On the Geometric Unification of Gravity and Dark Energy

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Abstract In the framework of Finslerian geometry, we propose a geometric unification between traditional gauge treatments of gravity, represented by a metric field, and dark energy, which arises as a corresponding gauge potential from the single $SU(2)$ group. Furthermore, we study the perturbation of gravitational waves caused by dark energy. This proposition may have far reaching applications in astrophysics and cosmology.

Keywords Alternative theories of gravity · Dark energy · Gravity waves

1 Introduction

Observations of type Ia supernovae and of large-scale structure (LSS), in combination with measurements of the characteristic angular size of fluctuations in the cosmic microwave background (CMB) [1–10] provide evidence that the expansion of the Universe is accelerating. This acceleration is attributed to the “dark energy”, a hypothetical energy with negative pressure [1–6]. Evidence for the presence of a dark energy is also provided by an independent, albeit more tentative, investigation of the integrated Sachs-Wolfe (ISW) [11].

The dark energy may result from Einstein’s cosmological constant (which has a phenomenally small value); from evolving scalar fields [12]; and from a weakening of gravity in our $3 + 1$ dimensions by leaking into the higher dimensions, as required in string theories [13]. These explanations may have crucial implications on the perturbation of gravitational waves caused by dark energy. This has stimulated further efforts to confirm the initial results on dark energy, test possible sources of error, and extend our empirical knowledge of this newly discovered component of the Universe.

The gauge theory of gravity is based on the gauge principle and was suggested immediately after the formulation of the gauge theory [14–23]. In the traditional gauge treatment of gravity, the Lorentz group is localized and the gravitational field is not represented by gauge potential, but by a metric field [18, 20].
In this paper we use the framework of Finslerian geometry [24–29, 37, 38] to propose a geometric unification between traditional gauge treatment of gravity, represented by the metric field $g_{\mu\nu}$, and dark energy, which appears as a corresponding gauge potential $B_\mu$, arising naturally from the gauge treatment of the single $SU(2)$ group [30–32]. We demonstrate that the dark energy would result naturally as a geometric effect of Randers space, rather than being an additional suggestion. Randers space, as a special kind of Finsler space, was first proposed by G. Randers [37]. A generalized Friedmann-Robertson-Walker (FRW) cosmology of Randers-Finsler geometry has been also suggested [39–41].

Furthermore, we study the dark energy perturbation of gravitational wave, and discuss some wider potential applications of this in astrophysics and cosmology.

2 On the Finslerian Geometric Unification of Gravity and Dark Energy

In the framework of a Riemannian approach, where two nearby particles are subject to the traditional gravitational field $g_{\mu\nu}$ (free-falling), the Equation of Deviations of Geodesics (EDG) takes the form:

$$\frac{D^2 n^\mu}{ds^2} + R^\mu_{\nu\kappa\lambda} n^\nu n^\kappa n^\lambda = 0 \quad (1)$$

where $n^\nu, n^\kappa$ represent the deviation vector; $\frac{D}{ds}$ is the covariant derivative; and $R^\mu_{\nu\kappa\lambda}$ is the Riemann tensor.

The equation of motion of two nearby particles subjected to the action of the massless vector B-field of dark energy is obtained by introducing the Lorentz term into the geodesics equation (1) and replacing the four-acceleration $a_\mu = \frac{du_\mu}{ds}$ by $\frac{D u_\mu}{ds}$.

$$\frac{D n^k}{ds} = \frac{d^2 x^k}{ds^2} + B^k_{\mu\nu} u^\mu u^\nu = \frac{1}{16\pi G} B_\mu^k u^\lambda \quad (2)$$

where $B_{\mu\nu}$ represents the dark energy field strength: $B_{\mu\nu} = B_{\nu,\mu} - B_{\mu,\nu};$ and $B_\mu$ is the dark energy gauge potential arising from the single $SU(2)$ group [30].

Equation (2) thus modifies (1) in the general form:

$$\frac{D^2 n^\mu}{ds^2} + R^\mu_{\nu\kappa\lambda} n^\nu n^\kappa n^\lambda = \Phi^\mu \quad (3)$$

where

$$\Phi^\mu = \beta \left( \frac{D B_\mu}{ds} u^\kappa + B^\mu_{\nu} B^\nu_\kappa u^\kappa \right);$$

and $\beta$ is a constant $\beta = \frac{1}{16\pi G}$.

