PATH-CONNECTIVITY OF THE SET OF UNIQUELY ERGODIC AND COBOUNDED FOLIATIONS

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Abstract. We show that if $S$ is a closed surface of genus $g \geq 5$ or a surface of genus $g \geq 2$ with at least $p \geq 1$ marked points, then the set of uniquely ergodic foliations and the set of cobounded foliations is path-connected and locally path-connected.

1. Introduction

Projective measured foliations play a prominent role in Teichmüller theory, dynamics and the study of mapping class groups. In addition to the structure of individual foliations, the set $\mathcal{PMF}(S)$ of all foliations on a given finite type surface $S$ has particular importance. $\mathcal{PMF}(S)$ carries a natural (weak-*) topology and is homeomorphic to a sphere of dimension $6g + 2p - 7$ if $S$ has genus $g$ and $p$ punctures. One reason for its importance stems from the fact that $\mathcal{PMF}(S)$ can be identified with both the sphere of directions, and the boundary of infinity of Teichmüller space. One can also use $\mathcal{PMF}(S)$ to describe the Gromov boundary of the curve graph.

In this article we study global topological properties of two dynamically motivated subsets of $\mathcal{PMF}(S)$. The first is the set $\mathcal{UE}(S)$ of uniquely ergodic foliations, where a foliation $F$ is called uniquely ergodic if it admits a unique transverse measure up to scale. The second is the set $\mathcal{COB}(S)$ of cobounded foliations, where $F$ is called cobounded if a Teichmüller geodesic ray with vertical foliation $F$ projects into a compact set of the moduli space of Riemann surfaces.

These sets have been intensely studied from a dynamical point of view, owing to their importance in Teichmüller theory. As a starting point, by a theorem of Masur [Mas], any cobounded foliation is uniquely ergodic, and we therefore have $\mathcal{COB}(S) \subset \mathcal{UE}(S) \subset \mathcal{PMF}(S)$.

Both $\mathcal{COB}(S)$ and $\mathcal{UE}(S)$ are dense in $\mathcal{PMF}(S)$ (but the same is also true for their complements). Masur and Veech [Mas, Vee] show that $\mathcal{UE}(S)$ has full measure in $\mathcal{PMF}(S)$. In contrast, the set $\mathcal{COB}(S)$ has measure zero.

It is known that there are many embedded circles in $\mathcal{COB}(S)$ [LS2]. On the other hand, Masur and Smillie [MS] have shown that the complement $\mathcal{PMF}(S) \setminus \mathcal{UE}(S)$ has Hausdorff dimension strictly bigger than $\dim \mathcal{PMF}(S) - 1$, and hence one cannot expect to naively locally deform paths in order to avoid $\mathcal{PMF}(S) \setminus \mathcal{UE}(S)$ by general position arguments.

Our main result shows that paths are nevertheless abundant in $\mathcal{COB}(S)$ and $\mathcal{UE}(S)$.
Theorem 1.1. Let $S$ be a closed surface of genus at least 5, or a surface of genus at least 2 with at least 1 puncture. Then the subsets $\mathcal{UE}(S), \mathcal{COB}(S)$ are path-connected, and locally path-connected. Moreover for any finite set $F$ we have that $\mathcal{UE}(S) \setminus F$, and $\mathcal{COB}(S) \setminus F$ are path connected.

In fact, the proof shows something slightly stronger: any two points in $\mathcal{UE}(S)$ can be joined by a continuous path which is contained in $\mathcal{COB}(S)$ except possibly at its endpoints.

Our result can also be used to show that through any finite number of points in $\mathcal{UE}$ or $\mathcal{COB}$ there is an embedded circle in $\mathcal{UE}$ or $\mathcal{COB}$. To ensure that the circle is embedded, one has to use the proof of Theorem 1.1 rather than just the statement. We omit details, as the claim is not central to our discussion.

Proof Strategy and Structure of this Article. To build our paths, we will connect a 'nice' (in the case of surfaces with punctures: stable foliation for a point-push Pseudo-Anosov) $p$ to an arbitrary uniquely ergodic foliation $\lambda$ by a sequence of paths $\gamma_0, \gamma_1, \ldots$ so that

1. the initial point on $\gamma_0$ is $p$.
2. The initial point on $\gamma_{j+1}$ is the terminal point of $\gamma_j$.
3. For every $\epsilon > 0$ there exists $k$ so that $\gamma_j$ is contained in an $\epsilon$-neighborhood of $\lambda$.
4. $\bigcup_j \gamma_j \subset \mathcal{COB}$.

These conditions give that the concatenation of the $\gamma_j$ extends to a path from $p$ to $\lambda$ (in particular it is continuous at $\lambda$).

We now highlight two main ingredients to accomplish this. On the one hand, we will develop in Section 3 a robust mechanism to construct paths of cobounded foliations in the sphere of projective measured foliations of a punctured surface. This construction was heavily inspired by the work in [LS1] (who showed that there is a dense path connected set of arational foliations in $\mathcal{PMF}$), and our main new contribution here is to use bad approximability of points under straight line flows on tori to certify coboundedness and to improve the paths built in [LS1] to consist of cobounded foliations. This will be done in Section 3.

Our second ingredient is to show how to link these paths to arbitrary uniquely ergodic foliations. Here, we use train track splitting sequences to define mapping class group sequences that exhibit contracting behaviour on $\mathcal{PMF}$, extending the contraction of the polyhedra of measures along the splitting sequence. The main technical work to make this work happens in Section 2 and uses the hyperbolic geometry of curve graphs to show that these sequences act on $\mathcal{PMF}$ in a contracting way.

Section 4 then combines these two parts and shows the path-connectivity statement in Theorem 1.1 for punctured surfaces. This is also the prerequisite for Section 5 in which the path-connectivity statement of Theorem 1.1 is proved for closed surfaces.

Finally, in Section 6 we show how to leverage the constructions of paths to show local path-connectivity.
Further Questions. Finally, we want to highlight a few questions for further research suggested by Theorem 1.1 and its proof.

**Question 1.** Are $\mathcal{UE}(S)$ and $\mathcal{COB}(S)$ simply connected, if the genus of $S$ is sufficiently large?

**Question 2 (Gabai [Gab]).** Is the set $\mathcal{AF}(S) \supset \mathcal{UE}(S)$ of arational foliations path-connected?

This question came up in Gabai’s analysis of connectivity properties of the Gromov boundary of the curve graph (which is the quotient of $\mathcal{AF}(S)$ by the map which “forgets” the measure on the foliation). Gabai proves that this boundary is path-connected, but his methods do not apply to $\mathcal{AF}(S)$ directly. Leininger and Schleimer [LS1] proved that the set $\mathcal{AF}(S)$ of arational foliations is connected, and contains a dense path-connected subset, but it is not clear that these paths can be extended to the closure. We suspect that our curve graph methods can recover Gabai’s result that ending lamination space is path connected in the case of a surface of genus at least 5. Partly because such genus bounds would not be optimal, we do not prove this here. However, we want to remark that this kind of strategy is used in [BCH] to prove path connectivity and local path connectivity of the boundary of the free factor graph.

Our methods are at the moment also unable to deal with the case of arational foliations, mainly because the contraction properties in Section 2. This is due to the fact that in order to certify contraction we use the curve graph boundary, which is unable to distinguish different measures supported on a topological foliation.

Next, one could consider more restrictive subsets of $\mathcal{COB}(S)$. Namely, suppose we fix a constant $\epsilon > 0$. Call a foliation $F$ $\epsilon$-cobounded if a Teichmüller ray with vertical foliation $F$ eventually stays in the $\epsilon$-thick part of Teichmüller space.

**Question 3.** Is the set $\mathcal{COB}_\epsilon(S)$ of $\epsilon$-cobounded foliations path-connected for any choice of small enough $\epsilon$?

Our methods do not yield this, since the paths (both in Section 4 and 5) need to degenerate very close to simple closed curves in order to apply the methods from Section 2. However, the basic paths from Section 3 can be guaranteed to have uniform thickness.

Finally, one motivating reason for studying paths of cobounded paths in the sphere of projective measured foliation stems from one of the central open questions in the study of mapping class groups and Teichmüller theory. Namely, Farb–Mosher [FM2] define *convex cocompact subgroups* in analogy to such Kleinian groups. At this time, all known examples of such groups are virtually free, and it is not clear if any other examples can exist. One touchstone question is therefore: is there a convex cocompact subgroup of the mapping class group, which is isomorphic to the fundamental group of a higher genus surface. Such a group $G$ would give rise to a $G$–invariant circle in $\mathcal{COB}(S)$.

**Question 4.** Are there embedded circles in $\mathcal{COB}(S)$ which are invariant under groups that are not free?

Most likely, this question requires significant new tools. A weaker version of this question arises if we relax the invariance condition, e.g.
Question 5. Is there a finite subset $F \subset \text{Mod}(S)$, and $P \subset F^2$ so that for $x$ in Teichmüller space we have that the limit in $\mathcal{PMF}$ of

$$\{s_1...s_n x : (s_1...s_n) \in F^n \text{ and } (s_i, s_{i+1}) \in P \text{ for all } i < n\}_{n \in \mathbb{N}}$$

is a circle in $\text{COB}(S)$. That is, is there a “convex cocompact shift of finite type” which has a circle limit set of cobounded foliations in $\mathcal{PMF}$?

One could also ask a similar question for semigroups.

2. Contractions on $\mathcal{PMF}$

We denote by $\mathcal{PMF}$ the sphere of projective measured foliations. Recall that a foliation is called minimal, if every regular leaf is dense. As mentioned in the introduction, we call a foliation $F$ uniquely ergodic, if $F$ admits a unique transverse measure up to scale. We call a foliation $F$ cobounded if a Teichmüller ray with vertical foliation $F$ is contained in some thick part of Teichmüller space. By Masur’s criterion [Mas], cobounded foliations are uniquely ergodic, and it is well known that uniquely ergodic foliations are minimal.

Throughout this article, we will use the notion of measured foliations, although most literature on train tracks uses measured geodesic laminations instead. We refer the reader to [Lev] for an excellent dictionary between foliations and laminations on surfaces. Most of the time this will not be cause for confusion. We only want to emphasise that a minimal foliation in our sense corresponds to a minimal and filling lamination. In particular, there are no simple closed curves which have intersection 0 with a minimal foliation.

2.1. From splitting sequences to mapping classes. This section sets out the framework connecting mapping class group elements and train track splitting sequences. We refer the reader to [PH] for a detailed treatment of the basic theory of train tracks, and [MM1] for some other concepts we use.

If $\tau$ is a train track and $F$ is a foliation, we write $F \prec \tau$ if $F$ is carried by $\tau$ (compare [PH, Section 1.6], noting that in [PH] the notion of measured geodesic laminations is used in place of foliations). We denote by $P(\tau) \subset \mathcal{MF} \setminus \{0\}$ the set of measured foliations which are carried by $\tau$. When it does not cause confusion, we will often identify $P(\tau)$ with the subset of the sphere $\mathcal{PMF}$ of projective measured foliations it defines. The set $P(\tau)$ naturally has the structure of a closed polyhedron, whose faces correspond to the polyhedra $P(\eta)$ of subtracks $\eta$ of $\tau$.

A train track is called recurrent, if for every branch there is a train path which traverses it. It is called birecurrent if in addition there is a multicurve hitting the train track efficiently (i.e. without generating bigons) which intersects every branch (compare [PH, Section 1.3] for details on these definitions). From now on, we will usually assume without mention that all train tracks we use are birecurrent. We say that a train track is large if every complementary component is simply connected, and maximal, if every complementary component is a triangle (which implies largeness).

For maximal, birecurrent train tracks $\tau$, the interior of $P(\tau)$ defines an open set in $\mathcal{PMF}$ [PH, Lemma 3.1.2]. For other train tracks this need not be the case. By the interior $\text{int} P(\tau)$ of $P(\tau)$ we will always mean the subset of $P(\tau)$ formed by all those measures which assign a positive weight to each branch. We stress again
that, in general, this is different from the topological interior of $P(\tau)$ as a subset of $\mathcal{PMF}$ or $\mathcal{MF} \setminus \{0\}$.

Given a train track $\tau$, a branch $b$ is large, if every train path through either of its endpoints runs through $b$. Recall that we can perform a left, right or central split at a large branch to obtain a new train track $\tau'$. Compare [PH §2.1] for details on this construction. We recall that a left or right split does not affect the number and type of complementary components of the train track, while a central split can join two complementary components into one.

Let $\tau$ be a fixed maximal, birecurrent train track. As noted above, the polyhedron $P(\tau)$ defines an open set in $\mathcal{PMF}$. We let $T(\tau)$ be the set of all large birecurrent train tracks which can be obtained from $\tau$ by any number of splits (left, right, or central). The set $T(\tau)$ can be stratified in the following way. Put $T_0(\tau) = \{\tau\}$, and inductively define $T_{n+1}(\tau)$ to be the set of large train tracks obtained from each $\sigma \in T_n(\tau)$ by splitting each large branch once (in one of the up to three possible ways). Note that a central split need not yield a large train track, so not all three possibilities are always allowed.

A large branch $b$ of a large birecurrent train track $\sigma$ defines a hyperplane $H$ in $P(\sigma)$ cutting $P(\sigma)$ into subpolyhedra $P_l, P_r$, which are exactly the polyhedra of the left and right splits of $\sigma$. The polyhedron of the central split of $\sigma$ at $b$ is the hyperplane $H$ [PH Proposition 2.2.2]. Hence, the interiors of the polyhedra $P(\sigma), \sigma \in T_n(\tau)$ define a decomposition of $P(\tau)$ into disjoint subpolyhedra.

Now, let $F \in \text{int } P(\tau)$ be given, and let
\[ T(\tau, F) = \{\sigma \in T(\tau), F \prec \sigma\} \]
be the subset of all those train tracks in $T(\tau)$ which carry $F$. We let $T_n(\tau, F)$ be the set of all those $\sigma \in T_n(\tau)$ which carry $F$. For the next lemma, we use the notion of diagonal extension. If $\tau$ is a train track, then we say that $\eta$ is a diagonal extension of $\tau$ if $\eta$ is obtained by adding branches inside simply connected complementary components. See [MM1] Section 4.1 for details.

**Lemma 2.1.** The sets $T_n(\tau, F)$ only contain diagonal extensions of the (large) train track $\eta_n \in T_n(\tau, F)$ with the fewest complementary components.

**Proof.** Consider the sequence $\eta_k$ of train tracks obtained by splitting $\tau$ in the direction of $F$ and always choosing a central split when possible. These have the property that they always carry $F$, and additionally, the weight defined by $F$ is positive on every branch of $\eta_k$ for all $k$ ($F$ fills $\eta_k$). Note that for any $\sigma \in T_n(\tau)$ there exists (at least one) $\eta_k$ (depending on $\sigma$) so that $\eta_k$ is a subtrack of $\sigma$ (this follows inductively, since if a foliation is carried by, and fills, a subtrack $\eta$ of $\sigma$ and $\sigma$ splits to $\sigma'$, then either $\eta$ or a split of $\eta$ is a subtrack of $\sigma'$). Since $F$ is minimal, and therefore there is no simple closed curve that doesn’t intersect $F$, it can only be carried by large train tracks. Therefore, $\sigma$ is a diagonal extension of $\eta_k$. The lemma now follows, since any set $T_n(\tau, F)$ contains at most one $\eta_k$, and the number of complementary components in a split decreases only during central splits. \[ \square \]

We put
\[ U_n(\tau, F) = \bigcup_{\sigma \in T_n(\tau, F)} \text{int } P(\sigma) \]
Lemma 2.2. $U_n(\tau, F)$ is an open neighborhood of $F$ in $PMF$ for every $n$.

Proof. We prove the lemma by induction. For $n = 0$ this is simply openness of $P(\tau)$ ([PTF Lemma 3.1.2]). Suppose now that $U_n(\tau, F)$ is an open neighborhood of $F$. From the description of the effect of splits on polyhedra given above we conclude that $U_{n+1}(\tau, F)$ is obtained from $U_n(\tau, F)$ by cutting at hyperplanes (corresponding to central splits) and retaining those polyhedra which contain $F$. If none of these hyperplanes contain $F$, it is clear that $U_{n+1}(\tau, F)$ is still an open neighborhood of $F$. However, suppose that one of them does contain $F$. This corresponds to the situation in which a track $\eta \in T_n$ has a large branch so that all three of the left, right and central splits of $\eta$ along that branch carry $F$ and therefore all three of these splits will contribute to $U_{n+1}(\tau, F)$, guaranteeing that the latter is still an open neighborhood of $F$. □

A splitting sequence $\tau_i$ in the direction of $F$ is a sequence $\tau_i$ of train tracks with $\tau_0 = \tau$ and so that each $\tau_i$ carries $F$, and $\tau_{i+1}$ is obtained from $\tau_i$ by splitting exactly one large of $\tau_i$ branch once. A full splitting sequence instead requires splitting each large branch of $\tau_i$ once when passing from $\tau_i$ to $\tau_{i+1}$. Hence, if $\tau_i$ is a full splitting sequence in the direction of $F$ starting in $\tau$, then $\tau_i \in T_i(\tau)$ for all $i$.

If $F$ is a foliation in the minimal stratum (i.e. each singularity is 3–pronged, and there are no saddle connections), then each split in a splitting sequence $\tau_i$ is a left or a right split, and furthermore the type is uniquely determined by $F$. If $F$ has saddle connections or $k$–prong singularities for $k > 3$, then it is possible that for some $n$, $F$ is carried by the left, right and central split of $\tau_n$. This is furthermore the last time $F$ is carried in the interior of a maximal train track $\tau_n$ along the splitting sequence.

