Coverings preserving the bottom of the spectrum

Panagiotis Polymerakis

Abstract

We prove that if a Riemannian covering preserves the bottom of the spectrum of a Schrödinger operator, which belongs to the discrete spectrum of the operator on the base manifold, then the covering is amenable.

1 Introduction

The spectrum of the Laplacian on a Riemannian manifold is a natural isometric invariant. However, its behavior under maps between Riemannian manifolds, which respect the geometry of the manifolds to some extent, remains largely unclear. In this paper, we study the behavior of the bottom (that is, the minimum) of the spectrum under Riemannian coverings.

Let \( p: M_2 \to M_1 \) be a Riemannian covering, \( S_1 = \Delta + V \) a Schrödinger operator on \( M_1 \), with \( V \) smooth and bounded from below, and \( S_2 = \Delta + V \circ p \) its lift on \( M_2 \). Then the bottoms of their spectra always satisfy the inequality \( \lambda_0(S_1) \leq \lambda_0(S_2) \). It is natural to examine when the equality holds. Brooks [7] proved that if the underlying manifold is closed (that is, compact without boundary), then a normal covering \( p \) preserves the bottom of the spectrum of the Laplacian if and only if \( p \) is amenable.

This theorem motivated the study of the behavior of the bottom of the spectrum under amenable coverings. Extending some former results [3, 6], it was proved in [2] that amenable Riemannian coverings preserve the bottom of the spectrum of Schrödinger operators, without any topological or geometric assumptions on the manifolds. In [14], it was proved that if, in addition, \( M_1 \) is complete, then the spectra of the operators satisfy \( \sigma(S_1) \subset \sigma(S_2) \). If, in addition, the covering is infinite sheeted, then \( \sigma(S_1) \subset \sigma_{\text{ess}}(S_2) \), where \( \sigma_{\text{ess}} \) stands for the essential spectrum of the operator.

Although amenability of the covering is a natural assumption for the preservation of the bottom of the spectrum, it is not clear to what extent it is optimal. In this direction, Brooks [6], and Roblin and Tapie [15], proved that under some quite restrictive
assumptions, if the bottom of the spectrum of the Laplacian is preserved, then the covering is amenable. These assumptions involve the spectrum of fundamental domains of the covering and in particular, imply that the bottom of the spectrum of the Laplacian on $M_1$ belongs to its discrete spectrum (that is, the bottom is an isolated point of the spectrum and in particular, an eigenvalue). Moreover, in both results, the covering is assumed to be normal, with finitely generated deck transformations group. Recently, in [1], these conditions were replaced with some more natural geometric assumptions. More precisely, it was proved that if the manifolds are complete, without boundary, with Ricci curvature bounded from below, $V$ and $\text{grad} V$ are bounded, the bottom of the spectrum is preserved, and belongs to the discrete spectrum of $S_1$, then the covering is amenable. A question raised in [1] is whether the assumption on the Ricci curvature is necessary. In this paper, we deal with this question and establish a generalization of all the above results.

Initially, using the result of [1] we prove the following analogue of Brooks’ theorem [7], involving the bottom of the Neumann spectrum of the Laplacian on manifolds with (smooth) boundary.

**Theorem 1.1.** Let $p: M_2 \to M_1$ be a Riemannian covering, with $M_1$ compact with boundary. Then $\lambda_0^N(M_2) = 0$ if and only if $p$ is amenable.

The fact that amenable coverings preserve the bottom of the Neumann spectrum was essentially established in [14]. The main point in the above theorem is the converse implication, which is the first result providing amenability of a covering of manifolds with boundary. This turns out to be quite useful in the study of arbitrary Riemannian coverings. More precisely, as an application of this theorem, we prove the following result.

**Theorem 1.2.** Let $p: M_2 \to M_1$ be a Riemannian covering. Let $S_1$ be a Schrödinger operator on $M_1$, with $\lambda_0(S_1) \notin \sigma_{\text{ess}}(S_1)$, and $S_2$ its lift on $M_2$. Then $\lambda_0(S_2) = \lambda_0(S_1)$ if and only if $p$ is amenable.

It is worth to point out that we do not impose any topological or geometric assumptions on the manifolds. Hence, Theorem 1.2 is more general than the results of [1, 6, 15], since their assumptions imply that $\lambda_0(S_1) \notin \sigma_{\text{ess}}(S_1)$. Examining the optimality of the assumption $\lambda_0(S_1) \notin \sigma_{\text{ess}}(S_1)$ in this theorem, we show that it cannot be replaced with $\lambda_0(S_1)$ being an eigenvalue.

Since the manifolds in the above theorem may be non-complete, we obtain immediately the corresponding result for Dirichlet spectra of Schrödinger operators on manifolds with boundary. For sake of completeness, we also establish the corresponding result for the Neumann spectra, obtaining a generalization of Theorem 1.1. In particular, we obtain analogues of Brooks’ result [7] for Schrödinger operators on manifolds with boundary. Namely, it follows that a Riemannian covering of a compact manifold is amenable if and only if it preserves the bottom of the Dirichlet/Neumann spectrum of some/any Schrödinger operator. In virtue of [14, Theorem 1], this is actually equivalent to the inclusion of the Dirichlet (Neumann) spectrum of a Schrödinger operator on $M_1$, in the
Dirichlet (Neumann, respectively) spectrum of its lift on $M_2$. The corresponding statement for manifolds without boundary has been established in [14, Theorem 1.5].

Finally, as another application of Theorem 1.1, we prove that if an infinite sheeted Riemannian covering preserves the bottom of the spectrum of a Schrödinger operator, then the bottom of the spectrum belongs to the essential spectrum of the operator on the covering space. This was observed for the Laplacian in [1]. For sake of completeness, we establish the analogous result for the Dirichlet and Neumann spectra of Schrödinger operators on manifolds with boundary.

The paper is organized as follows: In Section 2, we give some preliminaries. In Section 3, we present some properties of the spectrum of Schrödinger operators. In Section 4, we study Riemannian coverings of compact manifolds and establish Theorem 1.1. In Section 5, we study arbitrary Riemannian coverings and prove Theorem 1.2 and the corresponding results for manifolds with boundary. In Section 6, we show the aforementioned application of Theorem 1.1 for infinite sheeted coverings.

Acknowledgements. I would like to thank Werner Ballmann and Dorothee Schüth for their helpful comments and remarks. I am also grateful to the Max Planck Institute for Mathematics in Bonn for its support and hospitality.

2 Preliminaries

We begin by recalling some definitions and standard facts from functional analysis, which may be found for instance, in [13] and [17, Appendix A].

Let $L: \mathcal{D}(L) \subset \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$, over $\mathbb{R}$ or $\mathbb{C}$. The spectrum of $L$ is given by

$$\sigma(L) := \{ \lambda \in \mathbb{R} : (L - \lambda): \mathcal{D}(L) \subset \mathcal{H} \to \mathcal{H} \text{ not invertible} \}.$$ 

The essential spectrum of $L$ is defined as

$$\sigma_{\text{ess}}(L) := \{ \lambda \in \mathbb{R} : (L - \lambda): \mathcal{D}(L) \subset \mathcal{H} \to \mathcal{H} \text{ not Fredholm} \}.$$ 

Recall that an operator is called Fredholm if its kernel is finite dimensional and its range is closed and of finite codimension. The discrete spectrum of $L$ is given by $\sigma_d(L) := \sigma(L) \setminus \sigma_{\text{ess}}(L)$, and consists of isolated eigenvalues of $L$ of finite multiplicity.

The spectrum of a self-adjoint operator is a closed subset of $\mathbb{R}$. If $\sigma(L)$ is bounded from below, then its minimum is called the bottom of the spectrum of $L$ and is denoted by $\lambda_0(L)$. The following characterization is due to Rayleigh.

**Proposition 2.1.** If $\sigma(L)$ is bounded from below, then the bottom of the spectrum of $L$ is given by

$$\lambda_0(L) = \inf_{v \in \mathcal{D}(L) \setminus \{0\}} \frac{\langle Lv, v \rangle_{\mathcal{H}}}{\|v\|^2_{\mathcal{H}}}.$$
Let $T: \mathcal{D}(T) \subset \mathcal{H} \to \mathcal{H}$ be a densely defined, symmetric linear operator. Assume that $T$ is bounded from below, that is, there exists $c \in \mathbb{R}$, such that
\[
\langle Tv, v \rangle_{\mathcal{H}} \geq c \|v\|_{\mathcal{H}}^2,
\]
for all $v \in \mathcal{D}(T)$. Fix such a $c \in \mathbb{R}$ (not necessarily the supremum of all $c$ for which (1) holds) and consider the inner product
\[
\langle v_1, v_2 \rangle_{\mathcal{H}_1} := \langle Tv_1, v_2 \rangle_{\mathcal{H}} + (1 - c)\langle v_1, v_2 \rangle_{\mathcal{H}}
\]
on $\mathcal{D}(T)$. Let $\mathcal{H}_1$ be the completion of $\mathcal{D}(T)$ with respect this inner product. Then $\mathcal{H}_1$ can be identified with a dense subspace of $\mathcal{H}$, via a continuous injection. The domain of the Friedrichs extension $T^{(F)}$ of $T$ is given by
\[
\mathcal{D}(T^{(F)}) := \{ v \in \mathcal{H}_1 : \text{there exists } v' \in \mathcal{H}, \text{ such that } \langle v', w \rangle_{\mathcal{H}} = \langle v, w \rangle_{\mathcal{H}_1}, \text{ for all } w \in \mathcal{H}_1 \}.
\]
For $v \in \mathcal{D}(T^{(F)})$, we define $T^{(F)}v := v' + (c - 1)v$. Then $T^{(F)}$ is a self-adjoint extension of $T$ and is called the Friedrichs extension of $T$.

**Proposition 2.2.** The bottom of the spectrum of the Friedrichs extension of $T$ is given by
\[
\lambda_0(T^{(F)}) = c - 1 + \inf_{v \in \mathcal{H}_1 \setminus \{0\}} \frac{\|v\|_{\mathcal{H}_1}^2}{\|v\|_{\mathcal{H}}^2},
\]
where the infimum may be taken over any subspace $\mathcal{H}'$, with $\mathcal{D}(T) \subset \mathcal{H}' \subset \mathcal{H}_1$.

**Proof:** Evidently, for a non-zero $v \in \mathcal{D}(T^{(F)})$, we have
\[
c - 1 + \frac{\|v\|_{\mathcal{H}_1}^2}{\|v\|_{\mathcal{H}}^2} = \frac{\langle T^{(F)}v, v \rangle_{\mathcal{H}}}{\|v\|_{\mathcal{H}}^2}.
\]
From Proposition 2.1, we obtain the asserted equality, where the infimum is taken over all $v \in \mathcal{D}(T^{(F)}) \setminus \{0\}$. From the definition of $\mathcal{H}_1$, it is easy to see that we obtain the same infimum for $v \in \mathcal{D}(T) \setminus \{0\}$ and for $v \in \mathcal{H}_1 \setminus \{0\}$. 

### 2.1 Schrödinger operators

Throughout this paper, manifolds are assumed to be connected with not necessarily connected, possibly empty, smooth boundary, unless otherwise stated. Let $M$ be a possibly non-connected Riemannian manifold. A *Schrödinger operator* on $M$ is an operator of the form $S = \Delta + V$, where $\Delta$ is the (non-negative definite) Laplacian and $V: M \to \mathbb{R}$ is smooth and bounded from below. On the space $C_c^\infty(M)$ consider the inner product
\[
\langle f, g \rangle_{\mathcal{H}_V(M)} := \int_M (\langle \text{grad } f, \text{grad } g \rangle + (V - \inf_M V + 1)fg).
\]
If \( M \) has empty boundary, let \( H_V(M) \) be the completion of \( C^\infty_c(M) \) with respect to this inner product. If \( M \) has non-empty boundary, let \( H_V(M) \) be the completion of \( \{ f \in C^\infty_c(M) : \nu(f) = 0 \text{ on } \partial M \} \) with respect to this inner product, where \( \nu \) is the inward pointing normal to \( \partial M \). It is clear that \( H_V(M) \) can be identified with a dense subspace of \( L^2(M) \), via a continuous injection.

