UNIVERSAL CURVATURE IDENTITIES II

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Abstract. We show that any universal curvature identity which holds in the Riemannian setting extends naturally to the pseudo-Riemannian setting. Thus the Euh-Park-Sekigawa identity also holds for pseudo-Riemannian manifolds. We study the Euler-Lagrange equations associated to the Chern-Gauss-Bonnet formula and show that as in the Riemannian setting, they are given solely in terms of curvature (and not in terms of covariant derivatives of curvature) even in the pseudo-Riemannian setting.

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1. Introduction

The study of scalar and symmetric 2-tensor valued invariants of a metric is central in modern differential geometry. It also plays an important role in mathematical physics. The scalar curvature is the simplest such invariant and plays a central role not only in the Riemannian geometry [8, 23]. It also is important in the higher signature setting [2, 15, 22, 28]. The norm of the Weyl conformal curvature tensor $|W|^2$ appears in many settings, see for example the discussion in [7, 14, 16]. Turning to symmetric 2-tensor valued invariants, the trace free Ricci tensor is important [12] as is the Ricci tensor not only in the positive definite [24] but also the indefinite settings [3, 4]; see also [26] where the Weyl conformal tensor plays a crucial role. The Pfaffian (Gauss-Bonnet curvature) is a complicated invariant of the curvature tensor that defines the Einstein-Hilbert-Lovelock functional [21] and is important in Kazdan-Warner type identities [13]. It is also related to the Lipschitz-Killing curvature [25].

Motivated by these examples (and many more), we have decided to undertake a systematic study of scalar and symmetric 2-tensor valued invariants from an abstract point of view not only in the Riemannian but also the higher signature setting. We first proceed in the purely algebraic setting. A pair $(V, \varepsilon)$ is called an inner product space if $V$ is real vector space of dimension $m$ and if $\varepsilon$ is a non-degenerate inner product of signature $(p, q)$ on $V$ where $p + q = m$. An algebraic curvature tensor $A$ is an element of $\otimes^4(V^*)$ satisfying the relations of the Riemann curvature tensor, namely for all $x, y, z, w \in V$ one has the following relations:

\[ A(x, y, z, w) = -A(y, x, z, w) = A(z, w, x, y), \]
\[ A(x, y, z, w) + A(y, z, x, w) + A(z, x, y, w) = 0. \] (1.a)

Let $\mathfrak{A}(V) \subset \otimes^4(V^*)$ be the linear subspace of all such tensors. A triple $(V, \varepsilon, A)$ is said to be a curvature model if $(V, \varepsilon)$ is an inner product space and if $A \in \mathfrak{A}(V)$.

1.1. Geometric Realizations. We say that a curvature model $(V, \varepsilon, A)$ is geometrically realized at a point $\xi$ of a pseudo-Riemannian manifold $(M, g)$ if there exists an isomorphism $\Phi$ from $T_\xi M$ to $V$ so that $\Phi^*\varepsilon = g_\xi$ is the metric at $\xi$ and so that $\Phi^*A = R_\xi$ is the associated Riemann curvature tensor of the Levi-Civita connection at $\xi$. A useful result in the field, which we shall prove in Section 2 in the interests of completeness, is the following result:
Theorem 1.1. Every curvature model can be geometrically realized at some point of some compact pseudo-Riemannian manifold.

Theorem 1.1 shows that the symmetries of Equation (1.a) generate the universal curvature symmetries of the Riemann curvature tensor; there are no hidden additional symmetries. It lets us pass freely between the algebraic context and the geometric setting.

1.2. Scalar invariants. The orthogonal group $O(V, \varepsilon)$ acts on the space of algebraic curvature tensors $\mathfrak{A}(V)$ by pullback where one sets:

$$(T^*A)(v_1, v_2, v_3, v_4) := A(Tv_1, Tv_2, Tv_3, Tv_4).$$

If $P(A)$ is a polynomial of degree $\ell$ in the components of $A$ relative to some basis for $V$, then we shall say that $P$ is a scalar invariant if $P(T^*A) = P(A)$ for all $A \in \mathfrak{A}$ and for all $T \in O(V, \varepsilon)$. Let $O(V, \varepsilon)$ act trivially on $\mathbb{R}$. If we polarize $P$, we can regard, equivalently, $P$ as a linear invariant of $\otimes^\ell(\mathfrak{A}(V))$, i.e. as an equivariant linear map from $\otimes^\ell(\mathfrak{A}(V))$ to $\mathbb{R}$. We let $\mathcal{J}_\mathfrak{A}(V, \varepsilon)$ be the vector space of all such maps; these maps are homogeneous of degree $2\ell$ in the derivatives of the metric and this is a convenient indexing convention to use as there will be maps of odd order in the derivatives of the metric when we study manifolds with boundary presently.

Since any two inner product spaces of the same signature $(p, q)$ are isomorphic, we shall set $\mathcal{J}_\mathfrak{A}(p, q) := \mathcal{J}_\mathfrak{A}(V, \varepsilon)$ for any inner product space $(V, \varepsilon)$ of signature $(p, q)$. We shall see presently in Remark 1.1 that there is a natural way to identify $\mathcal{J}_\mathfrak{A}(p, q)$ with $\mathcal{J}_\mathfrak{A}(p_1, q_1)$ if $p_1 + q_1 = p + q$; this common space of invariants will then be denoted by $\mathcal{J}_\mathfrak{A}(m)$ since only the underlying dimension $m = p + q$ is normative.

Suppose that $P \in \mathcal{J}_\mathfrak{A}(p, q)$. Let $(V, \varepsilon)$ have signature $(p, q)$, let $(M, g)$ be a pseudo-Riemannian manifold of signature $(p, q)$, and let $R$ be the Riemann curvature tensor of the Levi-Civita connection on $(M, g)$. Given $\xi$ in $M$, we can find an isometry $\Phi$ which identifies $(T_\xi M, g_\xi)$ with $(V, \varepsilon)$. We then define $P(R_\xi) := P(\Phi^* R_\xi)$; the particular isometry $\Phi$ which is chosen being irrelevant as $P$ is invariant under the action of the orthogonal group. Thus elements of $\mathcal{J}_\mathfrak{A}$ give rise to geometric invariants; conversely, Theorem 1.1 lets us extend invariants from the geometric to the algebraic setting and identify the two contexts.

Let $\varepsilon^*$ denote the dual inner product on $V^*$; $\varepsilon^*$ is a linear map from $V^* \otimes V^*$ to $\mathbb{R}$ and, more generally, $\otimes^2(\varepsilon^*)$ is a linear map from $\otimes^2(V^*)$ to $\mathbb{R}$ which is invariant under the action of the orthogonal group. Let $\text{Perm}(4\ell)$ be the group of permutations of $4\ell$ elements and let $\sigma \in \text{Perm}(4\ell)$. We let $\sigma$ act on $\otimes^2(V^*)$ by permuting the factors and let $Q_{2\ell, \sigma, \varepsilon} := \{\otimes^2(\varepsilon^*)\} \circ \sigma$. This is clearly invariant under the orthogonal group and the restriction of $Q_{2\ell, \sigma, \varepsilon}$ to $\otimes^2(\mathfrak{A}(V))$ defines an element of $\mathcal{J}_\mathfrak{A}(V, \varepsilon)$ which we shall denote by $P_{2\ell, \sigma, \varepsilon}$.

There is a convenient formalism for describing the invariants $P_{2\ell, \sigma, \varepsilon}$. Choose a basis $\{e_i\}$ for $V$; let $\{e^i\}$ be the corresponding dual basis for $V^*$. Let $\varepsilon_{ij} := \varepsilon(e_i, e_j)$ and let $\varepsilon^{ij} = \varepsilon^*(e^i, e^j)$ denote the components of $\varepsilon$ and of $\varepsilon^*$ relative to this basis, respectively; $\varepsilon^{ij}$ is the inverse of the matrix $\varepsilon_{ij}$. If $x$ and $y$ are vectors (or co-vectors), we let $x \circ y := \frac{1}{2}(x \otimes y + y \otimes x)$ be the symmetric tensor product of $x$ with $y$. Adopt the Einstein convention and sum over repeated indices to express:

$$\varepsilon = \varepsilon_{ij} e^i \circ e^j \quad \text{and} \quad \varepsilon^* = \varepsilon^{ij} e_i \circ e_j.$$ 

