On the Equivalence of Affine $s\ell(2)$ and $N=2$ Superconformal Representation Theories*

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There exist two different languages, the $\hat{\mathfrak{sl}}(2)$ and $N=2$ ones, to describe similar structures; a dictionary is given translating the key representation-theoretic terms related to the two algebras. The main tool to describe the structure of $\hat{\mathfrak{sl}}(2)$ and $N=2$ modules is provided by diagrams of extremal vectors. The $\hat{\mathfrak{sl}}(2)$ and $N=2$ representation theories of certain highest-weight types turn out to be equivalent modulo the respective spectral flows.

1. INTRODUCTION

In this talk I address a representation-theoretic problem that originates in constructions as standard in conformal field theory as parafermions and the Kazama–Suzuki (KS) construction, which relate the affine algebra $\hat{\mathfrak{sl}}(2)$ and the $N=2$ superconformal (super-Lie) algebra in two dimensions. The results are conveniently expressed in the language of category theory.

Each of the two algebras has some relation to the bosonic string. The matter part of the string, which furnishes a representation of the Virasoro algebra, is known to be described as the Hamiltonian reduction of $\hat{\mathfrak{sl}}(2)$ [3]. At the same time, dressing the matter theory into a non-critical bosonic string gives rise to the $N=2$ superconformal algebra [3, 4]. On the other hand, one should keep in mind that, as Table 1 shows, there is hardly anything that these two algebras appear to have in common as regards their structure.

As regards the $N=2$ algebra, let me also note that besides its appearance in the bosonic string (hence in all other string theories, in that case as a subalgebra of larger superalgebras [3, 4]), it is the starting point in the construction of $N=2$ strings [3, 4, 8, 13, 11], which have recently been suggested to play an important role in M-theory [12].

In this talk, I describe some of the results of a work with Boris Feigin and Ilya Tipunin [1], in which certain categories of representations of the affine $\mathfrak{sl}(2)$ and $N=2$ superconformal algebras are shown to be equivalent. The tools essential for the analysis of $N=2$ and $\hat{\mathfrak{sl}}(2)$ modules include diagrams of extremal vectors and the spectral flow transform. The idea to consider extremal vectors was put forward in [3], where it was observed that many representation-theoretic problems can be naturally reformulated in terms of extremal vectors. An independent construction of [14] (and a similar one, [15]) can be considered as a manifestation of this general observation. I will illustrate in this talk several basic points related to extremal diagrams, first of all how their properties describe the structure of submodules of a given module (in particular, its (sub)singular vectors). I consider such $\hat{\mathfrak{sl}}(2)$ and $N=2$ modules that have isomorphic extremal diagrams, which results in the equivalence of certain categories built out of these modules. This requires introducing more general $\hat{\mathfrak{sl}}(2)$ modules than those usually considered.

The modules over the $N=2$ algebra that are generally [17] viewed as ‘standard’ Verma modules have an infinite number of equivalent highest-weight-like states. The affine $\mathfrak{sl}(2)$ modules that correspond to these $N=2$ modules, too, have an infinite number of ‘almost-highest-weight’ states. They are called the ‘relaxed’ Verma modules, since they differ from the standard Verma modules by somewhat ‘relaxed’ highest-weight conditions. The ordinary $\mathfrak{sl}(2)$ Verma modules consti-

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tute a class of submodules of relaxed Verma modules. Back to $N=2$, the corresponding ‘smaller’ class of modules are the so-called ‘topological’ [16, 14] (in fact, chiral) $N=2$ modules. It thus turns out that modules that look ‘standard’ on the $N=2$ side correspond to ‘less standard’ $\hat{s}lu(2)$ modules, and vice versa, which is one of the reasons for a ‘proliferation’ of different types of modules I am going to deal with. I will also have to twist (spectral-flow transform) both the $\hat{s}lu(2)$ and $N=2$ modules in order to compare highest-weight-type representation theories of the two algebras, since the nature of the correspondence between the two representation theories is such that it necessarily involves twisted modules even if one starts with ‘untwisted’ ones.

The main results are the pairwise equivalences of the respective corners of the diagrams of categories of $\hat{s}lu(2)$ representations on the one hand and $N=2$ representations on the other hand:

\[
\begin{array}{ccc}
\text{CHW} & \sim & \text{CRHW} \\
\text{CTOP} & \sim & \text{CMHW} \\
\text{CVER} & \sim & \text{CRVER} \\
\hat{s}lu(2) & & N=2 \\
\text{CTVER} & \sim & \text{CMVER}
\end{array}
\]

The categories are described as follows. ‘C’ always stands for chains, which I define below, of modules from the respective categories; $\text{VER}$ consists of the usual Verma modules over the affine $\hat{s}lu(2)$ and all their images under the spectral flow (i.e., the twisted Verma modules). Category $\text{HW}$ of twisted highest-weight-type modules is the corresponding analogue of the $\text{O}$ category: it is derived from $\text{VER}$ by taking all possible factor modules and ‘gluing’ to each other different modules (with the same twist); in what follows, I give an intrinsic definition of this category using a criterion that is invariant under twisting. Further, $\text{RVER}$ are the relaxed Verma modules, which differ from the usual Verma modules by one missing annihilation condition imposed on the highest-weight vector and as a result possess infinitely many ‘relaxed-highest-weight’ vectors. Then, $\text{RHW}$ is the corresponding analogue of the $\text{O}$ category of modules of the relaxed-highest-weight type.

On the $N=2$ side, $\text{TVER}$ are topological Verma modules and all their spectral flow transforms [16, 14], while $\text{TOP}$ are the corresponding topological-highest-weight type modules. The ‘massive’ Verma module category $\text{MVER}$ consists of all possible twists of those modules over the $N=2$ algebra that are commonly viewed as the “standard” $N=2$ Verma modules, while $\text{MHW}$ is made up of modules of the same highest-weight type, but not necessarily Verma modules.

Let me point out that, even though this is not indicated explicitly in the names of the categories, each of the above categories includes twisted modules (along with untwisted ones).

Taking chains of modules from these categories makes the respective corners of the two squares equivalent, e.g. $\text{CVER} \sim \text{CTVER}$ and, at the same time, $\text{CTVER} \sim \text{CVER}$, where the $\sim$ arrows are of course not intertwining operators, nor any kind of morphisms of modules, but rather, functors. Thus, any two modules related by a morphism in one of the categories are $\sim$-mapped into modules related by a morphism in the other category, and the claim of equivalence means in particular that whatever properties a chosen morphism may have (embedding, projection, . . . ), these are then preserved, while the composition of the direct and the inverse arrows takes any object (a representation) into an isomorphic object.
As long as submodules of Verma modules are associated with singular vectors, a part of the statement amounts to the isomorphism between singular vectors in the respective Verma modules. Singular vectors in topological $N=2$ modules allow a $1:1$ mapping into (well-known) singular vectors in $\hat{sl}(2)$ Verma modules (as was claimed in [26]); with the massive/relaxed Verma modules, this is also true, but the structure of singular vectors is more involved, and I will discuss it briefly for the $\hat{sl}(2)$ case (see [1] for the details).

The point is that a given singular vector does not necessarily generate a maximal submodule. This situation can be described in terms of subsingular vectors: constructing a system of sub-, subsub-, ...singular vectors is nothing but a way to describe the system of maximal submodules and of (non-maximal) submodules thereof generated by vectors of a particular type (those annihilated by a chosen set of operators from the algebra).

However, the structure of submodules can alternatively be described by specifying those vectors that generate maximal submodules. I will still call these vectors singular (and thus would no longer need the notion of subsingular vectors), for the following reasons: for the algebras under consideration, the vectors that generate maximal submodules turn out to satisfy vanishing relations that are nothing but the spectral flow transform of the ‘standard’ annihilation conditions (i.e., of those imposed on highest-weight vectors in untwisted modules). Moreover, as I have already mentioned, the relaxed/massive Verma modules can equally well be generated from an infinite number of vectors each of which satisfies precisely the thus twisted highest-weight conditions. This makes it natural to consider highest-weight conditions up to the spectral flow transform and to impose such highest-weight conditions on singular vectors. Among singular vectors understood in this broader sense, then, one can always find those that generate maximal submodules, even though the ‘standard’ singular vector may generate a smaller submodule.

