Odd triplet superconductivity induced by the moving condensate

M.A. Silaev,1,2 I. V. Bobkova,3,2,4 and A. M. Bobkov3

1Department of Physics and Nanoscience Center, University of Jyväskylä, P.O. Box 35 (YFL), FI-40014 University of Jyväskylä, Finland
2Moscow Institute of Physics and Technology, Dolgoprudny, 141700 Russia
3Institute of Solid State Physics, Chernogolovka, Moscow reg., 142432 Russia
4National Research University Higher School of Economics, Moscow, 101000 Russia

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It has been commonly accepted that magnetic field suppresses superconductivity by inducing the ordered motion of Cooper pairs. We demonstrate a mechanism which instead provides generation of superconducting correlations by inducing the motion of superconducting condensate. This effect arises in superconductor/ferromagnet heterostructures in the presence of Rashba spin-orbital coupling. We predict the odd-frequency spin-triplet superconducting correlations called the Berezinskii order to be switched on at large distances from the superconductor/ferromagnet interface by the application of a magnetic field. This is shown to result in the unusual behaviour of Josephson effect and local density of states in superconductor/ferromagnet structures.

The phenomenon of superconductivity dwells on the condensation of Cooper pairs each consisting of two electrons with opposite momenta. Magnetic field tends to break Cooper pairs by inducing their center-of-mass motion which makes the momenta of two paired electrons to be not exactly opposite. This seems to be the fundamental mechanism called the orbital depairing effect which exists in any superconducting system and leads to the suppression of superconductivity by the magnetic field.2

In this Letter we show that the magnetic field can in fact stimulate superconducting properties such as the Josephson current by generating correlations which are odd in the frequency domain. Our proposal is based on the observation that the combination of exchange field, spin-orbit coupling (SOC) and controllable condensate motion is generically enough for the effective manipulation of the odd-frequency spin-triplet pairing states2 which have attracted continual interest for several decades3,25. We show that the condensate motion adds an additional degree of freedom which can control the generation of odd-frequency superconductivity in currently available experimental setups26,32. In the context of superconductor (S)/ferromagnet (F) hybrid structures these correlations are known as long-range triplets (LRT) because they can penetrate at large distances into the ferromagnetic material9,35–42. We suggest a mechanism which leads to the stimulation of long-range Josephson current in SFS junctions by the external magnetic field thus opening great perspectives for low-dissipative spintronics43,44.

The prototypical setup where proposed effects can be observed is sketched in Fig.1. It consists of superconductor/ferromagnet (S/F) interface, where the superconductor and the ferromagnet are separated by a thin layer of heavy metal which induces45,46 the interfacial Rashba-type SOC $\alpha(\hat{\sigma} \times \mathbf{p}) \cdot \mathbf{n}$ where $\mathbf{n}$ is the interface normal, $\hat{\sigma}$ is the vector of spin Pauli matrices, $\mathbf{p}$ is the electron momentum. It is well-known that the presence of LRT Cooper pairs can be engineered combining usual superconductors such as Al or Nb with ferromagnets and SOC47,51. Although possible in principle, the generation of LRT requires specific type of SOC and superconductor/ferromagnet interface configuration47,48,50,52. That is LRT are absent in the generic S/F structures such as shown in Fig.1. Here the exchange field $\mathbf{h} \parallel z$ produces only short-range triplets (SRT) with $S_z = 0$ shown schematically by the blue arrows which decay at short ferromagnetic coherence length $\xi_F \sim 1$ nm in usual ferromagnets. In contrast, we demonstrate that in such a scenario setting the superconducting condensate to motion with momentum satisfying the condition $\mathbf{p}_s \times (\mathbf{n} \times \mathbf{h}) \neq 0$ generates LRT with $S_z = \pm 1$ shown schematically by red arrows in Fig.1. These correlations induce superconducting properties in the ferromagnet at the distance $\xi_N \gg \xi_F$. This effect is fully controllable by external parameters such as the in-plane supercurrent or the magnetic field $\mathbf{B} \perp \mathbf{h}$ which both induce the condensate...
flowing in the superconductor along the F/S interface as shown in Fig.1b.

