Convergence of Tâtonnement in Fisher Markets

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Abstract

Analyzing simple and natural price-adjustment processes that converge to a market equilibrium is a fundamental question in economics. Such an analysis may have implications in economic theory, computational economics, and distributed systems. Tâtonnement, proposed by Walras in 1874, is a process by which prices go up in response to excess demand, and down in response to excess supply. This paper analyzes the convergence of a time-discrete tâtonnement process, a problem that recently attracted considerable attention of computer scientists. We prove that the simple tâtonnement process that we consider converges (efficiently) to equilibrium prices and allocation in Eisenberg-Gale markets satisfying some mild and widely applicable assumptions. Our simple proof of convergence applies to essentially all utility functions for which previous results showed that discrete tâtonnement converges. It also applies to some utility functions for which there was no previous proof of convergence.

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1 Introduction

Motivation. General equilibrium theory, a cornerstone of microeconomics, deals with markets that consist of agents that are endowed with goods (or money). Each agent wants to maximize its utility. This may involve exchanging the initial endowment of goods for goods that other agents hold. The exchange of goods against other goods (or money) is governed by prices that are set to guarantee a market equilibrium where supply equals demand. In this context, an issue that was raised already by the founders of this theory is that justifying market equilibrium as a likely outcome of exchange requires suggesting a plausible market dynamic that leads to equilibrium. In other words, proving that some simple, natural, decentralized price-adjustment process converges quickly to a market equilibrium is a fundamental challenge of general equilibrium theory. Indeed, Walras [23] proposed such a process which he named tâtonnement. In this process, prices are adjusted upwards in response to excess demand and downwards in response to excess supply. (Notice that while buyers are expected to react to price changes by maximizing their utility subject to their budget constraints, the price changes themselves cannot be justified based on the incentives of the sellers.) In modern economic thought, tâtonnement is regarded as a model for a centrally planned economy rather than a decentralized market economy. In other words, it is considered a method of computation rather than a spontaneous market process. A scenario where this view of tâtonnement might be useful arises in the context of distributed large-scale computer systems, such as clouds. Artificial markets can be set to facilitate cooperative sharing of computing, storage, and communication resources among selfish agents (see, for example, [6, 26, 24]). Often, a service provider either controls the dispersed resources, or at least regulates the peer-to-peer interaction among the users, thus enforcing adherence to the price adjustment protocol (see, for instance, [11, 25]). On the other hand, the buying agents are still free to pursue their goals without having to formulate and disclose their utility functions beyond their reactions to the changing prices. (One of the reasons that this is important is that the conventional economic interpretation of utility functions is that they are a convenient expression of a preference order on the continuum of consumption baskets, rather than a meaningful and conscious numerical evaluation of each basket.)

Our results. We consider in this paper the following simple discrete tâtonnement rule. For some fixed $\epsilon > 0$, for every good, its price $p(t)$ at time $t = 1, 2, 3, \ldots$ is given by

$$p(t) = (1 - \epsilon) \cdot p(t - 1) + \epsilon \cdot f(t) \cdot p(t - 1),$$

where $f(t)$ is the total demand for this good assuming the prices at time $t - 1$ (the supply is always scaled to 1). We prove that this process converges to equilibrium prices and allocation in Eisenberg-Gale markets satisfying some mild and widely applicable assumptions. We also note that the rate of convergence gives a polynomial time algorithm for computing an approximate equilibrium—the precise bounds are stated in Section 4. Previous results on the convergence of discrete versions of tâtonnement apply to special cases, all of them satisfying our mild assumptions. Furthermore, we show utility functions (most notably nested CES-Leontief utilities) for which previous work did not establish convergence of tâtonnement and our results apply. We note that resource sharing in computer systems may require rather complicated utility functions that could not be handled by previous work (for instance, users may desire a combination of alternative bundles of resources).

Markets. A Fisher market (see [21]) is composed of a set of agents and a set of perfectly divisible goods, each of limited quantity. Each agent is endowed with a budget and a utility function on the possible baskets of goods. The goal of an agent is to spend the budget to buy an optimal basket of goods which maximizes the agent’s utility subject to the budget constraint. An equilibrium consists of an assignment of prices to goods and an optimal purchase of goods by each agent in which the market clears, i.e., the demand for each good equals to its supply. Eisenberg-Gale markets [16] are a rather general special case, where the utility functions are such that equilibrium prices and allocation can be formulated as the solutions to dual convex programs, originally due to Eisenberg and Gale [14].
and extended in [13, 19, 11, 17]. Fisher markets are a special case of the more general Walrasian model [23] of an exchange economy. Arrow and Debreu [4] proved, using the Kakutani fixed-point theorem, that in a Walrasian market, if the utility functions are continuous, strictly monotone, and quasi-concave, then an equilibrium always exists. However, their proof gives no indication of the market dynamics that might lead to an equilibrium.

Markets are often classified according to the type of utility functions used. Utility functions are usually assumed to be concave and monotonically non-decreasing. (Economically, the former assumption is the law of diminishing marginal utility, and the latter is implied by assuming free disposal.) A utility function of the form

$$u(x) = \left( \sum_{j=1}^{m} a_j^\rho x_j^\rho \right)^{1/\rho},$$

where \( \rho \in (-\infty, 0) \cup (0, 1] \), is called a utility with constant elasticity of substitution or a CES utility. CES utilities with \( \rho \in (0, 1] \) are a special case of utilities that satisfy the weak gross substitutes (WGS) property—increasing the price of a good does not decrease the demand for any other good. If \( \rho = 1 \), this is a linear utility. In CES utilities with \( \rho < 0 \), the goods are complementary. The limit of \( u(x) \) as \( \rho \to -\infty \) is called a Leontief utility, and the limit as \( \rho \to 0 \) is called a Cobb-Douglas utility. A utility function of the form

$$u(x) = \left( \sum_{j_1=1}^{k} \alpha_{j_1}^\rho \left( \sum_{j_2=1}^{m} \alpha_{j_1,j_2}^\rho x_{j_1,j_2}^\rho \right)^{\rho_1/\rho_2} \right)^{1/\rho_1},$$

is a (two-level) nested CES utility (see [18]). A resource allocation utility (see [19]) is a nested linear-Leontief utility \((\rho_1 = 1 \text{ and } \rho_2 = -\infty)\). An exponential utility (a.k.a. constant absolute risk aversion utility) is a simple example of a utility that does not exhibit constant elasticity of substitution. It has the form

$$u(x) = \sum_{j=1}^{m} a_j \left( 1 - e^{-\theta x_j} \right).$$

Tâtonnement. The idea of tâtonnement is due to Walras [23]. Arrow et al. [3] showed that a time-continuous version of tâtonnement converges to equilibrium for WGS utilities. Scarf [20] gave examples of Arrow-Debreu markets with non-WGS utilities (in particular Leontief utilities) where specific implementations of tâtonnement cycle and never converge to an equilibrium.

The more recent computer science literature considers discrete versions of tâtonnement in Fisher markets. (Discrete tâtonnement was also discussed in the economics literature, see [22].) Codenotti et al. [10] show that a tâtonnement-like process involving some price coordination converges in polynomial time for WGS utilities. Another tâtonnement-like process that involves coordination is analyzed in Fleischer et al. [15]. They show, for many interesting markets, a weak form of convergence of the process. They consider the average of the prices and allocations over the steps of the process and show that they converge to a weak notion of approximate equilibrium, which satisfies only the sum of budget constraints of the agents. On the other hand, their results apply to a wide range of markets. In particular, they apply to all the markets where we demonstrate explicit bounds on convergence to our much stronger notion of approximate equilibrium.

The first result to demonstrate the convergence to equilibrium of a true discrete tâtonnement process is due to Cole and Fleischer [12], who showed that the prices converge to equilibrium prices for non-linear CES utilities that satisfy WGS (i.e., \( \rho \in (0, 1) \)) and for Cobb-Douglas utilities. They analyze the same price-adjustment process given in Equation (1). Their analysis relies on a strong property that they prove: in the cases they analyze they analyze the price of each good moves towards the equilibrium value at each iteration. This is generally false for non-WGS utilities. Nevertheless, Cheung et al. [7] modified the analysis to apply to some non-WGS utilities, including complementary CES utilities with \( \rho \in (-1, 0) \), two-level nested CES utilities with \( \rho_1, \rho_2 > -1 \), and even multi-level nested CES
utilities with some restrictions on the elasticity (which do not breach the lower bound of $-1$). It should be noted that the analysis of [12, 9] applies also to a model of asynchronous price adjustments. Asynchrony is clearly a desirable property of a plausible dynamic for a market economy (though, as mentioned above, tatonnement is perhaps not the right process to consider in that case) or a true peer-to-peer system. Our results, while applicable to a wider range of utility functions, assume synchronous price adjustments.

Recently, Cheung et al. [7] show that a discrete tatonnement process that uses an exponential function update rule (Equation (1) can be thought of as a linearization of their rule) converges to the optimal value of the Eisenberg-Gale convex program, for complementary CES utilities and for Leontief utilities, i.e., for all $p \in [-\infty, 0)$. We note that our Lemma 10 shows that in the cases that they analyze, the convergence of the convex program objective function also implies convergence to equilibrium (in the same sense that we use in our results). The analysis of [7] relates their process to generalized gradient descent using a judicious choice of a Bregman divergence. In comparison, our analysis uses the multiplicative weights update paradigm (see [2]). It is well-known that multiplicative weights update using an exponential function is equivalent to gradient descent using KL-divergence. But there is no Bregman divergence that yields the linear descent step of Equation (1) (which is not to say that the analysis of [7] cannot be adapted somehow to apply to the linear update). We further note that like our analysis, the results in [7] only apply to synchronous price adjustments.