The term $\Phi^\mu$ in relation (3) describes the external interaction between two nearby mass particles due to dark energy. For $\Phi^\mu = 0$, relation (3) is reduced to (1), where only the gravitational field is present. $\Phi^\mu$ also governs the relative acceleration between two nearby particles in the flat space with $R^\mu_{\nu\kappa\lambda} = 0$.

From the above we conclude that Riemannian geometry does not provide a sufficient framework for the geometric unification between gravity and the dark energy. The disadvantage of Riemannian geometry is that the equation of motion of a particle subject to the action of a gravitational and dark energy field doesn’t occur physically from the geometry of space-time and it is necessary to be imposed as an independent axiom.
Riemannian geometry can be extended through the introduction of the Finsler space [25]. The metric function of the Finsler space is given by:

\[ F(x, V) = \sqrt{g_{\mu\nu}(x)V^\mu V^\nu + \beta B_\mu V^\mu} \]  

(4)

where \( g_{\mu\nu} \) is the Riemannian metric tensor and \( B_\mu \) is the dark energy vector potential. The metric \( f_{\mu\nu} \) of Finslerian space [29, 37], is given by

\[ f_{\mu\nu} = \frac{1}{2} \frac{\partial^2 F^2}{\partial V^\mu \partial V^\nu} \]  

(5)

\[ f_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}, \]  

(6)

where \( g_{\mu\nu} \) is the Riemannian metric tensor and \( h_{\mu\nu} \) a metric tensor, which is given by

\[ h_{\mu\nu} = \frac{2\beta}{\sigma} V^s g^s_{\mu B_v} + \beta^2 B_\mu B_v + \frac{\beta}{\sigma} V^l B_l r_{\mu\nu}, \]  

(7)

where \( \beta = \frac{1}{32\pi G}, \sigma = \sqrt{g_{\kappa\lambda}x^\kappa x^\lambda}, \)

\[ r_{\mu\nu} = g_{\mu\nu} - \sigma^{-1} g_{\kappa\mu} g_{\lambda\nu} x^\kappa x^\lambda, \quad a_{(ij)} = \frac{1}{2} (a_{ij} + a_{ji}) \]  

(8)

A space endowed with the metric tensor (4) is called a Randers space [29, 37].

The presence of the dark energy component \( h_{\mu\nu} \) in space-time causes the isotropy of space to break down. The geodesic equation for this space is:

\[ \frac{dx^\mu}{ds} + 2 \Lambda^\mu (x, x') = 0 \]  

(10)

\[ 2 \Lambda^\mu (x, x') = \left\{ \mu \kappa \lambda \right\} (x) x^\kappa x^\lambda + \beta B^\mu_{\kappa} x^\kappa \]  

(11)

Then, from \( \Lambda^\mu \) can be derived the connection coefficient \( \Lambda_{\kappa\lambda}^{\mu} \) of the space, similar to the Berwald connection coefficients discussed in [28]. By analogy to the Berwald curvature tensor [28], we may associate with connection coefficient \( \Lambda_{\kappa\lambda}^{\mu} \) a curvature tensor.

\[ \tilde{H}_{\nu j}^i = R_{\nu j k}^i + B_{\nu j k}^i, \]  

(12)

where \( R_{\nu j k}^i \) is the Riemannian curvature tensor that came from the metric \( g_{\mu\nu} \), and \( B_{\nu j k}^i \) is given by

\[ B_{\nu j k}^i = \frac{1}{2} (B_{\nu j k}^i + g_{h[kB_jl^m} B_{m]}^l - B_{h[j} B_{k]}^l) \]  

\[ + (u_k \nabla_{[k} B_{l]}^j + x^m g_{h[l]} \nabla_k B_{m]^l} + u_l \nabla_{(j} B_{k)}^i) \sigma^{-1} \]  

\[ - x^m u_k u_l \sigma^{-3} \nabla_{(j} B_{k)}^i + \]  

(13)

where

\[ u_i = \frac{\sigma^i_j}{\sigma}, \quad \sigma = \sqrt{g_{\kappa\lambda}x^\kappa x^\lambda} \]  

and \( 2t_{[ij]} = (t_{ij} - t_{ji}) \)

(14)

The equation of geodesic deviation is given by

\[ \frac{\delta^2 z^i}{\delta u^i} + \tilde{H}_{\nu j}^i (x, x') z^j = 0, \]  

