On a relation between classical and free
infinitely divisible transforms

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Abstract. We study two ways (levels) of finding free-probability
analogues of classical infinite divisible measures. More precisely,
we identify their Voiculescu transforms. For free-selfdecomposable
measures we found the formula (a differential equation) for their
background driving transforms. We illustrate our methods on
the hyperbolic characteristic functions. As a by-product our ap-
proach potentially may produce new formulas for definite inte-
grals.

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There are many notions of infinite divisibility that exhibit some similarities and as well some differences. Here we study the classical infinite divisibility with respect to the convolution $*$ and the free-infinite divisibility for the box-plus $⊞$ operation (Theorem 1). We introduced free-probability analogues of the Laplace (double exponential) and the hyperbolic distributions (on the real line). For the free-selfdecomposable Voiculescu transforms we found an ordinary differential equation for their background driving transforms (Theorem 2).

The hyperbolic distributions were studied from the infinite divisibility point of view by Pitman and Yor (2003); cf. [18]. While here we have utilized the fact that all of them are in the proper subclass of selfdecomposable distributions (also called the class $L$ distributions); Jurek (1996), cf. [9].

The program of the study can be viewed as a particular case of the following abstract set-up: there are two abstract semigroups $(S_1, ◦)$ and $(S_2, ◦)$, two $1 − 1$ and onto operators $A$ and $Z$ acting on domains $D_1$ and $D_2$, respectively and $1 − 1$ and onto mapping $j$ between the domains $D_1$ and $D_2$. That is we have

$$j : D_1 → D_2, \quad A : D_1 → S_1 \quad \text{and} \quad Z : D_2 → S_2.$$ 

Consequently, the diagram

$$\begin{array}{ccc}
D_1 & \xrightarrow{A} & (S_1, ◦) \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle r} \\
D_2 & \xrightarrow{Z} & (S_2, ◦)
\end{array}$$

allows us to define the identification $r$ between $(S_1, ◦)$ and $(S_2, ◦)$.

Namely, we say that $\tilde{s} ∈ (S_2, ◦)$ is $\circ$-analog or $\circ$-counterpart of an $s ∈ (S_1, ◦)$, if there exists $x ∈ D_1$ such that $A(x) = s$, $j(x) = y$ and $Z(y) = \tilde{s}$. That is, we have $r(A(x)) = r(s) := \tilde{s}$, or $Z(j(x)) = \tilde{s}$.

Similarly, $s ∈ (S_1, ◦)$ is $\circ$-analog of $\tilde{s} ∈ (S_2, ◦)$ if there exists $y ∈ D_2$ such that $Z(y) = \tilde{s}$, $j^{-1}(y) = x$ and $A(x) = s$.

1. INFINITE DIVISIBILITY.

1.1. In the setting of this paper, $(S_1, ◦) ≡ (ID, *)$ is the (classical) convolution semigroup ID of all infinitely divisible probability measures $μ$ on the real line with the convolution operation $∗$. The characteristic functions $φ$ (or the Fourier transforms) are functions given as

$$φ(t) := \int_{\mathbb{R}} e^{itx} μ(dx), \quad t ∈ \mathbb{R}, \quad \text{for some probability measure } μ.$$
Let $D_1 := \{ \phi \in ID : t \to (\phi)^{1/n}(t) \text{is characteristic function for } n = 2, 3, \ldots \}$, that is, $D_1$ consists of all $\ast$ - infinitely divisible characteristic functions.

Further, let $D_2 := \{ [a, m] : a \in \mathbb{R} \text{ and } m \text{ is a finite Borel measure on } \mathbb{R} \}$ so it is a family of pairs $[a, m]$. Because of the following fundamental Khintchine representation formula:

$$\phi \in D_1 \iff \phi(t) = \exp \{ ita + \int_{\mathbb{R}} (e^{itx} - 1 - \frac{itx}{1 + x^2}) \frac{1 + x^2}{x^2} m(dx) \}, \quad t \in \mathbb{R}, \quad (1)$$

for a uniquely determined parameters $a$ (a number) and $m$ (a finite measure), (for instance cf. [19], [17] or [1]) the mapping (an isomorphism) $j$ given as

$$j : D_1 \to D_2 \quad \text{and} \quad j(\phi) := [a, m] \quad \text{iff} \quad \phi \text{ is of the form } (1), \quad (2)$$

is well defined.

**Remark 1.** (i) In some situations and applications instead of the finite measure $m$, in (1), one uses a $\sigma$-finite measure $M(dx) := \frac{x^2}{1 + x^2} m(dx)$ on $\mathbb{R} \setminus \{0\}$ (equivalently: $m(dx) := \frac{1 + x^2}{x^2} M(dx)$) and slightly changed the integrand as given below. Then the equality (1) can be rewritten as follows:

$$\phi(t) = \exp \{ itb - \frac{1}{2} t^2 \sigma^2 + \int_{\mathbb{R} \setminus \{0\}} (e^{itx} - 1 - itx 1_{\{|x| \leq 1\}}(x)) M(dx) \} \quad (1a)$$

where $\sigma^2 := m(\{0\})$ and $b := a + \int_{\mathbb{R}} x 1_{\{|x| \leq 1\}}(x) - 1/(1 + x^2) M(dx)$.

(ii) The formula (1a) is called the Lévy-Khintchine representation of an infinitely divisible characteristic function (probability measures). The probability measure $\mu$ corresponding to (1a) is represented by the triple $\mu = [b, \sigma^2, M]$; cf. Parthasarathy (1967), Chapter VI; or [1], [17].

(iii) The measure $M$ has the following stochastic meaning: $M(A)$ is the expected number of jumps that occur up to time 1 and are of sizes in the set $A$, of the corresponding Lévy process $(Y(t), t \geq 0)$, where $\phi$ is the characteristic function of the random variable $Y(1)$.

1.2. Further, $(S_2, \circ) \equiv (ID, \boxplus)$ is the semigroup of all $\boxplus$ free-infinitely divisible probability measures. Namely for a probability measure $\nu$ on $\mathbb{R}$, one introduces its Voiculescu transform $V_\nu$ (an analogue of a characteristic function $\phi$) and an operation $\boxplus$ on measures in such a way that

$$V_{\mu \boxplus \nu}(z) = V_\mu(z) + V_\nu(z);$$

cf. Voiculescu (1999), cf.[20]. This in turn allows to introduce the notion of $\boxplus$-infinite divisibility and one has the following an analogue of the Khintchine representations:
\[ \nu \in (ID, \boxplus) \quad \text{iff} \quad V_\nu(z) = a + \int_\mathbb{R} \frac{1 + zx}{z - x} m(dx), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (3) \]

for uniquely determined a constant \( a \in \mathbb{R} \) and finite (Borel) measure \( m \); cf. [20] or [2], [3], [4].

