A MINIMAL VALUE PROBLEM AND THE PRESCRIBED $\sigma_2$ CURVATURE MEASURE PROBLEM

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Abstract. In this paper, we consider a minimal value problem and obtain an algebraic inequality. As an application, we prove the $C^2$ a priori estimate for a class of prescribed $\sigma_2$ curvature measure equations, which generalizes the results of the $\sigma_2$ case in Guan-Li-Li\cite{1} by a different method.

1. Introduction

In this paper, we consider the minimal value problem of the following polynomial,

\begin{equation}
  f(x_1, \cdots, x_N) = -b \sum_{i=1}^{N} x_i - \sum_{1 \leq i < j \leq N} x_i x_j,
\end{equation}

with

\begin{equation}
  \sum_{i=1}^{N} a_i x_i + C = 0,
\end{equation}

where $N \geq 2$ is an integer, $b, C, a_i$ are constants, and $a = (a_1, a_2, \cdots, a_N)$ satisfies the following two conditions

\begin{equation}
  \sum_{j=1}^{N} a_j \neq 0,
\end{equation}

\begin{equation}
  \left( \sum_{j=1}^{N} a_j \right)^2 - (N - 1) \sum_{j=1}^{N} a_j^2 > 0.
\end{equation}

Lemma 1.1. Under the above assumptions, we have

\begin{equation}
  f(x_1, \cdots, x_N) \geq \frac{b^2 [\left( \sum_{j=1}^{N} a_j \right)^2 - N \sum_{j=1}^{N} a_j^2] + 2bC \sum_{j=1}^{N} a_j - (N - 1)C^2}{2[\left( \sum_{j=1}^{N} a_j \right)^2 - (N - 1) \sum_{j=1}^{N} a_j^2]}.
\end{equation}

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Now, we recall the definition and some basic properties of elementary symmetric functions. For any \( k = 1, 2, \cdots, n \), we set
\[
\sigma_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}, \quad \text{for any} \quad \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n.
\]
We also set \( \sigma_0 = 1 \) and \( \sigma_k = 0 \) for \( k > n \).

In some sense, Lemma 1.1 is corresponding to the convexity of \( \sigma_2 \) operator (see Remark 1.2).

As an application, we consider the prescribed curvature measures problem, which is an important issue in convex geometry. There is a vast literature devoted to the study of this type of problems (see [3, 2, 1] and references therein).

Let \( X : M \to \mathbb{R}^{n+1} \) be a closed star-shaped hypersurface in \( \mathbb{R}^{n+1} \). The corresponding prescribed \( \sigma_k \) curvature measure equation is
\[
\sigma_k(\lambda(h_{ij})) = |X|^{-(n+1)}g\left( \frac{X}{|X|} \right) \langle X, N \rangle,
\]
where \( N \) is the unit outer normal of \( M \), \( g \in C^2(S^n) \) is a positive function and \( h_{ij} \) is the second fundamental form of \( M \). Recall that the Garding’s cone is defined as
\[
\Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_i(\lambda) > 0, \forall 1 \leq i \leq k \}.
\]
\( M \) is called \( k \)-convex if its principal curvature vector \( \lambda(h_{ij}) \in \Gamma_k \), and the corresponding solution to (1.6) is a \( k \)-admissible solution.

The existence theorem for \( \sigma_k \) curvature measure problems (\( 1 \leq k \leq n \)) has been solved. As the case \( k = n \) is the Alexandrov problem which was completely settled. Recently, Guan-Lin-Ma [2] proved the existence theorem to the convex solution case for \( 1 \leq k < n \) and the admissible solution for \( k = 1 \), and Guan-Li-Li [1] finished the admissible solution case for \( 1 < k < n \) by establishing the key \( C^2 \) a priori estimates.

The key \( C^2 \) a priori estimates in [1] is as follows.

**Theorem 1.2.** If \( M \) satisfies equation (1.6) and \( 2 \leq k \leq n \), then there exists a constant \( C \) depending only on \( n, k, \min_S g, ||g||_{C^1}, \text{ and } ||g||_{C^2} \), such that
\[
(1.7) \quad \max_M \sigma_1 \leq C.
\]

And Guan-Li-Li [1] get the following existence theorem of the admissible solution to (1.6).

**Theorem 1.3.** Let \( n \geq 2 \) and \( 1 \leq k \leq n \). Suppose \( g \in C^2(S^n) \) and \( g > 0 \). Then there exists a unique \( k \)-convex star-shaped hypersurface \( M \in C^3, \forall \alpha \in (0,1) \) such that it satisfies (1.6).

