Transposition Method for Backward Stochastic Evolution Equations Revisited, and Its Application

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Abstract

The main purpose of this paper is to improve our transposition method to solve both vector-valued and operator-valued backward stochastic evolution equations with a general filtration. As its application, we obtain a general Pontryagin-type maximum principle for optimal controls of stochastic evolution equations in infinite dimensions. In particular, we drop the technical assumption appeared in [11, Theorem 9.1].

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1 Introduction

Let \( T > \tau \geq 0 \), \( d \in \mathbb{N} \), and \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P}) \) be a complete filtered probability space (satisfying the usual conditions), on which a standard \( d \)-dimensional Brownian motion \( \{w(t)\}_{t \in [0,T]} \) is defined. Write \( \mathcal{F} = \{\mathcal{F}_t\}_{t \in [0,T]} \), and let \( X \) be a Banach space. For any \( t \in [0,T] \) and \( r \in [1, \infty) \), denote by \( L^r_{\mathcal{F}_t}(\Omega; X) \) the Banach space of all \( \mathcal{F}_t \)-measurable random variables \( \xi : \Omega \to X \) such that \( \mathbb{E} |\xi|^r_X < \infty \), with the canonical norm. Also, denote by \( L^r_{\mathcal{F}_T}(\Omega; D([\tau,T]; X)) \) the vector space of all \( X \)-valued càdlàg process \( \phi(\cdot) \) such that \( \mathbb{E} \left( \sup_{t \in [\tau,T]} |\phi(r)|_X^r \right)^{\frac{1}{r}} < \infty \). One can show that \( L^r_{\mathcal{F}_T}(\Omega; D([\tau,T]; X)) \) is a Banach space with the norm

\[
|\phi(\cdot)|_{L^r_{\mathcal{F}_T}(\Omega; D([\tau,T]; X))} \triangleq \left[ \mathbb{E} \left( \sup_{t \in [\tau,T]} |\phi(r)|_X^r \right)^r \right]^{\frac{1}{r}}.
\]

Denote by \( C_{\mathcal{F}}([\tau,T]; L^r(\Omega; X)) \) the Banach space of all \( X \)-valued \( \mathcal{F} \)-adapted processes \( \phi(\cdot) \) such that \( \phi(\cdot) : [\tau,T] \to L^r_{\mathcal{F}_T}(\Omega; X) \) is continuous, with the norm

\[
|\phi|_{C_{\mathcal{F}}([\tau,T]; L^r(\Omega; X))} \triangleq \sup_{t \in [\tau,T]} \left( \mathbb{E} |\phi(r)|_X^r \right)^{\frac{1}{r}}.
\]

Similarly, one can define the Banach space \( D_{\mathcal{F}}([\tau,T]; L^r(\Omega; X)) \).

Fix \( r_1, r_2, r_3, r_4 \in [1, \infty] \). Put

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\[
L^2_F(\Omega; L^r(\tau, T; X)) = \left\{ \varphi : (\tau, T) \times \Omega \to X \mid \varphi(\cdot) \text{ is } F\text{-adapted and } \mathbb{E} \left( \int_{\tau}^{T} |\varphi(t)|_X^{r_2} dt \right)^{\frac{r_1}{r_2}} < \infty \right\},
\]
\[
L^p_F(\tau, T; L^{r_1}(\Omega; X)) = \left\{ \varphi : (\tau, T) \times \Omega \to X \mid \varphi(\cdot) \text{ is } F\text{-adapted and } \int_{\tau}^{T} \left( \mathbb{E}|\varphi(t)|_X^{r_1} \right)^{\frac{r_2}{r_1}} dt < \infty \right\}.
\]

Both \(L^2_F(\Omega; L^r(\tau, T; X))\) and \(L^p_F(\tau, T; L^{r_1}(\Omega; X))\) are Banach spaces with the canonical norms. If \(r_1 = r_2\), we simply denote the above spaces by \(L^p_F(\tau, T; X)\). Let \(Y\) be another Banach space. Denote by \(L(X, Y)\) the (Banach) space of all bounded linear operators from \(X\) to \(Y\), with the usual operator norm (When \(Y = X\), we simply write \(L(X)\) instead of \(L(X, Y)\)). Further, we denote by \(L_{pd}(L^p_F(\tau, T; L^r(\Omega; X)), L^p_F(\tau, T; L^{r_1}(\Omega; Y)))\) (resp. \(L_{pd}(X, L^p_F(\tau, T; L^{r_1}(\Omega; Y)))\)) the vector space of all bounded, pointwisely defined linear operators \(G\) from \(L^p_F(\tau, T; L^r(\Omega; X))\) (resp. \(X\)) to \(L^p_F(\tau, T; L^{r_1}(\Omega; Y))\), i.e., for a.e. \((t, \omega) \in (\tau, T) \times \Omega\), there exists an \(L(t, \omega) \in L(X, Y)\) such that \((Gu)(t, \omega) = L(t, \omega)u(t, \omega), \forall u(\cdot) \in L^p_F(\tau, T; L^r(\Omega; X))\) (resp. \((Gx)(t, \omega) = L(t, \omega)x, \forall x \in X\)). In a similar way, one can define the spaces such as \(L_{pd}(L^p_{F\tau}(\Omega; X), L^p_F(\tau, T; L^{r_1}(\Omega; Y)))\) and \(L_{pd}(L^2_{F\tau}(\Omega; X), L^2_F(\tau, T; L^{r_1}(\Omega; Y)))\) for \(t \in [0, T]\), etc.

Let \(H\) be a complex Hilbert space. Denote by \(H^d\) the Cartesian product \(H \times H \times \cdots \times H\). Similarly, we will use the notations \(L(H)^d\), \(L_2(H)^d\) and so on, where \(L_2(H)\) stands for the (Hilbert) space of all Hilbert-Schmidt operators on \(H\).

Let \(A\) be an unbounded linear operator (with domain \(D(A) \subset H\)), which generates a \(C_0\)-semigroup \(\{S(t)\}_{t \geq 0}\) on \(H\). Denote by \(A^*\) the dual operator of \(A\). It is well-known that \(D(A)\) is a Hilbert space with the usual graph norm, and \(A^*\) generates a \(C_0\)-semigroup \(\{S^*(t)\}_{t \geq 0}\), which is the dual \(C_0\)-semigroup of \(\{S(t)\}_{t \geq 0}\).

First, we consider the following \(H\)-valued backward stochastic evolution equation (BSEE for short):

\[
\begin{aligned}
\left\{ \begin{array}{ll}
dy(t) = -A^*y(t)dt + f(t, y(t), Y(t))dt + Y(t)d\omega(t) & \text{in } [\tau, T), \\
y(T) = y_T,
\end{array} \right.
\end{aligned}
\]

(1.1)

where \(y_T \in L^p_{F\tau}(\Omega; H)\) with \(p \in (1, \infty]\), and \(f(\cdot, \cdot, \cdot) : [\tau, T] \times H \times H^d \to H\) satisfies, for some constant \(C_L > 0\),

\[
\left\{ \begin{array}{l}
f(\cdot, 0, 0) \in L^p_F(\tau, T; L^p(\Omega; H)), \\
f(t, x_1, y_1) - f(t, x_2, y_2) \leq C_L(|x_1 - x_2|_H + |y_1 - y_2|_{H^d}), \\
\text{a.e. } (t, \omega) \in [\tau, T] \times \Omega, \forall x_1, x_2 \in H, y_1, y_2 \in H^d.
\end{array} \right.
\]

(1.2)

Here neither the usual natural filtration condition nor the quasi-left continuity is assumed for the filtration \(\mathbb{F}\), and the unbounded operator \(A\) is only assumed to generate a general \(C_0\)-semigroup. Hence, we cannot apply the existing results on infinite dimensional BSEEs (e.g. \([1, 6, 12, 13]\)) to obtain the well-posedness of the equation (1.1).

Next, we consider the following \(L(H)\)-valued BSEE:

\[
\begin{aligned}
\left\{ \begin{array}{ll}
dP = -(A^* + J^*)Pdt - P(A + J)dt - K^*PKdt - (K^*Q + QK)dt + Fdt + Qd\omega(t) & \text{in } [\tau, T), \\
P(T) = P_T,
\end{array} \right.
\end{aligned}
\]

(1.3)

\[\text{1Throughout this paper, for any operator-valued process (resp. random variable) } R, \text{ we denote by } R^* \text{ its pointwisely dual operator-valued process (resp. random variable). For example, if } R \in L^p_F(\tau, T; L^2(\Omega; L(H))), \text{ then } R^* \in L^{p^*}(\tau, T; L^{r_2}(\Omega; L(H))), \text{ and } |R|^p_{L^p_F(\tau, T; L^{r_2}(\Omega; L(H)))} = |R^*|^p_{L^p_F(\tau, T; L^{r_2}(\Omega; L(H)))}.\]
where
\begin{align}
J \in L^p_b(\tau, T; L^\infty(\Omega; \mathcal{L}(H))), & \quad K \in L^p_b(\tau, T; L^\infty(\Omega; \mathcal{L}(H)^d)), \\
F \in L^p_b(\tau, T; \mathcal{L}(\Omega; \mathcal{L}(H))), & \quad P_T \in L^p_{F_T}(\Omega; \mathcal{L}(H)).
\end{align}
(1.4)

If \( H = \mathbb{R}^m \) for some \( m \in \mathbb{N} \), then (1.3) is an \( m \times m \) matrix-valued backward stochastic differential equation (BSDE for short), and hence, one can easily obtain its well-posedness for this special case. On the other hand, if \( \dim H = \infty \), \( F \in L^p_b(\tau, T; \mathcal{L}(\Omega; \mathcal{L}_2(H))) \) and \( P_T \in L^p_{F_T}(\Omega; \mathcal{L}(H)) \), then (1.3) is a special case of (1.1) (because \( \mathcal{L}_2(H) \) is a Hilbert space), and therefore in this case the well-posedness of (1.3) follows from that of (1.1). However, the situation is completely different when \( \dim H = \infty \) if one does not impose further assumptions on \( F \) and \( P_T \). Indeed, in the infinite dimensional setting, although \( \mathcal{L}(H) \) is still a Banach space, it is neither reflexive nor separable even if \( H \) itself is separable. Because of this, \( \mathcal{L}(H) \) is NOT a UMD space (needless to say a Hilbert space), and consequently, it is even a quite difficult problem to define the stochastic integral \( \int_T^T Qdw(t) \) (appeared in (1.3)) for an \( \mathcal{L}(H) \)-valued process \( Q \). We refer to [3, 11] for previous studies on the well-posedness of (1.3), by avoiding the definition of \( \int_T^T Qdw(t) \) in one way or another.

Similar to the finite dimensional case ([14]), both (1.1) and (1.3) play crucial roles in establishing the Pontryagin-type maximum principle for optimal controls of general infinite dimensional nonlinear stochastic systems with control-dependent diffusion terms and possibly nonconvex control regions ([3, 4, 11, 15]).

The main purpose of this paper is to improve our transposition method, developed in [11], to solve the equations (1.1) and (1.3). Especially, we shall give some well-posedness/regularity results for solutions to these two equations such that they can be conveniently used in the above mentioned Pontryagin-type maximum principle. In the stochastic finite dimensional setting, the transposition method (for solving BSDEs) was introduced in our paper [10], but one can find a rudiment of this method at [16, pp. 353–354].

We remark that, our method is also motivated by the classical transposition method to solve the non-homogeneous boundary value problems for deterministic partial differential equations (see [8] for a systematic introduction to this topic) and especially the boundary controllability problem for hyperbolic equations ([7]).

For the readers’ convenience, let us recall below the main idea of the classical transposition method to solve the following deterministic wave equation with non-homogeneous Dirichlet boundary conditions:
\begin{align}
\begin{cases}
y_{tt} - \Delta y = 0 & \text{in } Q \triangleq (0, T) \times G, \\
y = u & \text{on } \Sigma \triangleq (0, T) \times \Gamma, \\
y(0) = y_0, \quad y_t(0) = y_1 & \text{in } G,
\end{cases}
\end{align}
(1.5)

where \( G \) is a nonempty open bounded domain in \( \mathbb{R}^d \) with a \( C^2 \) boundary \( \Gamma \), \( (y_0, y_1) \in L^2(G) \times H^{-1}(G) \) and \( u \in L^2(\Sigma) \) are given, and \( y \) is the unknown.

When \( u \equiv 0 \), one can use the standard semigroup theory to prove the well-posedness of (1.5) in the solution space \( C([0, T]; L^2(G)) \cap C^1([0, T]; H^{-1}(G)) \).

When \( u \not\equiv 0 \), one needs to use the transposition method because \( y|_\Sigma = u \) does NOT make sense by the usual trace theorem. For this purpose, for any \( f \in L^1(0, T; L^2(G)) \) and \( g \in L^1(0, T; H^1_0(G)) \), consider the following adjoint equation of (1.5):
\begin{align}
\begin{cases}
\zeta_{tt} - \Delta \zeta = f + g_t, & \text{in } Q, \\
\zeta = 0, & \text{on } \Sigma, \\
\zeta(T) = \zeta_t(T) = 0, & \text{in } G.
\end{cases}
\end{align}
(1.6)
This equation admits a unique solution \( \zeta \in C([0, T]; H^1_0(G)) \cap C^1([0, T]; L^2(G)) \), which enjoys a hidden regularity \( \frac{\partial \zeta}{\partial T} \in L^2(\Sigma) \) (c.f. \[7\]). Here and henceforth, \( \nu \equiv \nu(x) \) stands for the unit outward normal vector of \( G \) at \( x \in \Gamma \).

