Computing The Complete Massless Spectrum Of A Landau-Ginzburg Orbifold

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We develop techniques to compute the complete massless spectrum in heterotic string compactification on N=2 supersymmetric Landau-Ginzburg orbifolds. This includes not just the familiar charged fields, but also the gauge singlets. The number of gauge singlets can vary in the moduli space of a given compactification and can differ from what it would be in the large radius limit of the corresponding Calabi-Yau. Comparison with exactly soluble Gepner models provides a confirmation of our results at Gepner points. Our methods carry over straightforwardly to (0, 2) Landau-Ginzburg models.

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1. Introduction

Landau-Ginzburg models have long been used as mean field models of critical phenomena. More recently it was realized that in two dimensions, much sharper results can be extracted from them. For instance, minimal conformal field theories can be described as Landau-Ginzburg models as shown for bosonic theories by Zamolodchikov [1]; this was extended for $N = 1$ supersymmetry in [2] and for $N = 2$ in [3] [4] [5].

The $N = 2$ case has many special simplifications related in part to the non-renormalization theorems for the superpotential. For instance, for $N = 2$ it is possible to calculate the minimal model characters directly from the Landau-Ginzburg model [3]. Also, for $N = 2$, certain orbifolds of Landau-Ginzburg models have a beautiful and unexpected relation to Calabi-Yau sigma models [4] [7] [8] [9] [10]. The Landau-Ginzburg model describes a certain “point,” or really a certain submanifold, in the Calabi-Yau moduli space.

The $N = 2$ models also have particularly interesting physical applications. $N = 2$ theories with the appropriate central charge can be used to construct compactifications of the heterotic string, and thereby to build models of particle physics, with unbroken space-time supersymmetry. Landau-Ginzburg models can in particular be used to build such compactifications – giving specializations of Calabi-Yau models [11] [12].

These specializations are technically natural, in the usual sense of particle physics, because of enhanced symmetries (involving twist fields; see [10], §3.4, for an explicit explanation). They are interesting because of calculable stringy effects (such as the enhanced symmetries or a deviation of the number of massless particles from what it would be in the field theory limit).

Also, Landau-Ginzburg models are special cases of Calabi-Yau models in which instanton corrections are turned off (see [10], §3.4). As the instanton corrections are the usual obstruction to forming $(0,2)$ deformations of sigma models [13], it would appear likely that $(0,2)$ Landau-Ginzburg models (which are easily constructed [10], §6) have conformally invariant infrared fixed points. This is then an interesting case in which conformally invariant $(0,2)$ models should be accessible for fairly detailed study. $(0,2)$ models are of course of considerable interest because of their use in constructing models of particle physics with effective four dimensional gauge groups more realistic than $E_6$.

Except for Gepner models, which are more or less fully constructed algebraically, most studies of these models have focussed on the chiral primary states. Those states
enter in many beautiful constructions and among other things determine the spectrum of massless charged particles. However, the massless gauge singlets are not (all) determined by the chiral primary states, and the notion of chiral primaries does not carry over to (0,2) models. (The two facts are related: the massless gauge singlets that do not come from chiral primaries are represented by vertex operators that break \( N = 2 \) or (2,2) supersymmetry down to (0,2).) Our intention in this paper is to develop methods for computing the complete massless spectrum of Landau-Ginzburg models, both (0,2) and (2,2) models, and including all of the gauge singlets.

In §2 we describe the necessary facts and methods. In §3 we study in detail a familiar model – the quintic. One virtue of this model is that (at a special point in the parameter space) the results can be compared to known results about the corresponding Gepner model. It should be clear, however, that our methods carry over without essential change to arbitrary Landau-Ginzburg models, including (0,2) models.

For Calabi-Yau manifolds, one can identify the particles which are massless in the field theory limit by computing suitable cohomology groups; but difficult questions then arise, in general, of whether instanton corrections might give non-vanishing (but exponentially small in the field theory limit) masses to some of these states. For Landau-Ginzburg models, however, one can argue – as we will do in §2.1 – that our results are actually exact. Intuitively, this is in keeping with the fact that the Landau-Ginzburg models have no instantons.

2. Background And Methods

We will work in \( N = 2 \) superspace with coordinates \( x^m, \theta^\alpha, \bar{\theta}^\dot{\alpha} \) (our conventions follow those of [10]). In an \( N = 2 \) superconformal theory, there are four supersymmetry charges \( Q_+, \, Q_-, \, \bar{Q}_+, \, \bar{Q}_- \), where \( - \) and \( + \) specify left- and right-movers on the worldsheet.\(^1\) The right moving supersymmetries satisfy

\[
Q_+^2 = \bar{Q}_+^2 = 0, \quad \{Q_+, \bar{Q}_+\} = 2 \, L_{0+} \tag{2.1}
\]

where \( L_{0+} \) is the coefficient of the zero mode in the Laurent expansion of the right moving stress-energy tensor \( T^{++} \).

\(^1\) We will use the terms left-moving and right-moving somewhat loosely to describe modes that in the conformally invariant limit are left-moving or right-moving.
The worldsheet “matter” that we are interested in will be chiral superfields \(\Phi\). Such fields satisfy
\[
[D_+, \Phi] = [D_-, \Phi] = 0
\]  
(2.2)
where \(D\) and \(\overline{D}\) are known as superspace covariant derivatives; the complex conjugates of the \(\Phi\)'s are anti-chiral fields \(\overline{\Phi}\) that satisfy equation (2.2) with \(\overline{D} \rightarrow D\). The chiral superfields have an expansion in terms of component fields
\[
\Phi(x, \theta) = \phi(y) + \sqrt{2} \theta^\alpha \psi(y) + \theta^\alpha \theta_\alpha F(y).
\]  
(2.3)

Recall that the most general renormalizable Lagrangian for an \(N = 2\) supersymmetric theory with chiral superfields \(\Phi_i\) and their anti-chiral conjugates \(\overline{\Phi}_i\) has the form
\[
L_1 = \int d^2x \ d^4\theta \ K(\Phi, \overline{\Phi}) - \int d\theta^+ d\theta^- W(\Phi) - \int d\overline{\theta}^+ d\overline{\theta}^- \overline{W}(\overline{\Phi})
\]  
(2.4)
where \(K\) is called the Kahler potential (its derivatives determine the metric on target space; the target spaces of \(N = 2\) models constructed from chiral superfields are always Kahler manifolds) and \(W\) is a holomorphic function of the fields, called the superpotential; we will choose \(K\) to have the form \(K = \overline{\Phi} \Phi\) corresponding to a flat metric. After performing the \(\theta\) integrals and integrating out the auxiliary fields, the Lagrangian becomes
\[
L_1 = \int \sum_i d^2x \left( -\partial_\alpha \overline{\Phi}_i \partial^\alpha \phi_i + i \overline{\psi}_{-,i}(\partial_0 + \partial_1)\psi_{-,i} + i \overline{\psi}_{+,i}(\partial_0 - \partial_1)\psi_{+,i} \right.
\]
\[
- \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2 - \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \psi_{-,i} \psi_{+,j} - \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \overline{\psi}_{+,j} \overline{\psi}_{-,i} \right)
\]  
(2.5)

The superpotential \(W(\Phi_i)\) is said to be quasi-homogeneous if for some integers \(n_i\) and \(d\) one has \(W(\lambda^{n_i} \Phi_i) = \lambda^d W(\Phi_i)\). Such quasi-homogeneity ensures the existence of left- and right-moving \(R\)-symmetries that play an important role. The models that are believed to be related to Calabi-Yau models are actually not Landau-Ginzburg models as introduced above but orbifolds in which one projects onto states with integral \(R\) charges. For future use, it is convenient to set
\[
\alpha_i = \frac{n_i}{d}.
\]  
(2.6)

The theory described by (2.4) is believed to flow in the infrared to a conformal field theory with central charge
\[
\hat{c} = \sum_i (1 - 2\alpha_i).
\]  
(2.7)
In applications in string theory, it is necessary to consider the model formulated in four sectors – (R,R), (NS,R), (R,NS), and (NS,NS), where R and NS refer to Ramond and Neveu-Schwarz boundary conditions; the two entries give the boundary conditions for left-movers and for right-movers. In applications to Type II superstrings, one would have (in models of this particular type) space-time supersymmetries coming from both left- and right-movers. These supersymmetries determine the spectrum in all four sectors in terms of the spectrum in, say, the (R,R) sector. In practice, this means that to identify massless particles in space-time, it suffices to find the (R,R) ground states. These have very special properties which have been much exploited in the literature on Landau-Ginzburg models and their applications. Their (NS,NS) cousins are represented by vertex operators that preserve (2,2) world-sheet supersymmetry.

We are actually interested in using the same models to describe compactifications of the heterotic string. In this case, we supplement (2.4) by ten left-moving free fermions

\[ L_2 = \int d^2x \sum_{I=1}^{10} \lambda_I (\partial_0 + \partial_1) \lambda_I \]  

and extra degrees of freedom representing an additional \( E_8 \) current algebra. The \( \lambda_I \) are given the same NS or R boundary conditions as the left-moving part of (2.4). The combined Lagrangian \( L_1 + L_2 \) is expected (as in Calabi-Yau compactification) to give an unbroken \( E_6 \) gauge group in space-time.

Space-time supersymmetries are now derived from right-movers only. Therefore, there are two sectors that must be studied – (R,R) and (NS,R). The study of the (NS,R) model is one of the main novelties in this paper. We are no longer interested only in states with a simple relation to (R,R) ground states, so new methods must be developed.

In fact, in the (NS,R) sector, there are massless gauge singlet states that are represented by vertex operators that (even if one suppresses the \( \lambda \)'s) break (2,2) world-sheet supersymmetry down to (0,2) supersymmetry. These are the modes that, in compactification on a Calabi-Yau manifold \( X \), arise from \( H^1(X, \text{End}(T)) \). Understanding these modes in the context of Landau-Ginzburg models is one of our main goals in this paper. In the process of doing this, we will automatically develop the techniques needed to compute the complete massless spectrum in more general (0,2) Landau-Ginzburg models.

An \( SO(10) \) symmetry acting on the ten \( \lambda \)'s is manifest in the above Lagrangian. \( SO(10) \) is not a maximal subgroup of \( E_6 \), which instead contains an \( SO(10) \times U(1) \) factor.

\[ \text{For some computations of this cohomology group in Calabi-Yau models see } [14], [15], [16], [17]. \]
The $U(1)$ generator is simply the left-moving $R$-current – call it $J_L$ – of the Landau-Ginzburg theory with Lagrangian $L_1$. The rest of $E_6$ is harder to see explicitly; the additional currents are twist fields coming from states in the left-moving Ramond sector.

2.1. The Born-Oppenheimer Approximation

Because we are looking for massless states in space-time, we can set the space-time momentum to zero and look for worldsheet wavefunctions which have only polynomial dependence on the lowest oscillator modes. In sectors with negative vacuum energy, we have to keep the lowest excited modes of the various fields. This truncation of the theory to a small finite number of modes, a worldsheet “Born-Oppenheimer” approximation, has been applied before in a string theory context in [18] and [19]. However, the focus there was on sigma models. In the Landau-Ginzburg context, it is easy to be more explicit.

