Finite Representability of Semigroups with Demonic Refinement

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Abstract

Composition and demonic refinement of binary relations are defined by

\[(x, y) \in (R; S) \iff \exists z ((x, z) \in R \land (z, y) \in S)\]
\[R \subseteq S \iff (\text{dom}(S) \subseteq \text{dom}(R) \land R|_{\text{dom}(S)} \subseteq S)\]

where \(\text{dom}(S) = \{ x : \exists y (x, y) \in S \}\) and \(R|_{\text{dom}(S)}\) denotes the restriction of \(R\) to pairs \((x, y)\) where \(x \in \text{dom}(S)\). Demonic calculus was introduced to model the total correctness of non-deterministic programs and has been applied to program verification [BZ86, DS90].

We prove that the class \(R(\preceq, ;)\) of abstract \((\preceq, \circ)\) structures isomorphic to sets of binary relations with inclusion and composition cannot be axiomatised by any finite set of first-order \((\preceq, \circ)\) formulas. We provide a fairly simple, infinite, recursive axiomatisation that defines \(R(\preceq, ;)\). We prove that a finite representable \((\preceq, \circ)\) structure has a representation over a finite base. This appears to be the first example of a signature for binary relations with composition where the representation class is non-finitely axiomatisable, but where the finite representations for finite representable structures property holds.

1 Introduction and Motivation

The simplest way of representing a \((\leq, \circ)\) structure is to interpret the binary relation \(\leq\) as set inclusion \(\subseteq\), and the binary function \(\circ\) as composition of binary relations \(;\). The class \(R(\leq, ;)\) of abstract \((\leq, \circ)\) structures isomorphic to sets of binary relations with inclusion and composition is defined exactly by the axioms of ordered semigroups [Zar59], i.e. associativity, partial order, left and right monotonicity. It is clear that these axioms are valid over \(R(\leq, ;)\). Conversely, given an ordered semigroup \(S = (S, \leq, \circ)\) we may extend the structure to the ordered semigroup \(S' = (S', \leq, \circ)\) by adding a single new two-sided identity element \(e\) where \(e \Preceq s\) and \(s \preceq e\) for \(s \in S\), and then defining a representation \(\theta\) of \(S\) over \(S'\) by

\[(x, y) \in s^\theta \iff y \leq x \circ s\]
illustrated in the first diagram of Figure 1. The extra identity element is used to prove faithfulness of \( \theta \): if \( s \not\leq t \in S \) then \( (c, s) \in s^{\theta} \cap t^{\theta} \). A dual representation \( \theta' \) of \( S \) over \( S' \), illustrated in the second part of Figure 1, is defined by

\[
(x, y) \in s^{\theta'} \iff x \leq s \circ y.
\]

Both inclusion and composition have demonic variants, written \((\leq, *)\) called demonic refinement (defined above in the abstract) and demonic composition defined by

\[
R \ast S = R; S \cap \{(x, y) : \forall z ((x, z) \in R \rightarrow z \in \text{dom}(S))\}.
\]

Closely related to the demonic refinement relation is the demonic join operator \( \sqcup \), defined by

\[
R \sqcup S = (R \cup S) \restriction_{\text{dom}(R) \cap \text{dom}(S)}
\]

In the set of all binary relations over a set \( X \), \( R \cup S \) is the meet (least upper bound) of \( R, S \) with respect to \( \subseteq \). Conversely, given the operator \( \sqcup \) we may recover the relation \( \leq \) by defining \( R \leq S \iff R \sqcup S = S \). Note, however, that an operator \( \sqcap \) that returns the greatest lower bound of two relations may not be defined, it is not in general the case that two binary relations have any common lower bound with respect to \( \subseteq \).

It is known that \( R(\leq, *) \) is also axiomatised by the three axioms of ordered semigroups. For \( R(\leq, *) \), although \( * \) remains associative, \( \leq \) is still a partial order and \( * \) is right monotonic with respect to \( \leq \), we find that left monotonicity fails. Recently it was shown that \( R(\leq, *) \) is not finitely axiomatisable [HMS].

The case we focus on here is \( R(\leq, ;) \), with ordinary composition and demonic refinement. This time, we find that both left and right monotonicity fail. How do we axiomatise \( R(\leq, ;) \)? Of course we may use the axioms of associativity and partial order, but what additional axioms should be included in order to fill the gap created by the omission of the two monotonicity axioms, and define the representation class? The main results here are a recursively defined infinite set of axioms that define \( R(\leq, ;) \) and a proof that no finite set of axioms can do it. Our proof that the recursive axiomatisation is complete also shows that a finite representable \((\leq, ;)\)-structure has a representation over a finite base set.