If \( M \) has empty boundary, we are interested in the Friedrichs extension of the operator

\[
S : C^\infty_c(M) \subset L^2(M) \to L^2(M).
\]

If \( M \) has non-empty boundary, we are interested in the Neumann extension of \( S \), that is, the Friedrichs extension of

\[
S : \{ f \in C^\infty_c(M) : \nu(f) = 0 \text{ on } \partial M \} \subset L^2(M) \to L^2(M).
\]

In any of these cases, we denote this Friedrichs extension by \( S^N \) and its domain by \( \mathcal{D}(S^N) \).

The spectrum and the essential spectrum of \( S^N \) are denoted by \( \sigma^N(S) \) and \( \sigma^N_{\text{ess}}(S) \), respectively, and their bottoms (that is, their minimums) by \( \lambda^N_0(S) \) and \( \lambda^N_{0,\text{ess}}(S) \), respectively. These sets and quantities for the Laplacian are denoted by \( \sigma^N(M) \), \( \sigma^N_{\text{ess}}(M) \) and \( \lambda^N_0(M) \), \( \lambda^N_{0,\text{ess}}(M) \), respectively. If \( M \) has empty boundary, we sometimes drop the superscript “\( N \)” in the notation of the spectrum, the essential spectrum and their bottoms.

If \( M \) has non-empty boundary, the Dirichlet extension \( S^D \) of \( S \) is the Friedrichs extension of the operator

\[
S : \{ f \in C^\infty_c(M) : f = 0 \text{ on } \partial M \} \subset L^2(M) \to L^2(M).
\]

We denote by \( \lambda^D_0(S) \) the bottom of the spectrum of this operator. The bottom of the spectrum of the Dirichlet extension of the Laplacian is denoted by \( \lambda^D_0(M) \). In virtue of the next remark, Dirichlet extensions of Schrödinger operators are closely related to Schrödinger operators on non-complete manifolds without boundary.

**Remark 2.3.** If \( M \) is a Riemannian manifold with non-empty boundary, then any \( f \in C^\infty_c(M) \) vanishing on \( \partial M \), can be approximated in \( H^1(M) \) with smooth functions, compactly supported in the interior of \( M \). Therefore, if \( S \) is a Schrödinger operator on \( M \), then the Dirichlet extension of \( S \) coincides with the Friedrichs extension of \( S \) viewed as an operator in the interior of \( M \).

We end this subsection with some already known properties of the spectrum, that will be used in the sequel. Since they will be used only for complete manifolds without boundary, we do not state them in their most general forms.

The next proposition characterizes the bottom of the spectrum of a Schrödinger operator as the maximum of its positive spectrum, and may be found in [8, Theorem 7], [12, Theorem 1] and [16, Theorem 2.1].
Proposition 2.4. Let $S$ be a Schrödinger operator on a complete Riemannian manifold $M$ without boundary. Then $\lambda_0(S)$ is the maximum of all $\lambda \in \mathbb{R}$, such that there exists a positive $\varphi \in C^\infty(M)$, with $S\varphi = \lambda \varphi$.

It is worth to point out that the positive functions involved in this proposition are not required to be square-integrable. The next expression of the bottom of the essential spectrum follows from the Decomposition Principle [9, Proposition 2.1].

Proposition 2.5 ([4, Proposition 3.2]). Let $S$ be a Schrödinger operator on a complete Riemannian manifold $M$ without boundary. Let $(K_n)_{n \in \mathbb{N}}$ be an exhausting sequence of $M$ consisting of compact sets. Then the bottom of the essential spectrum of $S$ is given by

$$\lambda_0^{\text{ess}}(S) = \lim_n \lambda_0(S, M \setminus K_n),$$

where $\lambda_0(S, M \setminus K_n)$ stands for the bottom of the spectrum of $S$ on $M \setminus K_n$.

### 2.2 Amenable coverings

In this subsection, we present the definition and some basic properties of amenable coverings. A right action of a countable group $\Gamma$ on a countable set $X$ is called amenable if there exists a $\Gamma$-invariant mean on $L^\infty(X)$.

Throughout the paper manifolds are assumed to be connected, unless otherwise stated. In particular, Riemannian coverings are assumed to be between connected manifolds, unless otherwise stated. For reasons that will become clear in the sequel, we must consider possibly non-connected covering spaces at some points.

Let $p : M_2 \to M_1$ be a Riemannian covering, with $M_2$ possibly non-connected, that is, $M_2$ has countably many connected components, the restriction of $p$ on any component is a Riemannian covering over $M_1$, and any point of $M_1$ has a neighborhood that is evenly covered with respect to the restriction of $p$ on any connected component of $M_2$. Fix $x \in M_1^0$ (that is, the interior of $M_1$) and consider the fundamental group $\pi_1(M_1)$ of $M_1$ with base point $x$. For $g \in \pi_1(M_1)$, let $\gamma_g : [0, 1] \to M_1$ be a representative loop based at $x$. For $y \in p^{-1}(x)$, lift $\gamma_g$ to a path $\tilde{\gamma}_g$, with $\tilde{\gamma}_g(0) = y$. We define $y \cdot g := \tilde{\gamma}_g(1)$. In this way, we obtain a right action of $\pi_1(M_1)$ on $p^{-1}(x)$. The covering $p$ is called amenable if this right action is amenable.

This definition coincides with the definition presented in [1,2,14] in terms of the right cosets of $\pi_1(M_2)$ in $\pi_1(M_1)$, when $M_2$ is connected. However, this definition allows us to extend the notion of amenable coverings in case $M_2$ is non-connected.

For instance, consider a Riemannian covering $p : M_2 \to M_1$, where $M_2$ has countably many connected components $M_2^{(n)}$, $n \in \mathbb{N}$. If the restriction $p : M_2^{(n)} \to M_1$ is amenable, for some $n \in \mathbb{N}$, then the covering $p : M_2 \to M_1$ is amenable. Indeed, if there exists a $\pi_1(M_1)$-invariant mean $\mu_n$ on $L^\infty(p^{-1}(x) \cap M_2^{(n)})$, for some $n \in \mathbb{N}$, then the linear functional $\mu : L^\infty(p^{-1}(x)) \to \mathbb{R}$, defined by

$$\mu(f) := \mu_n(f|_{p^{-1}(x) \cap M_2^{(n)}}),$$
for any \( f \in L^\infty(p^{-1}(x)) \), is a \( \pi_1(M_1) \)-invariant mean on \( L^\infty(p^{-1}(x)) \). However, the covering \( p: M_2 \to M_1 \) may be amenable, even when the restriction \( p: M_2^{(n)} \to M_1 \) is non-amenable, for any \( n \in \mathbb{N} \).

The following characterization of amenable coverings follows from Følner’s criterion (cf. [3, Section 2]).

**Proposition 2.6.** The covering \( p \) is amenable if and only if for any finite \( G \subset \pi_1(M_1) \) and \( \varepsilon > 0 \), there exists a finite \( F \subset p^{-1}(x) \), such that
\[
\#(F \setminus Fg) < \varepsilon \#(F),
\]
for all \( g \in G \).

In particular, the covering is amenable if and only if the right action of any finitely generated subgroup of \( \pi_1(M_1) \) on \( p^{-1}(x) \) is amenable. For a smoothly bounded, compact and connected neighborhood \( K \) of \( x \), we denote by \( i_*\pi_1(K) \) the image of the fundamental group of \( K \) in \( \pi_1(M_1) \). It is clear that \( p: p^{-1}(K) \to K \) is a Riemannian covering of manifolds with boundary, where \( p^{-1}(K) \) is possibly non-connected. Evidently, the covering \( p: p^{-1}(K) \to K \) is amenable if and only if the right action of \( i_*\pi_1(K) \) on \( p^{-1}(x) \) is amenable.

**Proposition 2.7.** The covering \( p: M_2 \to M_1 \) is amenable if and only if the covering \( p: p^{-1}(K) \to K \) is amenable, for any smoothly bounded, compact and connected neighborhood \( K \) of \( x \).

**Proof:** From Proposition 2.6, it suffices to prove that for any finite subset \( G \) of \( \pi_1(M_1) \), there exists a smoothly bounded, compact and connected neighborhood \( K \) of \( x \), such that \( G \subset i_*\pi_1(K) \). Let \( G \) be a finite subset of \( \pi_1(M_1) \) and consider a representative loop \( \gamma_g: [0, 1] \to M_1^g \), for each \( g \in G \). Let \( C \) be the union of the images of these loops and let \( U \) be a relatively compact, open neighborhood of \( C \) that does not intersect the boundary of \( M \) (if non-empty). Consider \( \chi \in C^\infty(M_1) \), with \( 0 \leq \chi \leq 1 \), \( \chi = 1 \) in \( C \) and \( \text{supp} \chi \subset U \). From Sard’s Theorem, it follows that for almost any \( t \in (0, 1) \), the level set \( \{ \chi = t \} \) is a smooth hypersurface of \( M_1 \). Consider such a \( t \), and the smoothly bounded, compact set \( K' := \{ \chi \geq t \} \). Then for the connected component \( K \) of \( K' \) containing \( x \), we have \( G \subset i_*\pi_1(K) \).

### 2.3 Manifolds with Ricci curvature bounded from below

In this subsection we recall the main result of [1] and point out that its proof, with some slight modifications, establishes this result for possibly non-connected covering spaces.

A non-connected Riemannian manifold \( M \) is complete if all of its connected components are complete. The distance of points of different connected components of \( M \) is considered to be infinite. In particular, any bounded subset of \( M \) is contained in a connected component of \( M \).
Theorem 2.8 ([1, Theorem 4.1]). Let $p: M_2 \to M_1$ be a Riemannian covering, with $M_2$ possibly non-connected. Assume that $M_1$ is complete, without boundary, and with Ricci curvature bounded from below. Let $S_1 := \Delta + V$ be a Schrödinger operator on $M_1$, with $V$ and $\text{grad}V$ bounded, and let $S_2$ be its lift on $M_2$. If $\lambda_0(S_2) = \lambda_0(S_1) \neq \lambda_0^{\text{ess}}(S_1)$, then the covering is amenable.

We begin with some definitions and remarks from [1]. Let $M$ be a possibly non-connected Riemannian manifold without boundary. A positive $\varphi \in C^\infty(M)$ satisfies a Harnack estimate if there exists a constant $c_\varphi \geq 1$, such that

$$\sup_{B(x,r)} \varphi^2 \leq c_\varphi^{r+1} \inf_{B(x,r)} \varphi^2,$$

for all $x \in M$ and $r > 0$. Assume that $M$ is complete, with Ricci curvature bounded from below, and let $S = \Delta + V$ be a Schrödinger operator on $M$, with $V$ and $\text{grad}V$ bounded. From [8, Theorem 6], if a positive function $\varphi \in C^\infty(M)$ satisfies $S\varphi = \lambda \varphi$, for some $\lambda \in \mathbb{R}$, then $\varphi$ satisfies a Harnack estimate.

The modified Cheeger’s constant of $M$ is defined as

$$h_\varphi(M) := \inf_A \frac{\int_{\partial A} \varphi^2}{\int_A \varphi^2},$$

where the infimum is taken over all bounded domains $A$ of $M$ with smooth boundary.

Lemma 2.9. Let $M$ be a possibly non-connected, complete Riemannian manifold, without boundary and with Ricci curvature bounded from below. Let $\varphi \in C^\infty(M)$ be a positive function, which satisfies a Harnack estimate. If $h_\varphi(M) = 0$, then for any $\varepsilon, r > 0$, there exists a bounded open subset $A$ of $M$, such that

$$\int_{A^r \setminus A} \varphi^2 < \varepsilon \int_A \varphi^2,$$

where $A^r := \{y \in M : d(y, A) < r\}$.

