Complementary nil detour eccentric domination number of a graph

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Abstract
In this paper, the complementary nil detour eccentric domination number of a graph is defined. Few results on complementary nil detour eccentric domination numbers are obtained. The relationship among complementary nil domination number, complementary nil eccentric domination number, detour eccentric domination number and complementary nil detour eccentric domination numbers are discussed. Few theorems related to the above said numbers are stated and proved.

Keywords
Complementary nil domination number, Complementary nil eccentric domination number, Complementary nil detour eccentric domination number.

AMS Subject Classification
05C12, 05C69.

1. Introduction
The terms dominating set and domination number of a graphs are characterized by O.Ore [10] in the year 1962 and T.W.Haynes et al.,[4] discussed many dominating parameters. In 2010, T.N. Janakiraman et. al., [5] introduced the concept of eccentric domination in graphs. M.Bhanumathi and S. Muthammal [1] discussed eccentric domination in trees and different bounds on these concepts. In 2019, M. Ismayil and Priyadharshini [8] introduced detour eccentric domination number of a graph. Maximal domination number in graphs ar introduced by Kulli and Janakiram [7] in the year 1997. T. Chelvam and R. Chellathurai [11] introduced the complementary nil domination number of a graph. M. Bhanumathi and Sudhasenthil [2] introduced the complementary nil eccentric domination number of a graph and its bounds in 2016. Papers [9, 11] are motivated us to study the complementary nil detour eccentric domination number of a graph.

For any graph $G = (V,E)$ where $V$ is the set of vertices and $E$ is the set of edges. The order and size of $G$ are $n = |V|$ and $m = |E|$ respectively. The length of the shortest $x-y$ path joining is the distance $d(x,y)$. The detour distance is the length of the longest $x-y$ path in graph $G$ and is denoted by $D(x,y)$. For any vertex $x$ in $G$, $e_D(x) = \max\{D(x,y) : y \in V\}$ is called detour eccentricity of $x$. If $e_D(x) = D(x,y)$ then $y$ is called a detour eccentric vertex of $x$ in $G$. The detour radius $R$ and detour diameter $D$ of $G$ are denoted and defined by $R_D = \min\{e_D(y) : y \in V\}$ and $D_D = \max\{e_D(y) : y \in V\}$ respectively. The vertex $y$ in $G$ is called detour central vertex if $e_D(y) = rad_D(G)$. The detour eccentric set of a vertex $y$ in $G$ is defined as $E_D(y) = \{x \in V / D(x,y) = e_D(y)\}$. A set $D \subseteq V$ is said to be a eccentric set if for all $y$ in $V - D$ is an eccentric vertex of at least one $x$ in $D$. A set $D \subseteq V$ is said to be a detour eccentric set if for all $y$ in $V - D$ is a detour eccentric vertex of at least one $x$ in $D$.

A set $D \subseteq V$ is said to be a dominating set in $G$, if every vertex in $V - D$ is dominated by a few vertex in $D$. $\gamma(G) = \min\{|D| / D$ is a dominating set $\}$ is called the domination number of $G$. If $D_1 \subseteq V$ in a graph $G$ is a dominating set and a detour set then $D_1$ is called a detour dominating
set (DD-set). $\gamma_{D}(G) = \min\{|D_1|/D_1\text{ is a detour dominating set}\}$ is called a detour domination number. If $D_2 \subseteq V$ is a dominating set and every $v \in V - D_2$, there exist at least one eccentric vertex $v$ in $D_2$ then $D_2$ is called an eccentric dominating set (ED-set). $\gamma_{D}(G) = \min\{|D_2|/(D_2\text{ is an eccentric dominating set})\}$ is called an eccentric domination number. If $D_3 \subseteq V$ is an ED-set and every $v \in V - D_3$, there exists at least one detour eccentric vertex $v$ of $u$ in $D_3$ then $D_3$ is called a detour eccentric dominating set (DED-set).

