Research Article

Existence and Uniqueness of Positive Solutions for a Class of Nonlinear Fractional Differential Equations with Singular Boundary Value Conditions

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This paper focuses on a singular boundary value (SBV) problem of nonlinear fractional differential (NFD) equation defined as follows: $D^\alpha_0, v(\tau) + f(\tau, v(\tau)) = 0$, $\tau \in (0, 1)$, $v(0) = v'(0) = v''(0) = 0$, where $2 < \alpha \leq 3$, $D^\alpha_0$ is the standard Caputo’s fractional derivative, and $f: [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ with $\lim_{\tau \rightarrow 0^+} f(\tau, \cdot) = +\infty$. Assuming some hypotheses on $f$, they gained positive solutions through the nonlinear Leray–Schauder-type alternative in a cone and the Guo–Krasnoselskii FP theory.

Xu [28] investigated the following SBV problem for NFD:

$$
\begin{align*}
D^\alpha_0 v(\tau) + f(\tau, v(\tau)) &= 0, & \tau \in (0, 1), \\
v(0) &= v'(0) = v''(0) = 0,
\end{align*}
$$

(2)

where $3 < \alpha \leq 4$, $D^\alpha_0$ represents the standard RLF derivative, and $f: [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ with $\lim_{\tau \rightarrow 0^+} f(\tau, \cdot) = +\infty$. They obtained the presence of multiple solutions under the condition of $f(\tau, v) = g(\nu) + h(\nu)$ and the uniqueness of solution for $f(\tau, v) = q(\tau)[g(\nu) + h(\nu)]$ using the Guo–Krasnoselskii FP theory, mixed monotone scheme, and Leray–Schauder’s nonlinear alternative.

Zhang and Zhong [31] investigated the boundary value problem of singular NFD written as

$$
\begin{align*}
D^\alpha_0 v(\tau) + f(\tau, v(\tau)) &= 0, & \tau \in (0, 1), \\
v(0) &= 0, D^\beta_0 v(0) = 0, D^\beta_0 v(1) = \sum_{i=1}^n \xi_i D^\beta_0 v(\eta),
\end{align*}
$$

(3)

where the function $f$ permits singularities at $\tau = 0$, $\tau = 1$, and $v = 0$. The presence of multiple positive solutions for $f(\tau,
the presence of multiple positive solutions of (4) for $\tau \in (0, 1)$, (4) where $3 < \beta \leq 4$, $D^\beta_0$ is the standard RLF derivative, and $f(\tau, v(\tau))$ becomes singular when $\tau = 0, \tau = 1$, and $v = 0$. In analogy with other works, the corresponding Green’s NFD and its positive characteristics are inferred. As application, analogy with other works, the corresponding Green’s NFD and its positive properties are deduced in this paper. (1) To the best of our knowledge, those in [26–34]. The remainder of this paper is structured as follows. Preliminaries are given in Section 2, including definitions, lemmas, the deduction of Green’s function for problem (4), and new positive properties. Section 3 proves the presence of positive solutions of (4) by the Guo–Krasnoselskii FP theory and demonstrates an example. Section 4 discusses the uniqueness of the positive solution of (4) by a mixed monotone operator and demonstrates another example.

2. Preliminaries

The lemmas and definitions from [3] are given for the convenience of the reader as follows:

**Definition 1** (see [3]). The RLF integral of the order $\beta > 0$ of a function $f(x): (0, +\infty) \rightarrow R$ is formulated:

$$I^\beta_0 f(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x - s)^{\beta - 1} f(s)ds,$$  

provided the right side is pointwise defined on $(0, +\infty)$. 

**Definition 2** (see [3]). The RLF derivative of the order $\beta > 0$ of a continuous function $f(x): (0, +\infty) \rightarrow R$ is formulated as

$$D^\beta_0 f(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x - s)^{\beta - 1} f(s)ds,$$  

where $n = [\beta] + 1$ and $[\cdot]$ represents the integer part of number.

**Lemma 1** (see [3]). The solution of the NFD equation defined as

$$D^\beta_0 \mu(\tau) = 0$$

is $\mu(\tau) = C_1 \tau^{\beta-1} + C_2 \tau^{\beta-2} + \cdots + C_N \tau^{\beta-N}, \ C_i \in \Re, \ i = 1, 2, \ldots, N$, where $N$ is the smallest integer greater than or equal to $\beta$ and $\mu \in C(0, 1) \cap L(0, 1)$.

