Higher regularity of homeomorphisms in the Hartman-Grobman theorem for semilinear evolution equations *

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Abstract

Hein and Prüss [20] presented a version of Hartman-Grobman type $C^0$ linearization result for semilinear hyperbolic evolution equations. They showed that the linearising map (homeomorphism) $H$ and its inverse $G = H^{-1}$ are Hölder continuous. An important question: is it possible to improve the regularity of the homeomorphisms $H$? In the present paper, we first formulate the result that the homeomorphisms $H$ in the Hartman-Grobman theorem is Lipchitzian, but the inverse $G$ is merely Hölder continuous. We also give a generalized local linearization result in this paper. Finally, some applications end the paper. As pointed out by Backes et al. [11], even if the diffeomorphism $F$ is $C^\infty$, the conjugacy (homeomorphism) can fail to be locally Lipschitz. The homeomorphisms

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are in general only locally Hölder continuous. In fact, it is proved that in all the previous works on Hartman-Grobaman theorem in $C^0$ linearization, the homeomorphisms are Hölder continuous. However, by establishing two effective dichotomy integral inequalities, we prove that the conjugacy is Lipchitzian, but the inverse is Hölder continuous. **keywords:** Hartman-Grobman theorem; semilinear evolution equations; homeomorphisms; linearization

1 Introduction

1.1 Motivations and novelty

Recently, Hein and Prüss [20] gave a version of $C^0$ linearization result for semilinear hyperbolic evolution equation on Banach space. Hein and Prüss [20] considered the following two evolution equations

$$\partial_t v = Av + r(v) \quad \text{and} \quad \partial_t u = Au,$$

where $A$ is the generator of $C_0$-group $e^{At}$ on Banach space $X$ and $r : X \rightarrow X$ is bounded and Lipschitzian. Hein and Prüss [20] successfully transferred the Hartman-Grobman theorem from ODEs to semilinear hyperbolic evolution equations. In fact, the general proof of finite dimension cannot directly be extended to the infinite dimension case.

Hein and Prüss [20] proved that both the homeomorphism $H$ and its inverse $G = H^{-1}$ are Hölder continuous. They believed that their estimates of Hölder exponent are optimal. Higher regularity of the homeomorphisms seems to be an interesting and delicate question. A question is: Is it possible to improve the regularity of the homeomorphisms? This paper gives an positive answer. Therefore, in this paper, by establishing two effective dichotomy integral inequalities, we improve the regularity of the homeomorphisms.

As pointed out by Backes et al. [1], even if the diffeomorphism $F$ is $C^\infty$, the conjugacy (homeomorphism) can fail to be locally Lipschitz. The homeomorphisms are in general only (locally) Hölder continuous (see [1] [2] [3] [4] [5] [8] [13] [20] [42] [54]). However, we prove that the conjugacy (homeomorphism) is Lipchitzian, but the inverse is Hölder continuous. Our result
is the first one to observe the higher regularity of homeomorphisms in the Hartman-Grobman theorem.

Now we summarize our goals of this paper as follows.

**The first purpose** is precisely to improve the regularity of the conjugacy. More specifically, if the mild solutions of semilinear system is bounded, then the linearising map $H$ can be Lipschitzian, but its inverse $G$ is merely Hölder continuous. Without this premise, we say that they are both Hölder continuous.

**The second purpose** is to weaken an important assumption in Hein and Prüss (2016) [20]. Hein and Prüss obtained the Hartman-Grobman theorem by setting that the whole $C_0$-group $e^{At}$ admits a dichotomy. In this paper, we reduce this assumption. In fact, it is enough to assume that the $C_0$-group $e^{At}$ partially satisfies the exponential dichotomy. More specifically, equations (1.1) can be rewritten as:

$$
\begin{align*}
\partial_t v_1 &= A v_1 + r(v_1, v_2), \\
\partial_t v_2 &= B v_2,
\end{align*}
$$

and

$$
\begin{align*}
\partial_t u_1 &= A u_1, \\
\partial_t u_2 &= B u_2,
\end{align*}
$$

where $A, B$ are the generators of $C_0$-groups $e^{At}, e^{Bt}$ on Banach spaces $X, Y$, respectively. The nonlinear term $r : X \times Y \to X$ is bounded and Lipschitz continuous. However, there is only required that $C_0$-group $e^{At}$ admits a dichotomy projection, but not requirement on $C_0$-group $e^{Bt}$.

**The third purpose** is to give a generalized version of local linearization. We suppose that the nonlinear term $r(x)$ admits a non-Lipschitz continuous with respect to $x$ on a small closed ball, that is:

$$
|r(x_1) - r(x_2)| \leq \mathcal{L}(\max\{|x_1|, |x_2|\})|x_1 - x_2|,
$$

where $\mathcal{L}(\cdot) : [0, \infty) \to [0, \infty)$ is a continuous, nondecreasing function and $\mathcal{L}(0) \equiv 0$. Hence, by a $C^\infty$ bump function, we obtain a local linearization result.

**The fourth purpose** is to establish two effective dichotomy integral inequalities in Section 2.3, which has a better estimate than Gronwall (Bellman) inequalities. It will be a novel and powerful tool to help us deal with the properties of linearising maps. To show its advantage over Bellman inequality, we also use Bellman inequality to prove the regularity, but we obtain that both the homeomorphisms are Hölder continuous.
The last purpose is to apply our results to some applications including the Hodekin-Huxley equations for the nerve axon.

1.2 Mechanism of our improvements

In general, to prove the regularity of the linearising map $H$, one employs the Bellman (Gronwall) inequality, see for examples [20, 41, 42, 46, 54]. However, the disadvantage of the Bellman inequality is that it will result in an exponential estimate of the form $e^{\alpha t} (\alpha > 0)$, one can refer to Lemma 3.10 in this paper. It is expansive, and the expansive estimate leads us to prove that the homeomorphism is Hölder continuous, not Lipschitzian. Therefore, most of the previous works on the regularity of conjugacy [13, 20, 41, 42, 46, 52, 54] is Hölder continuous.

On the contrary, the advantage of the dichotomy integral inequality (see Lemma 3.9 in Section 2.3) is that it yields an exponential decay of the form $e^{-\alpha_1 t} (\alpha_1 > 0)$. Thus, by dichotomy inequality, we can prove the Lipschitz continuity of the homeomorphism $H$ due to a better estimate (the exponential decay). However, if you use Bellman inequality, it is impossible to prove the Lipschitz continuity of the homeomorphisms due to the bad estimate $e^{\alpha t}$ with $\alpha > 0$.

1.3 History of linearization

The classical Hartman-Grobman theorem [16, 18, 36] states that if $x^*$ is a hyperbolic equilibrium point of a $C^1$ vector field $F(x)$ with flow $\varphi_t(x)$, then there exist a neighborhood $O$ of $x^*$ such that $\varphi$ is topologically conjugated to its linearization on $O$. The equivalent function $H$ in general is not in $C^1$ (see Chicone [10], Rodrigues and Solá-Morales [37]). Equivalently, it can be stated that if $x^*$ is a hyperbolic fixed point of a $C^1$ diffeomorphism $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$, then there exist a neighborhood $O$ of $x^*$ such that $F$ on $O$ is topologically conjugated to $DF(x^*)$. It also has a global version. Palmer [26] firstly extended the global version of Hartman-Grobman theorem to the nonautonomous differential equations in finite dimensional space. Palmer’s linearization theorem states that if the nonautonomous linear system admits an exponential dichotomy, and the nonlinear perturbation is bounded and Lipschitzian, then
nonlinear system can be linearized. To weaken the Palmer’s conditions, Jiang [23] presented a version of Hartman-Grobman theorem by setting that the linear system admits a generalized exponential dichotomy. Huerta [22] constructed a topological conjugacy between linear system and an unbounded nonlinear perturbation, while nonautonomous linear system admits a nonuniform contraction. Barreira and Valls [2] proved several versions of Hartman-Grobman theorem in different situations with the assumption that the linear systems admit a nonuniform dichotomy. Moreover, they proved that the topological conjugacy is Hölder continuous. Recently, Backes et al. [1] obtained a version of Hartman-Grobman theorem without the assumption that the linear system admits exponential dichotomy. Their results generalized those of Reinfelds and Steinberga [34]. Moreover, Backes [1] proved the Hölder continuity of the topological conjugacy and its inverse. Zgliczyński [52] established a version of Hartman-Grobman theorem for diffeomorphisms and ODEs by geometric proofs based on covering relations and cone conditions [53]. Different versions of the Hartman-Grobman theorem have been presented for the differential equations with piecewise constant argument [27, 28, 54], dynamic systems on time scales [29], the instantaneous impulsive system [15, 46].

Now we pay our attention to the Hartman-Grobman theorem in infinite dimensional space. In fact, some arguments (such as Brouwers fixed point theorem) for the finite-dimensional proofs are not valid for the infinite dimensional case. In 1969, Pugh [33] gave a global version of Hartman’s theorem on Banach space by providing that $A$ is a bounded operator. Later, Lu [24] successfully proved a Hartman-Grobman theorem for the scalar reaction-diffusion equations. Moreover, Bates and Lu [6] obtained a Hartman-Grobman theorem for Cahn-Hilliard equation and phase field equations. They proved the linearization of these semilinear partial differential equations based on the invariant manifold theory and the invariant foliation theory. A reduction theorem was proven in Reinfelds and Sermone [34]. Belitskil [8] studied a hyperbolic diffeomorphism in a Banach space and prove that the diffeomorphism admits local $\alpha$-Hölder linearization under some conditions. In bad situation, $C^0$ linearization is not enough to observe the dynamic behaviors, for instant, to distinguish the node from the focus. To this purpose, Sternberg [43, 44] initially investigated $C^r$ linearization for $C^k(1 \leq r \leq k \leq \infty)$ diffeomorphisms. Elbialy [14] and Rodrigues and Solà-Morales [38] improved Hartman’s result [19] to Banach space. Recently, Zhang et al. [47] improved the lower bound of $\alpha$ to lower the condition of $C^1$ linearization for planar contractions. They showed that the derivatives of the
transformations in their $C^1$ linearization are Hölder continuous and prove that the estimates for the Hölder exponent can not be improved anymore. Zhang et al. [48] obtained a set of sharpness conditions for the $C^1$ linearization of hyperbolic diffeomorphisms. They also proved that the $C^1$ linearization is actually a $C^{1,\beta}$ linearization and gave sharp estimates for $\beta$. Zhang et al. [49] studied the sharp regularity of linearization for $C^{1,1}$ hyperbolic diffeomorphisms with resonance. Futher, the $\alpha$ Hölder linearization of hyperbolic diffeomorphisms with resonance were studied in [49]. Zhang et al. [50] proved that the local homomorphism $H(x)$ is differentiable at the fixed point for a $C^1$ diffeomorphism $G(x)$ with $DG(x)$ being $\alpha$-Hölder continuous at the fixed point. Recently, Dragičević et al. [13] extend van Strien’s result [45] of simultaneously differentiable and Hölder linearization to nonautonomous differential equations with a nonuniform exponential dichotomy. Dragičević et al. [12] also studied the smooth linearization of nonautonomous difference equations with a nonuniform dichotomy. Some more delicate conditions for $C^r$ (or $C^1$)-smooth linearization obtained by Sell [40], Belitskii [7], Rodrigues and Solà-Morales [39].

