We have studied the conditions for quantum steering between output modes when squeezed Gaussian light and vacuum mix in a beamsplitter as a function of the amount of squeezing and purity of the incident state. For Gaussian measurements, we find that for sufficiently strong squeezing, steering is possible regardless of the purity of the incident state. Two different types of non-Gaussian measurements were also considered. For one of them, an increase with respect to Gaussian measurements of the set of two-way-steerable states is obtained only for unbalanced beamsplitters.

INTRODUCTION

The term “steering” was first used in the context of quantum mechanics by E. Schrödinger in 1935 [1], to refer to the ability of one system to influence the results of measurements carried in another when the two systems share an entangled state. Steering (also known as Einstein-Rosen-Podolsky entanglement) is a quantum correlation intermediate between entanglement and Bell-nongalicity and unlike these correlations, steering has a defined direction [2].

While entanglement is a necessary condition for steering, not all entangled states are steerable [3, 4]. However, Moroder and co-workers have shown that steerable states can be generated from a sufficiently large number of copies of any entangled state [5].

In addition to its fundamental interest in quantum theory [6], steering has been identified as an important resource for quantum information processing. Steering is required in two-parties quantum protocols, such as quantum key distribution [7, 8], quantum teleportation with continuous variables [9] and randomness certification [10], for which entanglement certification is needed although one of the parties cannot be trusted. The study of steering was extended to the multipartite scenario [11, 12] and its applications to quantum communication networks [13, 14].

A general framework for quantum steering was established in [4] in terms the joint-statics of the outcomes of measurements performed in two separate systems. However, in many situations this general characterization of steering is difficult to use as a practical criterion for the identification of steerable states. Previous works, such as [15], have suggested other steering criteria adapted to specific experimental schemes.

In this work we are concerned with Gaussian states of light [16]. Regarding fluctuations, a bipartite Gaussian state is fully characterized by its covariance matrix $V$ which can be written in block form as:

$$V = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix},$$

where $A$ and $B$ are the covariance matrices reduced to each of the two subsystems. As usual, we will refer to these subsystems as Alice’s (A) and Bob’s (B).

Kogias and co-workers [17] defined a steering quantifier (from Alice to Bob) by Gaussian measurements $G^{A\rightarrow B}$: that for a two-mode gaussian state with covariance matrix $V$, has the following expression:

$$G^{A\rightarrow B} = \max\left\{0, \frac{1}{2} \log \left[ \frac{\det(A)}{\det(V)} \right] \right\}.$$  

The purity $\mu$ of a state is related to its covariance matrix $V$ by $\mu = 1/2^m \sqrt{\text{Det}(V)}$, $m$ being the number of modes. In consequence, Eq. 2 implies that in the case of two-mode Gaussian states and Gaussian measurements, the sufficient and necessary condition for Alice to be able to steer Bob’s system is that her reduced state is less pure than the whole state.

Referring to non-Gaussian measurements, we shall say that a set of measurement operators outperforms Gaussian measurements when there are states (only Gaussian
states are considered here) which are not steerable (in a given direction) by Gaussian measurements but present steering under the set of operators considered.

It has been conjectured that Gaussian measurements are optimal for detecting steering in Gaussian states [18]. To the best of our knowledge this conjecture has not been disproved in the case of symmetrical states such as two-mode squeezed vacuum states (TMSV). However, recent work [19, 20] has shown that in the case where asymmetric losses act upon a TMSV state, the conjecture is not longer true. In this work we show that the non-Gaussian operators introduced in [20] also allow outperforming Gaussian measurements for the states under our consideration albeit only for asymmetric states.

**GAUSSIAN STATES ENTANGLED BY A BEAMSPLITTER**

Arguably, the simplest way to generate two entangled optical fields is to split a single optical wave in a beamsplitter (BS). In order to obtain entangled fields at the BS outputs the state of the incident field must be nonclassical [21], otherwise the two-mode output state is separable.

We consider in this work Gaussian states of light incident on a beamsplitter (Fig. 1). Gaussian states are easily generated through a large variety of physical mechanisms including thermal light sources, four-wave mixing, parametric down-conversion, polarization self-rotation, etc. [22, 23].

In order to be nonclassical a Gaussian state must be squeezed. That is one quadrature of the field must have a variance smaller than that of the vacuum. A study of entanglement obtained by splitting Gaussian states in a BS can be found in [24].

With the trivial exception of the vacuum, all experimentally prepared Gaussian states possess some amount of excess noise, that is the product of the variances of two orthogonal quadratures exceeds the lower limit imposed by the Heisenberg uncertainty relation. The state is therefore not pure.

In this work we study the requirement for steering at the outputs of a BS in terms of the purity and amount of squeezing characterizing the field incident on one input port of a BS. In most of the paper we assume that vacuum is incident on the second input port.