$\gamma_{D}(G) = \min\{|D_3|/D_3\text{ is a DED-set}\}$ is called a detour eccentric domination number. A dominating set $D_4 \subset V$ of $G$ is a complementary nil dominating set (CND-set) if $V - D_4$ is not a dominating set for $G$. $\gamma_{cND}(G) = \min\{|D_4|/D_4\text{ is a CND-set}\}$ is called a complementary nil eccentric domination number of $G$. An ED-set $D_5 \subset V$ of $G$ is a complementary nil eccentric dominating set (CND-set) if $V - D_5$ is not an ED-set for $G$. $\gamma_{cned}(G) = \min\{|D_5|/D_5\text{ is a CND-set}\}$ is called a complementary nil eccentric domination number of $G$ and $\Gamma_{cned}(G) = \max\{|D_5|/D_5\text{ is a CND-set}\}$ is called upper complementary nil eccentric domination number of $G$.

2. Prior Results

**Theorem 2.1.** [5]

\[ \gamma_{ed}(P_n) = \left\lfloor \frac{n}{3} \right\rfloor, \quad \text{if} \quad n = 3k + 1, \]
\[ \gamma_{ed}(P_n) = \left\lfloor \frac{n}{3} \right\rfloor + 1, \quad \text{if} \quad n = 3k \quad \text{or} \quad 3k + 2. \]

**Theorem 2.2.** [2]

\[ \gamma_{cned}(G) = \frac{n}{2} \]

or $\frac{n}{2} + 1$ where $n$ is even. 

**Theorem 2.3.** [2]

1. $\gamma_{ed}(K_{1,n}) = \gamma_{cned}(K_{1,n}) = 2$.
2. $\gamma_{cned}(K_{m,n}) = \min\{m,n\} + 1$, for $m,n \geq 2, m \geq n$.

In this paper we consider a non-trivial connected simple graphs only and all the undefined terms we can refer the book [3].

3. Complementary Nil Detour Eccentric Domination

In this section the complementary nil detour eccentric domination set and its numbers and given with appropriate illustration. The relations among the complementary nil domination numbers, the complementary nil detour dominating number and the complementary nil detour eccentric domination numbers are obtained.

**Definition 3.1.** A DED-set $C \subset V$ in a graph $G$ is said to be a complementary nil detour eccentric dominating set (CNDED-set) if $V - C$ is not a DED-set. A CNDED-set is called minimal if there is no set $C' \subset C$ is a CNDED-set. $\gamma_{cn-Ded}(G) = \min\{|C|/C\text{ is a CNDED-set}\}$ is called complementary nil detour eccentric domination number, $\Gamma_{cn-Ded}(G) = \max\{|C|/C\text{ is a CNDED-set}\}$ is called an upper complementary nil detour eccentric domination number. $\gamma_{cn-Ded}$-set is called minimum CNDED-set. Similarly, $\gamma_{ed}$-set, $\gamma_{Ded}$-set, $\gamma_{cned}$-set, $\gamma_{cn-Ded}$-set.

**Remark 3.2.** If $C$ is a CNDED-set, then $V - C$ not contain a dominating set or detour eccentric set but not both.

**Example 3.3.**

![Fig:3.1.](image1)

From the figure 3.1 $C = \{x_3, x_4, x_5, x_6, x_7\}$ is a minimum CNDED-set.

$C_1 = \{x_3, x_4, x_5\}$ is a minimum ED-set.

$C_2 = \{x_1, x_4, x_6, x_7\}$ is a minimum DED-set.

$C_3 = \{x_4, x_5, x_6, x_3\}$ is a minimum CND-set.

$C_4 = \{x_3, x_4, x_5, x_6\}$ is a minimum CNDED-set.

Therefore,

\[ \gamma_{ed} = 3, \gamma_{Ded} = 4, \gamma_{cned} = 4, \gamma_{cn-Ded} = 5. \]

$\gamma_{cned} < \gamma_{ned} < \gamma_{cn-Ded}$ and $\gamma_{ed} < \gamma_{Ded}$.