Discussion and comparison. Our results apply to the utilities analyzed in [12, 9, 7], and to additional utilities, such as nested CES-Leontief utilities and exponential utilities that we analyze explicitly. Thus, to the best of our knowledge, we give the most generally applicable proof of convergence of discrete tatonnement to date. It should be noted that excluding Leontief utilities analyzed in [7], all the other results show that the number of steps needed to reach prices that are $\delta$-close to equilibrium is proportional to $\log(1/\delta)$. Our results only give a bound proportional to $1/\delta^3$. (For Leontief utilities specifically, the analysis in [7] combined with our Lemma 10 gives a bound proportional to $1/\delta^2$.) However, the lower bound of $\Omega(1/\delta^2)$ for Leontief utilities in [8] demonstrates that one cannot hope to improve our bounds substantially without narrowing the scope of their applicability. (Their lower bound is expressed in terms of $\delta$-closeness to dual optimality, and it probably translates into a lower bound of $\Omega(1/\delta)$ when expressed in terms of $\delta$-closeness to equilibrium.)

The pervasive application of the multiplicative weights update method uses $\epsilon$ that depends on the desired accuracy of the outcome, and proves the convergence of a solution that averages over the iterations. Nevertheless, we show convergence of the sequence of prices (and allocations), and not just convergence of average prices (clearly implied by the former), and we show this for a fixed $\epsilon$ in Equation (1) that is independent of the desired closeness to equilibrium.

The techniques in [15] are somewhat similar to ours. However, they only prove the convergence of the average, and they use a much weaker notion of approximate equilibrium. On the other hand, that allows them to apply their results directly to utility functions for which our stronger notion of convergence is unlikely to hold, such as linear or resource allocation utilities. By tweaking the tatonnement process, we can handle these utilities as well. This is briefly discussed and analyzed in Section 5. In this context, we note that whether one is interested in convergence of the sequence or in convergence of the average depends on the phenomenon that is being modeled. One can think of tatonnement as a negotiation process—the price setters propose prices and collect purchase offers to study the market, while no exchange actually occurs during the process. In this case, convergence of the sequence seems more suitable. An alternative view of tatonnement is that of an “exploration” process—the price setters actually sell a fraction of the supply at each step of the process at the prices of that step. In this case, one would focus on convergence of the average. (A more rigorous treatment of this interpretation is given by the ongoing markets model of [12, 9].) Our proofs imply that taking the average over the steps of the prices and the allocations yields a somewhat relaxed notion of an approximate equilibrium.

Finally, we discuss the robustness of our results. For simplicity, we assume that throughout the process the prices satisfy the condition that their sum equals the total budget available to the agents. This condition is easy to satisfy initially, without explicit coordination of the prices. For example, set each price to be arbitrary (but strictly positive),

\[
\frac{\delta}{\epsilon} \leq \frac{1}{\Omega(1/\delta)}
\]
and at the first round reset the price of each good to be the total money spent on the good, given the initial arbitrary prices. Once the assumption is satisfied, it is maintained if the process is carried out precisely. However, we note that even if the initial prices deviate from satisfying this assumption, and the price adjustment or even the responses of the agents are perturbed, the convergence theorems would still hold under trivial modifications. (However, note that the initial prices must be strictly positive—tâtonnement cannot recover from zero prices). In particular, the proof of Proposition 6 also implies that the prices converge very quickly towards satisfying the above assumption. Once the sum of prices is close to the sum of budgets, the rest of the proof can be modified to the slight deviation from equality and also to slight deviations from optimality of the responses of the agents.

2 Preliminaries

Here we present the Fisher market model and some basic definitions. In a Fisher market there are \( m \) perfectly divisible goods, each with quantity scaled to 1, without loss of generality. There are \( n \) agents. Each agent \( i \) is endowed with a budget of \( b_i \), and aims at maximizing a concave utility function \( u_i : \mathbb{R}_+^m \to \mathbb{R}_+ \). We’ll assume monotonicity, so each agent spends all its budget. Given monotonicity, we may assume without loss of generality that for every agent \( i \), \( u_i(\bar{0}) = 0 \). Also without loss of generality, we will assume throughout the paper that the budgets are scaled so that \( \sum_{i=1}^{n} b_i = 1 \).

A market equilibrium is a pair \((p, x)\) where \( p : \{1, 2, \ldots, m\} \to \mathbb{R}_+ \) is an assignment of non-negative prices to the goods, and \( x : \{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\} \to \mathbb{R}_+ \) is an allocation of goods to agents, satisfying the following conditions: (i) The total spend \( \sum_{j=1}^{m} p_j x_{ij} \) of agent \( i \) is at most \( b_i \). (ii) The basket of goods \( x_i \) that agent \( i \) gets maximizes the utility \( u_i \) for any basket whose cost is at most \( b_i \). (iii) The total demand \( \sum_{i=1}^{n} x_{ij} \) for good \( j \) is at most 1. (iv) If the total demand for good \( j \) is less than 1, then \( p_j = 0 \).

Let \( x_i(p) \) denote the optimal basket of goods maximizing the utility \( u_i \) of agent \( i \), under the budget constraint \( b_i \) and the market prices \( p \). Notice that \( x_i(p) \) is given by a solution to the following convex program:

\[
  x_i(p) = \arg \max_{x_i \in \mathbb{R}_+^m} \left\{ u_i(x_i) : \sum_{j=1}^{m} p_j x_{ij} \leq b_i \right\}.
\]

We assume that computing an optimal basket is a tractable problem. Further denote by

\[
  z_j(p) = \sum_{i=1}^{n} x_{ij}(p) - 1
\]

the excess demand for good \( j \) under the prices \( p \). Notice that an equilibrium price vector \( p^* \) satisfies \( p^* \in B \), where

\[
  B = \left\{ p \in \mathbb{R}_+^m : \sum_{j=1}^{m} p_j = \sum_{i=1}^{n} b_i \right\}.
\]

We now define the notion of approximate equilibrium. We give two alternative definitions. In the first definition, each agent buys an optimal basket of goods subject to the prices, but the demand for each good may exceed the supply by a little. In the second definition, each agent buys a near-optimal basket of goods, and the supply constraints are satisfied. Definition 1 implies Definition 2 (possibly with a change of the approximation guarantee \( \delta \)) in many interesting markets. This is the case when a small change in allocation results in a small change in utility. Also, Definition 2 implies Definition 1 in many interesting markets. Specifically, if given specific prices, the optimal allocation for each player is unique and the utility functions are strongly concave, then this is the case. Definition 2 is more natural when the allocations are very sensitive to small changes in the prices (one such example is the case of linear utilities).

The definition that we use in most of the paper is:
Lemma 3. A price-demand pair \((p, x)\) is a \(\delta\)-approximate equilibrium iff

P1. For every agent \(i = 1, 2, \ldots, n\), the demand of \(i\) optimizes the utility of \(i\) given the prices \(p\): \(x_i = x_i(p)\).

P2. For every good \(j = 1, 2, \ldots, m\), the demand for \(j\) does not exceed the supply by much: \(z_j(p) \leq \delta\).

P3. The goods that are not purchased almost entirely do not cost much: For every \(j = 1, 2, \ldots, m\), if \(z_j(p) < -\delta\) then \(p_j \leq \delta\).

The alternative definition that is often equivalent to the first one is:

Definition 2. A price-demand pair \((p, x)\) is a \(\delta\)-approximate equilibrium iff

P1. For every agent \(i = 1, 2, \ldots, n\), the utility of \(i\) is near-optimal given the prices \(p\): \(u_i(x_i) \geq (1 - \delta) \cdot u_i(x_i(p))\).

P2. For every good \(j = 1, 2, \ldots, m\), the demand for \(j\) does not exceed the supply: \(z_j(p) \leq 0\).

P3. The goods that are not purchased almost entirely do not cost much: For every \(j = 1, 2, \ldots, m\), if \(z_j(p) < -\delta\) then \(p_j \leq \delta\).

For clarity, we use Dirac’s notation to do linear algebra in a vector space over \(\mathbb{R}\) endowed with the standard Euclidean norm. In particular \(\langle u \mid v \rangle\) denotes the inner product between two vectors \(u\) and \(v\), and \(\langle u \mid A \mid v \rangle\) denotes the bilinear map of a pair of vectors \(u, v\) using a linear operator \(A\). We denote by \(\mathbb{R}_+\) the set of non-negative real numbers, and by \(\mathbb{R}_{++}\) the set of strictly positive real numbers.

3 The Tâtonnement Process

In this section, we discuss arbitrary Eisenberg-Gale markets \([16]\). In an Eisenberg-Gale market, an equilibrium allocation of goods to agents can be computed using a specific convex program whose objective function is strictly concave in the utilities of the agents. The Lagrange variables corresponding to the supply constraints give the equilibrium prices. Thus, an equilibrium price vector can be computed by solving the dual program.

Let \(x_i\) denote a feasible allocation for agent \(i\), so \(x_{ij}\) is the quantity of good \(j\) that agent \(i\) gets in the allocation \(x_i\). An equilibrium allocation \(x^*\) is given by the solution to the following Eisenberg-Gale convex program.

\[
\begin{align*}
\max & \quad \sum_{i=1}^{n} b_i \ln u_i(x_i) \\
\text{s.t.} & \quad \sum_{i=1}^{n} x_{ij} \leq 1 \quad \forall j = 1, 2, \ldots, m \\
& \quad x_{ij} \geq 0 \quad \forall i = 1, 2, \ldots, n; \forall j = 1, 2, \ldots, m.
\end{align*}
\]

An equilibrium price vector \(p^*\) is given by the solution to the dual program:

\[
\begin{align*}
\min & \quad \sum_{j=1}^{m} p_j + \sum_{i=1}^{n} g_i^*(\mu_i) \\
\text{s.t.} & \quad p_j \geq -\mu_{ij} \quad i = 1, 2, \ldots, n; \forall j = 1, 2, \ldots, m \\
& \quad p_j \geq 0 \quad \forall j = 1, 2, \ldots, m,
\end{align*}
\]

where \(g_i^*\) is the convex conjugate of the convex function \(g_i(x_i) = -b_i \ln u_i(x_i)\). More specifically, \(g_i^*\) satisfies \(g_i^*(\mu_i) = \sup_{x \in \mathbb{R}^n} \{\langle \mu_i \mid x \rangle - g_i(x)\}\). Notice that \(g_i^*\) is monotonically non-decreasing in each coordinate of \(\mu_i\). Therefore, we can replace \(\mu_{ij}\) by \(-p_j\) for every \(i, j\). For future use, we denote the primal objective function by \(\psi(x) = \sum_{i=1}^{n} b_i \ln u_i(x_i)\), and the dual objective function by \(\phi(p) = \sum_{j=1}^{m} p_j + \sum_{i=1}^{n} g_i^*(-p)\).

