Topological classification of black Hole: Generic Maxwell set and crease set of horizon

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Abstract

The crease set of an event horizon or a Cauchy horizon is an important object which determines qualitative properties of the horizon. In particular, it determines the possible topologies of the spatial sections of the horizon. By Fermat’s principle in geometric optics, we relate the crease set and the Maxwell set of a smooth function in the context of singularity theory. We thereby give a classification of generic topological structure of the Maxwell sets and the generic topologies of the spatial section of the horizon.

1 Introduction

Topology is one of the fundamental qualitative properties of a black hole. It was investigated by many authors and now it is known that, under some reasonable conditions such as asymptotic flatness and the weak energy condition, each component of the black hole region is topologically trivial, i.e., simply connected [1]. On the other hand, there were numerical simulations which suggest non-trivial topologies of the horizons [2]. There has been some confusion, but it is now well understood that even though the black hole region in the spacetime is simply connected, there are many possible topologies of spatial sections. In particular, one of the authors [3] showed how topology of the spatial sections of a black hole is related to the endpoint set, or similarly, the crease set, of the event horizon. Therefore the crease set of an event horizon, or of a Cauchy horizon, is an important object which is independent from the choice of time slices and which determines qualitative properties of the horizon. Thus, to restrict the physically possible topologies of the black hole it is important to restrict the possible structure of the crease set.

In this paper we classify the possible topological structure of the crease set. “Possible” structures are important because they are the only ones that actually appear in the real world. There are at least two ways to define the “possible” structure: by stability and by genericity. In both approaches we consider spacetimes with different metrics. A stable structure is the one that is invariant against small change of the metric. A generic structure is, intuitively, the one that we find when we randomly pick up a spacetime. Physically, the small change of the metric may be interpreted as follows. First, in theoretical treatment of the world, we always assume some simple evolution equation and equation of state for matter. The precise equations may never be known. Second, when we are to determine the spacetime by observations, we can never collect perfectly accurate data of field strengths or matter density at all points of the spacetime. There must be errors and limits. Third, if we consider quantum mechanics, the fields and the metric can actually fluctuate at small scales. We shall give the precise mathematical definitions later. One interesting thing is that the both approaches lead the same conclusion (Theorem 3), namely, stability and genericity are equivalent.

Our stability/genericity approach is complementary to the analysis of exact solutions such as the Schwarzschild and Kerr spacetimes because the exact solutions are usually obtained by assuming high symmetry and many of its properties, especially topological ones, are not stable against perturbations [4]. One approach and the results will serve as guiding principles in the study of the black hole spacetime in mathematical relativity or in astrophysics.
The studies in the present paper are common for event horizons and Cauchy horizons, except for their direction of time. Therefore we only consider event horizons in the rest of the present paper. The same results hold for Cauchy horizons.

In Sect. 2, we define Fermat’s potential in a general nonstationary spacetime representing a gravitational collapse which is essential for the crease set and its classification. In Sect. 3, we give the precise definition of the Maxwell set which corresponds to the crease set of the event horizon. We also define stability and genericity of the Maxwell set, and introduce concepts necessary for our investigation. We show in Sect. 4 the equivalence of stability and genericity and obtain a list of stable Fermat potentials. We give in Sect. 5 the classification of the stable Maxwell sets. In Sect. 6, we discuss the cases of spacetimes of dimension other than four. Sect. 7 is for conclusion and discussions.

For the terms and notations about causal structure of a spacetime, see, e.g., Ref. [5].

2 Fermat’s principle and crease set

An event horizon is generated by null geodesics. A future event horizon $H$ cannot have future endpoints, but can have past endpoints if it is not eternal. As is pointed out in [3], the endpoint set $E$ of a horizon is an arc-wise connected acausal set. Points $u \in E$ are classified by the multiplicity $m(u)$ of $u$:

\[ C := \{ u \in E \mid m(u) > 1 \}, \]
\[ D := \{ u \in E \mid m(u) = 1 \}. \]

The set $C$ is called the crease set of the horizon. The crease set contains the interior of the endpoint set, i.e., the closure of $C$ contains $E$ [6]. The crease set $C$ equals the set of points of $E$ on which the horizon is not differentiable, i.e., the horizon is differentiable at $u \in E$ if and only if $u \in D$ [6, 7].

The horizon $H$ is the envelope of the light cone starting from the crease set $C$ which is an arc-wise connected acausal subset of $H$. In particular, if the spatial section of the horizon is a topological sphere at late times, the topology of the spatial section of the horizon can be nontrivial only at the crease set and the topology is completely determined by the time slicing of the crease set. This is studied in Ref. [3] in detail. In particular, when the crease set is a single point, all the possible spatial section of the horizon is a topological sphere. On the contrary, when the crease set has a disk-like structure, the horizon can have toroidal or higher-genus spatial sections. One would see the coalesce of horizons if the crease set has a line-like structure [2]. Therefore by classifying the structure of the crease set, we will know all possible topologies of the horizons. Here we do not assume that the spatial section of the horizon in the future is a sphere.

The crease set can be determined by Fermat’s principle in a simple stationary spacetime. In a nonstationary spacetime, we can extend Fermat’s principle and find a variational principle about light paths, imposing some appropriate causality condition such as global hyperbolicity. Here we show an example of the construction of the Fermat potential.

Let us assume that the spacetime $M$ is smooth and is globally hyperbolic from a smooth Cauchy surface $S$ which is diffeomorphic to $\mathbb{R}^3$. Furthermore, we consider a spacetime of gravitational collapse, namely, we assume that the event horizon $H$ is in the future of $S$. By global hyperbolicity, there are always an appropriate smooth global time coordinate $t : M \to \mathbb{R}$ and a timelike vector field $T$ such that $dt(T) = 1$. The spacetime $M$ is foliated by Cauchy surfaces $S_t = \{ q \in M | t(q) = t \}$. The vector field $T = \partial / \partial t$ defines a smooth projection $\pi$ from $M$ into the $S = S_{t_0}$ (see Fig. 1):

\[ \pi : M \to S, \]
\[ \pi^{-1}(q) = \{ \gamma(t) \mid \frac{\partial \gamma}{\partial t} = T, \gamma(t_0) = q, t \in \mathbb{R} \}. \]

Conversely, there is a diffeomorphism

\[ \phi : \mathbb{R} \times S \to M, \]
\[ t(\phi(t, u)) = t, \quad \pi(\phi(t, u)) = u. \]
Because $\mathcal{H}$ is achronal, the restriction of $\pi$ on $\mathcal{H}$ is injective and has an inverse, which we denote by $\psi$:

$$\psi: \pi(\mathcal{H}) \to \mathcal{H},
\psi(u) \in \mathcal{H}, \quad \pi(\psi(u)) = u. \quad (4)$$

The map $\psi$ is Lipschitz.

We take some (sufficiently large) $t = t_1$ and assume that $S_{t_1} \cap \mathcal{H}$ is diffeomorphic to a compact manifold $M$. We consider $M$ as a fixed submanifold embedded in $\mathcal{S}$ so that $S_{t_1} \cap \mathcal{H} = \psi(M)$. Consider a neighbourhood $\mathcal{U}$ of $\pi(\mathcal{H} \cap J^- (S_{t_1}))$ in $\mathcal{S}$. For $x \in M$ and $u \in \mathcal{U}$ we define Fermat’s potential as follows:

$$F(x, u) := -\sup\{t \in \mathbb{R} | \phi(t, u) \in J^- (\psi(x))\}. \quad (5)$$

The minimum points of $F$ corresponds to the generator of $\mathcal{H}$ through $x$. In particular, when the spacetime is static, the Fermat potential is the spatial geodesic distance, namely,

$$F(x, u) = \int_x^u \sqrt{\gamma_{ij}dx_idx_j} + \text{const.} \quad (6)$$

The projection $\pi$ is generated by the timelike killing vector and $\gamma_{ij}$ is an induced metric of the hypersurface $\mathcal{S}$ orthogonal to the timelike killing vector. Our definition is the generalization of this geodesic distance function to the non-static spacetime.

