Adaptive Consensus of Two Coupled Heterogeneous Networked Systems With Bidirectional Actions

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\textbf{ABSTRACT} The outer consensus between two coupled heterogeneous networked systems with bidirectional actions and nonidentical topologies is addressed in this paper. Based on the adaptive theory, a control protocol is designed for two coupled heterogeneous networked systems, where each system consists of double-integrator and single-integrator nodes with different nonlinear dynamics. In the light of Lyapunov method and LaSalle’s invariance principle, we prove that the coupled heterogeneous networked systems can reach outer consensus by using the reported controller. Meanwhile, the adaptive controller is also given for the second-order nonlinear systems under identical and nonidentical topologies. The correctness of the obtained theory is demonstrated by two simulation examples.

\textbf{INDEX TERMS} Adaptive consensus, nonlinear dynamics, heterogeneous networked system, bidirectional action.

\textbf{I. INTRODUCTION}

As a typical collective behavior, consensus or synchronization of agents or nodes in networked systems has received extensive attention from different fields. In practical applications, many factors can affect the consensus of networked systems, including nodes’ dynamics, topological structures, communication conditions and so on. During the past decades, there are a lot of good results on consensus or synchronization of multi-agent systems or complex networks, see [1]–[13] and references therein. In order to deal with the consensus problems for the various networked systems under different cases, adaptive control [14]–[17], pinning control [18]–[20], impulse control [21], [22] and other effective control techniques are used.

As known, almost all real systems are nonlinear. Compared with linear systems, the study of the synchronization of nonlinear systems is more complicated but more practical. Zhai and Li [20] propose a consensus protocol via a pinning control scheme to ensure that the systems with nonlinear dynamics and antagonistic interactions can reach bipartite synchronization under switching topologies. For the heterogeneous first-order systems with nonlinear dynamics, the authors in [6] investigate the quasi-synchronization problem under symmetric and asymmetric cases. For second-order networked systems, Liu \textit{et al.} [14] address the adaptive consensus of systems with time-varying delays and heterogeneous nonlinear dynamics by using a decentralized adaptive scheme. As a special nonlinear system, Lur’e system should be noted because many systems such as Lorenz system could be represented as Lur’e forms. In [23], by designing a proper impulsive pinning control strategy, the authors study the synchronization problem for the coupled Lur’e networks with derivative coupling and time-varying delays. Furthermore, the quasi-synchronization of Lur’e networks is studied in [24], where the networks consist of nonidentical Lur’e systems. Inspired by the idea of Lur’e systems, Wei \textit{et al.} [25] investigate the distributed synchronization problem of the nonlinear networked systems with adaptive nonlinear couplings.
There are some other important results on the consensus or synchronization of nonlinear systems [26], [27], just to name a few.

However, the above studies touch upon the consensus inside a networked system. For consensus or synchronization, it is extended to two coupled networked systems according to actual requirements [28]. For simplicity, the consensus between two coupled systems is also called outer consensus that corresponds to inner consensus. In our life, outer consensus really exists. Therefore, researchers start working on this problem to enhance or avoid it according to actual requirements. In [29], the authors apply an adaptive controller to deal with the outer synchronization problem for two networks, where the topological structures are nonidentical or identical. Furthermore, for two complex networks, Sun et al. [30] consider a hybrid synchronization problem. Some criteria are obtained to ensure that the anti-synchronization between two networks and inner synchronization in each network are achieved simultaneously by using the proposed controller. In [31], an impulsive controller with an adaptive and pinning scheme is used to analyze and solve outer synchronization problem. For the case that the nodes in the two complex networks have different dynamics, Wu et al. [32] address the generalized synchronization, and Han et al. [33] investigate projective synchronization between networks with unknown parameters.

In [29]–[33], the authors only take into account the unidirectional connection and internal coupling. Because of the complexity of system structure and communication interaction in practical applications, there may be the bidirectional actions between networked systems. In [34], Zheng et al. investigate outer consensus between two nonlinear coupled first-order networks under nonidentical topological and bidirectional actions. Sun et al. [35] address outer and inner consensus between two coupled networks with bidirectional actions simultaneously.

To the best of our knowledge, in the existing works about the outer synchronization problem for coupled networked systems, the nodes in each system have homogeneous or heterogeneous nonlinear dynamics, where the systems are the first-order systems. In the real world, however, the second-order systems or the hybrid systems with nonlinear dynamics are more practical. A typical example is a multi-robot system. Owing to the coexistence of the mix-order nodes with different nonlinear dynamics, the structure and interaction of heterogeneous networked systems are more complex. Thus, it is difficult to design an adaptive controller for the two coupled heterogeneous networked systems with bidirectional actions and nonidentical topologies.