As we will see, it is advantageous to consider the entire extremal diagram generated out of given highest-weight state and likewise, of a given singular vector. It turns out that vectors that generate maximal submodules – which are singular in the broader sense that I adopt from now on – and the ‘standard’ singular vectors belong to the same extremal diagram, and moreover, it is the properties of the extremal diagram that are responsible for whether or not the ‘standard’ singular vector would generate a maximal submodule. In general, all the states in a given extremal diagrams satisfy the same highest-weight conditions up to the spectral flow transform, however stronger annihilation conditions may occur for some states in the diagram, in which case the vectors in the extremal diagram may be divided into those which do, and which do not, generate a maximal submodule.

The equivalence claim for chains of modules means that $N=2$ and $\hat{sl}(2)$ representation theories are equivalent modulo the respective spectral flow transforms.

2. The algebras

2.1. The affine $\hat{sl}(2)$ algebra

The structure of $\hat{sl}(2)$ Verma modules is conveniently encoded in the extremal diagram

\[
\cdots \quad J^-_0 \rightarrow J^-_1 \rightarrow J^-_\infty \rightarrow J^+_1 \rightarrow J^+_\infty \rightarrow \cdots
\]  

(2.1)

which expresses the fact that $J^+_1$ and $J^-_0$ are the highest-level operators that do not yet annihilate the highest-weight state $\omega$. All the other states in the module should be thought of as lying in the interior of the wedge. In these conventions, e.g., $J^0_{-1}$ is represented as a downward vertical arrow. An important point is that the diagram is angle-shaped, which reformulates the following property: take a state $|v\rangle$ represented by a point inside the angle and, for a fixed $n \in \mathbb{Z}$, consider all the states $(J^+_n)^i |v\rangle$, $i \in \mathbb{N}$, and $(J^-_n)^i |v\rangle$, $i \in \mathbb{N}$. These states fill out a straight line in the diagram, which would necessarily intersect the edge of the diagram, and therefore, in one of the directions, the action with either $J^+_n$ or $J^-_n$ terminates (vanishes) after a certain number of steps.
Automorphisms of the affine $\mathfrak{sl}(2)$ algebra are the canonical involution and the spectral flow

$$\mathcal{U}_\theta : J^+_n \mapsto J^+_{n+\theta}, \quad J^-_n \mapsto J^-_{n-\theta}, \quad J^0_n \mapsto J^0_n + \frac{k}{2} \delta_{n,0},$$

(2.2)

where $\theta \in \mathbb{Z}$, and $k$ is the level (I assume $k \neq -2$ in what follows). In general, Verma over the $\hat{\mathfrak{sl}}(2)$ algebra are not invariant under the spectral flow and are mapped into ‘twisted’ modules. A twisted Verma module $\mathcal{M}_{j,k,\theta}$ is freely generated by

$$J^+_\leq j, \quad J^-\leq j, \quad J^+\leq 1$$

from a “twisted” highest-weight vector $|j, k; \theta\rangle_{\mathfrak{sl}(2)}$, defined by the conditions

$$J^+_\leq j, \quad J^-\leq j, \quad J^+\leq 1$$

(2.3)

Thus $\theta$ measures the gap between the mode numbers of $J^+$ and $J^-$ that annihilate the vacuum. The respective extremal diagrams are ‘rotations’ of (2.1). A crucial fact is that any straight line would still intersect the edge of the diagram, and thus the argument discussed below (2.1) still applies. I identify $|j, k\rangle_{\mathfrak{sl}(2)} = |j, k; 0\rangle_{\mathfrak{sl}(2)}$, and denote $\mathcal{M}_{j,k} = \mathcal{M}_{j,k,0}$.

All possible (integral) twists of Verma modules constitute the category $\mathcal{VE}\mathcal{R}$. However, already in the untwisted case, many (if not all) interesting representations are not Verma modules, but rather can be obtained from Verma modules by taking factors and ‘gluing’. This gives the category $\mathcal{O}$ (see [13]), in which the Verma modules are ‘universal’ objects in the sense that any irreducible representation is a factor of a Verma module. The standard definition of the category $\mathcal{O}$ singles out only the untwisted Verma modules (those with $\theta = 0$ in (2.3)). A remarkable fact is that there exists an intrinsic definition of the category $\mathcal{HW}$ of highest-weight type modules, which would include the twisted modules. First of all, to formalize the above observations, let $|X\rangle$ be an element of a module over the affine $\mathfrak{sl}(2)$ algebra and let us fix an integer $\theta$. For $J$ being either $J^+$ or $J^-$, we say that the $J_\theta$-chain terminates on $|X\rangle$, and write $\langle J_\theta |_\infty^+ |X\rangle = 0$, if

$$\exists N \in \mathbb{Z}, \quad n \geq N : \langle J_\theta |^n |X\rangle = 0.$$ 

Further, all the modules in what follows are assumed graded with respect to the Cartan subalgebra of the respective algebra. I will use the criterion of terminating chains to define categories of highest-weight-type representations. This and similar criteria will be applied to $J^+$ and $J^-$ generators in the $\mathfrak{sl}(2)$ case and to $G$ and $Q$ in the $N = 2$ case. For brevity, I will give explicitly only those parts of definitions that have to do with twisting, omitting explicit stipulations of the standard $\mathcal{O}$-category requirements with respect to the remaining generators ($J^0$, and $L$ and $H$ respectively), which state that acting with the annihilation operators spans out a finite-dimensional space. Then, an $\mathfrak{sl}(2)$ module $\mathcal{U}$ belongs to the category $\mathcal{HW}$ of $\mathfrak{sl}(2)$ twisted highest-weight-type representations if, for any element $|X\rangle$ of $\mathcal{U}$, $\forall n \in \mathbb{Z}$

either $\langle J_\theta^+ |_\infty^+ |X\rangle = 0$

or $\langle J_\theta^- |_\infty^+ |X\rangle = 0$ 

(2.4)

(and, in accordance with the above remarks, it is tacitly assumed that $\mathcal{U}$ is graded and that acting with $\langle J_\theta^m |^n, m, n \geq 1$, on any vector produces a finite-dimensional space).

Singular vectors in $\mathfrak{sl}(2)$ Verma modules $\mathcal{M}_{j,k}$ are defined in the standard way. To explicitly construct singular vectors, one introduces the objects $(J^-_1)^\alpha$ and $(J^+_0)^\alpha$ that implement the action of generators of the affine Weyl group on the space of highest-weights, see [14] for the details. These objects correspond to reflections with respect to two positive simple roots of the affine $\mathfrak{sl}(2)$ algebra. In fact $(J^-_1)^\alpha$ and $(J^+_0)^\alpha$ define the following Weyl group action on the line $k = \text{const}$ in the $kj$ plane of highest-weights:

$$\begin{align*}
(J^-_1)^{2j+1} : \langle j, k\rangle_{\mathfrak{sl}(2)} & \mapsto \langle -1 - j, k\rangle_{\mathfrak{sl}(2)} , \\
(J^+_1)^{k+1-2j} : \langle j, k\rangle_{\mathfrak{sl}(2)} & \mapsto \langle k + 1 - j, k\rangle_{\mathfrak{sl}(2)} .
\end{align*}$$

(2.5)

The action of $(J^-_1)^\alpha$ and $(J^+_0)^\alpha$ can be extended from the set of highest-weight vectors to the Verma modules over these vectors. Then,
Theorem 2.1

