University of Plovdiv Paisii Hilendarski
Faculty of Mathematics and Informatics
Department of Algebra and Geometry

Mancho Hristov Manev

On Geometry of Manifolds with Some Tensor Structures and Metrics of Norden Type

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To my wife Rositsa
Structure of the Dissertation

The present dissertation consists of an introduction, a main body, a conclusion and a bibliography. The introduction consists of two parts: a scope of the topic and the purpose of the dissertation. The main body includes two chapters containing 15 sections. The conclusion provides a brief summary of the main contributions of the dissertation, a list of the publications on the results given in the dissertation, a declaration of originality and acknowledgements. The bibliography contains a list of [155] publications used in the text.

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Introduction

*Scope of the Topic*

Among additional tensor structures on a smooth manifold, one of the most studied is *almost complex structure*, i.e. an endomorphism of the tangent bundle whose square, at each point, is minus the identity. The manifold must be even-dimensional, i.e. $\dim = 2n$. Usually it is equipped with a *Hermitian metric* which is a (Riemannian or pseudo-Riemannian) metric that preserves the almost complex structure, i.e. the almost complex structure acts as an isometry with respect to the (pseudo-)Riemannian metric. The associated $(0,2)$-tensor of the Hermitian metric is a 2-form and hence the relationship with symplectic geometry.

A relevant counterpart is the case when the almost complex structure acts as an anti-isometry regarding a pseudo-Riemannian metric. Such a metric is known as an anti-Hermitian metric or a *Norden metric* (first studied by and named after A. P. Norden [120, 121]). The Norden metric is necessary pseudo-Riemannian of neutral signature whereas the Hermitian metric can be Riemannian or pseudo-Riemannian of signature $(2n_1, 2n_2), n_1 + n_2 = n$. The associated $(0,2)$-tensor of any Norden metric is also a Norden metric. So, in this case we dispose with a pair of mutually associated Norden metrics, known also as twin Norden metrics. This manifold can be considered as an $n$-dimensional manifold with a complex Riemannian metric whose real and imaginary parts are the twin Norden metrics. Such a manifold is known as a *generalized B-manifold* [49, 108, 50, 45], an *almost complex manifold with Norden metric* [147, 35, 34, 14, 23, 123, 124, 131], an *almost complex manifold with B-metric* [36, 38], an *almost complex manifold with anti-Hermitian metric* [16, 17, 28] or a *manifold with complex Riemannian metric* [65, 105, 37, 64].

Supposing a manifold is of an odd dimension, i.e. $\dim = 2n + 1$, then
Introduction. Scope of the Topic

There exists a contact structure. The codimension one contact distribution can be considered as the horizontal distribution of the sub-Riemannian manifold. This distribution allows an almost complex structure which is the restriction of a contact endomorphism on the contact distribution. The vertical distribution is spanned by the corresponding Reeb vector field. Then the odd-dimensional manifold is equipped with an almost contact structure.

If we dispose of a Hermitian metric on the contact distribution then the almost contact manifold is called an almost contact metric manifold. In another case, when a Norden metric is available on the contact distribution then we have an almost contact manifold with B-metric. Any B-metric as an odd-dimensional counterpart of a Norden metric is a pseudo-Riemannian metric of signature \((n + 1, n)\).

A natural generalization of almost complex structure is almost hypercomplex structure. An almost hypercomplex manifold is a manifold which tangent bundle equipped with an action by the algebra of quaternions in such a way that the unit quaternions define almost complex structures. Then an almost hypercomplex structure on a 4n-dimensional manifold is a triad of anti-commuting almost complex structures whose triple composition is minus the identity.

It is known that, if the almost hypercomplex manifold is equipped with a Hermitian metric, the derived metric structure is a hyper-Hermitian structure. It consists of the given Hermitian metric with respect to the three almost complex structures and the three associated Kähler forms [2]. Almost hypercomplex structures (under the terminology of C. Ehresmann of almost quaternionic structures) with Hermitian metrics were studied in many works, e.g. [29, 122, 13, 150, 134, 18, 3].

An object of our interest is a metric structure on the almost hypercomplex manifold derived by a Norden metric. Then the existence of a Norden metric with respect to one of the three almost complex structures implies the existence of one more Norden metric and a Hermitian metric with respect to the other two almost complex structures. Such a metric is called a Hermitian-Norden metric on an almost hypercomplex manifold. Furthermore, the derived metric structure contains the given metric and three \((0,2)\)-tensors associated by the almost hypercomplex structure – a Kähler form and two Hermitian-Norden metrics for which the roles of the almost complex structures change cyclically. Thus, the derived manifold is called an almost hypercomplex manifold with Hermitian-Norden met-
Manifolds of this type are studied in several papers with the author participation \[47, 81, 103, 46, 82, 97\].

The notion of \textit{almost contact 3-structure} is introduced by Y. Y. Kuo in \[62\] and independently under the name \textit{almost coquaternion structure} by C. Udriște in \[146\]. Later, it is studied by several authors, e.g. \[62, 63, 137, 153\]. It is well known that the product of a manifold with almost contact 3-structure and a real line admits an almost hypercomplex structure \[62, 2\]. All authors have previously considered the case when there exists a Riemannian metric compatible with each of the three structures in the given almost contact 3-structure. Then the object of study is the so-called \textit{almost contact metric 3-structure} (see also \[5, 21, 19, 20\]).

Compatibility of an almost contact 3-structure with a B-metric is not yet considered before the publications that are part of this dissertation. In the present work we launch such a metric structure on a manifold with almost contact 3-structure.

* * *

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Purpose of the Dissertation

The object of study in the present dissertation are some topics in differential geometry of smooth manifolds with additional tensor structures and metrics of Norden type. There are considered four cases depending on dimension of the manifold: $2n$, $2n + 1$, $4n$ and $4n + 3$. The studied tensor structures, which are counterparts in the different related dimensions, are: the almost complex structure, the almost contact structure, the almost hypercomplex structure and the almost contact 3-structure. The considered metric on the $2n$-dimensional manifold is the Norden metric. The metrics on the manifolds in the other three cases are generated by the Norden metric and they are: the B-metric, the Hermitian-Norden metric and the metric of Hermitian-Norden type, respectively. The four types of tensor structures with metrics of Norden type are considered in their interrelationship.

The purpose of the dissertation is to carry out the following:

1. Further investigations of almost complex manifolds with Norden metric and, in particular, studying of natural connections with conditions for their torsion and invariant tensors under the twin interchange of Norden metrics.

2. Further investigations of almost contact manifolds with B-metric including studying of natural connections with conditions for their torsion and associated Schouten-van Kampen connections as well as classification of affine connections.

3. Introducing and studying of Sasaki-like almost contact complex Riemannian manifolds.

4. Further investigations of almost hypercomplex manifolds with Hermitian-Norden metrics including: studying of integrable structures of the considered type on 4-dimensional Lie algebra and tangent bundles with complete lift of the base metric; introducing and studying of associated
Nijenhuis tensors in relation with natural connections having totally skew-symmetric torsion as well as quaternionic Kähler manifolds with Hermitian-Norden metrics.

5. Introducing and studying of manifolds with almost contact 3-structures and metrics of Hermitian-Norden type and, in particular, associated Nijenhuis tensors and their relationship with natural connections having totally skew-symmetric torsion.

* * *
Chapter I.

ON MANIFOLDS WITH ALMOST
COMPLEX STRUCTURES AND ALMOST
CONTACT STRUCTURES, EQUIPPED
WITH METRICS OF NORDEN TYPE
§1. Almost complex manifolds with Norden metric

In the present section we recall some notions and knowledge for the almost complex manifolds with Norden metric [49, 34, 36, 113].

1.1. Almost complex structure and Norden metric

Let $(\mathcal{M}, J, g)$ be a $2n$-dimensional almost complex manifold with Norden metric or briefly an almost Norden manifold. This means that $J$ is an almost complex structure and $g$ is a pseudo-Riemannian metric on $\mathcal{M}$ such that

\begin{equation}
J^2 x = -x, \quad g(Jx, Jy) = -g(x, y).
\end{equation}

Here and further, $x, y, z, w$ will stand for arbitrary differentiable vector fields on the considered manifold (or vectors in its tangent space at an arbitrary point of the manifold).

On any almost Norden manifold, there exists an associated metric $\tilde{g}$ of its metric $g$ defined by

$$
\tilde{g}(x, y) = g(Jx, y).
$$

It is also a Norden metric since $\tilde{g}(Jx, Jy) = -\tilde{g}(x, y)$ and the manifold $(\mathcal{M}, J, \tilde{g})$ is an almost Norden manifold, too. Both metrics are necessarily of neutral signature $(n, n)$.

The elements of the pair of Norden metrics of an almost Norden manifold are also known as twin Norden metrics on $\mathcal{M}$ because of the associated metric of $g$ is $\tilde{g}$ and the associated metric of $\tilde{g}$ is $-g$, i.e.

\begin{equation}
\tilde{g}(x, y) = g(Jx, y), \quad \tilde{g}(Jx, y) = -g(x, y).
\end{equation}

Let the Levi-Civita connections of $g$ and $\tilde{g}$ be denoted by $D$ and $\tilde{D}$, respectively.
The structure group $\mathcal{G}$ of almost Norden manifolds is determined, according to [34], by the following way

\begin{equation}
\mathcal{G} = \mathcal{GL}(n; \mathbb{C}) \cap \mathcal{O}(n, n),
\end{equation}

i.e. it is the intersection of the general linear group of degree $n$ over the set of complex numbers and the indefinite orthogonal group for the neutral signature $(n, n)$. Therefore, $\mathcal{G}$ consists of the real square matrices of order $2n$ having the following type

\[
\begin{pmatrix}
A & B \\
-B & A
\end{pmatrix},
\]

such that the matrices $A$ and $B$ belongs to $\mathcal{GL}(n; \mathbb{R})$ and their corresponding transposes $A^\top$ and $B^\top$ satisfy the following identities

\begin{equation}
A^\top A - B^\top B = I_n, \\
B^\top A + A^\top B = O_n,
\end{equation}

where $I_n$ and $O_n$ are the unit matrix and the zero matrix of size $n$, respectively.

1.2. First covariant derivatives

1.2.1. Fundamental tensor $F$

The fundamental $(0, 3)$-tensor $F$ on $\mathcal{M}$ is defined by

\begin{equation}
F(x, y, z) = g((D_x J) y, z).
\end{equation}

It has the following basic properties: [49]

\begin{equation}
F(x, y, z) = F(x, z, y) = F(x, Jy, Jz)
\end{equation}

and their consequence

\[ F(x, Jy, z) = -F(x, y, Jz). \]

Let $\{e_i\}$ ($i = 1, 2, \ldots, 2n$) be an arbitrary basis of the tangent space of $\mathcal{M}$ at any its point and let $g^{ij}$ be the corresponding components of the inverse matrix of $g$. Then, the corresponding Lee forms of $F$ with respect to $g$ and $\tilde{g}$ are defined by

\[
\theta(z) = g^{ij} F(e_i, e_j, z), \quad \tilde{\theta}(z) = \tilde{g}^{ij} F(e_i, e_j, z),
\]

respectively. They imply the relation

\begin{equation}
\tilde{\theta} = \theta \circ J
\end{equation}
because of
\[ \tilde{g}^{ij} F(e_i, e_j, z) = -g^{ij} F(e_i, J e_j, z) = g^{ij} F(e_i, e_j, J z). \]

Somewhere, instead of \( \tilde{\theta} \) it is used the 1-form \( \theta^* \) associated with \( F \), which is defined by
\begin{equation}
\theta^*(z) = g^{ij} F(e_i, J e_j, z).
\end{equation}

Using \( \tilde{g} \), we have the following
\[ \theta^*(z) = g^{ij} F(e_i, J e_j, z) = J^j_k g^{ik} F(e_i, e_j, z) \]
\[ = -\tilde{g}^{ij} F(e_i, e_j, z). \]

Then the identity
\begin{equation}
\theta^* = -\tilde{\theta}
\end{equation}
holds by means of (1.6), because of (1.7) and
\[ \theta^*(z) = g^{ij} F(e_i, J e_j, z) = -g^{ij} F(e_i, e_j, J z) \]
\[ = -\theta(J z). \]

The almost Norden manifolds are classified into basic classes \( \mathcal{W}_i \) \((i = 1, 2, 3)\) with respect to \( F \) by G. Ganchev and A. Borisov in \[34\]. All classes are determined as follows:

\( \mathcal{W}_0 \): \( F(x, y, z) = 0; \)
\( \mathcal{W}_1 \): \( F(x, y, z) = \frac{1}{2n} \{ g(x, y)\theta(z) + \tilde{g}(x, y)\tilde{\theta}(z) \}
+ g(x, z)\theta(y) \\
+ \tilde{g}(x, z)\tilde{\theta}(y) \}; \)
\( \mathcal{W}_2 \): \( \mathcal{S}_{x,y,z} F(x, y, J z) = 0, \quad \theta = 0; \)
\( \mathcal{W}_3 \): \( \mathcal{S}_{x,y,z} F(x, y, z) = 0; \)
\( \mathcal{W}_1 \oplus \mathcal{W}_2 \): \( \mathcal{S}_{x,y,z} F(x, y, J z) = 0; \)
\( \mathcal{W}_1 \oplus \mathcal{W}_3 \): \( \mathcal{S}_{x,y,z} F(x, y, z) = \\
= \frac{1}{n} \mathcal{S}_{x,y,z} \{ g(x, y)\theta(z) + \tilde{g}(x, y)\tilde{\theta}(z) \}; \)
§1. Almost complex manifolds with Norden metric

\( W_2 \oplus W_3 : \ \theta = 0; \)
\( W_1 \oplus W_2 \oplus W_3 : \ \text{no conditions}. \)

The class \( W_0 \) is a special class that belongs to any other class, i.e. it is their intersection. It contains Kähler manifolds with Norden metric (known also as Kähler-Norden manifolds, a Kähler manifolds with B-metric or a holomorphic complex Riemannian manifolds).

1.2.2. Isotropic Kähler-Norden manifolds

The square norm \( \| D J \|^2 \) of \( D J \) with respect to the metric \( g \) is defined in [40] as follows

\( \| D J \|^2 = g^{ij} g^{kl} g((D_e J) e_k, (D_e J) e_l). \)

By means of (1.5) and (1.6), we obtain the following equivalent formula, which is given in terms of the components of \( F \)

\( \| D J \|^2 = g^{ij} g^{kl} g^{st} (F)_{iks} (F)_{jlt}, \)

where \( (F)_{iks} \) denotes \( F(e_i, e_k, e_s) \).

An almost Norden manifold satisfying the condition \( \| D J \|^2 = 0 \) is called an isotropic Kähler manifold with Norden metrics [113] or an isotropic Kähler-Norden manifold.

Let us remark that if a manifold belongs to \( W_0 \), then it is an isotropic Kähler-Norden manifold but the inverse statement is not always true. It can be noted that the class of isotropic Kähler-Norden manifolds is the closest larger class of almost Norden manifolds containing the class of Kähler-Norden manifolds.

1.2.3. Fundamental tensor \( \Phi \)

Let us consider the tensor \( \Phi \) of type \((1,2)\) defined in [36] as the difference of the Levi-Civita connections \( \tilde{D} \) and \( D \) of the corresponding Norden metrics as follows

\( \Phi(x, y) = \tilde{D}_x y - D_x y. \)

This tensor is known also as the potential of \( \tilde{D} \) regarding \( D \) because of the formula

\( \tilde{D}_x y = D_x y + \Phi(x, y). \)

Since both the connections are torsion-free, then \( \Phi \) is symmetric, i.e. \( \Phi(x, y) = \Phi(y, x) \) holds. Let the corresponding tensor of type \((0, 3)\) with
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respect to \( g \) be defined by

\[
\Phi(x, y, z) = g(\Phi(x, y), z).
\]

(1.14)

By virtue of properties (1.6), the following interrelations between \( F \) and \( \Phi \) are valid [36]

\[
\Phi(x, y, z) = \frac{1}{2}\{F(Jz, x, y) - F(x, y, Jz)
- F(y, x, Jz)\},
\]

(1.15)

\[
F(x, y, z) = \Phi(x, y, Jz) + \Phi(x, z, Jy).
\]

(1.16)

Let the corresponding 1-form of \( \Phi \) be denoted by \( f \) and let be defined as follows

\[
f(z) = g^{ij}\Phi(e_i, e_j, z).
\]

Using (1.15) and (1.7), we obtain the relation \( f = -\tilde{\theta} \).

An equivalent classification of the given one in (1.10) is proposed in [36], where all classes are defined in terms of \( \Phi \) as follows:

\[
\mathcal{W}_0 : \quad \Phi(x, y, z) = 0;
\]

\[
\mathcal{W}_1 : \quad \Phi(x, y, z) = \frac{1}{2n}\{g(x, y)f(z)
+ g(x, Jy)f(Jz)\};
\]

\[
\mathcal{W}_2 : \quad \Phi(x, y, z) = -\Phi(Jx, Jy, z), \quad f = 0;
\]

\[
\mathcal{W}_3 : \quad \Phi(x, y, z) = \Phi(Jx, Jy, z)
\]

(1.17)

\[
\mathcal{W}_1 \oplus \mathcal{W}_2 : \quad \Phi(x, y, z) = -\Phi(Jx, Jy, z),
\quad \Phi(Jx, y, z) = -\Phi(x, y, Jz);
\]

\[
\mathcal{W}_1 \oplus \mathcal{W}_3 : \quad \Phi(x, y, z) = \Phi(Jx, Jy, z)
+ \frac{1}{n}\{g(x, y)f(z) + g(x, Jy)f(Jz)\};
\]

\[
\mathcal{W}_2 \oplus \mathcal{W}_3 : \quad f = 0;
\]

\[
\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 : \quad \text{no conditions}.
\]
1.2.4. Pair of the Nijenhuis tensors

As it is well known, the Nijenhuis tensor (let us denote it by the brackets $[\cdot,\cdot]$) of the almost complex structure $J$ is defined by

$$[J,J](x,y) = [Jx,Jy] - [x,y] - J[Jx,y] - J[x,Jy].$$

Besides it, we give the following definition in analogy to (1.18), where the symmetric braces $\{x,y\} = D_x y + D_y x$ are used instead of the anti-symmetric brackets $[x,y] = D_x y - D_y x$. More precisely, the symmetric braces $\{x,y\}$ are determined by

$$g(\{x,y\}, z) = g(D_x y + D_y x, z)$$

$$= x(g(y,z)) + y(g(x,z)) - z(g(x,y)) - g([y,z], x) + g([z,x], y).$$

**Definition 1.1.** The symmetric $(1,2)$-tensor $\{J,J\}$ defined by

$$\{J,J\}(x,y) = \{Jx,Jy\} - \{x,y\} - J\{Jx,y\} - J\{x,Jy\}$$

is called the associated Nijenhuis tensor of the almost complex structure $J$ on $(\mathcal{M}, J, g)$.

The Nijenhuis tensor and its associated tensor for $J$ are determined in terms of the covariant derivatives of $J$ as follows:

$$[J,J](x,y) = (D_x J) Jy - (D_y J) Jx + (D_{Jx} J) y - (D_{Jy} J) x,$$

$$\{J,J\}(x,y) = (D_x J) Jy + (D_y J) Jx + (D_{Jx} J) y + (D_{Jy} J) x.$$

The tensor $\{J,J\}$ coincides with the associated tensor $\tilde{N}$ of $[J,J]$ introduced in [34] by the latter equality above.

The pair of the Nijenhuis tensors $[J,J]$ and $\{J,J\}$ plays a fundamental role in the topic of natural connections (i.e. such connections that $J$ and $g$ are parallel with respect to them) on an almost Norden manifold. The torsions and the potentials of these connections are expressed by these two tensors. Because of that we characterize the classes of the considered manifolds in terms of $[J,J]$ and $\{J,J\}$.

As it is known from [34], the class $\mathcal{W}_3$ of the quasi-Kähler manifolds with Norden metric is the only basic class of almost Norden manifolds with non-integrable almost complex structure $J$, because $[J,J]$ is non-zero there. Moreover, this class is determined by the condition $\{J,J\} = 0$. The class $\mathcal{W}_1 \oplus \mathcal{W}_2$ of the (integrable almost) complex manifolds with Norden
metric is characterized by $[J, J] = 0$ and $\{J, J\} \neq 0$. Additionally, the basic classes $\mathcal{W}_1$ and $\mathcal{W}_2$ are distinguish from each other according to the Lee form $\theta$: for $\mathcal{W}_1$ the tensor $F$ is expressed explicitly by the metric and the Lee form, i.e. $\theta \neq 0$; whereas for $\mathcal{W}_2$ the equality $\theta = 0$ is valid.

The corresponding $(0,3)$-tensors are determined as follows

$[J, J](x, y, z) = g([J, J](x, y), z), \quad \{J, J\}(x, y, z) = g(\{J, J\}(x, y), z)$.

Then, using (1.18) and (1.20), the $(0,3)$-tensors $[J, J]$ and $\{J, J\}$ can be expressed in terms of $F$ by:

\begin{align}
[J, J](x, y, z) &= F(x, Jy, z) - F(y, Jx, z) \\
&\quad + F(Jx, y, z) - F(Jy, x, z), \tag{1.22}
\end{align}

\begin{align}
\{J, J\}(x, y, z) &= F(x, Jy, z) + F(y, Jx, z) \\
&\quad + F(Jx, y, z) + F(Jy, x, z), \tag{1.23}
\end{align}

or equivalently

\begin{align}
\{J, J\}(x, y, z) &= F(Jx, y, z) + F(Jy, x, z) \\
&\quad - F(x, y, Jz) - F(y, x, Jz). \tag{1.24}
\end{align}

By virtue of (1.1), (1.6), (1.22) and (1.23), we get the following properties of $[J, J]$ and $\{J, J\}$:

\begin{align}
[J, J](x, y, z) &= [J, J](x, Jy, Jz) = [J, J](Jx, y, Jz) \\
&\quad = -[J, J](Jx, y, z), \tag{1.25}
\end{align}

\begin{align}
[J, J](Jx, y, z) &= [J, J](x, Jy, z) = -[J, J](x, y, Jz); \\
\end{align}

\begin{align}
\{J, J\}(x, y, z) &= \{J, J\}(x, Jy, Jz) = \{J, J\}(Jx, y, Jz) \\
&\quad = -\{J, J\}(Jx, Jy, z), \\
\{J, J\}(Jx, y, z) &= \{J, J\}(x, Jy, z) = -\{J, J\}(x, y, Jz). \tag{1.26}
\end{align}

**Theorem 1.1.** The fundamental tensor $F$ of an almost Norden manifold $(\mathcal{M}, J, g)$ is expressed in terms of the Nijenhuis tensor $[J, J]$ and its associated Nijenhuis tensor $\{J, J\}$ by the formula

\begin{align}
F(x, y, z) &= -\frac{1}{4}\{[J, J](Jx, y, z) + [J, J](Jx, z, y) \\
&\quad + \{J, J\}(Jx, y, z) + \{J, J\}(Jx, z, y)\}. \tag{1.27}
\end{align}
Proof. Taking the sum of (1.22) and (1.23), we obtain
\begin{align*}
F(Jx, y, z) + F(x, Jy, z) &= \frac{1}{2}\{[J, J](x, y, z) + \{J, J\}(x, y, z)\}.
\end{align*}

The identities (1.1) and (1.6) imply
\begin{align*}
F(x, z, Jy) &= -F(x, y, Jz).
\end{align*}

A suitable combination of (1.28) and (1.29) yields
\begin{align*}
F(Jx, y, z) &= \frac{1}{4}\{[J, J](x, y, z) + [J, J](x, z, y) + \{J, J\}(x, y, z) + \{J, J\}(x, z, y)\}.
\end{align*}

Applying (1.1) to (1.30), we obtain the stated formula. 

As direct corollaries of Theorem 1.1, for the classes of the considered manifolds with vanishing \([J, J]\) or \(\{J, J\}\), we have respectively:
\begin{align*}
\mathcal{W}_1 \oplus \mathcal{W}_2 : F(x, y, z) &= -\frac{1}{4}\{\{J, J\}(Jx, y, z) + \{J, J\}(Jx, z, y)\}, \\
\mathcal{W}_3 : F(x, y, z) &= -\frac{1}{4}\{[J, J](Jx, y, z) + [J, J](Jx, z, y)\}.
\end{align*}

According to Theorem 1.1, we obtain the following relation between the corresponding traces:
\begin{align*}
\theta &= \frac{1}{4}\vartheta \circ J,
\end{align*}
where we denote
\begin{align*}
\vartheta(z) &= g^{ij}\{J, J\}(e_i, e_j, z).
\end{align*}

For the traces \(\tilde{\theta}\) and \(\tilde{\vartheta}\) with respect to the associated metric \(\tilde{g}\) of \(F\) and \(\{J, J\}\), i.e.
\begin{align*}
\tilde{\theta}(z) &= \tilde{g}^{ij}F(e_i, e_j, z), \\
\tilde{\vartheta}(z) &= \tilde{g}^{ij}\{J, J\}(e_i, e_j, z),
\end{align*}

we obtain the following interrelations:
\begin{align*}
\tilde{\theta} &= -\frac{1}{4}\vartheta = \theta \circ J, \\
\tilde{\vartheta} &= 4\theta = \vartheta \circ J.
\end{align*}

Then, bearing in mind (1.10) and the subsequent comments on the pair of the Nijenhuis tensors, from Theorem 1.1 and (1.32) we obtain immediately the following
Theorem 1.2. The classes of almost Norden manifolds are characterized by the pair of Nijenhuis tensors \([J, J]\) and \(\{J, J\}\) as follows:

\[
\begin{align*}
\mathcal{W}_0 : & \ [J, J] = 0, \ \{J, J\} = 0; \\
\mathcal{W}_1 : & \ [J, J] = 0, \ \{J, J\} = \frac{1}{2n} \{g \otimes \vartheta + \tilde{g} \otimes \tilde{\vartheta}\}; \\
\mathcal{W}_2 : & \ [J, J] = 0, \ \vartheta = 0; \\
\mathcal{W}_3 : & \ \{J, J\} = 0; \\
\mathcal{W}_1 \oplus \mathcal{W}_2 : & \ [J, J] = 0; \\
\mathcal{W}_1 \oplus \mathcal{W}_3 : & \ \mathcal{G}\{J, J\} = \frac{1}{2n} \mathcal{G}\{g \otimes \vartheta + \tilde{g} \otimes \tilde{\vartheta}\}; \\
\mathcal{W}_2 \oplus \mathcal{W}_3 : & \ \vartheta = 0; \\
\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 : & \ no \ conditions.
\end{align*}
\]

1.3. Second covariant derivatives

Let \(R^D\) be the curvature tensor of \(D\) defined as usual by

\[
R^D(x, y)z = D_x D_y z - D_y D_x z - D_{[x,y]} z.
\]

As it is known, the latter formula can be rewritten as

\[
R^D(x, y)z = D^2_{x,y} z - D^2_{y,x} z,
\]

using the second covariant derivative given by

\[
D^2_{x,y} z = D_x D_y z - D_{D_{x,y}} z.
\]

The corresponding tensor of type \((0,4)\) with respect to the metric \(g\) is determined by

\[
R^D(x, y, z, w) = g(R^D(x, y)z, w).
\]

It has the following properties:

\[
R^D(x, y, z, w) = -R^D(y, x, z, w) = -R^D(x, y, w, z),
\]

\[
R^D(x, y, z, w) + R^D(y, z, x, w) + R^D(z, x, y, w) = 0.
\]

Any tensor of type \((0,4)\) satisfying (1.35) is called a curvature-like tensor. Moreover, if the curvature-like tensor \(R^D\) has the property

\[
R^D(x, y, Jz, Jw) = -R^D(x, y, z, w),
\]

it is called a Kähler tensor [36].
The Ricci tensor $\rho^D$ and the scalar curvature $\tau^D$ for the curvature tensor of $D$ (and similarly for every curvature-like tensor) are defined as usual by

$$
\rho^D(y, z) = g^{ij} R^D(e_i, y, z, e_j), \quad \tau^D = g^{ij} \rho^D(e_i, e_j).
$$

It is well-known that the Weyl tensor $C^D$ on a pseudo-Riemannian manifold $(\mathcal{M}, g)$, $\dim \mathcal{M} = 2n \geq 4$, is given by

(1.36) \quad C^D = R^D + \frac{1}{2(n-1)} g \otimes \rho^D - \frac{\tau^D}{4(n-1)(2n-1)} g \otimes g,

where $g \otimes \rho^D$ is the Kulkarni-Nomizu product of $g$ and $\rho^D$, i.e.

\begin{align*}
(g \otimes \rho^D)(x, y, z, w) &= g(x, z) \rho^D(y, w) - g(y, z) \rho^D(x, w) \\
&\quad + g(y, w) \rho^D(x, z) - g(x, w) \rho^D(y, z).
\end{align*}

Moreover, $C^D$ vanishes if and only if the manifold $(\mathcal{M}, g)$ is conformally flat, i.e. it is transformed into a flat manifold by an usual conformal transformation of the metric defined by $\overline{g} = e^{2u} g$ for a differentiable function $u$ on $\mathcal{M}$.

Let $R^\widetilde{D}$ be the curvature tensor of $\widetilde{D}$ defined as usually. Obviously, the corresponding curvature $(0,4)$-tensor with respect to the metric $\widetilde{g}$ is

$$
R^\widetilde{D}(x, y, z, w) = \widetilde{g}(R^\widetilde{D}(x, y)z, w)
$$

and it has the same properties as in (1.35). The Weyl tensor $C^\widetilde{D}$ is generated by $\widetilde{D}$ and $\widetilde{g}$ by the same way and it has the same geometrical interpretation for the manifold $(\mathcal{M}, J, \widetilde{g})$.

\* \* \*
§2. Invariant tensors under the twin interchange of Norden metrics on almost complex manifolds

The object of study in the present section are almost complex manifolds with a pair of Norden metrics, mutually associated by means of the almost complex structure. More precisely, a torsion-free connection and tensors with certain geometric interpretation are found which are invariant under the twin interchange, i.e. the swap of the counterparts of the pair of Norden metrics and the corresponding Levi-Civita connections. A Lie group depending on four real parameters is considered as an example of a 4-dimensional manifold of the studied type. The mentioned invariant objects are found in an explicit form.

The main results of this section are published in [89].

An interesting problem on almost Norden manifolds is the presence of tensors with some geometric interpretation which are invariant under the so-called twin interchange. This is the swap of the counterparts of the pair of Norden metrics and their Levi-Civita connections. Similar results for the considered manifolds in the basic classes $\mathcal{W}_1$ and $\mathcal{W}_3$ are obtained in [141] and [25], [114], respectively. The aim here is to solve the problem in general.

The present section is organised as follows. In Subsection 2.1 we present the main results on the topic about the invariant objects and their vanishing. In Subsection 2.2 we consider an example of the studied manifolds of dimension 4 by means of a construction of an appropriate family of Lie algebras depending on 4 real parameters. Then we compute the basic components of the invariant objects introduced in the previous subsection.
2.1. The twin interchange corresponding to the pair of Norden metrics and their Levi-Civita connections

Bearing in mind (1.2), we give the following

Definition 2.1. The interchange of the Levi-Civita connections $D$ and $\tilde{D}$ (and respectively their corresponding Norden metrics $g$ and $\tilde{g}$) we call the twin interchange.

2.1.1. Invariant classification

Let us consider the potential $\Phi$ of $\tilde{D}$ regarding $D$ on $(\mathcal{M}, J, g)$ defined by (1.12).

