On the equivalence of different presentations of Turner’s bracket abstraction algorithm

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Abstract

Turner’s bracket abstraction algorithm is perhaps the most well-known improvement on simple bracket abstraction algorithms. It is also one of the most studied bracket abstraction algorithms. The definition of the algorithm in Turner’s original paper is slightly ambiguous and it has been subject to different interpretations. It has been erroneously claimed in some papers that certain formulations of Turner’s algorithm are equivalent. In this note we clarify the relationship between various presentations of Turner’s algorithm and we show that some of them are in fact equivalent for translating lambda-terms in beta-normal form.

1 Introduction

Bracket abstraction is a way of converting lambda-terms into a first-order combinatory representation. This has applications to the implementation of functional programming languages [15, 14] or to the automation in proof assistants [12, 13]. The well-known simple bracket abstraction algorithms of Curry and Schönfinkel have the disadvantage of producing big combinatory terms in the worst case. Perhaps the most well-known improvement on these algorithms is the bracket abstraction algorithm of Turner [14]. It is also one of the most studied bracket abstraction algorithms, with analysis of its worst-case and average-case performance available in the literature.

The original definition of Turner’s algorithm in [15] is slightly ambiguous and it has been subject to many interpretations. It seems to be a relatively prevalent misconception that certain presentations of Turner’s algorithm based on “optimisation rules” and others based on recursive equations with side conditions are equivalent. This is explicitly (and erroneously) claimed e.g. in [8], and seems to be an implicit assumption in many other papers.

The non-equivalence between a presentation of Turner’s algorithm based on “optimisation rules” and a presentation based on “side conditions” was noted in [2]. In this paper we clarify the relationship between various formulations of Turner’s algorithm. In particular, we show the following.

• A certain formulation of Turner’s algorithm based on “optimisation rules” is equivalent for translating lambda-terms in β-normal form to a formulation based on equations with side conditions, provided one chooses the equations and the optimisations carefully. The formulations are not equivalent for translating lambda-terms with β-redexes.

• If one removes “η-rules” then certain presentations become equivalent in general.

• Analogous results hold for certain presentations of the Schönfinkel’s bracket abstraction algorithm.
2 Bracket abstraction

In this section we give a general definition of a bracket abstraction algorithm and present some of the simplest such algorithms. We assume basic familiarity with the lambda-calculus [1]. We consider lambda-terms up to $\alpha$-equivalence and we use the variable convention.

First, we fix some notation and terminology. By $\Lambda$ we denote the set of all lambda-terms, by $\Lambda_0$ the set of all closed lambda-terms, and by $V$ the set of variables. Given $B \subseteq \Lambda_0$ by $\text{CL}(B)$ we denote the set of all lambda-terms built from variables and elements of $B$ using only application. A \textit{(univariate) bracket (or combinatory) abstraction algorithm} $A$ for a basis $B \subseteq \Lambda_0$ is an algorithm computing a function $^1 A : V \times \text{CL}(B) \rightarrow \text{CL}(B)$ such that for any $x \in V$ and $t \in \text{CL}(B)$ we have $\text{FV}(A(x, t)) = \text{FV}(t) \setminus \{x\}$ and $A(x, t) =_{\beta\eta} \lambda x.t$. We usually write $[x]_A.t$ instead of $A(x, t)$. For an algorithm $A$ the \textit{induced translation} $H_A : \Lambda \rightarrow \text{CL}(B)$ is defined recursively by

$$H_A(x) = x$$
$$H_A(st) = (H_A(s))(H_A(t))$$
$$H_A(\lambda x.t) = [x]_A.H_A(t)$$

It follows by straightforward induction that $\text{FV}(H_A(t)) = \text{FV}(t)$ and $H_A(t) =_{\beta\eta} t$.

Abstraction algorithms are usually presented by a list of recursive equations with side conditions. It is to be understood that the first applicable equation in the list is to be used.

For instance, the algorithm $(\text{fab})$ of Curry [4, §6A] for the basis $\{S, K, I\}$ where

$$S = \lambda x y z. x (y z)$$
$$K = \lambda x y. x$$
$$I = \lambda x. x$$

may be defined by the equations

$$[x]_{(\text{fab})}.st = S([x]_{(\text{fab})}.s)([x]_{(\text{fab})}.t)$$
$$[x]_{(\text{fab})}.x = I$$
$$[x]_{(\text{fab})}.t = Kt$$

The last equation is thus used only when the previous two cannot be applied. For example $[x]_{(\text{fab})}.y t x = S(S(Ky)(Ky))t$. Note that we have $[x]_{(\text{fab})}.S = S$, $[x]_{(\text{fab})}.K = K$ and $[x]_{(\text{fab})}.I = I$, but $[x]_{(\text{fab})}.\lambda x.t$ is undefined if $\lambda x.t \notin \{S, K, I\}$. One easily shows by induction on the structure of $t \in \text{CL}(\{S, K, I\})$ that indeed $\text{FV}([x]_{(\text{fab})}.t) = \text{FV}(t) \setminus \{x\}$ and $[x]_{(\text{fab})}.t \rightarrow_{\beta\eta} t$, so $(\text{fab})$ is an abstraction algorithm. For all algorithms which we present, their correctness, i.e., that they are abstraction algorithms, follows by straightforward induction, and thus we will avoid mentioning this explicitly every time.