Proof: We may renormalize the Riemannian metric of $M$, so that $\text{Ric}_M \geq 1 - m$, where $m$ is the dimension of $M$. Since $h_\varphi(M) = 0$, for any $\varepsilon, r > 0$, there exists a non-empty, bounded domain $A$ of $M$ satisfying the estimate (3.2) of [1]. Evidently, $A$ is contained in a connected component of $M$ and the arguments of the proof of [1, Lemma 3.1] can be carried out in this connected component of $M$, establishing the asserted claim. \]

Lemma 2.10. In the setting of Theorem 2.8, there exists a compact set $K \subset M_1$, such that for any $\varepsilon, r > 0$, there exists $z \in K$ and a bounded open subset $A$ of $M_2$, such that

$$\#(p^{-1}(z) \cap (A^r \setminus A)) < \varepsilon \#(p^{-1}(z) \cap A).$$
Proof: Since $\lambda_0(S_1) \notin \sigma_{\text{ess}}(S_1)$, from Proposition 2.5, there exists a compact $K \subset M_1$, such that $\lambda_0(S_1, M_1 \setminus K) > \lambda_0(S_1)$. The proof is identical to the one of [1, Lemma 4.5], taking into account that [1, Lemma 3.1] has been extended to possibly non-connected manifolds in Lemma 2.9.

Proof of Theorem 2.8: Fix $x \in M_1$ and consider the fundamental group $\pi_1(M_1)$ with base point $x$. Consider a compact set $K \subset M_1$ as in Lemma 2.10, and let $R > 0$, such that $K \subset B(x, R)$. Let $\varepsilon > 0$ and $G$ be a finite subset of $\pi_1(M_1)$. For each $g \in G$, consider a smooth representative loop $\gamma_g$ based at $x$, and let

$$r > \max_{g \in G} \ell(\gamma_g) + 2R,$$

where $\ell(\cdot)$ stands for the length of a curve. From Lemma 2.10, there exists $z \in K$ and a bounded open subset $A$ of $M_2$, such that

$$\#(p^{-1}(z) \cap (A^r \setminus A)) < \varepsilon \#(p^{-1}(z) \cap A).$$

Consider a smooth path $\gamma: [0, 1] \to M_1$ from $x$ to $z$, of length less than $R$. For $y \in p^{-1}(x)$, lift $\gamma$ to a path $\tilde{\gamma}: [0, 1] \to M_2$, with $\tilde{\gamma}(0) = y$, and define $\Phi(y) := \tilde{\gamma}(1)$. Then the map $\Phi: p^{-1}(x) \to p^{-1}(z)$ is bijective. Let $F := \Phi^{-1}(p^{-1}(z) \cap A)$ and consider $y \in F \setminus Fg$, for some $g \in G$. Then $\Phi(y) \in A$ and $\Phi(y \cdot g^{-1}) \notin A$. Evidently, we have

$$d(\Phi(y), \Phi(y \cdot g^{-1})) \leq d(y, y \cdot g^{-1}) + 2\ell(\gamma) \leq \ell(\gamma_g) + 2R < r.$$

Therefore, $\Phi(y \cdot g^{-1}) \in A^r \setminus A$. Since $\Phi$ is bijective, it is clear that

$$\#(F \setminus Fg) = \#\{y \cdot g^{-1} : y \in F \setminus Fg\} = \#\{\Phi(y \cdot g^{-1}) : y \in F \setminus Fg\} \leq \#(p^{-1}(z) \cap (A^r \setminus A)) < \varepsilon \#(p^{-1}(z) \cap A) = \varepsilon \#(F).$$

From Proposition 2.6, it follows that the covering is amenable.

3 Properties of the Neumann spectrum

In this section, we establish some properties of the Neumann spectrum that will be used in the sequel. Let $M$ be a possibly non-connected Riemannian manifold and $S = \Delta + V$ a Schrödinger operator on $M$. It is worth to point out that we do not require $M$ to have non-empty boundary, which yields that the following results also hold for manifolds without boundary (and most of them are already known in this case). If $M$ has non-empty boundary, we denote by $\nu$ the inward pointing normal to $\partial M$. Throughout this section, we denote by $H_V(M)$ the space defined in Subsection 2.1.

First, we establish some convenient expressions for the bottom of the Neumann spectrum, and derive some straightforward applications to Riemannian coverings.
Proposition 3.1. Any compactly supported smooth function belongs to $H_V(M)$. Moreover, any compactly supported Lipschitz function is in $H_V(M)$.

Proof: If $M$ has empty boundary, then any compactly supported Lipschitz function $f$ belongs to $H^1_0(M)$. Since $V$ is smooth, it is easy to see that any such $f$ also belongs to $H_V(M)$. Therefore, it remains to prove the proposition for manifolds with non-empty boundary.

Let $f \in C^\infty_c(M)$. Then there exists a compact $K \subset \partial M$ and $\delta > 0$, such that the map $\Phi \colon K \times [0, \delta) \rightarrow M$, defined by $\Phi(x,t) := \exp_x(t\nu)$, is a diffeomorphism onto its image $W_\delta$, and $\text{supp} f \cap W_\delta \subset W_\delta$. For $0 < \delta_0 < \delta$, consider the Lipschitz function $f_{\delta_0}$, which is equal to $f$ outside $W_{\delta_0}$, and $f_{\delta_0}(\Phi(x,t)) = f(\Phi(x,\delta_0))$ in $W_{\delta_0}$. Let $K_1$ be a compact neighborhood of $\Phi(K \times \{\delta_0\})$ and $K_2$ a compact neighborhood of $K_1$, that does not intersect $\partial M$. Consider $\chi \in C^\infty_c(M)$, with $\chi = 1$ in $K_1$ and supp $\chi \subset K_2$. Since $\chi f_{\delta_0}$ is Lipschitz and compactly supported in the interior of $M$, it follows that $\chi f_{\delta_0} \in H_V(M)$. Moreover, $(1 - \chi)f_{\delta_0} \in C^\infty_c(M)$ and $\nu(f) = 0$ on $\partial M$. Therefore, $(1 - \chi)f_{\delta_0} \in H_V(M)$, which yields that $f_{\delta_0} \in H_V(M)$. It is clear that $f_{\delta_0} \rightarrow f$ in $H_V(M)$, as $\delta_0 \rightarrow 0$, and in particular, $f \in H_V(M)$.

Let $f$ be a compactly supported Lipschitz function on $M$. Consider a Riemannian manifold $N$ of the same dimension, without boundary, containing $M$ (for instance, glue cylinders along $\partial M$). Extend $f$ to a compactly supported Lipschitz function $f'$ in $N$ and let $K$ be a smoothly bounded, compact neighborhood of $\text{supp} f'$. Then there exists $(g_n)_{n \in \mathbb{N}} \subset C^\infty_c(N)$, with $\text{supp} g_n \subset K$ and $g_n \rightarrow f'$ in $H^1_0(K)$. Then $h_n := g_n|_M \in C^\infty_c(M)$ and from the first statement, it follows that $h_n \in H_V(M)$. Evidently, we have that $h_n \rightarrow f$ in $H_V(M)$, and in particular, $f \in H_V(M)$. \qed

For $f \in \text{Lip}_c(M) \setminus \{0\}$, the Rayleigh quotient of $f$ with respect to $S$, is defined as

$$\mathcal{R}_S(f) := \frac{\int_M (\|\text{grad} f\|^2 + Vf^2)}{\int_M f^2}.$$ 

Proposition 3.2. The bottom of the spectrum of $S^N$ is given by

$$\lambda^N_0(S) = \inf_{f \in C^\infty_c(M) \setminus \{0\}} \mathcal{R}_S(f) = \inf_{f \in \text{Lip}_c(M) \setminus \{0\}} \mathcal{R}_S(f).$$

Proof: It is clear that for any non-zero $f \in \text{Lip}_c(M)$, we have

$$\mathcal{R}_S(f) = \inf_M V - 1 + \frac{\|f\|^2_{H_V(M)}}{\|f\|^2_{L^2(M)}},$$

and the asserted equalities follow from Proposition 2.2. \qed

Proposition 3.3. Let $p \colon M_2 \rightarrow M_1$ be a Riemannian covering, with $M_2$ possibly non-connected. Let $S_1$ be a Schrödinger operator on $M_1$ and consider its lift $S_2$ on $M_2$. Then $\lambda^N_0(S_1) \leq \lambda^N_0(S_2)$. 

10
**Proof:** Let \( f \in C_c^\infty(M_2) \setminus \{0\} \) and consider its pushdown
\[
g(z) := \left( \sum_{y \in p^{-1}(z)} f(y)^2 \right)^{1/2}
\]
on \( M_1 \). Then \( g \in \operatorname{Lip}_c(M_1), \|g\|_{L^2(M_1)} = \|f\|_{L^2(M_2)} \) and \( \mathcal{R}_{S_1}(g) \leq \mathcal{R}_{S_2}(f) \) (cf. [2, Section 4]). The statement follows from Proposition 3.2.

**Theorem 3.4.** Let \( p : M_2 \to M_1 \) be a Riemannian covering. Let \( S_1 \) be a Schrödinger operator on \( M_1 \) and consider its lift \( S_2 \) on \( M_2 \). If \( p \) is infinite sheeted and amenable, then \( \lambda_0^N(S_1) = \lambda_0^{N,\text{ess}}(S_2) \).

**Proof:** Follows from [14, Theorem 1.2] and Corollary 3.3.

Next, we study properties of eigenfunctions corresponding to the bottom of the spectrum and minimizing sequences for the Rayleigh quotient of Schrödinger operators on connected Riemannian manifolds.

**Proposition 3.5.** Let \( S = \Delta + V \) be a Schrödinger operator on a Riemannian manifold \( M \), and consider \((f_n)_{n \in \mathbb{N}} \subset \operatorname{Lip}_c(M)\), with \( \|f_n\|_{L^2(M)} = 1 \) and \( \mathcal{R}_S(f_n) \to \lambda_0^N(S) \). If \( \lambda_0^N(S) \notin \sigma_{\text{ess}}^N(S) \), then there exists a subsequence \((f_{n_k})_{k \in \mathbb{N}}, \) such that \( f_{n_k} \to \varphi \) in \( L^2(M) \), for some \( \lambda_0^N(S) \)-eigenfunction \( \varphi \) of \( S^N \).

**Proof:** From Proposition 3.1, there exists \((f'_n)_{n \in \mathbb{N}} \subset C_c^\infty(M) \cap \mathcal{D}(S^N)\), with \( \|f'_n\|_{L^2(M)} = 1 \) and \( \|f_n - f'_n\|_{H^2(M)} \leq 1/n \), for any \( n \in \mathbb{N} \). Evidently, \( \mathcal{R}_S(f'_n) \to \lambda_0^N(S) \) and it suffices to prove the asserted statement for \((f'_n)_{n \in \mathbb{N}}\).

