Some Results from Arbitrage Opportunity on Nonlinear Wealth Processes*

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In this paper, we mainly concern about nonlinear BSDE which is often used to describe a case of constraint on the wealth of an investor. Unlike the linear case, we can show that under a certain situation, both buyer and seller can create arbitrage opportunities in the derivative market. We utilize a relation between BSDE and PDE in order to obtain the bounds of a solution. As a result, we succeed in establishing a sufficient condition which guarantees the existence of the arbitrage opportunities, and the limitation of the arbitrages as well. In addition, we are able to extend the results to a more general class of models and accomplish in strengthening the comonotonic theorem for BSDEs. Furthermore, by applying the results, we obtain a sufficient condition that ensures the additivity of $g$-expectation even when a generator of BSDE is nonlinear.

1. Introduction

Recently, many authors have concerned with the pricing of contingent claims with constraints on a wealth or portfolio process. In this case, the dynamics of wealth processes are given by a nonlinear Backward Stochastic Differential Equation (BSDE), which models more realistic situations in financial market than the linear ones. For example, (i) different interest rates for borrowing and lending, (ii) short-sales constrained situation, (iii) a large investor model, whose dynamic trading affects on prices of risky assets, (iv) derivative securities pricing with collateralization. Unlike the linear case, under some situations, both buyer and the writer can create arbitrage opportunities in the derivative market, by employing the certain investment strategies. The main objective is to construct them concretely using two BSDEs, which respectively describe seller and buyer’s hedging strategies, and provide a sufficient condition to ensure the existence of the arbitrage opportunities, which is explicitly described in terms of the drivers of BSDEs. Moreover, we will show that those arbitrages have the limitations which are dependent on the volume of the derivatives.

In this study, we focus on the system of forward stochastic differential equation

$$dP_s = \mu(s,P_s)ds + \sigma(s,P_s)dw_s, \quad P_0 = x,$$

and its associated BSDE

$$-dY_s = f(s,Y_s,Z_s)ds - Z_s^T dw_s, \quad Y_T = \Phi(P_T).$$

The above system effectively describes the relation between behavior of stock prices and a replicating portfolio composed of these stocks against the possible claim. In particular, in Markovian case, its solution can be written as a function of time and state process. According to the study of El Karoui, Peng and Quenez [3], under smoothness assumption on the coefficients, this function solves a certain quasi-linear Partial Differential Equation (PDE).

As a result, we succeed in establishing a lower bound and an upper bound of the solution of BSDE, and this lead us to our goal, a sufficient condition that guarantees the existence of arbitrage opportunities for both writer and buyer as well as the limitation of the arbitrage. Furthermore, we are also able to extend our results to a more general class of model and finally accomplish in strengthening the comonotonic theorem for BSDEs, the theorem that allows us to explore the properties of $Z$, the second part of a solution of BSDE. In addition, by applying the results, we obtain a sufficient condition that ensures the additivity of $g$-expectation, a pricing mechanism of financial derivatives, even when a generator of BSDE is nonlinear.

An outline of the paper is as follows. Survey of the main results from [1] are given in section 3. In section 4, we provide the generalized results and its applications to comonotonic theorem as well as $g$-expectation.

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2. Nonlinear Wealth Processes

We begin with specifying the notations. Let $T > 0$ be a constant, which represents a time horizon. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an $\mathbb{R}^n$-valued Brownian motion $w$, we consider

- $L^2_T(\mathbb{R}^d)$, the space of all $\mathcal{F}_T$-measurable random variables $\xi: \Omega \mapsto \mathbb{R}^d$ satisfying $\mathbb{E}[|\xi|^2] < \infty$.
- $\mathbb{H}_T^r(\mathbb{R}^d)$, the space of all predictable processes $\phi: \Omega \times [0,T] \mapsto \mathbb{R}^d$ such that $\mathbb{E}\int_0^T |\phi|^2 \, dt < \infty$.

For notation simplicity, we sometimes use $L^2_T(\mathbb{R}^d) = L^2_T$ and $\mathbb{H}_T^r(\mathbb{R}^d) = \mathbb{H}_T^r$.

2.1 The Model

Consider an agent who invests in the market contains $n+1$ assets. We introduce the following system

$$dP^0_t = R(t,X_t,\pi_t)P^0_t \, dt, \quad P^0_0 = 1,$$
$$dP^1_t = \mu^1_t dt + \sum_{i=1}^n \sigma^1_{i,t} dW^i_t, \quad P^0_0 = p^i, \quad 1 \leq i \leq n,$$
$$dX_t = \sum_{i=1}^n \delta^i_p dP^0_t + (X_t - \bar{1} \pi_t) R(t,X_t,\pi_t) dt, \quad X_0 = x_0. \quad (1)$$

Here $1 := (1, \ldots, 1)^\top \in \mathbb{R}^n$, and $\bar{1}^\top$ is a transpose of $A$. The process $P^0$ is interpreted as the price process of a riskless asset, the process $P^1$ is the price process of the risky asset (stock), the process $X$ is the wealth process of this (self-financing) investor whose initial wealth is $x_0$ and the dynamic (money-based) portfolio process $\pi = (\pi^1, \ldots, \pi^n)^\top$. The market coefficients are given as follows: $\mu = (\mu^1, \ldots, \mu^n)^\top$ is the stock return vector, $\sigma = \{\sigma^i\}^{1 \leq i,j \leq n}$ assumed to be invertible, is the volatility matrix, and $R$, which is a function in $X$ and $\pi$ is the interest rate. Note that all of them are also a function in $t$, predictable, and assumed to be bounded in all the variables.

