New Large-Rank Nichols Algebras
Over Nonabelian Groups With Commutator Subgroup $\mathbb{Z}_2$

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Abstract. In this article, we explicitly construct new finite-dimensional, link-indecomposable Nichols algebras with Dynkin diagrams of type $A_n, B_n, D_n, E_6, 7, 8, F_4$ over any group $G$ with commutator subgroup isomorphic to $\mathbb{Z}_2$. The construction is generic in the sense that the type just depends on the rank and center of $G$, and thus positively answers for all groups of this class a question raised by Susan Montgomery in 1995 [Mont95, AS02].

Our construction uses the new notion of a covering Nichols algebra as a special case of a covering Hopf algebra [Len12] and produces non-faithful Nichols algebras. However, we give faithful examples of Doi twists for type $A_3, B_3, D_4, F_4$ over several nonabelian groups of order 16 and 32. These are hence the first known examples of faithful, finite-dimensional, link-indecomposable Nichols algebras of rank >2 over nonabelian groups.

Contents

1. Preliminaries 6
2. Covering Nichols Algebras 8
  2.1. Twisted Symmetries 9
  2.2. Main Construction Theorem Over Abelian Groups 10
  2.3. Impact On The Dynkin Diagram: Folding 13
  2.4. Example: $A_2 \cup A_2$ over $\mathbb{Z}_2^2$ to $A_2$ over $D_4$ 18
3. Symplectic Root Systems 21
  3.1. A Symplectic $\mathbb{F}_p$-Vector Space On $\Gamma / \Gamma^p$ 21
  3.2. Symplectic Root Systems Over $\mathbb{F}_2$ For $ADE$ 22
4. Main Constructions For $[G, G] = \mathbb{Z}_2$ 24
  4.1. Unramified Cases $ADE \cup ADE \to ADE$ 27
  4.2. Example $A_4 \cup A_4 \to A_4$ 30
  4.3. Ramified Case $E_6 \to F_4$ 31
  4.4. Ramified Cases $A_{2n-1} \to B_n$ 33
  4.5. Disconnected Diagrams 34
5. Faithful Nichols algebras 36
  5.1. Doi Twists And Matsuo Mots Spectral Sequence 36
  5.2. Example: Type $A_3, B_3, D_4, F_4$ 37
Appendix: Nichols Algebras Over Groups Of Order 16 And 32 38
References 40
The Nichols algebra $B(M)$ of a Yetter-Drinfel’d module $M$ over a group $\Gamma$ is a quotient of the tensor algebra $\sum M$. It has a natural structure of a Hopf algebra in a braided category satisfying a certain universal property. Finite-dimensional Nichols algebras arise naturally e.g. as quantum Borel part in the classification of finite-dimensional pointed Hopf algebras [AS02], such as the small quantum groups $\bar{U}_q(g)$. Heckenberger classified all finite-dimensional Nichols algebras for $\Gamma$ abelian (e.g. [Heck05]) and Heckenberger and Schneider established the existence of a generalized root system and a Weyl groupoid for finite-dimensional Nichols algebras also over arbitrary finite groups [HS08]. However, the existence of a finite-dimensional Nichols algebra over a nonabelian group still seems to be a rather rare and difficult phenomenon. Andruskiewitsch et al. have constructed several finite-dimensional Nichols-algebras of rank 1 and established strong conditions to rule out the existence of an indecomposable finite-dimensional Nichols-algebras over many groups (e.g. [AFGV11]). Meanwhile, Schneider, Heckenberger and Vendramin have narrowed down the indecomposable finite-dimensional Nichols algebras of rank 2 to a list of 4 possible classes of groups ($G_2, G_3, G_4, \Sigma$) in recent papers [HS10] and [HV13], but the only known example of rank 2 is $\bar{M}S00$ over $\Gamma = D_4$ and more generally type $G_2$ in [HS10]. No examples of larger rank over nonabelian groups have been constructed so far.

In this article, we explicitly construct finite-dimensional link-indecomposable Nichols algebras with root systems of type $A_n, B_n, D_n, E_{6,7,8}, F_4$ and hence arbitrary rank over nonabelian groups $G$ that are central stem-extensions of an abelian group $\Gamma$, i.e.:

$$\Sigma^* = \mathbb{Z}_2 \to G \to \Gamma \quad \Sigma \subset [G, G] \cap Z(G)$$

As a side remark, we mention that the construction is a special case of the new notion of a covering Hopf algebra, applied to the bosonization of known finite-dimensional Nichols algebra over the abelian group $\Gamma$. The covering construction itself does not depend on $\Gamma$ being abelian: For example, in [Len12] we have constructed a covering Nichols algebra of dimension $24^2$ over $G = GL_2(\mathbb{F}_3)$, which is a $\mathbb{Z}_2$-stem-extension of $\Gamma = S_4$.

More concretely, we proceed in section 2.1 as follows: Suppose $M = \bigoplus_{i \in I} M_i$ is a semisimple Yetter-Drinfel’d module over the abelian finite group $\Gamma$ with simple 1-dimensional summands $M_i$, $i \in I$, diagonal braiding matrix $q_{ij}$ and known finite-dimensional Nichols algebra $B(M)$. Furthermore, suppose that a finite abelian group $\Sigma$ acts on the vector-space $M$, such that the $\Gamma$-gradation as well as the the self-braiding operators $c_{M_i, M_i} = q_{ii}$ and the monodromy operators $c_{M_i, M_j} c_{M_j, M_i} = q_{ij}q_{ji}$ are preserved. We usually assume the $\Sigma$-action induced from a permutation action on $I$. In this case we show that $\Sigma$ acts on the $q$-diagram and Dynkin-diagram of $M$ (having nodes $i \in I$) by graph automorphisms.
However, $\Sigma$ does not act on $M$ by Yetter-Drinfel’d module-automorphisms: The braiding matrix $q_{ij}$ itself is generally not preserved, but is supposed to be modified under the action of each $p \in \Sigma$ as prescribed by a bimultiplicative form $\langle \bar{g}_i, \bar{g}_j \rangle_p$ with respect to the $\Gamma$-graduation $\bar{g}_i, \bar{g}_j \in \Gamma$ of $M_i, M_j$. The bimultiplicative forms $\langle \bar{g}_i, \bar{g}_j \rangle_p$ will usually be induced from a group-2-cocycle $\sigma \in Z^2(\Gamma, \Sigma^*)$ via $\sigma(p) := \sigma(\bar{g}, \bar{h})(p)\sigma^{-1}(\bar{h}, \bar{g})(p)$. We call such an action a twisted symmetry action of $\Sigma$ on $M$ with respect to the 2-cocycle $\sigma$.

With these notions we construct in section 2.2 a covering Yetter-Drinfel’d module $\tilde{M}$ over a stem-extension $\Sigma^* \to G \to \Gamma$ with $\Gamma$ abelian as follows: We start with a $\Gamma$-Yetter-Drinfel’d module $M$ and a twisted permutation symmetry action of $\Sigma$ on $M$ (i.e. induced from $\Sigma$ permuting $I$) with respect to a group-2-cocycle $\sigma$ representing the given stem-extension. We decompose $M$ into simultaneous eigenspaces $M^{[\lambda]}$ for eigenvalues $\lambda \in \Sigma^*$ of the twisted symmetry action of $\Sigma$ and use this $\Sigma^*$-gradation to refine the $\Gamma$-gradation on $M$ to a $G$-gradation. Note that formerly $\Gamma$-homogeneous elements in $M$ are usually not $G$-homogeneous. By pulling back also the $\Gamma$-action on $M$ to a $G$-action, we obtain a covering Yetter-Drinfel’d module $\tilde{M}$ over the nonabelian group $G$. $\tilde{M}$ is isomorphic to $M$ as a braided vector-space. The Nichols algebra $\mathcal{B}(\tilde{M})$ of the covering Yetter-Drinfel’d module $\tilde{M}$ is called covering Nichols algebra and is isomorphic to $\mathcal{B}(M)$ as an algebra.

When we apply this construction in case $\Gamma$ abelian to a semisimple Yetter-Drinfel’d module $M = \bigoplus_{i \in I} M_i$ with $\Sigma$ acting by twisted permutation symmetries, then the different irreducible $\Gamma$-Yetter-Drinfel’d modules $M_i$ laying on an orbit of the twisted symmetry $\Sigma$ become a single irreducible $G$-Yetter-Drinfel’d module with increased dimension. This changes the Cartan matrix and hence the Dynkin diagram (see [HS08] Definition 6.4) of $\tilde{M}$ compared with $M$ as described in Theorem 2.16. Nodes of the Dynkin diagram of $M$ in a $\Sigma$-orbit give rise to a single node of the Dynkin diagram of $\tilde{M}$. The root system is reduced to the subsystem fixed by $\Sigma$ acting on the Dynkin diagram of $M$ by graph automorphisms. This behaviour is classically known as diagram folding of a Lie algebra by an outer automorphism (see e.g. the purely rootsystem approach in [Ginz06] p. 47f).

**Example.** (Section 4.3) There exists a 6-dimensional Yetter-Drinfel’d module $M$ over $\Gamma = \mathbb{Z}_2^4$ with $\Gamma$-homogeneous components of dimension 1, 1, 2, 2 which is the sum of 6 simple 1-dimensional Yetter-Drinfel’d modules $M_i$. The Dynkin diagram and Cartan matrix of $M$ are as the semisimple Lie algebra $E_6$ and the Nichols algebra $\mathcal{B}(M)$ has dimension $2^{36}$. Moreover, $M$ admits an action of $\Sigma = \mathbb{Z}_2$ by twisted symmetries, corresponding to a diagram automorphism of the $E_6$ rootsystem.

The covering Yetter-Drinfel’d module $\tilde{M}$ over the stem-extension $G = \mathbb{Z}_2^2 \times \mathbb{Z}_4$ is the sum of 4 simple Yetter-Drinfel’d modules $\tilde{M}_i$ of dimensions 1, 1, 2, 2. The covering Nichols algebra $\mathcal{B}(\tilde{M})$ is indecomposable, has also dimension $2^{36}$ and its Dynkin diagram is $F_4$. 
This corresponds to the classical fact, that there is an inclusion of the semisimple Lie algebras $F_4 \subset E_6$ as fixed points of the outer Lie algebra automorphism of $E_6$.

For twisted permutation symmetries of prime order $\Sigma = \mathbb{Z}_p$ we use the suggestive terms inert/split for $G$-nodes $\tilde{M}_i$ of dimension $1/p$. They correspond to $\Sigma$-orbits of length $1/p$ of $\Gamma$-nodes $M_i$ and by Lemma 2.15 to central/noncentral $G$-graduation (conjugacy class!). We also use the terms inert / ramified / split for $G$-edges between $G$-nodes $\tilde{M}_i, \tilde{M}_j$ that are both inert / one inert and one split / both split. They correspond to edges between $\Gamma$-nodes in different $\Sigma$-orbits of length $1, 1/p, p/p$ and by Theorem 2.16 to $G$-edges of type $A_2 / B_2 / A_2$. All these cases are shown in the preceding example for $p = 2$.

In order to summarize combinatorial considerations, we introduce in section 3.2 the notion of a symplectic root system and directly present the explicit examples relevant to this article. A symplectic root system for a given Cartan matrix resp. Dynkin diagram is a decoration of the diagram nodes by values in a finite symplectic vector-space, such that the Weyl group acts as symplectic isometries. Symplectic root systems provide non-trivial necessary and in our cases even sufficient conditions that allow to read off from the rank and center of a given group $G$ which diagrams admit a finite-dimensional covering Nichols algebra over $G$. For example, to realize the diagram $D_n$ the group $G$ needs to have a larger center than for other diagrams, and groups $G$ with even larger center can only support disconnected diagrams.

Note that we currently classify all symplectic root systems over $\mathbb{F}_2$ and found the one’s provided here to be the unique minimal symplectic root system for each Dynkin diagram. In [Len12] we also checked for all possible Nichols algebras over abelian groups $\Gamma$, whether they admit the covering construction for $\Sigma = \mathbb{Z}_p$: The Nichols algebras used in the present paper for $\Sigma \cong \mathbb{Z}_2$ are the only choices. However, there are additional diagrams not corresponding to semisimple Lie algebras that lead to new link-decomposable covering Nichols algebras with $\Sigma = \mathbb{Z}_2$ and even one $D_4 \rightarrow G_2$ for $\Sigma = \mathbb{Z}_3$. 
The application of the Construction Theorem proceeds case-by-case, depending on the assumed symmetric Dynkin diagram of $B(M)$ and the respective symplectic root system, which imposes certain restriction on $G$. For the constructions we assume $G$ to be of order $2^N$ to simplify statements. The general case then usually follows by $G = G_2 \times G_{odd}$ with $G_{odd}$ abelian, e.g. in Corollary 4.8. In each case we compactly describe dimension, root system and Hilbert series of the newly constructed covering Nichols algebra $B(\tilde{M})$.

- In section 4.1 we treat the generic, unramified case: Given a simply-laced Dynkin diagram of type ADE, we use the symplectic root system to define a $\Gamma$-Yetter-Drinfel’d module $M = N \oplus N_\sigma$ with finite-dimensional Nichols algebra $B(M)$, where $N \not\cong N_\sigma$ each have the given Dynkin diagram and the braiding matrices only contain entries $\pm 1$. The crucial aspect of the symplectic root system is that it ensures $N, N_\sigma$ have disconnected Dynkin subdiagrams in the Dynkin diagram of $M$ resp. $c_{NN_\sigma}c_{N_\sigma N} = id$. Then, interchanging $N, N_\sigma$ gives by construction an action of $\Sigma = \mathbb{Z}_2$ by twisted symmetries on $M$. Note that the well-known example of an indecomposable Nichols algebra over $G = D_4$ (see [MS00] resp. example section 2.4 in this article) is our model for this case and corresponds to the diagram $A_2$. We also give an example of this construction with diagram $A_4$.

- In section 4.3 we construct the exceptional example shown above, where the twisted symmetry acts on a single $\Gamma$-Yetter-Drinfel’d module $M$. The Nichols algebra $B(M)$ has Dynkin diagram $E_6$, while the covering Nichols algebra $B(\tilde{M})$ over $G$ has Dynkin diagram $F_4$. Thereby, ramified edges appear, that connect simple $G$-Yetter-Drinfel’d modules of different dimension. We use a symplectic root system to construct the split part of $M$ with diagram $A_2 \cup A_2$ and use an ad-hoc continuation by two inert nodes to $E_6$.

- In section 4.4 we construct an infinite family of ramified covering Nichols algebras of type $A_{2n-1} \to B_n$ similar to the previous case $E_6 \to F_4$. Again we use a symplectic root system for the split part of the diagram $A_{n-1} \cup A_{n-1}$ and an explicit continuation by one inert node to $A_{2n-1}$.