In [1], Guan-Li-Li raised a question regarding global \( C^2 \) estimates for general curvature equations. That is, suppose \( M \subset \mathbb{R}^{n+1} \) is a compact smooth hypersurface satisfying equation
\[
(1.8) \quad \sigma_k(\kappa(X)) = g(X, N), \kappa(X) \in \Gamma_k, \forall X \in M,
\]
where \( g \in C^2(\mathbb{R}^{n+1} \times S^n) \) is a general positive function and \( k \geq 2 \). Suppose there is a priori \( C^1 \) bound of \( M \), can one conclude a \( C^2 \) a priori bound of \( M \) in terms of \( C^1 \) norm of \( M, g, n, k \)?
In this paper, we give an affirmative answer to the following equations,
\begin{equation}
\sigma_2(h_{ij}) = \varphi(X) \langle X, N \rangle^\alpha.
\end{equation}
where $\varphi \in C^2(\mathbb{R}^{n+1})$ and $\alpha \in (-\infty, 1 + \delta)$ with $\delta = \inf_M \frac{(X, N)^2}{|X|^2}$. And we get the following $C^2$ a
priori estimate using Lemma 1.1.

**Theorem 1.4.** Let $n \geq 2$ and suppose $\varphi \in C^2(\mathbb{R}^{n+1})$ with $\varphi > 0$. If $M$ is 2-convex and satisfies
equation (1.9) with the $C^1$ a priori estimate holding, then for $\alpha \in (-\infty, 1 + \delta)$ with $\delta = \inf_M \frac{(X, N)^2}{|X|^2}$,
there is a constant $C$ depending only on $n, \alpha, \|\varphi\|_{C^2}, \|\varphi\|_{C^0}$ and the $C^1$ norm of $M$, such that
\begin{equation}
\sup_{\|\xi\|=1} n \sum_{i,j=1}^n h_{ij} \xi_i \xi_j \leq C.
\end{equation}
When $\alpha = 1$ and $\varphi(X) = |X|^{-(n+1)}g(|X|)$, the $C^1$ a priori estimate to (1.9) was proved in [2].
So the $\alpha = 1$ case of Theorem 1.4 is corresponding to Theorem 1.2.

The rest of this paper is organized as follows. In Section 2, we prove Lemma 1.1 by two methods.
In Section 3, we prove Theorem 1.4 using Lemma 1.1. And in Section 4, we give some remarks.

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2. Proof of Lemma 1.1

In this section, we will prove lemma 1.1 with two methods. One is the eigenvector decomposition
method, and the other is the Lagrange method of multipliers. They are all very useful, and may
be used for other problems.

**Method 1. the eigenvector decomposition method.** We will consider the eigenvalues and
eigenvectors of the quadratic terms of $f$.

By direct computations, we can know that the matrix
\begin{equation}
\frac{1}{2} \begin{pmatrix}
0 & -1 & \cdots & -1 \\
-1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & -1 \\
-1 & \cdots & -1 & 0
\end{pmatrix}
\end{equation}
has eigenvalues $-\frac{1}{2}(N-1)$ with a eigenvector $e_N = \frac{1}{\sqrt{N}}(1, \cdots, 1)$, and $\frac{1}{2}$ with eigenvectors
$e_1, \cdots, e_{N-1}$. In particular, we can assume that $\{e_1, \cdots, e_N\}$ is an orthonormal basis of $\mathbb{R}^N$
and
\begin{equation}
a = (a_1, \cdots, a_N) = (\mathbf{a}, e_N) e_N + de_1 = \frac{\sum_{i=1}^N a_i}{\sqrt{N}} e_N + de_1.
\end{equation}
So
\begin{equation}x = (x_1, \cdots, x_N) = \sum_{i=1}^N (x, e_i) e_i,
\end{equation}
and

\begin{equation}
(2.4)
    d^2 = \sum_{i=1}^{N} a_i^2 - \frac{(\sum_{i=1}^{N} a_i)^2}{N}.
\end{equation}

When \( d = 0 \), that is \( a_1 = a_2 = \cdots = a_N \neq 0 \), we can get

\begin{equation}
0 = \sum_{i=1}^{N} a_i x_i + C = \langle x, a \rangle + C = \sqrt{Na_1} \langle x, e_N \rangle + C,
\end{equation}

so

\begin{equation}
(2.5)
    \langle x, e_N \rangle = \frac{-C}{\sqrt{Na_1}}.
\end{equation}

Hence

\begin{equation}
(2.6)
    f(x_1, \cdots, x_N) = -b \sum_{i=1}^{N} x_i - \sum_{1 \leq i < j \leq N} x_i x_j
    \geq -\sqrt{N}b \langle x, e_N \rangle + \frac{1}{2} \sum_{i=1}^{N-1} \langle x, e_i \rangle^2 - \frac{1}{2} (N-1) \langle x, e_N \rangle^2
    \geq \frac{bC}{a_1} - \frac{1}{2} (N-1) \frac{C^2}{Na_1^2}.
\end{equation}

Hence \((1.5)\) holds in this case.