What is the degree of validity of the Born-Oppenheimer approximation? We will argue that for identifying the massless modes it is exact.

We will denote the right- and left-moving world-sheet Hamiltonians as $L_{0+}$ and $L_{0-}$. In the (R,R) and (NS,R) sectors that we will study, physical states have $L_{0+} = 0$; for massless particles on-shell, the “space-time” part of the string does not contribute to $L_{0+}$, so we can consider $L_{0+}$ to be the right-moving Hamiltonian of the “internal” theory only. In a right-moving Ramond sector, there are two right-moving global supersymmetries, say $Q_+$ and $\overline{Q}_+$, with
\[
\{Q_+, \overline{Q}_+\} = 2L_{0+}, \quad Q_+^2 = \overline{Q}_+^2 = 0.
\]
As in Hodge theory, it follows that the kernel of $L_{0+}$ is the same as the cohomology of $\overline{Q}_+$.

This simple fact is the starting point for all our computations: we identify the massless states with the cohomology of $\overline{Q}_+$ (or actually the subspace of that cohomology consisting of states with the correct eigenvalue of $L_{0-}$). This is a great advantage because – due to the simple properties of triangular matrices – cohomology is usually highly computable.

In the particular case at hand, the simplification comes mostly because the $\overline{Q}_+$ cohomology is naturally invariant under a rescaling of the superpotential by $W \to \epsilon W$. The reason for this is that, up to a rescaling of the fields by
\[
\Phi_i \to \epsilon^{-\alpha_i}\Phi_i,
\]
3 To be more precise, under $W \to \epsilon W$, the $\overline{Q}_+$ cohomology group of right-moving $U(1)$ charge $n$ is multiplied by $\epsilon^n$, because of the scaling introduced momentarily.
$W \to \epsilon W$ is equivalent to a certain modification of the kinetic energy. The whole kinetic energy is of the form \( \{ \overline{Q}_+, \ldots \} \) so the modification of the kinetic energy induced by the transformation (2.10) does not affect the $\overline{Q}_+$ cohomology. This means that in computing the $\overline{Q}_+$ cohomology, we can set $W$ to zero except when it is needed to lift degeneracies that are otherwise present. That fact is the basis for all of our calculations.

It is straightforward to write down the $\overline{Q}_+$ operator of the Landau-Ginzburg model:

$$\overline{Q}_+ = i \sqrt{2} \int dx^1 \left( i \overline{\psi}^+_{i,1} (\partial_0 + \partial_1) \phi_i + \frac{\partial W}{\partial \phi_i} \psi^-_{i,1} \right) \quad (2.11)$$

An additional simplification arises (as in \cite{6}) because of the principle stated in the last paragraph. Taking $W \to \epsilon W$ and trying to compute the $\overline{Q}_+$ cohomology perturbatively in $\epsilon$, the first step is to compute the cohomology of the part of $\overline{Q}_+$ that is independent of $W$:

$$\overline{Q}_+, R = i \sqrt{2} \int dx^1 \left( i \overline{\psi}^+_{i,1} (\partial_0 + \partial_1) \phi_i \right) . \quad (2.12)$$

The cohomology of this operator is the subspace of the full Hilbert space consisting of states in which the right-moving oscillators are all in their ground states and which depend holomorphically on the zero modes of the $\phi_i$; moreover the zero modes of $\psi^+$ and $\overline{\psi}^+$ can be omitted. This leaves a smaller Hilbert space, consisting of left-moving oscillators, zero modes of $\psi^-$ and $\overline{\psi}^-$, and holomorphic functions of boson zero modes. Let us call this the left-moving Hilbert space $\mathcal{H}_L$.

The next step, analogous to degenerate perturbation theory in quantum mechanics, is to compute the cohomology of the “perturbation”

$$\overline{Q}_+, L = i \sqrt{2} \int dx^1 \frac{\partial W}{\partial \phi_i} \psi^-_{i,1} \quad (2.13)$$

in $\mathcal{H}_L$. In quantum mechanics this would usually be only the beginning of a systematic expansion; but in the present situation we are actually at this stage finished (at least to all finite orders), because of the triangular nature of cohomology and the simplicity of the cohomology of the $\overline{Q}_+$ operator. The requisite argument is a standard “zig-zag” argument, as in \cite{20}, p. 95, using the following facts. Let $U$ be the operator that assigns the value 1 to $\overline{\psi}^+_{i,1}$, $-1$ to $\psi^+_{i,1}$, and 0 to other fields. Then $[U, \overline{Q}_+, R] = \overline{Q}_+, R$, $[U, \overline{Q}_+, L] = 0$, and the cohomology of $\overline{Q}_+, R$ is zero except at one value of $U$.

Let us use these facts to prove that the $\overline{Q}_+$ cohomology is naturally isomorphic to the cohomology of $\overline{Q}_+, L$ in the $\overline{Q}_+, R$ cohomology (which is isomorphic to $\mathcal{H}_L$). So to begin
with we have a state $\vert \alpha_0 \rangle$ that is annihilated by $\overline{Q}_{+,R}$ and annihilated by $\overline{Q}_{+,L}$ modulo $\overline{Q}_{+,R}(\ldots)$. We can assume that $\vert \alpha_0 \rangle$ has $U = 0$ since the $\overline{Q}_{+,R}$ cohomology vanishes for other values of $U$. The fact that $\vert \alpha_0 \rangle$ is annihilated by $\overline{Q}_{+,L}$ modulo $\overline{Q}_{+,R}(\ldots)$ means that there is some $\vert \alpha_{-1} \rangle$, necessarily of $U = -1$, such that

$$\overline{Q}_{+,L} \vert \alpha_0 \rangle = - \overline{Q}_{+,R} \vert \alpha_{-1} \rangle.$$  \hspace{1cm} (2.14)

Then $\overline{Q}_+ (\vert \alpha_0 \rangle + \vert \alpha_{-1} \rangle) = (\overline{Q}_{+,R} + \overline{Q}_{+,L})(\vert \alpha_0 \rangle + \vert \alpha_{-1} \rangle) = \overline{Q}_{+,L} \vert \alpha_{-1} \rangle$. Moreover

$$\overline{Q}_{+,R} (\overline{Q}_{+,L} \vert \alpha_{-1} \rangle) = - \overline{Q}_{+,L} \overline{Q}_{+,R} \vert \alpha_{-1} \rangle = \overline{Q}_{+,L} \overline{Q}_{+,L} \vert \alpha_0 \rangle = 0$$ \hspace{1cm} (2.15)

where the first step uses $\{ \overline{Q}_{+,R}, \overline{Q}_{+,L} \} = 0$, the second step uses (2.14), and the last step uses $\overline{Q}_{+,L}^2 = 0$. $\overline{Q}_{+,L} \vert \alpha_{-1} \rangle$ therefore represents a state in the cohomology of $\overline{Q}_{+,R}$ at $U = -1$; since the $\overline{Q}_{+,R}$ cohomology vanishes except at $U = 0$, this state is cohomologically trivial and there is a state $\vert \alpha_{-2} \rangle$ of $U = -2$ such that $\overline{Q}_{+,R} \vert \alpha_{-2} \rangle = - \overline{Q}_{+,L} \vert \alpha_{-1} \rangle$.

Continuing in this way, one inductively solves the equations

$$\overline{Q}_{+,R} \vert \alpha_{-n-1} \rangle = - \overline{Q}_{+,L} \vert \alpha_{-n} \rangle.$$ \hspace{1cm} (2.16)

The sum $\vert \alpha \rangle = \vert \alpha_0 \rangle + \vert \alpha_{-1} \rangle + \vert \alpha_{-2} \rangle + \ldots$ is then the desired state annihilated by $\overline{Q}_+ = \overline{Q}_{+,R} + \overline{Q}_{+,L}$. In defining $\vert \alpha \rangle$ and obeying the equations up to the first $n$ terms we have shown that the state which has zero energy in the Born-Oppenheimer approximation has zero energy up to $n^{th}$ order in perturbation theory in the superpotential $W$.

The question of whether the series converges is more subtle, but intuitively this should follow from the super-renormalizability of the Landau-Ginzburg model. The state $\alpha_{-n}$ has $U = -n$, and as $U$ is carried only by fermions, $\alpha_{-n}$ is a state with very high energy, roughly at least the energy of a degenerate fermi gas with fermi energy $n$. For such high energy states, $\overline{Q}_{+,R}$ dominates over $\overline{Q}_{+,L}$ because of being constructed from a current of higher dimension (containing an extra derivative), and in the relation (2.10), it should be possible to choose $\alpha_{-n-1}$ to be much smaller than $\alpha_{-n}$ in norm, ensuring convergence of the series. A rigorous proof of this assertion would be interesting.

The $\overline{Q}_+$ cohomology can be decomposed according to the action of certain operators that commute with $\overline{Q}_+$ or have simple commutation relations with it. In fact, $\overline{Q}_+$ commutes with the left-moving $U(1)$ charge but raises the right-moving $U(1)$ charge by one.
unit. Also obviously commutes with the λ’s, so states can be labeled by the number of λ oscillators.

Somewhat less obviously \[8\], in the Landau-Ginzburg theory (2.4), one can find an \( N = 2 \) superconformal algebra of left-moving fields that commute with \( Q_+ \). In components, one has

\[
J_L = \sum_i \left( (\alpha_i - 1) \psi_{-i} \bar{\psi}_{-i} + i \alpha_i \phi_i (\partial_0 - \partial_1) \phi_i \right)
\]

\[
G = -i\sqrt{2} \sum_i \psi_{-i} (\partial_0 - \partial_1) \phi_i
\]

\[
\overline{G} = i\sqrt{2} \sum_i \left( (1 - \alpha) (\partial_0 - \partial_1) \phi_i \cdot \bar{\psi}_{-i} - \alpha_i \phi_i (\partial_0 - \partial_1) \bar{\psi}_{-i} \right)
\]

\[
T = \sum_i |(\partial_0 - \partial_1) \phi_i|^2 + \frac{i}{2} (\psi_{-i} (\partial_0 - \partial_1) \bar{\psi}_{-i})
\]

\[4\] The statement that \( Q_+ \) raises the right \( U(1) \) charge is convention-dependent. Our conventions for \( U(1) \) charges are given in \( \S 2.2 \).
$H^1(X, \text{End}(T))$. We would like the analogous decomposition in the case of Landau-Ginzburg models. This can be done as follows. In the Calabi-Yau case, the three kinds of states can be described as states that are annihilated by $G_{-1/2}$, states that are annihilated by $\mathcal{G}_{-1/2}$, and states that are annihilated by neither. Since from (2.17) we can get an explicit and practical construction of $G_{-1/2}$ and $\mathcal{G}_{-1/2}$, we can make the decomposition into $H^1(X, T)$, $H^1(X, T^*)$, and $H^1(X, \text{End}(T))$ also in the Landau-Ginzburg case.

In addition to being of intrinsic interest, this decomposition can be of practical use in the following sense. The singlets coming from $H^1(X, T)$ and $H^1(X, T^*)$ are in one to one correspondence with 10’s of SO(10) which arise in the same twisted sectors. The concrete form of the correspondence is as follows. Consider a singlet which is created by a left chiral field, so its representative $|\Psi\rangle$ in the $Q_+, L$ cohomology satisfies $\mathcal{G}_{-1/2} |\Psi\rangle = 0$. Then the corresponding 10 of SO(10) is given by $\lambda 1/2 G_{1/2} |\Psi\rangle$. A similar construction applies to left anti-chiral singlets, with the role of $G$ and $\mathcal{G}$ reversed. We will illustrate this explicitly in the example of §3.

