To leading order in $\hbar$ it is the same as

$$\langle \psi_{x,\xi}^a \otimes \psi_{x,\xi}^{a*} \rangle_L \simeq \frac{1}{2L} \int_{-L}^L \psi_{G^t(x,\xi)}^{a_t} \otimes \psi_{G^t(x,\xi)}^{a_t*} dt,$$

where $a_t$ is a deformed symbol whose principal part is $a_0 \circ dg^t$ (cf. [CR, PU]).

As in the setting of analytic continuation to Grauert tubes, we restrict the diagonal matrix elements $|\langle \langle \psi_{x,\xi}^a \otimes \psi_{x,\xi}^{a*} \rangle_L \psi_j, \psi_j \rangle|^2$ to the orbit of $(x, \xi)$ and view them as a sequence of positive measures. When $(x, \xi)$ is a periodic point, we restrict to the pull back to $[0, L]$ and divide by the mass to obtain a sequence of probability measures on $[0, L]$. The weak* limits are then constant multiples of Lebesgue measure as in Proposition 4.1.

5. Proof of Proposition 4

This additional symmetry of Proposition 4 comes from the fact that $I^0(M \times M, C)$ is a right and left module over $\Psi^0(M)$ with respect to composition of operators, and that on the symbol level the left and right compositions commute. We thus consider the case where $\rho_j(AF) \sim \rho_j(FA)$ where $A \in \Psi^0(M)$, e.g. if $F^* = F$ as in the cases cos $t\sqrt{\Lambda}$ or $T_p$, or if $\varphi_j$ is an eigenfunction of $F$. We restate Proposition 4 and refer to the diagram (11).

**Proposition 5.1.** Suppose that $\rho_j(AF) \sim \rho_j(FA)$ and that the canonical relation $C$ is invariant under the geodesic flow $G^t$. Then the limit measures are among the $G^t$-invariant signed measures of mass $\leq 1$ on $C$ satisfying $\rho_\infty(\pi^*_X a \sigma) = \rho_\infty(\pi^*_1 a \sigma)$.
Corollary 5.2. Let $i : C \to C$ be the involution $i(\xi, \eta) = (\eta, \xi)$. Let $\nu_\infty$ be the limit on $\Psi^0$. Then $i^* \rho_\infty = \rho_\infty$. Equivalently,

$$\pi_1^* \rho_\infty = \pi_2^* \rho_\infty = \nu_\infty.$$ 

In the case where $C$ is the graph of a canonical transformation $\chi$, this says that if we identify $\rho_\infty$ with a measure on the domain, then $\chi^* \rho_\infty = \rho_\infty$. In the general case of a local canonical graph, it does not say that $\rho_\infty$ is invariant under the correspondence but rather than it is invariant under each local branch.

Proof. If we fix one (elliptic) element $F \in I^0(M \times M, C)$ (i.e. with nowhere vanishing symbol) then $I^0(M \times M, C)$ is spanned by sums of operators $AFB$ with $A, B \in \Psi^0(M)$. Thus $\rho_j$ induces functionals on $\Psi^0(M) \times \Psi^0(M)$ of the form

$$\rho_j(AFB) := \langle AFB \varphi_j, \varphi_j \rangle,$$

which can be normalized to have mass one by

$$\hat{\beta}_j(AFB) := \frac{\langle AFB \varphi_j, \varphi_j \rangle}{\langle F \varphi_j, \varphi_j \rangle}, \text{ if } F \varphi_j \neq 0.$$  

Thus, we can define right and left functionals

$$\beta_j^L(A) = \rho_j(AF), \quad \beta_j^R(A) = \rho_j(FA) \text{ on } \Psi^0(M)$$

and obtain signed limit measures on $S^*M$ from the weak* limits. Then $\{\beta_j^L\}$ and $\{\beta_j^R\}$ have weak* limits along the same subsequences and the limits are the same.

Then

$$\rho_\infty(\pi^*_X a \sigma) \sim \rho_j(AF) = \langle AF \varphi_j, \varphi_j \rangle = \overline{\langle F^* A^* \varphi_j, \varphi_j \rangle} = \langle FA^* \varphi_j, \varphi_j \rangle \sim \pi^*_Y a \sigma$$

since $\sigma_A^* \sim \overline{\sigma_A}$ and since $\overline{\sigma_F} = \sigma_F$.

