FINITE STATE AUTOMATA AND HOMEOMORPHISM OF
SELF-SIMILAR SETS

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ABSTRACT. The topological and metrical equivalence of fractals is an important
topic in analysis. In this paper, we use a class of finite state automata, called
Σ-automaton, to construct pseudo-metric spaces, and then apply them to the
study of classification of self-similar sets. We first introduce a notion of topology
automaton of a fractal gasket, which is a simplified version of neighbor automaton;
we show that a fractal gasket is homeomorphic to the pseudo-metric space induced
by the topology automaton. Then we construct a universal map to show that
pseudo-metric spaces induced by different automata can be bi-Lipschitz equivalent.
As an application, we obtain a rather general sufficient condition for two fractal
gaskets to be homeomorphic or Lipschitz equivalent.

1. Introduction

To determine whether two fractal sets are homeomorphic, quasi-symmetric or
Lipschitz equivalent is important in analysis. The study of homeomorphism of
fractal sets dated back to Whyburn [20]. For studies of quasi-symmetric equivalence
of fractal sets, see [4,19]. The study of Lipschitz equivalence of fractal sets derives
from 1990’s and it becomes a very active topic in recent years [5,6,9,14,15,18,22],
where most of the studies focus on self-similar sets which are totally disconnected.
For self-similar sets which are not totally disconnected, to construct homeomor-
phisms, quasi-symmetric maps or bi-Lipschitz maps is very difficult and there are
few results ( [4,16,20,23]). Whyburn [20] proved that all the Sierpinski curves
are homeomorphic, which can be applied to a class of connected fractal squares.
Solomyak [19] proved that a Julia set is always quasi-symmetric equivalent to a
planar self-similar set with two branches. Bonk and Merenkov [4] proved that the
quasi-symmetric map from Sierpinski carpet to itself must be an isometry.
There are several works devoted to the Lipschitz classification of non-totally dis-
connected fractal squares with contraction ratio 1/3, that is, a kind of Sierpinski
carpets ( [10,13,17,23]), but the problem is unsolved in case of the fractal squares
with 5 branches. Using neighbor automaton, Rao-Zhu [16] proved that $F_1 \simeq F_2$ in
Figure 1 but it is not known whether $F_j, j = 2, 3, 4, 5$ are Lipschitz equivalent or
homeomorphic.

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In this paper, we develop a more systematic theory to study the topological and metrical equivalence of self-similar sets by finite state automaton, which are mostly inspired by Rao and Zhu [16]. First, we recall the definition of finite state automaton.

**Definition 1.1** ([7]). A finite state automaton is a 5-tuple \((Q, A, \delta, q_0, P)\), where \(Q\) is a finite set of states, \(A\) is a finite input alphabet, \(q_0\) in \(Q\) is the initial state, \(P \subset Q\) is the set of final states, and \(\delta\) is the transition function mapping \(Q \times A\) to \(Q\). That is, \(\delta(q, a)\) is a state for each state \(q\) and input symbol \(a\).

Let \(\Sigma = \{1, \ldots, N\}\), which we call the set of alphabet. Inspired by the neighbor automaton of self-similar sets, we define

**Definition 1.2.** A finite state automaton \(M\) is called a \(\Sigma\)-automaton if

\[
M = (Q, \Sigma^2, \delta, Id, Exit),
\]

where the state set is \(Q = Q_0 \cup \{Id, Exit\}\), the input alphabet is \(\Sigma^2\), the initial state is \(Id\), the final state is \(Exit\), and the transition function \(\delta\) satisfies

\[
\delta(Id, (i, j)) = Id \iff i = j.
\]

By inputting symbol string \((x, y) \in \Sigma^\infty \times \Sigma^\infty\) to \(M\), we obtain a sequence of states \((S_i)_{i \geq 0}\) and call it the itinerary of \((x, y)\). If we arrive at the state \(Exit\), then we stop there and the itinerary is finite, otherwise, it is infinite. We define the surviving time of \((x, y)\) to be

\[
T_M(x, y) = (\text{length of the itinerary}) - 1.
\]

Let \(0 < \xi < 1\), we define a function \(\rho_{M, \xi}\) on \(\Sigma^\infty \times \Sigma^\infty\) as

\[
\rho_{M, \xi}(x, y) = \xi^{T_M(x, y)}.
\]

We are interested in the \(\Sigma\)-automaton such that \((\Sigma^\infty, \rho_{M, \xi})\) is a pseudo-quasimetric space for some \(0 < \xi < 1\) (see Section 2 for precise definition). In this case, we define...
\( x \sim y \) if \( \rho_{M,\xi}(x,y) = 0 \), then \( \sim \) is an equivalent relation. Set 
\[ A_M = \Sigma^\infty / \sim, \]
then \((A_M,\rho_{M,\xi})\) is a psuedo-metric space (Lemma 2.1), and we call it the psuedo-metric space induced by \( M \).

Our purpose is to use such spaces to study the homeomorphism or Lipschitz equivalence of fractal sets. (We remark that to construct bi-Lipschitz maps between two totally disconnected self-similar sets, a crucial idea is to use symbolic spaces with suitable metric as intermediate metric spaces, see [11,14,15,18,21]; in Luo and Lau [9], even a certain hyperbolic tree is used for such purpose.)

**Definition 1.3.** Two psuedo-metric spaces \((X,d_X)\) and \((Y,d_Y)\) are said to be Hölder equivalent, if there is a bijection \( f : X \to Y \), a number \( s > 0 \) and a constant \( C > 0 \) such that
\[ C^{-1}d_X(x_1,x_2)^{1/s} \leq d_Y(f(x_1),f(x_2)) \leq Cd_X(x_1,x_2)^s, \forall x_1,x_2 \in X; \]
in this case we say \( f \) is a bi-Hölder map with index \( s \).

If \( s = 1 \), we say \( X \) and \( Y \) are Lipschitz equivalent, denote by \( X \simeq Y \), and call \( f \) a bi-Lipschitz map.

In this paper, we confine ourselves to a special class of finite state automaton called the gasket automaton, where the state set \( Q \) contains eight states, see Definition 3.1 and 3.2. We show that

**Theorem 1.1.** Let \( M \) be a gasket automaton. Then \((\Sigma^\infty,\rho_{M,\xi})\) is a psuedo-metric space for every \( 0 < \xi < 1 \).

To construct bi-Hölder maps between induced psuedo-metric spaces of different automata is a difficult problem. To this end, we define a \( \gamma \)-isolated condition. Moreover, for a gasket automaton \( M \), we define the one-step simplification of \( M \) (see Section 5). The key result of this paper is the following.

**Theorem 1.2.** Let \( M \) be a gasket automaton satisfying the \( \gamma \)-isolated condition, and let \( M' \) be a one-step simplification of \( M \). Then \((A_M,\rho_{M,\xi}) \simeq (A_{M'},\rho_{M',\xi})\) for every \( \xi \in (0,1) \), i.e, they are Lipschitz equivalent.

Next, we apply the above results to study the classification of fractal gaskets defining as follows. An iterated function system (IFS) is a family of contractions \( \{\varphi_j\}_{j=1}^N \) on \( \mathbb{R}^d \), and the attractor of the IFS is the unique nonempty compact set \( K \) satisfying \( K = \bigcup_{j=1}^N \varphi_j(K) \) and it is called a self-similar set [8] if all \( \varphi_j \) are similitudes.

Let \( \Delta \subset \mathbb{R}^2 \) be the regular triangle with vertexes \((0,0), (1,0), \omega = (1/2, \sqrt{3}/2)\).

**Definition 1.4** (Fractal gasket). Let \((r_1,\ldots,r_N) \in (0,1)^N \) and \( \{d_1,\ldots,d_N\} \subset \mathbb{R}^2 \). Let \( K \) be a self-similar set generated by the IFS \( \{\varphi_j\}_{j=1}^N \) where \( \varphi_j(z) = r_j(z + d_j) \). We call \( K \) a fractal gasket if

(i) \( \bigcup_{j=1}^N \varphi_j(\Delta) \subset \Delta; \)

(ii) for any \( i \neq j \), \( \varphi_i(\Delta) \) and \( \varphi_j(\Delta) \) can only intersect at their vertices.
Notations of $\alpha, \beta, \gamma$. If $(0,0) \not\in K$, we set $\alpha = -1$, otherwise, we set $\alpha$ to be the symbol in $\Sigma = \{1, \ldots, N\}$ such that $\varphi_\alpha((0,0)) = (0,0)$. Similarly, we set $\beta = -2$ or $\varphi_\beta$ is the map with fixed point $(1,0)$, and set $\gamma = -3$ or $\varphi_\gamma$ is the map with fixed point $\omega$. Hereafter, we will denote $(0,0), (1,0)$ and $\omega$ by $\omega_\alpha, \omega_\beta$ and $\omega_\gamma$, respectively.

For a fractal gasket $K$, we introduce a notion of topology automaton of $K$, which we denote by $M_K$, in Section 4. Comparing to the neighbor automaton of self-similar sets, the topology automaton records less information since the information of size is ignored.

Denote $r_* = \min\{r_1, \ldots, r_N\}$ and $r^* = \max\{r_1, \ldots, r_N\}$. Let $\pi : \Sigma^\infty \to K$ be the well-known coding map, then $\pi$ induces a bijection from $A_{M_K}$ to $K$ which we still denote by $\pi$ (see Section 4 for details). Actually, $\pi$ is a bi-Hölder map.

**Theorem 1.3.** Let $K$ be a fractal gasket. Let $s = \sqrt{\log r^*/\log r_*}$ and $\xi = (r_*)^s$. Then $\pi : (A_{M_K}, \rho_{M_K, \xi}) \to K$ is a bi-Hölder map with index $s$.

**Remark 1.1.** We remark that if two of $\{\omega_\alpha, \omega_\beta, \omega_\gamma\}$ do not belong to $K$, then $\varphi_i(K) \cap \varphi_j(K) = \emptyset$ whenever $i \neq j$. In this case $K$ is totally disconnected and we will deal with it in another paper. From now on, we will always assume that $\omega_\alpha, \omega_\beta \in K$ without loss of generality, or equivalently, $\alpha, \beta \in \Sigma$.

A fractal gasket $K$ is said to satisfy the top isolated condition, if $\omega_\gamma \in K$, and $\varphi_\gamma(\Delta) \cap \varphi_j(\Delta) = \emptyset$ provided $j \neq \gamma$. This condition corresponds to the $\gamma$-isolated condition in Theorem 1.2.

We remark that if a fractal gasket satisfies the top isolated condition, then its non-trivial connected components are horizontal line segments. See Appendix A.

**Definition 1.5** (Horizontal block). Let $K$ be a fractal gasket. We call

$$I = \{i_1, i_2, \ldots, i_k\} \subset \Sigma$$

a horizontal block of $K$ if $\varphi_{i_j}(\omega_\beta) = \varphi_{i_{j+1}}(\omega_\alpha)$ for $1 \leq j \leq k - 1$ and $I$ is maximal with this property. We call $k$ the size of $I$.

