Dynamic Characteristics Analysis and Chaotic Synchronization of Fractional Order Financial Dynamic System

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Abstract. The equations of time-delay fractional order financial dynamic system are studied. Fractional calculus, as the expansion of the corresponding integer parts, has more complex definition and update laws. It is found that modeling by fractional-order differential equation in some nature phenomenon and engineering field can be consistent with essentiality of the phenomenon and matters, such as in mathematical modeling of viscoelastic materials, electromagnetic wave and other fields. Moreover, there exist more complex dynamical behaviors in fractional-order dynamical system than in integral-order one, and it has more preferable history memory ability. Therefore, it is of theoretical value and practical significance researching the fractional-order nonlinear dynamical systems on the base of the researches on the traditional integral-order nonlinear dynamical system.

1. Introduction
Chaotic solutions of delay differential equations are easier to obtain. In literature [1-7], the stability analysis of a kind of Brusselator model with time delay is studied. It is concluded that, when appropriate parameters are selected, Hopf bifurcation will not appear in the Brusselator model but the corresponding delay differential equation, and the dynamic characteristics will become complex. In literature [8], Chen system is taken as the study object to study the dynamic characteristics of it with disturbance terms, and the equilibrium point and the stability of Chen system with time-delay disturbance are analyzed. In literature [9], based on parameter identification and adaptive control. At present, most of the research on time-delay dynamic system is still focused on integral order dynamic system [7-12]. In literature [13], the asymptotic stability of fractional order neural network with discrete time-delay and distribution time-delay is studied in the meaning of Caputo derivative, and the sufficient conditions for asymptotic stability of fractional order neural network with discrete time-delay and distribution time-delay are obtained by constructing Lyapunov function and using fractional order Razumikhin theorem. In literature [14-17], the synchronization of time-delay fractional order chaotic system is studied. Based on the state observer method and the stability theory of fractional order system, a synchronization controller for fractional order time-delay chaotic system is designed to make the fractional order time-delay chaotic system synchronize.
Therefore, based on the stability theory of fractional order linear system, the control method is provided for the chaotic motion of time-delay fractional order dynamic system of single sliding mode, and the chaotic synchronization is realized by using that control method.

2. The fractional order financial system
that system can be described as follows:
\[
\begin{align*}
D_t^{\alpha_1} x(t) &= z + (y(t - \tau) - a)x, \\
D_t^{\alpha_2} y(t) &= 1 - by(t) - x^2(t - \tau), \\
D_t^{\alpha_3} z(t) &= -x(t - \tau) - cz,
\end{align*}
\]

In this paper, \(\alpha_1=0.93, \alpha_2=0.97, \alpha_3=0.95\) and \(\tau=0.06\), and the \(\lambda_{\text{max}}\) of the maximum Lyapunov index is equal to 0.0714187, the system (1) is in chaotic state under the above parameters.

3. Attractor change of financial system induced by disturbance
That process can be described as follows:
\[
\begin{align*}
D_t^{\alpha_1} x(t) &= z + (y(t - \tau) - a)x + \Delta f(x, y, z) + d(t), \\
D_t^{\alpha_2} y(t) &= 1 - by(t) - x^2(t - \tau), \\
D_t^{\alpha_3} z(t) &= -x(t - \tau) - cz,
\end{align*}
\]

Among them, \(\Delta f(x, y, z) = 0.5 \sin (\pi x) \cos (\pi y) \sin (2\pi z)\), \(d(t) = 0.1 \cos (2t)\), and the system (2) are shown in Fig. 1-2.

4. Chaotic synchronization financial system of single sliding mode control
Based on the stability theory of time-delay fractional order linear system, a control method is provided for chaos of the time-delay fractional order dynamic system. Research found that the control method could have good control effect and robustness. On that basis, the control method is further analyzed to realize chaotic synchronization system.
4.1. Self-synchronization of fractional order financial system controlled by the single sliding mode controller

Set the driving system as follows:

\[
\begin{align*}
D_\alpha^\alpha x_1(t) &= z_1 + (y_1(t-\tau) - ax_1), \\
D_\alpha^\alpha y_1(t) &= 1 - by_1(t) - x_2^2(t-\tau), \\
D_\alpha^\alpha z_1(t) &= -x_1(t-\tau) - cz_1,
\end{align*}
\] (3)

The system with controller is as follows:

\[
\begin{align*}
D_\alpha^\alpha x_2(t) &= z_2 + (y_2(t-\tau) - ax_2 + u, \\
D_\alpha^\alpha y_2(t) &= 1 - by_2(t) - x_2^2(t-\tau), \\
D_\alpha^\alpha z_2(t) &= -x_2(t-\tau) - cz_2,
\end{align*}
\] (4)

The problem of time-delay fractional order financial system is transformed into self-synchronization error to be solved, and the self-synchronization error is defined as follows:

\[ e = y - x \]

Among them, \( x = [x_1, y_1, z_1], y = [x_2, y_2, z_2], e = [e_1, e_2, e_3] \)

According to the above definition of synchronization error, the following equations can be obtained:

\[
\begin{align*}
D_\alpha^\alpha e_1(t) &= e_1 + e_2(t-\tau) - ae_1 + u, \\
D_\alpha^\alpha e_2(t) &= -be_2(t) - (x_1(t-\tau) + x_2(t-\tau))e_1(t-\tau), \\
D_\alpha^\alpha e_3(t) &= -e_1(t-\tau) - ce_3,
\end{align*}
\] (5)

Up to this point, the self-synchronization problem of the system in this paper has been transformed into designing an appropriate single sliding mode controller \( u \) to make the synchronization error meet the following:

\[
\lim_{t \to \infty} \| e \| = \lim_{t \to \infty} \| y - x \| = 0
\] (6)

To make the equation (6) at the zero point.

