Abstract. An overview of the braid group techniques in the theory of algebraic surfaces from Zariski to the latest results is presented. An outline of the Van Kampen algorithm for computing fundamental groups of complements of curves and the modification of Moishezon-Teicher regarding branch curves of generic projections are given. The paper also contains a description of a quotient of the braid group, namely \( \tilde{B}_n \) which plays an important role in the description of fundamental groups of complements of branch curves. It turns out that all such groups are “almost solvable” \( \tilde{B}_n \)-groups. Finally, the possible applications to study moduli spaces of surfaces of general type are described and new examples of positive signature spin surfaces whose fundamental groups can be computed using the above algorithm (Galois cover of Hirzebruch surfaces) are presented.

0. Introduction.

This manuscript is based on our talk in Santa Cruz, July 1995. It presents the applications of the braid group technique to the study of algebraic surfaces and curves in general and to the moduli space of surfaces and the topology of complements of curves in particular. These techniques started with Enriques, Zariski and Van Kampen in the 30’s (see [VK], [Z]) and were revived by Moishezon in the late 70’s (see, e.g., [Mo1]). The manuscript includes a survey on the topology of complements of branch curve starting with Zariski’s results, as well as new results (related to a quotient \( \tilde{B}_n \) of the braid group) and an open question on the topic.

The manuscript is divided as follows:

0. Introduction

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I. The connections between classification of algebraic surfaces and related fundamental groups.

In 1977 Gieseker prove that the moduli space of surfaces of general type is a quasi-projective variety (see [G]). Unlike the case for curves it is not irreducible. Catanese and Manetti proved results about the structure and the number of components of moduli spaces (see, e.g., [C1], [C2], [C3], [C4], [C5], [C6], [CCiLo], [CW], [Ma]). Not much is known about these moduli spaces. Nevertheless, unlike previous expectations, simply connected (and spin) surfaces exist also in the $\tau > 0$ area, $\tau = \frac{1}{4}(C_1^2 - 2C_2)$ (see [MoTe1], [MoTe2], [MoTe3], [Ch], [MoRoTe], [PPX]).

The fact that algebraic surfaces are nontrivial geometric objects was remarkably confirmed by S. Donaldson who showed that among algebraic surfaces one can find homeomorphic non-diffeomorphic (simply-connected) 4-manifolds. In particular, he produced the first counterexamples to the h-cobordism conjecture in dimension four.
Donaldson’s theory was also used to construct the first examples of homeomorphic non-diffeomorphic (simply-connected) algebraic surfaces of general type ([FMoM], [Mo2]). In 1994, Witten [W] and later Witten and Sieberg [SW] defined a new set of invariants for 4-manifolds (monopole invariants), and have shown the equivalence of this invariant with Donaldson’s polynomial. These invariants take a simple form for Kähler surfaces.

We expect that the connected components of moduli spaces of algebraic surfaces (of general type) correspond to the principal diffeomorphism classes of corresponding topological 4-manifolds. Thus, it is possible that Donaldson’s polynomial invariants will distinguish these connected components. However, we present here a more direct geometrical approach.

The ultimate goals of the braid group techniques are finding new invariants distinguishing connected components of the moduli space of surfaces of general type. For that we try to compute different fundamental groups related to the surface, groups which do not change when one moves in a connected component of the moduli space. The first groups we compute are \( \pi_1(\mathbb{C}^2 - S) \) and \( \pi_1(\mathbb{C}\mathbb{P}^2 - S) \) where \( S \) is the branch curve of a generic projection \( X \to \mathbb{C}\mathbb{P}^2 \). If \( \pi_1 \) is “big” then it can distinguish between connected components. If they are “small” there is hope to compute \( \pi_2 \) as a module over \( \pi_1 \). We can also compute fundamental groups of surfaces of general type. This is especially interesting in the positive signature area which is still rather wild.

For minimal surfaces of general type it turns out that all the information is contained in the canonical class: i.e. it is a diffeomorphism invariant and all other information about Donaldson’s polynomials must follow from it. Thus, for the problem of finding invariants of deformation types of surfaces of general type we are almost where we were 15 years ago (the only new invariant is divisibility of the canonical class). So fundamental groups of the complements to branching curves of generic projections might still be the best bet for this subject.

We want to recall here that computing fundamental groups of complements of
a plane curve is enough in order to understand the topology of a complement in $\mathbb{P}^N$ of any algebraic subset (as proven by Zariski). In fact, for a generic $\mathbb{P}^2$ in $\mathbb{P}^N$:

$$\pi_1(\mathbb{P}^N - V) \cong \pi_1(\mathbb{P}^2 - \mathbb{P}^2 \cap V).$$

Furthermore, we recall that lately there is also a growing interest in fundamental groups of algebraic varieties in general. A very partial list includes [BoKa], [CMan], [DOZa] [L1], [L2], [Si], [To].

The braid group appears in the formulation of the results and as an essential step of the algorithm for computing fundamental groups of complements of curves (see Section V).

II. Known results on fundamental groups of complements of branch curves; an open question.

Consider the following situation:

Surface $X \hookrightarrow \mathbb{C}\mathbb{P}^N$

$$\downarrow \text{generic projection}$$

$S \subseteq \mathbb{C}\mathbb{P}^2 \quad S = \text{branch curve}$

We denote: $G = \pi_1(\mathbb{C}^2 - S, *)$, $\overline{G} = \pi_1(\mathbb{C}\mathbb{P}^2 - S, *)$.