Remark: When $F$ is associated to a symplectic diffeomorphism $\chi$ of $T^*M \setminus 0$, then the left and right functionals are related as follows:

$$\beta_j^R(A) = \rho_j(FA) = \rho_j(FAF^{-1}F) = \beta_j^L(F^{-1}AF).$$  

By Egorov’s theorem $F^{-1}AF$ is a pseudo-differential operator with symbol $\sigma_A \circ \chi$, and so along any sequence with a weak* limit

$$\beta_\infty^R(\sigma_A) = \beta_\infty^L(\sigma_A \circ \chi).$$
6. Isometries and Hecke Operators

In this section we consider the weak* limits in Proposition I and Corollary I in the case of isometries and sums of isometries known as Hecke operators. By lifting the weak* limit problem to the canonical relation instead of $S^*M$ and using Proposition 5.1 we obtain a new invariance principle. We also make concrete identifications of the canonical relations.

In discussing the canonical relations of Hecke operators we make use of the well-known co-tangent lift $f_\sharp$ of a diffeomorphism $f : X_1 \to X_2$, defined by

$$f_\sharp(x_1, \xi_1) = (x_2, \xi_2), \quad \text{with} \quad \begin{cases} x_2 = f(x_1), \\ \xi_2 = df_{x_1}^*\xi_2 \in T_{x_1}^*X_1, \end{cases}$$

where

$$(df_{x_1})^* : T_{x_2}^*X_2 \to T_{x_1}^*X_1,$$

so that

$$f_\sharp|_{T_{x_1}^*} = (df_{x_1})^{-1}.$$ 

Then $f_\sharp$ is a symplectic diffeomorphism.

6.1. Isometries: $F = T_g$. We begin with the simplest example where $F = T_g$ is translation by an isometry $T_g f(x) = f(gx)$ of a Riemannian manifold $(M, ds^2)$. The canonical relation is the graph $\Gamma_g$ of the lift $[37]$ of $T_g$ to $T^*M$. Since $g$ is an isometry, $T_g$ commutes with $\Delta$ and we consider thier joint eigenfunctions with $T_g\varphi_j = e^{i\theta_j}\varphi_j$ for some $e^{i\theta_j} \in S^1$ (such as spherical harmonics $Y^l_m$ if $g$ is a rotation around the vertical axis). The relevant space of operators $P^0(M \times M, \Gamma_g)$ is spanned by sums of operators $AT_gB$ with $A, B \in \Psi^0(M)$ and so it suffices to consider the functionals

$$\hat{\rho}_j(AT_gB) = \langle AT_gB\varphi_j, \varphi_j \rangle = \langle AT_gB T^{-1}_g T_g\varphi_j, \varphi_j \rangle. \quad (38)$$

Note that $AT_gB T^{-1}_g \in \Psi^0(M)$. As one sees from the case $A = B = I$, a subsequence $\hat{\rho}_{j_k}$ can only have a unique weak* limit if the associated eigenvalues $e^{i\theta_{j_k}}$ have a limit. To define matrix elements with larger subsequential limits, we re-normalize the functionals by

$$\rho_j(AT_gB) := \langle AT_gB\varphi_j, \varphi_j \rangle = \langle AT_gB T^{-1}_g \varphi_j, \varphi_j \rangle. \quad (39)$$

Here, we assume $\langle T_g\varphi_j, \varphi_j \rangle \neq 0$. The reader may prefer the $\hat{\rho}_j$; the methods and results apply equally to them and to $[39]$.

We could regard the weak* limits as measures on $S^*M$ or on $S\Gamma_g$, the graph of the lift of $T_g$ on $S^*M$, which has the form $\{(\zeta, g \cdot \zeta) : \zeta \in S^*M\}$. To illustrate the lift, we write the quantum limit as

$$\rho_\infty(AT_gB) = \int_{S^*M} a(\zeta)b(g \cdot \zeta) d\nu(\zeta, g \cdot \zeta). \quad (40)$$

Proposition 6.1. Let $g \in Isom(M, ds^2)$ and let $\nu_g$ be a weak* limit measure for the functionals $\rho_j(F) = \langle F\varphi_j, \varphi_j \rangle$ on $P^0(M \times M, C_g)$. Suppose that $T_g\varphi_j = e^{-i\theta_j}\varphi_j$. Then under the identification $S^*M \to C_g, \zeta \to (\zeta, g \cdot \zeta)$, $\nu = \pi^*\tilde{\nu}$ is a signed measure of mass $\leq 1$ on $S^*M$ which is invariant under both $G^t$ and $g$. 