From (5), the crease set $\mathcal{C}$ is given by

$$\mathcal{C} = \psi(B_{\text{Maxwell}}(F)), \quad (7)$$

where $B_{\text{Maxwell}}(F)$ is the Maxwell set of $F$ where $F$ has two or more minimum points. We shall give a precise mathematical definition and a framework to study its properties later.

Let us consider a small change of the metric on $M$. To be precise, we may define the small change by a $C^\infty$ Whitney topology on the metric tensor field on $M$. Sufficiently small change of the metric leaves the vector field $T$ timelike and the Cauchy surfaces $S_t$ spacelike. The horizon near $S_{t_1} \cap \mathcal{H}$ changes only slightly in the spacetime $M$. This causes a small change of the differentiable map $\psi$ from $M$ to $\psi(M) \in \mathcal{H}$ but the topology of $M$ does not change. Also, the null geodesic system hence $J^-$ changes slightly. These cause the Fermat potential $F$ to be deformed slightly. In the sequel we shall study the stability and genericity of $F$ and $B_{\text{Maxwell}}(F)$ against this deformation of $F$.

In the formulation using such a potential function, actually a state space becomes infinite dimensional manifold, since we should not consider only the endpoints but also the path connecting them. However, if one can restrict the set of path into a finite dimensional family, the state space reduces to finite dimensional one. Indeed, in our case, two different points on event horizon cannot be connected by more than one generators by the fact that the generators of event horizon cannot have future endpoint.

On the other hand, in special cases one can take no spatial section on which there is no singular point of the horizon. Of course, such a difficulty is resolved in other formulations using “Lagrange manifold” and the result will not change. We are not concerned with this any more in the following.

The goal of our present study is to classify all generally possible structure about the singularities of the Fermat potential determining the horizon. As we shall see, the generic structure will be given by studying singularities of stable Fermat potentials concretely. The genericity or stability is defined as the property under small perturbations in a sense of Whitney $C^\infty$ topology. On the other hand, the stable Fermat potential is equivalent to the universal unfolding (parameter deformation family of a function) with the same number of deformation parameters. Then our main task is to give a classification of the universal unfoldings.

The above is a usual procedure when one discuss the bifurcation structure, so-called caustics, of a system. However, our main object here is not the bifurcation set but the Maxwell set (the difference will become clear later). The problem is not purely local but is rather semi-local. The definition of Maxwell set is local in control (parameter) space $U$ but is non-local in state (variable) space $M$. To treat this we introduce function multigerms. We will classify the universal multifoldings and the Maxwell sets.

Finally, we note that the catastrophe changes when some symmetry is present. An example is the toroidal event horizon studied in Refs. This is not a universal unfolding but an unfolding with infinite codimension.
3 The Maxwell set: Stability and genericity

Now we give the mathematical framework to study the Maxwell set. The Maxwell set of a function unfolding is the set of all values of the parameters for which the minimum is attained either at a non-Morse critical point or at two or more critical points. In the following we sometimes make \( x \) as a representative of the state variables and \( u \) the control variables. Of course, \( F \) is not a global function on \( \mathbb{R}^r \times \mathbb{R}^r \) rather a function on manifold \( M \times U \), where \( M \) is two-dimensional compact manifold and \( U \) is an open subset of \( \mathbb{R}^3 \). We will mainly deal with function germs instead of functions, since we are only concerned with local properties of a function on some neighbourhood of \( u_0 \in U \). To treat this, we introduce a germ of map which is an equivalence class of maps at a neighbourhood \( W \). A function \( F : M \times U \to \mathbb{R} \) can be considered as a family of functions \( f_u : M \to \mathbb{R} \) with \( f_u = F(\bullet, u) \).

Definition (Unfolding). A function \( F : M \times U \to \mathbb{R} \) is called an unfolding of a function \( f_{u_0} \) at \( u_0 \).

Definition (Maxwell set). For a function unfolding \( F : M \times U \to \mathbb{R} \) on a compact manifold \( M \), the Maxwell set \( B_{\text{Maxwell}}(F) \) of \( F \) is a subset of \( U \) given by

\[
B_{\text{Maxwell}}(F) := \{ u \in U \mid f_u \text{ has two or more global minimum points} \}.
\]

In the investigation of the Maxwell set we mainly focus on its local structure because the global structure is obtained by the combinations of local ones. Below we extensively use the notion of the germs of objects which provides the best way to characterize their local structure. Let \( M, N \) be \( C^\infty \)-manifolds. We denote the set of \( C^\infty \)-maps from \( M \) to \( N \) by \( C^\infty(M, N) \).

Definition (Map germ). Maps \( f, g \in C^\infty(M, N) \) are equivalent at \( x_0 \) if there is a neighbourhood \( W \) of \( x_0 \) such that \( f|_W = g|_W \). A map germ \( f \) at \( x_0 \), \( [f]_{x_0} \), is the equivalence class of \( f \). It is also denoted by \( f : (M, x_0) \to (N, f(x_0)) \).

Examples of map germs include function germs and diffeomorphism germs.

Definition (Set germ). Subsets \( X, Y \) of \( M \) are equivalent at \( x_0 \) if \( X \cap W = Y \cap W \) for each neighbourhood \( W \) of \( x_0 \). A set germ \((X, x_0)\) of \( X \) at \( x_0 \) is the equivalence class of \( X \). Set germs \((X, x_0)\) and \((Y, y_0)\) are diffeomorphic, \((X, x_0) \simeq (Y, y_0)\), if there is a diffeomorphism germ \( \phi : (M, x_0) \to M \) such that \((\phi(X), \phi(x_0)) = (Y, y_0)\).

Definition (Unfolding germ). Function unfoldings \( F \) and \( G \) are equivalent at \( u_0 \) if there is a neighbourhood \( W \) of \( u_0 \) in \( U \) such that \( F|_{M \times W} = G|_{M \times W} \). An unfolding germ \( F : (M \times U, M \times \{u_0\}) \to \mathbb{R} \), or \([F]_{u_0}\), of \( f_{u_0} \) at \( u_0 \) is the equivalence class of \( F \) defined by this equivalence relation.

We usually call unfolding germs defined above, which are germs with respect to \( U \) only, simply as unfoldings. Later we will define unfolding germs with respect to both \( M \) and \( U \), which we will call unfolding germs.

Below, we will determine the topological structure of \( C^\infty(M, N) \) where all the maps that we treat are included. We define a topology of \( C^\infty(M, N) \) by the r-jet space \( J^r(M, N) \) below.

Definition (Jet space). Let \( f \in C^\infty(M, N) \). The r-jet \( j^r f(x_0) \) of \( f \) at \( x_0 \) is the equivalence class of \( f \) in \( C^\infty(M, N) \) where two maps are equivalent if all of their s-th partial derivatives with \( 1 \leq s \leq r \), in some coordinate systems of \( M \) and \( N \), coincide. The r-jet space of \( C^\infty(M, N) \) is defined by

\[
J^r(M, N) := \{ j^r f(x_0) \mid f \in C^\infty(M, N) \}.
\]

The space of r-jets at a point is an \( n \left( \frac{m+r}{r} \right) \)-dimensional manifold, where \( m = \dim M \) and \( n = \dim N \).

Now we endow the space \( C^\infty(M, N) \) with the Whitney \( C^\infty \) topology.

Definition (Whitney \( C^\infty \) topology). For an open subset \( O \) of \( J^r(M, N) \), let

\[
W^r(O) := \{ f \in C^\infty(M, N) \mid j^r f(M) \subset O \}.
\]
The Whitney $C^\infty$ topology on $C^\infty(M, N)$ is the topology whose basis is

$$\bigcup_{r=0}^{\infty} \{W^r(O)\mid O \text{ is an open subset of } J^r(M, N)\}. \quad (11)$$

Hereafter we treat that $C^\infty(M, N)$ as a topological space with the Whitney $C^\infty$ topology. Now we can define stability of the Maxwell set using this topology.