- In section 4.5 we describe how to construct Nichols algebras $B(\tilde{M})$ with disconnected Dynkin diagrams and prove in particular, that any group $G$ with $[G,G] \cong \mathbb{Z}_2$ (regardless of the order) admits at least one finite-dimensional indecomposable Nichols algebra.

We summarize the properties of the constructed Nichols algebras with connected Dynkin diagram in the following table, where the first and second column give necessary and
sufficient conditions on the group $G$, which is again assumed of order $2^N$. We use the conventions $(n \mod 2) \in \{0, 1\}$ and $\Phi^+(X_n)$ for the set of positive roots in the root system $X_n$ (the dimensions are 2-powers, because all self-braidings are $q_{\alpha\alpha} = -1$).

| $\dim_{\mathbb{F}_2}(\Gamma/\Gamma^2)$ | $\dim_{\mathbb{F}_2}(Z(G)/[G,G]G^2)$ | Dynkin-D. of $\tilde{M}$ | $\dim(B(M)) = \dim\left(B(\tilde{M})\right)$ |
|-----------------------------------|-----------------------------------|----------------|-----------------------------------|
| $n$                              | $n \mod 2$                        | $A_{n\geq 2}$ | $2|\Phi^+(A_n\cup A_n)| = 2^n(n+1)$ |
| $n = 6, 7, 8$                    | $n \mod 2$                        | $E_{6,7,8}$   | $2|\Phi^+(E_n\cup E_n)| = 2^{72}, 2^{126}, 2^{240}$ |
| $n$                              | $2 - (n \mod 2)$                 | $D_{n\geq 4}$ | $2|\Phi^+(D_n\cup D_n)| = 2^{2n(n-1)}$ |
| $n$                              | $2 - (n \mod 2)$                 | $F_4$         | $2|\Phi^+(E_6)| = 2^{36}$ |
| $n$                              | $2 - (n \mod 2)$                 | $B_{n\geq 3}$ | $2|\Phi^+(A_{2n-1})| = 2^n(2n-1)$ |

Especially, we find indecomposable Nichols algebras (possibly with disconnected Dynkin diagram) over all groups $G$, that are $\mathbb{Z}_2$-stem-extensions of an abelian group $\Gamma$ (Corollary 4.8) and thus positively answer for such groups a respective question raised by Susan Montgomory in [Mont95] for pointed Hopf algebras by providing the bosonizations $H = k[G]\#B(\tilde{M})$. See [AS02] Question 3.17 for the Nichols algebra formulation.

By construction, a Nichols algebra $B(\tilde{M})$ obtained this way is non-faithful, diagonal and as an algebra isomorphic to the corresponding Nichols algebra $B(M)$ over the abelian $\Gamma$. However, the knowledge of $H^2(G,k^\times)$ together with Matsumoto’s spectral sequence often allows to obtain Doi twists that are truly new faithful, non-diagonal, finite-dimensional, link-indecomposable Nichols Algebras. We give such examples of type $A_2$, $A_3$, $B_3$, $D_4$, $F_4$ over several nonabelian groups of order 16 and 32 in section 5.2.

### 1. Preliminaries

Throughout this article we suppose $k = \mathbb{C}$, all groups are finite and all vector-spaces finite-dimensional. $\mathbb{F}_p$ denotes the field with $p$ elements, the letters $\mathbb{D}_4$, $\mathbb{Q}_8$, $S_n$, $A_n$ denote the groups and $A_n$, $B_n$, $C_n$, $D_n$, $E_n$ the Dynkin diagrams of the semisimple Lie algebras of rank $n$. $k^\times$ denotes the multiplicative group of the field and $\Gamma^* = \text{Hom}(\Gamma, k^\times)$ the dual group. We frequently call $\theta$ the generator of the multiplicatively denoted group $\mathbb{Z}_2$.

The following notions are standard. We summarize them to fix notation and refer to [HLecture08] for a detailed account.

**Definition 1.1.** A Yetter-Drinfel’d module $M$ over a group $\Gamma$ is a $\Gamma$-graded vector-space,

$$M = \bigoplus_{g \in \Gamma} M_g$$
with a $\Gamma$-action on $M$ such that
\[ g.M_h = M_{gh^{-1}}. \]
To restrict ourselves to the most relevant cases, we call $M$
- indecomposable, iff the support $\{g \mid M_g \neq 0\}$ generates all $\Gamma$.
- minimally indecomposable, iff $M$ is link-indecomposable and no proper sub-Yetter-
  Drinfel’d module is indecomposable. Note that every indecomposable Yetter-Drinfel’d
  module contains a minimally indecomposable Yetter-Drinfel’d module.
- faithful, iff the $\Gamma$-action on $M$ is faithful.

**Lemma 1.2.** The map $c_{MM} : M \otimes M \to M \otimes M$ defined on homogeneous elements by
\[ M_g \otimes M_h \ni v \otimes w \mapsto c_{MM}(v \otimes w) = g.w \otimes v \in M_h \otimes M_{gh^{-1}}. \]
fulfills the Yang-Baxter-equation
\[ (id \otimes c_{MM})(c_{MM} \otimes id)(id \otimes c_{MM}) = (c_{MM} \otimes id)(id \otimes c_{MM})(c_{MM} \otimes id) \]
turning $M$ into a braided vector-space.

**Example 1.3.** For abelian groups $\Gamma$, the compatibility condition implies the stability of
the homogeneous components $M_g$. For $k = \mathbb{C}$ all simple Yetter-Drinfel’d modules $M_i$
are 1-dimensional and isomorphic to some $O_{g_i}^\chi := x_i k$ with $\Gamma$-gradation $g_i$
and $\Gamma$-action defined by a 1-dimensional character $\chi_i : \Gamma \to k^\times$ via $g.x_i := \chi_i(g)x_i$. The braiding $c_{MM}$
is hence diagonal with braiding matrix $q_{ij} := \chi_j(g_i)$.

\[ x_i \otimes x_j \overset{c_{MM}}{\rightarrow} q_{ij}(x_j \otimes x_i) \]

**Definition 1.4.** Let $M$ be a Yetter-Drinfel’d module over an arbitrary group $\Gamma$
and let $x_i \in M_{g_i}$ be a fixed homogeneous basis. Consider the tensor algebra $\Sigma M$, which can be
identified with the algebra of words in the letters $\{x_i\}_i$ and is again a $\Gamma$-Yetter-Drinfel’d
module. We uniquely define skew derivations $\partial_i : \Sigma M \to \Sigma M$ on this algebra by
- $\partial_i(1) = 0$
- $\partial_i(x_j) = \delta_{ij} 1$
- $\partial_i(ab) = \partial_i(a)b + (g_i,a)\partial_i(b)$

The Nichols algebra $B(M)$ is the quotient of $\Sigma M$ by the largest homogeneous ideal $\mathfrak{I}$
in degree $\geq 2$, that is invariant under all $\partial_i$. It is a $\Gamma$-Yetter-Drinfel’d module as well.

Following [HLecture08] we draw a $q$-diagram for an abelian Yetter-Drinfel’d module $M$
by drawing a node for each basis element $x_i$ spanning a respective 1-dimensional simple
summand $M_i = O_{g_i}^\chi = x_i k$ of $M$. We draw an edges between $x_i, x_j$
whenever $q_{ij}q_{ji} \neq 1$ (i.e. $c_{MM}^2 \neq id$) and decorate the nodes by the complex numbers $q_{ii}$
and edges by $q_{ij}q_{ji}$. It turns out that this data is all needed to determine the respective Nichols algebra.
Definition 1.5. The braided commutator of elements $x, y \in \mathcal{B}(M)$ is defined by the following ($\mu_{\mathcal{B}(M)}$ denotes the multiplication, $c_{\mathcal{B}(M)\mathcal{B}(M)}$ the braiding of the $\Gamma$-Yetter-Drinfel’d module $\mathcal{B}(M)$):

$$[x, y]_c := \mu_{\mathcal{B}(M)} \circ (id_{\mathcal{B}(M)\otimes\mathcal{B}(M)} - c_{\mathcal{B}(M)\mathcal{B}(M)})$$

Especially $[x_i, x_j]_c := x_i x_j - q_{ij} x_j x_i$. The extend this definition to nonabelian groups, it is usual to instead consider the braided commutator space $[M_i, M_j]_c \subset \mathcal{B}(M)$ of simple Yetter-Drinfel’d modules, for abelian groups this reduces to $M_i = x_i \mathbb{k}$.

Theorem 1.6. For $M$ a Yetter-Drinfel’d module over $\Gamma$ abelian, it can be shown that we get a basis of iterated bradided commutators and $[x_i, x_j]_c = 0$ in $\mathcal{B}(M)$ iff $q_{ij} q_{ji} = 1$ (hence no edge is drawn). More generally, we get commutator relations ($M_i = x_i \mathbb{k}$)

$$\max_m \left( [M_i, [M_i, [M_i, M_j]_c \cdots]_c \neq 0 \right) = \min_m \left( q_{ii}^{-m} = q_{ij} q_{ji} \text{ or } q_{ii}^{m+1} = 1 \right)$$

The structure constants $-m$ are important and can be also defined for nonabelian $\Gamma$:

Definition 1.7. The Cartan matrix with indices $i \neq j \in I$ corresponding to simple $\Gamma$-Yetter-Drinfel’d modules is defined as follows:

$$C_{i,j} := -\max_m \left( [M_i, [M_i, [M_i, M_j]_c \cdots]_c \neq 0 \right) \quad C_{i,i} := 2$$

One may draw a Dynkin diagram with node set $I$ and edges decorated by $(C_{i,j}, C_{j,i})$. For some cases pictorial representations are custom (e.g. double line and arrow for $(-1, -2)$).

If especially the Cartan matrix $C_{ij}$ is the Cartan matrix of a semisimple Lie algebra, then the Nichols algebra is closely related to the respective root system. However, several additional exotic examples of finite-dimensional Nichols algebras exist, that possess unfamiliar Dynkin diagrams, such as a multiply-laced triangle, and where Weyl reflections may connect different $\Gamma$-Yetter-Drinfel’d modules (yielding a Weyl groupoid). Heckenberger completely classified all Nichols algebras over abelian $\Gamma$ in [Heck05].

2. Covering Nichols Algebras

In what follows, we will usually suppose a central extension of an abelian group $\Gamma$:

$$1 \to \Sigma^* \to G \xrightarrow{n} \Gamma \to 1 \quad \Sigma \subset Z(\Gamma)$$

where $\Sigma^* = \text{Hom}(\Sigma, \mathbb{k}^\times)$. We denote elements in $G$ by letters such as $g$, whereas elements in the abelian quotient $\Gamma$ are denoted by $\tilde{g}$. A $\Gamma$-Yetter-Drinfel’d module is denoted by $M$, whereas the covering $G$-Yetter-Drinfel’d module will be denoted by $\tilde{M}$. 
2.1. Twisted Symmetries.

**Definition 2.1.** Suppose a Yetter-Drinfel’d module $M$ over an abelian group $\Gamma$ with

\[ M = \bigoplus_{i \in I} M_i \]

\[ \bigoplus_{i \in I} O_{M_i}^{\chi_i} = \bigoplus_{i \in I} x_i k \]

\[ q_{ij} = \chi_j(\bar{g}_i) \]

written as a sum of simple 1-dimensional Yetter-Drinfel’d modules $M_i$, $i \in I$.

We call a linear bijection $f_0 : M \to M$ a twisted symmetry, iff

- $f_0$ preserves the $\Gamma$-grading (resp. is colinear)
- $f_0$ preserves self-braiding and monodromy:

\[ (f_0 \otimes f_0) c_{M_i M_i} = c_{M_i M_i} (f_0 \otimes f_0) \]

\[ (f_0 \otimes f_0) c_{M_i M_j} c_{M_j M_i} = c_{M_i M_i} c_{M_j M_j} (f_0 \otimes f_0) \]

Note that the braiding itself needs not to be preserved. However, we usually wish to control the modification by the following notions, which do not depend on $\Gamma$ being abelian:

**Definition 2.2.** Let $M = \bigoplus_{i \in I} M_i$ be a semisimple and indecomposable $\Gamma$-Yetter-Drinfel’d module written as a sum of simple 1-dimensional Yetter-Drinfel’d modules $M_i$.

- We call $f_0$ a twisted symmetry with respect to a given bimultiplicative form $\langle, \rangle : \Gamma \times \Gamma \to k^\times$, iff $f_0$ is a twisted symmetry and

\[ (f_0 \otimes f_0) c_{M_i M_i} = \langle \bar{g}_i, \bar{g}_j \rangle (f_0 \otimes f_0) c_{M_i M_i} \]

Note that since $f_0$ preserves the monodromy $c_{M_i M_i}$, the form is always antisymmetric $\langle \bar{g}_i, \bar{h} \rangle = \langle \bar{h}, \bar{g} \rangle^{-1}$ and since $f_0$ preserves the self-braiding $c_{M_i M_i}$, the form is isotropic $\langle \bar{g}, \bar{g} \rangle = 1$. Such a form can thus be generally called symplectic.

- We call $f_0$ a twisted symmetry with respect to a group-2-cocycle $\sigma_0 \in Z^2(\Gamma, k^\times)$, iff it is a twisted symmetry with respect to the form

\[ \langle \bar{g}_i, \bar{h} \rangle := \sigma_0(\bar{g}_i, \bar{h}) \sigma_0^{-1}(\bar{h}, \bar{g}) \]

**Corollary 2.3.** Any twisted symmetry $f_0 : M \to M$ of an indecomposable Yetter-Drinfel’d module $M$ with respect to a given form $\langle, \rangle : \Gamma \times \Gamma \to k$ is an isomorphism of Yetter-Drinfel’d modules $M \to M_{\langle, \rangle}$, where $M_{\langle, \rangle}$ is $M$ as $\Gamma$-graduated vector-space and the $\Gamma$-action on a homogeneous element $v \in M_{\bar{h}}$ is modified to

\[ \bar{g}_{\langle, \rangle} v := \langle \bar{g}, \bar{h} \rangle (\bar{g} \cdot v) \]

**Remark 2.4.** In particular, any twisted symmetry $f_0$ of an indecomposable Yetter-Drinfel’d module $M$ with respect to a given group-2-cocycle $\sigma_0 \in Z^2(\Gamma, k^\times)$ is an isomorphism of Yetter-Drinfel’d modules $M \to M_{\sigma_0}$, where $M_{\sigma_0}$ has accordingly modified $\Gamma$-action

\[ \bar{g}_{\sigma_0} v := \sigma_0(\bar{g}, \bar{h}) \sigma_0^{-1}(\bar{h}, \bar{g})(\bar{g} \cdot v) \]
or equivalently twisted characters $\chi_{i}^{\sigma_{0}}$

$$\chi_{i}^{\sigma_{0}}(\bar{g}) := \sigma_{0}(\bar{g}, \bar{h})\sigma_{0}^{-1}(\bar{h}, \bar{g})\chi_{i}^{\sigma_{0}}(\bar{g})$$

Note that by e.g. [M08] Prop 5.2 this condition precisely means that $f_{0}$ can be extended to an isomorphism of Nichols algebras $B(M) \to B(M_{\sigma_{0}})$ to the Doi twist $B(M_{\sigma_{0}}) \cong B(M)_{\sigma_{0}}$.