Proof. We have,
\[ \int_{S^*M} a(\zeta) d\nu(\zeta, g \cdot \zeta) = \int_{S^*M} a(\zeta) d\omega(\zeta), \]
and
\[ \int_{S^*M} b(\zeta) d\omega(\zeta) = \int_{S^*M} b(g \cdot \zeta) d\nu(\zeta, g \cdot \zeta) = \int_{S^*M} b(\zeta) d\nu(g^{-1} \zeta, \zeta), \]
which implies
\[ \int_{S^*M} a(\zeta) d\nu(\zeta, g \cdot \zeta) = \int_{S^*M} a(\zeta) d\nu(g^{-1} \zeta, \zeta), \]
i.e. \( d\nu(\zeta, g \cdot \zeta) = d\nu(g^{-1} \zeta, \cdot) \) or
\[ \int_{S^*M} a(\zeta) d\omega(\zeta) = \int_{S^*M} a(g\zeta) d\omega(\zeta). \]  
(41) □

6.2. Hecke operators on spheres. The now consider the interesting example of self-adjoint Hecke operators on \( S^n \), i.e. sums
\[ Tf(x) = \frac{1}{2d} \sum_{j=1}^{d} (f(g_jx) + f(g_j^{-1}x)), \quad g_j \in SO(n+1) \]  
(42)
of isometries on \( S^n \). Note that \( T \) is normalized so that \( T1 = 1 \) and \( ||T|| = 1 \). They are a helpful guide to Hecke correspondences on hyperbolic quotients in the next section, and have a considerable literature of their own (see \[LPS\] and subsequent articles). The main result of this section, Proposition 6.3, gives a new invariance principle for the quantum limits of joint eigenfunctions of \( T \) and \( \Delta \). A key point is to view the limit measures as measures on the canonical relation, which we may identify with \( \bigcup_{j=1}^{k} S^*S^n \), rather than on \( S^*S^n \).

A Hecke operator \( (42) \) is a discrete Radon transform \( T = \rho_*\pi^* \) corresponding to the trivial cover,
\[ \bigcup_{j=1}^{2d} S^n \]
\[ \pi \quad \rho \]
(43)
where \( \pi(x,j) = x \) and \( \rho(x,j) = g_jx \). The canonical relation is the cotangent lift of the graph
\[ \mathcal{G}_{T_{g_j}} = \{(z, g_jz) : x \in S^n\} \]
of the isometric correspondence
\[ C_T(x) = \{g_1x, \ldots, g_{2d}x\}. \]
Thus we have a second diagram
\[ \mathcal{G}_{T_g} \subset S^n \times S^n \]

The graph of the correspondence is an immersed submanifold of \( S^n \times S^n \) with self-intersections when the graphs of the individual isometries \( g_j \) intersect. This is clear since \( \iota(x, j) = \iota(x', j') \implies x = x' \) and \( g_j x = g_j x' \) or \( g_j^{-1} g_j x = x \). The diagrams (43) and (44) and are related as follows.

**Lemma 6.2.** The map

\[ \iota : \bigcup_{j=1}^{2d} S^n \to \mathcal{G}_{T_g} \]

defined by

\[ \iota(\hat{x}) = (\pi(\hat{x}), \rho(\hat{x})), \quad \text{or} \quad \iota(x, j) = (x, g_j x) \]

is an immersion whose image is the graph.

The canonical relation \( C_T \subset T^*(S^n \times S^n) \) of the Hecke operator \( T \) is the graph of the sum of cotangent lifts of the individual isometries, and the cotangent lift of the map in Proposition 6.2 gives a parametrization

\[ \iota : \bigcup_{j=1}^{2d} S^* S^n \to C_T \]

defined by

\[ \iota(\hat{x}) = (\pi_\sharp(\hat{x}), \xi, \rho_\sharp(\hat{x}), \xi), \quad \text{or} \quad \iota_\sharp(x, \xi, j) = (x, g_j x, \xi, D g_j^{-1} \xi). \quad (45) \]

Thus, we may consider quantum limit measures as measures on \( \bigcup_{j=1}^{2d} S^* S^n \), i.e. as a finite set \( \{ \frac{1}{2}(\nu_j + \nu_j) \}_{j=1}^{d} \) of real signed measures on \( S^* S^n \).

