Second weight codewords of generalized Reed-Muller codes

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1 Introduction

In this paper, we want to characterize the second weight codewords of generalized Reed-Muller codes.

We first introduce some notations:

Let $p$ be a prime number, $n$ a positive integer, $q = p^n$ and $\mathbb{F}_q$ a finite field with $q$ elements.

If $m$ is a positive integer, we denote by $B_q^m$ the $\mathbb{F}_q$-algebra of the functions from $\mathbb{F}_q^m$ to $\mathbb{F}_q$ and by $\mathbb{F}_q[X_1, \ldots, X_m]$ the $\mathbb{F}_q$-algebra of polynomials in $m$ variables with coefficients in $\mathbb{F}_q$.

We consider the morphism of $\mathbb{F}_q$-algebras $\varphi : \mathbb{F}_q[X_1, \ldots, X_m] \to B_q^m$ which associates to $P \in \mathbb{F}_q[X_1, \ldots, X_m]$ the function $f \in B_q^m$ such that

$$\forall x = (x_1, \ldots, x_m) \in \mathbb{F}_q^m, \ f(x) = P(x_1, \ldots, x_m).$$

The morphism $\varphi$ is onto and its kernel is the ideal generated by the polynomials $X_1^q - X_1, \ldots, X_m^q - X_m$. So, for each $f \in B_q^m$, there exists a unique polynomial $P \in \mathbb{F}_q[X_1, \ldots, X_m]$ such that the degree of $P$ in each variable is at most $q - 1$ and $\varphi(P) = f$. We say that $P$ is the reduced form of $f$ and we define the degree $\text{deg}(f)$ of $f$ as the degree of its reduced form. The support of $f$ is the set $\{ x \in \mathbb{F}_q^m : f(x) \neq 0 \}$ and we denote by $|f|$ the cardinal of its support (by identifying canonically $B_q^m$ and $\mathbb{F}_q^m$, $|f|$ is actually the Hamming weight of $f$).

For $0 \leq r \leq m(q-1)$, the $r$th order generalized Reed-Muller code of length $q^m$ is

$$R_q(r, m) := \{ f \in B_q^m : \text{deg}(f) \leq r \}.$$ 

For $1 \leq r \leq m(q-1) - 2$, the automorphism group of generalized Reed-Muller codes $R_q(r, m)$ is the affine group of $\mathbb{F}_q^m$ (see [1]).

For more results on generalized Reed-Muller codes, we can see for example [6].

We are now able to give precisely some results about minimum weight codewords and second weight codewords:
We write \( r = t(q - 1) + s, \) \( 0 \leq t \leq m - 1, \) \( 0 \leq s \leq q - 2. \)

In [9], interpreting generalized Reed-Muller codes in terms of BCH codes, it is proved that the minimal weight of the generalized Reed-Muller code \( R_q(r, m) \) is \( (q - s)q^{m-t-1}. \)

The following theorem gives the minimum weight codewords of generalized Reed-Muller codes and is proved in [9] or [10].

**Theorem 1.1** Let \( r = t(q - 1) + s < m(q - 1), \) \( 0 \leq s \leq q - 2. \) The minimal weight codewords of \( R_q(r, m) \) are codewords of \( R_q(r, m) \) whose support is the union of \( (q - s) \) distinct parallel affine subspaces of codimension \( t + 1 \) included in an affine subspace of codimension \( t. \)

In [8], Geil proves that the second weight of generalized Reed-Muller codes 
\[ R_q((m - 1)(q - 1) + s, m), \] \( 1 \leq s \leq q - 2 \) is \( q - s + 1 \) and that the second weight of generalized Reed-Muller codes \( R_q(r, m), \) \( 2 \leq r < q \) is \( (q - r + 1)(q - 1)q^{m-2}. \)

The other cases can be found in the following theorem. Rolland proves all the cases such that \( s \neq 1 \) in [11]. The case where \( s = 1 \) has been proved by Bruen in [4] using methods of Erickson (see [7]):

**Theorem 1.2** For \( m \geq 3, \) \( q \geq 3 \) and \( q \leq r \leq (m - 1)(q - 1) \) the second weight \( W_2 \) of the generalized Reed-Muller codes \( R_q(r, m) \) satisfies:

1. if \( 1 \leq t \leq m - 1 \) and \( s = 0, \)
   \[ W_2 = 2(q - 1)q^{m-t-1}; \]
2. if \( 1 \leq t \leq m - 2 \) and \( s = 1, \)
   (a) if \( q = 3, W_2 = 8 \times 3^{m-t-2}, \)
   (b) if \( q \geq 4, W_2 = q^{m-t}, \)
3. if \( 1 \leq t \leq m - 2 \) and \( 2 \leq s \leq q - 2, \)
   \[ W_2 = (q - s + 1)(q - 1)q^{m-t-2}. \]

In [3], Cherdieu and Rolland prove that the codewords of the second weight of \( R_q(s, m), \) \( 2 \leq s \leq q - 2, \) which are the product of \( s \) polynomials of degree 1 are of the following form.

**Theorem 1.3** Let \( m \geq 2, \) \( 2 \leq s \leq q - 2 \) and \( f \in R_q(s, m) \) such that \( |f| = (q - s + 1)(q - 1)q^{m-2}; \) we denote by \( S \) the support of \( f. \) Assume that \( f \) is the product of \( s \) polynomials of degree 1 then either \( S \) is the union of \( q - s + 1 \) parallel affine hyperplanes minus their intersection with an affine hyperplane which is not parallel or \( S \) is the union of \( (q - s + 1) \) affine hyperplanes which meet in a common affine subspace of codimension 2 minus this intersection.

In [12], Sboui proves that the only codewords of \( R_q(s, m), \) \( 2 \leq s \leq \frac{q}{2} \) whose weight is \( (q - s + 1)(q - 1)q^{m-2} \) are these codewords.

All the results proved in this paper are summarized in Section 2 and their proofs are in the following sections.
2 Results

In the following, except when an other affine space is specified, an hyperplane or a subspace is an affine hyperplane or an affine subspace of \( \mathbb{F}_q^m \).

2.1 Case where \( t = m - 1 \) and \( s \neq 0 \)

**Theorem 2.1** Let \( m \geq 2, q \geq 5, 1 \leq s \leq q - 4 \) and \( f \in R_q((m-1)(q-1)+s,m) \) such that \( |f| = q - s + 1 \). Then the support of \( f \) is included in a line.

**Proposition 2.2** Let \( m \geq 2 \). If \( q \geq 3 \) and \( f \in R_q((m-1)(q-1)+q-3,m) \) such that \( |f| = 4 \) or \( f \in R_q((m-1)(q-1+q-2,m) \) such that \( |f| = 3 \), then the support of \( f \) is included in an affine plane.

2.2 Case where \( 0 \leq t \leq m - 2 \) and \( 2 \leq s \leq q - 2 \)

**Theorem 2.3** Let \( q \geq 4, m \geq 2, 0 \leq t \leq m - 2, 2 \leq s \leq q - 2 \). The second weight codewords of \( R_q(t(q-1)+s,m) \) are codewords of \( R_q(t(q-1)+s,m) \) whose support \( S \) is included in an affine subspace of codimension \( t \) and either \( S \) is the union of \( q-s+1 \) parallel affine subspaces of codimension \( t+1 \) minus their intersection with an affine subspace of codimension \( t+1 \) which is not parallel or \( S \) is the union of \( (q-s+1) \) affine subspaces of codimension \( t+1 \) which meet in an affine subspace of codimension \( t+2 \) minus this intersection (see Figure [1]).

Figure 1: The possible support for a second weight codeword of \( R_4(5,3) \)

2.3 Case where \( s = 0 \)

**Theorem 2.4** Let \( m \geq 2, q \geq 3, 1 \leq t \leq m - 1 \). The second weight codewords of \( R_q(t(q-1),m) \) are codewords of \( R_q(t(q-1),m) \) whose support \( S \) is included in an affine subspace of codimension \( t-1 \) and either \( S \) is the union of \( 2 \) parallel affine subspaces of codimension \( t \) minus their intersection with an affine subspace of codimension \( t \) which is not parallel or \( S \) is the union of \( 2 \) non parallel affine subspaces of codimension \( t \) minus their intersection.
2.4 Case where $0 \leq t \leq m - 2$ and $s = 1$

**Theorem 2.5** For $q \geq 4$, $m \geq 1$, $0 \leq t \leq m - 1$, if $f \in R_q(t(q - 1) + 1, m)$ is such that $|f| = q^{m-t}$, the support of $f$ is an affine subspace of codimension $t$.

**Proposition 2.6** Let $m \geq 3$, $1 \leq t \leq m - 2$ and $f \in R_q(2t + 1, m)$ such that $|f| = 8.3^{m-t-2}$. We denote by $S$ the support of $f$. Then $S$ is included in $A$ an affine subspace of dimension $m-t+1$, $S$ is the union of two parallel hyperplanes of $A$ minus their intersection with two non parallel hyperplanes of $A$ (see Figure [2]).

![Figure 2: The support of a second weight codeword of $R_q(3,3)$](image)

3 Some tools

The following lemma and its corollary are proved in [6].

**Lemma 3.1** Let $m \geq 1$, $q \geq 2$, $f \in B_q^m$ and $a \in \mathbb{F}_q$. If for all $(x_2, \ldots, x_m)$ in $\mathbb{F}_q^{m-1}$, $f(a, x_2, \ldots, x_m) = 0$ then for all $(x_1, \ldots, x_m) \in \mathbb{F}_q^m$,

$$f(x_1, \ldots, x_m) = (x_1 - a)g(x_1, \ldots, x_m)$$

with $\deg_{x_1}(g) \leq \deg_{x_1}(f) - 1$.

**Corollary 3.2** Let $m \geq 1$, $q \geq 2$, $f \in B_q^m$ and $a \in \mathbb{F}_q$. If for all $(x_1, \ldots, x_m)$ in $\mathbb{F}_q^m$ such that $x_1 \neq a$, $f(x_1, \ldots, x_m) = 0$ then for all $(x_1, \ldots, x_m) \in \mathbb{F}_q^m$,

$$f(x_1, \ldots, x_m) = (1 - (x_1 - a)^q)(g(x_2, \ldots, x_m)).$$

**Lemma 3.3** Let $q \geq 3$, $m \geq 3$, and $S$ be a set of points of $\mathbb{F}_q^m$ such that $\#S = u.q^n < q^m$, with $u \neq 0 \mod q$. Assume that for all hyperplanes $H$ either $\#(S \cap H) = 0$ or $\#(S \cap H) = v.q^{n-1}$, $v < u$ or $\#(S \cap H) \geq u.q^{n-1}$ Then there exists $H$ an affine hyperplane such that $S$ does not meet $H$ or such that $\#(S \cap H) = vq^{n-1}$.

**Proof**: Assume that for all $H$ hyperplane, $S \cap H \neq \emptyset$ and $\#(S \cap H) \neq vq^{n-1}$. Consider an affine hyperplane $H$; then for all $H'$ hyperplane parallel to $H$, $\#(S \cap H') \geq u.q^{n-1}$. Since $u.q^n = \#S = \sum_{H' \parallel H} \#(S \cap H')$, we get that for all $H$ hyperplane, $\#(S \cap H) = u.q^{n-1}$.

Now consider $A$ an affine subspace of codimension 2 and the $(q+1)$ hyperplanes
through \( A \). These hyperplanes intersect only in \( A \) and their union is equal to \( \mathbb{F}_q^n \). So
\[
uq^n = \#S = (q + 1)u.q^{n-1} - q\#(S \cap A).
\]
Finally we get a contradiction if \( n = 1 \). Otherwise, \( \#(S \cap A) = u.q^{n-2} \). Iterating this argument, we get that for all \( A \) affine subspace of codimension \( k \leq n \), \( \#(S \cap A) = u.q^{n-k} \).

Let \( A \) be an affine subspace of codimension \( n + 1 \) and \( A' \) an affine subspace of codimension \( n - 1 \) containing \( A \). We consider the \( (q + 1) \) affine subspace of codimension \( n \) containing \( A \) and included in \( A' \), then
\[
uq = \#(S \cap A') = (q + 1)u - q\#(S \cap A)
\]
which is absurd since \( \#(S \cap A) \) is an integer and \( u \neq 0 \mod q \). So there exists \( H_0 \) an hyperplane such that \( \#(S \cap H_0) = eq^{n-1} \) or \( S \) does not meet \( H_0 \).

\[\square\]

4 Case where \( t = m - 1 \) and \( s \neq 0 \)

4.1 Proof of Theorem 2.1

Let \( \omega_1, \omega_2 \in S \) and \( H \) an affine hyperplane containing \( \omega_1 \) and \( \omega_2 \). Assume \( S \cap H \neq S \). We have \( \#S = q - s + 1 \leq q \) and \( \omega_1, \omega_2 \in S \cap H \), so there exists an affine hyperplane parallel to \( H \) which does not meet \( S \). By applying an affine transformation, we can assume that \( x_1 = 0 \) is an equation of \( H \) and we denote by \( H_a \) the affine hyperplane parallel to \( H \) of equation \( x_1 = a, a \in \mathbb{F}_q \). Let \( I := \{a \in \mathbb{F}_q : S \cap H_a = \emptyset \} \) and denote by \( k := \#I \); \( s \leq k \leq q - 2 \). Let \( c \notin I \), we define
\[
\forall x = (x_1, \ldots, x_m) \in \mathbb{F}_q^m, f_c(x) = f(x) \prod_{a \notin I, a \neq c} (x_1 - a)
\]
that is to say \( f_c \) is a function in \( B^c_n \) such that its support is \( S \cap H_c \). Since \( c \notin I \), \( f_c \) is not identically zero. Then \( |f| = \sum_{c \notin I} |f_c| \) and we consider two cases.

- Assume \( k > s \).
  Then the reduced form of \( f_c \) has degree at most \( (m - 1)(q - 1) + q - 1 + s - k \) and \( |f_c| \geq k - s + 1 \). Then,
  \[
  (q - s + 1) = |f| = \sum_{c \notin I} |f_c| \geq (q - k)(k - s + 1)
  \]
  which gives
  \[
  1 \geq (q - 1 - k)(k - s)
  \]
  this is possible if and only if \( k = q - 2 = s + 1 \) and we get a contradiction since \( s \leq q - 4 \).