Motivated by the above discussions, we investigate an adaptive consensus problem of two coupled heterogeneous networked systems with bidirectional actions and nonidentical topologies in this paper. The main contributions and distinct characteristics of this paper can be concluded as follows: (1) The outer consensus problem of two coupled heterogeneous networked systems is considered for the first time in this paper, where each system consists of double-integrator and single-integrator nodes, and each node has the different nonlinear dynamics. (2) The inner coupling in their own systems and the bidirectional actions between two systems are considered simultaneously. Meanwhile, effective adaptive controllers are designed for systems with nonidentical and identical topologies, respectively. (3) Based on the Lyapunov method and LaSalle’s invariance principle, the detailed theoretical proof of outer consensus is given. In addition, two simulation examples are presented to show that all the corresponding system states can achieve consensus by applying the proposed adaptive controller. Obviously, compared with the existing work [34], when the first-order system is extended to the second-order system and mix-order system, the dynamic behavior of the system becomes more complex, and thus more factors need to be considered for designing the controller.

This paper is structured as follows. Section II introduces a heterogeneous networked system model, some preliminaries and a definition of outer consensus. In Section III, the adaptive controllers and main results are provided. Section IV presents two simulation examples and Section V concludes this paper.

Notation: \( \mathbb{R}^p \) and \( \mathbb{R}^{p \times p} \) denote, respectively, the sets of \( p \times 1 \) real vectors and \( p \times p \) real matrices. \( I_r \) is an \( r \)-dimensional column vector with element 1, and \( I_r \) is an \( r \times r \) identity matrix. diag\( \{a_1, \cdots, a_n\} \) stands for a diagonal matrix. \( \Omega^T \) is the transpose of matrix \( \Omega \). \( \| \cdot \| \) represents the Euclidean norm. min\( \{w_1, \cdots, w_n\} \) (max\( \{w_1, \cdots, w_n\} \)) means the minimum (maximum) value of \( w_1 \) to \( w_n \). \( \mathcal{I}_m \) denotes set \( \{1, 2, \cdots, m\} \) and \( \mathcal{I}_m \setminus \mathcal{I}_n \) denotes set \( \{m+1, m+2, \cdots, n\} \). \( \otimes \) represents the Kronecker product.

II. MODEL AND PRELIMINARIES

Consider two coupled heterogeneous networked systems composed of \( N \) nodes with different nonlinear dynamics, where the nodes not only communicate with each other in the internal system, but also transmit information between two systems. At the same time, the topological structures or coupling strengths of two systems are non-identical. The model of a drive system is described as follows:

\[
\begin{align*}
\dot{x}_i(t) &= v_i(t), \\
\dot{v}_i(t) &= f_i(v_i, t) + \sum_{j=1}^{N} a_{ij} x_j(t) + \sum_{j=1}^{r} \tilde{a}_{ij} v_j(t) \\
&\quad + \sum_{j=1}^{N} b_{ij} w_j(t) + \sum_{j=1}^{r} \tilde{b}_{ij} y_j(t), i \in \mathcal{I}_r, \\
\dot{x}_i(t) &= g_i(x_i(t), t) + \sum_{j=1}^{N} a_{ij} x_j(t) + \sum_{j=1}^{N} b_{ij} w_j(t), \\
&i \in \mathcal{I}_N \setminus \mathcal{I}_r,
\end{align*}
\]
and the response system with control protocol has the following form:

\[
\begin{aligned}
\dot{w}_i(t) &= y_i(t), \\
\dot{y}_i(t) &= f_i(y_i, t) + \sum_{j=1}^{N} c_{ij}w_j(t) + \sum_{j=1}^{r} \tilde{c}_{ij}y_j(t) \\
&+ \sum_{j=1}^{N} d_{ij}x_j(t) + \sum_{j=1}^{r} d_{ij}v_j(t) + u_i(t), i \in \mathcal{I}_r.
\end{aligned}
\]

