Mittag-Leffler Input Stability of Fractional Differential Equations and Its Applications

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Abstract. This paper addresses the Mittag-Leffler input stability of the fractional differential equations with exogenous inputs. We continue the first note. We discuss three properties of the Mittag-Leffler input stability: converging-input converging-state, bounded-input bounded-state, and Mittag-Leffler stability of the unforced fractional differential equation. We present the Lyapunov characterization of the Mittag-Leffler input stability, and conclude by introducing the fractional input stability for delay fractional differential equations, and we provide its Lyapunov-Krasovskii characterization. Several examples are treated to highlight the Mittag-Leffler input stability.

1. Introduction. Control systems are often affected by noise, expressed, for instance, as perturbations on controls and errors on observations [31]. Modeling using fractional dynamical systems is not trivial as many types of errors occur. In this study, we gather these errors in the exogenous input. The perturbation term could result from modeling errors, aging, uncertainties, and disturbances that exist in many realistic problems [4, 7, 8, 17, 19, 20]. The idea of the new stability problem introduced in this special issue includes two properties: converging-input converging-state and bounded-input bounded-state. The fractional input stability imposes that the norm of the state at each time be bounded by the sum of a function of the amplitude of the applied exogenous input, and a term proportional to the initial state norm tending to zero over time. The Mittag-Leffler input stability is a particular case of the fractional input stability. In other words, that imposes like the fractional input stability the norm of the state be bounded by the sum of a class \( K_\infty \) function of the amplitude of the applied exogenous input and a term proportional to a Mittag-Leffler function. The fractional Mittag-Leffler input stability offers three properties:

- if the exogenous input is zero, then the trivial solution of the unforced fractional differential equation is Mittag-Leffler stable.
- if the exogenous input is convergent, then the generated solution is convergent.
- if the input is bounded, then the generated solution is bounded as well.

2010 Mathematics Subject Classification. Primary: 26A33, 93D05; Secondary: 93D25.

Key words and phrases. Fractional derivative, fractional differential equations with exogenous inputs, Mittag-Leffler input stable.

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These three properties offer both the fractional input stability and the Mittag-Leffler input stability a good method to analyze the stability of the fractional differential equations with exogenous inputs. One of the objectives of this paper is to present the properties above. The fractional input stability can be applied in many fields, particularly in fractional neutral networks. The last property offers a significant advance in stability issues. In many studies in stability analysis, the authors proved the asymptotic stability of the trivial solution of unforced fractional differential equations. The Mittag-Leffler input stability of fractional differential equations offers a new method to study the Mittag-Leffler stability of the unforced fractional differential equations. It is noteworthy that the fractional input stability exists in the integer order. Moreover, the input-to-state stability was introduced by Sontag in 1989 [27, 30, 31]. We conclude this paper by introducing the fractional input stability of the fractional differential equations with time delay.

Recently, significant advancements related to the Mittag-Leffler functions and fractional differential equations in the literature have been reported: in [9], Goufo developed new results on Mittag-Leffler functions. In [11], Goufo and Atangana developed some results on fractional derivative operators; they proposed new modifications on the Riemann-Liouville fractional derivative, and some simulations and applications for the new Riemann-Liouville fractional derivative were presented. A new and practical fractional derivative operator called the Atangana-Baleanu fractional derivative was presented in [3]. In [13], Goufo et al. described the transition to turbulence by introducing some fractional models and used numerical approximations to reveal the existence of attractor points. Recently, in [10], Goufo proposed an application of the Caputo-Fabrizio operator to replicator-mutator dynamics: Bifurcation, chaotic limit cycles, and control. In [12], Goufo et al. studied the existence and uniqueness of the solution to the model of a seventh-order Korteweg-de Vries equation with one perturbation level; additionally, it was established and proven to be continuous.

In Section 2, we present the necessary definitions and primary results of this paper. We study the Mittag-Leffler input stability for a particular class of the fractional differential equations with exogenous inputs. In Section 3, we present numerical examples. In Section 4, we introduce the fractional input stability for delay fractional differential equations. In Section 5, we present our conclusions and remarks.

**Notation.** $\mathcal{PD}$ denotes the set of all continuous functions $\chi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $\chi(0) = 0$ and $\chi(s) > 0$ for all $s > 0$. A class $\mathcal{K}$ function is an increasing $\mathcal{PD}$ function. The class $\mathcal{K}_\infty$ denotes the set of all unbounded $\mathcal{K}$ functions. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if $\beta(., t) \in \mathcal{K}$ for any $t \geq 0$, and $\beta(s,.)$ is non-increasing and tends to zero as its arguments tends to infinity. Given $x \in \mathbb{R}^n$, $\|x\|$ stands for its Euclidean norm: $\|x\| := \sqrt{x_1^2 + \ldots + x_n^2}$. $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is the vector saturation function defined as $\sigma(x) = (\sigma^0(x_1), \sigma^0(x_2), ..., \sigma^0(x_n))^T$, where $\sigma^0(s) = \min \{1;|s|\} \text{sign}(s)$ for each $s \in \mathbb{R}$. For a matrix $A$, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximal and minimal eigenvalue of $A$, respectively. If the condition $Re(\lambda_i) < 0, \forall i = 1, 2, ..., n$, holds, then the matrix $A$ is said to be Hurwitz.

2. **Problem statements and main results.** We present some mathematical tools in this section. Consider the following fractional differential equations with exogenous inputs:

$$D_\alpha^c x = f(x, u)$$ (1)
where $x(.)$ is the state variable taking values in $\mathbb{R}^n$ and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ a continuous locally Lipschitz function satisfying the characteodory conditions. The controls or inputs are measurable locally and are bounded functions $u : \mathbb{R}^+ \to \mathbb{R}^m$. The solution of (1) starting at $x_0$ at time $t = t_0$ is denoted by $x(.) = x(. , x_0, u)$.

