Existence of minimizers and convergence of critical points for a new Landau-de Gennes energy functional in nematic liquid crystals

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Abstract
The Landau-de Gennes energy in nematic liquid crystals depends on four elastic constants $L_1, L_2, L_3, L_4$. In the case of $L_4 \neq 0$, Ball and Majumdar (Mol. Cryst. Liq. Cryst., 2010) found an example that the original Landau-de Gennes energy functional in physics does not satisfy a coercivity condition, which causes a problem in mathematics to establish existence of energy minimizers. At first, we introduce a new Landau-de Gennes energy density with $L_4 \neq 0$, which is equivalent to the original Landau-de Gennes density for uniaxial tensors and satisfies the coercivity condition for all $Q$-tensors. Secondly, we prove that solutions of the Landau-de Gennes system can approach a solution of the $Q$-tensor Oseen-Frank system without using energy minimizers. Thirdly, we develop a new approach to generalize the Nguyen and Zarnescu (Calc. Var. PDEs, 2013) convergence result to the case of non-zero elastic constants $L_2, L_3, L_4$.

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1 Introduction

A liquid crystal is a state of matter between isotropic liquid and crystalline solid. Based on the molecular positional and orientational order of liquid crystals, there are three main types: smectic, cholesterics and nematic. The nematic liquid crystal is the most common type in which the general phases are uniaxial and biaxial. In 1971, de Gennes [11] used $Q$-tensor order parameters to formulate the elastic energy of liquid crystals with Landau’s bulk energy. The Landau-de Gennes theory has been verified in physics as a successful theory describing both uniaxial and biaxial phases in nematic liquid crystals. Indeed, Pierre-Gilles de Gennes
was awarded a Nobel prize for physics in 1991 for his discoveries in liquid crystals and polymers.

In the Landau-de Gennes framework, the space of $Q$-tensors in the Landau-de Gennes theory is a space of symmetric, traceless $3 \times 3$ matrices defined by

$$S_0 := \{ Q \in \mathbb{M}^{3 \times 3} : Q^T = Q, \ tr \ Q = 0 \},$$

where $\mathbb{M}^{3 \times 3}$ denotes the space of $3 \times 3$ matrices. When $Q \in S_0$ has two equal non-zero eigenvalues, a nematic liquid crystal is said to be uniaxial. When $Q$ has three unequal non-zero eigenvalues, a nematic liquid crystal is said to be biaxial. For material constants $a, b, c$, we define the constant order parameter

$$s_+ := \frac{b + \sqrt{b^2 + 24ac}}{4c}$$

and denote the identity matrix by $I$. The subspace of uniaxial $Q$-tensors is given by

$$S_u := \{ Q \in S_0 : Q = s_+ \left( u \otimes u - \frac{1}{3} I \right), \ u \in S^2 \}.$$

In this paper, we only consider the case of positive constants $a, b, c$, which corresponds to a lower temperature regime in liquid crystals (the constant $a$ could also be negative; see \cite{34, 35}).

Let $\Omega$ be a domain in $\mathbb{R}^3$. For a tensor $Q \in W^{1,2}(\Omega; S_0)$, the Landau-de Gennes energy is defined by

$$E_{LG}(Q; \Omega) = \int_\Omega f_{LG}(Q, \nabla Q) \, dx := \int_\Omega (f_E(Q, \nabla Q) + f_B(Q)) \, dx,$$

where $f_E$ is the elastic energy density with elastic constants $L_1, \ldots, L_4$ of the form

$$f_E(Q, \nabla Q) := \frac{L_1}{2} |\nabla Q|^2 + \frac{L_2}{2} \frac{\partial Q_{ij}}{\partial x_j} \frac{\partial Q_{ik}}{\partial x_k} + \frac{L_3}{2} \frac{\partial Q_{ik}}{\partial x_j} \frac{\partial Q_{ij}}{\partial x_k} + \frac{L_4}{2} Q_{lk} \frac{\partial Q_{ij}}{\partial x_l} \frac{\partial Q_{ij}}{\partial x_k},$$

and $f_B(Q)$ is the bulk energy density defined by

$$f_B(Q) := -\frac{a}{2} tr(Q^2) - \frac{b}{3} tr(Q^3) + \frac{c}{4} \left[ tr(Q^2) \right]^2$$

with positive material constants $a, b$ and $c$. Here and in the sequel, we adopt the Einstein summation convention for repeated indices.

In \cite{11}, de Gennes discovered the first two terms of the elastic energy density in (1.2) with $L_3 = L_4 = 0$. Since both the Oseen-Frank theory and the Landau-de Gennes theory should unify for modeling uniaxial liquid crystals, Schiele and Trimper \cite{38} pointed out that the early attempt of de Gennes’ work \cite{11} was incomplete since it would require the splay and bend Frank constants to be equal (i.e. $k_1 = k_3$) in the Oseen-Frank density (as defined below in (1.5)) of uniaxial tensors $Q = s_+(u \otimes u - \frac{1}{3} I)$. However, some experiments on liquid crystals showed that $k_3 > k_1$, so they added a third order term to original de Gennes’ elastic energy density by

$$\frac{L_1}{2} |\nabla Q|^2 + \frac{L_2}{2} \frac{\partial Q_{ij}}{\partial x_j} \frac{\partial Q_{ik}}{\partial x_k} + \frac{L_4}{2} Q_{lk} \frac{\partial Q_{ij}}{\partial x_l} \frac{\partial Q_{ij}}{\partial x_k}$$

with $L_4 = \frac{1}{2} \left( k_3 - k_1 \right) > 0$. Later, Berreman and Meiboom \cite{6} observed that the above two groups discarded the surface energy density in the Oseen-Frank density, which correlates
the blue phase theory for liquid crystals, so they proposed to recover a second order term \( \frac{L_3}{3} \frac{\partial Q_{ij}}{\partial x_k} \frac{\partial Q_{ij}}{\partial x_k} \) with four third order terms, but their density is over-determined with the Oseen-Frank density. Later, Longa et al. [31] gave an extension of the Landau-de Gennes density with 22 independent parameters. Finally, combining the work of Schiele and Trimper [38] with Berreman and Meiboom [2], Dickmann [13] found the full density (1.2), which is consistent with the Oseen-Frank density in (1.5) for uniaxial nematic liquid crystals. Since then, the general form (1.2) of the Landau-de Gennes energy density has been widely used in the study of nematic liquid crystals (e.g. [1, 33, 35]). Under the relation that one can show (c.f. [33]) that for a uniaxial tensor \( Q = s_+ (u \otimes u - \frac{1}{3} I) \) with \( u \in S^2 \),

\[
\begin{align*}
\begin{aligned}
\mathcal{E}_L(Q, \nabla Q) &= W(u, \nabla u), \\
W(u, \nabla u) &= \frac{k_1}{2} (\text{div } u)^2 + \frac{k_2}{2} (u \cdot \text{curl } u)^2 + \frac{k_3}{2} |u \times \text{curl } u|^2 \\
&\quad + \frac{k_2 + k_4}{2} (\text{tr}(\nabla u)^2 - (\text{div } u)^2)
\end{aligned}
\end{align*}
\]

for a unit director \( u \in W^{1,2}(\Omega; S^2) \). In (1.5), \( k_1, k_2, k_3 \) are the Frank constants for molecular distortion of splay, twist and bend respectively and \( k_4 \) is the Frank constant for the surface energy (c.f. [12]). In 1937, Zvetkov established numerical values for p-azoxyanisole (PAA) at 120°C (with the unit 10^{-12} m/J) as follows:

\[
\begin{align*}
&k_1 = 5, \quad k_2 = 3.8, \quad k_3 = 10.1.
\end{align*}
\]

Therefore, according to physical experiments on nematic liquid crystals, the elastic constant \( L_4 = \frac{1}{2s_+}(k_3 - k_1) \) is not equal to zero in general (c.f. [12]).

A fundamental problem in mathematics on the Landau-de Gennes theory is to establish existence of a minimizer of the energy functional \( \mathcal{E}_L(Q, \Omega) \) in \( W^{1,2}_0(\Omega; S_0) \) with \( L_4 \neq 0 \). If the functional density \( f_{LG}(Q, \nabla Q) \) satisfies the coercivity condition, one can prove existence of a minimizer of the functional \( \mathcal{E}_L(Q, \Omega) \) in \( W^{1,2}(\Omega; S_0) \). In 2010, Ball and Majumdar [2] found an example where for \( Q \in S_0 \), the general Landau-de Gennes energy density (1.2) with \( L_4 \neq 0 \) does not satisfy the coercivity condition. Very recently, Golovaty et al. [22] emphasized that “From the standpoint of energy minimization, unfortunately, such a version of Landau-de Gennes becomes problematic, since the inclusion of the cubic term leads to an energy which is unbounded from below”. Therefore, the Landau-de Gennes density (1.2) causes a knowledge gap between mathematical and physical theories on liquid crystals, since the energy functional \( \mathcal{E}_L(Q, \Omega) \) in \( W^{1,2}(\Omega; S_0) \) does not satisfy the coercivity condition and violates the existence theorem of minimizers (e.g. [1, 18]). In physics, concerning the third order term \( \frac{L_4}{2} Q_{ij} \frac{\partial Q_{ij}}{\partial x_k} \frac{\partial Q_{ij}}{\partial x_k} \) with \( L_4 \neq 0 \) in (1.2), Longa et al. [31] questioned that “In the presence of biaxial fluctuations the general third order theory in \( Q_{ij} \) becomes unstable and thus is thermodynamically incorrect”. In order to overcome the difficulty, they extended Landau-de Gennes densities through 22 independent second, third, fourth order terms, to preserve the stability of the free energy. Although their result is very interesting, all energy densities in [31] are complicated and have not addressed the above coercivity problem for all \( Q \)-tensors. In 2020, following similar spirit in [31], Golovaty et al. [22] proposed a new physical interpretation of the density through fourth order terms to address the above
coercivity problem for all $Q$-tensors. We would like to point out that the new density form in [22] is completely different from the original Landau-de Gennes density (1.2) although the density in [22] can recover the Oseen-Frank density for uniaxial $Q$-tensors.

In this paper, we will propose a new Landau-de Gennes energy density to solve the above coercivity problem with $L_4 \neq 0$. At first, we observe in Lemma 2.1 that for uniaxial tensors $Q \in S_*$, the original third order term on $L_4$ in (1.2), proposed by Schiele and Trimmer [38, p. 268] in physics, is a linear combination of a fourth order term and a second order term in the following:

$$Q_{lk} \frac{\partial Q_{ij}}{\partial x_l} \frac{\partial Q_{ij}}{\partial x_k} = \frac{3}{s_+} \left( Q_{ln} \frac{\partial Q_{ij}}{\partial x_l} \right) \left( Q_{kn} \frac{\partial Q_{ij}}{\partial x_k} \right) - \frac{2s_+}{3} |\nabla Q|^2. \quad (1.6)$$

In the case of $L_4 \geq 0$, we introduce a new elastic energy density

$$f_{E,1}(Q, \nabla Q) = \left( \frac{L_1}{2} - \frac{s_+ L_4}{3} \right) |\nabla Q|^2 + \frac{L_2}{2} \frac{\partial Q_{ij}}{\partial x_j} \frac{\partial Q_{ij}}{\partial x_k} + \frac{L_3}{2} \frac{\partial Q_{ik}}{\partial x_j} \frac{\partial Q_{ij}}{\partial x_k} + \frac{3L_4}{2s_+} Q_{ln} \frac{\partial Q_{ij}}{\partial x_l} \frac{\partial Q_{ij}}{\partial x_k} \quad (1.7)$$

for all $Q \in S_0$. We should point out that the fourth order term $Q_{ln} \frac{\partial Q_{ij}}{\partial x_l} \frac{\partial Q_{ij}}{\partial x_k}$ in (1.7) is a non-negative square term, so our new Landau-de Gennes density (1.7) for $Q \in S_0$ satisfies the coercivity condition in mathematics under suitable conditions on $L_1, \ldots, L_4$. The first three terms in (1.7) keep the original form (1.2) for $Q \in S_0$ and the new Landau-de Gennes density (1.7) is equivalent to the original density (1.2) for $Q \in S_*$. We also remark that our fourth order term $Q_{ln} \frac{\partial Q_{ij}}{\partial x_l} \frac{\partial Q_{ij}}{\partial x_k}$ is a linear combination of three fourth order terms $L_5^{(4)}, L_6^{(4)}, L_7^{(4)}$ in [31]; i.e., we verify in Lemma 2.2 that

$$Q_{ln} \frac{\partial Q_{ij}}{\partial x_l} \frac{\partial Q_{ij}}{\partial x_k} = \frac{8}{5} L_5^{(4)} - \frac{2}{5} L_6^{(4)} + \frac{2}{5} L_7^{(4)}.$$ 

For a $Q \in W^{1,2}(\Omega, S_0)$, we introduce a new Landau-de Gennes energy functional

$$E_L(Q; \Omega) = \int_\Omega f_{LG}(Q, \nabla Q) \, dx = \int_\Omega \left( f_{E,1}(Q, \nabla Q) + \frac{1}{L} \tilde{f}_B(Q) \right) \, dx, \quad (1.8)$$

where $f_{E,1}(Q, \nabla Q)$ has the form (1.7), $\tilde{f}_B(Q) := f_B(Q) - \min_{Q \in S_0} f_B(Q) \geq 0$ and $L > 0$ is a parameter to drive all elastic constants to zero [5, 17, 35].

Although there are many differences between the Oseen-Frank theory and the Landau-de Gennes theory, it is of great interest in mathematics whether minimizers of the Landau-de Gennes energy functional can approach a minimizer of the Oseen-Frank energy functional. When $L_2 = L_3 = L_4 = 0$ in (1.7), Majumdar and Zarnescu [34] first proved that as $L \to 0$, minimizers $Q_L$ of $E_L$ converges to $Q_* = s_+ (u^* \otimes u^* - \frac{1}{3} I)$, where $Q_*$ is a minimizer of the Dirichlet energy functional in $W^{1,2}_0(\Omega; S_*)$. Since then, there exist many developments on the one-constant approximation (c.f. [1]) and some special cases of unequal constants $L_2, L_3, L_4$ ([5, 29]). In theory of liquid crystals, the general expectation on the elastic constants is that $L_4$ is not always zero (c.f. [38, p. 268], [2]). For the case of $L_4 \neq 0$, we first prove

**Theorem 1**  Let $L_1, L_2, L_3$ and $L_4$ be elastic constants satisfying

$$L_1 - \frac{s_+ L_4}{6} > 0, \quad -L_1 - \frac{s_+ L_4}{6} < L_3 < 2L_1 - \frac{s_+ L_4}{3},$$

\[ \square \] Springer
\[ L_1 - \frac{s_+ L_4}{6} + \frac{5}{3} L_2 + \frac{1}{6} L_3 > 0, \quad L_4 \geq 0. \]  

(1.9)

Then, for each \( L > 0 \), \( f_{E,L}(Q, \nabla Q) \) in (1.8) satisfies the coercivity condition so that there exists a minimizer \( Q_L \) of the functional (1.8) in \( W^{1,2}_{\Omega_0}(\Omega; S_0) \) with boundary value \( Q_0 \in W^{1,2}(\Omega; S_\ast) \). As \( L \to 0 \), the minimizers \( Q_L \) of \( E_L(Q; \Omega) \) converge (up to a subsequence) strongly to \( Q_\ast \) in \( W^{1,2}_{\Omega_0}(\Omega; S_0) \) and satisfies

\[ \lim_{L \to 0} \frac{1}{L} \int_{\Omega} \tilde{f}_B(Q_L) \, dx = 0. \]

Furthermore, \( Q_\ast \) is a minimizer of the functional \( E(Q; \Omega) := \int_{\Omega} f_{E,1}(Q, \nabla Q) \, dx \) for all uniaxial \( Q \)-tensors in \( W^{1,2}_{\Omega_0}(\Omega; S_\ast) \).

**Remark 1** In the case of \( L_4 < 0 \), for each \( Q \in W^{1,2}(\Omega, S_0) \), we introduce an elastic energy density by

\[ f_{E,-}(Q, \nabla Q) := \left( \frac{L_1}{2} + \frac{s_+ L_4}{3} \right) |\nabla Q|^2 + \frac{L_2}{2} \partial Q_{ij} \partial Q_{ik} + \frac{L_3}{2} \partial Q_{ik} \partial Q_{ij} - \frac{3}{s_+} L_4 \left( |Q|^2 |\nabla Q|^2 - Q_{ln} Q_{kn} \partial Q_{ij} \partial Q_{ij} \right). \]

(1.10)

It is clear that \( |Q|^2 |\nabla Q|^2 - Q_{ln} \partial Q_{ij} \partial Q_{kn} \partial Q_{ij} \geq 0 \) for each \( Q \in W^{1,2}(\Omega, S_0) \), and that the elastic energy density \( f_{E}(Q, \nabla Q) \) in (1.2) is equal to \( f_{E,-}(Q, \nabla Q) \) for uniaxial tensors \( Q \in S_\ast \).

