THE MULTIDIMENSIONAL TRUNCATED MOMENT PROBLEM: GAUSSIAN AND LOG-NORMAL MIXTURES, THEIR CARATHÉODORY NUMBERS, AND SET OF ATOMS

PHILIPP J. DI DIO

Abstract. We study truncated moment sequences of distribution mixtures, especially from Gaussian and log-normal distributions and their Carathéodory numbers. For \( A = \{a_1, \ldots, a_m\} \) continuous (sufficiently differentiable) functions on \( \mathbb{R}^n \) we give a general upper bound of \( m - 1 \) and a general lower bound of \( \lceil 2m(2m+1)(n+1)n+2 \rceil \). For polynomials of degree at most \( d \) in \( n \) variables we find that the number of Gaussian and log-normal mixtures is bounded by the Carathéodory numbers in [DHS18]. Therefore, for univariate polynomials \( \{1, x, \ldots, x^d\} \) at most \( \lceil d+1 \rceil \) distributions are needed. For bivariate polynomials of degree at most \( 2d-1 \) we find that Gaussian distributions are sufficient. We also treat polynomial systems with gaps and find, e.g., that for \( \{1, x^2, x^3, x^5, x^6\} \) \( 3 \) Gaussian distributions are enough for almost all truncated moment sequences. We give an example of continuous functions where more Gaussian distributions are needed than Dirac delta measures. We show that any inner truncated moment sequence has a mixture which contains any given distribution.

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1. Introduction

In many applications, the distribution is a linear combination of simple distributions such as Gaussian distributions

\[
g_{\xi,\sigma}(x) := \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot e^{-\frac{(x-\xi)^2}{2\sigma^2}} \quad \text{with} \quad \xi \in \mathbb{R}, \sigma > 0
\]

or log-normal distributions

\[
l_{\xi,\sigma}(x) := \begin{cases} \frac{1}{\sqrt{2\pi} \cdot \sigma x} \cdot e^{-\frac{(\log x - \log \xi)^2}{2\sigma^2}} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0 \end{cases} \quad \text{with} \quad \xi, \sigma \in (0, \infty).\]

E.g., in the seminal paper of K. Pearson he investigates the distribution of the breadth of the foreheads of Naples Crabs and the length of Carapace of prawns [Pea94]. Since the data did not fit a single Gaussian distribution, he assumed that the distribution comes from a linear combination of two Gaussian distributions

\[
\frac{c_1}{\sigma_1 \sqrt{2\pi}} \cdot e^{-\frac{(x-x_1)^2}{2\sigma_1^2}} + \frac{c_2}{\sigma_2 \sqrt{2\pi}} \cdot e^{-\frac{(x-x_2)^2}{2\sigma_2^2}}.
\]

To determine \(c_1, c_2, \sigma_1, \sigma_2, x_1, \) and \(x_2\) he calculated the first five moments of \((3)\) (all are polynomials in \(c_1, \ldots, c_2\)) and after algebraic manipulations got a polynomial of degree 9. The zeros of this polynomial are the solution of fitting \((3)\) to the crab
data. This method is now well-known by the name method of moments, see e.g. [TSM85].

Another frequent distribution is the log-normal distribution (2). It appears e.g. in the study of option pricings in financial mathematics [Sto16], especially in the Black–Scholes model by Black, Scholes [BS73], and Merton [Mer73]. In that model it is found that the option pricing is given by

$$x \cdot g_{0,1}(d_1) - c \cdot e^{r(t-t^*)} \cdot g_{0,1}(d_2)$$

with

$$d_1 = \frac{\log(x/c) + (r + \frac{\sigma^2}{2})(t^* - t)}{\sqrt{t^* - t}} \quad \text{and} \quad d_2 = \frac{\log(x/c) + (r - \frac{\sigma^2}{2})(t^* - t)}{\sqrt{t^* - t}},$$

where \(t\) is the time variable and \(x, c, t^*, r,\) and \(v\) are parameters of the option/model. Despite the fact that in the Black–Scholes model the linear combination (4) depends on several parameters and is only related to the log-normal distribution, the log-normal distribution (2) is frequently used and one of the most important distributions in financial engineering [Sto16].

In the following article we treat the problem of mixtures of densities very general but we also derive more detailed results for the Gaussian (1) and log-normal distribution (2) because of their importance. We use the following general setting:

(a) \(\delta_{\xi,\sigma}\) are probability measures on a (topological) space \(\mathcal{X}\) with parameters \(\xi \in \mathcal{X}\) and \(\sigma \in \Sigma, \Sigma\) is the set of parameters (variance; in a larger metric space)

(b) \(A = \{a_1, \ldots, a_m\}\) is a set of linearly independent (real valued) continuous functions on the space \(\mathcal{X}\) s.t.

$$\left| \int_{x \in \mathcal{X}} a_i(x) \, d\delta_{\xi,\sigma}(x) \right| < \infty \quad \forall \xi \in \mathcal{X}, \sigma \in \Sigma;$$

(c) there exists a \(\sigma_0 \in \Sigma\) (closure of \(\Sigma\)) such that

$$\lim_{\Sigma \ni \sigma \rightarrow \sigma_0} \int_{x \in \mathcal{X}} a_i(x) \, d\delta_{\xi,\sigma}(x) = a_i(\xi) \quad \forall \xi \in \mathcal{X}, \quad i = 1, \ldots, m$$

(d) If the integral \(s_i := \int_{\mathcal{X}} a_i(x) \, d\mu(x)\) exists it is called ith (or \(a_i\)-)moment of the measure \(\mu\).

The name moment problem comes from

\(\mathcal{X} = \{1, x, x^2, \ldots, x^n\}\), i.e., the (classical) moments are

$$\int_{\mathcal{X}} x^i \, d\mu(x),$$

while the general moments are

$$\int_{\mathcal{X}} a_i(x) \, d\mu(x).$$

Truncated means that only finitely many moment of \(\mu\) are known (A is finite). Of course, since the integral is linear in the integrand, the moment problem rather depends on \(\text{lin } A\) than on \(A\). So we can always choose an appropriate basis \(A\) of \(\text{lin } A\).

Example 1. For the Gaussian distributions (1) we have \(\mathcal{X} = \mathbb{R}^n (n \in \mathbb{N})\), \(\Sigma \subset \mathbb{R}^{n \times n}\) is the set of all symmetric non-singular matrices, \(\sigma_0 = 0 \in \mathbb{R}^{n \times n}\) is the zero matrix. The Gaussian measure \(G_{2,\sigma}^G\) is then defined by

$$d\delta_{\xi,\sigma}(x) := G_{\xi,\sigma}(x) \, d\lambda^n(x) \quad \text{with} \quad G_{\xi,\sigma}(x) := \frac{\exp(-\frac{1}{2}(x-\xi)^T \sigma^{-2} (x-\xi))}{\sqrt{(2\pi)^n \det(\sigma)^2}}$$

and \(\lambda^n\) is the \(n\)-dimensional Lebesgue measure. \(A = \{a_1, \ldots, a_m\} \subset C(\mathbb{R}^n, \mathbb{R})\) is a linearly independent set of continuous functions s.t. (b) holds. By continuity of the
a, \( \delta \)'s (b) holds. Then (c) holds, i.e., the Dirac delta measure \( \delta_\xi \) is approximated by \( \delta^G_{\xi,\sigma} \) if \( \sigma \to \sigma_0 = 0 \).

Example 2. Similarly, for the log-normal distribution (2) we have \( X = R^n \) (or \( X = (0, \infty)^n \) with \( n \in \mathbb{N} \)), again \( \Sigma \subset \mathbb{R}^{n \times n} \) the set of all symmetric non-singular matrices, \( \sigma_0 = 0 \in \mathbb{R}^{n \times n} \) the zero matrix. We define the log-normal measure \( \delta_{L,\xi,\sigma} \) by

\[
d\delta_{L,\xi,\sigma}(x) := \begin{cases} \frac{\exp\left(-\frac{1}{2}(\log x - \log \xi)^T \sigma^{-2}(\log x - \log \xi)\right)}{\sqrt{(2\pi)^n \det(\sigma)^n} \prod_{i=1}^n x_i} & \text{for } x_1, \ldots, x_n > 0, \\ 0 & \text{else} \end{cases}
\]

where \( \log x := (\log x_i)_{i=1}^n \) and \( \lambda^n \) is again the n-dimensional Lebesgue measure.