**Lemma 2** (see [3]). Under the assumption that $\mu \in C(0, 1) \cap L(0, 1)$ with a NFD of order $\beta$, then

$$\int_0^\beta D^\beta_0 \mu(\tau) = \mu(\tau) C_1 \tau^{\beta-1} + C_2 \tau^{\beta-2} + \cdots + C_M \tau^{\beta-M},$$

where $C_i \in \Re, i = 1, 2, \ldots, M$, where $M$ is the smallest integer greater than or equal to $\beta$.

**Lemma 3.** Provided that $\kappa(\tau) \in C[0, 1]$ and $3 < \beta \leq 4$,

$$D^\beta_0 \mu(\tau) + \kappa(\tau) = 0, \ 0 < \tau < 1,$$

$$\mu(0) = \mu'(0) = \mu''(0) = \mu''(1) = 0.$$  

The solution of (10) is unique and as follows:

$$\mu(\tau) = \int_0^\tau G(\tau, s) \kappa(s)ds,$$  

where Green’s function $G(\tau, s)$ is denoted as

$$G(\tau, s) = \frac{1}{\Gamma(\beta)} \begin{cases} \tau^{\beta-1}(1-s)^{\beta-3} - (\tau - s)^{\beta-1}, & 0 \leq s \leq \tau \leq 1, \\ \tau^{\beta-1}(1-s)^{\beta-3}, & 0 \leq \tau \leq s \leq 1. \end{cases}$$  

**Proof.** (10) is rewritten as follows through Lemma 2:

$$\mu(\tau) = \frac{1}{\Gamma(\beta)} \int_0^\tau (\tau - s)^{\beta-1} \kappa(s)ds + C_1 \tau^{\beta-1} + C_2 \tau^{\beta-2} + C_3 \tau^{\beta-3} + C_4 \tau^{\beta-4},$$

where $C_i \in \Re, i = 1, 2, 3, 4$. From the boundary conditions $\mu(0) = \mu'(0) = \mu''(0) = 0$, we have $C_2 = C_3 = C_4 = 0$. Then,
\[ \mu(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \kappa(s) ds + C_1 t^{\beta-1}, \]
\[ \mu'(t) = -\frac{\beta-1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-2} \kappa(s) ds + C_1 (\beta-1) t^{\beta-2}, \]
\[ \mu''(t) = -\frac{(\beta-1)(\beta-2)}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-3} \kappa(s) ds + C_1 (\beta-1)(\beta-2) t^{\beta-3}. \]

By the condition \( u''(1) = 0 \), we have
\[ C_1 = \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-3} \kappa(s) ds. \] 

Accordingly, the unique solution of problem (10) and (11) is given as
\[ \mu(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \kappa(s) ds + \frac{t^{\beta-1}}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-3} \kappa(s) ds \]
\[ = \int_0^1 G(t,s) \kappa(s) ds. \]

Lemma 3 is proved with this. \( \square \)

\[ \Gamma(\beta)G(t,s) = t^{\beta-1}(1-s)^{\beta-3} - (t-s)^{\beta-1} = t^2 s^{\beta-3} - (t-s)^{\beta-1} \]
\[ = \left[ t^2 - (t-s)^2 \right] (t-s)^{\beta-3} = t^2 s(2-s)(t-s)^{\beta-3} \]
\[ = t^{\beta-1}s(2-s)(1-s)^{\beta-3}, \]
\[ \Gamma(\beta)G_{\tau}(t,s) = (\beta-1)t^{\beta-2}(1-s)^{\beta-3} - (\beta-1)(t-s)^{\beta-2}, \]
\[ \Gamma(\beta)G_{\tau_\tau}(t,s) = (\beta-1)(\beta-2)t^{\beta-3}(1-s)^{\beta-3} - (\beta-1)(\beta-2)(t-s)^{\beta-3} \]
\[ = (\beta-1)(\beta-2) \left[ (t-s)^{\beta-3} - (t-s)^{\beta-3} \right] \geq 0. \]

Obviously, \( \Gamma(\beta)G_{\tau}(t,s) \) is nondecreasing in \( t \), and thus, for \( 0 \leq s \leq \tau \leq 1 \), we have
\[ \Gamma(\beta)G(t,s) \leq \max_{0 \leq \tau \leq 1} \Gamma(\beta)G(1,s) = \Gamma(\beta)G(1,1) = (1-s)^{\beta-3} - 1 \]
\[ = \left[ 1 - (1-s)^2 \right] (1-s)^{\beta-3} = s(1-s)(2-s)^{\beta-3}. \] 