Next, we discuss critical points of the Landau-de Gennes energy functional (1.7) in \( W^{1,2}_{\Omega_0}(\Omega; S_0) \). One can write \( f_{E}(Q, \nabla Q) := \frac{\alpha}{2} |\nabla Q|^2 + V(Q, \nabla Q) \) for some \( \alpha > 0 \) so that \( V(Q, \nabla Q) \geq 0 \) for all \( Q \in W^{1,2}_{\Omega_0}(\Omega; S_0) \). Then, the Euler-Lagrange equation for the Landau-de Gennes energy functional (1.8) in \( W^{1,2}_{\Omega_0}(\Omega; S_0) \cap L^\infty(\Omega; S_0) \) is

\[ \alpha \Delta Q + \frac{1}{2} \nabla k \left( V_{Q_{sk}} + V_{Q_{sk}}^T \right) - \frac{1}{3} I \text{ tr} \left( \nabla k V_{Q_{sk}} \right) - \frac{1}{2} \left( V_Q + V_Q^T \right) + \frac{1}{3} I \text{ tr} \left( V_Q \right) \]

\[ = \frac{1}{L} \left( -a Q - b \left( Q Q - \frac{1}{3} I \text{ tr} \left( Q^2 \right) \right) + c Q \text{ tr} \left( Q^2 \right) \right) \]

(1.11)

in the weak sense, where \( A^T \) denotes the transpose of \( A \),

\[ V_Q := \frac{\partial V(Q, \nabla Q)}{\partial Q} \text{ and } V_{Q_{sk}} := \frac{\partial V(Q, \nabla Q)}{\partial Q_{sk}}. \]

For general elastic constants \( L_1, \cdots, L_4 \), we cannot find any reference having an explicit form of the Euler-Lagrange equation of \( E(Q; \Omega) \) for \( Q = s_+(u \otimes u - \frac{1}{3} I) \in S_\ast \) with \( u \in S^2 \), so we give an explicit form of the Euler-Lagrange equation in the following:

\[ \alpha \left( s_+ \Delta Q - 2 \nabla k Q \nabla k Q + 2 s_+^{-1} \left( Q + \frac{s_+}{3} I \right) |\nabla Q|^2 \right) \]

\[ + \nabla k \left( V_{Q_{sk}} \left( Q + \frac{s_+}{3} I \right) + \left( Q + \frac{s_+}{3} I \right) V_{Q_{sk}}^T \right) - 2 s_+^{-1} \left( Q + \frac{s_+}{3} I \right) \left( Q + \frac{s_+}{3} I, V_{Q_{sk}} \right) \]

\[ - V_{Q_{sk}} \nabla k Q - \nabla k Q V_{Q_{sk}}^T \]
\[
+ 2s^+ \left[ \langle V_{Qk}, \nabla_k Q \rangle \left( Q + \frac{s^+}{3} I \right) + \left\langle V_{Qk}, \left( Q + \frac{s^+}{3} I \right) \right\rangle \nabla_k Q \right]
- \left( Q + \frac{s^+}{3} I \right) V_Q^T - V_Q \left( Q + \frac{s^+}{3} I \right)
+ 2s^+ \left( V_Q, Q + \frac{s^+}{3} I \right) \left( Q + \frac{s^+}{3} I \right) = 0,
\]
which is equivalent to the Oseen-Frank system for \( u \in S^2 \), where \( \langle A, B \rangle = \text{tr}(B^T A) \) is the standard inner product of two matrices \( A \) and \( B \). In the case of \( L_2 = L_3 = L_4 = 0 \), the Euler-Lagrange Eq. (1.12) reduces to
\[
s^+ \Delta Q - 2 \frac{\partial Q}{\partial x_k} \frac{\partial Q}{\partial x_k} + 2s^+ \left( Q + \frac{s^+}{3} I \right) |\nabla Q|^2 = 0,
\]
which is equivalent to the harmonic map equation of \( u \) (c.f. [36]).

Since the Landau-de Gennes theory has been successfully used for modeling both uniaxial and biaxial states of nematic liquid crystals, it is of great interest whether the \( Q \)-tensor type of the Oseen-Frank system can be approximated by the Landau-de Gennes system (1.11) without using minimizers. In general, the problem of the convergence of solutions of the Landau-de Gennes Eq. (1.11) without using minimizers is open. Indeed, Gartland [17] pointed out that the convergence of solutions of the Landau-de Gennes Eq. (1.11) is similar to the convergence of solutions of the Ginzburg-Landau approximate equation from superconductivity theory. The Ginzburg-Landau functional was introduced in [20] to study the phase transition in superconductivity. For a parameter \( \varepsilon > 0 \), the Ginzburg-Landau system (1.14) weakly converge to a harmonic map in \( \mathbb{R}^3 \) as \( \varepsilon \to 0 \), solutions \( Q_L \) of the Ginzburg-Landau system (1.14) weakly converge to a harmonic map in \( W^{1,2}(\Omega; \mathbb{R}^3) \).

Chen and Struwe [9] proved global existence of partial regular solutions to the heat flow of harmonic maps using the Ginzburg-Landau approximation. In [3, 4], Bethuel, Brezis and Hélein obtained many results on asymptotic behavior for minimizers of \( E_\varepsilon \). In two dimensions as \( \varepsilon \to 0 \) (see also [39]). Recently, many works ([15, 25–27]) have examined the convergence of the Ginzburg-Landau approximation for the Ericksen-Leslie system with unequal Frank’s constants \( k_1, k_2, k_3 \). Motivated by the above results on the Ginzburg-Landau approximation, it is natural to investigate the converging problem on solutions of the Landau-de Gennes system (1.11) as \( L \to 0 \). By comparing with the result of Chen [7] (see also [8]) on the weak convergence of solutions of the Ginzburg-Landau equations, it is interesting to study whether solutions \( Q_L \) of the Landau-de Gennes equations (1.11) with uniform bound of the energy converge weakly to a solution \( Q_* \) of (1.12) in \( W^{1,2}_{Q_0}(\Omega; S_0) \). However, it seems that the problem is not clear when \( L_2, L_3, L_4 \) are not zero. Under a condition, we solve this problem and prove:

**Theorem 2** Let \( Q_L \) be a weak solution to the equation (1.11) with a uniform bound in \( L \). Assume that the solution \( Q_L \) converges strongly to \( Q_* \) in \( W^{1,2}_{Q_0}(\Omega; S_0) \) as \( L \to 0 \) and satisfies
\[
\lim_{L \to 0} \frac{1}{L} \int_\Omega f_B(Q_L) \, dx = 0.
\]
Then, $Q_*$ is a weak solution to (1.12).

For the proof of Theorem 2, we use a concept of a projection $\pi$ in a neighborhood $S_\delta$ of the space $S_\delta$, where

$$S_\delta := \{ Q \in S_0 : \text{dist}(Q; S_\delta) \leq \delta \}.$$ 

For a sufficiently small $\delta > 0$, there exists a smooth projection $\pi : S_\delta \to S_\delta$ so that for any $Q \in S_\delta$, $\pi(Q) \in S_\delta$ (c.f. [9]). By the projection $\pi$, we consider the modified bulk energy density

$$F(Q) := \tilde{f}_B(Q) + |Q - \pi(Q)|^2$$

so that the Hessian of $F(Q)$ is positive definite for each any $Q \in S_\delta$ with a sufficiently small $\delta > 0$. Then, choosing a suitable test function and using (1.15), we employ Taylor’s expansion of $F(Q)$ to cancel the limit term involving $\frac{1}{T} \nabla Q_{ij} f_B(Q_L)$. With these results, we divide the domain into three parts and then employ Egoroff’s theorem to prove Theorem 2.

When $L_2 = L_3 = L_4 = 0$, Majumdar and Zarnescu [34] first proved that minimizers $Q_L$ of $E_{LG}(Q; \Omega)$ uniformly converge to $Q_*$ away from the singular set of $Q_*$ since there exists a monotonicity formula for minimizers $Q_L$ of $E_{LG}(Q; \Omega)$ in $W^{1,2}(\Omega, S_0)$. Later, Nguyen and Zarnescu [36] improved the result by proving local smooth convergence of minimizers $Q_L$ away from the singular set of $Q_*$. For this question, we generalize Nguyen and Zarnescu’s result in [36] to the case of non-zero elastic constants $L_2, L_3, L_4$ as follows:

**Theorem 3** For each $L > 0$, let $Q_L$ be a weak solution to the equation (1.11) and let $Q_*$ be the limiting map of $Q_L$ in Theorem 2. Assume that $Q_L$ is smooth and converges to $Q_*$ uniformly inside $\Omega \setminus \Sigma$, where $\Sigma$ is the singular set of $Q_*$. Then, as $L \to 0$ (up to a subsequence), we have

$$Q_L \to Q_* \text{ in } C^k_{\text{loc}}(\Omega \setminus \Sigma)$$

for any positive integer $k \geq 0$.

We would like to point out that our proof of Theorem 3 is new and different from one in [36]. We outline main steps as follows:

Step I. For each $Q \in S_\delta$, there exists a rotation $R(Q) \in SO(3)$ such that $\tilde{Q} = R^T(Q)Q_R(Q)$ is diagonal. For any $\xi \in S_0$, we prove

$$\sum_{i,j=1}^3 \partial_{\tilde{Q}_{ii}} \partial_{\tilde{Q}_{jj}} f_B(Q)\xi_{ii}\xi_{jj} \geq \frac{\lambda}{2} \left( \xi_{11}^2 + \xi_{22}^2 + \xi_{33}^2 \right),$$

where $\lambda = \min\{3a, s_+, b\} > 0$. For each smooth $Q(x) \in S_0$, $R^T(Q(x))Q(x)R(Q(x))$ is diagonal. Then there exists a measure zero set $\Omega_0$ such that $Q(x)$ has a constant multiplicity of eigenvalues inside subdomains of $\Omega \setminus \Omega_0$ and $R(Q(x))$ is almost differentiable in $\Omega$. Using the geometric identity

$$\nabla \left( R^T(Q)QR(Q) \right)_{ii} = \left( R^T(Q)\nabla QR(Q) \right)_{ii}$$

with $i = 1, 2, 3$, we apply (1.17) to obtain

$$\int_{B_{\rho_0}(x_0)} |\nabla^2 Q_L|^2 \phi^2 \, dx \leq C \int_{B_{\rho_0}(x_0)} |\nabla Q_L|^2 |\nabla \phi|^2 \, dx,$$
for a uniform constant $C$ in $L$, where $\phi$ is a cutoff function in $B_{r_0}(x_0) \subseteq \Omega \setminus \Sigma$.

Step II. Using Step I with the technique of ‘filling hole’ on elliptic systems [18], we establish a uniform Caccioppoli inequality for solutions $Q_L$ of (2.10) in $L$; i.e., there exists a uniform constant $C$ independent of $L$ such that

$$\int_{B_{r/2}(x_0)} |\nabla Q_L|^2 \, dx \leq \frac{C}{r^2} \int_{B_r(x_0)} |Q_L - Q_{L,x_0,r}|^2 \, dx$$  \hspace{1cm} (1.18)

for any $x_0$ with $B_{r_0}(x_0) \subseteq \Omega \setminus \Sigma$ and any $r \leq r_0$, where $Q_{L,x_0,r} := \int_{B_r(x_0)} Q_L \, dx$.

Step III. Based on the uniform Caccioppoli inequality (1.18), we apply the well-known Gagliardo-Nirenberg interpolation (c.f. [28] or [15]) to obtain a control on the local $L^3$-estimate; i.e., there exists a uniform constant $r_0 > 0$ such that

$$\int_{B_{r_0}(x_0)} |\nabla Q_L|^3 \, dx \leq \varepsilon_0$$  \hspace{1cm} (1.19)

for some small $\varepsilon_0 > 0$. Then, combining (1.19) with (1.17), we apply an induction method to obtain uniform estimates on higher derivatives $\nabla^k Q_L$ in $L$ and prove Theorem 2.

**Remark 2** When $L_4$ is sufficiently small, the $L^p$-theory on the constant elliptic system (c.f. [18]) can assure that the weak solution $Q_L$ to the equation (1.11) is smooth in $\Omega$. In the case of $L_4 = 0$, Contreras and Lamy [10] proved that the minimizers $Q_L$ uniformly converge to $Q_*$ in $\Omega \setminus \Sigma$ by assuming that $Q_L$ is uniformly bounded.

**Remark 3** In a recent work [16] with Yu Mei, we can expand ideas on proofs of Theorems 2-3 to show that solutions for the Beris-Edward system for biaxial $Q$-tensors converge smoothly to the solution of the Beris-Edward system for uniaxial $Q$-tensors up to its maximal existence time.

Finally, we make some remarks about new forms of the Landau-de Gennes energy density through a strong Ericksen’s condition on the Oseen-Frank density. Recently, Golovaty et al. [22] proposed a novel form of the Landau-de Gennes energy density through the Oseen-Frank density. In addition to Ericksen’s inequalities that $k_2 \geq |k_4|$, $k_3 \geq 0$, $2k_1 \geq k_2 + k_4$, they also assumed that

$$k_1 \geq k_2 + k_4, \quad k_3 \geq k_2 + k_4, \quad k_4 \leq 0.$$  \hspace{1cm} (1.20)

However, their result is not optimal. In fact, Golovaty et al. [22] made a further remark that their assumption (1.20) could be relaxed if one includes more cubic terms in [31], but they did not do it. In Sect. 5, we improve their result to derive an implict form of $f_E(Q, \nabla Q)$. Assuming the strong Ericksen inequality with a weaker assumption that $2k_3 > k_2 + k_4$ instead of the condition (1.20), we write the Oseen-Frank density into a new form, which satisfies the coercivity condition for all $u \in \mathbb{R}^3$. In [12], de Gennes and Prost remarked that the bending constant $k_3$ is much larger than others $k_1$ and $k_2$. Therefore, the assumption $2k_3 > k_2 + k_4$ is satisfied through a strong Ericksen’s condition. We give an explicit form of $f_E(Q, \nabla Q)$ in Proposition 5.1, which satisfies the coercivity condition for all $Q \in S_0$.

The paper is organized as follows. In Sect. 2, we prove Theorem 1. In Sect. 3, we prove Theorem 2. In Sect. 4, we prove Theorem 3. In Sect. 5, we obtain new forms of the Landau-de Gennes energy density through a strong Ericksen’s condition.
### 2 The coercivity condition and existence of minimizers

At first, we note

**Lemma 2.1** For a uniaxial $Q = s_+ (u \otimes u - \frac{1}{3} I) \in S_*$ with $u \in S^2$, we have

$$Q_{lk} \frac{\partial Q_{ij}}{\partial x_l} \frac{\partial Q_{ij}}{\partial x_k} = \frac{3}{s_+} \left( Q_{ln} \frac{\partial Q_{ij}}{\partial x_l} \right) \left( Q_{kn} \frac{\partial Q_{ij}}{\partial x_k} \right) - \frac{2s_+}{3} |\nabla Q|^2 . \quad (2.1)$$

**Proof** Using the fact that $|u| = 1$, we have

$$Q_{ln} Q_{kn} = s_+^2 \left( u_k u_n - \frac{1}{3} \delta_{kn} \right) \left( u_l u_n - \frac{1}{3} \delta_{ln} \right)$$

$= s_+^2 \left( u_k u_l u_n u_n - \frac{1}{3} \delta_{kn} u_l u_n - \frac{1}{3} \delta_{ln} u_k u_n + \frac{1}{3} \delta_{ln} \delta_{kn} \right)$

$= s^2 \left( \frac{1}{3} u_k u_l + \frac{1}{9} \delta_{kl} \right) = \frac{s_+}{3} s_+ \left( u_k u_l - \frac{1}{3} \delta_{lk} \right) + \frac{2s_+^2}{9} \delta_{kl}$

$$= \frac{s_+}{3} Q_{kl} + \frac{2s_+^2}{9} \delta_{kl} . \quad (2.2)$$

Through the identity (2.2), we obtain

$$Q_{lk} \frac{\partial Q_{ij}}{\partial x_l} \frac{\partial Q_{ij}}{\partial x_k} = \left( \frac{3}{s_+} Q_{ln} Q_{kn} - \frac{2s_+}{3} \delta_{kl} \right) \frac{\partial Q_{ij}}{\partial x_l} \frac{\partial Q_{ij}}{\partial x_k}$$

$$= \frac{3}{s_+} \left( Q_{ln} \frac{\partial Q_{ij}}{\partial x_l} \right) \left( Q_{kn} \frac{\partial Q_{ij}}{\partial x_k} \right) - \frac{2s_+}{3} |\nabla Q|^2 .$$

Recall from Longa et al. [31] that

$$L^{(4)}_5 := Q_{\alpha \rho} Q_{\rho \beta} \frac{\partial Q_{\alpha \mu}}{\partial x_\beta} \frac{\partial Q_{\mu \nu}}{\partial x_\nu}, \quad L^{(4)}_6 := Q_{\alpha \rho} Q_{\rho \beta} \frac{\partial Q_{\alpha \mu}}{\partial x_\mu} \frac{\partial Q_{\beta \nu}}{\partial x_\nu} , \quad L^{(4)}_7 := Q_{\alpha \rho} Q_{\rho \beta} \frac{\partial Q_{\alpha \mu}}{\partial x_\nu} \frac{\partial Q_{\beta \mu}}{\partial x_\nu} .$$

Then we have

**Lemma 2.2** For a uniaxial $Q \in S_*$, we obtain

$$Q_{ln} Q_{kn} \frac{\partial Q_{ij}}{\partial x_l} \frac{\partial Q_{ij}}{\partial x_k} = \frac{8}{5} L^{(4)}_5 - \frac{2}{5} L^{(4)}_6 + \frac{2}{5} L^{(4)}_7 . \quad (2.3)$$

**Proof** Let $Q = s_+ (u \otimes u - \frac{1}{3} I)$ for $u \in S^2$. Noting that $u_i \nabla u_i = 0$, we calculate

$$(u \times \nabla u)_i^2 = [(u_2 \nabla u_2 - u_1 \nabla u_1) - u_3 (\nabla u_3 - \nabla u_1)]^2$

$$= (-u_1 \nabla u_1 - u_2 \nabla u_2 - u_3 \nabla u_3)^2 = [(u \cdot \nabla) u_i]_i^2 .$$

Similarly, we can calculate other terms to obtain

$$\sum_i [(u \cdot \nabla) u_i]^2 = \sum_i (u \times \nabla u)_i^2 = |u \times \nabla u|^2 .$$
Moreover, we calculate

$$Q_{ik} \frac{\partial Q_{ij}}{\partial x_i} \frac{\partial Q_{ij}}{\partial x_k} = s^3_+ \left( u_i u_k - \frac{1}{3} \delta_{i} \right) \nabla_i (u_i u_j) \nabla_k (u_i u_j)$$

$$= s^3_+ \left( u_i u_k - \frac{1}{3} \delta_{i} \right) (u_j \nabla_i u_i + u_i \nabla_i u_j) (u_j \nabla_k u_i + u_i \nabla_k u_k)$$

$$= s^3_+ \left( u_i u_k - \frac{1}{3} \delta_{i} \right) (\nabla_i u_i \nabla_k u_i + \nabla_i u_j \nabla_k u_j)$$

$$= 2s^3_+ \sum_i (u_i \cdot \nabla) u_i)^2 - \frac{2}{3} s^3_+ |\nabla u|^2 = 2s^3_+ |u \times \nabla u|^2 - \frac{2}{3} s^3_+ |\nabla u|^2.$$ 

(2.4)

It follows from using (2.1) and (2.4) that

$$Q_{ln} Q_{kn} \frac{\partial Q_{ij}}{\partial x_i} \frac{\partial Q_{ij}}{\partial x_k} = \frac{1}{3} s^3_+ Q_{ik} \frac{\partial Q_{ij}}{\partial x_i} \frac{\partial Q_{ij}}{\partial x_k} + \frac{2s^3_+}{9} |\nabla Q|^2$$

$$= \frac{2}{3} s^3_+ |u \times \nabla u|^2 + \frac{2}{9} s^3_+ |\nabla u|^2.$$ (2.5)

We can verify from [31] that

$$4L_5^{(4)} - L_6^{(4)} = \frac{5s^4_+}{3} |u \times \nabla u|^2, \quad L_7^{(4)} = \frac{5}{9} s^4_+ |\nabla u|^2.$$ (2.6)

Substituting (2.6) into (2.5), we have

$$Q_{ln} Q_{kn} \frac{\partial Q_{ij}}{\partial x_i} \frac{\partial Q_{ij}}{\partial x_k} = \frac{2}{5} (4L_5^{(4)} - L_6^{(4)}) + \frac{2}{5} L_7^{(4)}.$$ 

□

Under the condition (1.9), one can verify from Lemma 1.2 in [30] that there are two uniform constants $\alpha > 0$ and $C > 0$ such that the form $f_{E, 1}(Q, p)$ also satisfies

$$\frac{\alpha}{2} |p|^2 \leq f_{E, 1}(Q, p) \leq C (1 + |Q|^2) |p|^2, \quad |\partial_Q f_{E, 1}(Q, p)| \leq C (1 + |Q|) |p|^2$$ (2.7)

for any $Q \in M^{3 \times 3}$ and $p \in M^{3 \times 3} \times R^3$. Since $f_{E, 1}(Q, p)$ is quadratic in $p$ and satisfies (2.7), it can be checked (c.f. [28]) that $f_{E, 1}(Q, p)$ is uniformly convex in $p$; that is

$$\frac{\partial^2}{\partial p_{ij}^2} f_{E, 1}(Q, p) \xi^i_k \xi^j_m \geq \frac{\alpha}{2} |\xi|^2, \quad \forall \xi \in M^{3 \times 3} \times R^3.$$ (2.8)

Now we give a proof of Theorem 1.

