On two supercongruences of truncated hypergeometric series $\,_4F_3$

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Abstract
In this paper, we prove two supercongruences conjectured by Sun via the Wilf–Zeilberger method. One of them is, for any prime \( p > 3 \),

\[
\begin{align*}
\,_4F_3 \left[ \begin{array}{cccc} 
7 & 1 & 1 & 1 \\
6 & 6 & 6 & 6 \\
1 & 1 & 1 & 1 \\
4 & 4 & 4 & 4 
\end{array} \right]_{p-1} & \equiv p(-1)^{(p-1)/2} - p^3 E_{p-3} \pmod{p^4},
\end{align*}
\]

where \( E_{p-3} \) is the \((p - 3)\)th Euler number. In fact, this supercongruence is a generalization of a supercongruence of van Hamme.

Keywords Supercongruence · Truncated hypergeometric series · Wilf–Zeilberger method · Euler numbers

Mathematics Subject Classification Primary 11A07 · 33C20 · Secondary 05A10 · 11B65

1 Introduction
For \( n \in \mathbb{N} = \{0, 1, 2, \ldots\} \), define the truncated hypergeometric function:
\[ m+1 \mathrm{F}_m \left[ \begin{array}{c} \alpha_0, \alpha_1, \ldots, \alpha_m \\ \beta_1, \ldots, \beta_m \end{array} \right] z \right]_n := \sum_{k=0}^{n} \frac{(\alpha_0)_k (\alpha_1)_k \cdots (\alpha_m)_k}{(\beta_1)_k \cdots (\beta_m)_k} \cdot \frac{z^k}{k!}, \]

where \( \alpha_0, \ldots, \alpha_m, \beta_1, \ldots, \beta_m, z \in \mathbb{C} \) and

\[
(\alpha)_k = \begin{cases} 
\alpha(\alpha + 1) \cdots (\alpha + k - 1), & \text{if } k \geq 1, \\
1 & \text{if } k = 0.
\end{cases}
\]

In the past decade, many researchers studied supercongruences of truncated hypergeometric series via the Wilf–Zeilberger (WZ) method. For instance, Zudilin [22] proved several Ramanujan-type supercongruences by the WZ method. One of them, conjectured by van Hamme, says that for any odd prime \( p \),

\[
_{4}F_{3} \left[ \begin{array}{c} 5 \frac{1}{2} \frac{1}{2} \frac{1}{2} 1 \\ \frac{1}{4} \frac{1}{4} 1 1 \end{array} \right] _{p-1} \equiv \left( -1 \right)^{(p-1)/2} \left( \frac{1}{p} \right) \pmod{p^3}. \tag{1.1}
\]

**Remark 1.1** (1.1) was first proved by Mortenson [14] in 2008, he used a deep method involving \( p \)-adic Gamma function, Gauss sum and Jacobi sum.

Osburn and Zudilin [15] confirmed a conjecture of van Hamme by the WZ method, that is for any prime \( p > 3 \), we have

\[
_{4}F_{3} \left[ \begin{array}{c} 47 \frac{1}{2} \frac{1}{2} 1 \frac{1}{2} 1 \\ \frac{5}{42} \frac{1}{4} 1 1 1 \end{array} \right] _{p-1} \equiv \left( -1 \right) \left( \frac{1}{p} \right) \pmod{p^4}.
\]

Then Hu and the first author [9] extended the above congruence to

\[
_{4}F_{3} \left[ \begin{array}{c} 47 \frac{1}{2} \frac{1}{2} 1 \frac{1}{2} 1 \\ \frac{5}{42} \frac{1}{4} 1 1 1 \end{array} \right] _{p-1} \equiv \left( -1 \right) \left( \frac{1}{p} \right) - \frac{1}{5} p^3 E_{p-3} \pmod{p^4},
\]

which was conjectured by Sun [18], where \( E_n \) are the Euler numbers defined by

\[
E_0 = 1, \quad \text{and } E_n = - \sum_{k=1}^{\lfloor n/2 \rfloor} \left( \begin{array}{c} n \\ 2k \end{array} \right) E_{n-2k} \text{ for } n \in \{1, 2, \ldots\}.
\]