When \( 0 \leq \tau \leq s \leq 1 \), note that \( 0 \leq s(2-s) \leq 1 \), we have
\[ \Gamma(\beta)G(t,s) = t^{\beta-1}(1-s)^{\beta-3} \geq t^{\beta-1}s(2-s)(1-s)^{\beta-3}, \]
\[ \Gamma(\beta)G(t,s) = t^{\beta-1}(1-s)^{\beta-3} \leq s^{\beta-1}(1-s)^{\beta-3} = s^2 \cdot s^{\beta-3} \cdot (1-s)^{\beta-3} \leq s^2 (1-s)^{\beta-3}. \] 

Note that \( s(2-s) - s^2 = 2s - 2s^2 = 2s(1-s) \geq 0 \); then,

\textbf{Lemma 4.} The properties of \( G(t,s) \) defined by (13) are as follows:

\begin{enumerate}
\item \( t^{\beta-1}s(2-s)(1-s)^{\beta-3} \leq \Gamma(\beta)G(t,s) \leq s(2-s)(1-s)^{\beta-3} \), \( 0 \leq t, s \leq 1 \)
\item \( t^{\beta-1}s(1-s)^{\beta-3} \leq \Gamma(\beta)G(t,s) \leq t^{\beta-1}(1-s)^{\beta-3}, \ 0 \leq t, \ s \leq 1 \)
\item \( G(t,s) > 0, \ 0 < t, s < 1 \)
\item \( G(t,s) \in C([0,1] \times [0,1]) \)
\end{enumerate}

\textbf{Proof.} Property (4) is obvious and (3) holds from (1). Thus, here (1) and (2) will be proved.

(1) When \( s \leq t \),
\[
\Gamma(\beta)G(r, s) \leq s(2 - s)(1 - s)^{\beta-3}.
\]  
(21)

From (18)–(21), we have the first conclusion in Lemma 4 which holds.

(22)

\[
\Gamma(\beta)G(r, s) = r^{\beta-1}(1-s)^{\beta-3} - (r-s)^{\beta-1} = r^2(1-r)^{\beta-3} - (r-s)^{\beta-1} \\
= r^2 - (r-ts)^{\beta-3} - (r-s)^{\beta-1} = r^2(2-s)(r-ts)^{\beta-3} \\
= r^2s(2-s)(1-s)^{\beta-3},
\]

(23)

\[
\Gamma(\beta)G(r, s) = r^{\beta-1}(1-s)^{\beta-3} - (r-s)^{\beta-1} \leq r^{\beta-1}(1-s)^{\beta-3}.
\]

(24)

When \( r \leq s \),
\[
\Gamma(\beta)G(r, s) = r^{\beta-1}(1-s)^{\beta-3}.
\]

(25)

Since \( 0 \leq s(2-s) \leq 1 \) for \( 0 \leq s(2-s) \leq 1 \), so
\[
\Gamma(\beta)G(r, s) = r^{\beta-1}(1-s)^{\beta-3} \geq r^{\beta-1}s(2-s)(1-s)^{\beta-3}.
\]

(26)

Also,
\[
\Gamma(\beta)G(r, s) = r^{\beta-1}(1-s)^{\beta-3}.
\]

From (22)–(26), we have the second conclusion in Lemma 4 which holds.

Now, we give the following definitions and lemmas (see [34–39]), which are essential in proving the results.

**Lemma 5.** For a Banach space \( \Psi \), let \( \Lambda \subset \Psi \) denote a normal cone in \( \Psi \) and \( \Phi_1 \) and \( \Phi_2 \) denote open subsets of \( \Psi \) with \( \theta \in \Phi_1 \cap \Phi_2 \). Then, let a completely continuous operator \( F: \Lambda \rightarrow \Lambda \) satisfy either \( ||F\rho|| \leq ||\rho||, \rho \in \Lambda \cap \partial\Phi_1, ||F\rho|| \geq ||\rho||, \rho \in \Lambda \cap \partial\Phi_2 \) or \( ||F\rho|| \leq ||\rho||, \rho \in \Lambda \cap \partial\Phi_1, ||F\rho|| \geq ||\rho||, \rho \in \Lambda \cap \partial\Phi_2 \).

Then, \( F \) has an FP in \( \Lambda \cap \overline{\Phi}_1 \cap \overline{\Phi}_2 \).

Let \( c \in \Lambda \) with \( ||c|| \leq 1, c \neq \theta \) and \( \Omega_c = \{x \in \Lambda|x \neq \theta, \) there exists constants \( m, M > 0, \) such that \( mc \leq x \leq Mc \}, \) where \( \theta \) is the zero element in \( \Lambda \).