**Proof** Under the condition on $L_1, \cdots, L_4$ in Theorem 1, it is clear from using Lemma 1.2 in [30] that

$$f_{E, 1}(Q, \nabla Q) \geq \frac{\alpha}{2} |\nabla Q|^2, \quad \forall Q \in S_0.$$ 

By the standard theory in the calculus of variations (c.f. [17]), there exists a minimizer $Q_L$ of $E_L$ in $W^{1, 2}_{Q_0}(\Omega; S_0)$. For each $Q \in W^{1, 2}_{Q_0}(\Omega; S_0)$, we set

$$E(Q; \Omega) := \int_{\Omega} f_{E, 1}(Q, \nabla Q) \, dx.$$
It implies that

$$E(Q_L; \Omega) + \int_{\Omega} (f_B(Q_L) - \inf_{S_0} f_B) \, dx \leq E(Q; \Omega)$$

for any $Q \in W^{1,2}_{Q_0}(\Omega; S_a)$ with the fact that $\bar{f}_B(Q) = f_B(Q) - \inf_{S_0} f_B = 0$.

As $L \to 0$, minimizers $Q_L$ converge (possible passing subsequence) weakly to a tensor $Q_* \in W^{1,2}(\Omega; S_0)$ with that $f_B(Q_*) = 0$, which implies that $Q_* \in S_* \text{ a.e. in } \Omega$. Then, for any $Q \in W^{1,2}_{Q_0}(\Omega; S_*)$, we have

$$E(Q_\ast; \Omega) \leq \liminf_{L \to 0} E(Q_L; \Omega) \leq \limsup_{L \to 0} E(Q_L; \Omega) \leq E(Q; \Omega).$$

Therefore $Q_*$ is also a minimizer of $E$ in $W^{1,2}_{Q_0}(\Omega; S_*)$. Choosing $Q = Q_*$ in the above inequality, it implies that

$$E(Q_*; \Omega) = \lim_{L \to 0} E_L(Q_L; \Omega), \quad \lim_{L \to 0} \frac{1}{L} \int_{\Omega} \bar{f}_B(Q_L) \, dx = 0.$$  

Moreover, it is known that

$$\int_{\Omega} |\nabla Q_*|^2 \, dx \leq \liminf_{L \to 0} \int_{\Omega} |\nabla Q_L|^2 \, dx,$$

$$\int_{\Omega} V(Q_\ast, \nabla Q_\ast) \, dx \leq \liminf_{L \to 0} \int_{\Omega} V(Q_L, \nabla Q_L) \, dx.$$  

It implies that $\int_{\Omega} |\nabla Q_*|^2 \, dx = \liminf_{L \to 0} \int_{\Omega} |\nabla Q_L|^2 \, dx$. Otherwise, there is a subsequence $L_k \to 0$ such that

$$\int_{\Omega} |\nabla Q_*|^2 \, dx < \lim_{L_k \to 0} \int_{\Omega} |\nabla Q_{L_k}|^2 \, dx.$$  

Then

$$E(Q_*; \Omega) = \lim_{L_k \to 0} E_{L_k}(Q_{L_k}; \Omega),$$

$$= \frac{\alpha}{2} \lim_{L_k \to 0} \int_{\Omega} |\nabla Q_{L_k}|^2 \, dx + \lim_{L_k \to 0} \int_{\Omega} V(Q_{L_k}, \nabla Q_{L_k}) \, dx$$

$$< E(Q_*; \Omega).$$

This is impossible. Therefore, minimizers $Q_{L_k}$ converge strongly, up to a subsequence, to a minimizer $Q_* = s_+ (u_* \otimes u_* - \frac{1}{3} I)$ of $E$ in $W^{1,2}_{Q_0}(\Omega; S_0)$. Following from Lemma 3.1, $Q_*$ satisfies (1.12) and $u_*$ is a minimizer of the Oseen-Frank energy in $W^{1,2}(\Omega; S^2)$. Due to the well-known result of Hardt, Kinderlehrer and Lin [23], $u_*$ is partially regular in $\Omega$ (see also [24]). Thus $Q_*$ is partially regular.  

\[ \square \]

**Lemma 2.3** If $Q$ is a minimizer of $E_L(Q; \Omega)$ from (1.8) in $W^{1,2}_{Q_0}(\Omega; S_0)$, it satisfies

$$- \alpha \Delta Q_{ij} - \frac{1}{2} \nabla_k (V_{p_{ij}} + V_{p_{ij}}) + \frac{1}{3} \delta_{ij} \sum_{l=1}^3 \nabla_k V_{p_{li}}$$

$$+ \frac{1}{2} (V_{Q_{ij}} + V_{Q_{ij}}) - \frac{1}{3} \delta_{ij} \sum_{l=1}^3 V_{Q_{ll}}$$

\[ \square \]
\[ + \frac{1}{L} \left( -a Q_{ij} - b \left( Q_{ik} Q_{kj} - \frac{1}{3} \delta_{ij} \text{tr}(Q^2) \right) + c Q_{ij} \text{tr}(Q^2) \right) = 0 \]
in the weak sense, where \( V_{p_{ij}} := V_{p_{ij}}(Q, p) \) with \( p = (\nabla_k Q_{ij}) \).

**Proof** For any test function \( \phi \in C_0^\infty(\Omega; S_0) \), consider \( Q_t := Q + t \phi \) for \( t \in \mathbb{R} \). Then for all \( \phi \in C_0^\infty(\Omega; S_0) \), we calculate

\[
\int_\Omega \frac{d}{dt} \left( \tilde{f}_{E,1}(Q_t, \nabla Q_t) + \frac{1}{L} \tilde{f}_B(Q_t) \right)_{t=0} \, dx
\]

\[
= \int_\Omega \alpha \frac{\partial Q_{ij}}{\partial x_k} \frac{\partial \phi_{ij}}{\partial x_k} + V_{p_{ij}} \frac{\partial \phi_{ij}}{\partial x_k} + V_{Q_{ij}} \phi_{ij} \, dx
\]

\[
+ \frac{1}{L} \int_\Omega -a Q_{ij} \phi_{ij} - b Q_{ik} Q_{kj} \phi_{ij} + c (Q_{ij} \text{tr}(Q^2) \phi_{ij}) \, dx
\]

\[
= \int_\Omega \left( -a \Delta Q_{ij} - \frac{1}{2} \frac{\partial}{\partial x_k} \left( V_{p_{ij}} + V_{p_{ij}}^c \right) + \frac{1}{2} \left( V_{Q_{ij}} + V_{Q_{ji}} \right) \right) \phi_{ij} \, dx
\]

\[
+ \frac{1}{L} \int_\Omega \left( -a Q_{ij} - b Q_{ik} Q_{kj} + c Q_{ij} \text{tr}(Q^2) \right) \phi_{ij} \, dx
\]

\[
= \int_\Omega \left( -a \Delta Q_{ij} - \frac{1}{2} \nabla_k (V_{p_{ij}} + V_{p_{ij}}^c) - \frac{1}{3} \delta_{ij} \sum_l \nabla_k V_{p_{lj}}^c \right) \phi_{ij} \, dx
\]

\[
+ \int_\Omega \left( \frac{1}{2} (V_{Q_{ij}} + V_{Q_{ji}}) - \frac{1}{3} \delta_{ij} \sum_l V_{Q_{ll}} \right) \phi_{ij} \, dx
\]

\[
+ \frac{1}{L} \int_\Omega \left( -a Q_{ij} - b \left( Q_{ik} Q_{kj} - \frac{1}{3} \delta_{ij} \text{tr}(Q^2) \right) + c Q_{ij} \text{tr}(Q^2) \right) \phi_{ij} \, dx = 0,
\]

where we used the fact that \( \phi \) is traceless. This proves our claim. \( \square \)

In the case of \( L_2 = L_3 = L_4 = 0 \), Majumdar and Zarnescu [34] proved that the weak solution of \( (1.11) \) is bounded by using a maximum principle. However, when \( L_2, L_3, L_4 \) are non-zero, the system \( (1.11) \) is a nonlinear elliptic system, so there exists no such maximum principle for it (e.g. [17, 21]). Therefore, it is not clear whether each minimizer \( Q_L \) of \( E_L(Q; \Omega) \) in \( W^{1,2}_{Q_0}(\Omega, S_0) \) is bounded and the energy density \( f_{E,1}(Q, \nabla Q) \) in \( (1.8) \) can be bounded above by \( C |\nabla Q|^2 + C \). Without this above growth condition on the density, it is a well-known fact that a minimizer \( Q_L \) of the Landau-de Gennes energy functional in \( W^{1,2}_{Q_0}(\Omega, S_0) \) may not satisfy the Euler-Lagrange equation in \( W^{1,2}(\Omega, S_0) \). To overcome this difficulty, we can introduce a smooth cutoff function \( \eta(r) \) in \( [0, \infty) \) so that \( \eta(r) = 1 \) for \( r \leq M \) with a very large constant \( M > 0 \) and \( \eta(r) = 0 \) for \( r \geq M + 1 \). Then for each \( Q \in W^{1,2}(\Omega, S_0) \), one can modify the Landau-de Gennes density by

\[
\tilde{f}_E(Q, \nabla Q) := \frac{\alpha}{2} |\nabla Q|^2 + \tilde{V}(Q, \nabla Q) = \frac{\alpha}{2} |\nabla Q|^2 + \eta(|Q|) V(Q, \nabla Q)
\]

(2.9) with the property that

\[
\frac{\alpha}{2} |\nabla Q|^2 \leq \tilde{f}_E(Q, \nabla Q) \leq C |\nabla Q|^2.
\]

For a large \( M > 0 \) in (2.9), we consider a modified Landau-de Gennes functional

\[
\tilde{E}_L(Q; \Omega) = \int_\Omega \left( \tilde{f}_E(Q, \nabla Q) + \frac{1}{L} \tilde{f}_B(Q) \right) \, dx.
\]

(2.10)
Then we obtain

**Lemma 2.4** Let $Q_L$ be a weak solution to the equation \( (1.11) \) with the boundary value $Q_0 \in W^{1,2}(\Omega; S_*)$ associated to the functional $\tilde{f}_E(Q, \nabla Q)$ in (2.9). Then, $|Q_L| \leq M + 1$ for a sufficient large $M$.

**Proof** Recall from the definition of $\tilde{f}_E(Q, \nabla Q)$ in (2.9) that for a $Q \in S_0$ with $|Q| \geq M + 1$, 

$$ \tilde{f}_E(Q, \nabla Q) = \frac{\alpha}{2} |\nabla Q|^2. $$

Similarly to one in [8], choose a test function $\phi = Q(1 - \min\{1, \frac{M+1}{|Q|}\})$. Multiplying (1.11) by the test function $\phi$, we have

$$ \alpha \int_{|Q| \geq M+1} |\nabla Q|^2 \left(1 - \frac{M + 1}{|Q|}\right) - (M + 1) Q_{ij} \nabla_k Q_{i} \nabla_k \frac{1}{|Q|} \right) dx $$

$$ + \frac{1}{L} \int_{|Q| \geq M+1} \left( -a |Q|^2 - b Q_{ik} Q_{kj} Q_{ij} + c |Q|^4 \right) \left(1 - \frac{M + 1}{|Q|}\right) dx = 0. $$

Note the fact that $\nabla_k |Q|^2 = 2 Q_{ij} \nabla_k Q_{ij}$. The above second term is non-negative. For a sufficiently large $M > 0$, the third term is positive. This implies that the set $\{|Q| \geq M + 1\}$ is empty; i.e., $|Q| \leq M + 1$ a.e. in $\Omega$.

The following result is a variant result of Giaquinta-Giusti [19] (see more details in page 206 of [18]):

**Proposition 2.1** For each $L > 0$, let $Q_L$ be a bounded minimizer of (2.10) in $W^{1,2}_{Q_L}(\Omega; S_0)$. Then there exists an open set $\Omega_L \subset \Omega$ such that $Q_L \in C^{1,\alpha}_{loc}(\Omega \setminus \Omega_L)$ for each $\alpha < 1$. Moreover, there is a small constant $\varepsilon_0$ independent of $Q_L$ such that

$$ \Sigma_L := \Omega \setminus \Omega_L = \left\{ x_0 \in \Omega : \liminf_{R \to 0} \int_{B_r(x_0)} |\nabla Q_L|^2 dx > \varepsilon_0 \right\} $$

and the Hausdorff measure $\mathcal{H}^q(\Sigma_L) = 0$ with $0 < q < 1$.

### 3 Proof of Theorem 2

At first, let us recall that for a uniaxial tensor $Q \in W^{1,2}_{Q_0}(\Omega; S_*)$, its energy is given by

$$ E(Q; \Omega) := \int_{\Omega} f_E(Q, \nabla Q) \, dx, $$

where $f_E(Q, \nabla Q) = \frac{\alpha}{2} |\nabla Q|^2 + V(Q, \nabla Q)$. Then we have

**Lemma 3.1** If $Q$ is a minimizer of $E(Q; \Omega)$ in $W^{1,2}_{Q_0}(\Omega; S_*)$, it satisfies

$$ \alpha \left( -s_+ \Delta Q_{ij} + 2 \nabla_k Q_{i} \nabla_k Q_{j} - 2s_+^{-1} \left( Q_{ij} + \frac{s_+}{3} \delta_{ij} \right) |\nabla Q|^2 \right) $$

$$ - \nabla_k \left( \left( Q_{ij} + \frac{s_+}{3} \delta_{ij} \right) V_{p_{ij}}^k + \left( Q_{il} + \frac{s_+}{3} \delta_{il} \right) V_{p_{il}}^k \right) $$

$$ - 2s_+^{-1} \left( Q_{ij} + \frac{s_+}{3} \delta_{ij} \right) \left( Q_{lm} + \frac{s_+}{3} \delta_{lm} \right) V_{p_{lm}}^k $$

$$ + V_{p_{ij}}^k \nabla_k Q_{j} + V_{p_{jl}}^k \nabla_k Q_{i} $$
\[-2s_+^{-1}V_{pim} \left( \nabla_k Q_{lm} \left( Q_{ij} + \frac{s_+}{3} \delta_{ij} \right) + \left( Q_{lm} + \frac{s_+}{3} \delta_{lm} \right) \nabla_k Q_{ij} \right) \]

\[+ V_{Qij} \left( Q_{jl} + \frac{s_+}{3} \delta_{jl} \right) + V_{Qjl} \left( Q_{il} + \frac{s_+}{3} \delta_{il} \right) \]

\[-2s_+^{-1}V_{Qlm} \left( Q_{lm} + \frac{s_+}{3} \delta_{lm} \right) \left( Q_{ij} + \frac{s_+}{3} \delta_{ij} \right) = 0 \]

in the weak sense.

**Proof** Let \( \phi \in C^\infty_0(\Omega; \mathbb{R}^3) \) be a test function. For each \( u_t = \frac{u + \tau \phi}{|u + \tau \phi|} \) with \( \tau \in \mathbb{R} \), we define

\[ Q_t(x) := Q(u_t(x)) = s_+ \left( u_t(x) \otimes u_t(x) - \frac{1}{3} I \right) \in S_+ \quad (3.1) \]

For any \( \eta \in C^\infty_0(\Omega; S_0) \), we choose a test function \( \phi_i := u_k \eta_{ik} \). If \( Q \) is a minimizer, the first variation of the energy of \( Q \) is zero; that is

\[ \frac{d}{dt} \int_{\Omega} f(E, \nabla Q_t) \, dx \bigg|_{t=0} = \int_{\Omega} f_{Q_{ij}} \frac{dQ_{t:ij}}{dt} + f_{p_{ij}} \frac{dQ_{ij}}{dx^k} \, dx \bigg|_{t=0} = 0. \]

Note that

\[ \frac{dQ_{t:ij}}{dt} = \left( Q_{jl} + \frac{s_+}{3} \delta_{jl} + \tau (Q_{lm} + \frac{s_+}{3} \delta_{lm}) \eta_{il} + \left( Q_{il} + \frac{s_+}{3} \delta_{il} + \tau (Q_{lm} + \frac{s_+}{3} \delta_{lm}) \eta_{jl} \right) \right) + \frac{1}{1 + 2s_+^{-1} Q_{ij} \eta_{il} + \tau^2 s_+^{-1} (Q_{lm} + \frac{s_+}{3} \delta_{lm}) \eta_{im} \eta_{lm}} \]

\[ -2s_+^{-1} (Q_{ij} + \frac{s_+}{3} \delta_{ij}) \eta_{il} + \left( Q_{il} + \frac{s_+}{3} \delta_{il} \right) \eta_{jl}, \]

where we used the fact that \( |u| = 1 \) and \( \phi_i = u_k \eta_{ik} \). Then we have

\[ \left. \frac{dQ_{t:i,j}}{dt} \right|_{t=0} = s_+ (u_j \phi_i + u_i \phi_j - 2(u \cdot \phi)(u_i u_j)) \]

\[ = \left( Q_{jl} + \frac{s_+}{3} \delta_{jl} \right) \eta_{il} + \left( Q_{il} + \frac{s_+}{3} \delta_{il} \right) \eta_{jl} \]

\[ -2s_+^{-1} \left( Q_{ij} + \frac{s_+}{3} \delta_{ij} \right) \eta_{lm}. \]

Using the fact that \( \nabla_k |u + \tau \phi|^2 = 0 \) at \( t = 0 \) and substituting \( \phi_i := u_k \eta_{ik} \), a simple calculation shows

\[ \frac{d}{dt} \frac{\partial (Q_{t:ij})}{\partial x^k} \bigg|_{t=0} = \left( \frac{\partial}{\partial x^k} \frac{d}{dt} Q_{t:ij} \right) \bigg|_{t=0} \]

\[ = \frac{\partial Q_{jl}}{\partial x^k} \eta_{il} + \frac{\partial Q_{il}}{\partial x^k} \eta_{jl} \]

\[ -2s_+^{-1} \left( \frac{\partial Q_{ij}}{\partial x^k} Q_{lm} + \frac{\partial Q_{lm}}{\partial x^k} \left( Q_{ij} + \frac{s_+}{3} \delta_{ij} \right) \right) \eta_{lm} \]

\[ + \left( Q_{jl} + \frac{s_+}{3} \delta_{jl} \right) \frac{\partial \eta_{il}}{\partial x^k} + \left( Q_{il} + \frac{s_+}{3} \delta_{il} \right) \frac{\partial \eta_{jl}}{\partial x^k} \]

\[ -2s_+^{-1} \left( Q_{ij} + \frac{s_+}{3} \delta_{ij} \right) \eta_{lm} \frac{\partial \eta_{lm}}{\partial x^k}. \]

\[ \Box \text{ Springer} \]
In the special case of $\frac{1}{2} \int_{\Omega} |\nabla Q|^2 \, dx$, it follows from using (3.3) and $\langle Q, \nabla Q \rangle = 0$ that

$$\frac{d}{dt} \int_{\Omega} \frac{|\nabla Q|^2}{2} \, dx \bigg|_{t=0} = \int_{\Omega} \nabla_k Q_{i;j} \frac{d\nabla_k Q_{i;j}}{dt} \bigg|_{t=0} \, dx$$

$$= \int_{\Omega} 2\nabla_k Q_{i;l} \nabla_k Q_{j;l} \eta_{ij} - 2 \left( s_+^{-1} Q_{ij} + \frac{1}{3} \delta_{ij} \right) |\nabla Q|^2 \eta_{ij} \, dx$$

$$+ \frac{1}{2} s_+ \int_{\Omega} \left( \nabla_k Q_{i;l} \nabla_k \eta_{il} + \nabla_k Q_{ij} \nabla_k \eta_{ji} \right) \, dx$$

$$= \int_{\Omega} \left( -s_+ \Delta Q_{ij} + 2\nabla_k Q_{i;l} \nabla_k Q_{j;l} - 2 \left( s_+^{-1} Q_{ij} + \frac{1}{3} \delta_{ij} \right) |\nabla Q|^2 \right) \eta_{ij} \, dx$$

(3.4)

for all $\eta \in C_0^\infty(\Omega; S_0)$.

