Tsallis’ entropy, modified Newtonian accelerations and the Tully-Fisher relation

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received 13 August 2018; accepted in final form 5 November 2018
published online 5 December 2018

PACS 05.20.-y – Classical statistical mechanics
PACS 98.65.Cw – Galaxy clusters
PACS 51.10.+y – Kinetic and transport theory of gases

Abstract – In this paper we have shown that the connection between the number of bits and the area of the holographic screen, where both were established in Verlinde’s theory of entropic gravity, may depend on the thermostatistics theory previously chosen. Starting from the Boltzmann-Gibbs (BG) theory, we have reobtained the usual dependence of both, bits number and area. After that, using Tsallis’ entropy concept within the entropic gravity formalism, we have derived another relation between the bits number and the holographic screen area. We have used this new relation to derive three Newtonian-type accelerations in the context of Tsallis’ statistics which are a modified gravitational acceleration, a modified MOND theory and a modified Friedmann equation. Moreover we have obtained a nonextensive version of the Tully-Fisher (TF) relation which shows a dependence of the distance of the star in contrast to the standard TF expression. The BG limit gives the standard TF law.

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Introduction. – The recent formalism engineered by Verlinde [1] (see also [2] for previous ideas about the issue) obtained the gravitational acceleration by using the holographic principle [3,4] and the well-known equipartition law of energy. Verlinde’s ideas relied upon the fact that gravitation can be considered universal and independent of the details concerning the space-time microstructure. Besides, he introduced new concepts involving holography since the holographic principle must unify matter, gravity and quantum mechanics [3,4].

On the other hand, an important extension of the BG theory has been proposed by Tsallis [5] where a new form for the entropy has been given. Then one of the main objectives of this work is to redefine the usual number of bits, which is specified in Verlinde’s formalism, in order to be compatible with both Tsallis’ entropy and the Beckenstein-Hawking area entropy law. The physical consequence of this procedure will be discussed throughout this paper.

To describe our paper we will follow a sequence such that in the second section we have provided a brief review of Verlinde’s formalism. In the third section we have presented the Tsallis statistics also known as nonextensive (NE) statistics. In the fourth section we have obtained a new relation for the number of bits and the holographic screen area. From this relation we can derive a modified gravitational acceleration in the context of Tsallis’ statistics. In the fifth section we have obtained a modified MOND theory and the NE Tully-Fisher relation. In the sixth section we have derived an NE Friedmann equation. The conclusions and final remarks will be given in the last section.

The Verlinde formalism. – The geometrical structure related to the model is composed of a spherical
surface, which plays the role of an holographic screen, and a particle of mass \( M \) localized in its center. The holographic screen can be understood as a storage device for information. The number of bits (the basic unit of information in the holographic screen) is assumed to be proportional to the area \( A \) of the holographic screen and it is represented by

\[
N = \frac{A}{\ell_P^2}, \tag{1}
\]

where \( A = 4\pi r^2 \) and \( \ell_P \) is the Planck length. In Verlinde’s framework one can assume that the total number of bits for the energy on the screen is given by the equipartition law of energy

\[
E = \frac{1}{2} N k_B T. \tag{2}
\]

It is important to understand that the usual equipartition theorem in this last equation can be obtained from the standard Boltzmann-Gibbs (BG) thermostatistics. Let us consider that the energy of the particle inside the holographic screen can be understood as a storage device for information. The number of bits (the basic unit of information in the holographic screen) is assumed to be proportional to the area \( A \) of the holographic screen and it is represented by

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\[
E = \frac{1}{2} N k_B T. \tag{2}
\]

To calculate the gravitational acceleration, we can use both eq. (1) and the Unruh temperature equation [6] which is given by

\[
k_B T = \frac{1}{c} \frac{\hbar a}{2\pi}. \tag{4}
\]

So, one can be able to compute the (absolute) gravitational acceleration formula

\[
a = g = \frac{\ell_P^2 c^3 M}{\hbar} \frac{1}{R^2} = G \frac{M}{R^2}. \tag{5}
\]

