ON THE INDEX OF PSEUDO-DIFFERENTIAL OPERATORS
ON COMPACT LIE GROUPS

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ABSTRACT. In this note we study the analytical index of pseudo-differential
operators by using the notion of (infinite dimensional) operator-valued symbols
(in the sense of Ruzhansky and Turunen).

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1. Introduction
In this note we investigate index formulae for pseudo-differential operators on
compact Lie groups by using the notion of operator-valued symbol.

A pseudo-differential operator $A : C^\infty_0(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$, is an integral operator defined by

$$Af(x) = \int_{\mathbb{R}^n} e^{ix \xi} \sigma_A(x, \xi) \hat{f}(\xi) d\xi,$$

and associated to a smooth function $\sigma_A(x, \xi)$ – called the symbol of $A$ – satisfying some bounded conditions on its derivatives (see [18]). Here $\hat{f}$ denotes the euclidean Fourier transform of the function $f$. For every $m \in \mathbb{R}$ and every open set $U \subset \mathbb{R}^n$, the Hörmander class of symbols of order $m$, $S^m(U \times \mathbb{R}^n)$, (for a detailed description see [18]), is defined by functions satisfying the usual estimates

$$|\partial^\alpha_x \partial^\beta_\xi \sigma_A(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{m-|\beta|},$$

References
for all \((x, \xi) \in T^*U \cong U \times \mathbb{R}^n\) and \(\alpha, \beta \in \mathbb{N}^n\). These classes initially defined on open sets of \(\mathbb{R}^n\), can be defined on smooth manifolds by using charts. On a manifold \(M\) (orientable and without boundary), the corresponding operators associated to the Hörmander classes of order \(m\) will be denoted by \(\Psi^m(M)\). In our case we are interested when \(M = G\) is a compact Lie group.

It is well known that very elliptic pseudo-differential operator \(D\) on a closed manifold \(M\) (i.e. a compact manifold without boundary) acting in smooth functions, has kernel and cokernel of finite dimension and, in terms of the \(L^2\)-theory of Fredholm operators, to it can be associated an integer number, called the index of \(D\) and, defined by

\[
\text{ind}(D) := \dim \text{Kernel}(D) - \dim \text{Cokernel}(D). \tag{1.3}
\]

Now, let \(G\) be a compact Lie group, \(\mathcal{D}(G) = C^\infty(G)\) be the space of smooth functions on \(G\) endowed with the usual Frechet structure and \(\mathcal{D}'(G)\) be the space of Schwartz distributions. Let us consider a continuous operator \(A : C^\infty(G) \to C^\infty(G)\) and the right convolution operator \(r(f) \in C^\infty(G)\) (defined by \(r(f)(g) = g \ast f\), where \(f \in \mathcal{D}'(G), g \in C^\infty(G)\)). If \(\pi_R\) is the right regular representation on \(G\) (defined by \(\pi_R(x)f(y) = f(yx), x, y \in G, f \in C^\infty(G)\)) then, Ruzhansky and Turunen in [20] showed that

\[
Af(x) = \text{tr}(\sigma_A(x) r(f) \pi_R(x)), \tag{1.4}
\]

for some unique (operator-valued) symbol \(\sigma_A : G \to \mathcal{B}(C^\infty(G))\) from \(G\) into the space of continuous linear operators on \(C^\infty(G)\). The pseudo-differential term is justified because \(r(f)\) is sometimes, called the right global Fourier transform of \(f\).

In relation with our work, a characterization for the Hörmander classes \(\Psi^m(G)\) of pseudo-differential operators on arbitrary compact Lie groups, in terms of operator-valued symbols, was proved by M. Taylor (see Remark 10.11.22 of [20]). If \(A : C^\infty(G) \to C^\infty(G)\) extends to a Fredholm operator on \(L^2(G)\) (a necessary and sufficient condition for the Fredholmness of \(A\) is the ellipticity condition), in this paper we compute the index of \(A\) in terms of its operator valued symbol \(\sigma_A : G \to \mathcal{B}(C^\infty(G))\) and the operator valued symbol of its formal adjoint \(A^*\), \(\sigma_{A^*} : G \to \mathcal{B}(C^\infty(G))\). Complete references on index theory are the books [10, 15] and [16].