When \( \langle T u_j, u_j \rangle \neq 0 \), we can define the Wigner functionals by the following normalized matrix elements (cf. (39))

\[ \rho_{j, T}(A B) = \frac{\langle A B u_j, u_j \rangle}{\langle T u_j, u_j \rangle}, \quad A, B \in \Psi^0(S^n). \quad (46) \]

We denote by \( \hat{\rho}_j \) the other normalization as in (38).

We can also define the Wigner functionals for the individual isometries \( g_j \):

\[ \rho_{j, g_k}(A B) = \frac{\langle A T g_k B u_j, u_j \rangle}{\langle T g_k u_j, u_j \rangle}, \quad A, B \in \Psi^0(S^n). \quad (47) \]

Note that

\[ \rho_{j, g_k^{-1}}(A B) = \rho_{j, g_k}(B^* T A^*). \]

Exactly as in Proposition 6.1, the weak * limits of the \( \rho_{j, g_k} \) for each \( k \) are linear functionals of the form

\[ \nu_k(A B) = \int_{S^* S^n} a(\zeta) b(g_k \zeta) d\nu_k(\zeta, g_k \zeta), \quad (48) \]
Although they are not necessarily invariant under $g_k$. It follows that
\[ \int_{\Gamma_T} a(x)b(y)d\nu_T(x,y) = \sum_k \int_{S^*S^n} a(\zeta)b(g_k\zeta)d\nu_k(\zeta, g_k\zeta). \]

Lift $\{u_j\}$ to $\{\pi^*u_j\}$ on $\bigcup_{j=1}^{2d} S^n$ and let $\nu$ be a quantum limit measure of the sequence. If $A, B$ are microsupported on the $k$th component, then the quantum limit is $\nu_k$. That is,
\[ \int_{\Gamma_T} a(x)b(y)d\omega_T(x,y) = \sum_k \int_{S^*S^n} a_k(\zeta)b_k(g_k\zeta)d\nu_k(\zeta, g_k\zeta). \]

In Proposition 6.1, the measure $\nu_k$ is defined on the graph of the lift of $T_{g_k}$ to $S^*M$; it may (and will) be identified with a measure on $S^*M$. Applying Proposition 4 we get

**Proposition 6.3.** Suppose that $\{u_{jk}\}$ is a sequence of joint $\Delta - T$ eigenfunctions for which (1) has a unique weak * limit $\mu$ on $S^*S^n$ and (39) has a unique weak* limit $\nu$ on $\bigcup_{j=1}^{2d} S^*S^n$ as in Proposition 6.2. Then $\pi_*\nu = \rho_*\nu = \mu$. Moreover, each $\rho_{j,k,g,T}$ in (47) has a weak limit $\nu_k$, and
\[ \mu = \frac{1}{2d} \sum_{j=1}^{2d} \nu_j = \frac{1}{2d} \sum_{j=1}^{2d} (g_j)_*\nu_j = \frac{1}{2d} \sum_{j=1}^{2d} (g_j^{-1})_*\nu_j. \]

The measures $\nu_j$ are absolutely continuous with respect to $\mu$ and for each $g_j$, and also $T_{g_j} \nu_j$ is absolutely continuous with respect to $\mu$. Similarly for all powers $T^n$.