**Definition (Stable Maxwell set germ).** An unfolding,

$$F : (M \times U, M \times \{u_0\}) \to \mathbb{R}, \quad (12)$$

is stable with respect to the Maxwell set if for each neighbourhood $W$ of $u_0$ there exists a neighbourhood $U$ of $F$ in $C^\infty(M \times U, \mathbb{R})$ such that for each $G \in U$ there exists $v_0 \in W$ such that $(B_{\text{Maxwell}}(F), u_0) \simeq (B_{\text{Maxwell}}(G), v_0)$. We call $(B_{\text{Maxwell}}(F), u_0)$ a stable Maxwell set germ.

In this sense, our aim is to classify the stable Maxwell set germs of the Fermat potential $F$.

Since we have defined the topology of $C^\infty(M, N)$, now we can formulate the genericity of a class of smooth maps. A subset of a topological space $X$ is nowhere dense if its closure has no interior. A subset of $X$ is residual if its complement is a countable union of nowhere dense sets. The space $X$ is a Baire space if every residual set is dense.

**Definition (Genericity).** A property $P$ of $f \in C^\infty(M, N)$ is generic if the set

$$A_P := \{f \in C^\infty(M, N)\mid P(f) \text{ is true}\} \quad (13)$$

is residual in $C^\infty(M, N)$.

When $P$ is generic, $A_P$ is dense in $C^\infty(M, N)$ and any $g \in C^\infty(M, N)$ is approximated by a map $f$ satisfying $P$. $f \in A_P$. Furthermore, $A_P^c$, the set on which the negation of $P$ holds, is not generic. This is because of the following (e.g. [11]):

**Theorem 1.** $C^\infty(M, N)$ with the Whitney $C^\infty$ topology is a Baire space.

To prove a generic property, transversality theorems are fundamental. In particular, we make use of the Multitransversality Theorem by Mather below.

**Definition (Transversality).** Let $f \in C^\infty(M, N)$ and let $Q$ be a submanifold of $N$. The map $f$ is transversal to $Q$ at $x$ if either of the following holds:

1. $f_*(T_xM) \oplus T_{f(x)}Q = T_{f(x)}N$;
2. $f(x) \notin Q$.

The map $f$ is transversal to $Q$ if it is transversal to $Q$ at every $x \in Q$.

**Theorem 2 (Multitransversality Theorem by Mather [12]).** Let $M, N$ be $C^\infty$-manifolds and let $Q_1, Q_2, \ldots$ be a countable family of submanifolds of $j^r(M, N)$. Then the set

$$T := \{f \in C^\infty(M, N)\mid k_j f \text{ is transversal to } Q_1, Q_2, \ldots\} \quad (14)$$

is residual in $C^\infty(M, N)$.

This theorem is fundamental in our discussion. It states that transversality to a countable family of submanifolds is a generic property. Therefore the properties deduced from this theorem will also be generic. In the rest of the paper, we shall show that any stability and genericity of a Maxwell set is equivalent in a certain sense (Theorem 3) and carry out a topological classification of stable Maxwell sets.

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3This is true when $(\dim M, \dim N)$ are nice dimensions [13]. For example, if the $\dim U > 5$, stability does not imply genericity. In our case, however, we have $N = \mathbb{R}$ and it is known that $(\dim M, \dim R) = (1, 1)$ are always nice dimensions.
4 Classification of the Fermat potentials

Let us give the classification of stable Maxwell sets. The definition of the Maxwell set requires the global information of the function unfolding $F$. A simple but crucial observation is, however, that to determine the local structure of the Maxwell set, i.e., the Maxwell set germs, we only need the local information of $F$ around its global minimum points $p_1, ..., p_k \in M$. We first generalize the notion of germs to that of multigerms. Let $M^{(k)}$ be a $k$-tuple of distinct points of $M$, i.e.,

$$M^{(k)} := \{(x_1, ..., x_k) \in M^k | x_i \neq x_j \text{ for } i \neq j\}.\quad (15)$$

**Definition (Multigerm).** Let $(x_1, ..., x_k) \in M^{(k)}$. A $k$-fold map germ $f : (M, (x_1, ..., x_k)) \to N$, or $[f]_{x_1, ..., x_k}$, is the equivalence class of $f \in C^\infty(M, N)$, where two maps are equivalent if they coincide on some open subset of $M$ which contains $x_1, ..., x_k$. A $k$-fold unfolding germ $F : (M \times U, ((x_1, ..., x_k), u_0)) \to N$, or $[F]_{(x_1, ..., x_k), u_0}$, is the equivalence class of $F \in C^\infty(M, \mathbb{R})$ where two functions are equivalent if they coincide on some open subset of $M \times W$ which contains $(x_1, u_0), ..., (x_k, u_0)$. A $k$-fold germ is also called as a multigerm.

A multigerm can be considered as $k$-tuple of simple germs. For example, A map multigerm $[F]_{(x_1, ..., x_k), u_0}$ can be considered as $k$-tuple of function germs $([f_1]_{x_1}, ..., [f_k]_{x_k})$.

**Definition (Right equivalence).** Function germs $f : (M, x) \to \mathbb{R}$ and $g : (N, y) \to \mathbb{R}$ are right equivalent, $[f]_x \simeq [g]_y$, if there exist a diffeomorphism germ $\phi : (M, x) \to (N, y)$ and $a \in \mathbb{R}$ such that $f = g \circ \phi + a$ holds as an equality of function germs at $x$.

**Definition (Right equivalence at the minimum points).** Unfoldings $F : (M \times U, M \times \{u_0\}) \to \mathbb{R}$ and $G : (N \times V, N \times \{v_0\}) \to \mathbb{R}$ are right equivalent at the minimum points if the following conditions hold:

1. The functions $f_{u_0} = F(\bullet, u_0)$ and $g_{v_0} = G(\bullet, v_0)$ have the same number of global minimum points, $p_1, ..., p_k$ and $q_1, ..., q_k$, respectively.
2. There exist a diffeomorphism multigerm $\phi : (M \times U, ((p_1, ..., p_k), u_0)) \to (N \times V, ((q_1, ..., p_k), v_0))$, a diffeomorphism germ $\psi : (U, u_0) \to (V, v_0)$, and a function germ $\alpha : (U, u_0) \to \mathbb{R}$ such that

$$F(x, u) = G(\phi(x, u), \psi(u)) + \alpha(u)\quad (16)$$

holds with both sides being function multigerms at $((p_1, ..., p_k), u_0)$.

The Maxwell set germ of an unfolding is determined only by the unfolding multigerm at the minimum points:

**Proposition 1.** If unfoldings $F : (M \times U, M \times \{u_0\}) \to \mathbb{R}$ and $G : (M \times V, M \times \{v_0\}) \to \mathbb{R}$ are right equivalent at the minimum points, then their Maxwell set germs are diffeomorphic.

**Proof.** Follows directly from the definitions of right equivalence and of Maxwell set germs. \qed

Now we have completed the preparation. We will investigate stable structure of the function unfolding and its Maxwell set.

**Definition (Stability at the minimum points).** An unfolding $F : (M \times U, M \times \{u_0\}) \to \mathbb{R}$ is stable at the minimum points if for each neighbourhood $W$ of $u_0$ there exists a neighbourhood $\mathcal{U}$ of $F$ in $C^\infty(M \times U, \mathbb{R})$ such that for each $G \in \mathcal{U}$ there exists $v_0 \in W$ such that $[F]_{u_0} \simeq [G]_{v_0}$.

From Proposition 1 we immediately have the following proposition.

**Proposition 2.** If an unfolding $F : (M \times U, M \times \{u_0\}) \to \mathbb{R}$ is stable at the minimum points then $(B_{\text{Maxwell}}(F), u_0)$ is a stable Maxwell set germ.

To discuss the stability, we study the orbit of diffeomorphism on some standard functions in $J'(M, N)$ and its transversality. We will stratify $J'(M, N)$, i.e., decompose the $J'(M, N)$ into the union of submanifolds (strata) \[.\]
Definition (Strata).  
\[ A_0 := \{ j^r f(p) \in J^r(M, \mathbb{R}) \mid f \text{ is regular at } p \}, \]  
\[ A_k := \{ j^r f(p) \in J^r(M, \mathbb{R}) \mid |f|_p \simeq |\pm x_1^{k+1} \pm x_2^2|_0 \}, \]  
\[ D_4 := \{ j^r f(p) \in J^r(M, \mathbb{R}) \mid |f|_p \simeq |\pm x_1^4 \pm x_2^2|_0 \}, \]  
\[ E_5 := \{ j^r f(p) \in J^r(M, \mathbb{R}) \mid |f|_p \simeq |\pm x_1^5 \pm x_2^3|_0 \}. \]

The following is well discussed [14]:

Lemma 1. (1) \( A_0 \) is an open subset of \( J^r(M, \mathbb{R}) \) hence is a submanifold of codimension 0.