1.4 Notations and Basic concepts

Let $(X, | \cdot |_X)$ and $(Y, | \cdot |_Y)$ denote two arbitrary Banach spaces. For convenience, both norms $| \cdot |_X$ and $| \cdot |_Y$ will be denoted by $| \cdot |$. Let $J \subseteq \mathbb{R}$ be any real interval. Define

$$\mathcal{B}C(J, X) := \{x : J \to X | x(t) \text{ is continuous and } \sup_{t \in J} |x(t)| < \infty\}$$

and $\|x\| := \sup_{t \in J} |x(t)|$. Let $U$ be an open subset of $X$, and define

$$\mathcal{B}C(U, X) := \{f : U \to X | f(x) \text{ is continuous and } \sup_{x \in U} |f(x)| < \infty\}$$

and $|f|_{\infty} := \sup_{x \in U} |f(x)|$.

Obviously, $(\mathcal{B}C(J, X), \|x\|)$ and $(\mathcal{B}C(U, X), |f|_{\infty})$ are both Banach spaces with norms $\|\cdot\|$ and $|\cdot|_{\infty}$, respectively.

**Definition 1.1.** *(Topological Conjugacy, [42])* $x' = \varphi(x)$ and $y' = \phi(y)$ are said to be topologically conjugated if there exists a homeomorphism $H$ of $X$ into $X$ such that $H$ sends the solution of $x' = \varphi(x)$ onto the solution of $y' = \phi(y)$. 

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Definition 1.2. (Exponential Dichotomy, [30, 31, 32]) A projection $P_+ \in \mathbb{B}(X)$ is said to be a dichotomy projection for the $C_0$-semigroup $e^{At}$ in $X$, if there exist constants $k \geq 1, \alpha > 0$ such that the following conditions are satisfied:

(S1) $P_+ e^{At} = e^{At} P_+$, for all $t \geq 0$;
(S2) $|e^{At} P_+ x| \leq ke^{-\alpha t}|P_+ x|$, for all $x \in X, t \geq 0$;
(S3) $e^{At} P_-$ extends to a $C_0$-group on $R(P_-)$;
(S4) $|e^{At} P_- x| \leq ke^{\alpha t}|P_- x|$, for all $x \in X, t \leq 0$;

where $\mathbb{B}(X)$ is a bounded linear operator on $X$, $P_- = I_X - P_+$, $I_X$ is the identity operator.

Moreover, the Green kernel corresponding to the exponential dichotomy is denoted by

$$G_A(t) = \begin{cases} 
  e^{At} P_+, & t \geq 0, \\
  -e^{At} P_-, & t < 0.
\end{cases}$$

1.5 Outline of this paper

We organize this paper as follows: our main results are stated in Section 2, where we state the global linearization results and local linearization results, respectively. Some preliminary results are presented in Section 3. Rigorous proofs are given to show the regularity of the linearising maps in Section 4. Finally, several applications are given to demonstrate our results.

2 Statement of main results

In the present paper, we consider the following semilinear evolution equations:

$$\begin{cases} 
  \partial_t u_1 = Au_1 + f(u_1, u_2), \\
  \partial_t u_2 = Bu_2, \\
  u_1(0) = u_{10}, u_2(0) = u_{20},
\end{cases}$$

where $A, B$ are the generators of $C_0$-semigroups $e^{At}, e^{Bt}$ on the Banach spaces $X, Y$, respectively. The nonlinear term $f : X \times Y \to X$ is Lipschitzian. It is well known that the Cauchy
problem of equation (2.1) has a unique mild solution on the half line $\mathbb{R}^+$. In addition, if $e^{At}, e^{Bt}$ is even $C_0$-groups, then this unique mild solution exists globally, i.e., on $\mathbb{R}$.

### 2.1 Main results on the global linearization

Now, we are ready to present our main results on the global linearization in this paper. Firstly, we present the result on the existence of topological conjugacy.

**Theorem 2.1.** Let $A,B$ be the generators of $C_0$-groups $e^{At}, e^{Bt}$ on Banach spaces $X,Y$, respectively. Assume that $e^{At}$ admits an exponential dichotomy. If nonlinear term $f$ is bounded (denoted it by $|f|_\infty$), Lipschitzian (Lipschitz constant denoted it by $|f|_{Lip}$), and satisfies

$$4k\alpha^{-1} \cdot |f|_{Lip} < 1.$$  \hspace{1cm} (2.2)

Then system (2.1) is topologically conjugated to its linear equations

\[
\begin{aligned}
\partial_t v_1 &= Av_1, \\
\partial_t v_2 &= Bv_2, \\
v_1(0) &= v_{10}, v_2(0) = v_{20}.
\end{aligned}
\]  \hspace{1cm} (2.3)

Moreover, the linearising map $H(\cdot)$ and its inverse $G(\cdot)$ satisfy: $H(u) - u \in \mathbb{B}\mathbb{C}(X), G(v) - v \in \mathbb{B}\mathbb{C}(X)$, for any $u,v \in X$.

**Remark 2.1.** There is no requirement on the generator $B$ in Theorem 2.1. It means that $B$ can be a non-hyperbolic operator.

Next theorem is for the regularity of the transformation $H$ and its inverse $G$.

**Theorem 2.2.** Assume that all the conditions of Theorem 2.1 hold. If there exist $\xi_1 := x - \bar{x} \in P_+X$ and $\eta_1 := y - \bar{y} \in P_+Y$ (or $\xi_2 := x - \bar{x} \in P_-X$ and $\eta_2 := y - \bar{y} \in P_-Y$), then the transformation $H$ is Lipschitz continuous, but its inverse $G$ is Hölder continuous. More specifically, there exist positive constants $p_1, p_2 > 0$ and $0 < q < 1$ such that for $u = (x,y)^T$ and $\bar{u} = (\bar{x},\bar{y})^T$

\[
\begin{aligned}
|H(u) - H(\bar{u})| &\leq p_1 \cdot |u - \bar{u}|, \\
|G(u) - G(\bar{u})| &\leq p_2 \cdot |u - \bar{u}|^q.
\end{aligned}
\]
Remark 2.2. In general, it is well known that the linearising maps obtained by $C^0$ linearization are Hölder continuous, i.e., $C^{0,\alpha}(0 < \alpha < 1)$. Our result is the first one to observe that the linearising map $H$ is Lipschitzian, but the inverse $G = H^{-1}$ is merely Hölder continuous. The method is based on two important dichotomy inequalities and theory of stable (unstable) manifold. Furthermore, $G$ cannot be improved to be Lipschitzian due to the right side integral diverges in the proof, see Remark 4.3 for more detail.

For the sake of comparison, we also present a result on the Hölder continuity of the both linearising maps based on the Bellman inequalities.

Theorem 2.3. Assume that all the conditions of Theorem 2.1 hold. Then both the transformation $H$ and its inverse $G$ are both Hölder continuous, i.e., there exist positive constants $\tilde{p}_1, p_2 > 0$ and $0 < q, \tilde{q} < 1$ such that for $u = (x,y)^T$ and $\bar{u} = (\bar{x},\bar{y})^T$

\[
\begin{align*}
|H(u) - H(\bar{u})| & \leq \tilde{p}_1 \cdot |u - \bar{u}|^{\tilde{q}}, \\
|G(u) - G(\bar{u})| & \leq p_2 \cdot |u - \bar{u}|^{q}.
\end{align*}
\]

2.2 Local linearization

We present a generalized version of local linearization. It is well known that the classical local linearization can be achieved as long as the nonlinear term $f(x)$ satisfies: (1) $f(0) = f'(0) = 0$; (2) $f(x)$ has a small Lipschitz constant in some neighborhood of 0. In the present paper, we improve the second condition, that is, while $f(x)$ is not Lipschitz continuous, we can still perform the local linearization. We claim this facts as follows:

Let $f(u_1, u_2) : X \times Y \to X$ be continuous and $f(0,0) \equiv 0$. For $u_1, \bar{u}_1 \in X$, and $u_2, \bar{u}_2 \in Y$, assume that

\[
|f(u_1, u_2) - f(\bar{u}_1, \bar{u}_2)| \leq L(\max\{|u_1|, |\bar{u}_1|\}, \max\{|u_2|, |\bar{u}_2|\})(|u_1 - \bar{u}_1| + |u_2 - \bar{u}_2|),
\]

where $L(\cdot, \cdot) : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is continuous, nondecreasing and $L(0,0) \equiv 0$. In order to obtain a local linearization version, we shall first discuss the modified equation of
| δ > 0 is a given positive constant. $f_\delta(u_1, u_2)$ is the modified nonlinearity of $f(u_1, u_2)$ and defined as follows:

$$f_\delta(u_1, u_2) = f \left( \psi \left( \frac{|u_1|^2}{\delta^2} \right) u_1, \psi \left( \frac{|u_2|^2}{\delta^2} \right) u_2 \right),$$

where $\psi$ is a $C^\infty$ bump function, namely, $\psi(t) = 1$ as $t \in [0, 1]$; $0 < \psi(t) < 1$ as $t \in (1, 2)$; $\psi(t) = 0$ as $t \in [2, \infty)$ and $\psi'(t) \leq 2$. Obviously, $\psi(|u_1|^2), \psi(|u_2|^2)$ is a smooth bump function and

$$|D_{u_1} \left( \psi \left( \frac{|u_1|^2}{\delta^2} \right) u_1 \right)| \leq |u_1| \cdot \left| \psi' \left( \frac{|u_1|^2}{\delta^2} \right) \right| \cdot \frac{2|u_1|}{\delta^2} + \psi \left( \frac{|u_1|^2}{\delta^2} \right) \leq 2 \cdot \frac{2(\sqrt{2} \delta)^2}{\delta^2} + 1 = 9,$$

$$|D_{u_2} \left( \psi \left( \frac{|u_2|^2}{\delta^2} \right) u_2 \right)| \leq 9.$$ 

Hence, it is clear to see that the modified nonlinear term $f_\delta(u_1, u_2)$ has the following properties:

1. $f_\delta(u_1, u_2)|_{\mathcal{B}(0, \delta)} \equiv f(u_1, u_2); f_\delta(u_1, u_2)|_{\{u_1 \in X, u_2 \in Y \mid |u_i| \geq \sqrt{2} \delta, i = 1, 2\}} \equiv 0$, where $\mathcal{B}(0, \delta)$ is the closure of $\mathcal{B}(0, \delta)$ and $\mathcal{B}(0, \delta)$ is a spherical neighborhood.

