Deterministic and Randomized Actuator Scheduling
With Guaranteed Performance Bounds
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Abstract—In this paper, we investigate the problem of actuator selection for linear dynamical systems. We develop a framework to design a sparse actuator/sensor schedule for a given large-scale linear system with guaranteed performance bounds using deterministic polynomial-time and randomized approximately linear-time algorithms. We first introduce systemic controllability metrics for linear dynamical systems that are monotone, convex, and homogeneous with respect to the controllability Gramian. We show that several popular and widely used optimization criteria in the literature belong to this class of controllability metrics. Our main result is to provide a polynomial-time actuator schedule that on average selects only a constant number of actuators at each time step, independent of the dimension, to furnish a guaranteed approximation of the controllability/observability metrics in comparison to when all actuators/sensors are in use. We illustrate the effectiveness of our theoretical findings via several numerical simulations using benchmark examples.

I. INTRODUCTION

Over the past few years, controllability and observability properties of complex dynamical networks have been subjects of intense study in the controls community [1]–[12]. This interest stems from the need to steer or observe the state of large-scale, networked systems such as the power grids [13], social networks, biological and genetic regulatory networks [14]–[16], and traffic networks [17]. While the classical notion of controllability, introduced by Kalman in [18] is quite well understood, the question of controllability and the dependence of various measures of controllability or observability on number and location of sensors and actuators in networked systems are not fully understood [19]. Often times, there is a need to steer or estimate the state of a large-scale, networked control system with as few actuators/sensors as possible, due to issues related to cost and energy depletion. The desire to perform control/estimation using a sparse set of actuators/sensors spans various application domains, ranging from infrastructure networks (e.g., water and power networks) to multi-robot systems and the study of the human connectome. For example, energy conservation through efficient utilization of sensors and actuators can help extend the duration of battery life in networks of mobile sensors and multi-agent robotic networks; estimating the whole state of the power grid using fewer measurement units will help reduce the cost of monitoring the network for systemic failures, etc.

It is therefore desirable to have a limited number of sensors and actuators without compromising the control or estimation performance too much. Unfortunately, as the recent works in [1], [20] have shown, the problem of finding a sparse set of input variables such that the resulting system is controllable is NP-hard. Even the presumably easier problem of approximating the minimum number better than a constant multiplicative factor of \( \log n \) is also NP-hard. Other results in the literature have studied network controllability by exploring approximation algorithms for the closely related subset selection problem [1], [11], [12]. More recently, some of the authors showed that even the problem of finding a sparse set of actuators to guarantee reachability of a particular state is hard and even hard to approximate [21].

Previous studies have been mainly focused on solving the optimal sensor/actuator placement problem using the greedy heuristic, as approximations of the corresponding sparse-subset selection problem. While these results attempt to find approximation algorithms for finding the best sparse subset, our focus in this paper is to gain new fundamental insights into approximating various controllability metrics compared to the case when all possible actuators are chosen. Specifically, we are interested in actuator/sensor schedules that select a small number of actuators/sensors so as to save the energy while ensuring a suitable level of controllability (observability) performance for the entire network. Due to energy efficiency, we may want to minimize the number of active actuator/sensors at each time. At the same time, we would like to have a performance that closely resembles that of the original system, when all available sensor/actuators are active.

We investigate sparse sensor and actuator selection as particular instances where discrete geometric structures can be utilized to study network controllability and observability problems (cf. [20], [22], [23]). A key observation is the close connection between this problem and some classical problems in statistics such as outlier detection, active learning, and optimal experimental design. In recent years, there has been a renewed interest in optimal experiment design which has a long history going back at least 65 years [24], [25].

One of our main contributions is to show that the time-varying actuator selection problem, which goes back to a paper by Athans in 1972 [19], can be solved via random sampling. We propose an alternative to submodularity-based methods and instead use recent advances in theoretical computer science

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†From now on we will focus the paper on the actuator selection problem. The dual notion of sensor selection follows similar ideas.
to develop scalable algorithms for sparsifying control inputs. Current approaches based on polynomial time relaxations of the subset selection problem require an extra multiplicative factor of \( \log n \) sensors/actuators times the minimal number in order to just maintain controllability/observability. Using these recent advances \([25]–[31]\), we show that by carefully designing a scheduling strategy, one can choose on average a constant number of sensors and actuators at each time, to approximate the controllability/observability metrics of the system when all sensors and actuators are in use.

Some of our results appeared earlier in the conference version of this paper \([32]\), \([33]\); however, their proofs are presented here for first time. The manuscript also contains several new results, remarks, numerical examples, and proofs.

II. Preliminaries and Definitions

A. Mathematical Notations

Throughout the paper, discrete time index is denoted by \( k \). The sets of real (integer), positive real (integer), and strictly positive real (integer) numbers are represented by \( \mathbb{R} \) (\( \mathbb{Z} \)), \( \mathbb{R}_+ \) (\( \mathbb{Z}_+ \)) and \( \mathbb{R}_{++} \) (\( \mathbb{Z}_{++} \)), respectively. The set of natural numbers \( \{ i \in \mathbb{Z}_+ : i \leq n \} \) is denoted by \( [n] \). The cardinality of a set \( \sigma \) is denoted by \( \card(\sigma) \). Capital letters, such as \( A \) or \( B \), stand for real-valued matrices. For a square matrix \( X \), \( \det(X) \) and \( \text{Trace}(X) \) refer to the determinant and the summation of on-diagonal elements of \( X \), respectively, \( \mathbb{S}_n \) is the positive definite cone of \( n \)-by-\( n \) matrices. The \( n \)-by-\( n \) identity matrix is denoted by \( I \). Notation \( A \preceq B \) is equivalent to matrix \( B - A \) being positive semi-definite. The transpose of matrix \( A \) is denoted by \( A^\top \). The rank, kernel and image of matrix \( A \) are referred to by \( \text{rank}(A) \), \( \ker(A) \) and \( \text{Im}(A) \), respectively. The Moore-Penrose pseudo-inverse of matrix \( A \) is denoted by \( A^+ \). The ceiling function of \( x \in \mathbb{R} \) is denoted by \( \lceil x \rceil \) where it returns the least integer greater than or equal to \( x \). Finally, an actuator schedule is sparse if and only if on average a constant number of actuators, independent of the system dimension, are active each time.

B. Linear Systems and Controllability

We start with the canonical linear discrete-time, time-invariant dynamics

\[
x(k + 1) = Ax(k) + Bu(k),
\]

where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m} \) and \( k \in \mathbb{Z}_+ \). The state matrix \( A \) describes the underlying structure of the system and the interaction strength between the agents, and matrix \( B \) represents how the control input enters the system. Equivalently, the dynamics can be written as

\[
x(k + 1) = Ax(k) + \sum_{i \in [m]} b_i u_i(k),
\]

where \( b_i \)'s are columns of matrix \( B \in \mathbb{R}^{n \times m} \). Then, the controllability matrix at time \( t \) is given by

\[
C(t) = [B \ AB A^2 B \cdots A^{t-1} B].
\]

It is well-known that from a numerical standpoint it is better to characterize controllability in terms of the Gramian matrix at time \( t \) defined as follows:

\[
W(t) = \sum_{i=0}^{t-1} A^i B B^\top (A^i)^\top = C(t) C^\top (t).
\]

When looking at time-varying input/actuator schedules, we will consider the following linear system with time-varying input matrix \( B(\cdot) \):

\[
x(k + 1) = Ax(k) + B(k) u(k).
\]

For the above system, the controllability and Gramian matrices at time step \( t \) are defined as

\[
C_*(t) = [B(t-1) \ AB(t-2) A^2 B(t-3) \cdots A^{t-1} B(0)],
\]

and

\[
W_*(t) = \sum_{i=0}^{t-1} A^i B(t-i-1) B^\top (t-i-1) (A^i)^\top
\]

\[
= C_*(t) C_*^\top (t),
\]

respectively.

