Weak measurement and rapid state reduction in entangled bipartite quantum systems

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Abstract. In this paper, we consider feedback control algorithms for the rapid purification of a bipartite state consisting of two qubits, when the observer has access to only one of the qubits. We show (1) the algorithm maximizing the average purification rate is different from that for a projective measurement and (2) that the weak measurement protocol for bipartite systems contrasts with known single qubit protocols. We also reveal that the rate of increase in purity may be made deterministic for bipartite states.

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1. Introduction

Although quantum measurements are often treated as instantaneous, in practice measurement time-scales can be significant when compared to other relevant time-scales. Recent experimental advances have meant that it is now possible to observe continuous measurements of individual quantum systems. The measurement of an observable is not instantaneous, but takes place over a period of time [1]–[4]. This type of continuous measurement can be modelled by considering a series of projective measurements on an auxiliary system that is weakly coupled to the system of interest. The auxiliary system then acts as a conduit that extracts information from this system. This produces a continuous measurement record which contains information about the evolution of the quantum system of interest. The measurement record is then used to construct a best estimate of the underlying evolution—which is referred to as an ‘unraveling’ of the master equation for the system [5]–[9].

When continuous weak measurement is applied, it is possible to modify the evolution via Hamiltonian feedback, where the Hamiltonian applied to the system depends on the measurement record [7]–[15]. Hamiltonian feedback during measurement not only affects the final state of the system, but it can also affect the rate of state reduction. In a protocol described by Jacobs, the average rate of state reduction (as measured by the purity of an initially mixed state) for a single qubit is maximized [16]. This process is known as rapid state reduction [15], or as rapid purification [16]. Jacobs’ protocol is deterministic, but other protocols exist which are stochastic and minimize the average time for a single qubit to reach a given purity [17].

In this paper we consider the analogous situation of performing rapid state reduction in a two qubit system (shown in figure 1). There are two parties, identified as Alice and Bob, who may be separated and are not required to communicate. One observer (Alice) has access to a single qubit, which she can measure and manipulate using local Hamiltonian feedback. She does not have access to the second qubit, which is controlled by Bob. This corresponds to a physically realistic situation in which Bob’s qubit is either spatially separated from Alice, or—for architectural reasons—it is not possible to measure Bob’s qubit directly. Two qubit systems are important because they form the basis for many current applications in quantum information processing, and are the simplest system which exhibits entanglement.

In this paper, we highlight three aspects of rapid state reduction for bipartite qubit states: (i) we highlight a key difference between projective measurements and weak measurement. (ii) We show that the measurement which provides the maximum rate of state reduction for the unprobed qubit in a two qubit system is not necessarily the same as that for either of the known optimal single qubit protocols [16, 17]. (iii) We show that it is always possible to purify both qubits deterministically at the same time.

2. Weak measurement of entangled systems

The bipartite system which Alice and Bob share is described by the density matrix $\rho$. Without loss of generality, Alice’s system undergoes a constant weak measurement, giving a measurement record, $r(t)$. It is natural to expand $\rho$ in the two-qubit Pauli basis,

$$\rho = \frac{1}{4} \sum_{i,j=I,X,Y,Z} r_{ij} \sigma_i \otimes \sigma_j,$$

(1)
Figure 1. Conceptual layout of the qubits. Alice and Bob both have a single qubit. The bipartite system they share is described by the density matrix \( \rho \). Alice can make weak measurements on her system, but not Bob’s.

where \( r_{ij} \) are real (since \( \rho \) is Hermitian) and lie between \(-1\) and \(1\), \( \sigma_j \) are each of the three Pauli matrices and the identity. Each \( r_{ij} \) can be found, \( r_{ij} = \text{Tr} \rho \sigma_i \otimes \sigma_j \).

The stochastic master equation (SME) which governs the evolution of the density operator \( \rho \) in the presence of a weak measurement of a Hermitian observable, \( y \), is given by

\[
d\rho = -k[y, [y, \rho]]dt + \sqrt{2k(y\rho + \rho y - 2\langle y \rangle \rho)} dW.
\] (2)

Here \( k \) is the measurement strength. The first term in this equation describes the familiar drift towards the measurement axis. The second term in the equation is weighted by \( dW \), a Wiener increment with \( dW^2 = dt \). This term describes the update of knowledge of the density matrix conditioned on the measurement record [14].

Throughout this paper we will consider measurements on Alice’s qubit alone. A measurement of Alice’s qubit along a given axis \( \hat{n} \) means that \( y \) is given by

\[
y = \hat{n} \cdot \sigma I = n_x X I + n_y Y I + n_z Z I.
\] (3)

Here, \( X, Y \) and \( Z \) are the Pauli matrices, and \( I \) is the identity. The tensor product is implied. The measurement direction, \( \hat{n} \) may change at each time step. This is equivalent to the application of single qubit Hamiltonian feedback to Alice’s qubit, except that the measurement axis rotates, rather than Alice’s Bloch vector.

