Universal Cycles on 3–Multisets

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Abstract

Consider the collection of all $t$–multisets of \{1, \ldots, n\}. A universal cycle on multisets is a string of numbers, each of which is between 1 and $n$, such that if these numbers are considered in $t$–sized windows, every multiset in the collection is present in the string precisely once. The problem of finding necessary and sufficient conditions on $n$ and $t$ for the existence of universal cycles and similar combinatorial structures was first addressed by DeBruijn in 1946 (who considered $t$–tuples instead of $t$–multisets). The past 15 years has seen a resurgence of interest in this area, primarily due to Chung, Diaconis, and Graham’s 1992 paper on the subject. For the case $t = 3$, we determine necessary and sufficient conditions on $n$ for the existence of universal cycles, and we examine how this technique can be generalized to other values of $t$.

1 Introduction and Previous Work

Consider the collection of all $t$–multisets over the universe $[n] = \{1, \ldots, n\}$. A universal cycle (ucycle) on multisets is a cyclic string $X = a_1a_2\ldots a_k$ with $a_i \in [n]$ for which the collection \{\{a_1, a_2, \ldots, a_t\}, \{a_2, a_3, \ldots, a_{t+1}\}, \ldots, \{a_{k-t+1}, a_{k-t+2}, \ldots, a_k\}, \{a_{k-t+2}, a_{k-t+3}, \ldots, a_k, a_1\}, \ldots\} is precisely the collection of all $t$–multisets over $[n]$, i.e. each $t$–multiset over $[n]$ occurs precisely once in the above collection. For the remainder of this paper, the term universal cycle will refer to universal cycles on multisets unless noted otherwise.

Universal cycles do not exist for every value of $n$ and $t$. Indeed, simple symmetry arguments show that each of the numbers 1, \ldots, $n$ must occur an equal number of times in the ucycle. Since the length of the ucycle is equal to the number of $t$–multisets over $[n]$, which is $\binom{n+t-1}{t}$, we must have that $n \mid \binom{n+t-1}{t}$. While this condition is necessary, it is not sufficient for the existence of ucycles.

To date, the bulk of research on ucycles has been devoted to studying ucycles over sets (as opposed to multisets). Ucycles over sets are constructed in the same fashion as ucycles over multisets, except that we consider the collection of all $t$–sets over $[n]$ instead of the collection of all $t$–multisets, and our divisibility condition becomes $n \mid \binom{n}{t}$. In [1], Chung, Diaconis and Graham conjectured that for each value of $t$, there exists a number $n_0(t)$ such that universal cycles exist for $n \mid \binom{n}{t}$ and $n \geq n_0(t)$. In [2], Hurlbert consolidated and extended previous work, verifying the conjecture for $t = 2$ and 3 and developing partial results for $t = 4$ and 6. In [3], Godbole et al. considered universal cycles over multisets for the case $t = 2$, and verified the analogous form of the Chung–Diaconis–Graham conjecture (i.e. with the modified divisibility criterion) for this case. This work is of particular interest because

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Godbole et al. used a new inductive technique to arrive at their proof, and in this paper we extend this technique to the case \( t = 3 \). This new work suggests that the inductive method is a promising way of addressing the Chung–Diaconis–Graham conjecture. We also consider a second proof of the conjecture for the \( t = 3 \) multiset case, which builds off universal cycles on sets and lends itself more easily to generalization.

2 An Inductive Proof for Universal Cycles on 3–Multisets

For \( t = 3 \), the condition \( n \mid (\binom{n+t-1}{t}) \) implies that \( t \equiv 1 \) or \( 2 \) (mod 3). We will consider the case \( n \equiv 1 \) (mod 3), as the other case can be dealt with similarly. We will show that for \( n \geq 4 \), universal cycles exist whenever \( n \) satisfies \( n \mid (\binom{n+t-1}{t}) \)

Before describing the proof itself, we will define some terminology that will be useful for describing universal cycles. We say that a cyclic string \( X = a_1a_2...a_k \) contains the multiset collection \( \mathcal{I} \) if \( \mathcal{I} = \{\{a_1, a_2, a_3\}, \{a_2, a_3, a_4\}, ..., \{a_{k-2}, a_{k-1}, a_k\}, \{a_{k-1}, a_k, a_1\}, \{a_k, a_1, a_2\}\} \), where each of these sets must be distinct. Clearly \( k = \binom{n+2}{3} \), since this is the number of 3–multisets on \([n]\).

For a string \( X = a_1a_2...a_k \), we call the lead-in of \( X \) the substring \( a_1a_2 \) and the lead-out the substring \( a_{k-1}a_k \).

