STATE SPACE FORMULAS FOR A SUBOPTIMAL RATIONAL LEECH PROBLEM II: PARAMETRIZATION OF ALL SOLUTIONS

A.E. FRAZHO, S. TER HORST, AND M.A. KAASHOEK

Abstract. For the strictly positive case (the suboptimal case), given stable rational matrix functions $G$ and $K$, the set of all $H^\infty$ solutions $X$ to the Leech problem associated with $G$ and $K$, that is, $G(z)X(z) = K(z)$ and $\sup_{|z|<1} \|X(z)\| \leq 1$, is presented as the range of a linear fractional representation of which the coefficients are presented in state space form. The matrices involved in the realizations are computed from state space realizations of the data functions $G$ and $K$. On the one hand the results are based on the commutant lifting theorem and on the other hand on stabilizing solutions of algebraic Riccati equations related to spectral factorizations.

1. Introduction

The present paper is a continuation of the paper [10]. As in [10] we have given two stable rational matrix functions $G$ and $K$ of sizes $m \times p$ and $m \times q$, respectively, and we are interested in $p \times q$ matrix-valued $H^\infty$ solutions $X$ to the Leech problem:

$$(1.1) \quad G(z)X(z) = K(z) \quad (|z| < 1), \quad \|X\|_\infty = \sup_{|z|<1} \|X(z)\| \leq 1.$$ 

Here stable means that the poles of the functions belong to the set $|z| > 1$, infinity included. In particular, the given functions $G$ and $K$ (as well as the unknown function $X$) are matrix-valued $H^\infty$ functions.

As is well-known, a result by R.W. Leech dating from the early seventies, see [18] (and [17]), tells us that for arbitrary matrix-valued $H^\infty$ functions $G$ and $K$, not necessarily rational, the problem (1.1) is solvable if and only if the operator $T_GT_G^* - T_KT_K^*$ is nonnegative. Here

$$T_G : \ell^2_+ (\mathbb{C}^p) \to \ell^2_+ (\mathbb{C}^m) \quad \text{and} \quad T_K : \ell^2_+ (\mathbb{C}^q) \to \ell^2_+ (\mathbb{C}^m)$$

are the (block) Toeplitz operators defined by $G$ and $K$ respectively. Since then it has been shown by various authors that the Leech problem can be solved by using general methods for dealing with metric constrained completion and interpolation problems, including commutant lifting; see the review [17] and the references therein.

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In the present paper, as in [10], we deal with the suboptimal case where the operator
\begin{equation}
T_G T_G^* - T_K T_K^* \text{ is strictly positive.}
\end{equation}
Note that an $H^\infty$ solution to the Leech problem (1.1) exists if and only if the operator $T_G T_G^* - T_K T_K^*$ is positive, see [18]. In [10], using commutant lifting theory and state space methods from mathematical system theory, we proved that the maximum entropy solution to the Leech problem (1.1) with rational data is a stable rational matrix function and we computed a state space formula for this solution. The focus of the current paper is on computing all solutions.

In a few recent publications [21, 16, 9], a different approach to the Leech problem was presented, also leading to state space formulas for a solution. Although it is not hard to modify this approach to compute a set of rational matrix solutions, it remains unclear at this stage if the method is suitable to compute the set of all solutions, cf., [11].

One of the additional complications in describing the set of all solutions in our approach is that it requires an explicit description of the value at zero $\Theta$ associated with the model space $\text{Im} T_G$. Another difficulty, which already appears in [10], is the fact that the intertwining contraction $\Lambda = T_G (T_G T_G^*)^{-1} T_K$ appearing in the commutant lifting setting of the Leech problem is a rather complicated operator. If $K \neq 0$ this operator is not finite dimensional as in the classical Nevanlinna-Pick interpolation problem or a compact operator as in the Nehari problem for the Wiener class but, in general, $\Lambda$ is an infinite dimensional operator which can be Fredholm or invertible (cf., Proposition A.5 at the end of the present paper).

Before stating our main result, we need some preliminaries. As in [10], the starting point is the fact, well known from mathematical systems theory, that rational matrix functions admit finite dimensional state space realizations. We shall assume that the stable rational matrix function $[ G \ K ]$ is given in realized form:
\begin{equation}
\begin{bmatrix}
G(z) \\
K(z)
\end{bmatrix} = \begin{bmatrix} D_1 & D_2 \end{bmatrix} + zC(I_n - zA)^{-1} \begin{bmatrix} B_1 & B_2 \end{bmatrix}.
\end{equation}
Here $I_n$ is the $n \times n$ identity matrix and $A, B_1, B_2, C, D_1$ and $D_2$ are matrices of appropriate size. Without loss of generality we may assume $A$ is a stable matrix, i.e., all eigenvalues of $A$ are in the open unit disc $\mathbb{D}$, and the pair $\{ C, A \}$ is observable. The latter means that $CA^\nu x = 0$ for $\nu = 0, 1, 2, \ldots$ implies $x$ is the zero vector in $\mathbb{C}^n$. For $j = 1, 2$ let $P_j$ be the controllability gramians associated with the pair $\{ A, B_j \}$, i.e., $P_j$ is the unique solution to the Stein equation
\begin{equation}
P_j - AP_j A^* = B_j B_j^*.
\end{equation}

As Theorem 1.1 in [10] shows, since $G$ and $K$ are rational matrix $H^\infty$ functions, it is possible to present a solution criterion for the Leech problem in terms of matrices derived from the matrices appearing in the realization (1.3). This criterion involves an algebraic Riccati equation that appears in the spectral factorization of the rational $m \times m$ matrix function
\begin{equation}
R(z) = G(z) G^*(z) - K(z) K^*(z).
\end{equation}
Here $G^*(z) = G(\bar{z}^{-1})^*$ and $K^*(z) = K(\bar{z}^{-1})^*$. It was computed in [9] that $R$ admits the state space realization
\begin{equation}
R(z) = zC(I - zA)^{-1} \Gamma + R_0 + \Gamma^*(zI - A^*)^{-1} G^*.
\end{equation}
with $R_0$ and $\Gamma$ the matrices given by

$$R_0 = D_1D_1^* - D_2D_2^* + C(P_1 - P_2)C^*,$$

$$\Gamma = B_1D_1^* - B_2D_2^* + A(P_1 - P_2)C^*.$$ 

(1.6) \hspace{2cm} (1.7)

Under the hypothesis that $T_0T_0^* - T_KT_K^*$ is strictly positive, the Toeplitz operator $T_R$ defined by $R$ is also strictly positive. The latter is equivalent, see Remark 1.3 in [10], to the existence of a stabilizing solution $Q$ to the algebraic Riccati equation

$$Q = A^*QA + (C - \Gamma^*QA)^*(R_0 - \Gamma^*Q\Gamma)^{-1}(C - \Gamma^*QA).$$

(1.8)

In this context, for the solution $Q$ to (1.8) to be stabilizing means that the matrix $R_0 - \Gamma^*Q\Gamma$ must be strictly positive and that the matrix

$$A_0 = A - \Gamma\Delta^{-1}(C - \Gamma^*QA), \quad \Delta = R_0 - \Gamma^*Q\Gamma,$$

must be stable. These two stability conditions guarantee that there exists just one stabilizing solution $Q$ to (1.8). Furthermore, since the pair $(C, A)$ is observable, the stabilizing solution $Q$ is invertible, cf., [10, Eq. (1.18)]. Theorem 1.1 in [10] now states that $T_GT_G^* - T_KT_K^*$ is strictly positive if and only if there exists a stabilizing solution $Q$ to (1.8) such that

$$Q^{-1} + P_2 - P_1$$

is strictly positive.

To state our main theorem we need to consider an additional algebraic Riccati equation. Note that $T_GT_G^* \geq T_KT_K^* - T_KT_K^*$. Since $T_GT_G^* - T_KT_K^*$ is strictly positive, it follows that the same holds true for $T_GT_G^*$. This allows us to apply the results of the previous paragraph with the function $K$ identically equal to zero, and with $B_2 = 0$ and $D_2 = 0$. This leads to a second algebraic Riccati equation:

$$Q_0 = A^*Q_0A + (C - \Gamma_0^*Q_0A)^*(R_{10} - \Gamma_0^*Q_0\Gamma_0)^{-1}(C - \Gamma_0^*Q_0A).$$

(1.10)

Here

$$R_{10} = D_1D_1^* + CP_1C^*, \quad \Gamma_0 = B_1D_1^* + AP_1C^*.$$ 

Since $T_G$ is right invertible and the pair $(C, A)$ is observable, it follows that (1.10) has a unique stabilizing solution $Q_0$ such that $Q_0^{-1} - P_1$ is strictly positive.

Finally, since $T_GT_G^*$ is strictly positive, the projection on $\ker T_G = \ell_2^2(\mathbb{C}^p) \oplus \text{Im } T_G$ is given by $F_{\ker T_G} = I_p - T_G^*(T_GT_G^*)^{-1}T_G = T_0T_0^*$, with $\Theta$ the inner function associated with the model space $\text{Im } T_G$. This yields that the value $\Theta_0$ of $\Theta$ at zero is uniquely determined, up to a constant unitary matrix of order $p - m$ on the right, by

$$\Theta_0\Theta_0^* = I_p - E_p^*T_G^*(T_GT_G^*)^{-1}T_GE_p.$$

(1.11)

Here, for any positive integer $k$, we write $E_k$ for the canonical embedding of $\mathbb{C}^k$ onto the first coordinate space of $\ell_2^2(\mathbb{C}^k)$, see (1.15) below. The fact that the number of columns of $\Theta_0$ is $p - m$ is explained in Remark 1.2 below. Since the realization $G(z) = D_1 + zC(I_n - zA)^{-1}B_1$ is a stable state space realization, we can apply Theorem 1.1 in [13] to derive a formula for $\Theta_0$ in terms of the matrices $A, B_1, C, D_1$ and related matrices. Therefore in what follows we shall assume $\Theta_0$ is given. We shall refer to $\Theta_0$ as the left minimal rank factor determined by (1.11). See Lemma 2.1 in the next section for some further insight in the role of $\Theta_0$.

We are now ready to state our main theorem which provides a characterization of all solutions to the suboptimal rational Leech problem (1.1) in the form of the range of a linear fractional transformation.
Theorem 1.1. Let \( G \) and \( K \) be stable rational matrix functions of sizes \( m \times p \) and \( m \times q \), respectively, such that \( T_G T_G^* - T_K T_K^* \) is strictly positive, and assume that there is no non-zero \( x \in \mathbb{C}^p \) such that \( G(z)x \) is identically zero on the open unit disc \( \mathbb{D} \). Let \( \begin{bmatrix} G & K \end{bmatrix} \) be given by the observable stable realization [13]. Then the set of solutions to the Leech problem [11] appears as the range of the linear fractional transformation \( Y \mapsto X \) given by

\[
X(z) = (Y_{12}(z) + Y_{11}(z)Y(z))(Y_{22}(z) + Y_{21}(z)Y(z))^{-1}.
\]

Here the free parameter \( Y \) is any \( (p-m) \times q \) matrix-valued \( H^\infty \) function such that \( \|Y\|_\infty \leq 1 \), and

\[
\begin{align*}
Y_{11}(z) &= \Theta_0 \Delta^{-1} - z C(I - z A_0)^{-1} Q^{-1} (Q^{-1} + P_2 - P_1)^{-1} B_1 \Theta_0 \Delta^{-1}, \\
Y_{21}(z) &= -z C_2 (I - z A_0)^{-1} Q^{-1} (Q^{-1} + P_2 - P_1)^{-1} B_1 \Theta_0 \Delta^{-1}, \\
Y_{12}(z) &= (D_1^* \Delta^{-1} D_2 + D_1^* C_0 \Omega C_2^* + B_1^* Q B_0) \Delta^{-1} + z C_1 (I - z A_0)^{-1} B_0 \Delta^{-1}; \\
Y_{22}(z) &= \Delta_0 + z C_2 (I - z A_0)^{-1} B_0 \Delta^{-1},
\end{align*}
\]

where \( A_0 \) and \( \Delta \) are given by [10], the matrix \( \Theta_0 \) is the left minimal rank factor determined by (1.11), the matrices \( C_j, j = 0, 1, 2, \) and \( B_0 \) are given by

\[
C_0 = \Delta^{-1} (C - \Gamma^* QA), \quad C_j = D_j^* C_0 + B_j^* Q A_0, \quad j = 1, 2,
\]

\[
B_0 = B_2 - \Gamma \Delta^{-1} D_2 + A_0 \Omega C_2^*,
\]

with \( \Omega = (P_1 - P_2)(Q^{-1} + P_2 - P_1)^{-1} Q^{-1} \), where \( Q \) is the stabilizing solution of the Riccati equation (1.8), and \( \Delta_0 \) and \( \Delta_1 \) are the positive definite matrices determined by

\[
\begin{align*}
\Delta_0^2 &= I_q + C_2 \Omega C_2^* + (D_2 - \Gamma^*QB_2)^* \Delta^{-1} (D_2 - \Gamma^*QB_2) + B_1^* QB_2, \\
\Delta_1^2 &= I_{p-m} + \Theta_0^* B_1^* ((Q^{-1} + P_2 - P_1)^{-1} - (Q_0^{-1} - P_1)^{-1}) B_1 \Theta_0,
\end{align*}
\]

where \( Q_0 \) is the stabilizing solution of the Riccati equation (1.10).