Consider the evolution according to the SME as Alice’s qubit undergoes weak measurement along the \( \hat{n} \)-axis. The SME (equation (2)) can be expressed in terms of the real Pauli coefficients of the density matrix. If \( \hat{n}, \hat{m}_1 \) and \( \hat{m}_2 \) are mutually orthogonal unit vectors, then the corresponding SME is given by

\[
dr_{mj} = -(4kdr + r_{nj}\sqrt{8kdW})r_{mj}, \tag{4a}
\]

\[
dr_{nj} = (r_{lj} - r_{nlr_{nj}})\sqrt{8kdW}, \tag{4b}
\]

\[
dr_{lj} = (r_{nj} - r_{nlr_{lj}})\sqrt{8kdW}, \tag{4c}
\]
for \( m \in \{ \hat{m}_1, \hat{m}_2 \} \) and \( j \in \{ X, Y, Z, I \} \). Here, \( r_{nj} \) is given by

\[
r_{nj} = n_x r_{Xj} + n_y r_{Yj} + n_z r_{Zj},
\]

(5)

and similarly for \( \hat{m} \). This is a system of 16 stochastic differential equations.

In this paper, we will be particularly concerned with the evolution of both Alice and Bob’s reduced density matrices. Alice’s reduced density matrix is given by

\[
\rho_A = \text{Tr}_B (\rho) = I + r_{X1} X + r_{Y1} Y + r_{Z1} Z.
\]

(6)

Bob’s reduced density matrix can be described by a similar equation with the indices swapped.

### 3. Maximum rates of state reduction

The purity of a quantum state, \( \rho \) can be quantified by the purity, \( P(\rho) = \text{Tr} \rho^2 \). Purity has a minimum value of \( 1/d \) where \( d \) is the dimension of \( \rho \), and a maximum value of 1. The purity of Alice’s reduced density matrix is given by

\[
P_A = \text{Tr} \rho_A^2 = \frac{1}{2} \left( 1 + r_A^2 \right),
\]

(7)

where \( r_A \) is the Bloch vector of Alice’s reduced density matrix. A similar expression can be given for Bob’s reduced density matrix.

Consider the rate of state reduction of Alice’s qubit as she measures on her own system. The change in average purity of Alice’s reduced density matrix given in equation (7) under the evolution of the SME, is given by

\[
dP_A = (1 - r_{n1}^2) \left( 1 - r_A^2 \right) 4k \, dt + r_{nI} (1 - r_A^2) \sqrt{8k} \, dW.
\]

(8)

This expression does not depend on the state of Bob’s qubit, and not surprisingly, it is identical to the one qubit case. Hamiltonian feedback can be used to implement either of the known single qubit Hamiltonian protocols without modification [16, 17].

Now we consider the opposite situation, when Alice would like to find out the state of Bob’s qubit. Bob’s qubit is not being directly measured. It is only through the correlations in the initial system, \( \rho \), that Alice can learn the state of Bob’s system. For the most effective purification, Alice and Bob’s share an known initial entangled state.

The average change in purity (according to Alice) of Bob’s qubit when measured along the \( \hat{n} \)-axis is given by

\[
dP_B = \left[ (r_{nX} - r_{nl} r_{lX})^2 + (r_{nY} - r_{nl} r_{lY})^2 + (r_{nZ} - r_{nl} r_{lZ})^2 \right] 4k \, dt + \left[ r_{lX}(r_{nX} - r_{nl} r_{lX}) + r_{lY}(r_{nY} - r_{nl} r_{lY}) + r_{lZ}(r_{nZ} - r_{nl} r_{lZ}) \right] \sqrt{8k} \, dW,
\]

(9)

which can be written much more easily by identifying

\[
r_{ni} = (r_{nX}, r_{nY}, r_{nZ}),
\]

(10a)

\[
R_n = r_{nj} - r_{nl} r_{lj}.
\]

(10b)
$R_n$ is a vector with three components, one for each of $j \in \{X, Y, Z\}$. Then the change in average purity of Bob’s system

$$dP_B = R_n^2 4k \, dt + (r_B \cdot R_n) \sqrt{8k} \, dW.$$  \hfill (11)

We can write the nonlocal correlations as a matrix:

$$C_{ji} = r_{ij} - r_{ii}r_{jj},$$ \hfill (12)

for each $i, j \in \{X, Y, Z\}$. To retrieve $R_n$ from $C$ we multiply by the corresponding unit vector, $\hat{n}$. That is,

$$R_n = C\hat{n}.$$ \hfill (13)

The magnitude square of $R_n$ gives the rate of state reduction of Bob’s qubit, as seen by Alice, when she makes a weak measurement along the $\hat{n}$-direction.