We want to to find a general formula for $G$ and $\overline{G}$ which depends on known invariants of $X$. As we said in the our introduction, the topic started with Zariski who proved in the 30’s that if $X$ is a cubic surface in $\mathbb{C}\mathbb{P}^3$ then $\overline{G} \cong Z_2 \star Z_3$ (see [Z]). In the late 70’s Moishezon proved that if $X$ is a deg $n$ surface in $\mathbb{C}\mathbb{P}^3$ then $G \cong B_n$, $\overline{G} \cong B_n / \text{Center}$ (see [Mo1]). In fact, Moishezon’s result for $n = 3$ is the same as Zariski’s result since $B_3 / \text{Center} \cong Z_2 \star Z_3$.

The next example was $V_2$ (Veronese of order 2) (see [MoTe3]). In all the above examples we have $G \supset F_2$ where $F_2$ is a free noncommutative subgroup with 2 elements. We call a group $G$ “big” if $G \supset F_2$.

Since 1991 the following examples have been discovered: $V_3$, the Veronese of order 3 which was done by Moishezon and Teicher in [MoTe7], [MoTe8], [MoTe9],

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[MoTe10], [MoTe11], [Te2], and generalized later to general $V_n$ (preprint); $X_{ab}$, the embedding of $\mathbb{CP}^1 \times \mathbb{CP}^1$ into $\mathbb{CP}^N$ w.r.t. a linear system $|a\ell_1 + b\ell_2|$; CI, the complete intersection which was done by A. Robb in his Ph.D. Thesis in 1994, (see [Ro]).

Unlike previous expectations, in all the new example $G$ is not “big”. Moreover, $G$ is “small”, i.e., $G$ is “almost solvable”, i.e., it contains a subgroup of finite index which is solvable. It turned out that there exists a quotient of the braid group (by a subgroup of the commutant), namely $\tilde{B}_n$ s.t. all new results give $G = \tilde{B}_n$-group and $\overline{G} = G$/central element ($\tilde{B}_n$-group is a group on which $\tilde{B}_n$ act). For CI, $G$ is $\tilde{B}_n$ itself. So the old examples were exceptions ($V_2$ often turns out to be an exception) and fundamental groups of complements of branch curves are not “big”. They are surprisingly “small”. Moreover, in all the new examples $G, \overline{G}$ are an extension of a solvable group by a symmetric one. Based on that fact we ask the following

**Question.** For which families of simply connected algebraic surfaces of general type is the fundamental group of the complement of the branch curve of a generic projection to $\mathbb{CP}^2$ an extension of a solvable group by a symmetric group?

We believe that the answer to this question lies in the decomposable structure of the corresponding 4-manifold. One should also notice that if a group $G$ is “big” then it is not “small” and if it is “small” then it is not “big”.

**III. Presentation of $\tilde{B}_n$, a quotient of the braid group.**

The braid group is connected to fundamental groups of complements of branch curves in two ways. The first way is through the appearance of its quotient $\tilde{B}_n$ in the description of such groups (see Section II), and the second way is through the use of the braid group as a major tool in the algorithm for computing such groups (see Section IV).

We first review the definition of braid group (see also [A] and [B]), and then we shall define its quotient $\tilde{B}_n$. We will work with a geometric model of the braid group.
Definition: The braid group $B_n$.

Let $D$ be a topological disc, $K \subset D$ finite. Consider: $\{\beta|\beta: D \to D \text{ diffeomorphism}, \, \beta(K) = K, \, \beta|_{\partial D} = Id\}$. Clearly, $\{\beta\}$ is a group which acts naturally on $\pi_1(D-K)$. We define an equivalence relation on $\{\beta\}$ as follows: $\beta_1 \sim \beta_2 \Leftrightarrow$ the action of $\beta_1, \beta_2$ on $\pi_1(D-K)$ coincide. $B_n = \{\beta\} / \sim$

We have to distinguish certain elements in $B_n$.

Definition: Half-twist w.r.t. $\left[\frac{-1}{2}, \frac{1}{2}\right]$.

Consider $D_1$, the unit disc, $\pm \frac{1}{2} \in D_1$. Take $\rho: [0, 1] \to [0, 1]$ continuous s.t. $\rho(r) = \pi \quad r \leq \frac{1}{2} \rho(1) = 0$. Define $\delta: D_1 \to D_1: \delta(re^{i\theta}) = re^{i(\theta + \rho(r))}$. Clearly, $\delta\left(\frac{1}{2}\right) = -\frac{1}{2}, \, \delta\left(-\frac{1}{2}\right) = \frac{1}{2}$ and $\delta|_{\partial D_1} = Id$. The disc of radius $\frac{1}{2}$ rotates $180^\circ$ counterclockwise. Outside of this disc it rotates in smaller and smaller angles till it rests on the unit circle. Thus we get a braid $[\delta] \in B_2\left[D_1, \left\{\pm \frac{1}{2}\right\}\right]$. $[\delta]$ is called the half-twist w.r.t. the segment $\left[\frac{-1}{2}, \frac{1}{2}\right]$.

Using the above definition we define a generalized half-twist.

Definition: $H(\sigma)$, half-twist w.r.t. a path $\sigma$.

Let $D, \, K$ be as above, $a, b \in K$. Let $\sigma$ be a path from $a$ to $b$ which does not meet any other point of $K$. We take $D_2$ a small topological disc in $D$ s.t. $\sigma \subset D_2 \subset D, \, D_2 \cap K = \{a, b\}$. We take $\psi: D_2 \to D_1$ (unit disc) s.t. $\psi(\sigma) = \left[-\frac{1}{2}, \frac{1}{2}\right]$. $\psi(a) = -\frac{1}{2}, \psi(b) = \frac{1}{2}$. We consider a “rotation” $\psi\delta\psi^{-1}: D_2 \to D_2; \psi\delta\psi^{-1}$ is identity on the boundary of $D_2$. We extend it to $D$ by identity. $H(\sigma) = [\text{extension of } \psi\delta\psi^{-1}]$. 

\[
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\includegraphics[width=0.8\textwidth]{braid_diagram.png}
\end{array}
\]
We shall present now $\tilde{B}_n$, the quotient of the braid group by commutators of
the transversal half-twists. We define:

**Definition: Transversal half-twists.**
$H(\sigma_1)$ and $H(\sigma_2)$ are transversal if $\sigma_1 \cap \sigma_2 = \{\text{one point which is not an end point}\}$

**Definition: $\tilde{B}_n$.**
Let $X,Y$ be a pair of transversal half-twists. Let $[X,Y] = XYX^{-1}Y^{-1}$. Let $\langle[X,Y]\rangle$ be the subgroup normally generated by $[X,Y]$. $\tilde{B}_n = B_n / \langle[X,Y]\rangle$.