Now, consider the collection of all 3–multisets over \([n]\). We shall partition this collection into four subcollections. Let \( \mathcal{A} \) be the collection of all 3–multisets over \([n-3]\), and let \( \mathcal{B} \) be the collection of all 3–multisets over \([n-2,n-1,n]\) and \([n-6]\) which contain at least one element from \([n-2,n-1,n]\). Let \( \mathcal{C} \) be the collection of all 3–multisets with one or two elements from \([n-5,n-4,n-3]\) and one or two elements from \([n-2,n-1,n]\), and let \( \mathcal{D} \) be the collection of all 3–multisets with one element from each of \([n-6]\), \([n-5,n-4,n-3]\), and \([n-2,n-1,n]\). We can see that \( \mathcal{A}, \mathcal{B}, \mathcal{C}, \) and \( \mathcal{D} \) are disjoint, and that their union is the collection of all 3–multisets on \([n]\), as desired.

Now, let \( S \) be a universal cycle on \([n-6]\), and since \( 1, 1, 1 \) must occur somewhere in \( S \) and the beginning of \( S \) is arbitrary, we shall have \( S \) begin with \( 1, 1, 1 \). We shall also select \( S \) so that its lead-out is \( n-6, n-7 \). Thus \( S \), when considered as a cyclic string, contains all 3–multisets over \([n-6]\), and when considered as a non-cyclic string, contains all 3–multisets except \([1,n-7,n-6]\) and \([1,1,n-7]\). Let \( T \) be a string over \([n-3]\) such that \( ST \)—the concatenation of \( S \) and \( T \)—is a universal cycle over \([n-3]\). It is not clear that such a \( T \) must exist, but we shall find a specific example shortly. In the example we will find, \( T \) will begin with \( 1, 1 \) and will end with \( n-3, n-4 \). Since \( T \) begins with \( 1, 1 \), the string \( ST \) contains the multisets \( \{1,n-7,n-6\}, \{1,1,n-7\} \). We can see that the cyclic string \( ST \) contains all of the multisets in \( \mathcal{A} \), and that when \( ST \) is considered as a non-cyclic string, it contains \( \mathcal{A} \setminus \{\{1,n-4,n-3\}, \{1,1,n-4\}\} \). Now, consider the string \( T' \) obtained by taking \( T \) and replacing each instance of \( n-5 \) by \( n-2, n-4 \) by \( n-1 \), and \( n-3 \) by \( n \). Since \( T \) contained all multisets over \([n-3]\) which contained at least one element from \([n-5,n-4,n-3]\), we have that \( T' \) contains all multisets over \([n-2,n-1,n]\) and \([n-6]\) which contain at least one element from \([n-2,n-1,n]\), i.e. \( T' \) contains all the multisets in \( \mathcal{B} \). Since the lead-in of \( T \) is \( 1,1 \), the lead-in of \( T' \) is also \( 1,1 \), and since \( T \) ends with \( n-3, n-4 \), \( T' \) ends with \( n, n-1 \). If we consider the cyclic string \( STT' \), we can see that this string contains all the multisets in \( \mathcal{A} \cup \mathcal{B} \), while the non-cyclic version of this string is missing the multisets \( \{1,n-1,n\}, \{1,1,n-1\} \).

For notational convenience, we will use the following assignments: \( a := n-5, b := n-4, c := n-3, d := n-2, e := n-1, \) and \( f := n \). Now, consider the following string:
\[ V = \text{be}(n-6)\text{af}(n-7)\text{be}(n-8)\text{af}(n-9)\ldots \text{af}1\text{be} \]
\[ \ldots \text{af}1\text{be} \]
\[ \text{ad}(n-6)\text{ce}(n-7)\text{ad}(n-8)\text{ce}(n-9)\ldots \text{ce}1\text{ad} \]
\[ \ldots \text{ce}1\text{ad} \]
\[ \text{cf}(n-6)\text{bd}(n-7)\text{cf}(n-8)\text{bd}(n-9)\ldots \text{bd}1\text{ce}f. \]

We can see that this string contains every multiset in \( D \), as well as the multisets \( \{a, b, e\} \), \( \{a, d, e\} \), \( \{a, c, d\} \), and \( \{c, d, f\} \). Now, the following string (found with the aid of a computer) contains all of the multisets in \( C \setminus \{\{a, b, e\}, \{a, d, e\}, \{a, c, d\}, \{c, d, f\}\} \):

\[ U = \text{aaffc aeebb decec bddcc fbada dfbf} \]

Note that while the multisets \( \{b, b, f\} \) and \( \{b, e, f\} \) are not present in the above string \( U \), they are present in the concatenation of \( U \) with \( V \). Similarly, while \( U \) does not contain \( \{a, e, f\} \) and \( \{a, a, f\} \), these multisets are present in the concatenation of \( T' \) with \( U \).

