ALEXANDROV GROUPOIDS AND THE NUCLEAR DIMENSION OF TWISTED GROUPOID C*-ALGEBRAS

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Abstract. We consider a twist $E$ over an étale groupoid $G$. When $G$ is principal, we prove that the nuclear dimension of the reduced twisted groupoid C*-algebra is bounded by a number depending on the dynamic asymptotic dimension of $G$ and the topological covering dimension of its unit space. This generalizes an analogous theorem by Guentner, Willett, and Yu for the C*-algebra of $G$. Our proof uses a reduction to the unital case where $G$ has compact unit space, via a construction of “groupoid unitizations” $\tilde{G}$ and $\tilde{E}$ of $G$ and $E$ such that $\tilde{E}$ is a twist over $\tilde{G}$. The construction of $\tilde{G}$ is for r-discrete (hence for étale) groupoids $G$ which are not necessarily principal. When $G$ is étale, the dynamic asymptotic dimension of $G$ and $\tilde{G}$ coincide. We show that the minimal unitizations of the full and reduced twisted groupoid C*-algebras of the twist over $G$ are isomorphic to the twisted groupoid C*-algebras of the twist over $\tilde{G}$. We apply our result about the nuclear dimension of the twisted groupoid C*-algebra to obtain a similar bound on the nuclear dimension of the C*-algebra of an étale groupoid with closed orbits and abelian stability subgroups that vary continuously.

1. Introduction

Noncommutative generalizations of topological covering dimension have been employed in numerous contexts to capture rigidity phenomena. One that has received considerable attention in recent years in the study of C*-algebras is a refinement of nuclearity called nuclear dimension \[66\], which provided the necessary additional rigidity for the completion of the classification program: all simple, separable, unital C*-algebras with finite nuclear dimension which satisfy a universal coefficient theorem (UCT) are classified by their K-theoretic and tracial data \[58\]. While finite nuclear dimension is a prerequisite of the classification result, simplicity forces the nuclear dimension of a classifiable C*-algebra to be either 0 or 1 \[14\].

In recent years, the attention of C*-algebraists has turned towards both determining whether a given C*-algebra is classifiable (e.g., \[1, 19, 25, 26, 27, 32\]) and expanding classification to the non-simple setting (e.g., \[10, 18, 20, 21, 23, 55\]). Results and techniques...
for non-simple C*-algebras remain ad hoc, and a wealth of regularity properties are required for progress. Because of its permanence properties under various constructions, nuclear dimension remains an invariant of great interest in this context. A promising approach is the emerging correspondence between nuclear dimension and other noncommutative generalizations of topological covering dimension, including tower dimension [31], dynamic asymptotic dimension [29], and diagonal dimension [39].

Asymptotic dimension was originally introduced by Gromov as a coarse geometric analogue of covering dimension in topology [28]. In [29], Guentner, Willett, and Yu generalized the notion of asymptotic dimension of a group to locally compact, Hausdorff and étale groupoids. Groupoids and their associated C*-algebras have long been studied by C*-algebraists because they provide a rich class of natural and useful examples, and, more recently, because they have been shown to have deep connections to lingering problems and future tracks in the aforementioned classification program in C*-algebras.

A major remaining problem of the classification program is whether the UCT assumption is redundant. A remarkable breakthrough in [7], building on [59] and [36, 53], says that if a nuclear C*-algebra is isomorphic to a twisted groupoid C*-algebra, then it automatically satisfies the UCT. Moreover, by recent work of Li, any classifiable C*-algebra is isomorphic to a twisted groupoid C*-algebra [41]. This has contributed to a flurry of research in twisted groupoid C*-algebras (e.g., [6, 8, 11, 37, 40]), and it is hence of significant interest to detect, at the groupoid level, when an associated C*-algebra is classifiable.

For most of the assumptions, this has been explored: the twists E that give rise to classifiable C*-algebras are over second-countable, locally compact, Hausdorff and étale groupoids G which satisfy an aperiodicity condition called effectiveness, and the associated reduced twisted groupoid C*-algebra C*(E; G) is simple if and only if G is minimal [4]. If C*(E; G) is nuclear, then we know from [7, 59] that it automatically satisfies the UCT. A missing piece is conditions that translate to finite nuclear dimension (or, in the presence of nuclearity, other equivalent regularity properties of the C*-algebra; for example, see [14]). Such conditions will also propel developments in non-simple classification.

In Theorem 8.6 of [29], Guentner, Willett and Yu showed that the nuclear dimension of the reduced C*-algebra of a second-countable, locally compact, Hausdorff, principal and étale groupoid G is bounded by a number depending on the dynamic asymptotic dimension of the groupoid and the topological covering dimension of the unit space of the groupoid. Many interesting C*-algebras, such as separable AF algebras, fit into this picture. However, as far as we currently know, twists may be required to pick up all classifiable C*-algebras and there exist twisted groupoid C*-algebras that cannot be realized as non-twisted groupoid C*-algebras [13]. Thus, we want to detect finite nuclear dimension for a twisted groupoid C*-algebra.

Our main result (Theorem 4.1) shows that if E is a twist over a second-countable, locally compact, Hausdorff, principal and étale groupoid G, then the nuclear dimension of C*(E; G) is bounded by (N + 1)(d + 1) − 1 where N is the topological covering dimension of the unit space of G and d is the dynamic asymptotic dimension of G. To prove this, we show that this C*-algebra satisfies a colored version of local subhomogeneity that appears implicitly in various proofs in the literature, including [29, Theorem 8.6] and [31, Theorem 6.2]. We show in Proposition 4.2, that in all these proofs it is this colored local subhomogeneity that determines the nuclear dimension of the associated C*-algebra. To prove that our twisted groupoid C*-algebras satisfy the assumptions of Proposition 4.2, we take inspiration from [29].
but need to take into account the extra structure coming from the twist. The subgroupoids that feature in the definition of dynamic asymptotic dimension give rise to twists whose C*-algebras have primitive ideal spaces that may not be Hausdorff, and hence finding bounds on the topological dimension of these spaces is delicate (see Proposition 4.3). Crucial for this analysis is a description of the primitive ideal space of twisted groupoid C*-algebras by Clark and the fourth author in [15], and an understanding of the spaces of certain regular representations from work of Muhly and Williams in [43]. Building on these results, we show in Proposition 4.3 that the twisted-groupoid C*-algebras of these subgroupoids have a recursive subhomogeneous decomposition, as studied in [48], with topological dimension at most that of their unit spaces.

We also develop the Alexandrov compactifications of an étale groupoid $G$ and of a twist $E$ over $G$; these “compactifications” produce a twist $\tilde{E}$ over $\tilde{G}$ that witnesses the unitization of the twisted groupoid C*-algebra at the groupoid level. Thus the reduced (respectively, full) twisted groupoid C*-algebra of the twist $\tilde{E}$ over $\tilde{G}$ is isomorphic to the minimal unitization of the reduced (respectively, full) twisted groupoid C*-algebra of the twist over $G$. We then show that the dynamic asymptotic dimension of the unit space of $G$ coincides with that of $\tilde{G}$, and this allows us to reduce the proof of Theorem 4.1 to the unital case.

A surprising application of our main theorem is a bound on the nuclear dimension of the C*-algebra of an étale groupoid $G$ with closed orbits but potentially large abelian isotropy subgroups, provided these subgroups vary continuously (Corollary 5.5). Here $C^*_r(G)$ is isomorphic to a twisted groupoid C*-algebra over a principal groupoid by [15]. This principal groupoid is a transformation groupoid $\hat{A} \rtimes R$ obtained from an action of the quotient groupoid $R = G/A$ of $G$ by the isotropy subgroupoid $A$ on the spectrum $\hat{A}$ of the abelian C*-algebra of $A$. We show that if $R$ has finite dynamic asymptotic dimension at most $d$, then so does $\hat{A} \rtimes R$; thus our Theorem 4.1 applies, provided a bound on the topological covering dimension of the unit space $\hat{A}$ can be computed.

We illustrate Corollary 5.5 with two examples. The first, Example 5.6, is a transformation group whose C*-algebra has continuous trace but is not AF; here we find that the nuclear dimension is 1. The second, Example 5.8 is the groupoid of a directed graph $E$ whose C*-algebra is AF embeddable. Since $C^*(E)$ does not have a purely infinite ideal, it is outside the scope of the results on nuclear dimension of graph C*-algebras in [55]. Moreover, since $E$ has return paths, by Lemma 5.7 the dynamic asymptotic dimension of the graph groupoid is $\infty$, and hence is a poor predictor of the nuclear dimension of its C*-algebra. But in Corollary 5.5 we consider a groupoid modulo its isotropy groupoid, and this gives more information about the C*-algebra: we find that the nuclear dimension of $C^*(E)$ is actually 1.

Structure of the paper. In Section 2 we recall the definition of a groupoid twist, exhibit an induced Haar system on a twist over an étale groupoid and define the twisted groupoid C*-algebra. We also recall the definitions of nuclear dimension, decomposition rank and dynamic asymptotic dimension. In Section 3 we introduce the Alexandrov groupoid $\tilde{G}$ of an r-discrete groupoid $G$, construct a twist over $\tilde{G}$ from a twist over $G$, and study the associated C*-algebras. In Section 4 we prove our main theorem; in particular, for Theorem 4.1 and Proposition 4.3 we assume that $E$ is a twist over an étale and principal groupoid. In Section 5 we develop our application to non-principal groupoids and discuss the two examples mentioned above.
2. Preliminaries

2.1. Groupoids. Let $E$ be a locally compact, Hausdorff groupoid. We denote by $E^{(0)}$ the unit space of $E$. The range and source maps $r, s: E \rightarrow E^{(0)}$ are given by $r(e) = ee^{-1}$ and $s(e) = e^{-1}e$, respectively. The set of composable pairs $\{(d, e) : s(d) = r(e)\}$ is denoted by $E^{(0)}$. Then $E$ is principal if the map $e \mapsto (r(e), s(e))$ is injective. We say that $E$ is $r$-discrete if its unit space $E^{(0)}$ is open and that $E$ is étale if the range map is a local homeomorphism. Notice that every étale groupoid is $r$-discrete. For a subset $X \subset E$ we write $\langle X \rangle$ for the subgroupoid of $E$ generated by $X$.

Let $W \subset E^{(0)}$. We define the restriction of $E$ to $W$ to be

$$E|_{W} = \{ e \in E : s(e), r(e) \in W \}.$$ 

Note that $E|_{W}$ is an algebraic groupoid with unit space $W$. If, for example, $W$ is locally closed in $E^{(0)}$, then $E|_{W}$ is a locally compact, Hausdorff groupoid (see Lemma 2.3 below).

The saturation of $W$ is $r(s^{-1}(W)) = s(r^{-1}(W))$, and if $W = r(s^{-1}(W))$ then we say that $W$ is saturated. If $x \in E^{(0)}$, we call the saturation of $\{x\}$ the orbit of $x$ and denote it by $[x]$. We also set $xE := \{ e \in E : r(e) = x \}$ and $Ex := \{ e \in E : s(e) = x \}$. The group $xE \cap Ex$ is called the stability subgroup at $x$ and is also known as the isotropy subgroup at $x$.

Now suppose that $E$ has a left Haar system $\sigma = \{ \sigma^{x} : x \in E^{(0)} \}$; if $E$ is étale, then we choose the Haar system of counting measures. We recall the definition of the usual full and reduced $C^{*}$-algebras of $E$. Let $C_{c}(E)$ be the vector space of continuous and compactly supported $C^{*}$-valued functions on $E$, equipped with convolution and involution given for $e \in E$ and $f, g \in C_{c}(E)$ by

$$f * g(e) = \int_{E} f(d) g(d^{-1}e) d\sigma^{r(e)}(d) \quad \text{and} \quad f^{*}(e) = \overline{f(e^{-1})}.$$ 

A $*$-homomorphism $L: C_{c}(E) \rightarrow B(H_{L})$ into the bounded operators on some Hilbert space $H_{L}$ is a representation if it is $I$-norm bounded, or equivalently, if it is continuous when $C_{c}(E)$ has the inductive limit topology and $B(H_{L})$ the weak operator topology [61, Remark 1.46]. Then the full $C^{*}$-algebra $C^{*}(E, \sigma)$ of $E$ is the completion of $C_{c}(E)$ in the norm

$$\|f\|_{C^{*}(E, \sigma)} = \sup\{ \|L(f)\| : L \text{ is a representation of } C_{c}(E) \}.$$ 

As usual, we define another measure $\sigma_{x}$ on $E$ with support in $Ex$ by $\sigma_{x}(U) = \sigma^{x}(U^{-1})$. For each $x \in E^{(0)}$, define $L^{x}: C_{c}(E) \rightarrow B(L^{2}(Ex, \sigma_{x}))$ for $e \in Ex$ by

$$L^{x}(f)(\xi)(e) = \int_{E} f(d) \xi(d^{-1}e) d\sigma^{r(e)}(d)$$

(2.1)

Notice that if $\xi \in C_{c}(Ex) \subset L^{2}(Ex, \sigma_{x})$, then $L^{x}(f)\xi = f * \xi$. By [61, Proposition 1.41], $L^{x}$ is a representation of $C_{c}(E, \sigma)$ and hence extends to a representation $L^{x}: C^{*}(E) \rightarrow B(L^{2}(Ex, \sigma_{x}))$. The reduced $C^{*}$-algebra $C^{*}_{r}(E, \sigma)$ of $E$ is the completion of $C_{c}(E)$ in the norm

$$\|f\|_{C^{*}_{r}(E, \sigma)} = \sup\{ \|L^{x}(f)\| : x \in E^{(0)} \}.$$ 

2.2. Twists. There are various definitions of twists in the literature, for example, [36, Definition 2.4], [43, §2], [53, §4], [57, Definition 11.1.1] and [8, Definition 3.1], and they are not always consistent. Here we want to allow twists over possibly non-étale and non-principal groupoids. We start with the definition from [8].
Definition 2.1. Let $G$ be a locally compact and Hausdorff groupoid, and regard $G^{(0)} \times T$ as a trivial group bundle with fibers $T$. A twist $(E, \iota, \pi)$ over $G$ consists of a locally compact and Hausdorff groupoid $E$ and groupoid homomorphisms $\iota$, $\pi$ such that

$$G^{(0)} \times T \xrightarrow{\iota} E \xrightarrow{\pi} G$$

is a central groupoid extension, which means that

1. $\iota: G^{(0)} \times T \to \pi^{-1}(G^{(0)})$ is a homeomorphism, where $\pi^{-1}(G^{(0)})$ has the subspace topology from $E$, and it satisfies $\iota(\pi(u), 1) = u$ for all $u \in E^{(0)}$;
2. $\pi$ is a continuous, open surjection; and
3. $\iota(\pi(r(e)), z) e = e\iota(\pi(s(e)), z)$ for all $e \in E$ and $z \in T$.

The twists in [43] and [15] are twists in the sense of Definition 2.1 with the additional assumption that $G$ is principal. Because of Condition (1), we may identify $E^{(0)}$ with the homeomorphism $\pi|_{E^{(0)}}$.

Let $(E, \iota, \pi)$ be a twist over a locally compact and Hausdorff groupoid $G$. For $z \in T$ and $e \in E$, define

$$z \cdot e := \iota(r(e), z) e = e\iota(s(e), z).$$

This gives a continuous action of $T$ on $E$ which is free because $\iota$ is injective. If $d, e \in E$ such that $\pi(d) = \pi(e)$, then there exists a unique $z \in T$ such that $d = z \cdot e$.

We equip the orbit space $E/T$ with the quotient topology with respect to the orbit map $q: E \to E/T$. Then $q$ is open by [52, Lemma 4.57], and so by exactness, $\pi$ induces a homeomorphism of $E/T$ onto $G$. Thus, we may identify $E/T$ and $G$, and hence also $q$ and $\pi$. Since $E$ is locally compact and Hausdorff, it is completely regular [47, p. 151]. Now $T$ is a Lie group acting freely and, by compactness, properly on the completely regular space $E$, and it follows from [45, Theorem on p. 315] that $\pi$ is a locally trivial principal bundle, that is, for very $\alpha \in G$ there exist a neighborhood $U$ of $\alpha$ and a continuous map $S: U \to E$ such that $\pi \circ S = \text{id}_U$ (see [52, Proposition 4.65 and Hooptedoodle 4.68]). It is then routine to check that $(\beta, z) \mapsto z \cdot S(\beta)$ is a homeomorphism of $\bar{U} \times T$ onto $\pi^{-1}(\bar{U})$.

The existence of continuous local sections and the local trivializations are often assumed in the definition of a twist over an étale groupoid (see, for example, [57]), and it then follows that $\pi$ is automatically an open map. Indeed, when $G$ is étale, [57, Definition 11.1.1] and Definition 2.1 are equivalent, as outlined in [4, Remark 2.6].

If $G$ is r-discrete, then $G^{(0)} = E^{(0)}$ is open in $G$ and in $E$, and hence $\iota$ is an open map into $E$ as it is a homeomorphism onto an open subset. If $G$ is even étale, then the range and source maps on $E$ are open, and the multiplication map on $E^{(2)}$ is open (see, for example, [4, Lemma 2.7]).

The following lemma is well known, but we could not find a reference for it.

Lemma 2.2. Let $(E, \iota, \pi)$ be a twist over a locally compact, Hausdorff groupoid $G$. Then $\pi$ is a proper map.

Proof. Let $K \subset G$ be a compact set; we show that any net $\{e_i\}_i$ in $\pi^{-1}(K)$ has a convergent subnet. Since $G$ is Hausdorff, both $K$ and hence $\pi^{-1}(K)$ are closed; therefore, if a subnet of $\{e_i\}_i$ converges in $E$, its limit must be in $\pi^{-1}(K)$.

Let $\gamma_i := \pi(e_i)$. Since $\gamma_i \in K$ and $K$ is compact, the net has a convergent subnet; without loss of generality, we may assume that $\{\gamma_i\}_i$ itself converges to, say, $\gamma$. Let $e \in \pi^{-1}(\gamma)$ be arbitrary. Since $\pi$ is an open surjection, we can use Fell’s criterion (61, Proposition 1.1)): there exists a subnet $\{\gamma_{g(m)}\}_{m \in M}$ and a net $\{\tilde{e}_m\}_{m \in M}$, indexed by the same directed set $M$,
such that \( \tilde{e}_m \to e \) in \( E \) and \( \pi(\tilde{e}_m) = \gamma_{g(m)} \) for all \( m \). By choice of \( \gamma_i \), the latter implies that there exist \( z_m \in T \) such that \( \tilde{e}_m = z_m \cdot e_{g(m)} \). Since \( T \) is compact, there exists a subnet \( \{z_{h(n)}\}_{n \in N} \) of \( \{z_m\}_{m \in M} \) that converges to, say, \( z \). Since everything in sight is continuous, we conclude that
\[
\lim_n e_{g(h(n))} = \lim_n \overline{z_{h(n)}} \cdot \tilde{e}_{h(n)} = z \cdot e.
\]
In other words, we have found a subnet of \( \{e_i\}_i \) that converges.

A version of the following lemma was implicitly used in \([13, \text{Lemma 3.1}]\). Recall that a subset \( W \) of a topological space \( X \) is \textit{locally closed} if each point in \( W \) has an open neighborhood \( U \) in \( X \) such that \( U \cap W \) is closed in \( U \).

**Lemma 2.3.** Let \( G \) be a locally compact and Hausdorff groupoid, and let \( (E, \iota, \pi) \) be a twist over \( G \). Let \( W \) be a locally closed subset of \( G^{(0)} \). Then \( E \big|_W \) and \( G \big|_W \) are locally compact and Hausdorff groupoids, and \( (E \big|_W, \iota \big|_W \times T, \pi \big|_W) \) is a twist over \( G \big|_W \). If \( G \) is \( r \)-discrete, étale or principal, then so is \( G \big|_W \).