I. ([20]) A singular vector exists in the module $M_{j,k}$ iff $j = j^+(r,s,k)$ or $j = j^-(r,s,k)$, where
\[
\begin{align*}
    j^+(r,s,k) &= \frac{r + 1}{2} - (k + 2) \frac{1}{k} r \quad \text{for} \quad r, s \in \mathbb{Z} \quad k \in \mathbb{C} \\
    j^-(r,s,k) &= -\frac{r + 1}{2} + (k + 2) \frac{1}{k} r \\
\end{align*}
\]  

II. ([19]) All singular vectors $|S_{r,s,k}^{\text{MF}}\rangle$ in the Verma module $M_{j,k}$ over the affine $s(2)$ algebra are given by the explicit construction:
\[
|S_{r,s,k}^{\text{MF}}\rangle = \left((J_0^+)^{r+(s-1)(k+2)}(J_1^+)^{r+(s-2)(k+2)} \ldots \right. \\
\left. \cdot (J_{-1}^-)^{-(s-2)(k+2)}(J_0^-)^{-(s-1)(k+2)} \right) |j^+(r,s,k),k\rangle_{s(2)},
\]
\[
|S_{r,s,k}^{\text{MF}}\rangle = \left((J_0^-)^{r+(s-1)(k+2)}(J_0^+)^{r+(s-2)(k+2)} \ldots \right. \\
\left. \cdot (J_{0}^-)^{-(s-2)(k+2)}(J_0^+)^{-(s-1)(k+2)} \right) |j^-(r,s,k),k\rangle_{s(2)}.
\]

Singular vectors in twisted Verma modules follow by applying the spectral flow transform to (2.3).

A more general class of affine $s(2)$ modules can be introduced by relaxing the annihilation conditions (2.3): For $\theta \in \mathbb{Z}$, a relaxed twisted Verma module $R_{j,\Lambda,k,\theta}$ is freely generated by the operators $J_{\geq \theta}^+$, $J_{\leq -\theta}$, and $J_0^\theta$ from the state $|j,\Lambda,k,\theta\rangle_{s(2)}$ that satisfies the annihilation conditions
\[
J_{\geq \theta+1}^+ |j,\Lambda,k,\theta\rangle_{s(2)} = J_{\leq -\theta+1}^+ |j,\Lambda,k,\theta\rangle_{s(2)} = 0.
\]

and
\[
(J_0^\theta + \frac{k}{4} \theta |j,\Lambda,k,\theta\rangle_{s(2)} = j |j,\Lambda,k,\theta\rangle_{s(2)},

(J_{\geq \theta} J_{\leq -\theta} + (k + 2) \theta (j - \frac{k}{4} \theta)) |j,\Lambda,k,\theta\rangle_{s(2)} = \Lambda |j,\Lambda,k,\theta\rangle_{s(2)}.
\]

The corresponding extremal diagram opens up to the straight angle; in the untwisted case $\theta = 0$ it thus becomes
\[
\begin{align*}
\cdots \quad J_0^- \quad J_0^+ \quad J_0^- \quad J_0^+ \quad J_0^- \quad J_0^+ \quad \cdots (2.10)
\end{align*}
\]

The state marked with $\ast$ is the above $|j,\Lambda,k,\theta|_{s(2)}$. The other states $|j,\Lambda,k,\theta|_{n,s(2)}$, $n \in \mathbb{Z}$, from the extremal diagram are
\[
|j,\Lambda,k,\theta|_{n,s(2)} = \begin{cases} 
    (J_{-\theta}^-)^{-n} |j,\Lambda,k,\theta|_{s(2)}, & n < 0, \\
    (J_0^+)^n |j,\Lambda,k,\theta|_{s(2)}, & n > 0,
\end{cases}
\]

with $|j,\Lambda,k,\theta|_{0,s(2)} = |j,\Lambda,k,\theta|_{s(2)}$. I also define $|j,\Lambda,k,\theta|_{n,s(2)} = |j,\Lambda,k,\theta|_{s(2)}$.

In the generic case, one can travel both ways along the extremal diagram: for example, the ‘untwisted’ diagram (2.10) ($\theta = 0$) acquires a ‘fat’ form
\[
\begin{align*}
\cdots \quad J_0^- \quad J_0^+ \quad J_0^- \quad J_0^+ \quad J_0^- \quad J_0^+ \quad \cdots (2.11)
\end{align*}
\]

where the composition of the direct and the inverse arrows results in each case only in a factor:

$n \leq 0$:
\[
J_0^- |j,\Lambda,k|_{n,s(2)} = |j,\Lambda,k|_{n-1,s(2)},
\]
\[
J_0^+ |j,\Lambda,k|_{n-1,s(2)} = (\Lambda - n(n-1) - 2(n-1)j) \cdot |j,\Lambda,k|_{n,s(2)},
\]

$n \geq 0$:
\[
J_0^- |j,\Lambda,k|_{n+1,s(2)} = |j,\Lambda,k|_{n+1,s(2)},
\]
\[
J_0^+ |j,\Lambda,k|_{n+1,s(2)} = (\Lambda - n(n+1) - 2nj) \cdot |j,\Lambda,k|_{n,s(2)}
\]

However, this factor may vanish for some values of the parameters and this gives rise to standard Verma submodules. Thus, whenever the parameters are such that, e.g., $J_0^+ \approx 0$ at a certain step, properties of the extremal diagram change, and one cannot come back to the $\ast$ state by acting with $J_{\pm}^+$:
\[
\begin{align*}
J_0^+ |j,\Lambda,k|_{n,s(2)} = |j,\Lambda,k|_{n+1,s(2)},
\end{align*}
\]

\[
\begin{align*}
J_0^- |j,\Lambda,k|_{n+1,s(2)} = (\Lambda - n(n+1) - 2nj) \cdot |j,\Lambda,k|_{n,s(2)}
\end{align*}
\]

One keeps on acting with $J_{\pm}^+$ instead (one mode down), and thus the extremal diagram becomes
\[
\begin{align*}
\cdots \quad J_0^- \quad J_0^+ \quad J_0^- \quad J_0^+ \quad J_0^- \quad J_0^+ \quad \cdots (2.13)
\end{align*}
\]
where in the subdiagram one recognizes (the fat form of) the extremal diagram (2.11). Therefore, any Verma module can be thought of as a submodule of a relaxed Verma module.

Similarly, one may have \( J_0^- \approx 0 \) at a certain stage in the diagram (2.11),
\[
n \geq 1, \quad \Lambda = n(n - 1) + 2(n - 1)j \implies J_0^- | j, \Lambda, k | n \rangle_{s\ell(2)} = 0.
\] (2.14)

Then the branching of the extremal diagram is a mirror image of (2.11), and the Verma submodule is given by the spectral flow transform with \( \theta = 1 \) of a standard Verma module.

By a mere application of the spectral flow, the above results reformulate for twisted relaxed Verma modules.

As in the standard Verma case, the category \( \mathcal{RVER} \) of all twisted relaxed Verma modules can be extended to a larger category \( \mathcal{RHW} \) of arbitrary (twisted) relaxed-highest-weight type modules. Now that the angle in the extremal diagrams has opened up to the straight angle, there would certainly exist in the extremal diagrams straight lines infinite on both sides. However, a condition on the class of modules can still be given in the form of the requirement that, starting with any vector from a given module, the action with \( \theta \) would certainly exist in the extremal diagrams parallel to the edge of the diagram, with every state in the subdiagram satisfying the relaxed highest-weight conditions (2.8); interesting things start to happen when some of these states satisfy stronger, Verma, highest-weight conditions.

