Input-to-state stabilization of the perturbed systems in the generalized triangular form

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Abstract: We consider nonlinear control systems of the so-called generalized triangular form (GTF) with time-varying and periodic dynamics which linearly depends on some external disturbances. Our purpose is to construct a feedback controller which provides the global input-to-state stability of the corresponding closed-loop w.r.t. the disturbances. To do this, we combine the method proposed in the earlier work [23] devoted the global asymptotic stabilization of the GTF systems without disturbances with the ISS theory for time-varying systems proposed in [21]. Following this pattern we construct a feedback which provides the properties of uniform global stability and asymptotic gain w.r.t the disturbances. Then we obtain the semi-uniform ISS of the closed-loop system.

Keywords: input-to-state stability, triangular form, backstepping

1 INTRODUCTION

One of the most popular framework for design in nonlinear control theory is backstepping. Originally, this approach was proposed for constructing Lyapunov stabilizers [4,12,25]; very soon this technique was applied to solving adaptive control problems: firstly when the dynamics of a strict-feedback system is linear w.r.t. unknown parameters [10,14,15,26], then these results were extended to the cases of nonlinear parametrization [11], unknown control directions [33], backstepping for the systems with time delays [7,8], backstepping for the Volterra systems [18] etc. Let us remark that the classical version of this approach is applicable to the so-called strict-feedback form or, more generally, to the triangular form in the so-called regular case (the latter being introduced in 1973 in [16]), i.e., when the triangular system is feedback linearizable. As the exception we can mention works devoted to polynomial extensions of the strict-feedback forms [3,20,24,32] as well as a more general situation [28,31]. This leads to the concept of the so-called generalized triangular form - [17,19,23] (next called GTF). In the latter works the problem of global robust controllability and that of global asymptotic stabilization of generalized triangular form systems was successively solved.

On the other hand, in many applications one has to consider systems subject to disturbances. In this case the input-to-state stability (ISS) framework introduced in [29] is very fruitful for stability analysis. Therefore, having obtained the results on global asymptotic stabilization for the GTF systems [23], it is natural to consider a GTF system with some external disturbances in its dynamics and to ask whether it is possible to construct a feedback controller which provides a global input-to-state stability property
with respect to the disturbances. The goal of the current paper is to extend the result of the work [23] to this situation. In our case we will use the notion of uniform ISS developed for the case of time-varying systems [21]. A similar problem was considered for systems of the strict-feedback form in [5] (however the strict-feedback form systems under consideration were not only with external disturbances but also with unknown parameters). Since the GTF is an extension of the TF and strict-feedback forms, we extend the results of [5] in the current paper in this sense as well.

2 PRELIMINARIES

Throughout the paper, \( \mathbb{N} \) and \( \mathbb{Z} \) denote the sets of all natural and integer numbers respectively, \( \langle \cdot , \cdot \rangle \) denote the scalar product in \( \mathbb{R}^N \) (for any \( N \in \mathbb{N} \)); for \( A \subset \mathbb{R} \), \( \text{mes} A \) and \( \overline{A} \) denote the Lebesgue measure (if \( A \) is measurable) and the closure of \( A \) respectively. For a vector \( \xi \in \mathbb{R}^N \), by \( |\xi| \) we denote its quadratic norm, i.e., \( |\xi| = \langle \xi, \xi \rangle^{1/2} \).

A function \( \alpha \) of \( \mathbb{R}_+ \) to \( \mathbb{R}_+ \) is said to be of class \( \mathcal{K} \) if it is continuous and non-decreasing, is of class \( \mathcal{K}_\infty \) if it is continuous, positive definite and strictly increasing; and is of class \( \mathcal{K}_\infty \) if it is of class \( \mathcal{K} \) and unbounded.

A function \( \beta \) of \( \mathbb{R}_+ \times \mathbb{R}_+ \) to \( \mathbb{R}_+ \) is said to be of class \( \mathcal{KL} \) if for each fixed \( t \geq 0 \) the function \( \beta(\cdot, t) \) is of class \( \mathcal{K}_\infty \) and for each fixed \( s \geq 0 \), we have \( \beta(s, t) \to 0 \) as \( t \to +\infty \) and \( t \mapsto \beta(s, t) \) is decreasing. Given any \( \Delta(\cdot) \) in \( L_\infty \) by \( \| \Delta(\cdot) \| \) denote its \( L_\infty \) - norm on \( [0, +\infty[ \).

Consider the nonlinear system
\[
\dot{x} = f(t, x, \Delta) \tag{1}
\]
with states \( x \in \mathbb{R}^n \) inputs \( \Delta \in \mathbb{R}^m \), where \( f \) is continuous w.r.t \( (t, x, \Delta) \) and satisfies the local Lipschitz condition w.r.t. \( (x, \Delta) \), whose solution of the Cauchy problem \( x(t_0) = x^0 \) with \( \Delta = \Delta(t) \) is denoted by \( x(t, \xi, t_0, \Delta(\cdot)) \). Given any \( \Delta(\cdot) \) in \( L_\infty \) by \( \| \Delta(\cdot) \| \) denote its \( L_\infty \) - norm on \( [0, +\infty[ \).

The following three definitions and Theorem are borrowed from [21]

**Definition 1** System (1) is input-to-state stable (ISS) iff there are \( \beta \in \mathcal{KL}, \ U_0 \in \mathbb{N} \) and \( \gamma \in \mathcal{K} \) such that for each \( t_0 \), each \( \xi \) and each \( \Delta(\cdot) \), we obtain for all \( t \geq t_0 \)
\[
|x(t, \xi, t_0, \Delta(\cdot))| \leq \beta(U_0(t_0)|\xi|, t-t_0) + \gamma(\| \Delta(\cdot) \|_{L_\infty[t_0, +\infty[})
\]

System (1) is semi-uniformly input-to-state stable if it is ISS and furthermore there exists \( \Upsilon(\cdot) \in \mathcal{K} \) such that
\[
|x(t, \xi, t_0, \Delta(\cdot))| \leq \max\{\Upsilon(|\xi|), \Upsilon(\| \Delta(\cdot) \|)\} \quad \forall t \geq t_0 \tag{2}
\]
for all \( \xi \in \mathbb{R}^n \), and \( \Delta(\cdot) \) in \( L_\infty \).
Definition 2 We say that system (1) satisfies uniform local stability (ULS) property if there are $\gamma \in \mathcal{K}$, and $\delta > 0$ such that for all $|\xi| \leq \delta$ and all $\|\Delta(\cdot)\| \leq \delta$ we have (2). We say that system (1) satisfies uniform global stability (UGS) property if $\delta = +\infty$, i.e., (2) holds for all $\xi$ and $\Delta(\cdot)$.

Definition 3 We say that system (1) satisfies the asymptotic gain (AG) property if there is $\gamma(\cdot) \in \mathcal{K}$ such that

$$\limsup_{t \to +\infty} |x(t, \xi, t_0, \Delta(\cdot))| \leq \gamma(\|\Delta(\cdot)\|).$$

Theorem 1 [21] System (1) is semi-uniform ISS if and only if it is ULS and AG.

3 MAIN RESULT

We consider the following system

$$\begin{cases}
\dot{x}_1 = f_1(t, x_1, x_2) + \delta_1(t)\Phi_1(t, x_1) \\
\dot{x}_2 = f_2(t, x_1, x_2, x_3) + \delta_2(t)\Phi_2(t, x_1, x_2) \\
\vdots \\
\dot{x}_n = f_n(t, x_1, \ldots, x_n, u) + \delta_n(t)\Phi_n(t, x_1, \ldots, x_n)
\end{cases}$$

(3)

where $u \in \mathbb{R}^1$ is the control, $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$ is the state and $\delta_1(t), \delta_2(t), \ldots, \delta_n(t)$ are some external disturbances (in general $\delta_i(t)$ can be vectors of different finite dimensions). We assume that (3) satisfies the following assumptions:

A1: $f = (f_1, \ldots, f_n)^T$ and $\Phi_i$ are of class $C^{n+1}$ and $T$-periodic in time with some $T > 0$, i.e., $f(t + T, x, u) = f(t, x, u)$ and $\Phi_i(t + T, x) = \Phi_i(t, x)$ for all $[t, x, u]$ in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^1$.