For the Gaussian (1) and the log-normal distribution (2) all moment are known and finite (n = 1):

\[
\int_{R^n} (x - \xi)^i \, d\delta^G_{\xi,\sigma}(x) = \begin{cases} (i - 1)!! \cdot \sigma^i & \text{for } 2|i \\ 0 & \text{else} \end{cases}
\]

and

\[
\int_0^\infty x^i \, d\delta_{L,\xi,\sigma}(x) = \xi^i \cdot e^{\frac{1}{2}x^2}.
\]

For \( n > 1 \) similar formulas hold by diagonalizing \( \sigma \).

We investigate mixtures of distributions

\[
\sum_{i=1}^k \delta_{\xi,\sigma, i}
\]

with the moment method. In previous works and applications the number \( k \) of components is fixed and justified by the model or the data and one of the main questions is the identifiability (uniqueness/determinacy) of (9), see e.g. [Pea94], [BS73], [LSM85], [MMR05], [PFJ00], [Sto16], [AFS16], [ABB+17], [ARS17], and references therein. But in the present paper we want to investigate the moment cone (Section 3), the possible \( \delta_{\xi,\sigma} \) appearing in a representation (9) (Section 4), and the number \( k \) of components needed to represent a given finite number of moments (Section 5).

2. Preliminaries

The theory and application of moments is rich, see e.g. [KS53], [Ric57], [Rog58], [AK62], [Akh65], [Kem68], [KN77], [Sch91], [Mat92], [Rez92], [CF96a], [CF96b], [Sim98], [CF00], [Sch03], [FP05], [CF05], [PS08], [Mar08], [Lau09], [FN10], [CF13], [Lau13], [Sch14], [Sto15], [Fia17], [IKLS17], [Sto17], [Sch17], [RS18], [dDS18a], [dDS18b], and references therein. But in the present section we only present definitions and results needed in the following sections, especially from [dDS18a] and [dDS18b] with extensions to mixtures as presented in the introduction.

To efficiently deal with (linear combinations of) Dirac measures \( \delta_\xi \) and probability measures \( \delta_{\xi,\sigma} \) we introduce the following:
Definition 3. The moment map $s_{A}$ is defined by

$$s_{A} : X \rightarrow \mathbb{R}^{m}, \ x \mapsto s_{A}(x) := \begin{pmatrix} a_{1}(x) \\ \vdots \\ a_{m}(x) \end{pmatrix}$$

and for $k \in \mathbb{N}$ the moment map is defined by

$$S_{k,A} : R_{\geq 0}^{k} \times X^{k} \rightarrow \mathbb{R}^{m}, \ (C, X) \mapsto S_{k,A}(C, X) := \sum_{i=1}^{k} c_{i} \cdot s_{A}(x_{i})$$

where $C = (c_{1}, \ldots, c_{k})$ and $X = (x_{1}, \ldots, x_{k})$.

We denote by $M_{\mathbb{A}}$ the set of all (positive) measures $\mu$ on $X$ s.t. $|\int_{X} a_{i}(x) \ d\mu(x)| < \infty$ for all $i = 1, \ldots, m$.

Clearly, $s_{A}(x)$ is the moment sequence of the Dirac measure $\delta_{x}$ and $S_{k,A}(C, X)$ is the moment sequence of the measure $\mu = \sum_{i=1}^{k} c_{i} \cdot \delta_{x_{i}}$. This and further definitions of course depend on the choice and order of the $a_{i}$'s in $A$. But since the integral is linear in the integrand, reordering or changing the basis $A$ does not affect our results. We also write $\mu \equiv (C, X)$ for a finitely atomic measure and we have $\delta_{x_{i}}, (C, X) \in M_{A}$. To deal with $\delta_{x_{i}}$ we introduce the following.

Definition 4. We define

$$t_{A} : X \times \Sigma \rightarrow \mathbb{R}^{m}, \ (x, \sigma) \mapsto t_{A}(x, \sigma) := \left( \int_{X} a_{i}(y) \ d\delta_{x_{i}}(y) \right)_{i=1}^{m}$$

and

$$T_{k,A} : R_{\geq 0}^{k} \times X^{k} \times \Sigma^{k} \rightarrow \mathbb{R}^{m}, \ (C, X, \sigma) \mapsto T_{k,A}(C, X, \sigma) := \sum_{i=1}^{k} c_{i} \cdot t_{A}(x_{i}, \sigma_{i})$$

where $C = (c_{1}, \ldots, c_{k})$, $X = (x_{1}, \ldots, x_{k})$, and $\sigma = (\sigma_{1}, \ldots, \sigma_{k})$.

Clearly, $t_{A}(x, \sigma)$ is the moment sequence of $\delta_{x, \sigma} \in M_{A}$ and $T_{k,A}(C, X, \sigma)$ is the moment sequence of the mixture $\mu = (C, X, \sigma) = \sum_{i=1}^{k} c_{i} \cdot \delta_{x_{i}, \sigma_{i}} \in M_{A}$. From condition (c) we get

$$\lim_{\sum_{i=1}^{k} c_{i} \rightarrow \sigma_{i}} t_{A}(x, \sigma) = s_{A}(x).$$

Definition 5. We define the moment cone

$$\mathcal{S}_{A} := \left\{ \int_{X} s_{A}(x) \ d\mu(x) \middle| \mu \in M_{A} \right\} \subseteq \mathbb{R}^{m},$$

its boundary points

$$\partial \mathcal{S}_{A} := \partial \mathcal{S}_{A} \cap \mathcal{S}_{A},$$

and the set

$$\mathcal{T}_{A} := T_{m,A}(R_{\geq 0}^{m} \times \Sigma^{m}) = \text{range } T_{m,A}.$$
i.e., for every $\mu \in \mathcal{M}_A$ there is a finitely atomic measure $\mu' = (C, X) = \sum_{i=1}^k c_i \delta_{x_i}$ with the same moment sequence $\int_X a_i(x) \, d\mu(x) = \int_X a_i(x) \, d\mu'(x)$ and $k \leq m$.

By the Richter Theorem (theorem 10) every moment sequence $s \in \mathcal{S}_A$ has a finitely atomic representing measure and we can introduce the following number.

**Definition 7.** Let $s \in \mathcal{S}_A$. We call $C_A(s)$ defined by

$$C_A(s) := \min \{ k \in \mathbb{N} | s \in \text{range} \, S_{k,A} \}$$

the Carathéodory number of $s$. The Carathéodory number $C_A$ is

$$C_A := \max_{s \in \mathcal{S}_A} C_A(s).$$

For the special case of univariate polynomials Richter also proved the following famous result.

**Theorem 8** (H. Richter 1957 [Ric57, Satz 11]). Let $A = \{0, \ldots, x^d\}$ on an open, half-open, or closed interval of $\mathbb{R}$ (or $\mathcal{X} = \mathbb{R}$). Then

$$C_A = \left\lfloor \frac{d+1}{2} \right\rfloor = \left\lfloor \frac{d}{2} \right\rfloor.$$

In [dDS18a] we introduced the following important number.

**Definition 9.** Let $A = \{a_1, \ldots, a_m\} \subset C^1(U, \mathbb{R})$ be a linearly independent subset of $C^1$-functions on an open set $U \subseteq \mathbb{R}^n$. Define

$$N_A := \min \{ k \in \mathbb{N} | DS_{k,A} \text{ has full rank} \}$$

(11)

where $DS_{k,A}$ denotes the total derivative

$$DS_{k,A} = (\partial_{a_1} S_{k,A}, \partial_{x_1,a_1} S_{k,A}, \ldots, \partial_{x_1,a_1} S_{k,A}, \partial_{a_2} S_{k,A}, \ldots, \partial_{x_k,a_1} S_{k,A})$$

(12)

and the set of moment sequences $s$ with $C_A(s) < N_A$ has $m$-dimensional Lebesgue measure zero in $\mathbb{R}^m$.