If $\alpha$ and $\beta$ belong to the same horizontal block, then we call this block the $\alpha \beta$-block.

If $K$ satisfies the top isolated condition or $\omega_\gamma \not\in K$, and $\alpha$ and $\beta$ does not belong the same horizontal block, then $K$ is totally disconnected, and it is out of our consideration. Let $\mathcal{F}_{T,\alpha \beta}$ denote the collection of fractal gasket $K$ satisfying

(i) $K$ satisfies the top isolated condition or $\omega_\gamma \not\in K$;

(ii) $\alpha$ and $\beta$ belong to a same horizontal block of $K$.

**Theorem 1.4.** Let $E, F \in \mathcal{F}_{T,\alpha \beta}$. If there is a size-preserving bijection from the collection of horizontal-blocks of $E$ to that of $F$, and the $\alpha \beta$-block of $E$ have equal size with that of $F$, then $E$ is bi-Hölder equivalent (and homeomorphic) to $F$.

If in addition that both $E$ and $F$ are of uniform contraction ratio $r$, then $E \simeq F$.

Let us briefly explain the strategy of the proof of Theorem 1.4. Let $E$ and $F$ be two fractal gaskets in Theorem 1.4. Let $M_E$ be the topology automaton of $E$. We construct a sequence of automata

$$M_E = M_{E,0}, M_{E,1}, \ldots, M_{E,p} = M_E^*$$
such that $M_{E,i+1}$ is a one-step simplification of $M_{E,i}$ for each $i$, and $M^*_E$ only records the horizontal connective relations among $\varphi_j(\Delta), j \in \Sigma$. By Theorem 1.2, we have $A_{M_E} \simeq A_{M^*_E}$. We do the same thing for $F$ and we obtain an automaton $M^*_F$. The assumptions in Theorem 1.4 guarantee that $A_{M^*_E} \simeq A_{M^*_F}$. Therefore, $A_{M_E} \simeq A_{M_F}$.

Finally, by Theorem 1.3, we obtain that $E$ is bi-Hölder equivalent to $F$.

**Example 1.1.** Let $E$ and $F$ be two fractal gaskets indicated by Figure 2. There are four horizontal-blocks in both $E$ and $F$, and the sizes of the blocks are 1, 2, 3, 5 respectively. By Theorem 1.4, $E$ is homeomorphic to $F$. (In the fractal $E$, there are 3 points connecting $\varphi_j(\Delta)$ in different horizontal blocks, so we need 3 one-step simplifications to obtain $M^*_E$.)

**Open question:** Can we replace the $\gamma$-isolated condition by the following condition: $\varphi_\alpha(\Delta)$ and $\varphi_\beta(\Delta)$ belong to a same connected component of $\bigcup_{j=1}^N \varphi_j(\Delta)$, but $\varphi_\gamma(\Delta)$ does not?

This article is organized as follows: In Section 2, we discuss the psuedo-metric space induced by a finite state automaton. In Section 3, we introduce the gasket automaton, and Theorem 1.1 is proved there. In Section 4, we define the topology automaton of a fractal gasket, and Theorem 1.3 is proved there. In Section 5, we discuss the one-step simplification of a gasket automaton. In Section 6, we prove Theorem 1.2 and Theorem 1.4 by assuming Theorem 5.1. In Section 7, we construct a universal map $g$ on the symbolic space $\Omega$. Finally, in Section 8, we prove Theorem 5.1 which is technical.

**2. Psuedo-metric space induced by $\Sigma$-automaton**

Let us recall the definition of psuedo-metric space, see for instance [3,12].

**Definition 2.1.** A pseudo-quasimetric space is a pair $(\mathcal{A}, \rho)$ where $\mathcal{A}$ is a set and $\rho: \mathcal{A} \times \mathcal{A} \to \mathbb{R}_{\geq 0}$ satisfying for all $x, y, z \in \mathcal{A}$, it holds that

(i) $\rho(x, x) = 0$;
(ii) $\rho(x, y) = \rho(y, x)$;
(iii) (psuedo-triangle inequality) $\rho(x, z) \leq C(\rho(x, y) + \rho(y, z))$, where $C \geq 1$ is a constant independent of $x, y, z$. 

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**Figure 2.** Fractal gaskets in Example 1.1. $E$ is homeomorphic to $F$. 

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If in addition $x \neq y$ implies $\rho(x, y) > 0$, then we call $(\mathcal{A}, \rho)$ a psuedo-metric space.

Let $(\mathcal{A}, \rho)$ be a psuedo-quasimetric space. Define $x \sim y$ if $\rho(x, y) = 0$, then clearly $\sim$ is an equivalence relation. Denote the equivalent class containing $x$ by $[x]$. Set $\widehat{\mathcal{A}} := \mathcal{A}/ \sim$ to be the quotient space. For $[x], [y] \in \widehat{\mathcal{A}}$, define

$$\rho([x], [y]) = \inf \{ \rho(a, b); a \in [x], b \in [y] \}. $$

**Lemma 2.1.** The quotient space $(\widehat{\mathcal{A}}, \rho)$ is a psuedo-metric space.

*Proof.* The assertions $\rho([x], [x]) = 0$ and $\rho([x], [y]) = \rho([y], [x])$ are obvious.

If $a, a' \in [x]$ and $b, b' \in [y]$, by the psuedo-triangle inequality, one can show that $\rho(a, b) \leq C^2 \rho(a', b')$. Hence, if $a \in [x]$ and $b \in [y]$, we have

$$C^{-2} \rho(a, b) \leq \rho([x], [y]).$$

It follows that $\rho([x], [y]) > 0$ if $[x] \neq [y]$, and

$$\rho([x], [z]) \leq \rho(x, z) \leq C(\rho(x, y) + \rho(y, z)) \leq C^3 (\rho([x], [y]) + \rho([y], [z])).$$

The lemma is proved. □

Let $(\mathcal{A}, \rho)$ be a psuedo-metric space. In the same manner as the metric space, we can define convergence of sequence, dense subset and completeness of $\mathcal{A}$. (See [3,12].) The following lemma is obvious.

**Lemma 2.2.** Let $(\mathcal{A}, \rho)$ and $(\mathcal{A}', \rho')$ be two complete psuedo-metric spaces. Suppose $B \subset \mathcal{A}$ is $\rho$-dense in $\mathcal{A}$ and $B' \subset \mathcal{A}'$ is $\rho'$-dense in $\mathcal{A}'$. If $B \simeq B'$, then $\mathcal{A} \simeq \mathcal{A}'$.

Let $\Sigma = \{1, \ldots, N\}$. For $a \in \Sigma$, we use $a^k$ to denote the word consisting of $k$ numbers of $a$. Let $\Sigma^\infty$ and $\Sigma^k$ be the sets of infinite words and words of length $k$ over $\Sigma$ respectively. Let $\Sigma^* = \bigcup_{k \geq 0} \Sigma^k$.

Let $M$ be a $\Sigma$-automaton. If $(\Sigma^\infty, \rho_M)$ is a psuedo-quasimetric space, then we can define a psuedo-metric space by (2.1), which we denote by $(\mathcal{A}_M, \rho_M)$. Denote by $x \land y$ the maximal common prefix of $x$ and $y$. By (1.2), we see that

$$T_M(x, y) \geq |x \land y|,$$

where $|W|$ denotes the length of a word $W$.

**Lemma 2.3.** If $(\Sigma^\infty, \rho_M)$ is a psuedo-quasimetric space, then the induced psuedo-metric space $(\mathcal{A}_M, \rho_M)$ is complete.

*Proof.* We equip $\Sigma^\infty$ with the following metric: For any $x, y \in \Sigma^\infty$, define $d(x, y) = 2^{-|x \land y|}$. It is folklore that $(\Sigma^\infty, d)$ is a compact metric space.

For any Cauchy sequence $\{x_k\}_{k=1}^\infty$ of $\mathcal{A}_M$, let $\{x_{kp}\}_{p=1}^\infty$ be a subsequence of $\{x_k\}_{k=1}^\infty$ which converges to $y \in \Sigma^\infty$ in the metric $d$. Then $\lim_{p \to \infty} |x_{kp} \land y| = +\infty$, so we have $[x_{kp}] \to [y]$ with respect to $\rho_M$. Since $\{x_k\}_{k=1}^\infty$ is Cauchy in $\rho_M$, we conclude that $[x_k] \to [y]$ in $\rho_M$. The lemma is proved. □

**Lemma 2.4.** Suppose $(\mathcal{A}_M, \rho_M)$ is a psuedo-metric space. Let $\kappa \in \Sigma$. Then the set $\Omega = \{[\omega \kappa^\infty]; \omega \in \Sigma^*\}$ is $\rho_M$-dense in $\mathcal{A}_M$.

*Proof.* Pick $[x] \in \mathcal{A}_M$, denote $x = (x_i)_{i=1}^\infty$ and let $x_k = x_1 \ldots x_k \kappa^\infty$, then $[x_k] \in \Omega$. Clearly, $[x_k] \to [x]$. This finishes the proof. □
3. Triangle automaton and gasket automaton

In this section, we introduce the triangle automaton and gasket automaton.

3.1. Triangle automaton.

Let $\Sigma = \{1, \ldots, N\}$. Let $\{\alpha, \beta, \gamma\}$ be a subset of $\Sigma \cup \{-1, -2, -3\}$. For a pair $u, v \in \{\alpha, \beta, \gamma\}$ with $u \neq v$, we associate with it a state and denote it by $S_{uv}$. We set the state set $Q$ to be

\begin{equation}
Q = \{S_{\alpha\gamma}, S_{\beta\gamma}, S_{\alpha\beta}, S_{\beta\alpha}, S_{\gamma\beta}, S_{\gamma\alpha}\} \cup \{\text{Id}, \text{Exit}\}.
\end{equation}

**Definition 3.1 (Triangle automaton).** A $\Sigma$-automaton $M = \{Q, \Sigma^2, \delta, \text{Id}, \text{Exit}\}$ is called a triangle automaton if $Q$ is given by (3.1), and the transition function $\delta$ satisfies the following conditions: Let $u, v \in \{\alpha, \beta, \gamma\}$ and $u \neq v$.

(i) (Symmetry) If $\delta(\text{Id}, (i, j)) = S_{uv}$, then $\delta(\text{Id}, (j, i)) = S_{vu}$.

