LOCALIZATION OF INJECTIVE MODULES OVER VALUATION RINGS

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Abstract. It is proved that $E_J$ is injective if $E$ is an injective module over a valuation ring $R$, for each prime ideal $J \neq \mathcal{Z}$. Moreover, if $E$ or $\mathcal{Z}$ is flat, then $E_\mathcal{Z}$ is injective too. It follows that localizations of injective modules over h-local Prüfer domains are injective too.

If $S$ is a multiplicative subset of a noetherian ring $R$, it is well known that $S^{-1}E$ is injective for each injective $R$-module $E$. The following example shows that this result is not generally true if $R$ is not noetherian.

Example 1. Let $K$ be a field and $I$ an infinite set. We put $R = K^I$, $J = K^{(I)}$ and $S = \{1 - r \mid r \in J\}$. Then $R/J \cong S^{-1}R$, $R$ is an injective module, but $R/J$ is not injective by [5, Theorem].

However, we shall see that, for some classes of non-noetherian rings, localizations of injective modules are injective too. For instance:

Proposition 2. Let $R$ be a hereditary ring. For each multiplicative subset $S$ of $R$ and for every injective $R$-module $E$, $S^{-1}E$ is injective.

There exist non-noetherian hereditary rings.

Proof. Let $F$ be the kernel of the natural map: $E \to S^{-1}E$. Then $E/F$ is injective and $S$-torsion-free. Let $s \in S$. We have $(0 : s) = Re$, where $e$ is an idempotent of $R$. It is easy to check that $s + e$ is a non-zero divisor. So, if $x \in E$, there exists $y \in E$ such that $x = (s + e)y$. Clearly $eE \subseteq F$. Hence $x + F = s(y + F)$. Therefore the multiplication by $s$ in $E/F$ is bijective, whence $E/F \cong S^{-1}E$. □

In Proposition 2 and Example 1, $R$ is a coherent ring. By [3, Proposition 1.2] $S^{-1}E$ is fp-injective if $E$ is a fp-injective module over a coherent ring $R$, but the coherence hypothesis can’t be omitted: see [3, Example p.344].

The aim of this paper is to study localizations of injective modules and fp-injective modules over a valuation ring $R$. Let $\mathcal{Z}$ be the subset of its zerodivisors. Then $\mathcal{Z}$ is a prime ideal. We will show the following theorem:

Theorem 3. Let $R$ be a valuation ring, denote by $\mathcal{Z}$ the set of zero divisors of $R$ and let $E$ be an injective (respectively fp-injective) module. Then:

1. For each prime ideal $J \neq \mathcal{Z}$, $E_J$ is injective (respectively fp-injective).
2. $E_\mathcal{Z}$ is injective (respectively fp-injective) if and only if $E$ or $\mathcal{Z}$ is flat.

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In this paper all rings are associative and commutative with unity and all modules are unital. We say that an R-module E is divisible if, for every r ∈ R and x ∈ E, (0 : r) ⊆ (0 : x) implies that x ∈ rE, and that E is fp-injective (or absolutely pure) if Ext₁^R(F, E) = 0, for every finitely presented R-module F. A ring R is called self fp-injective if it is fp-injective as R-module. An exact sequence 0 → F → E → G → 0 is pure if it remains exact when tensoring it with any R-module. In this case we say that F is a pure submodule of E. Recall that a module E is fp-injective if and only if it is a pure submodule of every overmodule. A module is said to be uniserial if its submodules are linearly ordered by inclusion and a ring R is a valuation ring if it is uniserial as R-module. Recall that every finitely presented module over a valuation ring is a finite direct sum of cyclic modules [7, Theorem 1]. Consequently a module E over a valuation ring R is fp-injective if and only if it is divisible.

