Spin-Singlet to Spin Polarized Phase Transition
at $\nu = 2/3$: Flux-Trading in Action

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ABSTRACT

We analyze the phase transition between spin-singlet and spin-polarized states which occurs at $\nu = 2/3$. The basic strategy is to use adiabatic flux-trading arguments to relate this transition to the analogous transition at $\nu = 2$. The transition is found to be similar to a transition in ferromagnets. In our analysis, we find two possible scenarios. In one, the transition is first-order, in agreement with experimental and numerical studies of the $\nu = 2/3$ transition. In the other, we find a second-order transition to a partially polarized state followed by a second-order transition to a fully polarized state. This picture is in qualitative agreement with experiments on the $\nu = 4/3$ state, the particle-hole conjugate of $\nu = 2/3$. We analyze the edge modes which propagate at the boundaries between regions of different phases and show that these do not support gapless excitations. Finally, we consider the possibility of a finite-temperature compressible state with a Fermi surface which would explain the non-zero $\rho_{xx}$ seen in experiments.
1. Introduction

There is now convincing experimental and numerical evidence that a spin-singlet fractional quantum Hall state at $\nu = 2/3$ is realized at low densities in extremely pure samples. The spin-singlet state is of fundamental interest because its spin-symmetry is an emergent property (the microscopic theory is not accurately symmetric) that arises from special features of the correlated wavefunction. This symmetry is not even approximately valid in the generic case (e.g. at other filling fractions) and is not manifest in the standard effective theory. Such a possibility was first raised by Halperin [1], who noted long ago that band mass and $g$-factor corrections make the ratio of Zeeman to cyclotron energies $\sim \frac{1}{60}$. Correlations can compete with this small spin-dependent energy. The $\nu = 2/3$ state is the simplest fractional quantum Hall state which takes advantage of this circumstance. A simple trial wavefunction for this state has been proposed by Wu, Dev, and Jain [7]. This wavefunction has a large overlap with the ground state found by numerical diagonalization of the Hamiltonian for small numbers of particles[7,5,4]. In the tilted-field experiments of Clark, et al. [3], which were crucial to the identification of the spin-singlet character of this state, an in-plane magnetic field was introduced, thereby increasing the Zeeman energy but leaving the cyclotron energy unchanged. As the Zeeman energy was increased to a critical value, the quantum Hall state was destroyed. At still higher Zeeman energies, a quantum Hall state reappeared. This is nothing but the transition from spin-singlet to spin-polarized states as a function of the Zeeman energy. In this paper, we will study that transition in detail.

In an earlier paper [6], we noted that the $K$-matrices which describe the spin-polarized and spin-singlet states are precisely the same. As a result, we argued, the boundaries between regions of different phases in a first-order phase transition, or a second-order phase transition in the presence of disorder, will not support gapless excitations. Since these states are so similar, one might actually wonder whether there has to be a phase transition at all. However, despite the identity
of their $K$-matrices, the states are distinguished by other topological quantum numbers. For instance, the polarized state has shift $S = \frac{1}{\nu} N - N_\phi = 0$ on the sphere while the singlet state has shift $S = 1$. Still, one might doubt that the gap goes quite to zero since, as we noted above, there are no gapless excitations at phase boundaries. Experiments are unclear on this point since the non-vanishing $\rho_{xx}$ which they find could be indicative of a small but non-vanishing gap. In any case, the possibility of metallic states at the transition needs elucidation, whether or not they survive to $T = 0$. To address these questions, we exploit flux-trading arguments to relate the $\nu = 2/3$ transition to the $\nu = 2$ transition (as we suggested in our earlier paper [6]). At $\nu = 2$, in the vicinity of the transition, there are two nearly degenerate Landau levels. We then have, effectively, a two-component Hall system, similar to those analyzed in the context of double-layer or single-wide layer systems. We relate the transition at $\nu = 2$ to a ferromagnetic transition as a function of magnetic field below the critical temperature of such a model. In this picture, the charge transfer gap remains non-vanishing. Finally, we use flux-trading arguments again to construct a non-Fermi liquid effective field theory for a possible finite-temperature metallic state at the transition and describe its phenomenology. Some of the effects we suggest appear to have been observed in existing experimental and numerical work [3,5]. If further work confirms this, and especially if the predicted metallic state could be demonstrated, it would form an impressive demonstration of the fruitfulness of the flux-trading concept.

