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Cracks in strain gradient elasticity-distributed dislocation technique

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Abstract

The mode III fracture analysis of graded cracked plane in the framework of classical and strain gradient elasticity is presented in this work. Solutions to the problem of screw dislocation in plane are available for classical and strain gradient elasticity theories. Different approaches for the formulation of the strain gradient theory, especially considering the boundary conditions, result in singular and nonsingular stress fields at the crack tip. One of the applications of the dislocation is the analysis of cracked medium via the Distributed Dislocation Technique (DDT). The DDT has been applied extensively in the framework of the classical elasticity. In this article, this technique is generalized for the nonsingular strain gradient elasticity formulation available in the literature. For a system of interacting cracks in classical elasticity, DDT results in a system of Cauchy singular integral equations. In the framework of the gradient elasticity, due to the regularization of the classical singularity, a system of nonsingular integral equations is obtained. Plane with one crack is studied and the singular stress distribution in the classical elasticity is compared with the nonsingular stress components in gradient elasticity theories.

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1. Introduction

The generalized continuum elasticity theories were introduced at the beginning of the nineteenth century first by the brothers Cosserat and about half a century later they were revisited and revised by the founders of modern

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continuum mechanics (Toupin, Rivlin, Mindlin, Eringen and others) as reviewed by Aifantis (2003). In the classical elasticity, the strain energy is simply assumed to be a function of strain. Due to the lack of intrinsic length scale in classical elasticity, they represent scale-free continuum theories. In contrast, generalized continuum elasticity theories enrich the classical elasticity with additional material characteristic lengths in order to describe the scale effects resulting from the microstructures (Eringen, 1999).

In the strain gradient elasticity, the strain energy is generalized and is not simply a function of strain but may also depend on the gradient of strain. In this framework, different contributions have been made with specific forms of the strain energy. These contributions introduce internal lengths to consider scale effects. In the first strain gradient elasticity, the strain energy is assumed to be a quadratic function in terms of strain and first-order gradient strain, while in the second strain gradient elasticity, the strain energy is a function of strain, first-order and second-order gradient strain. In the generalized elasticity, hyperstress tensors such as double (couple) stress (in the first strain gradient elasticity) and triple stress tensors (in the second strain gradient elasticity) are defined to parameterize the higher order theory. However, due to many material constants in these theories, there used to be a gap between theories and experimental investigations.

In fracture mechanics, considering different approaches in the formulation of the strain gradient theories, singular and nonsingular stress fields are predicted ahead of the crack-tip. Paulino et al. (2003), Chan et al. (2008) and Gourgiotis and Georgiadis (2009) predicted the singular stresses at the crack-tip for the gradient theory. On the other hand, Gutkin and Aifantis (1996) elaborated that in the framework of gradient elasticity, the elastic singularity is eliminated from dislocation lines. Additionally, Lazar and Maugin (2005) reported the same behavior for a class of strain gradient as well as a stress gradient theory. Thus, the improved stress and strain fields present no singularity in the core region unlike the classical solutions. Currently, these approaches are under debate to determine correct boundary conditions and the compatibility conditions of gradient elasticity.

One of the applications of the dislocations is to be used as Green’s functions in the distributed dislocation technique (DDT) for the fracture analysis of the materials. In the classical elasticity, it has been proved that the cracked material with specific loading may be modeled by properly distributing the dislocations in the material (Korsunsky and Hills 1996). The DDT has been largely utilized in the classical elasticity for static (Fotuhi and Fariborz, 2006) and elastodynamic analyses (Mousavi and Fariborz, 2012). It has also been generalized for the analysis of piezoelectric (Mousavi and Paavola 2013) and magneto-electro-elastic materials (Mousavi and Paavola 2014). These investigations show that the DDT is useful in the interaction analysis of multiple straight and curved cracks. In these cases, the technique results in a system of Cauchy singular integral equations, which should be solved.

In the framework of generalized elasticity, DDT has been applied to a few problems. In the couple-stress elasticity, Gourgiotis and Georgiadis (2007, 2008) utilized DDT for the analysis of cracks in modes I, II and III. They studied the dislocation and disclination in the homogeneous couple-stress elasticity. In this case, the stress and hyperstresses possess singularity and hypersingularity. They proposed a technique to solve the hypersingular integral equations obtained in DDT.

In this study, the nonsingular approach in gradient theories is considered to analyze the antiplane fracture of a graded plane. The dislocation solution presented by Gutkin and Aifantis (1996) and Lazar et al. (2006) are applied in the DDT to analyze a cracked medium. The generalized DDT formulation covers classical and gradient elasticity. The stress distribution in a cracked graded plane under antiplane loading is determined for the classical and gradient theory.