2. $|f_\delta(u_1, u_2) - f_\delta(\bar{u}_1, \bar{u}_2)| \leq 9L(\sqrt{2} \delta, \sqrt{2} \delta)(|u_1 - \bar{u}_1| + |u_2 - \bar{u}_2|)$ for $u_1, \bar{u}_1 \in X$ and $u_2, \bar{u}_2 \in Y$.

3. $|f_\delta(u_1, u_2)| \leq 2\sqrt{2} \delta \cdot L(\sqrt{2} \delta, \sqrt{2} \delta).$

Now, we are in a position to present the **local linearization theorem**.

**Theorem 2.4.** Let $A, B$ be the generators of $C_0$-groups $e^{At}, e^{Bt}$ on Banach spaces $X, Y$, respectively. Assume that $e^{At}$ admits an exponential dichotomy. Further, for a given constant $\delta > 0$, if the nonlinearity $f_\delta$ satisfies properties (1)-(3) and such that

$$36k\alpha^{-1} \cdot L(\sqrt{2} \delta, \sqrt{2} \delta) < 1.$$

Then equation (2.4) is topologically conjugated to its linear parts in $\mathcal{B}(0, \delta)$. 

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2.3 Two important dichotomy integral inequalities

Next, we present a generalized version of dichotomy integral inequalities, which will play an important role in our main proofs. It will help us to prove Lipschitz continuity for the linearising maps.

**Lemma 2.1.** Assume that the function $T(t) : [0, s] \rightarrow [0, \infty)$ is continuous and bounded for any $s \in (0, \infty]$. If there exist non-negative constants $\alpha$ and $a_i, i = 1, ..., 4$ such that $a_3 + a_4 < \alpha$, then for any $t \in [0, s]$, the inequality

$$T(t) \leq a_1 + a_2 e^{-\alpha t} + a_3 \int_0^t e^{-\alpha(t-\tau)} T(\tau) d\tau + a_4 \int_t^s e^{\alpha(t-\tau)} T(\tau) d\tau$$

implies $T(t) \leq (1 - \varpi)^{-1}(a_1 + a_2 e^{-\alpha_1 t})$, where

$$\varpi := \sup_{t \in [0, s]} (a_3 \int_0^t e^{-\alpha(t-\tau)} d\tau + a_4 \int_t^s e^{\alpha(t-\tau)} d\tau) = (a_3 + a_4)/\alpha < 1,$$

$$\alpha_1 := \alpha - a_3 \cdot (1 - \varpi)^{-1}.$$

**Proof.** For any $t \geq 0$, suppose that the function

$$\hat{T}(t) = T(t) - \hat{a}_1$$

satisfies the following inequality

$$\hat{T}(t) \leq \hat{a}_2 e^{-\alpha t} + \hat{a}_3 \int_0^t e^{-\alpha(t-\tau)} \hat{T}(\tau) d\tau + \hat{a}_4 \int_t^s e^{\alpha(t-\tau)} \hat{T}(\tau) d\tau,$$

where $\hat{a}_i, i = 1, ..., 4$ are undetermined constants. From (2.6) and (2.7), we obtain

$$T(t) - \hat{a}_1 \leq \hat{a}_2 e^{-\alpha t} + \hat{a}_3 \int_0^t e^{-\alpha(t-\tau)} T(\tau) d\tau + \hat{a}_4 \int_t^s e^{\alpha(t-\tau)} T(\tau) d\tau - \hat{a}_1 \hat{a}_3 \int_0^t e^{-\alpha(t-\tau)} d\tau - \hat{a}_1 \hat{a}_4 \int_t^s e^{\alpha(t-\tau)} d\tau,$$

that is

$$T(t) \leq \hat{a}_1 (1 - (\hat{a}_3 + \hat{a}_4)/\alpha) + \hat{a}_2 e^{-\alpha t} + \hat{a}_3 \int_0^t e^{-\alpha(t-\tau)} T(\tau) d\tau + \hat{a}_4 \int_t^s e^{\alpha(t-\tau)} T(\tau) d\tau.$$ 

Comparing (2.8) with (2.5), we see that

$$\begin{cases} a_1 = \hat{a}_1 (1 - (\hat{a}_3 + \hat{a}_4)/\alpha), \\ a_2 = \hat{a}_2, a_3 = \hat{a}_3, a_4 = \hat{a}_4. \end{cases}$$
Thus it follows from (2.7) and dichotomy inequality [25] that
\[ \dot{T}(t) \leq \frac{a_2}{1 - \varpi} e^{-(a - \frac{a_3}{\varpi})t} = \frac{a_2}{1 - \varpi} e^{-a_1 t}. \]
Therefore,
\[ T(t) \leq (1 - \varpi)^{-1}(a_1 + a_2 e^{-a_1 t}). \]
This completes the proof.

Remark 2.1. If \( a_1 \equiv 0 \), then \( T(t) \leq (1 - \varpi)^{-1}a_2 e^{-a_1 t} \) for all \( t \geq 0 \). If \( a_1 := a_1(t) = \bar{a}_1 \cdot e^{-\bar{a}_1 t}(\bar{a}_1, \bar{\alpha} \geq 0) \), then \( T(t) \leq (1 - \varpi)^{-1}(\bar{a}_1 + a_2)e^{-\max(\bar{a}, a_1)t} \). Clearly, our results are a generalized version of classical dichotomy inequality in [25].

Next, we give a dichotomy inequality of negative time, the conditions and proof are the same as in Lemma 2.1.

Lemma 2.2. Assume that the function \( T(t) : [s, 0] \to [0, \infty) \) be continuous and bounded for any \( s \in [\infty, 0) \). If there exist non-negative constants \( \alpha \) and \( a_i, (i = 1, \ldots, 4) \) such that \( a_3 + a_4 < \alpha \), then for any \( t \in [s, 0] \), the inequality
\[ T(t) \leq a_1 + a_2 e^{\alpha t} + a_3 \int_s^0 e^{\alpha(t-\tau)}T(\tau)d\tau + a_4 \int_s^t e^{-\alpha(t-\tau)}T(\tau)d\tau \]
implies \( T(t) \leq (1 - \varpi)^{-1}(a_1 + a_2 e^{\alpha t}). \)

3 Preliminary results

3.1 Non-trivial bounded mild solution

We start with a fundamental lemma which is a key for the other lemmas. Idea follows from [30].

Lemma 3.1. Suppose that \( e^{At} \) admits an exponential dichotomy, then the Cauchy problem of the evolution equation \( u' = Au \) has non-trivial bounded mild solutions.
**Proof.** Suppose that $u(t;0,u_0) = e^{At}u_0$ is a bounded mild solution of $u' = Au$ with the initial condition $u(0) = u_0 \in X$. We want to show that $u(t) \equiv u(t;0,u_0) \equiv 0$. Firstly, for $t \geq 0$, $u(t+s) = e^{At}u(s)$, for all $s \in \mathbb{R}$. In particular, $u(s) = e^{At}u(s-t)$. Hence,

$$|P_+u(s)| \leq |e^{At}P| \cdot |P_+u(s-t)| \leq ke^{-\alpha t} \cdot |P_+u(s-t)|.$$ 

Let $s-t = n \cdot r$, where $n$ is an integer and $r > 0$ is a constant. Since $\|u\| < +\infty$, then

$$|u(s-t)| = |n \cdot u(r)| \leq |n| \cdot |u(r)|,$$

which implies that $\sup_{t \geq 0, s \in \mathbb{R}} |u(s-t)| < +\infty$. Therefore,

$$|P_+u(s)| \leq ke^{-\alpha t} \cdot \|u\| \to 0 \text{ as } t \to +\infty,$$

that is, $P_+u(s) \equiv 0$. Similarly,

$$|P_-u(s)| \leq |e^{-At}P_-| \cdot |P_-u(s+t)| \leq ke^{-\alpha t} \cdot \|u\| \to 0 \text{ as } t \to +\infty,$$

which implies that $P_-u(s) \equiv 0$. Hence, $u(s) \equiv 0$, for all $s \in \mathbb{R}$, which implies that the bounded mild solution $u(t;0,u_0)$ of $u' = Au$ is always equal to 0.

\[\square\]

### 3.2 Constructing the conjugacy

To construct the conjugacy in Theorem 2.1, we divide the proof of Theorem 2.1 into several preliminary results as follows.

For simplicity, let

$$
\begin{pmatrix}
U_1(t,0,u_{10},u_{20}) \\
U_2(t,0,u_{10},u_{20})
\end{pmatrix}
$$

be the mild solution of (2.1) and

$$
\begin{pmatrix}
V_1(t,0,v_{10},v_{20}) \\
V_2(t,0,v_{10},v_{20})
\end{pmatrix}
$$

be the mild solution of (2.3), where

$$
U_1(t,0,u_{10},u_{20}) = e^{At}u_{10} + \int_0^t e^{A(t-s)} \cdot f(U_1(s,0,u_{10},u_{20}),U_2(s,0,u_{10},u_{20})) ds,
$$

$$
U_2(t,0,u_{10},u_{20}) = e^{Bt}u_{20}, \quad V_1(t,0,v_{10},v_{20}) = e^{At}v_{10}, \quad V_2(t,0,v_{10},v_{20}) = e^{Bt}v_{20}.
$$

In what follows, we always suppose that all the conditions of Theorem 2.1 are satisfied.
Lemma 3.2. For each fixed \((\xi, \eta) \in X \times Y\), the linear inhomogeneous evolution equation
\[
\begin{cases}
z' = Az - f(U_1(t,0,\xi,\eta), U_2(t,0,\xi,\eta)), \\
z(0) = h(\xi, \eta) \in X,
\end{cases}
\tag{3.1}
\]
has a unique bounded mild solution.