**Example 3.4.**

![Fig:3.2.](image2)

From the figure 3.2 $C = \{x_1, x_2, x_3\}$ is a minimum CND, CNDE and CNDED-set.
Theorem 3.10. For any star graph, $\gamma_{cn-\text{Ded}}(G) = 2$.

Proof. By the observation 3.5 (3) lower bound obtained and every CNDED-set may be contain maximum of $n - 1$ vertices. Therefore upper bound also obtained.

Now, the bounds are sharp, Since

$$\gamma_{cn-\text{Ded}}(G) = 2 \Leftrightarrow G = K_{1,n}$$

and since

$$\gamma_{cn-\text{Ded}}(G) = n - 1 \Leftrightarrow G = (K_{1,n}^c).$$

\Box

4. Bounds for Complementary Nil Detour Eccentric Domination Number

In this section, the complementary nil detour eccentric domination numbers are obtained for few standard graphs and theorems related to the above said concepts are stated and proved.

Theorem 4.1. (i) $\gamma_{cn-\text{Ded}}(K_{1,n}) = 2, n \geq 2$.

(ii) $\gamma_{cn-\text{Ded}}(W_n) = 4, n \geq 5$.

Proof. We

(i) If $G = K_{1,n}$ and let $C = \{x,y\}$ and $y$ be a central vertex. Then the central vertex dominates all vertices which are in $V - C$ and $x$ is a DED-set. If $C$ is a CNDED-set, then $V - C$ not contain a dominating set or detour eccentric set or both. Hence $\gamma_{cn-\text{Ded}}(K_{1,n}) = 2, n \geq 2$.

(ii) If $G = W_n$, for $n \geq 5$, and let $x$ be a central vertex of $G$, then $x$ is a detour eccentric point of all other vertices of $G$ and $\{x\}$ is also a dominating set. Hence, $\{x\}$ is a minimum DED-set. The CNDED-set, not in $V - C$ or detour eccentric set. Hence, $\gamma_{cn-\text{Ded}}(W_n) = 4, n \geq 5$.

\Box

Theorem 3.7. For any path $P_n$ graph,

$$\gamma_{cn-\text{Ded}}(P_n) = \gamma_{ned}(P_n) = \frac{n}{2} or \frac{n}{2} + 1$$

where $n$ is even and $n > 4$.

Proof. By theorem 2.1(2), we obtain the result.

Theorem 3.8. For any complete bipartite graph, $\gamma_{cn-\text{Ded}}(K_{m,n}) = \gamma_{ned}(K_{m,n}) = \min\{m,n\} + 1$, for $m,n \geq 2, m \geq n$.

Proof. By theorem 2.3 (2), we obtain the result.

Theorem 3.9. For any star graph, $\gamma_{cn-\text{Ded}}(K_{1,n}) = \gamma_{ned}(K_{1,n}) = 2, n \geq 2$.

Proof. By theorem 2.3 (1), we obtain the result.

Theorem 3.10. If $G$ is not a complete graph, then $2 \leq \gamma_{cn-\text{Ded}} \leq n - 1$. 

Suppose $x$ is adjacent to all the vertices in $C$ and all the detour eccentric vertices of $x$ in $C$. Hence, $C - \{x\}$ is a DED-set and detour eccentric point set of $G$, which is contradiction to the minimality of $C$. 

$C_1 = \{x_1,x_3\}$ is a minimum ED-set, a minimum DED-set, a minimum CND-set and a minimum CNED-set.

Therefore,

$$\gamma_{ed} = 2, \gamma_{ded} = 2, \gamma_{ned} = 3, \gamma_{ned} = \gamma_{cn-\text{Ded}} = 3.$$

Observation 3.5. In a graph $G$

1. $\gamma_{ed}(G) \leq \gamma_{cn-\text{Ded}}(G)$

2. $\gamma_{cn-\text{Ded}}(G) \leq \Gamma_{cn-\text{Ded}}(G)$

3. Every CNDED-set contains at least 2 vertices.

4. There is no relationship between detour eccentric domination number and complementary nil detour eccentric domination number.

5. $\gamma_{cn-\text{Ded}}$ is not exists for complete graph.