We now consider how to maximize this rate, in a direct analogy to the optimizing strategy used by Jacobs for a single qubit [16]. In particular we find the largest average increase in purity of Bob’s reduced density matrix. Feedback could be applied to achieve this increase. While there are some circumstances in which this strategy will not give a globally optimal solution, because choosing a local maximum does not necessarily correspond to finding a global maximum, numerical simulations verify its use in this practice. The average increase in purity of Bob’s reduced density matrix is proportional to $|R_n|^2$. Expressing this in terms of the matrix, $C$, we find

$$dP_B^{(\text{max})} = 4k \max_{\hat{n}} |C\hat{n}|^2.$$ \hfill (14)

The maximum value is given by the largest singular value, $\sigma_1^2$, of $C$, therefore giving a maximum rate increase in purity of $4k\sigma_1^2$. The value of $\hat{n}$ which corresponds to this maximum rate of state reduction is given by the first right singular vector, $v_1$, of $C$.

### 4. Prototype examples

Point (i) of this paper contrasts weak measurement and projective measurement, using a specific example. The measurement which gives the greatest increase in purity of Bob’s qubit for a projective measurement is not the same as the measurement which gives the greatest rate of increase in purity of Bob’s qubit for a weak measurement.

Consider the state,

$$|\psi\rangle = \sqrt{\frac{1 + \beta}{2}} |00\rangle + \sqrt{\frac{1 - \beta}{2}} |11\rangle,$$ \hfill (15)

where $-1 \leq \beta \leq 1$. $\psi$ is in a coherent superposition, with both qubits aligned in the z-basis. If $\beta = 0$ this is a maximally entangled state, and is entangled unless $\beta = \pm 1$. If the system of interest undergoes a dephasing process, as is common in many quantum systems, then the...
off-diagonal terms of the density matrix decay. If the amount of dephasing is characterized by $\delta$ then the density matrix becomes

$$\rho_T = \frac{1}{4} \left( II + ZZ + \beta ZI + \beta IZ + \gamma(XX - YY) \right), \quad (16)$$

where $\gamma = \sqrt{1 - \beta^2 - \delta}$.

The maximum purity from a projective measurement is when Alice measures along the $z$-axis. Any projective measurement by Alice along the $z$-axis will project Bob’s state into a pure state (either $|0\rangle$ or $|1\rangle$), with purity, $P_B = 1$. As Alice rotates the measurement axis, the purity of Bob’s state reduces. The matrix $C$ is

$$C = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & -\gamma & 0 \\ 0 & 0 & 1 - \beta^2 \end{bmatrix}. \quad (17)$$

This matrix is diagonal. For small values of dephasing, $\delta$, the maximum rate of state reduction occurs in the $xy$-plane, as is shown in figure 2. As expected, since the specific state chosen is symmetric around the $z$-axis, figure 2 there is no $\theta$ dependence. Therefore the projective measurement which gives greatest purity is different from the weak measurement which gives the greatest rate of increase in purity.

Point (ii) of this paper contrasts weak measurement for bipartite systems, and existing single qubit protocols. Naively, one might expect that the best way for Alice to purify Bob’s system is to purify her own system fastest by applying known single qubit protocols to her own qubit. We show, by giving a specific counter-example, that this is not the case.

Consider the state,

$$\rho_t = \frac{1}{4} \left( II + \frac{1}{\sqrt{5}} (XI + XZ + ZZ) \right). \quad (18)$$
Figure 3. Graph showing the rate of average state reduction of Bob’s qubit for all possible orientations of the measurement axis, and a state given by $\rho_t$. $\phi$ is the zenith angle, and $\theta$ is the azimuthal angle.

This state is perfectly aligned for Jacobs’ one qubit feedback protocol; Alice’s reduced density matrix lies in the $xy$-plane [16]. However, the state $\rho_t$ was carefully chosen to also give a different two qubit feedback protocol. This is done by choosing two qubit terms ($X \otimes Z + Z \otimes Z = (X + Z) \otimes Z$) which give a maximum rate of purification at an angle to the $x$-axis. The factor $\frac{1}{\sqrt{5}}$ was chosen to make the effect as large as possible, but also keep the density matrix positive semi-definite.