Splitting sequences in the direction of minimal foliations have good contracting properties. In the following theorem, and below, we denote by $\Delta(F)$ the (closed) simplex of projective measured foliations which are topologically equivalent to $F$.

Theorem 2.3 (compare e.g. [Mos Theorem 5.1.1]). Suppose that $F$ is a minimal foliation and that $\tau_i$ is any splitting sequence in the direction of $F$. Then

$$\bigcap_{i=1}^{\infty} P(\tau_i) = \Delta(F).$$

As an immediate corollary, we have

Corollary 2.4. Let $F \in \text{int} P(\tau)$ be minimal. Then

$$\bigcap_{n \geq 0} U_n(\tau, F) = \Delta(F).$$

We now describe how to connect splitting sequences to sequences in the mapping class group. The first step is the following lemma.

Lemma 2.5. There is a finite number of sets

$$T^{(1)}, \ldots, T^{(M)}$$

\footnote{This means that the corresponding lamination has only triangles as its complementary components.}
so that for each maximal train track \( \tau \), each minimal \( F \), and each \( n \) there is a number \( k_{\tau,F,n} \) and a mapping class \( f_{\tau,F,n} \) with

\[
T_n(\tau,F) = f_{\tau,F,n} \left( T^{(k_{\tau,F,n})} \right).
\]

The number \( k_{\tau,F,n} \) is unique. The mapping class \( f_{\tau,F,n} \) is unique up to a finite indeterminacy.

We call the set of \( T^{(i)} \) standard neighborhood models and we call the number \( k_{\tau,F,n} \) the type of \( T_n(\tau,F) \).

**Proof of Lemma 2.5** By Lemma 2.1, \( T_n(\tau,F) \) consists of train tracks which are diagonal extensions of some large train track \( \eta_n \). Since the mapping class group \( \text{Mod}(S_g) \) acts on the set of (isotopy classes of) train tracks on \( S_g \) with finitely many orbits, there are finitely many choices for such a train track \( \eta_n \) up to the mapping class group action. Since the number of complementary components of \( \eta_n \) can be bounded from the Euler characteristic of \( S \) alone, there are a finite number of diagonal extensions of \( \eta_n \). This implies that the mapping class group also acts on the sets \( T_n(\tau,F) \) (over all \( \tau,F,n \)) with finitely many orbits. We can therefore choose the sets \( T^{(i)} \) to be orbit representatives of this action. This shows both the desired existence of \( k_{\tau,F,n} \) and \( f_{\tau,F,n} \), as well as the uniqueness of \( k_{\tau,F,n} \). The (coarse) uniqueness of the \( f_{\tau,F,n} \) follows since the set of mapping classes which fix a given train track is finite (compare e.g. [Ham2, Lemma 4.2]), and so the element \( f_{n,F} \) is also determined up to a finite choice. \( \square \)

Let \((\tau_i)\) be a full splitting sequence starting in a maximal train track \( \tau \) towards some minimal foliation \( F \). Then each \( \tau_i \in T(\tau,F) \), and in fact \( \tau_i \in T_i(\tau,F) \). We then get an associated Mod-sequence \((f_i, k_i)\) by applying Lemma 2.5 to \( T_i(\tau,F) \) for each \( i \). In particular, we then have

\[
T_n(\tau,F) = f_n(T^{(k_n)}).
\]

As before, the numbers \( k_i \) are uniquely determined by the splitting sequence, and the mapping classes \( f_n \) are determined up to a finite choice. We call the number \( k_n \) the type of the index \( n \).

Let \( U^{(k)} \) be the neighborhoods associated to our standard models \( T^{(k)} \), i.e.

\[
U^{(k)} = \bigcup_{\sigma \in T^{(k)}} \text{int} P(\sigma).
\]

We call the \( U^{(k)} \) the standard neighbourhoods.\(^2\) By the defining property of the associated sequence \((f_n, k_n)\) we can then relate the standard neighbourhoods to the neighbourhoods of \( F \) given by the splitting sequence in the following way:

\[
U_n(\tau,F) = f_n \left( U^{(k_n)} \right).
\]

The next lemma collects two crucial properties of the associated sequence.

\(^2\)Note that the model neighbourhoods \( U^{(k)} \) need not be contained in the polyhedra \( P(\tau_i) \) along the splitting sequence.
Lemma 2.6. There is a finite set $M \subset \text{Mod}(S_g)$ with $M = M^{-1}$ and so that the following holds. Suppose that $f_n, f_{n+1}$ are two consecutive terms of an associated Mod-sequence. Then we have

$$(4) \quad f_n^{-1} f_{n+1} \in M.$$ 

Furthermore,

$$f_n^{-1} f_{n+1} \left( \mathcal{U}^{(k_{n+1})} \right) \subset \mathcal{U}^{(k_n)}.$$

Proof. Let $T$ be the (finite) set of all those train tracks which can be obtained from one of the train tracks in $\cup_i T^{(i)}$ by full splits, and let $M_0$ be the set of all those mapping classes which map train tracks $\sigma \in T$ to train tracks in any $\cup_j T^{(j)}$. Note that since $T$ is finite, $M_0$ is finite by Lemma 2.5. We put $M = M_0 \cup M_0^{-1}$.

To see that it has property (4), observe that if $f_n, f_{n+1}$ are consecutive terms of an associated Mod-sequence, there are train tracks $\eta_n \in T^{(i_n)}, \eta_{n+1} \in T^{(i_{n+1})}$, so that $f_n^{-1} f_{n+1}$ is a full split of $f_n \eta_n$. This implies that $f_n^{-1} f_{n+1} \eta_{n+1}$ is a full split of $\eta_n$.

In other words, $f_n^{-1} f_{n+1}$ maps a train track in $T^{(i_{n+1})}$ to one in $T$, and is therefore an element of $M$ by definition.

Equation (5) follows immediately from the following:

$$f_{n+1} \left( \mathcal{U}^{(k_{n+1})} \right) = U_{n+1}(\tau, F) \subset U_n(\tau, F) = f_n \left( \mathcal{U}^{(k_n)} \right) .$$

The type-$k$ subsequence is the maximal subsequence $f_s^{(k)} = f_{r_s}$ so that $k_{r_s} = k$. We say that type $k$ is essential for the splitting sequence $(\tau_i)$, if the subsequence $f_s^{(k)}$ is an infinite sequence. At least one type is essential, but we suspect that the type of the initial train track need not repeat infinitely often.

By Lemma 2.6, we have that $f_{i+1} f_i^{-1} \in M$ for all $i$; but we warn the reader that the elements $f_s^{(k)} \left( f_i^{(k)} \right)^{-1}$ are not constrained to a finite set in the mapping class group.

2.2. Minimal Foliations and the Curve Graph. In this section we will prove that large terms in the associated Mod-sequence for a uniquely ergodic foliation send certain subsets of $PMF$ into small neighborhoods of the foliation, and will use this to prove contracting properties for associated Mod-sequences. Intuitively, we will show that all curves (and non-minimal foliations) are attracted to the foliations $F$ guiding the splitting sequence, and we will show that the speed of attraction can be controlled for certain geometrically constrained sets of curves.

We begin by rephrasing the contraction exhibited by train track polyhedra under splitting sequences (Theorem 2.3) in terms of associated Mod-sequences.

Corollary 2.7. Let $\tau_i$ be a splitting sequence towards a minimal foliation $F$ and let $(f_i, k_i)$ be an associated Mod-sequence. For any essential type $k$ we have that

$$\bigcap_s f_s^{(k)} \left( \mathcal{U}^{(k)} \right) = \Delta(F).$$
Proof. By Corollary 2.4 we have that
\[ \bigcap_{n \geq 0} U_n(\tau, F) = \Delta(F), \]
and therefore, by definition of essential type,
\[ \bigcap_{n \geq 0, k_n = k} U_n(\tau, F) = \Delta(F). \]
Now, using Equation (3) we see that
\[ U_n(\tau, F) = f_n(U(k)) \]
if the index \( n \) is of type \( k \), and therefore
\[ \bigcap_{n \geq 0, k_n = k} U_n(\tau, F) = \bigcap_{s \geq f} f_s(U(k)), \]
which shows the corollary. □

In other words, the mapping classes \( f_n(U(k)) \) eventually contract \( U(k) \) to a small neighborhood of \( \Delta(F) \). The rest of this section is concerned with studying the contraction properties of the mapping classes \( f_i \) outside the open sets \( U(k) \).

To this end, we use the geometry of the curve graph. Recall that the curve graph \( \mathcal{C}(S) \) of a surface is the graph whose vertex set is the set of isotopy classes of essential simple closed curves on \( S \), with edges between classes that admit representatives with intersection 0. We denote by \( d_{\mathcal{C}(S)} \) be the resulting metric on \( \mathcal{C}(S) \). The core feature of the geometry of the curve graph we need is the following.

**Theorem 2.8** (Masur-Minsky [MM1]). If \( S \) is a non-exceptional surface (i.e. \( \mathcal{C}(S) \) is connected), then the curve graph is hyperbolic in the sense of Gromov.

We will need two methods to produce quasigeodesics in the curve graph. The first one is the method employed to show hyperbolicity in [MM1].

**Theorem 2.9.** Let \( S \) be a surface of finite type. Then there are numbers \( K, K' \), depending on \( S \) with the following property: suppose that \( \rho : \mathbb{R} \to T(S) \) is a Teichmüller geodesic, and suppose that for each \( t \in \mathbb{R} \) the curve \( \alpha_t \) has smallest possible extremal length on \( \rho(t) \). Then the assignment
\[ t \to \alpha_t \]
is an unparametrised \( K \)-quasigeodesic in the curve graph. In particular, for any \( t < s \), the set \( \{ \alpha_r, t \leq r \leq s \} \) has Hausdorff distance at most \( K' \) from a curve graph geodesic joining \( \alpha_t \) to \( \alpha_s \).

**Proof.** Theorem 2.3 of [MM1] states that a coarsely transitive path family with the contraction property in a geodesic metric space consists of uniform unparametrised quasigeodesics (for the definitions, compare Section 2.4 of [MM1]). Theorem 2.6 of [MM1] then shows that the family of paths in the curve graph obtained by taking shortest extremal length curves has the contraction property (that these paths are coarsely transitive is easy to see). □

The second, related construction of quasigeodesics uses train tracks. It is proven in [MM2] Theorem 1.3, see also [Ham1] Corollary 2.6:

\[ \text{See e.g. [Ahl] for a definition. The precise definition of this does not matter too much to understand the theorem; it would remain true also for e.g. the shortest hyperbolic geodesic on } \rho(t). \]
Proposition 2.10. Let $S$ be a surface of finite type. Then there are numbers $K,K'$, depending on $S$ with the following property: suppose that $(\tau_i)_{i}$ is a splitting sequence and suppose that for each $i \in \mathbb{N}$ the curve $\alpha_i$ is a vertex cycle\(^{\text{(4)}}\) on $\tau_i$. Then the assignment

$$i \rightarrow \alpha_i$$

is an unparametrised $K$–quasigeodesic in the curve graph. In particular, for any $t<s$, the set $\{\alpha_r, t \leq r \leq s\}$ has Hausdorff distance at most $K'$ from a curve graph geodesic joining $\alpha_t$ to $\alpha_s$.

For a Gromov hyperbolic space, one can define a boundary at infinity, see e.g. [BH, III.H.3] for details. If $\alpha_0$ is some basepoint, recall the Gromov product

$$(x \cdot y)_{\alpha_0} = \frac{1}{2}(d(\alpha_0, x) + d(\alpha_0, y) - d(x, y)).$$

A sequence $(x_i)$ of points in $X$ converges at infinity if $(x_i \cdot y_j)_{\alpha_0} \rightarrow \infty$ as $i, j \rightarrow \infty$. The Gromov boundary is then defined as a set of equivalence classes of sequences converging to infinity, where two sequences $(x_i), (y_j)$ are equivalent if $(x_i \cdot y_j)_{\alpha_0} \rightarrow \infty$ as $i, j \rightarrow \infty$; see [BH, III.H.3.12] for details.

Also note that the Gromov product extends from the space to the boundary at infinity [BH, III.H.3.15]. The Gromov product has the property that

$$(6) \quad |(x \cdot y)_{\alpha_0} - (x' \cdot y)_{\alpha_0}| \leq d(x, x')$$

for any point $y$ and (finite) points $x, x'$.

In the case of the curve graph, the Gromov boundary can be identified explicitly with a different space. We define the set $\mathcal{EL}(S)$ to be the set of minimal foliations with the measure-forgetting topology. That is, we consider the subset $M \subset PMF$ of all minimal foliations, and let $\mathcal{EL}(S)$ be the quotient topological space $M/\sim$ under the equivalence relation which lets $F \sim F'$ if $F, F'$ are topologically equivalent.

**Theorem 2.11** ([Kla, Theorems 1.2, 1.3 and 1.4]). i) The Gromov boundary of $\mathcal{C}(S)$ is homeomorphic to the space $\mathcal{EL}(S)$.

ii) A sequence $\alpha_i$ of curves (interpreted as points in the curve graph) converges to the point at infinity defined by a minimal foliation $F$ if and only if every accumulation point of $\{\alpha_i, i \in \mathbb{N}\}$ in $PMF$ is contained in $\Delta(F)$.

iii) Suppose that $\rho$ is a Teichmüller geodesic ray whose vertical foliation is a minimal foliation $F$, and that for every $t$, the curve $\alpha_t$ is a curve of smallest extremal length on $\rho(t)$. Then the curves $\alpha_t$ (interpreted as points in the curve graph) converge to $F$ (interpreted as a point in the Gromov boundary)

As a consequence of Theorem 2.11 we have the following characterization of neighborhoods in $PMF$ using the curve graph.

**Lemma 2.12.** Suppose that $F$ is a minimal foliation, and $U$ is an open neighborhood of $\Delta(F)$ in $PMF$. Let $\gamma$ be an arbitrary simple closed curve. Then there is a

\(^{\text{(4)}}\)See e.g. [PH] or [MM1, Section 4.1] for a definition of vertex cycle. Again, the precise definition of this does not matter too much; the theorem would remain true for e.g. the shortest train path on $\tau_i$.\]
number $K$ with the following property: suppose that $\beta$ is a simple closed curve so that (as a point in the curve graph) we have $(F \cdot \beta)_\gamma > K$.

Then $\beta$ (seen as a projective measured foliation) is contained in $U$.

**Proof.** Suppose that the claim were false. Then we would find a sequence $(\beta_i)$ with $(\beta_i \cdot F)_\gamma > i$ but $\beta_i \notin U$. By the Gromov product condition, $(\beta_i)$ would then be a sequence converging at infinity to the boundary point $F$. So, by Theorem 2.11 ii), the sequence $\beta_i$ converges in the measure forgetting topology to $F$. Since $U$ is an open neighborhood of $\Delta(F)$ this is impossible as $\beta_i \notin U$. □

We also need the following partial converse.

**Lemma 2.13.** There is a number $k_0$, depending only on the topological type of the surface, with the following property. Suppose that $F$ is a minimal foliation, $\alpha$ is a simple closed curve, and $(F \cdot \alpha)_\gamma > B$. If $\mu_i$ is a sequence of minimal foliations converging to $\alpha$ in $\mathcal{PMF}$, then $(F \cdot \mu_i)_\gamma > B - k_0$ for all large $i$.

**Proof.** Denote by $\Phi : \mathcal{T}(S) \to \mathcal{C}(S)$ the map which assigns to a marked hyperbolic surface in Teichmüller space a curve of smallest extremal length. Pick a basepoint $X_0$ in Teichmüller space for which $\gamma$ is a curve of smallest extremal length, and consider the Teichmüller geodesic rays $\rho_i$ starting from $X_0$ in the direction of $\mu_i$. Since the $\mu_i$ converge to $\alpha$ in $\mathcal{PMF}$, the rays $\rho_i$ converge uniformly on compact subsets to the Teichmüller geodesic ray $\rho_\infty$ starting in $X_0$ with vertical foliation $\alpha$.

Theorem 2.9 implies that there is a constant $K$ (depending only on the topological type of the surface) so that the images $\Phi \circ \rho_i$ can be reparametrised to be $K$–quasigeodesics $q_i$ beginning in $\gamma$. By Theorem 2.11 iii), the quasigeodesic $q_i$ connects $\gamma$ to the point $\mu_i$ in the Gromov boundary of the curve graph.

There is a constant $T_0$ so that $\Phi \circ \rho_\infty(t)$ is coarsely equal to $\alpha$ for all $t \geq T_0$. As the $\rho_i$ converge to $\rho_\infty$ uniformly on compact sets in Teichmüller space, one concludes that $\Phi \circ \rho_i(T_0)$ is also coarsely equal to $\alpha$ for all large $i$. Hence, the $q_i$ pass uniformly close by $\alpha$ for all large $i$. This implies that there is a constant $k_0$, depending on $K$ and the hyperbolicity constant of the curve graph (and hence only the topological type of the surface), so that $(F \cdot \mu_i)_\gamma > (F \cdot \alpha)_\gamma - k_0$, which implies the lemma. □

The next lemma and corollary are well known and standard and included for completeness.

**Lemma 2.14.** Let $F$ be a minimal foliation, and $K$ a number. Then suppose that $x, y \in \mathcal{C}(S)$ with

$$(F \cdot x)_\gamma, (F \cdot y)_\gamma \geq K.$$
Let $z$ be a point on a geodesic between $x, y$. Then
\[(F \cdot z)\gamma \geq K - 4\delta,\]
where $\delta$ is the hyperbolicity constant of the curve graph.