The qualitative physics of the effect can be described as follows. The influence of SOC on the electron spin can be understood in terms of the Lorenz transformation between electric and magnetic fields in the moving coordinate frame. That is, an electric field \( E \), which is present in the laboratory frame, results in the magnetic field \( B = -v \times E / 2c \) in the frame of the electron moving with the velocity \( v \). Acting on the electron spin this magnetic field induces the Zeeman shift of the energy which depends on the velocity \( v \). This fact makes drastic difference with the external Zeeman field and strongly reduces effect of SOC on the spin-singlet Cooper pairs because the effective Zeeman field has opposite signs for electrons with velocities \( v \) and \(-v\) forming the Cooper pair. Hence the effect of SOC on spin-singlet correlations vanishes in the first approximation. Therefore the amplitude of spin-triplet correlations induced by SOC is determined by the small parameter \( v F_0 / E_F \ll 1 \), where \( v_F \) and \( E_F \) are Fermi velocity and energy.

We show that the effect of SOC on the superconducting state is strongly enhanced in the presence of the Zeeman field \( h \parallel z \). The role of \( h \) is to generate \( S_z = 0 \) spin triplet correlations which are converted to \( S_z = \pm 1 \) correlations due to the spin-rotation by SOC. The amplitude of \( S_z = \pm 1 \) correlations generated in such a way is determined by the parameter \( h v F_0 / \Delta^2 \gg v F_0 / E_F \) which in principle is not small. However these triplets are even-frequency and odd-parity correlations forming peculiar textures in momentum space but being suppressed by disorder scattering. We demonstrate that the condensate motion provides a mechanism of converting them into odd-frequency and even-parity correlations and propose an experimental setup where they can manifest itself as a long-range effect in the local density of states and a long-range Josephson effect switched by applying of the external magnetic field or a supercurrent parallel to the interfaces.

\textit{p-wave even frequency spin triplets} At first we consider the homogeneous superconductor and demonstrate that the combination of Zeeman field and Rashba SOC produces unusual structure of superconducting correlations in momentum space, which correspond to the spin-triplet odd-parity pairing similar to the one in p-wave superfluid \(^3\)He [58,60]. For that we use Eilenberger equations which allow for the study of anisotropic part of the Green’s function, that is momentum-resolved superconducting correlations. For the sake of simplicity we restrict ourselves by the ballistic limit at the moment.

For the spatially homogeneous system in the presence of constant in time vector potential, SOC and exchange field we use the quasiclassical propagator \( \hat{g} = \hat{g}(\omega, \mathbf{p}) \), which is a 4 \( \times \) 4 matrix in the direct product of particle-hole and spin spaces depending on the imaginary frequency \( \omega \) and momentum direction \( \mathbf{p} \). It is determined by the Eilenberger equation [61]

\[
v_F \mathbf{p}_s \{ \hat{r}_3, \hat{g} \} / 2 - \alpha [v_F \hat{A}, \hat{g}] = i [\hat{\Lambda}, \hat{g}]
\]

(1)

Here \( \hat{r}_i \) and \( \hat{\sigma}_i \) are Pauli matrices in particle-hole and spin spaces, respectively. We denote \( \hat{\Lambda} = \hat{r}_3 (\omega + ih \hat{\sigma}_i - i \Delta) \), where \( \Delta = |\Delta| e^{i \gamma x} \hat{r}_1 \) is the superconducting order parameter. We introduce the gauge-invariant condensate momentum \( \mathbf{p}_s = \nabla \chi - 2e \mathbf{A} / c \), and \( \hat{\mathbf{A}} = \mathbf{A} \hat{\sigma}_3 \) is the spin-dependent gauge field describing an arbitrary linear in momentum SOC. The quasiclassical equations are supplemented by the normalization condition \( \hat{g}^2 = 1 \).

First, let us consider the Cooper pair spin distribution in momentum space for zero applied current and magnetic field that is \( \mathbf{p}_s = 0 \). The GF can be written in the form [62]

\[
\hat{g} = \hat{g}_h + \hat{g}_{h\alpha}
\]

(2)

In the presence of Zeeman field the GF consists \( \hat{g}_h = \hat{g}_s + \hat{g}_t \) of spin-singlet \( \hat{g}_s \propto \hat{\sigma}_0 \) and \( \hat{g}_t = 0 \) spin-triplet \( \hat{g}_t \propto \hat{\sigma}_3 \) parts. The first-order correction by SOC is

\[
\hat{g}_{h\alpha} = i \alpha (\hat{g}_h (v_F \hat{A}) \hat{g}_h - v_F \hat{A}) / (s_+ + s_-)
\]

