Quantum capacity of lossy channel with additive classical Gaussian noise: a perturbation approach

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Abstract

For a quantum channel of additive Gaussian noise with loss, in the general case of $n$ copies input, we show that up to first order perturbation, any non-Gaussian perturbation to the product thermal state input has a less quantum information transmission rate when the input energy tends to infinitive.

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1 Introduction

Quantum capacity exhibits a kind of nonadditivity [1] that makes it extremely hard to deal with. The first example with calculable quantum capacity is quantum erasure channel[2]. Other examples are dephasing qubit channel[3], amplitude damping qubit channel[4], and continuous variable lossy channel[5], where the channels are either degradable or anti-degradable. Anti-degradable channel has null quantum capacity due to no clone theorem[6]. Degradable channel is a channel that when the sender transmits an unknown quantum state to the receiver with some quantum information leaks to the environment, the receiver can reconstruct what the environment received from the state himself received. Degradable quantum channels were first introduced in Ref. [3] where it was shown that their quantum capacity $Q$ can be expressed in terms of the single letter formula of $Q = \sup_{\sigma_n} \frac{1}{n} I_c(\sigma_n, \mathcal{E}^\otimes n)$. Where the coherent information (CI) $I_c(\sigma, \mathcal{E}) = S(\mathcal{E}(\sigma)) - S(\sigma)$. In Ref. [3] where $s$ is the input state, the application of the channel $\mathcal{E}$ results the output state $\mathcal{E}(\sigma) = \sigma^R \mathcal{E} \otimes I(\psi) \langle \psi | \psi \rangle$, with $\mathcal{E}$ referred to the 'reference' system[7] (the system under process is $Q$ system with annihilation and creation operators $a$ and $a^\dagger$, we denote $\sigma^R \mathcal{E}$ for simplicity), $|\psi \rangle$ is the purification of the input state $\sigma$. If a channel is not degradable, the regulation procedure should be applied to the quantum capacity, which is $\sup_{\sigma_n} \frac{1}{n} I_c(\sigma_n, \mathcal{E}^\otimes n)$.

Thus in the form of characteristic function, the additive property of the classical Gaussian noise $N_n$ is quite apparent.

A single mode thermal state $\rho$ has a characterization of the form $\chi(\mu) = \exp(-(N + 1/2)|\mu|^2)$, and we have $\rho = \int \frac{d\mu}{\pi} \chi(\mu) D(-\mu) = (1 - v) e^{\alpha^\dagger a}$, with $v = N/(N + 1)$, conventionally in the following, $\nu_x = N_x/(N_x + 1)]$, where $N$ is the average photon number. The noisy lossy state is $\rho' = \mathcal{E}(\rho) = (1 - v') e^{\alpha^\dagger a}$, with average photon number $N' = \eta N + N_n$. We denote the annihilation and creation operators of the 'reference' $R$ system as $b$ and $b^\dagger$. For thermal state input $\rho$, by a proper symplectic transformation, the joint output state $\rho' \mathcal{E}^\otimes n$ can be transformed to a direct product of two thermal states with average photon numbers $N_A$ and $N_B$, respectively, where $N_{A,B} = \frac{1}{2} |D \pm (N' - N) - 1|$. With

2 The channel and the single letter formula

The lossy channel with additive classical Gaussian noise can be described by

$$
\mathcal{E}(\sigma) = \frac{1}{N_n} \int \frac{d^2\alpha}{\pi} \exp(-|\alpha|^2/N_n) 
\times D(\alpha) \text{tr}_E[U(\sigma \otimes |0\rangle \langle 0|_E)U^\dagger]D(\alpha),
$$

where $N_n$ specifies the additive classical Gaussian noise, with $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ is the displacement operator. The unitary operator $U = \exp[i\theta(aa^\dagger_E - a^\dagger a_E)]$ with $a_E$ the annihilation operator of the environment, and $\eta = \cos^2 \theta$ is the quantum efficiency. Any quantum state $\sigma$ can be equivalently specified by its characteristic function $\chi^R(\mu) = tr[aD(\mu)]$, and inversely $\sigma = \int \prod_{\mu} \frac{d^2\mu}{\pi} \chi^R(\mu) D(-\mu)$. The characteristic function of the output state $\sigma' = \mathcal{E}(\sigma)$ is