Lemma 2.1. The potential $\Phi(x, y)$ is an anti-invariant tensor under the twin interchange, i.e.

$$(2.1) \quad \tilde{\Phi}(x, y) = -\Phi(x, y).$$

Proof. The equalities (1.6), (1.13), (1.14) and (1.15) imply the following relation between $F$ and its corresponding tensor $\tilde{F}$ for $(\mathcal{M}, J, \tilde{g})$, defined by

$$\tilde{F}(x, y, z) = \tilde{g}(\tilde{D}_x J y, z),$$

$$\tilde{F}(x, y, z) = \frac{1}{2} \left\{ F(Jy, z, x) - F(y, Jz, x) \right\}.$$  \hspace{1cm} (2.2)

Using (2.2), we write the corresponding formula for $\tilde{\Phi}$ and $\tilde{F}$ as

$$\tilde{\Phi}(x, y, z) = \frac{1}{2} \left\{ \tilde{F}(Jz, x, y) - \tilde{F}(x, y, Jz) \right\} - \tilde{F}(y, x, Jz).$$  \hspace{1cm} (2.3)

Using (2.2) and (2.3), we get an expression of $\tilde{\Phi}$ in terms of $F$ and then by (1.15) we obtain

$$\tilde{\Phi}(x, y, z) = -\Phi(x, y, Jz).$$  \hspace{1cm} (2.4)

Taking into account that $\tilde{\Phi}(x, y, z)$ is defined by

$$\tilde{\Phi}(x, y, z) = \tilde{g}(\tilde{\Phi}(x, y), z),$$

we accomplish the proof. \qed
In [35], for an arbitrary almost Norden manifold, it is given the following identity
\[
\Phi(x, y, z) - \Phi(Jx, Jy, z) - \Phi(Jx, y, Jz) - \Phi(x, Jy, Jz) = 0.
\]
(2.5)

The associated 1-forms \( f \) and \( f^* \) of \( \Phi \) are defined by
\[
f(z) = g^{ij} \Phi(e_i, e_j, z), \quad f^*(z) = g^{ij} \Phi(e_i, Je_j, z).
\]

Then, from (2.5) we get the identity
\[
f(z) = f^*(Jz).
\]
(2.6)

The latter identity resembles the equality \( \theta(z) = \theta^*(Jz) \), equivalent to (1.9). Indeed, there exists a relation between the associated 1-forms of \( \Phi \) and \( F \). It follows from (1.15) and has the form
\[
f(z) = \theta^*(z), \quad f^*(z) = -\theta(z).
\]
(2.7)

**Lemma 2.2.** The associated 1-forms \( f \) and \( f^* \) of \( \Phi \) are invariant under the twin interchange, i.e.
\[
\tilde{f}(z) = f(z), \quad \tilde{f}^*(z) = f^*(z).
\]

**Proof.** Taking the trace of (2.4) by \( \tilde{g}^{ij} = -J^i_k g^{jk} \) for \( x = e_i \) and \( y = e_j \), we have \( \tilde{f}(z) = f^*(Jz) \). Then, comparing the latter equality and (2.6), we obtain the statement for \( f \). The relation in the case of \( f^* \) is valid because of (2.6). \( \square \)

**Lemma 2.3.** The Lee forms \( \theta \) and \( \theta^* \) are invariant under the twin interchange, i.e.
\[
\theta(z) = \tilde{\theta}(z), \quad \theta^*(z) = \tilde{\theta}^*(z).
\]

**Proof.** It follows directly from Lemma 2.2 and (2.7). \( \square \)

**Theorem 2.4.** All classes \( \mathcal{W}_i \) of almost Norden manifolds are invariant under the twin interchange.

**Proof.** We use the classification by \( \Phi \) in [36], the definitions of all classes are given in (1.17).

Applying Lemma 2.1, Lemma 2.2 as well as equalities (2.4) and (1.14), we get the following conditions for the considered classes in terms of \( \tilde{\Phi} \):
\[
\mathcal{W}_0 : \ \tilde{\Phi} = 0;
\]
\[
\mathcal{W}_1 : \ \tilde{\Phi}(x, y, z) = \frac{1}{2n} \left\{ \tilde{g}(x, y)\tilde{f}(z) + \tilde{g}(x, Jy)\tilde{f}(Jz) \right\};
\]
\[
\mathcal{W}_2 : \ \tilde{\Phi}(x, y, z) = -\tilde{\Phi}(Jx, Jy, z), \quad \tilde{f} = 0;
\]
\[ \mathcal{W}_3 : \tilde{\Phi}(x, y, z) = \tilde{\Phi}(Jx, Jy, z); \]
\[ \mathcal{W}_1 \oplus \mathcal{W}_2 : \tilde{\Phi}(x, y, z) = -\tilde{\Phi}(Jx, Jy, z); \]
\[ \mathcal{W}_2 \oplus \mathcal{W}_3 : \tilde{f} = 0; \]
\[ \mathcal{W}_1 \oplus \mathcal{W}_3 : \tilde{\Phi}(x, y, z) = \tilde{\Phi}(Jx, Jy, z) \]
\[ + \frac{1}{n} \left\{ \tilde{g}(x, y)f(z) + \tilde{g}(x, Jy)f(Jz) \right\}; \]
\[ \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 : \text{no condition.} \]

Taking into account the latter characteristic conditions of the considered classes, we obtain the truthfulness of the statement. \( \square \)

Let us note that the invariance of \( \mathcal{W}_1 \) and \( \mathcal{W}_3 \) is proved in [35] and [114], respectively.

Actually, by Theorem 2.4 we establish that the classification with basic classes \( \mathcal{W}_i \) (\( i = 1, 2, 3 \)) has four equivalent forms: in terms of \( F, \tilde{F}, \Phi \) and \( \tilde{\Phi} \).

### 2.1.2. Invariant connection

Let us define an affine connection \( D^\circ \) by
\[ D^\circ_x y = D_x y + \frac{1}{2} \Phi(x, y). \]  
(2.8)

By virtue of (1.13), (2.1) and (2.8), we have the following
\[ \tilde{D}^\circ_x y = \tilde{D}_x y + \frac{1}{2} \tilde{\Phi}(x, y) = D_x y + \Phi(x, y) - \frac{1}{2} \Phi(x, y) \]
\[ = D_x y + \frac{1}{2} \Phi(x, y) = D^\circ_x y. \]

Therefore, \( D^\circ \) is an invariant connection under the twin interchange. Bearing in mind (1.12), we establish that \( D^\circ \) is actually the average connection of \( D \) and \( \tilde{D} \), because
\[ D^\circ_x y = D_x y + \frac{1}{2} \Phi(x, y) = D_x y + \frac{1}{2} \left\{ \tilde{D}_x y - D_x y \right\} \]
\[ = \frac{1}{2} \left\{ D_x y + \tilde{D}_x y \right\}. \]  
(2.9)

So, we obtain

**Proposition 2.5.** The average connection \( D^\circ \) of \( D \) and \( \tilde{D} \) is an invariant connection under the twin interchange.
Corollary 2.6. If the invariant connection $D^\circ$ vanishes then $(\mathcal{M}, J, g)$ and $(\mathcal{M}, J, \tilde{g})$ are Kähler-Norden manifolds and $D = \tilde{D}$ also vanishes.

Proof. Let us suppose that $D^\circ$ vanishes. Then we have the following relations $D = -\tilde{D}$ and $\Phi = -2 D$, because of (2.8) and (2.9). Hence we obtain

$$[x, y] = D_x y - D_y x = -\frac{1}{2} \{ \Phi(x, y) - \Phi(y, x) \} = 0$$

and consequently, using the corresponding Koszul formula for $g$ and $D$

$$2g(D_x y, z) = x (g(y, z)) + y (g(x, z)) - z (g(x, y)) + g([x, y], z) + g([z, x], y) + g([z, y], x),$$

we get that $D$ vanishes. Thus, $\tilde{D}$ and $\Phi$ vanish as well as $(\mathcal{M}, J, g)$ and $(\mathcal{M}, J, \tilde{g})$ belong to $\mathcal{W}_0$. \qed

2.1.3. Invariant tensors

As it is well-known, the Nijenhuis tensor $[J, J]$ of the almost complex structure $J$ is defined by (1.18). Besides $[J, J]$, in (1.20) it is defined the (1,2)-tensor $\{J, J\}$, called associated Nijenhuis tensor of $J$ and $g$.

Proposition 2.7. The Nijenhuis tensor is invariant and the associated Nijenhuis tensor is anti-invariant under the twin interchange, i.e.

$$[J, J](x, y) = [\tilde{J}, J](x, y), \quad \{J, J\}(x, y) = -\{\tilde{J}, J\}(x, y).$$

Proof. The relations of $[J, J]$ and $\{J, J\}$ with $\Phi$ are given in [35] as follows

$$[J, J](x, y, z) = 2\Phi(z, Jx, Jy) - 2\Phi(z, x, y),$$

$$\{J, J\}(x, y, z) = 2\Phi(x, y, z) - 2\Phi(Jx, Jy, z).$$

Using (2.4), the latter equalities imply the following

$$[\tilde{J}, J](x, y, z) = -[J, J](x, Jy, z),$$

$$\{\tilde{J}, J\}(x, y, z) = -\{J, J\}(x, y, Jz).$$

In (1.25), it is given the property $[J, J](x, y, z) = [J, J](x, Jy, Jz)$ which is equivalent to $[J, J](x, Jy, z) = -[J, J](x, y, Jz)$. Then (2.13) gets the form

$$[\tilde{J}, J](x, y, z) = [J, J](x, y, Jz).$$

The equalities (2.15) and (2.14) yield the relations in the statement. \qed
The following relation between the curvature tensors of $D$ and $\tilde{D}$, related by (1.13), is well-known:

\begin{equation}
R^{\tilde{D}}(x,y)z = R^D(x,y)z + P(x,y)z,
\end{equation}

where

\begin{equation}
P(x,y)z = (D_x \Phi)(y,z) - (D_y \Phi)(x,z) + \Phi(x,\Phi(y,z)) - \Phi(y,\Phi(x,z)).
\end{equation}

Let us consider the following tensor $A$, which is part of $P$:

\begin{equation}
A(x,y)z = \Phi(x,\Phi(y,z)) - \Phi(y,\Phi(x,z)).
\end{equation}

**Lemma 2.8.** The tensor $A(x,y)z$ is invariant under the twin interchange, i.e.

\begin{equation}
A(x,y)z = \tilde{A}(x,y)z.
\end{equation}

**Proof.** Since (2.1) is valid, we obtain immediately

\[
\Phi(x,\Phi(y,z)) - \Phi(y,\Phi(x,z)) = \tilde{\Phi}(x,\tilde{\Phi}(y,z)) - \tilde{\Phi}(y,\tilde{\Phi}(x,z)),
\]

which yields relation (2.19).

**Lemma 2.9.** The tensor $P(x,y)z$ is anti-invariant under the twin interchange, i.e.

\begin{equation}
\tilde{P}(x,y)z = -P(x,y)z.
\end{equation}

**Proof.** For the covariant derivative of $\Phi$ we have

\[
(D_x \Phi)(y,z) = D_x \Phi(y,z) - \Phi(D_x y, z) - \Phi(y, D_x z).
\]

Applying (1.13) and (2.1), we get

\[
(D_x \Phi)(y,z) = - (\tilde{D}_x \tilde{\Phi})(y,z) - \tilde{\Phi}(x, \tilde{\Phi}(y,z))
+ \tilde{\Phi}(y, \tilde{\Phi}(x,z)) + \tilde{\Phi}(z, \tilde{\Phi}(x,y)).
\]

As a sequence of the latter equality and (2.18) we obtain

\begin{equation}
(D_x \Phi)(y,z) - (D_y \Phi)(x,z)
= - (\tilde{D}_x \tilde{\Phi})(y,z) + (\tilde{D}_y \tilde{\Phi})(x,z) - 2\tilde{A}(x,y)z.
\end{equation}

Then, (2.17), (2.18), (2.19) and (2.21) imply relation (2.20). \qed
**Proposition 2.10.** The curvature tensor $R^D$ of the average connection $\bar{D}$ for $D$ and $\tilde{D}$ is an invariant tensor under the twin interchange, i.e. $R^D(x, y)z = R^{\bar{D}}(x, y)z$.

**Proof.** From (2.8), using the formulae (1.13), (2.16), (2.17) and (2.18), we get the following relation

$$R^D(x, y)z = R^D(x, y)z + \frac{1}{2} (D_x \Phi)(y, z) - \frac{1}{2} (D_y \Phi)(x, z) + \frac{1}{4} A(x, y)z,$$

which is actually

$$R^D(x, y)z = R^D(x, y)z + \frac{1}{2} P(x, y)z - \frac{1}{4} A(x, y)z. \quad (2.22)$$

By virtue of (2.16), (2.19), (2.20) and (2.21), we establish the relation $\bar{R}^D = R^{\bar{D}}$. □

As a consequence of (2.22), we obtain the following

**Corollary 2.11.** The invariant tensor $R^D$ vanishes if and only if the following equality is valid

$$R^D(x, y)z = -\frac{1}{2} P(x, y)z + \frac{1}{4} A(x, y)z.$$

Let us consider the average tensor $B$ of the curvature tensors $R^D$ and $R^{\tilde{D}}$, respectively, i.e. $B(x, y)z = \frac{1}{2} \{R^D(x, y)z + R^{\tilde{D}}(x, y)z\}$. Then by (2.16) we have

$$B(x, y)z = R^D(x, y)z + \frac{1}{2} P(x, y)z. \quad (2.23)$$

**Proposition 2.12.** The average tensor $B$ of $R^D$ and $R^{\tilde{D}}$ is an invariant tensor under the twin interchange, i.e. $B(x, y)z = \tilde{B}(x, y)z$.

**Proof.** Using (2.16), (2.23) and (2.1), we have the following

$$\tilde{B}(x, y)z = R^{\tilde{D}}(x, y)z + \frac{1}{2} \tilde{P}(x, y)z$$

$$= R^D(x, y)z + P(x, y)z - \frac{1}{2} P(x, y)z$$

$$= R^D(x, y)z + \frac{1}{2} P(x, y)z = B(x, y)z. \quad \Box$$
Corollary 2.13. The invariant tensor $B$ vanishes if and only if the following equality is valid
\[ R^D = -\frac{1}{2} P. \]

By virtue of (2.22) and (2.23), we have the following relation between the invariant tensors $R^D$, $B$ and $A$

(2.24) \[ R^D(x, y) z = B(x, y) z - \frac{1}{4} A(x, y) z. \]

Theorem 2.14. Any linear combination of the average tensor $B$ of the curvature tensors $R^D$ and $R^\tilde{D}$ and the curvature tensor $R^D$ of the average connection $D^\diamond$ for $D$ and $\tilde{D}$ is an invariant tensor under the twin interchange.

Proof. It follows from Proposition 2.10 and Proposition 2.12.

2.1.4. Invariant connection and invariant tensors on the manifolds in the main class

Now, we consider an arbitrary manifold $(\mathcal{M}, J, g)$ belonging to the basic class $\mathcal{W}_1$. This class is known as the main class in the classification in [34], because it is the only class where the fundamental tensor $F$ and the potential $\Phi$ are expressed explicitly by the metric. In this case, we have the form of $F$ and $\Phi$ in (1.10) and (1.17), respectively. Taking into account (2.2), (1.10) and (1.6), we obtain the following form of $F$ under the twin interchange

\[ \tilde{F}(x, y, z) = -\frac{1}{2n} \{ g(x, y)\theta(Jz) + g(x, z)\theta(Jy) \]
\[ - g(x, Jy)\theta(z) - g(x, Jz)\theta(y) \} . \]

Therefore, we get the following relation for a $\mathcal{W}_1$-manifold, i.e. an almost Norden manifold belonging to the class $\mathcal{W}_1$,

\[ \tilde{F}(x, y, z) = F(Jx, y, z). \]

The invariant connection $D^\diamond$ has the following form on a $\mathcal{W}_1$-manifold, applying the definition from (1.17) in (2.8),

\[ D^\diamond_x y = D_x y + \frac{1}{4n} \{ g(x, y)f^z + g(x, Jy)Jf^z \} , \]
where \( f^\sharp \) is the dual vector of the 1-form \( f \) regarding \( g \), i.e.
\[
f(z) = g(f^\sharp, z).
\]

The presence of the first equality in (1.17), the explicit expression of \( \Phi \) in terms of \( g \) for the case of a \( \mathcal{W}_1 \)-manifold, gives us a chance to find a more concrete form of \( P \) and \( A \) defined by (2.17) and (2.18), respectively. This expression gives results in the corresponding relations between \( R^D \) and \( \tilde{R}^D \), \( R^D \), \( B \), given in (2.16), (2.22), (2.23), respectively. A relation between \( R^D \) and \( \tilde{R}^D \) for a \( \mathcal{W}_1 \)-manifold is given in [141] but using \( \theta \).

**Proposition 2.15.** If \((\mathcal{M}, J, g)\) is an almost Norden manifold belonging to the class \( \mathcal{W}_1 \), then the tensors \( P \) and \( A \) have the following form, respectively:
\[
P(x, y)z = \frac{1}{2n} \left\{ g(y, z)p(x) + g(y, Jz)Jp(x) \\
- g(x, z)p(y) - g(x, Jz)Jp(y) \right\},
\]
\[
A(x, y)z = \frac{1}{4n^2} \left\{ g(y, z)a(x) + g(y, Jz)a(Jx) \\
- g(x, z)a(y) - g(x, Jz)a(Jy) \right\},
\]
where
\[
p(x) = D_x f^\sharp + \frac{1}{2n} \left\{ f(x)f^\sharp - f(f^\sharp)x - f(Jf^\sharp)Jx \right\}
\]
\[
a(x) = f(x)f^\sharp + f(Jx)Jf^\sharp.
\]

**Proof.** The formulae follow by direct computations, using (1.10), (1.17), (2.17) and (2.18). \( \Box \)

### 2.2. Lie group as a manifold from the main class and the invariant connection and the invariant tensors on it

In this subsection we consider an example of a 4-dimensional Lie group as a \( \mathcal{W}_1 \)-manifold given in [142].

Let \( L \) be a 4-dimensional real connected Lie group, and let \( \mathfrak{l} \) be its Lie algebra with a basis \( \{x_1, x_2, x_3, x_4\} \).

We introduce an almost complex structure \( J \) and a Norden metric by
\[
Jx_1 = x_3, \quad Jx_2 = x_4, \quad Jx_3 = -x_1, \quad Jx_4 = -x_2,
\]
(2.25)
\[
g(x_1, x_1) = g(x_2, x_2) = -g(x_3, x_3) = -g(x_4, x_4) = 1,
\]
(2.26)
\[
g(x_i, x_j) = 0, \quad i \neq j \quad (i, j = 1, 2, 3, 4).
\]
Then, the associated Norden metric \( \tilde{g} \) is determined by its non-zero components

\[
\tilde{g}(x_1, x_3) = \tilde{g}(x_2, x_4) = -1.
\]

Let us consider \((\mathcal{L}, J, g)\) with the Lie algebra \(I\) determined by the following nonzero commutators:

\begin{align*}
[x_1, x_4] &= [x_2, x_3] = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4, \\
[x_1, x_3] &= [x_4, x_2] = \lambda_2 x_1 - \lambda_1 x_2 + \lambda_4 x_3 - \lambda_3 x_4,
\end{align*}

where \(\lambda_i \in \mathbb{R} \ (i = 1, 2, 3, 4)\).

In [142], it is proved that \((\mathcal{L}, J, g)\) is a \(\mathcal{W}_1\)-manifold. Since the class \(\mathcal{W}_1\) is invariant under the twin interchange, according to Theorem 2.4, it follows that \((\mathcal{L}, J, \tilde{g})\) belongs to \(\mathcal{W}_1\), too.

**Theorem 2.16.** Let \((\mathcal{L}, J, g)\) and \((\mathcal{L}, J, \tilde{g})\) be the pair of \(\mathcal{W}_1\)-manifolds, determined by (2.25)–(2.28). Then both the manifolds:

(i) belong to the class of the locally conformal Kähler-Norden manifolds if and only if

\[
\lambda_1^2 - \lambda_2^2 + \lambda_3^2 - \lambda_4^2 = \lambda_1 \lambda_2 + \lambda_3 \lambda_4 = 0;
\]

(ii) are locally conformally flat by usual conformal transformations and the curvature tensors \(R^D\) and \(\tilde{R}^\tilde{D}\) have the following form, respectively:

\[
R^D = -\frac{1}{2} g \otimes \rho^D + \frac{1}{12} \tau^D g \otimes g,
\]

\[
\tilde{R}^\tilde{D} = -\frac{1}{2} \tilde{g} \otimes \tilde{\rho}^\tilde{D} + \frac{1}{12} \tilde{\tau}^\tilde{D} \tilde{g} \otimes \tilde{g}.
\]

(iii) are scalar flat and isotropic Kählerian if and only if the following conditions are satisfied, respectively:

\[
\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 = 0, \quad \lambda_1 \lambda_3 + \lambda_2 \lambda_4 = 0.
\]

**Proof.** According to (1.2), (2.25), (2.26), (2.28) and the Koszul formula for \(g, D\) and \(\tilde{g}, \tilde{D}\), we get the following nonzero components of \(D\) and \(\tilde{D}\):

\[
\begin{align*}
D_{x_1} x_1 &= D_{x_2} x_2 = \tilde{D}_{x_3} x_3 = \tilde{D}_{x_4} x_4 = \lambda_2 x_3 + \lambda_1 x_4, \\
D_{x_1} x_3 &= D_{x_4} x_2 = -\tilde{D}_{x_3} x_1 = \lambda_2 x_1 - \lambda_3 x_2, \\
D_{x_1} x_4 &= -D_{x_3} x_2 = \tilde{D}_{x_4} x_1 = \lambda_1 x_1 + \lambda_3 x_3, \\
D_{x_2} x_3 &= -D_{x_4} x_1 = \tilde{D}_{x_2} x_3 = -\tilde{D}_{x_4} x_1 = \lambda_2 x_2 + \lambda_4 x_4,
\end{align*}
\]
\[(2.31b)\]
\[
\begin{align*}
D_{x_2} x_4 & = D_{x_3} x_1 = -\tilde{D}_{x_1} x_3 = -\tilde{D}_{x_4} x_2 = \lambda_1 x_2 - \lambda_4 x_3, \\
D_{x_3} x_4 & = D_{x_4} x_3 = \tilde{D}_{x_1} x_1 = \tilde{D}_{x_2} x_2 = -\lambda_4 x_1 - \lambda_3 x_2.
\end{align*}
\]

The components of $D J$ and $\tilde{D} J$ follow from (2.31) and (2.25). Then, using (2.26), (2.27) and (1.5), we get the following nonzero components $(F)_{ijk} = F(x_i, x_j, x_k)$ and
\[
(\tilde{F})_{ijk} = \tilde{F}(x_i, x_j, x_k) = \tilde{g}(\tilde{D}_{x_i} J) x_j, x_k)
\]
of $F$ and $\tilde{F}$, respectively:
\[
\begin{align*}
\lambda_1 & = (F)_{112} = (F)_{121} = (F)_{134} = (F)_{143} \\
& = \frac{1}{2}(F)_{222} = \frac{1}{2}(F)_{244} \\
& = (F)_{314} = -(F)_{323} = -(F)_{332} = (F)_{341}, \\
\lambda_2 & = \frac{1}{2}(F)_{111} = \frac{1}{2}(F)_{133} \\
& = (F)_{212} = (F)_{221} = (F)_{234} = (F)_{243} \\
& = -(F)_{414} = (F)_{423} = (F)_{432} = -(F)_{441}, \\
\lambda_3 & = (F)_{114} = -(F)_{123} = -(F)_{132} = (F)_{141} \\
& = -(F)_{312} = -(F)_{321} = -(F)_{334} = -(F)_{343} \\
& = -\frac{1}{2}(F)_{422} = -\frac{1}{2}(F)_{444}, \\
\lambda_4 & = -(F)_{214} = (F)_{223} = (F)_{232} = -(F)_{241} \\
& = -\frac{1}{2}(F)_{333} = -\frac{1}{2}(F)_{311} \\
& = -(F)_{412} = -(F)_{421} = -(F)_{434} = -(F)_{443};
\end{align*}
\]
\[
(2.33a)
\]
\[
\begin{align*}
\lambda_1 & = (\tilde{F})_{114} = -(\tilde{F})_{123} = -(\tilde{F})_{132} = (\tilde{F})_{141} \\
& = -(\tilde{F})_{312} = -(\tilde{F})_{321} = -(\tilde{F})_{334} = -(\tilde{F})_{343} \\
& = -\frac{1}{2}(\tilde{F})_{422} = -\frac{1}{2}(\tilde{F})_{444}, \\
\lambda_2 & = -(\tilde{F})_{214} = (\tilde{F})_{223} = (\tilde{F})_{232} = -(\tilde{F})_{241} \\
& = -\frac{1}{2}(\tilde{F})_{311} = -\frac{1}{2}(\tilde{F})_{333} \\
& = -(\tilde{F})_{412} = -(\tilde{F})_{421} = -(\tilde{F})_{434} = -(\tilde{F})_{443},
\end{align*}
\]
\[ \lambda_3 = - (\tilde{F})_{112} = -(\tilde{F})_{121} = -(\tilde{F})_{134} = -(\tilde{F})_{143} \]
\[ = -\frac{1}{2} (\tilde{F})_{222} = -\frac{1}{2} (\tilde{F})_{244} \]
\[ = -(\tilde{F})_{314} = (\tilde{F})_{323} = (\tilde{F})_{332} = -(\tilde{F})_{341}, \]
\[ \lambda_4 = -\frac{1}{2} (\tilde{F})_{111} = -\frac{1}{2} (\tilde{F})_{133} \]
\[ = -(\tilde{F})_{212} = -(\tilde{F})_{221} = -(\tilde{F})_{234} = -(\tilde{F})_{243} \]
\[ = (\tilde{F})_{414} = -(\tilde{F})_{423} = -(\tilde{F})_{432} = (\tilde{F})_{441}. \]

Applying (1.11) for the components in (2.32) and (2.33), we obtain the square norms of \( DJ \) and \( \tilde{D}J \):
\[
\| DJ \|^2 = 16 (\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2),
\]
\[
\| \tilde{D}J \|^2 = \lambda_1 \lambda_3 + \lambda_2 \lambda_4.
\]

Let us consider the conformal transformations of the metric \( g \) defined by
\[
\bar{g} = e^{2u} \{ \cos 2v \ g + \sin 2v \ \tilde{g} \},
\]
where \( u \) and \( v \) are differentiable functions on \( \mathcal{M} \). Then, the associated metric \( \tilde{g} \) has the following image
\[
\tilde{g} = e^{2u} (\cos 2v \ \tilde{g} - \sin 2v \ g)
\]
The manifold \((\mathcal{M}, J, \bar{g})\) is again an almost Norden manifold. If \( v = 0 \), we obtain the usual conformal transformation. Let us remark that the conformal transformation for \( u = 0 \) and \( v = \pi/2 \) maps the pair \((g, \tilde{g})\) into \((\tilde{g}, -g)\).

According to [35], a \( \mathcal{W}_1 \)-manifold is locally conformal equivalent to a Kähler-Norden manifold if and only if its Lee forms \( \theta \) and \( \theta^* \) are closed. Moreover, the used conformal transformations are such that the 1-forms \( du \circ J \) and \( dv \circ J \) are closed.

Taking into account Lemma 2.3, we have
\[ \theta_k = (\tilde{\theta})_k, \quad (\theta^*)_k = (\tilde{\theta}^*)_k \]
for the corresponding components with respect to \( x_k \), \((k = 1, 2, 3, 4)\). Furthermore, the same situation is for \( d\theta = d\tilde{\theta} \) and \( d\theta^* = d\tilde{\theta}^* \). By (1.8), (1.9) and (2.32), we obtain \((\theta)_k\) and \((\theta^*)_k\) and thus we get the following
Chapter I | §2. Invariant tensors under the twin interchange of Norden metrics . . .

\[(\theta)_2 = (\theta^*)_4 = (\tilde{\theta})_2 = (\tilde{\theta}^*)_4 = 4\lambda_1,\]

\[\tilde{(\theta)}_1 = (\theta^*)_3 = (\tilde{\theta})_1 = (\tilde{\theta}^*)_3 = 4\lambda_2,\]

\[(\theta)_4 = -(\theta^*)_2 = (\tilde{\theta})_4 = -(\tilde{\theta}^*)_2 = 4\lambda_3,\]

\[(\theta)_3 = -(\theta^*)_1 = (\tilde{\theta})_3 = -(\tilde{\theta}^*)_1 = 4\lambda_4.\]

Using (2.28) and (2.36), we compute the components of \(d\theta\) and \(d\theta^*\) with respect to the basis \(\{x_1, x_2, x_3, x_4\}\). We obtain that \(d\theta^* = d\tilde{\theta}^* = 0\) and the nonzero components of \(d\theta = d\theta^*\) are

\[(d\theta)_{13} = (d\theta)_{42} = (d\tilde{\theta})_{13} = (d\tilde{\theta})_{42} = 4(\lambda_1^2 - \lambda_2^2 + \lambda_3^2 - \lambda_4^2),\]

\[(d\theta)_{14} = (d\theta)_{23} = (d\tilde{\theta})_{14} = (d\tilde{\theta})_{23} = -8(\lambda_1\lambda_2 + \lambda_3\lambda_4).\]

Therefore \((\mathcal{L}, J, g)\) and \((\mathcal{L}, J, \tilde{g})\) are locally conformal Kähler-Norden manifolds if and only if conditions (2.29) are valid. Then, the statement (i) holds.

By virtue of (2.26), (2.28) and (2.31), we get the basic components \(R_{ijkl}^D = R_{ijkl}^D(x_i, x_j, x_k, x_l)\) and \(\tilde{R}_{ijkl}^D = R_{ijkl}^D(x_i, x_j, x_k, x_l)\) of the curvature tensors for \(D\) and \(\tilde{D}\), respectively. The nonzero ones of them are determined by (1.35) and the following:

\[R_{1221}^D = \lambda_1^2 + \lambda_2^2, \quad R_{1331}^D = \lambda_4^2 - \lambda_2^2,\]

\[R_{1441}^D = \lambda_4^2 - \lambda_1^2, \quad R_{2332}^D = \lambda_3^2 - \lambda_2^2,\]

\[R_{2442}^D = \lambda_3^2 - \lambda_1^2, \quad R_{3443}^D = -\lambda_3^2 - \lambda_1^2,\]

\[R_{1341}^D = R_{2342}^D = -\lambda_1\lambda_2, \quad R_{2132}^D = -R_{4134}^D = -\lambda_1\lambda_3,\]

\[R_{1231}^D = -R_{4234}^D = \lambda_1\lambda_4, \quad R_{2142}^D = -R_{3143}^D = \lambda_2\lambda_3,\]

\[R_{1241}^D = -R_{3243}^D = -\lambda_2\lambda_4, \quad R_{3123}^D = R_{4124}^D = \lambda_3\lambda_4;\]

\[R_{1241}^D = -\lambda_3^2, \quad R_{2132}^D = -\lambda_3^2,\]

\[R_{3243}^D = -\lambda_4^2, \quad R_{4134}^D = -\lambda_4^2,\]

\[\tilde{R}_{1331}^D = 2\lambda_2\lambda_4, \quad R_{2442}^D = 2\lambda_1\lambda_3,\]

\[\tilde{R}_{1341}^D = R_{4124}^D = \lambda_2\lambda_3, \quad \tilde{R}_{1231}^D = \tilde{R}_{2142}^D = -\lambda_3\lambda_4,\]

\[R_{2342}^D = R_{3123}^D = \lambda_1\lambda_4, \quad R_{3143}^D = R_{4234}^D = -\lambda_1\lambda_2,\]

\[R_{1234}^D = \tilde{R}_{2341}^D = \lambda_1\lambda_3 + \lambda_2\lambda_4,\]

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Therefore, the components of the Ricci tensors and the values of the scalar curvatures for $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are:

\[
\begin{align*}
\rho_{11}^{\mathcal{D}} &= 2(\lambda_1^2 + \lambda_2^2 - \lambda_4^2), & \rho_{22}^{\mathcal{D}} &= 2(\lambda_2^2 + \lambda_3^2 - \lambda_1^2), \\
\rho_{33}^{\mathcal{D}} &= 2(\lambda_4^2 + \lambda_3^2 - \lambda_2^2), & \rho_{44}^{\mathcal{D}} &= 2(\lambda_1^2 + \lambda_3^2 - \lambda_2^2), \\
\rho_{11}^{\tilde{\mathcal{D}}} &= 2\lambda_1^2, & \rho_{22}^{\tilde{\mathcal{D}}} &= 2\lambda_2^2, \\
\rho_{33}^{\tilde{\mathcal{D}}} &= 2\lambda_3^2, & \rho_{44}^{\tilde{\mathcal{D}}} &= 2\lambda_4^2, \\
\rho_{13}^{\mathcal{D}} &= 2(\lambda_1\lambda_3 + 2\lambda_2\lambda_4), & \rho_{24}^{\mathcal{D}} &= 2(2\lambda_1\lambda_3 + \lambda_2\lambda_4), \\
\rho_{12}^{\mathcal{D}} = \rho_{12}^{\tilde{\mathcal{D}}} &= -2\lambda_3\lambda_4, & \rho_{23}^{\mathcal{D}} = \rho_{23}^{\tilde{\mathcal{D}}} &= 2\lambda_1\lambda_4, \\
\rho_{13}^{\mathcal{D}} &= -2\lambda_1\lambda_3, & \rho_{34}^{\mathcal{D}} = \rho_{34}^{\tilde{\mathcal{D}}} &= -2\lambda_1\lambda_2, \\
\rho_{14}^{\mathcal{D}} = \rho_{14}^{\tilde{\mathcal{D}}} &= 2\lambda_2\lambda_3, & \rho_{24}^{\mathcal{D}} &= -2\lambda_2\lambda_4;
\end{align*}
\]

(2.39) \(\tau^{\mathcal{D}} = 6(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2), \quad \tau^{\tilde{\mathcal{D}}} = -12(\lambda_1\lambda_3 + \lambda_2\lambda_4).\)

Applying (1.36) for the corresponding quantities of $\mathcal{D}$ and $\tilde{\mathcal{D}}$, we compute that the respective Weyl tensors $C$ and $\tilde{C}$ vanish. Then, we obtain the identities in (2.30). Furthermore, when the Weyl tensor vanishes then the corresponding manifold is conformal equivalent to a flat manifold by a usual conformal transformation. This completes the proof of (ii).

The truthfulness of (iii) follows immediately from (2.39) and the values of the square norms in (2.34). \(\square\)

Let us remark that the results in the latter theorem with respect to $\mathcal{D}$ are given in [142] besides (i), where it is shown a particular case of conditions (2.29).

