STRONG PEAK POINTS AND STRONGLY NORM ATTAINING POINTS WITH APPLICATIONS TO DENSENESS AND POLYNOMIAL NUMERICAL INDICES

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Abstract. Using the variational method, it is shown that the set of all strong peak functions in a closed algebra $A$ of $C_b(K)$ is dense if and only if the set of all strong peak points is a norming subset of $A$. As a corollary we can induce the denseness of strong peak functions on other certain spaces. In case that a set of uniformly strongly exposed points of a Banach space $X$ is a norming subset of $\mathcal{P}(nX)$, then the set of all strongly norm attaining elements in $\mathcal{P}(nX)$ is dense. In particular, the set of all points at which the norm of $\mathcal{P}(nX)$ is Fréchet differentiable is a dense $G_δ$ subset.

In the last part, using Reisner’s graph theoretic-approach, we construct some strongly norm attaining polynomials on a CL-space with an absolute norm. Then we show that for a finite dimensional complex Banach space $X$ with an absolute norm, its polynomial numerical indices are one if and only if $X$ is isometric to $\ell_\infty^n$.

Moreover, we give a characterization of the set of all complex extreme points of the unit ball of a CL-space with an absolute norm.

1. Introduction and Preliminaries

Let $K$ be a complete metric space and $X$ a (real or complex) Banach space. We denote by $C_b(K : X)$ the Banach space of all bounded continuous functions from $K$ to $X$ with the supremum norm. A nonzero function $f \in C_b(K : X)$ is said to be a strong peak function at $t \in K$ if every sequence $\{t_n\}$ in $K$ with $\lim_n \|f(t_n)\| = \|f\|$ converges to $t$. Given a subspace $A$ of $C_b(K : X)$, a point $t \in K$ called a strong peak point for $A$ if there is a strong peak function $f$ in $A$ with $\|f\| = \|f(t)\|$. We denote by $\rho_A$ the set of all strong peak points for $A$.

Let $B_X$ (resp. $S_X$) be the unit ball (resp. sphere) of a Banach space $X$. A nonzero function $f \in C_b(B_X : Y)$ is said to strongly attain its norm at $x$ if for every sequence $\{x_n\}$ in $B_X$ with $\lim_n \|f(x_n)\| = \|f\|$, there exist a scalar $\lambda$ with $|\lambda| = 1$ and a subsequence of $\{x_n\}$ which converges to $\lambda x$. Given a subspace $A$ of $C_b(B_X : Y)$, $x \in B_X$ is called a strongly norm-attaining point of $A$ if there exists a nonzero function $f$ in $A$ which strongly attain its norm at $x$. Denote by $\bar{\rho}_A$ the set of all strongly norm-attaining points of $A$.

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For complex Banach spaces $X$ and $Y$, we may use the following two subspaces of $C_b(B_X : Y)$:

\[ \mathcal{A}_b(B_X : Y) = \{ f \in C_b(B_X : Y) : f \text{ is holomorphic on the interior of } B_X \} \]

\[ \mathcal{A}_u(B_X : Y) = \{ f \in \mathcal{A}_b(B_X : Y) : f \text{ is uniformly continuous on } B_X \} . \]

We shall denote by $\mathcal{A}(B_X : Y)$ either $\mathcal{A}_b(B_X : Y)$ or $\mathcal{A}_u(B_X : Y)$. In case that $Y$ is the complex field $\mathbb{C}$, we write $\mathcal{A}(B_X)$, $\mathcal{A}_u(B_X)$ and $\mathcal{A}_b(B_X)$ instead of $\mathcal{A}(B_X : \mathbb{C})$, $\mathcal{A}_u(B_X : \mathbb{C})$ and $\mathcal{A}_b(B_X : \mathbb{C})$ respectively.

If $X$ and $Y$ are Banach spaces, an $k$-homogeneous polynomial $P$ from $X$ to $Y$ is a mapping such that there is an $k$-linear continuous mapping $L$ from $X \times \cdots \times X$ to $Y$ such that

\[ P(x) = L(x, \ldots, x) \quad \text{for every } x \in X. \]

$\mathcal{P}(kX : Y)$ denote the Banach space of all $k$-homogeneous polynomials from $X$ to $Y$, endowed with the polynomial norm $\| P \| = \sup_{x \in B_X} \| P(x) \|$. We also say that $P : X \to Y$ is a polynomial, and write $P \in \mathcal{P}(X : Y)$ if $P$ is a finite sum of homogeneous polynomials from $X$ into $Y$. In particular, replace $\mathcal{P}(kX : Y)$ by $\mathcal{P}(kX)$ and $\mathcal{P}(X : Y)$ by $\mathcal{P}(X)$ when $Y$ is a scalar field. We refer to [Din99] for background on polynomials.

The (polynomial) numerical index of a Banach space is a constant relating to the concepts of the numerical radius of functions on $X$. Actually, for each $f \in C_b(B_X : X)$, the numerical radius $v(f)$ is defined by

\[ v(f) = \sup \{|x^* f(x)| : x^*(x) = 1, x \in S_X, x^* \in S_{X^*} \}, \]

where $X^*$ is the dual space of $X$. For every integer $k \geq 1$, the $k$-polynomial numerical index of a Banach space $X$ is the constant defined by

\[ n^{(k)}(X) = \inf \{ v(P) : \| P \| = 1, P \in \mathcal{P}(kX : X) \} . \]

If $k = 1$, in particular, then it is called the numerical index of $X$ and we write $n(X)$. For more recent results about numerical index, see a survey paper [KMP06] and references therein.

Let’s briefly see the contents of the paper. In section 2, using the variational method in [DGZ93], we show that the set $\rho A$ is a norming subset of $A$ if and only if the set of all strong peak functions in $A$ is a dense $G_\delta$ subset of $A$, when $A$ is a closed subspace $C_b(K : X)$ which contains all elements of the form $t \mapsto (x^* f(t))^m x$ for all $x \in X$, $x^* \in X^*$, $f \in A$ and integers $m \geq 1$. Using this theorem and the variational methods, we investigate the denseness of the set of strong peak holomorphic functions and the denseness of the set of numerical strong peak functions on certain Banach spaces.

In section 3, we also apply the variation method to investigate the denseness of the set of strongly norm attaining polynomials when the set of all uniformly strongly exposed
points of a Banach space $X$ is a norming subset of $\mathcal{P}(^nX)$. As a direct corollary, the set of all points at which the norm of $\mathcal{P}(^nX)$ is Fréchet differentiable is a dense $G_δ$ subset if the set of all uniformly strongly exposed points of a Banach space $X$ is a norming subset of $\mathcal{P}(^nX)$.

In the last part, we will use the graph theory to get some strongly norm-attaining points or complex extreme points. Reisner gave a one-to-one correspondence between $n$-dimensional some Banach spaces and certain graphs with $n$ vertices. In detail he give a characterization of all finite dimensional real CL-spaces with an absolute norm using the graph-theoretic terminology. It gives a geometric picture of extreme points of the unit ball of CL-spaces and plays an important role to find the strongly norm-attaining points of $\mathcal{P}(^kX)$. Moreover we can find all complex extreme points on a complex CL-space with an absolute norm. These strongly norm-attaining points or complex extreme points help answering a problem about the numerical index of a Banach space. We give a partial answer to the Problem 43 in [KMP06]:

Characterize the complex Banach spaces $X$ satisfying $n(^kX) = 1$ for all $k \geq 2$.