Proof. For any fixed \((\xi, \eta)\), observe that a solution of (3.1) is given by the convolution
\[
z(t) := (GA * f)(t) = - \int_{\mathbb{R}} GA(s)f(U_1(t-s,0,\xi,\eta), U_2(s,0,\xi,\eta))ds
\]
= \[- \int_{\mathbb{R}} GA(t-s)f(U_1(s,0,\xi,\eta), U_2(s,0,\xi,\eta))ds.\]
We shall show that \(z(t)\) is the unique bounded mild solution of equation (3.1). Since \(f\) is bounded, we have
\[
\|z\| \leq 2k\alpha^{-1} \cdot \|f\|_{\infty} < +\infty.
\]
This shows that \(z(t)\) is bounded. From Lemma 3.1, we know, for its linear homogeneous part \(z' = Az\), that the bounded mild solution is a zero solution. Therefore, \(z(t)\) is a unique bounded mild solution.

In particular, if we take \(t = 0\), then
\[
h(\xi, \eta) = z(0) = - \int_{\mathbb{R}} GA(-s)f(U_1(s,0,\xi,\eta), U_2(s,0,\xi,\eta))ds.
\]
This mild solution \(h\) is also bounded with \(\|h(\xi, \eta)\| \leq 2k\alpha^{-1} \cdot \|f\|_{\infty}.\)

Lemma 3.3. For each fixed \((\xi, \eta) \in X \times Y\), the semilinear evolution equation
\[
\begin{cases}
w' = Aw + f(V_1(t,0,\xi,\eta) + w, V_2(t,0,\xi,\eta)), \\
w(0) = g(\xi, \eta) \in X,
\end{cases}
\tag{3.2}
\]
has a unique bounded mild solution.

Proof. Note that \(BC(\mathbb{R}, X) := \{w(t) \in X | w(t) \text{ is continuous and } \sup_{t \in \mathbb{R}} |w(t)| < \infty\}\), and define a map \(T\) as follows
\[
(Tw)(t) := \int_{\mathbb{R}} GA(t-s)f(V_1(s,0,\xi,\eta) + w(s), V_2(s,0,\xi,\eta))ds.
\]
Step 1. We shall show that $\mathcal{T}$ is a contraction map on $BC(\mathbb{R}, X)$, consequently, $\mathcal{T}$ has a unique fixed point. In fact, it is easy to obtain that

$$\|\mathcal{T}w\| \leq 2k \alpha^{-1} \cdot |f|_\infty.$$  

Hence, $\mathcal{T}$ is a self-map from $BC(\mathbb{R}, X)$ to $BC(\mathbb{R}, X)$. Note that $f$ is Lipschitz continuous with Lipschitz constant $|f|_{Lip}$, we have

$$|(\mathcal{T}w_1 - \mathcal{T}w_2)(t)| \leq \int_\mathbb{R} k e^{\alpha|t-s|} \cdot |f|_{Lip} \cdot |(w_1(s) - w_2(s))| ds$$

It follows from (2.2) that $\|\mathcal{T}w_1 - \mathcal{T}w_2\| \leq 2k \alpha^{-1} \cdot |f|_{Lip} \cdot \|w_1 - w_2\| < \frac{1}{2} \|w_1 - w_2\|$. Thus the map $\mathcal{T}$ has a unique fixed point, namely, $w^* = \mathcal{T}w^*$, and satisfying

$$w^*(t) = \int_\mathbb{R} G_A(t-s)f(V_1(s,0,\xi,\eta)+w^*(s),V_2(s,0,\xi,\eta))ds.$$  

Clearly, $w^*(t)$ is a bounded mild solution of (3.2).

Step 2. To prove uniqueness, suppose that there is another bounded mild solution $w^+(t)$ of (3.2). Then

$$w^+(t) = e^{At}w_0 + \int_0^t f(V_1(t,0,\xi,\eta)+w^+(s),V_2(t,0,\xi,\eta))ds$$

$$= e^{At} \left( w_0 - \int_\mathbb{R} G_A(-s)f(V_1(s,0,\xi,\eta)+w^+(s),V_2(s,0,\xi,\eta))ds \right)$$

$$+ \int_\mathbb{R} G_A(t-s)f(V_1(s,0,\xi,\eta)+w^+(s),V_2(s,0,\xi,\eta))ds.$$  

Since $\int_\mathbb{R} G_A(-s)f(V_1(s,0,\xi,\eta)+w^+(s),V_2(s,0,\xi,\eta))ds$ is bounded, it is convergent, denoted by $w_0^+$. It is easy to see that $e^{At}(w_0 - w_0^+)$ is a bounded mild solution of $w' = Av$ with the initial condition $w_0 - w_0^+ \in X$. From Lemma 3.1, it is a zero solution. Therefore,

$$w^+(t) = \int_\mathbb{R} G_A(t-s)f(V_1(s,0,\xi,\eta)+w^+(s),V_2(s,0,\xi,\eta))ds.$$  

Calculating $w^+(t) - w^*(t)$, we have

$$|w^+(t) - w^*(t)| \leq \int_\mathbb{R} k e^{\alpha|t-s|} \cdot |f|_{Lip} \cdot |w^+(s) - w^*(s)| ds.$$  

It follows from (2.2) that $\|w^+ - w^*\| \leq 2k \alpha^{-1} \cdot |f|_{Lip} \cdot \|w^+ - w^*\| < \frac{1}{2} \|w^+ - w^*\|$. This implies that $w^+(t) \equiv w^*(t)$, and the bounded mild solution is unique.

In particular, if we take $t = 0$, then $g(\xi, \eta) = w(0)$ is bounded with $\|g(\xi, \eta)\| \leq 2k \alpha^{-1} \cdot |f|_\infty.$
Lemma 3.4. Let \( \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \) be any mild solution of \( (2.1) \). Then the semilinear evolution equation

\[
z' = Az + f(u_1 + z, u_2) - f(u_1, u_2)
\]

has a unique bounded mild solution \( z = 0 \).

**Proof.** The proof is similar to that in Lemma 3.3.

Now, we define two maps \( H, G : X \times Y \to X \) as follows:

\[
H(u_1, u_2) = \begin{pmatrix} H_1(u_1, u_2) \\ H_2(u_1, u_2) \end{pmatrix} = \begin{pmatrix} u_1 + h(u_1, u_2) \\ u_2 \end{pmatrix},
\]

\[
G(v_1, v_2) = \begin{pmatrix} G_1(v_1, v_2) \\ G_2(v_1, v_2) \end{pmatrix} = \begin{pmatrix} v_1 + g(v_1, v_2) \\ v_2 \end{pmatrix},
\]

where \( u_1, v_1 \in X \) and \( u_2, v_2 \in Y \).

**Lemma 3.5.** \( \begin{pmatrix} H_1(U_1(t, 0, u_{10}, u_{20}), U_2(t, 0, u_{10}, u_{20})) \\ H_2(U_1(t, 0, u_{10}, u_{20}), U_2(t, 0, u_{10}, u_{20})) \end{pmatrix} \) is a mild solution of \( (2.3) \).

**Proof.** It is clear that

\[
H_1(U_1, U_2) = U_1 + h(U_1, U_2), \quad H_2(U_1, U_2) = e^{Bt}u_{20},
\]

and \( H_2 \) is a mild solution of the second equation in \( (2.3) \). Thus we only show that \( H_1 \) is a mild solution of the first equation in \( (2.3) \). Since \( U_1 \) is a mild solution of \( u'_1 = Au_1 + f(u_1, u_2) \), and \( h(U_1, U_2) \) is a mild solution of \( z' = Az - f(u_1, u_2) \), we have that \( H_1 \) is a mild solution of

\[
y' = u'_1 + z' = Au_1 + f(u_1, u_2) + Az - f(u_1, u_2) = A(u_1 + z) = Ay.
\]

**Lemma 3.6.** \( \begin{pmatrix} G_1(V_1(t, 0, v_{10}, u_{20}), V_2(t, 0, v_{10}, v_{20})) \\ G_2(V_1(t, 0, v_{10}, u_{20}), V_2(t, 0, v_{10}, v_{20})) \end{pmatrix} \) is a mild solution of \( (2.1) \).

**Proof.** The proof is similar to that of Lemma 3.5.

\[\square\]
Lemma 3.7. For any fixed $v_1 \in X$ and $v_2 \in Y$, 
\[
\begin{pmatrix}
H_1(G_1(v_1, v_2), G_2(v_1, v_2)) \\
H_2(G_1(v_1, v_2), G_2(v_1, v_2))
\end{pmatrix}
= \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.
\]

Proof. Let $(v_1(t), v_2(t))^T$ be any solution of (2.1). From Lemma 3.6, 
\[
\begin{pmatrix}
G_1(v_1(t), v_2(t)) \\
G_2(v_1(t), v_2(t))
\end{pmatrix}
\]
a solution of (2.3). Moreover, by Lemma 3.5, 
\[
\begin{pmatrix}
H_1(G_1(v_1(t), v_2(t)), G_2(v_1(t), v_2(t))) \\
H_2(G_1(v_1(t), v_2(t)), G_2(v_1(t), v_2(t)))
\end{pmatrix}
\]
is a solution of (2.3), denoted by $(\hat{v}_1(t), \hat{v}_2(t))^T$. Since 
\[
H_2(G_1(v_1(t), v_2(t)), G_2(v_1(t), v_2(t))) = G_2(v_1(t), v_2(t)) = v_2(t),
\]
we obtain that $\hat{v}_2(t) = v_2(t)$. Let $J(t) = \hat{v}_1(t) - v_1(t)$, then $J(t)$ is also a mild solution of $z' = Az$. It follows from the definition of $H, G$ that 
\[
|J(t)| = |H_1(G_1(v_1(t), v_2(t)), G_2(v_1(t), v_2(t))) - v_1(t)|
\leq |H_1(G_1(v_1(t), v_2(t)), G_2(v_1(t), v_2(t))) - G_1(v_1(t), v_2(t))|
+ |G_1(v_1(t), v_2(t)) - v_1(t)|.
\]
From Lemma 3.2 and Lemma 3.3, we have that $\|J\| \leq 2k\alpha^{-1} \cdot |f|_{\infty} + 2k\alpha^{-1} \cdot |f|_{\infty} = 4k\alpha^{-1} \cdot |f|_{\infty}$. Thus in view of Lemma 3.1, $J(t) \equiv 0$, namely, $\hat{v}_1(t) = v_1(t)$. Since $(v_1(t), v_2(t))^T$ is an arbitrary solution of (2.3), Lemma 3.7 holds. \hfill \Box

Lemma 3.8. For any fixed $u_1 \in X$ and $u_2 \in Y$, 
\[
\begin{pmatrix}
G_1(H_1(u_1, u_2), H_2(u_1, u_2)) \\
G_2(H_1(u_1, u_2), H_2(u_1, u_2))
\end{pmatrix}
= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.
\]

Proof. Combined with Lemma 3.4, the proof of Lemma 3.8 is similar to Lemma 3.7. \hfill \Box

3.3 Key lemma to prove the regularity

Next lemma is a key lemma to prove the regularity of the conjugacy.