Algebras of binary relations have been used extensively to model program semantics [MDM87, DS90], and the introduction of demonic choice (\( \sqcup \)) and
demonic composition (⋆) has extended this framework towards reasoning about
the total correctness of non-deterministic Turing Machines [DS90, BZ86].

The introduction of the demonic refinement predicate led to further verifica-
tion applications, for example utilising Refinement Algebras [W04, DCD08].
Furthermore, relaxing the requirement that composition is a total binary oper-
ator we obtain refined semigroupoids, which have been of interest in relation-
algebraic programming [Kah08].

The fairly extensive literature on demonic relations and operators includes a
variety of different notations. In the context of Kleene Algebra extensions, such
as Refinement Algebra, where the emphasis is on the behaviour of tests, □, △
are sometimes used in place of ⊃, ⊀.

2 Axiomatizing $R(⊒, ;)$

We focus on the signature $(⊒, ;)$, in the abstract case the corresponding symbols
will be $(≤, ◦)$. A binary relation over the base $X$ is a subset of $X \times X$. A
concrete $(⊒, ;)$ structure is a set of binary relations over some base, closed
under composition, with demonic refinement. An isomorphism from an abstract
$(≤, ◦)$ structure to a concrete $(⊒, ;)$ structure is called a representation. $R(⊒, ;)$
denotes the class of all $(≤, ◦)$ structures isomorphic to concrete $(⊒, ;)$ structures.

Given a $(≤, ◦)$ structure $S$ we let $S'$ be the structure obtained from $S$ by
adjoining a single new identity element $e$ where $e \circ x = x \circ e$, $e ≤ e = e \circ e$
but $x ≤ e$ and $e ≤ x$ for $x ∈ S$.

The signature does not include the domain operation, nor does it include
‘angelic’ (ordinary) set inclusion. However, we will define with infinitary
$(≤, ◦)$-formulas, the predicates $•, ⇐^s$ to signify the domain inclusion and inclusion of
the restriction to the domain of $s$ respectively.

Let

\[ a • b ⇔ \bigvee_{n<ω} a •_n b \]
\[ a ⇐^s b ⇔ \bigvee_{n<ω} a ⇐^s_n b \]

where

\[ a •_0 b ⇔ a ≥ b \lor \exists c(a ≥ b \circ c) \]
\[ a ⇐^s_0 b ⇔ (a ≤ b \wedge s = b) \]
\[ a •_{n+1} b ⇔ \left\{ \begin{array}{l}
(a ⇐^s_n b) \lor \\
\exists c(a •_n c \land c •_n b) \lor \\
\exists d, f, f' (a = d \circ f \land f •_n f' \land b = d \circ f') \end{array} \right\} \]
\[ a ⇐^s_{n+1} b ⇔ \left\{ \begin{array}{l}
(3c (a ⇐^s_n c \land c ⇐^s_n b)) \lor \\
\exists c, c', d, d' (a = c \circ d \land c ⇐^s_n c' \land d ⇐^s_n d' \land b = c' \circ d') \lor \\
\exists s' (a ⇐^s_n b \land s \cdot s') \end{array} \right\} \]
Lemma 1.

1. \( a \bullet_n b \land b \bullet_n c \rightarrow a \bullet_{n+1} c \), \( a \triangleleft_n b \land b \triangleleft_n c \rightarrow a \triangleleft_{n+1} c \), so \( \bullet \) and \( \triangleleft \) are transitive, for each \( s \in S \).

2. \( a \triangleleft_n b \circ c, b \triangleleft_n b' \) implies \( a \triangleleft_{n+1} b' \circ c' \).

3. \( d \bullet_n a \circ c, a \leq a' \), \( d \bullet_n a' \), \( c \bullet_n c' \) implies \( d \bullet_{n+3} a' \circ c' \).

4. \( s \bullet_n s', a \triangleleft_n b \) implies \( a \triangleleft_{n+1} b \).

