Operations with elements of transferred density matrix via unitary transformations at receiver side

A.I.Zenchuk

Institute of Problems of Chemical Physics RAS, Chernogolovka, Moscow reg., 142432, Russia

Abstract

Using the tool of unitary transformations of the extended receiver we perform simple operations with the non-diagonal elements of the initial sender’s density matrix after their transferring to the receiver. These operations are following: turning some matrix elements to zero, rearranging the matrix elements and preparing their linear combinations with required coefficients. We also propose a protocol for solving a system of linear algebraic equations with a system of two equations as an example. Such operations are numerically simulated on the basis of 42-node spin-1/2 chain with the two-qubit sender and receiver.

PACS numbers:
I. INTRODUCTION

The development of quantum counterparts of classical algorithms is a beneficial direction in quantum information processing. Since the basic tool of quantum mechanics is unitary transformations, the problem of their implementation and control is of general interest. In Ref. [1], a simple gates controlling the application of an arbitrary unitary transformation to the output qubit using one-, two- or more control qubits (generalization of CNOT) are proposed. Selecting a proper combination of unitary operators (perhaps, combined with certain measurements) allows to realize set of operations useful in various applications. Among others, we point out the quantum algorithm for solving the systems of linear algebraic equations that was proposed in Ref. [2] (so-called HHL-algorithm) and was first realized in photon systems [3, 4]. This algorithm is also effectively used in the quantum algorithm for data fitting over the exponentially large data set [5]. Recently this algorithm was implemented on the basis of superconducting qubit system to solve the system of two linear equations [6].

Review on quantum machine learning is represented in [7]. The problem of quantum speedup, which is crucial for justifying the advantages of quantum computations, is discussed in the above references as well. Basic quantum algorithms and their implementations in quantum information theory can be found in book [8]. The research on realization of quantum gates goes further to work out the effective remote control of unitary operations [9, 10], such as the multi-qubit controlled-phase gates [11].

In our work we deal with a spin-1/2 chain including sender, transmission line and receiver. During the evolution, due to the unavoidable spin-spin interactions, each element of the receiver’s density matrix become linear combinations of all the elements of the initial sender’s density matrix. This phenomena is an obstacle for the state transfer and remote manipulating with the matrix elements. To (partially) remove this mixture of elements we can impose a constraint on the evolution, for instance, the conservation of the $z$-projection of the total spin momentum [12]. This constraint provides the independent evolution of the multiple-quantum (MQ) coherence matrices, so that the matrix elements mix only inside of each particular MQ-coherence matrix. The importance of such independent evolution is associated with different meaning of the matrix elements. For instance, the diagonal elements represent the probabilities of certain state transfers and therefore must be non-negative. In addition, their sum equals one. This property of the diagonal elements appears to be
inconvenient in certain cases. For instance, it was shown that these elements require much more efforts (in comparison with non-diagonal elements) for their structural restoring at the remote side with the protocol proposed in Ref. [13].

In our paper we further develop the model of Ref. [13] and explore the power of the unitary transformation at the receiver side as a tool for performing simple quantum operations based on the mixed states of a quantum system. As such operations we consider some manipulations with non-diagonal elements of the sender’s density matrix transferred to the receiver. Such manipulations include turning some matrix elements to zero, rearrangement of matrix elements, constructing certain linear combinations of matrix elements and solving systems of linear algebraic equations via the protocol different from the above quoted HHL-algorithm based on the pure quantum states.

We use sender’s state as the input information, the receiver’s state as the output information and the unitary transformation of the extended receiver (the receiver supplemented with additional nodes called ancilla) as a tool allowing to manipulate with elements of the output state. The transmission line is an optional part, it connects the sender with the extended receiver and serves to perform the remote manipulation with the input data. It can be absent if we are interested in the local transformations. We emphasise that our protocols of unitary operations with a remotely transferred state do not include a classical communication channel and therefore can be referred to completely quantum processes.

The paper is organized as follows. In Sec. II we discuss the general protocol of operating with the transferred density matrix, including structure of the communication line, the initial state and a general form of the evolution operator. The set of manipulations with the receiver’s density matrix elements is introduced in Sec. III. In Sec. IV we briefly describe a particular model of communication line used in Sec. V for numerical simulations of operations with density matrix elements. The paper concludes with Sec. VI. The structure of the unitary transformation and examples of such transformation realizing the matrix element manipulations in Sec. V are represented in the Appendix, Sec. VII.

II. INITIAL STATE AND EVOLUTION OPERATOR

We consider the spin-1/2 communication line [13] consisting of the sender ($S$), transmission line ($TL$), receiver ($R$) and ancilla ($A$) in the strong external $z$-directed magnetic
FIG. 1: Spin-1/2 communication line consisting of the sender (S), transmission line (TL), ancilla (A) and receiver (R). The extended receiver combines the ancilla and receiver. The whole system is placed into the strong external magnetic field $\vec{H}$.

field, Fig. 1. The receiver and ancilla form the extended receiver (ER). Now let us discuss the general protocol, which performs certain manipulations with the non-diagonal elements of $\rho^{(R)}$. This protocol is schematically represented in Fig. 2. First of all, we assume the tensor-product initial state

$$\rho(0) = \rho^{(S)}(0) \otimes \rho^{(TL,ER)}(0),$$

where $\rho^{(S)}$ is the density matrix of the sender $S$, and $\rho^{(TL,ER)}$ is the density matrix of the subsystem $TL \cup ER$. Let both the sender and receiver have the same dimensionality. If $W(t)$ is the evolution operator describing the dynamics of a given quantum system, then the state of the receiver at some time instant $t$ reads

$$\rho^{(R)}_{N^3,M^3}(t) = \sum_{I^1,J^1} T_{N^3,M^3,I^1,J^1}(t) \rho^{(S)}_{I^1,J^1}(0),$$

$$T_{N^3,M^3,I^1,J^1} = \sum_{N^1,N^2} \sum_{I^2,J^2} W_{N^1,N^2,M^3,I^1,I^2,J^1,J^2} \rho^{(TL,ER)}_{I^2,J^2,J^3} W_{J^3,J^1,I^1,M^3}.$$ (3)