Assumption 1: Throughout the paper, we assume that the system \( (1) \) is controllable (i.e., the controllability matrix has full row rank and the Gramian is positive definite). However, all results presented in this paper can be modified/extended to uncontrollable systems.

C. Matrix Reconstruction and Sparsification

The key idea throughout the paper is to approximate the time-\( \tau \) controllability Gramian as a sparse sum of \( \tau \)-rank matrices, while controlling the approximation error. To this end, we present a key lemma from the sparsification literature which we use later in our algorithms to find sparse actuator schedules.

Lemma 1 (Dual Set Spectral Sparsification \([37]\)) Let \( V = \{v_1, \ldots, v_t\} \) and \( U = \{u_1, \ldots, u_t\} \) be two equal cardinality decompositions of identity matrices (i.e., \( \sum_{i=1}^{t} v_i v_i^\top = I_n \) and \( \sum_{i=1}^{t} u_i u_i^\top = I_\ell \)) where \( v_i \in \mathbb{R}^n \) \((n < t)\) and \( u_i \in \mathbb{R}^\ell \) \((\ell < t)\). Given an integer \( \kappa \) with \( n < \kappa < t \), Algorithm 1 computes a set of weights \( c_i \geq 0 \) where \( i \in [t] \), such that

\[
\lambda_{\min} \left( \sum_{i=1}^{t} c_i v_i v_i^\top \right) \geq \left( 1 - \sqrt{\frac{n}{\kappa}} \right)^2,
\]

\[
\lambda_{\max} \left( \sum_{i=1}^{t} c_i u_i u_i^\top \right) \leq \left( 1 + \sqrt{\frac{\ell}{\kappa}} \right)^2,
\]

and

\[
\card \{ i : c_i > 0, i \in [t] \} \leq \kappa.
\]

Algorithm 1 greedily selects vectors that satisfy a number of desired properties in each step. These properties will eventually imply the desired bounds on eigenvalues. In Algorithm 1, two
Algorithm 1: A Deterministic Dual Set Spectral Sparsification
DualSet($V,U,\kappa$).

**Input:** $V = \{v_1, \ldots, v_t\}$, with $VV^T = I_t$
$U = \{u_1, \ldots, u_t\}$, with $UU^T = I_t$
$\kappa \in \mathbb{Z}_+$, with $n < \kappa \leq t$

**Output:** $c = \{c_1, c_2, \ldots, c_t\} \in \mathbb{R}_+^{t \times t}$ with $\|c\|_0 \leq \kappa$

1. Set $c(0) = 0_{t \times t}$, $A(0) = 0_{n \times n}$, $\mathcal{A}(0) = 0_{t \times t}$, $\delta_0 = 1$,
   $\delta = \frac{1 + \sqrt{2}}{1 - \sqrt{2}}$
2. For $\tau = 0 : \kappa - 1$
   3. $\bar{\mu}(\tau) = \tau - \sqrt{\kappa n}$
   4. $\bar{\mu}(\tau) = \delta \left( \tau + \sqrt{\kappa n} \right)$
   5. Find an index $j$ such that
      $\mathcal{U}(u_j, \delta, \bar{\mu}(\tau)) \leq \mathcal{W}(v_j, \delta, \bar{\mu}(\tau))$
   6. Set $\Delta = 2(\mathcal{U}(u_j, \delta, \bar{\mu}(\tau)) + \mathcal{W}(v_j, \delta, \bar{\mu}(\tau)))^{-1}$
   7. Update the $j$-th component of $c(\tau)$:
      $c(\tau + 1) = c(\tau) + \Delta e_j$
8. $\mathcal{A}(\tau + 1) = \mathcal{A}(\tau) + \Delta u_j v_j^T$
9. $\bar{\mathcal{A}}(\tau + 1) = \bar{\mathcal{A}}(\tau) + \Delta u_j v_j^T$
10. end
11. return $c = \kappa^{-1} \left( 1 - \frac{1}{\sqrt{\kappa}} \right) c(\kappa)$

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instead of only finding an index $j$ such that

$$\mathcal{U}(u_j, \delta, \bar{\mu}(\tau)) \leq \mathcal{W}(v_j, \delta, \bar{\mu}(\tau)).$$

We should note that if an index $j$ maximizes $\mathcal{U}$, then it will satisfy $\mathcal{U}$. Therefore, Lemma 1 still holds for the modified algorithm, and hence the theoretical bounds are valid. We denote the application of the algorithm to $V$ and $U$ by

$$[c_1, c_2, \cdots, c_t] = \text{DualSet}^*(V,U,\kappa).$$

In the next section, we show how various controllability measures can be approximated by selecting a sparse set of actuators via the above algorithm.

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III. Systemic Controllability Metrics

Similar to the systemic notions introduced in [35]–[37], we define various controllability metrics. These measures are real-valued operators defined on the set of all linear dynamical systems governed by $\mathcal{A}$ and quantify various measures of the required control energy. All of the metrics depend on the controllability Gramian matrix of the system which is a positive definite matrix. Therefore, one can define a systemic controllability performance measure as an operator on the set of Gramian matrices of all controllable systems with $n$ states which we represent by $\mathcal{S}_n^\kappa$.

**Definition 1 (Systemic Criteria):** A controllability metric $\rho : \mathcal{S}_n^\kappa \to \mathbb{R}$ is systemic if and only if

1. **Homogeneity:** For all $\kappa > 1$,
   $$\rho(\kappa A) = \kappa^{-1} \rho(A);$$
2. **Monotonicity:** If $B \preceq A$, then
   $$\rho(A) \leq \rho(B);$$
3. **Convexity:** For all $0 \leq c \leq 1$,
   $$\rho(cA + (1-c)B) \leq c \rho(A) + (1-c) \rho(B).$$

For many popular choices of $\rho$, one can see that they satisfy the properties presented in Definition 1. Some of them are listed in Table 1. We note that similar criteria have been developed in [24, 25, 38] in the experiment design literature (cf. Table 1). In what follows, we will make this statement formal.

**Proposition 1:** For given dynamics $\mathcal{A}$ with Gramian matrix $\mathcal{W}(t)$, the metrics presented in Table 1 are systemic controllability measures.

**Proof:** One can easily see that all these measures satisfy the homogeneity, monotonicity, and convexity properties in Definition 1 (cf. [37, 39]).

In the next section, we show how various measures can be approximated by selecting a sparse set of actuators.

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IV. Sparse Actuator Selection Problems

For a given linear system $\mathcal{A}$ with a general underlying structure, the actuator scheduling problem seeks to construct a schedule of the control inputs that keeps the number of active actuators much less than the original system such that the
controllability matrices of the original and the new systems are similar in an appropriately defined sense. Specifically, given a canonical linear, time-invariant system \( \{1\} \) with \( m \) actuators and controllability Gramian matrix \( W(t) \) at time \( t \), our goal is to find a sparse actuator schedule such that the resulting system with controllability Gramian \( W_s(t) \) is well-approximated, i.e.,

\[
\frac{\rho(W(t)) - \rho(W_s(t))}{\rho(W(t))} \leq \epsilon,
\]

(9)

where \( \rho \) is any systemic controllability metric that quantifies the difficulty of the control problem for example as a function of the required control energy, and \( \epsilon \geq 0 \) is the approximation factor. The systemic controllability metrics are defined based on the controllability Gramian, therefore “close” Gramian matrices result in approximately the same values. Our goal here is to answer the following questions:

- What is the minimum number of actuators to be chosen to achieve a good approximation of the system with the full set of actuators utilized?
- What is the relation between the number of selected actuators and performance/controllability loss?
- Does a sparse approximation schedule exist with at most a constant number of active actuators at each time?
- What is the time complexity of choosing the subset of actuators with guaranteed performance bounds?