**Remark.** Since all transversal half-twists are conjugated, $\langle[X,Y]\rangle$ contains every
commutator of transversal half-twists and thus $\tilde{B}_n$ is independent of the choice of $X,Y$.

One can find a description of $\tilde{B}_n$ in [Te1].

IV. Two new theorems on the fundamental groups of complements of branch curves of $V_3$ (Veronese of order 3).

For example, we shall formulate exactly the structure theorem concerning $V_3$.

**Theorem 1.** [MoTe9]
Let $X$ be $V_3$ (the Veronese of order 3).
Let $S$ be the branch curve of a generic projection to $\mathbb{CP}^2$. Then:

$$\pi_1(\mathbb{C}^2 - S, \ast) \simeq \tilde{B}_9 \ltimes G_0(9) / N_9$$

where

$G_0(9) = \text{central extension of a free group with 8 elements} = \langle u_1, \ldots, u_8, \tau \rangle$

s.t.

$$[u_i, u_j] = \begin{cases} \tau & |i - j| = 1 \\ 1 & |i - j| \neq 1 \end{cases}$$
There exists a standard base of $\tilde{B}_9 : \tilde{X}_1, \ldots, \tilde{X}_8$ s.t. the action of $\tilde{B}_9$ on $G_0(9)$ is as follows:

$$(u_i)_x = \begin{cases} 
  u_i \tau & k = i \\
  u_i & |i - k| \geq 2 \\
  u_i u_i & |i - k| = 1.
\end{cases}$$

$N_9 = \langle u_i^3 = X_i^3, \quad \tau = c \rangle$ where $c \in \text{Center } \tilde{B}_9$, $c^2 = 1$.

The “almost solvable” theorem concerning $V_3$ is as follows:

**Theorem 2.** [MoTe10]

Let $X, S$ be as in the previous theorem.

Let $G = \pi_1(\mathbb{C}^2 - S)$. Then there exists a series $1 < H'_{9,0} < H_{9,0} < H_9 < G$ s.t.

- $G/H_9 \simeq S_9$
- $H_9/H_{9,0} \simeq \mathbb{Z}$
- $H_{9,0}/H'_{9,0} \simeq (\mathbb{Z} \oplus \mathbb{Z}/3)^8$
- $H'_{9,0} (= H'_9) = \{1, c\} \simeq \mathbb{Z}/2$ \quad ($c \in \text{Center } G$).

We did not discuss yet where does the braid group enter into the calculation of fundamental groups of complements of branch curves; we do this in the next section.

**V. An algorithm to compute fundamental groups of complements of branch curves.**

In this section we state the main steps used so far for computing such groups:

(a) Degeneration of the surface to a union of planes where no 3 planes meet in a line.

(b) Computing the braid monodromy of the branch curve (using the above degeneration).

(c) Enriques-Van Kampen method for getting a finite presentation of $\pi_1(\mathbb{C}P^2 - S)$ (using the braid monodromy).
(d) Invariance properties of the braid monodromy (to produce more relations in \( \pi_1(\mathbb{CP}^2 - S) \) than those induced from the Van Kampen method).

(e) Studying \( G \) as a \( \tilde{B}_n \)-groups and looking for prime elements.

(f) Proving “almost solvability” when available.

At the moment we work on eliminating the condition that no 3 planes meet in a line in order to enlarge the variety of surfaces to which we can apply our methods. The reason that we need the degeneration at all is to simplify the computations of the braid monodromy of the branch curve. If the surface is degenerated to a union of planes where no 3 planes meet in a line, then the degenerated object has a branch curve which is partial to an arrangement of lines known as “dual to a generic”.

An arrangement of lines “dual to generic” is an arrangement in which there are exactly 2 multiple points (where \( m \) lines meet, \( m \geq 3 \)) on every line. In [MoTe4] we presented an algorithm for computing the braid monodromy of arrangement “dual to a generic” In [MoTe6] we presented an algorithm how to get from the braid monodromy of the degenerated braid curve, the braid monodromy of the original curve. To eliminate the condition in (a) means to produce an algorithm for computing braid monodromies of arrangements of lines which are not “dual to generic”. This as explained earlier will enlarge the variety of surfaces for which we can compute \( \pi_1(\mathbb{CP}^2 - S) \).

VI. The braid monodromy (Step (b) of the algorithm).

Computing the braid monodromy is the main tool to compute fundamental groups of complements of curves (Step (b)). In this section we define the braid monodromy and compute some examples.

**Definition: The braid monodromy w.r.t. \( S, \pi, u \).**

Let \( S \) be a curve, \( S \subseteq \mathbb{C}^2 \).

Let \( \pi : S \rightarrow \mathbb{C}^1 \) be defined by \( \pi(x, y) = x \). We denote \( \text{deg} \pi \) by \( m \).