2. Relating $\nu = 2/3$ to $\nu = 2$

The states of interest at $\nu = 2/3$ are both related to integer quantum Hall states at $\nu = 2$ by adiabatically trading magnetic flux for statistical flux. In this familiar procedure, we simultaneously change the statistics of the particles and the magnetic field according to $n\Delta\left(\frac{\theta}{\pi}\right) = \Delta B$, thereby leaving the system unchanged in the mean-field approximation [7,8]. The change of statistics may be implemented with a Chern-Simons field which carries an average flux $\Delta B$. If we continue this
procedure until $\Delta \theta = 2\pi p$, where $p$ is an integer of either sign, then the statistics is again fermionic. This is a state of electrons in field $B_{\text{eff}} = B + \Delta B$, or, in other words, at inverse filling fraction $\frac{1}{\nu'} = \frac{1}{\nu} + 2p$, which, in mean-field approximation, is identical to the state at filling fraction $\nu$. The mean-field approximation is not exact, of course, so there are still gauge field fluctuations to be dealt with – ie. the Chern-Simons field is not really equal to its mean – but these shouldn’t be qualitatively important because both states have a gap. In particular, starting with a state at $\nu = 2$ and decreasing the $B$ field by two flux tubes per electron, we arrive at the state at $\nu = 2/3$, which should be qualitatively similar. Later, we will discuss the quantitative differences resulting from the gauge field fluctuations.

An ambiguity arises because there are, in fact, two possible integer quantum Hall states at $\nu = 2$. If the cyclotron energy is less that the Zeeman energy, $E_c < E_Z$, then the first and second Landau levels with spin aligned along $\mathbf{B}$ are filled. If, however, $E_c > E_Z$, the first spin-aligned Landau level and the first spin-reversed Landau level are filled; this is a spin-singlet state. As we mentioned in the introduction, the latter case is realized in GaAs-AlAs systems and the observed state at $\nu = 2$ is a spin-singlet. When an in-plane magnetic field, $B_{||} = B_{\perp} \tan \theta = B \sin \theta$, is turned on, $E_Z \propto B$ increases while $E_c \propto B_{\perp}$ is held fixed. At $\nu = 2$, band mass and $g$-factor corrections are so large that an enormous in-plane magnetic field would be necessary to favor the spin-polarized state, ie. $\cos \theta \sim \frac{1}{60}$. At $\nu = 2/3$, however, the aforementioned gauge field fluctuations can have the important quantitative effect of renormalizing the mass, thereby decreasing the cyclotron energy considerably. To get an idea of the magnitude of the mass renormalization, we can simply look at the measured quasiparticle gap at $\nu = 2/3$. First, it is important to remember that the quasiparticle gap is not the cyclotron energy, $E_c = \frac{eB_{\text{eff}}}{m_r}$, but the difference between cyclotron and Zeeman energies,

$$E_g = \frac{eB_{\text{eff}}}{m_r} - \frac{geB}{m_e}$$

because the lowest energy excitations are precisely from the lowest spin-reversed Landau level to the second spin-aligned Landau level. (Note that the second term
has $B$ rather than $B_{\text{eff}}$ because the Zeeman coupling is unaffected by the flux-trading procedure.) Fitting this expression to the measured gap at $\nu = 2/3$, we can obtain $m_r$. This value of the renormalized mass may be used to estimate the tilted-field angle at which the Zeeman and cyclotron energies are the same and the transition occurs:

$$\frac{eB_{\text{eff}} \cos \theta}{m_r} = \frac{geB}{m_e}$$

(2.2)

or, simply, $\cos \theta = 3g(m_r/m_e)$. Experiments indicate $m_r \approx 10m_b$, so we find $\cos \theta \sim 0.5$.

