SIMPLE $S_{r}$-HOMOTOPY TYPES OF HOM COMPLEXES AND BOX COMPLEXES ASSOCIATED TO $r$-GRAPHS

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Abstract. For a pair $(H_{1}, H_{2})$ of graphs, Lovász introduced a polytopal complex called the Hom complex $\text{Hom}(H_{1}, H_{2})$, in order to estimate topological lower bounds for chromatic numbers of graphs. The definition is generalized to hypergraphs. Denoted by $K_{r}$, the complete $r$-graph on $r$ vertices. Given an $r$-graph $H$, we compare $\text{Hom}(K_{r}, H)$ with the box complex $\mathcal{B}_{\text{edge}}(H)$, invented by Alon, Frankl and Lovász. We verify that $\text{Hom}(K_{r}, H)$ and $\mathcal{B}_{\text{edge}}(H)$, both are equipped with right actions of the symmetric group on $r$ letters $S_{r}$, are of the same simple $S_{r}$-homotopy type.

1. Introduction

In this paper, we consider homotopy types of cell complexes associated to $r$-graphs which are introduced in order to solve the problem on their chromatic numbers. The idea of assigning a cell complex to graphs was due to Lovász in [Lov78] in his proof of the Kneser’s conjecture [Kne56]. To a graph $G$, Lovász assigned a simplicial complex $N(G)$, called the neighborhood complex. By using its topological property, that is to say, the $k$-connectivity of $N(G)$, he succeeded in discovering a new lower bound for the chromatic number of $G$.

In the case of hypergraphs, the first topological lower bound for the chromatic number of an $r$-graph was derived by a simplicial complex $\mathcal{B}_{\text{edge}}(G)$ called the box complex, which was invented by Alon, Frankl and Lovász [AFL86]. It also played an important role in a proof of the Erdős’ conjecture [Erd76], which is a generalization of Kneser’s conjecture.

Lovász also introduced a polytopal complex associated to a pair $(G, H)$ of graphs, called the Hom complex $\text{Hom}(G, H)$. It is a generalization of $N(H)$ in view of $\text{Hom}(K_{2}, H)$ and $N(H)$ having the same homotopy type [Koz06]. Here $K_{2}$ denotes the complete graph on 2 vertices. There are also many researches on the homotopy type of $\text{Hom}(K_{r}, H)$, comparing with other simplicial complexes constructed for (hyper)graph coloring problems such as $\mathcal{B}_{\text{chain}}(G)$ by Kríž [Kri92] or $\mathcal{B}(G), \mathcal{B}_{0}(G)$ by Matoušek and Ziegler [MZ04]. However, there are still no results in the case of $r$-graphs. The motivation of this research is to find an $r$-graph which generalizes the results to the case of $r$-graphs.

The construction of the Hom complex is also extended to hypergraphs by Kozlov in [Koz07]. We notice here that the complete $r$-graph on $r$ vertices $K_{r}$ is the only $r$-graph having one edge as $K_{2}$, and that both $\text{Hom}(K_{r}', H)$ and $\mathcal{B}_{\text{edge}}(H)$ are equipped with right actions of the symmetric group on $r$ letters $S_{r}$. We obtain the following result on equivariant simple homotopy types by making use of equivariant acyclic partial matchings:

**Theorem** (Theorem 4.11). For any $r$-graph $H$, the Hom complex $\text{Hom}(K_{r}', H)$ and the box complex $\mathcal{B}_{\text{edge}}(H)$ have the same simple $S_{r}$-homotopy type.

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2. Preliminaries

In this section, we collect some definitions which are needed in our arguments. First, we write \([k]\) as the set \([0,1,\ldots,k]\).

**r-graphs.** A hypergraph is a triple \(H = (V(H), E(H), \varepsilon_H)\) of sets \(V(H), E(H)\) and a map \(\varepsilon_H : E(H) \to \bigsqcup_{r \geq 1} (V(H)^r/S_r)\). Here \(S_r\) is the symmetric group on \(r\)-letters acting on \(V(H)^r\) by permutation. Given two hypergraphs \(H_1\) and \(H_2\), a hypergraph homomorphism is a pair \((f_V, f_E)\) of \(f_V : V(H_1) \to V(H_2)\) and \(f_E : E(H_1) \to E(H_2)\) satisfying the following commutative diagram:

\[
\begin{array}{ccc}
E(H_1) & \xrightarrow{\varepsilon_{H_1}} & \bigsqcup_{r \geq 1} (V(H_1)^r/S_r) \\
\downarrow{f_E} & & \downarrow{\tilde{f}_V} \\
E(H_2) & \xrightarrow{\varepsilon_{H_2}} & \bigsqcup_{r \geq 1} (V(H_2)^r/S_r) ,
\end{array}
\]

where \(\tilde{f}_V\) is the map induced by \(f_V\). Then, we obtain the category \(\mathbf{H}\) of hypergraphs and hypergraph homomorphisms.

We denote here an equivalence class \([v_0, v_1, \ldots, v_{r-1}] \in V(H)^r/S_r\) simply by \(v_0v_1\ldots v_{r-1}\). A hypergraph \(H\) is \(r\)-uniform if \(\text{Im } \varepsilon_H \subset V(H)^r/S_r\). \(H\) is simple if \(\varepsilon_H\) is injective. Moreover, \(H\) is nondegenerate if

\[
\text{Im } \varepsilon_H \subset \bigsqcup_{r \geq 1} \{v_0 \cdots v_{r-1} \in V(H)^r/S_r \mid v_i \neq v_j \text{ whenever } i \neq j\} .
\]

For simplicity, simple nondegenerate \(r\)-uniform hypergraphs are called \(r\)-graphs. Denoted by \(\mathbf{H}_r\) the full subcategory of \(\mathbf{H}\) consisting of \(r\)-graphs. For example, the complete \(r\)-graph on \(m\) vertices, denoted by \(K_m^r\), is an \(r\)-graph with \(|V(K_m^r)| = m\) and \(\varepsilon_{K_m^r}\) being bijective. Since the map \(\varepsilon_H\) of an \(r\)-graph \(H\) is a bijection \(E(H) \to \text{Im } \varepsilon_H\), for simply, we identify \(E(H)\) with \(\text{Im } \varepsilon_H\), and write, for example, \(v_0\ldots v_{r-1} \in E(H)\).