When \( d \neq 0 \), we can get

\begin{equation}
0 = \sum_{i=1}^{N} a_i x_i + C = \langle x, a \rangle + C
    = \frac{\sum_{i=1}^{N} a_i}{\sqrt{N}} \langle x, e_N \rangle + d \langle x, e_1 \rangle + C,
\end{equation}

so

\begin{equation}
(2.8)
    \langle x, e_1 \rangle = \frac{-\sum_{i=1}^{N} a_i}{\sqrt{Nd}} \langle x, e_N \rangle - \frac{C}{d}.
\end{equation}
Then we can get
\[
f(x_1, \cdots, x_N) = -b \sum_{i=1}^{N} x_i - \sum_{1 \leq i < j \leq N} x_i x_j
\]
\[
= -\sqrt{N} b \langle x, e_N \rangle + \frac{1}{2} \sum_{i=1}^{N-1} \langle x, e_i \rangle^2 - \frac{1}{2} (N-1) \langle x, e_N \rangle^2
\]
\[
= \frac{1}{2} \left[ \frac{\sum_{i=1}^{N} a_i^2 - (N-1)Nd^2}{Nd^2} \langle x, e_N \rangle^2 + 2 \frac{\sum_{i=1}^{N} a_i C - Nbd^2}{\sqrt{Nd^2}} \langle x, e_N \rangle + \frac{C^2}{d^2} \right]
\]
\[
+ \frac{1}{2} \sum_{i=2}^{N-1} \langle x, e_i \rangle^2.
\]
By (1.4),
\[
\sum_{i=1}^{N} a_i^2 - (N-1)Nd^2 = N \left[ \sum_{j=1}^{N} a_j^2 - (N-1) \sum_{j=1}^{N} a_j^2 \right] > 0,
\]
so \( f \) has a minimal value. And
\[
f(x_1, \cdots, x_N) = \frac{1}{2} \left[ \frac{\sum_{i=1}^{N} a_i^2 - (N-1)Nd^2}{Nd^2} \langle x, e_N \rangle + \frac{\sqrt{N}(\sum_{i=1}^{N} a_i C - Nbd^2)}{(\sum_{i=1}^{N} a_i^2 - (N-1)Nd^2)} \right]^2
\]
\[
- \frac{\sum_{i=1}^{N} a_i C - bNd^2}{2d^2[\sum_{i=1}^{N} a_i^2 - (N-1)Nd^2]} + \frac{C^2}{2d^2} + \frac{1}{2} \sum_{i=2}^{N-1} \langle x, e_i \rangle^2
\]
\[
\geq - \frac{\sum_{i=1}^{N} a_i C - bNd^2}{2d^2[\sum_{i=1}^{N} a_i^2 - (N-1)Nd^2]} + \frac{C^2}{2d^2}
\]
\[
= \frac{b^2 \sum_{j=1}^{N} a_j^2 - N \sum_{j=1}^{N} a_j^2 + 2bC \sum_{j=1}^{N} a_j - (N-1)C^2}{2[\sum_{j=1}^{N} a_j^2 - (N-1) \sum_{j=1}^{N} a_j^2]}.
\]
So Lemma 1.1 holds.

**Method 2. the Lagrange method of multipliers.** Following the same argument (only consider the quadratic terms of \( \langle x, e_i \rangle \)), we can know \( f \) has a minimal value by (2.11). Now we
consider
\begin{equation}
(2.12) \quad f(x_1, \cdots, x_N, \mu) = -b \sum_{i=1}^{N} x_i - \sum_{1 \leq i < j \leq N} x_i x_j + \mu [\sum_{i=1}^{N} a_i x_i + C],
\end{equation}
so
\begin{equation}
(2.13) \quad 0 = f_x = -b - \sum_{j \neq i} x_j + \mu a_i, \quad \forall \quad i = 1, 2, \cdots, N.
\end{equation}
\begin{equation}
(2.14) \quad 0 = f_\mu = \sum_{i=1}^{N} a_i x_i + C.
\end{equation}