The following lemma gives an explicit expression for the gradient of \(\phi\). It appears in \([7]\) Lemma 3.3). In the appendix, we give the (short) proof for completeness.

Lemma 3 \([7]\). In every Eisenberg-Gale market,

\[\nabla \phi(p) = -z(p)\]
We also need the following property of the primal objective function. The proof appears in the appendix.

Claim 4. For every \( \alpha > 0 \), \( \psi\left(\frac{1}{1+\alpha} \cdot x\right) \geq \psi(x) - \alpha \).

The tâtonnement process proceeds as follows. Start with an arbitrary assignment of strictly positive prices \( p^0 \in \mathbb{R}^m_{++} \cap B \). Given the time \( t \) prices \( p^t \), each agent independently responds with its time \( t + 1 \) demand \( x^{t+1}_i \equiv x_i(p^t) \). Given the time \( t + 1 \) demands \( x^{t+1} \), the prices of goods are updated according to the excess demand — prices increase for goods whose demand exceeds supply and prices decrease for goods whose supply exceeds demand. More specifically,
\[
p^t_{j+1} = p^t_j \cdot (1 + \varepsilon z_j(p^t)),
\]
where \( \varepsilon \in (0, \frac{1}{2}) \) is a constant to be specified later.

We now give an alternative characterization of the optimal response of the agents to a price vector. The proof appears in the appendix.

Lemma 5. Let \( p \in B \) be a price vector. Then,
\[
x(p) = \arg\max_{x \in \mathbb{R}^n_{++}} \left\{ \sum_{i=1}^n b_i \ln u_i(x_i) : \frac{1}{\|p\|_1} \cdot \sum_{j=1}^m p_j \sum_{i=1}^n x_{ij} \leq 1 \right\}.
\]

We will use the following proposition whose proof appears in the appendix.

Proposition 6. For all \( t \), \( p^t \in B \).

4 A Simple Proof of Convergence

In this section we show a general proof of convergence of the discrete tâtonnement process. Under some simplifying and rather general assumptions, we further give an upper bound on the rate of convergence. In Section 5 we apply the bounds that we show here to derive concrete bounds on the convergence rate in specific types of markets. In order to state our main theorem we need the following definition. Let \( P \subset \mathbb{R}^m_{+} \) be a compact convex set which contains a dual optimal solution \( p^* \) and the sequence \( p^0, p^1, \ldots \). A judicious choice of \( P \) is crucial to the derivation of meaningful bounds on the convergence rate. Frequently, we can define \( P \) using lower bounds on the prices generated by tâtonnement. This, in turn, is achieved by observing that low prices imply positive excess demand which increases those prices, and by relying on the small change in prices in each step. We will use the notation \( p_{\max} = \max_{p \in P} \|p\|_\infty \), \( p_{\max} = \max_j p_{ij} \), and \( p_{\min} = \min_j p_{ij} \).

Definition 7. Let \( \beta \) be a function mapping \( \mathbb{R}_{+} \) to itself. An Eisenberg-Gale market is \( \beta \)-smooth iff for every \( \alpha > 0 \) and for every \( p \in P \) the following holds. If \( \phi(p) \leq \phi(p^*) + \beta(\alpha) \), then \( |z_j(p) - z_j(p^*)| \leq \alpha \) for every \( j = 1, 2, \ldots, m \).

Our main result is the following theorem.

Theorem 8. Consider a \( \beta \)-smooth Eisenberg-Gale market. Fix \( \delta > 0 \). Suppose that \( \varepsilon \) is sufficiently small and \( T \) is sufficiently large so that the following conditions hold.

C1. The sequence \( \phi(p^0), \phi(p^1), \ldots, \phi(p^T) \) is monotonically non-increasing.

C2. For every \( T' = T - \lfloor 3 \ln(1/\delta)/\delta \varepsilon \rfloor, \ldots, T \) and for every \( j = 1, 2, \ldots, m \),
\[
\frac{1}{T'+1} \sum_{t=0}^{T'} z_j(p^t) \leq \beta(\delta/3).
\]
Then, the price-demand pair \((p^T, x^{T+1})\) is a \(\delta\)-approximate equilibrium in the sense of Definition[7].

Proof. By construction, \(x^{T+1} = x(p^T)\).

Fix \(k \in \{0, 1, 2, \ldots, [3\ln(1/\delta)/\delta\epsilon]\}\). Let \(E x^t = \frac{1}{T-k+1} \cdot \sum_{t=1}^{T-k} x^t\) and let \(E p^t = \frac{1}{T-k+1} \cdot \sum_{t=0}^{T-k} p^t\). By condition C2, \(\frac{1}{1+\beta(\delta/3)} \cdot E x^t\) is a feasible primal solution. Therefore,

\[
\phi(p^*) \geq \psi \left( \frac{1}{1+\beta(\delta/3)} \cdot E x^t \right) \geq \psi(E x^t) - \beta(\delta/3) \geq \frac{1}{T-k+1} \cdot \sum_{t=1}^{T-k+1} \psi(x^t) - \beta(\delta/3) = \frac{1}{T-k+1} \cdot \sum_{t=0}^{T-k} \phi(p^t) - \beta(\delta/3) \geq \phi(E p^t) - \beta(\delta/3) \geq \phi(p^{T-k}) - \beta(\delta/3),
\]

where the first inequality follows from weak duality, the second inequality follows from Claim[4], the third inequality follows from the concavity of \(\psi\), the equality follows by construction (\(\psi(x^{t+1}) = \phi(p^t)\)), see the proof of Lemma[3], the fourth inequality follows from the convexity of \(\phi\), and the last inequality follows from condition C1 in the theorem. Concluding, we get that \(\phi(p^{T-k}) \leq \phi(p^*) + \beta(\delta/3)\). Thus, \(|z_j(p^{T-k}) - z_j(p^*)| \leq \frac{\delta}{3}\) for every \(j = 1, 2, \ldots, m\).

Consider \(j \in \{1, 2, \ldots, m\}\). Notice that \(z_j(p^*) \leq 0\). If \(z_j(p^*) \geq -\frac{2\delta}{3}\) then since for \(k = 0\) the above argument gives \(|z_j(p^T) - z_j(p^*)| \leq \frac{\delta}{3}\), we get that \(|z_j(p^T)| \leq \delta\). Otherwise, we have that \(z_j(p^{T-k}) < -\frac{\delta}{3}\), for \(k = 0, 1, 2, \ldots, [3\ln(1/\delta)/\delta\epsilon]\). But, \(p_j^{T-k}[3\ln(1/\delta)/\delta\epsilon] < 1\), and for every \(k \in \{0, 1, 2, \ldots, [3\ln(1/\delta)/\delta\epsilon]\} - 1\), \(p_j^{T-k} \leq (1 - \delta/3)p_j^{T-k-1}\). Thus, \(p_j^T \leq (1 - \delta/3)^{[3\ln(1/\delta)/\delta\epsilon]} \leq \delta\).

We now give general bounds on the rate of convergence using only the following assumption.

A. \(\phi(p)\) is twice continuously differentiable and strongly convex in the range \(P\).

Our proof below shows that this assumption implies \(\beta\)-smoothness and condition C1 of Theorem[8]. We also prove condition C2, independently of assumption A. Notice that assumption A implies that the equilibrium price vector is unique. Leontief (and obviously nested CES-Leontief) utilities violate the strong convexity assumption of A. However, we handle them in Section[5] by mapping the price vectors to a different space. When \(\phi\) is expressed in terms of the image variables, then it is strongly convex, and hence we can apply pretty much the same analysis as here to deduce \(\beta\)-smoothness and condition C1.

Let \(\nabla^2 \phi(p)\) denote the Hessian of \(\phi\) at \(p\), i.e., \(\nabla^2 \phi(p)\) is the symmetric real matrix whose entries are the second order partial derivatives of \(\phi\) at \(p\): \((\nabla^2 \phi)(p))_{jk} = \frac{\partial^2 \phi}{\partial p_j \partial p_k}(p)\). Let \(\lambda_{max}\) denote the supremum over \(p \in P\) of the largest eigenvalue of the Hessian \(\nabla^2 \phi(p)\) of \(\phi\) at \(p\). Let \(\lambda_{min}\) denote the infimum over \(p \in P\) of the smallest eigenvalue of \(\nabla^2 \phi(p)\). In cases where assumption A holds, applying our bounds on the convergence rate relies on finding constants \(\underline{\lambda}, \overline{\lambda}\) such that \(0 < \underline{\lambda} \leq \lambda_{min} \leq \lambda_{max} \leq \overline{\lambda}\). (Clearly, the lower bound is guaranteed by assumption A. The upper bound usually follows from the choice of \(P\).)

We first establish condition C1.

Lemma 9. Under assumption A, for \(\epsilon \leq \frac{1}{\lambda_{max} \lambda_{max}}\), the sequence \(\phi(p^t), t = 0, 1, 2, \ldots,\) is monotonically non-increasing.