**Proof.** First we observe that if $x, y, z$ are three points in $C(S)$ and $z$ lies on a geodesic between $x$ and $y$, we have
\[2(x \cdot z)\gamma = d(\gamma, x) + d(\gamma, y) - d(x, y) = 2(x \cdot y)\gamma.\]

By $\delta$–hyperbolicity, we have that for all triples $a, b, c$ of points in $C(S) \cup \partial_{\infty}C(S)$
\[(a \cdot c)\gamma \geq \min\{(a \cdot b)\gamma, (b \cdot c)\gamma\} - 2\delta,\]
compare e.g. [BH, III.H.3.17.(4)]. First, apply this to $x, F, y$ to conclude that
\[(x \cdot y)\gamma \geq K - 2\delta.\]

Now, apply this same estimate again, to conclude
\[(F \cdot z)\gamma \geq \min\{(F \cdot x)\gamma, (x \cdot z)\gamma\} - 2\delta \geq \min\{(F \cdot x)\gamma, (x \cdot y)\gamma\} - 2\delta \geq \min\{K, K - 2\delta\} - 2\delta \geq K - 4\delta\]
which is what we wanted to prove. \(\square\)

**Corollary 2.15.** Let $K, D > 0$ be numbers, $F$ be a minimal foliation. Suppose that $\tilde{x}, \tilde{y}$ are any two points in the curve complex or its boundary, satisfying
\[(F \cdot \tilde{x})\gamma, (F \cdot \tilde{y})\gamma \geq K.

Suppose that $z \in C(S)$ lies on a (possibly infinite) $D$–quasi-geodesic $q$ with endpoints $\tilde{x}$ and $\tilde{y}$. Then
\[(F \cdot z)\gamma \geq K - X,

where $X$ is a number depending only on the hyperbolicity constant of the curve graph and the quasi-geodesic constant $D$.

**Proof.** Choose points $x_i = q(r_i), y_i = q(s_i)$ in the curve complex on the quasi-geodesic $q$ which converge to $\tilde{x}, \tilde{y}$ respectively. If an endpoint of $q$ is finite, we assume that the corresponding sequence is eventually constant. Recall, e.g. from [BH, III.H.3.17.(5)], that
\[\liminf(F \cdot x_i)\gamma \geq (F \cdot \tilde{x})\gamma - 2\delta\]
and
\[\liminf(F \cdot y_i)\gamma \geq (F \cdot \tilde{y})\gamma - 2\delta.\]

By our assumption, we then conclude that
\[\min\{(F \cdot x_i)\gamma, (F \cdot y_i)\gamma\} \geq K - 2\delta - 1 \geq K - 4\delta - 1,

for large $i$. We furthermore assume that $i$ is large enough so that $z$ is contained in the subsegment $q_i$ of $q$ with endpoints $x_i, y_i$. By $\delta$–hyperbolicity, there is a number $B$ depending on $D$ (and $\delta$), so that the Hausdorff distance between $q_i$ and the
geodesic connecting $x_i$ to $y_i$ is at most $B$. Let $z'$ be a point on that geodesic of distance at most $B$ to $z$. By Lemma 2.14 we then have
\[(F \cdot z')_\gamma \geq K - 6\delta - 1,\]
and thus
\[(F \cdot z)_\gamma \geq K - 6\delta - B - 1.\]
Hence $X = 6\delta + B + 1$ satisfies the requirement. □

**Lemma 2.16.** Let $F$ be a minimal foliation, $\tau$ a train track and $(\tau_i)$ a splitting sequence in the direction of $F$ and let $(f_i, k_i)$ be an associated Mod-sequence. Suppose that $(\gamma_i)$ is a sequence of simple closed curves so that $\gamma_i$ is contained in $f_i^{(k_i)}(U^{(k_i)})$ for every $i$. Then, for any base point $\alpha_0$, we have
\[(\gamma_i \cdot F)_{\alpha_0} \rightarrow \infty.\]

**Proof.** By Corollary 2.4 and the assumption, any accumulation point of the curves $(\gamma_i)$ (interpreted as projective measured foliation) is contained in $\Delta(F) \subset PMF$. By Theorem 2.11 ii), the $\gamma_i$ therefore converge (interpreted as points in the curve graph) to $F$ in the Gromov boundary. By definition, this implies that the Gromov product condition claimed in the corollary. □

We can use this to show the following contraction behavior for finite-diameter subsets in the curve graph.

**Proposition 2.17.** Let $F$ be a minimal foliation, $\tau$ a train track and $(\tau_i)$ a splitting sequence in the direction of $F$ and let $(f_i, k_i)$ be an associated Mod-sequence.

Consider any neighborhood $V$ of $\Delta(F)$ in $PMF$, and let a simple closed curve $\beta_0$ and a number $d > 0$ be given.

Then there is a number $N = N(\tau, F, V, \beta_0, d) > 0$ so that the following holds: If $\beta$ is any simple closed curve with $d_{\mathcal{C}(S)}(\beta_0, \beta) \leq d$, then
\[f_n(\beta) \in V \quad \forall n > N.\]

**Proof.** As a first reduction, note that by Corollary 2.4 we may assume that $V$ is of the form $f_s^{(k, s)}(U^{(k(s))})$ for a large enough $s$. Fix, for concreteness, a vertex cycle $\alpha_0$ of $\tau$ as a basepoint in the curve graph (recall that there are finitely many such choices).

Apply Lemma 2.12 in order to obtain a number $D > 0$ with the property that if $\gamma$ is any curve so that the Gromov product satisfies
\[(\gamma \cdot F)_{\alpha_0} > D,\]
then $\gamma \in V$ as an element of $PMF$.

Now, for each $k$ choose a curve $\delta_k$ contained in $U^{(k)}$ and put $\gamma_n = f_n(\delta_{k(n)})$.

Observe that
\[d_{\mathcal{C}(S)}(f_n(\beta_0), \gamma_n) \leq \max_k d_{\mathcal{C}(S)}(\beta_0, \delta_k) = C_0\]
and, if $d_{\mathcal{C}(S)}(\beta, \beta_0) \leq d$ we therefore have
\[d_{\mathcal{C}(S)}(f_n(\beta), \gamma_n) \leq C_0 + d.\]
Thus, using Equation (6), we see
\[(f_n(\beta) \cdot F)_{o_0} \geq (\gamma_n \cdot F)_{o_0} - d_{C(S)}(f_n(\beta), \gamma_n) \geq (\gamma_n \cdot F)_{o_0} - (C_0 + d)\].

Applying Lemma 2.16 to the curves \(\gamma_n\) we see that there is a number \(N\) so that
\[(\gamma_n \cdot F)_{o_0} > D + C_0 + d \quad \forall n > N\].

Together with the previous inequality this implies that
\[(f_n(\beta) \cdot F)_{o_0} > D \quad \forall n > N\],
which finishes the proof. \(\square\)

The next lemma, which requires a definition, will allows us to obtain that large terms in the Mod-sequence to a uniquely ergodic foliation contract certain infinite diameter subsets of the curve graph (thought of as foliations) to a small neighborhood of the uniquely ergodic foliation.

Definition 2.18. Let \(D\) be a number, and \(\psi\) a pseudo-Anosov map. A \((D–) quasi-axis\) is a bi-infinite \(D–quasi-geodesic\) \(q : \mathbb{R} \rightarrow C(S)\) so that its image \(\psi^j q\) has (Hausdorff) distance at most \(D\) from the image of \(q\) for any power \(j \in \mathbb{Z}\).

Lemma 2.19. There are constants \(D, B > 0\), just depending on the surface, so that every pseudo-Anosov map \(\psi\) of \(S\) has a \(D\)–quasi-axis. Furthermore, any two such quasi-axes have Hausdorff distance at most \(B\).

Proof. Let \(\rho : \mathbb{R} \rightarrow T(S)\) be the Teichmüller geodesic invariant under \(\psi\), i.e. there is some \(T > 0\) so that for all \(t\) we have \(\psi^t \rho(t) = \rho(t + T)\). For each \(t \in [0, T)\), choose a curve \(\alpha_t\) of smallest extremal length on \(\rho(t)\). For \(t \in [iT, (i + 1)T)\) put \(\alpha_t = \psi^i(\alpha_{t-i})\). Then for all \(t\), the curve \(\alpha_t\) has smallest extremal length on \(\rho(t)\). By Theorem 2.9, the assignment \(t \rightarrow \alpha_t\) is an (unparametrised) quasigeodesic with quasigeodesic constant just depending on the topological type of the surface. By construction, \(t \rightarrow \alpha_t\) is invariant under the action of \(\psi\). This shows that quasi-axes exist.

The uniqueness statement follows since any quasiaxis for \(\psi\) converges in the Gromov boundary of the curve graph to the stable and unstable foliation of \(\psi\) by Theorem 2.11 (iii) and two \(D\)-quasigeodesics with the same endpoints in a Gromov hyperbolic space have bounded Hausdorff distance. \(\square\)

In the future, we will choose a \(D\) for which Lemma 2.19 holds once and for all, and simply refer to quasi-axes of pseudo-Anosov maps.

Also recall the definition of a Dehn twist \(T_\alpha\) about a simple closed curve \(\alpha\) (compare e.g. [F-MI, Section 3.1]). If \(\alpha\) is a multicurve, together with a choice of left/right for each component, then we denote by \(T_\alpha\) the product of the left/right Dehn twists about the curves in \(\alpha\).

Proposition 2.20. Let \(F\) be a minimal foliation, \(\tau\) a train track and \((\tau_i)\) a splitting sequence in the direction of \(F\) and let \((f_i, k_i)\) be an associated Mod-sequence.

Consider any neighborhood \(V\) of \(\Delta(F)\) in \(\mathcal{PMF}\). Let \(\psi\) be a pseudo-Anosov, and let \(\alpha\) be a multicurve which is within distance \(d\) of its quasi-axis in the curve graph. Let \(r > 0\) be any number. Suppose \(\beta_0\) is a curve.
Then there is a number \( N = N(\tau,F,V,\psi,\alpha,d,\beta_0) > 0 \) following property. Suppose that \( n > N \) is given. Then there is a number \( t_0 \) (which depends on \( n \)), so that for all \( t > t_0 \) the conjugate \( \hat{\psi} = (T_\alpha)^t \circ \psi \circ (T_\alpha)^{-t} \) satisfies the following:

If \( \beta \) is any simple closed curve with \( d_{\mathcal{C}(S)}(\beta_0, \beta) \leq d \), then

\[
f_n(\hat{\psi}^j \beta) \in V, \quad \forall j \in \mathbb{Z}
\]

Proof. We follow a similar strategy as in the previous proposition. Choose \( \alpha_0 \) a vertex cycle of \( \tau \). Apply Lemma \ref{lem:2.12} to find a number \( U \) so that if

\[
(\gamma \cdot F)_{\alpha_0} > U,
\]

then \( \gamma \in V \) as an element of \( \mathcal{P}M\mathcal{F} \).

Introduce the notation

\[
\hat{\psi}_t = (T_\alpha)^t \circ \psi \circ (T_\alpha)^{-t}.
\]

We therefore need to show, that there is a number \( N \) so that for all \( n > N \) there is a \( t_0 \) so that

\[
(f_n(\hat{\psi}_t^j \beta) \cdot F)_{\alpha_0} > U,
\]

for any curve \( \beta \) with \( d_{\mathcal{C}(S)}(\beta_0, \beta) \leq d \), and any \( t > t_0 \), any \( j \in \mathbb{Z} \).

The first stage of the proof consists of a (lengthy) reduction of this statement to a similar statement (Equation \ref{eq:7}) below about quasi-axes of the \( \hat{\psi}_t \). To begin showing this reduction, note that

\[
(f_n(\hat{\psi}_t^j \beta) \cdot F)_{\alpha_0} \geq (f_n(\hat{\psi}_t^j \beta_0) \cdot F)_{\alpha_0} - d(\beta, \beta_0) \geq (f_n(\hat{\psi}_t^j \beta_0) \cdot F)_{\alpha_0} - d
\]

and therefore it suffices to show

\[
(f_n(\hat{\psi}_t^j \beta_0) \cdot F)_{\alpha_0} > U + d.
\]

Arguing as above, we have that

\[
(f_n(\hat{\psi}_t^j \beta_0) \cdot F)_{\alpha_0} > (f_n(\hat{\psi}_t^j \alpha) \cdot F)_{\alpha_0} - d(\alpha, \beta_0).
\]

Hence, it suffices to show that

\[
(f_n(\hat{\psi}_t^j \alpha) \cdot F)_{\alpha_0} > U + d + d(\alpha, \beta_0) =: U_1,
\]

for any \( t > t_0 \), any \( j \in \mathbb{Z} \).

Now, let \( \rho \) be a \((D-)\)quasi-axis for \( \psi \). Since the mapping class group acts as isometries on the curve graph, we have that \( f_n T_\alpha^t \rho \) is a \((D-)\)quasi-axis for \( f_n \hat{\psi}_t f_n^{-1} \).

Furthermore,

\[
d(f_n \alpha, f_n T_\alpha^t \rho) = d(\alpha, T_\alpha^t \rho) = d(\alpha, \rho) = A,
\]

for all \( t \), since \( T_\alpha \) acts as an isometry fixing \( \alpha \). Hence, \( f_n \alpha \) is (for all choices of \( n \) and \( t \)) within \( A \) of the \( D \)-quasi-axis \( f_n T_\alpha^t \rho \) of \( f_n \hat{\psi}_t f_n^{-1} \). Let \( \eta \) be a point on \( f_n T_\alpha^t \rho \) with \( d(f_n \alpha, \eta) \leq A \). The \( D \)-quasi-axis property then implies that for any \( j \) we have that

\[
d((f_n \hat{\psi}_t f_n^{-1})^j \eta, f_n T_\alpha^d \rho) \leq D
\]

and therefore

\[
d((f_n \hat{\psi}_t f_n^{-1})^j f_n \alpha, f_n T_\alpha^d \rho) \leq A + D
\]

As such, we have that

\[
d(f_n(\hat{\psi}_t^j \alpha), f_n T_\alpha^t \rho) = d(f_n(\hat{\psi}_t^j f_n^{-1} f_n \alpha), f_n T_\alpha^t \rho) = d((f_n(\hat{\psi}_t f_n^{-1})^j f_n \alpha), f_n T_\alpha^t \rho) \leq A + D.
\]
Therefore, to prove the proposition, it suffices to show that there is a number $N$, so that for all $n > N$ there is a number $t_0$, so that for all $t > t_0$:

$$\forall x \in f_n T^a \rho : (x \cdot F)_{\alpha_0} > U_1 + A + D =: U_2.$$  

(7)

Now, use Lemma 2.16 as in the previous proof, to find a number $N$ so that

$$\forall x \in f_n T^a \rho : (x \cdot F)_{\alpha_0} > 2U_2 + X + k_0 \quad \forall n > N,$$

(8)

where $X$ is the number from Corollary 2.15 and $k_0$ is the number from Lemma 2.13 applied to the quasi-geodesic constant $D$. At this point, fix a number $n > N$.

Observe that if $\mu_+, \mu_-$ are the stable and unstable foliations of $\psi$, then $T^a_{\mu_+}, T^a_{\mu_-}$ are the stable and unstable foliations of $\psi_t$. Note that as $t \to \infty$, both of these foliations converge to $\alpha$ in $PMF$. Consider $f_n(\psi_t)^{-1}$, and observe that its stable and unstable foliations therefore converge to $f_n(\alpha)$ in $PMF$ as the number $t$ increases. By Lemma 2.13 this implies that we can choose $t_0$ large enough, so that for any $t > t_0$ we have

$$(f_n(T^a_{\mu_+}) \cdot F)_{\alpha_0} > U_2 + X$$

$$(f_n(T^a_{\mu_-}) \cdot F)_{\alpha_0} > U_2 + X$$

Let now $z$ be any point on a $D$–quasi-geodesic with endpoints $f_n(T^a_{\mu_+}), f_n(T^a_{\mu_-})$. Then Corollary 2.15 implies that

$$(z \cdot F)_{\alpha_0} > U_2$$

Since the quasi-axis $f_n T^a \rho$ is such a $D$–quasi-geodesic, the proposition follows. □

In the proof of local path-connectivity, we require uniform control over the constants $N$ appearing in the previous two results (Propositions 2.17 and 2.20). Before stating the corresponding lemma, suppose that $(\tau_i)$ is a full splitting sequence in the direction of some minimal foliation $F$.

Then consider, in Proposition 2.17 or 2.20, a neighbourhood $V = U_k(\tau, F)$, and observe that it is also a neighbourhood of $\Delta(E)$ for all minimal $E \in U_i(\tau, F), i \geq k$. Additionally, $E$ determines a full splitting sequence starting in $\tau$, whose first $i$ terms are identical with the one defined by $F$.

Hence, it makes sense to apply Proposition 2.17 or 2.20 for this neighbourhood $V$, and $E$ in place of $F$ with its full splitting sequence starting in $\tau$. The following lemma shows a boundedness of the resulting numbers $N$ that these propositions produce.

**Lemma 2.21.** Suppose that $(\tau_i)$ is a full splitting sequence in the direction of some minimal foliation $F$ with $\tau_1 = \tau$. Put $V = U_k(\tau, F)$ for some $k$.

Suppose we are given either

1. A curve $\beta_0$ and a number $d > 0$, or
2. A pseudo-Anosov $\psi$, a curve $\alpha$, a number $r > 0$ and a curve $\beta_0$.

Then there are numbers $M, N > 0$ with the property that the number

1. $N(\tau, E, V, \beta_0, d)$ from Proposition 2.17 or
2. $N(\tau, E, V, \psi, \alpha, d, r, \beta_0)$ from Proposition 2.20

can be chosen to be smaller than $N$ for all minimal $E \in U_M(\tau, F)$.  


Proof. We will describe the case of Proposition 2.20 in detail, the corresponding argument for Proposition 2.17 is similar and simpler.