(2) \( A_k \) is a submanifold of \( J^r(M, \mathbb{R}) \) of codimension \( k + 1 \).

(3) \( D_4 \) and \( E_5 \) are submanifolds of \( J^r(M, \mathbb{R}) \) of codimension 5.

(4) \( \Sigma := J^r(M, \mathbb{R}) \setminus \bigcup_{k=0}^{4} A_k - D_4 - E_5 \) is the union of a finite number of submanifolds of codimension 6 or greater:

\[ \Sigma = W_1 \cup \ldots \cup W_s. \]

Definition (Natural stratification). The natural stratification of \( J^r(M, \mathbb{R}) \), where \( r > 4 \), is the one given by

\[ S(J^r(M, \mathbb{R})) := \{ A_0, A_1, \ldots, A_4, D_4, E_5, W_1, \ldots, W_s \}. \]

We also extend the concept of jet space to multijet space.

Definition (Multijet space). The \( k \)-fold \( r \)-jet, or simply, \( r \)-multijet, \( k j^r f \) of \( f : M \to \mathbb{R} \) at \((x_1, \ldots, x_k) \in M^{(k)} \) is

\[ k j^r f(x_1, \ldots, x_k) := (j^r f(x_1), \ldots, j^r f(x_k)). \]

The \( r \)-multijet space \( k J^r(M, N) \) is given by

\[ k J^r(M, N) := \{ k j^r f(x_1, \ldots, x_k) \in (J^r(M, N))^k \mid (x_1, \ldots, x_k) \in M^{(k)} \}. \]

The map \( k j^r f : M^{(k)} \to k J^r(M, N) \) is called an \( r \)-multijet section.

To classify event horizons in a four-dimensional spacetime, we assume that \( M \) is a two-dimensional compact manifold and \( U \) is diffeomorphic to \( \mathbb{R}^4 \). Then, because \( \dim(M \times U) = 5 \), it is sufficient to consider the 5-fold 5-jet, \( J^5(M, \mathbb{R}) \). Let us define a natural stratification of the multijet space \( k J^5(M, \mathbb{R}) \).

Definition (Natural stratification of \( k J^5(M, \mathbb{R}) \)). The natural stratification of \( k J^5(M, \mathbb{R}) \) is the one given by

\[ S(k J^5(M, \mathbb{R})) := \{ \Delta_k \cap X_1 \times \ldots \times X_k | X_1, \ldots, X_k \in S(J^5(M, \mathbb{R})) \}, \]

where

\[ \Delta_k := \{ j^5 f_1(p_1), \ldots, j^5 f_k(p_k) \in k J^5(M, \mathbb{R}) \mid f_1(p_1) = \ldots = f_k(p_k) \}. \]

The following lemma is easily shown.

Lemma 2. Let \( M, N \) be \( C^\infty \)-manifolds and let \( f : M \to N \) be a \( C^\infty \)-map. Let \( S \) be a \( C^\infty \)-submanifold of codimension \( c \). Then for each \( x \in M \) there exists a neighbourhood \( W \) of \( f(p) \) in \( N \) and a \( C^\infty \)-map \( g : W \to \mathbb{R}^c \) such that

\[ \text{rank } dg_\xi = c, \quad S \cap W = g^{-1}(0), \quad 0 \in \mathbb{R}^c, \]

where \( dg_\xi : T_\xi \to T_{g(\xi)} \mathbb{R}^c \) is the differential map of \( g \) at \( \xi \). Moreover, the following two conditions are equivalent:

(1) \( f \) is transversal to \( S \) at \( x \).

(2) \( \text{rank } d(g \circ f)_x = k \).
Theorem 3. Let $F: (M \times U, M \times \{u_0\}) \to \mathbb{R}$ be an unfolding and let $p_1, \ldots, p_k$ be the minimum points of $f_{u_0} = F(\bullet, u_0)$. The unfolding $F$ is stable at the minimum points if and only if the multijet section

$$kJ^5 F : M^{(k)} \times U \to kJ^5(M, \mathbb{R})$$

is transversal to the natural stratification $S(kJ^5(M, \mathbb{R}))$ at $((p_1, \ldots, p_k), u_0) \in (\mathbb{R}^2)^k \times U$. Furthermore, such $F$ falls into one of the following four cases. (Below $0_i \in \mathbb{R}^2$ are the origins of local coordinate systems $(x_i, y_i)$).

(1) $k = 1$ and $F$ is right equivalent at the minimum points to either

$$A_1(x, y, u) = x^2 + y^2 \at (0, 0) \in \mathbb{R}^2 \times \mathbb{R}^3,$$

or

$$A_3(x, y, u) = x^4 + u_2 x^2 + u_1 x + y^2 \at (0, 0) \in \mathbb{R}^2 \times \mathbb{R}^3. \quad (29)$$

(2) $k = 2$ and $F$ is right equivalent at the minimum points to either

$$(A_1, A_1) = (x_1^2 + y_1^2 + u_1 x_1 + x_2^2 + y_2^2) \at ((0_1, 0_2), 0) \in (\mathbb{R}^2)^2 \times \mathbb{R}^3, \quad (30)$$

or

$$A_3(x, y, u) = (x_1^4 + u_2 x_1^2 + u_1 x_1 + y_1^2 + y_2^2 + x_2^2 + y_2^2) \at ((0_1, 0_2), 0) \in (\mathbb{R}^2)^2 \times \mathbb{R}^3. \quad (31)$$

(3) $k = 3$ and $F$ is right equivalent at the minimum points to

$$(A_1, A_1, A_1) = (x_1^2 + y_1^2 + y_1^2 + u_1, x_2^2 + y_2^2 + u_2, x_3^2 + y_3^2) \at ((0_1, 0_2, 0_3), 0) \in (\mathbb{R}^2)^3 \times \mathbb{R}^3. \quad (32)$$

(4) $k = 4$ and $F$ is right equivalent to

$$(A_1, A_1, A_1, A_1) \at ((0_1, 0_2, 0_3, 0_4, 0) \in (\mathbb{R}^2)^4 \times \mathbb{R}^3. \quad (33)$$

Remark 1. The theorem states that stability and genericity are equivalent in the following sense. Stability at the minimum points, or, equivalently, being in either one of the four cases above, is a generic property of $F$. On the contrary, when we classify unfoldings by right equivalence at the minimum points, the set of all generic unfoldings must include the four cases above. This equivalence exists for any spacetime dimensions.

Proof. (i) The stability implies the multitransversality.

Let $F : (M \times U, M \times \{u_0\}) \to \mathbb{R}$ be stable at the minimum points. Then there exists a neighbourhood $U$ of $F$ in $C^\infty(M \times U, \mathbb{R})$ such that for each $G \in U$ there exists $v_0 \in U$ such that

$$F(x, u) = G(\phi(x, u), \psi(u)) + \alpha(u) \quad (36)$$

with both sides being function multigerms at $((p_1, \ldots, p_k), u_0)$, where $\phi$, $\psi$ and $\alpha$ are as those in [10]. From the Multitransversality Theorem, the unfolding $G : (M \times U, M \times \{v_0\}) \to \mathbb{R}$ can be chosen so that its multijet section $kJ^5 G$ is transversal to $S(kJ^5(M, \mathbb{R}))$. By the definition of $S(kJ^5(M, \mathbb{R}))$, the action of $\phi$ on $kJ^5(M, \mathbb{R})$ preserves the stratification $S(kJ^5(M, \mathbb{R}))$. Moreover, the term $\alpha(u)$ in (36) is irrelevant concerning the transversality. Thus, from [10], $kJ^5 F$ is transversal to $S(kJ^5(M, \mathbb{R}))$.