where \(x_i \in \mathbb{R}^p\) (\(w_i \in \mathbb{R}^p\)) and \(v_i \in \mathbb{R}^p\) (\(y_i \in \mathbb{R}^p\)) are the system states of the \(i\)th node in the drive system (response system). \(u_i \in \mathbb{R}^p\) is the controller for the response system. \(A = [a_{ij}] \in \mathbb{R}^{N \times N}\) is the coupling configuration matrix of drive system (1) that represents the topological structure and coupling strength of nodes in system (1). If there exists a path from nodes \(j\) to \(i\) (\(i \neq j\)), we denote \(a_{ij} \neq 0\); otherwise \(a_{ij} = 0\). Meanwhile, the sum of every row in matrix \(A\) equals 0. \(C = [c_{ij}] \in \mathbb{R}^{N \times N}\) is the coupling configuration matrix of response system (2) which has the same meaning as that of \(A\). \(B = [b_{ij}] \in \mathbb{R}^{N \times N}\) is an interactive matrix between systems (1) and (2). The element \(b_{ij}\) represents that the interactional intensity from node \(j\) in system (2) to node \(i\) in system (1). If there exists a path from node \(j\) in system (2) to node \(i\) in system (1), we denote it by \(b_{ij} \neq 0\); otherwise \(b_{ij} = 0\). \(D = [d_{ij}] \in \mathbb{R}^{N \times N}\) is an interactive matrix between the response system and drive system that has the same meaning as that of \(A\). \(B = [b_{ij}] \in \mathbb{R}^{N \times N}\) is an interactive matrix between systems (1) and (2). The element \(b_{ij}\) represents that the interactional intensity from node \(j\) in system (2) to node \(i\) in system (1). If there exists a path from node \(j\) in system (2) to node \(i\) in system (1), we denote it by \(b_{ij} \neq 0\); otherwise \(b_{ij} = 0\). \(D = [d_{ij}] \in \mathbb{R}^{N \times N}\) is an interactive matrix between the response system and drive system that has the same meaning as that of \(A\).

Remark 1: From the models of the drive and response systems, we can see that each system is composed of double-integrator and single-integrator nodes when \(r < N\). In other words, each system contains the nodes with heterogeneous dynamical structure, and meanwhile each node in a system has different nonlinear dynamics. If \(r = N\), the drive networked system and response networked system only contain the double-integrator nodes with different nonlinear dynamics. Based on the above models, this paper considers two cases, i.e., \(r < N\) and \(r = N\). In addition, as mentioned earlier, the topological structures and coupling strength of two networked systems are hardly guaranteed to be identical in practice. Thus, this paper addresses a consensus problem of two coupled heterogeneous networked systems with nonidentical and identical topologies.

Assumption 1: Suppose that \(f_i: \mathbb{R}^p \to \mathbb{R}^p\) and \(g_i: \mathbb{R}^p \to \mathbb{R}^p\) are smooth functions. For some positive constants \(\rho_i\) and \(\sigma_i\), the following inequalities hold:

\[
\|f_i(y) - f_i(v)\| \leq \rho_i\|y - v\|, \quad i \in \mathcal{I}_r
\]

and

\[
\|g_i(w) - g_i(x)\| \leq \sigma_i\|w - x\|, \quad i \in \mathcal{I}_N \setminus \mathcal{I}_r.
\]

Assumption 2: The adaptive feedback gains are bounded.

**Lemma 1 [36]:** For any positive defined matrix \(\Omega \in \mathbb{R}^{p \times p}\), we have

\[
2x^T \Omega x \leq x^T \Omega x + y^T \Omega^{-1} y, \quad \forall x, y \in \mathbb{R}^p.
\]

Definition 1: Networked systems (1) and (2) are said to reach outer consensus, if

\[
\lim_{t \to \infty} \|w_i(t) - x_i(t)\| = 0, \quad i \in \mathcal{I}_N
\]

and

\[
\lim_{t \to \infty} \|y_i(t) - v_i(t)\| = 0, \quad i \in \mathcal{I}_r.
\]

In this paper, we aim at designing a controller to deal with the outer consensus problem between networked systems (1) and (2).

**III. MAIN RESULTS**

**A. CASE I: \(r < N\)**

In this subsection, we consider the case that \(r < N\), i.e., the drive networked and the response networked systems are both composed of double-integrator and single-integrator nodes with different nonlinear dynamics. To reach the objective of outer consensus, the following controller is proposed:

\[
\begin{aligned}
\dot{u}_i &= -m_i((w_i - x_i) + (y_i - v_i)) + \sum_{j=1}^{N} \theta_{ij}w_j \\
&+ \sum_{j=1}^{r} \tilde{\theta}_{ij}y_j + \sum_{j=1}^{N} \xi_{ij}x_j + \sum_{j=1}^{r} \tilde{\xi}_{ij}v_j, i \in \mathcal{I}_r,
\end{aligned}
\]

\[
\begin{aligned}
\dot{u}_i &= -n_i(w_i - x_i) + \sum_{j=1}^{N} \eta_{ij}w_j + \sum_{j=1}^{r} \tilde{\eta}_{ij}y_j, \\
i \in \mathcal{I}_N \setminus \mathcal{I}_r,
\end{aligned}
\]

