Abstract composition rule for relativistic kinetic energy in the thermodynamical limit

T. S. Bíró\(^{(a)}\)

KFKI Research Institute for Particle and Nuclear Physics - P.O. Box 49, H-1525 Budapest, Hungary, EU

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Abstract – We demonstrate by simple mathematical considerations that a power-law–tailed distribution in the kinetic energy of relativistic particles can be a limiting distribution seen in relativistic heavy-ion experiments. We prove that the infinite repetition of an arbitrary composition rule on an infinitesimal amount leads to a rule with a formal logarithm. As a consequence the stationary distribution of energy in the thermodynamical limit follows the composed function of the Boltzmann-Gibbs exponential with this formal logarithm. In particular, interactions described as solely functions of the relative four-momentum squared lead to kinetic energy distributions of the Tsallis-Pareto (cut power law) form in the high-energy limit.

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Why to investigate composition rules. – One of the theoretically challenging questions related to relativistic heavy-ion physics is how to establish the existence of a thermal or near thermal state of (quark) matter during high-energy collisions. Besides studies of the hadronic flavor composition [1–5], the shape and the steepness of transverse momentum spectra on different particles presents experimental information on this question [6–12]. Since for high energies of the out-coming particles (pions, kaons, antiprotons, etc.) these spectra show a power law tail, it is important to clarify, whether this prominent feature can be explained in the framework of general thermodynamic ideas, or whether this is a non-equilibrium effect. While in some earlier works we have demonstrated that such spectra can be obtained as stationary distributions by altering the energy composition rule in two-particle collisions from the simple addition to another rule [13,14], in the present paper we aim at understanding the general mechanism setting an effective rule and a non-exponential stationary distribution of the individual particle energies in the thermodynamical limit.

The classical thermodynamics using extensive and intensive quantities is being extremely successful in describing and understanding uncountable physical phenomena in nature. However, there are particular cases, where the observed distribution of individual energies in a complex system does not follow the exponential Boltzmann-Gibbs law. There are suggestions trying to go beyond the classical picture and to consider descriptions generalizing traditional thermodynamical concepts, in particular to consider the possibility that the composition rule deviates from the simple addition. Such quantities are often called “non-extensive”, although strictly speaking the extensivity property is required only in the thermodynamical limit, i.e. for large systems consisting of many particles.

In particular generalizations of the entropy formula, connecting the quantity of macroscopic entropy to probabilities of microstates of an extended system, have been repeatedly suggested in forms generalizing the Boltzmann-Gibbs-Shannon logarithmic formula [15–18]. The deformed (or generalized, extended) logarithm relates the abstract product rule to a summation or composition formula in a general way: the statistical independence of states is hence mapped to a non-additivity of the composition formula [19–23]. Reversely, the additivity of entropy is —in some cases— achieved by non-product probabilities, trying to grasp the essence of surviving correlations (surmised to occur due to long-range interactions) in systems, which are large in the thermodynamical sense [24]. The additivity of entropy also can be achieved by weakly non-local extensions [25,26]. The connection between generalized, among them non-Boltzmannian probability distributions and the thermodynamic entropy formula was clarified in a recent paper [27]. Naturally, the use of a formal composition rule is also applicable

\(^{(a)}\)E-mail: tsbiro@mail.kfki.hu
to the energy [28,29]. Two subsystems combine to a common larger system not necessarily additively with their energies; the interaction part may lead to a finite relative contribution in the infinite particle number limit.

**Repeated compositions and the proof of associativity.** In order to investigate arbitrary composition rules in the thermodynamical limit we repeat (functionally compose) abstract mathematical composition rules infinitely many times acting on infinitesimal amounts. We find that some rules transform to the simple addition in this limit, while others do not. We propose that the thermodynamical limit of an arbitrary pairwise, iterable composition rule is an associative rule. The importance of this statement becomes clear by referring to the known mathematical property that associative rules always possess a strict monotonic function, called here the formal logarithm, in terms of which they can be expressed [30,31].

