ON A MULTIDIMENTIONAL BROWNIAN MOTION WITH
PARTLY REFLECTING MEMBRANE ON A HYPERPLANE

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Abstract. A multidimensional Brownian motion with partial reflection on a hyperplane
$S$ in the direction $qN + \alpha$, where $N$ is the conormal vector to the hyperplane and $q \in [-1, 1], \alpha \in S$ are given parameters, is constructed and this construction is based on both
analytic and probabilistic approaches. The joint distribution of $d$-dimensional analogy to
skew Brownian motion and its local time on the hyperplane is obtained.

§1. Introduction.

In this paper we construct a generalized diffusion process in a $d$-dimensional Euclidean
space $\mathbb{R}^d$ for which the diffusion matrix $B$ is constant and the drift vector is equal to
$(qB\nu + \alpha)\delta_S(x)$ where $\nu \in \mathbb{R}^d$ is a given unit vector, $S$ is the hyperplane in $\mathbb{R}^d$ orthogonal
to $\nu$, the parameters $q \in [-1, 1]$ and $\alpha \in S$ are given and $\delta_S(x)$ is a generalized function
that is determined by the relation

$$\int_{\mathbb{R}^d} \varphi(y)\delta_S(y)dy = \int_S \varphi(y)d\sigma$$

valid for any test function $\varphi$.

This problem in a more general case (namely, the parameters $\alpha, q$ are some functions
of $x \in S$ and the diffusion matrix $B$ is a function of $x \in \mathbb{R}^d$) was considered by Kopytko
B.I. [1] and solved by an analytic method. More simple case ($B$ is an identical operator,
$\alpha, q$ are some constants) was considered by Kopytko B.I., Portenko N.I. [2] with use of
an analytic method too.

In this work a probabilistic approach to this problem will be proposed. We construct
the process desired as a solution of the following stochastic differential equation

$$dx(t) = (qB\nu + \alpha)\delta_S(x(t))dt + B^{1/2}dw(t)$$

where $B^{1/2}$ is the positive square root from the operator $B$ and $w(t)$ is a given Wiener
process in $\mathbb{R}^d$. If $\alpha = 0$ then the solution of this equation is known. Let us denote it by
$\tilde{x}(t)$. The idea is to add the process $\alpha \int_0^t \delta_S(\tilde{x}(\tau))d\tau$ to the projection of $\tilde{x}(t)$ on $S$.

The paper consists of two sections. In Section 2 we describe the analytic method. It
is similar to that of [2]. The probabilistic approach is presented in Section 3.

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\[2\] The Analytic Method.

Let \(\nu \in \mathbb{R}^d\) be a fixed vector with \(\|\nu\| = 1\) and \(S\) be the hyperplane in \(\mathbb{R}^d\) orthogonal to \(\nu\)
\[
S = \{x \in \mathbb{R}^d | (x, \nu) = 0\}
\]

And let \(x(t)\) be a continuous Markov process in \(\mathbb{R}^d\) with transition probability density \(g_0(t, x, y)\)
\[
g_0(t, x, y) = \left(\det(2\pi t B)\right)^{-1/2} \exp\left\{-\frac{1}{2t} (B^{-1}(y - x), y - x)\right\}, \quad t > 0, x \in \mathbb{R}^d, y \in \mathbb{R}^d
\]

where \(B\) is the given positive definite symmetric operator in \(\mathbb{R}^d\).

Let \(q \in [-1, 1], \alpha \in S\) be given. We are looking for a function \(u(t, x, \varphi)\) defined for \(t > 0, x \in \mathbb{R}^d, \varphi \in \mathcal{C}_b(\mathbb{R}^d)\) such that

1. \(u(t, x, \varphi)\) satisfies the heat equation in the domain \(t > 0, x \notin S\)
\[
\frac{\partial u}{\partial t} = \frac{1}{2} \text{Tr}(B \nabla^2 u)
\]

If we fix coordinates in \(\mathbb{R}^d\) then \(\nabla^2 u = (\frac{\partial^2 u}{\partial x_i \partial x_j})_{i,j=1}^d\) and this equation can be written as follows
\[
\frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i,j=1}^d b_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}
\]

where \(B = (b_{ij})_{i,j=1}^d\).