However, for the uniqueness questions, of the representation (3), is enough to consider Voiculescu transforms only on the imaginary axis; Jurek (2006), cf. [11]; (also [12], [13] and [8]).

The formulas (1) and (2) suggest to define the following mappings:

\[ A : D_1 \to (ID, *) \quad \text{given as} \quad A(\phi) := \mu \quad \text{iff} \quad \phi(t) = \int_\mathbb{R} e^{itx} \mu(dx) \quad t \in \mathbb{R}; \quad (4) \]

and then to define \( r : (ID, *) \to (ID, \boxplus) \) as

\[ r(\mu) := \tilde{\mu} \quad \text{iff} \quad V_{\tilde{\mu}}(it) = it^2 \int_0^\infty \log \phi(s)e^{-ts}ds, \quad t > 0. \quad (5) \]

Consequently, by (5), we get the composition \( r \circ A : D_1 \to (ID, \boxplus) \).

On the other hand, on the level \( Z \), using the mapping \( j \) (from (2)) we define

\[ Z : D_2 \to (ID, \boxplus) \quad \text{as} \quad Z([a,m]) := \nu \quad \text{iff} \quad V_\nu(it) = a + \int_\mathbb{R} \frac{1 + itx}{it - x} m(dx); \quad (6) \]

So we have the following question:

Does \( j(\phi) = [a,m] \) imply that \( r(A(\phi)) = Z(j(\phi)) \)? That is, \( \tilde{\mu} = \nu \)? \( (7) \)

**Remark 2.** The idea of inserting the same parameters \( a \) and \( m \) into two different integral kernels (1) and (3) is due to Bercovici - Paata (1999), Section 3; cf. [4]. [In our notation, it is the mapping on the level \( Z \).]

A different approach was proposed in Jurek (2006, 2007), cf. [11] and [12] and more recently repeated in Jurek (2016), cf. [16]. The key in those papers was the technique of the random integral representation. Here it is the mapping on the level \( A \).

**1.3.** Below we give straightforward connections between the formulas (1) and (3) and prove the equality (7). As in previous papers Jurek (2006), cf. [11] (or [12], [13]) we consider the transforms \( V_\nu \) only on the imaginary line.
Theorem 1. For each classical infinitely divisible $\mu \in (ID, \ast)$ with the characteristic function $\phi_{\mu}$ there exists its unique free-infinitely divisible analogue measure $\tilde{\mu}$ ($\tilde{\mu} \in (ID, \boxplus)$) such that its Voiculescu transform $V_{\tilde{\mu}}$ is given as

$$V_{\tilde{\mu}}(it) = it^2 \int_0^{\infty} \log \phi_{\mu}(s) e^{-ts} ds, \quad t > 0. \quad (A)$$

Furthermore, if $\mu$ has representation $\mu = [a, m]$ (in the Khintchine formula) then

$$V_{\tilde{\mu}}(it) = a + \int_{\mathbb{R}} \frac{1 + itx}{it - x} m(dx), \quad t > 0. \quad (Z)$$

For a symmetric $\mu = [0, m]$ (real $\phi_{\mu}$) we have

$$V_{\tilde{\mu}}(it) = it \int_0^{\infty} \int_{\mathbb{R}} \left( \frac{\cos(sx) - 1}{x^2} \right) e^{-ts} m(dx) \, ds,$$

Proof. The fact that the function given in (A), indeed, defines Voiculescu transform of an free-infinitely divisible measures was already shown in Jurek (2007), Corollary 6; cf. [12] (also repeated in [13]).

On the other hand, formula (Z) is obviously a Voiculescu transform of a free-infinitely measure in view of the characterization (3) above.

In order to show that both ways we get the same measure we show that

$$it^2 \int_0^{\infty} \log \phi_{[a,m]}(s) e^{-ts} ds = a + \int_{\mathbb{R}} \frac{1 + itx}{it - x} m(dx), \quad t > 0. \quad (8)$$

Taking the Lévy exponent, as it is given in Khintchine formula (1), computing the Laplace transform of the shift part $ita$ and then interchanging the order of integration we have that

$$LHS = it^2 \left( -\frac{ia}{t^2} + \int_{\mathbb{R}} \frac{1 + x^2}{x^2} \left[ \int_0^{\infty} \left( e^{-isx} - 1 + \frac{isx}{1 + x^2} \right) e^{-st} ds \right] m(dx) \right)$$

$$= it^2 \left( -\frac{ia}{t^2} + \int_{\mathbb{R}} \frac{1 + x^2}{x^2} \left[ \frac{1}{ix + t} - \frac{1}{t} + \frac{ix}{1 + x^2} \right] m(dx) \right)$$

$$= a + \int_{\mathbb{R}} \frac{1 + x^2}{x^2} \left[ \frac{-ix}{t(ix + t)} + \frac{ix}{(1 + x^2)t^2} \right] m(dx)$$

$$= a + \int_{\mathbb{R}} \frac{tx - i}{ix + t} m(dx) = a + \int_{\mathbb{R}} \frac{1 + itx}{it - x} m(dx) = RHS,$$

which concludes proof of the identity (8) and the first part of Theorem 1.
For the second part of the Theorem, we calculate as above, that is, we change the order of integration, utilize the fact that $m$ is symmetric measure and use the Laplace transform

$$
\int_0^\infty \cos(as)e^{-st}ds = \frac{t}{t^2 + a^2}.
$$

This conclude the argument for the second equality in Theorem 1. \hfill \square

**Corollary 1.** Let $\mathcal{E}_t, t > 0,$ denotes the exponential random variable with parameter $t$ and the probability density $t e^{-tx} 1_{(0,\infty)}(x)$. Then

$$
\mathbb{E}[\log \phi_{\mu}(-\mathcal{E}_t)] = \int_0^\infty \log(\phi_{\mu}(s)) \left( te^{-ts} \right) ds = (it)^{-1} V_{\hat{\mu}}(it), \quad \text{for} \quad t > 0.
$$

Furthermore, if $\mu = [0, m]$ is a symmetric (i.e., $\phi_{\mu}$ is real) and $m$ is a probability distribution of a random variable $X$ that is stochastically independent of $\mathcal{E}_t$ then

$$
\mathbb{E}\left[ (1 - \cos(\mathcal{E}_t X)) \frac{1 + X^2}{X^2} \right] = \mathbb{E}\left[ \frac{1 + X^2}{t^2 + X^2} \right], \quad \text{for} \quad t > 0. \tag{9}
$$

It follows from the second identity in Theorem 1.