By direct computations, we can get the minimal point,
\begin{equation}
(2.15) \quad \mu^0 = \frac{b \sum_{j=1}^{N} a_j - (N - 1)C}{[\sum_{j=1}^{N} a_j]^2 - (N - 1) \sum_{j=1}^{N} a_j^2},
\end{equation}
\begin{equation}
(2.16) \quad x_i^0 = -\frac{1}{N-1}b + \frac{\mu^0}{N-1} \sum_{j=1}^{N} a_j - \mu^0 a_i, \quad \forall \quad i = 1, 2, \cdots, N.
\end{equation}
So the minimum is
\begin{equation}
(2.17) \quad f(x_1^0, \cdots, x_n^0, \mu^0) = -b \sum_{i=1}^{N} x_i^0 - \sum_{1 \leq i < j \leq N} x_i^0 x_j^0 = -b \sum_{i=1}^{N} x_i^0 - \frac{1}{2} \sum_{i=1}^{N} x_i^0 \sum_{j \neq i} x_j^0 = -b \sum_{i=1}^{N} x_i^0 - \frac{1}{2} \sum_{i=1}^{N} x_i^0 [-b + \mu^0 a_i] = -b \frac{N}{2} x_i^0 - \mu^0 \sum_{i=1}^{N} a_i x_i^0 = -\frac{b}{2} \frac{N}{N-1} b + \frac{\mu^0}{N-1} \sum_{j=1}^{N} a_j + \mu^0 C = b^2[\sum_{j=1}^{N} a_j]^2 - N \sum_{j=1}^{N} a_j^2 + 2bC \sum_{j=1}^{N} a_j - (N - 1)C^2.
\end{equation}

Now we finish the proof of Lemma [1.1]

3. Proof of Theorem [1.4]

First we collect some well-known properties of elementary symmetric functions, which will be used in the proof of Theorem [1.4]
Proposition 3.1. Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and $k = 0, 1, \ldots, n$, then

\begin{equation}
\sigma_k(\lambda) = \sigma_k(\lambda|i) + \lambda_i \sigma_{k-1}(\lambda|i), \quad \forall 1 \leq i \leq n,
\end{equation}

\begin{equation}
\sum_{i}^{n} \lambda_i \sigma_{k-1}(\lambda|i) = k \sigma_k(\lambda),
\end{equation}

\begin{equation}
\sum_{i}^{n} \sigma_k(\lambda|i) = (n - k) \sigma_k(\lambda).
\end{equation}

And we also have,

Proposition 3.2. Suppose $W = (W_{ij})$ is diagonal, and $m$ ($1 \leq m \leq n$) is positive integer, then

\begin{equation}
\frac{\partial \sigma_m(W)}{\partial W_{ij}} = \begin{cases} 
\sigma_{m-1}(W|i), & \text{if } i = j, \\
0, & \text{if } i \neq j.
\end{cases}
\end{equation}

and

\begin{equation}
\frac{\partial^2 \sigma_m(W)}{\partial W_{ij} \partial W_{kl}} = \begin{cases} 
\sigma_{m-2}(W|ik), & \text{if } i = j, k = l, i \neq k, \\
-\sigma_{m-2}(W|ik), & \text{if } i = l, j = k, i \neq j, \\
0, & \text{otherwise}.
\end{cases}
\end{equation}

Now, we recall some relevant geometric quantities of a smooth closed hypersurface $M \subset \mathbb{R}^{n+1}$.

Throughout the paper, repeated indices denote summation and we assume the origin is inside the body enclosed by $M$.

Let $M^n$ be an immersed hypersurface in $\mathbb{R}^{n+1}$. For $X \in M \subset \mathbb{R}^{n+1}$, choose local normal coordinates in $\mathbb{R}^{n+1}$, such that $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$ are tangent to $M$ and $\partial_{n+1}$ is the unit outer normal of the hypersurface. We sometimes denote $\partial_i = \frac{\partial}{\partial x_i}$, and also use $N$ to denote the unit outer normal $\partial_{n+1}$. We use lower indices to denote covariant derivatives with respect to the induced metric.

For the immersion $X$, the second fundamental form is the symmetric $(2, 0)$-tensor given by the matrix $\{h_{ij}\}$,

\begin{equation}
h_{ij} = \langle \partial_i X, \partial_j N \rangle.
\end{equation}

Recall the following identities:

\begin{equation}
X_{ij} = -h_{ij} N \quad \text{(Gauss formula)},
\end{equation}

\begin{equation}
N_i = h_{ij} \partial_j \quad \text{(Weingarten equation)},
\end{equation}

\begin{equation}
h_{ijk} = h_{ikj} \quad \text{(Codazzi formula)},
\end{equation}

\begin{equation}
R_{ijkl} = h_{ikl} h_{j} - h_{il} h_{jk} \quad \text{(Gauss equation)},
\end{equation}

where $R_{ijkl}$ is the $(4, 0)$-Riemannian curvature tensor. We also have

\begin{equation}
h_{ijkl} = h_{klij} + (h_{mj} h_{il} - h_{mi} h_{lj}) h_{mk} + (h_{mj} h_{kl} - h_{mi} h_{kj}) h_{mk}.
\end{equation}

