CP violation in modular invariant flavor models

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Abstract

We study the spontaneous CP violation through the stabilization of the modulus $\tau$ in modular invariant flavor models. The CP-invariant potential has the minimum only at $\text{Re}[\tau] = 0$ or $1/2$. From this prediction, we study CP violation in modular invariant flavor models. The physical CP phase is vanishing. The important point for the CP conservation is the $T$ transformation in the modular symmetry. One needs the violation of $T$ symmetry to realize the spontaneous CP violation.
1 Introduction

Particle physics still has several mysteries. Mysteries related to the flavor origin are one of important issues to study, i.e. the family number, the hierarchy of quark and lepton masses, their mixing angles, and the origin of the CP violation.

Various studies have been carried out to solve these flavor mysteries. One of the interesting approaches is non-Abelian discrete flavor symmetries [1–9]. In this approach, a non-Abelian discrete flavor symmetry is imposed and its non-trivial representations are assigned to three families of quarks and leptons. These flavor symmetries are broken by vacuum expectation values of scalar fields, the so-called flavon fields such that one realizes quark and lepton masses and their mixing angles and CP phases. A great deal of models have been constructed by use of various discrete groups such as \( S_N, A_N, \Delta(3N^2), \Delta(6N^2) \), etc. for quarks and leptons.

Underlying theory may have an origin for these flavor symmetries. Within the framework of extra dimensional field theory and superstring theory, geometrical symmetries of compact space can provide us with the origin of these flavor symmetries. Torus and orbifold compactifications are simple compactifications, and these compactifications have the geometrical symmetry, i.e. the so-called modular symmetry \( SL(2,\mathbb{Z}) \), which corresponds to the change of the torus basis. The ratio of basis vectors is denoted by the modulus \( \tau \), that is, the modulus describes the shape of the torus and the orbifold. The modular group transforms the modulus \( \tau \) non-trivially. Yukawa couplings and other couplings depend on the modulus. Thus, the modular group transforms these couplings non-trivially. Note that zero-modes corresponding to quarks and leptons transform each other under the modular group within the framework of superstring theory [15–21]. The modular group is in this sense a flavor symmetry. It is further interesting that the modular group \( SL(2,\mathbb{Z}) \) includes finite groups such as \( \Gamma_2 \simeq S_3 \), \( \Gamma_3 \simeq A_4 \), \( \Gamma_4 \simeq S_4 \) and \( \Gamma_5 \simeq A_5 \) [22], although the modular group has infinite order.

Recently, inspired by these aspects, a new approach of flavor model building was proposed in Ref. [23]. The \( A_4 \) subgroup of the modular group is assumed in Ref. [23]. The three families of leptons are assigned to non-trivial representations. The couplings are non-trivial representations of the \( A_4 \) modular symmetry and depend on the modulus. The couplings are modular forms and transform under \( A_4 \) in this scenario. We can construct models which leads to a realistic result [24,25] by fixing appropriate values of the modulus \( \tau \) and other parameters. The modular forms of weight 2 are fundamental and modular forms of higher weights can be obtained by their products. In the last year, such fundamental modular forms of weight 2 have been constructed for \( S_3 \) [26], \( S_4 \) [27], \( A_5 \) [28], \( \Delta(96) \), and \( \Delta(384) \) [29]. Modular forms of odd weights are possible for double covering groups. For example, the modular forms of the weight 1 and higher weights are also given for \( T' \) doublet [30]. The new approach of flavor model building to the flavors, i.e. the flavor modular symmetric models, has been studied by use of these modular forms [24,25,31,53].

One of the important features in these flavor modular symmetric models is that the flavor
symmetry is broken when the value of the modulus $\tau$ is fixed. Thus, one does not need flavon fields to break flavor symmetries. Fixing the value of $\tau$ is one of most important issues, which is called as the modulus stabilization. The modulus stabilization was studied within the framework of supergravity theory. One can find the modular invariance of supergravity theory in the literature \cite{54}. The modulus stabilization was studied by assuming the $SL(2,\mathbb{Z})$ modular invariance for the non-perturbative superpotential in supergravity theory \cite{58,59}. Recently, such analysis was extended to one of the flavor modular symmetric models \cite{61}.