In order to give a reasonable definition for the solution to (1.5) by the transposition method, we consider first the case when \( y \) is sufficiently smooth. Assume that \( g \in C_0^\infty(0, T; H^1_0(G)), y_1 \in L^2(G), y \in H^2(Q) \) satisfies (1.5). Then, multiplying the first equation in (1.5) by \( \zeta \), integrating it in \( Q \), and using integration by parts, we find that

\[
\int_Q fy dx dt - \int_Q gy_t dx dt = \int_G \zeta(0)y_1 dx - \int_G \zeta_t(0)y_0 dx - \int_\Sigma \frac{\partial \zeta}{\partial \nu} ud\Sigma. \tag{1.7}
\]

Note that (1.7) still makes sense even if the regularity of \( y \) is relaxed as \( y \in C([0, T]; L^2(G)) \cap C^1([0, T]; H^{-1}(G)) \). This leads to the following notion:

**Definition 1.1** We call \( y \in C([0, T]; L^2(G)) \cap C^1([0, T]; H^{-1}(G)) \) a transposition solution to (1.5), if \( y(0) = y_0, y_t(0) = y_1 \), and for any \( f \in L^1(0, T; L^2(G)) \) and \( g \in L^1(0, T; H^1_0(G)) \), it holds that

\[
\int_Q fy dx dt - \int_0^T \langle g, y_t \rangle_{H^1_0(G), H^{-1}(G)} dt = (\zeta(0), y_1)_{H^1_0(G), H^{-1}(G)} + \int_\Omega \zeta_t(0)y_0 dx - \int_\Sigma \frac{\partial \zeta}{\partial \nu} ud\Sigma,
\]

where \( \zeta \) is the unique solution to (1.6).

One can show the well-posedness of (1.5) in the sense of Definition 1.1, by means of the transposition method ([7]). Clearly, the point of this method is to interpret the solution to a forward wave equation with non-homogeneous Dirichlet boundary conditions in terms of another backward wave equation with non-homogeneous source terms. Of course, in the deterministic setting, since the wave equation is time-reversible, there exists no essential difference between the forward problem and the backward one. Nevertheless, this reminds us to interpret BSDEs/BSEEs in terms of forward stochastic differential/equation evolutions, as we have done in [10, 11]. Clearly, the transposition method is a variant of the standard duality method, and in some sense it provides a way to see something which is not easy to be detected directly.

The rest of this paper is organized as follows. Section 2 is addressed to the well-posedness of the equation (1.1). Sections 3 and 4 are devoted to the well-posedness of the equation (1.3) and a regularity property for its solutions, respectively. Finally, in Section 5, we show a stochastic Pontryagin-type maximum principle for controlled stochastic evolution equations in infinite dimensions.

## 2 Well-posedness of the vector-valued BSEEs

In this section, we discuss the well-posedness of the equation (1.1) in the transposition sense.

Consider the following (forward) stochastic evolution equation:

\[
\begin{cases}
  dz = (Az + \psi_1)ds + \psi_2 dw(s) \quad \text{in } (t, T], \\
  z(t) = \eta,
\end{cases} \tag{2.1}
\]

where \( t \in [\tau, T], q = \frac{p}{p-1}, \psi_1 \in L^q_p(\Omega; L^1(t, T; H)), \psi_2 \in L^q_p(t, T; L^q(\Omega; H^d)) \) and \( \eta \in L^q_{\mathcal{F}_\tau}(\Omega; H) \).

Let us recall that \( z(\cdot) \in C_{\mathcal{F}}([t, T]; L^q(\Omega; H)) \) is a (mild) solution to the equation (2.1) if

\[
z(s) = S(s - t)\eta + \int_t^s S(s - \sigma)\psi_1(\sigma)d\sigma + \int_t^s S(s - \sigma)\psi_2(\sigma)dw(\sigma), \quad \forall s \in [t, T].
\]

We now introduce the following notion.
**Definition 2.1** We call \((y(\cdot), Y(\cdot)) \in L^p_\mathbb{F}(\Omega; D([\tau,T]; H)) \times L^q_\mathbb{F}(\tau,T; L^p(\Omega; H^d))\) a transposition solution to (1.1) if for any \(t \in [\tau,T], \psi_1(\cdot) \in L^2_\mathbb{F}(\Omega; L^1(t,T; H)), \psi_2(\cdot) \in L^2_\mathbb{F}(t,T; L^3(\Omega; H^d)), \eta \in L^q_\mathbb{F}(\Omega; H)\) and the corresponding solution \(z \in C_\mathbb{F}([t,T]; L^q(\Omega; H))\) to (2.1), it holds that

\[
\mathbb{E}\langle z(T), y_T \rangle_H = \mathbb{E} \int_t^T \langle z(s), f(s, y(s), Y(s)) \rangle_H \, ds
\]

\[
= \mathbb{E}\langle \eta, y(t) \rangle_H + \mathbb{E} \int_t^T \langle \psi_1(s), y(s) \rangle_H \, ds + \mathbb{E} \int_t^T \langle \psi_2(s), Y(s) \rangle_H \, ds.
\]  

(2.2)

In what follows, we will use \(C\) to denote a generic positive constant, which may be different from one place to another. We have the following result for the well-posedness of the equation (1.1).

**Theorem 2.1** For any \(y_T \in L^p_\mathbb{F}(\Omega; H), f(\cdot, \cdot, \cdot) : [\tau,T] \times H \times H^d \to H\) satisfying (1.2), the equation (1.1) admits one and only one transposition solution \((y(\cdot), Y(\cdot)) \in L^p_\mathbb{F}(\Omega; D([\tau,T]; H)) \times L^q_\mathbb{F}(\tau,T; L^p(\Omega; H^d))\). Furthermore,

\[
|\langle y(\cdot), Y(\cdot) \rangle|_{L^p_\mathbb{F}(\Omega; D([\tau,T]; H)) \times L^q_\mathbb{F}(\tau,T; L^p(\Omega; H^d))}
\leq C \left[ |f(\cdot, 0, 0)|_{L^1_\mathbb{F}(t,T; L^p(\Omega; H))} + |y_T|_{L^p_\mathbb{F}(\Omega; H)} \right], \quad \forall t \in [\tau,T].
\]  

(2.3)

**Remark 2.1** In [11, Theorem 3.1], it was assumed that \(1 < p \leq 2\) and \(n = 1\). Also, in Theorem 2.1 the solution space of the first unknown \(y\) in (1.1) is \(L^p_\mathbb{F}(\Omega; D([\tau,T]; H))\); while in [11] this space is \(D_\mathbb{F}([\tau,T]; L^p(\Omega; H))\). It is easy to see that

\[
L^p_\mathbb{F}(\Omega; D([\tau,T]; H)) \hookrightarrow D_\mathbb{F}([\tau,T]; L^p(\Omega; H)),
\]

algebraically and topologically. Hence, compared to [11, Theorem 3.1], Theorem 2.1 improves a little the regularity of solutions to (1.1).

Before proving Theorem 2.1, we first recall the following Riesz-type Representation Theorem (See [9, Corollary 2.3 and Remark 2.4]).

**Lemma 2.1** Fix \(t_1\) and \(t_2\) satisfying \(0 \leq t_2 < t_1 \leq T\). Assume that \(\mathcal{Y}\) is a reflexive Banach space. Then, for any \(r, s \in [1, \infty),\) it holds that

\[
(L^p_\mathbb{F}(t_2, t_1; L^s(\Omega; \mathcal{Y})))^* = L^p_\mathbb{F}(t_2, t_1; L^{s'}(\Omega; \mathcal{Y}')), \quad (L^q_\mathbb{F}(\Omega; L^r(t_2, t_1; \mathcal{Y})))^* = L^q_\mathbb{F}(\Omega; L^{r'}(t_2, t_1; \mathcal{Y}')), \n\]

where

\[
s' = \begin{cases} s/(s - 1), & \text{if } s \neq 1, \\ \infty & \text{if } s = 1; \end{cases} \quad r' = \begin{cases} r/(r - 1), & \text{if } r \neq 1, \\ \infty & \text{if } r = 1. \end{cases}
\]

**Proof of Theorem 2.1:** It suffices to consider a particular case for (1.1), i.e. the case that \(f(\cdot, \cdot, \cdot)\) is independent of the second and third arguments. More precisely, we consider the following equation:

\[
\begin{cases} 
  dy(t) = -A^s y(t) \, dt + f(t) \, dt + Y(t) \, dw(t) \quad \text{in } [\tau,T], \\
y(T) = y_T,
\end{cases}
\]  

(2.4)
where \( y_T \in L^p_{F_T}(\Omega; H) \) and \( f(\cdot) \in L^1_T(\tau, T; L^p_\Omega(\Omega; H)) \). The general case follows from the well-posedness for \((2.4)\) and the standard fixed point technique.

We divide the proof into several steps. Since the proof is very similar to that of [11, Theorem 3.1], we give below only a sketch.

**Step 1.** For any \( t \in [\tau, T] \), we define a linear functional \( \ell \) (depending on \( t \)) on the Banach space \( L^q_\Omega (\Omega; L^1(t, T; H)) \times L^2(t, T; L^q_\Omega (\Omega; H^d)) \times L^q_{F_t}(\Omega; H) \) as follows:

\[
\ell (\psi_1(\cdot), \psi_2(\cdot), \eta) = \mathbb{E} \langle z(T), y_T \rangle_H - \mathbb{E} \int_t^T \langle z(s), f(s) \rangle_H ds,
\]

\[
\forall \ (\psi_1(\cdot), \psi_2(\cdot), \eta) \in L^q_\Omega (\Omega; L^1(t, T; H)) \times L^2(t, T; L^q_\Omega (\Omega; H^d)) \times L^q_{F_t}(\Omega; H),
\]

where \( z(\cdot) \in C_{[t,T]}(L^q(\Omega; H)) \) solves the equation \((2.1)\). It is an easy matter to show that \( \ell \) is a bounded linear functional on \( L^q_\Omega (\Omega; L^1(t, T; H)) \times L^2(t, T; L^q_\Omega (\Omega; H^d)) \times L^q_{F_t}(\Omega; H) \). By Lemma 2.1, there exists a triple

\[
(y^l(\cdot), Y^l(\cdot), \xi^l) \in L^p_\Omega (\Omega; L^\infty (t, T; H)) \times L^2(t, T; L^p(\Omega; H^d)) \times L^p_{F_t}(\Omega; H)
\]

such that

\[
\mathbb{E} \langle z(T), y_T \rangle_H - \mathbb{E} \int_t^T \langle z(s), f(s) \rangle_H ds = \mathbb{E} \int_t^T \langle \psi_1(s), y^l(s) \rangle_H ds + \mathbb{E} \int_t^T \langle \psi_2(s), Y^l(s) \rangle_{H^d} ds + \mathbb{E} \langle \eta, \xi^l \rangle_H.
\]

It is clear that \( \xi^T = y_T \). Furthermore,

\[
| (y^l(\cdot), Y^l(\cdot), \xi^l) |_{L^p_\Omega (\Omega; L^\infty (t, T; H)) \times L^2(t, T; L^p(\Omega; H^d)) \times L^p_{F_t}(\Omega; H)} \leq C \left[ | f(\cdot) |_{L^p_\Omega (\Omega; L^p(\Omega; H))} + | y_T |_{L^p_{F_T}(\Omega; H)} \right], \quad \forall t \in [\tau, T].
\]

**Step 2.** Note that the \((y^l(\cdot), Y^l(\cdot))\) obtained in Step 1 may depend on \( t \). Now we show the time consistency of \((y^l(\cdot), Y^l(\cdot))\), that is, for any \( t_1 \) and \( t_2 \) satisfying \( \tau \leq t_2 \leq t_1 \leq T \), it holds that

\[
(y^{l_2}(s, \omega), Y^{l_2}(s, \omega)) = (y^{l_1}(s, \omega), Y^{l_1}(s, \omega)), \quad \text{a.e.} \ (s, \omega) \in [t_1, T] \times \Omega,
\]

by suitable choice of the \( \eta, \psi_1 \) and \( \psi_2 \) in \((2.1)\). In fact, for any fixed \( \rho(\cdot) \in L^q_\Omega (\Omega; L^1(t_1, T; H)) \) and \( \varsigma(\cdot) \in L^p_{F_T}(t_1, T; L^q(\Omega; H^d)) \), we choose first \( t = t_1, \eta = 0, \psi_1(\cdot) = \rho(\cdot) \) and \( \psi_2(\cdot) = \varsigma(\cdot) \) in \((2.1)\). From \((2.6)\), we obtain that

\[
\mathbb{E} \langle z^{t_1}(T), y_{t_1} \rangle_H - \mathbb{E} \int_{t_1}^T \langle z^{t_1}(s), f(s) \rangle_H ds = \mathbb{E} \int_{t_1}^T \langle \rho(s), y^{t_1}(s) \rangle_H ds + \mathbb{E} \int_{t_1}^T \langle \varsigma(s), Y^{t_1}(s) \rangle_{H^d} ds.
\]

Then, we choose \( t = t_2, \eta = 0, \psi_1(t, \omega) = \chi_{[t_1, T]}(t) \rho(t, \omega) \) and \( \psi_2(t, \omega) = \chi_{[t_1, T]}(t) \varsigma(t, \omega) \) in \((2.1)\). It follows from \((2.6)\) that

\[
\mathbb{E} \langle z^{t_1}(T), y_{t_1} \rangle_H - \mathbb{E} \int_{t_1}^T \langle z^{t_1}(s), f(s) \rangle_H ds = \mathbb{E} \int_{t_1}^T \langle \rho(s), y^{t_2}(s) \rangle_H ds + \mathbb{E} \int_{t_1}^T \langle \varsigma(s), Y^{t_2}(s) \rangle_{H^d} ds.
\]

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Combining (2.9) and (2.10), we obtain that
\[
\mathbb{E} \int_{t_1}^{T} \langle \varrho(s), y^{t_1}(s) - y^{t_2}(s) \rangle_H ds + \mathbb{E} \int_{t_1}^{T} \langle \zeta(s), Y^{t_1}(s) - Y^{t_2}(s) \rangle_{H^d} ds = 0,
\]
\[\forall \, \varrho(\cdot) \in L^p_{\mathcal{F}}(\Omega; L^1(t_1, T; H)), \quad \zeta(\cdot) \in L^q_{\mathcal{F}}(t_1, T; L^q(\Omega; H^d)).\]
This yields the desired equality (2.8).