**Definition 3.** A: \( Q_c \times Q_c \rightarrow Q_c \) is a mixed monotone operator when satisfying the monotone condition of \( A(x, y) \) in \( x, y \in Q_c \) such that \( A(x_1, y) \leq A(x_2, y) \) when \( x_1 \leq x_2 \) and \( A(x, y_1) \leq A(x, y_2) \) when \( y_1 \geq y_2. \) \( x^* \in Q_c \) is an FP of \( A \) when \( A(x^*, x^*) = x^* \).

(27)

\[
A\left(\frac{r_1, r_2}{\tau}, x, y\right) \geq \frac{r_1}{\tau}A(x, y), \quad \forall x, y \in Q_c, 0 < \tau < 1,
\]

where \( A \) is a mixed monotone operator.

**3. Presence of Positive Solutions of SVB**

The presence and multiplicity of positive solutions of (4) and (5) is investigated here. The nonlinear function \( f(\tau, x) \in C((0, 1) \times (0, +\infty)), (0, +\infty)), f \) may be singular when \( \tau = 0, \) \( \tau = 1, \) and \( x = 0. \)

For a Banach space \( \Psi = C[0, 1] \) with the maximum norm \( \max_{0 \leq \tau \leq 1} ||\mu||, \) let \( K \in \Psi \) denote a nonnegative cone defined as

\[
K = \{\mu \in \Psi||\mu(\tau) \geq r^{\beta-1}||\mu||, \tau \in [0, 1]\}.
\]

(28)

The operator \( T \) is defined as follows:

\[
(T\mu)(\tau) = \int_0^1 G(r, s)f(s, \mu(s))ds, \quad 0 \leq \tau \leq 1.
\]

(29)

Clearly, \( T: K - \{0\} \rightarrow C[0, 1]. \) Denote \( B_\tau = \{\mu(\tau) \in \Psi||\mu(\tau) || < r\} \)

(30)

and

\[
K_\tau = K \cap B_\tau = \{\mu \in K||\mu(\tau) || \leq \mu(\tau) < r\}.
\]

The following are assumed for later use:

(\( H_1 \)) \( f \in C((0, 1) \times (0, +\infty)), (0, +\infty)), f \)

(\( H_2 \)) \( There \ exist \ a_1, a_2 \in C((0, 1), [0, +\infty)) \ and \ f_1, f_2 \in C((0, +\infty), [0, +\infty)) \ satisfying \)

\[
f(\tau, \mu) \leq a_1(\tau)f_1(\mu) + a_2(\tau)f_2(\mu), \quad \forall \tau \in (0, 1), \mu \in (0, +\infty),
\]

(31)

and for any \( r > 0, \)

\[
\int_0^1 [a_1(\tau)f_1(\tau) + a_2(\tau)f_2(\tau)]d\tau < +\infty,
\]

(32)
where $f_{ir} (r) = \max \{f_i (\mu); \ r \beta^{-1} \leq \mu \leq r\}, i = 1, 2$.

(H3) There exist $R_i > 0, 0 < c_i < (1/2)$ and $[c_i, d_i] \subset (0, 1)$ and a nonnegative function $b_i (r) \in L^1 [0, 1]$ with $0 < 2^{\beta-1} \int_{c_i}^{d_i} G (1/2, s) b_i (s) ds$ satisfying

$$f (x, \mu) \geq b_1 (r) \mu, \ \forall (r, \mu) \in [c_i, d_i] \times (R_i, +\infty).$$

(33)

(H4) There exist $[c_2, d_2] \subset (0, 1)$ and a nonnegative function $b_2 (r) \in L^1 [0, 1]$ with $0 < \int_0^1 b_2 (s) ds < +\infty$ satisfying

$$f (x, \mu) \geq b_2 (r) \mu, \ \forall (r, \mu) \in [c_2, d_2] \times (R_i, +\infty).$$

$$\liminf_{\rho \rightarrow 0^+} \min_{r [c_i, d_i]} \frac{f (r, \mu)}{b_2 (r)} = +\infty. \quad (34)$$

Lemma 7. For any $r > 0, T: K \rightarrow B_r$ is completely continuous.

Proof. For any $\mu \in K / B_r$, we have $r^{\beta-1} \|\mu\| \leq \mu (r) \leq \|\mu\|$.