Higher dimensional theories such as higher dimensional super Yang–Mills theory and superstring theory conserve CP. The four-dimensional CP symmetry can be embedded into $(4+d)$ dimensions as higher dimensional proper Lorentz symmetry with positive determinant \cite{62,67}. That is, one can combine the four-dimensional CP transformation and $d$-dimensional transformation with negative determinant so as to obtain $(4+d)$ dimensional proper Lorentz transformation. For example in six-dimensional theory, we denote the two extra coordinates by a complex coordinate $z$. The four-dimensional CP symmetry with $z \to z^*$ or $z \to -z^*$ is a six-dimensional proper Lorentz symmetry. (See for CP symmetry from the viewpoint of flavor modular symmetries \cite{38}.) Extensions to other dimensions are straightforward. Thus, breaking of such a symmetry corresponds to the CP violation. The above transformation of the coordinate $z$ corresponds to the transformation of the modulus.

The modulus stabilization fixes the modulus value. The CP violation can occur through the modulus stabilization \cite{68,70}. The purpose of this paper is to study the spontaneous CP violation through the modulus stabilization in flavor modular symmetric models.

This paper is organized as follows. In section 2, we give a brief review on the modular symmetry and the CP symmetry, which is embedded in higher dimensions. We also study the modulus stabilization in the CP-invariant scalar potential in section 3. It is shown that the minimum of the CP-invariant potential corresponds to $\text{Re}[\tau] = 0$ or $1/2$. We study the CP violation at $\text{Re}[\tau] = 1/2$ in an $A_4$ flavor model, and study generic aspect in section 4. Section 5 is conclusion. Appendices A and B show modular forms of the levels 3 and 4, respectively.

## 2 Modular symmetry and CP

### 2.1 Modular symmetry

We briefly review on the modular symmetry. The two-dimensional torus is constructed by $\mathbb{R}^2/\Lambda$, where $\Lambda$ is a two-dimensional lattice. The lattice itself is spanned by two basis vectors, $\alpha_1$ and $\alpha_2$. We denote them as $\alpha_1 = 2\pi R$ and $\alpha_2 = 2\pi R\tau$, where $R$ is real and $\tau$ is complex. One can use other basis to span the same lattice $\Lambda$, that is, the same lattice $\Lambda$ is spanned by

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2 See for their applications e.g. \cite{55,57}.

3 See also \cite{60}.
the following two lattice vectors,
\[
\begin{pmatrix}
\alpha_2' \\
\alpha_1'
\end{pmatrix} = 
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
\alpha_2 \\
\alpha_1
\end{pmatrix},
\]
where \(a, b, c, d\) are integer with satisfying \(ad - bc = 1\). That is the \(SL(2, \mathbb{Z})\) transformation.

The modulus \(\tau = \alpha_2/\alpha_1\) transforms under the above change of bases,
\[
\tau \longrightarrow \tau' = \gamma \tau = \frac{a\tau + b}{c\tau + d}.
\]
This is the modular symmetry. Note that the modulus transforms identically, \(\tau \rightarrow \tau\), under the \(\mathbb{Z}_2\) transformation, i.e. \(a = d = -1\) and \(b = c = 0\). The symmetry is therefore \(PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\mathbb{Z}_2\), which is denoted by \(\bar{\Gamma}\). We restrict ourselves the upper half plane for \(\tau\): \(\text{Im}[\tau] > 0\).

The modular group has two generators, \(S\) and \(T\), which transform
\[
S: \tau \longrightarrow -\frac{1}{\tau}, \quad T: \tau \longrightarrow \tau + 1.
\]
They satisfy the following algebraic relations,
\[
S^2 = (ST)^3 = I.
\]

The congruence subgroups of level \(N\) are defined as
\[
\bar{\Gamma}(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}), \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}.
\]

Furthermore, the quotient subgroups \(\Gamma_N\) are given as \(\Gamma_N = \bar{\Gamma}/\bar{\Gamma}(N)\). These are finite groups for \(N = 2, 3, 4, 5\), and isomorphic to \(A_N\) or \(S_N\): \(\Gamma_2 \simeq S_3\), \(\Gamma_3 \simeq A_4\), \(\Gamma_4 \simeq S_4\), \(\Gamma_5 \simeq A_5\), where the algebraic relation \(T^N = 1\) is satisfied in addition to Eq.\((4)\).