Given a state \( | j, \Lambda, k \rangle \), consider \( (J_0^-)^{-\mu} | j, \Lambda, k \rangle \) with \( \mu = j^- (r, s, k) - j \). Whenever \( \Lambda \) is chosen as \( \Lambda (r, s, j, k) \) where
\[
\Lambda (r, s, j, k) = \frac{1}{\lambda} (\frac{1}{4} - 2j - r + 2s + ks)(1 + 2j - r + 2s + ks),
\]
the ‘continued’ state \( (J_0^-)^{-\mu} | j, \Lambda, k \rangle_{s\ell(2)} \) would satisfy the Verma highest-weight conditions, with the spin given by \( j^- (r, s, k) \). Therefore the usual MFF\(^{-}\) singular vector can be constructed on this state, as
\[
\mathcal{MFF}^- (r, s, k) (J_0^-)^{-j^- (r, s, k)} | j, \Lambda (r, s, j, k) \rangle_{s\ell(2)}
\]
where \( \mathcal{MFF}^- \) is the singular vector operator (read off by dropping the highest-weight state in (2.7)). This has to be mapped back to the original relaxed Verma module. In particular, no non-integral powers of \( J_0^- \) should remain, which is achieved by acting on (2.11) with \( (J_0^-)^{-j^- (r, s, k)} - j^+ N \), where \( N \) is an integer. However, to be left after the rearrangements with only positive integral powers, the integer \( N \) has to be \( \geq r + rs \). I thus choose
\[
\Sigma^- (r, s, j, k) = \left( (J_0^-)^{-j^- (r, s, k)} - j^+ r + rs \mathcal{MFF}^- (r, s, k) \right) \cdot (J_0^-)^{-j^- (r, s, k)} | j, \Lambda (r, s, j, k) \rangle_{s\ell(2)}
\]
as a representative of the singular vector in the relaxed Verma module \( R_{j, \Lambda (r, s, j, k), k, 0} \). The rules for dealing with non-integral powers are directly
analogous to those used in the standard, Verma, MFF construction.

Similarly, for \( \mu = j^+(r, s + 1, k) - j \), the state \((J_0^+)^{1+j^+(r,s+1,k)-j}|j, \Lambda(r,s,j,k),k\rangle_{s\ell(2)}\) is formally a Verma highest-weight state twisted by the spectral flow transform with \( \theta = 1 \), and thus the singular vector becomes

\[
\Sigma^+(r,s,j,k) = (J_0^+)^{j+r+s-1+j^+(r,s+1,k)} \mathcal{MFF}^{+,1}(r,s,k) \cdot (J_0^+)^{1+j^+(r,s+1,k)-j}|j, \Lambda(r,s,j,k),k\rangle_{s\ell(2)}
\]

These singular vectors can be acted upon with \( J_0^\pm \), which may allow one to map \( \Sigma^+ \) and \( \Sigma^- \) to the same grade. Whenever this is possible, the two vectors mapped to the same grade are linearly independent, and thus there exists a unique relaxed singular vector (in terms of extremal diagrams, \( \Sigma^+ \) and \( \Sigma^- \) then generate the same extremal subdiagram). However, the action of \( J_0^+ \) or \( J_0^- \) may give zero at some step, which would mean encountering a Verma highest-weight state in the subdiagram representing the singular vector. In that case, it may still be possible to act with \((J_0^-)^{-1}\) or \((J_0^+)^{-1}\) respectively, the latter being understood as one of the ‘continued’ operators from (2.5). The extremal diagram becomes, schematically,

\[
\begin{array}{c}
\Sigma^- \\
MFF \downarrow (J_0^-)^{-1} \\
\Sigma^+ \\
\end{array}
\]

where \( \Sigma^- \) happens to lie inside the Verma submodule built on the usual MFF singular vector; the latter is necessarily embedded into a Verma submodule of the type of the one pictured in (2.13).

A similar ‘slope’ may further be encountered when moving on the left from \( \Sigma^+ \), by acting on it with powers of \( J_0^- \). Whenever this happens, a further Verma highest-weight state would appear in the top floor of the diagram. Yet it may still be possible to generate the entire lower floor from \( \Sigma^+ \) by acting with \((J_0^+)^{-1}\) where the action of \( J_0^- \) vanishes.

By a generalized \( J_0^+ \)-descendant (respectively, generalized \( J_0^- \)-descendant) of a state \( |v\rangle \) in the relaxed Verma module \( R \), I will mean any state that can be obtained from \( |v\rangle \) by an arbitrary number of the following steps i) acting with \( J_0^\pm \) (resp. \( J_0^- \)) whenever the result is non-vanishing, and ii) acting with \((J_0^\pm)^{-1}\) (resp. \((J_0^-)^{-1}\)) whenever the result is defined as an element in \( R \) and step i) cannot be applied.

**Theorem 2.2**

I. In the general position, singular vectors \( \Sigma^+(r,s,j,k) \) and \( \Sigma^-(r,s,j,k) \) are different representatives for the same singular vector in the relaxed Verma module \( R_{j,\Lambda(r,s,j,k),k} \); the generalized \( J_0^\pm \)-descendants of \( \Sigma^+(r,s,j,k) \) and \( \Sigma^-(r,s,j,k) \) that are in the same grade are proportional to each other.

II. Whenever

\[
j = -\frac{1}{2}(1+m+n), \quad k + 2 = \frac{m+n+r}{2},
\]

with \( n \in \mathbb{N} \) and \( m \in \mathbb{N} \), there exist generalized \( J_0^\pm \)-descendants of \( \Sigma^+ \) and \( \Sigma^- \) in the same grade that are linearly independent.

In the second case, there thus exist two linearly independent singular vectors in the same grade. These linearly independent singular vectors then reside in a section of the length \( 2r - m + n \) of the lower floor of the extremal diagram. It follows that, whenever two different singular vectors in the same grade appear, these are necessarily the usual MFF singular vectors in the corresponding Verma submodules.

I refer to [4] for more pictures and a detailed description of all the possible cases.
2.2. Auxiliaries

I will need a fermionic system (bc ghosts), defined in terms of operator products as
\[ B(z) C(w) = \frac{1}{z-w} \text{ with the energy-momentum tensor } T^{GH} = -B \partial C. \]
Denote by \( \Omega \) the module generated from the vacuum \( |0\rangle_{GH} \) defined by the conditions
\[
C_{\geq 1} |0\rangle_{GH} = B_{\geq 0} |0\rangle_{GH} = 0 \quad (2.16)
\]
The thus defined vacuum is an \( sl_2 \)-invariant state \([21]\); one can choose other highest-weight states, \( |\lambda\rangle_{GH} \), which belong to the same module and are determined by
\[
C_{\geq -\lambda} |\lambda\rangle_{GH} = B_{\geq \lambda} |\lambda\rangle_{GH} = 0 \quad (2.17)
\]
The states \( |\lambda\rangle_{GH} \) with different \( \lambda \) can be connected by means of operators \( c(\mu, \nu) \) and \( b(\mu, \nu) \) which are products of fermionic modes
\[
c(\mu, \nu) = \prod_{n=1}^{\nu-\mu+1} C_{\mu+n}, \quad b(\mu, \nu) = \prod_{n=1}^{\nu-\mu+1} B_{\nu+n},
\]
with \( \nu - \mu + 1 \in \mathbb{N} \), and which map a vector \( |\lambda\rangle_{GH} \) as follows
\[
c(-\lambda - \ell + 1, -\lambda) : |\lambda\rangle_{GH} \mapsto |\lambda + \ell\rangle_{GH}, \quad \ell \in \mathbb{N}
\]
b\((-\lambda - \ell, \lambda - 1) : |\lambda\rangle_{GH} \mapsto |\lambda - \ell\rangle_{GH},
\]
Let me now introduce a “Liouville” scalar which will be used to ‘invert’ the KS mapping. This is just a free scalar, called ‘Liouville’ for its signature, \( \phi(z) \phi(w) = -\ln(z-w) \). I define vertex operators, referred to as ‘antifermions’, \( \psi = e^\phi \) and \( \psi^* = e^{-\phi} \). The energy-momentum tensor is taken to be
\[
T_\phi = -\tfrac{1}{2} \partial \phi \partial \phi + \tfrac{1}{2} \partial^2 \phi. \quad (2.18)
\]

