Gauging and Symplectic Blowing Up in Nonlinear Sigma Models: I. Point Singularities

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Abstract

In this paper a two dimensional non-linear sigma model with a general symplectic manifold with isometry as target space is used to study symplectic blowing up of a point singularity on the zero level set of the moment map associated with a quasi-free Hamiltonian action. We discuss in general the relation between symplectic reduction and gauging of the symplectic isometries of the sigma model action. In the case of singular reduction, gauging has the same effect as blowing up the singular point by a small amount. Using the exponential mapping of the underlying metric, we are able to construct symplectic diffeomorphisms needed to glue the blow-up to the global reduced space which is regular, thus providing a transition from one symplectic sigma model to another one free of singularities.

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I. Introduction

A general nonlinear sigma model with some Riemannian target manifold $M$ admits as a global symmetry the isometry group of $M$, which can be gauged [1] by introducing a minimal coupling to a gauge field. The same can be done in the presence of more structures on $M$, e.g., for the supersymmetric nonlinear sigma models [2]-[3]. In addition to its application to SUSY phenomenology, an important use of the sigma model gauging is to construct the quotient space and define on it a reduced nonlinear $\sigma$-model with fewer degrees of freedom (see [4] for a review of various quotient constructions including hyperkähler manifolds). One recent example is the gauging of a WZW model which leads to a quotient space describing the black hole geometry [5].

The nonlinear $\sigma$-model obtained by gauging can be very different from the original one, eventhough it is possible to return to the original number of degrees of freedom by introducing Lagrangian multiplier fields. Especially if the isometry group of the starting nonlinear $\sigma$-model has a nontrivial isotropy subgroup at certain points, the resulting quotient space in general contains singularity points. A less trivial construction [6] (which is related to the duality transformation) reveals further that there are equivalent gaugings in a single model, one of which leads to a singular quotient while another smooth. This raises the interesting question of whether a smooth change of topological structures in spacetime can be realized within a higher symmetric (string) theory [7].

In this paper, we give a detailed construction of gauging a quasi-free group action in a general nonlinear $\sigma$-model on a symplectic manifold. Usually, minimal coupling is not enough for gauging a general $\sigma$-model with a WZ term [8], but in the presence of an almost complex and symplectic structure which we assume, it is still possible to use minimal coupling to the gauge field. The nontrivial point comes when dealing with the quasi-free group action. In that case, the symplectic quotient is not necessarily smooth: it may contain point-like singularities. The singularities are precisely points where the isotropy subgroups are nontrivial. There are many reasons to believe that the physics near the singularity is far from trivial. In fact orbifolds (those singular spaces whose singularities have only discrete isotropy groups) in both quantum mechanics [9] and string theory [10] have been invoked as a major probe of higher (quantum) symmetries of the theories. Although singularities in e.g. stringy orbifolds look harmless due to the existence of winding modes, many theories lack efficient calculability when dealing with singularities. The question then arises whether we are able to make some predictions regarding behaviors of the theory near the singularities by looking at the blow-up of the singular space. Since blowing up is in general a mild operation (it is even a bi-rational transformation)
it may turn out that one obtains equivalent theories by going over to the blow-ups. However, unlike the previous examples of duality transformations in conformal field theory which can be taken as certain global discrete symmetries, at least at the tree level, blowing up operations usually involve some diffeomorphisms preserving the Kähler or symplectic structures on the manifold, and therefore must be a symmetry of only diffeomorphism-invariant theory. To find out this diffeomorphism-invariant theory is of course of utmost interests.

Blowing up a singularity in the complex category is a more or less familiar procedure [11]. Some of its rudiments with application to construct mirror pairs of Calabi-Yau spaces can be found in ref [12]. See also [13] for a use of blowing up orbifold points in constructing a geometric realization of the conjectured equivalence between Landau-Ginsburg and Calabi-Yau descriptions of the string backgrounds. For blowing up in the symplectic category, however, the relevant notions appear only recently [14]-[16]. Let us explain roughly what is involved in this construction. Since locally all symplectic manifolds look alike, symplectic invariants of a typical symplectic manifold are global in nature. Thus when we calculate for example the cohomology classes \([\omega_t] \in H^2(M)\) for a family of symplectic 2-forms parametrized by a real variable \(t\), the result does not vary much as we go along a smooth trajectory in \(M\) with the parameter \(t\). In fact, according to the Duistermaat-Heckman theorem [17], \([\omega_t]\) traces a line in the space \(H^2(M)\) if the trajectory of \(t\) does not cross a singular point. It is especially interesting when we think of the symplectic manifolds parametrized by \(t\) as a sequence of Marsden-Weinstein reduced symplectic spaces, then \(t\) is provided by (regular) values of the moment map sitting in a region in the dual space of the Lie algebra of the Hamiltonian vectors. If the corresponding group is abelian, e.g., a torus \(T^n\), by the well known theorem of Atiyah and Guillemin-Sternberg [18], the image of the symplectic manifold under the \(T^n\)-equivariant moment map is a convex body (polytope for example). In this case the regular values of the moment map at which the Marsden-Weinstein reduction is performed lie all in the interior region. When one tries to pass over the borderline of two separate subpolytopes the reduced space becomes singular. Thus a sequence of M-W reductions can be used to probe the local structures near the singular points. Now the M-W reduction for the local model of a symplectic manifold is extremely simple, in fact even a single generating symplectic space for a variety of reduced spaces can be constructed [15] which is much like the generalization of differential topological cobordism. Locally a symplectic blowing up contains nothing more than the same operation in complex analytic terms, i.e., replacing the linear space by a projective space (or more precisely a line bundle over the projective space made up by incidence relations). Globally, however, several symplectic diffeomorphisms are needed to glue the blow-ups smoothly back to the complement of the singular points.
of the original symplectic manifold. The resulting smooth manifold possesses a well defined reduced symplectic form, making it a genuine symplectic manifold which may differ from the original one by change of topological properties, for example.

The main concern of this work is to provide a gauged nonlinear sigma model version of the above constructions. In a subsequent paper, we will generalize this construction to the case of blowing up along a singular submanifold. The following is a brief outline of this paper. In section II, we collect the basic ingredients of nonlinear sigma model with a general symplectic target manifold $M$, suitable for gauging of the isometries preserving symplectic structure. In addition to the standard action, we also include a linear sigma model action in terms of normal coordinates centered at the specific point on the manifold which is a fixed point of a Hamiltonian group action of quasi-free type. We then discuss in section III in general terms the gauging of symplectic isometries leading to the symplectic quotient. This involves spelling out explicitly the moment map constraint in terms of a suitable coordinate system defined by solutions of the Pfaff equation associated to an integrable distribution. Gauging thus can be implemented as in the usual topological quotient construction on the zero level set of the moment map. In section IV, we deal with the quasi-free $U(1)$ action on the linear sigma model whose fields take values in a small neighborhood of the singular point which is diffeomorphic to $C^n$, we show the exact relationship between blowing up the singular point and the gauging of the corresponding group action. In section V, we discuss the relation of this linear model to the nonlinear one, emphasizing the importance of symplectic diffeomorphisms. We use the normal coordinate expansion method to derive a (weakly) coupled form of the total action, and argue that the decoupling is implemented by the one loop effective action of the original nonlinear sigma model. We also discuss how to compare gauge fields arising from gauging both linear and nonlinear $\sigma$-models. Section VI contains two examples of the application of our general results. Section VII summarizes our conclusions. An appendix contains a proof of the existence of the change of coordinates used in section III.

II. The Lagrangian

Our starting point is a nonlinear $\sigma$-model, with an arbitrary almost complex, symplectic target manifold $M$, over a Riemann surface $\Sigma$. Thus one introduces a complex structure on the tangent space of $M$ (dim $M=m$) with tensor field $F^i_j$ satisfying

$$F^i_k F^k_j = -\delta^i_j, \quad i,j = 1, \ldots, m. \quad (2.1)$$
This gives an almost complex structure to \( M \). A symplectic structure on \( M \) is provided if there exists a (globally defined) closed 2-form \( \omega \):

\[
d\omega = 0,
\]

\[
\omega = F_{ij} dx^i \wedge dx^j,
\]

\[ F_{ij} = G_{ik} F^k_{j}. \]  

(2.2)

where \( G_{ij} \) is a Riemannian metric on \( M \) compatible with the almost complex structure \( F^i_j \), i.e.

\[ G_{ij} = F^k_i F^l_j G_{kl}. \]  

(2.3)

The last property is equivalent to the antisymmetry of \( F_{ij} \), \( F_{ij} = -F_{ji} \). (A calibrated almost complex structure.) Let \( \epsilon^a_\beta \) be the natural complex structure on the Riemann surface \( \Sigma \), and \( \epsilon_{\mu \nu} = h_{\mu \alpha} \epsilon^\alpha_\nu \), \( \epsilon_{\mu \nu} = -\epsilon_{\nu \mu} \) be the antisymmetric rank-2 tensor on \( \Sigma \) with metric \( h_{\mu \nu} \). We adopt the notation where the integral of the 2-form \( \omega \) is expressed in terms of the \( \sigma \)-model scalar fields, i.e.

\[
\int F_{ij} dx^i \wedge dx^j = \int d^2 \sigma \epsilon^{\mu \nu} F_{ij} \partial_\mu x^i \partial_\nu x^j.
\]  

(2.4)

Let the manifold \( M \) be parametrized by \( m \) coordinate scalars \( \Phi^i \). Our general symplectic nonlinear \( \sigma \)-model has the following action (here, as usual, we have suppressed the 2d metric \( h_{\mu \nu} \), in order to avoid notational complexity):

\[
S = \frac{1}{2} \int d^2 \sigma G_{ij} \partial^\mu \Phi^i \partial_\mu \Phi^j + \frac{1}{2} \int d^2 \sigma \epsilon^{\mu \nu} F_{ij} \partial_\mu \Phi^i \partial_\nu \Phi^j,
\]  

(2.5)

with \( F_{ij} \) satisfying eq(2.2). Since the second term in eq(2.5) is topological the invariance of the action (2.5) is the same as that of the first term, i.e. the whole isometry group of \( M \) (as a riemannian manifold). However, when gauging is concerned, not all isometries can be gauged, but only those preserving the symplectic form (2.4). The condition for an isometry to preserve the symplectic form is familiar,

\[
\mathcal{L}_\xi \omega = di(\xi)\omega = 0,
\]  

(2.6)

where \( \xi^a_\alpha \) is a Killing vector generating the isometry (global) transformation

\[
\delta \Phi^i = \lambda^a \xi^i_\alpha(\Phi).
\]  

(2.7)

\( \xi^i_\alpha \) being a Killing vector means that the following Killing’s equation is satisfied

\[
\nabla_{(i} \xi_{j)a} = 0,
\]

(2.8)
where $\nabla_i$ denotes the covariant derivative with respect to the unique Riemannian connection $\Gamma^i_{jk}$ compatible with the metric $G_{ij}$ on $M$.