We will establish the curvature estimates to equation (1.9). Let

\begin{equation}
\phi(X, \xi) = \log(h_{kl} \xi_k \xi_l)(X) - \log \langle X, N \rangle,
\end{equation}
where $\xi \in S^n$. Suppose that $\phi(X, \xi)$ attains its maximum at some $X_0 \in M$ and $\xi_0 \in S^n$. We may assume $\xi_0$ is $e_1$ and the other directions $e_2, \cdots, e_n$ can be chosen such that $\{e_1, e_2, \cdots, e_n\}$ is a local orthonormal frame near $X_0$ and $h_{ij}(X_0)$ is diagonal. Then the function
\begin{equation}
(3.13) \quad \phi(X) = \log h_{11} - \langle X, N \rangle.
\end{equation}
attains its maximum at $X_0 \in M$. All the following computations is at the point $X_0$, and we denote $\lambda_i = h_{ii}$. So we have
\begin{equation}
(3.14) \quad \phi_i = \frac{h_{1i1}}{h_{11}} - \frac{\langle X, N \rangle_i}{\langle X, N \rangle} = 0,
\end{equation}
so
\begin{equation}
(3.15) \quad h_{1i1} = \frac{\langle X, N \rangle_i}{\langle X, N \rangle} = h_{ii} \langle X, e_i \rangle.
\end{equation}
And
\begin{equation}
(3.16) \quad \phi_{ii} = \frac{h_{1i1}}{h_{11}} h_{11}^{2} \frac{\langle X, N \rangle_{ii}}{\langle X, N \rangle} + \frac{\langle X, N \rangle_{ii}^2}{\langle X, N \rangle^2} = \frac{h_{1i1}}{h_{11}} - \frac{\langle X, N \rangle_{ii}}{\langle X, N \rangle},
\end{equation}
hence
\begin{equation}
(3.17) \quad 0 \geq \sum_{i=1}^{n} F_{ii} \phi_{ii} = \frac{1}{h_{11}} \sum_{i=1}^{n} F_{ii} h_{1i1} - \frac{1}{\langle X, N \rangle} \sum_{i=1}^{n} F_{ii} \langle X, N \rangle_{ii}.
\end{equation}
By (3.11), we have
\begin{equation}
(3.18) \quad h_{1i1} = h_{ii1} + h_{ii}^2 h_{ii} - h_{11} h_{ii}^2,
\end{equation}
so we obtain
\begin{align}
\sum_{i=1}^{n} F_{ii} h_{1i1} &= \sum_{i=1}^{n} F_{ii} h_{ii1} + h_{ii1} \sum_{i=1}^{n} F_{ii} h_{ii} - h_{11} \sum_{i=1}^{n} F_{ii} h_{ii}^2 \\
&= \langle \phi \langle X, N \rangle^\alpha \rangle_{11} - \sum_{ijkl=1}^{n} F_{ijkl} h_{ij1} h_{kl1} + 2 h_{11}^2 \phi \langle X, N \rangle^\alpha - h_{11} \sum_{i=1}^{n} F_{ii} h_{ii}^2 \\
&= \varphi_{11} \langle X, N \rangle^\alpha + 2 \alpha \phi_{1} \langle X, N \rangle^{\alpha-1} \langle X, N \rangle_1 + \alpha (\alpha - 1) \phi \langle X, N \rangle^{\alpha-2} \langle X, N \rangle_1^2 \\
&\quad + \alpha \phi \langle X, N \rangle^{\alpha-1} [h_{111} \langle X, e_i \rangle + h_{11} - h_{111}^2 \langle X, N \rangle)] \\
&\quad - \sum_{ijkl=1}^{n} F_{ijkl} h_{ij1} h_{kl1} + 2 h_{11}^2 \phi \langle X, N \rangle^\alpha - h_{11} \sum_{i=1}^{n} F_{ii} h_{ii}^2.
\end{align}
And
\begin{align}
\sum_{i=1}^{n} F_{ii} \langle X, N \rangle_{ii} &= \sum_{i=1}^{n} F_{ii} [h_{ii1} \langle X, e_i \rangle + h_{ii} - h_{ii}^2 \langle X, N \rangle] \\
&= (\phi \langle X, N \rangle^\alpha)_1 \langle X, e_i \rangle + 2 \phi \langle X, N \rangle^\alpha - \langle X, N \rangle \sum_{i=1}^{n} F_{ii} h_{ii}^2 \\
&= \varphi_{1} \langle X, N \rangle^\alpha \langle X, e_i \rangle + \alpha \phi \langle X, N \rangle^{\alpha-1} \langle X, N \rangle_1 \langle X, e_i \rangle \\
&\quad + 2 \phi \langle X, N \rangle^\alpha - \langle X, N \rangle \sum_{i=1}^{n} F_{ii} h_{ii}^2.
\end{align}
By (3.14), (3.19) and (3.20),