Next, we want to consider a family of twisted symmetries that respect a group law:

**Definition 2.5.** Let $M$ be a Yetter-Drinfel’d module over an abelian group $\Gamma$ and $\Sigma$ another finite abelian group. We say that $\Sigma$ acts on $M$ as twisted symmetries iff

- $\Sigma$ acts on the vector-space $M$, i.e. denoting the action of an $p \in \Sigma$ by $f_{p} : M \to M$ we demand $f_{1} = id_{M}$ and $f_{p}f_{q} = f_{pq}$ for all $p, q \in \Sigma$.
- For each $p \in \Sigma$ the action $f_{p}$ is a twisted symmetry.

For a given group-2-cocycle $\sigma \in Z^{2}(\Gamma, \Sigma^{*})$, we say that $\Sigma$ acts on $M$ as twisted symmetry with respect to $\sigma$, iff for all $p \in \Sigma$ the twisted symmetry $f_{p}$ of $M$ is a twisted symmetry with respect to the group-2-cocycle

$$\sigma_{p}(\bar{g}, \bar{h}) := \sigma(\bar{g}, \bar{h})(p)$$

### 2.2. Main Construction Theorem Over Abelian Groups.

Suppose a given central extension of an abelian group $\Gamma$:

$$1 \to \Sigma^{*} \to G \xrightarrow{\pi} \Gamma \to 1 \quad \Sigma \subset Z(\Gamma)$$

It can be described in terms of a cohomology class of 2-cocycles

$$[\sigma] \in H^{2}(\Gamma, \Sigma^{*})$$

We fix a set-theoretic section $s : \Gamma \to \Gamma$ of $\pi$ which is normalized, i.e. $s(1) = 1$. This corresponds to a choice of a specific representing 2-cocycle $\sigma \in Z^{2}(\Gamma, \Sigma^{*})$ with $s(\bar{g})s(\bar{h}) = \sigma(\bar{g}, \bar{h})s(\bar{gh})$. Different choices of $s, \sigma$ will in what follows produce identical forms $\langle \bar{g}, \bar{h}\rangle_{p} := \sigma(\bar{g}, \bar{h})(p)\sigma^{-1}(\bar{h}, \bar{g})(p)$ and hence identical notions of twisted symmetry.

**Theorem 2.6.** Suppose now $M$ to be a Yetter-Drinfel’d module over $\Gamma$ and an action of $\Sigma$ on $M$ as twisted symmetries with respect to the $\sigma \in Z^{2}(\Gamma, \Sigma^{*})$ fixed above. Because $\Sigma$ is abelian, we may simultaneously diagonalize the action and decompose $M$ into eigenspaces $M^{[\lambda]}$ with simultaneously eigenvalues $\lambda \in \Sigma^{*}$. Then the following structures define a $G$-Yetter-Drinfel’d module $\tilde{M}$, which we call the covering Yetter-Drinfel’d module of $M$:

- $\tilde{M} := M$ as vector-space
- The $G$-action is the pullback of the $\Gamma$-action via $\pi$.
  Especially $\Sigma^{*} \subset G$ acts trivially and thus $\tilde{M}$ is not faithful.
- The eigenspaces $M^{[\lambda]}$ give rise to the $G$-homogeneous layers via $\tilde{M}_{h} := M^{[hs(h)^{-1}]}_{h}$.
  Note that $\pi(hs(h)^{-1}) = 1$ so, $\lambda := hs(h)^{-1}$ is indeed an element of $\text{Ker}(\pi) = \Sigma^{*}$.
Proof. In order to prove $M$ to be a well-defined Yetter-Drinfel’d module, we have to check that $\tilde{M}$ fulfills the (nonabelian) Yetter-Drinfel’d condition $g.\tilde{M}_h = \tilde{M}_{ghg^{-1}}$.

Claim 1: We first prove, that in $M$ the $\Gamma$-action permutes the simultaneous $f_p$-eigenspaces $M^{[\lambda]}$ as follows:

$$\bar{g}.M_h^{[\lambda]} = M_h^{[\sigma(\bar{g},\bar{h})\sigma^{-1}(\bar{h},\bar{g}) \cdot \lambda]}$$

This can just be calculated: Let $v \in M_h^{[\lambda]}$, i.e. the twisted symmetry action of $\Sigma$ is $\forall p \in \Sigma f_p(v) = \lambda(p)v$, then by the defining property of a twisted symmetry

$$f_p(\bar{g}.v) = \sigma_p(\bar{g},\bar{h})\sigma_p^{-1}(\bar{h},\bar{g})\bar{g}.f_p(v) = \sigma_p(\bar{g},\bar{h})\sigma_p^{-1}(\bar{h},\bar{g}) \cdot \lambda(p)v = (\sigma(\bar{g},\bar{h})\sigma^{-1}(\bar{h},\bar{g}) \cdot \lambda)(p) \cdot v$$

and thus $\bar{g}.v$ is a simultaneous eigenvector for all $f_p$ with eigenvalue $\sigma_p(\bar{g},\bar{h})\sigma_p^{-1}(\bar{h},\bar{g}) \cdot \lambda$ as claimed.

Claim 2: We prove that because $\Gamma$ is abelian, the commutator in $G$ can be expressed as

$$[G,G] \ni [g,h] = \sigma(\bar{g},\bar{h})\sigma^{-1}(\bar{h},\bar{g}) \in \Sigma^*$$

This is by definition of $\sigma$ true for elements $s(\bar{g}), s(\bar{h})$ in the image of the section $\text{Im}(s)$:

$$[s(\bar{g}), s(\bar{h})] = s(\bar{g})s(\bar{h}) \cdot s(\bar{g})^{-1} \cdot s(\bar{h})^{-1} = s(\bar{g})s(\bar{h}) \cdot s(\bar{g}\bar{h})^{-1} \cdot s(\bar{g})^{-1} \cdot s(\bar{h})^{-1} \quad (\Gamma \text{ abelian})$$

General elements $g, h \in G$ differ from such elements in $\text{Im}(s)$ by a factor in $\text{Ker}(\pi) = \Sigma^* \subset G$. Because $\Sigma^*$ was supposed central in $G$, this does not change the commutator $[g,h]$, while the right hand side of the claim anyway only depends on the images $\bar{g}, \bar{h} \in \Gamma$. Thus the claim holds in the general case as well.
Claim 1 and 2 immediately imply the asserted Yetter-Drinfel’d condition $g.\tilde{M}_h = \tilde{M}_{ghg^{-1}}$

\[
g.\tilde{M}_h = \tilde{g}.M_{\tilde{h}}^{[hs(\tilde{h})^{-1}]}
\]

(claim 1)
\[
M_{\tilde{h}}^{[\sigma(g,h)\sigma^{-1}(h,g).hs(\tilde{h})^{-1}]}
\]

(claim 2)
\[
M_{\tilde{h}}^{[ghg^{-1}h^{-1}.hs(\tilde{h})^{-1}]}
\]

(Γ abelian)
\[
M_{\tilde{h}}^{[ghg^{-1}s(\tilde{h})^{-1}]} = \tilde{M}_{ghg^{-1}}
\]

\[\square\]

**Corollary 2.7.** If $M$ is indecomposable and $G \to \Gamma$ is a stem extension $\Sigma^* \subset [G,G]$, then any $\tilde{M}$ is again indecomposable, because any preimage of any generating system of the quotient $\Gamma$ generates already $G$ (see e.g. [Hup83]).

**Lemma 2.8.** $M,\tilde{M}$ are isomorphic as braided vector-spaces, hence the Nichols algebras $B(M), B(\tilde{M})$ are isomorphic as algebras.

**Proof.** Let $v, w$ be $G$-homogeneous elements in $\tilde{M}$:
\[
v \in \tilde{M}_g = M_{\tilde{g}}^{[gs(\tilde{g})^{-1}]}
\]
\[
w \in \tilde{M}_h = M_{\tilde{h}}^{[hs(\tilde{h})^{-1}]}
\]

Because the $G$-action is the pullback of the $\Gamma$-action we find
\[
c_{\tilde{M}\tilde{M}}(v \otimes w) = w \otimes h.v
\]

(pullback action)
\[
= w \otimes \tilde{h}.v
\]
\[
= c_{MM}(v \otimes w)
\]

\[\square\]

**Remark 2.9.** Note that in [Len12] we gave much more general definitions:

- A Hopf algebra $H$ and a group $\Sigma$ of Bigalois objects $H_p$ yield a Hopf algebra structure on the direct sum, the covering Hopf algebra (see [Len12] Thm. 1.6):
\[
\Omega := \bigoplus_{p \in \Sigma} H_p
\]

fitting into an exact sequence of Hopf algebras (see [Len12] Thm. 1.13)
\[
0 \to \mathbb{k}^\Sigma \to \Omega \to H \to 0
\]

The covering Hopf algebra $\Omega$ thereby can only be pointed, if among others $H$ is pointed and $\Sigma$ is an abelian group.
• If we specialize this to the bosonization $H = k[\Gamma] \# B(M)$ of a Nichols algebra of a Yetter-Drinfel’d module $M$ over an arbitrary group $\Gamma$, we yield a covering Nichols algebra $B(M)$ over a central extension.

$$1 \rightarrow \Sigma^* \rightarrow G \rightarrow \Gamma \rightarrow 1$$

The construction in [Len12] Thm. 4.3 uses a newly defined coaction. The direct formulation in the Construction Theorem 2.6 follows after diagonalizing the twisted symmetries. As an example for $\Gamma$ nonabelian, we have also constructed e.g. a Nichols algebra of dimension $24^2$ over $GL_2(\mathbb{F}_3) \rightarrow S_4$.

2.3. Impact On The Dynkin Diagram: Folding. First we observe, that if a twisted symmetry of a $\Gamma$-Yetter-Drinfel’d module $M$ directly permutes the simple summands $M_i$ (and thus the index set $I$), then it is already an automorphism of the $q$-diagram of $M$:

**Definition 2.10.** Let $M = \bigoplus_{i \in I} M_i = \bigoplus_{i \in I} x_i$ be a vector-space decomposed into 1-dimensional sub-vector-spaces according to some index set $I$, e.g. the decomposition of a Yetter-Drinfel’d module over an abelian group $\Gamma$ decomposed into simple summands. Then an action $(f_p)_{p \in \Sigma}$ of a group $\Sigma$ on $M$ is called permutation action, iff it is induced by a permutation representation $\rho$ on $I$, i.e.

$$\rho : \Sigma \rightarrow \text{Aut}(I) = S_{|I|}, \quad \forall i \in I f_p(x_i) = x_{p.i}$$

If $M$ is moreover a diagonally braided vector-space with braiding matrix $q_{ij}$ with respect to the basis $\{x_i\}_{i \in I}$, then we denote the permuted braiding matrix by $(q^{(p)})_{ij} := q_{p.i,p.j}$.

**Remark 2.11.** Note that for $\Sigma$ a cyclic group, the weaker condition $f_p(x_k) = x_{p,k}$ of a $\Sigma$-action permuting the 1-dimensional subspaces $M_i$ already suffices to find a rescaled basis $y_i \in x_{k_i}$ that is preserved under the $\Sigma$-action. The same holds for other abelian groups with $H^2(\Sigma, k^\times)$, e.g. $\mathbb{Z}_2 \times \mathbb{Z}_3$.

The following sufficient condition for a twisted symmetry to be a twisted permutation symmetry follows immediately. Especially it shows that twisted symmetries on a minimally indecomposable Yetter-Drinfel’d modules are permutation symmetries. It will not be directly used in what follows, because we construct our covering Nichols algebras from explicit permutation anyways, but it shows why this choice is sufficient for our purposes: we want to construct minimally indecomposable Nichols algebras. Otherwise (e.g in Remark 4.3), we could also consider non-premutation representations of $\Sigma$.

**Corollary 2.12.** Suppose $M = \bigoplus_{i \in I} M_i$ is a Yetter-Drinfel’d module over an abelian group $\Gamma$, such that all 1-dimensional simple summands $M_i$ of $M$ are mutually non-isomorphic. Then every action of some cyclic group $\Sigma$ as twisted symmetries on $M$ is an action by twisted permutation symmetries.
We may now for abelian $\Gamma$ express the twisted symmetry condition of a given permutation action in terms of the permuted braiding matrix. We especially recognize $\Sigma$ to consist necessarily of automorphism of the $q$-diagram:

**Corollary 2.13.** Let $M = \bigoplus_{i \in I} M_i = \bigoplus_{i \in I} x_i$ be a Yetter-Drinfel’d module over an abelian group $\Gamma$ and $\Sigma$ a group with permutation action on the decomposed $M$. Then the action is a action by twisted symmetries according to Definition 2.5 iff

- The $\Sigma$-action on $I$ only permutes elements $i, j$ with $M_i, M_j$ in the same homogeneous component of $M$.
- The $\Sigma$-action on $I$ preserves the self-braiding $(q^{(p)})_{ii} = q_{ii}$.
- The $\Sigma$-action on $I$ preserves the monodromy $(q^{(p)})_{ij}(q^{(p)})_{ji} = q_{ij}q_{ji}$

Moreover, the $\Sigma$-action is a twisted symmetry with respect to a 2-cocycle $\sigma \in Z^2(\Gamma, \Sigma^*)$, iff the braiding matrix transforms under the action of all $p \in \Sigma$ according to the prescribed bimultiplicative form

$$\langle \bar{g}, \bar{h} \rangle_p := \sigma(\bar{g}, \bar{h})(p)\sigma^{-1}(\bar{h}, \bar{g})(p)$$

$$q^{(p)}_{ij} = \langle \bar{g}_i, \bar{g}_j \rangle_p q_{ij}$$

Hence especially the permutation action is an automorphism of the $q$-diagram of $M$, that has by definition node set $I$ and is decorated with $q_{ii}$ and $q_{ij}q_{ji}$.

Next we calculate the Dynkin diagram of the covering Nichols algebra $\mathcal{B}(\tilde{M})$ of a Nichols algebra $\mathcal{B}(M)$ of a Yetter-Drinfel’d module $M$ over an abelian group $\Gamma$. We restrict to specific scenarios appearing in the present article (especially $p = 2$), but similar calculations can be carried out for other situations as well.