If $A \in \otimes^2(V^*)$, let $A_{ijk} := A(e_i, e_j, e_k, e_l)$ give the components of $A$ relative to the given basis. We then have that:

$$A = A_{ijk} e^i \otimes e^j \otimes e^k \otimes e^l.$$ 

Let $\sigma \in \text{Perm}(4\ell)$. Set $\sigma_i := \sigma(i)$ for $1 \leq i \leq 4\ell$. We may now express:

$$P_{2\ell, \sigma, \varepsilon}(A) = \varepsilon_1^{11} \cdots \varepsilon_1^{4\ell-4\ell} A_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} \cdots A_{\sigma_{4\ell-3} \sigma_{4\ell-2} \sigma_{4\ell-1} \sigma_{4\ell}}. \quad (1.b)$$
If we let $\beta = \sigma^{-1}$, then we may also express this invariant in the form:
\[
P_{2\ell,\sigma,\varepsilon}(A) = \varepsilon^{i_1 i_2} \ldots \varepsilon^{i_{4\ell-2} i_{4\ell-1} i_{4\ell}} A_{i_1 i_2 i_3 i_4} \ldots A_{i_{4\ell-3} i_{4\ell-2} i_{4\ell-1} i_{4\ell}}. \tag{1.c}
\]

The discussion above shows the value of Equation (1.b) is independent of the particular basis which was chosen. The usual scalar invariants of Riemannian geometry can be expressed in this notation. The scalar curvature $\tau$ is given by setting
\[
\tau(A) := \varepsilon^{i_1 i_2} \varepsilon^{i_3 i_4} A_{i_1 i_2 i_3 i_4}.
\]

Thus $\tau = P_{2\ell,\sigma,\varepsilon}$ where $\sigma_1 = 1$, $\sigma_2 = 3$, $\sigma_3 = 4$, and $\sigma_4 = 2$, i.e.
\[
\sigma = \begin{pmatrix} 1234 \\ 1342 \end{pmatrix}.
\]

Note that the permutation defining $\tau$ is not unique as the scalar curvature can also be a collection of real constants. We say that $\tau$ is a universal curvature identity on $(V, \varepsilon)$ if $P_{2\ell,\sigma,\varepsilon}(A) = 0$ for all $A \in \mathfrak{A}(V)$. Some identities hold for all dimensions. For example, the discussion given above shows
\[
0 = \varepsilon^{i_1 i_2} \varepsilon^{i_3 i_4} A_{i_2 i_3 i_4 i_1} - \varepsilon^{i_1 i_2} \varepsilon^{i_3 i_4} A_{i_1 i_2 i_3 i_4}.
\]

Here
\[
c_\sigma = \begin{cases} +1 & \text{if } \sigma = \begin{pmatrix} 1234 \\ 2431 \end{pmatrix} \\ -1 & \text{if } \sigma = \begin{pmatrix} 1234 \\ 1342 \end{pmatrix} \\ 0 & \text{otherwise} \end{cases}.
\]

Other identities are dimension specific. Let $\rho$ denote the Ricci tensor. For example, if $m = 3$, then we have the following identity (see Remark 1.2 for further details):
\[
\tau(A)^2 - 4|\rho(A)|^2 + |A|^2 = 0. \tag{1.d}
\]

Since $\tau(A)^2$, $|\rho(A)|^2$, and $|A|^2$ are scalar invariants, this relation can be expressed in the form $P_{2\ell,\sigma,\varepsilon}$ as follows. We have
\[
\tau(A)^2 = \varepsilon^{i_1 i_2} \varepsilon^{i_3 i_4} \varepsilon^{i_5 i_6} \varepsilon^{i_7 i_8} A_{i_1 i_2 i_3 i_4} A_{i_5 i_6 i_7 i_8},
\]
\[
|\rho(A)|^2 = \varepsilon^{i_1 i_2} \varepsilon^{i_3 i_4} \varepsilon^{i_5 i_6} \varepsilon^{i_7 i_8} A_{i_1 i_2 i_3 i_4} A_{i_5 i_6 i_7 i_8},
\]
\[
|A|^2 = \varepsilon^{i_1 i_2} \varepsilon^{i_3 i_4} \varepsilon^{i_5 i_6} \varepsilon^{i_7 i_8} A_{i_1 i_2 i_3 i_4} A_{i_5 i_6 i_7 i_8}.
\]
Thus the relation in Equation (1.d) can be defined by taking

\[ c_\sigma = \begin{cases} 
1 & \text{if } \sigma = \left( \begin{array}{c} 12345678 \\ 13425786 \end{array} \right) \\
-4 & \text{if } \sigma = \left( \begin{array}{c} 12345678 \\ 13452786 \end{array} \right) \\
1 & \text{if } \sigma = \left( \begin{array}{c} 12345678 \\ 13572468 \end{array} \right) \\
0 & \text{otherwise}
\end{cases}. \]

We note that this relation does not hold in dimension \( m = 4 \); fixing the underlying dimension of the vector space can be crucial.

1.4. Changing the signature. The identity of Equation (1.d) is not specific to the signature; it holds for any 3-dimensional pseudo-Riemannian manifold or curvature module – i.e. in signatures (0, 3), (1, 2), (2, 1), and (3, 0). More generally, the signature plays no role when considering universal curvature identities. We will establish the following result subsequently in Section 4:

**Theorem 1.3.** Let \( C := \{ c_\sigma \}_{\sigma \in \text{Perm}(4\ell)} \) be a collection of real constants. Let \((V_1, \varepsilon_1)\) be inner product spaces of signature \((p_1, q_1)\) where \( m = p_1 + q_1 = p_2 + q_2 \). Then

\[ P_{2\ell, C, \varepsilon_1}(A_1) = 0 \ \forall \ A_1 \in \mathfrak{A}(V_1) \iff P_{2\ell, C, \varepsilon_2}(A_2) = 0 \ \forall \ A_2 \in \mathfrak{A}(V_2). \]

**Remark 1.1.** Let \( P_{2\ell, \varepsilon} \in J_{2\ell}(V, \varepsilon) \) be a polynomial invariant of degree \( 2\ell \) which is defined for an inner product space \((V, \varepsilon)\) of signature \((p, q)\) in dimension \( m = p + q \). By Theorem 1.2, there is a collection of constants \( C \) so that may express:

\[ P_{2\ell, \varepsilon}(A) = \sum_{\sigma \in \text{Perm}(4\ell)} c_\sigma P_{2\ell, \sigma, \varepsilon}(A) \ \forall \ A \in \mathfrak{A}(V). \]

If \((V_1, \varepsilon_1)\) is an inner product space of signature \((p_1, q_1)\) where \( p_1 + q_1 = m \), we let

\[ P_{2\ell, \varepsilon_1}(A_1) := \sum_{\sigma \in \text{Perm}(4\ell)} c_\sigma P_{2\ell, \sigma, \varepsilon_1}(A_1) \ \forall \ A_1 \in \mathfrak{A}(V_1). \]

If we choose another collection of constants \( \tilde{C} \) so that

\[ P_{2\ell, \varepsilon}(A) = \sum_{\sigma \in \text{Perm}(4\ell)} \tilde{c}_\sigma P_{2\ell, \sigma, \varepsilon}(A) \ \forall \ A \in \mathfrak{A}(V), \]

then

\[ \sum_{\sigma \in \text{Perm}(4\ell)} (c_\sigma - \tilde{c}_\sigma) P_{2\ell, \sigma, \varepsilon}(A) = 0 \ \forall \ A \in \mathfrak{A}(V). \]

Hence, by Theorem 1.3,

\[ \sum_{\sigma \in \text{Perm}(4\ell)} (c_\sigma - \tilde{c}_\sigma) P_{2\ell, \sigma, \varepsilon_1}(A_1) = 0 \ \forall \ A_1 \in \mathfrak{A}(V_1). \]

Thus we may conclude that we may also express

\[ P_{2\ell, \varepsilon_1}(A_1) = \sum_{\sigma \in \text{Perm}(4\ell)} \tilde{c}_\sigma P_{2\ell, \sigma, \varepsilon_1}(A_1) \ \forall \ A_1 \in \mathfrak{A}(V_1). \]