Proof. Write \(p^{t+1} = p^t - q^t\), where for every \(j = 1, 2, \ldots, m\), \(q^t_j = -\epsilon p^t_j z_j(p^t)\). By Lemma[3] \(q^t_j = \epsilon p^t_j \nabla \phi(p^t_j)\). Consider the second order Taylor expansion of \(\phi(p^t)\) with respect to \(\phi(p^{t+1})\).

\[
\phi(p^t) = \phi(p^{t+1}) + \langle q^t \mid \nabla \phi(p^{t+1}) \rangle + \frac{1}{2} \langle q^t \mid \nabla^2 \phi(p) \mid q^t \rangle,
\]

where \(p = \gamma p^t + (1-\gamma) p^{t+1}\) for some \(\gamma \in [0, 1]\). As \(\phi\) is a convex function on \(P\) and \(p \in P\), the quadratic term in the Taylor expansion is non-negative. Thus, our proof is complete if we show that the linear term is also non-negative.
Write \( \nabla \phi(p^{t+1}) = \nabla \phi(p^t) + (\nabla \phi(p^{t+1}) - \nabla \phi(p^t)) \). Thus,
\[
\langle q^t \mid \nabla \phi(p^{t+1}) \rangle = \langle q^t \mid \nabla \phi(p^t) \rangle + \langle q^t \mid \nabla \phi(p^{t+1} - \nabla \phi(p^t)) \rangle
\]
\[
= \varepsilon \cdot \sum_{j=1}^{m} p_j^t (\nabla \phi(p^t))_j^2 - \varepsilon \cdot \sum_{j=1}^{m} p_j^t (\nabla \phi(p^t))_j \sum_{i=1}^{n} (x_{ij}^{t+1} - x_{ij}^{t+1}).
\]

Let \( f(p) \in \mathbb{R}_+^m \) denote the vector of total demands for the goods induced by \( x(p) \). In particular, \( f(p^t) = \sum_{i=1}^{m} x_i^{t+1} \).
In order to complete the proof we show that for sufficiently small \( \varepsilon \),
\[
\sum_{j=1}^{m} p_j^t (\nabla \phi(p^t))_j (f_j(p^{t+1}) - f_j(p^t)) \leq \sum_{j=1}^{m} p_j^t (\nabla \phi(p^t))_j^2.
\]

We use Taylor expansion again, this time a first-order expansion of \( f_j(p^{t+1}) \) with respect to \( f_j(p^t) \). For some interpolation point \( p = \gamma p^t + (1 - \gamma')p^{t+1} \),
\[
f_j(p^{t+1}) - f_j(p^t) = \langle -q^t \mid \nabla f_j(p) \rangle.
\]
Using Lemma 3, \( -(\nabla f_j(p))_j' = (\nabla^2 \phi(p))_{j,j'} \). Recall that by the strong convexity assumption \( \lambda_{\text{max}} > 0 \). We have that
\[
\sum_{j=1}^{m} p_j^t (\nabla \phi(p^t))_j (f_j(p^{t+1}) - f_j(p^t)) \leq \varepsilon \cdot \sum_{j=1}^{m} \sum_{j'=1}^{m} p_j^t (\nabla \phi(p^t))_j p_j^t (\nabla \phi(p^t))_{j'} (\nabla^2 \phi(p))_{j,j'}
\]
\[
\leq \varepsilon \cdot \lambda_{\text{max}} \cdot \sum_{j=1}^{m} (p_j^t (\nabla \phi(p^t))_j)^2
\]
\[
\leq \varepsilon \cdot \lambda_{\text{max}} \cdot \max_j p_j^t \cdot \sum_{j=1}^{m} (\nabla \phi(p^t))_j^2.
\]
Choosing \( \varepsilon \leq \frac{1}{\lambda_{\text{max}} p_{\text{max}}} \) completes the proof.

Next, we bound \( \beta \).

**Lemma 10.** Under assumption A, for every \( \alpha > 0 \), for every \( p \in P \), and for any dual optimal solution \( p^* \), if \( \phi(p) \leq \phi(p^*) + \min \left\{ 1, \frac{1}{\lambda_{\text{max}}} \right\} \cdot \frac{1}{2} \lambda_{\text{min}} \alpha^2 \), then \( |z_j(p) - z_j(p^*)| \leq \alpha \) for every \( j = 1, 2, \ldots, m \).

**Proof.** Consider the second order Taylor expansion of \( \phi(p) \) with respect to \( \phi(p^*) \). For an interpolation point \( p' = \gamma p + (1 - \gamma)p^* \),
\[
\phi(p) = \phi(p^*) + \langle p - p^* \mid \nabla \phi(p^*) \rangle + \frac{1}{2} \langle p - p^* \mid \nabla^2 \phi(p') \mid p - p^* \rangle.
\]
Notice that by the strong convexity assumption, \( \lambda_{\text{min}} > 0 \). Since \( \nabla \phi(p^*) = 0 \), we get that
\[
\min \left\{ 1, \frac{1}{\lambda_{\text{max}}} \right\} \cdot \frac{1}{2} \lambda_{\text{min}} \alpha^2 \geq \phi(p) - \phi(p^*) = \frac{1}{2} \langle p - p^* \mid \nabla^2 \phi(p') \mid p - p^* \rangle \geq \frac{1}{2} \cdot \lambda_{\text{min}} \cdot \| p - p^* \|_2^2.
\]
In particular, we get that \( \| p - p^* \|_2^2 \leq \frac{\alpha^2}{\lambda_{\text{max}}} \). On the other hand, \( |z_j(p) - z_j(p^*)| = |f_j(p) - f_j(p^*)| \leq \| f(p) - f(p^*) \|_2 \). Consider the first order Taylor expansion of \( f(p^T) \) with respect to \( f(p^*) \). For an interpolation point \( p'' = \gamma' p + (1 - \gamma')p^* \),
\[
f(p) = f(p^*) + \nabla f(p'') \mid p - p^* \rangle = f(p^*) + \nabla^2 \phi(p'') \mid p - p^* \rangle.
\]

\^1The proof here also shows that \( \| p - p^* \|_2 \leq \alpha \).
So
\[ \| f(p) - f(p^*) \|_2^2 = \| \nabla^2 \phi(p'') \|_2^2 \leq \lambda_{\text{max}}^2 \| p - p^* \|_2^2 \leq \alpha^2, \]
completing the proof.

Finally, we analyze the resulting bounds on \( \varepsilon \) and \( T \) that guarantee convergence based on Theorem 8. The following lemma is a variant of the multiplicative weights update method. Its proof appears in the appendix.

**Lemma 11.** Let \( w = \max_j \{ |z_j(p')| \} \), and let \( \varepsilon \leq \frac{1}{2w} \). Also let \( v = \max_j \frac{1}{T} \sum_{t=0}^{T-1} |z_j(p')|^2 \). Then,
\[
\frac{1}{T} \sum_{t=0}^{T-1} z_j(p') \leq \varepsilon v + \frac{\ln(1/p^0)}{\varepsilon T}.
\]

Let \( W = \max\{1, \max_{p \in P} \max_j z_j(p)\} \). We summarize the above analysis in

**Theorem 12.** Under assumption A, the following is true. Fix \( \varepsilon \leq \min \left\{ \frac{1}{\lambda_{\text{max}} P_{\text{max}}}, \frac{2W}{\lambda_{\text{min}}}, \frac{96 \lambda_{\text{max}}}{\lambda_{\text{min}}} \right\} \). For every \( \delta > 0 \), if \( T \geq \frac{27W \ln(1/p^0_{\text{min}})}{8 \varepsilon^2 \delta^4} + \frac{3 \ln(1/\delta)}{\varepsilon^2} \), the price-demand pair \((p^T, x^{T+1})\) is a \( \delta \)-approximate equilibrium in the sense of Definition 7.

**Proof.** We use Theorem 8 so we need to verify that the conditions stated in the theorem hold:

**Condition C1:** As \( \varepsilon \leq \frac{1}{2 \lambda_{\text{max}} P_{\text{max}}} \), by Lemma 8, condition C1 is satisfied.

**Condition C2:** As \( \varepsilon \leq \frac{1}{2W} \leq \frac{1}{2w} \) and for every \( T \),
\[
\frac{1}{T} \sum_{t=0}^{T-1} (z_j(p'))^2 \leq w^2,
\]
we can apply Lemma 11 to conclude that for every \( T \geq T_0 = \frac{\ln(1/p^0_{\text{min}})}{\varepsilon^2 w^2} \), we have that
\[
\frac{1}{T} \sum_{t=0}^{T-1} z_j(p') \leq 2 \varepsilon w^2 \leq \beta(w/4),
\]
where the last inequality follows from applying Lemma 10 + the bound \( \varepsilon \leq \frac{\lambda_{\text{min}}}{96 \lambda_{\text{max}}} \). Therefore, by the same argument as in the proof of Theorem 8, for every \( T \geq \frac{\ln(1/p^0_{\text{min}})}{\varepsilon^2 w^2} \), we know that \( \max_j |z_j(p^T) - z_j(p^*)| \leq w/4 \) (and in particular \( \max_j z_j(p^T) \leq w/4 \)). Next consider \( T \geq T_0 + T_1 \), where \( T_1 = 7 \cdot \frac{\ln(1/p^0_{\text{min}})}{\varepsilon^2 w^2} \). Consider \( j \in \{1, 2, \ldots, m\} \).

If \( z_j(p^*) < -w/4 \), then for every \( t > T_0 \), \( z_j(p^t) < 0 \), so
\[
\frac{1}{T} \sum_{t=0}^{T-1} z_j(p') < \frac{1}{T} \sum_{t=0}^{T_0} z_j(p') \leq 2 \varepsilon (w/2)^2 \leq \beta(w/8).
\]

Otherwise, for every \( t > T_0 \), \( z_j(p^t) \geq z_j(p^*) - w/4 \geq -w/2 \) (and for all \( j \), \( z_j(p^t) \leq w/4 \)). Therefore,
\[
\frac{1}{T} \sum_{t=0}^{T-1} (z_j(p'))^2 \leq \frac{1}{8} \cdot w^2 + \frac{7}{8} \cdot (w/2)^2 \leq 2(w/2)^2.
\]

Moreover, \( T \geq 8 \ln(1/p^0_{\text{min}})/\varepsilon^2 w^2 \), so \( \ln(1/p^0_{\text{min}})/\varepsilon T \leq \varepsilon (w/2)^2 \). Therefore, by Lemma 11
\[
\frac{1}{T} \sum_{t=0}^{T-1} z_j(p') \leq 3 \varepsilon (w/2)^2 \leq \beta(w/8).
\]
By the proof of Theorem\[8\] \( \max_j |z_j(p^T) - z_j(p^*)| \leq \frac{w}{8} \) and \( \max_j z_j(p^T) \leq w/8. \)