2. \(u(t, x, \varphi)\) satisfies the initial condition
\[
\lim_{t \downarrow 0} u(t, x, \varphi) = \varphi(x), \quad x \in \mathbb{R}^d, \varphi \in \mathcal{C}_b(\mathbb{R}^d)
\]

3. For \(x \in S\) and \(t > 0\) the relation
\[
u(t, x+, \varphi) = u(t, x-, \varphi)
\]
holds, where \(u(t, x \pm, \varphi) = \lim_{\varepsilon \downarrow 0} u(t, x \pm \varepsilon \nu, \varphi)\)

4. The relation
\[
\frac{1 + q}{2} \frac{\partial u(t, x+, \varphi)}{\partial N} - \frac{1 - q}{2} \frac{\partial u(t, x-, \varphi)}{\partial N} + (\alpha, \nabla u) = 0,
\]

is valid for \(x \in S\) and \(t > 0\), where \(\nabla u\) is gradient of \(u(t, x, \varphi)\), \(N\) is the conormal vector to \(S, N = B\nu\).

Let us denote by \(x_{\nu}(t)\) the projection of \(x(t)\) on \(\nu\), by \(\pi_S\) the operator of orthogonal projection on \(S\) such that \(x_{\nu}(t) = \pi_S x(t)\) is the projection of \(x(t)\) on \(S\). If \(\nu\) is not an eigen vector of the operator \(B\) then \(x_{\nu}(t)\) and \(x_{S}(t)\) are not independent processes and we can not represent \(g_0(t, x, y)\) as a product of \(x_{\nu}(t)\) and \(x_{S}(t)\) densities. But we may consider the following decomposition of \(g_0(t, x, y)\)
\[
g_0(t, x, y) = \frac{1}{\sqrt{2\pi t \sigma^2}} \exp\left\{-\frac{(y_{\nu} - x_{\nu})^2}{2t \sigma^2}\right\} \times
\]
\[
\times [\text{det}(2\pi t B_S)]^{-1/2} \exp \left\{ -\frac{1}{2t} (B_S^{-1}(y_S - x_S - \frac{y_\nu - x_\nu}{\sigma^2} \theta), y_S - x_S - \frac{y_\nu - x_\nu}{\sigma^2} b) \right\}
\]
where \( b = \pi_S B_\nu, B_S = (\pi_S B^{-1} \pi_S)^{-1}, \sigma^2 = (B_\nu, \nu) \).

The first factor of this decomposition is the density of the process \( x_\nu(t) \), the second one is the conditional density of the process \( x^S(t) \) under the condition \( x_\nu(t) = y_\nu \). Denoting by \( g^S(t, x, y) \) the second factor we may write

\[
g_0(t, x, y) = \frac{1}{\sqrt{2\pi t \sigma^2}} \exp \left\{ -\frac{(y_\nu - x_\nu)^2}{2t \sigma^2} \right\} g^S(t, x, y), t > 0, x \in \mathbb{R}^d, y \in \mathbb{R}^d
\]

The problem (1) – (4) may be solved in the following manner. Let us fix a system of coordinates so that \( \nu = e_d \) and make use of the Fourier transformation with respect to \( x_1, ..., x_{d-1} \) and the Laplace transformation with respect to \( t \) to the function \( u(t, x, \varphi) \).

It will reduce this problem to solving a pair of second order differential equations with constant coefficients. Then free coefficients may be found from boundary conditions (3), (4). Returning to the original function we get the following assertion.

**Theorem 1.** If \( |q| \leq 1 \) then there exists a continuous Markov process in \( \mathbb{R}^d \) with transition probability density \( G(t, x, y) \) such that

\[
u \varphi(y)G(t, x, y)dy, t > 0, x \in \mathbb{R}^d, \varphi \in C_0(\mathbb{R}^d)
\]

where

\[
G(t, x, y) = \frac{1}{\sqrt{2\pi t \sigma^2}} \left[ \exp \left\{ -\frac{(y_\nu - x_\nu)^2}{2t \sigma^2} \right\} - \exp \left\{ -\frac{(|y_\nu| + |x_\nu|)^2}{2t \sigma^2} \right\} \right] g^S(t, x, y) + \int_0^\infty (q \text{sign } y_\nu + 1) \frac{\sigma^2 \theta + |x_\nu| + |y_\nu|}{\sqrt{2\pi t^3 \sigma^2}} \exp \left\{ -\frac{(\sigma^2 \theta + |x_\nu| + |y_\nu|)^2}{2t \sigma^2} \right\} g^S(t, x + \alpha \theta, y)d\theta
\]

Remark. The condition \( |q| \leq 1 \) provides that \( G(t, x, y) \) is positive.