This shows that we can regard the collection \( \{ P_{2\ell, \varepsilon} \} \) as being defined for any inner product space of dimension \( m \); we shall denote this space of invariants by \( J_{2\ell}(m) \). The elements of \( J_{2\ell}(m) \) are functions; the elements \( P_{2\ell, \sigma, \varepsilon} \) are algebraic objects which define functions, but (as noted above) different elements \( P_{2\ell, \sigma, \varepsilon} \) can define the same function.
1.5. The restriction of scalar invariants. We wish to relate the spaces \( \mathcal{J}_{2\ell}(m) \) and \( \mathcal{J}_{2\ell}(m-1) \) by defining a restriction map \( r: \mathcal{J}_{2\ell}(m) \to \mathcal{J}_{2\ell}(m-1) \). We work in the geometric context for the moment; Theorem 1.1 permits us to then pass to the algebraic context. Let \( P \in \mathcal{J}_{2\ell}(p,q) \) where \( p > 0 \) and let \( (N,g_N) \) be a pseudo-Riemannian manifold of signature \((p-1,q)\). Let \( M := N \times S^1 \) and let \( g_M := g_N - d\theta^2 \) where \( \theta \) is the usual periodic parameter on the circle \( S^1 \). Then \((M,g_M)\) has signature \((p,q)\) and we set
\[
r(P)(N,g_N)(\xi) := P(M,g_M)(\xi,\theta_0)
\]
for any \( \theta_0 \in S^1 \), the particular point \( \theta_0 \) being irrelevant as the circle is a homogeneous space. This defines a map
\[
r_+: \mathcal{J}_{2\ell}(p,q) \to \mathcal{J}_{2\ell}(p,q-1) \quad \text{for} \quad q > 0.
\]
If \( q > 0 \) and if \( (N,g_N) \) has signature \((p,q-1)\), we may consider \( g_M := g_N + d\theta^2 \) to define a similar restriction map
\[
r_-: \mathcal{J}_{2\ell}(p,q) \to \mathcal{J}_{2\ell}(p,q-1) \quad \text{for} \quad q > 0.
\]
Let \( \varepsilon(p,q) \) have signature \((p,q)\). The invariants \( P_{2\ell,\sigma,\varepsilon(p,q)} \) are defined by summations in Equation (1.1) that range from 1 to \( m \). The product metric \( g_N \pm d\theta^2 \) is flat in the final coordinate and thus \( r_{\pm} \) is defined by summations which range from 1 to \( m-1 \). Consequently
\[
\begin{align*}
  r_-(P_{2\ell,\sigma,\varepsilon(p,q)}(\xi)) & = P_{2\ell,\sigma,\varepsilon(p,q-1)}(\xi) \quad \text{if} \quad p > 0, \\
r_+(P_{2\ell,\sigma,\varepsilon(p,q)}(\xi)) & = P_{2\ell,\sigma,\varepsilon(p,q-1)}(\xi) \quad \text{if} \quad q > 0.
\end{align*}
\]
Consequently, we can regard \( r_{\pm} \) as defining a unified and universally defined map
\[
r: \mathcal{J}_{2\ell}(m) \to \mathcal{J}_{2\ell}(m-1).
\]
Thus, for example, the scalar curvature in dimension \( m \) restricts naturally to the scalar curvature in dimension \( m-1 \); it is universally defined. It is necessary to first give a geometric definition and then invoke Theorem 1.1 to ensure that the subsequent algebraic characterization is well defined by showing
\[
\sum_{\sigma \in \text{Perm}(4\ell)} c_\sigma P_{2\ell,\sigma,\varepsilon_m}(A_m) = 0 \forall A_m \in \mathfrak{A}(V_m)
\]
\[
\Rightarrow \sum_{\sigma \in \text{Perm}(4\ell)} c_\sigma P_{2\ell,\sigma,\varepsilon_{m-1}}(A_{m-1}) = 0 \forall A_{m-1} \in \mathfrak{A}(V_{m-1})
\]
where \((V_m,\varepsilon_m)\) and \((V_{m-1},\varepsilon_{m-1})\) are arbitrary inner product spaces of dimensions \( m \) and \( m-1 \), respectively.

The following result follows from the discussion given above:

**Lemma 1.1.** Let \( \mathcal{C} = \{c_\sigma\}_{\sigma \in \text{Perm}(4\ell)} \) be a collection of real constants which defines an element \( P_{2\ell,\mathcal{C},m} \in \mathcal{J}_{2\ell}(m) \). Then \( r(P_{2\ell,\mathcal{C},m}) = P_{2\ell,\mathcal{C},m-1} \). Furthermore, if \( P_{2\ell,\mathcal{C},m} \) is a universal curvature identity in dimension \( m \), then \( r(P_{2\ell,\mathcal{C},m}) \) is a universal curvature identity in dimension \( m-1 \).

1.6. The Pfaffian. Define \( E_{2\ell,m,\varepsilon} \in \mathcal{J}_{2\ell}(V,\varepsilon) \) by setting:
\[
E_{2\ell,m,\varepsilon} := \frac{1}{(8\pi)^\ell \ell!} \sum_{i_1,...,i_{2\ell} = 1}^m A_{1i_1j_1}...A_{\ell i_{\ell-1}j_{\ell-1}} \varepsilon^*(e^{i_1} \wedge ... \wedge e^{i_{\ell-1}}) \varepsilon^*(e^{j_1} \wedge ... \wedge e^{j_{\ell-1}})
\]
where by definition one sets:
\[
\varepsilon^*(e^{i_1} \wedge ... \wedge e^{i_{\ell-1}}) := \det \begin{pmatrix} \varepsilon^*(e^{i_1}, e^{j_1}) & \cdots & \varepsilon^*(e^{i_1}, e^{j_{\ell-1}}) \\ \cdots & \cdots & \cdots \\ \varepsilon^*(e^{i_{\ell-1}}, e^{j_1}) & \cdots & \varepsilon^*(e^{i_{\ell-1}}, e^{j_{\ell-1}}) \end{pmatrix}.
\]
It is then immediate that
\[ r(E_{2\ell,m,\varepsilon}) = E_{2\ell,m-1,\varepsilon}. \]
Thus, in particular, this invariant is universal and will be denoted by \( E_{2\ell} \) when no confusion is likely to result. For example,
\[ E_2(A) = \frac{\tau(A)}{4\pi} \quad \text{and} \quad E_4(A) = \frac{\tau(A)^2 - 4|\rho(A)|^2 + |A|^2}{32\pi^2}. \]

**Theorem 1.4.**

1. \( r : \mathcal{J}_{2\ell}(m) \rightarrow \mathcal{J}_{2\ell}(m-1) \) is surjective for any \( m \).
2. If \( m > 2\ell \), then \( r : \mathcal{J}_{2\ell}(m) \rightarrow \mathcal{J}_{2\ell}(m-1) \) is injective.
3. If \( m = 2\ell \), then ker \( \{ r : \mathcal{J}_{2\ell}(m) \rightarrow \mathcal{J}_{2\ell}(m-1) \} = \text{Span}\{E_{2\ell,m}\} \).

**Proof.** The discussion above using Theorem 1.2 and Theorem 1.3 shows that it suffices to prove this result in the positive definite setting; this was done previously in [11]. \( \square \)

**Remark 1.2.** We have \( E_4(A) = \frac{1}{32\pi^2}(\tau(A)^2 - 4|\rho(A)|^2 + |A|^2); \) \( E_4 \) is non-zero on \((S^4, g_0)\) where \( g_0 \) is the round metric on the unit sphere \( S^4 \) in \( \mathbb{R}^5 \). This invariant vanishes identically on any 3-dimension pseudo-Riemannian manifold and thus provides the only quadratic universal curvature identity (modulo rescaling) in dimension 3 which does not hold in dimension 4.

1.7. **The Chern-Gauss-Bonnet Formula.** In addition to being (up to scaling) the only universal curvature identity in dimension \( 2\ell - 1 \) which is non-trivial in dimension \( 2\ell \), the invariants \( E_{2\ell} \) are the integrands of the Chern-Gauss-Bonnet formula. Let \( \chi(M) \) be the Euler-Poincaré characteristic of a compact manifold \( M \); since \( \chi(M) = 0 \) if \( m = \dim(M) \) is odd we may suppose that \( m = 2\ell \) is even. Let
\[ |\text{dvol}(g)| := |\det(g_{ij})|^{1/2} \ dx^1 \cdots dx^m \]
denote the Riemannian element of volume. We refer to [5] for the proof of the following result in the Riemannian (positive definite) setting and to [6] for the general case:

**Theorem 1.5.** Let \((M, g)\) be a compact pseudo-Riemannian manifold of dimension \( 2\ell \) with empty boundary. Then
\[ \int_M E_{2\ell}(M, g) \ |\text{dvol}(g)\| = \chi(M). \]