The operator $T$ in this formula maps the sender’s initial state into the receiver’s state at some time instant. Here the capital latin multi-indexes with superscripts 1, 2 and 3 are associated, respectively, with the sender, the transmission line joined with the ancilla, and the receiver. These indexes consist of a set of parameters each associated with the particular spin of the appropriate subsystem and can take either 0 or 1. Thus, for the $N$-qubit communication line and 2-qubit sender and receiver we write

$$I^i = \{j^i_{1},j^i_{2}\}, \ i = 1,3, \ I^2 = \{j^2_{1}, \ldots j^2_{N-4}\}.$$ (4)

Now we discuss the structure of the evolution operator $W$ including two blocks as shown in Fig. 2. First of all, it includes the inner spin dynamics described by some Hamiltonian $H$:
FIG. 2: General scheme for manipulating with the matrix elements of $\rho^{(R)}$. The initial state is $I$, the evolution operator $W$ consists of two parts: the evolution of the whole system under the Hamiltonian $H$ followed by the local unitary transformation $U^{(ER)}$ of the extended receiver. The final state $\rho^{(R)}$ is detected at the receiver.

$\tilde{V}(t) = e^{-iHt}$. In addition, the evolution operator $W$ includes the unitary transformation $U^{(ER)}$ of the extended receiver. This transformation includes the set of additional parameters $\varphi_i$ (hereafter we call them $\varphi$-parameters) needed to control the state of the receiver. Then

$$W = (E^{(S,TL)} \otimes U^{(ER)})\tilde{V}(t).$$  \hspace{1cm} (5)

Here $E^{(S,TL)}$ is the identity operator in the state space of the subsystem $S \cup TL$.

Hereafter we deal with such Hamiltonian $H$ and unitary transformation $U^{(ER)}$ that conserve the excitation number during the spin evolution, i.e.,

$$[H, I_z] = 0, \quad [U^{(ER)}, I_z^{(ER)}] = 0,$$  \hspace{1cm} (6)

where $I_z$ and $I_z^{(ER)}$ are the $z$-projections of the total spin momentum of, respectively, the whole communication line and the subsystem $ER$. Then, according to Ref. [12], the MQ-coherence matrices of the whole system do not interact during such evolution. In addition, to transfer the MQ-coherence matrices from the sender $S$ to the receiver $R$ without mixing, the initial density matrix $\rho^{(TL,ER)}(0)$ must include only the zero-order coherence matrix $[12]$. In particular, it can be the diagonal initial state $\rho^{(TL,ER)}(0)$ (for instance, the thermodynamic equilibrium state). Then eq. (3) becomes

$$T_{N_1 M_1; I_1, I_1} = \sum_{N_1 N_2} \sum_{I_1 I_2 I_3} W^{(TL,ER)}_{N_1 N_2; I_1 I_2 I_3} \rho^{(TL,ER)}(0) W^{(TL,ER)}_{I_1 I_2 I_3; N_1 N_2 M_3}. \hspace{1cm} (7)$$

Commutativity (6) implies the following constraint for the evolution operator $W$:

$$W_{N_1 N_2 N_3; I_1 I_2 I_3} = 0 \text{ if } \sum_{i=1}^3 |N_i| \neq \sum_{i=1}^3 |I_i|, \hspace{1cm} (8)$$
where $|N^i|$ and $|I^i|$ mean the sum of entries of the multi-indexes $N^i$ and $I^i$. We also set zero the energy of the ground state (the state without excitations), which means

$$W_{0^1 0^2, 0^2 0^2} = 1,$$

(9)

where $0^i, i = 1, 2, 3$, are the multi-indexes with all zero entries.

### III. Manipulations with Receiver’s Density Matrix Elements

Since there are free parameters in the unitary transformation $U^{(ER)}$ of the extended receiver, we can control the structure of the density matrix elements of the receiver. In particular, we can arrange the following operations with the elements inside of an MQ-coherence matrix of a certain order.

1. **Structural restoring of initial sender’s state** [13]. In this case, by choosing the $\varphi$-parameters, we have to satisfy the following constraints:

$$T_{N^3 M^3, I^3 J^3} = \lambda_{N^3 M^3} \delta_{N^3 M^3} , \quad (N^3, M^3) \neq (0^3, 0^3),$$

(10)

$$T_{0^3 0^3, I^3 J^3} = \delta_{I^3 J^3} (1 - \lambda_{I^3 J^3}), \quad (I^3, J^3) \neq (0^1, 0^1),$$

(11)

$$T_{0^3 0^3, 0^1 0^1} = 1.$$  

(12)

As the result, the structurally restored state reads

$$\rho_{N^3 M^3}^{(R)} = \lambda_{N^3 M^3} \rho_{N^3 M^3}^{(S)}, \quad (N^3, M^3) \neq (0^3, 0^3),$$

(13)

$$\rho_{0^3 0^3}^{(R)} = 1 - \sum_{I^3 \neq 0^1} \lambda_{I^3 J^3} \rho_{I^3 J^3}^{(S)},$$

(14)

2. **Turning some matrix elements to zero.** For some fixed multi-indexes $N^3 = N^3_0$ and $M^3 = M^3_0$ in (13) we can find the $\varphi$-parameters such that, along with system (10)-12), the additional constraint

$$\lambda_{N^3_0 M^3_0} = 0$$

is satisfied. This turns to zero the appropriate matrix element:

$$\rho_{N^3_0 M^3_0}^{(R)} = 0.$$  

(15)