**Remark 11.** Instead of $\mathcal{X}$ being an open subset of $\mathbb{R}^n$, we could extend the definition 10 and theorem 11 to (differentiable) manifolds $\mathcal{X}$. By choosing a chart $\varphi : U \subseteq \mathbb{R}^n \to \mathcal{X}$ of the manifold, $U$ open, we have again the previous definition and theorem for $A \circ \varphi = \{a_i \circ \varphi | i = 1, \ldots, m\}$. It therefore suffices to treat $\mathcal{X} \subseteq \mathbb{R}^n$ open or $\mathcal{X} = \mathbb{R}^n$.

For upper bounds we proved an $(m-1)$-Theorem, which we will tighten here.

**Theorem 10** (An extension of [dDS18a, Thm. 13]). Let $A$ and $\mathcal{X}$ s.t. there exists an $e \in \text{lin} \, A$ with $e(x) > 0$ for all $x \in \mathcal{X}$ and range $s_A \cdot \|s_A\|^{-1}$ consists of not more than $m-1$ path-connected components. Then

$$C_A \leq m - 1.$$

**Proof.** The proof is verbatim the same as in [dDS18a, Thm. 13]. \qed
In [118] we missed that we actually only need the assumptions in theorem [12]. We previously stated that \( A \) must be continuous, there is an \( e \in \text{lin} \ A \) s.t. \( e > 0 \) on \( X \) and \( X \) has not more than \( m - 1 \) components. This of course implies the assumptions in theorem [12]. The key step in the proof was that for any moment sequence \( s \) we find by Richter’s Theorem (theorem 6) a simplicial cone spanned by \( s_A(x_1), \ldots, s_A(x_m) \) containing \( s \). Then two \( s_A(x_i) \) and \( s_A(x_j) \) lie in the same component of range \( s_A \cdot \|s_A\|^{-1} \) and can therefore be connected by a path. Following this path shrinks the simplicial cone until \( s \) is contained in its boundary, i.e., \( s \) needs only \( m - 1 \) atoms.

3. The Moment Cones \( S_A \) and \( T_A \)

In definition 5 we defined the moment cones \( S_A \) and \( T_A \) and we already found \( T_A \subseteq S_A \) and \( S_A \) is a convex cone.

In the following we will “only” deal with moment sequences where we know that they have a representing mixture with finitely many components. That is the definition of \( T_A \) in definition 5. However, an application of the Richter Theorem (theorem 6) shows that this is enough.

Definition 13. Set \( \mathcal{B} := \{b_1, \ldots, b_m\} \) where \( b_i \) is a function on \( X \times \Sigma \) defined by
\[
(14) \quad b_i(x, \sigma) := \int_{X} a_i(y) \, d\delta_{x,\sigma} \quad \forall (x, \sigma) \in X \times \Sigma.
\]

Example 14 (Gaussian Distribution, example 1 revisited). From (9) and (7) we find for \( a_i(x) = x^\alpha \) that
\[
b_i(\xi, \sigma) := \int_{\mathbb{R}^n} x^\alpha \, d\delta_{\xi,\sigma}^L(x)
\]
with \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \), \( \alpha_i \in \mathbb{N}_0 \), is a polynomial in \( x_1, \ldots, x_n \) and \( \sigma_{i,j} \) of degree \( |\alpha| = \alpha_1 + \cdots + \alpha_n \).

Example 15 (Log-Normal Distribution, 2 revisited). From (5) we find for \( a_i(x) = x^i \) on \( (0, \infty) \) that
\[
b_i(\xi, \sigma) := \int_0^\infty x^i \, d\delta_{\xi,\sigma}^L(x) = \xi^i \cdot e^{\frac{\sigma^2}{2}}.
\]

\( \mathcal{B} \) is well-defined by condition (b). Then any finite or infinite sums of components or continuous versions are measures on \( X \times \Sigma \). The Richter Theorem for mixtures of distributions reads as follows.

Theorem 16. Let \( X \) and \( \Sigma \) be topological spaces, \( \mathcal{B} = \{b_1, \ldots, b_m\} \) be a finite set of functions on \( X \times \Sigma \), i.e., \( \delta_{x,\sigma} \in \mathcal{M}_{\mathcal{B}} \) for all \( (x, \sigma) \in X \times \Sigma \). Then for every \( \mu \in \mathcal{M}_{\mathcal{B}} \) there is a mixture with finitely many components \( \mu^* = (C, X, \bar{\sigma}) = \sum_{i=1}^k c_i \cdot \delta_{x_i,\sigma_i} \) with the same moment sequence and \( k \leq m \), i.e.,
\[
\left( \int_{X \times \Sigma} b_i(x, \sigma) \, d\mu(x, \sigma) \right)_{i=1}^m \in T_A = \text{range} \, T_{m,A} = T_{m,A}(\mathbb{R}^m_{\geq 0} \times X^m \times \Sigma^m).
\]

Proof. Apply the Richter Theorem (theorem 6) to \( \mathcal{B} = \{b_1, \ldots, b_m\} \) on \( X \times \Sigma \).

So it is sufficient to deal “only” with moment sequences coming from finite mixtures.

Theorem 17. Let \( A = \{a_1, \ldots, a_m\} \) be linearly independent continuous functions on \( X \). Then
i) \( T_A \) is a full-dimensional convex cone.
ii) \( \text{int} \, T_A = \text{int} \, S_A \).
iii) Assume that
1) \( \mathcal{X} \) is a locally compact Hausdorff space.
2) for every \( x \in \mathcal{X} \) and \( \sigma \in \Sigma \) there is a compact neighborhood \( U_{x,\sigma} \subseteq \text{supp} \delta_{x,\sigma} \) with \( \delta_{x,\sigma}(U_{x,\sigma}) > 0 \), and
3) for every \( f \in \text{lin} \mathcal{A} \) with \( f \geq 0 \) on \( \mathcal{X} \) and \( f|_U = 0 \) for a neighborhood \( U \) implies \( f = 0 \).

Then \( \mathcal{T}_A = \text{int} \mathcal{S}_A \cup \{0\} \).

Proof. i): That \( \mathcal{T}_A \) is a cone is clear. That \( \mathcal{T}_A \) is convex follows from the Carathéodory Theorem for cones, see e.g. [Rock72 Cor. 17.1.2]. To show that \( \mathcal{T}_A \) is full-dimensional, we take \( x_1, \ldots, x_m \in \mathcal{X} \) s.t. \( \sigma_A(x_1), \ldots, \sigma_A(x_m) \) are linearly independent (such \( x_i \)'s exist since \( A = \{a_1, \ldots, a_m\} \) is linearly independent). Let \( (\sigma_i)_{i \in \mathbb{N}} \subseteq \Sigma \) s.t. \( \sigma_i \to \sigma_0 \) as \( i \to \infty \). Then

\[
\lim_{i \to \infty} \text{det}(t_A(x_1, \sigma_i), \ldots, t_A(x_m, \sigma_i)) = \text{det}(s_A(x_1), \ldots, s_A(x_m)) \neq 0
\]

by condition (c) and continuity of the determinant, i.e., there is an \( N \in \mathbb{N} \) s.t. \( \text{det}(t_A(x_1, \sigma_N), \ldots, t_A(x_m, \sigma_N)) \neq 0 \) and therefore \( t_A(x_1, \sigma_N), \ldots, t_A(x_m, \sigma_N) \) are linearly independent in \( \mathbb{R}^m \) and \( \mathcal{T}_A \) is full-dimensional.