### 2.2.1. The invariant connection and invariant tensors under the twin interchange

We compute the basic components $B_{ijk} = B(x_i, x_j) x_k$ of the invariant tensor $B$, using that this tensor is the average tensor of $R^{\mathcal{D}}$ and $R^{\tilde{\mathcal{D}}}$ and equalities (2.37), (2.38). Thus, we get the components $B_{ijkl} = g(B(x_i, x_j) x_k, x_l)$. The nonzero of them are the following and the rest are determined by the properties (1.33) for $B$:

\[
\begin{align*}
\frac{1}{2}\lambda_1^2 &= B_{3421} = -B_{2341}, & \frac{1}{2}\lambda_2^2 &= -B_{3412} = -B_{1432}, \\
\frac{1}{2}\lambda_3^2 &= -B_{1243} = -B_{1423}, & \frac{1}{2}\lambda_4^2 &= B_{1234} = -B_{2314},
\end{align*}
\]

(2.40a)
\[ \frac{1}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) = -B_{1212}, \quad \frac{1}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_4^2) = B_{1221}, \]
\[ \frac{1}{2}(\lambda_1^2 + \lambda_3^2 - \lambda_4^2) = B_{1414}, \quad \frac{1}{2}(\lambda_1^2 - \lambda_2^2 - \lambda_4^2) = -B_{1441}, \]
\[ \frac{1}{2}(\lambda_2^2 - \lambda_3^2 + \lambda_4^2) = B_{2323}, \quad \frac{1}{2}(\lambda_1^2 - \lambda_2^2 + \lambda_3^2) = B_{2332}, \]
\[ \frac{1}{2}(\lambda_1^2 + \lambda_3^2 + \lambda_4^2) = B_{3434}, \quad \frac{1}{2}(\lambda_2^2 + \lambda_3^2 + \lambda_4^2) = -B_{3443}, \]
\[ \frac{1}{2}(\lambda_2^2 - \lambda_4^2) = B_{1313} = -B_{1331}, \]
\[ \frac{1}{2}(\lambda_1^2 - \lambda_3^2) = B_{2424} = -B_{2442}, \]
\[ \frac{1}{2}(\lambda_1 \lambda_2 + \lambda_3 \lambda_4) = B_{1234} = B_{1332} = B_{2423} = B_{2441}, \]
\[ \frac{1}{2}(\lambda_2 \lambda_3 - \lambda_1 \lambda_4) = B_{1312} = -B_{1334} = -B_{2421} = B_{2443}, \]
\[ \frac{1}{2}\lambda_1 \lambda_2 = -\frac{1}{2} B_{1341} = B_{1413} = -B_{1431} = B_{2324} = -B_{2342} \]
\[ = -\frac{1}{2} B_{2432} = B_{3411} = -B_{3422} = B_{3433} = -B_{3444}, \]
\[ \frac{1}{2}\lambda_3 \lambda_4 = -B_{1211} = B_{1222} = -B_{1233} = B_{1244} = -\frac{1}{2} B_{1323} \]
\[ = -B_{1424} = B_{1442} = -B_{2313} = B_{2331} = -\frac{1}{2} B_{2414}, \]
\[ \frac{1}{2}\lambda_1 \lambda_3 = -B_{1223} = B_{1241} = B_{1421} = B_{1443} = B_{2321} \]
\[ = B_{2343} = \frac{1}{2} B_{2422} = \frac{1}{2} B_{2444} = -B_{3423} = -B_{3441}, \]
\[ \frac{1}{2}\lambda_2 \lambda_4 = B_{1214} = -B_{1232} = \frac{1}{2} B_{1311} = \frac{1}{2} B_{1333} = B_{1412} \]
\[ = B_{1434} = B_{2312} = B_{2334} = B_{3414} = -B_{3432}, \]
\[ \frac{1}{2}\lambda_1 \lambda_4 = -B_{1213} = B_{1231} = \frac{1}{2} B_{1321} = B_{2311} = B_{2322} \]
\[ = B_{2333} = B_{2344} = \frac{1}{2} B_{2434} = B_{3424} = -B_{3442}, \]
\[ \frac{1}{2}\lambda_2 \lambda_3 = B_{1224} = -B_{1242} = \frac{1}{2} B_{1343} = B_{1411} = B_{1422} \]
\[ = B_{1433} = B_{1444} = \frac{1}{2} B_{2412} = -B_{3413} = B_{3431}. \]

The rest components are determined by the property \( B_{ijk} = -B_{jik} \). Let us remark that \( B \) is not a curvature-like tensor.

Obviously, \( B = 0 \) if and only if the corresponding Lie algebra is Abelian and \((\mathcal{L}, J, g)\) is a Kähler-Norden manifold.

Using \((1,12), (2,25), (2,26), (2,31)\), we get the components \((\Phi)_{ijk} =\)
Φ(x_i, x_j, x_k) of Φ as well as the components (f)_k = f(x_k) and (f^*_k = f^*(x_k) of its associated 1-forms. The nonzero of them are the following and the rest are obtained by the property (Φ)_{ijk} = (Φ)_{jik}:

\[-\lambda_1 = -(Φ)_{114} = -(Φ)_{224} = (Φ)_{334} = (Φ)_{444} = (Φ)_{132} = (Φ)_{242} = \frac{1}{4}(f)_4 = -\frac{1}{4}(f^*_2), \]

\[-\lambda_2 = -(Φ)_{113} = -(Φ)_{223} = (Φ)_{333} = (Φ)_{443} = (Φ)_{131} = (Φ)_{241} = -\frac{1}{4}(f)_3 = \frac{1}{4}(f^*_1), \]

\[-\lambda_3 = (Φ)_{112} = (Φ)_{222} = -(Φ)_{332} = -(Φ)_{442} = (Φ)_{134} = (Φ)_{244} = \frac{1}{4}(f)_2 = \frac{1}{4}(f^*_4), \]

\[-\lambda_4 = (Φ)_{111} = (Φ)_{221} = -(Φ)_{331} = -(Φ)_{441} = (Φ)_{133} = (Φ)_{243} = \frac{1}{4}(f)_1 = \frac{1}{4}(f^*_3). \]

(2.41)

The Nijenhuis tensor vanishes on (L, J, g) and (L, J, \tilde{g}) as on any \mathcal{W}_1-manifold. According to [35], [J, J] = 0 is equivalent to

\[ Φ(x_i, x_j) = -Φ(Jx_i, Jx_j). \]

Then, by means of (2.12) we obtain for the components of the associated Nijenhuis tensor \{J, J\}_{ijk} = 4(Φ)_{ijk}, where the components of Φ are given in (2.41).

Bearing in mind (2.8), (2.31) and (2.41), we get the components of the invariant connection D^o as follows

\[
\begin{align*}
D^o_{x_1} x_1 &= D^o_{x_2} x_2 = D^o_{x_3} x_3 = D^o_{x_4} x_4 = \frac{1}{4}(\lambda_4 x_1 + \lambda_3 x_2 + \lambda_2 x_3 - \lambda_1 x_4), \\
D^o_{x_1} x_3 &= -D^o_{x_2} x_4 = -D^o_{x_3} x_1 = D^o_{x_4} x_2 = \frac{1}{2}(\lambda_2 x_1 - \lambda_1 x_2 + \lambda_4 x_3 - \lambda_3 x_4), \\
D^o_{x_1} x_4 &= -D^o_{x_3} x_2 = \lambda_1 x_1 + \lambda_3 x_3, \\
D^o_{x_2} x_3 &= -D^o_{x_4} x_1 = \lambda_2 x_2 + \lambda_4 x_4.
\end{align*}
\]

(2.42)

After that we compute the basic components R^D_{ijk} = R^D(x_i, x_j)x_k of the invariant tensor R^D under the twin interchange, using (2.24), (2.40)
and (2.41). In other way, \( R^D_{ijk} \) can be computed directly from (2.42) as the curvature tensor of \( D^\circ \).

Then, we obtain for the basic components \( K_{ijkl} \), determined by the equality \( K_{ijkl} = g(R^D(x_i, x_j x_k, x_l)) \), the following quantities:

\[
\lambda_1^2 = K_{2424}, \quad \lambda_2^2 = K_{1313}, \quad \lambda_3^2 = K_{2442}, \quad \lambda_4^2 = K_{1331},
\]

\[
\frac{1}{2} \lambda_1 \lambda_3 = -K_{1223} = K_{1241} = K_{1421} = K_{1443} = K_{2321}
\]

\[
= K_{2343} = \frac{1}{2} K_{2422} = \frac{1}{2} K_{2444} = -K_{3423} = K_{3441},
\]

\[
\frac{1}{2} \lambda_2 \lambda_4 = K_{1214} = -K_{1232} = \frac{1}{2} K_{1311} = \frac{1}{2} K_{1333} = K_{1412}
\]

\[
= K_{1434} = K_{2312} = K_{2334} = K_{3414} = -K_{3432},
\]

\[
\lambda_1 \lambda_2 + \lambda_3 \lambda_4 = K_{1314} = K_{1332} = K_{2423} = K_{2441},
\]

\[
\frac{1}{4} (\lambda_1 \lambda_2 - \lambda_3 \lambda_4) = K_{1211} = -K_{1222} = K_{1424} = -K_{1431}
\]

\[
= K_{2313} = -K_{2342} = K_{3433} = -K_{3444},
\]

\[
\frac{1}{4} (\lambda_1 \lambda_4 + \lambda_2 \lambda_3) = -K_{1213} = K_{1224} = K_{1422} = K_{1433}
\]

\[
= K_{2311} = K_{2344} = K_{3431} = -K_{3442},
\]

\[
\frac{1}{4} (\lambda_1 \lambda_2 + 3 \lambda_3 \lambda_4) = -K_{1233} = K_{1244} = K_{1442} = K_{2331},
\]

\[
\frac{1}{4} (3 \lambda_1 \lambda_2 + \lambda_3 \lambda_4) = K_{1413} = K_{2324} = K_{3411} = -K_{3422},
\]

\[
\frac{1}{4} (\lambda_1 \lambda_4 - 3 \lambda_2 \lambda_3) = K_{1242} = -K_{1411} = -K_{1444} = K_{3413},
\]

\[
\frac{1}{4} (3 \lambda_1 \lambda_4 - \lambda_2 \lambda_3) = K_{1231} = K_{2322} = K_{2333} = K_{3424},
\]

\[
\frac{1}{4} (\lambda_1^2 - 2 \lambda_2^2 + \lambda_3^2) = K_{1432} = K_{3412}, \quad \lambda_1 \lambda_2 = -K_{1341} = -K_{2432},
\]

\[
\frac{1}{4} (\lambda_2^2 - 2 \lambda_3^2 + \lambda_4^2) = K_{1243} = K_{1423}, \quad \lambda_3 \lambda_4 = -K_{1323} = -K_{2414},
\]

\[
\frac{1}{4} (2 \lambda_1^2 - \lambda_2^2 - \lambda_3^2) = -K_{2341} = K_{3421}, \quad \lambda_1 \lambda_4 = K_{1321} = K_{2434},
\]

\[
\frac{1}{4} (\lambda_1^2 + \lambda_3^2 - 2 \lambda_4^2) = -K_{1234} = K_{2314}, \quad \lambda_2 \lambda_3 = K_{1343} = K_{2412},
\]

\[
\frac{1}{4} (\lambda_1^2 + 2 \lambda_2^2 + \lambda_3^2) = -K_{1212}, \quad \frac{1}{4} (2 \lambda_1^2 + \lambda_2^2 + \lambda_3^2) = K_{1221},
\]
\[
\frac{1}{4} (\lambda_1^2 + \lambda_3^2 + 2\lambda_4^2) = K_{3434}, \quad \frac{1}{4} (\lambda_2^2 + 2\lambda_3^2 + \lambda_4^2) = -K_{3443},
\]
\[
\frac{1}{4} (3\lambda_1^2 + 3\lambda_3^2 - 2\lambda_4^2) = K_{1414}, \quad \frac{1}{4} (2\lambda_1^2 - 3\lambda_2^2 - 3\lambda_4^2) = -K_{1441},
\]
\[
\frac{1}{4} (3\lambda_2^2 - 2\lambda_3^2 + 3\lambda_4^2) = K_{2323}, \quad \frac{1}{4} (3\lambda_1^2 - 2\lambda_2^2 + 3\lambda_3^2) = K_{2332}.
\]

The rest components are determined by property \( K_{ijk} = -K_{jik} \). Let us remark that \( K \) is not a curvature-like tensor.

Obviously, \( K = 0 \) if and only if the corresponding Lie algebra is Abelian and \( (\mathcal{L}, J, g) \) is a Kähler-Norden manifold.

\(* \ * \ * \)
§3. Canonical-type connections on almost complex manifolds with Norden metric

In the present section we give a survey with additions of results on differential geometry of canonical-type connections (i.e. metric connections with torsion satisfying a certain algebraic identity) on the considered manifolds.

The main results of this section are published in [88].

The differential geometry of affine connections with special requirements for their torsion on almost Hermitian manifolds \((\mathcal{M}^{2n}, J, g)\) is well developed. As it is known, P. Gauduchon gives in [42] a unified presentation of a so-called canonical class of (almost) Hermitian connections, considered by P. Libermann in [66].

Let us recall, an affine connection \(D^*\) is called Hermitian if it preserves the Hermitian metric \(g\) and the almost complex structure \(J\), i.e. the following identities are valid \(D^* g = D^* J = 0\).

The potential of \(D^*\) (with respect to the Levi-Civita connection \(D\)), denoted by \(Q\), is defined by the difference \(D^* - D\). The connection \(D^*\) preserves the metric and therefore is completely determined by its torsion \(T\). According to [22, 145, 128], the two spaces of all torsions and of all potentials are isomorphic as \(O(n)\) representations and an equivariant bijection is the following

\[
\begin{align*}
T(x, y, z) &= Q(x, y, z) - Q(y, x, z), \\
2Q(x, y, z) &= T(x, y, z) - T(y, z, x) + T(z, x, y).
\end{align*}
\]

Following E. Cartan [22], there are studied the algebraic types of the torsion tensor for a metric connection, i.e. an affine connection preserving the metric.

On an almost Hermitian manifold, a Hermitian connection is called canonical if its torsion \(T\) satisfies the following conditions: [42]
1) the component of $T$ satisfying the Bianchi identity and having the property $T(J\cdot, J\cdot) = T(\cdot, \cdot)$ vanishes;

2) for some real number $t$, it is valid $(\mathcal{S}T)^+ = (1-2t)(d\Omega)^+(J\cdot, J\cdot, J\cdot)$, where $(d\Omega)^+$ is the part of type $(2, 1) + (1, 2)$ of the differential $d\Omega$ for the Kähler form $\Omega = g(J\cdot, \cdot)$.

This connection is known also as the Chern connection \[24, 149, 155\].

According to \[42\], there exists an one-parameter family $\{\nabla^t\}_{t \in \mathbb{R}}$ of canonical Hermitian connections $\nabla^t = t\nabla^1 + (1-t)\nabla^0$, where $\nabla^0$ and $\nabla^1$ are the Lichnerowicz first and second canonical connections \[67, 68, 69\], respectively.

The connection $\nabla^t$ obtained for $t = -1$ is called the Bismut connection or the KT-connection, which is characterized with a totally skew-symmetric torsion \[11\]. The latter connection has applications in heterotic string theory and in 2-dimensional supersymmetric $\sigma$-models as well as in type II string theory when the torsion 3-form is closed \[41, 136, 59, 57\].

In \[31\] and \[32\], all almost contact metric, almost Hermitian and $G_2$-structures admitting a connection with totally skew-symmetric torsion tensor are described.

Similar problems are studied on almost hypercomplex manifold \[43, 55\] and Riemannian almost product manifolds in \[135, 51, 52, 53\].

An object of our interest in this section are almost Norden manifolds. The goal of the present section is to survey the research on canonical-type connections in the case of Norden-type metrics as well as some additions and generalizations are made.

### 3.1. Natural connections on an almost Norden manifold

Let $D^*$ be an affine connection with a torsion $T^*$ and a potential $Q^*$ with respect to the Levi-Civita connection $D$, i.e.

$$T^*(x, y) = D^*_x y - D^*_y x - [x, y], \quad Q^*(x, y) = D^*_x y - D^*_y y.$$  

The corresponding $(0,3)$-tensors are defined by

$$T^*(x, y, z) = g(T^*(x, y), z), \quad Q^*(x, y, z) = g(Q^*(x, y), z).$$

These tensors have the same mutual relations as in (3.1) and (3.2).

In \[38\], it is considered the space $\mathcal{T}$ of all torsion $(0,3)$-tensors $T$ (i.e. satisfying $T(x, y, z) = -T(y, x, z)$) on an almost Norden manifold $(\mathcal{M}, J, g)$. There, it is given a partial decomposition of $\mathcal{T}$ in the following form

$$\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3 \oplus \mathcal{T}_4.$$
The components $\mathcal{T}_i$ $(i = 1, 2, 3, 4)$ are invariant orthogonal subspaces with respect to the structure group $\mathcal{G}$ given in (1.3) and they are determined as follows

$\mathcal{T}_1 : \ T(x, y, z) = -T(Jx, Jy, z) = -T(Jx, y, Jz)$;

$\mathcal{T}_2 : \ T(x, y, z) = -T(Jx, Jy, z) = T(Jx, y, Jz)$;

$\mathcal{T}_3 : \ T(x, y, z) = T(Jx, Jy, z), \ \mathcal{G}_{x,y,z} T(x, y, z) = 0$,

$\mathcal{T}_4 : \ T(x, y, z) = T(Jx, Jy, z), \ \mathcal{G}_{x,y,z} T(Jx, y, z) = 0$.

Moreover, in [38] there are explicitly given the components $T_i$ of $T \in \mathcal{T}$ in $\mathcal{T}_i$ $(i = 1, 2, 3, 4)$ as follows

\[
T_1(x, y, z) = \frac{1}{4} \{ T(x, y, z) - T(Jx, Jy, z) \\
- T(Jx, y, Jz) - T(x, Jy, Jz) \},
\]

\[
T_2(x, y, z) = \frac{1}{4} \{ T(x, y, z) - T(Jx, Jy, z) \\
+ T(Jx, y, Jz) + T(x, Jy, Jz) \},
\]

\[
T_3(x, y, z) = \frac{1}{8} \{ 2T(x, y, z) - T(y, z, x) - T(z, x, y) \\
+ 2T(Jx, Jy, z) - T(Jy, z, Jx) \\
- T(z, Jx, Jy) - T(Jy, Jx, z) \\
- T(Jz, Jx, y) + T(y, Jz, Jx) \\
+ T(Jz, x, Jy) \},
\]

(3.3)

\[
T_4(x, y, z) = \frac{1}{8} \{ 2T(x, y, z) + T(y, z, x) + T(z, x, y) \\
+ 2T(Jx, Jy, z) + T(Jy, z, Jx) \\
+ T(z, Jx, Jy) + T(Jy, Jz, x) \\
+ T(Jz, Jx, y) - T(y, Jz, Jx) \\
- T(Jz, x, Jy) \}.
\]

An affine connection $D^*$ on an almost Norden manifold $(\mathcal{M}, J, g)$ is
called a *natural connection* if the structure tensors $J$ and $g$ are parallel with respect to this connection, i.e. $D^* J = D^* g = 0$. These conditions are equivalent to $D^* g = D^* \tilde{g} = 0$. The connection $D^*$ is natural if and only if the following conditions for its potential $Q^*$ are valid:

$$F(x, y, z) = Q^*(x, y, Jz) - Q^*(x, Jy, z),$$

$$Q^*(x, y, z) = -Q^*(x, z, y).$$

(3.4)

In terms of the components $T_i$, an affine connection with torsion $T$ on $(\mathcal{M}, J, g)$ is natural if and only if

$$T_2(x, y, z) = \frac{1}{4}[J, J](x, y, z),$$

$$T_3(x, y, z) = \frac{1}{8}(\{J, J\}(z, y, x) - \{J, J\}(z, x, y)).$$

The former condition is given in [38] whereas the latter one follows immediately by (3.1), (3.2), (1.23), (3.3) and (3.4).

### 3.2. The B-connection and the canonical connection

In [36], it is introduced the *B-connection* $D'$ only for the manifolds from the class $\mathcal{W}_1$ by relation

$$D'_x y = D_x y - \frac{1}{2} J(D_x J)y.$$  

(3.5)

Obviously, the B-connection is a natural connection on $(\mathcal{M}, J, g)$ and it exists in any class of the considered manifolds. The B-connection coincides with the Levi-Civita connection only on a $\mathcal{W}_0$-manifold (i.e. a Kähler-Norden manifold).

By virtue of (3.1), (1.25), (1.26), (1.27), from (3.5) we express the torsion $T'$ of the B-connection $D'$ in the following way

$$T'(x, y, z) = \frac{1}{8}([J, J](x, y, z) + \bigotimes_{x, y, z} [J, J](x, y, z)$$

$$+ \{J, J\}(z, y, x) - \{J, J\}(z, x, y)).$$

(3.6)

A natural connection $D''$ with a torsion $T''$ on an almost Norden manifold $(\mathcal{M}, J, g)$ is called a *canonical connection* if $T''$ satisfies the following condition [38]

$$T''(x, y, z) + T''(y, z, x) - T''(Jx, y, Jz) - T''(y, Jz, Jx) = 0.$$  

(3.7)

In [38], it is shown that (3.7) is equivalent to the condition $T_1'' = T_4'' = 0$, i.e. $T'' \in \mathcal{T}_2 \oplus \mathcal{T}_3$. Moreover, there it is proved that on every almost
Norden manifold exists a unique canonical connection $D''$. We express its torsion in terms of $[J, J]$ and $\{J, J\}$ as follows

$$T''(x, y, z) = \frac{1}{4}[J, J](x, y, z) + \frac{1}{8}\left(\{J, J\}(z, y, x) - \{J, J\}(z, x, y)\right).$$

Taking into account (3.8) and (3.6), it is easy to conclude that $D''$ coincides with $D'$ if and only if the condition $[J, J] = S_{[J, J]}$ holds. The latter equality is equivalent to the vanishing of $[J, J]$. In other words, on a complex Norden manifold, i.e. $(\mathcal{M}, J, g) \in \mathcal{W}_1 \oplus \mathcal{W}_2$, the canonical connection and the B-connection coincide.

Now, let $(\mathcal{M}, J, g)$ be in the class $\mathcal{W}_1$ containing the conformally equivalent manifolds of the Kähler-Norden manifolds. The conformal equivalence is made with respect to the general conformal transformations of the metric $g$ defined by (2.35). These transformations form the general group $C$. An important its subgroup is the group $C_0$ of the holomorphic conformal transformations, defined by the condition: $u + iv$ is a holomorphic function, i.e. the equality $d\ u = dv \circ J$ is valid.

Then the torsion of the canonical connection is an invariant of $C_0$, i.e. the relation $\overline{T''(x, y)} = T''(x, y)$ holds with respect to any transformation of $C_0$. It is proved that the curvature tensor of the canonical connection is a Kähler tensor if and only if $(\mathcal{M}, J, g) \in \mathcal{W}_1^0$, i.e. a $\mathcal{W}_1$-manifold with closed Lee forms $\theta$ and $\tilde{\theta}$. Moreover, there are studied conformal invariants of the canonical connection in $\mathcal{W}_1^0$.

Bearing in mind the conformal invariance of both the basic classes and the torsion $T''$ of the canonical connection, the conditions for $T''$ are used in [38] for other characteristics of all classes of the almost Norden manifolds as follows:

$$\mathcal{W}_0 : \quad T''(x, y) = 0;$$

$$\mathcal{W}_1 : \quad T''(x, y) = \frac{1}{2n}\left\{t''(x)y - t''(y)x + t''(Jx)Jy - t''(Jy)Jx\right\};$$

$$\mathcal{W}_2 : \quad T''(x, y) = T''(Jx, Jy), \quad t'' = 0;$$

$$\mathcal{W}_3 : \quad T''(Jx, y) = -JT''(x, y);$$

$$\mathcal{W}_1 \oplus \mathcal{W}_2 : \quad T''(x, y) = T''(Jx, Jy),$$

$$\mathcal{S}_{x,y,z} T''(x, y, z) = 0;$$

(3.9a)
\[ W_1 \oplus W_3 : \quad T''(Jx, y) + JT''(x, y) \]
\[ = \frac{1}{n} \{ t''(Jy)x - t''(y)Jx \}; \]

(3.9b)

\[ W_2 \oplus W_3 : \quad t'' = 0; \]

\[ W_1 \oplus W_2 \oplus W_3 : \text{no conditions,} \]

where the torsion form \( t'' \) of \( T'' \) is determined by \( t''(x) = g^{ij}T''(x, e_i, e_j) \).

The special class \( W_0 \) is characterized by the condition \( T''(x, y) = 0 \) and then \( D'' \equiv D \) holds.

The torsion \( T'' \) is known as a \textit{vectorial torsion}, because of its form on a \( W_1 \)-manifold. Let the subclass of \( \mathcal{T}_3 \) with vectorial torsions be denoted by \( \mathcal{T}_3^1 \) whereas \( \mathcal{T}_3^0 \) be the subclass of \( \mathcal{T}_3 \) with vanishing torsion forms \( t'' \).

The classes of the almost Norden manifolds are determined with respect to the Nijenhuis tensors in (1.33). The same classes are characterized by conditions for the torsion of the canonical connection in (3.9). By virtue of these results we obtain the following

**Theorem 3.1.** The classes of the almost Norden manifolds \((\mathcal{M}, J, g)\) are characterized by an expression of the torsion \( T'' \) of the canonical connection in terms of the Nijenhuis tensors \( [J, J] \) and \( \{J, J\} \) as follows:

\[ \mathcal{W}_1 : \quad T''(x, y, z) = \frac{1}{16n} \left( \vartheta(x)g(y, z) - \vartheta(y)g(x, z) \right) \]
\[ + \vartheta(Jx)g(y, Jz) \]
\[ - \vartheta(Jy)g(x, Jz) \];

(3.10a)

\[ \mathcal{W}_2 : \quad T''(x, y, z) = \frac{1}{8} \left( \{J, J\}(z, y, x) - \{J, J\}(z, x, y) \right), \]
\[ t'' = \vartheta = 0; \]

\[ \mathcal{W}_3 : \quad T''(x, y, z) = \frac{1}{4} [J, J](x, y, z); \]

\[ \mathcal{W}_1 \oplus \mathcal{W}_2 : \quad T''(x, y, z) = \frac{1}{8} \left( \{J, J\}(z, y, x) \right. \]
\[ \left. - \{J, J\}(z, x, y) \right); \]
\[ \mathcal{W}_1 \oplus \mathcal{W}_3 : \quad T''(x, y, z) = \frac{1}{4}[J, J](x, y, z) \]
\[ \quad + \frac{1}{16n}(\vartheta(x)g(y, z) \]
\[ \quad + \vartheta(Jx)g(y, Jz) \]
\[ \quad - \vartheta(y)g(x, z) \]
\[ \quad - \vartheta(Jy)g(x, Jz); \]
\[ \mathcal{W}_2 \oplus \mathcal{W}_3 : \quad T''(x, y, z) = \frac{1}{4}[J, J](x, y, z) \]
\[ \quad + \frac{1}{8}\{J, J\}(z, y, x) \]
\[ \quad - \{J, J\}(z, x, y), \]
\[ t'' = \vartheta = 0. \]

(3.10b)

The special class \( \mathcal{W}_0 \) is characterized by \( T'' = 0 \) and the whole class \( \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \) — by \((3.8)\) only.

Moreover, bearing in mind the classifications for almost Norden manifolds \((\mathcal{M}, J, g)\) with respect to the tensor \(F\) and the torsion \(T''\) in \([34]\) and \([38]\), respectively, we have:

\[ (\mathcal{M}, J, g) \in \mathcal{W}_1 \quad \Leftrightarrow \quad T'' \in \mathcal{T}_3^1; \]
\[ (\mathcal{M}, J, g) \in \mathcal{W}_2 \quad \Leftrightarrow \quad T'' \in \mathcal{T}_3^0; \]
\[ (\mathcal{M}, J, g) \in \mathcal{W}_3 \quad \Leftrightarrow \quad T'' \in \mathcal{T}_2; \]
\[ (\mathcal{M}, J, g) \in \mathcal{W}_1 \oplus \mathcal{W}_2 \quad \Leftrightarrow \quad T'' \in \mathcal{T}_3; \]
\[ (\mathcal{M}, J, g) \in \mathcal{W}_1 \oplus \mathcal{W}_3 \quad \Leftrightarrow \quad T'' \in \mathcal{T}_2 \oplus \mathcal{T}_3^1; \]
\[ (\mathcal{M}, J, g) \in \mathcal{W}_2 \oplus \mathcal{W}_3 \quad \Leftrightarrow \quad T'' \in \mathcal{T}_2 \oplus \mathcal{T}_3^0; \]
\[ (\mathcal{M}, J, g) \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \quad \Leftrightarrow \quad T'' \in \mathcal{T}_2 \oplus \mathcal{T}_3. \]

Proof. Let \((\mathcal{M}, J, g)\) be a complex Norden manifold, i.e. \((\mathcal{M}, J, g) \in \mathcal{W}_1 \oplus \mathcal{W}_2\). According to \((3.8)\) and \([J, J] = 0\) in this case, we have \(T'' = T''_3\), i.e. \(T'' \in \mathcal{T}_3\) and the expression

\[ T''(x, y, z) = \frac{1}{8}(\{J, J\}(z, y, x) - \{J, J\}(z, x, y)) \]
is obtained. Applying (1.33) to the latter equality, we determine the basic classes $W_1$ and $W_2$ as it is given in (3.10) and the corresponding subclasses $T_3^1$ and $T_3^0$, respectively. Taking into account the relation between the corresponding traces $\vartheta = 8t''$, which is a consequence of the equality for $W_1 \oplus W_2$, we obtain the characterization for these two basic classes in (3.9).

Let $(M, J, g)$ be a quasi-Kähler manifold with Norden metric, i.e. $(M, J, g) \in W_3$. By virtue of (3.8) and $\{J, J\} = 0$ for such a manifold, we have $T'' = T_2''$, i.e. $T'' \in T_2$ and therefore we give $T'' = \frac{1}{4}[J, J]$. Obviously, the form of $T''$ in the latter equality satisfies the condition for $W_3$ in (3.9).

In a similar way we get for the remaining classes $W_1 \oplus W_3$ and $W_2 \oplus W_3$. The conditions of these two classes, given in (3.9), are consequences of the corresponding equalities in (3.10). The case of the whole class $W_1 \oplus W_2 \oplus W_3$ was discussed above.

The canonical connections on quasi-Kähler manifolds with Norden metric are considered in more details in [111]. There are given the following formulae for the potential $Q''$ and the torsion $T''$ on a $W_3$-manifold:

$$Q''(x, y) = \frac{1}{4} \{(D_y J) Jx - (D_J y) x + 2 (D_J x) y\},$$

$$T''(x, y) = \frac{1}{2} \{(D_J x) Jy + (D_J y) y\}.$$

Moreover, some properties for the curvature and the torsion of the canonical connection are obtained.

### 3.3. The KT-connection

In [112], it is proved that a natural connection $D'''$ with totally skew-symmetric torsion, called a $KT$-connection, exists on an almost Norden manifold $(M, J, g)$ if and only if $(M, J, g)$ belongs to $W_3$, i.e. the manifold is quasi-Kählerian with Norden metric. Moreover, the KT-connection is unique and it is determined by its potential

$$Q'''(x, y, z) = -\frac{1}{4} \sum_{x, y, z} F(x, y, Jz).$$

As mentioned above, the canonical connection and the B-connection coincide on $(M, J, g) \in W_1 \oplus W_2$ whereas the KT-connection does not exist there.
Chapter I | §3. Canonical-type connections on almost complex manifolds with Norden metric

The following natural connections on \((\mathcal{M}, J, g)\) are studied on a quasi-Kähler manifold with Norden metric: the B-connection \(D' \, [109]\), the canonical connection \(D'' \, [111]\) and the KT-connection \(D'''' \, [110, 112]\).

Relations (3.8) and (3.6) of \(T'\) and \(T''\) in terms of the pair of Nijenhuis tensors are specialized for a \(\mathcal{W}_3\)-manifold in the following way

\[
T'(x, y, z) = \frac{1}{8} ([J, J](x, y, z) + \mathcal{R}_{x,y,z} [J, J](x, y, z)),
\]

(3.12)

\[
T''(x, y, z) = \frac{1}{4} [J, J](x, y, z).
\]

The equalities (3.11) and (3.11) yield

\[
T'''(x, y, z) = -\frac{1}{2} \mathcal{R}_{x,y,z} F(x, y, Jz),
\]

(3.13)

which by (1.31) for \(\mathcal{W}_3\) and (1.25) implies

\[
T'''(x, y, z) = \frac{1}{4} \mathcal{R}_{x,y,z} [J, J](x, y, z).
\]

(3.14)

Then from (3.12) and (3.14) we have the relation

\[
T' = \frac{1}{2} (T'' + T''''),
\]

which by (3.2) is equivalent to

\[
Q' = \frac{1}{2} (Q'' + Q'''').
\]

Therefore, as it is shown in [111], the B-connection is the average connection for the canonical connection and the KT-connection on a quasi-Kähler manifold with Norden metric, i.e.

\[
D' = \frac{1}{2} (D'' + D'''').
\]

* * *

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§4. Almost contact manifolds with B-metric

In the present section we recall some notions and knowledge for the almost contact manifolds with B-metric which are studied in [39, 78, 83, 95, 96, 99, 100, 116].

4.1. Almost contact structures with B-metric

Let \((M, \phi, \xi, \eta)\) be an almost contact manifold, i.e. \(M\) is a differentiable manifold of dimension \((2n + 1)\), provided with an almost contact structure \((\phi, \xi, \eta)\) consisting of an endomorphism \(\phi\) of the tangent bundle, a vector field \(\xi\) and its dual 1-form \(\eta\) such that the following algebraic relations are satisfied: [12]

\[
\begin{align*}
\phi \xi &= 0, \\
\phi^2 &= -I + \eta \otimes \xi, \\
\eta \circ \phi &= 0, \\
\eta(\xi) &= 1,
\end{align*}
\]

where \(I\) denotes the identity.

Further, let us equip the almost contact manifold \((M, \phi, \xi, \eta)\) with a pseudo-Riemannian metric \(g\) of signature \((n + 1, n)\) determined by

\[
g(\phi x, \phi y) = -g(x, y) + \eta(x)\eta(y).
\]

Then \((M, \phi, \xi, \eta, g)\) is called an almost contact manifold with B-metric or an almost contact B-metric manifold [39].

The associated metric \(\tilde{g}\) of \(g\) on \((M, \phi, \xi, \eta)\) is defined by the equality

\[
\tilde{g}(x, y) = g(x, \phi y) + \eta(x)\eta(y).
\]

Both metrics \(g\) and \(\tilde{g}\) are necessarily of signature \((n + 1, n)\). The manifold \((M, \phi, \xi, \eta, \tilde{g})\) is also an almost contact B-metric manifold.

Let us remark that the \(2n\)-dimensional contact distribution \(\mathcal{H} = \ker(\eta)\), generated by the contact 1-form \(\eta\), can be considered as the horizontal distribution of the sub-Riemannian manifold \(M\). Then \(\mathcal{H}\) is endowed with
an almost complex structure determined as $\varphi|_\mathcal{H}$ – the restriction of $\varphi$ on $\mathcal{H}$, as well as a Norden metric $g|_\mathcal{H}$, i.e.

$$g|_\mathcal{H}(\varphi|_\mathcal{H} \cdot, \varphi|_\mathcal{H} \cdot) = -g(\cdot, \cdot).$$

Moreover, $\mathcal{H}$ can be considered as an $n$-dimensional complex Riemannian manifold with a complex Riemannian metric $g^\mathbb{C} = g|_\mathcal{H} + i\tilde{g}|_\mathcal{H}$. By this reason we can refer to these manifolds as *almost contact complex Riemannian manifolds*.