$$
\chi'(\mu) = \chi(\sqrt{\eta}\mu)e^{-(N_n + 1/2)|\mu|^2}.
$$

Bosonic Gaussian channels [12] include all the physical transformations which preserve "Gaussian character" of the transmitted signals and can be seen as the quantum counterpart of the Gaussian channels in the classical information theory. A full classification of one-mode Bosonic Gaussian channels was presented in Ref. [14], where a very useful channel which is lossy accompanied by additive classical Gaussian noise is classified as weak degradable. As stressed by the Authors of Ref. [14], weak degradability is quite different from degradability. Thus for such a channel, a single letter formula of $Q$ may not be available. The regulation formula of (11) is needed.
where $g(s) = (s + 1) \log_2(s + 1) - s \log_2 s$ is the bosonic entropy function, and

$$N = N_B \cosh^2 r + (N_A + 1) \sinh^2 r,$$

(4)

with $r$ is the parameter of the symplectic transformation, and $\tanh 2r = 2\sqrt{\eta N(N + 1)/(N' + N_A + 1)}$. Based on the coherent information of single mode thermal state input, the quantum capacity of the channel has been conjectured as\cite{12}

$$Q = \max \{0, \lim_{N \to \infty} I_c(\rho, \mathcal{E})\}$$

$$= \max \{0, \log_2 |\eta| - \log_2 |1 - \eta| - g \left( \frac{N_n}{1 - \eta} \right) \},$$

(5)

Apart from the regulation, this single letter formula is doubtful for the input state is quite special. The procedure of maximization over all continuous variable input state (Gaussian or non-Gaussian) has not been taken yet. In the next section, we will prove that for all single mode Gaussian state input the single letter quantum capacity is really given by Eq. (3).

### 3 Gaussian state input to the one-mode channel

We now consider a single mode Gaussian state $\rho_G$ (which comprises thermal noise state as its special case) input to the single use of the channel. A single mode Gaussian state is described by its real correlation matrix $\alpha$ (we drop the first moments for they can be removed by local operations) which can be generated from that of thermal noise state $\rho$ with a symplectic transformation \cite{13}. We have

$$\alpha = \begin{bmatrix} \alpha_{qq} & \alpha_{qp} \\ \alpha_{pq} & \alpha_{pp} \end{bmatrix} \text{ with } \det(\alpha) = (N + \frac{1}{2})^2.$$ The energy of the Gaussian state is $E = Tr[(a^\dagger a + \frac{1}{2})\rho_G] = \frac{1}{2}(\alpha_{qq} + \alpha_{pp})$. For a Gaussian state input $\rho_G$, the output $\rho_G'$ and the joint output state $\rho_G''$ are still Gaussian. The symplectic eigenvalues \cite{12} of these states can be obtained. The coherent information is

$$I_c(\rho_G, \mathcal{E}) = g(d_0 - \frac{1}{2}) - g(d_1 - \frac{1}{2}) - g(d_2 - \frac{1}{2}),$$

(6)

with

$$d_0 = \sqrt{N_n^2 + 2\eta EN_n + \eta^2 E^2 x},$$

$$d_{1,2} = \sqrt{\frac{1}{2} x(\pm \sqrt{x^2 - 4Y})},$$

(7)

(8)

where $N_n' = N_n + \frac{1}{2}X, X = N_n^2 + 2\eta EN_n + \frac{1}{2} + (1 - \eta)^2 E^2 x$, $Y = \frac{\eta^2}{2}EN_n' + \frac{\eta^4}{2}E^2 N_n^2 x$, with $x = y^2$, and $y = E/(N + \frac{1}{2})$. With the condition $det(\alpha) = (N + \frac{1}{2})^2$, it is not difficult to prove that $y$ has its global minimum $y = 1$ when $\alpha_{qq} = \alpha_{pp} = N + \frac{1}{2}$, $\alpha_{qp} = 0$.