Chen et al. [2] confirmed a supercongruence conjectured by Sun [18], which says that for any prime \( p > 3 \), we have

\[
_{4}F_{3} \left[ \begin{array}{c} 3 \frac{1}{2} \frac{1}{2} 1 1 \\ \frac{1}{2} 1 1 1 \end{array} \right] _{p-1} \equiv p(-1)^{(p-1)/2} + p^3 E_{p-3} \pmod{p^4}.
\]
For $n \in \mathbb{N}$, define

$$H_n := \sum_{0 < k \leq n} \frac{1}{k}, \quad H_n^{(2)} := \sum_{0 < k \leq n} \frac{1}{k^2}, \quad H_0 = H_0^{(2)} = 0,$$

where $H_n$ with $n \in \mathbb{N}$ are often called the classical harmonic numbers. Let $p > 3$ be a prime. Wolstenholme [21] proved that

$$H_{p-1} \equiv 0 \pmod{p^2} \quad \text{and} \quad H_{p-1}^{(2)} \equiv 0 \pmod{p},$$

which imply that

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}. \quad (1.2)$$

Sun [19] proved the following supercongruence which is an extension of (1.1) by the WZ method, for any odd prime $p$,

$$4F3 \left[ \begin{array}{ccc} \frac{5}{4} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & 1 & 1 \end{array} \right] - 1 \right]_{p-1} \equiv (-1)^{p-1} p + p^3 E_{p-3} \pmod{p^4}. \quad (1.3)$$

Guo and Liu [5] showed that for any prime $p > 3$, we have

$$\sum_{k=0}^{(p+1)/2} (-1)^k (4k - 1) \frac{(-1)^3}{(1/2)^k} \equiv p(-1)^{(p+1)/2} + p^3 (2 - E_{p-3}) \pmod{p^4}. \quad (1.4)$$

Guo and his coauthors also researched $q$-analogues of Ramanujan-type supercongruences and $q$-analogues of supercongruences of van Hamme (see, for instance, [3,4,6,7]).

Long [11] proved a conjecture of van Hamme [20], which is, for any prime $p > 3$, we have

$$4F3 \left[ \begin{array}{ccc} \frac{7}{6} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & 1 & 1 \end{array} \right] \frac{1}{4^{E_{p-3}/2}} \equiv p(-1)^{(p-1)/2} \pmod{p^4}. \quad (1.5)$$

In this paper, we first obtain the following result which confirms a conjecture of Sun [18]:

**Theorem 1.2** Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{6k + 1}{256k} \binom{2k}{k}^3 = 4F3 \left[ \begin{array}{ccc} \frac{7}{6} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & 1 & 1 \end{array} \right] \frac{1}{4}\frac{1}{p-1} \equiv p(-1)^{(p-1)/2} - p^3 E_{p-3} \pmod{p^4}. \quad (1.5)$$
Via the WZ method, Zudilin [22] proved that for any odd prime \( p \),

\[
4 \left[ \begin{array}{ccccc}
\frac{23}{20} & \frac{3}{20} & 1 & 1 & 1 \\
\frac{1}{4} & \frac{3}{4} & \frac{3}{4}
\end{array} \right] \equiv p(-1)^{(p-1)/2} \pmod{p^3}. \tag{1.6}
\]

Sun [19] used the WZ pair which was used by Zudilin [22] to prove the following supercongruence: For any prime \( p > 3 \), we have

\[
4 \left[ \begin{array}{ccccc}
\frac{23}{20} & \frac{3}{20} & 1 & 1 & 1 \\
\frac{1}{4} & \frac{3}{4} & \frac{3}{4}
\end{array} \right] \equiv \frac{1}{3} p(-1)^{p-1/2} (2^{p-1} + 2 - (2^{p-1} - 1)^2) \pmod{p^4}.
\]

Our second result is the following congruence which generalizes (1.6).

**Theorem 1.3** For any odd prime \( p \), we have

\[
\sum_{k=0}^{p-1} \frac{2k + 3}{(-2)^k} \binom{4k}{k, k, k, k} = 3 \cdot 4 \left[ \begin{array}{ccccc}
\frac{23}{20} & \frac{3}{20} & 1 & 1 & 1 \\
\frac{1}{4} & \frac{3}{4} & \frac{3}{4}
\end{array} \right] \equiv 3p(-1)^{(p-1)/2} + 3p^3 E_{p-3} \pmod{p^4}.
\]

**Remark 1.4** In [19], Sun said that he ever proved the congruence in Theorem 1.3, but he lost the draft containing the complicated details. Recently, he asked us to prove this congruence.

Our main tool in this paper is the WZ method. We shall prove Theorem 1.2 in the next section, and the last section is devoted to prove Theorem 1.3.