3. **Rearrangement of matrix elements.** For a fixed pair of multi-indexes $N^3 = N^3_0$, $M^3 = M^3_0$ and $I^3 = I^3_0$, $J^3 = J^3_0$ in (2) we change formulas (10) as follows:

$$T_{N^3_0 M^3_0, I^3_0 J^3_0} = \delta_{N^3_0 J^3_0} \delta_{M^3_0 I^3_0} \lambda_{I^3_0 J^3_0}, \quad T_{I^3_0 J^3_0, N^3_0 M^3_0} = \delta_{N^3_0 J^3_0} \delta_{M^3_0 I^3_0} \lambda_{N^3_0 M^3_0},$$

(17)
and solve system (11), (12), (17) for the $\varphi$-parameters. Then, two equations in (13) which correspond to this fixed pair must be replaced with

$$\rho_{N_0^3;M_0^3}^{(R)} = \lambda_{f_0^3;j_0^3} \rho_{I_0^3;J_0^3}^{(S)}, \quad \rho_{I_0^3;J_0^3}^{(R)} = \lambda_{N_0^3;M_0^3} \rho_{N_0^3;M_0^3}^{(S)}. \quad (18)$$

Thus, we exchange the elements $\rho_{I_0^3;J_0^3}^{(S)}$ and $\rho_{N_0^3;M_0^3}^{(S)}$.

4. **Linear combination of matrix elements with required coefficients.** Since the elements of $n$-order coherence matrix $\rho^{(R;n)}$ are linear combinations of the elements of $\rho^{(S;n)}$, this operation means setting the required values to the corresponding elements of the $T$-operator (2) using the proper choice of the $\varphi$-parameters. In particular, these combinations can yield the polynomial expansion of certain degree for some functions, see Sec V D.

5. **Solving systems of linear algebraic equations**

$$A \vec{x} = \vec{b}, \quad (19)$$

where $A$ is a non-degenerate square matrix, $\vec{x}$ is a vector of unknowns and $\vec{b}$ is a constant vector. In this respect we shall remember that the algorithm solving linear matrix equation (19) with a given Hermitian $A$ via the tool of quantum mechanics is the well-known HHL-algorithm [2] based on pure states. The quantum system in that algorithm consists of the input qubits (where the input vector $\vec{b}$ is initialized), the additional register serving to get and store the eigenvalues of $A$ (using the phase estimation [14, 15]), and a one-qubit ancilla needed to extract the inverse eigenvalues of $A$ from the register. Then $\vec{b}$ transforms into the needed vector $\vec{x}$.

Here we consider the extended receiver (consisting of the 2-qubit ancilla and receiver, see Fig 1) as a single block for applying the unitary transformation establishing the desired operation. We do not use ancilla for a special role unlike the HHL-algorithm. Our protocol is based on mixed states and operates as follows. If there are $M$ elements $\rho_{N_i^3;M_i^3}^{(S;n)}$, $i = 1, \ldots, M$ in an $n$-order coherence matrix which map into $\tilde{M} \leq M$ linearly independent combinations at the receiver side (elements of $\rho^{(R;n)}$), then we can solve a system of $\tilde{M}$ equations for $\tilde{M}$ unknowns as follows. Let $\tilde{M} = M$ for simplicity. We first identify elements $\rho_{N_i^3;M_i^3}^{(S;n)}$ with the appropriate entries of $\vec{b}$:

$$\rho_{N_i^3;M_i^3}^{(S)} = b_i, \quad i = 1, \ldots, M. \quad (20)$$
From the other hand, we can write

$$\rho^{(S)}_{N_i^3;M_i^3} = \sum_{j=1}^{M} A_{ij} x_j, \ i = 1, \ldots, M. \quad (21)$$

Accordingly, we can represent the elements of the receiver’s $n$-order coherence matrix as

$$\rho^{(R)}_{N_i^3;M_i^3} = \sum_{j=1}^{M} \alpha_{ij} x_j, \ i = 1, \ldots, M, \quad (22)$$

where $\alpha_{ij}$ depend on the time $t$ and the $\varphi$-parameters of the unitary transformation $U^{(ER)}$. The coefficients $\alpha_{ij}$ are also linear expressions of the elements of $A$. Now we impose the constraints on the coefficients $\alpha_{ij}$:

$$\alpha_{ij} = \alpha \delta_{ij}, \ \alpha = const, \quad (23)$$

which can be satisfied by the proper choice of the $\varphi$-parameters for some value of the constant $\alpha$ at some time instant. Then eq. (22) yields

$$\rho^{(R)}_{N_i^3;M_i^3} = \alpha x_i, \ i = 1, \ldots, M, \quad (24)$$

i.e., the scaled values of $x_i$ appear in the entries of the receiver’s $n$-order coherence matrix.

Similarly, we can consider the case $\tilde{M} < M$, see Sec. [V E]. We notice, that the maximal absolute value of the arbitrary parameter $\alpha$ is restricted because of the dispersion during evolution.

IV. MODEL

We consider the communication line studied in Ref. [13]. It consists of the two-qubit sender $S$ (the first and second nodes), the two-qubit receiver $R$ (the 41st and 42nd nodes) and the two-qubit ancilla (the 39th and 40th nodes). The ancilla and receiver form the four-node extended receiver $ER$, which is connected to the sender through the transmission line $TL$. The spin dynamics is governed by the XX-Hamiltonian with the dipole-dipole interaction

$$H = \sum_{j>i} D_{ij}(I_{ix}I_{jx} + I_{iy}I_{jy}), \quad (25)$$

$$[H, I_z] = 0, \quad (26)$$
where $D_{ij} = \frac{\gamma^2}{r_{ij}^3}$ is the coupling constant between the $i$th and $j$th nodes, $\gamma$ is the gyromagnetic ratio, $\hbar$ is the Planck constant, $r_{ij}$ is the distance between the $i$th and $j$th nodes, $I_{i\alpha}$ ($\alpha = x, y, z$) is the projection operator of the $i$th spin on the $\alpha$ axis and $I_z = \sum_i I_{iz}$. We also consider the tensor-product initial state $1$ with an arbitrary $\rho^{(S)}(0)$, while the subsystem $TL \cup ER$ is in the state without excitations, i.e.,