Let \( N = \{ x \in \mathbb{C}^1 \mid \#\pi^{-1}(x) \leq m \} \). Take \( u \notin N \), s.t. \( x \ll u \ \forall x \in N \). Let \( \mathbb{C}^1_u = \{(u,y)\} \).
There is a natural defined homomorphism \( \pi_1(\mathbb{C}^1 - N, u) \xrightarrow{\varphi} B_m[\mathbb{C}_u^1, \mathbb{C}_u^1 \cap S] \) which is called the braid monodromy w.r.t. \( S, \pi, u \).

**Remark.** The classical monodromy factors through the braid monodromy

\[
\pi_1(\mathbb{C}^1 - N, u) \rightarrow B_m[\mathbb{C}_u^1, \mathbb{C}_u^1 \cap S] \xrightarrow{\varphi} S_m
\]

**Example of computing braid monodromy of a curve with only one singular point.**

Let \( S \) be defined by \( y^2 = x^m \).

For \( \pi : S \rightarrow \mathbb{C}^1 \) defined by \( \pi(x, y) = x \) we have \( \deg \pi = 2 \). \( S \) has only one singular point \((0, 0)\) and thus \( N = \{0\} \). We take \( u = 1 \). Clearly, \( \mathbb{C}_u^1 \cap S = \{-1, 1\} \).

Let \( \delta(t) = e^{2\pi it} \) (\( \delta(t) \) is a closed loop that starts in \( u \)). \( \delta \) is a generator of \( \pi_1(\mathbb{C}^1 - N, u) \).

We lift \( \delta(t) \) to \( S \). There are 2 liftings:

\[
\begin{align*}
\delta_1(t) &= \left(e^{2\pi it}, e^{\frac{2\pi i m}{2}}\right) \\
\delta_2(t) &= \left(e^{2\pi it}, -e^{\frac{2\pi i m}{2}}\right)
\end{align*}
\]
The projections of $\delta_1(t)$ and $\delta_2(t)$ to $C^1_u$ are:

$$a_1(t) = e^{\frac{2\pi it}{m}} = (e^{\pi it})^m$$
$$a_2(t) = -e^{\frac{2\pi it}{m}} = -(e^{\pi it})^m$$

By definition of the braid monodromy, $\varphi(\delta)$ is induced from the motion \(\{e^{\pi it}m, -(e^{\pi it})m\}\).

The braid induced from the motion \(\{e^{\pi it}, -(e^{\pi it})\}\) is $H = H([-1, 1])$ which is the half-twist in $C^1_u$ w.r.t. $[-1, 1]$. Clearly, $\varphi(\delta) = H^m$.

The above example is almost a proof for the following theorem of Zariski.

**Theorem.** (Zariski) Let $S$ be a cuspidal curve. Assume that above each point of $N$ there is only one singular point of $\pi$. Let $x_0 \in N$. Let $\delta$ be a loop in $\pi_1(C - N, u)$ around $x_0$ ($\delta$ is simple and no other point of $N$ is inside $\delta$). Let $\varphi : \pi_1(C - N, u) \to B_m$ be the braid monodromy. Then $\varphi(\delta) = H^\varepsilon$ where $H$ is a half-twist and

$$\varepsilon = \begin{cases} 1 & (x_0, y_0) \text{ is a branch point of } \pi \\ 2 & (x_0, y_0) \text{ is a node of } S \\ 3 & (x_0, y_0) \text{ is a cusp of } S \end{cases}$$

**Remark.** Clearly, the complexity in finding $\varphi(\delta)$ lies in finding $H$.

**VII. The Enriques-Van Kampen method (Step (c) of the algorithm).**

The Van Kampen method gives us a finite presentation in terms of generators and relation of plane complements of curves.

The categorical version of the Van-Kampen Theorem is as follows:

**Van Kampen Theorem.** [VK] Let $\overline{S} \subseteq \mathbb{CP}^2$ be a projective curve, which is transversal to the line at infinity. Let $S = \mathbb{C} \cap \mathbb{C}^2$. Let $\varphi_u : \pi_1(\mathbb{C} - N, u) \to B_m[\mathbb{C}_u^1, \mathbb{C}_u^1 \cap S]$ be the braid monodromy w.r.t. $S, \pi, u$. Then:

(a) $\pi_1(\mathbb{C}^2 - S, \ast) = \pi_1(\mathbb{C}^1_u - S, \ast)/\{V \mid \beta(\varphi_u). V \in \pi_1(\mathbb{C}^1_u - S)\}$.

(b) $\pi_1(\mathbb{CP}^2 - \overline{S}) \simeq \pi_1(\mathbb{C}^2 - S)/\{V \mid \Gamma \text{ is a simple loop in } \mathbb{C}^1_u - S \}$

where $\Gamma$ is a simple loop in $\mathbb{C}^1_u - S$ around $S \cap \mathbb{C}^1_u = \{q_1, \ldots, q_m\}$.

We want to rephrase the Van Kampen theorem in a way that it can be used with greater facility. To that end we need the notion of a good geometric base for
the fundamental group of a punctured disc. We recall that for $D - K$, a punctured disk, $\pi_1(D - K)$ is a free group and $B_n[D, K]$ acts naturally on $\pi_1(D - K)$. Before defining a good geometric base we need 2 additional definitions.

**Definition:** $\ell(q)$. Let $D$ be a topological disc, $K \subset D$, $K$ finite, $u \in \partial D$. Let $a \in K$, $q$ a simple path from $u$ to $a$ such that $q \cap K = a$. Let $c$ be a simple loop equal to the (oriented) boundary of a small neighborhood $V$ of $a$ chosen such that $q' = q - V \cap q$ is a simple path. Then $\ell(q) = q' \cup c \cup q'^{-1}$ (see figure).

We use the same notation $\ell(q)$ also for the element of $\pi_1(D - K, u)$ corresponding to $\ell(q)$.