Remark 1.2. The functions \( Y_{12} \) and \( Y_{22} \) already appear in [10]. More precisely, \( Y_{12}(z) \Delta_0 \) is the function \( U(z) \) given by [10] Eq. (5.14), and \( Y_{22}(z) \Delta_0 \) is the function \( V(z) \) given by [10] Eq. (5.13). Note that \( Y_{12}(z) Y_{22}(z)^{-1} = U(z)V(z)^{-1} \) is the solution which one obtains if the free parameter \( Y = 0 \); this solution is the maximum entropy solution given by [10] Eq. (1.12)]. Finally, the coefficient matrix

\[
Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}
\]

has a number of interesting properties which follow from the general theory derived in Section 3. For instance, \( Y \) is \( J_1,J_2 \)-inner, where \( J_1 = \text{diag}(I_p, -I_q) \), and \( J_2 = \text{diag}(I_{p-m}, -I_q) \).

Remark 1.3. All solutions can also be obtained as the range of a linear fractional map of Redheffer type:

\[
X(z) = \Phi_{22}(z) + \Phi_{21}(z)Y(z)(I - \Phi_{11}(z)Y(z))^{-1} \Phi_{12}(z),
\]

where, as in Theorem 1.1, the free parameter \( Y \) is any \( (p-m) \times q \) matrix-valued \( H^\infty \) function such that \( \|Y\|_\infty \leq 1 \), and the functions \( \Phi_{11}, \Phi_{12}, \Phi_{21} \) and \( \Phi_{22} \) are...
stable rational matrix functions given by stable state space realizations. In fact, as expected, these coefficients are uniquely determined by the identities

\[ \Phi_{11} = -\Phi_{12} \Upsilon_{21}, \quad \Phi_{12} = \Upsilon_{22}^{-1}, \]
\[ \Phi_{21} = \Upsilon_{11} - \Upsilon_{12} \Phi_{12} \Upsilon_{21}, \quad \Phi_{22} = \Upsilon_{12} \Phi_{12}. \]

We omit further details.

**Remark 1.4.** In terms of the realization \([\ref{eq:realization}](#eq:realization)\), the condition that there is no non-zero \(x \in \mathbb{C}^p\) such that \(G(z)x\) is identically zero on \(\mathbb{D}\) is equivalent to the requirement that \(\text{Ker} \begin{bmatrix} B_1 & D_1 \end{bmatrix}^\top\) consists of the zero vector only. To see this note that \(G(z)x = D_1x + zC(I_n - zA)^{-1}B_1x\). Hence

\[ G(z)x = 0 \ (z \in \mathbb{D}) \iff D_1x = 0 \text{ and } CA^\nu B_1x = 0 \ (\nu = 0, 1, 2, \ldots). \]

Since the pair \(\{C, A\}\) is observable, it follows that

\[ G(z)x = 0 \ (z \in \mathbb{D}) \iff D_1x = 0 \text{ and } B_1x = 0 \iff x \in \text{Ker} \begin{bmatrix} B_1 \\ D_1 \end{bmatrix}, \]

which yields the desired result. The condition that there is no non-zero \(x \in \mathbb{C}^p\) such that \(G(z)x\) is identically zero on \(\mathbb{D}\) can also be understood as a minimality condition on some isometric liftings; see Lemma \([\ref{lem:isometric_liftings}](#lem:isometric_liftings)\) in the next section.

The paper consists of five sections. The first is the present introduction. Section \([\ref{sec:preliminaries}](#sec:preliminaries)\) has a preliminary character. In this section \(G\) is an arbitrary matrix-valued \(H^\infty\) function, not necessarily rational. Among others we present the inner function \(\Theta\) describing the null space of \(T_G\). In Section \([\ref{sec:operators}](#sec:operators)\) the functions \(G\) and \(K\) are again just matrix-valued \(H^\infty\) functions, not necessarily rational. We derive infinite dimensional state space formulas for the two linear fractional representations of the set of all solutions to the sub-optimal Leech equation, starting from the abstract commutant lifting results in Section VI.6 of \([\ref{ref:1}](#ref:1)\). In Section \([\ref{sec:main_results}](#sec:main_results)\) we prove Theorem \([\ref{thm:main_result}](#thm:main_result)\). The final section, Section \([\ref{sec:appendix}](#sec:appendix)\) has the character of an appendix; in this section we present a version of the commutant lifting theorem, based on Theorem VI.6.1 in \([\ref{ref:1}](#ref:1)\). Theorem \([\ref{thm:main_result_appendix}](#thm:main_result_appendix)\), which follows Theorem VI.6.1 in \([\ref{ref:1}](#ref:1)\) but does not appear in \([\ref{ref:1}](#ref:1)\), serves as the abstract basis for the proofs of our main results.

Notation and terminology. We conclude this introduction with some notation and terminology used throughout the paper. As usual, we identify a \(k \times r\) matrix with complex entries with the linear operator from \(\mathbb{C}^r\) to \(\mathbb{C}^k\) induced by the action of the matrix on the standard bases. For any positive integer \(k\) we write \(E_k\) for the canonical embedding of \(\mathbb{C}^k\) onto the first coordinate space of \(\ell^2_+(\mathbb{C}^k)\), that is,

\[ E_k = [I_k \ 0 \ 0 \ \cdots]^\top : \mathbb{C}^k \rightarrow \ell^2_+(\mathbb{C}^k). \]

Here \(\ell^2_+(\mathbb{C}^k)\) denotes the Hilbert space of unilateral square summable sequences of vectors in \(\mathbb{C}^k\). By \(S_k\) we denote the unilateral shift on \(\ell^2_+(\mathbb{C}^k)\). For positive integers \(k\) and \(r\) we write \(H^\infty_{k \times r}\) for the Banach space of all \(k \times r\) matrices with entries from \(H^\infty\), the algebra of all bounded analytic functions of the open unit disc \(\mathbb{D}\). The supremum norm of \(F \in H^\infty_{k \times r}\) is given by \(\|F\|_\infty = \sup_{|z| < 1} \|F(z)\|\). By \(\mathcal{R}H^\infty_{k \times r}\) we denote the space of all stable rational \(k \times r\) matrix functions which we view as a subspace of \(H^\infty_{k \times r}\). The adjoint of \(F \in H^\infty_{k \times r}\) is the co-analytic function \(F^*\) which is defined by \(F^*(z) = F(1/\bar{z})^*, \ |z| < 1\). Finally, we write \(\bigcup_{i \in I} \mathcal{M}_i\) for the closure of the linear hull of the spaces \(\mathcal{M}_i\) ranging over the index set \(I\).
2. THE MODEL SPACE AND MODEL OPERATOR ASSOCIATED WITH THE KERNEL OF A SURJECTIVE ANALYTIC TOEPLITZ OPERATOR

Throughout this section let $G \in H_{\infty}^{\infty \times P}$. Then $S_m T_G = T_G S_p$ implies $\text{Ker} T_G$ is invariant under $S_p$, and hence $H' = \text{Im} T_G^\ell = \ell_2^k(C^p) \ominus \text{Ker} T_G$ is invariant under $S_p^*$. By the Beurling-Lax theorem, $H'$ is a model space, that is, there exists an inner function $\Theta \in H_{p \times k}^\infty$, for some $k \leq p$, such that $H' = \ell_2^k(C^p) \ominus T_{\Theta} \ell_2^k(C^k)$. We write $T'$ for the associated model operator $T' = P_{H'} S_p |_{H'}$.

We shall assume in addition that $T_G$ is a surjective operator, or equivalently, that $T_G T_G^*$ is an invertible operator on $\ell_2^k(C^m)$. In that case, we provide an explicit infinite dimensional state space representation for the inner function $\Theta$, along with some formulas that will be of use in the sequel.

Note that $S_p$ is an isometric lifting of $T'$, see the appendix for the definition of a (minimal) isometric lifting. In a second result in this section, Lemma 2.3 below, we present a condition which is equivalent to $S_p$ being a minimal isometric lifting of $T'$.

**Lemma 2.1.** The inner function $\Theta \in H_{p \times k}^\infty$ with $H' = \ell_2^k(C^p) \ominus T_{\Theta} \ell_2^k(C^k)$ is given by

\[
\Theta(z) = \Theta_0 - z E_p T_G (I - z S_m^\ast)^{-1} (T_G T_G^*)^{-1} N.
\]

Here $N$ is the operator from $C^k$ to $\ell_2^k(C^m)$ given by $N = S_m^* T_G E_p \Theta_0$, and $\Theta_0 = \Theta(0)$ is a one-to-one $p \times k$ matrix uniquely determined, up to multiplication with a constant unitary $k \times k$ matrix from the right, by

\[
\Theta_0 \Theta_0^* = I_p - E_p^* T_G^*(T_G T_G^*)^{-1} T_G E_p.
\]

Furthermore, $N = -T_G S_p^* T_{\Theta} E_k$ and for any $z \in \mathbb{D}$ we have

\[
\Theta(z) N^* (T_G T_G^*)^{-1} = E_p^* (I - z S_p^\ast)^{-1} T_G^*(T_G T_G^*)^{-1} (I - z S_m^\ast) S_m.
\]

**Remark 2.2.** Note that $\Theta_0$ is the analog of the left minimal rank factor introduced in the second paragraph preceding Theorem 1.1 in the rational case $k = p - m$; see Lemma 2.2 in [12]. However, it can be shown that the latter equality holds in general; see [13] Section 2.

**Proof of Lemma 2.1.** We first show that $N = -T_G S_p^* T_{\Theta} E_k$ holds. Using the fact that $T_G T_{\Theta} = 0$ and $\Theta_0 = E_p^* T_{\Theta} E_k$ we obtain that

\[
N = S_m^* T_G E_p \Theta_0 = S_m^* T_G E_p E_p^* T_{\Theta} E_k = S_m^* T_G (I - S_p S_p^\ast) T_{\Theta} E_k = -S_m^* S_m T_G S_p^* T_{\Theta} E_k = -S_m^* T_G S_p^* T_{\Theta} E_k = -T_G S_p^* T_{\Theta} E_k,
\]

as claimed.

Since $T_G$ is surjective, $T_G^*(T_G T_G^*)^{-1} T_G$ is the orthogonal projection onto $\text{Im} T_G^*$, so that

\[
T_{\Theta} T_{\Theta}^* = P_{\text{Ker} T_G} = I - P_{\text{Im} T_G} = I - T_G^*(T_G T_G^*)^{-1} T_G.
\]

Next observe that

\[
T_{\Theta} S_p^* T_{\Theta} E_k = S_k^* T_{\Theta} T_{\Theta} E_k = S_k^* E_k = 0.
\]
Together with the formula for $N$ we then obtain for each $z \in \mathbb{D}$ that
\[
\Theta(z) = E_p^*(I - zS_p^*)^{-1}T_\Theta E_k
\]
\[
= E_p^*T_\Theta E_k + zE_p^*(I - zS_p^*)^{-1}S_p^*T_\Theta E_k
\]
\[
= \Theta_0 + zE_p^*(I - zS_p^*)^{-1}(I - T_\Theta T_k)S_p^*T_\Theta E_k
\]
\[
= \Theta_0 + zE_p^*(I - zS_p^*)^{-1}T_G^*(T_G T_k^*)^{-1}T_G S_p^*T_\Theta E_k
\]
\[
= \Theta_0 - zE_p^*(I - zS_p^*)^{-1}T_G^*(T_G T_k^*)^{-1}N.
\]

This yields the desired state space representation (2.1) for $\Theta$.

Note that
\[
\text{Ker } \Theta_0 \subset \text{Ker } S_m^* T_G E_p \Theta_0 = \text{Ker } N.
\]

Thus, for $u \in \text{Ker } \Theta_0$, we have $\Theta(z)u = 0$ for all $z \in \mathbb{D}$, and hence also for a.e. $z \in \mathbb{T}$. Since $\Theta$ is inner, this implies $u = 0$. Hence $\text{Ker } \Theta_0 = \{0\}$.

Furthermore, since $E_p^*T_\Theta S_k = E_p^*S_p^*T_\Theta = 0$, we have
\[
E_p^*T_\Theta T_k^*E_p = E_p^*T_\Theta (E_k E_k^* + S_k S_k^*)T_\Theta E_p = E_p^*T_\Theta E_k E_k^* T_\Theta E_p = \Theta_0 \Theta_0^*.
\]

Along with (2.4), this yields
\[
\Theta_0 \Theta_0^* = E_p^*(I - T_G^*(T_G T_k^*)^{-1}T_G)E_p = I_p - E_p^* T_G^*(T_G T_k^*)^{-1}T_G E_p.
\]

Again using $T_G T_\Theta = 0$ and $N = -T_G S_p^* T_\Theta E_k$, we obtain that
\[
N E_k = -T_G S_p^* T_\Theta E_k E_k^* = \Theta_0 S_p^* T_\Theta (I - S_k S_k^*)
\]
\[
= -T_G S_p^* T_\Theta + T_G S_p^* T_\Theta S_k S_k^* = -T_G S_p^* T_\Theta + T_G T_\Theta S_k S_k^* = -T_G S_p^* T_\Theta.
\]

Fix $z \in \mathbb{D}$. Then we find
\[
\Theta(z)N^* = E_p^*(I - zS_p^*)^{-1}T_\Theta E_k N^* = -E_p^*(I - zS_p^*)^{-1}T_\Theta T_k^* S_m^* T_G^*.
\]

Using (2.4), yields
\[
T_\Theta T_k^* S_m^* T_G^* (T_G T_k^*)^{-1} = S_p^* T_G^* (T_G T_k^*)^{-1} - T_G^* (T_G T_k^*)^{-1} T_G S_p^* T_G^* (T_G T_k^*)^{-1}
\]
\[
= S_p^* T_G^* (T_G T_k^*)^{-1} - T_G^* (T_G T_k^*)^{-1} S_m.
\]

Combining this with the formula for $\Theta(z)N^*$ gives
\[
\Theta(z)N^*(T_G T_k^*)^{-1} = E_p^* (I - zS_p^*)^{-1} (T_G^* (T_G T_k^*)^{-1} S_m - S_p^* T_G^* (T_G T_k^*)^{-1})
\]
\[
= E_p^* (I - zS_p^*)^{-1} \times (T_G^* (T_G T_k^*)^{-1} S_m - zT_G^* (T_G T_k^*)^{-1} S_m^* S_m)
\]
\[
= E_p^* (I - zS_p^*)^{-1} (T_G^* (T_G T_k^*)^{-1} (I - zS_m^*) S_m).
\]

Hence the identity (2.3) holds. \hfill \Box

We now proceed with the second result of this section.