Using the Reeb vector field $\xi$ and its dual contact 1-form $\eta$ on an arbitrary almost contact B-metric manifold $(\mathcal{M}, \varphi, \xi, \eta, g)$, we consider two distributions in the tangent bundle $\mathcal{T}\mathcal{M}$ of $\mathcal{M}$ as follows

$$(4.3) \quad \mathcal{H} = \ker(\eta), \quad \mathcal{V} = \text{span}(\xi).$$

Then the horizontal distribution $\mathcal{H}$ and the vertical distribution $\mathcal{V}$ form a pair of mutually complementary distributions in $\mathcal{T}\mathcal{M}$ which are orthogonal with respect to both of the metrics $g$ and $\tilde{g}$, i.e.

$$(4.4) \quad \mathcal{H} \oplus \mathcal{V} = \mathcal{T}\mathcal{M}, \quad \mathcal{H} \perp \mathcal{V}, \quad \mathcal{H} \cap \mathcal{V} = \{0\},$$

where $o$ is the zero vector field on $\mathcal{M}$. Thus, there are determined the corresponding horizontal and vertical projectors

$$(4.5) \quad h : \mathcal{T}\mathcal{M} \mapsto \mathcal{H}, \quad v : \mathcal{T}\mathcal{M} \mapsto \mathcal{V}$$

having the properties $h \circ h = h$, $v \circ v = v$, $h \circ v = v \circ h = 0$. An arbitrary vector field $x$ in $\mathcal{T}\mathcal{M}$ has respective projections $x^h$ and $x^v$ so that

$$(4.6) \quad x = x^h + x^v,$$

where

$$(4.7) \quad x^h = -\varphi^2 x = x - \eta(x)\xi, \quad x^v = \eta(x)\xi$$

are the so-called horizontal and vertical components, respectively.

The structure group of $(\mathcal{M}, \varphi, \xi, \eta, g)$ is $\mathcal{G} \times \mathcal{I}$, where $\mathcal{G}$ is the group determined in (1.3) and $\mathcal{I}$ is the identity on $\mathcal{V}$. Consequently, the structure group consists of the real square matrices of order $2n + 1$ of the following type

$$(4.8) \quad \begin{pmatrix}
A & B & o^T \\
-B & A & o^T \\
o & o & 1
\end{pmatrix},$$
where $o$ and its transpose $o^\top$ are the zero row $n$-vector and the zero column $n$-vector; $A$ and $B$ are real invertible matrices of size $n$ satisfying the conditions in (1.4).

4.2. Fundamental tensors $F$ and $\tilde{F}$

The covariant derivatives of $\varphi$, $\xi$, $\eta$ with respect to the Levi-Civita connection $\nabla$ play a fundamental role in differential geometry on the almost contact manifolds. The fundamental tensor $F$ of type (0,3) on $(\mathcal{M}, \varphi, \xi, \eta, g)$ is defined by

\begin{equation}
F(x, y, z) = g((\nabla_x \varphi)y, z).
\end{equation}

It has the following basic properties:

\begin{equation}
F(x, y, z) = F(x, z, y)
= F(x, \varphi y, \varphi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi).
\end{equation}

The relations of $\nabla \xi$ and $\nabla \eta$ with $F$ are:

\begin{equation}
(\nabla_x \eta)(y) = g(\nabla_x \xi, y) = F(x, \varphi y, \xi).
\end{equation}

The 1-forms $\theta$, $\theta^*$ and $\omega$, called Lee forms on $(\mathcal{M}, \varphi, \xi, \eta, g)$, are associated with $F$ by the following way:

\begin{equation}
\theta(z) = g^{ij}F(e_i, e_j, z),
\end{equation}

\begin{equation}
\theta^*(z) = g^{ij}F(e_i, \varphi e_j, z),
\end{equation}

\begin{equation}
\omega(z) = F(\xi, \xi, z),
\end{equation}

where $g^{ij}$ are the components of the inverse matrix of $g$ with respect to a basis $\{e_i; \xi\}$ $(i = 1, 2, \ldots, 2n)$ of the tangent space $T_p\mathcal{M}$ of $\mathcal{M}$ at an arbitrary point $p \in \mathcal{M}$.

Obviously, the equality $\omega(\xi) = 0$ and the following relation are always valid:

\begin{equation}
\theta^* \circ \varphi = -\theta \circ \varphi^2.
\end{equation}

For the corresponding traces $\tilde{\theta}$ and $\tilde{\theta}^*$ with respect to $\tilde{g}$ we have

\begin{align*}
\tilde{\theta} &= -\theta^*, \\
\tilde{\theta}^* &= \theta.
\end{align*}

A classification of the almost contact B-metric manifolds with respect to the properties of $F$ is given by G. Ganchev, V. Mihova and K. Gribachev in [39]. This classification includes the basic classes $\mathcal{F}_1$, $\mathcal{F}_2$,
... $\mathcal{F}_{11}$. Their intersection is the special class $\mathcal{F}_0$ determined by the condition $F(x, y, z) = 0$. Hence $\mathcal{F}_0$ is the class of almost contact B-metric manifolds with $\nabla$-parallel structures, i.e.

$$\nabla \varphi = \nabla \xi = \nabla \eta = \nabla g = \nabla \tilde{g} = 0.$$  

The $\mathcal{F}_0$-manifolds are also known as *cosymplectic B-metric manifolds*. Further, we use the following characteristic conditions of the basic classes:

\[
\begin{align*}
\mathcal{F}_1 : & \quad F(x, y, z) = \frac{1}{2n} \left\{ g(x, \varphi y)\theta(\varphi z) + g(\varphi x, \varphi y)\theta(\varphi^2 z) \\
& \quad + g(x, \varphi z)\theta(\varphi y) + g(\varphi x, \varphi z)\theta(\varphi^2 y) \right\}; \\
\mathcal{F}_2 : & \quad F(\xi, y, z) = F(x, \xi, z) = 0, \quad \mathcal{S}_{x,y,z} F(x, y, \varphi z) = \theta = 0; \\
\mathcal{F}_3 : & \quad F(\xi, y, z) = F(x, \xi, z) = 0, \quad \mathcal{S}_{x,y,z} F(x, y, z) = 0; \\
\mathcal{F}_4 : & \quad F(x, y, z) = -\frac{\theta(\xi)}{2n} \left\{ g(\varphi x, \varphi y)\eta(z) + g(\varphi x, \varphi z)\eta(y) \right\}; \\
\mathcal{F}_5 : & \quad F(x, y, z) = -\frac{\theta^*(\xi)}{2n} \left\{ g(x, \varphi y)\eta(z) + g(x, \varphi z)\eta(y) \right\}; \\
\mathcal{F}_6 : & \quad F(x, y, z) = F(x, y, \xi)\eta(z) + F(x, z, \xi)\eta(y), \\
(4.14) & \quad F(x, y, \xi) = F(y, x, \xi) = -F(\varphi x, \varphi y, \xi), \\
& \quad \theta = \theta^* = 0; \\
\mathcal{F}_7 : & \quad F(x, y, z) = F(x, y, \xi)\eta(z) + F(x, z, \xi)\eta(y), \\
& \quad F(x, y, \xi) = -F(y, x, \xi) = -F(\varphi x, \varphi y, \xi); \\
\mathcal{F}_8 : & \quad F(x, y, z) = F(x, y, \xi)\eta(z) + F(x, z, \xi)\eta(y), \\
& \quad F(x, y, \xi) = F(y, x, \xi) = F(\varphi x, \varphi y, \xi); \\
\mathcal{F}_9 : & \quad F(x, y, z) = F(x, y, \xi)\eta(z) + F(x, z, \xi)\eta(y), \\
& \quad F(x, y, \xi) = -F(y, x, \xi) = F(\varphi x, \varphi y, \xi); \\
\mathcal{F}_{10} : & \quad F(x, y, z) = F(\xi, \varphi y, \varphi z)\eta(x); \\
\mathcal{F}_{11} : & \quad F(x, y, z) = \eta(x) \{ \eta(y)\omega(z) + \eta(z)\omega(y) \}.
\end{align*}
\]

In [70], it is proved that $(\mathcal{M}, \varphi, \xi, \eta, g)$ belongs to $\mathcal{F}_i$ $(i = 1, \ldots, 11)$ if and only if $F$ satisfies the condition $F = F^i$, where the components $F^i$
of $F$ are the following

\[
F^1(x, y, z) = \frac{1}{2n} \left\{ g(\varphi x, \varphi y)\theta(\varphi^2 z) + g(x, \varphi y)\theta(\varphi z) \right. \\
\left. + g(\varphi x, \varphi z)\theta(\varphi^2 y) + g(x, \varphi z)\theta(\varphi y) \right\},
\]

\[
F^2(x, y, z) = -\frac{1}{4} \left\{ F(\varphi^2 x, \varphi^2 y, \varphi^2 z) + F(\varphi^2 y, \varphi^2 z, \varphi^2 x) \\
- F(\varphi y, \varphi^2 z, \varphi x) + F(\varphi^2 x, \varphi^2 z, \varphi^2 y) \\
+ F(\varphi^2 z, \varphi^2 y, \varphi^2 x) - F(\varphi z, \varphi^2 y, \varphi x) \right\} \\
- \frac{1}{2n} \left\{ g(\varphi x, \varphi y)\theta(\varphi^2 z) + g(x, \varphi y)\theta(\varphi z) \\
+ g(\varphi x, \varphi z)\theta(\varphi^2 y) + g(x, \varphi z)\theta(\varphi y) \right\},
\]

\[
F^3(x, y, z) = -\frac{1}{4} \left\{ F(\varphi^2 x, \varphi^2 y, \varphi^2 z) - F(\varphi^2 y, \varphi^2 z, \varphi^2 x) \\
+ F(\varphi y, \varphi^2 z, \varphi x) + F(\varphi^2 x, \varphi^2 z, \varphi^2 y) \\
- F(\varphi^2 z, \varphi^2 y, \varphi^2 x) + F(\varphi z, \varphi^2 y, \varphi x) \right\},
\]

\[\text{(4.15a)}\]

\[
F^4(x, y, z) = -\frac{\theta(\xi)}{2n} \left\{ g(\varphi x, \varphi y)\eta(z) + g(\varphi x, \varphi z)\eta(y) \right\},
\]

\[
F^5(x, y, z) = -\frac{\theta^*(\xi)}{2n} \left\{ g(x, \varphi y)\eta(z) + g(x, \varphi z)\eta(y) \right\},
\]

\[
F^6(x, y, z) = \frac{\theta(\xi)}{2n} \left\{ g(\varphi x, \varphi y)\eta(z) + g(\varphi x, \varphi z)\eta(y) \right\} \\
+ \frac{\theta^*(\xi)}{2n} \left\{ g(x, \varphi y)\eta(z) + g(x, \varphi z)\eta(y) \right\} \\
+ \frac{1}{4} \left\{ F(\varphi^2 x, \varphi^2 y, \xi) + F(\varphi^2 y, \varphi^2 x, \xi) \\
- F(\varphi x, \varphi y, \xi) - F(\varphi y, \varphi x, \xi) \right\} \eta(z) \\
+ \frac{1}{4} \left\{ F(\varphi^2 x, \varphi^2 z, \xi) + F(\varphi^2 z, \varphi^2 x, \xi) \\
- F(\varphi x, \varphi z, \xi) - F(\varphi z, \varphi x, \xi) \right\} \eta(y),
\]

\[
F^7(x, y, z) = \frac{1}{4} \left\{ F(\varphi^2 x, \varphi^2 y, \xi) - F(\varphi^2 y, \varphi^2 x, \xi) \\
- F(\varphi x, \varphi y, \xi) + F(\varphi y, \varphi x, \xi) \right\} \eta(z)
\]
\[ F^8(x, y, z) = \frac{1}{4} \left\{ F(\varphi^2 x, \varphi^2 y, \xi) + F(\varphi y, \varphi^2 x, \xi) \right\} \eta(y), \]
\[ + \frac{1}{4} \left\{ F(\varphi^2 x, \varphi^2 z, \xi) + F(\varphi z, \varphi^2 x, \xi) \right\} \eta(y), \]
\[ F^9(x, y, z) = \frac{1}{4} \left\{ F(\varphi^2 x, \varphi^2 y, \xi) - F(\varphi y, \varphi^2 x, \xi) \right\} \eta(z), \]
\[ + \frac{1}{4} \left\{ F(\varphi^2 x, \varphi^2 z, \xi) - F(\varphi z, \varphi^2 x, \xi) \right\} \eta(z), \]
\[ F^{10}(x, y, z) = \eta(x) F(\xi, \varphi^2 y, \varphi^2 z), \]
\[ F^{11}(x, y, z) = -\eta(x) \left\{ \eta(y) F(\xi, \xi, \varphi^2 z) + \eta(z) F(\xi, \varphi^2 y, \xi) \right\}. \]

It is said that an almost contact B-metric manifold belongs to a direct sum of two or more basic classes, i.e. \((\mathcal{M}, \varphi, \xi, \eta, g) \in \mathcal{F}_i \oplus \mathcal{F}_j \oplus \cdots\), if and only if the fundamental tensor \(F\) on \((\mathcal{M}, \varphi, \xi, \eta, g)\) is the sum of the corresponding components \(F^i, F^j, \ldots\) of \(F\), i.e. the following condition is satisfied \(F = F^i + F^j + \cdots\) for \(i, j \in \{1, 2, \ldots, 11\}, i \neq j\).

For the minimal dimension 3 of an almost contact B-metric manifold, it is known that the basic classes are only seven, because \(\mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_6\) and \(\mathcal{F}_7\) are restricted to \(\mathcal{F}_0 [39, 70]\). The geometry of the considered manifolds in dimension 3 is recently studied in \([70, 75, 71, 72, 73, 74, 76]\).

Using (4.10) and taking the traces with respect to \(g\) denoted by \(\text{tr}\) and the traces with respect to \(\tilde{g}\) denoted by \(\text{tr}^*\), we obtain the following relations

\[ (4.16) \quad \theta(\xi) = \text{div}^*(\eta), \quad \theta^*(\xi) = \text{div}(\eta), \]

where \(\text{div}\) and \(\text{div}^*\) denote the divergence using a trace by \(g\) and by \(\tilde{g}\), respectively.

Since \(g(\xi, \xi) = 1\) implies \(g(\nabla_x \xi, \xi) = 0\), then we obtain \(\nabla_x \xi \in \mathcal{H}\). The
shape operator \( S : \mathcal{H} \to \mathcal{H} \) for the metric \( g \) is defined by

\[
S(x) = -\nabla_x \xi.
\]

As a corollary, the covariant derivative of \( \xi \) with respect to \( \nabla \) and the dual covariant derivative of \( \eta \) because of (4.11) are determined in each class as follows:

\[
\mathcal{F}_1 : \nabla \xi = 0; \quad \mathcal{F}_2 : \nabla \xi = 0; \quad \mathcal{F}_3 : \nabla \xi = 0;
\]

\[
\mathcal{F}_4 : \nabla \xi = \frac{1}{2n} \text{div}^*(\eta) \varphi; \quad \mathcal{F}_5 : \nabla \xi = -\frac{1}{2n} \text{div}(\eta) \varphi^2;
\]

\[
\mathcal{F}_6 : g(\nabla_x \xi, y) = g(\nabla_y \xi, x) = -g(\nabla \varphi_x \xi, \varphi y), \quad \text{div}(\eta) = \text{div}^*(\eta) = 0;
\]

\[
(4.18) \quad \mathcal{F}_7 : g(\nabla_x \xi, y) = -g(\nabla_y \xi, x) = -g(\nabla \varphi_x \xi, \varphi y);
\]

\[
\mathcal{F}_8 : g(\nabla_x \xi, y) = -g(\nabla_y \xi, x) = g(\nabla \varphi_x \xi, \varphi y);
\]

\[
\mathcal{F}_9 : g(\nabla_x \xi, y) = g(\nabla_y \xi, x) = g(\nabla \varphi_x \xi, \varphi y);
\]

\[
\mathcal{F}_{10} : \nabla \xi = 0; \quad \mathcal{F}_{11} : \nabla \xi = \eta \otimes \varphi \omega^\sharp,
\]

where \( \sharp \) denotes the musical isomorphism of \( T^* \mathcal{M} \) in \( T \mathcal{M} \) given by \( g \).

The latter characteristics of the basic classes imply the following

**Proposition 4.1.** The class of almost contact B-metric manifolds with vanishing \( \nabla \xi \) is \( \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3 \oplus \mathcal{F}_{10} \).

**Proof.** Let \((\mathcal{M}, \varphi, \xi, \eta, g)\) be in the general class \( \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3 \oplus \cdots \oplus \mathcal{F}_{11} \). Then the equality \( F = F^1 + F^2 + \ldots + F^{11} \) is valid, where \( F^i \) are given in (4.15). Suppose \( \nabla \xi \) vanishes, we obtain \( F^4 = F^5 = \cdots = F^9 = F^{11} = 0 \) by (4.11). Therefore, we have \( F = F^1 + F^2 + F^3 + F^{10} \), i.e. the manifold belongs to \( \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3 \oplus \mathcal{F}_{10} \).

Vice versa, let \((\mathcal{M}, \varphi, \xi, \eta, g)\) be in the class \( \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3 \oplus \mathcal{F}_{10} \), i.e. the fundamental tensor has the form \( F = F^1 + F^2 + F^3 + F^{10} \) for arbitrary \((x, y, z)\). Bearing in mind \( \varphi \xi = 0 \) and (4.15), we deduce that \( F^i(x, \varphi y, \xi) \) vanishes for \( i \in \{1, 2, 3, 10\} \) and then \( F(x, \varphi y, \xi) = 0 \) is valid. The latter equality is equivalent to the condition \( \nabla \xi = 0 \), according to (4.11). \( \square \)

Let us consider the tensor \( \Phi \) of type \((1,2)\) defined in [39] as the difference of the Levi-Civita connections \( \tilde{\nabla} \) and \( \nabla \) of the corresponding B-metrics \( \tilde{g} \) and \( g \) as follows

\[
(4.19) \quad \Phi(x, y) = \tilde{\nabla}_x y - \nabla_x y.
\]
This tensor is known also as the potential of $\widetilde{\nabla}$ regarding $\nabla$ because of the formula

\[(4.20)\quad \widetilde{\nabla}_xy = \nabla_x y + \Phi(x, y).\]

Since both the connections are torsion-free, then $\Phi$ is symmetric, i.e. $\Phi(x, y) = \Phi(y, x)$ holds. Let the corresponding tensor of type $(0, 3)$ with respect to $g$ be defined by

\[(4.21)\quad \Phi(x, y, z) = g(\Phi(x, y), z).\]

In [115], it is given a characterization of all basic classes in terms of $\Phi$ by means of the following relations between $F$ and $\Phi$ known from [39]

\[(4.22)\quad F(x, y, z) = \Phi(x, y, \varphi z) + \Phi(x, z, \varphi y) + \frac{1}{2} \eta(z)\{\Phi(x, y, \xi) - \Phi(x, \varphi y, \xi) + \Phi(\xi, x, y) - \Phi(\xi, x, \varphi y)\};\]

\[(4.23)\quad 2\Phi(x, y, z) = -F(x, y, \varphi z) - F(y, x, \varphi z) + F(\varphi z, x, y) + \eta(x)\{F(y, z, \xi) + F(\varphi z, \varphi y, \xi) + F(z, \varphi y, \xi) + F(\varphi z, \varphi x, \xi) + F(x, \varphi y, \xi) + F(y, \varphi x, \xi) - \omega(\varphi x)\eta(y) - \omega(\varphi y)\eta(x)\}.$

The corresponding fundamental tensor $\tilde{F}$ on $(\mathcal{M}, \varphi, \xi, \eta, \tilde{g})$ is determined by $\tilde{F}(x, y, z) = \tilde{g}((\tilde{\nabla}_x \varphi)y, z)$. In [79], it is given the relation between $F$ and $\tilde{F}$ as follows

\[(4.24a)\quad 2\tilde{F}(x, y, z) = -F(y, \varphi z, x)F(\varphi y, z, x) + \eta(x)\{F(y, z, \xi) + F(\varphi z, \varphi y, \xi) + F(z, y, \xi) + F(\varphi y, \varphi z, \xi)\}.$
\[ + \eta(y) \{ F(x, z, \xi) + F(\varphi z, \varphi x, \xi) + F(x, \varphi z, \xi) \} \]
\[ + \eta(z) \{ F(x, y, \xi) + F(\varphi y, \varphi x, \xi) + F(x, \varphi y, \xi) \}. \tag{4.24b} \]

Obviously, the special class \( \mathcal{F}_0 \) is determined by the following equivalent conditions: \( F = 0, \Phi = 0, \tilde{F} = 0 \) and \( \nabla = \tilde{\nabla} \).

The properties of \( \nabla \xi \) when \( (\mathcal{M}, \varphi, \xi, \eta, \tilde{g}) \) is in each of the basic classes are determined in a similar way as in (4.18).

### 4.3. Pair of the Nijenhuis tensors

An almost contact structure \( (\varphi, \xi, \eta) \) on \( \mathcal{M} \) is called normal and respectively \( (\mathcal{M}, \varphi, \xi, \eta) \) is a normal almost contact manifold if the corresponding almost complex structure \( J \) generated on \( \mathcal{M} \times \mathbb{R} \) is integrable (i.e. a complex manifold) [129]. An almost contact structure is normal if and only if the Nijenhuis tensor of \( (\varphi, \xi, \eta) \) is zero [12].

The Nijenhuis tensor \( N \) of the almost contact structure is defined by

\[ N = [\varphi, \varphi] + d\eta \otimes \xi, \tag{4.25} \]

where

\[ [\varphi, \varphi](x, y) = [\varphi x, \varphi y] + \varphi^2 [x, y] - \varphi [\varphi x, y] - \varphi [x, \varphi y] \tag{4.26} \]

is the Nijenhuis torsion of \( \varphi \) and \( d\eta \) is the exterior derivative of \( \eta \).

By analogy with the skew-symmetric Lie brackets \( [x, y] = \nabla_x y - \nabla_y x \), let us consider the symmetric braces \( \{x, y\} = \nabla_x y + \nabla_y x \) given by the same formula as in (1.19). Then we introduce the symmetric tensor

\[ \{\varphi, \varphi\}(x, y) = \{\varphi x, \varphi y\} + \varphi^2 \{x, y\} - \varphi \{\varphi x, y\} - \varphi \{x, \varphi y\}. \tag{4.27} \]

Additionally, we use the relation between the Lie derivative of the metric \( g \) along \( \xi \) and the covariant derivative of \( \eta \)

\[ (\mathcal{L}_\xi g)(x, y) = (\nabla_x \eta)(y) + (\nabla_y \eta)(x), \]

as an alternative of \( d\eta(x, y) = (\nabla_x \eta)(y) - (\nabla_y \eta)(x) \). Then, we give the following

**Definition 4.1.** The \((1,2)\)-tensor \( \hat{N} \) defined by

\[ \hat{N} = \{\varphi, \varphi\} + \xi \otimes \mathcal{L}_\xi g, \tag{4.28} \]

is called the associated Nijenhuis tensor of the almost contact B-metric structure \( (\varphi, \xi, \eta, g) \).
Obviously, $N$ is antisymmetric and $\hat{N}$ is symmetric, i.e.
\[ N(x, y) = -N(y, x), \quad \hat{N}(x, y) = \hat{N}(y, x). \]

From (4.25) and (4.26), using the expressions of the Lie brackets and $d\eta$, we get the following form of $N$ in terms of the covariant derivatives with respect to $\nabla$:
\[
N(x, y) = (\nabla_{\varphi x} \varphi) y - \varphi (\nabla_{x} \varphi) y - (\nabla_{\varphi y} \varphi) x + \varphi (\nabla_{y} \varphi) x \\
+ (\nabla_{x} \eta) (y) \cdot \xi - (\nabla_{y} \eta) (x) \cdot \xi.
\]

(4.29)

**Proposition 4.2.** The tensor $\hat{N}$ has the following form in terms of $\nabla \varphi$ and $\nabla \eta$:
\[
\hat{N}(x, y) = (\nabla_{\varphi x} \varphi) y - \varphi (\nabla_{x} \varphi) y + (\nabla_{\varphi y} \varphi) x - \varphi (\nabla_{y} \varphi) x \\
+ (\nabla_{x} \eta) (y) \cdot \xi + (\nabla_{y} \eta) (x) \cdot \xi.
\]

(4.30)

**Proof.** We obtain immediately
\[
\hat{N}(x, y) = \{\varphi, \varphi\}(x, y) + (\xi g)(x, y)\xi \\
= \{\varphi x, \varphi y\} + \varphi^{2}\{x, y\} - \varphi\{\varphi x, y\} - \varphi\{x, \varphi y\} \\
+ (\nabla_{x} \eta) (y) \cdot \xi + (\nabla_{y} \eta) (x) \cdot \xi \\
= \nabla_{\varphi x} \varphi y + \nabla_{\varphi y} \varphi x + \varphi^{2}\nabla_{x} y - \varphi \nabla_{\varphi} y \\
- \varphi \nabla_{y} \varphi x + \varphi^{2}\nabla_{y} x - \varphi \nabla_{\varphi} x \\
+ (\nabla_{x} \eta) (y) \cdot \xi + (\nabla_{y} \eta) (x) \cdot \xi \\
= (\nabla_{\varphi x} \varphi) y + (\nabla_{\varphi y} \varphi) x - \varphi (\nabla_{x} \varphi) y - \varphi (\nabla_{y} \varphi) x \\
+ (\nabla_{x} \eta) (y) \cdot \xi + (\nabla_{y} \eta) (x) \cdot \xi,
\]

which completes the proof. \qed

The corresponding tensors of type (0,3) are denoted by the same letters by the following way
\[ N(x, y, z) = g(N(x, y), z), \quad \hat{N}(x, y, z) = g(\hat{N}(x, y), z). \]

Then, by virtue of (4.29), (4.30) and (4.9), the tensors $N$ and $\hat{N}$ are expressed in terms of $F$ as follows:
\[
N(x, y, z) = F(\varphi x, y, z) - F(x, y, \varphi z) \\
- F(\varphi y, x, z) + F(y, x, \varphi z) \\
+ \eta(z)\{F(x, \varphi y, \xi) - F(y, \varphi x, \xi)\},
\]

(4.31)
\[ \hat{N}(x, y, z) = F(\varphi x, y, z) - F(x, y, \varphi z) \]
\[ + F(\varphi y, x, z) - F(y, x, \varphi z) \]
\[ + \eta(z)\{F(x, \varphi y, \xi) + F(y, \varphi x, \xi)\}. \]

(4.32)

Bearing in mind (4.1), (4.2) and (4.10), from (4.31) and (4.32) we obtain the following properties of the Nijenhuis tensors on an arbitrary almost contact B-metric manifold:

\[ N(x, \varphi y, \varphi z) = N(x, \varphi^2 y, \varphi^2 z), \]
\[ N(\varphi x, y, \varphi z) = N(\varphi^2 x, y, \varphi^2 z), \]
\[ N(\varphi x, \varphi y, z) = -N(\varphi^2 x, \varphi^2 y, z), \]
\[ \hat{N}(x, \varphi y, \varphi z) = \hat{N}(x, \varphi^2 y, \varphi^2 z), \]
\[ \hat{N}(\varphi x, y, \varphi z) = \hat{N}(\varphi^2 x, y, \varphi^2 z), \]
\[ \hat{N}(\varphi x, \varphi y, z) = -\hat{N}(\varphi^2 x, \varphi^2 y, z), \]
\[ N(\xi, \varphi y, \varphi z) + N(\xi, \varphi z, \varphi y) + \hat{N}(\xi, \varphi y, \varphi z) + \hat{N}(\xi, \varphi z, \varphi y) = 0. \]

(4.33)

(4.34)

The Nijenhuis tensors \(N\) and \(\hat{N}\) play a fundamental role in natural connections (i.e. such connections that the tensors of the structure \((\varphi, \xi, \eta, g)\) are parallel with respect to them) on an almost contact B-metric manifold. The torsions and the potentials of these connections are expressed by these two tensors. By this reason we characterize the classes of the considered manifolds in terms of \(N\) and \(\hat{N}\).

Taking into account (4.31) and (4.14), we compute \(N\) for each of the basic classes \(F_i\) \((i = 1, 2, \ldots, 11)\) of \(M = (M, \varphi, \xi, \eta, g)\):

\[ N(x, y) = 0, \quad M \in F_1 \oplus F_2 \oplus F_4 \oplus F_5 \oplus F_6; \]
and

\[ N(x, y) = 2 \left\{ (\nabla_{\varphi x} \varphi)_y - \varphi(\nabla_x \varphi)_y \right\}, \quad M \in F_3; \]
\[ N(x, y) = 4 (\nabla_x \eta)_y \cdot \xi, \quad M \in F_7; \]
\[ N(x, y) = 2 \left\{ \eta(x)\nabla_y \xi - \eta(y)\nabla_x \xi \right\}, \quad M \in F_8 \oplus F_9; \]
\[ N(x, y) = -\eta(x)\varphi(\nabla_{\varphi x} \varphi)_y + \eta(y)\varphi(\nabla_{\varphi y} \varphi)_x, \quad M \in F_{10}; \]
\[ N(x, y) = \{ \eta(x)\omega(\varphi y) - \eta(y)\omega(\varphi x) \}, \quad M \in F_{11}. \]

(4.35)  \(N(x, y) = 0, M \in F_1 \oplus F_2 \oplus F_4 \oplus F_5 \oplus F_6;\)

It is known that the class of the normal almost contact B-metric manifolds, i.e. \(N = 0\), is \(F_1 \oplus F_2 \oplus F_4 \oplus F_5 \oplus F_6\).
By virtue of (4.30) and the form of \( F \) in (4.14), we establish that \( \hat{N} \) has the following form on \( \mathcal{M} = (\mathcal{M}, \varphi, \xi, \eta, g) \) belonging to \( \mathcal{F}_i \) \((i = 1, 2, \ldots, 11)\), respectively:

\[
\begin{align*}
\hat{N}(x, y) &= \frac{2}{n} \{g(\varphi x, \varphi y) \varphi \theta^z + g(x, \varphi y) \theta^z\}, \quad \mathcal{M} \in \mathcal{F}_1; \\
\hat{N}(x, y) &= 2 \{(\nabla_{\varphi x} \varphi) y - \varphi (\nabla_x \varphi) y\}, \quad \mathcal{M} \in \mathcal{F}_2; \\
\hat{N}(x, y) &= 0, \quad \mathcal{M} \in \mathcal{F}_3 \oplus \mathcal{F}_7; \\
\hat{N}(x, y) &= \frac{2}{n} \theta(\xi) g(x, \varphi y) \xi, \quad \mathcal{M} \in \mathcal{F}_4; \\
\hat{N}(x, y) &= -\frac{2}{n} \theta^*(\xi) g(\varphi x, \varphi y) \xi, \quad \mathcal{M} \in \mathcal{F}_5; \\
\hat{N}(x, y) &= 4 (\nabla_x \eta)(y) \xi, \quad \mathcal{M} \in \mathcal{F}_6; \\
\hat{N}(x, y) &= -2 \{\eta(x) \nabla_y \xi + \eta(y) \nabla_x \xi\}, \quad \mathcal{M} \in \mathcal{F}_8 \oplus \mathcal{F}_9; \\
\hat{N}(x, y) &= -\eta(x) \varphi (\nabla_x \xi) \eta \\
&\quad - \eta(y) \varphi (\nabla_x \xi) x, \quad \mathcal{M} \in \mathcal{F}_{10}; \\
\hat{N}(x, y) &= -2 \eta(x) \eta(y) \varphi \omega^z \\
&\quad + \{\eta(x) \omega(\varphi y) + \eta(y) \omega(\varphi x)\} \xi, \quad \mathcal{M} \in \mathcal{F}_{11},
\end{align*}
\]

where \( \theta(z) = g(\theta^z, z) \) and \( \omega(z) = g(\omega^z, z) \).

Then, we obtain the truthfulness of the following

**Proposition 4.3.** The class of the almost contact B-metric manifolds with vanishing \( \hat{N} \) is \( \mathcal{F}_3 \oplus \mathcal{F}_7 \).

To characterize almost contact B-metric manifolds we need an expression of \( F \) by \( N \) and \( \hat{N} \).