For the term $V(Q, \nabla Q)$, using (3.2)-(3.3) and integrating by parts, we have

$$\int_{\Omega} \frac{d}{dt} V(Q, \nabla Q) \bigg|_{t=0} \, dx$$

$$= \int_{\Omega} \left[ V_{p;i} \frac{d\nabla_k Q_{i;j}}{dt} + V_{Q;j} \frac{dQ_{i;j}}{dt} \right] \bigg|_{t=0} \, dx$$

$$= \int_{\Omega} V_{p;i} \left( (Q_{jl} + \frac{s_+}{3} \delta_{jl}) \frac{\partial \eta_{il}}{\partial x_k} + (Q_{il} + \frac{s_+}{3} \delta_{il}) \frac{\partial \eta_{jl}}{\partial x_k} + \frac{\partial Q_{jl}}{\partial x_k} \frac{\partial \eta_{il}}{\partial x_j} + \frac{\partial Q_{il}}{\partial x_k} \frac{\partial \eta_{jl}}{\partial x_j} \right) \eta_{im} \, dx$$

$$- 2s_+^{-1} \int_{\Omega} V_{p;i} \left( \frac{\partial Q_{il}}{\partial x_k} (Q_{lm} + \frac{s_+}{3} \delta_{lm}) + \frac{\partial Q_{lm}}{\partial x_k} (Q_{il} + \frac{s_+}{3} \delta_{il}) \right) \eta_{im} \, dx$$

$$+ \int_{\Omega} V_{Q;i} \left( (Q_{il} + \frac{s_+}{3} \delta_{il}) \eta_{jl} - 2s_+^{-1} (Q_{ij} + \frac{s_+}{3} \delta_{ij}) (Q_{lm} + \frac{s_+}{3} \delta_{lm}) \eta_{im} \right) \eta_{lj} \, dx$$

$$= - \int_{\Omega} \frac{\partial}{\partial x_k} \left( (Q_{jl} + \frac{s_+}{3} \delta_{jl}) V_{p;i} + (Q_{il} + \frac{s_+}{3} \delta_{il}) V_{p;i} \right) \eta_{ij} \, dx$$

$$+ \int_{\Omega} \left( 2s_+^{-1} (Q_{ij} + \frac{s_+}{3} \delta_{ij}) (Q_{lm} + \frac{s_+}{3} \delta_{lm}) V_{p;i} \eta_{ij} + V_{p;i} \frac{\partial Q_{il}}{\partial x_k} \eta_{ij} \, dx$$

$$+ \int_{\Omega} \left( V_{p;i} \frac{\partial Q_{il}}{\partial x_k} - 2s_+^{-1} V_{p;i} \left( \frac{\partial Q_{lm}}{\partial x_k} (Q_{ij} + \frac{s_+}{3} \delta_{ij}) + (Q_{lm} + \frac{s_+}{3} \delta_{lm}) \frac{\partial Q_{ij}}{\partial x_k} \right) \right) \eta_{ij} \, dx$$

$$- 2s_+^{-1} \int_{\Omega} V_{Q;i} \left( Q_{lm} + \frac{s_+}{3} \delta_{lm} \right) (Q_{ij} + \frac{s_+}{3} \delta_{ij}) \eta_{ij} \, dx.$$ (3.5)

Combining above two identities (3.4) and (3.5), we prove Lemma 3.1. \hfill \Box

**Corollary 1** Assume that $Q = s_+ (u \otimes u - \frac{1}{2} I)$. Then $Q = (Q_{ij})$ is a solution of equation

$$\Delta Q_{ij} - 2s_+^{-1} \nabla_k Q_{i;l} \nabla_k Q_{j;l} + 2s_+^{-1} \left( s_+^{-1} Q_{ij} + \frac{1}{3} \delta_{ij} \right) |\nabla Q|^2 = 0$$ (3.6)

if and only if $u$ is a harmonic map from $\Omega$ into $S^2$; i.e., $-\Delta u = |\nabla u|^2 u$.

Now we give a proof of Theorem 2.

**Proof** For each $L > 0$, let $Q_L$ be a weak solution to the equation (1.11) with boundary value $Q_0 \in W^{1,2}(\Omega, S_0)$ and assume that $Q_L$ is uniformly bounded in $\Omega$. 

\[\text{Springer}\]
For each $\delta > 0$, define a set
\[ \Sigma_\delta = S_0 \setminus S_\delta = \{ Q \in S_0 : \text{dist}(Q, S_\delta) \geq \delta \}. \]

For each $Q \in \Sigma_\delta$, we have \( \pi(Q) \in S_\delta \); i.e., \( \pi(Q) = s_+ (u \otimes u - \frac{1}{3} I) \) with \( u \in S^2 \).

For a $\pi(Q)$, we have \( \pi(Q) = s_+ (u_L \otimes u_L - \frac{1}{3} I) \) with $u_L \in S^2$ and a test function $\phi \in C_0^\infty(\Omega; \mathbb{R}^3)$ with a small $\delta \in \mathbb{R}$, we set \( u_{L,t} := \frac{u_t + \phi}{|u_t + \phi|} \). Then we have
\[
(\pi(Q_L))_t := s_+ \left( u_{L,t} \otimes u_{L,t} - \frac{1}{3} I \right) \in S_\delta.
\] (3.7)

For any $Q \in S_\delta$, set
\[ F(Q) := f_B(Q) + |Q - \pi(Q)|^2. \]

Using the Taylor expansion of $F((\pi(Q_L))_t)$ at $Q_L \in S_\delta$, we derive
\[
\frac{F((\pi(Q_L))_t)}{L} = \frac{F(Q)}{L} + \frac{(\nabla F(Q))_t}{L} ((\pi(Q_L))_t - Q_L)_{ij}
+ \frac{1}{2L} \nabla^2 F_{ij}(Q_{\tau_1}) ((\pi(Q_L))_t - Q_L)_{ij} ((\pi(Q_L))_t - Q_L)_{kl},
\] (3.8)

where $Q_{\tau_1} := (1 - \tau_1)(\pi(Q_L))_t + \tau_1 Q_L$ for some $\tau_1 \in [0, 1]$. Note that
\[ |Q_{\tau_1} - (\pi(Q_L))_t| \leq |Q_L - \pi(Q_L)| + |(\pi(Q_L))_t - (\pi(Q_L))_t|. \]

Since $(\pi(Q_L))_t \in S_\delta$, it implies that $F((\pi(Q_L))_t) = 0$. Note that the function $F(Q)$ is smooth in $Q$. For sufficiently small $t$, we have $|Q_{\tau_1} - (\pi(Q_L))_t| \leq \delta$ for $Q_L \in S_{2\delta}$. For each $Q \in S_\delta$, it is known that the Hessian of $f_B(Q)$ is semi-positive definite at $Q = Q^*$. Therefore each $Q \in S_\delta$, the Hessian of $F_B(Q)$ is positive definite with sufficiently small $\delta > 0$; i.e., for any $Q_L \in S_{2\delta}$, we have
\[
\nabla^2 F(Q_{\tau_1})((\pi(Q_L))_t - Q_L)_{ij} ((\pi(Q_L))_t - Q_L)_{kl} \geq \frac{1}{2} |(\pi(Q_L))_t - Q_L|^2
\] (3.9)

with sufficiently small $t$ and $\delta$. Then it follows from (3.8)-(3.9) that
\[
\int_{\Omega_{L,2\delta}} \frac{(\nabla F(Q))_t}{L} ((\pi(Q_L))_t - Q_L)_{ij} \, dx
= \int_{\Omega_{L,2\delta}} \frac{\nabla Q_{ij} f_B(Q)}{L} ((\pi(Q_L))_t - Q_L)_{ij} \, dx
+ 2 \int_{\Omega_{L,2\delta}} \partial_{Q_{ij}}(Q_L - \pi(Q_L))_{mn} \frac{(Q_L - \pi(Q_L))_{mn}}{L} ((\pi(Q_L))_t - Q_L)_{ij} \, dx
\leq -\frac{1}{2} \int_{\Omega_{L,2\delta}} \frac{|(\pi(Q_L))_t - Q_L|^2}{L} \, dx
\] (3.10)

provided $\Omega_{L,2\delta} = \{ x \in \Omega : Q_L(x) \in S_{2\delta} \}$ for $\delta > 0$.

By using Young’s inequality, we have
\[
\int_{\Omega_{L,2\delta}} \frac{1}{L} \nabla Q_{ij} f_B(Q)((\pi(Q_L))_t - Q_L)_{ij} \, dx + \frac{1}{4} \int_{\Omega_{L,2\delta}} \frac{|(\pi(Q_L))_t - Q_L|^2}{L} \, dx
\leq C \int_{\Omega_{L,2\delta}} \frac{|Q_L - \pi(Q_L)|^2}{L} \, dx \leq C \int_{\Omega_{L,2\delta}} \frac{1}{L} f_B(Q) \, dx,
\] (3.11)
In fact, it follows from (3.13) that

\[
\hat{Q}_{L,t} := \begin{cases} 
\pi(Q_L)_t \cdot \frac{\pi(Q_L)_t}{\delta^2} \pi(Q_L)_t + \frac{\delta^2 - \pi(Q_L)_t^2}{\delta^2} Q_{*,t}, & \text{for } Q_L \in S_{\delta} \\
\frac{\pi(Q_L)_t}{\delta^2} \pi(Q_L)_t & \text{for } Q_L \in \Sigma_{\delta} \setminus \Sigma_{2\delta} \\
\end{cases} \quad (3.12)
\]

It can be checked that \( \hat{Q}_{L,t} \in W_{Q_0}^{1,2}(\Omega; S_0) \). Then

\[
\hat{Q}_{L,t} - Q_{*,t} = \begin{cases} 
\pi(Q_L)_t \cdot \frac{\pi(Q_L)_t}{\delta^2} \pi(Q_L)_t - Q_{*,t}, & \text{for } Q_L \in S_{\delta} \\
\frac{\pi(Q_L)_t}{\delta^2} \pi(Q_L)_t - Q_{*,t}, & \text{for } Q_L \in \Sigma_{\delta} \setminus \Sigma_{2\delta} \\
0, & \text{for } Q_L \in \Sigma_{2\delta} \\
\end{cases} \quad (3.13)
\]

On the other hand, there exists a uniform bound \( C(\delta) > 0 \) such that for all \( x \in \Omega \setminus \Omega_{L,\delta} \), \( \tilde{f}_B(Q_L(x)) \geq C(\delta) \). Using Lemma 2.4, we observe that

\[
\int_{\Omega \setminus \Omega_{L,\delta}} \frac{1}{L} \nabla Q_{ij} f_B(Q_L) (\hat{Q}_{L,t} - Q_L)_{ij} \, dx
\]

\[
= \int_{\Omega_{L,\delta}} \frac{1}{L} \nabla Q_{ij} f_B(Q_L) \left[ \frac{|Q_L - \pi(Q_L)_t^2}{\delta^2} (\pi(Q_L)_t - Q_{*,t}) + (Q_{*,t} - Q_L) \right]_{ij} \, dx
\]

\[
+ \int_{\Omega \setminus \Omega_{L,\delta}} \frac{1}{L} \nabla Q_{ij} f_B(Q_L) (Q_{*,t} - Q_L)_{ij} \, dx
\]

\[
\leq C \frac{|\Omega \setminus \Omega_{L,\delta}|}{L} \leq C(\delta) \int_{\Omega \setminus \Omega_{L,\delta}} \frac{\tilde{f}_B(Q_L)}{L} \, dx. \tag{3.14}
\]

By the assumption (1.15) in Theorem 2, we deduce from (3.11) and (3.14) that

\[
\lim_{L \to 0} \int_{\Omega} \frac{1}{L} \nabla Q_{ij} f_B(Q_L) (\hat{Q}_{L,t} - Q_L)_{ij} \, dx \leq 0. \tag{3.15}
\]

Multiplying (1.11) by \( (\hat{Q}_{L,t} - Q_L) \) and using (3.15) yield

\[
\lim_{L \to 0} \int_{\Omega} \left( \alpha \nabla_k Q_{L,ij} + \tilde{V}_{p_t}^* (Q_L, \nabla Q_L) - \tilde{V}_{Q_{ij}} (Q_L, \nabla Q_L) \right) \nabla_k (\hat{Q}_{L,t} - Q_L)_{ij} \, dx \geq 0. \tag{3.16}
\]

Here we used the fact that \( \hat{Q}_{L,t} - Q_L \) is symmetric and traceless.

In order to pass a limit, we claim that \( \hat{Q}_{L,t} \to Q_{*,t} \) strongly in \( W_{Q_0}^{1,2}(\Omega; S_0) \).

In fact, it follows from (3.13) that

\[
\int_{\Omega} |\nabla (\hat{Q}_{L,t} - Q_{*,t})|^2 \, dx = \int_{\Omega_{L,\delta}} |\nabla (\hat{Q}_{L,t} - Q_{*,t})|^2 \, dx
\]

\[
= \int_{\Omega_{L,\delta}} |\nabla (\hat{Q}_{L,t} - Q_{*,t})|^2 \, dx + \int_{\Omega_{L,\delta} \setminus \Omega_{L,\delta}} \left| \nabla \left( \frac{|Q_L - \pi(Q_L)_t^2}{\delta^2} (\pi(Q_L)_t - Q_{*,t}) \right) \right|^2 \, dx
\]

\[
\leq \int_{\Omega_{L,\delta}} |\nabla ((\pi(Q_L)_t - \pi(Q_{*,t}))|^2 \, dx + C \int_{\Omega_{L,\delta} \setminus \Omega_{L,\delta}} |\nabla ((\pi(Q_L)_t - \pi(Q_{*,t}))|^2 \, dx
\]

\[
+ C \int_{\Omega_{L,\delta} \setminus \Omega_{L,\delta}} \left( \frac{\pi(Q_L)_t - Q_{*,t}}{\delta^4} \right)^2 \left( |\nabla (Q_L - Q_{*,t})|^2 + |\nabla (\pi(Q_L) - \pi(Q_{*,t}))|^2 \right) \, dx. \tag{3.17}
\]
Note that
\[ \pi(Q_L) - \pi(Q_*) = \nabla \pi(Q_\xi)(Q_L - Q_*), \]
\[ (\pi(Q_L))_t - \pi(Q_*)_t = \nabla \pi(Q_\xi)_t(Q_L - Q_*)_t. \]

When \( Q_L \) approaches to \( Q_* \), \( \nabla \pi(Q_\xi) \) is close to the identity map \( I \) and \( \nabla \pi(Q_\xi)_t \) for small \( t \). Therefore
\[ |\nabla (\pi(Q_L) - \pi(Q_*))| \leq C|\nabla (Q_L - Q_*)| + C|\nabla Q_\xi||Q_L - Q_*|. \]

As \( Q_L \to Q_* \), the term \( (\pi(Q_L)_t - \pi(Q_*)_t) \) approaches to \( \pi(Q_\xi)_t \) and \( \nabla \pi(Q_\xi)_t \) for small \( t \). Note that \( \nabla^2 \pi(Q_\xi)_t \) is bounded. Then
\[ |\nabla (\pi(Q_L)_t - \pi(Q_*)_t)| \leq |\nabla \pi(Q_\xi)_t|\nabla (Q_L - Q_*)| + |\nabla^2 \pi(Q_\xi)_t||\nabla Q_\xi||Q_L - Q_*| \leq C|\nabla (Q_L - Q_*)| + C|\nabla Q_\xi||Q_L - Q_*|. \]

Then the inequality (3.17) reads as
\[
\int_\Omega |\nabla (\hat{Q}_{L,t} - Q_*)|^2 \, dx \\
\leq C \int_{\Omega_{L,2\delta}} |\nabla (Q_L - Q_*)|^2 + (|\nabla Q_L|^2 + |\nabla Q_*|^2)|Q_L - Q_*|^2 \, dx \\
\leq C \int_\Omega |\nabla (Q_L - Q_*)|^2 \, dx + C \left( \int_{\Omega_e \setminus \Sigma_e} + \int_{\Sigma_e} \right) |\nabla Q_*|^2 \, |Q_L - Q_*|^2 \, dx.
\]

Here we employ the Egoroff theorem; i.e., for all \( \varepsilon > 0 \), there exists a measurable subset \( \Sigma_\varepsilon \subset \Omega \) such that
\[ |\Sigma_\varepsilon| \leq \varepsilon \text{ and } Q_L \to Q_* \text{ uniformly on } \Omega \setminus \Sigma_\varepsilon \text{ as } L \to 0. \tag{3.18} \]

As \( \varepsilon \to 0 \) and \( L \to 0 \), we prove the claim that \( \hat{Q}_{L,t} \to Q_* \) strongly in \( W^{1,2}_{Q_0}(\Omega; S_0) \).