**Lemma 2.3.** The shift $S_p$ is a minimal isometric lifting of $T' = P_{\mathcal{H}'} S_{p_{\mathcal{H}'}}$ if and only there is no non-zero $x \in C^p$ such that $G(z)x$ vanishes identically, that is, $\cap_{z \in \mathbb{D}} \text{Ker } G(z) = \{0\}$.

**Proof.** Put
\[
\mathcal{X} = \bigvee_{\nu \geq 0} S_{p_{\mathcal{H}'}}^\nu \mathcal{H}', \quad \mathcal{X}_0 = \mathcal{X} \ominus \mathcal{H}', \quad \mathcal{X}_1 = l^2_\nu(C^p) \ominus \mathcal{X}.
\]
Since $\mathcal{X}$ is invariant under both $S_p$ and $S_p^*$, the same holds true for $\mathcal{X}_1$. Hence $S_p$ partitions as

$$S_p = \begin{bmatrix} T' & 0 & 0 \\ W_0 & Z_0 & 0 \\ 0 & 0 & Z_1 \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{H}' \\ \mathcal{X}_0 \\ \mathcal{X}_1 \end{bmatrix}$$

and the isometry

$$U' = \begin{bmatrix} T' & 0 \\ W_0 & Z_0 \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{H}' \\ \mathcal{X}_0 \end{bmatrix}$$

is a minimal isometric lifting of $T'$. In particular, the shift $S_p$ is a minimal isometric lifting of $T'$ if and only $\mathcal{X}_1$ consists of the zero element only.

Now take $h = (h_0, h_1, \ldots) \in \ell^2_1(\mathbb{C}^p)$. Then $h \in \mathcal{X}_1$ if and only if $h \perp S_p^* \text{Im } T_G^*$ for $\nu = 0, 1, 2, \ldots$. In other words

$$h \in \mathcal{X}_1 \iff T_G(S_p^*)^\nu h = 0, \quad \nu = 0, 1, 2, \ldots$$

$$\iff \begin{bmatrix} G_0 & 0 & 0 & \cdots \\ G_1 & G_0 & 0 & \cdots \\ G_2 & G_1 & G_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} h_\nu \\ h_{\nu+1} \\ h_{\nu+2} \\ \vdots \end{bmatrix} = 0, \quad \nu = 0, 1, 2, \ldots$$

$$\iff \begin{bmatrix} G_0 \\ G_1 \\ G_2 \end{bmatrix} h_\nu, \quad \nu = 0, 1, 2, \ldots$$

$$\iff G(z)h_\nu \equiv 0, \quad \nu = 0, 1, 2, \ldots$$

We conclude that $\mathcal{X}_1$ contains a non-zero element if and only if there exists a non-zero $x \in \mathbb{C}^p$ such that $G(z)x$ vanishes identically.

3. **Infinite dimensional state space formulas for the coefficients**

In this section $G \in H^\infty_{m \times p}$ and $K \in H^\infty_{m \times q}$, and we assume that $T_GT_G^* - T_KT_K^*$ is strictly positive. We do not require $G$ and $K$ to be rational matrix functions. Our aim is to describe all solutions to the Leech problem \[17\].

Note that $T_GT_G^* - T_KT_K^*$ strictly positive implies that $T_GT_G^*$ is strictly positive, and thus that $T_G$ is a surjective analytic Toeplitz operator. Hence the results of Section 2 apply. In particular, $\mathcal{H}' = \text{Im } T_G^*$ is a model space and the associated inner function $\Theta$ is given by \[2.1\]. As before, we write $T'$ for the model operator $T' = P_{\mathcal{H}', S_p} |_{\mathcal{H}'}$.

Next we recall some results from \[10\]. Set $\Lambda = T_G^*(T_GT_G^*)^{-1}T_K$, viewed as an operator mapping $\ell^2_1(\mathbb{C}^q)$ into $\mathcal{H}'$. According to Lemma 2.3 in \[10\], the operator $\Lambda$ is a strict contraction which satisfies

$$T'\Lambda = \Lambda S_q.$$  

These two facts make it possible to apply commutant lifting theory. Following the argumentation in the last paragraph of Section 2 from \[10\], the contractive liftings of $\Lambda$ that intertwine $S_p$ and $S_q$ are precisely the Toeplitz operators defined by the solutions $X$ to the Leech problem associated with $G$ and $K$. Hence, the solutions are described in the appendix by Theorem \[A.1\] as well as by Theorem \[A.4\] specified...
to the special choice of $\Lambda$ made here. Note that this require $S_p$ to be a minimal isometric lifting of $T'$. Therefore (cf., Lemma 2.3) in what follows we shall assume that $\cap_{z \in \mathbb{C}} \text{Ker} G(z) = \{0\}$.

The following theorem is based on Theorem A.4 specified for the case when the strict contraction $\Lambda$ is given by $\Lambda = T_G(T_G^* - K T_K^*)^{-1} T_K$. Its prove require a number of non-trivial operator manipulations.

**Theorem 3.1.** Let $G \in H_{m \times p}^\infty$ and $K \in H_{m \times q}^\infty$ be such that $T_G T_G^* - K T_K^*$ is strictly positive, and assume that there is no non-zero $x \in \mathbb{C}^p$ such that $G(z)x$ is identically zero on the open unit disc $\mathbb{D}$. Then the set of all solutions to the Leech problem (1) associated with $G$ and $K$ is given by the range of the linear fractional map

$$X(z) = (\Upsilon_{12}(z) + \Upsilon_{11}(z) Y(z)) (\Upsilon_{22}(z) + \Upsilon_{21}(z) Y(z))^{-1}, \quad |z| < 1.$$  

Here $Y$ is an arbitrary function in $H_{k \times q}^\infty$ with $\|Y\|_\infty \leq 1$, and

$$\Upsilon_{11}(z) = \Theta_0^{-1} - z E_p^* T_G (I - z S_m)^{-1} (T_G T_G^* - K T_K^*)^{-1} N \Delta_1^{-1},$$

$$\Upsilon_{21}(z) = -z E_p^* T_K^* (I - z S_m)^{-1} (T_G T_G^* - K T_K^*)^{-1} N \Delta_1^{-1},$$

$$\Upsilon_{12}(z) = E_p^* T_G (T_G T_G^* - K T_K^*)^{-1} T_K E_q \Delta_0^{-1} +$$

$$+ z E_p^* T_G (I - z S_m)^{-1} S_m (T_G T_G^* - K T_K^*)^{-1} T_K E_q \Delta_0^{-1},$$

$$\Upsilon_{22}(z) = \Delta_0 + z E^*_p T_K^* (I - z S_m)^{-1} S_m \times$$

$$\times (T_G T_G^* - K T_K^*)^{-1} T_K E_q \Delta_0^{-1}.$$  

Here $\Theta_0$ is a one-to-one $p \times k$ matrix uniquely determined, up to multiplication with a constant unitary $k \times k$ matrix from the right, by the identity (2.2), and $N = S_m^* T_G E_p \Theta_0$, as in Lemma 2.4. Furthermore, $\Delta_0$ and $\Delta_1$ are the positive definite matrices defined by

$$\Delta_0^2 = I_q + E_q^* T_K (T_G T_G^* - K T_K^*)^{-1} T_K E_q,$$

$$\Delta_1^2 = I_k + N^* \left( (T_G T_G^* - K T_K^*)^{-1} - (T_G T_G^*)^{-1} \right)^{-1} N.$$  

Before we prove the above theorem we recall two useful identities from [10] Lemma 3.2:

$$\Lambda (I - \Lambda^* \Lambda)^{-1} = I + T_K (T_G T_G^* - K T_K^*)^{-1} T_K,$$

$$\Lambda (I - \Lambda^* \Lambda)^{-1} = T_G^* (T_G T_G^* - K T_K^*)^{-1} T_K.$$  

**Proof.** We split the proof into three parts. In the first part we derive the identities (3.4) and (3.5) using formulas (3.7) and (3.8) in [10] Section 3. The final two parts contain the proofs of the formulas for $\Upsilon_{11}$ and $\Upsilon_{21}$.

**Part 1.** From Theorem A.4 we know that

$$\Upsilon_{12}(z) = U(z) \Delta_0^{-1} \quad \text{and} \quad \Upsilon_{22}(z) = U(z) \Delta_0^{-1},$$

where $U$ and $V$ are given by A.17 and A.18, respectively. From formulas (3.7) and (3.8) in [10] Section 3 we know that for our choice of $\Lambda$ the formulas (A.17).
and (A.18) lead to the following identities:
\[
U(z) = E_p^* T_G^* (T_G T_G^* - T_K T_K^*)^{-1} T_K E_q +
\]
\[
+ z E_p^* T_G^* (I - z S_m^* S_m^*)^{-1} S_m^* (T_G T_G^* - T_K T_K^*)^{-1} T_K E_q,
\]
\[
V(z) = I_q + E_q^* T_K^* (T_G T_G^* - T_K T_K^*)^{-1} E_q +
\]
\[
+ z E_q^* T_K^* (I - z S_m^* S_m^*)^{-1} S_m^* (T_G T_G^* - T_K T_K^*)^{-1} T_K E_q.
\]

Furthermore, according (3.8), for our choice of \( \Lambda \) the matrix \( \Delta_2 \) introduced in Lemma 2.1 we see that for our choice of \( \Lambda \), we have
\[
\Delta_2^2 = E_q \left( I + T_K^* (T_G T_G^* - T_K T_K^*)^{-1} T_K \right) E_q
\]
\[
= I_q + E_q^* T_K^* (T_G T_G^* - T_K T_K^*)^{-1} E_q.
\]

Hence \( \Delta_0 \) is the positive definite matrix determined by (3.6). Also note that \( V(0) = \Delta_0^2 \). But then multiplying (3.10) and (3.11) from the right by \( \Delta_0^{-1} \) we see that \( \Upsilon_{12} \) and \( \Upsilon_{22} \) are given by (3.12) and (3.13), respectively.

Part 2. In this part we derive the formula for \( \Upsilon_{21} \). Recall from Theorem A.3 that
\[
\Upsilon_{21}(z) = z E_p^* (I - z S_m^*)^{-1} B \Delta_1^{-1}.
\]
Using the adjoint of (A.3) and the operator \( N \) introduced in Lemma 2.1 we see that for our choice of \( \Lambda \), we have
\[
B \Theta = (I - \Lambda^* \Lambda)^{-1} \Lambda^* S_p^* T \Theta E_k
\]
\[
= T_K^* (T_G T_G^* - T_K T_K^*)^{-1} T_G S_p^* T \Theta E_k
\]
\[
= -T_K^* (T_G T_G^* - T_K T_K^*)^{-1} N.
\]

Since \( S_p^* T_k = T_k^* S_m^* \), it follows that
\[
\Upsilon_{21}(z) = -z E_p^* (I - z S_m^*)^{-1} T_k^* (T_G T_G^* - T_K T_K^*)^{-1} N \Delta_1^{-1}
\]
\[
= -z E_p^* T_k^* (I - z S_m^* S_m^*)^{-1} (T_G T_G^* - T_K T_K^*)^{-1} N \Delta_1^{-1}.
\]

This proves (3.8). It remains to show that \( \Delta_1 \) is determined by (3.7).