Since the considered two-mode states are issued from a beamsplitter it is worth signaling the singular case of the balanced BS (50% transmission and negligible losses). The generated output state is referred to as symmetrical and with not loss of generality, we can consider $A = B$ in [4]. In this case no unidirectional effect (directed only from Alice to Bob or the reverse) is to be expected. On the other hand if the BS is unbalanced, in which case the state is said asymmetrical, unidirectional steering could be possible.

It is worth mentioning that the two-mode Gaussian states considered here are different from the extensively studied TMSV (also known as EPR states). In the case of TMSV states the reduced state received by either Alice or Bob are thermal states. In the case considered here the reduced states at the two outputs of the BS correspond generally to (locally) squeezed states with entangled fluctuations.

The covariance matrix of the two-mode state at the BS input (Fig. 1) is:

$$ V = \frac{1}{2} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & \frac{1}{a^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (3) $$

where the vacuum fluctuations variance is set equal to $1/2$, $a \leq 1$ is the variance of the squeezed quadrature normalized to the vacuum fluctuations variance and $\mu \leq 1$ is the incident state purity.

The covariance matrix for the output modes of the BS is $V' = S V S^\dagger$, where $S$ is the matrix representing the BS

$$ S = \begin{pmatrix} \cos(\theta) I_2 & \sin(\theta) R(\phi) \\ -\sin(\theta) R(\phi) & \cos(\theta) I_2 \end{pmatrix}, \quad (4) $$

$I_2$ is the identity matrix in dimension 2, $R(\phi)$ is a rotation matrix of angle $\phi$ characterizing the phase difference introduced by the BS, and $y \theta$ characterizes its transmittance T and reflectance R: $T = \cos^2(\theta)$, $T + R = 1$ [25].

The phase difference introduced by the BS does not affect local measurements performed by either Alice or Bob, therefore it cannot influence the observation of entanglement or steering. In consequence, we will assume $\phi = 0$ with no loss of generality.

As we mentioned before, the necessary condition for entanglement is for the input Gaussian state to be squeezed ($a < 1$) [24], so we will restrict ourselves to $0 \leq a \leq 1$. For pure states, steering and entanglement are equivalent [3].

**STEERING BY GAUSSIAN MEASUREMENTS**

The steering from Alice to Bob by Gaussian measurements, quantified by [2] for a state of covariance matrix $V$ in the form of Eq. (3) is given by:
FIG. 2. Steering quantifier \( G^{A\rightarrow B} \) for a \( T = R = 0.5 \) BS as a function of the normalized squeezed-quadrature variance \( a \) and the purity \( \mu \) of the input state. The white line separates steerable from non-steerable states.

\[
G^{A\rightarrow B} = \max \{0, \tilde{G}(a)\}, \quad (5a)
\]

\[
\tilde{G}(a) = \frac{\log[(1/a - 1)(\mu^2 a - 1)(T + 1)^2]}{\log(2)} \quad \quad (5b)
\]

This quantifier is plotted in figure 2 for the case of a \( T = R = 0.5 \) BS.

As already mentioned, for pure states entanglement and steering are equivalent, so for pure states (\( \mu = 1 \)), all states with \( a < 1 \) are steerable as can be checked from Eqs. 5.

By imposing that the purity on Alice’s subsystem is equal to the purity of the whole \( \det(A) = \det(V) \) (see 1) we obtain the frontier surface \( \mu_f(a,T) \) separating the regions of steerable and non-steerable states:

\[
\mu_f(a,T) = \begin{cases} 
\sqrt{\frac{a(T+1)-1}{\sigma^2 + a(1-T)}} & \text{if } a \geq \frac{T}{1+T}, \\
0 & \text{if } a < \frac{T}{1+T}.
\end{cases} \qquad (6)
\]

The condition on purity for steering is therefore:

\[
\mu \geq \mu_f(a,T). \quad (7)
\]

If \( a < \frac{T}{1+T} \) steering from Alice to Bob is always possible regardless of the amount of excess noise in the input state.

Figure 3 shows the frontier surface for an arbitrary BS of transmission \( T \). The steerable states lie above the colored surface.