(3)

where \( s_\pm = \sqrt{(\omega \pm ih)^2 + \Delta^2} \). This corrections generates pairing in p-wave even-frequency spin-triplet channel which is known to exist in superfluid \(^3\)He and systems with Rashba SOC [63]. For Rashba SOC \( \hat{\mathbf{A}} = n \times \hat{\sigma} \) Eq. (21) can be written in the form taking into account the explicit spin structure

\[
\hat{g}_{h\alpha} = -\frac{\alpha v F_0}{(s_+ + s_-)} (\hat{\sigma} \mathbf{d}_p) (\hat{g}_t, \hat{g}_s) + 2\hat{g}_z^2 \hat{\sigma} \mathbf{n}_h
\]

(4)

The first term is the anomalous part since \( [\hat{g}_t, \hat{g}_s] = \hat{r}_3 (2\Delta h / s_+ s_-) \). It is an even function of frequency and odd in momentum space since the spin vector \( \mathbf{d}_p = n_x \times (n \times \hat{p}) \) satisfies \( \mathbf{d}_p (\hat{p}) = -\mathbf{d}_p (-\hat{p}) \) corresponding to the p-wave spin-triplet superconducting correlations. Depending on the relative orientation of the Zeeman field \( \mathbf{n}_h = h / h \) and Rashba vector \( \mathbf{n} \) the vector field \( \mathbf{d}_p \) can form different textures in momentum space. In Fig. 2(a,b,c) the two characteristic examples of the hedgehog and domain wall are shown, respectively.

\textit{s-wave spin-triplet correlations in the presence of supercurrent} The p-wave correlations can change observables especially in the ballistic systems [64]. However they are strongly suppressed in the presence of disorder. Here we show that in the presence of superflow \( \mathbf{p}_s \neq 0 \) in Eq. (1) the spin-triplet p-wave correlations in Eq. (4) are partially transformed into the s-wave ones. Mathematically this can be obtained by calculating the correction \( \hat{g}_{h\alpha} \propto \alpha \hat{A} \mathbf{p}_s \) from Eq. (1). In general such corrections have the spin structure similar to Eq. (4) but with momentum-independent spin vector \( \mathbf{d}_s = -\mathbf{n}_h \times (\mathbf{p}_s \times \mathbf{n}) / 2 \). Physically this can be understood as a result of the supercurrent-generated Fermi surface shift resulting in the suppression of pairing on one
The operator on the GF can be written in terms of the self-applied magnetic field or a supercurrent the action of this which can be rewritten in the form demonstrating the...

FIG. 2. Textures of the spin-triplet order parameter vector \( \mathbf{d} \) on the Fermi surface \( p_x, p_y \) cross-section. Left column: odd parity state \( \mathbf{d}_p \) for resting condensate \( \mathbf{p}_s = 0 \). (a) \( \mathbf{n} \parallel \mathbf{h} \), (c) \( \mathbf{n} \perp \mathbf{h} \) Right column: mixture of odd- and even-parity states \( \mathbf{d}_p + \mathbf{d}_s \) induced by the moving condensate \( \mathbf{p}_s \neq 0 \). (b) \( \mathbf{h} \parallel \mathbf{n} \parallel \mathbf{p}_s \) and (d) \( \mathbf{h} \parallel \mathbf{p}_s \parallel \mathbf{n} \).

The s-wave correlations are robust with respect to the impurity scattering so that they survive in dirty systems, which are more relevant to experiments, and we can analyze them using diffusive Usadel equation

\[
[\tilde{\Lambda}, \tilde{g}] = D \tilde{\partial}_k (\tilde{g} \tilde{\partial}_k \tilde{g})
\]

where \( D \) is the diffusion constant. The differential super-operator in Eq. (22) is

\[
\partial_k \tilde{g} = \nabla_k \tilde{g} - i[\alpha \tilde{A}_k + c \tilde{r}_3 \tilde{A}_k, \tilde{g}],
\]

For a homogeneous superconductor in the presence of applied magnetic field or a supercurrent the action of this operator on the GF can be written in terms of the self-energy \( [\tilde{\Lambda} + \tilde{\Sigma}, \tilde{g}] = 0 \). We obtain the following self-energy term induced by the condensate motion \([62]\)

\[
\tilde{g}_{A\alpha} = (\tilde{\Sigma}_{A\alpha} - \tilde{g}_h \tilde{\Sigma}_{A\alpha} \tilde{g}_h)/(s_+ + s_-),
\]

which can be rewritten in the form demonstrating the explicit spin structure of the GF:

\[
\tilde{g}_{A\alpha} = \frac{\alpha D}{s_+ + s_-} \times
\begin{align*}
&i(\sigma \mathbf{d}_s) \left( 2\tilde{g}_l^2 [\tilde{g}_s, \tilde{r}_3] + [\tilde{g}_s, \tilde{g}_l] [\tilde{g}_s, \tilde{r}_3] \right) - \\
&i(\sigma \mathbf{d}_s)(\sigma n_h) \left( 2\tilde{g}_l^2 [\tilde{g}_s, \tilde{r}_3] - [\tilde{g}_s, \tilde{g}_l] [\tilde{g}_s, \tilde{r}_3] \right)
\end{align*}
\]