A2: $f_i(t, x_1, \ldots, x_i)$ : $\mathbb{R}^1 \to \mathbb{R}^1$ is a surjection, i.e., $f_i(t, x_1, \ldots, x_i, \mathbb{R}^1) = \mathbb{R}^1$ for every $[x_1, \ldots, x_i] \in \mathbb{R}^1 \times \ldots \times \mathbb{R}^1$, and every $t \in [0, T]$, $i = 1, \ldots, n$.

A3: there exist $x_i^* \in \mathbb{R}^1$, $1 \leq i \leq n$, and $u^* = x_{n+1}^*$ in $\mathbb{R}^1$ such that $\frac{\partial f_i}{\partial x_{i+1}}(t, x_1^*, \ldots, x_i^*, x_{i+1}) \neq 0$ for every $t \in [0, T]$, $i = 1, \ldots, n$, and such that $f(t, x^*, u^*) = \Phi_i(t, x^*) = 0$ for all $t \in [0, T]$, $i = 1, \ldots, n$

The following example shows that even global asymptotic stabilization of the time-invariant triangular systems is not always possible if one wants to use a $C^1$ - feedback of the form $u = u(x)$. On the other hand, as we can see below, if we allow the feedback to be a time-varying, it can resolve the problem (even a periodic feedback will suit). That is why we start with the $T$ - periodic systems (of course, our result will be applicable to the time-invariant dynamics as a partial case as well).

Example 1. [19][23] Consider the system

$$\begin{cases}
\dot{x}_1 = x_2^3 - (1 - x_1^2)x_2 \\
\dot{x}_2 = u
\end{cases}$$

(4)
and suppose there is a feedback $u = u(x_1, x_2)$ of class $C^1$, which globally asymptotically stabilizes $x_i$ into $[0, 0]$. Put: $g(x) := [x_1^2 - (1 - x_2^2)x_2, u(x_1, x_2)]^T$, and $C := \{[x_1, x_2] \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$. Since the feedback $u = u(x)$ is continuous on $C$, and globally stabilizes (1), we have $u(x) \neq 0$ for all $x \in C$. Then, the map $C \ni x \mapsto \frac{g(x)}{|g(x)|} = [0, \frac{u(x)}{|u(x)|}]^T$ is well-defined. On the one hand there is a homotopy between the map and $C \ni x \mapsto (-x) \in C$ (see the proof of the famous Brockett theorem [1] given in [28], p. 184), but on the other these maps have different degrees. This contradiction proves that there is no a feedback $u = u(x)$ of class $C^1$ which globally stabilizes (1).

Our main result is as follows.

**Theorem 2** Assume that system (3) satisfies conditions A1-A3. Then, for any $\mu \in \mathbb{N} \cup \{+\infty\}$ system (3) is globally semi-uniformly input-to-state stabilizable into $x^*$ by means of a feedback law $u(t, x)$ of class $C^\mu(\mathbb{R} \times \mathbb{R}^n ; \mathbb{R}^1)$ such that $u(t + T, x) = u(t, x)$ for all $[t, x] \in \mathbb{R} \times \mathbb{R}^n$ and $u(t, x^*) = u^*$ for all $t \in \mathbb{R}$, where $T > 0$ is the period mentioned above in A1.

Let us remark that $x_i$ and $u$ can be vectors in general as in [23] and we assume them to be scalar for the simplicity only (for vectors, the argument will be similar).

## 4 BACKSTEPPING DESIGN

Let $k$ be in $\{0, ..., n-1\}$. For each $y_0 \in \mathbb{R}^{k+1}$, each $\omega_0 \in \mathbb{R}^1$, and each $r > 0$, let $B_r(y_0)$ and $\Omega_r(\omega_0)$ denote the open balls

$$B_r(y_0) := \{y \in \mathbb{R}^{k+1} \mid |y - y_0| < r\}; \quad \Omega_r(\omega_0) := \{\omega \in \mathbb{R}^1 \mid |\omega - \omega_0| < r\}$$

and $\overline{B_r(y_0)}$ and $\overline{\Omega_r(\omega_0)}$ be their closures.

Consider a control system

$$\dot{z} = g(t, z, z_{k+1}) + \sum_{j=1}^{N_k} \Delta_j(t) \varphi_j(t, z), \quad t \in \mathbb{R} \quad (5)$$

where $z_{k+1} \in \mathbb{R}^1$ is the control, $z = [z_1, ..., z_k]^T \in \mathbb{R}^k$, is the state, and $\delta(t) = [\Delta_1(t), ..., \Delta_{N_k}(t)]$ is some external disturbance.

Following [23], we also consider a dynamical extension of (5), i.e., the system

$$\begin{aligned}
\dot{z} &= g(t, z, z_{k+1}) + \sum_{j=1}^{N_k} \Delta_j(t) \varphi_j(t, z) \\
\dot{z}_{k+1} &= g_{k+1}(t, z, z_{k+1}, v) + \sum_{j=1}^{N_k} \Delta_j(t) \varphi_{k+1,j}(t, z) + \sum_{j=N_k+1}^{N_{k+1}} \Delta_j(t) \varphi_{k+1,j}(t, z, z_{k+1})
\end{aligned}$$

which we rewrite in the following vector form

$$\dot{y} = \psi(t, y, v) + \Delta(t) \phi(t, y), \quad t \in \mathbb{R}, \quad (6)$$
where \( y = [z, z_{k+1}]^T \in \mathbb{R}^{k+1} \) is the state, \( v \in \mathbb{R}^1 \) is the control, \( \Delta(t) = [\Delta_1(t), \ldots, \Delta_{N_k}(t), \Delta_{N_k+1}(t), \ldots, \Delta_{N_{k+1}}(t)] \) is its external disturbance (with \( N_{k+1} > N_k \)), and \( \psi(t, y, v) \) and \( \phi(t, z, z_{k+1}) \) are given by

\[
\psi(t, y, v) = \begin{bmatrix} g(t, y) \\ g_{k+1}(t, y, v) \end{bmatrix} \quad \text{and} \quad \phi(t, z, z_{k+1}) = \\
\begin{bmatrix}
\varphi_1 & \cdots & \varphi_{N_k} & 0 & \cdots & 0 \\
\varphi_{k+1,1} & \cdots & \varphi_{k+1,N_k} & \varphi_{k+1,N_k+1} & \cdots & \varphi_{k+1,N_{k+1}}
\end{bmatrix}
\]

for all \([t, y, v] \in \mathbb{R} \times \mathbb{R}^{k+1} \times \mathbb{R}^1\) (7)

As in \([23]\), if \( k=0 \), and system (5) consists of 0 equations, we define \( y := z_{k+1} = z_1 \); \( \psi(t, y, v) := g_{k+1}(t, y, v) = g_1(t, z_1, v) \) with \( v \in \mathbb{R}^1 \) and we say that (5) is empty or trivial and that \( \dot{z}_1 = g_1(t, z_1, z_2) \) with states \( z_1 = y \) and controls \( z_2 = v \) is the extension of the empty system (5).

We assume that \( \psi \) and \( \phi \) satisfy the following Assumptions:

A1': Functions \( \psi \) and \( \phi \) are of classes \( C^2(\mathbb{R} \times \mathbb{R}^{k+2}; \mathbb{R}^{k+1}) \) and \( C^2(\mathbb{R} \times \mathbb{R}^{k+1}; \mathbb{R}^{k+1}) \) respectively and there exists \( T > 0 \) such that \( \psi(t + T, y, v) = \psi(t, y, v) \) and \( \phi(t + T, y) = \phi(t, y) \) for all \([t, y, v] \in \mathbb{R} \times \mathbb{R}^{k+2}\).

A2': For every \( t \in \mathbb{R} \), we have: \( \psi(t, 0, 0) = 0 \); and \( \partial_{y_t} g_{k+1}(t, 0, 0) \neq 0 \).

