A lower bound on the fidelity between two states in terms of their Bures distance

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Abstract

Fidelity is a fundamental and ubiquitous concept in quantum information theory. Fuchs-van de Graaf’s inequalities deal with bounding fidelity from above and below. In this paper, we give a lower bound on the quantum fidelity between two states in terms of their Bures distance.

Keywords: Fidelity; Fuchs-van de Graaf’s inequality; Bures distance

1 Introduction

The fidelity between two quantum states, represented by density operators $\rho$ and $\sigma$, is defined as

$$F(\rho, \sigma) = \text{Tr} \left( \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}} \right).$$

Note that both density operators here are taken from $D(\mathcal{H}_d)$, the set of all positive semi-definite operator with unit trace on a $d$-dimensional Hilbert space $\mathcal{H}_d$. The squared fidelity above has been called transition probability [1, 2]. Operationally it is the maximal success probability of changing a state to another one by a measurement in a larger quantum system. The fidelity is also employed in a number of problems such as quantifying entanglement [3], and quantum error correction [4], etc.

For quantum fidelity, the well-known Fuchs-van de Graaf’s inequality states that: For arbitrary two density operators $\rho$ and $\sigma$ in $D(\mathcal{H}_d)$, it holds that

$$1 - \frac{1}{2} \|\rho - \sigma\|_1 \leq F(\rho, \sigma) \leq \sqrt{1 - \frac{1}{4} \|\rho - \sigma\|_1^2},$$

(1.1)

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which established a close relationship between the trace-norm of the difference for two density operators and their fidelity \([5]\).

Given two density operators \(\rho, \sigma \in \mathcal{D}(\mathcal{H}_d)\), \(\{\lambda \rho + (1 - \lambda)\sigma : \lambda \in [0, 1]\}\) is the affine mixed path of \(\rho\) and \(\sigma\). In the recent papers \([6, 7, 8]\), the author considered the estimate of relative entropy between a fixed state \(\rho\) and the affine mixed path \(\lambda \rho + (1 - \lambda)\sigma\). In this paper, we will consider similar problem for fidelity.

### 2 Main result

The Fuchs-van de Graaf’s inequality can not be improved because it is tight. For any value of the trace-norm between \(\rho\) and \(\sigma\), there exists a pair of states saturating the inequality. However, by supplying additional information about the pair it is possible to obtain a higher lower bound on the fidelity.

**Theorem 2.1.** Let \(\rho\) and \(\sigma\) be two pure states in \(\mathcal{D}(\mathcal{H}_d)\), \(\lambda \in [0, 1]\). Then

\[
F(\rho, \lambda \rho + (1 - \lambda)\sigma) \geq 1 - \frac{1}{2}(1 - \sqrt{\lambda}) \|\rho - \sigma\|_1. \tag{2.1}
\]

**Proof.** Let \(\rho = |\psi\rangle\langle\psi|\) and \(\sigma = |\phi\rangle\langle\phi|\) for normalized vectors \(|\psi\rangle, |\phi\rangle \in \mathcal{H}_d\). Without loss of generality, assume that \(\lambda \in (0, 1)\). Denote \(|\langle\psi|\phi\rangle| := r\). Then \(r \in [0, 1]\). Now

\[
F(\rho, \lambda \rho + (1 - \lambda)\sigma) = \sqrt{\langle\psi|\lambda|\psi\rangle\langle\psi| + (1 - \lambda)|\phi\rangle\langle\phi||\psi\rangle} \tag{2.2}
= \sqrt{\lambda + (1 - \lambda)r^2}. \tag{2.3}
\]

Note that

\[
|||\psi\rangle\langle\psi| - |\phi\rangle\langle\phi||_1 = 2\sqrt{1 - |\langle\psi|\phi\rangle|^2}. \tag{2.4}
\]

This implies that

\[
\|\rho - \sigma\|_1 = 2\sqrt{1 - r^2}. \tag{2.5}
\]

Next, all we have to do is to show that

\[
\sqrt{\lambda + (1 - \lambda)r^2} \geq 1 - (1 - \sqrt{\lambda})\sqrt{1 - r^2}. \tag{2.6}
\]

Denote by \(f(\lambda, r)\) the difference of the squared both sides as follows:

\[
f(\lambda, r) := \left(\sqrt{\lambda + (1 - \lambda)r^2}\right)^2 - \left(1 - (1 - \sqrt{\lambda})\sqrt{1 - r^2}\right)^2 \tag{2.7}
= 2(1 - \sqrt{\lambda})\left(\sqrt{1 - r^2} - (1 - r^2)\right) \tag{2.8}
\]
Since $r \in [0, 1]$, i.e. $1 - r^2 \in [0, 1]$, it follows that $\sqrt{1 - r^2} \geq 1 - r^2$. Therefore $f(\lambda, r) \geq 0$. We can conclude that

$$\sqrt{\lambda + (1 - \lambda)r^2} \geq 1 - (1 - \sqrt{\lambda})\sqrt{1 - r^2}.$$  \hfill (2.9)

The desired conclusion is obtained.

