Generating of additional force terms in Newton equation by twist-deformed Hopf algebras and classical symmetries

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Abstract

We compare two ways of force terms generating in the model of nonrelativistic particle moving in the presence of constant field force $\vec{F}$. First of them uses the twist-deformed acceleration-enlarged Newton-Hooke quantum space-times while the second one incorporates the doubly enlarged Newton-Hooke transformations of classical space. Particularly, we find the conditions for which the both treatments provide the same force terms.
1 Introduction

In the last time, there appeared a lot of papers dealing with classical and quantum mechanics (see e.g. [1]-[3]) as well as with field theoretical models (see e.g. [4]), in which the quantum space-time plays a crucial role. The idea to use noncommutative coordinates is quite old - it goes back to Heisenberg and was firstly formalized by Snyder in [5]. Recently, however, there were found new formal arguments based mainly on Quantum Gravity [6] and String Theory models [7], indicating that space-time at Planck scale should be noncommutative, i.e. it should have a quantum nature. Besides, the main reason for such considerations follows from the suggestion that relativistic space-time symmetries should be modified (deformed) at Planck scale, while the classical Poincare invariance still remains valid at larger distances [8], [9].

Presently, it is well known, that in accordance with the Hopf-algebraic classification of all deformations of relativistic and nonrelativistic symmetries, one can distinguish three basic types of quantum spaces [10], [11]. First of them corresponds to the well-known canonical type of noncommutativity

\[ [\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \]

with antisymmetric constant tensor \( \theta^{\mu\nu} \). Its relativistic and nonrelativistic Hopf-algebraic counterparts have been proposed in [12] and [13] respectively. The second kind of mentioned deformations introduces the Lie-algebraic type of space-time noncommutativity

\[ [\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}^\rho \hat{x}_\rho, \]

with particularly chosen coefficients \( \theta_{\mu\nu}^\rho \) being constants. The corresponding Poincare quantum groups have been introduced in [14]-[16], while the suitable Galilei algebras - in [17] and [13].

The last kind of quantum space, so-called quadratic type of noncommutativity

\[ [\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}^{\rho\tau} \hat{x}_\rho \hat{x}_\tau; \quad \theta_{\mu\nu}^{\rho\tau} = \text{const.}, \]

has been proposed in [18], [19] and [16] at relativistic and in [20] at nonrelativistic level.

Recently, in the series of papers [21]-[25], there has been demonstrated that the different (mentioned above) types of space-time noncommutativity produce in particle models the additional dynamical terms. Particularly, in article [21] it has been shown, that when two spatial coordinates commute to time in Lie-algebraically way, there is generated the additional acceleration of nonrelativistic particle moving in constant external field force. Similar investigations for quantum space with two spatial directions commuting to space in Lie-algebraically and quadratically way) have been studied in [21] and in article [24]. It should be noted, however, that the most interesting results have been obtained in paper [25], in which the authors considered nonrelativistic particle moving in gravitational potential on the well-known \( \kappa \)-Galilei
space-time [17]. Particulary, it has been demonstrated that the generated by such a type of space-time noncommutativity additional force term can be identified with so-called Pioneer anomaly [26]. The comparison of obtained result with the proper observational data [26], [27] permitted to fix the deformation parameter $\kappa$.

In this article we generate the (additional) time-dependent force terms in Newton equation of particle moving in constant force $\vec{F}$ in the completely new way, i.e. by the transformation of classical space of the form

$$x_i \longrightarrow x_i + a_i(t), \quad (4)$$

$$t \longrightarrow t. \quad (5)$$

Formally, the rules (4) and (5) can be realized in the framework of so-called doubly enlarged Newton-Hooke (undeformed) quantum group $U_0(\hat{NH}_+)$, for which [29]