More generally, suppose that we’ve verified that for every \( T \geq T_0 + T_1 + \cdots + T_i, \)

\[
\frac{1}{T} \cdot \sum_{t=0}^{T-1} z_j(p^t) \leq \beta(w/2^{i+2}).
\]

Then, by the proof of Theorem\[8\] \( \max_j |z_j(p^T) - z_j(p^*)| \leq w/2^{i+2} \) and \( \max_j z_j(p^T) \leq w/2^{i+2}. \) This immediately implies that for every \( j, \) one of the following two cases holds:

Case 1: \( z_j(p^T) < 0 \) for all \( T \geq T_0 + T_1 + \cdots + T_i. \)

Case 2: \(-w/2^{i+1} \leq z_j(p^T) \leq w/2^{i+2} \) for all \( T \geq T_0 + T_1 + \cdots + T_i. \)

Setting \( T_{i+1} = 7(T_0 + T_1 + \cdots + T_i), \) we satisfy the inductive hypothesis as follows. Consider \( T \geq T_0 + T_1 + \cdots + T_{i+1} \) and \( j \in \{1, 2, \ldots, m\}. \) If case 1 holds, then

\[
\frac{1}{T} \cdot \sum_{t=0}^{T-1} z_j(p^t) \leq \frac{1}{8} \cdot \beta(w/2^{i+2}) < \beta(w/2^{i+3}).
\]

If case 2 holds, then

\[
\frac{1}{T} \cdot \sum_{t=0}^{T-1} (z_j(p^t))^2 \leq \frac{T_0 w^2 + T_1 (w/2)^2 + \cdots + T_i (w/2^{i+1})^2 + T_{i+1} (w/2^{i+2})^2}{T_0 + T_1 + \cdots + T_{i+1}} \leq \frac{7}{8} \cdot (w/2^{i+1})^2 + \frac{7}{8} \cdot \frac{1}{8} \cdot (w/2^i)^2 + \cdots < 2(w/2^{i+1})^2.
\]

Also, \( T \geq 8(T_0 + \cdots + T_i) \geq \ln(1/p_{\min}^0)/\epsilon^2(w/2^{i+1})^2, \) so \( \ln(1/p_{\min}^0)/\epsilon T \leq \epsilon(w/2^{i+1})^2. \) Thus, by Lemma\[11\]

\[
\frac{1}{T} \cdot \sum_{t=0}^{T-1} z_j(p^t) \leq 3\epsilon(w/2^{i+1})^2 \leq \beta_2(w/2^{i+3}).
\]

By the proof of Theorem\[8\] this implies the inductive hypothesis.

Finally, notice that if we choose \( i_{\max} \geq \log_2(3w/\delta) - 2 \) we get that for every \( T \geq T_0 + T_1 + \cdots + T_{i_{\max}}, \)

\[
\frac{1}{T} \cdot \sum_{t=0}^{T-1} z_j(p^t) \leq \beta(\delta/3).
\]

This demonstrates that condition C2 is satisfied for \( T \geq T_0 + T_1 + \cdots + T_{i_{\max}} + \frac{3\ln(1/\delta)}{\epsilon^2 \delta}, \) Therefore, for every \( T \geq T_0 + T_1 + \cdots + T_{i_{\max}} + \frac{3\ln(1/\delta)}{\epsilon^2 \delta}, \) we have that \( (p^T, x^{T+1}) \) is a \( \delta \)-approximate equilibrium. Finally, notice that

\[
T_0 + \cdots + T_{i_{\max}} < 8^{i_{\max}+1} \cdot \frac{\ln(1/p_{\min}^0)}{\epsilon^2 w^2} \leq \frac{27}{8} \cdot \frac{w \ln(1/p_{\min}^0)}{\epsilon^2 \delta^3}.
\]

This concludes the proof of the theorem.

\[\blacksquare\]

**5 Examples**

Here we demonstrate the application of Theorem\[8\] in various markets. Our examples include markets that do not satisfy the strong convexity assumption A (for instance, markets with Leontief utilities). We note that Theorem\[8\] can be adapted to show that tâtonnement converges in markets with utilities that are an arbitrary nesting of CES utilities, though it does not seem trivial to derive general closed form bounds for the parameters that control the convergence
rate. Here we demonstrate explicit bounds for two-level nested CES-Leontief utilities. Linear utilities pose a challenge to the tâtonnement process, which seems unlikely to converge in that case. However, linear utilities can be approximated by non-linear CES utilities with elasticity of substitution $\rho$ approaching 1, and running tâtonnement with the distorted utilities yields an approximate equilibrium (in the sense of Definition 2) for the original market (see [12]). The same approach works for resource allocation utilities that we analyze here. Notice that in these cases the reactions of the agents are assumed to be optimal with respect to the distorted utilities and not the original utilities. (In the proportional response dynamics that converge to equilibrium in the case of linear utilities [5] the agents also do not respond optimally to the prices.)

Non-linear CES utilities. A market with (non-linear) CES utilities is defined by a constant $\rho \in (-\infty, 0) \cup (0, 1)$, and constants $c_{ij} \geq 0$, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. The utility function of agent $i$ is $u_i(x) = \left(\sum_{j=1}^{m} (c_{ij} x_{ij})^{\rho}\right)^{1/\rho}$. The following theorem is an easy corollary of Theorem 17 below, as non-linear CES utilities are special cases of the nested CES-Leontief utilities discussed there.\footnote{We note that the theorem can be generalized easily to the case that for each player $i$ there is a different constant $\rho_i$ instead of one uniform constant $\rho$.}

\begin{theorem}
Consider a market with non-linear CES utilities. There are constants $\kappa_1 = \kappa_1(b, c, \rho)$ and $\kappa_2 = \kappa_2(b, c, \rho)$ such that if $\varepsilon \leq \frac{1}{\kappa_1}$ and $T \geq \frac{\kappa_2 \ln(1/\rho_{\min})}{\varepsilon^{2\rho+1}}$, then the price-demand pair $(p^T, x^{T+1})$ is a $\delta$-approximate equilibrium in the sense of Definition 7.
\end{theorem}

Leontief utilities. The limit of $u_i(x)$ as $\rho \to -\infty$ is called a Leontief utility. This is given explicitly as follows. Let $J_i = \{j : c_{ij} > 0\}$. Put $u_i(x) = \min_{j \in J_i} \{c_{ij} x_{ij}\}$. Leontief utilities are also special cases of nested CES-Leontief utilities, so as a corollary of Theorem 17 we have:

\begin{theorem}
Consider a market with Leontief utilities. There are constants $\kappa_1 = \kappa_1(b, c)$ and $\kappa_2 = \kappa_2(b, c)$ such that if $\varepsilon \leq \frac{1}{\kappa_2}$ and $T \geq \frac{\kappa_2 \ln(1/\rho_{\min})}{\varepsilon^{2\rho+1}}$, then the price-demand pair $(p^T, x^{T+1})$ is a $\delta$-approximate equilibrium in the sense of Definition 7.
\end{theorem}

Cobb-Douglas utilities. The limit of a CES utility $u_i(x) = \left(\sum_{j=1}^{m} (c_{ij} x_{ij})^{\rho}\right)^{1/\rho}$ as $\rho_i \to 0$ is called a Cobb-Douglas utility. Explicitly, this puts $u_i(x) = \prod_{j=1}^{m} x_{ij}^{c_{ij}}$. Markets with Cobb-Douglas utilities converge to equilibrium in one step if we set $\varepsilon = 1$. For a fixed $\varepsilon < 1$, convergence to a $\delta$-approximate equilibrium in $O(\log(1/\delta))$ steps is easy to establish using the quantitative estimates of Banach’s fixed-point theorem, as the mapping $p^t \mapsto p^{t+1}$ is a contraction (using any choice of norm). Our bounds on the convergence rate are much weaker, but nevertheless we prove them as they give a very simple demonstration of the method.

\begin{theorem}
Theorem 2\footnote{We note that the theorem can be generalized easily to the case that for each player $i$ there is a different constant $\rho_i$ instead of one uniform constant $\rho$.} applies also to markets with Cobb-Douglas utilities.
\end{theorem}

\begin{proof}
W.l.o.g., we scale the coefficients $c_{ij}$ so that for every $i = 1, 2, \ldots, n$, $\sum_{j=1}^{m} c_{ij} = 1$. The Eisenberg-Gale convex program is:

$$\max \left\{ \sum_{i} b_i \sum_{j} c_{ij} \ln x_{ij} : \forall j = 1, \ldots, m, \sum_{i} x_{ij} \leq 1 \land \forall i = 1, \ldots, n, \forall j = 1, \ldots, m, x_{ij} \geq 0 \right\}.$$ 

The dual objective is

$$\phi(p) = \sum_{j=1}^{m} p_j - \sum_{i=1}^{n} b_i + \sum_{i=1}^{n} \sum_{j=1}^{m} b_i c_{ij} \ln \left( \frac{b_i c_{ij}}{p_j} \right).$$

The dual objective is

$$\phi(p) = \sum_{j=1}^{m} p_j - \sum_{i=1}^{n} b_i + \sum_{i=1}^{n} \sum_{j=1}^{m} b_i c_{ij} \ln \left( \frac{b_i c_{ij}}{p_j} \right).$$

Proof. W.l.o.g., we scale the coefficients $c_{ij}$ so that for every $i = 1, 2, \ldots, n$, $\sum_{j=1}^{m} c_{ij} = 1$. The Eisenberg-Gale convex program is:

$$\max \left\{ \sum_{i} b_i \sum_{j} c_{ij} \ln x_{ij} : \forall j = 1, \ldots, m, \sum_{i} x_{ij} \leq 1 \land \forall i = 1, \ldots, n, \forall j = 1, \ldots, m, x_{ij} \geq 0 \right\}.$$ 

The dual objective is

$$\phi(p) = \sum_{j=1}^{m} p_j - \sum_{i=1}^{n} b_i + \sum_{i=1}^{n} \sum_{j=1}^{m} b_i c_{ij} \ln \left( \frac{b_i c_{ij}}{p_j} \right).$$
Notice that the optimal allocation for each player, assuming a price vector $p$, is given by $x_{ij}(p) = \frac{b_i c_{ij}}{p_j}$. The Hessian $\nabla^2 \phi$ is a diagonal matrix. Its $j$-th coordinate is $\frac{\sum_i b_i c_{ij}}{p_j^2}$. If $\varepsilon \leq \frac{1}{2}$, it is easy to verify that for every $j = 1, 2, \ldots, m$, $\frac{1}{2} \sum_i b_i c_{ij} \leq p_j \leq \frac{3}{2} \sum_i b_i c_{ij}$ (assuming the initial price vector $p^0$ satisfies these bounds). We get that $\lambda_{\max} \leq \max_j \frac{1}{\sum_i b_i c_{ij}}$ and $\lambda_{\min} \geq \min_j \frac{4\varepsilon}{\sum_i b_i c_{ij}}$. Theorem 8 can be applied directly to complete the proof. 