The modular forms of weight \(k\) are the holomorphic functions of \(\tau\) and transform as
\[
f_i(\tau) \longrightarrow (c\tau + d)^k \rho(\gamma)_{ij} f_j(\tau),
\]
under the modular symmetry, where \(\rho(\gamma)_{ij}\) is a unitary matrix under \(\Gamma_N\). The chiral matter fields also transform non-trivially
\[
(\phi^{(i)})_i(x) \longrightarrow (c\tau + d)^{-k_i} \rho(\gamma)_{ij}(\phi^{(j)})_j(x).
\]
where \( q = e^{2\pi i \tau} \) and \( \eta(\tau) \) and \( \omega = e^{i\frac{2\pi}{3}} \), and \( \eta(\tau) \) denotes the Dedekind eta function:

\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).
\] (9)

They are triplet under \( A_4 \simeq \Gamma_3 \), where \( S \) and \( T \) are represented by

\[
\rho(S) = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad \rho(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}.
\] (10)

These modular forms are expanded by \( q \) as

\[
Y^{3,2}_3 = \begin{pmatrix} Y^{3,2}_1(\tau) \\ Y^{3,2}_2(\tau) \\ Y^{3,2}_3(\tau) \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + \ldots \\ -6q^{1/3}(1 + 7q + 8q^2 + \ldots) \\ -18q^{2/3}(1 + 2q + 5q^2 + \ldots) \end{pmatrix}.
\] (11)

The modular forms of higher weights are obtained by their products. For example, the \( A_4 \) trivial-singlet modular form of the weight 4 can be written by

\[
Y^{3,4}_1 = (Y^{3,2}_1)^2 + 2Y^{3,2}_2Y^{3,2}_3.
\] (12)

Other modular forms of the level 3 and the weight 4 are shown in Appendix A.

There are two modular forms of the level 2 and the weight 2 for \( \Gamma(2) \). These can be written by [26],

\[
Y^{2,2}_1(\tau) = \frac{i}{4\pi} \left( \frac{\eta'(\tau/2)}{\eta(\tau/2)} + \frac{\eta'((\tau + 1)/2)}{\eta((\tau + 1)/2)} - \frac{8\eta'(2\tau)}{\eta(2\tau)} \right),
\]

\[
Y^{2,2}_2(\tau) = \frac{\sqrt{3}i}{4\pi} \left( \frac{\eta'(\tau/2)}{\eta(\tau/2)} - \frac{\eta'((\tau + 1)/2)}{\eta((\tau + 1)/2)} \right).
\]

They are doublet under \( S_3 \simeq \Gamma_2 \), where \( S \) and \( T \) are represented by

\[
\rho(S) = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \quad \rho(T) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (13)

They can be expanded by \( q \),

\[
Y^{2,2}_2 = \begin{pmatrix} Y^{2,2}_1(\tau) \\ Y^{2,2}_2(\tau) \end{pmatrix} = \begin{pmatrix} 1 + 3q + 3q^2 + 12q^3 + 3q^4 + \ldots \\ \sqrt{3q^{1/2}(1 + 4q + 6q^2 + 8q^3 + \ldots)} \end{pmatrix}.
\] (14)

Furthermore, the \( S_3 \) trivial singlet of the weight 4 can be written by

\[
Y^{2,4}_1 = (Y^{2,2}_1(\tau))^2 + (Y^{2,2}_2(\tau))^2.
\] (15)

The five modular forms of the level 4 and weight 2 for \( \Gamma(4) \) are found in Ref. [27]. These are shown in Appendix B. They correspond to \( 2 \) and \( 3' \) under \( S_4 \). The \( S_4 \) trivial singlet of the weight 4 can be written by \( q \)-expansion [38],

\[
Y^{4,4}_1 = \frac{1}{64} + \frac{15}{4} q + \frac{135}{4} q^2 + 135q^3 + \cdots,
\] (16)

up to an overall factor.
2.2 CP

The four-dimensional CP can be embedded to higher dimensional symmetry. We focus on the six dimensions here. A six-dimensional proper Lorentz symmetry is the combination of the four-dimensional CP and a transformation with negative determinant in the extra two dimensions. Such extra dimensional transformations correspond to $z \rightarrow z^*$ and $z \rightarrow -z^*$. Note that $z = x + \tau y$, where $x$ and $y$ are real coordinates. The latter transformation $z \rightarrow -z^*$ maps the upper half plane $\text{Im}[\tau] > 0$ to the same half plane. Hence, we consider the transformation $z \rightarrow -z^*$ as the CP symmetry. That means that the CP transforms the modulus $\tau$

$$\tau \rightarrow -\tau^*. \quad (17)$$

The same transformation of $\tau$ was derived from the viewpoint of generalized CP symmetry embedded in modular symmetry [38]. We study the modulus stabilization in the next section. Once the value of $\tau$ is fixed at generic point, the above symmetry (17) is broken. The CP is also spontaneously violated through the modulus stabilization, although $\text{Re}[\tau] = 0$ is a symmetric point.