Put
\[y(t, \omega) = y^\tau(t, \omega), \quad Y(t, \omega) = Y^\tau(t, \omega), \quad \forall \, (t, \omega) \in [\tau, T] \times \Omega. \tag{2.11}\]
From (2.8), we see that
\[(y'(s, \omega), Y^t(s, \omega)) = (y(s, \omega), Y(s, \omega)), \quad \text{a.e.} \, (s, \omega) \in [t, T] \times \Omega. \tag{2.12}\]
Combining (2.6) and (2.12), we get that
\[
\mathbb{E} \langle z(T), y_T \rangle_H - \mathbb{E} \langle \eta, \xi^t \rangle_H = \mathbb{E} \int_{t}^{T} \langle z(s), f(s) \rangle_H ds + \mathbb{E} \int_{t}^{T} \langle \psi_1(s), y(s) \rangle_H ds + \mathbb{E} \int_{t}^{T} \langle \psi_2(s), Y(s) \rangle_{H^d} ds, \tag{2.13}\]
\[\forall \, (\psi_1(\cdot), \psi_2(\cdot), \eta) \in L^p_{\mathcal{F}}(\Omega; L^1(t, T; H)) \times L^q_{\mathcal{F}}(t, T; L^q(\Omega; H^d)) \times L^q_{\mathcal{F}, t}(\Omega; H).\]

**Step 3.** We show in this step that \(\xi^t\) has a càdlàg modification, and \(y(t, \omega) = \xi^t(\omega)\) for a.e. \((t, \omega) \in [\tau, T] \times \Omega\). The detail is lengthy and very similar to Steps 3–4 in the proof of [11, Theorem 3.1], and hence we omit it here. This completes the proof of Theorem 2.1.

### 3 Well-posedness of the operator-valued BSEEs

In this section, we consider the well-posedness of (1.3).

In order to define the transposition solution of (1.3), for any \(t \in [\tau, T]\), we introduce the following two (forward) stochastic evolution equations:
\[
\begin{align*}
&\begin{aligned}
&dx_1 = (A + J)x_1 ds + u_1 ds + Kx_1 dw(s) + v_1 dw(s) \quad \text{in} \, (t, T], \\
x_1(t) = \xi_1
\end{aligned} \\
&\begin{aligned}
&dx_2 = (A + J)x_2 ds + u_2 ds + Kx_2 dw(s) + v_2 dw(s) \quad \text{in} \, (t, T], \\
x_2(t) = \xi_2
\end{aligned}
\end{align*}
\tag{3.1}
\]
and
\[
\begin{align*}
&\begin{aligned}
&dx_1 = (A + J)x_1 ds + u_1 ds + Kx_1 dw(s) + v_1 dw(s) \quad \text{in} \, (t, T], \\
x_1(t) = \xi_1
\end{aligned} \\
&\begin{aligned}
&dx_2 = (A + J)x_2 ds + u_2 ds + Kx_2 dw(s) + v_2 dw(s) \quad \text{in} \, (t, T], \\
x_2(t) = \xi_2
\end{aligned}
\end{align*}
\tag{3.2}
\]
where \(\xi_1 \in L^2_{\mathcal{F}}(\Omega; H), u_i \in L^2_{\mathcal{F}}(\Omega; L^2(t, T; H)), v_i \in L^2_{\mathcal{F}}(t, T; L^2(\Omega; H^d))\) and \(i = 1, 2\). Also, we need to introduce the solution space for the equation (1.3). Put
\[
L^p_{\mathcal{F}, \omega}(\Omega; D([\tau, T]; \mathcal{L}(H))) \triangleq \left\{ P(\cdot, \cdot) \mid P(\cdot, \cdot) \in \mathcal{L}_{pd}(L^2_{\mathcal{F}}(\Omega; L^2(\tau, T; H)), L^{2p}_{q+1}(\Omega; L^2(\tau, T; H))), \right. \]
\[\text{and for every} \quad t \in [\tau, T] \text{ and } \xi \in L^2_{\mathcal{F}}(\Omega; H), P(\cdot, \cdot) \xi \in L^{2p}_{q+1}(\Omega; D([t, T]; H)) \]
\[\text{and } |P(\cdot, \cdot)\xi|_{L^{2p}_{q+1}(\Omega; D([t, T]; H))} \leq C|\xi|_{L^2_{\mathcal{F}}(\Omega; H)} \right\}
\]
and
\[
L^2_{\mathcal{F}, w}(\tau, T; L^p(\Omega; \mathcal{L}(H)))^d \triangleq \left[ \mathcal{L}_{pd}(L^2_{\mathcal{F}}(\tau, T; L^2(\Omega; H^d)), L^{2p}_{q+1}(\tau, T; L^{2p}_{q+1}(\Omega; H^d))) \right],
\]
where \(r \geq 2\). The transposition solution to the equation (1.3) is defined as follows:
Definition 3.1 We call \((P(\cdot), Q(\cdot)) \in L^p_{F,w}(\Omega; D([\tau, T]; \mathcal{L}(H))) \times L^2_{F,L}(\tau, T; L^p(\Omega; \mathcal{L}(H)^d))\) a transposition solution to the equation (1.3) if for any \(t \in [\tau, T]\), \(\xi_1, \xi_2 \in L^2_F(\Omega; H)\), \(u_1(\cdot), u_2(\cdot) \in L^2_F(\Omega; L^2(t, T; H))\) and \(v_1(\cdot), v_2(\cdot) \in L^2_F(t, T; L^2(\Omega; H^d))\), it holds that

\[
\mathbb{E}(P_T x_1(T), x_2(T))_H - \mathbb{E} \int_t^T \langle F(s)x_1(s), x_2(s) \rangle_H ds \\
= \mathbb{E}(P(t)\xi_1, \xi_2)_H + \mathbb{E} \int_t^T \langle P(s)u_1(s), x_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle P(s)x_1(s), u_2(s) \rangle_H ds \\
+ \mathbb{E} \int_t^T \langle P(s)K(s)x_1(s), v_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle P(s)v_1(s), K(s)x_2(s) + v_2(s) \rangle_H ds \\
+ \mathbb{E} \int_t^T \langle Q(s)v_1(s), x_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle x_1(s), Q(s)^*v_2(s) \rangle_H ds,
\]

where, \(x_1(\cdot)\) and \(x_2(\cdot)\) solve (3.1) and (3.2), respectively.

The well-posedness of the equation (1.3) in the sense of Definition 3.1 is still open. However, we can show the following uniqueness result for the transposition solution to (1.3).

Theorem 3.1 Assume that \(J, K, F\) and \(P_T\) satisfy (1.4). Then the equation (1.3) admits at most one transposition solution \((P(\cdot), Q(\cdot)) \in L^p_{F,w}(\Omega; D([\tau, T]; \mathcal{L}(H))) \times L^2_{F,w}(\tau, T; L^p(\Omega; \mathcal{L}(H)^d))\).

Proof: Assume that \((\overline{P}(\cdot), \overline{Q}(\cdot)) \in L^p_{F,w}(\Omega; D([\tau, T]; \mathcal{L}(H))) \times L^2_{F,w}(\tau, T; L^p(\Omega; \mathcal{L}(H)^d))\) is another transposition solution to (1.3). Then, by Definition 3.1, it follows that, for any \(t \in [\tau, T]\),

\[
0 = \mathbb{E}[\langle \overline{P}(t) - P(t), \xi_1, \xi_2 \rangle_H] + \mathbb{E} \int_t^T \langle \overline{P}(s) - P(s), u_1(s), x_2(s) \rangle_H ds \\
+ \mathbb{E} \int_t^T \langle \overline{P}(s) - P(s), x_1(s), u_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle \overline{P}(s) - P(s), K(s)x_1(s), v_2(s) \rangle_H ds \\
+ \mathbb{E} \int_t^T \langle \overline{P}(s) - P(s), v_1(s), K(s)x_2(s) + v_2(s) \rangle_H ds \\
+ \mathbb{E} \int_t^T \langle \overline{Q}(s) - Q(s), v_1(s), x_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle x_1(s), \overline{Q}(s)^* - Q(s)^* \rangle_H ds.
\]

Choosing \(u_1 = u_2 = 0\) and \(v_1 = v_2 = 0\) in the equation (3.1) and the equation (3.2), respectively, by (3.4), we obtain that

\[
0 = \mathbb{E}[\langle \overline{P}(t) - P(t), \xi_1, \xi_2 \rangle_H], \quad \forall t \in [\tau, T], \forall \xi_1, \xi_2 \in L^2_F(\Omega; H).
\]

Hence, we find that \(\overline{P}(\cdot) = P(\cdot)\). By this, it is easy to see that for any \(t \in [\tau, T]\),

\[
0 = \mathbb{E} \int_t^T \langle \overline{Q}(s) - Q(s), v_1(s), x_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle x_1(s), \overline{Q}(s)^* - Q(s)^* \rangle_H ds.
\]

Choosing \(t = \tau, \xi_2 = 0\) and \(v_2 = 0\) in (3.2), we see that (3.5) becomes

\[
0 = \mathbb{E} \int_\tau^T \langle \overline{Q}(s) - Q(s), v_1(s), x_2(s) \rangle_H ds.
\]

Similar to the proof of [11, Theorem 4.1], one can show that the set

\[
\Xi \triangleq \left\{ x_2(\cdot) \mid x_2(\cdot) \text{ solves (3.2)} \text{ with } t = \tau, \xi_2 = 0, v_2 = 0 \text{ and some } u_2 \in L^2_F(\Omega; L^2(\tau, T; H)) \right\}
\]
is dense in \( L^4_F(0, T; L^{2q}(\Omega; H)) \). By this fact, and noting (3.6), we see that
\[
[\bar{Q}(\cdot) - Q(\cdot)] v_1(\cdot) = 0, \quad \forall v_1(\cdot) \in L^4_F(0, T; L^{2q}(\Omega; H^d)).
\]

Therefore, we find that we see that for all \( r \geq 2 \),
\[
[\bar{Q}(\cdot) - Q(\cdot)] v_1(\cdot) = 0, \quad \forall v_1(\cdot) \in L^r_F(0, T; L^{2q}(\Omega; H^d)).
\]

Hence \( \bar{Q}(\cdot) = Q(\cdot) \). This completes the proof of Theorem 3.1.

Further, we have the following well-posedness result for (1.3) in a special case.

**Theorem 3.2** If \( H \) is a separable Hilbert space, \( P_T \in L^r_T(\Omega; \mathcal{L}_2(H)) \), \( F \in L^2_B(\tau, T; L^p(\Omega; \mathcal{L}_2(H))) \), \( J \in L^4_F(\tau, T; L^\infty(\Omega; \mathcal{L}(H))) \) and \( K \in L^4_F(\tau, T; L^\infty(\Omega; \mathcal{L}(H)^d)) \), then (1.3) admits a unique transposition solution \( \langle \bar{P}(\cdot, \cdot), \cdot \rangle \in L^2_F(\Omega; D([\tau, T]; \mathcal{L}_2(H))) \times L^2_F(\tau, T; L^p(\Omega; \mathcal{L}_2(H)^d)) \). Furthermore,
\[
|\langle P, Q \rangle|_{L^2_F(\Omega; D([\tau, T]; \mathcal{L}_2(H))) \times L^2_F(\tau, T; L^p(\Omega; \mathcal{L}_2(H)^d))} \leq C(|F|_{L^2_F(\tau, T; L^p(\Omega; \mathcal{L}_2(H)))} + |P_T|_{L^r_T(\Omega; \mathcal{L}_2(H))}). \tag{3.7}
\]

**Proof:** The proof is very similar to that for [11, Theorem 4.2], and hence we only give below a sketch.

First, we define a family of operators \( \{\mathcal{T}(t)\}_{t \geq 0} \) on \( \mathcal{L}_2(H) \) as follows:
\[
\mathcal{T}(t)O = S(t)OS^*(t), \quad \forall O \in \mathcal{L}_2(H).
\]

Then, \( \{\mathcal{T}(t)\}_{t \geq 0} \) is a \( C_0 \)-semigroup on \( \mathcal{L}_2(H) \). Denote by \( \mathcal{A} \) the infinitesimal generator of \( \{\mathcal{T}(t)\}_{t \geq 0} \). We consider the following \( \mathcal{L}_2(H) \)-valued BSEE:
\[
\begin{cases}
   dP = -\mathcal{A}^*Pdt + f(t, P, Q)dt + Qdw & \text{in } [\tau, T), \\
   P(T) = P_T,
\end{cases} \tag{3.8}
\]

where \( f(t, P, Q) = -J^*P - J*K^*PK - K^*Q - QK + F \). Since \( J \in L^2_F(\tau, T; L^\infty(\Omega; \mathcal{L}(H))) \), \( K \in L^2_F(\tau, T; L^\infty(\Omega; \mathcal{L}(H)^d)) \) and \( F \in L^2_B(\tau, T; L^p(\Omega; \mathcal{L}_2(H))) \), we see that \( f(\cdot, \cdot, \cdot) \) satisfies (1.2). Since \( \mathcal{L}_2(H) \) is a Hilbert space, by Theorem 2.1, one can find a pair \( \langle P, Q \rangle \in L^2_F(\Omega; D([\tau, T]; \mathcal{L}_2(H))) \times L^2_F(\tau, T; L^p(\Omega; \mathcal{L}_2(H)^d)) \) solving the equation (3.8) in the sense of Definition 2.1. Further, \( \langle P, Q \rangle \) satisfies (3.7).