**Theorem 4.4.** Let \( (\mathcal{M}, \varphi, \xi, \eta, g) \) be an almost contact B-metric manifold. Then the fundamental tensor \( F \) is given in terms of the pair of Nijenhuis tensors by the formula

\[
\begin{align*}
F(x, y, z) &= -\frac{1}{4} \{N(\varphi x, y, z) + N(\varphi x, z, y) \\
&\quad + \hat{N}(\varphi x, y, z) + \hat{N}(\varphi x, z, y)\} \\
&\quad + \frac{1}{2} \eta(x) \{N(\xi, y, \varphi z) + \hat{N}(\xi, y, \varphi z) \\
&\quad + \eta(z) \hat{N}(\xi, \xi, \varphi y)\}.
\end{align*}
\]
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Proof. Taking the sum of (4.31) and (4.32), we obtain

\[ F(\varphi x, y, z) - F(x, y, \varphi z) = \frac{1}{2}\left\{ N(x, y, z) + \hat{N}(x, y, z) \right\} \]

\[ - \eta(z)F(x, \varphi y, \xi). \]

(4.40)

The identities (4.10) together with (4.1) imply

\[ F(x, y, \varphi z) + F(x, z, \varphi y) = \eta(z)F(x, \varphi y, \xi) + \eta(y)F(x, \varphi z, \xi). \]

(4.41)

A suitable combination of (4.40) and (4.41) yields

\[ F(\varphi x, y, z) = \frac{1}{4}\left\{ N(x, y, z) + N(x, z, y) \right\} \]

\[ + \hat{N}(x, y, z) + \hat{N}(x, z, y) \} . \]

(4.42)

Applying (4.1), we obtain from (4.42)

\[ F(x, y, z) = \eta(x)F(\xi, y, z) \]

\[ - \frac{1}{4}\left\{ N(\varphi x, y, z) + N(\varphi x, z, y) \right\} \]

\[ + \hat{N}(\varphi x, y, z) + \hat{N}(\varphi x, z, y) \} . \]

(4.43)

Set \( x = \xi \) and \( z \to \varphi z \) into (4.40) and use (4.1) to get

\[ F(\xi, y, z) = \frac{1}{2}\left\{ N(\xi, y, \varphi z) + \hat{N}(\xi, y, \varphi z) \right\} + \eta(z)F(\xi, \xi, y). \]

(4.44)

Finally, set \( y = \xi \) into (4.44) and use the general identities \( N(\xi, \zeta) = F(\xi, \xi, \xi) = 0 \) to obtain

\[ F(\xi, \xi, z) = \frac{1}{2}\hat{N}(\xi, \xi, \varphi z). \]

(4.45)

Substitute (4.45) into (4.44) and the obtained identity insert into (4.43) to get (4.39). \( \square \)

As corollaries, in the cases when \( N = 0 \) or \( \hat{N} = 0 \), the relation (4.39) takes the following form, respectively:

\[ F(x, y, z) = - \frac{1}{4}\left\{ \hat{N}(\varphi x, y, z) + \hat{N}(\varphi x, z, y) \right\} \]

\[ + \frac{1}{2}\eta(x)\left\{ \hat{N}(\xi, y, \varphi z) + \eta(z)\hat{N}(\xi, \xi, \varphi y) \right\} , \]

\[ F(x, y, z) = - \frac{1}{4}\left\{ N(\varphi x, y, z) + N(\varphi x, z, y) \right\} + \frac{1}{2}\eta(x)N(\xi, y, \varphi z). \]

\* \* \*
§5. Canonical-type connections on almost contact manifolds with B-metric

In the present section, a canonical-type connection on the almost contact manifolds with B-metric is constructed. It is proved that its torsion is invariant with respect to a subgroup of the general conformal transformations of the almost contact B-metric structure. The basic classes of the considered manifolds are characterized in terms of the torsion of the canonical-type connection.

The main results of this section are published in [99].

In differential geometry of manifolds with additional tensor structures there are studied those affine connections which preserve the structure tensors and the metric, known also as natural connections on the considered manifolds.

Natural connections of canonical type are considered on the almost complex manifolds with Norden metric in §3 and [38, 35, 111].

Here, we are interested in almost contact B-metric manifolds. These manifolds are the odd-dimensional extension of the almost complex manifolds with Norden metric and the case with indefinite metrics corresponding to the almost contact metric manifolds. The geometry of some natural connections on almost contact B-metric manifolds are studied in [96, 78, 83, 101].

In the present section we consider natural connections of canonical type on the almost contact manifolds with B-metric. The section is organized as follows. In Subsection 5.1 we define a natural connection on an almost contact manifold with B-metric and we give a necessary and sufficient condition an affine connection to be natural. In Subsection 5.2 we consider a known natural connection (the $\varphi B$-connection) on these manifolds and give expressions of its torsion with respect to the pair of
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Nijenhuis tensors. In Subsection 5.3 we define a natural connection of canonical type (the $\varphi$-canonical connection) on an almost contact manifold with B-metric. We determine the class of the considered manifolds where the $\varphi$-canonical connection and the $\varphi$B-connection coincide. Then, we consider the group $G$ of the general conformal transformations of the almost contact B-metric structure and determine the invariant class of the considered manifolds and a tensor invariant of the group $G$. Also, we establish that the torsion of the canonical-type connection is invariant only regarding the subgroup $G_0$ of $G$. Thus, we characterize the basic classes of the considered manifolds by the torsion of the canonical-type connection. In the end of this subsection we supply a relevant example. In Subsection 5.4 we consider a natural connection with totally skew-symmetric torsion (the $\varphi$KT-connection) and give a necessary and sufficient condition for existence of this connection. Finally, we establish a linear relation between the three considered connections.

5.1. Natural connection on almost contact B-metric manifold

Let us consider an arbitrary almost contact B-metric manifold $(\mathcal{M}, \varphi, \xi, \eta, g)$.

**Definition 5.1.** An affine connection $\nabla^*$ is called a natural connection on $(\mathcal{M}, \varphi, \xi, \eta, g)$ if the almost contact structure $(\varphi, \xi, \eta)$ and the B-metric $g$ are parallel with respect to $\nabla^*$, i.e.

$$\nabla^* \varphi = \nabla^* \xi = \nabla^* \eta = \nabla^* g = 0.$$ 

As a corollary, the associated metric $\tilde{g}$ is also parallel with respect to the natural connection $\nabla^*$ on $(\mathcal{M}, \varphi, \xi, \eta, g)$, i.e. $\nabla^* \tilde{g} = 0$.

Therefore, an arbitrary natural connection $\nabla^*$ on $(\mathcal{M}, \varphi, \xi, \eta, g) \notin \mathcal{F}_0$ plays the same role as $\nabla$ on $(\mathcal{M}, \varphi, \xi, \eta, g) \in \mathcal{F}_0$. Obviously, $\nabla^*$ and $\nabla$ coincide when $(\mathcal{M}, \varphi, \xi, \eta, g) \in \mathcal{F}_0$. Because of that, we are interested in natural connections on $(\mathcal{M}, \varphi, \xi, \eta, g) \notin \mathcal{F}_0$.

**Theorem 5.1.** An affine connection $\nabla^*$ is natural on $(\mathcal{M}, \varphi, \xi, \eta, g)$ if and only if $\nabla^* \varphi = \nabla^* g = 0$.

**Proof.** It is known, that an affine connection $\nabla^*$ is a natural connection on $(\mathcal{M}, \varphi, \xi, \eta, g)$ if and only if the following properties for the potential $Q$ of $\nabla^*$ with respect to $\nabla$ are valid [83]:

$$Q(x, y, \varphi z) - Q(x, \varphi y, z) = F(x, y, z),$$

$$Q(x, y, z) = -Q(x, z, y).$$  \hspace{1cm} (5.1)
These conditions are equivalent to $\nabla^*\varphi = 0$ and $\nabla^*g = 0$, respectively. Moreover, $\nabla^*\xi = 0$ is equivalent to the relation

$$Q(x, \xi, z) = -F(x, \xi, \varphi z),$$

which is a consequence of the first equality of (5.1). Finally, since $\eta = \xi \lrcorner g$, then supposing $\nabla^*g = 0$ we have $\nabla^*\xi = 0$ if and only if $\nabla^*\eta = 0$. Thus, the statement is truthful. 

5.2. $\varphi$B-connection

In [96], it is introduced a natural connection on $(\mathcal{M}, \varphi, \xi, \eta, g)$. In [98], this connection is called a $\varphi$B-connection. It is studied for all main classes $\mathcal{F}_1$, $\mathcal{F}_4$, $\mathcal{F}_5$, $\mathcal{F}_{11}$ of almost contact B-metric manifolds in [95, 96, 77, 78, 98] with respect to properties of the torsion and the curvature as well as the conformal geometry. A basic class is called a main class if the fundamental tensor $F$ is expressed explicitly by the metric $g$. Main classes contain the conformally equivalent manifolds of cosymplectic B-metric manifolds by transformations of $G$. The restriction of the $\varphi$B-connection $\nabla'$ on $\mathcal{H}$ coincides with the B-connection $D'$ on the corresponding almost Norden manifold, given in (3.5) and studied for the class $\mathcal{W}_1$ in [36].

Further, we use (4.39) and the orthonormal decomposition given in (4.4), (4.6) and (4.7). Then we give the expression of the torsion of the $\varphi$B-connection in terms of the pair of Nijenhuis tensors regarding the horizontal and vertical components of vector fields:

$$T'(x, y, z) = \frac{1}{8}\left\{N(x^h, y^h, z^h) + \bigoplus_{x, y, z} N(x^h, y^h, z^h)ight\}$$

$$+ \hat{N}(z^h, y^h, x^h) - \hat{N}(z^h, x^h, y^h)$$

$$+ \frac{1}{2}\left\{N(x^h, y^h, z^v) + N(x^v, y^h, z^h) - N(y^v, x^h, z^h)ight\}$$

$$- \hat{N}(z^v, x^v, y^h) + \hat{N}(z^v, y^v, x^h)$$

$$+ \frac{1}{4}\left\{N(z^h, x^h, y^v) - N(z^h, y^h, x^v) + N(z^v, x^h, y^h)ight\}$$

$$- N(z^v, y^h, x^h) - \hat{N}(z^h, x^h, y^v) + \hat{N}(z^h, y^h, x^v)$$

$$- \hat{N}(z^v, x^h, y^h) + \hat{N}(z^v, y^h, x^h)\right\}.

Taking into account (5.7), (5.2) and (4.37), we obtain for the manifolds
from $\mathcal{F}_3 \oplus \mathcal{F}_7$ the following

\[ T'(x, y, z) = \frac{1}{8} \left\{ N(x^h, y^h, z^h) + \mathcal{S} N(x^h, y^h, z^h) \right\} + \frac{1}{4} \left\{ N(x^h, y^h, z^v) + \mathcal{S} N(x^h, y^h, z^v) \right\}. \]

(5.3)

Therefore, using the notation $N^h(x, y, z) = N(x^h, y^h, z^h)$, for the basic classes with vanishing $\hat{N}$ we have:

\[ \mathcal{F}_3: \ T' = \frac{1}{8} \{ N^h + \mathcal{S} N^h \}, \]

(5.4)

\[ \mathcal{F}_7: \ T' = \frac{1}{2} \{ d\eta \otimes \eta + \eta \wedge d\eta \}. \]

On any almost contact manifold with B-metric $(\mathcal{M}, \varphi, \xi, \eta, g)$, it is introduced in [96] a natural connection $\nabla'$, defined by

\[ \nabla'_x y = \nabla_x y + Q'(x, y), \]

where its potential with respect to the Levi-Civita connection $\nabla$ has the following form

\[ Q'(x, y) = \frac{1}{2} \{ (\nabla_x \varphi) \varphi y + (\nabla_x \eta) y \cdot \xi \} - \eta(y)\nabla_x \xi. \]

Therefore, for the corresponding $(0,3)$-tensor $Q'(x, y, z) = g(Q'(x, y), z)$ we have

\[ Q'(x, y, z) = \frac{1}{2} \left\{ F(x, \varphi y, z) + \eta(z)F(x, \varphi y, \xi) \right\} - 2\eta(y)F'(x, \varphi z, \xi). \]

(5.6)

The torsion of $\nabla'$ is expressed by the fundamental tensor $F$ by the following way

\[ T'(x, y, z) = -\frac{1}{2} F(x, \varphi y, \varphi^2 z) + \frac{1}{2} F(y, \varphi x, \varphi^2 z) \]

\[ + \eta(x)F(y, \varphi z, \xi) - \eta(y)F(x, \varphi z, \xi) \]

\[ + \eta(z)\{ F(x, \varphi y, \xi) - F(y, \varphi x, \xi) \}. \]

(5.7)

The torsion forms associated with the torsion $T$ of an arbitrary affine connection are defined as follows:

\[ t(x) = g^{ij} T(x, e_i, e_j), \]

(5.8)

\[ t^*(x) = g^{ij} T(x, e_i, \varphi e_j), \]

\[ \hat{t}(x) = T(x, \xi, \xi). \]
regarding an arbitrary basis \( \{ e_i; \xi \} \) \((i = 1, 2, \ldots, 2n)\) of \( T_p \mathcal{M} \). Obviously, \( \hat{t}(\xi) = 0 \) is always valid.

Applying (4.12) and (5.8), we have the following relations for the torsion forms of \( T' \) and the Lee forms:

\[
\begin{align*}
t' &= \frac{1}{2} \left\{ \theta^* + \theta^*(\xi) \eta \right\}, \\
t'^* &= -\frac{1}{2} \left\{ \theta + \theta(\xi) \eta \right\}, \\
\hat{t}' &= -\omega \circ \varphi.
\end{align*}
\]

The equality (4.13) and (5.9) imply the following relation:

\[
t'^* \circ \varphi = -t' \circ \varphi^2.
\]

### 5.3. \( \varphi \)-Canonical connection

**Definition 5.2.** A natural connection \( \nabla'' \) is called a \( \varphi \)-canonical connection on the manifold \((\mathcal{M}, \varphi, \xi, \eta, g)\) if the torsion tensor \( T'' \) of \( \nabla'' \) satisfies the following identity

\[
\begin{align*}
T''(x, y, z) - T''(x, \varphi y, \varphi z) - T''(x, z, y) + T''(x, \varphi z, \varphi y) \\
-\eta(x) \left\{ T''(\xi, y, z) - T''(\xi, \varphi y, \varphi z) \\
- T''(\xi, z, y) + T''(\xi, \varphi z, \varphi y) \right\} \\
-\eta(y) \left\{ T''(x, \xi, z) - T''(x, \varphi z, \varphi y) - \eta(x)T''(z, \xi, \xi) \right\} \\
+\eta(z) \left\{ T''(x, \xi, y) - T''(x, \varphi y, \varphi z) - \eta(x)T''(y, \xi, \xi) \right\} = 0.
\end{align*}
\]

Let us remark that the restriction of the \( \varphi \)-canonical connection \( \nabla'' \) on \((\mathcal{M}, \varphi, \xi, \eta, g)\) to the contact distribution \( \mathcal{H} \) is the unique canonical connection \( \nabla'' \) on the corresponding almost complex manifold with Norden metric, studied in [38] and §3.

We construct an affine connection \( \nabla'' \) as follows:

\[
g(\nabla''_{x} y, z) = g(\nabla_{x} y, z) + Q''(x, y, z),
\]

where

\[
Q''(x, y, z) = Q'(x, y, z)
\]

\[
-\frac{1}{8} \left\{ N(\varphi^2 z, \varphi^2 y, \varphi^2 x) + 2N(\varphi z, \varphi y, \varphi \xi) \eta(x) \right\}.
\]

By direct computations, we check that \( \nabla'' \) satisfies conditions (5.1) and therefore it is a natural connection on \((\mathcal{M}, \varphi, \xi, \eta, g)\). Its torsion is given
in the following equality

\[ T''(x, y, z) = T'(x, y, z) - \frac{1}{8} \left\{ N(\varphi^2 z, \varphi^2 y, \varphi^2 x) - N(\varphi^2 z, \varphi^2 x, \varphi^2 y) \right\} \]

(5.14)

\[ - \frac{1}{4} \left\{ N(\varphi z, \varphi y, \xi(x)) - N(\varphi z, \varphi x, \xi)(y) \right\}, \]

where \( T' \) is the torsion tensor of the \( \varphi \)-B-connection from (5.7).

The relation (5.14) is equivalent to

\[ T''(x, y, z) = T'(x, y, z) + \frac{1}{8} \left\{ N(x^h, y^h, z^h) - \mathcal{S}_{x,y,z} N(x^h, y^h, z^h) \right\} \]

(5.15)

\[ + \frac{1}{4} \left\{ N(x^h, y^h, z^v) - \mathcal{S}_{x,y,z} N(x^h, y^h, z^v) \right\}. \]

We verify immediately that \( T'' \) satisfies (5.11) and thus \( \nabla'' \), determined by (5.12) and (5.13), is a \( \varphi \)-canonical connection on \((M, \varphi, \xi, \eta, g)\).

The explicit expression (5.12), supported by (5.6) and (4.31), of the \( \varphi \)-canonical connection by the tensor \( F \) implies that the \( \varphi \)-canonical connection is unique.

Moreover, the torsion forms of the \( \varphi \)-canonical connection coincide with those of the \( \varphi \)-B-connection \( \nabla' \) given in (5.9).

Immediately we get the following

**Proposition 5.2.** A necessary and sufficient condition for the \( \varphi \)-canonical connection to coincide with the \( \varphi \)-B-connection is \( N(\varphi \cdot, \varphi \cdot) = 0 \).

**Lemma 5.3.** The class \( U_0 = \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \oplus \mathcal{F}_8 \oplus \mathcal{F}_9 \oplus \mathcal{F}_{10} \oplus \mathcal{F}_{11} \) of the almost contact B-metric manifolds is determined by the condition \( N(\varphi \cdot, \varphi \cdot) = 0 \).

**Proof.** It follows directly from (4.36) and (4.37). \( \square \)

Thus, Proposition 5.2 and Lemma 5.3 imply

**Corollary 5.4.** The \( \varphi \)-canonical connection and the \( \varphi \)-B-connection coincide on an almost contact B-metric manifold \((M, \varphi, \xi, \eta, g)\) if and only if \((M, \varphi, \xi, \eta, g)\) is in the class \( U_0 \).

Then, bearing in mind (5.2), we obtain that the torsions of the \( \varphi \)-canonical connection and the \( \varphi \)-B-connection on a manifold from \( U_0 \) have
the form

\[ T'''(x, y, z) = T'(x, y, z) \]

\[ = \frac{1}{8} \left\{ \hat{N}(z^h, y^h, x^h) - \hat{N}(z^h, x^h, y^h) \right\} \]

\[ + \frac{1}{2} \left\{ N(x^v, y^h, z^h) - N(y^v, x^h, z^h) \right\} \]

\[ + \hat{N}(x^v, y^h, z^h) - \hat{N}(y^v, x^h, z^h) \]

\[ - \hat{N}(z^v, x^v, y^h) + \hat{N}(z^v, y^v, x^h) \right\} \]

\[ + \frac{1}{4} \left\{ N(z^v, x^h, y^h) - N(z^v, y^h, x^h) \right\} \]

\[ - \hat{N}(z^v, x^h, y^h) + \hat{N}(z^v, y^v, x^h) \right\}. \]

The torsions \( T' \) and \( T''' \) are different to each other on a manifold that belongs to the basic classes \( \mathcal{F}_3 \) and \( \mathcal{F}_7 \) as well as to their direct sums with other classes. For \( \mathcal{F}_3 \oplus \mathcal{F}_7 \), using (5.3) and (5.15), we obtain the form of the torsion of the \( \varphi \)-canonical connection as follows

\[ T'''(x, y, z) = \frac{1}{4} N(x^h, y^h, z^h) + \frac{1}{2} N(x^h, y^h, z^v). \]

Therefore, using (4.37), the torsion of the \( \varphi \)-canonical connection for \( \mathcal{F}_3 \) and \( \mathcal{F}_7 \) is expressed by

(5.16) \hspace{1cm} \mathcal{F}_3 : \hspace{0.5cm} T''' = \frac{1}{4} N^h, \hspace{1cm} \mathcal{F}_7 : \hspace{0.5cm} T''' = d\eta \otimes \eta.

5.3.1. \( \varphi \)-Canonical connection and general contact conformal group \( G \)

Now we consider the group of transformations of the \( \varphi \)-canonical connection generated by the general contact conformal transformations of the almost contact B-metric structure.

According to [78], the general contact conformal transformations of the almost contact B-metric structure are defined by

(5.17) \[ \bar{\xi} = e^{-w}\xi, \quad \bar{\eta} = e^{w}\eta, \]

\[ \bar{g}(x,y) = \alpha g(x,y) + \beta g(x,\varphi y) + (\gamma - \alpha)\eta(x)\eta(y), \]

where \( \alpha = e^{2u}\cos 2v, \beta = e^{2u}\sin 2v, \gamma = e^{2w} \) for differentiable functions \( u, v, w \) on \( \mathcal{M} \). These transformations form a group denoted by \( G \).
Chapter I § 5. Canonical-type connections on almost contact manifolds with B-metric

If \( w = 0 \), we obtain the contact conformal transformations of the B-metric, introduced in [95]. By \( v = w = 0 \), the transformations (5.17) are reduced to the usual conformal transformations of \( g \).

Let us remark that \( G \) can be considered as a contact complex conformal gauge group, i.e. the composition of an almost contact group preserving \( \mathcal{H} \) and a complex conformal transformation of the complex Riemannian metric \( \tilde{g}^C = e^{2(u+iv)}g^C \) on \( \mathcal{H} \).

Note that the normality condition \( N = 0 \) is not preserved under the action of \( G \). We have

**Proposition 5.5.** The tensor \( N(\varphi \cdot, \varphi \cdot) \) is an invariant of the group \( G \) on any almost contact B-metric manifold.

**Proof.** Taking into account (4.25) and (5.17), we obtain

\[
\overline{N} = N + (dw \wedge \eta) \otimes \xi
\]

and clearly we have \( \overline{N}(\varphi x, \varphi y) = N(\varphi x, \varphi y) \). □

According to Lemma 5.3, we establish the following

**Corollary 5.6.** The class \( U_0 \) is closed by the action of the group \( G \).

Let \((\mathcal{M}, \varphi, \xi, \eta, g)\) and \((\mathcal{M}, \varphi, \bar{\xi}, \bar{\eta}, \bar{g})\) be contactly conformally equivalent with respect to a transformation from \( G \). The Levi-Civita connection of \( \bar{g} \) is denoted by \( \bar{\nabla} \). Using the general Koszul formula for the metric \( g \) and the corresponding Levi-Civita connection \( \nabla \)

\[
2g(\nabla xy, z) = x(g(y, z)) + y(g(z, x)) - z(g(x, y))
+ g([x, y], z) - g([y, z], x) + g([z, x], y),
\]

by straightforward computations we get the following relation between \( \nabla \) and \( \bar{\nabla} \):

\[
2(\alpha^2 + \beta^2)g(\nabla xy - \bar{\nabla} xy, z) =
\]

\[
= \frac{1}{2}\left\{-\alpha\beta\left[2F(x, y, \varphi^2 z) - F(\varphi^2 z, x, y)\right]
\right.
\]

\[
- \beta^2\left[2F(x, y, \varphi z) - F(\varphi z, x, y)\right]
\]

\[
+ \frac{\beta}{\gamma}\left(\alpha^2 + \beta^2\right)\left[2F(x, y, \xi) - F(\xi, x, y)\right]\eta(z)
\]

\[
+ 2\left(\frac{\alpha}{\gamma} - 1\right)(\alpha^2 + \beta^2)F(\varphi^2 x, \varphi y, \xi)\eta(z)
\]

(5.19a)
+ 2\alpha(\gamma - \alpha) \left[ F(x, \varphi z, \xi) + F(\varphi^2 z, \varphi x, \xi) \right] \eta(y) \\
- 2\beta(\gamma - \alpha) \left[ F(x, \varphi^2 z, \xi) - F(\varphi z, \varphi x, \xi) \right] \eta(y) \\
- 2(\alpha \, d\alpha(x) + \beta \, d\beta(x)) \, g(\varphi y, \varphi z) \\
+ 2(\alpha \, d\beta(x) - \beta \, d\alpha(x)) \, g(y, \varphi z) \\
\tag{5.19b} \\
- \left[ \alpha \, d\alpha(\varphi^2 z) + \beta \, d\alpha(\varphi z) \right] g(\varphi x, \varphi y) \\
+ \left[ \alpha \, d\beta(\varphi^2 z) + \beta \, d\beta(\varphi z) \right] g(x, \varphi y) \\
+ \left[ \alpha \, d\gamma(\varphi^2 z) + \beta \, d\gamma(\varphi z) \right] \eta(x) \eta(y) \\
+ \frac{1}{\gamma}(\alpha^2 + \beta^2) \left\{ d\alpha(\xi)g(\varphi x, \varphi y) - d\beta(\xi)g(x, \varphi y) \right\} \eta(z) \\
+ \frac{1}{\gamma}(\alpha^2 + \beta^2) \left\{ 2d\gamma(x)\eta(y) - d\gamma(\xi)\eta(x)\eta(y) \right\} \eta(z) \right\} \bigg|_{(x \leftrightarrow y)}, \\
where (for the sake of brevity) we use the notation \{A(x, y, z)\}_{(x \leftrightarrow y)} instead of the sum \{A(x, y, z) + A(y, x, z)\} for any tensor A(x, y, z).

Using (4.9) and (5.19), we obtain the following formula for the transformation by G of the tensor F:

\[ 2\overline{F}(x, y, z) = 2\alpha F(x, y, z) \]

\[ + \left\{ \beta \left\{ F(\varphi y, z, x) - F(y, \varphi z, x) + F(x, \varphi y, \xi) \eta(z) \right\} \right. \\
\left. + (\gamma - \alpha) \left\{ [F(x, y, \xi) + F(\varphi y, \varphi x, \xi)] \eta(z) \right. \\
\right. \\
\left. + [F(y, z, \xi) + F(\varphi z, \varphi y, \xi)] \eta(x) \right\} \right\} \bigg|_{(x \leftrightarrow y)}. \tag{5.20} \]

**Proposition 5.7.** Let the almost contact B-metric manifolds \((\mathcal{M}, \varphi, \xi, \eta, g)\) and \((\mathcal{M}, \varphi, \xi, \eta, g)\) be contactly conformally equivalent with respect to a
transformation from $G$. Then the corresponding $\varphi$-canonical connections $\nabla''$ and $\nabla''$ as well as their torsions $\mathcal{T}''$ and $\mathcal{T}''$ are related as follows:

$$
\nabla''_{x}y = \nabla''_{x}y - du(x)\varphi^{2}y + dv(x)\varphi y + dw(x)\eta(y)\xi
$$

\begin{equation}
+ \frac{1}{2}\{[du(\varphi^{2}y) - dv(\varphi y)] \varphi^{2}x - [du(\varphi y) + dv(\varphi^{2}y)] \varphi x
\end{equation}

$$
- g(\varphi x, \varphi y) [\varphi^{2}p - \varphi q] + g(x, \varphi y) [\varphi p + \varphi^{2}q] \}.
$$

(5.21)

where $p = \text{grad} u$, $q = \text{grad} v$;

$$
\mathcal{T}''(x,y) = \mathcal{T}''(x,y) + \{dw(x)\eta(y) - dw(y)\eta(x)\}\xi
$$

\begin{equation}
- \frac{1}{2}\{[du(\varphi^{2}y) + dv(\varphi y) - 2du(\xi)\eta(y)] \varphi^{2}x
\end{equation}

$$
- [du(\varphi^{2}x) + dv(\varphi x) - 2du(\xi)\eta(x)] \varphi^{2}y
$$

\begin{equation}
+ [du(\varphi y) - dv(\varphi^{2}y) + 2dv(\xi)\eta(y)] \varphi x
\end{equation}

$$
- [du(\varphi x) - dv(\varphi^{2}x) + 2dv(\xi)\eta(x)] \varphi y \}.
$$

(5.22)

Proof. Taking into account (5.5), we have the following equality on $(\mathcal{M}, \varphi, \xi, \eta, g)$:

$$
g(\nabla'_x y - \nabla_x y, z) = \frac{1}{2}\{F(x, \varphi y, z) + F(x, \varphi y, \xi)\eta(z)
$$

\begin{equation}
- 2F(x, \varphi z, \xi)\eta(y) \}.
\end{equation}

(5.23)

Then we can rewrite the corresponding equality on the manifold $(\mathcal{M}, \varphi, \bar{\xi}, \bar{\eta}, \bar{g})$, which is the image of $(\mathcal{M}, \varphi, \xi, \eta, g)$ by a transformation belonging to $G$:

$$
\bar{g}(\nabla'_x y - \nabla_x y, z) = \frac{1}{2}\{\bar{F}(x, \varphi y, z) + \bar{F}(x, \varphi y, \bar{\xi})\bar{\eta}(z)
$$

\begin{equation}
- 2\bar{F}(x, \varphi z, \bar{\xi})\bar{\eta}(y) \}.
\end{equation}

(5.24)

By virtue of (5.23), (5.21), (5.20) and (5.19), we get the following formula of the transformation by $G$ of the $\varphi$B-connection:

$$
g(\nabla'_x y - \nabla_x y, z) = \frac{1}{8}\sin 4v N(\varphi z, \varphi y, \varphi x)
$$

\begin{equation}
- \frac{1}{4}\sin^{2} 2v N(\varphi^{2}z, \varphi^{2}y, \varphi^{2}x)
\end{equation}

(5.25a)
$- \frac{1}{4} e^{2(w-u)} \sin 2v \, N(\varphi^2 z, \varphi y, \xi) \eta(x) \\
- \frac{1}{4} \left(1 - e^{2(w-u)} \cos 2v\right) \, N(\varphi z, \varphi y, \xi) \eta(x) \\
- du(x)g(\varphi y, \varphi z) + dv(x)g(y, \varphi z) \\
+ dw(x)\eta(y)\eta(z) \\
(5.25b) \\
+ \frac{1}{2} \left[du(\varphi^2 y) - dv(\varphi y)\right] \, g(\varphi x, \varphi z) \\
- \frac{1}{2} \left[du(\varphi y) + dv(\varphi^2 y)\right] \, g(x, \varphi z) \\
- \frac{1}{2} \left[du(\varphi^2 z) - dv(\varphi z)\right] \, g(\varphi x, \varphi y) \\
+ \frac{1}{2} \left[du(\varphi z) + dv(\varphi^2 z)\right] \, g(x, \varphi y).$

From (4.31), (5.20), (4.10) and (5.17), it follows the formula for the transformation by $G$ of the Nijenhuis tensor:

$$\overline{N}(\varphi x, \varphi y, z) = \alpha \, N(\varphi x, \varphi y, z) + \beta \, N(\varphi x, \varphi y, \varphi z) + (\gamma - \alpha) \, N(\varphi x, \varphi y, \xi) \eta(z).$$

(5.26)

Taking into account (5.13), (5.26), (5.17) and (5.25), we get (5.21). As a consequence of (5.21), the torsions $T''$ and $\overline{T}''$ of $\nabla''$ and $\overline{\nabla}'$, respectively, are related as in (5.22).

The torsion forms associated with $T''$ of the $\varphi$-canonical connection are defined by the same way as in (5.8).

Using (5.8), (5.14), (5.7), (4.10) and (4.33), we obtain that the torsion forms of the $\varphi$-canonical connection are expressed with respect to the Lee forms by the same way as in (5.9) for the torsion forms of the $\varphi$B-connection, namely:

$$t'' = \frac{1}{2} \left\{\theta^* + \theta^*(\xi)\eta\right\},$$

(5.27)

$$t''* = -\frac{1}{2} \left\{\theta + \theta(\xi)\eta\right\},$$

$$\hat{t}'' = -\omega \circ \varphi.$$
5.3.2. \(\varphi\)-Canonical connection and general contact conformal subgroup \(G_0\)

Let us consider the subgroup \(G_0\) of \(G\) defined by the conditions
\[
\begin{align*}
du \circ \varphi^2 + dv \circ \varphi &= du \circ \varphi - dv \circ \varphi^2 \\
= du(\xi) &= dv(\xi) = dw \circ \varphi = 0.
\end{align*}
\]
(5.28)

By direct computations, from (4.14), (5.17), (5.20) and (5.28), we prove the truthfulness of the following

**Theorem 5.8.** Each of the basic classes \(F_i\) \((i=1,2,\ldots,11)\) of the almost contact B-metric manifolds is closed by the action of the group \(G_0\). Moreover, \(G_0\) is the largest subgroup of \(G\) preserving the Lee forms \(\theta, \theta^*\), \(\omega\) and the special class \(F_0\).

**Theorem 5.9.** The torsion of the \(\varphi\)-canonical connection is invariant with respect to the general contact conformal transformations if and only if these transformations belong to the group \(G_0\).

**Proof.** Proposition 5.7 and (5.28) imply immediately
\[
\begin{align*}
\nabla''_{xy} &= \nabla''_{x}y - du(x)\varphi^2y + dv(x)\varphi y + dw(\xi)\eta(x)\eta(y)\xi \\
&\quad - du(y)\varphi^2x + dv(y)\varphi x + g(\varphi x, \varphi y)p - g(x, \varphi y)q.
\end{align*}
\]
(5.29)

The statement follows from (5.29) or alternatively from (5.22) and (5.28).