(15)
where $\tilde{H}^i_{hjk} = \tilde{H}^i_{hjk} x^{h} x^{k}$, $z^j$ is the deviation vector and $x'$ is the tangent vector. As $\tilde{H}^i_{hjk}$ has a part independent of velocity $x'^n$, we have the relation

$$ (u_h \nabla_{[l} B^i_{j]} + x^m \tilde{g}_{h[l} B^i_{m] k} + u_{[l} \nabla_{k]} B^i_{h]} ) \sigma^{-1} - x^m u_h u_{[j} \sigma^{-3} \nabla_{k]} B^i_{m] = 0} \quad (16) $$

By (16), (12) becomes

$$ \tilde{H}^i_{hjk}(x) = R^i_{hjk} + \frac{1}{2} \left( B^i_{hjk} + g_{h[k} B^i_{j] m} - B_{h[l} B^i_{j]} k] \right) \quad (17) $$

By (17) we derive the action of the system

$$ S = H^i_{hjk}(x) g^{hj} = R + \beta B_{mn} B^{mn} \quad (18) $$

The variation of action’s integral

$$ \delta I = \delta \int d^4 x S \sqrt{|g|}, \quad g = \text{det} (g_{ij}) \quad (19) $$

leads to the field equations

$$ R_{mn} - \frac{1}{2} g_{mn} R + \beta \left( B_{nr} B^{r} - \frac{1}{4} g_{mn} B_{rs} B^{rs} \right) = 8\pi T_{mn} \quad (20) $$

where $R_{mn} = R^i_{mni}$, $R = g^{mn} R_{mn}$ and $T_{mn}$ is the energy-momentum tensor. The (20) are the field equations of the “gravito—dark energy” of the Randers space with respect to the condition (16). From a physical point of view the curvature $H^i_{hjk}$ can be considered as a “gravito—dark energy” curvature of the space.

For (12) and (15) we observe that the deviation equation $z^j$ has two terms: a pure gravitational deviation, represented by $R^i_{hjk}$ in the curvature tensor equation (12), which we would observe if there was no dark energy field, and the admixture of gravitational and dark energy deviations, represented by $B^i_{hjk}$ in the curvature tensor equation (12). We examine the following cases:

For $B^i_{hjk} = 0$ we have

$$ \frac{\delta^2 z^i}{\delta u^2} + R^i_{hjk}(x, x') x'^i x^l z^k = 0 \quad (21) $$

where the deviation equation (15) becomes a Riemannian one. For $B^i_{hjk} \neq 0$, $R^i_{hjk} \neq 0$ the associated curvature tensor of Randers space $\tilde{H}^i_{hjk}$ is derived from the connection coefficients

$$ H^i_{mn} = \left\{ \begin{array}{l} l_{mn} \end{array} \right\} + \frac{1}{2} \left( g_{mn} x^k B^i_{k} + u_m B^i_{n} \right) / \sigma - \frac{1}{2} u_m u_n x^k B^i_{k} / \sigma^3 \quad (22) $$

The last relation shows that dark energy field is incorporated in the geometry of space. The second part $B^i_{hjk}$ of the full curvature $\tilde{H}^i_{hjk}$ (12) describes the dark force that two freely falling particles of masses $m_1$ and $m_2$ would exercise on each other. In such a case, the dark force would result naturally as a geometrical effect and it would not be necessary for us to impose it in addition.

Finally, for $R^i_{hjk} = 0$ the equations of the geodesic deviations are governed by the dark energy and relation (15) is reduced to:

$$ \frac{\delta^2 z^i}{\delta u^2} + B^i_{hjk}(x, x') x'^i x^l z^k = 0 \quad (23) $$
In this case the first term of the Randers tensor corresponds to the Lorenz metric. The metric function $F(x, \dot{x})$ can, then, be expressed in the form:

$$F(x, \dot{x}) = \sqrt{n_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + \beta B_i(x)\dot{x}^i, \quad \beta = \frac{1}{16\pi G}$$

(24)

This metric function is interesting for a possible linear theory caused by the dark energy field.

### 3 The Friedman Equation for a Linearized Dark Energy Vector Field

In some cases it is useful from a physical point of view to consider a vector field in the form $y_i/i, i = 1, 2, 3, 4$ and the induced Finslerian metric tensor gives rise to the osculating Riemannian metric tensor $g_{\mu\nu}(x) = f_{\mu\nu}(x, y(x))$ [42].

The Osculating Riemannian approach (for details see [42]), can be specialized for the tangent vector field $y(x)$ of the cosmological fluid flow lines.