**1.4. Illustration of the application of Theorem 1.** Let $C$ stands for hyperbolic-cosh variable or its probability distribution. Then it is *-infinitely divisible and its characteristic function is equal $\phi_C(t) = (\cosh t)^{-1}$. From Theorem 1, on the level of the characteristic functions (the mapping $A$), its free-infinitely divisible analog $\tilde{C}$ (of the hyperbolic-cosh) has the following Voiculescu transform:

$$
V_{\tilde{C}}(it) = -it^2 \int_0^\infty \log \cosh(s) e^{-ts} ds = i[1 - t\beta(t/2)], \quad t > 0, \tag{10}
$$

where $\beta$ is a special function defined in (19), formula (vii). Equality (10) is shown in Section 3.1 below.

On the other, on the level of the parameters $[a,m]$ (the mapping $Z$), the hyperbolic-cosh has $a = 0$ and $m(dx) = \frac{1}{2} \frac{|x|}{1+x^2} \frac{1}{\sinh(\pi|x|/2)} dx$. Consequently, by (8), its $\boxplus$-free infinitely divisible analog $\hat{C}$ has the following Voiculescu transform

$$
V_{\hat{C}}(it) = -it \int_0^\infty \frac{|x|}{t^2 + x^2} \frac{1}{\sinh(\pi|x|/2)} dx = i[t \beta(t/2 + 1) - 1], \tag{11}
$$

for computational details see Section 3.1 below.
As a byproduct of (10) and (11) we get the following functional relation for the special function $\beta$:

$$\beta(s) + \beta(s + 1) = 1/s, s > 0,$$

(12)

However, the formula (12) can also be obtained from other known representation of the special function $\beta$; cf. Section 3.0, formula (ix).

**Remark 3.** Our two ways (two levels, two mappings) of getting free-infinitely divisible analogues of classical infinitely divisible characteristic function may produce new unknown before explicite relation between some special functions.

## 2. SELFDECOMPOSABILITY.

### 2.1. An important and a proper subclass of the class $(ID, \ast)$, of all infinitely divisible measures, is the class $L$, also known as the class of selfdecomposable probability measures; cf. Jurek-Vervaat (1983), cf. [15] or Jurek and Mason (1993), cf. [14], Chapter 3.

Let us recall that the class $L$ contains, among others, all stable probability measures, exponential distributions, t-Student distribution, chi-square, gamma, Laplace, hyperbolic-sine and hyperbolic-cosine measures (characteristic functions), etc; cf. Jurek (1997), cf. [10].

For the purposes of this paper let us recall that for $\mu \in L$ (or equivalently for characteristic function $\phi \in L$) there exists an unique $\nu \in ID_{\log}$, i.e., infinitely divisible measures with finite logarithmic moment (or equivalently there exists an unique $\psi \in ID_{\log}$ ) such that

$$\log \psi(t) = t \frac{d}{dt} \log \phi(t); \quad \text{equivalently} \quad \log \phi(t) = \int_0^t \log \psi(s) \frac{ds}{s}$$

(13)

The above relations follow from the random integral representation of a selfdecomposable distributions: for each distribution $\mu \in L$ there exists a unique Lévy process $Y_\nu$ such that

$$\mu = \mathcal{L}\left(\int_0^\infty e^{-s}dY_\nu(s)\right), \quad Y_\nu(s), s \geq 0, \quad \mathcal{L}(Y_\nu(1)) = \nu \in ID_{\log};$$

(14)

cf. Jurek and Mason (1993), cf. [14] Theorems 3.4.6, 3.6.8 and Remark 3.6.9(4) or Jurek and Vervaat (1983), cf. [15].

The characteristic function $\psi$ (in (13)) is referred to as the background driving characteristic function (BDCF) of $\phi \in L$ and $Y_\nu$ the background driving Lévy process (BDLP) of $\mu$. 
Remark 4. A very nice argument, based on the random integral representation (14), for the existence of densities for all real-valued selfdecomposable variables is due to Jacod (1985), cf. [7]. His proof is repeated in Jurek (1997), cf.[10], pp.104-105.

2.2. Here is a technical property (limit at infinity) of Lévy exponent $\Phi$ of an infinitely divisible characteristic function with a real parameter $a$ and a finite measure $m$, that is,

$$
\Phi(t) := ita + \int_{\mathbb{R}} (e^{itx} - 1 - \frac{itx}{1 + x^2}) \frac{1 + x^2}{x^2} m(dx) = itb + \int_{\mathbb{R}} (e^{itx} - 1 - itx 1_{|x| \leq 1}(x)) \frac{1 + x^2}{x^2} m(dx), \quad (15)
$$

where $b := a + \int_{\mathbb{R}} x 1_{\{|x| \leq 1\}}(x) - 1/(1 + x^2) \frac{1 + x^2}{x^2} m(dx)$ and the finiteness of the measure $m$ guarantees the existence of the integral.

Lemma 1. For any constants $c_1 > 0$ and $c_2 > 0$ and any Lévy exponent $\Phi$ we have

$$
\lim_{t \to \infty} t^{c_1} e^{-c_2 t} \Phi(t) = \lim_{t \to \infty} t^{c_1} e^{-c_2 t} \int_{\mathbb{R}} (e^{itx} - 1 - itx 1_{|x| \leq 1}(x)) \frac{1 + x^2}{x^2} m(dx) = 0.
$$

Proof. For pure degenerate $\Phi$, i.e., when $m = 0$ in (15), Lemma 1 is obvious.

Let us assume that $b = 0$. Since

$$
|e^{itx} - 1 - itx| \leq \min(\frac{|tx|^2}{2}, 2|tx|), \quad \text{and} \quad |e^{itx} - 1| \leq \min(|tx|, 2) \leq 2;
$$

(for instance, Billingsley (1986), cf. [5], pp. 352 and 353) therefore from (15) we get

$$
t^{c_1} e^{-c_2 t} |\Phi(t)| \leq t^{c_1} e^{-c_2 t} \int_{|x| \leq 1} \frac{t^2}{2} x^2 \frac{1 + x^2}{x^2} m(dx) + t^{c_1} e^{-c_2 t} \int_{|x| > 1} \frac{1 + x^2}{x^2} m(dx)
\leq \frac{1}{2} t^{c_1} e^{-c_2 t} \int_{|x| \leq 1} (1 + x^2) m(dx) + 2 t^{c_1} e^{-c_2 t} \int_{|x| > 1} (1 + x^{-2}) m(dx) \to 0 \text{ as } t \to \infty,
$$

which completes a proof of the lemma.