- Assume that \( k = s \).
  Then \( S \) meets \( (q - s - 1) \) affine hyperplanes parallel to \( H \) in 1 point and \( H \) in 2 points. Consider the function \( g \) in \( B^c_n \) defined by
  \[
  \forall x = (x_1, \ldots, x_m) \in \mathbb{F}_q^m, g(x) = x_1 f(x).
  \]
The reduced form of $g$ has degree at most $(m-1)(q-1)+s+1$ and
\[ |g| = (q - s - 1). \]
So $g$ is a minimum weight codeword of $R_q((m-1)(q-1)+s+1,m)$ and its support is included in a line. This line is not included in $H$. So consider $H_1$ an affine hyperplane which contains this line but does not contain both $\omega_1$ and $\omega_2$. Then $S \cap H_1 \neq S$ and $H_1$ contains at least 3 points of $S$ since $s \leq q - 4$ which gives a contradiction by applying the previous argument to $H_1$.

So $S$ is included in all affine hyperplanes through $\omega_1$ and $\omega_2$ which gives the result.

4.2 Proof of Theorem 2.2

- If $f \in R_q((m-1)(q-1)+q-2,m)$ is such that $|f| = 3$, we have the result since 3 points are always included in an affine plane.

- Assume $f \in R_q((m-1)(q-1)+q-3,m)$ is such that $|f| = 4$. Let $a, b, c, d \in \mathbb{F}_q^*$ and $\omega^{(a)} = (\omega_1^{(a)}, \ldots, \omega_m^{(a)})$, $\omega^{(b)} = (\omega_1^{(b)}, \ldots, \omega_m^{(b)})$, $\omega^{(c)} = (\omega_1^{(c)}, \ldots, \omega_m^{(c)})$, $\omega^{(d)} = (\omega_1^{(d)}, \ldots, \omega_m^{(d)})$ 4 distinct points of $\mathbb{F}_q^m$ such that for all $x = (x_1, \ldots, x_m) \in \mathbb{F}_q^m$,
\[
f(x) = a \prod_{i=1}^m (1 - (x_i - \omega_i^{(a)})^{q-1}) + b \prod_{i=1}^m (1 - (x_i - \omega_i^{(b)})^{q-1}) + c \prod_{i=1}^m (1 - (x_i - \omega_i^{(c)})^{q-1}) + d \prod_{i=1}^m (1 - (x_i - \omega_i^{(d)})^{q-1}).
\]
So,
\[
f(x) = \begin{cases} (-1)^m(a + b + c + d) \prod_{i=1}^m x_i^{q-1} \\
+ (-1)^{m-1} \sum_{i=1}^m (a\omega_i^{(a)} + b\omega_i^{(b)} + c\omega_i^{(c)} + d\omega_i^{(d)})x_i^{q-2} \prod_{j \neq i} x_j^{q-1} + r \end{cases}
\]
with $\deg(r) \leq (m-1)(q-1)+q-3$. Since $f \in R_q((m-1)(q-1)+q-3,m)$,
\[
\begin{cases} a + b + c + d = 0 \\
a\omega^{(a)} + b\omega^{(b)} + c\omega^{(c)} + d\omega^{(d)} = 0 
\end{cases}
\]
So, $\omega^{(d)}\omega^{(a)} + b\omega^{(d)}\omega^{(b)} + c\omega^{(d)}\omega^{(c)} \rightarrow 0$ which gives the result.

Remark 4.1 In both cases we cannot prove that the support of $f$ is included in a line. Indeed,

- Let $\omega_1, \omega_2, \omega_3$ 3 points of $\mathbb{F}_q^m$ not included in a line. For $q \geq 3$ we can find $a, b \in \mathbb{F}_q^*$ such that $a + b \neq 0$. Let $f = a1_{\omega_1} + b1_{\omega_2} - (a + b)1_{\omega_3}$ where for $\omega \in \mathbb{F}_q^m$, $1_{\omega}$ is the function from $\mathbb{F}_q^m$ to $\mathbb{F}_q$ such that $1_{\omega}(\omega) = 1$ and $1_{\omega}(x) = 0$ for all $x \neq \omega$. Then, since $\sum_{x \in \mathbb{F}_q^m} f(x) = a + b - (a + b) = 0$, $f \in R_q((m-1)(q-1)+q-2,m)$. 

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• Let \( \omega_1, \omega_2, \omega_3 \) 3 points of \( \mathbb{F}_q^m \) not included in a line and set

\[ \omega_4 = \omega_1 + \omega_2 - \omega_3. \]

Then \( f = 1_{\omega_1} + 1_{\omega_2} - 1_{\omega_3} \leq 1_{\omega_4} \in R_q((m - 1)(q - 1)q - 3, m). \)

\section{Case where \( 0 \leq t \leq m - 2 \) and \( 2 \leq s \leq q - 2 \)}

\subsection{Case where \( t = 0 \)}

In this subsection, we write \( r = a(q-1) + b \) with \( 0 \leq a \leq m - 1 \) and \( 0 \leq b \leq q - 1 \).

\textbf{Lemma 5.1} Let \( q \geq 3, m \geq 2, 0 \leq a \leq m - 2, 2 \leq b \leq q - 1 \) and \( f \in R_q(a(q-1) + b, m) \) such that \( |f| = (q-b+1)(q-1)q^{m-a-2}; \) we denote by \( S \) the support of \( f \). If \( H \) is an affine hyperplane of \( \mathbb{F}_q^m \) such that \( S \cap H \neq \emptyset \) and \( S \cap H \neq \emptyset \) then either \( S \) meets \( q - 1 \) affine hyperplanes parallel to \( H \) or \( S \) meets \( q - 1 \) affine hyperplanes parallel to \( H \) in \( (q-b+1)q^{m-a-2} \) points or \( S \) meets \( q - 1 \) affine hyperplanes parallel to \( H \) in \( (q-b+1)q^{m-a-2} \) points.

\textbf{Proof:} By applying an affine transformation, we can assume that \( x_1 = 0 \) is an equation of \( H \) and consider the \( q \) affine hyperplanes \( H_w \) of equation \( x_1 = w, w \in \mathbb{F}_q \), parallel to \( H \). Let \( I := \{ w \in \mathbb{F}_q : S \cap H_w = \emptyset \} \) and denote by \( k := |I| \). Assume that \( k \geq 1 \). Since \( S \cap H \neq \emptyset \) and \( S \cap H \neq S, k \leq q - 2 \). For all \( c \in \mathbb{F}_q, c \not\in I \), we define

\[ \forall x = (x_1, \ldots, x_n) \in \mathbb{F}_q^m, \quad f_c(x) = f(x) \prod_{w \in \mathbb{F}_q, w \neq c, w \not\in I} (x_1 - w). \]

• Assume \( b < k \).

Then \( 2 \leq q - 1 + b - k \leq q - 2 \) and for all \( c \not\in I \), the reduced form of \( f_c \) has degree at most \( a(q-1) + q - 1 + b - k \). So \( |f_c| \geq (k-b+1)q^{m-a-2}. \)

Hence

\[ (q - 1)(q - b + 1)q^{m-a-2} \geq (q - k)(k - b + 1)q^{m-a-1} \]

which means that \( (b-k)q(q-k-1)+b-1 \geq 0. \) However \( (b-k) \leq -1 \) and \( q-k+1 \leq 1 \) so \( (b-k)q(q-k-1)+b-1 < 0 \) which gives a contradiction.

• Assume \( b \geq k \).

Then \( 0 \leq b-k \leq q-2 \) and for all \( c \not\in I \), the reduced form of \( f_c \) has degree at most \( (a+1)(q-1) + b - k \). So \( |f_c| \geq (q-b+k)q^{m-a-2}. \)

Hence

\[ (q - 1)(q - b + 1)q^{m-a-2} \geq (q - k)(q - b + k)q^{m-a-2} \]

with equality if and only if for all \( c \not\in I \), \( |f_c| = (q-b+k)q^{m-a-2}. \) Finally, we obtain that \( (k-1)(k-b+1) \geq 0 \) which is possible if and only if \( k = 1 \) or \( 1 \geq b-k \geq 0 \). Now, we have to show that \( k = s \) is impossible to prove the lemma. If \( b = q-1 \), since \( k \leq q - 2 \), we have the result. Assume that \( b \leq q - 2 \) and \( b = k \). Then, for all \( c \not\in I \), \( f_c \in R_q((a+1)(q-1), m) \). The minimum weight of \( R_q((a+1)(q-1), m) \) is \( q^{m-a-1} \) and its second weight is \( 2(q-1)q^{m-a-2} \). We denote by \( N_1 := \# \{ c \not\in I : |f_c| = q^{m-a-1} \} \). Since \( k = b, N_1 \leq q - b \). Furthermore, we have

\[ (q-b+1)(q-1)q^{m-a-2} \geq N_1 q^{m-a-1} + (q-b-N_1)(q-1)q^{m-a-2} \]
which means that $N_1 \geq \frac{(q-1)(q-b-1)}{q-2} > q-b-1$. Finally, $N_1 = q-b$ and for all $c \notin \mathcal{I}$, $|f_c| = q^{m-a-1}$. However $(q-1)(q-b+1)q^{m-a-2} > (q-b)q^{m-a-1}$ which gives a contradiction.

\[ \square \]

**Lemma 5.2** For $m = 2$, $q \geq 3$, $2 \leq b \leq q-1$. The second weight codewords of $R_q(b, 2)$ are codewords of $R_q(b, 2)$ whose support $S$ is the union of $q-b+1$ parallel lines minus their intersection with a line which is not parallel or $S$ is the union of $(q-b+1)$ lines which meet in a point minus this point.

**Proof**: To prove this lemma, we use some results on blocking sets proved by Erickson in [7] and Bruen in [4]. All these results are recalled in the Appendix of this paper. By Theorem 1.3 which is also true for $b = q-1$ (see [7, Lemma 3.12]), it is sufficient to prove that $f \in R_q(b, 2)$ such that $|f| = (q-b+1)(q-1)$ is the product of linear factors.

Let $f \in R_q(b, 2)$ such that $|f| \leq (q-b+1)(q-1) = q(q-b)+b-1$. We denote by $S$ its support. Then, $S$ is not a blocking set of order $(q-b)$ of $\mathbb{F}_q^2$ (Theorem 3.3) and $f$ has a linear factor (Lemma 3.2).

We proceed by induction on $b$. If $b = 2$ and $f \in R_q(b, 2)$ is such that $|f| \leq (q-b+1)(q-1)$, then $f$ has a linear factor and by Lemma 3.1 $f$ is the product of 2 linear factors. Assume that if $f \in R_q(b-1, 2)$ is such that $|f| \leq (q-b+2)(q-1)$ then $f$ is a product of linear factors. Let $f \in R_q(b, 2)$ such that $|f| \leq (q-b+1)(q-1)$; then $f$ has a linear factor. By applying an affine transformation, we can assume that for all $(x, y) \in \mathbb{F}_q^2$, $f(x, y) = y\tilde{f}(x, y)$ with $\deg(\tilde{f}) \leq b-1$. So, $L$ the line of equation $y = 0$ does not meet $S$ the support of $f$. Since $(q-b+1)(q-1) > q$, $S$ is not included in a line and by Lemma 5.1 either $S$ meets $(q-b+1)$ lines parallel to $L$ in $(q-1)$ points or $S$ meets $(q-1)$ lines parallel to $L$ in $(q-b+1)$ points.

In the first case, by Lemma 3.1 we can write for all $(x, y) \in \mathbb{F}_q^2$, $f(x, y) = y(y-a_1)\ldots(y-a_{q-2})g(x, y)$ where $a_i$, $1 \leq i \leq q-2$ are $q-2$ distinct elements of $\mathbb{F}_q$ and $\deg(g) \leq 1$ which gives the result.

In the second case, we denote by $a \in \mathbb{F}_q$ the coefficient of $x^{q-1}$ in $\tilde{f}$. Then for any $\lambda \in \mathbb{F}_q^*$, since $S$ meets all lines parallel to $L$ but $L$ in $q-s+1$ points, we get for all $x \in \mathbb{F}_q$, $f(x, \lambda) = a\lambda(x-a_1(\lambda))\ldots(x-a_{b-1}(\lambda))$

So there exists $a_1, \ldots, a_{b-1} \in \mathbb{F}_q[Y]$ of degree at most $q-1$ such that for all $(x, y) \in \mathbb{F}_q^2$, $f(x, y) = ay(x-a_1(y))\ldots(x-a_{b-1}(y))$.

Then for all $x \in \mathbb{F}_q$, $\tilde{f}_0(x) = \tilde{f}(x, 0) = a(x-a_1(0))\ldots(x-a_{b-1}(0))$ and $|\tilde{f}_0| \leq q-1$. So, $|\tilde{f}| \leq |f| + |\tilde{f}_0| \leq (q-b+2)(q-1)$.

By recursion hypothesis, $\tilde{f}$ is the product of linear factors which finishes the proof of Lemma 5.2.
Proposition 5.3 For $m \geq 2$, $q \geq 3$, $2 \leq b \leq q - 1$. The second weight codewords of $R_q(b, m)$ are codewords of $R_q(b, m)$ whose support $S$ is the union of $q - b + 1$ parallel hyperplanes minus their intersection with an affine hyperplane which is not parallel or $S$ is the union of $(q - b + 1)$ hyperplanes which meet in an affine subspace of codimension 2 minus this intersection.

Proof: We say that we are in configuration $A$ if $S$ is the union of $(q - b + 1)$ parallel hyperplanes minus their intersection with an affine hyperplane which is not parallel (see Figure 1a) and that we are in configuration $B$ if $S$ is the union of $(q - b + 1)$ hyperplanes which meet in an affine subspace of codimension 2 minus this intersection (see Figure 1b).

We prove this proposition by induction on $m$. The Lemma 5.2 proves the case where $m = 2$. Assume that $m \geq 3$ and that second weight codeword of $R_q(b, m - 1)$, $2 \leq b \leq q - 1$ are of type $A$ or type $B$. Let $f \in R_q(b, m)$ such that $|f| = (q - 1)(q - b + 1)q^{m-2}$ and we denote by $S$ its support.