In the most general situation, we will assume that the Killing vectors generate a group $K$,

$$[\xi_a, \xi_b] = \mathcal{L}_a \xi_b = f_{ab}^c \xi_c$$

(2.9)

with $f_{ab}^c$ the structure constants of $K$.

Isometries fulfilling conditions (2.6) and (2.8) will be called symplectic isometries. We will consider a subgroup $H \subset K$ of the symplectic isometries which leave one point $x$ in $M$ fixed. The isotropy group at $x, K_x$ is the whole of $H$ and we will gauge the subgroup $H$ by minimal coupling to some gauge fields. In section III, we will argue that minimal coupling is appropriate in this situation and also describe in some details the main points which arise in obtaining the symplectic quotient by gauging the symplectic isometries.

In general, an isometry acts on $\Phi^i$ nonlinearly, while at each point of the manifold $M$, local coordinates can be introduced such that the isoptropy group at that point acts linearly (i.e, as in flat case). This is related to the fact that there always exists a normal coordinate neighborhood of a given point which is diffeomorphic to a small neighborhood of the origin in the tangent space at that point [19]. The diffeomorphism in question is the exponential mapping defined by the underlying Riemannian structure (metric) of $M$. We will say more on this diffeomorphism in section V. Here we content ourselves by choosing a normal coordinate system $\{\phi^i, i = 1, \ldots, m\}$ in an $\epsilon$-small neighborhood $U_\epsilon$ of $x \in M$, in terms of which we have a free, linear $\sigma$-model in addition to the one given by (2.5),

$$S_0 = \frac{1}{2} \int d^2x \partial_\mu \phi^i \partial_\mu \phi^j.$$

(2.10)

We remark that, when treated as high energy modes of our nonlinear model this linear $\sigma$-model action can be conveniently integrated out resulting in some world sheet effect which might in the present context be completely ignored. However, we will not pursue this line of reasoning in this paper, and instead treat this linear model completely on equal footing. This is in fact a basic practice in symplectic dynamics, and we will directly deal with this linear model when performing symplectic blowing up in section IV.
III. Symplectic quotient and gauging

In this section, we describe in some details the gauged $\sigma$-model realization of the regular symplectic reduction, which serves as the reduced space outside the singular locus. The methods are quite general and may have their own applications.

According to the general philosophy of $\sigma$-model isometry gauging, the global invariance of eq(2.5) and eq(2.7) can be promoted to a local invariance (with $\lambda^a(x)$ arbitrary functions of $x$) by introducing a gauge field $A^a_\mu$ transforming as

$$\delta A^a_\mu = \partial_\mu \lambda^a + f^a_{bc} A^b_\mu \lambda^c.$$  \hspace{1cm} (3.1)

Under the transformation (3.1) and (2.7), the gauge covariant derivative defined by

$$D_\mu \Phi^i = \partial_\mu \Phi^i - A^a_\mu \xi^i_a(\Phi)$$  \hspace{1cm} (3.2)

transforms like a tangent vector,

$$\delta D_\mu \Phi^i = \lambda^a \partial \xi^i_a / \partial \Phi^k D_\mu \Phi^k.$$  \hspace{1cm} (3.3)

This, together with eq(2.8) and the closedness of the symplectic 2-form, guarantee the local gauge invariance of the minimally coupled, gauged action

$$S_{gauged} = \frac{1}{2} \int d^2x G_{ij} D^\mu \Phi^i D_\mu \Phi^j + \frac{1}{2} \int d^2x \epsilon^{\mu \nu} F_{ij} D_\mu \Phi^i D_\nu \Phi^j.$$  \hspace{1cm} (3.4)

(Note that after the substitution of covariant derivatives, the second term in (3.4) is no longer topological, but the reduced form after integrating out the gauge fields will still be topological.) As usual, we do not include kinetic terms for the gauge fields, so that the gauged model eq(3.4), after eliminating gauge fields, describes a nonlinear $\sigma$-model on the space of orbits of the group action on $M$. This is the usual geometric interpretation of gauging a (Riemannian) isometry.

Gauging of symplectic isometries, leading to a symplectic quotient, has to meet additional constraints, i.e. eq(2.6). Locally (2.6) implies the existence of a function $\mu$ such that

$$\omega_{ij} \xi^j_a = \partial_i \mu_a.$$  \hspace{1cm} (3.5)

Globally, the solution to eq(3.5) may not always exist. But under certain assumptions, such as triviality of the first cohomology group of $h(= \text{Lie}(H))$, global functions $\mu_a$ exist and are unique up to central elements of $h$ [20]. These $\mu_a$ fit together to form the moment map

$$\mu : M \rightarrow h^*.$$  \hspace{1cm} (3.6)
The symplectic quotient is the usual topological quotient of the subspace \( N = \mu^{-1}(0) \) by the group \( H \). In our case, we have to implement the constraint \( \mu(\Phi) = 0 \) to the gauged action so that it really describes \( N = \mu^{-1}(0) \subset M \).

Explicitly solving the moment map constraints is in general a difficult task, especially when the hamiltonian group action is nonlinearly realized. However, when the condition on cohomology groups of the Lie algebra, e.g., \( H^1(h) = H^2(h) = 0 \) is satisfied, one can convince oneself quickly that components of the moment map can be found in terms of Killing vectors. Indeed, since \( \xi_a \in h \) preserve the symplectic form \( \omega = F_{ij} d\Phi_i \wedge d\Phi_j \),

\[
\mathcal{L}_a(i(\xi_b)\omega) = i(\mathcal{L}_a \xi_b)\omega + i(\xi_b)\mathcal{L}_a \omega
= i([\xi_a, \xi_b])\omega = f_{ab}^{c} i(\xi_c)\omega.
\tag{3.7}
\]

Thus the moment map components corresponding to Killing vectors \( \xi_a \) can be written as

\[
\mu_a = (f_{bc}^{a})^{-1} i(\xi_b)i(\xi_c)\omega = (f_{bc}^{a})^{-1} \xi_b^c \xi_j \omega_{ij}.
\tag{3.8}
\]

Note that \( \mu_a \) in (3.8) is well defined if the Lie algebra is semi-simple, that is if \( f \) is invertible. So we have in principle \( \dim H \) relations expressed by the vanishing of the components of the moment map (3.8). Although reasonable, it is in practice impossible to impose these constraints directly into the Lagrangian, because of lack knowledge of the explicit dependence of the Killing vectors on the coordinate fields \( \Phi^i \). We will devote the following paragraphs to construct a different way to impose the moment map constraints directly into the Lagrangian.

Let us denote by \( N \) the constraint set \( \mu^{-1}(0) \) in \( M \), i.e. the set of points in \( M \) which are mapped into the same point \( 0 \in h^* \). To describe the submanifold \( N = \mu^{-1}(0) \), first observe that because of the equivariance of the moment map \( \mu \),

\[
\mu(g \cdot x) = g \cdot \mu(x) \in \mathcal{O} \subset h^*,
\tag{3.9}
\]

for any point \( x \in M \). Here \( \mathcal{O} \) is an orbit in \( h^* \) by the coadjoint action. If \( x \) lies in \( N \), we must have \( \mu(g \cdot x) = g \cdot \mu(x) = 0 \) for all \( x \in N \), since the \( H \) action certainly leaves \( 0 \in h^* \) fixed. Consider the tangent space to the orbit \( \mathcal{O} \) at any point \( a \in \mathcal{O}, T_a \mathcal{O} \), the lift to the tangent space \( T_x M \) of the moment map \( d\mu_x : T_x M \to h^* \) is defined as the transpose of the linear map \( h \to T_x M \) (with \( T_x M \) identified with \( T_x^* M \) via the symplectic structure), sending each element \( \xi \in h \) into the Hamiltonian (tangent) vectors \( \xi_M \) on \( M \). If we set

\[
T_x N' = d\mu_x^{-1}(T_{\mu(x)=a} \mathcal{O}),
\tag{3.10}
\]

...
as a submanifold of $T_xM$, it must be spanned by the basis of $T_a\mathcal{O}$ and those of $Kerd\mu_x$. Now $Kerd\mu_x$ can be taken as a definition for a vector $Y$ to be an element of $T_xN$

$$d\mu_x(Y) = 0 \quad \forall Y \in T_xN. \quad (3.11)$$

Thus the basis vectors in $T_a\mathcal{O}$ are in one to one correspondence with the vectors lying in the orthogonal complement of $T_yN$ in $T_xM$. Obviously, the complement of the tangent vectors coming from $h$ in $T_xM$ is contained in $T_xN$.