**Proof.** In the special case where \( W \) is either open or closed in \( G^{(0)} \), we have that \( E \big|_W = r_E^{-1}(W) \cap s_E^{-1}(W) \) is an open (respectively, closed) subset of a locally compact space, hence is locally compact. Next, suppose that \( W \) is locally closed in \( G^{(0)} \), so that \( W \) is open in its closure \( W \) by \([62, \text{Lemma 1.25}]\). Now \( E \big|_W \) is locally compact and \( W \) is an open subset of its unit space \( W \), and so \( E \big|_W \) is locally compact by the previous argument. Similarly, \( G \big|_W \) is locally compact. Subsets of Hausdorff spaces are Hausdorff and restrictions of continuous maps are continuous, and so both \( E \big|_W \) and \( G \big|_W \) are locally compact and Hausdorff groupoids with unit spaces \( W \). It remains to show that \( (E \big|_W, \iota \big|_W \times T, \pi \big|_W) \) is a twist over \( G \big|_W \).

The restrictions \( \iota \big|_W \times T \) and \( \pi \big|_W \) are still continuous groupoid homomorphisms, and \( \iota \big|_W \times T \) is still injective and its range is contained in \( E \big|_W \). Restricting a homeomorphism to a subset of its domain yields a homeomorphism onto the image of that subset; thus, \( \iota \big|_W \times T \) is a homeomorphism onto \( \iota(W \times T) = \pi^{-1}(W) = \pi^{-1}(E \big|_W) \). It is immediate that \( \iota \big|_W \times T(W \times T) \) is central in \( E \big|_W \) because it is central in \( E \). To see that \( \pi \big|_W \) is open, let \( U \) be open in \( E \big|_W \). Then there exists an open subset \( U' \) of \( E \) such that \( U = U' \cap E \big|_W \). Then
\[
\pi \big|_W(U) = \pi(U' \cap E \big|_W) = \{\pi(e) : e \in U', \iota_E(e), s_E(e) \in W\} = \pi(U') \cap G \big|_W
\]
because \( \pi \) is range and source preserving. Since \( \pi \) is open, \( \pi(U') \) is open in \( G \), and hence \( \pi \big|_W(U) \) is open in \( G \big|_W \). Thus, \( \pi \big|_W \) is open and \( (E \big|_W, \iota \big|_W \times T, \pi \big|_W) \) is a twist over \( G \big|_W \).

Clearly, if \( G \) is principal, then so is \( G \big|_W \). Now suppose that \( G \) is \( r \)-discrete, i.e., \( G^{(0)} \) is open in \( G \). Hence, \( W = G \big|_W \cap G^{(0)} \) is open in \( G \big|_W \) and so \( G \big|_W \) is \( r \)-discrete. If \( G \) is even étale, then since the counting measures form a Haar system for \( G \), they also form a Haar system for \( G \big|_W \). It follows from \([61, \text{Proposition 1.29}]\) that \( G \big|_W \) is also étale.

2.2.1. **An induced Haar system on a twist.** In this paper we consider \( r \)-discrete as well as étale groupoids \( G \). To consider a \( C^* \)-algebra of \( G \), we need to assume that \( G \) admits a Haar system. But an \( r \)-discrete groupoid with a Haar system is étale by \([61, \text{Propositions 1.23 and 1.29}]\), and then we may as well use the Haar system of counting measures. So whenever we consider a \( C^* \)-algebra of \( G \) or a twisted groupoid \( C^* \)-algebra over \( G \), then we will assume that \( G \) is étale and we will write \( C^*_r(G) \) and \( C^*(G) \) for its reduced respectively full \( C^* \)-algebra. Moreover, for a twist over \( G \), we will also always use the induced Haar system that we now construct. See also Remark \([3.2]\).
Let \((E, \iota, \pi)\) be a twist over a locally compact, Hausdorff groupoid \(G\), let \(\lambda\) be normalized Haar measure on \(T\), and assume that \(G\) is étale. Fix \(x \in E^{(0)} = G^{(0)}\) and let \(\gamma \in xG\). For each \(e \in \pi^{-1}(\gamma)\) there is a homeomorphism \(\rho_e : T \to \pi^{-1}(\gamma)\) given by \(z \mapsto z \cdot e\). We equip \(\pi^{-1}(\gamma)\) with the measure \(\sigma^{x,\gamma}\) defined by \(\sigma^{x,\gamma}(U) = \lambda(\rho_e^{-1}(U))\), where \(e \in \pi^{-1}(\gamma)\) is arbitrary. This notation is justified, for if \(\pi(d) = \pi(e)\), then \(d = w \cdot e\) for some \(w \in T\), and since \(\lambda\) is rotation invariant it follows that \(\sigma^{x,\gamma}\) indeed does not depend on the choice of \(e \in \pi^{-1}(\gamma)\). Because \(xE\) is the disjoint union \(\bigcup_{\gamma \in xG} \pi^{-1}(\gamma)\) we obtain a measure \(\sigma^x\) on \(E\), taking values in \([0, \infty]\) and with support in \(xE\), such that
\[
(2.2) \hspace{1cm} \sigma^x(U) = \sum_{\gamma \in xG} \sigma^{x,\gamma}(U).
\]

The following lemma is well known; we provide a proof for completeness.

**Lemma 2.4.** Let \((E, \iota, \pi)\) be a twist over a locally compact, Hausdorff and étale groupoid \(G\). Let \(s : G \to E\) be a (not necessarily continuous) section of \(\pi\) and for \(x \in G^{(0)}\) let \(\sigma^x\) be the measure in \((2.2)\). Then for \(f \in C_c(E)\) we have
\[
(2.3) \hspace{1cm} \int f(e) \, d\sigma^x(e) = \sum_{\gamma \in xG} \int_T f(z \cdot s(\gamma)) \, d\lambda(z).
\]

Moreover, \(\{\sigma^x : x \in G^{(0)}\}\) is a left Haar system on \(E\).

**Proof.** Fix \(x \in G^{(0)}\). By definition, each \(\sigma^x\) has support contained in \(xE\) and the formula \((2.3)\) holds. Next, fix \(e \in xE\) and let \(U\) be a precompact neighborhood of \(e\). For each \(\gamma \in xG\), choose \(e_\gamma \in \pi^{-1}(\gamma)\). Since \(\rho_{e_\gamma}^{-1}(U)\) is open in \(T\) and since \(\lambda\) is a Haar measure, we get
\[
\sigma^x(U) = \sum_{\gamma \in xG} \sigma^{x,\gamma}(U) = \sum_{\gamma \in xG} \lambda(\rho_{e_\gamma}^{-1}(U)) > 0.
\]

Thus, the support of \(\sigma^x\) is \(xE\).

Next, we show that the collection of measures is left-invariant, i.e., that for \(d \in E\) and \(f \in C_c(E)\) we have
\[
\int f(de) \, d\sigma^{s(d)}(e) = \int f(e) \, d\sigma^{r(d)}(e).
\]

The integrand on the left-hand side is the function \(e \mapsto f(de)\). Thus, using \((2.3)\)
\[
\int f(de) \, d\sigma^{s(d)}(e) = \sum_{\gamma \in s(d)G} \int_T f(d(z \cdot s(\gamma))) \, d\lambda(z)
\]
\[
= \sum_{\gamma \in s(d)G} \int_T f(z \cdot (ds(\gamma))) \, d\lambda(z).
\]

Since \(\pi(ds(\gamma)) = \pi(d)\gamma = \pi(s(\pi(d)\gamma))\) there exists a unique \(z_\gamma \in T\) such that \(ds(\gamma) = z_\gamma \cdot s(\pi(d)\gamma)\). Together with our above computation, we get
\[
\int f(de) \, d\sigma^{s(d)}(e) = \sum_{\gamma \in s(d)G} \int_T f(z \cdot (z_\gamma \cdot s(\pi(d)\gamma))) \, d\lambda(z)
\]
\[
= \sum_{\gamma \in s(d)G} \int_T f(w \cdot s(\pi(d)\gamma)) \, d\lambda(w)
\]
by the change of variable \( w = zz' \). Since \( \gamma \mapsto \pi(d)\gamma \) is a bijection of \( s(d)G \) onto \( r(d)G \) we get

\[
\int f(e) \, d\sigma^{s(d)}(e) = \sum_{\alpha \in r(d)G} \int f(w \cdot s(\alpha)) \, d\lambda(w) = \int f(e) \, d\sigma^{r(d)}(e),
\]
as needed.

Next, we show that

\[
E^{(0)} \to \mathbf{C}, \quad x \mapsto \int f(e) \, d\sigma^x(e),
\]
is continuous for any fixed \( f \in C_c(E) \). Let \( x_n \to x \) in \( E^{(0)} \); we need to show that

\[
(2.4) \quad \left| \sum_{\gamma \in x_nG} \int_T f(z \cdot s(\gamma)) \, d\lambda(z) - \sum_{\alpha \in xG} \int_T f(z \cdot s(\alpha)) \, d\lambda(z) \right| \to 0
\]
as \( n \to \infty \). We will show that \( \hat{f} : G \to \mathbf{C} \) defined by \( \hat{f}(\gamma) = \int_T f(z \cdot s(\gamma)) \, d\lambda(z) \) is in \( C_c(G) \); then (2.4) follows because the counting measures form a Haar system on \( G \).

Fix \( \epsilon > 0 \) and \( \gamma \in G \), and suppose that \( \gamma_n \to \gamma \) in \( G \). Since \( E \) is a twist over \( G \) there exist a neighborhood \( U_\gamma \) and a continuous \( S : U_\gamma \to E \) such that \( \pi \circ S = \text{id}_{U_\gamma} \). Let \( \beta \in U_\gamma \). Then \( \pi(s(\beta)) = \beta = \pi(S(\beta)) \), and hence there exists \( z_\beta \in T \) such that \( s(\beta) = z_\beta \cdot S(\beta) \). Now

\[
(2.5) \quad \hat{f}(\beta) = \int_T f(zz_\beta \cdot S(\beta)) \, d\lambda(z) = \int_T f(w \cdot S(\beta)) \, d\lambda(w)
\]
via the change of variable \( zz_\beta \mapsto w \). In other words, in the neighborhood \( U_\gamma \) of \( \gamma \), we were able to write \( \hat{f} \) using the continuous but only locally defined section \( S \) instead of the not necessarily continuous but global section \( s \). For fixed \( z \in T \), the function \( \beta \mapsto f(z \cdot S(\beta)) \) is continuous on \( U_\gamma \). So there exists a neighborhood \( V \subset U_\gamma \) of \( \gamma \) such that

\[
\eta \in V \implies |f(z \cdot S(\eta)) - f(z \cdot S(\gamma))| < \epsilon.
\]
Then for all \( n \) large enough such that \( \gamma_n \in V \subset U_\gamma \), it follows with Equation (2.5)

\[
|\hat{f}(\gamma_n) - \hat{f}(\gamma)| = \left| \int_T (f(z \cdot S(\gamma_n)) - f(z \cdot S(\gamma))) \, d\lambda(z) \right| \\
\leq \int_T |f(z \cdot S(\gamma_n)) - f(z \cdot S(\gamma))| \, d\lambda(z) < \epsilon.
\]
Thus, \( \hat{f} \) is continuous on \( G \). To see that \( \hat{f} \) has compact support, suppose that \( \hat{f}(\beta) \neq 0 \). Then there exists \( z \in T \) such that \( z \cdot s(\beta) \in \text{supp } f \) and so \( \beta = \pi(z \cdot s(\beta)) \in \pi(\text{supp } f) \). It follows that \( \hat{f} \) has support contained in \( \pi(\text{supp } f) \), which is compact since \( f \in C_c(E) \). Thus, \( \hat{f} \in C_c(G) \) and (2.4) follows. We have shown that \( \{ \sigma^x : x \in G^{(0)} \} \) is a Haar system. \( \square \)

Given a twist \((E, \iota, \pi)\) over an étale groupoid \( G \), we will always choose the above Haar system on \( E \) and therefore write \( C^*(E) \) and \( C^*_l(E) \) instead of \( C^*(E, \sigma) \) and \( C^*_r(E, \sigma) \).

2.2.2. The \( C^* \)-algebras associated to a twist. Let \((E, \iota, \pi)\) be a twist over a locally compact, Hausdorff and étale groupoid \( G \), let \( \{ \sigma^x : x \in G^{(0)} \} \) be the left Haar system defined in (2.2) and let \( s : G \to E \) be a (not necessarily continuous) section of \( \pi \). The full and reduced
twisted groupoid $C^*$-algebras $C^*_c(E; G)$ and $C^*_t(E; G)$ are obtained from $C^*(E)$ and $C^*_t(E)$ as follows. Set $C^*_c(E; G) := \{ f \in C_c(E) : f(z \cdot e) = zf(e) \text{ for all } z \in T \text{ and } e \in E \}$.

It is straightforward to check that $C^*_c(E; G)$ is an ideal in $C^*_c(E)$: for example, if $f \in C^*_c(E)$ and $g \in C^*_c(E; G)$, then for $e \in E$ and $z \in T$ we have

$$g \ast f(z \cdot e) = \int_E g(d) f(d^{-1}(z \cdot e)) \, d\sigma^{r(z \cdot e)}(d)$$

which, using the change of variable $c = d^{-1}(z \cdot e)$ becomes

$$= \int_E g(z \cdot e c^{-1}) f(c) \, d\sigma_{s(e)}(c)$$

$$= z \int_E g(ec^{-1}) f(c) \, d\sigma_{s(e)}(c)$$

$$= z(g \ast f)(e)$$

by another change of variable. Thus, $C^*_c(E; G)$ extends to an ideal $C^*(E; G)$ of $C^*(E)$. By [54, Lemma 3.3], the map $\Upsilon : C^*_c(E) \rightarrow C^*_c(E; G)$ defined by

$$\Upsilon(f)(e) = \int_T f(z \cdot e) \bar{z} \, dz$$

is the identity on $C^*_c(E; G)$, a surjective $*$-homomorphism which is continuous in the inductive limit topology, and hence extends to a homomorphism $\Upsilon : C^*(E) \rightarrow C^*(E; G)$. Thus, $C^*(E; G)$ is also a quotient of $C^*_t(E)$, and hence is a direct summand.

Similarly, $C^*_t(E; G)$ is the completion of $C^*_t(E; G)$ in $C^*_t(E)$, and is a direct summand of $C^*_t(E)$. Let

$$L^2(Ex; Gx) = \{ \xi \in L^2(Ex, \sigma_x) : \xi(z \cdot e) = z\xi(e) \text{ for all } z \in E \text{ and } e \in T \}. $$

For each $x \in G^0$, let $L^2 : C^*(E) \rightarrow B(L^2(Ex; \sigma_x))$ be the extension to $C^*(E)$ of the representation defined at (2.1). We write $\pi^x$ for the nondegenerate part of $L^2|_{C^*(E; G)}$, so that (2.6)

$$\pi^x : C^*(E; G) \rightarrow B(L^2(Ex; Gx)).$$

In particular, $C^*_t(E; G)$ is the completion of $C^*_c(E; G)$ with respect to the norm defined by

$$\| f \|_{C^*_t(E; G)} = \sup \{ \| \pi^x(f) \| : x \in G^0 \}$$

for $f \in C^*_c(E; G)$.

It is useful to note that the integral formula for convolution simplifies to a sum: for $f, g \in C^*_c(E; G)$ we have

$$f \ast g(e) = \int_E f(d)g(d^{-1}e) \, d\sigma^{r(e)}(d)$$

$$= \sum_{\gamma \in r(e)G} \int_T f(z \cdot s(\gamma)) g(\bar{z} \cdot s(\gamma)^{-1}e) \, dz$$

$$= \sum_{\gamma \in r(e)G} f(s(\gamma)) g(s(\gamma)^{-1}e)$$

In other places in the literature, $C^*_c(E; G)$ is the subset of $C^*_c(E)$ whose elements satisfy $f(z \cdot e) = \bar{z}f(e)$ instead. Our convention is consistent with [13, 32, 33].
since $f, g$ are $T$-equivariant and the measure of $T$ is 1. Similarly, for $\xi \in L^2(Ex; Gx)$ we have
\[ ||\xi||^2 = \int_E |\xi(e)|^2 \, d\sigma_x(e) = \sum_{\gamma \in G_x} \int_T |\xi(z \cdot s(\gamma))|^2 \, d(z) = \sum_{\gamma \in G_x} |\xi(s(\gamma))|^2. \]

Finally, for $h \in C_c(G^{(0)})$ define a function $f_h$ on $E$ by
\[ (2.7) \quad f_h(e) = \begin{cases} zh(x) & \text{if } e = \iota(x, z) \text{ for some } (x, z) \in G^{(0)} \times T \\ 0 & \text{else.} \end{cases} \]

Then $f_h \in C^*_c(E; G)$ and $h \mapsto f_h$ extends to an isomorphism from $C_0(G^{(0)})$ onto the closure of the $*$-subalgebra $\{f \in C^*_c(E; G) : \text{supp } f \subset \iota(G^{(0)} \times T)\}$ of $C^*_r(E; G)$.

We will use the following lemma.

**Lemma 2.5** ([11, Lemma 2.7]). Let $G$ be a locally compact, Hausdorff and étale groupoid and let $(E, \iota, \pi)$ be a twist over $G$. Suppose that $H$ is an open subgroupoid of $G$. Then the open subgroupoid $\pi^{-1}(H)$ of $E$ gives a twist $(\pi^{-1}(H), \iota|_{H^{(0)} \times T}, \pi|_{\pi^{-1}(H)})$ over $H$ and there is an injective homomorphism from $C^*_r(\pi^{-1}(H); H)$ into $C^*_r(E; G)$ induced by inclusion and extension by zero.

The above twist over $H$ is often denoted by $E_H$. But to avoid confusion with $E|_W = r^{-1}(W) \cap s^{-1}(W)$ for $W \subseteq E^{(0)}$, we use the notation $\pi^{-1}(H)$ instead.

### 2.3. Topological, nuclear, and dynamic asymptotic dimension.

Nuclear dimension and its predecessor, decomposition rank, arose as key structural properties in the classification program for separable nuclear $C^*$-algebras. Both were modeled on Lebesgue covering dimension for topological spaces.

Let $X$ be a topological space. An open cover $U$ of $X$ has order $m$ if each element of $X$ belongs to at most $m$ elements of $U$, i.e., $\bigcap_{i=1}^{m+1} U_i = \emptyset$ for any distinct $U_1, \ldots, U_{m+1} \in U$. The Lebesgue (or topological) covering dimension of $X$, written $\dim X$, is the smallest integer $N$ such that every open cover of $X$ admits an open refinement of order $N + 1$.

We let $X_+ := X \cup \{\infty\}$ denote the Alexandrov (or minimal one-point) compactification of the topological space $X$. Its topology consists of open sets in $X$ and all sets of the form $(X \setminus K) \cup \{\infty\}$, where $K$ is a closed and compact subset of $X$, making $X_+$ compact and $X$ an open subspace. If $X$ is locally compact and Hausdorff, then $X_+$ is Hausdorff. We will need the following lemma, which may be well known but does not seem to be well documented.

**Lemma 2.6.** Let $X$ be a second-countable, locally compact and Hausdorff space, and let $X_+$ be its one-point compactification. Then $\dim X = \dim X_+$.

**Proof.** Since $X_+$ is second-countable, compact and Hausdorff, it is normal [44, Theorem 32.1] and metrizable [44, Theorem 34.1]. Now $X$ is a subspace of the metrizable space $X_+$, and hence $\dim X \leq \dim X_+ [16, Theorem 1.8.3]$. Further, in a metrizable space every open set is an $F_\sigma$ set [16, Lemma 1.8.2]. Now $X_+ = X \cup \{\infty\}$ is a union of two $F_\sigma$ sets of dimension at most $\dim X$. Thus, $\dim X_+ \leq \dim X$ by [44, Proposition 5.3, Chapter 3].

