An Effective Field Theory for the Three-Body System

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Abstract. We study the scattering of a particle from a bound pair in an effective field theory using a distorted-wave renormalisation group method to find the power-counting for the three-body force terms. We find that three-body terms appear at lower orders than naively expected. They start with a marginal term that varies as a logarithm rather than a power of the energy scales in the problem. The marginal term has important implications for the three-body problem in nuclear physics.

1 Introduction

The need for a model-independent approach to the three-nucleon system is clear. The plethora of two-body potentials, each capable of describing two-body data extremely well, fail when applied to the three-body problem without an additional three-body force constructed to fit the data.

Effective Field Theories (EFTs) offer a systematic and model-independent treatment of nuclear and hadronic physics at low energies [1]. In an EFT we hope to take advantage of a separation of scales by expanding observables in powers of the ratio $Q/\Lambda_0$, where $Q$ denotes a generic low-energy scale and $\Lambda_0$ a typical scale of the underlying physics. The expansion will be useful if the separation of scales is large enough to ensure quick convergence.

An EFT is defined by a Lagrangian containing all possible local terms consistent with the symmetries of the underlying theory. Although this will invariably lead to an infinite number of terms, we hope to organise them according to some "power-counting", related to the number of low-energy scales in each term. EFTs are well-developed and understood for two-body physics and now the three-body problem is an area of keen interest.

The mathematical tool that allows us to determine the power-counting is the renormalisation group (RG). If we are concerned with problems where the wavelengths of the particles are far longer than the range of the interaction, the scattering is insensitive to the detailed structure of that interaction. Replacing the interaction with contact interactions and forming an EFT Lagrangian
leads to UV divergences from the high-momentum modes of loop diagrams. Accordingly we regulate the loop diagrams with a floating UV cut-off $\Lambda$. Since all observables should be independent of this cut-off we renormalise the theory by absorbing the resulting $\Lambda$-dependence into the EFT couplings. Solving the resulting RG equation provides a frame-work for constructing power-counting schemes. The resulting EFT absorbs the effects of the short-range physics and parameterises it on a power-series in $Q$.

The key to constructing power-counting schemes is fixed points. After rescaling all quantities in terms of $\Lambda$, we may look for solutions to the RG equation which are independent of $\Lambda$. Such solutions are referred to as fixed points. Since these have no scale attached to them solutions of the RG equation tend to one of them as $\Lambda \to 0$. Study of the scaling behaviour of solutions close to the fixed points leads to different power-counting schemes.

### 2 Two-body EFT

The simplest EFT has only one field, the field of the asymptotic particles being considered. Such a theory is known as a “pionless” EFT in nuclear physics. For a two-body system the Lagrangian takes the form,

$$
L_{2B} = \psi^\dagger \left( i \partial_0 + \frac{\nabla^2}{2M} \right) + \frac{C_0}{4} |\psi|^4 + \frac{C_2}{4} (\psi^\dagger (\nabla \cdot \nabla) \psi^\dagger) \psi^2 + H.c. + \ldots
$$

Using the RG method outlined above one may construct two different power-counting schemes for such a system. The RG equation for the leading term $C_0$ is,

$$
\Lambda \frac{\partial \hat{C}_0}{\partial \Lambda} = \hat{C}_0 (1 + \hat{C}_0),
$$

where the hat signifies a dimensionless rescaled coupling. There are clearly two fixed point solutions to this equation, $\hat{C}_0 = 0$ and $\hat{C}_0 = -1$. These lead to the two different schemes.

Perturbing about the fixed point $\hat{C}_0 = 0$ leads to Weinberg or naive counting in which the leading perturbation scales with $Q$. This counting system is useful for weakly interacting systems.

Perturbing about the second fixed point $\hat{C}_0 = -1$ leads to KSW counting, in which the leading perturbation scales with $Q^{-1}$. The fixed-point solution corresponds to a system with a bound-state at exactly zero energy, consequently this expansion is useful for strongly attractive systems with a shallow bound-state and is the expansion most useful in nuclear and atomic physics. The counting scheme is equivalent to the effective range expansion. The couplings, $C_0$ and $C_2$, can be directly related to the scattering length, $a_2$, and effective range, $r_2$, respectively.