Now, let us consider the situation where this investor wants to create a replicating portfolio against a European derivative security whose time-$T$ payout is $\xi$. In other words, he/she wants to have a self-financing portfolio $(x_0, \pi)$ so that

$$\xi = X_T = x_0, \pi,$$

equivalently, wants to solve the equation (2) with the terminal condition $X_T = \xi$, which is a Forward-Backward SDE (FBSDE), $(x_0, \pi)$ being a part of the solution.

$$dP^0_t = R(t,X_t,\pi_t)P^0_t \, dt, \quad P^0_0 = 1,$$
$$dP^1_t = \mu^1_t dt + \sum_{i=1}^n \sigma^1_{i,t} dW^i_t, \quad P^0_0 = p^i, \quad 1 \leq i \leq n,$$
$$dX_t = \sum_{i=1}^n \delta^i_p dP^0_t + (X_t - \bar{1} \pi_t) R(t,X_t,\pi_t) dt, \quad X_T = \xi. \quad (2)$$

Indeed, with the invertibility of $\sigma$, FBSDE (2) is equivalent to the following BSDE

$$-dY_t = f(t,Y_t,Z_t) dt - Z^\top_t \, dW_t, \quad Y_T = \xi, \quad (3)$$

with an expression

$$f(t,y,z) := -R(t,y,\sigma^{-1}z)y - \Theta(t,y,\sigma^{-1}z)^\top z, \quad (4)$$

where

$$y := x,$$
$$z := \sigma^\top \pi,$$
$$\Theta(t,y,\sigma^{-1}z) := \sigma^{-1}\left\{\mu_t - R(t,y,\sigma^{-1}z)1\right\}.$$

Therefore, BSDE (3) with driver (4) can be regarded as the “general equation for a wealth process”. We provide some examples for the wealth equations.

**Example 2.1** (Standard linear wealth)

$$R(t,y,z) = r t.$$

Here, $r$ is a risk-free interest, independent of $(y, z)$.

**Example 2.2** (Portfolio-dependent interest rate I)

$$R(t,y,z) = r_1 \{y - 1^\top \sigma^{-1}z \geq 0\} + r_1 \{y - 1^\top \sigma^{-1}z < 0\}.$$

Here, and $R$ respectively represent lending and borrowing rates. This model appears in [3], [4] and [5].

**Example 2.3** (Portfolio-dependent interest rate II)

$$R(t,y,z) = r_1 \{y - \sigma^{-1}z \geq H\} + r_1 \{y - \sigma^{-1}z < H\},$$

$$\Theta(t,y,z) = \frac{\mu_t - R(t,y,z)}{\sigma_t},$$

where $H$ is a constant. This is an example of one-dimensional case for different in borrowing and lending rates with a constant collateralization.

From now, we concentrate on BSDE (3) and the generator function $f$ in a form of (4). To make sure that the BSDE has a unique solution, we make some classical assumptions.

**Assumption 2.1** $f$ satisfies the assumptions of standard driver. Precisely, $f: \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following assumptions.

- Uniformly Lipschitz condition i.e., there exists a constant $K > 0$ such that for any $(y,z), (\tilde{y}, \tilde{z})$

$$\|f(t,y,z) - f(t,\tilde{y},\tilde{z})\| \leq K(|y-\tilde{y}| + |z-\tilde{z}|) \, d\mathbb{P} \otimes dt \text{ a.s.}.$$

- Integrability, $f(\cdot, 0, 0) \in L^2_T$.

According to [4], the above assumptions guarantee that for any $\xi \in L^2_T$, there exists a unique solution for BSDE (3).

2.2 BSDE for Writer, Buyer and Arbitrage

In this section, we derive a couple of BSDEs, which respectively describe seller’s hedging strategy and buyer’s hedging strategy against the payoff of the same derivative.

2.2.1 BSDE for Writer

Consider the writer of a derivative security $(T, \xi)$. For hedging, he/she solves BSDE (3) to construct a
self-financing portfolio

\[(Y_0, \pi) = (Y_0, \{\pi_t\}_{t \in [0,T]}), \text{ where } \pi_t = (\sigma_t^T)^{-1} \xi_t.\]

Then, the terminal wealth of the hedger is

\[-\xi + X_T^{Y_0, \pi} = -\xi + Y_T = 0.\]

So, \((Y_0, \pi)\) is a writer’s super-hedging strategy. Moreover, we can claim that this is a minimal super-hedging strategy. Since if writer performs another super-hedging strategy \(\tilde{Y}_0\) such that \(X_T^{\tilde{Y}_0, \pi'} \geq \xi\), then the comparison theorem for BSDE (see Section 2.2 of [3]) gives

\[\tilde{Y}_0 \geq Y_0.\]

\[2.2.2 \text{ BSDE for Buyer}\]

For the buyer, to hedge this contingent claim, he/she solves the following BSDE

\[-dY'_t = f(t, Y'_t, Z'_t) dt - Z'_t \theta_t \, dw_t, \quad Y'_T = -\xi.\]  \[5\]

Then construct a self-financing portfolio

\[(Y'_0, \pi').\]

At maturity, buyer has

\[\xi + X_T^{Y'_0, \pi'} = \xi + Y'_T = 0.\]

Therefore, \((Y'_0, \pi')\) is a buyer’s sub-hedging strategy. Again by the comparison theorem, we can claim that, it is a maximal sub-hedging strategy.