The algorithm $(\text{abf}')$ is defined by the equations for $(\text{fab})$ plus the optimisation rule

$$S(Ks)(Kt) \rightarrow K(st).$$

More precisely the algorithm $(\text{abf}')$ is defined by

$$[x]_{(\text{abf}')} . st = \text{Opt}[S([x]_{(\text{abf}')} . s)([x]_{(\text{abf}')} . t)]$$
$$[x]_{(\text{abf}')} . x = I$$
$$[x]_{(\text{abf}')} . t = Kt$$

where the function $\text{Opt}[]$ is defined by

$$\text{Opt}[S(Ks)(Kt)] = K(st)$$
$$\text{Opt}[Sat] = Sst$$

\footnote{Whenever convenient we confuse algorithms with the functions they compute.}
Of course, it is to be understood that an earlier equation takes precedence when more than one equation applies. This conforms to the interpretation of “optimisation rules” in [17, Chapter 16], but e.g. Bunder [2] interprets them as rewrite rules. For example we have \([x]_{(abf')}\) if \(yyx = S(K(yy))I\). It is a recurring pattern that certain bracket abstraction algorithms are defined by the equations for \((abf')\), differing only in the definition of the function \(Opt[]\). In such a case we will not repeat the equations of \((abf')\), and only note that an algorithm is defined by the optimisations given by a function \(Opt[]\).

The algorithm \((abf')\) is defined by the following optimisations.

\[
\begin{align*}
Opt[S(Ks)(Kt)] &= K(st) \\
Opt[S(Ks)] &= s \\
Opt[Sst] &= Sst
\end{align*}
\]

Note that e.g. \([x]_{(abf')}\) \(S(Ky)(Ky)x = S(Ky)(Ky)\).

It follows by induction on \(t\) that \([x]_{(abf')}t = Kt\) if \(x \not\in FV(t)\). Note that this would not be true if optimisations could be applied as rewrite rules: below the root and recursively to results of optimisations (consider e.g. abstracting \(x\) from \(S(Ka)(Ka)\)). This seems to disprove\footnote{Like with many presentations of abstraction algorithms based on optimisation rewrite rules, it is not completely clear what the precise algorithm actually is, but judging by some examples given in [2] recursive optimisations below the root of optimisation results are allowed.} claim (5) in [2].

Using the above fact one easily shows that \((abf')\) is equivalent (i.e. gives identical results) to the following algorithm (abf).

\[
\begin{align*}
[x]_{(abf)}x &= I \\
[x]_{(abf)}t &= Kt & \text{if } x \not\in FV(t) \\
[x]_{(abf)}st &= S([x]_{(abf)}s)([x]_{(abf)}t)
\end{align*}
\]

The algorithm \((abf)\) is perhaps the most widely known and also one of the simplest bracket abstraction algorithms, but it is not particularly efficient. A natural measure of the efficiency of an abstraction algorithm \(A\) is the translation size – the size of \(H_A(t)\) as a function of the size of \(t\). For \((fab)\) the translation size may be exponential, while for \((abf)\) it is \(O(n^3)\). See [8, 9] for an analysis of the translation size for various bracket abstraction algorithms. For a fixed finite basis \(\Omega(n \log n)\) is a lower bound on the translation size [8, 9]. This bound is attained in [11] (see also [3] and [8, Section 4]).

Schönfinkel’s bracket abstraction algorithm \(S\) is defined for the basis \{\(S, K, I, B, C\)\} where:

\[
\begin{align*}
B &= \lambda xyz.x(yz) \\
C &= \lambda xyz.xzy
\end{align*}
\]

The algorithm \(S\) is defined by the following equations.

\[
\begin{align*}
(1) \quad [x]_S.t &= Kt & \text{if } x \not\in FV(t) \\
(2) \quad [x]_S.x &= I \\
(3) \quad [x]_S.wx &= s & \text{if } x \not\in FV(s) \\
(4) \quad [x]_S.st &= Bs([x]_S.t) & \text{if } x \not\in FV(s) \\
(5) \quad [x]_S.st &= C([x]_S.s)t & \text{if } x \not\in FV(t) \\
(6) \quad [x]_S.st &= S([x]_S.s)([x]_S.t)
\end{align*}
\]