In other words, we have estimated the cyclotron energy to be of the same order of magnitude as the Zeeman energy, in agreement with the experiments of Clark, et al., who find a transition at $\cos \theta \sim 0.9$. This estimate was extremely crude, but it is possible that a careful calculation of gauge-field corrections to the effective mass, perhaps taking into account disorder, could predict this value more accurately. However, if such a calculation fails to accurately locate the transition, then the discrepancy might be due to a more fundamental problem. The flux-trading arguments depend for their validity on the existence of a gap, or at least a Fermi surface. We assume that this condition is met. While it is a logical possibility that the gap does go to zero and a Fermi surface does not develop, in which case gauge-field fluctuations can completely destabilize the mean-field theory and we cannot relate the physics at $\nu = 2$ to the physics at $\nu = 2/3$, we believe that the semi-quantitative consistency of our description makes such a possibility unlikely.

Given the assumptions and qualifications just expressed, we are led to anticipate that the basic physics of the $\nu = 2/3$ states be visible in the $\nu = 2$ system, which is a priori easier to understand. We will adopt this point of view here.
3. Physical Picture at $\nu = 2$

Let’s now consider the $\nu = 2$ system in the vicinity of the transition. The second spin-aligned and the first spin-reversed Landau levels are nearly degenerate (the filled first spin-aligned Landau level is essentially inert; we ignore these electrons here and in what follows). Naively, there will be many low-energy excitations because there are twice as many available states as there are electrons. Hence, interactions will be crucial even at integer filling fraction. This system is very similar to a double-layer or single wide quantum well system at $\nu = 1$ (again, ignoring the inert filled Landau level). The two Landau levels play the role of the two layers. Tuning the magnetic field away from the transition is analogous to unbalancing the two wells. The strong Coulomb repulsion leads to a ferromagnetic exchange interaction ($J \sim \frac{e^2}{\epsilon \hbar} \gtrsim 10K$) which will order the spins at low temperature. There are then two possibilities, depending on whether inter-level or intra-level interactions are stronger (the latter are typically stronger in real double layer systems, of course).

*Ising case.* When the inter-level interactions are stronger than intra-level interactions, it is energetically favorable to have all of the electrons in the lower of the two Landau levels (if inter-layer interactions were weaker, then it might be energetically favorable to have some electrons in the higher level in order to decrease the interaction energy). At the transition, the electrons all switch into the other level. Let us consider, for a moment, the Ising model at zero temperature. In the Ising model, the spins are all aligned along the field. As the field is reversed, the spins become reversed. This is a first-order transition (with a possibility of hysteresis). At finite temperature, there is a transition to a paramagnetic phase. Returning, now, to the case of two nearly degenerate Landau levels we see the following analogy. The two low-temperature ferromagnetic phases of the Ising model (spin up and spin down) are analogous to the two states of electrons at
$\nu = 2$. These are related by a $Z_2$ symmetry which is broken in the Ising model by a non-zero field and in the $\nu = 2$ system by a non-zero splitting between the two levels in question. If we hypothesize that both of these $Z_2$ models are in the same universality class, then we would predict a second-order phase transition at finite temperature which terminates the line of first-order phase transitions. To observe it, one could tune to the first-order phase transition at low-temperature and then raise the temperature. Of course, the Coulomb interactions set the scale for this transition temperature, so the $\nu = 2/3$ state would probably be destroyed before this temperature is reached. If there were a separation of scales, however, and this transition could be observed, then we would observe a logarithmically diverging specific heat, a power law spin-polarization, and the other characteristic behaviors of the Ising universality class.