3 Modulus stabilization

We study the modulus stabilization within the framework of supergravity theory following Ref. [61]. We focus only on the CP violation, that is, the value of $\text{Re}[\tau]$. We use a unit $M_p = 1$.

Supergravity theory Lagrangian can be written by $G$,

$$G = K + \ln |W|^2, \quad (18)$$

where $K$ and $W$ denote the Kähler potential and the superpotential. The scalar potential is written by

$$V = e^G(G^{-1}_{\tau \tau}|G_\tau|^2 - 3) = e^K(K^{-1}_{\tau \tau}|D_\tau W|^2 - 3|W|^2), \quad (19)$$

where

$$D_\tau W = K_\tau W + W_\tau, \quad (20)$$

with $G_\tau = \partial G/\partial \tau$, $K_\tau = \partial K/\partial \tau$ and $W_\tau = \partial W/\partial \tau$.

The typical Kähler potential of the modulus field $\tau$ is obtained as

$$K = -\ln[i(\bar{\tau} - \tau)], \quad (21)$$

and it transforms as

$$-\ln[i(\bar{\tau} - \tau)] \rightarrow -\ln[i(\bar{\tau} - \tau)] + \ln |c\tau + d|^2, \quad (22)$$

See for generalized CP [71,73].

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4 See for generalized CP [71,73].

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under the modular transformation. The superpotential should transform as

$$W \rightarrow \frac{W}{c\tau + d}, \quad (23)$$

since $G$ must be invariant. That is, the superpotential must be a holomorphic function of the modular weight $-1$.

Note that the Kähler potential in $G$ is invariant under the transformation, $\tau \rightarrow -\tau^*$. Thus, $G$ and the scalar potential $V$ are CP-invariant if $|W|^2$ is invariant:

$$W(\tau) \rightarrow W(-\tau^*) = e^{i\chi}W(\tau), \quad (24)$$

under the CP with $\tau \rightarrow -\tau^*$ including the CP transformation of chiral matter fields (See Ref. [38]).

Let us study the $A_4$ modular invariant model. When the modulus-dependent superpotential $W(\tau)$ is generated by some non-perturbative mechanism, the modulus can be stabilized at a potential minimum (See for early studies [58, 59]). The superpotential $W(\tau)$ must have the modular weight $-1$ as mentioned above. However, the $A_4$ modular invariant theory has no modular form of odd weight. We need some mechanism to compensate the difference of modular weights. We assume that the condensation $\langle Q\bar{Q} \rangle \neq 0$ occurs in the hidden sector by strong dynamics such as supersymmetric QCD; and then the following superpotential is induced,

$$W = \Lambda_d(Y_1^{(3,4)}(\tau))^{-1}, \quad (25)$$

where $\Lambda_d$ is the dynamical scale which is related to the condensation, e.g. $\Lambda_d = m\langle Q\bar{Q} \rangle$. We also assume that $\Lambda_d$ has the modular weight 3. Note that $|W|^2$ is invariant under $\tau \rightarrow -\tau^*$.

We analyze the potential minimum of the scalar potential $V$ with the above ansatz for the superpotential Eq. (25). The supersymmetric vacuum corresponds to $D_\tau W = 0$. However, there is no solution for $D_\tau W = 0$ at a finite value of $\tau$. There is no supersymmetric vacuum, but there is a non-supersymmetric vacuum. Figure 1 shows the potential minima at

$$\tau_{\text{min}} = 1.09i + n, \quad (26)$$

where $n$ is integer. Thus, the stabilized point is $\text{Re}[\tau] = 0$ (mod 1). CP is not violated at these minima. This result can be found reasonable by the $q$-expansion of the singlet modular form $Y_1^{(3,4)}(\tau) = 1 + a_1q + a_2q^2 + \cdots$, where $a_i$ are positive. The scalar potential $V$ depends on $\cos 2\pi n\text{Re}[\tau]$ as well as $\sin 2\pi n\text{Re}[\tau]$, and the potential becomes minimum at $\cos 2\pi\text{Re}[\tau] = 1$.