We are interested in producing the Einstein field equations as in [40]. After calculating the connection and the curvature for the Riemannian osculating metric $g_{\mu\nu}(x)$ we are lead to [43]

$$L_{\mu\nu} - \frac{1}{2}Lg_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu}$$

(25)

where all the quantities in (25) are functions of $(x, y), y = y(x)$. The energy-momentum tensor for the signature $(+, -, -, -)$ is defined to be

$$T_{\mu\nu} = -Pg_{\mu\nu} + (\mu + P)y_\mu y_\nu$$

(26)

where $P$ is the pressure and $\mu$ is the energy density of an ideal cosmic fluid.

In order to investigate the FRW cosmology, we set the Riemannian metric in (24) to be the Robertson-Walker one

$$a_{\mu\nu} = \text{diag}(1, -\frac{R^2(t)}{1-kr^2}, -R^2(t)r^2, -R^2(t)r^2 \sin^2 \theta)$$

(27)

where $t$ is the cosmic proper time $r, \theta, \phi$, the comoving spherical coordinates, $k = 0, \pm 1$ and a $R(t)$ the scale factor of the expanding volume. The new metric function

$$F(x, y) = \sigma(x, y) + \beta B_i y^i, \quad \beta = \frac{1}{16\pi G},$$

(28)

$$\sigma(x, y) = \sqrt{a_{\mu\nu}(x)y^\mu y^\nu}$$

(29)

If we fix the direction $y = \dot{x}$ then $\sigma(x, \dot{x}) = 1$. The vector field $B_i$ stands for a weak dark energy vector field $|B_i| \ll 1$, incorporated to the geometry of space-time as an intrinsic characteristic. This field would most naturally be expected to point in the same direction with the tangent vectors of the fluid flow lines [44]. As a result it will have only a time like component which can be expressed as a function of the proper time $B_i = (B_0, 0, 0, 0)$.

We can approximate the 0-component of the dark energy vector field $B_i$ at first order of Taylor type approximation [39, 40]:

$$B(t) = B(t_0) + \dot{B}(t_0 - t)$$

(30)

Since all of the other components of the dark energy vector field vanish, only the diagonal elements of the metric and the Ricci tensor survive. Under the assumption of a weak
Lorentz violation (LV) as in [39, 41] we can restrict \( \dot{B}(t_0) \) to be small enough (\( \dot{B}(t_0) \to 0 \)) considering an almost constant value of the field.

In virtue of the metric \( g_{\mu\nu}(x) = f_{\mu\nu}(x, y(x)) \) [42], we are able to calculate the Christoffel symbols and the curvature (for details see [39, 40]). The Ricci tensors \( L_{\mu\nu} \) can be approximated for (\( \dot{B}(t_0) \to 0 \)) and this implies the following components:

\[
\begin{align*}
L_{00} &= 3(\ddot{R}/R + 3/4\dot{R}/R\dot{B}_0) \\
L_{11} &= -(\ddot{R}R + 2\dddot{R} + 2k + 11/4\dot{R}\dot{B}_0)/(1 - kr^2) \\
L_{22} &= -(\ddot{R}R + 2\dddot{R} + 2k + 11/4\dot{R}\dot{B}_0)/r^2 \\
L_{33} &= -(\ddot{R}R + 2\dddot{R} + 2k + 11/4\dot{R}\dot{B}_0)/r^2 \sin^2 \theta
\end{align*}
\] (31)

The substitution of (26) to the field equations (25) implies the following equations at the weak field limit

\[
\ddot{R}R + 3/4\dot{R}R\dot{B}_0 = -4\pi G/3(\mu + 3P)
\] (32)

\[
\ddot{R}R + 2\dddot{R} + 2k/R^2 + 11/4\dot{R}R\dot{B}_0 = 4\pi G(\mu - P)
\] (33)

after subtracting (32) from (33) we obtain the Friedman-like equation

\[
\left(\frac{\dot{R}}{R}\right)^2 + \frac{\dot{R}}{R}\dot{B}_0 = \frac{8\pi G}{3}\mu - \frac{k}{R^2}
\] (34)

The previous equation is similar to the one derived from the Robertson-Walker metric in the Riemannian framework, apart from the extra term \( \dot{R}/R\dot{B}_0 \). We associate this extra term to the present anisotropic Universe’s dark energy. The \( \dot{B}_i \) vector field reflects a preferred direction in every tangent space and mimic’s possible LV [39, 41].