Here, in Theorem 2, we have a free-selfdecomposability analogue of the differential relations (13), for the background driving equation, known for the classical selfdecomposability.
Theorem 2. Let \( \tilde{\phi} \) and \( \tilde{\psi} \) be free-analogues of a self-decomposable characteristic function \( \phi \) and its background driving characteristic function \( \psi \), respectively. Then their Voiculescu transforms \( V_{\tilde{\phi}} \) and \( V_{\tilde{\psi}} \) satisfy the differential equation:

\[
V_{\tilde{\psi}}(it) = V_{\tilde{\phi}}(it) - t \frac{d}{dt}[V_{\tilde{\phi}}(it)], \quad t > 0. \tag{16}
\]

Equivalently, in terms of \( V_{\tilde{\psi}} \), we get

\[
V_{\tilde{\phi}}(it) - t V_{\tilde{\phi}}(i) = -t \int_{1}^{t} s^{-2} V_{\tilde{\psi}}(is) ds = t \int_{1}^{t} V_{\tilde{\psi}}(is) d(s^{-1}), \quad t > 0. \tag{17}
\]

Proof. Note that using the definition (A) from Theorem 1, the relation (13) for classical self-decomposability and then Lemma 1 we have

\[
V_{\tilde{\psi}}(it) := it^2 \int_{0}^{\infty} \log \psi(-v)e^{-tv} dv = it^2 \int_{0}^{\infty} (\log \phi(-v))'(-v) e^{-tv} dv
\]

\[
= it^2 \int_{0}^{\infty} (\log \phi)'(-v)(-1) (-v) e^{-tv} dv = it^2 \int_{0}^{\infty} (\log \phi)'(-v) ve^{-tv} dv
\]

\[
= it^2 \left[ \log \phi(-v)ve^{-tv} \bigg|_{v=\infty}^{v=\infty} - \int_{0}^{\infty} \log \phi(-v) (1 - tv) e^{-tv} dv \right]
\]

\[
= it^2 \left[ \int_{0}^{\infty} \log \phi(-v)e^{-tv} dv + t \int_{0}^{\infty} \log \phi(-v) ve^{-tv} dv \right]
\]

\[
= -V_{\tilde{\phi}}(it) - it^3 \frac{d}{dt}[\int_{0}^{\infty} \log \phi(-v)e^{-tv} dv] = -V_{\tilde{\phi}}(it) - it^3 \frac{d}{dt}[(it^2)^{-1}V_{\tilde{\phi}}(it)]
\]

\[
= -V_{\tilde{\phi}}(it) - t^3 \frac{d}{dt}[t^{-2}V_{\tilde{\phi}}(it)] = -V_{\tilde{\phi}}(it) - t^3[-2t^{-3}V_{\tilde{\phi}}(it) + t^{-2} \frac{d}{dt}V_{\tilde{\phi}}(it)]
\]

\[
= V_{\tilde{\phi}}(it) - t \frac{d}{dt}V_{\tilde{\phi}}(it),
\]

which completes a proof of equality (16).

For the equality (17), note that (16) is a first-order linear differential equation that we can solve by the integrating factor method. More explicitly, note that (16) can be rewritten as follows

\[
t^{-1}V_{\tilde{\psi}}(it) = t^{-1}V_{\tilde{\phi}}(it) - \frac{d}{dt}[V_{\tilde{\phi}}(it)] = -t \frac{d}{dt} \left[ \frac{V_{\tilde{\phi}}(it)}{t} \right].
\]

Hence, dividing by \( t \) and then integrating both sides over the interval \([1,t]\) (or \([t,1]\)) , we get

\[
\frac{V_{\tilde{\phi}}(it)}{t} - V_{\tilde{\phi}}(i) = -\int_{1}^{t} s^{-2} V_{\tilde{\psi}}(is) ds,
\]

which completes the proof of the Theorem 2. \(\square\)
2.3. Illustration of the application of Theorem 2. The hyperbolic-cosh function \( \phi_C(t) = (\cosh t)^{-1} \) is selfdecomposable; Jurek (1996), cf. [9]. From (10) and (16) (in Theorem 2) we obtain \( V_{\tilde{\psi}_C} \), free-probability analog of the background driving characteristic function \( \psi_C \), as

\[
V_{\tilde{\psi}_C}(it) = i \left[ 1 + \frac{1}{2} t^2 \beta'(\frac{1}{2}t) \right] = i \left[ 1 + \frac{t^2}{2} \zeta(2, \frac{t}{2}) - \frac{t^2}{4} \zeta(2, \frac{t}{4}) \right]; \quad t > 0.
\] (18)

For the first equality one needs to put (10) into (16) and then use the formula that expresses \( \beta' \) in terms of Riemann’s function \( \zeta(2, a) \); for details See Section 3.0. [Two more ways of getting the above formula are discussed in Section 4.]

3. FREE - PROBABILITY ANALOGUES OF THE HYPERBOLIC CHARACTERISTIC FUNCTIONS.

3.0. For an ease of reference we recall definitions and basic properties of some special function. All formulas followed by a boldface reference number
are taken from I. S. Gradshteyn, I. M. Ryzhik (1994), cf. [6].

(i) \( (a) \) \( \Gamma(z) := \int_0^\infty x^{z-1}e^{-x}dx, \ \Re z > 0; \) (Euler function)

\( (b) \) \( \psi(z) := \frac{d}{dz} \ln \Gamma(z), \ \Re z > 0 \) (digamma function)

(ii) \( \psi_n(z) \equiv \psi^{(n)}(z) = (-1)^{n+1} n! \zeta(n+1, z); \) \( \text{8.363(8)}, \)

( n-th derivative; called also as polygamma).