1.8. **Symmetric 2-tensor valued invariants.** Let \( S^2(V^*) \) denote the space of symmetric 2-cotensors. Let \( \mathcal{J}_{2\ell}^{(2)}(V, \varepsilon) \) denote the set of all \( O(V, \varepsilon) \) equivariant maps from \( \otimes^4(V) \) to \( S^2(V^*) \) or, equivalently, polynomials of degree \( 2\ell \) in the components of \( A \in \mathfrak{g}(V) \) which are \( S^2(V^*) \) valued and invariantly defined. We can extend Theorem 1.2, Theorem 1.3, and Theorem 1.4 to this setting. We adopt the following notational conventions. There are two fundamental invariants. If \( \eta \in \otimes^4(V^*) \), define \( Q^{(2)}_{2\ell,1,\varepsilon} \in \mathcal{J}_{2\ell}^{(2)}(V, \varepsilon) \) and \( Q^{(2)}_{2\ell,2,\varepsilon} \in \mathcal{J}_{2\ell}^{(2)}(V, \varepsilon) \) by setting:
\[ Q^{(2)}_{2\ell,1,\varepsilon}(\eta) := \varepsilon^{i_1i_2} \cdots \varepsilon^{i_{4\ell-3i_{4\ell-2}}i_{4\ell-1i_{4\ell}}e^{i_{4\ell-1}e^{i_{4\ell}}}, \]
\[ Q^{(2)}_{2\ell,2,\varepsilon}(\eta) := \varepsilon^{i_1i_2} \cdots \varepsilon^{i_{4\ell-3i_{4\ell-2}}i_{4\ell-1i_{4\ell}}e^{i_{4\ell-1}e^{i_{4\ell}}}. \]
If \( \sigma \in \text{Perm}(4\ell) \), then as before, we shall define
\[ Q^{(2)}_{2\ell,1,\sigma,\varepsilon} := Q^{(2)}_{2\ell,1,\varepsilon} \circ \sigma, \quad P^{(2)}_{2\ell,1,\sigma,\varepsilon} := Q^{(2)}_{2\ell,1,\sigma,\varepsilon}
\]
\[ \otimes^4(V), \]
\[ Q^{(2)}_{2\ell,2,\sigma,\varepsilon} := Q^{(2)}_{2\ell,2,\varepsilon} \circ \sigma, \quad P^{(2)}_{2\ell,1,\sigma,\varepsilon} := Q^{(2)}_{2\ell,1,\sigma,\varepsilon}
\]
\[ \otimes^4(V). \]
We will establish the following extension of Theorem 1.2 in Section 3:
Theorem 1.6. \( \mathcal{J}^{(2)}_{2\ell}(V, \varepsilon) = \text{Span}_{\sigma \in \text{Perm}(4\ell)} \left\{ P^{(2)}_{2\ell,1,\sigma,\varepsilon}, P^{(2)}_{2\ell,2,\sigma,\varepsilon} \right\} \).

If \( D := \{ d^{(2)}_{1,\sigma}, d^{(2)}_{2,\sigma} \}_{\sigma \in \text{Perm}(4\ell)} \) is a collection of real constants, we set

\[
P^{(2)}_{2\ell,D,\varepsilon} := \sum_{\sigma \in \text{Perm}(4\ell)} \left\{ d^{(2)}_{1,\sigma} P^{(2)}_{2\ell,1,\sigma,\varepsilon} + d^{(2)}_{2,\sigma} P^{(2)}_{2\ell,2,\sigma,\varepsilon} \right\}
\]

We shall establish the following extension of Theorem 1.3 in Section 4:

**Theorem 1.7.** Let \( D := \{ d^{(2)}_{1,\sigma}, d^{(2)}_{2,\sigma} \}_{\sigma \in \text{Perm}(4\ell)} \) be a collection of real constants. Let \((V_i, \varepsilon_i)\) be inner product spaces of the same dimension \( m = p_1 + q_1 = p_2 + q_2 \). Then

\[
P^{(2)}_{2\ell,D,\varepsilon_1}(A_1) = 0 \forall A_1 \in \mathfrak{A}(V_1) \iff P^{(2)}_{2\ell,D,\varepsilon_2}(A_2) = 0 \forall A_2 \in \mathfrak{A}(V_2).
\]

As in the scalar case, we use Theorem 1.6 and Theorem 1.7 to identify \( \mathcal{J}^{(2)}_{2\ell}(p, q) \) with \( \mathcal{J}^{(2)}_{2\ell}(p_1, q_1) \) if \( p + q = p_1 + q_1 = m \) and to define a universal space of invariants \( \mathcal{J}^{(2)}_{2\ell}(m) \) in dimension \( m \). The restriction maps

\[
r_{\pm}^{(2)} : \mathcal{J}^{(2)}_{2\ell}(p, q) \to \mathcal{J}^{(2)}_{2\ell}(p - 1, q) \quad \text{for} \quad p > 0,
\]

\[
r_{\pm}^{(2)} : \mathcal{J}^{(2)}_{2\ell}(p, q) \to \mathcal{J}^{(2)}_{2\ell}(p, q - 1) \quad \text{for} \quad q > 0,
\]

are defined geometrically as before by setting:

\[
\{ \iota_{\pm}^{(2)}(P_{2\ell}^{(2)}) \}(N, g_N)(\xi) = i_N^{(2)} \{ P_{2\ell}^{(2)}(N \times S^1, g_N \pm d\theta^2)(\xi, \theta_1) \}
\]

where \( i_N^{(2)} \) is the dual map induced by the inclusion \( N \to N \times \{ \theta_1 \} \subset N \times S^1 \). The additional bit of technical fuss defined in using \( i_N^{(2)} \) is required as it is necessary to restrict a symmetric 2-cotensor from \( N \times S^1 \) to \( N \) (this is not necessary for scalar valued invariants). The restriction map \( r_{\pm}^{(2)} \) is defined similarly if \( q > 0 \). As before, the restriction maps patch together to define a coherent map \( r^{(2)} \) from \( \mathcal{J}^{(2)}_{2\ell}(m) \) to \( \mathcal{J}^{(2)}_{2\ell}(m - 1) \) so that

\[
r_{\pm}^{(2)}(P^{(2)}_{2\ell,1,\sigma,\varepsilon}) = P^{(2)}_{2\ell,1,\sigma,\varepsilon(p-1,q)} \quad \text{if} \quad p > 0,
\]

\[
r_{\pm}^{(2)}(P^{(2)}_{2\ell,2,\sigma,\varepsilon}) = P^{(2)}_{2\ell,2,\sigma,\varepsilon(p,q-1)} \quad \text{if} \quad q > 0.
\]

The reason to give a geometric definition first, of course, was to ensure that the image of a universal curvature identity was again a universal curvature identity so that \( r^{(2)}(P_{2\ell}^{(2)}) \) was defined independent of the representation of \( P_{2\ell}^{(2)} \) in terms of the fundamental invariants \( \{ P_{2\ell,1,\sigma,\varepsilon}, P_{2\ell,2,\sigma,\varepsilon} \} \).

In analogy with the Pfaffian, we define \( T^{(2)}_{2\ell,m,\varepsilon} \in \mathcal{J}^{(2)}_{2\ell}(V, \varepsilon) \) by setting:

\[
T^{(2)}_{2\ell,m,\varepsilon} := \sum_{i_1, \ldots, i_{\ell+1}, j_1, \ldots, j_{\ell+1} = 1}^m A_{i_1i_2j_2j_1 \ldots A_{i_{\ell+1}j_{\ell+1}}} e^{i_1+1} \wedge \ldots \wedge e^{i_{\ell+1}} \wedge e^{j_1} \wedge \ldots \wedge e^{j_{\ell+1}}.
\]

We let \( T^{(2)}_{2\ell,m} \) denote the invariants \( \{ T^{(2)}_{2\ell,m,\varepsilon} \} \) in dimension \( m \). We then have that

\[
r^{(2)}(T^{(2)}_{2\ell,m}) = T^{(2)}_{2\ell,m-1}.
\]

Consequently, once again, these invariants are universal. Theorem 1.4 generalizes to this setting to become:

**Theorem 1.8.**

1. \( r^{(2)} : \mathcal{J}^{(2)}_{2\ell}(m) \to \mathcal{J}^{(2)}_{2\ell}(m - 1) \) is surjective for any \( m \).

2. If \( m > 2\ell + 1 \), then \( r^{(2)} : \mathcal{J}^{(2)}_{2\ell}(m) \to \mathcal{J}^{(2)}_{2\ell}(m - 1) \) is injective.
If \( m = 2\ell + 1 \), then \( \ker \{ e^{(2)} : \mathcal{J}^{(2)}_{2\ell} (m) \to \mathcal{J}^{(2)}_{2\ell} (m - 1) \} = \text{Span} \{ T^{(2)}_{2\ell,m} \} \).

**Proof.** The discussion above using Theorem 1.6 and Theorem 1.7 shows that it suffices to prove Theorem 1.8 in the positive definite context. This was done previously in [11].