Intuitively the associativity of an abstract composition rule means that —no matter what is the order of compositions—the result of several repeated compositions is the same. Regarding a two-step composition of three subsystems, either the composition of number 1 and 2, i.e. 12 can be composed with the subsystem number 3, or the number 1 subsystem with the composition of systems 2 and 3, i.e. with 23. We obtain the same composite system 123 both ways. So even non-trivial correlations and memory effects do not distinguish at the end these two ways of composition. We actually expect that this property is a test (if not the most general definition) of the largeness of a composed system with respect to the thermodynamical limit. Let us denote an abstract pairwise composition rule by the mapping \((x,y) \rightarrow h(x,y)\). Whenever \(h(x,y)\) is an element of the same set as \(x\) and \(y\), the composition is iterable arbitrarily long. The associativity of such a rule is formulated by the function equation

\[
h(h(x,y),z) = h(x,h(y,z))
\] (1)

for \(x, y\) and \(z\) being elements of the same group. For our purpose we shall consider energies or entropies of physical subsystems. The general solution of the associativity equation (1) is given by

\[
h(x,y) = X^{-1}(X(x) + X(y))
\] (2)

with \(X(x)\) being a strict monotonic function. It is referred to as the “formal logarithm”, because it maps the arbitrary composition rule \(h(x,y)\) to the addition by taking the \(X\)-function of eq. (2):

\[
X(h(x,y)) = X(x) + X(y).
\] (3)

Due to this construction the generalized analogs to classical extensive (and additive) quantities are formal logarithms, whenever the composition rule is associative. As a consequence, stationary distributions, in particular solutions of generalized Boltzmann equations [14], are the Gibbs exponentials of the formal logarithm,

\[
f(x) = \frac{1}{Z} e^{-\beta X(x)}.
\] (4)

Now we investigate a large number of iterations, \(N\), of the composition rule applied to an infinitesimal amount \(y/N\) in each step:

\[
x_N(y) := h \circ \ldots \circ h \left( \frac{y}{N}, \ldots, \frac{y}{N} \right)
\] (5)

Whenever the limit,

\[
\lim_{N \to \infty} x_N(y) < \infty,
\] (6)
is finite for a finite \(y\), we are dealing with an extensive (but not necessarily additive) system. Our purpose is to study such systems and to obtain their asymptotic composition rule

\[
x_{N_1 + N_2} = \varphi(x_{N_1}, x_{N_2})
\] (7)

in the limit \(N_1, N_2 \to \infty\). The repetitive composition can be formulated as a recursion at an arbitrary step \(n\) between 0 and \(N\) as follows:

\[
x_n = h \left( x_{n-1}, \frac{y}{N} \right),
\] (8)

with \(x_0 = 0\). It is a natural requirement, but important for what follows, to consider only such rules which satisfy \(h(x,0) = x\). Subtracting \(x_{n-1} = h(x_{n-1},0)\) from this formula we arrive at

\[
x_n - x_{n-1} = h \left( x_{n-1}, \frac{y}{N} \right) - h(x_{n-1},0).
\] (9)

In the large-\(N\) limit \(y/N\) is an infinitesimally small quantity. In this case the composition rule in the above equation can be Taylor expanded:

\[
x_n - x_{n-1} = \frac{y}{N} h'_2(x_{n-1},0^+) + O\left(\frac{y^2}{N^2}\right).
\] (10)

In this expression on the right-hand side \(h'_2(x,0^+)\) denotes the partial derivative of the rule \(h(x,y)\) with respect to its second argument taken at a value approaching zero from above. Introducing the variable \(t = n/N\), which describes how far one has proceeded in the composition process, a single step belongs to a change of \(\Delta t = 1/N\). The \(x(t) = x_{n-1}\) values evolve due to the composition following the rule

\[
x(t + \Delta t) - x(t) = y \Delta t h'_2(x(t),0^+) + O(y^2 \Delta t^2).
\] (11)

In the large-\(N\) limit we are considering \(\Delta t \to 0\) and obtain an evolution equation similar to the renormalization flow:

\[
\frac{dx}{dt} = y h'_2(x,0^+).
\] (12)

Note that the uniformity of subdivisions to \(y/N\) is not really necessary; all infinitesimal divisions summing up to a finite \(y\) lead to the same differential flow equation.
The solution of eq. (12) is given by
\[ L(x) = \int_0^x \frac{dz}{h_2'(z, 0^+)} = y t. \] \hspace{1cm} (13)