**The category \(\mathbf{C-G}\).** Let \(G\) be a group. Denoted by \(G^\text{opp}\) the group whose elements are elements of \(G\) and multiplication defined by \(gh\) (in \(G^\text{opp}\)) = \(hg\) (in \(G\)). For an object \(X\) of a category \(\mathbf{C}\), a right action of \(G\) on \(X\) is a homomorphism \(\rho : G^\text{opp} \to \text{Hom}_\mathbf{C}(X,X)\). We denote by \(\mathbf{C-G}\) the category whose objects are all pairs \((X, \rho)\) of object \(X\) of \(\mathbf{C}\) and a right action \(\rho\). A morphism from \((X_1, \rho_1)\) to \((X_2, \rho_2)\) is a morphism \(f \in \text{Hom}_\mathbf{C}(X_1, X_2)\) such that \(f \circ \rho_1(g) = \rho_2(g) \circ f\) for any \(g \in G^\text{opp}\). We note here that, for for example \(v_0\ldots v_{r-1} \in E(H)\).

**Simplicial complexes and polytopal complexes.** An (abstract) simplicial complex is a pair \((V, K)\) of a set \(V\) and a collection \(K\) of subsets of \(V\) closed under taking subsets. We denote a simplicial complex \((V, K)\) briefly by \(K\) and write \(V\) as \(V(K)\). Each elements in \(V\) is called a simplex or a cell of \(K\). If \(F \in K\) and \(F' \subset F\), we say that \(F'\) is a face of \(F\), and, at the same time, \(F\) is a coface of \(F'\). A subcomplex of \(K\) is a simplicial complex \(K'\) such that \(F \in K'\) implies that \(F \in K\).

For two simplicial complex \(K\) and \(K'\), a simplicial map \(f : K \to K'\) is a map \(f : V(K) \to V(K')\) satisfying that \(f(F) \in K'\) if \(F \in K\). Let \(\text{ASC}\) denote the category of simplicial complexes and simplicial maps. In particular, an object in the category \(\text{ASC-G}\) is called a simplicial \(G\)-complex.

Let \(P\) be a convex polytope. A proper face of \(P\) is of the form \(\text{conv}(V(P) \cap h)\), where \(h\) is a hyperplane satisfying \((\text{Int } P) \cap h = \emptyset\) and \(V(P)\) denotes the vertex set of \(P\). The term “coface” for convex polytopes is also defined analogously. Note here that the empty set is also a proper face of any polytopes.

A polytopal complex is a collection \(K\) of convex polytopes in some \(\mathbb{R}^N\) satisfying that (1) every face of \(P \in K\) is also in \(K\), and (2) the intersection of \(P_1, P_2 \in K\) is a face of both. Elements in \(K\) are
called cells of $K$. The underlying space of a polytopal complex $K$ is the subspace of $\mathbb{R}^N$ defined by $|K| = \bigcup_{P \in K} P$. A subcomplex of $K$ is a subcollection $K'$ of $K$ which is itself a polytopal complex.

For two polytopal complexes $K_1$ and $K_2$, a polytopal map $f : K_1 \rightarrow K_2$ is a map $f : |K_1| \rightarrow |K_2|$ satisfying that the restrictions $f|_F$ to each $F \in K_1$ is affine. Moreover, a polytopal map $f : K_1 \rightarrow K_2$ is said to be regular if $F \in K_1$ implies that $f(F) \in K_2$. In this paper, we will make use of the category $\text{PTC}$ consisting of polytopal complexes and polytopal maps and its subcategory $\text{PTC}_{\text{reg}}$ consisting of polytopal complexes and regular polytopal maps. An object of the category $\text{PTC}_{\text{G}}$ (or $\text{PTC}_{\text{reg,G}}$) is called a polytopal $G$-complex.

**Posets.** Let $\text{Poset}$ denote the category of posets and poset maps (i.e. a map $f : P \rightarrow Q$ satisfying $f(x) \leq_Q f(y)$ whenever $x \leq_P y$). An object in the category $\text{Poset}-G$ is called a $G$-poset.

Given a poset $P$, we call a totally ordered subset $A = \{A_0, A_1, \ldots, A_k\}$, where each $A_i \in P$ and $A_0 <_P A_1 <_P \cdots <_P A_{k-1}$ a $k$-chain in $P$. The number $k$ is called the length of $A$, denoted by $|A|$. In this paper, elements in a chain $A$ in $P$ are written by $A_i$ ($i \in |A|$). The order complex of $P$, denoted by $\Delta(P)$, is the simplicial complex on $P$ whose $k$-simplices are the $k$-chains in $P$. A poset map $f : P \rightarrow Q$ induces a simplicial map $\Delta(f) : \Delta(P) \rightarrow \Delta(Q)$, and so $\Delta(\cdot)$ is a covariant functor $\text{Poset} \rightarrow \text{ASC}$.

The face poset of a simplicial (polytopal) complex $K$, denoted by $\mathcal{F}(K)$, is a poset of all nonempty cells of $K$ ordered by inclusion. Each simplicial (polytopal) map $f : K \rightarrow K'$ induces a poset map $\mathcal{F}(f) : \mathcal{F}(K) \rightarrow \mathcal{F}(K')$. So we obtain covariant functors $\mathcal{F}(\cdot) : \text{ASC} \rightarrow \text{Poset}$ and $\mathcal{F}(\cdot) : \text{PTC} \rightarrow \text{Poset}$.

For $x, y \in P$, we call $x$ covers $y$, and write $x \succ y$, if $y <_P x$ and there is no $z \in P$ such that $y <_P z <_P x$.

3. **Equivariant simple homotopy types**

Now let $K$ be a simplicial or a polytopal complex. Maximal cells of $K$ are called facets. A cell $\sigma \in K$ is free if $\sigma$ is a proper face of only one facet $\varphi_{\sigma} \in K$. A collection $\mathcal{F}$ of free cells of $K$ is said to be independently free if, for any $\sigma, \sigma' \in \mathcal{F}, \sigma \neq \sigma'$ implies that there is no cell in $K$ which is a coface of both $\sigma$ and $\sigma'$.

The deletion of a cell $F \in K$, denoted by $\text{dl}_F(K)$, is the subcomplex of $K$ consisting of all $F' \in K$ such that $F$ is not a face of $F'$. We also define the deletion $\text{dl}_S(K)$ of a set $S$ of cells of $K$ from $K$ as the intersection of $\text{dl}_F(K)$ over all $F \in S$.