2.3. \( N=2 \)

Nonvanishing commutation relations of the \( N=2 \) superconformal algebra \( \mathcal{A} \) can be chosen as
\[
[L_m, L_n] = (m-n)L_{m+n}, \quad [H_m, H_n] = \frac{c}{3} m \delta_{m+n},
\]
\[
[L_m, G_n] = (m-n)G_{m+n}, \quad [H_m, G_n] = G_{m+n},
\]
\[
[L_m, Q_n] = -nQ_{m+n}, \quad [H_m, Q_n] = -Q_{m+n},
\]
\[
[L_m, H_n] = -nH_{m+n} + \frac{c}{6}(m^2 + m)\delta_{m+n},
\]
\[
\{G_m, Q_n\} = 2L_{m+n} - 2nH_{m+n} + \frac{c}{3}(m^2 + m)\delta_{m+n}, \quad m, n \in \mathbb{Z}.
\]

When applied to the generators of \([21]\), the spectral flow transform \([27, 28]\), \( U_\theta \) acts as
\[
L_n \mapsto L_n + \theta H_n + \frac{c}{6}(\theta^2 + \theta)\delta_{n,0},
\]
\[
H_n \mapsto H_n + \frac{c}{6} \theta \delta_{n,0}, \quad \theta \mapsto \theta + \theta_0 \quad (2.20)
\]
\[
Q_n \mapsto Q_{n-\theta}, \quad \theta \mapsto \theta \quad (2.20)
\]
This gives the algebra \( \mathcal{A}_\theta \), which is isomorphic to the \( N=2 \) superconformal algebra and whose generators \( L^\theta_n, Q^\theta_n, H^\theta_n \) and \( G^\theta_n \) can be taken as the RHSs of \((2.20)\). One thus obtains the Neveu–Schwarz and Ramond \( N=2 \) algebras, as well as the algebras in which the fermion modes range over \( \pm \theta + \mathbb{Z}, \theta \in \mathbb{C} \).

Now I define twisted topological \([3]\) Verma modules \( \mathcal{V}_{h,t;\theta} \) over the \( N=2 \) algebra. This is the module generated from the topological highest-weight vector \( |h, t; \theta\rangle_{\text{top}} \) defined by
\[
L_m|h, t; \theta\rangle_{\text{top}} = 0, \quad H_m|h, t; \theta\rangle_{\text{top}} = 0, \quad m \geq 1,
\]
\[
Q_\lambda|h, t; \theta\rangle_{\text{top}} = 0, \quad \lambda \in -\theta + \mathbb{N}_0, \quad \theta \in \mathbb{Z}; \quad (2.21)
\]
and for the ‘Cartan’ generators,
\[
(H_0 + \frac{c}{6} \theta) |h, t; \theta\rangle_{\text{top}} = h |h, t; \theta\rangle_{\text{top}},
\]
\[
(L_0 + \theta H_0 + \frac{c}{6}(\theta^2 + \theta)) |h, t; \theta\rangle_{\text{top}} = 0 \quad (2.22)
\]
\[
C |h, t; \theta\rangle_{\text{top}} = \frac{3(t-2)}{c} |h, t; \theta\rangle_{\text{top}}
\]
The \( \theta = 0 \) case describes the ‘ordinary’ topological Verma modules \( \mathcal{V}_{h,t} \equiv \mathcal{V}_{h,t;0} \), with the corresponding topological highest-weight vector \( |h, t\rangle_{\text{top}} \equiv |h, t; 0\rangle_{\text{top}} \).

The extremal diagram of a topological Verma module reads

![Diagram](image)

\( \theta \) The name is inherited from the non-critical bosonic string, where matter vertices can be dressed into \( N=2 \) primaries that satisfy the highest-weight conditions \((2.20)\); in that context, the algebra \([21]\) is viewed as a topological algebra.
An important point here is the existence of a ‘cusp’, i.e. a state that satisfies stronger highest-weight than the other states in the diagram.

Next, I need the concept of terminating fermionic chains. Let \( F \) denote either \( Q \) or \( G \), and \( |X\rangle \) an element of a module over the \( N = 2 \) algebra. Fix also an integer \( n \). We say that the fermionic \( F \)-chain terminates on \( |X\rangle \), and write \( \cdots F_{n-3} F_{n-2} F_{n-1} F_n |X\rangle = 0 \) if \( \exists N \in \mathbb{Z} \), \( N \leq n : F_N F_{n+1} \cdots F_n |X\rangle = 0 \). Now, an \( N = 2 \) module \( \mathcal{U} \) is said to belong to the topological \( N = 2 \) category \( \mathcal{TOP} \) if, for any element \( |X\rangle \) of \( \mathcal{U} \), \( \forall n \in \mathbb{Z} \)

\[
either \cdots Q_{n-3} Q_{n-2} Q_{n-1} Q_n |X\rangle = 0 \quad \text{or} \quad \cdots G_{n-4} G_{n-3} G_{n-2} G_{n-1} |X\rangle = 0
\]

This condition works by excluding those diagrams that have no ‘cusps’ and thus, being wider than the diagram (2.23), necessarily intersect the edge, at which point the fermionic chain terminates.

Positions of topological singular vectors can be obtained \[ \| \text{26} \]. A topological singular vector exists in the topological Verma module \( V_{h,\ell} \) iff either \( h = h^+(r, s, t) \) or \( h = h^-(r, s, t) \), where

\[
h^+(r, s, t) = - \frac{r - 1}{t} + s - 1, \quad r, s \in \mathbb{N} \tag{2.24}
\]

\[
h^-(r, s, t) = \frac{r + 1}{t} - s \]

I now introduce two operators \( g(\mu, \nu) \) and \( q(\mu, \nu) \), with \( \mu, \nu \in \mathbb{C} \), that represent the action of two \( N = 2 \) Weyl group” generators when \( \mu \) and \( \nu \) are special (see \[ \| \text{43} \] for the details). These operators act on the plane \( t = \text{const} \) as follows

\[
g(h t + \theta - 1, \theta - 1) : |h, t; \theta\rangle \mapsto \left| \frac{2}{t} - h, t; h t + \theta - 1\right\rangle \tag{2.25}
\]

\[
g(-(h + 1)t - \theta + 1, -\theta - 1) : |h, t; \theta\rangle \mapsto \left| \frac{2}{t} - 2 - h, t; (h + 1)t + \theta - 1\right\rangle \tag{2.26}
\]

The action of \( g(a, b) \) and \( q(a, b) \) on highest-weight vectors can be extended to the corresponding topological Verma modules, which allows one to explicitly construct singular vectors in the topological Verma modules \( V_{h^+(r, s, t); t} \):

\[
E(r, s, t)^+ = g(-r, (s - 1)t - 1) \cdot
\]

\[
q(-(s - 1)t, r - 1 - t, \ldots) |g((s - 2)t - r, t - 1)\cdot
\]

\[
g(-t, (s - 1)t - t(s - 1)) \cdot
\]

\[
g((s - 1)t - r, -1) \cdot
\]

\[
|E(r, s, t)\rangle^+ = q(-r, (s - 1)t - 1) \cdot
\]

\[
g(-(s - 1)t, r - 1 - t, \ldots) |g((s - 2)t - r, t - 1)\cdot
\]

\[
g(-t, (s - 1)t - t(s - 1)) \cdot
\]

\[
g((s - 1)t - r, -1) \cdot
\]

One also introduces the ‘massive’ \( N = 2 \) Verma modules \( W_{h,\ell;\theta} \), in which the highest-weight states satisfy the following annihilation and eigenvalue conditions:

\[
L_m|h, \ell, t; \theta\rangle = H_m|h, \ell, t; \theta\rangle = 0, \; m \geq 1,
\]

\[
Q_\lambda|h, \ell, t; \theta\rangle = 0, \; \lambda \in -\theta + \mathbb{N}
\]

\[
G_\nu|h, \ell, t; \theta\rangle = 0, \; \nu = \theta + N_0 \tag{2.28}
\]

\[
(H_0 + \frac{\lambda}{6} \theta) |h, \ell, t; \theta\rangle = h |h, \ell, t; \theta\rangle,
\]

\[
(L_0 + \theta H_0 + \frac{\lambda}{6} (2 + \theta)) |h, \ell, t; \theta\rangle = \ell |h, \ell, t; \theta\rangle.
\]

The untwisted module, freely generated from \( |h, \ell, t\rangle = |h, \ell, t; 0\rangle \), is denoted by \( U_{h,\ell,\ell} \). They are called massive because of their property to have a dimension \( \ell \) generally different from zero.