An R-module F is pure-injective if for every pure exact sequence

0 → N → M → L → 0

of R-modules, the following sequence

0 → Hom₃₄(L, F) → Hom₃₄(M, F) → Hom₃₄(N, F) → 0

is exact. Then a module is injective if and only if it is pure-injective and fp-injective. A ring R is said to be an IF-ring if every injective module is flat. By [4, Theorem 2] R is an IF-ring if and only if R is coherent and self fp-injective.

In the sequel R is a valuation ring whose maximal ideal is P and Z is its subset of zerodivisors. Some preliminary results are needed to show Theorem 3.

**Proposition 4.** Let R be a valuation ring, let E be an injective module and r ∈ P. Then E/rE is injective over R/rR.

**Proof.** Let J be an ideal of R such that Rr ⊆ J and g : J/rr → E/rE be a nonzero homomorphism. For each x ∈ E we denote by ̄x the image of x in E/rE. Let a ∈ J \ Rr such that ̄y = g(̄a) ̸= 0. Then (Rr : a) ⊆ (rE : y). Let t ∈ R such that r = at. Thus ty = rz for some z ∈ E. It follows that t(y − az) = 0. So, since at = r ̸= 0, we have (0 : a) ⊆ Rr ⊆ (0 : y − az). The injectivity of E implies that there exists x ∈ E such that y = a(x + z). We put xa = x + z. If b ∈ J \ Ra then a(xa − xb) ∈ rE. Hence xb ∈ xa + (rE : E a). Since E is pure-injective, by [6, Theorem 4] there exists x ∈ ∩a∈J xa + (rE : E a). It follows that g(̄a) = āx for each a ∈ J.

**Lemma 5.** Let R be a valuation ring, let U be a module and F a flat module. Then, for each r, s ∈ R, F ⊗₄₃(U : r) ∼= (F ⊗₄₃ sU : F ⊗₄₃(U : r)).

**Proof.** We put E = F ⊗₄₃ U. Let φ be the composition of the multiplication by r in U with the natural map \( \mathcal{U} \to U/sU \). Then \( (sU : \mathcal{U} : r) = \ker(\phi) \). It follows that \( F \otimes₄₃ (sU : \mathcal{U} : r) \) is isomorphic to \( \ker(1_F \otimes \phi) \) since F is flat. We easily check that \( 1_F \otimes \phi \) is the composition of the multiplication by r in E with the natural map \( E \to E/sE \). It follows that \( F \otimes₄₃ (sU : \mathcal{U} : r) \) ∼= \( (sE : \mathcal{E} : r) \).

**Proposition 6.** Let R be a valuation ring. Then every pure-injective R-module F satisfies the following property: if \( (x_i)_{i \in I} \) is a family of elements of F and \( (A_i)_{i \in I} \) a family of ideals of R such that the family \( F = (x_i + A_i F)_{i \in I} \) has the finite intersection property, then F has a non-empty intersection. The converse holds if F is flat.
Proof. Let $i \in I$ such that $A_i$ is not finitely generated. By Lemma 29 either $A_i = P \cap r$ or $A_i = \cap_{r \in R, i \in A_i} r$. If, $\forall i \in I$ such that $A_i$ is not finitely generated, we replace $x_i + A_i F$ by $x_i + r_i F$ in the first case, and by the family $(x_i + cF)_{r \in R, i \in A_i}$ in the second case, we deduce from $F$ a family $G$ which has the finite intersection property. Since $F$ is pure-injective, it follows that there exists $x \in F$ which belongs to each element of the family $G$ by Theorem 4. We may assume that the family $(A_i)_{i \in I}$ has no smallest element. So, if $A_i$ is not finitely generated, there exists $j \in I$ such that $A_j \subset A_i$. Let $c \in A_i \setminus P A_j$ such that $x_j + cF \in G$. Then $x - x_j \in cF \subseteq A_i F$ and $x_j - x_i \in A_i F$. Hence $x - x_i \in A_i F$ for each $i \in I$.

Conversely, if $F$ is flat then by Lemma 5 we have $(sF : r) = (sR : r) F$ for each $s, r \in R$. We use Theorem 4 to conclude.