(ii) The multitransversality implies that one of the conditions (1)–(4) holds.

Let the multijet section $kJ^5 F : M^{(k)} \times U \to kJ^5(M, \mathbb{R})$ be transversal to $S(kJ^5(M, \mathbb{R}))$ at $((p_1, \ldots, p_k), u_0)$. 

Suppose $k \geq 5$. Then the transversality of the multijet section $k j^5 F$ to $S(k J^5 (M, \mathbb{R}))$ at $((p_1, ..., p_k), u_0)$ would imply
\[
k j^5 F((p_1, ..., p_k), u_0) \notin \Delta_k \cap A_1 \times ... \times A_1.
\]
(37)
because $\text{codim} (\Delta_k \cap A_1 \times ... \times A_1) = k+1 > 5$ and $\text{dim}(M \times U) = 5$. For the other strata $\Delta_k \cap X_1 \times ... \times X_k$ with $X_i \neq A_0$, the codimension is larger and we would have $k j^5 F((p_1, ..., p_k), u_0) \notin \Delta_k \cap X_1 \times ... \times X_k$. This would contradict with the fact that $p_1, ..., p_k$ are minimum points. Thus the cases with $k \geq 5$ cannot occur.

When $k = 1$, the statement is a direct consequence of Thom’s elementary catastrophe theory. When $2 \leq k \leq 4$, let the multijet section $k j^5 F$ be transversal to $S(k J^5 (M, \mathbb{R}))$ at $((p_1, ..., p_k), u_0)$, where $p_1, ..., p_k$ are the minimum points of $f_{u_0}$. Then the jet section $j^5 F$ must be transversal to $S(J^5 (M, \mathbb{R}))$ at $(p_i, u_0)$, $i = 1, ..., k$. It follows from the case $k = 1$ that $j^5 F$ must be in either $A_1$ or $A_3$. Thus by considering the codimensions of $A_k$, $D_4$, $E_5$ and $\Sigma$, the dimension of $M \times U$ $(=5)$, and the fact that $p_1, ..., p_k$ are minimum points, we find that the only possibilities are the following, up to addition of a function of $u$:

(a) $k = 2, 3, 4$

\[
F(x_i, y_i, u_1, u_2, u_3) = x_i^2 + y_i^2 + \alpha_i(u), \quad (i < k)
\]
\[
F(x_k, y_k, u_1, u_2, u_3) = x_k^4 + y_k^2.
\]

(b) $k = 2$

\[
F(x_1, y_1, u_1, u_2, u_3) = x_1^4 + y_1^2 + u_2 x_1^4 + u_1 x_1 + \alpha_1(u),
\]
\[
F(x_2, y_2, u_1, u_2, u_3) = x_2^2 + y_2^2.
\]

where for $i = 1, ..., k$, $(x_i, y_i, u_1, u_2, u_3)$ is some local coordinate system in a neighbourhood of $(p_i, u_0)$ in $M \times U$.

In the following we omit detailed calculations and sketch the proof. In the case (a), one can easily construct a map $g : k J^5 (M, \mathbb{R}) \rightarrow \mathbb{R}^{3k-1}$ such that $\Delta_{(p_1, ..., p_k)} \cap A_1 \times ... \times A_1 = g^{-1}(0)$, where $\Delta_{(p_1, ..., p_k)}$ is $\Delta_k$ with the minimum points fixed to $(p_1, ..., p_k)$. One can verify that the differential map $dg_\xi$ is nondegenerate for $\xi \in k J^5 (M, \mathbb{R})$. By transversality of $k j^5 F$ to $S(k J^5 (M, \mathbb{R}))$ at $((p_1, ..., p_k), u_0)$ and by Lemma 2, we find that $d(g \circ k j^5 F)((p_1, ..., p_k), u_0)$ is nondegenerate. This implies that the Jacobi matrix of the map $(u_1, u_2, u_3) \mapsto (\alpha_1(u), ..., \alpha_{k-1}(u))$ is nondegenerate. This allows a coordinate transformation $(u_1, u_2, u_3) \mapsto (v_1, v_2, v_3)$ given by

\[
v_i = \alpha_i(u), \quad i \leq k - 1, \quad v_i = u_i, \quad i > k - 1.
\]

This gives the normal forms in the theorem.

In the case (b), we construct $g : k J^5 (M, \mathbb{R}) \rightarrow \mathbb{R}$ such that $\Delta_2 = g^{-1}(0)$ where $dg_\xi$ is nondegenerate. By transversality of $k j^5 F$ to $\Delta_2 \cap A_3 \cap A_1$ at $((p_1, p_2), u_0)$ and by Lemma 2, we find that $d(g \circ k j^5 F)((p_1, p_2), u_0)$ is nondegenerate. This implies that the following coordinate transformation is possible:

\[
v_1 = u_1, \quad v_2 = u_2, \quad v_3 = \alpha_1(u).
\]

This gives the normal form in the theorem.

(iii) Each of conditions (1)–(4) implies the stability at the minimum points.

Let us consider the case $k = 2$, i.e., let $F : (M \times U, M \times \{u_0\}) \rightarrow \mathbb{R}$ satisfy (2) in the theorem. When $F$ is right equivalent to $(A_3, A_1)$, the double 5-jet section $2 j^5 F$ intersects $\Delta_2 \cap A_3 \times A_1 \in S(2 J^5 (M, \mathbb{R}))$ transversally at $((p_1, p_2), u_0) \in M^{(2)} \times U$. Then, for $G : M \times U \rightarrow \mathbb{R}$ sufficiently close to $F$, there exists $v_0 \in U$ close to $u_0$ such that the number of minimum points of $g_{v_0} = G(\bullet, v_0)$ is two and that $2 j^5 G$ intersects $\Delta_2 \cap A_3 \times A_1$ transversally at $((q_1, q_2), v_0) \in M^{(2)} \times U$, where $q_1, q_2$ are the minimum points of $g_{v_0}$. Then from (ii) again, $G$ is right equivalent at the minimum points to $(A_3, A_1)$. Since both $F$ and $G$ are right equivalent at the minimum points to $(A_3, A_1)$, $G$ is right equivalent to $F$ at the minimum points. Therefore $F$ is stable at the minimum points. For the other cases, the assertion can be shown similarly.

\[ \square \]
Remark 2. In the book of Arnold [15] there is a statement in general dimensions without proof: The Maxwell set of a generic \( l \leq 6 \)-parameter family of functions is locally stably diffeomorphic to one of the sets \( \{A_{\mu_i}\} \), where the \( \mu_i \) are odd and \( \sum \mu_i \leq l + 1 \), i.e., either to the set with the singularity \( A^m_{\mu} (m \leq l + 1) \) or to one of the sets of the following table (the types of minimum points are indicated):

\[
\begin{array}{cccccc}
 l & 2 & 3 & 4 & 5 & 6 \\
\hline
 \text{Type} & A_3 & A_1 A_3 & A_2 A_3 & A_4 A_3, A_1 A_5 & A_1 A_3, A_4 A_3, A_7 \\
\end{array}
\]

The notation is according to his book. \( A_1^2 \) means \( (A_1, A_1) \), etc.

5 Classification of Maxwell sets

In this section we reveal the concrete structure of the Maxwell set for each \( F \) appeared in Theorem 5.

Definition (Minimum function). The minimum function \( \mu_F : U \to \mathbb{R} \) of an unfolding \( F : M \times U \to \mathbb{R} \) is given by

\[ \mu_F(u) := \min \{ F(x,u) | x \in M \}. \] (42)

There are two cases when the minimum function \( \mu_F \) has a singularity at \( u \):
(1) the function \( f_u \) has several minimum points in \( M \), or
(2) the number of minimum points changes there.

In the case (1) \( u \) is a point of the Maxwell set. In the case (2) \( u \) is not a point of the Maxwell set but corresponds to a point of the endpoint set \( \mathcal{E} \).

