ON RATIONAL CURVES IN n-SPACE WITH GIVEN NORMAL BUNDLE

HERBERT CLEMENS

Abstract. The stable rationality of components of the moduli space of (unparametrized) rational curves in projective n-space with fixed normal bundle is proved, provided these components dominate the moduli space of immersed rational curves in the plane.

1. Introduction

In two papers almost twenty years ago, Eisenbud and Van de Ven studied the variety of parametrized rational curves in \(\mathbb{P}^3\) and showed that it is stratified according to the isomorphism classes of the normal bundle of the curve and that each stratum is rational. They posed the problem of rational strata in the unparametrized case and rationality was then proved for “half” the cases by Ballico in [B]. Also in 1984 Katsylo [H] implicitly proved rationality of the full moduli space of unparametrized rational curves in \(\mathbb{P}^n\). There seems to have been little attention to the problem of rationality of strata of rational curves in higher-dimensional projective spaces, where Eisenbud and Van de Ven said that the straightforward extension of the their results to higher dimension was “to be feared.”

Let \(\tilde{S}_d^n\) denote the space of (unparametrized) immersed rational curves in \(\mathbb{P}^n\), that is

\[
\tilde{S}_d^n = \{ f : \mathbb{P}^1 \to \mathbb{P}^n : f \text{ immersive, } \deg f = d \}
\]

The purpose in writing this short note is to draw attention to the unparametrized case in \(\mathbb{P}^n\), and in particular to make an observation that each stratum of the natural stratification of \(\tilde{S}_d^n\) by normal-bundle-type has a natural stratification with strata which are birationally vector bundles over strata of \(\tilde{S}_d^2\). Using Ballico’s result, it is easy to show that
\( \tilde{S}_d \) is stably rational, in fact that
\[ \tilde{S}_d \times \mathbb{A}^{d+1} \]
is rational. This allows us to conclude stably rational for all components
of strata of \( \tilde{S}_d \) which dominate \( \tilde{S}_d \), and so to conclude rationality whenever
\[ \dim S' - \dim \tilde{S}_d \geq d + 1. \]

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2. Immersed rational curves

Let
\[ f = (s^0, \ldots, s^n) : \mathbb{P}^1 \to \mathbb{P}^n \]
be an immersion. We have an exact sequence
\[ 0 \to \mathfrak{D}_1 (\mathcal{O}_{\mathbb{P}^1} (1)) \to f^* \mathfrak{D}_1 (\mathcal{O}_{\mathbb{P}^n} (1)) \to N_f \to 0 \]
where \( \mathfrak{D}_1 (L) \) is the sheaf of first-order holomorphic differential operators on sections of the line bundle \( L \) and
\[ \mu \left( a^i \frac{\partial}{\partial U^i} \right) = a^i \frac{\partial s^j}{\partial U^i} \frac{\partial}{\partial X^j}. \]
Considering \( \mathfrak{D}_1 \) as a left \( \mathcal{O} \)-module, apply the functor
\[ \text{Hom}(\ , \mathcal{O}_{\mathbb{P}^1}) \]
to the exact sequence
\[ 0 \to \mathfrak{D}_1 (\mathcal{O}_{\mathbb{P}^1} (1)) \to f^* \mathfrak{D}_1 (\mathcal{O}_{\mathbb{P}^n} (1)) \to N_f \to 0 \]
to obtain
\[ 0 \to N_f^\vee \to f^* \mathcal{O}_{\mathbb{P}^n} (-1)^{\otimes (n+1)} \to \mathcal{O}_{\mathbb{P}^1} (-1)^{\otimes 2} \to 0. \]

Now a first-order deformation of the map \( f \) gives a first-order deformation of the map \( \mu^\vee \) and so an element of
\[ \text{Hom}(N_f^\vee, \mathcal{O}_{\mathbb{P}^1} (-1)^{\otimes 2}). \]
Thus, by the exact sequence (3), we obtain an element
\[ \nu \in \text{Ext}^1 (N_f^\vee, N_f^\vee) = H^1 (\text{End}(N_f)) \]
which measures the first-order deformation of the normal bundle \( N_f \) as a vector bundle over \( \mathbb{P}^1 \).