Now let $A$ and $B$ be $C^*$-algebras. Recall that a linear map $\phi : A \to B$ is **positive** if it maps positive elements in $A$ to positive elements in $B$. It is **completely positive (c.p.)** if this also holds for all matrix amplifications $\phi^{(n)} : M_n(A) \to M_n(B)$ where $\phi^{(n)}([a_{ij}]) = [\phi(a_{ij})]$. A completely positive norm-decreasing map is called **completely positive contractive (c.p.c.)**.
A c.p. map $\phi: A \to B$ between C*-algebras is called order zero if for any positive elements $a, b \in A$ with $ab = 0$, we have $\phi(a)\phi(b) = 0$.

One indispensable result for c.p.c. maps is Arveson’s extension theorem \cite{arveson1976extension}. For easy reference, we state the particular consequence that we use in this article.

**Theorem 2.7** (Arveson). Let $A$ and $D$ be unital C*-algebras, and let $B \subset A$ be a C*-subalgebra. Then any c.p.c. map $\varphi: B \to D$ extends to a c.p.c. map $\psi: A \to D$.

In case $1_A \in B$, the reader is referred to \cite[Theorem 1.6.1]{brown2008cuntz} for a proof. If $B$ is not unital, then we may replace $B$ with $C^*(B, 1_A)$, which is canonically isomorphic to the minimal unitization $\overline{B}$ of $B$, and replace $\varphi$ with its c.p.c. unitization $\tilde{\varphi}: C^*(B, 1_A) \to D$ given by $\tilde{\varphi}(x + \lambda 1_A) = \varphi(x) + \lambda 1_D$ for $x \in B, \lambda \in \mathbb{C}$, as in \cite[Proposition 2.2.1]{brown2008cuntz}. If $1_A \notin B$ but $B$ still has a unit $1_B \in B$, then we consider the “forced unitization” $C^*(B, 1_A) \cong B \oplus \mathbb{C}$. Since the map $B \oplus C \to D$ given by $b \oplus \lambda \mapsto \varphi(b)$ is clearly c.p.c., it follows that there exists a c.p.c. extension $\tilde{\varphi}: C^*(B, 1_A) \to D$ of $\varphi$. Then we may apply \cite[Theorem 1.6.1]{brown2008cuntz} to extend $\tilde{\varphi}: C^*(B, 1_A) \to D$ to a c.p.c. map $\psi: A \to D$ where $\psi|_B = \tilde{\varphi}|_B = \varphi$.

**Definition 2.8** (\cite[Definition 2.1]{brown2008cuntz}). Let $A$ be a C*-algebra and let $d \in \mathbb{N}$. Then $A$ has nuclear dimension at most $d$, written $\dim_{\text{nuc}}(A) \leq d$, if for any finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$, there exist a finite-dimensional C*-algebra $F = \bigoplus_{j=0}^{d} F_j$, a c.p.c. map $\psi: A \to F$ and a c.p.m. map $\phi: F \to A$ such that $\phi_j := \phi|_{F_j}$ is c.p.c. and order zero for each $0 \leq j \leq d$, and for all $a \in \mathcal{F}$,

$$\|\phi(\psi(a)) - a\| < \varepsilon.$$ 

A subtle yet stark strengthening of nuclear dimension is decomposition rank, where we add the assumption that $\phi$, and not just each $\phi_j$, is contractive:

**Definition 2.9** (\cite[Definition 3.1]{brown2008cuntz}). A C*-algebra $A$ has decomposition rank at most $d \in \mathbb{N}$, written $\text{dr}(A) \leq d$, if for any finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$, there exists a finite-dimensional C*-algebra $F = \bigoplus_{j=0}^{d} F_j$ and c.p.c. maps $\psi: A \to F$ and $\phi: F \to A$ such that $\phi_j := \phi|_{F_j}$ is order zero for each $0 \leq j \leq d$ and such that for all $a \in \mathcal{F}$,

$$\|\phi(\psi(a)) - a\| < \varepsilon.$$ 

There are numerous reformulations of nuclear dimension and decomposition rank. In \cite[Definition 8.1]{guentner2013dimension}, for example, Guentner, Willett, and Yu use a version of finite nuclear dimension which is also more fitting to our purposes. For the reader’s benefit, we provide a proof that \cite[Definition 8.1]{guentner2013dimension} and Definition 2.8 are equivalent.

**Lemma 2.10.** Let $A$ be a C*-algebra and let $d \in \mathbb{N}$. Then $\dim_{\text{nuc}}(A) \leq d$ if and only if for any finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$, there exist finite-dimensional C*-algebras $F_0, \ldots, F_d$ and c.p.c. maps $A \xrightarrow{\psi_j} F_j \xrightarrow{\phi_j} A$ with each $\phi_j$ order zero such that for all $a \in \mathcal{F}$,

$$\left\| \sum_{j=0}^{d} (\phi_j \circ \psi_j)(a) - a \right\| < \varepsilon.$$ 

**Proof.** Fix $\mathcal{F} \subset A$ finite and $\varepsilon > 0$.

First, assume that $\dim_{\text{nuc}}(A) \leq d$. Then there exist a finite-dimensional C*-algebra $F$, c.p.m maps $A \xrightarrow{\psi} F \xrightarrow{\phi} A$ such that $\psi$ is c.p.c., $F$ decomposes as $\bigoplus_{j=0}^{d} F_j$, $\phi_j := \phi|_{F_j}$ is c.p.c. and order zero for each $0 \leq j \leq d$, and for all $a \in \mathcal{F}$,

$$\|\phi(\psi(a)) - a\| < \varepsilon.$$
For each $0 \leq j \leq d$, let $1_{F_j}$ be the projection onto the $j$th summand and set $\psi_j(\cdot) := 1_{F_j} \psi(\cdot) 1_{F_j}$. Then each $\psi_j$ is c.p.c., and for $a \in F$,
$$
\psi(a) = \sum_{j=0}^{d} \psi_j(a) = \bigoplus_{j=0}^{d} \psi_j(a),
$$
so that
$$
\varepsilon > \|\phi(\psi(a)) - a\| = \|\phi\left(\sum_{j=0}^{d} \psi_j(a)\right) - a\| = \|\sum_{j=0}^{d} (\phi_j \circ \psi_j)(a) - a\|,
$$
as needed.

Second, assume that there exist finite-dimensional $C^*$-algebras $F_0, \ldots, F_d$ and c.p.c. maps $A \xrightarrow{\psi_j} F_j \xrightarrow{\phi_j} A$ with each $\phi_i$ order zero such that for all $a \in F$,
$$
\|\sum_{j=0}^{d} (\phi_j \circ \psi_j)(a) - a\| < \varepsilon.
$$
Set $F := \bigoplus_{j=0}^{d} F_j$ and $\psi := \bigoplus_{j=0}^{d} \psi_j$. Extend each $\phi_j$ to $F$ by setting $\phi_j \equiv 0$ off $F_j$, and set $\phi := \sum_{j=0}^{d} \phi_j$. Then for $a \in F$,
$$
\|\phi(\psi(a)) - a\| = \|\phi\left(\sum_{j=0}^{d} \psi_j(a)\right) - a\| = \|\sum_{j=0}^{d} \phi_j(\psi_j(a)) - a\| < \varepsilon.
$$
Thus, $\text{dim}_{\text{mac}}(A) \leq d$, as needed.  

**Definition 2.11** ([29, Definition 5.1]). Let $G$ be a locally compact, Hausdorff and étale groupoid. Then $G$ has dynamic asymptotic dimension $d \in \mathbb{N}$ if $d$ is the smallest natural number with the property that for every open and precompact subset $V \subset G$, there are open subsets $U_0, U_1, \ldots, U_d$ of $G^{(0)}$ that cover $s(V) \cup r(V)$ such that for each $i \in \{0, \ldots, d\}$ the set $\{\gamma \in V : s(\gamma), r(\gamma) \in U_i\}$ is contained in a precompact subgroupoid of $G$. If no such $d$ exists, then we say that $G$ has infinite dynamic asymptotic dimension. We write $\text{DAD}(G)$ for the dynamic asymptotic dimension of $G$.

In Definition 2.11 we could have equivalently asked that for each $0 \leq i \leq d$ the subgroupoid $\langle V \cap G_{U_i} \rangle$ generated by the set $\{\gamma \in V : s(\gamma), r(\gamma) \in U_i\}$ is precompact. Notice that each $\langle V \cap G_{U_i} \rangle$ is open in $G$. If the unit space of $G$ is compact, then we can relax the definition of dynamic asymptotic dimension by only considering precompact sets $V$ that contain $G^{(0)}$.

**Remark 2.12.** To compute the dynamic asymptotic dimension of a locally compact, Hausdorff and étale groupoid $G$, many helpful results have been established. By [29, Example 5.3], $\text{DAD}(G) = 0$ if and only if $G$ is locally finite in the sense that it is an increasing union of open, precompact subgroupoids. (In particular, the dynamic asymptotic dimension of a compact, Hausdorff and étale groupoid is 0.)

In [29], Guentner, Willett and Yu also defined the notion of dynamic asymptotic dimension for an action of a discrete group $\Gamma$ on a locally compact, Hausdorff space $X$, and they showed that the associated transformation-group groupoid $X \rtimes \Gamma$ has dynamic asymptotic dimension equal to the dynamic asymptotic dimension of the group action [29, Lemma 5.4]. In particular, if $\mathbb{Z}$ acts minimally on an infinite compact space $X$, then $\text{DAD}(X \rtimes \mathbb{Z}) = 1$ by [29, Theorem 3.1]. There are more results about dynamic asymptotic dimension of
transformation-group groupoids in, for example, \[ \{2, 51, 49, 50\} \]; see also Proposition 5.3 below for results about a transformation groupoid where a groupoid acts on a space.

When \( X \) is a coarse space, the associated coarse groupoid has dynamic asymptotic dimension equal to the asymptotic dimension of \( X \) by [29, Theorem 6.4] and the dynamic asymptotic dimension of a graph groupoid is either 0 or \( \infty \) by Lemma 5.7 below.

Recently, some very useful reduction results have appeared in [9, Theorem A]: equivalent étale Hausdorff groupoids have the same dynamic asymptotic dimension. Moreover, if \( U_1, \ldots, U_n \) is an open cover of the unit space of such a groupoid \( G \), then \( \text{DAD}(G) \) is the maximum of the \( \text{DAD}(G|_{U_i}) \). Finally, by Theorem 3.1 below, for any locally compact, Hausdorff étale groupoid \( G \) there exists a locally compact, Hausdorff étale groupoid \( \tilde{G} \) with compact unit space such that \( \text{DAD}(G) = \text{DAD}(\tilde{G}) \), and that is a key tool in the proof of our main theorem, Theorem 4.1.

3. Unitizations

For every C*-algebra \( A \) there exists a unique unital C*-algebra \( \tilde{A} \), called the minimal unitization of \( A \), that contains \( A \) as an ideal with \( \tilde{A}/A \) isomorphic to \( C \) via \( 1 + A \mapsto 1 \). A favorite technique among C*-algebraists is to reduce a proof to the unital setting using the minimal unitization. In this section, we describe how such a unitization for a (twisted) groupoid C*-algebra is realized at the level of the groupoid by taking the Alexandrov one-point-compactification of the unit space. Our construction in the non-twisted case agrees with the Alexandrov groupoid of [30]. Just as for the associated C*-algebras, the original groupoid will share many key properties with its associated Alexandrov groupoid, such as dynamic asymptotic dimension and topological covering dimension. The following theorem summarizes our main results for étale groupoids, though some of our results are shown for r-discrete groupoids (see Remark 3.2 on our hypotheses).

**Theorem 3.1.** Let \( (E, \iota, \pi) \) be a twist over a locally compact, Hausdorff and étale groupoid \( G \) with non-compact unit space \( G(0) \). Then there exists a locally compact, Hausdorff and étale groupoid \( \tilde{G} \) with compact unit space and a twist \( (\tilde{E}, \tilde{\iota}, \tilde{\pi}) \) over \( \tilde{G} \) such that the minimal unitization of \( C^*_r(E; G) \) is isomorphic to \( C^*_r(\tilde{E}; \tilde{G}) \). Similarly, the minimal unitization of \( C^*_r(E; G) \) is isomorphic to \( C^*_r(\tilde{E}; \tilde{G}) \). Moreover, \( \text{DAD}(G) = \text{DAD}(\tilde{G}) \) and \( \text{dim}(G(0)) = \text{dim}(\tilde{G}(0)) \).

**Remark 3.2.** We state and prove Theorem 3.1 for C*-algebras of étale groupoids instead of r-discrete groupoids for the following reason. To consider a groupoid C*-algebra of \( G \), we need to assume that \( G \) carries a Haar system, and to consider \( \tilde{G} \), we need to assume that \( G \) is r-discrete. These two assumptions combined imply that \( G \) is étale by [61, Propositions 1.23 and 1.29].

We call the groupoid \( \tilde{G} \) the Alexandrov groupoid of \( G \) after [30, Definition 7.7], and we call the twist \( (\tilde{E}, \tilde{\iota}, \tilde{\pi}) \) over \( \tilde{G} \) the Alexandrov twist. We will prove Theorem 3.1 via a series of lemmas below. In Section 3.1 we give precise descriptions of the Alexandrov groupoid and the Alexandrov twist and establish their relevant groupoid properties (Lemma 3.4 and Lemma 3.6). In Section 3.2 we show that the correspondence between the unitizations of the groupoid C*-algebras and the Alexandrov groupoid C*-algebras. Finally, in Section 3.3 Proposition 3.13 proves our claim about dynamic asymptotic dimension.
3.1. The Alexandrov groupoid and twist. In order to elucidate some topological minutiae that will feature in subsequent proofs, we state the following lemma for easy reference.

**Lemma 3.3.** Let $X$ be a topological space and let $Y$ be a subspace. Then for $C \subset Y$, we have the following.

1. $C$ is compact in $X$ if and only if it is compact in $Y$.
2. If $C^X \subset Y$, then $C$ is precompact in $X$ if and only if it is precompact in $Y$.
3. If $X$ is Hausdorff and $C$ is precompact in $Y$, then $C^X = C^Y \subset Y$ and $C$ is precompact in $X$.

**Proof.** A subset of a topological space is compact if and only if it is compact with respect to the subspace topology. Since the topologies on $C$ that are induced from $X$ resp. from $Y$ coincide, Part (1) follows.

For Part (2), note that if $C^X \subset Y$, then $C^Y = C^X$, and it then follows from Part (1) that $C$ is precompact in $X$ if and only if it is precompact in $Y$.

For Part (3), we note that if $C$ is precompact in $Y$, then $C^Y$ is compact in $X$ by Part (1). Since $X$ is Hausdorff it follows that $C^Y$ is closed in $X$, and hence $C^Y = C^X$. Then Part (2) finishes the claim.

**Lemma 3.4.** Let $G$ be a locally compact, Hausdorff and $r$-discrete groupoid with non-compact unit space $G^{(0)}$, and let $G_+^{(0)} := G^{(0)} \cup \{\infty\}$ denote the Alexandrov one-point compactification of $G^{(0)}$. Set $\tilde{G} := G \cup \{\infty\}$, $\tilde{G}^{(2)} := G^{(2)} \cup \{(\infty, \infty)\}$, and $\tilde{G}^{(0)} := G_+^{(0)}$. We define composition of tuples in $G^{(2)} \subset \tilde{G}^{(2)}$ as in $G$, and extend $r$ and $s$ to $\tilde{G}$ by setting $r^{-1}(\infty) = s^{-1}(\infty) = \{\infty\}$. The set $\mathcal{U}$ consisting of $\tilde{G}$, all open sets in $G^{(0)}$ and all open sets in $G$ is a basis for a topology on $\tilde{G}$. Equipped with this basis, $\tilde{G}$ is a locally compact, Hausdorff and $r$-discrete groupoid with compact unit space $\tilde{G}^{(0)}$, and it contains $G$ as a subgroupoid. If in addition $G$ is étale, then so is $\tilde{G}$.

**Proof.** To see that $\mathcal{U}$ is a basis for a topology, the only non-trivial claim is that if $\alpha \in U_1 \cap U_2$ for $U_1$ an open subset of $G$ and $U_2$ an open subset of $\tilde{G}^{(0)} = G^{(0)} \cup \{\infty\}$, then there exists $U$ in $\mathcal{U}$ with $\alpha \in U \subset U_1 \cap U_2$. Since compact sets in the Hausdorff space $G^{(0)}$ are closed, we always have that $U_2 \setminus \{\infty\} = G^{(0)} \setminus \alpha$ for some closed subset $A$ of $G^{(0)}$, independent of whether $\infty$ is or is not an element of $U_2$. Since $G$ is $r$-discrete, the unit space $G^{(0)}$ is open in $G$, and so $U_1 \cap U_2 = U_1 \cap (G^{(0)} \setminus A)$ is automatically open in $G$ and thus the required element $U \in \mathcal{U}$.

By definition of $\mathcal{U}$, both $G$ and $\tilde{G}^{(0)} = G^{(0)} \cup \{\infty\}$ are open in $\tilde{G}$. In particular, $\tilde{G}$ is $r$-discrete. Using nets in $\tilde{G}$ that either converge to a point in $G$ or to $\infty$, it is easy to verify that the operations and the range and source maps on $\tilde{G}$ are continuous.

Since $G$ is Hausdorff, to check that $\tilde{G}$ is Hausdorff, it suffices to separate $\infty$ from an arbitrary $\beta \in G$. Since $G$ is locally compact, we can find an open set $V$ around $\beta$ in $G$ which is precompact in $G$. Since $G^{(0)}$ is closed in $G$, the closure $K$ of $V \cap G^{(0)}$ in $G^{(0)}$ is a compact subset of $G^{(0)}$. In particular, $(G^{(0)} \setminus K) \cup \{\infty\}$ is an open neighborhood of $\infty$ in $\tilde{G}$ which is disjoint from the open neighborhood $V$ of $\beta$.

Next we check that $\tilde{G}$ is locally compact. Since $\tilde{G}^{(0)}$ is a compact open neighborhood of $\infty$ in $\tilde{G}$, we only need to check that any $\alpha \in \tilde{G} \setminus \{\infty\}$ has a precompact open neighborhood in $\tilde{G}$. Let $U$ be a neighborhood of $\alpha$ that is precompact and open in $G$. Then $U \in \mathcal{U}$, and
since $G \subset \tilde{G}$ is equipped with the subspace topology, Lemma 8.3 says that $U$ is precompact in $\tilde{G}$. Thus, $\tilde{G}$ is locally compact, Hausdorff and $r$-discrete.

Finally, suppose that $G$ is étale. Since $G$ is $r$-discrete, this is equivalent to $G$ having a basis of open bisections. Since $\mathcal{U}$ contains all open sets of $G$, all open bisections of $G$ are open bisections of $\tilde{G}$; since $\mathcal{U}$ contains all open sets of $G^{(0)}_+$ and since the restriction of the range map to these sets is the identity, any open subset of $G^{(0)}_+$ containing $\infty$ is an open bisection in $\tilde{G}$. Thus, by adding all open subsets of $G^{(0)}_+$ containing $\infty$ to a basis of open bisections of $G$ we obtain a basis of open bisections of $\tilde{G}$. This concludes our proof. \qed

If $G$ is étale, then $\tilde{G}$ agrees with the Alexandrov groupoid $G^+$ from [30, Definition 7.7]. We reserve the notation $X_+$ for the Alexandrov compactification of a topological space $X$ and use $\tilde{G}$ for the Alexandrov groupoid of a groupoid $G$.

**Example 3.5.** Let $\Gamma$ be a discrete group with identity $e$ which acts continuously on a locally compact, non-compact, Hausdorff space $X$ and let $G = \Gamma \ltimes X$ be the transformation-group groupoid. Let $G$ be the Alexandrov groupoid as above, so that $\tilde{G} = G \cup \{\infty\}$. Denote by $X_+$ the one-point compactification of $X$. It is easy to check that the action of $\Gamma$ on $X_+$ given by

$$\gamma \cdot x = \begin{cases} \gamma \cdot x & \text{if } x \in X \\ \infty & \text{if } x = \infty \end{cases}$$

is continuous. As sets, we have

$$\tilde{G} = \{(\gamma, x) : \gamma \in \Gamma, x \in X\} \cup \{(e, \infty)\}$$

and $\Gamma \ltimes X_+ = \{(\gamma, x) : \gamma \in \Gamma, x \in X_+\}$.