ii): From \( \mathcal{T}_A \subseteq \mathcal{S}_A \) we get \( \text{int} \mathcal{T}_A \subseteq \text{int} \mathcal{S}_A \). So we have to prove the reverse inclusion \( \text{int} \mathcal{T}_A \supseteq \text{int} \mathcal{S}_A \). Let \( s \in \text{int} \mathcal{S}_A \). Then there are \( x_1, \ldots, x_m \in \mathcal{X} \) and \( c_i > 0 \) s.t. \( s = \sum_{i=1}^m c_i \sigma_A(x_i) \). So \( s \in \text{int} \text{cone}(\sigma_A(x_1), \ldots, \sigma_A(x_m)) \). By (c) there exists \( \sigma \in \Sigma \) s.t. \( s \in \text{int} \text{cone}(t_A(x_1, \sigma), \ldots, t_A(x_m, \sigma)) \subseteq \text{int} \mathcal{T}_A \).

iii): Since \( 0 \in \mathcal{T}_A \) and \( \text{int} \mathcal{S}_A = \text{int} \mathcal{T}_A \) by ii) we have \( \text{int} \mathcal{S}_A \cup \{0\} \subseteq \mathcal{T}_A \). So it is sufficient to prove the reverse inclusion \( \mathcal{T}_A \subseteq \text{int} \mathcal{S}_A \cup \{0\} \).

Assume this inclusion does not hold, i.e., \( \mathcal{T}_A \cap \partial^* \mathcal{S}_A \neq \emptyset \) since \( \mathcal{T}_A \subseteq \mathcal{S}_A \). Let \( s \in \mathcal{T}_A \cap \partial^* \mathcal{S}_A \), \( s \neq 0 \), then \( \mu = \sum_{i=1}^k c_i t_A(x_i, \sigma_i) \) is a non-trivial representing measure of \( s \) (since \( s \in \mathcal{T}_A \)) and there is a \( f \geq 0 \) in \( \text{lin} \mathcal{A} \setminus \{0\} \) s.t. \( \int_X f(x) \, d\mu(x) = 0 \) (since \( s \in \partial^* \mathcal{S}_A \); \( f \) is a separating hyperplane supporting \( \mathcal{S}_A \) at \( s \)). Let \( U_{x_1, \sigma_1} \subseteq \text{supp} \delta_{x_1, \sigma_1} \) be a compact neighborhood, then by continuity of \( f \) and 3) we have \( c := \max_{x \in U_{x_1, \sigma_1}} f(x) \in (0, \infty) \). Therefore, \( U := U_{x_1, \sigma_1} \cap f^{-1}((c/2, 2c)) \) is open in \( U_{x_1, \sigma_1} \) by continuity of \( f \), i.e., \( \delta_{x_1, \sigma_1}(U) > 0 \). Then

\[
0 = \int_X f(x) \, d\mu(x) \geq \int_{U_{x_1, \sigma_1}} f(x) \, d\sigma_{x_1, \sigma_1} \geq \frac{c}{2} \delta_{x_1, \sigma_1}(U) > 0.
\]

This is a contradiction, i.e., \( \mathcal{T}_A \cap \partial^* \mathcal{S}_A = \{0\} \) and therefore \( \mathcal{T}_A \subseteq \text{int} \mathcal{S}_A \cup \{0\} \). \( \square \)

In iii) in the previous theorem we actually proved the following. It is a reformulation of Lemma 3 in [DPS13].

**Lemma 18.** Assume
1) \( \mathcal{X} \) is a locally compact Hausdorff space,
2) for every \( x \in \mathcal{X} \) and \( \sigma \in \Sigma \) there is a compact neighborhood \( U_{x,\sigma} \subseteq \text{supp} \delta_{x,\sigma} \) with \( \delta_{x,\sigma}(U_{x,\sigma}) > 0 \), and
3) for every \( f \in \text{lin} \mathcal{A} \) with \( f \geq 0 \) on \( \mathcal{X} \) and \( f|_U = 0 \) for a neighborhood \( U \) implies \( f = 0 \).

Then \( \mu \) is a representing measure of \( s \in \mathcal{S}_A \) with \( \text{supp} \mu \neq \emptyset \) \( \Rightarrow \) \( s \in \text{int} \mathcal{S}_A \).

**Example 19** (Gaussian Mixtures, example [P] revisited). For the Gaussian mixtures we have \( \mathcal{X} = \mathbb{R}^n \) (a locally compact Hausdorff space), \( \text{supp} \delta_{x,\sigma} = \mathcal{X} = \mathbb{R}^n \) for all \( x \in \mathcal{X} = \mathbb{R}^n \) and \( \sigma \in \Sigma \subseteq \mathbb{R}^{n \times n} \), the set of all symmetric non-singular matrices. Let \( A \) be a linearly independent set of holomorphic functions (e.g., poly/monomials). Lemma [P] applies and every moment sequence \( s \) is an inner point.
of the moment cone $S\Lambda$, i.e., the set of non-zero moment sequences from Gaussian mixtures is open.

**Example 20** (Log-Normal Mixtures, example revisited). For the Gaussian mixtures we have $X = (0, \infty)^n$ (a locally compact Hausdorff space), supp $\delta_{x,\sigma} = X = (0, \infty)^n$ for all $x \in X = (0, \infty)^n$ and $\sigma \in \Sigma \subseteq \mathbb{R}^{n \times n}$, the set of all symmetric non-singular matrices. Let $A$ be a linearly independent set of holomorphic functions (e.g., poly-/monomials). lemma 18 applies and every moment sequence $s$ is an inner point of the moment cone $S\Lambda$, i.e., the set of non-zero moment sequences from log-normal mixtures is open.

Of course, $A$ being holomorphic can be weakened to condition 3) in lemma 18.

In theorem 17 ii) we actually showed that any $s \in \text{int } T\Lambda$ has a mixture representation with (at most) $m$ components and all components have the same $\sigma$. In the following theorem we will show that we can represent large parts of $T\Lambda$ by mixture representations with (at most) $m$ components and all components have the same $\sigma$. $\sigma$ must “just” be close enough to $\sigma_0$.

**Definition 21.** Set

$$T\Lambda,\sigma := T_{m,A}(\mathbb{R}_{\geq 0}^m \times X^m \times \{(\sigma, \ldots, \sigma)\}).$$

So $T\Lambda,\sigma$ is the (convex) set of all moment sequences $s$ s.t. every $s$ possesses a mixture representation $\sum_{i=1}^k c_i \delta_{x_i,\sigma}$ ($k \leq m$) with at most $m$ components and all components have the same $\sigma$.

**Theorem 22.** i) $T\Lambda,\sigma$ is a convex cone for all $\sigma \in \Sigma$.

ii) Let $(\sigma_i)_{i \in \mathbb{N}} \subseteq \Sigma$ s.t. $\sigma_i \to \sigma_0$ as $i \to \infty$. Then

$$\text{int } T\Lambda \cup \{0\} \subseteq \bigcup_{i \in \mathbb{N}} T\Lambda,\sigma_i \subseteq \bigcup_{\sigma \in \Sigma} T\Lambda,\sigma.$$

iii) Let $s_1, \ldots, s_k \in \text{int } T\Lambda$ be points and $(\sigma_i)_{i \in \mathbb{N}} \subseteq \Sigma$ s.t. $\sigma_i \to \sigma_0$ as $i \to \infty$. Then there exists an $N \in \mathbb{N}$ s.t.

$$s_1, \ldots, s_k \in \text{int } T\Lambda,\sigma_i \quad \forall i \geq N.$$

iv) Let $K \subset \text{int } T\Lambda$ be compact and $(\sigma_i)_{i \in \mathbb{N}} \subseteq \Sigma$ s.t. $\sigma_i \to \sigma_0$ as $i \to \infty$. Then there exists an $N \in \mathbb{N}$ s.t.