This algorithm is called \((abcdef)\) in [3, §6A] and it is actually the Schönfinkel’s algorithm implicit in [13]. Like for \((abf)\), the translation size for \(S\) is also \(O(n^3)\) but with a smaller constant [8].
A variant $S'$ of Schönfinkel’s algorithm is defined by the optimisations:

1. $\text{Opt}[S(\text{K}s)(\text{K}t)] = K(st)$
2. $\text{Opt}[S(\text{K}s)] = s$
3. $\text{Opt}[S(\text{K}s)t] = Bst$
4. $\text{Opt}[S(\text{K}s)t] = Cst$
5. $\text{Opt}[S] = Sst$

The algorithm $S'$ seems to have been introduced by Turner in [14] where he calls it “an improved algorithm of Curry”, but this algorithm is not equivalent to any of Curry’s algorithms [2]. In fact, it is a common misconception (claimed e.g. in [8]) that the translations induced by the algorithms $S$ and $S'$ are equivalent. As a counterexample consider $\lambda y. (\lambda z.x)yy$. We have

$$H_S(\lambda y. (\lambda z.x)yy) = [y]S.Kxyy = S([y]S.Kxy)y = S(Kx)$$

but

$$H_{S'}(\lambda y. (\lambda z.x)yy) = [y]_{S'}Kxyy = x$$

because

$$\text{Opt}[S([y]S.Kxy)y] = \text{Opt}[S(Kxy)] = x.$$ 

The difference is that the algorithm $S'$ may effectively contract some β-redexes already present in the input term. That the algorithms themselves are not equivalent has already been observed by Bunder in [2] with the following counterexample: $K\text{S}x(K\text{S}x)$. We have $[x]S.KSx(KSx) = S(KS)(KS)$ but $[x]S'.KSx(KSx) = K(SS)$. The term of Bunder’s counterexample is indeed also a counterexample for the equivalence of the induced translations, which is not completely immediate, but it is easy to show using the following identities (which do not hold e.g. for $(\text{fab})$):

$$H_S(\text{K}) = H_{S'}(\text{K}) = K$$
$$H_S(\text{S}) = H_{S'}(\text{S}) = S$$

As another counterexample consider the term $\lambda y. z((\lambda x.x)y)$. We have $H_{S'}(\lambda y. z((\lambda x.x)y)) = z$ but $H_S(\lambda y. z((\lambda x.x)y)) = BzI$. This shows that it may be impossible to rewrite $H_S(t)$ to $H_{S'}(t)$ using the optimisations of the algorithm $S'$ as rewrite rules.

3 Turner’s algorithm

Turner’s algorithm [14] is perhaps the most widely known improvement on Schönfinkel’s algorithm. The basis for Turner’s algorithm is $\{S, K, I, B, C, S', B', C'\}$ where

$$S' = \lambda kxyz.k(xz)(yz)$$
$$B' = \lambda kxyz.kx(yz)$$
$$C' = \lambda kxyz.k(xz)y$$
Turner’s algorithm $T$ is defined by the following equations.

\[
\begin{align*}
(1) \quad [x]_{T}.t &= Kt & \text{if } x \notin \text{FV}(t) \\
(2) \quad [x]_{T}.x &= I \\
(3) \quad [x]_{T}.sx &= s & \text{if } x \notin \text{FV}(s) \\
(4) \quad [x]_{T}.uxt &= Cu & \text{if } x \notin \text{FV}(ut) \\
(5) \quad [x]_{T}.uxt &= Su([x]_{T}.t) & \text{if } x \notin \text{FV}(u) \\
(6) \quad [x]_{T}.ust &= B'^{us}([x]_{T}.t) & \text{if } x \notin \text{FV}(us) \\
(7) \quad [x]_{T}.ust &= C'u([x]_{T}.st)t & \text{if } x \notin \text{FV}(ut) \\
(8) \quad [x]_{T}.ust &= S'u([x]_{T}.s)t & \text{if } x \notin \text{FV}(ut) \\
(9) \quad [x]_{T}.st &= Bs([x]_{T}.t) & \text{if } x \notin \text{FV}(s) \\
(10) \quad [x]_{T}.st &= C([x]_{T}.s)t & \text{if } x \notin \text{FV}(t) \\
(11) \quad [x]_{T}.st &= S([x]_{T}.t)([x]_{T}.t)
\end{align*}
\]

The translation size of $T$ is worst-case $O(n^2)$ \[8\] and average-case $O(n^{3/2})$ \[3\].