**Definition 2.14.** Let $M = \bigoplus_{i \in I} M_i = \bigoplus_{i \in I} O_{\bar{g}_i}^{X_i}$ be some $\Gamma$-Yetter-Drinfel’d module and let $\Sigma = \mathbb{Z}_2 = \langle \theta \rangle$ act on $M$ as twisted permutation symmetries. Then $I$ decomposes into orbits of length 1 resp. 2. We call such nodes $i \in I$ “inert” resp. “split”.

In this situation the splitting behaviour of a node corresponds to centrality in $G$: 
Lemma 2.15. Let $\Sigma$ acts as twisted permutation symmetries: If a node $i \in I$ is inert, then necessarily its decoration $\bar{g}_i \in \Gamma$ has central image $s(\bar{g}_i) \in G$. If moreover all 1-dimensional simple summands $M_i$ of $M$ are mutually non-isomorphic (see Corollary 2.12), then the converse holds also: A node $i \in I$ is inert resp. split iff $s(\bar{g}_i) \in G$ is central resp. noncentral in $G$.

Hence $\tilde{M}$ decomposes into simple $G$-Yetter-Drinfel’d modules $\tilde{M}_k$, where the new nodes $\tilde{k} \subset I$ are $\Sigma$-orbits of cardinality 1 resp. 2 and are called inert resp. split as well. These subsets of $I$ hence form the new nodes set $\tilde{I}$.

Proof. Because we consider an action of $\Sigma = \mathbb{Z}_2$ on a set $I$, the orbits have length 1, 2.

For the first claim, we assume some $i_1 \in I$ with $s(\bar{g}_{i_1}) \notin Z(G)$ and prove $i_1$ to be a split node: By assumption of noncentrality there exist some $g \in G$ with commutator

$$[g, s(\bar{g}_{i_1})] = \theta^* \in \Sigma^* \subset G$$

This is, because $\Gamma$ is abelian and hence every commutator lays in the kernel $\Sigma^*$ of the central extension; if the commutator is nontrivial, it has to coincide with the generator $\theta^*$ of $\Sigma^* \cong \mathbb{Z}_2$, i.e. the element with $\theta^*(\theta) = -1_k$. By claim 2 in the proof of Theorem 2.6 we then have

$$\sigma(\bar{g}, \bar{g}_{i_1})\sigma^{-1}(\bar{g}_{i_1}, \bar{g}) = [g, s(\bar{g}_{i_1})] = \theta^*$$

We assumed $\Sigma$ to act by twisted permutation symmetries $f_p$, hence $f_{\theta}(M_{i_1}) =: M_{i_2}$ is another summand of $M$ and we wish to prove $i_1 \neq i_2$. This finally follows from Remark 2.4 as the twisted $\Gamma$-action on $M_{i_2}$ is

$$\bar{g}, \sigma_{\theta} v := \sigma_{\theta}(\bar{g}, \bar{h})\sigma^{-1}_{\theta}(\bar{h}, \bar{g}) \cdot \bar{g}, v = \sigma(\bar{g}, h)\sigma^{-1}(\bar{h}, \bar{g})(\theta) \cdot \bar{g}, v = \theta^*(\theta) \cdot \bar{g}, v = -\bar{g}, v$$

This is a different $\Gamma$-action, hence $M_{i_1} \not\cong M_{i_2}$ and $i_1 \neq i_2$ and thus the node is split.

For the second claim, assume now moreover that all 1-dimensional simple summands $M_i$ of $M$ are mutually non-isomorphic, then we also prove the converse: Suppose some $i_1 \in I$ with $s(\bar{g}_{i_1}) \in Z(G)$, then we prove $i_1$ to be inert: By assumption of centrality the commutator $[g, s(\bar{g}_{i_1})] = 1$ for all $g \in G$. By claim 2 in the proof of Theorem 2.6 we have

$$\sigma(\bar{g}, \bar{g}_{i_1})\sigma^{-1}(\bar{g}_{i_1}, \bar{g}) = [g, s(\bar{g}_{i_1})] = 1_{\Sigma^*}$$
We assumed \( \Sigma \) acts as twisted permutation symmetries, hence \( f_\theta(M_{i_1}) \) is also a summand \( M_{i_2} \) of \( M \); we wish to prove \( i_1 = i_2 \). By Remark \( 2.4 \), the twisted \( \Gamma \)-action on \( M_{i_2} \) is:

\[
\bar{g} \cdot \sigma_p v := \sigma_p(\bar{g}, \bar{h})\sigma_p^{-1}(\bar{h}, \bar{g})(\bar{g}, v) = \bar{g}, v
\]

Thus \( M_{i_1} \cong M_{i_2} \) and by the additional assumption hence \( i_1 = i_2 \) and \( i_1 \) is inert. \( \square \)

**Theorem 2.16.** Suppose again all \( M_i \) to be mutually nonisomorphic. We then determine the Cartan matrix \( \tilde{C}_{k,l} \) and hence Dynkin diagram of \( \tilde{M} \) in cases relevant to this article:

1. **Disconnected:** Let \( \tilde{k}, \tilde{l} \in \tilde{I} \) be arbitrary nodes (split or inert) and for all \( k \in \tilde{k} \subset I \) and \( l \in \tilde{l} \subset I \) let the \( q \)-diagram on these two elements be disconnected i.e. \( q_{kl} = 1 \) and \( [x_k, x_l]_c = 0 \). This corresponds by Theorem 1.6 to a diagonal Cartan submatrix of type \( A_1 \cup A_1 \) for \( B(M) \). Then the Cartan matrix of \( B(\tilde{M}) \) is also diagonal \( \tilde{C}_{kl} = 0 \), i.e. the covering nodes \( \tilde{k}, \tilde{l} \) of \( \tilde{M} \) are disconnected as well.

2. **Inert Edge:** Let \( \tilde{k} = \{k\}, \tilde{l} = \{l\} \subset I \) be inert nodes. Then the Cartan matrix entry in the covering Nichols algebra is of identical type \( \tilde{C}_{k,l} = C_{k,l} \)

3. **Split Edge:** Let \( \tilde{k} = \{k_1, k_2\} \) and \( \tilde{l} = \{l_1, l_2\} \) both be split and (after possible renumbering) let the Dynkin diagram of \( B(\tilde{M}) \) restricted to the 4 simple \( \Gamma \)-Yetter-Drinfel’d modules \( M_{k_1}, M_{k_2}, M_{l_1}, M_{l_2} \) be of type \( A_2 \cup A_2 \), i.e.

\[
[M_{k_1}, M_{l_1}]_c =: N_i \neq \{0\} \quad i = 1, 2
\]

where \( N_i \subset B(M) \) and all other braided commutators trivial.

Then the Cartan matrix entry in the covering Nichols algebra over \( G \) is of type \( A_2 \) with \( [\tilde{M}_k, \tilde{M}_l]_c = N_1 \oplus N_2 \subset B(\tilde{M}) \)

\[
\begin{pmatrix}
C_{kk} & C_{kl} \\
C_{lk} & C_{ll}
\end{pmatrix} =
\begin{pmatrix}
2 & -1 \\
-1 & 2
\end{pmatrix}
\]

See the example over \( G \cong \text{D}_4 \rightarrow \mathbb{Z}_2^2 \cong \Gamma \) in section 2.4.

4. **Ramified Edge:** Let \( \tilde{k} = \{k_1, k_2\} \in \tilde{I} \) be split and \( \tilde{l} = \{l\} \) be inert. Let the Dynkin diagram of \( B(\tilde{M}) \) restricted to the 3 simple \( \Gamma \)-Yetter-Drinfel’d modules \( M_{k_1}, M_{k_2}, M_l \) be of type \( A_3 \) with \( l \) the middle node, i.e.

\[
[M_{k_1}, M_l]_c =: N_i \neq \{0\} \quad i = 1, 2
\]

\[
[M_{k_2}, N_1]_c = [M_{k_1}, N_2]_c =: N_\sigma \neq \{0\}
\]
and all other braided commutators trivial. Then the Cartan matrix entry in the covering Nichols algebra over $G$ is of type $B_2$ with the split node $k$ the longer root:

$$
\begin{pmatrix}
C_{kk} & C_{kl} \\
C_{lk} & C_{ll}
\end{pmatrix} = \begin{pmatrix}
2 & -1 \\
-2 & 2
\end{pmatrix}
$$

This case can never occur isolated in indecomposable coverings, but the reader may check for example the ramified edge in $E_6 \to F_4$ in section 4.3.

**Proof.** By definition the nondiagonal Cartan matrix entries are

$$
\tilde{C}_{kl} := -\max_m \left( \prod_m \tilde{M}_i, \prod_m \tilde{N}_i \right)_{c} \neq \{0\}
$$

so we calculate the braided commutator spaces to verify all claims case-by-case. Thereby we use, that $B(\tilde{M}) \cong B(M)$ as an algebra (Lemma 2.8).

1. In this case all braided commutators are trivial:

$$
[\tilde{M}_i, \tilde{M}_j]_c = \bigoplus_{k \in \tilde{k}} \bigoplus_{i \in \tilde{i}} M_k, \bigoplus_{i \in \tilde{i}} M_l
$$

$$
= \sum_{k,l} [M_k, M_l]_c
$$

$$
= \{0\}
$$

2. We calculate the braided commutator spaces of $\tilde{M}$ over $G$ from the assumed type $A_2 \cup A_2$ of $M$ over $\Gamma$, i.e. both $N_i$ central:

$$
[\tilde{M}_k, \tilde{M}_l]_c = [M_{k_1} \oplus M_{k_2}, M_{l_1} \oplus M_{l_2}]_c
$$

$$
= [M_{k_1}, M_{l_1}]_c + [M_{k_1}, M_{l_2}]_c + [M_{k_2}, M_{l_1}]_c + [M_{k_2}, M_{l_2}]_c
$$

$$
= N_1 \oplus N_2
$$

$$
[\tilde{M}_k, \tilde{M}_l]_c = [M_{k_1} \oplus M_{k_2}, N_1 \oplus N_2]_c
$$

$$
= \{0\}
(3) We calculate the braided commutator spaces of \( \tilde{M} \) over \( G \) from the assumed type \( A_3 \) of \( M \) over \( \Gamma \), i.e. \( N_\sigma \) central:

\[
\left[ \tilde{M}_k, \tilde{M}_l \right]_c = [M_{k_1} \oplus M_{k_2}, M_l]_c
\]

\[
= N_1 \oplus N_2
\]

\[
\left[ \tilde{M}_k, \tilde{M}_l, \tilde{M}_j \right]_c = [M_{k_1} \oplus M_{k_2}, N_1 \oplus N_2]_c
\]

\[
= N_\sigma
\]

\[
\left[ \tilde{M}_k, \tilde{M}_l, \tilde{M}_j, \tilde{M}_i \right]_c = [M_{k_1} \oplus M_{k_2}, N_\sigma]_c
\]

\[
= \{0\}
\]

\[
\left[ \tilde{M}_i, \tilde{M}_j, \tilde{M}_k \right]_c = [M_i, N_1 \oplus N_2]_c
\]

\[
= \{0\}
\]

\[\Box\]

2.4. Example: \( A_2 \cup A_2 \) over \( \mathbb{Z}_2^2 \) to \( A_2 \) over \( \mathbb{D}_4 \). In [MS00] Milinski and Schneider gave examples of indecomposable Nichols algebras over the non-abelian Coxeter groups \( G = \mathbb{D}_4, S_3, S_4, S_5 \). We want to show how the first case \( G = \mathbb{D}_4 \) may be constructed as a covering Nichols algebra of a certain \( M \) with \( q \)-diagram \( A_2 \cup A_2 \) over \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) (case 2 in Theorem 2.16).

The example is interesting in our context at several points:

- Grañahad remarked, that the Nichols algebra possesses a “strange” non-homogeneous basis of precise type \( A_2 \cup A_2 \), which allowed Schneider and Milinski to more easily write down the relations, see [MS00] p. 21. The above construction precisely reproduces this basis as the formerly homogeneous basis of \( M \) that is no longer homogeneous in \( \tilde{M} \).
- The construction of an \( M \) with a twisted symmetry of order 2 is a model for the unramified construction case in section 4.1. The diagram consists of disconnected copies of a \( q \)-diagram interchanged by the twisted symmetry.
- The Nichols algebra itself is non-faithful and even diagonal. On the other hand, there is a Doi twist of \( B(M) \), which is a faithful indecomposable Nichols algebra over \( \mathbb{D}_4 \). We will produce faithful Doi twists as well in section 5.2.
For this exemplary construction we take the 4-dimensional diagonal Yetter-Drinfel'd module over $\Gamma := \mathbb{Z}_2^2 = \langle v, w \rangle$

$$M = \bigoplus_{i=0}^{4} \mathcal{O}_{\tilde{g}_i}^{\chi_i} = \bigoplus_{i=0}^{4} y_i \bar{g}$$

$$\tilde{g}_1 = \tilde{g}_3 = v \quad \tilde{g}_2 = \tilde{g}_4 = w$$

$$\chi_1 = \chi_4 = (-1, -1) \quad \chi_3 = (-1, +1) \quad \chi_2 = (+1, -1)$$

where the tuples denote the character value on $v, w$.

According to [Heck05], this $M$ is of type $A_2 \cup A_2$ and hence has relations as follows:

- Trivial braided commutators between the disjoint $A_2$ copies:
  $$[y_1, y_3]_c = 0 \quad [y_1, y_4]_c = 0 \quad [y_2, y_3]_c = 0 \quad [y_2, y_4]_c = 0$$

- Serre relations for the two $A_2$ copies
  $$[y_1, [y_1, y_2]]_c = [[y_1, y_2], y_2]_c = 0$$
  $$[y_3, [y_3, y_4]]_c = [[y_3, y_4], y_4]_c = 0$$

- Truncations from the rank 1 Nichols subalgebras $B(y_i k)$
  $$y_1^2 = y_2^2 = 0 \quad y_3^2 = y_4^2 = 0$$
  as well as from the additional rank 1 subalgebras $B([y_1, y_2]_c)$ and $B([y_3, y_4]_c)$:
  $$[y_1, y_2]_c^2 = 0 \quad [y_3, y_4]_c^2 = 0$$

Altogether, the multiplication in $B(M)$ gives rise to an isomorphism of vectorspaces

$$B(M) \cong B(y_1) \otimes B(y_2) \otimes B([y_1, y_2]_c)$$

$$\otimes B(y_3) \otimes B(y_4) \otimes B([y_3, y_4]_c)$$

$$\cong \mathbb{K}[y_1]/(y_1^2) \otimes \mathbb{K}[y_2]/(y_2^2) \otimes \mathbb{K}[y_3, y_4]/(y_3, y_4)^2$$

$$\otimes \mathbb{K}[y_3]/(y_3^2) \otimes \mathbb{K}[y_4]/(y_4^2) \otimes \mathbb{K}[y_3, y_4]/(y_3, y_4)^2$$

and hence the Hilbert series of the Nichols algebra is

$$\mathcal{H}(t) = (1 + t)(1 + t^2)(1 + t^2)(1 + t)(1 + t^2) = [2]_t^4 [2]_t^2$$

where $[n]_t := \frac{t^n - 1}{t - 1}$ and thus $[2]_t = 1 + t$. Especially $\dim(B(M)) = \mathcal{H}(1) = 2^6 = 64$. 
To perform a covering construction, consider the stem-extension of \( \Gamma \)
\[
\mathbb{Z}_2 \to D_4 \to \mathbb{Z}_2^2
\]
With \( a^4 = b^2 = 1 \) the usual generators of \( D_4 \) we choose a splitting \( s : D_4 \to \Gamma \) by sending the elements \( 1, v, w, vw \) to \( 1, b, ab, a^3 = bab \).