**Exponential utilities.** Here we discuss utilities of the form $u_i(x) = \sum_{j=1}^{m} a_{ij} (1 - e^{-x_{ij}})$, where $\theta > 0$ and $a_{ij} \geq 0$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Here, too, we get a very simple proof of convergence.

**Theorem 16.** Consider a market with exponential utilities. There are constants $\kappa_1 = \kappa_1(a, b, \theta)$ and $\kappa_2 = \kappa_2(a, b, \theta)$ such that if $\varepsilon \leq \frac{1}{\kappa_1}$ and $T \geq \frac{\kappa_2 \ln(1/p_{\min})}{\varepsilon^2 \theta^4}$, then the price-demand pair $(p^T, x^{T+1})$ is a $\delta$-approximate equilibrium in the sense of Definition 1.

**Proof.** With exponential utilities, we have that for every agent $i$ and for every good $j$ such that $a_{ij} > 0$, $x(p)_{ij} = \frac{1}{\theta} \cdot \ln \left( \frac{a_{ij}(1+b_i)}{p_j} \right)$. Therefore, the Hessian $\nabla^2 \phi$ is a diagonal matrix which is given by $(\nabla^2 \phi(p))_{jj} = \frac{1}{\theta^2} \cdot \left[ \{ (a_{ij} > 0) \} \right]$. Notice that for every $p \in P$ and $j = 1, 2, \ldots, m$, $p_j \leq 1$. Also, if $p_j < \frac{a_{ij}(1+b_i)}{e^\theta}$ then the demand of agent $i$ for good $j$ is more than 1, so $p_{j+1} > p_j$. Therefore, because $\varepsilon \leq \frac{1}{2}$ we may assume that $p_j \geq \max_i \frac{a_{ij}(1+b_i)}{2e^\theta}$ for all $t$. So, $\max_i \frac{a_{ij}(1+b_i)}{2e^\theta} \leq \lambda_{\min} \leq \lambda_{\max} \leq 1$ and we can use Theorem 12 to derive this theorem.

**Nested CES-Leontief utilities.** In a market with nested CES-Leontief utilities every agent $i = 1, 2, \ldots, n$ needs a set of “objects” $J_i$. Each object $J \in J_i$ consists of a utility coefficient $c_{ij} > 0$, a set of goods (which we also denote by $J$), and utility coefficients $a_{ij} > 0$ for all $j \in J$. The utility functions are formally given by $u_i(x) = \left( \sum_{j \in J_i} \left( c_{ij} \min_{j \in J_i} \left( \frac{x_{ij}}{a_{ij}} \right) \right)^{\rho} \right)^{1/\rho}$, for some $\rho \in (-\infty, 0) \cup (0, 1)$.

**Theorem 17.** There are constants $\kappa_1 = \kappa_1(a, b, c, \rho)$ and $\kappa_2 = \kappa_2(a, b, c, \rho)$ such that if $\varepsilon \leq \frac{1}{\kappa_1}$ and $T \geq \frac{\kappa_2 \ln(1/p_{\min})}{\varepsilon^2 a^3}$, then the price-demand pair $(p^T, x^{T+1})$ is a $\delta$-approximate equilibrium in the sense of Definition 1.

**Proof.** W.l.o.g. we scale each $a_i$ and $c_i$ so that $\|a_i\|_\infty = 1$ and $\sum_{j \in J_i} (c_{ij}^J)^{\rho/(1-\rho)} = 1$. (Notice that the behavior of agent $i$ depends only on the relative values of the coordinates of $a_i$ and $c_i$.) The Eisenberg-Gale convex program is as follows:

$$\max \left\{ \sum_i b_i \ln \left( \sum_{j \in J_i} \left( c_{ij}^J u_i^J \right)^{\rho} \right)^{1/\rho} : \forall j = 1, \ldots, m, \sum_{i=1}^{n} \sum_{j \in J_i} a_{ij}^J u_i^J \leq 1 \land \forall i = 1, \ldots, n, \forall J \in J_i, u_i^J \geq 0 \right\}.$$ 

The dual objective is

$$\phi(p) = \sum_{j=1}^{m} p_j - \sum_{i=1}^{n} b_i + \sum_{i=1}^{n} b_i \ln \left( \sum_{J \in J_i} \left( \frac{b_i c_{ij}^J}{\sum_{j=1}^{m} a_{ij}^J p_j} \right)^{\rho/(1-\rho)} \right)^{(1-\rho)/\rho}.$$ 

In order to apply Theorem 8 we need to verify the “smoothness” parameter of the market and the two conditions C1 and C2. For proving $\beta$-smoothness and the monotonicity condition we bound $\langle x \mid \nabla^2 \phi(p) \mid x \rangle$ for every $p \in P$. It
is easy to verify that the Hessian $\nabla^2 \phi$ of the dual objective function is given by

$$(\nabla^2 \phi(p))_{ij} = \frac{1}{1 - \rho} \cdot \sum_{i=1}^{n} b_i \cdot \left( \frac{\sum_{J \in \mathcal{J}_i} a_{ij}^J a_{iJ}^J ((c_i^J)^\rho/\tilde{p}_i^J)^{1-\rho}}{\sum_{J \in \mathcal{J}_i} (c_i^J/\tilde{p}_i^J)^{1-\rho}} \right) - \rho \cdot \left( \frac{\sum_{J \in \mathcal{J}_i} a_{ij}^J ((c_i^J)^\rho/\tilde{p}_i^J)^{1-\rho}}{\sum_{J \in \mathcal{J}_i} (c_i^J/\tilde{p}_i^J)^{1-\rho}} \right) \cdot \left( \frac{\sum_{J \in \mathcal{J}_i} a_{iJ}^J ((c_i^J)^\rho/\tilde{p}_i^J)^{1-\rho}}{\sum_{J \in \mathcal{J}_i} (c_i^J/\tilde{p}_i^J)^{1-\rho}} \right),$$

where $\tilde{p}_i^J = \sum_{j=1}^{m} a_{ij}^J p_j$. Therefore,

$$\langle x | \nabla^2 \phi(p) | x \rangle = \frac{1}{1 - \rho} \cdot \sum_{i=1}^{n} b_i \cdot \left( \frac{\sum_{J \in \mathcal{J}_i} (c_i^J/\tilde{p}_i^J)^{1-\rho}}{\left( \sum_{J \in \mathcal{J}_i} (c_i^J/\tilde{p}_i^J)^{1-\rho} \right)^2} \right) \cdot \left( \sum_{J \in \mathcal{J}_i, J' \neq J} \sum_{J' \in \mathcal{J}_i} (c_i^J c_i^{J'})^{1-\rho} \left( \frac{(\tilde{x}_i^J)^2}{(\tilde{p}_i^J)^{1-\rho} (\tilde{p}_i^{J'})^{1-\rho} - \rho \cdot \tilde{x}_i^J \tilde{x}_i^{J'}}{(\tilde{p}_i^J)^{1-\rho} (\tilde{p}_i^{J'})^{1-\rho}} \right) \right),$$

where $\tilde{x}_i^J = \sum_{j=1}^{m} a_{ij}^J x_j$.

Fix $i$, denote $X_J = \tilde{x}_i^J$ and $q_J = \frac{1}{\tilde{p}_i^J}$, then consider the term

$$Z_{J,J'} = q_{J,J'} \frac{\tilde{x}_i^J}{\tilde{p}_i^{J}} \frac{\tilde{x}_i^{J'}}{\tilde{p}_i^{J'}} X_J^2 - \rho \cdot \frac{1}{\tilde{p}_i^J} \frac{1}{\tilde{p}_i^{J'}} X_J X_{J'}. $$

Put $A_{J,J'} = \frac{q_{J,J'} \tilde{x}_i^J}{\tilde{p}_i^{J}} \frac{\tilde{x}_i^{J'}}{\tilde{p}_i^{J'}}$. Because we sum over all $J, J' \in \mathcal{J}_i$, we can replace $Z_{J,J'}$ by

$$\frac{1}{2} \cdot (A_{J,J} X_J^2 + A_{J,J'} X_{J'}^2) - \rho A_{J,J} A_{J',J} X_J X_{J'} =$$

$$= \frac{\rho}{2} \cdot (A_{J,J} X_J + A_{J',J} X_{J'})^2 + \frac{1 - \rho}{2} \cdot (A_{J,J} X_J^2 + A_{J',J} X_{J'}^2).$$

Denote

$$L = \sum_{i=1}^{n} b_i \cdot \frac{\sum_{J \in \mathcal{J}_i, J' \in \mathcal{J}_i} \left( (c_i^J c_i^{J'})^{1-\rho} / (\tilde{p}_i^J)^{1-\rho} (\tilde{p}_i^{J'})^{1-\rho} \right) \cdot (\tilde{x}_i^J)^2}{\sum_{J \in \mathcal{J}_i, J' \in \mathcal{J}_i} (c_i^J c_i^{J'})^{1-\rho} / (\tilde{p}_i^J)^{1-\rho} (\tilde{p}_i^{J'})^{1-\rho}}. $$

we have that for $\rho > 0$,

$$L \leq \langle x | \nabla^2 \phi(p) | x \rangle \leq \frac{1}{1 - \rho} \cdot L.$$

Similarly, for $\rho < 0$,

$$\frac{1}{1 - \rho} \cdot L \leq \langle x | \nabla^2 \phi(p) | x \rangle \leq L.$$
Proof. Notice that for every agent $i$, the demand that $i$ has for the object $J \in \mathcal{J}_i$ given the prices $p$ is $\frac{b_i}{\tilde{p}^{\rho}_i} \cdot \frac{(c'_{ij}/\tilde{p}_i^\rho)^{1-\rho}}{\sum_{J \in \mathcal{J}_i} (c'_{ij}/\tilde{p}_i^\rho)^{1-\rho}}$. Consider $J_0 \in \mathcal{J}_i$ that minimizes $\tilde{p}^{J_0}_i$, and let $J_1 \in \mathcal{J}$ be any object for which $\tilde{p}^{J_1}_i \leq 2\tilde{p}^{J_0}_i$.