The idea with Turner’s combinators $S'$, $B'$, $C'$ is that they allow to leave the structure of the abstract of an application $st$ unaltered in the form $\kappa st'$ where $\kappa$ is a “tag” composed entirely of combinators. For instance, if $x_1, x_2, x_3 \in \text{FV}(s) \cap \text{FV}(t)$ (and $s$ has a form such that the equations 3-5 are not used) then

\[x_1, x_2, x_3]_{T}.st = S'(S'S)\left([x_1, x_2, x_3]_{T}.s\right)\left([x_1, x_2, x_3]_{T}.t\right)\]

while

\[x_1, x_2, x_3]_{S}.st = SBS(B'BS)\left([x_1, x_2, x_3]_{S}.s\right)\left([x_1, x_2, x_3]_{S}.s\right)\]

In \[14\] Turner formulates his algorithm in terms of “optimisation rules”. There is some ambiguity in the original definition and it has been subject to different interpretations. Whatever the interpretation, the original formulation is not equivalent to the algorithm $T$ as defined above. Perhaps the most common interpretation is like in \[7\ Chapter 16\]. We thus define the algorithm $T'$ by the following optimisations (see Section 2).

\[
\begin{align*}
(1) \quad \text{Opt}\llbracket S(Ks)(Kt)\rrbracket &= K(st) \\
(2) \quad \text{Opt}\llbracket S(Ks)I\rrbracket &= s \\
(3) \quad \text{Opt}\llbracket S(K(us))t\rrbracket &= B'^{ust} \\
(4) \quad \text{Opt}\llbracket S(Ks)t\rrbracket &= Bst \\
(5) \quad \text{Opt}\llbracket S(Bus)(Kt)\rrbracket &= C'^{ust} \\
(6) \quad \text{Opt}\llbracket S(B'u_1u_2s)(Kt)\rrbracket &= C'(u_1u_2)st \\
(7) \quad \text{Opt}\llbracket S(st)(Kt)\rrbracket &= Cst \\
(8) \quad \text{Opt}\llbracket S(Bus)t\rrbracket &= S'^{ust} \\
(9) \quad \text{Opt}\llbracket S(B'u_1u_2s)t\rrbracket &= S'(u_1u_2)st \\
(10) \quad \text{Opt}\llbracket Sst\rrbracket &= Sst
\end{align*}
\]

It is easily seen that the translations induced by $T$ and $T'$ are not equivalent by reusing the counterexamples for $S$ and $S'$ from the previous section. We will later show that the translations induced by $T$ and $T'$ are equivalent for terms in $\beta$-normal form.

That $T$ and $T'$ are not equivalent is because they may effectively perform $\eta$-contractions. If we disallow this, then the algorithms become equivalent. Let $T'_{\eta}$ be $T$ without the equations (3), (4) and (5), and let $T''_{\eta}$ be $T'$ without the optimisation (2). We shall show later that $T'_{\eta}$ and $T''_{\eta}$ are equivalent.

Another popular formulation of Turner’s algorithm is $T''$ which is like $T$ except that $u$ is required to be closed. This formulation is used e.g. in \[3\ \[10\]. The translations induced by the
algorithms $T'$ and $T''$ are not equivalent even for terms in $\beta$-normal form. For instance, we have

$$H_{T''}(\lambda xyz.y(xz)x) = [x,y]_{T''}.C(Byx)x = [x]_{T''}.C(CBx)x = S'(C'C)(CB)l$$

but

$$H_{T'}(\lambda xyz.y(xz)x) = [x,y]_{T'}.C(yxx) = [x]_{T'}.C(C'x)x = S(BC(CC'))l.$$ 

In [8] it is erroneously claimed that an algorithm Abs/Dash/1, which is essentially (4) equations to $\mathcal{T}$ terms in removed, the following equation is added after the third one

$$H_{T''}(\lambda xyz.y(xz)x) = [x,y]_{T''}.C(yxx) = [x]_{T''}.C(C'x)x = S(BC(CC'))l.$$ 

In [7, Chapter 16] it is suggested that Turner’s algorithm may be improved by using instead of $B'$ the combinator $B^*$ defined by

$$B^* = \lambda fxyz.f(x(yz))$$

Modified Turner’s algorithm $T_*$ is defined like algorithm $T$ except that the equation (6) is removed, the following equation is added after the third one

$$[x]_{T_*}.st = B^*st_1t_2 \text{ if } x \notin FV(s) \text{ and } [x]_{T_*}.t = Bt_1t_2$$

and the equation (9) is moved after the added one. A variant $T'_*$ of modified Turner’s algorithm is defined by the optimisations below.