Since, $G = K_n$ then $rad_{D} = diam_{D} = n - 1$.

Hence any vertex $x$ in $K_n$ dominates all other vertices and is also a detour eccentric point of other vertices.

Hence, complementary nil detour eccentric domination does not exist in complete graph.

Theorem 3.6. For any tree $T_n, n \geq 3$, $\gamma_{cn-\text{Ded}} \leq n - \Delta(T_n) + 1$.

Proof. In any tree $T_n$, $V - S$ is a DED-set of vertices adjacent to a vertex with maximum degree. Therefore, $|V - S| + 1$ is a CNDED-set and hence

$$\gamma_{cn-\text{Ded}} \leq n - \Delta(T_n) + 1.$$

Therefore,

$$\gamma_{cn-\text{Ded}} \leq n - \Delta(T_n) + 1 = \min\{m,n\} + 1.$$

Theorem 3.7. For any path $P_n$ graph,

$$\gamma_{cn-\text{Ded}}(P_n) = \gamma_{ned}(P_n) = \frac{n}{2} or \frac{n}{2} + 1$$

where $n$ is even and $n > 4$.

Proof. By theore 2.1(2), we obtain the result.

Theorem 3.8. For any complete bipartite graph, $\gamma_{cn-\text{Ded}}(K_{m,n}) = \gamma_{ned}(K_{m,n}) = \min\{m,n\} + 1$, for $m,n \geq 2, m \geq n$.

Proof. By theorem 2.3 (2), we obtain the result.

Theorem 3.9. For any star graph, $\gamma_{cn-\text{Ded}}(K_{1,n}) = \gamma_{ned}(K_{1,n}) = 2, n \geq 2$.

Proof. By theorem 2.3 (1), we obtain the result.

Theorem 3.10. If $G$ is not a complete graph, then $2 \leq \gamma_{cn-\text{Ded}} \leq n - 1$. 

Proof. By the observation 3.5 (3) lower bound obtained and every CNDED-set may be contain maximum of $n - 1$ vertices. Therefore upper bound also obtained.
(ii) Suppose there exists \( x \in V - C \) shows that \( N(y) \cap C = \emptyset \). Then \( C \) is not a dominating set and \( E_D(V) \cap C = \emptyset \). Therefore, \( C \) is not an ED-set.

(iii) Suppose \( V - (C - \{x\}) \) is not a DED-set. This implies that \( C - \{x\} \) is a CNDED-set of \( G \), which is a contradiction to the minimality of \( D \).

Hence for each \( x \in D \) any one of the three condition (i), (ii), (iii), satisfied.

On the other hand, Assume \( C \) may be a CNDED-set and for each \( x \in C \), one of the conditions holds, we show that \( C \) is a minimal CNDED-set.

Assume that \( C \) is not a minimal CNDED-set, that is, there exists a vertex \( x \in C \) such that \( C - \{x\} \) is a CNDED-set.

Hence, \( x \) is adjacent to at least one vertex \( y \in (C - \{x\}) \) and \( x \) has a detour eccentric point in \( C - \{x\} \).

Therefore, condition (i) does not hold. Also if, \( C - \{x\} \) is a CNDED-set, every element \( u \in V - (C - \{x\}) \) is adjacent to at least one vertex in \( C - \{x\} \) and \( u \) has a detour eccentric point in \( C - \{x\} \).

Hence condition (ii) does not hold. Since \( C - \{x\} \) is a CNDED-set, \( V - (C - \{x\}) \) is not a DED-set, that is, condition (iii) does not hold. Therefore, there exists \( x \in C \) such that \( x \) does not satisfy conditions (i), (ii), (iii) which is a contradiction to our assumption.

**Theorem 4.3.** For any graph \( G \) with an end vertex, then

\[
\gamma_{cn-Ded}(G) = \gamma_{Ded}(G) \text{ or } \gamma_{Ded}(G) + 1.
\]

**Proof.** Let \( C \) be a \( \gamma_{Ded} \)-set of \( G \). Let \( x \) be an end vertex in \( G \). If \( x \) and its support vertex is in \( C \), then \( V - C \) is not a dominating set.