Now the condition for a vector $Y$ to be tangent to $N$ at $x$ can be written as

$$i(\xi)\omega(Y) = d\mu(Y) = 0, \quad \forall \xi \in h. \quad (3.12)$$

Because $\xi$ is a symplectic vector field, i.e. $di(\xi)\omega = 0$, the above equation (3.12) defines an integrable distribution or, in slightly different terms, a foliation [20]. The equivariance of the moment map guarantees that this foliation is in fact fibrating, i.e., in a certain coordinate system, the solution (integration) of the equation (3.12) defines a submanifold $N$ in $M$. The desired coordinate system is provided by the Frobenius theorem [20], which says that there exists a coordinate system $\{w^i, i = 1, \ldots, m\}$ in the neighborhood $W$ of each point in $M$, so that the leaves of the foliation are given by

$$w^1 = \text{const.}, \ldots, w^r = \text{const.}, \quad r = m - \dim H = \dim N, \quad (3.13)$$

and that the tangent space at that point to the foliation is spanned by

$$\frac{\partial}{\partial w^{r+1}}, \ldots, \frac{\partial}{\partial w^m}. \quad (3.14)$$

In this case the (co)tangent space of the submanifold $N \subset M$ is the null subspace of $T^*_mM$ with respect to the distribution. In the coordinate system $\{w^i\}$ the symplectic form is $\omega' = \sum \omega'_{ij}dw^i \wedge dw^j$, with the functions $\omega'_{ij}$ satisfying

$$\omega'_{ij} = 0, \quad i, j = r + 1, \ldots, m, \quad (3.15)$$

$$\frac{\partial \omega'_{ij}}{\partial w^a} = 0, \quad a = r + 1, \ldots, m. \quad (3.16)$$

The first is due to the condition (3.12), while the second is due to $d\omega = 0$. If the foliation is fibrating (which is the case here), then $w^s, s = 1, \ldots, r$ can be chosen as the coordinates of $N = \mu^{-1}(0)$ on which there exists a closed 2-form $\omega^*$ with $f^*\omega' = \omega^*$ where $f: (w^1, \ldots w^m) \rightarrow (w^1, \ldots w^r)$ is a submersion. Now, the question boils down to find the system of coordinates dictated by the Frobenius theorem.
In the above discussion, we have used the fact that the set of Hamiltonian vectors \( \xi_a \) are identified with (smooth) tangent vectors to \( M \), with respect to an arbitrary coordinate system. In order to apply the Frobenius theorem, \( \xi_a \) have to be converted into the form (3.14) by a suitable change of coordinates. Note that in an arbitrary coordinate system \( (U; u^i) \), \( \xi_a \) looks as

\[
\xi_a|_U = \sum \xi^i_a \frac{\partial}{\partial u^i}.
\]  
(3.17)

Comparing with (3.14), we see that we must find a change of coordinates which smears out a sufficient number of components of \( \xi_a \). In fact, if there exists a change of coordinates, \( u^i \rightarrow v^i \), such that, in the coordinate system \( (V; v^i) \), \( \xi_a \) looks like

\[
\xi_a|_V = \xi^1_a(v) \frac{\partial}{\partial v^1},
\]
(3.18)

for each \( a \), then, by defining \( w^1 = \int_0^{v^1} dv^1/\xi^1, \ w^\alpha = v^\alpha, \alpha = 2, \ldots, m \), we have

\[
\xi_a|_W = \frac{\partial}{\partial w^1}
\]
(3.19)

for each \( a \). Continuing this step for every \( \xi_a, a = r + 1, \ldots, m \), we thus have provided the desired system of coordinates \( (W; w^i) \) in terms of which the Hamiltonian vectors \( \xi_a \) are converted into the form (3.14). In Appendix A, we prove that there indeed exists such a change of coordinates with the above properties. Now in the new coordinate system, we are able to write down the gauged action directly in terms of coordinates \( w^i \) on \( N = \mu^{-1}(0) \). Using equations (3.15), (3.16) together with

\[
F'_{ij} = G'_{ik} F^k_j,
\]
(3.20)

it is straightforward to verify that the resulting gauged action, in the new coordinate system, takes the same form as in (3.4) except now the indices \( i, j \) run from 1, \ldots, \( \dim N \), and the functions \( G_{ij}, F_{ij} \) become

\[
G'_{ij} = G_{ij} \frac{\partial w^\alpha(0)}{\partial w^i(a)} \frac{\partial w^\beta(0)}{\partial w^j(a)} = G_{ij} \frac{\partial w^\alpha(0)}{\partial w^i(1)} \frac{\partial w^\beta(1)}{\partial w^j(1)} \cdots \frac{\partial w^\alpha(a-1)}{\partial w^i(a-1)} \times \frac{\partial w^\beta(a-1)}{\partial w^j(a-1)} \times \frac{\partial w^\beta(1)}{\partial w^i(2)} \cdots \frac{\partial w^\beta(a-1)}{\partial w^j(2)} \cdots \frac{\partial w^\beta(a)}{\partial w^j(a)},
\]
(3.21)

\[
F'_{ij} = -G'_{i,j+\frac{m}{2}}, \ j \leq m/2, \quad F'_{ij} = G'_{i,j-\frac{m}{2}}, \ j > m/2,
\]
(3.22)
where the partial derivatives are given by the relations
\[
\frac{\partial w^s_{(s-1)}}{\partial w^s_{(s)}} = \xi^s_{(s)}, \quad \frac{\partial w^\alpha_{(s)}}{\partial w^s_{(s)}} = \xi^\alpha_{(s)}, \quad \frac{\partial w^s_{(s-1)}}{\partial w^\beta_{(s)}} = \delta^\alpha_{\beta},
\]
\[s = 1, \ldots, \dim H; \quad \alpha, \beta = s + 1, \ldots, m. \tag{3.23}\]

We remark that after the coordinate transformation, the Killing vectors are now represented by a set of null vectors in \(\dim H\) directions on the submanifold \(N^\circ\):
\[
\delta \Phi^s_{(a)} = \lambda^s, \quad s = 1, \ldots, \dim H. \tag{3.24}\]

In the case of an abelian group \(U(1)\), the above formulae simplify to
\[
G'_{11} = G_{11}(\xi^1)^2 + G_{1a}\xi^1\xi^a + G_{ab}\xi^a\xi^b,
\]
\[
G'_{a1} = G_{a1}\xi^1 + G_{ab}\xi^b,
\]
\[
G'_{ab} = G_{ab}, \quad a, b \neq 1, 1 + m/2. \tag{3.25}\]

and similarly for \(F'_{ij}(F'_{aa} = 0)\).

If we denote
\[
(\xi_m, \xi_n) = G_{ij}\xi_m^i\xi_n^j = G_{mn},
\]
\[
(\xi_m, \xi_n) = F_{ij}\xi_m^i\xi_n^j = F_{mn},
\]
\[
\xi_{ma} = G_{ia}\xi_m^i, \quad \xi_{ma} = F_{ia}\xi_m^i. \tag{3.26}\]

where \(m, n = 1, \ldots, \dim H\), and \(a, b = 1, \ldots, m - 2\dim H\), the general gauged action takes the form
\[
S_{gauged} = \frac{1}{2} \int d^2\sigma [(\xi_m, \xi_n)D_\mu \Phi^m D_\nu \Phi^n + \xi_{ma}D_\mu \Phi^m \partial_\nu \Phi^a]
\]
\[
+ \frac{1}{2} \int d^2\sigma \epsilon^{\mu\nu}(\xi_m, \xi_n)D_\mu \Phi^m D_\nu \Phi^n + \epsilon^{\mu\nu}\xi_{ma}D_\mu \Phi^m \partial_\nu \Phi^a]
\]
\[
\quad + S[G_{ab}, F_{ab}],
\]
\[
D_\mu \Phi^m = \partial_\mu \Phi^m - A^m_\mu. \tag{3.27}\]

Solving the equation of motion for the nondynamical gauge fields
\[
[\eta^{\mu\nu}(\xi_m, \xi_n) + \epsilon^{\mu\nu}(\xi_m, \xi_n)]A^n_\nu = \eta^{\mu\nu}(\xi_m, \xi_n)\partial_\nu \Phi^n
\]
\[
+ \epsilon^{\mu\nu}(\xi_m, \xi_n)\partial_\nu \Phi^a + \frac{1}{2} \eta^{\mu\nu}\xi_{ma}\partial_\nu \Phi^a + \frac{1}{2} \epsilon^{\mu\nu}\xi_{ma}\partial_\nu \Phi^a, \tag{3.28}\]
and choosing a gauge, leads us back to a symplectic $\sigma$-model defined on the symplectic quotient space. It is easily checked that the solution of (3.28) transforms as a connection of the principal $H$ bundle.

In view of (3.24), what we have done above amounts to converting the general $H$ action of isometries on $M$ into the Abelian torus (taking into account the periodic boundary conditions for $\Phi^m$) action on $N$, with simplified gauge connections. This is important in order to compare with the quotient in the linear case, to be considered in next section, which arises from gauging only the Abelian subgroup of the whole $U(n)$ isometries.