Using the definition of \( \Delta_1^2 \) in (A.27), our choice of \( \Lambda \) and the operator \( N \) introduced in Lemma 2.1 we obtain
\[
\Delta_1^2 = I_k + E_k^* T_G^* S_p^* (I - \Lambda^* \Lambda)^{-1} \Lambda^* S_p^* T \Theta E_k
\]
\[
= I_k + E_k^* T_G^* S_p^* T_G^* (T_G T_G^* - T_K T_K^*)^{-1} T_K (I - \Lambda^* \Lambda)^{-1} T_K^* (T_G T_G^*)^{-1} \times
\]
\[
\times T_G S_p^* T \Theta E_k
\]
\[
= I_k + N^* (T_G T_G^*)^{-1} T_K (I - \Lambda^* \Lambda)^{-1} T_K^* (T_G T_G^*)^{-1} N.
\]

To complete the proof of (3.7) it remains to show that
\[
(T_G T_G^*)^{-1} T_K (I - \Lambda^* \Lambda)^{-1} T_K^* (T_G T_G^*)^{-1} =
\]
\[
= (T_G T_G^* - T_K T_K^*)^{-1} - (T_G T_G^*)^{-1}.
\]

This will be done in a few steps. We first show that for our choice of \( \Lambda \) we have
\[
T_K (I - \Lambda^* \Lambda)^{-1} T_K^* = T_G T_G^* (T_G T_G^* - T_K T_K^*)^{-1} T_K T_K^*.
\]

To see this note that
\[
T_K \left( I - T_K^* (T_G T_G^*)^{-1} T_K \right) = \left( I - T_K^* (T_G T_G^*)^{-1} \right) T_K,
\]
and hence
\[
T_K \left( I - T_K (T_G T_G^*)^{-1} T_K \right)^{-1} = \left( I - T_K T_K^* (T_G T_G^*)^{-1} \right)^{-1} T_K
= T_G T_G^* \left( T_G T_G^* - T_K T_K^* \right)^{-1} T_K.
\]

Thus, again using our choice of $\Lambda$, we see that
\[
T_K (I - \Lambda^* \Lambda)^{-1} T_K^* = T_K \left( I - T_K (T_G T_G^*)^{-1} \left( T_G T_G^* (T_G T_G^*)^{-1} - T_K T_K^* \right) \right)^{-1} T_K^*
= T_K \left( I - T_K^* (T_G T_G^*)^{-1} T_K \right)^{-1} T_K^*
= T_G T_G^* \left( T_G T_G^* - T_K T_K^* \right)^{-1} T_K T_K^*.
\]

which proves (3.10). But then
\[
(T_G T_G^*)^{-1} T_K (I - \Lambda^* \Lambda)^{-1} T_K^* (T_G T_G^*)^{-1} = \left( T_G T_G^* \right)^{-1} \left( T_G T_G^* - T_K T_K^* \right) T_K T_K^* \left( T_G T_G^* \right)^{-1}
= \left( T_G T_G^* - T_K T_K^* \right)^{-1} T_K T_K^* \left( T_G T_G^* \right)^{-1}
= \left( T_G T_G^* - T_K T_K^* \right)^{-1} \left( T_G T_G^* - T_K T_K^* \right) \left( T_G T_G^* \right)^{-1}
= \left( T_G T_G^* - T_K T_K^* \right)^{-1} - (T_G T_G^*)^{-1}.
\]

This proves (3.14). Using the identity (3.14) in (3.13) yields (3.7).

Part 3. In this part we derive the formula for $\Theta_1$. Using our choice of $\Lambda$, the formula for $\Theta$ given by (2.1), and the first identity in (A.26) we see that
\[
\Theta_1(z) - \Theta_0 \Delta_1^{-1} = A(z) + B(z) + C(z),
\]
where
\[
A(z) = -z E_p^* T_G^* (I - z S_m^*)^{-1} (T_G T_G^*)^{-1} N \Delta_1^{-1},
B(z) = z E_p^* (I - z S_p^*)^{-1} T_G (T_G T_G^*)^{-1} T_K E_q^* (I - z S_q^*)^{-1} B_\nu \Delta_1^{-1},
C(z) = -z \Theta(z) E_q^* S_p T_G (T_G T_G^*)^{-1} T_K (I - z S_q^*)^{-1} S_q B_\nu \Delta_1^{-1}.
\]

Here $B_\nu = -T_K^* (T_G T_G^* - T_K T_K^*)^{-1} N$, with $N = -T_G S_p^* T_\theta E_k$ as in Lemma 2.1 and $\Delta_1$ is the positive definite matrix determined by (3.7).

First we deal with $C(z)$. Using the formula for $N$ and the identity (2.26) we see that
\[
C(z) = z \Theta(z) N^* (T_G T_G^*)^{-1} T_K (I - z S_q^*)^{-1} S_q B_\nu \Delta_1^{-1}
= z E_p^* (I - z S_p^*)^{-1} T_G (T_G T_G^*)^{-1} T_K (I - z S_m^*) S_m \times
\times T_K (I - z S_q^*)^{-1} S_q B_\nu \Delta_1^{-1}.
\]

Note that $(I - z S_m^*) S_m = S_m - z I$, and hence $C(z) = C_1(z) + C_2(z)$, where
\[
C_1(z) = z E_p^* (I - z S_p^*)^{-1} T_G T_G^* (T_G T_G^*)^{-1} S_m T_K (I - z S_q^*)^{-1} S_q B_\nu \Delta_1^{-1},
C_2(z) = -z^2 E_p^* (I - z S_p^*)^{-1} T_G T_G^* (T_G T_G^*)^{-1} T_K (I - z S_q^*)^{-1} S_q B_\nu \Delta_1^{-1}.
\]

Next we use the intertwining relation $T_K S_q = S_m T_K$ and the identity
\[
S_q (I - z S_q^*)^{-1} S_q^* = S_q S_q^* (I - z S_q^*)^{-1}.
\]
This yields
\[ C_1(z) = zE_p(I - zS_p^*)^{-1}T_G(T_G^*)^{-1}T_KS_qS_q^*(I - zS_q^*)^{-1}B_\ast \Delta_1^{-1}, \]
and hence, using \( E_qE_q^* + S_qS_q^* = I \), we obtain
\[ B(z) + C_1(z) = zE_p(I - zS_p^*)^{-1}T_G(T_G^*)^{-1}T_K(I - zS_q^*)^{-1}B_\ast \Delta_1^{-1}. \]
Next observe that
\[ C_2(z) = zE_p^*(I - zS_p^*)^{-1}T_G^*(T_GT_G^*)^{-1}T_K(I - zS_q^*)^{-1}(-zS_q^*)B_\ast \Delta_1^{-1} \]
\[ = zE_p^*(I - zS_p^*)^{-1}T_G^*(T_GT_G^*)^{-1}T_K(I - zS_q^*)^{-1} \]
\[ \times (I - zS_q^*) \] \[ B_\ast \Delta_1^{-1} \]
\[ = C_{21}(z) + C_{22}(z), \]
where
\[ C_{21}(z) = zE_p^*(I - zS_p^*)^{-1}T_G^*(T_GT_G^*)^{-1}T_KB_\ast \Delta_1^{-1}, \]
\[ C_{22}(z) = -zE_p^*(I - zS_p^*)^{-1}T_G^*(T_GT_G^*)^{-1}T_K(I - zS_q^*)^{-1}B_\ast \Delta_1^{-1}. \]
We conclude that \( B(z) + C_1(z) + C_{22}(z) = 0, \) and hence
\[ \Upsilon_{ij}(z) - \Theta_0 \Delta_1^{-1} = A(z) + C_{21}(z). \]
Next, using the intertwining relation \( S_mT_G = T_GS_p \) and the formula for \( B_\ast \) given by (3.12), we see that
\[ C_{21}(z) = -zE_p^*T_G^*(I - zS_m^*)^{-1}(T_GT_G^*)^{-1}T_KT_K^*(T_GT_K^* - T_KT_K^*)^{-1}N \Delta_1^{-1}. \]
But then
\[ A(z) + C_{21}(z) = -zE_p^*T_G^*(I - zS_m^*)^{-1}MN \Delta_1^{-1}, \]
where
\[ M = (T_GT_G^*)^{-1} + (T_GT_G^*)^{-1}T_KT_K^*(T_GT_K^* - T_KT_K^*)^{-1} \]
\[ = (T_GT_G^*)^{-1}(T_GT_G^* - T_KT_K^*)(T_GT_K^* - T_KT_K^*)^{-1} \]
\[ = (T_GT_G^* - T_KT_K^*)^{-1}. \]
Thus \( \Upsilon_{ij}(z) = \Theta_0 \Delta_1^{-1} + A(z) + C_{21}(z) \) is equal to the right hand sight of the (3.2), and hence the identity (3.2) is proved.

**Remark 3.2.** We conclude this section with a remark about the coefficients \( \Upsilon_{ij} \), \( 1 \leq i, j \leq 2 \), in the linear fractional map (1.12). Since each \( X \) given by (3.1) is a solution to the Leech problem (1.14) associated with \( G \) and \( K \) we see that
\[ G(z) \left( \Upsilon_{12}(z) + \Upsilon_{11}(z)Y(z) \right) = K(z) \left( \Upsilon_{22}(z) + \Upsilon_{21}(z)Y(z) \right) \]
for each \( Y \) in \( H_{k\times q}^\infty \) with \( ||Y||_\infty \leq 1 \). The previous identity can be rewritten as
\[ G(z)\Upsilon_{12}(z) - K(z)\Upsilon_{22}(z) = -\left( G(z)\Upsilon_{11}(z) - K(z)\Upsilon_{21}(z) \right)Y(z). \]
Using the freedom in the choice of \( Y \), we see that the following proposition holds.

**Proposition 3.3.** The functions \( \Upsilon_{ij} \), \( 1 \leq i, j \leq 2 \), given by (5.2) – (5.5), satisfy the following identities:
\[ G(z)\Upsilon_{ij}(z) - K(z)\Upsilon_{2j}(z) = 0, \quad z \in \mathbb{D} \quad (j = 1, 2). \]
We use the remaining part of this section to give a direct proof of the two identities in (3.16). We begin with two lemmas.

**Lemma 3.4.** The following identities hold:

\[(3.17) \quad T_G E_p E_p^* T_G^* = T_G T_G - S_m T_G T_K^* S_m^* , \]
\[(3.18) \quad T_K E_q E_q^* T_K^* = T_K T_K - S_m T_K T_K^* S_m^* , \]
\[(3.19) \quad E_m^* (I - z S_m^*)^{-1} S_m = z E_m^* (I - z S_m^*)^{-1} \quad (z \in \mathbb{D}) . \]

Furthermore, for any \( z \in \mathbb{D} \) and any bounded linear operator \( X \) on \( L_1^2 (\mathbb{C}^m) \) we have

\[(3.20) \quad E_m^* (I - z S_m^*)^{-1} \left( X - S_m X S_m^* \right) (I - z S_m^*)^{-1} = E_m^* (I - z S_m^*)^{-1} X . \]

**Proof.** Note that \( E_p E_q^* I = S_p S_q^* \). Since \( T_G \) is a block lower triangular operator \( T_G S_p = S_m T_G \), and \( S_q^* T_K^* = T_K^* S_m^* \) by duality. From these remarks (3.17) is clear.

The identity (3.18) is proved in the same way.

The identity (3.19) follows from \( S_m^* S_m = I \) and \( E_m^* S_m = 0 \). Indeed, using the latter two identities, we see that

\[
E_m^* (I - z S_m^*)^{-1} S_m = E_m^* \left( I + z(I - z S_m^*)^{-1} S_m \right) S_m
= E_m^* \left( S_m + z(I - z S_m^*)^{-1} S_m S_m \right) = z E_m^* (I - z S_m^*)^{-1} .
\]

Finally, to obtain (3.20) we use (3.19). Indeed

\[
E_m^* (I - z S_m^*)^{-1} \left( X - S_m X S_m^* \right) (I - z S_m^*)^{-1} =
= E_m^* (I - z S_m^*)^{-1} X (I - z S_m^*)^{-1} +
- E_m^* (I - z S_m^*)^{-1} S_m X S_m^* (I - z S_m^*)^{-1}
= E_m^* (I - z S_m^*)^{-1} X (I - z S_m^*)^{-1} +
- z E_m^* (I - z S_m^*)^{-1} X S_m^* (I - z S_m^*)^{-1}
= E_m^* (I - z S_m^*)^{-1} X (I - z S_m^*)^{-1} +
+ E_m^* (I - z S_m^*)^{-1} X (I - z S_m^*)^{-1}
= E_m^* (I - z S_m^*)^{-1} X ,
\]

which completes the proof. \( \square \)

**Lemma 3.5.** Put \( \Delta = T_G T_G^* - T_K T_K^* \), and let

\[(3.21) \quad A(z) = E_p^* T_G^* (I - z S_m^*)^{-1} , \quad B(z) = E_q^* T_K^* (I - z S_m^*)^{-1} . \]

Then

\[(3.22) \quad G(z) A(z) - K(z) B(z) = E_m^* (I - z S_m^*)^{-1} \Delta \quad (z \in \mathbb{D}) . \]

**Proof.** First note that since \( G \) and \( K \) admit the following infinite dimensional realizations:

\[(3.23) \quad G(z) = E_m^* (I - z S_m^*)^{-1} T_G E_p \quad (z \in \mathbb{D}) , \]
\[(3.24) \quad K(z) = E_m^* (I - z S_m^*)^{-1} T_K E_q \quad (z \in \mathbb{D}) . \]

Using (3.23), the definition of \( A(z) \) in (3.21), and the identity (3.17), we see that

\[
G(z) A(z) = E_m^* (I - z S_m^*)^{-1} T_G E_p E_p^* T_G^* (I - z S_m^*)^{-1}
= E_m^* (I - z S_m^*)^{-1} (T_G T_G^* - S_m T_G T_K^* S_m^* ) (I - z S_m^*)^{-1} .
\]
Similarly, using (3.21), the definition of $B(z)$ in (3.21), and the identity (3.18), we get
\[
K(z)B(z) = E_m^*(I - zS_m^*)^{-1}T_K E_q^* T_K^*(I - zS_m^*)^{-1}
= E_m^*(I - zS_m^*)^{-1}(T_K T_K^* - S_m T_K S_m^*)(I - zS_m^*)^{-1}.
\]
Applying (3.20), first with $X = T_G T_G^*$ and next with $X = T_K T_K^*$, we conclude that
\begin{align}
G(z)A(z) &= E_m^*(I - zS_m^*)^{-1}T_G T_G^* \quad (z \in \mathbb{D}), \\
K(z)B(z) &= E_m^*(I - zS_m^*)^{-1}T_K T_K^* \quad (z \in \mathbb{D}).
\end{align}
Taking the difference yields (3.22).