2.2. Symmetries And Quantum Numbers

Consider an $N = 2$ Landau-Ginzburg theory with chiral superfields $\Phi_i$ and quasi-homogeneous superpotential $W$ such that

$$W(\lambda^{n_i} \Phi_i) = \lambda^d W(\Phi_i)$$

and again set $\alpha_i = n_i/d$. The superpotential $W$ will then have left- and right-moving charges $(1, 1)$ – as befits a marginal operator – if the superfields $\Phi_i$ have charges $(\alpha_i, \alpha_i)$. In components the charges are therefore as in Table 1.

| Field  | $q_-$  | $q_+$  |
|--------|--------|--------|
| $\phi_i$ | $\alpha_i$ | $\alpha_i$ |
| $\bar{\phi}_i$ | $-\alpha_i$ | $-\alpha_i$ |
| $\psi^-_i$ | $\alpha_i - 1$ | $\alpha_i$ |
| $\psi^+_i$ | $\alpha_i$ | $\alpha_i - 1$ |
| $\bar{\psi}^-_i$ | $1 - \alpha_i$ | $-\alpha_i$ |
| $\bar{\psi}^+_i$ | $-\alpha_i$ | $1 - \alpha_i$ |

5 In fact, the signs of both $U(1)$ charges are mere conventions. Flipping the convention for one leads to an exchange of 27’s and $\overline{27}$’s; this simple observation motivated the discovery of mirror symmetry.
At this point, the attentive reader might worry about the following point. The $J_L$ operator that transforms the fields according to the charges given in the table is

$$J_L = \sum_i \int dx^1 \left( (\alpha_i - 1)\bar{\psi}_{-i}\psi_{-i} + i\alpha_i \phi_i \bar{\phi}_i + \alpha_i \psi_{+i}\bar{\psi}_{+i} \right). \quad (2.19)$$

The density that is being integrated in (2.19) does not commute with $Q_+$, but the integrated expression does. On the other hand, in equation (2.17) we have written down a left-moving $U(1)$ charge density that does commute with $Q_+$. Using this density, we have a second candidate for the left-moving $U(1)$ charge, namely

$$J'_L = \sum_i \int dx^1 \left( (\alpha_i - 1)\bar{\psi}_{-i}\psi_{-i} + i\alpha_i \phi_i (\partial_0 - \partial_1)\bar{\phi}_i \right). \quad (2.20)$$

Using the commutation relations

$$\{Q_+, \psi_{+,i}\} = -\sqrt{2}(\partial_0 + \partial_1)\phi_i$$
$$[Q_+, \bar{\phi}_i] = i\sqrt{2}\psi_{+,i}$$
$$\{Q_+, \bar{\psi}_{-,i}\} = i\sqrt{2}\frac{\partial W}{\partial \phi^i}$$

(with other components vanishing), one finds that

$$J_L = J'_L + \left\{ Q_+, \frac{i}{\sqrt{2}} \int dx^1 \left( \sum_i \alpha_i \bar{\phi}_i \psi_{+,i} \right) \right\}. \quad (2.22)$$

This shows that as regards the action on the $Q_+$ cohomology, it does not matter whether we use $J_L$ or $J'_L$. $J'_L$ arises naturally in the simplest description of the $N = 2$ algebra that acts on the cohomology, while $J_L$ is distinguished because it generates a symmetry even before taking the $Q_+$ cohomology.

A similar question, which we might as well dispose of now, arises for the left-moving energy operator $L_{0-}$. The Landau-Ginzburg theory (2.4), even away from criticality, has a conserved Hamiltonian $H$ and momentum $P$. The conventional $L_{0-}$ operator would be $L_{0-} = H - P$ or concretely

$$L_{0-} = \int dx^1 \left( |(\partial_0 - \partial_1)\phi_i|^2 - i\bar{\psi}_{-i}\partial_0\bar{\psi}_{-i} \right.$$
$$\left. + \frac{\partial^2 W}{\partial \phi_i \partial \phi_j}\psi_{-i}\psi_{+j} + \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \bar{\psi}_{+j}\bar{\psi}_{-i} + \left| \frac{\partial W}{\partial \phi_i} \right|^2 \right). \quad (2.23)$$
The $L_{0-}$ operator that we would form from the stress tensor in (2.17) is instead

$$L'_{0-} = \int dx^{1} \left( |(\partial_{0} - \partial_{1})\phi_{i}|^2 + \frac{i}{2} \psi_{-i}(\partial_{0} - \partial_{1})\overline{\psi}_{-i} + \frac{\alpha_{i}}{2}(\partial_{0} - \partial_{1})(i\psi_{-i}\overline{\psi}_{-i} - \phi_{i}(\partial_{0} - \partial_{1})\overline{\phi}_{i}) \right). \tag{2.24}$$

In fact, $L'_{0-} = L_{0-} + \{Q, \ldots\}$, though a slightly lengthy calculation is needed to show this. For instance, to reduce (2.24) to a more recognizable form, one first writes $(\partial_{0} - \partial_{1})\psi_{-i} = (\partial_{0} + \partial_{1})\psi_{-i} - 2\partial_{1}\psi_{-i}$, and then evaluates $(\partial_{0} + \partial_{1})\psi_{-i}$ via the equations of motion. $(\partial_{0} - \partial_{1})\overline{\psi}_{-i}$ can be treated similarly. Discarding a total derivative, the term $(\partial_{0} - \partial_{1})(i\alpha_{i}\psi_{-i}\overline{\psi}_{-i} - \phi_{i}(\partial_{0} - \partial_{1})\overline{\phi}_{i})$ in (2.24) can be replaced by

$$(\partial_{0} + \partial_{1})(i\alpha_{i}\psi_{-i}\overline{\psi}_{-i} - \phi_{i}(\partial_{0} - \partial_{1})\overline{\phi}_{i}). \tag{2.25}$$

Using the fact that $\partial_{0} + \partial_{1} \sim \{Q_{+} + \overline{Q}_{+}\}$, it follows that if $\overline{Q}_{+}X = 0$, then $(\partial_{0} + \partial_{1})X = \{Q_{+}, \ldots\}$. Applying this principle with $X$ being the current in the first line in (2.17), we find that up to $\{\overline{Q}_{+}, \ldots\}$, (2.25) can be replaced by $(\partial_{0} + \partial_{1})(i\psi_{-i}\overline{\psi}_{-i})$. This in turn can be evaluated using the equations of motion. After adding one last correction term

$$\left\{\overline{Q}_{+}, -\frac{i}{\sqrt{2}} \int dx^{1} \frac{\partial W}{\partial \overline{\phi}_{i}} \psi_{-i} \right\} \tag{2.26}$$

to (2.24) one obtains the desired results that $L_{0-} = L'_{0-}$ modulo $\{\overline{Q}_{+}, \ldots\}$. The significance of this is similar to the significance of the analogous statement demonstrated for the currents in the last paragraph: $L'_{0-}$ is more closely related to the $N = 2$ algebra that acts on the cohomology, but $L_{0-}$ is natural because it generates a symmetry even before taking the cohomology.

The equivalence of the two $J_{L}$ operators and of the two $L_{0-}$ operators means that the ground state quantum numbers are independent of $W$ (which does not appear in $J'_{L}$ and $L'_{0-}$) and can be computed using the standard formulas associated with normal-ordering of $J_{L}$ and $L_{0-}$.

2.3. Construction Of The Orbifold

Calabi-Yau sigma models are related not quite to Landau-Ginzburg models but to certain Landau-Ginzburg orbifolds. These are orbifolds in which one projects on integral
values of $J_L$; $J_R$ then automatically also becomes integral. The projection is made by dividing by the group generated by

$$e^{-2\pi i \oint J_L(z)} = e^{-2\pi i J_L}$$

with a due modification which we will now explain when certain fermion zero modes are present.

In physical applications of the Landau-Ginzburg orbifold, one wishes to sum over left-moving Ramond and Neveu-Schwarz sectors. (This is the GSO-like projection that enters in constructing $E_8$ current algebra.) In $N = 2$ models, the GSO projection [22] can be interpreted as a projection onto states for which $J_L$ is even. We are not quite dealing here with an $N = 2$ model but with a $(0, 2)$ model containing also the left-moving free fermions $\lambda_I$. Hence, in the left-moving NS sectors, the GSO projection that we want is the one that projects onto states in which $J_L$ plus the number of $\lambda_I$ excitations is even. So we project onto states with $g = 1$ where

$$g = \exp(-i\pi J_L) \cdot (-1)^\lambda.$$  \hfill (2.28)

The necessary statement in R sectors is more subtle because of fermion zero modes. Let $q_-$ and $q_+$ be the left-moving and right-moving $U(1)$ charges of the “internal” Landau-Ginzburg theory. Then in left-moving Ramond sectors, the GSO projection (on states that are in the ground state of the $SO(10)$ sector) can be summarized by saying that the value of $q_-$ determines whether states transform in the $16$ or the $\overline{16}$ of $SO(10)$. One (standard) way to understand this in more detail is to organize the ten $SO(10)$ fermions of (2.8) into five complex fermions

$$\eta_I = \frac{1}{\sqrt{2}} (\lambda_{2I-1} + i\lambda_{2I})$$ \hfill (2.29)

where $I = 1, \cdots 5$. The complex fermi fields have zero modes $\eta_{0,I}$ and $\eta_{0*}^I$ which satisfy the standard anti-commutation relations

$$\{\eta_{0,I}, \eta_{0,J}\} = \{\eta_{0*}^I, \eta_{0*}^J\} = 0, \quad \{\eta_{0,I}, \eta_{0*}^J\} = \delta_{IJ}.$$ \hfill (2.30)

Then acting on the Fock vacuum $|0\rangle$ which satisfies $\eta_{0,I} |0\rangle = 0$, a 32 dimensional representation of $SO(10)$ is furnished by the 32 states

$$\eta_{0*}^{j_1} \cdots \eta_{0*}^{j_k} |0\rangle.$$ \hfill (2.31)
It is well known that this is a reducible representation of $SO(10)$ which decomposes into two 16 dimensional irreducible representations, the $16$ and the $\overline{16}$; the $16$ is composed of the states in (2.31) with $k$ even, while the $\overline{16}$ is given by the states in (2.31) with $k$ odd. Notice from (2.28) that the gauge fermions should be thought of as carrying an extra $U(1)$ charge of 1, for the purposes of the projection onto even left-moving $U(1)$ charge. Then the states in (2.31) with a given value of $k$ carry a left $U(1)$ charge of $-\frac{5}{2}+k$ (the $-\frac{5}{2}$ being the charge of the Fock vacuum $|0\rangle$; see §2.4). The conclusion is that states with $q_- - \frac{5}{2}$ even are projected onto $16$’s of $SO(10)$, while states with $q_- - \frac{5}{2}$ odd are associated with $\overline{16}$’s of $SO(10)$.