IV. The local model

In a neighborhood of each point on a symplectic manifold, the symplectic form can be brought into the standard form and the Hamiltonian group action is linearly realized. Gauging thus becomes extremely simple. We will consider in this section the consequences of gauging a quasi-free $S^1$ action in a small neighborhood of a point (a singular point of the group action, to be precise) where we have a linear $\sigma$-model given by (2.10). The origin of this linear $\sigma$-model may be looked at from several viewpoints, physically, it can be regarded as the result of performing a normal coordinate expansion around a certain classical configuration, see Section 5 for more. We only consider the abelian $U(1)$ action, since this case is typical and sufficient to demonstrate the essence of symplectic blowing up.

An almost complex, symplectic manifold has on its tangent space at a point a natural complex structure. Thus we can use complex coordinates, $z^i = 1/\sqrt{2}(x^i + iy^i)$, $\bar{z}^i = 1/\sqrt{2}(x^i - iy^i)$, where $x, y$ are real fields and the index runs now $i = 1, ..., n = m/2$. In terms of the complex fields $z^i, \bar{z}^i$, our linear $\sigma$-model action takes the form

$$L = \sum_{i=1}^{n} \int d^2\sigma \partial_\mu z^i \partial^\mu \bar{z}^i = \sum \int d^2\sigma \partial_+ z^i \partial_- \bar{z}^i.$$  \hspace{1cm} (4.1)

The complex fields $z^i, \bar{z}^i$ span a linear space $C^m$ on which $U(1)$ acts linearly and analytically, preserving the symplectic structure

$$\Omega = -i\partial\bar{\partial}|z|^2 = -i \sum dz^i \wedge d\bar{z}^i.$$  \hspace{1cm} (4.2)

(Actually, the maximal analytic invariance of (4.2) is $U(n)$.) Our assumption of the quasi-free action implies in this case the existence of an isolated singularity. Without loss of generality, we can take the global $U(1)$ action acting diagonally on
$z^i$, as follows
\[ z^1 \rightarrow e^{-i\theta} z^1, \quad z^i \rightarrow e^{i\theta} z^i, \quad i = 2, \ldots, n, \]  
with $\theta \in \mathbb{R}$ constant.

The moment map associated to this action is
\[ \mu(z) = -|z^1|^2 + \sum_{i=2}^{n} |z^i|^2. \]  
(4.4)

It is clear that $\mu^{-1}(0)$ contains an isolated critical point at $0 \in C^n$. Because of this singularity, pathology is to be envisaged when one tries to perform the symplectic reduction. However, a genuine blow up makes sense and relates the strata of the singular level set to some Marsden-Weinstein reduced spaces.

First we need to recall some facts concerning the mathematical notion of blowing up\footnote{The following few paragraphs are probably well known to experts. The reader can find most of the mathematical statements in ref[15][16], or consult [22] for a similar discussion including some physical motivations}. Suppose the origin 0 in $C^{n+1}$ is a singular point, to be blown up. Due to the well known relation in complex geometry between $C^{n+1} - \{0\}$ and $CP^n$ \cite{21}, there is a line bundle $L$ over $CP^n$ whose fibers are copies of $C^* = C - \{0\}$. The blowing up of $C^{n+1}$ at the origin amounts to a map which sends the complement of the zero section of $L$ bijectively onto the complement of the origin in $C^{n+1}$, and sends the whole zero section to the complex projective space $CP^n$. Then $L$ is called the blow up of $C^{n+1}$ at origin.

In the situation of (4.4), if we slightly perturb the value of the moment map by $\pm \epsilon$ with $\epsilon > 0$ sufficiently small, and consider the Marsden-Weinstein reduced spaces at both $+\epsilon$ and $-\epsilon$, we obtain two different reduced spaces. On $-\epsilon$, it is a copy of $C^{n-1}$ through identification of the global cross section of the $U(1)$ action. On $+\epsilon$, it is the line bundle $L$ over $CP^{n-2}$. In fact, for $\mu = +\epsilon$, the level set of the moment map is
\[ -|z^1|^2 + \sum_{i=2}^{n} |z^i|^2 = \epsilon. \]  
(4.5)

It becomes $S^{2n-3} \times C$ after a change of coordinates
\[ w^1 = z^1, \quad w^i = (\epsilon + |z^1|^2)^{-1/2} z^i. \]  
(4.6)

The $S^1$ action in terms of the new coordinates $w^1, w^i$ sends $w^1$ into $e^{-i\theta'} w^1$ and $w^i$ into $e^{i\theta'} w^i$, and the quotient is exactly the line bundle $L$.

Let us see what this whole procedure of blowing up amounts to in terms of the $\sigma$-model gauging. The action (4.1) is obviously invariant under the transformations
However, as its zero level set is singular, directly imposing local invariance and gauging will not be a good attitude. Let us therefore perform the desingularization first. As observed before, this lifts the zero level an $\epsilon$ amount. We will consider only the $+\epsilon$ direction since the other case is less illustrative.

Looking back at the Lagrangian, note that under the change of coordinates (4.6), the standard symplectic form on $C^n$, (4.2), becomes

$$-i\sum_{i=1}^{n} dz^i \wedge d\bar{z}^i = -i[ dw^1 \wedge d\bar{w}^1 + (|w^1|^2 + \epsilon) \sum_{i=2}^{n} dw^i \wedge d\bar{w}^i + d(|w^1|^2) \wedge \alpha]$$

$$= -i[ dw^1 \wedge d\bar{w}^1 + \epsilon \pi^*\Omega_{F-S} + d(|w^1|^2 \alpha)]$$

(4.7)

where

$$\alpha = \frac{1}{2} \sum_{i=2}^{n} (w^i d\bar{w}^i - \bar{w}^i dw^i)$$

(4.8)

is the $U(n)$ invariant connection 1-form on $S^{2n-3}$ regarded as a circle bundle over $CP^{n-2}$, and we have denoted its curvature by $d\alpha = \pi^*\Omega_{F-S}$, the pull back via the blowing up map $\pi: L \rightarrow CP^{n-2}$, of the Fubini-Study 2-form on $CP^{n-2}$. Similarly, the action (4.1) transforms into

$$L = 2 \int d^2\sigma \partial_+ w^1 \partial_- \bar{w}^1 + \sum_{i=2}^{n} \int d^2\sigma (\epsilon + |w^1|^2) \partial_+ w^i \partial_- \bar{w}^i$$

$$+ \frac{1}{2} \sum_{i=2}^{n} \int d^2\sigma (\bar{w}^i \partial_+ w^i \partial_- |w^1|^2 + \partial_+ |w^1|^2 w^i \partial_- \bar{w}^i),$$

(4.9)

here we have used the fact that in the new coordinates, the moment map constraint (4.5) becomes $\sum_{i=2}^{n} |w^i|^2 = 1$. Upon using the equations of motion, this can be rewritten as

$$L = 2 \int d^2\sigma \partial_+ w^1 \partial_- \bar{w}^1 + \epsilon \sum_{i=2}^{n} \int d^2\sigma \partial_+ w^i \partial_- \bar{w}^i$$

$$+ \text{surface term.}$$

(4.10)

The surface term is in complete analogy with the last term of the transformed symplectic form (4.7). We will assume here that this surface term can be dropped. However, we should like to mention one interesting situation when this surface term can not be dropped and plays a prominent role in Floer's study of symplectic diffeomorphism and holomorphic curves [23].

If we drop the surface term in (4.10), then the Lagrangian becomes completely decoupled between $w^1$ and $w^i$, with $w^i$ obeying the constraint $\sum |w^i|^2 = 1$, i.e.,
constrained on a sphere $S^{2n-3}$. Since the $S^1$ action on $w^i$ is free, we can apply the usual steps of gauging the linear isometry [4], i.e., first promoting to local $U(1)$ transformation $\theta \rightarrow \theta(x)$, then substituting the minimal coupling of a gauge field

$$\partial_\pm \rightarrow \partial_\pm + i A_\pm.$$  

(4.11)

Solving the equation of motion for the gauge field $A_\pm$ in terms of the 1-form constructed from $w^i$:

$$A_\pm = \frac{i}{2} \sum (\bar{w}^i \partial_\pm w^i - w^i \partial_\pm \bar{w}^i) = -\frac{i}{2} \alpha,$$  

(4.12)

and after some trivial manipulations, the final form of the Lagrangian is

$$L = 2 \int d^2x \partial_+ w^1 \partial_- \bar{w}^1 + \epsilon \int d^2x \{ \frac{1}{(1 + |w|^2)^2} w^i \partial_+ \bar{w} \partial_- w - \frac{\delta_{ij}}{(1 + |w|^2)^2} \partial_+ w^i \partial_- w^j \}, \quad i, j = 1, ..., n-2,$$  

(4.13)

where $|w|^2 = \sum |w^i|^2$, and $w^i \partial_\pm \bar{w} = \sum w^i \partial_\pm \bar{w}^i$ (here we have used inhomogenous coordinates on $CP^{n-2}$). This is a set of $2(n-2)$ real fields parametrizing $CP^{n-2}$, together with a free, decoupled complex scalar field.

It is interesting to note the remarkable coincidence of the gauge field $A_\pm$ we introduced through minimal coupling and the pre-existing 1-form of the Hopf bundle $S^{2n-3} \rightarrow CP^{n-2}$.

We have chosen to work with a quasi-free $S^1$ action in this section, while in the last section the reduction carried out could be more general, i.e. for any (possibly nonabelain) compact group of isometries. The reason for this is that, when we treat a general quasi-free $G$ action, it is always possible to first desingularize the action, passing over to the nearby regular values of the moment map. Using an observation given in [17] that, for a regular value of the moment map, $J : M \rightarrow g^*$, the reduced symplectic space with respect to $G$ is equal to the corresponding reduced space, with respect to the maximal torus $T$, of a submanifold $M'$ of $M$, and for the $T$-action the regular values all lie in the interior of some convex polytope, we are able to arrange the image set of the moment map so that a one-parameter-subgroup of $T$ is generated by a non-vanishing element of $t$, which points perpendicularly to the wall separating two nearby regions of the polytope of regular values. This $S^1$ is what we are discussing in this section.