Now, we can see that the string \( ST'UV \) is a universal cycle over \([n]\) because the non-cyclic string \( ST' \) contained all the multisets in \( A \cup B \setminus \{\{1, n-1, n\}, \{1, 1, n-1\}\} \), and it is precisely the multisets \( \{1, n-1, n\} \) and \( \{1, 1, n-1\} \) which are obtained by the wrap-around of the lead-out of \( V \) with the lead-in of \( S \). The lead-in and lead-out of the other strings has been engineered so as to ensure that each multiset occurs precisely once.

This completes the induction proof, since the string \( ST \) is a universal cycle over \([n-3]\) (taking the place of \( S \) in the previous iteration of the induction), and the string \( T'UV \) extends this cycle to \([n]\) (taking the place of \( T \) in the previous iteration of the induction). Also note that \( T'UV \) begins with \( 1,1 \) and ends with \( n, n-1 \), as required for the induction hypothesis.

Thus, all that remains is the find a base case from which the induction can proceed. A possible base case (there are many) for \( n-6 = 4 \), \( n-3 = 7 \) is

\[ S = 11144 42223 33121 24343 \]
\[ T = 11522 63374 45166 27732 57366 77135 34641 71555 36127 42556 66477 75526 4576, \]

which would lead to

\[ T' = 11822 93304 48199 20032 80399 00138 34941 01888 39120 42889 99400 08829 4809 \]
\[ U = 55007 59966 89797 68877 06585 8060 \]
\[ V = 69450 36925 01695 84793 58279 15870 46837 02681 709 \]

Where “0” denotes 10 and the spacings have been added to increase readability.

All of the work up to this point has dealt with \( n \equiv 1 \pmod{3} \). The proof for \( n \equiv 2 \pmod{3} \) is similar, so it has been omitted for the sake of brevity.

3 A Second Proof of the Existence of Ucycles on 3–Multisets

In this proof, we construct a ucycle on 3–multisets of \([n]\) by modifying a ucycle on 3–subsets of \([n]\).

(We know from \([2]\) that ucycles on 3–subsets of \([n]\) exist for all \( n \geq 8 \) not divisible by 3.) Before giving the proof, we introduce two terms. We call each element of \([n]\) a letter, and each \( a_i \) in the
ucycle $X = a_1 \ldots a_k$ a character. To summarize, a ucycle on 3–multisets of $[n]$ is made up of $\binom{n+t-1}{t}$ characters, each of which equals one of $n$ letters.

To demonstrate the proof’s technique, we will first use an argument similar to it to create ucycles on 2–multisets from ucycles on 2–subsets. We start with this ucycle on 2–subsets of $[5]$:

$$1234513524$$

Then, we repeat the first instance of every letter to create the following ucycle on 2–multisets:

$$112233445513524$$

The technique works because repeating a character $a_i$ as above adds the multiset $\{a_i, a_i\}$ to the ucycle and has no other effect.

To use this technique on ucycles on 3–subsets, we repeat not single characters, but pairs of characters. For example, changing

$$\ldots a_{i-1}a_ia_{i+1}a_{i+2} \ldots$$

to

$$\ldots a_{i-1}a_ia_{i+1}a_{i+1}a_{i+2} \ldots$$

has only has the effect of adding the 3–multisets $\{a_i, a_i, a_{i+1}\}$ and $\{a_i, a_{i+1}, a_{i+1}\}$ to the cycle. In order to use this technique, we will need to know which consecutive pairs of letters appear in a ucycle on 3–subsets. For instance, the following ucycle (generated using methods from [2]) on 3–subsets of $[8]$ contains every unordered pair of letters as consecutive characters but $\{1,5\}, \{2,6\}, \{3,7\}$, and $\{4,8\}$:

$$1235783 6782458 3457125 8124672 5671347 2346814 7813561 45 68236$$

(The spaces in the cycle are added only for readability.) This ucycle is missing 4 pairs, which happens to be $n/2$. This is no coincidence: in fact, this is the most pairs that a ucycle on 3–subsets can fail to contain.

**Lemma.** No two unordered pairs not appearing as consecutive characters in a ucycle on 3–subsets have a letter in common. A ucycle can hence be missing at most $n/2$ pairs of letters.

**Proof.** Suppose that we have a ucycle on 3–subsets that contains neither $a$ and $b$ as consecutive characters, nor $a$ and $c$ as consecutive characters, where $a, b, c \in [n]$. Then the ucycle does not contain the 3–subset $abc$, for all permutations of $abc$ contain either $a$ and $b$ consecutively, or $a$ and $c$ consecutively. But this is a contradiction, as a ucycle by definition contains all 3–subsets.

Hence, no two pairs of characters missing in the ucycle can have a letter in common. By the pigeonhole principle, the ucycle can be missing at most $n/2$ pairs of letters.