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By Grothendieck’s lemma
\[ N_f = \bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^1} (a'_i) \]
where, by adjunction,
\[ \sum_{i=1}^{n-1} a'_i = d(n + 1) - 2. \]
Let \( L \subseteq \mathbb{P}^n \) be a general linear space of codimension 3. Then the family of projective spaces
\[ \overline{L f (x)}, \ x \in \mathbb{P}^1, \]
gives a sub-bundle
\[ \mathcal{O}_{\mathbb{P}^1} (d)^{n-2} \subseteq N_f. \]
Varying the choice of \( L \), these sub-bundles span \( N_f \). Thus each \( a'_i \geq d \) and we put
\[ a_i = a'_i - d. \]
Then
\[ N_f = \bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^1} (d + a_i) \]
where, for all \( i \),
\[ 0 \leq a_i \leq 3d - 2 \]
and
\[ \sum_{i=1}^{n-1} a_i = 2d - 2. \]

3. Projection to \( \mathbb{P}^2 \)

Suppose we begin with an immersion
\[ f_0 = (s^0, \ldots, s^n, 0, \ldots, 0) : \mathbb{P}^1 \to \mathbb{P}^{n+r}. \]
We wish to study the deformations of \( f_0 \) in \( \mathbb{P}^{n+r} \). Such a deformation is given by choosing \( t^k \in H^0 (\mathcal{O}_{\mathbb{P}^1} (d)) \) for \( k = 1, \ldots, r \) and defining
\[ f_\varepsilon = (f_0, \varepsilon t^1, \ldots, \varepsilon t^r). \]
We wish to compute the associated element of \( H^1 (\operatorname{End} (N_{f_0})) \) in \( \mathbb{P} \).
Applying the functor
\[ R\operatorname{Hom} (N_{f_0}^\vee, ) \]
to the sequence \( \mathbb{B} \), the infinitesimal deformation \( N_{f_\epsilon} \) of \( N_{f_0} \) is given by the image of the matrix

\[
(4) \quad \begin{pmatrix}
0 & \cdots & 0 & \frac{\partial s_0}{\partial U_0} & \cdots & \frac{\partial s_n}{\partial U_0} \\
0 & \cdots & 0 & \frac{\partial s_0}{\partial U_1} & \cdots & \frac{\partial s_n}{\partial U_1}
\end{pmatrix} \in \text{Hom} \left( N_{f_0}^\vee, \mathcal{O}_{\mathbb{P}^1} (-1)^{\oplus 2} \right)
\]

in \( \text{Ext}^1 (N_{f_0}^\vee, N_{f_0}^\vee) = \text{Ext}^1 (N_{f_0}, N_{f_0}) \), that is, by the equivalence class of \( (4) \) in

\[
\text{Hom} \left( N_{f_0}^\vee, \mathcal{O}_{\mathbb{P}^1} (-1)^{\oplus 2} \right) / \text{Hom} \left( N_{f_0}^\vee, f^* \mathcal{O}_{\mathbb{P}^n} (-1)^{\oplus (n+r+1)} \right) \circ \mu^\vee.
\]