where \(m_i\) and \(n_i\) are adaptive feedback gains that their initial values are non-negative. \(\dot{m}_i = k_i[(w_i - x_i)^T(w_i - x_i) + (y_i - v_i)^T(y_i - v_i)]\) for \(i \in \mathcal{I}_r\) and \(\dot{n}_i = k_i(w_i - x_i)^T(w_i - x_i)\) for \(i \in \mathcal{I}_N \setminus \mathcal{I}_r\) with \(k_i > 0\). \(\dot{\theta}_{ij} = -[(w_i - x_i) + (y_i - v_i)]^T w_j, \dot{\theta}_{ij} = -[(w_i - x_i) + (y_i - v_i)]^T v_j, \dot{\xi}_{ij} = -[(w_i - x_i) + (y_i - v_i)]^T x_j, \dot{\tilde{\xi}}_{ij} = -[(w_i - x_i) + (y_i - v_i)]^T v_j, \dot{\eta}_{ij} = -(w_i - x_i)^T w_j, \dot{\eta}_{ij} = -(w_i - x_i)^T v_j, \dot{\tilde{\eta}}_{ij} = -(w_i - x_i)^T x_j, \) Then, we obtain the first main result.

**Theorem 1:** Consider two coupled heterogeneous networked systems (1) and (2) with nonidentical topological structures \((C \neq A, D \neq B)\). Systems (1) and (2) with controller (3) can reach the outer consensus under Assumptions 1 and 2.
Proof: Let the outer consensus errors between systems (1) and (2) be $e_{pi} = w_i - x_i (i \in I_r)$ and $e_{si} = y_i - v_i (i \in I_r)$. The error system is given as follows:

$$
\begin{align*}
\dot{e}_{pi} &= e_{si}, \\
\dot{e}_{si} &= f_i(y_i, t) - f_i(v_i, t) + \sum_{j=1}^{N} c_{ij}w_j + \sum_{j=1}^{r} \tilde{c}_{ij}y_j \\
&+ \sum_{j=1}^{N} d_{ij}x_j - \sum_{j=1}^{r} a_{ij}x_j - \sum_{j=1}^{r} a_{ij}v_j \\
&- \sum_{j=1}^{N} b_{ij}w_j - \sum_{j=1}^{N} b_{ij}y_j + u_i, \ i \in I_r, \\
\dot{e}_{pi} &= g_i(w_i, t) - g_i(x_i, t) + \sum_{j=1}^{N} c_{ij}y_j + \sum_{j=1}^{r} d_{ij}x_j \\
&- \sum_{j=1}^{N} a_{ij}x_j - \sum_{j=1}^{N} b_{ij}y_j + u_i, \ i \in I_N \setminus I_r,
\end{align*}
$$

where $u_i = -m_i(e_{pi} + e_{si}) + \sum_{j=1}^{N} \theta_{ij}w_j + \sum_{j=1}^{r} \tilde{\theta}_{ij}y_j + \sum_{j=1}^{N} \xi_{ij}x_j + \sum_{j=1}^{r} \tilde{\xi}_{ij}y_j$ for $i \in I_r$. \(\dot{m}_i = k_i(e_{pi}^T e_{pi} + e_{si}^T e_{si})\), \(\dot{\theta}_{ij} = -(e_{pi} + e_{si})^T w_j\), \(\dot{\tilde{\theta}}_{ij} = -(e_{pi} + e_{si})^T y_j\), \(\dot{\xi}_{ij} = -(e_{pi} + e_{si})^T x_j\), \(\dot{\tilde{\xi}}_{ij} = -(e_{pi} + e_{si})^T y_j\). For system (4), we can construct a Lyapunov function as follows:

$$
V(t) = V_1(t) + V_2(t) + V_3(t),
$$

where

$$
V_1(t) = \frac{1}{2} \sum_{i=1}^{r} (e_{pi} + e_{si})^T (e_{pi} + e_{si}) + \frac{1}{2} \sum_{i=r+1}^{N} e_{pi}^T e_{pi},
$$

$$
V_2(t) = \frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{N} (c_{ij} - a_{ij} + \theta_{ij})^2 + \frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{N} (\tilde{c}_{ij} - \tilde{a}_{ij} + \tilde{\theta}_{ij})^2 \\
+ \frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{N} (d_{ij} - b_{ij} + \xi_{ij})^2 + \frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{N} (\tilde{d}_{ij} - \tilde{b}_{ij} + \tilde{\xi}_{ij})^2 \\
+ \frac{1}{2} \sum_{i=r+1}^{N} \sum_{j=1}^{N} (c_{ij} - a_{ij} + \xi_{ij})^2 + \frac{1}{2} \sum_{i=r+1}^{N} \sum_{j=1}^{N} (d_{ij} - b_{ij} + \xi_{ij})^2,
$$

$$
V_3(t) = \frac{1}{2} \sum_{i=1}^{r} \frac{1}{k_i} (m_i - m_i^*)^2 + \frac{1}{2} \sum_{i=r+1}^{N} \frac{1}{k_i} (n_i - n_i^*)^2.
$$

$m_i^* > 0$ and $n_i^* > 0$ are sufficiently large constants; $\tilde{m}_i$ is the upper bound of adaptive gain $m_i$.