Since \( \lambda_0^N(S) \) is not in the essential spectrum, it is an isolated eigenvalue of finite multiplicity. Let \( E \) be the eigenspace corresponding to \( \lambda_0^N(S) \), and \( g_n \) be the projection of \( f'_n \) on \( E \) with respect to the \( L^2(M) \)-inner product, \( n \in \mathbb{N} \). Since \( E \) is finite dimensional, after passing to a subsequence, we may assume that \( g_n \to \varphi \) in \( L^2(M) \), for some \( \varphi \in E \). Consider \( h_n := f'_n - g_n \in \mathcal{D}(S^N) \). Since \( h_n \) is perpendicular to \( E \), from the Spectral Theorem (cf. for instance [18, Chapter 8]), it follows that there exists \( c_0 > 0 \), such that
\[
\|h_n\|^2_{H^2(M)} - (1 - \inf_M V)\|h_n\|^2_{L^2(M)} = \langle S^N h_n, h_n \rangle_{L^2(M)} \geq (\lambda_0^N(S) + c_0)\|h_n\|^2_{L^2(M)},
\]
for any \( n \in \mathbb{N} \). It is clear that
\[
\langle h_n, g_n \rangle_{H^2(M)} = \langle h_n, S^N g_n \rangle_{L^2(M)} + (1 - \inf_M V)\langle h_n, g_n \rangle_{L^2(M)} = (\lambda_0^N(S) + 1 - \inf_M V)
\]
\[
\langle h_n, g_n \rangle_{H^2(M)} = 0.
\]
Let \( \varepsilon > 0 \). Then, for \( n \) sufficiently large, we have \( \mathcal{R}_S(f_n) \leq \lambda_0^N(S) + \varepsilon \), and thus
\[
\|h_n\|^2_{H^2(M)} - (1 - \inf_M V)\|h_n\|^2_{L^2(M)} = (\|f'_n\|^2_{H^2(M)} - (1 - \inf_M V)\|f'_n\|^2_{L^2(M)}) - (\|g_n\|^2_{H^2(M)} - (1 - \inf_M V)\|g_n\|^2_{L^2(M)})
\]
\[
\leq (\lambda_0^N(S) + \varepsilon)\|f'_n\|^2_{L^2(M)} - \lambda_0^N(S)\|g_n\|^2_{L^2(M)}
\]
\[
= \varepsilon + \lambda_0^N(S)\|h_n\|^2_{L^2(M)}.
\]
From (2), this yields that \( h_n \to 0 \) in \( L^2(M) \). Therefore, \( f'_n \to \varphi \) in \( L^2(M) \).
Lemma 3.6. Let $S$ be a Schrödinger operator on a (connected) Riemannian manifold $M$ and let $\varphi \in C^\infty(M) \setminus \{0\}$ be a non-negative function satisfying $S\varphi = \lambda \varphi$, for some $\lambda \in \mathbb{R}$. Then $\varphi$ is positive in the interior of $M$. If, in addition, $M$ has non-empty boundary, and $\nu(\varphi) = 0$ on $\partial M$, then $\varphi$ is positive on $\partial M$.

Proof: Assume that there exists a point $x$ in the interior of $M$, such that $\varphi(x) = 0$. Let $\delta > 0$, such that $\exp_x : B(0, 2\delta) \subset T_x M \to M$ is a diffeomorphism onto its image. Then $B(x, \delta)$ may be considered as a geodesic ball of radius $\delta$ in a complete Riemannian manifold without boundary. In $B(x, \delta)$, for any $\varepsilon > 0$, we have

$$|\Delta(\varphi + \varepsilon)| \leq (\varphi + \varepsilon) \sup_{B(x, \delta)} |\lambda - V|$$

and

$$\|\text{grad} \Delta(\varphi + \varepsilon)\| \leq \|\text{grad}(\varphi + \varepsilon)\| \sup_{B(x, \delta)} |\lambda - V| + (\varphi + \varepsilon) \sup_{B(x, \delta)} \|\text{grad} V\|.$$ 

From [8, Theorem 6], it follows that there exists $c > 0$, independent from $\varepsilon$, such that

$$\sup_{B(x, \delta/2)} (\varphi + \varepsilon) \leq c \inf_{B(x, \delta/2)} (\varphi + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, it follows that if $\varphi(x) = 0$, then $\varphi = 0$ in $B(x, \delta/2)$. In particular, the set $\{x \in M^\circ : \varphi(x) = 0\}$ is open and closed. Since $M$ is connected and $\varphi$ is not identically zero, it follows that $\varphi$ is positive in $M^\circ$.

Assume that $M$ has non-empty boundary and $\nu(\varphi) = 0$ on $\partial M$. Assume that there exists $x \in \partial M$, such that $\varphi(x) = 0$. Since $\nu(\varphi) = 0$ on $\partial M$ and $\varphi|_{\partial M}$ attains a minimum at $x$, it follows that $\text{grad} \varphi(x) = 0$. Consider a coordinate system $\Phi : U := B(0, r) \cap \mathbb{H}^m \to M$, with $\Phi(0) = x$, where $m$ is the dimension of $M$ and $\mathbb{H}^m$ is the upper half-space of dimension $m$. Consider $c_0 \in \mathbb{R}$, with $c_0 \geq -\inf_M V$ and $c_0 \geq -\lambda$. Then $\phi := \varphi \circ \Phi$ is non-negative, smooth and satisfies

$$L\phi := -\frac{1}{\sqrt{\det g}} \sum_{i,j=1}^m \frac{\partial}{\partial x_i} (g^{ij} \sqrt{\det g} \frac{\partial \phi}{\partial x_j}) + (V + c_0)\phi = (\lambda + c_0)\phi \geq 0.$$ 

Since $V + c_0 \geq 0$, $\phi(0) = 0 < \phi(y)$ for all $y \in U^\circ$, and $U$ satisfies the interior ball condition at the origin, from Hopf’s Lemma (cf. for instance [11, p. 330]), it follows that

$$\frac{\partial \phi}{\partial x_m}(0) \neq 0,$$

which is a contradiction, since $\text{grad} \phi(0) = 0$. Therefore, $\varphi$ is positive on $\partial M$.

Proposition 3.7. Let $M$ be a (connected) Riemannian manifold and $S = \Delta + V$ a Schrödinger operator on $M$. If $\varphi \in \mathcal{D}(S^N) \setminus \{0\}$ is a $\lambda_0^N(S)$-eigenfunction of $S^N$, then $\varphi$ is smooth and nowhere vanishing. Moreover, if $M$ has non-empty boundary, then $\nu(\varphi) = 0$ on $\partial M$. 

12
Proof: Since \( \varphi \in \mathcal{D}(S^N) \), there exists \((f_n)_{n \in \mathbb{N}} \subset C_c^\infty(M)\), such that \( f_n \to \varphi \) in \( H_V(M) \). Clearly, \(|f_n|\) is Lipschitz and compactly supported. From Proposition 3.1, it follows that \(|f_n| \in H_V(M)\). From Rademacher’s Theorem, \(|f_n|\) is almost everywhere differentiable. Therefore, we have \( \| \text{grad } |f_n| \| = \| \text{grad } f_n \| \) almost everywhere, and in particular, \( \mathcal{R}_S(|f_n|) = \mathcal{R}_S(f_n) \). Since \((|f_n|)_{n \in \mathbb{N}}\) is bounded in \( H_V(M) \), it has a weakly convergent subsequence in \( H_V(M) \). Since \(|f_n| \to |\varphi|\) in \( L^2(M) \), it follows that \(|\varphi| \in H_V(M)\), and after passing to a subsequence, we have that \(|f_n| \to |\varphi|\) in \( H_V(M) \). Hence, \( \mathcal{R}_S(|\varphi|) = \lambda^N_0(S) \).

In particular, for any \( f \in C_c^\infty(M) \), the function \( t \mapsto \mathcal{R}_S(|\varphi| + tf) \), with \(|t| < \varepsilon\), is differentiable and attains minimum for \( t = 0 \). This yields that

\[
\int_M (\langle \text{grad } \varphi, \text{grad } f \rangle + V|\varphi|f) = \lambda^N_0(S) \int_M |\varphi|f, \tag{3}
\]

for any \( f \in C_c^\infty(M) \). From Elliptic Regularity Theory, it follows that \(|\varphi| \in C^\infty(M^c)\) and \( S|\varphi| = \lambda^N_0(S) |\varphi| \) in \( M^c \). From Lemma 3.6, \(|\varphi|\) is nowhere vanishing in the interior of \( M \), and so is \( \varphi \). If \( M \) has empty boundary, this completes the proof.

If \( M \) has non-empty boundary, then without loss of generality, we may assume that \( \varphi \) is positive in the interior of \( M \). Since \( \varphi \in \mathcal{D}(S^N) \) and \( S^N \varphi = \lambda^N_0(S) \varphi \), from Elliptic Regularity Theory, it follows that \( \varphi \in C^\infty(M) \). Moreover, from (3) we have that

\[
\int_{\partial M} \nu(\varphi)f = \int_M fS\varphi - \int_M (\langle \text{grad } \varphi, \text{grad } f \rangle + V\varphi f) = 0,
\]

for any \( f \in C_c^\infty(M) \). Therefore, \( \nu(\varphi) = 0 \) on \( \partial M \), and from Lemma 3.6, it follows that \( \varphi \) is positive on \( \partial M \). \( \blacksquare \)

**Proposition 3.8.** Let \( S \) be a Schrödinger operator on a (connected) Riemannian manifold \( M \), with \( \lambda^N_0(S) \notin \sigma^N_{ess}(S) \). Then for any compact \( K \subset M \) of positive measure, we have

\[
\inf_f \mathcal{R}_S(f) > \lambda^N_0(S),
\]

where the infimum is taken over all non-zero \( f \in \text{Lip}_c(M) \), with \( \text{supp } f \cap K = \emptyset \).

**Proof:** Assume to the contrary that there exists a compact subset \( K \) of \( M \) of positive measure, such that for any \( \varepsilon > 0 \), there exists a non-zero \( f \in \text{Lip}_c(M) \), with \( \mathcal{R}_S(f) < \lambda^N_0(S) + \varepsilon \) and \( \text{supp } f \cap K = \emptyset \). Evidently, there exists \((f_n)_{n \in \mathbb{N}} \subset \text{Lip}_c(M)\), with \( \|f_n\|_{L^2(M)} = 1 \), \( \text{supp } f_n \cap K = \emptyset \) and \( \mathcal{R}_S(f_n) \to \lambda^N_0(S) \). From Proposition 3.5, after passing to a subsequence, we have that \( f_n \to \varphi \) in \( L^2(M) \), for some \( \lambda^N_0(S) \)-eigenfunction \( \varphi \) of \( S^N \). Since \( \|\varphi\|_{L^2(M)} = 1 \), from Proposition 3.7, it follows that \( \varphi \) is nowhere vanishing in \( M \). This is a contradiction, since

\[
\|\varphi - f_n\|_{L^2(M)}^2 \geq \int_K \varphi^2 > 0,
\]

while \( f_n \to \varphi \) in \( L^2(M) \). This proves the asserted claim. \( \blacksquare \)
We end this section with the notion of renormalized Schrödinger operators. This notion was introduced for the Laplacian in [6] and for Schrödinger operators on manifolds without boundary in [14].

Let \( M \) be a possibly non-connected Riemannian manifold and \( S = \Delta + V \) a Schrödinger operator on \( M \). Let \( \varphi \in C^\infty(M) \) be a positive function, satisfying \( S \varphi = \lambda \varphi \), for some \( \lambda \in \mathbb{R} \). If \( M \) has non-empty boundary, assume that \( \nu(\varphi) = 0 \) on \( \partial M \). Consider the space \( L^2_\varphi(M) := \{ [v] : \varphi v \in L^2(M) \} \), where two measurable functions are equivalent if they are almost everywhere equal, endowed with the inner product \( \langle v_1, v_2 \rangle_{L^2_\varphi(M)} := \int_M v_1 v_2 \varphi^2 \).

Then the map \( \mu_\varphi : L^2_\varphi(M) \to L^2(M) \), defined by \( \mu_\varphi v := \varphi v \), is an isometric isomorphism. In particular, \( L^2_\varphi(M) \) is a separable Hilbert space. The renormalization \( S_\varphi \) of \( S \) with respect to \( \varphi \) is defined by

\[
S_\varphi v := \mu_\varphi^{-1}(S^N - \lambda)(\mu_\varphi v), \quad \text{for all } v \in \mathcal{D}(S_\varphi) := \mu_\varphi^{-1}(\mathcal{D}(S^N)).
\]

It is clear that \( S_\varphi : \mathcal{D}(S_\varphi) : L^2_\varphi(M) \to L^2_\varphi(M) \) is self-adjoint and \( \sigma(S_\varphi) = \sigma(S^N) - \lambda \). For a non-zero \( f \in \text{Lip}_c(M) \), the Rayleigh quotient of \( f \) with respect to \( S_\varphi \) is defined as

\[
\mathcal{R}_{S_\varphi}(f) := \frac{\int_M \| \nabla f \|^2 \varphi^2}{\int_M f^2 \varphi^2}.
\]

**Proposition 3.9** ([14, Proposition 7.1],[1, Subsection 2.1]). In the above situation, if \( M \) has empty boundary, then the bottom of the spectrum of \( S_\varphi \) is given by

\[
\lambda_0(S) - \lambda = \lambda_0(S_\varphi) = \inf_{f \in C^\infty_c(M) \setminus \{0\}} \mathcal{R}_{S_\varphi}(f) = \inf_{f \in \text{Lip}_c(M) \setminus \{0\}} \mathcal{R}_{S_\varphi}(f).
\]

**Proposition 3.10.** In the above situation, if \( M \) has non-empty boundary, then the bottom of the spectrum of \( S_\varphi \) is given by

\[
\lambda_0^N(S) - \lambda = \lambda_0(S_\varphi) = \inf_f \mathcal{R}_{S_\varphi}(f),
\]

where the infimum is taken over all non-zero \( f \in C^\infty_c(M) \), with \( \nu(f) = 0 \) on \( \partial M \).