At a first-order phase transition point, there can be phase coexistence. At the boundaries between the two phases (regions of spin-up or spin-down in the Ising analogy), one might expect gapless edge modes. As we pointed out in [6], this expectation is seen to be incorrect, once inter-edge interactions are taken into account. Let now us analyze this more carefully. At $\nu = 2$, the $K$-matrix describing excitations at the boundary between two phase regions is simply:

$$K = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} \quad (3.1)$$

At $\nu = 2/3$, it is

$$K = \begin{pmatrix}
1 & 2 & 0 & 0 \\
2 & 1 & 0 & 0 \\
0 & 0 & -1 & -2 \\
0 & 0 & -2 & -1
\end{pmatrix} \quad (3.2)$$

In both cases the electron tunneling operator can be a relevant operator when inter-edge interactions are sufficiently strong. (In the former case, the tunneling
operator is relevant even in the absence of these interaction.) Consider the effective Lagrangian for these edge modes:

\[ S_0 = \int d\tau dx \left( K_{ij} \partial_{\tau} \phi^i \partial_x \phi^j + V_{ij} \partial_x \phi^i \partial_x \phi^j \right) \]  

(3.3)

At \( \nu = 2 \), \( K_{ij} \) is given by (3.1) and the electron tunneling operators of interest are \( e^{i(\phi^1 - \phi^3)} \) and \( e^{i(\phi^2 - \phi^4)} \). The first tunneling operator corresponds to tunneling between the lowest spin-aligned Landau level on either side of the phase boundary. The second tunneling operator corresponds to tunneling between the lowest spin-reversed Landau level and the second spin-aligned Landau level. If a gap forms, it is presumably because these pairs of modes form gaps. The tunneling operator

\[ S_t = \int d\tau dx \left( te^{ik_{13} x} e^{i(\phi^1 - \phi^3)} + c.c. \right) \]  

(3.4)

(and the same operator with \( 1 \rightarrow 2 \) and \( 3 \rightarrow 4 \)) can lead to the formation of a gap if it exists. However, this is a momentum non-conserving process unless \( k_{13} = 0 \). This will not be a problem if there are umklapp processes at wavevector \( k_{13} \), but in general there won’t be, so we will need \( k_{13} = 0 \). This is just the statement that the two edges be separated by a distance less than a magnetic length because \( k_{13} \) is equal to the magnetic flux passing through the region between the two edges. This condition is almost certainly satisfied by the edges of the two lowest Landau levels on either side of the boundary (the lowest Landau level is the same in both phases, of course). It is highly plausible that this is also true of the edges of the second Landau levels (which are different in the two phases), at least once the lowest levels have paired off and formed a gap. However, it is possible that there is some aspect of the microscopic physics which distinguishes between the lowest and second Landau levels. Such effects could prevent \( k_{13} = 0 \) from being satisfied, but we will assume that this does not happen. Numerical investigations might shed valuable light on this issue.

The second condition that must be met in order for a gap to form is that the tunneling operator (3.4) be relevant (ie. have positive dimension in momentum...
space). The scaling dimension of the tunneling operator may be obtained from the scalar field two-point function. To obtain the scalar field two-point function, we diagonalize the Lagrangian (3.3). The result of these calculations is that the scaling dimension of the operator

\[ S_t = \int d\tau dx \left( t e^{i_1 \phi^i} + c.c. \right) \]  

is \( 2 - \Delta_l \), where \( \Delta_l = \frac{1}{2} a^i_\mu a^j_\mu l_i l_j \). Here, the \( a_\mu \)'s are the simultaneous eigenvectors of the \( K_{ij} \) and \( V_{ij} \) matrices of eq. (3.3); they are orthonormal with respect to \( K_{ij} \). \( \Delta_l \) satisfies the inequality:

\[ \Delta_l \geq \frac{1}{2} K^{-1}_{ij} l_i l_j \]  

Hence, the maximum scaling dimension of the tunneling operator (3.5) is \( 2 - \frac{1}{2} K^{-1}_{ij} l_i l_j \).