The group-2-cocycle \( \sigma \in Z^2(\Gamma, \Sigma^*) \) and especially the evaluation on the generator \( \theta \in \mathbb{Z}_2 \cong \Sigma \) can hence be calculated explicitly (rows, columns are labeled \( 1, v, w, vw \)):

\[
\sigma_\theta = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1
\end{pmatrix}
\]

To apply the Construction Theorem 2.6 we need an action of \( \Sigma \) on \( M \) by twisted symmetries with respect to \( \sigma \). Hence we first calculate from \( \sigma \) the nontrivial bimultiplicative form \( \langle \cdot, \cdot \rangle_\theta \) in Definition 2.2:

\[
\langle - , v \rangle_\theta = \sigma_\theta (- , v) \sigma_\theta^{-1} (v, -) = (+1, -1) \\
\langle - , w \rangle_\theta = \sigma_\theta (- , w) \sigma_\theta^{-1} (w, -) = (-1, +1)
\]

This form immediately determines the twisted characters by Corollary 2.3:

\[
\chi^\sigma_1 (-) = (+1, -1) \chi_1 (-) = \chi_3 (-) \\
\chi^\sigma_3 (-) = (+1, -1) \chi_3 (-) = \chi_1 (-) \\
\chi^\sigma_2 (-) = (-1, +1) \chi_2 (-) = \chi_4 (-) \\
\chi^\sigma_4 (-) = (-1, +1) \chi_4 (-) = \chi_2 (-)
\]

Hence switching \( y_1, y_3 \) respectively \( y_2, y_4 \) is a twisted symmetry with respect to the bimultiplicative form \( \langle - , - \rangle_\theta \) (Definition 2.2). Moreover, taking this map as \( f_\theta \) (and \( f_1 := id \)) defines an action of \( \Sigma \) on \( M \) by twisted symmetries with respect to the cocycle \( \sigma \) (Definition 2.5). The covering construction (Theorem 2.6) hence yields an indecomposable Nichols algebra of dimension \( \dim \mathcal{B}(\tilde{M}) = \dim \mathcal{B}(M) = 64 \) over \( G = D_4 \).
To connect to the notation in [MS00] we now also calculate the $G$-homogeneous components, as they follow from the construction theorem as $f_p$-eigenvectors to the trivial eigenvalue $1^* = \epsilon_{\Sigma}$ or the unique nontrivial eigenvalue $\theta^* \in \Sigma^*$ with $\theta^*(\theta) = -1$:

$$x_1 := y_1 + y_3 \in M^1_v = \tilde{M}_b$$

$$x_2 := y_2 + y_4 \in M^1_w = \tilde{M}_{ab}$$

$$x_3 := y_1 - y_3 \in M^{\theta^*}_v = \tilde{M}_{1,a2b}$$

$$x_4 := y_2 - y_4 \in M^{\theta^*}_w = \tilde{M}_{1,a3b}$$

3. Symplectic Root Systems

3.1. A Symplectic $\mathbb{F}_p$-Vector Space On $\Gamma/\Gamma_p$. Suppose we are given a finite group $G$ with commutator subgroup $[G, G] = \mathbb{Z}_p$. Such a group is clearly always a stem-extension of its abelianization $\Gamma = G/[G, G]$. As usual e.g. for p-groups (see e.g. [Hup83]) we consider the commutator map $[,]$, which is skew-symmetric and isotropic:

$$G \times G \xrightarrow{[,]} [G, G] = \mathbb{Z}_p$$

$$g, h \mapsto [g, h] = ghg^{-1}h^{-1}$$

$$[h, g] = [g, h]^{-1} \quad [g, g] = 1$$

Because $[G, G]$ is central, the map is multiplicative in both arguments:

$$[g, h][g', h] = (ghg^{-1}h^{-1})(g'hg'^{-1}h'^{-1})$$

$$= g(g'hg'^{-1}h^{-1})hg^{-1}h^{-1}$$

$$= gg'hg'^{-1}h^{-1}$$

$$= [gg', h]$$

and factors to $\langle , \rangle : \Gamma \times \Gamma \to \mathbb{Z}_p$. Because of bimultiplicativity, $[g^p, h] = [g, h]^p = 1$ holds and thus the commutator map even factorizes one step further to $V := \Gamma/\Gamma_p \cong \mathbb{F}_p^n$

$$V \times V \xrightarrow{\langle , \rangle} \mathbb{F}_p \quad \text{denoted additively}$$

Remark 3.1. Note that by claim 2 in the proof of Theorem 2.6, this bimultiplicative form coincides with the form

$$\langle g, \tilde{h} \rangle_\theta := \sigma(\bar{g}, \bar{h})(\theta)\sigma^{-1}(\bar{h}, \bar{g})(\theta)$$

associated by Definition 2.5 to any 2-cocycle $\sigma$ representing the present stem-extension

$$\langle \theta \rangle \cong \mathbb{Z}_p = \Sigma \to G \to \Gamma$$

Thus, the form $\langle , \rangle$ determines directly the notion of twisted symmetry in this situation.
Theorem 3.2 (Burnside Basis Theorem). For $|G| = 2^N$ every minimal generating set of $G$ consists precisely of $n = \dim(V)$ elements $g_1, \ldots, g_n$ with images a basis of $V$. This holds more generally for any $p$-group $G$ and $V := G/\Phi(G)$ for $\Phi(G)$ the Frattini group.

In what follows, we shall consider $V = G/([G, G]G^p)$ as a symplectic vector-space $\mathbb{F}_p^n$ with (possibly degenerate!) symplectic form $\langle v, w \rangle$. For a sub-vector-space $W \subset V$ we define the orthogonal complement:

$$ W^\perp := \{ v \in V \mid \forall w \in W \ \langle v, w \rangle = 0 \} $$

Especially $V^\perp = Z(G)/([G, G]^p)$ is the nullspace of vectors orthogonal on all of $V$ (note that always $\langle v, v \rangle = 0$). For $V^\perp = \{0\}$ we call $V$ nondegenerate.

It is well known (see e.g. [Hup83]) that there is always a symplectic basis $\{x_i, y_i, z_j\}_{i,j}$ consisting of mutually orthogonal nullvectors $z_j \in V^\perp$ and symplectic base pairs $\langle x_i, y_i \rangle = 1$ generating a maximal nondegenerate subspace. Note especially, that nondegenerate symplectic vector-spaces hence always have even dimension! They lead for example to extraspecial groups $G = p^{\dim(V)+1}$, especially for $p = 2$ and $\dim(V) = 2$ to $G = D_4, Q_8$.

3.2. Symplectic Root Systems Over $\mathbb{F}_2$ For ADE.

Definition 3.3. Given a symplectic vector-space $V$ over $\mathbb{F}_2$ and a graph $D$, we define a symplectic root system for this graph as a decoration $\phi : \text{Nodes}(D) \rightarrow V$, such that $\text{Im}(\phi)$ generates $V$ and nodes $i \neq j$ are connected iff $\langle \phi(i), \phi(j) \rangle = 1_{\mathbb{F}_2}$ (note that always $\langle v, v \rangle = 0$). If $\text{Im}(\phi)$ is even a $\mathbb{F}_2$-basis of $V$, we call the symplectic root system minimal.

Remark 3.4. We will use the notion for one directly on simply-laced Dynkin diagrams $D$, but also as tool for the ramified case $D'$, where only a part of the diagram $D \subset D'$ is split (such as $D = A_2, A_{n-1}$ for ramified $E_6 \rightarrow F_4$ and $A_{2n-1} \rightarrow B_n$). Minimal symplectic root systems thereby correspond to minimally indecomposable covering Nichols algebras.

Note that we currently classify all symplectic root systems over $\mathbb{F}_2$ and found unique minimal symplectic root systems for Cartan type Dynkin diagrams, that are given below. However, that the reader may directly verify that the following decorations indeed do form symplectic root systems for all simply-laced Dynkin diagrams.

Theorem 3.5. The graph of a simply laced Dynkin diagram of rank $n$ admits a minimal symplectic root system over the symplectic vector-space $V$ of dimension $n$ and typically minimal nullspace dimension $k = \dim(V^\perp)$:

- $k = 0$ for $n$ even, i.e. $V = \langle \{x_i, y_i\}_i \rangle_k$
- $k = 1$ for $n$ odd, i.e. $V = \langle \{x_i, y_i\}_i, z \rangle_k$
- $k = 2$ for type $D_n$ and $n$ even, i.e. $V = \langle \{x_i, y_i\}_i, z_1, z_2 \rangle_k$. 
Proof. Explicit symplectic root systems are given by the following decorations $\varphi$. One checks easily in every instance, that indeed $\langle \varphi(i), \varphi(j) \rangle = 1$ iff $i, j$ are connected:

$A_n$, $2|n$

$A_n$, $2 \nmid n$

$D_{n-2}$, $2 \nmid n$

$D_n$, $2|n$
4. Main Constructions For $[G, G] = \mathbb{Z}_2$

Suppose a nonabelian group $G$ with commutator subgroup $[G, G] = \mathbb{Z}_2$, which is hence a stem-extension $\Sigma^* = \mathbb{Z}_2 \to G \to \Gamma$ of an abelian group $\Gamma$. Using Heckenberger’s classification [Heck08] of finite-dimensional Nichols algebras $B(M)$ over abelian groups $\Gamma$ and symplectic root systems, we now construct finite-dimensional minimally indecomposable covering Nichols algebras $B(\tilde{M})$ with connected Dynkin diagram, depending on 2-rank and 2-center of $G$.

In section 5.2 we will furthermore give nondiagonal (and especially some faithful!) Doi twists with rank $\leq 4$ over various groups of order 16 and 32 and hence many new large-rank indecomposable faithful Nichols algebras over 2-groups.
Theorem 4.1. For any group $G$ of order $2^N$ with $[G, G] \cong \mathbb{Z}_2$ consider the following invariants

$$V := G/[G, G][G]^2 \cong \Gamma/\Gamma^2 \quad \text{dim}_{\mathbb{F}_2}(V) =: \text{2-rank}$$

$$V^\perp = Z(G)/[G, G][G]^2 \quad \text{dim}_{\mathbb{F}_2}(V^\perp) =: \text{2-center}$$

and denote $(n \mod 2) \in \{0, 1\}$. Then $G$ admits a finite-dimensional minimally indecomposable Nichols algebra $\mathcal{B}(\tilde{M})$ with the following connected Dynkin diagram depending on 2-rank and 2-center of $G$. They are covering Nichols algebras of a suitable $\mathcal{B}(M)$ over $\Gamma$:

• **Unramified** (generic) simply-laced components from a disconnected double with a symplectic root system (section 4.1):

| $\text{dim}_{\mathbb{F}_2}(V)$ | $\text{dim}_{\mathbb{F}_2}(V^\perp)$ | $M$ | $\tilde{M}$ |
|-----------------------------|-----------------------------------|-----|----------|
| $n$                         | $n \mod 2$                        | $A_n \cup A_n$ | $A_{n \geq 2}$ |
| $n$                         | $n \mod 2$                        | $E_n \cup E_n$ | $E_{n=6,7,8}$ |
| $n$                         | $2 - (n \mod 2)$                  | $D_n \cup D_n$ | $D_{n \geq 4}$ |

• **Ramified** components from a single diagram with an order 2 automorphism and a symplectic root system for the split part of the diagram $A_2, A_{n-1}$ (sections 4.3 and 4.4) leading to a decomposition $V := V_{\text{inert}} \oplus V_{\text{split}}$:

| $\text{dim}_{\mathbb{F}_2}(V)$ | $\text{dim}_{\mathbb{F}_2}(V^\perp)$ | $M$ | $\tilde{M}$ |
|-----------------------------|-----------------------------------|-----|----------|
| $2 + 2$                     | $= 4$                             | $2 + (2 \mod 2)$ | $E_6$ |
| $1 + (n - 1)$               | $= n$                             | $1 + (n - 1 \mod 2)$ | $A_{2n - 1}$ |
| $A_n \cup A_n$             | $A_{n \geq 2}$                   | $A_{n \geq 2}$ | $B_{n \geq 3}$ |

Note that the prescribed 2-rank and 2-center for each split part of a diagram precisely corresponds to the dimension and nullspace-dimension of the respective symplectic root system in Theorem 3.5.

Link-decomposable Nichols algebras of the following additional types with only one split node may appear:

• unramified $A_1 \cup A_1 \to A_1$
• ramified $A_3 \to B_2$
• ramified $D_{n+1} \to C_n$
• several non-Cartan diagrams alike $A_2, D_4, D_n$
• an isolated loop diagrams $A_2 \to A_1, q \in \mathbb{k}_3$.
• ramified $D_4 \to \Gamma_2$ which is the only covering with $\Sigma \cong \mathbb{Z}_3$.

Disconnected diagrams for any group $G$ of arbitrary order and arbitrary 2-rank $n = \text{dim}_{\mathbb{F}_2}(V)$ and 2-center $k = \text{dim}_{\mathbb{F}_2}(V^\perp)$: For every decomposition $(n, k) = \sum_i (n_i, k_i)$ with
all \((n_i, k_i)\) appearing in the list above, we can construct a covering Nichols algebra, which has connected components according to the table for each \((n_i, k_i)\).