It is probably worth mentioning that, generalizing the quasi-free group action of (4.3), to the more general case of $(p,q)$ signature action (i.e., with $p$ minus and $q$ plus signs in front of $\theta$), the result of this section gives rise to a sequence of fibred
manifolds over $CP^{p-1}$ or $CP^{q-1}$ for negative or positive values of $\epsilon$ parameters, respectively. Near $\epsilon = 0$, the desingularized space looks like a disc bundle over $CP^{p-1} \times CP^{q-1}$. Passing from $-\epsilon$ to $+\epsilon$, as explained in [15], thus consists in first blowing up and then blowing down. The phenomenon is what is called symplectic cobordism.

V. The gluing diffeomorphism

According to the last section, symplectic blowing up can be regarded locally as gauging the quasi-free $S^1$ action of a linear $\sigma$-model defined in a small neighborhood of the singular point, which is diffeomorphic to the linear space $C^n, n = \dim M/2$. To actually accomplish the blowing up of the symplectic manifold $M$ by gauging, we have to glue the linear blow-up model back to the complement of the singular point in $M$ by a certain diffeomorphism (preserving the symplectic structure). This is accomplished by using a diffeomorphism on the complement of the singular locus, and the diffeomorphism arising from the normal coordinate expansion at the singular point. We will also discuss some subtleties of gluing of the gauge connections.

Let us first of all make a remark on the role of symplectic diffeomorphisms in the present context. The usual (bosonic) nonlinear sigma model with a (Riemannian, or symplectic) target manifold contains no extra degrees of freedom except those scalar fields parametrizing the respective manifolds. The topological sigma model [23] is one essential example in which extra dynamical degrees of freedom are introduced so that it really describes a theory which possesses a high(est) symmetry and admits, in a sophisticated way, a generalization to a sigma model with a symplectic diffeomorphism. As emphasized by Witten, his theory includes one important situation where one treats a one parameter family of symplectic manifolds related by a symplectic diffeomorphism, and which enables him to relate the global observables of the topological sigma model to the Floer cohomology group. Now, in our case a one parameter family of symplectic manifolds certainly arises in the process of blowing up as in Section 4. These are Marsden-Weinstein reduced spaces with respect to different values of the moment map other than zero. However, as we have mentioned in Section 2, the linear sigma model fields cannot be treated as independent degrees of freedom of the whole symplectic sigma model, rather, it is related to the nonlinear sigma model by a certain diffeomorphism. Thus, the situation in which we are involved here is actually a symplectic nonlinear sigma model together with a diffeomorphism. It is quite plausible that by going over to the symplectic generating space (or a cobordism) for a family of reduced spaces, as described in [15], one can find a diffeomorphism invariant theory which effectively calculates the global transition.
between different geometric and topological structures on both sides of the blown up. Leaving aside this interesting possibility, we restrict ourselves in the present paper only to the case where we treat the symplectic diffeomorphism as gluing data for forming the globally defined blown up sigma model. We will first take a closer look at the diffeomorphism on the complement of the singular point, which is used to glue together the linear symplectic forms, and then use the exponential mapping to discuss some further properties of the symplectic diffeomorphisms.

**Diffeomorphisms on** $L - \text{(zero section)}$

As we have seen in the last section, the blowing up of the origin in the linear space $C^n$ results in a symplectic form $\epsilon \pi^* \Omega_{F-S}$ defined in a small neighborhood of the blown up point, where $\pi$ is (one part of) the blowing up map, $\pi : L \to CP^{n-1}$, the other part of this map is called $\beta : L \to C^n$, which maps the zero section into one point (the origin) in $C^n$. On the complement of the zero section, i.e., on $L - \text{(zero section)}$, the map $\beta$ is a bijection, or in general terms, a $G$ equivariant symplectic diffeomorphism. The role of this diffeomorphism is to pull back the $G$-invariant symplectic form on $C^n - (0)$ to $L$ and glue it to the blown up $\pi^* \Omega$. Let us see how to construct symplectic diffeomorphisms in the linear case.

Remember we have obtained our blown up manifold (a line bundle over $CP^{n-1}$) as a Marsden-Weinstein reduced space with respect to a nonzero value $\epsilon$ of the moment map. The value $\epsilon$ lies in the dual of the Lie algebra of $S^1$, i.e., $R$. Now the moment map $\mu$ of the $S^1$ action does not uniquely correspond to the symplectic 2-form on $C^n - (0)$, but leaves undetermined an arbitrary constant. Indeed [15], let $f(s), s = |z|^2$ be an arbitrary diffeomorphism of $R^+ \subset R$. Viewed as a function of $z$, $g(z) = f(|z|^2)$, an arbitrary $U(n)$ invariant symplectic form on $C^n - (0)$ has the form

$$-i\partial \bar{\partial} g = -i[f''(|z|^2)\bar{z}dz \wedge zd\bar{z} + f'(|z|^2)dz \wedge d\bar{z}] . \quad (5.1)$$

By taking the interior product of this form with an $S^1$-valued vector field, we obtain the moment map in terms of the function $f$,

$$i(\xi)[-i\partial \bar{\partial} g] = d\mu(|z|^2),$$

where

$$\mu'(s) = f'(s) + sf''(s), \quad (5.2)$$

or

$$\mu(s) = sf'(s) + c_1 . \quad (5.2a)$$

The solution of this equation for $f$ in terms of $\mu$ contains therefore as integration constants

$$c_1 + c_2 \log s .$$
This means that any two closed 2-forms with the same moment map differ by a constant multiple of the form

$$\Omega_{F-S} = -i\partial\bar{\partial}\log(|z|^2).$$

(5.3)

In particular, the linear symplectic form (corresponding to the choice of \( f = s \)), when perturbed by a term \( \epsilon \Omega_{F-S} \), is a well defined, \( U(n) \) invariant symplectic form on \( C^n - (0) \). The question then arises what symplectic diffeomorphism can bring this symplectic form into the corresponding one on \( L \). It turns out [15] that there always exists such a symplectic diffeomorphism if there is a diffeomorphism \( h \) of \( R^+ \) onto itself such that the integral constant \( c_1 \) in (5.2a) is left invariant. Then the required symplectic diffeomorphism is

$$\psi(z) = h(|z|^2)z.$$  

(5.4)

A simple calculation reveals that \( h(|z|^2) \) can be taken to be

$$h(s) = (1 + \epsilon(\sigma - 1)s^{-1} + \epsilon\sigma'\log s)^{\frac{1}{2}},$$

(5.5)

then,

$$\psi^*\omega_1 = \omega_2,$$

$$\omega_1 = -i\partial\bar{\partial}(s + \epsilon\log s),$$

$$\omega_2 = -i\partial\bar{\partial}(s + \epsilon\sigma(s)\log s),$$

(5.6)

where \( \sigma(s) \) is a non-negative smooth function of compact support (a cut-off function) which is identically one in some neighborhood of the origin.

This consideration can be generalized to the case where the quadratic moment map \( \mu(z) \) is replaced by \( \phi(z) = f(z)/|f(z)| \), with \( f(z) \) a polynomial function of \( n \)-complex variables \( z = (z_1, z_2, ..., z_n) \). \( \phi(z) \) defines a mapping from the zero level set of \( f \) to \( S^1 \). According to a theorem of Milnor [25, 26], in the complement of a small neighborhood of the singular point in the zero level set, \( \phi(z) \) is a projection map of a smooth fiber bundle over \( S^1 \).

Normal Coordinate Expansion

As observed in the last subsection, a symplectic diffeomorphism associated to the blowing up map usually arises as a consequence of the indeterminacy of the \( G \)-equivariant closed 2-form by the moment map. Globally this indeterminacy gets fixed by specifying the symplectic invariants, such as the additive constant \( c_1 \) and
the interval of the image of the moment map. Then the globally defined symplectic diffeomorphism is given by some diffeomorphism of $R^+$ onto a subinterval $I$.

The aim of this subsection is to use the method of normal coordinate expansion [27], [19] to clarify some further properties of the symplectic diffeomorphism in the context of blowing up. Application of this method to multi-loop calculations in the nonlinear sigma model has been initiated by Honerkamp in the early 70's [28], and considerably elaborated by Alvarez-Gaumé et al a decade later [29]. It is useful to note that whereas the references [28, 29] deal with perturbation expansions of the quantum theory, therefore specifying to a situation where the normal coordinate variables are treated as quantum fields, as opposed to the fields sitting over the origin of the normal coordinate system which are treated classically, we will here treat the normal coordinate variables simply as a change of coordinates in a small neighborhood containing the singular point. In this way we are able to provide a local version of our blowing up diffeomorphism. Together with the global one, given by (5.4) in the last subsection, this completes the whole set of (symplectic) diffeomorphisms needed to glue the blow-up back to the global reduced space.