Actually, inter-edge interactions are unnecessary at \( \nu = 2 \); the tunneling operator (3.4) is a dimension 1 operator even when \( V_{ij} \) is diagonal. Non-zero off-diagonal elements of \( V_{ij} \) can make the tunneling operator even more relevant, up to a maximum dimension of 2, according to the arguments of the previous paragraph. At \( \nu = 2/3 \), however, these interactions are necessary; in their absence, (3.4) is an irrelevant operator. However, \( K^{-1}_{ij} l_i l_j = 0 \), so sufficiently strong inter-edge interactions can make this operator relevant, again to a maximum dimension 2. Hence, we find that so long as the edges of the different phase regions are in close proximity – close enough to make \( k_{13} = 0 \) and to make inter-edge interactions strong enough to make (3.4) relevant – these phase boundaries do not support gapless excitations.

**XY case.** When inter-level interactions are weaker than intra-level interactions, it is energetically favorable, near the transition, to excite some electrons to the higher level because the interaction energy is thereby lowered. There is a critical
level-splitting at which this begins to occur. When this happens, the spin vector rotates away from the $z$-axis into the $x - y$ plane (the spin remains maximal as a result of the strong ferromagnetic exchange interaction). The entire system, including the electrons in the lowest Landau level, is then neither a spin-singlet nor a spin-polarized state. Hence, the transition occurs in two steps in this scenario: spin-singlet $\rightarrow$ partially polarized $\rightarrow$ fully polarized.

This may be described phenomenologically with the following effective Hamiltonian.

$$ H = S_x^2 + S_y^2 + \eta S_z^2 - h S_z + \ldots $$ \hspace{1cm} (3.7)

The ellipses indicate gradient and higher-order terms. Here, $\eta > 1$ is a measure of the anisotropy between inter- and intra-layer interactions. For $\eta < 1$, (3.7) describes the Ising case we discussed earlier. $h$ is a measure of the splitting between the two levels. We have neglected the term $(S^2 - S_{\text{max}}^2)^2$ (where $S_{\text{max}}$ is $N/2$, and $N$ is the number of electrons excluding the “inert” electrons in the lowest Landau level) which leads to symmetry breaking, but at low temperature this term just enforces the constraint $S^2 = S_{\text{max}}^2$. Substituting this constraint into (3.7), we find:

$$ H = (\eta - 1)S_z^2 - h S_z + \text{const.} + \ldots $$ \hspace{1cm} (3.8)

The minimum of this Hamiltonian is at $S_z = \min(\frac{h}{2(\eta - 1)}, S_{\text{max}})$. In other words, when the splitting between levels is large compared to the anisotropy in interaction strengths the system is either in a fully spin-polarized state or a spin-singlet. When the splitting is small, the system is partially polarized. Clearly, the order parameter, $S_z$, is continuous, although its first derivative is not. Hence, the spin-singlet to partially polarized and partially polarized to fully polarized transitions are both second-order phase transitions transitions.

As has been discussed in the context of double-layer systems, the partially polarized state has a $U(1)$ symmetry corresponding to rotations in the $S_x - S_y$ plane. The $U(1)$ charge is just $n_\uparrow - n_\downarrow$, the number of spin-up electrons minus
the number of spin-down electrons. This symmetry is broken at $T = 0$, and there are Goldstone bosons. At finite temperature, there is a Kosterlitz-Thouless phase transition. The most natural wavefunction for the partially polarized state is the $(1, 1, 1)$ wavefunction for a two-component Hall system. Some modification of this wavefunction is required, however, because one of the “layers” is actually a second Landau level.

As in the Ising case, the phase boundaries between spin-singlet or fully polarized states and the partially polarized state, which can occur in the presence of disorder, do not support gapless excitations. The arguments are completely analogous to those we used in that case. This is, in fact, quite intuitive. The transitions we are considering are changes of the spin configuration of the state. The charge transfer gap may become small, but it does not close. One might worry that the closing of the gap, due to the existence of Goldstone bosons in the partially polarized state, invalidates the flux-trading procedure. However, the charged modes – which are the only modes affected by the flux-trading procedure – are not gapless, so we expect that this does not occur.