The main result in index theory is the Atiyah-Singer index theorem. This theorem was conjectured by I. M. Gelfand and several of its versions or extensions can be found in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] for several classes of manifolds (and non-commutative structures). For a general elliptic pseudo-differential operator \(D\), acting on smooth sections of a closed manifold \(M\), the Atiyah-Singer index formula has the following structure. First, the principal symbol of the operator \(D\) defines a Chern character \(\chi(\sigma_D)\) which is a cohomology class with compact support in \(TM\), the tangent bundle on \(M\). In addition, there exists a cohomology class \(\text{td}(TM \otimes \mathbb{C})\), called the Todd class, which give rise to the following integral expression for the index of \(D\)

\[
\text{ind}(D) = (-1)^n \int_{TM} \chi(\sigma_D) \text{td}(TM \otimes \mathbb{C}), \tag{1.5}
\]
where \( n = \dim(M) \), see [4]. Although the general Atiyah-Singer index theorem applies for compact Lie groups, our main goal is to write the index of elliptic operators as an integral expression taking advantage of the Ruzhansky-Turunen operator-valued calculus. So, if \( \delta_g \in \mathcal{D}'(G) \) is the Dirac point mass distribution at \( g \in G \), and \( D = A \in \Psi^0(G) \) is an elliptic operator on \( G \), we prove that

\[
\text{ind}(A) = \int_G \mu_\gamma(g) \, dg, \tag{1.6}
\]

where

\[
\mu_\gamma(g) := \exp(-\gamma \sigma A^\ast(g) \sigma A(g)) \delta_g(g) - \exp(-\gamma \sigma A(g) \sigma A^\ast(g)) \delta_g(g), \tag{1.7}
\]

for all \( \gamma > 0 \). The right hand side of (1.6) is understood in the sense of distributions. The corresponding index theorem for operator of general order will be given in Theorem 3.6. As it will be observed, our instrumental tool will be the McKean-Singer lemma

\[
\text{ind}(A) = \text{tr}(e^{-tA^\ast A}) - \text{tr}(e^{-tAA^\ast}), \quad t > 0. \tag{1.8}
\]

In our case, the McKean-Singer lemma implies a local index formula of the form

\[
\text{ind}(A) = \int_G \mu_0(g) \, dg \tag{1.9}
\]

for some density \( \mu_0 \) on \( G \) defined by certain geometrical invariants (see the classical work [2] of Atiyah, Bott and Patody). In this case, the operator-valued symbolic calculus of Ruzhansky and Turunen simplifies the McKean-Singer formula (1.8) to the expression (1.6). Certainly, the density \( \mu_\gamma \) is defined by the operator valued symbols associated to \( A \) and \( A^\ast \) and consequently by the global Fourier analysis associated to every compact Lie group. On the other hand, if \( \widehat{G} \) denotes the set of equivalence classes of strongly continuous, irreducible and unitary representations on \( G \), we show that every elliptic operator \( A \in \Psi^m(G) \) has Fredholm index given by

\[
\text{ind}(A) = \int_G \sum_{[\xi] \in \widehat{G}} \dim(\xi) \text{Tr}((\mu_\gamma(x)\xi)(e_G)) \, dx, \tag{1.10}
\]

where

\[
(\mu_\gamma(x)\xi)(e_G) = (\exp(-\gamma \sigma A^\ast(x) \sigma A(x))\xi)(e_G) - (\exp(-\gamma \sigma A(x) \sigma A^\ast(x))\xi)(e_G), \tag{1.11}
\]

here \( e_G \) is the identity element of \( G \). In this case, \( \tilde{\mu}_\gamma(x)\xi := \dim(\xi) \text{Tr}((\mu_\gamma(x)\xi)(e_G)) \) can be viewed as a density on the non-commutative phase space \( G \times \widehat{G} \). So, the index of elliptic operators on compact Lie groups can be written in terms of the algebraic information in the representation theory of the group and the operator-valued symbol of these operators.

This paper is organized as follows. In Section 2 we present some basics on the Fourier analysis used in our context and the global quantization of operators trough of operator-valued symbols. Finally in Section 3 we prove our index formulas.
2. Preliminaries

In this section we present some topics on compact Lie groups, the Fourier analysis used here, and the operator valued calculus of Ruzhansky and Turunen. For this we follow [17] and [20]. The reference [18] include a complete background on the theory of pseudo-differential operators.

2.1. The operator-valued quantization. Throughout of this paper \( G \) is a compact Lie group endowed with its normalised Haar measure \( dg \). For \( f \in \mathcal{D}'(G) \), the respective right-convolution operator \( r(f) : C^\infty(G) \to C^\infty(G) \) is defined by

\[
  r(f)g = g \ast f
\]

if \( f \in L^2(G) \), the (right) global Fourier transform is given by

\[
  r(f) = \int_G f(y)\pi_R(y)^*dy,
\]

where \( \pi_R \) is the right regular representation on \( G \), defined by \( \pi_R(x)f(y) = f(yx) \) and \( \pi_R(x)^* = \pi_R(x^{-1}) \). In this case, the Fourier inversion formula gives

\[
  f(x) = \text{tr}(r(f)\pi_R(x)), \quad f \in C^\infty(G), x \in G.
\]