(iii) \( \psi(2z) = \frac{1}{2}(\psi(z) + \psi(z + \frac{1}{2})) + \ln 2; \) \( \text{8.365(6)}; \)

(iv) \( \zeta(s,a) := \sum_{k=0}^\infty \frac{1}{(k+a)^s}, \ \Re s > 1, \ -a \notin \mathbb{N}, \) (Riemann’s zeta functions);

(v) \( \zeta(s,a+1) = \zeta(s,a) - \frac{1}{a^s}; \ \zeta(s,a+1/2) = 2^s \zeta(s,2a) - \zeta(s,a); \)

(vi) \( \zeta(2,t) - \frac{1}{4} \zeta(2, t/2) = \frac{1}{4} \zeta(2, t + 1/2); \) (form (v));

(vii) \( \beta(x) := \frac{1}{2} \left[ \psi \left( \frac{x+1}{2} \right) - \psi \left( \frac{x}{2} \right) \right], \ \beta(x) = \sum_{k=0}^\infty \frac{(-1)^k}{x+k}, \ -x \notin \mathbb{N}, \) \( \text{8.732(1)}; \)

(viii) \( \beta'(x) = -\sum_{k=0}^\infty \frac{(-1)^k}{(x+k)^2} = \zeta(2,x) - \frac{1}{2} \zeta(2, \frac{x}{2}); \) \( \text{8.374}; \)

(ix) \( \beta(t) = \int_0^\infty \frac{1}{1 + e^{-x} e^{-tx}} dx, \ \Re t > 0; \) \( \text{8.371(2)}; \)

(x) \( ci(x) \equiv Ci(x) := -\int_x^\infty \frac{\cos u}{u} du; \)

\( si(x) := -\int_x^\infty \frac{\sin u}{u} du = -\frac{\pi}{2} + Si(x), \) where \( Si(x) := \int_0^x \frac{\sin u}{u} du; \) \( \text{(19)} \)

3.1. Hyperbolic-cosine random variable.

Let \( C \) stands for the standard hyperbolic cosine variable, that is, the random variable with the characteristic function

\[ \phi_C(t) := \frac{1}{\cosh t} = \exp \int_\mathbb{R} (\cos(tx) - 1) \frac{1 + x^2}{x^2} \left[ \frac{1}{2} \frac{|x|}{1 + x^2} \sinh(\pi|x|/2) \right] (dx) \]

\[ = \exp \int_\mathbb{R} (\cos(tx) - 1) \left[ \frac{1}{2} \frac{|x|}{\sinh(\pi|x|/2)} \right] (dx), \quad \text{(20)} \]

where in the first bracket \( [\ldots] \) is the density of the Khintchine finite measure \( m_C \) corresponding to \( \phi_C \) in the representation (1) and in the second one, is
the density of the sigma-finite Lévy (spectral) measure \( M \) in (1a); Jurek and Yor (2004), cf. [16].

**Corollary 2.** The free-probability analog of the hyperbolic cosine characteristic function \( \phi_C \) has the following Voiculescu transform

\[
V_{\phi_C}(it) = i[1 - t\beta(t/2)], \quad t > 0. \tag{21}
\]

As a by-product we infer the following identity for the function \( \beta \):

\[
\beta(s) + \beta(s + 1) = 1/s, \quad s > 0.
\]

**Proof.** (First proof.)

For first proof one needs the identity:

\[
\int_0^\infty e^{-\xi x} \ln(cosh x) dx = \frac{1}{\xi} \left[ \beta(\xi/2) - 1/\xi \right], \quad \Re \xi > 0; \quad \text{cf. 4.342(2) in [6].} \tag{22}
\]

Hence and Theorem 1, level (A),

\[
V_{\phi_C}(it) = it^2 \int_0^\infty \log \phi_C(-v)e^{-tv} dv = -it^2 \int_0^\infty \log \cosh(v)e^{-tv} dv
\]

\[
= -it^2 \left( \frac{1}{t} \left( \beta(t/2) - 1/t \right) \right) = i[1 - t\beta(t/2)], \quad t > 0;
\]

which completes the calculation of (21).

(Second proof.)

This time the integral identity needed below is as follows:

\[
\int_0^\infty \frac{x dx}{(b^2 + x^2) \sinh(\pi x)} = \frac{1}{2b} - \beta(b+1), \quad b > 0; \quad \text{cf. 3.522(2) in [6].} \tag{23}
\]

From first line in (20) we have that \( \phi_C \) has finite Khintchine measure

\[
m_C(dx) = \frac{1}{2} |x| \frac{1}{1 + x^2 \sinh(\pi |x|/2)} dx
\]

Consequently, from Theorem 1 we get

\[
V_{\phi_C}(it) = -it \int_\mathbb{R} \frac{1 + x^2}{t^2 + x^2} \frac{1}{2} \frac{1}{1 + x^2 \sinh(\pi |x|/2)} dx
\]

\[
= -it \int_0^\infty \frac{x}{t^2 + x^2 \sinh(\pi x/2)} \frac{1}{2} dy = -it \int_0^\infty \frac{y}{(t/2)^2 + y^2 \sinh \pi y} dy
\]

\[
= -it \left[ \frac{1}{t} - \beta(t/2 + 1) \right] = i \left[ t \beta(t/2 + 1) - 1 \right], \quad \text{for } t > 0.
\]
From those two proofs we must have
\[ t\beta \left( \frac{t}{2} + 1 \right) - 1 = 1 - t\beta \left( \frac{t}{2} \right) \quad \text{or} \quad \beta(s) + \beta(s + 1) = 1/s; \quad (24) \]
and this completes the proof of Corollary 2. \[ \square \]

**Remark 5.** The identity (24) also follows from the fact that
\[ \beta(t) = \int_0^\infty (1 + e^{-x})^{-1} e^{-tx} dx, \ t > 0; \ cf. \ 8.371(2) \ in \ [6] \ or \ Section \ 0, \ (ix). \]

### 3.2. Hyperbolic-sine variable.

Let \( S \) stands for the standard hyperbolic-sine variable, that is, the random variable with the characteristic function
\[ \phi_S(t) := \frac{t}{\sinh t} = \exp \int_\mathbb{R} (\cos(tx) - 1) \left\{ \frac{1}{2} \frac{|x|}{1 + x^2} \sinh(\pi |x|/2) \right\} dx \]
\[ = \exp \int_\mathbb{R} (\cos(tx) - 1) \left[ \frac{e^{-\pi |x|/2}}{2|x| \sinh(\pi |x|/2)} \right] (dx), \quad (25) \]
where in the first bracket \([\ldots]\) is the density of the (Khintchine) finite measure \( m_S \) corresponding to \( \phi_S \) in (1) and in the second one, is the density of the sigma-finite Lévy (spectral) measure \( M_S \) in (1a).