Let us recall the some definitions and lemmas. For more details about the Caputo fractional order derivative and fractional integral, refer to [5, 22, 23, 25].

**Definition 2.1.** Given a function $f : [a, +\infty[ \to \mathbb{R}$, then the Caputo fractional derivative of $f$ of order $\alpha$ is defined by

$$D^c_\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_t^\infty f'(s) (t-s)^{\alpha-1} ds$$

for all $t > a$, $\alpha \in (0,1)$ and $\Gamma(.)$ is a gamma function.

**Definition 2.2.** [16] Given a function $f : [a, +\infty[ \to \mathbb{R}$, then the Riemann-Liouville integral of $f$ of order $\alpha$ is defined by

$$I^RL_\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds$$

for all $t > a$, $\alpha \in (0,1)$ and $\Gamma(.)$ is a gamma function.

The generalization of the definition above was introduced in [16]. A generalized fractional derivative according to the works in [16] was proposed in the literature. For more information, see [14, 15]. Let us recall the definition of the Mittag-Leffler function.

**Definition 2.3.** [2] Let $\alpha > 0$, $\beta \in \mathbb{R}$, and $z \in \mathbb{C}$. The Mittag-Leffler function is defined by the series

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(k\alpha + \beta)}$$

for more new results on the Mittag-Leffler functions, refer to [9]. We recall the following definitions and lemmas, which we use later in the text.

**Definition 2.4.** [26] The function $g : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and locally Lipschitz with Lipschitz constant $L$, that is

$$\|g(t, x) - g(t, y)\| \leq L \| x - y \|.$$  

The following lemmas present conditions in which the fractional differential equations can be studied in the presence of the saturation function.

**Lemma 2.5.** [26] Given any scalar $\epsilon > 0$ and $u, v \in \mathbb{R}^n$, the following holds:

$$u^T v + v^T u \leq \epsilon^{-1} u^T u + \epsilon v^T v.$$  

**Lemma 2.6.** [6] Let $x(t) \in \mathbb{R}^n$ be a vector of differentiable functions. Then, for any time instant $t \geq 0$, the following relationship holds

$$D^c_\alpha (x(t)^T P x(t)) \leq 2x(t)^T P D^c_\alpha x(t) \quad \forall \alpha \in (0,1)$$

where $P \in \mathbb{R}^{n \times n}$ is a constant, square, symmetric, and positive definite matrix.

According to Lemma 2.6, we obtain the following lemma.

**Lemma 2.7.** Let there exist a constant, square, symmetric, and positive definite matrix $P \in \mathbb{R}^{n \times n}$ such that the matrix $A^T P + PA$ is negative definite; then, the matrix $A \in \mathbb{R}^{n \times n}$ satisfies the condition $|\arg(\lambda(A))| > \frac{\pi}{2}$ for all $\alpha \in (0,1)$. 

Proof. Consider the fractional differential equation associated to the matrix $A$ defined as follows: $D^\alpha_x = Ax$. The asymptotic stability for linear differential equation $D^\alpha_x = Ax$ can be established using the Lyapunov direct method. To visualize, we consider the Lyapunov function candidate $V(t,x) = x^T P x$ where $P$ is a positive symmetric matrix. The $\alpha$-derivative of the function $V$ along the trajectories of the fractional differential equation $D^\alpha_x = Ax$ is given by

$$D^\alpha_x V(t,x) \leq 2x^T PD^\alpha_x x = x^T (A^T P + PA)x,$$

from which we have the following relation:

$$D^\alpha_x V(t,x) \leq 2x^T PD^\alpha_x x = x^T (A^T P + PA)x \leq -\lambda_{\min}(P) \|x\|^2,$$

where $\lambda_{\min}(P)$ is the minimum eigenvalue of the matrix $P$ satisfying the condition $A^T P + PA = -Q$ as negative definite. Using Theorem 3.5 in [29], it follows that the fractional differential equation $D^\alpha_x = Ax$ is asymptotically stable. Furthermore, the solution of the fractional differential $D^\alpha_x = Ax$ is given by $x(t) = x_0 E_\alpha (\lambda(t - t_0)^\alpha)$. As shown, when the state matrix $A$ does not satisfy the condition $|\arg(\lambda(A))| > \frac{\alpha\pi}{2}$, then the solution $x(t)$ diverges when $t$ tends to infinity [24]. This contradicts the fact that the fractional differential equation $D^\alpha_x = Ax$ is asymptotically stable. Otherwise, we know that the asymptotic stability for the linear fractional differential equation $D^\alpha_x = Ax$ is equivalent using the Lyapunov direct method to the existence of a constant, square, symmetric, and positive definite matrix $P \in \mathbb{R}^{n \times n}$ such that the matrix $A^T P + PA$ is negative definite. Finally, when the condition $|\arg(\lambda(A))| > \frac{\alpha\pi}{2}$ does not hold, then the fractional differential equation $D^\alpha_x = Ax$ is not asymptotically stable. In other words, when the condition $|\arg(\lambda(A))| > \frac{\alpha\pi}{2}$ does not hold, then the assumption that “there exists a constant, square, symmetric, and positive definite matrix $P \in \mathbb{R}^{n \times n}$ such that the matrix $A^T P + PA$ is negative definite” does not hold. That ends the proof of Lemma 2.7. $\square$

We recall some definitions and properties related to the fractional differential equations with exogenous inputs.