**Proof:** Let \( f \in C^\infty_c(M) \setminus \{0\} \), with \( \nu(f) = 0 \) on \( \partial M \). Since \( \varphi \) is smooth and \( \nu(\varphi) = 0 \) on \( \partial M \), it follows that \( f \in \mathcal{D}(S_\varphi) \). It is easy to see that

\[
S_\varphi f = \Delta f - \frac{2}{\varphi} \langle \nabla \varphi, \nabla f \rangle.
\]

Hence, we have

\[
\langle S_\varphi f, f \rangle_{L^2_\varphi(M)} = \int_M (\varphi^2 \Delta f - 2f \varphi \langle \nabla \varphi, \nabla f \rangle)
\]

\[
= \int_M (\langle \nabla (\varphi^2 f), \nabla f \rangle - 2f \varphi \langle \nabla \varphi, \nabla f \rangle) + \int_{\partial M} \varphi^2 f \nu(f)
\]

\[
= \int_M \| \nabla f \|^2 \varphi^2,
\]

14
where we used that $\nu(f) = 0$ on $\partial M$. In particular, we have that

$$R_{S_\varphi}(f) = \frac{\langle S_\varphi f, f \rangle_{L^2_\varphi(M)}}{\|f\|_{L^2_\varphi(M)}^2}.$$  

From Proposition 2.1, it follows that $R_{S_\varphi}(f) \geq \lambda_0(S_\varphi)$. From Proposition 2.2, there exists $(g_n)_{n \in \mathbb{N}} \subset C_c^\infty(M) \setminus \{0\}$, with $\nu(g_n) = 0$ on $\partial M$ and $R_S(g_n) \to \lambda_0^N(S)$. Consider $f_n := \mu_1^{-1} g_n$. Then $f_n \in C_c^\infty(M)$, $\nu(f_n) = 0$ on $\partial M$, and $R_{S_\varphi}(f_n) \to \lambda_0(S_\varphi)$. This proves the asserted equality.

4 Coverings of compact manifolds

Throughout this section, for simplicity of notation, we denote by $R(f)$ the Rayleigh quotient of a Lipschitz function $f$ with respect to the Laplacian. The aim of this section is to prove Theorem 1.1. Since a part of it follows from Theorem 3.4, it remains to prove the converse implication. For reasons that will become clear in the sequel, we need to establish it also for non-connected covering spaces.

**Theorem 4.1.** Let $p: M_2 \to M_1$ be a Riemannian covering, with $M_1$ compact with non-empty boundary, and $M_2$ possibly non-connected. If $\lambda_0^N(M_2) = 0$, then $p$ is amenable.

Let $\nu_i$ be the inward pointing normal to $\partial M_i$, $i = 1, 2$. Then there exists $\delta > 0$, such that the map $\Phi: \partial M_1 \times [0, 2\delta) \to M_1$, defined by $\Phi(x, t) := \exp_x(t \nu_1)$, is a diffeomorphism onto its image. By definition, any point of $M_1$ has an evenly covered neighborhood with respect to the restriction of $p$ on any connected component of $M_2$. Therefore, we may assume that $\delta$ is sufficiently small, so that for any $x \in \partial M_1$ and $y_1, y_2 \in p^{-1}(x)$, with $y_1 \neq y_2$, we have $d(y_1, y_2) \geq 2\delta$. It is worth to point out that we consider the distance of points of different connected components of $M_2$ to be infinite.

**Lemma 4.2.** The map $\Psi: \partial M_2 \times [0, \delta) \to M_2$, defined by $\Psi(y, t) := \exp_y(t \nu_2)$, is a diffeomorphism onto its image.

**Proof:** Since $(p \circ \Psi)(y, t) = \Phi(p(y), t)$, for any $y \in \partial M_2$ and $t \in [0, \delta)$, it is clear that $\Psi$ is a local diffeomorphism. So, it suffices to prove that it is injective. Consider $y_1, y_2 \in \partial M_2$ and $t_1, t_2 \in [0, \delta)$, such that $\Psi(y_1, t_1) = \Psi(y_2, t_2) =: z$. Then $d(y_i, z) < \delta$, $i = 1, 2$, which yields that $d(y_1, y_2) < 2\delta$. Moreover, it follows that $\Phi(p(y_1), t_1) = \Phi(p(y_2), t_2)$. Since $\Phi$ is a diffeomorphism onto its image, this yields that $t_1 = t_2$, $p(y_1) = p(y_2)$, and in particular, $y_1 = y_2$.

**Lemma 4.3.** There exists a Riemannian metric $g'$ on $M_1$, such that $\Phi$ restricted to $\partial M_1 \times [0, \delta)$ is an isometry onto its image.
Proof: Let $g_c$ be the push-forward of the product metric of $\partial M_1 \times [0,2\delta)$ via $\Phi$. Denote by $g$ the original Riemannian metric of $M_1$. Consider a smooth $\tau: [0,2\delta) \to [0,1]$, with $\tau(t) = 1$ for $t \leq \delta$, and $\tau(t) = 0$ for $t \geq 3\delta/2$. Consider the function $\tau' \in C^\infty(M_1)$, defined by $\tau'(\Phi(x,t)) = \tau(t)$ in $\Phi(\partial M_1 \times [0,2\delta))$, and $\tau' = 0$ otherwise. Evidently, the Riemannian metric

$$g' := \tau'g_c + (1-\tau')g.$$

on $M_1$ satisfies the desired property.

Consider $M_1$ and $M_2$ endowed with $g'$ and its lift, respectively. Since $(p \circ \Psi)(y,t) = \Phi(p(y),t)$, for any $y \in \partial M_2$ and $t \in [0,\delta)$, it follows that $\Phi$ restricted on $\partial M_2 \times [0,\delta)$ is a local isometry, with respect to the lift of $g'$. From Lemma 4.2, this map is also injective, which yields that it is an isometry onto its image. Denote by $U_t$ the open set $\Psi(\partial M_2 \times [0,t))$, and by $C_t$ the closed set $\Psi(\partial M_2 \times \{t\})$.

Evidently, there exist $c_1, c_2 > 0$, such that for any $f \in C^\infty(M_2)$, the norms of the gradients of $f$ with respect to the lifts of $g$ and $g'$, are related by

$$c_1\|\text{grad}_g f\|_g \leq \|\text{grad}_{g'} f\|_{g'} \leq c_2\|\text{grad}_g f\|_g.$$

Moreover, there exists a positive, smooth $\nu: M_1 \to \mathbb{R}$, such that the volume elements induced by the lifts of $g$ and $g'$, satisfy

$$\frac{d\text{Vol}_{g'}}{d\text{Vol}_g} = \nu \circ \rho.$$

Therefore, for any non-zero $f \in C^\infty_c(M_2)$, the Rayleigh quotients of $f$ with respect to the Laplacians induced by the lifts of $g$ and $g'$, satisfy

$$\mathcal{R}_{g'}(f) = \frac{\int_{M_2} \|\text{grad}_{g'} f\|_{g'}^2 d\text{Vol}_{g'}}{\int_{M_2} f^2 d\text{Vol}_{g'}} \leq c_2 \max \nu \mathcal{R}_g(f).$$

From Proposition 3.2, since $\lambda_0^N(M_2) = 0$ with respect to the lift of $g$, it follows that $\lambda_0^N(M_2) = 0$ with respect to the lift of $g'$. From now on, we will be working with $g'$ and its lift. It is worth to point out that the maps $\Phi$ and $\Psi$ are defined in terms of the exponentials with respect to the original Riemannian metrics.

Lemma 4.4. For any $\varepsilon > 0$, there exists $f \in \text{Lip}_c(M_2)$, smooth on $M_2 \setminus C_{t_0}$, for one $t_0 \in (0,\delta)$, with $f|_{\partial M_2}$ non-zero, such that $\mathcal{R}(f) \leq \varepsilon$ and

$$\frac{\int_{\partial M_2} \|\text{grad} f\|^2}{\int_{\partial M_2} f^2} \leq \varepsilon.$$

Proof: Since $\lambda_0^N(M_2) = 0$, from Proposition 3.2, there exists $(f_n)_{n \in \mathbb{N}} \subset C^\infty_c(M_2)$, with $\|f_n\|_{L^2(M_2)} = 1$, such that $\mathcal{R}(f_n) \to 0$. Assume that there exists $\varepsilon > 0$, such that for any $n \in \mathbb{N}$ and $t \in [0,\delta)$, we have

$$\int_{C_t} \|\text{grad} f_n\|^2 > \varepsilon \int_{C_t} f_n^2.$$  \hfill (4)
Then
\[ \int_{U_\delta} \| \nabla f_n \|^2 \geq \varepsilon \int_{U_\delta} f_n^2, \]
which yields that \( \int_{U_\delta} f_n^2 \to 0 \) and \( \int_{M_2 \setminus U_\delta} f_n^2 \to 1 \). Let \( \chi \in C^\infty(M_1) \), with \( \chi(x) = 1 \) for \( d(x, \partial M_1) \geq \delta \), and \( \chi(x) = 0 \) for \( d(x, \partial M_1) < \delta/2 \). Let \( \bar{\chi} \in C^\infty(M_2) \) be the lift of \( \chi \). Then \( \bar{\chi} = 0 \) in \( U_{\delta/2} \) and \( \bar{\chi} = 1 \) outside \( U_\delta \). For \( g_n := \bar{\chi} f_n \in C^\infty_c(M_2) \), we have
\[ \| g_n \|_{L^2(M_2)}^2 = \int_{U_\delta} \bar{\chi}^2 f_n^2 + \int_{M_2 \setminus U_\delta} f_n^2 \to 1, \]
and
\[ \int_{M_2} \| \nabla g_n \|^2 \leq 2 \int_{U_\delta} (\bar{\chi}^2 \| \nabla f_n \|^2 + f_n^2 \| \nabla \bar{\chi} \|^2) + \int_{M_2 \setminus U_\delta} \| \nabla f_n \|^2 \to 0. \]
In particular, we have that \( \mathcal{R}(g_n) \to 0 \). Since \( g_n \) is supported in the interior of \( M_2 \), for any \( n \in \mathbb{N} \), from Proposition 3.2 and Remark 2.3, it follows that \( \lambda_0^D(M_2) = 0 \). This is a contradiction, since from Proposition 3.3 and Remark 2.3, we have \( \lambda_D^D(M_2) \geq \lambda_D^D(M_1) > 0 \).

Hence, (4) cannot hold, that is, for any \( \varepsilon > 0 \), there exists \( n \in \mathbb{N} \) and \( t \in [0, \delta) \), such that
\[ \int_{C_t} \| \nabla f_n \|^2 \leq \varepsilon \int_{C_t} f_n^2. \]
Let \( 0 < \varepsilon < \lambda_0^D(M_2) \) and consider \( f_n \in C^\infty_c(M_2) \), with \( \| f_n \|_{L^2(M_2)} = 1 \), \( \mathcal{R}(f_n) < \varepsilon \), satisfying (5) for some \( t \in [0, \delta) \). Let \( t_0 \) be the minimum of all \( t \in [0, \delta) \), for which (5) holds. If \( t_0 = 0 \), then \( f_n \) is the desired function. Otherwise, define \( f \in C_c(M_2) \) by \( f = f_n \) outside \( U_{t_0} \), and \( f(\Psi(x, t)) = f_n(\Psi(x, t_0)) \) for \( t \leq t_0 \). It is clear that \( f \in \text{Lip}_c(M) \) and is smooth on \( M_2 \setminus C_{t_0} \).