$$a_i(t) = a_i \cosh \left( \frac{t}{\tau} \right) + v_i \tau \sinh \left( \frac{t}{\tau} \right) + 2b_i \tau^2 \left( \cosh \left( \frac{t}{\tau} \right) - 1 \right) + 6c_i \tau^3 \left( \sinh \left( \frac{t}{\tau} \right) - \frac{t}{\tau} \right),$$

and for which, in $\tau$ approaching infinity limit, we have

$$a_i(t) = a_i + v_i t + b_i t^2 + c_i t^3. \quad (7)$$

It should be also noted, that by the proper contraction schemes ($\tau \to \infty$) one can get from $U_0(\hat{NH}_+)$ the other nonrelativistic symmetry groups, such as: acceleration-enlarged Newton-Hooke, acceleration-enlarged Galilei and Galilei classical Hopf algebras, respectively.

In the next step of our investigations, we compare the obtained result with the mentioned above model of nonrelativistic particle defined on the quantum space of the form

$$[t, \bar{x}_i] = [\bar{x}_1, \bar{x}_3] = [\bar{x}_2, \bar{x}_3] = 0, \quad [\bar{x}_1, \bar{x}_2] = i f(t) ; \quad i = 1, 2, 3, \quad (8)$$

with

$$f(t) = f_{\kappa_1} \left( \frac{t}{\tau} \right) = \kappa_1 \cosh^2 \left( \frac{t}{\tau} \right), \quad (9)$$

$$f(t) = f_{\kappa_2} \left( \frac{t}{\tau} \right) = \kappa_2 \tau \cosh \left( \frac{t}{\tau} \right) \sinh \left( \frac{t}{\tau} \right), \quad (10)$$

1In [29] there is considered only the hyperbolic (De-Sitter) case. However, the trigonometric (anti-De-Sitter) transformation can be easily obtained by changing cosmological constant $\tau$ into $i\tau$ in all above equations.

2Here, we take under consideration the most general (known) type of nonrelativistic transformations of classical space.

3The formula (7) defines the transformation rules for so-called doubly enlarged Galilei Hopf structure $U_0(\hat{G})$.

4We consider particle moving in the presence of external constant field force $\vec{F}$. 

3
\[ f(t) = f_{\kappa_3} \left( \frac{t}{\tau} \right) = \kappa_3 \tau^2 \sinh^2 \left( \frac{t}{\tau} \right) , \quad (11) \]

\[ f(t) = f_{\kappa_4} \left( \frac{t}{\tau} \right) = 4\kappa_4 \tau^4 \left( \cosh \left( \frac{t}{\tau} \right) - 1 \right)^2 , \quad (12) \]

\[ f(t) = f_{\kappa_5} \left( \frac{t}{\tau} \right) = \kappa_5 \tau^2 \left( \cosh \left( \frac{t}{\tau} \right) - 1 \right) \cosh \left( \frac{t}{\tau} \right) , \quad (13) \]

\[ f(t) = f_{\kappa_6} \left( \frac{t}{\tau} \right) = \kappa_6 \tau^3 \left( \cosh \left( \frac{t}{\tau} \right) - 1 \right) \sinh \left( \frac{t}{\tau} \right) , \quad (14) \]

and (for \( \tau \to \infty \))

\[ f(t) = f_{\kappa_1}(t) = \kappa_1 , \quad (15) \]

\[ f(t) = f_{\kappa_2}(t) = \kappa_2 t , \quad (16) \]

\[ f(t) = f_{\kappa_3}(t) = \kappa_3 t^2 , \quad (17) \]

\[ f(t) = f_{\kappa_4}(t) = \kappa_4 t^4 , \quad (18) \]

\[ f(t) = f_{\kappa_5}(t) = \frac{1}{2} \kappa_5 t^2 , \quad (19) \]

\[ f(t) = f_{\kappa_6}(t) = \frac{1}{2} \kappa_6 t^3 . \quad (20) \]

The commutation relations (8) has been provided in the framework of twist procedure of acceleration-enlarged Newton-Hooke Hopf algebra \( U_0(\tilde{\mathcal{NH}}_+) \) \[ \text{[28]} \]. Here, by the comparison of both treatments, we find the direct link between transformation functions \( a_i(t) \) and time-dependent noncommutativity \( f(t) \).