We also assume and analyze the potential minimum of the following superpotential alternatively,

$$W = \Lambda_d(Y_1^{(3,4)}(\tau)), \quad (27)$$

where we assume that $\Lambda_d$ has the modular weight $-5$. Again, there is no supersymmetric vacuum satisfying $D_\tau W = 0$ at a finite value of $\tau$. Figure 2 shows the potential minima at

$$\tau_{\text{min}} = 1.09i + p/2, \quad (28)$$

where $p$ is integer.
where $p$ is odd. Thus, we have obtained $\text{Re}[\tau] = 1/2 \pmod{1}$. This result also can be understood by the $q$-expansion. The symmetry Eq. (17) is violated at this minimum. The modular forms have CP phases at $\text{Re}[\tau] = 1/2$. Such phases may appear as physical CP phases. We will discuss its meaning in the next section.

We can also discuss the $S_3$ invariant model. The scalar potential $V$ with the following superpotential:

$$ W = \Lambda_d (Y_{1}^{(2,4)}(\tau))^{-1}, $$

has the minimum at $\text{Re}[\tau] = 0$. On the other hand, the scalar potential $V$ with the following superpotential:

$$ W = \Lambda_d (Y_{1}^{(2,4)}(\tau)), $$

has the minimum at $\text{Re}[\tau] = 1/2$.

We discuss the $S_4$ invariant model. The scalar potential $V$ with the following superpotential:

$$ W = \Lambda_d (Y_{1}^{(4,4)}(\tau))^{-1}, $$

has the minimum at $\text{Re}[\tau] = 0$, while the scalar potential $V$ with the following superpotential:

$$ W = \Lambda_d (Y_{1}^{(4,4)}(\tau)), $$

has the minimum $\text{Re}[\tau] = 1/2$.

The minima of CP-invariant scalar potential are obtained at $\text{Re}[\tau] = 0$ or $1/2$ as the results of explicit potential analyses. We can explain these results by using the $q$-expansions of the singlet modular forms. We give another explanation for generic model. Suppose that the scalar potential $V(\tau, \bar{\tau})$ is invariant under the CP transformation,

$$ \text{Re}[\tau] \rightarrow -\text{Re}[\tau], $$

$$ \text{Im}[\tau] \rightarrow -\text{Im}[\tau], $$

$$(33)$$
which is the reflection symmetry at \( \text{Re}[\tau] = 0 \). That implies that
\[
\frac{\partial V}{\partial a} = 0,
\] at \( \text{Re}[\tau] = 0 \), where \( a = \text{Re}[\tau] \). The point \( \text{Re}[\tau] = 0 \) corresponds to a minimum or maximum of the CP-invariant scalar potential \( V \). Note that the scalar potential \( V(\tau, \bar{\tau}) \) is a trivial singlet under the \( T \)-transformation: \( \tau \to \tau + 1 \). Hence, the scalar potential must have the periodicity, \( \text{Re}[\tau] = \text{Re}[\tau] + 1 \) although the generic modular form of \( \Gamma(N) \) has the periodicity, \( \text{Re}[\tau] = \text{Re}[\tau] + N \): \( T^N = \mathbb{I} \). It means that if \( \text{Re}[\tau] = 0 \) (mod 1) leads to the minimum (maximum), \( \text{Re}[\tau] = 1/2 \) is maximum (minimum). Therefore, the minimum of the CP-invariant scalar potential corresponds to \( \text{Re}[\tau] = 0 \) or 1/2. That is a quite clear result. This statement is also true for non-supersymmetric potential although we have studied the scalar potential in supergravity theory. We emphasize that the important point is the \( T \)-transformation in this result: \( \text{Re}[\tau] = \text{Re}[\tau] + 1 \), while the CP-invariant potential has the minimum at \( \text{Re}[\tau] = 0 \) or 1/2.

4 CP violation in \( A_4 \) models

We have shown that the CP-invariant scalar potential has the minimum \( \text{Re}[\tau] = 0 \) or 1/2 in the previous section. That is the quite strong prediction. It is obvious that the CP is not violated when \( \text{Re}[\tau] = 0 \). On the other hand, the symmetry of Eq. (17) is violated when \( \text{Re}[\tau] = 1/2 \). Indeed, the modular forms have non-vanishing phase at \( \text{Re}[\tau] = 1/2 \) in general because the modular forms of the level \( N \) are expanded by \( q^{1/N} \). We study the meaning of this phase for \( \text{Re}[\tau] = 1/2 \) by using an \( A_4 \) flavor model in Refs. [25] and discuss a generic aspect.