**Definition:** A bush. Let $D, K, u$ be as above. Let $K = \{a_1, \ldots, a_n\}$. Consider in $D$ ordered sets of simple paths $(T_1, \ldots, T_n)$ connecting $a_i$’s with $u$ such that

1. $\forall i = 1, \ldots, n$ $t_i \cap w_j = \emptyset$ if $i \neq j$;
2. $\bigcap_{i=1}^n T_i = u$;
3. for a small circle $c(u)$ around $u$ each $u'_i = T_i \cap c(u)$ is a single point and the order in $(u'_1, \ldots, u'_n)$ is consistent with the positive (“counterclockwise”) orientation of $c(u)$.

We say that two such sets $(T_1, \ldots, T_n)$ and $(T'_1, \ldots, T'_n)$ are equivalent if $\forall i = 1, \ldots, n$: $\ell(T_i) = \ell(T'_i)$ (in $\pi_1(D - K, u)$). An equivalence class of such sets is called a bush in $(D - K, u)$. The bush represented by $(T_1, \ldots, T_n)$ is denoted by $\langle T_1, \ldots, T_n \rangle$. 

Definition: A good geometric base (g-base).

Let $D, K$ be as above. A good geometric base of $\pi_1(D - K, u)$ is an ordered free base of $\pi_1(D - K, u)$ of the form $(\ell(T_1), \ldots, \ell(T_n))$ where $(T_1, \ldots, T_n)$ is a bush in $D - K$.

We are now able to formulate the Van Kampen theorem.

Van Kampen theorem. (Working format) [VK] Let $S \subseteq \mathbb{C}P^2$ be a projective curve, which is transversal to the line at infinity. Let $S = S \cap \mathbb{C}^2$. Let $\varphi$ be the braid monodromy w.r.t. $S, \pi, u$. $\varphi : \pi_1(\mathbb{C} - N, u) \to B_m[\mathbb{C}_u^1, \mathbb{C}_u^1 - S]$. Let $\{\delta_i\}$ be a good geometric base of $\pi_1(\mathbb{C} - N, u)$. Let $\{\Gamma_j\}$ be a good geometric base of $\pi_1(\mathbb{C}_u^1 - S, *)$. Then:

(a) $\pi_1(\mathbb{C}^2 - S, *)$ is generated by images of $\{\Gamma_j\}$ in $\pi_1(\mathbb{C}^2 - S, *)$ with the following relations: $\varphi(\delta_i)\Gamma_j = \Gamma_j \quad \forall i \forall j$.

(b) $\pi_1(\mathbb{C}P^2 - S) \simeq \pi_1(\mathbb{C}^2 - S)/(\prod \Gamma_j).

To be able to apply the Van Kampen method one has to know the actions of $B_n[D, K]$ on $\pi_1(D - K)$ (in order to be able to compute $\varphi(\delta_i)\Gamma_j$).

One can learn how to compute the action of $B_n[D, K]$, just by considering a simple situation as follows.

Claim.

Assume $K = \{a, b\}$, $\sigma$ a simple path from $a$ to $b$.

Let $H = H(\sigma) \in B_n[D, K]$ be the half-twist w.r.t. to $\sigma$.

Let $\Gamma_a = a$ loop around a counterclockwise, $\Gamma_b = a$ loop around $b$ counterclockwise.

Then:

(a) $(\Gamma_a)H = \Gamma_b.$
(b) \((\Gamma_b)H = \Gamma_b\Gamma_a\Gamma_b^{-1}\).

\[
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\beta
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\]

Proof.

(a) Trivial.

(b) Clearly, \((\Gamma_b)H = \)

Now:

\[
\Gamma_b\Gamma_a = 
\]

\[
\Gamma_b\Gamma_a\Gamma_b^{-1} =
\]

\((\Gamma_b^{-1} \text{ goes clockwise around } b.)\)

Corollary. (Van Kampen for cuspidal curves) Let \(S\) be a cuspidal curve. The relations on \(\pi_1(C^2 - S, *)\) induced by the braid monodromy are of the form:

\[
A = B
\]

or

\[
AB = BA
\]

or

\[
ABA = BAB
\]

where \(A\) and \(B\) are connected to a braid \(\rho(\delta)\) by the relation \(B = \rho(\delta)A\) and \(A, B\) can be part of a good geometric base. The first relation appears when \(\varphi(\delta) = H\), the second one when \(\varphi(\delta) = H^2\) and the third one when \(\varphi(\delta) = H^3\) (\(H\) a half-twist).
Proof. Let $\delta$ be an element of a $g$-base of $\pi_1(C^1 - N)$. We want to determine the type of relation that $\varphi(\delta)$ is inducing on $\pi_1(C^2 - S, \ast)$. By Van Kampen $\varphi(\delta)$ induces the relations:

$$\varphi(\delta)\Gamma_j = \Gamma_j$$

where $\{\Gamma_j = \ell(\gamma_j)\}$ is a good geometric base for $\pi_1(C^1_u - S, \ast)$.

Since $S$ is cuspidal, $\varphi(\delta) = H^\varepsilon$ for $\varepsilon = 1, 2$ or $3$. (See Zariski’s Theorem in the previous section.) Thus the induced relations are $H^\varepsilon(\Gamma_j) = \Gamma_j \forall j$.

We write $H = H(\sigma)$ where $\sigma$ connects $a$ and $b$.

Case 1. $\sigma$ is a straight line connecting $a$ and $b$, $K = \{a, b\}$ and $\gamma_a\sigma\gamma_b^{-1}$ does not contain any point of $K$ in its interior. We take $A = \Gamma_a$, $B = \Gamma_b$.

From the previous claim we know that in $\pi_1(C^1_u - S, \ast)$:

$H(\Gamma_a) = \Gamma_b$

$H(\Gamma_b) = \Gamma_b\Gamma_a\Gamma_b^{-1}$.