Now we define the notion of $G$-collapsings, following Larrión et. al. in [LPVF08]. Note here that, for a simplicial (polytopal) $G$-complex $K$, the orbit $\sigma G$ of a free cell $\sigma \in K$ is a collection of free cells in $K$. Let $\sigma$ be a free cell of $K$ with dim $\varphi_{\sigma} = \dim \sigma + 1$. Suppose $\sigma G$ being independently free. An elementary $G$-collapsing of $K$ with respect to $\sigma$ is defined as the process to obtain $\text{dl}_{\sigma G}(K)$ from $K$. Conversely, an elementary $G$-expanding of $K$ with respect to $\sigma$ is defined to be the process to obtain $K$ from $\text{dl}_{\sigma G}(K)$.

We denote by $K \backslash_{\sigma G} K'$ if there exists an elementary $G$-collapsing of $K$ onto its $G$-subcomplex $K'$. Moreover, we say that $K$ $G$-collapses onto a $G$-subcomplex $K'$ if there is a sequence of elementary $G$-collapsings leading from $K$ to $K'$. Two simplicial (polytopal) $G$-complex $K$ and $L$ are said to have the same simple $G$-homotopy type if there is a sequence of elementary $G$-collapsings and elementary $G$-expansions leading from $K$ to $L$. Such a sequence is called a formal $G$-deformation.

3.1. **Simple $G$-homotopy types of subdivisions.** It is well-known on a relationship between a simplicial (polytopal) complex $K$ and its barycentric subdivision $sdK$ that they are of the same simple homotopy type. However, we need an equivariant version of this result in our argument.
Following the construction of a formal deformation by Kozlov in [Koz06], it is useful to define an equivariant stellar subdivision of $K$.

**Definition 3.1.** Let $K$ be a simplicial $G$-complex and $\sigma$ be a simplex of $K$ such that, in $\sigma G$, $g \neq g'$ implies that no simplex in $K$ being a coface of both $\sigma g$ and $\sigma g'$. The **stellar $G$-subdivision** of $K$ at the orbit $\sigma G$, denoted by $\text{sd}(K, \sigma G)$, is the simplicial $G$-complex on $V(K) \cup \sigma G$ with the following set of simplices:

$$\text{sd}(K, \sigma G) = \bigcap_{g \in G} \{ F \in K \mid \sigma g \text{ is not a face of } F \} \cup \bigcup_{g \in G} \{ F \} \bigcup \{ \sigma g \} \mid F \in K, \sigma g \text{ is not a face of } F, \text{ and } \sigma g \cup F \in K \}.$$

We can define the stellar subdivision for a polytopal $G$-complex $K$ analogously by replacing elements in $\sigma G$ with their barycenters.

Making use of stellar subdivisions, we obtain our desired result:

**Proposition 3.2.** Let $K$ be a simplicial or polytopal $G$-complex. Then $K$ and its barycentric subdivision $\text{sd}K$ have the same simple $G$-homotopy type.

**Proof.** Choose a cell $\sigma$ from each orbit such that they preserves inclusion order in $F(K)$ and construct a totally ordered set $L$ of these $\sigma$’s, such that $\bigcup_{\sigma \in L} \sigma G = F(K)$ as sets. Then a simplicial $G$-complex obtained by a sequence of stellar $G$-subdivisions of $K$ at the orbits of simplices in decreasing order with respect to $L$ is isomorphic to $\text{sd}K$. Hence, it suffices to consider a formal deformation leading from $K$ to the stellar $G$-subdivision $\text{sd}(K, \sigma G)$ at the orbit of the maximum cell $\sigma \in L$.

First, add cones over each stern($\sigma g$), $g \in G$. This construction implies that, for each face $\sigma'$ of $\sigma$, $\sigma'G$ is a collection of free cells which is independently free. Hence, we obtain a sequence of elementary $G$-expansions leading to cones. Here we obtain the unique facet containing $\sigma g \in \sigma G$ in each added cone. Then we obtain our desired result by taking an elementary $G$-collapsing with respect to $\sigma G$. \( \square \)

4. **Hom complexes**

The construction the Hom complexes was extended to hypergraphs by Kozlov [Koz07]. In this paper, however, we will consider only the one associated to a pair of $r$-graphs.

**Definition 4.1.** Let $H_1, H_2$ be $r$-graphs. A map $f : V(H_1) \to 2^{V(H_2)} \setminus \{\emptyset\}$ is called a hypergraph multihomomorphism if every map $f_0 : V(H_1) \to V(H_2)$ satisfying $f_0(v) \in f(v)$ for any $v \in V(H_1)$ induces a hypergraph homomorphism.

For hypergraphs $H_1, H_2$, we write $P_{H_1, H_2}$ as the poset of all hypergraph multihomomorphisms ordered by $f \leq g$ if and only if $f(v) \subseteq g(v)$ for any $v \in V(H_1)$. The Hom complex $\text{Hom}(H_1, H_2)$ is construed from this poset as follows:

**Definition 4.2.** Let $H_1, H_2$ be $r$-graphs. The **Hom complex** is the polytopal complex

$$\text{Hom}(H_1, H_2) = \left\{ \prod_{v \in V(H_1)} \Delta^{f(v)} \right\}_{f \in P_{H_1, H_2}}.$$

Here $\Delta^S$ denotes a simplex with the vertex set $S$. 
Denoted by $H'_r$ a subcategory of $H_r$ consisting of $r$-graphs and injective hypergraph homomorphisms. By definition, we obtain the following commutative diagrams concerning functorial proper-

\[
\begin{array}{ccc}
H_r & \xrightarrow{P_{H_r}} & \text{Poset} \\
\downarrow & & \downarrow \pi_\mathcal{F} \\
H'_r & \xrightarrow{\text{Hom}(H,-)} & \text{PTC}_{\text{reg}}
\end{array}
\]

In particular, we obtain right Aut($H_1$)-actions on both the poset $P_{H_1,H_2}$ and the polytopal complex $\text{Hom}(H_1,H_2)$. Furthermore, we can see that $f < g$ in $P_{H_1,H_2}$ if and only if $\prod_{v \in \mathcal{V}(H_1)} \Delta^{j(v)}$ is a proper face of $\prod_{v \in \mathcal{V}(H_1)} \Delta^{r(v)}$. Therefore, $\mathcal{F}(\text{Hom}(H_1,H_2))$ and $P_{H_1,H_2}$ are Aut($H_1$)-isomorphic as posets, and $\text{sd} \text{Hom}(H_1,H_2)$ and $\Delta(P_{H_1,H_2})$ are Aut($H_1$)-isomorphic as simplicial complexes.