Let’s first consider $\rho > 0$. Using the scaling of $c_i$, we have that $\sum_{J \in \mathcal{J}_i} (c'_{ij}/\tilde{p}_i^\rho)^{1-\rho} \leq |\mathcal{J}_i|/(\tilde{p}^{J_0}_i)^{1-\rho}$ and $c_{\min} \leq \frac{1}{|\mathcal{J}_i|}$. Therefore, the demand that $i$ has for $J_1$ is at least $\frac{b_i c_{\min}^{2-\rho}}{2^{1/(1-\rho)} \tilde{p}^{J_0}_i}$. This is a lower bound on $x(p)_{ij}$ for every $j \in J_1$. Notice that if for all $j \in J_1$, $x(p)_{ij} > 1$, then $\tilde{p}^{J_1}_i$ must increase in the next time step. Thus, we conclude that if $\tilde{p}^{J_0}_i \leq 2^{1/(1-\rho)} \cdot b_i \cdot c_{\min}^{2-\rho}$, then $\tilde{p}^{J_1}_i$ increases in the next time step.

A similar analysis applies to $\rho < 0$. In this case we bound the demand that agent $i$ has for $J_1$ by using the fact that for every $J \in \mathcal{J}_i$, $\tilde{p}^{J_1}_i \leq 1$. Therefore, $\sum_{J \in \mathcal{J}_i} (\tilde{p}^{J_0}_i/c'_i)^{1-\rho} \leq |\mathcal{J}_i| \cdot c_{\min} \leq 1$. Using the fact that $(c'_i)^{\rho/(1-\rho)} \leq 1$ and $\tilde{p}^{J_0}_i \leq \tilde{p}^{J_1}_i \leq 2\tilde{p}^{J_0}_i$, we get that the demand is at least $\frac{b_i}{2(\tilde{p}^{J_0}_i)^{1/(1-\rho)}}$. So if $\tilde{p}^{J_0}_i \leq 2^{\rho-1} \cdot b_i^{1-\rho}$ then $\tilde{p}^{J_1}_i$ increases in the next time step.

Now the rest of the proof follows by induction on the number of time steps, assuming that the inequality holds initially. Notice that in one iteration the prices never drop by more than a factor of 2 (because $\varepsilon \leq \frac{1}{2}$). So let $q$ denote the new prices, and let $J_i$ denote the object minimizing $\tilde{q}^i_1$ (over $J \in \mathcal{J}_i$). If $\tilde{p}^{J_1}_i > 2\tilde{p}^{J_0}_i$, then the induction hypothesis holds trivially. Otherwise, we showed that there exists $\gamma$ such that if $\tilde{p}^{J_0}_i < \gamma$ then $\tilde{q}^{J_1}_i > \tilde{p}^{J_1}_i$. If indeed $\tilde{p}^{J_0}_i < \gamma$, then the induction hypothesis holds trivially. Otherwise, $\tilde{p}^{J_1}_i \geq \tilde{p}^{J_0}_i \geq \gamma$, so $\tilde{q}^{J_1}_i \geq \frac{q}{2}$, and this completes the proof.

We are now ready to complete the proof of Theorem 17. Using Claim 18 the trivial upper bound $\|\tilde{p}\|_{\infty} \leq 1$ (which follows from the bounds $\|p\|_{\infty} \leq 1$ and $\|a\|_{\infty} \leq 1$), the notation $b_{\min}$, $c_{\min}$, and the fact that for all $i$, $1 \leq |\mathcal{J}_i| \leq c_{\min}^{2-\rho}$, we get the following lower and upper bounds on $L$. If $\rho > 0$ we get that

$$2^{\frac{\rho(\rho-2)}{1-\rho}} \cdot b_i^{1-\rho} \cdot c_{\min}^{2-\rho} \cdot \|\tilde{x}\|_2^2 \leq L \leq 2^{\frac{\rho(\rho-2)}{1-\rho}} \cdot b_i^{2-\rho} \cdot c_{\min}^{1-\rho} \cdot \|\tilde{x}\|_2^2.$$

Similarly, if $\rho < 0$ we get that

$$2^{\frac{\rho(\rho-2)}{1-\rho}} \cdot b_i^{1-\rho} \cdot \|\tilde{x}\|_2^2 \leq L \leq 2^{\frac{3\rho-2}{1-\rho}} \cdot b_i^{3\rho-2} \cdot c_{\min}^{\rho} \cdot \|\tilde{x}\|_2^2.$$

Using the semi-norm $\|p\| = \|\tilde{p}\|_2$ instead of the Euclidean norm $\|p\|_2$, the proofs of Lemma 9 and Lemma 10 go through without assumption A. (Notice that if two price vectors $p, q$ satisfy $\|p - q\| = 0$ then $\phi(p) = \phi(q)$ and $x(p) = x(q)$.) The parameters $\lambda_{\min}$ and $\lambda_{\max}$, respectively, are replaced with the above lower and upper bounds, respectively, for $\frac{\langle x, \nabla \phi(x) \rangle}{\|x\|_2^2}$. This allows us to apply Theorem 12 without assumption A, and this completes the proof.

**Resource allocation utilities.** A market with resource allocation utilities is similar to a market with nested CES-Leontief utilities, except that $\rho$ is set to 1 in the case of resource allocation utilities. In other words, the utility functions are given by $u_i(x) = \sum_{J \in \mathcal{J}_i} c_{ij}^\rho \min_{J \in \mathcal{J}_i} \left\{ \frac{x^{\rho}_{ij}}{c_{ij}^\rho} \right\}$. As resource allocation markets generalize both Leontief utilities and linear utilities, we need to deal with assumption A not holding, and also we need to apply the tâtonnement process to distorted utilities. We will approximate the utility function of each agent as a nested CES-Leontif utility.

**Theorem 19.** Let $k = \max_i |\mathcal{J}_i|$. For every $\delta > 0$ there are constants $\kappa_0 = \kappa_0(c), \kappa_1 = \kappa_1(a, b)$, and $\kappa_2 = \kappa_2(a, b, c)$ such that the following holds. For $\varepsilon \leq \frac{1}{k \log^2 k/\delta \cdot \kappa_1(c)}$, and $T \geq \frac{\kappa_2 \ln(1/\rho_{\max})}{\varepsilon \delta^2}$, the price-demand pair $(p^T, x^{T+1})$ is a $\delta$-approximate equilibrium in the sense of Definition 2.
Proof. We replace the utility functions by their distorted versions \( \tilde{u}_i(x) = \left( \sum_{J_i \in J_i} \left( c_i^J \min_{j \in J_i} \left\{ \frac{x_{ij}^J}{a_{ij}^J} \right\} \rho \right) \right)^{1/\rho} \), for \( \rho = 1 - \frac{\delta}{1 + \delta} \). We apply Theorem 17 to get prices \( p \) and allocations \( x \) that are a \( \frac{\delta}{2} \)-approximate equilibrium in the sense of Definition 1 for the utilities \( \tilde{u}_i \). Notice that by properties P2 and P3 of Definition 1, the allocations \( \tilde{x} = \frac{1}{1 + \delta/2} x \) satisfy properties P2 and P3 of Definition 2. Moreover, as the distorted utility functions are \( 1 \)-homogeneous (this is also true of resource allocation utilities), \( \tilde{u}_i(\tilde{x}) = \frac{1}{1 + \delta/2} \tilde{u}_i(x) \). To complete the proof, denote by \( x^* \) the optimal allocations with respect to the prices \( p \) for the original resource allocation utilities. We have that for every \( i = 1, 2, \ldots, n \), \( u_i(\tilde{x}) \geq k^{1-1/\rho} \cdot \tilde{u}_i(\tilde{x}) \geq (1 - \delta/2) \cdot \tilde{u}_i(\tilde{x}) \geq \frac{1 - \delta/2}{1 + \delta/2} \cdot \tilde{u}_i(x) \geq \frac{1 - \delta/2}{1 + \delta/2} \cdot u_i(x^*) \geq (1 - \delta) \cdot u_i(x^*) \). This establishes property P1 of Definition 2.

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Appendix: Proofs

PROOF OF LEMMA $3$ By the definition of $g_i^*$,

$$
\phi(p) = \max_{x \in \mathbb{R}_{++}^{n \times m}} \left\{ \sum_{i=1}^{n} b_i \ln u_i(x_i) + \sum_{j=1}^{m} p_j \left( 1 - \sum_{i=1}^{n} x_{ij} \right) \right\}.
$$

Let $\phi_x(q) = \sum_{i=1}^{n} b_i \ln u_i(x_i) + \sum_{j=1}^{m} q_j \left( 1 - \sum_{i=1}^{n} x_{ij} \right)$. This is a linear function of $q$. Notice that $\phi(q) = \max_x \phi_x(q)$. Fix $x$ to be a maximizing assignment for $q = p$. By a well-known fact, $\nabla \phi(p) = \nabla \phi_x(p)$. But $\phi_x(p)$ is a linear function of $p$ and its gradient is given by $(\nabla \phi_x(p))_j = 1 - \sum_{i=1}^{n} x_{ij}$.