$H(\Gamma_j) = \Gamma_j, \ j \neq a, b$.

The relation imposed on $\pi_1(C^2 - S)$ from $\Gamma_a = H^\varepsilon(\Gamma_a)$ depends on $\varepsilon$.

$\varepsilon = 1 \Rightarrow \Gamma_a = H(\Gamma_a) \Rightarrow \Gamma_a = \Gamma_b$.

$\varepsilon = 2 \Rightarrow \Gamma_a = H^2(\Gamma_a) \Rightarrow \Gamma_a = H(\Gamma(\Gamma_a)) = H(\Gamma_b) = \Gamma_b\Gamma_a\Gamma_b^{-1} \Rightarrow \Gamma_a\Gamma_b = \Gamma_b\Gamma_a$

$\varepsilon = 3 \Rightarrow \Gamma_a = H^3(\Gamma_a) \Rightarrow \Gamma_a = H(\Gamma(\Gamma_a)) = H(\Gamma_b\Gamma_a\Gamma_b^{-1}) = \Gamma_b\Gamma_a\Gamma_b^{-1}\Gamma_b\Gamma_a\Gamma_b^{-1} = \Gamma_b\Gamma_a\Gamma_b\Gamma_a\Gamma_b = \Gamma_b\Gamma_a\Gamma_b$

It is easy to see that writing $H^\varepsilon(\Gamma_b) = \Gamma_b$ in $\pi_1(C^2 - S, \ast)$ will impose the same relation between $\Gamma_a$ as $\Gamma_b$ as the relation imposed from $H^\varepsilon(\Gamma_a) = \Gamma_a$. The relation $H^\varepsilon(\Gamma_j) = \Gamma_j$ for $j \neq a, b$ is the trivial relation since $H(\Gamma_j)$ equals $\Gamma_j$ already in $\pi_1(C^1_u - S)$. Thus the realtions are of the type quoted in the lemma.

Case 2. $\sigma$ is any path connecting $a$ and $b$ s.t. $K \cap \sigma = \{a, b\}$.

Choose a point $x$ on $\sigma$. It divides $\sigma$ into 2 paths $\sigma_1$ and $\sigma_2$. We choose a connection of $x$ to $\ast$ in $D - K$. We call this connection $\mu(\sigma)$. Clearly, $\mu(\sigma)\sigma_1^{-1}\sigma_2^{-1}\mu(\sigma)^{-1} = (\mu(\sigma)\sigma_1)(\mu(\sigma)\sigma_2)^{-1}$ has no point of $K$ in its interior and locally we are in the situation of case 1.
Let
\[ A = \ell(\mu(\sigma)\sigma_1^{-1}) \quad B = \ell(\mu(\sigma)\sigma_2) \]

Since we are locally in the situation of case 1 we have \( H(A) = B \) and \( H(B) = BAB^{-1} \). Moreover, as in case 1
\[
\begin{align*}
\varepsilon &= 1 \Rightarrow H(A) = A \Rightarrow A = B \\
\varepsilon &= 2 \Rightarrow H^2(A) = A \Rightarrow AB = BA \\
\varepsilon &= 3 \Rightarrow H^3(A) = A \Rightarrow ABA = BAB
\end{align*}
\]

Example of simple computation of \( G = \pi(\mathbb{C}^2 - S, \ast) \) using the Van Kampen method.

Let \( S : y^2 = x^3 \). Clearly, \( N = \{0\} \) and we take \( u = 1 \). \( \mathbb{C}_1 \cap S = \{-1, 1\} \) and thus \( \pi_1(\mathbb{C}_1 - S, u) \) is generated by \( \Gamma_1 \) and \( \Gamma_{-1} \). \( \pi_1(\mathbb{C}^1 - N) \simeq \langle \delta \rangle \) where \( \delta(t) = e^{2\pi it} \).

Thus the group \( \pi_1(\mathbb{C}^1 - N) \) induces only one relation in \( \pi_1(\mathbb{C}^2 - S, \ast) \). In Section V we computed the braid monodromy of \( \delta \) and got \( \varphi(\delta) = H^3 \) where \( H = H[-1, 1] \).

To compute the relation induced on \( \pi_1(\mathbb{C}^1 - S) \) from \( \varphi(\delta) \) we notice that we are in a simple case where: \( A = \Gamma_{-1} \quad B = \Gamma_1 \). Since \( \varepsilon = 3 \) the relation is \( ABA = BAB \).

Thus, \( \pi_1(\mathbb{C}^2 - S) \simeq \langle \Gamma_{-1}, \Gamma_1 \rangle / ABA = BAB \simeq \langle A, B \rangle / ABA = BAB \). By Artin’s structure theorem we get \( \pi_1(\mathbb{C}^2 - S, \ast) \simeq B_3 \).

Remark. This example is very simple in the sense that we have only one relation while for interesting branch curves we have many relations (\( S \) has many singular points). In all our previous works (see [MoTe2], [MoTe6], [MoTe8], [MoTe9]) we could not minimize the list of relations without first adding more relations using
invariance properties. An invariance property of the braid monodromy is a rule with which we can replace $A$ (and $B$) in a certain relation by a loop close to it (close enough that they almost coincide in the degeneration). Invariance properties are proven using the degeneration process (see, e.g., [MoTe2]).

VIII. Some facts on the structure of $\tilde{B}_n$ and $\tilde{B}_n$-groups (steps (e) and (f) of the algorithm).

As pointed out in Section II, it turned out that all the new examples of $G$ and $\overline{G}$ are $\tilde{B}_n$-groups, i.e., groups which admits an action of $\tilde{B}_n$. Moreover, like $\tilde{B}_n$ they are extensions of a solvable group by a symmetric one. We shall formulate the “almost solvability” theorem for $\tilde{B}_n$.