**Proof.** The constructions asserted by this theorem are carried out case-by-case in the following sections [4.1-4.5].

Before we proceed to the proof, we comment on the classificatory value of this result, as well as possible outlooks to groups of higher nilpotency class:

**Remark 4.2.** Note that by the Reconstruction Theorem 5.1 in [Len12] every finite-dimensional indecomposable Nichols algebra \(\mathcal{B}(\tilde{M})\) over \(G\) with \([G, G] \cong \mathbb{Z}_2\) and trivial \([G, G]\)-action on \(\tilde{M}\) contains a connected component described by the preceding theorem.

To attempt a full classification for specific \(G\), we need to assume a nontrivial \(\Sigma^*\)-action. The action is severely restricted by general results of Schneider and Heckenberger [HS08] for finite-dimension Nichols algebras. On the other hand, certain actions may correspond to Doi-twists over \(G\) of Yetter-Drinfel’d modules with trivial action, for which our result applies.

By enumerating the possible \(\Sigma\)-actions using Matsumoto’s spectral sequence (see section 5.2), we may indeed show for specific \(G\) that Doi twists already exhaust all possible actions and hence coverings classify all finite-dimensional Nichols algebras. Successful examples of this approach for some groups of order 16 and 32 are [Len12] section 7.2-7.4.

**Remark 4.3.** Note that non-minimally indecomposable covering Nichols algebras over \(G\) might be interesting as well, especially because in this case there might exist a secondary covering Nichols algebra \(\mathcal{B}(\tilde{\tilde{M}})\) over a group \(\tilde{G}\) of nilpotency class 3, that use a twisted symmetry of the primary covering Nichols algebra \(\mathcal{B}(\tilde{M})\) over \(G\). Such a group would match the open case \(G_4\) in [HS10].

We have found non-minimal symplectic rootsystems as well, e.g. quotients of \(D_n\). As a conjecture, this might give rise to covering Nichols algebras

\[
D_n \cup D_n \rightarrow D_n \rightarrow C_n
\]

over a group \(\tilde{G}\) of type \(G_2\) and nilpotency class 3. In this case, \(\tilde{M}\) would be the sum of a unique simple Yetter-Drinfel’d module of dimension 4 and \(n - 1\) simple Yetter-Drinfel’d modules of dimension 2. The Nichols algebra \(\mathcal{B}(\tilde{M})\) would exhibit a \(C_n\)-Dynkin diagram, while the Hilbert series were the square of a Hilbert series of a Nichols algebra with diagram \(D_n\) over an abelian group \(\Gamma\).
4.1. **Unramified Cases** $ADE \cup ADE \to ADE$. The most natural and generic way to construct a Yetter-Drinfel’d module with twisted symmetry $\mathbb{Z}_2$ (or $\mathbb{Z}_p$) has already been demonstrated on the case $D_4$ in section 2.4; we take $\Gamma$-Yetter-Drinfel’d modules $N$ as well as $N_\sigma$ with modified $\Gamma$-action (Remark 2.4) and force twisted symmetry by considering $M := N \oplus N_\sigma$. The symplectic root system thereby assures that the diagram of $M$ consists of disconnected identical subdiagrams for $N, N_\sigma$ (while $N \not\cong N_\sigma$). We will subsequently calculate an explicit example for $A_4 \cup A_4 \to A_4$ in section 4.2.

**Theorem 4.4.** Suppose a simply-laced Dynkin diagram $X_n$ of rank $n$ and $G$ an arbitrary group of order $2^N$ with $[G, G] = \mathbb{Z}_2$ and $\Gamma := G/[G, G]$, such that

- $\dim_{\mathbb{F}_2}(V) = \dim_{\mathbb{F}_2}(\Gamma/\Gamma^2) = n \geq 2$
- $\dim_{\mathbb{F}_2}(V^\perp) = \dim_{\mathbb{F}_2}(Z(G)/[G, G]G^2) = (n \bmod 2)$ resp. $\frac{n}{2}$ for diagrams $D_{2m}$

Then there exists a $\Gamma$-Yetter-Drinfel’d module $M = N \oplus N_\sigma$ with $N, N_\sigma$ disconnected in $M$ and an twisted permutation action of $\Sigma \cong \mathbb{Z}_2$ interchanging $N \leftrightarrow N_\sigma$. The covering Yetter-Drinfel’d module over $G$ is hence $\tilde{M} = \bigoplus_{i=1}^n \tilde{M}_i$ of dimension $2n$ with:

- $[G, G]$ acts trivially on $\tilde{M}$, which is hence diagonal, but $V$ acts faithfully.
- $\tilde{M}$ is minimally indecomposable.
- $\mathcal{B}(\tilde{M})$ is finite-dimensional, with Hilbert series and dimension the square of the Hilbert series single diagram in the diagonal case over $\mathbb{Z}_2^n$, especially of dimension

$$\dim \left( \mathcal{B}(\tilde{M}) \right) = \mathcal{H}(1) = 2^{|\Phi(X_n \cup X_n)|}$$

- $\tilde{M}$ has the prescribed Cartan matrix and Dynkin diagram with all nodes $\tilde{M}_i$ of dimension 2 (i.e. underlying conjugacy class of length 2).

Several faithful Doi twist and hence nondiagonal Nichols algebras for small rank $D_4, A_2, A_3, A_4$ over various $G$ are given in section 5.2.
Proof. The strategy has been outlined above:

**Step 1:** We first construct a $\Gamma$-Yetter-Drinfel’d module $N := \bigoplus_{i=1}^{n} O_{\bar{g}_i}^{X_i}$, such that
- $N$ is minimally indecomposable
- The braiding matrix only contains $\pm 1$
- The quotient $V$ acts faithfully
- Nodes $i, j$ are connected iff $\langle \bar{g}_i, \bar{g}_j \rangle \neq 0$ (i.e. any lifts $g_i, g_j \in G$ discommute)
- The Nichols algebra is finite-dimensional and has the prescribed Dynkin diagram

This is done by using precisely the symplectic root systems constructed in sections 3.1-3.2: $V := G/G^2$ is a symplectic vector-space as described in the cited section with dimension $\dim_{\mathbb{F}_2}(\Gamma/\Gamma^2)$ and nullspace dimension $\dim_{\mathbb{F}_2}(Z(G)/[G, G]G^2)$. Hence the assumptions of the present theorem exactly match those of cit. loc. and we get a symplectic root system basis $\phi(i)(1 \leq i \leq n)$ of $V$, i.e. $\langle \phi(i), \phi(j) \rangle \neq 0$ iff $i, j$ are connected. Choose maximal cyclic $\bar{g}_i := \phi(i) \in \Gamma$ to define an indecomposable Yetter-Drinfel’d module $N (=coaction)$. We yet have to construct suitable characters $\chi_i : \Gamma \rightarrow k^\times (=action)$ that realize the given diagram with braiding matrix $\pm 1$. Because the $\phi(i)$ were a basis of $\Gamma/\Gamma^2$, there is exactly one $\chi_i$ such that $\chi_i(\bar{g}_j) = -1$ if $i = j$ or $i < j$ are connected and $+1$ otherwise. Then $N := \bigoplus_{i} O_{\bar{g}_i}^{X_i}$ has a braiding matrix with monodromy $q_{ij}q_{ji} = \pm 1$ depending on whether lifts $g_i, g_j$ discommute in $G$. Note by construction, as $\mathbb{F}_2$-matrix $\chi_1, \ldots, \chi_n$ is triangular, hence $V$ acts faithful, which also proves this part of the statement.

**Step 2:** The central extension in question is $\Sigma^* = \mathbb{Z}_2 \rightarrow G \rightarrow \Gamma$

Take a section $s$ and $\sigma \in Z^2(\Gamma, \Sigma^*)$ the respective cocycle, that especially describes the commutator map to $\Sigma^*$:

$$\sigma(a, b)\sigma^{-1}(\bar{b}, \bar{a}) = [a, b]$$

Thus the symplectic form describes the demand of the twisted symmetry on a $\Gamma$-Yetter-Drinfel’d module $M$ – take $\theta$ the generator of $\Sigma = \mathbb{Z}_2$, then the twisted $\Gamma$-action after applying $f_\theta$ on an element $v_\bar{b} \in M_\bar{b}$ reads as:

$$\bar{a}.f_\theta(v_\bar{b}) \overset{!}{=} \sigma_\theta(\bar{a}, \bar{b})\sigma_\theta^{-1}(\bar{b}, \bar{a})f_\theta(\bar{a}.v_\bar{b})$$

$$= (u(\bar{a}, \bar{b})u^{-1}(\bar{b}, \bar{a})) (\theta)f_\theta(\bar{a}.v_\bar{b})$$

$$= ((\bar{a}, \bar{b})) (p)f_\theta(\bar{a}.v_\bar{b})$$

Hence any decorating character on some decorating group element $\chi_k(g_i)$ picks up an additional $-1$ iff $[g_k, g_i] \neq 1$ iff $\langle \bar{g}_k, \bar{g}_i \rangle \neq 0$. 

Step 3: We now construct a $\Gamma$-Yetter-Drinfel’d module with an action $\Sigma = \mathbb{Z}_2$ by twisted permutation symmetries as in the example $D_4$ in section 2.4. We start with the indecomposable $N = \bigoplus_{i=1}^n N_i$ constructed in step 1. Then we add the necessary twisted image $f_\theta(N)$ for $\theta$ the generator of $\Sigma = \mathbb{Z}_2$ (see remark 2.4):

$$N_\sigma := N_{\sigma\theta} = N_\sigma(\theta)$$

$N_\sigma$ is hence the sum of simple Yetter-Drinfel’d modules $N_{\sigma i}$ given by the same group elements $\phi(i)$ but with twisted $\Gamma$-action:

$$\chi_{\sigma\theta}^i(\bar{b}) := \left(\langle \phi(i), \bar{b} \rangle \right)(\theta)\chi_i(\bar{b})$$

$\bar{a}_{\sigma\theta} v^i_\theta = \left(\langle \phi(i), \bar{b} \rangle \right)(\theta)$

By construction $M := N \oplus N_\sigma$ admits a twisted symmetry $f_\theta$ interchanging $N_i \leftrightarrow N_{\sigma i}$.

Step 4: We yet have to check that $M$ still has a finite Nichols algebra, so we determine its full Dynkin diagram – as intended, we prove now, that it really consists of two disconnected copies of the given one. First be reminded on Corollary 2.13 that twisted symmetries leave Dynkin diagrams and q-diagram invariant.

Hence the tricky part is, that there are no additional mixed edges between any $N_i \leftrightarrow N_{\sigma j}$. This is precisely where we need the specific base choice $\phi(i)$ and the fact that all $q_{ij} = \pm 1$. We have to calculate their mixed braiding factors:

$$q := q_{N_i, N_{\sigma j}, N_{\sigma j}, N_i}$$

$$= \chi_i(\phi(j))\chi_j(\phi(i))$$

$$= \chi_i(\phi(j)) \cdot \sigma_\theta(\phi(j), \phi(i))\sigma_\theta^{-1}(\phi(i), \phi(j))\chi_j(\phi(i))$$

$$= \langle \phi(i), \phi(j) \rangle(\theta)\chi_i(\phi(j))\chi_j(\phi(i))$$

$$= \langle \phi(i), \phi(j) \rangle(\theta)q_{ij} q_{ji}$$

We have to distinguish two cases that yield $q = 1$ in different ways:

- Suppose $i, j$ disconnected in the original diagram. Then $q_{ij} q_{ji} = 1$ and at the same time by construction $\langle \phi(i), \phi(j) \rangle = 0$, hence $q = 1$.
- Suppose $i, j$ connected by a single edge. Then $q_{ij} q_{ji} = -1$ and at the same time by construction $\langle \phi(i), \phi(j) \rangle \neq 0$, hence $= \theta^*(\theta) = 1$. Hence we again get $q = 1$.

Step 5: Thus we are done: We constructed a twist-symmetric indecomposable Yetter-Drinfel’d module $M$ over $\Gamma$ with finite-dimensional Nichols algebra and Hilbert series $H_M(t) = H_N(t)H_{\sigma}(t) = H_N(t)^2$. Hence the covering $G$-Yetter-Drinfel’d module $\tilde{M}$ is indecomposable and the Nichols algebra $B(\tilde{M})$ over $G$ has Hilbert series $H_{\tilde{M}}(t) = H_M(t)$, especially it is finite-dimensional. □
4.2. Example $A_4 \cup A_4 \rightarrow A_4$. We realize $A_4$ as prescribed over a group $G$ with 2-rank $\dim_{\mathbb{F}_2}(\Gamma/\Gamma^2) = 4$ and no 2-center $\dim_{\mathbb{F}_2}(\mathbb{Z}(G)/[G,G]G^2) = 0$, such as the extraspecial group $G = D_4^{+1} = D_4 \ast D_4$ (the central product identifies the two dihedral centers), which is generated by mutually discommuting involutions $x, y$ and $x', y'$, corresponding to a symplectic basis of the nondegenerate symplectic vector-space $V = \Gamma = \mathbb{F}_2^4$. We need a $\Gamma$-Yetter-Drinfel’d module of type $A_4 \cup A_4$ admitting an involutory twisted symmetry

$$M = N \oplus N_\sigma =: (M_1 \oplus M_2 \oplus M_3 \oplus M_4) \oplus (M_5 \oplus M_6 \oplus M_7 \oplus M_8)$$

where each $M_k = \mathcal{O}_{\tilde{g}_k}$ is 1-dimensional. The group elements are determined by the respective symplectic root system in Theorem 3.5:

$$\tilde{g}_1 = \tilde{g}_5 = x \quad \tilde{g}_2 = \tilde{g}_6 = y \quad \tilde{g}_3 = \tilde{g}_7 = xx' \quad \tilde{g}_4 = \tilde{g}_8 = y'$$

Then the characters $\chi_k$ for $k \leq 4$ were defined in such a way that $\chi_k(\tilde{g}_k) = -1$, and $\chi_k(\tilde{g}_l) = -1$ for edges $k < l$ and +1 else. This has to be basis-transformed to be expressed as row vector showing the values in the original basis $(\chi(x), \chi(y), \chi(x'), \chi(y'))$:

$$\chi_1 = (-1, -1, -1, +1) \quad \chi_2 = (+1, -1, -1, +1) \quad \chi_3 = (+1, +1, +1, -1) \quad \chi_4 = (+1, +1, +1, -1)$$

As generally calculated, the twisted characters $\chi_{4+k} = \chi_k^\sigma$ catch an additional $-1$ on every element $G$-discommuting with lifts of $\tilde{g}_k$ resp. non-orthogonal in $V$:

$$\chi_1 = (-1, +1, -1, +1) \quad \chi_2 = (-1, -1, +1, +1) \quad \chi_3 = (+1, -1, -1, +1) \quad \chi_4 = (+1, +1, -1, -1)$$

Altogether we find the following covering Nichols algebra $\mathcal{B}(\tilde{M})$ with Hilbert series $\mathcal{H}(t)$ and dimension $\dim(\mathcal{B}(\tilde{M}))$ as for $M$ and hence the square of the Hilbert series of the prescribed diagram $A_4$ of $M$ for $q = -1$ [Heck05]:

$$\mathcal{H}(t) = \left(2^4 \cdot 2^3 \cdot 2^2 \cdot 2^1 \cdot 2^0 \right)^2 = \left(2^{16} \cdot 2^6 \cdot 2^4 \cdot 2^2 \cdot 2^0 \right) \quad \dim(\mathcal{B}(\tilde{M})) = \mathcal{H}(1) = 2^{20} = 2^{2|\Phi^{+}(A_4 \cup A_4)|}$$

It even has a faithful and hence nondiagonal Doi twist by section 5.2.
4.3. **Ramified Case** $E_6 \to F_4$. The examples of the last two sections are “generic” in the sense, that they exploit a disconnected doubling of a rather arbitrary Dynkin diagram, and the very same diagram is reproduced in the nonabelian setting. Especially, every (nonabelian) edge corresponds to the $D_4$ example above; it is not allowed for the Dynkin diagrams to connect conjugacy classes of different length (e.g. abelian and nonabelian).