If \( \rho : G \to \mathcal{B}(C^\infty(G)) \) is a continuous operator, the pseudo-differential operator \( A \) associated to \( \rho \), is defined by

\[
  Af(x) = \text{tr}(\rho(x)r(f)\pi_R(x)), \quad f \in C^\infty(G).
\]

Conversely, if \( A : C^\infty(G) \to C^\infty(G) \) is a continuous operator, then there exists an unique \( \sigma_A : G \to \mathcal{B}(C^\infty(G)) \) (called the symbol of \( A \)) satisfying

\[
  Af(x) = \text{tr}(\sigma_A(x)r(f)\pi_R(x)), \quad f \in C^\infty(G).
\]

The symbol \( \sigma_A \) is defined as follows. Let \( K_A \in C^\infty(G) \otimes \mathcal{D}'(G) \) be the distributional Schwartz kernel of \( A \) and \( R_A(x,y) = K(x,y^{-1}x) \) is the right-convolution kernel associated to \( A \). If we fixed \( x \in G \), and \( R_A(x) \in \mathcal{D}'(G) \) is defined by \( (R_A(x))(y) = R_A(x,y) \) for every \( y \in G \), the (right) operator valued symbol \( \rho = \sigma_A \) associated to \( A \) is defined by \( \sigma_A(x) = r(R_A(x)), \ x \in G \). In our further analysis will be useful the following composition theorem.

**Proposition 2.1.** Let us assume that \( A \) and \( E \) are continuous operators on \( C^\infty(G) \). Then, for every \( x \in G \) we have

\[
  \sigma_{AE}(x) = \sigma_A(x)\sigma_E(x).
\]

**Proof.** Let us note that for \( f \in C^\infty(G) \),

\[
  AEf(x) = \text{tr}(\sigma_A(x)r(Ef(\cdot))\pi_R(x)).
\]

Since

\[
  r(Ef(\cdot)) = r(\pi_R(\cdot)(\cdot)) = r(\pi_R(\cdot))r(f) = \sigma_E(\cdot)r(f).
\]

Thus

\[
  AEf(x) = \text{tr}(\sigma_A(x)\sigma_E(x)r(f)\pi_R(x)),
\]

and by uniqueness we have \( \sigma_{AE}(x) = \sigma_A(x)\sigma_E(x) \). So, we finish the proof. \( \square \)
2.2. **Fredholm operators.** The index is defined for a broad class of operators called Fredholm operators. Now, we introduce this notion in more detail. For $X, Y$ normed spaces $B(X, Y)$ is the set of bounded linear operators from $X$ into $Y$.

**Definition 2.2.** If $H_1$ and $H_2$ are Hilbert spaces, the closed and densely defined operator $A : H_1 \to H_2$ is Fredholm if only if $\ker(A)$ is finite dimensional and $A(H_1) = \text{Rank}(A)$ is a closed subspace of $H_2$ with finite codimension. In this case, the index of $A$ is defined by $\text{ind}(A) = \dim \ker(A) - \dim \coker(A)$. The index formula also can be written as

$$\text{ind}(A) = \dim \ker(A) - \dim \coker(A^*) .$$

Now, we end this subsection with a result now known as McKean-Singer index formula. We present the proof by completeness.

**Lemma 2.3.** Let us assume that $T : H_1 \to H_2$ is a Fredholm operator, $TT^*$ and $T^*T$ have a discrete spectrum, and for all $t > 0$ $e^{-tT^*T}$ and $e^{-tt^*T}$ are trace class. Then

$$\text{ind}(T) = \text{tr}(e^{-tt^*T}) - \text{tr}(e^{-tt^*T}).$$  \hfill (2.9)

**Proof.** Let $\lambda \in \Sigma(T^*T)$, $\lambda \neq 0$, then there exists a non-zero vector $\phi \in H_1$ such that $T^*T(\phi) = \lambda \phi$. Then, $TT^*T(\phi) = \lambda T\phi$ so that if $T(\phi)$ is non-zero, then it is an eigenvalue of $TT^*$. It follows that the non-zero eigenvalues of $TT^*$ and $TT^*$ are the same. Thus

$$\text{tr}(e^{-tt^*T}) - \text{tr}(e^{-tt^*T}) = \dim \ker(T^*T) - \dim \ker(TT^*).$$

Since $\ker(A^*A) = \ker(A)$ and $\ker(AA^*) = \ker(A^*)$ we end the proof. \hfill $\square$