Obviously, $V(t) \geq 0$. Then, we calculate the derivatives of $V_1(t)$, $V_2(t)$ and $V_3(t)$, respectively.

$$
\dot{V}_1(t) = \sum_{i=1}^{r} e_{pi}^T e_{pi} + \sum_{i=1}^{r} e_{si}^T e_{si} + \sum_{i=1}^{r} (e_{pi} + e_{si})^T \dot{e}_{pi} + \sum_{i=r+1}^{N} e_{pi}^T \dot{e}_{pi},
$$

$$
\dot{V}_2(t) = \sum_{i=1}^{r} (e_{pi} + e_{si})^T (e_{pi} + e_{si})^T (f_i(y_i, t) - f_i(v_i, t)) \\
+ \sum_{i=1}^{r} (e_{pi} + e_{si})^T (c_{ij}w_j + \tilde{c}_{ij}y_j) \\
+ \sum_{i=1}^{r} (e_{pi} + e_{si})^T (d_{ij}x_j + \tilde{d}_{ij}y_j) \\
+ \sum_{i=1}^{r} (e_{pi} + e_{si})^T (a_{ij}x_j - \tilde{a}_{ij}y_j) \\
+ \sum_{i=1}^{r} (e_{pi} + e_{si})^T (b_{ij}w_j - \tilde{b}_{ij}y_j) \\
+ \sum_{i=r+1}^{N} \sum_{j=1}^{r} (c_{ij} - a_{ij} + \theta_{ij})^2 (e_{pi} + e_{si})^T (e_{pi} + e_{si}) \\
+ \sum_{i=r+1}^{N} \sum_{j=1}^{r} (d_{ij} - b_{ij} + \xi_{ij})^2 (e_{pi} + e_{si})^T (e_{pi} + e_{si}) \\
+ \sum_{i=r+1}^{N} \sum_{j=1}^{r} (c_{ij} - a_{ij} + \xi_{ij})^2 (e_{pi} + e_{si})^T (e_{pi} + e_{si}),
$$

$$
\dot{V}_3(t) = \sum_{i=1}^{r} (m_i - \tilde{m}_i - m_i^*) (e_{pi}^T e_{pi} + e_{si}^T e_{si}) \\
+ \sum_{i=r+1}^{N} (n_i - n_i^*) e_{pi}^T e_{pi}.
$$

From Eqs. (5), (6) and (7), one has

$$
\dot{V}(t) = \sum_{i=1}^{r} (-\tilde{m}_i - m_i^*) (e_{pi}^T e_{pi} + e_{si}^T e_{si}) - \sum_{i=r+1}^{N} n_i^* e_{pi}^T e_{pi} \\
- 2 \sum_{i=1}^{r} m_i e_{pi}^T e_{si} + \sum_{i=1}^{r} e_{pi}^T e_{si} + \sum_{i=1}^{r} e_{si}^T e_{si},
$$

where $r \in I_r$ and $N \in I_N \setminus I_r$.
Similarly, we define the following matrices

\[ \tilde{A} = \begin{pmatrix} a_{11}^2 & a_{12}^2 & \cdots & a_{1N}^2 \\ a_{21}^2 & a_{22}^2 & \cdots & a_{2N}^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1}^2 & a_{r2}^2 & \cdots & a_{rN}^2 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} b_{11}^2 & b_{12}^2 & \cdots & b_{1N}^2 \\ b_{21}^2 & b_{22}^2 & \cdots & b_{2N}^2 \\ \vdots & \vdots & \ddots & \vdots \\ b_{r1}^2 & b_{r2}^2 & \cdots & b_{rN}^2 \end{pmatrix} \]

and

\[ \tilde{\tilde{A}} = \begin{pmatrix} \tilde{a}_{11}^2 & \tilde{a}_{12}^2 & \cdots & \tilde{a}_{1N}^2 \\ \tilde{a}_{21}^2 & \tilde{a}_{22}^2 & \cdots & \tilde{a}_{2N}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{r1}^2 & \tilde{a}_{r2}^2 & \cdots & \tilde{a}_{rN}^2 \end{pmatrix}, \quad \tilde{\tilde{B}} = \begin{pmatrix} \tilde{b}_{11}^2 & \tilde{b}_{12}^2 & \cdots & \tilde{b}_{1N}^2 \\ \tilde{b}_{21}^2 & \tilde{b}_{22}^2 & \cdots & \tilde{b}_{2N}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{b}_{r1}^2 & \tilde{b}_{r2}^2 & \cdots & \tilde{b}_{rN}^2 \end{pmatrix} \]