Steerable and non-steerable states are shown in Fig. 4 in the specific case of a 50% transmission BS. In such case steering is always possible if \( a < 1/3 \). This observation is reminiscent of the result established in 26 stating that symmetrical losses by a factor larger than 1/3 prevents steering in states such as the ones considered here. The two results can be related by noticing that symmetrical losses after the BS are equivalent to the same loss fraction \( \varepsilon \) on the input state before the BS. In consequence, the two-mode covariance matrix \( W \) at the BS input for an initially pure state is:

\[
W = \frac{1}{2} \begin{pmatrix} 
(1-\varepsilon)a + \varepsilon & 0 & 0 & 0 \\
0 & (1-\varepsilon)a + \varepsilon & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad (8)
\]

Introducing the normalized squeezed variance of this
state \( a' \equiv (1 - \varepsilon)a + \varepsilon \) we notice that the state purity is given by:

\[
\mu^2 = \frac{a' - \varepsilon}{(1 - \varepsilon)^2 + \varepsilon(a' - \varepsilon)}
\]

which satisfies \( \mu < \mu_f(a', T = 0.5) \) for \( \varepsilon > 1/3 \) in which case the state is unsteerable [27].

**STEERING BY NON-GAUSSIAN MEASUREMENTS**

The previous analysis considered only Gaussian measurements. One can therefore inquire whether non-Gaussian measurements can extend the ensemble of states within the ones considered in this work, for which steering is possible. A similar inquiry was addressed in [20, 28] for the two-mode squeezed vacuum (TMSV) state.

A steering quantifier analogous to (2) is not available for arbitrary non-Gaussian measurement operators. Specific steering criteria must be derived for each type of measurements.

A first example of non-Gaussian measurements for which a steering condition could be established was presented in [20]. The set of measurement operators was constructed from projectors \(|n\rangle\langle n|\) and transition operators \(|n\rangle\langle m|\) involving Fock basis states bounded to \(n, m \leq N\). A specific steering criterion for these states was derived [violation of inequality (10) in [20]]. This steering condition is different for \(A \rightarrow B\) and \(B \rightarrow A\) steering.

A second sufficient condition for steering based on moments was established in [17] that can be applied to a broad range of measurement operators Gaussian or not. This condition was applied in [28] to two families of pseudospin operators named type 1 and type 2. While it was shown that type 2 measurements cannot outperform Gaussian measurements for the observation of steering in arbitrary Gaussian states, it was demonstrated that type 1 measurements can increase the steering range in the case of a TMSV with asymmetric losses on the two modes.

Both studies show no increase in the ensemble of states for which steering is possible relative to Gaussian measurements when the light modes on Alice and Bob sides are symmetrical. However an extension of the steering range is observed in the case of asymmetric states obtained by introducing additional loss on Bob’s mode. In this case, a steering-range increase occurs for steering from Bob to Alice therefore increasing the set of states presenting bi-directional steering.

We have used the non-Gaussian observables introduced in [20] and [28] to explore the possibility of steering between the two outputs of the BS in the scheme presented in Fig. 1. Notice that in this case asymmetric two-mode states can be trivially generated by using an unbalanced BS.

The evaluation of steering conditions for the two sets of observables requires the knowledge of the total and reduced density matrices elements in the Fock state basis. These were computed from the respective covariance matrices using multivariate Hermite polynomials [19, 20].

Following [20] we have first considered non-Gaussian operators arising from a truncated Fock state basis restricted to states \(|0\rangle\) and \(|1\rangle\).

![Contour plot of the steering inequality (10) in [20]](image)

FIG. 5. Contour plot of the steering inequality (10) in [20] using a two-state Fock basis for a BS with \(T = 0.5\). Steering is possible in the region above the contour line 0. Thick solid line: Limit of the steerability region for Gaussian measurements (see Fig. 1).

Figure 5 shows a contour plot of the steering condition established in [20] in the case of a symmetric BS. Steering is possible in the region above the contour line 0. No increase in the steerable states region is obtained with respect to Gaussian measurements. The same conclusion is reached in the case of an asymmetric BS with 90\% transmission (Fig. 6). We also noticed that no increase in the steerability region is obtained by enlarging the size of the truncated Fock states basis.

We next considered pseudospin observables (PSO) sim-
FIG. 6. Contour plot evaluating inequality (10) in [20] using $n = 2$ for an asymmetric BS with $T = 0.9$ (towards Alice). The inequality is violated (steering is possible) in the region above the contour line $0$. Thick solid lines: Limit of the steerability region for Gaussian measurements. a) $A \to B$ steering. b) $B \to A$ steering.

FIG. 7. Limit of the region where steering is possible, according to inequality (11) for a BS with $T = 0.5$. Steering is possible in the region above the limit line. Thin solid lines correspond to the limits for increasing values of $N = [0, 1, \ldots, 8]$ (from top to bottom) in the truncation of the pseudospin observables in Eqs. (10). Thick solid line: Limit of the steerability region using Gaussian measurements.

FIG. 8. Contour plot of the LHS of inequality (11) for an asymmetric BS with $T = 0.9$ (towards Alice) using pseudospin operators (10) truncated at $N = 8$. Steering is possible in the region above the contour line 1. Thick solid lines: Limits of the steerable regions for Gaussian measurements, $A \to B$ (red) and $B \to A$ (blue). The hatched area shows an increase with respect to Gaussian measurements of the set of states where $B \to A$ steering is possible.