Let $i : \tilde{G} \rightarrow \Gamma \ltimes X_+$ be the inclusion; it is trivial to check this is a groupoid homomorphism. Now notice that

$$(\Gamma \ltimes X_+) \setminus i(\tilde{G}) = \{(\gamma, \infty) : \gamma \neq e\}$$

is closed: for if $(\gamma_n, \infty) \rightarrow (\alpha, y)$ in $\Gamma \ltimes X_+$ with $\gamma_n \neq e$, then $\gamma_n \rightarrow \alpha$ and $\infty \rightarrow y$; thus $y = \infty$ and $\alpha \neq e$ since $\Gamma$ is discrete, and hence $\tilde{G}$ is a wide and open subgroupoid of $\Gamma \ltimes X_+$. It follows that the “inclusion” $i : C_c(\tilde{G}) \rightarrow C_c(\Gamma \ltimes X_+)$ given by extending functions to be 0 off of $\tilde{G}$ is a well-defined homomorphism which is isometric for the reduced norms. Thus, $i$ extends to an injective homomorphism $i : C_c^*(\tilde{G}) \rightarrow C_c^*(\Gamma \ltimes X_+)$. \hfill 

Next, we discuss the Alexandrov twist.

**Lemma 3.6.** Let $(E, \iota, \pi)$ be a twist over a locally compact, Hausdorff, $r$-discrete groupoid $G$ with non-compact unit space $G^{(0)}$. Set $\tilde{E} := E \cup \{\infty_z : z \in T\}$ and $\tilde{E}^{(0)} := E^{(0)} \cup \{(\infty_w, \infty_z) : z, w \in T\}$ with composition of pairs in $E^{(0)} \subset \tilde{E}^{(0)}$ defined as in $E$ and $\infty_w \infty_z = \infty_{wz}$ for all $w, z \in T$. Define $\tilde{r}, \tilde{s} : \tilde{E} \rightarrow G^{(0)} \cup \{\infty\}$ by $\tilde{r}|_E = r$, $\tilde{s}|_E = s$ and $\tilde{r}(\infty_w) = \infty = \tilde{s}(\infty_w)$ for all $w \in T$, and define $\tilde{i} : \tilde{E}^{(0)} \times T \rightarrow \tilde{E}$ by $\tilde{i}|_{G^{(0)} \times T} = \iota$ and $\tilde{i}(\infty, z) = \infty_z$. Then we have the following:

1. The collection $\mathcal{M}$ consisting of $\tilde{E}$, the open sets of $E$ and the sets $\tilde{i}(V)$ where $V$ is an open subset of $G^{(0)} \times T$, is a basis for a topology on $\tilde{E}$ with respect to which $\tilde{i}$ is open and continuous.

2. The topology on $E$ coincides with the subspace topology and $\tilde{E}$ is a locally compact and Hausdorff groupoid.

3. Define $\tilde{\pi} : \tilde{E} \rightarrow \tilde{G}$ by $\tilde{\pi}|_E = \pi$ and $\tilde{\pi}(\infty_z) = \infty$. Then $(\tilde{E}, \tilde{r}, \tilde{s}, \tilde{i})$ is a twist over $\tilde{G}$. \hfill 

Proof. For \([1]\) let \(U_1, U_2 \in \mathcal{V}\); we will show that

\[ U := U_1 \cap U_2 \in \mathcal{V}. \]

If either \(U_i\) is \(\tilde{E}\), or if both \(U_i\) are subsets of \(E\), then the claim is trivial. If both \(U_i\) are of the form \(i(V)\), then since \(\tilde{i}\) is injective we again have \(U = \tilde{i}(V_1 \cap V_2) \in \mathcal{V}\).

So suppose that \(U_1\) is open in \(E\) and that \(U_2 = i(V)\) where \(V\) is open in \(\tilde{G}^{(0)} \times \mathcal{T}\). Without loss of generality, \(V = W \times S\) where \(W\) is open in \(\tilde{G}^{(0)}\) and \(S\) is open in \(\mathcal{T}\). Then since \(U_1 \subset E\), we have \(U = U_1 \cap \iota(W \setminus \{\infty\} \times S)\). As mentioned earlier, since \(G\) is \(r\)-discrete, \(\pi^{-1}(G^{(0)})\) is open in \(E\), and since \(\iota\) is a homeomorphism onto \(\pi^{-1}(G^{(0)})\), it follows that \(\iota\) is an open map into \(E\). Since \(W \setminus \{\infty\} \times S\) is open in \(G^{(0)} \times \mathcal{T}\) we get that \(\iota(W \setminus \{\infty\} \times S)\) is open in \(E\). So \(U\) is open in \(E\) and hence in \(\tilde{E}\) as well. Hence, \(\mathcal{V}\) is a basis for a topology \(\tilde{E}\). Note that \(\tilde{i}\) is an open, injective map by construction.

To see that \(\tilde{i}\) is continuous, let \(U \subset \tilde{E}\) be an open set. If \(U\) is open in \(E\), then \(\tilde{i}^{-1}(U) = \iota^{-1}(U)\) is open in \(G^{(0)} \times \mathcal{T}\) and hence in \(\tilde{G}^{(0)} \times \mathcal{T}\). If \(U\) is of the form \(i(V)\) for some open set \(V\) in \(\tilde{G}^{(0)} \times \mathcal{T}\), then \(\tilde{i}^{-1}(U) = V\) is open. Thus, \(\tilde{i}\) is continuous.

For \([2]\) to see that the topology \(\tau\) on \(E\) is exactly the subspace topology \(\mathcal{V} \cap E\), note first that \(\tau \subset \mathcal{V}\). For the other inclusion, it suffices to check that, whenever \(W\) and \(S\) are open subsets of \(\tilde{G}^{(0)}\) and \(\mathcal{T}\), respectively, then \(\tilde{i}(W \times S) \cap E\) is open in \(E\). Note that \(\tilde{i}(W \times S) \cap E = \iota([W \setminus \{\infty\}] \times S)\). Since \(W \setminus \{\infty\}\) is open in \(G^{(0)}\) and \(\iota\) is open as a map into \(E\), it follows that \(\tilde{i}(W \times S) \cap E\) is open in \(E\).

It is easy to verify that \(\tilde{E}\) is an algebraic groupoid with unit space \(\tilde{G}^{(0)} = G^{(0)} \cup \{\infty\}\); it remains to show that \(\tilde{s}, \tilde{r}\), inversion and multiplication are continuous. To that end, let \(\{e_\lambda\}_\lambda\) be a net in \(\tilde{E}\) that converges to \(e \in \tilde{E}\). Assume first that \(e \in E\). Then since \(E\) is open in \(\tilde{E}\), we eventually have \(e_\lambda \in E\) and hence

\[ \tilde{s}(e_\lambda) = s(e_\lambda) \to s(e) = \tilde{s}(e), \quad \tilde{r}(e_\lambda) = r(e_\lambda) \to r(e) = \tilde{r}(e) \quad \text{and} \quad e_\lambda^{-1} \to e^{-1} \]

by continuity of the source, range, and inversion map of \(E\). Second, assume that \(e = \infty_z\) for some \(z \in \mathcal{T}\). Then \(e = \tilde{i}(\infty, z)\) is in the image of \(\tilde{i}\). If we take a neighborhood around \(e\), say \(\tilde{i}(V)\) for \(V = [(G^{(0)} \setminus K) \cup \{\infty\}] \times S\) where \(K\) is a compact subset of \(G^{(0)}\) and \(S\) is an open neighborhood around \(z\) in \(\mathcal{T}\), then eventually \(e_\lambda \in \tilde{i}(V)\). Since \(\tilde{i}\) is, when restricted to \(V\), still an open map that is surjective onto its range, Fell’s criterion ([61, Proposition 1.1]) implies that there exists a subnet \(\{e_{f(\mu)}\}_\mu\) of \(\{e_\lambda\}_\lambda\) such that for each \(\mu\), \(e_{f(\mu)} = \tilde{i}(x_\mu, z_\mu)\) for some \(x_\mu \in \tilde{G}^{(0)}\) and some \(z_\mu \in \mathcal{T}\) where \(x_\mu \to \infty\) and \(z_\mu \to z\). Since

\[ \tilde{r}(e_{f(\mu)}) = \tilde{s}(e_{f(\mu)}) = x_\mu \to \infty = \tilde{s}(e) = \tilde{r}(e) \quad \text{in} \quad \tilde{G}^{(0)} \]

and

\[ e_{f(\mu)}^{-1} = \tilde{i}(x_\mu, z_\mu) \to \tilde{i}(\infty, z) = e^{-1} \]

by continuity of \(\tilde{i}\), we have shown that \(\{\tilde{s}(e_\lambda)\}_\lambda = \{\tilde{r}(e_\lambda)\}_\lambda\) has a subnet that converges to \(\tilde{s}(e) = \tilde{r}(e)\) and that \(\{e_\lambda^{-1}\}_\lambda\) has a subnet that converges to \(e^{-1}\). This shows that \(\tilde{s}, \tilde{r}\), and the inversion map are continuous by, for example, [44, Theorem 18.1].

Suppose next that we have \((d_\lambda, e_\lambda) \to (d, e)\), a convergent net in \(\tilde{E}^{(2)}\). Since \((d, e)\) is composable, we either have \((d, e) \in E^{(2)}\) or \((d, e) = (\infty_w, \infty_z)\). In the first case, note that since \(E\) is open in \(\tilde{E}\), \(E^{(2)}\) is open in \(\tilde{E}^{(2)}\). Thus, we eventually have \((d_\lambda, e_\lambda) \in E^{(2)}\) and continuity of the multiplication on \(E\) implies that \(d_\lambda e_\lambda \to de\) in \(E\) and hence in \(\tilde{E}\). In the case where
Consider a basic open neighborhood of \((d,e)\), say \([\overline{T}(V_1) \times \overline{T}(V_2)] \cap \overline{E}\). Eventually, \((d,\lambda)\) must be in that set, so we must have \(d_\lambda = \overline{i}(x_\lambda, w_\lambda)\) and \(e_\lambda = \overline{i}(x_\lambda, z_\lambda)\) with \(x_\lambda \to \infty\) in \(\mathcal{G}^0 \cup \{\infty\}\), and with \(w_\lambda \to w\) and \(z_\lambda \to z\) in \(T\). By continuity of \(\overline{i}\), we conclude
\[
\begin{align*}
 d_\lambda e_\lambda &= \overline{i}(x_\lambda, w_\lambda) \overline{i}(x_\lambda, z_\lambda) \\
 &= \overline{i}(x_\lambda, w_\lambda z_\lambda) \\
 &= \overline{i}(\infty, wz) = \infty_w \infty_z = de.
\end{align*}
\]

Because \(E\) and \(T\) are Hausdorff and the map \(i\) is injective, to check that \(\tilde{E}\) is Hausdorff, we need only to check that we can separate a point \(e \in E\) from a point \(\infty_z \in \tilde{E} \setminus E\). Since \(s(e) \neq \infty\), there exists an open neighborhood \(V\) of \(s(e)\) in \(\mathcal{G}^0\) with compact closure \(K\). Then \(X := (\mathcal{G}^0 \setminus K) \cup \{\infty\}\) is an open neighborhood of \(\infty\) in \(\tilde{\mathcal{G}}^0\) with \(V \cap X = \emptyset\). Then \(s^{-1}(V) = \tilde{s}^{-1}(V)\) is an open neighborhood of \(e\) in \(\tilde{E}\) and \(\overline{i}(X \times T)\) is an open neighborhood of \(\infty_z\). Since \(\tilde{s}(s^{-1}(V) \cap \overline{i}(X \times T)) \subset V \cap X = \emptyset\) we must have \(\tilde{s}^{-1}(V) \cap \overline{i}(X \times T) = \emptyset\) as well.

Since \(E \subset \tilde{E}\) is a subspace, Lemma \([3,3](1)\) tells us that a neighborhood of a point \(e \in E\) that is open and precompact in \(E\) is open and precompact in \(\tilde{E}\) as well. For \(z \in T\), \(\overline{i}(\tilde{\mathcal{G}}^0 \times T)\) is an open compact neighborhood of \(\infty_z \in \tilde{E}\). Thus, \(\tilde{E}\) is locally compact and Hausdorff.

For \([3]\), we note first that, clearly, \(\overline{i}\) is a homomorphism. By construction, \(\overline{i}\) is an injective map which is open into \(\tilde{E}\), and we have shown above that \(\overline{i}\) is continuous. It follows that \(\overline{i}\) is a homeomorphism onto its range
\[
\overline{i}(\tilde{\mathcal{G}}^0 \times T) = \iota((\mathcal{G}^0 \times T) \cup \{\infty_z : z \in T\}) = \pi^{-1}(\mathcal{G}^0) \cup \{\infty_z : z \in T\} = \pi^{-1}(\tilde{\mathcal{G}}^0).
\]

Next, \(\pi\) is clearly a surjective homomorphism. To see that \(\pi\) is continuous, observe that for any open set \(U \subset G\) the set \(\pi^{-1}(U) = \pi^{-1}(U)\) is open in \(E\) and hence in \(\tilde{E}\). For a basic open neighborhood \((G^0 \setminus K) \cup \{\infty\}\) around \(\infty\) with \(K\) compact in \(\mathcal{G}^0\), we compute
\[
\begin{align*}
\pi^{-1}((G^0 \setminus K) \cup \{\infty\}) &= \pi^{-1}(G^0 \setminus K) \cup \{\infty_z : z \in T\} \\
&= \iota((G^0 \setminus K) \times T) \cup \{\infty_z : z \in T\} \\
&= \overline{i}(((G^0 \setminus K) \cup \{\infty\}) \times T);
\end{align*}
\]

since \([(G^0 \setminus K) \cup \{\infty\}] \times T\) is open in \(\tilde{\mathcal{G}}^0 \times T\), this is an open set in \(\tilde{E}\), and \(\pi\) is continuous.

To see that \(\pi\) is open, let \(U\) be a basic open set in \(\tilde{E}\). If \(U = \tilde{E}\), then \(\pi(U) = \tilde{G}\) is open. If \(U\) is open in \(E\), then \(\pi(U) = \pi(U)\) is open in \(E\) and hence in \(\tilde{E}\). Otherwise, \(U = \overline{i}(V)\) where \(V\) is open in \(\tilde{\mathcal{G}}^0 \times T\). We may assume that \(V = W \times S\) where \(W\) is open in \(\tilde{\mathcal{G}}^0\) and \(S\) is open in \(T\). Then
\[
\pi(U) = \pi(\overline{i}(W \times S)) = W
\]
because \(\pi \circ \iota(x,z) = x\) and \(\pi \circ \overline{i}(\infty, z) = \infty\). Thus, \(\pi\) is open.

To see that \(\overline{i}(\tilde{\mathcal{G}}^0 \times T)\) is central, first take \(e \in E\). We have \(\overline{i}(\tilde{r}(e), z) = \iota(r(e), z)\) and so the corresponding property of \(i\) implies \(\overline{i}(\tilde{r}(e), z)e = e\iota(\tilde{s}(e), z)\), as needed. If \(e = \infty_w\), then the claim follows from commutativity of \(T\):
\[
\overline{i}(\tilde{r}(e), z)e = \overline{i}(\infty, z)e = \infty_z \infty_w = \infty_w \infty_z = e\iota(\tilde{s}(e), z).
\]

Thus, \((\tilde{E}, \iota, \pi)\) is a twist over \(\tilde{G}\). \(\square\)
3.2. The twisted $C^*$-algebra of the Alexandrov groupoid. Next we will show that the minimal unitization of a non-unital twisted groupoid $C^*$-algebra coincides with the twisted groupoid $C^*$-algebra of the Alexandrov twist. We need the following lemma.

Lemma 3.7. Let $(E, i, \pi)$ be a twist over a locally compact, Hausdorff and étale groupoid $G$. Let $U$ be an open and invariant subset of $G^{(0)}$ and set $W = G^{(0)} \setminus U$. Then the inclusion map on $C_c(E|_U)$ and the restriction map on $C_c(E; G)$ induce the short exact sequence

$$0 \to C^*(E|_U; G|_U) \overset{i}{\to} C^*(E; G) \overset{j}{\to} C^*(E|_W; G|_W) \to 0. \tag{3.1}$$

If the reduced norm on $C_c(E|_W)$ equals the universal norm, then the inclusion and restriction maps induce the short exact sequence

$$0 \to C^*_r(E|_U; G|_U) \overset{i}{\to} C^*_r(E; G) \overset{j}{\to} C^*_r(E|_W; G|_W) \to 0. \tag{3.2}$$

Proof. By [61, Theorem 5.1], the inclusion and restriction maps induce the short exact sequence

$$0 \to C^*(E|_U) \overset{i}{\to} C^*(E) \overset{j}{\to} C^*(E|_W) \to 0,$$

and if the reduced norm on $C_c(E|_W)$ equals the universal norm, by [61, Proposition 5.2] they also induce

$$0 \to C^*_r(E|_U) \overset{i}{\to} C^*_r(E) \overset{j}{\to} C^*_r(E|_W) \to 0.$$

Since $U$ is invariant, for $z \in T$ we have $z \cdot e \in E|_U$ if and only if $e \in E|_U$.

It follows that the inclusion map takes $C_c(E|_U; G|_U)$ to $C_c(E; G)$. Similarly, $W$ is invariant, and the restriction map takes $C_c(E; G)$ to $C_c(E|_W; G|_W)$. Since $C^*(E|_U; G|_U)$, $C^*(E; G)$ and $C^*(E|_W; G|_W)$ are direct summands of $C^*(E|_U)$, $C^*(E)$ and $C^*(E|_W)$, respectively, it follows that Equation (3.1) is exact. Similarly, Equation (3.2) is exact. □

Lemma 3.8. Let $(E, i, \pi)$ be a twist over a locally compact, Hausdorff and étale groupoid $G$. Then $C^*(\tilde{E}; G) \cong C^*(E; G)^\sim$ and $C^*_r(\tilde{E}; G) \cong C^*_r(E; G)^\sim$.

Proof. We will apply Lemma 3.7 to the twist $(\tilde{E}, \tilde{i}, \tilde{\pi})$ with the open and invariant subset $U = G^{(0)}$ and its complement $W = \{\infty\}$. Since the restricted groupoid $\tilde{E}|_W = \{\infty_w : w \in T\}$ is isomorphic to $T$, the reduced and universal norms on $C_c(\tilde{E}|_W)$ coincide, and so Lemma 3.7 gives short exact sequences on the level of both the full and reduced $C^*$-algebras.