\[
0 \geq \langle X, N \rangle \sum_{i=1}^{n} F_{i}^{i} h_{i11i} - h_{11} \sum_{i=1}^{n} F_{i}^{i} \langle X, N \rangle_{ii} \]

\[
= (2 - \alpha) \varphi \langle X, N \rangle^{1+\alpha} h_{11}^{2} + h_{11}^{2}[2\alpha \varphi_{1} \langle X, e_{1} \rangle - \varphi_{1} \langle X, e_{1} \rangle] - (2 - \alpha) \varphi \langle X, N \rangle^{\alpha}
\]

(3.21)

\[\varphi_{11} \langle X, N \rangle^{1+\alpha} + \alpha(\alpha - 1) \varphi \langle X, N \rangle^{\alpha-1} \langle X, e_{1} \rangle^{2} h_{11}^{2} - \langle X, N \rangle \sum_{i,j,k,l=1}^{n} F_{i}^{i} F_{j}^{j} F_{k}^{k} F_{l}^{l} h_{i1j1} h_{k1l}.\]

In the following, we will deal with the last term in (3.21), that is

\[
- \sum_{i,j,k,l=1}^{n} F_{i}^{i} F_{j}^{j} F_{k}^{k} F_{l}^{l} h_{i1j1} h_{k1l} = \sum_{i \neq j} h_{i1j1}^{2} - \sum_{i \neq j} h_{i1} h_{j1}
\]

(3.22)

\[
= \sum_{i \neq j} h_{i1j1}^{2} - 2h_{111} \sum_{i=2}^{n} h_{i1i} - 2 \sum_{2 \leq i < j \leq n} h_{i1i} h_{j1j}.
\]

Differentiating (1.9) once, we can get,

(3.23)

\[\sum_{i=1}^{n} \sigma_{1}(\lambda|i) h_{i1i} = (\varphi \langle X, N \rangle^{\alpha})_{i}.\]

Denote

(3.24)

\[f(x_{2}, \cdots, x_{n}) = -h_{111} \sum_{i=2}^{n} x_{i} - \sum_{2 \leq i < j \leq n} x_{i} x_{j},\]

and we consider the minimal value problem of \(f(x_{2}, \cdots, x_{n})\) under (3.24), that is

(3.25)

\[\sum_{i=2}^{n} \sigma_{1}(\lambda|i)x_{i} + [\sigma_{1}(\lambda|1) h_{111} - (\varphi \langle X, N \rangle^{\alpha})_{1}] = 0.\]

By direct computations, we get

(3.26)

\[\sum_{i=2}^{n} \sigma_{1}(\lambda|i) = (n - 2) \sigma_{1}(\lambda) + \lambda_{1} > 0,
\]

(3.27)

\[\sum_{i=2}^{n} \sigma_{1}(\lambda|i)^{2} - (n - 2) \sum_{i=2}^{n} \sigma_{1}(\lambda|i)^{2} = (n - 1) \lambda_{1}^{2} + 2(n - 2) \sigma_{2}(\lambda) > 0.
\]

So from lemma 1.1, we can get \(f(x_{2}, \cdots, x_{n})\) has a minimum

(3.28)

\[h_{111}^{2} \left[ \left( \sum_{i=2}^{n} \sigma_{1}(\lambda|i)^{2} - (n - 1) \sum_{i=2}^{n} \sigma_{1}(\lambda|i)^{2} \right) + 2h_{111} C \sum_{i=2}^{n} \sigma_{1}(\lambda|i) - (n - 2) C^{2} \right] / \left[ 2[(n - 1) \lambda_{1}^{2} + 2(n - 2) \sigma_{2}(\lambda)] \right],
\]

where

(3.29)

\[C = \sigma_{1}(\lambda|1) h_{111} - (\varphi \langle X, N \rangle^{\alpha})_{1} = \sigma_{1}(\lambda|1) h_{111} - \varphi_{1} \langle X, N \rangle^{\alpha} - \alpha \varphi \langle X, N \rangle^{\alpha-1} \langle X, N \rangle_{1} - \alpha \varphi \langle X, N \rangle^{\alpha-1} \langle X, N \rangle_{1} - \varphi_{1} \langle X, N \rangle^{\alpha}.
\]
So

\[
h_{111}^2 \left[ \sum_{i=2}^{n} \sigma_1 (\lambda|i)^2 - (n-1) \sum_{i=2}^{n} \sigma_1 (\lambda|i)^2 \right] + 2h_{111}C \sum_{i=2}^{n} \sigma_1 (\lambda|i) - (n-2)C^2
\]

\[
eq \left( \frac{h_{111}}{h_{11}} \right)^2 \left[ \sum_{i=2}^{n} \sigma_1 (\lambda|i)^2 - (n-1) \sum_{i=2}^{n} \sigma_1 (\lambda|i)^2 \right] \lambda_1^2
\]

\[
+ 2 \frac{h_{111}}{h_{11}} \left[ \sigma_1 (\lambda|1)^2 - \alpha \sigma_2 (\lambda) \frac{h_{111}}{h_{11}} - \varphi_1 (X, N) \right] \lambda_1 \sum_{i=2}^{n} \sigma_1 (\lambda|i)
\]