The derivative of the coherent information with respect to $x$ is $\frac{dL(\rho, \mathcal{E})}{dx} = f(d_0) - f(d_1) - f(d_2)$, with $f(z) = \frac{d(\eta z - 1/2)}{dx} = \frac{1}{\eta} \log_2 \frac{z + 1/2 + \eta^2 E^2 x}{z - 1/2}$. When $E \to \infty$, we have $d_0 \eta E \sqrt{\eta x} \to \infty$, $d_1 (1 - \eta) \sqrt{\eta x} \to \infty$, then

$$f(d_0) - f(d_1) = -\frac{N_n}{x E \ln 2} \left( \frac{1}{\eta} - \eta \frac{N_n}{1 - \eta} \right).$$

(9)

Note that even when $E \to \infty$, we have $d_2 = N_n/(1 - \eta)$, thus

$$f(d_2)_{E \to \infty} = \frac{3\eta[N_n^2 - \frac{1}{2}(1 - \eta)N_n^2]}{4x^2(1 - \eta)^3} \log_2 \frac{N_n' + \frac{1}{2}(1 - \eta)}{N_n - \frac{1}{2}(1 - \eta)},$$

(10)

which is always positive for nonzero noise $N_n$. Hence for sufficiently large input energy $E$, we have $\frac{dL(\rho, \mathcal{E})}{dx} < 0$. While $x$ has its global minimum value $x = 1$, so the coherent information achieves its maximum at $x = 1$ which corresponds to thermal noise state input. Hence we can conclude that for sufficient large but definite input energy, the one-shot quantum information capacity of the channel is achieved by thermal noise state input of all Gaussian state inputs.

### 4 Perturbation to the n use of the channel

In the $n$ use of the channel with an input Gaussian state $\rho_G$, the algebraic equations of the symplectic eigenvalues \cite{12} are not analytically solvable. And for non-Gaussian state input, it is even worse in calculating the coherent information. So, in this paper, we turn to perturbation of the conjectured extremal state of the product thermal state $\rho_\otimes^n$. To treat the problem with perturbation theory, we need the following lemmas:

**Lemma 1:** $(\mathcal{E} \otimes I)(a^k \rho_\otimes^n b^m \otimes \phi) = \frac{(a^k \rho_\otimes^n b^m \otimes \phi)}{v^{(k+m)/2} k^k b^m} \cdot \ldots \cdot \rho_{\otimes^n} b^m$.

**Proof:** (1) Both of the characteristic functions of the lhs and the rhs are equal to $(1 - v \exp[-\frac{1}{2} |\mu_k|^2 + (\eta - \frac{1}{2} - N_n) |\mu_l|^2] \cdot \int \frac{dx^2}{2} \alpha^k x \alpha^m \exp[-(1 - v) |x|^2 + (\eta |\mu_k - \sqrt{\eta} |\mu_l - \sqrt{\eta} |\mu_\alpha^*| \alpha^*]$. (2) can be proved similarly.

We consider the first order multi-mode perturbation to the input product thermal state $\rho_\otimes^n$. A typical case is $\chi_{\alpha \times n}(\mu) = \chi_{\alpha}(\mu)[1 + \varepsilon(c_{k1} \mu_{k1} \cdots \mu_{k_n} + c_{l1} \mu_{l1} \cdots \mu_{l_n} + \cdots + c_{l_n} \mu_{l_1} \cdots \mu_{l_n})]$, with $\sum_{i=1}^n k_i = \sum_{i=1}^n l_i = m$ (the requirement of first order perturbation). We may denote the perturbation as $(k, l)$, with vectors $k = (k_1, k_2, \cdots, k_n), l = (l_1, l_2, \cdots, l_n)$. The perturbed input state is $\rho_{\alpha \times n} = \rho_\otimes^n + \phi$. The general form of the perturbation should be a linear combination of this typical $\phi$. We will prove that each $\phi$ contributes independently to the coherent information a negative quantity. To simplify the calculation, we introduce a generation function $I_\phi(\tau, \sigma) = \int \frac{d\mu}{2\pi} \chi_{\alpha}(\mu) D(-\mu) \exp[\mu \cdot \tau + \mu^* \cdot \sigma]$, \[2\]
\[
\phi = \left( c^{2\tau_{n} I_{\phi}(\tau, \sigma)} + c^{2\tau_{n} I_{\phi}(\tau, \sigma)} \right)_{\tau_{n} = 0}.
\]