Recall from the proof of Proposition 2.20 that what one needs to show is the estimate in (7). This in turn is implied by (8), which is purely a statement about Gromov product growth of images of $\alpha$ under the associated mapping class group sequence $f_n$ of the given splitting sequence. Hence, to finish the proof, we will argue that the number $N$ in (8) can be uniformly bounded for the associated mapping class sequences $f_n$ defined by splitting sequences arising from any minimal $E \in U_M(\tau, F)$ independent of $E$ itself.

If $E \in U_M(\tau, F)$, then by definition the first $M$ terms of the associated Mod-sequence for $E$ and $F$ agree. Hence, to show this lemma, we have to show that the existence of a number $N$ making (8) true can already be guaranteed by knowing a large initial segment of the associated Mod-sequence. The remainder of this proof is concerned with showing that.

Similar to the proof of Proposition 2.17, choose for each $k$ a curve $\alpha_k$ which is carried by each $\sigma \in T^{(k)}$ as a vertex cycle. By Proposition 2.10, the path $n \mapsto f_n\alpha_{k(n)}$ is then uniformly Hausdorff close to a uniform quasi-geodesic in the curve graph which converges to $F$.

In particular, this implies that for any $K_0$ there is an $N$ with the property that

$$(f_n\alpha_{k(n)} \cdot F)_\gamma > K_0 \quad \text{for all } n > N.$$  

If now $F' \in U_N(\tau, F)$ and $(f'_i)$ is an associated Mod-sequence for $F'$, then we may assume $f'_i = f_i$ for all $i \leq N$ by definition. Thus, for some uniform constant $c$ (depending on the quasi-geodesic constant $k_1$ Proposition 2.10 and the hyperbolicity constant of the curve graph) we have that

$$(f'_n\alpha_{k'(n)} \cdot F)_\gamma > K_0 - c \quad \text{for all } n > N.$$  

Since the distance between the curve $\alpha$ and the (finitely many) $\alpha_k$ is bounded, there is a further constant $d$ so that

$$(f'_n\alpha \cdot F)_\gamma > K_0 - c - d \quad \text{for all } n > N.$$  

Choosing $K_0 - c - d > 2U_2 + X + k_0$ then yields that the corresponding $N$ works in (8) for the sequences of all $F' \in U_N(\tau, F)$, proving the lemma. \hfill \Box

3. Paths by pushing points

In this section we will construct many special paths of cobounded foliations for punctured surfaces, which will serve as building blocks for all subsequent constructions. The paths we will eventually use to connect uniquely ergodic foliations will be concatenations of paths of this form, except possibly at a countable set of points which will be stable foliations of pseudo-Anosovs (or the endpoints).

The construction described in this section is crucially inspired by the work of Leininger and Schleimer in [LSII], where they build paths of minimal foliations. Our main contribution is that we modify their construction to produce paths of uniquely ergodic (and in fact cobounded) foliations, and obtain some extra control over how these paths follow a “combinatorial skeleton” given by a finite set of curves.
3.1. Preliminaries on Covers, and on Adding Points. Our notation follows [LS1] and we refer the reader to that article for a very good and readable source for background information on the methods used here.

A smooth surface will denote a smooth, connected, compact, oriented 2–manifold without boundary. All maps between smooth surfaces will be assumed to be smooth unless specified. By a slight abuse of notation, a (holomorphic) Abelian differential on $S$ is a smooth 1–form $\omega$ which is holomorphic with respect to some complex structure on $S$ (compatible with orientation and smooth structure). We denote by $d_\omega$ the (singular) flat metric on the surface defined by integrating $\omega$.

We let $\tilde{\Omega}(S)$ be the set of all such Abelian differentials. Note that $\tilde{\Omega}(S)$ is a path-connected set (in fact, a vector bundle over a contractible base; compare [LS1, Section 2.6]).

The quotient $\Omega(S) = \tilde{\Omega}(S)/\text{Diff}_0(S)$ is the Hodge bundle of Abelian differentials over Teichmüller space of $S$. We need a variant for surfaces with marked points (which is, crucially, the point of this whole discussion). Namely, if $z \subset S$ is a finite, ordered set of distinct points, we let $\text{Diff}_0(S, z)$ denote the group of diffeomorphisms of $S$, fixing each point in $z$, which are homotopic to the identity through such maps. We let $\Omega(S, z) = \tilde{\Omega}(S)/\text{Diff}_0(S, z)$.

As in [LS1], the central idea is that any Abelian differential $\omega \in \tilde{\Omega}(S)$ defines projections $\hat{\omega} \in \Omega(S, z)$ and $\bar{\omega} \in \Omega(S)$ (in the notation of [LS1]).

There is an action of $\text{SL}_2(\mathbb{R})$ on $\tilde{\Omega}(S)$ defined in the usual way (e.g. by postcomposing canonical flat charts) which descends to the usual $\text{SL}_2(\mathbb{R})$–action on $\Omega(S)$. We denote by $g_t$ the action of diagonal matrices, i.e. Teichmüller geodesic flow.

3.2. Torus Covers and Badly Approximable Points. In this section, we begin to construct Abelian differentials with desirable horizontal foliations.

To begin, we say that a Abelian differential $\omega \in \tilde{\Omega}(S)$ is (eventually) $\epsilon$–thick if there exists $N$ so that for all $t > N$ we have that every essential simple closed curve on $S$ has length $\geq \epsilon$ with respect to the singular flat metric $g_t\omega$. We say that $\omega$ is strongly (eventually) $\epsilon$–thick with respect to $z$ if the same is true for any arc with endpoints in $z$. Note that (strong) eventual thickness is invariant under the $\text{Diff}_0(S, z)$–action, and therefore the notion also makes sense for differentials in $\Omega(S, z)$.

The purpose of this section is to give a robust criterion that we will use to construct many paths of thick Abelian differentials.

We make the following (slightly idiosyncratic) definitions, which will be one of the core mechanisms in our construction.

**Definition 3.1.**

i) Let $(X, d)$ be a metric space and $T : X \to X$ be a dynamical system. We say a pair of points $(x, y) \in X$ is B-badly approximable if there exists $N$ so that for all $t > N$ we have that for every essential simple closed curve on $S$ has length $\geq \epsilon$ with respect to the singular flat metric $g_t\omega$. We say that $(x, y)$ is B-badly approximable for some (equivalently every) $x \in \mathbb{R}$ for the dynamical
is eventually strongly δ
there is a regular branched cover, branched over one point,
|b
horizontal component is given by the definition of the set of badly approximable α.

iii) Similarly, if F^t : X → X is a measurable flow of a metric space, we say a pair of points (x, y) is B-badly approximable if there exists N so that t • d(F^t x, y) ≥ B for all t ≥ N and moreover, F^t x ≠ y for all t ≠ 0. We say a straight line flow on a torus is B-badly approximable if the pair (x, x) is B-badly approximable for some x.

The following lemma shows why we are interested in badly approximable points.

Lemma 3.2. If q and q’ are distinct B-badly approximable points on a torus then any trajectory γ from q to q’ has |g_t γ| ≥ √B for all large enough t.

Proof. Let t_0 satisfy d(F^{L_0} q, q’) > B for all L ≥ t_0. Because q and q’ are not in the same orbit, by the definition of B-badly approximable, \( \lim_{t \to \infty} |g_t \gamma| = \infty \) for every γ a trajectory from q to q’. Thus, we may restrict our attention to the cofinite set of such γ with vertical component at least t_0. Let γ be such a geodesic from q to q’. Because the torus is flat, if the vertical component of γ is a and the horizontal component is b we have that d(F^a q, q’) = b. Since we assume that (q, q’) are B-badly approximable, b is at least \( \frac{B}{2} \) if a is large enough. Since the product of the horizontal and vertical components of curves are preserved by \( g_t \), we have \( |g_t \gamma| \) is at least \( \sqrt{2ab} \geq \sqrt{B} \) for all t. (We are also using the elementary fact that the shortest vector in the positive cone in \( \mathbb{R}^2 \) with fixed product of horizontal and vertical components has angle \( \frac{\pi}{4} \)).

\[ \square \]

Definition 3.3. Let S be a closed surface of genus \( g \geq 2 \). An Abelian differential \( \omega \in \Omega(S) \) is called \( (\epsilon, B) \)-torus good with respect to marked points \( q_1, \ldots, q_k \) if there is a regular branched cover, branched over one point,

\[ p: S \to T \]

of S to a torus T and an Abelian differential \( \omega_T \) on T so that

1. \( \omega_T \) is eventually \( \epsilon \)-thick.
2. \( \omega \) is the pullback of \( \omega_T \).
3. The images \( p(q_i) \) of \( q_i \) in T are pairwise B-badly approximable with respect to the flat structure defined by \( \omega_T \).

The associated data to the \( (\epsilon, B) \)-torus good \( \omega \) comprise the cover \( p \) and the base differential \( \omega_T \).

The notion of being torus good is invariant under the action of Diff_0(S, \{q_1, \ldots, q_k\}) by pulling back differentials, and therefore is also defined for differentials in \( \Omega(S, \{q_1, \ldots, q_k\}) \).

The following connects the above definition to Teichmüller dynamics.

Proposition 3.4. For any \( (\epsilon, B) \) and S there is a number \( \delta > 0 \) with the following property. If \( \omega \) is \( (\epsilon, B) \)-torus good with respect to marked points \( q_1, \ldots, q_k \), then \( \omega \) is eventually strongly \( \delta \)-thick with respect to \( z = (q_1, \ldots, q_k) \).

In particular, the horizontal foliation of \( \omega \) is cobounded as a foliation on \( (S, z) \).
Proof. To prove that $\omega$ is eventually strongly $\delta$-thick with respect to $z$, by definition we have to show that there is a $t_0$ so that:

- if $\gamma$ is a simple closed curve on $\omega$ then $|g_t\gamma| \geq \delta$ for all $t > t_0$ and
- if $\gamma$ is a trajectory from $q_i$ to $q_j$ with $j \neq i$ we have that $|g_t\gamma| > \delta$ for all $t > t_0$.

The first condition follows for any $\delta \leq \epsilon$ because we are assuming that $\omega_T$ is eventually $\epsilon$-thick and any simple closed curve on $\omega$ projects to a closed curve of the same length on $\omega_T$ because we are branched over a single point. Similarly we have that $\pi(\gamma)$ is a trajectory from $\pi(q_i)$ to $\pi(q_j)$ and $\pi(g_t\gamma) = g_t\pi(\gamma)$ and so any such trajectory has length at least $B$ by the Lemma 3.2. This implies the two conditions above, and therefore eventual strong $\delta$–thickness of $\omega$.

To see the second claim, note that as $t \to \infty$, the differentials $g_t\omega$ all lie in a compact set of the moduli space of flat surfaces by the first part. This in turn implies that Teichmüller flow in the direction of the horizontal foliation of $\omega$ also only defines Riemann surfaces which lie in a compact set of the moduli space of $S - z$. This shows the proposition. □

Next, we will show that these torus good differentials are in fact dense in the set of all differentials. The proof of this uses Schmidt games, a technique from Diophantine approximation, which we briefly define and discuss in the next section.

3.3. Schmidt game digression. Suppose we are given a set $E \subset \mathbb{R}^n$. Suppose two players Bob and Alice take turns choosing a sequence of closed Euclidean balls

$$B_0 \supset A_1 \supset B_1 \supset A_2 \supset B_2 \ldots$$

(Bob choosing the $B_i$ and Alice the $A_i$) whose diameters satisfy, for fixed $0 < \alpha, \beta < 1$, and all $i > 0$

$$|B_i| = \beta|A_i| \quad \text{and} \quad |A_{i+1}| = \alpha|B_i|.$$  \hfill (9)

The only requirement on $B_0$ is that it has positive diameter. Following Schmidt [Sch] we make the following definition.

**Definition 3.5.** We say $E$ is an $(\alpha, \beta)$-winning set if Alice has a strategy so that no matter what Bob does, $\bigcap_{i=1}^{\infty} B_i \in E$. It is $\alpha$-winning if it is $(\alpha, \beta)$-winning for all $0 < \beta < 1$. $E$ is a winning set for Schmidt game if it is $\alpha$-winning for some $\alpha > 0$.

Because Bob’s first move is unconstrained we have:

**Lemma 3.6.** If $S$ is an $(\alpha, \beta)$ winning set for any $\alpha, \beta$ then $S$ is dense in $X$.

**Lemma 3.7.** (Sch Theorem 2)](Sch) If $S_1, \ldots, S_k$ are $\alpha$-winning sets then $\bigcap_{i=1}^{k} S_i$ is $\alpha$-winning.

In fact the previous lemma is true for countable intersections as well, but we do not need this stronger statement.

**Theorem 3.8.** [Tse first line of Section 2] Let $\xi \in [0, 1)$, $R$ denote rotation by $\xi$ and $x \in [0, 1)$. The set of $y$ so that $x, y$ is $(\alpha\beta^3\xi)$-badly approximable (for $R$) is a $(\alpha, \beta)$ winning set.

From the previous two results we obtain:
Corollary 3.9. Given any rotation $R$ and a finite number of points $p_1, \ldots, p_r$ in $[0, 1)$ we have that the set of $q$ so that $p_i, q$ are $B$-badly approximable for some $B > 0$ are $\alpha$-winning for some $\alpha > 0$.

By iterating the previous result and Lemma 3.6, we get:

Corollary 3.10. Given any rotation on $[0, 1)$, the set of $p_1, \ldots, p_k$ that are pairwise $B$-badly approximable for some $B > 0$ is dense in $[0, 1)^k$.

Remark 3.11. Note that the above discussion can be modified to treat fixed $B$, as Lemma 3.7 can be modified to show the finite intersection of $(\alpha, \beta)$-winning is $(\alpha, \tilde{\beta})$ (which depends on $\alpha, \beta$ the number of intersections). While this may be useful for some applications (such as answering Question 3) it plays no role in this work and so we do not discuss it here.

3.4. Density of torus good differentials.

Proposition 3.12. Let $S$ be a closed surface of genus $g \geq 2$ and $q_1, \ldots, q_k$ be a set of marked points. Suppose that we fix a regular branched cover, branched over one point,

$$p : S \to T$$

of $S$ to a torus $T$ and an Abelian differential $\omega_T$ on $T$ so that $\omega$ is the pullback of $\omega_T$.

Then for every $\omega_T$, every neighborhood $U$ of $\omega_T$ in $\overline{\Omega}(T)$, and every $\delta > 0$ there exists $\omega'_T \in U$ and points $q'_i \in S$ with $d_{\omega}(q_i, q'_i) < \delta$, for all $i$, so that the pullback of $\omega'_T$ is $(\epsilon, B)$-torus good with respect to marked points $(q'_i)$. In this, $\epsilon$ can be chosen independent of $\omega$ and $B$ can be chosen to just depend on $k$.

Proof. We will work throughout with the canonical flat charts defined by $\omega$, $\omega_T$ realizing $p$ as a holomorphic map. We will then show that we can move the $q_i$ by a small amount (in these charts!) and modify $\omega_T$ by a small rotation to obtain an $(\epsilon, B)$-torus good differential. This is enough to show the proposition.

Given a straight line flow on a flat torus, there are many (geodesic) transversals so that the first return map of the flow to the transversal is a rotation. Moreover there exists $C$ so that every aperiodic straight line flow on a flat torus of area 1 has infinitely many transversals $\gamma$, so that the first return map to $\gamma$ is a rotation and the return times to $\gamma$ are between $\frac{1}{C \gamma^t}$ and $\frac{C}{\gamma^t}$. To see this, note that there is a compact set $K$ in the moduli space of flat tori, so that if the orbit $g_i \omega_T$ of a torus $\omega_T$ under Teichmüller flow does not diverge to infinity (without recurring), then there exist arbitrarily large $t$ so that $g_i \omega_T \in K$. In the case of an aperiodic straight line flow the first case does not happen. In the second case, a side of the fundamental domain of the torus $g_i \omega_T \in K$ will work as a transversal.

Sublemma: Let $p, q$ be points on a torus $T$ and $F^t$ a minimal straight line flow on $T$. Suppose that $\gamma$ is a transversal for $F^t$, and let $T_0$ be the minimal first return time of $F^t$ to the transversal $\gamma$. Assume further that the first return of $F^t$ defines a rotation $R_{s_t}$ on $\gamma$. Suppose that $s_1, s_2 > 0$ are minimal so that $F^{s_1}p, F^{s_2}q \in \gamma$, and that the points $F^{s_1}p, F^{s_2}q$ are $B$-badly approximable for $R_{s_t}$. Then $p$ and $q$ are $B'$-badly approximable for $F^t$ for any $B' < B \cdot T_0$. 
Proof of Sublemma. We prove the statement by contradiction. Assume that there exists $\epsilon > 0$ and arbitrarily large $L$ so that
\[
d(F^L p, q) < \frac{BT_0 - \epsilon}{L}.
\]
Assume that the straight line flow is vertical. We may assume that $F^L p$ is on the same horizontal as $q$. Let $T_1$ be the maximal return time of the flow to $\gamma$. Then there is some $0 \leq \ell \leq 3T_1$, so that $F^{\ell} q \in \gamma \setminus \partial \gamma$ (since at most two returns can hit a boundary point of $\gamma$). Furthermore, after fixing $\ell$, we have that for all large enough $L$:
\[
d(F^L p, q) < d(F^{\ell} q, \partial \gamma)
\]
Let $h \subset \gamma$ be the shortest horizontal segment connecting $F^L p$ to $q$. Then $F^{\ell}(h) \subset \gamma$, and it is a horizontal segment of length $d(F^L p, q)$ joining $F^{\ell+L} p$ to $F^{\ell} q$. Since $F^{\ell+L} p$ and $F^{\ell+1} p$ are in the same $R_\alpha$–orbit, there is a power $k$ so that $F^{\ell+L} p = R^k p, F^{\ell+1} p$. Since $s_1$ is the first time that the flow line through $p$ hits $\gamma$, we know that $k \leq 3 + \frac{L}{C}$. In other words, for this $k$ we have:
\[
d(R^k p, F^{s_1} p, F^{\ell} q) = d(F^L p, q)
\]
Since rotations are isometries, and $F^{\ell} q$ is in the $R_\alpha$ orbit of $F^{s_2} q$, there exists some $j \geq 0$ so that
\[
d(R^{k-j} p, F^{s_2} q) = d(F^L p, q).
\]
If $L$ is large enough (depending on $\epsilon$), we then have a contradiction of our claim that $F^{s_1} p, F^{s_2} q$ are $B$-badly approximable. \hfill $\square$

Next, observe that there exists $B > 0$ so that the rotation $R_\xi$ for any $\xi$ whose continued fraction expansion terminates in all 1’s is $B$-badly approximable. Note that such $\xi$ are dense in the reals.