A3': \( g_{k+1}(t, y, \mathbb{R}^1) = \mathbb{R}^1 \) for every \([t, y] \in [0, T] \times \mathbb{R}^{k+1}\).

Given an initial state \( z_0 \in \mathbb{R}^k \), a feedback control \( \omega(t, z) \) of \( \mathbb{R} \times \mathbb{R}^{k} \) to \( \mathbb{R}^1 \) a disturbance \( \delta(\cdot) \) and \( t_0 \in \mathbb{R} \), let \( t \mapsto z(t, t_0, z_0, \omega(\cdot, \cdot), \delta(\cdot)) \) denote the trajectory, of system (5) that is defined by this control \( \omega(\cdot, \cdot) \), by this disturbance \( \delta(\cdot) \) and by the initial condition \( z(t_0) = z_0 \). Similarly, for system (6), given an initial state \( y_0 \in \mathbb{R}^{k+1} \), a feedback \( v(t, y) \) of \( \mathbb{R} \times \mathbb{R}^{k+1} \) to \( \mathbb{R}^1 \), a disturbance \( \Delta(\cdot) \) and \( t_0 \in \mathbb{R} \), let \( y(t, t_0, y_0, v(\cdot, \cdot), \Delta(\cdot)) \) denote the trajectory, of (6), that is defined by the control \( v(\cdot, \cdot) \), by the disturbance \( \Delta(\cdot) \), and by the initial condition \( y(t_0) = y_0 \). In addition, we presume that the existence and the uniqueness of the solution of the corresponding Cauchy problem are ensured in this definition. Of course, if \( \omega \) and \( v \) are at least of class \( C^1 \), and if the disturbances are of class \( L_\infty \), then it guarantees the existence and the uniqueness of the corresponding solution automatically.

Following \([23]\), for systems (5) and (6), we consider the following Lyapunov pairs:

\[
V_k(z) := \langle z, z \rangle, \quad V_{k+1}(y) := \langle y, y \rangle \quad \text{for all} \quad z \in \mathbb{R}^k; \quad y \in \mathbb{R}^{k+1}
\]

We reduce Theorem 2 to the following Theorem.

**Theorem 3** Assume that systems (5) and (6) satisfy Assumptions A1’-A3’. Suppose there exist sequences \( \{r_q\}_{q=2}^{+\infty} \subset \mathbb{R} \), \( \{\rho_q\}_{q=1}^{+\infty} \subset \mathbb{R} \) and \( \{d_q\}_{q=1}^{+\infty} \subset \mathbb{R} \) such that \( 0 < \rho_q < r_{q+1} < \rho_{q+1} \) and \( 0 < d_q < d_{q+1} \)
for all $q \in \mathbb{N}$ such that $r_q \to +\infty$, $\rho_q \to +\infty$ and $d_q \to +\infty$ as $q \to \infty$. Assume that there exists a function

$$\gamma_k(\cdot) \in \mathcal{K}_\infty$$

such that $d_1 < \max V_k(z)$ and the following conditions hold:

$$C1: \frac{\partial V_k(z)}{\partial z}(g(t,z,0) + \sum_{j=1}^{N_i} \Delta_j \varphi_j(t,z)) \leq -V_k(z) + \gamma_k(|\delta|) \text{ for all } \delta \in \mathbb{R}^{N_k}, \text{ whenever } |z|^2 < r_k^2, \ z \in \mathbb{R}^k, \ t \in [0,T].$$

$$C2: \text{For every } z_0 \in \mathbb{R}^k, \text{ and every } t_0 \in [0,T] \text{ if } |z_0|^2 \leq r_k^2 \text{ with some } q \in \mathbb{N} \text{ then}$$

$$|z(t,t_0,z_0,0,\delta(\cdot))|^2 \leq \rho_{q+2} - \frac{t-t_0}{T}(\rho_{q+2} - \rho_q^2) \text{ for all } t \in [t_0,t_0+T],$$

whenever $\delta(\cdot)$ in $L_{\infty}[t_0,t_0+T]$ satisfies $\gamma_k(||\delta(\cdot)||_{L_{\infty}[t_0,t_0+T]}) \leq d_q$.

Then, for every $\mu \in \mathbb{N} \cup \{\infty\}$, there exist $q_0 \geq 0$ ($q_0 \in \mathbb{Z}$) positive real numbers $r_1$, $r_0$, ..., $r_{-q_0}$, positive real $d_0$, $d_{-1}$, ..., $d_{-q_0-1}$, a sequence of positive real numbers $\{R_q\}_{q=-q_0-1}^{\infty}$, a function $\gamma_{k+1}(\cdot) \in \mathcal{K}_\infty$ such that $\gamma_{k+1}(|\Delta|) \geq \gamma_k(|\delta|) + |\Delta|^2$ and a feedback controller $v(\cdot, \cdot)$ of class $C^\mu(\mathbb{R} \times \mathbb{R}^{k+1}; \mathbb{R}^1)$ such that $0 < R_q < r_{q+1} < R_{q+1}$ and $0 < d_q < d_{q+1}$ for all $q \geq -q_0 - 1$, $q \in \mathbb{Z}$ and $d_{-q_0-1} < \max_{|y| \leq R_{-q_0-1}} V_k(y)$ and such that the following conditions hold:

(i) $v(T+t,y) = v(t,y)$ for all $[t,y]$ in $\mathbb{R} \times \mathbb{R}^{k+1}$, and $v(t,0) = 0 \in \mathbb{R}^1$ for all $t \in \mathbb{R}$.

(ii) For each $t \in \mathbb{R}$, each $y = [z, z_{k+1}]^T \in \mathbb{R}^{k+1}$, and each $\Delta \in \mathbb{R}^{N_{k+1}}$, we have:

$$\frac{\partial V_{k+1}(y)}{\partial y}(\psi(t,y,v(t,y)) + \Delta \phi(t,y)) \leq -V_{k+1}(y) + \gamma_{k+1}(|\Delta|)$$

(iii) For every $y_0 \in \mathbb{R}^{k+1}$, and every $t_0 \in \mathbb{R}$ if $|y_0|^2 \leq r_q^2$ with some $q \geq -q_0 - 1$, $q \in \mathbb{Z}$ then

$$|y(t,t_0,y_0,v(\cdot,\cdot),\Delta(\cdot))|^2 \leq R_{q+2}^2 - \frac{t-t_0}{T}(R_{q+2}^2 - R_q^2) \text{ for all } t \in [t_0,t_0+T],$$

whenever $\Delta(\cdot) \in L_{\infty}[t_0,t_0+T]$ satisfies $\gamma_{k+1}(||\Delta(\cdot)||_{L_{\infty}[t_0,t_0+T]}) \leq d_q$.

(If $k=0$, i.e., system (5) is empty, we say that Conditions C1, C2 hold by definition, and the Theorem states that, for the corresponding extension (6), there is a control $v(\cdot, \cdot)$ such that Conditions (i), (ii), (iii) hold with $\gamma_1(|\Delta|) = |\Delta|^2$).

It is easy to prove that Theorem 3 implies Theorem 2. Indeed, assume that system (3) satisfies Assumptions A1-A3 and without loss of generality assume that $x^* = 0$, $u^* = 0$.