Extemal diagrams of massive Verma modules have the form (in the untwisted case for simplicity)

\[
Q_{-\theta}|h, \ell, t; \theta\rangle = 2|\ell h - \frac{2}{3}, \ell + h - \frac{2}{3}, t; \theta + 1\rangle
\]

\[
G_{\theta - 1}|h, \ell, t; \theta\rangle = 2|\ell - h| h + \frac{2}{3}, \ell - h, t; \theta - 1\rangle
\]

for \( \theta < 0 \) and \( \theta > 0 \) respectively; this may lead to the vanishing result, which gives the conditions for the so-called ‘charged’ singular vectors \[ \| \text{47} \] to appear:

\[
(|h_{\cdots - 2}, \ell_{\cdots - 2}, t_{\cdots - 2} \rangle)
\]

\[
|h_{\cdots 3}, \ell_{\cdots 3} \rangle
\]
Theorem 2.3 A massive Verma module $U_{h,t,t}$ contains a twisted topological Verma submodule iff $\ell = l_{ch}(r, h, t)$, where

$$l_{ch}(r, h, t) = r(h + \frac{t-1}{2}), \quad r \in \mathbb{Z} \setminus \{0\}; \quad (2.31)$$

The corresponding singular vector reads

$$|E(r, h, t)\rangle_{ch} = \begin{cases} Q_r \dots Q_0 |h, l_{ch}(r, h, t), t\rangle, & r \leq -1 \\ G_{-r} \dots G_{-1} |h, l_{ch}(r, h, t), t\rangle, & r \geq 1 \end{cases}$$

These ‘charged’ singular vectors are analogous to Verma points encountered in extremal diagrams of relaxed $\hat{sl}(2)$ modules, see (2.13). Moreover, extremal diagrams of $N = 2$ modules are nothing but a deformation (of straight lines into parabolas) of $\hat{sl}(2)$ extremal diagrams. This follows from the statement of equivalence of categories given in Sect. 4, or can be derived by a case-by-case analysis of all possible branchings of extremal diagrams for each of the two algebras; in view of the advertised result, we omit the analysis of $N = 2$ singular vectors in the cases when extremal diagrams branch and several singular vectors coexist in the module.

The above condition for the fermionic chains to terminate is not satisfied for massive Verma modules. Instead, a fermionic chain terminates whenever it is yet wider than the diagram (2.29), i.e. it contains two or more arrows along the same straight line. Thus, an $N = 2$ module $\mathcal{U}$ belongs to the category $\mathcal{MHW}$ of $N = 2$ modules if, for any element $|X\rangle$ of $\mathcal{U}$, $\forall n \in \mathbb{Z}$,

- either $\ldots Q_{n-3} Q_{n-2} Q_{n-1} Q_n |X\rangle = 0$
- or $G_{-n-1} G_{-n-3} G_{-n-2} G_{-n} |X\rangle = 0$.

3. Kazama–Suzuki and related mappings

The simplest KS construction uses a couple of spin-1 $BC$ ghosts, which allows one to build up the topological algebra generators as $[22, 23, 24, 25]:$

$$Q = CJ^+, \quad G = \frac{2}{k+2} BJ^-, \quad H = \frac{k+2}{k+2} BC - \frac{2}{k+2} J^0,$$

$$T = \frac{1}{k+2} (J^+ J^-) - \frac{k}{k+2} B \partial C - \frac{2}{k+2} BC J^0. \quad (3.1)$$

Generators (3.1) close to the algebra (2.19), the central elements being related by $c = \frac{k}{k+2}$. Thus, eqs. (3.1) define a mapping $F_{KS}: \mathcal{A} \to U\hat{sl}(2)_k \otimes [BC], \quad (3.2)$

where $\mathcal{A}$ is the $N = 2$ algebra (2.19) and $\mathcal{U}$ denotes the universal enveloping (and $[BC]$ is the free fermion theory).

The KS mapping produces also a bosonic current

$$I^+ = \sqrt{\frac{2}{k+2}} (BC + J^0) \quad (3.3)$$

whose modes commute with the $N = 2$ generators (3.1). Its energy-momentum tensor reads

$$T^+ = \frac{1}{2} (I^+)^2 - \frac{1}{2} \frac{1}{\sqrt{2(k+2)}} \partial I^+ \quad (3.4)$$

Expanding as $I^+(z) = \sum_{n \in \mathbb{Z}} I^+_n z^{-n-1}$, I get a Heisenberg algebra $[I^+_n, I^-_m] = m \delta_{m+n,0}$. I define a module $H^+_p$ over the Heisenberg algebra by the highest-weight conditions

$$I_{\geq 1} |p\rangle = 0, \quad I_0 |p\rangle = p |p\rangle \quad (3.5)$$

Under the KS mapping (3.1), one has the identities

$$T_{Sug} + T_{GH} = T + T^+ \quad (3.6)$$

where $T$ is the energy-momentum tensor (3.1)), and

$$J^0 - BC = -2H + \frac{k-2}{\sqrt{2(k+2)}} I^+. \quad (3.7)$$

A mapping in the inverse direction to the KS mapping is constructed using the above ‘antifermions’ $\psi = e^\phi$, $\psi^* = e^{-\phi}$:

$$J^+ = Q \psi, \quad J^- = \frac{1}{3} \frac{1}{c} G \psi^*, \quad J^0 = -\frac{3}{3-c} H + \frac{3}{2} \partial \phi \quad (3.8)$$

For $c \neq 3$, generators (3.8) close to the affine $\hat{sl}(2)$ algebra of the level $k = \frac{3-c}{2}$ where $c$ is the $N = 2$ central charge. Thus, eqs. (3.3) define a mapping $F_{KS}^{-1}: U\hat{sl}(2) \to \mathcal{U} \mathcal{A} \otimes [\psi \psi^*]. \quad (3.9)$

One also has a free scalar with signature $-1$, whose modes commute with the $\hat{sl}(2)$ generators:

$$I^- = \sqrt{\frac{3}{3-c}} (H - \partial \phi). \quad (3.10)$$

The modes of $I^-(z) = \sum_{n = -\infty}^{-1} I^-_n z^{-n-1}$ generate a Heisenberg algebra. The module $H^-_p$ is defined as a Verma module over the Heisenberg algebra with the highest-weight vector defined by $I_0^- |q\rangle = 0, \quad n \geq 1, \quad I_0^+ |q\rangle = q |q\rangle \quad (3.11)$
Under the anti-KS mapping (3.8) one has the identities
\[ T + T_\phi = T^{\text{Sus}} + T^- \] (3.12)

where \( T \) is the energy-momentum tensor of the \( N = 2 \) algebra and \( T_\phi \) is the energy-momentum tensor of the Liouville system (2.18), and
\[ -2H + \partial^2 \phi = J^0 + \left( \frac{\hbar}{2} - 1 \right) \sqrt{\frac{k+2}{2}} I. \] (3.13)

