Is It Reasonable to Substitute Discontinuous SMC by Continuous HOSMC?

Comments to Discussion Paper by V. Utkin

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Abstract—Professor Utkin in his discussion paper proposed an example showing that the amplitude of chattering caused by the presence of parasitic dynamics in systems governed by First-Order Sliding-Mode Control (FOSMC) is lower than the obtained using Super-Twisting Algorithm (STA). This example served to motivate this research reconsidering the problem of comparison of chattering magnitude in systems governed by FOSMC that produces a discontinuous control signal and by STA that produces a continuous one, using Harmonic Balance (HB) methodology. With this aim the Averaged Power (AP) criteria for chattering measurements is revisited. The STA gains are redesigned to minimize amplitude or AP of oscillations predicted by HB. The comparison of the chattering produced by FOSMC and STA with redesigned gains is analyzed taking into account their amplitudes, frequencies and values of AP allowing to conclude that: (a) for any value of upperbound of disturbance and Actuator Time Constant (ATC) there exist a bounded disturbance for which the amplitude of possible oscillations produced by FOSMC is lower than the caused by STA; (b) if the upperbound of disturbance and upperbound of time-derivative disturbance are given, then for all sufficiently small values of ATC the amplitude of chattering and AP produced by STA will be smaller than the caused by FOSMC; (c) critical values of ATC are predicted by HB for which the parameters, amplitude of chattering and AP, produced by FOSMC and STA are the same. Also the frequency of self-excited oscillations caused by FOSMC is always greater than the produced by STA.

I. INTRODUCTION

Sliding-Mode Control (SMC) is an efficient technique used for bounded matched uncertainty compensation [18]. The First-Order Sliding-Mode Control (FOSMC) keeps a desired constraint \( \sigma \) of relative degree one, by means of theoretically infinite-frequency switching control. However, infinite-frequency switching control is not feasible due to the presence of parasitic dynamics as actuators and sensors [7] hysteresis effects [17], [5], and other non-idealities. Hence, the sliding set converges to a real sliding motion with finite (high) frequency, this effect is well-known as chattering effect and it is the main drawback of the sliding-mode control theory.

Higher-Order Sliding Mode Control (HOSMC) algorithms were proposed as an attempt to adjust the chattering by substituting (intuitively) discontinuous control inputs by continuous ones [6], [10], [2]. One of the most efficient algorithms is the Super-Twisting Algorithm (STA) compensating theoretically exactly matched Lipschitz uncertainties in finite-time [10], [11]. This idea was very attractive but in the paper [3] was shown that in systems driven by STA (as well as by any other controller with infinite gain at the origin) the chattering also appears. Moreover, Professor Utkin [19], [20], [16], presented some examples showing that some systems governed by STA exhibit bigger amplitude of chattering that the systems driven by First-Order Sliding Mode Control (FOSMC), when parasitic dynamics affects the SMC/HOSMC closed-loop.

In this paper we will try to answer the question: Is It Expedient to Substitute Discontinuous SMC by Continuous HOSMC, when parasitic dynamics are presented in the control loop? For that, we analyze the amplitude and frequency of possible self-excited oscillations and its effect on the Average Power (AP). Harmonic Balance (HB) is widely used to estimate the amplitude and frequency of possible oscillations for dynamically perturbed SMC/HOSMC systems [17], [3], where a Describing Function (DF) characterizes the non-linearities effects on the parameters of periodic motions [9], [1].

The contributions of this work are listed below:

- We confirm the hypothesis of Professor V. Utkin: for any value of the actuator time-constant (ATC) there exist a bounded disturbance for which the amplitude of possible oscillations produced by FOSMC is lower than the obtained applying STA.
- Given the upperbound of disturbance and the upper-bound of time-derivative disturbance, for a sufficiently small value of ATC the amplitude of chattering and AP produced by STA will be smaller than the caused by FOSMC.

With this aim, we propose the following:

1. Reformulation of A. Levant “energy like” criterion to compute the AP based on HB methodology.
2. Selection of STA gains to minimize the amplitude of possible oscillations.
3. Selection of STA gains to minimize the AP.