For $k=0$, define $y := x_1$, $v := x_2$, $\psi := f_1(t, y, v)$, $\phi := \Phi_1(t, y)$, $\Delta := \delta_1$ and find the feedback $u_1(t, y) := v(t, y)$, the $\mathcal{K}_\infty$ - function $\gamma_{k1}(|\Delta|) := |\Delta|^2$ and positive numbers $r_q$ ($q \geq -q_0$) and $R_q$, $d_q$ ($q \geq -q_0 - 1$) satisfying all the statement of Theorem 3 including (i)-(iii). Without loss of generality, assume that $q_0 = 2$ (otherwise shift the indexation accordingly).
Then, for \( k = 1 \) redefine:

\[
z = z_1 := x_1, \quad z_{k+1} = z_2 := x_2 - \alpha_1(t, x_1), \quad y := [z_1, z_2], \quad v := x_3
\]

\[
g(t, z, z_{k+1}) := f_1(t, z_1, z_2 + \alpha_1(t, z_1));
\]

\[
[\Delta_1, \ldots, \Delta_{N_k}] := \delta_1, \quad [\Delta_{N_k+1}, \ldots, \Delta_{N_{k+1}}] := \delta_2, \quad [\varphi_1, \ldots, \varphi_{N_k}](t, z) := \Phi_1(t, z),
\]

\[
g_{k+1}(t, z, z_{k+1}, v) := f_2(t, z, z_2 + \alpha_1(t, z_1), v) - \frac{\partial \alpha_1(t, z_1)}{\partial t} - \frac{\partial \alpha_1(t, z_1)}{\partial z} g(t, z, z_2);
\]

\[
[\varphi_{k+1,1}, \ldots, \varphi_{k+1,N_k}](t, z) := -\frac{\partial \alpha_1(t, z_1)}{\partial z} \Phi_1(t, z)
\]

\[
[\varphi_{k+1,N_k+1}, \ldots, \varphi_{k+1,N_{k+1}}](t, z, z_{k+1}) := \Phi_2(t, z_1, z_2 + \alpha_1(t, z_1))
\]

Then, for \( k = 1 \) system (5) satisfies Assumptions C1-C2 of Theorem 3 and, applying Theorem 3, we obtain the existence of \( r_q, R_q, d_q, v(t, y) \), and \( \gamma_2(\cdot) \in \mathcal{K}_\infty \) satisfying the statement of Theorem 3 for \( k = 1 \) (and satisfying (i)-(iii)). Similarly, after \( n \) coordinate transformations and \( n \) steps of the backstepping procedure, we obtain system (5) of dimension \( k+1 = n \) and the existence of the corresponding \( T \)-periodic feedback \( v(t, y) \) of \( \mathbb{R} \times \mathbb{R}^n \) to \( \mathbb{R} \), \( \gamma_n(\cdot) \in \mathcal{K}_\infty \) and the existence of positive numbers \( r_q \) \( (q > -q_0) \) and \( R_q \), \( d_q \) \( (q < -q_0) \) satisfying (i)-(iii) and the statement of Theorem 3. Then one proves that the \( n \)-dimensional closed-loop system

\[
\dot{y} = \psi(t, y, v(t, y)) + \Delta(t)\phi(t, y), \quad t \in \mathbb{R},
\]

satisfies the asymptotic gain (AG) property and the global uniform stability (UGS) property.

(AG): Take any \( \{d_q\}_{q=-\infty}^{-q_0-2} \) such that \( 0 < d_q < d_{q+1} \) for all \( q \in \mathbb{Z} \) and such that \( d_q \to 0 \) as \( q \to -\infty \). From (iii) we obtain:

\[
\gamma_n(\| \Delta(\cdot) \|) \leq d_q \Rightarrow \limsup_{t \to +\infty} \| y(t, t_0, y^0, v(\cdot, \cdot), \Delta(\cdot)) \|^2 \leq R_q^2,
\]

whenever \( q > -q_0 - 1 \)

\[
\gamma_n(\| \Delta(\cdot) \|) \leq d_q \Rightarrow \limsup_{t \to +\infty} \| y(t, t_0, y^0, v(\cdot, \cdot), \Delta(\cdot)) \|^2 \leq d_q < d_{-q_0-1} < R_{-q_0-1}^2,
\]

whenever \( q < -q_0 - 1 \)

for all \( y^0 \in \mathbb{R}^n \). Find any \( \gamma(\cdot) \in \mathcal{K}_\infty \) such that

\[
d_q < \gamma_n(\| \Delta \|) \leq d_{q+1} \Rightarrow \gamma(\| \Delta \|) \geq R_{q+1}^2 \text{ for each } q \geq -q_0 - 2
\]

\[
d_q < \gamma_n(\| \Delta \|) \leq d_{q+1} \Rightarrow \gamma(\| \Delta \|) \geq d_{q+2} \text{ for each } q < -q_0 - 2
\]
Then from (5)-(11), we obtain the AG property:

$$\limsup_{t \to +\infty} |y(t, t_0, y^0, v(\cdot, \cdot), \Delta(\cdot))|^2 \leq \gamma(\|\Delta(\cdot)\|)$$

whatever $y^0 \in \mathbb{R}^n$.

(UGS): Take any \( \{d_q\}_{q=-\infty}^{q=0} \) such that $0 < d_q < d_{q+1}$ for all $q \in \mathbb{Z}$ and such that $d_q \to 0$ as $q \to -\infty$. Also take $\{R_q\}_{q=-\infty}^{q=0} \to 0$ as $q \to -\infty$ and such that $\gamma_1 < R_q^2$ for all $q \in \mathbb{Z}$ (note that for $q \geq -q_0 - 1$ the latter follows from Theorem 3). From (iii), we obtain that for each $q \geq -q_0 - 1$, if $|y_0|^2 \leq r_{q+2}^2$ and $\gamma_n(\|\Delta(\cdot)\|) \leq d_q$ then

$$|y(t, t_0, y^0, v(\cdot, \cdot), \Delta(\cdot))|^2 \leq R_{q+2}^2 \quad \forall t \geq t_0.$$ 

Similarly, from (ii) and from the inequalities $\gamma_1 < R_q^2 < R_{q+2}^2$, which hold true for all $q \in \mathbb{Z}$, we obtain that for each $q < -q_0 - 1$, if $|y_0|^2 \leq r_{q+2}^2$ and $\gamma_n(\|\Delta(\cdot)\|) \leq d_q$ then

$$|y(t, t_0, y^0, v(\cdot, \cdot), \Delta(\cdot))|^2 \leq R_{q+2}^2 \quad \forall t \geq t_0.$$ 

Combining these two implications, we obtain the following one:

$$|y(t, t_0, y^0, v(\cdot, \cdot), \Delta(\cdot))|^2 \leq \max\{R_q^2, R_{q+2}^2\}$$

Find any $\Upsilon(\cdot) \in \mathcal{K}_\infty$ such that

$$r_{q+1} < |y_0| \leq r_{q+2} \Rightarrow \Upsilon(|y_0|) \geq R_{q+2}^2$$

$$d_{q-1} < \gamma_n(\|\Delta(\cdot)\|) \leq d_q \Rightarrow \Upsilon(\|\Delta(\cdot)\|) \geq R_{q+2}^2$$

for all $q \in \mathbb{Z}$. Combining the latter with (12), we obtain the UGS:

$$|y(t, t_0, y^0, v(\cdot, \cdot), \Delta(\cdot))|^2 \leq \max\{\Upsilon(|y_0|), \Upsilon(\|\Delta(\cdot)\|)\}$$

for all $t \geq t_0, y^0 \in \mathbb{R}^n$.

Since our transformation of coordinates was triangular, $T$ - periodic, and is a global diffeomorphism of states, we see that the original system (3) will also be UGS and AG with this feedback. The proof of Theorem 2 is complete, it remains to prove Theorem 3.