Next, denote by \( O(\cdot) \) the tensor product of \( x_1(\cdot) \) and \( x_2(\cdot) \), solutions to (3.1) and (3.2), respectively. As usual, \( O(s, \omega)x = \langle x, x_1(s, \omega) \rangle_H x_2(s, \omega) \) for a.e. \( (s, \omega) \in [t, T] \times \Omega \) and \( x \in H \). Hence, \( O(s, \omega) \in \mathcal{L}_2(H) \) for a.e. \( (s, \omega) \in [t, T] \times \Omega \). It can be proved that
\[
\begin{cases}
   dO(s) = \mathcal{A}O(s)ds + uds + vdw(s) & \text{in } (t, T], \\
   O(t) = \xi_1 \otimes \xi_2,
\end{cases} \tag{3.9}
\]

where
\[
\begin{cases}
   u = JO(\cdot) + O(\cdot)J^* + u_1 \otimes x_2 + x_1 \otimes u_2 + KO(\cdot)K^* + (Kx_1) \otimes v_2 + v_1 \otimes (Kx_2) + v_1 \otimes v_2, \\
   v = KO(\cdot) + O(\cdot)K^* + v_1 \otimes x_2 + x_1 \otimes v_2.
\end{cases}
\]

Noting that \( \langle P(\cdot), Q(\cdot) \rangle \) solves the equation (3.8) in the transposition sense and by (3.9), we have that
\[
\begin{align*}
   &\mathbb{E}\langle O(T), P_T \rangle_{\mathcal{L}_2(H)} - \mathbb{E} \int_t^T \langle O(s), f(s, P(s), Q(s)) \rangle_{\mathcal{L}_2(H)} ds \\
   &= \mathbb{E} \langle \xi_1 \otimes \xi_2, P(t) \rangle_{\mathcal{L}_2(H)} + \mathbb{E} \int_t^T \langle u(s), P(s) \rangle_{\mathcal{L}_2(H)} ds + \mathbb{E} \int_t^T \langle v(s), Q(s) \rangle_{\mathcal{L}_2(H)^d} ds. \tag{3.10}
\end{align*}
\]
Finally, by (3.10) and some direct computation, one can show that \((P(\cdot), Q(\cdot))\) satisfies (3.3), and therefore it is a transposition solution to the equation (1.3) (in the sense of Definition 3.1). The uniqueness of \((P(\cdot), Q(\cdot))\) follows from Theorem 3.1.

Since we are not able to prove the well-posedness of the equation (1.3) in the sense of Definition 3.1 at this moment, we need to introduce a weaker notion of solution, i.e., relaxed transposition solution to this equation. For this purpose, we write

\[
Q^p[\tau, T] = \left\{ (Q^{(i)}, \hat{Q}^{(i)}) \mid Q^{(i)} = (Q^{1,(i)}, \ldots, Q^{d,(i)}), \hat{Q}^{(i)} = (\hat{Q}^{1,(i)}, \ldots, \hat{Q}^{d,(i)}) \right\}.
\]

For \((Q^{(i)}, \hat{Q}^{(i)}) \in Q^p[\tau, T]\), put

\[
\left\| (Q^{(i)}, \hat{Q}^{(i)}) \right\|_{Q^p[\tau, T]} \triangleq \sum_{i=1}^d \sup_{t \in [\tau, T]} \left\| (Q^{(i)}(t), \hat{Q}^{(i)}(t)) \right\|_{\mathcal{L}(L^q_{\tau}(\Omega; H) \times L^q_{\tau}(\Omega; L^2(t,T;H)) \times L^q_{\tau}(\Omega; L^2(t,T;H)))}^2.
\]

The relaxed transposition solution to (1.3) is defined as follows:

**Definition 3.2** We call \((P(\cdot), Q^{(\cdot)}, \hat{Q}^{(\cdot)}) \in L^p_{\tau,T}(\Omega; D([\tau, T]; \mathcal{L}(H))) \times Q^p[\tau, T]\) a relaxed transposition solution to the equation (1.3) if for any \(t \in [\tau, T]\), \(\xi_1, \xi_2 \in L^2(\Omega; H)\), \(u_1(\cdot), u_2(\cdot) \in L^2(\Omega; L^2(t,T;H))\) and \(v_1(\cdot), v_2(\cdot) \in L^2(\Omega; L^2(t,T;H))\), it holds that

\[
\begin{align*}
\mathbb{E} \langle P_T x_1(T), x_2(T) \rangle_H &- \mathbb{E} \int_t^T \langle F(s) x_1(s), x_2(s) \rangle_H ds \\
&= \mathbb{E} \langle P(t) \xi_1, \xi_2 \rangle_H + \mathbb{E} \int_t^T \langle P(s) u_1(s), x_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle P(s) x_1(s), u_2(s) \rangle_H ds \\
&\quad + \mathbb{E} \int_t^T \langle P(s) K(s) x_1(s), v_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle P(s) v_1(s), K(s) x_2(s) + v_2(s) \rangle_H ds \\
&\quad + \mathbb{E} \int_t^T \langle v_1(s), \hat{Q}^{(t)}(\xi_1, u_1(s), v_2(s)) \rangle_{H^2} ds + \mathbb{E} \int_t^T \langle \hat{Q}^{(t)}(\xi_1, u_1(s), v_2(s)) \rangle_{H^2} ds,
\end{align*}
\]

where, \(x_1(\cdot)\) and \(x_2(\cdot)\) solve the equations (3.1) and (3.2), respectively.

We have the following well-posedness result for the equation (1.3).

**Theorem 3.3** Assume that \(H\) is a separable Hilbert space, and \(L^p_{\tau,T}(\Omega; \mathbb{C})\) \((1 \leq p < \infty)\) is a separable Banach space. Then, for any \(J, K, F\) and \(P_T\) satisfying (1.4), the equation (1.3) admits one and only one relaxed transposition solution \((P(\cdot), Q^{(\cdot)}, \hat{Q}^{(\cdot)})\). Furthermore,

\[
|P|_{L^p_{\tau,T}(\Omega; D([\tau, T]; \mathcal{L}(H)))} + \left\| (Q^{(\cdot)}, \hat{Q}^{(\cdot)}) \right\|_{Q^p[\tau, T]} \leq C \left[ |F|_{L^p_{\tau}(\Omega; L^p(\Omega; \mathcal{L}(H)))} + |P_T|_{L^p_{\tau}(\Omega; \mathcal{L}(H))} \right].
\]

**Proof:** The proof of this theorem is very lengthy and technical, and it is very similar to that of [11, Theorem 6.1]. Hence, we only give here a sketch.
In Theorem 3.1, we have obtained the well-posedness of (1.3) with \( P_T \in L_{F_T}^p(\Omega; L_2(H)) \) and \( F \in L_2^p(\tau,T; L^p(\Omega; L_2(H))) \). Noting that \( L_2(H) \) is dense in the space \( L(H) \) (with the usual strong operator topology), we may approximate the general data \( P_T \in L_{F_T}^p(\Omega; L(H)) \) and \( F \in L_2^p(\tau,T; L^p(\Omega; L_2(H))) \) by \( \{P_{m}^p\}_{m=1}^\infty \subset L_{F_T}^p(\Omega; L_2(H)) \) and \( \{F_{m}^p\}_{m=1}^\infty \subset L_2^p(\tau,T; L^p(\Omega; L_2(H))) \), respectively. Denote by \( (P_{m}^p, Q_{m}^p) \) the corresponding solution to (1.3) with \( P_T \) and \( F \) replaced respectively by \( P_{m}^p \) and \( F_{m}^p \). Then, we obtain the desired \( P(\cdot) \) as the weak limit of \( \{P_{m}^p(\cdot)\}_{m=1}^\infty \) and \( (Q_{m}^p(\cdot), \hat{Q}_{m}^p(\cdot)) \) as the weak limit of \( \{(Q_{m}^p(\cdot), Q_{m}^p(\cdot)^*)\}_{m=1}^\infty \). The most difficult part is to show that \( \{P_{m}^p(\cdot)\}_{m=1}^\infty \) and \( \{(Q_{m}^p(\cdot), Q_{m}^p(\cdot)^*)\}_{m=1}^\infty \) converge respectively to some elements in \( L_{F,T,n}^p(\Omega; D([\tau,T]; L(H))) \times Q^p[\tau,T], \) in some weak sense. All of these are guaranteed by some Banach-Alaoglu-type theorems established in [11]. \[ \square \]

4 A regularity property for relaxed transposition solutions to the operator-valued BSEEs

In this section, we shall derive a regularity property for relaxed transposition solutions to the equation (1.3). This property will play key roles in the proof of our general Pontryagin-type stochastic maximum principle, presented in Section 5. To simplify the notations, we assume that \( d = 1 \) in this section.

We need some preliminaries. First of all, as an immediate consequence of the well-posedness result for (3.2), it is easy to prove the following result.

**Lemma 4.1** If \( u_2 = v_2 = 0 \) in the equation (3.2), then for each \( t \in [0,T] \), there exists an operator \( U(\cdot,t) \in \mathcal{L}(L_{F_T}^p(\Omega; H), C_p([t,T]; L^p(\Omega; H))) \) such that the corresponding solution to this equation can be represented as \( x_2(\cdot) = U(\cdot,t)x_2. \) Further, for any \( t \in [0,T] \), \( \varepsilon > 0 \) and \( \xi \in L_{F_T}^p(\Omega; H) \), there is a \( \delta > 0 \) such that for all \( s \in [t,t+\delta] \),

\[
|U(s,t)\xi - \xi|_{L_{F_T}^p(\Omega; H)} < \varepsilon.
\]

Next, we recall the following known result.

**Lemma 4.2** ([11, Corollary 5.1]) Let \( X \) and \( Y \) be respectively a separable and a reflexive Banach space, and let \( L_{F_T}^q(\Omega) \), with \( 1 \leq q < \infty \), be separable. Let \( 1 < q_1, q_2 < \infty \). Assume that \( \{G_n\}_{n=1}^\infty \) is a sequence of uniformly bounded, pointwisely defined linear operators from \( X \) to \( L_{F_T}^{q_1}(0,T; L_{F_T}^{q_2}(\Omega; Y)) \). Then, there exist a subsequence \( \{G_{n_k}\}_{k=1}^\infty \subset \{G_n\}_{n=1}^\infty \) and an \( \mathcal{G} \in \mathcal{L}_{pd}(X, L_{F_T}^{q_1}(0,T; L_{F_T}^{q_2}(\Omega; Y))) \) such that

\[
\mathcal{G}x = (w) - \lim_{k \to \infty} G_{n_k}x \quad \text{in} \quad L_{F_T}^{q_1}(0,T; L_{F_T}^{q_2}(\Omega; Y)), \quad \forall \ x \in X.
\]

Moreover, \( \|\mathcal{G}\|_{\mathcal{L}(X, L_{F_T}^{q_1}(0,T; L_{F_T}^{q_2}(\Omega; Y)))} \leq \sup_{n \in \mathbb{N}} \|G_n\|_{\mathcal{L}(X, L_{F_T}^{q_1}(0,T; L_{F_T}^{q_2}(\Omega; Y)))}. \)

Further, let \( \{\Delta_n\}_{n=1}^\infty \) be a sequence of partitions of \([0,T]\), that is,

\[
\Delta_n \triangleq \left\{ t_i^n \mid i = 0, 1, \cdots, n, \text{ and } 0 = t_0^n < t_1^n < \cdots < t_n^n = T \right\}
\]

such that \( \Delta_n \subset \Delta_{n+1} \) and \( \delta(\Delta_n) \triangleq \max_{0 \leq i \leq n-1} (t_{i+1}^n - t_i^n) \to 0 \) as \( n \to \infty \). We introduce the following subspaces of \( L_{F_T}^{q_2}(0,T; L_{F_T}^{2q}(\Omega; H)) \):

\[
\mathcal{H}_n = \left\{ \sum_{i=0}^{n-1} \chi_{[t_i^n, t_{i+1}^n]}(\cdot)U(\cdot, t_i^n)h_i \mid h_i \in L_{F_T}^{2q}(\Omega; H) \right\}.
\]

Here \( U(\cdot, \cdot) \) is the operator introduced in Lemma 4.1. We have the following result.
Proposition 4.1 The set $\bigcup_{n=1}^{\infty} \mathcal{H}_n$ is dense in $L^2_T(0,T;L^{2q}(\Omega;H))$.

Proof: We introduce the following subspace of $L^2_T(0,T;L^{2q}(\Omega;H))$:

$$\tilde{\mathcal{H}}_n = \left\{ \sum_{i=0}^{n-1} \chi_{[t_i^n, t_{i+1}^n]}(\cdot) h_i^n \mid h_i^n \in L^2_T(\Omega;H) \right\}. \quad (4.2)$$

It is clear that $\bigcup_{n=1}^{\infty} \tilde{\mathcal{H}}_n$ is dense in $L^2_T(0,T;L^{2q}(\Omega;H))$.

For any $n \in \mathbb{N}$ and $h_i^n \in L^2_T(\Omega;H)$, $i \in \{0,1,\cdots, n-1\}$, write $\tilde{v}_n = \sum_{i=0}^{n-1} \chi_{[t_i^n, t_{i+1}]}(\cdot) h_i^n$. Clearly, $\tilde{v}_n \in \mathcal{H}_n$. We claim that for any $\varepsilon > 0$, there exist $m \in \mathbb{N}$ and $v_m \in \mathcal{H}_m$ such that

$$\left| \tilde{v}_n - v_m \right|_{L^2_T(0,T;L^{2q}(\Omega;H))} < \varepsilon. \quad (4.3)$$

Indeed, by Lemma 4.1, for each $h_i^n$, there is a $\delta_i^n > 0$ such that for all $t \in [t_i^n, T - \delta_i^n)$ and $s \in [t, t + \delta_i^n]$, it holds that

$$\left| U(s,t)h_i^n - h_i^n \right|_{L^2_T(\Omega;H)} < \varepsilon \sqrt{T}. \quad (4.4)$$

Now we choose a partition $\Delta_m$ of $[0,T]$ such that $\Delta_n \subset \Delta_m$ and $\max_{0 \leq j \leq m-1} \{t_{j+1} - t_j\} \leq \min_{0 \leq i \leq n-1} \{\delta_i^n\}$. Let

$$v_m = \sum_{j=0}^{m-1} \chi_{[t_j^m, t_{j+1}^m]}(\cdot) U(\cdot, t_j^m) h_j^m,$$

where $h_j^m = h_i^n$ whenever $[t_j^m, t_{j+1}^m] \subset [t_i^n, t_{i+1}^n]$. From (4.4), we find that

$$\left| \tilde{v}_n - v_m \right|_{L^2_T(0,T;L^{2q}(\Omega;H))} = \left| \sum_{j=0}^{m-1} \chi_{[t_j^m, t_{j+1}^m]}(\cdot) U(\cdot, t_j^m) h_j^m - \sum_{i=0}^{n-1} \chi_{[t_i^n, t_{i+1}]}(\cdot) h_i^n \right|_{L^2_T(0,T;L^{2q}(\Omega;H))}$$

$$= \left| \sum_{j=0}^{m-1} \chi_{[t_j^m, t_{j+1}^m]}(\cdot) \left( U(\cdot, t_j^m) h_j^m - h_j^m \right) \right|_{L^2_T(0,T;L^{2q}(\Omega;H))} < \sqrt{T} \frac{\varepsilon}{\sqrt{T}} = \varepsilon.$$

This proves (4.3). Hence, $\bigcup_{n=1}^{\infty} \mathcal{H}_n$ is dense in $L^2_T(0,T;L^{2q}(\Omega;H))$. \hfill $\Box$

Our regularity result for solutions to (1.3) can be stated as follows.