From (H2) and (1) of Lemma 4

$$(T\mu) (r) = \int_0^1 G (r, s) f (s, \mu (s)) ds \leq \int_0^1 (2-s) (1-s)^{\beta-3} f (s, \mu (s)) ds$$

$$\leq \int_0^1 s (2-s) (1-s)^{\beta-3} [a_1 (s) f_{1r} (s) + a_2 (s) f_{2r} (s)] ds < +\infty, \quad (35)$$

meaning that $T$ is well defined.

And, by (1) of Lemma 4,

$$(T\mu) (r) \leq \int_0^1 s (2-s) (1-s)^{\beta-3} [a_1 (s) f_{1r} (s) + a_2 (s) f_{2r} (s)] ds < +\infty, \quad (36)$$

So, $T$ maps $K, B_r$ into $K$.

$$\left|G (r', s) - G (r'', s)\right| \leq \frac{\epsilon}{1 + \int_0^1 [a_1 (s) f_{1r} (s) + a_2 (s) f_{2r} (s)] ds} \quad (37)$$

which means $T (D)$ is uniformly continuous. $G (r, s)$ is uniformly continuous on $[0, 1] \times [0, 1]$. Accordingly, $\delta > 0$ exists for any $\epsilon > 0$, such that $|r' - r''| < \delta, s \in [0, 1]$, for $r', r'' \in [0, 1]$.

Consequently,

$$\|T\mu (r') - T\mu (r'')\| \leq \int_0^1 \left|G (r', s) - G (r'', s)\right| f (s, \mu (s)) ds$$

$$\leq \int_0^1 \left|G (r', s) - G (r'', s)\right| [a_1 (s) f_{1r} (s) + a_2 (s) f_{2r} (s)] ds$$

$$\leq \int_0^1 \left[\frac{\epsilon}{1 + \int_0^1 [a_1 (s) f_{1r} (s) + a_2 (s) f_{2r} (s)] ds} \right] [a_1 (s) f_{1r} (s) + a_2 (s) f_{2r} (s)] ds$$

$$< \epsilon, \quad (39)$$

implying $T (D)$ is equicontinuous. According to the Arzelâ–Ascoli theorem, $T: K \rightarrow B_r$ is compact.
\[
\begin{align*}
\|T(\mu_n) - T(\mu_0)\| & \leq \max_{0 \leq s \leq 1} \int_0^1 G(r, s) f(s, \mu_n(s)) - f(s, \mu_0(s)) ds \\
& \leq \int_0^1 s(2 - s)(1 - s)\frac{\beta - 3}{2} [a_1(s) f_1(s) + a_2(s) f_2(s)] ds \\
& \leq 2 \int_0^1 s(2 - s)(1 - s)\frac{\beta - 3}{2} [a_1(s) f_1(s) + a_2(s) f_2(s)] ds \\
& \quad + \int_{\delta}^{1-\delta} s(2 - s)(1 - s)\frac{\beta - 3}{2} [a_1(s) f_1(s) + a_2(s) f_2(s)] ds \\
& \quad + 2 \int_{1-\delta}^1 s(2 - s)(1 - s)\frac{\beta - 3}{2} [a_1(s) f_1(s) + a_2(s) f_2(s)] ds \\
& \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
& = \epsilon.
\end{align*}
\]

So, \( T \) is continuous. The proof is finished. \( \square \)

**Theorem 1.** Under the conditions \((H_1) - (H_3)\) and the existence of a positive constant \( r > 0 \) and \( \lambda > \beta > 0 \) such that
\[
\int_0^1 [a_1(s) f_1(s) + a_2(s) f_2(s)] ds < 2^3 r,
\]
(44)

at least two positive solutions \( \mu_1 \) and \( \mu_2 \) of (4) exists with \( 0 < \|\mu_1\| < 2^3 r < \|\mu_2\| \).

**Proof.** According to Lemma 7, the complete continuity of \( T \) is valid from \( K - B \), into \( K \) for any \( r > 0 \). Then, the existence of two fixed points \( \mu_1 \) and \( \mu_2 \) with \( 0 < \|\mu_1\| < 2^3 r < \|\mu_2\| \) is proved here. For any \( r > 0 \) and condition \((H_3)\), we choose
\[
r_1 > \max \{2^3 r, 2^4 r\}.
\]
(45)

When \( \mu \in K \) and \( \|\mu\| = r_1 \), we have
\[
\mu(r) \geq r_1 \geq \left(\frac{1}{2}\right)^{\beta - 1} r_1 > r_1, \forall r \in \left[\frac{1}{2}, 1\right].
\]
(46)