We show that for a finite dimensional complex Banach space $X$ with an absolute norm, its polynomial numerical indices are one if and only if $X$ is isometric to $\ell^\infty_n$.

For later use, recall the definitions of real and complex extreme points of a unit ball. Let $X$ be a real or complex Banach space. Recall that $x \in B_X$ is said to be an extreme point of $B_X$ if whenever $y + z = 2x$ for some $y, z \in B_X$, we have $x = y = z$. Denote by $\text{ext}(B_X)$ the set of all extreme points of $B_X$. When $X$ is a complex Banach space, an element $x \in B_X$ is said to be a complex extreme point of $B_X$ if $\sup_{0 \leq \theta \leq 2\pi} \|x + e^{i\theta}y\| \leq 1$ for some $y \in X$ implies $y = 0$. The set of all complex extreme points of $B_X$ is denoted by $\text{ext}_C(B_X)$.

### 2. Denseness of the set of strong peak functions

Let $X$ be a Banach space and $A$ a closed subspace $C_b(K : X)$. A subset $F$ of $K$ is said to be a norming subset for $A$ if for each $f \in A$, we have

$$\|f\| = \sup\{\|f(t)\| : t \in F\}.$$ 

Following the definition of Globevnik in [Glo79], the smallest closed norming subset of $A$ is called the Shilov boundary for $A$ and it is shown in [CLS] that if the set of strong peak functions is dense in $A$ then the Shilov boundary of $A$ exists and it is the closure of $\rho A$. The variation method used in [DGZ93] gives the partial converse of the above mentioned result.
Theorem 2.1. Let $A$ be a closed subspace $C_b(K : X)$ which contains all elements of the form $t \mapsto (x^* f(t))^m x$ for all $x \in X$, $x^* \in X^*$, $f \in A$ and integers $m \geq 1$. Then the set $\rho A$ is a norming subset of $A$ if and only if the set of all strong peak functions in $A$ is a dense $G_\delta$ subset of $A$.

Proof. Let $d$ be the complete metric on $K$. Fix $f \in A$ and $\epsilon > 0$. For each $n \geq 1$, set

$$U_n = \left\{ g \in A : \exists z \in \rho A \text{ with } \|(f - g)(z)\| > \sup\{\|(f - g)(x)\| : d(x, z) > 1/n\} \right\}.$$

Then $U_n$ is open and dense in $A$. Indeed, fix $h \in A$. Since $\rho A$ is a norming subset of $A$, there is a point $w \in \rho A$ such that

$$(2.1) \quad \|(f - h)(w)\| > \|f - h\| - \epsilon/2. \quad \forall t \in K.$$

Choose a peak function $q \in A$ at $w$ with $\|q(w)\| = 1$ and $w^* q(w) = 1$. Then it is easy to see that $w^* \circ q$ is also a strong peak function in $C_b(K)$. So there is an integer $m \geq 1$ such that $|w^* q(x)|^m < 1/3$ for all $x \in K$ with $d(x, w) > 1/n$. Now define the function by

$$p(t) = -(w^* q(t))^m \cdot \frac{(f - h)(w)}{\|(f - h)(w)\|}, \quad \forall t \in K.$$

Set $g(x) = h(x) + \epsilon \cdot p(x)$. Then $\|f(w) - h(w) - \epsilon p(w)\| = \|f(w) - h(w)\| + \epsilon$ and $\|g - h\| \leq \epsilon$. The equation (2.1) shows that

$$\|(f - g)(w)\| = \|f(w) - h(w) - \epsilon p(w)\| = \|f(w) - h(w)\| + \epsilon$$

$$> \|f - h\| + \epsilon/2$$

$$\geq \sup\{\|(f - h)(x) - \epsilon p(x)\| : d(x, w) > 1/n\}.$$  

$$= \sup\{\|(f - g)(x)\| : d(x, w) > 1/n\}.$$ 

Therefore $g \in U_n$.

By the Baire category theorem there is a $g \in \bigcap U_n$ with $\|g\| < \epsilon$, and we shall show that $f - g$ is a strong peak function. Indeed, $g \in U_n$ implies that there is $z_n \in X$ such that

$$\|(f - g)(z_n)\| > \sup\{\|(f - g)(x)\| : d(x, z_n) > 1/n\}.$$ 

Thus $d(z_p, z_n) \leq 1/n$ for every $p > n$, and hence $\{z_n\}$ converges to a point $z$, say. Suppose that there is another sequence $\{x_k\}$ in $B_X$ such that $\{\|(f - g)(x_k)\|\}_k$ converges to $\|f - g\|$. Then for each $n \geq 1$, there is $M_n \geq 1$ such that for every $m \geq M_n$,

$$\|(f - g)(x_m)\| > \sup\{\|(f - g)(x)\| : d(x, z_n) > 1/n\}.$$ 

Then $d(x_m, z_n) \leq 1/n$ for every $m \geq M_n$. Hence $\{x_m\}_m$ converges to $z$. This shows that $f - g$ is a strong peak function at $z$.

By Proposition 2.19 in [KL], the set of all strong peak functions in $A$ is a $G_\delta$ subset of $A$. This proves the necessity.
Concerning the converse, it is shown in [CLS] that if the set of all strong peak functions is dense in $A$, then the set of all strong peak points is a norming subset of $A$. This completes the proof. □

Let $B_X$ be the unit ball of the Banach space $X$. Recall that the point $x \in B_X$ is said to be a *smooth point* if there is a unique $x^* \in B_{X^*}$ such that $\text{Re} x^*(x) = 1$. We denote by $\text{sm}(B_X)$ the set of all smooth points of $B_X$. We say that a Banach space is *smooth* if $\text{sm}(B_X)$ is the unit sphere $S_X$ of $X$. The following corollary shows that if $\rho A$ is a norming subset, then the set of smooth points of $B_A$ is dense in $S_A$.

**Corollary 2.2.** Suppose that $K$ is a complete metric space and $A$ is a closed subalgebra of $C_b(K)$. If $\rho A$ is a norming subset of $A$, then the set of all smooth points of $B_A$ contains a dense $G_\delta$ subset of $S_A$.

*Proof.* It is shown in [CLS] that if $f \in A$ is a strong peak function and $\|f\| = 1$, then $f$ is a smooth point of $B_A$. Then Theorem 2.1 completes the proof. □

Recall that a Banach space $X$ is said to be a locally uniformly convex if whenever there is a sequence $\{x_n\} \in B_X$ with $\lim_n \|x_n + x\| = 2$ for some $x \in S_X$, we have $\lim_n \|x_n - x\| = 0$. It is shown in [CLS] that if $X$ is locally uniformly convex, then the set of norm attaining elements is dense in $\mathcal{A}(B_X)$. That is, the set consisting of $f \in \mathcal{A}(B_X)$ with $\|f\| = |f(x)|$ for some $x \in B_X$ is dense in $\mathcal{A}(B_X)$.

The following corollary gives a stronger result. Notice that if $X$ is locally uniformly convex, then every point of $S_X$ is the strong peak point for $\mathcal{A}(B_X)$ [Glo79]. Theorem 2.1 and Corollary 2.2 implies the following.