We start by proving that $C_c(\tilde{E}|_W; \tilde{G}|_W)$ is isomorphic to $C$. Here $C_c(\tilde{E}|_W; \tilde{G}|_W)$ consists of functions $f : \{\infty_w : w \in T\} \to C$ such that $zf(\infty_w) = f(z \cdot \infty_w)$. Define $\rho : C_c(\tilde{E}|_W; \tilde{G}|_W) \to C$ by $\rho(f) = f(\infty_1)$. Let $f, g \in C_c(\tilde{E}|_W; \tilde{G}|_W)$. Then for any section $s : \tilde{G} \to \tilde{E}$ of $\tilde{\pi}$ we have

$$\rho(f \ast g) = (f \ast g)(\infty_1) = \sum_{\gamma \in \infty \tilde{G}} f(s(\gamma))g(s(\gamma)^{-1}\infty_1)$$

$$= f(\infty_2)g(\infty_2 \infty_1) = z\bar{z} f(\infty_1)g(\infty_1) = \rho(f)\rho(g)$$

and $\rho(f^*) = \overline{\rho(f)} = \overline{f(\infty_1)} = \overline{\rho(f)}$. Thus, $\rho$ is a $\ast$-homomorphism. Since $\tilde{G}$ is étale, so is $\tilde{G}|_W$ by Lemma 2.3. Thus $\rho$ extends to a homomorphism $\rho : C^*(\tilde{E}|_W; \tilde{G}|_W) \to C$ by [4, Lemma 2.14]. It is clear that $\rho$ is surjective. To see that $\rho$ is injective, we note that since the reduced and universal norm on $C_c(\tilde{E}|_W)$ coincide they also coincide on $C_c(\tilde{E}|_W; \tilde{G}|_W)$. \hfill \square
Thus, $\|f\| = \|f\|_r = \|\pi^\infty(f)\|$ where $\pi^\infty$ is as defined in (2.6). In order to compute $\pi^\infty(f)$, note that for $\xi \in L^2(\tilde{E}|_W; \tilde{G}|_W \infty) = L^2(\tilde{E}|_W; \infty)$, we have

$$
\|\xi\|^2 = \int_{\tilde{E}} |\xi(e)|^2 d\sigma_\infty = \sum_{\gamma \in \tilde{G}} \int_{T} |\xi(z \cdot s(\gamma))| d\lambda(z)
$$

$$
= \int_{T} |\xi(z \cdot s(\infty))| d\lambda(z) = \int_{T} |\xi(\infty z)| d\lambda(z) = |\xi(\infty_1)|^2.
$$

A computation analogous to that in Equation (3.3) then shows that

$$
\|\pi^\infty(f)\| = |(f \ast \xi)(\infty_1)| = |f(\infty_1)| |\xi(\infty_1)| = |\rho(f)| \|\xi\|,
$$

so that $\|\pi^\infty(f)\| = |\rho(f)|$. It follows that $\rho$ is an isometric isomorphism of $C^*(\tilde{E}|_W; \tilde{G}|_W) = C^r(\tilde{E}|_W; \tilde{G}|_W)$ onto $C_r\ast$.

As mentioned previously, Lemma 3.7 gives short exact sequences; by what we have shown so far, they can be rewritten to

$$
0 \longrightarrow C^r(\tilde{E}|_U; \tilde{G}|_U) \xrightarrow{i} C^r(\tilde{E}; \tilde{G}) \xrightarrow{j} C \longrightarrow 0;
$$

$$
0 \longrightarrow C^r(\tilde{E}|_U; \tilde{G}|_U) \xrightarrow{i} C^r(\tilde{E}; \tilde{G}) \xrightarrow{j} C \longrightarrow 0.
$$

Since $\tilde{E}|_U = E$ and $\tilde{G}|_U = G$, this shows that $C^r(E; G)$ and $C^r_r(E; G)$ are ideals in $C^r(\tilde{E}; \tilde{G})$ and $C^r_r(\tilde{E}; \tilde{G})$, respectively, both of codimension 1. This completes the proof.

Let $E = T \times G$ be the trivial twist, which as a groupoid is the Cartesian product. Then the full (or reduced) twisted groupoid $C^\ast$-algebra is isomorphic to the full (or reduced) $C^\ast$-algebra of $G$ and we obtain the following corollary of Lemma 3.8.

**Corollary 3.9.** Let $G$ be a second-countable, locally compact, Hausdorff and étale groupoid with non-compact unit space, and let $\tilde{G}$ be its Alexandrov groupoid. Then $C^\ast_r(\tilde{G}) \cong C^\ast_r(G)\sim$ and $C^\ast_r(\tilde{G}) \cong C^\ast_r(G)\sim$.

**Remark 3.10.** Let $G$ be a locally compact, not necessarily Hausdorff, étale groupoid with Hausdorff unit space. In [23], Exel and Pitts have proposed that a certain quotient of $C^\ast_r(G)$, called the essential groupoid $C^\ast$-algebra and denoted by $C^\ast_{\text{ess}}(G)$, is the right replacement for the reduced $C^\ast$-algebra of $G$ when $G$ is topologically principal. Evidence towards this is that if $G$ is topologically principal, then (1) $C_0(G^0)$ detects ideals of $C^\ast_{\text{ess}}(G)$ and (2) $C^\ast_{\text{ess}}(G)$ is simple if and only if $G$ is minimal [24, Theorem 22.6]. Again if $G$ is topologically principal, then the ideal $I$ of $C^\ast_r(G)$ giving the quotient $C^\ast_{\text{ess}}(G)$ consists of singular functions, and $I$ vanishes when $G$ is Hausdorff [24, Proposition 18.9]. The essential groupoid $C^\ast$-algebras have been studied further, for example in [38, 30]. In particular, there is a version of the Alexandrov groupoid $\tilde{G}$ for a non-Hausdorff $G$, and [30, Proposition 7.8] says that $C^\ast_{\text{ess}}(\tilde{G})$ is isomorphic to $C^\ast_{\text{ess}}(G)\sim$; applying this to the Hausdorff case, this gives another proof of Corollary 3.9 for the reduced groupoid $C^\ast$-algebras.

### 3.3. Dynamic asymptotic dimension of the Alexandrov groupoid

Finally, we are ready to show that dynamic asymptotic dimension is preserved by the Alexandrov construction. We begin with a brief digression which establishes, in particular, that the passage of finite nuclear dimension to quotients and ideals can be witnessed at the groupoid level with dynamic asymptotic dimension.
Lemma 3.11. Let $G$ be a locally compact, Hausdorff and étale groupoid. Suppose that either

1. $H = G\big|_U$ for some open set $U \subset G^0$, or
2. $H$ is a closed subgroupoid of $G$.

If $\text{DAD}(G) = d$, then $\text{DAD}(H) \leq d$.

Remark 3.12. Before proving Lemma 3.11 we remark on its significance. Suppose that $\text{DAD}(G)$ and $\dim(G^0)$ are finite, so that $C^*_r(G)$ is nuclear by [29, Theorem 8.6]. In particular, $G$ is amenable by [3, Corollary 6.2.14] and any closed subgroupoid of $G$ is amenable as well [3, Proposition 5.11]. Let $U \subset G^0$ be an open invariant subset and let $W := G^0 \setminus U$. Then $G\big|_W$ is amenable, and inclusion and extension by zero and restriction induce a short exact sequence

$$0 \to C^*_r(G\big|_U) \to C^*_r(G) \to C^*_r(G\big|_W) \to 0$$

by [61, Proposition 5.2]. Then by [66, Proposition 2.9],

$$\max\{\dim_{\text{nuc}}(C^*_r(G\big|_U)), \dim_{\text{nuc}}(C^*_r(G\big|_W))\} \leq \dim_{\text{nuc}}(C^*_r(G)).$$

Lemma 3.11 tells us that this $C^*$-algebraic statement is already witnessed at the groupoid level by finite dynamic asymptotic dimension.

Proof of Lemma 3.11. Suppose first that $H = G\big|_U$ for some open subset $U$ of the unit space. Let $V$ be an open and precompact subset of $H$. Since $s$ and $r$ are open and continuous maps, we can replace $V$ by $V \cup s(V) \cup r(V)$ if necessary, so assume without loss of generality that $s(V) \cup r(V) \subset V$. Since $H$ is open, $V$ is also open in $G$, and is precompact in $G$ with $\overline{V^G} = \overline{V^H} \subset H$ by Lemma 3.3(3). Since $\overline{V^G} = \overline{V^H}$, we drop the superscript and simply write $V$.

Since $\text{DAD}(G) = d$ there exist $U_0, \ldots, U_d \subset G^0$ which are open in $G$, cover $s(V) \cup r(V)$ and such that the subgroupoid

$$G_i = \langle V \cap G|_{V_i} \rangle = \langle V \cap G|_{U_i \cap V} \rangle$$

of $G$ is precompact in $G$. For $0 \leq i \leq d$, let $V_i = U_i \cap V$. Then $\{V_i\}_{i=0}^d$ is an open cover of $s(V) \cup r(V)$ in $H$. Notice that $G_i = \langle V \cap G|_{V_i} \rangle$ is a subgroupoid of $H$. If $\alpha \in \overline{V^G}$, then $s(\alpha) \in \overline{V^H}$ by continuity of the source map. This implies that $s(\alpha) \in s(V) \subset U$ because $V$ is a precompact subset of $H$. Similarly, $r(\alpha) \in U$, and hence $\alpha \in H$. Thus, the closure of $G_i$ in $G$ is contained in $H$ and hence $G_i$ is precompact in $H$ by Lemma 3.3(2). This proves that $\text{DAD}(H) \leq d$.

Now suppose that $H$ is a closed subgroupoid of $G$. Let $V \subset H$ be a precompact open subset of $H$. Let $V'$ be open in $G$ such that $V = H \cap V'$. By Lemma 3.3(3) $V'$ is precompact in $G$, and since $G$ is locally compact, we can find a precompact open set $W$ containing $\overline{V^G}$; replacing $V'$ by $V' \cap W$ if necessary we can assume without loss of generality that $V'$ is precompact.

We now apply $\text{DAD}(G) \leq d$ to $V'$ to get open sets $U'_0, \ldots, U'_d$ of $G^0$ such that $s(V') \cup r(V') \subset \bigcup_{i=0}^d U'_i$ and such that $\langle G|_{U'_i \cap V'} \rangle$ is precompact. Then the collection of $U_i := U'_i \cap H$ is an open cover in $H$ of $s(V) \cup r(V)$. Furthermore, $\langle H|_{U_i \cap V} \rangle$ is contained in $\langle G|_{U_i \cap V'} \rangle$, and is therefore precompact in $G$. Since $H$ is closed, Lemma 3.3(2) implies that $\langle H|_{U_i \cap V} \rangle$ is also precompact in $H$. As $V$ was arbitrary, it follows that $\text{DAD}(H) \leq d$.

Proposition 3.13. Suppose that $G$ is a locally compact, Hausdorff and étale groupoid. Then $\text{DAD}(G) = \text{DAD}(\hat{G})$. 
Note that since \( \bigcup_i W_i \) is open in \( \tilde{G} \), the closure of \( W \) is contained in \( \tilde{G} \setminus \tilde{G}^{(0)} = G \setminus G^{(0)} \); by Lemma 3.3 \((2)\) \( W \) is hence precompact in \( G \). Since \( G \) is locally compact and Hausdorff and \( G \setminus G^{(0)} \) is open in \( G \), there exists an open precompact subset \( W' \) of \( G \) with \( \overline{W} \subset W' \subset \overline{W'} \subset G \setminus G^{(0)} \).

Since \( \text{DAD}(G) = d \), there exist open subsets \( U_0, \ldots, U_d \) of \( G \) such that \( W' \cap G \setminus U_i \) is a precompact subgroupoid of \( G \). By replacing each \( U_i \) with \( U_i \cap (s(W') \cup r(W')) \), we can without loss of generality assume that the \( U_i \)'s are precompact in \( G \). Since \( W \cap G \setminus U_i \) \( \subset \langle W' \cap G \setminus U_i \rangle \), we have that \( G_i := \langle W \cap G \setminus U_i \rangle \) is also a precompact subgroupoid of \( G \) for each \( 0 \leq i \leq d \). Since \( W \) is compact, we know that \( s(W') \cup r(W') = s(G \setminus U') \cap r(W) = s(W) \cup r(W) \cap (s(W') \cup r(W')) \) is closed in \( \tilde{G}^{(0)} \). Since \( \tilde{G}^{(0)} \) is normal and \( s(W') \cup r(W) \subset s(W') \cup r(W') \subset \bigcup_{i=0}^d U_i \), we can thus find an open set \( U \subset \tilde{G}^{(0)} \) such that

\[
\begin{align*}
    s(W) \cup r(W) & \subset U \subset U^{(0)} = \overline{U}^G \subset U^d \subset \bigcup_{i=0}^d U_i.
\end{align*}
\]

Note that since \( \bigcup_{i=0}^d U_i \subset G^{(0)} \), \( U \) contains no neighborhood of \( \infty \) and is hence also open in \( G \). Since each \( U_i \) is precompact in \( G \), it follows that \( U \) is precompact in \( G \), so \( \overline{U} := \overline{U}^G = U^G \).

Now, if we let \( U'_0 := U_0 \cup [\tilde{G}^{(0)} \setminus U] \), then \( \{U'_0, U_1, \ldots, U_d\} \) is a cover of \( G^{(0)} = s(V) \cup r(V) \) consisting of open subsets of \( \tilde{G}^{(0)} \). Since \( \tilde{G}^{(0)} \subset V \) and \( W = V \setminus \tilde{G}^{(0)} \), we have for any \( F \subset G^{(0)} \)

\[
\begin{align*}
    V \cap G \mid_F = \langle V \cap G \setminus F \rangle \cup F,
\end{align*}
\]

and so \( \langle V \cap G \mid_{U_i} \rangle = G_i \cap U_i \). Moreover, since \( U \) contains \( s(W) \cup r(W) \), we have

\[
\begin{align*}
    [s(W) \cup r(W)] \cap U'_0 = [s(W) \cup r(W)] \cap U_0.
\end{align*}
\]

Hence, \( W \cap G \mid_{U'_0} = W \cap G \mid_{U_0} \), and thus we likewise have \( \langle V \cap G \mid_{U'_0} \rangle = G_0 \cup U'_0 \). Since each \( G_i \) and \( U_i \) is precompact in \( G \), it follows that \( G_0 \cup U'_0 \) and each \( G_i \cup U_i \) is precompact in \( G \).

Hence, \( U'_0, U_1, \ldots, U_d \) satisfies Definition 2.11, and so \( \text{DAD}(G) \leq d = \text{DAD}(G) \).

Now, suppose \( \tilde{G} \) has DAD \( d' \). Since \( \tilde{G}^{(0)} \) is open in \( \tilde{G} \) and \( G = \tilde{G} \mid_{G^{(0)}} \) we can apply Lemma 3.11 \((1)\) to conclude that \( \text{DAD}(G) \leq d' = \text{DAD}(\tilde{G}) \).

\[\]
By taking $E$ to be the Cartesian product $T \times G$ we recover [29, Theorem 8.6]. Before giving the proof of Theorem 4.1 (below after Lemma 4.5), we need to establish some preliminary results. In particular, our proof of Theorem 4.1 is inspired by arguments in the proofs of [29, Theorem 8.6] and [31, Theorem 6.2]: both pick up on a colored version of local subhomogeneity [22, Definition 1.5]. The following proposition makes this explicit. In the proof of Theorem 4.1 we apply this proposition to subhomogeneous $C^*$-algebras $B_i$ that are the $C^*$-algebras of the subgroupoids arising from applying the definition of finite asymptotic dimension.

**Proposition 4.2.** Let $A$ be a unital $C^*$-algebra and let $X \subset A$ be such that $\text{span}(X)$ is dense in $A$. Let $d, N \in \mathbb{N}$. Suppose that for every finite $F \subset X$ and every $\varepsilon > 0$ there exist $C^*$-subalgebras $B_0, \ldots, B_d$ of $A$ with

$$\max_{0 \leq i \leq d} \dim_{\text{nuc}}(B_i) \leq N$$

and $b_0, \ldots, b_d \in A$ of norm at most 1 such that $b_iFb_i^* \subset B_i$ for $0 \leq i \leq d$ and such that

$$\max_{x \in F} \left\| x - \sum_{i=0}^{d} b_i x b_i^* \right\| < \varepsilon.$$

Then the nuclear dimension of $A$ is at most $(d + 1)(N + 1) - 1$.

**Proof.** We will use the characterization of nuclear dimension from Lemma 2.10. Since span$(X)$ is dense in $A$, it suffices to show that the criteria from Lemma 2.10 hold for any fixed finite subset $F \subset X$ and $\varepsilon > 0$.

By assumption, there exist $C^*$-subalgebras $B_0, \ldots, B_d \subset A$ with $\dim_{\text{nuc}}(B_i) \leq N$ and $b_0, \ldots, b_d \in A$ of norm at most 1 such that $b_iFb_i^* \subset B_i$ for $0 \leq i \leq d$, and such that

$$\left\| x - \sum_{i=0}^{d} b_i x b_i^* \right\| < \varepsilon \quad \text{for all } x \in F. \quad (4.1)$$

Since each $B_i$ has nuclear dimension at most $N$ and $b_iFb_i^*$ is a finite subset of $B_i$, for $0 \leq k \leq N$ there exist finite-dimensional $C^*$-algebras $F_{i,k}$ and c.p.c. maps $B_i \xrightarrow{\rho_{i,k}} F_{i,k} \xrightarrow{\phi_{i,k}} B_i$ with $\phi_{i,k}$ order zero such that for all $x \in F$,

$$\left\| \sum_{k=0}^{N} (\phi_{i,k} \circ \rho_{i,k})(b_i x b_i^*) - b_i x b_i^* \right\| < \frac{\varepsilon}{2(d + 1)}. \quad (4.2)$$

Using Arveson’s Extension Theorem, as stated in Theorem 2.7, we extend $\rho_{i,k} : B_i \to F_{i,k}$ to c.p.c. maps $A \to F_{i,k}$, which we also denote by $\rho_{i,k}$. For $0 \leq i \leq d$ and $0 \leq k \leq N$, define $\psi_{i,k} : A \to F_{i,k}$ by $\psi_{i,k}(a) = \rho_{i,k}(b_i a b_i^*)$ for all $a \in A$. Since $\|b_i\| \leq 1$ for each $i$, these are all c.p.c. maps. Thus, for $0 \leq i \leq d$ and $0 \leq k \leq N$, we have c.p.c. maps

$$A \xrightarrow{\psi_{i,k}} F_{i,k} \xrightarrow{\phi_{i,k}} B_i \subset A$$
with each $\phi_{i,k}$ order zero such that for each $x \in F$,

$$
\left\| \sum_{i=0}^{d} \sum_{k=0}^{N} (\phi_{i,k} \circ \psi_{i,k})(x) - x \right\| \\
\leq \left\| \sum_{i=0}^{d} \sum_{k=0}^{N} (\phi_{i,k} \circ \rho_{i,k})(b_{i}xb_{i}^{*}) - \sum_{i=0}^{d} b_{i}xb_{i}^{*} \right\| + \left\| \sum_{i=0}^{d} b_{i}xb_{i}^{*} - x \right\| \\
< \left( \sum_{i=0}^{d} \left( \sum_{k=0}^{N} (\phi_{i,k} \circ \rho_{i,k})(b_{i}xb_{i}^{*}) - b_{i}xb_{i}^{*} \right) \right) + \frac{\varepsilon}{2} \\
< (d + 1) \frac{\varepsilon}{2(d + 1)} + \frac{\varepsilon}{2} = \varepsilon.
$$

(\text{by (4.2)})

Thus, $\text{dim}_{\text{nuc}}(A) \leq (d + 1)(N + 1) - 1$. □

Next, we examine the properties of the subgroupoids arising from the definition of dynamic asymptotic dimension: that they are precompact groupoids translates to them having a finite open cover of bisections.

**Proposition 4.3.** Let $H$ be a second-countable, locally compact, Hausdorff, principal and étale groupoid and let $(F, \iota, \pi)$ be a twist over $H$. Let $M \in \mathbb{N}$ and suppose that $H$ has a finite open cover of $M$ bisections.

1. Then $H$ is amenable.

2. The primitive ideal space of $C^*(F; H)$ is homeomorphic to the orbit space $H^{(0)}/H$ and each irreducible representation of $C^*(F; H)$ has dimension at most $M$.

3. For $m \in \mathbb{N}$, let $\text{Prim}_m(C^*(F; H))$ denote the set of primitive ideals of irreducible representations of dimension $m$ and let $H^{(0)} m = \{ x \in H^{(0)} : \|x\| = m \}$. The homeomorphism of (2) takes $H^{(0)} m / H$ to $\text{Prim}_m(C^*(F; H))$.

4. Let $m \in \mathbb{N}$. Then $H^{(0)} m / H$ is locally compact and Hausdorff, and $\text{dim}(H^{(0)} m) = \text{dim}(H^{(0)} m / H)$.