4.1. **Comparison between Hom complexes and box complexes.** Let $(G,H)$ be a pair of $r$-graphs. As stated before, we are interested in homotopy type of the Hom complex $\text{Hom}(G,H)$, comparing with simplicial complexes associated to an $r$-graph $H$. We now give the definition of the box complex $\mathcal{B}_{\text{edge}}(H)$ invented by Alon, Frankl and Lovász in [AFL86]:

Recall that the collection $\{A_j\}_{j=0}^{r-1}$ of subsets of $V(H)$ generates the complete $r$-partite sub-$r$-graph in $H$ if, for any $x_j \in A_j$, $j \in [r-1]$, $x_0x_1 \cdots x_{r-1}$ is an edge of $H$. In particular, if $V(H) = \bigcup_{j=1}^{r} A_j$, $H$ itself is said to be the complete $r$-partite $r$-graph, denoted by $K_r^{m_0, \ldots, m_{r-1}}$ if $|A_j| = m_j$, $j \in [r-1]$.

**Definition 4.3** (See [AFL86]). Let $H$ be an $r$-graph. A simplicial complex $\mathcal{B}_{\text{edge}}(H)$ is defined to be a pair $(V, \mathcal{B}_{\text{edge}}(H))$ of the vertex set $V$ consisting of all $(v_1, \ldots, v_r) \in V(H)^r$ such that $v_1 \cdots v_r \in E(H)$ and the set of simplices $\mathcal{B}_{\text{edge}}(H)$ consisting of all subsets $F \subseteq V$ such that $\{pr_j(F)\}_{j=1}^{r}$ is the collection of pairwise disjoint sets generating the complete $r$-partite sub-$r$-graph in $H$. Here $pr_j(F)$ denotes the projection of $F$ onto its $j$-th factor.

Now we consider relationships between the Hom complexes and the box complexes. As stated before, $\text{Hom}(K_2, H)$ has the same (simple) homotopy type as the neighborhood complex $\mathcal{N}(H)$ and other box complexes. In the case of $r$-graph, since $K_2$ has only one edge, we thought that the complete $r$-graph $K'_r$, which also has only one edge, may play an important role in determining homotopy types of the Hom complexes. Thus, we now compare homotopy types between $\text{Hom}(K'_r, H)$ and $\mathcal{B}_{\text{edge}}(H)$. However, we cannot do it directly because $\text{Hom}(K'_r, H)$ is a polytopal while $\mathcal{B}_{\text{edge}}(H)$ is a simplicial complex. We consider their face posets and construct two maps between them as follows:

\[
p : \mathcal{F}(\mathcal{B}_{\text{edge}}(H)) \to P_{K'_r,H}; \quad p(F)(j) = pr_j(F);
\]

\[
i : P_{K'_r,H} \to \mathcal{F}(\mathcal{B}_{\text{edge}}(H)); \quad i(\varphi) = \prod_{j=1}^{r} \varphi(j).
\]
Notice here that both \( \text{Hom}(K'_r, H) \) and \( B_{\text{edge}}(H) \) are equipped with right \( S_r \)-actions. We claim that both \( p \) and \( i \) are \( S_r \)-equivariant poset maps whose composition \( p \circ i \) is the identity on \( P_{K'_r, H} \).

Indeed, for the \( S_r \)-equivariance of \( p \), given a simplex \( S = \{ (v'_1, \ldots, v'_r) \}_{j \in J} \in \mathcal{F}(B_{\text{edge}}(H)) \) and \( \sigma \in S_r \), we have \( S \sigma = \{ (v'_{\sigma(j)}, \ldots, v'_{\sigma(r)}) \}_{j \in J} \). Recall that the right \( S_r \)-action on \( P_{K'_r, H} \) is given as, for \( \sigma \in S_r \), \( \sigma : P_{K'_r, H} \to P_{K'_r, H}; \ \varphi \mapsto \varphi \circ \sigma \). Hence, for all \( l \in [r] \),
\[
p(S \sigma(l)) = \{ v'_{\sigma(j)} \}_{j \in J} = p(S)(\sigma(l)) = (p(S)\sigma)(l).
\]

For the \( S_r \)-equivariant of \( i \), given \( f \in P_{K'_r, H} \) and \( \sigma \in S_r \), we have
\[
i(f)\sigma = \left( \prod_{j=1}^{r} f(j) \right) \sigma = \{ (v_1, \ldots, v_r) | v_j \in f(j), \ j \in [r] \} \sigma = \{ (v_{\sigma(j)}, \ldots, v_{\sigma(r)}) | v_{\sigma(j)} \in f(\sigma(j)), \ j \in [r] \}
\]
\[
= \{ (v_1, \ldots, v_r) | f \circ \sigma(j) = i(f) \sigma \}.
\]

The injectivity of \( i \) implies that the order complex \( \Delta(i(P_{K'_r, H})) \), which can be identified with the barycentric subdivision \( \text{sd} \text{Hom}(K'_r, H) \), is an \( S_r \)-subcomplex of \( \text{sd} B_{\text{edge}}(H) \).

Here we remark that, in general, the composition \( i \circ p \) may not be the identity, as shown in the following example.

**Example 4.4.** Consider the complete \( r \)-partite \( r \)-graph \( K'_{1, \ldots, 1, 2, 2} \) generated by the collection
\[
\{ \{a_0\}, \ldots, \{a_{r-3}\}, \{b_1, b_2\}, \{c_1, c_2\} \}.
\]
For instance, taking a simplex
\[
F = \{ \{a_0, \ldots, a_{r-3}, b_1, c_1\}, \{a_0, \ldots, a_{r-3}, b_2, c_1\} \} \in \mathcal{F}(B_{\text{edge}}(K'_{1, \ldots, 1, 2, 2})),
\]
we find that
\[
\text{pr}_j(F) = \begin{cases} \{a_j\} & \text{if } j \in [r-3] \\ \{b_1, b_2\} & \text{if } j = r-2 \\ \{c_1, c_2\} & \text{if } j = r-1. \end{cases}
\]
Hence, \( i \circ p(F) \neq F \). With this example, we can conclude that there is an example of \( r \)-graph \( H \) whose poset \( i(P_{K'_r, H}) \) is a proper \( S_r \)-subposet of \( \mathcal{F}(B_{\text{edge}}(H)) \).