$$
\rho^{(TL,ER)}(0) = \text{diag}(1, 0, \ldots).
$$

In addition, we consider the spin dynamics in the two-excitation subspace of the whole state space of a quantum system. Then eq. (7) with setting $N_3 = \{n_1 n_2\}$, $M_3 = \{m_1 m_2\}$, $I_1 = \{i_1 i_2\}$, $J_1 = \{j_1 j_2\}$,

$$
\{N^1 N^2\} = J_{N-2} = \{j_1 \ldots j_{N-2}\}, \quad \rho^{(TL,ER)}_{j_1 j_2 i_1 i_2} = \delta_{j_1 j_2} \delta_{i_1 i_2},
$$

gets the following form:

$$
T_{n_1 n_2 m_1 m_2; i_1 i_2 j_1 j_2} = \sum_{J_{N-2}} W_{J_{N-2} n_1 n_2; i_1 i_2} \rho_{0 N-2; J_{N-2} m_1 m_2}^+, \quad (29)
$$

where $0_{N-2}$ is the set of $N-2$ zeros. In this case $W_{J_{N-2} n_1 n_2; i_1 i_2}$ can be considered as the elements of the Kraus operators [16], because they satisfy the constraint

$$
\sum_{m_1, m_2} \sum_{J_{N-2}} W_{J_{N-2} m_1 m_2; i_1 i_2}^+ W_{J_{N-2} m_1 m_2; i_1 i_2} = \delta_{i_1 j_1} \delta_{i_2 j_2}, \quad (30)
$$

which follows from the definition (5) of the operator $W$.

In our protocol we use the expansion of the density matrices in the sums of the multiple-qubit (MQ) coherence matrices [17] which read in the two-qubit case as

$$
\rho^{(S)} = \sum_{k=-2}^2 \rho^{(S,k)}, \quad \rho^{(R)} = \sum_{k=-2}^2 \rho^{(R,k)}, \quad (31)
$$

where the MQ coherence matrices $\rho^{(S,k)}$ collect terms responsible for the state transfers changing the $z$-projection of the total spin momentum by $k$.

Follow Ref. [13], we introduce notation

$$
D_{i(i+1)} \equiv \delta_i, \quad i = 1, \ldots, N-1, \quad (32)
$$

and consider the chain of $N = 42$ nodes with the following coupling constants:

$$
\delta_k = \delta, \quad 3 \leq k \leq N-3, \quad \delta_1 = \delta_{N-1}, \quad \delta_2 = \delta_{N-2}. \quad (33)
$$
Here $\delta_1$, $\delta_2$ and $t$ are chosen to maximize the intensity of the second-order coherence. For $N = 42$ their optimal values read [13]:

$$\delta_1 = 0.3005\delta, \quad \delta_2 = 0.5311\delta, \quad \delta t = 58.9826. \quad (34)$$

V. NUMERICAL SIMULATION OF OPERATIONS WITH NONDIAGONAL ELEMENTS OF DENSITY MATRIX

In this section we simulate the operations with density matrix elements introduced in Sec[III]. According to eqs. (2) and (29), the general structure of the upper nondiagonal elements of the receiver’s density matrix is following:

$$\rho^{(R)}_{00;11} = d\rho^{(S)}_{00;11}, \quad (35)$$
$$\rho^{(R)}_{00;n_1n_2;01\rho^{(S)}_{00;01} + a_{n_1n_2;01}\rho^{(S)}_{00;10} + b_{n_1n_2;10}\rho^{(S)}_{01;11} + b_{n_1n_2;10}\rho^{(S)}_{01;11}, \quad n_1 + n_2 = 1, (36)$$
$$\rho^{(R)}_{n_1n_2;11} = c_{n_1n_2;01}\rho^{(S)}_{01;11} + c_{n_1n_2;10}\rho^{(S)}_{10;11}, \quad n_1 + n_2 = 1, (37)$$
$$\rho^{(R)}_{01;10} = \sum_{n_1+n_2=1} f_{n_1n_2;m_1m_2}\rho^{(S)}_{n_1n_2;n_1n_2} + f_{01;10}\rho^{(S)}_{01;10} + f_{10;01}\rho^{(S)}_{10;01} + f_{11;11}\rho^{(S)}_{11;11}, \quad (38)$$

where

$$d = W^+_{110N-2;0N-211}, \quad (39)$$
$$a_{n_1n_2;i_1i_2} = W^+_{i_1i_20N-2;0N-2n_1n_2}, \quad b_{n_1n_2;i_1i_2} = \sum_{|J_{N-2}|=1} W_{J_{N-2}00;ii_20N-2} W^+_{110N-2;J_{N-2}n_1n_2},$$
$$c_{n_1n_2;i_1i_2} = W_{0N-2;2n_1n_2;ii_20N-2} d = a_{n_1n_2;i_1i_2} d,$$
$$f_{n_1n_2;m_1m_2} = W_{0N-2;01;n_1n_20N-2} W^+_{m_1m_20N-2;0N-210} = a_{n_1n_2;01} a_{10;m_1m_2}, \quad n_1 + n_2 = m_1 + m_2 + 2 = 1,$$
$$f_{11;11} = \sum_{|J_{N-2}|=1} W_{J_{N-2}01;110N-2} W^+_{110N-2;J_{N-2}10}.$$

A. Partial structural restoring of initial sender’s state

In view of the structure of matrix elements (35)-(38), system (10)-(12) for the structural restoring of the upper nondiagonal elements [13] now reads

$$a_{n_1n_2;n_1n_2} = b_{n_1n_2;m_1m_2} = c_{n_1n_2;n_1n_2} = f_{n_1n_2;n_1n_2} = f_{10;01} = f_{11;11} = 0, \quad (40)$$
$$n_1 + n_2 = m_1 + m_2 = 1,$$
which must be solved for the $\varphi$-parameters. This system is satisfied if

$$W_{0N-2n_1n_2} = 0, \quad n_1 + n_2 = 1,$$

$$\sum_{|J_{N-2}|=1} W_{J_{N-2}0m_1m_2} W^+_{110N-2;J_{N-2}n_1n_2} = 0, \quad m_1 + m_2 = n_1 + n_2 = 1,$$

$$\sum_{|J_{N-2}|=1} W_{J_{N-2}0110N-2} W^+_{110N-2;J_{N-1}10} = 0, \quad n_1 + n_2 = 1.$$  (41)  (42)  (43)