$$K \subset \text{int } T\Lambda,\sigma_i \quad \forall i \geq N.$$

v) Let $C \subset \text{int } T\Lambda$ be a closed cone and $(\sigma_i)_{i \in \mathbb{N}} \subseteq \Sigma$ s.t. $\sigma_i \to \sigma_0$ as $i \to \infty$. Then there exists an $N \in \mathbb{N}$ s.t.

$$C \subset \text{int } T\Lambda,\sigma_i \quad \forall i \geq N.$$

vi) Assume conditions 1), 2), and 3) from theorem 17 iii) hold and let $(\sigma_i)_{i \in \mathbb{N}} \subseteq \Sigma$ s.t. $\sigma_i \to \sigma_0$ as $i \to \infty$. Then

$$T\Lambda = \bigcup_{i \in \mathbb{N}} T\Lambda,\sigma_i = \bigcup_{\sigma \in \Sigma} T\Lambda,\sigma.$$

**Proof.** i): That $T\Lambda$ is a cone is clear. That $T\Lambda$ is convex follows from the Carathéodory Theorem for cones, see e.g. [Roc72, Cor. 17.1.2].

ii): The proof follows the proof of theorem 17 ii). Of course, $0 \in T\Lambda,\sigma$ for all $\sigma \in \Sigma$ and the second inclusion holds. So let $s \in \text{int } T\Lambda$. Then $s \in \text{int } S\Lambda$ by theorem 17 ii) and there are $x_1, \ldots, x_m \in X$ and $c_i > 0$ s.t. $s = \sum_{i=1}^m c_i \delta_{x_i}$, so $s \in \text{int } \text{cone}(s\Lambda(x_1), \ldots, s\Lambda(x_m))$. By (c) there exists $\sigma_1 \in \Sigma$ s.t.

$$s \in \text{int } \text{cone}(t\Lambda(x_1, \sigma), \ldots, t\Lambda(x_m, \sigma)) \subseteq \text{int } T\Lambda.$$
iii): As in ii) let \((\sigma_i)_{i\in \mathbb{N}} \subseteq \Sigma\) s.t. \(\sigma_i \to \sigma_0\) as \(i \to \infty\). By ii) for \(s_i\) there is an \(N_i \in \mathbb{N}\) s.t. \(s_i \in \text{int} \mathcal{T}_{\mathcal{A},\sigma_i}\) for all \(l \geq N_i\). Set \(N := \max\{l_1, \ldots, l_k\}\). Then \(s_1, \ldots, s_k \in \text{int} \mathcal{T}_{\mathcal{A},\sigma_i}\) for all \(i \geq N\).

iv): \(\text{conv} K\) is compact since \(K\) is compact and \(\text{conv} K \subseteq \text{conv} \mathcal{T}_\mathcal{A} = \text{int} \mathcal{T}_\mathcal{A}\) since \(\text{int} \mathcal{T}_\mathcal{A}\) is convex. Therefore, \(\text{dist}(\partial \mathcal{T}_\mathcal{A}, \text{conv} K) > 0\) and there are \(s_1, \ldots, s_k \in \text{int} \mathcal{T}_\mathcal{A}\) s.t. \(\text{conv} K \subseteq \text{conv} \{s_1, \ldots, s_k\}\). By iii) there is an \(N \in \mathbb{N}\) s.t. \(s_1, \ldots, s_k \in \text{int} \mathcal{T}_{\mathcal{A},\sigma_i}\) for all \(i \geq N\). Since all \(\text{int} \mathcal{T}_{\mathcal{A},\sigma_i}\) are convex, we have conv \(\{s_1, \ldots, s_k\} \subseteq \text{int} \mathcal{T}_{\mathcal{A},\sigma_i}\) for all \(i \geq N\). In conclusion we have

\[
K \subseteq \text{conv} K \subseteq \text{conv} \{s_1, \ldots, s_k\} \subseteq \text{int} \mathcal{T}_{\mathcal{A},\sigma_i} \quad \forall i \geq N.
\]

v): Let \(S^m\) be the unit sphere in \(\mathbb{R}^m\). Then \(K = C \cap S^m\) is closed and bounded (i.e., compact by the Heine–Borel Theorem) and generates \(C\) (i.e., cone \(K = C\)). By iv) there is an \(N \in \mathbb{N}\) s.t. \(K \subseteq \text{int} \mathcal{T}_{\mathcal{A},\sigma_i}\) for all \(i \geq N\). Since \(\mathcal{T}_{\mathcal{A},\sigma_i}\) are (convex) cones by i) we have the \(C = \text{cone} K \subseteq \text{cone} \text{int} \mathcal{T}_{\mathcal{A},\sigma_i} = \text{int} \mathcal{T}_{\mathcal{A},\sigma_i}\) for all \(i \geq N\).

vi): From theorem \(17\) ii) and iii) we have \(\mathcal{T}_\mathcal{A} = \text{int} S_\mathcal{A} \cup \{0\} = \text{int} \mathcal{T}_\mathcal{A} \cup \{0\}\). Then with ii) in this theorem we have

\[
\mathcal{T}_\mathcal{A} = \text{int} \mathcal{T}_\mathcal{A} \cup \{0\} \subseteq \bigcup_{i \in \mathbb{N}} \mathcal{T}_{\mathcal{A},\sigma_i} \subseteq \bigcup_{\sigma \in \Sigma} \mathcal{T}_{\mathcal{A},\sigma} \subseteq \mathcal{T}_\mathcal{A}. \quad \square
\]

4. Set of Atoms and Identifiability/Uniqueness/Determinacy

We have seen that any \(s \in \mathcal{T}_\mathcal{A}\) has a finite mixture representation (by definition) and we want to know the possible positions \((x, \sigma)\) s.t. \(\delta_{x,\sigma}\) appears in any such representation.

Definition 23. Let \(s \in \mathcal{T}_\mathcal{A}\). The set of atoms (components) \(\mathcal{W}(s)\) is defined by

\[
\mathcal{W}(s) := \left\{ (x, \sigma) \mid \exists c_i, c > 0, (x_i, \sigma_i) \in \mathcal{X} \times \Sigma : s = c \cdot t_{\mathcal{A}}(x, \sigma) + \sum_{i=1}^k c_i t_{\mathcal{A}}(x_i, \sigma_i) \right\}.
\]

So \(\mathcal{W}(s)\) is the set of all \((x, \sigma)\) in \(\mathcal{X} \times \Sigma\) s.t. \(\delta_{x,\sigma}\) appears in a mixture representation of \(s\).

Definition 24. Let \(s \in \mathcal{T}_\mathcal{A}\). \(s\) is called determined if it has only one mixture representation. Otherwise \(s\) is called indeterminate.

The following theorem summarizes properties of \(\mathcal{W}(s)\) and determinacy. It is a reformulation of Theorems 16 and 19 in [dDS18a].

Theorem 25. Let \(s \in \mathcal{T}_\mathcal{A}\).

i) \(s \in \text{int} \mathcal{T}_\mathcal{A} \Leftrightarrow \mathcal{W}(s) = \mathcal{X} \times \Sigma\).

ii) \(s\) is indeterminate \(\Leftrightarrow \{t_{\mathcal{A}}(x, \sigma) \mid (x, \sigma) \in \mathcal{W}(s)\}\) is linearly dependent.

iii) Assume that

1) \(\mathcal{X}\) is a locally compact Hausdorff space,
2) for every \(x \in \mathcal{X}\) and \(\sigma \in \Sigma\) there is a compact neighborhood \(U_{x,\sigma} \subseteq \text{supp} \delta_{x,\sigma}\) with \(\delta_{x,\sigma}(U_{x,\sigma}) > 0\), and
3) for every \(f \in \text{lin} \mathcal{A}\) with \(f \geq 0\) on \(\mathcal{X}\) and \(f|_{U} = 0\) for a neighborhood \(U\) implies \(f = 0\).

Then every \(s \in \mathcal{T}_\mathcal{A}\) is indeterminate and \(\mathcal{W}(s) = \mathcal{X} \times \Sigma\) for all \(s \in \mathcal{T}_\mathcal{A}\) \(\setminus \{0\}\).