From eq. (5) we can see that the Newton constant \( G \) can be written in terms of the standard constants,

\[
G = \frac{\ell_P^2 c^3}{\hbar}.
\]

In the NE scenario, it is possible to derive the NE equipartition theorem which is written as [7]

\[
E_q = \frac{1}{5 - 3q} N k_B T. \tag{6}
\]

In Verlinde’s formalism one can apply the NE equipartition equation, namely, eq. (6). In this way, we can obtain a modified acceleration formula [8,9]

\[
a = G_q \frac{M}{R^2}, \tag{7}
\]

where \( G_q \) is an effective gravitational constant which can be given by

\[
G_q = \frac{5 - 3q}{2} G. \tag{8}
\]

From this last equation, we can note that the effective gravitational constant depends on the NE parameter \( q \). For instance, when \( q = 1 \) we have that \( G_q = G \) (BG scenario) and for \( q = 5/3 \) we have the curious and hypothetical result \( G_q = 0 \), which defines the point \( q = 5/3 \) as a singular one.

**The Tsallis statistics.** – Tsallis’ statistics [5,10], which is the NE extension of BG statistical theory, defines the NE and a nonadditive entropy as

\[
S_q = k_B \frac{1 - \sum_{i=1}^{W} p_i^q}{q - 1} \left( \sum_{i=1}^{W} p_i = 1 \right), \tag{9}
\]

where \( p_i \) is the probability of a system to exist within a microstate, \( W \) is the total number of configurations (microstates) and \( q \), known in the current literature as being the Tsallis parameter or NE parameter, is a real parameter which measures the degree of nonextensivity.

The definition of entropy in Tsallis statistics carries the standard properties of positivity, equiprobability, concavity and irreversibility. This approach has been successfully used in many different physical systems. For instance, we can mention the Levy-type anomalous diffusion [11], turbulence in a pure-electron plasma [12] and, gravitational and cosmological systems [13–25]. It is noteworthy to affirm that Tsallis thermostatistics formalism has the BG statistics as a particular case in the limit \( q \to 1 \) where the standard additivity of entropy can be recovered. In other words, it is mandatory to obtain BG statistics when we have \( q \to 1 \) in Tsallis approach equations.

In the microcanonical ensemble, where all the states have the same probability, Tsallis entropy reduces to [13]

\[
S_q = k_B \frac{W^{1-q} - 1}{1 - q}, \tag{10}
\]

where in the limit \( q \to 1 \) we recover the usual BG entropy formula, \( S = k_B \ln W \).

**Modified gravitational acceleration.** – We will start our proposal by considering initially the BG statistics. From eq. (1) we will write a general relation between the bits number and the area of the holographic screen as

\[
N = \frac{A}{Q}, \tag{11}
\]

where \( Q \) is, at first sight, a parameter. We will consider the number of microstates \( W \) given by

\[
W = c_1 Q, \tag{12}
\]
where \( c_1 \) is the number of internal degrees of freedom of the holographic screen. In a BG statistics the entropy is given by \( s = k_B \ln c_1 \). Using eq. (12) we have
\[
S = k_B N \ln c_1. \tag{13}
\]

Using the Beckenstein-Hawking formula that says that \( S = k_B A/4l_p^2 \), we can obtain the number of bits, \( N \), as a function of \( A \) to derive that
\[
N = \frac{A}{4l_p^2 \ln c_1} = \frac{A}{Q}. \tag{14}
\]

where \( Q \equiv 4l_p^2 \ln c_1 \). If we make \( 4 \ln c_1 = 1 \) then we recover the usual definition of the bits number, eq. (1). It is important to observe that, in a BG statistics scenario, the \( q \) parameter is a constant, i.e., it does not depend on the holographic screen area.