**§3. The Probabilistic Method.**

Let \( w(t) \) be a given \( d \)-dimensional Wiener process, \( B \) be a symmetric positive definite operator in \( \mathbb{R}^d \) and parameters \( q \in [-1, 1], \alpha \in S \) be given. Consider the stochastic differential equation

\[
dx(t) = (q B\nu + \alpha) \delta_S(x(t))dt + B^{1/2}dw(t)
\]

where \( \delta_S(x) \) is the generalized function on \( \mathbb{R}^d \) defined above.

When \( \alpha = 0 \) then the solution of the equation (1) is known. It may be called a multidimensional analogy to skew Brownian motion. Let us denote it by \( \tilde{x}(t) \). The density of \( \tilde{x}(t) \) equals to

\[
g(t, x, y) = \frac{1}{\sqrt{2\pi t \sigma^2}} \left[ \exp \left\{ -\frac{(y_\nu - x_\nu)^2}{2t \sigma^2} \right\} + q \text{sign } y_\nu \exp \left\{ -\frac{(|y_\nu| + |x_\nu|)^2}{2t \sigma^2} \right\} \right] g^S(t, x, y)
\]

where \( t > 0, x \in \mathbb{R}^d, y \in \mathbb{R}^d \)
The equation (1) means that we have to find the process \( x(t) \) that satisfies the following properties. The projection of \( x(t) \) on \( \nu \) is skew Brownian motion, the projection of \( x(t) \) on \( S \) is the Gaussian process in \( S \) (with mean \( x^S + \frac{y_\nu - x_\nu}{\sigma^2} \) and correlation operator \( tB_S \)) plus the process \( \alpha \eta \), where \( \eta \) is the functional of the process \( x(t) \) such that

\[
\eta = \int_0^t \delta_S(x(\tau))d\tau
\]

**Remark.** Actually \( \eta \) depends only on \( x_\nu(t) \). Therefore we may write \( \eta = \int_0^t \delta_S(\tilde{x}(\tau))d\tau \)

The functional \( \eta \) is a nonnegative continuous homogeneous additive functional, it increases at those instants of time for which the process \( \tilde{x}(t) \) hits the hyperplane \( S \).

To solve the equation (1) we have to find the joint distribution of \( \tilde{x}(t) \) and \( \eta_t \).

**Lemma 1.** The joint distribution of \( \tilde{x}(t) \) and \( \eta_t \) has the form

\[
\mathbb{P}_x \{ \tilde{x}(t) = dy, \eta_t \in d\theta \} = \left\{ \frac{\delta(\theta)}{\sqrt{2\pi}\tau} \left[ \exp \left\{ \frac{(y_\nu - x_\nu)^2}{2\tau^2} \right\} - \exp \left\{ \frac{(|y_\nu| + |x_\nu|)^2}{2\tau^2} \right\} \right] + (1 + q \text{sign } y_\nu) \sigma^2 \theta + \frac{x_\nu + y_\nu}{\sqrt{2\pi}\tau^2} \exp \left\{ \frac{-(\sigma^2 \theta + |x_\nu| + |y_\nu|)^2}{2\tau^2} \right\} \right\} g^S(t, x, y)dyd\theta
\]

**Remark.** \( \mathbb{P}_x \{ \tilde{x}(t) = dy, \eta_t \in d\theta \} \) has the atom at the point \( \theta = 0 \).

**Proof.** Consider

\[
u(t, x, \lambda, \mu) = \mathbb{E}_x e^{i(t(\lambda, \mu) + i\lambda \theta)}, \quad t > 0, x \in \mathbb{R}^d, \lambda \in \mathbb{R}, \mu \in \mathbb{R}^d
\]

where \( \mathbb{E}_x \) is the averaging operator corresponding \( \tilde{g}(t, x, y) \).