**Corollary 2.3.** Suppose that $X$ is a locally uniformly convex, complex Banach space. Then the set of all strong peak functions in $\mathcal{A}(B_X)$ is a dense $G_\delta$ subset of $\mathcal{A}(B_X)$. In particular, the set of all smooth points of $B_{\mathcal{A}(B_X)}$ contains a dense $G_\delta$ subset of $S_{\mathcal{A}(B_X)}$.

It is shown [CHL07] that the set of all strong peak points for $\mathcal{A}(B_X)$ is dense in $S_X$ if $X$ is an order continuous locally uniformly $c$-convex sequence space. (For the definition see [CHL07]). Then by Theorem 2.1, we get the denseness of the set of all strong peak functions.

**Corollary 2.4.** Let $X$ be an order continuous locally uniformly $c$-convex Banach sequence space. Then the set of all strong peak functions in $\mathcal{A}_u(B_X : X)$ is a dense $G_\delta$ subset of $\mathcal{A}_u(B_X : X)$.

Let $\Pi(X) = \{(x, x^*) \in B_X \times B_{X^*} : x^*(x) = 1\}$ be the topological subspace of $B_X \times B_{X^*}$, where $B_X$ (resp. $B_{X^*}$) is equipped with norm (resp. weak-$*$ compact) topology.
The numerical radius of holomorphic functions was deeply studied in [Har71] and the denseness of numerical radius holomorphic functions is studied on the classical Banach spaces [AK07]. Recently the numerical strong peak function is introduced in [KL] and the denseness of holomorphic numerical strong peak functions in $\mathcal{A}(B_X : X)$ is studied in various Banach spaces. The function $f \in \mathcal{A}(B_X : X)$ is said to be a numerical strong peak function if there is $(x, x^*)$ such that $\lim_n |x_n^* f(x_n)| = v(f)$ for some $\{(x_n, x_n^*)\}_n$ in $\Pi(X)$ implies that $(x_n, x_n^*)$ converges to $(x, x^*)$ in $\Pi(X)$. The function $f \in \mathcal{A}(B_X : X)$ is said to be numerical radius attaining if there is $(x, x^*)$ in $\Pi(X)$ such that $v(f) = |x^* f(x)|$. An element in the intersection of the set of all strong peak functions and the set of all numerical strong peak functions is called a norm and numerical strong peak function of $\mathcal{A}(B_X : X)$. Using the variational method again we obtain the following.

**Proposition 2.5.** Suppose that the set $\Pi(X)$ is complete metrizable and the set $\Gamma = \{(x, x^*) \in \Pi(X) : x \in \rho \mathcal{A}(B_X) \cap sm(B_X)\}$ is a numerical boundary. That is, $v(f) = \sup_{(x, x^*) \in \Gamma} |x^* f(x)|$ for each $f \in \mathcal{A}(B_X : X)$. Then the set of all numerical strong peak functions in $\mathcal{A}(B_X : X)$ is a dense $G_\delta$ subset of $\mathcal{A}(B_X : X)$.

**Proof.** Let $A = \mathcal{A}(B_X : X)$ and let $d$ be a complete metric on $\Pi(X)$. In [KL], it is shown that if $\Pi(X)$ is complete metrizable, then the set of all numerical peak functions in $A$ is a $G_\delta$ subset of $A$. We need prove the denseness. Let $f \in A$ and $\epsilon > 0$. Fix $n \geq 1$ and set

$$U_n = \{g \in A : \exists (z, z^*) \in \Gamma \text{ with } |z^*(f - g)(z)| > \sup\{|x^*(f - g)(x)| : d((x, x^*), (z, z^*)) > 1/n\}\}.$$ 

Then $U_n$ is open and dense in $A$. Indeed, fix $h \in A$. Since $\Gamma$ is a numerical boundary for $A$, there is a point $(w, w^*) \in \Gamma$ such that

$$|w^*(f - h)(w)| > v(f - h) - \epsilon/2.$$ 

Notice that $d((x, x^*), (w, w^*)) > 1/n$ implies that there is $\delta_n > 0$ such that $\|x - w\| > \delta_n$. Choose a peak function $p \in \mathcal{A}(B_X)$ such that $\|p\| = 1 = |p(w)|$ and $|p(x)| < 1/3$ for $\|x - w\| > \delta_n$ and $|w^*(f - h)(w) - \epsilon p(w)| = |w^* f(w) - w^* h(w)| + \epsilon |p(w)| = |w^* f(w) - w^* h(w)| + \epsilon/2$.

Put $g(x) = h(x) + \epsilon \cdot p(x) w$. Then

$$|w^*(f - g)(w)| = |w^* f(w) - w^* h(w) - \epsilon p(w)| = |w^* f(w) - w^* h(w)| + \epsilon$$

$$> v(f - h) + \epsilon/2$$

$$\geq \sup\{|x^*(f - h)(x) - \epsilon p(x) x^*(w)| : d((x, x^*), (w, w^*)) > 1/n\}.$$ 

$$= \sup\{|x^*(f - g)(x)| : d((x, x^*), (w, w^*)) > 1/n\}.$$
That is, \( g \in U_n \).

By the Baire category theorem there is a \( g \in \bigcap U_n \) with \( \|g\| < \epsilon \), and we shall show that \( f - g \) is a strong peak function. Indeed, \( g \in U_n \) implies that there is \((z_n, z_n^*) \in \Gamma\) such that

\[
|z_n^*(f - g)(z_n)| > \sup\{|x^*(f - g)(x)| : d((x, x^*), (z_n, z_n^*)) > 1/n\}.
\]

Thus \( d((z_p, z_p^*), (z_n, z_n^*)) \leq 1/n \) for every \( p > n \), and hence \( \{ (z_n, z_n^*) \} \) converges to a point \( (z, z^*) \), say. Suppose that there is another sequence \( \{ (x_k, x_k^*) \} \) in \( \Pi(X) \) such that \( \{|x_k^*(f - g)(x_k)|\} \) converges to \( v(f - g) \). Then for each \( n \geq 1 \), there is \( M_n \geq 1 \) such that for every \( m \geq M_n \),

\[
|x_m^*(f - g)(x_m)| > \sup\{|x^*(f - g)(x)| : d((x, x^*), (z_n, z_n^*)) > 1/n\}.
\]

Then \( d((x_m, x_m^*), (z_n, z_n^*)) \leq 1/n \) for every \( m \geq M_n \). Hence \( \{ (x_m, x_m^*) \} \) converges to \( (z, z^*) \). This shows that \( f - g \) is a numerical strong peak function at \( (z, z^*) \). □

**Corollary 2.6.** Suppose that \( X \) is separable, smooth and locally uniformly convex. Then the set of norm and numerical strong peak functions is a dense \( G_\delta \) subset of \( \mathcal{A}_u(B_X : X) \).