Moreover, we can conclude that \( \Delta(i(P_{K'_r, K'_{1, \ldots, 1, 2, 2}})) \) and \( \text{sd} B_{\text{edge}}(K'_{1, \ldots, 1, 2, 2}) \) are not isomorphic.

We also introduce an example of \( r \)-graph implying that \( i \circ p \) being the identity, and hence, two cell complexes are \( S_r \)-isomorphic:

**Example 4.5.** Considering the complete \( r \)-partite \( r \)-graph \( K'_{1, \ldots, 1, n} \) \((n \in \mathbb{N})\), we find that each simplex \( F \) of \( B_{\text{edge}}(K'_{1, \ldots, 1, n}) \) can be written as the product of sets, \( r-1 \) sets of them having cardinality 1. Therefore, \( i \circ p = 1 \).

Remark here that the structures of both \( \text{Hom}(K'_r, H) \) and \( B_{\text{edge}}(H) \), associated to an \( r \)-graph \( H \), depend on the containment of complete \( r \)-partite \( r \)-subgraphs in \( H \). If an \( r \)-graph \( H \) containing \( K'_{m_1, \ldots, m_r} \), where \( |\{i \mid m_i \geq 2\}| \geq 2 \), then it also contains the complete \( r \)-partite \( r \)-graph \( K'_{1, \ldots, 1, 2, 2} \). Together with the above examples, we obtain the following criterion of determining whether the Hom complexes and the box complexes are isomorphic:

**Proposition 4.6.** Let \( H \) be an \( r \)-graph. Then \( \Delta(i(P_{K'_r, H})) \equiv \text{sd} B_{\text{edge}}(H) \) if and only if \( H \) does not contain the complete \( r \)-partite sub-\( r \)-graph \( K'_{1, \ldots, 1, 2, 2} \).
Example 4.7. Note that the complete $r$-partite $r$-graph $K'_1,\ldots,K'_n$ has $r+2$ vertices. Then, for the complete $r$-graph $K'_n$, two simplicial complexes $\Delta(i(P_{K'_i},K'_j))$ and $\text{sd} B_{\text{edge}}(K'_n)$ are isomorphic if and only if $n \leq r+1$.

4.2. Simple $S_r$-homotopy type of $\text{Hom}(K'_r, H)$ and $B_{\text{edge}}(H)$. Now we return to the argument of verifying that $\text{Hom}(K'_r, H)$ and $B_{\text{edge}}(H)$ have the same simple homotopy type. Our strategy is to show that

1. both $\text{Hom}(K'_r, H)$ and $B_{\text{edge}}(H)$ have the same simple homotopy type with their barycentric subdivisions, and
2. $\text{sd} B_{\text{edge}}(H)$ $S_r$-collapses onto $\text{sd} \text{Hom}(K'_r, H)$.

The statements in the first step are proved by Proposition 3.2. To prove the second one, we will verify the existence of $S_r$-collapsing of $\text{sd} B_{\text{edge}}(H)$ onto $\Delta(i(P_{K'_r},H))$ by making use of an equivariant acyclic partial matching. We give here its definition and its relationships between an equivariant collapsing:

Definition 4.8. Let $G$ be a finite group and $K$ be a simplicial $G$-complex. A partial $G$-matching on $\mathcal{F}(K)$ is a pair $(\Sigma, \mu)$ of a $G$-subset $\Sigma$ of $\mathcal{F}(K)$ and a $G$-equivariant injection $\mu : \Sigma \to \mathcal{F}(K) \setminus \Sigma$ such that $\mu(x) > x$ for any $x \in \Sigma$. Elements in $\mathcal{F}(K) \setminus (\Sigma \cup \mu(\Sigma))$ are called critical. Such a partial $G$-matching is acyclic if there is no sequence of distinct elements $x_0, x_1, \ldots, x_t \in \Sigma$ ($t \geq 1$) such that $\mu(x_0) > x_1$, $\mu(x_1) > x_2, \ldots$, $\mu(x_{t-1}) > x_t$ and $\mu(x_t) > x_0$.

Proposition 4.9. Let $G$ be a finite group, $K$ a simplicial $G$-complex and $K'$ a $G$-subcomplex of $K$. Then $G$-collapses onto $K'$ if and only if there is an acyclic partial $G$-matching on $\mathcal{F}(K)$ whose set of critical elements is just $\mathcal{F}(K')$.

Proof. First, we assume that $K$ $G$-collapses onto $K'$. Then we have a sequence of elementary $G$-collapsings

$$K = K_0 \searrow G K_1 \searrow G K_2 \searrow G \cdots \searrow G K_k = K'$$

and we can find simplices $\sigma_0, \sigma_1, \ldots, \sigma_k$ in $K$ such that, for each $i \in [k]$, $\sigma_i$ is free in $K_i$; $\dim \varphi_{\sigma_i} = \dim \sigma_i + 1$; $\sigma_i G$ is independently free; and $K_{i+1} = K_i \setminus (\sigma_i G \cup \varphi_{\sigma_i} G)$. Let $\Sigma = \bigcup_{i=0}^k \sigma_i G$; and $\mu : \Sigma \to \mathcal{F}(K) \setminus \Sigma$ be defined by $\mu(\sigma_i g) = \varphi_{\sigma_i} g$. Then the pair $(\Sigma, \mu)$ is an acyclic partial $G$-matching of $\mathcal{F}(K)$ whose set of critical elements is $\mathcal{F}(K')$.

We state here only a proof of $\mu$ being injective: note first that, if we let $i < j$, we find that, for any $g, g' \in G$, $\varphi_{\sigma_i} g \notin K_j$ while $\varphi_{\sigma_j} g' \in K_j$, so $\varphi_{\sigma_i} g \neq \varphi_{\sigma_j} g'$. Hence, $\mu(G\sigma_j) \cap \mu(G\sigma_i) = \emptyset$. Then it suffices to verify the injectivity of each restriction $\mu|_{\sigma_i G}$.

Suppose that there exist $g, g' \in G$ such that $\mu(\sigma_i g) = \mu(\sigma_j g')$, that is, $\varphi_{\sigma_i} g = \varphi_{\sigma_j} g'$. Then, $\varphi_{\sigma_i} g$ is a simplex in $K_i$ containing both $\sigma_i g$ and $\sigma_i g'$. Since $\sigma_i G$ is independently free, we must have $\sigma_i g = \sigma_i g'$.

Let us prove the converse. Let $(\Sigma, \mu)$ be an acyclic $G$-matching on $\mathcal{F}(K)$ whose set of critical elements is $\mathcal{F}(K')$. We give here an algorithm to construct $K$ from its subcomplex $K'$.