The Bures distance is a very useful quantity in quantum information theory, the Bures distance between two states is defined as

$$B(\rho, \sigma) = \sqrt{1 - F(\rho, \sigma)^2}.$$  

It is seen easily that $\frac{1}{2} \|\rho - \sigma\|_1 \leq B(\rho, \sigma)$ by the Fuchs-van de Graaf's inequality. We have the following weaker inequality like the conjectured inequality (3.1):

**Corollary 2.2.** Let $\rho$ and $\sigma$ be two density operators in $D(\mathcal{H}_d)$, $\lambda \in [0, 1]$. Then

$$F(\rho, \lambda\rho + (1 - \lambda)\sigma) \geq 1 - (1 - \sqrt{\lambda})B(\rho, \sigma).$$  \hfill (2.10)

**Proof.** Let $|\Psi\rangle$ and $|\Phi\rangle$ be any purifications of $\rho$ and $\sigma$, respectively such that $F(\rho, \sigma) = |\langle \Psi | \Phi \rangle|$. Then

$$F(\rho, \lambda\rho + (1 - \lambda)\sigma) \geq F(|\Psi\rangle\langle \Psi |, \lambda|\Psi\rangle\langle \Psi | + (1 - \lambda)|\Phi\rangle\langle \Phi |) \geq 1 - (1 - \sqrt{\lambda})\frac{1}{2} \|\langle \Psi | - |\Phi\rangle\|_1$$

$$= 1 - (1 - \sqrt{\lambda})\sqrt{1 - |\langle \Psi | \Phi \rangle|^2},$$  \hfill (2.13)

implying the desired inequality. \hfill □

### 3 Discussion

Naturally, we have the following conjecture: Let $\rho$ and $\sigma$ be any two states in $D(\mathcal{H}_d)$, $\lambda \in [0, 1]$. Then

$$F(\rho, \lambda\rho + (1 - \lambda)\sigma) \geq 1 - \frac{1}{2}(1 - \sqrt{\lambda}) \|\rho - \sigma\|_1.$$  \hfill (3.1)

If this conjecture were true, then we would supply the max-relative entropy between the states as additional information in the lower bound of fidelity. The max-relative entropy is defined as

$$S_{\text{max}}(\rho||\sigma) := \inf_\gamma \{\gamma : \rho \leq e^\gamma \sigma\}.$$  

Clearly $e^{S_{\text{max}}(\rho||\sigma)} = \max_\gamma (\sigma^{-1/2}\rho \sigma^{-1/2})$. The inequality (3.1) is about the special pair of states $\rho$ and $\lambda\rho + (1 - \lambda)\sigma$ and seems to be of rather restricted importance. However it is possible
to reformulate the inequality as an inequality about any pair of states, we will now show the following conclusion:

\[
F(\rho, \sigma) \geq 1 - \frac{1}{2} \sqrt{\frac{e^{S_{\max}(\rho|\sigma)}}{e^{S_{\max}(\rho|\sigma)} + 1}} \|\rho - \sigma\|_1
\] (3.2)

for \( \rho, \sigma \in D(H_d) \). Indeed, given any two density operators \( \rho \) and \( \sigma \), we know that if \( \rho\sigma \neq 0 \), then

\[
\min\{\lambda > 0 : \rho \leq \lambda \sigma\} = \lambda_{\max}(\sigma^{-1/2} \rho \sigma^{-1/2}) := \lambda_0,
\]

where the notation \( \lambda_{\max}(X) \) is used to denote the maximum eigenvalue of the operator \( X \), we define also \( \min\{\lambda > 0 : \rho \leq \lambda \sigma\} = +\infty \) if \( \rho\sigma = 0 \). Clearly \( \lambda_0 > 0 \). If denote \( \hat{\sigma} := \frac{\sigma - \lambda_0^{-1} \rho}{1 - \lambda_0^{-1}} \), then

\[
\sigma = \lambda_0^{-1} \rho + (1 - \lambda_0^{-1}) \hat{\sigma}.
\]

Therefore

\[
F(\rho, \sigma) = F(\rho, \lambda_0^{-1} \rho + (1 - \lambda_0^{-1}) \hat{\sigma}) \geq 1 - \frac{1}{2} \left(1 - \sqrt{\frac{1}{\lambda_0}}\right) \|\rho - \sigma\|_1
\]

implies that

\[
F(\rho, \sigma) \geq 1 - \frac{1}{2} \sqrt{\frac{\lambda_0}{\lambda_0 + 1}} \|\rho - \sigma\|_1 \text{ for } \lambda_0 > 0.
\]

The lower bound in the above inequality is indeed tighter than one in Fuchs-van de Graaf’s inequality. Thus, we get a state-dependent factor in the lower bound for fidelity. That is, when \( \lambda_0 = +\infty \), the above lower bound is reduced to the lower bound in Fuchs-van de Graaf’s inequality.

For the related problems along this line such as min- and max- (relative) entropy, it is referred to [9].

4 Conclusion

In this paper, we obtained a lower bound on the fidelity between a fixed state and its a mixed path with another state. Based on this result, we derived a lower bound on the fidelity between two states. The result may shed new light on some related problems in quantum information theory, for example, ones can consider bounding the input-output fidelity of transpose channel.

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