**Remark:** If we use $\hat{\rho}_j$ (cf. (38)) instead of (46), then the statement should be: the sequence of eigenvalues $\rho_{jk}(T)$ of $T$-eigenvalues has a limit $\rho(T)$ and
\[ \mu = \frac{1}{2d} \sum_{j=1}^{2d} \nu_j = \frac{\rho(T)}{2d} \sum_{j=1}^{2d} (g_j)_*\nu_j. \]

**Proof.** First, the statement that $\mu = \frac{1}{2d} \sum_{j=1}^{2d} \nu_j$ follows from the fact that the quantum limit $\nu$ is calculated on $\bigcup_{j=1}^{2d} S^n$ using the pullbacks of $\{u_{jk}\}$. If we test against an operator which is also pulled back we get $2d$ times the quantum limit on the base. The statement that
\[ \mu = \frac{1}{2d} \sum_{j=1}^{2d} (g_j)_*\nu_j \]
comes the fact that $\{u_{jk}\}$ is a sequence of $T$-eigenfunctions. Then the second equality is obvious by taking limits of $\langle TAu_{jk}, u_{jk} \rangle$ for operators pulled back from the base.

Since $T_{g_k}$ is unitary,
\[ |\langle T_{g_k}A\varphi_j, \varphi_j \rangle| \leq |\langle A^*A\varphi_j, \varphi_j \rangle| \]
and that implies $\nu_k << \mu$. Indeed, (51) implies that along the relevant subsequence,
\[ \lim |\langle T_{g_k}A\varphi_j, \varphi_j \rangle| = \int_{S^*M} \sigma_A d\nu_k \leq \int_{S^*M} \sigma_A^2 d\mu. \]

Also, if $A^* = A$,
\[ |\langle T_{g_k}A\varphi_j, \varphi_j \rangle| \leq |\langle T_{g_k}AT_{g_k}^{-1}T_{g_k}\varphi_j, \varphi_j \rangle| \leq \langle (T_{g_k}AT_{g_k}^{-1})^2 \varphi_j, \varphi_j \rangle \]
and
\[ \lim |\langle T_{g_k}A\varphi_j, \varphi_j \rangle| \leq \int_{S^*M} \sigma_A d\nu_k \leq \int_{S^*M} (T_{g_k}\sigma_A)^2 d\mu. \]
and therefore $d\nu_k << g_k^{-1} d\mu$. Also
\[
|\langle AT_{g_k}^* \varphi_j, \varphi_j \rangle|^2 = |\langle T_{g_k} T_{g_k}^{-1} AT_{g_k} \varphi_j, \varphi_j \rangle|^2 \leq \langle (T_{g_k}^{-1} AT_{g_k})^2 \varphi_j, \varphi_j \rangle
\] (55)
and
\[
\lim |\langle T_{g_k}^{-1} A \varphi_j, \varphi_j \rangle|^2 = |\int_{S^*M} \sigma_A d\nu_k|^2 \leq \int_{S^*M} (T_{g_k}^{-1} \sigma_A)^2 d\mu,
\] (56)
and therefore $d\nu_k << g_k d\mu$.

Here we are using \[\text{(35)-(36)}\] termwise on each term and the fact that $\varphi_j$ is a $T$-eigenfunction. That is, $\nu_j$ is essentially the left limit and $g_j^* \nu_j$ is the right limit. Inequality (51) is the inequality for the right functional and (53) is the inequality for the left functional.

\[\square\]

Remark: If instead of $\rho_j$ we use $\hat{\rho}_j$, then in order for the weak limit to exist, we need a fortiori that the sequence $\langle Tu_j, v_j \rangle$ tends to a limit, i.e. that the associated sequence of eigenvalues tends to a limit. The rest proceeds as above.

**Corollary 6.4.** In the notation of Proposition 6.3, let $\Lambda_j$ be the singular support of $\nu_j$ and let $\Lambda$ be the singular support of $\mu$. Then $\Lambda = \bigcup_j \Lambda_j = \bigcup_j g_j(\Lambda_j)$.

**Proof.** Let $\Lambda_j$ be the singular support of $\nu_j$. In view of (19), $\Lambda \subset \bigcup_{j=1}^d \Lambda_j$. Strict inclusion is a priori possible since cancellations may occur among the terms. On the other hand, since all $\nu_j << \mu$, $\Lambda_j \subset \Lambda$ for all $j$ and therefore $\bigcup_{j=1}^{2d} \Lambda_j \subset \Lambda$. It follows that
\[
\Lambda = \bigcup_{j=1}^{2d} \Lambda_j.
\] (57)
It also follows from (19) that $\Lambda \subset \bigcup_{j=1}^{2d} g_j \Lambda_j$, since the singular support of $g_j \nu_j$ is $g_j \Lambda_j$. But also by (24), $g_j^* \nu_j << \mu$, so $g_j^* \nu_j$ has singular support in $\Lambda$ for all $j$, i.e. $g_j \Lambda_j \subset \Lambda$ and therefore
\[
\Lambda = \bigcup_{j=1}^{2d} g_j \Lambda_j.
\] (58)

**Remark:** If $T \varphi_j = \rho_j(T) \varphi_j$ then also $T^\ell \varphi_j = \rho_j(T)^{\ell} \varphi_j$. The Proposition and Corollary apply equally to all powers of $T$.