We observe that
\[
\int_\Omega |\tilde{V}_{p_{ij}}(Q_L, \nabla Q_L)\nabla (\hat{Q}_{L,t} - Q_L)_{ij} - \tilde{V}_{p_{ij}}(Q_*, \nabla Q_*)\nabla (Q_{*,t} - Q_*)_{ij}| \, dx \\
\leq \int_\Omega |\tilde{V}_{p_{ij}}(Q_L, \nabla Q_L)||\nabla (\hat{Q}_{L,t} - Q_*\xi_{t,j})| + (|\nabla Q_*| - |\nabla Q_L|)|\nabla Q_*||\nabla Q_L| \, dx \\
+ \left( \int_{\Omega_{\setminus \Sigma_e}} + \int_{\Sigma_e} \right) |\tilde{V}_{p_{ij}}(Q_L, \nabla Q_L)\nabla (Q_{*,t} - Q_*)_{ij} - \tilde{V}_{p_{ij}}(Q_*, \nabla Q_*)\nabla (Q_{*,t} - Q_*)_{ij}| \, dx
\]
and
\[
\int_\Omega |\tilde{V}_{Q_{ij}}(Q_L, \nabla Q_L)(\hat{Q}_{L,t} - Q_L)_{ij} - \tilde{V}_{Q_{ij}}(Q_*, \nabla Q_*)(Q_{*,t} - Q_*)_{ij}| \, dx \\
\leq \left( \int_{\Omega_{\setminus \Sigma_e}} + \int_{\Sigma_e} \right) |\tilde{V}_{Q_{ij}}(Q_L, \nabla Q_L)(\hat{Q}_{L,t} - Q_L)| + |\tilde{V}_{Q_{ij}}(Q_*, \nabla Q_*)(\hat{Q}_{L,t} - Q_L)| \, dx \\
+ \int_\Omega |\tilde{V}_{Q_{ij}}(Q_*, \nabla Q_L)(\hat{Q}_{L,t} - Q_L)_{ij} - \tilde{V}_{Q_{ij}}(Q_*, \nabla Q_*)(\hat{Q}_{*,t} - Q_*)_{ij}| \, dx.
\]

Using the uniform convergence of \( Q_L \) in \( \Omega \setminus \Sigma_\varepsilon \) and strong convergence of \( \hat{Q}_{L,t}, Q_L \) in \( W^{1,2}_{Q_0}(\Omega, S_0) \), we derive
\[
\lim_{L \to 0} \int_\Omega |\tilde{V}_{Q_{ij}}(Q_L, \nabla Q_L)(\hat{Q}_{L,t} - Q_L)_{ij} - \tilde{V}_{Q_{ij}}(Q_*, \nabla Q_*)(Q_{*,t} - Q_*)_{ij}| \, dx = 0.
\]
As $L \to 0$, the estimate (3.16) yields
\begin{equation}
\int_{\Omega} \left( \alpha \nabla k \mathbf{Q}_{*,ij} + \tilde{V}_{p_{ij}}(\mathbf{Q}_*, \nabla \mathbf{Q}_*) \right) \nabla k(\mathbf{Q}_{*,t} - \mathbf{Q}_*)_{ij} \, dx + \int_{\Omega} \tilde{V}_{ij}(\mathbf{Q}_*, \nabla \mathbf{Q}_*)(\mathbf{Q}_{*,t} - \mathbf{Q}_*)_{ij} \, dx \geq 0.
\end{equation}
(3.19)

For each $\eta \in C^0(\Omega, S_0)$, we define
\begin{equation}
\varphi_{ij}(\mathbf{Q}, \eta) := \left( \frac{1}{\alpha} \right) \left( s_{ij}^{-1} \mathbf{Q} \right)_{ij} = \frac{1}{3} \tilde{k} \eta_{ij} + \frac{1}{3} \tilde{k} \eta_{il} - 2 \left( \frac{1}{\alpha} \right) \left( s_{ij}^{-1} \mathbf{Q} \right)_{ij} + \frac{1}{3} \tilde{k} \eta_{im} \eta_{lm}.
\end{equation}
(3.20)

In view of (3.2) and (3.3), we have
\begin{align*}
\lim_{t \to 0} \frac{(\mathbf{Q}_t - \mathbf{Q}_*)}{t} &= \varphi(\mathbf{Q}_*, \eta), \\
\lim_{t \to 0} \nabla \frac{(\mathbf{Q}_t - \mathbf{Q}_*)}{t} &= \nabla \varphi(\mathbf{Q}_*, \eta).
\end{align*}

For the estimate (3.19), the limit in $t$ exists. Dividing (3.19) by $t$ then as $t \to 0^+$ and $t \to 0^-$, we have
\begin{equation}
\int_{\Omega} \left( \alpha \nabla k \mathbf{Q}_{*,ij} + \tilde{V}_{p_{ij}}(\mathbf{Q}_*, \nabla \mathbf{Q}_*) \right) \nabla k \varphi_{ij}(\mathbf{Q}_*, \eta) + V_{Q_{ij}}(\mathbf{Q}_*, \nabla \mathbf{Q}_*) \varphi_{ij}(\mathbf{Q}_*, \eta) \, dx = 0.
\end{equation}

Repeating the same steps in (3.4) and (3.5), we prove that $\mathbf{Q}_*$ satisfies (1.12). \hfill \Box

### 4 Smooth convergence of solutions

In this section, we will prove Theorem 3. At first, we derive some key lemmas.

For any tensor $\mathbf{Q} \in S_0$, there exists a rotation $R(\mathbf{Q}) \in SO(3)$ such that $\tilde{\mathbf{Q}} := R^T(\mathbf{Q})Q R(\mathbf{Q})$ is diagonal. Moreover, the space $S^*$ has only three diagonal tensors so for each $\mathbf{Q} \in S_*$, we assume that
\begin{equation}
R^T(\mathbf{Q})Q R(\mathbf{Q}) = \begin{pmatrix}
\frac{2-s_+}{3} & 0 & 0 \\
0 & -\frac{s_+}{3} & 0 \\
0 & 0 & \frac{2-s_+}{3}
\end{pmatrix} := \mathbf{Q}^+.
\end{equation}
(4.1)

**Lemma 4.1** For any $\mathbf{Q} \in S_0$ and $\xi \in S_0$ with a sufficiently small $\delta > 0$, the Hessian of the bulk density $f_B(\mathbf{Q})$ satisfies the following estimate
\begin{equation}
\sum_{i,j=1}^{3} \partial_{\tilde{Q}_{ii}} \partial_{\tilde{Q}_{jj}} f_B(\tilde{\mathbf{Q}}) \xi_{ii} \xi_{jj} \geq \frac{\lambda}{2} \left( \xi_{11}^2 + \xi_{22}^2 + \xi_{33}^2 \right),
\end{equation}
(4.2)
where $\lambda = \min \{3a, s_+ b\} > 0$ and $\tilde{\mathbf{Q}} := R^T(\mathbf{Q})Q R(\mathbf{Q})$ is diagonal.

**Proof** For a fixed $\pi(\mathbf{Q}_0) \in S_*$, there exists a rotation $R(\pi(\mathbf{Q}_0)) \in SO(3)$ in (4.1) such that $R^T(\pi(\mathbf{Q}_0))\pi(\mathbf{Q}_0) R(\pi(\mathbf{Q}_0)) = \mathbf{Q}^+$. For $i = 1, 2, 3$, we calculate the first derivative of $f_B(\tilde{\mathbf{Q}})$ by
\begin{equation}
\partial_{\tilde{Q}_{ii}} f_B(\tilde{\mathbf{Q}}) = \left( -a \tilde{Q}_{ii} - b \tilde{Q}_{ik} \tilde{Q}_{ki} + c \tilde{Q}_{ii} |\tilde{Q}|^2 \right).
\end{equation}
Then the second derivative of \( f_B(\tilde{Q}) \) with \( i, j = 1, 2, 3 \) is
\[
\partial_{Q_{ii}} \partial_{Q_{jj}} f_B(\tilde{Q}) = -a\delta_{ij} - 2b\delta_{ij} \tilde{Q}_{ii} + c(\delta_{ij}|\tilde{Q}|^2 + 2\tilde{Q}_{ii} \tilde{Q}_{jj}).
\] (4.3)

For the case of \( i = j \) at \( Q = Q_0 \), \( \tilde{Q} = Q^+ \). From the equality \( \frac{2}{3}c \frac{s^2}{2} = \frac{1}{3}bs_+ + a \) (c.f. [34]), we find
\[
\partial_{Q_{ii}} \partial_{Q_{ii}} f_B(\tilde{Q}) = -a - 2\tilde{Q}_{ii} b + (|\tilde{Q}|^2 + 2\tilde{Q}_{ii}^2) c = -(2\tilde{Q}_{ii} - \frac{s_+}{3})b + 2\tilde{Q}_{ii}^2 c.
\]

Then, at \( \tilde{Q} = Q^+ \), we have
\[
\partial_{Q_{11}} \partial_{Q_{11}} f_B(\tilde{Q}) = \left(s + b + \frac{2s^2}{9} c\right) = \frac{1}{3}a + \frac{10s_+}{9} b, \quad (4.4)
\]
\[
\partial_{Q_{22}} \partial_{Q_{22}} f_B(\tilde{Q}) = \frac{1}{3}a + \frac{10s_+}{9} b, \quad (4.5)
\]
\[
\partial_{Q_{33}} \partial_{Q_{33}} f_B(\tilde{Q}) = -s + b + \frac{8s_+}{9} c = \frac{4}{3}a - \frac{5s_+}{9} b. \quad (4.6)
\]

For the case of \( i \neq j \), at \( \tilde{Q} = Q^+ \), we observe that
\[
\begin{align*}
2\partial_{Q_{11}} \partial_{Q_{22}} f_B(\tilde{Q}) &= 4\tilde{Q}_{11} \tilde{Q}_{22} c = \frac{4s^2}{9} c = \frac{2}{3}a + \frac{2s_+}{9} b, \\
2\partial_{Q_{11}} \partial_{Q_{33}} f_B(\tilde{Q}) &= 4\tilde{Q}_{11} \tilde{Q}_{33} c = -\frac{8s^2}{9} c = -\left(\frac{4}{3}a + \frac{4s_+}{9} b\right), \\
2\partial_{Q_{22}} \partial_{Q_{33}} f_B(\tilde{Q}) &= 4\tilde{Q}_{22} \tilde{Q}_{33} c = -\frac{8s^2}{9} c = -\left(\frac{4}{3}a + \frac{4s_+}{9} b\right).
\end{align*}
\] (4.7) (4.8) (4.9)

In conclusion, using the fact that \( \xi_{33} = -(\xi_{11} + \xi_{22}) \), we have at \( \tilde{Q} = Q^+ \)
\[
\partial_{Q_{ii}} \partial_{Q_{jj}} f_B(\tilde{Q})\xi_{ii}\xi_{jj} = \left(\frac{1}{3}a + \frac{10s_+}{9} b\right)\left(\xi_{11}^2 + \xi_{22}^2\right) + \left(\frac{2}{3}a + \frac{2s_+}{9} b\right)\xi_{11} \xi_{22} + \left(\frac{4}{3}a - \frac{5s_+}{9} b\right)\xi_{33}^2 - \left(\frac{4}{3}a + \frac{4s_+}{9} b\right)\xi_{33}(\xi_{11} + \xi_{22})
\]
\[
= bs_+ (\xi_{11}^2 + \xi_{22}^2) + 3a \xi_{33}^2 \geq \lambda (\xi_{11}^2 + \xi_{22}^2 + \xi_{33}^2)
\]

with \( \lambda = \min\{3a, s_+ b\} > 0 \). Then
\[
\partial_{Q_{ii}} \partial_{Q_{jj}} f_B(\tilde{Q})\xi_{ii}\xi_{jj} \geq \partial_{Q_{ii}} \partial_{Q_{jj}} f_B(Q^+)\xi_{ii}\xi_{jj} = C|\tilde{Q} - Q^+|^3 \left| \sum_{i=1}^3 \xi_{ii}\right|^2.
\]

Due to the fact that \( |\tilde{Q} - Q^+| = |Q - \pi(Q)| \), we prove (4.2) for a sufficiently small \( \delta > 0 \).

\[ \square \]

**Corollary 2** For any \( Q \in S_b \) with a sufficiently small \( \delta > 0 \), there exists constants \( C_1, C_2, C_3 > 0 \) such that
\[
C_1 \tilde{f}_B(Q) \leq |Q - \pi(Q)|^2 \leq C_2 \tilde{f}_B(Q), \quad (4.10)
\]
\[
|Q - \pi(Q)| \leq C_3 |g_B(Q)|. \quad (4.11)
\]
Proof It follows from the Taylor expansion of \( \tilde{f}_B(Q) \) at \( \pi(Q) \) that
\[
\tilde{f}_B(Q) = \tilde{f}_B(\pi(Q)) + \nabla_{Q_{ij}} \tilde{f}_B(\pi(Q))(Q - \pi(Q))_{ij} + \partial_{Q_{ij}} \partial_{Q_{kl}} \tilde{f}_B(\pi(Q))(Q - \pi(Q))_{ij}(Q - \pi(Q))_{kl}, \tag{4.12}
\]
where \( Q_\tau \) is an intermediate point between \( Q_L \) and \( \pi(Q) \).

Since \( Q \) commutes with \( \pi(Q) \) (c.f. [36]), they can be simultaneously diagonalized. Note that \( \tilde{f}_B(\pi(Q)) = 0 \), \( \nabla_{Q_{ij}} \tilde{f}_B(\pi(Q)) = 0 \) and \( Q_\tau \) is sufficiently close to \( \pi(Q) \). Using Lemma 4.1 with the fact that
\[
\sum_{i,j=1}^{3} \partial_{Q_{ij}} \partial_{Q_{ij}} \tilde{f}_B(\pi(Q)) (Q - \pi(Q))_{ij}(Q - \pi(Q))_{kl} \geq \frac{\lambda}{2} \sum_{i=1}^{3} (\tilde{Q} - Q^+)_{ii}^2, \tag{4.13}
\]
we have
\[
\partial_{Q_{ij}} \partial_{Q_{ij}} \tilde{f}_B(\pi(Q))(Q - \pi(Q))_{ij}(Q - \pi(Q))_{kl} \geq \frac{\lambda}{2} \sum_{i=1}^{3} (\tilde{Q} - Q^+)_{ii}^2.
\]

Then we obtain
\[
\tilde{f}_B(Q) \geq \frac{\lambda}{2} \sum_{i=1}^{3} (\tilde{Q} - Q^+)_{ii}^2 = \frac{\lambda}{2} |Q - \pi(Q)|^2.
\]

The left-hand side of (4.10) is a direct consequence of (4.12) by using Young’s inequality. Taking Taylor expansion of \( g_B(Q) \) at \( \pi(Q) \) yields
\[
(g_B(Q))_{ij} = \partial_{Q_{ij}} \partial_{Q_{kl}} f_B(Q_{\tau_1})(Q - \pi(Q))_{kl}.
\]

Multiplying both side by \( (Q - \pi(Q))_{ij} \) and using (4.13), we obtain (4.11). \( \square \)

From now on, for each \( L > 0 \), let \( Q_L \) be a solution to the equation (1.11) and assume that \( Q_L \) is smooth and converges to \( Q_\ast \) uniformly inside \( \Omega \setminus \Sigma \), where \( \Sigma \) is the singular set of \( Q_\ast \). For a sufficiently small \( L \), \( \text{dist}(Q_L; S_\ast) \leq \delta \) inside \( \Omega \setminus \Sigma \).

Set
\[
H(Q, \nabla Q) := a \Delta Q + \frac{1}{2} \nabla_k (V_{Q_{sk}} + V_{Q_{sk}}^T) - \frac{1}{3} I \text{ tr}(\nabla_k V_{Q_{sk}}) - \frac{1}{2} (V_Q + V_Q^T) + \frac{1}{3} I \text{ tr}(V_Q) \tag{4.14}
\]
and
\[
g_B(Q) = \left( -aQ - b \left( QQ - \frac{1}{3} I \text{ tr}(Q^2) \right) - cQ \text{ tr}(Q^2) \right). \tag{4.15}
\]

Due to the fact that \( \tilde{Q} := R^T(Q)Q R(Q) \) is diagonal, \( g_B(\tilde{Q}) = R^T(Q) g_B(Q) R(Q) \) is also diagonal for a rotation \( R(Q) \in SO(3) \).