For a symmetric BS ($T = 0.5$) PSO do not increase the region where steering is possible. For large purities the limit of the steering region approaches the limit obtained for Gaussian observables. As $N$ is increased the steering region obtained from PSO approaches the steerable states ensemble obtained from Gaussian observables (Fig. 7).

For an asymmetric BS (Fig. 8), PSO result in an increase of the region where steering is possible from $B$ to $A$ (most of the light goes towards $A$). No increase in the steering region is observed from $A$ to $B$. 

\begin{align*}
\hat{S}_x^A &= \sum_{n=0}^{n=N} (|2n\rangle\langle 2n+1| + |2n+1\rangle\langle 2n|) \quad (10a) \\
\hat{S}_y^A &= \sum_{n=0}^{n=N} i (|2n\rangle\langle 2n+1| - |2n+1\rangle\langle 2n|) \quad (10b) \\
\hat{S}_z^A &= \sum_{n=0}^{n=N} (|2n+1\rangle\langle 2n+1| - |2n\rangle\langle 2n|) \quad (10c)
\end{align*}

Notice that unlike in [28] the sums on the RHS of Eqs. (10) are truncated at $n = N$. However, since the angular momentum commutation rules are preserved the inequality

$$\langle \hat{S}_x^A \otimes \hat{T}_x^B \rangle^2 + \langle \hat{S}_y^A \otimes \hat{T}_y^B \rangle^2 + \langle \hat{S}_z^A \otimes \hat{T}_z^B \rangle^2 > 1$$

remains a sufficient condition for steering (either $A \to B$ or $B \to A$) [28]. Here $\hat{S}_i^A$ and $\hat{T}_i^B$ are pseudospin operators acting on Alice’s and Bob’s Hilbert spaces respectively.
CONCLUSIONS

We have studied the conditions for steering in the entangled states obtained by mixing a squeezed Gaussian state with vacuum in a beamsplitter. We have described the steerability of the state in terms of the maximally squeezed quadrature variance and the purity of the state.

For Gaussian measurements we have determined the requirement for steering on the state purity for a given amount of squeezing. We found that any input state with squeezed quadrature variance smaller than a critical value allows steering of one output mode by the other regardless of the input state purity. Such critical value is $1/3$ of the vacuum quadrature fluctuations variance in the case of a symmetric BS. The importance of this result for experimental applications based on steering deserves to be stressed since it signifies that state-purity requirements can be disregarded provided a sufficient degree of squeezing is achieved.

In the case of an asymmetric BS, where most of the light is directed towards Alice, non-Gaussian measurements of the form of Eqs. (10) can result in an increase, with respect to Gaussian measurements, of the ensemble of states allowing steering from Bob to Alice. This enlarged set is nevertheless included in the ensemble of states for which steering by Gaussian measurements is possible from Alice to Bob.

We notice that for the states considered in this work as well as for the previously studied TMSV [28, 29] no gain in the possibility of steering with respect to Gaussian measurements is obtained if the two-mode state is symmetric. One can therefore speculate that Gaussian measurements are optimal for (two-way) steering in symmetric states. A conjecture that deserves to be further explored.

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Appendix: Added thermal noise

Departing from the situation previously studied, we consider now for completeness the case of a thermal state entering the BS instead of vacuum on Fig. 1. The covariance matrix of the input state is now given by:

$$\sigma_{in} = \frac{1}{2} \begin{pmatrix}
da & 0 & 0 & 0 \\
0 & \frac{1}{\mu_a} & 0 & 0 \\
0 & 0 & 2n + 1 & 0 \\
0 & 0 & 0 & 2n + 1
\end{pmatrix},$$

(A.12)

where $n$ is the mean photon number of the thermal state [25]. We only consider the $T = 0.5$ BS as an example.

The frontier between steerable and non-steerable states now depends also on $n$. Previous results are recovered for $n = 0$.

For the output state to be steerable, the purity of the incident squeezed Gaussian state has to be equal or higher than

$$\mu_f(a, T = 1/2, n) = \sqrt{\frac{4a(2n + 1)^2 - (2n + 1 + a)}{a(2n + 1)(2n + 1 + a)}}.$$

(A.13)

When

$$a \leq \frac{2n + 1}{4(2n + 1)^2 - 1}$$

(A.14)

all states are steerable regardless of their purity.

As expected, the squeezed quadrature variance required for steering independently of purity decreases as the thermal field mean photon number increases (Fig. 9).

FIG. 9. Normalized squeezed quadrature variance required for steering independently of purity.

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