Lemma 3.9. Let $(U_1(t, 0, u_{10}, u_{20}), U_2(t, 0, u_{10}, u_{20}))$ be a mild solution of (2.1).

(i) For any $\xi_1 \in P_+(X), \eta_1 \in P_+(Y)$, then (2.1) has a unique bounded solution $(U_1(t), U_2(t))$ satisfying for $t \geq 0$, $P_+(U_1(0), U_2(0)) = (\xi_1, \eta_1)$;
(ii) For any \( \xi_2 \in P_-(X), \eta_2 \in P_-(Y) \), then (2.1) has a unique bounded solution \((U_1(t), U_2(t))\) satisfying for \( t \leq 0 \), \( P_-(U_1(0), U_2(0)) = (\xi_2, \eta_2) \). Moreover, if exponent \( \varpi = (2k/\alpha) \cdot |f|_{Lip} < 1 \) and \( \alpha_1 = \alpha - k|f|_{Lip}/(1 - \varpi) > 0 \), then the following conclusions hold:

(1) for any \((u_{10} - \bar{u}_{10}) \in P_+(X)\) and \((u_{20} - \bar{u}_{20}) \in P_+(Y)\), we deduce that

\[
|U_1(t, 0, u_{10}, u_{20}) - U_1(t, 0, \bar{u}_{10}, \bar{u}_{20})| + |U_2(t, 0, u_{10}, u_{20}) - U_2(t, 0, \bar{u}_{10}, \bar{u}_{20})| \leq (1 - \varpi)^{-1} \cdot (ke^{-\alpha_1 t} \cdot |P_+(u_{10} - \bar{u}_{10})| + MB|P_+(u_{20} - \bar{u}_{20})|), \quad t \geq 0; \tag{3.4}
\]

(2) for any \((u_{10} - \bar{u}_{10}) \in P_-(X)\) and \((u_{20} - \bar{u}_{20}) \in P_-(Y)\), we have that

\[
|U_1(t, 0, u_{10}, u_{20}) - U_1(t, 0, \bar{u}_{10}, \bar{u}_{20})| + |U_2(t, 0, u_{10}, u_{20}) - U_2(t, 0, \bar{u}_{10}, \bar{u}_{20})| \leq (1 - \varpi)^{-1} \cdot (ke^{\alpha_1 t} \cdot |P_-(u_{10} - \bar{u}_{10})| + MB|P_-(u_{20} - \bar{u}_{20})|), \quad t \leq 0. \tag{3.5}
\]

**Proof.** We claim the first part by means of Banach Contraction Principle. Let \( \mathbb{BC} \) be the subset of all bounded continuous functions defined for \( t \geq 0 \). Define

\[
\Phi(t) := U_1(t, 0, u_{10}, u_{20}) = e^{At}u_{10} + \int_0^t e^{A(t-s)} \cdot f(U_1(s, 0, u_{10}, u_{20}), U_2(s, 0, u_{10}, u_{20}))ds,
\]

\[
\Psi(t) := U_2(t, 0, u_{10}, u_{20}) = e^{Bt}u_{20}.
\]

If \( \mathcal{J} \) is the map defined by

\[
\mathcal{J}(\Phi, \Psi)(t) = e^{At}\xi_1 + e^{Bt}\eta_1 + \int_0^t e^{A(t-s)} P_+f(\Phi(s), \Psi(s))ds - \int_t^{\infty} e^{A(t-s)} P_-f(\Phi(s), \Psi(s))ds,
\]

where \( \xi_1 \in P_+(X) \) and \( \eta_1 \in P_+(Y) \). Then \( \mathcal{J}(\Phi, \Psi) \) is continuous and bounded:

\[
|\mathcal{J}(\Phi, \Psi)| \leq ke^{-\alpha t}|\xi_1| + MB|\eta_1| + \int_0^\infty |G_A(t-s)||f|_{\infty}ds
\]

\[
\leq k|\xi_1| + MB|\eta_1| + (2k|f|_{\infty}/\alpha) < \infty.
\]

Hence, \( \mathcal{J} \) maps \( \mathbb{BC} \) into itself. Note that \( 2k|f|_{Lip} < \alpha \), for any \( \Phi_1, \Phi_2, \Psi_1, \Psi_2 \in \mathbb{BC} \), we have

\[
|\mathcal{J}(\Phi_1, \Psi_1) - \mathcal{J}(\Phi_2, \Psi_2)| \leq \int_0^\infty ke^{-\alpha(t-s)}|f|_{Lip}(|\Phi_1(s) - \Phi_2(s)| + |\Psi_1(s) - \Psi_2(s)|)ds
\]

\[
\leq (2k|f|_{Lip}/\alpha)(\|\Phi_1 - \Phi_2\| + \|\Psi_1 - \Psi_2\|),
\]

which implies that \( \mathcal{J} \) is a contraction mapping in \( \mathbb{BC} \), that is, there exists a unique fixed point \( (\Phi^*, \Psi^*) = \mathcal{J}(\Phi^*, \Psi^*) \) such that \( (\Phi^*, \Psi^*) \) is bounded for \( t \geq 0 \).
Similar to the procedure just shown, \((\Phi(t), \Psi(t))\) is a mild solution of (2.1) with the required property (ii) for \(t \leq 0\).

We now claim the second part by means of the generalized dichotomy inequalities, (i.e., Lemmas 2.1 and 2.2). Define
\[
\Omega_i(t) := U_i(t, 0, u_{10}, u_{20}) - U_i(t, 0, \bar{u}_{10}, \bar{u}_{20}), \quad i = 1, 2.
\]
It follows that
\[
\sum_{i=1}^{2} |\Omega_i(t)| \leq ke^{-\alpha t}|P_+(u_{10} - \bar{u}_{10})| + M_B|P_+(u_{20} - \bar{u}_{20})|
\]
\[
+ \int_{0}^{\infty} ke^{-\alpha|t-s|} \cdot |f|_{Lip} \cdot \left( \sum_{i=1}^{2} |\Omega_i(s)| \right) ds.
\]
Using dichotomy inequality in Lemma 2.1, we conclude that
\[
\sum_{i=1}^{2} |\Omega_i(t)| \leq (1 - \varpi)^{-1} \cdot (ke^{-\alpha_1 t}|P_+(u_{10} - \bar{u}_{10})| + M_B|P_+(u_{20} - \bar{u}_{20})|), \quad t \geq 0,
\]
where \(\varpi = (2k|f|_{Lip}/\alpha) < 1\) and positive constant \(\alpha_1\) satisfies \(\alpha_1 = \alpha - k|f|_{Lip}/(1 - \varpi)\).

Finally, analogous analysis for \(t \leq 0\), dichotomy inequality in Lemma 2.2 ends the proof.

\[\square\]

Remark 3.1. If \(B\) is a hyperbolic operator, i.e., \(e^{Bt}\) admits a dichotomy projection, then by the condition of Lemma 3.1—\(U_2\) is bounded, we obtain \(U_2 \equiv 0\). It follows from Remark 2.1 that
\[
\sum_{i=1}^{2} |\Omega_i(t)| = |\Omega_1(t)| \leq (1 - \varpi)^{-1}ke^{-\alpha_1 t}|P_+(u_{10} - \bar{u}_{10})|.
\]

Remark 3.2. If \(e^{Bt}\) is an exponential stable group, i.e., there exists \(M_B > 0, \omega_B > 0\) such that \(|e^{Bt}| \leq M_B e^{-\omega_B |t|}\). It is obvious that \(e^{Bt}\) is bounded, then
\[
\sum_{i=1}^{2} |\Omega_i(t)| \leq ke^{-\alpha t}|P_+(u_{10} - \bar{u}_{10})| + M_B e^{-\omega_B |t|}|P_+(u_{20} - \bar{u}_{20})|
\]
\[
+ \int_{0}^{\infty} ke^{-\alpha|t-s|} \cdot |f|_{Lip} \cdot \left( \sum_{i=1}^{2} |\Omega_i(s)| \right) ds
\]
\[
\leq De^{-\omega|t|}(|P_+(u_{10} - \bar{u}_{10})| + |P_+(u_{20} - \bar{u}_{20})|)
\]
\[
+ \int_{0}^{\infty} ke^{-\alpha|t-s|} \cdot |f|_{Lip} \cdot \left( \sum_{i=1}^{2} |\Omega_i(s)| \right) ds.
\]
By using dichotomy inequality in Lemma 2.2 and Lemma 3.9, we obtain
\begin{equation}
\sum_{i=1}^{2} |\Omega_i(t)| \leq (1 - \overline{\omega})^{-1} D e^{-d_1|t|} \cdot (|P_+(u_{10} - \overline{u}_{10})| + |P_+(u_{20} - \overline{u}_{20})|),
\end{equation}
where \( D = \max\{k, M_B\} \), \( d = \min\{\alpha, \omega_B\} \) and \( 0 < d_1 < d \).