Bearing in mind the invariance of \(F_i\) \((i=1,2,\ldots,11)\) and \(T''\) with respect to the transformations of \(G_0\), we establish that each of the eleven basic classes of the manifolds \((\mathcal{M}, \varphi, \xi, \eta, g)\) is characterized by the torsion of the \(\varphi\)-canonical connection. Then we give this characterization in the following

**Proposition 5.10.** The basic classes of the almost contact B-metric manifolds are characterized by conditions for the torsion of the \(\varphi\)-canonical connection as follows:

\[
\begin{align*}
F_1 : \quad &T''(x, y) = \frac{1}{2n} \left\{ t''(\varphi^2 x)\varphi^2 y - t''(\varphi^2 y)\varphi^2 x \\
&\quad + t''(\varphi x)\varphi y - t''(\varphi y)\varphi x \right\}; \\
F_2 : \quad &T''(\xi, y) = 0, \quad \eta(T''(x, y)) = 0, \\
&\quad T''(x, y) = T''(\varphi x, \varphi y), \quad t'' = 0;
\end{align*}
\]
\[ \mathcal{F}_3 : \quad T''(\xi, y) = 0, \quad \eta(T''(x, y)) = 0, \]
\[ T''(x, y) = \varphi T''(x, \varphi y); \]
\[ \mathcal{F}_4 : \quad T''(x, y) = \frac{1}{2n} t''(\xi) \{ \eta(y) \varphi x - \eta(x) \varphi y \}; \]
\[ \mathcal{F}_5 : \quad T''(x, y) = \frac{1}{2n} t''(\xi) \{ \eta(y) \varphi^2 x - \eta(x) \varphi^2 y \}; \]
\[ \mathcal{F}_6 : \quad T''(x, y) = \eta(x) T''(\xi, y) - \eta(y) T''(\xi, x), \]
\[ T''(\xi, y, z) = T''(\xi, z, y) = -T''(\xi, \varphi y, \varphi z); \]
\[ \mathcal{F}_7 : \quad T''(x, y) = \eta(x) T''(\xi, y) - \eta(y) T''(\xi, x) + \eta(T''(x, y)) \xi, \]
\[ T''(\xi, y, z) = -T''(\xi, z, y) = -T''(\xi, \varphi y, \varphi z) \]
\[ = \frac{1}{2} T''(y, z, \xi) = -\frac{1}{2} T''(\varphi y, \varphi z, \xi); \]
\[ \mathcal{F}_8 : \quad T''(x, y) = \eta(x) T''(\xi, y) - \eta(y) T''(\xi, x) + \eta(T''(x, y)) \xi, \]
\[ T''(\xi, y, z) = -T''(\xi, z, y) = T''(\xi, \varphi y, \varphi z) \]
\[ = \frac{1}{2} T''(y, z, \xi) = \frac{1}{2} T''(\varphi y, \varphi z, \xi); \]
\[ \mathcal{F}_9 : \quad T''(x, y) = \eta(x) T''(\xi, y) - \eta(y) T''(\xi, x), \]
\[ T''(\xi, y, z) = T''(\xi, z, y) = T''(\xi, \varphi y, \varphi z); \]
\[ \mathcal{F}_{10} : \quad T''(x, y) = \eta(x) T''(\xi, y) - \eta(y) T''(\xi, x), \]
\[ T''(\xi, y, z) = -T''(\xi, z, y) = T''(\xi, \varphi y, \varphi z); \]
\[ \mathcal{F}_{11} : \quad T''(x, y) = \left\{ \hat{t}''(x) \eta(y) - \hat{t}''(y) \eta(x) \right\} \xi. \]

**Proof.** According to Proposition 5.2, Corollary 5.4, equalities (5.7) and (5.14), we have the following form of the torsion of the \( \varphi \)-canonical connection when \( (\mathcal{M}, \varphi, \xi, \eta, g) \) belongs to \( \mathcal{F}_i \) for \( i \in \{1, 2, \ldots, 11\}; i \neq 3, 7 \):

\[ T''(x, y) = \frac{1}{2} \left\{ (\nabla_x \varphi) \varphi y + (\nabla_x \eta) y \cdot \xi + 2\eta(x) \nabla_y \xi \right\} \]
\[ - (\nabla_y \varphi) \varphi x + (\nabla_y \eta) x \cdot \xi + 2\eta(y) \nabla_x \xi \} \]

For the classes \( \mathcal{F}_3 \) and \( \mathcal{F}_7 \), we use (5.14) and equalities (4.36) and (4.37).

Then, using (4.10), (5.27), (5.10) and (4.14), we obtain the characteristics in the statement. \( \square \)
5.3.3. An example of an almost contact B-metric manifold with coinciding $\varphi B$-connection and $\varphi$-canonical connection

In [39], it is given an example of the considered manifolds as follows. Let the vector space $\mathbb{R}^{2n+2} = \{ (u^1, \ldots, u^{n+1}; v^1, \ldots, v^{n+1}) | u^i, v^i \in \mathbb{R} \}$ be considered as a complex Riemannian manifold with the canonical complex structure $J$ and the metric $g$ defined by

$$g(x, x) = -\delta_{ij} \lambda^i \lambda^j + \delta_{ij} \mu^i \mu^j$$

for $x = \lambda^i \frac{\partial}{\partial u^i} + \mu^i \frac{\partial}{\partial v^i}$. Identifying the point $p \in \mathbb{R}^{2n+2}$ with its position vector, it is considered the time-like sphere $S$:

$$S : g(U, U) = -1$$

of $g$ in $\mathbb{R}^{2n+2}$, where $U$ is the unit normal to the tangent space $T_pS$ at $p \in S$. It is set

$$g(U, JU) = \tan \psi, \quad \psi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Then the almost contact structure is introduced by

$$\xi = \sin \psi U + \cos \psi JU, \quad \eta = g(\cdot, \xi), \quad \varphi = J - \eta \otimes J\xi.$$

It is shown that $(S, \varphi, \xi, \eta, g)$ is an almost contact B-metric manifold in the class $\mathcal{F}_4 \oplus \mathcal{F}_5$.

Since the $\varphi$-canonical connection coincides with the $\varphi B$-connection on any manifold in $\mathcal{F}_4 \oplus \mathcal{F}_5$, according to Corollary [5.4], then by virtue of (5.7) we get the torsion tensor and the torsion forms of the $\varphi$-canonical connection as follows:

$$T''(x, y, z) = \cos \psi \left\{ \eta(x)g(y, \varphi z) - \eta(y)g(x, \varphi z) \right\}$$

$$- \sin \psi \left\{ \eta(x)g(\varphi y, \varphi z) - \eta(y)g(\varphi x, \varphi z) \right\},$$

$$t'' = 2n \sin \psi \eta, \quad t''^* = -2n \cos \psi \eta, \quad \hat{t}'' = 0.$$

These equalities are in accordance with Proposition [5.10].

5.4. $\varphi KT$-connection

In [83], on an almost contact B-metric manifold $(M, \varphi, \xi, \eta, g)$, it is introduced a natural connection $\nabla'''$ called a $\varphi KT$-connection, which torsion $T'''$ is totally skew-symmetric, i.e. a 3-form. There, it is proved that
the \(\varphi\)KT-connection exists only on \((\mathcal{M}, \varphi, \xi, \eta, g)\) belonging to \(\mathcal{F}_3 \oplus \mathcal{F}_7\), i.e. the considered manifold has a Killing vector field \(\xi\) and a vanishing cyclic sum \(\mathcal{G}\) of \(F\).

**Corollary 5.11.** The \(\varphi\)KT-connection exists on an almost contact B-metric manifold if and only if the tensor \(\hat{N}\) vanishes on this manifold.

**Proof.** According to Proposition 4.3, the class \(\mathcal{F}_3 \oplus \mathcal{F}_7\) is characterized by the condition \(\hat{N} = 0\). Bearing in mind the statement above, the proof is completed.

The \(\varphi\)KT-connection is the odd-dimensional analogue of the KT-connection \(\nabla'''\) discussed in Subsection 3.3 on the corresponding class of quasi-Kähler manifolds with Norden metric.

According to [83], the unique \(\varphi\)KT-connection \(\nabla'''\) is determined by

\[
g(\nabla'''x y, z) = g(\nabla x y, z) + \frac{1}{2} T'''(x, y, z),
\]

where the torsion tensor is defined by

\[
T'''(x, y, z) = \frac{1}{2} (\mathcal{G} \{ F(x, y, \varphi z) - 3\eta(x)F(y, \varphi z, \xi) \} \\
= \frac{1}{2} (\eta \wedge d\eta)(x, y, z) + \frac{1}{4} \mathcal{G} N(x, y, z).
\]

(5.30)

Obviously, the torsion forms of the \(\varphi\)KT-connection are zero.

From (5.30) and (4.37), for the classes \(\mathcal{F}_3\) and \(\mathcal{F}_7\) we obtain

\[
(5.31) \quad \mathcal{F}_3: \quad T''' = \frac{1}{4} \mathcal{G} N^h, \quad \mathcal{F}_7: \quad T''' = \eta \wedge d\eta.
\]

As it is stated in Corollary 5.4, the \(\varphi\)B-connection and the \(\varphi\)-canonical connection coincide if and only if \((\mathcal{M}, \varphi, \xi, \eta, g)\) belongs to \(\mathcal{F}_i, i \in \{1, 2, \ldots, 11\} \setminus \{3, 7\}\), i.e. we have \(\nabla' \equiv \nabla''\) if and only if the \(\varphi\)KT-connection \(\nabla'''\) does not exist.

For the rest basic classes \(\mathcal{F}_3\) and \(\mathcal{F}_7\) (where the \(\varphi\)KT-connection exists), we obtain

**Proposition 5.12.** Let \((\mathcal{M}, \varphi, \xi, \eta, g)\) be an arbitrary manifold belonging to \(\mathcal{F}_i, i \in \{3, 7\}\). The \(\varphi\)B-connection \(\nabla'\) is the average connection of the \(\varphi\)-canonical connection \(\nabla''\) and the \(\varphi\)KT-connection \(\nabla'''\), i.e. the following relation is valid

\[
\nabla' = \frac{1}{2} \{ \nabla'' + \nabla''' \}.
\]
Proof. By virtue of (6.9), (6.10) and (5.14) we obtain:

1) for $\mathcal{F}_3$

$$p_{1,2}(T')(x,y,z) = p_{1,2}(T'')(x,y,z) = p_{1,2}(T'''')(x,y,z) \nonumber$$

$$= -\frac{1}{2}\left\{ F(\varphi^2x, \varphi^2y, \varphi z) + F(\varphi^2y, \varphi^2z, \varphi x) \right. \nonumber$$

$$- F(\varphi^2z, \varphi^2x, \varphi y) \left. \right\}, \nonumber$$

$$p_{1,4}(T')(x,y,z) = \frac{1}{2}p_{1,4}(T'')(x,y,z) = -\frac{1}{2}F(\varphi^2z, \varphi^2x, \varphi y), \nonumber$$

$$p_{1,4}(T'''')(x,y,z) = 0; \nonumber$$

2) for $\mathcal{F}_7$

$$p_{2,1}(T')(x,y,z) = p_{2,1}(T'')(x,y,z) = p_{2,1}(T'''')(x,y,z) \nonumber$$

$$= 2\eta(z)F(x, \varphi y, \xi), \nonumber$$

$$p_{3,2}(T')(x,y,z) = \frac{1}{2}p_{3,2}(T'')(x,y,z) \nonumber$$

$$= \eta(x)F'(y, \varphi z, \xi) - \eta(y)F(x, \varphi z, \xi), \nonumber$$

$$p_{3,2}(T'''')(x,y,z) = 0. \nonumber$$

Thus, we establish the equality $2T' = T'' + T'''$ for $\mathcal{F}_3$ and $\mathcal{F}_7$. Then, using (3.2), we obtain the expression $2Q' = Q'' + Q'''$ for the corresponding potentials with respect to $\nabla$, defined by

$$Q'(x,y,z) = g(\nabla_x^y y - \nabla_x y, z), \nonumber$$

$$Q''(x,y,z) = g(\nabla''_x y - \nabla_x y, z), \nonumber$$

$$Q'''(x,y,z) = g(\nabla'''_x y - \nabla_x y, z). \nonumber$$

Therefore, we have the statement. \qed

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§6. Classification of affine connections on almost contact manifolds with B-metric

In the present section the space of the torsion (0,3)-tensors of the affine connections on almost contact manifolds with B-metric is decomposed in 15 orthogonal and invariant subspaces with respect to the action of the structure group. This decomposition gives a rise to a classification of the corresponding affine connections. Three known connections, preserving the structure, are characterized regarding this classification.

The main results of this section are published in [100].

The investigations of affine connections on manifolds take a central place in the study of the differential geometry of these manifolds. The affine connections preserving the metric are completely characterized by their torsion tensors. In accordance with our goals, it is important to describe affine connections regarding the properties of their torsion tensors with respect to the structures on the manifold. Such a classification of the space of the torsion tensors is made in [38] by G. Ganchev and V. Mihova in the case of almost complex manifolds with Norden metric.

The idea of decomposition of the space of the basic (0,3)-tensors, generated by the covariant derivative of the fundamental tensor of type (1,1), is used by different authors in order to obtain classifications of manifolds with additional tensor structures. For example, let us mention the classification of almost Hermitian manifolds given in [44], of almost complex manifolds with Norden metric – in [34], of almost contact metric manifolds – in [4], of almost contact manifolds with B-metric – in [39], of Riemannian almost product manifolds – in [119], of Riemannian manifolds with traceless almost product structure – in [135], of almost paracontact metric manifolds – in [118], of almost paracontact Riemannian manifolds of type \((n, n)\) – in [104].
The affine connections preserving the structure (also known as natural connections) are particularly interesting in differential geometry. Canonical Hermitian connections on almost Hermitian manifolds are discussed in the beginning of §3.

Natural connections of canonical type are considered on the Riemannian almost product manifolds in [51, 53, 52] and on the almost complex manifolds with Norden metric in [38, 35, 111]. The Tanaka-Webster connection on a contact metric manifold is introduced ([140, 139, 148]) in the context of CR-geometry. A natural connection with minimal torsion on the quaternionic contact structures, introduced in [10], is known as the Biquard connection.

The goal of the present section is to describe the torsion space with respect to the almost contact B-metric structure, which can be used to study some natural connections on these manifolds.

This section is organized as follows. Subsection 6.1 is devoted to the decomposition of the space of torsion tensors on almost contact manifolds with B-metric. On this basis, in Subsection 6.2, we classify all affine connections on the considered manifolds. In Subsection 6.3, we find the position of three known natural connections from §5 in the obtained classification.

6.1. A decomposition of the space of torsion tensors

The object of our considerations are the affine connections with torsion. Thus, we have to study the properties of the torsion tensors with respect to the almost contact structure and the B-metric.

If $T$ is the torsion tensor of an affine connection $\nabla^*$, i.e.

$$T(x, y) = \nabla^*_x y - \nabla^*_y x - [x, y],$$

then the corresponding tensor of type (0,3) is determined as usually by $T(x, y, z) = g(T(x, y), z)$.

Let us consider $T_p \mathcal{M}$ at arbitrary $p \in \mathcal{M}$ as a $(2n+1)$-dimensional vector space with almost contact B-metric structure $(V, \varphi, \xi, \eta, g)$. Moreover, let $\mathcal{T}$ be the vector space of all tensors $T$ of type (0,3) over $V$ having skew-symmetry by the first two arguments, i.e.

$$\mathcal{T} = \{T(x, y, z) \in \mathbb{R}, x, y, z \in V \mid T(x, y, z) = -T(y, x, z)\}.$$

The metric $g$ induces an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{T}$ defined by

$$\langle T_1, T_2 \rangle = g^{iq}g^{jr}g^{ks}T_1(e_i, e_j, e_k)T_2(e_q, e_r, e_s)$$
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§6. Classification of affine connections on almost contact manifolds with B-metric

for any $T_1, T_2 \in \mathcal{T}$ and a basis $\{e_i\}$ ($i = 1, 2, \ldots, 2n + 1$) of $V$.

The structure group $\mathcal{G} \times \mathcal{I}$ consisting of matrices of the form (4.8) has a standard representation in $V$ which induces a natural representation $\lambda$ of $\mathcal{G} \times \mathcal{I}$ in $\mathcal{T}$ as follows

$$((\lambda a)T)(x, y, z) = T(a^{-1}x, a^{-1}y, a^{-1}z)$$

for any $a \in \mathcal{G} \times \mathcal{I}$ and $T \in \mathcal{T}$, so that

$$\langle (\lambda a)T_1, (\lambda a)T_2 \rangle = \langle T_1, T_2 \rangle, \quad T_1, T_2 \in \mathcal{T}.$$  

Using the projectors $h$ and $v$ on $V$, which are introduced as in (4.5) and (4.7), we have an orthogonal decomposition of $V$ in the form

$$V = h(V) \oplus v(V).$$

Then we construct a partial decomposition of $\mathcal{T}$ as follows.

At first, we define the operator $p_1 : \mathcal{T} \to \mathcal{T}$ by

$$p_1(T)(x, y, z) = -T(\varphi^2 x, \varphi^2 y, \varphi^2 z), \quad T \in \mathcal{T}.$$  

It is easy to check the following

**Lemma 6.1.** The operator $p_1$ has the following properties:

(i) $p_1 \circ p_1 = p_1$;

(ii) $\langle p_1(T_1), T_2 \rangle = \langle T_1, p_1(T_2) \rangle$, $T_1, T_2 \in \mathcal{T}$;

(iii) $p_1 \circ (\lambda a) = (\lambda a) \circ p_1$.

According to Lemma [6.1], we have the following orthogonal decomposition of $\mathcal{T}$ by the image and the kernel of $p_1$:

$$\mathcal{P}_1 = \text{im}(p_1) = \{T \in \mathcal{T} \mid p_1(T) = T\},$$

$$\mathcal{P}_1^\perp = \text{ker}(p_1) = \{T \in \mathcal{T} \mid p_1(T) = 0\}.$$

Further, we consider the operator $p_2 : \mathcal{P}_1^\perp \to \mathcal{P}_1^\perp$, defined by

$$p_2(T)(x, y, z) = \eta(z)T(\varphi^2 x, \varphi^2 y, \xi), \quad T \in \mathcal{P}_1^\perp.$$  

We obtain immediately the truthfulness of the following

**Lemma 6.2.** The operator $p_2$ has the following properties:

(i) $p_2 \circ p_2 = p_2$;

(ii) $\langle p_2(T_1), T_2 \rangle = \langle T_1, p_2(T_2) \rangle$, $T_1, T_2 \in \mathcal{P}_1^\perp$.
(iii) \( p_2 \circ (\lambda a) = (\lambda a) \circ p_2. \)

Then, bearing in mind Lemma 6.2, we obtain
\[
\mathcal{P}_2 = \text{im}(p_2) = \{ T \in \mathcal{P}_1 \mid p_2(T) = T \},
\]
\[
\mathcal{P}_2^\perp = \ker(p_2) = \{ T \in \mathcal{P}_1 \mid p_2(T) = 0 \}.
\]

Finally, we consider the operator \( p_3 : \mathcal{P}_2^\perp \to \mathcal{P}_2^\perp \) defined by
\[
p_3(T)(x, y, z) = \eta(x)T(\xi, \varphi^2 y, \varphi^2 z) + \eta(y)T(\varphi^2 x, \xi, \varphi^2 z), \quad T \in \mathcal{P}_2^\perp
\]
and we get the following

**Lemma 6.3.** The operator \( p_3 \) has the following properties:

(i) \( p_3 \circ p_3 = p_3; \)

(ii) \( \langle p_3(T_1), T_2 \rangle = \langle T_1, p_3(T_2) \rangle, \quad T_1, T_2 \in \mathcal{P}_2^\perp; \)

(iii) \( p_3 \circ (\lambda a) = (\lambda a) \circ p_3. \)

By virtue of Lemma 6.3, we have
\[
\mathcal{P}_3 = \text{im}(p_3) = \{ T \in \mathcal{P}_2^\perp \mid p_3(T) = T \},
\]
\[
\mathcal{P}_4 = \ker(p_3) = \{ T \in \mathcal{P}_2^\perp \mid p_3(T) = 0 \}.
\]

From Lemma 6.1, Lemma 6.2 and Lemma 6.3 we have immediately

**Theorem 6.4.** The decomposition \( \mathcal{T} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \mathcal{P}_3 \oplus \mathcal{P}_4 \) is orthogonal and invariant under the action of \( \mathcal{G} \times \mathcal{I} \). The subspaces \( \mathcal{P}_i \) \( (i = 1, 2, 3, 4) \) are determined by

\[
\begin{align*}
\mathcal{P}_1 & : \quad T(x, y, z) = -T(\varphi^2 x, \varphi^2 y, \varphi^2 z), \\
\mathcal{P}_2 & : \quad T(x, y, z) = \eta(z)T(\varphi^2 x, \varphi^2 y, \xi), \\
\mathcal{P}_3 & : \quad T(x, y, z) = \eta(x)T(\xi, \varphi^2 y, \varphi^2 z) + \eta(y)T(\varphi^2 x, \xi, \varphi^2 z), \\
\mathcal{P}_4 & : \quad T(x, y, z) = -\eta(z)\{ \eta(y)T(\varphi^2 x, \xi, \xi) + \eta(x)T(\xi, \varphi^2 y, \xi) \}
\end{align*}
\]

for arbitrary vectors \( x, y, z \in V \).

**Corollary 6.5.** The subspaces \( \mathcal{P}_i \) \( (i = 1, 2, 3, 4) \) are characterized as follows:

\[
\mathcal{P}_1 = \{ T \in \mathcal{T} \mid T(x^x, y, z) = T(x, y, z^y) = 0 \},
\]
\[
\mathcal{P}_2 = \{ T \in \mathcal{T} \mid T(x^y, y, z) = T(x, y, z^h) = 0 \},
\]
\[ P_3 = \{ T \in T \mid T(x, y, z^v) = T(x^h, y^h, z) = 0 \}, \]
\[ P_4 = \{ T \in T \mid T(x, y, z^h) = T(x^h, y^h, z) = 0 \}, \]

where \( x, y, z \in V \).

According to Corollary 6.5, (6.1) and (5.8), we obtain the following

**Corollary 6.6.** The torsion forms of \( T \) have the following properties in each of the subspaces \( P_i \) \((i = 1, 2, 3, 4)\):

(i) If \( T \in P_1 \), then \( t \circ v = t^* \circ v = \hat{t} = 0 \);
(ii) If \( T \in P_2 \), then \( t = t^* = \hat{t} = 0 \);
(iii) If \( T \in P_3 \), then \( t \circ h = t^* \circ h = \hat{t} = 0 \);
(iv) If \( T \in P_4 \), then \( t = t^* = 0 \).

Further we continue the decomposition of the subspaces \( P_i \) \((i = 1, 2, 3, 4)\) of \( T \).

**6.1.1. The subspace \( P_1 \)**

Since the endomorphism \( \varphi \) induces an almost complex structure on \( \mathcal{H} \) (which is the orthogonal complement \( \{ \xi \}^\perp \) of the subspace \( \mathcal{V} \)) and the restriction of \( g \) on \( \mathcal{H} \) is a Norden metric (because the almost complex structure causes an anti-isometry on \( \mathcal{H} \)), then the decomposition of \( P_1 \) is made as the decomposition of the space of the torsion tensors on an almost complex manifold with Norden metric known from [38].

Let us consider the linear operator \( L_{1,0} : \mathcal{P}_1 \to \mathcal{P}_1 \) defined by
\[ L_{1,0}(T)(x, y, z) = -T(\varphi x, \varphi y, \varphi^2 z). \]

Then, it follows immediately

**Lemma 6.7.** The operator \( L_{1,0} \) is an involutive isometry on \( P_1 \) and it is invariant with respect to the group \( \mathcal{G} \times \mathcal{I} \), i.e.

(i) \( L_{1,0} \circ L_{1,0} = \text{Id}_{\mathcal{P}_1} \);
(ii) \( \langle L_{1,0}(T_1), L_{1,0}(T_2) \rangle = \langle T_1, T_2 \rangle \);
(iii) \( L_{1,0}((\lambda a)T) = (\lambda a)(L_{1,0}(T)) \),

where \( T_1, T_2 \in \mathcal{P}_1 \), \( a \in \mathcal{G} \times \mathcal{I} \).
Therefore, $L_{1,0}$ has two eigenvalues $+1$ and $-1$, and the corresponding eigenspaces

\[ P_1^+ = \{ T \in P_1 \mid L_{1,0}(T) = T \} , \]
\[ P_1^- = \{ T \in P_1 \mid L_{1,0}(T) = -T \} \]

are invariant orthogonal subspaces of $P_1$.

In order to decompose $P_1^-$, we consider the linear operator $L_{1,1} : P_1^- \to P_1^-$ defined by

\[ L_{1,1}(T)(x, y, z) = -T(\varphi x, \varphi^2 y, \varphi z) . \]

Let us denote the eigenspaces

\[ P_{1,1} = \{ T \in P_1^- \mid L_{1,1}(T) = -T \} , \]
\[ P_{1,2} = \{ T \in P_1^- \mid L_{1,1}(T) = T \} . \]

We have

**Lemma 6.8.** The operator $L_{1,1}$ is an involutive isometry on $P_1^-$ and it is invariant with respect to $G \times I$.

According to the latter lemma, the eigenspaces $P_{1,1}$ and $P_{1,2}$ are invariant and orthogonal.

To decompose $P_1^+$, we define the linear operator $L_{1,2} : P_1^+ \to P_1^+$ as follows:

\[ L_{1,2}(T)(x, y, z) = -\frac{1}{2} \left\{ T(\varphi^2 z, \varphi^2 x, \varphi^2 y) + T(\varphi^2 z, \varphi x, \varphi y) \right. \]
\[ -T(\varphi^2 z, \varphi^2 y, \varphi^2 x) - T(\varphi^2 z, \varphi y, \varphi x) \right\} . \]

**Lemma 6.9.** The operator $L_{1,2}$ is an involutive isometry on $P_1^+$ and it is invariant with respect to $G \times I$.

Thus, the eigenspaces

\[ P_{1,3} = \{ T \in P_1^+ \mid L_{1,2}(T) = T \} , \]
\[ P_{1,4} = \{ T \in P_1^+ \mid L_{1,2}(T) = -T \} . \]

are invariant and orthogonal.

Using Lemma 6.7, Lemma 6.8 and Lemma 6.9, we get the following

**Theorem 6.10.** The decomposition $P_1 = P_{1,1} \oplus P_{1,2} \oplus P_{1,3} \oplus P_{1,4}$ is orthogonal and invariant with respect to the structure group.

Bearing in mind the definition of the subspaces $P_{1,i}$ ($i = 1, 2, 3, 4$), we obtain
Proposition 6.11. The subspaces \( \mathcal{P}_{1,i} \) \( (i = 1, 2, 3, 4) \) of \( \mathcal{P}_1 \) are determined by:

\[
\begin{align*}
\mathcal{P}_{1,1} : & \quad T(\xi, y, z) = T(x, y, \xi) = 0, \\
& \quad T(x, y, z) = -T(\varphi x, \varphi y, z) = -T(x, \varphi y, \varphi z); \\
\mathcal{P}_{1,2} : & \quad T(\xi, y, z) = T(x, y, \xi) = 0, \\
& \quad T(x, y, z) = -T(\varphi x, \varphi y, z) = T(\varphi x, y, \varphi z); \\
\mathcal{P}_{1,3} : & \quad T(\xi, y, z) = T(x, y, \xi) = 0, \\
& \quad T(x, y, z) - T(\varphi x, \varphi y, z) = \mathcal{S}_{x,y,z} T(x, y, z) = 0; \\
\mathcal{P}_{1,4} : & \quad T(\xi, y, z) = T(x, y, \xi) = 0, \\
& \quad T(x, y, z) - T(\varphi x, \varphi y, z) = \mathcal{S}_{x,y,z} T(\varphi x, y, z) = 0.
\end{align*}
\]

Using Corollary 6.6 (i), Proposition 6.11 and (5.8), we obtain

Corollary 6.12. The torsion forms \( t \) and \( t^* \) of \( T \) have the following properties in the subspaces \( \mathcal{P}_{1,i} \) \( (i = 1, 2, 3, 4) \):

(i) If \( T \in \mathcal{P}_{1,1} \), then \( t = -t^* \circ \varphi, \ t \circ \varphi = t^* \);

(ii) If \( T \in \mathcal{P}_{1,2} \), then \( t = t^* = 0 \);

(iii) If \( T \in \mathcal{P}_{1,3} \), then \( t = t^* \circ \varphi, \ t \circ \varphi = -t^* \);

(iv) If \( T \in \mathcal{P}_{1,4} \), then \( t = t^* = 0 \).

Let us remark that each of the subspaces \( \mathcal{P}_{1,1} \) and \( \mathcal{P}_{1,3} \) can be additionally decomposed to a couple of subspaces — one of zero traces \( (t, \ t^*) \) and one of non-zero traces \( (t, \ t^*) \), i.e.

\[
\begin{align*}
\mathcal{P}_{1,1} &= \mathcal{P}_{1,1,1} \oplus \mathcal{P}_{1,1,2}, \\
\mathcal{P}_{1,3} &= \mathcal{P}_{1,3,1} \oplus \mathcal{P}_{1,3,2},
\end{align*}
\]

where

\[
\begin{align*}
\mathcal{P}_{1,1,1} &= \{ T \in \mathcal{P}_{1,1} \mid t \neq 0 \}, \\
\mathcal{P}_{1,1,2} &= \{ T \in \mathcal{P}_{1,1} \mid t = 0 \}, \\
\mathcal{P}_{1,3,1} &= \{ T \in \mathcal{P}_{1,3} \mid t \neq 0 \}, \\
\mathcal{P}_{1,3,2} &= \{ T \in \mathcal{P}_{1,3} \mid t = 0 \}.
\end{align*}
\]

Proposition 6.13. Let \( T \in \mathcal{T} \) and \( p_{1,i} \) \( (i = 1, 2, 3, 4) \) be the projection operators of \( \mathcal{T} \) in \( \mathcal{P}_{1,i} \), generated by the decomposition above. Then we
have
\[ p_{1,1}(T)(x, y, z) = -\frac{1}{4} \{ T(\varphi^2 x, \varphi^2 y, \varphi^2 z) - T(\varphi x, \varphi y, \varphi^2 z) \]
\[ - T(\varphi x, \varphi^2 y, \varphi z) - T(\varphi^2 x, \varphi y, \varphi z) \}; \]
\[ p_{1,2}(T)(x, y, z) = -\frac{1}{4} \{ T(\varphi^2 x, \varphi^2 y, \varphi^2 z) - T(\varphi x, \varphi y, \varphi^2 z) \]
\[ + T(\varphi x, \varphi^2 y, \varphi z) + T(\varphi^2 x, \varphi y, \varphi z) \}; \]
\[ p_{1,3}(T)(x, y, z) = -\frac{1}{4} \{ T(\varphi^2 x, \varphi^2 y, \varphi^2 z) + T(\varphi x, \varphi y, \varphi^2 z) \}
\[ + \frac{1}{8} \{ T(\varphi^2 z, \varphi^2 x, \varphi^2 y) + T(\varphi^2 z, \varphi x, \varphi y) \]
\[ + T(\varphi z, \varphi x, \varphi^2 y) - T(\varphi z, \varphi^2 x, \varphi y) \]
\[ - T(\varphi^2 z, \varphi^2 y, \varphi^2 x) - T(\varphi^2 z, \varphi y, \varphi x) \]
\[ - T(\varphi z, \varphi y, \varphi^2 x) + T(\varphi z, \varphi^2 y, \varphi x) \}, \]
\[ p_{1,4}(T)(x, y, z) = -\frac{1}{4} \{ T(\varphi^2 x, \varphi^2 y, \varphi^2 z) + T(\varphi x, \varphi y, \varphi^2 z) \}
\[ - \frac{1}{8} \{ T(\varphi^2 z, \varphi^2 x, \varphi^2 y) + T(\varphi^2 z, \varphi x, \varphi y) \]
\[ + T(\varphi z, \varphi x, \varphi^2 y) - T(\varphi z, \varphi^2 x, \varphi y) \]
\[ - T(\varphi^2 z, \varphi^2 y, \varphi^2 x) - T(\varphi^2 z, \varphi y, \varphi x) \]
\[ - T(\varphi z, \varphi y, \varphi^2 x) + T(\varphi z, \varphi^2 y, \varphi x) \}. \]

Proof. Let us show the calculations about \( p_{1,1} \) for example, using \[ \text{Lemma } 6.7 \].

Lemma 6.7 implies that the tensor \( \frac{1}{2} \{ T - L_{1,0}(T) \} \) is the projection of \( T \in \mathcal{P}_1 \) in \( \mathcal{P}_1^- = \mathcal{P}_{1,1} \oplus \mathcal{P}_{1,2} \). Using Lemma 6.8, we find the expression of \( p_{1,1} \) in terms of the operators \( L_{1,0} \) and \( L_{1,1} \) for \( T \in \mathcal{P}_1 \), namely
\[ p_{1,1}(T) = \frac{1}{4} \{ T - L_{1,0}(T) - L_{1,1}(T) + L_{1,1} \circ L_{1,0}(T) \}, \]
which implies the stated expression of \( p_{1,1} \), taking into account that \( T \in \mathcal{P}_1 \).
\( \mathcal{P}_1 \) is the image of \( T \in \mathcal{T} \) by \( p_1 \). In a similar way we prove the expressions for the other projectors under consideration.

We verify the following equalities for \( i = 1, 2, 3, 4 \)
\[
p_{1,i} \circ p_{1,i} = p_{1,i}, \quad \sum_{i} p_{1,i} = \text{Id}_{\mathcal{P}_1}.
\]

\subsection*{6.1.2. The subspace \( \mathcal{P}_2 \)}

Following the demonstrated procedure for \( \mathcal{P}_1 \), we continue the decomposition of the other main subspaces of \( \mathcal{T} \) with respect to the almost contact B-metric structure.

**Lemma 6.14.** The operator \( L_{2,0} \), defined by
\[
L_{2,0}(T)(x, y, z) = \eta(z)T(\varphi x, \varphi y, \xi),
\]
is an involutive isometry on \( \mathcal{P}_2 \) and invariant with respect to \( G \times I \).

Hence, the corresponding eigenspaces
\[
\mathcal{P}_{2,1} = \{T \in \mathcal{P}_2 \mid L_{2,0}(T) = -T\},
\]
\[
\mathcal{P}_{2,2} = \{T \in \mathcal{P}_2 \mid L_{2,0}(T) = T\}.
\]
are invariant and orthogonal. Therefore, we have

**Theorem 6.15.** The decomposition \( \mathcal{P}_2 = \mathcal{P}_{2,1} \oplus \mathcal{P}_{2,2} \) is orthogonal and invariant with respect to the structure group.