2. Solution to screw dislocation: classical and strain gradient elasticity

The gradation direction of the graded plane is assumed to be parallel to y-axis. A screw dislocation is imposed at the origin of the infinite \((x,y)\)-plane with Burgers vector \((0,0,b_\nu)\). The line of the dislocation is assumed to be perpendicular to the gradation direction of the graded plane. In the classical continuum mechanics, stress components are given by (Gutkin and Aifantis, 1996, Lazar, 2007)
\[ \sigma_{\alpha} = -b_2 \frac{\mu_0 \gamma^2 y}{2\pi} \left( \frac{\gamma y}{r} K_i(\gamma y) - K_0(\gamma y) \right) \]
\[ \sigma_{\alpha} = b_2 \frac{\mu_0 \gamma^2 x}{2\pi} K_i(\gamma y), \]

while \( K_n \) denotes the modified Bessel function of the second kind with order \( n \) and \( r = \sqrt{x^2 + y^2} \). These stress components possess a Cauchy \((1/r)\) as well as logarithmic singularities. It is worth mentioning that the screw dislocation in a graded layer has been reported by Fotuhi and Fariborz. It is obvious that the presence of the strip boundary conditions increases the mathematical efforts needed to solve the resulting PDE and will result in complicated integral expressions.

Considering a special formulation in gradient elasticity, Lazar (2007) solved the problem of a screw dislocation in a graded material within the first strain gradient elasticity. He provides the nonsingular stress field of the screw dislocation in a graded material for first gradient elasticity as

\[ \sigma_{\alpha} = \sigma_{\alpha}^0 - b_2 \frac{\mu_0 \gamma^2 y}{2\pi} \left[ -\frac{y}{r} \frac{1 + \gamma^2 l^2}{l} K_i\left(\frac{1 + \gamma^2 l^2}{l} r\right) + \gamma K_0\left(\frac{1 + \gamma^2 l^2}{l} r\right) \right], \]
\[ \sigma_{\alpha} = \sigma_{\alpha}^0 - b_2 \frac{\mu_0 \gamma^2 x}{2\pi} K_i\left(\frac{1 + \gamma^2 l^2}{l} r\right). \]  

It should be noted that there exist other formulation in generalized elasticity which results in singular stress components (Gourgiotis and Georgiadis, 2009).

Furthermore, it is a challenging task to determine the correct form of the higher order condition for the dislocation, as there are different suggestions available in the literature. (Paulino et al., 2003; Georgiadis, 2003; Gourgiotis and Georgiadis, 2007, 2008). The first two references provided the higher order condition for the solution of the crack problem directly, while the second two references implied the higher order condition for the dislocation problem. Polizzotto (2003) provided a thorough discussion about nonstandard boundary conditions in the gradient elasticity.

### 3. Distributed dislocation technique for classical and gradient elasticity

Distributed dislocation technique (DDT) is a method to analyze a medium containing multiple cracks. Classical stress fields of the dislocations contain singularity which results in singular integral equations in DDT. In the framework of the gradient elasticity presented in the previous section, the stress components are nonsingular. The graded plane, with a varying shear modulus \( \mu = \mu_0 \exp(2\gamma y) \), is assumed to contain a screw dislocation with situated at a point with coordinates \((\eta, \zeta)\). The line of the dislocation is assumed to be parallel to \( x \)-axis. The stress components at a point with coordinates \((x, y)\) due to the dislocation at \((\eta, \zeta)\) may be obtained by making the conversions \( x \rightarrow (x-\eta), \ y \rightarrow (y-\zeta) \) in Eqs (1, 2).

The moveable orthogonal coordinate system \((n, t)\) is chosen such that the origin may move on the crack while \( t \)-axis remains tangent to the crack surface. The antiplane traction on the surface of \( k \)th crack in terms of stress components in the Cartesian coordinates \((x, y)\) becomes

\[ \sigma_{\alpha}^j(x_k, y_k) = \sigma_{\alpha}^j(\theta_k) - \sigma_{\alpha}^j(\theta_k), \]