### 5 PROOF OF THEOREM 3

Following [23], we prove the existence of numbers $r \in ]0, \rho_1[$, $d_{-q_0 - 1}$ in $]0, d_1[$, feedback $\nu(\cdot, \cdot)$ of class $C^\infty(\mathbb{R} \times \mathcal{B}_{2r}(0); \mathbb{R}_1)$ and function $\gamma_{k+1}(\cdot)$ of class $\mathcal{K}_\infty$ such that $\gamma_{k+1}(|\Delta|) \geq \gamma_k(|\delta|) + |\Delta|^2$ and $d_{-q_0 - 1} \leq \max_{|y| \leq r} V_{k+1}(y)$ and such that

$$\nu(t, 0) = 0; \quad \nu(t + T, y) = \nu(t, y) \quad \text{for all } t \in \mathbb{R}, y \in \mathbb{R}^{k+1}$$

(13)
Define

\[ \frac{\partial V_{k+1}(y)}{\partial y} (\psi(t, y, \nu(t, y)) + \Delta \phi(t, y)) \leq -V_{k+1}(y) \]
\[ + \gamma_{k+1}(|\Delta|) \text{ for all } \Delta \in \mathbb{R}^{N_{k+1}}, \ y = [z, z_{k+1}] \in \overline{B}_{2r}(0), \ t \in \mathbb{R} \] (14)

Indeed, by condition C1 of Theorem 3, the derivative of \( V_{k+1} \) along the trajectories of (6) is

\[
\frac{dV_{k+1}}{dt} = \frac{\partial V_{k+1}}{\partial y} (\psi(t, y, \nu) + \Delta \phi(t, y))
\]
\[
= \frac{\partial V_k(z)}{\partial z} (g(t, z, 0) + \sum_{j=1}^{N_{k+1}} \Delta_j \varphi_j(t, z) + \frac{\partial V_k(z)}{\partial z})
\]
\[
(g(t, z, z_{k+1}) - g(t, z, 0)) + 2z_{k+1}(g_{k+1}(t, y, v) + \sum_{j=1}^{N_{k+1}} \Delta_j \varphi_{k+1,j}(t, z, z_{k+1})) \leq -V_k(z) + \gamma_k(|\delta|)
\]
\[
+ z_{k+1}(2g_{k+1}(t, y, v) + \frac{\partial V_k(z)}{\partial z}) J(t, z, z_{k+1})
\]
\[
+ 2 \sum_{j=1}^{N_{k+1}} \Delta_j \varphi_{k+1,j}(t, z, z_{k+1})) \leq -V_k(z) + \gamma_k(|\delta|) + |\Delta|^2
\]
\[
+ z_{k+1}(2g_{k+1}(t, y, v) + \frac{\partial V_k(z)}{\partial z}) J(t, z, z_{k+1})
\]
\[
+ z_{k+1} \sum_{j=1}^{N_{k+1}} \varphi_{k+1,j}^2(t, z, z_{k+1}))
\]

for all \( \Delta \in \mathbb{R}^{N_{k+1}} \), whenever \(|z|^2 < r_{2k}^2, z \in \mathbb{R}^k, t \in [0, T] \), where

\[
J(t, z, z_{k+1}) = \int_0^1 \frac{\partial g(t, z, \theta z_{k+1})}{\partial z_{k+1}} d\theta
\]

Then we obtain the existence of \( r \in [0, \rho_1] \) and \( T - \text{ periodic feedback } \nu(\cdot, \cdot) \) in \( C^\infty(\mathbb{R} \times \overline{B}_{2r}(0); \mathbb{R}^1) \) such that

\[
z_{k+1}(2g_{k+1}(t, y, \nu(t, y)) + \frac{\partial V_k(z)}{\partial z}) J(t, z, z_{k+1}) + z_{k+1} \sum_{j=1}^{N_{k+1}} \varphi_{k+1,j}^2(t, z, z_{k+1}))
\]
\[
\leq -|z_{k+1}|^2 \quad \text{for all } [t, y] \in \mathbb{R} \times \overline{B}_{2r}(0).
\]

Take any \( d_{-q_0-1} \) in \( [0, d_1] \) that satisfy the inequality \( d_{-q_0-1} < \frac{1}{6} \max_{|y| \leq r} V_{k+1}(y) \) and any function \( \gamma_{k+1}(\cdot) \) of class \( \mathcal{K}_\infty \) such that \( \gamma_{k+1}(|\Delta|) \geq \gamma_k(|\delta|) + |\Delta|^2 \). Then (13), (14) are satisfied, and \( \nu(\cdot, \cdot) \) satisfies Condition (ii) of Theorem 3.

Let us point out that for \( k = 0 \) all these arguments will be simplified (the terms corresponding to \( z, g(t, z, z_{k+1}) \) and to their scalar product will be absent) - similar remark can be made for the next steps.

Next, we extent our control onto the whole state space to satisfy condition (iii).

Define

\[
\varkappa := \min \left\{ \frac{1}{6} \min_{|z| \geq r} V_k(z), \ \frac{1}{4} \left( \max_{|z| \leq \rho_1} V_k(z) - d_1 \right) \right\}
\]
Using the Gronwall-Bellman lemma and Condition C2 of Theorem 3, we find positive numbers $R_q > 0$, $-q_0 - 1 \leq q \leq 3$, $\sigma_{-q_0} = \sigma_{-q_0+1} = \ldots = \sigma_1 = \sigma_2 = \sigma_3 = \sigma$ and $d_0 > d_{-1} > \ldots > d_{-q_0}$ with $d_{-q_0} > d_{-q_0-1}$ (where $d_{-q_0-1}$ was chosen above) such that first,

$$\frac{\partial V_k(z)}{\partial z}(g(t, z, z_{k+1}) + \sum_{j=1}^{N_k} \Delta_j \varphi_j(t, z))$$

$$= 2(z, g(t, z, z_{k+1}) + \sum_{j=1}^{N_k} \Delta_j \varphi_j(t, z)) \leq -2\kappa,$$ whenever

$$|z|^2 + |z_{k+1}|^2 < R_q \text{ and } \gamma_k(\|\delta\|) < d_q \quad \text{for all}$$

$$y = [z, z_{k+1}] \in \left(\overline{B}_r(0) \setminus B_{R_{-q_0-1}}(0)\right) \cap \left(\mathbb{R}^k \times \Omega_{3\sigma}(0)\right),$$

$$-q_0 - 1 \leq q \leq 1, \quad t \in \mathbb{R}$$

$$r_q < \rho_q < R_q < r_{q+1} \quad \text{for all } q = 1, 2, 3;$$

$$R_{-q_0} < 2r = 2r_{-q_0}, \quad r_{-q_0} = r, \quad \text{and } R_{q_0-1} < r_q < R_q < r_{q+1}$$

$$\text{for all } -q_0 \leq q \leq 1$$

second, for every $z_0 \in \mathbb{R}^k$, every $t_0 \in [0, T]$, every $\omega(\cdot)$ in $C([t_0, t_0 + T]; \mathbb{R}^1)$, and every $\delta(\cdot)$ in $L_\infty[t_0, t_0 + T]$ if

$$\max_{t_0 \leq t \leq t_0 + T} |\omega(t)| \leq 3\sigma_3, \quad |z_0|^2 \leq r_3^2, \quad \text{and } \gamma_k(\|\delta(\cdot)\|_{L_\infty[t_0, t_0 + T]}) \leq d_1,$$ then

$$|z(t, t_0, z_0, \omega(\cdot), \delta(\cdot))|^2 + |\omega(t)|^2 \leq R_3^2 - \frac{t - t_0}{T} (R_3^2 - R_1^2) \quad \text{for all } t \in [t_0, t_0 + T];$$

and

$$\max_{t_0 \leq t \leq t_0 + T} |\omega(t)| \leq 3\sigma_{q+2}, \quad |z_0|^2 \leq r_{q+2}^2, \quad \text{and } \gamma_k(\|\delta(\cdot)\|_{L_\infty[t_0, t_0 + T]}) \leq d_q,$$ then

$$|z(t, t_0, z_0, \omega(\cdot), \delta(\cdot))|^2 + |\omega(t)|^2 \leq R_{q+2}^2 - \frac{t - t_0}{T} (R_{q+2}^2 - R_q^2)$$

for all $t \in [t_0, t_0 + T]; \quad -q_0 - 1 \leq q \leq 0, \quad q \in \mathbb{Z}.$

and, third,

$$-2\kappa < -\frac{r_{q+2}^2 - R_{q-1}^2}{T} \quad \text{for all } -q_0 \leq q \leq 0, \quad q \in \mathbb{Z}$$

$$\frac{\partial V_{k+1}(y)}{\partial y}(\psi(t, y, \nu(t, y)) + \Delta \phi(t, y)) - 2(y, \psi(t, y, \nu(t, y)) + \Delta \phi(t, y)) < -2\kappa$$

whenever $t \in [0, T], \quad R_{-q_0-1} \leq |y| \leq R_{-q_0+1}, \quad y \in \mathbb{R}^{k+1}$

and $\gamma_k(\|\Delta(\cdot)\|_{L_\infty[t_0, t_0 + T]}) < d_q$

Then, using Condition C2 of Theorem 3 and the induction over $q \geq -q_0$, $q \in \mathbb{Z}$, if $R_{-q_0}, R_{-q_0+1}, \ldots, R_1, \ldots, R_{q_0+1}$, and $\sigma_{-q_0}, \sigma_{-q_0+1}, \ldots, \sigma_1, \ldots, \sigma_{q_0+1}$, are already constructed for some $q \geq 2$, we find $R_{q+2} > 0$ and $\sigma_{q+2} > 0$ such that

$$r_{q+2} < \rho_{q+2} < R_{q+2} < r_{q+3}; \quad 0 < \sigma_{q+2} \leq \sigma_{q+1}$$