The structure of the paper is as follows: a motivation example is discussed in section II; section III contains the
preliminaries about HB approach for FOSMC and STA; chatter parameters obtained by HB are analyzed in section IV; comparison examples in section V; and section VI presents the conclusions about the obtained results.

II. MOTIVATION EXAMPLE

Consider the disturbed first-order system shown in the Figure 1, the plant can be modeled as

$$\dot{x}(t) = u(t) + F(t),$$  \hspace{1cm} (1)

where $x \in \mathbb{R}$ is the output and $\bar{u} \in \mathbb{R}$ is the control input. The disturbance term $F$ has the form

$$F = \alpha \sin(\Omega t) \Rightarrow \begin{cases} |F| \leq \delta = \alpha \Omega \\ |F| \leq \Delta = \alpha \Omega \end{cases}$$  \hspace{1cm} (2)

Let us apply two control laws:

- **Discontinuous FOSMC** [18]:

  $$u = -M \text{sign}(x),$$  \hspace{1cm} (3)

  where the control gain is chosen $M = 1.1 \delta$ from the upperbound of disturbance $\delta$, ensuring the global finite-time convergence to the first order sliding-mode ($\exists t_r : x(t) = 0, \forall t \geq t_r$) when the actuator dynamics is fast enough.

- **Continuous STA** [10], [11]:

  $$\begin{align*}
  &u = -k_1|x|^{1/2} \text{sign}(x) + v, \\
  &\dot{v} = -k_2 \text{sign}(x),
  \end{align*}$$  \hspace{1cm} (4)

  where the control gains are chosen $k_1 = 1.5 \sqrt{\Delta}$, $k_2 = 1.1 \Delta$ from the upperbound of time-derivative disturbance $\Delta$, ensuring the global finite-time convergence to the second order sliding-mode ($\exists t_r : x(t) = \dot{x}(t) = 0, \forall t \geq t_r$) when the actuator dynamics is fast enough.

**Remark 1**: FOSMC can reject bounded disturbances but STA can compensate Lipschitz disturbances (not necessarily bounded). In order to compare both algorithms we consider a bounded and Lipschitz disturbance $\Delta$.

Following [19], we consider the same actuator model as Professor V. Utkin which consists of a 2\textsuperscript{nd}-order linear system

$$\begin{align*}
  \dot{z}(t) &= \begin{bmatrix} 0 & 1 \\ \frac{1}{\mu^2} & -\frac{1}{\mu^2} \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ \frac{1}{\mu^2} \end{bmatrix} u(t), \\
  \bar{u}(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} z(t),
\end{align*}$$  \hspace{1cm} (5)

where $\mu > 0$ is the actuator time-constant (ATC). Thus, the effects of parasitic dynamics can be parameterized through ATC. Let us notice that FOSMC and STA were designed for the system (1) with relative degree one but the presence of actuator (5) increases the relative degree of the system (1).

![Figure 1. Control Scheme.](image-url)
Consider the nominal case \( F = 0 \), the dynamically perturbed system \([5]–[11]\) has the transfer function
\[
W(s) = G_a(s)G(s) = \frac{1}{s(\mu s + 1)^2},
\]
whose relative degree is \( r = 3 \).

**Assumption 1:** Due the parasitic dynamics \([5]\), the output of the system converges to a periodic solution \([3], [19]\), which can be approximated by its first-harmonic,
\[
x(t) = A \sin(\omega t),
\]
\[
\dot{x}(t) = A \omega \cos(\omega t),
\]
where \( A \) and \( \omega \) are the chattering parameters, amplitude and frequency, respectively.

### A. Amplitude and Frequency Estimation

Let us apply the DF method \([9], [10]\) to predict periodic oscillations. Parameters of a possible limit cycle may be found as an intersection point of the actuator-plant dynamics \( W(s) \) Nyquist plot and the negative reciprocal DF \(-N^{-1}(A, \omega)\) of the SMC/HOSMC algorithm, which corresponds to the Harmonic Balance equation (HBE)
\[
N(A, \omega)W(j\omega) = -1,
\]
whose solution is an estimate of chattering parameters: amplitude \( A \) and frequency \( \omega \).