**Remark 1.3.** If \( \ell = 2 \), we define \( Q^{(2)}_{4,\varepsilon,m} \in \mathcal{J}^{(2)}_4 (m) \) by setting:

\[
Q^{(2)}_{4,\varepsilon,m} (A) := -\frac{1}{4} \varepsilon^{i_1 i_2} \varepsilon^{i_3 i_4} \varepsilon^{i_5 i_6} \varepsilon^{i_7 i_8} \varepsilon^{i_9 i_{10}} \\
\times (A_{i_1 i_2 i_3 i_4} A_{i_5 i_7 i_8 i_9} - 4 A_{i_1 i_2 i_3 i_4} A_{i_5 i_7 i_8 i_9} + A_{i_1 i_2 i_3 i_4} A_{i_5 i_7 i_8 i_9}) e^{i_9} \circ e^{i_{10}} \\
+ \delta^{i_3}_{i_4} \delta^{i_5}_{i_6} \varepsilon^{i_7 i_8} \varepsilon^{i_9 i_{10}} \\
\times (A_{i_5 i_7 i_8 i_9} A_{i_3 i_4 i_5} - 2 A_{i_5 i_7 i_8 i_9} A_{i_3 i_4 i_5}) e^{i_2} \circ e^{i_4} \\
+ \delta^{i_3}_{i_4} \delta^{i_5}_{i_6} \varepsilon^{i_7 i_8} \varepsilon^{i_9 i_{10}} \\
\times (-2 A_{i_5 i_7 i_8 i_9} A_{i_1 i_2 i_3} + A_{i_5 i_7 i_8 i_9} A_{i_1 i_2 i_3}) e^{i_2} \circ e^{i_4}.
\]

Euh, Park, and Sekigawa [10] showed that if \( \varepsilon \) is positive definite and if \( m = 4 \), then

\[
Q^{(2)}_{4,\varepsilon,4} (A) = 0 \quad \forall \quad A \in \mathfrak{X}(V).
\]

Note that this does not hold true if \( m \geq 5 \) so this is a universal curvature identity which holds in dimension 4 but not in dimension 5. Theorem 1.7 shows this relation holds in signature \((0, 4), (1, 3), (2, 2), \) and \((1, 3)\) as well. By Theorem 1.8, there is a universal constant \( c \) so that \( Q^{(2)}_{4,\varepsilon,4} = c T^{(2)}_{4,4} \). We refer to [9] for an evaluation of the normalizing constants which arise when this invariant is expressed in terms of a Weyl basis using the methods of universal examples.

1.9. Euler-Lagrange Equations. Let \( h \) be an arbitrary symmetric 2-cotensor field on a compact pseudo-Riemannian manifold \((M, g)\) of dimension \( m \). We form the 1-parameter family \( g(\vartheta) := g + \vartheta h; \) this is non-degenerate for \( \vartheta \) small. Since the Pfaffian \( E_{2\ell} \) only involves the first and second derivatives of the metric, the variation only involves the first and second derivatives of \( h \). We may therefore express:

\[
\partial_{\vartheta} \left\{ E_{2\ell,m} (g(\vartheta)) \, |\text{dvol}| (g(\vartheta)) \right\} \bigg|_{\vartheta=0} = \left\{ Q^{ij}_{2\ell,m} h_{ij} + Q^{ijkl}_{2\ell,m} h_{ijk;l} + Q^{ijkl}_{2\ell,m} h_{ij;k;l} \right\} |\text{dvol}| (g)
\]

where \( h_{ij;k} \) and \( h_{ij;k;l} \) give the components of the covariant derivative of \( h \) with respect to the Levi-Civita connection of \( g \). Let \( Q^{ij}_{2\ell,m;ij} = \) and \( Q^{ijkl}_{2\ell,m;ijkl} \) be the components of the first and second covariant derivatives of these tensors, respectively. Define:

\[
S^{ij}_{2\ell,m} := Q^{ij}_{2\ell,m} - Q^{ij}_{2\ell,m;k} + Q^{ijkl}_{2\ell,m;ijkl}.
\]

The tensor \( S^{ij}_{2\ell,m} e_i \circ e_j \in S^2 (V) \) is characterized by the property that if \((M, g)\) is any compact pseudo-Riemannian manifold of dimension \( m \) with empty boundary, then we may integrate by parts to see that:

\[
\partial_{\vartheta} \left\{ \int_M E_{2\ell,m} (g(\vartheta)) \, |\text{dvol}| (g(\vartheta)) \right\} \bigg|_{\vartheta=0} = \int_M S^{ij}_{2\ell,m} h_{ij} \, |\text{dvol}| (g).
\]

We use the metric to raise and lower indices and to define the corresponding symmetric 2-cotensor \( S^{(2)}_{2\ell,m} \) by the identity:

\[
S^{ij}_{2\ell,m} h_{ij} = g(S^{(2)}_{2\ell,m}, h).
\]

By Theorem 1.5, this vanishes if \( m = 2\ell \). In Section 5, we shall establish a conjecture of Berger [1] in the pseudo-Riemannian setting:
Theorem 1.9. \( S^{(2)}_{2\ell,m} = c_{2\ell} T^{(2)}_{2\ell,m} \in \mathcal{J}^{(2)}_{2\ell,m} \) for any \((\ell, m)\).

A-priori, Equation (1.1) involves the second covariant derivatives of the curvature tensor. There are appropriate restriction maps in this category. As

\[ r^{(2)}_{\pm}(S^{(2)}_{2\ell,m}) = S^{(2)}_{2\ell,m-1}, \]

these invariants are universally defined. We will establish Theorem 1.9 by showing that in fact \( S^{(2)}_{2\ell,m} \) only depends on the curvature tensor and not on its covariant derivatives and thus \( S^{(2)}_{2\ell,m} \in \mathcal{J}^{(2)}_{2\ell,m} \). By Theorem 1.5, the Euler-Lagrange equations vanish if \( m = 2\ell \) and thus \( S^{(2)}_{2\ell,m} \) belongs to \( \ker(r_{\pm}) \). We may then use Theorem 1.8 to established the desired identity

\[ S^{(2)}_{2\ell,m} = c_{2\ell} T^{(2)}_{2\ell,m}. \]

In the Riemannian setting, this result is not new. It was first established by Kuz’mina [17] and subsequently established using different methods by Labbi [18, 19, 20]. The present paper was motivated in part by a desire to extend this result from the Riemannian setting to the pseudo-Riemannian setting.

2. The proof of Theorem 1.1

Let \((V, \varepsilon)\) be an inner product space and let \( A \in \mathfrak{A}(V) \). Fix a basis \( \{e_i\} \) for \( V \) to identify \( V \) with \( \mathbb{R}^m \). Define:

\[ g_{ik} := \varepsilon_{ik} - \frac{1}{3} A_{ijk} x^j x^l. \]

Clearly \( g_{ik} = g_{ki} \). As \( g_{ik}(0) = \varepsilon_{ik} \) is non-singular, \( g \) is a pseudo-Riemannian metric on some neighborhood of the origin in \( \mathbb{R}^m \). Let

\[ g_{ijkl} := \partial_{x_k} g_{ij} \quad \text{and} \quad g_{ijkl} := \partial_{x_k} \partial_{x_l} g_{ij}. \]

The Christoffel symbols of the first kind are given by:

\[ \Gamma_{ijk} := g(\nabla_{\partial_{x_i}} \partial_{x_j}, \partial_{x_k}) = \frac{1}{2} (g_{ik/j} + g_{jk/i} - g_{ij/k}). \]

As \( g = \varepsilon + O(|x|^2) \) and \( \Gamma = O(|x|) \), we may compute:

\[ R_{ijkl} = \{ \partial_{x_i} \Gamma_{jkl} - \partial_{x_j} \Gamma_{ikl} \} + O(|x|^2) \]

\[ = \frac{1}{2} \{ g_{jkl} g_{ik} + g_{iklj} - g_{jkl} g_{ik} - g_{iklj} \} + O(|x|^2) \]

\[ = \frac{1}{2} \{ - A_{jikl} - A_{jkl} - A_{ijkl} \} + O(|x|^2) \]

\[ = \frac{1}{2} \{ 4 A_{ijkl} - 2 A_{ijlk} - 2 A_{iklj} \} + O(|x|^2) \]

\[ = A_{ijkl} + O(|x|^2). \]