2.3. **The matrix-valued quantization.** Let us consider for every compact Lie group $G$ its unitary dual $\hat{G}$, that is the set the set of continuous, irreducible and unitary representations on $G$. If $[\xi] \in \hat{G}$, and $\xi : G \to U(\mathbb{C}^{d_\xi})$, the following equalities follow from the Fourier transform on $G$

$$\widehat{f}(\xi) = \int_G \varphi(x)\xi(x)^* dx, \quad f(x) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}(\xi(x)\widehat{f}(\xi)) ,$$

and the Peter-Weyl Theorem on $G$ implies the Plancherel Theorem on $L^2(G)$,

$$\|f\|_{L^2(G)} = \left( \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}(\widehat{f}(\xi)\widehat{f}(\xi)^*) \right)^{\frac{1}{2}} = \|\widehat{f}\|_{L^2(\hat{G})} .$$

Notice that, since $\|A\|_{HS} = \text{Tr}(AA^*)$, the term within the sum is the Hilbert-Schmidt norm of the matrix $\widehat{f}(\xi)$. Any linear operator $A$ on $G$ mapping $C^\infty(G)$ into $\mathcal{D}'(G)$ gives rise to a matrix-valued global (or full) symbol $\sigma_A(x, \xi) \in \mathbb{C}^{d_\xi \times d_\xi}$ given by

$$\sigma_A(x, \xi) = \xi(x)^*(A\xi)(x),$$  \hfill (2.10)
which can be understood from the distributional viewpoint. Then it can be shown that the operator $A$ can be expressed in terms of such a symbol as \[ Af(x) = \sum_{|\xi| \leq \hat{G}} d_\xi \text{Tr}[\xi(x)\sigma_A(x, \xi)\hat{f}(\xi)]. \] (2.11)

The Hilbert space $L^2(\hat{G})$ is defined by the norm \[
\|\Gamma\|^2_{L^2(\hat{G})} = \sum_{|\xi| \leq \hat{G}} d_\xi \|\Gamma(\xi)\|^2_{HS}.
\]

Now, we want to introduce Sobolev spaces and, for this, we give some basic tools. Let $\xi \in \text{Rep}(G) := \cup \hat{G}$, if $x \in G$ is fixed, $\xi(x) : H_\xi \to H_\xi$ is an unitary operator and $d_\xi := \dim H_\xi < \infty$. There exists a non-negative real number $\lambda_{[\xi]}$ depending only on the equivalence class $[\xi] \in \hat{G}$, but not on the representation $\xi$, such that $-\mathcal{L}_G \xi(x) = \lambda_{[\xi]} \xi(x)$; here $\mathcal{L}_G$ is the Laplacian on the group $G$ (in this case, defined as the Casimir element on $G$). Let $\langle \xi \rangle$ denote the function $\langle \xi \rangle = (1 + \lambda_{[\xi]})^{\frac{1}{2}}$.

**Definition 2.4.** For every $s \in \mathbb{R}$, the Sobolev space $H^s(G)$ on the Lie group $G$ is defined by the condition: $f \in H^s(G)$ if only if $\langle \xi \rangle^s \hat{f} \in L^2(\hat{G})$.

The Sobolev space $H^s(G)$ is a Hilbert space endowed with the inner product $\langle f, g \rangle_s = \langle \Lambda_s f, \Lambda_s g \rangle_{L^2(G)}$, where, for every $r \in \mathbb{R}$, $\Lambda_s : H^r \to H^{r-s}$ is the bounded pseudo-differential operator with symbol $\langle \xi \rangle^s \mathcal{I}_\xi$. In this paper the notion of Sobolev spaces $H^s(G)$ is essential. Indeed, every elliptic operator $T \in \Psi^m(G)$ of order $m$ is a bounded operator from $H^s(G)$ into $H^{s-m}(G)$ and, more importantly, its index – as an operator from $C^\infty(G)$ to $C^\infty(G)$ – agrees with the index of $T$ as operator acting from $H^s(G)$ into $H^{s-m}(G)$, for every $s \in \mathbb{R}$.

**Definition 2.5.** Let $(Y_j)_{j=1}^{\dim G}$ be a basis for the Lie algebra $\mathfrak{g}$ of $G$, and let $\partial_j$ be the left-invariant vector fields corresponding to $Y_j$. We define the differential operator associated to such a basis by $D_{Y_j} = \partial_j$ and, for every $\alpha \in \mathbb{N}^n$, the differential operator $\partial_\alpha$ is the one given by $\partial_\alpha^a = \partial_1^a \cdots \partial_n^a$. Now, if $\xi_0$ is a fixed irreducible representation, the matrix-valued difference operator is the given by $D_{\xi_0} = (D_{\xi_0,i,j})_{i,j=1}^{d_{\xi_0}} = \xi_0(\cdot) - I_{d_{\xi_0}}$. If the representation is fixed we omit the index $\xi_0$ so that, from a sequence $D_1 = D_{\xi_0,1;1}, \cdots, D_n = D_{\xi_0,1;n}$ of operators of this type we define $D^a = D_1^a \cdots D_n^a$, where $\alpha \in \mathbb{N}^n$.