In the rest of this paper, we show how some fairly recent advances in theoretical computer science and the probabilistic method can be utilized to answer these questions. The probabilistic method is one of the most important tools of modern combinatorics which was introduced by Erdős. The idea is that a deterministic solution is shown to exist by constructing a random candidate satisfying all the requirements of the problem with positive probability. Recently, Marcus, Spielman, and Srivastava introduced a new variant of the probabilistic method which ends up solving the so-called Kadison-Singer (KS) conjecture [30]. We use the solution approach to the KS conjecture together with a combination of tools from Subsections \( \text{V} \) and \( \text{VI} \) to find a sparse approximation of the actuator selection problem with algorithms that have favorable time-complexity.

Later, in Section \( \text{VII} \) we use time-varying actuator schedules to sparsify the selected subset of control inputs using a sub-sampling method, according to which the actuators are selected or unselected according to probabilities that encode the relative importance of each input.

### V. A WEIGHTED SPARSE ACTUATOR SCHEDULE

As a starting point, we allow for scaling of the input signals at chosen inputs while keeping the input scaling bounded. The input scaling allows for an extra degree of freedom that could allow for choosing a sparser set of inputs. Given \( \{1\} \), we define a weighted actuator schedule by \( \sigma = \{\sigma_k\}_{k=0}^{t} \) and scalings \( s_i(k) \geq 0 \) where \( i \in [m] \), \( k + 1 \in [t] \), and \( \sigma_k = \{i | s_i(k) > 0\} \subseteq [m] \). The resulting system with this schedule is

\[
x(k + 1) = Ax(k) + \sum_{i \in \sigma_k} s_i(k) b_i u_i(k), \quad k \in \mathbb{Z}_+
\]

(10)

where \( s_i(k) \geq 0 \) shows the strength of the \( i \)-th control input at time \( k \). The controllability Gramian \( W_s(t) \) at time \( t \) for this system can be rewritten as

\[
W_s(t) = \sum_{k=0}^{t-1} \sum_{j \in \sigma_k} s_j^2(k) (A^{t-k-1}b_j) (A^{t-k-1}b_j)^\top.
\]

(11)

Our goal is to reduce the number of active actuators on average \( d \), where

\[
d := \frac{\sum_{k=0}^{t-1} \text{card} \{\sigma_k\}}{t},
\]

(12)

such that the controllability Gramian of the fully actuated and the new sparsely actuated system are “close.” Of course, this approximation will require horizon lengths that are potentially longer than the dimension of the state. The definition below formalizes this approximation.

**Definition 2:** Given a time horizon \( t \geq n \), system \( \{1\} \) with a weighted actuator schedule is \((\epsilon, d)\)-approximation of system \( \{1\} \) if and only if

\[
(1 - \epsilon) W(t) \preceq W_s(t) \preceq (1 + \epsilon) W(t),
\]

(13)

where \( W(t) \) and \( W_s(t) \) are the controllability Gramian matrices of \( \{1\} \) and \( \{1\} \), respectively, and parameter \( d \) is defined by (12) as the average number of active actuators, and \( \epsilon \in (0, 1) \) is the approximation factor.

**Remark 2:** While it might appear that allowing for the choice of \( s_i(k) \) might lead to amplification of input signals, we note that the scaling cannot be too large because the approximation is two-sided. Specifically, by taking the trace from both sides of (13), we can see that the weighted summation of \( s_i^2(k) \)'s is bounded. Moreover, based on Definition 2 the ranks of matrices \( W(t) \) and \( W_s(t) \) are the same. Thus, the

| Optimality-criteria | Systemic Controllability Measure | Matrix Operator Form |
|---------------------|---------------------------------|---------------------|
| A-optimality        | Average control energy          | Trace(\( W^{-1}(t) \)) |
| D-optimality        | The volume of the ellipsoid     | (det \( W(t) \))^{-1/n} |
| T-optimality        | Inverse of the trace            | 1/Trace(\( W(t) \)) |
| E-optimality        | Inverse of the minimum eigenvalue | \( 1/\lambda_{\min}(W(t)) \) |
| V-optimality        | Average variance                | Trace(\( C^{-1}(t)W(t)^{-1}C(t) \)) |
| G-optimality        | Maximum entry in the diagonal   | max diag \( C^{-1}(t)W(t)^{-1}C(t) \) |

**TABLE 1:** Some important examples of systemic controllability metrics. For V- and G- optimalities, matrix \( C(t) \) is the design pool; which in our case is the full controllability matrix.
resulting $(\epsilon, d)$-approximation remains controllable (recall that we assume that the original system is controllable).

Remark 3: The results presented in this paper also work for the case of linear time-varying systems, and it is straightforward to extend them for nonlinear discrete-time systems as well.

Existence Results: The next theorem uses results from the graph sparsification literature to prove the existence of a sparse actuator set for a given linear system.

Theorem 1: Given the time horizon $t \geq n$, model $\{1\}$, and $d > 1$, there exists an actuator schedule such that the resulting system $\{10\}$ is a $(\epsilon, d)$-approximation of $\{1\}$ with $\epsilon = \frac{2}{\sqrt{\frac{dt}{n}} + \sqrt{\frac{t}{n}}}$.

Proof: The controllability Gramian of $\{1\}$ at time $t$ is given by

$$W(t) = \sum_{i=0}^{t-1} \sum_{j=1}^{m} (A^i b_j)(A^i b_j)^T$$

By multiplying $W^{-\frac{1}{2}}(t)$ on both sides of $14$, it follows that

$$I = \sum_{i=0}^{t-1} \sum_{j=1}^{m} (W^{-\frac{1}{2}}(t) A^i b_j)(W^{-\frac{1}{2}}(t) A^i b_j)^T$$

We now apply $\{28\}$, which shows that there exist scalars $\bar{s}_{ij} \geq 0$ with

$$\text{card} \{ (i, j) : i + 1 \in [t], j \in [m], \bar{s}_{ij} > 0 \} \leq \frac{dt}{n} \times n, \quad (16)$$

such that

$$I \leq \sum_{i=0}^{t-1} \sum_{j=1}^{m} \bar{s}_{ij} \bar{v}_{ij} \bar{v}_{ij}^T \leq \left( \frac{\sqrt{\frac{dt}{n}} + 1}{\sqrt{\frac{dt}{n}} - 1} \right)^2 I,$$

or equivalently,

$$W(t) \leq \sum_{i=0}^{t-1} \sum_{j=1}^{m} \bar{s}_{ij} \bar{v}_{ij} \bar{v}_{ij}^T \leq \left( \frac{\sqrt{\frac{dt}{n}} + 1}{\sqrt{\frac{dt}{n}} - 1} \right)^2 W(t). \quad (17)$$

We write the controllability Gramian of $\{10\}$ at time $t$ as

$$W_s(t) = \sum_{i=0}^{t-1} \sum_{j=1}^{m} s_j^2(t - i - 1)(A^i b_j)(A^i b_j)^T$$

where $s_j(t - i - 1) := \sqrt{\bar{s}_{ij}(1 - \epsilon)}$. We get

$$W(t) \leq \left( \frac{1}{1 - \epsilon} \right) W_s(t) \leq \left( \frac{1 + \epsilon}{1 - \epsilon} \right) W(t). \quad (19)$$

Finally, using $\{19\}$, $\{16\}$, and Definition $2$, we obtain the desired result.

Tradeoffs: Theorem $1$ illustrates a tradeoff between the average number of active actuators $d$ and the time horizon $t$ (also known as the time-to-control). This implies that the reduction in the average number of active actuators comes at the expense of increasing time horizon $t$ in order to get the same approximation factor $\epsilon$. Moreover, the approximation becomes more accurate as $t$ and $d$ are increased. Of course, increasing $d$ will require more active actuators and larger $t$ requires a larger control time window.

Fig. $1$ depicts the approximation ratio $\epsilon$ given by Theorem $1$ versus the average number of active actuators $d$ and the normalized time horizon $t/n$. We note that the approximation factor improves as $t$ becomes larger than $n$. Moreover, because of $\frac{2}{x + \epsilon} \leq 1$ for $x > 0$, the approximation factor $\epsilon = \frac{2}{\sqrt{\frac{dt}{n}} + \sqrt{\frac{t}{n}}}$ is always less than or equal to one. Hence, the upper bound ratio in $\{13\}$ is at most two.