For generic $T$, we can obtain some simple restrictions on weak* limits from Corollary 6.4 and Remark 6.2. For simplicity, we assume that $n = 2$, so that each maximal abelian subgroup is one dimensional. We assume that each $g_j$ is a topological generator in a maximal abelian subgroup, i.e. that it is a rotation with an irrational angle. We also assume that the $g_j$ are pairwise independent in the sense that no two generate the same circle of rotations (unless they are inverses).

As a sample application, we show that $\Lambda$ cannot be a single closed geodesic $\gamma$, i.e. $d\mu$ cannot be a constant multiple of the invariant probability measure $\mu_\gamma$ on $\gamma$. Otherwise, we would have $\nu_j = c_j \mu_\gamma$ with $|c_j| \leq 1$ for all $j$. Certainly $c_{j_0} \neq 0$ for some $j_0$. Corollary 6.4 then forces $g_{j_0} \gamma = \gamma$, i.e. $g_{j_0}$ is a generator of $\gamma$. But by the assumption that the $g_j$ are
independent, this forces the other $c_j = 0$ (i.e. except for $g_{j_0}, g_{j_0}^{-1}$). By Proposition 6.3 we would then have $\mu_\gamma = \frac{1}{2d}(c_{j_0} + \gamma c_{j_0})\mu_\gamma = \frac{\rho(T)}{2d}(c_{j_0} + \gamma c_{j_0})\mu_\gamma$. This leads to several contradictions. First, $|c_{j_0}| \leq 1$ and $d \geq 1$, so it is impossible that $1 = \frac{1}{2d}(c_{j_0} + \gamma c_{j_0})$. The equation also forces $\rho(T) = 1$. This is a contradiction when $T$ has a spectral gap [LPS]. Further contradictions arise if we consider powers of $T$, since almost all terms in the weak * limit sums must vanish for similar reasons.

6.3. Hecke operators on hyperbolic quotients. We now generalize Proposition 6.3 to arithmetic hyperbolic quotients. The arguments are essentially the same, with $H$ replacing $S^n$ everywhere. The one difference is that we have an additional discrete group $\Gamma$ operating which commutes with the Hecke operator so that it acts on the quotient. However, we only need to use it to ensure that the relevant cover of $H$ is finite sheeted.

Let $\Gamma$ is a co-compact (or cofinite) discrete subgroup of $G = \text{PSL}(2, \mathbb{R})$ and let $X_\Gamma$ be the corresponding compact (or finite area) hyperbolic surface. An element $g \in G, g \notin \Gamma$ is said to be in the commensurator $\text{Comm}(\Gamma)$ if $\Gamma' := \Gamma \cap g^{-1}\Gamma g$ is of finite index in $\Gamma$ and $g^{-1}\Gamma g$. More precisely,

$$\Gamma = \bigcup_{j=1}^d \Gamma'(g)\gamma_j, \text{ (disjoint)},$$

or equivalently

$$\Gamma g \Gamma = \bigcup_{j=1}^d \Gamma \alpha_j, \text{ where } \alpha_j = g\gamma_j.$$ 

It is also possible to choose $\alpha_j$ so that $\Gamma g \Gamma = \bigcup_{j=1}^d \alpha_j\Gamma$. We refer to [Sh] for background, or [Z1] for notation of this section.