(ii) (Loop) $\delta(S_{uv}, (i, j)) = \begin{cases} S_{uv}, & \text{if } (i, j) = (v, u); \\ \text{Exit}, & \text{otherwise}. \end{cases}$

Let us call $S_{uv}$ the mirror state of $S_{vu}$, and let the mirror state of $\text{Id}$ and $\text{Exit}$ to be themselves. Clearly, if $S$ is translated to $S'$ by $(i, j)$, then the mirror state of $S$ is translated to the mirror state of $S'$ by $(j, i)$. Therefore, we have

**Lemma 3.1.** Let $(S_k)_{k \geq 1}$ and $(S'_k)_{k \geq 1}$ be the itineraries of $(x, y)$ and $(y, x)$, respectively. Then $S'_k$ is the mirror state of $S_k$. Consequently, $T_M(x, y) = T_M(y, x)$.

We denote

$$ P_{uv} = \{(i, j) \in \Sigma^2; \delta(\text{Id}, (i, j)) = S_{uv}\}. $$

Then by symmetry of the automaton, $P_{vu} = \{(i, j); (j, i) \in P_{uv}\}$. The transition diagram of a triangle automaton $M$ is illustrated in Figure 3. Clearly, $M$ is completely determined by $P_{\alpha\beta}, P_{\alpha\gamma}, P_{\beta\gamma}$.

![Figure 3. The transition diagram of triangle automaton $M$.](image)

If $(i, j) \in P_{\alpha\gamma}$, then we denote $i \prec_{\alpha\gamma} j$ and say $i$ is the $\alpha\gamma$-predecessor of $j$ and $j$ is the $\alpha\gamma$-successor of $i$; similarly, we define $i \prec_{\beta\gamma} j$, $i \prec_{\alpha\beta} j$. Moreover, according to Definition 3.1(i), we make the convention that $i \prec_{uv} j$ if and only if $j \prec_{vu} i$. 


3.2. Gasket automaton.

Firstly, we recall some notions of graph theory, see [1]. Let \( G = (V, E) \) be a directed graph, where \( V \) is the vertex set and \( E \) is the edge set. Each edge \( e \) is associated to an ordered pair \((u, v)\) in \( V \), we say \( e \) is incident out of \( u \) and incident into \( v \). The number of edges incident out of a vertex \( v \) is the outdegree of \( v \) and is denoted by \( \deg^+(v) \). The number of edges incident into a vertex \( v \) is the indegree of \( v \) and is denoted by \( \deg^-(v) \). If \( \deg^-(v) = 0 \), then we say \( v \) is minimal; if \( \deg^+(v) = 0 \), then we say \( v \) is maximal. If \( v \) is both minimal and maximal, then we say \( v \) is isolated.

A directed walk joining vertex \( v_1 \) to vertex \( v_k \) in \( G \) is a sequence \((v_1, v_2, \ldots, v_k)\) with \((v_i, v_{i+1}) \in E\). In addition, if all \( v_i (1 \leq i \leq k) \) are distinct, then we call it a path. If all \( v_i (1 \leq i \leq k-1) \) are distinct and \( v_k = v_1 \), then we call it a cycle. A path \((v_1, v_2, \ldots, v_k)\) is called a chain, if \( v_1 \) is minimal and \( v_k \) is maximal.

For a triangle automaton \( M \), we will regard \((\Sigma, \mathcal{P}_{\alpha\gamma}), (\Sigma, \mathcal{P}_{\beta\gamma})\) and \((\Sigma, \mathcal{P}_{\alpha\beta})\) as three graphs. A symbol \( j \in \Sigma \) is said to be \( \alpha\beta \)-minimal (resp. maximal) if it is minimal (resp. maximal) in \((\Sigma, \mathcal{P}_{\alpha\beta})\). Similarly, we can define \( \alpha\gamma \)-minimal (maximal) and \( \beta\gamma \)-minimal (maximal). Moreover, we make the convention that \( j \) is \( uv \)-minimal if it is \( vu \)-maximal.

**Definition 3.2** (Gasket automaton). A triangle automaton \( M = \{Q, \Sigma^2, \delta, Id, Exit\} \) is called a gasket automaton if \( \delta \) satisfies the following conditions: Let \( u, v \in \{\alpha, \beta, \gamma\} \) and \( u \neq v \).

(i) (Uniqueness) If \( i <_{uv} j \) and \( i <_{uv} j' \), then \( j = j' \).

(ii) (Gathering condition) Any two of the following statements imply the third one: ① \( a <_{\alpha\gamma} c \); ② \( a <_{\beta\gamma} b \); ③ \( b <_{\alpha\beta} c \).

(iii) (Boundary condition) If \( \alpha \in \Sigma \), then \( \alpha \) is \( \alpha\gamma \)-minimal and \( \alpha\beta \)-minimal; similarly, if \( \beta \in \Sigma \), then \( \beta \) is \( \beta\gamma \)-minimal and \( \beta\alpha \)-minimal; if \( \gamma \in \Sigma \), then \( \gamma \) is \( \gamma\alpha \)-minimal and \( \gamma\beta \)-minimal.

By the uniqueness property, we see that for any \( u, v \in \{\alpha, \beta, \gamma\} \), the graph \((\Sigma, \mathcal{P}_{uv})\) is a union of disjoint chains and cycles.

![Figure 4. An illustration of the boundary condition: the six positions with crosses are forbidden.](image-url)
Remark 3.1. Let $K$ be a fractal gasket. If $\varphi_a(\omega_\gamma) = \varphi_b(\omega_\beta) = \varphi_c(\omega_\alpha)$, then $\varphi_a(\triangle)$, $\varphi_b(\triangle)$ and $\varphi_c(\triangle)$ gather at one point. This is the motivation of gathering condition.

3.3. Induced pseudo metric space.

Let $M$ be a triangle automaton. For $(x, y) \in \Sigma^\infty \times \Sigma^\infty$, let $(S_{M,i})_{i \geq 0}$ be the itinerary of $(x, y)$; recall that the surviving time $T_M(x, y)$ is the largest $k$ such that $S_{M,k} \neq \text{Exit}$.

We will use $S \xrightarrow{(i,j)} S'$ as an alternative notation for $\delta(S, (i,j)) = S'$, and denote the initial state by $id$ instead of $Id$ for clarity.

Proposition 3.1. Let $M$ be a gasket automaton. For any $x, y, z \in \Sigma^\infty$ we have

$$\min\{T_M(x, y), T_M(x, z)\} \leq T_M(y, z) + 1. \tag{3.2}$$

Proof. We will denote $T := T_M$ for simplicity. Clearly, (3.2) holds if either $T(y, z) = \infty$ or any two of $x, y, z$ are equal. So we assume that $T(y, z) < \infty$ and $x, y, z$ are distinct.

Denote $\ell := |y \land z|$. Then $T(y, z) \geq \ell$, and at least one of $|x \land y| \leq \ell$ and $|x \land z| \leq \ell$ holds. Without loss of generality, assume that

$$k = |x \land y| \leq |x \land z|.$$

Suppose on the contrary that (3.2) is false, then $T(x, y) > \ell + 1$ and $T(x, z) > \ell + 1$.

Case 1. $k < \ell$.

In this case we have $|x \land y| = |x \land z| = k$, which together with $T(x, y) > \ell + 1$ imply that the first $(\ell + 3)$-states (including $id$) of the itinerary of $(x, y)$ are

$$id \rightarrow (Id)^k \rightarrow (S_{uv})^{\ell+2-k}, \text{ where } u, v \in \{\alpha, \beta, \gamma\}. \tag{3.3}$$

So $(x_{k+1}, y_{k+1}) \in P_{uv}$ and

$$(\sigma^k(x), \sigma^k(y)) = (x_{k+1}u^{\ell+1-k} \cdots, y_{k+1}u^{\ell+1-k} \cdots).$$

By $|y \land z| = \ell$, the first $\ell + 1$ states (including $id$) of the itinerary of $(x, z)$ are the same as that of $(x, y)$; in particular, the $(\ell + 1)$-th state is $S_{uv}$. Moreover, since $T(x, z) > \ell + 1$, a prefix of the itinerary of $(x, z)$ is also given by (3.3). It follows that $\sigma^k(z) = y_{k+1}u^{\ell+1-k} \cdots$ and $|y \land z| \geq \ell + 2$, which is a contradiction. Hence (3.2) holds in this case.

Case 2. $k = \ell = |x \land z|$.

Then $x_1 \ldots x_k = y_1 \ldots y_k = z_1 \ldots z_k$ and $x_{k+1}, y_{k+1}$ and $z_{k+1}$ are distinct. So the itinerary of $(x, y)$ is initialised by

$$id \rightarrow (Id)^k \rightarrow (S_{uv})^2, \text{ where } u, v \in \{\alpha, \beta, \gamma\}. \tag{3.4}$$

and the itinerary of $(x, z)$ is initialised by

$$id \rightarrow (Id)^k \rightarrow (S_{uv'})^2, \text{ where } w, v' \in \{\alpha, \beta, \gamma\}. \tag{3.5}$$

The $(k+2)$-th transitions of (3.4) and (3.5) imply $(x_{k+2}, y_{k+2}) = (v, u)$ and $(x_{k+2}, z_{k+2}) = (v', w)$, and it follows that $w = u'$. The $(k+1)$-th transitions imply $x_{k+1} <_{uv} y_{k+1}$ and $x_{k+1} <_{uv} z_{k+1}$, then $w \neq u$ by the uniqueness property, and $y_{k+1} <_{uv} z_{k+1}$ by the gathering condition.
Let $p$ be the largest integer such that
\[(\sigma^k(x), \sigma^k(y), \sigma^k(z)) = (x_{k+1} v^p, \ldots, y_{k+1} u^p, \ldots, z_{k+1} u^p, \ldots).\]

Then all of $T(x, y), T(y, z), T(x, z)$ are no less than $k + p + 1$, and two of them are no larger than $k + p + 2$ since $(x_{k+p+2}, y_{k+p+2}, z_{k+p+2}) \neq (v, u, w)$. The lemma holds in this case.

**Case 3.** $k = \ell < |x \land z|$.

Since $x_{k+1} \neq y_{k+1}$, equation (3.4) still holds. Let $p$ be the largest integer such that
\[(\sigma^k(x), \sigma^k(y)) = (x_{k+1} v^p, \ldots, y_{k+1} u^p, \ldots).\]

Then $T(x, y) = k + 1 + p$.

Let $q$ be the largest integer such that $\sigma^k(z) = x_{k+1} v^q, \ldots$. If $q \geq p - 1$, then $T(y, z) \geq k + p$ since $|x \land z| \geq k + p$, and the lemma holds in this case. If $q \leq p - 2$, then $T(y, z) = k + q + 1$ and
\[(\sigma^{k+q+1}(x), \sigma^{k+q+1}(z)) = (v^2, \ldots, v, \eta, \ldots), \text{ where } \tilde{v} \neq v.\]

Suppose $Id$ is not translated to $Exit$ by $(v, \tilde{v})$, say, $Id \xrightarrow{(v, \tilde{v})} S$. If $S = S_{wav}$ for some $w \in \{\alpha, \beta, \gamma\}$, then $\tilde{v} <_{wav} v$, which violates the boundary condition. Thus $S \xrightarrow{(v, \eta)} Exit$ and $T(x, z) \leq k + q + 2$. The lemma holds in this scenario. This finishes the proof of (3.2). \hfill \Box

Now we can prove Theorem 1.1

**Proof of Theorem 1.1.** Let $x, y, z \in \Sigma^\infty$. Denote $\rho := \rho_{M, \xi}$.

First, it is obvious that $T_M(x, x) = \infty$, so $\rho(x, x) = 0$.