Theorem 4.1 Suppose that the assumptions in Theorem 3.3 hold and let $(P(\cdot), Q(\cdot), \tilde{Q}(\cdot))$ be the relaxed transposition solution to the equation (1.3). Then, there exist an $n \in \mathbb{N}$ and two pointwisely defined linear operators $Q^n$ and $\tilde{Q}^n$, both of which are from $\mathcal{H}_n$ to $L^2_T(0,T;L^{2q}(\Omega;H))$, such that, for any $\xi_1, \xi_2 \in L^2_T(\Omega;H)$, $u_1(\cdot), u_2(\cdot) \in L^2_T(\Omega;L^2(0,T;H))$ and $v_1(\cdot), v_2(\cdot) \in \mathcal{H}_n$, it holds that

$$E \int_0^T \langle v_1(s), \tilde{Q}^{(0)}(\xi_2, u_2, v_2)(s) \rangle_H ds + E \int_0^T \langle Q^{(0)}(\xi_1, u_1, v_1)(s), v_2(s) \rangle_H ds$$

$$= E \int_0^T \left[ \langle (Q^n v_1)(s), x_2(s) \rangle_H + \langle x_1(s), (\tilde{Q}^n v_2)(s) \rangle_H \right] ds, \quad (4.5)$$

where, $x_1(\cdot)$ and $x_2(\cdot)$ solve accordingly (3.1) and (3.2) with $t = 0$. Further, there is a positive constant $C$, independent of $n$, such that

$$|Q^n v_1|_{L^2_T(0,T;L^{2q}(\Omega;H))} + |\tilde{Q}^n v_2|_{L^2_T(0,T;L^{2q}(\Omega;H))} \leq C \left( |\tilde{v}_1|_{L^2(0,T;L^{2q}(\Omega;H))} + |\tilde{v}_2|_{L^2(0,T;L^{2q}(\Omega;H))} \right), \quad (4.6)$$
where
\[ \hat{v}_1 = \sum_{i=0}^{n-1} \chi_{[\tau^i_{m-1}, \tau^i_m]}(\cdot) h_i \quad \text{for} \quad v_1 = \sum_{i=0}^{n-1} \chi_{[\tau^i_{m-1}, \tau^i_m]}(\cdot) U(\cdot, t_i) h_i \]
and
\[ \hat{v}_2 = \sum_{j=0}^{n-1} \chi_{[\tau^j_{m-1}, \tau^j_m]}(\cdot) h_j \quad \text{for} \quad v_2 = \sum_{j=0}^{n-1} \chi_{[\tau^j_{m-1}, \tau^j_m]}(\cdot) U(\cdot, t_j) h_j. \]

Proof: Let \( \{e_m\}_{m=1}^\infty \) be an orthonormal basis of \( H \) and \( \{\Gamma_m\}_{m=1}^\infty \) be the standard projection operator from \( H \) onto its subspace span \( \{e_1, e_2, \ldots, e_m\} \). Let \( P^m_T = \Gamma_m P_T \Gamma_m \) and \( F_m(\cdot) = \Gamma_m F(\cdot) \Gamma_m \).

Clearly, \( P^m_T \in L^2_P(\Omega; L_2(H)) \) and \( F_m \in L^2_P(0, T; L^2(\Omega; L_2(H))) \). By Theorem 3.2, the equation (1.3) with \( P_T \) and \( F \) replaced respectively by \( P^m_T \) and \( F_m \) admits a unique transposition solution \( (P^m, Q^m) \in L^2_P(\Omega; D(\tau, T); L_2(H)) \times L^2_P(\tau, T; L^2(\Omega; L_2(H))) \) such that

\[
\mathbb{E}\langle P^m_T x_1(T), x_2(T) \rangle_H - \mathbb{E}\int_t^T \langle F_m(s)x_1(s), x_2(s) \rangle_H ds \\
= \mathbb{E}\langle P^m(t) \xi_1, \xi_2 \rangle_H + \mathbb{E}\int_t^T \langle P^m(s)u_1(s), x_2(s) \rangle_H ds + \mathbb{E}\int_t^T \langle P^m(s)x_1(s), u_2(s) \rangle_H ds \\
+ \mathbb{E}\int_t^T \langle P^m(s)K(s)x_1(s), v_2(s) \rangle_H ds + \mathbb{E}\int_t^T \langle P^m(s)v_1(s), K(s)x_2(s) + v_2(s) \rangle_H ds \\
+ \mathbb{E}\int_t^T \langle Q^m(s)v_1(s), x_2(s) \rangle_H ds + \mathbb{E}\int_t^T \langle Q^m(s)x_1(s), v_2(s) \rangle_H ds. \tag{4.7}
\]

Here, \( x_1(\cdot) \) and \( x_2(\cdot) \) solve (3.1) and (3.2), respectively.

For any \( i \in \{1, 2, \ldots, n-1\} \), \( \xi_1 \in L^2_{P_{\tau^i}}(\Omega; H) \) and \( \xi_2 \in L^2_{P_{\tau^i}}(t^i_i, T; L^2(\Omega; H)) \), letting \( u_1 = 0 \) and \( v_1 = 0 \) in the equation (3.1), and letting \( \xi_2 = 0 \) and \( u_2 = 0 \) in the equation (3.2), by (4.7) with \( t = t^i_i \), we find that

\[
\mathbb{E}\langle P^m_T x_1(T), x_2(T) \rangle_H - \mathbb{E}\int_{t^i_i}^T \langle F_m(s)x_1(s), x_2(s) \rangle_H ds \\
= \mathbb{E}\int_{t^i_i}^T \langle P^m(s)K(s)x_1(s), v_2(s) \rangle_H ds + \mathbb{E}\int_{t^i_i}^T \langle Q^m(s)U(s, t^i_n)\xi_1, v_2(s) \rangle_H ds. \tag{4.8}
\]

For these data \( \xi_1, u_1, v_1, \xi_2, u_2 \) and \( v_2 \), from the variational equality (4.7) with \( t = t^i_{n+1} \), we obtain that

\[
\mathbb{E}\langle P^m_T x_1(T), x_2(T) \rangle_H - \mathbb{E}\int_{t^i_{n+1}}^T \langle F_m(s)x_1(s), x_2(s) \rangle_H ds \\
= \mathbb{E}\langle P^m(t^i_{n+1})x_1(t^i_{n+1}), x_2(t^i_{n+1}) \rangle_H + \mathbb{E}\int_{t^i_{n+1}}^T \langle P^m(s)K(s)x_1(s), v_2(s) \rangle_H ds \\
+ \mathbb{E}\int_{t^i_{n+1}}^T \langle Q^m(s)U(s, t^i_n)\xi_1, v_2(s) \rangle_H ds. \tag{4.9}
\]

From (4.8) and (4.9), it follows that

\[
\mathbb{E}\langle P^m(t^i_{n+1})\xi_1, x_2(t^i_{n+1}) \rangle_H - \mathbb{E}\int_{t^i_i}^{t^i_{n+1}} \langle F_m(s)x_1(s), x_2(s) \rangle_H ds \\
= \mathbb{E}\int_{t^i_i}^{t^i_{n+1}} \langle P^m(s)K(s)x_1(s), v_2(s) \rangle_H ds + \mathbb{E}\int_{t^i_i}^{t^i_{n+1}} \langle Q^m(s)U(s, t^i_n)\xi_1, v_2(s) \rangle_H ds, \tag{4.10}
\]

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holds for any $i \in \{1, 2, \cdots, n-1\}$, $\xi_1 \in L^2_{\overline{F}^{t_n}_{t_i}}(\Omega; H)$ and $v_2 \in L^2_{\overline{F}^{t_n}_{t_i}}(t^n_i, t^n_{i+1}; L^{2q}(\Omega; H))$.

We choose a $\xi_1 \in L^2_{\overline{F}^{t_n}_{t_i}}(\Omega; H)$ with $|\xi_1|_{L^2_{\overline{F}^{t_n}_{t_i}}(\Omega; H)} = 1$ such that

$$|Q^m (\cdot) U (\cdot, t^n_i) \xi_1|_{L^2_{\overline{F}^{t^n_i}_{t_i+1} L^{2q}} (\Omega; H)} \geq \frac{1}{2} \|Q^m (\cdot) U (\cdot, t^n_i)\|_{L(L^2_{\overline{F}^{t^n_i}_{t_i+1}} (\Omega; H), L^2_{\overline{F}^{t^n_i}_{t_i+1} L^{2q}} (\Omega; H))}.$$

Next, we find a $v_2 \in L^2_{\overline{F}^{t^n_m}_{t_{i+1}}} L^{2q}(\Omega; H))$ with $|v_2|_{L^2_{\overline{F}^{t^n_m}_{t_{i+1}}}} L^{2q}(\Omega; H)) = 1$ so that

$$\mathbb{E} \int_{t^n_i}^{t^n_{i+1}} \langle Q^m(s) U(s, t^n_i) \xi_1, v_2(s) \rangle_H ds \geq \frac{1}{2} \|Q^m (\cdot) U (\cdot, t^n_i)\|_{L(L^2_{\overline{F}^{t^n_m}_{t_{i+1}} L^{2q}} (\Omega; H))}.$$

Hence,

$$\mathbb{E} \int_{t^n_i}^{t^n_{i+1}} \langle Q^m(s) U(s, t^n_i) \xi_1, v_2(s) \rangle_H ds \geq \frac{1}{2} \|Q^m (\cdot) U (\cdot, t^n_i)\|_{L(L^2_{\overline{F}^{t^n_m}_{t_{i+1}} L^{2q}} (\Omega; H))}.$$

Also, it is easy to see that

$$\left| P^m (t^n_{i+1}) \xi_1, x_2(t^n_{i+1}) \right|_H - \mathbb{E} \int_{t^n_i}^{t^n_{i+1}} \langle F_m(s)x_1(s), x_2(s) \rangle_H ds - \mathbb{E} \int_{t^n_i}^{t^n_{i+1}} \langle P^m(s) K(s)x_1(s), v_2(s) \rangle_H ds \right|_H \leq C \left| P_T L^p_{\overline{F}^{t^n}_x} (\omega; L^q_H) \right| + \left| F L^p_{\overline{F}^{t^n}_x} (\omega; L^q_H) \right|_H (1 + |(J, K)|_{L^q(0, T; L^{\infty}(\Omega; L^q_H))})^2. \quad (4.12)$$

Combining (4.8), (4.11) and (4.12), we find that

$$\left| P^m (t^n_{i+1}) \xi_1, x_2(t^n_{i+1}) \right|_H - \mathbb{E} \int_{t^n_i}^{t^n_{i+1}} \langle F_m(s)x_1(s), x_2(s) \rangle_H ds - \mathbb{E} \int_{t^n_i}^{t^n_{i+1}} \langle P^m(s) K(s)x_1(s), v_2(s) \rangle_H ds \right|_H \leq C \left| P_T L^p_{\overline{F}^{t^n}_x} (\omega; L^q_H) \right| + \left| F L^p_{\overline{F}^{t^n}_x} (\omega; L^q_H) \right|_H (1 + |(J, K)|_{L^q(0, T; L^{\infty}(\Omega; L^q_H))})^2. \quad (4.13)$$

By (4.13) and Lemma 4.2, there exist a bounded, pointwisely defined linear operator $Q^m_{t^n_i}$ from $L^2_{\overline{F}^{t^n_m}_{t_i}} (\Omega; H)$ to $L^2_{\overline{F}^{t^n_m}_{t_{i+1}} L^{2q}} (\Omega; H)$, and a subsequence $\{ m^{(k)}_i \}_{k=1}^\infty$ of $\{ m_i \}_{i=1}^\infty$ such that

$$\lim_{k \to \infty} Q^{m^{(k)}_i} (\cdot) U (\cdot, t^n_i) \xi = Q^m_{t^n_i} (\cdot) \xi \quad \text{in} \quad L^2_{\overline{F}^{t^n_m}_{t_{i+1}} L^{2q}} (\Omega; H), \quad \forall \xi \in L^2_{\overline{F}^{t^n_m}_{t_i}} (\Omega; H). \quad (4.14)$$

Since $Q^m_{t^n_i}$ is pointwisely defined, for a.e. $(t, \omega) \in (t^n_i, t^n_{i+1}) \times \Omega$, there is a $q^n_{t^n_i} (t, \omega) \in L^q_H (\omega; H)$ such that

$$(Q^m_{t^n_i} \xi)(t, \omega) = q^n_{t^n_i} (t, \omega) \xi(\omega), \quad \forall \xi \in L^2_{\overline{F}^{t^n_m}_{t_i}} (\Omega; H).$$