**Definition 2.8.** [18] The origin of the unforced fractional differential equation defined by $D^\alpha_x = f(x,0)$ is said to be Mittag-Leffler stable, if for any initial condition $x_0$, its solution satisfies

$$\|x(t,0)\| \leq \left[ m(\|x_0\|) E_\alpha (\lambda(t - t_0)^\alpha) \right]^\frac{1}{\beta}$$

where $b > 0$, and $m$ is locally Lipschitz on a domain contained in $\mathbb{R}^n$ with a Lipschitz constant $K$, and satisfies $m(0) = 0$.

**Definition 2.9.** [28] The fractional differential equation defined by (1) is said to be fractional input stable if for any input $u \in \mathbb{R}^m$, there exists a class $K\mathcal{L}$ function $\beta$, and a class $K_{\infty}$ function $\gamma$ such that for any initial condition $x_0$, its solution satisfies

$$\|x(t,x_0,0)\| \leq \beta(\|x_0\|, t - t_0) + \gamma(\|u\|_\infty).$$

The fractional input stability imposes an asymptotic gain ($\gamma$) decay of the norm of the state up to a function of the $\|u\|_{\infty}$ of the signal. The fractional input stability considers three properties: the convergent-input convergent-state (CICS) and the bounded-input bounded-state (BIBS) properties, and the asymptotic stability of the fractional differential equation without input. It is noteworthy that if the $K\mathcal{L}$
function is in the form \( \beta(\|x_0\|, t - t_0) = [K \|x_0\| E_\alpha (\lambda(t - t_0)^\alpha)]^{1/\gamma} \), we obtain the Mittag-Leffler input stability defined as follows:

**Definition 2.10.** [28] The fractional differential equation defined by (1) is said to be Mittag-Leffler input stable if for any input \( u \in \mathbb{R}^m \), there exists a class \( K_\infty \) function \( \gamma \) such that for any initial condition \( x(t_0) \), its solution satisfies

\[
\|x(t, x_0, u)\| \leq [K \|x_0\| E_\alpha (\lambda(t - t_0)^\alpha)]^{1/\gamma} + \gamma(\|u\|_\infty).
\]

where \( K \) and \( b > 0 \) are nonnegative constants.

The function \( \gamma \) is \( K_\infty \); thus, \( \gamma(0) = 0 \). In other words, if the input \( u = 0 \), we recover the definition of the Mittag-Leffler stability. We conclude that the fractional Mittag-Leffler input stability of the fractional differential equation with exogenous input (1) implies the Mittag-Leffler stability of the trivial solution of the fractional differential equation without input \( D_\alpha x = f(x, 0) \).

Next, we highlight the BIBS and CICS properties. For more information, refer to the first note. Consider the fractional linear differential equation that is represented mathematically by the following form

\[
D_\alpha^\alpha x = Ax + Bu
\]

where \( x \in \mathbb{R}^n \) is the state variable, \( A \) is \( n \) matrix in \( \mathbb{R}^{n \times n} \), \( B \) is \( n \) matrix in \( \mathbb{R}^{n \times m} \), and \( u \in \mathbb{R}^m \) is the exogenous input. The solution of the fractional differential equation (10) is given by

\[
x(t) = x_0 E_\alpha (A(t - t_0)^\alpha) + \int_{t_0}^{t} (t - s)^{\alpha - 1} E_{\alpha, \alpha} (A(t - s)^\alpha) Bu(s) ds,
\]

from which it follows that

\[
\|x(t)\| \leq \|x_0\| \|E_\alpha (A(t - t_0)^\alpha)\| + \|B\| \|u\|_\infty \int_{t_0}^{t} \|(t - s)^{\alpha - 1} E_{\alpha, \alpha} (A(t - s)^\alpha)\| ds.
\]

If the state matrix \( A \) satisfies the condition \( |\text{arg}(\lambda(A))| > \frac{2\pi}{\Gamma} \), according to [24], there exists a positive constant \( M \) such that

\[
\int_{t_0}^{t} \|(t - s)^{\alpha - 1} E_{\alpha, \alpha} (A(t - s)^\alpha)\| ds \leq M
\]

Therefore, we have the following inequality

\[
\|x(t)\| \leq \|x_0\| \|E_\alpha (A(t - t_0)^\alpha)\| + \|B\| \|u\|_\infty M
\]

By the inequality in (11), we observe that the solution of (10) satisfies the inequality in (9) with

\[
\beta(\|x_0\|, t - t_0) = \|x_0\| \|E_\alpha (A(t - t_0)^\alpha)\| \quad \text{and} \quad \gamma(\|u\|_\infty) = \|B\| \|u\|_\infty M
\]

This proves the fractional differential equation (10) is Mittag-Leffler input stable.

- As shown, if we take a converging input \( u \) in (11), then the solution of the fractional differential equation converges. This is the CICS property.
- By the condition (11), we can observe that a bounded input generates a bounded state. This is the BIBS property. To visualize, we continue the majoration using the fact that \( \beta(\|x_0\|, t - t_0) \leq \beta(\|x_0\|, 0) \) and we obtain

\[
\|x(t, x_0, u)\| \leq \beta(\|x_0\|, 0) + \gamma(\|u\|_\infty).
\]
For a bounded input \( \|u\| \leq \epsilon \), we obtain \( \gamma (\|u\|) \leq \gamma (\epsilon) \). Thus, we obtain the following majoration

\[
\|x(t, x_0, u)\| \leq \beta (\|x_0\|, 0) + \gamma (\epsilon).
\]

- If \( u = 0 \) in (11), then the solution of the fractional differential equation (10) satisfies \( \|x(t)\| \leq \|x_0\| \|E_\alpha (A(t - t_0)^\alpha)\| \). It follows that the trivial solution of \( D_\alpha^\alpha x = Ax \) is Mittag-Leffler stable.