Similarly,

\[ \tilde{C} = \begin{pmatrix} c_{11}^2 & c_{12}^2 & \cdots & c_{1N}^2 \\ c_{21}^2 & c_{22}^2 & \cdots & c_{2N}^2 \\ \vdots & \vdots & \ddots & \vdots \\ c_{r1}^2 & c_{r2}^2 & \cdots & c_{rN}^2 \end{pmatrix}, \quad \tilde{D} = \begin{pmatrix} d_{11}^2 & d_{12}^2 & \cdots & d_{1N}^2 \\ d_{21}^2 & d_{22}^2 & \cdots & d_{2N}^2 \\ \vdots & \vdots & \ddots & \vdots \\ d_{r1}^2 & d_{r2}^2 & \cdots & d_{rN}^2 \end{pmatrix} \]

Since \( m_i \) is the upper bound of adaptive gain \( m_i \), we have \( m_i \leq m_i \leq 0 \). Combining with Lemma 1, one can obtain

\[
\dot{V}(t) \leq -\sum_{i=1}^{r} m_i e_{pi}^T e_{pi} - \sum_{i=1}^{r} m_i e_{pi}^T e_{si} - \sum_{i=r+1}^{N} n_i e_{pi}^T e_{pi} \\
+ \sum_{i=1}^{r} (2N + 2 + \tilde{\tilde{A}}_i \tilde{B}_i + \tilde{\tilde{B}}_i \tilde{A}_i + \tilde{B}_i \tilde{B}_i) e_{pi}^T e_{pi} \\
+ \sum_{i=r+1}^{N} (2N + 2 + \tilde{\tilde{C}}_i \tilde{D}_i + \tilde{D}_i \tilde{C}_i) e_{pi}^T e_{pi}
\]

where \( e_s = (e_{s_1} e_{s_2} \cdots e_{s_N})^T \), and \( \tilde{A}_i \) represents the \( i \)th row of matrix \( \tilde{A} \). Symbols \( \tilde{B}_i, \tilde{C}_i, \tilde{D}_i, \tilde{A}_i, \) and \( \tilde{B}_i \) have the similar meanings to \( \tilde{A}_i \).

Suppose that \( 0 < e_1 < 1 \) and \( 0 < e_i - 1(i = 1, 2) \) satisfy \( e_1 + e_2 = 1 \) and \( \kappa_1 + \kappa_2 = 1 \), respectively. Then, we obtain

\[
\dot{V}(t) \leq -\sum_{i=1}^{r} \epsilon_1 m_i e_{pi}^T e_{pi} + e_{pi}^T e_{pi} - \sum_{i=r+1}^{N} \kappa_1 n_i e_{pi}^T e_{pi} - \sum_{i=r+1}^{N} \kappa_0 n_i e_{pi}^T e_{pi}
\]

where \( c_0 = 2N + 2 + \tilde{\tilde{A}}_1 \tilde{B}_1 + \tilde{\tilde{B}}_1 \tilde{A}_1 + \tilde{B}_1 \tilde{B}_1 \), \( c_1 = 3 + 2r + 2N + \tilde{\tilde{C}}_i \tilde{D}_i + \tilde{D}_i \tilde{C}_i \), and \( c_2 = 1 + 2r + 2N + \kappa_1 n_i e_{pi}^T e_{pi} - \kappa_0 n_i e_{pi}^T e_{pi} - e_{pi}^T Q e_{pi}
\]
controller:
\[
\begin{align*}
    \dot{v}_i(t) &= v_i(t), \\
    \dot{y}_i(t) &= f_i(y_i, t) + \sum_{j=1}^{N} a_{ij}y_j(t) + \sum_{j=1}^{N} b_{ij}y_j(t), \\
    \dot{w}_i(t) &= y_i(t), \\
    \dot{y}_i(t) &= f_i(y_i, t) + \sum_{j=1}^{N} c_{ij}w_j(t) + \sum_{j=1}^{N} c_{ij}y_j(t), \\
    \dot{w}_i(t) &= y_i(t) + \sum_{j=1}^{N} d_{ij}y_j(t) + \sum_{j=1}^{N} d_{ij}y_j(t) + u_i(t), \\
\end{align*}
\]

and
\[
\begin{align*}
    \dot{v}_i(t) &= v_i(t), \\
    \dot{y}_i(t) &= f_i(y_i, t) + \sum_{j=1}^{N} a_{ij}y_j(t) + \sum_{j=1}^{N} a_{ij}y_j(t), \\
    \dot{w}_i(t) &= y_i(t), \\
    \dot{y}_i(t) &= f_i(y_i, t) + \sum_{j=1}^{N} c_{ij}y_j(t) + \sum_{j=1}^{N} c_{ij}y_j(t), \\
    \dot{w}_i(t) &= y_i(t) + \sum_{j=1}^{N} d_{ij}y_j(t) + \sum_{j=1}^{N} d_{ij}y_j(t) + u_i(t), \\
\end{align*}
\]

where \( u_i = -m_i(e_{pi} + e_{si}) + \sum_{j=1}^{N} \theta_{ij}w_j + \sum_{j=1}^{N} \theta_{ij}y_j + \sum_{j=1}^{N} \xi_{ij}x_j + \sum_{j=1}^{N} \xi_{ij}v_j \) for \( i \in \mathcal{I}_N \).

