A Practical Method for Relativistic 3N-Scattering Calculations with Realistic Potentials

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Abstract. Using a scale transformation in momentum space a phase equivalent relativistic potential is generated from the nonrelativistic potential. By that transformation a practical method for the relativistic 3N scattering with realistic nucleon-nucleon potentials is introduced. The formalism can be applied to any realistic nonrelativistic potential. We also discuss the locations of the moving logarithmic singularities of the free relativistic three-body Green’s function. It enters in the relativistic 3N Faddeev equations, which have been formulated long time ago for a 3-boson bound state and which we propose to also use for 3N scattering. Finally we compare relativistic deuteron wave functions to nonrelativistic ones.

1 Introduction

Recently the total cross section for nd scattering has been measured for incident neutron energies from 50 MeV to 600 MeV [1]. The data have been analyzed [2] up to 300 MeV by rigorous 3N Faddeev calculations [3] based on the CD-Bonn potential [4] and the Tucson-Melbourne three-nucleon force (3NF) [5]. Two nucleon forces alone are not sufficient to describe the data above about 100 MeV. The discrepancy is very likely filled by 3NF effects and relativistic corrections. In [2] we simply estimated one relativistic kinematical effect related to the incident flux. This leads to an increase of the total cross section by about 3% at 100 MeV and about 7% at 250 MeV. The rest of the discrepancy of about the same amount turned out to be understandable as a 3NF contribution. 3NF effects can be seen already at lower energies around 65 MeV nucleon lab energy in the minimum of the differential cross section in elastic pd scattering. This is more pronounced at higher energies as recent measurements at RIKEN [6, 7]

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and IUCF\cite{5} show. Those discrepancies to NN force predictions alone are known as Sagara discrepancy\cite{9,10,11}. This very simplistic estimate of a relativistic correction just mentioned is not a substitute for a consistent treatment. The dynamical equation should be relativistic, not only the kinematics.

Relativistic effects in a three-boson bound state have been discussed in\cite{12}. Following those concepts several questions arise:

(1) How do we formulate the relativistic three-body equations for scattering processes?

(2) To what extent does relativity require modifications of the conventional nuclear forces?

With respect to (1) the answer is given in\cite{12} for three bosons by appropriately supplementing the relativistic Faddeev equation presented in\cite{12} for scattering. An important ingredient in these equations is the boost transformation of the two-body t-matrices. What is not treated in\cite{12} are the Wigner rotations of the nucleon spin. This has to be added. This formalism in\cite{12} is of the instant form of relativistic dynamics and has been proposed by F. Coester in\cite{14}.

In order to answer the second question one has to introduce new forces which inserted into the relativistic two-body Schrödinger equation reproduce the two-body observables as precisely as the nonrelativistic potentials.

Recently Glöckle and the author found an analytical momentum-transformation \cite{15} which provides a mathematical relation between the relativistic and nonrelativistic two-body Schrödinger equation and thus between the potentials in the two equations. This transformation is such that the NN phase shifts and the binding energy of the deuteron do not change. In\cite{16} the parameters of a given NN potential (AV18) have been readjusted when used together with the relativistic form of the kinetic energy in order to guarantee the same NN phase shifts to some degree of accuracy. Our transformation given in\cite{15} guarantees exactly the same phase shifts and no refitting is required. These transformed NN forces will be the dynamical input for the properly supplemented relativistic 3N equation in our treatment of relativistic 3N scattering.

In section 2 we will briefly review the formalism proposed in\cite{12} for a relativistic 3N Faddeev equation and extend it from bound state to scattering calculations. Thereby the treatment of the logarithmic moving singularities arising from the free 3N propagator requires special considerations. In section 3 we address the transformation which generates from the nonrelativistic NN potential a new one adequate to the relativistic form of the Schrödinger equation and compare the resulting relativistic deuteron wave function with the one established in a recent study of the Urbana group\cite{16}. The summary is given in section 4.
\section{Glöckle - Lee - Coester Relativistic Faddeev Equation}

In the article \cite{12}, a relativistic Faddeev equation is introduced for a bound state of three bosons. In the Bakamjian-Thomas scheme \cite{13} the two-body potential \( v_{ij} \) which is defined in the two-body c.m. system (2CM), is transformed into \( V_{ij} \) which belongs to the three-body c.m. system (3CM), as

\[
V_{ij} \equiv \sqrt{(\omega_{ij} + v_{ij})^2 + p_{ij}^2} - \sqrt{\omega_{ij}^2 + p_{ij}^2}
\]  

(1)

where \( \omega_{ij} = 2\sqrt{m^2 + k^2} \) and \( p_{ij} \) are the energy and total momentum of the noninteracting pair of particles \( ij \). The usual nonrelativistic relative momentum \( \frac{1}{2} (k_i - k_j) \), where \( k_{i,j} \) are the individual momenta, is replaced now by the Lorentz tranformed individual momenta \( k \) and \( -k \) in the 2CM :

\[
k = \frac{1}{2} (k_i - k_j) - \frac{1}{2} \frac{E_i - E_j}{E_i + E_j + \sqrt{(E_i + E_j)^2 - p_{ij}^2}}
\]

(2)