Construction Results: The next theorem constructs a solution for the sparse weighted actuator schedule problem in polynomial time.

Let us define

$$\epsilon := \frac{2}{\sqrt{\frac{dt}{n}} + \sqrt{\frac{t}{n}}},$$

and

$$s_j(t - i - 1) := \sqrt{\bar{s}_{ij}(1 - \epsilon)}.$$
Algorithm 2: A deterministic greedy-based algorithm to construct a sparse weighted actuator schedule (Theorem 2).

Input: \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, t \) and \( d \)

Output: \( s_i(k) \geq 0 \) for \((i, k + 1) \in [m] \times [t] \)

1. \( C(t) := [B \ AB \ A^2 B \ \cdots \ A^{t-1} B] \)
2. Set \( V = (C(t)C^T(t))^{-\frac{1}{2}} C(t) \)
3. Set \( U = V \)
4. Run \([c_1, \cdots, c_{mt}] = \text{DualSet}^*(V, U, dt)\)
5. return \( s_i(k) := \sqrt{c_{i+mk}/(1 + \frac{n}{dt})} \) for \((i, k + 1) \in [m] \times [t] \)

Algorithm 3: A deterministic greedy-based algorithm to construct a sparse weighted actuator schedule (Theorem 3).

Input: \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, t \) and \( d \)

Output: \( s_i(k) \geq 0 \) for \((i, k + 1) \in [m] \times [t] \)

1. \( C(t) := [B \ AB \ A^2 B \ \cdots \ A^{t-1} B] \)
2. Set \( V = (C(t)C^T(t))^{-\frac{1}{2}} C(t) \)
3. Set \( U = \begin{bmatrix} e_1, \cdots, e_{mt} \end{bmatrix} = \text{DualSet}^*(V, U, dt) \)
4. Run \([c_1, \cdots, c_{mt}] = \text{DualSet}^*(V, U, dt)\)
5. return \( s_i(k) := \sqrt{c_{i+mk}} \) for \((i, k + 1) \in [m] \times [t] \)

Theorem 2: Given the time horizon \( t \geq n, \) model \([12]\), and \( d > 1, \) Algorithm 2 deterministically constructs an actuator schedule such that the resulting system \([16]\) is a \((\epsilon, d)\)-approximation of \([12]\) with \( \epsilon = \frac{\sqrt{n}}{\sqrt{d^2} + \sqrt{n}} \) in at most \( O(drn(t)n^2) \) operations.

Proof: Similar to the proof of Theorem 1, we get equality \([15]\). Using that, we define \( U := \{\bar{e}_j | i + 1 \in [t], j \in [m]\} \) and \( V := U. \) According to \([9, 14, \) and \([15]\), elements of \( U \) are the columns of matrix \((C(t)C^T(t))^{-\frac{1}{2}} C(t).\) We now apply Lemma 1, which shows that there exist scalars \( \bar{c}_{ij} \geq 0 \) with \( \text{card} \{(i, j) : i + 1 \in [t], j \in [m], \bar{c}_{ij} > 0\} \leq \frac{dt}{n} \times n, \) \( (20) \) such that \( \left(1 - \sqrt{\frac{n}{dt}} \right)^2 I \geq \sum_{i=0}^{t-1} \sum_{j=1}^{m} \bar{c}_{ij} \bar{v}_{ij} \bar{v}_{ij}^T, \)

and \( \sum_{i=0}^{t-1} \sum_{j=1}^{m} \bar{c}_{ij} \bar{v}_{ij} \bar{v}_{ij}^T \leq \left(1 + \sqrt{\frac{n}{dt}} \right)^2 I, \)

or equivalently, \( \left(1 - \sqrt{\frac{n}{dt}} \right)^2 \mathcal{W}(t) \leq \sum_{i=0}^{t-1} \sum_{j=1}^{m} \bar{c}_{ij} \bar{v}_{ij} \bar{v}_{ij}^T \leq \left(1 + \sqrt{\frac{n}{dt}} \right)^2 \mathcal{W}(t). \) \( (21) \)

We can of course write the controllability Gramian of \([10]\) at time \( t \) as \( \mathcal{W}_s(t) = \sum_{i=0}^{t-1} \sum_{j=1}^{m} s_{ij}^2 (t - i - 1) (A^i b_j)(A^i b_j)^T \)

\( \mathcal{W}_s(t) = \sum_{i=0}^{t-1} \sum_{j=1}^{m} s_{ij}^2 (t - i - 1) v_{ij} v_{ij}^T. \)

Define \( \epsilon := \frac{\sqrt{2}}{\sqrt{n^2 + \sqrt{n^2}}}, \) and \( s_{ij} (t - i - 1) := \sqrt{\bar{c}_{ij}/(1 + \frac{n}{dt})}. \) \( (22) \)

Then, by substituting \( (1 + \frac{n}{dt}) \mathcal{W}_s(t) \leq (1 + \epsilon) \mathcal{W}(t). \) \( (23) \)

Finally, using \([22, 20, \) and Definiton 2, we obtain the desired result. Moreover, this algorithm runs in \( d \) \( t \) iterations; In each iteration, the functions \( \mathcal{U} \) and \( \mathcal{L} \) are evaluated at most \( nt \) times. All \( mt \) evaluations for both functions need at most \( O(n^3 + mtn^2) \) time, because for all of them the matrix inversions can be calculated once. Finally, the updating step needs an additional \( O(n^2) \) time. Overall, the complexity of the algorithm is of the order \( O(dn^2 t^2). \)

Remark 4: For a given \( d \geq 1, \) while choosing \( d n \) columns of the controllability matrix that form a full row rank matrix (i.e., the system is controllable) is an easy task but finding \( d n \) columns of the controllability matrix that approximate the full Gramian matrix is what we are interested in here. To do so, we should note that approximating the full Gramian matrix while keeping the number of active actuators less than a constant \( d \) at each time is not possible in general. For example, in the case that \( A = 0_{n,n} \) and \( B = I_n, \) at least all actuators at time \( k = 0 \) are needed to form a full row rank matrix (or to approximate the full Gramian matrix). However, as we mentioned earlier, the number of active actuators on average can be kept constant in order to approximate the full Gramian matrix. Furthermore, condition \( dt \geq n \) is needed for any algorithm that has a hope of success. Indeed, taking \( B = I_n \) and \( A = I_n, \) it is straightforward to see that if \( dt < n, \) then we cannot hope to approximate the controllability Gramian because the controllability matrix of any schedule with \( d \) active actuators on average is not full rank.

A. Sparse Actuator Schedules with Energy Constraints

In this subsection, based on the energy/budget constraints on the scalings \( s_i(k)'s \) where \( i \in [m] \) and \( k + 1 \in [t]; \) three cases are considered as follows.
on average at most $d$ active actuators, and the following

$$\rho(W_s(t)) \leq \left( 1 - \sqrt{\frac{n}{dt}} \right)^{-2} \rho(W(t))$$

holds for all systemic controllability measures. Moreover, the sum of scaling ratios for all inputs is bounded by

$$\max_{k=0}^{t-1} \sum_{i=0}^{m} s_i^2(k) \leq \gamma,$$

where $\gamma = t \left( 1 + \sqrt{\frac{t}{m}} \right)^2$.

Proof: The proof is a simple variation on the proof of Theorem 2 and is not repeated here.

Theorem 5: Given the time horizon $t \geq n$, model (1), and $d > 1$, Algorithm 5 deterministically constructs an actuator schedule for (10) in $O(\text{dm}(tn)^2)$ operations such that it has on average at most $d$ active actuators, and the following

$$\rho(W_s(t)) \leq \left( 1 - \sqrt{\frac{n}{dt}} \right)^{-2} \rho(W(t))$$

holds for all systemic controllability measures. Moreover, the sum of scaling ratios at each time is bounded by

$$\max_{k=0}^{t-1} \sum_{i=0}^{m} s_i^2(k) \leq \gamma,$$

where $\gamma = m \left( 1 + \sqrt{\frac{m}{t}} \right)^2$.