**B. CASE II: \( r = N \)**

We consider the case that \( r = N \) in this subsection. In this case, systems (1) and (2) can be rewritten as
\[
\begin{align*}
    \dot{x}_i(t) &= v_i(t), \\
    \dot{v}_i(t) &= f_i(v_i, t) + \sum_{j=1}^{N} a_{ij}x_j(t) + \sum_{j=1}^{N} a_{ij}v_j(t), \\
    \dot{w}_i(t) &= y_i(t), \\
    \dot{y}_i(t) &= f_i(y_i, t) + \sum_{j=1}^{N} c_{ij}w_j(t) + \sum_{j=1}^{N} c_{ij}y_j(t), \\
    \dot{w}_i(t) &= y_i(t) + \sum_{j=1}^{N} d_{ij}y_j(t) + \sum_{j=1}^{N} d_{ij}y_j(t) + u_i(t), \\
\end{align*}
\]

According to the discussions in Section III. A, the following adaptive controller is proposed:
\[
\begin{align*}
    u_i &= -m_i[(w_i - x_i) + (y_i - v_i)] + \sum_{j=1}^{N} \theta_{ij}w_j + \sum_{j=1}^{N} \theta_{ij}y_j \\
    &+ \sum_{j=1}^{N} \xi_{ij}x_j + \sum_{j=1}^{N} \xi_{ij}v_j, \\
\end{align*}
\]

where \( m_i = k_i[(w_i - x_i)^T(w_i - x_i) + (y_i - v_i)^T(y_i - v_i)] \) for \( i \in \mathcal{I}_N \) with \( k_i > 0 \).

**Remark 2:** It can be seen from the system models that systems (10) and (11) are the simple case of systems (1) and (2). Based on the design of controller (3), we remove the corresponding part of the first-order system and modify the updated laws according to the system characteristics.

Let the outer consensus errors between systems (10) and (11) be \( e_{pi} = w_i - x_i \) and \( e_{si} = y_i - v_i \) for \( i \in \mathcal{I}_N \).

Then, the error system is represented as
\[
\begin{align*}
    \dot{e}_{pi} &= e_{si}, \\
    \dot{e}_{si} &= f_i(y_i, t) - f_i(y_i, t) + \sum_{j=1}^{N} c_{ij}w_j(t) + \sum_{j=1}^{N} c_{ij}y_j(t) \\
    &+ \sum_{j=1}^{N} d_{ij}y_j(t) + \sum_{j=1}^{N} d_{ij}y_j(t) - \sum_{j=1}^{N} a_{ij}x_j(t) - \sum_{j=1}^{N} a_{ij}v_j(t) \\
    &- \sum_{j=1}^{N} b_{ij}w_j(t) - \sum_{j=1}^{N} b_{ij}y_j(t) + u_i, \\
\end{align*}
\]

**IV. SIMULATION RESULTS**

To verify the effectiveness of the proposed theory, two simulation examples are performed in this section.

**Example 1:** Consider two coupled heterogeneous networked systems (1) and (2) composed of two double-integrator nodes and a single-integrator node, where nodes 1, 2, 3, and 4 are second-order integrators and nodes 0 and 3 are first-order integrators. Fig. 1 depicts the interaction topology. The weight coupling matrices are given as follows:

\[
A = \begin{pmatrix}
-2 & 1 & 1 \\
0 & -1 & 1 \\
1 & 0 & -1
\end{pmatrix},
B = \begin{pmatrix}
-1 & 0 & 1 \\
1 & -2 & 1 \\
0 & 2 & -2
\end{pmatrix}.
\]
In this example, similar to [6], [25] and [35], Chua’s circuit and Lorenz system are chosen as the nonlinear dynamics of different nodes. The dynamic behaviors are described by the following equations:

\[
\begin{align*}
\dot{x}_i(t) &= v_i(t), \\
\dot{v}_i(t) &= f_i(v_i, t), & i &= 1, 2, \\
\dot{x}_i(t) &= g_i(x_i, t), & i &= 3,
\end{align*}
\]