**Proposition 6.16.** The subspaces of \( \mathcal{P}_2 \) are determined by:
\[
\mathcal{P}_{2,1} : \quad T(x, y, z) = \eta(z)T(\varphi^2 x, \varphi^2 y, \xi),
\]
\[
T(x, y, \xi) = -T(\varphi x, \varphi y, \xi),
\]
\[
\mathcal{P}_{2,2} : \quad T(x, y, z) = \eta(z)T(\varphi^2 x, \varphi^2 y, \xi),
\]
\[
T(x, y, \xi) = T(\varphi x, \varphi y, \xi).
\]

Then the tensors \( \frac{1}{2}\{T - L_{2,0}(T)\} \) and \( \frac{1}{2}\{T + L_{2,0}(T)\} \) are the projections of \( \mathcal{P}_2 \) in \( \mathcal{P}_{2,1} \) and \( \mathcal{P}_{2,2} \), respectively. Moreover, we have the truthfulness of the properties
\[
p_{2,1} \circ p_{2,1} = p_{2,1}, \quad p_{2,2} \circ p_{2,2} = p_{2,2}, \quad p_{2,1} + p_{2,2} = \text{Id}_{\mathcal{P}_2}.
\]
Therefore, taking into account \( p_2 \), we obtain
Proposition 6.17. Let \( p_{2,1} \) and \( p_{2,2} \) be the projection operators of \( T \) in \( \mathcal{P}_{2,1} \) and \( \mathcal{P}_{2,2} \), respectively, generated by the decomposition above. Then we have for \( T \in \mathcal{T} \) the following

\[
p_{2,1}(T)(x, y, z) = \frac{1}{2} \eta(z) \left\{ T(\varphi^2 x, \varphi^2 y, \xi) - T(\varphi x, \varphi y, \xi) \right\},
\]

\[
p_{2,2}(T)(x, y, z) = \frac{1}{2} \eta(z) \left\{ T(\varphi^2 x, \varphi^2 y, \xi) + T(\varphi x, \varphi y, \xi) \right\}.
\]

According to Corollary 6.6 (ii), Proposition 6.16 and (5.8), we obtain

Corollary 6.18. The torsion forms of \( T \) are zero in each of the subspaces \( \mathcal{P}_{2,1} \) and \( \mathcal{P}_{2,2} \), i.e. if \( T \in \mathcal{P}_{2,1} \oplus \mathcal{P}_{2,2} \), then \( t = t^* = \hat{t} = 0 \).

6.1.3. The subspace \( \mathcal{P}_3 \)

Lemma 6.19. The following operators \( L_{3,k} \) (\( k = 0, 1 \)) are involutive isometries on \( \mathcal{P}_3 \) and invariant with respect to \( \mathcal{G} \times \mathcal{I} \):

\[
L_{3,0}(T)(x, y, z) = \eta(x)T(\xi, \varphi y, \varphi z) - \eta(y)T(\xi, \varphi x, \varphi z),
\]

\[
L_{3,1}(T)(x, y, z) = \eta(x)T(\xi, \varphi^2 z, \varphi^2 y) - \eta(y)T(\xi, \varphi^2 z, \varphi^2 x).
\]

By virtue of their action, we obtain consecutively the corresponding invariant and orthogonal eigenspaces:

\[
\mathcal{P}_3^- = \{ T \in \mathcal{P}_3 \mid L_{3,0}(T) = -T \},
\]

\[
\mathcal{P}_3^+ = \{ T \in \mathcal{P}_3 \mid L_{3,0}(T) = T \},
\]

\[
\mathcal{P}_{3,1} = \{ T \in \mathcal{P}_3^- \mid L_{3,1}(T) = T \},
\]

\[
\mathcal{P}_{3,2} = \{ T \in \mathcal{P}_3^- \mid L_{3,1}(T) = -T \},
\]

\[
\mathcal{P}_{3,3} = \{ T \in \mathcal{P}_3^+ \mid L_{3,1}(T) = T \},
\]

\[
\mathcal{P}_{3,4} = \{ T \in \mathcal{P}_3^+ \mid L_{3,1}(T) = -T \}.
\]

In such a way, we get

Theorem 6.20. The decomposition \( \mathcal{P}_3 = \mathcal{P}_{3,1} \oplus \mathcal{P}_{3,2} \oplus \mathcal{P}_{3,3} \oplus \mathcal{P}_{3,4} \) is orthogonal and invariant with respect to the structure group.

Proposition 6.21. The subspaces of \( \mathcal{P}_3 \) are determined by:

\[
\mathcal{P}_{3,1} : \quad T(x, y, z) = \eta(x)T(\xi, \varphi^2 y, \varphi^2 z) - \eta(y)T(\xi, \varphi^2 x, \varphi^2 z),
\]

\[
T(\xi, y, z) = T(\xi, z, y) = -T(\xi, \varphi y, \varphi z);
\]
\( \mathcal{P}_{3,2} : \ T(x, y, z) = \eta(x)T(\xi, \varphi^2y, \varphi^2z) - \eta(y)T(\xi, \varphi^2x, \varphi^2z), \)
\( T(\xi, y, z) = -T(\xi, z, y) = -T(\xi, \varphi y, \varphi z); \)

\( \mathcal{P}_{3,3} : \ T(x, y, z) = \eta(x)T(\xi, \varphi^2y, \varphi^2z) - \eta(y)T(\xi, \varphi^2x, \varphi^2z), \)
\( T(\xi, y, z) = T(\xi, z, y) = T(\xi, \varphi y, \varphi z); \)

\( \mathcal{P}_{3,4} : \ T(x, y, z) = \eta(x)T(\xi, \varphi^2y, \varphi^2z) - \eta(y)T(\xi, \varphi^2x, \varphi^2z), \)
\( T(\xi, y, z) = -T(\xi, z, y) = T(\xi, \varphi y, \varphi z). \)

By virtue of Corollary 6.6 (iii), Proposition 6.21 and equalities (5.8), we obtain

**Corollary 6.22.** The torsion forms \( t \) and \( t^* \) of \( T \) are zero in \( \mathcal{P}_{3,k} \subset \mathcal{P}_3 \) \((k = 2, 3, 4)\).

Let us remark that \( \mathcal{P}_{3,1} \) can be additionally decomposed to three subspaces determined by conditions \( t = 0 \), \( t^* = 0 \) and \( t = t^* = 0 \), respectively, i.e.

\[ \mathcal{P}_{3,1} = \mathcal{P}_{3,1,1} \oplus \mathcal{P}_{3,1,2} \oplus \mathcal{P}_{3,1,3}, \]

where

\[ \mathcal{P}_{3,1,1} = \{ T \in \mathcal{P}_{3,1} \mid t \neq 0, \ t^* = 0 \}, \]

\[ \mathcal{P}_{3,1,2} = \{ T \in \mathcal{P}_{3,1} \mid t = 0, \ t^* \neq 0 \}, \]

\[ \mathcal{P}_{3,1,3} = \{ T \in \mathcal{P}_{3,1} \mid t = 0, \ t^* = 0 \}. \]

**Proposition 6.23.** Let \( T \in \mathcal{T} \) and \( p_{3,k} \) \((k = 1, 2, 3, 4)\) be the projection operators of \( \mathcal{T} \) in \( \mathcal{P}_{3,k} \), generated by the decomposition above. Then we have

\[ p_{3,k}(T)(x, y, z) = \frac{1}{4} \{ \eta(x)A_{3,k}(y, z) - \eta(y)A_{3,k}(x, z) \}, \]

where

\[ A_{3,1}(y, z) = T(\xi, \varphi^2y, \varphi^2z) + T(\xi, \varphi^2z, \varphi^2y) \]
\[ -T(\xi, \varphi y, \varphi z) - T(\xi, \varphi z, \varphi y), \]

\[ A_{3,2}(y, z) = T(\xi, \varphi^2y, \varphi^2z) - T(\xi, \varphi^2z, \varphi^2y) \]
\[ -T(\xi, \varphi y, \varphi z) + T(\xi, \varphi z, \varphi y), \]

\[ A_{3,3}(y, z) = T(\xi, \varphi^2y, \varphi^2z) + T(\xi, \varphi^2z, \varphi^2y) \]
\[ + T(\xi, \varphi y, \varphi z) + T(\xi, \varphi z, \varphi y), \]

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\[ A_{3,4}(y, z) = T(\xi, \varphi^2 y, \varphi^2 z) - T(\xi, \varphi^2 z, \varphi^2 y) + T(\xi, \varphi y, \varphi z) - T(\xi, \varphi z, \varphi y). \]

\subsection*{6.1.4. The subspace \( P_4 \)}

Finally, we only denote \( P_4 \) as \( P_{4,1} \) and it is determined as follows

\[ P_{4,1} : \quad T(x, y, z) = \eta(z) \left\{ \eta(y)\hat{t}(x) - \eta(x)\hat{t}(y) \right\}. \]

Obviously, the projection operator \( p_{4,1} : \mathcal{T} \rightarrow P_{4,1} \) has the form

\[ (6.3) \quad p_{4,1}(T)(x, y, z) = \eta(z) \left\{ \eta(y)\hat{t}(x) - \eta(x)\hat{t}(y) \right\}. \]

\subsection*{6.1.5. The fifteen subspaces of \( \mathcal{T} \)}

In conclusion of the decomposition explained above, we combine Theorems \ref{thm6.4}, \ref{thm6.10}, \ref{thm6.15} and \ref{thm6.20}. We denote the subspaces \( P_{i,j} \) and \( P_{i,j,k} \) by \( \mathcal{T}_s, s \in \{1, 2, \ldots, 15\} \), as follows:

\[ (6.4) \quad \begin{align*}
\mathcal{T}_1 &= P_{1,1,1}, & \mathcal{T}_2 &= P_{1,1,2}, & \mathcal{T}_3 &= P_{1,2}, \\
\mathcal{T}_4 &= P_{1,3,1}, & \mathcal{T}_5 &= P_{1,3,2}, & \mathcal{T}_6 &= P_{1,4}, \\
\mathcal{T}_7 &= P_{2,1}, & \mathcal{T}_8 &= P_{2,2}, & \mathcal{T}_9 &= P_{3,1,1}, \\
\mathcal{T}_{10} &= P_{3,1,2}, & \mathcal{T}_{11} &= P_{3,1,3}, & \mathcal{T}_{12} &= P_{3,2}, \\
\mathcal{T}_{13} &= P_{3,3}, & \mathcal{T}_{14} &= P_{3,4}, & \mathcal{T}_{15} &= P_{4,1}.
\end{align*} \]

Then we obtain the following main statement in the present section

\textbf{Theorem 6.24.} \textit{Let \( \mathcal{T} \) be the vector space of the torsion tensors of type (0, 3) over the vector space \( V \) with almost contact B-metric structure \((\varphi, \xi, \eta, g)\). The decomposition

\[ (6.5) \quad \mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \cdots \oplus \mathcal{T}_{15} \]

is orthogonal and invariant with respect to the structure group \( \mathcal{G} \times \mathcal{I} \).}

\subsection*{6.2. The fifteen classes of affine connections}

It is well known that any metric connection \( \nabla^* \) (i.e. \( \nabla^* g = 0 \)) with a potential \( Q \) regarding \( \nabla \), defined by \( Q(x, y, z) = g(\nabla^*_x y - \nabla_y x, z) \), is completely determined by its torsion tensor \( T \) by means of \( (3.2) \), according to the Hayden theorem \cite{54}.}
For an almost contact B-metric manifold \((M, \varphi, \xi, \eta, g)\), there exist infinitely many affine connections on the tangent space \(T_pM, p \in M\). Then the subspace \(T_s (s \in \{1, 2, \ldots, 15\})\), where \(T\) belongs, is an important characteristic of \(\nabla^*\). In such a way the conditions for \(T\) described as the subspace \(T_s\) give rise to the corresponding class of the connection with respect to its torsion tensor. Therefore, the conditions of torsion tensors determine corresponding classes of the connections on the tangent bundle derived by the almost contact B-metric structure.

**Definition 6.1.** It is said that an affine connection \(\nabla^*\) on an almost contact B-metric manifold belongs to a class \(C_s, s \in \{1, 2, \ldots, 15\}\), if the torsion tensor \(T^*\) of \(\nabla^*\) belongs to the subspace \(T_s\) in the decomposition \((6.5)\) of \(T\).

Bearing in mind Theorem 6.24, we obtain the following classifying

**Theorem 6.25.** The set of affine connections \(C\) on an almost contact B-metric manifold is divided into 15 basic classes \(C_s, s \in \{1, 2, \ldots, 15\}\), by the decomposition

\[(6.6)\]

\[C = C_1 \oplus C_2 \oplus \cdots \oplus C_{15}.\]

The special class \(C_0\) contains all symmetric connections and it corresponds to the zero vector subspace \(T_0\) of \(T\) determined by the condition \(T = 0\). This class belongs to any other class \(C_s\). The Levi-Civita connections \(\nabla\) and \(\tilde{\nabla}\) are symmetric and therefore they belong to the class \(C_0\).

The classes \(C_i \oplus C_j \oplus \ldots\), which are direct sums of basic classes, are defined in a natural way by the corresponding subspaces \(T_i \oplus T_j \oplus \ldots\), following Definition 6.1. According to \((6.6)\), the number of all classes of affine connections on \((M, \varphi, \xi, \eta, g)\) is \(2^{15}\) and their definition conditions are readily obtained from those of the basic 15 classes.

**6.3. Some natural connections in the introduced classification**

Further in the present section we discuss the three mentioned natural connections with torsion on \((M, \varphi, \xi, \eta, g)\). Natural connections are a generalization of the Levi-Civita connection.

**Proposition 6.26.** Let \(\nabla^*\) be a natural connection with torsion \(T\) on an almost contact B-metric manifold \((M, \varphi, \xi, \eta, g)\). Then the following
implies hold true:

\[ T \in T_1 \oplus T_2 \oplus T_6 \oplus T_{12} \Rightarrow \mathcal{M} \in \mathcal{F}_0; \quad T \in T_3 \Rightarrow \mathcal{M} \in \mathcal{F}_3; \]
\[ T \in T_4 \Rightarrow \mathcal{M} \in \mathcal{F}_1; \quad T \in T_5 \Rightarrow \mathcal{M} \in \mathcal{F}_2; \]
\[ T \in T_7 \Rightarrow \mathcal{M} \in \mathcal{F}_7; \quad T \in T_8 \Rightarrow \mathcal{M} \in \mathcal{F}_8 \oplus \mathcal{F}_{10}; \]
\[ T \in T_9 \Rightarrow \mathcal{M} \in \mathcal{F}_5; \quad T \in T_{10} \Rightarrow \mathcal{M} \in \mathcal{F}_4; \]
\[ T \in T_{11} \Rightarrow \mathcal{M} \in \mathcal{F}_6; \quad T \in T_{13} \Rightarrow \mathcal{M} \in \mathcal{F}_9; \]
\[ T \in T_{14} \Rightarrow \mathcal{M} \in \mathcal{F}_{10}; \quad T \in T_{15} \Rightarrow \mathcal{M} \in \mathcal{F}_{11}. \]

**Proof.** The implications follow from (3.2), (5.1), (4.14), (6.4) and the corresponding characteristic conditions of \( P_{i,j} \) and \( P_{i,j,k} \) as well as the projection operators \( p_{i,j} \). We show the proof in detail for some classes and the rest follow in a similar way.

By virtue of (3.2) and (5.1) we have

\[
2F(x, y, z) = T(x, y, \varphi z) - T(y, \varphi z, x) + T(\varphi z, x, y)
- T(x, \varphi y, z) + T(\varphi y, z, x) - T(z, x, \varphi y).
\]

(6.7)

Let us consider \( T \in \mathcal{P}_{1,1} = T_1 \oplus T_2 \), which is equivalent to \( T = p_{1,1}(T) \). Then, according to Proposition 6.13, we have

\[
T(x, y, z) = -\frac{1}{4} \{ T(\varphi^2 x, \varphi^2 y, \varphi^2 z) - T(\varphi x, \varphi y, \varphi^2 z)
- T(\varphi x, \varphi^2 y, \varphi z) - T(\varphi^2 x, \varphi y, \varphi z) \},
\]

which together with (6.7) imply \( F(x, y, z) = 0 \). Therefore, we obtain \( \mathcal{M} \in \mathcal{F}_0 \).

Now, let us suppose \( T \in \mathcal{P}_{1,2} = T_3 \) and hence \( T = p_{1,2}(T) \), which has the following form, taking into account Proposition 6.13:

\[
T(x, y, z) = -\frac{1}{4} \{ T(\varphi^2 x, \varphi^2 y, \varphi^2 z) - T(\varphi x, \varphi y, \varphi^2 z)
+ T(\varphi x, \varphi^2 y, \varphi z) + T(\varphi^2 x, \varphi y, \varphi z) \}.
\]

Then, according to the latter equality and (6.7), we obtain

\[
F(x, y, z) = -\frac{1}{4} \{ -T(\varphi^2 x, \varphi^2 y, \varphi z) + T(\varphi x, \varphi y, \varphi z)
+ T(\varphi^2 x, \varphi y, \varphi^2 z) + T(\varphi x, \varphi^2 y, \varphi z)
- T(\varphi z, \varphi^2 x, \varphi^2 y) - T(\varphi^2 z, \varphi x, \varphi^2 y) \}
\]

(6.8a)
and consequently \( F(\xi, y, z) = F(x, y, \xi) = 0 \). Next, we take the cyclic sum of (6.8) by the arguments \( x, y, z \) and the result is \( \sum_{x,y,z} F(x, y, z) = 0 \). Therefore, \((\mathcal{M}, \varphi, \xi, \eta, g)\) belongs to \( \mathcal{F}_3 \). □

Bearing in mind the class of the almost contact B-metric manifolds with \( N = 0 \) and Proposition 6.26, we obtain immediately

**Corollary 6.27.** An almost contact B-metric manifold \((\mathcal{M}, \varphi, \xi, \eta, g)\) is normal, i.e. it has vanishing \( N \), if any natural connection on \((\mathcal{M}, \varphi, \xi, \eta, g)\) belongs to the class \( \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10} \oplus \mathcal{C}_{11} \).

Similarly, Proposition 4.3 and Proposition 6.26 imply

**Corollary 6.28.** An almost contact B-metric manifold \((\mathcal{M}, \varphi, \xi, \eta, g)\) has vanishing \( \hat{N} \), if any natural connection on \((\mathcal{M}, \varphi, \xi, \eta, g)\) belongs to the class \( \mathcal{C}_3 \oplus \mathcal{C}_7 \).

### 6.3.1. The \( \varphi B \)-connection in the classification

The \( \varphi B \)-connection \( \nabla' \) is discussed in §5 and it has a torsion tensor \( T' \) and corresponding torsion 1-forms, given in (5.7) and (5.9), respectively.

Applying Propositions 6.13, 6.17, 6.23 and equation (6.3) for the torsion tensor \( T' \) from (5.7), we obtain the components of \( T' \) in each of the subspaces \( \mathcal{P}_{i,j} \):

\[
\begin{align*}
\text{p}_{1,1}(T')(x, y, z) &= 0, \\
p_{1,2}(T')(x, y, z) &= \frac{1}{4} \left\{ F(\varphi x, \varphi y, \varphi z) - F(\varphi^2 x, \varphi^2 y, \varphi z) \\
 & \quad - F(\varphi y, \varphi x, \varphi z) + F(\varphi^2 y, \varphi^2 x, \varphi z) \right\}, \\
p_{1,3}(T')(x, y, z) &= -\frac{1}{8} \left\{ F(\varphi^2 z, \varphi^2 y, \varphi x) + F(\varphi^2 x, \varphi^2 y, \varphi z) \\
 & \quad + F(\varphi x, \varphi y, \varphi z) - F(\varphi^2 z, \varphi^2 x, \varphi y) \\
 & \quad - F(\varphi^2 y, \varphi^2 x, \varphi z) - F(\varphi y, \varphi x, \varphi z) \right\}, \\
\end{align*}
\]

(6.9a)
\[ p_{1,4}(T')(x, y, z) = -\frac{1}{8}\{F(\varphi^2 z, \varphi^2 y, \varphi x) - F(\varphi^2 x, \varphi^2 y, \varphi z) \]
\[ - F(\varphi x, \varphi y, \varphi z) - F(\varphi^2 z, \varphi^2 x, \varphi y) \]
\[ + F(\varphi^2 y, \varphi^2 x, \varphi z) + F(\varphi y, \varphi x, \varphi z)\}, \]
\[ p_{2,1}(T')(x, y, z) = -\frac{1}{2}\eta(z)\{F(\varphi^2 x, \varphi y, \xi) - F(\varphi x, y, \xi) \]
\[ - F(\varphi^2 y, \varphi x, \xi) + F(\varphi y, x, \xi)\}, \]
\[ p_{2,2}(T')(x, y, z) = -\frac{1}{2}\eta(z)\{F(\varphi^2 x, \varphi y, \xi) + F(\varphi x, y, \xi) \]
\[ - F(\varphi^2 y, \varphi x, \xi) - F(\varphi y, x, \xi)\}, \]
\[ p_{3,1}(T')(x, y, z) = \frac{1}{4}\{\eta(y)\{F(\varphi^2 x, \varphi z, \xi) + F(\varphi^2 z, \varphi x, \xi) \]
\[ - F(\varphi x, z, \xi) - F(\varphi z, x, \xi)\]}
\[ \] (6.9b)
\[ - \eta(x)\{F(\varphi^2 y, \varphi z, \xi) + F(\varphi^2 z, \varphi y, \xi) \]
\[ - F(\varphi y, z, \xi) - F(\varphi z, y, \xi)\}\}, \]
\[ p_{3,2}(T')(x, y, z) = \frac{1}{4}\{\eta(y)\{F(\varphi^2 x, \varphi z, \xi) - F(\varphi^2 z, \varphi x, \xi) \]
\[ - F(\varphi x, z, \xi) + F(\varphi z, x, \xi)\]}
\[ \] - \eta(x)\{F(\varphi^2 y, \varphi z, \xi) - F(\varphi^2 z, \varphi y, \xi) \]
\[ - F(\varphi y, z, \xi) + F(\varphi z, y, \xi)\}\}, \]
\[ p_{3,3}(T')(x, y, z) = \frac{1}{4}\{\eta(y)\{F(\varphi^2 x, \varphi z, \xi) + F(\varphi^2 z, \varphi x, \xi) \]
\[ + F(\varphi x, z, \xi) + F(\varphi z, x, \xi)\] \]
− \eta(x)[F(\varphi^2 y, \varphi z, \xi) + F(\varphi^2 z, \varphi y, \xi) \\
+ F(\varphi y, z, \xi) + F(\varphi z, y, \xi)],
\}
\]

p_{3,4}(T')(x, y, z) = \frac{1}{4}\{\eta(y)[F(\varphi^2 x, \varphi z, \xi) - F(\varphi^2 z, \varphi x, \xi) \\
+ F(\varphi x, z, \xi) - F(\varphi z, x, \xi) \\
+ 2F(\xi, \varphi x, \varphi^2 z)] \\
- \eta(x)[F(\varphi^2 y, \varphi z, \xi) - F(\varphi^2 z, \varphi y, \xi) \\
+ F(\varphi y, z, \xi) - F(\varphi z, y, \xi) \\
+ 2F(\xi, \varphi y, \varphi^2 z)]\},

p_{4,1}(T')(x, y, z) = \eta(z)\{\eta(x)\omega(\varphi y) - \eta(y)\omega(\varphi x)\}.

Such a way we establish that the torsion \( T' \) of the \( \varphi B \)-connection \( \nabla' \) on \( (\mathcal{M}, \varphi, \xi, \eta, g) \) belongs to \( T_3 \oplus T_4 \oplus \cdots \oplus T_{15} \). Thus, the position of \( \nabla' \) in the classification (6.6) is determined as follows

**Proposition 6.29.** The \( \varphi B \)-connection on \( (\mathcal{M}, \varphi, \xi, \eta, g) \) belongs to the class \( \mathcal{C}_3 \oplus \mathcal{C}_4 \oplus \cdots \oplus \mathcal{C}_{15} \).

6.3.2. The \( \varphi \)-canonical connection in the classification

According to the classification of the torsion tensors given above in the present section and Proposition 5.10, we get the following

**Proposition 6.30.** The correspondence between the classes \( \mathcal{F}_i \) \( (i \in \{1, 2, \ldots, 11\}) \) of the manifolds \( (\mathcal{M}, \varphi, \xi, \eta, g) \) and the classes \( \mathcal{C}_s \) \( (s \in \{1, 2, \ldots, 15\}) \) of the \( \varphi \)-canonical connection \( \nabla'' \) on \( (\mathcal{M}, \varphi, \xi, \eta, g) \) is given as follows:

\((\mathcal{M}, \varphi, \xi, \eta, g) \in \mathcal{F}_0 \iff \nabla'' \in \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_6 \oplus \mathcal{C}_{12};\)

\((\mathcal{M}, \varphi, \xi, \eta, g) \in \mathcal{F}_1 \iff \nabla'' \in \mathcal{C}_4;\)

\((\mathcal{M}, \varphi, \xi, \eta, g) \in \mathcal{F}_2 \iff \nabla'' \in \mathcal{C}_5;\)

\((\mathcal{M}, \varphi, \xi, \eta, g) \in \mathcal{F}_3 \iff \nabla'' \in \mathcal{C}_3;\)
For the example in Subsection 5.3.3 on page 75, it follows that the statement $T'' \in P_{3,1,1} \oplus P_{3,1,2}$ is valid, which supports Proposition 6.30, bearing in mind (6.4).

### 6.3.3. The $\varphi KT$-connection in the classification

In a similar way as for (6.9), from (5.30) we get the following non-zero components of $T'''$:

\[
p_{1,2}(T''')(x, y, z) = -\frac{1}{2}\left\{F(x, y, \varphi z) + F(y, z, \varphi x) - F(z, x, \varphi y) - \eta(x)F(y, \varphi z, \xi) + \eta(y)F(z, \varphi x, \xi) + \eta(z)F(x, \varphi y, \xi)\right\},
\]

(6.10)\[
p_{1,4}(T''')(x, y, z) = -F(z, x, \varphi y) - \eta(x)F(y, \varphi z, \xi),
\]
\[
p_{2,1}(T''')(x, y, z) = 2\eta(z)F(x, \varphi y, \xi),
\]
\[
p_{3,2}(T''')(x, y, z) = 2\eta(x)F(y, \varphi z, \xi) + 2\eta(y)F(z, \varphi x, \xi).
\]

Therefore, we have that $T'''$ belongs to $\mathcal{T}_3 \oplus \mathcal{T}_6 \oplus \mathcal{T}_7 \oplus \mathcal{T}_{12}$ and the following

**Proposition 6.31.** The $\varphi KT$-connection $\nabla'''$ on $(\mathcal{M}, \varphi, \xi, \eta, g)$ in the class $\mathcal{F}_3 \oplus \mathcal{F}_7$ belongs to $\mathcal{C}_3 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_{12}$. Moreover, if $(\mathcal{M}, \varphi, \xi, \eta, g) \in \mathcal{F}_3$ (resp. $\mathcal{F}_7$) then $\nabla''' \in \mathcal{C}_3 \oplus \mathcal{C}_6$ (resp. $\mathcal{C}_7 \oplus \mathcal{C}_{12}$).

***
§7. Pair of associated Schouten-van Kampen connections adapted to an almost contact B-metric structure

In the present section there are introduced and studied a pair of associated Schouten-van Kampen affine connections adapted to the contact distribution and an almost contact B-metric structure generated by the pair of associated B-metrics and their Levi-Civita connections. By means of the constructed non-symmetric connections, the basic classes of almost contact B-metric manifolds are characterized. Curvature properties of the considered connections are obtained.

The main results of this section are published in [90].

The Schouten-van Kampen connection has been introduced for a studying of non-holonomic manifolds. It preserves by parallelism a pair of complementary distributions on a differentiable manifold endowed with an affine connection [130, 56, 9]. This connection is also used for investigations of hyperdistributions in Riemannian manifolds (e.g., [133]).

On the other hand, any almost contact manifold admits a hyperdistribution, the known contact distribution. In [125], it is studied the Schouten-van Kampen connection adapted to an almost (para)contact metric structure. On these manifolds, the studied connection is not natural in general because it preserves the structure tensors except the structure endomorphism.

We consider almost contact B-metric structures. As it is mentioned above, an important characteristic of the almost contact B-metric structure, which differs from the metric one, is that the associated (0,2)-tensor of the B-metric is also a B-metric. Consequently, this pair of B-metrics generates a pair of Levi-Civita connections.
The present section is organized as follows. In Subsection 7.1 we introduce a pair of Schouten-van Kampen connections which is associated to the pair of Levi-Civita connections and adapted to the contact distribution of an almost contact B-metric manifold. Then we determine conditions these connections to coincide and to be natural for the corresponding structures. In Subsection 7.2 we study basic properties of the potentials and the torsions of the pair of the constructed connections. Finally, in Subsection 7.3 we give some curvature properties of the studied connections.

7.1. Remarkable metric connections regarding the contact distribution on the considered manifolds

Let us consider the horizontal and the vertical distributions $\mathcal{H}$ and $\mathcal{V}$ in the tangent bundle $TM$ on an arbitrary almost contact B-metric manifold $(M, \varphi, \xi, \eta, g)$ given in (4.3) and (4.4). Further, we use the corresponding projections $x^h$ and $x^v$ of an arbitrary vector field $x$ in $TM$ bearing in mind (4.6) and (4.7).

7.1.1. Schouten-van Kampen connection $\nabla^o$ associated to $\nabla$

Let us consider the Schouten-van Kampen connection $\nabla^o$ associated to the Levi-Civita connection $\nabla$ and adapted to the pair $(\mathcal{H}, \mathcal{V})$. This connection is defined (locally in [130], see also [56]) by

$$(7.1) \quad \nabla^o_{x} y = (\nabla_{x} y^h)^h + (\nabla_{x} y^v)^v.$$ 

The latter equality implies the parallelism of $\mathcal{H}$ and $\mathcal{V}$ with respect to $\nabla^o$. From (4.7) we obtain

$$(\nabla_{x} y^h)^h = \nabla_{x} y - \eta(y)\nabla_{x} \xi - \eta(\nabla_{x} y)\xi,$$

$$(\nabla_{x} y^v)^v = \eta(\nabla_{x} y)\xi + (\nabla_{x} \eta)(y)\xi.$$ 

Then we get the expression of the Schouten-van Kampen connection in terms of $\nabla$ as follows (cf. [133])

$$(7.2) \quad \nabla^o_{x} y = \nabla_{x} y - \eta(y)\nabla_{x} \xi + (\nabla_{x} \eta)(y)\xi.$$ 

According to (7.2), the potential $Q^o$ of $\nabla^o$ with respect to $\nabla$ and the torsion $T^o$ of $\nabla^o$, defined by $Q^o(x, y) = \nabla^o_{x} y - \nabla_{x} y$ and $T^o(x, y) = \nabla^o_{x} y - \nabla_{y} x - [x, y]$, respectively, have the following form

$$(7.3) \quad Q^o(x, y) = -\eta(y)\nabla_{x} \xi + (\nabla_{x} \eta)(y)\xi,$$
(7.4) \[ T^\circ(x, y) = \eta(x) \nabla_y \xi - \eta(y) \nabla_x \xi + d\eta(x, y)\xi. \]

**Theorem 7.1.** The Schouten-van Kampen connection \( \nabla^\circ \) is the unique affine connection having a torsion of the form (7.4) and preserving the structures \( \xi, \eta \) and the metric \( g \).

**Proof.** Taking into account (7.2), we compute directly that the structures \( \xi, \eta \) and \( g \) are parallel with respect to \( \nabla^\circ \), i.e. \( \nabla^\circ \xi = \nabla^\circ \eta = \nabla^\circ g = 0 \). The connection \( \nabla^\circ \) preserves the metric and therefore is completely determined by its torsion \( T^\circ \). According to [22], the two spaces of all torsions and of all potentials are isomorphic and the bijection is given by (3.1) and (3.2).

Then, the connection \( \nabla^\circ \) determined by (7.2) and its potential \( Q^\circ \) given in (7.3) are replaced in (3.1) to determine its torsion \( T^\circ \) and the result is (7.4). Vice versa, the form of \( T^\circ \) in (7.4) yields by (3.2) the equality for \( \nabla^\circ \) in (7.2).

Obviously, the connection \( \nabla^\circ \) exists on \((\mathcal{M}, \varphi, \xi, \eta, g)\) in each class, but \( \nabla^\circ \) coincides with \( \nabla \) if and only if the condition

\[ \eta(y) \nabla_x \xi - (\nabla_x \eta)(y)\xi = 0 \]

holds. The latter equality is equivalent to vanishing of \( \nabla_x \xi \) for each \( x \). This condition is satisfied only for the manifolds belonging to the class \( \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3 \oplus \mathcal{F}_{10} \), according to Proposition 4.1. Let us denote this class briefly by \( \mathcal{U}_1 \), i.e.

\[ \mathcal{U}_1 = \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3 \oplus \mathcal{F}_{10}. \]

Thus, we prove the following

**Theorem 7.2.** The Schouten-van Kampen connection \( \nabla^\circ \) coincides with \( \nabla \) if and only if \((\mathcal{M}, \varphi, \xi, \eta, g)\) belongs to the class \( \mathcal{U}_1 \).

**7.1.2. The conditions \( \nabla^\circ \) to be natural for \((\varphi, \xi, \eta, g)\)**

Using (7.2), we express the covariant derivative of \( \varphi \) as follows

(7.5) \[ (\nabla^\circ_{x}\varphi) y = (\nabla_{x}\varphi) y + \eta(y) \varphi \nabla_{x} \xi - \eta(\nabla_{x} \varphi y)\xi. \]

Therefore, \( \nabla^\circ \varphi = 0 \) if and only if

\[ (\nabla_{x} \varphi) y = -\eta(y) \varphi \nabla_{x} \xi + \eta(\nabla_{x} \varphi y)\xi, \]

which yields the following equality by (4.10)

(7.6) \[ F(x, y, z) = F(x, y, \xi) \eta(z) + F(x, z, \xi) \eta(y). \]
According to (4.14) and [80], the latter condition determines the direct sum $\mathcal{F}_4 \oplus \cdots \oplus \mathcal{F}_9 \oplus \mathcal{F}_{11}$, which we denote by $\mathcal{U}_2$ for the sake of brevity, i.e.

$$\mathcal{U}_2 = \mathcal{F}_4 \oplus \cdots \oplus \mathcal{F}_9 \oplus \mathcal{F}_{11}.$$ 

Thus, we find the kind of the considered manifolds where $\nabla^\circ$ is a natural connection, i.e. the tensors of the structure $(\varphi, \xi, \eta, g)$ are covariantly constant regarding $\nabla^\circ$. In this case it follows that $(\nabla^\circ_x \varphi)(y) = (\nabla_x \eta)(y) \cdot \xi$ holds. Then the Schouten-van Kampen connection $\nabla^\circ$ coincides with the $\varphi B$-connection defined by (5.5). Such a way, we establish the truthfulness of the following

**Theorem 7.3.** The Schouten-van Kampen connection $\nabla^\circ$ is a natural connection for the structure $(\varphi, \xi, \eta, g)$ if and only if $(\mathcal{M}, \varphi, \xi, \eta, g)$ belongs to the class $\mathcal{U}_2$. In this case $\nabla^\circ$ coincides with the $\varphi B$-connection.