1. Assume that $S$ meets all affine hyperplanes.

Then, by Lemma 5.3 there exists an affine hyperplane $H$ such that 

$\#(S \cap H) = (q - b)q^{m-2}$. By applying an affine transformation, we can assume that $x_1 = 0$ is an equation of $H$. We denote by $1_H$ the function in $B^q_m$ such that

$$\forall x = (x_1, \ldots, x_m) \in F^m_q, 1_H(x) = 1 - x_1^{q-1}$$

then the reduced form $f.1_H$ has degree at most $(t + 1)(q - 1) + s$ and the support of $f.1_H$ is $S \cap H$ so $S \cap H$ is the support of a minimal weight codeword of $R_q(q - 1 + b, m)$ and $S \cap H$ is the union of $(q - b)$ parallel affine subspaces of codimension 2. Consider $P$ an affine subspace of codimension 2 included in $H$ such that $\#(S \cap P) = (q - b)q^{m-3}$. Assume that there are at least 2 hyperplanes through $P$ which meet $S$ in $(q - b)q^{m-2}$ points. Then, there exists $H_1$ an affine hyperplane through $P$ different from $H$ such that $\#(S \cap H_1) = (q - b)q^{m-2}$. So, $S \cap H_1$ is the union of $(q - b)$ parallel affine subspaces of codimension 2. Consider $G$ an affine hyperplane which contains $Q$ an affine subspace of codimension 2 included in $H$ which does not meet $S$ and the affine subspace of codimension 2 included in $H_1$ which meets $Q$ but not $S$ (see Figure 3).

Figure 3

\[\text{Figure 3}\]
By applying an affine transformation, we can assume that $x_m = \lambda$, $\lambda \in \mathbb{F}_q$ is an equation of an hyperplane parallel to $G$. For all $\lambda \in \mathbb{F}_q$, we define $f_\lambda \in B_{m-1}^2$ by

$$\forall(x_1, \ldots, x_{m-1}) \in \mathbb{F}_q^{m-1}, \quad f_\lambda(x_1, \ldots, x_{m-1}) = f(x_1, \ldots, x_{m-1}, \lambda).$$

If all hyperplanes parallel to $G$ meets $S$ in $(q-b+1)(q-1)q^{m-3}$ then for all $\lambda \in \mathbb{F}_q$, $f_\lambda$ is a second weight codeword of $R_q(b, m-1)$ and its support is of type $A$ or $B$. We get a contradiction if we consider an hyperplane parallel to $G$ which meets $S \cap H$ and $S \cap H_1$. So, there exits $G_1$ an hyperplane parallel to $G$ which meets $S$ in $(q-b)q^{m-2}$ points and $S \cap G_1$ is the union of $(q-b)$ parallel affine subspaces of codimension 2 which is a contradiction. Then for all $H'$ hyperplane through $P$ different from $H$ $\#(S \cap H') \geq (q-1)(q-b+1)q^{m-3}$. Furthermore,

$$(q-b)q^{m-2} + q \cdot (q-1)(q-b+1)q^{m-3} - q \cdot (q-b)q^{m-3} = (q-1)(q-b+1)q^{m-2}.$$ 

Finally, by applying the same argument to all affine hyperplanes of codimension 2 included in $H$ parallel to $P$, we get $q$ parallel hyperplanes $(G_i)_{1 \leq i \leq q}$ such that for all $1 \leq i \leq q$, $\#(S \cap G_i) = (q-b+1)(q-1)q^{m-3}$ and $\#(S \cap G_i \cap H) = (q-b)q^{m-3}$. Then by recursion hypothesis, $S \cap G_i$ is either of type $A$ or of type $B$.

If there exists $i_0$ such that $S \cap G_{i_0}$ is of type $A$. Consider $F$ an affine hyperplane containing $R$ an affine subspace of codimension 2 included in $H$ which does not meet $S$ and the affine subspace of codimension 2 included in $G_{i_0}$ which does not meets $S$ but meets $R$. If for all $F'$ hyperplane parallel to $F$, $\#(S \cap F') > (q-b)q^{m-2}$ then $\#(S \cap F') = (q-1)(q-b+1)q^{m-3}$. So $S \cap F'$ is the support of a second weight codeword of $R_q(b, m-1)$ and is either of type $A$ or of type $B$ which is absurd is we consider an hyperplane parallel to $F$ which meets $S \cap H$. So there exits $F_1$ an affine hyperplane parallel to $F$ which meets $S$ in $(q-b)q^{m-2}$ points. So $S \cap F_1$ is the union of $(q-s)$ parallel affine subspaces of codimension 2 which is absurd since $S \cap G_{i_0}$ is of type $A$ (see Figure 4).

Figure 4

If for all $1 \leq i \leq q$, $S \cap G_i$ is of type $B$. Let $H_i$ be the affine hyperplane parallel to $H$ which contains the affine subspace of codimension 3 intersection of the affine subspaces of codimension 2 of $S \cap G_i$. We consider $R$ an affine subspace of codimension 2 included in $H$ which does not meet $S$. Then there is $(q-b+1)$ affine hyperplanes through $R$ which meet $S \cap G_1$.
in \((q - b)q^{m-3}\). However, if we denote by \(k\) the number of hyperplanes through \(R\) which meet \(S\) in \((q - b)q^{m-2}\) points, we have

\[
k(q - b)q^{m-2} + (q + 1 - k)(q - 1)(q - b + 1)q^{m-3} \leq (q - 1)(q - b + 1)q^{m-2}
\]

which implies that \(k \geq q - b + 2\). For all \(H'\) hyperplane through \(R\) such that \(\#(S \cap H') = (q - b)q^{m-2}\), \(S \cap H'\) is the union of \((q - b)\) affine subspaces of codimension 2 parallel to \(R\) and then \(\#(S \cap H' \cap G_1) = (q - b)q^{m-3}\) which is absurd (see Figure 5).

Figure 5

![Diagram](image)

- So, there exists \(H\) an affine hyperplane such that \(H\) does not meet \(S\).

Then, by Lemma 5.1 either \(S\) meets \((q - 1)\) hyperplanes parallel to \(H\) in \((q - b + 1)q^{m-2}\) points or \(S\) meets \((q - b + 1)\) hyperplanes parallel to \(H\) in \((q - 1)q^{m-2}\) points.

If \(S\) meets \((q - b + 1)\) hyperplanes parallel to \(H\) in \((q - 1)q^{m-2}\) points, then, for all \(H'\) hyperplane parallel to \(H\) such that \(S \cap H' \neq \emptyset\), \(S \cap H'\) is the support of a minimal weight codeword of \(R_q(q, m)\) and is the union of \((q - 1)\) parallel affine subspaces of codimension 2. Let \(H'\) be an affine hyperplane parallel to \(H\) such that \(S \cap H' \neq \emptyset\). We denote by \(P\) the affine subspace of codimension 2 of \(H'\) which does not meet \(S\). Consider \(H_1\) an affine hyperplane which contains \(P\) and a point not in \(S\) of an affine hyperplane \(H''\) parallel to \(H\) which meets \(S\). Then

\[
\#(H_1 \setminus S) \geq bq^{m-2} + 1.
\]

However, if \(S \cap H_1 \neq \emptyset\), \(\#(H_1 \setminus S) \leq bq^{m-2}\). So, \(S \cap H_1 = \emptyset\) and we are in configuration \(A\).

If \(S\) meets \((q - 1)\) hyperplanes parallel to \(H\) in \((q - b + 1)q^{m-2}\) points. Then for all \(H'\) parallel to \(H\) different from \(H\), \(S \cap H'\) is the support of a minimal weight codeword of \(R_q((q - 1) + b - 1, m)\) and is the union of \((q - b + 1)\) parallel affine subspaces of codimension 2. Let \(H_1\) be an affine hyperplane parallel to \(H\) different from \(H\) and consider \(P\) an affine subspace of codimension 2 included in \(H_1\) such that

\[
\#(S \cap P) = (q - b + 1)q^{m-3}.
\]

Assume that there exists \(H_2\) an affine hyperplane through \(P\) such that \(\#(S \cap H_2) = (q - b)q^{m-2}\). Then \(S \cap H_2\) is the support of a minimal
weight codeword of $R_q(q-1+b,m)$ and is the union of $(q-b)$ parallel affine subspaces of codimension 2 which is absurd since $S \cap H_2$ meets $H_1$ in $S \cap P$ (see Figure 6).

Figure 6

![Diagram of parallel hyperplanes](image)

Then, for all $H'$ through $P \#(S \cap H') \geq (q-1)(q-b+1)q^{m-3}$. Furthermore,

$$(q-b+1)q^{m-2} + q(q-1)(q-b+1)q^{m-3} - q(q-b+1)q^{m-2} = (q-1)(q-b+1)q^{m-2}. $$

So for all $H'$ hyperplane through $P$ different from $H_1$, 

$$\#(S \cap H') = (q-1)(q-b+1)q^{m-3}. $$

By applying the same argument to all affine subspaces of codimension 2 included in $H_1$ parallel to $P$, we get $q$ parallel hyperplanes $(G_i)_{1 \leq i \leq q}$ such that for all $1 \leq i \leq q$, $\#(S \cap G_i) = (q-b+1)(q-1)q^{m-3}$ and $\#(S \cap G_i \cap H_1) = (q-s+1)q^{m-3}$. By recursion hypothesis, for all $1 \leq i \leq q$, either $S \cap G_i$ is of type $A$ or $S \cap G_i$ is of type $B$.

Assume that there exists $i_0$ such that $S \cap G_{i_0}$ is of type $A$. Consider $F$ an affine hyperplane containing $Q$ an affine subspace of codimension 2 included in $H_1$ which does not meet $S$ and the affine subspace of codimension 2 included in $G_{i_0}$ which does not meets $S$ but meets $Q$. Assume that $S$ meets all hyperplanes parallel to $F$ in at least $(q-b)q^{m-2}$. If for all $F'$ parallel to $F$, $\#(S \cap F') > (q-b)q^{m-2}$ then

$$\#(S \cap F') \geq (q-1)(q-b+1)q^{m-3}. $$

So $S \cap F'$ is the support of a second weight codeword of $R_q(b,m-1)$ and is either of type $A$ or of type $B$ which is absurd we consider an hyperplane parallel to $F$ which meets $S \cap H_1$ and $S \cap G_{i_0}$. So, there exits $F_1$ an affine hyperplane parallel to $F$ such that $\#(S \cap F_1) = (q-b)q^{m-2}$. Then, $S \cap F_1$ is the union of $(q-b)$ parallel affine subspaces of codimension 2, which is absurd. Finally, there exists an affine hyperplane parallel to $F$ which does not meet $S$. By Lemma 5.1 either $S$ meets $(q-b+1)$ hyperplanes parallel to $F$ in $(q-1)q^{m-2}$ points and we have already seen that in this case $S$ is of type $A$ or $S$ meets $(q-1)$ hyperplanes parallel to $F$ in $(q-b+1)q^{m-2}$ points. In this case, for all $F'$ parallel to $F$ such that $S \cap F' \neq \emptyset$, $S \cap F'$ is the support of a minimal weight codeword of $R_q(q-1+b-1,m)$ and is the union of $q-b+1$ parallel affine subspaces of codimension 2, which is absurd since $S \cap G_{i_0}$ is of type $A$ (see Figure 7).
Now, assume that for all $1 \leq i \leq q$, $G_i \cap S$ is of type $B$. Let $Q$ be an affine subspace of codimension 2 included in $H_1$ which does not meet $S$. Assume that $S$ meets all affine hyperplanes through $Q$ and denote by $k$ the number of these hyperplanes which meet $S$ in $(q - b)q^{m-2}$ points. Then,

$$k(q - b)q^{m-2} + (q + 1 - k)(q - 1)(q - b + 1)q^{m-3} \leq (q - 1)(q - b + 1)q^{m-2}$$

which means that $k \geq q - b + 2$. These $(q - b + 2)$ hyperplanes are minimal weight codewords of $R_q(q - 1 + b, m)$. So, they meet $S$ in $(q - b)$ affine subspaces of codimension 2 parallel to $Q$, that is to say, they meet $S \cap G_1$ in $(q - b)q^{m-3}$ points. This is absurd since $S \cap G_1$ is of type $B$ and so there are at most $(q - b + 1)$ affine hyperplanes through $Q$ which meet $S \cap G_1$ in $(q - b)q^{m-3}$ points (see Figure 8). So there exists an affine hyperplane through $Q$ which does not meet $S$.

By applying the same argument to all affine subspaces of codimension 2 included in $H_1$ which does not meet $S$, since $S \cap G_i$ is of type $B$ for all $i$, we get that $S$ is of type $B$.

5.2 The support is included in an affine subspace of codimension $t$.

The two following lemmas are proved in [7].

**Lemma 5.4** Let $m \geq 2$, $q \geq 3$, $1 \leq t \leq m - 1$, $1 \leq s \leq q - 2$. Assume that $f \in R_q(t(q - 1) + s, m)$ is such that $\forall x = (x_1, \ldots, x_m) \in \mathbb{F}_q^m$,

$$f(x) = (1 - x_1^{q-1})\tilde{f}(x_2, \ldots, x_m)$$
and that $g \in R_q(t(q-1)+s-k), 1 \leq k \leq q-1$, is such that $(1-x_1^{q-1})$ does not divide $g$. Then, if $h=f+g$, either $|h| \geq (q-s+k)q^{m-t-1}$ or $k=1$.

**Lemma 5.5** Let $m \geq 2, q \geq 3, 1 \leq t \leq m-1, 1 \leq s \leq q-2$ and $f \in R_q(t(q-1)+s,m)$. For $a \in \mathbb{F}_q$, the function $f_a$ of $B_{m-1}^q$ defined for all $(x_2,\ldots,x_m) \in \mathbb{F}_q^m$ by $f_a(x_2,\ldots,x_m) = f(a,x_2,\ldots,x_m)$. Assume that for $a, b \in \mathbb{F}_q$ $f_a$ is different from the zero function and $(1-x_2^{q-1})$ divides $f_a$ and that

$$0 < |f_a| < (q-s+1)q^{m-t-2}.$$ 

Then there exists $T$ an affine transformation, fixing $x_i$ for $i \neq 2$ such that $(1-x_2^{q-1})$ divides $(f \circ T)_a$ and $(f \circ T)_b$.