4 The Dark Energy Perturbation of Gravity Wave

As an extension of the theory of gravitational waves described by General Relativity, we introduce a Finslerian metric, representing the Finslerian perturbation of Riemannian metric [33–35]

\[
f_{\mu\nu}(x, y) = g_{\mu\nu}(x) + \varepsilon \theta_{\mu\nu}(x, y), \quad |\varepsilon| \ll 1
\] (35)

where \( g_{\mu\nu} \) is the Riemannian metric tensor and \( \theta_{\mu\nu}(x, y) \) is the Finslerian perturbation to the Riemannian metric tensor. Metric tensor (35) can be called a post Riemannian metric tensor [36].

Here, the Finslerian perturbation of Riemannian metric represents the dark energy perturbation of the gravity wave. This observation invites us to consider a Finslerian manifold, whose metric function contains two massless dark energy fields with 4-potential vectors, \( B^{(1)}_i \) and \( B^{(2)}_i \), in the following form:

\[
F(x, V) = \sqrt{g_{ij}(x)V^iV^j} + \beta(B^{(1)}_i + B^{(2)}_i)V^i + \phi A(x, V),
\] (36)

where \( V^i = dx^i/ds \) is a 4-velocity of a particle, \( \beta \) is a constant, \( \phi = \lambda B^{(1)}(t)B^{(2)}(t) \) is the interaction term of two dark energy fields, \( \lambda \) is a constant, and \( A(x, V) \) is an homogeneous function of 1st degree, assumed to be scalar in the Finslerian manifold [25, 27].

The last term of (36) corresponds to the gravitational field induced by the interaction between the dark energy fields. It contains the information of the gravitational field caused
by the interaction of the dark energy field. This gravitational field affects the motion of every physical object in space-time.

There are regions in space-time similar to those discussed in [27]. There is a region U on the manifold “bent” by the gravitational fluctuations between the dark energy vector fields. This additional curvature causes other gravitational effects like: test particle-region U, dark energy fields-region U and dark energy fields-test particle-region U effects.

The gauge transformation that relates a pair of local dark energy vector fields, whose regions of definition in space-time are typically different, is given as follows

\[ B_i^{(2)}(x) = \Lambda(x, V)^{-1} B_i^{(1)}(x) \Lambda(x, V) + \Lambda(x, V)^{-1} \partial^i \Lambda(x, V) \]  

(37)

\( \partial^i \) denotes the partial differentiation with respect to \( V^i \). Applying (5) to (36) we obtain the following metric tensor for this field:

\[ f_{ij} = g_{ij} + \varepsilon h_{ij}, \quad |\varepsilon| \ll 1, \]  

(38)

where \( g_{\mu\nu} \) is the Riemannian metric tensor corresponding to the gravitational wave perturbation (\( |g_{\mu\nu}| \ll 1 \)) to Minkowski metric \( n_{\mu\nu} = \text{diag}(-1, -1, -1, +1) \) [33–35], and \( h_{\mu\nu} \) describes the interactions between the gravity wave and dark energy, and dark energy to itself.

\[ h_{\mu\nu} = \frac{2\beta}{\sigma} V^i g_i(B_j) + \beta^2 B_i B_j + \frac{\beta}{\sigma} V^i B_i r_{ij} \]
\[ + \frac{2\phi}{\sigma} x^i g_i(\theta_j) \Lambda + 2\phi^2 \beta (\theta_j) \Lambda + \frac{\phi \Lambda}{\sigma} h_{ij} \]
\[ + (\sigma \phi + \beta \phi V^i B_i) \partial^2 \Lambda + \phi^2 \lambda_{ij} \]  

(39)

where

\[ \beta = \frac{1}{16\pi G}, \quad \sigma = \sqrt{g_{\kappa\lambda} x^k x^\lambda}, \]
\[ r_{\mu\nu} = g_{\mu\nu} - \sigma^{-1} g_{\kappa\mu} g_{\lambda\nu} x^k x^\lambda, \quad a_{(ij)} = \frac{1}{2} (a_{ij} + a_{ji}), \]

and \( B_i = B_i^{(1)} + B_i^{(2)} \) and \( \lambda_{ij} = \frac{1}{2} \partial^2 \Lambda \).

5 Conclusion

The equation of deviations of geodesics in Finsler space allows incorporation of dark energy in the geometry of space. We demonstrate that the dark energy would result naturally as a geometric effect and it would not be necessary for us to impose it in addition. In a way the geometric unification between gravity and dark energy massless vector field is achieved. We also find that, whenever the dark energy component \( h_{\mu\nu} \) is present in space-time, the isotropy of space breaks down. In the framework of Finsler space, we can also predict a dark energy perturbation of gravitational waves.

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