We review first the classical Artin presentation of the braid group.

**Theorem.** Let $X = H(\sigma_1)$ and $Y = H(\sigma_2)$ be 2 half-twists. Then

$$\sigma_1 \cap \sigma_2 = \emptyset \Rightarrow [X,Y] = XYX^{-1}Y^{-1} = 1$$

$$\sigma_1 \cap \sigma_2 = \{\text{end point}\} \Rightarrow \langle X,Y \rangle = XYXY^{-1}X^{-1}Y^{-1} = 1.$$  

In other words, disjoint half-twists commute; adjacent half-twists satisfy the triple relation.

Using half-twists we build a set of generators for $B_n$:

**Definition: Frame of a Braid Group.**

Take $K = \{a_1, \ldots, a_n\}$. Let $\sigma_i$ be a simple path from $a_i$ to $a_{i+1}$ s.t.:

$\sigma_i \cap \sigma_{i+1} = \{a_{i+1}\}$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $|i-j| > 1$.

Let $X_i = H(\sigma_i) = \text{half-twist w.r.t. } \sigma_i$. $\{X_i\}_{i=1}^{n-1}$ is called a frame of the braid group $B_n$.

**Remark.** There is a natural epimorphism $B_n \xrightarrow{\psi} S_n$ where $\psi(X_i) = \text{the transposition } (i \ i+1)$. This epimorphism “forgets” the diffeomorphism and only remembers the permutation of $K$. It is well known that a frame generation $B_n$ (Artin’s theorem).
Artin’s Structure Theorem. A frame $\{X_i\}_{j=1}^n$ generates the braid group $B_n$, with only the following relations:

\[
X_i X_j = X_j X_i \quad |i - j| > 1
\]
\[
X_i X_j X_i = X_j X_i X_j \quad |i - j| = 1.
\]

We need the following definitions for presenting the structure theorem for $\tilde{B}_n$.

Definitions.

$P_n = \ker(B_n \to S_n)$ where $\psi_n(X_i) = (i i + 1)$ for some frame $\{X_i\}$ of $B_n$.

$P_{n,0} = \ker(P_n \to \text{Ab}(B_n))$

$\tilde{P}_n, \tilde{P}_{n,0}$ the images of $P_n$ and $P_{n,0}$ in $\tilde{B}_n$.

Theorem. Consider $\tilde{P}_n$ as a $\tilde{B}_n$-group.

(a) $\tilde{P}_{n,0}$ is generated by a $\tilde{B}_n$-orbit of $\tilde{X}^2 \tilde{Y}^{-2}$ where $X$ and $Y$ are consecutive half-twists.

(b) There exist:

\[
1 \to (\tilde{P}'_n =) \tilde{P}'_{n,0} < \tilde{P}'_{n,0} < \tilde{P}_n < \tilde{B}_n \to S_n
\]

(c) $\tilde{P}'_{n,0} = \{1, c\}$, $c \in \text{Center} \tilde{B}_n$ ($c^2 = 1$)

Corollary. $\tilde{B}_n$ is “almost solvable”. Moreover, it is an extension of a solvable group by a symmetric group.

We shall not prove this theorem here. We only mention that the first step of the proof was the following observation. If we have a “good” quadrangle in $\tilde{B}_n$, i.e.

$X_i = H(x_i)$ where $x_i$ are as above, then $\tilde{X}_1^2 \tilde{X}_4^2 = \tilde{X}_2^2 \tilde{X}_4^2$.

As we pointed out earlier fundamental groups turn out to be $\tilde{B}_n$-groups and they are also “almost solvable”. When studying $\tilde{B}_n$-groups we distinguish certain
elements which we call “prime elements” (i.e., \( \tilde{X}^2 \tilde{Y}^{-2} \) in \( \tilde{P}_n \)). Finding prime elements in a group (e.g., a fundamental group) is the first step in proving that it is “almost solvable”.

**Definition: Prime element.**

Let \( G \) be a \( \tilde{B}_n \)-group. We denote \( b(g) \) by \( g_b \). An element \( g \in G \) is called prime if there exists a half-twist \( X \in B_n \) and \( \tau \in \text{Center}(G) \) s.t. \( \tau b = \tau \forall b \in \tilde{B}_n \), \( \tau^2 = 1 \) and

1. \( g_{\tilde{X}^{-1}} = g^{-1} \tau \)
2. \( g_{\tilde{X} \tilde{Y} \tilde{X}^{-1}} = g_{\tilde{X}}^{-1} g_{\tilde{X} \tilde{Y} \tilde{X}^{-1}} \forall Y \text{ consecutive to } X \)
3. \( g_Z = g \) \( \forall Z \text{ disjoint to } X \)

\( X \) is called the supporting half-twist of \( g \).
\( \tau \) is called the corresponding central element.

We call these elements prime since they satisfy an existence and a uniqueness property. We first introduce a polarization on half-twists, i.e., a direction which determines the beginning and end points of the path.

**Existence and Uniqueness Theorem.**

*Let \( g \) be prime supporting half-twist \( X \). Let \( T \) be another half-twist. Then: \( \exists! h \in G \text{ prime and } \tilde{b} \in \tilde{B}_n \text{ s.t. } g_b = h \ X_{\tilde{b}} = T \text{ preserving the polarization.} \)*

We have proved several criteria for an element to be prime (see [MoTe9] and [Te1]).

**IX. The connection between fundamental groups of complements of branch curves and Galois covers.**

As pointed out in the introduction, our techniques also allow us to compute some fundamental groups of surfaces of general type (see [MoTe1], [MoTe2], [MoTe5], [MoRoTe]). These surfaces are Galois covers of generic projection to \( \mathbb{CP}^2 \).