Secondly, $T_M(x, y) = T_M(y, x)$ by Lemma 3.1, so $\rho(x, y) = \rho(y, x)$.

Thirdly, by Proposition 3.1, we have
\[\rho(x, y) + \rho(y, z) = \xi^{T_M(x, y)} + \xi^{T_M(y, z)} \geq \xi^{T_M(x, z) + 1} = \xi \rho(x, z).\]

The theorem is proved. \hfill \Box

Hence a gasket automaton $M$ induces a psuedo metric space, which we denote by $(\mathcal{A}_M, \rho_M)$. By (2.2) and (2.3), we have
\[\rho_M(x, y) \leq \xi^{-2} \rho_M([x], [y]) \tag{3.6}\]
and
\[\rho_M([x], [z]) \leq \xi^{-3} (\rho_M([x], [y]) + \rho_M([y], [z])). \tag{3.7}\]

4. **Topology automaton of fractal gasket**

The neighbor graph (automaton) of self-similar sets is an important tool in fractal geometry, see [2, 16, 23]. The topology automaton is a simplified version of the neighbor automaton.
Definition 4.1 (Topology automaton). Let \( K \) be a fractal gasket generated by \( \{\varphi_j\}_{j=1}^N \). Let \( \{\alpha, \beta, \gamma\} \) be a subset of \( \Sigma \cup \{-1, -2, -3\} \) defined in Section 1. Let \( M_K \) be a triangle automaton satisfying: For \( i \neq j \),
\[
\delta(Id, (i, j)) = \begin{cases} 
S_{uv}, & \text{if } u, v \in \Sigma \text{ and } \varphi_i(\omega_u) = \varphi_j(\omega_u), \\
Exit, & \text{if } \varphi_i(K) \cap \varphi_j(K) = \emptyset.
\end{cases}
\]
We call \( M_K \) the topology automaton of \( K \).

Lemma 4.1. The topology automaton of a fractal gasket is always a gasket automaton.

Proof. Let \( K \) be a fractal gasket generated by \( \{\varphi_j\}_{j=1}^N \). Since the functions \( \varphi_i \) are all distinct, \( \varphi_i(K) \) can has at most one neighbor in \( \theta \)-direction for each \( \theta \in \{\exp(2\pi i k/6); k = 0, 1, \ldots, 5\} \). This verifies the uniqueness property of the gasket automaton. The gathering condition and the boundary condition are obvious. \( \square \)

For \( A, B \subseteq \mathbb{R}^2 \), let \( \text{dist}(A, B) = \min\{\|a - b\|; a \in A, b \in B\} \). Zhu and Yang \cite{23} defined the sharp separation condition for self-similar sets with uniform contraction ratio. We extend it to general self-similar sets.

Definition 4.2 (Sharp separation condition). A self-similar set \( K \) is said to satisfy the sharp separation condition, if there exists a constant \( C' > 0 \) such that for any \( k \geq 1 \) and \( I, J \in \Sigma^k \), \( \varphi_I(K) \cap \varphi_J(K) = \emptyset \) implies that
\[
\text{dist}(\varphi_I(K), \varphi_J(K)) \geq C' \min\{\text{diam } \varphi_I(K), \text{diam } \varphi_J(K)\}.
\]

Lemma 4.2. A fractal gasket always satisfies the sharp separation condition.

Proof. Let \( K \) be a fractal gasket with IFS \( \Phi = \{\varphi_i\}_{i=1}^N \). Denote \( K_i = \varphi_i(K) \) for \( 1 \leq i \leq N \). Let
\[
C_1 = \min\{\text{dist}(K_i, K_j); i, j \in \Sigma \text{ and } K_i \cap K_j = \emptyset\};
\]
\[
C_2 = \min\{\text{dist}(K_i, z); i \in \Sigma, z \in \{\omega_\alpha, \omega_\beta, \omega_\gamma\} \text{ and } z \notin K_i\}.
\]
For any \( I = x_1 \ldots x_k, J = y_1 \ldots y_k \in \Sigma^k \), suppose that \( \varphi_I(K) \cap \varphi_J(K) = \emptyset \). Let \( 0 \leq \ell \leq k - 1 \) be the largest integer such that \( \varphi_{x_1 \ldots x_{\ell}}(K) \cap \varphi_{y_1 \ldots y_{\ell}}(K) \neq \emptyset \).
If \( x_1 \ldots x_\ell = y_1 \ldots y_\ell \), then
\[
\text{dist}(\varphi_I(K), \varphi_J(K)) = r_{x_1 \ldots x_\ell} \text{dist}(K_{x_{\ell+1}}, K_{y_{\ell+1}}) \geq \frac{C_1}{\text{diam}(K)} \text{diam}(\varphi_I(K)).
\]
If \( x_1 \ldots x_\ell \neq y_1 \ldots y_\ell \), let \( z_0 \) be the unique element of \( \varphi_{x_1 \ldots x_{\ell}}(K) \cap \varphi_{y_1 \ldots y_{\ell}}(K) \). Let us assume that \( z_0 \notin \varphi_{x_1 \ldots x_{\ell+1}}(K) \) without loss of generality. Then
\[
\text{dist}(\varphi_I(K), \varphi_J(K)) \geq \text{dist}(\varphi_{x_1 \ldots x_{\ell+1}}(K), z_0) \geq \frac{C_2}{\text{diam}(K)} \text{diam}(\varphi_I(K)).
\]
The lemma is proved. \( \square \)

Define \( \pi : \Sigma^\infty \to K \), which we call the coding map, by
\[
\{\pi(x)\} = \bigcap_{i \geq 1} \varphi_{x_1 \ldots x_i}(K).
\]
If \( \pi(x) = x \in K \), then the sequence \( x \) is called a coding of \( x \).
Proof of Theorem 1.3. Take \(x, y \in K\). Let \(x\) and \(y\) be a coding of \(x\) and \(y\), respectively. Let \(k = T(x, y)\) be the surviving time in the topology automaton \(M_K\). Then the \(k\)-th cylinders containing \(x\) and that containing \(y\) either coincide or have non-empty intersection. It follows that
\[
\|x - y\| \leq 2(r^*)^k.
\]
On the other hand, the \((k + 1)\)-th cylinders containing \(x\) and that containing \(y\) are disjoint, so by the sharp separation condition, we have
\[
\|x - y\| \geq C'(r^* )^{k+1}
\]
where \(C'\) is the constant in the sharp separation condition. Recall that \(s = \sqrt{\log r^*/\log r_*}\) and \(\xi = (r_*)^s\). So
\[
\rho_{M_K, \xi}(x, y) = \xi^k = (r_*)^{sk} = (r_*)^{k/s}.
\]
Set \(C = \max\{2, 1/(r_* C')\}\), we obtain the theorem. \(\Box\)

5. \(\gamma\)-isolated gasket automaton and simplification

In this section we imposing additional conditions to the gasket automaton so that the automaton can be simplified.

Definition 5.1 (\(\gamma\)-isolated condition). Let \(M\) be a gasket automaton. We say \(M\) satisfies the \(\gamma\)-isolated condition if
(i) \(\{\alpha, \beta, \gamma\} \subset \Sigma\);
(ii) The graph \((\Sigma, \mathcal{P}_{\alpha\gamma} \cup \mathcal{P}_{\beta\gamma})\) has no cycle;
(iii) \(\gamma\) is isolated, in the sense that it is \(\alpha\gamma\)-isolated, \(\beta\gamma\)-isolated and \(\alpha\beta\)-isolated.

Lemma 5.1. If \(K\) is a fractal gasket such that \(\{\omega_\alpha, \omega_\beta, \omega_\gamma\} \subset K\) and satisfying the top isolated condition, then the topology automaton \(M_K\) satisfies the \(\gamma\)-isolated condition.

Proof. Let \(K\) be a fractal gasket with IFS \(\Phi = \{\varphi_i\}_{i=1}^N\). If \(i <_\alpha \gamma j\), then \(\varphi_j(\omega_\alpha) = \varphi_i(\omega_\gamma)\); denote by \(c\) the contraction ratio of \(\varphi_i\), we have
\[
\varphi_j(\omega_\alpha) - \varphi_i(\omega_\alpha) = \varphi_i(\omega_\gamma) - \varphi_i(\omega_\alpha) = c(\omega_\gamma - \omega_\alpha),
\]
and it follows that \(\varphi_j(0, 0)\) has larger second coordinate than that of \(\varphi_i(0, 0)\). The same conclusion holds if \(i <_\beta \gamma j\). This verifies (ii) in Definition 3.1. Clearly the top isolated condition implies that \(\gamma\) is isolated. \(\Box\)

In the rest of this section, we always assume that \(M\) is a gasket automaton satisfying the \(\gamma\)-isolated condition such that
\[
\mathcal{P}_{\alpha\gamma} \cup \mathcal{P}_{\beta\gamma} \neq \emptyset.
\]
For any \(b \in \Sigma\), we say \(b\) is double-maximal in \(M\) if \(b\) is both \(\alpha\gamma\)-maximal and \(\beta\gamma\)-maximal.

Lemma 5.2. There exists \((\tau, \kappa) \in \mathcal{P}_{\alpha\gamma} \cup \mathcal{P}_{\beta\gamma}\) such that \(\kappa\) is double-maximal.

Proof. Since \(\mathcal{P}_{\alpha\gamma} \cup \mathcal{P}_{\beta\gamma} \neq \emptyset\), there exists \((b_1, b_2) \in \mathcal{P}_{\alpha\gamma} \cup \mathcal{P}_{\beta\gamma}\). Let \((a_1, \ldots, a_k)\) be the chain in \(\mathcal{P}_{\alpha\gamma} \cup \mathcal{P}_{\beta\gamma}\) containing \((b_1, b_2)\), then \((\tau, \kappa) = (a_{k-1}, a_k)\) is the desired edge. \(\Box\)
From now on, we fix a pair \((\tau, \kappa)\) satisfying Lemma 5.2. Moreover, we assume that \((\tau, \kappa) \in P_{\alpha \gamma}\) without loss of generality.