**Proof.** It is shown [KL] that if \( X \) is separable, \( \Pi(X) \) is complete metrizable. In view of [Pal04, Theorem 2.5], \( \Gamma \) is a numerical boundary for \( \mathcal{A}_u(B_X : X) \). Hence Proposition 2.5 shows that the set of all numerical strong peak functions is dense in \( \mathcal{A}_u(B_X : X) \). Finally, Corollary 2.3 implies that the set of all norm and numerical peak functions is a dense \( G_\delta \) subset of \( \mathcal{A}_u(B_X : X) \). □

**Corollary 2.7.** Let \( X \) be an order continuous locally uniformly \( c \)-convex, smooth Banach sequence space. Then the set of all norm and numerical strong peak functions in \( \mathcal{A}_u(B_X : X) \) is a dense \( G_\delta \) subset of \( \mathcal{A}_u(B_X : X) \).

**Proof.** Notice that \( X \) is separable since \( X \) is order continuous. Hence the set of all smooth points of \( B_X \) is dense in \( S_X \) by the Mazur theorem and \( \Pi(X) \) is complete metrizable [KL]. In view of [Pal04, Theorem 2.5], \( \Gamma \) is a numerical boundary for \( \mathcal{A}_u(B_X : X) \). Hence Proposition 2.5 shows that the set of all numerical strong peak functions is a dense \( G_\delta \) subset of \( \mathcal{A}_u(B_X : X) \). Theorem 2.1 also shows that the set of all strong peak functions is a dense \( G_\delta \) subset of \( \mathcal{A}_u(B_X : X) \). This completes the proof. □
3. Denseness of strongly norm attaining polynomials

Recall that the norm $\| \cdot \|$ of a Banach space is said to be Fréchet differentiable at $x \in X$ if

$$\lim_{\delta \to 0} \sup_{\|y\| = 1} \frac{\|x + \delta y\| + \|x - \delta y\| - 2\|x\|}{\delta} = 0.$$ 

It is well-known that the set of Fréchet differentiable points in a Banach space is a $G_\delta$ subset [BL00, Proposition 4.16]. Ferrera [Fer02] shows that in a real Banach space $X$, the norm of $P(nX)$ is Fréchet differentiable at $Q$ if and only if $Q$ strongly attains its norm.

A set $\{x_\alpha\}$ of points on the unit sphere $S_X$ of $X$ is called uniformly strongly exposed (u.s.e.), if there are a function $\delta(\epsilon)$ with $\delta(\epsilon) > 0$ for every $\epsilon > 0$, and a set $\{f_\alpha\}$ of elements of norm 1 in $X^*$ such that for every $\alpha$, $f_\alpha(x_\alpha) = 1$, and for any $x$,

$$\|x\| \leq 1 \text{ and } \Re f_\alpha(x) \geq 1 - \delta(\epsilon) \text{ imply } \|x - x_\alpha\| \leq \epsilon.$$ 

Lindenstrauss [Lin63, Proposition 1] showed that if $B_X$ is the closed convex hull of a set of u.s.e. points, then $X$ has property $A$, that is, for every Banach space $Y$, the set of norm-attaining elements is dense in $L(X, Y)$, the Banach space of all bounded operators from $X$ into $Y$. The following theorem gives stronger result.

**Theorem 3.1.** Let $\mathbb{F}$ be the real or complex scalar field and $X$, $Y$ Banach spaces over $\mathbb{F}$. For $k \geq 1$, suppose that the u.s.e. points $\{x_\alpha\}$ in $S_X$ is a norming subset of $P(kX)$. Then the set of strongly norm attaining elements in $P(kX : Y)$ is dense. In particular, the set of all points at which the norm of $P(nX)$ is Fréchet differentiable is a dense $G_\delta$ subset.

**Proof.** Let $\{x_\alpha\}$ be a u.s.e. points and $\{x_\alpha^*\}$ be the corresponding functional which uniformly strongly expose $\{x_\alpha\}$. Let $A = P(kX : Y)$, $f \in A$ and $\epsilon > 0$. Fix $n \geq 1$ and set

$$U_n = \{g \in A : \exists z \in \rho A \text{ with } \|(f - g)(z)\| > \sup\{(f - g)(x) : \inf_{|\lambda| = 1} \|x - \lambda z\| > 1/n\}\}.$$ 

Then $U_n$ is open and dense in $A$. Indeed, fix $h \in A$. Since $\{x_\alpha\}$ is a norming subset of $A$, there is a point $w \in \{x_\alpha\}$ such that

$$\|(f - h)(w)\| > \|f - h\| - \delta(1/n)\epsilon.$$ 

Set

$$g(x) = h(x) - \epsilon \cdot p(x)^k \frac{f(w) - h(w)}{\|f(w) - h(w)\|}.$$
where \( p \) is a strongly exposed functional at \( w \) such that \(|p(x)| < 1 - \delta(1/n)\) for 
\[ \inf_{|\lambda|=1} \|x - \lambda w\| > 1/n, \ p(w) = 1. \]
Then \( \|g - h\| \leq \epsilon \) and 
\[ \|f - g)(w)\| = \left\| f(w) - h(w) + \epsilon p(w)^k \frac{f(w) - h(w)}{\|f(w) - h(w)\|} \right\| = \|f(w) - h(w)\| + \epsilon \]
\[ > \|f - h\| + \epsilon(1 - \delta(1/n)) \]
\[ \geq \sup \{ \|f - h)(x) - \epsilon p(x)^k \frac{f(w) - h(w)}{\|f(w) - h(w)\|} \| : \inf_{|\lambda|=1} \|x - \lambda w\| > 1/n \}. \]
\[ = \sup \{ \|f - g)(x)\| : \inf_{|\lambda|=1} \|x - \lambda w\| > 1/n \}. \]
That is, \( g \in U_n \).

By the Baire category theorem there is a \( g \in \bigcap U_n \) with \( \|g\| < \epsilon \), and we shall show that \( f - g \) is a strong peak function. Indeed, \( g \in U_n \) implies that there is \( z_n \in X \) such that 
\[ \|(f - g)(z_n)\| > \sup \{ \|f - g)(x)\| : \inf_{|\lambda|=1} \|x - \lambda z_n\| > 1/n \}. \]
Thus \( \inf_{|\lambda|=1} \|z_p - \lambda z_n\| \leq 1/n \) for every \( p > n \), and \( \inf_{|\lambda|=1} |z^*_p(z_n) - \lambda| = 1 - |z^*_p(z_p)| \leq 1/n \) for every \( p > n \). So \( \lim_{n} \inf_{p>n} |z^*_p(z_p)| = 1 \). Hence there is a subsequence of \( \{z_n\} \) which converges to \( z \), say by [Aco91, Lemma 6]. Suppose that there is another 
sequence \( \{x_k\} \) in \( B_X \) such that \( \{\|(f - g)(x_k)\|\} \) converges to \( \|f - g\| \). Then for each \( n \geq 1 \), there is \( M_n \) such that \( M_n \geq n \) and for every \( m \geq M_n \), 
\[ \|(f - g)(x_m)\| > \sup \{ \|f - g)(x)\| : \inf_{|\lambda|=1} \|x - \lambda z_n\| > 1/n \}. \]
Then \( \inf_{|\lambda|=1} \|x_m - \lambda z_n\| \leq 1/n \) for every \( m \geq M_n \). So \( \inf_{|\lambda|=1} \|x_m - \lambda z\| \leq \inf_{|\lambda|=1} \|x_m - \lambda z_n\| + \|z - z_n\| \leq 2/n \) for every \( m \geq M_n \). Hence we get a convergent subsequence of \( x_n \) of which limit is \( \lambda z \) for some \( \lambda \in S_C \). This shows that \( f - g \) strongly norm attains at \( z \).