The composition \( F_{\text{KS}}^{-1} \circ F_{\text{KS}} \) maps the \( \hat{\mathfrak{s}\ell}(2) \) algebra into an \( \hat{\mathfrak{s}\ell}(2) \) algebra in the tensor product \( \mathcal{U} \hat{\mathfrak{s}\ell}(2) \otimes [BC] \otimes [\psi \psi^*] \)
\[ \mathcal{T}' = J^+ e^\phi C, \mathcal{T}^- = J^- e^{-\phi} B, \] (3.14)
\[ \mathcal{T}^0 = J^0 + \frac{\hbar}{2} (\partial \phi - BC). \]

The same happens with the \( N = 2 \) algebra under the action of \( F_{\text{KS}} \circ F_{\text{KS}}^{-1} \), which maps the \( N = 2 \) algebra \( \mathcal{A} \) into an \( N = 2 \) algebra in the tensor product \( \mathcal{U} \mathcal{A} \otimes [BC] \otimes [\psi \psi^*] \)
\[ \mathcal{Q} = Q e^\phi C, \mathcal{G} = G e^{-\phi} B, \]
\[ \mathcal{H} = H + \frac{k}{k+2} (BC - \partial \phi), \]
\[ \mathcal{T} = T + H (BC - \partial \phi) + \frac{k}{2(k+2)} ((\partial \phi)^2 - 2\partial \phi \cdot BC + \partial^2 \phi - 2B \partial C). \] (3.15)

The above identities allow one to deduce

**Theorem 3.1**

I. The KS mapping induces an isomorphism of \( N = 2 \) representations
\[ \mathcal{M}_{j,k,\theta} \otimes \Omega \cong \bigoplus_{\lambda \in \mathbb{Z}} \mathcal{V} \mathcal{V}^\perp_{-\frac{j}{2}, k+2; \lambda+\theta} \otimes \mathcal{H}^+ \bigoplus_{\lambda \in \mathbb{Z}} \mathcal{V} \mathcal{V}^\perp_{\frac{j}{2}, -k-2; \lambda-\theta} - \lambda \]

where on the LHS the \( N = 2 \) algebra acts by the generators (3.7), while on the RHS it acts on \( \mathcal{V} \mathcal{V}^\perp_{-\frac{j}{2}, k+2; \lambda+\theta} \) as on its twisted Verma module.

II. The anti-KS mapping induces an isomorphism of \( \hat{\mathfrak{s}\ell}(2) \) representations
\[ \mathcal{V}_{h,t,\theta} \otimes \Xi \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{M}_{-\frac{j}{2}, h,t-2; n-\theta} \otimes \mathcal{H}^- \bigoplus_{n \in \mathbb{Z}} \mathcal{V} \mathcal{V}^\perp_{h,t, 2; n+\theta} \]

where on the LHS the \( \hat{\mathfrak{s}\ell}(2) \) algebra acts by the generators (3.8), while on the RHS it acts on \( \mathcal{M}_{-\frac{j}{2}, h,t-2; n-\theta} \) as on its twisted Verma module.

As a corollary, observe that singular vectors in \( \hat{\mathfrak{s}\ell}(2) \) Verma modules and in topological \( N = 2 \) Verma modules occur (or do not occur) simultaneously.

A ‘relaxed’ version of the above result reads

**Theorem 3.2**

I. The KS mapping induces an isomorphism of \( N = 2 \) representations
\[ \mathcal{R}_{j,\lambda, k, t, \theta} \otimes \Omega \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{W} \mathcal{V}^\perp_{-\frac{j}{2}, \lambda+\theta} \otimes \mathcal{H}^+ \bigoplus_{n \in \mathbb{Z}} \mathcal{V} \mathcal{V}^\perp_{\frac{j}{2}, \lambda-\theta} - \lambda \]

where on the LHS the \( N = 2 \) algebra acts by the generators (3.9), while on the RHS it acts naturally on \( \mathcal{W} \mathcal{V}^\perp_{-\frac{j}{2}, k+2; \lambda+\theta} \) as on a twisted massive Verma module.

II. The anti-KS mapping induces an isomorphism of \( \hat{\mathfrak{s}\ell}(2) \) representations
\[ \mathcal{W}_{h, t, \theta} \otimes \Xi \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{R} \mathcal{R}^-_{-\frac{j}{2}, \lambda(t), t-2; n-\theta} \otimes \mathcal{H}^- \bigoplus_{n \in \mathbb{Z}} \mathcal{V} \mathcal{V}^\perp_{h, t, 2; n+\theta} - \lambda \]

where on the LHS the \( \hat{\mathfrak{s}\ell}(2) \) algebra acts by generators (3.8), while on the RHS it acts naturally on \( \mathcal{R} \mathcal{R}^-_{-\frac{j}{2}, h, t-2; n-\theta} \) as on a twisted relaxed Verma module.

This is illustrated by the following diagram:
4. Categorial equivalences

The above theorems suggest that one extends the mappings $F_{KS}$ and $F_{KS}^{-1}$ to a functor that would establish the correspondence between categories of $\hat{\mathfrak{s}\mathfrak{l}}(2)$ and $N = 2$ modules. However, a difficulty in defining such a functor can be seen already when attempting to relate modules $M_{j,k,\theta}$ and $V_{h,t,\theta'}$ for fixed $\theta$ and $\theta'$. While on the $\hat{\mathfrak{s}\mathfrak{l}}(2)$ side any submodule of $M_{j,k,\theta}$ is again a twisted Verma module with the same value of $\theta$, this is not so on the $N = 2$ side, where submodules of a (twisted) topological Verma module are the twisted topological Verma modules with different values of the 'spectral' parameter $\theta$.

Thus, an equivalence between some categories of $\hat{\mathfrak{s}\mathfrak{l}}(2)$ and $N = 2$ modules can only be established for those categories that effectively allow for a factorization with respect to the spectral flow. These categories can be defined as follows. Consider the objects that are infinite chains $(M_{j,k,\theta})_{\theta \in \mathbb{Z}}$, where $M_{j,k,\theta}$ are twisted $\hat{\mathfrak{s}\mathfrak{l}}(2)$ Verma modules. As morphisms between $(M_{j,k,\theta})_{\theta \in \mathbb{Z}}$ and $(M'_{j',k',\theta'})_{\theta' \in \mathbb{Z}}$, take any Verma module morphism $M_{j,k,\theta_1} \to M'_{j',k',\theta_2}$. Call this category the $\hat{\mathfrak{s}\mathfrak{l}}(2)$ Verma chain category $\mathcal{CVER}$.

The meaning of the definition of morphisms of chains is that, given a morphism between any two modules, one spreads it over the entire chains by spectral flow transforms.

On the $N = 2$ side, the topological Verma chain category $\mathcal{CTVER}$ is defined similarly: one takes chains of twisted topological Verma modules, every such chain consisting of twisted topological Verma modules with all $\theta \in \mathbb{Z}$. Morphisms of the chains are defined similarly to the $\hat{\mathfrak{s}\mathfrak{l}}(2)$ case $^3$.