## 2 Proof of Theorem 1.2

We will use the following WZ pair which appears in [8] to prove Theorem 1.2. For nonnegative integers \( n, k \), define

\[
F(n, k) = \frac{(6n - 2k + 1) \binom{2n}{n} \binom{2n + 2k}{n+k} \binom{2n - 2k}{n-k} \binom{n+k}{n}}{2^{8n-2k} \binom{2k}{k}}
\]

and

\[
G(n, k) = \frac{n^2 \binom{2n}{n} \binom{2n + 2k}{n+k} \binom{2n - 2k}{n-k} \binom{n+k}{n}}{2^{8n-2k-4} (2n + 2k - 1) \binom{2k}{k}}.
\]

Clearly, \( F(n, k) = G(n, k) = 0 \) if \( n < k \). It is easy to check that

\[
F(n, k - 1) - F(n, k) = G(n + 1, k) - G(n, k) \tag{2.1}
\]
for all nonnegative integer \( n \) and \( k > 0 \).

Summing (2.1) over \( n \) from 0 to \( p - 1 \), we have

\[
\sum_{n=0}^{p-1} F(n, k - 1) - \sum_{n=0}^{p-1} F(n, k) = G(p, k) - G(0, k) = G(p, k).
\]

Furthermore, summing both side of the above identity over \( k \) from 1 to \( p - 1 \), we obtain

\[
\sum_{n=0}^{p-1} F(n, 0) = F(p - 1, p - 1) + \sum_{k=1}^{p-1} G(p, k).
\]  

(2.2)

**Lemma 2.1** Let \( p > 3 \) be a prime. Then

\[
F(p - 1, p - 1) \equiv -3p^2 - 12p^3 + 18p^3 q_p(2) \pmod{p^4},
\]

where \( q_p(2) \) stands for the Fermat quotient \((2^{p-1} - 1)/p\).

**Proof** By the definition of \( F(n, k) \), we have

\[
F(p - 1, p - 1) = \frac{4p - 3}{2^{6p-6}} \binom{4p - 4}{2p - 2} \binom{2p - 2}{p - 1} = \frac{p(2p-1)(4p-3)}{2^{6p-6}} \binom{4p-1}{2p-1} \binom{2p-1}{p-1} \cdot \frac{1}{(4p - 1)2^{6p-6}}.
\]

It is known that Jocobsthal’s congruence (see [1]) is as follows: For primes \( p > 3 \), integers \( a, b \) and integers \( r, s \geq 1 \), we have

\[
\binom{ap^r}{bp^s} / \binom{ap^r-1}{bp^s-1} \equiv 1 \pmod{p^{r+s+\min\{r,s\}}}.
\]  

(2.3)

Thus,

\[
\binom{4p - 1}{2p - 1} = \frac{1}{2} \binom{4p}{2p} = \frac{1}{2} \binom{4}{2} \equiv 3 \pmod{p^3}.
\]

This, with (1.2) and \( 2^{p-1} = 1 + p q_p(2) \), yields that

\[
F(p - 1, p - 1) \equiv \frac{3p^2}{(4p - 1)2^{6p-6}} \equiv -3p^2 - 12p^3 + 18p^3 q_p(2) \pmod{p^4}.
\]

Therefore, the proof of Lemma 2.1 is complete.

\[\square\]
By the definition of $G(n, k)$, we have

$$G(p, k) = \frac{p^2 \binom{2p}{p} \binom{2p+2k}{p+k} \binom{2p-2k}{p-k}}{2^{8p-4-2k} (2p+2k-1) \binom{2k}{k}},$$

where we used the binomial transformation:

$$\binom{n}{k} \binom{k}{j} = \binom{n}{j} \binom{n-j}{k-j}.$$

**Lemma 2.2** ([17, (3.1)])

$$\sum_{k=1}^{(p-1)/2} \frac{4k}{k(2k-1) \binom{2k}{k}} \equiv 2E_{p-3} \pmod{p}.$$ 

**Remark 2.3** The congruence in Lemma 2.2 is often used when we use the WZ method to prove some supercongruences. For instance, see [2,9].