(3.30) \(- (n-2) \left[ (\sigma_1 (\lambda|1)^2 - \alpha \sigma_2 (\lambda)) \frac{h_{111}}{h_{11}} - \varphi_1 (X, N) \right]^2.
\]

By direct computations, the coefficient of \(\left( \frac{h_{111}}{h_{11}} \right)^2\) in (3.30) is

\[
\lambda_1^2 \left[ (\sum_{i=2}^{n} \sigma_1 (\lambda|i)^2 - (n-1) \sum_{i=2}^{n} \sigma_1 (\lambda|i)^2) + 2[\sigma_1 (\lambda|1)^2 - \alpha \sigma_2 (\lambda)] \lambda_1 \sum_{i=2}^{n} \sigma_1 (\lambda|i) - (n-2)\alpha^2 \sigma_2^2 (\lambda)\right]
\]

\[
- (n-2)[\sigma_1 (\lambda|1)^2 - \alpha \sigma_2 (\lambda)]^2
\]

\[
eq \lambda_1^2 \left[ (\sum_{i=1}^{n} \sigma_1 (\lambda|i)^2 - (n-1) \sum_{i=1}^{n} \sigma_1 (\lambda|i)^2) - 2(n-1)\alpha \sigma_2 (\lambda) \lambda_1^2 - (n-2)\alpha^2 \sigma_2^2 (\lambda)\right]
\]

\[
eq \lambda_1^2 \left[ (n-1)\sigma_1 (\lambda)^2 - (n-1)(n-1)\sigma_1 (\lambda)^2 - 2\sigma_2 (\lambda)\right]
\]

\[
- 2(n-1)\alpha \sigma_2 (\lambda) \lambda_1^2 - (n-2)\alpha^2 \sigma_2^2 (\lambda)
\]

\[
eq 2(1-\alpha)(n-1)\sigma_2 (\lambda) \lambda_1^2 - (n-2)\alpha^2 \sigma_2^2 (\lambda).
\]

So

\[
h_{111}^2 \left[ \sum_{i=2}^{n} \sigma_1 (\lambda|i)^2 - (n-1) \sum_{i=2}^{n} \sigma_1 (\lambda|i)^2 \right] + 2h_{111}C \sum_{i=2}^{n} \sigma_1 (\lambda|i) - (n-2)C^2
\]

\[
eq [2(1-\alpha)(n-1)\sigma_2 (\lambda) \lambda_1^2 - (n-2)\alpha^2 \sigma_2^2 (\lambda)] \frac{h_{111}}{h_{11}}^2
\]

\[
- 2 \frac{h_{111}}{h_{11}} [(n-1)\lambda_1^2 + (n-2)\alpha \sigma_2 (\lambda)] \varphi_1 (X, N) - (n-2)\varphi_1^2 (X, N)^2 \alpha
\]

\[
eq 2(1-\alpha)(n-1)\varphi (X, N)^{\alpha-2} (X, N) \lambda_1^2 + (n-2)\alpha^2 \varphi^2 (X, e_1)^2 \lambda_1^2
\]

\[
+ 2 \varphi_1 (X, e_1) [(n-1)\lambda_1^2 + (n-2)\lambda_1 \alpha \varphi (X, N)^{\alpha}] - (n-2)\varphi_1^2 (X, N)^{2\alpha}.
\]
Hence

\[-(X, N) \sum_{ijkl=1}^{n} F^{ijkl} h_{ijkl} h_{ijkl} \geq (X, N) \left[ -2h_{1111} \sum_{i=2}^{n} h_{ii} - 2 \sum_{2 \leq i < j \leq n} h_{ii} h_{jj} \right] \]

\[
\geq (X, N) \frac{n \sigma_1(\lambda i) - (n - 1) \sum_{i=2}^{n} \sigma_1(\lambda i)^2 + 2h_{1111} C \sum_{i=2}^{n} \sigma_1(\lambda i) - (n - 2)C^2}{(n - 1)\lambda_1^2 + 2(n - 2)\sigma_2(\lambda)}
\]

\[
\geq (1 - \alpha) \varphi (X, N)^{\alpha - 1} \varphi (X, e_1)^2 h_{1111}^2 \left[ 2 - \frac{4(n - 2)\sigma_2(\lambda)}{(n - 1)\lambda_1^2 + 2(n - 2)\sigma_2(\lambda)} \right] - C_1h_{1111} - C_2
\]

\[
= 2(1 - \alpha) \varphi (X, N)^{\alpha - 1} \varphi (X, e_1)^2 h_{1111}^2 - C_1h_{1111} - C_3.
\]