Now we introduce, for every $m \in \mathbb{R}$, the Hörmander class $\Psi^m(G)$ of pseudo-differential operators of order $m$ on the compact Lie group $G$. As a compact manifold we consider $\Psi^m(G)$ as the set of those operators which, in all local coordinate charts, give rise to pseudo-differential operators in the Hörmander class $\Psi^m(U)$ for an open set $U \subseteq \mathbb{R}^n$, characterized by symbols satisfying the usual estimates \[ |\partial_{\xi}^\alpha \partial_{\xi}^\beta \sigma(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{m-|\beta|}, \] (2.12)
for all $(x, \xi) \in T^*U \cong \mathbb{R}^{2n}$ and $\alpha, \beta \in \mathbb{N}^n$. This class contains, in particular, differential operator of degree $m > 0$ and other well-known operators in global analysis such as heat kernel operators.
$\Psi^m(G \times \widehat{G}) = \Psi^m(G)$ of pseudo-differential operators of order $m$. The class of the Hörmander classes $\Psi^m(G)$ where characterized in [20] by the condition: $A \in \Psi^m(G)$ if only if its matrix-valued symbol $\sigma_A(x, \xi)$ satisfies the inequalities
\[
\|\partial_x^\alpha \partial_\xi^\beta \sigma_A(x, \xi)\|_{op} \leq C_{\alpha, \beta} \langle \xi \rangle^{m - |\beta|}, \tag{2.13}
\]
for every $\alpha, \beta \in \mathbb{N}^n$. For a rather comprehensive treatment of this quantization process we refer to [20]. In this paper we are interested in the index of elliptic operators in $\Psi^m(G)$, where $m \in \mathbb{R}$. Now, we present the following theorem on elliptic pseudo-differential operators.

**Theorem 2.6.** An operator $A \in \Psi^m(G)$ is elliptic if and only if its matrix-valued symbol $\sigma_A(x, \xi)$ is an invertible matrix for all but finitely many $[\xi] \in \widehat{G}$, and for all such $\xi$ and $x \in G$ satisfies
\[
\|\sigma_A(x, \xi)^{-1}\|_{op} \leq C \langle \xi \rangle^{-m}.
\]
Thus both statements are equivalent to the existence of $B \in \Psi^{-m}(G)$ such that $R_1 = I - AB$ and $R_2 = I - BA$ are smoothing. This means that $R_i \in \Psi^{-\infty}(G) := \cap_m \Psi^m(G)$, for $i = 1, 2$.

3. **The index of operators on compact Lie groups**

In this section we prove our main result. Since the prototype of Fredholm operators are elliptic operators, we classify such condition in terms of the operator-valued quantization.

### 3.1. Ellipticity in terms of the operator-valued quantization.

In terms of the representation theory of a compact Lie group and the notion of operator valued symbol, the ellipticity of operators can be characterized as follows.

**Theorem 3.1.** Let $G$ be a compact Lie group and $e_G$ its identity element. An operator $A \in \Psi^m(G)$ with operator valued symbol $\sigma_A : G \to \mathcal{B}(C^\infty(G))$ is elliptic if and only if the matrix-valued function $\sigma_A(x)\xi(e_G)$ is an invertible matrix for all but finitely many $[\xi] \in \widehat{G}$, and for all such $\xi$ and $x \in G$ satisfies
\[
\|\sigma_A(x)\xi\|_{op} \leq C \langle \xi \rangle^{-m}.
\]
Thus both statements are equivalent to the existence of $B \in \Psi^{-m}(G)$ such that $R_1 = I - AB$ and $R_2 = I - BA$ are smoothing. This means that $R_i \in \Psi^{-\infty}(G) := \cap_m \Psi^m(G)$, for $i = 1, 2$.