If \(\kappa\) has no \(\alpha \beta\)-predecessor, we set
\[
(5.2) \quad P'_{\alpha \beta} = P_{\alpha \beta}, \quad P'_{\alpha \gamma} = P_{\alpha \gamma} \setminus \{ (\tau, \kappa) \}, \quad \text{and} \quad P'_{\beta \gamma} = P_{\beta \gamma}.
\]

If \(\kappa\) has a \(\alpha \beta\)-predecessor, we denote it by \(\lambda\) and set
\[
(5.3) \quad P'_{\alpha \beta} = P_{\alpha \beta}, \quad P'_{\alpha \gamma} = P_{\alpha \gamma} \setminus \{ (\tau, \kappa) \}, \quad \text{and} \quad P'_{\beta \gamma} = P_{\beta \gamma} \setminus \{ (\tau, \lambda) \}.
\]

Let \(M'\) be the triangle automaton determined by \(P'_{\alpha \beta}\), \(P'_{\alpha \gamma}\) and \(P'_{\beta \gamma}\), and we call it a one-step simplification of \(M\). If (5.2) holds, we call \(M'\) a \((\tau, \kappa)\)-simplification, otherwise, we call \(M'\) a \((\tau, \kappa, \lambda)\)-simplification.

\[
\begin{array}{c}
\text{(a)} \\
\begin{array}{c}
\bigcirc \kappa \\
\bigcirc \tau \\
\bigcirc \eta
\end{array} \\
(\Rightarrow) \\
\begin{array}{c}
\bigcirc \kappa \\
\bigcirc \tau \\
\bigcirc \eta
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\text{(b)} \\
\begin{array}{c}
\bigcirc \kappa \\
\bigcirc \lambda \\
\bigcirc \eta
\end{array} \\
(\Rightarrow) \\
\begin{array}{c}
\bigcirc \lambda \\
\bigcirc \eta
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\text{(c)} \\
\begin{array}{c}
\bigcirc \tau \\
\bigcirc \lambda \\
\bigcirc \eta
\end{array} \\
(\Rightarrow) \\
\begin{array}{c}
\bigcirc \tau \\
\bigcirc \eta
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\text{(d)} \\
\begin{array}{c}
\bigcirc \kappa \\
\bigcirc \tau \\
\bigcirc \eta
\end{array} \\
\begin{array}{c}
\bigcirc \kappa \\
\bigcirc \lambda \\
\bigcirc \eta
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\text{\(a_{\alpha \beta}\):} \\
\text{\(a_{\alpha \gamma}\):} \\
\text{\(a_{\beta \gamma}\):}
\end{array}
\]

**Figure 5.** \((a) \Rightarrow (b)\) illustrates a \((\tau, \kappa)\)-simplification, while \((c) \Rightarrow (d)\) illustrates a \((\tau, \kappa, \lambda)\)-simplification.

**Lemma 5.3.** Let \(M'\) be the one-step simplification of \(M\), then \(M'\) is also a gasket automaton satisfying the \(\gamma\)-isolated condition.

**Proof.** Notice that \((\Sigma, P'_{\alpha \beta}) = (\Sigma, P_{\alpha \beta})\), moreover, \((\Sigma, P'_{\alpha \gamma})\) and \((\Sigma, P'_{\beta \gamma})\) are subgraphs of \((\Sigma, P_{\alpha \gamma})\) and \((\Sigma, P_{\beta \gamma})\) respectively, so \(M'\) satisfies the unique property and the boundary condition in the definition of gasket automaton. Also, item (ii) in Definition 5.1 holds.

If three edges with vertices \(a, b, c\) satisfies the assumptions 1\(\overline{2}\overline{3}\) in gathering condition, then we call the set \(\{(a, b), (b, c), (c, a)\}\) a family of \(M\). Since one edge in a family can determine the other two edges, we obtain that any two families are edge-disjoint.

If \(M'\) is a \((\tau, \kappa)\)-simplification, then \((\tau, \kappa)\) does not belong to any family, so the simplification does not affect any family, and \(M'\) satisfies the gathering condition. If \(M'\) is a \((\tau, \kappa, \lambda)\)-simplification, then the simplification deletes two members of a family, so \(M'\) still satisfies the gathering condition. The lemma is proved. \(\square\)

\[
\begin{array}{c}
b \\
\bigcirc
\end{array}
\]

\[
\begin{array}{c}
c \\
\bigcirc
\end{array}
\]

\[
\begin{array}{c}
a \\
\bigcirc
\end{array}
\]

\[
\begin{array}{c}
\text{\(a_{\alpha \beta}\):} \\
\text{\(a_{\alpha \gamma}\):} \\
\text{\(a_{\beta \gamma}\):}
\end{array}
\]

**Figure 6.** A family in \(M\).
Lemma 5.4. Let $M'$ be the one-step simplification of $M$. Then
(i) both $\tau$ and $\kappa$ are double-maximal in $M'$, and $\kappa$ is $\alpha\gamma$-isolated in $M'$.
(ii) $\kappa \notin \{\alpha, \gamma\}$.

Proof. (i) That $\kappa$ is double-maximal in $M'$ since it is double-maximal in $M$. The one-step simplification deletes the edges connecting $\tau$ to its $\alpha\gamma$-successor and $\beta\gamma$-successor (if exists), so the other assertions in (i) hold. (See Figure 5 (b) or (d) for an illustration.)
(ii) Since $\tau \prec_{\alpha\gamma} \kappa$, that $\alpha$ is $\alpha\gamma$-minimal and $\gamma$ is $\alpha\gamma$-isolated imply $\kappa \notin \{\alpha, \gamma\}$. □

The following theorem plays a crucial role in this paper.

Theorem 5.1. Let $M'$ be the one-step simplification of $M$. Then for any $x, y \in \Omega := \{\omega \kappa^\infty; \omega \in \Sigma^*\}$ there exists a bijection $g : \Omega \rightarrow \Omega$ such that

\[
|T_M(x, y) - T_{M'}(g(x), g(y))| \leq 5.
\]

In Section 7, we give the construction of the map $g$, and we prove Theorem 5.1 in Section 8.

6. Proof of Theorem 1.2 and Theorem 1.4

Let $M$ be a gasket automaton satisfying the $\gamma$-isolated condition.

We denote by $M^\ast$ the gasket automaton determined by $\mathcal{P}_{\alpha\gamma}^* = \mathcal{P}_{\beta\gamma}^* = \emptyset$ and $\mathcal{P}_{\alpha\beta}^* = \mathcal{P}_{\alpha\beta}$, and we call it the final-simplification of $M$.

Proof of Theorem 1.2. Recall that $\Omega = \{\omega \kappa^\infty; \omega \in \Sigma^*\}$. Let $g : \Omega \rightarrow \Omega$ be the map given in Theorem 5.1. Since for any $x, y \in \Omega$,

\[
T_M(x, y) - 5 \leq T_{M'}(g(x), g(y)) \leq T_M(x, y) + 5,
\]

which implies that $\xi^5 \rho_M(x, y) \leq \rho_{M'}(g(x), g(y)) \leq \xi^{-5} \rho_M(x, y)$. Hence $g$ is bi-Lipschitz.

Define $[x]_M = \{y \in \Sigma^\infty; \rho_M(x, y) = 0\}$ and set $\Omega_M = \{[x]_M; x \in \Omega\}$; similarly we define $[x]_{M'}$ and $\Omega_{M'}$. Define $\tilde{g} : \Omega_M \rightarrow \Omega_{M'}$ by $\tilde{g}([x]_M) = [g(x)]_{M'}$. We claim that $\tilde{g}$ is bi-Lipschitz. First,

\[
\rho_{M'}(\tilde{g}([x]_M), \tilde{g}([y]_M)) = \rho_{M'}([g(x)]_{M'}, [g(y)]_{M'}) \leq \rho_{M'}(g(x), g(y)) \leq \xi^{-5} \rho_M(x, y) \leq (\xi^{-5})(\xi^{-2})\rho_M([x]_M, [y]_M). \quad \text{(By (3.6).)}
\]

In the same manner, we can prove the other direction inequality. The claim is proved.

Moreover, by Lemma 2.4, $\Omega_M$ is dense in $(\mathcal{A}_M, \rho_M)$ and $\Omega_{M'}$ is dense in $(\mathcal{A}_{M'}, \rho_{M'})$. So $\mathcal{A}_M \simeq \mathcal{A}_{M'}$ by Lemma 2.2. □

Using Theorem 1.2 repeatedly, we obtain

Corollary 6.1. Let $M$ be a gasket automaton and $M^\ast$ be the final-simplification of $M$. Then $(\mathcal{A}_M, \rho_M) \simeq (\mathcal{A}_{M^\ast}, \rho_{M^\ast})$. \hfill 14
Proof. Notice that a gasket automaton $M$ admits a one-step simplification provided $\mathcal{P}_{\alpha\gamma} \cup \mathcal{P}_{\beta\gamma} \neq \emptyset$ (see Section 5). Then there exists a sequence

$$M = M_0, M_1, \ldots, M_q = M^*$$

such that $M_{j+1}$ is the one-step simplification of $M_j$ for each $0 \leq j \leq q - 1$. So the result is a consequence of Theorem 1.2.

Let $E$ and $F$ be the fractal gaskets defined in Theorem 1.4 and let $M_E$ and $M_F$ be the topology automata of $E$ and $F$, respectively. Without loss of generality, we assume that $\alpha = 1$ and $\beta = 2$ for both $E$ and $F$.

If $\omega_j \in E$, we set $M_E^* = (M_E)^*$ to be the final-simplification of $M_E$, otherwise, we set $M_E^* = M_E$. Similarly we define $M_F^*$.

Remark 6.1. Actually, if $\omega_j \notin E$, then the topology automaton $M_E$ only records the horizontal connective relations among $\varphi_j(E), j \in \Sigma$; more precisely, for any $i, j \in \Sigma$, $\varphi_i(E) \cap \varphi_j(E) \neq \emptyset$ if and only if $i \triangleleft_{\alpha\beta} j$ or $j \triangleleft_{\alpha\beta} i$. Thus $\mathcal{P}_{\alpha\gamma} \cup \mathcal{P}_{\beta\gamma} = \emptyset$.

Lemma 6.1. There exists an isometry $f : (A_{M_E^*}, \rho_{M_E^*}) \to (A_{M_F^*}, \rho_{M_F^*})$.