### B. Averaged Power Criteria

In the paper by A. Levant \([13]\) an “energy like” criteria for chattering measurement are presented
\[
E = \left( \int_0^T \dot{x}^p(\tau) d\tau \right)^{1/p},
\]
inspired by \( L_p \) norm. Unfortunately, it is only a qualitative criterion to understand the chattering effects because it has no physical sense and require information of \( \dot{x} \) (knowledge of disturbance) to compute it.

HB approach allows to compute the Averaged Power (AP)

\[
P = \frac{\omega}{2\pi} \int_0^{2\pi} \left( A \omega \cos(\omega t) \right)^2 dt = \frac{A^2\omega^2}{2},
\]
due to the output \( x \) and its time-derivative \( \dot{x} \) are assumed of the form \([7]\).

### C. HB Analysis of First Order Sliding Mode Control

The describing function of the non-linearity \([8]\) has the form
\[
N(A) = \frac{4M}{\pi A}.
\]
The HBE \([8]\) can be separated as real and imaginary parts
\[
\frac{4M}{\pi A} = 2 \mu \omega^2,
\]
\[
0 = \omega(\mu^2 \omega^2 - 1),
\]
whose analytic solution is \([16]\)
\[
A = \frac{2M\mu}{\pi},
\]
\[
\omega = \frac{1}{\mu}.
\]
Substituting the limit cycle parameters \([12], [13]\) in the expression \([10]\), the AP has the form
\[
P = \frac{2M^2}{\pi^2}.
\]

### D. HB Analysis of Super-Twisting Algorithm

The describing function of the non-linearity \([4]\) has the form
\[
N(A, \omega) = \frac{1.1128k_1}{A^{1/2}} - j \frac{4k_2}{\pi A \omega}.
\]

Once again, the HBE \([8]\) can be separated as real and imaginary parts
\[
\frac{1.1128k_1}{A^{1/2}} = 2 \mu \omega^2,
\]
\[
\frac{4k_2}{\pi A \omega} = \omega(1 - \mu^2 \omega^2),
\]
whose analytic solution is \([15]\)
\[
A = \mu^2 \mathbb{K}_A,
\]
\[
\omega = \frac{\mathbb{K}_\omega}{\mu},
\]
where
\[
\mathbb{K}_A = \left( \frac{1}{2} \cdot \frac{(1.1128k_1)^2 + 4k_2}{1.1128k_1} \right)^2 = \left( \frac{1.1128k_1}{2\pi} \right)^2,
\]
\[
\mathbb{K}_\omega = \left( \frac{(1.1128k_1)^2 + 4k_2}{1.1128k_1^2 + \pi k_2} \right)^{1/2}.
\]

Substituting the limit cycle parameters \([16], [17]\) in the expression \([10]\), the AP has the form
\[
P = \frac{\mu^2}{32} \cdot \left( \frac{(1.1128k_1)^2 + 16k_2}{32k_1^2} \right).
E. Design of STA Gains to Minimize the Amplitude of Possible Oscillations

The expression (16) allows to obtain the values of STA gains, $k_1$ and $k_2$, such that the amplitude is minimized. Consider the following normalization

$$A = \left( \frac{1}{2} \right) \frac{(1.1128k_1)^2 + \frac{16}{\pi}k_2}{1.1128k_1} \mu^2,$$  \hspace{1cm} (19)

it is possible to compute the value of the gain $k_1$ which minimizes the amplitude (16) for each $k_2 > \Delta$,

$$k_1 = \left( \frac{16k_2}{\pi(1.1128)^2} \right)^{1/2} = 2.028\sqrt{k_2}.$$  \hspace{1cm} (20)

Remark 2: The selection of STA gains that minimize the amplitude of possible oscillations based on HB approach is

$$k_1 = 2.127\sqrt{\Delta}, \hspace{0.5cm} k_2 = 1.1\Delta.$$  \hspace{1cm} (21)

For these STA gains, the chattering parameters become

$$A = 5.6023\Delta\mu^2, \hspace{0.5cm} \omega = \frac{1}{\mu\sqrt{2}}, \hspace{0.5cm} P = 7.8464\Delta^2\mu^2.$$  

Figure 2 shows the critical value of the amplitude normalization (19) for the STA gains (21) with $\Delta = 1$. Table II contains the simulation results for $\Delta = 10$ and $\mu = 10^{-2}$, the STA gains are selected to compare the amplitude of oscillations and confirms the selection criterion (21).