For the state, $\rho_t$, the matrix $C$ is given by

$$C = \begin{bmatrix}
0 & 0 & \frac{1}{\sqrt{5}} \\
0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{5}}
\end{bmatrix}, \quad (19)$$

which has a single nonzero singular value of $\sigma_1 = 1/\sqrt{5}$, and a corresponding vector of $n_1 = \pm \frac{1}{\sqrt{2}}(\hat{x} + \hat{z})$. We therefore expect that the fastest rate of state reduction for Bob’s qubit occurs when Alice measures her own system at $45^\circ$ to the $z$-axis and to the $x$-axis, as shown in figure 3. Although at first glance it appears to be three local maxima, there are only two, as the axes are cyclic. The two local maxima correspond to the positive and negative solutions for the measurement axis, both represent the same physical measurement. The numerical values for this plot were $k = 0.1$, $dt = 0.1$ and $N = 10,000$ repetitions.

This measurement does not correspond to either the Jacobs’ single qubit scheme [16], or the scheme proposed by Wiseman and Ralph [17]. Purifying the unprobed qubit is dependant on the correlations and the nature of those correlations. As we show here, it is purified fastest by choosing a measurement axis to make use of those correlations, and not by simply using a one qubit protocol.
5. Deterministic state reduction

Point (iii) of this paper is that it is possible to make bipartite state reduction deterministic. Increasing the purity of systems deterministically is a desirable property. From a theoretical standpoint it makes the equations easier, but the main advantages are practical. If the evolution of purity is deterministic, then each qubit is guaranteed to reach a set target purity in a given time. When weak measurement is being used for state preparation in a multiple qubit system, they will all reach the target purity together.

The condition required for the purity of Alice’s reduced density matrix to evolve deterministically is that

\[ r_A \cdot \hat{n} = 0. \]  

That is, the measurement should be taken in the plane orthogonal to the direction described by Alice’s reduced density matrix.

The condition for Bob’s system to be deterministic is clear from equation (11). It is given by

\[ r'_B \cdot \hat{n} = 0, \]  

where \( r'_B = C^T r_B \). Therefore for the evolution of the purity of Bob’s reduced density matrix to be deterministic, the measurement axis must be chosen in a plane orthogonal to \( r'_B \). Let \( \hat{p}_1, \hat{p}_2 \) be two arbitrary vectors spanning the plane.

Deterministic state reduction of Bob’s qubit is desirable, but we would still like to find the maximum rate of state reduction. Although the measurement axis, \( \hat{n} \) can be chosen anywhere in the plane, not all orientations of \( \hat{n} \) will purify Bob’s reduced density matrix at the same rate.

The maximum deterministic rate of state reduction of Bob’s qubit is found by taking the singular value decomposition of \( CP_p \), where \( P_p \) is given by

\[ P_p = p_1 p_1^T + p_2 p_2^T. \]  

The fastest rate of average deterministic purification of Bob’s reduced density matrix is given by \( 4k \sigma_1^2 \) where \( \sigma_1 \) is the largest singular value of the product \( CP_p \), and the axis of measurement is given by \( \hat{n}_{\text{det}} = v_1 \), the first right singular vector of \( CP_p \).

It is possible to choose a weak measurement which Alice can make on her qubit so that the purity of both her reduced density matrix, and also Bob’s reduced density matrix both evolve deterministically. If the measurement axis is chosen to be

\[ \hat{n}_{\text{det}} \propto r_A \times r'_B, \]  

then the evolution of the purity of both Alice and Bob’s reduced density matrices is deterministic. The conditions for the evolution to be deterministic depend on \( r_A \), which is the Bloch vector of Alice’s reduced density matrix. It also depends on \( r'_B \), which depends in turn on the correlations between qubits, characterized by the matrix \( C \) and also Bob’s reduced density matrix Bloch vector, \( r_B \). By choosing the measurement axis, \( \hat{n} \) to be the cross product of both sets of axes, Alice guarantees that it perpendicular to both, and that the purity of both her reduced density matrix, and that of Bob’s reduced density matrix both evolve deterministically.
6. Conclusion

We have found the rates of purification of both Alice’s reduced density matrix in equation (8) and of Bob’s reduced density matrix in equation (11) evolving under weak measurement of Alice’s qubit. These equations each contain a drift term of order $dt$, describing the average increase in purity of each system under measurement. A feedback strategy applied to maximize this term, which increases the rate the state of the system, is determined. The equations also contain a stochastic term, which we showed—for both Alice’s and Bob’s qubit—can be made to vanish. Timeliness of events is important, and a feedback strategy to provide deterministic evolution of the purity means that the qubits to reach a given purity in a consistent and predictable time.

In this paper we investigated the effect of weak measurements on a bipartite system consisting of two qubits—Alice’s qubit and Bob’s qubit. We allowed measurement on Alice’s qubit alone. We gave expressions for the rate of state reduction of both systems, based on the measurement record. We have shown how to maximize the average rate of state reduction of either system, and how to achieve deterministic evolution of the purity. We have demonstrated that weak measurement of two qubits can be very different from both projective measurements and the weak measurement of a single qubit. Many interesting effects occur in bipartite quantum systems under measurement.

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