Proposition 7. Let $R$ be a valuation ring and let $F$ be a flat pure-injective module. Then:

1. $F \otimes_R U$ is pure-injective if $U$ is a uniserial module.
2. For each prime ideal $J$, $F_J$ is pure-injective.

Proof.

(1). Let $E = F \otimes_R U$. We use Theorem 4 to prove that $E$ is pure-injective. Let $(x_i)_{i \in I}$ be a family of elements of $F$ such that the family $F = (x_i + N_i)_{i \in I}$ has the finite intersection property, where $N_i = (s_i F : r_i)$ and $r_i, s_i \in R, \forall i \in I$.

First we assume that $U = R/A$ where $A$ is a proper ideal of $R$. So $E \cong F/AF$. If $s_i \notin A$ then $N_i = (s_i F : r_i)/AF = (R r_i : r_i) F/AF$. We set $A_i = (R s_i : r_i)$ in this case. If $s_i \in A$ then $N_i = (AF : r_i)/AF = (A : r_i) F/AF$. We put $A_i = (A : r_i)$ in this case. For each $i \in I$, let $y_i \in F$ such that $x_i = y_i + AF$. It is obvious that the family $(y_i + A_i F)_{i \in I}$ has the finite intersection property. By Proposition 6 this family has a non-empty intersection. Then $F$ has a non-empty intersection too.

Now we assume that $U$ is not finitely generated. It is obvious that $F$ has a non-empty intersection if $x_i + N_i = E, \forall i \in I$. Now assume there exists $i_0 \in I$ such that $x_{i_0} + N_{i_0} \neq E$. Let $I' = \{ i \in I | N_i \subseteq N_{i_0} \}$ and $F' = (x_i + N_i)_{i \in I'}$. Then $F$ and $F'$ have the same intersection. By Lemma 6 $N_{i_0} = F \otimes_R (s_{i_0} U : r_{i_0})$. It follows that $(s_{i_0} U : r_{i_0}) \subseteq U$ because $N_{i_0} \neq E$. Hence $\exists u \in U$ such that $x_{i_0} + N_{i_0} \subseteq F \otimes_R R u$. Then, $\forall i \in I'$, $x_i + N_i \subseteq F \otimes_R R u$. We have $F \otimes_R R u \cong F/(0 : u) F$. From the first part of the proof $F/(0 : u) F$ is pure-injective. So we may replace $R$ with $R/(0 : u)$ and assume that $(0 : u) = 0$. Let $A_i = ((s_i U : r_i) : u), \forall i \in I'$. Then $N_i = A_i F, \forall i \in I'$. By Proposition 4 $F'$ has a non-empty intersection. So $F$ has a non-empty intersection too.

(2). We apply (1) by taking $U = R_J$.

Proof of Theorem 3

Let $J$ be a prime ideal and $E$ a module. If $E$ is fp-injective, $E$ is a pure submodule of an injective module $M$. It follows that $E_J$ is a pure submodule of $M_J$. So, if $M_J$ is injective we conclude that $E_J$ is fp-injective. Now we assume that $E$ is injective.