**Proof of Proposition 3.3** We split the proof into two parts. As in the preceding lemma, $\Delta = T_G T_G^* - T_K T_K^*$. Furthermore, throughout $z \in \mathbb{D}$.

Part 1. We prove the identity (3.16) for $j = 1$. Using the formula for $\Theta$ in (2.1) we see that $\Upsilon_{11}$ can be rewritten in the following equivalent form:
\[
\Upsilon_{11}(z) = \Theta(z) \Delta^{-1} + zE_p^* T_G^*(I - zS_m^*)^{-1}(T_G T_G^*)^{-1} N \Delta^{-1} + zE_p^* T_G^*(I - zS_m^*)^{-1}(T_G T_G^* - T_K T_K^*)^{-1} N \Delta^{-1}.
\]
The fact that $\text{Im} T_{\Theta} = \text{Ker} T_G$ implies that $G(z)\Theta(z) = 0$, and hence, using the definition of $A(z)$ in (3.21), we see that
\[
G(z)\Upsilon_{11}(z) = zG(z)A(z) \left(T_G T_G^* - (T_G T_G^* - T_K T_K^*)^{-1} N \Delta^{-1}\right).
\]
Next, using the definition of $B(z)$ in (3.21), we obtain
\[
K(z)\Upsilon_{11}(z) = -z K(z) B(z) (T_G T_G^* - T_K T_K^*)^{-1} N \Delta^{-1}.
\]
Taking the difference, applying (3.22) and using (3.21), we get
\[
G(z)\Upsilon_{12}(z) - K(z)\Upsilon_{22}(z) = zG(z)A(z) \left(T_G T_G^* - (T_G T_G^* - T_K T_K^*)^{-1} N \Delta^{-1}\right) = zG(z)A(z) \left(T_G T_G^* - (T_G T_G^* - T_K T_K^*)^{-1} N \Delta^{-1}\right)
\]
(3.27)
\[
= zG(z)A(z) \left(T_G T_G^* - (T_G T_G^* - T_K T_K^*)^{-1} N \Delta^{-1}\right) = zE_m^*(I - zS_m^*)^{-1} N \Delta^{-1}.
\]
According to (3.26) we have $G(z)A(z)(T_G T_G^*)^{-1} = E_m^*(I - zS_m^*)^{-1}$. Using the latter identity in (3.27), we see that (3.16) holds for $j = 1$.

Part 2. We prove the identity (3.16) for $j = 2$. Note that (3.4) and (3.5) can be rewritten in the following equivalent form:
\[
\Upsilon_{12}(z) = E_p^* T_G^*(I - zS_m^*)^{-1}(T_G T_G^* - T_K T_K^*)^{-1} T_K E_q \Delta_0^{-1},
\]
\[
\Upsilon_{12}(z) = \Delta_0^{-1} + E_p^* T_K^*(I - zS_m^*)^{-1}(T_G T_G^* - T_K T_K^*)^{-1} T_K E_q \Delta_0^{-1}.
\]
Using (3.21) and the above formulas for $\Upsilon_{12}$ and $\Upsilon_{22}$, we see that
\[
G(z)\Upsilon_{12}(z) = G(z)A(z) \Delta^{-1}(T_K E_q \Delta_0^{-1}),
\]
\[
K(z)\Upsilon_{22}(z) = K(z) \Delta_0^{-1} + K(z) B(z) \Delta^{-1}(T_K E_q \Delta_0^{-1}).
\]
Taking the difference, applying \(3.22\) and using \(3.24\), we obtain
\[
G(z)\Upsilon_{12}(z) - K(z)\Upsilon_{22}(z) = \\
= (G(z)A(z) - K(z)B(z))\Delta^{-1}(T_K E_q \Delta_0^{-1}) - K(z)\Delta_0^{-1} \\
= E_m^* (I - zS_m^*)^{-1} \Delta^{-1}(T_K E_q \Delta_0^{-1}) - K(z)\Delta_0^{-1} \\
= E_m^* (I - zS_m^*)^{-1}(T_K E_q \Delta_0^{-1}) - E_m^* (I - zS_m^*)^{-1}T_K E_q \Delta_0^{-1} = 0.
\]
This completes the proof. \(\square\)

4. State space computations

In this section we prove Theorem 1.1. To this end, we first recall some formulas derived in [10]. Let \(G \in \mathcal{R}H_\infty^{m \times p}\) and \(K \in \mathcal{R}H_\infty^{m \times q}\) be given by the realization of \([G \quad K]\) in (1.3). Assume \(T_G T_G^* - T_K T_K^*\) is strictly positive. Then there exist stabilizing solutions \(Q\) and \(Q_0\) to the Riccati equations (1.8) and (1.10), respectively. Let \(P_1\) and \(P_2\) be the controllability gramians that solve the Stein equations (1.4) for \(j = 1, 2\). Define \(\Delta\) and \(A_0\) by (1.3), the matrices \(C_j\), for \(j = 0, 1, 2, B_0\), and \(\Delta_j\), for \(j = 0, 1\), as in Theorem 1.1. Furthermore, as in Theorem 1.1 the matrix \(\Omega\) is given by
\[
\Omega = (P_1 - P_2)(Q^{-1} + P_2 - P_1)^{-1}Q^{-1}.
\]

Now, write \(W_{obs}\) and \(W_0\) for the observability operators defined by the pairs \(\{C, A\}\) and \(\{C_0, A_0\}\), respectively, that is,
\[
W_{obs} = \begin{bmatrix}
C \\
CA \\
CA^2 \\
\vdots
\end{bmatrix}, \quad W_0 = \begin{bmatrix}
C_0 \\
C_0A_0 \\
C_0A_0^2 \\
\vdots
\end{bmatrix}.
\]
The following identities are covered by [10, Eq.(5.9)] and [10, Eq.(5.5)]
\[
(4.1) \quad E_m^* T_K^* W_0 = C_1, \quad E_q^* T_K^* W_0 = C_2, \quad Q = W_{obs}^* W_0.
\]
Moreover, according to the comment directly after [10, Eq.(5.7)] we have
\[
(4.2) \quad S_m^* (T_G T_G^* - T_K T_K^*)^{-1} T_K E_q = W_0 B_0.
\]
Finally, let \(R\) be the function given by (1.3) and \(T_R\) the Toeplitz operator associated with \(R\). Recall that \(T_G T_G^* - T_K T_K^*\) strictly positive implies \(T_R\) is strictly positive. Then Theorem 1.1 in [10] yields
\[
(T_G T_G^* - T_K T_K^*)^{-1} = T_R^{-1} + T_R^{-1} W_{obs} \Omega W_{obs}^* T_R^{-1}.
\]
Along with
\[
(4.3) \quad W_0 = T_R^{-1} W_{obs},
\]
which was proved in [10, Lemma 5.1], this shows that
\[
(4.4) \quad (T_G T_G^* - T_K T_K^*)^{-1} = T_R^{-1} + W_0 \Omega W_0^*.
\]
Note that (4.3) also shows that \(Q = W_{obs}^* T_R^{-1} W_{obs}\), by the third identity in (4.1).

Using the formulas in (4.1) and (4.2) the state space representations of \(\Upsilon_{12}\) and \(\Upsilon_{22}\) in Theorem 1.1 follow immediately. In fact, as we have seen before (Part 1 of the proof of Theorem 1.1), \(\Upsilon_{12}\) and \(\Upsilon_{22}\) are related to \(U\) and \(V\) in [10] through \(\Upsilon_{12} \equiv U \Delta_0^{-1}\) and \(\Upsilon_{22} \equiv V \Delta_0^{-1}\), and the formulas for \(\Upsilon_{12}\) and \(\Upsilon_{22}\) in Theorem 1.1
above follow directly from the formulas for $U$ and $V$ derived in \cite{10}; see \cite{10} Eq. (5.14) and \cite{10} Eq. (5.13), respectively.

In order to show that $\Upsilon_{11}$ and $\Upsilon_{21}$, the two remaining functions in Theorem 3.1 admit the desired finite dimensional state space realizations requires a bit more work.

**Proof of Theorem 1.1.** In order to complete the proof of Theorems 1.1 it suffices to show that $\Upsilon_{11}$ in (3.2) and $\Upsilon_{21}$ in (3.3) admit finite dimensional state space representations as in (1.13) and that the positive definite matrices $\Delta_0$ and $\Delta_1$ defined by (3.6) and (3.7) are also given by (1.14). Note that in Theorem 1.1 as well as in Theorem 3.1 we assume that there is no non-zero $x \in \mathbb{C}^p$ such that $G(z)x$ is identically zero on the open unit disc $\mathbb{D}$.

In order to compute the remaining state space formulas, we prove the following identity:

$$
(4.5) \quad (T_G T_G^* - T_K T_K^*)^{-1} N = W_0 Q^{-1}(Q^{-1} + P_2 - P_1)^{-1} B_1 \Theta_0.
$$

First observe that

$$
(4.6) \quad N = S^*_m T_G E_p \Theta_0 = W_{obs} B_1 \Theta_0.
$$

Now, combining (4.4) and (4.3) along with the third identity in (4.1) we obtain that

$$(T_G T_G^* - T_K T_K^*)^{-1} W_{obs} =$$

$$= T_R^{-1} W_{obs} + W_0 (P_1 - P_2)(Q^{-1} + P_2 - P_1)^{-1} Q^{-1} W_0^* W_{obs}$$

$$= W_0 + W_0 (P_1 - P_2)(Q^{-1} + P_2 - P_1)^{-1}$$

$$= W_0 (I + (P_1 - P_2)(Q^{-1} + P_2 - P_1)^{-1})$$

$$= W_0 (Q^{-1} + P_2 - P_1 + P_1 - P_2)(Q^{-1} + P_2 - P_1)^{-1}$$

$$= W_0 (Q^{-1} + P_2 - P_1)^{-1}.$$

Together with (4.6) this gives (4.5).

Using (4.6) along with $S^*_m W_0 = W_0 A_0$ we obtain

$$\Upsilon_{11}(z) = \Theta_0 \Delta^{-1}_1 - z E^*_p T_G^* (I - z S^*_m)^{-1} (T_G T_G^* - T_K T_K^*)^{-1} N \Delta^{-1}_1$$

$$= \Theta_0 \Delta^{-1}_1 - z E^*_p T_G^* (I - z S^*_m)^{-1} W_0 Q^{-1}(Q^{-1} + P_2 - P_1)^{-1} B_1 \Theta_0 \Delta^{-1}_1$$

$$= \Theta_0 \Delta^{-1}_1 - z \Upsilon_{21} W_0 (I - z A_0)^{-1} Q^{-1} (Q^{-1} + P_2 - P_1)^{-1} B_1 \Theta_0 \Delta^{-1}_1.$$

To obtain the last equality we used the first equality in (4.11). Similarly

$$\Upsilon_{21}(z) = -z E^*_p T_K^* (I - z S^*_m)^{-1} (T_G T_G^* - T_K T_K^*)^{-1} N \Delta^{-1}_1$$

$$= -z E^*_p T_K^* W_0 (I - z A_0)^{-1} Q^{-1} (Q^{-1} + P_2 - P_1)^{-1} B_1 \Delta^{-1}_1$$

$$= -z C_2 (I - z A_0)^{-1} Q^{-1} (Q^{-1} + P_2 - P_1)^{-1} B_1 \Delta^{-1}_1.$$

In the final step of the above computation we used the second equality in (1.14). The computations above show that $\Upsilon_{11}$ and $\Upsilon_{21}$ admit the state space representation given by (1.13). It remains to show that $\Delta_0$ and $\Delta_1$ are the positive definite matrices determined by (1.14). The matrix $\Delta_0$ in fact appears in (10).
denoted by \( D_Y \) in [10, Eq.(3.4)], and a formula in terms of the state space realization (3.3) and related matrices is given in [10, Eq.(1.16)]. We derive here a different formula, given in (4.4) above, which better exhibits the positive definite character.

Recall from (3.6) that
\[
\Delta_0^2 = I_q + E_q T_K^* (T_G T_G^* - T_K T_K^*)^{-1} T_K^* E_q.
\]

Using (4.5) and the second identity in (4.1) we obtain that
\[
\Delta_0^2 = I_q + C_2 \Omega C_2^* + E_q T_K^* T_R^{-1} T_K^* E_q.
\]

Using (4.4) and the second identity in (4.1) we obtain that
\[
\Delta_0^2 = I_q + C_2 \Omega C_2^* + E_q T_K^* T_R^{-1} T_K^* E_q.
\]

Recall that on page 14 of [10] it was shown that
\[
\Delta_0^2 = I_q + C_2 \Omega C_2^* + E_q T_K^* T_R^{-1} T_K^* E_q.
\]

Recall from (3.6) that
\[
\Delta_0^2 = I_q + E_q T_K^* (T_G T_G^* - T_K T_K^*)^{-1} T_K^* E_q.
\]

Using (4.5) we obtain that
\[
\Delta_0^2 = I_q + C_2 \Omega C_2^* + E_q T_K^* T_R^{-1} T_K^* E_q.
\]

Using (4.5) and the second identity in (4.1) we obtain that
\[
\Delta_0^2 = I_q + C_2 \Omega C_2^* + E_q T_K^* T_R^{-1} T_K^* E_q.
\]

Recall that on page 14 of [10] it was shown that
\[
\Delta_0^2 = I_q + C_2 \Omega C_2^* + E_q T_K^* T_R^{-1} T_K^* E_q.
\]

Recall that (4.7) and (4.8) into the formula for \( \Delta_1^2 \) derived above gives the formula for \( \Delta_2^2 \) in (1.14).