(22)
and such that for every \( z_0 \in \mathbb{R}^k \), every \( t_0 \in [0, T] \), every \( \omega(\cdot) \) in \( C([t_0, t_0+T]; \mathbb{R}^1) \) and every \( \delta(\cdot) \) in \( L_\infty[t_0, t_0+T] \)

\[
\text{if } \max_{t_0 \leq t \leq t_0+T} |\omega(t)| \leq 3\sigma_{q+2}, \ |z_0|^2 \leq r_{q+2}^2, \\
\text{and } \gamma_k(\|\delta(\cdot)\|_{L_\infty[t_0, t_0+T]}) < d_q, \text{ then }
\]

\[
|z(t, t_0, z_0, \omega(\cdot), \delta(\cdot))|^2 + |\omega(t)|^2 \leq R_{q+2}^2 - \frac{t-t_0}{T} (R_{q+2}^2 - R_q^2) \\
\text{for all } t \in [t_0, t_0+T]; \ q \geq -q_0-1, \ q \in \mathbb{Z}. \tag{23}
\]

Define

\[
\Xi_{-q_0+1} := \overline{B}_{r_{q_0+1}}(0)
\]

and \( \Xi_{q+1} := \overline{B}_{r_{q+1}}(0) \setminus B_{r_q}(0) \), \( q \geq -q_0 + 1, \ q \in \mathbb{Z}; \tag{24} \)

\[
P_{-q_0+1} := \Xi_{-q_0+1} \cap \left( \mathbb{R}^k \times \overline{\Omega}_{\sigma_3}(0) \right) \quad \text{and}
\]

\[
P_{q+1} := \Xi_{q+1} \cap \left( \mathbb{R}^k \times \overline{\Omega}_{\sigma_{q+4}}(0) \right), \ q \geq -q_0 + 1, \ q \in \mathbb{Z}; \tag{25} \)

\[
E_{-q_0+1} := \Xi_{-q_0+1} \cap \left( \mathbb{R}^k \times (\overline{\Omega}_{2\sigma_2}(0) \setminus \Omega_{\sigma_4}(0)) \right) \quad \text{and}
\]

\[
E_{q+1} := \Xi_{q+1} \cap \left( \mathbb{R}^k \times (\overline{\Omega}_{2\sigma_{q+2}}(0) \setminus \Omega_{\sigma_{q+4}}(0)) \right), \\
q \geq -q_0 + 1, \ q \in \mathbb{Z}; \tag{26} \)

\[
G_{-q_0+1} := \Xi_{-q_0+1} \cap \left( \mathbb{R}^k \times (\mathbb{R}^1 \setminus \Omega_{\sigma_3}(0)) \right); \quad \text{and}
\]

\[
G_{q+1} := \Xi_{q+1} \setminus \left( \mathbb{R}^k \times \Omega_{2\sigma_{q+2}}(0) \right), \ q \geq -q_0 + 1, \ q \in \mathbb{Z}; \tag{27} \)

\[
K_{q+1} := \bigcup_{i=-q_0+1}^{q} (E_{i+1} \cup P_{i+1}) \quad \text{and} \quad H_{q+1} := \bigcup_{i=-q_0+1}^{q} P_{i+1}, \\
q \geq -q_0 + 1, \ q \in \mathbb{Z}; \tag{28} \)

Then

\[
H_{q+1} \subset K_{q+1}, \ q \geq -q_0 + 1, \ q \in \mathbb{Z}.
\]

Define

\[
\varepsilon_{q+1} := \min \left\{ \frac{r_{-q_0}}{2}; \ \frac{\sigma_{q+4}^2}{2}; \ \min_{-q_0-1 \leq i \leq q+1} \left\{ \frac{r_{i+1} - R_i}{5} \right\} \right\};
\]

\[
\min_{-q_0 \leq i \leq q+1} \left\{ \frac{R_i - r_i}{5} \right\}; \ \min_{-q_0-1 \leq i \leq q+1} \left\{ \frac{r_{i+1}^2 - R_i^2}{5} \right\}; \tag{29} \)

\[
\min_{-q_0 \leq i \leq q+1} \left\{ \frac{R_i^2 - r_i^2}{5} \right\}, \ q \geq -q_0 + 1, \ q \in \mathbb{Z};
\]

\[
m_{q+1} := \max_{\[t, z, z_{k+1}] \in [0, T] \times K_{q+3}} \left( 2(|z, g(t, z, z_{k+1})| + \sum_{j=1}^{N_k} \Delta_j \varphi_j (t, z)) \right) + 1 \quad \text{for all } t \in [t_0, t_0+T]; \tag{30} \)

\[
\gamma_k(\|\delta(\cdot)\|_{L_\infty[t_0, t_0+T]}) < d_{q+3}.
\]
Using Assumption A3' and the compactness of \([0,T] \times (G_{q+1} \cup E_{q+1})\), for every \(q \geq -q_0 + 1, q \in \mathbb{Z}\), one gets the existence of \(M_1(q) \in \mathbb{N}\) such that

\[
\forall[t,z,z_{k+1}] \in [0,T] \times (G_{q+1} \cup E_{q+1}) \ \exists v_{t,z,z_{k+1}} \in \mathbb{R}^1 \text{ such that } |v_{t,z,z_{k+1}}| \leq M_1(q) \quad \text{and} \quad (\forall \Delta \in \mathbb{R}^{N_{k+1}} \quad \gamma_k(\Delta) < d_{q+1} \Rightarrow \langle z, g(t,z,z_{k+1}) + \sum_{j=1}^{N_{k+1}} \Delta_j \varphi_j(t,z) \rangle + \langle z_{k+1}, g_{k+1}(t,z,z_{k+1},v_{t,z,z_{k+1}}) + \sum_{j=1}^{N_{k+1}} \Delta_j \varphi_{k+1,j}(t,z,z_{k+1}) \rangle < -2D_{q+1})
\]

In addition, using Assumption A3' and the compactness of \([0,T] \times P_{q+1}\), for every \(q \geq -q_0 + 1, q \in \mathbb{Z}\), we obtain the existence of \(M_2(q) \in \mathbb{N}\) such that

\[
\forall[t,z,z_{k+1}] \in [0,T] \times P_{q+1} \ \exists w_{t,z,z_{k+1}} \in \mathbb{R}^1 \text{ such that } |w_{t,z,z_{k+1}}| \leq M_2(q) \quad \text{and} \quad |\langle z_{k+1}, g_{k+1}(t,z,z_{k+1},w_{t,z,z_{k+1}}) \rangle| = 0
\]

For each \(q \geq -q_0 + 1, q \in \mathbb{Z}\), define

\[
M(q) := \max\{M_1(q), M_2(q), \max_{0 \leq t \leq T, |y| \leq 2r-q_0} |v(t,y)|\}
\]

\[
U_q := \{u \in \mathbb{R}^1 \mid |u| \leq M(q)\}, \quad q \geq -q_0 + 1, q \in \mathbb{Z},
\]

Without loss of generality, we assume that

\[
M(q) \leq M(q+1), \quad \text{i.e.,} \quad U_q \subset U_{q+1}
\]

for all \(q \geq -q_0 + 1, q \in \mathbb{Z}\).