It is shown in [CLS] that the norm is Fréchet differentiable at \( P \) if and only if whenever there are sequences \( \{t_n\}, \ \{s_n\} \) in \( B_X \) and scalars \( \alpha, \beta \) in \( S_F \) such that 
\[ \lim_n \alpha P(t_n) = \lim_n \beta P(s_n) = \|P\|, \]
we get 
\[ \lim_n \sup_{\|Q\|=1} |\alpha Q(t_n) - \beta Q(s_n)| = 0. \]
We have only to show that every nonzero element \( P \) in \( A \) which strongly attains its norm satisfies the condition (3.1). Suppose that \( P \) strongly attains its norm at \( z \) and \( P \neq 0 \).

For each \( Q \in A \), there is a \( k \)-linear form \( L \) such that \( Q(x) = L(x, \ldots, x) \) for each \( x \in X \). The polarization identity [Din99] shows that \( \|Q\| \leq \|L\| \leq (k^k/k!)\|Q\| \). Then for each \( x, y \in B_X \), 
\[ \|Q(x) - Q(y)\| \leq n\|L\|\|x - y\| \quad \text{and} \quad \|Q(x) - Q(y)\| \leq \frac{k^k + 1}{k!}\|Q\|\|x - y\|. \]
Suppose that there are sequences \( \{t_n\}, \{s_n\} \) in \( B_X \) and scalars \( \alpha, \beta \) in \( S_F \) such that 
\[
\lim_n \alpha P(t_n) = \lim_n \beta P(s_n) = \|P\|,
\]
then for any subsequences \( \{s'_n\} \) of \( \{s_n\} \) and \( \{t'_n\} \) of \( \{t_n\} \), there are convergent further subsequences \( \{t''_n\} \) of \( \{t'_n\} \) and \( \{s''_n\} \) of \( \{s'_n\} \) and scalars \( \alpha'' \) and \( \beta'' \) in \( S_F \) such that 
\[
\lim_n t''_n = \alpha''z \quad \text{and} \quad \lim_n s''_n = \beta''z.\]
Then \( \alpha P(\alpha''z) = \beta P(\beta''z) = 1 \). So \( \alpha(\alpha'')^k = \beta(\beta'')^k \).

Then we get
\[
\lim_n \sup_{\|Q\|=1} |\alpha Q(t''_n) - \beta Q(s''_n)| \leq \lim_n \sup_{\|Q\|=1} \left( |\alpha Q(t''_n) - \alpha Q(\alpha''z)| + |\beta Q(\beta''z) - \beta Q(s''_n)| \right)
\leq \lim_n \frac{k^{k+1}}{k!} (\|t''_n - \alpha''z\| + \|\beta''z - s''_n\|) = 0.
\]
This implies that \( \lim_n \sup_{\|Q\|=1} |\alpha Q(t_n) - \beta Q(s_n)| = 0 \). Therefore the norm is Fréchet differentiable at \( P \). This completes the proof. \( \square \)

**Remark 3.2.** Suppose that the \( B_X \) is the closed convex hull of a set of u.s.e points, then the set of u.s.e. points is a norming subset of \( X^* = \mathcal{P}(1_X) \). Hence the elements in \( X^* \) at which the norm of \( X^* \) is Fréchet differentiable is a dense \( G_δ \) subset.

### 4. Polynomial numerical index and Graph theory

A norm \( \|\cdot\| \) on \( \mathbb{R}^n \) or \( \mathbb{C}^n \) is said to be an absolute norm if \( \|(a_1, \ldots, a_n)\| = \|(|a_1|, \ldots, |a_n|)\| \) for every scalar \( a_1, \ldots, a_n \), and \( \|(1,0,\ldots,0)\| = \cdots = \|(0,\ldots,0,1)\| = 1 \). We may use the fact that the absolute norm is less than or equal to the \( \ell_1 \)-norm and it is nondecreasing in each variable.

A real or complex Banach space \( X \) is said to be a CL-space if its unit ball is the absolutely convex hull of every maximal convex subset of the unit sphere. In particular, if \( X \) is finite dimensional, then it is equivalent to the condition that \( |x^*(x)| = 1 \) for every \( x^* \in \text{ext } B_{X^*} \) and every \( x \in \text{ext } B_X \) [Lim78].

Let \( X \) be a \( n \)-dimensional Banach space with an absolute norm \( \|\cdot\| \). In this section, \( X \) as a vector space is considered \( \mathbb{R}^n \) or \( \mathbb{C}^n \) and we denote by \( \{e_j\}_{j=1}^n \) and \( \{e^*_j\}_{j=1}^n \) the canonical basis and the coordinate functionals, respectively. We also denote by \( \text{ext } B_X \) the set of all extreme points of \( B_X \).

Now define the following mapping between \( n \)-dimensional Banach spaces with an absolute norm and certain graphs with \( n \) vertices:
\[
X \longmapsto G = G(X),
\]
where \( G \) is a graph with the vertex set \( V = \{1,2,\ldots,n\} \) and the edge set \( E = \{(i,j): e_i + e_j \notin B_X\} \). For example, if \( X = \ell_1^n \), then \( G(X) \) is a complete graph of order \( n \), that is, a graph in which every pair of \( n \) vertices is connected by an edge. Conversely, \( G(X) \) is a graph in which no pairs of \( n \) vertices are connected by an edge if \( X = \ell_\infty^n \). Using these graphs and their theory, Reisner gave the exact characterization of all
finite dimensional CL-spaces with an absolute norm. Prior to Reisner’s theorem, we
give some basic definitions in the graph theory.

Given a graph \( G = (V, E) \), we say that \( \sigma \subset V \) is a clique of \( G \) if the edge set \( E \) of
\( G \) contains all pairs consisting of any two vertices in \( \sigma \). Conversely, \( \tau \subset V \) is called a
stable set of \( G \) if \( E \) contains no pairs consisting of two vertices in \( \tau \). A graph \( G \) is said
to be perfect if

\[
\omega(H) = \chi(H) \quad \text{for every induced subgraph } H \, \text{of} \, G,
\]

where \( \omega(G) \) denotes the clique number of \( G \) (the largest cardinality of a clique of \( G \))
and \( \chi(G) \) is the chromatic number of \( G \) (the smallest number of colors needed to color
the vertices of \( G \) so that no two adjacent vertices share the same color).

**Theorem 4.1.** [Rei91, Reisner] Let \( X \) be a finite dimensional Banach space with an
absolute norm. Then \( X \) is a CL-space if and only if \( G = G(X) \) is a perfect graph and
there exists a unique common element between every maximal clique and each maximal
stable set of \( G \). In particular, for every \( n \)-dimensional CL-space \( X \), the following
characterizations of the set of extreme points of \( B_X \) and \( B_{X^*} \) hold respectively:

1. \((x_1, x_2, \ldots, x_n) \in \text{ext } B_X \) if and only if \( |x_j| \in \{0, 1\} \) for all \( j = 1, 2, \ldots, n \) and
   the support \( \{j \in V : x_j \neq 0\} \) is a maximal stable set of \( G \).
2. \((x_1, x_2, \ldots, x_n) \in \text{ext } B_{X^*} \) if and only if \( |x_j| \in \{0, 1\} \) for all \( j = 1, 2, \ldots, n \) and
   the support \( \{j \in V : x_j \neq 0\} \) is a maximal clique of \( G \).