It is quite direct to generalize the BG entropy and, consequently, to obtain a more general relation for the number of bits, eq. (11). To complete the task (and we can see a similar way described in refs. [26] and [27]), we will use Tsallis entropy in the microcanonical ensemble. Then, using eq. (12) and Tsallis entropy, eq. (10), we have
\[
k_B c_1^{N(1-q)} - 1 = k_B A \frac{A}{4l_p^2} \Rightarrow N = \frac{\ln \left[ 1 + (1-q) \frac{A}{4l_p^2} \right]}{(1-q) \ln c_1}. \tag{15}
\]

We can recover the usual bits number formula, eq. (1), for the limit \( q \rightarrow 1 \) if we use \( \ln c_1 = 1/4 \) in eq. (16). A similar result for the value of \( c_1 \) was obtained by Padmanabhan in [2]. Hence we can generalize eq. (14) in the context of Tsallis statistics as
\[
N = \frac{\ln \left[ 1 + (1-q) \frac{A}{4l_p^2} \right]}{4(1-q)}. \tag{17}
\]

It is important to mention that, for a given value of \( A \), the number of bits \( N \) in eq. (17) is now a function of the NE parameter \( q \).

Using the generalized bits number, eq. (17), together with eqs. (6) and (4) and defining \( E_q = M c^2 \), we can obtain a modified absolute gravitational acceleration in Tsallis statistics
\[
a = G M \pi \frac{\sqrt[2]{\pi}}{l_p} \ln \left[ 1 + \frac{A}{4l_p^2} \frac{1-q}{1} \right],
\]
\[
= G M \pi \frac{\sqrt[2]{\pi}}{l_p} \ln \left[ 1 + \frac{A}{4l_p^2} \frac{1-q}{1} \right]. \tag{18}
\]

where we have used that \( A \), the area of the holographic screen, is equal to \( 4 \pi R^2 \). We can see that in the limit \( q \rightarrow 1 \) we reobtain \( a = G M / R^2 \), which is just eq. (5). In fig. 1 we have plotted two-dimensionally, the normalized modified gravitational acceleration, eq. (18), as a function of the normalized radial distance \( \tilde{R} \) for only three different values of \( q \). In fig. 2 we have the corresponding three-dimensional graphic. Namely, the set of curves obtained in fig. 1 for the continuum group of \( q \)-values.

From figs. 1 and 2 we can observe clearly that the absolute value of the gravitational acceleration decreases as the value of the NE parameter \( q \) increases. Notice that this is a new result. Based only on eq. (18), an observer could conclude that the NE parameter is the underlying parameter that designs the way massive objects attract each other.

**Modified MOND theory and the nonextensive Tully-Fisher relation.** MOND’s success [28–30] is due mainly to its capacity to explain the majority of the galaxies rotations. It reproduces the well-known Tully-Fisher (TF) relation [31]. In this way, it can be an alternative...
to dark matter model. However, MOND has problems to explain both the galaxy clusters temperature pattern and the confrontation with cosmology as well. Besides, if we can say that the standard TF relation is the $q \to 1$ limit of its NE version, what is its NE version? We will respond this question in a jiffy.

In few words, MOND is basically a modification of Newton’s second law where the force can be written as

$$F = m\mu \left( \frac{a}{a_0} \right) a,$$  \hfill (19)

where $a_0$ is a constant that will be defined just below and $\mu(x) \approx 1$ for $x > 1$ and $\mu(x) \approx x$ for $x \ll 1$. For simplicity, it is usual to assume that the variation of $\mu(x)$ between the asymptotic limits happens abruptly at $x = 1$ or $a = a_0$.

Concerning the rotational movement of the galaxies we have that the standard TF relation is given by $v^2 = \sqrt{GMa_0}$, where $a_0$, the well-known MOND’s constant (an acceleration scale), value is $a_0 \approx 1.2 \cdot 10^{-8}$ cm/s$^2$.

One of us [32] has obtained MOND’s theory by using Verlinde’s entropic gravity approach. Basically it was considered that the fraction of bits with zero energy is given by

$$\frac{N_0}{N} = 1 - \frac{T}{T_c},$$  \hfill (20)

where $N$ is the total bits number, $N_0$ is the number of bits with zero energy and $T_c$ is the critical temperature. For $T \geq T_c$ we have $N_0 = 0$. Consequently, in MOND’s regime, the number of bits with energy different from zero for a given temperature $T < T_c$ is

$$N - N_0 = N \frac{T}{T_c}.$$  \hfill (21)