The paper is organized as follows. In second section we consider the noncommutative model of nonrelativistic particle moving in constant field force \( \vec{F} \). Further, we provide its commutative counterpart which incorporates the transformation rules (4) and (5). In section three we compare the both (described above) force term generating procedures. The final remarks are mentioned in the last section.

## 2  Generating of the additional force terms in Newton equation

### 2.1  Generating by the space-time noncommutativity (8) - the first treatment

Let us now turn to the dynamical models in which the additional force terms are generated by space-time noncommutativity. Firstly, we start with the following phase space\[ ^6\]

\[ \{ t, \bar{x}_i \} = 0 , \quad \{ \bar{x}_1, \bar{x}_2 \} = f(t) , \quad \{ \bar{x}_1, \bar{x}_3 \} = 0 = \{ \bar{x}_2, \bar{x}_3 \} , \quad (21) \]

\[ ^5\text{In this article we consider the most general (known) twist deformation of nonrelativistic symmetries.} \]

\[ ^6\text{We use the correspondence relation } \{ a, b \} = \frac{1}{\hbar} [\hat{a}, \hat{b}] (\hbar = 1). \]
\{ \bar{x}_i, \bar{p}_j \} = \delta_{ij} , \quad \{ \bar{p}_i, \bar{p}_j \} = 0 , \quad (22)
corresponding to the commutation relations [5]. One can check that the relations (21), (22) satisfy the Jacobi identity and for deformation parameters \( \kappa_a \) running to zero become classical. Next, we define the Hamiltonian function for nonrelativistic particle moving in constant field force \( \bar{F} \) as follows

\[ H(\bar{p}, \bar{x}) = \frac{1}{2m} \left( \bar{p}_1^2 + \bar{p}_2^2 + \bar{p}_3^2 \right) - \sum_{i=1}^{3} F_i \bar{x}_i . \quad (23) \]

In order to analyze the above system we represent the noncommutative variables \((\bar{x}_i, \bar{p}_i)\) on classical phase space \((x_i, p_i)\) as (see e.g. [30]-[32])

\[ \bar{x}_1 = x_1 - \frac{f_{\kappa_a}(t)}{2} p_2 , \quad \bar{x}_2 = x_2 + \frac{f_{\kappa_a}(t)}{2} p_1 , \quad \bar{x}_3 = x_3 , \quad \bar{p}_i = p_i , \quad (24) \]

where

\[ \{ x_i, x_j \} = 0 = \{ p_i, p_j \} , \quad \{ x_i, p_j \} = \delta_{ij} . \quad (25) \]

Then, the Hamiltonian (23) takes the form

\[ H(p, x) = H_f(t) = \frac{1}{2m} \left( p_1^2 + p_2^2 + p_3^2 \right) - \sum_{i=1}^{3} F_i x_i + F_1 \frac{f_{\kappa_a}(t)}{2} p_2 - F_2 \frac{f_{\kappa_a}(t)}{2} p_1 . \quad (26) \]

Using the formulas (25) and (26) one gets the following canonical Hamiltonian equations of motions \((\dot{x}_i = \frac{d}{dt} x_i = \{ o_i, H \})\)

\[ \dot{x}_1 = \frac{p_1}{m} - \frac{f_{\kappa_a}(t)}{2} F_2 , \quad \dot{p}_1 = F_1 , \quad (27) \]

\[ \dot{x}_2 = \frac{p_2}{m} + \frac{f_{\kappa_a}(t)}{2} F_1 , \quad \dot{p}_2 = F_2 , \quad (28) \]

\[ \dot{x}_3 = \frac{p_3}{m} , \quad \dot{p}_3 = F_3 , \quad (29) \]

which when combined yield the proper Newton law

\[
\begin{align*}
\begin{cases}
m \ddot{x}_1 &= F_1 - \frac{mf_{\kappa_a}(t)}{2} F_2 = G_1(t) \\
m \ddot{x}_2 &= F_2 + \frac{mf_{\kappa_a}(t)}{2} F_1 = G_2(t) \\
m \ddot{x}_3 &= F_3 = G_3 .
\end{cases}
\end{align*}
\quad (30)
\]