5. We have $\text{dim}(H^{(0)}) = \max_{1 \leq m \leq M} \text{dim}(H^{(0)} m)$.

6. The decomposition rank of $C^*(F; H)$ is at most $\text{dim}(H^{(0)})$.

Proof. Let $x \in H^{(0)}$. Then $Hx$ meets each bisection at most once. Thus, $\|Hx\| \leq M$ for all $x \in H^{(0)}$. Hence, each orbit $[x] := r(Hx)$ has at most $M$ elements. In particular, $[x]$ is closed in $H^{(0)}$ whence the orbit space $H^{(0)} / H$ is $T_1$. Since $H$ is principal, its stability subgroups are trivially abelian, and it follows from [60, Lemma 5.4] that $H$ is amenable, giving (1).

Since $H$ is principal, it further follows from [15, Proposition 3.3] that $C^*(F; H)$ is liminal. In particular, the spectrum and the primitive ideal space of $C^*(F; H)$ are homeomorphic. By [15, Theorem 3.4], the orbit space $H^{(0)} / H$ is homeomorphic to $\text{Prim}(C^*(F; H))$. The homeomorphism is given by $[x] \mapsto \ker \pi^x$ where $\pi^x$ is the representation described at (2.6). But in the proof of [43, Lemma 3.2], which assumes that $H$ is principal, Muhly and Williams show that $\pi^x$ is unitarily equivalent to a representation on $\ell^2(Hx)$ which they denote by $M^x$. 
Since \(|Hx| \leq M\) it now follows that each irreducible representation of \(C^*(F; H)\) has dimension at most \(M\) and that the homeomorphism above takes \(H_m^{(o)}/H\) to \(\text{Prim}_m(C^*(F; H))\). This gives (2) and (3).

Let \(m \in \mathbb{N}\). By \[17\], Proposition 3.6.4 and its proof, \(\text{Prim}_m(C^*(F; H))\) is locally compact, Hausdorff and hence \(H_m^{(o)}/H\) is locally compact, Hausdorff by (3).

If \(\text{Prim}_m(C^*(F; H)) = \emptyset\), then \(H_m^{(o)}/H = \emptyset\) by (3) and their dimensions are both \(-1\). Suppose that there exists a largest \(N_1\) in \(\{1, \ldots, M\}\) such that \(\text{Prim}_{N_1}(C^*(F; H))\) is non-empty (otherwise we are done with (4)). By (2) \(\text{Prim}(C^*(F; H))\) consists of exactly all irreducible representations of dimension at most \(N_1\). Thus, \(\text{Prim}_{N_1}(C^*(F; H))\) is open in \(\text{Prim}(C^*(F; H))\) by \[17\], Proposition 3.6.3. Using (3), it follows that \(H_m^{(o)}/H\) is open in \(H^{(o)}/H\).

Let \(q: H^{(o)} \to H^{(o)}/H\) be the quotient map; it is open by \[61\], Proposition 2.12. Write \(U := H_{N_1}^{(o)}\) and note that \(U = q^{-1}(q(U))\). Thus, \(U\) and \(W := H^{(o)} \setminus U\) are respectively, open and closed, and invariant subsets of \(H^{(o)}\). By Lemma 2.3, \(F|_U\) is a twist over \(H|_U\) and \(F|_W\) is a twist over \(H|_W\). By \[15\], Lemma 3.1 we get the exact sequence

\[
0 \to C^*(F|_U; H|_U) \to C^*(F; H) \to C^*(F|_W; H|_W) \to 0.
\]

The ideal \(C^*(F|_U; H|_U)\) is homogeneous and (2) implies it has primitive ideal space homeomorphic to \(H_{N_1}^{(o)}/H = U/H\). Similarly, the quotient \(C^*(F|_W; H|_W)\) is subhomogeneous such that irreducible representations have dimension at most \(N_1 - 1\) and has primitive ideal space homeomorphic to \(W/H\).

Now consider the restriction \(q_{N_1}: H_{N_1}^{(o)} = U \to U/H\) of the quotient map \(q\) to the open subset \(U\). We now verify the hypotheses of \[14\], Chapter 9, Proposition 2.16. First, since \(q\) is open and \(U\) is open, so is \(q_{N_1}\). Thus, \(q_{N_1}\) is a continuous, open surjection. Second, since \(U\) and \(U/H\) are locally compact, Hausdorff and second-countable, they are both weakly paracompact and normal. Finally, \(q_{N_1}([y]) = N_1\) for all \([y] \in U/H\). Hence, it follows from \[46\], Chapter 9, Proposition 2.16 that \(\dim(H_{N_1}^{(o)}) = \dim(H_{N_1}^{(o)}/H)\).

If \(\text{Prim}_m(C^*(F|_W; H|_W)) = \emptyset\) for all \(1 \leq m < N_1\), then we are done because the claim is trivial. Otherwise, let \(N_2\) be the largest in \(\{1, \ldots, N_1 - 1\}\) such that \(\text{Prim}_{N_2}(C^*(F|_W; H|_W))\) is non-empty. Again \(\text{Prim}_{N_2}(C^*(F|_W; H|_W))\) is open in \(\text{Prim}(C^*(F|_W; H|_W))\) by \[17\], Proposition 3.6.3. Since \(W\) is closed, \(H|_W\) has the same topological properties as \(H\). Also, for each \(1 \leq m < N_1\), we have \(H_m^{(o)} = (H|_W)^{(o)}\). We may therefore replace \(H\) by \(H|_W\) in the above argument to get that \(\dim(H_{N_2}) = \dim(H_{N_2}^{(o)}/H)\). Continue to get (4).

Let \(1 \leq m \leq M\) and let \(\text{Prim}_{\leq m}(C^*(F; H))\) be the space of primitive ideals of irreducible representations of dimension at most \(m\) and set \(H_{\leq m}^{(o)} = \{x \in H^{(o)} : \|x\| \leq m\}\). From \[17\], Proposition 3.6.3 and (2) it follows that \(H_{\leq m}^{(o)}/H\) is closed in \(H^{(o)}/H\) and that \(H_m^{(o)}/H\) is open in \(H_{\leq m}^{(o)}/H\). Thus, \(H_{\leq m}^{(o)} = q^{-1}(H_{\leq m}^{(o)}/H)\) is closed in \(H^{(o)}\) and \(H_m^{(o)} = q^{-1}(H_m^{(o)}/H)\) is open in \(H_{\leq m}^{(o)}\). Since each \(H_{\leq m}^{(o)}\) is a closed subspace of a second-countable, locally compact and Hausdorff space, it is normal. Now the normal space \(H_{\leq m}^{(o)}\) is the disjoint union of the closed subset \(H_{\leq m-1}^{(o)}\) and the open subset \(H_m^{(o)}\), and \[46\], Chapter 3, Corollary 5.8] gives

\[
\dim(H_{\leq m}^{(o)}) \leq \max\{\dim(H_m^{(o)}), \dim(H_{\leq m-1}^{(o)})\}.
\]
It follows that
\[
\dim(H^{(0)}) = \dim(H_{\leq M}^{(0)}) \leq \max\{\dim(H_{M_1}^{(0)}), \dim(H_{M_2}^{(0)})\}
\]
\[
\leq \max\{\dim(H_{M_1}^{(0)}), \dim(H_{M_2}^{(0)}), \dim(H_{M_3}^{(0)})\}
\]
\[
\vdots
\]
\[
\leq \max_{1 \leq m \leq M} \dim(H_{m}^{(0)}).
\]

To see that \(\dim(H^{(0)}) = \max_{1 \leq m \leq M} \dim(H_{m}^{(0)})\), we note that \(H^{(0)}\) is metrizable, so that \(\dim(H_{m}^{(0)}) \leq \dim(H^{(0)})\) for each \(m\) by [16, Theorem 1.8.3]. This gives (5).

For (6) we apply [65, Theorem 1.6] to get that
\[
\dim(\text{Prim}_m(C^*(F; H))) = \max_{1 \leq m \leq M} \dim(H_{m}^{(0)}/H) = \max_{1 \leq m \leq M} \dim(H_{m}^{(0)}) = \dim(H^{(0)})
\]
using (4) and (5). \(\square\)

Remark 4.4. In the language of [48, Theorem 2.16] Proposition 4.3 implies that \(C^*(F; H)\) has a recursive subhomogeneous decomposition with maximum matrix size at most \(M\) and topological dimension at most \(\dim(H^{(0)})\).

We need a version of [29, Lemma 8.20]. Let \(K\) be a compact subset of \(E\). We write \(C_K(E; G) := \{f \in C_c(E; G) : \text{supp } f \subset K\}\). For \(f \in C_c(E; G)\) and \(h \in C_c(G^{(0)})\) we define a “commutator” \([f, h]\) by
\[
[f, h](e) = f(e)(h(s_E(e)) - h(r_E(e))
\]
for \(e \in E\). Notice that \([f, h] \in C_c(E; G)\).

Lemma 4.5. Let \(G\) be a second-countable, locally compact, Hausdorff and étale groupoid, and let \((E, \iota, \pi)\) be a twist over \(G\).

1. Let \(f \in C_c(E; G)\) and \(g \in C_c(E)\) such that the point-wise product \(f \cdot g\) is an element of \(C_c(E; G)\) and \(\pi(\text{supp } g)\) is a bisection in \(G\). Then
\[
\|f \cdot g\|_{C^*_G(E; G)} \leq \|f\|_{C^*_G(E; G)} \|g\|_{\infty}.
\]

2. Let \(\epsilon > 0\), let \(K\) be a compact subset of \(E\) and let \(V\) be an open, precompact neighborhood of \(\pi(K)\) in \(G\). There exists \(\delta > 0\) such that if \(h \in C_c(G^{(0)})\) satisfies \(\sup_{\gamma \in V} |h(s_G(\gamma)) - h(r_G(\gamma))| < \delta\), then for any \(f \in C_K(E; G)\),
\[
\|[f, h]\|_{C^*_G(E; G)} \leq \epsilon \|f\|_{C^*_G(E; G)}.
\]

Proof. Fix a unit \(x \in G^{(0)}\) and let \(\pi^x\) be the representation of \(C_c(E; G)\) on \(L^2(Ex; Gx)\) defined at (2.6). By definition of the norm on \(C^*_G(E; G)\), it suffices to show that \(\|\pi^x(f \cdot g)\| \leq \|f\|_{C^*_G(E; G)} \|g\|_{\infty}\) to conclude our first claim. Let \(s : G \to E\) denote a fixed but arbitrary, not necessarily continuous, section of \(\pi\). We compute for \(\xi \in C_c(E; G) \cap L^2(Ex; Gx)\) of norm at most one:
\[
\|\pi^x(f \cdot g)\xi\|^2 = \|(f \cdot g) \ast \xi\|^2 = \sum_{\gamma \in Gx} |((f \cdot g) \ast \xi)(s(\gamma))|^2.
\]
For each $\gamma \in Gx$ we have

$$((f \cdot g) \ast \xi)(s(\gamma)) = \sum_{\alpha \in \pi(\gamma)G} (f \cdot g)(s(\alpha)) \xi(s(\alpha)^{-1}s(\gamma))$$

$$= \int_E (f \cdot g)(e) \xi(e^{-1}s(\gamma)) \, d\sigma^x(e)$$

by our choice of Haar system $\sigma$ on $E$. The change of variable $d = e^{-1}s(\gamma)$ yields

$$((f \cdot g) \ast \xi)(s(\gamma)) = \int_E (f \cdot g)(s(\gamma)d^{-1}) \xi(d) \, d\sigma_x(d)$$

$$= \sum_{\alpha \in Gx} (f \cdot g)(s(\gamma)s(\alpha)^{-1}) \xi(s(\alpha)).$$

Suppose that $s(\gamma)s(\alpha)^{-1} \in \text{supp } g$. Then $\alpha \in \pi(\text{supp } g)^{-1}\gamma$. Since $\pi(\text{supp } g)^{-1}$ is a bisection, it contains at most one element with source $r_G(\gamma)$. So for each $\gamma \in Gx$, there is at most one $\alpha_\gamma$ such that $s(\gamma)s(\alpha_\gamma)^{-1} \in \text{supp } g$. Thus,

$$\|\pi^x(f \cdot g)\xi\|^2 = \sum_{\exists \alpha_\gamma \in \pi(\text{supp } g)^{-1}\gamma} |(f \cdot g)(s(\gamma)s(\alpha_\gamma)^{-1}) \xi(s(\alpha_\gamma))|^2$$

$$\leq \|f\|^2 \|g\|^2 \sum_{\exists \alpha_\gamma \in \pi(\text{supp } g)^{-1}\gamma} |\xi(s(\alpha_\gamma))|^2.$$

We claim that $\alpha_\gamma$ for $\gamma$ ranging over $Gx$ is a non-repetitive subset of $Gx$. For if $\alpha_\gamma = \alpha_\delta$, then they are both in $\pi(\text{supp } g)^{-1}\gamma$ and in $\pi(\text{supp } g)^{-1}\delta$. Say $\alpha_\gamma = \xi_\gamma = \eta_\delta$ where $\xi, \eta \in \pi(\text{supp } g)^{-1}$. Since $\xi, \eta$ have the same range and $\pi(\text{supp } g)^{-1}$ is a bisection, we get that $\xi = \eta$, and then $\gamma = \delta$ as well, giving the claim. It follows that

$$\sum_{\exists \alpha_\gamma \in \pi(\text{supp } g)^{-1}\gamma} |\xi(s(\alpha_\gamma))|^2 \leq \sum_{\beta \in Gx} |\xi(s(\beta))|^2 = \|\xi\|^2.$$

Thus,

$$\|\pi^x(f \cdot g)\xi\|^2 \leq \|f\|^2 \|g\|^2 \|\xi\|^2,$$

where we have used that $\|f\|_\infty \leq \|f\|_{C^1(G;E;G)}$ by [11, Proposition 2.8]. This gives [1].

For [2], let $L$ be the compact closure of $V$ in $G$. By Urysohn’s lemma, there exists a continuous function $b: G \to [0,1]$ with compact support in $L$ such that $b$ is 1 on $\pi(K)$ and is 0 on $G \setminus V$.

Fix $h \in C_c(G^0)$ and $f \in C_K(E;G)$. We set $g := (b \circ \pi) \cdot (h \circ s_E - h \circ r_E)$; then since $\pi^{-1}(L)$ is compact by Lemma 2.2, $g \in C_c(E)$ with compact support in $\pi^{-1}(L)$ and

$$\|g\|_\infty \leq \sup_{\gamma \in V} |(h \circ s_G - h \circ r_G)(\gamma)|.$$
Since $g(t \cdot e) = g(e)$ for all $e \in E$ and $t \in T$, we have $f \cdot g \in C_c(E; G)$. Since the support of $f$ is contained in $K$ by assumption, we have

$$[f, h](e) = f(e)(h(s_E(e)) - h(r_E(e))$$

$$= \begin{cases} f(e)b(\pi(e))(h(s_E(e)) - h(r_E(e)) & \text{if } e \in K \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} f(e)g(e) & \text{if } e \in K \\ 0 & \text{else} \end{cases}$$

$$= (f \cdot g)(e).$$

Since the support of $b$ is compact and $G$ is étale, there exist $n \in \mathbb{N} \setminus \{0\}$ and $b_1, \ldots, b_n \in C_c(G)$ such that the support of each $b_i$ is a bisection, $b = \sum_{i=1}^{n} b_i$ and $\|b_i\|_\infty \leq \|b\|_\infty$. Then $g = \sum_{i=1}^{n} g_i$ where $g_i = (b_i \circ \pi) \cdot (h \circ s_E - h \circ r_E)$ and $\pi(\text{supp } g_i)$ is a bisection, and $\|g_i\|_\infty \leq \|g\|_\infty$. Now, our above computation yields

$$\|\|f, h\|_{C^*_r(E;G)} = \|f \cdot g\|_{C^*_r(E;G)}$$

$$\leq \sum_{i=1}^{n} \|f \cdot g_i\|_{C^*_r(E;G)}$$

$$\leq \sum_{i=1}^{n} \|f\|_{C^*_r(E;G)} \|g_i\|_\infty$$

$$\leq n \|f\|_{C^*_r(E;G)} \|g\|_\infty$$

$$\leq n \|f\|_{C^*_r(E;G)} \sup_{\gamma \in V} |(h \circ s_G - h \circ r_G)(\gamma)|$$

using (1.4). Since $b$ and hence $n$ depend only on $K$ and $V$, and are independent of $h$ and $f$, we conclude that $\delta = \epsilon/n$ works, giving (2).

\[\Box\]

Remark 4.6. By taking $E$ to be the Cartesian product $T \times G$, so that $C^*_r(G) \cong C^*_r(E;G)$, Lemma 4.5 makes it clear that [29, Lemmas 8.19 and 8.20] do not require $G$ to be principal as assumed in [29].

Proof of Theorem 4.1. First assume that the unit space $G^{(0)}$ is compact. Let $F$ be a finite subset of $C_c(E;G) \setminus \{0\}$ and let $\varepsilon > 0$. There exists a compact subset $K$ of $E$ such that $f \in F$ implies $\text{supp } f \subset K$. Since both $K^{-1}$ and $\pi^{-1}(\pi(K))$ are compact sets (see Lemma 2.2), we may assume that $K = K^{-1}$ and that $K = \pi^{-1}(\pi(K))$. Since $G^{(0)} = E^{(0)}$ is compact and open in $E$, we may also assume that $G^{(0)} \subset K$ and hence that the characteristic function of $G^{(0)}$ is in $F$, i.e., that $1 \in F$.

Let $V \subset G$ be an open and precompact neighborhood of $\pi(K)$, and let $\delta$ be as in Lemma 4.5 (2) for $\max_{f \in F} \|f\|_{C^*_r(E;G)}$, $K$ and $V$. Since $G$ has dynamic asymptotic dimension $d$, applying [29, Proposition 7.1] to $\delta$ and the precompact, open subset $V$ of $G$ gives

1. open sets $U_0, \ldots, U_d$ covering $G^{(0)} = r_G(V) \cup s_G(V)$ such that the subgroupoids $H_i$ generated by $\{\gamma \in V : s_G(\gamma), r_G(\gamma) \in U_i\}$ are open and precompact in $G$ for $0 \leq i \leq d$. 

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Note first that $\left| \left( f \ast f \right) \right| = \left| \left( f \ast f \right) \ast 1 \right| = \left| \left( f \ast f \right) \right|$. Since $G$ is second-countable, locally compact, Hausdorff, and $G^0$ is metrizable. Now $H^0_i$ is a subspace of $G^0$, and it follows from [10, Theorem 1.8.3] that $N_i = \dim(H^0_i) \leq \dim(G^0) = N$. Thus, the decomposition rank and hence also the nuclear dimension of $C^*(\pi^{-1}(H_i); H_i)$ is at most $N$. Using Lemma 2.5, we identify $C^*(\pi^{-1}(H_i); H_i)$ with a $C^*$-subalgebra $B_i$ of $C^*_c(E; G)$.

Let $f_i := f_{h_i} \in C_c(E; G)$ be the image of $h_i$ under the isomorphism described at (2.7) so that supp $f_i \subset \iota(G^0 \times T)$ and $f_i(\iota(x, z)) = z h_i(x)$ for $(x, z) \in G^0 \times T$. Then $f_i$ is self-adjoint since $h_i$ is. We aim to show that $f_i$ and $B_i$ satisfy the hypotheses of Proposition 4.2 with $f \in C_c(E; G), e \in E$, and $s : G \to E$ a (not necessarily continuous) section of $\pi$, we have

\[
(f_i * f)(e) = \sum_{\gamma \in \pi(E)(e)G} f_i(s(\gamma)) f(s(\gamma)^{-1}e) = \sum_{\gamma \in \pi(E)(e)G : s(\gamma) \in \iota(G^0 \times T)} f_i(s(\gamma)) f(s(\gamma)^{-1}e).
\]

(4.5)

similarly, $(f * f_i)(e) = f(e) h_i(s_E(e)) = f(e) h_i(s_E(e))$ and

\[
f_i * f * f_i = h_i(r_E(e)) f(e) h_i(s_E(e)).
\]

Thus, for $f \in \mathcal{F} \subset C_K(E; G)$ and $e \in E$,

\[
(f_i * f * f_i)(e) \neq 0 \implies e \in K \text{ and } r_E(e), s_E(e) \in U_i \implies \pi(e) \in V \text{ and } r_G(\pi(e)), s_G(\pi(e)) \in U_i \implies \pi(e) \in H_i.
\]

This shows that $f_i * f * f_i \in B_i$ and thus $f_i * \mathcal{F} * f_i \subset B_i$ as claimed.