Proof. (1), (2), (3) follow directly from the definitions of \( \bullet, \triangleleft \). For (3), observe how from \( a \leq a' \) we have \( a \triangleleft_n a' \) which, together with \( a \circ c' \bullet_n a' \) give us \( a \triangleleft_n c' \). From this and \( c' \triangleleft_n c' \) we get \( a \circ c' \triangleleft_n c' \) and thus \( a \circ c' \triangleleft_{n+1} a \circ c' \). We also have \( c \bullet_n c' \) and hence \( a \circ c \bullet_{n+3} a \circ c' \). So, by the transitive steps \( d \bullet_n a \circ c \bullet_{n+3} a \circ c' \) we obtain \( d \bullet_{n+3} a' \circ c' \). \( \square \)

Lemma 2. Let \( S \in R(\subseteq,;) \) and let \( \theta \) be a representation of \( S \). For all \( a, b, s \in S \)

\[
\begin{align*}
  a \bullet b & \Rightarrow \text{dom}(a^0) \subseteq \text{dom}(b^0), \text{ and} \\
  a \triangleleft b & \Rightarrow a^0|_{\text{dom}(s^0)} \subseteq b^0.
\end{align*}
\]

Proof. We prove the claim by induction over \( n \). In the base case, if \( a \bullet_0 b \) then either \( a^0 \equiv b^0 \) or \( a^0 \equiv b^0, c^0 \) (for some \( c \)) hence \( \text{dom}(a^0) \subseteq \text{dom}(b^0) \). And if \( a \triangleleft_0 b \) then \( s = b, a \leq b \), so \( a^0|_{\text{dom}(s^0)} = a^0|_{\text{dom}(s^0)} \subseteq b^0 \).

For the inductive step, suppose \( a \bullet_{n+1} b \), from the recursive definition, there are three alternatives. In the first case, \( a \triangleleft_n b \) then inductively \( a^0 = a^0|_{\text{dom}(s^0)} \subseteq b^0 \) so \( \text{dom}(a^0) \subseteq \text{dom}(b^0) \). In the second case, inductively \( \text{dom}(a^0) \subseteq \text{dom}(c^0) \subseteq \text{dom}(b^0) \). In the third case, there are \( d, f, f' \) where \( a = d \circ f \), \( f \bullet_n f' \) and \( b = d \circ f' \). For any \( x \in \text{dom}(a^0) \), there is \( y \) such that \( (x, y) \in a^0 \) and there is \( z \) such that \( (x, z) \in d^0, (z, y) \in f^0 \). Inductively, \( z \in \text{dom}(f^0) \subseteq \text{dom}(((f')^0)) \) so there is \( w \) such that \( (z, w) \in (f')^0 \), hence \( (x, w) \in d^0; (f')^0 = b^0 \), so \( x \in \text{dom}(b^0) \), proving \( \text{dom}(a^0) \subseteq \text{dom}(b^0) \).

Now suppose \( a \triangleleft_{n+1} b \). There are three alternatives in the recursive definition. In the first case, inductively \( a^0|_{\text{dom}(s^0)} \subseteq c^0 \) and \( c^0|_{\text{dom}(s^0)} \subseteq b^0 \), so \( a^0|_{\text{dom}(s^0)} \subseteq b^0 \). In the second case, there are \( c, c', d, d' \) as in the definition. If \( x \in \text{dom}(s^0) \) and \( (x, y) \in a^0 \) then there is \( z \) such that \( (x, z) \in c^0, (z, y) \in d^0 \). Inductively, \( (x, z) \in (c')^0 \) and \( (z, y) \in (d')^0 \), hence \( (x, y) \in (c' \circ d')^0 = b^0 \). In the third case, \( \text{dom}(s^0) \subseteq \text{dom}(s^0) \), so \( a|_{\text{dom}(s^0)} \subseteq a|_{\text{dom}(s^0)} \subseteq b^0 \). This proves \( a^0|_{\text{dom}(s^0)} \subseteq b^0 \), as required. \( \square \)

Let

\[
\begin{align*}
  \sigma_n &= ((b \bullet_n a \land a \triangleleft_n b) \rightarrow a \leq b) \\
  \sigma &= ((b \bullet a \land a \triangleleft b) \rightarrow a \leq b)
\end{align*}
\]

For finite \( n \), \( \sigma_n \) is a first-order formula, while \( \sigma \) is infinitary and is equivalent to \( \bigwedge_{n < \omega} \sigma_n \).
Lemma 3.

\[ R(\subseteq,;) \models \sigma. \]

Proof. Let \( S \in R(\subseteq,;) \) and let \( \theta \) be a representation. Assume the premise of \( \sigma \), \( S \models (b \bullet a \land a \prec^b b) \). By the previous Lemma, \( \text{dom}(b^\theta) \subseteq \text{dom}(a^\theta) \) and \( a^\theta \upharpoonright \text{dom}(b^\theta) \subseteq b^\theta \), i.e. \( a^\theta \subseteq b^\theta \). Since \( \theta \) represents \( \subseteq \) as \( \subseteq \) we must have \( S \models a \leq b \). Thus \( S \models \sigma \).

The following definition is used to prove completeness of our axioms.