This solution, when strict monotonic and hence invertible, defines the following asymptotic composition rule in the thermodynamic limit:
\[ x_{12} := \varphi(x_1, x_2) = L^{-1}(L(x_1) + L(x_2)) \] \hspace{1cm} (14)

Here we interpret \( x_1 = x(t_1) \) and \( x_2 = x(t_2) \) as being composed by \( N_1 = N t_1 \) and \( N_2 = N t_2 \) steps. The final system \( x_{12} = x(t_1 + t_2) \) could have been made of \( N_1 + N_2 \) steps. Note that \( t_i = N_i/N \) are the extensivity shares of the respective subsystems. This asymptotic composition rule is associative and commutative.

**Classification, important examples.** – Now we turn to the analysis of important particular rules and their asymptotic pendants in the thermodynamic limit.

The trivial (and classical) addition is the simplest composition rule: \( h(x, y) = x + y \). In this case \( h_2'(x, 0^+) = 1 \) and one obtains
\[ L(x) = \int_0^x \frac{dz}{1 + a z} = \frac{1}{a} \ln(1 + a x). \] \hspace{1cm} (15)

and with that the original Gibbs exponentials, \( e^{-\beta E}/Z \), for stationary distributions of any Monte Carlo type algorithm using the original composition rule. The corresponding asymptotic rule is \( \varphi(x, y) = x + y \).

The rule leading to the \( q \)-exponential [19] (or Pareto, or Tsallis) distribution is given by \( h(x, y) = x + y + a x y \) with the parameter \( a \) proportional to \( q - 1 \). In this case one obtains \( h_2'(x, 0^+) = 1 + a x \) and
\[ L(x) = \int_0^x \frac{dz}{1 + a z} = \frac{1}{a} \ln(1 + a x). \] \hspace{1cm} (16)

This formal logarithm leads to a stationary distribution with the power law tail as the function composition \( \exp \circ L \) on the power \(-\beta\):
\[ f(E) = \frac{1}{Z} e^{-a\ln(1+aE)} = \frac{1}{Z} (1 + aE)^{-\beta/a}. \] \hspace{1cm} (17)

A generalized entropy formula, on the other hand, can be constructed as the expectation value of the inverse of this function, \( L^{-1} \circ \ln \):
\[ S = \int f \frac{e^{-a\ln(f)} - 1}{a} = \frac{1}{a} \int (f^{1-a} - f). \] \hspace{1cm} (18)

The asymptotic composition rule again coincides with the original one: \( \varphi(x, y) = x + y + a x y \).

A simple rule suggested by Kaniadakis [32] is based on the sinh function. The formal logarithm is given as
\[ L(x) = \frac{1}{\kappa} \text{Arsh}(\kappa x), \] \hspace{1cm} (19)

and its inverse becomes \( L^{-1}(t) = \sinh(\kappa t)/\kappa \). The stationary distribution, composed by \( \exp \circ L \), is
\[ f_{\text{eq}}(p) = \frac{1}{Z} \left( \kappa p + \sqrt{1 + \kappa^2 p^2} \right)^{\beta/\kappa}. \] \hspace{1cm} (20)

For large arguments it gives a power law in the momentum absolute value \( p \) and hence also in the relativistic energy. The corresponding entropy formula is the average of \( L^{-1} \circ \ln \) over the allowed phase space:
\[ S_K = - \int f \sinh(\kappa \ln f) = \int \frac{f^{1-\kappa} - f^{1+\kappa}}{2\kappa}. \] \hspace{1cm} (21)

The composition formula can be reduced to
\[ h(x, y) = x \sqrt{1 + \kappa^2 y^2} + y \sqrt{1 + \kappa^2 x^2}. \] \hspace{1cm} (22)

For low arguments it is additive, \( h(x, y) \approx x + y \), for high ones it is multiplicative, \( h(x, y) \approx 2xy \). It has been motivated by the relativistic kinematics of massive particles. Regarding \( \kappa = 1/mc, \) \( \kappa p = \sinh \eta \), the formal logarithm is proportional to the rapidity, \( L(p) = mc \eta \). This implies a stationary distribution like \( \exp(-\beta mc \eta) \), which has not yet been observed in relativistic particle systems. Therefore it is wishful to consider some further scenarios based on other quantities deduced from relativistic kinematics (see next section).