Then the restored system reads

$$\rho^{(R)}_{00;11} = d \rho^{(S)}_{00;11},$$  (44)

$$\rho^{(R)}_{00;1n_2} = a_{n_1n_2;11} \rho^{(S)}_{00;1n_2}, \quad n_1 + n_2 = 1,$$  (45)

$$\rho^{(R)}_{n_1n_2;11} = c_{n_1n_2;11} \rho^{(S)}_{n_1n_2;11}, \quad n_1 + n_2 = 1,$$  (46)

$$\rho^{(R)}_{01;10} = f_{01;10} \rho^{(S)}_{01;10}.$$  (47)

The structural restoring was considered in [13], so that we do not discuss it hereafter.

**B. Turning matrix elements to zero**

It is interesting to notice that not any single element of the restored non-diagonal part of $\rho^{(R)}$ can be turned to zero in our model, because turning to zero one of the elements in (44) - (47) implies vanishing of some others. We consider three variants of this operation.

1. The coefficients $a_{01;10}, c_{01;10}$ and $f_{01;10}$ have the common factor $W^+_{010N-2;0N-201}$ or its conjugate $W_{0N-201;001N-2}$. Therefore, if we choose the $\varphi$-parameters in eqs. (44)-(47) such that

$$W_{0N-201;010N-2} = 0,$$  (48)

then

$$\rho^{(R)}_{00;01} = \rho^{(R)}_{01;11} = \rho^{(R)}_{01;10} = 0.$$  (49)

2. Similarly, the coefficients $a_{10;10}, c_{10;10}$ and $f_{01;10}$ have the common factor $W^+_{100N-2;0N-210}$ or its conjugate $W_{0N-210;100N-2}$. Therefore, if the $\varphi$-parameters in (44)-(47) are such that

$$W_{0N-210;100N-2} = 0,$$  (50)
then

\[ \rho_{00;10}^{(R)} = \rho_{10;11}^{(R)} = \rho_{01;10}^{(R)} = 0. \]  

(51)

3. The coefficients \( c_{01;01} \) and \( c_{10;10} \) are proportional to \( d = W_{110N-2:0N-211}^+ \). Therefore, if the \( \phi \)-parameters in (44)-(47) are such that

\[ W_{110N-2:0N-211}^+ = 0, \]  

(52)

then

\[ \rho_{01;11}^{(R)} = \rho_{10;11}^{(R)} = \rho_{00;11}^{(R)} = 0. \]  

(53)

Thus, in the numerical simulations of operations (49), (51) and (53), we have to solve eqs. (41) - (43) and one of equations, respectively, (48), (50) or (52) for the \( \phi \)-parameters. The solution of this system is not unique. Therefore, to reveal the optimal one, we find 1000 different solutions and choose one corresponding to the maximal sum of the absolute values of the scale factors in the non-diagonal elements of the receiver’s density matrix. Results of numerical simulation of these transformations are collected in Table I, the first three lines. The list of the \( \phi \)-parameters of the unitary transformation corresponding to the reduction (49) is given in Appendix, the first line in Table II.

|   | \( \rho_{01;10}^{(R)} \) | \( \rho_{00;01}^{(R)} \) | \( \rho_{00;10}^{(R)} \) | \( \rho_{01;11}^{(R)} \) | \( \rho_{10;11}^{(R)} \) | \( \rho_{00;11}^{(R)} \) |
|---|------------------|------------------|------------------|------------------|------------------|------------------|
| 0 | 0                | 0                | 0.9182e\(-1.5135\) | 0                | 0.4393e\(1.1232\) | 0.4784e\(-0.3903\) |
| 0 | 0.6993e\(-1.4322\) | 0                | 0.4129e\(0.9917\) | 0                | 0.5904e\(-1.3405\) | 0                |
| 0.3464e\(-2.9264\) | 0.3846e\(-2.2924\) | 0.9009e\(0.0040\) | 0.2210e\(1.3317\) | 0.4095e\(1.6726\) | 0.0987e\(0.3835\) | 0.4464e\(1.7153\) |
| 0.2028e\(1.2800\) | 0.9175e\(0.0427\) | 0.3846e\\(\) | 0.2210e\(1.3317\) | 0.4095e\(1.6726\) | 0.0987e\(0.3835\) | 0.4464e\(1.7153\) |

TABLE I: Results of numerical simulation of turning to zero some elements (rows 1-3) and rearranging of the matrix elements (the 4th row)

C. Rearrangement of matrix elements

We consider the rearrangement of the elements \( \rho_{00;01}^{(S)} \) and \( \rho_{00;10}^{(S)} \), which implies equations

\[ \rho_{00;01}^{(R)} = a_{01;10}\rho_{00;10}^{(S)}; \quad \rho_{00;10}^{(R)} = a_{10;01}\rho_{00;01}^{(S)}. \]  

(54)
instead of (45). Therefore we must replace $a_{n_1n_2;n_2n_1}$ with $a_{n_1n_2;n_1n_2}$ in system (40), which implies

$$W_{0_{N-2}n_1n_2} = 0$$

(55) instead of (41). Consequently, the elements $\rho_{01;11}^{(S)}$ and $\rho_{10;11}^{(S)}$ interchange as well, i.e., we have

$$\rho_{01;11}^{(R)} = c_{01;10} \rho_{10;11}^{(S)}, \quad \rho_{10;11}^{(R)} = c_{10;01} \rho_{01;11}^{(S)}$$

(56) instead of (46).