Recall: If \( f : X \to \mathbb{CP}^2 \) is a generic projection of deg \( n \) then \( \tilde{X} \), the Galois cover, is
defined as follows:
\[ \tilde{X} = \left( X \times \cdots \times X \right) - \Delta \]

We can compute fundamental groups of Galois covers since it can be proven that they are quotients of subgroups of the fundamental group of the complement of the branch curve (see [MoTe5]). So the first steps of computing \( \pi_1(\tilde{X}) \) are the same as computing \( \pi_1(\mathbb{C}P^2 - S) \). In our early attempts to find new discrete invariants for components of moduli spaces of surfaces in the \( \tau > 0 \) zone, we tried to use the fundamental groups of the surface itself, believing that it can not be trivial.

In fact, until 1984 the “Bogomolov Watershed Conjecture”:
\[ \tau > 0 \Rightarrow \pi_1(X) \neq \{1\} \]

was widely believed to be true for surfaces of general type (see [FH]). In 1984 (in the process of such computations) Moishezon-Teicher disproved the Bogomolov conjecture by constructing counter-examples ([MoTe2]). (The surfaces we used are Galois covers of \( \mathbb{C}P' \times \mathbb{C}P' \). They appeared in [Mi], in which it was pointed out that these surfaces are of positive signature.) The proof was based on computing quotients of a fundamental group of a complement of curves. In 1986 Chen produced (see [Ch]) new examples, all of which were non-spin. Till lately the only known examples of spin surfaces with \( \pi_1(X) = 1, \ \tau = 0 \) or \( \tau > 0 \) were the 1984 examples. Using the Hirzebruch surfaces we succeeded to produce infinitely many new examples of simply connected surfaces with \( \tau > 0 \) and three new examples with \( \tau = 0 \) [MoRoTe]. (Lately, new examples with \( \tau > 0 \) were also produced by Xiao, Persson and Peters [PPX]).
X. Galois covers of Hirzebruch surfaces: new examples.

$F_k$ = Hirzebruch surface of order $k$ can be schematically described as

\[
\begin{array}{c}
\text{C}_+ \\
\hline
l
\end{array}
\]

where $C_+l = 1 \quad l^2 = 0 \quad C_+^2 = k$.

Let $f_{ab} = f|_{a\ell + bC_+}$ be the embedding of $F_k$ in $\mathbb{CP}^N$ w.r.t. the full linear system $|a\ell + bC_+|$. $F_k \hookrightarrow \mathbb{CP}^N$. We denote $F_{k(a,b)} = f_{ab}(F_k)$. Let $f$ be a generic projection of $F_{k(a,b)}$ to $\mathbb{CP}^2$. Let $\tilde{F}_{k(a,b)}$ be the Galois cover of $F_{k(a,b)}$ w.r.t. $f$. $\tilde{F}_{k(a,b)}$ are the new examples.

**Theorem.** (Moishezon, Robb, Teicher) in [MoRoTe]

(a) For each $k$ there are infinitely many $F_{k(a,b)}$ s.t. $\tilde{F}_{k(a,b)}$ are simply connected surfaces of general type which are spin manifolds with $\tau > 0$.

(b) There are 5 surfaces which are spin with $\tau = 0$. Four of them are simply connected and the other one has fundamental group $\simeq \mathbb{Z}_5^{48}$.

Exact lists of $k, a, b$ and proofs can be found in [MoRoTe]. The hardest part is computing $\pi_1(\tilde{F}_{k(a,b)})$. Since $\pi_1(\tilde{X}) = \ker \left( \pi_1(\mathbb{C}^2 - S_{k(a,b)}) \to \mathbb{S}_n \right)$, the first steps of computing $\pi_1(\tilde{F}_{k(a,b)})$ coincide with the first steps of computing $\pi(\mathbb{C}^2 - S_{k(a,b)})$ where $S_{k(a,b)}$ is the branch curve of $F_{k(a,b)} \to \mathbb{CP}^2$. In particular, the first step is the degeneration of the surfaces into union of planes (Step (a) of the algorithm).

We shall only present here 2 examples of degeneration.

In fact, we present a schematic description of the degenerated object where a plane is presented by a triangle and an intersection line between planes by an edge of a triangle. One can see that no 3 planes meet in a line. The branch curve of the degenerated object is represented by the union of the edges of the triangles, and the singular points are the intersection points of lines. There are 2 types of
singular points, depending on the number of planes/lines that come together. In [MoTe6] we describe how to determine the type of singular points of the original branch curve that arise from a singular point of the degenerated object. We also presented there the associated braid monodromies. The degeneration of $V_3$ to the union of 9 planes is described in [MoTe7], the degeneration of $F_k(a,b)$ to the union of $2ab + kb^2$ planes is described in [MoRoTe].

\begin{center}
\begin{tikzpicture}
\node at (0,0) {$V_3$};
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
\node at (0,0) {$F_{2(2,3)}$};
\end{tikzpicture}
\end{center}

Remarks.

1. $\tilde{F}_{0(a,b)}$ are the examples from 1984 ([MoTe1],[MoTe2],[MoTe5]).
2. Kotschik used $\tilde{F}_{0(a,b)}$ to build examples of orientation-reversing homeomorphic surfaces which are not diffeomorphic ([K]).
3. $\tilde{F}_{1(a,b)} = V_b$ is the Veronese of order $b$.
4. There is another procedure in progress to produce such examples ([MoTe11]).
5. Together with Robb and Freitag we proved lately that all other $\tilde{F}_{k(a,b)}$ have finite fundamental groups which are products of cyclic groups [FRoTe].
(6) We used [MoRoTe] to produce spin surfaces of positive signature with the same Chern classes and different fundamental groups (see [RoTe]).

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