**Lemma 5.6** Let $m \geq 3, q \geq 4, 1 \leq t \leq m-2$ and $2 \leq s \leq q-2$. If $f \in R_q(t(q-1)+s,m)$ is such that $|f| = (q-s+1)(q-1)q^{m-t-2}$, then the support of $f$ is included in an affine hyperplane of $\mathbb{F}_q^m$.

**Proof** : We denote by $S$ the support of $f$. Assume that $S$ is not included in an affine hyperplane. Then, by Lemma 5.5 there exists an affine hyperplane $H$ such that either $H$ does not meet $S$ or $H$ meets $S$ in $(q-s)q^{m-t-2}$. Now, by Lemma 5.6 since $S$ is not included in an affine hyperplane, either $S$ meets all affine hyperplanes parallel to $H$ or $S$ meets $(q-1)$ affine hyperplanes parallel to $H$ in $(q-s+1)q^{m-t-2}$ or $S$ meets $(q-s+1)$ affine hyperplanes parallel to $H$ in $(q-1)q^{m-t-2}$ points. By applying an affine transformation, we can assume that $x_1 = \lambda$, $\lambda \in \mathbb{F}_q$ is an equation of $H$. We define $f_{\lambda} \in B_{m-1}^q$ by

$$\forall (x_2,\ldots,x_m) \in \mathbb{F}_q^{m-1}, \quad f_{\lambda}(x_2,\ldots,x_m) = f(\lambda,x_2,\ldots,x_m).$$

We set an order $\lambda_1,\ldots,\lambda_q$ on the elements of $\mathbb{F}_q$ such that

$$|f_{\lambda_1}| \leq \ldots \leq |f_{\lambda_q}|.$$ 

Then either $|f_{\lambda_1}| = 0$ or $|f_{\lambda_1}| = (q-s)q^{m-t-2}$, that is to say either $f_{\lambda_1}$ is null or $f_{\lambda_1}$ is the minimal weight codeword of $R_q(t(q-1)+s,m-1)$ and its support is included in an affine subspace of codimension $t+1$. Since $t \geq 1$, in both cases, the support of $f_{\lambda_1}$ is included in an affine hyperplane of $\mathbb{F}_q^m$ different from the hyperplane parallel to $H$ of equation $x_1 = \lambda_1$. By applying an affine transformation that fixes $x_1$, we can assume that $(1-x_2^{q-1})$ divides $f_{\lambda_1}$. Since $S$ is not included in an affine hyperplane, there exists $2 \leq k \leq q$ such that $1-x_2^{q-1}$ does not divide $f_{\lambda_k}$. We denote by $k_0$ the smallest such $k$.

Assume that $S$ meets all affine hyperplanes parallel to $H$ and that

$$|f_{\lambda_{k_0}}| \geq (q-s+k_0-1)q^{m-t-2}.$$ 

Then

$$|f| = \sum_{k=1}^{q} |f_{\lambda_k}|$$

$$\geq (q-s)q^{m-t-2}(k_0-1) + (q-k_0+1)(q-s+k_0-1)q^{m-t-2}$$

$$= (q-s)q^{m-t-1} + (k_0-1)(q-k_0+1)q^{m-t-2}$$

$$> (q-s)q^{m-t-1} + (s-1)q^{m-t-2}$$

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which gives a contradiction. In the cases where \( S \) meets \((q - s')\), \( s' = 1 \) or \( s' = s - 1 \), for \( 1 \leq i \leq s' \), \(|f_{\lambda_i}| = 0 \) and the support of \( f_{\lambda_i} \) is \( S \cap H_{x_{\lambda_i+1}} \), where \( H_{x_{\lambda_i+1}} \) is the hyperplane of equation \( x_1 = \lambda_{i+1} \). Since \( S \cap H_{x_{\lambda_i+1}} \) is the support of a minimum weight codeword of \( R_q((t + 1)(q - 1) + s', m) \), it is included in an affine subspace of codimension \( t + 1 \). So in those cases, we can assume that \( k_0 \geq s' + 2 \). Finally, \(|f_{\lambda_0}| < (q - s + k_0 - 1)q^{m-t-2} \).

We write
\[
\begin{align*}
  f(x_1, x_2, x_3, \ldots, x_m) &= \sum_{i=0}^{m-1} x_i^i g_i(x_1, x_2, x_3, \ldots, x_m) \\
  &= h(x_1, x_2, x_3, \ldots, x_m) + (1 - x_2^{q-1})g(x_1, x_3, \ldots, x_m).
\end{align*}
\]

Since for all \( 1 \leq i \leq k_0 - 1, 1 - x_2^{q-1} \) divides \( f_{\lambda_i} \), for all \((x_2, \ldots, x_m) \in \mathbb{F}_q^{m-1} \), for all \( 1 \leq i \leq k_0 - 1, h(\lambda_i, x_2, \ldots, x_m) = 0 \). So, by Lemma 5.4, \( f(x_1, x_2, x_3, \ldots, x_m) = (x_1 - \lambda_1) \ldots (x_1 - \lambda_{k_0-1})h(x_1, x_2, x_3, \ldots, x_m) + (1 - x_2^{q-1})g(x_1, x_3, \ldots, x_m) \)

with \( \deg(h) \leq r - k_0 + 1 \). Then by applying Lemma 5.3 to \( f_{\lambda_{k_0}} \), since
\[
|f_{\lambda_0}| < (q - s + k_0 - 1)q^{m-t-2},
\]
\( k_0 = 2 \). This gives a contradiction in the cases where \( S \) does not meet all hyperplanes parallel to \( H \). In the case where \( S \) meets all hyperplanes parallel to \( H \), by applying Lemma 5.3 there exists \( T \) an affine transformation which fixes \( x_1 \) such that \((1 - x_2^{q-1}) \) divides \((f \circ T)_{\lambda_1} \) and \((f \circ T)_{\lambda_2} \), we set \( k_0' \) the smallest \( k \) such that \((1 - x_2^{q-1}) \) does not divide \((f \circ T)_{\lambda_k} \). Then \( k_0' \geq 3 \) and by applying the previous argument to \( f \circ T \), we get a contradiction.

\[\ Boxed\]

**Proposition 5.7** Let \( m \geq 3 \), \( q \geq 4 \), \( 1 \leq t \leq m - 2 \) and \( 2 \leq s \leq q - 2 \). If \( f \in R_q(t(q - 1) + s, m) \) is such that \(|f| = (q - 1)(q - s + 1)q^{m-t-2} \), then the support of \( f \) is included in an affine subspace of codimension \( t \).

**Proof :** We denote by \( S \) the support of \( f \). By Lemma 5.3, \( S \) is included in \( H \) an affine hyperplane. By applying an affine transformation, we can assume that \( x_1 = 0 \) is an equation of \( H \). Let \( g \in B_{m-1}^d \) defined by
\[
\forall x = (x_2, \ldots, x_m) \in \mathbb{F}_q^{m-1}, \quad g(x) = f(0, x_2, \ldots, x_m)
\]
and denote by \( P \in \mathbb{F}_q[X_2, \ldots, X_m] \) its reduced form. Since
\[
\forall x = (x_1, \ldots, x_m) \in \mathbb{F}_q^{m}, \quad f(x) = (1 - x_1^{q-1})P(x_2, \ldots, x_m)
\]
the reduced form of \( f \in R_q(t(q - 1) + s, m) \) is
\[
(1 - X_1^{q-1})P(X_2, \ldots, X_m).
\]

Then \( g \in R_q((t-1)(q - 1) + s, m - 1) \) and
\[
|g| = |f| = (q - s + 1)(q - 1)q^{m-t-2} = (q - 1)(q - s + 1)q^{m-1-(t-1)-2}.
\]

Then, by Lemma 5.3 if \( t \geq 2 \), the support of \( g \) is included in an affine hyperplane of \( \mathbb{F}_q^{m-1} \). By iterating this argument, we get that \( S \) is included in an affine subspace of codimension \( t \).

\[\ Boxed\]
5.3 Proof of Theorem 2.3

Let $0 \leq t \leq m - 2$, $2 \leq s \leq q - 2$ and $f \in R_q(t(q-1) + s, m)$ such that

$$|f| = (q-s+1)(q-1)q^{m-t-2};$$

we denote by $S$ the support of $f$. Assume that $t \geq 1$. By Proposition 5.7, $S$ is included in an affine subspace $G$ of codimension $t$. By applying an affine transformation, we can assume that

$$G = \{x = (x_1, \ldots, x_m) \in \mathbb{F}_q^m : x_i = 0 \text{ for } 1 \leq i \leq t\}.$$

Let $g \in B_{m-t}^q$ defined for all $x = (x_{t+1}, \ldots, x_m) \in \mathbb{F}_q^{m-t}$ by

$$g(x) = f(0, \ldots, 0, x_{t+1}, \ldots, x_m)$$

and denote by $P \in \mathbb{F}_q[X_{t+1}, \ldots, X_m]$ its reduced form. Since

$$\forall x = (x_1, \ldots, x_m) \in \mathbb{F}_q^m, f(x) = (1 - x_1^{q-1}) \ldots (1 - x_t^{q-1})P(x_{t+1}, \ldots, x_m),$$

the reduced form of $f \in R_q(t(q-1) + s, m)$ is

$$(1 - X_1^{q-1}) \ldots (1 - X_t^{q-1})P(X_{t+1}, \ldots, X_m).$$

Then $g \in R_q(s, m - t)$ and $|g| = |f| = (q-s+1)(q-1)q^{m-t-2}$. Thus, using the case where $t = 0$, we finish the proof of Theorem 2.3.

6 Case where $s = 0$

6.1 The support is included in an affine subspace of dimension $m - t + 1$

Proposition 6.1 Let $q \geq 3$, $m \geq 2$ and $f \in R_q((m-1)(q-1), m)$ such that $|f| = 2(q-1)$. Then, the support of $f$ is included in an affine plane.

In order to prove this proposition, we need the following lemma.

Lemma 6.2 Let $m \geq 3$, $q \geq 4$ and $f \in R_q((m-1)(q-1), m)$ such that $|f| = 2(q-1)$. If $H$ is an affine hyperplane of $\mathbb{F}_q^m$ such that $S \cap H \neq S$, $|S \cap H| = N$, $3 \leq N \leq q - 1$ and $S \cap H$ is not included in a line then there exists $H_1$ an affine hyperplane of $\mathbb{F}_q^m$ such that $S \cap H_1 \neq S$, $|S \cap H_1| \geq N + 1$ and $S \cap H_1$ is not included in a line

Proof: Since $S \cap H \neq S$, by Lemma 5.1 either $S$ meets $(q-1)$ hyperplanes parallel to $H$ or $S$ meets 2 hyperplanes parallel to $H$ or $S$ meets all affine hyperplanes parallel to $H$. If $S$ does not meet all affine hyperplanes parallel to $H$ then $S \cap H$ is the support of a minimal weight codeword of $R_q((m-1)(q-1)+s', m)$, $s' = 1$ or $s' = q - 2$. In both cases, $S \cap H$ is included in a line which is absurd. So, $S$ meets all affine hyperplanes parallel to $H$.

By applying an affine transformation, we can assume that $x_1 = 0$ is an equation of $H$. Let $I := \{a \in \mathbb{F}_q : \#(\{x_1 = a\} \cap S) = 1\}$ and $k := \#I$. Since $\#S = 2(q-1)$ and $\#(S \cap H) = N$, $k \geq N$. We define

$$\forall x = (x_1, \ldots, x_m) \in \mathbb{F}_q^m, \quad g(x) = f(x) \prod_{a \in I}(x_1 - a).$$

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Then, \( \deg(g) \leq (m - 1)(q - 1) + q - k \) and \( |g| = k \). So, \( g \) is a minimal weight codewords of \( R_q((m - 1)(q - 1) + q - k, m) \) and its support is included in a line \( L \) which is not included in \( H \). We denote by \( \vec{w} \) a directing vector of \( L \). Let \( b \) be the intersection point of \( H \) and \( L \) and \( \omega_1, \omega_2, \omega_3 \) 3 points of \( S \cap H \) which are not included in a line. Then there exists \( \vec{v} \) and \( \vec{w} \in \{b \omega_1, b \omega_2, b \omega_3\} \) which are linearly independent. Since \( L \) is not included in \( H \), \( \{ \vec{v}, \vec{w} \} \) are linearly independent. We choose \( H_1 \) an affine hyperplane such that \( b \in H_1, b + \vec{v} \in H_1, L \subset H_1 \) but \( b + \vec{w} \notin H_1 \).

□

Now we can prove the proposition:

**Proof**: If \( m = 2 \), we have the result. Assume \( m \geq 3 \). Let \( S \) be the support of \( f \). Since \( |S| = 2(q - 1) > q \), \( S \) is not included in a line. Let \( \omega_1, \omega_2, \omega_3 \) 3 points of \( S \) not included in a line. Let \( H \) be an hyperplane such that \( \omega_1, \omega_2, \omega_3 \notin H \). Assume that \( S \cap H \neq S \). Then there exists an affine hyperplane \( H_1 \) such that \( |S \cap H_1| \geq q \), \( S \cap H_1 \) is not included in a line and \( S \cap H_1 \neq S \). Indeed, if \( q = 3 \), we take \( H_1 = H \) and for \( q \geq 4 \), we proceed by induction using the previous Lemma. Then by Lemma 4.4, either \( S \) meets 2 hyperplanes parallel to \( H_1 \) in 2 points or \( S \) meets 2 hyperplanes parallel to \( H_1 \) in \( q - 1 \) points or \( S \) meets all affine hyperplanes parallel to \( H_1 \). Since \( |S \cap H_1| \geq q \), \( S \) meets all hyperplanes parallel to \( H_1 \). Then, we must have

\[
q + q - 1 \leq 2(q - 1)
\]

which is absurd.

□

The two following lemmas are proved in [7].