Physical applications also involve a right-moving GSO projection, onto states with the appropriate mod 2 right-moving fermion number. We will be interested in massless states, which are always right-moving ground states; for such states the GSO projection in right-moving Ramond sectors means the following. States with $q_+ + \frac{3}{2}$ even give left-handed spin one-half massless fermions in space-time; states with $q_+ + \frac{3}{2}$ odd give right-handed ones. The detailed explanation involves exactly the same sort of reasoning that we have just carried out for left-movers. (The description of the right-moving GSO projection in right moving NS sectors is standard but we need not give it here as we only consider right moving R sectors in this paper.)

Since in constructing the spectrum, we project onto states with a particular eigenvalue of the operator $g$ of equation (2.28), modular invariance forces us to add twisted sectors constructed with twists by arbitrary powers of $g$. The operator $g$ is a version of the $(-1)^F$ operator that counts fermions modulo two. So, starting with the completely untwisted (R,R) sector, a twist by an even power of $g$ makes a left-moving Ramond sector; a twist by an odd power makes a left-moving Neveu-Schwarz sectors. With $d$ being the least common denominator of the charges of the $\Phi_i$, $g^{2d} = 1$, so there are $2d$ sectors twisted by $1, g, g^2, \ldots, g^{2d-1}$.

2.4. Ground State Quantum Numbers

As is well known in analogous computations, one of the main steps in determining the spectrum of one of these models is to determine the quantum numbers of the ground state in each twisted sector. To be precise, in the sector twisted by $g^k$, we wish to determine the left- and right-moving $U(1)$ charges (i.e., $J_L$ and $J_R$ eigenvalues), and the left-moving energy ($L_{0-}$ eigenvalue) of the ground state. We will always consider right-moving Ramond sectors, so the $L_{0+}$ eigenvalue of the ground state will always be zero.
First, we determine the $U(1)$ charges. Our viewpoint is that of [23]: the reason the twisted sectors have fractional $U(1)$ charges is that when the fermions satisfy twisted boundary conditions, the vacuum has a fractional fermion number. Formally, the charge carried by a filled fermi sea with fermions of charge $e$ is

$$Q = e \int_{-\infty}^{0} dE \, \rho(E)$$

(2.32)

where $\rho(E)$ is the density of states. This is of course divergent, and must be regulated. Since we are really interested in the change in $Q$ as a function of the twisted boundary conditions on the fermions, we can subtract an (infinite) constant $\frac{1}{2} \int_{-\infty}^{\infty} dE \, \rho(E)$ without doing any harm; we also introduce a convergence factor:

$$Q = \frac{1}{2} \lim_{s \to 0} \int_{-\infty}^{\infty} dE \, \text{sign}(E) \, \rho(E) \, e^{-s|E|}.$$  

(2.33)

For our case of interest, which is left moving fermions on a circle of circumference $2\pi$ (and coordinate $0 \leq \sigma < 2\pi$) with Hamiltonian $-i \frac{\partial}{\partial \sigma}$, the integral in (2.33) is easily evaluated for arbitrary choice of boundary conditions. In particular, for fermions with boundary conditions

$$\psi(\sigma + 2\pi) = e^{-i \theta} \psi(\sigma)$$

(2.34)

with $0 \leq \theta < 2\pi$, one finds

$$Q = e \left( \frac{\theta}{2\pi} - \frac{1}{2} \right)$$

(2.35)

(so the vacuum has a fractional fermion number of $\frac{\theta}{2\pi} - \frac{1}{2}$). The above formula is valid for $0 < \theta < 2\pi$. It becomes valid for all $\theta$ after the obvious modification to

$$Q = e \left( \frac{\theta}{2\pi} - \left[ \frac{\theta}{2\pi} \right] - \frac{1}{2} \right)$$

(2.36)

where $[x]$ denotes the greatest integer less than $x$. There is an important subtlety here. The expression $[\theta/2\pi]$ has a discontinuity when $\theta$ is an integral multiple of $2\pi$. At such values of $\theta$, both values of $Q$ should be kept. The reason for this is that precisely when $\theta = 2\pi n$, with integer $n$, there are fermion zero modes; upon quantizing them, one finds (for a single complex fermion) a pair of ground states. One of these is the limit of the ground state as $\theta$ approaches $2\pi n$ from above; the other is the limit as $\theta$ approaches $2\pi n$ from below. So the charges of the two ground states are the two limiting values of (2.36).

The analogous formula for right-moving fermions is easily derived, with the result that for the same boundary conditions (2.34) the right-moving fermion would contribute $-Q$. 

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Since the right-moving worldsheet fermions do carry non-vanishing left $U(1)$ charge, it is important to take into account their contribution when computing the left $U(1)$ charges of the twisted vacua.

We know the $U(1)$ charges $q$ of the fermions from Table 1, and in the sector twisted by $g^k$ they pick up phases $\psi \to e^{-i\pi kq} \psi$ when going around the circle. So without further ado, we can write the general formula for the left $U(1)$ charges of the vacua:

$$q_{k,-} = \sum_i \left\{ (\alpha_i - 1) \left( \frac{k(\alpha_i - 1)}{2} + \left[ \frac{k(1 - \alpha_i)}{2} \right] + \frac{1}{2} \right) + \alpha_i \left( -\frac{k\alpha_i}{2} + \left[ \frac{k\alpha_i}{2} \right] + \frac{1}{2} \right) \right\}$$

(2.37)

The analogous formula for the right-moving $U(1)$ charges is simply

$$q_{k,+} = \sum_i \left\{ \alpha_i \left( \frac{k(\alpha_i - 1)}{2} + \left[ \frac{k(1 - \alpha_i)}{2} \right] + \frac{1}{2} \right) + (\alpha_i - 1) \left( -\frac{k\alpha_i}{2} + \left[ \frac{k\alpha_i}{2} \right] + \frac{1}{2} \right) \right\}$$

(2.38)

We also need to determine the ground state eigenvalues of $L_{0-}$ ($L_{0+}$ always vanishes in the ground state by right-moving supersymmetry). In the (R,R) sectors, the vacuum eigenvalue of $L_{0-}$ vanishes. Indeed, the contribution of the fields in the “internal” Landau-Ginzburg theory vanishes by supersymmetry, since the bosons and fermions satisfy the same boundary conditions in (R,R) sectors. The contribution of the 16 $E_8$ fermions (in their ground state, which is in the NS sector, that is with antiperiodic boundary conditions) is $-\frac{16}{48}$ while the contribution of the 10 $SO(10)$ fermions is $\frac{10}{24}$ and the contribution of the remaining 2 spacetime bosons (in light-cone gauge) is $-\frac{2}{24}$. Simply doing the arithmetic, this sums to 0.

The (NS,R) sectors, on the other hand, can have negative vacuum energies. The 10 $SO(10)$ fermions, 16 $E_8$ fermions, and 2 spacetime bosons contribute $-\frac{5}{8}$ to the vacuum energy. The contribution of the internal Landau-Ginzburg theory can be determined by using the standard formulae for the energy of a twisted boson or fermion. The contribution to the ground state energy (normal ordering constant of $L_0$) for a complex fermion twisted by $\theta$ ($-\pi \leq \theta \leq \pi$) with respect to being antiperiodic

$$\psi \to e^{i(\pi+\theta)} \psi$$

(2.39)

is given by

$$E_\theta = -\frac{1}{24} + \frac{1}{8} \left( \frac{\theta}{\pi} \right)^2 .$$

(2.40)
A boson with the same boundary conditions would contribute the negative of (2.40) to the vacuum energy.

We are interested in bose-fermi pairs with left $U(1)$ charges $\alpha_i$ and $\alpha_i - 1$. Therefore, if in some (NS,R) sector the fermion has boundary condition $\psi \rightarrow e^{i(\pi + \theta)}\psi$ (with $\theta$ between $-\pi$ and $\pi$) then the boson is $\pi - |\theta|$ away from being antiperiodic. Simply using the formula (2.40) we see that the fermion-boson pair then contributes

$$E_\theta = \frac{1}{4} \frac{|\theta|}{\pi} - \frac{1}{8}$$

(2.41)

to the vacuum energy.

Using these formulae and the fermion and boson $U(1)$ charges from table 1, we find that the vacuum energy of the sector twisted by $g^k$ with $k$ odd is given in general by

$$E_k = -\frac{5}{8} + \sum_i \left( \frac{1}{4} |\beta^i_k| - \frac{1}{8} \right).$$

(2.42)

$\beta^i_k$ is $k\alpha_i$, reduced mod 2 to lie between $-1$ and 1.

Now that we know the quantum numbers of the twisted vacua $|0\rangle_k$, we must determine the spectrum of physical states in each twisted sector. In the next section, we will do this in detail in a familiar example: The Landau-Ginzburg model that corresponds to a quintic hypersurface in $\mathbb{CP}^4$.

2.5. $E_6$ And Supersymmetry Multiplets And $U(1)$ Charges

Certain symmetries of these systems – $E_6$ symmetry and space-time supersymmetry – are not manifest in the formalism. The proper assembly of states into $E_6$ multiplets and supermultiplets can be carried out using the $U(1)$ charges.

Let us consider first the construction of $E_6$ multiplets. The $27$ and $\overline{27}$ of $E_6$ decompose under $SO(10) \times U(1)$ as $27 = 16_{1/2} \oplus 10_{-1} \oplus 1_2$ and $\overline{27} = 16_{-1/2} \oplus 10_1 \oplus 1_{-2}$. Therefore, singlets of $SO(10)$ with $q_- = \pm 2$ are parts of $27$s and $\overline{27}$s of $E_6$, while singlets of $SO(10)$ with $q_- = 0$ are also singlets of $E_6$. The decomposition of the adjoint representation of $E_6$ as $78 = 45 \oplus 16_{-3/2} \oplus 16_{3/2} \oplus 1$ is also helpful in studying gluinos.

The right-moving $U(1)$ charge plays a similar role in identifying supermultiplets [24]. For right-moving NS states, one can understand the values of $q_+$ by considering unitarity
constraints. For example, if we consider a state of right conformal weight \( h_+ \) and right-moving \( U(1) \) charge \( q_+ \), denoted by \( |h_+, q_+ \rangle \), then using
\[
\{ G_{1/2, +}, \overline{G}_{-1/2, +} \} = 2L_{0,+} + J_{0,R} \tag{2.43}
\]
\[
\{ G_{-1/2, +}, \overline{G}_{1/2, +} \} = 2L_{0,+} - J_{0,R}
\]
and requiring that the states \( G_{-1/2, +} |h_+, q_+ \rangle \) and \( \overline{G}_{-1/2, +} |h_+, q_+ \rangle \) have non-negative norm we find that
\[
h_+ \geq \frac{1}{2} |q_+| . \tag{2.44}
\]
This is useful because we know that massless right NS states must have \( h_+ = \frac{1}{2} \). Then also requiring locality means that \( q_+ = \pm 1 \): if \( q_+ = 1 \), the state is right chiral (annihilated by \( \overline{G}_{-1/2, +} \)) and if \( q = -1 \) the state is right antichiral (annihilated by \( G_{-1/2, +} \)).