Now, to complete the proof, we show that the maximizing assignment $x$ for $q = p$ is $x = x(p)$. Notice that

$$
\arg \max_{x \in \mathbb{R}_{++}^{n \times m}} \left\{ \sum_{i=1}^{n} b_i \ln u_i(x_i) + \sum_{j=1}^{m} p_j \left( 1 - \sum_{i=1}^{n} x_{ij} \right) \right\} = \arg \max_{x \in \mathbb{R}_{++}^{n \times m}} \left\{ \sum_{i=1}^{n} b_i \ln u_i(x_i) + \sum_{j=1}^{m} p_j \left( \sum_{i=1}^{n} x_{ij}(p) - \sum_{i=1}^{n} x_{ij} \right) \right\},
$$

because the expressions on both sides of the equation differ only by an additive constant that does not depend on $x$. Finally, notice that the solution to the right-hand side is an equilibrium demand for the same agents, but where the supply of each good $j$ is equal to $\sum_{i=1}^{n} x_{ij}(p)$. The equilibrium demand in this market is precisely $x(p)$.

PROOF OF CLAIM $4$ For every agent $i$,

$$
u_i \left( \frac{1}{1 + \alpha} \cdot x \right) = \nu_i \left( \frac{1}{1 + \alpha} \cdot x + \alpha \cdot \hat{\alpha} \right) \geq \frac{1}{1 + \alpha} \cdot \nu_i(x) + \frac{\alpha}{1 + \alpha} \cdot \nu_i(\hat{\alpha}) = \frac{1}{1 + \alpha} \cdot \nu_i(x),$$

where the inequality follows from the concavity of $\nu_i$. Thus,

$$
\psi \left( \frac{1}{1 + \alpha} \cdot x \right) = \sum_{i=1}^{n} b_i \ln \left( \nu_i \left( \frac{1}{1 + \alpha} \cdot x \right) \right) \geq \sum_{i=1}^{n} b_i \ln \left( \frac{1}{1 + \alpha} \cdot \nu_i(x) \right) \geq \psi(x) - \alpha,
$$

where the second inequality follows from $\ln(1 + \alpha) \leq \alpha$ and the scaling $\sum_{i=1}^{n} b_i = 1$.

PROOF OF LEMMA $5$ The proof uses essentially the same argument as in the proof of Lemma $3$. Given a price vector $p \in B$. Consider the following program,

$$
\max \left\{ \sum_{i=1}^{n} b_i \ln u_i(x_i) : \frac{1}{\|p\|_1} \cdot \sum_{j=1}^{m} p_j \sum_{i=1}^{n} x_{ij} \leq 1 \right\}.
$$

By Lagrange duality we get that the maximum is achieved at

$$
\lambda_{\text{max}}, x_{\text{max}} = \arg \max_{x \in \mathbb{R}_{++}^{n \times m}, \lambda \geq 0} \left\{ \sum_{i=1}^{n} b_i \ln u_i(x_i) - \lambda \left( 1 - \sum_{j=1}^{m} \frac{p_j}{\|p\|_1} \sum_{i=1}^{n} x_{ij} \right) \right\}.
$$

We will show that $\lambda_{\text{max}} = \|p\|_1$ and $x_{\text{max}} = x(p)$ by showing that they satisfy the KKT conditions. The first condition is that if $\lambda_{\text{max}} > 0$ then $1 = \sum_{j=1}^{m} \frac{p_j}{\|p\|_1} \sum_{i=1}^{n} (x_{\text{max}})_{ij}$. By the definition of $x(p)$, $\sum_{j=1}^{m} p_j x(p)_{ij} = b_i$ (as $u_i(x)$ is monotonically non-decreasing). Thus, $x(p)$ satisfies $\sum_{i=1}^{n} \sum_{j=1}^{m} p_j x(p)_{ij} = \sum_{i=1}^{n} b_i = \|p\|_1$. For the
second KKT condition notice the following. In a market in which every good \( j \) has quantity \( \sum_{i=1}^{n} x(p)_{ij} \), the price-demand pair \((p, x(p))\) is an equilibrium (each agent optimizes its demand and the market clears). Thus, \( x(p) \) is a primal optimal solution to the Eisenberg-Gale convex program for this market. Therefore,

\[
x(p) = \arg \max_{x \in \mathbb{R}^{n \times m}_{+}} \left\{ \sum_{i=1}^{n} b_i \ln u_i(x_i) : \sum_{i=1}^{n} x_{ij} = \sum_{i=1}^{n} x(p)_{ij} \forall j \right\}
\]

\[
= \arg \max_{x \in \mathbb{R}^{n \times m}_{+}} \left\{ \sum_{i=1}^{n} b_i \ln u_i(x_i) - \sum_{j=1}^{m} p_j \sum_{i=1}^{n} (x(p)_{ij} - x_{ij}) \right\}
\]

\[
= \arg \max_{x \in \mathbb{R}^{n \times m}_{+}} \left\{ \sum_{i=1}^{n} b_i \ln u_i(x_i) - \sum_{j=1}^{m} p_j \sum_{i=1}^{n} (1 - x_{ij}) \right\}.
\]

The first equation follows from the definition of the Eisenberg-Gale market with quantities \( \sum_{i=1}^{n} x(p)_{ij} \). The second equation follows from Lagrange duality and the fact that the prices \( p \) are a dual optimal solution. The third equation follows as \( \sum_{j=1}^{m} p_j \sum_{i=1}^{n} x(p)_{ij} \) is a constant independent of \( x \).

Going back to the second KKT condition, notice that putting \( \lambda_{\text{max}} = \|p\|_1 \) gives exactly the last equation. Therefore, \( x_{\text{max}} = x(p) \), \( \lambda_{\text{max}} = \|p\|_1 \) satisfy the second KKT condition as well. ■

**Proof of Proposition 6.** The proof is by induction on \( t \). For \( t = 0 \), the claim is satisfied by choosing the initial prices \( p^0 \in B \). For the inductive step, notice that

\[
\sum_{j=1}^{m} p_{j}^{t+1} = (1 - \epsilon) \sum_{j=1}^{m} p_{j}^{t} + \epsilon \sum_{j=1}^{m} p_{j}^{t} \sum_{i=1}^{n} x_{ij}^{t+1} = (1 - \epsilon) \sum_{j=1}^{m} p_{j}^{t} + \epsilon \sum_{i=1}^{n} \sum_{j=1}^{m} p_{j}^{t} x_{ij}(p^t) = \sum_{j=1}^{m} b_j
\]

As we assume that \( u_i(x_i) \) is monotonically non-decreasing, by the definition of \( x_i(p) \) for every \( i \), \( \sum_{j=1}^{m} p_{j}^{t} x_{ij}(p^t) = b_i \) (every agent spends all its budget at every round). Thus the last equality follows (together with the induction hypothesis on \( p^t \)). ■

**Proof of Lemma 11.** The proof uses the standard potential function argument analyzing the multiplicative weights update method. We begin with the following inequality that holds for every \( t = 0, 1, 2, \ldots \)

\[
\sum_{j=1}^{m} p_{j}^{t+1} = \sum_{j=1}^{m} p_{j}^{t} \left( 1 + \varepsilon z_j(p^t) \right)
\]

\[
= \sum_{j=1}^{m} p_{j}^{t} + \varepsilon \cdot \left( \sum_{j=1}^{m} p_{j}^{t} \right) \cdot \sum_{j=1}^{m} \sum_{j'=1}^{m} p_{j'}^{t} z_{j}(p^t)
\]

\[
= \left( \sum_{j=1}^{m} p_{j}^{t} \right) \cdot \left( 1 + \varepsilon \sum_{j=1}^{m} \sum_{j'=1}^{m} p_{j'}^{t} z_{j}(p^t) \right)
\]

\[
\leq \left( \sum_{j=1}^{m} p_{j}^{t} \right) \cdot e^{\varepsilon \sum_{j=1}^{m} \sum_{j'=1}^{m} p_{j'}^{t} z_{j}(p^t)} / \sum_{j'=1}^{m} p_{j'}^{t}.
\]

So, on the one hand,

\[
\sum_{j=1}^{m} p_{j}^{T} \leq \left( \sum_{j=1}^{m} P_{j}^{0} \right) \cdot e^{\varepsilon \sum_{i=0}^{T-1} \sum_{j=1}^{m} p_{j}^{i} z_{j}(p^{i}) / \sum_{j'=1}^{m} p_{j'}^{i}}.
\]
On the other hand, for every $k \in \{1, 2, \ldots, m\}$,

$$\sum_{j=1}^{m} p_j T \geq p_k^T = p_k^0 \cdot \prod_{t=0}^{T-1} (1 + \varepsilon z_k(p^t)) .$$

Taking the logarithms of the lower and upper bounds for $\sum_{j=1}^{m} p_j^T$, we get that

$$\sum_{t=0}^{T-1} \ln (1 + \varepsilon z_k(p^t)) \leq \ln \left( \frac{\sum_{j=0}^{m} p_j^0}{p_k^0} \right) + \varepsilon \cdot \sum_{t=0}^{T-1} \sum_{j=1}^{m} \sum_{j' = 1}^{m} p_j^t z_j(p^t).$$

The second term on the right-hand side equals 0 (see the proof of Lemma 5 and recall that $z_j(p^t) = 1 - \sum_{i=1}^{n} x(p^t)_{ij}$). Using the fact that $\ln(1 + \xi) \geq \xi - \xi^2$ for every $\xi \in [-\frac{1}{2}, \frac{1}{2}]$, we get that

$$\sum_{t=0}^{T-1} \varepsilon z_k(p^t) - \sum_{t=0}^{T-1} \varepsilon^2 (z_k(p^t))^2 \leq \ln \left( \frac{\sum_{j=1}^{m} p_j^0}{p_k^0} \right).$$

Averaging over $t$, we get that

$$\frac{1}{T} \cdot \sum_{t=0}^{T-1} \varepsilon z_k(p^t) \leq \frac{1}{T} \cdot \frac{\ln(1/p_k^0)}{\varepsilon} + \frac{1}{T} \cdot \sum_{t=0}^{T-1} \varepsilon \cdot (z_k(p^t))^2 \leq \frac{\ln(1/p_k^0)}{\varepsilon T} + \varepsilon v.$$

(Recall that we scale $b$ so that $\sum_{j=1}^{m} p_j^0 = \sum_{i=1}^{n} b_i = 1.$)  

[19]