Definition 4. Let \( S \) be a \((\leq,\circ)\)-structure. Consider the base set

\[ X = X_i \cup X_f \cup X_\beta \]

where

\[ X_i = \{ (d_i, s_i) : d, s \in S, d \bullet s \} \]
\[ X_f = \{ s_f : s \in S \} \]
\[ X_\beta = \{ d_\beta : d \in S \} \]

we refer to the points in \( X_i, X_f, X_\beta \) as initial points, following points and branch points, respectively, it may help to visualise these points using Figure 2.

For each \( x \in X \) let \( \delta(x) \in S \) be defined by \( \delta(x) = d \) if and only if \( x \in \{(d_i, s_i), d_f, d_\beta : d \bullet s \in S\} \). It may help the reader to think of \( \delta(x) \) as an element with \( \bullet \)-minimal domain in the representation we construct, illustrated as a vertical outgoing arrow in Figure 2. For \( x \in X_i \cup X_f \), define \( \lambda(x) \in S \) by letting \( \lambda(x) = s \) iff \( x \in \{(d_i, s_i), s_f : d \bullet s \} \) (undefined if \( x \in X_\beta \), illustrated as the label of the edge \((x, e_f)\) in Figure 2). \( \lambda(x) \) will be used as the \( \prec^b(x) \)-minimal label of the edge \((x, e_f)\) when \( x \in X_i \cup X_f \), there are no labels on \((x, e_f)\) when \( x \in X_\beta \). Observe \( \delta(x) = \lambda(x) \) for \( x \in X_f \).

For each \( a \in S \) define a binary relation \( a^\theta \subseteq X \times X \) by letting \((x, y) \in a^\theta \) if and only if

I. \( y \notin X_i \),
II.  $x \in X_\beta \Rightarrow y \in X_\beta,$

III.  $\delta(x) \bullet a \circ \delta(y)$ and

IV.  $x \in X_i \cup X_f, y \in X_f \Rightarrow \lambda(x) \prec \delta(x) a \circ \lambda(y).$

The $\lambda$ part of this definition is loosely based on the dual representation $\theta'$ of ordered semigroups (see the second part of Figure 1 and is visualised in Figure 3)

**Lemma 5.** Let $S = (S, \leq, \circ)$ be a structure where $\circ$ is associative, $\leq$ is a partial order, $S \models \sigma$ and suppose there is an identity $e \in S$. Let $\theta$ be from Definition 4. Then $a \leq b \in S$ if and only if $a^\theta \subseteq b^\theta$.

Proof. Assume $a \leq b$, so either $\neg b \bullet a$ or $\neg a \prec b^\theta$, by $\sigma$. In the former case $(b_\beta, e_\beta) \in b^\theta$ but $b_\beta \notin \text{dom}(a^\theta)$. Otherwise $b \bullet a$ and the latter case holds, but then $((b_1, a_1), e_f) \in a^\theta \setminus b^\theta$. Either way, $a^\theta \subseteq b^\theta$.

Now suppose $a \leq b$. First we check that $\text{dom}(b^\theta) \subseteq \text{dom}(a^\theta)$. If $x \in \text{dom}(b^\theta)$ there is $y \in X$ where $(x, y) \in b^\theta$. It follows that $\delta(x) \bullet b \bullet a$, so $(x, e_\beta) \in a^\theta$ and $x \in \text{dom}(a^\theta)$. Secondly, if $x \in \text{dom}(b^\theta)$ (so $\delta(x) \bullet b$) and $(x, y) \in a^\theta$ we know that (III) - (IV) hold for $a$, in particular $\delta(x) \bullet a \circ \delta(y)$. It follows that $\delta(x) \bullet b \circ \delta(y)$, by Lemma 13, as required by (III). Conditions (I), (II) remain true for $b^\theta$. For (IV) if $x \in X_f$ then $\lambda(x) \prec \delta(x) a \circ \lambda(y) \prec \delta(x) b \circ \lambda(y)$, by Lemma 12. Hence $(x, y) \in b^\theta$, thus $a^\theta \subseteq b^\theta$.

**Lemma 6.** Let $S = (S, \leq, \circ)$ be a structure where $\circ$ is associative, $\leq$ is a partial order, $S \models \sigma$ and let $\theta$ be from Definition 4. For any $a, b \in S$, we have $(a \circ b)^\theta = a^\theta \circ b^\theta$.