The constructed sliding mode surface is as follows:

\[ s = D_\alpha^\alpha e_1(t) + \int_0^t ae_1(\tau) d\tau \] (7)

The derivation of the above equation is as follows:

\[
\begin{align*}
\dot{s} &= D_\alpha^\alpha e_1(t) + ae_1(t) \\
\dot{s} &= 0, s = 0 \\
u_{eq} &= -e_1(t) + e_2(t-\tau)
\end{align*}
\] (8)

The inconsecutive switching rate is designed as follows:

\[ u_\tau = k \text{sign}(s) \quad (k < 0) \]
By combining the above two parts, a complete sliding mode controller can be obtained as follows:

\[ u = -e_3(t) + e_2(t - \tau) + k\text{sign}(s) \]  

(10)

Construct Lyapunov function of \( V = s^2 / 2 \) to prove the rationality of the design of sliding mode controller

\[
\dot{V} = s(s(D_t^\alpha e_3(t) + ae_1(t)))
\]

\[
= s(e_3(t) - e_2(t - \tau) - ae_1 + u + ae_1)
\]

\[
= s(k\text{sign}(s))
\]

\[
= k|s| \leq 0
\]

(11)

The proof process is finished.

The above proofs show that the sliding mode controller designed, so that the driving system and the response system can achieve chaotic synchronization. In order to further verify the effectiveness of the control scheme, the numerical simulation of the control scheme is carried out. The step length is \( h = 0.01 \), the gain is \( k = -1 \), and the initial values of \([x_1, y_1, z_1, x_2, y_2, z_2]\) are \([-2, -3, -1, 1, 2, 3]\). It can be seen in Fig. 5 that the sliding mode motion can reach and stay on the sliding mode surface. It can be seen in Fig. 6 that the synchronization error gradually converges to zero with evolution over time. The simulation results verify the effectiveness of the control method in the paper.

4.2. Synchronization mode under disturbance act

Consider \( \Delta f(x, y, z) \) and \( d(t) \) in the system (2) occur in the self-synchronization process of the same system, study the synchronous state and mode change of driving system and response system to test the anti-interference ability of synchronization controller (single sliding mode controller) provided in this paper, the obtained synchronization control results are used to get the influence of the chaotic synchronization of time-delay fractional order chaotic system under the specific disturbance by inverse method. Based on the induction process in response system and that in driving system respectively, the analyses are as follows:

4.2.1. Disturbance act on response system

At this point, assume that the driving system (1) is chaotic motion, and the response system (2) is periodic motion. A single sliding mode controller is applied to exert control over the response system. After adding the controller, the response system is as follows:

\[
D_t^\alpha x_2(t) = z_2 + (y_2(t - \tau) - a)x_2 + \Delta f(x, y, z) + d(t) + u,
\]

\[
D_t^\alpha y_2(t) = 1 - by_2(t) - x_2^2(t - \tau),
\]

\[
D_t^\alpha z_2(t) = -x_2(t - \tau) - cz_2,
\]

(12)

To describe the system synchronization, the effectiveness of sliding mode controller of the equation (10) is proved theoretically as follows. According to the expressions of \( \Delta f(x, y, z) \) and \( d(t) \),
there are $|\Delta f(x, y, z)| \leq 0.5$ and $|d(t)| \leq 0.1$. The error system (12) and the driving system (1) is brought into Lyapunov function to obtain the following:

\[
V = \frac{s^2}{2} \dot{V} = s \dot{s} = s(\Delta f(x, y, z) + d(t) + k\text{sign}(s)) \leq (k + 0.6)|s| \leq 0 \quad (k \leq -0.6)
\]

The proof process is finished.

It can be proved that the above closed-loop system conforms to the stability theory of Lyapunov, which indicates that the original synchronization control method is still effective, and requirements on the gain $k$ are slightly different.

4.2.2. Disturbance act on driving system

When the disturbance occurs in driving system, assume that the driving system (14) turns to be periodic motion, and the response system (15) is chaotic motion. The response system (15) changes from chaotic motion to periodic motion under the act of the synchronization controller (10) and finally realizes periodic synchronization with the driving system.