(1). Suppose that $J \subset Z$. Let $s \in Z \setminus J$. Then there exists $0 \neq r \in J$ such that $s r = 0$. Hence $r E$ is contained in the kernel of the natural map: $E \to E_J$. Moreover $R_J = (R/r R)_J$ and $E_J = (E/r E)_J$. By Proposition 4 $E/r E$ is injective over $R/r R$ and by Theorem 11 $R/r R$ is an IF-ring. So $E/r E$ is flat over $R/r R$. From Proposition 4 we deduce that $E_J$ is pure-injective and by Proposition 1.2 $E_J$ is fp-injective. So $E_J$ is injective.
Assume that \( Z \subseteq J \). We set
\[
F = \{ x \in E \mid J \subseteq (0 : x) \} \quad \text{and} \quad G = \{ x \in E \mid J \subseteq (0 : x) \}.
\]
Let \( x \in E \) and \( s \in R \setminus J \) such that \( sx \in F \) (respectively \( G \)). Then \( sJ \subseteq (0 : x) \) (respectively \( sJ \subseteq (0 : x) \)). Since \( s \notin J \) we have \( sJ = J \). Consequently \( x \in F \) (respectively \( G \)). Thus the multiplication by \( s \) in \( E/F \) (and \( E/G \)) is bijective because \( E \) is injective. So \( E/F \) and \( E/G \) are modules over \( R/J \) and \( E/F \) \( \cong \) \( E/G \). We have \( G \cong \text{Hom}_R(R/J, E) \). It follows that \( E/G \cong \text{Hom}_R(J, E) \). But \( J \) is a flat module. Thus \( E/G \) is injective. Let \( A \) be an ideal of \( R/J \) and \( f : A \to E/F \) an homomorphism. Then there exists an homomorphism \( g : R_J \to E/G \) such that \( g \circ u = p \circ f \) where \( u : A \to R_J \) and \( p : E/F \to E/G \) are the natural maps. It follows that there exists an homomorphism \( h : R_J \to E/F \) such that \( g = p \circ h \). It is easy to check that \( p \circ (f - h \circ u) = 0 \). So there exists an homomorphism \( \ell : A \to G/F \) such that \( v \circ \ell = f - h \circ u \) where \( v : G/F \to E/F \) is the inclusion map. First assume that \( A \) is finitely generated over \( R/I \). We have \( A = R/I \). If \( 0 \neq \ell(a) = y + F \), where \( y \in G \), then \( (0 : a) \subseteq Z \subseteq J = (0 : y) \). Since \( E \) is injective there exists \( x \in E \) such that \( y = ax \). Hence \( f(a) = a(h(1) + (x + F)) \). Now suppose that \( A \) is not finitely generated over \( R/I \). If \( a \in A \) then there exist \( b \in A \) and \( r \in J \) such that \( a = rb \). We get that \( \ell(a) = r\ell(b) = 0 \). Hence \( f = h \circ u \).

(2). Let the notations be as above. Then \( E_Z = E/F \). If \( Z \) is flat, we do as above to show that \( E_Z \) is injective. If \( E \) is flat then \( F = 0 \), whence \( E_Z = E \). Now, assume that \( E_Z \) is \( \text{fp-injective and } Z \) is not flat. By [2] Theorem 10) \( R_Z \) is an IF-ring. It follows that \( E_Z \) is flat. Consequently \( F \) is a pure submodule of \( E \). Suppose there exists \( 0 \neq x \in F \). If \( 0 \neq s \in Z \) then \( (0 : s) \subseteq Z \subseteq (0 : x) \). So, there exists \( y \in E \) such that \( x = sy \). By [2] Lemma 2) \( (0 : y) = s(0 : x) \subseteq Z \). Since \( F \) is a pure submodule, we may assume that \( y \in F \). Whence \( Z \subseteq (0 : y) \). We get a contradiction. Hence \( F = 0 \) and \( E \) is flat. \( \square \)

Now we give a consequence of Theorem 4. Recall that a domain \( R \) is said to be **h-local** if \( R/I \) is semilocal for every nonzero ideal \( I \), and if \( R/P \) is local for every nonzero prime ideal \( P \), [4].

**Corollary 8.** Let \( R \) be a h-local Prüfer domain. For each multiplicative subset \( S \) of \( R \) and for every injective \( R \)-module \( E \), \( S^{-1}E \) is injective.

**Proof.** By [4] Theorem 24) \( E_P \) is injective for each maximal ideal \( P \). Since \( R_P \) is a valuation domain, we deduce from Theorem 4 that \( E_J \) is injective for each prime ideal \( J \). It is easy to check that \( S^{-1}R \) is a h-local Prüfer domain. So, by [4] Theorem 24) \( S^{-1}E \) is injective. \( \square \)

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