Let $Q$ be the set of elements of $\Sigma$ already added to $K'$ and $W$ the set of minimal elements in $\mathcal{F}(K) \setminus \mathcal{F}(K')$. Suppose first $Q = \emptyset$. We can find $\tau \in W$ such that, for any $g \in G$, $\mu(\tau g) = \mu(\tau) g$ is the only simplex covering $\tau g$; if not, we can choose elements of $W$ contradicting the assumption that $(\Sigma, \mu)$ is acyclic.

Set $K = K' \cup \tau G \cup \mu(\tau) G$. This $K$ is a simplicial $G$-complex: if there were a proper face of $\tau g$ in $\mathcal{F}(K) \setminus \mathcal{F}(K')$, then $\tau g$ cannot be minimal in $\mathcal{F}(K) \setminus \mathcal{F}(K')$, contradicting $\tau g \in W$. Moreover, the orbit $\tau G$ is a collection of free faces which is independently free: since $\mu$ is injective and $G$-equivariant,
τg ≠ τg' implies that μ(τ)g ≠ μ(τ)g', that is, no facets in \( F(\tilde{K}) \) cover both τg and τg' if g ≠ g'. So we can conclude that \( \tilde{K} \) elementary G-collapses onto \( K' \).

Delete all elements in τG from W, let \( Q := Q \cup τG \cup \mu(τ)G \), \( K' = \tilde{K} \), and repeat our argument until \( W = \emptyset \). If \( W = \emptyset \), take a new W of minimal elements in \( F(K) \setminus (F(K') \cup Q) \) and continue our argument until \( Q = F(K) \setminus F(K') = Σ \cup μ(Σ) \); and we obtain a sequence of elementary G-collapsings leading from K to K'. □

By this proposition, if one wants to verify that two simplicial G-complexes have the same simple homotopy types, it suffices to construct an acyclic partial G-matching on their face posets. Now we give a construction for our main result:

**Lemma 4.10.** For an \( r \)-graph \( H \), \( sd B_{\text{edge}}(H) \) \( S_r \)-collapses onto \( \Delta(i(P_{K,H})) \).

**Proof.** Since \( \Delta(i(P_{K,H})) \) is a \( S_r \)-subcomplex of \( sd B_{\text{edge}}(H) \), we will construct an acyclic partial \( S_r \)-matching on \( F(sd B_{\text{edge}}(H)) \) whose set of critical elements is \( F(\Delta(i(P_{K,H}))) \).

Note first that, for any chain A of \( sd B_{\text{edge}}(H) \), A is a chain of \( \Delta(i(P_{K,H})) \) if and only if \( i \circ p(A_k) = A_k \) for any \( k \in [\#A] \). Indeed, if A is a chain of \( \Delta(i(P_{K,H})) \), then we can choose \( φ_k \in P_{K,H} \) and write \( A_k = i(φ_k) \) for each \( k \in [\#A] \). Since \( p \circ i = 1 \), we obtain \( i \circ p(A_k) = A_k \). The converse holds by the definitions of i and p.

To achieve our purpose, it suffices to construct an acyclic partial \( S_r \)-matching which matches chains not belonging to \( \Delta(i(P_{K,G})) \). First, we define a subset \( D \) of \( F(sd B_{\text{edge}}(H)) \) by

\[
D = \{ F \in F(sd B_{\text{edge}}(H)) \mid i \circ p(F_j) ≠ F_1 \text{ for some } j \in [\#F] \}.
\]

\( D = \emptyset \) implies that \( \Delta(i(P_{K,H})) \) and \( sd B_{\text{edge}}(H) \) are the same. We assume \( D ≠ \emptyset \). For any \( F \in D \), we let \( l(F) \) denote the minimal index \( s \) such that \( i \circ p(F_1) ≠ F_s \), and \( r(F) \) the maximal index \( r \) such that \( F_{l(F)+r} \) is included in \( i \circ p(F_{l(F)}) \). With these indices, we define \( Σ_1, Σ_2 ⊂ D \) as follows:

\[
Σ_1 = \{ F \in D \mid l(F) + r(F) = \#F, i \circ p(F_{l(F)}) ∈ F_1 \};
\]

\[
Σ_2 = \{ F \in D \mid l(F) + r(F) < \#F, i \circ p(F_{l(F)}) ∩ F_{l(F)+r(F)+1} ∈ F \}.
\]

Now we define a map \( μ : Σ_1 ∪ Σ_2 → F(sd B_{\text{edge}}(H)) \setminus (Σ_1 ∪ Σ_2) \) as

\[
μ(F) = \begin{cases} 
F \cup \{ i \circ p(F_{l(F)}) \} & \text{if } F \in Σ_1; \\
F \cup \{ i \circ p(F_{l(F)}) ∩ F_{l(F)+r(F)+1} \} & \text{if } F \in Σ_2.
\end{cases}
\]

We claim that the pair \( (Σ_1 ∪ Σ_2, μ) \) is an acyclic partial \( S_r \)-matching on \( F(sd B_{\text{edge}}(H)) \).

We first check that \( Σ_1 ∪ Σ_2 \) is an \( S_r \)-subset of \( F(sd B_{\text{edge}}(H)) \): let \( F = \{ F_0, F_1, \ldots, F_{\#F} \} \) be an element of \( Σ_1 ∪ Σ_2 \subset F(sd B_{\text{edge}}(H)) \) satisfying \( F_0 ⊂ F_1 ⊂ \ldots ⊂ F_{\#F} \) and \( σ \in S_r \). Then \( Fσ = \{ F_0σ, F_1σ, \ldots, F_{\#F}σ \} \) is a chain of \( sd B_{\text{edge}}(H) \). Since \( F ∈ D \), we can take an index \( j \) with \( i \circ p(F_j) ≠ F_j \). Then \( S_r \)-equivariance of \( i \circ p \) implies that \( i \circ p(F_σ) ≠ F_j σ \). So \( Fσ ∈ D \).

Now suppose \( F ∈ Σ_1 \). The condition \( l(Fσ) + r(Fσ) = \#F \) holds because of the bijectivity of \( σ \).