In this theorem, the maximality of cliques and stable sets comes from the partial
order of inclusion.

Let \( X \) be a finite dimensional CL-space. If \( \tau \) is a maximal clique of \( G = G(X) \), then
a subspace \( \text{span}\{e_j : j \in \tau\} \) of \( X \) is isometrically isomorphic to \( \ell_1^{|	au|} \). Indeed, since the
absolute norm is less than or equal to the \( \ell_1 \)-norm, we have \( \| \sum_{j \in \tau} a_j e_j \| \leq \sum_{j \in \tau} |a_j| \)
for every scalar \( a_j \). For the inverse inequality, let \( x_\tau^* = \sum_{j \in \tau} \text{sign}(a_j) \cdot e_j^* \). Then \( x_\tau^* \) is
in \( \text{ext } B_{X^*} \) by Theorem 4.1 and hence \( x_\tau^*(\sum_{j \in \tau} a_j e_j) = \sum_{j \in \tau} |a_j| \), which completes the
proof.

**Remark 4.2.** Originally, Reisner just proved the above theorem for the real case.
However, it can be extended to the general case (real or complex). For this, we need
the following comments and proposition.

There is a natural one-to-one correspondence between the absolute norm of \( \mathbb{R}^n \)
and the one of \( \mathbb{C}^n \). Specifically, given real Banach space \( X = (\mathbb{R}^n, \| \cdot \|) \) with an
absolute norm, we can find the complexification \( \tilde{X} = (\mathbb{C}^n, \| \cdot \|) \) of \( X \) defined by
\( \| (z_1, \ldots, z_n) \|_C := \| (|z_1|, \ldots, |z_n|) \| \) for each \((z_1, \ldots, z_n) \in \tilde{X}. \) Then \( \tilde{X} \) is clearly the
complex Banach space with the absolute norm. Moreover we get the following basic proposition.

**Proposition 4.3.** Let $X$ be a real Banach space with an absolute norm and $\tilde{X}$ the complexification of $X$. Then, for an element $x$ of $X$, $x \in \text{ext} B_X$ if and only if $x \in \text{ext} B_{\tilde{X}}$. In particular, $X$ is a CL-space if and only if $\tilde{X}$ is a CL-space.

**Proof.** The sufficiency is clear. Suppose that $x$ is an extreme point of $B_X$ and $2x = y + z$ for some $y, z$ in $B_{\tilde{X}}$. We claim that $y$ and $z$ are in $X$. For the contrary, suppose that some $j^{th}$-coordinate of $y$ is not a real number. That is, $e_j^*(y) = a + bi$, $a, b \in \mathbb{R}$ and $b \neq 0$. Since $2x = y + z = \text{Re}(y) + \text{Re}(z) + i(\text{Im}(y) + \text{Im}(z))$, we have $\text{Re}(y) = \text{Re}(z) = x$, where $\text{Re}(x) = \sum_{k=1}^{n} \text{Re}(e_k^*(x))e_k$ and $\text{Im}(x) = \sum_{k=1}^{n} \text{Im}(e_k^*(x))e_k$. Take a positive real number $\delta$ less than $\sqrt{a^2 + b^2} - |a|$. Note that $\|x \pm \delta e_j\| \leq \|y\| \leq 1$ by the basic property of an absolute norm. So $2x = (x + \delta e_j) + (x - \delta e_j)$ contradicts to the fact that $x$ is an extreme point of $B_X$. \hfill \Box

Using the graph-theoretic technique on CL-spaces, we are about to find the strongly norm attaining points of $\rho \mathcal{P}^m(X)$. For this, let us consider the following lemma.

**Lemma 4.4.** Let $Y = \ell_1^N$ and let $m$ be a positive integer. For any $j_1, j_2, \ldots, j_m \in \{1, 2, \ldots, N\}$, define a $m$-homogeneous polynomial $Q_{j_1,j_2,\ldots,j_m}$ of $Y$ by

$$Q_{j_1,j_2,\ldots,j_m}(x) = \prod_{k=1}^{m} e_{j_k}^*(x) + \left[ \sum_{j \in \{j_1,\ldots,j_m\}} e_j^*(x) \right]^m.$$ 

Then $Q_{j_1,j_2,\ldots,j_m}$ attains its norm only at $\frac{1}{m} \sum_{k=1}^{m} e_{j_k}$, $|c| = 1$. Hence $Q_{j_1,j_2,\ldots,j_m}$ strongly attains its norm at $\frac{1}{m} \sum_{k=1}^{m} e_{j_k}$.

**Proof.** For positive integers $m_1, \ldots, m_n$, consider the product

$$(x_1, \ldots, x_n) \mapsto \prod_{k=1}^{n} x_k^{m_k}$$

on the compact subset $\mathbb{R}_+^n \cap S_{\ell_1^N}$. Then it is easy to see by induction that the product has the unique maximum at $((m_1, \ldots, m_n), \ldots, (m_1, \ldots, m_n))$. Hence the norm of the polynomial $\prod_{k=1}^{m} e_{j_k}^*(x)$ is attained only at $x = (x_1, \ldots, x_N)$, where $|x_j| = \frac{1}{m} \sum_{k=1}^{m} e_{j_k}(e_{j_k})$ for each $1 \leq j \leq N$. Notice also that the norm of the polynomial $(\sum_{j=1}^{N} e_j^*(x))^m$ is attained only at $x = (x_1, \ldots, x_N)$, where $\text{sign}(x_1) = \cdots = \text{sign}(x_N)$ and $x \in S_{\ell_1^N}$. Hence $Q_{j_1,j_2,\ldots,j_m}$ attains its norm only at $\frac{1}{m} \sum_{k=1}^{m} e_{j_k}$ for some $c \in S_{\mathbb{C}}$. This completes the proof. \hfill \Box

**Theorem 4.5.** Let $X$ be a finite dimensional (real or complex) CL-space with an absolute norm and let $m$ be a positive integer. Then $\frac{1}{m} \sum_{j=1}^{m} y_j \in \rho \mathcal{P}^m(X)$ whenever $y_1, y_2, \ldots, y_m$ are extreme points of $B_X$ whose coordinates are nonnegative real numbers.
Proof. Denote by $M(G)$ the family of all maximal cliques of $G = G(X)$ and let $y_1, y_2, \ldots, y_m$ be extreme points of $B_X$ whose coordinates are nonnegative real numbers. For each $J \in M(G)$, define the $m$-homogeneous polynomial $Q_J$ and linear functional $L_J$ as the following

$$Q_J = Q_{j_1, j_2, \ldots, j_m}, \quad L_J = \sum_{j \in \{j_1, \ldots, j_m\}} e_j^*,$$

where each $j_k$ $(k = 1, 2, \ldots, m)$ is a unique common element between a maximal clique $J$ and the support of an extreme point $y_k$. Now define a $m$-homogeneous polynomial $Q$ of $X$ by

$$Q = \sum_{J \in M(G)} Q_J + \left[ \sum_{J \in M(G)} L_J \right]^m.$$

For every maximal clique $J$ of $G$, denote by $P_J$ the projection of $X$ onto $\text{span}\{e_j : j \in J\}$. Then, it is clear that

$$P_J \left( \frac{1}{m} \sum_{j=1}^m y_j \right) = \frac{1}{m} \sum_{j=1}^m P_J(y_j) = \frac{1}{m} \sum_{k=1}^m e_{j_k}.$$