In this paper, we define the boundary of a Maxwell set as follows. An interior point \( u \) of a Maxwell set \( B_{\text{Maxwell}}(F) \) is such that there exists a \( C^0 \)-submanifold \( V \) of codimension 1 of \( U \) which contains \( q \) and which is entirely contained in \( B_{\text{Maxwell}}(F) \). The boundary of \( B_{\text{Maxwell}}(F) \) is the complement of the interior of \( B_{\text{Maxwell}}(F) \) in \( B_{\text{Maxwell}}(F) \), where \( B_{\text{Maxwell}}(F) \) is the closure of \( B_{\text{Maxwell}}(F) \) in \( U \).

5.1 \( A_1 \)

In this case the number of minimum point is one. Obviously, the Maxwell set \( B_{\text{Maxwell}}(A_1) \) is empty.

The minimum function \( \mu_F \) does not have singularities.

5.2 \((A_1, \ldots, A_1)\)

In this case of \((A_1, \ldots, A_1)\), \( k \) minimum points compete, where \( k \) can be 2, 3, or 4. When \( k = 2 \), \( F \) is right equivalent at the minimum points to

\[ (A_1, A_1) = (x_1^2 + y_1^2 + u_1, x_2^2 + y_2^2) \text{ at } ((0_1, 0_2), 0) \in (\mathbb{R}^2)^{(2)} \times \mathbb{R}^3. \] (43)

The minimum function is given by

\[ \mu_F(u) = \min \{ u_1, 0 \} \] (44)

This is singular on the plane \( u_1 = 0 \). The Maxwell set germ is given by

\[ (B_{\text{Maxwell}}(F), 0) = \{ (u_1, u_2, u_3) | u_1 = 0 \}. \] (45)

This is a surface (germ) through the origin. The points of \((A_1, A_1)\) form a surface.

When \( k = 3 \), \( F \) is right equivalent at the minimum points to

\[ (A_1, A_1, A_1) = (x_1^2 + y_1^2 + u_1, x_2^2 + y_2^2 + u_2, x_3^2 + y_3^2) \text{ at } ((0_1, 0_2, 0_3), 0) \in (\mathbb{R}^2)^{(3)} \times \mathbb{R}^3. \] (46)

The minimum function is

\[ \mu_F(u) = \min \{ u_1, u_2, 0 \}. \] (47)
The structure is given by the condition of the quartic function above having two minimum points below. The point \( A \) changes as (5.4)

In the case where \( A \) there are two minimum points (5.3)

This is an intersecting point of three half planes. The points of \((A_1, A_1, A_1)\) form a submanifold of codimension 2 which is a curve when \( \dim U = 3 \).

When \( k = 4, F \) is right equivalent at the minimum points to

\[
(A_1, A_1, A_1, A_1)
\]

\[
= (x_1^2 + y_1^2 + u_1, x_2^2 + y_2^2 + u_2, x_3^2 + y_3^2 + u_3, x_4^2 + y_4^2)
\]

at \(((0_1, 0_2, 0_3, 0_4), 0) \in (\mathbb{R}^2)^4 \times \mathbb{R}^3.\) (49)

The minimum function is

\[
\mu_F(u) = \min \{u_1, u_2, u_3, 0\}.
\]

The Maxwell set is

\[
B_{\text{Maxwell}}(F) = \{(u_1, u_2, u_3)|0 \geq u_1 = u_2 \leq u_3\}
\]

\[
\cup \{(u_1, u_2, u_3)|0 \geq u_2 = u_3 \leq u_1\}
\]

\[
\cup \{(u_1, u_2, u_3)|0 \geq u_3 = u_1 \leq u_2\}
\]

\[
\cup \{(u_1, u_2, u_3)|u_2 \geq u_1 = 0 \leq u_3\}
\]

\[
\cup \{(u_1, u_2, u_3)|u_1 \geq u_2 = 0 \leq u_3\}
\]

\[
\cup \{(u_1, u_2, u_3)|u_1 \geq 0 \leq u_2\}.
\]

(51)

This is an intersecting point of six pieces of surfaces.

The embedded images for \( k = 2, 3, 4 \) are illustrated in Figure 2.

### 5.3 \( A_3 \)

In the case where \( F \) is right equivalent at the minimum points to

\[
A_3(x, y, u_1, u_2) = x^4 + u_2 x^2 + u_1 x + y^2 \text{ at } (0, 0) \in \mathbb{R}^2 \times \mathbb{R}^3.
\]

However, there is only one minimum point exist for \( u = 0 \), there \( u \)'s in the vicinity of the origin where there are two minimum points \((A_1, A_1)\). The shape of the graph of the function \( f_u(x) = F(x, 0, u_1, u_2) \) changes as \((u_1, u_2)\) changes around the origin of \( A_3 \). This is depicted in Figure 3. We have

\[
\mu_F(u) = \min_x(x^4 + u_2 x^2 + u_1 x).
\]

(53)

The maxwell set germ is given by the condition of the quartic function above having two minimum points:

\[
(B_{\text{Maxwell}}(F), 0) = \{(u_1, u_2, u_3) \in \mathbb{R}^3|u_1 = 0, u_2 < 0\}, 0).
\]

(54)

The structure \( A_3 \) appears on the boundary of the surface formed by points of \((A_1, A_1)\) of the Maxwell set, where two minimum points \((A_1, A_1)\) become degenerate. The structure also appears near \((A_3, A_1)\) below. The point \( A_3 \) itself (the origin) is not contained in \( B_{\text{Maxwell}}(F) \). The structure appears at the boundary of \( B_{\text{Maxwell}}(F) \).

### 5.4 \((A_3, A_1)\)

In the case where \( F \) is right equivalent at the minimum points to

\[
(A_3, A_1) = (x_1^4 + u_2 x_1^2 + u_1 x_1 + y_1^2 + u_3, x_2^2 + y_2^2)
\]

(55)
The minimum function is given by
\[
\mu_F(u) = \min \{ \min_x h(x), 0 \}.
\] (56)

where \( h(x) := x^4 + u_2 x^2 + u_1 x + u_3 \). Let \( x_m \) the minimum point of \( h(x) \). Then we have
\[
h'(x_m) = 4x_m^3 + 2u_2 x_m + u_1 = 0
\] (57)
so that
\[
u_1 = -2x_m(2x_m^2 + u_2) =: b(x_m),
\] (58)
\[
h(x_m) = -3x_m^4 - u_2 x_m^2 + u_3.
\] (59)

A point of the Maxwell set must satisfy either of the following:
(a) \( h \) has two minimum points and \( h(x_m) \leq 0 \),
(b) \( h \) has one minimum point and \( h(x_m) = 0 \).

The case (a) is the same as the case of single \( A_3 \). The function \( h \) has two minimum points if and only if \( u_1 = 0 \), \( u_2 < 0 \). The minimum value \( h(\pm \sqrt{-u_2/2}) = u_3 - u_2^2/4 \) must not be positive. Thus the part of the Maxwell set for the case (a) is
\[
M_1 = \{ (u_1, u_2, u_3) | u_1 = 0, u_2 < 0, u_3 \leq \frac{u_2^2}{4} \},
\] (60)

Let us consider the case (b). If \( u_1 = 0 \), then \( u_2 \geq 0 \) must hold. The minimum value is \( h(0) = u_3 \). Thus we have \( u_3 = 0 \). If \( u_1 \neq 0 \), then there is always a unique solution of \( 57 \) which satisfies \( u_1 x_m < 0 \). This solution \( x_m \) gives the unique minimum point of \( h \). From \( 58 \), the condition \( u_1 x_m < 0 \) can be satisfied when
\[
2x_m^2 + u_2 > 0.
\] (61)

From \( h(x_m) = 0 \) and \( 59 \), we have
\[
x_m^2 = -u_2 \pm \sqrt{u_2^2 + 12u_3}. \tag{62}
\]

From \( 61 \) and \( 62 \) (and \( u_1 x < 0 \)), we have
\[
-2u_2 < \pm \sqrt{u_2^2 + 12u_3} > u_2. \tag{63}
\]

Thus the plus sign always has to be taken. When \( u_2 \geq 0 \), we have \( u_3 > 0 \). When \( u_2 < 0 \), we have \( u_3 > u_2^2/4 \). In both cases, from \( 58 \), \( u_1 \) is expressed by \( u_2 \) and \( u_3 \) as
\[
u_1 = \pm b \left( \sqrt{\frac{u_2^2 + 12u_3 - u_2}{6}} \right). \tag{64}
\]

Therefore the part of the Maxwell set for the case (b) is
\[
M_2 \setminus \{ (u_1, u_2, u_3) | u_1 = 0, u_2 < 0, u_3 = \frac{u_2^2}{4} \}, \tag{65}
\]
where
\[
M_2 = \{ (u_1, u_2, u_3) \}
\]
\[
|u_1| = \frac{\sqrt{2}}{3\sqrt{3}} \left( \sqrt{u_2^2 + 12u_3 + 2u_2} \sqrt{u_2^2 + 12u_3 - u_2} \right). \tag{66}
\]
The point \((A_3, A_1)\) is not on the boundary of the Maxwell set according to our definition above.