**Lemma 3.10.** Let \((U_1, U_2)\) be a mild solution of (2.1). If there exist positive constants \(M_A, M_B > 0\) and \(\omega_A, \omega_B > 0\) such that \(|e^{At}| \leq M_A e^{\omega_A|t|}, |e^{Bt}| \leq M_B e^{\omega_B|t|}, t \in \mathbb{R}\), then the inequality
\begin{equation}
\sum_{i=1}^{2} |\Omega_i(t)| \leq M_A e^{\omega_A|t|} |u_{10} - \overline{u}_{10}| + M_B e^{\omega_B|t|} |u_{20} - \overline{u}_{20}| + \int_0^t |e^{A(t-s)}| \cdot |f|_{\text{Lip}} \cdot \sum_{i=1}^{2} |\Omega_i(s)| ds
\end{equation}
implies
\begin{equation}
\sum_{i=1}^{2} |\Omega_i(t)| \leq M_c (|u_{10} - \overline{u}_{10}| + |u_{20} - \overline{u}_{20}|) e^{(M_c |f|_{\text{Lip}} + \omega_c) t},
\end{equation}
where \( M_c = \max\{M_A, M_B\}, \omega_c = \max\{\omega_A, \omega_B\} \).

**Proof.** It is obvious that
\begin{equation}
\sum_{i=1}^{2} |\Omega_i(t)| \leq |e^{At}| |u_{10} - \overline{u}_{10}| + |e^{Bt}| |u_{20} - \overline{u}_{20}| + \int_0^t |e^{A(t-s)}| \cdot |f|_{\text{Lip}} \cdot \sum_{i=1}^{2} |\Omega_i(s)| ds
\end{equation}
\[ \leq M_A e^{\omega_A t} (|u_{10} - \overline{u}_{10}| + |u_{20} - \overline{u}_{20}|) + \int_0^t M_B e^{\omega_B(s-t)} \cdot |f|_{\text{Lip}} \cdot \sum_{i=1}^{2} |\Omega_i(s)| ds. \]
That is
\[ \sum_{i=1}^{2} e^{\omega_c t} |\Omega_i(t)| \leq M_c (|u_{10} - \overline{u}_{10}| + |u_{20} - \overline{u}_{20}|) + \int_0^t M_c e^{-\omega_c s} \cdot |f|_{\text{Lip}} \cdot \sum_{i=1}^{2} |\Omega_i(s)| ds. \]
By using Bellman inequality, it implies that
\begin{equation}
\sum_{i=1}^{2} |\Omega_i(t)| \leq M_c (|u_{10} - \overline{u}_{10}| + |u_{20} - \overline{u}_{20}|) e^{(M_c |f|_{\text{Lip}} + \omega_c) t}.
\end{equation}

**Remark 3.3.** In Lemma 3.9, the dichotomy inequality will help us to prove the Lipschitz continuity of equivalent functions under the premise of constrained bounded solutions. Without this premise, we can only rely on the Bellman inequality to obtain the Hölder continuity of equivalent functions.
4 Proofs of main results

Now we are in position to prove our main results.

4.1 Proof of Theorem 2.1.

Proof. From Lemma 3.7 and 3.8, we have proved that $H$ and $G$ are homeomorphism. From Lemma 3.2 and 3.3, we have derived that $h,g \in BC(X)$. From Lemma 3.5 and 3.6, $H$ sends the mild solution of semilinear evolution equation (2.1) onto the mild solution of linear evolution equation (2.3) and vice versa. Hence, (2.1) and (2.3) are topologically conjugated. This completes the proof of Theorem 2.1.

4.2 Proof of Theorem 2.2

We split the proof of Theorem 2.2 into two steps.

Proof. The aim in this part is to claim that $H$ is Lipschitzian, and $G$ is Hölder continuous.

Step 1-1. We will use dichotomy inequality in Lemma 3.9 to prove the Lipschitz continuity of the linearising map. We show that $|H(u) - H(\bar{u})| \leq p_1|u - \bar{u}|$, where $p_1 \geq 1$ is a constant. From Lemma 3.2, it follows that

$$h(\xi, \eta) = -\int_{\mathbb{R}} GA(-s)f(U_1(s,0,\xi,\eta),U_2(s,0,\xi,\eta))ds,$$

which is equivalent to

$$P_+ h(\xi, \eta) = -\int_{-\infty}^{0} e^{-As}P_+ f(U_1(s,0,\xi,\eta),U_2(s,0,\xi,\eta))ds$$

and

$$P_- h(\xi, \eta) = \int_{0}^{\infty} e^{-As}P_- f(U_1(s,0,\xi,\eta),U_2(s,0,\xi,\eta))ds.$$
Thus we get
\begin{align*}
R_1 & \triangleq P_+ h(\xi, \eta) - P_+ h(\xi, \eta) \\
& = \int_{-\infty}^{0} e^{-As} P_+[f(U_1(s, 0, \xi, \eta), U_2(s, 0, \xi, \eta)) - f(U_1(s, 0, \xi, \eta), U_2(s, 0, \xi, \eta))]ds,
\end{align*}
\begin{align*}
R_2 & \triangleq P_- h(\xi, \eta) - P_- h(\xi, \eta) \\
& = \int_{0}^{\infty} e^{-As} P_-[f(U_1(s, 0, \xi, \eta), U_2(s, 0, \xi, \eta)) - f(U_1(s, 0, \xi, \eta), U_2(s, 0, \xi, \eta))]ds.
\end{align*}
In view of Lemma 3.9, for any initial condition on $P_+(X)$ and $P_+(Y)$ (or $P_-(X), P_-(Y)$) of (2.1) is bounded on semiaxis $[0, \infty)$ (or $(-\infty, 0]$). Then by using Lemma 3.9 part $t \leq 0$, we deduce that
\begin{align*}
|R_1| & \leq \int_{-\infty}^{0} ke^{\alpha s} \cdot |f|_{Lip} \cdot \left(\sum_{i=1}^{2} |\Omega_i(s)|\right)ds \\
& \leq \int_{-\infty}^{0} ke^{\alpha s} \cdot |f|_{Lip} \cdot \frac{1}{1 - \frac{\alpha}{\alpha_1}} (ke^{-\alpha_1 s} \cdot |P_+(\xi - \tilde{\xi})| + M_B|P_+(\eta - \tilde{\eta})|)ds,
\end{align*}
and similarly, by using Lemma 3.9 part $t \geq 0$,
\begin{align*}
|R_2| & \leq \int_{0}^{\infty} ke^{-\alpha s} \cdot |f|_{Lip} \cdot \frac{1}{1 - \frac{\alpha}{\alpha_1}} (ke^{-\alpha_1 s} \cdot |P_-(\xi - \tilde{\xi})| + M_B|P_-(\eta - \tilde{\eta})|)ds.
\end{align*}
Hence we conclude that
\begin{align*}
|R_1| + |R_2| & \leq \int_{-\infty}^{0} ke^{\alpha s} \cdot |f|_{Lip} \cdot \frac{1}{1 - \frac{\alpha}{\alpha_1}} (ke^{-\alpha_1 s} \cdot |P_+(\xi - \tilde{\xi})| + M_B|P_+(\eta - \tilde{\eta})|)ds \\
& \quad + \int_{0}^{\infty} ke^{-\alpha s} \cdot |f|_{Lip} \cdot \frac{1}{1 - \frac{\alpha}{\alpha_1}} (ke^{-\alpha_1 s} \cdot |P_-(\xi - \tilde{\xi})| + M_B|P_-(\eta - \tilde{\eta})|)ds \\
& \leq \frac{k^2 \cdot |f|_{Lip}}{(\alpha - \alpha_1)(1 - \frac{\alpha}{\alpha_1})} \cdot \left( |P_+(\xi - \tilde{\xi})| + |P_-(\xi - \tilde{\xi})| \right) \\
& \quad + \frac{k \cdot |f|_{Lip} \cdot M_B}{\alpha(1 - \frac{\alpha}{\alpha_1})} \cdot \left( |P_+(\eta - \tilde{\eta})| + |P_-(\eta - \tilde{\eta})| \right) \\
& \leq \max \left\{ \frac{2k^2 \cdot |f|_{Lip}}{(\alpha_1 + \alpha)(1 - \frac{\alpha}{\alpha_1})}, \frac{2k \cdot |f|_{Lip} \cdot M_B}{\alpha(1 - \frac{\alpha}{\alpha_1})} \right\} (|\xi - \tilde{\xi}| + |\eta - \tilde{\eta}|).
\end{align*}
By the definition of $H(u)$, $u = (u_1, u_2)^T$,
\begin{align*}
|H(u) - H(\bar{u})| & \leq |u - \bar{u}| + \max \left\{ \frac{2k \cdot |f|_{Lip}}{\alpha(1 - \frac{\alpha}{\alpha_1})}, \frac{2k \cdot |f|_{Lip} \cdot M_B}{\alpha(1 - \frac{\alpha}{\alpha_1})} \right\} |u - \bar{u}| \\
& \leq \left( 1 + \max \left\{ \frac{2k^2 \cdot |f|_{Lip}}{(\alpha_1 + \alpha)(1 - \frac{\alpha}{\alpha_1})}, \frac{2k \cdot |f|_{Lip} \cdot M_B}{\alpha(1 - \frac{\alpha}{\alpha_1})} \right\} \right) |u - \bar{u}| \\
& := p_1 |u - \bar{u}|.
\end{align*}
This completes the proof of Step 1-1.
Remark 4.1. If $B$ is a hyperbolic operator, in view of Remark 3.1, we obtain
\[
|h(\xi, \eta) - h(\bar{\xi}, \bar{\eta})| \leq \int_{\mathbb{R}} k e^{\alpha|s|} \cdot |f|_{Lip} \cdot \frac{k e^{-\alpha_1|s|} |\xi - \bar{\xi}|}{1 - \omega} ds
\leq 2 \int_{0}^{\infty} \frac{k^2 |f|_{Lip} |\xi - \bar{\xi}| \cdot e^{(\alpha_1 - \alpha)s}}{1 - \omega} ds
\leq \frac{2k^2 |f|_{Lip}}{(1 - \omega)(\alpha - \alpha_1)} \cdot |\xi - \bar{\xi}|.
\]
Hence, $h$ is Lipschitzian, consequently, $H = x + h$ is also Lipschitzian.