Proof. i) “⇒”: Let \((x, \sigma) \in \mathcal{X} \times \Sigma\). Since \(s \in \text{int} \mathcal{T}_\mathcal{A}\) there is an \(\epsilon > 0\) s.t. \(s' := s - \epsilon \cdot t_{\mathcal{A}}(x, \sigma) \in \text{int} \mathcal{T}_\mathcal{A}\). Then \(s'\) has a finite mixture representation \(\mu'\) and \(\mu := \mu' + \epsilon \cdot \delta_{x,\sigma}\) is a mixture presentation of \(s\) containing \(\delta_{x,\sigma}\).

---

1Every closed convex set is the intersection of all halfspaces containing it, see e.g. [Rog08 Thm. 11.5]. So taking only finitely many halfspaces gives a polytope \(P\) with \(\text{conv} K \subseteq P\) and \(\sup_{y \in P} \inf_{x \in \text{conv} K} \|x - y\| \leq \frac{\text{dist}(\partial \mathcal{T}_\mathcal{A}, \text{conv} K)}{2} > 0\). Then take \(s_1, \ldots, s_k\) as the vertices of \(P\).
i) “ε”-Let \((x_i, \sigma_i) \in W(s) = \mathcal{X} \times \Sigma (i = 1, \ldots, m)\) s.t. \((t_A(x_1, \sigma_1), \ldots, t_A(x_m, \sigma_m))\) has full rank. Let \(\mu_i\) be representing mixtures of \(s\) s.t. every \(\mu_i\) contains the component \(\delta_{x_i, \sigma_i}\). Then

\[
\mu := \frac{1}{m} \sum_{i=1}^{m} \mu_i = \sum_{i=1}^{m} c_i \delta_{x_i, \sigma_i} + \sum_{j=1}^{k} d_j \delta_{y_j, \sigma_j}'
\]
is a representing mixture of \(s\) which contains all \(\delta_{x_i, \sigma_i}\). Then the map

\[
S(\gamma_1, \ldots, \gamma_m) := \sum_{i=1}^{m} \gamma_i t_A(x_i, \sigma_i) + \sum_{j=1}^{k} d_j t_A(y_j, \sigma_j')
\]
maps a neighborhood \(B_\varepsilon((c_1, \ldots, c_m)) \subset (0, \infty)^n\) with \(0 < \varepsilon < \min\{c_1, \ldots, c_m\}\) to a neighborhood of \(s\) since the \(t_A(x_i, \sigma_i)\) are linearly independent, i.e., \(s \in \text{int}\, T_A\).

ii) Apply (i) \(\Leftrightarrow\) (ii) in [dDS18a, Thm. 19].

iii): By theorem 25 iii) \(T_A \setminus \{0\} = \text{int}\, S_A\) is open and i) and ii) apply. \(\square\)

**Example 26** (Examples 1 and 2 revisited). For the Gaussian and log-normal distributions the conditions 1), 2), and 3) are fulfilled, i.e., every moment sequence is indeterminate and any component \(\delta_{x, \sigma}^G\) or \(\delta_{x, \sigma}^L\), respectively, can appear in a representing mixture.

That for any \(s \in \text{int}\, T_A\) any \(\delta_{x, \sigma}\) appears in a representing mixture is only possible since the number of components is not restricted. So we need to learn more about the number of components.

5. THE CARATHÉODORY NUMBER \(C^M_A\) FOR MIXTURES OF DISTRIBUTIONS

Since (by definition or theorem 16) every \(s \in T_A\) has a mixture representation we can define the Carathéodory number of mixtures similar to definition 4.

**Definition 27.** Let \(s \in T_A\). Define the Carathéodory number \(C^M_A(s)\) of mixtures of \(s\) by

\[
C^M_A(s) := \min \{ k \in \mathbb{N} \mid s \text{ has a mixture representation with } k \text{ components} \}.
\]

The Carathéodory number \(C^M_A\) of mixtures is defined by

\[
C^M_A := \max_{s \in T_A} C^M_A(s).
\]

\(C^M_A(s)\) and \(C^M_A\) are well-defined by theorem 3 or equivalently theorem 16 since \(0 \leq C_A(s) \leq C^M_A \leq m\) and \(C_A(s), C_A \in \mathbb{N}_0\). We will shortly see in example 51 that not necessarily \(C^M_A \leq C_A\) even though \(T_A \subseteq S_A = \text{range } S_{c,A}\) holds.

In important cases, e.g., Gaussian and log-normal mixtures (Examples 1 and 2), the moment cone has no boundary points despite 0. So “standard” methods to bound \(C^M_A\) in [dDS18a] and [RS18] can not be applied. These “standard” methods are, e.g., “taking an inner point, removing an atom to get to the boundary and describe the boundary” or “close the moment cone by going from \(\mathbb{R}^n\) to projective space \(\mathbb{P}^n\) and to homogeneous polynomials”. In all these cases, a boundary point \(s \neq 0\) would imply that there is an \(f \in \text{lin}\, A\), \(f \geq 0\), such that \(\text{supp } \mu \subseteq W(s) \subseteq Z(f) := \{ x \in \mathcal{X} \mid f(x) = 0 \} \neq \mathcal{X}\) for all representing measures \(\mu\) of \(s\). But this is not possible as long as conditions 1), 2), and 3) shall hold and int \(\text{supp } \delta_{x, \sigma} \neq \emptyset\), see theorem 25 iii). So recent methods in [dDS18a] and [RS18] do not apply. Theorem 22 fills the gap. But let us start with the lower bounds on \(C^M_A\).

**Definition 28.** Let \(\mathcal{X} \subseteq \mathbb{R}^{n_1}\) and \(\Sigma \subseteq \mathbb{R}^{n_2}\) be open. Furthermore, let \(b_i\) from definition 23 be \(C^1\)-functions. We define

\[
N^M_A := \min \{ k \in \mathbb{N} \mid DT_{k,A} \text{ has full rank} \}.
\]
DT_{k,A} is the total derivative of T_{k,A}.

**Example 29.** For the Gaussian distribution we have $X = \mathbb{R}^n$ and for the log-normal distribution we have $X = (0, \infty)^n$, see Examples 1 and 3. In both cases $\Sigma$ is the set of all symmetric non-singular matrices in $\mathbb{R}^{n \times n}$, e.g., $\Sigma$ is a open $\frac{n(n+1)}{2}$-dimensional smooth manifold, i.e., remark 11 applies.

We have the following lower bound on $C^M_A$.

**Theorem 30.** Let $X \subseteq \mathbb{R}^{n_1}$ and $\Sigma \subseteq \mathbb{R}^{n_2}$ be open. Furthermore, let $b_i$ from definition (1) be $C^r$-functions with $r > \mathcal{N}_A(n_1 + n_2 + 1) - m$. Then

$$\left| \frac{m}{n_1 + n_2} \right| \leq \mathcal{N}_A^M \leq C^M_A.$$

**Proof.** Apply theorem 10 [dDS18b, Thm. 27] with $X \times \Sigma \subseteq \mathbb{R}^{n_1+n_2}$. □

See also remark 11 for extensions of $X$ and $\Sigma$ to differentiable manifolds. The previous theorem then implies that there are cases where $C^M_A \not\subseteq C_A$.

**Example 31 (C^M \not\subseteq C_A, see [dDS18b] Exm. 16 and Rem. 17)).** Let $\varphi = (\varphi_i)^{m}_{i=1}$ be the coordinate functions of a space filling curve $\sigma_{\varphi}$, see [dDS18b] Thm. 27], i.e., $\sigma : [0, 1]^m \to [0, 1]^m$ are continuous. Extend all $\varphi_i$ continuously to $\mathbb{R}$ s.t. $\text{supp} \varphi_i \subseteq [-1, 2]$. For the Gaussian distributions (example 7) we can then interchange differentiation and integration in (1) by applying a result of Lebesgue (see e.g. [Gru00] Lem. 2.8]) and we get that all $b_i$ are $C^\infty$. Therefore theorem 30 holds and we get with example 29 that

$$\left| \frac{2m}{(n+2)(n+1)} \right| \leq C^M_A,$$

and for $2m > (n+2)(n+1)$ we have $C_A = 1 < C^M_A$.