**Lemma 6.3** Let \( m \geq 2, q \geq 3, 1 \leq t \leq m \) and \( f \in R_q(t(q - 1), m) \) such that \( |f| = q^{m-t} \) and \( g \in R_q(t(q - 1) - k, m), 1 \leq k \leq q - 1 \), such that \( g \neq 0 \). If \( h = f + g \) then either \(|h| = kq^{m-t}\) or \(|h| \geq (k + 1)q^{m-t}\).

**Lemma 6.4** Let \( m \geq 2, q \geq 3, 1 \leq t \leq m - 1 \) and \( f \in R_q(t(q - 1), m) \). For \( a \in \mathbb{F}_q \), we define the function \( f_a \) of \( B_{m-1}^n \) by for all \((x_2, \ldots, x_m) \in \mathbb{F}_q^m\), \( f_a(x_2, \ldots, x_m) = f(a, x_2, \ldots, x_m) \). If for some \( a, b \in \mathbb{F}_q, |f_a| = |f_b| = q^{m-t-1}\), then there exists \( T \) an affine transformation fixing \( x_1 \) such that \( (f \circ T)_a = (f \circ T)_b \).

**Proposition 6.5** Let \( q \geq 3, m \geq 2, 1 \leq t \leq m - 1 \). If \( f \in R_q(t(q - 1), m) \) is such that \( |f| = 2(q - 1)q^{m-t-1} \) then the support of \( f \) is included in an affine subspace of dimension \( m - t + 1 \).

**Proof**: For \( t = 1 \), this is obvious. For the other cases we proceed by recursion on \( t \). Proposition 6.3 gives the case where \( t = m - 1 \).

If \( m \leq 3 \) we have considered all cases. Assume \( m \geq 4 \). Let \( 2 \leq t \leq m - 2 \). Assume that for \( f \in R_q((t + 1)(q - 1), m) \) such that \( |f| = 2(q - 1)q^{m-t-2} \) the support of \( f \) is included in an affine subspace of dimension \( m - t \). Let \( f \in R_q(t(q - 1), m) \) such that \( |f| = 2(q - 1)q^{m-t-1} \). We denote by \( S \) the support of \( f \).
Assume that $S$ is not included in an affine subspace of dimension $m - t + 1$. Then there exists $H$ an affine hyperplane of $\mathbb{F}_q^m$ such that $S \cap H \neq S$ and $S \cap H$ is not included in an affine space of dimension $m - t$. By Lemma 5.4, either $S$ meets all affine hyperplanes parallel to $H$ or $S$ meets $(q - 1)$ affine hyperplanes parallel to $H$ in $2q^{m-t-1}$ or $S$ meets $2$ affine hyperplanes parallel to $H$ in $(q - 1)q^{m-t-1}$ points.

If $S$ does not meet all hyperplanes parallel to $H$ then $S \cap H$ is the support of a minimal weight codeword of $R_q(t(q - 1) + s', m)$, $s' = 1$ or $s' = q - 2$. So $S \cap H$ is included in an affine subspace of dimension $m - t$ which gives a contradiction. So, there exists $H_1$ an affine hyperplane parallel to $H$ such that $\#(S \cap H_1) = q^{m-t-1}$.

By applying an affine transformation, we can assume that $x_1 = \lambda$, $\lambda \in \mathbb{F}_q$ is an equation of $H$. For $\lambda \in \mathbb{F}_q$, we define $f_\lambda \in B^q_{m-1}$ by

$$\forall(x_2, \ldots, x_m) \in \mathbb{F}_q^{m-1}, \quad f_\lambda(x_2, \ldots, x_m) = f(\lambda, x_2, \ldots, x_m).$$

We set an order $\lambda_1, \ldots, \lambda_q$ on the elements of $\mathbb{F}_q$ such that

$$|f_{\lambda_1}| \leq \ldots \leq |f_{\lambda_q}|.$$ 

Since $\#(S \cap H_1) = q^{m-t-1}$ and $S$ meets all hyperplanes parallel to $H$,

$$|f_{\lambda_1}| = q^{m-t-1}$$

and $f_{\lambda_1}$ is a minimum weight codeword of $R_q(t(q - 1) + m - 1)$. Let $k_0$ be the smallest integer such that $|f_{\lambda_{k_0}}| > q^{m-t-1}$. Since $|f| > q^{m-t}$, $k_0 \leq q$. Then by Lemma 6.3 and applying an affine transformation that fixes $x_1$, we can assume that for all $2 \leq i \leq k_0 - 1$, $f_{\lambda_i} = f_{\lambda_1}$. If we write for all $x = (x_1, \ldots, x_m) \in \mathbb{F}_q^m$,

$$f(x) = f_{\lambda_1}(x_2, \ldots, x_m) + (x_1 - \lambda_1)\hat{f}(x_1, \ldots, x_m).$$

Then for all $2 \leq i \leq k_0 - 1$, for all $\mathfrak{f} = (x_2, \ldots, x_m) \in \mathbb{F}_q^{m-1}$,

$$f_{\lambda_i}(\mathfrak{f}) = f_{\lambda_1}(\mathfrak{f}) + (\lambda_i - \lambda_1)\hat{f}_{\lambda_1}(\mathfrak{f}).$$

Since for all $2 \leq i \leq k_0 - 1$, $f_{\lambda_i} = f_{\lambda_1}$, by Lemma 5.3 we can write for all $x = (x_1, \ldots, x_m) \in \mathbb{F}_q^m$,

$$f(x) = f_{\lambda_1}(x_2, \ldots, x_m) + (x_1 - \lambda_1)\ldots(x_1 - \lambda_{k_0-1})\hat{f}(x_1, \ldots, x_m)$$

with $\deg(\hat{f}) \leq t(q - 1) - k_0 + 1$. Now, we have $f_{\lambda_{k_0}} = f_{\lambda_1} + \lambda'\hat{f}_{\lambda_{k_0}}, \lambda' \in \mathbb{F}_q^*$. Then, by Lemma 6.3 either $|f_{\lambda_{k_0}}| > k_0q^{m-t-1}$ or $|f_{\lambda_{k_0}}| = (k_0 - 1)q^{m-t-1}$. Assume that $|f_{\lambda_{k_0}}| > k_0q^{m-t-1}$. Then

$$|f| = \sum_{i=1}^{q} |f_{\lambda_i}|$$

$$\geq (k_0 - 1)q^{m-t-1} + (q + 1 - k_0)k_0q^{m-t-1}$$

$$= q^{m-t} + (k_0 - 1)(q - k_0 + 1)q^{m-t-1}$$

$$> 2(q - 1)q^{m-t-1}.$$
Let \( g \) then \( f \) case where \( t \)

Finally, we denote by \( S \) is included in an affine subspace of dimension \( m - t + 1 \).

\[ 6.2 \hspace{1em} \text{Proof of Theorem 2.4} \]

Let \( 1 \leq t \leq m - 1 \) and \( f \in R_q(t(q - 1), m) \) such that

\[ |f| = 2(q - 1)q^{m-t-1}; \]

we denote by \( S \) the support of \( f \). Assume that \( t \geq 2 \). By proposition 6.3, \( S \) is included in an affine subspace \( G \) of codimension \( t - 1 \). By applying an affine transformation, we can assume that

\[ G = \{ x = (x_1, \ldots, x_m) \in \mathbb{F}_q^m : x_i = 0 \text{ for } 1 \leq i \leq t - 1 \}. \]

Let \( g \in \mathbb{F}_q^m \) defined for all \( x = (x_t, \ldots, x_m) \in \mathbb{F}_q^{m-t+1} \) by

\[ g(x) = f(0, \ldots, 0, x_t, \ldots, x_m) \]

and denote by \( P \in \mathbb{F}_q[X_t, \ldots, X_m] \) its reduced form. Since

\[ \forall x = (x_1, \ldots, x_m) \in \mathbb{F}_q^m, \]  \[ f(x) = (1 - x_1^{q-1}) \cdots (1 - x_t^{q-1})P(x_t, \ldots, x_m), \]

the reduced form of \( f \in R_q(t(q - 1) + s, m) \) is

\[ (1 - X_1^{q-1}) \cdots (1 - X_t^{q-1})P(X_t, \ldots, X_m). \]

Then \( g \in R_q(q - 1, m - t + 1) \) and \( |g| = |f| = 2(q - 1)q^{m-t-1} \). Thus, using the case where \( t = 1 \), we finish the proof of Theorem 2.4.
7 Case where $0 \leq t \leq m - 2$ and $s = 1$

7.1 Case where $q \geq 4$

Lemma 7.1 Let $m \geq 2$, $q \geq 4$, $0 \leq t \leq m - 2$ and $f \in R_q(t(q - 1) + 1,m)$ such that $|f| = q^{m-t}$. We denote by $S$ the support of $f$. Then, if $H$ is an affine hyperplane of $\mathbb{F}_q^m$ such that $S \cap H \neq \emptyset$ and $S \cap H \neq S$, $S$ meets all affine hyperplanes parallel to $H$.

Proof: By applying an affine transformation, we can assume that $x_1 = 0$ is an equation of $H$. Let $H_a$ be the $q$ affine hyperplanes parallel to $H$ of equation $x_1 = a$, $a \in \mathbb{F}_q$. We denote by $I := \{a \in \mathbb{F}_q : S \cap H_a = \emptyset\}$. Let $k := \#I$ and assume that $k \geq 1$. Since $S \cap H \neq \emptyset$ and $S \cap H \neq S$, $k < q - 2$. For all $c \notin I$ we define

$$\forall x = (x_1, \ldots, x_m) \in \mathbb{F}_q^m, \quad g_c(x) = f(x) \prod_{a \in \mathbb{F}_q \setminus I, a \neq c} (x_1 - a).$$

Then $|f| = \sum_{c \notin I} |g_c|$.

- Assume $k \geq 2$.
  Then for all $c \notin I$, $\deg(g_c) \leq t(q - 1) + q - k$ and $2 \leq q - k \leq q - 2$. So, $|g_c| \geq k q^{m-t-1}$. Let $N = \# \{c \notin I : |g_c| = k q^{m-t-1} \}$. If $|g_c| > k q^{m-t-1}$, $|g_c| \geq (k + 1)(q - 1)q^{m-t-2}$. Hence
  $$q^{m-t} \geq N k q^{m-t-1} + (q - k - N)(k + 1)(q - 1)q^{m-t-2}.$$ Since $k \geq 2$, we get that $N \geq q - k$. Since $(q - k)k q^{m-t-1} \neq q^{m-t}$, we get a contradiction.

- Assume $k = 1$.
  Then, for all $c \notin I$, $\deg(g_c) \leq t(q - 1) + 1 + q - 2 = (t + 1)(q - 1)$. So $|g_c| \geq q^{m-t-1}$. Let $N = \# \{c \notin I : |g_c| = q^{m-t-1} \}$. If $|g_c| > q^{m-t-1}$, $|g_c| \geq 2(q - 1)q^{m-t-2}$. Since for $q \geq 4$, $2(q - 1)q^{m-t-2} > q^{m-t}$. $N \geq 1$. Furthermore, since $(q - 1)q^{m-t-1} < q^{m-t}$, $N \leq q - 2$. For $\lambda \in \mathbb{F}_q$, we define $f_\lambda \in B_{m-1}^q$ by

$$\forall (x_2, \ldots, x_m) \in \mathbb{F}_q^{m-1}, \quad f_\lambda(x_2, \ldots, x_m) = f(\lambda, x_2, \ldots, x_m).$$

We set $\lambda_1, \ldots, \lambda_q$ an order on the elements of $\mathbb{F}_q$ such that for all $i \leq N$, $|f_{\lambda_i}| = q^{m-t-1}$, $|f_{\lambda_{N+1}}| = 0$ and $q^{m-t-1} < |f_{\lambda_{N+2}}| \leq \ldots \leq |f_{\lambda_N}|$.

Since $f_{\lambda_{N+1}} = 0$, we can write for all $(x_1, \ldots, x_m) \in \mathbb{F}_q^m$,

$$f(x_1, \ldots, x_m) = (x_1 - \lambda_{N+1}) h(x_1, \ldots, x_m)$$

with $\deg(h) \leq t(q - 1)$. Then, for all $1 \leq i \leq q$, $i \neq N + 1$ and $(x_2, \ldots, x_m) \in \mathbb{F}_q^{m-1}$,

$$f_{\lambda_i}(x_2, \ldots, x_m) = (\lambda_i - \lambda_{N+1}) h_{\lambda_i}(x_2, \ldots, x_m).$$

So $\deg(f_{\lambda_i}) \leq t(q - 1)$ and $h_{\lambda_i} = \frac{f_{\lambda_i}}{\lambda_i - \lambda_{N+1}}$. 

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Since $h \in R_q(t(q-1), m)$, by Lemma 6.3, there exists an affine transformation such that for all $i \leq N$, $h_{\lambda_i} = h_{\lambda_i}$. Then, for all $(x_1, \ldots, x_m) \in \mathbb{F}_q^m$, 

$$h(x_1, \ldots, x_m) = h_{\lambda_1}(x_2, \ldots, x_m) + (x_1 - \lambda_1) \ldots (x_1 - \lambda_N)\tilde{h}(x_1, \ldots, x_m)$$

with $\deg(\tilde{h}) \leq t(q-1) - N$. Hence, for all $(x_1, \ldots, x_m) \in \mathbb{F}_q^m$,

$$f(x_1, \ldots, x_m) = \frac{x_1 - \lambda_{N+1}}{\lambda_1 - \lambda_{N+1}} f_{\lambda_1}(x_2, \ldots, x_m) + (x_1 - \lambda_1) \ldots (x_1 - \lambda_{N+1})\tilde{h}(x_1, \ldots, x_m).$$

Then, for all $(x_2, \ldots, x_m) \in \mathbb{F}_q^{m-1}$,

$$f_{\lambda_{N+2}}(x_2, \ldots, x_m) = \lambda f_{\lambda_1}(x_2, \ldots, x_m) + \lambda \tilde{h}_{\lambda_{N+2}}(x_2, \ldots, x_m)$$

with $\lambda, \lambda' \in \mathbb{F}_q^*$.