Consider a spin zero physical state \( s \) of \( q_+ = 1 \). It is represented by the spin zero part of a chiral superfields \( S \) with component expansion
\[
S(x, \theta) = s(x) + \theta \eta(x) + \theta \theta F(x) \tag{2.45}
\]
Likewise a scalar \( \overline{s} \) of \( q_+ = -1 \) is represented by a supermultiplet
\[
\overline{S}(x, \overline{\theta}) = \overline{s}(x) + \overline{\theta} \overline{\eta}(x) + \overline{\theta} \overline{\theta} \overline{F}(x) . \tag{2.46}
\]

We are most interested in the worldsheet quantum numbers of the vertex operators for \( \eta \) and \( \overline{\eta} \), since we are going to be finding the spectrum of spacetime fermions. The fermions are obtained by acting with the spacetime supersymmetries on (2.45) and (2.46). In particular, with the information derived above and a knowledge of \( U(1) \) charges of the spacetime supersymmetry generators, we can infer the expected values of \( q_+ \) for the fermions which are part of chiral or antichiral multiplets. Recall that the explicit form of the spacetime supersymmetries is
\[
Q_\alpha = \oint dz \ e^{-\frac{z}{\theta}} S_\alpha \Sigma(z)
\]
\[
Q_{\dot{\alpha}} = \oint dz \ e^{-\frac{z}{\theta}} S_{\dot{\alpha}} \Sigma^\dagger(z) \tag{2.47}
\]
where \( e^{-\frac{z}{\theta}} \) is a spin field for the superconformal ghosts, \( S_\alpha \) and \( S_{\dot{\alpha}} \) are spin fields for the world sheet “spacetime” fermions \( \psi^\mu \), and \( \Sigma \) and \( \Sigma^\dagger \) are Ramond sector fields which
essentially implement right spectral flow by $e^\pm i\pi J_{0,R}$. Therefore, we see that $Q_\alpha$ and $\bar{Q}_\bar{\alpha}$ leave the value of $q_-$ unchanged, while they change $q_+$ by $\pm \frac{3}{2}$.

Now using the fact that $s$ is constrained to have $q_+ = 1$ by the representation theory of the right moving N=2 algebra, we see that $\eta$ must have $q_+ = -\frac{1}{2}$, while the vertex operator for the auxiliary field $F$ must have $q_+ = -2$. Similarly, $\bar{\eta}$ must have $q_+ = \frac{1}{2}$, while $\bar{F}$ must have $q_+ = 2$.

The same argument can be applied to find the quantum numbers of the gauginos. We know that generically in heterotic string theory the spacetime gauge symmetry must be generated by (NS,NS) vector bosons, which correspond to states of the form

$$\mathcal{J}_{-1,L} \psi_{\frac{1}{2},+}^\mu |0\rangle$$

where $\mathcal{J}_L$ is a left-moving symmetry generator and $\psi_{\frac{1}{2},+}^\mu$ is one of the right-moving “spacetime” fermions. In particular, the state (2.48) always has $q_+ = 0$. The gauginos arise by applying the supersymmetries (2.47) to the vector superfields, which have the same quantum numbers as (2.48). Therefore, in particular gauginos always have $q_+ = \pm 3/2$. For the gaugino partners of the $U(1)$ symmetries of Gepner models, which are also neutral under the spacetime $E_6$ gauge symmetry, $q_- = 0$ as well.

So in summary: We expect to find fermions with $q_+ = \pm \frac{1}{2}$ which are parts of spacetime antichiral and chiral supermultiplets, and fermions with $q_+ = \pm \frac{3}{2}$ which are part of spacetime vector supermultiplets. The latter are in correspondence with generators of spacetime gauge symmetries.

3. The Quintic

Let us now use the technology developed in §2 to study the massless spectrum of string theory compactified on a quintic hypersurface $\mathbb{P}_4(5) \subset \mathbb{C}P^4$, in the Landau-Ginzburg orbifold formulation. We consider a quintic defined by the zeroes of a generic quintic polynomial

$$W = \frac{1}{5} \sum_{i_1 \ldots i_5} w_{i_1 \ldots i_5} \Phi^{i_1} \ldots \Phi^{i_5}.$$  \hspace{1cm} (3.1)

In practice, that means that we consider a Landau-Ginzburg orbifold with $W$ as superpotential. The general results involve a reduction to a description involving finite matrices.
When we want to make the results completely explicit, we will consider the example of the Fermat quintic, with

\[ W = \sum_{i=1}^{5} \frac{1}{5} \Phi_i^5 \]  

(3.2)

which has enhanced symmetry and corresponds to a soluble Gepner point [25]. We will carry out the discussion for a (2, 2) model with superpotential \( W \), but no essential modification is required for the (0, 2) case, as we will explain in §3.9.

We must obtain the spectrum in 10 sectors, which arise, starting with the untwisted (R,R) sector, by twisting by \( \exp(-ik\pi J_{0L}) \), with \( 0 \leq k \leq 9 \). In practice, it suffices to consider \( 0 \leq k \leq 5 \), as CPT exchanges \( k \) with \( 10 - k \).

The (R,R) sector is the sum of the twisted sectors of even \( k \), and the (NS,R) sector is the sum of the twisted sectors of odd \( k \). Happily, the (R,NS) and (NS,NS) sectors need not be studied explicitly, as they are related to (R,R) and (NS,R) by space-time supersymmetry.

As a preliminary, let us review the fields and their quantum numbers here. In addition to the bosons \( \phi_i \) and \( \overline{\phi}_i \), there are left moving fermions \( \psi_i^- \) and \( \overline{\psi}_i^- \), and right moving fermions \( \psi_i^+ \) and \( \overline{\psi}_i^+ \). Their left and right moving \( U(1) \) charges are summarized in Table 2:

| Field          | q−   | q+   |
|----------------|------|------|
| \( \phi_i \)   | 1/5  | 1/5  |
| \( \overline{\phi}_i \) | −1/5 | −1/5 |
| \( \psi_i^- \)  | −4/5 | 1/5  |
| \( \psi_i^+ \)  | 1/5  | −4/5 |
| \( \overline{\psi}_i^- \) | 4/5  | −1/5 |
| \( \overline{\psi}_i^+ \) | −1/5 | 4/5  |

So, using the general formula for \( U(1) \) charges of ground states developed in the last section, we find that the left and right \( U(1) \) charges \( q_{k,-} \) and \( q_{k,+} \) of the twisted sector vacua are

\[
q_{k,-} = 5 \left\{ -\frac{4}{5} \left( -\frac{2k}{5} + \left[ \frac{2k}{5} \right] + \frac{1}{2} \right) + \frac{1}{5} \left( -\frac{k}{10} + \left[ \frac{k}{10} \right] + \frac{1}{2} \right) \right\} \tag{3.3}
\]
\[ g_{k,+} = 5 \left\{ \frac{1}{5} \left( \frac{-2k}{5} + \left\lfloor \frac{2k}{5} \right\rfloor + \frac{1}{2} \right) - \frac{4}{5} \left( \frac{-k}{10} + \left\lfloor \frac{k}{10} \right\rfloor + \frac{1}{2} \right) \right\} \]  \hspace{1cm} (3.4)

where the fractional fermion numbers in (3.3) and (3.4) arise because of the boundary conditions on the fermions in the sector twisted by \( g^k \)

\[
\psi_0^{-} \rightarrow e^{i\frac{2\pi}{5}k}\psi_0^{+}, \quad \psi_0^{+} \rightarrow e^{-i\frac{2\pi}{5}k}\psi_0^{+}.
\]  \hspace{1cm} (3.5)

We also need to know the ground state energies of the vacua in the various twisted sectors. Using the normal formulae for the zero-point energies of twisted bosons and fermions as discussed in §2, we see that the even \( k \) sectors have vanishing vacuum energy, as expected from supersymmetry, while the odd \( k \) sectors have ground state energies

\[
E_1 = -1, \quad E_3 = -\frac{1}{2}, \quad E_5 = 0, \quad E_7 = -\frac{1}{2}, \quad E_9 = -1.
\]  \hspace{1cm} (3.6)

We recall that upon taking the \( \overline{Q}_{+,R} \) cohomology, the right-moving fermions are eliminated, so in this analysis the left-moving fermions are the only ones of interest and will be usually denoted as \( \psi_+^i \), not \( \psi_0^{-} \). Also, upon taking the \( \overline{Q}_{+,R} \) cohomology, the zero modes of \( \overline{\phi} \) are eliminated. So in practice, we need to compute in the sectors \( 0 \leq k \leq 5 \) and in the reduced Hilbert space the cohomology of the \( \overline{Q}_{+,L} \) operator \( \overline{Q}_{+,L} = \sum_{i=1}^{5} \oint \frac{\partial W}{\partial \phi_i} \psi_+^i. \)  \hspace{1cm} (3.7)

After carrying out the analysis, we will summarize the resulting spectrum at the end of this section. We also will assemble the 1’s, 10’s and 16’s of \( SO(10) \) into 27’s and \( \overline{27} \)’s of \( E_6 \), using the values of the left \( U(1) \) charge as explained in §2.5.

3.1. \( k = 0 \) Sector

This corresponds to the normal untwisted (R,R) sector. The ground state energy vanishes. Since all of the fields are untwisted, the relevant lowest energy modes are (from the comment in the last paragraph) the zero modes

\[
\phi_0^i, \quad \psi_0^i, \quad \overline{\psi}_0^i.
\]  \hspace{1cm} (3.8)

\footnote{In our analysis of models, we will drop the \( i\sqrt{2} \) prefactor of \( \overline{Q}_{+,L} \) in (2.13), which is obviously irrelevant in computing the cohomology.}
The commutation relations of the fermion zero modes are
\[ \{ \psi^i_0, \bar{\psi}^j_0 \} = \delta^{ij}. \] (3.9)

We let \( |0\rangle \) denote a Fock vacuum with
\[ \psi^i_0 |0\rangle = 0. \] (3.10)

This state has left and right moving \( U(1) \) charges \((q_-, q_+) = (-3/2, -3/2)\).

Since the ground state energy is zero, in studying zero energy states we can altogether ignore the oscillator modes in the definition of \( \bar{Q}_{+,L} \), so that \( \bar{Q}_{+,L} \) reduces to
\[ \bar{Q}_{+,L} = \psi^i_0 \frac{\partial W(\phi_0)}{\partial \phi^i_0}. \] (3.11)

The cohomology of \( \bar{Q}_{+,L} \) is generated entirely by states of the form
\[ F(\phi_0) |0\rangle \] (3.12)
and the projection onto half-integral \( U(1) \) charges means that we need consider only functions \( F \) of degree \( 5j \) for \( j = 0, 1, 2, \ldots \). But also, note that
\[ \bar{Q}_{+,L} \psi^i_0 |0\rangle = \frac{\partial W}{\partial \phi^i_0} |0\rangle \] (3.13)
so we must mod out by the ideal generated by the \( \{ \frac{\partial W}{\partial \phi^i_0} \} \). What we have found here is of course just the famous result that the chiral ring \( \mathcal{R} \) of a Landau-Ginzburg theory is given by the “singularity ring” of the superpotential
\[ \mathcal{R} \simeq \frac{\mathbb{C}[\phi^i]}{\{ \partial_j W(\phi) \}}. \] (3.14)

It is easy to enumerate the resulting states. At \((q_-, q_+) = (-3/2, -3/2)\), we simply get \( |0\rangle \). At \((-1/2, -1/2)\), we get the quintic functions of \( \phi \) modulo the ideal generated by derivatives of \( W(\Phi) \) – 101 states in all according to a standard counting. At \((1/2, 1/2)\), we get the tenth order polynomials modulo those in the ideal generated by the derivatives – again 101 states. At \((3/2, 3/2)\), there is a single state; for instance, for the Fermat polynomial, it can be represented by \( \prod_{i=1}^5 \phi_i^3 |0\rangle \).