It is noteworthy that this previous example has motivated the introduction of the fractional input stability in the stability problems for fractional differential equations with exogenous inputs in the first note [28].

We now present the Lyapunov characterization of the Mittag-Leffler input stability of the fractional differential equations with exogenous inputs. Several examples are provided to illustrate the main result.

**Theorem 2.11.** If there exists a positive function \( V : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R} \) that is continuous and differentiable, a \( K_\infty \) function \( \chi_1, \chi_2 \), and class \( K \) function \( \chi \), satisfying the following conditions

1. \( \chi_1 (\|x\|) = \|x\|^a \leq V(t, x) \leq \chi_2 (\|x\|) \)
2. \( V(t, x) \) has Caputo fractional derivative of order \( \alpha \) for all \( t > t_0 \geq 0 \)
3. If for any \( \|x\| \geq \chi_4 (\|u\|) \) \( \implies D_\alpha^\alpha V(t, x) \leq -kV(t, x) \)

where \( a > 0 \). Then, the fractional differential equations (1) are Mittag-Leffler input stable.

**Proof of Theorem 2.11.** Let the set \( S = \{ \eta : V(\eta) \leq c = \chi_2 \circ \chi_4 (\|u\|) \} \). The use of the fractional differential comparison lemma is fundamental in the first part of the proof. We make two claims. The proof of claim 1 can be found in the first note [28]. The proof of the first claim is inspired by [30].

**Claim 1.** If there exists \( t_0 \geq 0 \) such that \( x_0 \in S \), then \( x(t) \in S \) for all \( t \geq t_0 \).

**Proof.** Suppose that the claim does not apply. Then, there exists \( \epsilon > 0 \) such that \( V(t) > \epsilon + c \). We consider \( \tau = \inf \{ t \geq t_0 : V(t) \geq \epsilon + c \} \). Subsequently, \( \|x(\tau)\| \geq \chi_4 (\|u\|) \), from which we obtain under assumption (3) \( D_\alpha^\alpha V(t_0) < 0 \) for all \( t \in [t_0, \tau] \). Using the fractional differential comparison Lemma [28], we have \( V(t) \geq V(\tau) \). We notice that this contradicts the minimality of \( \tau \). Thus, \( x(t) \in S \) for all \( t \geq t_0 \).

Let \( t_1 = \inf \{ t \geq 0 : x(t) \in S \} \); according to the reasoning above, it follows that \( V(t) \leq \chi_2 \circ \chi_4 (\|u\|) \) for all \( t \geq t_1 \). Under the first assumption, we have \( \chi_1 (\|x\|) \leq V(t) \leq \chi_2 \circ \chi_4 (\|u\|) \), from which we obtain

\[
\|x(t)\| \leq \chi_1^{-1} \circ \chi_2 \circ \chi_4 (\|u\|) \quad (12)
\]

Let \( \gamma = \chi_1^{-1} \circ \chi_2 \circ \chi_4 \); according to the comparison function, it is a \( K_\infty \) function.

**Claim 2.** There exists a \( KL \) function \( \beta \) such that for each \( x_0 \) and each bounded control \( u \), there exists a time instant \( T > 0 \) (necessarily \( T = t_1 \)) such that \( \|x(t)\| \leq \beta (\|x_0\|, t - t_0) \) for all \( t \leq t_1 \) and \( x(t) \in S \) for all \( t \geq t_1 \).

**Proof.** For all \( t \leq t_1 \), \( x(t) \) is clearly not in the set \( S \), from which it follows that \( \|x(t)\| \geq \chi_4 (\|u\|) \); additionally, under assumption (3), we have \( D_\alpha^\alpha V(t) \leq -kV(t) \), from which we have

\[
V(t) = V(t_0) E_\alpha (-k(t - t_0)^\alpha) - \int_{t_0}^{t} (t - s)^{\alpha - 1} E_{\alpha, \alpha} (-k(t - s)^\alpha) m(s) \, ds
\]
where \( m \) is a positive and continuous function, see in [25]. Subsequently, we obtain \( V(t) \leq V(t_0) E_\alpha (-k(t-t_0)^\alpha) \) by neglecting the second term. Now, from the inequality in assumption (1), we obtain for all \( t \leq t_1 \) that

\[
\|x(t)\| \leq \{ V(t_0) E_\alpha (-k(t-t_0)^\alpha) \}^{\frac{1}{\alpha}} = \beta (\|x(t_0)\|, t-t_0)
\]

Superposing the solutions (12) and (13), we obtain for the following equality for all \( t \geq t_0 \):

\[
\|x(t)\| \leq \{ V(t_0) E_\alpha (-k(t-t_0)^\alpha) \}^{\frac{1}{\alpha}} + \gamma (\|u\|)
\]

This ends the proof of Theorem 2.11. \( \square \)

In Theorem 2.11, if we choose the input \( u = 0 \), we recover the Lyapunov characterization of the Mittag-Leffler stability. In other words, the Mittag-Leffler input stability implies the Mittag-Leffler stability of the trivial solution of the unforced fractional differential equation defined by \( D_\alpha^c x = f(x,0) \).