Since $f_{\lambda_1} \in R_q(t(q-1), m-1)$ and $\tilde{h}_{\lambda_{N+2}} \in R_q(t(q-1) - N, m-1)$, by Lemma 6.3 either $|f_{\lambda_{N+2}}| = Nq^{m-t-1}$ or $|f_{\lambda_{N+2}}| \geq (N+1)q^{m-t-1}$.

If $N = 1$, since $|f_{\lambda_{N+2}}| > q^{m-t-1}$, we get

$$q^{m-t-1} + (q - 2)2q^{m-t-1} \leq q^{m-t}$$

which means that $q \leq 3$. So $N \geq 2$. Then,

$$Nq^{m-t-1} + (q - 1 - N)Nq^{m-t-1} \leq q^{m-t}.$$ 

Since $N(q - N) \geq 2(q - 2)$, we get that $q \leq 4$. So, the only possibility is $q = 4$ and $N = q - 2 = 2$.

If $t = 0$, $H_{\lambda_1}$ contains $2.4^{m-1}$ points which is absurd. Assume $t \geq 1$. Since $h_{\lambda_1} = h_{\lambda_2}$ and for $i \in \{1, 2\}, f_{\lambda_i} = (\lambda_i - \lambda_3)h_{\lambda_3}$, $S \cap H_{\lambda_1}$ and $S \cap H_{\lambda_2}$ are both included in $A$ an affine subspace of dimension $m - t$. If $t = 1$, by applying an affine transformation which fixes $x_1$, we can assume that $x_2 = 0$ is an equation of $A$. If $S$ is included in $A$, then

$$\#(S \cap H_{\lambda_1} \cap A) = 2.4^{m-2}$$

which is absurd since $H_{\lambda_1} \cap A$ is an affine subspace of codimension 2. So there exists an affine hyperplane $H'$ containing $A$ but not $S$. By applying an affine transformation which fixes $x_1$, we can assume that $x_2 = 0$ is an equation of $H'$. Now, consider $g$ defined for all $(x_1, \ldots, x_m) \in \mathbb{F}_q^m$ by $g(x_1, \ldots, x_m) = x_2f(x_1, \ldots, x_m)$. Then $|g| \leq 2.4^{m-t-1}$. Furthermore, since $S$ is not included in $H'$ and $\deg(g) \leq 3t + 2$, $|g| \geq 2.4^{m-t-1}$. So $g$ is a minimum weight codeword of $R_q(3t + 2, m)$ and its support is the union of 2 parallel affine subspaces of codimension $t + 1$ included in an affine subspace of codimension $t$. Then, since $H' \cap H_{\lambda_3} = \emptyset$, there exists $H'_1$ a hyperplane parallel to $H'$ such that $S \cap H'_1 = \emptyset$. Now, consider $G$ the hyperplane through $H_{\lambda_1} \cap H'_1$ and $H' \cap H_{\lambda_3}$ and $G'$ the hyperplane through $H' \cap H_{\lambda_3}$ parallel to $G$ (see Figure 9).

Then $G$ and $G'$ does not meet $S$ but $S$ is not included in an hyperplane parallel to $G$ which is absurd by the previous case.

$\square$
Lemma 7.2 For $m \geq 3$, if $f \in R_4(3(m - 2) + 1, m)$ is such that $|f| = 16$, the support of $f$ is an affine plane.

Proof: We denote by $S$ the support of $f$.

First, we prove the case where $m = 3$. To prove this case, by Lemma 7.1, we only have to prove that there exists an affine hyperplane which does not meet $S$.

Assume that $S$ meets all affine hyperplanes. Let $H$ be an affine hyperplane. Then for all $H'$ affine hyperplane parallel to $H$, $(S \cap H') \geq 3$. Assume that for all $H'$ parallel to $H$, $(S \cap H') = 4$. For reason of cardinality, for all $H'$ parallel to $H$, $(S \cap H') \geq 3$.

For reason of cardinality, for all $H'$ parallel to $H$.

1. $H_1$ is the only hyperplane through $L'$ such that $(S \cap H_1) = 3$ and $S \cap H_1$ is included in one of the affine hyperplane parallel to $H_1$.

Since $S \cap H_1 \cap H'_1 = \emptyset$ there exists an affine hyperplane parallel to $H_1$ which meets $S \cap H'_1$ in at least 2 points. Assume for example that it is $H_2$. Since $m = 3$, these 2 points are included in $L_1$ a line which is a translation of $L$. Consider $H$ the hyperplane containing $L_1$ and $L$. Then, $H$ meets $S \cap H_1$ and $S \cap H_4$ in 1 point (see Figure 10a).

2. There are exactly 2 hyperplanes through $L'$ which meets $S$ in 3 points and such that its intersection with $S$ is included in one of the affine hyperplane parallel to $H_1$.

Assume that $H_2$ contains $S \cap H_1$ where $H_1$ is the hyperplane through $L'$ different from $H_1$ such that $(S \cap H') = 3$ and $S \cap H'$ is included in an hyperplane parallel to $H_1$, say $H_2$. We denote by $L_1 = H \cap H_2$. Since for
all $H'$ hyperplane $\#(S \cap H') \geq 3$, $S \cap H'_1$ meets $H_3$ and $H_4$ in at least one point. Then consider $H$ the hyperplane through $L$ and $L_1$. Since $H$ is different from the hyperplane through $L'$ and $L_1$, $H$ meets $H_3$ and $H_4$ in at least 1 point each (see Figure [10a]). So $\#(S \cap H) \geq 7$.

3. There are exactly 3 hyperplanes through $L'$ which meet $S$ in 3 points and such that its intersection with $S$ is included in one of the affine hyperplane parallel to $H_1$.

If 2 such hyperplanes have their intersection with $S$ included in the same hyperplane parallel to $H_1$, say $H_2$, then $\#(S \cap H_2) \geq 7$. Now, assume that they are included in 2 different hyperplanes, $H_2$ and $H_3$. If $S \cap H'_1$ is included in $H_4$ then we consider $H$ the hyperplane through $L$ and $S \cap H'_1$ and $\#(S \cap H) \geq 7$. Otherwise, we can assume that $S \cap H'_1$ meets $H_2$ in at least 1 point. Let $H$ be the hyperplane through $L$ and $L_1$ the line containing the minimum weight codeword included in $H_3$. Since $H$ is different from the hyperplane through $L'$ and $L_1$, $H$ meets $S \cap H_2$ in at least 1 point and $\#(S \cap H) \geq 7$ (see Figure [10c]).

4. There are 4 hyperplanes through $L'$ which meets $S$ in 3 points and such that its intersection with $S$ is included in one of the affine hyperplane parallel to $H_1$.

If 3 such hyperplanes have their intersection with $S$ included in the same hyperplane parallel to $H_1$, say $H_2$, then $\#(S \cap H_2) \geq 7$. Assume that 2 such hyperplanes have their intersection included in the same hyperplane parallel to $H_1$, say $H_2$ and the last one has its intersection with $S$ included in $H_3$. Then, since $\#(S \cap H_1) \geq 3$, $\#(S \cap H'_1 \cap H_4) \geq 3$.

If $\#(S \cap H_4 \cap H'_1) = 4$, we consider $H$ the hyperplane through $L$ and $S \cap H'_1$ and $\#(S \cap H) \geq 7$. Otherwise, there is one point of $S \cap H_4$ included in $H_2$ or $H_3$. If this point is included in $H_2$ then $\#(S \cap H_2) \geq 7$. If it is included in $H_3$, we consider $L_1$ and $L_2$ the 2 lines in $H_2$ containing $S$ which are a translation of $L$. Then either the hyperplane through $L$ and $L_1$ or the hyperplane through $L$ and $L_2$ meets $S \cap H_3$ or $S \cap H_4$ (see Figure [10d]). So there is an hyperplane $H$ such that $\#(S \cap H) \geq 7$.

Now assume that for each hyperplane $H'$ parallel to $H_1$, there is only one hyperplane through $L'$ which meets $S$ in 3 points such that its intersection with $S$ included in $H'$. If $S \cap H'_1$ is included in an affine hyperplane parallel to $H_4$ then we consider $H$ the hyperplane through $L$ and $S \cap H'_1$ and $\#(S \cap H) \geq 7$. Otherwise, $S \cap H'_1$ meets at least 2 hyperplanes parallel to $H_1$, say $H_2$ and $H_3$ in at least 1 point. For $i \in \{2, 3, 4\}$, we denote by $H'_i$ the hyperplane through $L'$ such that $S \cap H'_i \subset H_i$. If $H$ the hyperplane through $L$ and $S \cap H'_4$ does not meet $S \cap H_2$ and $S \cap H_3$, then $H$ the hyperplane through $S \cap H'_4$ and $S \cap H'_3$ meet $S \cap H_2$. Indeed, if $H$ does not meet $S \cap H_2$ we consider 4 hyperplanes through $S \cap H'_4$ different from $H_4$, which intersect $H_2$ in 4 distinct parallel lines. However 2 of these lines meet $S$ (see Figure [10e]). So there is an hyperplane $H$ such that $\#(S \cap H) \geq 7$.

In all cases, there exists an affine hyperplane $H$ such that $\#(S \cap H) \geq 7$. If $\#(S \cap H) > 7$, since $S$ meets all affine hyperplanes in at least 3 points, $\#S > 7 + 3.3 = 16$ which gives a contradiction. If $\#(S \cap H) = 7$, then
for all $H'$ parallel to $H$ different form $H$ $\#(S \cap H') = 3$. By applying an affine transformation, we can assume that $x_1 = 0$ is an equation of $H$. Then $g = x_1 f \in R_4(3(m-2) + 2, m)$ and $|g| = 9$. So, $g$ is a second weight codeword of $R_4(3(m-2) + 2, m)$ and by Theorem 2.3 the support of $g$ is included in a plane $P$. Since $S$ meets all hyperplanes, $S$ is not included in $P$. Then, $S$ meets all hyperplanes parallel to $P$ in at least 3 points. However $3.3 + 9 = 18 > 16$ which is absurd.

Now, assume that $m \geq 4$. Assume that $S$ is not included in an affine subspace of dimension 3. Then there exists $H$ an affine hyperplane such that $S \cap H$ is not included in a plane and $S$ is not included in $H$. So, by Lemma 7.1 $S$ meets all affine hyperplanes parallel to $H$ in at least 3 points.

Assume that for all $H'$ parallel to $H$, $\#(S \cap H') \geq 4$, then for reason of cardinality, $\#(S \cap H) = 4$. So $S \cap H$ is the support of a second weight codeword of $R_4(3(m-1) + 1, m)$ and is included in a plane which is absurd. So there exists $H_1$ an affine hyperplane parallel to $H$ such that $\#(S \cap H_1) = 3$. Then, $S \cap H_1$ is the support of the minimum weight codeword of $R_4(3(m-1) + 1, m)$ and is included in a line $L$. We denote by $\vec{u}$ a directing vector of $L$ and $a$ the point of $L$ which is not in $S$. 

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Let \( w_1, w_2, w_3 \) 3 points of \( S \cap H \) which are not included in a line. Then, there are at least 2 vectors of \( \{ \tilde{w}_1 \tilde{w}_2', \tilde{w}_1 \tilde{w}_3', \tilde{w}_2 \tilde{w}_3' \} \) which are not collinear to \( \tilde{u} \). Assume that they are \( \tilde{w}_1 \tilde{w}_2' \) and \( \tilde{w}_1 \tilde{w}_3' \). Let \( a \) be an affine subspace of codimension 2 included in \( H_1 \) which contains \( a, a+\tilde{w}_1 w_2', a+\tilde{w}_1 w_3' \) but not \( a+\tilde{u} \). Then \( S \) does not meet \( A \). Assume that \( S \) does not meet one hyperplane through \( A \). Then \( S \) is included in an affine hyperplane parallel to this hyperplane which is absurd by definition of \( A \). So, \( S \) meets all hyperplanes through \( A \) and since \( 3.4 + 4 = 16 \), There exists \( H_2 \) an hyperplane through \( A \) such that \#(\( S \cap H_2 \)) = 4 and \( S \cap H_2 \) is included in a plane. For all \( H' \) hyperplane through \( A \) different from \( H_2 \), \#(\( S \cap H' \)) = 3 and \( S \cap H' \) is included in a line. Consider \( H_2' \) the hyperplane through \( A \) such that \( w_1 \in H_2' \). Then \( w_1, w_2, w_3 \in H_2' \). Since for all \( H' \) hyperplane through \( A \) different from \( H_2 \), \( S \cap H' \) is included in a line and \( w_1, w_2, w_3 \) are not included in a line \( H_2' = H_2 \). Further more \( S \cap H_2 \) is included in a plane, so \( S \cap H_2' \subset H \).

For all \( H' \) hyperplane through \( A \) different from \( H_2 \), \( S \cap H' \) is the support of a minimum weight codeword of \( R_4(3(m - 1) + 1, m) \) which does not meet \( H_1 \), so either \( S \cap H' \) is included an affine hyperplane parallel to \( H_1 \) or \( S \cap H' \) meets all affine hyperplanes parallel to \( H \) but \( H_1 \) in 1 point. Since \( S \cap H_2 \) is included in \( H \) and all hyperplanes parallel to \( H \) meets \( S \) in at least 3 points, there are only 3 possibilities:

1. For all \( H_2' \) hyperplane through \( A \), \( S \cap H_2' \) is included in an affine hyperplane parallel to \( H \).

2. For \( H_2' \) hyperplane through \( A \) different from \( H_2 \) and \( H_1 \), \( S \cap H_2' \) meets all affine hyperplanes parallel to \( H \) different from \( H_1 \) in 1 points.

3. There is 4 hyperplanes through \( A \) such that their intersection with \( S \) is included in an affine hyperplane parallel to \( H \) and 1 hyperplane through \( A \) which meets all hyperplanes parallel to \( H \) but \( H_1 \) in 1.