Now

\[
N_{f_0}^\vee = \mathcal{O}_{\mathbb{P}^1} (-d)^r \oplus N_g^\vee
\]

where

\[
g = (s_0, \ldots, s_n) : \mathbb{P}^1 \to \mathbb{P}^n
\]

and the projection

\[
\text{Hom} \left( N_{f_0}^\vee, \mathcal{O}_{\mathbb{P}^1} (-1)^{\oplus 2} \right) \to \text{Hom} \left( \mathcal{O}_{\mathbb{P}^1} (-d)^r, \mathcal{O}_{\mathbb{P}^1} (-1)^{\oplus 2} \right)
\]

takes \( \text{Hom} \left( N_{f_0}^\vee, f^* \mathcal{O}_{\mathbb{P}^n} (-1)^{\oplus (n+r+1)} \right) \) to the homomorphisms generated by the columns of

\[
\begin{pmatrix}
\frac{\partial s_0}{\partial U_0} & \cdots & \frac{\partial s_n}{\partial U_0} \\
\frac{\partial s_0}{\partial U_1} & \cdots & \frac{\partial s_n}{\partial U_1}
\end{pmatrix}.
\]

Thus, for example, every first-order deformation of \( N_{f_0} \) which does not change the normal bundle is a combination of a deformation of \( g \) in \( \mathbb{P}^n \) combined with a deformation of \( \mathbb{P}^n \) in \( \mathbb{P}^n+r \).

Finally now suppose that \( n = 2 \). Then we have that

\[
N_{f_0}^\vee = \mathcal{O}_{\mathbb{P}^1} (-d)^r \oplus N_g^\vee = \mathcal{O}_{\mathbb{P}^1} (-d)^{\oplus r} \oplus \mathcal{O}_{\mathbb{P}^1} (2-3d).
\]

Also

\[
\text{Hom} \left( N_{f_0}^\vee, \right) = \text{Hom} \left( \mathcal{O}_{\mathbb{P}^1} (-d)^r, \right) \oplus \text{Hom} \left( N_g^\vee, \right)
\]

and

\[
\text{Ext}^1 (N_{f_0}^\vee, N_{f_0}^\vee) = \text{Ext}^1 (\mathcal{O}_{\mathbb{P}^1} (-d)^{\oplus r}, N_{f_0}^\vee) \]

\[
= \text{Ext}^1 (\mathcal{O}_{\mathbb{P}^1} (-d), N_g^\vee)^{\oplus r}
\]

\[
= H^1 (\mathcal{O}_{\mathbb{P}^1} (2-2d))^{\oplus r}.
\]
4. The case of $\mathbb{P}^{2+1}$

From what we have just seen in the last section, the $j$-th entry in the element of $\text{Ext}^1(N^\vee_{f_0},N^\vee_{f_0})$ corresponding to the deformation (3) is given by applying

$$R\text{Hom} (\mathcal{O}_{\mathbb{P}^1} (-d) , )$$

to the sequence

$$0 \to N^\vee_g \to \mathcal{O}_{\mathbb{P}^1} (-d)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^1} (-1)^{\oplus 2} \to 0$$

and finding the image $\nu$ of

$$\left( \frac{\partial t}{\partial U^0}, \frac{\partial t}{\partial U^1} \right) \in \text{Hom} \left( \mathcal{O}_{\mathbb{P}^1} (-d) , \mathcal{O}_{\mathbb{P}^1} (-1)^{\oplus 2} \right)$$

in

$$\text{Hom} \left( \mathcal{O}_{\mathbb{P}^1} (-d) , \mathcal{O}_{\mathbb{P}^1} (-1)^{\oplus 2} \right) = H^1 \left( \mathcal{O}_{\mathbb{P}^1} (2 - 2d) \right).$$

On the other hand, for all integers $a$ we have

$$\text{Ext}^1 \left( N^\vee_{f_0},N^\vee_{f_0} \right) = \text{Ext}^1 \left( N^\vee_{f_0} (a) , N^\vee_{f} (a) \right)$$

and so we have a pairing

$$H^0 \left( N^\vee_{f_0} (a) \right) \otimes H^1 \left( \text{End} \left( N^\vee_{f_0} (a) \right) \right) \to H^1 \left( N^\vee_{f_0} (a) \right)$$