The rule leading to a stretched exponential stationary distribution is given by \( h(x, y) = (x^b + y^b)^{1/b} \). Here some care has to be taken, the partial derivative has to be evaluated not at zero, but at a small positive argument, \( \epsilon \). Regarding \( \epsilon = y/2N \). We get \( h_2'(x, \epsilon) = c(\epsilon)x^{1-b} \) with a factor depending on \( \epsilon \) and —depending on \( \beta \)—being possibly diverging in the \( \epsilon \to 0 \) limit. However, this does not spoil our procedure; we obtain \( L(x) = c(\epsilon)x^{b}/b \) and with that the asymptotic rule: \( \varphi(x, y) = (x^{b} + y^{b})^{1/b} \). The reason is that constant factors in the formal logarithm can be eliminated without loss of any information.

Our next example is a non-associative rule; its asymptotic pendant cannot be itself. We consider
\[ h(x, y) = x + y + a \frac{xy}{x + y} \] \hspace{1cm} (23)

(a combination of arithmetic and harmonic means).

The fiducial derivative is given by \( h_2'(x, 0^+) = 1 + a \) and —being a constant— it leads to \( L(x) = x/(1 + a) \) and with that to the addition as asymptotic rule: \( \varphi(x, y) = x + y \).

As a last example in this train we discuss Einstein’s formula for velocity addition,
\[ h(x, y) = \frac{x + y}{1 + xy/c^2}. \] \hspace{1cm} (24)

This rule is associative, and it also preserves its form in the thermodynamic limit. The fiducial derivative is given by \( h_2'(x, 0^+) = 1 - x^2/c^2 \) and the formal logarithm, \( L(x) = c \text{atanh}(x/c) \) turns out to be the rapidity. The asymptotic composition rule recovers the original one.
In the case of a general second-order polynomial for the fiducial derivative \( h'_2(z, 0) \) the asymptotic composition rule turns out to be:

\[
\varphi(x, y) = \frac{x + y + axy}{1 + xy/c^2}
\]  

with \( c^2 = -z_1z_2 \) and \( a = -(z_1 + z_2)/z_1z_2 \); \( z_1 \) and \( z_2 \) being the algebraic roots of \( h'_2(z, 0) \). It is a generalization of the Tsallis and Einstein rules. A similar composition rule has been found for the parallel transmittivity in certain Potts models, for a review see ref. [33].

Finally, we prove that all associative rules are mapped to themselves by the thermodynamic limit. Given an original composition rule, \( h(x, y) \), which is associative, it can be expressed by its formal logarithm:

\[
h(x, y) = X^{-1}(X(x) + X(y)).
\]

Then letting to act the strict monotonic function \( X \) on both sides and derive with respect to the second argument, we obtain \( X'(h)\partial h/\partial y = X'(y) \) and

\[
h'_2(x, 0^+) = \frac{X'(0)}{X'(h(x, 0))}.
\]

Due to the property \( h(x, 0) = x \) (equivalently \( X(0) = 0 \)) the formal logarithm of the asymptotic composition rule is given by

\[
L(x) = \int_0^x \frac{X'(z)}{X'(0)}\,dz = \frac{X(x)}{X'(0)};
\]

it is proportional to the formal logarithm of the starting rule. Therefore the asymptotic rule is the same as we begin with: \( \varphi(x, y) = h(x, y) \). Actually the freedom in a factor of the formal logarithm can always be used to set \( X'(0) = 1 \).

This way any associative composition rule describes a thermodynamical limit of a class of non-associative rules. Associativity is a synonym to the thermodynamical limit.

In figure 1 we show the composition of \( h(x, y) = x + y + xy/(1 + 5xy) \) up to the amount \( 1/N \) (boxes), for finite \( N \)-sized pieces by the rule \( h(x, y) = x + y + axy/(1 + 5xy) \) with \( a = -1, 0 \) and 1. The 21st values are the asymptotic compositions of half-systems: \( \varphi(x_{10}, x_{10}) \).