3. Numerical examples and applications. We now illustrate Theorem 2.11.

We establish the Mittag-Leffler input stability of the fractional bilinear differential equation expressed by

\[
D_\alpha^c x = \left( A + \sum_{i=1}^m u_i A_i \right) x + Bu
\]

where \( x \in \mathbb{R}^n \) is a state variable, the matrix \( A \in \mathbb{R}^{n \times n} \) is Hurwitz or satisfies the classical condition \( |\arg(\lambda(A))| > \frac{\alpha \pi}{2} \), the matrix \( A_i \in \mathbb{R}^{n \times m} \) for all \( i = 1, 2, ..., m \), and the input vector \( u = (u_1, u_2, ..., u_m)^T \in \mathbb{R}^m \).

- First, a particular fractional differential equation that we obtain with (14) is defined by

\[
D_\alpha^c x = -x + xu
\]

where \( C \in \mathbb{R}^{n \times n} \) and \( u \in \mathbb{R} \). We prove the fractional differential equation (15) is not in general Mittag-Leffler input stable. Therefore, we construct a counterexample.

It is straightforward that if the exogenous input is a constant, then the matrix \( A + Cu \) does not satisfy the condition \( |\arg(\lambda(A + Cu))| > \frac{\alpha \pi}{2} \).

To illustrate this, we consider the fractional bilinear differential equations defined by \( D_\alpha^c x = -x + xu \). If we choose \( u = 2 \), we obtain the fractional differential equation defined by \( D_\alpha^c x = x \) in which the solution is given by \( x(t) = x_0 E_\alpha ((t - t_0)^\alpha) \). All trajectories starting at \( x_0 = 1 \) diverge. Thus, the fractional differential equation is not BIBS. Therefore, the fractional differential fractional \( D_\alpha^c x = -x + xu \) is not Mittag-Leffler input stable. We thus conclude that the fractional bilinear differential equation with exogenous input (15) is not Mittag-Leffler input stable.

- Another particular fractional differential equation that we obtain with (14) is defined by

\[
D_\alpha^c x = Ax + Bu
\]

where \( A \in \mathbb{R}^{n \times n} \) is Hurwitz, and the matrix \( B \in \mathbb{R}^{n \times n} \). The equation (16) is already studied in the previous section.

To conclude, we illustrate Theorem 2.11 by studying the Mittag-Leffler input stability of the fractional bilinear differential equations defined by (14). Therefore,
we consider the Lyapunov function defined by \( V(t, x) = x^T P x = \| x \|^2_p \). The \( \alpha \)-derivative of the Lyapunov function candidate along the trajectories is given by

\[
D^\alpha_x V(t, x) \leq 2x^T PT_\alpha x = \left[ \left( A + \sum_{i=1}^{m} u_i A_i \right) x + Bu \right]^T P x + x^T P \left[ \left( A + \sum_{i=1}^{m} u_i A_i \right) x + Bu \right] = x^T (A^T P + PA) x + x^T \left( \sum_{i=1}^{m} u_i A_i \right)^T P + P \left( \sum_{i=1}^{m} u_i A_i \right) x + (Bu)^T P x + x^T P(Bu)
\]

We know that there exists a positive constant \( c \) such that

\[
\left| \left( \sum_{i=1}^{m} u_i A_i \right)^T P + P \left( \sum_{i=1}^{m} u_i A_i \right) \right| \leq \lambda_{max}(P) c \| u \| \| x \|
\]

Then, the \( \alpha \)-derivative of the Lyapunov candidate function yields

\[
D^\alpha_x V(t, x) \leq 2x^T PT_\alpha x \leq -\lambda_{min}(Q) \| x \|^2 + (2 \| B \| + c) \lambda_{max}(P) \| u \| \| x \|
\]

We choose any \( \theta \in (0, 1) \) and set

\[
k = \frac{(2 \| B \| + c) \lambda_{max}(P)}{\lambda_{min}(Q) - \theta} \quad \chi(x) = k r.
\]

Thus, if \( \| x \| \geq \chi(\| u \|) \), it implies that \( D^\alpha_x V(t, x) \leq -\theta \| x \|^2_p = -\theta V(t, x) \). Finally, we conclude with Theorem 2.11 that the fractional bilinear differential equation defined by (14) is Mittag-Leffler input stable.

- In this section, we study a particular class of the fractional differential equations with exogenous inputs. We consider the fractional differential equation defined by

\[
D^\alpha_x x = Ax + \psi(t, x) + Bu
\]

where the matrix \( A \in \mathbb{R}^{n \times n} \) satisfies \( |\arg(\lambda(A))| > \frac{\alpha \pi}{2} \), the perturbation term \( \psi : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n \) is locally Lipschitz and satisfies in particular the condition \( \psi(t, 0) = 0 \), the matrix \( B \in \mathbb{R}^{n \times m} \), and \( u \in \mathbb{R}^m \) represents the exogenous input.

- We first investigate the unforced differential equation obtained if the exogenous input is null \( u = 0 \). The unforced fractional differential equation that we obtain is expressed by

\[
D^\alpha_x x = Ax + \psi(t, x)
\]

As is known, the solution of the fractional differential equation (18) exists and is unique. Because \( \psi(t, 0) = 0 \), \( x = 0 \) is an equilibrium point of the fractional differential equation (18). The asymptotic stability conditions for the fractional linear equations (18) were already treated in the literature when the perturbation term \( \psi \) was not a saturation function, see [24]. Further, the asymptotic stability of (18) was obtained under the conditions \( |\arg(\lambda(A))| > \frac{\alpha \pi}{2} \) and \( \| \psi(t, x) \| \leq \gamma \| x \| \), where \( \gamma \) is a positive small constant.