First, let us study the \( A_4 \) modular invariant model in Ref. [25]. Table 1 shows the modular weights and \( A_4 \) representations of the left-handed leptons and right-handed charged leptons in the model of Ref. [25]. The model in Ref. [25] is the global supersymmetric model where the superpotential has the vanishing modular weight. We re-arrange the modular weights of chiral superfields such that the supergravity superpotential has the modular weight \(-1\).

|      | \( L \) | \( e_R, \mu_R, \tau_R \) | \( H_u \) | \( H_d \) |
|------|---------|--------------------------|---------|---------|
| \( SU(2) \) | 2       | 1                        | 2       | 2       |
| \( A_4 \) | 3       | 1, 1", 1'                | 1       | 1       |
| \( -k_I \) | -1      | -1                       | -1/2    | -1      |

Table 1: The charge assignment of \( SU(2) \), \( A_4 \), and the modular weights \(-k_I\) for fields.

The Yukawa couplings terms in the superpotential are written by
\[
W_e = \alpha e_R H_d(L Y^{3,2}_3) + \beta \mu_R H_d(L Y^{3,2}_3) + \gamma \tau_R H_d(L Y^{3,2}_3),
\]
(35)
and the Weinberg operator terms are written by

\[ W_\nu = -\frac{1}{\Lambda} (H_u H_u^{\dagger})_{11} L, \]  

(36)

where \( \alpha, \beta, \gamma, \) and \( \Lambda \) are real. These superpotential terms are CP-invariant satisfying Eq. 24. The phase appears only in \( q = e^{2\pi i \tau} \) of the modular forms. The charged lepton mass matrix is written by

\[ M_E = v_d \text{diag}[\alpha, \beta, \gamma] \begin{pmatrix} Y_{12}^2 & Y_{13}^2 & Y_{23}^2 \\ Y_{13}^2 & Y_{23}^2 & Y_{12}^2 \\ Y_{23}^2 & Y_{12}^2 & Y_{13}^2 \end{pmatrix}_{RL}, \]  

(37)

and the neutrino mass matrix is written by

\[ M_\nu = -\frac{v_v^2}{\Lambda} \begin{pmatrix} 2Y_{12}^3 & -Y_{13}^3 & -Y_{23}^3 \\ -Y_{13}^3 & 2Y_{23}^3 & -Y_{12}^3 \\ -Y_{23}^3 & -Y_{12}^3 & 2Y_{13}^3 \end{pmatrix}_{LL}, \]  

(38)

where we renormalize the coefficients \( \alpha, \beta, \gamma \) and \( \Lambda \) by normalization factors of kinetic terms, because such normalization factors are irrelevant to the CP phase.

We set \( \text{Re}[\tau] = 1/2 \) and the mass matrices have the following phase behavior as shown in Eq. 11:

\[ M_E = \begin{pmatrix} m_{e11} & m_{e12} e^{2i\phi} & m_{e13} e^{i\phi} \\ m_{e21} e^{i\phi} & m_{e22} & m_{e23} e^{2i\phi} \\ m_{e31} e^{2i\phi} & m_{e32} e^{i\phi} & m_{e33} \end{pmatrix}, \]  

(39)

and

\[ M_\nu = \begin{pmatrix} m_{\nu11} & m_{\nu12} e^{2i\phi} & m_{\nu13} e^{i\phi} \\ m_{\nu21} e^{i\phi} & m_{\nu22} e^{i\phi} & m_{\nu23} \\ m_{\nu31} e^{2i\phi} & m_{\nu32} & m_{\nu33} e^{2i\phi} \end{pmatrix}, \]  

(40)

where \( \phi = \pi/3 \), and \( m_{eij} \) and \( m_{\nu ij} \) are real. These phase structure is very specific and we can rephase \( L_i \) and \( E_i \) such that all the phases vanish in these mass matrices. Thus, there is no physical CP phase.