Remark 4.2. If $e^{Bt}$ is an exponential stable group, it follows from Remark 3.2 that
\[
|h(\xi, \eta) - h(\bar{\xi}, \bar{\eta})| \leq \int_{\mathbb{R}} k e^{\alpha|s|} \cdot |f|_{Lip} \cdot \frac{D(|\xi - \bar{\xi}| + |\eta - \bar{\eta}|) e^{-d_1|s|}}{1 - \omega} ds
\leq 2kD |f|_{Lip} \cdot ((|\xi - \bar{\xi}| + |\eta - \bar{\eta}|).
\]
It is easy to see that $H$ is Lipschitzian.

Step 1-2. We are going to prove that the inverse $G = H^{-1}$ is Hölder continuous. By Lemma 3.3 we know that $g(\xi, \eta)$ is a fixed point of the following map $\mathcal{T}$
\[
(\mathcal{T} z)(0) = \int_{\mathbb{R}} G_{A}(-s) \cdot f(V_{1}(s, 0, \xi, \eta) + z(s), V_{2}(s, 0, \xi, \eta)) ds
= \int_{\mathbb{R}} G_{A}(-s) \cdot f(e^{A_{s} \xi} + z(s), e^{B_{s} \eta}) ds.
\]
(4.1)
Let $g_{0}(\xi, \eta) \equiv 0$, and by recursion define
\[
g_{m+1}(\xi, \eta) = \int_{\mathbb{R}} G_{A}(-s) \cdot f(e^{A_{s} \xi} + g_{m}(\xi, \eta), e^{B_{s} \eta}) ds.
\]
It is not difficult to show that
\[
g_{m}(\xi, \eta) \to g(\xi, \eta), \text{ as } m \to +\infty,
\]
uniformly with respect to $\xi, \eta$.

Note that $g_{0}(\xi, \eta) = g_{0}(t, (t, e^{A_{t} \xi}, e^{B_{t} \eta})) \equiv 0$. Thus, by induction, it is clear that for all $m(m \in \mathbb{N})$, $g_{m}(\xi, \eta) = g_{m}(t, (t, e^{A_{t} \xi}, e^{B_{t} \eta}))$. Choose $p > 0$ sufficiently large and $q > 0$ sufficiently small such that
\[
\begin{align*}
p > (8k/\alpha) \cdot |f|_{\infty} + \frac{4k |f|_{Lip} M_{e}}{\omega_{c} - \alpha},
q \omega_{c} < \alpha,
0 < \frac{2k |f|_{Lip} M_{e}^{q}}{\alpha - q \omega_{c}} < \frac{1}{2},
\end{align*}
\]
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Then, we want to show that

$$|g_m(\xi, \eta) - g_m(\bar{\xi}, \bar{\eta})| \leq p(|\xi - \bar{\xi}| + |\eta - \bar{\eta}|)^q.$$  \hspace{1cm} (4.2)

Obviously, inequality (4.2) holds if \( m = 0 \). Now making the inductive assumption that (4.2) holds. From (4.1), it follows that

\[
g_{m+1}(\xi, \eta) - g_{m+1}(\bar{\xi}, \bar{\eta})
= 2 \int_0^\infty G_A(-s) \cdot [f(e^{As}\xi + g_m(\xi, \eta), e^{Bs}\eta) - f(e^{As}\bar{\xi} + g_m(\bar{\xi}, \bar{\eta}), e^{Bs}\bar{\eta})] ds
= 2 \int_0^{\tau_2} G_A(-s) \cdot [f(e^{As}\xi + g_m(\xi, \eta), e^{Bs}\eta) - f(e^{As}\bar{\xi} + g_m(\bar{\xi}, \bar{\eta}), e^{Bs}\bar{\eta})] ds
+ 2 \int_{\tau_2}^\infty G_A(-s) \cdot [f(e^{As}\xi + g_m(\xi, \eta), e^{Bs}\eta) - f(e^{As}\bar{\xi} + g_m(\bar{\xi}, \bar{\eta}), e^{Bs}\bar{\eta})] ds
= 2(I_1 + I_2),
\]

where \( \tau_2 = \frac{1}{\omega_c} \cdot \ln \frac{1}{|\xi - \bar{\xi}| + |\eta - \bar{\eta}|} \). In view of \( \alpha < \omega_c \), we have

$$|I_2| \leq \int_{\tau_2}^\infty ke^{-\alpha s} \cdot 2|f|_\infty ds \leq (2k/\alpha) \cdot |f|_\infty \cdot e^{-\alpha \tau_2} = (2k/\alpha) \cdot |f|_\infty \cdot ||\xi - \bar{\xi}| + |\eta - \bar{\eta}||^{\alpha/\omega_c}.$$ 

Notice that \( A, B \) are the generators of \( C_0 \)-groups \( e^{At}, e^{Bt} \) in \( X \) with \( |e^{At}| \leq M_A e^{\omega_A|t|}, |e^{Bt}| \leq M_B e^{\omega_B|t|} \), respectively. Then we have the following estimates

$$|e^{At}x - e^{At}\bar{x}| \leq M_A|x - \bar{x}|e^{\alpha|t|}, \quad |e^{Bt}x - e^{Bt}\bar{x}| \leq M_B|x - \bar{x}|e^{\omega_B|t|},$$

where \( M_A, M_B \geq 1, \omega_A, \omega_B > 0 \). In view of Lemma 3.10, we have

\[
|g_m(\xi, \eta) - g_m(\bar{\xi}, \bar{\eta})| = |g_m(t, (t, e^{At}\xi, e^{Bt}\eta)) - g_m(t, (t, e^{At}\bar{\xi}, e^{Bt}\bar{\eta})| \\
\leq p[|e^{At}\xi - e^{At}\bar{\xi}| + |e^{Bt}\eta - e^{Bt}\bar{\eta}|]^q \\
\leq p[M_A e^{\omega_A|t| |\xi - \bar{\xi}|} + M_B e^{\omega_B|t| |\eta - \bar{\eta}|}]^q \\
\leq p[M e^{\alpha|t|(|\xi - \bar{\xi}| + |\eta - \bar{\eta}|)}]^q.
\]
Therefore,

\[
|I_1| \leq \int_0^{T_2} ke^{-\alpha s} \cdot |f|_{\text{Lip}} \cdot \{ M_A e^{\omega_A s} |\xi - \xi| + M_B e^{\omega_B s} |\eta - \bar{\eta}| \\
+ p[M_c e^{\omega_c s}(|\xi - \xi| + |\eta - \bar{\eta}|)]^q \} ds \\
\leq \int_0^{T_2} ke^{-\alpha s} \cdot |f|_{\text{Lip}} \cdot \{ M_c e^{\omega_c s}(|\xi - \xi| + |\eta - \bar{\eta}|) \\
+ p[M_c e^{\omega_c s}(|\xi - \xi| + |\eta - \bar{\eta}|)]^q \} ds \\
\leq \int_0^{T_2} \frac{k|f|_{\text{Lip}} \cdot M_c e^{(\omega_c - \alpha)s}(|\xi - \xi| + |\eta - \bar{\eta}|)}{\omega_c - \alpha} \cdot e^{(\omega_c - \alpha)s}|s = T_2 \\
+ \frac{k|f|_{\text{Lip}} \cdot pM_c^q e^{(\omega_c - \alpha)s}(|\xi - \xi| + |\eta - \bar{\eta}|)^q}{\omega_c - \alpha} \cdot e^{(\omega_c - \alpha)s}|s = T_2.
\]

Note that \( \omega_c - \alpha > 0, q\omega_c - \alpha < 0 \) (0 < \( q < 1 \), then

\[
|I_1| \leq \frac{k|f|_{\text{Lip}} \cdot M_c}{\omega_c - \alpha} (|\xi - \xi| + |\eta - \bar{\eta}|) e^{\omega_c s} + \frac{k|f|_{\text{Lip}} \cdot pM_c^q}{\omega_c - \alpha} (|\xi - \xi| + |\eta - \bar{\eta}|)^q.
\]

Therefore, we obtain

\[
|g_{m+1}(\xi, \eta) - g_{m+1}(\xi, \bar{\eta})| \\
\leq 2(|I_1| + |I_2|) \\
\leq \frac{2k|f|_{\text{Lip}} \cdot M_c}{\omega_c - \alpha} (|\xi - \xi| + |\eta - \bar{\eta}|) e^{\omega_c s} + \frac{2k|f|_{\text{Lip}} \cdot pM_c^q}{\omega_c - \alpha} (|\xi - \xi| + |\eta - \bar{\eta}|)^q \\
+ (4k/\alpha) \cdot |f|_{\infty} \cdot (|\xi - \xi| + |\eta - \bar{\eta}|) e^{\omega_c s} \\
\leq p \cdot (|\xi - \xi| + |\eta - \bar{\eta}|)^q.
\]

It implies that

\[
|g(\xi, \eta) - g(\xi, \bar{\eta})| \leq p \cdot (|\xi - \xi| + |\eta - \bar{\eta}|)^q, \text{ as } m \to \infty.
\]

By the definition of \( G(v) \), \( v = (v_1, v_2)^T \), we obtain that

\[
|G(v) - G(\bar{v})| \leq (1 + p) \cdot |v - \bar{v}|^q := p_2 \cdot |v - \bar{v}|^q.
\]

This completes the proof of Step 1-2.
Remark 4.3. We now explain that $G$ cannot be improved to be Lipschitzian. In fact, in proving the Hölder regularity of $G$, we divide $g_{m+1}(\xi, \eta) - g_{m+1}(\bar{\xi}, \bar{\eta})$ into two parts: $I_1, I_2$. If we directly prove that $G$ is Lipschitzian, then the contradiction appears as follows:

$$|g_{m+1}(\xi, \eta) - g_{m+1}(\bar{\xi}, \bar{\eta})| \leq \int_0^\infty k e^{-\alpha s} \cdot |f|_{Lip} \cdot \{M_A e^{\omega_A s} |\xi - \bar{\xi}| + M_B e^{\omega_B s} |\eta - \bar{\eta}|$$

$$+ p[M e^{\omega_s}(|\xi - \xi| + |\eta - \bar{\eta}|)]^q ds$$

$$\leq \int_0^\infty k |f|_{Lip} \cdot M e^{(\omega_c - \alpha) s} (|\xi - \xi| + |\eta - \bar{\eta}|) ds$$

$$+ \int_0^\infty k |f|_{Lip} \cdot p M^q e^{(\omega_c q - \alpha) s} (|\xi - \xi| + |\eta - \bar{\eta}|)^q ds.$$

Since $\omega_c - \alpha > 0$, the integral $\int_0^\infty e^{(\omega_c - \alpha) s} ds$ is divergent in the above first right-side estimation. Hence, $G$ cannot be improved to be Lipschitzian.

\[\square\]

4.3 Proof of Theorem 2.3

Proof. The goal in this part is to claim that $H$ and $G$ are Hölder continuous based on Bellman inequality.