If we introduce

\[U := -Y', \quad V := -Z',\]

and

\[g(t,u,v) := -f(t,-u,-v),\]

then, we can rewrite (5) as

\[-dU_t = g(t, U_t, V_t) dt - V_t \theta_t \, dw_t, \quad U_T = \xi.\]  \[6\]

Thus, we have obtained a couple of BSDEs which describe wealth processes of two counter parties who attempt to hedge against the payout of the same derivative.

\[2.2.3 \text{ Arbitrage}\]

In this section, we will see that by using BSDE (3) and (6), one can derive an arbitrage opportunity in the market. First, let us introduce a formal definition of the arbitrage here.

**Definition 2.1** An arbitrage opportunity (free-lunch) is a self-financing strategy \((X, \pi)\) such that

\[X_0 = 0, \quad X_T \geq 0 \quad \text{with} \quad \mathbb{P}(X_T > 0) > 0.\]

Observe that, in the linear case given in Example 2.1, the two BSDEs coincide and we have that minimal super-hedging price \((Y_0) = \) maximal sub-hedging price \((U_0)\), and hence, a unique arbitrage-free price is determined and so, an arbitrage does not exist.

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On the other hand, in the case of nonlinear drivers, they are not necessary to be identical as illustrated in Figs. 2 and 3. For Fig. 2, we usually encounter in the real market, for instance, when the bid-ask spread has been included into the prices. However, in Fig. 3, all the prices between \((Y_0, U_0)\) become acceptable for both sides since \(Y_0\) (resp. \(U_0\)) is minimal (resp. maximal) initial investment to perform a hedging against \(\xi\), and this leads us to the appearance of an arbitrage.

**3. Main Results**

**3.1 Sufficient Condition for the Arbitrage**

In this section, we provide the main results from [1], a sufficient condition which ensures the existence of the arbitrage opportunity. We begin with the following theorem.

**Theorem 3.1** Let \(A = [a_1, a_2]\) be an interval such that

\[0 \leq a_1 e^{KT} < a_2 e^{-KT},\]

where \(K\) is the Lipschitz constant. Suppose that

\[f(t, y, z) < g(t, y, z), \quad (y, z) \in A \times \mathbb{R}^n \times d\mathbb{P} \times dt \, a.s.\]

Then, there exists an arbitrage opportunity for the both writer and buyer.

The theorem indicates that with an “appropriate choice” of derivative security, both writer and seller can create the arbitrage opportunities in the market.

**Applications**

To provide a clearer understanding, let us begin with a fictitious model where the borrowing rate and
the lending one are different. We have already known that if the lending rate is greater than the borrowing rate for every amount of lending/borrowing, it is an obvious arbitrage. But now, what we have learned from the Theorem 3.1 is, if the lending rate is greater than the borrowing rate even in a very small range of lending, with a clever choice of derivative, one can construct the arbitrage opportunity.

**Example 3.1** Let \( \Theta(t,y,z) = \theta(t,y,z) \), and

\[
R(t,y,z) = R_1 1_{\{y < 0\}} + R_2 1_{\{0 \leq y \leq a\}} + (R_1 + c(y)) 1_{\{y > a\}},
\]

where \( c(y) = \frac{2}{y} (R_2 - R_3) \) and \( R_3 < R_1 < R_2 \).

For this case, one can create an arbitrage opportunity, by employing a derivative whose payout at maturity is less than \( a \left( \frac{R_2 - R_3}{R_1 - R_3} \right) e^{-Kt} \). Here \( a \) is the amount of money which there is a shift in an interest rate, \( K \) is the Lipschitz constant depending on the bounds of market coefficients.

For a more realistic model, we consider a simplified version of Example 2.3. Observe that, one way to fulfill a sufficient condition in Theorem 3.1 is to find a nice interval (let’s call it “M”) where “\( g - f \)” is positive for any \( z \) on this interval.

**Example 3.2** Let \( f \) be a driver in Example 2.3 with \( \Theta(t,y,z) = \theta(t,y,z) \). Suppose that there exists an interval \( M \) such that, for any \( z \in M \)

\[
a_1 - \sigma^{-1} z \leq -H.
\]

Then, there exists an arbitrage opportunity for the both writer and buyer.

Intuitively, it tells us to go further to explore about this interval. As a result, we can obtain such an interval by consider a more specific model like Markovian model as in [2] and [3].