Making the GSO projections as described in §2.3, the states in this sector with \((q_-, q_+) = (-1/2, -1/2)\) correspond to right handed fermions in the \( \mathbf{16} \) of \( SO(10) \), while those with \((q_-, q_+) = (1/2, 1/2)\) correspond to left handed fermions in the \( \mathbf{16} \) of \( SO(10) \). In fact, the former are the \( \mathbf{16} \) components of the 101 right-handed \( \mathbf{27} \)'s, while the latter are the \( \mathbf{16} \) components of the 101 left handed \( \mathbf{27} \)'s. The \((-3/2, -3/2)\) and \((3/2, 3/2)\) states are gluinos, according to the discussion of the right-moving \( U(1) \) charge in §2.5. The GSO projections cause the \((-3/2, -3/2)\) states to be left-handed in space-time and a \( \mathbf{16} \) of \( SO(10) \), while the \((3/2, 3/2)\) are a right-handed \( \mathbf{16} \).
3.2. $k=1$ Sector

The ground state energy is $E_1 = -1$ and the ground state $U(1)$ charges are $(0, -3/2)$. Because of the twist by $e^{-i\pi J_0}$, there are no zero modes. However, since we are looking for states of energy 0 and the vacuum has negative energy, we should keep the lowest excited modes of the fields. These are

$$\phi_{-1/10}^i, \bar{\phi}_{-9/10}^i, \psi_{-3/5}^i, \bar{\psi}_{-2/5}^i.$$ \hfill (3.15)

Now, we simply need to write down all the states that have zero energy in the free field approximation and find the $Q_{+, L}$ cohomology. The $Q_{+, L}$ operator restricted to the relevant modes is

$$Q_{+, L} = \psi_{-3/5}^{i_1} \psi_{-2/5}^{i_2} \phi_{-1/10}^{i_3} \phi_{-1/10}^{i_4} \phi_{-1/10}^{i_5} + 4 \psi_{-3/5}^{i_1} \psi_{-3/5}^{i_2} \phi_{-1/10}^{i_3} \phi_{-1/10}^{i_4} \phi_{-1/10}^{i_5}.$$ \hfill (3.16)

Other terms in $Q_{+, L}$ have zero matrix elements among states of zero energy. Since $Q_{+, L}$ does not change left $U(1)$ charge, we can compute its cohomology separately in spaces of states of different $q_-$. Since $Q_{+, L}$ increases the value of $q_+$ by one, so we have a sequence of maps

$$0 \to V_{-3/2} \overset{Q_{+, L}}{\to} V_{-1/2} \overset{Q_{+, L}}{\to} V_{1/2} \to 0.$$ \hfill (3.18)

Here, $V_{q_+}$ denotes the space spanned by the states of right $U(1)$ charge $q_+$.

In the general case, one can write down a similar sequence to (3.18) above. For each fixed value of the left $U(1)$ charge, one gets a sequence

$$0 \to V_{q_+} \overset{Q_{+, L}}{\to} V_{q_+ + 1} \overset{Q_{+, L}}{\to} \cdots \overset{Q_{+, L}}{\to} V_{q_+ + n} \to 0.$$ \hfill (3.19)
where in general $V_{q+}$ is the space of states with right $U(1)$ charge $q_+$. Then the $\overline{Q}_+$ cohomology is the cohomology of (3.19).

The concrete case of (3.18) can be analyzed as follows. The states of $q_+ = -3/2$ can be written in the form $\overline{\psi}_{-2/5, i} \overline{\psi}_{-2/5, j} A^{ij}(\phi_{-1/10})|0\rangle$ where $A^{ij}(\phi_{-1/10})$ are homogeneous quadratic functions of $\phi_{-1/10}$, in components $A^{ij} = A^{ij}_{kl} \phi_{-1/10}^k \phi_{-1/10}^l$. The states of $q_+ = -1/2$ are of the form $\overline{\phi}_{-2/5, i} B^i(\phi_{-1/10})|0\rangle$, with the $B_i$ being homogeneous sixth order functions, and the states of $q_+ = 1/2$ are of the form $C(\phi_{-1/10})|0\rangle$, with $C$ being a homogeneous tenth order function. The action of the $\overline{Q}_+$ operator is

$$\overline{Q}_{+, L} \left( \overline{\psi}_{-2/5, i} \overline{\psi}_{-2/5, j} A^{ij} |0\rangle \right) = \overline{\psi}_{-2/5, j} A^{ij} \frac{\partial W}{\partial \phi_i} |0\rangle \tag{3.20}$$

$$\overline{Q}_{+, L} \left( \overline{\psi}_{-2/5, i} B^i |0\rangle \right) = B^i \frac{\partial W}{\partial \phi_i} |0\rangle.$$

This is precisely isomorphic to a piece of the effective $\overline{Q}_{+, L}$ operator that we met in the $k = 0$ sector, except that the variables are now called $\phi_{-1/10}$ instead of $\phi_0$ and $\overline{\psi}_{-2/5}$ instead of $\overline{\psi}_0$. In particular, the cohomology vanishes at $q_+ = -3/2$ and $-1/2$, and at $q_+ = 1/2$, the cohomology consists of the tenth order polynomials in $\phi_{-1/10}$ modulo the ideal generated by $\partial_i W$. This is a 101 dimensional space, in natural one-to-one correspondence with the Ramond ground states of $k = 0$ that were constructed from tenth order polynomials. This is expected from $E_6$ symmetry: these states will combine with some of the $k = 0$ states into $E_6$ multiplets.

**$E_6$ Singlets**

$E_6$ singlets arise as $SO(10)$ singlets of $q = 0$. These take the following form:

$$q_+ = -3/2 : \overline{\psi}_{-2/5, i} \overline{\psi}_{-3/5}^i |0\rangle \tag{25}$$

and also $\overline{\phi}_{-9/10, j} \phi_{-1/10}^j |0\rangle \tag{25}$

$$q_+ = -1/2 : \phi_{-1/10}^i \phi_{-1/10}^j \cdots \phi_{-1/10}^j \overline{\psi}_{-3/5}^i |0\rangle \tag{350}.$$

The number in parentheses is the number of states of a given type.

The maps in the resulting sequence of the form (3.19) are given by

$$\overline{Q}_{+, L} \left( A^i_j \overline{\psi}_{-2/5, j} \overline{\psi}_{-3/5}^i |0\rangle \right) = A^i_j \psi_{-3/5}^i \partial_j W(\phi_{-1/10})|0\rangle \tag{3.22}$$

$$\overline{Q}_{+, L} \left( B^i_j \overline{\phi}_{-9/10, j} \phi_{-1/10}^i |0\rangle \right) = B^i_j \phi_{-1/10}^i \psi_{-3/5}^k \partial_j \partial_k W(\phi_{-1/10})|0\rangle. \tag{3.23}$$
In the particular case of the Fermat quintic (3.2), one can see that at $q_+ = -3/2$, $\mathcal{Q}_+$ has a five dimensional kernel, spanned by the states $(\frac{1}{10}\phi_{-1/10}^{i-9/10,i} - \bar{\psi}_-2/5, i\psi_{-3/5}^{i})|0\rangle$ (no sum over $i$). This means that 45 of the states at $q_+ = -1/2$ are trivial in $\mathcal{Q}_{+L}$ cohomology while the other 305 must represent nontrivial cohomology classes. Hence, one finds 310 singlets of $\text{SO}(10)$ in this sector for the Fermat quintic: 5 at $q_+ = -3/2$ and 305 at $q_+ = -1/2$. One of the states at $q_+ = -3/2$ is present for generic $W$ and is in the adjoint representation of $E_6$; the other 4 at $q_+ = -3/2$ and all 305 at $q_+ = -1/2$ are singlets of $E_6$.

In general, to summarize equations (3.22),(3.23), the $E_6$ singlets at $q_+ = -1/2$ are represented by a collection of five quartic functions $P_i(\phi_{-1/10})$ subject to the equivalence relation
\begin{equation}
P_i \sim P_i + A_i^j \partial_j W + \phi^k B_{kl} \partial^2 W_{\phi^l \phi^j}.
\end{equation}
(3.24)
After redefining $A$, this can alternatively be written
\begin{equation}
P_i \simeq P_i + A_i^j \partial_j W + \partial_i \left( \phi^k B_{k}^l \partial_l W \right).
\end{equation}
(3.25)

### Finer Classification Of $E_6$ Singlets

In the field theory limit, there are three types of massless $E_6$ singlets at $q_+ = -1/2$, namely states that originate in $H^1(\mathbb{P}_4(5), T)$, $H^1(\mathbb{P}_4(5), T^*)$, and $H^1(\mathbb{P}_4(5), \text{End}(T))$. These may be distinguished as follows. States $|\Psi\rangle$ which satisfy the chiral condition
\begin{equation}
G_{-1/2} |\Psi\rangle = 0
\end{equation}
correspond to elements of $H^1(\mathbb{P}_4(5), T)$ while those which satisfy the anti-chiral condition
\begin{equation}
\overline{G}_{-1/2} |\Psi\rangle = 0
\end{equation}
correspond to elements of $H^1(\mathbb{P}_4(5), T^*)$. The singlets which are orthogonal to those obeying the chiral or anti-chiral condition correspond to elements of $H^1(\mathbb{P}_4(5), \text{End}(T))$. We want to implement this classification in the Landau-Ginzburg model, using the explicit forms of $G_{-1/2}$ and $\overline{G}_{-1/2}$ from (2.17).

First of all, using the above explicit description of the $E_6$ singlets at $q_+ = -1/2$, and the fact that $\overline{G}_{-1/2}$ has a term proportional to $\psi'_{-1/10} \psi_{-2/5,i}$, none of the $E_6$ singlets of $q_+ = -1/2$ are annihilated by $\overline{G}_{-1/2}$. On the other hand, one finds that
\begin{equation}
G_{-1/2} \left( \psi_{-3/5}^i P_i(\phi_{-1/10})|0\rangle \right) \sim \psi_{-3/5}^i \psi_{-3/5}^j \partial_i P_j(\phi_{-1/10})|0\rangle.
\end{equation}
(3.26)
This therefore vanishes precisely if $\partial_i P_j - \partial_j P_i = 0$, or in other words if $P_i = \partial_i S$ for some quintic polynomial $S$. From the homogeneity of the $P_i$ it follows that $S = \phi^i P_i / 5$. The equivalence relation (3.24) then amounts to

$$S \cong S + \phi^i A_i^j \partial_j W + \phi^i \phi^k B_{k}^{i} \frac{\partial^2 W}{\partial \phi^i \partial \phi^i}.$$  

(3.27)

From the homogeneity of $W$ it follows that $\partial^i \partial_i W = 4 \partial_i W$, and finally then the space of $E_6$ singlets annihilated by $G_{-1/2}$ is the 101 dimensional space of quintic polynomials $S$ modulo the usual ideal generated by the $\partial_i W$.