(1) Opt[$S(Ks)(Kt)$] = $K(st)$
(2) Opt[$S(Ks)t$] = $s$
(3) Opt[$S(Ku)(Bst)$] = $B^*ust$
(4) Opt[$S(Ks)t$] = $Bst$
(5) Opt[$S(Bus)(Kt)$] = $C'ust$
(6) Opt[$S(B^*ust_1s_2)(Kt)$] = $C'u(Bs_1s_2)t$
(7) Opt[$Ss(Kt)$] = $Cst$
(8) Opt[$S(Bus)t$] = $S'ust$
(9) Opt[$S(B^*ust_1s_2)t$] = $S'u(Bs_1s_2)t$
(10) Opt[$Ss(t)$] = $Sst$

Of course, the algorithms $T_*$ and $T'_*$ are not equivalent, which may be seen by again considering the counterexamples against the equivalence of $S$ and $S'$. We will show that $T_*$ and $T'_*$ are equivalent as far as translating lambda-terms in $\beta$-normal form is concerned. Actually, another variant $T''_*$ of Turner’s modified algorithm presented in [7, Chapter 16], which is $T'_*$ with the equations (6) and (9) removed. This algorithm $T''_*$ is not equivalent to $T_*$ even for translating closed $\beta$-normal forms. For instance, we have

$$H_{T_*}(\lambda xy.x(x(y))x) = [x]_{T_*}.C'(x(Bxx)x = S(SC'(SBI))l$$

but

$$H_{T''_*}(\lambda xy.x(x(y))x) = [x]_{T''_*}.Opt[S(B^*xxx)(Kx)] = [x]_{T''_*}.C(B^*xxx)x = S'C(S(SB^*l))l.$$ 

\[ \text{In [8] it is not completely clear what the precise algorithm based on “optimisation rules” actually is, but the ambiguity does not affect the fact that the formulations are not equivalent.} \]
4 Equivalence of $H_T$ and $H_{T'}$ for $\beta$-normal forms

In this section we show that the translations induced by the algorithms $T$ and $T'$ are equivalent for terms in $\beta$-normal form.

Lemma 4.1.
1. If $x /\notin \text{FV}(t)$ then $[x]_{T'} t = K t$.
2. If $x /\notin \text{FV}(t)$ then $[x]_{T'} t x = t$.

Proof.
1. Induction on the structure of $t$. If $t$ is not an application then $[x]_{T'} t = K t$ by the definition of $T'$. So assume $t = t_1 t_2$. Then $[x]_{T'} t = \text{Opt}[S([x]_{T'} t_1) ([x]_{T'} t_2)]$. By the inductive hypothesis $[x]_{T'} t_1 = K t_1$ and $[x]_{T'} t_2 = K t_2$. Thus $[x]_{T'} t = \text{Opt}[S(K t_1)(K t_2)] = K(t_1 t_2) = K t$.
2. Using the previous point we have $[x]_{T'} t x = \text{Opt}[[S([x]_{T'} t)] I] = \text{Opt}[[S(K t)] I] = t$.

Definition 4.2. A term is $T$-normal if it does not contain subterms of the form $K t_1 t_2$, $B t_1 t_2 t_3$ or $B' t_1 t_2 t_3 t_4$.

Lemma 4.3. Let $t$ be $T$-normal.
1. If $[x]_{T'} t = K s$ then $s = t$ and $x /\notin \text{FV}(t)$.
2. If $[x]_{T'} t = 1$ then $t = x$.
3. If $[x]_{T'} t = B t_1 t_2$ then $t = t_1 t_2'$, $x /\notin \text{FV}(t_1)$ and $t_2 = [x]_{T'} t_2'$.
4. If $[x]_{T'} t = B' t_1 t_2 t_3$ then $t = t_1 t_2 t_3'$, $x /\notin \text{FV}(t_1 t_2)$ and $t_3 = [x]_{T'} t_3'$.

Proof. We show the first point. It follows directly from the definition of $T$ that either $s = t$ and $x /\notin \text{FV}(t)$, or $t = K s x$ with $x /\notin \text{FV}(s)$. The second case is impossible because $t$ is $T$-normal.

The proofs for the remaining points are analogous.

Lemma 4.4. If $t$ is $T$-normal then $[x]_{T'} t = [x]_{T'} t$.

Proof. Induction on $t$. We distinguish the cases according to which equation in the definition of $T$ is used. If $x /\notin \text{FV}(t)$ then $[x]_{T'} t = K t$ by Lemma 4.1. If $t = x$ then $[x]_{T'} t = 1 = [x]_{T'} t$.

If $t = s x$ with $x /\notin \text{FV}(s)$ then $[x]_{T'} t = s = \text{Opt}[[S(K s)] I] = \text{Opt}[[S([x]_{T'} s) ([x]_{T'} x)] I] = [x]_{T'} t$, where in the penultimate equation we use Lemma 4.1.

Assume $t = u x s$ and $x /\notin \text{FV}(u s)$. Then $[x]_{T'} t = \text{Cus}$. On the other hand $[x]_{T'} t = \text{Opt}[[S([x]_{T'} u x) ([x]_{T'} s)] I]$. We have $[x]_{T'} s = K s$ and $[x]_{T'} u x = u$ by Lemma 4.1. It suffices to show $\text{Opt}[[S u(K s)] I] = \text{Cus}$, for which it suffices that $u$ does not have the form $K u'$ or $B u_1 u_2$. But this is the case because $u x$ is $T$-normal.