Once a power-counting scheme is chosen the free parameters in the couplings can be fixed using experimental data. The resulting theory is then predictive for other processes.
3 Building a Three-Body EFT

Our aim is to extend the KSW EFT to describe three-body interactions, by using the already determined two-body couplings and constructing a counting scheme for the three-body couplings,

\[ \mathcal{L}_{3B} = \frac{D_0}{36} \psi^6 + \frac{D_2}{36} \left( \psi^\dagger (\overrightarrow{\nabla} - \overleftarrow{\nabla})^2 \psi \right) \psi^3 + \text{H.c.} + \ldots \]  

(3)

In scattering of a third particle from a bound pair the two-body contact potential leads to a particle exchange force with a range \( \sim a_2 \). In nuclear physics in particular, the length scale set by the scattering length is far longer than any other scale in the problem. We hope to use this separation of scales to construct an EFT for three-bodies.

We may treat the long-range physics, \( Q \sim 1/a_2 \) exactly by working in terms of the distorted waves (DWs) of the particle-exchange potential [3]. The tool for constructing the power-counting schemes is then the distorted wave renormalisation group (DWRG) in which the cut-off \( \Lambda \) is applied in the basis of the distorted waves. This method ensures that all non-analytic behaviour resulting from the long-range potential is factored out in terms constructed from the DWs.

To apply the DWRG method to the three-body problem we must find the three-body wavefunction for a system with a finite two-body scattering length and zero effective-range. This two-body interaction can be expressed simply as a boundary condition on the wavefunction at zero separation. For the three-body wavefunction this takes the form,

\[
\left[ \frac{\partial \psi(r_{23}, r_1)}{\partial r_{23}} \right]_{r_{23}=0} = -\frac{1}{a_2} \left[ \psi(r_{23}, r_1) \right]_{r_{23}=0},
\]

(4)

where we are using the three-body Jacobian coordinate system. When this is applied to the integral equation for the three-body wavefunction we obtain an integro-differential equation for the projection of the wavefunction, \( \phi(r_1) = \psi(r_{23} = 0, r_1) \). This is equivalent to the equation given by Skorniakov and Ter-Martirosian [4]. Generally, the equations for \( \phi \) and \( \psi \) must be solved numerically. In the zero-energy limit we can solve them analytically. These analytic solutions give the short-range asymptotic behaviour of the full solutions

\[
\psi(r_{23}, r_1) = N(k a_2) f(\theta) \sin(s_0 \ln(k R) + \eta(R_0)),
\]

(5)

where \( R \) and \( \theta \) are the hyperradius and hyperangle respectively, \( f \) is some known function, \( k = \sqrt{4(a_2^2 + ME)/3} \), \( s_0 \approx 1 \), \( N \) is some numerically computed normalisation. Here, \( \eta \) is an arbitrary phase that must be fixed by applying a boundary condition at some \( R_0 \), \( \eta(R_0) = -s_0 \ln(k R_0) \). This solution is precisely that found by Efimov [5].

Using the DWRG method, we may use this result to construct the power-counting for the three-body system. One finds that the DWRG equation has no true fixed point but a logarithmic evolution. Perturbations around this lead
to an expansion of the form $D_{n,m}p^{2n}a_2^{-m}\Lambda_0^{-2n-m}$, where $p$ is the on-shell momentum, $n, m \geq 0$, $D_{n,m}$ are free dimensionless parameters that should be of order one and $\Lambda_0$ is the scale at which the EFT breaks down. This expansion provides the power-counting for the system.

The leading term is marginal, i.e. it is dimensionless and does not scale with $Q/\Lambda_0$. As expected, such a marginal perturbation leads to logarithmic behaviour in $Q$ which necessitates the introduction of a new scale. In this case the scale is in fact $R_0$, the scale appearing in the boundary condition on the three-body wavefunction.

It is important to notice that since the leading three-body term is marginal, it occurs at next-to-leading order in the expansion (two-body leading term occurs at $Q^{-1}$) rather than the naively expected next-to-next-to-leading order.

The final result may be expressed as a DW effective range expansion,

$$N(pa_2) \cot(\delta_{3B}) = -M(pa_2) + \sum_{n,m} D_{n,m}p^{2n}a_2^{-m}\Lambda_0^{-2n-2m}.$$  \hspace{1cm} (6)

$\delta_{3B}$ is the correction to the phaseshift due to the three-body force and $N$ and $M$ are numerically computed functions.

4 Conclusion

This analysis shows the importance of the three-body force in this system. The marginal behaviour of this force means that it must be included in the low-energy EFT. It, or equivalently the boundary condition on the three-body wave function at short distances, is required to resolve the ambiguities in the three-body system with zero-range two-body forces. In nuclear physics, its role can be seen from the fact that different phenomenological nucleon-nucleon potentials lead to quite different predictions for three-nucleon binding energies and scattering lengths. However, these predictions all lie on a single “Phillips line” showing that their differences can be expressed in a single effective parameter $R_0$, the scale appearing the marginal term of the three-body potential.

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