Proof: The proof is a simple variation on the proof of Theorem 2 and is not repeated here.

VI. AN UNWEIGHTED SPARSE ACTUATOR SCHEDULE

In the previous section, we allowed for re-scaling of the input to come up with a sparse approximation of the Gramian. Here, we assume that the actuator/signal strength cannot be arbitrarily set for individual active actuators and only can be 0
or 1. Given a time horizon $t \geq n$, our problem is to compute an actuator schedule $\sigma = \{\sigma_k\}_{k=0}^{t-1}$ where $\sigma_k \subset [m]$ for the system (1), i.e.,

$$x(k + 1) = Ax(k) + \sum_{i \in \sigma_k} b_i u_i(k), \ k \in \mathbb{Z}_+.$$  \hfill (24)

As before, the controllability Gramian at time $t$ for schedule (24) is given by

$$\mathcal{W}_\sigma(t) := \sum_{i=0}^{t-1} \sum_{j \in \sigma_i} (A^{t-i-1} b_j)(A^{t-i-1} b_j)^\top.$$  \hfill (25)

Optimal actuator selection can now be formulated as a combinatorial optimization problem. We consider both static and dynamic actuator schedules, corresponding to time-invariant and time-varying input matrices.

1. **The Static Scheduling Problem**: In this case, all sets $\sigma_i \subset [m]$ for $i + 1 \in [t]$ are identical, which means we keep the same schedule at every point in time for the whole time horizon $t$:

$$\min_{\sigma \in \mathcal{S}(m,d)} \rho \left( \sum_{i=0}^{t-1} \sum_{j \in \sigma_i} (A^i b_j)(A^i b_j)^\top \right),$$  \hfill (26)

where

$$\mathcal{S}(m,d) := \{\sigma : \sigma \subset [m], \text{card}(\sigma) \leq d\},$$  \hfill (27)

where $d$ is the number of active actuators at each time, and $m$ is the total number of actuators.

2. **The Time-varying Scheduling Problem**: In this case, the optimal dynamic strategy is given as:

$$\min_{\{\sigma_i\}_{t-1}^0 \in \mathcal{S}(m,d,t)} \rho \left( \sum_{i=0}^{t-1} \sum_{j \in \sigma_i} (A^{t-i} b_j)(A^{t-i} b_j)^\top \right),$$  \hfill (28)

where

$$\mathcal{S}(m,d,t) := \left\{\{\sigma_i\}_{i=0}^{t-1} : \sigma_i \subset [m], \sum_{i=0}^{t-1} \text{card}(\sigma_i) \leq td \right\},$$  \hfill (29)

and $d$ is the average number of active actuators at each time, i.e., $d = \sum_{i=0}^{t-1} \text{card}(\sigma_i)/t$, where $t$ is a time horizon, and $m$ is the total number of actuators.

This optimal actuator selection problem also can be formulated as follows:

$$\text{minimize } s_i(k)$$  \hfill (30)

subject to:

$$s_i(k) \in \{0, 1\} \text{ for all } (i, k + 1) \in [m] \times [t],$$  

$$\sum_{i=1}^{m} \sum_{k=0}^{t-1} s_i(k) \leq dt.$$  

The exact combinatorial optimization problems (26) and (30) are intractable and NP-hard optimization problems; however, it is straightforward to solve a continuous relaxation of these optimization problems because of the convexity property in Definition (1) To find a near-optimal solution of optimization problems (26) and (30), one can use a variety of standard methods for optimal experimental design (greedy methods, sampling methods, the classical pipage rounding method combined with SDP). Specifically, in the case of submodular systemic controllability measures (e.g., D- and T-optimality), the classical rounding method (e.g., pipage and randomized rounding) combined with SDP relaxation methods with computationally fast algorithms with a constant approximation ratio (38). These approaches are not applicable to non-submodular systemic measures, such as A-, E-, V- or G-optimality (25), (40).

In the following result, we use a result based on regret minimization of the least eigenvalues of positive semi-definite matrices (cf. (25)) to obtain a constant approximation ratio for all systemic controllability metrics.

**Theorem 6**: Assume that time horizon $t \geq n$, dynamics (1), systemic controllability metric $\rho : \mathbb{S}^n_{++} \rightarrow \mathbb{R}$, and $d > 2$ are given. Then there exists a polynomial-time algorithm which computes a schedule $\hat{\sigma} = \{\hat{\sigma}_i\}_{i=0}^{t-1}$ that satisfies

$$\rho(\mathcal{W}_\hat{\sigma}(t)) \leq \gamma \left( \frac{dt}{n} \right) \min_{\{\sigma_i\}_{i=0}^{t-1} \in \mathcal{S}(m,d,t)} \rho(\mathcal{W}(t)),$$

where $\gamma(dt/n)$ is a positive constant depending only on $dt/n$.

**Proof**: The proof is a simple variation on the proof of (25) thm. 1.1, and is not repeated here.

Next, we use the results from Section V to obtain an unweighted sparse actuator schedule with guaranteed performance bound.

**Theorem 7**: Assume that time horizon $t \geq n$, dynamics (1), and $d > 1$ are given. Then polynomial-time Algorithm (2) deterministically constructs an actuator schedule for (10) with $s_i(k) \in \{0, 1\}$ such that it has an average at most $d$ active
actuators, and the following
\[
\rho(W_s(t)) \leq \left( \frac{1 + \sqrt{\frac{\pi}{2}}}{1 - \sqrt{\frac{\pi}{2}}} \right)^2 \rho(W(t)),
\]
holds for all systemic controllability measures.

**Proof:** The proof is a simple variation on the proof of Theorem \[2\] and is not repeated here.

In view of this result, one can choose any constant number greater than one as the number of active actuators on average to construct a sparse unweighted actuator schedule in order to approximate controllability measures. This, however, comes at the cost of an extra \((1 + \sqrt{\frac{\pi}{2}})^2\) factor in terms of the energy cost compared to the weighted sparse actuator schedule (cf. Theorem \[3\]).

Alternatively, one can use the solution of the Kadison-Singer problem which can be cast as follows:

**Conjecture 1 (Kadison-Singer):** There are universal constants \(\epsilon > 0\), \(\delta > 0\), and \(r \in \mathbb{N}\) for which the following statement holds. If \(v_1, \cdots, v_m \in \mathbb{R}^n\) satisfy \(\|v_i\|_2 \leq \delta\) for all \(i\) and
\[
\sum_{i=1}^m v_i v_i^\top = I,
\]
then, there is a partition \(X_1, \cdots, X_r\) of \([m]\) for which
\[
\left\| \sum_{i \in X_j} v_i v_i^\top \right\| \leq 1 - \epsilon,
\]
for every \(j \in [r]\).

Originally posed in a different form in the functional analysis literature in the 1950s, the conjecture was recently solved by Marcus, Spielman, and Srivastava \[30\], using similar techniques to earlier work on sparsification \[28\]. However, the proof is only existential and not constructive.

**Theorem 8 (Marcus-Spielman-Srivastava):** Given a set of vectors \(v_1, \cdots, v_m \in \mathbb{R}^n\) in isotropic position (i.e., \(\sum_{i=1}^m v_i v_i^\top = I\)), if \(\max_{i \in [m]} \|v_i\|_2^2 \leq \epsilon\) then there is a two-partitioning \(S_1, S_2\) of \([m]\) such that for each \(j \in \{1, 2\}\),
\[
\frac{1}{2} - \mathcal{O}(\sqrt{\epsilon}) \leq \left\| \sum_{i \in S_j} v_i v_i^\top \right\| \leq \frac{1}{2} + \mathcal{O}(\sqrt{\epsilon}).
\]

We can therefore use the above result to prove the existence of a sparse actuator schedule that does not require rescaling. The following result is a direct corollary of Theorem \[8\].