Let us define an operator $Q^n$ from $\mathcal{H}_n$ to $L^2_{\overline{F}^{t^n_m}_{t_i}} (0, T; L^{2q}_{\overline{F}^{t^n_m}_{t_i}} (\Omega; H))$ as follows:

$$(Q^n v)(t, \omega) = \sum_{i=0}^{n-1} \chi_{[t^n_i, t^n_{i+1})}(t) q^n_{t^n_i} (t, \omega) \xi_i, \quad \text{a.e.} \quad (t, \omega) \in (0, T) \times \Omega,$$

where $v = \sum_{i=0}^{n-1} \chi_{[t^n_i, t^n_{i+1})}(\cdot) U (\cdot, t^n_i) \xi_i \in \mathcal{H}_n$ with $\xi_i \in L^2_{\overline{F}^{t^n_m}_{t_i}} (\Omega; H)$. It is easy to check that $Q^n v \in L^2_{\overline{F}^{t^n_m}_{t_i}} (0, T; L^{2q}_{\overline{F}^{t^n_m}_{t_i}} (\Omega; H))$, $Q^n$ is a pointwisely defined linear operator from $\mathcal{H}_n$ to $L^2_{\overline{F}^{t^n_m}_{t_i}} (0, T; L^{2q}_{\overline{F}^{t^n_m}_{t_i}} (\Omega; H))$ and

$$|Q^n v|_{L^2_{\overline{F}^{t^n_m}_{t_i}} (0, T; L^{2q}_{\overline{F}^{t^n_m}_{t_i}} (\Omega; H))} \leq C|v|_{L^2_{\overline{F}^{t^n_m}_{t_i}} (0, T; L^{2q}_{\overline{F}^{t^n_m}_{t_i}} (\Omega; H))}.$$
Consider the following controlled (forward) stochastic evolution equation:

\[ Q^{m_k}(s)u(s) = \sum_{i=0}^{n-1} \chi_{[t_i,t_{i+1})}(s)Q^{m_k}(s)U(s,t_i)h_i. \]

Hence,

\[ Q^{m_k}(s)u(s) - Q^n v(s) = \sum_{i=0}^{n-1} \chi_{[t_i,t_{i+1})}(s) \left[ Q^{m_k}(s)U(s,t_i)h_i - (Q^n h_i)(s) \right]. \]

This gives that

\[ (w) - \lim_{k \to \infty} Q^{m_k}(s)u(s) = Q^n v \quad \text{in } L^2_\mathbb{F}(0,T;L^{2p+1}_\mathbb{F}^\ast(\Omega; H)), \quad \forall v \in \mathcal{H}_n. \quad (4.15) \]

Similarly, one can find a subsequence \( \{m^{(2)}_k\}_{k=1}^{\infty} \subset \{m^{(1)}_k\}_{k=1}^{\infty} \) and a pointwise linear operator \( \hat{Q}^n \) from \( \mathcal{H}_n \) to \( L^2_\mathbb{F}(0,T;L^{2p+1}_\mathbb{F}^\ast(\Omega; H)) \) such that

\[ (w) - \lim_{k \to \infty} Q^{m_k}(s)v(s) = \hat{Q}^n v \quad \text{in } L^2_\mathbb{F}(0,T;L^{2p+1}_\mathbb{F}^\ast(\Omega; H)), \quad \forall v \in \mathcal{H}_n. \quad (4.16) \]

For any \( \xi_1, \xi_2 \in L^2_\mathbb{F}(\Omega; H), u_1(s), u_2(s) \in L^2_\mathbb{F}(\Omega;L^2(0,T;H)) \) and \( v_1(s), v_2(s) \in \mathcal{H}_n \), by (4.15)–(4.16), it is easy to see that

\[ \lim_{k \to \infty} \mathbb{E} \int_0^T \left[ \langle Q^{m_k}(s)v_1(s), x_2(s) \rangle_H + \langle Q^{m_k}(s)x_1(s), v_2(s) \rangle_H \right] ds = \mathbb{E} \int_0^T \left[ \langle Q^n v_1(s), x_2(s) \rangle_H + \langle x_1(s), (\hat{Q}^n v_2)(s) \rangle_H \right] ds. \quad (4.17) \]

On the other hand, from the proof of [11, Theorem 6.1], one can show that there exists a subsequence \( \{m^{(3)}_k\}_{k=1}^{\infty} \) of \( \{m^{(2)}_k\}_{k=1}^{\infty} \) such that

\[ \lim_{k \to \infty} \mathbb{E} \int_0^T \left[ \langle Q^{m_k}(s)v_1(s), x_2(s) \rangle_H + \langle Q^{m_k}(s)x_1(s), v_2(s) \rangle_H \right] ds = \mathbb{E} \int_0^T \langle v_1(s), \hat{Q}^{(0)}(\xi_2, u_2, v_2)(s) \rangle_H ds + \mathbb{E} \int_0^T \langle \hat{Q}^{(0)}(\xi_1, u_1, v_1)(s), v_2(s) \rangle_H ds. \quad (4.18) \]

Combining (4.17) and (4.18), we obtain (4.5). This completes the proof of Theorem 4.1.

### 5 Pontryagin-type maximum principle for controlled stochastic evolution equations

In this section, for simplicity of the presentation, we only consider the case that \( \{w(t)\}_{t \geq 0} \) is a standard one dimensional Brownian motion.

Let \( U \) be a separable metric space with metric \( d(\cdot, \cdot) \). Put

\[ \mathcal{U}[0,T] = \left\{ u(\cdot) : [0,T] \to U \bigg| u(\cdot) \text{ is } \mathbb{F}\text{-adapted} \right\}. \]

Consider the following controlled (forward) stochastic evolution equation:

\[
\begin{align*}
\begin{cases}
\displaystyle dx(t) &= [Ax(t) + a(t, x(t), u(t))] \, dt + b(t, x(t), u(t)) \, dw(t) \\ x(0) &= x_0,
\end{cases} 
\end{align*}
\quad (5.1)
\]
where \( u(\cdot) \in \mathcal{U}[0, T] \) and \( x_0 \in L^8_{\mathcal{F}_0}(\Omega; H) \). Similar to (2.1), \( x(\cdot) \equiv x(\cdot; x_0, u(\cdot)) \in C_F([0, T]; L^8(\Omega; H)) \) is understood as a mild solution to the equation (5.1).

Similar to [11], we assume the following three conditions:

(S1) Suppose that \( a(\cdot, \cdot, \cdot), b(\cdot, \cdot, \cdot) : [0, T] \times H \times U \to H \) are two maps such that for \( \varphi(t, x, u) = a(t, x, u), b(t, x, u) \), it holds that: i) For any \( (x, u) \in H \times U \), the map \( \varphi(\cdot, x, u) : [0, T] \to H \) is Lebesgue measurable; ii) For any \( (t, x) \in [0, T] \times H \), the map \( \varphi(t, \cdot, \cdot) : U \to H \) is continuous, and

\[
\begin{align*}
|\varphi(t, x_1, u) - \varphi(t, x_2, u)|_H &\leq C_L|x_1 - x_2|_H, \quad \forall (t, x_1, x_2, u) \in [0, T] \times H \times H \times U, \\
|\varphi(t, 0, u)|_H &\leq C_L, \quad \forall (t, u) \in [0, T] \times H \times H \times U;
\end{align*}
\]

(S2) Suppose that \( g(\cdot, \cdot, \cdot) : [0, T] \times H \times U \to \mathbb{R} \) and \( h(\cdot) : H \to \mathbb{R} \) are two functionals such that for \( \psi(t, x, u) = g(t, x, u), h(x) \), it holds that: i) For any \( (x, u) \in H \times U \), the function \( \psi(\cdot, x, u) : [0, T] \to \mathbb{R} \) is Lebesgue measurable; ii) For any \( (t, x) \in [0, T] \times H \), the function \( \psi(t, x, \cdot) : U \to \mathbb{R} \) is continuous, and

\[
\begin{align*}
|\psi(t, x_1, u) - \psi(t, x_2, u)|_H &\leq C_L|x_1 - x_2|_H, \quad \forall (t, x_1, x_2, u) \in [0, T] \times H \times H \times U, \\
|\psi(t, 0, u)|_H &\leq C_L, \quad \forall (t, u) \in [0, T] \times H \times U;
\end{align*}
\]

(S3) The map \( a(t, x, u) \) and \( b(t, x, u) \), and the functional \( g(t, x, u) \) and \( h(x) \) are \( C^2 \) with respect to \( x \), such that for \( \varphi(t, x, u) = a(t, x, u), b(t, x, u) \), \( \psi(t, x, u) = g(t, x, u), h(x) \), it holds that \( \varphi_x(t, x, u), \varphi_{xx}(t, x, u) \) and \( \psi_x(t, x, u) \) are continuous with respect to \( u \). Moreover,

\[
\begin{align*}
|\varphi_x(t, x, u)|_{L(H)} + |\psi_x(t, x, u)|_{H} &\leq C_L, \quad \forall (t, x, u) \in [0, T] \times H \times U, \\
|\varphi_{xx}(t, x, u)|_{L(H \times H, H)} + |\psi_{xx}(t, x, u)|_{L(H)} &\leq C_L, \quad \forall (t, x, u) \in [0, T] \times H \times U.
\end{align*}
\]

Define a cost functional \( J(\cdot) \) (for the controlled equation (5.1)) as follows:

\[
J(u(\cdot)) \triangleq \mathbb{E}\left[ \int_0^T g(t, x(t), u(t)) dt + h(x(T)) \right], \quad \forall u(\cdot) \in \mathcal{U}[0, T].
\]

Let us consider the following optimal control problem for (5.1):

**Problem (OP)** Find a \( \bar{u}(\cdot) \in \mathcal{U}[0, T] \) such that

\[
J(\bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[0, T]} J(u(\cdot)). \tag{5.6}
\]

Any \( \bar{u}(\cdot) \) satisfying (5.6) is called an optimal control. The corresponding state process \( \bar{x}(\cdot) \) is called an optimal state process. \( (\bar{x}(\cdot), \bar{u}(\cdot)) \) is called an optimal pair.

There exist some works addressing the Pontryagin-type maximum principle for optimal controls of infinite dimensional stochastic evolution equations (e.g. [1, 2, 5, 15, 17] and the references therein). However, most of the previous works in this respect addressed only to the case that either the diffusion term does NOT depend on the control variable (i.e., the map \( b(t, x, u) \) in (5.1) is independent of \( u \)) or the control region \( U \) is convex. Recently, this restriction was relaxed in [3, 4, 11]. In both [3] and [4], the filtration \( \mathcal{F} \) is assumed to be the natural one (generated by the Brownian motion \( \{w(t)\}_{t \in [0, T]} \) and augmented by all of the \( \mathbb{P} \)-null sets). Also, in [3], the authors assume that \( A \) is a strictly monotone operator; while in [4], the authors assume that \( H = L^2(D, \mathcal{D}, \mu) \) (for a measure space \((D, \mathcal{D}, \mu)\) with finite measure \( \mu \)), and the restriction of
{S(t)}_{t \geq 0} to the space $L^4(D, \mathcal{D}, \mu)$ is a strongly continuous analytic semigroup and the domain of its infinitesimal generator is compactly embedded in $L^4(D, \mathcal{D}, \mu)$. On the other hand, in [11, Theorem 9.1], a technical assumption $b_x(\cdot, \bar{x}(\cdot), \bar{u}(\cdot)) \in L^4(0, T; L^\infty(\Omega; \mathcal{L}(D(A))))$ is imposed. The purpose of this section is to establish a Pontryagin-type maximum principle without any of the above mentioned assumptions.

Define a function $H : [0, T] \times H \times \mathcal{U} \times H \times H \to \mathbb{R}$ as follows:

$$
H(t, x, u, k_1, k_2) \triangleq \langle k_1, a(t, x, u) \rangle_H + \langle k_2, b(t, x, u) \rangle_H - g(t, x, u),
$$

$$(t, x, u, k_1, k_2) \in [0, T] \times H \times \mathcal{U} \times H \times H. \quad (5.7)
$$

We have the following result.