Notice that $Q_J \circ P_J = Q_J$ and $L_J \circ P_J = L_J$ for each $J \in M(G)$. It follows by Theorem 5.1 that

$$Q \left( \frac{1}{m} \sum_{j=1}^m y_j \right)$$

$$= \sum_{J \in M(G)} Q_J \left( \frac{1}{m} \sum_{j=1}^m y_j \right) + \left[ \sum_{J \in M(G)} L_J \left( \frac{1}{m} \sum_{j=1}^m y_j \right) \right]^m$$

$$= \sum_{J \in M(G)} Q_J \left( \frac{1}{m} \sum_{j=1}^m e_{j_k} \right) + \left[ \sum_{J \in M(G)} L_J \left( \frac{1}{m} \sum_{j=1}^m e_{j_k} \right) \right]^m$$

$$= \sum_{J \in M(G)} \|Q_J\| \left[ \sum_{J \in M(G)} \|L_J\| \right]^m$$

$$= \|Q\|,$$

and that $Q$ attains its norm at $\frac{1}{m} \sum_{j=1}^m y_j$.

Then we claim that the above polynomial $Q$ strongly attains its norm at $\frac{1}{m} \sum_{j=1}^m y_j$. Now, suppose that $|Q(y)| = \|Q\|$ and we need to show that $y = \frac{1}{m} \sum_{j=1}^m y_j$ for some
Then we have the following inequalities.
\[ |Q(y)| = \left| \sum_{J \in M(G)} Q_J(P_Jy) \right| + \left| \sum_{J \in M(G)} L_J(P_Jy) \right|^m \]
\[ \leq \sum_{J \in M(G)} |Q_J(P_Jy)| + \left[ \sum_{J \in M(G)} |L_J(P_Jy)| \right]^m \]
\[ \leq \sum_{J \in M(G)} \|Q_J\| + \left[ \sum_{J \in M(G)} \|L_J\| \right]^m \]
\[ = \|Q\|. \]
Hence \( |Q_J(P_Jy)| = \|Q_J\| \), \( L_J(P_Jy) = \|L_J\| \) for each \( J \in M(G) \) and sign \( L_J(P_Jy) \) are all the same for all \( J \in M(G) \). By Theorem 5.1, \( P_Jy = \frac{c_J}{m} \sum_{k=1}^m e_{j_k} \) for each \( J \in M(G) \). Since sign \( (L_J(P_Jy)) = c_J \) and they are all the same for all \( J \in M(G) \), take \( c_J = c \) for all \( J \in M(G) \).

Since each maximal clique \( J \) induces every extreme point \( x_J^* \) of \( B_{X^*} \), the above equation implies that
\[ x_J^*(y) = x_J^*(P_Jy) = x_J^* \left( \frac{c}{m} \sum_{j=1}^m y_j \right) = x_J^* \left( \frac{c}{m} \sum_{j=1}^m y_j \right) \]
Consequently \( y = \frac{c}{m} \sum_{j=1}^m y_j \). This completes the proof.

It was shown in [Lee08] that a necessary and sufficient condition for a Banach space with the Radon-Nikodym Property to have the polynomial numerical index one can be stated as follows: \( n^{(k)}(X) = 1 \) if and only if
\[ |x^*(x)| = 1 \text{ for all } x \in \partial P^{(k)}X \text{ and } x^* \in \text{ext } B_{X^*}. \]

The following theorem is a partial answer to Problem 43 in [KMP06]:

\text{Characterize the complex Banach spaces } X \text{ satisfying } n^{(k)}(X) = 1 \text{ for all } k \geq 2.

**Theorem 4.6.** Let \( X \) be a \( n \)-dimensional complex Banach space with an absolute norm and let \( k \) be an integer greater than or equal to 2. Then \( n^{(k)}(X) = 1 \) if and only if \( X \) is isometric to \( \ell^n_\infty \).

**Proof.** Suppose that \( n^{(k)}(X) = 1 \). Since \( n^{(k)}(X) = 1 \) implies \( n(X) = 1 \) (i.e. \( X \) is a CL-space), we can apply Theorem 4.1. To show that \( X \) is isometric to \( \ell^n_\infty \), it suffices to prove that \( B_X \) has only one extreme point whose coordinates are nonnegative real number. Suppose that there exist distinct two extreme points \( x, y \) of \( B_X \) whose coordinates are all nonnegative. Take a maximal clique \( \tau \) of \( G = G(X) \). Then, by Theorem 4.1, there exists a unique common element \( i \) between a maximal clique \( \tau \) and the support of \( x \).
Similarly, let $j$ be a common element between $\tau$ and the support of $y$. Now consider an $x^*_\tau \in \text{ext } B_X^*$ defined by

$$x^*_\tau(e_k) = \begin{cases} -1, & \text{if } k = i \\ 1, & \text{if } k \in \tau \setminus \{i\} \\ 0, & \text{if } k \notin \tau. \end{cases}$$

Then, using Theorem 4.1, we get

$$x^*_\tau \left( \frac{x + y}{2} \right) = \frac{x^*_\tau(x) + x^*_\tau(y)}{2} = -1 + 1 = 0.$$

However, since $\frac{x + y}{2} \in \tilde{\rho}P(\tilde{X})$ by Theorem 4.5, this contradicts to (4.1).

For the converse, note that $\tilde{\rho}P(kX) \subset \text{ext}_C B_X$ by [Lee08, Proposition 2.1]. It is also easy to see that $\text{ext}_C B_X = \text{ext } B_X$ when $X = \ell^m_\infty$. After all, it follows from (4.1) that $n(k)(\ell^m_\infty) = 1$. \qed

5. Characterization of complex extreme points

**Theorem 5.1.** Let $X = (\mathbb{C}^n, \| \cdot \|)$ be a $n$-dimensional complex CL-space with an absolute norm. Then an element $(a_1, a_2, \ldots, a_n)$ in $X$ is an complex extreme point of $B_X$ if and only if $(|a_1|, |a_2|, \ldots, |a_n|)$ is a convex combination of extreme points of $B_X$ whose coordinates all are positive real numbers. In short, $\mathbb{R}_+^n \cap \text{ext } B_X = \text{co}(\mathbb{R}_+^n \cap \text{ext } B_X)$.