**Lemma 2.4** For any prime $p > 3$, we have

$$\sum_{k=1}^{(p-1)/2} G(p, k) \equiv -p^3 E_{p-3} \pmod{p^4}.$$ 

**Proof** It is easy to see that $\binom{2p-2k}{p-k} \equiv 0 \pmod{p}$ for each $1 \leq k \leq (p-1)/2$. Then by (2.4), (2.3) and Lucas congruence (see [12]), we have

$$G(p, k) \equiv \frac{4p^2 \binom{2p-2k}{p-k}}{(2p+2k-1)2^{8p-4-2k}} \pmod{p^4}.$$ 

In view of [18, Lemma 2.1]

$$k \binom{2k}{k} \binom{2(p-k)}{p-k} \equiv (-1)^{\lfloor 2k/p \rfloor - 1} 2p \pmod{p^2} \text{ for all } k = 1, \ldots, p-1.$$ 

Then for each $1 \leq k \leq (p-1)/2$, we have

$$\binom{2p-2k}{p-k} \equiv -\frac{2p}{k \binom{2k}{k}} \pmod{p^2}.$$ 

Hence,

$$G(p, k) \equiv \frac{-p^3}{2^{8p-7-2k} k(2k-1) \binom{2k}{k}} \equiv -\frac{p^3}{2} \frac{4k}{k(2k-1) \binom{2k}{k}} \pmod{p^4}.$$
Therefore, we immediately obtain the desired result with Lemma 2.2.

\section*{Lemma 2.5} Let $p > 3$ be a prime. Then

$$G(p, (p + 1)/2) \equiv (-1)^{(p-1)/2} p \left(1 - 3pq_p(2) + 6p^2q_p(2)^2\right) \pmod{p^4}.$$ 

\section*{Proof} In view of (2.4) and (2.3), we have the following congruence modulo $p^4$:

$$G(p, (p + 1)/2) = \frac{p^2(2p)^1((p-1)/2)(2p+1)}{2^{7p-5}(2p + p)(p+1)(2p+1)} = \frac{p^2(2p)^1((p-1)/2)(2p+1)}{2^{7p-5}(2p + p)(p+1)(2p+1)}.$$

It is easy to see that

$$\binom{2p-1}{(p-1)/2} \equiv (-1)^{(p-1)/2} \left(1 - 2pH_{(p-1)/2} + 2p^2H^2_{(p-1)/2} - 2p^2H^2_{(p-1)/2}\right) \pmod{p^3}.$$

In view of [10,16], we have

$$H_{(p-1)/2} \equiv -2q_p(2) + pq_p(2)^2 \pmod{p^2} \text{ and } H^2_{(p-1)/2} \equiv 0 \pmod{p}. \quad (2.5)$$

Thus,

$$\binom{2p-1}{(p-1)/2} \equiv (-1)^{(p-1)/2} \left(1 + 4pq_p(2) + 6p^2q_p(2)^2\right) \pmod{p^3}.$$

Therefore,

$$G(p, (p + 1)/2) \equiv (-1)^{(p-1)/2} p \left(1 - 3pq_p(2) + 6p^2q_p(2)^2\right) \pmod{p^4}$$

since $1/2^{7p-7} \equiv 1 - 7pq_p(2) + 28p^2q_p(2)^2 \pmod{p^3}$.

\section*{Lemma 2.6} ([17, (1.1) and (1.7)]) Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{(p-3)/2} \frac{(2k)^1}{(2k+1)^4} \equiv (-1)^{(p-1)/2}q_p(2) \pmod{p^2},$$

$$\sum_{k=0}^{(p-3)/2} \frac{(2k)^1}{(2k+1)^2} \equiv -\frac{(-1)^{(p-1)/2}q_p(2)^2}{2} \pmod{p}.$$
Lemma 2.7 ([9]) For any positive integer \( n \), we have
\[
\sum_{k=1}^{n} \frac{(-1)^{k} \binom{n}{k} H_k}{(2k+1)} = -\frac{4^n}{(2n+1)\binom{2n}{n}} \sum_{k=1}^{n} \frac{(2k)}{k4^k},
\]
\[
\sum_{k=1}^{n} \frac{(-1)^{k} \binom{n}{k} H_{2k}}{(2k+1)} = -\frac{4^n}{(2n+1)\binom{2n}{n}} \left( \frac{H_n}{2} + \frac{1}{2} \sum_{k=1}^{n} \frac{(2k)}{k4^k} \right).
\]

Lemma 2.8 ([13]) For any prime \( p > 3 \), we have
\[
\left( \frac{p-1}{(p-1)/2} \right) \equiv (-1)^{(p-1)/2} 4^{p-1} \pmod{p^3}.
\]