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Figure 3: $(x, y) \in \theta(a)$ with (i) $x \in X_i \cup X_f$ and $y \in X_f$, (ii) $x \in X_i \cup X_f$ and $y \in X_\beta$, and (iii) $x \in X_\beta$, $y \in X_f$. In each case $\delta(x) \bullet a \circ \delta(y)$, in case (i) only $\lambda(x) \prec \delta(x) a \circ \lambda(y)$. In (i) and (ii) if $x \in X_f$ then the $\delta(x)$ and $\lambda(x)$ arrows coincide.
Figure 4: Witness for \((x, y) \in (a \circ b)^\theta\) where \(x \in X_i \cup X_f\) and \(y \in X_f\) (left), \(x \in X, y \in X_\beta\) (right).

Proof. First, let’s show that \(a^\theta; b^\theta \subseteq (a; b)^\theta\). Take any \((x, y) \in a^\theta\) and \((y, z) \in b^\theta\). We have \(\delta(x) \ast a \circ \delta(y)\) and \(\delta(y) \ast b \circ \delta(z)\), so \(\delta(x) \ast a \circ b \circ \delta(z)\), by Lemma 13. By (III) if \(x \notin X_i\) and by (II) if \(x \in X_\beta\) then \(y \in X_\beta\) and then \(z \in X_\beta\). Also by (II) \(y \notin X_i\) so if \(z \in X_f\) then \(y \in X_f\) and \(\lambda(y) = \delta(y)\). Then \(x \in X_i \cup X_f\) and \(\lambda(x) \ast \delta(x)\ast a \circ \lambda(y), \lambda(y) \ast \lambda(z)\), so \(\lambda(x) \ast \delta(x)\ast a \circ b \circ \lambda(z)\), by Lemma 12. Hence \((x, z) \in (a \circ b)^\theta\).

Conversely, to show that \((a \circ b)^\theta \subseteq a^\theta; b^\theta\), take any \((x, y) \in (a \circ b)^\theta\). By (II) \(y \notin X_i\). If \(y \in X_\beta\) let \(z = (b \circ \delta(y))_\beta \in X_\beta\) (Figure 4 right) and then \((x, z) \in a^\theta\); \((z, y) \in b^\theta\). Otherwise, \(y \in X_f\) and we let \(z = (b \circ \lambda(y))_f \in X_f\) (Figure 4 left) and again we have \((x, z) \in a^\theta\); \((z, y) \in b^\theta\), as required. 

\[\square\]

Theorem 7. \(R(\subseteq ;)\) is axiomatised by partial order, associativity and \(\{\sigma_n : n < \omega\}\). Finite structures \(S \in R(\subseteq ;)\) are representable over a finite base \(X\) with \(|X| \leq (1 + |S|)^2 + 2 \cdot (1 + |S|)\).

Proof. Soundness of partial order, associativity is clear, soundness of \(\sigma_n\) is from Lemma 3. For completeness, take any associative, partially ordered \((\subseteq, \circ)\)-structure \(\mathcal{S} \models \{\sigma_n : n < \omega\}\). We may define \(\mathcal{S}'\) be adding a new identity \(e\) to \(\mathcal{S}\) unordered with other elements. By Lemmas 5 and 6 the map \(\theta\) of Definition 4 is a \((\subseteq, \circ)\)-representation of \(\mathcal{S}'\), hence it restricts to a \((\subseteq, \circ)\)-representation of \(\mathcal{S}\).

The representation \(\theta\) has base contained in a disjoint union of a copy of \((\mathcal{S}')^2\) and two copies of \(\mathcal{S}'\). 

\[\square\]

3 \(R(\subseteq ;)\) is not Finitely Axiomatisable

Definition 8. Let \(n < \omega\), \(N = 1 + 2^n\) and let \(S_n\) be a \((\subseteq, \circ)\)-structure whose underlying set \(S_n\) has \(3 + 3N\) elements

\[S_n = \{0, b, c\} \cup \{a_i, a_i b, a_i c : i < N\}\]
Lemma 9. For \( n \geq 2 \), \( \mathcal{S}_n \) is not representable, but \( \mathcal{S}_n \models \sigma_k \) for \( k < n \).

Proof. Since \( s \leq 0 \) we have \( 0 \cdot s \). Also, for \( i < N \), since \( a_{i+1} \circ b \leq a_i \) have \( a_i \cdot b_0 a_{i+1} \cdot b \cdot a_{i+1} \), so \( a_i \cdot b_\ast k a_{i+2} \) for \( k \geq 1 \), using Lemma 1(1). Hence \( \{a_i, a_{i+1} b : i < N\} \) is a clique of \( \ast_n \), but for \( k < n \) we do not have \( a_{i+1} \cdot b \).