In the following, we discard the topological term from our discussions and consider the term containing the metric tensor only. This is reasonable because in the normal coordinate expansion, everything is expressed in terms of solutions of the geodesic equation, and it does not matter whether we use the Riemannian connection or a general affine connection with torsion. We choose a small neighborhood $U$ of an arbitrary point $p$ in $M$, such that any two points in $U$ are connected by a single geodesic. Thus, there exists a coordinate transformation from the coordinate system $x^i$ with origin $x^i_0$ being the coordinates of the point $p$, to $u^i$ with $u^i_0 = 0$ as the origin in the tangent space at $p$. This transformation is given by the solution of the geodesic equation

$$\frac{d^2 x^i}{d t^2} + \Gamma^i_{jk} \frac{d x^j}{d t} \frac{d x^k}{d t} = 0,$$

(5.7)

here $\Gamma^i_{jk}$ is the affine connection on $M$, (note that only the symmetric part of this connection appears in this equation,) and $x^i(t)|_{t=0} = x^i_0$, $dx^i(t)/dt|_{t=0} = u^i$, as initial conditions. The solution $x^i(t)$ in terms of the initial values can be expressed as a Taylor series near $x^i_0$

$$x^i(t) = x^i_0 + u^i t - \frac{1}{2} \Gamma^i_{jk} u^j u^k t^2 - ....$$

(5.8)

with higher power terms containing the derivatives of $\Gamma^i_{jk}$.

Since the Jacobian $(\partial x^i/\partial u^i)_{u^i=0} = \delta^i_j$, the transformation is regular at the origin $u^i_0 = 0$. The coordinate system $(u^i)$ in a small neighborhood of $p$ is called
the normal coordinate system. The map between the two coordinate systems is the exponential map we have mentioned before. The exponential mapping is in fact a local diffeomorphism in the sense that there exists a neighborhood \( U \) of a point \( x_0 \), which is contained in the normal coordinate neighborhoods of all points in \( U \). With \( x^i \) and \( u^i \) viewed as the scalar fields in the \( \sigma \)-model, the actions in terms of the field \( u^i \) and of \( x^i = x^i(t = 1) \) possess the same form up to a redefinition of the metric tensor. Thus when we do not think of \( x_0^i \) as a classical field upon which one was to perform the background field expansion, passing from \( x^i \) to \( u^i \) simply amounts to a local change of coordinates, with \( x_0^i \) being constant, thus disappearing from the action. Now consider the situation where we have a nonlinear sigma model with a (symplectic) diffeomorphism, then generally the statement that \( x_0^i \) is a constant configuration remains true only if \( x^i(t)|_{t=0} = x_0^i \) is preserved by the diffeomorphism \( \psi \), i.e., if the point \( p \) is a fixed point of \( \psi \). When \( p \) is not a fixed point of \( \psi \), \( x_0^i \) acquires a nontrivial dependence on spacetime via the diffeomorphism \( \psi \). In this case, if we perform the normal coordinate expansion for the action, we get [29] \((\phi = \psi^*(x_0^i))\),

\[
S = \int d^2 \sigma G_{ij} \partial_\mu x^i \partial^\mu x^j = S_0 + S_1
\]

\[
= \int d^2 \sigma G_{ij} \partial_\mu \phi^i \partial^\mu \phi^j + \int d^2 \sigma G_{ij} \partial_\mu \phi^i D^\mu u^j
\]

\[
+ \frac{1}{2} \int d^2 \sigma (G_{ij} D_\mu u^i D^\mu u^j + R_{ijkl} u^k u^l \partial_\mu \phi^i \partial^\mu \phi^j)
\]

\[+ O(u^3),\] (5.9)

where

\[
D_\mu u^i = \partial_\mu u^i + \Gamma^i_{jk} u^j \partial_\mu \phi^k.
\]

Note that we have used the fact that under a generic coordinate change, the \( u^i \) transforms as a contravariant vector while the form of the geodesic equation (5.7) remains unchanged if the diffeomorphism \( \psi \) is such that it reduces to a linear function of \( t \) when restricted to the geodesic curve \( t \rightarrow x(t) \). The above expansion is valid at any point of the manifold \( M \). Especially, it is valid at the singular point when it is not a fixed point of the diffeomorphism \( \psi \).

Now we can see the role of the normal coordinate expansion. It embodies the diffeomorphism \( \psi \) on the large (which reduces to a linear function of the geodesic length) as new dynamical degrees of freedom which disappear when the singular point is a fixed point of \( \psi \), and provides a (weakly) coupled form of the total action (5.9). It is weakly coupled in the following sense: the coupling between \( u^i \) and \( \phi^j \) in (5.9) is in a form such that it contributes to the one-loop divergences of the theory
and must be decoupled when we add the one loop renormalization counter terms of the form

\[
\frac{1}{4\pi\epsilon} \int d^2\sigma [R_{ij}\partial_\mu \phi^i \partial_\mu \phi^j + \frac{G_{ijk}}{\partial_\phi^j} \delta S_0].
\] (5.10)

Then the action is completely decoupled modulo equations of motion by the following equality (which is easily proved):

\[
\frac{\delta S_0}{\delta \phi^i} u^i = \int d^2\sigma G_{ij} \partial_\mu \phi^i D_\mu u^j.
\] (5.11)

Thus, the remaining total action, when the field \( u^i \) is lifted to the tangent frame by using the \( m \)-bein, \( e^a_i, \ a = 1, \ldots, m \), can be put into the form

\[
\int d^2\sigma G_{ij} \partial_\mu \phi^i \partial_\mu \phi^j + \frac{1}{2} \int d^2\sigma D_\mu u^a D_\mu u^a.
\] (5.12)

Note that the action for \( u^a = e^a_i u^i \) is already gauge invariant with a maximal gauge invariance \( SO(m) \) and the gauge field \( (\omega_i)^{ab}_i \partial_\mu \phi^i = A^{ab}_\mu \), is matrix valued in \( so(m) \). It is the pull back to spacetime of a Yang-Mills connection on the (orthogonal) frame bundle over \( M \), i.e., a one-form \( A^{ab}_i d\phi^i \).

**Reduction of Connections**

The connection form appearing in (5.12) for the linear model is apparently \( so(m) \) valued, but usually we can not gauge the maximal isometry group, instead, for the linear model in the last section, only the \( U(1) \) subgroup is to be gauged. To carry out gauging by associating a gauge field to the sigma model is thus a problem of reducing the gauge connection of an orthogonal frame bundle to that of a subbundle of the bundle of unitary frames. Here we discuss in what sense the connection which appeared above is related to the gauge field arising from gauging.

It is well known that if the Hamiltonian group action on the symplectic manifold is free, then the regular level set is a principal fiber bundle over the reduced symplectic space. Any connection can be used to define the integrable distribution which is equivalent to the tangent space decomposition. In the case of a singular reduction, instead of a fiber bundle, one has a stratified space, each of whose strata is a regular Marsden-Weinstein reduced space, under a rather general assumption [30]. In the case of the point singularity, there are only two strata: the complement of the singular point \( p \) in \( M \), and \( p \) itself. The isotropy group at \( p \) is the same as the subgroup of the isometry group we choose to gauge. It follows from the results of [30] that the fiber types of both the global regular reduced space and the blown
up at the singular point \( p \) are the same, hence their respective principal \( G \)-bundles admit the same decomposition of tangent vectors. Thus the following discussions for the linear case are equally applicable to the nonlinear case with minor modifications.

Given a connection on a principal fiber bundle \( P(M, G) \), which is the same as given a direct sum decomposition of the tangent space at \( x \in P \)

\[
T_x P = \text{Vert} \oplus \text{Hor},
\]

(5.13)

into the vertical and horizontal subspaces (for details see the second reference of [19]). A connection \( \Gamma \) on \( P(M, G) \) is reducible to a connection \( \Gamma' \) on a subbundle \( Q(M, H) \) with \( H \subset G \), if for any point \( x \) on \( Q \), the horizontal subspace \( \text{Hor}(P) \) of \( T_x P \) is tangent to \( Q \). And if \( \phi \) is a homomorphism of groups \( \phi : G \to H \), (which induces a homomorphism on the Lie algebra level, still denoted by \( \phi \)) then \( \phi \cdot \omega = \phi^* \omega' \), where \( \omega, \omega' \) are connection 1-forms with respect to \( \Gamma, \Gamma' \).

In the cotangent space approach for connections, in terms of Lie algebra valued 1-forms, a connection 1-form admits the general expression:

\[
\omega = \omega_0 + g^{-1}A g = g^{-1}dg + g^{-1}A_\mu gdx^\mu, \quad g \in G
\]

(5.14)

where \( A = A^a_\mu \lambda_a dx^\mu \), and \( \omega_0 \) is the Maurer-Cartan canonical 1-form. At the point \((1, x)\) on the locally trivialized principal \( G \)-bundle \( P_\alpha = G \times U_\alpha \), the tangent space \( T_{(1,x)} P_\alpha \) is spanned by vectors \((A_\mu, \partial/\partial x^\mu = \partial_\mu)\), with \( A_\mu \) viewed as an element of the Lie algebra of \( G \). An arbitrary tangent vector in \( T_{(1,x)} P_\alpha \), \( X = X_H + X_V \) can be expanded in terms of the basis \((A_\mu, \partial_\mu)\). Now we can deduce the reduced connection on the subbundle \( Q(M, H) \), \( H \subset G \) as follows. Rewrite \( \phi \cdot \omega^G = \phi^* \omega^H \) as \( \phi(\omega^G(X_V)) = \omega^H(\phi(X_V)) \). Let \( g = h \oplus m \) be the direct sum decomposition of the Lie algebra of \( G \), therefore \( \phi : g \to h \) is a projection. We have two conditions determining \( \omega^H \) uniquely from \( \omega^G \), the first implies that, \( \omega^G(X_V)|_h = \omega^H(X'_V) \), where \( X'_V \) on \( T_{(1,\phi(x))} Q \) are fundamental vector fields corresponding to elements of the Lie algebra of \( H \). The second says simply that certain vertical components \( X_V \) are mapped by \( \phi \) into some linear combinations of horizontal vectors of \( TQ \). Using the tangent basis \((A_\mu, \partial_\mu)\), the first condition gives

\[
\omega^G(A_\mu, \partial_\mu)|_h = A_\mu|_h = A'_\mu = \omega^H(X'_V) = \omega'(X'_V),
\]

and the second condition just expresses the fact that \((A'_\mu, \partial_\mu)\) can be chosen as the basis for the horizontal subspace of \( TQ \). Thus the horizontal part of the connection 1-form on the reduced principal bundle is (using the projection \( S : T_g Q \to T_g M \))

\[
S\omega'(A'_\mu, \partial_\mu) = \omega'(S(A'_\mu, \partial_\mu)) = g^{-1}A'_\mu g, \quad g \in H.
\]

(5.15)
Note that its curvature is by definition horizontal: \( \Omega = Sd\omega = g^{-1}(dA + [A, A])g \).