Fig. 2: Approach to the asymptotic extensivity rule as a function of the number of repetitions of the non-associative rule, \( h(x, y) = x + y + xy/(1 + 5xy) \), to the amount \( 1/N \) (boxes). Plotted are \( x_{2N} \) and for comparison the amount \( \varphi(x_{2N}, x_{2N}) \) as well as the curve corresponding to the \( 2N \)-fold composition of \( \varphi \).
rule $\varphi(x, y)$. It is given by the formula

$$x_{2N} = \varphi \circ \cdots \circ \varphi \left( \frac{1}{2N}, \ldots, \frac{1}{2N} \right) = \left( \frac{1}{2N} - 1 \right)^{2N}. \quad (29)$$

The asymptotic rule is occurring slowly in this case, the convergence rate is given by the convergence of the Euler formula to the Euler number ($e_\infty = e - 1$).

**Deriving composition rules from interaction energy and kinematics.** In this section we review a few general considerations which may relate the interaction energy due to some kind of correlation to the use of formal logarithms and abstract composition rules. Our basic assumption is that the interaction energy between two subsystems (“particles”) can be expressed as a function of the individual energies without the interaction (in asymptotic free states). This way $E_{12} = E_1 + E_2 + U(E_1, E_2)$ is a general energy composition rule.

In most cases, discussed in physics, the interaction is given as a function of the relative distance, and this form is not directly related to kinematic data. In the sense of medium long-time behavior, however, the interaction energy can often be expressed by the relative momentum: either due to a virial theorem or due to the direct quantum-mechanical solution for the relative wave function, like, e.g., in the two-body Coulomb problem.

In the following we shall assume that the interaction energy is a function of the kinematic variable $Q^2$, the square of the relative four-momentum. We shall study whether relativistic speeds alone can cause “non-extensivity”, i.e. a power-law–tailed kinetic energy distribution. The relativistic formula for $Q^2$ is given by the following Lorentz-invariant quantity:

$$Q^2 = (\vec{p}_1 - \vec{p}_2)^2 - (E_1 - E_2)^2 \quad (30)$$

with $\vec{p}_i, E_i$ being, respectively, the relativistic momenta and full energies of interacting bodies. Expressed by the energies and the angle $\Theta$ between the two momenta this becomes a linear expression of $\cos \Theta$:

$$Q^2 = 2 \left( E_1 E_2 - m^2 - p_1 p_2 \cos \Theta \right) \quad (31)$$

with $p_i = \sqrt{E_i^2 - m^2}$ for $i = 1, 2$. Here we use relativistic units ($c = 1$) and assume the same mass for both interacting partners, for simplicity. It is useful to note that writing eq. (31) as $Q^2 = 2(A - B \cos \Theta)$ we have

$$A \pm B = E_1 E_2 - m^2 \pm p_1 p_2 = m^2 (\cosh(\eta_1 \pm \eta_2) - 1) \quad (32)$$

using the rapidities $\eta_i$. For equal momenta or rapidities $Q^2 = 0$ and $A = B; A^2 - B^2 = m^2 (E_1 - E_2)^2$ measures the energy difference.

In order to estimate the interaction contribution we subtract the zero-momentum terms, and assume

$$E_{12} = E_1 + E_2 + U(Q^2) - U(Q_1^2) - U(Q_2^2) + U(0), \quad (33)$$

with $Q_i^2 = 2m(E_i - m)$. This construction ensures that for equal momenta, i.e. for $Q^2 = 0$, no interaction correction occurs to the addition law for the energy.