Similar to (43), we then have a diagram of finite (non-Galois) covers:

$$\Gamma'(g) \setminus H \overset{\pi}{\underset{\rho}{\leftarrow \rightarrow}} g^{-1}\Gamma g \setminus H.$$ 

Here,

$$\pi(\Gamma'(g)z) = \Gamma z, \quad \rho(\Gamma'(g)z) = g\Gamma g^{-1}(g\gamma_j z) = g\Gamma z,$$

where in the definition of $\rho$ any of the $\gamma_j$ could be used. The horizontal map is $z \rightarrow g^{-1}z$. The Radon transform $\rho_* \pi^*$ of the diagram defines a Hecke operator $T_g : L^2(X_\Gamma) \rightarrow L^2(X_\Gamma)$,

$$T_g f(x) = \frac{1}{d} \sum_{j=1}^d f(g\gamma_j x).$$

Then $T_{g^{-1}} u(z) = u(g^{-1}z)$ takes $\Gamma$-invariant functions to $g^{-1}\Gamma g$-invariant functions. Hecke operators are self-adjoint and commute with the hyperbolic Laplacian and we can consider joint eigenfunctions of $\Delta$ and $T_g$,

$$T_g u_j = \rho_j(T_g) u_j.$$
The main difference to the case of $S^n$ is that the covering is not trivial, i.e. not a disjoint union of $d$ copies of the base. However, below we uniformize so that it does become trivial.

The Hecke operator is a kind of averaging operator over orbits of the Hecke correspondence, which is the multi-valued holomorphic map $C_g(z) = \{\alpha_1 z, \ldots, \alpha_d z\}$. Its graph $\mathcal{G}_g = \{(z, \alpha_j z) : z \in X_\Gamma\}$ is an algebraic curve in the product $X_\Gamma \times g^{-1} \Gamma g \backslash H$. Similar to (44) we have a second diagram

$$\mathcal{G}_g \subset X_\Gamma \times g^{-1} \Gamma g \backslash H$$

\[ \pi_1 \leftarrow \downarrow \rho \]

$(60)$

Here, $\pi_1, \rho_1$ are the natural projections. The following is a kind of analogue of Lemma 6.2.

**Lemma 6.5.** The map

$$\iota : \Gamma'(g) \backslash H \to \Gamma_g$$

defined by

$$\iota(z) = (\pi(z), \rho(z)),$$

is a local diffeomorphic parametrization, and $\pi_1 \iota = \pi, \rho_1 \iota = \rho$.

*Proof.* It is obvious that $\iota$ is well-defined, takes its values in $\Gamma_g$ and intertwines the projections. Each of the maps $\pi, \rho$ is itself a local diffeomorphism and therefore $\iota$ also is. The ‘fiber’ over $(z, \alpha_j z)$ is multiple if and only if there exist $\alpha_j, \alpha_k, j \neq k$ such that $\alpha_j z = \alpha_k z$ and this can only happen for a finite set of $z$. $\square$

We now uniformize and consider Hecke operators on $H$ as $\Gamma$-periodic versions of the Hecke operators on spheres. We then obtain a picture similar to that of $S^n$, where

$$\bigcup_{j=1}^d H \times \{j\}$$

\[ \pi \leftarrow \downarrow \rho \]

$(61)$

where $\pi(x, j) = x$ and $\rho(x, j) = g\gamma_j x$. The Radon transform $\rho_* \pi^*$ of the diagram defines the Hecke operator $T_g : L^2(H) \to L^2(H)$. Exactly as in (6.2) in the case of $S^n$, the map

$$\iota : \bigcup_{j=1}^d H \times \{j\} \to \mathcal{G}_{T_g} \subset H \times H$$

defined by

$$\iota(\hat{x}) = (\pi(\hat{x}), \rho(\hat{x})), \quad \text{or} \quad \iota(x, j) = (x, g\gamma_j x)$$

$(62)$

is an immersion whose image is the graph $\Gamma_T$. The self-intersection points occur when $\iota(x, j) = \iota(x', j') \implies x = x'$ and $g_j x = g_{j'} x$ or $g_j^{-1} g_j x = x$. If we take the quotient by $\Gamma$ of diagram (61) we get diagram (59). But it is preferable to regard all of the functionals as defined on pseudo-differential operators on $H$ with compact spatial support. Taking the quotient by $\Gamma$ amounts to cutting off $\Gamma$-invariant pseudo-differential operators to fundamental
domains, but nothing is lost (and some generality is gained) by using non $\Gamma$-invariant pseudo-differential operators.