Since \( i \circ p(F_{l(F)}) \notin F \), \( (Fσ)_{l(F)} = F_{l(F)}σ \) and \( i \circ p \) is \( S_r \)-equivariant, we have \( i \circ p((Fσ)_{l(F)}) \notin Fσ \), and so \( Fσ ∈ Σ_1 \). Next let \( F ∈ Σ_2 \). The condition \( l(Fσ) + r(Fσ) < \#F \) is obvious. The second condition comes from the following calculation:

\[
i \circ p((Fσ)_{l(F)}) ∩ (Fσ)_{l(F)+r(Fσ)+1} = i \circ p(F_{l(F)}σ) ∩ F_{l(F)+r(F)+1} σ
\]

\[
i \circ p(F_{l(F)}σ) ∩ F_{l(F)+r(F)+1} σ
\]

\[
i \circ p(F_{l(F)}σ) ∩ F_{l(F)+r(F)+1} σ \notin Fσ.
\]

So \( Fσ ∈ Σ_2 \). Summing up, \( Σ_1 ∪ Σ_2 \) is an \( S_r \)-subset.

Next, we must verify that \( μ \) satisfies the condition for being a partial \( S_r \)-matching: First we find that both \( i \circ p(F_{l(F)}) \) for \( F ∈ Σ_1 \) and \( i \circ p(F_{l(F)}) ∩ F_{l(F)+r(F)+1} \) for \( F ∈ Σ_2 \) are simplices of \( B_{\text{edge}}(H) \).
Finally, we find that

\[ F_0 \subset \ldots \subset F_{r(F)} \subset \ldots \subset F_{\#F} \subset i \circ p(F_{r(F)}) \]

for \( F \in \Sigma_1 \), and

\[ F_0 \subset \ldots \subset F_{r(F)} \subset \ldots \subset F_{r(F)+r(F)+1} \subset i \circ p(F_{r(F)}) \cap F_{r(F)+r(F)+1} \subset F_{r(F)+r(F)+1} \subset \ldots \subset F_{\#F}. \]

for \( F \in \Sigma_2 \). We can see from the relations (1) and (2) that, for any \( F \in \Sigma_1 \cup \Sigma_2 \), \( \mu(F) \) covers \( F \) but is not a chain in \( \Sigma_1 \cup \Sigma_2 \); moreover, \( F_1 \in \Sigma_1 \) and \( F_2 \in \Sigma_2 \) imply that \( \mu(F_1) \neq \mu(F_2) \). If we suppose that both \( F_1 \) and \( F_2 \) belong to \( \Sigma_j \) \( (j = 1 \text{ or } 2) \) satisfying \( \mu(F_1) = \mu(F_2) \), then we find that the inserted terms to obtain \( \mu(F_1) \) and \( \mu(F_2) \) are in the same index. This yields that \( F_1 = F_2 \), and so \( \mu \) is injective. This \( \mu \) is \( S_r \)-equivariant because of the following calculations: if \( F \in \Sigma_1 \),

\[
\mu(F) = F \sigma \cup \{ i \circ p(F_{r(F)}) \} = F \sigma \cup \{ i \circ p(F_{r(F)}) \} = (F \cup \{ i \circ p(F_{r(F)}) \}) \sigma = \mu(F) \sigma.
\]

If \( F \in \Sigma_2 \), we have

\[
\mu(F) = F \sigma \cup \{ i \circ p(F_{r(F)}) \} \cap (F \sigma)_{r(F)+r(F)+1} = (F \cup \{ i \circ p(F_{r(F)}) \}) \sigma = \mu(F) \sigma.
\]

Finally, we find that \( \Sigma_1 \cup \Sigma_2 \cup \mu(\Sigma_1 \cup \Sigma_2) = D \), and we can conclude that the pair \( (\Sigma_1 \cup \Sigma_2, \mu) \) is a partial \( S_r \)-matching on \( F \) (sd \( B_{edge}(H) \)) whose set of critical elements is \( F(\Delta(i(P_{K^i}; \mu))) \).

It remains to prove that the matching is acyclic: suppose that there exists a sequence of distinct elements \( F^0, F^1, \ldots, F^t \in \Sigma_1 \cup \Sigma_2 \) \( t \geq 1 \) such that

\[
\mu(F^0) > F^1, \mu(F^1) > F^2, \ldots, \mu(F^{t-1}) > F^t \text{ and } \mu(F^t) > F^0.
\]

For each \( j \in [t-1] \), since \( \mu(F^j) \) covers both \( F^j \) and \( F^{j+1} \) which are distinct, we can choose a simplex \( A_j \in F^j \) such that \( F^{j+1} = \mu(F^j) \setminus \{ A_j \} \). Similarly, \( A_j \in F^t \) can be chosen such that \( F^0 = \mu(F^t) \setminus \{ A_t \} \).

It is useful if we know what are \( A_j, j \in [t] \): we claim here that

\[
A_j = \begin{cases} F^j_{r(F)} & \text{if } F^j \in \Sigma_1; \\ F^j_{r(F)+r(F)+1} \text{ or } F^j_{k(F)} & \text{if } F^j \in \Sigma_2. \end{cases}
\]

In fact, for \( F^j \in \Sigma_1 \), if \( A_j \) were not \( F^j_{r(F)} \), it follows from the equation (1) that \( F^{j+1}_{r(F)+1} = F^{j}_{r(F)} \), and so \( i \circ p(F^{j+1}_{r(F)+1}) \) is \( F^{j+1} \); hence \( F^{j+1} \notin \Sigma_1 \). Since \( i \circ p(F^j_{r(F)}) \) contains all simplices in \( F^j \), we obtain \( F^{j+1} \notin \Sigma_2 \). Therefore \( F^{j+1} \notin \Sigma_1 \cup \Sigma_2 \), contradicting to the assumption of \( F^{j+1} \). For \( F^j \in \Sigma_2 \), if \( A_j \) were not \( F^j_{r(F)} \) and \( F^j_{r(F)+r(F)+1} \), it follows from the equation (2) that \( F^{j+1}_{r(F)+1} = F^{j}_{r(F)} \). So \( F^{j+1} \notin \Sigma_1 \) because the simplex \( F^{j+1}_{r(F)+r(F)+1} \) still exists. Moreover, we obtain \( F^{j+1}_{r(F)+r(F)+1} = F^{j}_{r(F)+r(F)+1} \) and then \( i \circ p(F^{j+1}_{r(F)+1}) \cap F^{j+1}_{r(F)+r(F)+1} \in F^{j+1} \). Hence \( F^{j+1} \notin \Sigma_2 \). Summing up, \( F^{j+1} \notin \Sigma_1 \cup \Sigma_2 \), which contradicts to the assumption of \( F^{j+1} \).