This argument shows that \((\mathbb{R}^m, g)\) is the germ of a pseudo-Riemannian manifold which provides a geometric realization of the model \((V, \varepsilon, A)\) at the origin; it is non-degenerate on some neighborhood \( \mathcal{U} \) of the origin. Let \( \mathbb{T}^m := \mathbb{R}^m / \mathbb{Z}^m \) be the torus and let \( (\theta^1, ..., \theta^m) \) be the usual periodic parameters. We define the flat metric \( b(\partial_{\theta_i}, \partial_{\theta_j}) = \varepsilon_{ij} \). We may regard \( \mathcal{U} \) as a neighborhood of 0 in \( \mathbb{T}^m \) as well. Let \( \phi \) be a plateau function which is identically 1 near 0 in \( \mathbb{T}^m \) and which has compact support in \( \mathcal{U} \). We consider \( g^\phi := g + (1 - \phi) h \). Since \( g^\phi \) agrees with \( g \) on a smaller neighborhood of 0, \((\mathbb{T}^m, g)\) provides a geometrical realization of \((V, \varepsilon, A)\) at 0. Since \( g(0) = h(0) \), \( g^\phi \) is non-degenerate for \( \phi \) with sufficiently small support. \( \square \)

Remark 2.1. The role of the torus is inessential; any pseudo-Riemannian manifold of the proper signature could have been used; the crucial point is that the manifold in question should admit some background pseudo-Riemannian manifold of the given signature.
3. The proof of Theorem 1.2 and of Theorem 1.6

Let \((V, \varepsilon)\) be an inner product space and let \((V^*, \varepsilon^*)\) be the dual inner product space. If \(\vec{v} = (v^1, \ldots, v^k)\) and if \(\vec{w} = (w^1, \ldots, w^k)\) are elements of \(x^k(V^*)\), the map

\[
\vec{v} \times \vec{w} \rightarrow \varepsilon^*(v^1, w^1) \cdots \varepsilon^*(v^k, w^k)
\]

is a bilinear symmetric map from \(\{x^kV^*\} \times \{x^kV^*\}\) to \(\mathbb{R}\) which extends to a symmetric inner product on \(\varepsilon^*\) to \(\otimes^k(V^*)\). If \(\{e_i\}\) is an orthonormal basis for \(V\), let \(\{e^i\}\) be the corresponding orthonormal basis for \(V^*\). If \(I = (i_1, \ldots, i_k)\) is a multi-index, let \(e^I := e^{i_1} \otimes \cdots \otimes e^{i_k}\). The collection \(\{e^I\}_{|I|=k}\) forms a basis for \(\otimes^k(V^*)\) with

\[
\varepsilon^*(e^I, e^K) = \begin{cases} 
\varepsilon^*(e^{i_1}, e^{i_1}) \cdots \varepsilon^*(e^{i_k}, e^{i_k}) & \text{if } I \neq K \\
0 & \text{if } I = K 
\end{cases}.
\]

Since \(\varepsilon^*(e^I, e^I) = \pm 1\), \(\varepsilon^*\) is non-degenerate on \(\otimes^k(V^*)\). The orthogonal group \(O(V, \varepsilon)\) extends to act naturally on \(\otimes^k(V^*)\) by pull-back and preserves this inner product. We have the following useful, if elementary, observation:

**Lemma 3.1.** Let \(W\) be an \(O(V, \varepsilon)\) invariant subspace of \(\otimes^k(V^*)\). Then the restriction of \(\varepsilon^*\) to \(W\) is non-degenerate.

**Proof.** Find an orthogonal direct sum decomposition \(V = V_+ \oplus V_-\) where \(V_+\) is spacelike and \(V_-\) is timelike, i.e. the restriction of the inner product to \(V_+\) is positive definite and the restriction of the inner product to \(V_-\) is negative definite. Let \(\Theta = \pm \text{Id}\) on \(V_\pm\) define an element of \(O(V, \varepsilon)\). Let \(\{e_1, \ldots, e_p\}\) be an orthonormal basis for \(V_-\) and let \(\{e_{p+1}, \ldots, e_n\}\) be an orthonormal basis for \(V_+\). We have that:

\[
\Theta e^I = \{\varepsilon^*(e^{i_1}, e^{i_1}) \cdots \varepsilon^*(e^{i_k}, e^{i_k})\} e^I = \varepsilon^*(e^I, e^I) e^I = \pm e^I.
\]

Let \(0 \neq w\). If \(\Theta^* w = w\), then \(w\) is a spacelike vector in \(\otimes^k(V^*)\) (i.e. \(\varepsilon^*(w, w) > 0\)) while if \(\Theta^* w = -w\), then \(w\) is a timelike vector in \(\otimes^k(V^*)\) (i.e. \(\varepsilon^*(w, w) < 0\)). Thus, in particular, the induced inner product on \(\otimes^k(V^*)\) is non-degenerate.

Let \(W\) be a non-trivial \(O(V, \varepsilon)\) invariant subspace of \(\otimes^k(V^*)\). Since \(\Theta \in O(V, \varepsilon)\), \(\Theta\) preserves \(W\) by assumption. Thus we may decompose \(W = W_+ \oplus W_-\) into the \(\pm 1\) eigenspaces of \(\Theta\). Since \(\Theta\) acts orthogonally, \(W_+ \bot W_-\). Since \(W_+\) is spacelike and \(W_-\) is timelike, the induced inner product on \(W\) is non-degenerate. \(\square\)

### 3.1. The proof of Theorem 1.2

Let \(P_{2\ell}\) be an invariant map from \(\otimes^\ell(\mathfrak{A}(V))\) to \(\mathbb{R}\). By Lemma 3.1, we have an equivariant orthogonal decomposition

\[
\otimes^{4\ell}(V^*) = \{\otimes^\ell(\mathfrak{A}(V))\} \oplus \{\otimes^\ell(\mathfrak{A}(V))\}^\perp.
\]

Extend \(P_{2\ell}\) to be 0 on \(\{\otimes^\ell(\mathfrak{A}(V))\}^\perp\) to define an invariant map \(Q_{2\ell}\) from \(\otimes^{4\ell}(V^*)\) to \(\mathbb{R}\) which restricts to yield \(P_{2\ell}\) on \(\otimes^\ell(\mathfrak{A}(V))\); there are, of course, many possible such extensions. We may now use H. Weyl’s theorem [27] concerning the invariants of the orthogonal group to express \(Q_{2\ell}\) in terms of the invariants \(\{Q_{2\ell, \sigma, \varepsilon}\}\); restricting once again to \(\otimes^\ell(\mathfrak{A}(V))\) then establishes Theorem 1.2 by expressing the original invariant \(P_{2\ell}\) in terms of the invariants \(\{P_{2\ell, \sigma, \varepsilon}\}\). \(\square\)

### 3.2. The proof of Theorem 1.6

Let \(P^{(2)}_{2\ell}\) be an \(O(V, \varepsilon)\) map from \(\otimes^\ell(\mathfrak{A}(V))\) to \(S^2(V^*)\). Contracting with \(\varepsilon^*\) then defines an invariant map

\[
\eta^{(2)}: \otimes^\ell(\mathfrak{A}(V)) \otimes S^2(V^*) \rightarrow \mathbb{R}.
\]

The subspace \(\otimes^\ell(\mathfrak{A}(V)) \otimes S^2(V^*)\) is an \(O(V, \varepsilon)\) invariant subspace of \(\otimes^{4\ell+2}(V^*)\). Consequently we may argue as above and use Lemma 3.1 to extend \(\eta^{(2)}\) to an invariant map \(\tilde{\eta}^{(2)}\) from \(\otimes^{4\ell+2}(V^*)\) to \(\mathbb{R}\). Again, H. Weyl’s theorem shows that \(\tilde{\eta}^{(2)}\)
can be expressed in terms of contractions of indices. Thus using the notation of Equation (1.c), we can express $\eta^{(2)}$ as a sum of invariants of the form

$$
\eta^{(2)}(A, h) = \varepsilon^*(P^{(2)}_{2\ell}(A), h) = \sum_{\beta \in \text{Perm}(4\ell + 2)} d_\beta Q_{2\ell, \beta, \varepsilon}(A, h)
$$

where

$$
Q_{2\ell, \beta, \varepsilon}(A, h) := \varepsilon^{i_1i_2} \cdots \varepsilon^{i_{4\ell+1}i_{4\ell+2}} \times A_{i_1i_2i_3i_4} \cdots A_{i_{4\ell-3}i_{4\ell-2}i_{4\ell-1}i_{4\ell}} h_{i_{4\ell+1}i_{4\ell+2}}.
$$

We distinguish 2 cases:

1) Suppose that $\{\beta_{2j-1}, \beta_{2j}\} = \{4\ell + 1, 4\ell + 2\}$ for some $j$. Then $h_{ij}$ is contracted against itself and all the indices of $A$ are contracted against indices of $A$. By renumbering things, we may in fact assume that $\beta_{4\ell+1} = 4\ell + 1$ and $\beta_{4\ell+2} = 4\ell + 2$ so that

$$
Q_{2\ell, \beta, \varepsilon}(A, h) = \varepsilon^{i_1i_2} \cdots \varepsilon^{i_{4\ell-1}i_{4\ell}} A_{i_1i_2i_3i_4} \cdots A_{i_{4\ell-3}i_{4\ell-2}i_{4\ell-1}i_{4\ell}} \varepsilon^{ij} h_{ij}
$$

for some $\sigma \in \text{Perm}(4\ell)$. We then have

$$
Q_{2\ell, \beta, \varepsilon}(A, h) = \varepsilon^*(P_{2\ell, 2, \sigma, \varepsilon}^{(2)}(A), h) \quad \text{where} \quad \tilde{\sigma} = \sigma^{-1}.
$$

2) If $h_{ij}$ is contracted against indices of $A$, then we obtain:

$$
Q_{2\ell, \beta, \varepsilon}(A, h) = \varepsilon^*(P_{2\ell, 1, \sigma, \varepsilon}^{(2)}(A), h)
$$

for some suitably chosen $\sigma \in \text{Perm}(4\ell)$.