Driving system:

\[
\begin{align*}
D^\alpha_x x_1(t) &= z_1 + (y_1(t - \tau) - a)x_1 + \Delta f(x, y, z) + d(t), \\
D^\alpha_y y(t) &= 1 - by_1(t) - x_1^2(t - \tau), \\
D^\alpha_z z(t) &= -x_1(t - \tau) - cz_1,
\end{align*}
\]

Response system:

\[
\begin{align*}
D^\alpha_x x_2(t) &= z_2 + (y_2(t - \tau) - a)x_2 + u, \\
D^\alpha_y y_2(t) &= 1 - by_2(t) - x_2^2(t - \tau), \\
D^\alpha_z z_2(t) &= -x_2(t - \tau) - cz_2,
\end{align*}
\]

The theoretical proof of the effectiveness of the applied controller is consistent with the above. The numerical simulation results are shown in Fig. 3. It can be seen in the figures that the original driving-response system underwent chaotic motion for a period of time and then turned into periodic synchronization.
6. Conclusion

On that basis, for a kind of fractional-order nonlinear dynamical systems with time delay, single controller, by which the fractional-order nonlinear financial system and Liu system with time delay can be controlled to the fixed point from chaos, is designed by the method of classical sliding model to control. Furthermore, the single controller, by which the fractional-order nonlinear Liu system can be controlled from chaos to limit cycle, is explored. The fractional-order financial systems with time delay as an example, the controlling project of chaotic synchronization of the fractional-order chaotic systems with time delay is provided by single controller, and the robustness of the controlling project is analyzed. Finally, the chaotic phenomenon of the fractional-order Lorenz system with time-varying lags is verified by numerical simulation.

References

[1] Grigorenko I, Grigorenko E. Chaotic dynamics of the fractional Lorenz system[J]. Physical Review Letters, 2003, 91 (3): 034101-4.
[2] Li C P, Peng G J. Chaos in Chen’s system with a fractional order[J]. Chaos Solitons & Fractals, 2004, 22(2): 443-450.
[3] T. T. Hartley, C. F. Lorenzo, H. K. Qammer. Chaos In A Fractional Order Chua’s System. IEEE Transactions on Circuits and Systems I. 1995, 42(8):485-490.
[4] Sachin Bhalekar, Varsha Daftardar-Gejji. Differential equations of fractional order[J]. Journal of Fractional Calculus and Applications, 2011,(5): 1-9.
[5] S.Bhalekar, V. Daftardar-Gejji, Fractional ordered Liu system with time-delay[J]. Communications in Nonlinear Science and Numerical Simulation. 2010,15 (8) : 2178 – 2191.
[6] Ma J. G. and Chen Y.S., Study for the bifurcation topological structure and the global complicated character of a kind of nonlinear finance system[J]. Applied mathematical and mechanics,2001,22,1375-1382.
[7] Yang Xiaoyan, Wang Qiaoyu, XU Huijie, YI Xiu. Stability analysis of a class of Brusselator models with time delay[J]. Journal of Shanxi University of Technology, 2020, 36(3) : 88-92.
[8] Ding Xu, Cui Yan, He Hongjun. Analysis and control of Hopf bifurcation in Chen system with delayed perturbation[J]. Chinese Journal of Applied Mechanics, 2020, 37 (3) : 1239-1244.
[9] Hodgkin A L, Huxley A F. A Quantitative Description Of Membrane Current And Its Application To Conduction And Excitation In Nerve[J]. J. Physiol., 1952, 117:500–544.
[10] Fitzhugh R. Impulses and physiological states in theoretical states in theoretical models of nerve membrane[J]. Biophys.J.1961, 1:445-466.
[11] K.Aihara,G Matsumoto, Two stable steady in the Hodgkin-Huxley axons[J]. Biophysical Journal 41(1983)87-89.
[12] G Matsumoto, S.Aihara, Y.Hanyu, N.Takahashi,S. Yoshizawa, J.Nagumo, chaos and phase locking in normal squid axons[J].Physics Letters A 123(1987)162-166.
[13] Liu Jian , Zhang Zhixin, Jiang wei. Asymptotic stability analysis of fractional neural networks with discrete delays [J]. Applied Mathematics and Mechanics, 2020, 41 (6) : 646-657.
[14] K.Aihara, G Matsumoto, Chaotic oscillations and bifurcations in squid giant axons,in:A.V. Holden (Ed), chaos, Manchester University Press, Manchester, UK,1986 PP.257-269.
[15] Zhang Ruo-Xun,Yang Shi-Ping, Modified adaptive controller for synchronization of incommensurate fractional-order chaotic systems[J]. Chin.Phys.B. 2012,21(03): 030505-05.
[16] Wang Zhen, Huang Xia, Shi Guodong. Analysis of nonlinear dynamics and chaos in a fractional order financial system with time delay [J]. Computers and Mathematics with Applications 62 (2011) 1531–1539.

[17] Yang Li-xin, Jiang Jun. Adaptive synchronization of drive-response fractional-order complex dynamical networks with uncertain parameters [J]. Commun Nonlinear Sci Numer Simulat 2014, 19: 1496-1506

[18] Le Hoa Nguyen, Keum-Shik Hong. Adaptive synchronization of two coupled chaotic Hindmarsh–Rose neurons by controlling the membrane potential of a slave neuron [J]. Applied Mathematical Modelling, 2013, 37: 2460-2468.