With this lemma, we can finish our proof, creating a ucycle on 3–multisets of $[n]$ whenever $n$ is not divisible by 3. First, we consider the case when $n$ is even. Let $X$ be a ucycle on 3–subsets of $[n]$. Let $x_1, \ldots, x_n$ be a permutation of $[n]$ such that

- $x_1$ equals the first character in $X$.
- $x_n$ equals the last character in $X$. 

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• The list \( \{x_1, x_2\}, \{x_3, x_4\}, \ldots, \{x_{n-1}, x_n\} \) contains all unordered pairs of letters not contained as consecutive characters in \( X \), which is possible by our lemma. (If \( X \) is missing exactly \( n/2 \) pairs of letters, these pairs will be exactly the pairs missing from \( X \). If \( X \) is missing fewer than \( n/2 \) pairs of letters, then the pairs consist of all missing pairs of letters, plus the remaining letters paired arbitrarily.)

Make \( X' \) by repeating the first instance of every unordered pair of letters in \( X \) except for \( \{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}, \{x_n, x_1\} \). The cycle \( X' \) now contains all multisets except

\[ \{x_1, x_1, x_1\}, \ldots, \{x_n, x_n, x_n\} \]

\[ \{x_1, x_1, x_2\}, \{x_1, x_2, x_2\}, \{x_2, x_2, x_3\}, \{x_2, x_3, x_3\}, \ldots, \{x_n, x_n, x_1\}, \{x_n, x_1, x_1\} \]

Now, add the string \( x_1x_1x_1x_2x_2x_2x_2 \ldots x_nx_nx_n \) to the end of \( X' \) to create \( X'' \). This provides exactly the missing multisets, creating a cycle on 3-multisets.

For example, when \( n = 8 \), we start with the following cycle on 3-subsets:

\[
X = 1235783 \ 6782458 \ 3457125 \ 8124672 \\
5671347 \ 2346814 \ 7813561 \ 4568236
\]

The cycle on 3-subsets \( X \) does not contain the pairs \( \{1, 5\}, \{2, 6\}, \{3, 7\}, \) and \( \{4, 8\} \). Hence, we set

\[
x_1 = 1, \ x_2 = 5, \ x_3 = 3, \ x_4 = 7 \\
x_5 = 4, \ x_6 = 8, \ x_7 = 2, \ x_8 = 6
\]

Note that \( x_1 \) equals the first character of \( X \), and \( x_8 \) equals the last.

Now, we repeat the first instance of every unordered pair except for \( \{1, 5\}, \{5, 3\}, \{3, 7\}, \{7, 4\}, \{4, 8\}, \{8, 2\}, \{2, 6\}, \) and \( \{6, 1\} \). (Note that four of these pairs actually did appear in \( X \), because \( X \) was missing fewer than \( n/2 \) pairs of letters, it would not affect the proof.)

\[
X' = 12123235757878383 \ 63676782424545858 \ 3434571712525 \ 81812464672 \\
56567131347 \ 2723468681414 \ 7813561 \ 4568236
\]

Finally, we add the string \( x_1x_1x_1 \ldots x_nx_nx_n \) to complete the cycle:

\[
X'' = 12123235757878383 \ 63676782424545858 \ 3434571712525 \ 81812464672 \\
56567131347 \ 2723468681414 \ 7813561 \ 4568236 \\
11155533377744448888222666
\]

The proof is similar when \( n \) is odd, and we omit it for the sake of brevity.

## 4 Further Directions and Remarks

Both of the proofs given above suggest natural extensions to the \( t = 4 \) and larger cases, and it is simple to use the techniques described above to create a proof sketch. In personal correspondence,
Glenn Hurlbert indicated that his technique for creating ucycles on sets in [2] can also be used to create ucycles on multisets. Though this provides a more concise proof for the existence of ucycles on 3-multisets, the two proofs presented may prove useful by their introduction of new techniques for approaching ucycles. The first proof is notable for its use of induction, a technique which has not been used before to create ucycles. The second, while it is tied to ucycles on sets, is not tied to any particular approach for creating ucycles on sets; it could perhaps be extended to situations to which Hurlbert’s technique cannot.

For values of \( n \) and \( t \) for which ucycles do exist, one interesting question is how many ucycles exist. Clearly each ucycle has \( n! \) representations, since there are \( n! \) permutations of 1, \ldots, \( n \). However, when searching for ucycles using a computer, vast numbers of distinct (i.e. not differing merely by a permutation of 1, \ldots, \( n \)) ucycles were found. Currently, it is not clear whether \( N(n,t) \), the number of distinct ucycles for a given value of \( n \) and \( t \), is a function that has a simple description.

References

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