The interchange between the pairs $\rho_{00;01}^{(S)}$, $\rho_{00;10}^{(S)}$ and $\rho_{01;11}^{(S)}$, $\rho_{10;11}^{(S)}$ is impossible because of the structure of the appropriate elements of $\rho^{(R)}$ (see eqs. (36) and (37), there are no elements $\rho_{00;01}^{(S)}$ and $\rho_{00;10}^{(S)}$ in the linear combination (37)).

Thus, we have to solve system (42), (43) and (55) for the $\varphi$-parameters. Again, the solution is not unique. Therefore, we find 1000 different solutions and choose one corresponding to the maximal sum of the absolute values of the scale factors in the non-diagonal elements of the density matrix.

Results of numerical simulation of this rearrangement are represented in Table II the last line. The list of the $\varphi$-parameters of the unitary transformation corresponding to this rearrangement is given in Appendix, the second line in Table II.

D. Linear combinations of matrix elements

The structure of matrix elements (35) - (38) shows that, among the elements of the first order coherence matrix, only $\rho_{00;01}^{(R)}$ and $\rho_{00;10}^{(R)}$ can be two independent linear combinations of all the elements of $\rho^{(S;1)}$ with arbitrary coefficients. Also the coefficient in $\rho_{00;11}^{(R)}$ (second-order coherence matrix) is independent on the coefficients in $\rho_{00;01}^{(R)}$ and $\rho_{00;10}^{(R)}$. In the element $\rho_{01;10}^{(R)}$, only the coefficient $f_{11;11}$ is independent on other coefficients in $\rho_{00;01}^{(R)}$ and $\rho_{00;10}^{(R)}$, therefore $\rho_{01;10}^{(R)}$ can not be used for creating additional combination of the elements of $\rho^{(S;0)}$ with independent coefficients. Thus, we can arrange the following independent linear combinations:

$$\rho_{00;01}^{(R)} = \alpha_1 \rho_{00;01}^{(S)} + \alpha_2 \rho_{00;01}^{(S)} + \alpha_3 \rho_{01;11}^{(S)} + \alpha_4 \rho_{10;11}^{(S)}$$

(57)

$$\rho_{00;10}^{(R)} = \beta_1 \rho_{00;01}^{(S)} + \beta_2 \rho_{00;01}^{(S)} + \beta_3 \rho_{01;11}^{(S)} + \beta_4 \rho_{10;11}^{(S)}$$

(58)

$$\rho_{00;11}^{(R)} = \gamma \rho_{00;11}^{(S)}$$

(59)
where $\alpha_i$, $\beta_i$ and $\gamma$ are arbitrary (but such that $\rho^{(R)}$ remains the non-negatively definite matrix with unit trace) scalar coefficients.

In particular, if

$$
\rho^{(S)}_{00;01} = x, \quad \rho^{(S)}_{00;10} = x^2, \quad \rho^{(S)}_{01;11} = x^3, \quad \rho^{(S)}_{10;11} = x^4, \quad \rho^{(S)}_{00;11} = x^5,
$$

the above combinations represent the power series in $x$. For instance, let

$$
\alpha_1 = \alpha, \quad \alpha_2 = \frac{\alpha}{2}, \quad \alpha_3 = \frac{\alpha}{3!}, \quad \alpha_4 = \frac{\alpha}{4!}, \quad \beta_1 = -\alpha, \quad \beta_2 = \frac{\alpha}{2}, \quad \beta_3 = -\frac{\alpha}{3!}, \quad \beta_4 = \frac{\alpha}{4!}, \quad \gamma = \frac{\alpha}{5!},
$$

then we have the power expansions of $\alpha(e^x - 1)$ and $\alpha(e^{-x} - 1)$ in the entries $\rho^{(R)}_{00;01}$ and $\rho^{(R)}_{00;10}$ respectively, while the entry $\rho^{(R)}_{00;10}$ yields a fifth-power term in the expansion of $\alpha(e^x - 1)$.

In the numerical simulation we find 1000 different solutions of system (61) and choose one that maximizes $|\alpha|$, which yields $\alpha_{\text{max}} = 0.5399$. The list of the $\varphi$-parameters of the unitary transformation corresponding to this manipulation is given in Appendix, the third line in Table II.

**E. Solving system of linear algebraic equations**

We show that the $2 \times 2$ linear algebraic system (19)

$$
\begin{pmatrix}
 a_{11} & a_{12} \\
 a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
 x_1 \\
 x_2
\end{pmatrix}
=
\begin{pmatrix}
 b_1 \\
 b_2
\end{pmatrix}
$$

(62)

can be solved using the tool of unitary transformations at the extended receiver. Let

$$
\rho^{(S)}_{01;11} = \rho^{(S)}_{10;11} = 0,
$$

(63)

$$
\rho^{(S)}_{00;01} = b_1, \quad \rho^{(S)}_{00;10} = b_2.
$$

We also assume that

$$
\rho^{(S)}_{00;01} = a_{11}x_1 + a_{12}x_2, \quad \rho^{(S)}_{00;10} = a_{21}x_1 + a_{22}x_2.
$$

(64)

After evolution with the operator $W$ we have

$$
\rho^{(R)}_{00;01} = \alpha_{11}x_1 + \alpha_{12}x_2,
$$

(65)

$$
\rho^{(R)}_{00;10} = \alpha_{21}x_1 + \alpha_{22}x_2.
$$
where \( \alpha_{ij} \) depend on the \( \varphi \)-parameters and on the parameters \( a_{ij} \) of linear system (62). We choose the \( \varphi \)-parameters to satisfy the conditions

\[
\alpha_{12} = \alpha_{21} = 0, \quad \alpha_{11} = \alpha_{22} = c = \text{const.}
\]  
(66)

Then system (65) gets the following form

\[
\begin{align*}
\rho^{(R)}_{00;01} & = cx_1, \\
\rho^{(R)}_{00;10} & = cx_2.
\end{align*}
\]  
(67)

Thus, the elements \( \rho^{(R)}_{00;01} \) and \( \rho^{(R)}_{00;10} \) of the first-order coherence matrix represent solution of the original system (62) up to the factor \( c \).