Here the anomalous part is given by the first term \( \propto \tilde{r}_1 \) and the normal part is the second term \( \propto \tilde{r}_0 \). Therefore, the spin structure of the corresponding s-wave superconducting correlations is determined by the vector \( \mathbf{d}_s \). Since \( \mathbf{d}_s \perp \mathbf{n}_h \) these correlations are equal-spin. In contrast to the p-wave component \([1]\) the s-wave component of the anomalous function is an odd-frequency one.

Suppose that we apply a supercurrent to our homogeneous superconductor in the direction of the exchange field \( \mathbf{p}_s \parallel \mathbf{h} \). In this case the equal-spin superconducting condensate with \( \mathbf{d}_s \neq 0 \parallel \mathbf{n} \) is generated. On the contrary, if the supercurrent is applied in plane of the superconductor perpendicular to the exchange field then \( \mathbf{d}_s = 0 \) and the equal-spin correlations are not generated. Therefore, the spin structure of the superconducting correlations is controlled by the direction of externally applied supercurrent or an in-plane magnetic field. This remarkable feature allows for obtaining a controllable long-range triplet proximity effect since equal-spin triplet correlations are not antagonistic to ferromagnetism and, consequently, decay slowly into the depth of a ferromagnet. This effect and its experimental manifestations are discussed below.

**LRT generated by a magnetic field** The most striking demonstration of the spin-triplet correlations with \( \mathbf{d}_s \perp \mathbf{h} \) can be achieved in a S/F structure with Rashba SOC shown in Fig. 1. First, let us notice that for \( \mathbf{B} = 0 \) we have \( \mathbf{d}_s = 0 \) and therefore despite the presence of SOC only \( S_z = 0 \) Cooper pairs are generated. Such correlations decay in the ferromagnet at the length scale \( \xi_F = \sqrt{D}/\hbar \) which is rather short - \( \xi_F \sim 1 \) nm in usual materials. Thus with exponential accuracy the superconductivity is absent at the distances \( x \gg \xi_F \) from the S/F interface.

At the same time the correlations with \( \mathbf{d}_s \neq 0 \) that appear at \( \mathbf{B} \neq 0 \) have \( S_z = \pm 1 \). Therefore they are LRT which are robust to the spin depairing and the only correlations that survive at the distances \( x \gg \xi_F \) from the S/F interface in the ferromagnet. In the setup shown in Fig. 1 such pairs appear only upon applying the magnetic field \( \mathbf{B} \perp \mathbf{h} \). Hence we claim to find the mechanism of the odd triplet superconductivity generated by the magnetic field. Formation of LRTs has important consequence in the transport properties\([40, 42, 65, 67]\) which can be directly measured using the electrical probes. Below we discuss two of them - the tunnel conductance and Josephson current.

For the tunnelling S/F interface the superconducting
correlations are small in the ferromagnet so that we linearize the Usadel equation by assuming $\tilde{\sigma} = \text{sign}(\omega)\tilde{\gamma} + \tilde{\sigma}$ where the anomalous part $\tilde{\sigma} = \tilde{f}_s + \tilde{f}_t\sigma$ can be written as the sum of spin-singlet $\tilde{f}_s$ and spin-triplet $\tilde{f}_t$ components. We assume that the SOC is localized near FS interfaces at the lengthscale much smaller than $\xi_F$, so that $\alpha(x > \xi_F) = 0$. Then we obtain the linearized equations

$$\frac{D}{2}\nabla^2 \tilde{f}_s = (|\omega| + q)\tilde{f}_s + \text{sgn}(\omega)(h\tilde{f}_t)$$

(9)

$$\frac{D}{2}\nabla^2 \tilde{f}_t = (|\omega| + q)\tilde{f}_t + \text{sgn}(\omega)(h\tilde{f}_s)$$

(10)

where we introduce the orbital depairing energy $q = 2e^2DA^2$. This equation is supplemented by the boundary condition at SF interface

$$n_x\nabla_x \tilde{f}_s = \gamma\tilde{F}_{bc\alpha}$$

(11)

$$n_x\nabla_x \tilde{f}_t = 4i\tilde{\gamma}\tilde{f}_t \times (p_s \times n)$$

(12)

where $\tilde{F}_{bc\alpha} = i\tilde{\gamma}\tilde{\Delta}/s$ and the surface SOC strength $\tilde{\alpha} = \int dx\alpha(x)$ and $\gamma$ is the S/F interface transparency $[9, 68]$. 