**Corollary 1:** Assume that time horizon \(t \geq n\), dynamics \[1\], and parameter \(d \geq 1\) are given. Then there exists an actuator schedule \(\sigma = \{\sigma_t\}_{t=1}^T\) such that for \(\sigma\), its complement \(\bar{\sigma}\) (i.e., \(\bar{\sigma} := \{[m]\setminus\sigma\}_{t=1}^T\)), and any systemic metric \(\rho: S^n_+ \to \mathbb{R}_+\), we have
\[
\left| \frac{\rho\left(\frac{1}{2}W(t) - \rho(W_{\sigma}(t))\right)}{\rho\left(\frac{1}{2}W(t)\right)} \right| \leq \mathcal{O}(\sqrt{\epsilon}),
\]
and
\[
\left| \frac{\rho\left(\frac{1}{2}W(t) - \rho(W_{\bar{\sigma}}(t))\right)}{\rho\left(\frac{1}{2}W(t)\right)} \right| \leq \mathcal{O}(\sqrt{\epsilon}),
\]
where \(\epsilon\) is the maximum leverage score of inputs (i.e.,
\[
\epsilon = \max_{i+1 \in [1], j \in [m]} \ell(A^i b_j),
\]
see \[37\] in Section \[VII\].

**Proof:** Here we show how the sparse actuator selection problem can be cast as the Kadison-Singer theorem given above. We first define vector \(\bar{v}_{ij}\) for \(i + 1 \in [t]\) and \(j \in [m]\) as follows
\[
\bar{v}_{ij} = \mathcal{W}^{-\frac{1}{2}}(t) A^i b_j,
\]
where \(\mathcal{W}(t)\) is given by \[3\] and \(b_j\) is the \(j\)-th column of matrix \(B \in \mathbb{R}^{n \times m}\). Then, using \[3\] and \[33\], it follows that
\[
I = \sum_{i=0}^{t-1} \sum_{j=1}^m \bar{v}_{ij} \bar{v}_{ij}^\top.
\]
Based on Theorem \[8\] \((i, j) : i + 1 \in [t], j \in [m]\) can be partitioned into two sets \(S_1\) and \(S_2\) such that
\[
\left(\frac{1}{2} - \mathcal{O}(\sqrt{\epsilon})\right) I \preceq \sum_{(i, j) \in S_a} \bar{v}_{ij} \bar{v}_{ij}^\top \preceq \left(\frac{1}{2} + \mathcal{O}(\sqrt{\epsilon})\right) I,
\]
where \(a \in \{1, 2\}\) and \(\epsilon = \max_{i+1 \in [1], j \in [m]} \|\bar{v}_{ij}\|^2\). Accordingly, we now partition the columns of \(\mathcal{C}(t)\) into two sets \(S_1\) and \(S_2\) such that
\[
\mathcal{W}(t) \preceq \mathcal{W}_{S_a}(t) \preceq \left(\frac{1}{2} + \mathcal{O}(\sqrt{\epsilon})\right) \mathcal{W}(t),
\]
where \(a \in \{1, 2\}\), and \(\epsilon\) is given by
\[
\epsilon = \max_{i+1 \in [1], j \in [m]} \|\bar{v}_{ij}\|^2 = \max_{i+1 \in [1], j \in [m]} \bar{v}_{ij}^\top \bar{v}_{ij} = \max_{i+1 \in [1], j \in [m]} (A^i b_j)^\top \mathcal{W}^{-1}(t) A^i b_j = \max_{i+1 \in [1], j \in [m]} \ell(A^i b_j).
\]
Moreover, we have
\[
\mathcal{W}(t) = \mathcal{W}_{S_1}(t) + \mathcal{W}_{S_2}(t).
\]

Then, according to the monotonicity property in Definition \[1\] and \[36\], we get \[31\] and \[32\].

In view of this result, one can take any time-varying actuator schedule and split it into two sets, each of which provides a crude approximation of the Gramian in terms of its spectrum. By recursively dividing each set further into two sets, we can repeat this procedure until the desired number of inputs or the desired approximation factor is reached.

We use a different idea in Section \[VII\] to develop scalable algorithms that sparsify control inputs by employing a subsampling method for a time-varying actuator schedule. This however come at the cost of an extra log factor in terms of
the average number of selected actuators.

VII. SAMPLING BASED ON THE LEVERAGE SCORE

In this part, we focus on a computationally tractable method for the weighted sparse actuator scheduling problem that achieve near optimal solution.

Definition 3: The leverage score of the i-th column of matrix $P \in \mathbb{R}^{n \times m}$ is defined as
$$
\ell_i = p_i^T (PP^T)^{-1} p_i,
$$
where $p_i$ is the i-th column of matrix $P$.

This quantity encodes the importance of the i-th column compared to the other columns. A larger leverage score shows that the corresponding column has more influence on the spectrum of $P$. Based on the leverage score definition, we get $\ell_i \in [0,1]$ for all $i \in [m]$. Because $\ell_i$’s are the diagonal elements of the projection matrix $P^T (PP^T)^{-1} P$ and the diagonal elements of the projection matrix are between zero and one. Hence score $\ell_i = 1$ means that the i-th column has a component orthogonal to the rest of the columns. Therefore, eliminating that column will decrease the rank of matrix $P$. On the other hand, $\ell_i = 0$ means that the i-th column is parallel to the rest of the columns. When the corresponding matrix is the graph Laplacian, this quantity reduces to the effective resistance of each link in a graph $[27]$.

We group the columns of $C(t)$ in the following form
$$
C(t) = \begin{bmatrix}
    [b_1 \ Ab_1 \ldots A^{t-1}b_1] & 
    \ldots 
    & 
    [b_m \ Ab_m \ldots A^{t-1}b_m]
\end{bmatrix},
$$
where $b_j$ is the j-th column of matrix $B$. Matrix $C_j(t)$ presents the controllability matrix of input $j$ at time $t$. The leverage score for each column of $C(t)$ is defined as
$$
\ell(A^jb_j) = (A^jb_j)^T (C(t)C(t)^T)^{-1}A^jb_j,
$$
where $(i+1) \in [t]$ and $j \in [m]$. For these scores, we have
$$
\sum_{i=0}^{t-1} \sum_{j=1}^{m} \ell(A^jb_j) = \text{Trace} (C(t)^T C(t) C(t)^T C(t))^T C(t))
$$
$$
= \sum_{i=1}^{n} \lambda_i (C(t) C(t)^T (C(t))^T C(t))
$$
$$
= \text{rank} (C(t)^T) C(t)^T (C(t))^T C(t))
$$
$$
= \text{rank} (C(t)) = n,
$$
where $\lambda_i$’s are eigenvalues of matrix $C^T(t) C(t) C(t)^T C(t)$. In (38), we use the fact that $C^T(t) C(t) C(t)^T C(t)$ is a projection matrix, and $\text{rank}(C(t)) = n$ (i.e., the system is controllable).

We now randomly sample the actuators with probabilities proportional to their leverage scores to sparsify control inputs. This sampling occurs across time and over all possible actuators at each time (see Algorithm 7). At every time, each actuator is kept active or inactive according to probability $\ell(A^jb_j)/n$ where $(i + 1) \in [t]$ and $j \in [m]$. Using [27] Thm. 1, we can construct a sampling strategy that utilizes the leverage score to probabilistically choose actuators. The catch is that there is an extra $\log n$ factor in the average number of selected actuators, and potentially different actuators are chosen at different times.

Theorem 9: Assume that dynamics (1), time horizon $t \geq n$, and approximation factor $\epsilon \in [1/\sqrt{n}, 1]$ are given. Choose a real number $d$ of order $n \log n / \epsilon^2$. Then, Algorithm 7 produces scheduling (10) which is $(\epsilon, d)$-approximation of (11) with probability of at least 0.5.