We have presented two scenarios for the spin-singlet to spin-polarized phase transition at $\nu = 2/3$. It is natural to ask whether one or the other of these possibilities is better suited to describing experiments. Numerical studies indicate that the phase transition at $\nu = 2/3$ is a first-order phase transition with a simple level crossing [5]. Experiments indicate that the transition is directly from a spin-singlet to a spin-polarized state; there is no evidence of a partially polarized state [3]. Thus, if our flux-trading arguments are correct, the Ising-type transition analyzed at the beginning of this section describes the $\nu = 2/3$ transition. However, experiments at $\nu = 4/3$ – the particle-hole conjugate of $\nu = 2/3$ – find a two-step transition with a partially polarized state at the intermediate step [3]. This transition appears to be described by the $XY$ picture analyzed in the latter
part of this section. However, experiments find a non-zero $\rho_{xx}$, indicative of a small or vanishing gap, at the $\nu = 2/3$ and $4/3$ transitions. Presumably, the experiments are conducted at temperatures higher than a possible gap, so the state at the transition appears to be metallic.

4. A Possible Non-Fermi Liquid Metallic State

Using flux-trading arguments yet again, we can construct a mean-field theory with a Fermi surface for the state at the transition. Gauge-field fluctuations lead to corrections of a non-Fermi liquid form, but they do not destabilize the Fermi surface. This is completely analogous to the procedure used by Halperin, Lee, and Read [12] to construct the non-Fermi liquid theory of the half-filled Landau level. The new feature is the degeneracy between two Landau levels. As a result, we have a non-Fermi liquid theory with spin.

Of course, this mean-field theory is only an ansatz. If it is not energetically favored compared to the possible incompressible states, which is what we expect, then it will not be realized at $T = 0$. In particular, we will find that there is a marginally relevant operator in the effective field theory of this model which is a special feature of models with spin. This operator leads to the formation of a gap and condensation at low temperature. The low-temperature state is presumably the state which we discussed in the previous section. At temperatures above the transition temperature, we expect a non-Fermi liquid metallic state, however.

The starting point is two degenerate Landau levels – the second spin-aligned level and the first spin-reversed level – and one electron per magnetic flux tube. In this mean field theory, we will assume that half of the electrons are in each Landau level. We then introduce two Chern-Simons gauge fields which are coupled to the electrons in each of the two levels. These gauge fields attach two flux tubes to each electron in the direction antiparallel to the magnetic field. As usual, these gauge fields have no effect; in particular, they do not change the statistics of the electrons. However, in mean field theory, the system is now one of electrons with spin in zero
net field. Of course, the gauge-field fluctuations cannot be neglected because the mean-field theory is gapless. Elsewhere [9], we have used the renormalization group to tame the gauge-field fluctuations in the context of the half-filled Landau level. The same framework may be applied to this problem, but with a few twists.

We begin with the effective action:

\[
S = \int d\omega d^2 k \left\{ \psi_\sigma^\dagger (i\omega - \epsilon(k)) \psi_\sigma \right\} + \int d\omega d^2 k a_{\sigma 0} \epsilon_{ij} k_i a_{\sigma j} \\
+ g \int d\omega d\omega' d^2 k d^2 q \left\{ \psi_\sigma^\dagger (k + q, \omega + \omega') \psi_\sigma (k, \omega) \left( a_{\sigma i} (q, \omega') \frac{\partial}{\partial k_i} \epsilon(q + 2k) + a_{\sigma 0} (q, \omega') \right) \right\} \\
+ V_0 \int d\omega d\omega' d\omega'' d^2 k d^2 k' d^2 k'' \psi_\sigma^\dagger (k + k', \omega + \omega') \psi_\sigma (k', \omega') \frac{1}{k^x} \times \\
\psi_\sigma^\dagger (-k + k'', -\omega + \omega'') \psi_\sigma (k'', \omega'') \tag{4.1}
\]