The individual energies are \( E_i = \sqrt{m^2 + k_i^2} \). The "boosted" potential \( V_{ij} \) depends on the total momentum of the two-body subsystem as is obvious from Eq. (1). Using the complete set of two-body states, \( < k | \phi_0 > \) and \( < k | \phi(k_0) > \equiv < k | \phi(k_0)^{(+)} \), \( V_{ij} \) is explicitly given as,

\[
< p_{ij} k | V_{ij} | p_{ij} ' k' > = \\
\delta (p_{ij} - p_{ij}') \int d k_0 < k | \phi(k_0)^{(+)} \sqrt{\omega^2(k_0) + p_{ij}^2} < k_0 | k' > \\
- \delta (p_{ij} - p_{ij}') \delta (k - k') \sqrt{\omega^2(k) + p_{ij}^2} + \ldots
\]

(3)

where

\[
<k | (\omega(k) + v_{ij}) | \phi(k_0) > = \omega(k_0) < k | \phi(k_0) >.
\]

(4)

The dots denote the two-body bound state contribution. Instead of using \( \delta \) one can also express directly the two-body boosted transition operator \( T_{ij} \) in the 3CM by the t-matrix in the 2CM, as shown in \cite{12}.

The free three-body Greens function is singular above the three-body breakup threshold. It can be written as

\[
G_0^{-1} = \frac{E - \sqrt{\omega^2(k) + p_{ij}^2} - \sqrt{p_{ij}^2 + m^2} + i \epsilon}{E - \sqrt{p_{ij}^2 + m^2} - \sqrt{p_{ij}^2 + m^2 - \sqrt{p_{ij}^2 + p_{jk}^2 + 2p_{ij} p_{jk} x + m^2}}}
\]

(5)

which exhibits an angular dependence \( (x = \hat{p}_{ij} \cdot \hat{p}_{kj}) \). After integration over \( x \) logarithmic singularities appear in the Faddeev equation. By substituting \( y = G_0^{-1}(x) \) the integration can be carried through with the result \( \int dx/y = \int y(\sqrt{p_{ij}^2 + m^2} + \sqrt{p_{jk}^2 + m^2 - E} \ln(y))/p_{ij} p_{jk} \). The singularities occur under the conditions \( x = \pm 1 \) and \( G_0^{-1} = 0 \). Their locations are displayed in Fig. 1 for the example of \( E_{lab} = 250 \text{ MeV} \). We see a shift of the relativistic singularity lines in comparison to the nonrelativistic ones.
Figure 1. The loci of the Green's function singularity at $E_{lab} = 250$ MeV. The solid (dashed) line is for relativistic (nonrelativistic) kinematics.

3 Momentum Scale Transformation

In [15] a momentum scale transformation has been introduced by which the nonrelativistic potential $v^{NR}$ can be rewritten into a "relativistic" one, $v$, as used in Eq. (1) without changing the on-the-energy-shell properties in the two-body system. The nonrelativistic potential $v^{NR}$ enters the nonrelativistic Schrödinger equation,

$$\langle k^{NR}|(k^{NR^2}/m + 2m + v^{NR}_{ij})\psi(k^{NR})\rangle = (k_0^2/m + 2m) \langle k^{NR}\psi(k_0^{NR})\rangle$$

(6)

The corresponding relativistic equation is given in Eq. (4). The momentum transformation connecting relativistic and nonrelativistic momenta.

$$\omega(k) = 2\sqrt{k^2 + m^2} \equiv \frac{k^{NR^2}}{m} + 2m$$

(7)

leads to the relation between the relativistic and nonrelativistic wave functions:

$$\phi(k) = \frac{\sqrt{2m}}{\sqrt{\sqrt{m^2 + k^2} \sqrt{2m(m + \sqrt{m^2 + k^2})}}} \psi^{NR}(\sqrt{2m(\sqrt{m^2 + k^2} - m)})$$

(8)

Fig. 2. shows the S- and D- wave components of the deuteron wave function for the AV18 [17] model potential in configuration space.

At short distances the relativistic wave function becomes larger, a tendency which agrees with the result of a recent study [16]. In [16] an approximation to the ratio between the relativistic and nonrelativistic D- wave functions (see Eq. (2.27)) has been considered:

$$R = \frac{\phi_D(k)}{\psi_D^{NR}(k)} = \frac{k^2}{2(\sqrt{m^2 + k^2} - m)\sqrt{m^2 + k^2}}$$

(9)
We compare that ratio to the one given in Eq. (8) in Fig. 3 and thereby neglect the momentum transformation of the argument in $\psi^{NR}$. Both results agree well, especially if one takes into account, that Eq. (9) is only an approximation and that our result is even closer to the real calculated ratio in [16].

![Figure 2](image1.png)

**Figure 2.** The deuteron wave functions. The solid (dotted) lines are the relativistic (nonrelativistic) wave functions.

![Figure 3](image2.png)

**Figure 3.** The ratio between the relativistic D-wave deuteron wave function and the nonrelativistic one. The solid and dashed curves are due to Eq. (8) and Eq. (9), respectively.

### 4 Summary

We reviewed briefly the formalism in [12], which was devoted to a relativistic treatment of a three-boson bound state in the instant form of relativistic dynamics. The relativistic Faddeev equation given there is extended to three-nucleon scattering and the necessary changes in the free propagator singularities are pointed out. This scheme requires NN forces which are tuned to NN phaseshifts together with the relativistic form of the kinetic energy. We proposed to use the momentum scale transformation from [13], which provides an
analytical connection between the potentials in the nonrelativistic and relativistic two-body Schrödinger equations. No refitting like in [16] is required. We did not yet address the Wigner rotation of the spin states, but we expect this to be achieved along the line given in [18]. We also plan to proceed without partial wave decomposition and then the first steps formulated by [19] will be useful.

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