**Proof.** Let us denote by $B(x, \xi)$ the matrix-valued symbol associated to $A$. This means that $B : G \times \widehat{G} \to \cup_{[\xi] \in \widehat{G}}(\mathbb{C}^{d_G})$ satisfies $B(x, \xi) = \xi(x)^*(A\xi)(x)$. Theorem 10.11.16 in [20] gives the identity $\xi(y)^*\sigma_A(x)\xi(y) = B(x, \xi)$, $x, y \in G$, $[\xi] \in \widehat{G}$. In particular, if $y = e_G$ is the identity element in $G$,
\[
\sigma_A(x)\xi(e_G) = B(x, \xi), \quad x \in G, \quad [\xi] \in \widehat{G}. \tag{3.1}
\]
So, from Theorem 2.6 we finish the proof. \qed
Remark 3.2. A similar analysis as in the previous result gives the following characterization of Hörmander classes in terms of operator-valued symbols. In fact, \( A \in \Psi^m(G) \) if and only if
\[
\| \partial_x^\alpha \nabla^\beta (\sigma_A(x) \xi(e_G)) \|_{op} \leq C_{\alpha, \beta} \langle \xi \rangle^{m-|\beta|},
\] (3.2)
for all \( \alpha, \beta \in \mathbb{N}^n \). It is well known that pseudo-differential operators in \( \Psi^0(G) \) are bounded operators on \( L^2(G) \) (see [18]).

3.2. Index formulae for elliptic operators. Our main result is the following.

Theorem 3.3. Let \( G \) be a compact Lie group and \( A \in \Psi^0(G) \) be an elliptic operator. Then the analytical index of \( A \) is given by
\[
\text{ind}(A) = \int_G \mu_\gamma(g) dg,
\] (3.3)
where
\[
\mu_\gamma(g) := \exp(-\gamma \sigma_A^*(g) \sigma_A(g)) \delta_g(g) - \exp(-\gamma \sigma_A(g) \sigma_A^*(g)) \delta_g(g),
\] (3.4)
for all \( \gamma > 0 \), where \( \delta_g \) is the Dirac point mass at \( g \in G \).

Proof. Let us assume that \( A \in \Psi^0(G) \) is an elliptic operator. Let us denote for all \( x \in G \), by \( B_{\sigma_A(x)} \) the matrix-valued symbol associated to \( \sigma_A(x) : C^\infty(G) \to C^\infty(G) \). If \( B(x, \xi) \) is the matrix-valued symbol associated to \( A \), then Theorem 10.11.16 in [20] gives
\[
B_{\sigma_A(x)}(y, \xi) = B(x, \xi), \ x, y \in G, [\xi] \in \hat{G}.
\] (3.5)
Consequently for every \( x \in G \), \( \sigma_A(x) \in \Psi^0(G) \) is an elliptic operator. So, for every \( x \in G \), \( \sigma_A(x) \) extends to a bounded and Fredholm operator on \( L^2(G) \) and the operators \( A, A^* A, AA^*, \sigma_A(x), \sigma_A(x)^* \sigma_A(x), \sigma_A(x) \sigma_A(x)^*, \sigma_A(x)^* \), \( e^{-\gamma A^* A} \), \( e^{-\gamma AA^*} \), \( e^{-\gamma \sigma_A(x)^* \sigma_A(x)} \), and \( e^{-\gamma \sigma_A(x) \sigma_A(x)^*} \) have discrete spectrum. In order to compute the index of \( A \) we need to compute the operator-valued symbol of every operator \( e^{-\gamma \sigma_A(x)^* \sigma_A(x)} \) and \( e^{-\gamma \sigma_A(x) \sigma_A(x)^*} \). Hence, let us note that for \( f \in C^\infty(G) \) and \( \gamma > 0 \), Proposition 2.1 gives
\[
e^{-\gamma A^* A} f(x) = \left( \sum_{k=0}^\infty (-\gamma)^k (A^* A)^k \right) f(x) = \sum_{k=0}^\infty (-\gamma)^k ((A^* A)^k f)(x)
\]
\[
= \sum_{k=0}^\infty (-\gamma)^k (A^* A)^k f(x)
\]
\[
= \sum_{k=0}^\infty (-\gamma)^k (\sigma_A A^* A) f(x)
\]
\[
= \sum_{k=0}^\infty (-\gamma)^k (\sigma_A A^* A) f(x)
\]
\[
e^{-\gamma \sigma_A(x)^* \sigma_A(x)} f(x),
\]
where we have used that $A$ and $\sigma_A(x)$ are bounded operators on $L^2(G)$ justifying so the convergence computations with the exponentials operators. So, the operator valued symbol associated to $e^{-\gamma A^* A}$ is given by $\sigma_{e^{-\gamma A^* A}}(x) = e^{-\gamma \sigma_A(x)}\sigma_A(x)$.