Since \( \mathcal{R}(f_n) < \lambda_0^D(M_2) \), from Proposition 3.2 and Remark 2.3, it follows that \( f_n \) is not identically zero on \( U_{t_0} \). Since \( \mathcal{R}(f_n) < \varepsilon \), from the definition of \( t_0 \), it follows that \( f_n \) is not identically zero on \( M_2 \setminus U_{t_0} \). In particular, this yields that \( f \) is non-zero. Since (5) holds for \( t = t_0 \), we have
\[ \int_{\partial M_2} \| \nabla f \|^2 = \int_{C_{t_0}} \| \nabla (f_n|_{C_{t_0}}) \|^2 \leq \varepsilon \int_{C_{t_0}} f_n^2 = \varepsilon \int_{\partial M_2} f^2. \]
Furthermore, we have
\[
\mathcal{R}(f) = \frac{\int_{t_0}^{t_0} \int_{C_t} \| \nabla f \|^2 + \int_{M_2 \setminus U_{t_0}} \| \nabla f_n \|^2}{\int_{t_0}^{t_0} \int_{C_t} f^2 + \int_{M_2 \setminus U_{t_0}} f_n^2} \\
\leq \frac{\varepsilon \int_{t_0}^{t_0} \int_{C_t} f^2 + \int_{M_2 \setminus U_{t_0}} \| \nabla f_n \|^2}{\int_{t_0}^{t_0} \int_{C_t} f^2 + \int_{M_2 \setminus U_{t_0}} f_n^2} \\
\leq \max \left\{ \varepsilon, \frac{\int_{M_2 \setminus U_{t_0}} \| \nabla f_n \|^2}{\int_{M_2 \setminus U_{t_0}} f_n^2} \right\}. \tag{6} \]
It is clear that

\[
\varepsilon > \mathcal{R}(f_n) = \frac{\int_{t_0}^{t_0} \|\nabla f_n\|^2 + \int_{M_2 \setminus U_{t_0}} \|\nabla f_n\|^2}{\int_{t_0}^{t_0} \int_{C_t} f_n^2 + \int_{M_2 \setminus U_{t_0}} f_n^2} \\
\geq \min \left\{ \frac{\int_{t_0}^{t_0} \int_{C_t} \|\nabla f_n\|^2}{\int_{t_0}^{t_0} \int_{U_{t_0}} f_n^2}, \frac{\int_{M_2 \setminus U_{t_0}} \|\nabla f_n\|^2}{\int_{M_2 \setminus U_{t_0}} f_n^2} \right\}.
\]

From the definition of \( t_0 \), the first term is greater than \( \varepsilon \), which yields that the second term is smaller than \( \varepsilon \). From (6), it follows that \( \mathcal{R}(f) \leq \varepsilon \). Since \( \varepsilon < \lambda_0^0(M_2) \), from Remark 2.3 and Proposition 3.2, it is clear that \( f \) cannot vanish identically on \( \partial M_2 \). \( \blacksquare \)

Glue the cylinder \( \partial M_1 \times [0, +\infty) \), with the product metric, along \( \partial M_1 \), so that \( \partial / \partial t \) is the outward pointing normal to \( \partial M_1 \). Denote by \( N_1 \) the obtained Riemannian manifold. The covering \( p: M_2 \to M_1 \) can be extended to a Riemannian covering \( p: N_2 \to N_1 \), where \( N_2 \) is the Riemannian manifold obtained by gluing \( \partial M_2 \times [0, +\infty) \) along \( \partial M_2 \) in the analogous way. Evidently, \( p: M_2 \to M_1 \) is amenable if and only if \( p: N_2 \to N_1 \) is amenable. Points in \( N_i \setminus M_i \) will be written in the form \((x, t)\), with \( x \in \partial M_i \) and \( t \geq 0 \), \( i = 1, 2 \).

Consider a positive smooth \( \varphi: [0, +\infty) \to \mathbb{R} \), with \( \varphi(t) = 1 \) for \( t \leq 1/2 \), and \( \varphi(t) = e^{-t} \) for \( t \geq 1 \). Let \( \varphi \in C^\infty(N_1) \) be the square-integrable function defined by \( \varphi = 1 \) in \( M_1 \), and \( \varphi(x, t) = \varphi(t) \) in \( N_1 \setminus M_1 \). Consider the function \( V \in C^\infty(N_1) \), defined by \( V = 0 \) in \( M_1 \), and \( V(x, t) = \varphi''(t)/\varphi(t) \) in \( N_1 \setminus M_1 \). It is worth to point out that outside the compact set \( M_1 \cup (\partial M_1 \times [0, 1]) \), we have that \( V = 1 \) and in particular, \( V \) is bounded from below. Consider the Schrödinger operator \( S_1 = \Delta + V \) on \( N_1 \) and its lift \( S_2 \) on \( N_2 \). It is clear that \( S_1 \varphi = 0 \).

**Remark 4.5.** Evidently, \( N_1 \) is complete, without boundary and with Ricci curvature bounded from below. Since \( V = 1 \) outside the compact set \( M_1 \cup (\partial M_1 \times [0, 1]) \), from Propositions 2.5 and 3.2, it follows that \( \lambda_0^{\text{ess}}(S_1) \geq 1 \). Moreover, it is clear that \( V \) and \( \text{grad} V \) are bounded.

**Lemma 4.6.** The function \( \varphi \) belongs to the domain of the Friedrichs extension of \( S_1 \) and in particular, \( \lambda_0(S_1) = 0 \).

**Proof:** For \( T > 0 \), consider the compactly supported Lipschitz function \( \chi_T \) defined by \( \chi_T = 1 \) in \( M_1 \), \( \chi_T(x, t) = 1 \) for \( t \leq T \), \( \chi_T(x, t) = T + 1 - t \) for \( T \leq t \leq T + 1 \), and \( \chi_T(x, t) = 0 \) for \( t \geq T \). Then \( \chi_T \varphi \in H^1_0(N_1) \), for any \( T > 0 \), and

\[
\| \varphi - \chi_T \varphi \|_{L^2(N_1)}^2 \leq \int_{\partial M_1 \times [T, +\infty)} \varphi^2.
\]
Moreover, we have
\[
\int_{N_1} \| \operatorname{grad}(\varphi - \chi_T \varphi) \|^2 \leq 2 \int_{N_1} \left( (1 - \chi_T)^2 \| \operatorname{grad} \varphi \|^2 + \varphi^2 \| \operatorname{grad}(1 - \chi_T) \|^2 \right)
\]
\[
\leq 2 \int_{\partial M_1 \times [T, +\infty)} \| \operatorname{grad} \varphi \|^2 + 2 \int_{\partial M_1 \times [T, T+1]} \varphi^2.
\]
Evidently, \( \varphi(x, t) = \| \operatorname{grad} \varphi(x, t) \| = e^{-t} \) for \( t \geq 1 \), which yields that \( \chi_T \varphi \to \varphi \) in \( H^1_0(N_1) \), as \( T \to +\infty \). Since \( V \) is bounded, it follows that \( \varphi \in H_V(N_1) \). Since \( S_1 \varphi = 0 \), it is clear that \( \varphi \) is an eigenfunction of the Friedrichs extension of \( S_1 \), which yields that \( \lambda_0(S_1) \leq 0 \). From Proposition 2.4, since \( \varphi \) is positive, it follows that \( \lambda_0(S_1) = 0 \). \[\qed\]

Denote by \( \tilde{\varphi} \) the lift of \( \varphi \) on \( N_2 \) and consider the renormalization \( S_{\tilde{\varphi}} \) of \( S_2 \) with respect to \( \tilde{\varphi} \). Let \( g \in \operatorname{Lip}_c(M_2) \), such that \( g \) restricted on \( \partial M_2 \) is non-zero and smooth, and \( h: [0, +\infty) \to \mathbb{R} \) be a compactly supported, smooth function, with \( h(t) = 1 \) for \( t \leq 1/2 \). Extend \( g \) in the glued ends \( \partial M_2 \times [0, +\infty) \) by
\[
g(x, t) := g(x) h(t).
\]
(7)

It is clear that \( g \in \operatorname{Lip}_c(N_2) \), and in the glued ends, we have
\[
\operatorname{grad} g(x, t) = g(x) h'(t) \frac{\partial}{\partial t} + h(t) \operatorname{grad} g(x).
\]
In particular, it follows that
\[
\| \operatorname{grad} g(x, t) \|^2 = g^2(x) h'(t)^2 + h^2(t) \| \operatorname{grad} g(x) \|^2,
\]
which yields that
\[
\frac{\int_{N_2 \setminus M_2} \| \operatorname{grad} g \| \varphi^2}{\int_{N_2 \setminus M_2} g^2 \varphi^2} = \frac{\int_{\partial M_2} \int_0^{+\infty} \| \operatorname{grad} g \| \varphi^2}{\int_{\partial M_2} \int_0^{+\infty} g^2 \varphi^2} = \frac{\int_{\partial M_2} \| \operatorname{grad} g \|^2}{\int_{\partial M_2} g^2} + \frac{\int_0^{+\infty} (h')^2 \varphi^2}{\int_0^{+\infty} h^2 \varphi^2},
\]
(8)
where we used that in the glued ends \( \partial M_2 \times [0, +\infty) \), we have \( \tilde{\varphi}(x, t) = \phi(t) \).

**Proposition 4.7.** The renormalized operator \( S_{\tilde{\varphi}} \) satisfies \( \lambda_0(S_{\tilde{\varphi}}) = 0 \), which yields that \( \lambda_0(S_2) = \lambda_0(S_1) \).

**Proof:** Let \( \varepsilon > 0 \). From Lemma 4.4, there exists \( g \in \operatorname{Lip}_c(M_2) \), smooth on \( M_2 \setminus C_{t_0} \), for one \( t_0 \in (0, \delta) \), not vanishing identically on the boundary, such that
\[
\frac{\int_{M_2} \| \operatorname{grad} g \|^2}{\int_{M_2} g^2} < \frac{\varepsilon}{2} \quad \text{and} \quad \frac{\int_{\partial M_2} \| \operatorname{grad} g \|^2}{\int_{\partial M_2} g^2} < \frac{\varepsilon}{2}.
\]
19
Let $T > 1$ and consider a compactly supported, smooth $h: [0, +\infty) \to \mathbb{R}$, with $h(t) = 1$ for $t \leq T$, $h(t) = 0$ for $t \geq T + 1$, and $|h'| \leq 2$. Extend $g \in \text{Lip}_c(M_2)$ to the compactly supported $g \in \text{Lip}_c(N_2)$ as in (7). Evidently, we have

$$
\int_0^{+\infty} (h')^2 \varphi^2 \, dt \leq 4 \int_T^{T+1} e^{-2t} \, dt = \frac{4}{e^2 - e^{2T}} < \frac{\varepsilon}{2},
$$

for some sufficiently large $T$. From (8), it follows that

$$
\frac{\int_{N_2 \setminus M_2} \|\text{grad} \, g\|^2 \varphi^2 \, d\mu}{\int_{N_2 \setminus M_2} g^2 \varphi^2 \, d\mu} < \varepsilon.
$$

Hence, we have

$$
\mathcal{R}_{S_\varphi}(g) = \frac{\int_{M_2} \|\text{grad} \, g\|^2 \, d\mu + \int_{N_2 \setminus M_2} \|\text{grad} \, g\|^2 \varphi^2 \, d\mu}{\int_{M_2} g^2 \, d\mu + \int_{N_2 \setminus M_2} g^2 \varphi^2 \, d\mu} \leq \max \left\{ \frac{\int_{M_2} \|\text{grad} \, g\|^2 \, d\mu}{\int_{M_2} g^2 \, d\mu}, \frac{\int_{N_2 \setminus M_2} \|\text{grad} \, g\|^2 \varphi^2 \, d\mu}{\int_{N_2 \setminus M_2} g^2 \varphi^2 \, d\mu} \right\} < \varepsilon.
$$

Since $\varepsilon > 0$ is arbitrary, from Proposition 3.9, it follows that $\lambda_0(S_\varphi) = 0$ and in particular, $\lambda_0(S_2) = \lambda_0(S_1)$. ■

Proof of Theorem 4.1: Consider a Riemannian metric on $M_1$ as in Lemma 4.3 and its lift on $M_2$. Glue cylinders along the boundaries and extend the covering $p: M_2 \to M_1$ to a Riemannian covering $p: N_2 \to N_1$ as above. From Remark 4.5, $N_1$ is complete, without boundary, and with Ricci curvature bounded from below. Consider the Schrödinger operator $S_1 = \Delta + V$ on $N_1$, as above, and its lift $S_2$ on $N_2$. From Remark 4.5, we have that $V$ and grad $V$ are bounded. From Lemma 4.6 and Proposition 4.7, we obtain that $\lambda_0(S_2) = \lambda_0(S_1) = 0$, and Remark 4.5 yields that $\lambda_0^{\text{ess}}(S_1) \geq 1$. From Theorem 2.8, it follows that the covering $p: N_2 \to N_1$ is amenable, and so is the covering $p: M_2 \to M_1$. ■

Proof of Theorem 1.1: Follows from Theorems 4.1 and 3.4. ■

5 Arbitrary Riemannian coverings

In this section, we prove Theorem 1.2 and present some immediate consequences of it. As stated in the Introduction, we establish the following more general version of this theorem, involving manifolds with possibly non-empty boundary.