Proof: The structure of the proof follows from the proof of [27] Thm. 4. Let us start with the following projection matrix
$$
\Pi = C(t)^T W^{-1}(t) C(t),
$$
where $C(t)$ is n-by-tm controllability matrix (2) and matrix $W(t) = C(t)^T C(t)$ is given by (3). The tm-by-tm matrix $\Pi$ is a projection matrix and has eigenvalue at 0 with multiplicity $t \times m - n = n$ and eigenvalue at 1 with multiplicity $n$. We use a concentration lemma to prove this theorem. Therefore, first we need to translate our problem to have $M = \mathcal{O}(n \log n / \epsilon^2)$ independent samples drawn from a probability distribution $\pi$ over set $X$. The set $X$ is obtained based on columns of $\Pi$. However, we need to rescale columns of $\Pi$ to guarantee that $\mathbb{E}(y^T y) \leq 1$ where vector $y$ is an event in $X$ (we need this for the concentration lemma). Based on (39), each column of $\Pi$ corresponds to $(i,k) \in [m] \times [t]$. If we rescale each column of $\Pi$ with its corresponding $(\pi(i,k))^{-1/2}$ where $(i,k) \in [m] \times [t]$, then we can easily see that the expected value is exactly one. Assume $(i,k)$ corresponds to the j-th column of $\Pi$. Then $\pi_j := \pi(i,k)$ is the probability of selecting j-th column. Therefore, we get the desired probability distribution, selecting $y = (\pi_j)^{-1/2} \Pi(:,j)$ with probability $\pi_j$ where $j \in [mt]$. Then by the concentration lemma of Rudelson [29] Thm. 3.1 and
Markov’s inequality, with probability of at least 0.5, we get
\[ \|\Pi - \Pi\Pi\|_2 \leq \epsilon, \tag{40} \]
where \( \Gamma \) is a non-negative diagonal matrix (with weights \( s_i^2(k) \)) on its diagonal such that \( W_s(t) = C(t) \Gamma C^\top(t) \). Then, it is straightforward to show that for every systemic controllability measure \( \rho : \mathbb{S}_+^n \to \mathbb{R}_+ \), we have
\[
\left| \rho(W(t)) - \rho(W_s(t)) \right| \leq \epsilon \rho(W(t))
\]
in which \( W_s(t) = C(t) \Gamma C^\top(t) \). Based on \cite{27} Lemma 4], the inequality (40) is equivalent to
\[
\sup_{x \in \mathbb{C}^m} \frac{|x^\top (\Pi - \Pi\Pi)x|}{x^\top x} \leq \epsilon. \tag{41}
\]
Since we have \( \text{Im}\{C^\top(t)\} \subset \mathbb{R}^{nt} \), it follows that
\[
\sup_{x \neq 0} \frac{|x^\top (\Pi - \Pi\Pi)x|}{x^\top x} \leq \epsilon.
\]
Let us define \( x = C^\top(t)x' \). Then, we rewrite (41) as follows
\[
\sup_{x \in \mathbb{C}^m} \frac{|x'^\top (W(t) - W_s(t))x'|}{x'^\top W(t)x'} \leq \epsilon. \tag{42}
\]
As a result, it follows that
\[
\sup_{x \in \mathbb{C}^m} \frac{|x'^\top (W(t) - W_s(t))x'|}{x'^\top W(t)x'} \leq \epsilon, \tag{43}
\]
which implies that
\[
(1 - \epsilon)W(t) \preceq W_s(t) = C(t)\Gamma C^\top(t) \preceq (1 + \epsilon)W(t). \tag{44}
\]
Finally, using (44) and Definition 2] we conclude the desired result.

This result shows that with a simple randomized sampling strategy, one can choose on average less than \( O(\log n/\epsilon^2) \) number of actuators at each time, to approximate any of the controllability metrics when \( t = n \). Moreover, this result shows that it is possible to have a time-varying actuator schedule with a constant number of active actuators on average over a time horizon a little longer than \( n \) (i.e., \( t = O(n \log n) \)) via random sampling. According to Theorem 3, the average number of active actuators can be reduced to \( O(1/\epsilon^2) \), at the expense of either solving SDPs \cite{26} or greedily handling certain eigenvalue bounds (see Algorithm 2). Algorithm 7] is conceptually simpler than Algorithm 2 and the SDP-based algorithm presented in \cite{26}, which provide \( d = O(1/\epsilon^2) \) in \( O(mn^2/\epsilon^2) \) and \( O(\log n/\epsilon^2) \) time, respectively.

The concept of a leverage score for each column can be generalized to a group of columns as follows
\[
\ell_{C_i} = \text{Trace} \left( C_i^\top(t) C(t) C^\top(t) \right) C_i(t). \tag{45}
\]
Using group leverage scores, one can also use a greedy heuristic algorithm to obtain an approximation solution for the static scheduling problem. We note that the problem of approximation of the controllability Gramian with a sparse, static actuator set is considerably more challenging as it doesn’t lend itself to a sampling-based strategy: any choice made at one time has to be consistent with the next.

When using a time-varying schedule, the contribution of each actuator to the Gramian at each time is a rank-one matrix. Therefore, we can use the machinery developed for the Kadison-Singer conjecture to find a sparse subset of actuators over time to approximate the (potentially very large) sum of rank-one matrices. In the static case, however, the choices of actuators at different times are all the same. As a result, the Gramian can be written as a sum of positive semi-definite matrices corresponding to the selected actuators at each time. Finding a sparse approximation in this case would require a generalization of the Kadison-Singer conjecture from sums of rank-one to sums of higher ranked positive semidefinite matrices. Such a result has remained elusive as of yet.

**VIII. Numerical Examples**

In this section, we consider three numerical examples to demonstrate the results.

We compare our results with a greedy heuristic that sequentially picks control inputs to maximize the systemic metric decrease of the controllability matrix (see Algorithm 8). The selected inputs are active at all times. It is shown that the greedy method works well and matches the inapproximability

\begin{algorithm}[h]
\caption{A greedy heuristic for given \( \rho(.) \) which sequentially picks inputs \( \text{GreedyStatic}(A, B, t, d) \).}
\begin{algorithmic}[1]
\State \( W_s := 0 \times 0 \)
\For {\( k = 1 \) to \( d \)}
\State \( j \leftarrow \text{find a column of } B \text{ that returns the maximum value for } \rho(W_s + \sum_{i=0}^{t-1} A^i B(:,j) B(:,j)^\top (A^i)^\top + \alpha I_n) \) \hfill // \( \alpha > 0 \) is sufficiently small to avoid singularity
\State \( B_s \leftarrow [B_s, B(:,j)] \)
\State \( W_s = \sum_{i=0}^{t-1} A^i B_s B_s^\top (A^i)^\top \)
\State \( B(:,j) \leftarrow [ ] \)
\EndFor
\State \( B_s, \rho(W_s) \)
\end{algorithmic}
\end{algorithm}
Fig. 2: Six unweighted actuator schedules for Example 1: (a) all actuators are active at time 7 (b) actuator one is active at each time (c) the schedule is obtained Algorithm 6 (d) three actuators are active at all time and each actuator is used three times (e) three fixed actuators \{1, 2, 8\} are active at all time (f) the proposed sparse schedule based on Algorithm 6 with less than two active actuators at each time on average. The color of element \((i,k)\) is red when \(s_i(k) = 1\) and white otherwise where \(i \in [8], k + 1 \in [8]\) and \(s_i(k) \in \{0, 1\}\). For Figs. 2(c)&(f), which are obtained based on Algorithm 6, we can observe that the actuator schedule has procrastination in actuator activations (i.e., more active actuators at the end of the time horizon); however, in Example 3 we can see “front-loaded” behavior (i.e., more active actuators early in the time horizon) due to different dynamics in this example.