where \( \theta_k(s) = \tan^{-1}(\beta_k^j(s)/\alpha_k^j(s)) \) is the angle between \( x \)- and \( t \)-axes and prime denotes differentiation with respect to the argument. Additionally, \( j=0,1 \) denote the components in the classical and strain gradient elasticity, respectively. A crack is constructed by a continuous distribution of dislocations. Suppose dislocations with unknown density \( B_z \) are distributed on the infinitesimal segment \( \sqrt{(\alpha_k^j)^2 + (\beta_k^j)^2} \ dt \) at the surface of the \( j \)th crack where \(-1 \leq t \leq 1\) and prime denotes differentiation with respect to the relevant argument. The antiplane traction on the surface of \( k \)th crack due to the presence of the above-mentioned distribution of dislocations on all \( N \) cracks yields integral equations for the dislocation densities.
\[ \sigma^i_{xz}(\alpha_k(s), \beta_k(s)) = \sum_{j=1}^{N} \int_{-1}^{1} K_j(s,t,k) \left[ \alpha'_j(t) \right]^2 + \left[ \beta'_j(t) \right]^2 B'_j(t) \, dt, \]

while the kernels of the integral can be determined using (1,2). Equations (4) are Cauchy singular integral equations for \( i=0 \) (classical elasticity), and are nonsingular integral equations for \( i=1 \) (strain gradient elasticity).

Now consider a plane with the following loadings.
\[ \sigma_{yz}^{\infty} = \sigma_0, \quad \sigma_{xz}^{\infty} = 0. \]

The graded plane in the absence of cracks under loading (5) is in a state of pure antiplane shear stress field \( \sigma_{yz}(x,y) = \sigma_0 \). Substituting this stress field in (3), the traction in the location of cracks in the plane in the absence of cracks yields
\[ \sigma_{nz}(x_k, y_k) = \sigma_0 \cos(\theta_k) \quad k \in \{1,2,\ldots,N\}. \]

By virtue of the Buckner’s principal, the traction in (6) with opposite sign should be substituted in left-hand side of Eq. (4). Employing the definition of dislocation density function, the equation for the crack opening displacement across the \( j \)th crack becomes
\[ w_j(s) - w_j(s) = \int_{-1}^{1} \left[ \alpha'_j(t) \right]^2 + \left[ \beta'_j(t) \right]^2 B'_j(t) \, dt. \]

Further, the displacement field is single-valued out of crack surfaces. Thus, considering the singularity in the case of the classical solution, the dislocation density for an embedded crack is subjected to the following closure requirement
\[ \int_{-1}^{1} \left[ \alpha'_j(t) \right]^2 + \left[ \beta'_j(t) \right]^2 B'_j(t) \, dt = 0, \]

where \( i=0,1 \) and \( j \in \{1,2,\ldots,N\} \), is the crack number. To obtain the dislocation density for cracks, the integral equations (4) and (8) are to be solved simultaneously. This system of integral equations is an ill-posed problem. In the case of the classical elasticity, this system of equations is Cauchy singular, while in the cases of the gradient elasticity, they are nonsingular.

4. Numerical results and discussion: Graded plane with a horizontal straight crack

The plane graded in \( y \)-direction is assumed to be weakened by a horizontal crack (Fig. 1). The plane is under constant remote loading \( \sigma_{yz} = \mu_0 \).

![Fig. 1. Geometry of the functionally graded plane with a horizontal crack](image)

Using the DDT, equations (4, 8) are written for the current configuration. This system of integral equations, after proper discretization, is solved by means of the singular value decomposition (Golub and Reinsch, 1970) and the dislocation densities are determined for classical and gradient theories. The dislocation density is used to determine the stress distribution in the plane.
Fig. 2 shows the variation of the stress component $\sigma_y^i$ in the vicinity of the crack tip, while $i=0, 1$ depict the classical and gradient theories, respectively. According to the stress distribution in the classical framework ($\sigma_y^0$), the singularity exists in the crack tip, while in the gradient theory, the classical singular behavior is regularized and there is no singularity in the crack tip. A three-dimensional stress distribution for gradient elasticity is shown in Fig. 3.

Fig. 3. Three-dimensional normalized stress distribution for a horizontal crack in a graded plane ($\gamma=2$) in gradient elasticity ($l/l=0.02a$)

5. Conclusion

The distributed dislocation technique (DDT) is successfully applied to the nonsingular dislocation solutions for
gradient elasticity available in the literature. General formulation is derived for the DDT to analyze the planes with various configurations of cracks in the classical and strain gradient theories. Analysis of a cracked graded plane depicts that the singular behavior of crack tip is regularized in the gradient strain theory considered in this article. Due to the generality of the present DDT formulation, it is capable of the analysis of any other configurations as well as non-horizontal cracks in the graded plane. This is quite helpful in the analysis of the interaction of planes containing multiple cracks in gradient strain theories.

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