(6)

which measures the obstruction to deforming sections of $N^\vee_{f_0} (a)$ to first order with a first-order deformation of $N^\vee_{f_0}$ given by an element of

$$H^1 \left( \text{End} \left( N^\vee_{f_0} \right) \right) = H^1 \left( \text{End} \left( N^\vee_{f_0} (a) \right) \right).$$

But we are in a situation in which the only obstructions to deforming sections of $N^\vee_{f_0} (a)$ to sections of $N^\vee_{f_0} (a)$ for all $\varepsilon$ are of first order. To see this use (2) to write

$$0 \to N^\vee_{f_\varepsilon} (a) \xrightarrow{\nu} \mathcal{O}_{\mathbb{P}^n} (a - d)^{\oplus 4} \xrightarrow{\mu^\vee} \mathcal{O}_{\mathbb{P}^1} (a - 1)^{\oplus 2} \to 0$$

with the matrix $\mu^\vee$ is given by

$$\left( \begin{array}{cccc} \frac{\partial s^0}{\partial U^0} & \frac{\partial s^1}{\partial U^0} & \frac{\partial s^2}{\partial U^0} & \frac{\partial s^3}{\partial U^0} \\ \frac{\partial s^0}{\partial U^1} & \frac{\partial s^1}{\partial U^1} & \frac{\partial s^2}{\partial U^1} & \frac{\partial s^3}{\partial U^1} \end{array} \right).$$
So the equations for extension of sections become
\[
\begin{pmatrix}
\frac{\partial s^0}{\partial U^0} & \frac{\partial s^1}{\partial U^1} & \frac{\partial s^0}{\partial U^1} & \frac{\partial s^1}{\partial U^0} & \varepsilon \frac{\partial}{\partial U^0} & \alpha_{00} + \varepsilon \alpha_{01} \\
\frac{\partial s^1}{\partial U^1} & \frac{\partial s^2}{\partial U^2} & \frac{\partial s^1}{\partial U^2} & \frac{\partial s^2}{\partial U^1} & \varepsilon \frac{\partial}{\partial U^1} & \alpha_{10} + \varepsilon \alpha_{11} \\
\frac{\partial s^2}{\partial U^2} & \frac{\partial s^3}{\partial U^3} & \frac{\partial s^2}{\partial U^3} & \frac{\partial s^3}{\partial U^2} & \varepsilon \frac{\partial}{\partial U^2} & \alpha_{20} + \varepsilon \alpha_{21} \\
\frac{\partial s^3}{\partial U^3} & \frac{\partial s^4}{\partial U^4} & \frac{\partial s^3}{\partial U^4} & \frac{\partial s^4}{\partial U^3} & \varepsilon \frac{\partial}{\partial U^3} & \alpha_{30}
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
which have no terms of degree > 1 in \( \varepsilon \).

Now we rewrite the pairing (6) vertically as
\[
(H^0(\mathcal{O}_{\mathbb{P}^1}(a-d)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(a + 2 - 3d))) \otimes H^1(\mathcal{O}_{\mathbb{P}^1}(2 - 2d)) \downarrow
\]
\[
H^1(\mathcal{O}_{\mathbb{P}^1}(a-d)) \oplus H^1(\mathcal{O}_{\mathbb{P}^1}(a + 2 - 3d))
\]
Thus for \( d - 1 \leq a \leq 3d - 2 \) the pairing (6) becomes the multiplication map
\[
H^0(\mathcal{O}_{\mathbb{P}^1}(a-d)) \otimes H^1(\mathcal{O}_{\mathbb{P}^1}(2 - 2d)) \downarrow
\]
\[
H^1(\mathcal{O}_{\mathbb{P}^1}(a + 2 - 3d))
\]
which we restrict to the element
\[
\text{image} \left( \frac{\partial t}{\partial U} \right) \in H^1(\mathcal{O}_{\mathbb{P}^1}(2 - 2d)).
\]