**Theorem 5.1.** Let $p: M_2 \to M_1$ be a Riemannian covering. Let $S_1$ be a Schrödinger operator on $M_1$, with $\lambda_0^N(S_1) \notin \sigma_0^{\text{ess}}(S_1)$, and $S_2$ its lift on $M_2$. Then $\lambda_0^N(S_2) = \lambda_0^N(S_1)$ if and only if the covering is amenable.
The following lemma, which is a consequence of Theorem 4.1, is essential for the proof of this theorem.

**Lemma 5.2.** Let \( p: M_2 \to M_1 \) be a non-amenable Riemannian covering. Let \( S_1 \) be a Schrödinger operator on \( M_1 \), with \( \lambda_0^N(S_1) \) being an eigenvalue of \( S_1^N \), and \( S_2 \) its lift on \( M_2 \). If \( \lambda_0^N(S_2) = \lambda_0^N(S_1) \), then there exists a compact \( K \subset M_1 \) with non-empty interior, and \((f_n)_{n \in \mathbb{N}} \subset \mathcal{C}^{\infty}(M_2)\), with \( \|f_n\|_{L^2(M_2)} = 1 \), \( \operatorname{supp} f_n \cap p^{-1}(K) = \emptyset \), for any \( n \in \mathbb{N} \), and \( \mathcal{R}_{S_2}(f_n) \to \lambda_0^N(S_2) \).

**Proof:** If \( M_1 \) has non-empty boundary, then we denote by \( \nu_i \) the inward pointing normal to \( \partial M_i \), \( i = 1, 2 \). From Proposition 2.7, since \( p: M_2 \to M_1 \) is non-amenable, there exists a smoothly bounded, compact domain \( K' \), with non-empty interior, such that the covering \( p: p^{-1}(K') \to K' \) is non-amenable, where \( p^{-1}(K') \) may be non-connected. From Theorem 4.1, it follows that \( \lambda_0^N(p^{-1}(K')) > 0 \).

Since \( \lambda_0^N(S_1) \) is an eigenvalue of \( S_1^N \), from Proposition 3.7, there exists a positive function \( \varphi \in \mathcal{C}^{\infty}(M_1) \), with \( S_1 \varphi = \lambda_0^N(S_1) \varphi \) and \( \nu_1(\varphi) = 0 \) on \( \partial M_1 \) (if non-empty). Consider the lift \( \tilde{\varphi} \) of \( \varphi \) on \( M_2 \) and the renormalization \( S_2 \tilde{\varphi} \) of \( S_2 \) with respect to \( \tilde{\varphi} \). Since \( \lambda_0^N(S_2) = \lambda_0^N(S_1) \), from Propositions 3.9 and 3.10, it follows that

\[
0 = \lambda_0(S_\tilde{\varphi}) = \inf_{f} \frac{\int_{M_2} \| \nabla f \|^2 \tilde{\varphi}^2}{\int_{M_2} f^2 \tilde{\varphi}^2},
\]

where the infimum is taken over all non-zero \( f \in \mathcal{C}^{\infty}(M_2) \), with \( \nu_2(f) = 0 \) on \( \partial M_2 \) (if non-empty). In particular, there exists \((f_n)_{n \in \mathbb{N}} \subset \mathcal{C}^{\infty}(M_2)\), with \( \|f_n\|_{L^2(M_2)} = 1 \), \( \mathcal{R}_{S_2}(f_n) \to 0 \) and \( \nu_2(f_n) = 0 \) on \( \partial M_2 \) (if non-empty).

Since \( \varphi \) is smooth and positive and \( K' \) is compact, there exist \( c_1, c_2 > 0 \), such that \( c_1 \leq \varphi \leq c_2 \) in \( K' \). From Proposition 3.2, it follows that

\[
\frac{\int_{p^{-1}(K')} \| \nabla f \|^2 \tilde{\varphi}^2}{\int_{p^{-1}(K')} f^2 \tilde{\varphi}^2} \geq \frac{c_1^2}{c_2^2} \lambda_0^N(p^{-1}(K')), \quad > 0,
\]

for any \( f \in \mathcal{C}^{\infty}(p^{-1}(K')) \setminus \{0\} \). Since \( \|f_n\|_{L^2_\tilde{\varphi}(M_2)} = 1 \) and \( \mathcal{R}_{S_\tilde{\varphi}}(f_n) \to 0 \), it follows that

\[
\int_{p^{-1}(K')} f_n^2 \tilde{\varphi}^2 \to 0 \quad \text{and} \quad \int_{M_2 \setminus p^{-1}(K')} f_n^2 \tilde{\varphi}^2 \to 1.
\]

Let \( K \subset M_2^\circ \) be a compact set, with non-empty interior, contained in the interior of \( K' \). Let \( \chi \in \mathcal{C}^{\infty}(M_1) \), with \( \chi = 1 \) in a neighborhood of \( K \), and \( \text{supp} \chi \subset K' \cap M_2^\circ \). Consider the lift \( \tilde{\chi} \) of \( \chi \) on \( M_2 \), and let \( g_n := (1 - \tilde{\chi}) f_n \in \mathcal{C}^{\infty}(M_2) \). It is clear that if \( M_1 \) has non-empty boundary, then \( \nu_2(g_n) = 0 \) on \( \partial M_2 \). Moreover, we have

\[
\|g_n\|^2_{L^2_\tilde{\varphi}(M_2)} = \int_{p^{-1}(K')} (1 - \tilde{\chi})^2 f_n^2 \tilde{\varphi}^2 + \int_{M_2 \setminus p^{-1}(K')} f_n^2 \tilde{\varphi}^2 \to 1.
\]
and
\[
\int_{M_2} \| \text{grad } g_n \|^2 \varphi^2 \leq 2 \int_{p^{-1}(K')} (f_n^2) \| \text{grad } \tilde{\chi} \|^2 + (1 - \tilde{\chi})^2 \| \text{grad } f_n \|^2 \varphi^2 \\
+ \int_{M_2 \backslash p^{-1}(K')} \| \text{grad } f_n \|^2 \varphi^2 \to 0.
\]

Therefore, \( R_{S_n}(g_n) \to 0 \) and \( \text{supp } g_n \cap p^{-1}(K) = \emptyset \). We may normalize \( g_n \) in \( L^2_p(M) \), so that \( \| g_n \|_{L^2_p(M_2)} = 1 \), for any \( n \in \mathbb{N} \).

Consider \( h_n := \tilde{\varphi} g_n \in C_c^\infty(M_2) \). If \( M_2 \) has non-empty boundary, since \( \nu_2(\tilde{\varphi}) = 0 \) and \( \nu_2(g_n) = 0 \), it follows that \( \nu_2(h_n) = 0 \) on \( \partial M_2 \). Evidently, \( \| h_n \|_{L^2(M_2)} = \| g_n \|_{L^2_p(M_2)} = 1 \).

Moreover, from the definition of the renormalized Schrödinger operator, it is clear that
\[
R_{S_n}(h_n) = \langle S_2 h_n, h_n \rangle_{L^2(M_2)} = \langle S_\varphi g_n, g_n \rangle_{L^2_p(M_2)} + \lambda_0^N(S_2) \\
= R_{S_n}(g_n) + \lambda_0^N(S_2) \to \lambda_0^N(S_2),
\]

which completes the proof.

**Proof of Theorem 5.1:** From Theorem 3.4, if the covering is infinite sheeted and amenable, then \( \lambda_0^N(S_1) = \lambda_0^N(S_2) \). If the covering is finite sheeted, then for \( f \in C_c^\infty(M_1) \), we have that \( f \circ p \in C_c^\infty(M_2) \), and the equality of the bottoms follows from Proposition 3.2 and Corollary 3.3. Hence, it remains to prove the converse implication.

Assume to the contrary that the covering is non-amenable. Since \( \lambda_0^N(S_2) = \lambda_0^N(S_1) \notin \sigma^N_{\text{ess}}(S_1) \), from Lemma 5.2 there exists a compact \( K \subset M_1 \) with non-empty interior, and \( (f_n)_{n \in \mathbb{N}} \subset C_c^\infty(M_2) \), with \( \| f_n \|_{L^2(M_2)} = 1 \), \( \text{supp } f_n \cap p^{-1}(K) = \emptyset \), for any \( n \in \mathbb{N} \), and \( R_{S_n}(f_n) \to \lambda_0^N(S_2) \). For \( n \in \mathbb{N} \), consider the pushdown \( g_n \) of \( f_n \), defined by
\[
g_n(z) := \left( \sum_{y \in p^{-1}(z)} f_n(y)^2 \right)^{1/2},
\]
for any \( z \in M_1 \). Then \( g_n \in \text{Lip}_c(M_1) \), \( \| g_n \|_{L^2(M_1)} = 1 \) and \( R_{S_1}(g_n) \leq R_{S_2}(f_n) \), for any \( n \in \mathbb{N} \) (cf. [2, Section 4]). From Proposition 3.2, since \( \lambda_0^N(S_2) = \lambda_0^N(S_1) \), it follows that \( R_{S_1}(g_n) \to \lambda_0^N(S_1) \). From Proposition 3.8, since \( \lambda_0^N(S_1) \notin \sigma^N_{\text{ess}}(S_1) \) and \( \text{supp } g_n \cap K = \emptyset \), this is a contradiction. Hence, the covering is amenable.

**Proof of Theorem 1.2:** Follows from Theorem 5.1, since the manifolds involved may have empty boundary.

**Remark 5.3.** In Theorem 1.2, the manifolds do not have to be complete. Therefore, from Remark 2.3, we obtain the corresponding result for the Dirichlet spectrum of Schrödinger operators on manifolds with boundary.

**Corollary 5.4.** Let \( p: M_2 \to M_1 \) be a Riemannian covering, with \( M_1 \) compact. Then the covering is amenable if and only if it preserves the bottom of the Dirichlet/Neumann spectrum of some/any Schrödinger operator.
Proof: Follows from Theorem 5.1 and Remark 5.3, since the Dirichlet and the Neumann spectrum of a Schrödinger operator on a compact manifold is discrete.

The next example shows that the assumption $\lambda_0(S_1) \notin \sigma_{\text{ess}}(S_1)$ in Theorem 1.2 cannot be replaced with $\lambda_0(S_1)$ being an eigenvalue of the Friedrichs extension of $S_1$.