Lemma 2.9 Let \( p > 3 \) be a prime. Then
\[
\sum_{k=\frac{p+3}{2}}^{\frac{p-1}{2}} G(p, k) \equiv 3p^2(1 + 4p - 6pq_p(2))
\]
\[
+ (-1)^{(p-1)/2} 3p^2 q_p(2)(1 - 2pq_p(2)) \pmod{p^4}.
\]

Proof In view of (2.4), we have the following congruence modulo \( p^4 \):
\[
\sum_{k=\frac{p+3}{2}}^{\frac{p-1}{2}} G(p, k) \equiv \frac{2p^2}{2^{8p-4}} \sum_{k=\frac{p+3}{2}}^{\frac{p-1}{2}} \frac{4^k \binom{2p+2k}{p} \binom{p+2k}{k} \binom{2p-2k}{p-k}}{(2p+2k-1)\binom{2k}{k}}
\]
\[
= \frac{2p^2}{2^{6p-4}} \sum_{k=1}^{p-3} \frac{4^k \binom{2p-2k}{p} \binom{p-2k}{k} \binom{2k}{k}}{4^k(4p-2k-1)\binom{2p-2k}{p-k}}.
\]

It is easy to check that for each \( 1 \leq k \leq (p-3)/2 \),
\[
\binom{4p-2k}{p} \binom{3p-2k}{p-k} = \frac{(4p-2k) \ldots (2p-k+1)}{p!(p-k)!}
\]
\[
= 6p \frac{(3p + p - 2k) \ldots (3p + 1)(3p - 1) \ldots (2p + 1)(2p - 1) \ldots (2p - (k-1))}{(p-1)!(p-k)!}
\]
\[
\equiv 6p \frac{(p-2k)!(1 + 3pH_{p-2k})(p-1)!(1 + 2pH_{p-1})(-1)^{k-1}(k-1)!}{(p-1)!(p-k)!}
\]
\[
\equiv (-1)^{k-1} 6p(1 + 3pH_{p-2k})(-1)^{k-1}(k-1)! \frac{1}{k(p-k)} \pmod{p^3}.
\]
Hence,
\[
\sum_{k=(p+3)/2}^{p-1} G(p, k) \equiv -12p^3 \sum_{k=1}^{(p-3)/2} \frac{(1 + 3pH_{p-2k} - 2pH_{k-1})^{2k}}{(4p - 2k - 1)k^{(2p-2k)}p^{(p-k)}} \pmod{p^4}.
\]

By simple computation, for each integer \(k\) with \(1 \leq k \leq (p - 3)/2\), we have
\[
\frac{\binom{2p-2k}{p-k}}{p} = \frac{(2p - 2k) \cdots (p + 1)(p - 1) \cdots (p - 2k + 1)}{(p - k)k!} = \frac{(-1)^k (2k - 1)! (1 - p(H_{2k-1} - H_{k-1}))}{k!(k-1)!} \equiv \frac{(-1)^k}{2} \binom{2k}{k} (1 - pH_{2k-1} + pH_{k-1}) \pmod{p^2}, \tag{2.6}
\]
and
\[
\binom{p-k}{k} \equiv \frac{(p - k) \cdots (p - 2k + 1)}{k!} \equiv \frac{(-1)^k (2k - 1)! (1 - p(H_{2k-1} - H_{k-1}))}{k!(k-1)!} \pmod{p^2}.
\tag{2.7}
\]
Thus,
\[
\sum_{k=(p+3)/2}^{p-1} G(p, k) \equiv \frac{3p^2}{26p-6} \sum_{k=1}^{(p-3)/2} \frac{(1 + 3pH_{p-2k} - 2pH_{k-1} - pH_{2k-1} + pH_{k-1})^{2k}}{4^k(4p - 2k - 1)} \equiv \frac{3p^2}{26p-6} \sum_{k=1}^{(p-3)/2} \frac{(1 + 2pH_{2k} - pH_k)^{2k}}{4^k(4p - 2k - 1)} \pmod{p^4},
\]
where we used \(H_{p-1-k} \equiv H_k \pmod{p}\) for all \(k \in \{0, 1, \ldots, p - 1\}\).