It will be important to note one structural property of the map (6). Let \( p \in \mathbb{P}^1 \) with local coordinate \( z \). Using the exact sequence
\[
0 \to \mathcal{O}_{\mathbb{P}^1}((2 - 2d) \cdot p) \to \mathcal{O}_{\mathbb{P}^1} \to \left\{ \sum_{j=1}^{\infty} C_j z^j \right\} \frac{z^{2d - 2}}{z^{2d - 2}} \to 0
\]
we rewrite (6) as
\[
\left\{ \sum_{i=0}^{a-d} C_i z^i \right\} \otimes \left\{ \sum_{j=1}^{\infty} C_j z^j \right\} \frac{z^{2d - 2}}{z^{2d - 2}} \to 0
\]
and make the important remark that the dimension of the kernel of any map
\[
\xi : \left\{ \sum_{i=0}^{a-d} C_i z^i \right\} \to \left\{ \sum_{j=1}^{\infty} C_j z^j \right\} \frac{z^{3d - a - 2}}{z^{3d - a - 2}}
\]
given by multiplication by $\xi \in \left\{ \sum_{j=1}^{\infty} C \cdot z^j \right\}$ depends only on the degree of the leading term in $\xi$. Also, since the map

$$H^0(\mathcal{O}_{\mathbb{P}^1}(d)) \to H^1(\mathcal{O}_{\mathbb{P}^1}(2 - 2d))$$

$$t \mapsto \xi$$

is linear, the leading coefficient must vanish on a codimension-one subspace of $H^0(\mathcal{O}_{\mathbb{P}^1}(d))$.

5. The theorem for $\mathbb{P}^{2+r}$

So, returning to the general situation of $n = 2$ and $r$ arbitrary, we can completely characterize the extension $N^\vee_{f_\varepsilon}$ by knowing

$$h^0\left(N^\vee_{f_\varepsilon}(a)\right)$$

for each integer $a$. But, for a given first-order deformation $\Xi \in H^1(End(N_{f_0})) = H^1(End(N_{f_0}(a)))$, these latter numbers are given by the dimensions of the subspaces of $H^0\left(N^\vee_{f_\varepsilon}(a)\right)$ consisting of sections which deform to first-order, and hence to all orders, with $\varepsilon$. But these are just the dimension of $\ker\left(H^0(N_{f_0}(a)) \to H^1(N_{f_0}(a))\right)$.

So these dimensions are given by the kernels of the $r$ maps (8). Now use the map

$$\Phi : H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\oplus r \to \left\{ \sum_{j=1}^{\infty} C \cdot z^j \right\}^\oplus r$$

$$(t^1, \ldots, t^r) \mapsto \left( \frac{\partial t^1}{\partial U_0} \quad \cdots \quad \frac{\partial t^r}{\partial U_0} \right)$$

to define a map

$$H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\oplus r \to \left\{ 1, \ldots, 2d - 3 \right\}^\oplus r$$

which associates to each $(t^1, \ldots, t^r)$ the degrees of the leading coefficients of $\Phi(t^1, \ldots, t^r)$. This is a semi-continuous map for which the preimage of each element is a linear space minus a linear subspace. And as we vary the immersed curve $g(\mathbb{P}^1) \subseteq \mathbb{P}^2$, the dimensions of each linear space is locally constant on a Zariski open set. Thus:
Theorem 5.1. i) Let $S$ be the set of all smooth rational curves in $\mathbb{P}^{2+r}$ whose image via the standard projection
\[ \mathbb{P}^{2+r} \to \mathbb{P}^2 \]
is a fixed immersed curve and whose normal bundle is
\[ \bigoplus_{i=1}^{r+1} \mathcal{O}_{\mathbb{P}^1}(d + a_i) \]
for some fixed value of $(a_1, \ldots, a_{r+1})$. Then $S$ is rational.

ii) Suppose the general curve in $\tilde{S}_d^{r+2}$ deforms to a curve in $\tilde{S}_{d,a_1,\ldots,a_{r+1}}$. The set
\[ \tilde{S}_{d,a_1,\ldots,a_{r+1}}^{r+2} \]
consisting of all smooth rational curves in $\mathbb{P}^{2+r}$ with normal bundle (9) has a component which is birationally isomorphic to a vector bundle over the set of immersed rational curves in $\mathbb{P}^2$, and all other components consists of curves projecting into proper subvarieties of $S_d^2$.