### Algorithm 9: A greedy heuristic for given \(\rho(.)\) which sequentially picks inputs and activation times

**GreedyTimeVarying**\((A,B,t,d)\).

**Input:** \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, t\) and \(d\)

**Output:** \(\rho(W_s)\)

1. \(C := [B \ AB \ A^2B \ \ldots \ A^{t-1}B]\)
2. \(C_s := \mathbf{0}_{n \times mt}\)
3. for \(k = 1\) to \(M := \lceil dt \rceil\) do
   4. \(j \leftarrow \text{find a column of } C \text{ that returns the maximum value for}
      \[
      \rho(W_s + \alpha I_n) - \rho \left( W_s + C(:,j)C(:,j)^\top + \alpha I_n \right)
      \]
      \(/ \alpha > 0 \text{ is sufficiently small to avoid singularity}
   5. \(C_s \leftarrow [C_s, C(:,j)]\)
   6. \(W_s = C_sC_s^\top\)
   7. \(C(:,j) \leftarrow [\ ]\)
4. end
9. return \(\rho(W_s)\)

**Example 1 (\textsuperscript{3})**: Assume that the state space matrices of system (1) are given by

\[
A = \begin{bmatrix}
    1 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{7}{5} \\
    0 & 2 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{5} \\
    0 & 0 & 3 & 0 & 0 & 0 & 0 & -\frac{5}{3} \\
    0 & 0 & 0 & 4 & 0 & 0 & 0 & -\frac{2}{3} \\
    1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\
    1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \\
    0 & 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{3} \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}
\end{bmatrix}
\] (46)

and

\[
B_{\min} = \text{diag} \{1, 1, 0, 0, 0, 0, 0, 1\}
\] (47)

Direct computation shows that choosing (47) makes the system controllable and no diagonal-matrix sparser than \(B_{\min}\) renders \(A\) controllable. For this case \((B = B_{\min})\), the performance is:

\[
\text{Trace} \left( \sum_{i=0}^{n-1} A^iB_{\min}B_{\min}^\top(A^i)^\top \right)^{-1} = 0.503,
\]
TABLE II: The values of controllability performance and average number of active actuators at each time for the unweighted actuator schedule presented in Fig. 2 and based on greedy algorithms 8 and 9. The unweighted schedules presented in Figs. 2(c)&(f) are obtained based on Algorithm 6. It is not possible to greedily select three inputs (active at all time) to make the system in Example 1 controllable.

| d   | Trace ($W^{-1}(n)$) | Algorithm 8 | Algorithm 9 | Fully Actuated |
|-----|---------------------|--------------|--------------|----------------|
| 1   | 0.628               | 0.503        | 0.161        | 0.294          | 0.132          |
| 3   | 1.125               | 1.875        | 3            |                |

TABLE III: The values of controllability performance for three different actuator schedules in Example 2: 1) the weighted actuator schedule in Fig. 4 based on Algorithm 7, 2) the static leader schedule with 160 leaders active at all time, 3) the fully actuated case. To have a fair comparison, we normalize the resulting schedule of Algorithm 7 such that the sum of the scalings satisfies \( \sum_{i=0}^{n-1} \sum_{j=1}^m s_i^j(k) = dn \) where \( d = 40 \). The value of the controllability metric for the materialized result of Algorithm 7 is 18.54, which is much closer to the controllability metric of the fully actuated case.

| d   | Trace ($W^{-1}(n)$) | Static Leader Schedule | Fully Actuated |
|-----|---------------------|------------------------|---------------|
| 40  | 93.64               | 676.68                 | 18.16         |
| 160 | 676.68              | 966.68                 | 18.16         |
| 200 | 18.16               | 18.16                  | 18.16         |

Example 2: Let us consider a dynamic network consisting of \( n = 200 \) agents/nodes, which are randomly distributed in a \( 1 \times 1 \) square-shape area in space and are coupled over a proximity graph. Every agent is connected to all of its spatial neighbors within a closed ball of radius \( r = 0.125 \). Assume that the state space matrices of this network are given by

\[
A = I_n - \frac{1}{n} L, \quad B = I_n, \tag{48}
\]

where \( L \) is the Laplacian matrix of the underlying graph given by Fig. 3. Now, we consider the actuator scheduling problem discussed in Section V. For undirected consensus networks, a similar problem arises in assignment of a pre-specified number of active agents, as leaders, in order to minimize the controllability metric, e.g., the average controllability energy (cf. [41], [42]). In our setup, each leader \( i \) in addition to relative information exchange with its neighbors (based on Laplacian matrix \( L \)), it also has access to a control input \( u_i(.) \). This system is controllable with only a few inputs/leaders.

*The system is not controllable with only one input, because \( A \) does not have distinct eigenvalues [42].
power network, and the links represent the transmission lines between buses. The state space model of the swing equation used for frequency control in power networks can be written as follows:

$$
\begin{bmatrix}
\dot{\theta}(t) \\
\dot{w}(t)
\end{bmatrix}
= 
\begin{bmatrix}
0 & I \\
-M^{-1}L & -M^{-1}D
\end{bmatrix}
\begin{bmatrix}
\theta(t) \\
w(t)
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
M^{-1}
\end{bmatrix} u(t)
$$

where $M$ and $D$ are diagonal matrices with inertia coefficients and damping coefficients of generators and their diagonals, respectively.

We assume that both rotor angle and frequency are available for measurement at each generator. This means each subsystem in the power network has a phase measurement unit (PMU). The PMU is a device that measures the electrical waves on an electricity grid using a common time source for synchronization. The system is discretized to the discrete-time LTI system with state matrices $A$, $B$, and $C$ and the sampling time of 0.2 second (the matrices are borrowed from [45]).

Fig. 5 depicts nine sparse schedules based on the proposed deterministic method (Algorithms 3) for different values of $d$. The sparsity degree of each schedule is captured by $d$. As $d$ increases the number of non-zero scalings (i.e., activations) increases while the controllability metric decreases (improves). Fig. 7 compares the results of Algorithms 3, 6, 8, and 9. The plot presents the values of the average controllability energy (A-optimality) versus the average number of active actuators. To have a fair comparison, we normalize the resulting schedules of all the methods such that the sum of all the scalings satisfies $\sum_{k=0}^{n-1} \sum_{i=1}^{m} s_i^2(k) = nd$.

As one expects, Algorithms 3, 6, and 9 outperform Algorithm 8. One observes that Algorithms 3 and 6 perform nearly as optimal as the time-varying greedy method 9; however, based on our results, we have theoretical guaranteed performance bounds for Algorithms 3 and 6. Furthermore, the usefulness of Algorithms 3 and 6 accentuates itself when the number of active actuators on average is not too small; and potentially can result in a better solution compare to Algorithm 9 (see Fig. 7).

**IX. CONCLUDING REMARKS**

In this paper, we have shown how recent advances in matrix reconstruction and graph sparsification literature can be utilized to develop subset selection tools for choosing a relatively small subset of actuators to approximate certain controllability measures. Current approaches based on polynomial time relaxations of the subset selection problem require an extra multiplicative factor of $\log n$ sensors/actuators times the minimal number in order to just maintain controllability/observability. Furthermore, when the control energy is chosen as the cost, submodularity-based approaches fail to guarantee the performance using greedy methods. In contrast, we show that there exists a polynomial-time actuator schedule that on average selects only a constant number of actuators at each time, to approximate controllability measures. Similar approaches
Fig. 6: Subplots (a)-(i) presents nine weighted sparse schedules for Example 3 based on the proposed deterministic method (Algorithm 3) where $d \in \{1.05, 1.75, 2.30, 3.10, 3.95, 4.60, 5.25, 5.75, 6.35\}$ is the average number of active actuators at each time, respectively. The color of element $(i,k)$ is proportional to the scaling factor $s^2_i(k)$ where $i \in [10]$ and $k + 1 \in [20]$.

Fig. 7: This plot compares four different methods (Algorithms 3, 6, 8 and 9) for obtaining sparse actuator schedules of the 10-machine New England Power System in Example 3. The plot presents the values of average controllability energy (A-optimality) versus the average number of active actuators at each time (d).

results can be developed for the sensor selection problem. A potential future direction is to see whether this approach can be used to develop an efficient scheme for minimal reachability problems.

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