It has been proved in Ref. [15] that the perturbation to the entropy is
\[
S(\rho_{n}) - S(\rho_{n}^{\otimes n}) = -\frac{1}{2} \varepsilon^{2} T_{2} (\phi^{2}/\rho_{n}^{\otimes n}) + o(\varepsilon^{3}).
\]

For
\[
T_{2} (\phi^{2}/\rho_{n}^{\otimes n}) = \exp\left[ -\frac{\tau - \sigma}{N} - \frac{\tau^{\prime} - \sigma^{\prime}}{N + 1} \right]
\]

Thus
\[
T_{2} (\phi^{2}/\rho_{n}^{\otimes n}) = 2c_{0} \frac{\Pi_{i}(k_{i} l_{i})}{[N(N + 1)]^{m}} + o(\varepsilon^{3}).
\]

where \(c_{0} = |c|^{2}\) for \(k \neq 1\) and \(c_{0} = 4c_{R}^{2}\) for \(k = 1, c_{R}\) is the real part of \(c\). The perturbation to the entropy of the output state \(\rho_{n}^{\otimes n}\) is
\[
S(\rho_{n}^{\otimes n}) - S(\rho_{n}^{\otimes n}) = -\varepsilon^{2} c_{0} \frac{\Pi_{i}(k_{i} l_{i})}{[N(N + 1)]^{m}} + o(\varepsilon^{3}).
\]

Eq. (13) exhibits that any intercross item of \(T_{2} (\phi^{2}/\rho_{n}^{\otimes n})\) type will be nullified for \((k,1) \neq (k,1)\). Thus each perturbation item contributes to the entropy separately.

The perturbation to the joint QR state is more sophisticated. We may express the perturbed joint input state as \(\rho_{n}^{QR} = \rho_{n}^{QR} + \varepsilon \Phi_{0}\) and \(\Phi = \frac{1}{2}(\Phi_{0} + \Phi_{0}')\). The generation function of \(\Phi_{0}\) is
\[
I_{\Phi_{0}} = \exp\left[ \frac{1}{N + 1} \right] \exp\left[ \frac{\tau \cdot \sigma}{N} \rho_{n}^{QR} \right].
\]

The action of the channel then is
\[
I_{\Phi_{0}} = (\mathcal{E} \otimes 1) I_{\Phi_{0}} = \exp(\mathcal{E} R \cdot b) \exp(\mathcal{E} \phi R \cdot b) \rho_{n}^{QR}
\]