Solution of this system can be obtained in the compact form in the realistic limit $\xi_F \ll d_F \ll \xi_N$, where $d_F$ is the thickness of the ferromagnet and $\xi_N = \sqrt{D/|\omega|}$. Further we are only interested in the LRT component of the anomalous Green’s function $\tilde{f}_{L,R}$, which is approximately constant in the ferromagnet for the case under consideration. It can be found by integrating Eq. (10) over the ferromagnet. Up to the first order in the parameter $\alpha(p_s\xi_N)(\xi_N/d_F)$, which is assumed to be small, we obtain

$$\tilde{f}_t = -\frac{4iD}{d_F(|\omega| + q_0)}\tilde{\gamma}[\tilde{\alpha}^L \tilde{f}_t^L + \tilde{\alpha}^R \tilde{f}_t^R] \cdot n_h d_s,$$  

(13)

where indices $L, R$ correspond to the values of the corresponding surface SOC and the short-range triplet component $(\tilde{f}_t n_h)$ taken at the left (right) boundary of the ferromagnet, $q_0 = (1/d_F) \int dx q$.

First let us consider the configuration with single F/S interface shown in Fig. 1. Calculating the short-range triplet component from Eqs. (9,12) and substituting it into Eq. (13) we obtain

$$\tilde{f}_t = -\frac{2\gamma\xi_F\tilde{\alpha}}{d_F(|\omega| + q_0)}\tilde{\gamma}\tilde{F}_{bc\alpha}\cdot d_s$$

(14)

Now we are ready to calculate the enhancement of DOS in the ferromagnet due to the odd-frequency LRT [14,30] [69]

$$\delta N(\varepsilon) = \frac{1}{2}(|\tilde{f}_t^L|^2(\omega = i\varepsilon))$$

(15)

For $d_F \gg \xi_F$ the contribution to this correction resulting from the short-range triplets already vanishes and, therefore, $\delta N = 0$ at $B = 0$. However, at $B \neq 0$ the LRTs expressed by Eq. (14) start to appear as $\delta N \propto B^2$. 

Next let us consider the long-range Josephson effect induced by magnetic field in the setup shown in the inset in Fig[3]. We assume similar conditions on the thickness of F layer $d_F$ as above. Having in hand the solution for LRT $\tilde{f}_t$ we calculate [62] the Josephson current-phase relation in the form $j = I_c \sin \phi$ with the critical current

$$I_c/I_{00} = \xi^2(p_s \times n)^2 \sum_{\omega > 0} \frac{2\pi T \Delta^2}{(\omega^2 + \Delta^2)(\omega + q_0)}$$

(16)

where $I_{00} = 4\sigma_F \Delta (\gamma\tilde{\alpha}_F)^2/c d_F$, $p_s$, $n$ are taken either at right or left S/F interface and $p_s(d_F/2) = -p_s(-d_F/2) = eBd_F$, $\alpha_F$ is the ferromagnet conductivity and $\xi = \sqrt{D/\Delta}$ is the coherence length in superconductor. One can see that for small fields the current grows as $j \propto p_s^2 \propto B^2$. In additions we need to take into account phase variation along the junction which leads to the usual factor $L \sin \phi/\phi$ in the critical current where $\phi = 2\pi \Phi/\Phi_0$ and $\Phi = 2\lambda_L LB$ is the total flux through the junction area. Here $L$ is the length of the junction and $\lambda_L$ is the London penetration length. In result we obtain $I_c \propto B$ envelope dependence of the critical current shown in Fig[3]. This growth is bounded from above by the depairing effects in superconducting electrode when the field becomes closer to the second critical one $B \sim H_{c2}$. Quite importantly, although the interference pattern in Fig[3] has zero-field minimum it is determined by the unusual magnetic field behaviour of the single-channel Josephson current distributed homogeneously along the junction. This makes a drastic difference with other systems [70] which can demonstrate zero-field minimum of $I_c(B)$ curves due to the interference between different Josephson current channels. In
Odd-frequency even-parity correction to the Green’s function induced by the moving condensate in the diffusive limit