Now, suppose we are given the torus $\omega_T$. Pick a transversal $\gamma$ so that the first return map for the vertical straight line flow on $\omega_T$ defines a rotation on $\gamma$, and furthermore the return time is between $\frac{1}{C[5]}$ and $\frac{C}{[7]}$.

By changing the preferred direction on the torus we may assume that this rotation (when rescaling the transversal to have length 1) is in fact $B$-badly approximable by the density observation above.

These flows are now $B$-badly approximable by the Sublemma.

It remains to modify the points. Given $q_1, \ldots, q_k$ we choose $p_1, \ldots, p_k$ that are the first times the vertical flows from the $q_i$ intersect our transversal. By Corollary 3.10 we may choose $p_1', \ldots, p_k'$ in a $\delta$ neighborhood of these points and on the transversal that are $(\frac{1}{4+\varepsilon})^{3(k-1)}$-badly approximable for the rotation (thought of as being on $[0, 1]$). Applying the vertical flow (which is minimal) in the backwards direction, we can obtain $q_1', \ldots, q_k'$, in a $\delta$ neighborhood for $q_1, \ldots, q_k$, which are pairwise $c$-badly approximable for the flow for any $c < (\frac{1}{4+\varepsilon})^{3(k-1)} \frac{1}{B}$ (by our choice of transversal). \hfill $\square$

Finally, we need the following density statement for $(\epsilon, B)$–torus good $\omega$:

\footnote{technically, postcomposing the flat charts with a rotation matrix in $\text{SL}_2(\mathbb{R})$ close to the identity.}
Proposition 3.13. There exists $\epsilon > 0$ so that for any $q_1, \ldots, q_k \in S$ there exists $B > 0$ so that the set of $(\epsilon, B)$–torus good $\omega$ with respect to $q_1, \ldots, q_k$ is dense in $\tilde{\Omega}(S)$.

Proof. First note that the set of all $\omega$ which are lifts of Abelian differentials on tori branched over one point are dense in the space $\tilde{\Omega}(S)$. Namely, this notion is invariant under the action of $\text{Diff}(S, \{q_i\})$, and the desired density is true for strata of Abelian differentials in the Hodge bundle over Teichmüller space.

The desired density now follows from Proposition 3.12, since being torus good is invariant under pullback by differentials: if $\omega$ is torus good, and $\phi$ is a diffeomorphism, then $\phi^* \omega$ is also torus good. □

3.5. Point-pushing and torus good differentials. Next, we describe constructions which allows us to modify a given $(\epsilon, B)$–torus good $\omega$ in a simple way. In its description, we think of simple closed curves as actual smooth, nonsingular maps from $S^1 = \mathbb{R}/\mathbb{Z}$ to $S$, and not their isotopy classes.

Definition 3.14. We say that a simple closed curve $\alpha$ on $S$ is clean for $\omega \in \tilde{\Omega}(S)$ if

1. $\alpha$ is disjoint from all zeroes of $\omega$.
2. $\alpha$ is transverse to the horizontal and vertical foliation of $\omega$.

Observe that if $\alpha$ is clean, it intersects every horizontal or vertical segment (in the metric given by $\omega$) in at most finitely many points, since the angle to the horizontal or vertical direction is bounded away from zero on the compact curve $\alpha$.

Lemma 3.15. For every $\omega \in \tilde{\Omega}(S)$ there is a clean $\alpha$. Given any clean $\alpha$ there is an open neighbourhood $U_{\omega, \alpha}$ of $\omega \in \tilde{\Omega}(S)$, so that $\alpha$ is clean for every $\eta \in U_{\omega, \alpha}$.

Proof. This follows from Lemmas 4.2 and 4.7 of [LS1]. □

Definition 3.16. Suppose that $\alpha$ is a differentiable, simple closed curve on $S$ which is clean for some $\omega \in \tilde{\Omega}(S)$. We say that $\alpha$ is parametrised with constant horizontal speed if, in the flat charts defined by $\omega$, the horizontal derivative $\alpha'(t)$ is constant in $t$.

Observe that any clean $\alpha$ admits a parametrisation with constant horizontal speed, since by the definition of clean $\alpha$ is nowhere vertical in the flat charts.

Proposition 3.17. Suppose that $\omega$ is $(\epsilon, B)$–torus good with respect to marked points $q_1, \ldots, q_k$, and that $\alpha$ is a simple closed curve on $S$ with the following properties

1. $\alpha$ is clean for $\omega$, and parametrised with constant horizontal speed.
2. There are $t_i$ so that $q_i = \alpha(t_i)$.

Then for all $B' < B$ and any $s \in \mathbb{R}$, we have that $\omega$ is $(\epsilon, B')$–torus good with respect to the marked points $\alpha(t_1 + s), \ldots, \alpha(t_k + s)$.

Proof. Put $q'_i = \alpha(t_i + s)$. Since the torus is a homogeneous space we may assume without loss of generality $q_1 = q'_1$ and that therefore, by the choice of our parametrization, $q'_j$ is a translate along a vertical leaf from $q_j$ for all $j > 1$; Let $\gamma'$ be
a curve connecting $q'_i$ to $q'_j$ and $\gamma$ be the curve connecting $q_i = q'_i$ to $q_j$ by traversing $\gamma'$ and then the vertical segment of length $\ell$. Because $|g_u \gamma'| \geq |g_u \gamma| - e^{-\frac{2}{3}} \ell$ we have the proposition.

Next, we want to re-interpret the families of $(\epsilon, B)$–torus good differentials constructed in Proposition 3.17 as paths in $\tilde{\Omega}(S)$. It will be useful to describe this construction slightly more generally.

To begin, recall from e.g. [LS1, Section 4.2] that associated to a simple closed curve $\alpha$ there is an isotopy $D_{\alpha,t} : S \to S$ which “pushes along the curve $\alpha$”, i.e. $D_{\alpha,t}(\alpha(s)) = \alpha(t+s)$. Observe that such a diffeomorphism $D_{\alpha,t}$ preserves the curve $\alpha$ setwise. Furthermore, note that any diffeomorphism $F : S \to S$ defines by pullback a map $\tilde{\Omega}(S) \to \tilde{\Omega}(S)$, which induces a map

$$\tilde{\Omega}(S) \to \tilde{\Omega}(S)$$

that preserves geometric properties like being (eventually) strongly $\epsilon$–thick, or having a vertical foliation with all leaves closed.

Since $D_{\alpha,t}$ is a smoothly varying family of diffeomorphisms, for any Abelian differential $\omega$, the assignment $t \mapsto D_{\alpha,t}^{-1} \omega$ defines a continuous path $C(\alpha, \omega)$ of Abelian differentials in $\tilde{\Omega}(S)$. Furthermore, this path depends continuously on the initial differential $\omega$.

As $\alpha$ is a closed curve, the endpoint $D_{\alpha,1}^{-1}$ is actually a diffeomorphism fixing $(q_1, \ldots, q_k)$. Hence, the endpoint of $C(\alpha, \omega)$ is obtained from the initial point by pulling back the differential by that diffeomorphism. This path in $\tilde{\Omega}(S)$ depends on the choice of the isotopy $D_{\alpha,t}$. Note that the mapping class of $S - \{q_1, \ldots, q_k\}$ defined by $D_{\alpha,1}^{-1}$ depends only on the homotopy class of $\alpha$ relative to the set $\{q_1, \ldots, q_k\}$ and not the actual curve. We call this mapping class a multi-point-push, and denote it by $P_\alpha$. Observe that if $\alpha$ is embedded, then $P_\alpha$ is a product of Dehn twists about curves to either side of $\alpha$. In particular, results in Section 2 proved for (multi-)Dehn twists also apply for these $P_\alpha$.

We summarize some more basic properties of these paths in the following proposition.

**Proposition 3.18.** Suppose that $\omega$ is an Abelian differential on $S$, and $\alpha$ is a clean simple closed curve which is parametrised with constant horizontal speed. Suppose that $q_i = \alpha(t_i)$ are points on the curve. Then there is a continuous path $c : [0,1] \to \Omega(S, \{q_1, \ldots, q_k\})$, whose endpoint $c(1)$ is the image of $c(0)$ under the multi-point-push along $\alpha$. Furthermore, we have

1. If $\omega$ is $(\epsilon, B)$–torus good with respect to $q_1, \ldots, q_k$, then any point on $c$ $(\epsilon, B')$-thick for any $B' < B$ and thus it is eventually strongly $\delta$–thick with respect to $q_1, \ldots, q_k$.

2. If $\omega$ has vertical foliation a weighted multicurve, then the same is true for every point on $c$ (though the multi-curve may change).

3. The path $c$ depends (for a fixed $\omega$) continuously on the initial differential $\omega$.

---

7Strictly, the curve is only fixed up to reparametrisation; for any $\omega$ one has to choose a constant horizontal speed parametrisation.
on

Lemma 3.19. Suppose that \( q_\alpha \) are as in Proposition 3.18 and \( \omega \) is an Abelian differential whose vertical foliation is a multicurve \( \delta \). Consider the path in \( \Omega(S, \{q_1, \ldots, q_k\}) \) from Proposition 3.18. Then, only a finite number of weighted multicurves appear along this path as vertical foliations.

Proof. The fact that every vertical foliation along the path is a (weighted) multicurves appear along this path as vertical foliations.

Lemma 3.20. A twisting pair for an Abelian differential \( \omega \) and a number of marked points \( q_1, \ldots, q_k \) is a pair of curves \( \alpha, \beta \) so that

1. \( \alpha, \beta \) fill \( S \).
2. \( \alpha, \beta \) are clean.
3. There are numbers \( t_i, s_i \) so that \( q_i = \alpha(t_i) = \beta(s_i) \).

Arguing exactly as in the proof of [LS1 Lemma 4.2], we see the following.

Lemma 3.21. Let \( S \) be a closed surface of genus at least 2, and with some number of marked points \( q_1, \ldots, q_k \). For every \( \omega \in \Omega(S) \) there is a countable set of twisting pairs \( (\alpha, \beta) \). Given any twisting pair \( (\alpha, \beta) \) there is an open neighbourhood \( U_{\omega, \alpha, \beta} \) of \( \omega \in \overline{\Omega}(S) \), so that \( (\alpha, \beta) \) is a twisting pair for every \( \eta \in U_{\omega, \alpha, \beta} \).
Indeed, [LS1]’s argument involves a sequence of simple closed curves converging to minimal foliations (which are obtained by real parts of a rotated \( \omega \)), and they show that any simple closed curves sufficiently close to the limits will have the desired property – in particular there will be countably many such possible choices.

As an immediate consequence of Proposition 3.18 and the fact that the product of multi-point-pushes around filling curves are pseudo-Anosov, we have the following result, analogous to [LS1] Lemma 4.5.

**Corollary 3.22.** Let \((q_1, \ldots, q_k)\) be points on \( S \). Suppose that \( \omega \) is an Abelian differential and \( \alpha, \beta \) is a twisting pair for \( \omega \). Then, for any \( j \), define the diffeomorphism \( \psi^{(j)} = P_{\alpha}^j(P_{\alpha}^{-1}P_{\beta})^j \).

Let \( F \) be the vertical foliation of \( \omega \). Then, there is a point push path in \( \mathcal{PMF} \)
\[ P(F, \psi^{(j)}F) \]
joining \( F \) to \( \psi^{(j)}F \). If \( \omega \) is \((\epsilon, B)-\)torus good with respect to \((q_1, \ldots, q_k)\), then any point on \( P(F, \psi^{(j)}F) \) is a cobounded foliation. If \( F \) is a multicurve, then there is a finite set of multicurves \( F_i \) so that every point on \( P(F, \psi^{(j)}F) \) consists of a weighted multicurve homotopic to one of the \( F_i \) (with varying weights).

**Proof.** The idea is to apply Proposition 3.18 \( 2j + 2 \) times to join \( \omega \) to \( \Psi^{(j)}\omega \) (where \( \Psi^{(j)} \) is a diffeomorphism defining the multi-point-pushing \( \psi^{(j)} \)), and then obtain \( P \) as the associated path of vertical foliations. To make this precise, denote by \( P_\alpha, P_\beta \) point pushing diffeomorphisms around \( \alpha, \beta \), and denote by \( C(\eta, P_\eta) \) the path of Abelian differentials guaranteed by applying Proposition 3.18. We now form the concatenated path

\[
C := C(\omega, P_\alpha \omega) \ast C(P_\alpha \omega, P_\alpha^2 \omega) \ast \cdots
\]
\[
C(P_\alpha^j \omega, P_\alpha^{j+1} \omega) \ast C(P_\alpha^{j+1} \omega, P_\alpha^{j+1} P_\beta^{-1} \omega) \ast \cdots
\]
\[
C(P_\alpha^{j+1} P_\beta^{-1} \omega, P_\alpha^{j+1} P_\beta^{-1} P_\beta^{-1} P_\alpha^{-1} \omega) \ast \cdots
\]
\[
C(P_\alpha^{j+1} P_\beta^{-1} P_\beta^{-j+1} \omega, P_\alpha^{j+1} P_\beta^{-1} P_\beta^{-j+1} P_\alpha^{-j} \omega).
\]

Taking the vertical foliations, we then obtain a path in \( \mathcal{PMCL} \)
\[
P := P(F, P_\alpha F) \ast P(P_\alpha F, P_\alpha^2 F) \ast \cdots
\]
\[
P(P_\alpha^j F, P_\alpha^{j+1} F) \ast P(P_\alpha^{j+1} F, P_\alpha^{j+1} P_\beta^{-1} F) \ast \cdots
\]
\[
P(P_\alpha^{j+1} P_\beta^{-1} F, P_\alpha^{j+1} P_\beta^{-1} P_\beta^{-1} F) \ast \cdots
\]
\[
P(P_\alpha^{j+1} P_\beta^{-1} P_\beta^{-j+1} F, P_\alpha^{j+1} P_\beta^{-1} P_\beta^{-j+1} P_\beta^{-j} F)
\]
of foliations joining the vertical foliation \( F \) of \( \omega \) to \( \psi^{(j)}(F) \). Proposition 3.18 says that if \( \omega \) was \((\epsilon, B)-\)torus good, the same is true for any point on the path \( C \), hence \( P \) consists of cobounded foliations. If \( F \) was a multicurve, the claim follows from Lemma 3.19. \[ \square \]

**Corollary 3.23.** Suppose that \( q_1, \omega, \alpha, \beta \) and \( \psi^{(j)} \) are as in Corollary 3.22, and suppose that \( \omega \) is \((\epsilon, B)-\)torus good. Then the concatenation
\[
P(F, \psi^{(j)}F) \ast \psi^{(j)} P(F, \psi^{(j)}F) \ast \left(\psi^{(j)}\right)^2 P(F, \psi^{(j)}F) \ast \cdots
\]
extends to a continuous path of cobounded foliations connecting \( F \) to the stable foliation of \( \psi^{(j)} \).
Proof. First observe that \( P(F, \psi^{(j)}F) \) is disjoint from the unstable foliation of \( \psi^{(j)} \) for all \( j \). Namely, the unstable foliation of \( \psi^{(j)} \) has an angle-\( \pi \) singularity, since it is a point-pushing map (compare [LS1 Lemma 2.2]), whereas \( F \) (and any point push of it) as a lift of a foliation on a torus has no such singularities. Now, the corollary is an immediate consequence of the fact that pseudo-Anosov maps act on \( \mathcal{P}M_F \) with north-south dynamics. \( \square \)

**Proposition 3.24.** Let \((q_1, \ldots, q_k)\) be points on \( S \). Suppose that \( \omega \) is an Abelian differential whose vertical foliation is a multicurve, and let \( \alpha, \beta \) be a twisting pair for \( \omega \). Then, for any \( j \in \mathbb{Z} \), define the mapping class \( \psi^{(j)} = P_\alpha(P_\alpha P_\beta^{-1} P_\alpha^{-j}). \)

1. There is a constant \( C = C(\omega, \alpha, \beta) > 0 \), so that the union of the sets of multicurves appearing in paths \( P(F, \psi^{(j)}F) \) from Corollary 3.22 (over all \( j \)) has diameter at most \( C \) in the curve graph.

2. If \( \omega_n \) is a sequence of Abelian differentials converging to \( \omega \), with vertical foliations \( F_n \), then the paths \( P(F_n, \psi^{(j)}F_n) \) converge to \( P(F, \psi^{(j)}F) \).