F. Design of STA Gains to Minimize the Averaged Power

The expression (18) allows to obtain the values of STA gains, $k_1$ and $k_2$, such that the AP is minimized. Consider the following normalization

$$P = \frac{1}{32} \frac{(1.1128k_1)^2 + \frac{16}{\pi}k_2}{(1.1128k_1)^2} \mu^2,$$  \hspace{1cm} (22)

it is possible to compute the value of the gain $k_1$ which minimizes the AP for each $k_2 > \Delta$,

$$k_1 = \left( \frac{8k_2}{\pi(1.1128)^2} \right)^{1/2} = 1.434\sqrt{k_2}.$$  \hspace{1cm} (23)

Remark 3: The selection of STA gains that minimize the AP based on HB approach is

$$k_1 = 1.504\sqrt{\Delta}, \hspace{0.5cm} k_2 = 1.1\Delta.$$  \hspace{1cm} (24)

For these STA gains, the parameters of periodic motion become

$$A = 6.3025\Delta\mu^2, \hspace{0.5cm} \omega = \frac{1}{\mu\sqrt{3}}, \hspace{0.5cm} P = 6.6203\Delta^2\mu^2.$$  

Figure 2 shows the critical value of the AP normalization (22) with $\Delta = 1$. Table III contains the simulation results for $\Delta = 10$ and $\mu = 10^{-2}$, the STA gains are selected to compare the AP and confirms the selection criterion (24).

Remark 4: Sufficient conditions to guarantee finite-time stability are presented in [14], where the STA gains have to satisfy

$$k_1 > 1.414\sqrt{k_2}, \hspace{0.5cm} k_2 > \Delta.$$  \hspace{1cm} (25)

where $\Delta$ is the upperbound of the time-derivative disturbance [2]. STA proposed gains in the expressions (21) and (24) ensure finite-time stability when the actuator dynamics is just enough.
IV. ANALYSIS OF CHATTERING PARAMETERS ESTIMATED BY HARMONIC BALANCE

A. Amplitude Analysis

In order to analyze the chattering parameters with respect to the ATC, consider $\delta = \Delta = 1$ and the control gain $M = 1.1\delta$ for FOSMC and the selection proposed in (21) for STA. Figure 3 shows the chattering parameters for several values of ATC, there is a value of the ATC such that the amplitude of the output is the same [15], despite the use of FOSMC (3) or STA (4) on the dynamically perturbed system (6),

$$\mu^* = 8M(1.1128k_1)^2 \frac{\pi}{(1.1128k_1)^2 + 16\pi k_2)^{3/2}}.$$  (26)

for this example, the value of ATC to have the same amplitude is

$$\mu^* = 0.125\frac{\delta}{\Delta}.$$  (27)

**Remark 5:** Figure 3 confirms that to substitute FOSMC by STA, we should consider that the amplitude of oscillations may be greater(lower) when the ATC,

$$\mu > \mu^* \Rightarrow A_{FOSMC} < A_{STA},$$
$$\mu < \mu^* \Rightarrow A_{FOSMC} > A_{STA}.$$  (28)

B. Frequency Analysis

The order of high-frequency oscillations is $O(1/\mu)$ when it is applied FOSMC (3) or STA (4). However, the frequency is always lower for the STA than the obtained using FOSMC, as it is shown in the graphical solution of the HB equation (8) of Figure 4.