We can see from the above remark on \( A_j \) that, if \( F^j \in \Sigma_2 \), \( F^{j+1} \) can be a chain in either \( \Sigma_1 \) or \( \Sigma_2 \), while, if \( F^j \in \Sigma_1 \), \( F^{j+1} \) can be a chain only in \( \Sigma_1 \) because \( i \circ p(F^{j+1}_{r(F)+1}) \) contains \( i \circ p(F^j_{r(F)}) \), which contains all \( F^j_{k} \) \( (k \in \{#F^j\}) \). Similarly, \( F^t \in \Sigma_1 \) implies that \( F^0 \in \Sigma_1 \). Then we can conclude that there are three cases on a set to which the chains \( F^0, \ldots, F^t \) belongs, as follows:

(a) All \( F^0, \ldots, F^t \) belong to \( \Sigma_1 \);
(b) All \( F^0, \ldots, F^t \) belong to \( \Sigma_2 \);
(c) There exists \( j \in [t-1] \) such that \( F^j \in \Sigma_2 \) but \( F^{j+1} \in \Sigma_1 \).
We can find a contradiction for the case (c) at once because the fact that \( F^k \in \Sigma_1 \) whenever \( F^{k-1} \in \Sigma_1 \) implies that \( F^j \in \Sigma_1 \). For the case (a), considering the number \( t(F^j) \) of indices \( i \) such that \( F^j \neq i \odot p(F^j) \), we obtain a contradiction \( t(F^0) < t(F^0) \).

For the case (b), we let us denote \( s(F^j) \) the number of simplices in \( F^j \) not contained in \( i \odot p(F^j) \). By assumption, we have \( s(F^j) \geq 1 \) for any \( j \in [t] \). By the assumption, each \( A_j = F^j_{H(F^j)} \) or \( F^j_{H(F^j)+r(F^j)+1} \). If \( A_j = F^j_{H(F^j)} \), the fact that \( i \odot p(F^j_{H(F^j)+1}) \supseteq i \odot p(F^j_{H(F^j)}) \) implies that \( s(F^j+1) \leq s(F^j) \). If \( A_j = F^j_{H(F^j)+r(F^j)+1} \), then we have \( s(F^j+1) = s(F^j) - 1 < s(F^j) \). Summing up, \( F^0, \ldots, F^t \in \Sigma_2 \) implies the following inequalities:

\[
(3) \quad s(F^0) \leq s(F^1) \leq \cdots \leq s(F^t) \leq s(F^0).
\]

We will get a contradiction if there exists a “less than or equal to” sign which is really the “less than” sign. We obtain the assertion at once if there is \( j \in [t] \) with \( A_j = F^j_{H(F^j)+r(F^j)+1} \).

Assume that \( A_j = F^j_{H(F^j)} \) for all \( j \in [t] \). By definition, we can choose \( F^{j+1} \in H(F^{j+1}) \) and \( F^{j+1} \in H(F^{j+1}+r(F^{j+1}+1) \) in each \( F^j \). However, we will get a contradiction

\[
F^{j+1} \ni i \odot p(F^{j+1} \cap F^{j+1} \cap F^{j+1}+r(F^{j+1}+1) \cap F^{j+1}+1)
\]

if there exists \( j \in [t] \) such that either of these conditions holds:

\[
(c1) \quad i \odot p(F^j \cap F^{j+1} \cap F^{j+1} \cap F^{j+1}+1) = i \odot p(F^j \cap F^{j+1} \cap F^{j+1}+1) \text{ or}
\]

\[
(c2) \quad i \odot p(F^j \cap F^{j+1} \cap F^{j+1} \cap F^{j+1}+1) \text{ is distinct from } F^j \text{ and is in } F^j.
\]

Then we can assume that all \( j \in [t] \) do not satisfy both conditions. Suppose that \( s(F^0) = s(F^1) = \cdots = s(F^t) \). We find that \( F^0_{H(F^0)+r(F^0)+1} \) is the minimal simplex not included in \( i \odot p(F^0_{H(F^0)}) \) for any \( j \in [t] \). Paying attention to the simplices inserted to each chain, we find by our assumption that

\[
i \odot p(F^0) \cap F^0_{H(F^0)+r(F^0)+1} \subseteq i \odot p(F^0) \cap F^0_{H(F^0)+r(F^0)+1} \subseteq \cdots.
\]

\[
(4) \quad i \odot p(F^0) \cap F^0_{H(F^0)+r(F^0)+1} \subseteq i \odot p(F^0_{H(F^0)}).
\]

Since \( F^0_{H(F^0)+r(F^0)+1} \) is the minimal simplex not included in \( i \odot p(F^0_{H(F^0)}) \), we obtain

\[
i \odot p(F^0) \cap F^0_{H(F^0)+r(F^0)+1} \subseteq i \odot p(F^0_{H(F^0)}).
\]

Then,

\[
i \odot p(F^0) \cap F^0_{H(F^0)+r(F^0)+1} \subseteq i \odot p(F^0_{H(F^0)}).\]

With (4), we thus obtain a contradiction \( i \odot p(f_{H(F^0)}) \cap f_{H(F^0)+r(F^0)+1} \subseteq i \odot p(f_{H(F^0)}) \cap f_{H(F^0)+r(F^0)+1}. \)

Therefore, in (3), there exists a “less than or equal to” sign which is really the “less than” sign, and so we get a contradiction \( s(F^0) < s(F^0). \)

Summing up, our argument contradicts itself if we suppose that \( \Sigma_1 \cup \Sigma_2, \mu \) is not acyclic. \( \square \)

We depict an \( S_r \)-collapsing constructed by the above acyclic partial \( S_r \)-matching for a part of \( sd B_{edge}(H) \), \( H = K_{2,1}^3 \) as the following figure. Here we draw a hypergraph by edge-based drawings, see [KKS09].

We now complete our argument in all steps, obtaining a construction of a formal \( S_r \)-deformation between \( \text{Hom}(K', H) \) and \( B_{edge}(H) \). So the following conclusion holds:

**Theorem 4.11.** For an \( r \)-graph \( H \), the Hom complex \( \text{Hom}(K', H) \) and the box complex \( B_{edge}(H) \) have the same simple \( S_r \)-homotopy type.
Figure 1. $K^3_{2,2,1}$ and a part of the $S^3$-collapsing of $sdB_{\text{edge}}(K^3_{2,2,1})$ onto $\Delta(i(P_{K^3_{1,1,1}K^3_{2,1,2}}))$.

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