Seeking for an effective energy composition rule as an isotropic average over the relative directions of the respective momenta, one averages over the angle $\Theta$:

$$\langle U(Q^2) \rangle = \frac{1}{2} \int_0^\pi U(2A - 2B \cos \Theta) \sin \Theta \, d\Theta = \frac{F(2A + 2B) - F(2A - 2B)}{4B}, \quad (34)$$

with $U(w) = dF/dw$. This is easy to see upon the substitution $w = 2(A - B \cos \Theta)$. Since at zero momenta the total relativistic energies are non-zero, it is more physical to consider the kinetic energy only, $K_i = E_i - m$. The rule for the kinetic energy composition is hence given by

$$K_{12} = K_1 + K_2 + \frac{F(2A + 2B) - F(2A - 2B)}{4B} - U(2mK_1) - U(2mK_2) + U(0). \quad (35)$$

The coefficients $A$ and $B$ are also expressed by the respective kinetic energies:

$$A = m(K_1 + K_2) + K_1 K_2, \quad B = K_1 K_2 (1 + 2m/K_1)^{1/2} (1 + 2m/K_2)^{1/2}. \quad (36)$$

One observes that the product of kinetic energies occurs due to kinematic reasons.

Taylor expanding the integral of the unknown function $U(w)$ around $w = 2A$ we obtain the following composition rule for the relativistic kinetic energies:

$$h(x, y) = x + y - U(2mx) - U(2my) + U(0) + \sum_{j=0}^{\infty} U^{(2j)}(2A) \left( \frac{4B^2 j}{(2j + 1)!} \right) \quad (37)$$

with $A = m(x + y) + xy$ and $4B^2 = 4xy(x + 2m)(y + 2m)$. The fiducial derivative becomes an expression with a finite number of terms

$$h'_j(x, 0) = 1 - 2mU'(0) + 2(m + x) U''(2mx) + \frac{4}{3} mx (2m + x) U'''(2mx). \quad (38)$$

For all traditional approaches the interaction energy $U$ is not considered as dependent on $Q^2$. In such cases $h'_j(x, 0) = 1$ and one arrives at the simple addition as composition rule. As a consequence the stationary energy distribution is of Boltzmann-Gibbs type, which in the relativistic case is referred to as the Jüttner distribution [34]. Recently a renewed interest has been aroused in the thermodynamics of relativistic systems [35]. For the $Q^2$-dependent interaction it is enlightening to analyze two particular kinematical cases: the extreme relativistic and
the non-relativistic ones. In the first case \( m = 0 \) has to be replaced and one obtains

\[
h_2'(x,0) = 1 + 2x U'(0). \tag{39}
\]

As discussed above this leads to a Tsallis-Pareto distribution in the kinetic energy (at zero rapidity in the variable \( m_T - m \)). The opposite extreme, \( m \gg x \) leads to an undetermined asymptotic composition rule due to

\[
h_2'(x,0) \approx 1 + 2m(U'(2mx) - U'(0)) + \frac{8}{3} m^2 x U''(2mx). \tag{40}
\]

This result includes for \( U' = 0 \) the traditional momentum-independent interaction case leading to the addition as asymptotic rule for non-relativistic kinetic energies, and hence to the Boltzmann-Gibbs distribution. We note that in the relativistic kinematics the next simplest assumption, \( U' = a = \text{const} \) leads to a Tsallis-Pareto distribution due to \( h_2'(x,0) = 1 + 2ax \) obtained from eq. (38), while its non-relativistic approximation derived from eq. (40) is still the Boltzmann-Gibbs exponential.

**Conclusion.** — Summarizing we have proved that in the thermodynamical limit, by composing a finite amount of an extensive physical quantity as a repeated composition of infinitesimally small amounts of the same quantity one arrives at an effective asymptotic composition rule which is derivable from a formal logarithm. This formal logarithm, \( L \), is constructed from the original composition rule uniquely and serves as a basis for describing a stationary distribution, \( \exp \circ L \) and the corresponding formal expression for the entropy it canonically maximizes, \( (L^{-1} \circ \ln) \). Associative rules lead to themselves in this limit, and are attractors for other rules.

The addition is the simplest composition rule, the formal logarithm being the identity map. It leads to the Boltzmann-Gibbs distribution. The next simplest one leads to the Pareto-Tsallis power-law–tailed stationary distribution. We also have shown that considering interaction energies dependent on the relative four-momentum, in general a non-trivial asymptotic rule arises. In particular, the Pareto-Tsallis distribution — and the corresponding non-additive asymptotic composition rule — emerges generally from extreme relativistic kinematics. In fact high-energy particle spectra often show a power-law–like tail in their kinetic energy. Besides that also non-relativistic systems may show such thermodynamical limit, but here the general case can also be different.

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