C. Averaged Power Analysis

In order to analyze the chattering parameters with respect to the ATC, consider $\delta = \Delta = 1$ and the control gain $M = 1.1\delta$ for FOSMC and the selection proposed in (24) for STA. Figure 5 shows the chattering parameters for several values of ATC, there is a value of ATC such that the AP is the same despite the use of FOSMC (3) or STA (4) on the dynamically perturbed system (6),

$$\mu^* = 0.1924\frac{\delta}{\Delta}.$$  (30)

**Remark 6:** Figure 5 confirms that to substitute FOSMC by STA, we should consider that the AP may be greater(lower) when the ATC,

$$\mu > \mu^* \Rightarrow P_{FOSMC} < P_{STA},$$
$$\mu < \mu^* \Rightarrow P_{FOSMC} > P_{STA}.$$  (31)

V. COMPARISON EXAMPLES

A. High-Frequency Disturbances

The previously simulation examples are for the nominal case ($F = 0$) but the control gains are selected according to the upperbound of disturbance in the case of FOSMC (3), or the upperbound of time-derivative disturbance for the STA (4), then

- **FOSMC (3):**
  $$M = 1.1\delta = 1.1\alpha.$$  

- **STA (4):**
  $$k_1 = 2.127\sqrt{\Delta} = 2.127\sqrt{\alpha\Omega},$$
  $$k_2 = 1.1\Delta = 1.1\alpha\Omega.$$  

The value of ATC predicted by HB for which the amplitude of chattering is the same despite the use of discontinuous FOSMC (3) or continuous STA (4), on the dynamically perturbed system (6) is

$$\mu^* = 0.125\frac{\delta}{\Delta} = 0.125\frac{1}{\Omega}.$$  (32)
In order to compare the system behavior for some values of the ATC, consider
\[ \mu_1 = 0.25 \frac{1}{\Omega}, \quad \mu_2 = 0.0833 \frac{1}{\Omega}, \] (33)

Table IV contains the sliding-mode output accuracy for some values of disturbance frequency \( \Omega \), taking into account the critical value of ATC \( \mu \) and \( \mu_1 > \mu^* \), \( \mu_2 < \mu^* \) from (33).

Remark 7: Simulation results confirm that for any disturbance frequency \( \Omega \) should be a critical value of ATC \( \mu^* \) for which the magnitude of chattering is the same when FOSMC or STA are applied. If ATC is greater than \( \mu^* \) (for example \( \mu_1 \)) the amplitude of oscillations is lower using FOSMC than the obtained applying STA. But if ATC is lower than \( \mu^* \) (for example \( \mu_2 \)) the amplitude of oscillations is higher using FOSMC than the obtained applying STA.

B. Professor V. Utkin Example
The following example was taken from the paper \[19\], they propose that the upperbound of disturbance and the upperbound of time-derivative disturbance have the same value \( \delta = \Delta = 60 \). Taking into account the FOSMC \( \delta \) gain \( M = 1.16 \) and the STA \( \delta \) proposed gains \( 21 \), the following chattering parameters are obtained by HB:

- **FOSMC**
  \[ A = 42.017\mu, \quad \omega = \frac{1}{\mu}, \quad P = 882.7102. \]

- **STA**
  \[ A = 336.135\mu^2, \quad \omega = \frac{1}{\mu\sqrt{2}}, \quad P = 28246.93\mu^2. \]

Hence the critical values of ATC become
\[ \mu^* = 0.125, \quad \mu^* = 0.1768, \] (34)

for same amplitude and same AP, respectively. Table V shows the chattering parameters obtained in simulation for some values of ATC and the critical values (34). Note that when the ATC \( \mu > \mu^* \) the amplitude of oscillations generated by FOSMC is lower than the produced by STA, this situation is reversed when \( \mu < \mu^* \). On the other hand, when the ATC \( \mu > \mu^* \) the AP generated by FOSMC is lower than the produced by STA, and when \( \mu < \mu^* \) the AP caused by FOSMC is grater than the produced by STA.

VI. CONCLUSIONS
A methodology for analysis based on Harmonic Balance approach is proposed to study the chattering for dynamically perturbed systems driven by FOSMC and STA. HB approach and simulation results allows to confirm the Professor V. Utkin hypothesis, given the value of ATC and the upperbound of disturbance there exist a bounded disturbance for which the amplitude of possible oscillations produced by FOSMC is lower than the obtained applying STA. On the other hand, given the upperbound of disturbance and the upperbound of time-derivative disturbance there exist an actuator fast enough for which the STA amplitude of oscillations or(and) AP are lower than the generate by FOSMC. Selection criteria of STA gains are presented to adjust the chattering effects on the amplitude of possible oscillations and AP.

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