Therefore, \( \gamma_{cn-Ded}(G) = \gamma_{Ded}(G) \). If \( x \) or its support vertex \( y \) is in \( C \), then \( C_1 = C \cup \{y\} \) or \( C_1 = C_1 \cup \{x\} \) is a DED-set and \( V - C_1 \) is not a dominating set.

Therefore,

\[
\gamma_{cn-Ded}(G) = \gamma_{Ded}(G) + 1.
\]

**Theorem 4.4.** If \( G \) is of radius two, then

\[
\gamma_{cn-Ded}(G) \leq \Delta(G) + 1.
\]

**Proof.** Let the radius of a graph \( G \) be 2. Let \( x \in V(G) \) be such that maximum \( \deg x(\Delta(G)) \). Now take \( C = \{x\} \cup N(x) = N[x] \). Every vertex in \( V - C \) is adjacent to elements of \( N(x) \) and are detour eccentric to \( x \). This implies that \( C \) is a DED-set, and \( V - C = V - N(x) \), this \( V - C \) has no dominating set. Since \( x \) cannot be dominated by any element of \( -C \).

Therefore, \( C \) is a CNDED-set. Hence

\[
\gamma_{cn-Ded}(G) \leq \Delta(G) + 1.
\]

**Theorem 4.5.** If \( G \) is not complete, then

\[
\gamma_{cn-Ded}(G) \leq \gamma_{Ded}(G) + \delta(G).
\]

**Proof.** Let \( C \) be the \( \gamma_{Ded} \)-set of \( G \) and \( x \in V \) such that \( \deg x = \delta(G) \). If \( x \in V - C \), there exists \( y \in N(x) \) such that \( y \in D \). \( |N(x)| = \delta(G) \). Now \( C_1 = C \cup N[x] \) is a DED-set and \( V - C_1 \) is not a dominating set. Hence,

\[
\gamma_{cn-Ded}(G) \leq \gamma_{Ded}(G) + \delta(G).
\]

**Theorem 4.6.** Let \( T \) be a tree such that each non end vertex is adjacent to minimum one end vertex. Then

\[
\gamma_{cn-Ded}(T) \leq t + 1,
\]

where \( t \) is the number of support vertices.

**Proof.** Let \( s \) be the number of support vertices. All the non end vertices form a dominating set. To form a complementary nil detour dominating set we have to add at least one end vertex. Therefore,

\[
\gamma_{cn-Ded}(T) \leq t + 1.
\]

**Theorem 4.7.** If \( G \) is a graph with \( rad_D(G) \geq 3 \), then

\[
\gamma_{cn-Ded}(G) \leq q - \delta(G).
\]

**Proof.** Let \( y \) be a vertex such that \( \deg y = \delta(G) \), where \( q \) is even. Since \( rad_D(G) \geq 3 \), there exists a vertex \( x \) in \( V - N[y] \) but \( x \) is not adjacent to any vertex in \( N[y] \) and every vertex in \( N[y] \) has a detour eccentric point in \( V - N(y) \). Now, \( V - N(y) \) is a DED-set, but vertices in \( N(y) \) has no detour eccentric points in \( V - N(y) \). So, \( V - N(y) \) is a DED-set, but \( N(v) \) is not a DED-set. Therefore,

\[
\gamma_{cn-Ded}(G) \leq |V - N(y)| = q - \delta(G).
\]

**Theorem 4.8.** If \( G \) is a caterpillar such that each non end vertex is of degree three, then

\[
\gamma_{cn-Ded}(G) = \frac{q}{2} + 1.
\]

**Proof.** Since degree of each non end vertex is three, \( G \) is of the following form given in fig 4.1.

![Fig 4.1](image)
All the non-pendent vertices form a dominating set but not a detour dominating set. Therefore all the pendent vertices form a DED-set.

Hence,

$$\gamma_{cn-Ded}(G) = \frac{q}{2} + 1.$$