It turns out, that this ramified case is far more restrictive! We shall now give an example of this type, where the $\mathbb{Z}_2$-automorphism of a single $E_6$-diagram has a covering Nichols algebra with the non-simply-laced Dynkin diagram $F_4$:

![Dynkin Diagram](image)

**Theorem 4.5.** Suppose a group $G$ of order $2^N$ with $[G, G] = \mathbb{Z}_2$ and $\Gamma := G/[G, G]$ s.t.

- $\dim_{\mathbb{F}_2}(V) = \dim_{\mathbb{F}_2}(\Gamma/\Gamma^2) = 4$
- $\dim_{\mathbb{F}_2}(V^\perp) = \dim_{\mathbb{F}_2}(Z(G)/[G, G]Z(G)^2) = 2$

Then there exists a suitable $\Gamma$-Yetter-Drinfel’d module $M$ of type $E_6$ with an involutory diagram automorphisms. The covering $G$-Yetter-Drinfel’d module $\tilde{M}$ decomposes into 4 simple Yetter-Drinfel’d module like $\tilde{M} = \bigoplus_{k=1}^4 \tilde{M}_k$, has dimension 6 and moreover:

- $[G, G]$ acts trivially on $M$, which is hence diagonal, but $V$ acts faithfully.
- $\tilde{M}$ is minimally indecomposable.
- $\mathcal{B}(\tilde{M})$ has Hilbert series
  \[ H(t) = [2^6]t[2]t^2[2^5]t^3[2]t^5[2]t^6[2]t^7[2]t^8[2]t^9[2]t^10[2]t^{11} \]
  and is thus especially of dimension
  \[ \dim (\mathcal{B}(\tilde{M})) = H(1) = 2^{36} = 2^{\Phi(E_6)} \]
- $\tilde{M}$ has the Dynkin diagram $F_4$, where the long roots corresponds to conjugacy classes of length 2 and the short roots to a central elements (length 1).

There exists also a faithful Doi twist and hence a nondiagonal Nichols algebra over $\Gamma = \mathbb{Z}_2^2 \times D_4$, see section 5.2.
Proof. Denote by \( \bar{z}, \bar{z}', \bar{x}, \bar{y} \in \Gamma \) some lifts of a basis of the 4-dimensional symplectic vector-space \( V = \Gamma/\Gamma^2 \) with 2-dimensional nullspace, such that \( \bar{z}, \bar{z}' \) were nullvectors and \( \bar{x}, \bar{y} \) was a symplectic base pair in \( V \). This means that any further lifts to \( z, z', x, y \in G \) will obey:

\[
z, z' \in Z(\Gamma) \quad [x, y] \neq 1
\]

We directly construct the \( \Gamma \)-Yetter-Drinfel’d module \( \bigoplus_{k=1}^6 O_{\chi_k}^\Gamma \) of type \( E_6 \), but otherwise proceed as in the unramified case. Note that the following could also be derived systematically using the (rather trivial) symplectic root system \( \bar{x}, \bar{y} \) for the aspired split part of \( V \) and character via some ordering of the nodes, as it is done for the remaining ramified case below; but here we want to keep everything explicit! Further denote any character \( \chi \in \Gamma^* \) as row-vectors containing the basis images \( (\chi(\bar{z}), \chi(\bar{z}'), \chi(\bar{x}), \chi(\bar{y})) \), then \( M \) shall be (we’ve introduced additional signs for the faithfulness-statement):

One can check directly, that \( q_{ii} = -1 \) and the \( q_{ij}q_{ji} = \pm 1 \) exactly match the given diagram. Furthermore, already \( \chi_1, \chi_2, \chi_3, \chi_4 \) is \( \mathbb{F}_2 \)-linearly independent and \( z, z' \) have been constructed to act as \(-1\) on \( x, y \) resp. \( y \), hence the faithfulness assertions hold. This defined a proper Nichols algebra \( \mathcal{B}(M) \) of dimension \( 2^{36} \) and the prescribed Hilbert series by [HS10] Theorem 4.5.

We calculate now, that the \( E_6 \) diagram automorphisms \( f_\theta \) is here a twisted symmetry:

\[
\chi_1^\sigma(g_k) = \sigma_\theta(g_k, g_1)\sigma_\theta^{-1}(g_1, g_k)\chi_1(g_k) = \langle g_k, z \rangle \chi_1(g_k) = \chi_1(g_k)
\]

\[
\chi_3^\sigma(z) = \langle z, x \rangle \chi_3(z) = \chi_3(z) = +1 = \chi_5(z)
\]

\[
\chi_3^\sigma(z') = \langle z', x \rangle \chi_3(z') = \chi_3(z') = -1 = \chi_5(z')
\]

\[
\chi_3^\sigma(x) = \langle x, x \rangle \chi_3(x) = \chi_3(x) = -1 = \chi_5(x)
\]

\[
\chi_3^\sigma(y) = \langle y, x \rangle \chi_3(z') = -\chi_3(y) = +1 = \chi_5(y)
\]

This shows \( \chi_1^\sigma = \chi_1 \) and \( \chi_3^\sigma = \chi_5 \). The same calculations prove \( \chi_2^\sigma = \chi_2 \) and \( \chi_4^\sigma = \chi_6 \), hence the generator \( f_\theta : M \to M \) defines a twisted symmetry action of \( \Sigma = \mathbb{Z}_2 \) on \( M \). The covering Nichols algebra \( \mathcal{B}(M) \) over \( G \) then has the asserted properties. \( \square \)
4.4. **Ramified Cases** $A_{2n-1} \to B_n$. The second ramification will be treated more systematically, by completely reducing it to the unramified case $A_{n-1} \cup A_{n-1} \to A_{n-1}$ and an additional inert node causing an additionally ramified edge.

![Diagram](https://via.placeholder.com/150)

**Theorem 4.6.** Suppose a group $G$ of order $2^N$ with $[G,G] = \mathbb{Z}_2$ and $\Gamma := G/[G,G]$, s.t.
- $\dim \mathbb{F}_2(V) = \dim \mathbb{F}_2(\Gamma/\Gamma^2) = n \geq 3$
- $\dim \mathbb{F}_2(V^\perp) = \dim \mathbb{F}_2(Z(G)/[G,G]Z(G)^2) = 1 + 1_{n-1}$

Then there exists a suitable $\Gamma$-Yetter-Drinfel’d module of type $A_{2n-1}$ with an involutory diagram automorphisms. The covering Yetter-Drinfel’d module $\tilde{M}$ over $G$ has rank $n$, dimension $2n - 1$ and moreover:
- $[G,G]$ acts trivially on $\tilde{M}$, which is hence diagonal, but $V$ acts faithfully.
- $\tilde{M}$ is minimally indecomposable.
- $\mathcal{B}(\tilde{M})$ has Hilbert series
  $$\mathcal{H}(t) = [2]_{t_1}^{2n-1}[2]_{t_2}^{2n-2}[2]_{t_3}^{2n-3} \cdots [2]_{t_{2n-1}}^{1}$$
  and is thus especially of dimension
  $$\dim \left( \mathcal{B}(\tilde{M}) \right) = \mathcal{H}(1) = 2^{n(2n-1)} = 2^{\Phi(A_{2n-1})}$$
- $\tilde{M}$ has the nonabelian Dynkin diagram $B_n$ where the long roots corresponds to conjugacy classes of length 2 and the unique short root to a central element (i.e. a conjugacy class of length 1).

Exemplary nondiagonal and even faithful Doi twists of $B_3$ over various $\Gamma$ are given in section 5.2.

**Proof.** As in the ramified case $E_6 \to F_4$ above, we use the prescribed dimension $1 + (n - 1 \mod 2)$ nullspace of $V = \Gamma/\Gamma^2$ to decompose $V = \mathbb{F}_2 \oplus W$ with $\dim(W^\perp) = n - 1 \mod 2$ for the split nodes and $z \in Z(\Gamma)$ for the inert node.
Our main goal is to construct a $\Gamma$-Yetter-Drinfel’d module $M$ of dimension $1+2(n-1)$ and Dynkin diagram $A_{2n-1}$ with the involutory diagram automorphism a twisted symmetry. The starting point is the Yetter-Drinfel’d module constructed in the proof of section 4.1 of dimension $2(n-1)$ and Dynkin diagram $A_{n-1} \cup A_{n-1}$, numbered $2 \ldots 1+2(n-1)$, with an involutory twisted symmetry over the subgroup $\Gamma' \subset \Gamma$ generated by any lifts of $W$. Denote the leftmost nodes 2, 3 of both copies by $O_\chi', O_{\bar{g}}$. We extend all used characters trivially to $\Gamma$ except

$$\chi(\bar{z}) = -1 \quad \chi(\bar{g}) = -1 \quad \chi(\bar{g}_k) = +1$$

for all other $\bar{g}_k$, which is possible because $\bar{g} = \bar{g}_1, \ldots \bar{g}_n$ was a $W$-basis. Note that the former Yetter-Drinfel’d module had already been proven to be faithful over the $\Gamma$-quotient $W$, with $\bar{z}$ now acting trivial on all but the new node $M_1$, hence faithfulness of $V$ again holds.

**First** we have to check that $M$ indeed has decorated diagram $A_{1+2(n-1)}$ and hence the asserted Hilbert series by [HS10] Theorem 4.5. We’ve shown that already for the subdiagram $A_{n-1} \cup A_{n-1}$, and the additional node $M_1$ obeys for $k \geq 4$:

- $q_{11} = \chi(\bar{z}) = -1$
- $q_{12}q_{21} = \chi(\bar{g})\chi'(\bar{z}) = (-1)(+1) = -1$
- $q_{13}q_{31} = \chi(\bar{g})\chi''(\bar{z}) = (-1)(+1) = -1$
- $q_{1k}q_{k1} = \chi(\bar{g}_k)\chi_k(\bar{z}) = (+1)(+1) = +1$

**Secondly** we have to extend the established twisted symmetry $f_p$ of $A_{n-1} \cup A_{n-1}$ by $f_p(x_1) := x_1$, which is possible by $z$’s centrality in $G$:

$$\chi^\sigma(\bar{h}) = \sigma(\bar{z}, \bar{h})\sigma_p^{-1}(\bar{h}, \bar{z})\chi(\bar{h})$$

$$= (\bar{h}, \bar{z})\chi(\bar{h}) = \chi(\bar{h})$$

The covering Yetter-Drinfel’d module over $G$ then has the asserted properties. □

4.5. **Disconnected Diagrams.** So far we have constructed covering Nichols algebras $\mathcal{B}(M)$ with connected Dynkin diagram. We will now show, how disconnected diagrams can be realized. As a corollary, we will note that every group $G$ with $[G, G] = \mathbb{Z}_2$ and no restrictions on 2-rank and 2-center admit possibly disconnected finite-dimensional link-indecomposable Nichols algebras $\mathcal{B}(M)$. Note that in the next lemma we actually construct a whole family of covering Nichols algebras for arbitrary 2-rank and 2-center (without claiming these are all), but for the existence corollary, very simple choices suffice.
Lemma 4.7. Let $G$ be a group of order $2^N$ with $[G,G] \cong \mathbb{Z}_2$ and arbitrary 2-rank and 2-center
\[ n := \dim(V) = \dim_{\mathbb{F}_2}(\Gamma/\Gamma^2) \]
\[ k := \dim(V^\perp) = \dim_{\mathbb{F}_2}(Z(G)/[G,G]G^2) \]
For every numerical decomposition $(n,k) = \sum_i (n_i,k_i)$, where all $(n_i,k_i)$ appear as 2-rank and 2-center in the list of Theorem 4.1, we can construct a covering Nichols algebra, which has connected components as prescribed above.

Proof. Consider again $V = \Gamma/\Gamma^2$ as a symplectic $\mathbb{F}_2$-vector-space. The assumed decomposition $(\dim(V), \dim(V^\perp)) = \sum_i (n_i,k_i)$ implies an orthogonal decomposition of $V$ as symplectic vector-space into
\[ V \cong \bigoplus_i V_i \]
where $(n_i,k_i) = (\dim(V_i), \dim(V_i^\perp))$. Apply the constructions of sections 4.1–4.4 that yield $\Gamma$-Yetter-Drinfel’d modules $M^{(i)}$ with $\mathcal{B}(M^{(i)})$ having a connected Dynkin diagram of the respective type and consider
\[ M := \bigoplus_i M^{(i)} \]
where the action of $V_i$ on $M^{(j)}$ for $i \neq j$ is trivial. Hence $c_{M^{(i)}M^{(j)}c_{M^{(j)}M^{(i)}}} = id$, thus we have trivial braided commutator $[\mathcal{B}(M^{(i)}), \mathcal{B}(M^{(j)})]_c = 0$ and the multiplication in $\mathcal{B}(M)$ yields an isomorphism of vector-spaces
\[ \mathcal{B}(M) \cong \bigotimes_i \mathcal{B}(M^{(i)}) \]
and the Dynkin diagram of $\mathcal{B}(M)$ consists of mutually disconnected components, each of the respective type of $\mathcal{B}(M^{(i)})$.