The Maxwell set is given by

\[ B_{\text{Maxwell}}(F) = M_1 \cup M_2. \]  

The subset \(M_2\) is diffeomorphic to a part of the bifurcation set of the swallow-tail catastrophe.

The Maxwell set around \((A_3, A_1)\) is depicted in Figure 4. The stable Maxwell set is diffeomorphic to the union of the broken swallow-tail \(\{x^4 + u_2x^2 + u_1x \text{ has exactly one real root}\}\) and the quadrant \(\{u_1 = 0, u_3 \geq u_2^2/4\}\) bounded by its line of self-intersection and its transversal. Since the boundary (the bold white line in Figure 4) is the same as that of \(A_3\), it has only one minimum point and is not contained in the Maxwell set. On surfaces around \((A_3, A_1)\) the structure is diffeomorphic to \((A_1, A_1)\) and the two minimum values are degenerate. On the vertex of the surface \((A_1, A_1, A_1)\), three minimum values are degenerate. At \((A_3, A_1)\) these three minimum points are degenerate into two minimum points.

5.5 The whole structure

We have enumerated the stable/generic local structure of the Maxwell set. Now it is easy to see that the whole of any stable/generic Maxwell set is obtained by connecting the parts in accord with the following rules. This shows which Maxwell sets with less minimum points surround the Maxwell set.

\[
\begin{array}{c c c c}
(A_3, A_1) \searrow & (A_1, A_1) \downarrow & (A_1, A_1, A_1) \swarrow \\
(A_1, A_1, A_1) \rightarrow & (A_1, A_1) \leftarrow & (A_1, A_1, A_1, A_1) \\
\end{array}
\]  

An arrow means that the structure at the origin of the arrow has the structures pointed by the arrow in the vicinity. The box means that the structure appears at the boundary of the Maxwell set and the structure is not contained in the Maxwell set. Though the structure is not a point of the Maxwell set but corresponds to a point of the endpoint set of the event horizon.

In particular, we have the following:

Proposition 3. The stable/generic Maxwell set does not contain its boundary. Any point on the boundary has the structure \(A_3\). Accordingly, no boundary point of the stable/generic endpoint set \(E\) is contained in the crease set \(C\). Any boundary point intersects a unique null generator of \(H\) so that \(H\) is differentiable there. On the stable/generic crease set, the multiplicity of null tangent is no more than four.

Here a boundary point \(q\) of the crease set \(C\) (or the endpoint set \(E\)) is defined such that there is no \(C^0\)-submanifold of codimension 2 of \(M\) which contains \(q\) and which is contained entirely in \(C\) (or \(E\)). An interior point is a point at which such a submanifold exists.

We conclude that for stable/generic horizons,

\[
C = \text{interior}(E) = \psi(B_{\text{Maxwell}}(F)),
\]

\[
D = \text{boundary}(E) = \psi(B_{\text{Maxwell}}(F) \cap B(F)),
\]

\[
E = \psi(B_{\text{Maxwell}}(F)),
\]

where \(\psi\) is the map defined in Sect. 2. The set \(B(F)\) is the bifurcation set of \(F\) which commonly appears in singularity theory. Local minimum points bifurcate on \(B(F)\).

The generic embedding of the crease set is depicted in Fig. 5. The endpoint set \(E\) of the horizon is smooth and possesses a null tangent plane at the boundary. Since all structure is two-dimensional, every generic horizon admits time slices by tori or by higher genus surfaces.

We demonstrate an example of generic horizon. The generic crease set is composed of the Maxwell sets above and an example is depicted in figure 6. They are combinations of two-dimensional segment and the boundary is always \(A_3\) which is not contained in the crease set. The topology of the spatial section of the horizon can be a torus or higher genus surface or can have many components and its crease set has a boundary with regular null tangent plane.
6 Other spacetime dimensions

So far we have assumed the spacetime is four-dimensional. Now we discuss other dimensions. By the same line of argument as we have presented, for any dimension, the equivalence of the stability and genericity holds. Up to six spacetime dimensions, all Maxwell set germs are enumerated by combination of finite elementary catastrophes. For higher dimensions, this does not hold and there will be parametrized family of stable Maxwell set germs.

In a three-dimensional spacetime, $U$ is two-dimensional. The stable Fermat potential unfoldings and the Maxwell set germs are those found in the previous sections which include at most two parameters $u_1$ and $u_2$. Then there are only two structures. One is $(A_1, A_1)$. The Maxwell set is given by

$$B_{\text{Maxwell}}(F) = \{(u_1, u_2)|u_1 = 0\}. \quad (72)$$

The other is $(A_1, A_1, A_1)$. The Maxwell set is given by

$$B_{\text{Maxwell}}(F) = \{(u_1, u_2)|u_1 = u_2 \leq 0\} \cup \{(u_1, u_2)|u_1 = 0 \leq u_2\} \cup \{(u_1, u_2)|u_2 = 0 \leq u_1\}. \quad (73)$$

The connection rule is given by

$$(A_1, A_1, A_1) \rightarrow (A_1, A_1) \leftarrow A_3. \quad (74)$$

Thus only possible structure of the endpoint set $E$ is a binary tree. In particular, the boundary of $E$ does not have an intersection with the crease set $C$ and consists of the points of multiplicity one. The multiplicity of the endpoints of the horizon does not exceed three.

In five dimensions, where $U$ is four-dimensional, the stable Maxwell set germs include the direct product of $\mathbb{R}$ and each stable Maxwell set germ in a four-dimensional spacetime. New types of the stable Fermat potential unfolding $F$ emerging in five-dimensional spacetime are the following three cases (Remark 2).

1. The unfolding $F$ is right equivalent at the minimum points to

$$(A_1, A_1, A_1, A_1) = (x_1^2 + y_1^2 + u_1, ..., x_4^2 + y_4^2 + u_4, x_5^2 + y_5^2)$$

at $((0_1, ..., 0_5), 0) \in (\mathbb{R}^2)^5 \times \mathbb{R}^3$. \quad (75)

The minimum function is

$$\mu_F(u) = \min\{u_1, u_2, u_3, u_4, 0\}. \quad (76)$$

The Maxwell set is

$$B_{\text{Maxwell}}(F) = (\bigcup_{1 \leq i < j \leq 4} M_{ij}) \cup (\bigcup_{1 \leq i \leq 4} N_i), \quad (77)$$

where

$$M_{ij} = \{(u_1, ..., u_4)|u_i = u_j \leq 0 \text{ the other } u_i\text{'s}, 0\}, \quad (78)$$

$$N_i = \{(u_1, ..., u_4)|u_i = 0 \leq 0 \text{ the other } u_i\text{'s\}. \quad (79)$$

2. The unfolding $F$ is right equivalent at the minimum points to

$$(A_3, A_1, A_1) = (x_1^2 + u_2 x_2^2 + u_1 x_1 + y_1^2 + u_3, x_2^2 + y_2^2 + u_4, x_3^2 + y_3^2)$$

at $((0_1, 0_2, 0_3), 0) \in (\mathbb{R}^2)^3 \times \mathbb{R}^3$. \quad (80)