To define a functor relating such chains, I first construct correspondences between individual modules in the chains. Given a topological Verma module $V_{h,t,\theta}$, and an arbitrary $\theta' \in \mathbb{Z}$, take the Heisenberg modules $H^+ - \sqrt{\frac{1}{2} (j + \frac{1}{4} \theta' - \theta)}$ and construct

$$V_{h,t,\theta} \otimes H^+ - \sqrt{\frac{1}{2} (j + \frac{1}{4} \theta' - \theta)} \oplus \bigoplus_{m \in \mathbb{Z}, m \neq 0} V_{h,t,\theta + m} \otimes H^+ - \sqrt{\frac{1}{2} (j + \frac{1}{4} \theta' - \theta + m)}$$

This is isomorphic to the tensor product of an $\hat{\mathfrak{s}\mathfrak{l}}(2)$ Verma module $M_{\frac{1}{2}, h,t, \theta}$ with a ghost module. Define the result of applying $F_{KS}(\theta, \theta')$ to $V_{h,t,\theta}$ to be the module $M_{\frac{1}{2}, h,t, \theta}$:

$$F_{KS}(\theta, \theta') : V_{h,t,\theta} \mapsto M_{\frac{1}{2}, h,t, \frac{1}{2}, \theta'}, \theta, \theta' \in \mathbb{Z}.$$ Similarly,

$$F_{KS}^{-1}(\theta, \theta') : M_{j,k, \theta} \mapsto V_{\frac{1}{k+j}, j,k, 2, \theta'}, \theta, \theta' \in \mathbb{Z}$$

is defined as follows. Given a Verma module $M_{j,k, \theta}$ and $\theta' \in \mathbb{Z}$, construct the sum

$$M_{j,k, \theta} \otimes H^- - \sqrt{\frac{1}{2} (j + \theta' - \frac{k+2}{2})} \oplus \bigoplus_{n \in \mathbb{Z}, n \neq 0} M_{j+\frac{1}{k+j}, k, \theta+n} \otimes H^- - \sqrt{\frac{1}{2} (j + \theta' - \frac{k+2}{2} + n)}$$

which is isomorphic to the module $V_{\frac{1}{k+j}, j,k, 2, \theta'}$ tensored with a module of antifermions. This twisted topological $N = 2$ module is by definition the result of applying $F_{KS}^{-1}(\theta, \theta')$ to $M_{j,k, \theta}$.

While $F_{KS}$ and $F_{KS}^{-1}$ depend on chosen $\theta$ and $\theta'$, the $\theta$-dependence disappears when applied to the elements of $\mathcal{CVER}$ and $\mathcal{CTVER}$:

$$F_{KS} : \mathcal{CTVER} \mapsto \mathcal{CVER}$$

$$F_{KS}^{-1} : \mathcal{CVER} \mapsto \mathcal{CTVER}$$

$^3$It is important to note that any submodule of a (twisted) topological Verma module is a twisted topological Verma module.
Evidently, the composition of $F_{KS}$ and $F_{KS}^{-1}$ maps each chain of twisted Verma modules into an isomorphic chain. Therefore, $F_{KS}$ and $F_{KS}^{-1}$ would become the direct and inverse functors as soon as I define how $F_{KS}$ and $F_{KS}^{-1}$ act on morphisms.

Recall that in Verma module categories, morphisms are naturally identified with singular vectors. As can be seen from the above isomorphisms, an $\hat{\mathfrak{s}(2)}$ singular vector exists in at least one twisted topological Verma module $\hat{\mathfrak{m}}_{\theta,\theta}$ and the (twists of) corresponding spectral flows.

Theorem 4.1 The KS and anti-KS mappings give rise to isomorphisms, denoted again by $F_{KS}(\theta,\theta')$ and $F_{KS}^{-1}(\theta,\theta')$ respectively, between the (twisted) $\hat{\mathfrak{s}(2)}$ singular vectors in the Verma modules $\mathcal{M}_{\theta,\theta}$ and the (twists of) $N=2$ singular vectors in the twisted massive Verma modules $\mathcal{W}_{\theta,\theta} = \mathcal{W}_{\theta,1}$.

\[ F_{KS}(\theta,\theta') : |E(s, t, k + 2)\rangle_{\pm, \theta} \mapsto |S_{\pm}^{MFF}(s, k)\rangle_{\theta', \theta}, \]  
\[ F_{KS}^{-1}(\theta,\theta') : |S_{\pm}^{MFF}(s, k)\rangle_{\theta, \theta'} \mapsto |E(s, t, k + 2)\rangle_{\pm, \theta'}, \]

where $|E(s, t)\rangle_{\pm, \theta}$ and $|S_{\pm}^{MFF}(s, k)\rangle_{\theta', \theta}$ denote the respective singular vectors transformed by the corresponding spectral flows.

Evidently, $F_{KS}(\cdot, \cdot)$ and $F_{KS}^{-1}(\cdot, \cdot)$ applied to chains of the respective Verma modules take morphisms (between chains) into morphisms. Thus, finally,

Theorem 4.2 The functors $F_{KS}$ and $F_{KS}^{-1}$ are covariant functors which are inverse to each other and which therefore establish the equivalence of the Verma chain category on the $\hat{\mathfrak{s}(2)}$ side and the topological Verma chain category on the $N=2$ side.

An extension of the above theorems to the relaxed and massive Verma modules is as follows. For uniformity, I call the Verma highest-weight states in extremal diagrams of relaxed $\mathfrak{s}(2)$ modules (see (2.13)) the charged $\mathfrak{s}(2)$ singular vector.

Theorem 4.3 The KS and anti-KS mappings gives rise to identifications, which we denote again by $F_{KS}(\theta,\theta')$ and $F_{KS}^{-1}(\theta,\theta')$ respectively, between

1) singular vectors $\Sigma^{-}(r, s, j, k)$ and $\Sigma^{+}(r, s, j, k)$ in the relaxed Verma module $\mathcal{R}_{r,s,j,k}$ and $N=2$ singular vectors in the twisted massive Verma modules $\mathcal{W}_{\theta,\theta}$ and $2) \text{ charged } \hat{\mathfrak{s}(2)} \text{ singular vectors and charged } N=2 \text{ singular vectors:}

\[ F_{KS}(\theta,\theta') : |C(r, j, k)\rangle_{\theta, \theta'} \mapsto |E(r, h, t)\rangle_{\theta', \theta}, \]  
\[ F_{KS}^{-1}(\theta,\theta') : |E(r, h, t)\rangle_{\theta, \theta'} \mapsto |C(r, j, k)\rangle_{\theta', \theta}, \]

where $|\Sigma^{\pm}(r, s, k)\rangle_{\theta, \theta'}$ and $|\Sigma^{\pm}(r, s, h, t)\rangle_{\theta, \theta'}$ denote the respective singular vectors transformed by the corresponding spectral flows.

Thus,

Theorem 4.4 The functors $F_{KS}$ and $F_{KS}^{-1}$ are covariant functors which are inverse to each other and which therefore establish the equivalence of the relaxed Verma chain category $\mathcal{CRVER}$ on the $\hat{\mathfrak{s}(2)}$ side and the massive Verma chain category $\mathcal{CMVER}$ on the $N=2$ side.

To conclude, one of the central notions relating the $\hat{\mathfrak{s}(2)}$ and $N=2$ parts of the story are the extremal diagrams. As I have already mentioned, extremal diagrams of $N=2$ modules are
nothing but a deformation of $\hat{sl}(2)$ extremal diagrams when straight lines become ‘parabolas’. Yet in the $\hat{sl}(2)$ case, where extremal diagrams consist of straight lines, it appears more ‘obvious’ that different states on the same floor are equivalent (generically, unless charged singular vectors – Verma highest-weight states – appear). In the $\mathcal{N}=2$ case, on the other hand, it may be tempting to assign a more fundamental status to vectors at the top level of (untwisted) extremal diagrams. However, working with top-level representatives of $\mathcal{N}=2$ extremal diagrams conceals the true nature of massive $\mathcal{N}=2$ modules and the ‘topological’ structure of some of their submodules. Back in the $\hat{sl}(2)$ terms, the top-level representatives of a singular vector would correspond to those states where the $\Lambda$ parameter reaches an extremum among the states in the same extremal subdiagram, which does not seem to be of much practical use.

Thus, there exist two different languages, the $\hat{sl}(2)$ and $\mathcal{N}=2$ ones, to describe essentially the same structure; a very interesting task is to extend the dictionary relating representation-theoretic terms for the two algebras to include a fusion-related vocabulary.

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4Factoring with respect to the spectral flows makes the fusion rules isomorphic, thus the difference between $\hat{sl}(2)$ and $\mathcal{N}=2$ fusion rules is related to different periodicity of the respective modules under the spectral flows.