**Corollary 3.** The free-probability analog \( \tilde{\phi}_S \) of the hyperbolic sine characteristic function \( \phi_S \) has the following Voiculescu transform
\[ V_{\tilde{\phi}_S}(it) = it[\psi(t/2) - t \ln(t/2) + 1], \ t > 0. \quad (26) \]

**Proof.** (First proof.) The key integral identity for (26) is the following one:
\[ \int_0^\infty e^{-\xi x} (\ln(\sinh x) - \ln x) dx = \frac{1}{\xi} [\ln(\xi/2) - 1/\xi - \psi(\xi/2)], \ \Re \xi > 0; \]
cf. 4.342(3) in [6].
( NOTE the misprint in [6]; comp. www.mathtable.com/errata/gr6errata.pdf).
Hence and from Theorem 1, equality (A), we get
\[ V_{\tilde{\phi}_S}(it) = it^2 \int_0^\infty \log \phi_S(-v)e^{-tv} dv = -it^2 \int_0^\infty [\log \sinh v - \log v] e^{-tv} dv \]
\[ = -it^2 \frac{1}{t} [\ln(t/2) - 1/t - \psi(t/2)] = it[\psi(t/2) - t \ln(t/2) + 1], \quad (27) \]
which proves (26).
(Second proof.) This time we need the formula

$$\int_0^\infty \frac{x \, dx}{(x^2 + \beta^2)(e^{\mu x} - 1)} = \frac{1}{2} \left[ \log \left( \frac{\beta \mu}{2\pi} \right) - \frac{\pi}{\beta \mu} - \psi \left( \frac{\beta \mu}{2\pi} \right) \right], \Re \beta > 0, \Re \mu > 0;$$

cf. 3.415(1) in [6].

From first line in (25) we have that the Khintchine (finite) measure $m_S$ (for $\phi_S$) is equal to

$$m_S(dx) = \frac{1}{2} \frac{|x|}{1 + x^2} \frac{e^{-\pi|x|/2}}{\sinh(\pi|x|/2)} \, dx = \frac{|x|}{1 + x^2} \frac{1}{1 - e^{-\pi|x|}} \, dx$$

(also Jurek-Yor (2004), cf. [16].) Thus the above identity and Theorem 1, equality (Z), give

$$V_{\tilde{\phi}_S}(it) = -2it \int_0^\infty \frac{x}{t^2 + x^2} \frac{1}{e^{\pi x} - 1} \, dx = -it \left[ \ln \left( \frac{t}{2} \right) - \frac{1}{t} - \psi \left( \frac{t}{2} \right) \right]$$
$$= it \left[ t \psi(t/2) - t \log(t/2) + 1 \right], \quad t > 0,$$

which coincides with (26). This completes a proof of Corollary 3. \hfill \Box

3.3. The hyperbolic-tangent variable.

Let $T$ stands for the standard hyperbolic-tangent variable, that is, the random variable with the characteristic function $\phi_T(t) = \frac{\tanh t}{t}$. It’s Khintchine representation is as follows:

$$\phi_T(t) = \frac{\tanh t}{t} = \exp \left( \int_{-\infty}^{\infty} (\cos tx - 1) \left[ \frac{1}{2} \frac{|x|}{1 + x^2} \frac{e^{-\pi|x|/2}}{\cosh(\pi|x|/2)} \right] dx \right);$$

where in the bracket [...] there is the density of the finite Khintchine measure $m_T$ from the formula (1).

**Corollary 4.** The free-probability analog $\tilde{\phi}_T$ of hyperbolic tangent characteristic function has the following Voiculescu transform

$$V_{\tilde{\phi}_T}(it) = it \left[ \ln \left( \frac{t}{2} \right) - \beta \psi \left( \frac{t}{2} \right) - \psi \left( \frac{t}{4} + \frac{1}{2} \right) \right] = it \left[ \ln \left( \frac{t}{4} \right) - \psi \left( \frac{t}{4} + \frac{1}{2} \right) \right], \quad t > 0.$$ (30)

Consequently, we get the identity for Euler’s function

$$2 \psi(2s) - \psi(s) - \psi(s + 1/2) = 2 \ln 2 \quad s > 0.$$ (31)
Proof. (First proof.)

From the equality \(\phi_C(t) = \phi_S(t) \cdot \phi_T(t)\), Theorem 1, (A), and Corollaries 2 and 3 we get

\[
V_{\phi_T}(it) = it^2 \int_0^\infty \left[ \log \phi_C(t) - \log \phi_S(t) \right] e^{-ts} ds = V_{\phi_C}(it) - V_{\phi_S}(it)
\]

\[
i[t - t \beta(t/2)] - i[t \psi(t/2) - t \log(t/2) + 1] = it[\ln(t/2) - \beta(t/2) - \psi(t/2)],
\]

which gives the first equality in (30).

(Second proof.)

This time we need the formula 3.415(3) in [6], that is,

\[
\int_0^\infty \frac{x}{(x^2 + \beta^2)(e^{\mu x} + 1)} dx = \frac{1}{2} \left[ \psi\left(\frac{\beta \mu}{2 \pi} + \frac{1}{2}\right) - \ln\left(\frac{\beta \mu}{2 \pi}\right) \right], \quad \Re \beta > 0, \quad \Re \mu > 0.
\]

Since \(1 - \tanh x = \frac{e^{-x}}{\cosh x} = \frac{2}{e^{x} + 1}\), using the above and Theorem 1, equality (Z), we have

\[
V_{\phi_T}(it) = -it \int \frac{1 + x^2}{t^2 + x^2} \frac{|x|}{2} \frac{e^{-\pi|x|/4}}{\cosh(\pi|x|/4)} dx
\]

\[
= -2it \int_0^\infty \frac{x}{(t^2 + x^2)(e^{\pi x/2} + 1)} dx = -2it \frac{1}{2} \left[ \psi\left(\frac{t \pi/2}{2 \pi} + \frac{1}{2}\right) - \ln\left(\frac{t \pi/2}{2 \pi}\right) \right]
\]

\[
= it \left[ \ln\left(\frac{t}{4}\right) - \psi\left(\frac{t}{4} + \frac{1}{2}\right) \right],
\]

that is the second equality in (30). Consequently, by Theorem 1,

\[
\psi(2s) - 1/2\psi(s) - 1/2\psi(s + 1/2) = \ln 2 \quad s > 0.
\]

which completes the proof.

Remark 6. (a) Note that the above identity (33) concides with the formula 8.365(6) in [6], for \(n = 2\), in [6]. (cf. also Section 3.0, formula (iii)).

(b) By reasoning as in the seconds proofs of Corollaries 2 and 3, and using (32), we get for \(t > 0\)

\[
\int_0^\infty \frac{x}{t^2 + x^2}(1 - \tanh(\pi x)) dx = \psi(2t) + \beta(2t) - \log(2t) = \psi(t + 1/2) - \log t.
\]

(c) In particular, \(\int_0^\infty \frac{x}{1 + x^2}(1 - \tanh(\pi x)) dx = \psi(3/2)\).
4. FREE - PROBABILITY ANALOGUES OF BACKGROUND DRIVING FUNCTIONALS OF HYPERBOLIC DISTRIBUTIONS.