**3.1.2 Result on Markovian Case**

Next, we provide the results associated with Markovian FBSDE which play an important role in localizing a sufficient condition in Theorem 3.1. Instead of (2), we consider the following system.

For any given \( (t,x) \in [0,T] \times \mathbb{R} \), the following system defined on \([0,T]\):

\[
dP^0_s = R(s,Y_s,Z_s)P^0_s ds, \quad s \in [t,T],
\]

\[
P^0_s = 1, \quad s \in [0,t],
\]

\[
dP_s = \mu(s)P_s ds + \sigma^T(s)P_s dw_s, \quad s \in [t,T],
\]

\[
P_s = x, \quad s \in [0,t],
\]

With its associated BSDE:

\[
-Y_s = f(s,Y_s,\bar{Z}_s)ds - \bar{Z}_s^T dw_s,
\]

\[
Y_T = \Phi(P_T)
\]

The solution of the above system will be denoted by \( \{(P^t_{s,x,y},Y^t_{s,x,y},\bar{Z}^t_{s,x,y})\}, 0 \leq t \leq T \). Here \( R \) is \( \mathbb{R} \)-valued bounded deterministic function defined on \([0,T] \times \mathbb{R} \times \mathbb{R}^n \). \( f \) (resp. \( \Phi \)) is an \( \mathbb{R} \)-valued (resp. non-negative valued) Borel function defined on \([0,T] \times \mathbb{R} \times \mathbb{R}^n \) (resp. on \( \mathbb{R} \)) and \( \mu \) (resp. \( \sigma \)) is a bounded \( \mathbb{R} \)-valued (resp. bounded \( \mathbb{R}^n \)-valued) deterministic function defined on \([0,T] \). Standard Lipschitz condition is required on coefficients.

**Assumption 3.1** There exists a constant \( K > 0 \) such that for any \( (y,z),(\bar{y},\bar{z}) \)

\[
|f(t,y,z) - f(t,\bar{y},\bar{z})| \leq K(|y - \bar{y}| + |z - \bar{z}|),
\]

and also

\[
|\Phi(x)| \leq K(1 + |x|^p),
\]

for \( p \geq 1/2 \).

Like in [2], we employ a relation between BSDE and PDE to localize a sufficient condition of Theorem 3.1 and obtain the following result.

**Proposition 3.1** Suppose that functions \( f \) and \( \Phi \) are in \( C^3 \) with all partial derivatives are bounded. Then, there exists a constant \( R \) such that for \( 0 \leq t \leq s \leq T, \quad x \in \mathbb{R}, 1 \leq i \leq n \)

\[
-\text{BER} \leq Z^t_{s,x,i} \leq \text{BER} \\
\text{d}P \otimes \text{d}s \text{ a.s.,}
\]

where

\[
R = \max \left\{ \frac{1}{2} \left( \frac{2L + B^2 + 2C + D^2}{\sqrt{T}} \right), \frac{1}{2} \left( \frac{(2K)^2 + 6K^2 + 4LT + B^2T}{\sqrt{T}} \right) \right\},
\]

and \( B,L,C,D,E \) be the constants such that for all \( s, x, y, z \)

\[
|\sigma(s)|^2 \leq B^2, \\
|\mu(s)| \leq L, \\
f''_s(s,y,z) \leq C, \\
|f'_s(s,y,z)|^2 \leq D^2, \\
|\Phi(x)| \leq E, \\
K' = \max \{C,D\}
\]

**Remark.** Proposition 3.1 roughly asserts that if the parameters appearing in the assumptions are bounded, so is the hedging portfolio.
We will now utilize Proposition 3.1 in order to create a more local sufficient condition, to ensure the existence of arbitrage.

**Theorem 3.2** Let \( f \) and \( \Phi \) be the functions in \( C^3 \) with all partial derivatives are bounded and \( A = [a_1,a_2] \) be an interval so that \( 0 \leq a_1 e^{KT} < a_2 e^{-KT} \),

where \( K \) is the Lipschitz constant. Suppose that

\[
f(t,y,z) < g(t,y,z), \quad (t,y,z) \in [0,T] \times A \times M^n,
\]

where \( M = [-BER, BER] \) and \( B, E, R \) are the constants defined in Proposition 3.1. Then, there exists an arbitrage opportunity for the both writer and buyer.

### 3.2 Limitation of Arbitrage

In this section, we study about a limitation of the arbitrage. By limitation, we mean the condition without which the arbitrage vanishes.

First, we modify Theorem 3.2 as follows:

Let \( f \) and \( \Phi \) be the functions in \( C^3 \) with all partial derivatives are bounded and \( A = [a_1,a_2] \) be an interval so that

\[
0 \leq a_1 e^{KT} < a_2 e^{-KT},
\]

where \( K \) is the Lipschitz constant. Suppose that

\[
f(t,y,z) < g(t,y,z) \quad \text{if} \quad (t,y,z) \in [0,T] \times A \times M^n,
\]

\[
f(t,y,z) \geq g(t,y,z) \quad \text{otherwise}.
\]

Now, consider BSDEs

\[
-dY^{(k)} = f(t,Y^{(k)},Z^{(k)}) dt - (Z^{(k)})^\top dw, \quad Y^{(k)} T = \kappa, \\
-dU^{(k)} = g(t,U^{(k)},V^{(k)}) dt - (V^{(k)})^\top dw, \quad U^{(k)} T = \kappa,
\]

where \( k \geq 0 \) is a parameter represents an amount of financial derivatives.