Assume $t = u x s$, $x /\notin \text{FV}(u)$ and $x \in \text{FV}(s)$. Then $[x]_{T'} t = S u([x]_{T'} s)$. We have $[x]_{T'} s = [x]_{T'} s$ by the inductive hypothesis, and $[x]_{T'} u x = u$ by Lemma 4.1. Since $[x]_{T'} t = \text{Opt}[[S([x]_{T'} u x) ([x]_{T'} s)] I] = \text{Opt}[[S u([x]_{T'} s)] I]$, it suffices to show that $u$ does not have the form $K u'$ or $B u_1 u_2$, and $[x]_{T'} s$ does not have the form $K s'$. This follows from the fact that $u x$ is $T$-normal and from Lemma 4.3.

Assume $t = t_1 t_2 t_3$, $x /\notin \text{FV}(t_1 t_2)$ and $x \in \text{FV}(t_3), t_3 \neq x$. Then $[x]_{T'} t = B' t_1 t_2 ([x]_{T'} t_3)$. We have $[x]_{T'} t_3 = [x]_{T'} t_3$ by the inductive hypothesis, and $[x]_{T'} t_1 t_2 = K(t_1 t_2)$ by Lemma 4.1. Thus $[x]_{T'} t = \text{Opt}[[S(K(t_1 t_2)) ([x]_{T'} t_3)] I]$, so it suffices to show that $[x]_{T'} t_3$ does not have the form $K t_3'$.
or I. But if this is not the case then \( x \notin \text{FV}(t_3) \) or \( t_3 = x \) by Lemma 4.3 which contradicts our assumptions.

Assume \( t = t_1t_2t_3 \), \( x \notin \text{FV}(t(t_3)) \), \( x \in \text{FV}(t_2) \), \( t_2 \neq x \). Then \([x]_T.t = C\cdot t_1([x]_T.t_2)\cdot t_3\). We have \([x]_T.t_2 = [x]_T.t_2\) by the inductive hypothesis, and \([x]_T.t_1 = Kt_1\) by Lemma 4.1. Thus \([x]_T.t_2\) does not have the form \( Kt_0 \) or I. First assume \( t_1 \) is not an application. Then \([x]_T.t_1t_2 = \text{Opt}[[S(Kt_1)]([x]_T.t_2)] = Bt_1([x]_T.t_2)\). Thus \([x]_T.t = \text{Opt}[[S(Bt_1([x]_T.t_2))](Kt_3)] = C\cdot t_1([x]_T.t_2)\cdot t_3 = [x]_T.t\). If \( t_1 = u_1u_2 \) then \([x]_T.t_1t_2 = \text{Opt}[[S(K(u_1u_2))](x)_T.t_2)] = B'\cdot u_1u_2([x]_T.t_2)\). Thus \([x]_T.t = \text{Opt}[[S(B'\cdot u_1u_2([x]_T.t_2))](Kt_3)] = C\cdot (u_1u_2)([x]_T.t_2)\cdot t_3 = [x]_T.t\).

Assume \( t = t_1t_2t_3 \), \( x \notin \text{FV}(t_1) \), \( x \in \text{FV}(t_2) \), \( t_2 \neq x \). Then \([x]_T.t = S't_1([x]_T.t_2)\cdot [x]_T.t_3\). We have \([x]_T.t_2 = [x]_T.t_2\), \([x]_T.t_3 = [x]_T.t_3\) by the inductive hypothesis, and \([x]_T.t_1 = Kt_1\) by Lemma 4.1. First assume \( t_1 \) is not an application. Then like in the previous paragraph we obtain \([x]_T.t_1t_2 = Bt_1([x]_T.t_2)\). Thus \([x]_T.t = \text{Opt}[[S(Bt_1([x]_T.t_2))](x)_T.t_3)] = B'\cdot u_1u_2([x]_T.t_2)\). In each case it suffices to show that \([x]_T.t_3 \) does not have the form \( Kt_0 \). This follows from Lemma 4.3 and \( x \in \text{FV}(t_3) \).

Assume \( t = t_1t_2 \) and \([x]_T.t = Bt_1([x]_T.t_2)\). Then \( x \notin \text{FV}(t_1) \), \( t_1 \) is not an application (otherwise equation (6) would apply), \( x \in \text{FV}(t_2) \) and \( t_2 \neq x \). We have \([x]_T.t_2 = [x]_T.t_2\) by the inductive hypothesis, and \([x]_T.t_1 = Kt_1\) by Lemma 4.1. Hence \([x]_T.t = \text{Opt}[[S(Kt_1)]([x]_T.t_2)]\). Since \( t_1 \) is not an application, it thus suffices to show that \([x]_T.t_2\) does not have the form \( Kt_0 \) or I. But this follows from Lemma 4.3 and \( x \in \text{FV}(t_2) \) and \( t_2 \neq x \).