The reduction process concerned here corresponds to first passing from the \( SO(2n) \) bundle of the linear frames to the \( U(n) \) bundle of the complex linear frames, and then reducing the \( U(n) \) bundle to the abelian \( U(1) \) (or rather the maximal torus \( T \)) sub-bundle. As the symplectic manifold \( M \) has an almost complex structure, its tangent space is equipped with the standard complex structure \( J \), which enables us to complexify the coordinates on \( T_xM \sim C^n(z = x + iy, \bar{z} = x - iy) \). The complexification induces naturally a direct sum decomposition of the Lie algebra \( so(2n) = u(n) \oplus m \) with \( m \) the orthogonal complement of \( u(n) \) in \( so(2n) \). According to our discussion above, we see that the (horizontal part of the) connection 1-form on the \( U(n) \)-bundle as reduced from that of the \( SO(2n) \) bundle is simply

\[
A^a = A_i^a dz^i + A_{\bar{i}}^a d\bar{z}^i,
\]

where \( a \) is the index for the basis of the Lie algebra of \( U(n) \), and \( i, \bar{i} \) run from 1 to \( n \).

For the reduction of the \( U(n) \) connection to the \( H = U(1) \) connection, it suffices to consider the reduction to \( T = U(1)^n \), the maximal torus of \( U(n) \), since from \( T \) to \( U(1) \) the process is simply by taking diagonals. Thus assume the Lie algebra \( u(n) \) admits the decomposition \( u(n) = t \oplus m \), let \( \phi : u(n) \rightarrow t \) be the projection. \( \phi \) maps the horizontal tangent vectors to horizontal vectors. Denote an arbitrary tangent vector in \( u(n) \times T_{(z, \bar{z})}U_\alpha \) as \( \tau = (A_i + A_{\bar{i}}, \partial_i + \bar{\partial}_i) \), therefore

\[
\phi : \tau = (A_i + A_{\bar{i}}, \partial_i + \bar{\partial}_i) \rightarrow (\phi(A_i) + \phi(A_{\bar{i}}), \partial_i + \bar{\partial}_i) \in t \times T_{(z, \bar{z})}U_\alpha.
\]

We thus have the following equations determining the unique \( T \)-connection:

\[
\omega^T (\cdot, \partial_i + \bar{\partial}_i) = \phi(A_i) + \phi(A_{\bar{i}}) = A_i' + A_{\bar{i}}'
\]

\[
\omega^T (\xi^i A'_i + \bar{\xi}^i A'_{\bar{i}}, \cdot) = 0, \quad A' \in t,
\]

(5.16)

where \( \cdot \) means "for any" vertical (horizontal) components of the tangent vector, \( \xi^i, \bar{\xi}^i \) are arbitrary holomorphic (anti-holomorphic) functions of \((g, z)\). From the last equation of (5.16), we deduce that

\[
g^{-1}A'_igdz^i + g^{-1}A'_{\bar{i}}gd\bar{z}^i = \omega_0 = g^{-1}dg, \quad g \in T.
\]

(5.17)

If we choose the parametrization of the group \( T \) as follows

\[
T = U(1)^n = \{(z^1, z^2, ..., z^n) \in (C^*)^n | \ |z^i|^2 = 1 \},
\]

it is not difficult to see that the connection thus obtained coincides, for the diagonal \( U(1) \), with the one which has appeared in (4.12). This completes our discussion of
reduction of the connection in the linear $\sigma$-model. On the other hand, the general connection used to gauge the nonlinear $\sigma$-model belongs to the subgroup of the whole isometry group of its tangent space. By the same arguments as above, one can deduce that it can be reduced to the connection of the principal $T$-bundle, with $T$ the maximal torus of the gauge group $H$. This, when expressed in terms of the $\sigma$-model scalar fields, is related to the connection of the linear $\sigma$-model precisely by the gluing diffeomorphisms discussed before.

We now comment briefly on the process of integrating out gauge fields. It is known that this process receives quantum corrections at the sigma model loop level. Since the one loop effect has been vital in our discussion of the normal coordinate expansion, it is expected that it is also important to include the quantum corrections in the blowing up construction. Unlike the case of the conformally invariant sigma models where the one loop corrections can be conveniently summarized into the dilaton shift, in the present situation, there is no place for a dilaton, neither the particular notion of conformal invariance, but instead, we have the symplectic diffeomorphism which might have nontrivial fixed point structures. It may happen that these fixed point structures manifest themselves into some unforeseeable dynamical modes of the (gauged) nonlinear sigma model, e.g. the appearance of the kinetic term for the gauge fields, in much like the way it arises in ref[31], where an interesting mechanism for generating dynamics for the gauge field by the $1/N$ corrections to the $CP^N$ model is suggested. We leave the discussion of this possibility for future work.

VI. Applications

Toric $\sigma$-models

A toric manifold associated with an integral or rational polyhedron can be obtained as a symplectic reduction of $C^N$ by the Hamiltonian action of the subtorus of $T_C^N = (C^*)^N$, at a regular level of the corresponding moment map (see [16] for more properties of the toric manifolds). Obviously a toric $\sigma$-model (i.e. a $\sigma$-model with target space being a toric manifold) can be viewed as a suitable symplectic reduction of the linear $\sigma$-model, or as the gauged nonlinear $\sigma$-model studied in section III. It is a nontrivial fact that some Hamiltonian subtorus actions on the toric manifold can have fixed points when the image of these points are exactly the vertices of the convex polyhedron. We are interested in the effect of blowing up a point in the toric $\sigma$-model.
Let us take the simplest example of $\mathbb{CP}^2$, constructed as a toric manifold whose associated polyhedron is the standard 2-simplex, i.e. a triangle $\Delta \subset \mathbb{R}^2$. If $e_1, e_2$ denote the basis vectors of $(\mathbb{Z}^2)^* \subset \mathbb{R}^2$, which are two edge vectors of $\Delta$, we can form a fan whose 1-skeletons (edges of the 2-cones) are all of the form $tx_i$, $0 \leq t < \infty$, $i = 1, 2, 3$, where $x_i = e_i$, $x_3 = -(e_1 + e_2)$. One can take $x_i$ to be the basis vectors in $\mathbb{Z}^3$, thus there exists a natural map $\mathbb{Z}^3 \to \mathbb{Z}^2$, $x \mapsto e$ which induces the corresponding map $\mathbb{R}^3 \to \mathbb{R}^2$ and the quotient map $T^3 \to T^2 \to 0$ with kernel $S^1$. The realization of $\mathbb{CP}^2$ as symplectic reduction of $\mathbb{C}^3$ is carried out by reducing $\mathbb{C}^3$ by the (smooth) Hamiltonian action of this $S^1$. From this construction it is obvious that $T^2 \subset T^3$ acts on $\mathbb{CP}^2$ in a Hamiltonian fashion and the image of the $\mathbb{CP}^2$ under its moment map is exactly $\Delta$. (The same steps work for other simple toric manifolds, giving rise to Hamiltonian spaces of dimension twice of that of the corresponding torus.)

We already know that the nonlinear $\sigma$-model of $\mathbb{CP}^2$ can be expressed as a gauged $\sigma$-model with the gauge field

$$A_\mu = \frac{i}{2} \sum_{i=1}^{3} \bar{z}_i \partial_\mu z_i - z_i \partial_\mu \bar{z}_i,$$

and the action of the form $\sum D_\mu z_i \bar{D}_\mu z_i$, $D_\mu = \partial_\mu + i A_\mu$. The integral of the symplectic form over a homology cycle in $\mathbb{CP}^2$ equals a topological invariant $1/2 \pi \epsilon \mu \nu \partial_\mu A_\nu$ which is the first Chern number of the tangent bundle. The $\mathbb{CP}^2$ $\sigma$-model has a global $SU(3)$ invariance, of which the maximal torus $T^2$ acts in the Hamiltonian fashion. We know that this $T^2$ action is not free at some points whose image under the moment map are the vertices of $\Delta$. This may cause serious problems in the quantum theory, even though it is harmless classically, as far as one does not perform the quotient (which is a point in this case). A possible resolution is to blow up the fixed point on $\mathbb{CP}^2$. Application of our general procedure leads to a nonlinear $\sigma$-model whose target space may be identified as a connected sum of $\mathbb{CP}^2$ and another $\mathbb{CP}^2$, considered as the projectivization of a line bundle over $\mathbb{CP}^1$, with the symplectic form of the latter $\mathbb{CP}^2$ multiplied by a small real number $\epsilon$. One implication of this example is, that quantum mechanically, the blown up $\mathbb{CP}^2$ model automatically overcomes the zero area limit, as the transition from $-\epsilon$ to $+\epsilon$ in this construction is smooth. Details of the blown up toric $\sigma$ model will be reported separately.