\( \psi_\sigma \) are the fermion fields; \( \sigma \) is the spin or, alternatively, the Landau level index. The \( a_{\sigma} \) are the gauge fields coupled to the two fermion fields. The final term is a four-fermion interaction representing electron-electron interactions. If \( x = 1 \), it is just the Coulomb interaction, but we will let \( x \) be arbitrary for now. It will be convenient to rewrite the action in terms of the gauge fields \( a_c = \frac{1}{2} (a_\uparrow + a_\downarrow) \) and \( a_s = \frac{1}{2} (a_\uparrow - a_\downarrow) \). Furthermore, we will need to use the \( a_{\sigma 0} \) equation of motion, or Chern-Simons constraint:

\[
\epsilon_{ij} k_i a_{\sigma j} (k) = g \int d^2 q d\omega' \psi_\sigma^\dagger (k + q, \omega + \omega') \psi_\sigma (q, \omega) . \tag{4.2}
\]

(no sum over \( \sigma \)). If we substitute this constraint back into the non-local four-fermion interaction, the important role of this interaction becomes clear. It takes the form

\[
S_a = \int d\omega d^2 k \epsilon_{ij} \epsilon_{mn} k_i k_m k^{-x} a_{cj} (k, \omega) a_{cn} (-k, -\omega) . \tag{4.3}
\]

Thus as \( x \) is increased, the long-range fluctuations of the gauge field \( a_c \) are suppressed. The four-fermion interaction does not affect the gauge field \( a_s \). However, there should also be terms in (4.1) representing spin-spin interactions which will
lead to a term of the form of (4.3) but with $x = 0$ since spin-spin interactions are local. Even otherwise, such a term will arise at one-loop anyway, so we should include such a term in the action. Introducing renormalization counterterms and using a regularization procedure analogous to dimensional regularization, as in [9], we have the action:

$$S = \int d\omega d^2 k \psi_\sigma^\dagger \left( iZ\omega - ZZ_{\nu F} \epsilon(k) \right) \psi_\sigma + \int d\omega d^2 k a_{\sigma 0} \epsilon_{ij} k_i a_{\sigma j}$$

$$+ \int d\omega d^2 k \epsilon_{i j m n} k_i k_m k_n \epsilon a_{c j} (k, \omega) a_{c n} (-k, -\omega)$$

$$+ \int d\omega d^2 k \epsilon_{i j m n} k_i k_m a_{s j} (\omega) a_{s n} (-k, -\omega)$$

$$+ \mu \frac{g_c}{\pi} g_c Z_{g_c} \int d\omega d\omega' d^2 q \, \epsilon_{\sigma}^\dagger (k + q, \omega + \omega') \psi_\sigma (k, \omega) \left( a_{c i} (q, \omega') \frac{\partial}{\partial k_i} \epsilon (q + 2k) + a_{c 0} (q, \omega') \right)$$

$$+ \mu \frac{g_s}{\pi} g_s Z_{g_s} \int d\omega d\omega' d^2 q \, \epsilon_{\sigma}^\dagger (k + q, \omega + \omega') \tau^3_{\sigma \sigma'} (k, \omega) \left( a_{s i} (q, \omega') \frac{\partial}{\partial k_i} \epsilon (q + 2k) + a_{s 0} (q, \omega') \right)$$

(4.4)

$a_c$ couples to the total charge while $a_s$ couples to the $z$-component of the quasiparticle spin; $\tau^3_{\sigma \sigma'}$ is the appropriate Pauli matrix. The action (4.4) is not $SU(2)$ invariant because the magnetic field determines a preferred quantization axis. There is no quadratic coupling of $a_s$ to $a_c$; such a term will not be generated in perturbation theory because the contributions of spin-up and spin-down quasiparticles will cancel.

In [9], we showed that the fermion-gauge field interaction is relevant for $x < 1$ and found a non-Fermi liquid fixed point for $1 - x$ small. At $x = 1$, we found logarithmic corrections to Fermi liquid behavior. We hypothesized that this fixed point exists even at $x = 0$. A number of authors have claimed to have constructed this fixed point using resummations of perturbation theory. Here we have an example of fermions interacting with two gauge fields, one of which has $x = 1$, the other $x = 0$. We expect that for most correlation functions the gauge field coupling to $S_z$ with $x = 0$ will dominate because it has the more singular low-energy behavior, so we ignore the other gauge field in what follows.