The above behavior of the CP phase appears not only in the above model. That is a rather generic property. Let us study the flavor model with the \( \Gamma_N \) flavor modular symmetry. We use the basis that \( \rho(T) \) is represented by a diagonal matrix. It means that the chiral field \( \Phi_i \) transforms

\[ \Phi_i \rightarrow e^{2\pi ik_i/N} \Phi_i, \]  

(41)

under the \( T \) transformation \( \tau \rightarrow \tau + 1 \), where \( k_i \) is integer because \( \rho(T)^N = I \). That is, the \( Z_N \) symmetry. We study the mass terms in the superpotential,

\[ W = M_{ij} \Phi_i \Phi_j. \]  

(42)
These mass terms may originate from the Yukawa coupling terms or Weinberg operator terms in the superpotential. The mass matrix $M_{ij}$ depends on the modulus $\tau$, and the mass matrix $\tilde{M}_{ij}$ must transform under the $T$ transformation. At any rate, $M_{ij} \Phi_i \Phi_j$ should be invariant under the $T$ transformation because it is a trivial singlet of $\Gamma_N$. It transforms as $M_{ij} \rightarrow e^{-2\pi i(k_i+k_j)/N} M_{ij}$ to cancel the transformation of $\Phi_i \Phi_j$. Hence, the mass matrix must have the following form:

$$M_{ij} = m_{ij}(q) e^{-\pi i (k_i+k_j)/N},$$

(43)

where $m_{ij}(q)$ include only integer powers of $q$, i.e. $q^n$. When $\text{Re}[\tau] = 1/2$, the phase behavior of the mass matrix must be written by

$$M_{ij} = \tilde{m}_{ij} e^{-\pi i (k_i+k_j)/N},$$

(44)

where $\tilde{m}_{ij} = m_{ij} e^{2\pi i(k_i+k_j) \text{Im}[\tau]/N}$ and they are real. Such phases can be canceled by rephasing $\Phi_i \rightarrow e^{\pi i k_i/N} \Phi_i$. Thus, there is no physical CP phase.

As a result, the CP conserves in the modular invariant flavor model at $\text{Re}[\tau] = 1/2$ as well as $\text{Re}[\tau] = 0$; and these values of $\text{Re}[\tau]$ are realized as the minimum of the modular invariant potential. The spontaneous CP violation does not occur in general. An important point of this result is the $T$ invariance. The $T$ invariance prevents the CP violation. The spontaneous CP violation may occur if the $T$ symmetry is violated.

For example, some representations of $\Gamma_N$ have $\det \rho(T) \neq 1$. Fermions with such representations can lead to anomalies. Non-perturbative effects can induce breaking terms for anomalous symmetries. If the $T$ symmetry is anomalous, mass terms corresponding to non-trivial singlets can appear the superpotential, e.g.

$$W = (M_{ij}^{(0)} + M_{ij}^{(1)} + M_{ij}^{(2)} \cdots) \Phi_i \Phi_j,$$

(45)

where $M_{ij}^{(0)} = M_{ij}$ in the mass term. The mass matrix $M_{ij}^{(k)}$ has the $Z_N$ charge different from $M_{ij}^{(0)}$ by $k$, and transforms as

$$\frac{M_{ij}^{(k)}}{M_{ij}^{(0)}} \rightarrow e^{2\pi i k/N} M_{ij}^{(k)},$$

(46)

We can not cancel the phases in the mass matrices by rephasing and the physical CP appear in this case. Thus, the violation of the $T$ symmetry is important to realize the spontaneous CP violation.

The $T$ symmetry in the above $A_4$ model is anomaly-free. The $T$ symmetry becomes anomalous if we consider a specific assignment: for example, $\mathbf{1}$ and two $\mathbf{1}'$ to the three families of right-handed leptons. However, it is another problem whether such assignment can lead to realistic masses and mixing angles.

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5 See also [21].
5 Conclusion

We have studied the CP violation through the modulus stabilization. The CP-invariant potential has the minimum at Re[τ] = 0 or 1/2. That is a quite strong prediction. It is clear that the CP is not violated at Re[τ] = 0. However, some modular forms have non-vanishing phases at Re[τ] = 1/2, and they may lead to the physical CP phase. We have studied explicitly the $A_4$ flavor model at Re[τ] = 1/2. This $A_4$ model has a specific structure in the CP phase. The phases of the modular forms at Re[τ] = 1/2 does not appear as the physical CP phase. This behavior is not a special case unique to this model. It is a rather generic property of the CP-invariant and modular invariant flavor models. In particular, the $T$ transformation is important. The scalar potential has the periodicity $\tau \sim \tau + 1$ since the potential is trivial singlet of $T$. Such periodicity leads to the strong prediction: the minimum is realized at Re[τ] = 0 or 1/2. The modular forms have phases at Re[τ] = 1/2. The $T$ invariance, the $Z_N$ symmetry, leads to a phase behavior such that phases can be canceled by rephasing of the fields and the physical CP phase does not appear.