According to Theorem 3.2, one can create an arbitrage, using a derivative (\( T, \Phi(P_T) \)) which satisfies (9). However, with this model, we can claim that this arbitrage is “limited” in the following sense:

**Proposition 3.2** Assume (10) holds and consider a derivative security (\( T, \Phi(P_T) \)) satisfies (9). Then, there exists \( k_0 > 1 \) so that for \( k \geq k_0 \)

\[
Y^{(k)}_0 \geq Y^{(k)}_0
\]

In other words, one can create arbitrage, using 1-unit of a derivative, however, cannot create arbitrage, using \( k \)-units (\( k \geq k_0 \)) of the derivative. That is the profit without risk which is obtained from the arbitrage is limited.

Proposition 3.2 is one of the examples which explains why the bulk sales and buying do not yield an arbitrage. In this case, the most important factor that affects to the limitation is an “area of the arbitrage”, i.e., \( A \times M^n \) in (10) which is a result from the bounds of economical factors, such as payout of financial derivative, return and volatility of stocks, as well as a maturity. The larger this area is, the higher profit can be made.

To see this, we revisit Example 3.1. As a result, buyer and seller can not attain the arbitrage opportunity, if they sell or buy the derivatives as mentioned in the example more than 1 unit.

### 4. Additional Results

In the previous section, we focused on the case of linear SDE in order to attain Lipschitz condition on its associated BSDE. We noticed that when a driver of BSDE is indepent of the parameters of SDE, the result can be extended to a more general models, for instance, a class of local volatility model.

For \((t,x) \in [0,T] \times \mathbb{R} \), consider the following SDE:

\[
\begin{align*}
  dP_s &= \mu(s,P_s) dt + \sigma(s,P_s) dw_s, \quad s \in [0,T], \\
  P_s &= x, \quad s \in [0,t]
\end{align*}
\]

where \( \mu : [0,T] \times \mathbb{R} \to \mathbb{R} \) and \( \sigma : [0,T] \times \mathbb{R} \to \mathbb{R}^n \) satisfy Lipschitz condition and linear growth. Then, we consider its associated BSDE:

\[
\begin{align*}
  -dY_s &= g(s,Y_s,Z_s) ds - (Z_s)_{\top} dw_s, \\
  Y_T &= \Phi(P_T^x)
\end{align*}
\]

where \( \Phi \) is a continuous function defined on \( \mathbb{R} \). The generating function \( g : [0,T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) satisfies Assumption 3.1.

Here, we state the lemma that shall be used in this section (the full version of the lemma can be found in Section 4 of [3]).

**Lemma 4.1** Suppose that the functions \( f \) and \( \Phi \) are in \( C^3 \) with all partial derivatives are bounded. Then, for \( t \leq s \leq T \), the solution \((Y,Z)\) of the BSDE (13) satisfies

(i) \( Y_s = u(s,P_s^x) \quad dP \otimes ds \text{ a.s.} \),

(ii) \( Z_s = \sigma(s,P_s^x) \partial_x u(s,P_s^x) \quad dP \otimes ds \text{ a.s.} \)

Here, \( u(t,x) \) belongs to \( C^{1,2}(0,T] \times \mathbb{R} \) and it is a regular solution of the following PDE:

\[
\partial_t u(t,x) + L u(t,x) + f(t,u(t,x),\sigma(t,x)) = 0,
\]

where \( L \) denotes the second-order differential operator, i.e.,

\[
L u(t,x) = \frac{1}{2} (\sigma(t,x)^\top \sigma(t,x)) u(t,x) + (\mu(t,x)) u(t,x).
\]

Now, instead of \( \mu \) is bounded, assume that \( \mu, \sigma \in C^3 \) with all partial derivatives are bounded. Then, we can obtain a result for a more general model.

**Theorem 4.1** Let \( g, \mu, \sigma \) and \( \Phi \) be the functions in \( C^3 \) with all partial derivatives are bounded. Suppose that there exists a constant \( S \) such that for all \( s, x \)

\[
|\sigma(s,x)|^2 \leq S^2.
\]
Then, there exists a constant $R$ so that for $t \leq s \leq T$, $x \in \mathbb{R}$, $1 \leq i \leq n$
\[ -SER \leq Z^{t,x,i}_s \leq SER \quad d\mathbb{P} \otimes ds \text{ a.s.}, \]
where
\[ R = \max \left\{ \frac{1}{\varepsilon^2 (2L^2 + 2C + D^2) T}, \sqrt{T} \frac{1}{\varepsilon^2 (2K'^2 + 6K' + 2LT + B^2 T)} \right\} \]
and $B, L, C, D, E$ be constants such that for any $s, x, y, z$
\[ |\partial_x \sigma(s,x)|^2 \leq B^2, \]
\[ |\partial_x \mu(s,x)| \leq L, \]
\[ f'_y(s,y,z) \leq C, \]
\[ |f'_z(s,y,z)|^2 \leq D^2, \]
\[ |\Phi(x)| \leq E, \]
\[ K' = \max \{ C, D \} \]

To prove this theorem, similar with Proposition 3.1, we need the following lemmas (see [1] for their proofs).