Substituting the number of bits $N$ in eq. (6) by eq. (21) and defining $E_q = Mc^2$, we have that

$$Mc^2 = \frac{1}{5 - 3q} N \frac{T}{T_c} k_B T.$$  \hfill (22)

Then, using eqs. (17) and (4) into (22) we obtain

$$a \left( \frac{a}{a_0} \right) = \frac{GM}{2l_p} \frac{\pi}{\ln \left( 1 + \frac{2\pi}{l_p} \right) (1 - q)},$$  \hfill (23)

and consequently we have that

$$a = \sqrt{\frac{GMa_0}{2l_p} \frac{\pi}{\ln \left( 1 + \frac{2\pi}{l_p} \right) (1 - q)}},$$  \hfill (24)

where $a_0$ is given by $a_0 = \pi c k_B T_c / h$ and we have used $A = 4\pi R^2$ in eq. (23). Therefore, we have obtained an acceleration’s formula for MOND’s theory in the context of Tsallis’ statistics in eq. (24). Let us explain the physics behind this last result with more details.

Since we have the BG limit given by

$$\lim_{q \to 1} \frac{\pi k_B T_c (5 - 3q)}{1 + (1 - q) \frac{\pi l_p^2}{l_p}} = \frac{2l_p^2}{\pi R^2},$$  \hfill (25)

and substituting this result into eq. (24) we have that

$$\lim_{q \to 1} a = \frac{\sqrt{GMa_0}}{R},$$  \hfill (26)

which is the TF relation divided by $R$. Hence

$$\lim_{q \to 1} a = \frac{v^2}{R},$$  \hfill (27)

which is an expected result since it confirms MOND’s primary assumption. So, we can conclude from this last result that the acceleration given in eq. (24) is the galaxy rotation acceleration which, in the BG limit, is the galaxy centripetal acceleration. So, let us write that

$$\frac{v^2}{R} = \sqrt{\frac{GMa_0}{2l_p} \frac{\pi}{\ln \left( 1 + \frac{2\pi}{l_p} \right) (1 - q)}}$$  \hfill (28)

which can be understood as the NE version of the TF relation. In the BG limit, using eq. (25) we have that $v^2 = \sqrt{GMa_0}$. Notice that in eq. (28), the NE TF relation is dependent on $R$. This $R$-dependence is not a feature given by MOND, but eq. (28), its NE version, is $R$-dependent. This is a new result in the literature. We believe that we can use eq. (28) to obtain valid intervals for the $q$-value. It is an ongoing research.

Relatively recent astronomical observations have demonstrated that, for disk galaxies, the fourth power of the stars circular speed moving around the galaxy core, $v^4_{\text{core}}$, is proportional to the galaxy total luminosity $L$ having an accuracy of more than two orders of magnitude in $L$. In other words, $v^4_{\text{core}} \propto L$. Since $L$ is proportional to the galaxy mass, $M$, one can obtain $v^4_{\text{core}} \propto M$, i.e., the TF law. However, we have dependence on the distance from the star up to the center of the galaxy as the Newtonian law $v^2_{\text{core}} = GM/r$ requires concerning circular motion. In order to correct this deviation from Newton’s law, astronomers suppose that there are halos around the galaxy, which are stuffed with dark matter and distributed in such ways so that they satisfy the TF law for each specific situation. We can see clearly a different scenario from the NE version of the TF relation in eq. (28) where the rotation velocity depends on the $R$ distance. To investigate the dark matter consequences that eq. (28) can bring is another ongoing research that will be published elsewhere.

In fig. 3 we can see clearly that the NE version of the TF relation generates an $R$-dependence that does not exist in the standard TF law. We can notice that the rotation...
velocity \((v^2)\) increases as \(R\) increase and \(q\) has low values. Near the BG limit, the velocity decreases as the \(R\) also increases together with \(q\).