Thus, we get from \((H_2)\) and (45) and (46):

\[
(T(\mu)) \left(\frac{1}{2}\right) = \int_0^1 G \left(\frac{1}{2}, s\right) f(s, \mu(s)) ds \geq \int_{c_1}^{d_1} G \left(\frac{1}{2}, s\right) f(s, \mu(s)) ds \\
\geq \int_{c_1}^{d_1} G \left(\frac{1}{2}, s\right) b_1(s) \mu(s) ds \geq \int_{c_1}^{d_1} G \left(\frac{1}{2}, s\right) b_1(s) \mu(s) ds \geq \left(\frac{1}{2}\right)^{\beta - 1} r_1
\]
(47)
Therefore,
\[
\|T\mu\| = \max_{\tau \in [0,1]} \|T(\mu)(\tau)\| \geq \left|\left(T\mu\right)\left(\frac{1}{2}\right)\right| \geq r_1 = \|\mu\|,
\]
\[\forall \mu \in K, \|\mu\| = r_1. \tag{48}\]

The condition (H4) guarantees that, for \( M = r \int_{c_2}^{d_2} G((1/2, s)b_2(s)ds)^{-1} \), there exists \( R_2 \in (0, 1) \) satisfying
\[
\begin{align*}
(T\mu)\left(\frac{1}{2}\right) &= \int_0^1 G(\frac{1}{2}, s)f(s, \mu(s))ds \geq r \left[ \int_{c_2}^{d_2} G(\frac{1}{2}, s)b_2(s)ds \right]^{-1} \int_{c_2}^{d_2} G(\frac{1}{2}, s)b_2(s)ds = r > r_2.
\end{align*}
\]

Therefore,
\[
\|T\mu\| = \max_{\tau \in [0,1]} \|T(\mu)(\tau)\| \geq \left|\left(T\mu\right)\left(\frac{1}{2}\right)\right| \geq r_2 = \|\mu\|,
\]
\[\forall \mu \in K, \|\mu\| = r_2. \tag{52}\]

\[\begin{align*}
(T\mu)(\tau) &= \int_0^1 G(\tau, s)f(s, \mu(s))ds \leq \int_0^1 s(2-s)(1-s)\beta f(s, \mu(s))ds \\
&\leq \int_0^1 s(2-s)(1-s)\beta \left[ a_1(s)f_{1r}(s) + a_2(s)f_{2r}(s) \right]ds \\
&\leq \int_0^1 \left[ a_1(s)f_{1r}(s) + a_2(s)f_{2r}(s) \right]ds \\
&< 2^{\beta}r.
\end{align*}\]

Thus, we have
\[
\|T\mu\| \leq \|\mu\|, \quad \forall \mu \in K, \|\mu\| = 2^{\beta}. \tag{54}\]

From Lemma 5, (48), (52), and (54), two fixed points \( \mu_1, \mu_2 \) of \( T \) satisfy
\[
\begin{aligned}
D_0^{(7/2)} \mu(\tau) + \frac{r^2}{2} \mu^{1/2} + \frac{33\Gamma(7/2)}{\sqrt{3} - \sqrt{2}} (1-\tau)^{(1/2)} \mu = 0, & \quad 0 < \tau < 1, \\
\mu(0) = \mu'(0) = \mu''(0) = \mu''(1) = 0.
\end{aligned}\]
\[\tag{56}\]

Choose
\[
0 < r_2 < \min(r, R_2). \tag{50}\]

When \( \mu \in K \) and \( \|\mu\| = r_2 \), we have
\[
f(\tau, \mu) > Mb_2(\tau), \quad \forall (\tau, \mu) \in [c_2, d_2] \times (0, R_2]. \tag{49}\]

For \( \mu \in K \), where \( \|\mu\| = 2^4r \), (H3), (1) of Lemma 4, and
\[0 < s(2-s)(1-s)^{\beta - 3} < 1, \quad \text{it is similar to (35) and (44), one can get}\]
\[
0 < r_2 \leq \|\mu_1\| < 2^4r \leq \|\mu_2\| \leq r_1. \tag{55}\]

The proof is completed. \[\square\]

Example 1. Consider the following SBV problem:

\[\begin{align*}
\beta = (7/2) \quad \text{and} \quad f(\tau, \mu) &= (r^2/2)\mu^{1/2} + ((33\Gamma(7/2))/\sqrt{3} - \sqrt{2}) (1-\tau)^{(1/2)} \mu = 0, \\
&\quad 0 < \tau < 1, \\
&\mu(0) = \mu'(0) = \mu''(0) = \mu''(1) = 0.
\end{align*}\]
\[\tag{56}\]