Step 2-1. From the points of our mechanism, we say that the linearising map is merely Hölder continuous based on the Bellman inequality in Lemma 3.10. Without loss of generality, assume that $|\xi - \bar{\xi}| + |\eta - \bar{\eta}| < 1$. Set $\tau_1 = \frac{1}{\omega_c + M e^{|f|_{Lip}}} \cdot \ln \frac{1}{|\xi - \bar{\xi}| + |\eta - \bar{\eta}|}$, we see that

$$h(\xi, \eta) - h(\bar{\xi}, \bar{\eta})$$

$$= \int_\mathbb{R} G_A(-s)[f(U_1(s, 0, \xi, \bar{\eta}), U_2(s, 0, \xi, \bar{\eta})) - f(U_1(s, 0, \xi, \eta), U_2(s, 0, \xi, \eta))] ds$$

$$= 2 \int_0^{\tau_1} G_A(-s)[f(U_1(s, 0, \xi, \bar{\eta}), U_2(s, 0, \xi, \bar{\eta})) - f(U_1(s, 0, \xi, \eta), U_2(s, 0, \xi, \eta))] ds$$

$$+ 2 \int_{\tau_1}^{\infty} G_A(-s)[f(U_1(s, 0, \xi, \bar{\eta}), U_2(s, 0, \xi, \bar{\eta})) - f(U_1(s, 0, \xi, \eta), U_2(s, 0, \xi, \eta))] ds$$

$$= 2(J_1 + J_2),$$
Note that $\alpha < \omega_c$, we have
\[
|J_2| \leq \int_\tau_1^\infty ke^{-\alpha s} \cdot |f|_\infty ds \leq (2k/\alpha) \cdot |f|_\infty \cdot e^{-\alpha \tau_1}
=(2k/\alpha) \cdot |f|_\infty \cdot (|\xi - \xi| + |\eta - \bar{\eta}|)^\omega_c^{\omega_c/M_c |f|_{Lip}}.
\]
Since $M_c \cdot |f|_{Lip} + \omega_c > \alpha$, and it follows from Lemma 3.10 that
\[
|J_1| \leq \int_0^{\tau_1} ke^{-\alpha s} \cdot |f|_{Lip} \cdot M_c \cdot (|\xi - \xi| + |\eta - \bar{\eta}|) e^{(M_c |f|_{Lip} + \omega_c - \alpha)s} ds
\leq k \cdot |f|_{Lip} \cdot M_c \cdot (|\xi - \xi| + |\eta - \bar{\eta}|) \int_0^{\tau_1} e^{(M_c |f|_{Lip} + \omega_c - \alpha)s} ds
\leq \frac{k \cdot |f|_{Lip} \cdot M_c}{M_c \cdot |f|_{Lip} + \omega_c - \alpha} \cdot (|\xi - \xi| + |\eta - \bar{\eta}|)^{1 - \frac{M_c |f|_{Lip} + \omega_c - \alpha}{\omega_c + M_c |f|_{Lip}}}.
\]
Therefore,
\[
|h(\xi, \eta) - h(\xi, \bar{\eta})| \leq 2(|J_1| + |J_2|)
\leq \left[(4k/\alpha) \cdot |f|_\infty + 2k |f|_{Lip} M_c \frac{M_c |f|_{Lip} + \omega_c - \alpha}{\omega_c + M_c |f|_{Lip}}\right] (|\xi - \xi| + |\eta - \bar{\eta}|)^\omega_c^{\omega_c/M_c |f|_{Lip}}.
\]
Further, it implies that $H$ is Hölder continuous, i.e.,
\[
|H(u) - H(\bar{u})| \leq \tilde{p}_1 \cdot |u - \bar{u}|^q.
\]

**Step 2-2.** We show that $G$ is also Hölder continuous. From Step 1-2, we immediately obtain it. Hence, we complete the proof of Theorem 2.3. 

### 4.4 Proof of Theorem 2.4

**Proof.** We just verify that it satisfies all the conditions of Theorem 2.1. Since $\delta$ is said to be locality radius of a spherical neighborhood $B(0, \delta)$, it is clear to see that $L(\sqrt{2}\delta, \sqrt{2}\delta)$ is a small constant and
\[
|f_\delta|_{Lip} = 9L(\sqrt{2}\delta, \sqrt{2}\delta),
|f_\delta|_\infty = 2\sqrt{2}\delta \cdot L(\sqrt{2}\delta, \sqrt{2}\delta) < \infty.
\]
In addition, from the other conditions of Theorem 2.4 we can obtain its local linearization in $B(0, \delta)$.
5 Applications

We consider the coupled *parabolic* equation and *ordinary* differential equation:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + f(u, v), \quad (0 < x < \pi), \\
\frac{dv}{dt} &= Bv, \\
u(0, t) &= 0, \quad u(\pi, t) = 0, \\
u(x, 0) &= u_0, \quad v(0) = v_0.
\end{align*}
\]  

(5.1)

It is easy to see that \( B \) is the generator of \( C_0 \)-group \( e^{Bt} \) on Banach space \( X \), and

\[ D(B) = \{ v_0 \in X \mid \text{there exists a constant } h \text{ such that } \lim_{h \to 0} \frac{e^{Bh}v_0 - v_0}{h} \text{ holds} \}. \]

Define the linear operator \( A \) by

\[ A\psi(x) := -\frac{\partial^2 \psi(x)}{\partial x^2}, \quad 0 < x < \pi, \]

where \( \psi \) is a smooth function on \([0, \pi]\) with \( \psi(0) = 0, \psi(\pi) = 0 \). Using Friedrichs theorem in [21], \( A \) can be extended to a self-adjoint densely defined linear operator in \( L^2(0, \pi) \). In this cases,

\[ D(A) = \{ \psi \in L^2(0, \pi) \mid A\psi \in L^2(0, \pi) = H^1_0(0, \pi) \cap H^2(0, \pi) \}, \]

and the spectrum \( \sigma(A) \) of \( A \) consists only of simple eigenvalues \( \lambda_n = n^2, (n = 1, 2, \cdots) \) with corresponding eigenfunctions \( \psi_n(x) = (2/\pi)^{\frac{1}{2}} \sin nx \). Then the coupled system can be rewritten as a differential equation on Banach space:

\[ \partial_t u = -Au + f(u, e^{Bt}v_0). \]

(5.2)

Now, we give a version of linearization theorem for equations (5.1).

**Theorem 5.1.** Assume that \( \partial_t u = -Au \) admits an exponential dichotomy. If the nonlinearity \( f \) is bounded and Lipschitzian, and \( |f|_{\text{Lip}} < 1 \). Then equations (5.1) is topologically conjugated to its linear parts.

**Proof.** We just verify that this theorem satisfies all of the conditions for Theorem [2.1]. It is known that \( f \) is bounded and Lipschitzian. It is easy to see that equation (5.2) has a global
existence and uniqueness result, i.e.,
\[ u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)} \cdot f(u(s), e^{B_s}v_0) ds, \] (5.3)

which comes from the variation of constants formula. Since the spectrum \( \sigma(A) \) consists of simple eigenvalues \( \lambda_n \), we clear that \( A \) admits a dichotomy projection (in fact, we know that it is a trivial dichotomy). Without loss of generality, taking \( 4k\alpha^{-1} = |\max\{-\lambda_n\}| = 1 \). Hence, condition (2.2) can be rewritten as: \( |\max\{-\lambda_n\}| \cdot |f|_{Lip} = |f|_{Lip} < 1 \). Therefore, all the conditions of Theorem 2.1 are satisfied. We say that equations (5.1) is topologically conjugated to its linear equations.

Example. Consider the Hodekin-Huxley equations for the nerve axon:

\[
\begin{align*}
C \frac{\partial V}{\partial t} &= \frac{1}{r_e + r_i} \frac{\partial^2 V}{\partial x^2} - g_k n^4 (V - E_k) - g_{Na} m^3 h (V - E_{Na}), \\
\frac{\partial n}{\partial t} &= \alpha_n (v)(1 - n) - \beta_n (v)n, \\
\frac{\partial m}{\partial t} &= \alpha_m (v)(1 - m) - \beta_m (v)m, \\
\frac{\partial h}{\partial t} &= \alpha_h (v)(1 - h) - \beta_h (v)h,
\end{align*}
\] (5.4)

where \( V \) denote the electrical potential and \( C \) denote the membrane capacitance. The quantities \( m, n, h \), which vary between 0 and 1, and describe the changes in the conductance of the axon membrane for sodium (\( Na \)) and potassium (\( K \)). One can see Cole [11] for more details. Here we take \( \alpha_n, \beta_n, \) etc. are all positive constants. Therefore, equations (5.4) is a coupled equations with a parabolic equation and ODEs, it can be written as:

\[
\begin{align*}
\frac{\partial V}{\partial t} &= -AV - g_k n^4 (V - E_k) - g_{Na} m^3 h (V - E_{Na}), \\
\frac{\partial n}{\partial t} &= \gamma_n n - \alpha_n, \\
\frac{\partial m}{\partial t} &= \gamma_m m - \alpha_m, \\
\frac{\partial h}{\partial t} &= \gamma_h h - \alpha_h,
\end{align*}
\] (5.5)

where \( -AV = \frac{C^{-1}}{r_e + r_i} \frac{\partial^2 V}{\partial x^2} \), and \( C, g_k, E_k, g_{Na}, E_{Na} \), etc. are constants. Define \( f(V, n, m, h) := -g_k n^4 (V - E_k) - g_{Na} m^3 h (V - E_{Na}) \), since \( m, n, h \in [0, 1] \), we can check that \( f \) is Lipschitzian. Further if \( f \) is bounded. Since \( -A \) has a dichotomy projection, \( \gamma_n, \gamma_m, \gamma_h \) are constants. It follows from Theorem 5.1 that equations (5.5) can be completely linearised.
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