**Proof.** Let $(a_1, a_2, \ldots, a_n)$ be an complex extreme point of $B_X$. If $\tilde{X} = (\mathbb{R}^n, \| \cdot \|)$, then all extreme points of $B_X$ are extreme points of $B_X$ by Proposition 4.3. So an element $x := (|a_1|, |a_2|, \ldots, |a_n|)$ has an expression $x = \sum_{j=1}^m \lambda_j y_j$ where $\sum_{j=1}^m \lambda_j = 1$, $\lambda_j > 0$ and each $y_j$ is an extreme point of $B_X$ whose coordinates are real numbers. Now we claim that all coordinates of each $y_j$ are positive. For the contradiction, suppose that the first coordinate of some $y_j$ is negative. More specifically, assume that the first coordinate of $y_j$ is $-1$ if $1 \leq j \leq r$ and $1$ otherwise. Then $|a_1| < 1$. Note that for $y'_j = y_j + 2e_1$, $j = 1, 2, \ldots, r$,

$$(1, |a_2|, \ldots, |a_n|) = \sum_{j=1}^r \lambda_j y'_j + \sum_{j=r+1}^m \lambda_j y_j \in \text{co}(\text{ext } B_X) = B_X.$$ 

It follows that for all $\varepsilon \in \mathbb{C}$ with $|\varepsilon| < 1 - |a_1|$, 

$$\|x + \varepsilon e_1\| = \|(1, |a_2|, \ldots, |a_n|)\| \leq \|(1, |a_2|, \ldots, |a_n|)\| \leq 1.$$ 

So $(|a_1|, |a_2|, \ldots, |a_n|)$ is not a complex extreme point, which is a contradiction.
For the converse, let \( x = \sum_{j=1}^{m} \lambda_j y_j \) where \( \sum_{j=1}^{m} \lambda_j = 1 \) and \( y_j \in \text{ext} \, B_X \) for all \( j = 1, 2, \ldots, m \). Suppose that \( x \) is not a complex extreme point of \( B_X \). Then there exists a nonzero \( y \) in \( X \) such that \( x + \varepsilon y \in B_X \) whenever \( |\varepsilon| \leq 1 \). Take a maximal clique \( \tau \) in \( V \) containing some vertices which are nonzero coordinates of \( y \). Consider the projection \( P_\tau \) of \( X \) onto the linear span \( Y \) of \( \{ e_j : j \in \tau \} \). Then it follows from assumption that \( P_\tau x \) is also not a complex extreme point of \( B_Y \). Moreover, we can check that \( P_\tau x \) is on the unit sphere of \( X \). Indeed, if an extreme point \( x^* \tau \) of \( B_X \) is defined by \( x^*_\tau(e_j) = 1 \) for \( j \in \tau \) and \( x^*_\tau(e_j) = 0 \) for \( j \notin \tau \), then we have \( x^*_\tau(y_j) = 1 \) for all \( j = 1, 2, \ldots, m \) by Theorem 4.1. Consequently,
\[
x^*_\tau(P_\tau x) = x^*_\tau(x) = \sum_{j=1}^{m} \lambda_j x^*_\tau(y_j) = \sum_{j=1}^{m} \lambda_j = 1.
\]
Now, from the fact that \( Y = \text{span}\{ e_j : j \in \tau \} \) is isometrically isomorphic to \( \ell_1^{\vert \tau \vert} \) since \( \tau \) is a clique, we get that every element of norm one in \( Y \) is a complex extreme point of \( B_Y \). [Glo75] This is a contradiction. \( \square \)

The above characterization of complex extreme points have some application in the theory about the numerical index of Banach spaces. Specifically we want to apply to the analytic numerical index of \( X \) defined by
\[
n_a(X) = \inf\{ v(P) : \| P \| = 1, P \in \mathcal{P}(X : X) \}
\]
It is easily observed that \( 0 \leq n_a(X) \leq n(1)(X) \leq n(X) \leq 1 \) for every \( k \geq 2 \). Note that a necessary and sufficient condition for a finite dimensional complex Banach space to have the analytic numerical index 1 can be stated using complex extreme points as follows (see [Lee08]):
\[
n_a(X) = 1 \text{ if and only if } \quad (5.1) \quad |x^*(x)| = 1 \text{ for all } x \in \text{ext} \, B_X \text{ and } x^* \in \text{ext} \, B_{X^*}.
\]

After applying the characterization of complex extreme points to the above theorem, we have the following corollary by the same argument as the proof of Theorem 4.6.

**Corollary 5.2.** Let \( X \) be an \( n \)-dimensional complex Banach space with an absolute norm. Then \( n_a(X) = 1 \) if and only if \( X \) is isometric to \( \ell_\infty^n \).

As an immediate consequence of the above theorem, we get a corollary. For further details, we need some definitions about the Daugavet property. A function \( \Phi \in \ell_\infty(B_X, X) \) is said to satisfy the (resp. alternative) Daugavet equation if the norm equality
\[
\| I_d + \Phi \| = 1 + \| \Phi \|
\]
(resp. \( \sup_{\omega \in S_C} \| I_d + \omega \Phi \| = 1 + \| \Phi \| \))
holds. If this happens for all weakly compact polynomials in $\mathcal{P}(X : X)$, we say that $X$ has the (resp. alternative) $p$-Daugavet property. Similarly, $X$ is said to have the $k$-order Daugavet property if the Daugavet equation is satisfied for all rank-one $k$-homogeneous polynomials in $\mathcal{P}^{(k)}(X : X)$.

**Corollary 5.3.** Let $X$ be a finite dimensional complex Banach space with an absolute norm. Then the followings are equivalent.

(a) $X$ has the alternative $p$-Daugavet property.

(b) $X$ has the $k$-order Daugavet property for some $k \geq 2$.

(b') $X$ has the $k$-order Daugavet property for every $k \geq 2$.

(c) $n^{(k)}(X) = 1$ for some $k \geq 2$.

(c') $n^{(k)}(X) = 1$ for every $k \geq 2$.

(d) $n_a(X) = 1$

(e) $X = \ell^\infty_n$

**Proof.** Theorem 4.6 and Theorem 5.2 imply (c) ⇔ (e) and (d) ⇔ (e) respectively. (b) ⇒ (c) or (b') ⇒ (c') is induced by [CGMM07, Proposition 1.3]. It is also easy to check that

$$(b) \Rightarrow (c) \Leftrightarrow (e) \quad (a) \Rightarrow (b') \Rightarrow (c') \Leftarrow (d)$$

Corollary 2.10 in [CGMM07] shows (e) ⇒ (a). The proof is done. □

**Remark 5.4.** Let $X$ be a finite dimensional (real or complex) Banach space with an absolute norm. Then it is impossible that $X$ has the $p$-Daugavet property. Indeed, if $X$ is a real (resp. complex) Banach space, then $n^{(k)}(X) = 1$ for all $k \geq 2$ and $X = \mathbb{R}$ (resp. $X = \ell^\infty_n$ for some $n$) (see [Lee08] and Corollary 4.6). Then it is easy to see that both $\mathbb{R}$ and complex $\ell^\infty_n$ do not have $p$-Daugavet property.

It is worth to note that the complex extreme points play an important role in the study of norming subsets of $A(B_X)$. In particular, $\rho_A(B_X) = \text{ext}_C B_X$ when $X$ is a finite dimensional complex Banach space (see [CHL07]). Using the result in [HN05], we obtain the following corollary. For the further references about complex convexity and monotonicity, see [Lee05, Lee07a, Lee07b].

**Corollary 5.5.** Let $X = (\mathbb{R}^n, \|\cdot\|)$ be an $n$-dimensional real $CL$-space with an absolute norm. Then an element $(|a_1|, |a_2|, \ldots, |a_n|)$ in $S_X$ is a point of upper monotonicity of $X$ if and only if $(|a_1|, |a_2|, \ldots, |a_n|)$ is a convex combination of extreme points of $B_X$ whose coordinates all are positive real numbers.
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