From (3.21), we can get

\[
0 \geq (2 - \alpha) \varphi (X, N)^{1+\alpha} h_{1111}^2 - Ch_{1111} - C
\]

\[
+ \alpha(\alpha - 1) \varphi (X, N)^{\alpha - 1} \varphi (X, e_1)^2 h_{1111}^2 - (X, N) \sum_{ijkl=1}^{n} F^{ijkl} h_{ijkl} h_{ijkl}
\]

(3.31)

\[
\geq (2 - \alpha) \varphi (X, N)^{\alpha - 1} \varphi (X, e_1)^2 h_{1111}^2 (X, N)^2 + (1 - \alpha) \langle X, e_1 \rangle^2 - Ch_{1111} - C.
\]

When \(\alpha \in (-\infty, 1]\), we can get

\[
0 \geq (2 - \alpha) \varphi (X, N)^{\alpha + 1} h_{1111}^2 - Ch_{1111} - C,
\]

so Theorem 1.4 holds.

When \(\alpha \in (1, 1 + \delta]\) with \(\delta = \inf_M \frac{(X, N)^2}{|X|^2}\), we can get

(3.32)

\[
(X, N)^2 + (1 - \alpha) \langle X, e_1 \rangle^2 \geq (X, N)^2 + (1 - \alpha)|X|^2
\]

\[
= (X, N)^2 - \delta|X|^2 + (1 + \delta - \alpha)|X|^2
\]

\[
\geq (1 + \delta - \alpha)|X|^2,
\]

so we obtain

(3.34)

\[
0 \geq (2 - \alpha)(1 + \delta - \alpha) \varphi (X, N)^{\alpha - 1} |X|^2 h_{1111}^2 - Ch_{1111} - C.
\]

So Theorem 1.4 holds.

### 4. Some Remarks

**Remark 4.1.** For the minimal value problem (1.1), (1.2), there is a unique minimal point such that "=" holds in (1.5). So (1.5) is optimal. And we can get the unique minimal point from the proof.

**Remark 4.2.** The minimal value problem (1.1), (1.2) (with \(a_i = \sigma_1(\lambda i), i = 2, \cdots, n\)) is corresponding to the convexity of \(\sigma_2\). To be precise, the corresponding minimal value problems is

(4.1)

\[
f^2(x_2, \cdots, x_n) = -b \sum_{i=2}^{n} x_i - \sum_{2 \leq i < j \leq n} x_i x_j,
\]
with
\[ (4.2) \sum_{i=2}^{n} \sigma_1(\lambda |i)x_i + [\sigma_1(\lambda |1)b + C] = 0. \]

And we can get
\[ (4.3) f^2(x_2, \cdots, x_n) \geq \frac{2(n-1)\sigma_2(\lambda)\beta^2 + 2(n-1)\lambda_1 bC - (n-2)C^2}{2[(n-1)\lambda_1^2 + 2(n-2)\sigma_2(\lambda)]}. \]

For general \( \sigma_k \) \((k > 2)\), we guess there are similar minimal value problems, that is
\[ (4.4) f^k(x_2, \cdots, x_n) = -b \sum_{i=2}^{n} \sigma_{k-2}(\lambda |1i)x_i - \sum_{2 \leq i < j \leq n} \sigma_{k-2}(\lambda |ij)x_ix_j, \]

with
\[ (4.5) \sum_{i=2}^{n} \sigma_{k-1}(\lambda |i)x_i + [\sigma_{k-1}(\lambda |1)b + C] = 0. \]

But we cannot get the minimum by the methods in Section 2. There are some techniques which cannot be used here.

Remark 4.3. For a general prescribed \( \sigma_2 \) curvature measure equation
\[ (4.6) \sigma_2(h_{ij}) = \varphi(X) \langle X, N \rangle^{\alpha}, \]
where \( \alpha \in (-\infty, 1 + \delta) \), the \( C^1 \) a priori estimate was proved in \([2]\) when \( \alpha = 1 \). In fact, the proof of \( C^1 \) a priori estimate in \([2]\) holds for \( \alpha \in (0, 1] \) with a small modification. For general \( \alpha \), I don’t know the corresponding results because of my limitation of knowledge.

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