Continuing to calculate the above congruence, we have the following congruence modulo \(p^4\):
\[
\sum_{k=\frac{p+3}{2}}^{p-1} G(p, k) \equiv -\frac{3p^2}{26p-6} \left( \sum_{k=1}^{\frac{p-3}{2}} \binom{2k}{k} \frac{1}{(2k + 1)4^k} + 4p \sum_{k=1}^{\frac{p-3}{2}} \frac{\binom{2k}{k}}{(2k + 1)^24^k} + p \sum_{k=1}^{\frac{p-3}{2}} \binom{2k}{k} \frac{(2H_{2k} - H_k)}{(2k + 1)4^k} \right).
\]

In view of Lemma 2.7, we have
\[
\sum_{k=1}^{\frac{p-3}{2}} (-1)^k \binom{\frac{p-1}{2}}{k} (2H_{2k} - H_k) \pmod{p^4},
\]
\[
\binom{\frac{p-1}{2}}{k} (2H_{2k} - H_k)
\]
$$\sum_{k=1}^{p-1} \frac{(-1)^k \binom{p-1}{k}}{2k+1} (2H_{2k} - H_k) = \frac{(-1)^{p-1}}{p} (2H_{p-1} - H_{p-1}^\pi)$$

$$\equiv - \frac{2^{p-1}}{p^{(p-1)/2}} H_{(p-1)/2} + \frac{(-1)^{p-1}}{p} H_{p-1}^\pi \pmod{p}.$$ 

Hence, by Lemma 2.8, $2^{p-1} = 1 + pq p$ and $H_{(p-1)/2} \equiv -2q p(2) \pmod{p}$, we have

$$\sum_{k=1}^{p-3} \frac{(-1)^k \binom{p-1}{k}}{2k+1} (2H_{2k} - H_k) \equiv -2(-1)^{(p-1)/2} q p(2)^2 \pmod{p}. \quad (2.8)$$

Therefore, we immediately get the desired result with Lemma 2.6. \(\Box\)

**Proof of Theorem 1.2** Combining (2.2) with Lemmas 2.1, 2.4, 2.5 and 2.9, we immediately obtain that

$$4F_3\left[\begin{array}{c} 7/6 \\ 1/6 \\ 1/4 \\ 1/2, 1/2, 1/2, 1 \\
6, 1, 1, 1 \\
\end{array}\right]_{p-1} = \sum_{k=0}^{p-1} \frac{6k + 1}{256^k} \binom{2k}{k}^3 \equiv (-1)^{(p-1)/2} p - p^3 E_{p-3} \pmod{p^4}.$$ 

Therefore, the proof of Theorem 1.2 is finished. \(\Box\)

### 3 Proof of Theorem 1.3

To prove Theorem 1.3, we should use the following WZ pair which appears in [19] (see also [22]). For nonnegative integers $n, k$, define

$$F(n, k) = (-1)^{n+k} \frac{(2n - 2k + 3)(2n)(4n+2k)_{n+k}(2n)_{n+k}(2n+k)_{2k}}{4^{5n-k} \binom{2k}{n+k}}$$

and

$$G(n, k) = (-1)^{n+k} n(\frac{2n-1}{n})_{n-1}(\frac{4n+2k-2}{2n+k-1})_{n-1}(\frac{2n-k-1}{2k})_{2k}.$$ 

Clearly, $F(n, k) = G(n, k) = 0$ if $n < k$. It is easy to check that

$$F(n, k - 1) - F(n, k) = G(n + 1, k) - G(n, k) \quad (3.1)$$

for all nonnegative integer $n$ and $k > 0$. 

\(\Box\) Springer
Summing (3.1) over $n$ from 0 to $p - 1$ and then over $k$ from 1 to $p - 1$, we have
\[
\sum_{n=0}^{p-1} F(n, 0) = F(p - 1, p - 1) + \sum_{k=1}^{p-1} G(p, k). \tag{3.2}
\]

**Lemma 3.1** Let $p > 3$ be a prime. Then
\[
F(p - 1, p - 1) \equiv 15p^2(-1 - 6p + 8pq_p(2)) \pmod{p^4}.
\]

**Proof** By the definition of $F(n, k)$, we have
\[
F(p - 1, p - 1) = \frac{18p - 15}{4^{4p-4}} \cdot \frac{(6p - 6)(3p - 3)}{(3p - 3)(p - 1)} \cdot \frac{3p}{4^{4p-4}} \cdot \frac{(3p-1)(6p-3)}{2 \cdot 4^{4p-4}} = \frac{p^2(3p)(6p)}{(6p-1)4^{4p-3}}.
\]

In view of (2.3), we have
\[
\binom{3p}{2p} \binom{6p}{3p} = \binom{3}{2} \binom{6}{3} \equiv 60 \pmod{p^3}.
\]