Proof. Let $I = \{a_1, a_2, \ldots, a_k\} \subset \Sigma$ be a horizontal-block of $E$. By the definition of horizontal-block and neighbor automaton $M_E$, we have

$$a_j \triangleleft_{\alpha\beta} a_{j+1} \text{ in } M_E, \ 1 \leq j \leq k - 1.$$

By assumptions of Theorem 1.4 there is a size-preserving bijection from the collection of horizontal-blocks of $E$ to that of $F$, which we denote by $\hat{h}$. That is,

$$\hat{h}(I) = \{b_1, b_2, \ldots, b_k\}$$

is a horizontal-block of $F$. Define $h : \Sigma \to \Sigma$ by $h(a_j) = b_j$, that is, if $a_j$ is the $j$-th element of a horizontal block $I$ of $E$, then we define $h(a_j)$ to be the $j$-th element of $\hat{h}(I)$. Then, for any $r, s \in \Sigma$,

$$r \triangleleft_{\alpha\beta} s \text{ in } M_E^* \text{ if and only if } h(r) \triangleleft_{\alpha\beta} h(s) \text{ in } M_F^*.$$

Now we define $f : \Sigma^\infty \to \Sigma^\infty$ by $f((x_i)_{i=1}^\infty) = (h(x_i))_{i=1}^\infty$. Clearly, $f$ is a bijection and for any $x, y \in \Sigma^\infty$, $T_{M_E^*}(x, y) = T_{M_F^*}(f(x), f(y))$. It follows that $[x] \mapsto [f(x)]$ is an isometry from $A_{M_E^*}$ to $A_{M_F^*}$.

Proof of Theorem 1.4. Fix $\xi \in (0, 1)$. We have

$$A_{M_E} \simeq A_{M_E^*} \simeq A_{M_F^*} \simeq A_{M_F},$$

where the first and third relations are due to Corollary 6.1 and the secondly relation is by Lemma 6.1. Next, by Theorem 1.3, we have that $E$ is bi-Hölder equivalent to $A_{M_E}$ and $F$ is bi-Hölder equivalent to $A_{M_F}$. Thus $E$ is bi-Hölder equivalent to $F$. (Here we use the fact that $(A_M, \rho_{M, \xi})$ is bi-Hölder equivalent to $(A_M, \rho_{M, \xi'})$ provided $\xi' \in (0, 1)$.)

Especially, if both $E$ and $F$ have uniform contraction ratio $r$, then setting $\xi = r$, we obtain $E \simeq F$. 

\[\square\]
7. A universal map from $\Omega$ to $\Omega$

Let $\Sigma = \{1, 2, \ldots, N\}$ with $N \geq 4$. Let $\tau, \kappa, \alpha, \gamma \in \Sigma$ be distinct except that $\tau = \alpha$ is allowed. Set

$$(7.1) \quad \Omega = \{\omega \kappa^\infty; \omega \in \Sigma^*\}.$$ 

In this section, we construct a bijection $g : \Omega \to \Omega$, which can be realized by a transducer (see Appendix B). In next section, we will show that the map $g$ is the desired map in Theorem 5.1.

**Remark 7.1.** (i) The discussion of this section is purely symbolic: it is irrelevant to metric or automaton.

(ii) If $\tau$ and $\kappa$ come from a one-step simplification, then $\kappa \not\in \{\alpha, \gamma\}$ by Lemma 5.4, and $\tau \neq \gamma$ since $\gamma$ is isolated.

7.1. Segment decomposition.

First we introduce two decompositions of sequences in $\Omega$. Set

$$(7.2) \quad C_M := \{\tau \gamma^k; k \geq 2\} \cup \{\kappa \alpha^k \kappa \gamma; k \geq 0\}.$$ 

**Definition 7.1** (M-decomposition). Let $x = (x_i)_{i=1}^\infty \in \Omega$. The longest prefix $X_1$ of $x$ satisfying $X_1 \in C_M \cup \Sigma$ is called the $M$-initial segment of $x$.

Inductively, each $x = (x_i)_{i=1}^\infty \in \Omega$ can be uniquely written as $x = \prod_{j=1}^\infty X_j := X_1 X_2 \cdots X_k \cdots$, where $X_k$ is the $M$-initial segment of $\prod_{j=2}^k X_j$. We call $(X_j)_{j=1}^\infty$ the $M$-decomposition of $x$.

Next we define $M'$-decomposition. Set

$$(7.3) \quad C_{M'} = \{\kappa \alpha^k \kappa \gamma; k \geq 0\} \cup \{\kappa \alpha^k \kappa \gamma \gamma; k \geq 0\} \cup \{\tau \gamma \gamma\},$$ 

**Definition 7.2** (M'-decomposition). Let $u = (u_i)_{i=1}^\infty \in \Omega$. A word $U_1$ is called the $M'$-initial segment of $u$, if it is the longest prefix of $u$ such that $U_1 \in C_{M'} \cup \Sigma$.

Similar as above, we define the $M'$-decomposition of $u$.

Two words are said to be comparable, if one is a prefix of the other.

**Remark 7.2.** Here are two useful observations.

(i) If two elements in $C_M$ are comparable, then both of them are of the form $\tau \gamma^k$. If two elements in $C_{M'}$ are comparable, then one of them is $\kappa \alpha^k \kappa \gamma$ and another one is $\kappa \alpha^k \kappa \gamma \gamma$.

(ii) Let $W \in C_M \cup C_{M'}$. Then $W$ is initialled by a word in $\{\kappa \alpha, \kappa \kappa, \tau \gamma\}$. Moreover, these words cannot appear in $W$ except as a prefix.

7.2. Construction of $g$. First we define $g_0 : C_M \cup \Sigma \to C_{M'} \cup \Sigma$ by

$$g_0 : \begin{cases} 
\tau \gamma^k & \mapsto \kappa \alpha^{k-2} \kappa \gamma, \ k \geq 2; \\
\kappa \alpha^k \kappa \gamma & \mapsto \kappa \alpha^{k-1} \kappa \gamma \gamma, \ k \geq 1; \\
\kappa \kappa \gamma & \mapsto \tau \gamma \gamma; \\
 i & \mapsto i, \forall i \in \Sigma. 
\end{cases}$$
Clearly $g_0 : \mathcal{C}_M \cup \Sigma \to \mathcal{C}_{M'} \cup \Sigma$ is a bijection. We define $g : \Omega \to \Omega$ by
\begin{equation}
(7.4) \quad g(x) = \prod_{j=1}^{\infty} g_0(X_j),
\end{equation}
where $(X_j)_{j=1}^{\infty}$ is the $M$-decomposition of $x$. (Notice that any $x \in \Omega$ has a $M$-
decomposition of the form $(X_j)_{j=1}^{\ell} (\kappa)_{\infty}$, so $g(x) = (\prod_{j=1}^{\ell} g_0(X_j))(\kappa)_{\infty} \in \Omega$.)

**Remark 7.3.** Rao and Zhu [16] constructed the first map of this type related to two neighbor automata of fractal squares, based on geometrical observations. The map $g$ above is an improvement of the map in [16].

**Proposition 7.1.** Let $x = x_1x_2 \ldots, u = u_1u_2 \ldots = g(x)$.
\begin{enumerate}[(i)]
\item If $(X_j)_{j \geq 1}$ is the $M$-decomposition of $x$, then the $M'$-decomposition of $g(x)$ is $(g_0(X_j))_{j \geq 1}$.
\item Similarly, if $(U_j)_{j \geq 1}$ is the $M'$-decomposition of $u$, then the $M$-decomposition of $h(u)$ is $(g_0^{-1}(U_j))_{j \geq 1}$, where $h(u) = \prod_{j=1}^{\infty} g_0^{-1}(U_j)$.
\item The map $g : \Omega \to \Omega$ is a bijection.
\end{enumerate}

**Proof.** We denote $U \triangleleft W$ if $U$ is a prefix of $W$.
\begin{enumerate}[(i)]
\item To prove the first statement, we only need to show that $U_1 = g_0(X_1)$.
Suppose on the contrary that $U_1 \neq g_0(X_1)$. First, $U_1 \triangleleft g_0(X_1)$ is impossible, since $g_0(X_1) \in \mathcal{C}_{M'}$ and we always choose the longest one to be the initial segment. Let $p$ be the least integer such that $U_1 \triangleleft g_0(X_1) \cdots g_0(X_p)$, then $p \geq 2$.
If $|X_j| = 1$ for all $j = 1, \ldots, p$, then $x$ is initialled by $x_1 \ldots x_{|U_1|} = U_1$, which forces that $U_1 = \kappa \alpha^k \kappa \gamma$, and $U_1 = \kappa \alpha^k \kappa \gamma$. But then $X_1 = \tau \gamma^{k+2}$ and $x_{k+4} = \gamma$, which contradicts that $X_1$ is an initial segment. Item (i) is proved.
\item For the second assertion, we only need to show that $X_1 = g_0^{-1}(U_1)$, which is almost the same as the proof of (i).
Suppose on the contrary that $X_1 \neq g_0^{-1}(U_1)$. Then $X_1 \triangleleft g_0^{-1}(U_1)$ is impossible since $g_0^{-1}(U_1) \in \mathcal{C}_M$. Let $p \geq 2$ be the least integer such that $X_1 \triangleleft g_0^{-1}(U_1) \cdots g_0^{-1}(U_p)$.
If $|U_j| = 1$ for all $j = 1, \ldots, p$, then $u$ is initialled by $X_1$, so $X_1 \in \mathcal{C}_M \setminus \mathcal{C}_{M'}$ and $X_1 = \tau \gamma^k (k \geq 3)$ is the only choice. But then the initial segment of $u$ should be $\kappa \alpha^k \kappa \gamma$, a contradiction.
\item By the same reason as item (i), we have $U_1 \in \mathcal{C}_{M'}$. Thus $g_0^{-1}(U_1)$ is a proper prefix of $X_1$, which forces that $g_0^{-1}(U_1) = \tau \gamma^k (k \geq 2)$ and $X_1 = \tau \gamma^\ell (\ell > k)$. But then $U_1 = \kappa \alpha^{k-2} \kappa \gamma$ and $u_{k+2} = \gamma$, which is a contradiction. Item (ii) is proved.
\item From (i) and (ii) we have $h \circ g = g \circ h = id$, so $g$ is a bijection. \hfill \square
\end{enumerate}

Let $\sigma : \Sigma^\infty \to \Sigma^\infty$ be the shift operator defined by $\sigma((x_k)_{k \geq 1}) = (x_k)_{k \geq 2}$.

**Lemma 7.1.** Let $x = (x_k)_{k \geq 1}, y = (y_k)_{k \geq 1} \in \Omega$. Then
\[
|g(x) \wedge g(y)| \geq |x \wedge y| - 2.
\]
In other words, \( u_1 \cdots u_k \) is determined by \( x_1 \cdots x_{k+2} \), where \( k \geq 1 \).

Proof. Denote \( u = g(x) \), \( v = g(y) \). Let \((X_j)_{j=1}^\infty \) and \((Y_j)_{j=1}^\infty \) be the \( M \)-decompositions of \( x \) and \( y \) respectively. Denote \( U_1 = g_0(X_1) \) and \( V_1 = g_0(Y_1) \). Let \( k = |x \land y| - 2 \geq 1 \).

We first prove the lemma in case of \( X_1 \neq Y_1 \). Without loss of generality, we assume that \( |Y_1| \geq |X_1| \).