Now we calculate the correction induced by the moving condensate. The s-wave correlations are robust with respect to the impurity scattering so that they survive in dirty systems and we can analyse them using diffusive Usadel equation

$$[\hat{\Lambda}, \hat{g}] = D\partial_k (\hat{g} \partial_k \hat{g})$$

where $D$ is the diffusion constant. The differential super-operator in Eq. (22) is

$$\partial_k \hat{g} = \nabla_k \hat{g} - i[\alpha \hat{A}_k + \frac{e}{c} \hat{\tau}_3 \hat{A}_k, \hat{g}]$$

For a homogeneous superconductor in the presence of applied magnetic field or a supercurrent the action of this operator on the GF can be written in terms of the self-energy $[\hat{\Lambda} + \hat{\Sigma}, \hat{g}] = 0$ where the self-energy can be represented as a sum of three terms:

$$\hat{\Sigma} = \hat{\Sigma}_{\alpha^2} + \hat{S}_{A^2} + \hat{\Sigma}_{A^2}$$

The components include electromagnetic self-energy

$$\hat{\Sigma}_{A^2} = (D/4)p_3^2 \hat{\tau}_3 \hat{\gamma}_3$$

addition to spin relaxation $\hat{\Sigma}_{\alpha^2} = D\alpha^2 \hat{A}_k \hat{A}_k$, and the most interesting for us part converting condensate motion into the deformation of the spin texture

$$\hat{\Sigma}_{A^2} = -\frac{\alpha D}{2} p_s (\hat{A} \hat{\gamma}_3 + \hat{\gamma}_3 \hat{A})$$

The correction to the GF due to the presence of the self-energy can be found also as a sum of the corresponding terms:

$$\hat{g} = \hat{g}_h + \hat{g}_{\alpha^2} + \hat{g}_{A^2} + \hat{g}_{A^2}$$

If we restrict ourselves by $\hat{n} \perp \hat{n}_h$, that is by the case on in-plane exchange field, then the corrections $\hat{g}_{\alpha^2}$ and $\hat{g}_{A^2}$ do not contain equal-spin correlations, which we are interested at the moment. Therefore, we only calculate $\hat{g}_{A^0}$. Taking into account the normalization condition $\hat{g}_h \hat{g}_h + \hat{g}_{A^0} \hat{g}_h = 0$ we obtain the expression for this correction to GF expressed by Eq.(8) of the main text.

Details of the Josephson current calculation

For the problem under consideration the simplest way is to calculate the current at one of the S/F interfaces. Then it can be expressed as

$$j = i \gamma \sum_{\omega > 0} \frac{e}{\hbar} \text{Tr} \hat{\tau}_3 (f^e \hat{F}_{bcx}(x = d_F),$$

where $\text{Tr} \hat{\tau}_3$ is the matrix trace of the Pauli matrix $\hat{\tau}_3$. The sum is taken over all frequencies $\omega > 0$.
where we need to calculate \( \hat{f}_s \) up to the leading (second) order with respect to \( \tilde{\alpha}(p_\alpha \xi_N) \). The singlet and short-range triplet components decay rapidly into the depth of the ferromagnet and can be calculated at each of the S/F interfaces separately. For definiteness we consider the right S/F interface. The singlet and SRT anomalous Green’s functions are to be calculated from Eqs. (9)-(12) of the main text. In order to find them up to the second order with respect to \( \tilde{\alpha}(p_\alpha \xi_N) \) we need to substitute the first-order correction to the triplet anomalous Green’s function Eq. (13) of the main text into the right-hand side of Eq. (12) of the main text with \( f^{L,R}_t = (i\gamma \xi_F/2) \hat{F}^{L,R}_{\text{bcs}} \). Then solving Eqs. (9)-(12) we obtain the following result for \( \hat{f}_s \):

\[
\hat{f}_s = -\frac{\gamma}{2} \xi_F \hat{F}^{R}_{\text{bcs}} - \tilde{\alpha}^2(p_s \times n)^2 \frac{4D\gamma|\Delta|\tau_2\xi_F^2 \cos(\chi/2)}{db_s/\omega + q_0}.
\]

Substitution of this result into Eq. (27) gives the sinusoidal current-phase relation with the critical current expressed by Eq. (16) of the main text.

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