Assume \( t = t_1t_2 \) and \([x]_T.t = C\cdot t_1([x]_T.t_2)\). Then \( x = \text{FV}(t_1) \), \( x \notin \text{FV}(t_2) \), and if \( t_1 = u_1u_2 \) then \( x \in \text{FV}(u_1) \). We have \([x]_T.t_1 = [x]_T.t_1\) by the inductive hypothesis, and \([x]_T.t_2 = Kt_2\) by Lemma 4.1. Thus \([x]_T.t = \text{Opt}[[S(C\cdot t_1)](Kt_2)]\). It suffices to show that \([x]_T.t_1 \) does not have the form \( Kt_1 \), \( Bt_1u_2 \) or \( B'\cdot u_1u_2u_3 \). But this follows from Lemma 4.3 and our assumptions on \( t_1 \) and \( t_2 \).

**Lemma 4.5.** If \( t \) is \( T \)-normal then so is \([x]_T.t\).

**Proof.** By induction on the structure of \( t \).

**Lemma 4.6.** If \( t \) is in \( \beta \)-normal form then \( H_T(t) \) is \( T \)-normal.

**Proof.** Induction on \( t \). Because \( t \) is in \( \beta \)-normal form, either \( t = xt_1\ldots t_n \) or \( t = \lambda x.s \). In the first case the claim follows directly from the inductive hypothesis. In the second case we have \( H_T(t) = [x]_T.H_T(s) \). By the inductive hypothesis \( H_T(s) \) is \( T \)-normal. Hence \( H_T(t) \) is \( T \)-normal by Lemma 4.3.

**Theorem 4.7.** If \( t \) is in \( \beta \)-normal form then \( H_T(t) = H_T'(t) \).

**Proof.** Induction on \( t \). If \( t = x \) then \( H_T(t) = x = H_T'(t) \). If \( t = t_1t_2 \) then the claim follows directly from the inductive hypothesis. So assume \( t = \lambda x.s \). Then \( H_T(t) = [x]_T.H_T(s) \). By the inductive hypothesis \( H_T'(s) = H_T(s) \). By Lemma 1.6 we have that \( H_T'(s) \) is \( T \)-normal. Thus \( H_T(t) = [x]_T.H_T'(s) = [x]_T.H_T(s) = H_T'(t) \) by Lemma 1.4.
5 Equivalence of $T_{-\eta}$ and $T'_{-\eta}$

The reason for the non-equivalence of $T$ and $T'$ is the fact that these algorithms may effectively perform some $\eta$-contractions. We show that if this is disallowed, then the algorithms become equivalent. In other words, we show that $T_{-\eta}$ and $T'_{-\eta}$ are equivalent (see Section 3).

**Theorem 5.1.** For every lambda-term $t$ and every variable $x$ we have $[x]_{T_{-\eta}}.t = [x]_{T'_{-\eta}}.t$.

**Proof.** Induction on $t$. We consider possible cases according to which equation in the definition of $[x]_{T_{-\eta}}.t$ is used. If $x \notin \text{FV}(t)$ or $t = x$ then $[x]_{T_{-\eta}}.t = [x]_{T'}$.t follows directly from definitions.

The remaining cases are shown by a straightforward modification of the proof of Lemma 4.3 noting that for $T_{-\eta}$ the first point of Lemma 4.1 still holds, Lemma 4.3 holds without the assumption that $t$ is $T$-normal, and the only places in the proof of Lemma 4.3 where the second point of Lemma 4.1 (which does not hold), $T$-normality or optimisation (2) are used directly is for the equations (3), (4) and (5) not present in $T_{-\eta}$.

By way of an example we consider the case when $t = t_1t_2t_3$ and $[x]_{T_{-\eta}}.t = \text{B}'t_1t_2([x]_{T_{-\eta}}.t_3).$ Then $x \notin \text{FV}(t_1t_2)$ and $x \in \text{FV}(t_3)$. By the inductive hypothesis $[x]_{T'_{-\eta}}.t_3 = [x]_{T_{-\eta}}.t_3$ and $[x]_{T_{-\eta}}.t_1t_2 = [x]_{T_{-\eta}}.t_1t_2 = \text{K}(t_1t_2)$ because $x \notin \text{FV}(t_1t_2).$ Thus $[x]_{T'_{-\eta}}.t = \text{Opt}(\text{S}(\text{K}(t_1t_2))([x]_{T_{-\eta}}.t_3))$, so it suffices to show that $[x]_{T_{-\eta}}.t_3$ does not have the form $\text{K}t'^3.$ But if $[x]_{T_{-\eta}}.t_3 = \text{K}t'^3$ then $x \notin \text{FV}(t_3)$, which contradicts our assumption.