So, the conditions (H1) and (H2) hold.
Next, we set \( b_1 (\tau) = a_2 (\tau), b_2 (\tau) = a_1 (\tau) \), \([c_1, d_1] = [c_2, d_2] = [(1/4), (3/4)]\). Then, it is obviously \( f(\tau, \mu) \geq b_1 (\tau) \mu, \forall (\tau, \mu) \in [c_1, d_1] \times (R_1, +\infty) \) for any \( R_1 > 0 \). By simple computation, we have \( \int_0^{R_1} G((1/2), s) b_1 (s) ds = 3 \times 2^{-2(5/2)} > 2^{(7/2)-1} \). And \( \lim_{\mu \to 0+} \inf \min_{y \in [1/4, (3/4)]} (f(\tau, \mu)/ b_2 (\tau)) = +\infty \). So, the conditions \((H_3)\) and \((H_4)\) also hold. Taking \( r = 1, \lambda = 10.25 \), we have by (57)

\[
\int_0^1 [a_1 (s) f_1 (s) + a_2 (s) f_2 (s)] ds = 1084.4 < 2^{10.25} \cdot 1.
\]

Consequently, condition (44) holds. Then, from Theorem 1 in Example 1 at least two positive solutions \( \mu_1 \) and \( \mu_2 \) exist with \( 0 < ||\mu_1|| < 2^{1/4} ||\mu_2|| \).

### 4. Uniqueness of Singular Problem Solution

By property (2) of Green’s function,

\[
\zeta (\tau) p (s) \leq \Gamma (\beta) G (\tau, s) \leq \zeta (\tau) q (s), \quad \tau, s \in [0, 1],
\]

where \( \zeta (\tau) = \tau^{1-\beta}, p (s) = s (1-s)^{\beta-2}, \) and \( q (s) = (1-s)^{\beta-3} \).

From Lemma 3 and (59), a solution of (4) \( \mu (\tau) \) is

\[
\mu (\tau) = \int_0^1 G (\tau, s) f (s, \mu (s)) ds, \quad \tau \in [0, 1],
\]

\[
\zeta (\tau) \int_0^1 p (s) f (s, \mu (s)) ds \leq \mu (\tau) \leq \zeta (\tau) \int_0^1 q (s) f (s, \mu (s)) ds.
\]

**Proof.** Taking the similar process in [34], for \( \tau \in (0, 1), \mu > 0, \)

\[
f_2 (\tau \mu) \leq \tau^{-\gamma} f_2 (\mu),
\]

\[
f_2 (\tau) \leq \tau^{-\gamma} f_2 (1),
\]

\[
f_2 (\tau^{-1}) \geq \tau^{\gamma} f_2 (1),
\]

\[
f_1 (\tau) \geq \tau^{\gamma} f_1 (1),
\]

\[
f_1 (\mu) \leq \mu^{\gamma} f_1 (1), \quad \mu > 1.
\]

Define

\[
Q_{\zeta} = \left\{ \mu \in \Lambda : \frac{1}{M} \zeta (\tau) \lambda (\tau) \leq \mu (\tau) \leq M \zeta (\tau), \tau \in [0, 1] \right\},
\]

where \( \zeta (\tau) = \tau^{1-\beta} \), and \( M > 1 \) is defined as

\[
M = \max \left\{ \left\{ \left( \int_0^1 \frac{1}{\Gamma (\beta)} q (s)^{-\gamma} (a_1 (s) f_1 (1) + a_2 (s) f_2 (1)) ds \right)^{1/(1-\gamma)} \right\}, \left\{ \left( \int_0^1 \frac{1}{\Gamma (\beta)} p (s)^{-\gamma} (a_1 (s) f_1 (1) + a_2 (s) f_2 (1)) ds \right)^{-1/(1-\gamma)} \right\} \right\}.
\]

For \( \mu, \nu \in Q_{\zeta} \), we define

\[
T_{\zeta} (\mu, \nu) (\tau) = \xi \int_0^1 G (\tau, s) [a_1 (s) f_1 (\mu (s)) + a_2 (s) f_2 (\nu (s))] ds, \quad \forall \tau \in [0, 1].
\]
We firstly show that $T_\xi: Q_\zeta \times Q_\zeta \rightarrow Q_\zeta$. Let $\mu, \nu \in Q_\zeta$, and from (65) and (66), we have then from (69),

$$f_1(\mu) \leq f_1(M\zeta(\tau)) \leq f_1(M) \leq M^\gamma f_1(1), \quad \forall \tau \in (0, 1), \quad (73)$$