Proof. (1) We inductively consider the terms used in the proof of Equation (10) in Corollary 3.22. Let \( \delta \) be one of the curves in the multicurve \( F \). By Lemma 3.19 only finitely many curves appear in \( P(F, P_\alpha F) \); call that set of curves \( G_0 \). In the next terms

\[
P(P_\alpha^i F, P_\alpha^{i+1} F) = P_\alpha^i P(F, P_\alpha F)
\]

the curves that appear are the images of \( G_0 \) under powers of a Dehn multitwist, and as these act on the curve graph by isometries with fixed points (elliptically), all curves that appear are contained in a \( C_0 \)-neighbourhood of \( \delta \). The curves appearing in the next two terms:

\[
P(P_\alpha^j F, P_\alpha^{j+1} F) * P(P_\alpha^{j+1} F, P_\alpha^{j+1} P_\beta^{-1} F) = P_\alpha^j(P(F, P_\alpha F) * P(P_\alpha F, P_\alpha P_\beta^{-1} F))
\]

are images under \( P_\alpha^j \) of the (finitely many) curves appearing in \( P(F, P_\alpha F) * P(P_\alpha F, P_\alpha P_\beta^{-1} F) \), and are therefore also contained in some \( C_1 \)-neighbourhood of \( \delta \). Finally, in the terms of the third type

\[
P(P_\alpha^{j+1} P_\beta^{-1} P_\alpha^{-i} F, P_\alpha^{j+1} P_\beta^{-1} P_\alpha^{-i-1} F) = P_\alpha^{j+1} P_\beta^{-1} P_\alpha^{-i} P(F, P_\alpha F)
\]

we argue similarly. The path \( P(F, P_\alpha^{-1} F) \) involves finitely many curves, which remain in a \( C_3 \)-neighbourhood of \( \delta \) by application of any power \( P_\alpha^{-1} \). Let \( C_4 = 2C_3 + d(\delta, P_\alpha^{-1} P_\beta \delta) \). Then the image of the \( C_3 \)-neighbourhood around \( \delta \) is mapped by the pseudo-Anosov \( P_\alpha^{-1} P_\beta \) into the \( C_4 \)-neighbourhood around \( \delta \).

Finally, letting \( C_5 = 2C_4 + d(\delta, \alpha) \), the \( C_4 \)-neighbourhood around \( \delta \) is sent to a \( C_5 \)-neighbourhood around \( \delta \) by the application of any further power of \( P_\alpha \). Hence, \( C_5 \) has the desired property.

(2) This is a consequence of the fact that diffeomorphisms act continuously on the space of Abelian differentials; compare Proposition 3.18 (3).

\( \square \)

4. **Paths in the punctured case, and Islands of point-pushes**

We now come to the main technical connectivity result for \((\epsilon, B)\)-torus good foliations. Let \( S \) be a surface, and fix a finite set of points \( z \neq \emptyset \).
Suppose \((\epsilon, B)\) are given so that there is a \((\epsilon, B)\)-torus good \(\omega\) on \(S\) with respect to \(z\). We begin by defining \(\mathcal{T}\mathcal{G}\) to be the set of all vertical foliations on \(S\) of all \(\omega\) which are \((\epsilon, B)\)-torus good. Since being \((\epsilon, B)\)-torus good is invariant under diffeomorphisms preserving \(z\), \(\mathcal{T}\mathcal{G}\) is a mapping-class-group invariant set, and thus dense in \(\mathcal{F}(S, z)\).

Note that we have quantified our base point here a little differently than we did in the introduction (and is often done) when we defined \(\epsilon\)-cobounded foliations. Here we choose a best possible base point and an \((\epsilon, B)\) which are \((\omega)\)-torus good \(\omega\). To \(z\) eventuall

**Lemma 4.1.** Suppose that \(F, F' \in \mathcal{T}\mathcal{G}\) are arbitrary. Then, there are

1. A finite number of simple closed curves \(\alpha_i, \beta_i, i = 1, \ldots, N - 1\),
2. numbers \(\epsilon, B > 0\),
3. Abelian differentials \(\omega_1, \ldots, \omega_N\),
4. Abelian differentials \(\hat{\omega}_1, \ldots, \hat{\omega}_N\),
5. and multicurves \(\delta_1, \ldots, \delta_N\),

so that the following hold:

i) The vertical foliation of \(\omega_1\) is \(F\), and the vertical foliation of \(\omega_N\) is \(F'\).

ii) The curves \((\alpha_i, \beta_i)\) are a twisting pair for both \(\omega_i, \hat{\omega}_i\) and \(\omega_{i+1}, \hat{\omega}_{i+1}\).

iii) All \(\omega_i\) are \((B, \epsilon)\)-torus good, given by pullbacks of \(\omega_i^T\) along a cover \(p_i : S \to T\),

iv) all \(\hat{\omega}_i\) are pullbacks of Abelian differentials \(\hat{\omega}_T\) whose vertical foliation is a single cylinder.

v) the multicurves \(\delta_i\) are the core curves of the vertical cylinders of the \(\hat{\omega}_i\).

Moreover, we can choose a countable family as above that only overlap as necessary. That is, for each such sequence the first of the \(\omega_i\) has vertical foliation \(F\) and the last has vertical foliation \(F'\), but all other chosen vertical foliations, differentials and curves are distinct.

**Proof.** We first produce a single collection as in (1)-(3) that satisfy i)-iii). Let \(F, F' \in \mathcal{T}\mathcal{G}\) be given. By definition of \(\mathcal{T}\mathcal{G}\), they are vertical foliations of \((\epsilon, B)\)-torus good Abelian differentials \(\omega_1, \omega_N\). Note that these satisfy i) and iii) by choice. Choose a path \(\gamma(t)\) from \(\omega_1\) to \(\omega_N\) in \(\Omega(S)\). For every \(\gamma(t)\) there is a twisting pair \(\alpha(t), \beta(t)\) for \(\gamma(t)\) by Lemma 3.21. In fact, by the same lemma, there is a small open neighbourhood \(U_t\) so that the curves \(\alpha(t), \beta(t)\) are twisting pairs for all differentials in \(U_t\). By compactness of the path \(\gamma\), a finite number \(U_1, \ldots, U_N\) of such neighborhoods suffice to cover the path \(\gamma\). We let \(\omega_i \subset U_i \cap U_{i-1}\) be \((\epsilon, B)\)-torus good (which is possible since \((\epsilon, B)\)-torus good differentials are dense), and \(p_i : S \to T\) the defining covering. This implies the existence of the desired objects (1) through (3) with properties i) through iii).

We now inductively produce a countable collection as in (1)-(3) that satisfy i)-iii) and additionally only overlap as necessary. Note that because Lemma 3.21 produces a countable family of curves, if we are given

\[\left\{\left(U_i^{(j)}, (\alpha_i^{(j)}, \beta_i^{(j)}), \omega_i^{(j)}\right)_{i=1}^N\right\}_{j=1}^k\]
Corollary 4.3. \( K \) really depend only on the choice of suitable \( F \) differential \( \hat{p} \) so that the pullback \( \omega \) of \( i \) / does not appear in the above list for any \( i \notin \{1, N_{k+1}\} \) and neither \( \alpha_i^{(k+1)} \) nor \( \beta_i^{(k+1)} \) appear in the previous list for any \( i \). Indeed for each \( \omega \in \gamma(t) \) there are a countable choice of \( \alpha \) and \( \beta \) and torus good differentials are dense.

It remains to show the existence of Abelian differentials and curves as in (4), (5) with properties iv) and v). Let \( F \) cobounded foliation which we call the number iii). Furthermore, the vertical foliation of \( \hat{\alpha} \) choice of \( \alpha \) \( \beta \) and for each \( \alpha, \beta \) \( \omega \) \( \{U, \ldots, F, N_{k+1}\} \) (1) \( P \gamma \) \( \gamma \) be the differential so that \( \omega \) \( \hat{p} \) by \( p \). As the \( U_i \) are open, and cylinder directions are dense, there is a differential \( \hat{\omega} \) on \( T \) which has vertical direction a simple closed curve \( \delta_i \subset T \), and so that the pullback \( p_i^* \hat{\omega} = \hat{\omega} \) is also contained in \( U_i \cap U_{i-1} \). By the definition of that set, \( (\alpha_i, \beta_i) \) and \( (\alpha_{i-1}, \beta_{i-1}) \) are then twisting pairs for \( \hat{\omega} \), so they satisfy iii). Furthermore, the vertical foliation of \( \hat{\omega} \) is the multicurve \( p_i^{-1}(\delta_i) = \delta_i \). Hence, \( \hat{\omega}, \delta_i \) satisfy properties iv) and v). We can run this argument for each of the \( \{U_1^{(j)}, (\alpha_1^{(j)}, \beta_1^{(j)}), \omega_1^{(j)}\}_{j=1}^{N_j} \) produced above and using the density of single cylinder surfaces, we can ensure the \( \hat{\omega}_i^{(j)} \) only overlap as necessary. □

Using the output of Lemma 4.1, we can construct paths between torus good foliations in the following way.

Definition 4.2. Let \( \omega_i, (\alpha_i, \beta_i), p_i, \omega_i \) be as in Lemma 4.1 For each \( i \) choose a number \( K_i \) and a corresponding mapping class
\[
\psi_i^{(K_i)} = P_{\alpha_i}^1 (P_{\alpha_i} P_{\beta_i}^{-1}) P_{\alpha_i}^{K_i}.
\]
which we call the peak pseudo-Anosov, and for each \( i = 2, \ldots, N - 1 \) choose a cobounded foliation \( F_i \) which is a lift of a foliation under \( p_i \), which we call the base foliations, so that the \( (\alpha_i, \beta_i) \) are a twisting pair for that lift. Put \( F = F_1, F' = F_N \). The associated push-and-peak-path is then the path \( \gamma \) obtained as a concatenation
\[
\gamma = \gamma_1^+ \gamma_2^+ \cdots \gamma_{N-1}^+ \gamma_N^+ \gamma_1^- \gamma_2^- \cdots \gamma_{N-1}^- \gamma_N^-,
\]
where \( \gamma \) denotes the path with opposite orientation, in the following way:

1. \( \gamma_1^+ \) is the path starting in \( F_1 \), and ending in the stable foliation of \( \psi_i^{(K_i)} \) which is obtained as the concatenation
\[
P(F_1, \psi_i^{(K_i)} F_1) \ast P(F_1, \psi_i^{(K_i)} F_1) \ast \left( \psi_i^{(K_i)} \right)^2 \ast P(F_1, \psi_i^{(K_i)} F_1) \ast \ldots
\]
of images of the point-push path \( P(F_1, \psi_i^{(K_i)} F_1) \) (compare Corollary 3.22 and 3.23) under \( \psi_i^{(K_i)} \).

2. \( \gamma_i^- \) is the path starting in \( F_i \), and ending in the stable foliation of \( \psi_i^{(K_i-1)} \) which is similarly obtained as the concatenation
\[
P(F_1, \psi_i^{(K_i-1)} F_1) \ast \psi_i^{(K_i-1)} \ast \left( \psi_i^{(K_i-1)} \right)^2 \ast P(F_1, \psi_i^{(K_i-1)} F_1) \ast \ldots
\]

Observe that peak-and-push paths are defined using Abelian differentials, but really depend only on the choice of suitable \( F_1, \ldots, F_N; \alpha_1, \beta_1, \ldots, \alpha_N, \beta_N \) and \( K_1, \ldots, K_N \).

Corollary 4.3. Any two points in \( TG \subset PMF \) can be joined by countably many paths of cobounded foliations which only intersect at the first and last foliations.
Proof. Let \( F, F' \in \mathcal{T} \mathcal{G} \) be given. Apply Lemma 4.1 to obtain \( \omega_i, (\alpha_i, \beta_i), p_i, \omega_i' \). Construct the push-and-peak path as in Definition 4.2. First observe that by using Corollary 3.23, we see that this is indeed a continuous path of cobounded foliations. It joins \( F \) to \( F' \) by construction. To obtain disjoint paths, we apply Lemma 4.1 to obtain a sequence of different \( \omega_i^{(j)} \), and we further stipulate that \( \omega_i^{(j)} \) and \( \omega_i^{(j')} \) project to different foliations on the torus if \( j \neq j' \). This ensures that the path segments, \( \gamma_i^\pm \) for different \( j \) do not overlap except possibly at the stable foliations of the pseudo-Anosovs. As \( K_i \) goes to infinity the stable foliations of the \( \psi_i^{(K_i)} \) converge to \( \alpha_i \). Given these paths for \( \{\{\omega_i^{(j)}, (\alpha_i^{(j)}, \beta_i^{(j)})\}_{i=1}^{N_j}\}_{j=1}^{r} \), by choosing \( K_i \) large enough given these we can ensure the path we build from \( \{\omega_i^{(r+1)}, (\alpha_i^{(r+1)}, \beta_i^{(r+1)})\}_{i=1}^{N_{r+1}} \) is disjoint from the previous \( r \) paths.

We now use the machinery developed in Section 2.2 in order to contract suitable point-push-paths into small neighbourhoods of uniquely ergodic foliations. We briefly recall the setup from that section. Namely, suppose that \((\tau, E)\) be an associated \( \text{Mod} \)-sequence. By choosing \( K \) large enough given these we can ensure the path we build from \( \{\omega_i^{(r+1)}, (\alpha_i^{(r+1)}, \beta_i^{(r+1)})\}_{i=1}^{N_{r+1}} \) is disjoint from the previous \( r \) paths.

The following theorem is concerned with finding paths \( P \) which connect two points in a model neighbourhood \( \mathcal{U}^{(k)} \), and which are also moved by the \( f_n \) into smaller and smaller neighbourhoods of \( E \) (even though the path \( P \) may leave \( \mathcal{U}^{(k)} \)).

Theorem 4.4. Suppose that \((\tau_n)\) is a full splitting sequence in the direction of a uniquely ergodic foliation \( E \), and let \((f_m, k_m)\) be an associated \( \text{Mod} \)-sequence.

Fix an essential type \( k \), and let \( F, F' \in \mathcal{T} \mathcal{G} \cap \mathcal{U}^{(k)} \) be two foliations defined by torus good Abelian differentials \( \omega, \omega' \). Furthermore let \( \delta, \delta' \) be lifts of simple closed curves on the base tori. Assume that

\[
(*): \mathcal{U}^{(k)} \text{ contains every foliation which is a lift of the torus covers defined by } \omega, \omega'.
\]

Then for any \( n \) and \( r \) there is a number \( m_0 \) with the following property. For any \( m > m_0 \) with \( k_m = k \) there are peak-and-push paths \( \gamma_1, ..., \gamma_r \) connecting \( F \) to \( F' \), which intersect only at \( F, F' \) and so that \( f_m \gamma_i \) is completely contained in \( U_n(\tau, E) \).

Without property (*) the conclusion remains true for \( F, F' \) which are sufficiently close (depending on \( m \)) to the curves \( \delta, \delta' \).

Proof. We begin by noting that due to property (*), the initial segment \( \gamma_i^+ \) and terminal segment \( \gamma_i^- \) are automatically contained in \( \mathcal{U}^{(k)} \), independent of all other choices. Hence, for any \( m > n \) with \( k_m = k \), the images of these segments under \( f_m \) are contained in \( U_n(\tau, F) \), by Equation 4 of the associated sequence. If (*) does not hold, we will argue for the initial/terminal segment exactly as below.
We will now explain how to construct the path segments \( \gamma_i^+ \) of the push-and-peak-path; the segments \( \gamma_i^- \) will be constructed analogously. Whenever a constant \( K_i \) is chosen, it needs to be chosen to be large enough for the construction of both \( \gamma_i^+ \) and \( \gamma_i^- \).

**Definition 4.2** produces a set of bounded diameter in \( C(S) \): Let \( \omega_i, (\alpha_i, \beta_i), \delta_i \) be the objects guaranteed by Lemma 4.1 applied to \( F, F' \). Consider the point-pushing pseudo-Anosov map

\[
\psi_i^{(K_i)} = P_{\alpha_i}^{K_i} P_{\alpha_i}^{-1} P_{\alpha_i}^{-K_i}.
\]

By Proposition 3.24, there are numbers \( C_i \), depending only on \( \alpha_i, \beta_i \) and \( \delta_i \) so that for any \( L \in \mathbb{Z} \) every point on the peak-and-push paths \( P(\delta_i, \psi_i^{(L)} \delta_i) \) corresponds to a multicurve which is contained in the \( C_i \)-neighbourhood of \( \delta_i \) in the curve graph.

Let \( G_i \) be the set of all multicurves appearing on such paths. Because \( \delta_1, ..., \delta_N \) are given and each \( G_i \) is contained in a \( C_i \)-neighbourhood of \( \delta_i \) in the curve graph, the (finite) union

\[
G = \bigcup_{i=1}^{N} G_i
\]

also has finite diameter in the curve graph. We can therefore choose a number \( d \) large enough so that for all \( i \), every curve in \( G \) has distance at most \( d \) from \( \alpha_i \).

**Obtaining \( m \) from Proposition 2.20** By increasing \( d \), we may also assume that (for any choice of powers \( K_i \) in the push-and-peak-paths), the quasi-axes of \( \psi_i^{(K_i)} \) pass within distance \( d \) of \( \alpha_i \) as well. Indeed, if \( \rho \) is a quasi-axis for \( \psi_i^{(0)} \) then \( P_{\alpha_i} \rho \) is a quasi-axis for \( \psi_i^{(K_i)} \), and the claim follows since \( P_{\alpha_i} \) fixes \( \alpha_i \).

Apply Proposition 2.20 with this \( d \) to \( P_{\alpha_i} P_{\beta_i}^{-1} \) as the pseudo-Anosov, and \( \mathcal{V} = U_n(\tau, F) \) as the neighbourhood and any curve in \( G \) as the curve \( \beta_0 \) for every \( i \) to get a constant \( N = N_i \). Let \( m_0 \) be the maximum of these constants.