One needs violation of $T$ symmetry to realize the spontaneous CP violation. For example, anomaly of the $T$ symmetry may lead to the spontaneous CP violation by non-perturbative effects.

Acknowledgement

This work is supported by MEXT KAKENHI Grant Number JP19H04605 (TK), and JSPS Grants-in-Aid for Scientific Research 18J11233 (THT). The work of YS is supported by JSPS KAKENHI Grant Number JP17K05418 and Fujyukai Foundation.

Appendix

A Modular forms for $A_4$

The modular forms of the level 3 and weight 4 are obtained by products of the modular forms of the weight 2 [23]. There are five modular forms of the weight 4, and they correspond to 1, $1'$, and $3$ of $A_4$,

\[ Y_{1}^{3,4} = (Y_{1}^{3,2})^2 + 2Y_{2}^{3,2}Y_{3}^{3,2}, \quad Y_{1}^{3,4} = (Y_{1}^{3,2})^2 + 2Y_{1}^{3,2}Y_{2}^{3,2}, \]

\[ Y_{3}^{3,4} = \begin{pmatrix} Y_{31}^{3,4}(	au) \\ Y_{32}^{3,4}(	au) \\ Y_{33}^{3,4}(	au) \end{pmatrix} = \begin{pmatrix} (Y_{1}^{3,2})^2 - Y_{2}^{3,2}Y_{3}^{3,2} \\ (Y_{3}^{3,2})^2 - Y_{1}^{3,2}Y_{2}^{3,2} \\ (Y_{2}^{3,2})^2 - Y_{1}^{3,2}Y_{3}^{3,2} \end{pmatrix}. \]
B Modular forms for $S_4$

The modular forms of the level 4 and the weight 2 are written by [27],

\[
Y_1(\tau) = Y(1, 1, \omega, \omega^2, \omega^2|\tau),
Y_2(\tau) = Y(1, 1, \omega^2, \omega, \omega^2|\tau),
Y_3(\tau) = Y(1, -1, -1, -1, 1|\tau),
Y_4(\tau) = Y(1, -1, -\omega^2, -\omega, \omega^2|\tau),
Y_5(\tau) = Y(1, -1, -\omega, -\omega^2, \omega, \omega^2|\tau),
\]

(49)

where

\[
Y(a_1, a_2, a_3, a_4, a_5, a_6 \tau) = a_1 \eta'(\tau + 1/2) + a_2 \eta'(4\tau) + a_3 \eta'((\tau + m)/4).
\]

(50)

These five modular forms correspond to 2 and 3' representations under $\Gamma_4 \cong S_4$

\[
Y_{S_42}(\tau) = \left( \begin{array}{c} Y_1(\tau) \\ Y_2(\tau) \end{array} \right), \quad Y_{S_43'}(\tau) = \left( \begin{array}{c} Y_3(\tau) \\ Y_4(\tau) \\ Y_5(\tau) \end{array} \right).
\]

(51)

They represent the generators, $S$ and $T$ as

\[
\rho(S) = \left( \begin{array}{cc} 0 & \omega \\ \omega^2 & 0 \end{array} \right), \quad \rho(T) = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right),
\]

(52)

for 2, and

\[
\rho(S) = -\frac{1}{3} \left( \begin{array}{ccc} -1 & 2\omega^2 & 2\omega \\ 2\omega & 2 & -\omega^2 \\ 2\omega^2 & -\omega & 2 \end{array} \right), \quad \rho(T) = -\frac{1}{3} \left( \begin{array}{ccc} -1 & 2\omega & 2\omega^2 \\ 2\omega & 2 & -1 \\ 2\omega^2 & -1 & 2\omega \end{array} \right),
\]

(53)

for 3'. These are not symmetric. The modular form of weight 4 corresponding to the $S_4$ trivial singlet is written by

\[
Y_{4}^{1,4} = Y_1 Y_2.
\]

(54)

Note that the trivial singlet is obviously symmetric.

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