Using the compactness of all \(U_q, \overline{B}_{r_q}(0)\), take any sequence \(\{L_q\}_{q=-q_0+1}^{\infty} \subset \mathbb{R}\) such that

\[
0 < L_{q+1} \leq L_q, \quad q \geq -q_0 + 1, q \in \mathbb{Z},
\]

\[
2L_q(|\psi(t,y,u)| + |\Delta||\phi(t,y)| + 1) \leq 1 \quad \text{for all } t \in [0,T],
\]

whenever \(y \in \overline{B}_{r_{q+3}}(0), u \in U_{q+3}, \gamma_{k+1}(\Delta) < d_{q+3}, \Delta \in \mathbb{R}^{N_{k+1}}, q \geq -q_0 + 1, q \in \mathbb{Z}\).
For every $L > 0$, by $\mathcal{S}_L$ denote the system of all the sets given by
\[
\Gamma_{\emptyset, \varnothing}, A_{\emptyset, A_{\emptyset}} := \{ [s, y] \in \mathbb{R} \times \mathbb{R}^{k+1} \mid \vartheta(y) \leq s \leq \Theta(y) \} \setminus \left( \{ [s, y] \in \mathbb{R} \times \mathbb{R}^{k+1} \mid (s = \Theta(y)) \land (y \in A_{\emptyset}) \} \cup \{ [s, y] \in \mathbb{R} \times \mathbb{R}^{k+1} \mid (s = \vartheta(y)) \land (y \in A_{\emptyset}) \} \right),
\]
where $\Theta(\cdot)$ and $\vartheta(\cdot)$ range over the set of all the functions from class $C(\mathbb{R}^{k+1}; [0, T])$ such that
\[
|\Theta(y_1) - \Theta(y_2)| \leq L|y_1 - y_2| \quad \text{and} \quad |\vartheta(y_1) - \vartheta(y_2)| \leq L|y_1 - y_2|
\]
for all $y_1, y_2 \in \mathbb{R}^{k+1}$, (38)
and such that $A_{\emptyset} \subset \mathbb{R}^{k+1}$, $A_{\emptyset} \subset \mathbb{R}^{k+1}$ range over the set of all subsets of $\mathbb{R}^{k+1}$. It is straightforward that for each $L > 0$, $\mathcal{S}_L$ is a semi-ring of sets, i.e., first, $\emptyset \in \mathcal{S}_L$; second, $\Gamma' \cap \Gamma'' \in \mathcal{S}_L$ for each $\Gamma' \in \mathcal{S}_L$, and each $\Gamma'' \in \mathcal{S}_L$; third, for each $\Gamma \in \mathcal{S}_L$, and each $\Gamma_1 \in \mathcal{S}_L$, if $\Gamma_1 \subset \Gamma$, then there is a finite sequence
\[
\{ \Gamma_i \}_{i=2}^l \subset \mathcal{S}_L \text{ such that } \Gamma = \bigcup_{j=1}^l \Gamma_j \text{ and } \Gamma_i \cap \Gamma_j = \emptyset, \text{ whenever } i \neq j, \{ i, j \} \subset \{ 1, 2, \ldots, l \}.
\]
Given $[t, y] = [t, z, z_{k+1}] \in [0, T] \times (\mathbb{R}^{k+1} \setminus B_{r_{-q_0+1}}(0))$, let $q \geq -q_0 + 1$, $q \in \mathbb{Z}$ be such that $y \in \Xi_{q+1}$. By the construction (see (24)-27), we obtain
\[
\Xi_{q+1} \subset P_{q+1} \cup E_{q+1} \cup G_{q+1} \quad \text{for all } q \geq -q_0 + 1, q \in \mathbb{Z}
\]
(39)
Then, the following situations are possible.

1) $y \in (G_{q+1} \cup E_{q+1})$. Then, by (32) and (31), there exist $v_{t, z, z_{k+1}} \in U_q$ and a set $T_{t, z, z_{k+1}} \subset \mathcal{S}_{L_q+2}$ such that $T_{t, z, z_{k+1}} \subset [0, T] \times \mathbb{R}^{k+1}$, $[t, z, z_{k+1}] \in T_{t, z, z_{k+1}}$, and $T_{t, z, z_{k+1}}$ is open in $[0, T] \times \mathbb{R}^{k+1}$ with respect to its standard topology and such that
\[
|y' - y''| \leq \varepsilon_{q+1} \quad \text{for all } |t', y'| \in T_{t, z, z_{k+1}}, |t'', y''| \in T_{t, z, z_{k+1}}
\]
(40)
and
\[
\langle z', g(t', z', z_{k+1}'') + \sum_{j=1}^{N_k} \Delta_j \varphi_j(t', z') \rangle + \langle z_{k+1}'', g_{k+1}(t', z', z_{k+1}'', v_{t, z, z_{k+1}}) \rangle + \sum_{j=1}^{N_k+1} \Delta_j \varphi_{k+1, j}(t', z', z_{k+1}'') \rangle \leq -2D_{q+1} \quad \text{and}
\]
\[
\langle z_{k+1}'', g_{k+1}(t', z', z_{k+1}'', v_{t, z, z_{k+1}}) \rangle + \sum_{j=1}^{N_k+1} \Delta_j \varphi_{k+1, j}(t', z', z_{k+1}'') \rangle < -\frac{3\sigma^2}{T}
\]
for all $[t', z', z_{k+1}'] \in T_{t, z, z_{k+1}}$, whenever $\gamma_{k+1}(|\Delta|) \leq d_{q+1}$.
(41)
2) \(y \in P_{q+1}\). Then, by \((33), (24)\), there exist \(w_{t,z,z_{k+1}} \in U_q\) and a set \(S_{t,z,z_{k+1}} \subset \mathcal{S}_{Lq+2}\) such that \(S_{t,z,z_{k+1}} \subset [0, T] \times \mathbb{R}^{k+1}\), \([t, z, z_{k+1}] \in S_{t,z,z_{k+1}},\) and \(S_{t,z,z_{k+1}}\) is open in \([0, T] \times \mathbb{R}^{k+1}\) with respect to its standard topology and such that

\[
|y' - y'| < \varepsilon_{q+1} \quad \text{for all } [t', y'] \in S_{t,z,z_{k+1}}, \quad [t'', y''] \in S_{t,z,z_{k+1}}
\]

(If \(y \in \Xi_{q+1} \cap \Xi_{q+2}\), i.e., \(y = r_{q+1}\), then we choose \(T_{t,z,z_{k+1}}\) (or \(S_{t,z,z_{k+1}}\)) and \(v_{t,z,z_{k+1}}\) (respectively \(w_{t,z,z_{k+1}}\)) which correspond to the \(\Xi_{q+2}\). Then, by \((22), (33), (34)\), there exist \(\lambda_{q+1} = 0 < \lambda_{q+1}, \quad 0 = \eta_{q+1} < \eta_{q+1},\)

and \(T_{t,z,z_{k+1}} \subset [0, T] \times \mathbb{R}^{k+1}\), therefore by \((22), (23)\), and by \((40)-(42)\), there exist sequences of sets \(\{T_{t,z,z_{k+1}}\}_{\lambda=1}^{\infty}\), \(\{S_{t,z,z_{k+1}}\}_{\eta=1}^{\infty}\), \(\{v_{t,z,z_{k+1}}\}_{\lambda=1}^{\infty}\), \(\{w_{t,z,z_{k+1}}\}_{\eta=1}^{\infty}\) such that

\[
0 = \lambda_{-q_0 + 1} < \lambda_q < \lambda_{q+1}, \quad 0 = \eta_{-q_0 + 1} < \eta_q < \eta_{q+1},
\]

for all \(q \geq -q_0 + 2, \quad q \in \mathbb{Z}\); and

\[
[0, T] \times (E_{q+1} \cup G_{q+1}) \subset \bigcup_{\lambda=1}^{\lambda+1} T_{t,z,z_{k+1}}, \quad [0, T] \times P_{q+1} \subset \bigcup_{\eta=1}^{\eta+1} S_{t,z,z_{k+1}}, \quad \text{for all } q \geq -q_0 + 1, \quad q \in \mathbb{Z}
\]