In order to apply Proposition 4.2, it remains to show that for each $f \in \mathcal{F}$,

\[
\left\| f - \sum_{i=0}^d f_i * f * f_i \right\|_{C^*_c(E; G)} \leq \varepsilon.
\]

Let $f \in \mathcal{F} \subset C_K(E; G)$. Item (2) on page 28, the choice of $\delta$ and Lemma 4.5 imply that

\[
\| [f, h_i] \|_{C^*_c(E; G)} \leq \| f \|_{C^*_c(E; G)} \leq \frac{\varepsilon}{d + 1}.
\]
An easy computation gives \( f \ast f_i = f_i \ast f + [f, h_i] \). Also, using Equation (4.5) and that \( \sum_{i=0}^{d} h_i (r_E(e))^2 = 1 \) gives \( (\sum_{i=0}^{d} f_i \ast f_i) \ast f = f \). Hence,
\[
\sum_{i=0}^{d} f_i \ast f = \sum_{i=0}^{d} f_i \ast (f_i \ast f + [f, h_i]) = f + \sum_{i=0}^{d} f_i \ast [f, h_i].
\]

Thus
\[
\left\| f - \sum_{i=0}^{d} f_i \ast f \right\|_{C^*_r(E;G)} = \left\| \sum_{i=0}^{d} f_i \ast [f, h_i] \right\|_{C^*_r(E;G)} \\
\leq (d + 1) \|h_i\|_{\infty} \| [f, h_i] \|_{C^*_r(E;G)} \\
\leq \varepsilon.
\]

It now follows from Proposition 4.2 that if the unit space of \( G \) is compact, then the nuclear dimension of \( C^*_r(E;G) \) is at most \((d + 1)(N + 1) - 1\).

Now we suppose that \( G^{(0)} \) is not compact. Let \( \tilde{G} \) be the Alexandrov groupoid described in Lemma 3.4 and let \((\tilde{E}, \tilde{i}, \tilde{\pi})\) be the Alexandrov twist over \( \tilde{G} \) described in Lemma 3.6. Since \( G \) has dynamic asymptotic dimension \( d \), so does \( \tilde{G} \) by Proposition 3.13. By Lemma 2.6, we have \( \dim(\tilde{G}^{(0)}) = \dim(G^{(0)}) \leq N \). From the compact case we have that \( \dim_{\text{max}}(C^*_r(\tilde{E}; \tilde{G})) \) is at most \((N + 1)(d + 1) - 1\). By Lemma 5.5, \( C^*_r(\tilde{E}; \tilde{G}) \) is the minimal unitization of \( C^*_r(E;G) \), and hence it follows from [66, Remark 2.11] that the nuclear dimension of \( C^*_r(E;G) \) is also at most \((N + 1)(d + 1) - 1\). \( \square \)

5. Application to \( C^\ast \)-algebras of non-principal groupoids

In this section we consider a groupoid \( G \) with potentially large stability subgroups. Corollary 5.5 gives a bound on the nuclear dimension of \( C^*_r(G) \) involving the topological dimension of \( G^{(0)} \) and the dynamic asymptotic dimension of the quotient of \( G \) by its isotropy subgroupoid. Its proof is based on an isomorphism from [15] of \( C^*(G) \) with a twisted groupoid \( C^\ast \)-algebra to which Theorem 4.1 applies.

The isotropy subgroupoid \( \mathcal{A} \) of \( G \) is
\[
\mathcal{A} := \{ \alpha \in G : r_G(\alpha) = s_G(\alpha) \};
\]

it is closed and hence locally compact, and acts on \( G \) on the right via \( \gamma \cdot \alpha = \gamma \alpha \) when \( s_G(\gamma) = r_G(\alpha) \). The quotient \( \mathcal{R} := G/\mathcal{A} \) is a principal groupoid. We write \( \pi : G \to \mathcal{R} \) for the quotient map.

Throughout this section, \( G \) is a second-countable, locally compact, Hausdorff groupoid, with abelian stability subgroups that vary continuously, i.e., the map
\[
u \mapsto \mathcal{A}_u := \{ \alpha \in G : r_G(\alpha) = u = s_G(\alpha) \}
\]
is continuous from \( G^{(0)} \) into the space of closed subgroups of \( G \) in the Fell topology (see [61, §3.4]). By [61, Theorem 6.12], this continuity is equivalent to \( \mathcal{A} \) having a Haar system. For étale groupoids we can say more:

Lemma 5.1. Let \( G \) be a locally compact, Hausdorff and étale groupoid. Then the stability subgroups of \( G \) vary continuously if and only if \( \mathcal{A} \) is open in \( G \).
Lemma 5.2. Let $G$ be a second-countable, locally compact and Hausdorff groupoid (not necessarily étale) with abelian and continuously varying stability subgroups. Let $\mathcal{A}$ be the isotropy groupoid and let $\mathcal{R} = G/\mathcal{A}$ be the quotient groupoid.

(1) Then $\mathcal{R}$ is locally compact and Hausdorff.
(2) The subgroupoid $\mathcal{A}$ is open in $G$ if and only if $\mathcal{R}$ is $r$-discrete.
(3) If $G$ is étale, then $\mathcal{R}$ is étale.

Proof of Lemma 5.2. For (1), the action of $G$ on itself is proper by [61, Example 2.16]. Since $\mathcal{A}$ is closed in $G$, it follows easily from [61, Proposition 2.17] that the action of $\mathcal{A}$ on $G$ is proper. Moreover, $\mathcal{A}$ is itself locally compact and Hausdorff. Since the stability subgroups vary continuously, the restriction of the range map to $\mathcal{A}$ is open by [61, Theorem 6.12]. It now follows from [61, Proposition 2.18] applied to $\mathcal{A}$ that $\mathcal{R} = G/\mathcal{A}$ is locally compact and Hausdorff.

For (2), we observe that $\pi^{-1}(\mathcal{R}^{(0)}) = \mathcal{A}$ because
\[
\gamma \in \pi^{-1}(\mathcal{R}^{(0)}) \iff \pi(\gamma) = \pi(\gamma)\pi(\gamma)^{-1} \iff \pi(\gamma) = \pi(r(\gamma))
\iff \exists \alpha \in \mathcal{A}, \gamma \alpha = r(\gamma) \iff \gamma \in \mathcal{A}.
\]

Now $\mathcal{R}$ is $r$-discrete if and only if $\mathcal{R}^{(0)}$ is open, and by definition of the quotient topology, this happens if and only if $\pi^{-1}(\mathcal{R}^{(0)}) = \mathcal{A}$ is open in $G$.

For (3), suppose that $G$ is étale. Since the stability subgroups vary continuously, $\mathcal{A}$ is open by Lemma 5.1. Then $\mathcal{R}$ is $r$-discrete by Part (2). For $U \subset \mathcal{R}$, one checks that
\[
\pi^{-1}(r(\mathcal{R}(U))) = \mathcal{A} \cap r_G^{-1}\left(r_G(\pi^{-1}(U))\right).
\]
Since $G$ is étale, $r_G$ is an open map. So if $U$ is open in $\mathcal{R}$, then $\pi^{-1}(r(\mathcal{R}(U)))$ is open in $G$, using the above equality and that $\mathcal{A}$ is open. Since $\mathcal{R}^{(0)}$ is open and $\mathcal{R}$ has the quotient topology, this proves that $r(\mathcal{R}) : \mathcal{R} \to \mathcal{R}^{(0)}$ is open. Thus $\mathcal{R}$ is étale by [61, Proposition 1.29].

In the following, we consider a general transformation groupoid rather than the specific $\tilde{\mathcal{A}} \rtimes \mathcal{R}$; we thank the referee for bringing to our attention that our proofs work in that generality. Item (3) of the next proposition was subsequently proved in greater generality in [7, Proposition 2.4].

Proposition 5.3. Let $X$ be a locally compact, Hausdorff space and let $\mathcal{K}$ be a locally compact, Hausdorff and étale groupoid. Suppose that $\mathcal{K}$ acts continuously on the right of $X$ with surjective anchor map $\sigma : X \to \mathcal{K}^{(0)}$.

(1) The transformation groupoid $X \rtimes \mathcal{K}$ is étale.
(2) If $\sigma$ is open, then the projection map $\text{pr}_X : X \rtimes \mathcal{K} \to X$, $(x, k) \mapsto k$, is continuous and open.
(3) If $\sigma$ is open and if DAD($\mathcal{K}$) $\leq d$, then DAD($X \rtimes \mathcal{K}$) $\leq d$.

Proof. We set $\mathcal{H} := X \rtimes \mathcal{K}$ for convenience.

For (1), $X \times \mathcal{K}^{(0)}$ is open in $X \times \mathcal{K}$, so that
\[
\mathcal{H}^{(0)} = (X \times \mathcal{K}^{(0)}) \cap \mathcal{H}
\]
is open in $\mathcal{H}$, proving that $\mathcal{H}$ is $r$-discrete. We henceforth identify $\mathcal{H}^{(0)}$ with $X$ via the map $(x, \sigma(x)) \mapsto x$. To see that $\mathcal{H}$ is étale, we will show that the range map $r_X : \mathcal{H} \to \mathcal{H}^{(0)}$ is open.

Let $\{x_\lambda\}_\lambda$ be a net in $\mathcal{H}^{(0)}$ that converges to $r_X(x, k) = x$ in $\mathcal{H}^{(0)}$. Then $\sigma(x_\lambda) \to \sigma(x) = r_\mathcal{K}(k)$. Since $\mathcal{K}$ is étale, $r_\mathcal{K}$ is open by [61, Proposition 1.29]. It follows from Fell’s criterion ([61, Proposition 1.1]) that there exists a subnet $\{x_{f(\mu)}\}_\mu$ of $\{x_\lambda\}_\lambda$ and a net $\{k_\mu\}_\mu$ in $\mathcal{K}$ such that $r_\mathcal{K}(k_\mu) = \sigma(x_{f(\mu)})$ and $k_\mu \to k$. Thus, $(x_{f(\mu)}, k_\mu)$ is a lift of $x_{f(\mu)} \in \mathcal{H}^{(0)}$ in $\mathcal{H}$ under $r_X$ which
converges to \((x, k)\). By another application of Fell’s criterion, this implies that \(r_{\mathcal{H}}\) is open. Thus, \(\mathcal{H}\) is étale.

For \([2]\), we first observe that continuity of \(\text{pr}_\mathcal{K}\) is immediate because \(\times \mathcal{K}\) has the subspace topology from \(\times X\). To see that \(\text{pr}_\mathcal{K}\) is open, we will use Fell’s criterion twice. Let \(\{k_\lambda\}_\lambda\) be a net in \(\mathcal{K}\) that converges to \(k = \text{pr}_\mathcal{K}(x, k)\). Then \(u_\lambda := r_\mathcal{K}(k_\lambda)\) is a net converging to \(u := r_\mathcal{K}(k) = \sigma(x)\). Since \(\sigma\) is open and surjective, by Fell’s criterion there exists a subnet \(\{u_{j(\mu)}\}_\mu\) of \(\{u_\lambda\}_\lambda\) and a lift \(\{x_\mu\}_\mu\) of that subnet under \(\sigma\) (i.e., \(\sigma(x_\mu) = u_{j(\mu)}\)) such that \(x_\mu \rightarrow x\). Then \(\{(x_\mu, k_{j(\mu)})\}_\mu\) is a net in \(\times \mathcal{K}\) that is a lift of the subnet \(\{k_{j(\mu)}\}_\mu\) under \(\text{pr}_\mathcal{K}\) and that converges to \((x, k)\). By Fell’s criterion, this proves that \(\text{pr}_\mathcal{K}\) is open.

For \([3]\) let \(W \subset \mathcal{H}\) be precompact and open; without loss of generality assume \(W = W^{-1}\), so that \(r_{\mathcal{H}}(W) \cup s_{\mathcal{H}}(W) = r_{\mathcal{H}}(W)\). By Part \([2]\) the projection map \(\text{pr}_\mathcal{K}: \mathcal{H} \rightarrow \mathcal{K}\) onto the second component is a continuous open map, so that \(W' := \text{pr}_\mathcal{K}(W) = (W^{-1})^{-1}\) is open in \(\mathcal{K}\). Recall that images of precompact sets under continuous maps are precompact, so long as the codomain is Hausdorff. Therefore, \(W'\) is precompact. Since \(\text{DAD}(\mathcal{K}) \leq d\), there exist open \(U_0, \ldots, U_d \subset \mathcal{K}^{(0)}\) which cover \(r_{\mathcal{K}}(W')\) and such that

\[
\mathcal{K}_i := \langle W' \cap \mathcal{K} \mid U_i \rangle
\]

is precompact in \(\mathcal{K}\). Let \(V_i := \sigma^{-1}(U_i) \cap r_{\mathcal{H}}(W) \subset X\). Since \(\sigma\) is continuous, the \(V_i\)'s are open subsets of \(X\). We claim that since the \(U_i\)'s cover \(r_{\mathcal{K}}(W')\), the \(V_i\)'s cover \(r_{\mathcal{H}}(W)\). For an arbitrary \(x \in r_{\mathcal{H}}(W)\), take \(k \in \mathcal{K}\) such that \((x, k) \in W\); in particular, \(\sigma(x) = r_{\mathcal{K}}(k)\) and \(k = \text{pr}_\mathcal{K}(x, k) \in W'\). Thus \(\sigma(x) = r_{\mathcal{K}}(k) \in U_i\) for some \(0 \leq i \leq d\), i.e., \(x \in \sigma^{-1}(U_i) \cap r_{\mathcal{H}}(W) = V_i\), as claimed.

To conclude that \(\text{DAD}(\mathcal{H}) \leq d\), it now suffices to show that \(\mathcal{H}_i := \langle W \cap \mathcal{H} \mid V_i \rangle\) is precompact. Towards this, note first that each \(V_i\) is precompact. Indeed, since \(X\) is Hausdorff, \(W\) is precompact and \(r_{\mathcal{H}}\) is continuous, it follows, as argued before, that \(r_{\mathcal{H}}(W)\) and hence \(V_i\) is precompact.

We claim that \(\mathcal{H}_i\) is contained in \(X \times \mathcal{K}_i\). To see this, let \((x, k) \in \mathcal{H}_i\). There exist finitely many \((x_j, k_j) \in W \cap \mathcal{H} \mid V_i\) such that \((x, k) = (x_1, k_1) \cdots (x_m, k_m)\). As \((x_j, k_j) \in W\), we know that \(k_j \in W'\). As \((x_j, k_j) \in \mathcal{H} \mid V_i\), we have that both \(x_j = r_{\mathcal{H}}(x_j, k_j)\) and \(x_j \cdot k_j = s_{\mathcal{H}}(x_j, k_j)\) is in \(V_i\); in particular, \(r_{\mathcal{K}}(k_j) = \sigma(x_j)\) and \(s_{\mathcal{K}}(k_j) = \sigma(x_j \cdot k_j)\) are both elements of \(\sigma(V_i) \subset U_i\). Thus, \(k_j \in \mathcal{K} \mid U_i\) and so \(k = k_1 \cdots k_m \in \mathcal{K}_i\), as claimed.

To see that \(\mathcal{H}_i\) is precompact, suppose \(\{(x_\lambda, k_\lambda)\}_\lambda\) is a net in \(\mathcal{H}_i\); we must show that there exists a subnet that converges (in \(\mathcal{H}_i\), not necessarily in \(\mathcal{H}_i\)). By the above argument, \(k_\lambda\) is in the precompact \(\mathcal{K}_i\), so by passing to a subnet, we can without loss of generality assume that \(\{k_\lambda\}_\lambda\) converges (again, in \(\mathcal{K}\) and not necessarily in \(\mathcal{K}_i\)). By definition of \(\mathcal{H}_i\), we further have that all \(x_\lambda = r_{\mathcal{H}}(x_\lambda, k_\lambda)\) are elements of the precompact \(V_i\). Again, passing to a subnet allows us without loss of generality to assume that \(\{x_\lambda\}_\lambda\) converges in \(X\). Since \(\mathcal{H}\) has the subspace topology inherited from \(X \times \mathcal{K}\), we conclude that \(\{(x_\lambda, k_\lambda)\}_\lambda\) (has a subnet that) converges in \(\mathcal{H}_i\), as required. This proves that \(\mathcal{H}_i\) is precompact. Thus, \(\text{DAD}(\mathcal{H}) \leq d\).

**Corollary 5.4.** Let \(G\) be a second-countable, locally compact, Hausdorff and étale groupoid. Suppose that the stability subgroups of \(G\) are abelian and vary continuously. Let \(\mathcal{R}\) be the quotient groupoid of \(G\) by the isotropy groupoid \(\mathcal{A}\). The transformation groupoid \(\hat{\mathcal{A}} \times \mathcal{R}\) is étale. Moreover, if \(\text{DAD}(\mathcal{R}) \leq d\), then \(\text{DAD}(\hat{\mathcal{A}} \times \mathcal{R}) \leq d\).

**Proof.** Since the abelian stability subgroups of \(G\) vary continuously, we may invoke Lemma \([5,2]\). \(\mathcal{R}\) is locally compact and Hausdorff by Part \([1]\), and it is étale by Part \([3]\) since \(G\) is étale.
Since \( \hat{A} \) is the spectrum of the commutative \( C^* - \)algebra \( C^*(\mathcal{A}) \), it is locally compact and Hausdorff. Since the anchor map \( p: \hat{A} \to R^0 \) is open ([42, page 3631]), we may thus further invoke Proposition 5.3 \( \hat{A} \times R \) is étale by Part (1) and \( \text{DAD}(\hat{A} \times R) \leq \text{DAD}(R) \) by Part (3).

**Corollary 5.5.** Let \( G \) be a second-countable, locally compact, Hausdorff and étale groupoid. Suppose that the orbits of \( G \) are closed in \( G \) and that the stability subgroups of \( G \) are abelian and vary continuously. Let \( A \) be the isotropy groupoid, let \( \hat{A} \) be the spectrum of the commutative \( C^* - \)algebra \( C^*(A) \), and let \( R \) be the quotient groupoid of \( G \) by \( A \). Suppose that the topological dimension of \( \hat{A} \) is at most \( N \) and that \( R \) has finite dynamic asymptotic dimension at most \( d \). Then the nuclear dimension of \( C^r_r(D) = C^r_r(G) \) is at most \((N+1)(d+1)-1\).

**Proof.** Let \( D \) be the \( T \)-groupoid over \( \hat{A} \times R \) constructed in [42, §4, page 3636]. Since the abelian stability subgroups vary continuously and \( G \) is étale, so are \( R \) and \( \hat{A} \times R \) by Lemma 5.2. Since \( R \) has finite dynamic asymptotic dimension at most \( d \), so does \( \hat{A} \times R \) by Corollary 5.4. By Theorem 4.1, \( C^r_r(D; \hat{A} \times R) \) has nuclear dimension at most \((N+1)(d+1)-1\).

Since \( \hat{A} \times R \) is principal and has finite dynamic asymptotic dimension, it is amenable by [29, Corollary 8.25]. Since \( D \) is an extension of \( \hat{A} \times R \) by \( T \), it follows that \( D \) is topologically amenable by [3, Proposition 5.1.2]. Hence, \( C^r_r(D) = C^r_r(D) \), and then also \( C^r_r(D; \hat{A} \times R) = C^r_r(D; \hat{A} \times R) \).

Since the stability subgroups are abelian and the orbits are closed, \( G \) is amenable by [60, Lemma 5.4], and hence \( C^*(G) = C^r_r(G) \). Since the stability subgroups also vary continuously, \( C^*(D; \hat{A} \times R) \) and \( C^*(G) \) are isomorphic by [15, Proposition 6.3]. Now the corollary follows.

We now give two examples which illustrate Corollary 5.5.