For \( \ast \), we have

- \( t \circ_0 u \) if \( t \leq u \) and \( s = u \), i.e. \( s \circ_0 s, s \circ_0 0 \) (all \( s \)), \( a_{i+1} b \circ_0 a_i, a_i \circ_0 a_{i+1} c \) and \( a_i b \circ_0 a_i c \) (all \( i < N \)), but \( \circ_0 \) holds in no other cases.

- Since \( a_{i+1} b \circ c = 0 \) and \( a_{i+1} b \circ_0 a_i \), it follows by Lemma 1(2) that \( 0 \circ_1 a_i c \), similarly, \( 0 \circ_1 a_i b \). Also by Lemma 1(2), since \( a_i \circ_0 a_i + c \) and \( a_{i+1} c \circ b = 0 \) we get \( a_i b \circ_0 a_i + c \), similarly \( a_i c \circ_0 a_i + c \). And from \( a_i \circ_0 a_i c \) and \( c \circ_0 c \) we get \( a_i c \circ_0 a_i c \), similarly \( a_i b \circ_0 a_i b \). The only non-zero products are \( a_i \circ b \) and \( a_i \circ c \), so the only remaining case of \( \circ_1 \) we obtain from Lemma 1(2) is \( 0 \circ_1 0 \), which follows since \( s \circ_0 s \), for all \( s \in \mathcal{S}_n \) and \( 0 \circ_0 0 \). By Lemma 1(3), from \( a_{i+1} b \circ_0 a_i \) we get \( a_{i+1} b \circ_1 a_i \) for \( s \circ_0 a_i \). This concludes the exhaustive enumeration of elements in \( \circ_1 \), not covered by \( \circ_0 \).

- If \( a \circ_1 b \) and \( s \cdot \ast_1 s \) we get \( a \circ_1 b \), in particular \( 0 \circ_2 a_i c \).

- Since \( a_i b \circ_1 a_{i+1} c \circ_2 a_i c \), it follows by Lemma 1(1) that \( a_i b \circ_3 a_{i+1} c \).

- The remaining cases of \( \ast \) can be enumerated as follows. We have \( 0 \ast a_i + c \), \( 0 \ast a_i + b \), \( a_i c \ast a_i c \) for \( s \ast a_0 \), by Lemma 1(3). Additionally, since \( a_{i+1} b \ast a_i \), we get \( 0 \ast a_i \), by Lemma 1(1). Also by Lemma 1(1), for any \( s \in \mathcal{S}_n \) since \( s \ast 0 \) and \( 0 \ast a_i b, 0 \ast a_i c, 0 \ast a_i \), we have \( s \ast a_i b, s \ast a_i c, s \ast a_i \), and if \( a \in \{a_i b, a_i c : i < N\} \), and \( b \in \{a_i, a_i b, a_i c : i < N\} \) we have \( a \ast a_i b \).

This covers all triples \((a, s, b)\) where \( a \ast b \). It follows that \( \mathcal{S}_n \models \sigma_{n+1} \) for \( n \geq 2 \), since \( a_i + c \cdot a_i + a_i b, a_i b \ast a_i + c \ast a_i + c \) but \( \mathcal{S}_n \models a_i b \leq a_i + c \). By Theorem 7, \( \mathcal{S}_n \) is not representable. The only cases where \( a \ast b \) and \( a \neq b \) are \( a_i b \ast a_i + c \ast a_i + c \), but for \( k < n \) we do not have \( a_i + c \cdot a_i b \), hence \( \mathcal{S}_n \models \sigma_1 \). \( \square \)

Theorem 10. \( R(\ast, ;) \) cannot be defined by finitely many axioms.
Proof. Each structure $S_n \notin R(\subseteq,;)$. For any $k < \omega$ almost all $S_n$ satisfy $\sigma_k$ (in fact, all $S_n$ where $n > k$) and they are all associative and partially ordered, hence any non-principal ultraproduct $S = \Pi U S_n$ is associative, partially ordered and satisfies all $\sigma_k$s, so by Theorem 7 $S \in R(\subseteq,;)$. By Łoś’ theorem, $R(\subseteq,;) \implies \) has no finite axiomatisation.

4 Finite axiomatisability and representability

For any relation algebra signature $\Sigma$, the representation class $R(\Sigma)$ may be finitely axiomatisable or not, and it may be that finite representable structures have finite representations or not. All four combinations of these two properties are possible.