1. Suppose \( |X_1| = 1 \). In this case we must have \( Y_1 = \kappa a^q k \gamma \land q \geq k \). Moreover, we have \( |X_1| = 1 \) as long as \( i \leq |Y_1| - 2 \), since a word in \( C_M \) is initialised by \( \kappa k \), \( \kappa \alpha \) or \( \tau \gamma \). Since, \( k + 1 \leq |Y_1| - 2 \), this implies that \( x_1 \cdots x_{k+1} \) (which is the same as \( y_1 \cdots y_{k+1} \)) is a prefix of \( u \). Since \( V_1 = \kappa a^{q-1} k \gamma \gamma \) is initialised by \( y_1 \cdots y_q \), we have \(|u \land v| = |y_1 \cdots y_{k+1} \land y_1 \cdots y_q| = \min\{k+1,q\} \geq k \).

2. Suppose \( |X_1| > 1 \). If \( X_1 = \tau \gamma \ell \), then \( Y_1 = \tau \gamma q \) with \( q > \ell \), so \(|u \land v| = |U_1 \land V_1| = |\kappa a^{\ell-2} k \gamma \land \kappa a^{q-2} k \gamma| = \ell - 1 = k \). If \( X_1 = k \alpha^q k \gamma \), then \( Y_1 = \kappa a^q k \gamma \) with \( q > \ell \), so \(|u \land v| = |\kappa a^{q-1} k \gamma \land \kappa a^{q-1} k \gamma| = \ell = |x \land y| - 1 = k + 1 \).

Hence the lemma is valid if \( X_1 \neq Y_1 \).

If \( X_1 = Y_1 \), denote \( p = |X_1| \). Denote \( a = \sigma^p(x) \) and \( b = \sigma^p(y) \). Hence the lemma holds for \( x \) and \( y \) if and only if it holds for \( a \) and \( b \). So the lemma can be proved by induction. \( \Box \)

In Appendix B, we give a transducer which can realize the map \( g \). The transducer provides an alternative proof of Lemma 7.1.

8. Proof of Theorem 5.1

Let \( M \) be a gasket automaton satisfying the \( \gamma \)-isolated condition, and let \( M' \) be a one-step simplification of \( M \). For \( x, y \in \Omega = \{\omega k^\infty; \omega \in \Sigma^* \} \), denote \( u = g(x) \) and \( v = g(y) \). Let \((X_j)_{j=1}^\infty \), \((Y_j)_{j=1}^\infty \) be the \( M \)-decompositions of \( x, y \) respectively, and \((U_j)_{j=1}^\infty \), \((V_j)_{j=1}^\infty \) be the \( M' \)-decompositions of \( u, v \) respectively. Clearly, we always have

\[
T_{M'}(x, y) \leq T_M(x, y).
\]

Recall that \( C_M = \{\tau \gamma k; k \geq 2\} \cup \{k \alpha^k k \gamma; k \geq 0\} \) and \( C_{M'} = \{k \alpha^k k \gamma; k \geq 0\} \cup \{k \alpha^k k \gamma \gamma; k \geq 0\} \cup \{\tau \gamma \gamma\} \). One should keep in mind that

\[
S_{u v} \xrightarrow{(i,j)} S_{u v} \text{ if and only if } (i,j) = (v,u)
\]

in both \( M \) and \( M' \). Since \( \gamma \) is isolated, if \( \tilde{\gamma} \in \Sigma \setminus \{\gamma\} \), then

\[
Id \xrightarrow{(\gamma, \tilde{\gamma})} \text{Exit and } Id \xrightarrow{(\tilde{\gamma}, \gamma)} \text{Exit}
\]

in both \( M \) and \( M' \).

Lemma 8.1. Let \( a, b \in \Omega \) such that \( a_1 \neq b_1 \). If \( a = \alpha^k k \gamma \cdots (k \geq 0) \), then \( T_M(a, b) \leq 2 \).

Proof. Suppose \((a_1, b_1) \in P_M \), then \( Id \xrightarrow{(a_1, b_1)} S_{u v} \) for some \( u, v \in \{\alpha, \beta, \gamma\} \). To prove the lemma, by \( \Box \), we only need to show that \( a_2 \neq v \) or \( a_3 \neq v \).

If \( k = 0 \), then \((a_1, b_1) = (k, b_1) \) and \( a_2 = \gamma \). Since \( \kappa \) is double-maximal in both \( M \), that is, \( \kappa \) is \( \alpha \gamma \)-maximal and \( \beta \gamma \)-maximal, we have \( v \neq \gamma \). So \( a_2 \neq v \).
If \( k = 1 \), then \( a_2 = \kappa \) and \( a_3 = \gamma \). Since \( \kappa \neq \gamma \), clearly either \( a_2 \neq v \) or \( a_3 \neq v \).

If \( k > 1 \), then \((a_1, b_1) = (\alpha, b_1)\) and \( a_2 = \alpha \). Since \( \alpha \) is \( \alpha \gamma \)-minimal and \( \alpha \beta \)-minimal \( M \), we have \( v \neq \alpha \). The lemma is proved. \( \square \)

**Lemma 8.2.** (i) Let \( x, y \in \Omega \). If \( x_1 = y_1 \) and \( X_1 \neq Y_1 \), then

\[
T_M(x, y) \leq |x \wedge y| + 2.
\]

(ii) Let \( u, v \in \Omega \). If \( u_1 = v_1 \) and \( U_1 \neq V_1 \), then

\[
T'_{M'}(u, v) \leq |u \wedge v| + 2.
\]

**Proof.** (i) Let \( k = |x \wedge y| \), then \( k \geq 1 \) and \( x_{k+1} \neq y_{k+1} \). By \( X_1 \neq Y_1 \) we know that at least one of \( X_1 \) and \( Y_1 \) is in \( C_M \), say \( X_1 \in C_M \). We consider two cases according to \( X_1 \).

1. \( X_1 = \tau \gamma^\ell (\ell \geq 2) \). In this case, we have \( k \leq \ell + 1 \), for otherwise \( Y_1 = X_1 \).

   If \( k \leq \ell \), then \( x_{k+1} = \gamma \), so \((x_{k+1}, y_{k+1}) = (\gamma, \tilde{\gamma})\), where \( \tilde{\gamma} \in \Sigma \setminus \{\gamma\} \); if \( k = \ell + 1 \), then \( Y_1 = \tau \gamma^s \) with \( s > \ell \), which implies \((x_{k+1}, y_{k+1}) = (\tilde{\gamma}, \gamma)\). Hence, by formula \( [8.3] \), \( Id \) is transferred to \( Exit \) by \((x_{k+1}, y_{k+1})\), so \( T_M(x, y) = k \).

2. \( X_1 = \kappa \ell \kappa \gamma^\ell (\ell \geq 0) \). In this case, we have \( k \leq \ell + 2 \), for otherwise \( X_1 = Y_1 \).

   If \( k \leq \ell + 1 \), then \( x_{k+1} \cdots x_{k+p} = \alpha \beta \kappa \gamma \cdots (p \geq 0) \), and we have \( T_M(x, y) \leq k + 2 \) by Lemma \( 8.3 \). If \( k = \ell + 2 \), then \( x_{k+1} = \gamma \) and \((x_{k+1}, y_{k+1}) = (\gamma, \tilde{\gamma})\), so \( T_M(x, y) = k \).

This completes the proof of (i).

(ii) Using item (i) we have just proved, we have

\[
|u \wedge v| \leq T'_{M'}(u, v) \leq T_M(u, v) \leq |u \wedge v| + 2.
\]

The lemma is proved. \( \square \)

**Lemma 8.3.** Let \( x = x_1 s^k \cdots \), where \( k \geq 2 \) and \( s \in \{\alpha, \beta, \gamma\} \). Then

\[
g(x) = \begin{cases} 
\kappa \alpha^{-2} \cdots, & \text{if } x_1 s^k = \tau \gamma^k; \\
x_1 s^{k-2} \cdots, & \text{otherwise}. 
\end{cases}
\]

**Proof.** By Lemma \( 7.1 \), \( u_1 \cdots u_{k-1} \) is determined by \( x_1 \cdots x_{k+1} \). So the lemma holds since \( g(\tau \gamma^k \kappa^\infty) = \kappa \alpha^{k-2} \kappa \gamma \kappa^\infty \) and \( g(x_1 s^k \kappa^\infty) = x_1 s^k \kappa^\infty \). \( \square \)

**Lemma 8.4.** Let \( x, y \in \Omega \). If \( X_1 \neq Y_1 \), then

\[
(8.4) \quad T_M(x, y) - T'_{M'}(u, v) \leq 5.
\]

**Proof.** Let \( S_{M,1} \) be the first state of the itinerary of \((x, y)\) in \( M \) (after the initial state \( id \)). If \( S_{M,1} = Exit \), obvious \( [8.4] \) holds.

If \( S_{M,1} = Id \), then \( x_1 = y_1 \), so \( T_M(x, y) \leq |x \wedge y| + 2 \) by Lemma \( 8.2 \). By Lemma \( 7.1 \), we have \( T'_{M'}(u, v) \geq |u \wedge v| \geq |x \wedge y| - 2 \). Hence \( [8.4] \) holds in this case.

Finally, we deal with the case \( S_{M,1} \in Q \setminus \{Id, Exit\} \). Suppose \( \tau \) has a \( \beta \gamma \)-successor \( \lambda \) in \( M \), then \( M' \) is the \((\tau, \kappa, \lambda)\)-simplification of \( M \) with \( \mathcal{P}_{\alpha \beta} = \mathcal{P}_{\alpha \hat{\beta}}, \mathcal{P}_{\alpha \gamma} = \mathcal{P}_{\alpha \gamma} \setminus \{(\tau, \kappa)\} \) and \( \mathcal{P}_{\beta \gamma} = \mathcal{P}_{\beta \gamma} \setminus \{(\tau, \lambda)\} \). (See Figure \( 7 \))
Denote \( k = T_M(x, y) \). If \( k \leq 5 \), (8.4) holds trivially. Now we assume \( k \geq 6 \). Denote \( S_{M,1} = S_{rs} \), where \( r, s \in \{ \alpha, \beta, \gamma \} \). Then the itinerary of \((x, y)\) is \( id \rightarrow (S_{rs})^k \rightarrow \text{Exit} \), so \((x_1, y_1) \in \mathcal{P}_{rs} \) in \( M \) and

\[
x = x_1 s^{k-1} \ldots, \quad y = y_1 r^{k-1} \ldots.
\]

**Case 1.** \( x_1 s^{k-1}, y_1 r^{k-1} \neq \tau \gamma^{k-1} \).

By Lemma 8.3 we have \( u = x_1 s^{k-3} \ldots, \quad v = y_1 r^{k-3} \ldots \). We claim that

\[
(8.5) \quad (x_1, y_1) \notin \{(\tau, \kappa), (\tau, \lambda)\} \cup \{(\kappa, \tau), (\lambda, \tau)\}.
\]

If \((x_1, y_1) = (\tau, \kappa)\), then \( x_1 <_{\alpha \gamma} y_1 \), which implies that \( s = \gamma \), which contradicts \( x_1 s^{k-1} \neq \tau \gamma^{k-1} \). Similarly, we can eliminate the other scenarios. The claim is proved.