As an example we take \( a_{11} = 0.4, a_{12} = 0.3, a_{21} = 0.6, a_{22} = 0.2 \) and solve the system (66) for the \( \varphi \)-parameters. Again, the solution is not unique. We find that the maximal value of \( c \) for which system (66) has a solution is \( c_{\text{max}} = 0.1094 \). The list of \( \varphi \)-parameters of the unitary transformation corresponding to this operation is given in Appendix, the 4th line in Table II.

VI. CONCLUSIONS

The unitary transformations at the receiver side can be an effective tool not only for structural restoring of the transferred state (as shown in Ref. [13]) but also for realizing a set of other manipulations with elements of the receiver’s density matrix. Being larger than the receiver, the extended receiver admits larger number of free parameters in the applied unitary transformation. This is another benefit of the extended receiver which first was used to improve the characteristics of the communication line in the protocol of remote state creation [18]. It is important that our protocols do not use the classical channel for information transfer because they are based on the quantum state transfer rather than teleportation.

Of a special interest is a protocol for solving linear systems of algebraic equations, which is based on mixed states and defers from the HHL algorithm [2]. The importance of such algorithms is justified by the wide application area of linear systems in various branches of physics and mathematics.

We shall notice that certain restrictions in manipulating with matrix elements in Sec. III (for instance, only triplets of matrix elements can be turned to zero rather than any particular
element, restricted possibilities for rearranging the matrix elements) are associated with the initial state of the transmission line and receiver. Using a different initial state (for instance, the thermal equilibrium one) we might extend possibilities of such manipulations.

This work is partially supported by the program of the Presidium of RAS No. 5 "Electron resonance, spin-dependent electron effects and spin technology" and by the Russian Foundation for Basic Research (Grants No.16-03-00056).

VII. APPENDIX: UNITARY TRANSFORMATION AS A TOOL FOR MANIPULATING WITH MATRIX ELEMENTS

For the matrix representation of the unitary transformation $U^{(ER)}$ and density matrix $\rho^{(ER)}$ we use the basis of eigenstates of $I_z^{(ER)}$, 

$$|0000\rangle, |0001\rangle, |0010\rangle, |0011\rangle, |0100\rangle, |0101\rangle, |1000\rangle, |1001\rangle, |1010\rangle, |1100\rangle,$$

and write the basis of the Lie algebra associated with this unitary transformation taking into account commutativity $[6]$. This basis consists of 42 non-diagonal elements $\gamma^{(1;ij)}$, $\gamma^{(2;ij)}$, $j > i$ (diagonal elements are not useful in our transformations) with the following non-zero elements $[13]$:

$$\gamma^{(1;ij)}_{ij} = \gamma^{(1;ji)}_{ji} = 1, \quad \gamma^{(2;ij)}_{ij} = -\gamma^{(1;ji)}_{ji} = -i, \quad (68)$$

$$(i, j) \in \{(2, 3), (2, 5), (2, 8), (3, 5), (3, 8), (4, 6), (4, 7), (4, 9), (4, 10), (4, 11), (5, 8), \}$$

$$(6, 7), (6, 9), (6, 10), (6, 11), (7, 9), (7, 10), (7, 11), (9, 10), (9, 11), (10, 11}\}.$$

The unitary operator $U^{(ER)}$ gets the form

$$U^{(ER)} = \prod_{n=1}^{11} \prod_{m>n} e^{i\varphi^{(2)}_{n,m} \gamma^{(2;nm)}} e^{i\varphi^{(1)}_{n,m} \gamma^{(1;nm)}}, \quad (70)$$

where $\varphi^{(k)}_{n,m}$ are scalar parameters and the product is ordered in such a way that $n$ and $m$ increase from the right to the left.

In Table $[\text{II}]$ we represent the families of $\varphi$-parameters of the unitary transformations corresponding to the manipulations with matrix elements discussed in Secs. $[\text{V B - V E}]$ The first line in this table corresponds to transformation $[49]$. The second line corresponds to rearrangements $[54], [56]$. The third line corresponds to the linear combination $[57] - [61]$.  

16
TABLE II: Families of the $\varphi$-parameters of unitary transformation (70) corresponding to the manipulations with the receiver’s density matrix elements discussed in Secs. V B - V E.

The lines 1 – 4 correspond, respectively, to the transformation (49), rearrangements (54) and (56), linear combination of elements (57) - (61) and solving the system of two linear algebraic equations (62).

The parameters of the unitary transformation solving the system of two linear algebraic equations (62) are collected in the fourth line of this table.

| # | $\varphi_{2,3}$ | $\varphi_{3,5}$ | $\varphi_{3,8}$ | $\varphi_{4,6}$ | $\varphi_{4,7}$ | $\varphi_{4,9}$ | $\varphi_{4,10}$ | $\varphi_{5,8}$ | $\varphi_{6,7}$ | $\varphi_{6,9}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.2871 | 1.2703 | 5.3837 | 0.0157 | 4.4081 | 4.3842 | 1.4712 | 0.7807 | 5.7079 | 5.7107 | 5.0750 |
| 2 | 6.2612 | 1.3199 | 4.6848 | 2.7895 | 3.1939 | 4.9799 | 1.0863 | 3.7417 | 0.3117 | 5.7079 | 5.7107 | 5.0750 |
| 3 | 1.4280 | 1.1782 | 3.5900 | 0.3906 | 0.8475 | 1.1926 | 5.9163 | 5.5803 | 4.1273 | 5.7371 | 5.7107 | 5.0750 |
| 4 | 3.7020 | 3.5217 | 3.9665 | 5.0063 | 2.3526 | 5.2017 | 1.0258 | 0.3117 | 1.0826 | 5.9441 | 0.9333 | 2.0108 |