Then, from (62), (67), and (68),

$$T_\xi(\mu, \nu)(\tau) \leq f_2\left(\frac{1}{M} \zeta(\tau)\right) \leq f_2\left(\frac{1}{M} \zeta^{-\gamma}(\tau)\right) \leq M^\gamma f_1(1)\zeta^{-\gamma} f_2(1), \quad \forall \tau \in (0, 1). \quad (74)$$

So, we have

$$T_\xi(\mu, \nu)(\tau) \leq M^\gamma(\tau)\zeta^{-\gamma} f_1(1) \zeta^{-\gamma} f_2(1) = M\zeta(\tau), \quad \forall \tau \in [0, 1]. \quad (75)$$

And, from (62), (67), and (68),

$$f_1(\mu(\tau)) \geq f_1\left(\frac{1}{M} \zeta(\tau)\right) \geq f_1\left(\frac{1}{M} \zeta\right) \geq M^{-\gamma} f_1(1) \zeta f_1(1), \quad \forall \tau \in (0, 1), \quad (76)$$

$$f_2(\nu(\tau)) \geq f_2(M\zeta(\tau)) \geq f_2(M) \geq M^{-\gamma} f_2(1), \quad \forall \tau \in (0, 1),$$

for all $\mu, \nu \in Q_\zeta$. So,

$$T_\xi(\mu, \nu)(\tau) \geq M^{-\gamma}(\tau)\zeta^{-1} f_2(1) = M^{-1}\zeta(\tau), \quad \forall \tau \in [0, 1]. \quad (77)$$

It is easy to check that $T_\xi(\mu, \nu)$ is nondecreasing in $\mu$ and nonincreasing in $\nu$. 
Next, for any $\sigma \in (0, 1)$ and $\mu, \nu \in Q_{\sigma}$, we have

\[
T_{\xi}(\sigma \mu, \sigma^{-1} \nu)(\tau) = \xi \int_{0}^{1} G(\tau, s) \left[ a_{1}(s) f_{1}(\sigma \mu(s)) + a_{2}(s) f_{2}(\sigma^{-1} \nu(s)) \right] ds
\]

\[
\geq \xi \int_{0}^{1} G(\tau, s) \left[ a_{1}(s) \sigma^{\gamma} f_{1}(\mu(s)) + a_{2}(s) \sigma^{\gamma} f_{2}(\nu(s)) \right] ds
\]

\[
= \sigma^{\gamma} T_{\xi}(\mu, \nu)(\tau), \quad \forall \tau \in [0, 1].
\]

Lemma 6 accordingly holds. Also, a unique $\mu^{*} \in Q_{\sigma}$ satisfying $T_{\xi}(\mu^{*}, \mu^{*}) = \mu^{*}$ exists. Theorem 2 is proved. \qed

**Example 2.** The following example for SBV is considered here:

\[
\begin{aligned}
D_{0}^{(7/2)} & \mu(\tau) + \frac{1}{\sqrt{T}} \mu^{1/3} + \tau \mu^{(-1/4)} = 0, \quad 0 < \tau < 1, \\
\mu(0) &= \mu^{'}(0) = \mu^{''}(0) = \mu^{'''}(1) = 0.
\end{aligned}
\]

We let $f_{1}(\mu(\tau)) = \mu^{1/3}(\tau)$, $a_{1}(\tau) = (1/\sqrt{T})$, $f_{2}(\mu(\tau)) = \mu^{(-1/4)}(\tau)$, $a_{2}(\tau) = \tau$, and $\gamma = 1/3$. Accordingly,

\[
f_{1}(\tau \mu) = \tau^{1/3} \mu^{1/3} \geq \tau^{\gamma} f_{1}(\mu), \quad f_{2}(\tau^{-1} \mu) = \tau^{1/4} \mu^{(-1/4)} \geq \tau^{\gamma} f_{2}(\mu),
\]

\[
\int_{0}^{1} q(s) \xi^{\gamma} \left[ a_{1}(s) f_{1}(1) + a_{2}(s) f_{2}(1) \right] ds = \int_{0}^{1} (1-s)^{(7/2)-3} s^{-(7/2)-1/3} \left[ s^{-(1/9)} + s \right] ds
\]

\[
= 18 + \frac{6}{7} < + \infty.
\]

So, Theorem 2 is validated, indicating the presence of a unique positive solution $\mu^{*}$.

**Data Availability**

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

**Conflicts of Interest**

The author declares no conflicts of interest.

**Authors’ Contributions**

The author has read and approved the final manuscript.

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