**Modified Friedmann equation.** From eq. (18) we have the thermodynamical expression for the Newtonian acceleration. Hence,

\[
\ddot{R} = \frac{\dot{a}(t)}{a} r = -GM \frac{\pi}{2l_p^2} \frac{(5 - 3q)(1 - q)}{1 + \frac{\pi a^2 r^2}{l_p^2} (1 - q)}, \quad (29)
\]

where \(R\) is the physical radius of the holographic screen which can be written as \(R(t,r) = a(t)r\), \(a(t)\) is the scale factor of the Friedmann-Robertson-Walker (FRW) metric and \(r\) is the radial comoving coordinate. Based on [33] the acceleration in eq. (29) results from the active gravitational mass, which is the well-known Tolman-Komar mass [34,35] given by

\[
M = \frac{4\pi}{3} a^3 r^3 \left( \rho + \frac{3p}{c^2} \right), \quad (30)
\]

which is proportional to the scale function. Substituting eq. (30) into eq. (29) we have a modified Friedmann equation in the context of Tsallis’ statistics which is

\[
\frac{\ddot{a}}{a} = -\frac{2\pi^2}{3l_p^2} a^2 r^2 G \left( \rho + \frac{3p}{c^2} \right) \frac{(5 - 3q)(1 - q)}{\ln \left[ 1 + \frac{\pi a^2 r^2}{l_p^2} (1 - q) \right]} \quad (31)
\]

Carrying out the limit \(q \to 1\) in eq. (31) we obtain the usual Friedmann equation, as expected, which is

\[
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left( \rho + \frac{3p}{c^2} \right). \quad (32)
\]

Here we would like to mention that in order to reproduce the usual Friedmann equation in the limit \(q \to 1\), eq. (32), we have introduced a negative sign on the right-hand side of eq. (29). Besides, let us comment that the NE Friedmann equations obtained in refs. [15] and [16] are constructed by replacing the usual gravitational constant by the effective gravitational constant which was defined in eq. (8). In [15,16], the usual number of bits, which is defined in eq. (1), was not modified.

The radial comoving coordinate \(r\) in eq. (31) can be eliminated using the dynamical apparent horizon radius [36,37] defined as

\[
R = ar = \frac{c}{\sqrt{H^2 + \frac{k}{a^2}}} \quad (33)
\]

where \(H = \dot{a}/a\) is the Hubble parameter and \(k\) denotes the curvature of space with \(k = 0, 1, -1\), corresponding to plane, open and closed universes, respectively. Then, using eq. (33) into (31) and the fact that \(\ddot{a}/a = \ddot{H} + H^2\), we obtain

\[
(\dot{H} + H^2) \ln \left[ 1 + \frac{\pi c^2}{l_p^2} \frac{1}{(H^2 + \frac{k}{a^2})} (1 - q) \right] = -\frac{2\pi^2 c^2}{3l_p^2} (5 - 3q)(1 - q)G \left( \rho + \frac{3p}{c^2} \right), \quad (34)
\]

which shows us an extremely nonlinear differential equation for the Hubble parameter as a function of \(q\). **Conclusions.** By choosing that the microstates number defined by \(W = c_1 N\), where \(c_1\) is the number of internal degrees of freedom and \(N\) is the number of bits, we have derived a nontrivial relation for the number of bits as a function of the holographic screen area. We have used an important non-Gaussian statistics defined by Tsallis’ entropy.

If we consider that the entropy of the holographic screen has a black hole form, i.e., \(S = k_B A/4l_p^2\), this new number of bits leads to an interesting modification of Newtonian...
acceleration, MOND theory and Friedmann equation. Our paper clearly shows that when the number of bits is directly proportional to the area of the holographic screen, then BG statistics is behind the previously chosen theory.

The NE version of the galaxies rotation acceleration was also obtained. After that we have obtained the NE version of the TF relation, which relates the rotation velocity of the galaxies to their mass. Although in the standard TF relation there is not a dependence on the distance of the star to the center of the galaxy, its NE version shows such distance dependence. The BG limit applied in this NE TF relation has recovered the standard TF relation. One can ask if this NE format, i.e., the $q$-dependent format, can bring any light to the dark matter problem. This is also an ongoing research.

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The authors thank CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico), Brazilian scientific support federal agency, for partial financial support, Grants Nos. 302155/2015-5 (EMCA) and 303140/2017-8 (JAN).