| # | $\varphi_{2,3}$ | $\varphi_{3,5}$ | $\varphi_{3,8}$ | $\varphi_{4,6}$ | $\varphi_{4,7}$ | $\varphi_{4,9}$ | $\varphi_{4,10}$ | $\varphi_{5,8}$ | $\varphi_{6,7}$ | $\varphi_{6,9}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1.5038 | 6.3390 | 3.6119 | 3.4403 | 1.5498 | 1.6824 | 1.7554 | 4.9749 | 3.0129 | 1.6850 | 0.8209 | 3.3091 | 4.7280 |
| 2 | 3.2009 | 4.7175 | 4.7844 | 4.2364 | 4.4368 | 2.3704 | 4.7478 | 3.8223 | 0.0666 | 4.3057 | 2.4175 | 3.7802 | 5.0880 |
| 3 | 3.5958 | 4.7751 | 3.3132 | 5.7055 | 5.2464 | 5.7296 | 3.2206 | 6.1773 | 2.7911 | 1.8675 | 1.1603 | 0.2330 | 2.7525 |
| 4 | 5.0184 | 3.4904 | 3.8684 | 5.7569 | 6.2019 | 4.5391 | 3.6655 | 3.9450 | 0.5175 | 6.2573 | 5.9570 | 3.0617 | 2.9621 |

| # | $\varphi_{6,10}$ | $\varphi_{6,11}$ | $\varphi_{7,9}$ | $\varphi_{7,10}$ | $\varphi_{9,10}$ | $\varphi_{10,11}$ |
|---|---|---|---|---|---|---|
| 1 | 4.2237 | 5.9065 | 2.8883 | 2.5863 | 5.3478 | 0.8597 | 4.3707 | 1.4844 |
| 2 | 3.7829 | 3.0648 | 0.4342 | 0.1331 | 1.6031 | 5.1080 | 4.3414 | 4.7175 |
| 3 | 2.3509 | 0.2806 | 2.1732 | 2.5927 | 0.7668 | 2.1064 | 0.4130 | 5.7157 |
| 4 | 3.9155 | 4.5563 | 3.5955 | 3.1836 | 4.6770 | 2.1579 | 4.3653 | 1.0752 |

| # | $\varphi_{6,10}$ | $\varphi_{6,11}$ | $\varphi_{7,9}$ | $\varphi_{7,10}$ | $\varphi_{9,10}$ | $\varphi_{10,11}$ |
|---|---|---|---|---|---|---|
| 1 | 4.2920 | 2.5717 | 0.7667 | 0.2932 | 5.0036 | 5.2766 | 6.1909 | 2.5412 |
| 2 | 5.0943 | 0.0743 | 1.8950 | 0.8286 | 5.3777 | 2.7487 | 1.9213 | 2.3468 |
| 3 | 5.8947 | 1.2263 | 0.1978 | 2.4723 | 3.5912 | 1.2593 | 1.2938 | 2.9502 |
| 4 | 4.5605 | 2.3953 | 3.0425 | 2.0803 | 4.7866 | 4.6980 | 0.8416 | 4.1266 |
[1] A. Barenco, Ch. H. Bennett, R. Cleve, D. P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. A. Smolin, H. Weinfurter, Phys. Rev. A 52, 3457 (1995)

[2] A. W. Harrow, A. Hassidim, and S. Lloyd, Phys. Rev. A 103, 150502 (2009)

[3] X.-D. Cai, C. Weedbrook, Z.-E. Su, M.-C. Chen, M. Gu, M.-J. Zhu, L. Li, N.-L. Liu, Ch.-Y. Lu, and J.-W. Pan, Phys. Rev. Lett. 110, 230501 (2013)

[4] S. Barz, I. Kassal, M. Ringbauer, Y. O. Lipp, B. Daki, A. Aspuru-Guzik, and P. Walther, Sci. Rep. 4, 6115 (2014)

[5] N. Wiebe, D. Braun, and S. Lloyd, Phys. Rev. Lett. 109 (2012) 050505

[6] Y. Zheng, Ch. Song, M.-Ch. Chen, B. Xia, W. Liu, Q. Guo, L. Zhang, D. Xu, H. Deng, K. Huang, Y. Wu, Zh. Yan, D. Zheng, L. Lu, J.-W. Pan, H. Wang, Ch.-Y. Lu, and X. Zhu, Phys. Rev. Lett. 118, 210504 (2017)

[7] J. Biamonte, P. Wittek, N. Pancotti, P. Rebentrost, N. Wiebe, and S. Lloyd, Nature 549, 195 (2017)

[8] M. A. Nielsen and I. L. Chuang, Quantum computation and quantum information (Cambridge Univ. Press, 2000)

[9] S. F. Huelga, M. B. Plenio, G.-Y. Xiang, J. Li, and G.-C. Guo, J. Opt. B: Quantum Semiclass. Opt. 7 (2005) S384

[10] X. Qiang, X. Zhou, K. Aungskunsiri, H. Cable, and J. L O’Brien, Quantum Sci. Technol. 2 (2017) 045002

[11] D. Wang, L. Ye, Int. J. Theor. Phys. 53 (2014) 350

[12] E. B. Fel’dman, A. I. Zenchuk, JETP 125, 1042 (2017)

[13] A. I. Zenchuk, Phys. Lett. A 382 (2018) 3244

[14] A. Luis, J. Peñina, Phys. Rev. A 54, 4564 (1996)

[15] D. W. Berry, G. Ahokas, R. Cleve, B. C. Sanders, Commun. Math. Phys. 270, 359 (2007)

[16] K. Kraus, States, Effects and Operations (Spring-Verlag, Berlin, 1983)

[17] E. B. Fel’dman, S. Lacelle, Chem. Phys. Lett. 253, 27 (1996)

[18] G. Bochkin and A. Zenchuk, Qunt. Inf. Comp., 16 1349 (2016)