Example 5.5. Let $M_1$ be a two dimensional torus with a cusp attached, endowed with a Riemannian metric, such that $M_1$ is complete and outside a compact set, the cusp is the surface of revolution generated by $1/t^2$, with $t \geq 1$. Since $M_1$ has finite volume, it follows that $\lambda_0(M_1) = 0$ and constant functions are $\lambda_0(M_1)$-eigenfunctions of the Friedrichs extension of the Laplacian on $M_1$. Let $x$ be a point of the torus and consider the non-negative quantity

$$\mu := - \lim_{r \to +\infty} \frac{1}{r} \ln(\text{Vol}(M_1) - \text{Vol}(B(x, r))) \leq - \lim_{r \to +\infty} \frac{1}{r} \ln(2\pi \int_{r+1}^{+\infty} \frac{1}{t^2} dt) = 0.$$  

From [5, Theorem 1], it follows that $\lambda_0^{\text{ess}}(M_1) = 0$. Consider the universal covering $p: M_2 \to M_1$. Since $\pi_1(M_1)$ is the free group with two generators, it follows that $p$ is non-amenable. Since the fundamental group of the cusp is amenable, from [14, Corollary 1.6], it follows that $\lambda_0(M_2) = 0$.

It is clear that Theorem 1.2 is more general than the results of [1, 6, 15]. For sake of completeness, we present an example demonstrating this fact. Let $p: M_2 \to M_1$ be a Riemannian covering, with $M_1$ non-compact, complete, without boundary, and with $\sigma_{\text{ess}}(M_1) = \emptyset$. Then, from Theorem 1.2, it follows that $\lambda_0(M_2) = \lambda_0(M_1)$ if and only if the covering is amenable. It is worth to point out that since we do not require the covering to be normal, the results of [6, 15] cannot be applied in this case. Moreover, from [10, Theorem 3.1], it follows that the Ricci curvature of $M_1$ is not bounded from below. Hence, also the result of [1] cannot be applied in this case.

6 An application

The aim of this section is to prove the following proposition, which was established for the Laplacian on manifolds without boundary in [1].

Proposition 6.1. Let $p: M_2 \to M_1$ be an infinite sheeted Riemannian covering. If $\lambda_0^N(S_1) = \lambda_0^N(S_2)$, then $\lambda_0^N(S_2) \in \sigma_{\text{ess}}^N(S_2)$.

The main point of this proposition is that the covering is not required to be normal (or to have infinite deck transformations group), since in this case, according to [14, Corollary 1.4], the spectrum of $S_2^N$ coincides with its essential spectrum. It is worth to point out that the manifolds in this proposition may have empty boundary. Moreover, since they may be non-complete, from Remark 2.3, the analogous statement holds for the Dirichlet spectrum of Schrödinger operators on manifolds with boundary.
Proposition 6.2. Let $S = \Delta + V$ be a Schrödinger operator on a Riemannian manifold $M$ and $(f_n)_{n \in \mathbb{N}} \subset \text{Lip}_c(M)$, with $\|f_n\|_{L^2(M)} = 1$ and $\mathcal{R}_S(f_n) \to \lambda^N_0(S)$. If $\lambda^N_0(S)$ is not an eigenvalue of $S^N$, then there exists a subsequence $(f_{n_k})_{k \in \mathbb{N}}$, such that $f_{n_k} \to 0$ in $L^2(M)$.

Proof: From Proposition 3.1, there exists $(f'_n) \in C^\infty_c(M) \cap \mathcal{D}(S^N)$, with $\|f'_n\|_{L^2(M)} = 1$ and $\|f_n - f'_n\|_{H^V(M)} \leq 1/n$, for any $n \in \mathbb{N}$, where $H^V(M)$ is the space defined in Subsection 2.1. It is clear that $\mathcal{R}_S(f'_n) \to \lambda^N_0(S)$ and it suffices to prove the statement for $(f'_n)_{n \in \mathbb{N}}$.

From the Spectral Theorem (cf. [18, Chapter 8]), there exists a measure space $X$, such that $L^2(M)$ is isometrically isomorphic to $L^2(X)$, and under this identification, $S^N$ corresponds to a multiplication operator with a measurable function $f : X \to \mathbb{R}$; that is, an operator of the form $\mu_f : \mathcal{D}(\mu_f) \subset L^2(X) \to L^2(X)$, with $\mathcal{D}(\mu_f) := \{g \in L^2(X) : fg \in L^2(X)\}$ and $\mu_f(g) = fg$, for any $g \in \mathcal{D}(\mu_f)$. The spectrum of $S^N$ coincides with the essential range of $f$ and in particular, $f \geq \lambda^N_0(S)$ almost everywhere.

Let $(g_n)_{n \in \mathbb{N}} \subset \mathcal{D}(\mu_f)$ be the sequence corresponding to $(f'_n)_{n \in \mathbb{N}}$ under this identification. Since $\|g_n\|_{L^2(X)} = 1$, after passing to a subsequence, we have that $g_n \to g$ in $L^2(X)$, for some $g \in L^2(X)$. It is clear that

$$\int_X (f - \lambda^N_0(S))g_n^2 = \langle \mu_f g_n, g_n \rangle_{L^2(X)} - \lambda^N_0(S) = \mathcal{R}_S(f'_n) - \lambda^N_0(S) \to 0.$$

For $\varepsilon > 0$, consider the measurable set $A_\varepsilon := \{f \geq \lambda^N_0(S) + \varepsilon\}$. Evidently, we have

$$\int_{A_\varepsilon} g_n^2 \leq \frac{1}{\varepsilon} \int_{A_\varepsilon} (f - \lambda^N_0(S))^2 g_n^2 \to 0.$$

Since $g_n \to g$ in $L^2(X)$, this yields that $g = 0$ almost everywhere in $A_\varepsilon$. In particular, $g = 0$ almost everywhere in $X \setminus f^{-1}(\{\lambda^N_0(S)\})$, which yields that $\mu_f g = \lambda^N_0(S)g$. Since $\lambda^N_0(S)$ is not an eigenvalue of $S^N$, it follows that $g = 0$. Therefore, $g_n \to 0$ in $L^2(X)$, which yields that $f'_n \to 0$ in $L^2(M)$.

Lemma 6.3. Let $p : M_2 \to M_1$ be a Riemannian covering. If $\lambda^N_0(S_2) = \lambda^N_0(S_1) \notin \sigma_{\text{ess}}^N(S_2)$, then $\lambda^N_0(S_1)$ is an eigenvalue of $S^N_1$.

Proof: Assume to the contrary that $\lambda^N_0(S_1)$ is not an eigenvalue of $S^N_1$. From Proposition 3.7, there exists a square-integrable, $\lambda^N_0(S_2)$-eigenfunction $\varphi$ of $S^N_2$, which is smooth and positive in $M_2$. Without loss of generality, we may assume that $\|\varphi\|_{L^2(M_2)} = 1$. Since $\varphi \in H^V_{\text{top}}(M_2)$, there exists $(f_n)_{n \in \mathbb{N}} \subset C^\infty_c(M_2)$, with $\|f_n\|_{L^2(M_2)} = 1$ and $f_n \to \varphi$ in $H^V_{\text{top}}(M_2)$, where $H^V_{\text{top}}(M_2)$ is the space defined in Subsection 2.1. Evidently, we have that $\mathcal{R}_{S_2}(f_n) \to \lambda^N_0(S_2)$.

Consider the pushdowns

$$g_n(z) := \left( \sum_{y \in p^{-1}(z)} f_n(y)^2 \right)^{1/2}.$$

24
Therefore, \( \lambda \rightarrow 0 \) in \( L^2(M_1) \).

Consider a non-negative \( \chi_2 \in C_c^\infty(M_2) \setminus \{0\} \), and its pushdown \( \chi_1 \in \text{Lip}_c(M_1) \) on \( M_1 \). Then

\[
\langle \chi_2, f_n \rangle_{L^2(M_2)} = \int_{M_1} \sum_{y \in p^{-1}(z)} \chi_2(y) f_n(y) dz \\
\leq \int_{M_1} \left( \sum_{y \in p^{-1}(z)} \chi_2(y)^2 \right)^{1/2} \left( \sum_{y \in p^{-1}(z)} f_n(y)^2 \right)^{1/2} dz \\
= \langle \chi_1, g_n \rangle_{L^2(M_1)}.
\]

This is a contradiction, since \( \langle \chi_1, g_n \rangle_{L^2(M_1)} \to 0 \) and \( \langle \chi_2, f_n \rangle_{L^2(M_2)} \to \int_{M_2} \chi_2 \varphi > 0 \). Therefore, \( \lambda^N_0(S_1) \) is an eigenvalue of \( S^N_1 \).

**Proof of Proposition 6.1:** If the covering is amenable, then the claim follows from Theorem 3.4. Hence, it remains to prove the statement for \( p \)-non-amenable. Assume to the contrary that \( \lambda^N_0(S_2) \notin \sigma^N_{\text{ess}}(S_2) \). From Lemma 6.3, it follows that \( \lambda^N_0(S_1) \) is an eigenvalue of \( S^N_1 \).

Since \( \lambda^N_0(S_2) = \lambda^N_0(S_1) \), from Lemma 5.2, there exists a compact set \( K \subset M_1 \) with non-empty interior, and \( (f_n)_{n \in \mathbb{N}} \subset C_c^\infty(M_2) \setminus \{0\} \), such that \( \mathcal{R}_S(f_n) \to \lambda^N_0(S_2) \) and \( \text{supp } f_n \cap p^{-1}(K) = \emptyset \), for any \( n \in \mathbb{N} \). From Proposition 3.8, since \( \lambda^N_0(S_2) \notin \sigma^N_{\text{ess}}(S_2) \) and \( p^{-1}(K) \) contains compact sets of positive measure, this is a contradiction.

**References**

[1] Werner Ballmann, Henrik Matthiesen, and Panagiotis Polymerakis, *Bottom of spectra and amenability of coverings*, MPI-Preprint 2018, arxiv.org/abs/1803.07353.

[2] Werner Ballmann, On the bottom of spectra under coverings, Math. Z. **288** (2018), no. 3-4, 1029–1036.

[3] Pierre Bérard and Philippe Castillon, *Spectral positivity and Riemannian coverings*, Bull. Lond. Math. Soc. **45** (2013), no. 5, 1041–1048.

[4] G. Pacelli Bessa, J. Fabio Montenegro, and Paolo Piccione, *Riemannian submersions with discrete spectrum*, J. Geom. Anal. **22** (2012), no. 2, 603–620.

[5] Robert Brooks, *On the spectrum of noncompact manifolds with finite volume*, Math. Z. **187** (1984), no. 3, 425–432.

[6] Robert Brooks, The bottom of the spectrum of a Riemannian covering, J. Reine Angew. Math. **357** (1985), 101–114.

[7] Robert Brooks, The fundamental group and the spectrum of the Laplacian, Comment. Math. Helv. **56** (1981), no. 4, 581–598.

[8] S. Y. Cheng and S. T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **28** (1975), no. 3, 333–354.
[9] Harold Donnelly and Peter Li, *Pure point spectrum and negative curvature for noncompact manifolds*, Duke Math. J. **46** (1979), no. 3, 497–503.

[10] Harold Donnelly, *On the essential spectrum of a complete Riemannian manifold*, Topology **20** (1981), no. 1, 1–14.

[11] Lawrence C. Evans, *Partial differential equations*, 2nd ed., Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 2010.

[12] Doris Fischer-Colbrie and Richard Schoen, *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math. **33** (1980), no. 2, 199–211.

[13] Tosio Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[14] Panagiotis Polymerakis, *On the spectrum of differential operators under Riemannian coverings*, MPI-Preprint 2018, arxiv.org/abs/1803.03223.

[15] T. Roblin and S. Tapie, *Exposants critiques et moyennabilité*. Géométrie ergodique, 61–92, Monogr. Études Math., 43, Études Math., Geneva, 2013, MR3220551, Zbl 1312.53060.

[16] Dennis Sullivan, *Related aspects of positivity in Riemannian geometry*, J. Differential Geom. **25** (1987), no. 3, 327–351.

[17] Michael E. Taylor, *Partial differential equations I. Basic theory*, 2nd ed., Applied Mathematical Sciences, vol. 115, Springer, New York, 2011.

[18] ———, *Partial differential equations II. Qualitative studies of linear equations*, 2nd ed., Applied Mathematical Sciences, vol. 116, Springer, New York, 2011.

Max Planck Institute for Mathematics
Vivatsgasse 7, 53111, Bonn
E-mail address: polymerp@mpim-bonn.mpg.de