REFERENCES

[1] Verlinde E., JHEP, 04 (2011) 029.
[2] Padmanabhan T., Mod. Phys. Lett. A, 25 (2010) 1129.
[3] 't Hooft G., Dimensional Reduction in Quantum Gravity, arXiv:gr-qc/9310026.
[4] Susskind L., J. Math. Phys., 36 (1995) 6377.
[5] Tsallis C., J. Stat. Phys., 52 (1988) 479.
[6] Unruh W. G., Phys. Rev. D, 14 (1976) 870.
[7] Plastino A. R. and Lima J. A. S., Phys. Lett. A, 260 (1999) 46.
[8] Neto Jorge Ananias, Physica A, 391 (2012) 4320.
[9] Abreu Everton M. C., Neto Jorge Ananias, Mendes Albert C. R. and Oliveira W., Physica A, 392 (2013) 5154.
[10] Tsallis C., Introduction to Nonextensive Statistical Mechanics: Approaching a Complex World (Springer) 2009; Tsallis C., Braz. J. Phys., 29 (1999) 1.
[11] Alemay P. A. and Zanette D. H., Phys. Rev. Lett., 75 (1995) 366.
[12] Anteneodo C. and Tsallis C., J. Mol. Liq., 71 (1997) 255.
[13] Tsallis C., Chaos, Solitons Fractals, 13 (2002) 371.
[14] Silva R. and Alcaniz J. S., Physica A, 341 (2004) 208.
[15] Barboza Edélio M. jr., Nunes Rafael C., Abreu Everton M. C. and Neto Jorge Ananias, Physica A, 436 (2015) 301.
[16] Nunes Rafael C., Barboza Edélio M. jr., Abreu Everton M. C. and Neto Jorge Ananias, JCAP, 08 (2016) 051.
[17] Abreu Everton M. C., Neto Jorge Ananias, Mendes Albert C. R. and Souza Daniel O., EPL, 120 (2018) 20003.
[18] Abreu Everton M. C., Neto Jorge Ananias, Mendes Albert C. R. and Bonilla Alexander, EPL, 121 (2018) 45002.
[19] Jahromi A. Sayahian, Moosavi S. A., Moradpour H., Morais Graça J. P., Lobo I. P., Salako I. G. and Jawad A., Phys. Lett. B, 780 (2021) 21.
[20] Tayyef M., Sheykhi A., Bamba Kazuharu and Moradpour H., Phys. Lett. B, 781 (2018) 195.
[21] Ghaffari S., Moradpour H., Lobo I. P., Morais Graça J. P. and Bezerra Valdir B., Eur. Phys. J. C, 78 (2018) 706.
[22] Sheykhi A., Phys. Lett. B, 785 (2018) 18.
[23] Saridakis Emmanuel N., Bamba Kazuharu and Myrzakulov R., Holographic dark energy through Tsallis entropy, arXiv:1806.01301.
[24] da Silva W. J. C., Gimenes H. S. and Silva R., Astrophys. Part., 105 (2019) 37.
[25] Plastino A. and Rocca M. C., Phys. A, 505 (2018) 190.
[26] Majhi A., Phys. Lett. B, 775 (2017) 32.
[27] Sagui A. and Sarandy M. S., Phys. Lett. A, 374 (2010) 3384.
[28] Milgrom M., Astrophys. J., 270 (1983) 371.
[29] Milgrom M., Astrophys. J., 270 (1983) 365.
[30] Milgrom M., Astrophys. J., 270 (1983) 384.
[31] Tully R. B. and Fisher J. R., Astron. Astrophys., 54 (1977) 661.
[32] Neto J. A., Int. J. Theor. Phys., 50 (2011) 3552.
[33] Padmanabhan T., Class. Quantum Grav., 21 (2004) 4855.
[34] Tolman R. C., Phys. Rev., 35 (1930) 875.
[35] Komar A., Phys. Rev., 113 (1959) 934.
[36] Sheykhi A. and Hendi S. H., Phys. Rev. D, 84 (2011) 044023.
[37] Komatsu N., Eur. Phys. J. C, 77 (2017) 229.