This analysis permits us to express:

$$
\varepsilon^*(P_{2\ell}^{(2)}(A), h) = \varepsilon^* \left( \sum_{\sigma \in \text{Perm}(4\ell)} \left( d_{1, \sigma, \varepsilon}^{(2)} P_{2\ell, 1, \sigma, \varepsilon}^{(2)}(A) + d_{2, \sigma, \varepsilon}^{(2)} P_{2\ell, 2, \sigma, \varepsilon}^{(2)}(A) \right) , h \right).
$$

Since this holds for all $h$, we conclude

$$
P_{2\ell}^{(2)}(A) = \sum_{\sigma \in \text{Perm}(4\ell)} \left( d_{1, \sigma, \varepsilon}^{(2)} P_{2\ell, 1, \sigma, \varepsilon}^{(2)}(A) + d_{2, \sigma, \varepsilon}^{(2)} P_{2\ell, 2, \sigma, \varepsilon}^{(2)}(A) \right).
$$

4. THE PROOF OF THEOREM 1.3 AND OF THEOREM 1.7

**Proof.** An inner product space is determined, up to isomorphism, by its signature. Fix a positive definite inner product space $(V, \varepsilon)$ of dimension $m$. Theorem 1.3 will follow if given any signature $(p, q)$ with $p + q = m$, we can find an inner product space $(V(p, q), \varepsilon(p, q))$ of signature $(p, q)$ so that $P_{2\ell, c, \varepsilon}(A) = 0$ for all $A \in \mathfrak{A}(V)$ if and only if $P_{2\ell, c, \varepsilon}(p, q)(A) = 0$ for all $A \in \mathfrak{A}(V(p, q))$; Theorem 1.7 will follow similarly.

We shall use analytic continuation. Let $W := V \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $V$, and let $\mathfrak{A}(W) := \mathfrak{A}(V) \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $\mathfrak{A}(V)$; $\mathfrak{A}(W)$ is the complex vector space consisting of all elements of $\otimes^4(W^*)$ satisfying Equation (1.a). Let $\varepsilon_W$ be the complexification of $\varepsilon_V$; $\varepsilon_W$ is a symmetric complex bilinear form. Let $\mathcal{O}_C(W, \varepsilon_W)$ be the complex orthogonal group; this is the group of all complex linear maps of $W$ preserving $\varepsilon_W$. This group acts naturally on $\mathfrak{A}(W)$ by pullback. We use Equation (1.b) to extend $P_{2\ell, c, \varepsilon}$ to $\mathfrak{A}(W)$ to be invariant under the structure group $\mathcal{O}_C(W, \varepsilon_W)$; this is independent of the particular complex basis chosen for the complex vector space $W$.

We suppose given a signature $(p, q)$ with $p + q = m$. Let $V(p, q) := \text{Span}_{\mathbb{R}} \{ \sqrt{-1}e_1, \ldots, \sqrt{-1}e_p, e_{p+1}, \ldots, e_{p+q} \}$.
and let \( \varepsilon(p, q) \) be the restriction of \( \varepsilon_W \) to \( V(p, q) \); \( \varepsilon(p, q) \) is a real inner product of signature \( (p, q) \) on the real vector space \( V(p, q) \). Note that

\[
(V, \varepsilon) = (V(0, m), \varepsilon(0, m)), \quad V(0, q) \otimes \mathbb{R} \subset W, \quad \mathfrak{A}(V(0, q)) \otimes \mathbb{R} \subset \mathfrak{A}(W).
\]

Let \( U \) be a real vector space of dimension \( r \) and let \( U_\mathbb{C} := U \otimes \mathbb{R} \mathbb{C} \). Suppose that \( P = P(u_1, ..., u_r) \) is holomorphic on \( \times^r(U_\mathbb{C}) \). By the identity theorem, one then has that \( P(u) = 0 \) for all \( u \in U \) if and only if \( P(u) = 0 \) for all \( u \in U_\mathbb{C} \). We apply this observation as follows. Let \( C = \{ c_\sigma \}_{\sigma \in \text{Perm}(4t)} \) be a collection of complex constants. Then \( P_{2t,C,\varepsilon_W} \) is a holomorphic on \( \mathfrak{A}(W) \). Consequently, the following assertions are equivalent:

1. \( P_{2t,C,\varepsilon}(A) = 0 \) for all \( A \in \mathfrak{A}(V) \)
2. \( P_{2t,C,\varepsilon}(A) = 0 \) for all \( A \in \mathfrak{A}(V) \otimes \mathbb{R} \mathbb{C} \)
3. \( P_{2t,C,\varepsilon(p,q)}(A) = 0 \) for all \( A \in \mathfrak{A}(V(p, q)) \otimes \mathbb{R} \mathbb{C} \)
4. \( P_{2t,C,\varepsilon(p,q)}(A) = 0 \) for all \( A \in \mathfrak{A}(V(p, q)) \).

Theorem 1.3 now follows; the proof of Theorem 1.7 is similar and is therefore omitted in the interests of brevity.

\[ \square \]

5. The proof of Theorem 1.9

A-priori the invariant \( S_{2t,m}^{(2)} \) of Section 1.9 can involve first and second covariant derivatives of the curvature tensor. Our first task is to see that this does not happen. We fix a signature \( (p, q) \). For the moment, we work with a coordinate based formalism. Let \( \xi \) be a point of a pseudo-Riemannian manifold \( (M, g) \) of signature \( (p, q) \) and dimension \( m = p + q \). Let \( X = (x^1, ..., x^m) \) be a system of local coordinates on \( M \) which are defined in a neighborhood of \( \xi \) in \( M \). Let

\[
g_{ij}(X, g) := g(\partial_{x_i}, \partial_{x_j})
\]

give the components of the metric tensor. If \( \alpha = (a_1, ..., a_m) \), is a multi-index, set

\[
g_{ij/\alpha}(X, g) := \partial_{x_{i_1}}^{a_{i_1}} ... \partial_{x_{i_m}}^{a_{i_m}} g_{ij}(X, g).
\]

A local formula \( P(g_{ij}, g_{ij/\alpha})(X, g) \) is a polynomial in the \( g_{ij/\alpha} \) variables with coefficients which are smooth in the \( g_{ij} \) variables where \( \{ g_{ij} \} \) is assumed to define a non-singular bilinear form of signature \( (p, q) \). We say that \( P \) is invariant if \( P(X, g, \xi) \) depends only on \( (g, \xi) \) and not on the particular coordinate system chosen; it is to be universally and polynomially defined in the category of pseudo-Riemannian manifolds of signature \( (p, q) \). We let \( \mathcal{I}(p, q) \) be the vector space of all such invariants. The space of symmetric 2-tensor invariants \( \mathcal{I}^{(2)}(p, q) \) is defined similarly.