**Example 5.6.** In [64, Example 5.4] there is an action of \( R \) on \( C \) which is \( \sigma \)-proper (that is, proper relative to the stability subgroups); the orbits are closed and the stability subgroups vary continuously. In the following, we restrict this action to \( Z \) and show that this gives an example of an étale groupoid \( G \) which satisfies the hypotheses of Corollary 5.5.

Set

\[
X = \{ z \in C : |z| \in \mathbb{N} \} = \{0\} \cup \bigsqcup_{n \geq 1} n \mathbb{T};
\]

then \( Z \) acts on \( X \) via

\[
n \cdot z = \begin{cases} 
0 & \text{if } z = 0 \\
 e^{2\pi i n / |z|} z & \text{if } z \neq 0.
\end{cases}
\]

The orbits are concentric circles about the origin and the stability subgroups are

\[
S_z := \{ n \in Z : n \cdot z = z \} = \begin{cases} 
Z & \text{if } z = 0 \\
|z|Z & \text{if } z \neq 0.
\end{cases}
\]

Let \( G = Z \times X \) be the transformation-group groupoid, which is étale since \( Z \) is discrete. Then the isotropy groupoid is

\[
\mathcal{A} = (Z \times \{0\}) \cup \bigsqcup_{n \geq 1} (nZ \times n \mathbb{T})
\]
which is open in $\mathbb{Z} \times X$, i.e., open in $G$. Thus, $R := G/A$ is étale by Lemma 5.2(3). For $(n, z) \in G$ write $[(n, z)]$ for the equivalence class in $R$. Note that if $(k, w) \in A$, then the product $(n, z)(k, w)$ is defined if and only if $z = k \cdot w = w$. We have $[(0, 0)] = \mathbb{Z} \times \{0\}$ and if $z \neq 0$ we have

$$[(n, z)] = \{(n, z)(m|z), z : m \in \mathbb{Z}\} = (n + |z|\mathbb{Z}) \times \{z\}.$$  

In particular, $(n, z) \sim (n + |z|m, z)$ for every $m \in \mathbb{Z}$.

We claim that $\mathcal{R}$ is a proper groupoid, and hence DAD$(\mathcal{R}) = 0$. Fix a compact subset $K$ of the unit space $\mathcal{R}^{\infty}$ (identified with $X$). Without loss of generality we can make $K$ larger, and so we can assume that $K = \{z \in X : |z| \leq r\}$ for some $r \in \mathbb{N}$. Since $\mathbb{Z} \cdot K = K$ we have

$$\mathcal{R}|_{K} = \{((n, z)) : r((n, z)), s((n, z)) \in K\} = \{(n, z) : z \in K\}.$$  

Consider a sequence $[(n_{i}, z_{i})]$ in $\mathcal{R}|_{K}$; we will show that it admits a subsequence that converges. Since $K$ is compact, we may assume that $z_{i} \to z$ for some $z \in K$. Then $|z_{i}| = |z|$ eventually. For each $n$, choose $m_{i} \in \mathbb{Z}$ such that $n_{i} + |z|m_{i}$ is in $\{0, \ldots, |z| - 1\}$. Then $[(n_{i}, z_{i})] = [(n_{i} + |z|m_{i}, z_{i})]$ has a subsequence in which the first component is constant; thus, we may without loss of generality assume that our sequence is of the form $[(n, z_{i})]$ for some $n \in \{0, \ldots, |z| - 1\}$. Since $(n, z_{i}) \to (n, z)$ in $G$, this sequence converges to $[(n, z)]$ in $\mathcal{R}|_{K}$. Thus, $\mathcal{R}|_{K}$ is compact. It follows that $\mathcal{R}$ is a proper groupoid. In particular, if $V \subset \mathcal{R}$ is precompact open, then since the unit space of $\mathcal{R}$ is open, $U_{0} := r(V) \cup s(V)$ is precompact, open in $\mathcal{R}^{\infty}$. Now choose a compact $K$ large enough to contain the closure of $U_{0}$. Then $\langle V \cap \mathcal{R}|_{U_{0}} \rangle$ is contained in $\mathcal{R}|_{K}$, which is compact by the above argument, and so it follows that DAD$(\mathcal{R}) = 0$.

We claim that the topological dimension of $\hat{\mathcal{A}}$ is 1. Since $\hat{\mathcal{A}}$ is the spectrum of a separable, commutative $C^{*}$-algebra, it is locally compact, Hausdorff and second-countable, and hence is normal. Since the stability subgroups are constant along orbits, we have

$$\hat{\mathcal{A}} = \{((\chi, z)) : z \in X, \chi \in \hat{S}_{z}\} = \bigcup_{n \in \mathbb{N}} \hat{S}_{n} \times n\mathbb{T},$$  

which is a countable union of closed subsets of $\hat{\mathcal{A}}$. By the countable-union theorem [16, Theorem 1.7.1], $\dim \hat{\mathcal{A}} = \sup_{n \in \mathbb{N}} \dim \langle \hat{S}_{n} \times n\mathbb{T} \rangle$. The dimension of $\hat{S}_{0} \times 0\mathbb{T} = \mathbb{T} \times \{0\}$ is 1. Each $\hat{S}_{n} \times n\mathbb{T}$ for $n \neq 0$ is homeomorphic to $\mathbb{T} \times \mathbb{T}$, which has topological dimension 1 by writing it as the union $\mathbb{T} \times \{1\} \cup \{1\} \times \mathbb{T}$ of closed subsets of dimension 1. Thus, $\dim(\hat{\mathcal{A}})$ is 1.

So this example fits our hypotheses: an étale groupoid $G$ with closed orbits, abelian and continuously varying stability subgroups such that $\mathcal{R}$ is étale with finite dynamic asymptotic dimension and such that $\hat{\mathcal{A}}$ has finite topological dimension. In particular, Corollary 5.5 gives

$$\dim_{\text{nuc}}(C^{*}(G)) \leq (1 + 1)(0 + 1) - 1 \leq 1.$$  

Here $C^{*}(G) \cong C_{0}(X) \rtimes \mathbb{Z}$ has continuous trace by [64, Theorem 5.1]. Thus

$$\dim_{\text{nuc}}(C^{*}(G)) = \dim \langle C^{*}(G) \rangle = \dim \langle \text{Prim}(C_{0}(X) \rtimes \mathbb{Z}) \rangle,$$

and we could have used the description of the primitive ideal space from [63, Theorem 5.3] to determine the topological dimension of Prim$(C_{0}(X) \rtimes \mathbb{Z})$.

Our second example, Example 5.8 below, is a $C^{*}$-algebra of a graph groupoid. We start with the required background on directed graphs.
Let \( E = (E^0, E^1, r, s) \) be a row-finite directed graph, so that \( r^{-1}(v) \) is finite for every \( v \in E^0 \). A finite path is a finite sequence \( \mu = \mu_1\mu_2 \cdots \mu_k \) of edges \( \mu_i \in E^1 \) with \( s(\mu_j) = r(\mu_{j+1}) \) for \( 1 \leq j \leq k - 1 \); write \( s(\mu) = s(\mu_k) \) and \( r(\mu) = r(\mu_1) \), and call \( |\mu| := k \) the length of \( \mu \). An infinite path \( x = x_1x_2 \cdots \) is defined similarly; note that \( s(x) \) is undefined. Let \( E^* \) and \( E^\infty \) denote the set of all finite paths and infinite paths in \( E \), respectively. If \( \mu = \mu_1 \cdots \mu_k \) and \( \nu = \nu_1 \cdots \nu_j \) are finite paths with \( s(\mu) = r(\nu) \), then \( \mu \nu \) is the path \( \mu_1 \cdots \mu_k \nu_1 \cdots \nu_j \). When \( x \in E^\infty \) with \( s(\mu) = r(x) \) define \( \mu x \) similarly. A return path is a finite path \( \mu \) of non-zero length such that \( s(\mu) = r(\mu) \). By [35, Corollary 2.2], the cylinder sets
\[
Z(\mu) := \{ x \in E^\infty : x_1 = \mu_1, \ldots, x|\mu| = \mu|\mu| \},
\]
parameterized by \( \mu \in E^* \), form a basis of compact, open sets for a locally compact, totally disconnected, Hausdorff topology on \( E^\infty \).

Given a row-finite directed graph \( E \), in [35], Kumjian, Pask, Raeburn and Renault define a groupoid \( G_E \) as follows. Two paths \( x, y \in E^\infty \) are shift equivalent with lag \( k \in \mathbb{Z} \) (written \( x \sim_k y \)) if there exists \( N \in \mathbb{N} \) such that \( x_i = y_{i+k} \) for all \( i \geq N \). Then the groupoid is
\[
G_E := \{(x, k, y) \in E^\infty \times \mathbb{Z} \times E^\infty : x \sim_k y \},
\]
with composable pairs
\[
G_E^{(2)} := \{ ((x, k, y), (y, l, z)) : (x, k, y), (y, l, z) \in G_E \},
\]
and composition and inverse given by
\[
(x, k, y) \cdot (y, l, z) := (x, k + l, z) \quad \text{and} \quad (x, k, y)^{-1} := (y, -k, x).
\]
For \( \mu, \nu \in E^* \) with \( s(\mu) = s(\nu) \), let \( Z(\mu, \nu) \) be the set
\[
\{ (x, k, y) : x \in Z(\mu), y \in Z(\nu), k = |\nu| - |\mu|, x_i = y_{i+k} \text{ for } i > |\mu| \}.
\]
By [35, Proposition 2.6], the collection of sets
\[
\{ Z(\mu, \nu) : \mu, \nu \in E^*, s(\mu) = s(\nu) \}
\]
is a basis of compact, open sets for a second-countable, locally compact, Hausdorff topology on \( G_E \) such that \( G_E \) is an étale groupoid. After identifying \((x, 0, x) \in G_E^{(0)} \) with \( x \in E^\infty \), [35, Proposition 2.6] says that the topology on \( G_E^{(0)} \) is identical to the topology on \( E^\infty \).

To discuss Example 5.8 below we will need item (1) of the following lemma which was proved by the third author and Lisa Orloff Clark.

**Lemma 5.7.** Let \( E \) be a row-finite directed graph with no sources.

1. If \( E \) has no return paths, then \( \text{DAD}(G_E) = 0 \).
2. If \( E \) has a return path, then \( \text{DAD}(G_E) = \infty \).

**Proof.** For (1), suppose that \( E \) has no return paths. Fix an open, precompact subset \( K \) of \( G_E \). Without loss of generality, we can replace \( K \) by \( K \cup K^{-1} \). Let \( U_0 = E^\infty \). There exists a finite subset \( F \) of the fibered product \( E^* \times_s E^* := \{ (\mu, \nu) \in E^* \times E^* : s(\mu) = s(\nu) \} \) such that \( K \subset \cup_{(\mu, \nu) \in F} Z(\mu, \nu) \). Without loss of generality, we may assume that \( (\mu, \nu) \in F \) implies \((\nu, \mu) \in F \). For a path \( \mu \), write \( V(\mu) \) for the subset of vertices on \( \mu \). Set \( V_F = \cup \{ V(\mu) : (\mu, \nu) \in F \} \) and let \( E_F \) be the subgraph of \( E \) of paths with vertices in \( V_F \).

Since there are no return paths in \( E \), the set \( \{ (\alpha, \beta) \in E_F^* \times_s E_F^* \} \) is finite. We claim that the groupoid
\[
G_K = \bigcup_{n \in \mathbb{N}} \{ k_1 \cdots k_n : k_i \in K \text{ for } 1 \leq i \leq n \}
\]
There exist open neighborhoods \( G \) generated by \( \{a, b\} \) \( \in E^*_F \times E^*_F \) for each \( n \). Now consider a product \( l_1 \cdots l_i \cdots l_{n+1} \) with \( l_i \in K \). By the induction hypothesis there exist \( (\eta, \zeta) \in E^*_F \times E^*_F \) such that \( l_1 \cdots l_n = (\eta x, |\eta| - |\zeta|, \zeta x) \). Since \( l_{n+1} \in K \), there exists \( (\mu, \nu) \in F \) such that \( l_{n+1} \in Z(\mu, \nu) \). Thus, \( \mu, \nu \in E^*_F \) and there exists \( y \in E^\infty \) such that \( l_{n+1} = (\mu y, |\mu| - |\nu|, \nu y) \). We have

\[
l_1 \cdots l_i l_{n+1} = (\eta x, |\eta| - |\zeta|, \zeta x)(\mu y, |\mu| - |\nu|, \nu y).
\]

Thus, \( \zeta x = \mu y \). Suppose that \( |\mu| \geq |\zeta| \). Then there exists \( \mu' \in E^*_F \) such that \( \mu = \zeta \mu' \) and \( x = \mu' y \). Now

\[
(\eta x, |\eta| - |\zeta|, \zeta x)(\mu y, |\mu| - |\nu|, \nu y) = (\eta \mu' y, |\eta| - |\zeta|, \mu y)(\mu y, |\mu| - |\nu|, \nu y) = (\eta \mu' y, |\mu| - |\nu| + |\eta| - |\zeta|, \nu y).
\]

Now notice that both \( \eta \mu' \) and \( \nu \) are in \( E^*_F \). Thus, \( l_1 \cdots l_i l_{n+1} \) is in the union at (5.2), which proves the claim. This shows that \( \text{DAD}(G_E) = 0 \).

For (2), suppose that \( E \) has a return path \( \mu \). We argue by contradiction: suppose that \( G_E \) has finite dynamic asymptotic dimension \( d \). The length \( |\mu| \) of \( \mu \) is at least 1. We write \( \mu^\infty \) for the infinite path obtained by concatenating \( \mu \) with itself infinitely many times. Then \( (\mu^\infty, |\mu|, \mu^\infty) \in G_E \). Let \( K \) be an open, precompact subset of \( G_E \) containing \( (\mu^\infty, |\mu|, \mu^\infty) \). There exist open neighborhoods \( U_0, \ldots, U_d \) which cover \( r(K) \cup s(K) \) and the groupoid \( G_i \) generated by \( \{g \in K : s(g) = r(r) \in U_i \} \) is precompact. Without loss of generality, \( \mu^\infty \in U_0 \). Then for each \( n \geq 1 \) the composition \( (\mu^\infty, n|\mu|, \mu^\infty) \) of \( (\mu^\infty, |\mu|, \mu^\infty) \) with itself \( n \) times is in \( G_0 \). But \( \{(\mu^\infty, n|\mu|, \mu^\infty)\}_{n \geq 1} \) has no convergent subsequence, which is a contradiction. \( \square \)

**Example 5.8.** Let \( E \) be the graph of Figure 1 which was considered in [15, Example 1 in §7]: We write \( G_E \) for the graph groupoid and identify its \( C^* \)-algebra with the \( C^* \)-algebra \( C^*(E) \) of the graph. We know a lot about \( C^*(G_E) \): it has bounded trace but is not a Fell algebra by [15, p. 188], hence does not have continuous trace. Since \( E \) has return paths, \( C^*(G_E) \) is not AF by [34, Theorem 2.4], and since the return paths do not have entries,
$C^*(G_E)$ is AF-embeddable by [56, Theorem 1.1]. Here we will use Corollary 5.5 to show that $C^*(G_E)$ has nuclear dimension 1.

For $x \in E^\infty$, we write $A_x = \{ g \in G_E : r(g) = x = s(g) \}$ for the stability subgroup at $x$. Let $\alpha^{k,n}$ be a return path of length $n+1$ with range and source some $w_{n,k}$. If $x_{\alpha^{k,n}} = \alpha^{k,n}_{\alpha^{k,n}} \ldots$ then the stability subgroup at $x_{\alpha^{k,n}}$ is isomorphic to $(n+1)\mathbb{Z}$; all other stability subgroups are trivial. The stability subgroups vary continuously and the orbits are closed.

We first claim that the topological dimension of $\hat{A}$ is 1. We write $\hat{A}_x$ for the dual of the stability subgroup $A_x$. Since $E^\infty$ is a countable set, $\hat{A}_x \times \{x\}$ is a countable union of closed subsets. Each $\hat{A}_x \times \{x\}$ is either $\{(0,x)\}$ or homeomorphic to $T$. Since $T$ has topological dimension 1, the countable-union theorem [46, Chapter 2, Theorem 2.5] or [16, Theorem 1.7.1] gives $\dim(\hat{A}) = 1$, as claimed.

Second, we claim that the dynamic asymptotic dimension of $\mathcal{R} := G_E/A$ is zero. To see this, let $F$ be the graph shown in Figure 2.

![Figure 2. The graph $F$](image-url)

Consider the subset $U = \bigcup_{i \geq 0} (Z(v'_i) \cup \bigcup_{j \leq i} Z(w'_{i,j}))$ of $F^\infty$. Notice that $U$ is both open and closed. (But $U$ is not an invariant subset. To see this, let $x$ be the infinite path ending with the edge $f'_{0,0}$. Let $\sigma$ be the shift map on $E^\infty$. Then $(x,2,\sigma^2(x)) \in G_F$, its range $x$ is in $U$ but its source $\sigma^2(x)$ is not in $U$.)

By [15, p. 189], $\mathcal{R}$ is isomorphic to the reduction $(G_F)_U$ to $U$ of $G_F$. In particular, $\mathcal{R}$ is étale because $U$ is open and because $G_F$ is étale. Since $F$ is row-finite with no sources or return paths, $G_F$ has dynamic asymptotic dimension 0 by Lemma 5.7. Since $U$ is both open and closed, it follows that the dynamic asymptotic dimension of $(G_F)_U$ is also 0 by
Lemma 3.11. Thus, $DAD(R) = 0$ as well. Now Corollary 5.5 applies to give $\dim_{\text{nuc}}(C^*(E)) \leq 2 \cdot 1 - 1 = 1$; we know it is at least 1 since $C^*(E)$ is not AF.

APPENDIX A. NOTATION INDEX

- $G^{(0)}$: unit space of $G$
- $r: G \rightarrow G^{(0)}$: range map
- $s: G \rightarrow G^{(0)}$: source map
- $G^{(2)}$: composable pairs in $G$
- $\langle X \rangle$: subgroupoid generated by $X$
- $G|_W$: $\{ e \in G : s(e), r(e) \in W \subset G^{(0)} \}$
- $[x]$: orbit of $x \in G^{(0)}$
- $xG$: $\{ e \in G : r(e) = x \}$
- $Gx$: $\{ e \in G : s(e) = x \}$
- $C_c(G)$: compactly supported $C^*$-valued functions on $G$
- $\sigma^*$: measure in left Haar system for $x \in G^{(0)}$
- $\sigma_x(U)$: $\sigma^*(U^{-1})$
- $C^*(G, \sigma)$: full $C^*$-algebra of $G$ with respect to Haar system $\sigma$
- $C^*_r(G, \sigma)$: reduced $C^*$-algebra of $G$ with respect to Haar system $\sigma$
- $(E, \iota, \pi)$: a twist $G^{(0)} \times T \xrightarrow{\iota} E \xrightarrow{\pi} G$, see Definition 2.1
- $C_c(E; G)$: $f \in C_c(E)$ such that $f(z \cdot e) = zf(e)$ for $z \in T$ and $e \in E$ for a twist $(E, \iota, \pi)$ over $G$
- $C^*(E; G)$: full twisted groupoid $C^*$-algebra for a twist $(E, \iota, \pi)$ over $G$
- $C^*_r(E; G)$: reduced twisted groupoid $C^*$-algebra for a twist $(E, \iota, \pi)$ over $G$
- $C_K(E; G)$: $\{ f \in C_c(E; G) : \text{supp } f \subset K \}$ where $K \subset E$
- $X_+$: Alexandrov one-point-compactification of $X$
- $\tilde{A}$: minimal unitization of a $C^*$-algebra $A$
- $\tilde{G}$: Alexandrov groupoid for a groupoid $G$, see Lemma 3.4
- $(\tilde{E}, \tilde{\iota}, \tilde{\pi})$: Alexandrov twist for a twist $(E, \iota, \pi)$, see Lemma 3.6
- $[f, h]$: commutator, as in Equation (4.3)

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