Proof. We consider the set
\[ \tilde{g} \cdot GL(2) \]
of reparametrizations of a fixed map
\[ \tilde{g} : \mathbb{A}^2 \to \mathbb{A}^3 \]
with projectivization
\[ g : \mathbb{P}^1 \to \mathbb{P}^2. \]
We need only check the action of the group $GL(2)$ on the space
\[ (\tilde{g} \cdot GL(2)) \times H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\oplus r \]
parametrization of maps
\[ \tilde{f} : \mathbb{A}^2 \to \mathbb{A}^{3+r}. \]
But, considering (10) as a (trivial) vector bundle over the affine variety $(\tilde{g} \cdot GL(2))$, this is a free action of the group
\[ \frac{GL(2)}{\mu_d I} \]
on the vector bundle (10) over a free action on the base space $(\tilde{g} \cdot GL(2))$ and this group acts as a group of vector bundle isomorphisms. So, by descent theory for coherent sheaves and faithful flatness, the quotient is a vector bundle with fiber isomorphic to $H^0(\mathcal{O}_{\mathbb{P}^1}(d))^\oplus r$. \qed
Corollary 5.2. Suppose the general curve in $\tilde{S}_d^2$ deforms to a curve in $\tilde{S}_{d,a_1,...,a_{r+1}}^{r+2}$. Then each component $S'$ of $\tilde{S}_{d,a_1,...,a_{r+1}}^{r+2}$ over the generic curve of $\tilde{S}_d^2$ is stably rational. Furthermore $S'$ is rational if
\[
\left( \dim S' - \dim \tilde{S}_d^2 \right) \geq (d + 1).
\]

Proof. Katsylo proved [H] that $\tilde{S}_d^3 = \tilde{S}_d^{3,d-1,d-1}$ is rational. But by Theorem 5.1, $\tilde{S}_d^{3,d-1,d-1}$ is birationally a vector bundle over $\tilde{S}_d^2$. Since vector bundles are locally trivial in the Zariski topology, we conclude that the product of $\tilde{S}_d^2$ with the vector-bundle fiber $F \cong \mathbb{A}^{d+1}$ is rational. That is
\[
\tilde{S}_{d,d-1,d-1}^{3} \xrightarrow{\text{biration.}} \tilde{S}_d^2 \times \mathbb{A}^{d+1}.
\]
So $\tilde{S}_d^2$ is stably rational. Now suppose a component $S'$ of $\tilde{S}_{d,a_1,...,a_{r+1}}^{r+2}$ dominates $\tilde{S}_d^2$ via the map
\[
\tilde{S}_{d,a_1,...,a_{r+1}}^{r+2} \rightarrow \tilde{S}_d^2
\]
induced from the projection map
\[
\mathbb{P}^{r+2} \rightarrow \mathbb{P}^2.
\]
Again by Theorem 5.1, $S'$ is birationally a (locally trivial) vector bundle over $\tilde{S}_d^2$ with fiber $F' \cong \mathbb{A}^s$. So $S'$ is always stably rational and is rational whenever
\[
s \geq d + 1.
\]

Notice that some of the $\tilde{S}_{d,a_1,...,a_{r+1}}^{r+2}$ may not dominate $\tilde{S}_d^2$, in which case we can conclude nothing.

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Mathematics Department, University of Utah, Salt Lake City, UT, 84112, USA
E-mail address: clemens@math.utah.edu