There is a natural grading which is defined on \( \mathcal{I}(p, q) \) and on \( \mathcal{I}^{(2)}(p, q) \) by setting \( \text{ord}(g_{ij/\alpha}) := |\alpha| \). We may then decompose

\[
\mathcal{I}(p, q) = \bigoplus \mathcal{I}_{2\ell}(p, q) \quad \text{and} \quad \mathcal{I}^{(2)}(p, q) = \bigoplus \mathcal{I}_{2\ell}^{(2)}(p, q)
\]

where the spaces \( \mathcal{I}_{2\ell}(p, q) \) and \( \mathcal{I}_{2\ell}^{(2)}(p, q) \) consist of invariants which are of order \( 2\ell \) in the derivatives of the metric; there are no invariants of odd order. This grading can also be expressed in a coordinate free fashion by noting that if \( P \in \mathcal{I}(p, q) \) and if \( P^{(2)} \in \mathcal{I}^{(2)}(p, q) \), then

\[
P \in \mathcal{I}_{2\ell}(p, q) \quad \iff \quad P(c^2 g) = c^{-2\ell} P(g),
\]

\[
P^{(2)} \in \mathcal{I}_{2\ell}^{(2)}(p, q) \quad \iff \quad P^{(2)}(c^2 g) = c^{-2\ell - 2} P^{(2)}(g).
\]

Furthermore, it is clear that

\[
\mathcal{J}_{2\ell}(p, q) \subset \mathcal{I}_{2\ell}(p, q) \quad \text{and} \quad \mathcal{J}_{2\ell}^{(2)}(p, q) \subset \mathcal{I}_{2\ell}^{(2)}(p, q).
\]
On the other hand, the invariant
\[ \gamma^i_\alpha g^{jk} R_{ijkl;ab} dx^a \circ dx^b \]
is an element of \( \mathcal{I}_4^2(p, q) \) which does not belong to \( \mathcal{J}_4^2(p, q) \) since it involves the 4th derivatives of the metric.

H. Weyl's theorem [27] shows that all the elements of \( \mathcal{I}_{2\ell}(p, q) \) and of \( \mathcal{I}_{2\ell}^2(p, q) \) are given by suitable contractions of indices involving covariant derivatives of the curvature tensor. We shall not introduce the necessary formalism as it plays no role in our analysis. The discussion of the restriction map in the geometric context given in Section 1.5 then extends the restriction maps defined previously to define maps:
\[
\begin{align*}
  r_{-}^{(2)} : \mathcal{I}_{2\ell}^2(p, q) & \to \mathcal{I}_{2\ell}^2(p - 1, q) \quad \text{if} \quad p > 0,
  r_{+}^{(2)} : \mathcal{I}_{2\ell}^2(p, q) & \to \mathcal{I}_{2\ell}^2(p, q - 1) \quad \text{if} \quad q > 0.
\end{align*}
\]
These maps are characterized, as previously, by the property that
\[
(r_{\pm}^{(2)} P_{2\ell}^{(2)}(N, g_N))i = i_N \{ P_{2\ell}^{(2)}(N \times S^1; g_N \pm d\theta^2)(\xi, \theta_1) \}.
\]
Again, if we express the invariants in terms of contractions of indices, the restriction maps simply let the range of summation be from 1 to \( m-1 \) rather than from 1 to \( m \).

Fix a signature \((p, q)\) and let \( 0 \neq P \in \mathcal{I}_{2\ell}^2(p, q) \). Let \( \varepsilon \) be a given quadratic form of signature \((p, q)\). Let \( \xi \) be a point of a pseudo-Riemannian manifold \((M, g)\) of signature \((p, q)\). We can choose coordinates \( x = (x^1, ..., x^m) \) which are centered at \( \xi \) (i.e. \( x(\xi) = 0 \)) and which are normalized so that
\[
g_{ij} = \varepsilon_{ij} + O(|x|^2) \quad \text{where} \quad \varepsilon_{ij} = \begin{cases} 
  \pm 1 & \text{if} \quad i = j \\
  0 & \text{if} \quad i \neq j
\end{cases}.
\]

With this normalization, the tensor \( g_{ij} \) and the first derivatives of the metric play no role and we can regard \( P^{(2)} = P^{(2)}(g_{ij}/\alpha) \) as a polynomial in the derivatives (where \( |\alpha| \geq 2 \)) which is symmetric 2-tensor valued. Thus a typical monomial takes the form
\[
A^{(2)} = \frac{\varepsilon_{ij}}{\alpha_1} ... \frac{\varepsilon_{rj}}{\alpha_r} dx^k \circ dx^l. \tag{5.a}
\]
We let \( c(A^{(2)}, P^{(2)}) \) be the coefficient of \( A^{(2)} \) in \( P^{(2)} \). We say \( A^{(2)} \) is a monomial of \( P \) if \( c(A^{(2)}, P^{(2)}) \neq 0 \). We let \( \deg_{n}(A^{(2)}) \) be the number of times that the index \( n \) appears in the monomial \( A^{(2)} \):
\[
\deg_{n}(A^{(2)}) = \delta_{11, n} + \delta_{1j, n} + \alpha_1(n) + ... + \delta_{ir, n} + \alpha_r(n) + \delta_{k, n} + \delta_{r, n} \tag{5.b}
\]

**Lemma 5.1.** Let \( 0 \neq P^{(2)}_{2\ell} \in \mathcal{I}_{2\ell}^2(p, q) \). Assume that \( r_{-}^{(2)}(P^{(2)}_{2\ell}) = 0 \) if \( p > 0 \) and that \( r_{+}^{(2)}(P^{(2)}_{2\ell}) = 0 \) if \( q > 0 \).

1. \( \deg_{n}(A^{(2)}) \geq 2 \) for every monomial \( A^{(2)} \) of \( P^{(2)}_{2\ell} \).
2. \( p + q \leq 2\ell + 1 \).
3. If \( p + q = 2\ell + 1 \), then \( P^{(2)}_{2\ell} \in \mathcal{J}_{2\ell}^2(p, q) \).

**Proof.** If \( p > 0 \), we choose an orthonormal basis for the model space \( \{e_i\} \) so \( e_m \) is timelike. Then \( r_{-}^{(2)}(P^{(2)}_{2\ell}) \) is defined by evaluating \( P^{(2)}_{2\ell} \) on a metric of the form \( ds_N^{(2)} \). The only additional relation that is imposed by restricting our attention to such metrics is to set \( A^{(2)} = 0 \) if \( \deg_{m}(A^{(2)}) > 0 \). Thus we may conclude that \( \deg_{m}(A^{(2)}) \geq 2 \) for every monomial \( A^{(2)} \) of \( P^{(2)}_{2\ell} \). By replacing \( \partial x_m \) by \( -\partial x_m \), we may conclude that \( \deg_{m}(A^{(2)}) \) is even and thus \( \deg_{m}(A^{(2)}) \geq 2 \) for every monomial \( A^{(2)} \) of \( P^{(2)}_{2\ell} \). We can permute the coordinate indices; this may replace a space-like coordinate vector field by a time-like coordinate vector field and thus it is important.\]
to work with $r_+^{(2)}$ and $r_-^{(2)}$ simultaneously if both $p$ and $q$ are positive and it was for this reason that we introduced both $r_+^{(2)}$ and $r_-^{(2)}$. Thus we have

$$\deg_k(A^{(2)}) \geq 2 \text{ for } 1 \leq k \leq p + q.$$ 

We adopt the notation of Equation (5.a) and use Equation (5.b) together with the fact that $|\alpha_j| \geq 2$ to estimate:

$$2(p + q) \leq \sum_{n=1}^{p+q} \deg_n(A^{(2)}) = 2r + 2 + \sum_{j=1}^r |\alpha_j| \leq 2 + 2 \sum_{j=1}^r |\alpha_j| = 2 + 4\ell.$$ 

Assertion (2) now follows. In the limiting case that $p + q = 2\ell + 1$, all of the equalities must have been equalities. In particular, $|\alpha_j| = 2$ for all $j$. This establishes Assertion (3).\)

5.1. The proof of Theorem 1.9. It is clear from the definition of the Euler-Lagrange equations that since $E^{(2)}_{2\ell}$ is universal, we have that $S^{(2)}_{2\ell,2\ell}$ is universal as well. By Theorem 1.5 $S^{(2)}_{2\ell,2\ell} = 0$. Thus $r^{(2)}(S^{(2)}_{2\ell,2\ell+2}) = 0$ so by Lemma 5.1,

$$S^{(2)}_{2\ell,2\ell+1} \in J^{2\ell,2\ell+1}_{2\ell,2\ell+1}.$$ 

We may therefore apply Theorem 1.8 to conclude

$$S^{(2)}_{2\ell,2\ell+1} = cT^{(2)}_{2\ell,2\ell+1}$$

for some constant $c$. Therefore $r^{(2)}(S^{(2)}_{2\ell,2\ell+2} - cT^{(2)}_{2\ell,2\ell+2}) = 0$ so by Lemma 5.1,

$$S^{(2)}_{2\ell,2\ell+2} - cT^{(2)}_{2\ell,2\ell+2} = 0.$$ 

We consider in this fashion to show

$$S^{(2)}_{2\ell,m} - cT^{(2)}_{2\ell,m} = 0 \text{ for } m \geq 2\ell + 1.$$ 

The equality if $m < 2\ell + 1$ is of course trivial since both $S^{(2)}_{2\ell,m}$ and $T^{(2)}_{2\ell,m}$ vanish in that instance. \)

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