Firstly, by trivial integration one can find the solution of above system; it looks as follows

\[
\begin{align*}
x_1(t) &= \frac{F_1 t^2}{2m} + v_1^0 t + \frac{F_2}{2m} \int_0^t f_{\kappa_a}(t') dt' \\
x_2(t) &= \frac{F_2 t^2}{2m} + v_2^0 t + \frac{F_1}{2m} \int_0^t f_{\kappa_a}(t') dt' \\
x_3(t) &= \frac{F_3 t^2}{2m} + v_3^0 t + x_3^0,
\end{align*}
\quad (31)
\]

5
with $v_0$ and $x_0$ denoting the initial velocity and position of particle respectively. Next, one should observe that the noncommutativity [8] generates the new, time-dependent force term $\vec{G}(t) = [G_1(t), G_2(t), G_3]$, which for deformation parameters $\kappa_\alpha$ approaching zero reproduces undeformed force $\vec{F}$. Finally, it should be noted that for $f(t) = \lim_{r \to \infty} f_{\kappa_1}(t) = \kappa_1 = \theta$ and $f(t) = \lim_{r \to \infty} f_{\kappa_2}(t) = \kappa_2 t$ (see formulas (15) and (16) respectively) we recover two models provided in [21]. First of them does not introduce any modification of Newton equation, while the second one generates the constant acceleration of particle.

### 2.2 Generating by the transformation of classical space (4)-(6) - the second treatment

Let us now turn to the second model in which the force terms are generated by the transformation of classical space (4). Firstly, we start with the following equation of motion

$$
\begin{cases}
  m\ddot{x}_1 = F_1 \\
  m\ddot{x}_2 = F_2 \\
  m\ddot{x}_3 = F_3,
\end{cases}
$$

(32)

defined on the commutative (standard) space-time. Next, by using transformation rules (11) with function $a_i(t)$ equal zero, we get the following Newton law in the nonrelativistic space-time with changed space coordinates (see (11))

$$
\begin{cases}
  m\ddot{x}_1 = F_1 + m\ddot{a}_1(t) = H_1(t) \\
  m\ddot{x}_2 = F_2 + m\ddot{a}_2(t) = H_2(t) \\
  m\ddot{x}_3 = F_3 = H_3,
\end{cases}
$$

(33)

i.e. there appeared in Newton equation the additional force term given by the function $\ddot{a}_i(t)$. Moreover, let us notice that only for Galileian transformation

$$
a_i(t) = a_i + v_i t,
$$

(34)

the equation of motion (32) remains unchanged. Besides, one should observe that the solution of (33) is given by

$$
\begin{cases}
  x_1(t) = \frac{F_1}{2m} t^2 + v_1^0 t + x_1^0 + a_1(t) \\
  x_2(t) = \frac{F_2}{2m} t^2 + v_2^0 t + x_2^0 + a_2(t) \\
  x_3(t) = \frac{1}{2m} F_3 t^2 + v_3^0 t + x_3^0,
\end{cases}
$$

(35)
and (for example) for function (7), it takes the form

\[
\begin{align*}
    x_1(t) &= c_1 t^3 + \left(\frac{F_1}{2} + b_1\right) t^2 + (v_1^0 + v_1) t + x_1^0 + a_1 \\
    x_2(t) &= c_2 t^3 + \left(\frac{F_2}{2} + b_2\right) t^2 + (v_2^0 + v_2) t + x_2^0 + a_2 \\
    x_3(t) &= \frac{1}{2m} F_3 t^2 + v_3^0 t + x_3^0 .
\end{align*}
\]

(36)

Consequently, we see that as in the previous treatment there appeared in model the new, time-dependent force \( \vec{H}(t) = [H_1(t), H_2(t), H_3] \) defined by the equation (33).