**Remark 4.1.** Two important special cases of the Leech problem are the Toeplitz corona problem, which can be reduced to the case where \( q = m \) and \( K \) is identically equal to the identity matrix \( I_m \) (\( K = I_m \)), and the case where \( K \) is identically equal to the zero matrix (\( K \equiv 0 \)). On the level of the state space representation (1.3) these correspond to the cases \( B_2 = 0 \) and \( D_2 = I_m \), and \( B_2 \) is 0 and \( D_2 = 0 \), respectively. Recall that the scalar corona problem was proved by Carlson [5] and the matrix case by Fuhrmann [14]; see [19] for a discussion of the problem. For
the Toeplitz corona problem, Theorem [1.1] leads to a description of the solutions via a similar linear fractional transformation. We omit the precise formulas for the coefficients $\Upsilon_{ij}$, $i, j = 1, 2$, and only mention some of the matrices appearing in Theorem [1.1] that simplify:

\[
P_2 = 0, \quad \Gamma = \Gamma_0, \quad C_2 = C_0, \quad B_0 = A_0 \Omega - \Gamma \Delta^{-1}, \quad \Delta_0^2 = I_q + C_0 \Omega C_0^* + \Delta^{-1}, \quad \Delta_1 = I_{m-p}.
\]

The situation is different for the case $K \equiv 0$, i.e., $B_2 = 0$ and $D_2 = 0$. Then

\[
P_2 = 0, \quad \Gamma = \Gamma_0, \quad C_2 = 0, \quad B_0 = 0, \quad \Delta_0 = I_q, \quad \Delta_1 = I_{m-p}.
\]

From these formulas one immediately obtains that

\[
\Upsilon_{12}(z) = 0, \quad \Upsilon_{21}(z) = 0, \quad \Upsilon_{22}(z) = I_q \quad (z \in \mathbb{D}).
\]

The formula for $\Upsilon_{11}$ reduces to

\[
\Theta_0 - z C_1 (I - z A_0)^{-1} Q_0^{-1} (Q_0^{-1} - P_1) B_1 \Theta_0 \quad (z \in \mathbb{D})
\]

where $Q_0$ is the stabilizing solution to the Riccati equation (1.10) and

\[
A_0 = A - \Gamma_0 (R_{10} - \Gamma_0^* Q_0 \Gamma_0)^{-1} (C - \Gamma_0^* Q_0 A).
\]

On inspection of the formula for $\Upsilon_{11}$ given in Section 3, we see that

\[
\Upsilon_{11}(z) = \Theta_0 - z E_\nu T_G^* (I - z S_{\nu}^*)^{-1} (T_G^* T_G)^{-1} N = \Theta(z),
\]

where $\Theta$ is the inner function in $H^\infty_{(p-m) \times m}$ such that $\text{Ker} T_G = \text{Im} T_\Theta$, see Lemma [2.1]. Hence, as expected, the solutions to the Leech problem (1.1) with $K \equiv 0$ are given by $X = \Theta Y$ with $Y$ an arbitrary function in $H^\infty_{(p-m) \times m}$ with $\|Y\|_{\infty} \leq 1$.

**Appendix A. Commutant Lifting**

In this appendix we derive a version of the commutant lifting theorem, based on Theorem VI.6.1 in [3], which we need for the proof of our main results.

We begin with some notation. Throughout this appendix $\mathcal{H}'$ is a subspace of $\ell_2^2(\mathbb{C}^p)$, invariant under the backward shift $S_\nu^*$ on $\ell_2^2(\mathbb{C}^p)$. The latter means there exists an inner function $\Theta \in H^\infty_{p \times k}$ for some positive integer $k \leq p$ such that $\mathcal{H}' = \text{Ker} T_\Theta^*$, that is,

\[
\ell_2^2(\mathbb{C}^p) = \mathcal{H}' \oplus T_\Theta \ell_2^2(\mathbb{C}^k).
\]

By $T'$ we denote the compression of the forward shift $S_p$ on $H^2_p$ to $\mathcal{H}'$. It follows that $S_p$ admits the following operator $2 \times 2$ block operator matrix representation for appropriate choices of $W$ and $Z$:

\[
S_p = \begin{bmatrix} T' & 0 \\ W & Z \end{bmatrix} \quad \text{on} \quad \begin{bmatrix} \mathcal{H}' \\ \text{Im} T_\Theta \end{bmatrix}.
\]

Hence $S_p$ is an isometric lifting of $T'$. The first theorem in this appendix is the following variation on Theorem VI.6.1 in [3] for the isometric lifting $S_p$ of $T'$. We shall assume that $S_p$ is a minimal isometric lifting of $T'$, that is,

\[
\ell_2^2(\mathbb{C}^p) = \bigvee_{\nu \geq 0} S_p^\nu \mathcal{H}.
\]
Theorem A.1. Assume $S_p$ is a minimal isometric lifting of $T'$, and let $\Lambda$ be a strict contraction mapping $\ell^2_+(C^q)$ into $\mathcal{H}' \subset \ell^2_+(C^p)$ satisfying the intertwining relation $T'\Lambda = \Lambda S_q$. Then all functions $X$ in $H^\infty_{p,q,s}$ satisfying

\[ \Lambda = P_{\mathcal{H}'X} \quad \text{and} \quad \|X\|_\infty \leq 1 \]

are given by

\[ X(z) = \Phi_{22}(z) + \Phi_{21}(z)Y(z)(I - \Phi_{11}(z)Y(z))^{-1}\Phi_{12}(z). \quad |z| < 1. \]

Here $Y$ is an arbitrary function in $H^\infty_{k,q,s}$ with $\|Y\|_\infty \leq 1$, and

\[ \Phi_{11}(z) = -z\Delta_0^{-1}E^*_q(I - zM)^{-1}B\psi$, \]

\[ \Phi_{12}(z) = \Delta_0^{-1}E^*_q(I - zM)^{-1}E_q, \]

\[ \Phi_{21}(z) = \Theta(z)\Delta_1 - \Theta(z)E^*_qT_0S_p\Lambda(I - zM)^{-1}B\psi \Delta_1^{-1}, \]

\[ \Phi_{22}(z) = E^*_q(I - zS^*_p)^{-1}AE_q + \Theta(z)E^*_qT_0S_p\Lambda M(I - zM)^{-1}E_q. \]

Here $M$ is the operator on $\ell^2_q(C^q)$, with spectral radius $r_{\text{spec}}(M) \leq 1$, given by

\[ M = S^*_q - S^*_q(I - \Lambda^*\Lambda)^{-1}E_q\Delta_0^{-2}E^*_q. \]

Furthermore, $B\psi = (I - \Lambda^*\Lambda)^{-1}\Lambda^*S^*_pT_0E_k$, which maps $C^k$ into $\ell^2_+(C^q)$, and $\Delta_0$ and $\Delta_1$ are the positive definite matrices given by

\[ \Delta_0^2 = E^*_q(I - \Lambda^*\Lambda)^{-1}E_q \]

\[ \Delta_1^2 = I_k + E^*_qT_0S_p\Lambda(I - \Lambda^*\Lambda)^{-1}\Lambda^*S^*_pT_0E_k. \]

Moreover, the $(k + m) \times (m + p)$ coefficient matrix $\Phi$ defined by

\[ \Phi = [ \Phi_{11} \Phi_{12} \Phi_{21} \Phi_{22} ], \]

with $\Phi_{11}, \Phi_{12}, \Phi_{21}, \Phi_{22}$ defined above, is inner.

It is useful to first prove some preliminary results.

The description of intertwining liftings in Theorem VI.6.1 in [8] is with respect to the Sz-Nagy-Schäffer isometric lifting $U'_{NS}$ of $T'$, which is given by

\[ U'_{NS} = \begin{bmatrix} T' & 0 \\ E'D' & S_{D'} \end{bmatrix} \quad \text{on} \quad \begin{bmatrix} \mathcal{H}' \\ \ell^2_+(D') \end{bmatrix}. \]

Here $D'$ is the defect operator defined by $T'$, and $D'$ is the corresponding defect space, i.e., $D' = (I - T'^*T')^{\frac{1}{2}}$ and $D' \subset \text{closure of } \text{Im } D'$. Furthermore, $E' : D' \to \ell^2_+(D')$ is the canonical embedding defined by $(E'd') = (d',0,0,...)$ for each $d' \in D'$. It is well known that $U'_{NS}$ is a minimal isometric lifting of $T'$. Since $S_p$ is assumed to be a minimal isometric lifting, there exists a unique unitary operator $\Psi_0$ mapping $\ell^2_+(D')$ onto $\text{Im } T_0 = \ell^2_+(C^q) \ominus \mathcal{H}'$ such that

\[ \begin{bmatrix} I_{\mathcal{H'}} & 0 \\ 0 & \Psi_0 \end{bmatrix} \begin{bmatrix} T' & 0 \\ E'D' & S_{D'} \end{bmatrix} = \begin{bmatrix} T' & 0 \\ W & Z \end{bmatrix} \begin{bmatrix} I_{\mathcal{H'}} & 0 \\ 0 & \Psi_0 \end{bmatrix}. \]

The next lemma provides a description of the unitary operator $\Psi_0$.

Lemma A.2. Assume $S_p$ is a minimal isometric lifting of $T'$. Let $\Psi_0$ be the unitary operator defined by (A.10), and let $\Xi$ be the unitary operator defined by

\[ \Xi : \ell^2_+(C^k) \to \ell^2_+(C^p) \ominus \mathcal{H}', \quad \Xi g = T_0g \quad (g \in \ell^2_+(C^k)). \]
Then there exists a unitary operator $N_0$ from $\mathcal{D}'$ onto $\mathbb{C}^k$ such that $\Psi_0 = \Xi T_{N_0}$, with $T_{N_0}$ the diagonal Toeplitz operator defined by the constant function with value $N_0$, i.e.,

(A.12) \[ \Psi_0 f = T_\Theta T_{N_0} f \quad (f \in \ell_\Theta^2(\mathcal{D}')). \]

Moreover,

(i) the matrix $N_0$ is uniquely determined by the identity

(A.13) \[ N_0 D' = E_k^* \Xi^* W; \]

(ii) the operator $W$ in (A.2) is given by $W = T_\Theta E_k N_0 D'$.

Proof. From the definition of $\Xi$ and the fact that $\Theta$ is inner we see that $T_\Theta$ admits the following partitioning:

\[ T_\Theta = \begin{bmatrix} 0 \\ \Xi \end{bmatrix} : \ell_\Theta^2(\mathbb{C}^k) \to \begin{bmatrix} \mathcal{H}' \\ \text{Im} T_\Theta \end{bmatrix}. \]

Since $S_p T_\Theta = T_\Theta S_k$, this implies that

\[ \begin{bmatrix} I_{\mathcal{H}'} & 0 \\ 0 & \Xi^* \end{bmatrix} \begin{bmatrix} T' \\ \Xi^* W \\ 0 \end{bmatrix} = \begin{bmatrix} T' \\ \Xi^* W \\ 0 \end{bmatrix} \begin{bmatrix} I_{\mathcal{H}'} & 0 \\ 0 & \Xi^* \end{bmatrix}. \]

But then (A.10) yields

\[ \begin{bmatrix} I_{\mathcal{H}'} & 0 \\ 0 & \Xi^* \Psi_0 \end{bmatrix} \begin{bmatrix} T' \\ E' D' \\ S_{D'} \end{bmatrix} = \begin{bmatrix} T' \\ \Xi^* W \\ 0 \end{bmatrix} \begin{bmatrix} I_{\mathcal{H}'} & 0 \\ 0 & \Xi^* \Psi_0 \end{bmatrix}. \]

In particular, $(\Xi^* \Psi_0) S_{D'} = S_k (\Xi^* \Psi_0)$. Since the operator $\Xi^* \Psi_0$ is unitary, the latter intertwining relation implies that $\Xi^* \Psi_0$ is a block diagonal Toeplitz operator $T_{N_0} = \text{diag} (N_0, N_0, \ldots)$, where $N_0$ is a unitary operator from $\mathcal{D}'$ onto $\mathbb{C}^k$. The identity $T_{N_0} = \Xi^* \Psi_0$ and the fact that $\Xi$ is unitary imply that $\Xi T_{N_0} = \Psi_0$. Using the definition of $\Xi$ in (A.11) the latter identity yields (A.12). Finally, from $T_{N_0} E' D' = \Xi^* \Psi_0 E' D' = \Xi^* W$ we obtain (A.13). \(\square\)

Proof of Theorem A.1 The characterization of all solutions in (A.4) follows by applying Theorem VI.6.1 from [8] to the commutant lifting data described above. Note that $\|A\| < \gamma = 1$ implies $\Lambda$ is a strict contraction. Directly applying the formulas from [8], using $A = \Lambda$, $T = S_q$ and $\Pi_0 = E_q^*$ and multiplying with $\Theta(z) N_0$ on the right, as noted in Lemma A.2, we obtain that the functions $X$ in $H^\infty_{p \times q}$ satisfying (A.3) are given by (A.4) with