according to the lemma, where \(p = \frac{N(N + 1)}{N + 1/2}\), and we have used the fact that \(\exp(\mathcal{E} R) = \exp(\mathcal{R} R \cdot b) \rho^{QR}\). The contribution to the entropy should be evaluated in the eigenbasis of \(\rho_{n}^{QR}\). We may denote the subspace of \(\rho_{n}^{QR}\) as \(i, j\) which has eigenvalue \(\lambda_{ij} = (1 - v_{A})^{n} (1 - v_{B})^{n} v_{A}^{1/2} v_{B}^{1/2}\), where \(i = (i_{1}, i_{2}, \ldots, i_{n})\), \(j = (j_{1}, j_{2}, \ldots, j_{n})\), \(j = (j_{1}, j_{2}, \ldots, j_{n})\). In this subspace, we denote \(\Phi_{0}\) as \(M_{ij}\), the elements of \(M_{ij}\) are \(i, j, m_{ij}, m_{ij}'\), the sum of the large particle of the eigenvalue of is \(\text{Tr} M_{ij}\). We obtain the contribution to the entropy by first evaluating \(i, j, m_{ij} \text{'}\) \(I_{\Phi_{0}} = (i, j, m_{ij}) V^{\otimes n} I_{\Phi_{0}} V^{\otimes n} \cdot p_{ij}, j, m_{ij}\)', which is \(\exp(\mathcal{E} \phi R / (N + 1)) \langle i, j, m_{ij}, j \rangle V_{\Phi_{0}}^{\otimes n} \cdot p_{ij}, j, m_{ij}\)', Here \(V\) is the unitary transformation that transforms \(\rho^{QR}\) to its product form of \(A\) and \(B\) parts. We expand the exponent of the operators to drop the terms that do not keep the total particle numbers in \(A\) and \(B\) parts respectively. Denote
\[
I_{B}(\tau, \sigma) = \sum_{k=0}^{\infty} \left( -\frac{1}{k!} \right)^{k} (p \cosh r)^{2k}(\tau \cdot b)^{k}(\sigma \cdot b)^{k},
\]

Then in the calculation of the contribution to the entropy become a trace on the whole space, the restriction on the subspace is removed. We have the generation function
\[
F = \exp[(\tau \cdot \sigma + \tau^{\prime} \cdot \sigma^{\prime})/(N + 1)]
\]

Thus
\[
\sum_{m=0}^{\infty} \frac{B^{m-j} (m-j)!}{(N(N + 1))^{2m}} \times (\tau \cdot \sigma)^{m}(\tau^{\prime} \cdot \sigma)^{m}.
\]

with \(B = N(B + 1) \cosh^{4} r, A = N(A + 1) \sinh^{4} r\). Where we have used another generation function in evaluating \(F\), and at the final step we exchange the orders of summation and make use of Eq. (4) to conceal the factor \(\exp[(\tau \cdot \sigma + \tau^{\prime} \cdot \sigma^{\prime})/(N + 1)]\). The generation function is about that the two ingredient both come from \(\Phi_{0}\), if both come from \(\Phi_{0}'\), the result will be the same. In the case intercross of \(\Phi_{0}\) and \(\Phi_{0}'\), we should substitute \((\tau \cdot \sigma)^{m}(\tau^{\prime} \cdot \sigma)^{m}\) in Eq. (20) by \((\tau \cdot \sigma')^{m}(\tau^{\prime} \cdot \sigma')^{m}\).

\[
\sum_{i,j} \sum_{m=0}^{\infty} \frac{B^{m-j} (m-j)!}{(N(N + 1))^{2m}} \times (\tau \cdot \sigma)^{m}(\tau^{\prime} \cdot \sigma)^{m}.
\]

When \(N' < N\), that is \(N > N_{n} / (1 - \eta)\), we have \(N/(N+1) > 1\). With Eq. (4), it is not difficult to prove that
\[
\frac{1}{N(N + 1)^{m}} \sum_{i,j}^{m} \left( \frac{m}{j} \right)^{2} B^{j} A^{m-j} < 1.
\]

Thus
\[
\lim_{N \to \infty} \left| I_{\Phi_{0}} (\rho_{n}, \mathcal{E}^{\otimes n}) - I_{\Phi_{0}} (\rho_{n}, \mathcal{E}^{\otimes n}) \right| < 0.
\]

Eq. (20) indicates that each perturbation term contributes to the entropy separately. In expanding \(\rho_{n}^{QR}\), there is the \(\varepsilon^{2}\) term which will also contributes to the entropy up to \(\varepsilon^{2}\). But this is negligible at large input energy.

5 Conclusions

We have shown that all first order perturbation to the input product identical thermal state can only decrease the coherent information at large input energy in the most general case of \(n\) use of channel. Any perturbation to the
input state contributes the coherent information independently. A linear combination of perturbations to the input state results a linear combination of the perturbations of the coherent information. In the sense of first order perturbation, the channel capacity of additive Gaussian quantum channel with loss is described by the long standing conjecture formula \[ \text{formula} \].

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