**Choosing \( K_i \) large enough to obtain contraction**: Let now \( m > m_0 \) be given. Then Proposition 2.20 yields\(^9\) that if we choose the powers \( K_i \) in the definition of \( \psi_i^{(K_i)} \) large enough, the images of the point-pushing paths \( \left( \psi_i^{(K_i)} \right)^3 P(\delta_i, \psi_i^{(K_i)} \delta_i), \left( \psi_i^{(K_i-1)} \right)^3 P(\delta_i, \psi_i^{(K_i-1)} \delta_i) \) under \( f_m \) are contained in \( U_n(\tau, F) \) for all \( j \).

As we let \( K_i \to \infty \), the stable foliation of \( \psi_i^{(K_i)} \) converges to \( \alpha_i \). Hence, we can choose numbers \( K_i \) large enough, so that the stable foliation of \( \psi_i^{(K_i)} \) is sent into \( U_n(\tau, E) \) by \( f_m \) (in addition to the previous constraints).

Since the pseudo-Anosov \( \psi_i^{(K_i)} \) acts on \( \mathcal{PMF} \) with north-south-dynamics, and the (compact) path \( P(\delta_i, \psi_i^{(K_i)} \delta_i) \) does not intersect the unstable foliation of \( \psi_i^{(K_i)} \) (as the path consists only of multicurves), there is a number \( \epsilon > 0 \) and \( J > 0 \) so that the \( \epsilon \)-neighbourhood of \( P(\delta_i, \psi_i^{(K_i)} \delta_i) \) is mapped into \( U_n(\tau, E) \) by \( f_m \left( \psi_i^{(K_i)} \right)^3 \) for all \( j > J \).

\(^9\)noting that since multi-point pushing maps are multitwists, we can apply that Proposition in this situation.
By the continuity of the maps $f_m \psi_i^{(K_i)}, ..., f_m \left( \psi_i^{(K_i)} \right)^j$, we may therefore choose $F_i$ close enough to $\delta_i$ so that in fact the path $P(F_i, \psi_i^{(K_i)} F_i)$ is contained in the $\epsilon$–neighbourhood of $P(\delta_i, \psi_i^{(K_i)} \delta_i)$, and therefore

$$f_m \left( \psi_i^{(K_i)} \right)^j P(F_i, \psi_i^{(K_i)} F_i)$$

is contained in $U_n(\tau, E)$ for all $j > J$.

By the continuity of the maps $\psi_i^{(K_i)}, \left( \psi_i^{(K_i)} \right)^2, \left( \psi_i^{(K_i)} \right)^j$, we can choose the foliation $F_i$ even closer to $\delta_i$, to ensure that for all $j \geq 0$, since the paths $f_m \left( \psi_i^{(K_i)} \right)^j P(\delta_i, \psi_i^{(K_i)} \delta_i)$ are all contained in $U_n(\tau, F)$. Repeating the same argument for all $i$, and analogously for the paths for $\gamma_i^{-}$ finishes the argument.

A finite number of curves: To obtain the result for curves $\gamma_1, ..., \gamma_r$, we use Lemma 4.1 and Corollary 4.3 to produce $\{\omega_i^{(k)}, (\alpha_i^{(k)}, \beta_i^{(k)}), (\delta_i^{(k)}), \}_{k=1}^J$. We then apply the step. Definition 4.2 produces a set of bounded diameter in $C(S)$, to each $\{\omega_i^{(k)}, (\alpha_i^{(k)}, \beta_i^{(k)}), (\delta_i^{(k)}), \}_{k=1}^J$ to produce $G^{(k)}$. Let $\bar{G} = \cup_{k=1}^J G^{(k)}$ and observe that it is a set of bounded diameter in $C(S)$. So there exists $d$ so that $\bar{G}$ is in a $d$ neighborhood of $\alpha_i^{(j)}$ for all $i, j$. We may apply Proposition 2.20 to obtain a uniform $m$ for all possible $P_{\hat{\alpha_i^{(k)}}} P_{\hat{\beta_i^{(k)}}}$. Once we have this $m$ we can treat each $\gamma_i^{\hat{k}}$ separately in the step Choosing $K_i$ large enough to obtain contraction. We can arrange the disjointness of the paths as in Corollary 4.3.

The following corollary will be used in the next section to build paths of foliations on surfaces without punctures.

**Corollary 4.5.** Suppose that $S$ is a surface with marked points, which is a branched cover over a torus. Suppose further that $\hat{S}$ is a closed surface and $p: \hat{S} \rightarrow S$ is a properly branched cover, with branching set $\mathbf{z}$ equal to the marked points of $S$. Suppose that $\tau_n$ is a splitting sequence of train tracks on $\hat{S}$ in the direction of a uniquely ergodic foliation $\mathbf{E}$. Let $f_1, ...$ be an associated Mod–sequence.

Fix an essential type $k$, and let $\hat{F} = p^{-1}(F), \hat{F'} = p^{-1}(F') \in \mathcal{U}^{(k)}$ be lifts under $p$ of torus good foliations $F, F'$ on $(S, \mathbf{z})$, defined by Abelian differentials $\omega, \omega'$, and let $\delta, \delta'$ be lifts of simple closed curves on the base tori. Assume that

$$(*) \quad \mathcal{U}^{(k)} \text{ contains every lift under } p \text{ of a foliation on } S \text{ which is a lift of the torus covers defined by } \omega, \omega'.$$

Then for any $n$ and $r$ there is a number $m_0$ with the following property. For any $m > m_0$ with $k_m = k$ there are peak-and-push paths $\hat{\gamma}_1, ..., \hat{\gamma}_r$, connecting $F$ to $F'$, which lifts under $p$ to a paths $\hat{\gamma}_1, ..., \hat{\gamma}_r$ of cobounded foliations, and so that each $f_m \hat{\gamma}_i$ is completely contained in $U_n(\tau, F)$. Moreover the $\hat{\gamma}_i$ intersect only at $f_m F$, $f_m F'$.

Without property $(*)$ the conclusion remains true for $\hat{F}, \hat{F}'$ which are sufficiently close (depending on $\tau$) to lifts $\delta, \delta'$ of the curves $\delta, \delta'$.

**Proof.** This follows exactly like the previous proof, using that the lifting map $\mathcal{PMF}(S) \rightarrow \mathcal{PMF}(\hat{S})$ is a continuous embedding. \qed
4.1. Proof of the main theorem in the punctured case. As an application of Theorem 4.4 we can now prove the main theorem in the case of punctured surfaces.

**Theorem 4.6.** Suppose that \( \Sigma \) is a surface of genus \( g \geq 2 \) and with \( p \geq 1 \) punctures. Then the set of uniquely ergodic foliations on \( \Sigma \), \( \mathcal{UE}(\Sigma) \), is path-connected. Moreover, given any finite set \( S \), we have that \( \mathcal{UE}(\Sigma) \setminus S \) is path connected.

To prove the theorem, the main step is to show that one can connect an arbitrary uniquely ergodic foliation \( E \) to a torus good foliation. By Corollary 4.3 this will be enough to show the theorem.

Before beginning the proof in earnest, let us remark quickly about the case that \( \Sigma \) has only a single marked point. In the previous section, we usually thought of the surfaces to have at least two marked points (since this is much harder, and the naturally occurring case in the proof of the main theorem in the closed case). However, for a single marked point we simply set the “pairwise badly approximable” condition in the definition of torus good differentials to be empty. A torus good differential in this sense has cobounded vertical foliation, and (any) point-push (of the single point) preserves this property. Hence, the desired results also hold in this case.

In order to prove the main step, we use the connection to splitting sequences described in Section 2.

To this end, let \( \tau \) be a maximal train track carrying \( E \), and \( \tau_s \) a full splitting sequence in direction of \( E \). We let \( (f_n, k_n) \) be an associated Mod-sequence. First, we need the following statement, purely about the model neighbourhoods.

**Lemma 4.7.** Given any \( k \) there is a torus cover \( p_k: \Sigma \rightarrow T \), so that the lift of every foliation from \( T \) via \( p_k \) is contained in \( \mathcal{U}(k) \).

**Proof.** Let \( p: \Sigma \rightarrow T \) be any branched torus cover, and let \( L \subset PMF \) be the set of all lifts of foliations on \( T \) via \( p \). Precomposing the cover \( p \) by a mapping class \( \varphi^{-1} \) replaces \( L \) by \( \varphi(L) \).

Choose a pseudo-Anosov \( \varphi \) whose attracting foliation is contained in the (open) set \( \mathcal{U}(k) \), and whose repelling foliation is not contained in \( L \). As pseudo-Anosovs act on \( PMF \) with north-south dynamics, there is a power \( N \) so that \( \varphi^N(L) \subset \mathcal{U}(k) \), which shows the existence of the desired cover. \( \Box \)

From now on, we fix for each \( k \) covers \( p_k \) as in Lemma 4.7. Furthermore we choose, once and for all, torus good foliations \( \mathcal{F}(k) \in \mathcal{U}(k) \setminus \text{Mod}(\Sigma)S \) which are defined by these covers \( p_k \). Note that these exist, since there are uncountably many torus-good foliations.

Recall that the associated Mod-sequence has the property that

\[ U_s(\tau, E) = f_s(U(k_{s-1})). \]

In particular, the (torus good) foliations \( f_s(F(k_s)) \) converge to \( E \). Our strategy will be to find paths \( \gamma_s \) of cobounded foliations which connect \( f_s(F(k_s)) \) to \( f_{s+1}(F(k_{s+1})) \), so that the concatenated paths

\[ c_n = \gamma_1 \ast \gamma_2 \ast \cdots \gamma_n \]

converge, as \( n \rightarrow \infty \), to a path connecting the torus good foliation \( f_1(F(k_1)) \) to \( E \).
Recall from Lemma 2.6 that there is a finite set $M$ of mapping classes, so that for all $n$ we have

$$f_n^{-1}f_{n+1} \in M.$$  

The following corollary of Theorem 4.4 is what makes our construction of paths work:

**Corollary 4.8.** Given any $n$, and $r$ there is a number $m$ with the following property: if $s > m$, then there are paths $\gamma_s^{(1)}, \ldots, \gamma_s^{(r)}$ with the following properties:

1. $\gamma_s^{(j)}$ joins $f_s(F^{(k_s)})$ to $f_{s+1}(F^{(k_{s+1})})$ for all $1 \leq j \leq r$,
2. $\gamma_s^{(j)}$ consists only of cobounded foliations for all $1 \leq j \leq r$, and
3. $\gamma_s^{(j)} \subset U_n(\tau, E)$ for all $1 \leq j \leq r$.
4. For all $j, \neq j'$, $\gamma_s^{(j)} \cap \gamma_s^{(j')} = \{f_s(F^{(k_s)}), f_{s+1}(F^{(k_{s+1})})\}$ for all $1 \leq j \leq r$.

**Proof.** Since we only make a claim about large $s$, we may assume without loss of generality that every type $k_s$ for $s > m$ is essential.

We will then find $\gamma_s$ as

$$\gamma_s = f_s\iota_s$$

for a suitable path $\iota_s$. In order to satisfy (1), the path $\iota_s$ needs to join $F^{(k_s)}$ to $f_s^{-1}f_{s+1}(F^{(k_{s+1})})$. Note that by the second claim of Lemma 2.6 (Equation (5)) we have

$$f_s^{-1}f_{s+1}(U^{(k_{s+1})}) \subset U^{(k_s)}$$

and therefore we have that

$$F^{(k_s)}, f_s^{-1}f_{s+1}(F^{(k_{s+1})}) \in U^{(k_s)}.$$

In fact, as the foliations $F^{(k_s)}$ are defined by the covers from Lemma 4.7 the foliations $F = F^{(k_s)}$, $F' = f_s^{-1}f_{s+1}(F^{(k_{s+1})})$ are defined by Abelian differentials $\omega, \omega'$ which satisfy condition $(\ast)$ in Theorem 4.4 by the comment right after the proof of Lemma 4.4.

Hence, for any essential type $k$ and $s$ with $k_s = k$, we can apply Theorem 4.4 to $r$, $U_n(\tau, E)$ and pairs of foliations $(F^{(k)}, f_s^{-1}f_{s+1}(F^{(k_{s+1})}))$, to obtain thresholds $m_0(k, F^{(k)}, f_s^{-1}f_{s+1}(F^{(k_{s+1})}))$. Note that since there are finitely many $F^{(i)}$ and for all $s$, $f_s^{-1}f_{s+1} \in M$ for the finite set $M$ from Lemma 2.6 there is a number

$$m = \max m_0(k, F^{(k)}, f_s^{-1}f_{s+1}(F^{(k_{s+1})})).$$

We claim that this has the desired property. Namely, suppose that $s > m$. Then, let $k = k_s$ be the type of the index $s$. By our choice of $m$ the foliations $F^{(k_s)}$, $f_s^{-1}f_{s+1}(F^{(k_{s+1})})$ and the number $s$ then satisfy the prerequisites of Theorem 4.4, and we can choose $\iota_s$ to be the path guaranteed by that theorem. Since peak-and-push-paths consist only of cobounded foliations, and this property is invariant under the mapping class group, $f_s\iota_s$ then satisfies (1) and (2). Property (3) and (4) are directly guaranteed by Theorem 4.4.

$\Box$

**Proof of Theorem 4.6.** In order to show the theorem, in light of Corollary 4.3 it suffices to show that, any uniquely ergodic foliation $E \notin S$ can be joined to a torus...
good foliation by a path that doesn’t intersect $S$. We will do this by using the construction outlined above.

Namely, apply Corollary 4.8 for every $n$ to get a sequence $m_n$ of threshold indices. We may assume without loss of generality that $m_n$ is increasing in $n$. For $s \leq m_1$, choose $\gamma_s$ to be any path of cobounded foliations connecting $f_s(F^{(k_1)})$ to $f_{s+1}(F^{(k_{s+1})})$ that does not intersect $f_{s-1}^{-1}S$. Indeed, we apply Corollary 4.8 with $r = |S| + 1$. For $m_{n+1} \geq s > m_n$, let $\gamma_s$ be the result of applying Corollary 4.8. We then have that $\gamma_s \subset U_n(\tau, E)$ for $m_{n+1} \geq s > m_n$.

Consider now the paths

$$c_r = \gamma_1 * \cdots * \gamma_r,$$

and note that they join the torus good foliation $f_1(F^{(k_1)})$ to $f_{r+1}(F^{(k_{r+1})})$. For any $s < r$, let

$$i_{s,r} = \gamma_{s+1} * \cdots * \gamma_r,$$

so that

$$c_r = c_s * i_{s,r}.$$ 

By our construction of the $\gamma_s$, we have that for any $n$ there is some $m_n$, so that for all $r > s > m_n$:

$$i_{s,r} \subset U_n(\tau, E)$$

As by Corollary 2.3 we have that

$$\bigcap_n U_n(\tau, E) = \{E\},$$

this shows that since $c_r \subset U_n$ for all $r > m_n$, the infinite concatenation

$$c_\infty = c_1 * c_2 * \cdots * c_n * \cdots$$

extends to a continuous path with endpoints $f_1(F^{(k_1)})$, $E$, finishing the proof. □

5. Paths in the closed case, and Islands of branched covers

**Theorem 5.1.** Suppose that $\Sigma$ is a closed surface of genus $g \geq 5$. Then the set $UE(\Sigma)$ of uniquely ergodic foliations on $\Sigma$ is path-connected. Moreover, for any finite set $S$, $UE(\Sigma) \setminus S$ is path connected.

To prove this theorem, we want to run the strategy of the proof of Theorem 4.6 with the addition of using branched covers to lift paths from punctured to closed surfaces.

The first ingredient is the following theorem, which follows from the methods developed in [LS1].

**Proposition 5.2.** Suppose that $g \geq 5$. Then there is an involution $\sigma$ of the closed surface $\Sigma_g$ with the following properties.

i) $\Sigma_g/\sigma$ is a surface of genus at least 2 with several marked points.

ii) For any conjugate $\hat{\sigma}$ of $\sigma$ in the mapping class group there is a sequence $\sigma_i$ so that

$$\sigma = \sigma_1, \ldots, \sigma_n = \hat{\sigma},$$

and for any $i$ the group $G_i = \langle \sigma_i, \sigma_{i+1} \rangle$ is a finite group so that $\Sigma_g/G_i$ is a torus with four marked points. In that case we also say that $\sigma, \hat{\sigma}$ are a good pair.
In the proof we need the notion of Humphries generators for the mapping class group. We refer the reader to the textbook [FM1, Chapter 4] for a detailed discussion, and only recall the definition for convenience. Namely, a Humphries generating set for the mapping class group of a genus $g$ surface consists of Dehn twists about curves $\alpha_i, i = 1, \ldots, 2g + 1$ so that
- $\alpha_1, \ldots, \alpha_{2g}$ form a chain, i.e. $\alpha_i, \alpha_j$ intersect in one point if $|i - j| = 1$, and are disjoint otherwise.
- $\alpha_{2g+1}$ is disjoint from all $\alpha_i$ except $\alpha_4$, which it intersects in a single point.

The crucial result [FM1, Theorem 4.14] is that Dehn twists about any such set of curves generate the mapping class group.

**Proof of Proposition 5.2.** When $g$ is even this is [LS1, Theorem 5.3]. The case of odd genus is a fairly straightforward modification which is below.

The strategy is as follows. We show that for $f_1, \ldots, f_n$ a suitably chosen generating set for $\text{Mod}(\Sigma_g)$ and $\sigma$ a suitably chosen involution we have that $\sigma, f_i\sigma f_i^{-1}$ are a good pair. Since whenever $\sigma, \sigma'$ are good pair, $g\sigma g^{-1}$ and $g\sigma'g^{-1}$ are as well, we have that by induction of the word length in $f_1, \ldots, f_n$, $\sigma$ can be joined to $f\sigma f^{-1}$ for any mapping class $f$.