Now we want to look at the analog of $H^1(\mathbb{P}_4(5), \text{End}(T))$ – the states orthogonal to the chiral and anti-chiral states. Since we have already taken account of the states of the form $\partial_i S$, we now look at states $P_i$ with anything of the form $\partial_i S$ considered trivial. Hence the analog of $H^1(\mathbb{P}_4(5), \text{End}(T))$ in the $k = 1$ sector is the space of five quartic polynomials $P_i$ subject to

$$P_i \to P_i + A_i^j \partial_j W + \partial_i S.$$  

(3.28)

(This is similar to (3.25) but $\phi^i C_i^j \partial_k W$ is now replaced by an arbitrary quintic $S$.) By homogeneity of $S$, $\phi^i \partial_i S = 5 S$, so $S$ can be uniquely fixed by normalizing $P$ so that $\phi^i P_i = 0$. So the analog of $H^1(\mathbb{P}_4(5), \text{End}(T))$ can be identified with the space of five quartic polynomials $P_i$, with $\phi^i P_i = 0$, and the equivalence relation

$$P_i \cong P_i + A_i^j \partial_j W - \frac{1}{5} \partial_i \left( \phi^k A_k^j \partial_j W \right).$$  

(3.29)

Now let us compare this to the computation of $H^1(\mathbb{P}_4(5), \text{End}(T))$ in the field theory limit. A tangent vector to the quintic hypersurface $W = 0$ in $\mathbb{C} \mathbb{P}^4$ can be represented by a collection of five complex numbers $V^i$ obeying an equivalence relation

$$V^i \to V^i + \lambda \phi^i$$  

(3.30)

(with $\phi^i$ being the homogeneous coordinates on $\mathbb{C} \mathbb{P}^4$) and a constraint

$$V^i \partial_i W = 0$$  

(3.31)

(so that the vector field on $\mathbb{C} \mathbb{P}^4$ represented by the $V^i$ is tangent to the hypersurface $W = 0$). The constraint (3.31) and equivalence relation (3.30) are compatible because
\( \phi^i \partial_i W = 5W \) vanishes at \( W = 0 \). To deform the tangent bundle of the quintic, one can replace (3.31) by

\[
V^i(\partial_i W + P_i) = 0
\]

where the \( P_i \) are homogeneous quartic polynomials and (to maintain compatibility with (3.30)) \( \phi^i P_i = 0 \). So in field theory, \( H^1(\mathbb{P}_4(5), \text{End}(T)) \) is the space of \( P_i \)'s subject to \( \phi^i P_i = 0 \). The answer is almost the same in the Landau-Ginzburg model, but in the Landau-Ginzburg theory there is an additional equivalence relation (3.29), so some states are missing. We will return to this point after examining the spectrum for other values of \( k \).

For the time being, let us just quantify the discrepancy. For generic \( W \), the equation

\[
A_{i}^{\, j} \partial_j W - \frac{1}{5} \partial_i \left( \phi^k A_k^{\, j} \partial_j W \right) = 0
\]

is obeyed only if \( A_{i}^{\, j} = \delta_{i}^{\, j} \). In that case, the Landau-Ginzburg theory is missing 24 states with the quantum numbers of a traceless matrix \( A_{i}^{\, j} \). It can happen that for particular \( W \)'s there are other \( A \)'s for which (3.33) vanishes. In that case the equivalence relation (3.29) is less powerful, so the Landau-Ginzburg theory has extra massless \( E_6 \) singlets (for example, we saw that in the case of the Fermat quintic there are five extra states \((\frac{1}{5} \phi_{-1/10}^i \phi_{-9/10,i} - \bar{\psi}_{-2/5,i} \psi_{-3/5,i})|0\) in the \( \overline{Q}_{+,L} \) cohomology). When this happens, there are extra massless \( E_6 \) singlets at \( q_{+} = -3/2 \) that are supersymmetric partners of extra gauge bosons that occur for this particular \( W \), and extra massless singlets at \( q_{+} = -1/2 \) that are supersymmetric partners of Higgs bosons that will give mass to the extra gauge bosons when \( W \) is perturbed. Apart from this possibility of extra gauge symmetries and scalar partners for particular \( W \)'s, the discrepancy between field theory and \( k = 1 \) Landau-Ginzburg theory consists of 24 missing states with the quantum numbers of a traceless matrix \( A_{i}^{\, j} \).

**SO(10) 10 Components Of 27’s**

The states we have been considering so far have all been 1’s of \( SO(10) \), but we also need to consider 10’s of \( SO(10) \). Such states will contain an excitation of the gauge fermions \( \lambda_{i/1/2}^I \) and will correspond to cohomology classes of \( \overline{Q}_{+,L} \) with total energy \(-1/2\) in the internal theory. We find two patterns of such states, both with \( q_{-} = 1 \):

\[
q_{+} = -3/2 : \; \lambda_{-1/2,I} \phi_{-1/10}^i \bar{\psi}_{-2/5,j} |0\rangle
\]

(25)

\[
q_{+} = -1/2 : \; \lambda_{-1/2,I} \phi_{-1/10}^{i_1} \cdots \phi_{-1/10}^{i_5} |0\rangle
\]

(3.34)
The states of \( q_+ = -1/2 \) can thus be written as \( \lambda_{-1/2,i} S(\phi_{-1/10})|0\rangle \) with \( S \) a homogeneous quintic function. The map in the associated sequence (3.19) is given by

\[
\overline{Q}_{+,L} \left( \lambda_{-1/2,i} C^i_j \phi^j_{-1/10} \overline{\psi}_{-2/5,j} \right)|0\rangle = \lambda_{-1/2,i} \phi^i_{-1/10} C^i_j \partial_j W(\phi_{-1/10})|0\rangle \tag{3.35}
\]

The cohomology is thus the space of quintic homogeneous polynomials \( S(\phi_{-1/10}) \) modulo the ideal generated by the \( \partial_j W \). This is the familiar space of Ramond ground states at \( k = 0 \) – to which these are indeed related by \( E_6 \) symmetry. In fact, these 10’s of \( SO(10) \) have a simple (and standard) relation to the \( E_6 \) singlets with \( P_i = \partial_i S \) that are annihilated by \( G_{-1/2} \) and derived from \( H^1(\mathbb{P}_4(5), T) \) in the field theory limit. The \( E_6 \) singlets arise by acting on \( S(\phi_{-1/10})|0\rangle \) with \( G_{-1/2} \), and the 10’s of \( SO(10) \) arise by acting on the same states with \( \lambda_{-1/2} \).

**Other States**

The states \( \lambda_{-1/2,i} \lambda_{-1/2,j}|0\rangle \) have \( U(1) \) charges \((0, -3/2)\) and are left-handed gluinos in the adjoint representation of \( SO(10) \).

The states \( \partial_- X_{\mu}|0\rangle \), where \( X^\mu \) are the Minkowski space bosons, represent the left-handed gravitino and dilatino.

Gluinos of the second \( E_8 \) have the form \( \tilde{J}^a_{-1}|0\rangle \), where \( \tilde{J}^a \) are the left-moving worldsheet currents generating the second \( E_8 \).

This completes the analysis of the massless fermions for \( k = 1 \).

**3.3. \( k = 2 \) Sector**

This sector has vanishing ground state energy and \((q_-, q_+) = (3/2, -3/2)\) as the ground state \( U(1) \) charges. All of the fields are twisted, so there are no zero modes and hence we get only one state of total energy zero, the ground state. This single element of \( \overline{Q}_{+,L} \) cohomology from the \( k = 2 \) sector is a left handed 16 of \( SO(10) \); these are gluinos forming part of the adjoint representation of \( E_6 \).

**3.4. \( k = 3 \) Sector**

The ground state energy is \(-1/2\) and the ground state \( U(1) \) charges are \((q_-, q_+) = (-1, -1/2)\). The lowest modes of the various fields are

\[
\phi^i_{-3/10}, \overline{\phi}_{-7/10,i}, \psi^i_{-4/5}, \overline{\psi}_{-1/5}.
\]
These values ensure the important fact that $G_{-1/2}$ annihilates the ground state in this sector. However,

$$\mathcal{O}_{-1/2}|0\rangle \sim \sum_{i=1}^{5} \psi_{-1/5,i} \phi_{-3/10}^i|0\rangle$$

(3.36)
does not vanish. As it has zero energy and $q_-=0$, and is obviously annihilated by $\mathcal{O}_{-1/2}$, it is an $E_6$ singlet related to $H^1(\mathbb{P}_4(5), T^*)$.

The only other states of vanishing energy built out of “internal” excitations are the $(q_-,q_+)= (0,-1/2)$ states

$$A_j^i \overline{\psi}_{-1/5,i} \phi_{-3/10}^j|0\rangle$$

with a traceless matrix $A_{ij}$. These are annihilated by neither $G_{-1/2}$ nor $\mathcal{O}_{-1/2}$ so they are analogous to $H^1(\mathbb{P}_4(5), \text{End}(T))$ in field theory. Indeed, we have found the piece of $H^1(\mathbb{P}_4(5), \text{End}(T))$ that was missing in the $k=1$ sector.

Actually, because of instanton effects, a precise correspondence between the classical $H^1(\mathbb{P}_4(5), \text{End}(T))$ and the Landau-Ginzburg contribution was not guaranteed and does not occur in general; we do not know why it occurs in the particular case of the quintic hypersurface. However, one is guaranteed that the “character-valued” index (the imaginary part of the character of any discrete symmetries that may be present in field theory, for a particular $W$) should be the same for field theory or Landau-Ginzburg, since this index is a topological invariant.\footnote{More generally, the element in the $K$ theory of the moduli space of complex structures represented by the left-handed singlets minus the right-handed ones is a topological invariant.}

The missing piece that we have just found was the simplest possibility compatible with this topological invariance.

We can also act with the gauge fermions $\lambda_{-1/2,I}$ on the vacuum $|0\rangle$ to produce a single $10$ of $SO(10)$ which is also in the cohomology of $\mathcal{Q}_{+,L}$. Since this state has $q_+=-1/2$, it corresponds to a right handed fermion. Of course, this state has the usual relation to the anti-chiral state $3.36$; one is obtained by acting on a suitable state (here the vacuum) by $\mathcal{O}_{-1/2}$, while the other is obtained by acting on the same state with $\lambda_{-1/2,I}$.

3.5. $k=4$ Sector

The ground state has zero energy and $(q_-,q_+)= (1/2,-1/2)$. Since all fields are twisted, there are no zero modes and the ground state is the only state of zero energy we can construct. So this sector contributes to the $\mathcal{Q}_{+,L}$ cohomology one right handed $16$ of $SO(10)$, which is part of a $27$ of $E_6$. 