Adopting the results of [9], we have the $\beta$-functions for the coupling constant,
\[ \alpha_s = \frac{g^2 v_F}{(2\pi)^2} . \]

\[ \beta(\alpha_s) = -\frac{1}{2} (1 - x_s) \alpha_s + 2\alpha_s^2 + O(\alpha_s^3) \] (4.5)

and the anomalous dimensions,

\[ \eta_{v_F}(\alpha_s) = -\eta(\alpha_s) = -2\alpha_s + O(\alpha^2) . \] (4.6)

This leads to the anomalous scaling form for the fermion two-point function:

\[ G^{(2)}(\omega, v_{F r}, \alpha, \mu) = \omega^{-1+\eta} G^{(2)}(1, \frac{v_{F r}}{\omega^{1+\eta_{v_F}}}, \alpha^*, \mu) \] (4.7)

Such a non-Fermi liquid metallic state has dramatic experimental consequences. Surface acoustic wave propagation will exhibit anomalies at the Fermi wavevector \( k_F \), which is given by \( \pi k_F^2 = \frac{1}{6} B \) at \( \nu = 2/3 \) (half the electrons are inert and the remaining half are divided into spin-up and spin-down). Furthermore, magnetic focusing experiments of the type conducted by Goldman, et al. [10] in the vicinity of \( \nu = \frac{1}{2} \) would reveal the existence of the Fermi wavevector \( k_F \). Time-of-flight measurements in such an experiment, as discussed in [11], will reflect the Fermi velocity renormalization due to \( a_s \).

As we mentioned at the beginning of this section, there are new possibilities for pairing as a result of the additional quantum number, spin, and its gauge field. This is because spin-up and spin-down electrons are oppositely charged with respect to the gauge field \( a_s \). As Bonesteel has observed in the case of double-layer systems [13], this gauge-field mediated interaction is attractive. The enhanced pairing interaction presumably leads to a finite temperature transition to the incompressible state of the previous section.

We mention now briefly a fundamental and at first sight disturbing point, that we will explore in reater depth elsewhere. We have relied heavily, in our analysis, on coupling gauge fields to \( S_z \) – a quantity that is not strictly conserved. Local gauge invariance must be exact, however. What is going on here?
Local gauge invariance can be maintained, without its usual implication of associated conservation laws, if (and only if) non-local terms are allowed in the action. Thus, for example in our context, spin-flip processes must be accommodated by appropriate non-local interactions.

5. Discussion

In the preceding sections, we have applied the method of adiabatic flux-trading and analogies with double-layer systems to analyze the phase transition between spin-singlet and spin-polarized states at the same filling fraction. We argued that the phase transition at $\nu = 2/3$ is qualitatively similar to that at $\nu = 2$. A simple way of testing the validity of this picture would be to see how the critical tilt angle varies with the $g$-factor (which can be altered by changing the Al concentration in AlAs-GaAs systems). Even at $\nu = 2$, we found two possibilities. In the first, the transition between polarized and singlet states is first-order. The transition at $\nu = 2/3$ appears to be of this type. At $\nu = 4/3$, the other possibility is realized. Two first-order phase transitions occur; first from a fully polarized state to a partially polarized state and thence to a spin-singlet. This rich phase structure is reminiscent of that of the double-layer Hall states [14].

As in the double-layer systems [13], there is the possibility of a metallic state. This phenomenology of this state is qualitatively similar to that of the half-filled Landau level. Two important differences result, however, from the presence of a second gauge field. First, the gauge field which couples to spin leads to a pairing instability. Second, spin-flip processes are non-local in terms of the low-energy quasiparticles. This latter fact can result in a dramatically altered response of the spin degrees of freedom.

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