**Theorem 11.** The representation class $R(\Sigma)$ is finitely axiomatisable, and finite structures in $R(\Sigma)$ have finite representations, according to the following incomplete table.

| Fin. Rep. | Fin. Ax. | Not Fin. Ax. |
|-----------|----------|--------------|
| $\land,;,$ | $(\subseteq,;)$ | | $(\subseteq,;), (\subseteq,*)$ |
| Not Fin. Rep. | $(\land,;)$ | $(\land,;), (\land,;) \subseteq \Sigma$ | $(\subseteq,;), (\subseteq,;) \subseteq \Sigma$ |

where $\Sigma' \subseteq \Sigma$ signifies the language characterised by $\Sigma$ being an expansion of the language characterised by $\Sigma'$.

Proof. Finite axiomatisability of $R(\subseteq,;)$ is proved in [Bre77] and the finite representation property for this signature is proved in [HM13]. Both $R(\subseteq,;) \land R(\subseteq,;)$ are defined by the axioms of ordered semigroups and have the finite representation property [Zar59, HMS].

The finite representation property is proved for $R(\subseteq,;) \land R(\subseteq,;)$ in Theorem 7, non-finite axiomatisability is proved in Theorem 10. The failure of the finite representation property for signatures containing $(\land,;)$ is proved in [Neu17], finite axiomatisability of $R(\land,;)$ is proved in Proposition 12 below.

For the final quadrant of the diagram, if the representation problem for finite structures in $R(\Sigma)$ is undecidable, we know that there can be no finite axiomatisation, and since the set of formulas valid over $R(\Sigma)$ is recursively enumerable the finite representation property cannot hold. The representation problem for finite structures is proved undecidable for signatures containing $(\land,;)$ in [HJ12] and for signatures containing $(\subseteq,;)$, where negation is interpreted as complementation relative to a universal relation $X \times X$, in [Neu16]. We extend that result to prove failure of the finite representation property for representations where $-$ denotes complementation relative to an arbitrary maximal binary relation in Proposition 13 below.


Figure 5: Node Addition in a Representation Game for $(\cdot, \circ)$

**Proposition 12.** $R(\cap, ;)$ is finitely axiomatisable.

**Proof.** A $(\cap, ;)$-representable $(\cdot, \circ)$-structure clearly satisfies the semilattice laws, associativity and monotonicity. Conversely, in a representation game played over an associative, monotonic semilattice $S$, $\exists$ plays a sequence of networks — graphs $N$ whose edges are labelled by upward closed subsets of $S$, such that $N(x, y); N(y, z) \subseteq N(x, z)$ for all $x, y, z \in N$. [See Definition 7.7 of [HH02] for more details of a representation game for the full signature of relation algebra, and Chapter 9 for representation games in a more general setting.] In the initial round suppose $\forall$ picks $a \neq b$. By antisymmetry either $a \leq b$ or $b \leq a$, without loss assume the former. $\exists$ plays a network $N_0$ with two nodes labelled $N(x, y) = a^\uparrow$, all other edges have empty labels, note that $b \notin N(x, y)$. In a subsequent round let $N$ be the current network. $\exists$ adds a single new node $z$ and lets $N'(w, z) = N(w, x); a^\uparrow$, $N'(z, w) = b^\uparrow; N(y, w)$ for all $w \in N$ to define $N'$. Edges within $N$ are not refined. If $u, w \in N$ then $N'(u, z); N'(w, z) = N(u, x); a^\uparrow; N(y, w) \subseteq N(u, x); N(x, y); N(y, w)$, since $a; b \in N(x, y)$, using associativity, left and right monotonicity (see Figure 5). It is easily seen that $N'$ is a consistent network, a legal response to $\forall$’s move not refining the initial edge. It follows that $S \in R(\cap, ;)$

Let $S$ be a $(\leq, -\circ)$-structure. A $(\leq, \setminus; \cdot)$-representation of $S$ over base $X$ is a map $\theta: S \to \wp(X \times X)$ such that for all $a, b \in S$,

- $a \leq b \to a^\theta \subseteq b^\theta$,
- $(x, y) \in a^\theta \to (x, y) \in \Delta(b^\theta, (-b)^\theta)$ (the symmetric difference of $b^\theta$ and $(-b)^\theta$),
- $(x, y) \in (a \circ b)^\theta \iff \exists z((x, z) \in a^\theta \land (z, y) \in b^\theta)$.