**Theorem 5.1** Suppose that $H$ is a separable Hilbert space, $L^p_{\mathcal{F}_T}(\Omega; \mathbb{C})$ ($1 \leq p < \infty$) is a separable Banach space, and $U$ is a separable metric space. Let the conditions (S1), (S2), and (S3) hold, and $(\bar{x}(\cdot), \bar{u}(\cdot))$ be an optimal pair for Problem (OP). Let $(y(\cdot), Y(\cdot))$ be the transposition solution to the equation (1.1) with $p = 2$, and $y_T$ and $f(\cdot, \cdot, \cdot)$ given by

$$
\begin{align*}
  y_T &= -h_x(\bar{x}(T)), \\
  f(t, y_1, y_2) &= -a_x(t, \bar{x}(t), \bar{u}(t))y_1 - b_x(t, \bar{x}(t), \bar{u}(t))y_2 + g_x(t, \bar{x}(t), \bar{u}(t)).
\end{align*} \quad (5.8)
$$

Assume that $(P(\cdot), Q^{(\cdot)}, \hat{Q}^{(\cdot)})$ is the relaxed transposition solution to the equation (1.3) in which $P_T$, $J(\cdot)$, $K(\cdot)$, and $F(\cdot)$ are given by

$$
\begin{align*}
  P_T &= -h_{xx}(\bar{x}(T)), \\
  J(t) &= a_x(t, \bar{x}(t), \bar{u}(t)), \\
  K(t) &= b_x(t, \bar{x}(t), \bar{u}(t)), \\
  F(t) &= -H_{xx}(t, \bar{x}(t), \bar{u}(t), y(t), Y(t)).
\end{align*} \quad (5.9)
$$

Then,

$$
\begin{align*}
  \text{Re} H(t, \bar{x}(t), \bar{u}(t), y(t), Y(t)) - \text{Re} H(t, \bar{x}(t), u, y(t), Y(t)) \\
  -\frac{1}{2} \langle P(t)[b(t, \bar{x}(t), \bar{u}(t)) - b(t, \bar{x}(t), u)] + b(t, \bar{x}(t), \bar{u}(t)) - b(t, \bar{x}(t), u) \rangle_H \\
  \geq 0, \quad \text{a.e. } [0, T] \times \Omega, \quad \forall u \in U.
\end{align*} \quad (5.10)
$$

**Proof:** We divide the proof into two steps.

**Step 1.** For each $\varepsilon > 0$, let $E_\varepsilon \subset [0, T]$ be a measurable set with measure $\varepsilon$. Put

$$
u^\varepsilon(t) = \begin{cases} \\ 
  \bar{u}(t), & t \in [0, T] \setminus E_\varepsilon, \\
  u(t), & t \in E_\varepsilon,
\end{cases}
$$

where $u(\cdot)$ is an arbitrary given element in $\mathcal{U}(0, T)$. Write

$$
\begin{align*}
  a_1(t) &= a_x(t, \bar{x}(t), \bar{u}(t)), \\
  b_1(t) &= b_x(t, \bar{x}(t), \bar{u}(t)), \\
  g_1(t) &= g_x(t, \bar{x}(t), \bar{u}(t)), \\
  a_{11}(t) &= a_{xx}(t, \bar{x}(t), \bar{u}(t)), \\
  b_{11}(t) &= b_{xx}(t, \bar{x}(t), \bar{u}(t)), \\
  g_{11}(t) &= g_{xx}(t, \bar{x}(t), \bar{u}(t)), \\
  \delta a(t) &= a(t, \bar{x}(t), u(t)) - a(t, \bar{x}(t), \bar{u}(t)), \\
  \delta b(t) &= b(t, \bar{x}(t), u(t)) - b(t, \bar{x}(t), \bar{u}(t)), \\
  \delta g(t) &= g(t, \bar{x}(t), u(t)) - g(t, \bar{x}(t), \bar{u}(t)), \\
  \delta b_1(t) &= b_x(t, \bar{x}(t), u(t)) - b_x(t, \bar{x}(t), \bar{u}(t)).
\end{align*} \quad (5.11)
$$

We introduce the following two stochastic evolution equations:

$$
\begin{align*}
  dx^\varepsilon_2 &= [Ax^\varepsilon_2 + a_1(t)x^\varepsilon_2]dt + [b_1(t)x^\varepsilon_2 + \chi_{E_\varepsilon}(t)\delta b(t)]dw(t) \quad \text{in } (0, T], \\
  x^\varepsilon_2(0) &= 0.
\end{align*} \quad (5.12)
$$
and
\[
\begin{cases}
   dx_3^\varepsilon = \left[Ax_3^\varepsilon + a_1(t)x_3^\varepsilon + \chi E_\varepsilon(t)\delta a(t) + \frac{1}{2}a_{11}(t)(x_2^\varepsilon, x_2^\varepsilon)\right]dt \\
   + \left[b_1(t)x_3^\varepsilon + \chi E_\varepsilon(t)\delta b_1(t)x_2^\varepsilon + \frac{1}{2}b_{11}(t)(x_2^\varepsilon, x_2^\varepsilon)\right]dw(t) \quad \text{in } (0, T), \\
   x_3^\varepsilon(0) = 0.
\end{cases}
\] (5.13)

When \(\varepsilon \to 0\), by the proof of [11, Theorem 9.1], we have
\[
\begin{cases}
   |x_2^\varepsilon|_{L_2^\infty(0, T; L^2(\Omega; H))} = O(\sqrt{\varepsilon}), \\
   |x_3^\varepsilon|_{L_2^\infty(0, T; L^2(\Omega; H))} = O(\varepsilon),
\end{cases}
\] (5.14)

and
\[
\begin{align*}
    J(u^\varepsilon(\cdot)) - J(\bar{u}(\cdot)) &= \text{Re} \mathbb{E} \int_0^T \left[ \langle g_1(t), x_2^\varepsilon(t) \rangle_H + \frac{1}{2} \langle g_{11}(t)x_2^\varepsilon(t), x_2^\varepsilon(t) \rangle_H + \chi E_\varepsilon(t)\delta g(t) \right] dt \\
    &+ \text{Re} \mathbb{E} \langle h_x(\bar{x}(T)), x_3^\varepsilon(T) \rangle_H + \frac{1}{2} \text{Re} \mathbb{E} \langle h_{xx}(\bar{x}(T))x_2^\varepsilon(T), x_2^\varepsilon(T) \rangle_H + o(\varepsilon). \tag{5.15}
\end{align*}
\]

By the definition of the transposition solution to the equation (1.1) (with \(y_T \) and \(f(\cdot, \cdot, \cdot)\) given by (5.8)), we obtain that
\[
-\mathbb{E} \langle h_x(\bar{x}(T)), x_2^\varepsilon(T) \rangle_H - \mathbb{E} \int_0^T \langle g_1(t), x_2^\varepsilon(t) \rangle_H dt = \mathbb{E} \int_0^T \langle Y(t), \delta b(t) \rangle_H \chi E_\varepsilon(t) dt \tag{5.16}
\]
and
\[
-\mathbb{E} \langle h_x(\bar{x}(T)), x_3^\varepsilon(T) \rangle_H - \mathbb{E} \int_0^T \langle g_1(t), x_3^\varepsilon(t) \rangle_H dt \\
= \mathbb{E} \int_0^T \left\{ \frac{1}{2} \left[ \langle y(t), a_{11}(t)(x_3^\varepsilon(t), x_2^\varepsilon(t)) \rangle_H + \langle Y(t), b_{11}(t)(x_3^\varepsilon(t), x_2^\varepsilon(t)) \rangle_H \right] \right. \\
+ \chi E_\varepsilon(t) \left[ \langle y(t), \delta a(t) \rangle_H + \langle Y(t), \delta b_1(t)x_2^\varepsilon(t) \rangle_H \right] \right\} dt. \tag{5.17}
\]

According to (5.14)–(5.17), we conclude that
\[
\begin{align*}
    &J(u^\varepsilon(\cdot)) - J(\bar{u}(\cdot)) \\
    &= \frac{1}{2} \text{Re} \mathbb{E} \int_0^T \left[ \langle g_{11}(t)x_2^\varepsilon(t), x_2^\varepsilon(t) \rangle_H - \langle y(t), a_{11}(t)(x_3^\varepsilon(t), x_2^\varepsilon(t)) \rangle_H \right. \\
    &- \langle Y(t), b_{11}(t)(x_3^\varepsilon(t), x_2^\varepsilon(t)) \rangle_H \right] dt + \text{Re} \mathbb{E} \int_0^T \chi E_\varepsilon(t) \left[ \delta g(t) - \langle y(t), \delta a(t) \rangle_H \right. \\
    &+ \langle Y(t), \delta b(t) \rangle_H \right\} dt + \frac{1}{2} \text{Re} \mathbb{E} \langle h_{xx}(\bar{x}(T))x_2^\varepsilon(T), x_2^\varepsilon(T) \rangle_H + o(\varepsilon), \quad \text{as } \varepsilon \to 0. \tag{5.18}
\end{align*}
\]

**Step 2.** By the definition of the relaxed transposition solution to the equation (1.3) (with \(P_T, 

\)

\footnote{Recall that, for any \(C^2\)-function \(f(\cdot)\) defined on a Banach space \(X\) and \(x_0 \in X\), \(f_{xx}(x_0) \in \mathcal{L}(X \times X, X)\). This means that, for any \(x_1, x_2 \in X\), \(f_{xx}(x_0)(x_1, x_2) \in X\). Hence, by (5.11), \(a_{11}(t)(x_2^\varepsilon, x_2^\varepsilon)\) (in (5.13)) stands for \(a_{xx}(t, \bar{x}(t), \bar{u}(t))(x_2^\varepsilon(t), x_2^\varepsilon(t))\). One has a similar meaning for \(b_{11}(t)(x_2^\varepsilon, x_2^\varepsilon)\) and so on.}
\[ J(\cdot), K(\cdot) \text{ and } F(\cdot) \text{ given by (5.9)}, \] we obtain that
\[
-\mathbb{E}\langle h_{xx}(\bar{x}(T)) \bar{x}_2^x(T), \bar{x}_2^x(T) \rangle_H + \mathbb{E} \int_0^T \langle \mathbb{H}_{xx}(t, \bar{x}(t), \bar{u}(t), y(t), Y(t)) \bar{x}_2^x(t), \bar{x}_2^x(t) \rangle_H dt \\
= \mathbb{E} \int_0^T \chi_{E_\varepsilon}(t) \langle b_1(t) \bar{x}_2^x(t), P(t)^* \delta b(t) \rangle_H dt + \mathbb{E} \int_0^T \chi_{E_\varepsilon}(t) \langle P(t) \delta b(t), b_1(t) \bar{x}_2^x(t) \rangle_H dt \\
+ \mathbb{E} \int_0^T \chi_{E_\varepsilon}(t) \langle P(t) \delta b(t), \bar{x}_2^x(t) \rangle_H dt + \mathbb{E} \int_0^T \chi_{E_\varepsilon}(t) \langle P(t) \delta b(t), \bar{x}_2^x(t) \rangle_H dt.
\]
\[(5.19)\]

Now, we estimate the terms in the right hand side of (5.19). By (5.14), we have
\[
\left| \mathbb{E} \int_0^T \chi_{E_\varepsilon}(t) \langle b_1(t) \bar{x}_2^x(t), P(t)^* \delta b(t) \rangle_H dt + \mathbb{E} \int_0^T \chi_{E_\varepsilon}(t) \langle P(t) \delta b(t), b_1(t) \bar{x}_2^x(t) \rangle_H dt \right| = o(\varepsilon).
\]
\[(5.20)\]

In what follows, for any \( \tau \in [0, T] \), we choose \( E_\varepsilon = [\tau, \tau + \varepsilon] \subset [0, T] \).

By Proposition 4.1, we can find a sequence \( \{\beta_n\}_{n=1}^\infty \) such that \( \beta_n \in \mathcal{H}_n \) (Recall (4.1) for the definition of \( \mathcal{H}_n \) and \( \lim_{n \to \infty} \beta_n = \delta b \) in \( L^2_F(0; T; L^4(\Omega; H)) \)). Hence, for some positive constant \( C(x_0) \) (depending on \( x_0 \)),
\[
|\beta_n|_{L^2_F(0; T; L^4(\Omega; H))} \leq C(x_0) < \infty, \quad \forall n \in \mathbb{N},
\]
and there is a subsequence \( \{n_k\}_{k=1}^\infty \subset \{n\}_{n=1}^\infty \) such that
\[
\lim_{k \to \infty} |\beta_{n_k}(t) - \delta b(t)|_{L^4_F(\Omega; H)} = 0 \quad \text{for a.e. } t \in [0, T].
\]
\[(5.22)\]

Denote by \( Q^{n_k} \) and \( \hat{Q}^{n_k} \) the corresponding pointwisely defined linear operators from \( \mathcal{H}_{n_k} \) to \( L^2_F(0; T; L^4(\Omega; H)) \), given in Theorem 4.1.

Consider the following equation:
\[
\begin{align*}
&dx_{2,n_k}^x = [A x_{2,n_k}^x + a_1(t)x_{2,n_k}^x] dt + [b_1(t) x_{2,n_k}^x + \chi_{E_\varepsilon}(t) \beta_{n_k}(t)] dw(t) \quad \text{in } (0, T], \\
x_{2,n_k}^x(0) = 0.
\end{align*}
\]
\[(5.23)\]

We have
\[
\mathbb{E} |x_{2,n_k}^x(t)|_H^4 \\
= \mathbb{E} \left| \int_0^t S(t-s)a_1(s)x_{2,n_k}^x(s) ds + \int_0^t S(t-s)b_1(s)x_{2,n_k}^x(s) dw(s) + \int_0^t S(t-s)\chi_{E_\varepsilon}(s)\beta_{n_k}(s) dw(s) \right|_H^4 \\
\leq C \left[ \mathbb{E} \left| \int_0^t S(t-s)a_1(s)x_{2,n_k}^x(s) ds \right|_H^4 + \mathbb{E} \left| \int_0^t S(t-s)b_1(s)x_{2,n_k}^x(s) dw(s) \right|_H^4 \right. \\
\left. + \mathbb{E} \left| \int_0^t S(t-s)\chi_{E_\varepsilon}(s)\beta_{n_k}(s) dw(s) \right|_H^4 \right] \\
\leq C \left[ \int_0^t \mathbb{E} |x_{2,n_k}^x(s)|_H^4 ds + \varepsilon \int_{E_\varepsilon} \mathbb{E} |\beta_{n_k}(s)|_H^4 ds \right].
\]
\[(5.24)\]
By (5.21) and thanks to Gronwall’s inequality, (5.24) leads to

\[ |x^{\varepsilon}_{2,n_k}(\cdot)|^2_{L^\infty_T(0,T;L^4(\Omega; H))} \leq C(x_0, k)\varepsilon^2. \]  

(5.25)

Here and henceforth, \( C(x_0, k) \) is a generic constant (depending on \( x_0, k, T, A \) and \( C_L \)), which may be different from line to line. For any fixed \( k \in \mathbb{N} \), since \( Q^{n_k} \beta_{n_k} \in L^2_T(0,T;L^4(\Omega; H)) \), by (5.25), we find that

\[ \left|\mathbb{E} \int_0^T \chi_{E_\varepsilon}(t) \langle (Q^{n_k} \beta_{n_k})(t), x^{\varepsilon}_{2,n_k}(t) \rangle_H dt \right| \leq |x^{\varepsilon}_{2,n_k}(\cdot)|_{L^\infty_T(0,T;L^4(\Omega; H))} \int_{E_\varepsilon} \left| (Q^{n_k} \beta_{n_k})(t) \right|_{L^4(\Omega; H)} \frac{4}{T} dt \]

\[ \leq C(x_0, k)\sqrt{T} \int_{E_\varepsilon} \left| (Q^{n_k} \beta_{n_k})(t) \right|_{L^4(\Omega; H)} \frac{4}{T} dt \]

\[ = o(\varepsilon), \quad \text{as } \varepsilon \to 0. \]  