(A.14) \[
\begin{align*}
\Phi_{11}(z) &= -z \Delta_0^{-1} E_q^*(I - z M)^{-1} (I - \Lambda^* \Lambda)^{-1} \Lambda^* D' \Delta_1^{-1} N_0^* \\
\Phi_{12}(z) &= \Delta_0^{-1} E_q^*(I - z M)^{-1} E_q \\
\Phi_{21}(z) &= \Theta(z) N_0 (\Delta_1^2 - D' \Lambda (I - z M)^{-1} (I - \Lambda^* \Lambda)^{-1} \Lambda^* D') \Delta_1^{-1} N_0^* \\
\Phi_{22}(z) &= E_q^*(I - z S_q^*)^{-1} \Lambda E_q + \Theta(z) N_0 D' \Lambda M (I - z M)^{-1} E_q,
\end{align*}
\]

where $\Delta_0$ (in [8] denoted by $N$) is as in (A.7) and $M$ and $\Delta_1$ (in [8] denoted by $T_A^*$ and $N_1$, respectively) are given by

(A.15) \[
M = (I - S_q^* \Lambda^* \Lambda S_q)^{-1} S_q^*(I - \Lambda^* \Lambda) \quad \text{and} \quad \Delta_1^2 = I_k + D' \Lambda D_{\Lambda}^{-2} \Lambda^* D'.
\]

Here we multiplied the formulas in [8] for $\Phi_{11}$ and $\Phi_{21}$ with the unitary operator $N_0^* : \mathbb{C}^k \to \mathcal{D}'$ from Lemma A.2 so that the free parameter function $Y$ maps into the right space.
Using the fact that $N_0$ is a unitary operator satisfying (A.13), it is obvious that $N_0\Delta^2 = \Delta^2 N_0$. Then, also $N_0\Delta_1 = \Delta_1 N_0$ and $N_0\Delta_1^{-1} = \Delta_1^{-1} N_0$. It remains to show that the formulas for $M$ in (A.15) and (A.7) coincide. Indeed, once this fact is established, it easily follows from the intertwining relations for $\Delta_1$ and $\Delta_1$, together with (A.13), that the functions $\Phi_{ij}$ in (A.14) are also given by (A.8).

To see that the two formulas for $M$ coincide, note that

\begin{align*}
M &= (I - S_q^*\Lambda^*\Lambda S_q)^{-1}S_q^*(I - \Lambda^*\Lambda) \\
   &= S_q^*(I - \Lambda^*\Lambda S_q S_q^*)^{-1}(I - \Lambda^*\Lambda) \\
   &= S_q^*(I - \Lambda^*\Lambda (I - E_q E_q^*))^{-1}(I - \Lambda^*\Lambda) \\
   &= S_q^*((I - \Lambda^*\Lambda) + \Lambda^*\Lambda E_q E_q^*)^{-1}(I - \Lambda^*\Lambda) \\
   &= S_q^*(I + (I - \Lambda^*\Lambda)^{-1}\Lambda^*\Lambda E_q E_q^*)^{-1}.
\end{align*}

Now set

\[ A = I, \quad B = E_q^*, \quad C = (I - \Lambda^*\Lambda)^{-1}\Lambda^*\Lambda E_q, \quad D = I. \]

Since $I + (I - \Lambda^*\Lambda)^{-1}\Lambda^*\Lambda E_q E_q^* = D + CA^{-1}B$ is invertible, so is

\[ A^* := A + BD^{-1}C = I + E_q^*(I - \Lambda^*\Lambda)^{-1}\Lambda^*\Lambda E_q \\
   = E_q^*(I + (I - \Lambda^*\Lambda)^{-1}\Lambda^*\Lambda)E_q = E_q^*(I - \Lambda^*\Lambda)^{-1}E_q = \Delta_0^2. \]

By standard inversion formulas, cf., [3], we obtain that

\[ M = S_q^*(D^{-1} - D^{-1}C(A^*)^{-1}BD^{-1}) \\
   = S_q^*(I - (I - \Lambda^*\Lambda)^{-1}\Lambda^*\Lambda E_q E_q^*) \\
   = S_q^* - S_q^*(I - \Lambda^*\Lambda)^{-1}(I - (I - \Lambda^*\Lambda))E_q E_q^* \\
   = S_q^* - S_q^*(I - \Lambda^*\Lambda)^{-1}E_q E_q^* + S_q^* E_q E_q^* \\
   = S_q^* - S_q^*(I - \Lambda^*\Lambda)^{-1}E_q E_q^*. \]

Here we used that $S_q E_k = 0$. The latter identity implies $E_q S_q = 0$, and hence $MS_q = I$. Hence $M$ is given by (A.6). Therefore

\[ (A.16) \quad M = (I - S_q^*\Lambda^*\Lambda S_q)^{-1}S_q^*(I - \Lambda^*\Lambda) = S_q^* - S_q^*(I - \Lambda^*\Lambda)^{-1}E_q E_q^*. \]

As in [10] we shall need the following functions:

\[ (A.17) \quad U(z) = E_p^*(I - z S_q^*)^{-1} \Lambda (I - \Lambda^*\Lambda)^{-1} E_q, \]
\[ (A.18) \quad V(z) = E_q^*(I - z S_q^*)^{-1} (I - \Lambda^*\Lambda)^{-1} E_q. \]

As mentioned in Theorem 2.1 in [10], det $V(z) \neq 0$ for $|z| < 1$, the function $V^{-1}$ belongs to $H_{q\times q}^\infty$ and is an outer function.

**Proposition A.3.** Let $\Phi_{12}$ and $\Phi_{22}$ be as in (A.5), and let $U$ and $V$ be given by (A.17) and (A.18), respectively. Then

\[ (A.19) \quad \Phi_{12}(z) = \Delta_0 V(z)^{-1} \quad \text{and} \quad \Phi_{22}(z) = U(z)V(z)^{-1} \quad (z \in \mathbb{D}). \]

**Proof.** First we prove the first identity in (A.19). From the definition of $\Phi_{12}$ in (A.5) it is clear that

\[ \Phi_{12}(z) = \Delta_0^{-1} + z \Delta_0^{-1} E_q^* (I - z M)^{-1} M E_q. \]
Using [4, Theorem 2.1], it follows that in a neighborhood of zero we have

$$\Phi_{12}(z)^{-1} = \Delta_0 - zE_q^* (I - zM^\alpha)^{-1} M E_q \Delta_0.$$ 

This with (A.16) yields

(A.20) \[ M^\alpha = M - (ME_q)\Delta_0 (\Delta_0^{-1} E_q^*) = M - ME_q E_q^* \]

(A.21) \[ M S_q S_q^* = (I - S_q^* \Lambda^* S_q)^{-1} S_q^* (I - \Lambda^* \Lambda) S_q S_q^* = S_q^*. \]

Using (A.21) it follows that

(A.22) \[ \Phi_{12}(z)^{-1} = \Delta_0 - zE_q^* (I - zS_q^*)^{-1} M E_q \Delta_0, \quad z \in \mathbb{D}. \]

Next, note that

$$M E_q \Delta_0^2 = (S_q^* - S_q^* (I - \Lambda^* \Lambda)^{-1} E_q \Delta_0^{-2} E_q^*) E_q \Delta_0^2$$

$$= -S_q^* (I - \Lambda^* \Lambda)^{-1} E_q \Delta_0^{-2} E_q^* E_q \Delta_0^2$$

$$= -S_q^* (I - \Lambda^* \Lambda)^{-1} E_q.$$

Hence

$$\Phi_{12}(z)^{-1} \Delta_0 = \Delta_0^2 - zE_q^* (I - zS_q^*)^{-1} M E_q \Delta_0^2$$

$$= \Delta_0^2 + zE_q^* (I - zS_q^*)^{-1} S_q^* (I - \Lambda^* \Lambda)^{-1} E_q$$

$$= \Delta_0^2 + E_q^* (I - zS_q^*)^{-1} (I - (I - zS_q^*)) (I - \Lambda^* \Lambda)^{-1} E_q$$

$$= \Delta_0^2 + E_q^* (I - zS_q^*)^{-1} (I - \Lambda^* \Lambda)^{-1} E_q - E_q^* (I - \Lambda^* \Lambda)^{-1} E_q$$

$$= E_q^* (I - zS_q^*)^{-1} (I - \Lambda^* \Lambda)^{-1} E_q = V(z).$$

This proves the first identity in (A.19).

To prove the second identity in (A.19), note that \( \Phi_{22} \) is the so-called central solution, i.e., the solution that one obtains if the free parameter \( Y \) in (A.4) is taken to be zero. But then [3, Theorem IV.7.1] tells us that \( \Phi_{22} \) is the maximum entropy solution and we can apply [10, Proposition 3.1] to show that the second identity in (A.19) holds true. For the sake of completeness we also give a direct proof.

We take \( \Phi_{22} \) as in (A.14). This formula can be rewritten as

$$\Phi_{22}(z) = E_p^* (I - zS_p^*)^{-1} \left( \Lambda (I - zM) + T_\theta E_k N_0 D' \Lambda M \right) (I - zM)^{-1} E_q$$

$$= E_p^* (I - zS_p^*)^{-1} \left( \Lambda (I - zM) + W \Lambda M \right) (I - zM)^{-1} E_q.$$

Here we used the identity \( \Theta(z) = E_p^* (I - zS_p^*)^{-1} T_\theta E_k \) and item (ii) in Lemma A.2. Put \( M(z) = \Lambda (I - zM) + W \Lambda M \). This operator function admits the following partitioning:

$$M(z) = \begin{pmatrix} \Lambda (I - zM) & 0 \\ W \Lambda M & 0 \end{pmatrix} : \mathbb{C}^n \rightarrow \mathbb{H}'.$$
Using this partitioning, formula (A.22), the intertwining relation \( T'\Lambda = \Lambda S_q \), and the fact that \( M S_q = I \), we see that

\[
M(z)S_q = \begin{bmatrix} \Lambda S_q - z\Lambda \\ WA \end{bmatrix} = \begin{bmatrix} T'\Lambda \\ WA \end{bmatrix} - z \begin{bmatrix} \Lambda \\ 0 \end{bmatrix} = S_p \begin{bmatrix} \Lambda \\ 0 \end{bmatrix} - z \begin{bmatrix} \Lambda \\ 0 \end{bmatrix} = (I - zS_p^*)S_p \begin{bmatrix} \Lambda \\ 0 \end{bmatrix}.
\]

If follows that

(A.23) \[ E_p^*(I - zS_p^*)^{-1}M(z)S_q = E_p^*(I - zS_p^*)^{-1}(I - zS_p^*)S_p \begin{bmatrix} \Lambda \\ 0 \end{bmatrix} = 0. \]

Applying this to our formula for \( \Phi_{22} \) we obtain

\[
\Phi_{22}(z) = E_p^*(I - zS_p^*)^{-1}M(z)(I - zM)^{-1}E_q
\]

\[ = E_p^*(I - zS_p^*)^{-1}M(z)(E_q E_q^* + S_q S_q^*)(I - zM)^{-1}E_q \]

\[ = E_p^*(I - zS_p^*)^{-1}M(z)E_q E_q^*(I - zM)^{-1}E_q. \]

Using the definition of \( \Phi_{12} \) in (A.14), and the definition of \( \Delta_0 \) in (A.7), we see that

\[ E_q^* (I - zM)^{-1}E_q = \Delta_0 \Phi_{12} = \Delta_0^2 V(z)^{-1} = E_q^*(I - \Lambda^*\Lambda)^{-1}E_q V(z)^{-1}. \]

Together with \( E_q E_q^* = I - S_q S_q^* \) and the identity (A.23) the previous identity yields

\[
\Phi_{22}(z) = E_p^*(I - zS_p^*)^{-1} \begin{bmatrix} \Lambda(I - zM) \\ WA \end{bmatrix} (I - \Lambda^*\Lambda)^{-1}E_q V(z)^{-1}.
\]

Finally, using the formula for \( M \) given by the left hand side of (A.16) and \( S_q^* E_q = 0 \), we see that

\[ M(I - \Lambda^*\Lambda)^{-1}E_q = (I - S_q^* \Lambda^*\Lambda S_q) - S_q^* E_q = 0. \]

Hence the above formula for \( \Phi_{22} \) simplifies to

\[
\Phi_{22}(z) = E_p^*(I - zS_p^*)^{-1} \begin{bmatrix} \Lambda \\ 0 \end{bmatrix} (I - \Lambda^*\Lambda)^{-1}E_q V(z)^{-1}.
\]

Using the definition of \( U \) in (A.17), this yields the second identity in (A.19). \( \square \)

The following result in the analogue of Theorem A.1 with the Redheffer representation of all solution (A.4) being replaced by a linear fractional map.