An interesting question in this section is as follows: how can the stability of the perturbed fractional differential equation (17) be studied when \( \psi \) is a saturation function. The classical method consists of obtaining the solution of equation
(17). However, this method appears difficult to be applied in our study. A natural alternative is to use the Lyapunov direct method.

If the linear fractional differential equation without an input is perturbed by a saturation function, we have the following theorem.

**Theorem 3.1.** Let the perturbation function be of the form \( \psi(t, x) = C \sigma(Kx) \), where \( C \in \mathbb{R}^{n \times n} \) is a real constant matrix. If there exists a positive definite and symmetric matrix \( P \), and a positive scalar \( \epsilon \) such that the matrix

\[
A^T P + PA + \epsilon^{-1} PCC^T P + \epsilon L^2_\sigma I_n < 0,
\]

then the trivial solution of the perturbed fractional differential equation (18) is asymptotically stable.

**Proof of Theorem 3.1.** Consider the Lyapunov candidate function defined by \( V(t, x) = x^T Px \). The \( \alpha \) derivative of \( V \) along the trajectories yields

\[
D_\alpha^\alpha V(t, x) \leq (Ax + C\sigma(Kx))^T Px + x^T (A^T P + PA)x + \sigma^T(Kx) C^T Px + \epsilon^{-1} x^T PCC^T P x + \epsilon L^2_\sigma x^T I_n x
\]

Using Lemma 2.5, we obtain

\[
D_\alpha^\alpha V(t, x) \leq x^T (A^T P + PA)x + \epsilon^{-1} x^T PCC^T P x + \epsilon \sigma^T(Kx) \sigma(Kx)
\]

Because \( \sigma^T(Kx) \sigma(Kx) - L^2_\sigma x^T I_n x \leq 0 \) as \( \sigma \) is locally Lipchitz, we have

\[
D_\alpha^\alpha V(t, x) \leq x^T (A^T P + PA)x + \epsilon^{-1} x^T PCC^T P x + \epsilon L^2_\sigma x^T I_n x
\]

Thus, to obtain the asymptotic stability of the fractional differential equation (18), we impose that the matrix \( A^T P + PA + \epsilon^{-1} PCC^T P + \epsilon L^2_\sigma X_n \) is negative definite.

- To end this section, we investigate the Mittag-Leffler input stability of a particular class of the fractional differential equation, with the exogenous input defined by

\[
D_\alpha^\alpha x = Ax + C\sigma(Kx) + Bu.
\]

We know that the Mittag-Leffler input stability implies, in particular, the Mittag-Leffler stability of the trivial solution of the unforced fractional differential equation. In other words, when the fractional differential (20) is Mittag-Leffler input stable, then the origin of the fractional differential equation \( D_\alpha^\alpha x = Ax + C\sigma(Kx) \) becomes globally asymptotically stable. The fractional input stability presents many other consequences, thus rendering this stability notion highly practical. Therefore, we establish the following theorem.

**Theorem 3.2.** If there exists a Lyapunov candidate function \( V(t, x) = x^T Px = \|x\|^2_P \) with a positive definite symmetric matrix \( P \) and a positive scalar \( \epsilon \) such that

\[
\|x\| \geq \chi_4(||u||) \implies D_\alpha^\alpha V(t, x) \leq - (1 - \theta) \lambda_{\text{min}}(R) \|x\|^2
\]

where \( R = A^T P + PA + \epsilon^{-1} PCC^T P + \epsilon L^2_\sigma I_n < 0 \) and \( \chi_4(s) = \frac{2\lambda_{\text{max}}(P) \|B||u\|}{\lambda_{\text{min}}(R) - \theta} \) where \( \theta \in (0, 1) \), then the fractional differential equation (20) is Mittag-Leffler input stable.
Proof of Theorem 3.2. Let the Lyapunov candidate function be defined by \( V(t,x) = x^T P x \). The \( \alpha \) derivative of \( V \) along the trajectories yields

\[
D_\alpha^c V(t,x) \leq (Ax + C\sigma(Kx) + Bu)^T P x + x^T P (Ax + C\sigma(Kx) + Bu) \\
\leq x^T (A^T P + PA) x + \sigma^T(Kx) C^T P x + x^T PC\sigma(Kx) + (Bu)^T P x + x^T P (Bu)
\]

Thus, we can observe that if \( \epsilon \in (0,1) \) and set

\[
\begin{align*}
\lambda_{max}(P) & = \frac{2\lambda_{max}(P) B}{\lambda_{min}(R)-\theta} \\
\chi_4(r) & = kr.
\end{align*}
\]

We consider the matrix \( R = A^T P + PA + \epsilon^{-1} PCC^T P + \epsilon L^2 \sigma^T I_n \) to be negative definite. Thus, using the maximal and minimal eigenvalues, we obtain

\[
D_\alpha^c V(t,x) \leq -\lambda_{min}(R) \|x\|^2 + 2\lambda_{max}(P) \|B\| \|u\| \|x\|
\]

We choose any \( \theta \in (0,1) \) and set

\[
k = \frac{2\lambda_{max}(P) B}{\lambda_{min}(R)-\theta}
\]

Thus, we can observe that if \( \|x\| \geq \chi_4(\|u\|) \), we have \( D_\alpha^c V(t,x) \leq -\theta \|x\|^2 \). Then, the fractional differential equation (20) is Mittag-Leffler input stable.

Theorem 3.2 provides a sufficient characterization for the Mittag-Leffler input stability of the fractional differential equations with exogenous inputs in the presence of the saturation functions. It follows that if \( u = 0 \), then the trivial solution of the fractional differential equation \( D_\alpha^c x = Ax + C\sigma(Kx) \) is globally asymptotically stable. In other words, the fractional differential equation with input \( D_\alpha^c x = Ax + C\sigma(u) \) can be stabilized under feedback \( u = Kx \).