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3.6. \( k = 5 \) Sector

The ground state has zero energy again. Actually, the ground state is not unique because \( \psi_- \) and \( \bar{\psi}_- \) are untwisted and have zero modes. The state \( |0\rangle \) annihilated by \( \bar{\psi}_{0,i} \) has \( U(1) \) charges \((2, -1/2)\). Other states are obtained by acting with factors \( \psi_0^i \). As this field has quantum numbers \((-4/5, 1/5)\), the only state other than \( |0\rangle \) that has integral \( q_- \) and hence survives in the orbifold is the state \( \prod_{i=1}^5 \psi_0^i |0\rangle \), with quantum numbers \((-2, 1/2)\). So this sector contributes two states, both singlets of \( SO(10) \), one part of a right handed \( 27 \) and one part of a left handed \( \overline{27} \) of \( E_6 \).

3.7. Summary: Spectrum Of String Theory On \( M^4 \times \mathbb{P}_4(5) \)

The rest of the massless spectrum (for \( k > 5 \)) follows by complex conjugation from the above results.

Assembling the pieces, for the Fermat superpotential there are 330 \( E_6 \) singlets that are superpartners of massless scalars (1 coming from \( H^1(\mathbb{P}_4(5), T^*) \), 101 coming from \( H^1(\mathbb{P}_4(5), T) \), and 228 coming from \( H^1(\mathbb{P}_4(5), \text{End}(T)) \)), 4 that are superpartners of neutral gluinos, 101 left handed \( 27 \)'s of \( E_6 \), and 1 left handed \( \overline{27} \) of \( E_6 \) (along with their right handed anti-particles). Those numbers agree with those found by Gepner in his analysis of this model as the product of five level three minimal models. In particular, the enhanced gauge symmetry (\( U(1)^4 \) associated with the four neutral gluinos) agrees with that found by Gepner.

One can see the \( SO(10) \) multiplets combine into multiplets of \( E_6 \) more explicitly, as follows. Under \( SO(10) \times U(1) \), the \( \overline{27} \)'s decompose as \( 16_{-1/2} \oplus 10_1 \oplus 1_{-2} \). The way the \( \overline{27} \)'s arise in this model is indicated in Table 3:
### Table 3

| Sector | 1–2   | 16–1/2 | 10_1  |
|--------|-------|--------|-------|
| 0      | 0     | 101    | 0     |
| 1      | 0     | 0      | 101   |
| 2      | 0     | 0      | 0     |
| 3      | 0     | 0      | 0     |
| 4      | 0     | 0      | 0     |
| 5      | 1     | 0      | 0     |
| 6      | 0     | 1      | 0     |
| 7      | 0     | 0      | 1     |
| 8      | 0     | 0      | 0     |
| 9      | 101   | 0      | 0     |

The table shows the number of $SO(10) \times U(1)$ multiplets of given type arising in each sector. In general, starting from the 10, one obtains the 16 by spectral flow by $e^{i\pi J_0}$ and the 1 by spectral flow by another $e^{i\pi J_0}$; the sector number $k$ shifts by 1 each time.

One important point is not indicated in the above table: The 27’s coming from sectors 0, 1, and 9 are right-handed in space-time while the 2\overline{7} coming from sectors 5, 6, and 7 is left-handed. The corresponding table for 27’s comes by complex conjugation, and the analogous table for gluinos can be similarly constructed.

### 3.8. Absence Of Anomalies In The $\mathbb{Z}_5$ Symmetry

Part of the fascination of the Landau-Ginzburg models is that they have a “quantum” symmetry, not present for other choices of the Kahler class, which keeps track of the sector number $k$. This $\mathbb{Z}_{10}$ symmetry is the product of a $\mathbb{Z}_2$ symmetry (which counts fermion number modulo two and is always present) and a quantum $\mathbb{Z}_5$ symmetry. It can be seen that this symmetry is actually an $R$ symmetry in space-time.

A natural question is whether the quantum symmetry suffers from an anomaly at the level of space-time instantons. To answer this question, it suffices to consider only instantons contained inside $SO(10)$. In units in which a left-handed fermion multiplet in the 10 of $SO(10)$ contributes 1 to the anomaly, the 16 and 16 contribute 2 and the 45 contributes 8. Working out the values of $k$ for the various left-handed multiplets (and remembering to include the gluinos), one finds that the quantum symmetry has no anomaly for $E_6$ instantons (and also no anomaly for instantons in the second $E_8$).
3.9. \((0,2)\) Deformations

\((0,2)\) deformations of the quintic can be constructed by deforming the tangent bundle as a holomorphic vector bundle over \(X\). As we have recalled in (3.32), this is done by substituting \(\partial_i W \rightarrow \partial_i W + G_i\) (where \(G_i\) are quartic polynomials obeying \(\phi^i G_i = 0\)) in the definition of the tangent bundle. As one can see from [10], §6, the effect of this on the \(Q_{+,L}\) operator will be just the obvious substitution; the \(Q_{+,L}\) operator of the \((0,2)\) model is simply

\[
Q_{+,L} = i \sqrt{2} \oint \sum_i \left( \frac{\partial W}{\partial \phi_i} + G_i \right) \psi_i.
\] (3.37)

Our techniques then carry over to the \((0,2)\) case without any conceptual difficulties. The physical spectrum of the \((0,2)\) model is given by the cohomology of \(Q_{+,L}\), which can be computed by the same methods that we have used at \(G = 0\).

4. Directions For Future Research

It should be apparent that our methods carry over without essential modification for the analysis of more general Landau-Ginzburg models, including \((0,2)\) models. The detailed analysis of the \(Q_{+,L}\) cohomology can be more elaborate, but the principles are the same. One novelty (already known from the special case of Gepner models) is that in general the number of massless \(E_6\) singlets at the Landau-Ginzburg “point” differs from what it is in the field theory limit.

A number of interesting additional issues about these models are worth pursuing. In particular, it should be possible to compute at least the unnormalized Yukawa couplings; this would assist in the investigation of real phenomenology based on Landau-Ginzburg orbifolds. It should also be straightforward to generalize our approach to Landau-Ginzburg orbifolds with discrete torsion \([12]\).

One of the most interesting prospects lies in the detailed exploration of \((0,2)\) models. Their rather complicated geometrical description makes them hard to study by traditional techniques, but we have shown that their Landau-Ginzburg description makes them amenable to quite detailed analysis. One can write down \((0,2)\) models with gauge groups like \(SO(10)\) or \(SU(5)\), which are much less cumbersome than \(E_6\). This makes \((0,2)\) models perhaps the most promising class of models for realistic phenomenology. In addition, it is quite plausible that a better understanding of \((0,2)\) models could lead to progress in the understanding of topology-changing processes in string theory.
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References

[1] A.B. Zamolodchikov, “Conformal Symmetry and Multicritical Points in Two Dimensional Quantum Field Theory,” Sov. J. Nucl. Phys. 44 (1986) 529.

[2] D. Kastor, E. Martinec, and S. Shenker, “RG Flow in N=1 Discrete Series,” Nucl. Phys. B316 (1989) 590.

[3] C. Vafa and N. Warner, “Catastrophes and the Classification of Conformal Theories,” Phys. Lett. 218B (1989) 51.

[4] E. Martinec, “Algebraic Geometry and Effective Lagrangians,” Phys. Lett. 217B (1989) 431, “Criticality, Catastrophes, and Compactifications,” in Physics and Mathematics of Strings, ed. L. Brink, D. Friedan, and A. Polyakov (World Scientific, 1990).

[5] S. Cecotti, L. Girardello, and A. Pasquinucci, “Non-perturbative Aspects and Exact Results for the N=2 Landau-Ginzburg Models,” Nucl. Phys. B338 (1989) 701, “Singularity Theory and N=2 Supersymmetry,” Int. J. Mod. Phys. A6 (1991) 2427.

[6] E. Witten, “On the Landau-Ginzburg Description of N=2 Minimal Models,” Institute for Advanced Study Preprint IASSNS-HEP-93/10.

[7] B. Greene, C. Vafa and N. Warner, “Calabi-Yau Manifolds and Renormalization Group Flows,” Nucl. Phys. B324 (1989) 371.

[8] W. Lerche, C. Vafa and N. Warner, “Chiral Rings in N=2 Superconformal Theories,” Nucl. Phys. B324 (1989) 427.

[9] S. Cecotti, “N=2 Landau-Ginzburg v.s. Calabi-Yau σ Models: Non-Perturbative Aspects,” Int. J. Mod. Phys. A6 (1991) 1749.

[10] E. Witten, “Phases of N=2 Theories in Two Dimensions,” Institute for Advanced Study Preprint IASSNS-HEP-93/3, to appear in Nucl. Phys. B.

[11] C. Vafa, “String Vacua and Orbifoldized LG Models,” Mod. Phys. Lett. A4 (1989) 1169.

[12] K. Intriligator and C. Vafa, “Landau-Ginzburg Orbifolds,” Nucl. Phys. B339 (1990) 95.

[13] M. Dine, N. Seiberg, X.G. Wen and E. Witten, “Non-Perturbative Effects on the String World Sheet I,” Nucl. Phys. B278 (1986) 769, “Non-Perturbative Effects on the String World Sheet II,” Nucl. Phys. B289 (1987) 319.

[14] J. Distler, B. Greene, K. Kirklin and P. Miron, “Calculating Endomorphism Valued Cohomology: Singlet Spectrum in Superstring Models,” Comm. Math. Phys. 122 (1989) 117.

[15] M. Eastwood and T. Hubsch, “Endormorphism Valued Cohomology and Gauge Neutral Matter,” Comm. Math. Phys. 132 (1990) 383.

[16] P. Berglund, T. Hubsch and L. Parkes, “Gauge Neutral Matter in Three Generation Superstring Compactifications,” Mod. Phys. Lett. A5 (1990) 1485.

[17] T. Hubsch, Calabi-Yau Manifolds: A Bestiary For Physicists (World Scientific, 1992).
[18] J. Distler and B. Greene, “Aspects of (0,2) String Compactifications,” *Nucl. Phys.* B304 (1988) 1.
[19] R. Rohm and E. Witten, “The Antisymmetric Tensor Field in Superstring Theory,” *Ann. Phys.* 170 (1986) 454.
[20] R. Bott and L. Tu, *Differential Forms In Algebraic Topology* (Springer-Verlag, 1982).
[21] P. Fré, F. Gliozzi, M. Monteiro, and A. Piras, “A Moduli-Dependent Lagrangian For (2, 2) Theories On Calabi-Yau n-Folds,” *Class. Quant. Grav.* 8 (1991) 1455; P. Fré, L. Girardello, A. Lerda, and P. Soriani, “Topological First-Order Systems With Landau-Ginzburg Interactions,” *Nucl. Phys.* B387 (1992) 333.
[22] F. Gliozzi, J. Scherk, and D. Olive, “Supersymmetry, Supergravity Theories, and the Dual Spinor Model,” *Nucl. Phys.* B122 (1977) 253.
[23] X.G. Wen and E. Witten, “Electric and Magnetic Charges in Superstring Models,” *Nucl. Phys.* B261 (1985) 651.
[24] L. Dixon, “Some World-Sheet Properties of Superstring Compactifications on Orbifolds and Otherwise,” in *Proceedings of the 1987 ICTP Summer Workshop in High Energy Physics and Cosmology*, e.d. G. Furlan et. al.
[25] D. Gepner, “Exactly Solvable String Compactification on Manifolds of SU(N) Holonomy,” *Phys. Lett.* 199B (1987) 380.