It follows that \((u_1, v_1) = (x_1, y_1) \in \mathcal{P}_{rs}' \). Therefore, the first \( k - 1 \) states of the itinerary of \((u, v)\) are \( id \rightarrow (S_{rs})^{k-2} \), so \( T_{M'}(u, v) \geq k - 2 = T_M(x, y) - 2 \).

**Case 2.** \( x_1 s^{k-1} = \tau \gamma^{k-1} \) or \( y_1 r^{k-1} = \tau \gamma^{k-1} \).

Without loss of generality, we assume \( x_1 s^{k-1} = \tau \gamma^{k-1} \). Then \( \tau <_{\alpha \gamma} y_1 \) in \( M \). By Lemma 8.3 we have \( u = \kappa \alpha^{k-3} \ldots \) and \( v = y_1 r^{k-3} \ldots \).

If \( r = \alpha \), then \( \tau <_{\alpha \gamma} y_1 \), which forces \( y_1 = \kappa \) and \( v = \kappa \alpha^{k-3} \ldots \). Hence \( T_{M'}(u, v) \geq |u \land v| \geq k - 2 \). Similarly, if \( r = \beta \), then \( \tau <_{\beta \gamma} y_1 \), which forces \( y_1 = \lambda \) and \( v = \lambda \beta^{k-3} \ldots \). So \( T_{M'}(u, v) = T_{M'}(\kappa \alpha^{k-3} \ldots, \lambda \beta^{k-3} \ldots) \geq k - 2 \).

If \( M' \) is a \((\kappa, \tau)\) simplification of \( M \), we can prove the lemma in the same manner as above. The proof is finished.

For \( u, v \in \Omega \), we claim that if \( U_1 \neq V_1 \), then

\[
(8.6) \quad T_{M'}(u, v) - T_M(x, y) \leq 5.
\]

Let \( S_{M',1} \) be the first state of the itinerary of \((u, v)\) in \( M' \) (after the initial state). We will prove (8.6) in Lemmas 8.6 and 8.7.

**Lemma 8.5.** Let \( v = v_1 \eta^\ell \tilde{\eta} \ldots \) where \( \tilde{\eta} \neq \eta \). If \( V_1 = v_1 \), then \( g^{-1}(v) = v_1 \eta^{\ell-2} \ldots \).

**Proof.** Notice that if \( \eta^p \) is a prefix of an element of \( C_{M'} \), then we must have \( p = 1 \) or \( 2 \) (in the latter case the element must be \( \kappa \kappa \gamma \)). Hence, we have \(|V_1| = 1 \) holds for \( 1 \leq i \leq \ell - 1 \), so \( g^{-1}(v) = v_1 \eta^{\ell-2} \ldots \).

**Lemma 8.6.** Equation (8.6) holds if \( U_1 \neq V_1 \) and \( S_{M',1} = Id \).

**Proof.** That \( S_{M',1} = Id \) implies \( u_1 = v_1 \), so at least one of \( U_1 \) and \( V_1 \) is in \( C_{M'} \), say \( U_1 \in C_{M'} \). Now we divide the proof into two cases.

**Case 1.** \( U_1 = \tau \gamma \gamma \).
In this case, we have $|u \land v| \leq 2$, so $T_{M'}(u, v) \leq 4$ and (8.6) follows.

Case 2. $U_1 = \kappa \alpha^k \kappa \gamma$ or $\kappa \alpha^k \kappa \gamma \gamma (k \geq 0)$.

1. If $V_1 = \kappa \alpha^l \kappa \gamma$ or $\kappa \alpha^l \kappa \gamma \gamma (l \geq 0)$, then $l \neq k$ when $U_1$ and $V_1$ have the same form. It is easy to see that $|u \land v| \leq 3 + \min\{k, l\}$.

Applying $g_0^{-1}$ to $U_1$ and $V_1$, we have

$$X_1 \in \{\tau \gamma^{k+2}, \kappa \alpha^{k+1} \kappa \gamma\}$$

Therefore, $T_M(x, y) \geq 1 + \min\{k + 1, l + 1\}$. Thus

$$T_{M'}(u, v) - T_M(x, y) \leq |u \land v| + 2 - T_M(x, y) \leq 3.$$

2. If $V_1 = \kappa$, write $v$ as $\kappa \alpha^l \alpha \ldots$ where $\alpha \neq \alpha$, then $|u \land v| \leq 2 + \min\{k, l\}$.

Applying $g_0^{-1}$ to $U_1$ and using Lemma 8.5 to $v$, we get

$$X_1 \in \{\tau \gamma^{k+2}, \kappa \alpha^{k+1} \kappa \gamma\}, \quad y = \kappa \alpha^{l-2} \ldots.$$

Therefore, $T_M(x, y) \geq 1 + \min\{k + 1, l - 2\}$. So (8.6) holds by the same reason as above. The lemma is proved. □

**Lemma 8.7.** Equation (8.6) holds if $U_1 \neq V_1$ and $S_{M', 1} \in Q \setminus \{\text{Id, Exit}\}$.

**Proof.** Let $k = T_M(u, v)$. If $k \leq 5$, (8.6) holds trivially. So we assume $k \geq 6$.

Denote $S_{M', 1} = S_{rs}$, where $r, s \in \{\alpha, \beta, \gamma\}$. Then the itinerary of $(u, v)$ is $id \rightarrow (S_{rs})^k \rightarrow \text{Exit}$, so $(u_1, v_1) \in P_{rs}$ and

$$u = u_1 s^{k-1} \ldots, \quad v = v_1 r^{k-1} \ldots.$$

Since $\tau$ is double-maximal in $M'$ (Lemma 5.4), we have $u_1 s^{k-1}, v_1 r^{k-1} \neq \tau \gamma^{k-1}$.

**Case 1.** $u_1 s^{k-1}, v_1 r^{k-1} \neq \kappa \alpha^{k-1}$.

In this case, neither $u$ nor $v$ can be initialled by $\tau \gamma$ or $\kappa \alpha$, so $|U_1| = |V_1| = 1$. Hence, by Lemma 8.5 we have $x = u_1 s^{k-3} \ldots, y = v_1 r^{k-3} \ldots$. Thus

$$T_M(x, y) \geq T_M(x, y) \geq k - 2 = T_M(u, v) - 2.$$

**Case 2.** $u_1 s^{k-1} = \kappa \alpha^{k-1}$ or $v_1 r^{k-1} = \kappa \alpha^{k-1}$.

Without loss of generality, assume that $u_1 s^{k-1} = \kappa \alpha^{k-1}$, then $v_1 \preceq_{\alpha\tau} \kappa$. Since $\kappa$ is $\alpha\gamma$-isolated in $M'$ (Lemma 5.4), we have $v_1 = \lambda$ and $r = \beta$. So $M'$ is a $(\tau, k, \lambda)$-simplification of $M$. On one hand, $y = g_0^{-1}(\lambda \beta^{k-1} \ldots) = \lambda \beta^{k-1} \ldots$. On the other hand, if $U_1 = \kappa \alpha^k \kappa \gamma (l \geq k - 1)$, then $X_1 = g_0^{-1}(U_1) = \tau \gamma^{l+2}$; otherwise, by Lemma 8.5 we have $x = \kappa \alpha^{k-3} \ldots$ no matter $U_1 = \kappa \alpha^k \kappa \gamma \gamma (l \geq k - 1)$ or $|U_1| = 1$. Thus

$$x \in \{\tau \gamma^{k+1}, \ldots, \kappa \alpha^{k-3} \ldots\} \quad \text{and} \quad y = \lambda \beta^{k-1} \ldots,$$

which imply that $T_M(x, y) \geq k - 2$. The lemma is proved. □

**Proof of Theorem 5.1.** For $x, y \in \Omega$, if $x_1 \ldots x_k = y_1 \ldots y_k$ and $x_{k+1} \neq y_{k+1}$, then $U_1 \ldots U_k = V_1 \ldots V_k$ and $U_{k+1} \neq V_{k+1}$. Let $\ell = |X_1 \ldots X_k|$, then

$$(8.7) \quad T_M(x, y) - T_{M'}(u, v) = T_M(\sigma^\ell(x), \sigma^\ell(y)) - T_{M'}(\sigma^\ell(u), \sigma^\ell(v)),$$

where $\sigma$ is the shift operator. By (8.4) and (8.6), Theorem 5.1 holds for $\sigma^\ell(x)$ and $\sigma^\ell(y)$, so it also holds for $x$ and $y$. □
Appendix A. Connected components of a fractal gasket

Define $\pi(x, y) = y$.

Lemma A.1. Let $K$ be a fractal gasket. If $K$ satisfies the top isolated condition or $\omega_1 \notin K$, then any nontrivial connected component of $K$ is a line segment. If $\alpha$ and $\beta$ are not in the same horizontal block in addition, then $K$ is totally disconnected.

Proof. Let $U$ be a connected component of $K$. Notice that the collection of the vertices of $\varphi_i(\Delta)$, $i \in \Sigma$, forms a cut set of $K$.

Let $I$ be a horizontal block of $\Sigma$ (see Definition 1.5), we denote $B(I) = \bigcup_{i \in I} \varphi_i(\Delta)$.

If $\omega_1 \notin K$, then all $B(I)$ are disjoint, so either $U \subset B(I)$ or they are disjoint; if $K$ satisfies the top isolated condition, it is easy to show that the same conclusion still holds. It follows that $|\pi(U)| \leq r^*$.

Now we regard $K$ as the invariant set of the IFS $\{\varphi_i\}_{i \in \Sigma}$, and $\varphi_i(\Delta)$ is a set of points. The map $g$ defined in (7.4) can be realized by the transducer indicated in Figure 8. The state set is $\{\text{Id}, \tau, \tau', \tau'', \kappa, \kappa', \kappa''\}$ where ‘Id’ is the initial state. Each edge is labeled by $x_i/\omega_i$, where $x_i \in \Sigma$ is the input letter, and $\omega_i \in \Sigma^*$ is the output word. For any input symbol string $x = (x_i)_{i=1}^{\infty}$, there is a unique sequence determined by the transducer, which is denoted by $\omega_1\omega_2\cdots = g(x)$.

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Figure 8. The transducer of \( g \), where \( a \in \Sigma \setminus \{ \tau, \kappa \} \), \( b \in \Sigma \setminus \{ \tau, \kappa, \gamma \} \),
\( c \in \Sigma \setminus \{ \tau, \kappa, \alpha \} \), \( d \in \Sigma \setminus \{ \tau, \kappa, \alpha, \gamma \} \).

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