Let \( Q \) be differentiable in \( \Omega \). Then there exists a set \( \Sigma_Q \), which has measure zero, such that \( R(Q) \) is differentiable in \( \Omega \setminus \Sigma_Q \) (c.f. Corollary 2 [34, 37]). Therefore, we have the following geometric identity of rotations:
Lemma 4.2. Assume that for any \( x \in \Omega \setminus \Sigma_Q \), there exists a differentiable rotation \( R(Q) \) such that both \( R^T(Q)QR(Q) \) and \( R^T(Q)h(Q)R(Q) \) are diagonal. Then, for each \( i \), we have

\[
\nabla \left( R^T(Q)h(Q)R(Q) \right)_{ii} = \left( R^T(Q)\nabla h(Q)R(Q) \right)_{ii}.
\]

\( (4.16) \)

**Proof** Let \( x_0 \) be a fixed point in \( \Omega \setminus \Sigma_Q \) and fix \( i = 1, 2, 3 \). For \( Q_0 = Q(x_0) \in S_0 \), there exists \( R_0 := R(Q_0) \in SO(3) \) such that \( R_0^T Q_0 R_0 \) is diagonal. Denote \( \bar{R}(Q) = R_0^T R(Q) \) with \( \bar{R}(Q_0) = I \). Fix \( Q_0 \in S_0 \), there is \( R_0 := R(Q_0) \in SO(3) \) such that \( R_0^T Q_0 R_0 \) and \( R_0^T h(Q_0)R_0 \) diagonal. Denote \( \tilde{R}(Q) = R_0^T R(Q) \), so \( \tilde{R}(Q_0) = I \). Since \( R(Q) \in SO(3) \), \( R_{ki}(Q)R_{kj}(Q) = \delta_{ij} \). Then, for each \( i \), we have at \( x_0 \)

\[
\nabla \left( R^T(Q)h(Q)R(Q) \right)_{ii} = \nabla \left( R^T(Q)R_0 R_0^T h(Q)R_0 R_0^T R(Q) \right)_{ii}
\]

\[
= \sum_{k,l=1}^3 \nabla \tilde{R}_{ki}(Q)(R_0^T h(Q)R_0)_{kl} \tilde{R}_{li}(Q)
\]

\[
+ \sum_{k,l=1}^3 \tilde{R}_{ki}(Q)(R_0^T h(Q)R_0)_{kl} \nabla \tilde{R}_{li}(Q)
\]

\[
= \sum_{k,l=1}^3 \tilde{R}_{ki}(Q)\nabla (R_0^T h(Q)R_0)_{kl} \tilde{R}_{li}(Q). \quad (4.17)
\]

Note that \( R_0^T h(Q_0)R_0 \) is diagonal, \( \tilde{R}_{ik}(Q_0) = \delta_{ik} \) and \( \nabla \tilde{R}_{ii}(Q_0) = 0 \). It can be seen that the term \( \nabla \bar{R}_{ki}(Q)(R_0^T Q_0 R_0)_{kl} \tilde{R}_{li}(Q) \) at \( Q = Q_0 \) is zero. Therefore

\[
\nabla \left( R^T(Q)h(Q)R(Q) \right)_{ii} \big|_{Q=Q_0} = \left( R^T(Q_0)\nabla h(Q)\big|_{Q=Q_0} R(Q_0) \right)_{ii}.
\]

Since \( x_0 \) is any point, we prove \( (4.16) \). \( \square \)

Denote the inner product by \( \langle A, B \rangle = A_{ij}B_{ij} \) for \( A, B \in \mathbb{M}^{3\times3} \). Using the above geometric identity, we have

**Lemma 4.3** Let \( k \) and \( l \) be two integers, for \( x \in \Omega \setminus \Sigma_Q \), \( Q = Q(x) \) and any smooth scalar function \( \phi \), we have

\[
\langle \nabla^k (g_B(\tilde{Q})\phi^2), R^T(Q)\nabla^{l+1} Q R(Q) \rangle
\]

\[
= \langle \nabla^k (g_B(\tilde{Q})\phi^2), R^T \nabla^l Q R(Q) \rangle
\]

\[
- \langle \nabla^{k+1} (g_B(\tilde{Q})\phi^2), R^T(Q)\nabla^l Q R(Q) \rangle. \quad (4.18)
\]

**Proof** For \( x_0 \in \Omega \setminus \Sigma_Q \), \( R(x) \) is differentiable in the neighborhood of \( x_0 \). Fixing \( Q_0 = Q(x_0) \in S_0 \), there exists \( R_0 = R(Q(x_0)) \) such that \( R_0^T Q_0 R_0 \) is diagonal. Recall that \( \bar{R}(Q) = R_0^T R(Q) \), \( \tilde{R}_{ik}(Q_0) = \delta_{ik} \) and \( \nabla \tilde{R}_{ii}(Q_0) = 0 \). For any matrix \( A \), let \( A_D \) be the diagonal part of \( A \) and \( A_N \) the non-diagonal part of \( A \) such that \( A = A_D + A_N \). Recall that \( \nabla^k g_B(\tilde{Q}) \) and \( \nabla^{k+1} g_B(\tilde{Q}) \) are diagonal. By employing an analogous argument in the proof of Lemma 4.2,
we obtain
\[
\nabla (\nabla^k (g_B(\tilde{Q})\phi^2), \tilde{R}^T (Q)(R_0^T \nabla^l Q R_0)_D \tilde{R}(Q))
\]
\[
= \left( \nabla^{k+1} (g_B(\tilde{Q})\phi^2), \tilde{R}^T (Q)(R_0^T \nabla^l Q R_0)_D \tilde{R}(Q) \right) + \left( \nabla^k (g_B(\tilde{Q})\phi^2), \tilde{R}^T (Q)(R_0^T \nabla^{l+1} Q R_0)_D \tilde{R}(Q) \right)
\]
\[
= \left( \nabla^{k+1} (g_B(\tilde{Q})\phi^2), \tilde{R}^T (Q)\nabla^l Q R(Q) \right) + \left( \nabla^k (g_B(\tilde{Q})\phi^2), \tilde{R}^T (Q)\nabla^{l+1} Q R(Q) \right)
\]
\[
= \nabla (\nabla^k (g_B(\tilde{Q})\phi^2), \tilde{R}^T (Q)\nabla^l Q R(Q)).
\]

(4.19)

Here we used that
\[
\left( \nabla^{k+1} (g_B(\tilde{Q})\phi^2), \tilde{R}^T (Q)(R_0^T \nabla^l Q R_0)_N \tilde{R}(Q) \right) = 0.
\]

Similarly, using (4.19), at \(Q = Q_0\), we find
\[
\left( \nabla^k (g_B(\tilde{Q})\phi^2), \tilde{R}^T (Q)\nabla^l Q R(Q) \right)
\]
\[
= \left( \nabla^k (g_B(\tilde{Q})\phi^2), \tilde{R}^T (Q)(R_0^T \nabla^l Q R_0)_D \tilde{R}(Q) \right)
\]
\[
= \left( \nabla^k (g_B(\tilde{Q})\phi^2), \ln (\tilde{R}^T (Q)_D \tilde{R}(Q)) \right)
\]
\[
= \nabla (\nabla^k (g_B(\tilde{Q})\phi^2), \tilde{R}^T (Q)(R_0^T \nabla^l Q R_0)_D \tilde{R}(Q))
\]
\[
= \nabla (\nabla^k (g_B(\tilde{Q})\phi^2), \tilde{R}^T (Q)\nabla^l Q R(Q)) - \left( \nabla^{k+1} (g_B(\tilde{Q})\phi^2), \tilde{R}^T (Q)\nabla^l Q R(Q) \right).
\]

(4.20)

Since \(x_0 \in \Omega \setminus \Sigma_Q\) is arbitrary, this completes the proof. □

Let \(R_L = R(Q_L)\) be a rotation such that \(R_L^T Q_L R\) is diagonal. Then we have

**Lemma 4.4** Let \(x_0 \in \Omega\) with some \(B_{r_0}(x_0) \subset \Omega \setminus \Sigma\) for a sufficiently small \(r_0\). Then, for any \(\phi \in C^2(B_{r_0}(x_0))\) and \(L \in S_3\) with sufficiently small \(\delta\), we have

\[
\int_{\Omega} \left( |\nabla^2 Q_{\phi}|^2 + \frac{|(R_L^T \nabla Q L)_D^2|}{L} \right) \phi^2 \, dx \leq C \int_{\Omega} |\nabla^2 Q_{\phi}|^2 |\nabla\phi|^2 \, dx,
\]

where \(C\) is a constant independent of \(L\).

**Proof** Let \(\varphi_\varepsilon\) be a cutoff function such that \(\varphi_\varepsilon(x) = 0\) for \(\text{dist}(x, \Sigma_{Q_L}) \leq \varepsilon\) and \(\varphi_\varepsilon(x) = 1\) for \(\text{dist}(x, \Sigma_{Q_L}) \geq 2\varepsilon\). Multiplying (1.11) by \(\nabla (\phi^2 \varphi_\varepsilon^2 \nabla Q_L)\) yields

\[
\int_{\Omega} (H(Q_L, \nabla Q_L), \nabla (\phi^2 \varphi_\varepsilon^2 \nabla Q_L)) \, dx = \int_{\Omega} \frac{1}{L} g_B(Q_L), \nabla (\phi^2 \varphi_\varepsilon^2 \nabla Q_L)) \, dx.
\]

(4.22)

Utilizing Lemma 4.1 with a sufficiently small \(\delta > 0\), we derive

\[
\left( \nabla g_B(\tilde{Q}_L), \nabla \tilde{Q}_L \right) = \nabla \left[ \delta_{\tilde{Q}_L} f_B(\tilde{Q}_L) \right] \nabla \left[ \tilde{Q}_L \right]_{ii}
\]
\[
= \sum_{i,j} \partial_{\tilde{Q}_L, \tilde{Q}_L} f_B(\tilde{Q}_L) \nabla \left[ \tilde{Q}_L \right]_{ii} \nabla \left[ \tilde{Q}_L \right]_{jj} \geq \frac{\lambda}{2} \sum_{i=1}^{3} |\nabla a(\tilde{Q}_L)_{ii}|^2 = \frac{\lambda}{2} |\nabla a(\tilde{Q}_L)|^2.
\]

(4.23)
Using Lemma 4.2 and (4.23), we have
\[
\frac{1}{L} \int_{\Omega} \langle g_B(Q_L), \nabla(\phi^2 \varphi^2 \nabla Q_L) \rangle \, dx \\
= \frac{1}{L} \int_{\Omega} \langle g_B(\tilde{Q}_L), R_L^T \nabla(\phi^2 \varphi^2 \nabla R_L) \rangle \, dx \\
= \frac{1}{L} \int_{\Omega} \langle g_B(\tilde{Q}_L), \nabla(\phi^2 \varphi^2 R_L^T \nabla R_L) \rangle \, dx \\
= - \frac{1}{L} \int_{\Omega} \langle \nabla g_B(\tilde{Q}_L), R_L^T \nabla R_L \rangle \, dx \\
= - \frac{1}{2} \int_{\Omega} \frac{|\nabla \tilde{Q}_L|^2}{L} \phi^2 \varphi^2 \, dx \\
\leq - \frac{\lambda}{2} \int_{\Omega} \frac{|R_L^T \nabla R_L| D|^2}{L} \phi^2 \varphi^2 \, dx. \tag{4.24}
\]

As \( \epsilon \) tends to zero, we observe that
\[
\lim_{\epsilon \to 0} \int_{\Omega} \langle H(Q_L, \nabla Q_L), \nabla(\phi^2 \varphi^2 \nabla Q_L) \rangle \, dx \\
= - \lim_{\epsilon \to 0} \int_{\Omega} \langle \nabla(H(Q_L, \nabla Q_L)), \phi^2 \varphi^2 \nabla Q_L \rangle \, dx \\
= - \int_{\Omega} \langle \nabla_i H(Q_L, \nabla Q_L), \nabla_i Q_L \rangle \phi^2 \, dx.
\]

It follows from using integrating by parts, (2.8) and Young’s inequality that
\[
- \int_{\Omega} \langle \nabla_i H(Q_L, \nabla Q_L), \nabla_i Q_L \rangle \phi^2 \, dx \\
= - \int_{\Omega} \nabla_i \left( \nabla_j \frac{\partial f_E(1)}{\partial (Q_L)_{ij}} - \frac{\partial f_E(1)}{\partial (Q_L)} \right) \nabla_i (Q_L)_{ij} \phi^2 \, dx \\
\geq \int_{\Omega} \frac{\partial^2 f_E(1)}{\partial (Q_L)_{ij} \partial (Q_L)_{mn}} \nabla^2 (Q_L)_{ij} \nabla^2 (Q_L)_{mn} \phi^2 \, dx \\
- C \int_{\Omega} (|\nabla Q_L|^2 + |\nabla Q_L|^2) |\nabla Q_L|^2 \phi^2 \\
+ (|\nabla Q_L|^2 |\nabla Q_L|^2 + |\nabla Q_L|^3) |\nabla \phi| |\phi| \, dx \\
\geq \frac{\alpha}{4} \int_{\Omega} |\nabla Q_L|^2 \phi^2 \, dx - C \int_{\Omega} |\nabla Q_L|^4 \phi^2 + |\nabla Q_L|^2 |\nabla \phi|^2 \, dx.
\]

Combining (4.24) with (4.25) yields
\[
\int_{\Omega} \left( \frac{\alpha}{4} |\nabla Q_L|^2 + \frac{\lambda}{2} \frac{|R_L^T \nabla R_L| D|^2}{L} \right) \phi^2 \, dx \leq C \int_{\Omega} |\nabla Q_L|^4 \phi^2 + |\nabla Q_L|^2 |\nabla \phi|^2 \, dx.
\]

Integrating by parts and using Young’s inequality, we deduce
\[
\int_{\Omega} |\nabla Q_L|^4 \phi^2 \, dx = \int_{\Omega} \langle \nabla Q_L, |\nabla Q_L|^2 \nabla Q_L \rangle \phi^2 \, dx \\
= - \int_{\Omega} \langle Q_L - Q_{L,x_0,r}, \nabla(|\nabla Q_L|^2 \nabla Q_L) \rangle \phi^2 \, dx.
\]
Here $Q_{x,r} := \int_{B_r(x)} Q \, dx$. Note that

$$|Q_L(x) - Q_{L;x_0,r}| \leq |Q_L(x) - Q_*(x)| + |Q_{L;x_0,r} - Q_*| + |Q_*(x) - Q_*|,$$

and for $x \in B_r(x_0) \subset \Omega \setminus \Sigma$, $Q_L(x)$ uniformly converges to $Q_*(x)$. For a sufficiently small $r_0$ and $L$, we see that

$$\int_{\Omega} |\nabla Q_L|^4 \phi^2 \, dx \leq \int_{\Omega} |Q_L - Q_{L;x_0,r}|^2 |\nabla^2 Q_L|^2 \phi^2 \, dx + C \int_{\Omega} |\nabla Q_L|^2 |\nabla \phi|^2 \, dx \leq \frac{\alpha}{4} \int_{\Omega} |\nabla^2 Q_L|^2 \phi^2 \, dx + C \int_{\Omega} |\nabla Q_L|^2 |\nabla \phi|^2 \, dx.$$

Then we conclude that

$$\int_{\Omega} \left( |\nabla^2 Q_L|^2 + \frac{|(R_L^T \nabla Q_L)^L D|^2}{L} \right) \phi^2 \, dx \leq C \int_{\Omega} |\nabla Q_L|^2 |\nabla \phi|^2 \, dx.$$

As an application of Lemma 4.4, we obtain a uniform Caccioppoli inequality for minimizer $Q_L$ as follows.

**Lemma 4.5** Let $x_0 \in \Omega$ with $B_{r_0}(x_0) \subset \Omega \setminus \Sigma$ for a sufficiently small $r_0 > 0$. Then for any $r \leq r_0$, we have

$$\int_{B_{r/2}(x_0)} |\nabla Q_L|^2 \, dx \leq \frac{C}{r^2} \int_{B_r(x_0)} |Q_L - Q_{L;x_0,r}|^2 \, dx,$$

(4.26)

where $Q_{L;x_0,r} := \int_{B_r(x_0)} Q \, dx$ and $C$ is a constant independent of $L$.

**Proof** For two $s, t$ such that $\frac{r}{2} \leq t < s \leq r$, choose a cutoff function $\phi \in C_0^\infty(B_s(x_0))$ such that $0 \leq \phi \leq 1$, $\phi = 1$ on $B_t$ and $|\nabla \phi| \leq C/(s-t)$.

Integrating by parts and using Young’s inequality, we have

$$\int_{\Omega} |\nabla Q_L(x)|^2 \phi^2 \, dx = -\int_{\Omega} (\Delta Q_L, Q_L(x) - Q_{L;x_0,r}) \phi^2 \, dx$$

$$- \int_{\Omega} (\nabla Q, (Q(x) - Q_{x_0,r}) \nabla \phi^2) \, dx \leq \frac{1}{2} \int_{\Omega} |\nabla Q_L(x)|^2 \phi^2 \, dx - \int_{\Omega} (\Delta Q_L, Q_L(x) - Q_{L;x_0,r}) \phi^2 \, dx$$

$$+ C \int_{\Omega} |Q_L(x) - Q_{L;x_0,r}|^2 |\nabla \phi|^2 \, dx.$$

Then, by Young’s inequality and Lemma 4.4, we obtain

$$\int_{B_t} |\nabla Q_L|^2 \, dx \leq (s-t)^2 \int_{\Omega} |\nabla^2 Q_L|^2 \, dx + \frac{C}{(s-t)^2} \int_{B_r(x_0)} |Q_L(x) - Q_{L;x_0,r}|^2 \, dx.$$
\[ \leq C_1 \int_{B_s \setminus B_t} |\nabla Q_L|^2 \, dx + \frac{C}{(s-t)^2} \int_{B_t(x_0)} |Q_L(x) - Q_L; x_0, r|^2 \, dx. \]

Through the standard technique of ‘filling hole’, we have

\[ \int_{B_t} |\nabla Q_L|^2 \phi^2 \, dx \leq \theta \int_{B_s} |\nabla Q_L|^2 \, dx + \frac{C}{(s-t)^2} \int_{B_t(x_0)} |Q_L(x) - Q_L; x_0, r|^2 \, dx \]

for \( \theta = \frac{C_1}{1+C_1} < 1 \) and two \( s, t \) such that \( r/2 \leq t \leq s \leq r \). In view of Lemma 3.1 in Chapter V of [18], the relation (4.21) follows.

Using Lemma 4.5, we have local uniform estimates on higher derivatives.

**Lemma 4.6** Let \( x_0 \in \Omega \) with \( B_{r_0}(x_0) \subset \Omega \setminus \Sigma \). Assume that there exists a constant \( \varepsilon_0 > 0 \) such that

\[ \int_{B_{r_0}(x_0)} |\nabla Q_L|^3 \, dx \leq \varepsilon_0^3. \]  

(4.27)

Then, for any integer \( k \geq 1 \), there exist a constant \( r_k \geq r_0/2 \) and a positive constant \( C_k \) independent of \( L \) such that

\[ \int_{B_{r_k}(x_0)} |\nabla^{k+1} Q_L|^2 + \frac{1}{L} |(R_L^T \nabla^k Q_L R_L)_D|^2 \, dx \leq C_k. \]  

(4.28)

**Proof** For simplicity of notations, we denote \( Q = Q_L \) and \( R = R_L \). The claim (4.28) is true for \( k = 1 \). At first, we show the case of \( k = 2 \).