The minimum function is given by

$$\mu_F(u) = \min\{\min h(x), u_4, 0\}. \quad (81)$$
where \( h(x) := x^4 + u_2x^2 + u_1x + u_3 \). In the region \( u_4 \geq 0 \) the Maxwell set is the same as that of \((A_3, A_1)\). In the region \( u_4 \leq 0 \) it is the same as that of \((A_3, A_1)\) but \( u_3 \) is replaced by \( u_3 - u_4 \). We have

\[
B_{\text{Maxwell}}(F) = \{(u_1, u_2, u_3, u_4) | u_1 = 0, u_2 < 0, u_3 \leq \frac{u_2^2}{4}, u_4 \geq 0\}
\]

\[
\cup \left\{ (u_1, u_2, u_3, u_4) \big| |u_1| = \sqrt{\frac{2}{3\sqrt{3}}(\sqrt{u_2^2 + 12u_3 + 2u_2})} \times \sqrt{u_2^2 + 12u_3 - u_2, u_4 \geq 0} \right\}
\]

\[
\cup \left\{ (u_1, u_2, u_3, u_4) | u_1 = 0, u_2 < 0, u_3 - u_4 \leq \frac{u_2^2}{4}, u_4 \leq 0\right\}
\]

\[
\cup \left\{ (u_1, u_2, u_3, u_4) \big| |u_1| = \sqrt{\frac{2}{3\sqrt{3}}(\sqrt{u_2^2 + 12(u_3 - u_4) + 2u_2})} \times \sqrt{u_2^2 + 12(u_3 - u_4) - u_2, u_4 \leq 0} \right\}
\]

(3) The Unfolding \( F \) is right equivalent at the minimum points to

\[
A_3(x, y, u_1, \ldots, u_4) = x^6 + u_4x^4 + u_3x^3 + u_2x^2 + u_1x + y^2
\]

at \((0_1, \ldots, 0_5, 0) \in (\mathbb{R}^2)^5 \times \mathbb{R}^3\).

The minimum function is

\[
\mu_F(u) = \min h(x),
\]

\[
h(x) = x^6 + u_4x^4 + u_3x^3 + u_2x^2 + u_1x.
\]

We obtain the Maxwell set by considering the function \( h(x) \) to have two or more global minimum points. The condition is that \( h(x) \) must be of the form

\[
h(x) = (x - \alpha)^2(x - \beta)^2(x^2 + ax + b) + c,
\]

\[
a^2 - 4b \leq 0, \quad a, b, c \in \mathbb{R}.
\]

Because \( h(0) = 0 \) and \( h(x) \) does not have a \( x^5 \)-term, we have

\[
c = -b\alpha^2\beta^2, \quad a = -2(\alpha + \beta).
\]

Thus, setting \( s = \alpha + \beta \) and \( p = \alpha\beta \), we have

\[
h(x) = (x^2 - sx + p)^2(x^2 + 2sx + b) - b\alpha^2\beta^2.
\]

From this equation we obtain a parametric expression for the Maxwell set:

\[
B_{\text{Maxwell}}(F) = \{(u_1, \ldots, u_4) | u_1 = -2bps + 2p^2s, u_2 = 2bp + p^2 + bs^2 - 4ps^2, u_3 = -2bs + 2ps + 2s^3, u_4 = b + 2p - 3s^2, 4p \leq s^2 \leq b\}.
\]

The connection rule is given by the union of the diagram \( \boxed{\text{DS}} \) and the following ones. Below only connections from the new three types of Maxwell sets are shown.

\[
(A_3, A_1) \quad \leftarrow \quad (A_3, A_1, A_1) \quad \rightarrow \quad \boxed{A_3}
\]

\[
(A_1, A_1, A_1) \quad \swarrow \quad (A_1, A_1, A_1) \quad \downarrow \quad (A_1, A_1) \quad \nearrow \quad (A_1, A_1, A_1)
\]

\[
(A_1, A_1, A_1) \quad \swarrow \quad (A_1, A_1) \quad \downarrow \quad (A_1, A_1) \quad \nearrow \quad (A_1, A_1, A_1)
\]
Again, the boundary is always $A_3$. The boundary of a generic $\mathcal{E}$ consists of points of multiplicity one. The multiplicity on $\mathcal{E}$ does not exceed five.

7 Conclusion and discussions

In this paper, we relate the crease set of the event horizon to the Maxwell set of a function unfolding through extended Fermat’s principle. We have shown equivalence of stability and genericity of the Maxwell set. We have classified the stable Maxwell set, hence the crease set of the horizon, for spacetimes of dimension 3, 4, and 5. We have enumerated the parts to construct a stable crease set and have shown how we can connect the parts. In particular, the multiplicity is always one on the boundary the endpoint set and the multiplicity of the endpoint is less than or equal to the dimension of the spacetime.

In a four-dimensional spacetime, all generic horizon is possible to be realized as a toroidal or higher genus one, since allowed structure is two-dimensional. Furthermore, the number of null generators which belong to a point of the crease set is determined. It is concluded that the generic crease set does not contain any of its boundary points.

In some cases or purposes, it may be useful or necessary to treat the stability and the genericity under some exact symmetry. Such a symmetry restricts the space of deformations of $F$ and our classification here gives only a sufficient conditions for stability or genericity in the new function space. An example is numerically generated event horizons in the study of gravitational collapse where some exact symmetry such as axial one is imposed. Another example is black hole in the brane universe scenario, where the five-dimensional spacetime has an exact $Z_2$ symmetry. On those situations, the framework of this paper would require some modifications. The classification of Maxwell sets under exact symmetries is our future problem.

Acknowledgments

We are greatly indebted to Professor Takuo Fukuda for detailed advices on mathematical concepts.

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Figure 1: An example of construction of a Fermat potential. The projection \( \pi \) from the spacetime \( \mathcal{M} \) to a Cauchy surface \( \mathcal{S} \) is defined by a timelike vector field \( T = \partial / \partial t \). A spatial section of the horizon is given by \( \mathcal{H} \cap \mathcal{S}_{t_1} = \psi(M) \), where \( M \) is a compact subset of \( \mathcal{S} \) and \( \psi = (\pi|_{\mathcal{H}})^{-1} \). The Fermat potential \( F(x, u) \) is defined by minus the supremum of \( t \) such that there is a causal curve from \( \phi(t, u) \) to \( \psi(x) \). Let \( x \) is a minimum point of \( F(\bullet, u) \) over \( M \). Then the null geodesic through \( \psi(x) \) which intersects \( \phi(t, u) \) gives a null generator of the horizon.

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Figure 2: Topology of the Maxwell sets for $(A_1,\ldots,A_1)$. For $(A_1,A_1)$ the Maxwell set germ at 0 is a surface (plane) $u_1 = 0$. For $(A_1,A_1,A_1)$, the Maxwell set is a direct product of a Y-junction and $\mathbb{R}$. In the case of $(A_1,A_1,A_1,A_1)$, the Maxwell set is intersection of six pieces of planes each of which contains an edge of the tetrahedron whose center is the origin. The numbers of minimum points at each parameter are indicated by $k$. 
Figure 3: Structure of $A_3$ and the Maxwell set. The shape of the deformation of $F(x, 0, u_1, u_2)$ in the parameter space $(u_1, u_2)$ is shown. At the point of the Maxwell set (bold line) two local minimum values are degenerate. On the cusp, one of the two local minimum points vanishes. At the top of the cusp (the origin), two local minimum points become degenerate. In each case, $k$ is the number of minimum points. The origin $A_3$ is not contained in the Maxwell set.
Figure 4: Structure of $(A_3, A_1)$. In this case, $A_3$ competes with $A_1$. They balance on the surface $M_2$, which is not smooth on $u_1 = 0, u_2 < 0$. Above the surface the Maxwell set becomes that of $A_3$.

$k$ indicates the number of minimum points.

Figure 5: A generic Maxwell set embedded into a spacetime as a crease set (the spatial dimension is suppressed to two.). Since its boundary is open, the crease set becomes null there. $k$ indicates the number of minimum points.
Figure 6: An example of generic crease set. This is composed of two \((A_3, A_1)\), which are connected to each other through \((A_1, A_1)\) and \((A_1, A_1, A_1)\). Since the boundary is that of \(A_3\) and \((A_3, A_1)\), it is open. \(k\) indicates the number of minimum points.