**Corollary 5.2.** For every lambda-term $t$ we have $H_{T_{-\eta}}(t) = H_{T'_{-\eta}}(t)$.

6 Other equivalences

By modifying the proof of Theorem 4.7 from Section 4 we can show the following.

**Theorem 6.1.** If $t$ is in $\beta$-normal form then $H_S(t) = H_{S'}(t)$.

**Proof sketch.** The proof is a simplification of the proof of Theorem 5.7. One defines a term to be $S$-normal if it does not contain subterms of the form $\text{K}t_1t_2$ or $lt$. Then one proves lemmas analogous to the lemmas proved in Section 4 essentially by simplifying the proofs in Section 4. Theorem then follows from the lemmas in exactly the same way.

**Theorem 6.2.** If $t$ is in $\beta$-normal form then $H_T(t) = H_{T'}(t)$.

**Proof sketch.** The proof is analogous to the proof of Theorem 4.7 in Section 4. We define a term to be $T$-normal if it does not contain subterms of the form $\text{K}t_1t_2$, $lt$, $Bt_1t_2t_3$ or $B^*t_1t_2t_3t_4$, and we modify all lemmas appropriately. Only Lemma 4.1 requires significant adjustments. First, we show the new case when $t = us$ and $[x]_{T}t = B^*us_1s_2$. Then $x \notin \text{FV}(u)$ and $[x]_{T}t = B^*s_1s_2$. We have $[x]_{T}t = \text{Opt}(\text{S}(\text{B}^*s_1s_2))$. By the inductive hypothesis $[x]_{T}t = [x]_{T'}.t$. Also $[x]_{T'}u = Ku$ by (an appropriate restatement of) Lemma 4.1. Thus $[x]_{T}t = \text{Opt}(\text{S}(\text{K}u)(\text{B}^*s_1s_2)) = \text{B}^*us_1s_2 = [x]_{T}.t$.

We also need to reconsider the case when $[x]_{T}.t = \text{B}^*u$. Then $t = us$, $x \notin \text{FV}(u)$, $x \in \text{FV}(s)$, $s \neq x$ and $s' = [x]_{T}.s$ does not have the form $\text{B}^*s_1s_2$. We have $[x]_{T}.t = \text{Opt}(\text{S}(\text{B}^*s_1s_2))$. By Lemma 4.1 and $x \notin \text{FV}(u)$ we have $[x]_{T}.u = Ku$. By the inductive hypothesis $[x]_{T}.s = [x]_{T}.s = s'$. Hence $[x]_{T}.t = \text{Opt}(\text{S}(\text{K}u)s')$. Since $s'$ does not have the form $\text{B}^*s_1s_2$, the third optimisation of $T'$ (the one for $B^*$) does not apply to $\text{S}(\text{K}u)s'$. Because $x \in \text{FV}(s)$ and $s \neq x$, by (an appropriate restatement of) Lemma 4.3, the first two optimisations do not apply either. Thus $[x]_{T}.t = \text{B}^*s' = [x]_{T}.t$.

Consider the case when $[x]_{T}t = C't_1t_2t_3$. Then $t = t_1t_2t_3$, $t_2' = [x]_{T}.t_2$, $x \notin \text{FV}(t_1t_3)$, $x \in \text{FV}(t_2)$, $t_2 \neq x$. We have $[x]_{T}.t_2 = [x]_{T}t_2 = t_2'$ by the inductive hypothesis, and $[x]_{T}t_1 = \text{C'}t_1t_2t_3$.
Thus \( t'_2 \) does not have the form \( Kt'_2 \) or \( I \). First assume \( t'_2 \) does not have the form \( Bu_1u_2 \). Then \( [x]_{T'_1}t_1t_2 = \text{Opt}(S(Kt_1)Bu_1u_2) = Bt'_1u_2 \) and thus \( [x]_{T'_1}t_1t_2 = \text{Opt}(S(Bt'_1Bu_1u_2)(Kt_3)) = C't_1Bu_1u_2t_3 = [x]_{T'_1}t_1t_2 \).

The case when \( [x]_{T'_1}t_1t_2 = St'_1t'_2t'_3 \) needs adjustments similar to the case above. The remaining cases go through as before. □

Let \( S_{-\eta} \) be \( S \) without the equation (3), and let \( S'_{-\eta} \) be \( S' \) without the optimisation (2). By modifying the proof of Theorem 5.1 from Section 5 we can show the following.

**Theorem 6.3.** For every lambda-term \( t \) and every variable \( x \) we have \( [x]_{S_{-\eta}}t = [x]_{S'_{-\eta}}t \).

**Proof sketch.** The proof is a straightforward simplification of the proof of Theorem 5.1 □

**Corollary 6.4.** For every lambda-term \( t \) we have \( H_{S_{-\eta}}(t) = H_{S'_{-\eta}}(t) \).

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