According to this definition, $-$ is represented as complementation in the union of all the represented binary relations.
Claim 1: If \( p \leq x \) for all \( p \in \mathbb{R} \), yet in order to be a lower bound its domain should contain both the atoms \( e, l, g \), where \( e \) is the identity, the converse of \( l \) is \( g \), composition for atoms is given by

\[
\begin{array}{c|ccc}
\circ & e & l & g \\
\hline
e & e & l & g \\
l & l & l & g \\
g & g & 1 & g \\
\end{array}
\]

and the operators extend to arbitrary elements by additivity. A representation of \( \mathcal{P} \) over \( \mathbb{Q} \) may be obtained by representing \( e, l, g \) as the identity \( \{(q, q) : q \in \mathbb{Q}\} \), less than \( \{(q, q') : q < q'\} \) and greater than, respectively. It follows that the reduct of \( \mathcal{P} \) to \((\leq, -, \circ)\) is \((\leq, \\setminus, \;)\)-representable. We claim it has no finite \((\leq, \\setminus, \;)\)-representation.

Let \( \theta \) be any \((\leq, \\setminus, \;)\)-representation of \( \mathcal{P} \) over the base \( X \).

Claim 1: If \((x, y) \in g^\theta\) then \( x \neq y \). To prove the claim, suppose for contradiction that there is a point \( x \in X \) with \((x, x) \in g^\theta\). As \( g \leq e \), \((x, x) \in g^\theta \subseteq (e)^\theta \).

And since \( g = g \circ e \), there exists a \( y \) s.t. \((x, y) \in g^\theta \), \((y, x) \in e^\theta \). Since \( e \circ g = g \), we also have \((y, x) \in g^\theta \). But \( e \leq -g \), so \((y, x) \in (-g)^\theta \). Since \((y, x) \in g^\theta \) we have reached a contradiction and proved claim 1.

Claim 2: For \( n \geq 0 \) there is \( x \in X \) and distinct points \( y_0, \ldots, y_n \in X \) such that for all \( i \leq n \) we have \((x, y_i) \in (-g)^\theta \) and for all \( i < j \leq n \) we have \((y_j, y_i) \in g^\theta \).

See Figure 6. Claim 2 is proved by induction over \( n \). For the base case, \( n = 0 \), since \(-g \notin 0\) there are \( x, y_0 \) where \((x, y_0) \in (-g)^\theta \). Assume the hypothesis for some \( n \geq 0 \). Since \((x, y_n) \in (-g)^\theta \) and \(-g \leq 1 = (g) \circ g \), there must be \( y_{n+1} \in X \) where \((x, y_{n+1}) \in (-g)^\theta \) and \((y_{n+1}, y_n) \in g^\theta \). Since \((y_n, y_i) \in g^\theta \) it follows that \((y_{n+1}, y_i) \in (g \circ g)^\theta = g^\theta \), for \( i \leq n \). By the previous claim, \( y_{n+1} \) is distinct from \( y_i \), for \( i \neq n \), as required. This proves claim 2.

Since \( X \) contains a set of \( n \) distinct points, for all \( n < \omega \), it follows that \( X \) must be infinite.

\[\square\]

5 Demoniac Lattice and Semilattice

We have seen in the introduction that demonic join \( \sqcup \) is the meet operation for demonic refinement \( \sqsubseteq \). A demonic meet \( \sqcap \), acting as a least upper bound of its two arguments, may not in general be defined, as there are binary relations having no common lower bound at all. If a point \( x \) is in the domain of two binary relations \( R, S \), but not in the domain of \( R \cap S \), then any lower bound of \( R, S \) would be below the intersection \( R \cap S \), hence \( x \) would be outside its domain, yet in order to be a lower bound its domain should contain both the domain of \( R \) and the domain of \( S \), a contradiction. This problem could solved
Figure 6: Induction showing a new node is needed for representation of $P$

be adding a single new point $\infty$ to the base $X$ of the representation $\theta$ and letting $\theta'(R) = \theta(R) \cup \{(x, \infty) : x \in \text{dom}(R)\}$ to obtain an alternative representation of the refinement algebra, with $\sqsubseteq$-least element $\{(x, \infty) : x \in X \cup \{\infty\}\}$. Over such a representation, a greatest lower bound may be defined by

$$R \cap S = \{(x, y) : (x, y) \in R, x \notin \text{dom}(S)\}$$
$$\cup (R \cap S)$$
$$\cup \{(x, y) : (x, y) \in S, x \notin \text{dom}(R)\}$$

Hence, every representable $(\sqsubseteq, ;)$-structure embeds into a representable $(\cap, \cup, ;)$-structure forming a distributive lattice with composition. We expect that additional properties are required to ensure that such a representation exists.

**Problem 14.** Is the class of representable semigroups with demonic semilattice $R(\sqcup, ;)$ finitely axiomatisable and are the finite structures in $R(\sqcup, ;)$ representable over finite bases?

**Problem 15.** Find axioms for the class of all $(\sqcup, \cap, ;)$-structures of binary relations with demonic join and meet under composition.

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