3 Comparison of the both approaches

Let us now compare the formulated above treatments. Firstly, one can observe that the new, time-dependent forces \( \vec{G}(t) \) and \( \vec{H}(t) \) are exactly the same when

\[
\ddot{a}_1(t) = -\frac{\dot{f}_{\kappa}(t)}{2} F_2 , \quad \ddot{a}_2(t) = \frac{\dot{f}_{\kappa}(t)}{2} F_1 .
\]

(37)

Unfortunately, such a situation appears only for (see (7) and (16))

\[
\begin{align*}
    f_{\kappa_2}(t) &= \kappa_2 t , \quad a_1(t) = a_1 + v_1 t + b_1 t^2 ; \quad b_1 = -\frac{\kappa_2}{4} F_2 , \\
    a_2(t) &= a_2 + v_2 t + b_2 t^2 ; \quad b_2 = \frac{\kappa_2}{4} F_1 ,
\end{align*}
\]

(38)

as well as for (see formulas (7), (17) and (19))

\[
\begin{align*}
    a_1(t) &= a_1 + v_1 t + c_1 t^3 ; \quad c_1 = -\frac{\kappa}{6} F_2 , \\
    a_2(t) &= a_2 + v_2 t + c_2 t^3 ; \quad c_2 = \frac{\kappa}{6} F_1 ,
\end{align*}
\]

(40)

\[
\begin{align*}
    f_{\kappa_3}(t) &= \kappa_3 t^2 = f_{\kappa_3}(t) = \frac{1}{2} \kappa_3 t^2 = \kappa t^2 .
\end{align*}
\]

(42)

Then, for the choices (38), (39) and (40)-(42), we get the following equalities

\[
\begin{align*}
    H_1(t) &= G_1(t) = F_1 - \frac{m \kappa_2}{2} F_2 , \\
    H_2(t) &= G_2(t) = F_2 + \frac{m \kappa_2}{2} F_1 , \\
    H_3 &= G_3 = F_3 ,
\end{align*}
\]

(43)

and

\[
\begin{align*}
    H_1(t) &= G_1(t) = F_1 - m \kappa F_2 t , \\
    H_2(t) &= G_2(t) = F_2 + m \kappa F_1 t , \\
    H_3 &= G_3 = F_3 ,
\end{align*}
\]

(46)

7 The equality of force terms in both treatments appears only for cosmological constant \( \tau \) approaching infinity.
respectively. Besides, it seems that the second treatment (by the classical groups) has one advantage - it follows from (30) and (33) that the second approach (contrary to the first one) does not need the presence of initial constant force $\vec{F} = [F_1, F_2, F_3]$ to generate the additional dynamical terms. In fact, when one puts $\vec{F} = [0, 0, 0]$ in the equation of motion (30) then its right side vanishes, while in the equation (33) there are still presented the additional forces of the form

$$H_1(t) = m\ddot{a}_1(t) \quad , \quad H_2(t) = m\ddot{a}_2(t) .$$

(49)

4 Final remarks

In this article we compare two ways of force term generating in the model of nonrelativistic particle moving in constant external field force. First of them uses the space-time noncommutativity (5) while the second one is based on the transformation rules of classical space (4). Particularly, we find for which functions $f(t)$ and $a_i(t)$ the generated force terms are the same in both treatments. Finally, it should be noted that performed in this article considerations concern only the simplest model of nonrelativistic particle moving in constant force $\vec{F}$. However, they can be extended to the arbitrary norelativistic system and then, the necessary calculations become much more complicated but the general mechanism remains the same.

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