The function \( u(t, x, \lambda, \mu) \) satisfies the following integral equation (see [3])

\[
u(t, x, \lambda, \mu) = \int_{\mathbb{R}^d} e^{i(y, \mu)} \tilde{g}(t, x, y)dy + i\lambda \int_0^t \int_S u(t - \tau, z, \lambda, \mu) \tilde{g}(\tau, x, z)dz
\]

Note that the function \( u(t, \cdot, \lambda, \mu) \) on the right hand side of (2) depends on its values on \( S \). Then using Fourier-Laplace transformation we can find \( u(t, x, \lambda, \mu) \) for \( x \in S \)

\[
u(t, x, \lambda, \mu) = \int_0^\infty d\theta \int_{\mathbb{R}^d} e^{i(y, \mu) + i\lambda \theta} (1 + q \text{sign } y_\nu) \times \frac{1}{\sqrt{2\pi}\tau^2} \exp \left\{ \frac{-(\sigma^2 \theta + |x_\nu| + |y_\nu|)^2}{2\tau^2} \right\} \right\} g^S(t, x, y)dy,
\]

Substituting this relation in (2) we will obtain

\[
u(t, x, \lambda, \mu) = \int_{\mathbb{R}^d} e^{i(y, \mu)} \left[ \exp \left\{ \frac{-(y_\nu - x_\nu)^2}{2\tau^2} \right\} - \exp \left\{ \frac{(|y_\nu| + |x_\nu|)^2}{2\tau^2} \right\} \right] g^S(t, x, y)dy + \int_0^\infty d\theta \int_{\mathbb{R}^d} e^{i(y, \mu) + i\lambda \theta} \sigma^2 \theta + \frac{x_\nu + y_\nu}{\sqrt{2\pi}\tau^2} \times \frac{1}{\sqrt{2\pi}\tau^2} \exp \left\{ \frac{-(\sigma^2 \theta + |x_\nu| + |y_\nu|)^2}{2\tau^2} \right\} \right\} g^S(t, x, y)dy
\]

where \( t > 0, x \in \mathbb{R}^d, \lambda \in \mathbb{R}, \mu \in \mathbb{R}^d \). This gives us \( \mathbb{P}_x \{ \tilde{x}(t) \in dy, \eta_t \in d\theta \} \).

If we know the joint distribution of \( \tilde{x}(t) \) and \( \eta_t \), then the equation (1) can be solved.
Theorem 2. The solution of the equation (1) is a continuous Markov process $x(t)$ with transition probability density $G(t, x, y)$.

Proof. We have

$$
\mathbb{P}_x \{x(t) \in dy\} = \int_{[0, +\infty)} \mathbb{P}_x \{x(t), \eta_t \in d\theta\}
$$

When $\theta = 0$ then $x(t) = \tilde{x}(t)$ and joint distribution $\mathbb{P}_x \{x(t) \in dy, \eta_t \in d\theta\}$ is equal to $\mathbb{P}_x \{\tilde{x}(t) \in dy, \eta_t \in d\theta\}$ at the point $\theta = 0$. For each $\theta \in (0, \infty)$ the process $x(t)$ is equal to $\tilde{x}(t) + \alpha \theta$. Integrating (3) on $\theta$ we obtain that

$$
\mathbb{P}_x \{x(t) \in dy\} = \frac{1}{\sqrt{2\pi t \sigma^2}} \left[ \exp \left\{ \frac{(y_\nu - x_\nu)^2}{2t \sigma^2} \right\} - \exp \left\{ \frac{(|y_\nu| + |x_\nu|)^2}{2t \sigma^2} \right\} \right] g^S(t, x, y)dy
$$

$$
+ \int_0^\infty (q \text{sign } y_\nu + 1) \frac{\sigma^2 \theta + |x_\nu| + |y_\nu|}{\sqrt{2\pi t \sigma^2}} \exp \left\{ \frac{(\sigma^2 \theta + |x_\nu| + |y_\nu|)^2}{2t \sigma^2} \right\} g^S(t, x + \alpha \theta, y)d\theta dy
$$

It coincides with the result of the Theorem 1.

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