**Theorem A.4.** Assume \( S_p \) is a minimal isometric lifting of \( T' \), and let \( \Lambda \) be a strict contraction mapping \( \ell^2_p(\mathbb{C}^q) \) into \( \mathcal{H}' \subset \ell^2_p(\mathbb{C}^p) \) satisfying the intertwining relation \( T'\Lambda = \Lambda S_q \). Then all functions \( X \) in \( H^\infty_{p\times q} \) satisfying

(A.24) \[ \Lambda = P_{\mathcal{H}'}T_X \quad \text{and} \quad ||X||_\infty \leq 1 \]

are given by

(A.25) \[ X(z) = \left( \Upsilon_{12}(z) + \Upsilon_{11}(z) Y(z) \right) \left( \Upsilon_{22}(z) + \Upsilon_{21}(z) Y(z) \right)^{-1}, \quad |z| < 1. \]
Here \( Y \) is an arbitrary function in \( H_\infty^{\infty} \) with \( \|Y\|_\infty \leq 1 \), and
\[
\begin{align*}
\Upsilon_{11}(z) &= zE_p^*(I - zS_p)E_q^*(I - zS_q)^{-1}B_\nabla \Delta_1^{-1} + \\
&\quad \Theta(z)\Delta_1^{-1} - z\Theta(z)E_p^*T_0^*S_p\Lambda(I - zS_q)^{-1}S_q^*B_\nabla \Delta_1^{-1},
\end{align*}
\]
(A.26)
\[
\begin{align*}
\Upsilon_{21}(z) &= zE_p^*(I - zS_p)^{-1}B_\nabla \Delta_1^{-1} \\
\Upsilon_{12}(z) &= U(z)\Delta_0^{-1}, \\
\Upsilon_{22}(z) &= V(z)\Delta_0^{-1}.
\end{align*}
\]

Here \( B_\nabla = (I - \Lambda^*\Lambda)^{-1}\Lambda^*S_p^*T_0E_k \), the functions \( U \) and \( V \) are given by (A.17) and (A.18), respectively, and \( \Delta_0 \) and \( \Delta_1 \) are the positive definite matrices given by
\[
\begin{align*}
\Delta_0^2 &= E_q^*(I - \Lambda^*\Lambda)^{-1}E_q \\
\Delta_1^2 &= I_k + E_p^*T_0^*S_p\Lambda(I - \Lambda^*\Lambda)^{-1}\Lambda^*S_p^*T_0E_k.
\end{align*}
\]
(A.27)

Moreover, the \((p+k) \times (q+p)\) coefficient matrix \( \Upsilon \) defined by
\[
\Upsilon = \begin{bmatrix}
\Upsilon_{11} & \Upsilon_{12} \\
\Upsilon_{21} & \Upsilon_{22}
\end{bmatrix},
\]
with \( \Upsilon_{11}, \Upsilon_{12}, \Upsilon_{21} \) and \( \Upsilon_{22} \) as above, is \( J_1 \), \( J_2 \)-inner, where \( J_1 \) and \( J_2 \) are given by \( J_1 = \text{diag}(I_p, -I_q) \), and \( J_2 = \text{diag}(I_k, -I_q) \).

**Proof.** The fact that \( \Phi_{12}(z) \) is invertible for each \( z \in \mathbb{D} \), with an analytic inverse, implies that we can apply the Potapov-Ginzburg transform pointwise, cf., Section 2.5 in [2], defining analytic matrix valued functions \( \Upsilon_{ij} \), \( i, j = 1, 2 \), on \( \mathbb{D} \) via
\[
\begin{align*}
\Upsilon_{11} &= \Phi_{21} - \Phi_{22}\Phi_{12}^{-1}\Phi_{11}, \\
\Upsilon_{12} &= \Phi_{22}\Phi_{12}^{-1}, \\
\Upsilon_{21} &= -\Phi_{12}^{-1}\Phi_{11}, \\
\Upsilon_{22} &= \Phi_{12}^{-1}.
\end{align*}
\]
(A.28)

Following [2], we obtain that the identity
\[
\Phi_{22} + \Phi_{21}Y(I - \Phi_{11}Y)^{-1}\Phi_{12} = (\Upsilon_{12} + \Upsilon_{11}Y)(\Upsilon_{22} + \Upsilon_{21}Y)^{-1}
\]
holds point wise on \( \mathbb{D} \) for any function \( Y \) in \( H_\infty^{\infty} \) with \( \|Y\|_\infty \leq 1 \). Moreover, since \( \Phi \) in (A.8) is inner, we obtain that the coefficient matrix
\[
\Upsilon = \begin{bmatrix}
\Upsilon_{11} & \Upsilon_{12} \\
\Upsilon_{21} & \Upsilon_{22}
\end{bmatrix},
\]
(A.29)
is \( J_1 \), \( J_2 \)-inner, where \( J_1 = \text{diag}(I_p, -I_q) \), and \( J_2 = \text{diag}(I_k, -I_q) \), that is, for almost any \( z \in \mathbb{T} \) we have \( \Upsilon(z)^*J_1\Upsilon(z) = J_2 \).

From the results in the previous paragraph we conclude that in order to prove the theorem it suffices to show that the functions \( \Upsilon_{ij}, 1 \leq i, j \leq 2 \), defined in (A.28), are also given by the right hands of the formulas in (A.26). For \( \Upsilon_{12} \) and \( \Upsilon_{22} \) this follows directly from the two identities in (A.19). So it remains to consider the functions \( \Upsilon_{11} \) and \( \Upsilon_{21} \).
We begin with \( \Upsilon_{21} \). Using the definition of \( \Upsilon_{21} \) in (A.28), the identity (A.22), and the first identity in (A.5), we see that

\[
\Upsilon_{21}(z) = -\Phi_{12}(z)^{-1}\Phi_{11}(z)
\]

which yields

\[
\Upsilon_{21}(z) = -\left( I_q - zE_q^*(I - zS_q^*)^{-1}ME_q \right) \Delta_0 \times \\
\times \left( -z\Delta_0^{-1}E_q^*(I - zM)^{-1}B\nabla \Delta_1^{-1} \right)
\]

\[
= zE_q^*(I - zM)^{-1}B\nabla \Delta_1^{-1} + \\
- zE_q^*(I - zS_q^*)^{-1}(zME_qE_q^*)(I - zM)^{-1}B\nabla \Delta_1^{-1}.
\]

From (A.20) and (A.21) we see that

\[
M - S_q^* = ME_qE_q^*.
\]

Using the latter identity we obtain

\[
(I - zS_q^*)^{-1}(zME_qE_q^*)(I - zM)^{-1} = \\
= (I - zS_q^*)^{-1}(zM - zS_q^*)(I - zM)^{-1} \\
= (I - zS_q^*)^{-1}((I - zS_q^*) - (I - zM))(I - zM)^{-1} \\
= (I - zM)^{-1} - (I - zS_q^*)^{-1}.
\]

It follows that

\[
\Upsilon_{21}(z) = zE_q^*(I - zM)^{-1}B\nabla \Delta_1^{-1} + \\
- zE_q^*(I - zM)^{-1}B\nabla \Delta_1^{-1} + zE_q^*(I - zS_q^*)^{-1}B\nabla \Delta_1^{-1} \\
= zE_q^*(I - zS_q^*)^{-1}B\nabla \Delta_1^{-1}.
\]

This proves the second identity in (A.27).

Next we deal with \( \Upsilon_{11} \). According to (A.28), we have

\[
\Upsilon_{11}(z) = \Phi_{22}(z) - \Phi_{22}(z)\Phi_{12}(z)^{-1}\Phi_{11}(z) = \Phi_{21} + \Phi_{22}(z)\Upsilon_{21}(z).
\]

We first compute \( \Phi_{22}\Upsilon_{21} \) using the first identity in (A.5) and the second in (A.26). This yields

\[
\Phi_{22}(z)\Upsilon_{21}(z) = z\Phi_{22}(z)E_q^*(I - zS_q^*)^{-1}B\nabla \Delta_1^{-1} \\
= A(z) + B(z),
\]

where

\[
A(z) = zE_p^*(I - zS_p^*)^{-1}AE_qE_q^*(I - zS_q^*)^{-1}B\nabla \Delta_1^{-1},
\]

\[
B(z) = z\Theta(z)E_p^*T_0^*S_p\Lambda M(I - zM)^{-1}zE_q^*(I - zS_q^*)^{-1}B\nabla \Delta_1^{-1}.
\]

Again using the identity in (A.30) we obtain

\[
zM(I - zM)^{-1}E_qE_q^*(I - zS_q^*)^{-1} = (I - zM)^{-1} - (I - zS_q^*)^{-1}.
\]

This yields

\[
B(z) = \Theta(z)E_p^*T_0^*S_p\Lambda (I - zM)^{-1}B\nabla \Delta_1^{-1} + \\
- \Theta(z)E_p^*T_0^*S_p\Lambda (I - zS_q^*)^{-1}B\nabla \Delta_1^{-1}.
\]
Recall that $\Phi_{21}$ is given by the third identity in (A.5). If follows that

$$
\Phi_{21}(z) + B(z) = \Theta(z)\Delta_1 - \Theta(z)E_k^*T_\Theta^*S_p\Lambda(I - zS_q^*)^{-1}B_\varphi\Delta_1^{-1}.
$$

Hence

$$
\Upsilon_{11}(z) = \Phi_{21}(z) + \Phi_{22}(z)\Upsilon_{21}(z) = \Phi_{21}(z) + A(z) + B(z)
= zE_p^*(I - zS_q^*)^{-1}\Lambda E_q^*E_k^*(I - zS_q^*)^{-1}B_\varphi\Delta_1^{-1} +
$$

$$
\Theta(z)\Delta_1 - \Theta(z)E_k^*T_\Theta^*S_p\Lambda(I - zS_q^*)^{-1}B_\varphi\Delta_1^{-1}.
$$

(A.31)

To get the first identity in (A.31) we have to do one additional step. Note that $I = (I - zS_q^*)^{-1} - zS_q^*$. Hence

$$
E_k^*T_\Theta^*S_p\Lambda(I - zS_q^*)^{-1}B_\varphi\Delta_1^{-1} =
$$

$$
= E_k^*T_\Theta^*S_p\Lambda(I - zS_q^*)^{-1}(I - zS_q^*) - zS_q^*B_\varphi\Delta_1^{-1} +
$$

$$
- zE_k^*T_\Theta^*S_p\Lambda(I - zS_q^*)^{-1}S_q^*B_\varphi\Delta_1^{-1}.
$$

Next, using the definitions of $B_\varphi$ and $\Delta_1$ in Theorem A.4 we have

$$
E_k^*T_\Theta^*S_p\Lambda B_\varphi\Delta_1^{-1} = E_k^*T_\Theta^*S_p\Lambda(I - \Lambda^*\Lambda)^{-1}\Lambda^*S_q^*T_\Theta\Lambda E_k\Delta_1^{-1}
$$

$$
= (\Delta_1^2 - I_k)\Delta_1^{-1} = \Delta_1 - \Delta_1^{-1}.
$$

It follows that

$$
\Theta(z)E_k^*T_\Theta^*S_p\Lambda(I - zS_q^*)^{-1}B_\varphi\Delta_1^{-1} =
$$

$$
= \Theta(z)\Delta_1 - \Theta(z)\Delta_1^{-1} - z\Theta(z)E_k^*T_\Theta^*S_p\Lambda(I - zS_q^*)^{-1}S_q^*B_\varphi\Delta_1^{-1}.
$$

Using the latter identity in (A.31), we obtain the first identity in (A.31). \qed

Comment on the Toeplitz corona problem. The Toeplitz corona problem can be reduced to the special case of the Leech problem where $q = m$ and $K$ is identically equal to $I_m$. In that case the solvability condition is that $T_GT_G^* \geq I$, and thus $T_GT_G^*$ is strictly positive. Being a special case of the Leech problem, the Toeplitz corona problem can be formulated as a commutant lifting problem of the form considered in this section, where $\Lambda = T_G^*(T_GT_G^*)^{-1}$ viewed as an operator mapping $\ell^2_+(C^m)$ into $\mathcal{H}' = \text{Im} T_G^*$. Note that in this case $\Lambda$ is an invertible contraction.

**Proposition A.5.** Let $\Lambda$ be an invertible contraction mapping $\ell^2_+(\mathbb{C}^m)$ into $\mathcal{H}' = \ker T_G^*$, with $\Theta \in H^\infty_{p\times k}$ an inner function, and assume that $\Lambda$ intertwines $S_q$ with the compression of $S_p$ to $\mathcal{H}'$. Then there exists a function $G \in H^\infty_m\times p$ such that $T_G$ is right invertible, the space $\mathcal{H}' = \text{Im} T_G^*$, and $\Lambda = T_G^*(T_GT_G^*)^{-1}$ viewed as an operator mapping $\ell^2_+(\mathbb{C}^m)$ into $\mathcal{H}'$. In fact, $T_G = \Lambda^{-1}\Pi'$, where $\Pi' : \ell^2_+(\mathbb{C}^p) \to \mathcal{H}'$ denotes the orthogonal projection onto $\mathcal{H}'$.

**Proof.** Put $T := \Lambda^{-1}\Pi'$. It suffices to show that $T$ is a Toeplitz operator since clearly $T$ is left invertible, $\text{Im} T^* = \mathcal{H}'$, and

$$
T^*(TT^*)^{-1} = \Pi'\Lambda^{-*}(\Lambda^{-1}\Pi'\Pi'^*\Lambda^{-*})^{-1} = \Pi'\Lambda^{-*}(\Lambda^{-1}\Lambda^{-*})^{-1} = \Pi'\Lambda.
$$

To see that $T$ is Toeplitz, note that $T'\Lambda = \Lambda S_m$ implies $\Lambda^{-1}T' = S_m\Lambda^{-1}$. Using that $S_p$ is an isometric lifting of $T'$, we find

$$
S_mT = S_m\Lambda^{-1}\Pi' = \Lambda^{-1}T'\Pi' = \Lambda^{-1}\Pi'S_p = TS_p.
$$


which proves our claim.

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