Since three hyperbolic characteristic functions \( \phi_C, \phi_S \) and \( \phi_T \) (of the random variables \( C, S \) and \( T \)) are selfdecomposable (in other words, in Lévy class \( \mathbb{L} \)) therefore they admit infinitely divisible background driving characteristic functions (BDCF) \( \psi_C, \psi_S \) and \( \psi_T \), respectively. Further, if \( N_C, N_S \) and \( N_T \) are their Lévy spectral measures then

\[
\psi_C(t) = \exp[-t \tanh t]; \quad N_C(dx) = \frac{\pi}{4} \frac{\cosh((\pi x)/2)}{\sinh^2((\pi x)/2)} dx \text{ on } \mathbb{R} \setminus \{0\}; \quad (34)
\]

\[
\psi_S(t) = \exp[1 - t \coth t]; \quad N_S(dx) = \frac{\pi}{4} \frac{1}{\sinh^2((\pi x)/2)} dx \text{ on } \mathbb{R} \setminus \{0\} \quad (35)
\]

\[
\psi_T(t) = \exp \left[ \frac{2t}{\sinh(2t)} - 1 \right]; \quad N_T(dx) = \frac{\pi}{8} \frac{1}{\cosh^2(\pi x/4)} dx \text{ on } \mathbb{R} \setminus \{0\}; \quad (36)
\]

Jurek-Yor (2004), cf. [16].

Remark 7. (a) Note that \( \psi_T \) is a characteristic function of a compound Poisson distribution.

(b) Elementary calculations give \( \psi_C/\psi_S = \psi_T \).

Let \( \tilde{\psi}_C, \tilde{\psi}_S \) and \( \tilde{\psi}_T \) be the free-probability analogues of BDCF for \( \psi_C, \psi_S, \psi_T \), respectively. Note that in those cases we have three possible ways of finding them: two because of the levels A and Z from Theorem 1 and, if possible, the third one by the differential equation (16) in Theorem 2.

Corollary 5. Let \( \tilde{\psi}_C, \tilde{\psi}_S \) and \( \tilde{\psi}_T \) be the free-analogues the corresponding BDCF. Then their Voiculescu transforms are as follows:

(a) \( V_{\tilde{\psi}_C}(it) = i \left[ t^2/2 \zeta(2, t/2) - t^2/4 \zeta(2, t/4) + 1 \right] \quad (37) \)

(b) \( V_{\tilde{\psi}_S}(it) = i \left[ 1 + t - (1/2) t^2 \zeta(2, t/2) \right]. \quad (38) \)

(c) \( V_{\tilde{\psi}_T}(it) = it[t \zeta(2, t/2) - t/4 \zeta(2, t/4) - 1] = it[t/4 \zeta(2, t+2)/4) - 1] \quad (39) \)

Proof. (a) From Theorem 1, using the Mathematica and formulas (v) ad (vi) from Section 3.0 we get

\[
V_{\tilde{\psi}_C}(it) = -it^2 \int_0^\infty v \tanh ve^{-tv} dv = -it^2 \left[ \frac{1}{8} \zeta(2, \frac{t}{4} + 1) - \frac{1}{8} \zeta(2, \frac{t}{4} + \frac{1}{2}) + \frac{1}{t^2} \right]
\]

\[
= -it^2 \left[ 1/4 \zeta(2, t/4) - 1/2 \zeta(2, t/2) - 1/t^2 \right] = it^2/2 \zeta(2, t/2) - 1/4t^2 \zeta(2, t/4) + 1. \]

(Note that equality (a), by a different way (Theorem 2), was already computed in Illustration 2.3, formula (18).)
(b) From the equality 3.551(3) in [6]:
\[
\int_0^\infty x^{\mu-1}e^{-\beta x} \coth x \, dx = \Gamma(\mu) \left[ 2^{1-\mu} \zeta(\mu, \beta/2) - \beta^{-\mu} \right], \Re\mu > 1, \Re\beta > 0;
\]
Putting \( \mu = 2 \) and \( \beta = t \) we get
\[
V_{\psi_S}(it) = it^2 \int_0^\infty (1 - v \coth v)e^{-vt} \, dv = it^2[t^{-1} - \int_0^\infty v \coth ve^{-vt} \, dv]
\]
\[
= it^2[t^{-1} - \frac{1}{2} \zeta(2, \frac{t}{2}) + \frac{1}{t^2}] = it[1 - \frac{1}{2}t^2(\zeta(2; \frac{t}{2}) + 1) - i[1 + t - \frac{1}{2}t^2\zeta(2; \frac{t}{2})];
\]
that gives (b).

(c) By Remark 7(b), \( \log \psi_T = \log \psi_C - \log \psi_S \). Thus using (a) and (b) we get
\[
V_{\psi_T}(it) = -it^2 \int_0^\infty [\log \psi_C(v) - \log \psi_S(v)]ve^{-tv} \, dv
\]
\[
i[t^2/2 \zeta(2, t/2) - (t^2/4) \zeta(2, t/4) + 1] - i[1 + t - (1/2) t^2\zeta(2, t/2)]
\]
\[
= it^2\zeta(2, t/2) - t^2/4\zeta(2, t/4) - t = it[\zeta(2, t/2) - t/4\zeta(2, t/4) - 1]
\]
\[
= it\left[\frac{t}{4} \zeta(2, \frac{t + 2}{4}) - 1\right],
\]
which gives (c).

(An alternative proof is possible by using the differential equation from Theorem 2 and the formulas in Corollary 5.)

5. FREE LAPLACE (or FREE DOUBLE EXPONENTIAL) MEASURE.

All the hyperbolic characteristic functions \( \phi_C, \phi_S, \phi_T \), discussed in the previous sections, are infinite products of the Laplace (called also double exponential) distributions; Jurek (1996), cf. [9]. Thus we include it in this paper as well.

5.1. Recall that that the double exponential (2e) (or Laplace) distribution has the probability density \( f(x) := 2^{-1}e^{-|x|}, \quad x \in \mathbb{R} \) and the characteristic function
\[
\phi_{2e}(t) = \frac{1}{1 + t^2} = \exp \int_{\mathbb{R}} (\cos tx - 1) \frac{1 + x^2}{1 + x^2} \left[ \frac{e^{-|x|}}{|x|} \right] dx
\]
\[
= \exp \int_{\mathbb{R} \setminus \{0\}} (\cos tx - 1) \left[ \frac{e^{-|x|}}{|x|} \right] dx, \quad (40)
\]
where in the first square bracket $[...]$ there is the finite Khintchine spectral
measure $m_2$ and in the second one there is the Lévy spectral measure $M_2$.