Assume that there exists a constant \( C_1 > 0 \) such that

\[ \int_{B_{r_1}(x_0)} \left( |\nabla Q|^4 + |\nabla^2 Q|^2 + \frac{1}{L} |(R^T \nabla Q R)_D|^2 \right) \, dx \leq C_1. \]

Let \( \phi \) be a cutoff function in \( C_0^\infty(B_{r_1}(x_0)) \), where \( r_2 \) satisfies \( \frac{r_2}{2} < r_2 < r_1 < r_0 \) and \( \phi = 1 \) in \( B_{r_2}(x_0) \). Let \( \varphi_\varepsilon \) be another cutoff function such that \( \varphi_\varepsilon(x) = 0 \) for \( \text{dist}(x, \Sigma_Q) \leq \varepsilon \) and \( \varphi_\varepsilon(x) = 1 \) for \( \text{dist}(x, \Sigma_Q) \geq 2\varepsilon \). We differentiate (1.11) twice and multiply by \( \nabla^2 Q \phi^2 \varphi_\varepsilon^2 \) to get

\[ \int_{B_{r_0}(x_0)} \left\{ \nabla^2_{\bar{\gamma} \bar{\gamma}} H(Q, \nabla Q), \nabla^2_{\bar{\gamma} \bar{\gamma}} Q \phi^2 \varphi_\varepsilon^2 \right\} \, dx = \int_{B_{r_0}(x_0)} \left\{ \frac{1}{L} \nabla^2_{\bar{\gamma} \bar{\gamma}} g_B(Q), \nabla^2_{\bar{\gamma} \bar{\gamma}} Q \phi^2 \varphi_\varepsilon^2 \right\} \, dx. \]

(4.29)
Applying Lemma 4.3 and Lemma 4.1 to the right-hand side of (4.29), we find

\[
\frac{1}{L} \int_{B_{\rho}(x_0)} \langle \nabla_{\beta\gamma}^2 g_B(Q), \nabla_{\beta\gamma} Q \phi^2 \phi^2_e \rangle \, dx
\]

\[
= \frac{1}{L} \int_{B_{\rho}(x_0)} \langle g_B(Q), \nabla_{\beta\gamma}^2 (\nabla_{\beta\gamma} Q \phi^2 \phi^2_e) \rangle \, dx
\]

\[
= \frac{1}{L} \int_{B_{\rho}(x_0)} \langle g_B(\tilde{Q}), R^T \nabla_{\beta\gamma}^2 (\nabla_{\beta\gamma} Q \phi^2 \phi^2_e) \rangle \, dx
\]

\[
= \frac{1}{L} \int_{B_{\rho}(x_0)} \langle \nabla_{\beta\gamma}^2 (g_B(\tilde{Q})), R^T \nabla_{\beta\gamma}^2 Q \phi^2 \phi^2_e \rangle \, dx
\]

\[
= \frac{1}{L} \int_{B_{\rho}(x_0)} \langle \nabla_{\beta\gamma}^2 g_B(\tilde{Q}), R^T \nabla_{\beta\gamma}^2 Q \phi^2 \phi^2_e \rangle \, dx \tag{4.30}
\]

\[
= \frac{1}{L} \int_{B_{\rho}(x_0)} \sum_{i,j} \frac{\partial^2 \tilde{Q}_{ii} \tilde{Q}_{jj}}{\partial x_i \partial x_j} f_B(\tilde{Q}) \nabla_{\gamma\beta}^2 \tilde{Q}_{ii} \nabla_{\gamma\beta} \tilde{Q}_{jj} \phi^2 \phi^2_e \, dx
\]

\[
+ \frac{1}{L} \int_{B_{\rho}(x_0)} \sum_{i,j,m} \frac{\partial^3 \tilde{Q}_{ii} \tilde{Q}_{jj} \tilde{Q}_{mm}}{\partial x_i \partial x_j \partial x_m} f_B(\tilde{Q}) \nabla_{\gamma\beta} \tilde{Q}_{ii} \nabla_{\gamma\beta} \tilde{Q}_{jj} \nabla_{\gamma\beta} \tilde{Q}_{mm} \phi^2 \phi^2_e \, dx
\]

\[
\geq \frac{\lambda}{4L} \int_{B_{\rho}(x_0)} \frac{|\nabla^2 \tilde{Q}|^2 \phi^2 \phi^2_e}{\phi^2 \phi^2_e} \, dx - \frac{C}{L} \int_{B_{\rho}(x_0)} \frac{|\nabla \tilde{Q}|^4 \phi^2 \phi^2_e}{\phi^2 \phi^2_e} \, dx.
\]

\[
\geq \frac{\lambda}{4L} \int_{B_{\rho}(x_0)} \frac{|(R^T \nabla^2 Q R)_D|^2 \phi^2 \phi^2_e}{\phi^2 \phi^2_e} \, dx - \frac{C}{L} \int_{B_{\rho}(x_0)} \frac{|\nabla Q|^4 \phi^2 \phi^2_e}{\phi^2 \phi^2_e} \, dx \tag{4.31}
\]

where we used that

\[
\int_{B_{\rho}(x_0)} |\nabla^2 \tilde{Q}|^2 \phi^2 \phi^2_e \, dx = - \int_{B_{\rho}(x_0)} \langle \nabla (R^T \nabla Q R)_D, \nabla (\nabla^2 \tilde{Q} \phi^2 \phi^2_e) \rangle \, dx
\]

\[
= \int_{B_{\rho}(x_0)} \langle (R^T \nabla^2 Q R)_D \phi^2 \phi^2_e, \nabla \tilde{Q} \rangle \, dx
\]

\[
= \int_{B_{\rho}(x_0)} \langle (R^T \nabla^2 Q R)_D \phi^2 \phi^2_e, \nabla \tilde{Q} \rangle \, dx.
\]

Observer that, for any fixed \( \varepsilon > 0 \), \( R(x) \) in (4.30) is differentiable. Then it follows from (4.1) that \( Q^+ = R^T \pi(Q) R \) and

\[
C \int_{B_{\rho}(x_0)} \frac{|\nabla Q|^4 \phi^2 \phi^2_e}{\phi^2 \phi^2_e} \, dx
\]

\[
= C \int_{B_{\rho}(x_0)} \frac{|(R^T \nabla(Q) - \pi(Q)) R|_D|^2 \phi^2 \phi^2_e}{\phi^2 \phi^2_e} \, dx
\]

\[
= C \int_{B_{\rho}(x_0)} \frac{|\nabla(Q - \pi(Q)) R|_D|^2 \phi^2 \phi^2_e}{\phi^2 \phi^2_e} \, dx
\]
In view of (4.32)-(4.32), we deduce (4.30) to

\[
\leq C \int_{B_0(x_0)} |\nabla Q|^2 \frac{|\nabla (Q - \pi(Q))|^2}{L} \phi^2 \, dx. \tag{4.32}
\]

Letting \( \varepsilon \to 0 \) in (4.32), using (1.11) and the fact that \( |Q - \pi(Q)| \leq C |g_B(Q)| \) in (4.11), we find

\[
C \int_{B_0(x_0)} |\nabla Q|^2 \frac{|\nabla (Q - \pi(Q))|^2}{L} \phi^2 \, dx
\]

\[
= - \frac{C}{L} \int_{B_0(x_0)} \langle (Q - \pi(Q), \nabla_{\beta} (\nabla_{\beta} (Q - \pi(Q)) |\nabla Q|^2 \phi^2) \rangle \, dx
\]

\[
\leq C \int_{B_0(x_0)} \frac{|Q - \pi(Q)|}{L} |\nabla Q| |\nabla Q|^2 \phi^2 \, dx + C \int_{B_0(x_0)} \frac{|Q - \pi(Q)|}{L} |\nabla Q|^2 |\nabla \phi||\phi| \, dx
\]

\[
\leq C \int_{B_0(x_0)} |H(Q, \nabla Q)| |\nabla Q| |\nabla Q|^2 \phi^2 \, dx + C \int_{B_0(x_0)} |H(Q, \nabla Q)| |\nabla Q|^2 |\nabla \phi||\phi| \, dx
\]

\[
\leq C \int_{B_0(x_0)} (|\nabla Q|^6 + |\nabla Q|^2 |\nabla Q|^2) \phi^2 \, dx + C \int_{B_0(x_0)} (|\nabla Q|^4 + |\nabla Q|^2) |\nabla \phi|^2 \, dx. \tag{4.33}
\]

In view of (4.32)-(4.32), we deduce (4.30) to

\[
\int_{B_0(x_0)} \left\{ \frac{1}{L} \nabla_{\beta} g_B(Q), \nabla_{\beta} Q \phi^2 \right\} \, dx
\]

\[
\geq \frac{\lambda}{4L} \int_{B_0(x_0)} |(R^T \nabla Q R)D|^2 \phi^2 \, dx - C \int_{B_0(x_0)} (|\nabla Q|^6 + |\nabla Q|^2 |\nabla Q|^2) \phi^2 \, dx
\]

\[
- C \int_{B_0(x_0)} (|\nabla Q|^4 + |\nabla Q|^2) |\nabla \phi|^2 \, dx. \tag{4.34}
\]

Applying (2.8) and Young’s inequality to the left-hand side of (4.29), we obtain

\[
\int_{B_0(x_0)} \left\{ \nabla_{\beta}^2 H(Q, \nabla Q), \nabla_{\beta}^2 Q \phi^2 \right\} \, dx
\]

\[
= \int_{B_0(x_0)} \nabla_{\beta}^2 \left( \nabla_{\beta} \frac{\partial f_{E,1}(Q, \nabla Q)}{\partial Q_{ij,k}} - \frac{\partial f_{E,1}(Q, \nabla Q)}{\partial Q_{ij}} \right) \nabla_{\beta} Q_{ij} \phi^2 \, dx
\]

\[
\leq - \int_{B_0(x_0)} \frac{\partial^2 f_E}{\partial Q_{ij,k} \partial Q_{mn,l}} \nabla_{\alpha} Q_{mn} \nabla_{\alpha} Q_{ij} \phi^2 \, dx
\]

\[
+ C \int_{B_0(x_0)} \left| \nabla \left( \frac{\partial^2 f_{E,1}(Q, \nabla Q)}{\partial p \partial Q} \nabla Q \right) \right| |\nabla^3 Q| \phi^2 \, dx
\]

\[
+ C \int_{B_0(x_0)} \left| \nabla \frac{\partial^2 f_{E,1}(Q, \nabla Q)}{\partial p \partial p} \nabla Q \right| |\nabla^2 Q| |\nabla \phi||\phi| + \left| \nabla \frac{\partial^2 f_{E,1}(Q, \nabla Q)}{\partial Q} \right| |\nabla^2 Q| \phi^2 \, dx
\]

\[
\leq \int_{B_0(x_0)} \frac{\alpha}{4} |\nabla^3 Q|^2 \phi^2 \, dx + C \int_{B_0(x_0)} (|\nabla Q|^4 + |\nabla Q|^2) |\nabla \phi|^2 \, dx
\]

\[
+ C \int_{B_0(x_0)} (|\nabla Q|^6 + |\nabla Q|^2 |\nabla Q|^2) \phi^2 \, dx. \tag{4.35}
\]
Using Hölder’s inequality, we have
\[
\int_{B_{r_0}(x_0)} (|\nabla^2 Q|^2 |\nabla Q|^2 + |\nabla Q|^6) \phi^2 \, dx \leq C \varepsilon_0 \int_{B_{r_0}(x_0)} |\nabla^3 Q|^2 \phi^2 + |\nabla^2 Q|^2 |\nabla \phi|^2 \, dx.
\]  
(4.36)

Combining (4.34) with (4.36) and choosing \( \varepsilon_0 \) sufficiently small, we obtain
\[
\int_{B_{r_2}(x_0)} |\nabla^3 Q|^2 + \frac{1}{L} |(R^T \nabla^2 Q R)|^2 \, dx \leq C_2.
\]
Set \( r_k := (1 - \sum_{i=1}^k 2^{-(i+1)}) r_0 > \frac{r_0}{2} \). For any \( k \leq l \), we can assume that there is a constant \( C_k \) such that
\[
\int_{B_{r_k}(x_0)} (|\nabla^{k+1} Q|^2 + \frac{1}{L} |(R^T \nabla^k Q R)|^2) \, dx \leq C_k.
\]  
(4.37)

As a consequence of the Sobolev inequality, we have
\[
\|\nabla^{k-1} Q\|_{L^\infty(B_{r_k}(x_0))} \leq C_k
\]  
(4.38)
for any \( k \leq l - 1 \).

Next, we prove it for \( k = l + 1 \). Let \( r_{l+1} \) be the constant satisfying
\[
\frac{r_0}{2} < r_{l+1} < r_l < \cdots < r_1 < r_0.
\]

Let \( \phi \) be a cutoff function in \( C_0^\infty(B_r(x_0)) \) with \( \phi = 1 \) in \( B_{r_{l+1}}(x_0) \). We apply \( \nabla^{l+1} \) to (1.11) and multiply by \( \nabla^{l+1} Q \phi^2 \varphi^2_\varepsilon \) to have
\[
\int_{B_{r_0}(x_0)} \langle \nabla^{l+1} H(Q, \nabla Q), \nabla^{l+1} Q \phi^2 \varphi^2_\varepsilon \rangle \, dx
\]
\[
= \int_{B_{r_0}(x_0)} \left\langle \frac{1}{L} \nabla^{l+1} g_B(Q), \nabla^{l+1} Q \phi^2 \varphi^2_\varepsilon \right\rangle \, dx.
\]  
(4.39)

It follows from Lemma 4.3 that
\[
\frac{1}{L} \int_{B_{r_0}(x_0)} \langle \nabla^{l+1} g_B(Q), \nabla^{l+1} Q \phi^2 \varphi^2_\varepsilon \rangle \, dx
\]
\[
= (-1)^{l+1} \frac{1}{L} \int_{B_{r_0}(x_0)} \langle g_B(\tilde{Q}), R^T \nabla^{l+1} (\nabla^{l+1} Q \phi^2 \varphi^2_\varepsilon) R \rangle \, dx
\]
\[
= \frac{1}{L} \int_{B_{r_0}(x_0)} \langle \nabla^{l+1} g_B(\tilde{Q}) \phi^2 \varphi^2_\varepsilon, R^T \nabla^{l+1} Q R \rangle \, dx
\]
\[
= (-1)^l \frac{1}{L} \int_{B_{r_0}(x_0)} \langle \nabla(\nabla^{l+1} g_B(\tilde{Q}) \phi^2 \varphi^2_\varepsilon), \nabla \tilde{Q} \rangle \, dx
\]
\[
= \frac{1}{L} \int_{B_{r_0}(x_0)} \langle \nabla^{l+1} g_B(\tilde{Q}), \nabla^{l+1} \tilde{Q} \phi^2 \varphi^2_\varepsilon \rangle \, dx
\]
\[
= \frac{1}{L} \int_{B_{r_0}(x_0)} \sum_{i,j} \nabla^l (\partial_{Q_{iij}}^2 \tilde{Q}_{jj} f_B(\tilde{Q}) \nabla \tilde{Q}_{jj}) \nabla^{l+1} \tilde{Q}_{ii} \phi^2 \varphi^2_\varepsilon \, dx.
\]  
(4.40)
Using a similar argument in (4.31), for $i \geq 3$, one can check that

$$\int_{B_{r_0}(x_0)} |\nabla^i \tilde{Q}|^2 Z(x) \phi_e^2 \, dx = (-1)^{i-1} \int_{B_{r_0}(x_0)} \langle \nabla \tilde{Q}, \nabla^{i-1}(\nabla^i \tilde{Q}Z(x)\phi_e^2) \rangle \, dx$$

$$= \int_{B_{r_0}(x_0)} \langle (R^T \nabla^i Q R)_D Z(x) \phi_e^2, \nabla^i \tilde{Q} \rangle \, dx$$

$$= \int_{B_{r_0}(x_0)} |(R^T \nabla^i Q R)_D|^2 Z(x) \phi_e^2 \, dx \quad (4.41)$$

for some scalar function $Z(x)$. Observe that $\partial_j f_B(\tilde{Q}) = 0$ for $j \geq 5$. Applying Lemma 4.1 with a sufficiently small $\delta$ and (4.41) to (4.40), we obtain

$$\frac{1}{L} \int_{B_{r_0}(x_0)} (\nabla^{l+1} g_B(Q), \nabla^{l+1} Q \phi^2 \phi_e^2) \, dx$$

$$\geq \frac{1}{2L} \int_{B_{r_0}(x_0)} |\nabla^{l+1} \tilde{Q}|^2 \phi^2 \phi_e^2 \, dx$$

$$- \frac{C}{L} \int_{B_{r_0}(x_0)} \sum_{\mu_1 \leq \mu_2 \leq \mu_3 \leq l} |\partial_3^2 f_B(\tilde{Q})|^2 |\nabla^{\mu_1} \tilde{Q}|^2 |\nabla^{\mu_2} \tilde{Q}|^2 |\nabla^{\mu_3} \tilde{Q}|^2 |\nabla^2 \phi^2 \phi_e^2 \, dx$$

$$- \frac{C}{L} \int_{B_{r_0}(x_0)} \sum_{\mu_1 \leq \mu_2 \leq \mu_3 \leq l+1} |\partial_4^2 f_B(\tilde{Q})|^2 |\nabla^{\mu_1} \tilde{Q}|^2 |\nabla^{\mu_2} \tilde{Q}|^2 |\nabla^{\mu_3} \tilde{Q}|^2 |\nabla^2 \phi^2 \phi_e^2 \, dx$$

$$\geq \frac{1}{2L} \int_{B_{r_0}(x_0)} |(R^T \nabla^{l+1} Q R)_D|^2 |\nabla^2 \phi^2 \phi_e^2 \, dx$$

$$- \frac{C}{L} \int_{B_{r_0}(x_0)} \sum_{\mu_1 \leq \mu_2 \leq \mu_3 \leq l} |\partial_3^3 f_B(\tilde{Q})|^2 |(R^T \nabla^{\mu_1} Q R)_D|^2 |(R^T \nabla^{\mu_2} Q R)_D|^2 |\nabla^2 \phi^2 \phi_e^2 \, dx$$

$$- \frac{C}{L} \int_{B_{r_0}(x_0)} \sum_{\mu_1 \leq \mu_2 \leq \mu_3 \leq l+1} |\partial_4^3 f_B(\tilde{Q})|^2 |(R^T \nabla^{\mu_1} Q R)_D|^2$$

$$\times |(R^T \nabla^{\mu_2} Q R)_D|^2 |(R^T \nabla^{\mu_3} Q R)_D|^2 |\nabla^2 \phi^2 \phi_e^2 \, dx. \quad (4.42)$$

As $\varepsilon$ tends to zero, we obtain from (4.38) and (4.42) that

$$\lim_{\varepsilon \to 0} \frac{1}{L} \int_{B_{r_0}(x_0)} (\nabla^{l+1} g_B(Q), \nabla^{l+1} Q \phi^2 \phi_e^2) \, dx$$

$$\geq \frac{1}{4L} \int_{B_{r_0}(x_0)} |(R^T \nabla^{l+1} Q R)_D|^2 |\nabla^2 \phi^2 \, dx - \frac{C}{L} \int_{B_{r_0}(x_0)} |(R^T \nabla Q R)_D|^2 |(R^T \nabla^l Q R)_D|^2 |\nabla^2 \phi^2 \, dx$$

$$- \frac{C}{L} \int_{B_{r_0}(x_0)} |(R^T \nabla^2 Q R)_D|^2 |(R^T \nabla^{l-1} Q R)_D|^2 |\nabla^2 \phi^2 \, dx - C_k$$

$$\geq \frac{1}{4L} \int_{B_{r_0}(x_0)} |(R^T \nabla^{l+1} Q R)_D|^2 |\nabla^2 \phi^2 \, dx - C_k. \quad (4.43)$$
Using Young’s inequality and integration by parts, we have

\[
\int_{B_{r_0}(x_0)} \langle \nabla^{l+1} H(Q, \nabla Q), \nabla^{l+1} Q \phi^2 \rangle \, dx \\
= - \int_{\mathbb{R}^3} \frac{\partial^2 f_E}{\partial Q_{ij, \gamma} \partial Q_{mn, \beta}} \nabla^{l+1} \nabla_{\beta} Q_{mn} \nabla^{l+1} \nabla_{\gamma} Q_{ij} \phi^2 \, dx \\
+ C \int_{B_{r_0}(x_0)} \left| \nabla^{l+1} \frac{\partial f_{E,1}(Q, \nabla Q)}{\partial p} \right| |\nabla^{l+1} Q| |\nabla \phi||\phi| \, dx \\
+ C \int_{B_{r_0}(x_0)} \left| \nabla^{l+1} \frac{\partial f_{E,1}(Q, \nabla Q)}{\partial Q} \right| |\nabla^{l+1} Q| \phi^2 \, dx \\
+ C \int_{B_{r_0}(x_0)} \left| \nabla^{l} \left( \frac{\partial^2 f_{E,1}(Q, \nabla Q)}{\partial p \partial Q} \nabla Q \right) \right| |\nabla^{l+2} Q| \phi^2 \, dx \\
+ C \int_{B_{r_0}(x_0)} \left| \nabla^{l-1} \left( \frac{\partial^2 f_{E,1}(Q, \nabla Q)}{\partial p \partial p} \nabla^{2} Q \right) \right| |\nabla^{l+2} Q| \phi^2 \, dx \\
\leq - \int_{B_{r_0}(x_0)} \frac{\alpha}{4} |\nabla^{l+2} Q|^2 \phi^2 \, dx \\
+ C \int_{B_{r_0}(x_0)} \sum_{\mu_1 + \cdots + \mu_l = l+2} |\nabla^{\mu_1} Q|^2 |\nabla^{\mu_2} Q|^2 \cdots |\nabla^{\mu_{l-1}} Q|^2 |\nabla^{\mu_l} \phi|^2 \, dx.
\]

In view of (4.37)-(4.38), we have

\[
C \int_{B_{r_0}(x_0)} \sum_{\mu_1 + \cdots + \mu_l = l+2} |\nabla^{\mu_1} Q|^2 |\nabla^{\mu_2} Q|^2 \cdots |\nabla^{\mu_{l-1}} Q|^2 |\nabla^{\mu_l} \phi|^2 \leq C_l. \tag{4.45}
\]

Combining (4.44) with (4.45), our claim (4.28) follows for \( k = l + 1 \). \( \square \)

Now we give a proof of Theorem 3.