If we are interested in representations with fixed $\sigma$, then we need at least $\left[ \frac{m}{n_1 + 1} \right]$ distributions $\delta_{x,\sigma}$. And when we want a presentation s.t. all $\sigma_1 = \cdots = \sigma_k = \sigma$ are the same but we are allowed to chose $\sigma$ freely, then we need at least $\left[ \frac{m-n}{n_1 + 1} \right]$ distributions. Apply theorem 10 or modify the proof in [dDS18b] Thm. 27] to prove these.

Let us now treat the upper bound estimates. We already established $C^M_A \leq m$ in theorem 10. We can tighten this.

**Theorem 32.** Let $A, X, \Sigma, (\sigma_i)_{i \in \mathbb{N}} \subseteq \Sigma$, and $\delta_{x,\sigma}$ s.t. $\sigma_i \to \sigma_0$, there exists an $e \in \text{lin} \{b_1, \ldots, b_m\}$ from definition (1) and $N \in \mathbb{N}$ with $e(x, \sigma_i) > 0$ for all $x \in X$ and $i \geq N$, and range $t_A \cdot \|t_A\|^{-1}$ consists of not more than $m - 1$ path-connected components. Then

$$C^M_A \leq m - 1.$$

**Proof.** Let $s \in T_{A,\sigma}$. Then by theorem 22 ii) there is a $N' \in \mathbb{N}$ s.t. $s \in T_{A,\sigma_i}$ for all $i \geq N'$. Apply theorem 12 to $T_{A,\sigma_i}$ for some $i \geq \max\{N, N'\}$. □

Theorem 22 can be used to bound $C^M_A$.

**Theorem 33.** Let $(\sigma_i)_{i \in \mathbb{N}} \subseteq \Sigma$ s.t. $\sigma_i \to \sigma_0$ as $i \to \infty$ and there exist $C, N \in \mathbb{N}$ s.t. $C(b_1(\cdot, \sigma_i), \ldots, b_m(\cdot, \sigma_i)) \leq C$ for all $i \geq N$. Then

$$C_A^M \leq C.$$

**Proof.** Let $s \in T_{A,\sigma}$. Then by theorem 22 ii) there is a $N' \in \mathbb{N}$ s.t. $s \in T_{A,\sigma_i}$ for all $i \geq \max\{N, N'\}$. I.e., $C_A^M(s) \leq C$ since $C(b_1(\cdot, \sigma_i), \ldots, b_m(\cdot, \sigma_i)) \leq C$ for all $i \geq N$. Since $s$ was arbitrary, we have $C_A^M \leq C$. □
Let us give an application to the most common cases: Gaussian and log-normal distributions (Examples 1 and 2). Let us start with the following remark.

Remark 34. Let

\[ A_{n,d} := \{ x^\alpha \mid \alpha \in \mathbb{N}_0^n \land |\alpha| \leq d \} \]

be the monomials of degree at most \( d \) in \( n \) variables and \((\sigma_i)_{i \in \mathbb{N}} \subset \Sigma \) with \( \sigma_i := i^{-1} \text{id} \), \( \text{id} \) the identity matrix. For the Gaussian distributions \( \delta^G_{x,\sigma} \) we find from (7) that

\[ b_\alpha(x, i^{-1} \text{id}) := \int_{\mathbb{R}^n} y^\alpha \delta^G_{x,i^{-1} \text{id}}(y) \]

is a polynomial in \( x_1, \ldots, x_n \) with leading term \( x^\alpha \). So (15) holds for \( \{ b_\alpha(x, i^{-1} \text{id}) \mid \alpha \in \mathbb{N}_0^n \land |\alpha| \leq d \} = \text{lin} A_{n,d} \).

Since the Carathéodory number does not depend on the choice of basis functions spanning \( \text{lin} \{ b_\alpha(x, i^{-1} \text{id}) \} \) we can apply theorem 33 with results from previous studies of Carathéodory numbers from Dirac measures, see e.g. [dDS18a] and [RS18].

For the log-normal distribution \( \delta^L_{x,\sigma} \) we find the same: (15) holds by (8). But we have \( X = (0, \infty) \).

Let us apply the previous remark.

Theorem 35. Let

\[ A_{n,d} = \{ x^\alpha \mid \alpha \in \mathbb{N}_0^n \land |\alpha| \leq d \} \]

be the monomials of degree at most \( d \) in \( n \) variables. Then for Gaussian and log-normal mixtures we have

\[ C^M_{A_{n,d}} \leq C^A_{n,d} \]

Proof. Follows from (15) and theorem 33 \( \square \)

Let us give some explicit applications of the previous theorem. For the one-dimensional Gaussian mixture we have

Corollary 36. Let \( A = \{ 1, x, \ldots, x^d \} \) on \( \mathbb{R} \), \( d \in \mathbb{N} \). For the Gaussian mixtures we have

\[ \left\lceil \frac{d+1}{3} \right\rceil \leq C^M_A \leq \left\lfloor \frac{d+1}{2} \right\rfloor \]

and every moment sequence \( s \) coming from a linear combination of Gaussian measures can be written as

\[ s = \sum_{i=1}^k c_i s_{A,\sigma}(x_i) \quad \text{with} \quad k \leq \left\lceil \frac{d+1}{2} \right\rceil \quad \text{and some} \quad \sigma = \sigma(s) > 0. \]

Equivalently, every moment sequence \( s \) from a Gaussian mixture has a Gaussian mixture representation

\[ F(x) = \sum_{i=1}^k c_i e^{-\frac{(x-x_i)^2}{2\sigma^2}} \quad \text{with} \quad k \leq \left\lfloor \frac{d+1}{2} \right\rfloor \quad \text{and some} \quad \sigma = \sigma(s) > 0. \]

Proof. \( C^M_A \geq \left\lceil \frac{d+1}{3} \right\rceil \) follows from theorem 30 with \( n = 1 \) and the upper bound follows from theorem 35 with theorem 8 \( \square \)

For the one-dimensional log-normal distribution we will even have a more general result since it only lives on \((0, \infty)\), see theorem 10.

For systems \( A \subset \mathbb{R}[x_1, \ldots, x_n] \) with gaps, the application of previous results is more involved. \( \square \) no longer holds. E.g. for \( A = \{ 1, x^2, x^3, x^0 \} \) on \( \mathbb{R} \) we get

\[ b_0(x, \sigma) = 1, \quad b_2(x, \sigma) = x^2 + \sigma^2, \]
\[ b_3(x, \sigma) = x^3 + 3\sigma^2 x, \]
\[ b_5(x, \sigma) = x^5 + 10\sigma^2 x^3 + 15\sigma^4 x, \]
\[ b_6(x, \sigma) = x^6 + 15\sigma^2 x^4 + 45\sigma^4 x^2 + 15\sigma^6, \]

so
\[ \text{lin } \{ b_i(x, \sigma) \} = \{1, x^2, x^3 + 3\sigma^2 x, x^5 - 15\sigma^4 x^3, x^6 + 15\sigma^2 x^4 \}, \]
i.e., we always have contributions from \( x \) and \( x^2 \).

Systems with gaps, especially the univariate case, were treated in [IDS18]. For \( A = \{1, x^2, x^3, x^5, x^6\} \) on \( \mathbb{R} \) we found that \( C_A = 3 \) [IDS18] Exm. 46. theorem 16 gives \( C_A^M \leq 5 \) while theorem 12 gives a bound of \( C_A^M \leq 4 \). We will show with the following results, at least \( C_A^M(s) \leq 3 \) for almost every \( s \in \mathcal{T}_A \), see example 39. At first we will show that a k-atomic Dirac measure \( (C, X, \sigma) \) s.t. \( DS_{k,A}(C, X) \) has full rank gives an mixture with at most \( k \) components.