(5.26)

Similarly,

\[ \left|\mathbb{E} \int_0^T \chi_{E_\varepsilon}(t) \langle x^{\varepsilon}_{2,n_k}(t), (\widehat{Q}^{n_k} \beta_{n_k})(t) \rangle_H dt \right| = o(\varepsilon), \quad \text{as } \varepsilon \to 0. \]  

(5.27)

From (4.5) in Theorem 4.1, and noting that both \( Q^{n_k} \) and \( \widehat{Q}^{n_k} \) are pointwisely defined, we arrive at the following equality:

\[ \mathbb{E} \int_0^T \langle x^{\varepsilon}_{E_\varepsilon}(t), (\widehat{Q}^{(0)}(0,0,\chi_{E_\varepsilon} \beta_{n_k})(t))_H dt \rangle + \mathbb{E} \int_0^T \langle (Q^{(0)}(0,0,\chi_{E_\varepsilon} \beta_{n_k})(t), t, \chi_{E_\varepsilon} \beta_{n_k}(t))_H dt \rangle \]

\[ = \mathbb{E} \int_0^T \chi_{E_\varepsilon} \left[ \langle (Q^{n_k} \beta_{n_k})(t), x^{\varepsilon}_{2,n_k}(t) \rangle_H + \langle x^{\varepsilon}_{2,n_k}(t), (\widehat{Q}^{n_k} \beta_{n_k})(t) \rangle_H \right] dt. \]  

(5.28)

Hence,

\[ \mathbb{E} \int_0^T \langle x^{\varepsilon}_{E_\varepsilon}(t), (\widehat{Q}^{(0)}(0,0,\chi_{E_\varepsilon} \beta_{n_k})(t))_H dt \rangle + \mathbb{E} \int_0^T \langle (Q^{(0)}(0,0,\chi_{E_\varepsilon} \beta_{n_k})(t), t, \chi_{E_\varepsilon} \beta_{n_k}(t))_H dt \rangle \]

\[ - \mathbb{E} \int_0^T \chi_{E_\varepsilon} \left[ \langle (Q^{n_k} \beta_{n_k})(t), x^{\varepsilon}_{2,n_k}(t) \rangle_H + \langle x^{\varepsilon}_{2,n_k}(t), (\widehat{Q}^{n_k} \beta_{n_k})(t) \rangle_H \right] dt \]

\[ = \mathbb{E} \int_0^T \langle x^{\varepsilon}_{E_\varepsilon}(t), (\widehat{Q}^{(0)}(0,0,\chi_{E_\varepsilon} \beta_{n_k})(t))_H dt \rangle + \mathbb{E} \int_0^T \langle (Q^{(0)}(0,0,\chi_{E_\varepsilon} \beta_{n_k})(t), t, \chi_{E_\varepsilon} \beta_{n_k}(t))_H dt \rangle \]

\[ - \mathbb{E} \int_0^T \langle x^{\varepsilon}_{E_\varepsilon}(t), (\widehat{Q}^{(0)}(0,0,\chi_{E_\varepsilon} \beta_{n_k})(t))_H dt \rangle - \mathbb{E} \int_0^T \langle (Q^{(0)}(0,0,\chi_{E_\varepsilon} \beta_{n_k})(t), t, \chi_{E_\varepsilon} \beta_{n_k}(t))_H dt \rangle. \]  

(5.29)

It is easy to see that

\[ \left| \mathbb{E} \int_0^T \langle x^{\varepsilon}_{E_\varepsilon}(t), (\widehat{Q}^{(0)}(0,0,\chi_{E_\varepsilon} \beta_{n_k})(t))_H dt \rangle - \mathbb{E} \int_0^T \langle x^{\varepsilon}_{E_\varepsilon}(t), (\widehat{Q}^{(0)}(0,0,\chi_{E_\varepsilon} \beta_{n_k})(t))_H dt \rangle \right| \]

\[ \leq \left| \mathbb{E} \int_0^T \langle x^{\varepsilon}_{E_\varepsilon}(t), (\widehat{Q}^{(0)}(0,0,\chi_{E_\varepsilon} \beta_{n_k})(t))_H dt \rangle - \mathbb{E} \int_0^T \langle x^{\varepsilon}_{E_\varepsilon}(t), (\widehat{Q}^{(0)}(0,0,\chi_{E_\varepsilon} \beta_{n_k})(t))_H dt \rangle \right| \]

\[ + \mathbb{E} \int_0^T \langle x^{\varepsilon}_{E_\varepsilon}(t), (\widehat{Q}^{(0)}(0,0,\chi_{E_\varepsilon} \beta_{n_k})(t))_H dt \rangle - \mathbb{E} \int_0^T \langle x^{\varepsilon}_{E_\varepsilon}(t), (\widehat{Q}^{(0)}(0,0,\chi_{E_\varepsilon} \beta_{n_k})(t))_H dt \rangle. \]  

(5.30)
From (5.22) and the density of the Lebesgue points, we find that for a.e. \( \tau \in [0, T) \), it holds that

\[
\lim_{k \to \infty} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left| \mathbb{E} \int_0^T \langle \chi_{E_\epsilon}(t) \delta b(t), \tilde{Q}^{(0)}(0, 0, \chi_{E_\epsilon} \beta_{n_k})(t) \rangle_H \, dt \right|
\]

\[
- \mathbb{E} \int_0^T \langle \chi_{E_\epsilon}(t) \delta b(t), \tilde{Q}^{(0)}(0, 0, \chi_{E_\epsilon} \beta_{n_k})(t) \rangle_H \, dt \right|
\]

\[
\leq \lim_{k \to \infty} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left| \int_0^T \chi_{E_\epsilon}(t) \left( \mathbb{E} |\delta b(t)|^4_H \right)^{\frac{1}{2}} \, dt \right| \left[ \mathcal{Q}^{(0)}(0, 0, \chi_{E_\epsilon} (\delta b - \beta_{n_k})) \right]_{L^2_T(0, T; L^4 \Omega)}
\]

\[
\leq C \lim_{k \to \infty} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left| \int_0^T \chi_{E_\epsilon}(t) \left( \mathbb{E} |\delta b(t)|^4_H \right)^{\frac{1}{2}} \, dt \right| \left| \mathcal{Q}^{(0)}(0, 0, \chi_{E_\epsilon} (\delta b - \beta_{n_k})) \right|_{L^2_T(0, T; L^4 \Omega)}
\]

\[
\leq C \lim_{k \to \infty} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left| \int_0^T \chi_{E_\epsilon}(t) \left( \mathbb{E} |\delta b(t)|^4_H \right)^{\frac{1}{2}} \, dt \right| \left| \mathcal{Q}^{(0)}(0, 0, \chi_{E_\epsilon} (\delta b - \beta_{n_k})) \right|_{L^2_T(0, T; L^4 \Omega)}
\]

\[
= C \lim_{k \to \infty} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left| \int_0^T \chi_{E_\epsilon}(t) \left( \mathbb{E} |\delta b(t)|^4_H \right)^{\frac{1}{2}} \, dt \right| \left| \mathcal{Q}^{(0)}(0, 0, \chi_{E_\epsilon} (\delta b - \beta_{n_k})) \right|_{L^2_T(0, T; L^4 \Omega)}
\]

\[
= C \lim_{k \to \infty} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left| \int_0^T \chi_{E_\epsilon}(t) \left( \mathbb{E} |\delta b(t)|^4_H \right)^{\frac{1}{2}} \, dt \right| \left| \mathcal{Q}^{(0)}(0, 0, \chi_{E_\epsilon} (\delta b - \beta_{n_k})) \right|_{L^2_T(0, T; L^4 \Omega)}
\]

\[
= 0.
\]

Similarly,

\[
\lim_{k \to \infty} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left| \mathbb{E} \int_0^T \langle \chi_{E_\epsilon}(t) \delta b(t), \tilde{Q}^{(0)}(0, 0, \chi_{E_\epsilon} \beta_{n_k})(t) \rangle_H \, dt \right|
\]

\[
- \mathbb{E} \int_0^T \langle \chi_{E_\epsilon}(t) \delta b(t), \tilde{Q}^{(0)}(0, 0, \chi_{E_\epsilon} \beta_{n_k})(t) \rangle_H \, dt \right|
\]

\[
\leq \lim_{k \to \infty} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left| \int_0^T \chi_{E_\epsilon}(t) \left( \mathbb{E} |\delta b(t)|^4_H \right)^{\frac{1}{2}} \, dt \right| \left[ \int_0^T \chi_{E_\epsilon}(t) \left( \mathbb{E} |\delta b(t) - \beta_{n_k}(t)|^4_H \right)^{\frac{1}{2}} \, dt \right]^{\frac{1}{2}}
\]

\[
\leq C \lim_{k \to \infty} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left| \int_0^T \chi_{E_\epsilon}(t) \left( \mathbb{E} |\delta b(t) - \beta_{n_k}(t)|^4_H \right)^{\frac{1}{2}} \, dt \right| \left| \mathcal{Q}^{(0)}(0, 0, \chi_{E_\epsilon} (\delta b - \beta_{n_k})) \right|_{L^2_T(0, T; L^4 \Omega)}
\]

\[
\leq C \lim_{k \to \infty} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left| \int_0^T \chi_{E_\epsilon}(t) \left( \mathbb{E} |\delta b(t) - \beta_{n_k}(t)|^4_H \right)^{\frac{1}{2}} \, dt \right| \left| \mathcal{Q}^{(0)}(0, 0, \chi_{E_\epsilon} (\delta b - \beta_{n_k})) \right|_{L^2_T(0, T; L^4 \Omega)}
\]

\[
= C \lim_{k \to \infty} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left| \int_0^T \chi_{E_\epsilon}(t) \left( \mathbb{E} |\delta b(t) - \beta_{n_k}(t)|^4_H \right)^{\frac{1}{2}} \, dt \right| \left| \mathcal{Q}^{(0)}(0, 0, \chi_{E_\epsilon} (\delta b - \beta_{n_k})) \right|_{L^2_T(0, T; L^4 \Omega)}
\]

\[
= C \lim_{k \to \infty} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left| \int_0^T \chi_{E_\epsilon}(t) \left( \mathbb{E} |\delta b(t) - \beta_{n_k}(t)|^4_H \right)^{\frac{1}{2}} \, dt \right| \left| \mathcal{Q}^{(0)}(0, 0, \chi_{E_\epsilon} (\delta b - \beta_{n_k})) \right|_{L^2_T(0, T; L^4 \Omega)}
\]

\[
= C \lim_{k \to \infty} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left| \int_0^T \chi_{E_\epsilon}(t) \left( \mathbb{E} |\delta b(t) - \beta_{n_k}(t)|^4_H \right)^{\frac{1}{2}} \, dt \right| \left| \mathcal{Q}^{(0)}(0, 0, \chi_{E_\epsilon} (\delta b - \beta_{n_k})) \right|_{L^2_T(0, T; L^4 \Omega)}
\]

\[
= 0.
\]
From (5.30)–(5.32), we find that
\[
\lim_{k \to \infty} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left| \mathbb{E} \int_0^T \langle \chi_{E_\varepsilon}(t) \delta b(t), \tilde{Q}^{(0)}(0, 0, \chi_{E_\varepsilon} \delta b)(t) \rangle_H dt - \mathbb{E} \int_0^T \langle \chi_{E_\varepsilon}(t) \beta_{n_k}(t), \tilde{Q}^{(0)}(0, 0, \chi_{E_\varepsilon} \beta_{n_k})(t) \rangle_H dt \right| = 0.
\] (5.33)

By a similar argument, we obtain that
\[
\lim_{k \to \infty} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left| \mathbb{E} \int_0^T \langle Q^{(0)}(0, 0, \chi_{E_\varepsilon} \delta b)(t), \chi_{E_\varepsilon}(t) \delta b(t) \rangle_H dt - \mathbb{E} \int_0^T \langle Q^{(0)}(0, 0, \chi_{E_\varepsilon} \beta_{n_k})(t), \chi_{E_\varepsilon}(t) \beta_{n_k}(t) \rangle_H dt \right| = 0.
\] (5.34)

From (5.26)–(5.29) and (5.33)–(5.34), we obtain that
\[
\left| \mathbb{E} \int_0^T \chi_{E_\varepsilon}(t) \langle \delta b(t), \tilde{Q}^{(0)}(0, 0, \chi_{E_\varepsilon} \delta b)(t) \rangle_H dt + \mathbb{E} \int_0^T \chi_{E_\varepsilon}(t) \langle Q^{(0)}(0, 0, \delta b)(t), \delta b(t) \rangle_H dt \right| = o(\varepsilon), \quad \text{as } \varepsilon \to 0.
\] (5.35)

Combining (5.18), (5.19), (5.20) and (5.35), we end up with
\[
\mathcal{J}(u^{\varepsilon}(-)) - \mathcal{J}(\tilde{u}(-)) = \text{Re} \mathbb{E} \int_0^T \left[ \delta g(t) - \langle y(t), \delta a(t) \rangle_H - \langle Y(t), \delta b(t) \rangle_H - \frac{1}{2} \langle P(t) \delta b(t), \delta b(t) \rangle_H \right] \chi_{E_\varepsilon}(t) dt + o(\varepsilon).
\] (5.36)

Since \( \tilde{u}(-) \) is the optimal control, \( \mathcal{J}(u^{\varepsilon}(-)) - \mathcal{J}(\tilde{u}(-)) \geq 0 \). Thus,
\[
\text{Re} \mathbb{E} \int_0^T \chi_{E_\varepsilon}(t) \left[ \langle y(t), \delta a(t) \rangle_H + \langle Y(t), \delta b(t) \rangle_H - \delta g(t) + \frac{1}{2} \langle P(t) \delta b(t), \delta b(t) \rangle_H \right] dt \leq o(\varepsilon),
\] (5.37)

as \( \varepsilon \to 0 \).

Finally, by (5.37), we obtain (5.10). This completes the proof of Theorem 5.1. \[\Box\]

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