6.4. Quantum limits on $H$. On $H$ we can define the Wigner functionals for the individual isometries $g_j$:

$$\rho_{j,g}(ATB) = \frac{\langle ATg_kBu_j, u_j \rangle}{\langle Tu_j, u_j \rangle}, \quad A, B \in \Psi^0_c(H).$$

By Proposition 6.1, the limits have the form

$$\int_{S^*H} a(\zeta)b(g_k\zeta)d\nu_k(\zeta, g_k\zeta).$$

As with $S^n$ it follows that

$$\int_{\hat{\Gamma}} a(x)b(y)d\nu_T(x,y)) = \sum_k \int_{S^*H} a(\zeta)b(g_k\zeta)d\nu_k(\zeta, g_k\zeta).$$

Here, $d\nu_k$ is a measure on the $k$th copy of $H$. The Hecke limit measure is the sum

$$\nu = \sum_j \nu_{g_j} \quad \text{on} \quad \bigcup_j S^*H \times \{j\},$$

with $\nu_{g_j}$ living on the $j$th copy $S^*H \times \{j\}$.

When $\langle T_p u_j, u_j \rangle \neq 0$, we can define the Wigner functionals by the following normalized matrix elements:

$$\rho_{j,g}(ATpB) = \frac{\langle ATpBu_j, u_j \rangle}{\langle T_pu_j, u_j \rangle}, \quad A, B \in \Psi^0_c(H).$$

Completely analogously to Proposition 6.3, we have:

**Proposition 6.6.** Suppose that $\{u_{jk}\}$ is a sequence of joint $\Delta - T$ eigenfunctions for which (1) has a unique weak * limit $\mu$ on $S^*H$ and (39) has a unique weak* limit $\nu$ on $\bigcup_{j=1}^d S^*H$. Then $\pi_*\nu = \rho_*\mu = \omega$. Moreover, each $\rho_{j,k,g,T}$ in (47) has a weak limit $\nu_k$ and

$$\mu = \frac{1}{d} \sum_{j=1}^d \nu_j = \frac{1}{d} \sum_{j=1}^{2d} (g_j)_*\nu_j.$$  \hspace{1cm} (65)

The measures $\nu_j$ are absolutely continuous with respect to $\mu$ and for each $g_j$, and also $T_{g_j}\nu_j$ is absolutely continuous with respect to $\mu$. Similarly for all powers $T^n$. Moreover, $\mu$ is $\Gamma$-invariant.

The proof is essentially the same as for Proposition 6.3 and is omitted. The fact that $H$ is of infinite volume and the pseudo-differential operators are spatially compactly supported does not change the proof in any significant way. The only new statement is that $\mu$ is $\Gamma$-invariant, which is obvious from the fact that $T$ commutes with $\Gamma$.

**Corollary 6.7.** In the notation of Proposition 6.3, let $\Lambda_j$ be the singular support of $\nu_j$ and let $\Lambda$ be the singular support of $\mu$. Then $\Lambda = \bigcup_j \Lambda_j = \bigcup_j g_j(\Lambda_j)$.

Again the proof is the same as for Corollary 6.4.
6.5. **Quantum limits on** $S^*\Gamma'(g)\backslash H$. Instead of viewing $\nu$ as a measure on $\bigcup_j S^*H \times \{j\}$ as in Proposition 6.6, we may take the quotient by $\Gamma$ and view it as a measure on $S^*(\Gamma'(g)\backslash H) = \Gamma'(g)\backslash G$. Note that the quotient by $\Gamma$ glues together the $d$ disjoint copies of $H$ and then takes the quotient of the resulting $H$ by $\Gamma'(g)$. To explain the $\Gamma$-invariance properties we prove

**Lemma 6.8.** $\sum_j \nu_j$ is a $\Gamma$-invariant measure on $S^*H \cong G$.

**Proof.** Let $A, B$ be compactly supported pseudo-differential operators on $H$. Since $T_g L_\gamma = L_\gamma T_g$,

$$
\langle AT_g Bu_j, u_j \rangle = \langle AT_g BL_\gamma u_j, L_\gamma u_j \rangle
= \langle L_\gamma^* AT_g BL_\gamma u_j, u_j \rangle
= \langle L_\gamma^* AL_\gamma T_g L_\gamma^* BL_\gamma u_j, u_j \rangle.
$$

Taking the limit gives $\gamma_* \nu = \nu$. 

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