On the other hand, let us denote $K_{e^{-\gamma A^* A}}$ to the distributional kernel associated to $e^{-\gamma A^* A}$. Then,

$$\text{tr}(e^{-\gamma A^* A}) = \int_G K_{e^{-\gamma A^* A}}(g, g)dg.$$  \hspace{1cm} (3.6)

Because $K_\gamma(g, g) = R_{e^{-\gamma A^* A}}(g, e_G)$, for every $g \in G$, (here $e_G$ is the identity element of $G$) we have the distributional identity

$$e^{-\gamma A^* A}\delta_g(g) = \sigma_{e^{-\gamma A^* A}}(g)\delta_g(g) = \int_G \delta_g(y)R_{e^{-\gamma A^* A}}(y^{-1}g)dy = R_{e^{-\gamma A^* A}}(g, e_G).$$  \hspace{1cm} (3.7)

Taking into account the first part of the proof, we deduce

$$e^{-\gamma \sigma_{A^*}(g)\sigma_A(g)}\delta_g(g) = R_{e^{-\gamma A^* A}}(g, e_G)$$  \hspace{1cm} (3.8)

and consequently

$$\text{tr}(e^{-\gamma A^* A}) = \int_G e^{-\gamma \sigma_{A^*}(g)\sigma_A(g)}\delta_g(g)dg.$$  \hspace{1cm} (3.9)

Similarly, an analogous analysis applied to $A^*$ instead of $A$ gives

$$\text{tr}(e^{-\gamma AA^*}) = \int_G e^{-\gamma \sigma_{A^*}(g)\sigma_A(g)}\delta_g(g)dg.$$  \hspace{1cm} (3.10)

So, by Lemma (2.3) we have

$$\text{ind}(A) = \int_G (e^{-\gamma \sigma_{A^*}(g)\sigma_A(g)}\delta_g(g) - e^{-\gamma \sigma_A(g)\sigma_{A^*}(g)}\delta_g(g))dg,$$  \hspace{1cm} (3.11)

where in the last line we have used Lemma 2.3. So, we finish the proof. \hfill \Box

Remark 3.4. The main advantage here is Proposition 2.1 for the composition of operators, where we have a closed formula, instead of the usual global calculus where is used the notion of asymptotic expansions.

Remark 3.5. If $A \in \Psi^0(G)$ is a left invariant operator on a compact Lie group, its operator valued symbol is the constant mapping $\sigma_A(x) = A$, $x \in G$; in this case, $A$ is a right convolution operator and from (3.3), $\text{ind}(A) = 0$. Operators on compact Lie groups with non vanishing index can be found in [20, Chapter 4].

Now, we prove an index theorem for operators of arbitrary order.

**Theorem 3.6.** Let $G$ be a compact Lie group, $m \in \mathbb{R}$ and $A \in \Psi^m(G)$ be an elliptic operator. Then the analytical index of $A$ is given by

$$\text{ind}(A) = \int_G \mu(\gamma)(g)dg,$$  \hspace{1cm} (3.10)

where

$$\mu_{\gamma,m}(g) := \exp(-\gamma\sigma_{A^*}(g)\Lambda_{-2m}\sigma_A(g))\delta_g(g) - \exp(-\gamma\Lambda_{-m}\sigma_A(g)\sigma_{A^*}(g)\Lambda_{-m})\delta_g(g),$$  \hspace{1cm} (3.11)

for all $\gamma > 0$, where $\delta_g$ is the Dirac point mass at $g \in G$. 


Proof. For the proof we apply Theorem 3.3 to the operator $E = \Lambda_{-m}A \in \Psi^0(G)$. In fact, by using that $\Lambda_m$ is self-adjoint, from the logarithmic property of the index we have

$$\text{ind}(A) = \text{ind}(\Lambda_m) + \text{ind}(E) = \text{ind}(E).$$

Because $\Lambda_{-m}$ is left invariant, $\sigma_{\Lambda_{-m}}(x) = \Lambda_{-m}$, and $\sigma_E(x) = \Lambda_{-m}\sigma_A(x)$. Now, $E^* = A\Lambda_{-m}$, $\sigma_{E^*}(x) = \sigma_A(x)\Lambda_{-m}$ and we have

$$\text{ind}(E) = \int_G \mu_\gamma(g)dg,$$

where

$$\mu_{\gamma,m}(g) := \exp(-\gamma\sigma_{A^*}(g)\Lambda_{-2m}\sigma_A(g))\delta_g(g) - \exp(\gamma\Lambda_{-m}\sigma_A(g)\sigma_{A^*}(g)\Lambda_{-m})\delta_g(g),$$

for all $\gamma > 0$. So, we finish the proof.

Now, we need some preliminary results in order to prove our third index theorem.

**Proposition 3.7.** Let $G$ be a compact Lie group. Every elliptic pseudo-differential operator $T : C^\infty(G) \to C^\infty(G)$ of order $m \geq 0$ extends to a closed operator $T$ on $L^2(G)$.