This, with $2^{p-1} = 1 + pq_p(2)$, yields that
\[
F(p - 1, p - 1) \equiv \frac{15p^2}{(6p - 1)4^{4p-4}} \equiv 15p^2(-1 - 6p + 8pq_p(2)) \pmod{p^4}.
\]

Therefore, the proof of Lemma 3.1 is complete. \qed

By the definition of $G(n, k)$, we have
\[
G(p, k) = \frac{(-1)^{k+1} p \binom{2p-1}{p-1} \binom{4p+2k-2}{2p+k-1}}{4^{5p-4-k} \binom{2k}{k}} \cdot \frac{2p-k-1}{2p-k} \cdot \frac{1}{p-1} \cdot \frac{2p+k-1}{2k}.
\]

where we used the binomial transformation:
\[
\binom{n}{k} \binom{k}{j} = \binom{n}{j} \binom{n-j}{k-j}.
\]
Note that \( \binom{n}{k} = (-1)^k \binom{-n+k-1}{k} \), then we have
\[
\binom{4p + 2k - 2}{2k} = \binom{-4p + 1}{2k} = \frac{4p(4p - 1)}{2k(2k - 1)} \binom{-4p - 1}{2k - 2}.
\]
So by (2.3), we have
\[
G(p, k) \equiv \left( -1 \right)^{k+1} 6p^3 \binom{-4p - 1}{2k-2} \binom{3p-1}{p-k} \mod p^4. \tag{3.3}
\]

**Lemma 3.2** For any primes \( p > 3 \), we have
\[
\sum_{k=1}^{(p-1)/2} G(p, k) \equiv 3p^3 E_{p-3} \mod p^4.
\]

**Proof** It is easy to see that
\[
\binom{-4p - 1}{2k - 2} \binom{3p - 1}{p - k} \equiv (-1)^{p-k} \mod p.
\]
Then by (3.3), we have
\[
G(p, k) \equiv \frac{3p^3 4^k}{2k(2k - 1)\binom{2k}{k}} \mod p^4.
\]
Therefore, we immediately obtain the desired result with Lemma 2.2. \(\square\)

**Lemma 3.3** Let \( p > 3 \) be a prime. Then
\[
G(p, (p + 1)/2) \equiv (-1)^{(p-1)/2} 3p \left( 1 - 5p q_p(2) + 15p^2 q_p(2)^2 \right) \mod p^4.
\]

**Proof** In view of (3.3), we have the following congruence modulo \( p^4 \):
\[
G(p, (p + 1)/2) \equiv \frac{(-1)^{(p-1)/2} 6p^3 (-4p-1) \binom{3p-1}{p-1/2}}{2^9 p^9 \binom{p+1}{p+1/2}^2} = \frac{(-1)^{(p-1)/2} 3p (-4p-1) \binom{3p-1}{p-1/2}}{2^9 p^9 \binom{p-1}{p-1/2}^2}.
\]
It is easy to see that
\[
\binom{3p - 1}{(p - 1)/2} \equiv (-1)^{(p-1)/2} \left( 1 - 3p H_{(p-1)/2} + 3p^2 H_{(p-1)/2}^2 - 3p^2 H_{(p-1)/2}^2 \right) \mod p^3.
\]
In view of (2.5), we have

\[
\left(\frac{3p-1}{(p-1)/2}\right) \equiv (-1)^{(p-1)/2} \left(1 + 6pq_p(2) + 15p^2q_p(2)^2\right) \pmod{p^3}.
\]

Therefore, by Lemma 2.8, (2.3) and \(1/2^{11p-11} \equiv 1 - 11pq_p(2) + 66p^2q_p(2)^2 \pmod{p^3}\), we have

\[
G(p, (p + 1)/2) \equiv (-1)^{(p-1)/2} 3p \left(1 - 5pq_p(2) + 15p^2q_p(2)^2\right) \pmod{p^4}.
\]

Now, the proof of Lemma 3.3 is completed. \[\square\]

Lemma 3.4 For any prime \(p > 3\), we have

\[
\sum_{k=(p+3)/2}^{p-1} G(p, k) \equiv 15p^2(1 + 6p - 8pq_p(2)) + (-1)^{(p-1)/2} 15p^2(q_p(2) - 3pq_p(2)^2) \pmod{p^4}.
\]