where

\[
\begin{align*}
f_1(v_1, t) &= \begin{cases} 10(v_{12} - v_{11} - q_1(v_{11})), & \\
v_{11} - v_{12} + v_{13}, & \\
-18v_{12}, & \\
\end{cases} \\
f_2(v_2, t) &= \begin{cases} 7(v_{22} - q_2(v_{21})), & \\
v_{21} - v_{22} + v_{23}, & \\
-10v_{22}, & \\
\end{cases} \\
g_3(x_3, t) &= \begin{cases} 10(x_{32} - x_{31}), & \\
28x_{31} - x_{32} - x_{31}x_{33}, & \\
\frac{8}{\sqrt{3}}x_{33} + x_{31}x_{32}. & \\
\end{cases}
\]

with \(q_1(v_{11}) = -0.75v_{11} - \frac{1}{2}(|v_{11} + 1| - |v_{11} - 1|), q_2(v_{21}) = 0.4v_{21} - 0.3(|v_{21} + 1| - |v_{21} - 1|),\) and

Obviously, the above systems satisfy Assumption 1. In the simulation, the initial values of systems and feedback gains \(m_1(0), m_2(0), n_3(0)\) are chosen randomly in \((0, 1)\). Let \(k_1 = 5, k_2 = 10\) and \(k_3 = 5\). Then, Figs. 2–4 depict the simulation results.

From Figs. 2 and 3, we can see that the outer consensus between systems (1) and (2) can be achieved by using controller (3). The system state errors are defined as \(\|e_{pi}\| = \sqrt{(w_{i1} - x_{i1})^2 + (w_{i2} - x_{i2})^2 + (w_{i3} - x_{i3})^2}(i = 1, 2, 3)\) and \(\|e_{pi}\| = \sqrt{(y_{i1} - v_{i1})^2 + (y_{i2} - v_{i2})^2}(i = 1, 2)\). Fig. 4 shows variation tendency of the adaptive feedback gains. It is obvious that the gains tend to different constants over time.

In particular, if the interaction topologies of systems (1) and (2) are the same, i.e.,

\[
A = C = \begin{pmatrix} -2 & 1 & 1 \\
0 & -1 & 1 \\
1 & 0 & -1 \end{pmatrix}
\]

and

\[
B = D = \begin{pmatrix} -1 & 0 & 1 \\
1 & -2 & 1 \\
0 & 2 & -2 \end{pmatrix},
\]

the outer consensus of systems (1) and (2) can be achieved according to Corollary 1. The trajectories of state errors are shown in Figs. 5 and 6.
Example 2: Consider two coupled networked systems (10) and (11) composed of three double-integrator nodes. Suppose that the interaction weight matrices of systems (10) and (11) are the same, i.e.,

\[
A = C = \begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \quad B = D = \begin{pmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix},
\]

and the nonlinear functions \( f_i (i = 1, 2, 3) \) are chosen as the following unified chaotic systems that are similar to the example in [34] and satisfy Assumption 1.

\[
f_i(v_i, t) = \begin{cases} 
(25|x_i| + 10)(v_{i2} - v_{i1}), \\
(28 - 35|x_i|)v_{i1} + (29|x_i| - 1)v_{i2} - v_{i1}v_{i3}, \\
(3|x_i| + 8)v_{i3} + v_{i1}v_{i2},
\end{cases}
\]

where \( x_i = \cos(i - 1) \). Let \( k_i = 5(i = 1, 2, 3) \). Other initial conditions are the same as in Example 1. Then, the outer consensus between systems (10) and (11) is achieved and the trajectories of consensus errors are shown in Figs. 7 and 8.

Further, we consider the case that the topological structures are nonidentical. Suppose that matrices \( A \) and \( B \) are chosen as the same as those in the above case, and

\[
C = \begin{pmatrix} -2 & 0 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 1 & 1 & -2 \end{pmatrix}.
\]

Figs. 9–11 represent the simulation results. In the view of Figs. 9 and 10, the states errors tend to zero, which implies that the outer consensus between networked systems
(10) and (11) is reached. Meanwhile, the adaptive feedback gains \( m_i(t) \) tend to constants.

V. CONCLUSION

This paper deals with an outer consensus problem of two coupled heterogeneous networked systems, where each system consists of the double-integrator and single-integrator nodes with different nonlinear dynamics. To ensure outer consensus between systems under the identical and nonidentical topologies, the adaptive controllers are designed. By using Lyapunov theory and LaSalle’s invariance principle, the theory proofs of outer consensus are provided. In particular, when each system only contains the second-order nodes with different nonlinear dynamics, the adaptive controllers are also given. Two examples demonstrate the validity of the reported method. In future work, we would like to optimize the proposed controllers and apply them to the more general networked systems. Moreover, motivated by the works in [37], [38], the more practical communication factors can be considered in the consensus problem, such as communication restriction or unreliable link. In addition, the consensus problem in discrete event systems are of much interest [39–45].

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