**Lemma 4.2** For $(t,x) \in [0,T] \times \mathbb{R}$,
\[ \partial_x u(t,x) \leq \sqrt{T} e^{c} \frac{1}{2} (2K'^2 + 6K' + 2LT + B^2 T). \]

**Lemma 4.3** For $(t,x) \in [0,T] \times \mathbb{R}$,
\[ \partial_x u(t,x) \geq -e \frac{1}{2} (2L^2 + 2C + D^2) T. \]

**Proof of Theorem 4.1** From Lemma 4.2 and 4.3, let
\[ R = \max \left\{ \frac{1}{\varepsilon^2 (2L^2 + 2C + D^2) T}, \sqrt{T} \frac{1}{\varepsilon^2 (2K'^2 + 6K' + 2LT + B^2 T)} \right\} \]
Then, for $(t,x) \in [0,T] \times \mathbb{R}$, we have
\[ -ER \leq \partial_x u(t,x) \leq ER. \]
Utilizing Lemma 4.1, we obtain the result.

**4.1 Further Results on Comonotonic Theorem**

Comonotonic theorem for BSDEs is a theorem concerning about the properties of $Z$, the second part of the the solution of BSDE, studied in [2]. Let us start this section with the definition of comonotonicity.

**Definition 4.1** The functions $\xi$ and $\psi$ are said to be comonotonic, if both $\xi$ and $\psi$ are of the same monotonocity, that is, if $\xi$ is increasing (or decreasing), so is $\psi$. Furthermore, $\xi$ and $\psi$ are said to be strictly comonotonic, if $\xi$ and $\psi$ are strictly monotonic.

For $j = 1, 2$, we consider a system in the same class as (12) and (13), i.e., SDEs:
\[ dP^j_t = \mu^j(t,P^j_t) dt + \sigma^j(t,P^j_t) dW_t, \quad P^j_0 = x_j \quad (14) \]
Their associated BSDEs:
\[ Y^j_t = \Phi^j(P^j_T) + \int_t^T g^j(s,Y^j_s,Z^j_s) ds - \int_t^T (Z_s^j)^\top dw_s \quad (15) \]
Now, by applying a result in the previous section, we can now strengthen the comonotonic theorem.

**Theorem 4.2** For $j = 1, 2$, let $P^j$ be the solutions of SDEs (14), $(Y^j, Z^j)$ be the solutions of BSDEs (15) corresponding to terminal values $\Phi^j(P^j_T)$, all the functions $\mu^j, \sigma^j, g^j$ and $\Phi^j$ are in $C^3$ with all partial derivatives are bounded and $S_j$ be the constants such that for all $t, x$
\[ |\sigma^j(t,x)|^2 \leq S_j^2. \]

(i) Suppose that $\Phi^1$ and $\Phi^2$ are comonotonic and for $1 \leq i \leq n$, $\sigma^{1,i}(t,P^1_t) \sigma^{2,i}(t,P^2_t) \geq 0 \quad d\mathbb{P} \otimes dt \text{ a.s.}$, then
\[ 0 \leq Z^1_{t,i} Z^2_{t,i} \leq S_1 S_2 E_1 E_2 R_1 R_2 \quad d\mathbb{P} \otimes dt \text{ a.s.}, \]
where $E_j$ and $R_j$ are constants corresponding to $j$, defined in Theorem 4.1.

(ii) In particular, if $\Phi^1$ and $\Phi^2$ are strictly comonotonic, then for $1 \leq i \leq n$
\[ \sigma^{1,i}(t,P^1_t) \sigma^{2,i}(t,P^2_t) > 0 \quad d\mathbb{P} \otimes dt \text{ a.s.} \]

Proof. We follow the same argument as in the proof Theorem 1 in [2]. For any given $0 \leq t \leq T$ and $x \in \mathbb{R}$, let $\{P^j_{t,s,j}, 0 \leq t \leq s \leq T \}$ be the solutions of following SDEs:
\[ dP^j_s = \mu^j(s,P^j_s) ds + \sigma^j(s,P^j_s) dW_s \quad s \in [t,T], \]
\[ P^j_t = x_j, \quad s \in [0,t], \quad (16) \]
and $\{1 \leq i \leq n, Y^j_{t,s,j}, 0 \leq t \leq s \leq T \}$ be the solutions of their associated BSDEs:
\[ Y^j_t = \Phi^j(P^j_T) + \int_t^T g^j(s,Y^j_s,Z^j_s) ds - \int_t^T (Z^j_s)^\top dw_s \quad (17) \]
Set $t = 0$, we see that solutions of (16) become the solutions of (14), that is, $P^{0,x,j} = P^j$, and also for (17) and (15), $Z^{0,x,j} = Z^j$. Therefore, for $1 \leq i \leq n$, we have
\[ Z^1_{t,i} Z^2_{t,i} = (\partial_s u^1)\partial_s u^2 (s,P^1_s) (\partial_s u^1)\partial_s u^2 (s,P^2_s) - \sigma^{1,i}(s,P^1_s) \sigma^{2,i}(s,P^2_s) \]
Here, by Lemma 4.2 - 4.3 and comonotonicity of $\Phi^1$ and $\Phi^2$, we can conclude that
\[ 0 \leq \partial_s u^1 (s,P^1_s) \partial_s u^2 (s,P^2_s) \leq E_1 E_2 R_1 R_2 \quad d\mathbb{P} \otimes ds \text{ a.s.} \]
Thus, we can deduce the results.