We present a numerical example to illustrate Theorem 3.2. Let the fractional differential equation in the presence of a saturation function be defined by

\[
\begin{align*}
D_\alpha^c x_1 & = -x_1 + \sigma^0(x_2) \\
D_\alpha^c x_2 & = -x_2 + u
\end{align*}
\]

where \( x = (x_1, x_2) \in \mathbb{R}^2 \), and \( u \in \mathbb{R} \) is the exogenous input. The objective of this example is to prove the unforced fractional differential equation expressed by

\[
\begin{align*}
D_\alpha^c x_1 & = -x_1 + \sigma^0(x_2) \\
D_\alpha^c x_2 & = -x_2
\end{align*}
\]

is Mittag-Leffler stable. Therefore, we will use the property stipulating that the Mittag-Leffler input stability of (21) implies the Mittag-Leffler stable of the trivial solution of the fractional differential without input (22). The technique adopted in this study is a good method to prove the Mittag-Leffler stability. In our study, we consider the Lyapunov candidate function defined by \( V(t,x) = \frac{1}{2} (x_1^2 + x_2^2) = \frac{1}{2} \|x\|^2 \). The \( \alpha \) derivative of the Lyapunov function candidate along the trajectories yields

\[
D_\alpha^c V(t,x) \leq -x_1^2 + x_1 \sigma^0(x_2) - x_2^2 + x_2 u \\
\leq -x_1^2 - x_2^2 + \frac{1}{2} x_1^2 + \frac{1}{2} \sigma^0(x_2) \sigma^0(x_2) + x_2 u
\]

The saturation function is locally Lipschitz \( (\sigma^0)^T(x_2) \sigma^0(x_2) - x_2^T x_2 < 0 \); therefore,

\[
D_\alpha^c V(t,x) \leq -\frac{1}{2} x_1^2 - \frac{1}{2} x_2^2 + x_2 u \\
\leq -2 \|x\|^2 + x_2 u
\]
We choose any $\theta \in (0,1)$ and set
\begin{equation*}
k = \frac{\|u\|}{2 - \theta} \quad \text{and} \quad \chi_4(r) = kr.
\end{equation*}
If $\|x\| \geq \chi_4(\|u\|)$, it implies that $D_\alpha^\gamma V(t,x) \leq -\theta \|x\|^2$. According to Theorem 2.11, it follows that the fractional differential equation in the presence of saturation function (21) is Mittag-Leffler input stable. Using the property that the Mittag-Leffler input stability implies the Mittag-Leffler stability of the trivial solution of the unforced fractional differential equation, the trivial solution of the fractional differential equation defined by (22) is Mittag-Leffler stable. The Mittag-Leffler input stability of the fractional differential equation (21) can be obtained by using Theorem 3.2. Hence, we consider the matrices
\begin{equation*}
A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad K = \begin{pmatrix} 1 & 0 \end{pmatrix}.
\end{equation*}
It is straightforward to verify by calculation, and we recover with Lyapunov candidate $V(t,x) = \frac{1}{2} \|x\|^2$ the following condition: $\|x\| \geq \frac{\|u\|}{2 - \theta}$ implies $D_\alpha^\gamma V(t,x) \leq -\theta \|x\|^2$, under which we have the Mittag-Leffler input stability of the fractional differential equations (21). Another method to prove the Mittag-Leffler input stability using the Lyapunov direct method can be found in the first note [28].

4. Fractional input stability for fractional differential equation with time delay. Consider the following systems of fractional-order nonlinear dynamic systems:
\begin{equation}
D_\alpha^\gamma x(t) = f(t, x_t, u)
\end{equation}
where $x : \mathbb{R} \to \mathbb{R}^n$ is a function, $f : \mathbb{R} \times C \times F^m \to \mathbb{R}^n$ is continuous and Lipschitz on each bounded set of $\mathbb{R} \times C \times F^m$, and $u \in F^m$ represents the exogenous input. It is noteworthy that $x_t \in \mathbb{R}^n$ represents the state of this system: at each time instant $t$, it is a function of the space variable rather than a single point of $\mathbb{R}^n$. Further, for the set $C$ design, the set of the continuous functions maps to $[-\tau, 0]$ to $\mathbb{R}^n$, and the $C_\alpha$ design maps to the set of functions in $C$, such that $\psi \in C_\alpha$. Furthermore, $\|\psi\|_C \leq a$, with $a$ a positive real number. It is noteworthy that $x_{t_0} = x_0$ is the initial state at time $t_0$; in other words, $x(t_0 + \eta) = \psi(\eta)$, and the following norm will be used later $\|\psi\|_C = \max_{\eta \in [-\tau, 0]} \|\psi(\eta)\|$.

The contribution of this section is to develop a Lyapunov-Krasovskii theorem in the context of fractional order systems. We introduce the definition of the fractional input stability for time-delay fractional differential equations parallel to the one presented for nonlinear delay free systems in [21, 32, 33].