**Corollary 6.** The free analogue $\tilde{\phi}_{2e}$ of the double exponential distribution
has the following Voiculescu transform

$$V_{\tilde{\phi}_{2e}}(it) = 2it \left[ ci(t) \cos t + si(t) \sin t \right] = -2it \int_0^\infty \frac{\cos w}{w + t} dw, \quad t > 0.$$  

**Proof.** (First the argument via the equality (A) in Theorem 1.)

For $\Re \beta > 0$ and $\Re \xi > 0$ we have the identity

$$\int_0^\infty e^{-tx} \ln(\beta^2 + x^2) dx = \frac{2}{\xi} \ln \beta - ci(\beta \xi) \cos(\beta \xi) - si(\beta \xi) \sin(\beta \xi)],$$

where $si(x) := -\int_x^\infty \frac{\sin t}{t} dt$; $ci(x) := -\int_x^\infty \frac{\cos t}{t} dt$, are the integral-sine and integral-cosine functions, respectively; cf. 4.338(1) in [6] or Section 0, formula (x).

Therefore

$$V_{\tilde{\phi}_{2e}}(it) = it^2 \int_0^\infty \frac{\log \phi_{2e}(v)e^{-tv} dv}{e^{1+v^2}e^{-tv} dv} \text{ (by 4.338(1))}$$

$$-it^2 \left[ 2t^{-1}(-ci(t) \cos t - si(t) \sin t) \right] = -2it(\cos t \int_t^\infty \frac{\cos x}{x} dx + \sin t \int_t^\infty \frac{\sin x}{x} dx)$$

$$= -2it \int_t^\infty \frac{\cos t \cos x + \sin t \sin x}{x} dx = -2it \int_t^\infty \frac{\cos(x-t)}{x} dx \quad (w := x-t)$$

$$= -2it \int_0^\infty \frac{\cos w}{w+t} dw, \quad t > 0. \quad (41)$$

(Second argument via formula (Z) in Theorem 1)

By second part of Theorem 1 and by 3.354(2) in [6] we get

$$V_{\tilde{\phi}_{2e}}(it) = -it \int_\mathbb{R} \frac{1 + x^2}{t^2 + x^2} \left[ \frac{|x|}{1 + x^2} e^{-|x|} \right] dx = -2it \int_0^\infty \frac{x}{t^2 + x^2} e^{-x} dx =$$

$$= 2it \left[ ci(t) \cos t + si(t) \sin t \right] = -2it \int_0^\infty \frac{\cos w}{w+t} dw, \quad (42)$$

which coincides with previous calculations.

**5.2.** The Laplace (or double exponential) characteristic function $\phi_{2e} = (1 + t^2)^{-1}$ is selfdecomposable. Therefore it has the background driving characteristic function $\psi_{2e}$ related to $\phi_{2e}$ via (13). Hence

$$\psi_{2e}(t) = \exp(t \frac{\phi_{2e}(t)}{\phi_{2e}(t)}) = \exp(-\frac{2t^2}{1+t^2}) = \exp 2(\frac{1}{1+t^2} - 1) \text{ (compound Poisson)}$$

$$= \exp 2(\int_{\mathbb{R}} (e^{itx} - 1) \frac{1}{2} e^{-|x|} dx = \exp \int_{\mathbb{R}} (e^{itx} - 1 - \frac{itx}{1 + x^2}) e^{-|x|} dx; \quad (43)$$
from which we infer that $m_{\psi_{2e}}(dx) := \frac{x^2}{1+x^2}e^{-|x|}dx$ is its finite Khintchine measure in (1). Here is the Voiculescu transform of the free analog of $\psi_{2e}$.

**Corollary 7.** The free analog of BDCF $\psi_{2e}$ has Voiculescu transform $V_{\psi_{2e}}$ given as

$$V_{\psi_{2e}}(it) = 2it \left[ t(ci(t)\sin(t) - si(t)\cos(t)) - 1 \right],$$  \hspace{1cm} (44)

where $si$ and $ci$ are the integral sine and cosine functions.

**Proof.** (First proof.) From Corollary 6, $V_{\phi_{2e}}(it) = 2it\alpha(t)$ where $\alpha(t) := ci(t)\cos t + si(t)\sin t$. Then Theorem 2 gives that

$$V_{\psi_{2e}}(it) = V_{\phi_{2e}}(it) - t\frac{d}{dt}[V_{\phi_{2e}}(it)] = 2it\alpha(t) - t(2i\alpha(t) + 2it\alpha'(t)) = -2it^2\alpha'(t) = -2it^2(t^{-1} - ((ci(t)\sin(t) - si(t)\cos(t)))) = 2it[t(ci(t)\sin(t) - si(t)\cos(t)) - 1],$$

that completes the reasoning for the formula (44).

(Second proof.) Here we will use the equality

$$\int_0^{\infty} \frac{e^{-\xi x}}{\beta^2 + x^2} dx = \frac{1}{\beta}[ci(\xi\beta)\sin(\xi\beta) - si(\xi\beta)\cos(\xi\beta)], \hspace{1cm} \Re\xi > 0, \hspace{1cm} \Re\beta > 0;$$

cf. 3.354 (1) in [6] or the identity used in the first proof of Corollary 6. Hence and from Theorem 1 (level A) and (43)

$$V_{\psi_{2e}}(it) = it^2\int_0^{\infty} \log \psi_{2e}(s)e^{-ts} ds = 2it^2\int_0^{\infty} \left[ \frac{1}{1+s^2} - 1 \right] e^{-st} ds$$

$$= 2it^2 \left[ |ci(t)\sin(t) - si(t)\cos(t)| - t^{-1} \right] = 2it \left[ t(ci(t)\sin(t) - si(t)\cos(t)) - 1 \right],$$

which coincides with (44).

(Third proof.) Now we use the level Z from Theorem 1 and the finite Khintchine measures $m_\psi(dx) = \frac{x^2}{1+x^2}e^{-|x|}dx$ from representation (43). Thus

$$V_{\psi_{2e}}(it) = -it\int_\mathbb{R} \frac{x^2}{t^2 + x^2}e^{-|x|}dx = -it\int_\mathbb{R} (1 - \frac{t^2}{t^2 + x^2})e^{-|x|}dx$$

$$= 2it \left( t^2 \int_0^{\infty} \frac{1}{t^2 + x^2}e^{-x} dx - 1 \right) = 2it \left[ t\int_1^\infty \left( ci(t)\sin(t) - si(t)\cos(t) \right) - 1 \right],$$

that completes the last argument in the proof of Corollary 7.

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