Now, consider (14) and (15) with $j = 1$. We can
obtain the following, the comonotonic result for a forward-backward system.

**Corollary 4.1** Let $\mu, \sigma, g$ and $\Phi$ be the functions in $C^3$ with all partial derivatives are bounded and $S$ be a constant such that for all $t, x$ and $1 \leq i \leq n$

$$0 \leq \sigma^i(t,x) \leq S$$

(i) If $\Phi$ is an increasing function, then for $1 \leq i \leq n$

$$0 \leq Z_i \sigma^i(t,P_t) \leq S^2 ER \quad d\mathbb{P} \otimes dt \ a.s.$$  

(ii) If $\Phi$ is a decreasing function, then for $1 \leq i \leq n$

$$-S^2 ER \leq Z_i \sigma^i(t,P_t) \leq 0 \quad d\mathbb{P} \otimes dt \ a.s.$$  

**Proof.** Consider an integral form of SDE (14)

$$P_t = P_T - \int_t^T \mu(s,P_s)ds - \int_t^T \sigma^T(s,P_s)dw_s$$  

Since $\Phi(x)$ and $x$ are increasing, applying Theorem 4.2, we obtain (i).

When $\Phi(x)$ is decreasing, BSDE (18) can be written as

$$P_t = -P_T + \int_t^T \mu(s,P_s)ds + \int_t^T \sigma^T(s,P_s)dw_s$$  

We see that $-x$ is also decreasing. Again, Theorem 4.2 implies the result.

### 4.2 Application to $g$-Expectation

In [11], Peng introduced a typical model of dynamic pricing mechanisms of financial derivatives, so-called $g$-expectation which is defined by a solution of a BSDE with $g$ as its generating function. Here, we give the definition of $g$-expectation.

**Definition 4.2** Let $g$ be a function satisfies assumption 3.1. For any $\xi \in L^2(\mathcal{F}_T)$, let $Y$ be a solution of the following BSDE:

$$Y_t = \xi + \int_t^T g(u,Y_u,Z_u)du - \int_t^T (Z_u)^TdW_u.$$  

Consider a mapping denoted by

$$E_{t,T}^g[\xi] = Y_t.$$  

($E_{t,T}^g[\xi]$)$_{0 \leq t \leq T < \infty}$ is called the $g$-expectation or $g$-pricing mechanism.

The following properties can be found in [11], which are the reasons for why this $g$-expectation is a good candidate to model a dynamic pricing mechanism of financial derivatives.

**Properties.** For each $t \leq T < \infty$ and $\xi, \xi' \in L^2(\mathcal{F}_T)$

(P1) $E_{t,T}^g[\xi] \leq E_{t,T}^g[\xi']$, a.s. for $\xi \leq \xi'$,

(P2) $E_{t,T}^g[\xi] = \xi$,

(P3) $E_{s,T}^g[E_{t,T}^g[\xi]] = E_{s,T}^g[\xi]$, for $s \leq t$.

**Remark.** (P1) and (P2) are economically obvious conditions for a pricing mechanism. Property (P3) means that, at the time $s$, the random value $E_{s,T}^g[\xi]$ can be regarded as a maturity value with maturity $t$.

The price of this derivative at $s$ is $E_{s,T}^g[\xi]$ and it must be the same as the price $E_{s,T}^g[\xi]$ at time $s$.

In general, when a generator of the BSDE is nonlinear, additive property (w.r.t. terminal condition) is not necessary to hold for its associated $g$-expectation. However, by considering (15) with $g^1 = g^2 = g$, Theorem 4.2 yields us a result which guarantee the additivity of $g$-expectation even when a driver of BSDE is nonlinear.

**Corollary 4.2** Consider (14) and (15). Let $\mu^1, \sigma^1, g^1$ and $\Phi^1$ be the functions in $C^3$ with all partial derivatives are bounded and $S_j$ be the constants such that for all $t,x$

$$0 \leq |\sigma^i(t,x)|^2 \leq S_j^2$$  

Moreover, $g$ is also positively additive i.e., for any $(y_1,z_1),(y_2,z_2)$ such that $y_1, y_2 \geq 0$ and for $1 \leq i \leq n$, $0 \leq z_1 \leq S_j = S_1 S_2 E^1_1 E^1_2 R^1_1 R^1_2$,

$$g(t,y_1+y_2,z_1+z_2) = g(t,y_1,z_1)+g(t,y_2,z_2)$$

(i) Suppose that $\Phi^1$ and $\Phi^2$ are comonotonic with $\Phi^1(1), \Phi^2(2) \geq 0$ (or $\leq 0$) and

$$\sigma^1 \sigma^2(t,P^1_t) \sigma^2(t,P^2_t) \geq 0, 1 \leq i \leq n,$$  

$$E_{T,T}^g[\Phi^1(P^1_T)] + \Phi^2(P^2_T) = E_{T,T}^g[\Phi^1(P^1_t)] + E_{T,T}^g[\Phi^2(P^2_t)]$$

(ii) If $g$ does not depend on $y$, then an assumption $\Phi^1(1), \Phi^2(2) \geq 0$ (or $\leq 0$) in (i) can be dropped.