Proof Again by (3.3), we have

\[
\sum_{k=(p+3)/2}^{p-1} G(p, k) \equiv -6p^3 \sum_{k=(p+3)/2}^{p-1} \frac{(-4)^k}{k(2k-1)} \left(\frac{3p-1}{p-k}\right) \left(\frac{3p-1}{p-k}\right) \pmod{p^4}.
\]

\[
= -6p^3 \sum_{k=1}^{(p-3)/2} \frac{(-4)^{p-k}(-4p-1)^{3p-1}}{(p-k)(2p-2k-1)(2p-2k-1)} \pmod{p^4}.
\]

\[
= 6p^3 \sum_{k=1}^{(p-3)/2} \frac{(-4)^{p-k}(-4p-1)^{3p-1}}{(2p-2k-2)^2k} \pmod{p^4}.
\]

It is easy to check that for each \(1 \leq k \leq (p - 3)/2\)

\[
\left(\frac{3p-1}{k}\right) = \prod_{j=1}^{k} 3p - j = (-1)^k \prod_{j=1}^{k} \left(1 - \frac{3p}{j}\right)
\]

\[
\equiv (-1)^k (1 - 3pH_{p-2k}) \pmod{p^2}
\]

and

\[
\left(\frac{-4p-1}{2p-2k-2}\right) = \prod_{j=1}^{2p-2k-2} -4p - j = \prod_{j=1}^{2p-2k-2} \left(1 + \frac{4p}{j}\right)
\]

\[
= 5 \prod_{j=p+1}^{p-1} \left(1 + \frac{4p}{j}\right) \prod_{j=p+1}^{p-1} \left(1 + \frac{4p}{j}\right)
\]

\[\Box\]
\[ \equiv 5(1 + 4p H_{p-1})(1 + 4p H_{p-2k-2}) \]
\[ \equiv 5(1 + 4p H_{p-2k-2}) \pmod{p^2}. \]

Hence,
\[ \sum_{k=0}^{p-1} G(p, k) \equiv \frac{30p^3}{4^3 p - 4} \sum_{k=1}^{(p-3)/2} \frac{(1 - 3p H_k + 4p H_{p-2k-2})}{4^k (p - k)(2p - 2k - 1)} \left( \frac{2p - 2k - 1}{p - k} \right) \pmod{p^4}. \]

This, with (2.6) and (2.7), yields that the following congruence holds modulo \( p^4 \):
\[ \sum_{k=0}^{p-1} G(p, k) \equiv -\frac{15 p^2}{4^3 p - 4} \sum_{k=1}^{(p-3)/2} \frac{(1 - 3p H_k + 4p H_{2k+1} - 2p H_{2k-1} + 2p H_{k-1})k(2k)}{4^k (p - k)(2p - 2k - 1)} \]

where we used \( H_{p-1-k} \equiv H_k \pmod{p} \) for all \( k \in \{0, 1, \ldots, p-1\} \).
It is easy to see that
\[ \sum_{k=1}^{(p-3)/2} \frac{k(2k)}{4^k (p - k)(2p - 2k - 1)} \equiv \sum_{k=1}^{(p-3)/2} \frac{2p(2k)}{4^k (2k + 1)} + \sum_{k=1}^{(p-3)/2} \frac{p(2k)}{4^k k (2k + 1)} \pmod{p^2}. \]

So modulo \( p^4 \), we have
\[ \sum_{k=0}^{p-1} G(p, k) \equiv -\frac{15 p^2}{4^3 p - 4} \left( \sum_{k=1}^{p-3} \frac{2k}{4^k (2k + 1)} + \sum_{k=1}^{p-3} \frac{p(2k)}{4^k (2k + 1)} + \sum_{k=1}^{p-3} \frac{6p(2k)}{4^k (2k + 1)^2} \right). \]

Then we immediately get the desired result with Lemma 2.6 and (2.8). \( \Box \)

**Proof of Theorem 1.3** Combining (3.2) with Lemmas 3.1, 3.2, 3.3 and 3.4, we immediately get that for any prime \( p > 3 \),
\[
4 \binom{\tfrac{23}{20}}{\tfrac{3}{20}}^{\binom{12}{4}} - \binom{-\tfrac{1}{4}}{1} = \frac{1}{3} \sum_{n=0}^{p-1} 20n + 3 \left( \begin{array}{c} 4n \\ n, n, n, n \end{array} \right) 
\equiv (-1)^{p-1} - 1 \pmod{p^4}.
\]

The case of \( p = 3 \) is easy to check. Therefore, the proof of Theorem 1.3 is complete. □

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