**Definition 4.1.** The fractional differential equation (23) is said to be fractional input stable if there exists a $KL$ function $\beta$, and a $K_\infty$ function $\gamma$ such that, for any initial state $\psi_0$ and any measurable, locally essentially bounded input $u$, the solution exists for all $t \geq t_0 \geq 0$. Furthermore, it satisfies
\begin{equation}
\|x(t, t_0, \psi_0)\| \leq \beta(\|\psi_0\|_C, t - t_0) + \gamma(\|u\|_\infty).
\end{equation}

The function $\gamma$ is called the asymptotic gain. We notice that, in inequality (24), if we choose the converging input, because $\gamma$ is a $K_\infty$ function and $\beta$ is a $KL$ function, the generated solution is converging. Further, as described in the motivation section, we recover the CICS as well. Further, we obtain with inequality
Then the fractional differential equation (23) is fractional input stable with 

\( \gamma \)

and functions \( \chi_1, \chi_2 \) of class \( K_\infty \) such that

1. \( \chi_1(\|\psi(0)\|) \leq V(t, \psi) \leq \chi_2(\|\psi\|_C) \).
2. \( V(t, \psi, u) \) has a Caputo fractional derivative of order \( \alpha \) for all \( t > t_0 \geq 0 \).
3. If for any \( \|\psi\|_C \geq \chi_2(\|u\|) \) then \( D_0^\alpha V(t, \psi, u) \leq -\chi_3(\|\psi\|_C) \).

then the fractional differential equation (23) is fractional input stable with \( \gamma = \chi_1^{-1} \circ \chi_2 \circ \chi_4 \).

It is noteworthy that Theorem 4.2 provides the sufficient condition for the fractional input stability of fractional differential equation (23) with time delay.

**Proof of Theorem 4.2**

Let \( v(t) = V(t, x_t) \) be a locally and absolutely continuous function in this context.

Let the set \( S = \{ \eta : v(\eta) \leq c = \chi_2 \circ \chi_4(\|u\|) \} \). We make two claims.

**Claim 1.** If there exists \( t_0 \geq 0 \) such that \( \|\psi_0\|_C \in S \), then \( x(t) \in S \) for all \( t \geq t_0 \).

**Proof.** Suppose the above does not apply. Then, there exists \( \epsilon > 0 \) such that \( v(t) > \epsilon + c \). Consider \( \tau = \inf\{ t \geq t_0 : v(t) > \epsilon + c \} \). Subsequently, it follows that \( \|\psi(\tau)\|_C \geq \chi_4(\|u\|) \), from which we obtain under the assumption (3) \( D_0^\alpha v(t)_{t=\tau} \leq -\chi_2 \circ \chi_3(\|u\|)_{t=\tau} < 0 \) for all \( t \in [t_0, \tau] \). Using the fractional differential comparison lemma, we have \( v(t) \geq v(\tau) \). We notice that this contradicts the minimality of \( \tau \).

Thus, \( x(t) \in S \) for all \( t \geq t_0 \). We now let \( t_1 = \inf\{ t \geq 0 : x(t) = S \} \). According to the reasoning above, it follows that \( v(t) \leq \chi_2 \circ \chi_4(\|u\|) \) for all \( t \geq t_1 \). Under the first assumption of the theorem, we obtain

\[
\|x(t)\|_C \leq \chi_1^{-1} \circ \chi_2 \circ \chi_4(\|u\|) \tag{25}
\]

Let \( \gamma = \chi_1^{-1} \circ \chi_2 \circ \chi_4 \); according to the comparison functions, it is clear that a \( K_\infty \) function exists.

**Claim 2.** There exists a KL function \( \beta \) such that, for each \( \|\psi(0)\|_C \) and each bounded control \( u \), there exists a time instant \( T > 0 \) (necessarily \( T = t_1 \)) such that \( \|x(t)\|_C \leq \beta (\|\psi_0\|_C, t - t_0) \) for all \( t \leq t_1 \) and \( x(t) \in S \) for all \( t \geq t_1 \).

**Proof.** For all \( t \leq t_1 \), \( x(t) \notin S \), implying \( \|\psi\|_C \geq \chi_4(\|u\|) \), and under assumption (3), we have \( D_0^\alpha v(t) \leq -\chi_3(\|u\|) = -g(v(t)) \). Further, let \( y(t) \) be the solution of the equation defined by \( D_0^\alpha y = -g(y) \), when \( y(t_0) \geq v(t_0) \) applies. By Sene in [29], there exist a class KL function \( \sigma \) such that

\[ v(t) \leq \sigma (v(t_0), t - t_0) \]

From the inequality in assumption (1), we obtain

\[ \|x(t)\| \leq \chi_1^{-1} \circ \sigma (v(t_0), t - t_0) \]
We let the function \( \beta(\|\psi_0\|_C, t-t_0)) = \chi_1^{-1} \circ \sigma(v(t_0), t-t_0) \), which is a KL function. We conclude that
\[
\|x(t)\|_C \leq \beta(\|\psi_0\|_C, t-t_0) \quad \text{for all} \quad t \in [t_0, t_1]
\]  
(26)
Subsequently, superposing the solutions in (25) and (26), we obtain
\[
\|x(t)\|_C \leq \beta(\|\psi_0\|_C, t-t_0) + \gamma(\|u\|_\infty)
\]
for all \( t \geq t_0 \), where the asymptotic gain is given by \( \gamma = \chi_1^{-1} \circ \chi_2 \circ \chi_4 \).

5. Conclusions and remarks. We herein discussed the Mittag-Leffler input stability of the fractional differential equations with exogenous inputs and used the Caputo fractional derivative. Further, we introduced the Lyapunov characterization of the Mittag-Leffler input stability and illustrated it through examples. We also discussed the asymptotic stability of fractional differential equations in the presence of saturation functions. The fractional input stability and Mittag-Leffler input stability are new stability problems to study the stability of fractional differential equations with exogenous inputs. Finally, we discussed the fractional input stability for delay fractional differential equations.

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Received August 2018; revised October 2018.

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