**Proof.** Let us assume that $T \in \Psi^m(G)$ is an elliptic operator. We will show that $T$ is closed on $L^2(G)$. Let $f_n \to f$ and assume that $Tf_n \to g$ where the convergence is in the $L^2(G)$-norm. We will prove that $Tf = g$. From the Theorem 2.6 there exists $S \in \Psi^{-m}(G)$ such that $TS = I + R$ where $R \in \Psi^{-\infty}(G)$. Since operators in $\Psi^r(G)$ are bounded on $L^2(G)$ for $r \leq 0$, we have that $STf_n \to Sg$ and $(ST)f_n \to (ST)f$. Hence $Sg = STf$. Notice that $g \in L^2(G)$ and $TS(g) = TST(f) = TSTf = Tf + RTf = g + Rg$. On the other hand

$$R(Tf) = RT(\lim_{n \to \infty}f_n) = \lim_{n \to \infty} RTf_n = R(\lim_{n \to \infty}Tf_n) = Rg.$$

Now from the equality $Tf + RTf = g + Rg$ we deduce that $Tf = g$. \hfill \Box

The corresponding statement for trace class pseudo-differential operators is the following (for the proof, we follow the approach of the recent works by J. Delgado and M. Ruzhansky [12, 13, 14]).

**Theorem 3.8.** Let $A$ be a pseudo-differential operator on $\Psi^m(G)$, $m < -\dim(G)$ with distributional kernel $K(x, y)$. Then $A$ is trace class on $L^2(G)$ and

$$\text{tr}(A) = \int_G K(x, x)dx = \int_G \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}[[\sigma_A(x, \xi)]dx,$$

where $\sigma_A(x, \xi)$ is the matrix-valued symbol of $A$.

**Proof.** It is well known that if $A$ is a pseudo-differential operator of order less than $\neq$ then $A$ is trace class (see [19]). Now, the trace $\text{Tr}(A)$ of $A$ is given by $\text{Tr}(A) = \int_G K(x, x)dx$ where $K(x, y)$ is the Schwartz kernel of $A$. In the case of compact Lie groups we have $K(x, y) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}([[\sigma(x, y)\xi(y)]^\ast)$. Thus, $K(x, x) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}([[\sigma_A(x, \xi)]].$ This implies the theorem. \hfill \Box
We end this section with the following result.

**Theorem 3.9.** Let us consider \( m \geq 0 \) and let \( A \in \Psi^m(G) \) be an elliptic operator. Then the index of \( A \) is given by

\[
\text{ind}(A) = \int_G \sum_{[\xi] \in \hat{G}} d\xi \text{Tr}( (\mu_\gamma(g)\xi)(e_G) ) dg,
\]

where

\[
(\mu_\gamma(g)\xi)(e_G) = (\exp(-\gamma\sigma A^*(g)\sigma A(g))\xi)(e_G) - (\exp(-\gamma\sigma A(g)\sigma A^*(g))\xi)(e_G),
\]

for all \( \gamma > 0 \).

**Proof.** By taking into account that \( A \) has discrete spectrum, by Proposition 3.7 we can use Theorem 2.3 in order to compute the index of \( A \). If \( B_\gamma(x, \xi) \) is the matrix-valued symbol associated to \( e^{-\gamma A^*A} \) then

\[
(\sigma e^{-\gamma A^*\sigma A}(x)\xi)(e_G) = (e^{-\gamma\sigma A(x)\sigma A(x)}(x)\xi)(e_G) = B_\gamma(x, \xi),
\]

and consequently

\[
\text{tr}(e^{-\gamma A^*A}) = \int_G \sum_{[\xi] \in \hat{G}} d\xi \text{Tr}(B_\gamma(x, \xi)) dx,
\]

where we have used Theorem 3.8. So, we have

\[
\text{tr}(e^{-\gamma A^*A}) = \int_G \sum_{[\xi] \in \hat{G}} d\xi \text{Tr}( (e^{-\gamma\sigma A^*(x)\sigma A(x)}(x)\xi)(e_G) ) dx.
\]

A similar analysis gives

\[
\text{tr}(e^{-\gamma AA^*}) = \int_G \sum_{[\xi] \in \hat{G}} d\xi \text{Tr}( (e^{-\gamma\sigma A(x)\sigma A^*(x)}(x)\xi)(e_G) ) dx.
\]

Thus, we obtain

\[
\text{ind}(A) = \int_G \sum_{[\xi] \in \hat{G}} d\xi \text{Tr}( (e^{-\gamma\sigma A^*(x)\sigma A(x)}(x)\xi)(e_G) ) - (e^{-\gamma\sigma A(x)\sigma A^*(x)}(x)\xi)(e_G) ) dx.
\]

With the last line we finish the proof. \( \square \)

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