$N=2$ Supersymmetric $\sigma$-model

In this case, there exists a general procedure [3,6] of performing the $N = 2$ quotient by gauging the (holomorphic) isometries of the $N = 2$ superspace action of the form

$$S = \frac{1}{2} \int d^2 \sigma D_+ D_- \bar{D}_+ \bar{D}_- K(\Phi, \bar{\Phi}, \Lambda, \bar{\Lambda}),$$

(6.2)
with arbitrary chiral and twisted chiral superfield multiplets $\Phi, \Lambda$. The gauged action takes the general form of a new Kähler potential $K'$ which is the original potential $K$ with $\Phi$ and $\Lambda$ minimally coupled to some gauge multiplet $V$, plus terms which are trivially gauge invariant, such as the Fayet-Iliopoulos terms which are present when the isometry group contains $U(1)$ subgroups. In the same spirit as the bosonic $\sigma$-model and its symplectic reduction studied in the previous sections, we can carry out the symplectic blowing up for the $N = 2$ $\sigma$-model as well. The basic ingredients are a superspace analogue of the normal coordinate expansion on the one hand, and the identification (and the interpretation) of the blowing up parameter $\epsilon$ in the linear model as the coupling constant in front of the Fayet-Iliopoulos terms, on the other hand. We will not describe both these important points here. However, the picture of the blown up $N = 2$ $\sigma$-model is clear: to each $N = 2$ supersymmetric $\sigma$-model, arising from gauging an appropriate holomorphic isometry group, at any (isolated) configuration which is the fixed point of a subgroup of the isometry group, one can associate a gauged version of the $N = 2$ linear $\sigma$-model. The linear $N = 2$ $\sigma$-model has been studied intensively in [22]. Ref [32] contains also discussions of the flops in the Calabi-Yau spaces which are in fact a sequence of blow-ups and blow-downs.

VII. Summary and conclusions

What has been done in the previous sections can be actually viewed as giving concrete explanations to the various pieces of information contained in the following formula for the symplectic form on the reduced space which is diffeomorphic to a connected sum of the global reduced space and a copy of $CP^n$:

$$\Omega_\epsilon = \Omega_0 + \epsilon \pi^* \Omega_{F-S}. \quad (7.1)$$

Perhaps we should add that it is a well-defined quantity. Although tedious, it can be checked directly that the 2-form $\Omega_0$ as obtained from (3.27), after integrating out gauge fields, is closed. From the arguments given before (5.9) it follows that the fields in the nonlinear $\sigma$-model have no support near the singular locus. The connected sum is formed by gluing the linear $\sigma$-model back to the gauged nonlinear one along an annular region (smoothly) diffeomorphic to $B_{2\epsilon} - B_\epsilon$ of balls in $C^n$. Thus the reduced symplectic form is well-defined on the blow-up.

From the symplectic geometric point of view, blowing up is a useful tool to obtain numerous interesting examples of symplectic manifolds [33]. The symplectic forms are classified into the equivalence classes up to diffeomorphisms of certain type. It is a challenging problem to calculate some invariants on the space of equivalence classes.
of symplectic forms physically. For example, a generalization of the Duistermaat-Heckman theorem to the singular reductions [16] states that the cohomology class of \( \Omega_\epsilon \), \([\Omega_\epsilon]\) is continuous as an affine function of the parameter \( \epsilon \) (even at \( \epsilon = 0 \)), the slope of the line segment in \( H^2(M) \) spreaded by \([\Omega_\epsilon]\) for all \( \epsilon < 0 \), gets a jump when going through \( \epsilon = 0 \). It is hoped that the results of this paper may point a way to effectively calculating this interesting invariant.

We have suggested in this paper a gauged nonlinear sigma model for the symplectic blowing up, and discussed its various aspects as a well defined model for describing the non-singular symplectic manifold resulting from blowing up a singular point. The model consists of gauging both nonlinear and linear parts of the action, resulting in the respective symplectic quotients. The linear sigma model has been used to provide a local version of the symplectic blowing up (which is in the linear case identical to the corresponding complex blowing up, i.e. passing from the affine varieties to the projective ones), whereas the nonlinear part of the action describes the complementary region of the singular point of the target manifold. The two pieces of the construction are glued together by the symplectic diffeomorphism which appears as part of the blowing up maps (the birational morphisms). Our main conclusions are, the symplectic blowing up is a process that is well defined in a gauged nonlinear sigma model with symplectic diffeomorphism, and the blow-up which is diffeomorphic to a connected sum of a symplectic manifold and a copy of the complex projective space, is described by the resulting classical configurations (instantons).

One motivation of the present work is to try to understand in physical terms what is involved in the construction of the symplectic cobordism described in [15]. While it is relatively easy to convince oneself that the gauged sigma model is the appropriate arena here, it is far from trivial to identify the blow up modes in the gauged sigma model. Classically, it turns out, the blow up modes can be viewed as the consequence of the existence of the symplectic diffeomorphism. Quantum mechanically, it is quite plausible that the fluctuation around the classical blow up modes might be much enhanced, or in the language of the sigma model, the instanton corrections might become more significant. The advantage of working with the nonlinear sigma models is that here the instanton effects can be conveniently handled, as they have long been explored. Another aspect of the result is that it seems to provide a concrete construction of the so-called topology changing process in terms of the nonlinear sigma model. Comparing with the recent work [7, 32], our result seems to bring together the local and global analyses separately pursued by those authors.
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Appendix

We present in this Appendix the proof that there exists a change of coordinates so that in a small neighborhood of a point in $M$, any smooth vector takes the form (3.18) or (3.19).

Let an arbitrary smooth vector in an arbitrary coordinate system in which it is nonvanishing be

$$X = \sum_{i=1}^{m} \xi^i \frac{\partial}{\partial u^i}. \quad (A1)$$

Because it is nonvanishing, at least one of its components cannot be zero. Let $\xi^1$ be a nonzero component. Consider the differential equations

$$\frac{du^\alpha}{du^1} = \frac{\xi^\alpha(u^1, u^m)}{\xi^1(u^1, u^m)}, \quad 2 \leq \alpha \leq m, \quad (A2)$$

with $u^\alpha$ arbitrary functions (of variable $u^1$). According to the theory of ordinary differential equations [34], there exists a unique set of solutions of (A2) in a sufficiently small neighborhood, $|u^1| < \delta$ in $U$

$$u^\alpha = \phi^\alpha(u^1), \quad |u^1| < \delta, \quad (A3)$$

obeying the prescribed initial conditions $\phi^\alpha(0) = v^\alpha$. $\phi^\alpha$ depend smoothly on $u^1$ and the initial values $v^\alpha$, therefore can be taken as functions $\phi^\alpha(u^1, v^2, ... v^m)$. Make the change of coordinates

$$u^1 = v^1$$

$$u^\alpha = \phi^\alpha(v^1, v^2, ..., v^m), \quad 2 \leq \alpha \leq m. \quad (A4)$$

Because the Jacobi is equal to 1 at $v^1 = 0$, there is a coordinate neighborhood $\mathcal{V} \subset \mathcal{U}$, $\{\mathcal{V}; v^1\}$, such that

$$X|_\mathcal{V} = \sum \xi^i \frac{\partial}{\partial w^i} = \xi^1 \frac{\partial}{\partial u^1} + \sum_{\alpha=2}^{m} \xi^\alpha \frac{\partial}{\partial w^\alpha}$$

$$= \xi^1 \sum_{i=1}^{m} \frac{\partial u^i}{\partial v^1} \frac{\partial}{\partial u^i} = \xi^1 \frac{\partial}{\partial v^1}. \quad (A5)$$
Thus by defining
\[ w^1 = \int_0^v \frac{d\nu^1}{\xi^1}, \quad w_\alpha = \nu^\alpha, \quad \alpha = 2, \ldots, m, \] (A6)
in the new coordinate system \((W; w^i)\), the vector \(X\) takes the form
\[ X|_W = \frac{\partial}{\partial w^i}. \] (A7)

We can perform a chain of changes of coordinates, until all the Hamiltonian vectors \(\xi_a\) are transformed into the form (A7). To write out the general formulae, note that in the case of two vectors \(\xi(1), \xi(2)\), in the coordinate system in which \(\xi(1) = \partial/\partial w^1(1)\), due to the linear independence of \(\xi(1)\) and \(\xi(2)\), the coefficients of \(\xi(2)\) can not depend on \(w^1(1)\), thus
\[ \xi(2) = \sum_{\alpha=2}^m \xi(2)_\alpha \frac{\partial}{\partial w^\alpha(1)} = \ldots \]
\[ = \xi(2)_2 \frac{\partial}{\partial w^2(1)}, \] (A8)
when
\[ \xi(2)_\alpha / \xi(2)_2 = dw^\alpha(1)/dw^2(1) = dw^\alpha(1)/dw^2(2) \] (A9)
are satisfied by the new coordinates \(w^i(2)\).

In general, after a chain of coordinate changes
\[ w^i(0) \rightarrow w^i(1) \rightarrow \ldots \rightarrow w^i(a), \]
the vectors \(\xi_a\) are transformed into the form \(\{\partial/\partial w^1(a), \partial/\partial w^2(a), \ldots, \partial/\partial w^a(a)\}\). It is easy to derive the following relations
\[ \frac{\partial w^a(s-1)}{\partial w^s(s)} = \xi^s(s), \quad \frac{\partial w^\alpha(s-1)}{\partial w^\alpha(s)} = \xi^\alpha(s), \quad \frac{\partial w^\alpha(s-1)}{\partial w^\beta(s)} = \delta^\alpha_\beta, \]
\[ s = 1, \ldots, a; \quad \alpha, \beta = s + 1, \ldots, m. \] (A10)
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