Take $f_\theta$ the sum of the twisted symmetries employed in the construction of each $M^{(i)}$. Then the covering Nichols algebra $\tilde{M}$ has as Dynkin diagram the mutually disconnected Dynkin diagrams of each $\mathcal{B}(\tilde{M}^{(i)})$ as they follow from the respective construction. □

Observe especially, that a group $G$ of arbitrary order and $[G,G], \mathbb{Z}_2$ decomposes into $G = G_2 \times G_{\text{odd}}$. Every $(n,k)$ may be decomposed e.g. into $(n-k,0) + (0,k)$ for $n$ even resp. $(n-k+1,1) + (0,k-1)$ for $n$ odd, leading to a disconnected Dynkin diagram with a split part of type $A_n$ and an arbitrary inert part over $G_2$. We can add an arbitrary inert part over $G_{\text{odd}}$. Hence we’ve proven

Corollary 4.8. Every group $G$ of arbitrary order with $[G,G] \cong \mathbb{Z}_2$ admits a finite-dimensional link-indecomposable Nichols algebra $\mathcal{B}(\tilde{M})$. 


5. Faithful Nichols algebras

The covering Nichols algebras $B(\tilde{M})$ over $G$ constructed in this article are by construction non-faithful, because the $G$-action is the pullback of the action of the quotient $\Gamma$. Especially the commutators $[G,G]$ act trivially, so the braiding of $\tilde{M}$ is still diagonal.

However, over $G$ there may exist Doi twists $B(\tilde{M})_\eta$ by a $G$-group-2-cocycle $\eta \in Z^2(G, k^\times)$, such that the action of the subgroup $\Sigma^* = [G,G]$ we extended $G$ with is nontrivial. Then $\tilde{M}$ has a nondiagonal braiding, and it even may be faithful by ad-hoc arguments.

5.1. Doi Twists And Matsumotets Spectral Sequence. We already noted in Remark 2.4, that a Doi twist of the Nichols algebra produces the following twisted action on the twisted Yetter-Drinfel’d module $\tilde{M}_\eta$:

$$a_\eta v_h = \eta(aga^{-1}, a)\eta^{-1}(a,g)a.v_h$$

Hence the central $\Sigma^*$ with trivial action on $\tilde{M}$ acts by multiplication of the scalar

$$\gamma(\eta)(a,g) := \eta(g,a)\eta^{-1}(a,g)$$

$$\gamma(\eta) \in \Sigma^* \otimes G$$

This expression appears already in literature on group cohomology, namely in Matsumoto’s extension [IM64] for central group extensions of the general Lyndon-Hochschild-Serre spectral sequence:

$$1 \rightarrow \Gamma^* \rightarrow G^* \rightarrow \Sigma \rightarrow H^2(\Gamma, k^\times) \rightarrow H^2(G, k^\times)_{\Sigma} \rightarrow \Sigma^* \otimes G$$

Here, $H^2(G, k^\times)_{\Sigma}$ denotes the kernel of the restriction map to $\Sigma^*$ and the map $\gamma$ yields as expected a bimultiplicative pairing that exactly matches the expression above!

**Theorem 5.1.** Let $\Sigma^* = \mathbb{Z}_p \rightarrow G \rightarrow \Gamma$ be a stem-extension, $M$ a $\Gamma$-Yetter-Drinfel’d module with finite-dimensional Nichols algebra $B(M)$ and $\tilde{M}$ be the covering $G$-Yetter-Drinfel’d module. If we assume that the following holds

$$\frac{p^n|H^2(G, k^\times)|}{|H^2(\Gamma, k^\times)|} > 1$$

then there exists a group-2-cocycle $\eta \in Z^2(G, k^\times)$, such that the Doi twist $\tilde{M}_\eta$ has nontrivial action of $\Sigma^*$ and is hence nondiagonal.

**Proof.** We use Matsumoto’s sequence to enumerate the number of different $\Sigma^*$-actions $|\text{Im}(\gamma)|$, that can be achieved by Doi twisting. Note that

- For stem-extensions we have $G^* = \Gamma^*$, so the first terms disappear.
- In our case $\Sigma^* = \mathbb{Z}_p$ we have $H^2(\Sigma^*, k^\times) = 1$, so the restriction kernel is all $H^2(G, k^\times)_{\Sigma} = H^2(G, k^\times)$
so Matsumoto’s sequence takes the following form
\[ 1 \rightarrow \Sigma \rightarrow H^2(\Gamma, k^\times) \rightarrow H^2(G, k^\times) \xrightarrow{\sim} \Sigma^* \otimes G \]

In particular, counting elements shows the claim:
\[ |\text{Im}(\gamma)| = |H^2(G, k^\times)| \cdot |H^2(\Gamma, k^\times)|^{-1} \cdot |\mathbb{Z}_p| > 1 \]

□

Remark 5.2. This approach has also classificatory value in special cases: In \[\text{[Len12]}\] sections 7.2-7.4 we prove for several exemplary groups \(G\) of order 16 and 32, that these Doi twists already exhaust all \(\Sigma^*\)-actions, that are possible on a \(G\)-Yetter-Drinfel’d module with finite dimensional Nichols algebra by \[\text{[HS08]}\]. Thereby, by Remark 4.2, all finite-dimensional Nichols algebras over \(G\) are Doi twists of covering Nichols algebras, which we can write down explicitly.

Especially for the last cases in \[\text{[Len12]}\] section 7.4, having \(G = \mathbb{Z}_2^3\) and a certain commutator structure, there is no possible covering Nichols algebra and this disproves existence of finite-dimensional link-indecomposable Nichols algebras over these groups at all.

5.2. Example: Type \(A_3, B_3, D_4, F_4\). In addition to the known \(A_2\) Nichols algebra over \(G = \mathbb{D}_4 \rightarrow \mathbb{Z}_2^3 = \Gamma\) (example section 2.4 and already in \[\text{[MS00]}\]), we can construct the following faithful link-indecomposable finite-dimensional covering Nichols algebras \(\mathcal{B}(\tilde{M})\) with connected Dynkin diagram for certain stem-extensions \(\mathbb{Z}_2 ightarrow G ightarrow \mathbb{Z}_2^n=3,4\)

- Unramified \(A_3 \cup A_3 \rightarrow A_3\) over group \#10 in \[\text{Group16}\] of order 16.
- Unramified \(A_3 \cup A_3 \rightarrow A_3\) over \(G = \mathbb{D}_4 \times \mathbb{Z}_2\)
- Ramified \(A_5 \rightarrow B_3\) over \(G = \mathbb{D}_4 \times \mathbb{Z}_2\)
- Unramified \(D_4 \cup D_4 \rightarrow D_4\) over \(G = \mathbb{D}_4 \times \mathbb{Z}_2^3\)
- Ramified \(E_6 \rightarrow F_4\) over \(G = \mathbb{D}_4 \times \mathbb{Z}_2^3\)
- Unramified \(A_3 \cup A_3 \rightarrow A_3\) over group \#12 in \[\text{Group32}\] of order 32.
- Ramified \(A_5 \rightarrow B_3\) over group \#12 in \[\text{Group32}\] of order 32.

Several other groups appear possess at least a nondiagonal Doi-twist, where the action of \(\Sigma\) is nontrivial. The full table for groups of order 16 and 32 is given in the appendix.

In all cases, the covering Nichols algebras was constructed in Theorem 2.6, while existence of a nontrivial Doi twist and faithfulness follows as in the preceding paragraph:

- The application of Theorem 5.1 requires a large enough cohomology. This has been checken in \[\text{[Len12]}\] and leaves all nondiagonal examples given above.
- Faithfulness of the \(\Sigma\)-action follows by one of the following ad-hoc arguments:
  - Suppose e.g. \(\Sigma^* = [G, G] = Z(G)\), then the entire center acts faithful on \(\tilde{M}\), while noncentral elements cannot act diagonal after all, so \(G\) acts faithful.
  - The same argument holds, if \(Z(G)\) is cyclic and \(\Sigma^* \subset Z(G)\), because then for \(\Sigma^*\) to act nontrivial, already a generator of the center has to do so.
**Appendix: Nichols Algebras Over Groups Of Order 16 And 32**

In [Len12] we have used Theorem 2.6 to find covering Nichols algebras and even determine nondiagonal and faithful Doi twists on all groups $G$ of order 16 and 32 – note that most of these 2-groups are stem-extensions. We also point to the other cases discussed as outlook in Remark 4.3 and 5.2.

We give this list in the following and denote by the symbol $X_n$ any Dynkin diagram and by the superscript $[I], [U], [R]$ inert, split or ramified coverings. Note that the absence of the statement “faithful” does not necessarily exclude the existence of faithful Nichols algebras, but the existence of faithful Doi twists of covering Nichols algebras.

The list has been calculated by hand using the relations in the classification [Group16] and [Group32]. Each group was recognized by first the rank $\dim_{\mathbb{F}_2}(G/[G,G]G^2)$ and then the commutator structure $[G,G]$. Most $G$ are stem extensions of an abelian group $\Gamma = G/[G,G]$. Then Theorem 4.1 provides Nichols algebra of the types given in the table.

In [Len12] we calculated also all cohomology groups $H^2(G,k^\times)$ in order to derive the “nondiagonal”-statement. The “faithful”-statements are derived by ad-hoc arguments (for both see section 5.2).

| Group $\Gamma$ (a) | Known as.. | Nichols algebra | Orbifolds $G = \Sigma, (\Gamma)$ section |
|---------------------|-------------|-----------------|---------------------------------------|
| #1 – 5              | abelian     | (YES, faithful) $X_n^{[I]}$, $n = 1 \ldots 4$ |             | Heck05 |
| #6                  | $\mathbb{Z}_2 \times D_4$ | YES, faithful $\mathbb{Z}_2.(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2)$ | $A_3^{[U]}$, $X_1^{[I]} \cup A_2^{[U]}$ | 4.1 |
| #7                  | $\mathbb{Z}_2 \times Q_8$ | YES $\mathbb{Z}_2.(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2)$ | $B_3^{[R]}$ | 4.4 |
| #8                  | $\mathbb{Z}_4 \star D_4 \cong \mathbb{Z}_4 \star Q_8$ | YES $\mathbb{Z}_2.(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2)$ | $A_3^{[U]}$, $X_1^{[I]} \cup A_2^{[U]}$ | 4.1 |
| #9                  | $(\Gamma, \Gamma) \not\subset \Gamma^2$ | YES, nondiag. $\mathbb{Z}_2.(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$ | $B_3^{[R]}$ | 4.4 |
| #10                 | $(\Gamma, \Gamma) \subset \Gamma^2$ | YES, faithful $\mathbb{Z}_2.(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$ | $A_2^{[U]}$ | 4.1 |
| #11                 | $(\Gamma, \Gamma) \subset \Gamma^4$ | YES $\mathbb{Z}_2.(\mathbb{Z}_2 \oplus \mathbb{Z}_4)$ | $A_2^{[U]}$ | 4.1 |
| #12 – 14            | $D_8, D_8, Q_{16}$ | (?) Class 3 $\mathbb{Z}_2(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ | $A_3^{[U]}$, $A_2^{[U]}$ | (b) |

(a) From the classification [Group16]  
(b) Noncentral extension. This is case $G_4$ in [HS10], see Remark 4.3.
| Group $\Gamma$ (c) | Known as.. | Nichols algebra | Orbifolds $G = \Sigma.(\Gamma)$ | section |
|-------------------|------------|----------------|-------------------------------|---------|
| #1 - 7 abelian    | (YES, faithful) | $X^{[f]}_n$, $n = 1 \ldots 5$ | | 
| #8                | $\mathbb{Z}_2 \times D_4$ YES, faithful | $\mathbb{Z}_2.(\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2)$ | | 
| #9                | $\mathbb{Z}_2 \times Q_8$ YES | $\mathbb{Z}_2.(\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2)$ | | 
| #10               | $\mathbb{Z}_2 \times (\mathbb{Z}_4 \ast D_4)$ YES | $\mathbb{Z}_2.(\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2)$ | | 
| #11               | $\mathbb{Z}_2 \times #16^9$ YES, nondiag. | $\mathbb{Z}_2.(\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2)$ | | 
| #12               | $\mathbb{Z}_2 \times #16^{10}$ YES, faithful | $\mathbb{Z}_2.(\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_4)$ | | 
| #13               | $\mathbb{Z}_2 \times #16^{11}$ YES | $\mathbb{Z}_2.(\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_4)$ | | 
| #14               | $\mathbb{Z}_4 \times D_4$ YES, non-diag. | $\mathbb{Z}_2.(\mathbb{Z}_4 + \mathbb{Z}_2 + \mathbb{Z}_2)$ | | 
| #15               | $\mathbb{Z}_4 \times Q_8$ YES | $\mathbb{Z}_2.(\mathbb{Z}_4 + \mathbb{Z}_2 + \mathbb{Z}_2)$ | | 
| #16, 17           | YES | $\mathbb{Z}_2.(\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_4)$ | | 
| #18               | $([\Gamma, \Gamma] \not\subset \Gamma^2)$ YES, non-diag. | $\mathbb{Z}_2.(\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_4)$ | | 
| #19               | $([\Gamma, \Gamma] \subset \Gamma^2)$ YES | $\mathbb{Z}_2.(\mathbb{Z}_4 + \mathbb{Z}_4)$ | | 
| #20, 21           | $([\Gamma, \Gamma] \not\subset \Gamma^2)$ YES, non-diag. | $\mathbb{Z}_2.(\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_4)$ | | 
| #22               | $([\Gamma, \Gamma] \subset \Gamma^4)$ YES | $\mathbb{Z}_2.(\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_4)$ | | 
| #23 - 25          | $\mathbb{Z}_2 \times D_8, D_8, Q_{16}$ (?) Class 3 | $\mathbb{Z}_4^{Aut}.(\mathbb{Z}_2 + \mathbb{Z}_4)$ | | (e) |
| #26 - 32          | $\mathbb{Z}_2 \times D_8, D_8, Q_{16}$ (?) Class 3 | $\mathbb{Z}_4^{Aut}.(\mathbb{Z}_2 + \mathbb{Z}_4)$ | | (e) |
| #33 - 35          | fibre products NO | $\mathbb{Z}_2^2.(\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2)$ | | 
| #36 - 41          | NO | $\mathbb{Z}_2^2.(\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2)$ | | 
| #42, 43           | $\mathbb{D}_4 \ast \mathbb{D}_4$, $\mathbb{D}_4 \ast Q_8$ YES | $\mathbb{Z}_2.(\mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2)$ | | 
| #44 - 48          | (?) Class 3 | $\mathbb{Z}_4^{Aut}.(\mathbb{Z}_2 + \mathbb{Z}_4)$ | | (e) |
| #49 - 51          | $\mathbb{D}_{16}, \tilde{\mathbb{D}}_{16}, Q_{32}$ NO Class 4 | | |

(c) From the classification [Group32]
(d) These are discarded by showing them to be Doi twists of covering Nichols algebras with Dynkin diagram a 4-cycle, see Remark 4.2.
(e) Noncentral extension. This is case $\mathcal{G}_4$ in [HS10], see Remark 4.3.
(f) Noncentral extension of higher class, discarded by [HS10].
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