**Theorem 37.** Let \( A \in \mathbb{C}^1 \) s.t. \( b_i(x, \sigma \text{id}) \) and \( \partial b_i(x, \sigma \text{id}) \) are continuous in \( \sigma \in [0, \infty) \) and \( x \in \mathbb{R}^n \) for all \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \), and let \( s \in \mathcal{T}_A \) s.t. has a k-atomic representing measure \( (C, X, \sigma) \) with \( DS_{k,A}(C, X) \) has full rank. Then \( C_A^M(s) \leq k \).

**Proof.** Since \( s \) has a k-atomic representing measure \( (C, X, \sigma) \) s.t. \( DS_{k,A}(C, X) \) has full rank, \( s \in \text{int } \mathcal{T}_A \) by theorem 17. Since \( DS_{k,A}(C, X) \) has full rank, pick \( m \) variables \( y = (y_1, \ldots, y_m) \) from \( c_1, \ldots, c_k \) and \( x_1, x_2, \ldots, x_{k,n} \) s.t. \( D_y S_{k,A}(C, X) \in \mathbb{R}^{m \times m} \) is non-singular. Since \( b_i(x, \sigma \text{id}) \) and \( \partial b_i(x, \sigma \text{id}) \) are continuous in \( \sigma \in [0, \infty) \) and \( x \in \mathbb{R}^n \) for all \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \), \( D_y T_{k,A}(C, X, \sigma \text{id}, \ldots, \text{id}) = D_y S_{k,b_i(\cdot \sigma \text{id})},(C, X) \) is continuous in \( \sigma \in [0, \infty) \), \( c_i \in [0, \infty) \), and \( x \in \mathbb{R}^n \). Therefore, there is an \( \varepsilon > 0 \) s.t. \( D_y S_{k,b_i(\cdot \sigma \text{id})},(C, X) \) is non-singular for all \( (\sigma, C, X) \in K_\varepsilon := [0, \varepsilon] \times B_{\varepsilon}(C, X) \) since the determinant is continuous in the entries of the matrix. Denote by \( \tau_1(\sigma, C, X) \leq \cdots \leq \tau_n(\sigma, C, X) \) the singular values of \( D_y S_{k,b_i(\cdot \sigma \text{id})},(C, X) \). Since the singular values also depend continuously on the matrix, they depend continuously on \( (\sigma, C, X) \in K_\varepsilon \). Since \( \tau_1(\sigma, C, X) \) are continuous, they are bounded from above on the compact set \( K_\varepsilon \). But since
\[
\det(D_y S_{k,A \text{id}}(C, X)) = \pm \tau_1(\sigma, C, X) \cdots \tau_n(\sigma, C, X) \neq 0,
\]
\[
\tau_1(\sigma, C, X) = \inf_{(\sigma, C, X) \in K_\varepsilon} \tau_1(\sigma, C, X) \geq \tau_{\min}.
\]

But this means that
\[
B_r(S_{k,b_i(\cdot \sigma \text{id})},(C, X)) \subseteq S_{k,b_i(\cdot \sigma \text{id})},(B_{\varepsilon}(C, X)) \quad \forall r \in (0, \min \{\varepsilon, \tau_{\min}\}) \wedge \sigma \in [0, \varepsilon].
\]

Fix \( r \in (0, \min \{\varepsilon, \tau_{\min}\}) \). Then \( s \in B_r(S_{k,b_i(\cdot \sigma \text{id})},(C, X)) \subseteq S_{k,b_i(\cdot \sigma \text{id})},(B_{\varepsilon}(C, X)) \) for all \( \sigma \in (0, \varepsilon) \) s.t. \( \|s - S_{k,b_i(\cdot \sigma \text{id})},(C, X)\| < r \), i.e.,
\[
s = S_{k,b_i(\cdot \sigma \text{id})},(C, X) = T_{k,A}(C, X, \sigma \text{id}, \ldots, \text{id})\]
for a \((C, X) \in B_{\varepsilon}(C, X)\). \( \square \)

Note, that in the proof of theorem 37 the use of the multiple of id is arbitrary, any non-singular symmetric matrix will do, just insert a basis transformation on \( \mathbb{R}^n \).

From theorem 37 we get the following.
Theorem 38. Let \( A = \{a_1, \ldots, a_m\} \) in \( C^r(\mathbb{R}^n, \mathbb{R}) \) with \( r > N_A \cdot (n + 1) - m \) s.t. \( b_i(x, \sigma \text{id}) \) and \( \partial_j b_i(x, \sigma \text{id}) \) are continuous in \( \sigma \in [0, \infty) \) and \( x \in \mathbb{R}^n \) for all \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \). Then
\[
C_M^A(s) \leq C_A(s) \leq C_A \quad \forall s \in T_A \lambda^n - \text{a.e.}
\]
and the interior of the set where eq. (16) holds is dense in \( T_A \).

Proof. From Sard’s Theorem [Sar42] we know that the set of singular values is of \( n \)-dimensional Lebesgue measure zero and theorem 37 applies to the regular values, i.e., moment sequences s.t. all representing measures \( (C, X) \) have full rank \( DS_k, A(C, X) \).

The open problem is: Can we ensure that any moment sequence has a representing measure \( (C, X) \) with full rank \( DS_k, A(C, X) \) with at most \( C_A \) atoms. If we allow more atoms, this is true by [dDS18a, Lem. 36]. But this raises the Carathéodory bound.

Let us give an example of theorem 38.

Example 39. Let \( A = \{1, x^2, x^3, x^5, x^6\} \) on \( \mathbb{R} \), see [dDS18b, Exm. 46]. There we found that \( C_A = 3 \). It is easily seen that \( A \) fulfills all condition in theorem 38 (resp. theorem 37) and therefore
\[
C_M^A(s) \leq 3 \quad s \in T_A \lambda^n - \text{a.e.}
\]
and the lower bound
\[
\left\lceil \frac{5}{3} \right\rceil = 2 \leq C_M^A
\]
holds because of theorem 37.

We end this study with the general one-dimensional result for log-normal mixtures. Is uses the fact that \( x \in (0, \infty) \) and therefore a prior one-dimensional result [dDS18b, Thm. 45] can be applied.

Theorem 40. Let \( m, d_1, d_2, \ldots, d_m, d \in \mathbb{N} \) be such that \( 0 = d_1 < \cdots < d_m = 2d \) and \( A = \{x^{d_1}, \ldots, x^{d_m}\} \subset \mathbb{R}[x] \). Then for the log-normal distribution we have
\[
\left\lceil \frac{m}{3} \right\rceil \leq C_A^M \leq \left\lceil \frac{m}{2} \right\rceil.
\]

Proof. The lower bound follows from theorem 30.

For the upper bound we use [dDS18b, Thm. 45]. The assumptions are the same as in [dDS18b, Thm. 45] and we only have the check condition (32) in [dDS18b, Thm. 45]. It is a condition about the zero set of a Schur-like polynomial. But since \( X = (0, \infty) \), Schur polynomials and the Schur-like polynomial in (32) in [dDS18b, Thm. 45] have no zeros in \((0, \infty)^k\). This additional condition is therefore trivially fulfilled and [dDS18b, Thm. 45] gives the upper \( \left\lceil \frac{m}{2} \right\rceil \).

Example 41. Let \( A = \{1, x, x^2, x^{17}, x^{1863}, x^{25,376}\} \). Then by using theorem 37 we find that every moment sequence from a log-normal mixture has another log-normal mixture representation with at most 3 components.

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Universität Leipzig, Mathematisches Institut, Augustusplatz 10/11, D-04109 Leipzig, Germany
Max Planck Institute for Mathematics in the Sciences, Inselstraße 22, D-04103 Leipzig, Germany
E-mail address: didio@mi-leipzig.de