**Sketch of the Proof.** By the comparison theorem for BSDE, an assumption on terminal conditions implies that $Y^1 Y^2 \geq 0$. Moreover, Theorem 4.2 also gives a bound of $Z$. Therefore, the solutions of BSDEs are always in the desired range and hence, the results follow.

**References**

[1] K. Thoednithi: Arbitrage opportunity on nonlinear wealth processes; Proceeding, The 45th ISCIE International Symposium on Stochastic Systems Theory and Its Applications (2013)

[2] Z. Chen, R. Kulperger and G. Wei: A comonotonic theorem for BSDEs; Stochastic Processes and their Applications, Vol. 115, pp. 41-54 (2005)

[3] N. El Karoui, S. Peng and M. C. Quenez: Backward stochastic differential equations in finance; Mathematical Finance, Vol. 7, pp. 1–71 (1997)

[4] N. El Karoui, S. Peng and M. C. Quenez: A dynamic maximum principle for the optimization of recursive utilities under constraints; The Annals of Applied Probability, Vol. 11, No. 3, pp. 664–693 (2001)

[5] N. El Karoui and M. C. Quenez: Non-linear pricing theory and backward stochastic differential equations; Lecture Note of Mathematics, Springer, Volume 1656, pp. 191–246 (1997)

[6] M. Fujii and A. Takahashi: Derivative pricing under asymmetric and imperfect collateralization and CVA, working paper, Department of Economics of
the University of Tokyo (2011)

[7] J. P. Lepeltier and J. San Martin: Backward stochastic differential equations with continuous coefficient; Statistics & Probability Letters, Vol. 32, pp. 425–430 (1997)

[8] X. Mao: Adapted solutions of backward stochastic differential equations with non-Lipschitz coefficients; Stoch. Proc. Appl., Vol. 58, pp. 281–292 (1995)

[9] H. Xiao-Qin, W. Mian-Sen and J. Jun-Guo: Two comparison theorems of BSDEs; J. Appl. Math. & Computing, Vol. 24, Nos. 1-2, pp. 377–385 (2007)

[10] E. Pardoux and S. Peng: Backward stochastic differential equations and quasilinear parabolic partial differential equations; Lecture Notes in CIS, Vol 176, Springer-Verlag, pp. 200–217 (1992)

[11] S. Peng: Modelling derivatives pricing mechanisms with their generating functions; Lecture note of Cornell University (2006)

Appendix

In [1], Lemma 3.3 is required in order to prove Lemma 4.2 and 4.3 (Lemma 3.4 and 3.5 in [1]). However, for the class of models in Section 4, we need to adapt Lemma 3.3 as follows.

**Lemma 3.3** For any given $0 \leq t \leq T$ and $x \in \mathbb{R}$, let $\{\partial_x P^t_x, 0 \leq t \leq s \leq T\}$ be the solution of the following SDE

\[
d\partial_x P^t_x = \partial_x \mu(s)\partial_x P^t_x ds + \partial_x \sigma^T(s)\partial_x P^t_x dw_s, \]

\[
\partial_x P^t_t = 1.
\]

Then, it holds that

\[
E[|\partial_x P^t_x|^2] \leq e^{(2L+B^2)T},
\]

and

\[
E[|\partial_x P^t_x|] \geq e^{-(L+\frac{1}{2}B^2)T}.
\]

Here

\[
\partial_x \mu(u) = \partial_x \mu(u, P^t_u) \quad \text{and} \quad \partial_x \sigma(u) = \partial_x \sigma(u, P^t_u).
\]

**proof.** The solution can be obtained explicitly as

\[
\partial_x P^t_x = \exp \left\{ \int_t^s (\partial_x \mu(u) - \frac{1}{2}|\partial_x \sigma(u)|^2)du + \int_t^s \partial_x \sigma^T(u)dw_u \right\}
\]

If we set

\[
M_s = 2\int_t^s |\partial_x \sigma(u)|^2du,
\]

then, $\langle M \rangle_s = 4\int_t^s |\partial_x \sigma(u)|^2du$. Hence

\[
E[|\partial_x P^t_x|^2] = E \left[ e^{\int_t^s (2\partial_x \mu(u) + |\partial_x \sigma(u)|^2)du + M_s - \frac{1}{2}\langle M \rangle_s } \right]
\]

\[
\leq e^{(2L+B^2)(s-t)}E \left[ e^{M_s - \frac{1}{2}\langle M \rangle_s } \right]
\]

\[
\leq e^{(2L+B^2)T}.
\]

For the second claim, Jensen’s inequality yields the result. **

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