In the two first cases, since \( S \cap H \) is not included in a plane and \( S \) meets all hyperplanes parallel to \( H \) in at least 3 points, \#(\( S \cap H \)) = 7 and for all \( H' \) parallel to \( H \) different form \( H \), \#(\( S \cap H' \)) = 3. By applying an affine transformation, we can assume that \( x_1 = 0 \) is an equation of \( H \). Then \( g = x_1 f \in R_4(3(m - 2) + 2, m) \) and \(|g| = 9 \). So, \( g \) is a second weight codeword of \( R_4(3(m - 2) + 2, m) \) and by Theorem 2.3 the support of \( g \) is included in a plane \( P \). Since \( S \) is not included in \( P \), there exists \( H_1' \) an affine hyperplane which contains \( P \) but not \( S \). Then, \( S \) meets all hyperplanes parallel to \( H_1' \) in at least 3 points. However \( 3.3 + 9 = 18 > 16 \) which is absurd.

Assume we are in the third case. Since \( S \cap H \) is the union of a point and \( S \cap H_2 \) which is included in a plane and \( m \geq 4 \), there exist \( B \) an affine subspace of codimension 2 included in \( H \) such that \( S \) does not meet \( B \) and \( S \cap H \) is not included in affine hyperplane parallel to \( B \). Then \( S \) meets all affine hyperplanes through \( B \) in at most 4 points which is a contradiction since \#(\( S \cap H \)) = 5.

So \( S \) is included in \( G \) an affine subspace of dimension 3. By applying an affine transformation, we can assume that

\[
G := \{(x_1, \ldots, x_m) \in \mathbb{F}_q^m : x_4 = \ldots = x_m = 0\}.
\]

Let \( g \in B_2^3 \) defined for all \( x = (x_1, x_2, x_3) \in \mathbb{F}_q^3 \) by

\[
g(x) = f(x_1, x_2, x_3, 0, \ldots, 0)
\]

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and denote by $P \in \mathbb{F}_q[X_1, X_2, X_3]$ its reduced form. Since
\[ \forall x = (x_1, \ldots, x_m) \in \mathbb{F}_q^m, \quad f(x) = (1 - x_{m-1}^q) \ldots (1 - x_1^q)P(x_1, x_2, x_3), \]
the reduced form of $f \in R_q(3(m-2)+1, m)$ is
\[ (1 - X_{m-1}^q) \ldots (1 - X_1^q)P(X_1, X_2, X_3). \]
Then $g \in R_q(4, 3)$ and $|g| = |f| = 16$. Thus, using the case where $m = 3$, we finish the proof of Lemma 7.2.

\[ \square \]

**Theorem 7.3** For $q \geq 4$, $m \geq 2$, $0 \leq t \leq m-2$, if $f \in R_q(t(q-1)+1, m)$ is such that $|f| = q^{m-t}$, the support of $f$ is an affine subspace of codimension $t$.

**Proof:** If $t = 0$, the second weight is $q^m$ and we have the result.

For other cases, we proceed by recursion on $t$.

If $q \geq 5$, we have already proved the case where $t = m-1$ (Theorem 2.1); if $m = 2$ and $t = m-2 = 0$, we have the result. Assume that $m \geq 3$.

For $q = 4$, if $m = 2$, $t = m-2 = 0$ and we have the result. If $m \geq 3$, we have already proved the case $t = m-2$ (Lemma 7.2). Furthermore, if $m = 3$ we have considered all cases. Assume $m \geq 4$.

Let $1 \leq t \leq m-2$ (or for $q = 4$, $1 \leq t \leq m-3$). Assume that the support of $f \in R_q((t+1)(q-1)+1, m)$ such that $|f| = q^{m-t-1}$ is an affine subspace of codimension $t+1$.

Let $f \in R_q(t(q-1)+1, m)$ such that $|f| = q^{m-t}$. We denote by $S$ its support. Assume that $S$ is not included in an affine subspace of codimension $t$. Then there exists $H$ an affine hyperplane such that $S \cap H$ is not included in an affine subspace of codimension $t+1$ and $S \cap H \neq S$. Then, by Lemma 7.1, $S$ meets all affine hyperplanes parallel to $H$ and for all $H'$ hyperplane parallel to $H$,
\[ \#(S \cap H') \geq (q-1)q^{m-t-2}. \]
If for all $H'$ hyperplane parallel to $H$, $\#(S \cap H') > (q-1)q^{m-t-2}$ then, for reason of cardinality, $\#(S \cap H) = q^{m-t-1}$. So $S \cap H$ is the support of a second weight codeword of $R_q((t+1)(q-1)+1, m)$ and is included in an affine subspace of codimension $t+1$ which is a contradiction.

So there exists $H_1$ parallel to $H$ such that $\#(S \cap H_1) = (q-1)q^{m-t-2}$. Then $S \cap H_1$ is the support of a minimal weight codeword of $R_q((t+1)(q-1)+1, m)$. Hence, $S \cap H_1$ is the union of $q-1$ affine subspaces of codimension $t+2$ included in an affine subspace of codimension $t+1$.

Let $A$ be an affine subspace of codimension 2 included in $H_1$ such that $A$ meets the affine subspace of codimension $t+1$ which contains $S \cap H_1$ in the affine subspace of codimension $t+2$ which does not meet $S$. Assume that there is an affine hyperplane through $A$ which does not meet $S$. Then, by Lemma 7.1.
$S$ is included in an affine hyperplane parallel to this hyperplane which is absurd by construction of $A$. So, $S$ meets all hyperplanes through $A$. Furthermore,

$$q^{m-t} = q^{m-t-1} + q(q-1)q^{m-t-2}.$$ 

So $S$ meets one of the hyperplane through $A$ in $q^{m-t-1}$ points, say $H_2$, and all the others in $(q-1)q^{m-t-2}$ points.

Since $H_2 \neq H_1$, $H_2 \cap H_1 = A$ and $S \cap H_2 \cap H_1 = \emptyset$. So, $S \cap H_2$ is the support of a second weight codewords of $R_q((t+1)(q-1)+1, m)$ which does not meet $H_1$. Hence, $S \cap H_2$ is included in one of the affine hyperplanes parallel to $H$.

Furthermore, for all $H'_2$ hyperplane through $A$ different from $H_2$ and $H_1$, $S \cap H'_2$ is the support of a minimum weight codeword of $R_q((t+1)(q-1)+1, m)$ which does not meet $H_1$, so it meets all affine hyperplanes parallel to $H_1$ different from $H_1$ in $q^{m-t-2}$ points or is included in an affine hyperplane parallel to $H_1$. Since $S \cap H_2$ is included in one of the affine hyperplanes parallel to $H$ and all hyperplanes parallel to $H$ meets $S$ in at least $(q-1)q^{m-t-2}$ points, there are only 3 possibilities:

1. For all $H'_2$ hyperplane through $A$, $S \cap H'_2$ is included in an affine hyperplane parallel to $H$.

2. For $H'_2$ hyperplane through $A$ different from $H_2$ and $H_1$, $S \cap H'_2$ meets all affine hyperplanes parallel to $H$ different from $H_1$ in $q^{m-t-2}$ points.

3. There is $q$ hyperplanes through $A$ such that their intersection with $S$ is included in an affine hyperplane parallel to $H$ and 1 hyperplane through $A$ which meets all hyperplanes parallel to $H$ but $H_1$ in $q^{m-t-2}$.

In the two first cases, if $S \cap H_2$ is not included in $H'$ parallel to $H$, 

$$\#(S \cap H') = (q-1)q^{m-t-2}$$

and $S \cap H'$ is the support of a minimum weight codewords of $R_q((t+1)(q-1)+1, m)$. So $S \cap H'$ is included in an affine subspace of codimension $t+1$. Then, necessarily, $S \cap H_2$ is included in $H$. For all $H'$ parallel to $H$ but $H$, $\#(S \cap H') = (q-1)q^{m-t-2}$. In the third case, for all $H'$ hyperplane parallel to $H$ different from $H_1$ which does not contain $S \cap H_2$,

$$\#(S \cap H') = q^{m-t-1}.$$ 

So $S \cap H'$ is the support of a second weight codeword of $R_q((t+1)(q-1)+1, m)$ and is an affine subspace of dimension $m-t-1$. Then, $S \cap H_2 \subset H$ and $\#(S \cap H) = q^{m-t-2} + q^{m-t-1}$, $\#(S \cap H_1) = (q-1)q^{m-t-2}$.

So if we are in the last case for reason of cardinality, for all $A'$ affine subspace of codimension 2 included in $H_1$ such that $A'$ meets the affine subspace of codimension $t+1$ which contains $S \cap H_1$ in the affine subspace of codimension $t+2$ which does not meet $S$ we are also in case 3. Then $S$ is the union of affine subspaces of dimension $m-t-2$ which are a translation of the affine subspace of codimension $t+2$ which does not meet $S$ in $S \cap H_1$. Then, since $S \cap H_2$ is the support of a second weight codeword of $R_q((t+1)(q-1)+1, m)$, it is an affine subspace of dimension $m-t-1$. So $S \cap H$ is the union of an affine subspace of dimension $m-t-1$ and an affine subspace of dimension $m-t-2$. Since $S$ is the union of affine subspaces of dimension $m-t-2$ which are a translation of an affine subspace of codimension $t+2$, there exists $B$ an affine subspace of codimension 2 such that $B$ does not meet $S$ and $S \cap H$ is not included in an affine subspace of codimension 2 parallel to $B$. Now, we consider all affine hyperplanes through $B$. Assume that there exists $G$ an affine hyperplane through $B$ which does not meet $S$. Then, $S$ is included in an affine hyperplane parallel to $G$.
which is absurd by construction of $B$. So, $S$ meets all hyperplanes through $B$ and there exists $G_1$ hyperplane through $B$ such that $\#(S \cap G_1) = q^{m-t-1}$ and for all $G$ through $B$ but $G_1$, $\#(S \cap G) = (q - 1)q^{m-t-2}$ which is absurd since $\#(S \cap H) = q^{m-t-1} + q^{m-t-2}$. Finally, we are in case 1 or 2.

By applying an affine transformation, we can assume that $x_1 = 0$ is an equation of $H$. Now, consider $g$ the function defined by

$$\forall x = (x_1, \ldots, x_m) \in \mathbb{F}_q^m \quad g(x) = x_1f(x).$$

Then $\deg(g) \leq t(q - 1) + 2$ and $|g| = (q - 1)^2q^{m-t-2}$. So, $g$ is a second weight codeword of $R_q(t(q - 1) + 2, m)$ and by Theorem 2.3 the support of $g$ is included in an affine subspace of codimension $t$.

Let $H_3$ be an affine hyperplane containing the support of $g$ but not $S$. Then, $\#(S \cap H_3) \geq (q - 1)^2q^{m-t-2}$. Furthermore, since $S \not\subset H_3$, $S$ meets all affine hyperplanes parallel to $H_3$ in at least $(q - 1)q^{m-t-2}$. Finally,

$$\#S \geq 2(q - 1)^2q^{m-t-2} > q^{m-t}.$$ 

We get a contradiction. So $S$ is included in an affine subspace of codimension $t$. For reason of cardinality, $S$ is an affine subspace of codimension $t$.

\[\square\]

7.2 Case where $q = 3$, proof of Theorem 2.6

Lemma 7.4 Let $m \geq 2$, $0 \leq t \leq m - 2$, $f \in R_3(2t + 1, m)$ such that $|f| = 8.3^{m-t-2}$. If $H$ is an affine hyperplane of $\mathbb{F}_3^m$ such that $S \cap H \neq \emptyset$ and $S \cap H \neq S$ then either $S$ meets 2 hyperplanes parallel to $H$ in 4.3^{m-t-2}$ points or $S$ meets all affine hyperplanes parallel to $H$.

Proof : By applying an affine transformation, we can assume that $x_1 = 0$ is an equation of $H$. We denote by $H_a$ the affine hyperplanes parallel to $H$ of equation $x_1 = a$, $a \in \mathbb{F}_q$. Let $I := \{a \in \mathbb{F}_q : S \cap H_a = \emptyset\}$ and $k := |I|$. Since $S \cap H \neq \emptyset$ and $S \cap H \neq S$, $k \leq q - 2 = 1$. Assume $k = 1$. For all $c \not\in I$ we define

$$\forall x = (x_1, \ldots, x_m) \in \mathbb{F}_q^m, \quad f_c(x) = f(x) \prod_{a \in I, a \neq c} (x_1 - a).$$

Then $\deg(f_c) = (t + 1)2$ and $|f_c| \geq 3^{m-t-1}$. Assume that there exists $H'$ an affine hyperplane parallel to $H$ such that $\#(S \cap H') = 3^{m-t-1}$ and $S \cap H'$ is the support of a minimal weight codeword of $R_3(2(t + 1), m)$. Then consider $A$ an affine subspace of codimension 2 included in $H'$ containing $S \cap H'$ and $A'$ an affine subspace of codimension 2 included in $H'$ parallel to $A$. We denote by $k$ the number of hyperplanes through $A$ which meet $S$ and by $k'$ the number of affine hyperplanes through $A'$ which meet $S$ in $3^{m-t-1}$ points. Then

$$k'3^{m-t-1} + (k - k')4.3^{m-t-2} \leq 8.3^{m-t-2}.$$ 

Since $\#S > \#(S \cap H')$ and $k' \leq k$, we get $k = 2$. Then, if we denote by $H''$ the other hyperplane parallel to $H'$ which meets $S$, $S \cap H''$ is included in an affine subspace of codimension 2 which is a translation of $A$. By applying this
argument to all affine subspaces of codimension 2 included in \( H' \) and containing \( S \cap H' \), we get that \( S \cap H'' \) is included in a an affine subspace of dimension \( m - t - 1 \). For reason of cardinality this is absurd. If \( |f_c| > 3^{m-t-1} \) then \( |f_c| \geq 4.3^{m-t-2} \). For reason of cardinality, we have the result.

\[ \square \]

Now, we prove Proposition 2.6.