(44)

and such that

\[
([0, T] \times (E_{q+1} \cup G_{q+1})) \cap T_{t,z,z_{k+1}} = \emptyset, \quad \lambda_{q+1} \leq \lambda \leq \lambda_{q+1}
\]

\[
([0, T] \times P_{q+1}) \cap S_{t,z,z_{k+1}} = \emptyset, \quad \eta_{q+1} \leq \eta \leq \eta_{q+1}
\]

(45)

and

\[
T_{t,z,z_{k+1}} \subset \mathcal{S}_{Lq+2}, \quad \text{for all } \lambda_{q+1} \leq \lambda \leq \lambda_{q+1}
\]

\[
S_{t,z,z_{k+1}} \subset \mathcal{S}_{Lq+2}, \quad \text{for all } \eta_{q+1} \leq \eta \leq \eta_{q+1}
\]

(46)

Define

\[
T_{\lambda} := T_{t,z,z_{k+1}}, \quad S_{\eta} := S_{t,z,z_{k+1}}, \quad v_{\lambda} := v_{t,z,z_{k+1}}, \quad w_{\eta} := w_{t,z,z_{k+1}} \quad \text{for all } \lambda \in \mathbb{N}, \quad \eta \in \mathbb{N}.
\]

(47)

Since \(\mathcal{S}_{Lq}\) are semirings of sets and \(\mathcal{S}_{Lq+1} \subset \mathcal{S}_{Lq}\), we use \((29), (40), (42), (46)\) and Lemma 2 from \([3], p.40\), and the induction over \(q \geq -q_0 + 1, \quad q \in \mathbb{Z}\), and obtain the existence of non-empty sets \(\{\Gamma_i\}_{i=1}^{\infty} = \{\Gamma_{t(z), \eta(z), \mathcal{A}_{q_0}, \mathcal{A}_{q_1}}\}_{i=1}^{\infty}\) and a strictly increasing sequence \(\{l_{q}\}_{q=-q_0}^{+\infty} \subset \mathbb{Z}\) (with \(l_{-q_0 + 1} = l_{-q_0} = 0\) and with \(l_{q+1} > l_q\) for all \(q \geq -q_0 + 1\)) such that
Now we define the feedback $v(\cdot, \cdot)$, which satisfies conditions (i), (ii) and (iii) of Theorem 3, as follows

**Definition 4** Take any $l \in \mathbb{N}$, and let $q \geq -q_0 + 1$, $q \in \mathbb{Z}$ be such that $l_q + 1 \leq l \leq l_{q+1}$. Then, by (c) (and by (b), (d)) $l \in \bigcup_{\lambda=1}^{\lambda_{q+1}} C(\lambda) \cup \bigcup_{\eta=1}^{\eta_{q+1}} D(\eta)$. If $l \in \bigcup_{\lambda=1}^{\lambda_{q+1}} C(\lambda)$ then by the construction (see (b)-(d)) there exists $\lambda(l) \in \mathbb{N}$ such that $\lambda_q + 1 \leq \lambda(l) \leq \lambda_{q+1}$ and such that $\Gamma_l \subset T_{\lambda(l)}$, and in this case we define

$$\chi_l := v_{\lambda(l)}$$

If $l \notin \bigcup_{\lambda=1}^{\lambda_{q+1}} C(\lambda)$, then by (48), $l \in \bigcup_{\eta=1}^{\eta_{q+1}} D(\eta)$, and, by (b)-(d) there exists $\eta(l) \in \mathbb{N}$ such that $\eta_q + 1 \leq \eta(l) \leq \eta_{q+1}$ and such that $\Gamma_l \subset S_{\eta(l)}$, and in this case we define

$$\chi_l := w_{\eta(l)}.$$ 

Using the induction over $q \geq -q_0 + 1$, $q \in \mathbb{Z}$, take a sequence $\{h_l\}_{l=1}^{\infty} \subset ]0, +\infty[$ of positive and small enough numbers.

For every $l \in \mathbb{N}$, define $\Gamma'_l$, and $\Gamma''_l$ as follows

$$\Gamma'_l := \{ [s, y] \in \mathbb{R} \times \mathbb{R}^{k+1} \mid \vartheta_l(y) + \frac{h_l}{2} \leq s \leq \Theta_l(y) - \frac{h_l}{2} \}$$

$$\Gamma''_l := \{ [s, y] \in \mathbb{R} \times \mathbb{R}^{k+1} \mid \vartheta_l(y) + h_l \leq s \leq \Theta_l(y) - h_l \},$$

$$l \in \mathbb{N}$$

(49)

(In general, some $\Gamma'_l$, and $\Gamma''_l$ are allowed to be empty).
Let \( p_l(\cdot, \cdot), l \in \mathbb{N} \), be functions of class \( C^\infty(\mathbb{R} \times \mathbb{R}^{k+1}; [0, 1]) \) such that

\[
0 \leq p_l(t, y) \leq 1 \quad \text{for all } [t, y] \in \mathbb{R} \times \mathbb{R}^{k+1}
\]

\[
p_l(t, y) = 0, \quad \text{whenever } [t, y] \notin \Gamma'_l
\]

\[
p_l(t, y) = 1, \quad \text{whenever } [t, y] \in \Gamma''_l
\]

Let \( p(\cdot) \) be any function of class \( C^\infty(\mathbb{R}^{k+1}; [0, 1]) \) such that

\[
p(y) = 1, \quad \text{whenever } y \in \mathbb{B}_{r_{q_0}+1}(0)
\]

\[
p(y) = 0, \quad \text{whenever } y \in \mathbb{R}^{k+1} \setminus \mathbb{B}_r(0)
\]

with some \( r \in ]r_{q_0+1}, 2r[ \)

Define the feedback \( v(\cdot, \cdot) \) of class \( C^\mu([0, T] \times \mathbb{R}^{k+1}; \mathbb{R}^1) \) as follows

\[
v(t, y) := \sum_{l=1}^{\infty} p_l(t, y) \chi_l + p(y)(1 - \sum_{l=1}^{\infty} p_l(t, y)) \nu(t, y)
\]

for all \( [t, y] \in [0, T] \times \mathbb{R}^{k+1} \)

and extend it smoothly and \( T \)-periodically onto \( \mathbb{R} \times \mathbb{R}^{k+1} \).

Using (51), (b), and the inclusion \( \Gamma''_l \subset \Gamma'_l \subset \Gamma_l, l \in \mathbb{N} \), we obtain that, if \( \gamma_l(t, y) \neq 0 \) for some \( l \in \mathbb{N} \) and some \( [t, y] \in [0, T] \times \mathbb{R}^{k+1} \), then \( \gamma_{l'}(t, y) = 0 \) for all \( l' \neq l \); therefore \( v(\cdot, \cdot) \) given by (55) is well-defined. Furthermore, by the construction, \( v(\cdot, \cdot) \) can be \( T \)-periodically smoothly extended onto the whole \( \mathbb{R} \times \mathbb{R}^{k+1} \). (Indeed, by (51), for each \( y \in \mathbb{R}^{k+1} \), is \( h > 0 \) such that \( \gamma_l(t, y') = 0, l \in \mathbb{N} \), and therefore \( v(t, y') = \gamma(y') \beta(t, y') \) for all \( t \in [0, h] \cup [T-h, T] \) and all \( y' \) in some small neighborhood of \( y \). Then the \( T \)-periodic extension of \( v(\cdot, \cdot) \) given by (55) is of class \( C^\infty \).

Arguing as in [23], Step 5 it is possible to prove that \( \{h_l\}_{l=1}^{\infty} \subset ]0, +\infty[ \) can be chosen so small that this feedback extended \( T \)-periodically onto the whole \( \mathbb{R} \times \mathbb{R}^{k+1} \) is well-defined belongs to \( C^\mu \) and solves the problem, i.e. globally asymptotically stabilizes system (6). This completes the proof of Theorem 2 and Theorem 3.

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