To construct $\sigma$ and $\sigma'$, we use the following setup (compare Figure 1). We realise

![Figure 1](image_url)

**Figure 1.** The setup for Proposition 5.2 realising the dihedral group action

the surface $S$ of genus $2k + 1$ as a union

$$S = \bigcup_{i=0}^{2k-1} H_i$$

where each $H_i$ is a torus with two boundary components, and the two boundaries of $H_i$ are glued to $H_{i+1}, H_{i-1}$ in a ring (compare Figure 1). Denote by $\delta_0, \ldots, \delta_{2k-1}$
the boundary curves of the $H_i$, so that $\partial H_i = \delta_i \cup \delta_{i+1}$. The dihedral group of order $4k$ embeds into the mapping class group of $S$, generated by an order $2k$ element $r$ and an order 2 element $\sigma$. We have that $r(H_i) = H_{i+1}, r(\delta_i) = \delta_{i+1}$ (where indices are taken mod $2k$), and $\sigma$ can be described in the following way: the curves $\delta_0, \delta_k$ cut $S$ into two subsurfaces $S_+, S_-$, each of which has genus $(g-1)/2$ and has two boundary components. The involution $\sigma$ will exchange $S^+$ and $S^-$ and fix both boundary components of $S^+$ setwise.

Intuitively, we imagine $S$ as a symmetric, thickened $2k$-gon in three-space, with a torus in each corner. The element $r$ then rotates the $2k$-gon by $\pi/k$ around its center, while $\sigma$ rotates by $\pi$ about an axis through $\delta_0, \delta_k$ (compare Figure 1).

We then define $\sigma' = r\sigma r^{-1}$. We claim that $\Sigma_g/\langle \sigma, \sigma' \rangle$ is a torus with four marked points. Indeed, $\langle \sigma, \sigma' \rangle$ contains $r^2$ (recall that $\sigma, r$ generate a dihedral group), and thus

$$H_0 \cup H_1 \to \Sigma_g/\langle \sigma, \sigma' \rangle$$

is already surjective. Since $\sigma'$ exchanges $H_0$ and $H_1$, even

$$H_0 \to \Sigma_g/\langle \sigma, \sigma' \rangle$$

is already surjective. In fact, $\Sigma_g/\langle \sigma, \sigma' \rangle$ is obtained from $H_0$ by identifying two halves of $\delta_0$ with each other (via the action of $\sigma$) and identifying two halves of $\delta_1$ with each other (via the action of $\sigma'$). This shows that $\Sigma_g/\langle \sigma, \sigma' \rangle$ is indeed a torus with four marked points (coming from the fixed points of $\sigma, \sigma'$ in $H_0$).

![Figure 2](image)

**Figure 2.** The setup for Proposition 5.2. $\gamma_1, \gamma_2, \gamma_3$ are the curves which are not in $S^\pm$ and they are invariant under $\sigma$. The curve $\gamma'$ is invariant under $\sigma'$.

Next, we claim that there are simple closed curves $\alpha_i$ with the following properties:
a) Dehn twists about the $\alpha_i$ form a (Humphries) generating set for the mapping class group of $\Sigma_g$.
b) Each $\alpha_i$ is either contained in one of the $S^\pm$, or is invariant under $\sigma$.
c) If $\alpha_i \subset S^\pm$, then it is nonseparating in that subsurface.
d) There is one $\alpha_{j_0}$ which is contained in $S^-$ and which is invariant under $\sigma'$.

That such a set of curves exists is an exercise using Figure 2.

Now, from property c) we get the following:

(11) $\forall \alpha_i$ not invariant under $\sigma$ $\exists \phi_i \in \text{Mcg}(\Sigma_g) : [\phi_i, \sigma] = 1, \phi_i(\alpha_{j_0}) = \alpha_i$.

Namely, suppose first that $\alpha_i \subset S^-$. Then, since both $\alpha_i, \alpha_{j_0}$ are nonseparating in $S^-$, there is a mapping class $f$ of $S^-$ fixing $\partial S^-$ which sends $\alpha_{j_0}$ to $\alpha_i$. Extend $f$ to a mapping class $\phi_i$ of $S$ by setting it to be $\sigma f \sigma$ on $S^+$. This has the desired property. In the case where $\alpha_i \subset S^+$, we start with $f$ which sends $\sigma \alpha_i$ to $\alpha_{j_0}$ as above, and let $\phi_i = \sigma f$ on $S^-$ and $f \sigma$ on $S^+$.

We claim that for any of the Humphries generators $T = T_{\alpha_i}$ we can connect $\sigma$ to $T \sigma T^{-1}$ with a path as in ii) of the statement of the Proposition.

For twists about curves $\alpha_i$ which are invariant under $\sigma$ there is nothing to show, as such twists commute with $\sigma$, and therefore the trivial path connects $\sigma$ and $T_{\alpha_i} \sigma T_{\alpha_i}^{-1} = \sigma$. If $\alpha_i$ is not invariant, let $\phi_i$ be the mapping class guaranteed by (11). We claim that

\[
\begin{align*}
\sigma_1 &= \sigma, \\
\sigma_2 &= \phi_i \sigma' \phi_i^{-1}, \\
\sigma_3 &= T_{\alpha_i} \sigma T_{\alpha_i}^{-1}
\end{align*}
\]

is a path as desired. To begin with, note that

\[
G_1 = (\sigma, \phi_i \sigma' \phi_i^{-1}) = (\phi_i \sigma \phi_i^{-1}, \phi_i \sigma' \phi_i^{-1}) = \phi_i \langle \sigma, \sigma' \rangle \phi_i^{-1},
\]

since $\phi_i$ commutes with $\sigma$. As by assumption $\sigma, \sigma'$ is a good pair, $G_1$ is a group as desired.

Next, observe that

\[
\phi_i T_{\alpha_{j_0}} \phi_i^{-1} = T_{\phi_i \alpha_{j_0}} = T_{\alpha_i}
\]

and therefore

\[
[T_{\alpha_i}, \phi_i \sigma' \phi_i^{-1}] = [\phi_i T_{\alpha_{j_0}} \phi_i^{-1}, \phi_i \sigma' \phi_i^{-1}] = \phi_i [T_{\alpha_{j_0}}, \sigma'] \phi_i^{-1} = 1,
\]

since $\sigma'$ preserves $\alpha_{j_0}$ and therefore commutes with the Dehn twist about $\alpha_{j_0}$. As $G_1$ is generated by a good pair, so is

\[
T_{\alpha_{j_0}} G_1 T_{\alpha_{j_0}}^{-1} = \langle T_{\alpha_{j_0}} \sigma T_{\alpha_{j_0}}^{-1}, T_{\alpha_{j_0}} \phi_i \sigma' \phi_i^{-1} T_{\alpha_{j_0}}^{-1} \rangle = \langle T_{\alpha_{j_0}} \sigma T_{\alpha_{j_0}}^{-1}, \phi_i \sigma' \phi_i^{-1} \rangle = \langle \sigma_3, \sigma_2 \rangle.
\]

Hence, $\sigma_1, \sigma_2, \sigma_3$ is indeed a path as desired.

For the remainder of this section, we fix $\sigma$ to be as in the conclusion of Proposition 5.2. Say that a foliation $F$ is lifted torus good, if $F$ is the lift of a torus good foliation on $\Sigma_g/\hat{\sigma}$ for $\hat{\sigma}$ a conjugate of $\sigma$ in $\text{Mod}(S)$ (possibly by the identity).

The following will replace Lemma 4.1.

**Lemma 5.3.** Suppose that $F, F'$ are lifted torus good. Then there are

1. involutions $\sigma_1, \ldots, \sigma_N$, which are conjugate to $\sigma$,
2. Abelian differentials $\omega_i, i = 1, \ldots, N$ on $\Sigma_g$. 

so that the following hold:

i) For any \( i \), the group \( \langle \sigma_i, \sigma_{i+1} \rangle \) is finite and \( T_i = \Sigma_g / \langle \sigma_i, \sigma_{i+1} \rangle \) is a torus with four marked points.

ii) The differential \( \omega_i \) is a lift of a torus good differential on the torus \( T_i \) (with marked points).

Proof. Suppose that \( F \) is a lift of a foliation on \( \Sigma_g / \sigma \) and \( F' \) is a lift of a foliation on \( \Sigma_g / \sigma' \). Apply Proposition 5.2 to \( \sigma, \sigma' \) to find the involutions \( \sigma_i \) with property i). The differentials \( \omega_1, \omega_N \) are chosen to be the ones defining \( F, F' \); the other \( \omega_i \) can be chosen as arbitrary lifts of torus good differentials on \( T_i \). \( \square \)

Finally, the following will replace Theorem 4.4.

Theorem 5.4. Suppose that \( (\tau_n) \) is a full splitting sequence in the direction of a uniquely ergodic foliation \( E \), and let \( f_m \) be an associated \( \text{Mod} \)-sequence.

Fix an essential type \( k \) and let \( F,F' \in U(k) \) be two lifted torus good foliations lifted from covers \( \Sigma_g / \sigma, \Sigma_g / \sigma' \). Assume that \( U(k) \) contains every foliation which is a lift of the cover defined by \( \Sigma_g / \sigma, \Sigma_g / \sigma' \).

Then for any \( n \) there is a number \( m_0 \) with the following property. Suppose that \( m > m_0 \) and that \( k_m = k \). Then there is a path \( \gamma \) connecting \( F \) to \( F' \), so that \( f_m \gamma \) is completely contained in \( U_n(\tau, E) \), and consists only of cobounded foliations. Moreover, given any finite set \( S \) we may assume that \( f_m \gamma \) does not intersect \( S \setminus \{ f_m F, f_m F' \} \).

Proof. Suppose that \( F, F' \) are given as in the theorem. First, apply Lemma 5.3 to obtain a sequence of involutions \( \sigma_1, \ldots, \sigma_N \). We now have two sequences of covers

\[
p_i : \Sigma_g \to \Sigma_g / \sigma_i
\]

and

\[
t_i : \Sigma_g \to \Sigma_g / \langle \sigma_i, \sigma_{i+1} \rangle
\]

which are compatible in the sense that \( t_i \) factors through both \( p_i \) and \( p_{i+1} \):

Now for each \( i \), let \( \delta_i \) be a lift of a simple closed curve on the four times punctured torus \( \Sigma_g / \langle \sigma_i, \sigma_{i+1} \rangle \) by the map \( t_i \), and let \( \mu_i \) be a lift of a simple closed curve from \( \Sigma_g / \sigma_i \) by the map \( p_i \). We will next construct lifted torus good foliations \( B_i, I_j^+, I_j^- \), and the desired path as a concatenation

\[
\gamma = \gamma_1^+ \gamma_2^- \gamma_2^+ \cdots \gamma_{N-1}^- \gamma_{N-1}^+ \gamma_N^+
\]

where \( \gamma \) denotes the path with opposite orientation, and
(1) $\gamma_j^+$ is a path starting in $I_j^+$, and ending in $B_j+1$.
(2) $\gamma_j^0$ is a path starting in $I_j^-$, and ending in $I_j^+$.
(3) $\gamma_j^-$ is a path starting in $I_j^-$, and ending in $B_j-1$.

All $\gamma_j^*$ will be produced by using Corollary 4.5.

Namely, put $I_0^+ = F, I_N^- = F'$. For the remaining $B_i, I_j^+, I_j^-$ choose lifted torus good foliations (for the covers $p_i, t_j, t_j$ respectively) close enough to $\delta_i, \mu_j$ so that we can apply Corollary 4.5 (to Theorem 4.4), in the version without $(\ast)$. Note that this closeness depends on $m \geq m_0$ (and not just $m_0$). This can e.g. be achieved by starting with any lifted torus good foliations, and Dehn twisting them about the $\delta_i, \mu_j$. Note that we can produce disjoint $B_i, I_j^+, I_j^-$ by twisting different numbers of times, and produce distinct paths as in Corollary 4.3.

Now, for $\gamma_1^+$ and $\gamma_N^-$, we will apply Corollary 4.5 with $(\ast)$. Indeed, as both $I_0^+$ and $B_1$ are both given by the same lifting maps $(\ast)$ in the statement of this theorem gives the assumption of $(\ast)$ in Corollary 4.5. (Similarly for $I_n^-$ and $B_{N-1}$.) Note that by Corollary 4.5 these paths can be chosen to overlap only at $F, F'$.

With this in place, we can finish the proof of Theorem 5.1 exactly as in the case of Theorem 4.6.

In fact, the proof shows something a little bit stronger, which will be useful to show local path-connectivity.

**Corollary 5.5.** Suppose $\tau$ is a train track carrying a uniquely ergodic foliation $E$, and suppose that $\tau_n$ is a splitting sequence in the direction of $E$. Then for any $n$ there is a $m = m(\tau, n, E)$ with the following property. If $E'$ is any uniquely ergodic foliation contained in $U_m(\tau, E)$, then there is a path of uniquely ergodic laminations connecting $E'$ to $E$ completely contained in $U_n(\tau, E)$.

**Proof.** In the case of a punctured surface, i.e. Theorem 4.6 all bounds on $m$ come from applying Proposition 2.17 or 2.20 within the proof of Theorem 4.4. By Lemma 2.21 we can choose these bounds to be independent of the actual foliation guiding the splitting sequence, as long as the foliation is contained in $U_k(\tau, E)$ for $k$ large enough. The bounds in Theorem 5.1 come from applying Theorem 4.4 and its Corollary 4.5 and so the same is true there. □

### 6. Local Path Connectivity

In this section, we improve the Theorem from the last section to the following.

**Theorem 6.1.** If $g \geq 5$ or $g \geq 2, p \geq 1$, the set of uniquely ergodic foliations on $S_{g,p}$ is locally path-connected.

Given a uniquely ergodic foliation $F$ and a full splitting sequence $(\tau_n)_n$ towards $F$. For any $n$, we let $m(\tau, n, F)$ the number guaranteed by Corollary 5.5. Define

$$G_n(\tau, F) = U_{m(\tau, n, F)}(\tau, F).$$

Corollary 5.5 guarantees that for any $F' \in G_n(\tau, F)$ there exists a path $P_{F, F'}$ of cobounded foliations joining $F$ to $F'$, which is contained in $U_n(\tau, F)$. 
Let $\hat{G}_n(\tau, F)$ be the intersection of $G_n(\tau, F)$ with the set of uniquely ergodic foliations. For any point $p \in P_{F,F'}$, we can define a neighbourhood

$$G_n(p, \tau)$$

as above, i.e. with the property that $p$ can be joined to any $p' \in G_n(p, \tau)$ by a path of cobounded foliations which is contained in $U_n(\tau, F)$.

Also observe that

$$(12) \quad U_i(\tau, p) \subset U_i(\tau, F)$$

for all $i \leq n$.

Define

$$N^{(1)}(F, n) := \bigcup_{F' \in \hat{G}_n(\tau, F)} \bigcup_{p \in P_{F,F'}} G_n(p, \tau).$$

Inductively, put

$$N^{(r+1)}(F, n) = \bigcup_{p \in N^{(r)}(F, n)} N^{(1)}(p, n).$$

Also observe that we have $N^{(r)}(F, n) \subset U_n(\tau, F)$ by Equation (12), whenever $F' \in \hat{G}_n(\tau, F)$.

**Proposition 6.2.** Any point in $N^{(r)}(F, n)$ is connected to $F$ by a path of uniquely ergodic foliations, which is contained in in $N^{(r+1)}(F, n)$.

**Proof.** We prove this by induction.

**Base case:** If $p \in N^{(1)}(F, n)$ then we can connect it to $F$ by a path in $N^{(2)}(F, n)$.

Proof. If $p \in P_{F,F'}$ this is obvious. Otherwise $p \in G_n(\hat{p}, \tau)$ for some $\hat{p} \in P_{F,F'}$ where $F' \in G_n(\tau, F)$. By definition we have that there exists a path of cobounded foliations contained in $G_n(\hat{p}, \tau)$ connecting $p$ to $\hat{p}$. Concatenating this with the segment of the path is in $N^{(1)}(\hat{p}, n)$ and so the whole thing is in $N^{(2)}(F, n)$. \hfill \Box

**Inductive step:** Assume $p \in N^{(r)}(F, n)$ and that any point in $N^{(r-1)}(F, n)$ is connected to $F$ by a path of cobounded foliations in $N^{(r)}(F, n)$. We will now show that $p$ is path connected by cobounded foliations in $N^{(r+1)}(F, n)$ to $F$.

Proof. Because $p \in N^{(r)}(F, n)$ we know (by definition of $N^{(r+1)}$) $p \in N^{(1)}(\hat{p}, n)$ for some $\hat{p} \in N^{(r-1)}(F, n)$. By the base case of induction applied to $\hat{p}$ it is connected to $\hat{p}$ by a path in $N^{(2)}(\hat{p}, n) = \bigcup_{p' \in N^{(1)}(\hat{p}, n)} N^{(1)}(p', n)$. This is contained in $\bigcup_{p' \in N^{(r)}(F, n)} N^{(1)}(p', n) = N^{(r+1)}(F, n)$. To finish linking $p$ to $F$ we use our inductive assumption to link $\hat{p}$ to $F$ by a path in $N^{(r-1+1)}(F, n)$. \hfill \Box

**Corollary 6.3.** For any uniquely ergodic foliation $F$, and any $n$, the set

$$\left( \bigcup_{r \geq 1} N^{(r)}(F, n) \right) \cap \mathcal{UE}$$

is an open neighbourhood of $F$ in $\mathcal{UE}$, which is path-connected and contained in $U_n(\tau, F)$.\hfill \Box
Proof. The set is open as a union of open subsets. It is contained in $U_n(\tau, F)$, since all $N^{(\tau)}(F, n)$ have this property. It is path-connected by Proposition 6.2. □

By Corollary 2.4 the $U_n(\tau, F)$ are a basis for neighbourhoods of $F$ in $U\mathcal{E}$, and thus this finishes the proof of Theorem 6.1.

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