- First, we prove the case where \( t = 1 \). Obviously, \( S \) is included in an affine subspace of dimension \( m \). Assume that \( S \) meets all affine hyperplanes of \( \mathbb{F}_q^m \). Then for all \( H' \) affine hyperplane of \( \mathbb{F}_q^m \), \( \#(S \cap H') \geq 2.3^{m-3} \) and by Lemma 3.3 there exists \( H \) an affine hyperplane such that

\[ \#(S \cap H) = 2.3^{m-3}. \]

Then \( S \cap H \) is the support of a minimum weight codeword of \( R_3(5, m) \). So it is the union of \( P_1, P_2 \) 2 parallel affine subspaces of dimension \( m - 3 \) included in an affine subspace of dimension \( m - 2 \). Let \( A \) be an affine subspace of codimension 2 included in \( H \), containing \( P_1 \) and different from the affine subspace of codimension 2 containing \( S \cap H \). Then there exists \( A' \) an affine hyperplane of codimension 2 included in \( H \) parallel to \( A \) which does not meet \( S \). We denote by \( k \) the number of affine hyperplanes through \( A' \) which meet \( S \) in \( 2.3^{m-3} \) points. Then, if \( m \geq 4 \),

\[ k2.3^{m-3} + (4 - k)8.3^{m-4} \leq 8.3^{m-3} \]

which means that \( k \geq 4 \). If \( m = 3 \), \( 2k + (4 - k)3 \leq 8 \) which also means that \( k \geq 4 \). Then for all \( H' \) hyperplane through \( A \) different from \( H \), \( S \cap H' \) is a minimal weight codeword of \( R_3(5, m) \) which does not meet \( H \) and either \( S \cap H' \) is included in one of the hyperplanes parallel to \( H \) or \( S \cap H' \) meets the 2 hyperplanes parallel to \( H \) different from \( H \). In all cases, \( S \) is the union of 8 affine subspaces of dimension \( m - 3 \). By applying this argument to all affine subspaces of codimension 2 included in \( H \), containing \( P_1 \) and different from the affine subspace of codimension 2 containing \( S \cap H \), we get that these 8 affine subspaces are a translation of \( P_1 \).

Choose \( H_1 \) one of the hyperplanes through \( A' \) and consider \( H_2 \) and \( H_3 \) the 2 hyperplanes parallel to \( H_1 \). Since \( \#(S \cap H_1) = 2.3^{m-3} \) and \( S \) meets all hyperplanes in at least \( 2.3^{m-3} \) points, either \( \#(S \cap H_2) = 3.3^{m-3} \) and \( \#(S \cap H_3) = 3.3^{m-3} \) or \( \#(S \cap H_2) = 2.3^{m-3} \) and \( \#(S \cap H_3) = 4.3^{m-3} \).

First consider the case where \( \#(S \cap H_2) = 3.3^{m-3} \) and \( \#(S \cap H_3) = 3.3^{m-3} \). Then there exists an affine subspace of codimension 2 in \( H_2 \) which does not meet \( S \). We denote by \( k' \) the number of hyperplanes through \( A \) which meet \( S \) in \( 2.3^{m-3} \) points. Then, we have \( k' \geq 4 \) which is absurd since \( \#(S \cap H_2) = 3.3^{m-3} \).

Now, consider the case where \( \#(S \cap H_2) = 2.3^{m-3} \) and \( \#(S \cap H_3) = 4.3^{m-3} \). By applying an affine transformation, we can assume that \( x_1 = 0 \) is an equation of \( H_3 \). Then \( x_1.f \) is a codeword of \( R_3(4, m) \) and \( |x_1.f| = 4.3^{m-3} \). So, by Theorem 2.4 its support is included in an affine hyperplane \( H'_1 \) and \( S \cap H'_1 \cap H_3 = \emptyset \). So \( S \) is included \( H'_1 \) and \( H_3 \).
and there exists an affine hyperplane through $H'_1 \cap H_3$ which does not meet $S$ which is absurd.

Finally there exists an affine hyperplane $G_1$ which does not meet $S$. So, by Lemma 7.4 $S$ meets $G_2$ and $G_3$ the 2 hyperplanes parallel to $G_1$ in $4.3^{m-3}$ points. Then, Theorem 2.4, $G_2 \setminus S$ and $G_3 \setminus S$ are the union of two non parallel affine subspaces of codimension 2. Consider $A$ one of the affine subspaces of codimension 2 in $G_2 \setminus S$. Assume that all hyperplanes through $A$ meet $S$. So for all $G'$ hyperplane through $A$, $(G' \setminus S) \leq 7.3^{m-3}$.

Furthermore, one of the hyperplanes through $A$, say $G$, meets $G_3 \setminus S$ in at least $2.3^{m-3}$, then $(G \setminus S) \geq 2.3^{m-2} + 2.3^{m-3}$ which is absurd (see Figure 11). So there exists $G'$ through $A$ which does not meet $S$. By applying the same argument to the other affine subspace of dimension 2 of $G_2 \setminus S$, we get the result for $t = 1$.

Figure 11

- We prove by recursion on $t$ that $S$ is included in an affine subspace of dimension $m - t + 1$. Consider first the case where $t = m - 2$. If $m = 3$ then $t = 1$ and we have already consider this case. Assume that $m \geq 4$. Let $f \in R_3(2(m - 2) + 1, m)$ such that $|f| = 8$. Assume that $S$ is not included in an affine subspace of dimension 3. Let $w_1, w_2, w_3, w_4$ 4 points of $S$ which are not included in a plane. Since $S$ is not included in an affine subspace of dimension 3, there exists $H$ an affine hyperplane such that $H$ contains $w_1, w_2, w_3, w_4$ but $S$ is not included in $H$. Then by Lemma 7.4 either $S$ meets 2 hyperplanes parallel to $H$ in 4 points or $S$ meets all hyperplanes parallel to $H$.

If $S$ meets 2 hyperplanes parallel to $H$ then $S \cap H$ is the support of a second weight codeword of $R_3(2(m - 1), m)$ so is included in a plane which is absurd since $w_1, w_2, w_3, w_4 \in S \cap H$. So $S$ meets all hyperplanes parallel to $H$ and for all $H'$ hyperplane parallel to $H$, $(S \cap H') \geq 2$. Since $S = 8$ and $(S \cap H) \geq 4$, for all $H'$ hyperplane parallel to $H$ different from $H$ $(S \cap H') = 2$ and $(S \cap H) = 4$.

By applying an affine transformation, we can assume that $x_1 = 0$ is an equation of $H$. Then $x_1, f \in R_3(2(m - 1), m)$ and $|x_1 f| = 4$ so $x_1, f$ is a second weight codeword of $R_3(2(m - 1), m)$ and its support is included in a plane $P$ not included in $H$. Let $H'$ be an affine hyperplane which contains $P$ and a point of $(S \cap H) \setminus P$ but not all the points of $S \cap H$. Then, $(S \cap H') \geq 5$ and $S \cap H' \neq S$. By applying the same argument to $H'$ than to $H$ we get a contradiction for reason of cardinality.
If \( m \leq 4 \), we have already considered all the possible values for \( t \). Assume that \( m \geq 5 \). Let \( 2 \leq t \leq m - 3 \). Assume that if \( f \in R_{3}(2(t + 1) + 1, m) \) is such that \( |f| = 8.3^{m-t-3} \) then its support is included in an affine subspace of dimension \( m - t \). Let \( f \in R_{3}(2t + 1, m) \) such that \( |f| = 8.3^{m-t-2} \) and denote by \( S \) its support. Assume that \( S \) is not included in an affine subspace of dimension \( m - t + 1 \). Then, there exists \( H \) an affine hyperplane such that \( S \cap H \neq S \) and \( S \cap H \) is not included in an affine subspace of dimension \( m - t \). So, by Lemma 7.4 either \( S \) meets \( 2 \) affine hyperplanes parallel to \( H \) in \( 4.3^{m-t-2} \) points or \( S \) meets all affine hyperplanes parallel to \( H \).

If \( S \) meets \( 2 \) affine hyperplanes in \( 4.3^{m-t-2} \) points, \( S \cap H \) is the support of a second weight codeword of \( R_{3}(2(t + 1), m) \) and is included in an affine subspace of dimension \( m - t \) which is absurd. So \( S \) meets all affine hyperplanes parallel to \( H \) and for all \( H' \) hyperplane parallel to \( H \),

\[
\#(S \cap H') \geq 2.3^{m-t-2}.
\]

Assume that for all \( H' \) parallel to \( H \), \( \#(S \cap H') > 2.3^{m-t-2} \). Then, for reason of cardinality \( \#(S \cap H) = 8.3^{m-t-3} \) and \( S \cap H \) is the support of a second weight codeword of \( R_{3}(2(t + 1) + 1, m) \) which is absurd since \( S \cap H \) is not included in an affine subspace of dimension \( m - t \). So there exists \( H_{1} \) parallel to \( H \) such that \( \#(S \cap H_{1}) = 2.3^{m-t-2} \) and \( S \cap H_{1} \) is the support of a minimal weight codeword of \( R_{3}(2(t + 1) + 1, m) \) so \( S \cap H_{1} \) is the union of \( P_{1} \) and \( P_{2} \) \( 2 \) parallel affine subspaces of dimension \( m - t - 2 \) included in an affine subspace of dimension \( m - t - 1 \).

Let \( A \) be an affine subspace of codimension \( 2 \) included in \( H_{1} \) and containing \( P_{1} \) and such that \( A \cap P_{2} = \emptyset \). Let \( A' \) be an affine subspace of codimension \( 2 \) included in \( H_{1} \) parallel to \( A \) which does not meet \( S \). Assume that there exists \( H_{1}' \) an affine hyperplane through \( A' \) which does not meet \( S \). Then, \( S \) meets \( H_{1}' \) and \( H_{1}' \) the \( 2 \) hyperplanes parallel to \( H_{1}' \) different from \( H_{1}' \) in \( 4.3^{m-t-2} \) points. For example, we can assume that \( A \subset H_{2}' \). Then, \( S \cap H_{1}' \) is the support of a second weight codeword of \( R_{3}(2(t + 1), m) \). So \( S \cap H_{1}' \) meets \( H \) in \( 0, 3^{m-t-2}, 2.3^{m-t-2} \) or \( 4.3^{m-t-2} \) points. Since \( S \) meets all hyperplanes parallel to \( H \) in at least \( 2.3^{m-t-2} \) points, if

\[
\#(S \cap H \cap H_{1}') \geq 4.3^{m-t-2},
\]

\( S \cap H \cap H_{2}' = \emptyset \). So \( S \cap H \) is included in an affine subspace of dimension \( m - t \) which is absurd. So \( S \cap H_{2}' \) and \( S \cap H_{3}' \) are the support of second weight codewords of \( R_{3}(2(t + 1), m) \) not included in \( H \), then their intersection with \( H \) is the union of at most \( 2 \) disjoint affine subspaces of dimension \( m - t - 2 \).

Now assume that \( S \) meets all hyperplanes through \( A' \). We denote by \( k \) the number of the hyperplanes through \( A \) which meet \( S \) in \( 2.3^{m-t-2} \) points. Then

\[
k2.3^{m-t-2} + (4 - k)8.3^{m-t-3} \leq 8.3^{m-t-2}
\]

which means that \( k \geq 4 \). So for all \( H' \) affine hyperplane through \( A' \) different from \( H_{1} \), \( S \cap H' \) is the support of minimum weight codeword of \( R_{3}(2(t + 1) + 1, m) \) which does not meet \( H_{1} \). So either \( S \cap H' \) is included.
in $H$ or $S \cap H'$ meets $S$ in an affine subspace of dimension $m - t - 2$. In both cases, $S \cap H$ is the union of at most 4 disjoint affine subspaces of dimension $m - t - 2$. By applying this argument to all affine subspaces of dimension 2 included in $H_1$ containing $P_1$ but not $P_2$, we get that $S \cap H$ is the union of 4 affine subspaces of dimension $m - t - 2$ which are a translation of $P_1$. This gives a contradiction since $S \cap H$ is not included in an affine subspace of dimension $m - t$. So $S$ is included in an affine subspace of dimension $m - t + 1$.

- Let $f \in R_3(2t + 1, m)$ such that $|f| = 8.3^{m-t-2}$ and $A$ the affine subspace of dimension $m - t + 1$ containing $S$. By applying an affine transformation, we can assume

$$A := \{(x_1, \ldots, x_m) \in \mathbb{F}_q^m : x_1 = \ldots = x_{t-1} = 0\}. $$

Let $g \in B_3^{m-t+1}$ defined for all $x = (x_t, \ldots, x_m) \in \mathbb{F}_3^{m-t+1}$ by

$$g(x) = f(0, \ldots, 0, x_t, \ldots, x_m)$$

and denote by $P \in \mathbb{F}_3[X_t, \ldots, X_m]$ its reduced form. Since

$$\forall x = (x_1, \ldots, x_m) \in \mathbb{F}_q^m, f(x) = (1 - x_1^2) \ldots (1 - x_{t-1}^2)P(x_t, \ldots, x_m),$$

the reduced form of $f \in R_3(t(q - 1) + s, m)$ is

$$(1 - X_1^2) \ldots (1 - X_{t-1}^2)P(X_t, \ldots, X_m).$$

Then $g \in R_3(3, m - t + 1)$ and $|g| = |f| = 8.3^{m-t-2}$. Thus, using the case where $t = 1$, we finish the proof of Proposition 2.6.

A Appendix : Blocking sets

Blocking sets have been studied by Bruen in [3, 2, 4] in the case of projective planes. Erickson extends his results to affine planes in [7].

**Definition A.1** Let $S$ be a subset of the affine space $\mathbb{F}_q^2$. We say that $S$ is a blocking set of order $n$ in $\mathbb{F}_q^2$ if for all line $L$ in $\mathbb{F}_q^2$, $\#(S \cap L) \geq n$ and $\#((\mathbb{F}_q^2 \setminus S) \cap L) \geq n$.

**Proposition A.2 (Lemma 4.2 in [7])** Let $q \geq 3$, $1 \leq b \leq q - 1$ and $f \in R_q(b, 2)$. If $f$ has no linear factor and $|f| \leq (q - b + 1)(q - 1)$, then the support of $f$ is a blocking set of order $(q - b)$ of $\mathbb{F}_q^2$.

In [7] Erickson make the following conjecture. It has been proved by Bruen in [4].

**Theorem A.3 (Conjecture 4.14 in [7])** If $S$ is a blocking set of order $n$ in $\mathbb{F}_q^2$, then $\#S \geq nq + q - n$. 

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