**Proof** For any \( x_0 \in \Omega \setminus \Sigma \), let \( B_{2r_0}(x_0) \) be a ball such that \( B_{2r_0}(x_0) \subset \Omega \setminus \Sigma \). From Lemma 4.5, we deduce the following estimate

\[
\frac{1}{r_0} \int_{B_{4r_0}(x_0)} |\nabla Q_L|^2 \, dx \leq C \int_{B_{2r_0}(x_0)} |Q_L - Q_{L,x_0,2r_0}|^2 \, dx \\
\leq C \sup_{x \in B_{2r_0}(x_0)} |Q_L(x) - Q_{L;x_0,\rho}|^2. \tag{4.46}
\]

It follows from (4.21) that

\[
\int_{B_{r_0}(x_0)} |\nabla^2 Q_L|^2 \, dx \leq \frac{C}{r_0} \int_{B_{4r_0}(x_0)} |\nabla Q_L|^2 \, dx.
\]

As a consequence of the Gagliardo-Nirenberg interpolation (c.f. [15]), we have

\[
\int_{B_{r_0}(x_0)} |\nabla Q_L|^3 \, dx \leq C \left( \int_{B_{r_0}(x_0)} r_0 |\nabla^2 Q_L|^2 + \frac{1}{r_0} |\nabla Q_L|^2 \, dx \right)^{3/2}
\]
+ C \left( r_0^{-1} \int_{B_{r_0}(x_0)} |\nabla Q_L|^2 \, dx \right)^{3/2} \leq \epsilon_0^3.

Using Lemma 4.5 with any $k \geq 1$, we obtain

$$\|Q_L\|_{W^{k,2}_{\text{loc}}(\Omega \setminus \Sigma)} < C_k. \quad (4.47)$$

Then, $Q_L$ converges smoothly to $Q_* \in \Omega \setminus \Sigma$.

## 5 The Landau-de Gennes density through the Oseen-Frank density

In this section, we will obtain a new form of the Landau-de Gennes energy density through the Oseen-Frank density. Under the condition (1.9), it was shown in [33] that for each $Q = s_+(u \otimes u - \frac{1}{3} I) \in S_n$, one has

$$W(u, \nabla u) = f_E(Q, \nabla Q).$$

Assuming the strong Ericksen condition

$$k_2 > |k_4|, \quad k_3 > 0, \quad 2k_1 > k_2 + k_4, \quad (5.1)$$

it was pointed out in [28] (see also [1, 14, 15]) that there are positive constants $\lambda$ and $C$ such that the density $W(u, \nabla u)$ is equivalent to a new form that $\tilde{W}(u, p)$ satisfies

$$\lambda |p|^2 \leq \tilde{W}(u, p) \leq C(1 + |u|^2)|p|^2$$

for any $u \in \mathbb{R}^3$ and any $p \in \mathbb{M}^{3 \times 3}$. However, there seems no reference for an explicit form of $\tilde{W}(u, \nabla u)$, so we give an explicit form $\tilde{W}(u, \nabla u)$ here. For $u \in \mathbb{R}^3$, it can be checked that

$$2|\nabla u|^2 - |\text{curl} u|^2 - (\text{div} u)^2$$

$$= (\nabla_1 u_1 + \nabla_2 u_2)^2 + (\nabla_1 u_2 + \nabla_3 u_3)^2 + (\nabla_1 u_1 + \nabla_3 u_3)^2$$

$$+ (\nabla_2 u_3 + \nabla_3 u_2)^2 + (\nabla_1 u_3 + \nabla_3 u_1)^2 + (\nabla_1 u_3 + \nabla_2 u_1)^2. \quad (5.2)$$

**Lemma 5.1** Assume the Frank constants $k_1, \cdots, k_4$ satisfy

$$k_2 > |k_4|, \quad \min\{k_1, k_3\} > \frac{1}{2}(k_2 + k_4). \quad (5.3)$$

Then the density $W(u, \nabla u)$ of the form (1.5) for each $u \in S^2$ is equivalent to the new form

$$\tilde{W}(u, \nabla u) = \frac{\tilde{\alpha}}{2} |\nabla u|^2 + \frac{2k_1 - k_2 - k_4 - \alpha}{4} (\text{div} u)^2$$

$$+ \frac{k_2 - k_4 - \alpha}{4} (u \cdot \text{curl} u)^2 + \frac{2k_3 - k_2 - k_4 - \alpha}{4} |u \times \text{curl} u|^2$$

$$+ \frac{k_2 + k_4 - \alpha}{4} \sum_{i \neq j} \left( (\nabla_i u_i + \nabla_j u_j)^2 + (\nabla_i u_j + \nabla_j u_i)^2 \right) \quad (5.4)$$

where $\alpha = \min\{k_2 - |k_4|, 2k_1 - k_2 - k_4, 2k_3 - k_2 - k_4\} > 0$.

**Proof** Note that $W(u, \nabla u)$ is rotational invariant (c.f. [24]); i.e., for each $R \in SO(3)$, $\tilde{x} = R(x - x_0)$ and $\tilde{u} = Ru(x) = Ru$. Then we have

$$W(\tilde{u}, \nabla \tilde{u}) = W(Ru, R\nabla u R^T) = W(u, \nabla u).$$
Then for any \( u \in S^2 \), we can find some \( R = R(u(x_0)) \in SO(3) \) at each point \( x_0 \in \Omega \) such that
\[
\tilde{u}(0) := Ru(x_0) = (0, 0, 1)^T.
\]

Using the relation
\[
\frac{\partial \tilde{u}_3}{\partial \tilde{x}_i} = - (\tilde{u}_1 \frac{\partial \tilde{u}_1}{\partial \tilde{x}_i} + \tilde{u}_2 \frac{\partial \tilde{u}_2}{\partial \tilde{x}_i}) = 0
\]
for all \( i = 1, 2, 3 \), we evaluate four terms of the Oseen-Frank energy density at \( \tilde{x}_0 \)
\[
(\tilde{\nabla} \cdot \tilde{u})^2 = (\tilde{\nabla}_1 \tilde{u}_1 + \tilde{\nabla}_2 \tilde{u}_2)^2,
\]
\[
(\tilde{u} \cdot \text{curl} \tilde{u})^2 = (\tilde{\nabla}_1 \tilde{u}_2 - \tilde{\nabla}_2 \tilde{u}_1)^2,
\]
\[
|\tilde{u} \times \text{curl} \tilde{u}|^2 = |\tilde{\nabla}_3 \tilde{u}_1|^2 + |\tilde{\nabla}_3 \tilde{u}_2|^2,
\]
\[
(\text{tr}(\tilde{\nabla} \tilde{u})^2 - (\tilde{\nabla} \cdot \tilde{u})^2) = 2 \tilde{\nabla}_1 \tilde{u}_2 \tilde{\nabla}_2 \tilde{u}_1 - 2 \tilde{\nabla}_1 \tilde{u}_1 \tilde{\nabla}_2 \tilde{u}_2.
\]

(5.5)

Substituting above identities into the density, we have
\[
W(\tilde{u}, \tilde{\nabla} \tilde{u}) = \frac{k_2}{2} (\tilde{\nabla}_1 \tilde{u}_1 + \tilde{\nabla}_2 \tilde{u}_2)^2 + \frac{k_2}{2} (|\tilde{\nabla}_1 \tilde{u}_2|^2 + |\tilde{\nabla}_2 \tilde{u}_1|^2)
\]
\[
+ \frac{k_3}{2} (|\tilde{\nabla}_3 \tilde{u}_1|^2 + |\tilde{\nabla}_3 \tilde{u}_2|^2) + k_4 \tilde{\nabla}_1 \tilde{u}_2 \tilde{\nabla}_2 \tilde{u}_1 - (k_2 + k_4)(\tilde{\nabla}_1 \tilde{u}_1 \tilde{\nabla}_2 \tilde{u}_2)
\]
\[
= \frac{\tilde{\alpha}}{2} |\tilde{\nabla} \tilde{u}|^2 + \frac{2k_1 - k_2 - k_4 - \tilde{\alpha}}{4} (\tilde{\nabla}_1 \tilde{u}_1 + \tilde{\nabla}_2 \tilde{u}_2)^2
\]
\[
+ \frac{k_2 + k_4 - \tilde{\alpha}}{4} (\tilde{\nabla}_1 \tilde{u}_1 - \tilde{\nabla}_2 \tilde{u}_2)^2 + \frac{k_2 - k_4 - \tilde{\alpha}}{2} (|\tilde{\nabla}_1 \tilde{u}_2|^2 + |\tilde{\nabla}_2 \tilde{u}_1|^2)
\]
\[
+ \frac{(k_3 - \tilde{\alpha})}{2} (|\tilde{\nabla}_3 \tilde{u}_1|^2 + |\tilde{\nabla}_3 \tilde{u}_2|^2) + \frac{k_4}{2} (\tilde{\nabla}_1 \tilde{u}_2 + \tilde{\nabla}_2 \tilde{u}_1)^2
\]
\[
= \frac{\tilde{\alpha}}{2} |\nabla u|^2 + \frac{2k_1 - k_2 - k_4 - \alpha}{4} (\text{div} u)^2 + \frac{k_2 - k_4 - \alpha}{2} (u \cdot \text{curl} u)^2
\]
\[
+ \frac{k_3 - \alpha}{2} |u \times \text{curl} u|^2 + \frac{k_2 + k_4 - \alpha}{4} \left( |\tilde{\nabla}_1 \tilde{u}_1 - \tilde{\nabla}_2 \tilde{u}_2|^2 + |\tilde{\nabla}_1 \tilde{u}_2 + \tilde{\nabla}_2 \tilde{u}_1|^2 \right),
\]

(5.6)

where \( \tilde{\alpha} \) is a positive constant due to the strong Ericksen condition (5.1). Using (5.5), we find
\[
|\tilde{\nabla}_1 \tilde{u}_1 - \tilde{\nabla}_2 \tilde{u}_2|^2 + |\tilde{\nabla}_1 \tilde{u}_2 + \tilde{\nabla}_2 \tilde{u}_1|^2 + |\tilde{\nabla}_3 \tilde{u}_1|^2 + |\tilde{\nabla}_3 \tilde{u}_2|^2
\]
\[
= |\tilde{\nabla} \tilde{u}|^2 + \text{tr}(\tilde{\nabla} \tilde{u})^2 - (\tilde{\nabla} \cdot \tilde{u})^2 = 2 |\nabla u|^2 - |\text{curl} u|^2 - (\text{div} u)^2.
\]

If we further assume that \( 2k_3 > k_2 + k_4 \), then we can rewrite (5.6) into
\[
\tilde{W}(u, \nabla u) = \frac{\tilde{\alpha}}{2} |\nabla u|^2 + \frac{2k_1 - k_2 - k_4 - \alpha}{4} (\text{div} u)^2
\]
\[
+ \frac{k_2 - k_4 - \alpha}{4} (u \cdot \text{curl} u)^2 + \frac{2k_3 - k_2 - k_4 - \alpha}{4} |u \times \text{curl} u|^2
\]
\[
+ \frac{k_2 + k_4 - \alpha}{4} (2 |\nabla u|^2 - |\text{curl} u|^2 - (\text{div} u)^2).
\]

(5.7)

From (5.2), we prove (5.4). \( \square \)

It is clear that the new form \( \tilde{W}(u, p) \) in (5.4) with \( p = \nabla u \) satisfies
\[
\frac{\tilde{\alpha}}{2} |p|^2 \leq \tilde{W}(u, p) \leq C (1 + |u|^2) |p|^2.
\]
for all \( u \in \mathbb{R}^3 \) and \( p \in M_3^{3 \times 3} \).

Through the relation (5.4), we can have the new Landau-de Gennes energy density satisfying the coercivity in the following:

**Proposition 5.1** Assume that \( \hat{L}_1, \hat{L}_2, \hat{L}_3 \) and \( \hat{L}_4 \) satisfy the condition

\[
\begin{align*}
\hat{L}_1 &:= s_+^{-2} \frac{2k_1 - k_2 - k_4}{2}, \\
\hat{L}_2 &:= s_+^{-2} \frac{2k_3 - k_2 - k_4}{2}, \\
\hat{L}_3 &:= s_+^{-2} \frac{k_2 - k_4}{2}, \\
\hat{L}_4 &:= s_+^{-2} \frac{k_2 + k_4}{2}.
\end{align*}
\]

Then for each \( Q \in S_n \), we obtain

\[
\begin{align*}
f_{E,2}(Q, \nabla Q) := & \frac{\alpha}{2} |\nabla Q|^2 + \frac{\hat{L}_1 - \tilde{\alpha}}{2} \sum_{i=1}^{3} \left( (s_+^{-1} Q + \frac{1}{3} I)_{ij} (\nabla \cdot Q_{ij}) \right)^2 \\
+ & \frac{\hat{L}_2 - \tilde{\alpha}}{2} \left| (s_+^{-1} Q + \frac{1}{3} I)_{i} \times \text{curl} Q_{i} \right|^2 \\
+ & \frac{\hat{L}_3 - \tilde{\alpha}}{2} \sum_{i=1}^{3} \left( (s_+^{-1} Q + \frac{1}{3} I)_{ij} (\text{curl} Q_{ij}) \right)^2 \\
+ & \frac{\hat{L}_4 - \tilde{\alpha}}{2} \sum_{i \neq j} \sum_{k=1}^{3} \left( (\nabla_i Q_{ik} + \nabla_j Q_{jk})^2 + (\nabla_i Q_{jk} + \nabla_j Q_{ik})^2 \right).
\end{align*}
\]

where \( Q_i \) is the \( i \)-th column of the \( Q \) matrix and \( \tilde{\alpha} \) is given by

\[
\alpha = \min\{\hat{L}_1, \hat{L}_2, \hat{L}_3, \hat{L}_4\} > 0.
\]

**Proof** Due to the fact that \( |u|^2 = 1 \), a direct calculation yields

\[
\nabla_k u_i = u_j \nabla_k (u_i u_j) = s_+^{-1} (u_1 \nabla_k Q_{i1} + u_2 \nabla_k Q_{i2} + u_3 \nabla_k Q_{i3}) = s_+^{-1} u_j \nabla_k Q_{ij}.
\]

One can verify that

\[
u_j \nabla_k u_i = s_+^{-1} (s_+^{-1} Q + \frac{1}{3} I)_{ij} \nabla_k Q_{ij}.
\]

Here we used the fact that \( |Q| = \sqrt{\frac{2}{3}} s_+ \). Then we can derive \( f_{E}(Q, \nabla Q) \) from (5.4) that

\[
\begin{align*}
s_+^2 (\text{div} u)^2 &= s_+^2 \sum_i (u_i \text{div} u)^2 = \sum_i \left( (s_+^{-1} Q + \frac{1}{3} I)_{ij} (\nabla \cdot Q_{ij}) \right)^2, \\
s_+^2 |u \times \text{curl} u|^2 &= s_+^2 |u \times (s_+^{-1} u_j (\text{curl} Q_{ij}))|^2 = \left| (s_+^{-1} Q + \frac{1}{3} I)_{i} \times \text{curl} Q_{i} \right|^2, \\
s_+^2 (u \cdot \text{curl} u)^2 &= s_+^2 (s_+^{-1} u_i u_j (\text{curl} Q_{ij}))^2 = \left( (s_+^{-1} Q + \frac{1}{3} I)_{ij} (\text{curl} Q_{ij}) \right)^2.
\end{align*}
\]

It then follows from (5.2) and (5.10) that

\[
s_+^2 \left( 2|\nabla u|^2 - |\text{curl} u|^2 - (\text{div} u)^2 \right)
\]

\[
= \frac{1}{2} \sum_{i,k=1}^{3} \sum_{i \neq j} (s_+^{-1} Q + \frac{1}{3} I)_{lk}^2 \left( (\nabla_i Q_{ik} + \nabla_j Q_{jk})^2 + (\nabla_i Q_{jk} + \nabla_j Q_{ik})^2 \right).
\]

Substituting the identities (5.11)-(5.14) into the equation (5.4), we complete a proof. \( \square \)
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Data availability.  No datasets were generated or analyzed during the current study.

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