Let us remark that in the case when $(\mathcal{M}, \varphi, \xi, \eta, g)$ belongs to a class which has a nonzero component in both of the direct sums $\mathcal{U}_1$ and $\mathcal{U}_2$, then the connection $\nabla^\circ$ is not a natural connection and it does not coincide with $\nabla$. Then the class of all almost contact B-metric manifolds can be decomposed orthogonally to $\mathcal{U}_1 \oplus \mathcal{U}_2$.

### 7.1.3. Schouten-van Kampen connection $\tilde{\nabla}^\circ$ associated to $\tilde{\nabla}$

In a similar way as for $\nabla^\circ$, let us consider the Schouten-van Kampen connection $\tilde{\nabla}^\circ$ associated to the Levi-Civita connection $\tilde{\nabla}$ for $\tilde{g}$ and adapted to the pair $(\mathcal{H}, \mathcal{V})$. We define this connection as follows

$$\tilde{\nabla}^\circ_x y = (\tilde{\nabla}_x y^h)^h + (\tilde{\nabla}_x y^v)^v.$$ 

Then the hyperdistribution $(\mathcal{H}, \mathcal{V})$ is parallel with respect to $\tilde{\nabla}^\circ$, too. Analogously, we express the Schouten-van Kampen connection $\tilde{\nabla}^\circ$ in terms of $\tilde{\nabla}$ by

$$\tilde{\nabla}^\circ_x y = \tilde{\nabla}_x y - \eta(y) \tilde{\nabla}_x \xi + (\tilde{\nabla}_x \eta)(y) \xi.$$ 

(7.7)

By virtue of (7.7), the potential $\tilde{Q}^\circ$ of $\tilde{\nabla}^\circ$ with respect to $\tilde{\nabla}$ and the torsion $\tilde{T}^\circ$ of $\tilde{\nabla}^\circ$ have the following form

$$\tilde{Q}^\circ(x, y) = -\eta(y) \tilde{\nabla}_x \xi + (\tilde{\nabla}_x \eta)(y) \xi,$$

(7.8)

$$\tilde{T}^\circ(x, y) = \eta(x) \tilde{\nabla}_y \xi - \eta(y) \tilde{\nabla}_x \xi + d\eta(x, y) \xi.$$ 

(7.9)

Similarly to Theorem 7.1 we have the following
Theorem 7.4. The Schouten-van Kampen connection $\tilde{\nabla}^\circ$ is the unique affine connection having a torsion of the form (7.9) and preserving the structures $\xi$, $\eta$ and the associated metric $\tilde{g}$.

It is clear that the connection $\tilde{\nabla}$ exists on $(\mathcal{M}, \varphi, \xi, \eta, \tilde{g})$ in each class, but $\tilde{\nabla}$ coincides with $\tilde{\nabla}$ if and only if the condition

$$\eta(y)\tilde{\nabla}_x\xi - (\tilde{\nabla}_x\eta)(y)\xi = 0$$

is valid or equivalently $\tilde{\nabla}\xi$ vanishes. Similarly to Proposition 4.1, we prove that the condition $\tilde{\nabla}\xi = 0$ holds if and only if $\tilde{\eta}$ satisfies the condition (4.14) of $\tilde{F}$ for $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3 \oplus \mathcal{F}_9$, which we denote by $\tilde{\mathcal{U}}_1$. Thus, we prove the following

Theorem 7.5. The Schouten-van Kampen connection $\tilde{\nabla}^\circ$ coincides with $\tilde{\nabla}$ if and only if $(\mathcal{M}, \varphi, \xi, \eta, \tilde{g})$ belongs to the class $\tilde{\mathcal{U}}_1$.

Taking into account (4.24), we establish immediately

Lemma 7.6. The manifold $(\mathcal{M}, \varphi, \xi, \eta, g)$ belongs to $\mathcal{U}_1$ if and only if the manifold $(\mathcal{M}, \varphi, \xi, \eta, \tilde{g})$ belongs to $\tilde{\mathcal{U}}_1$.

Then, Theorem 7.2, Theorem 7.5 and Lemma 7.6 imply the following

Theorem 7.7. Let $\nabla^\circ$ and $\tilde{\nabla}^\circ$ be the Schouten-van Kampen connections associated to $\nabla$ and $\tilde{\nabla}$ and adapted to the pair $(\mathcal{H}, V)$ on $(\mathcal{M}, \varphi, \xi, \eta, g)$ and $(\mathcal{M}, \varphi, \xi, \eta, \tilde{g})$, respectively. Then the following assertions are equivalent:

(i) $\nabla^\circ$ coincides with $\nabla$;
(ii) $\tilde{\nabla}^\circ$ coincides with $\tilde{\nabla}$;
(iii) $(\mathcal{M}, \varphi, \xi, \eta, g)$ belongs to $\mathcal{U}_1$;
(iv) $(\mathcal{M}, \varphi, \xi, \eta, \tilde{g})$ belongs to $\tilde{\mathcal{U}}_1$.

Corollary 7.8. Let $\nabla^\circ$ and $\tilde{\nabla}^\circ$ be the Schouten-van Kampen connections associated to $\nabla$ and $\tilde{\nabla}$ and adapted to the pair $(\mathcal{H}, V)$ on $(\mathcal{M}, \varphi, \xi, \eta, g)$ and $(\mathcal{M}, \varphi, \xi, \eta, \tilde{g})$, respectively. If $\tilde{\nabla}^\circ \equiv \nabla$ or $\nabla^\circ \equiv \tilde{\nabla}$ then the four connections $\nabla^\circ$, $\tilde{\nabla}^\circ$, $\tilde{\nabla}$ and $\nabla$ coincide. The coinciding of $\nabla^\circ$, $\tilde{\nabla}^\circ$, $\nabla$ and $\tilde{\nabla}$ is equivalent to the condition $(\mathcal{M}, \varphi, \xi, \eta, g)$ and $(\mathcal{M}, \varphi, \xi, \eta, \tilde{g})$ to be cosymplectic $B$-metric manifolds.

We obtain from (7.14) the following relation between $\nabla^\circ$ and $\tilde{\nabla}^\circ$

$$\tilde{\nabla}^\circ x y = \nabla^\circ x y + \Phi(x, y) - \eta(\Phi(x, y))\xi - \eta(y)\Phi(x, \xi).$$
It is clear that $\tilde{\nabla}^\circ = \nabla^\circ$ if and only if
\[ \Phi(x, y) = \eta(\Phi(x, y))\xi + \eta(y)\Phi(x, \xi) \]
which is equivalent to
\[ \Phi(x, y) = \eta(\Phi(x, y))\xi + \eta(x)\eta(y)\Phi(\xi, \xi) \]
because $\Phi$ is symmetric. Using relation (4.22), we obtain condition (7.6) which determines the class $\mathcal{U}_2$. Then, the following assertion is valid.

**Theorem 7.9.** The Schouten-van Kampen connections $\tilde{\nabla}^\circ$ and $\nabla^\circ$ associated to $\tilde{\nabla}$ and $\nabla$, respectively, and adapted to the pair $(\mathcal{H}, \mathcal{V})$ coincide with each other if and only if the manifold belongs to the class $\mathcal{U}_2$.

### 7.1.4. The conditions $\tilde{\nabla}^\circ$ to be natural for $(\varphi, \xi, \eta, \tilde{g})$

Using (7.10), we have the following relation between the covariant derivatives of $\varphi$ regarding $\tilde{\nabla}^\circ$ and $\nabla^\circ$
\[
(\tilde{\nabla}^\circ_x \varphi)y = (\nabla^\circ_x \varphi)y + \Phi(x, \varphi y) - \varphi \Phi(x, y) \\
+ \eta(y)\varphi \Phi(x, \xi) - \eta(\Phi(x, \varphi y))\xi.
\]
(7.11)
By virtue of the latter equality, we establish that $\tilde{\nabla}^\circ \varphi$ and $\nabla^\circ \varphi$ coincide if and only if the condition
\[ \Phi(x, \varphi^2 y, \varphi^2 z) = -\Phi(x, \varphi y, \varphi z) \]
holds. Then, using the classification of almost contact B-metric manifolds regarding $\Phi$, given in [115], the latter condition is satisfied only when $(\mathcal{M}, \varphi, \xi, \eta, g)$ is in the class $\mathcal{F}_3 \oplus \mathcal{U}_3$, where $\mathcal{U}_3$ denotes the direct sum $\mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \oplus \mathcal{F}_7 \oplus \mathcal{F}_11$. By direct computations we establish that $(\mathcal{M}, \varphi, \xi, \eta, \tilde{g})$ belongs to the same class. Therefore, we obtain

**Theorem 7.10.** The covariant derivatives of $\varphi$ with respect to the Schouten-van Kampen connections $\nabla^\circ$ and $\tilde{\nabla}^\circ$ coincide if and only if both the manifolds $(\mathcal{M}, \varphi, \xi, \eta, g)$ and $(\mathcal{M}, \varphi, \xi, \eta, \tilde{g})$ belong to the class $\mathcal{F}_3 \oplus \mathcal{U}_3$.

Using (4.23), (7.5) and (7.11), we obtain that $\tilde{\nabla}^\circ \varphi = 0$ is equivalent to the condition
\[ F(\varphi y, \varphi z, x) + F(\varphi^2 y, \varphi^2 z, x) - F(\varphi z, \varphi y, x) - F(\varphi^2 z, \varphi^2 y, x) = 0. \]
Then, by virtue of (4.14) we get the following
Theorem 7.11. The Schouten-van Kampen connection \( \tilde{\nabla}^\circ \) is a natural connection for the structure \((\varphi, \xi, \eta, \tilde{g})\) if and only if \((\mathcal{M}, \varphi, \xi, \eta, \tilde{g})\) belongs to the class \( \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{U}_3 \).

Consequently, bearing in mind Theorem 7.3, Theorem 7.10, Theorem 7.11, we have the validity of the following

**Theorem 7.12.** The Schouten-van Kampen connections \( \nabla^\circ \) and \( \tilde{\nabla}^\circ \) are natural connections on \((\mathcal{M}, \varphi, \xi, \eta, g, \tilde{g})\) if and only if \((\mathcal{M}, \varphi, \xi, \eta, g)\) and \((\mathcal{M}, \varphi, \xi, \eta, \tilde{g})\) belong to the class \( \mathcal{U}_3 \).

### 7.2. Properties of the potentials and the torsions of the pair of connections \( \nabla^\circ \) and \( \tilde{\nabla}^\circ \)

Bearing in mind (7.3), (7.4), (3.1), (3.2) and (4.17), we establish that the properties of the torsion, the potential and the shape operator for \( \nabla^\circ \) are related. Analogously, similar linear relations between the respective torsion, potential and shape operator for \( \tilde{\nabla}^\circ \) are valid.

According to the expressions (7.3) and (7.4) of \( Q^\circ \) and \( T^\circ \), respectively, their horizontal and vertical components have the following form

\[
\begin{align*}
Q^{\circ h} &= - (\nabla \xi) \otimes \eta, & Q^{\circ v} &= (\nabla \eta) \otimes \xi, \\
T^{\circ h} &= \eta \wedge (\nabla \xi), & T^{\circ v} &= d\eta \otimes \xi.
\end{align*}
\]  

Then, the corresponding \((0,3)\)-tensors

\[Q^\circ (x, y, z) = g(Q^\circ (x, y), z), \quad T^\circ (x, y, z) = g(T^\circ (x, y), z)\]

are expressed in terms of \( S \) as follows

\[
\begin{align*}
Q^\circ (x, y, z) &= -\pi_1(\xi, S(x), y, z), \\
T^\circ (x, y, z) &= -\pi_1(\xi, S(x), y, z) + \pi_1(\xi, S(y), x, z),
\end{align*}
\]

where it is denoted

\[
\pi_1(x, y, z, w) = g(y, z)g(x, w) - g(x, z)g(y, w).
\]

Moreover, their horizontal and vertical components have the form

\[
\begin{align*}
Q^{\circ h} &= S \otimes \eta, & Q^{\circ v} &= -S^\flat \otimes \xi, \\
T^{\circ h} &= -\eta \wedge S, & T^{\circ v} &= -2\text{Alt}(S^\flat) \otimes \xi,
\end{align*}
\]

where we use \( S^\flat(x, y) = g(S(x), y) \) and \( \text{Alt} \) for the alternation.

By virtue of the equalities for the vertical components of \( Q^\circ \) and \( T^\circ \) in (7.12) and (7.14), we obtain immediately
Chapter I

§7. Pair of associated Schouten-van Kampen connections adapted to . . .

Theorem 7.13. The following equivalences are valid:

(i) \( \nabla \eta \) is symmetric \( \Leftrightarrow \) \( \eta \) is closed \( \Leftrightarrow \) \( Q^o \) is symmetric \( \Leftrightarrow \) \( T^o \) vanishes \( \Leftrightarrow \) \( S \) is self-adjoint regarding \( g \) \( \Leftrightarrow \) \( S^b \) is symmetric \( \Leftrightarrow \) \( \mathcal{M} \in U_1 \bigoplus \mathcal{F}_4 \bigoplus \mathcal{F}_5 \bigoplus \mathcal{F}_6 \bigoplus \mathcal{F}_9 \);

(ii) \( \nabla \eta \) is skew-symmetric \( \Leftrightarrow \) \( \xi \) is Killing with respect to \( g \) \( \Leftrightarrow \) \( Q^o \) is skew-symmetric \( \Leftrightarrow \) \( S \) is anti-self-adjoint regarding \( g \) \( \Leftrightarrow \) \( S^b \) is skew-symmetric \( \Leftrightarrow \) \( \mathcal{M} \in U_1 \bigoplus \mathcal{F}_7 \bigoplus \mathcal{F}_8 \);

(iii) \( \nabla \eta = 0 \Leftrightarrow \) \( d \eta = \mathcal{L}_\xi g = 0 \Leftrightarrow \) \( \nabla \xi = 0 \Leftrightarrow \) \( S = 0 \Leftrightarrow \) \( S^b = 0 \)
\( \Leftrightarrow \) \( \nabla^o = \nabla \Leftrightarrow \) \( \mathcal{M} \in U_1 \).

The horizontal and vertical components of \( \tilde{Q}^o \) and \( \tilde{T}^o \) of \( \tilde{\nabla}^o \) are respectively

\[
\begin{align*}
\tilde{Q}^{oh} &= - (\tilde{\nabla} \xi) \otimes \eta, \\
\tilde{Q}^{ov} &= (\tilde{\nabla} \eta) \otimes \xi, \\
\tilde{T}^{oh} &= \eta \wedge (\tilde{\nabla} \xi), \\
\tilde{T}^{ov} &= d \eta \otimes \xi.
\end{align*}
\]

From \( \tilde{g}(\xi, \xi) = 1 \) we have \( \tilde{g}(\tilde{\nabla}_x \xi, \xi) = 0 \) and therefore \( \tilde{\nabla} \xi \in \mathcal{H} \). The shape operator \( \tilde{S} : \mathcal{H} \mapsto \mathcal{H} \) for the metric \( \tilde{g} \) is defined by \( \tilde{S}(x) = -\tilde{\nabla}_x \xi \).

Since we have
\[
(\tilde{\nabla}_x \eta)(y) = (\nabla_x \eta)(y) - \eta(\Phi(x, y)), \quad \tilde{\nabla}_x \xi = \nabla_x \xi + \Phi(x, \xi),
\]
then we obtain

\[
\begin{align*}
\tilde{S}(x) &= S(x) - \Phi(x, \xi), \quad \tilde{S}^b(x, y) = S^b(x, \varphi y) - \Phi(\xi, x, \varphi y),
\end{align*}
\]
where \( \tilde{S}^b(x, y) = \tilde{g}(\tilde{S}(x), y) \).

Moreover, (7.3), (7.4), (7.12), (7.8), (7.9) and (7.15) imply the following relations

\[
\begin{align*}
\tilde{Q}^o(x, y) &= Q^o(x, y) - \eta(y)\Phi(x, \xi) - \eta(\Phi(x, y))\xi, \\
\tilde{T}^o(x, y) &= T^o(x, y) + \eta(x)\Phi(y, \xi) - \eta(y)\Phi(x, \xi); \\
\tilde{Q}^{oh} &= Q^{oh} - (\xi \downarrow \Phi) \otimes \eta, \\
\tilde{Q}^{ov} &= Q^{ov} - (\eta \circ \Phi) \otimes \xi, \\
\tilde{T}^{oh} &= T^{oh} + \eta \downarrow (\xi \downarrow \Phi), \\
\tilde{T}^{ov} &= T^{ov}.
\end{align*}
\]

Using the latter equalities and (7.16), we obtain the following formulae

\[
\begin{align*}
\tilde{Q}^o &= Q^o + (\tilde{S} - S) \otimes \eta - (\tilde{S}^b - S^b) \otimes \xi, \\
\tilde{T}^o &= T^o + (\tilde{S} - S) \wedge \eta;
\end{align*}
\]
\[ \tilde{Q}^{\text{ch}} = Q^{\text{ch}} + (\tilde{S} - S) \otimes \eta, \quad \tilde{Q}^{\text{ov}} = Q^{\text{ov}} - (\tilde{S}^b - S^b) \otimes \xi, \]
\[ \tilde{T}^{\text{ch}} = T^{\text{ch}} + (\tilde{S} - S) \wedge \eta, \quad \tilde{T}^{\text{ov}} = T^{\text{ov}}. \]

**Theorem 7.14.** The following equivalences are valid:

(i) $\tilde{\nabla} \eta$ is symmetric $\iff$ $\eta$ is closed $\iff$ $\tilde{Q}^{\text{ov}}$ is symmetric $\iff$ $\tilde{T}^{\text{ov}}$ vanishes $\iff$ $\tilde{S}$ is self-adjoint regarding $\tilde{g}$ $\iff$ $\tilde{S}^b$ is symmetric $\iff (M, \varphi, \xi, \eta, \tilde{g}) \in \tilde{U}_1 \oplus F_4 \oplus F_5 \oplus F_6 \oplus F_{10}$;

(ii) $\tilde{\nabla} \eta$ is skew-symmetric $\iff$ $\xi$ is Killing with respect to $\tilde{g}$ $\iff$ $\tilde{Q}^{\text{ov}}$ is skew-symmetric $\iff$ $\tilde{S}$ is anti-self-adjoint regarding $\tilde{g}$ $\iff$ $\tilde{S}^b$ is skew-symmetric $\iff (M, \varphi, \xi, \eta, \tilde{g}) \in \tilde{U}_1 \oplus F_7$;

(iii) $\tilde{\nabla} \eta = 0$ $\iff$ $d\eta = L_\xi \tilde{g} = 0$ $\iff$ $\tilde{\nabla} \xi = 0$ $\iff$ $\tilde{S} = 0$ $\iff$ $\tilde{S}^b = 0$ $\iff$ $\tilde{\nabla}^\circ = \tilde{\nabla} \iff (M, \varphi, \xi, \eta, \tilde{g}) \in \tilde{U}_1$.

### 7.3. Curvature properties of the pair of connections $\nabla^\circ$ and $\tilde{\nabla}^\circ$

On an almost contact B-metric manifold $(\mathcal{M}, \varphi, \xi, \eta, g)$, let $R$ be the curvature tensor of $\nabla$, i.e. $R = [\nabla, \nabla] - \nabla [\cdot, \cdot]$, and the corresponding $(0, 4)$-tensor is determined by $R(x, y, z, w) = g(R(x, y)z, w)$. The Ricci tensor $\rho$ and the scalar curvature $\tau$ are defined as usual by $\rho(y, z) = g^{ij}R(e_i, y, z, e_j)$ and $\tau = g^{ij}\rho(e_i, e_j)$, where $g^{ij}$ are the components of the inverse matrix of $g$ regarding an arbitrary basis $\{e_i\}$ ($i = 1, \ldots, 2n + 1$) of $T_p\mathcal{M}, p \in \mathcal{M}$.

Each non-degenerate 2-plane $\alpha$ in $T_p\mathcal{M}$ with respect to $g$ and $R$ has the following sectional curvature

\[ k(\alpha; p) = \frac{R(x, y, y, x)}{\pi_1(x, y, y, x)}, \]

where $\{x, y\}$ is an arbitrary basis of $\alpha$.

A 2-plane $\alpha$ is said to be a $\xi$-section, a $\varphi$-holomorphic section or a $\varphi$-totally real section if $\xi \in \alpha$, $\alpha = \varphi\alpha$ or $\alpha \perp \varphi\alpha$ regarding $g$, respectively. The latter type of sections exist only for $\dim \mathcal{M} \geq 5$.

In [85], some curvature properties with respect to $\nabla$ are studied in several subclasses of $U_2$.

Let us denote the curvature tensor, the Ricci tensor, the scalar curvature and the sectional curvature of the connection $\nabla^\circ$ by $R^\circ$, $\rho^\circ$, $\tau^\circ$ and $k^\circ$, respectively. The corresponding $(0, 4)$-tensor of $R^\circ$, denoted by the same letter, as well as $\rho^\circ$, $\tau^\circ$ and $k^\circ$ are determined by $g$. 

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Analogously, let the corresponding quantities for the connections $\tilde{\nabla}$ and $\tilde{\nabla}^\circ$ be denoted by $\tilde{R}$, $\tilde{\rho}$, $\tilde{\tau}$, $\tilde{k}$ and $\tilde{R}^\circ$, $\tilde{\rho}^\circ$, $\tilde{\tau}^\circ$, $\tilde{k}^\circ$, respectively. The corresponding $(0,4)$-tensors of $\tilde{R}$ and $\tilde{R}^\circ$, denoted by the same letters, as well as $\tilde{\rho}$, $\tilde{\tau}$, $\tilde{k}$ and $\tilde{\rho}^\circ$, $\tilde{\tau}^\circ$, $\tilde{k}^\circ$ are obtained by $\tilde{g}$.

**Theorem 7.15.** The curvature tensors of $\nabla^\circ$ and $\nabla$ (respectively, of $\tilde{\nabla}^\circ$ and $\tilde{\nabla}$) are related as follows

\begin{align*}
R^\circ(x, y, z, w) &= R(x, y, \varphi^2 z, \varphi^2 w) + \pi_1(S(x), S(y), z, w), \\
\tilde{R}^\circ(x, y, z, w) &= \tilde{R}(x, y, \varphi^2 z, \varphi^2 w) + \tilde{\pi}_1(\tilde{S}(x), \tilde{S}(y), z, w),
\end{align*}

where $\tilde{\pi}_1$ is constructed as in (7.13) by $\tilde{g}$.

**Proof.** Using (7.22), we compute $R^\circ$. Taking into account that $g(\nabla_x \xi, \xi)$ vanishes for each $x$ and $\nabla^\circ \xi = 0$, we obtain the equality

\begin{align*}
R^\circ(x, y)z &= R(x, y)z - \eta(z)R(x, y)\xi - \eta(R(x, y)z)\xi \\
&- g(\nabla_x \xi, z) \nabla_y \xi + g(\nabla_y \xi, z) \nabla_x \xi.
\end{align*}

The latter equality implies the first relation in (7.17).

The second equality in (7.17) follows as above but in terms of $\tilde{\nabla}^\circ$, $\tilde{\nabla}$ and their corresponding metric $\tilde{g}$. \qed

Then, Theorem 7.15 has the following

**Corollary 7.16.** The Ricci tensors of $\nabla^\circ$ and $\nabla$ (respectively, of $\tilde{\nabla}^\circ$ and $\tilde{\nabla}$) are related as follows

\begin{align*}
\rho^\circ(y, z) &= \rho(y, z) - \eta(z)\rho(y, \xi) - R(\xi, y, z, \xi) \\
&- g(S(S(y)), z) + \text{tr}(S)g(S(y), z), \\
\tilde{\rho}^\circ(y, z) &= \tilde{\rho}(y, z) - \eta(z)\tilde{\rho}(y, \xi) - \tilde{R}(\xi, y, z, \xi) \\
&- \tilde{g}(\tilde{S}(\tilde{S}(y)), z) + \tilde{\text{tr}}(\tilde{S})\tilde{g}(\tilde{S}(y), z),
\end{align*}

where $\tilde{\text{tr}}$ denotes the trace with respect to $\tilde{g}$.

Let us remark that we have $\text{tr}(S) = \tilde{\text{tr}}(\tilde{S}) = -\text{div}(\eta)$, because of (4.16), the definitions of $S$ and $\tilde{S}$ as well as $g^{ij}\Phi(\xi, e_i, e_j) = 0$, using (4.23) and (4.10).

From the definition of the shape operator we get

\[ R(x, y)\xi = - (\nabla_x S) y + (\nabla_y S) x. \]

Then, the latter formula and $S(\xi) = -\nabla_\xi = -\varphi^k$ lead to the following expression of one of the components in the right-hand side of the first
equality in \((7.18)\)
\[
R(\xi, y, z, \xi) = g((\nabla_\xi S) y - (\nabla_y S) \xi, z) \\
= g((\nabla_\xi S) y - \nabla_y S(\xi) - S(y), z).
\]
Therefore, taking the trace of the latter equalities and using the relations
\[
\text{div}(\omega \circ \varphi) = g^{ij}(\nabla_e \omega \circ \varphi) e_j = g^{ij} g(\nabla_e \varphi \omega^e, e_j) = -\text{div}(S(\xi))
\]
for the divergence of the 1-form \(\omega \circ \varphi\), we obtain
\[
(7.19) \quad \rho(\xi, \xi) = \text{tr}(\nabla_\xi S) - \text{div}(S(\xi)) - \text{tr}(S^2).
\]
Similar equalities for the quantities with a tilde are valid with respect to \(\tilde{g}\), i.e.
\[
(7.20) \quad \tilde{\rho}(\xi, \xi) = \tilde{\text{tr}}(\tilde{\nabla}_\xi \tilde{S}) - \tilde{\text{div}}(\tilde{S}(\xi)) - \tilde{\text{tr}}(\tilde{S}^2).
\]
Bearing in mind the latter computations, from Corollary 7.16 we obtain the following

**Corollary 7.17.** The scalar curvatures of \(\nabla^\circ\) and \(\nabla\) (respectively, of \(\tilde{\nabla}^\circ\) and \(\tilde{\nabla}\)) are related as follows
\[
\tau^\circ = \tau - 2\rho(\xi, \xi) - \text{tr}(S^2) + (\text{tr}(S))^2,
\]
\[
\tilde{\tau}^\circ = \tilde{\tau} - 2\tilde{\rho}(\xi, \xi) - \tilde{\text{tr}}(\tilde{S}^2) + (\tilde{\text{tr}}(\tilde{S}))^2,
\]
where \(\rho(\xi, \xi)\) and \(\tilde{\rho}(\xi, \xi)\) are expressed by \(S\) and \(\tilde{S}\) in \((7.19)\) and \((7.20)\), respectively.

From Theorem 7.15 we obtain the following

**Corollary 7.18.** The sectional curvatures of an arbitrary 2-plane \(\alpha\) at \(p \in \mathcal{M}\) regarding \(\nabla^\circ\) and \(\nabla\) (respectively, \(\tilde{\nabla}^\circ\) and \(\tilde{\nabla}\)) are related as follows
\[
k^\circ(\alpha; p) = k(\alpha; p) + \frac{\pi_1(S(x), S(y), y, x)}{\pi_1(x, y, y, x)} \\
- \frac{\eta(x) R(x, y, y, \xi) + \eta(y) R(x, y, \xi, x)}{\pi_1(x, y, y, x)},
\]
\[
(7.21) \quad \tilde{k}^\circ(\alpha; p) = \tilde{k}(\alpha; p) + \frac{\tilde{\pi}_1(\tilde{S}(x), \tilde{S}(y), y, x)}{\tilde{\pi}_1(x, y, y, x)} \\
- \frac{\eta(x) \tilde{R}(x, y, y, \xi) + \eta(y) \tilde{R}(x, y, \xi, x)}{\tilde{\pi}_1(x, y, y, x)},
\]
where \(\{x, y\}\) is an arbitrary basis of \(\alpha\).
If \( \alpha \) is a \( \xi \)-section at \( p \in \mathcal{M} \) denoted by \( \alpha_{\xi} \) and \( \{x, \xi\} \) is its basis, then from (7.21) and \( g(S(x), \xi) = 0 \) for each \( x \) we obtain that the sectional curvature of \( \alpha \) regarding \( \nabla^o \) is zero, i.e. \( k^o(\alpha_{\xi}; p) = 0 \). Analogously, we have \( \tilde{k}^o(\alpha_{\xi}; p) = 0 \).

If \( \alpha \) is a \( \varphi \)-section at \( p \in \mathcal{M} \) denoted by \( \alpha_{\varphi} \) and \( \{x, y\} \) is its arbitrary basis, then from (7.21) and \( \eta(x) = \eta(y) = 0 \) we obtain that the sectional curvatures of \( \alpha_{\varphi} \) regarding \( \nabla^o \) and \( \nabla \) are related as follows

\[
k^o(\alpha_{\varphi}; p) = k(\alpha_{\varphi}; p) + \frac{\pi_1(S(x), S(y), y, x)}{\pi_1(x, y, y, x)}.\]

Analogously, we have

\[
\tilde{k}^o(\alpha_{\varphi}; p) = \tilde{k}(\alpha_{\varphi}; p) + \frac{\tilde{\pi}_1(\tilde{S}(x), \tilde{S}(y), y, x)}{\tilde{\pi}_1(x, y, y, x)}.\]

If \( \alpha \) is a \( \varphi \)-totally real section orthogonal to \( \xi \) denoted by \( \alpha_{\perp} \) and \( \{x, y\} \) is its arbitrary basis, then from (7.21) and \( \eta(x) = \eta(y) = 0 \) we obtain that the sectional curvatures of \( \alpha_{\perp} \) regarding \( \nabla^o \) and \( \nabla \) are related as follows

\[
k^o(\alpha_{\perp}; p) = k(\alpha_{\perp}; p) + \frac{\pi_1(S(x), S(y), y, x)}{\pi_1(x, y, y, x)}.\]

Analogously, we have

\[
\tilde{k}^o(\alpha_{\perp}; p) = \tilde{k}(\alpha_{\perp}; p) + \frac{\tilde{\pi}_1(\tilde{S}(x), \tilde{S}(y), y, x)}{\tilde{\pi}_1(x, y, y, x)}.\]

In the case when \( \alpha \) is a \( \varphi \)-totally real section non-orthogonal to \( \xi \) regarding \( g \) or \( \tilde{g} \), the relation between the corresponding sectional curvatures regarding \( \nabla^o \) and \( \nabla \) (respectively, \( \tilde{\nabla}^o \) and \( \tilde{\nabla} \)) is just the first (respectively, the second) equality in (7.21).

The equalities in the present section are specialised for the considered manifolds in the different classes since \( S \) and \( \tilde{S} \) have a special form in each class, bearing in mind (4.18).

\[
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\]
§8. Sasaki-like almost contact complex Riemannian manifolds

In the present section a Sasaki-like almost contact complex Riemannian manifold is defined as an almost contact complex Riemannian manifold which complex cone is a holomorphic complex Riemannian manifold. Explicit compact and non-compact examples are given. A canonical construction producing a Sasaki-like almost contact complex Riemannian manifold from a holomorphic complex Riemannian manifold is presented and it is called an $S^1$-solvable extension.

The main results of this section are published in [58].

The almost contact complex Riemannian manifold is an $(2n + 1)$-dimensional pseudo-Riemannian manifold equipped with a 1-form $\eta$ and a codimension one distribution $\mathcal{H} = \ker(\eta)$ endowed with a complex Riemannian structure. More precisely, the $2n$-dimensional distribution $\mathcal{H}$ is provided with a pair of an almost complex structure and a pseudo-Riemannian metric of signature $(n, n)$ compatible in the way that the almost complex structure acts as an anti-isometry on the metric. Almost contact complex Riemannian manifolds are investigated and studied in [39, 78, 83, 95, 96, 99, 100, 116].

The main goal of this section is to find a class of almost contact complex Riemannian manifolds which characteristics resemble to some basic properties of the well known Sasakian manifolds. We define this class of Sasaki-like spaces as an almost contact complex Riemannian manifold which complex cone is a holomorphic complex Riemannian manifold. We note that a holomorphic complex Riemannian manifold is a complex manifold endowed with a complex Riemannian metric whose local components in holomorphic coordinates are holomorphic functions (see [105]). We determine the Sasaki-like almost contact complex Riemannian structure with an explicit expression of the covariant derivative of the structure tensors and construct explicit compact and non-compact examples. We also present a canonical construction producing a Sasaki-like
almost contact complex Riemannian manifold from any holomorphic complex Riemannian manifold which we call an $S^1$-solvable extension. When we study the curvature of Sasaki-like spaces, we show that it is completely determined by the curvature of the underlying holomorphic complex Riemannian manifold. We develop gauge transformations of Sasaki-like spaces, i.e. we find the class of contact conformal transformations of an almost contact complex Riemannian manifolds which preserve the Sasaki-like condition.

We introduce the following convention for the present section. Let $(\mathcal{M}, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact complex Riemannian manifold with a pseudo-Riemannian metric $g$ of signature $(n+1, n)$. We shall use $x, y, z, u$ to denote smooth vector fields on $\mathcal{M}$, i.e. $x, y, z, u \in \mathfrak{X}(\mathcal{M})$. We shall use $X, Y, Z, U$ to denote smooth horizontal vector fields on $\mathcal{M}$, i.e. $X, Y, Z, U \in \mathcal{H} = \ker(\eta)$. The $2n$-tuple $\{e_1, \ldots, e_{n+1} = \varphi e_1, \ldots, e_{2n} = \varphi e_n\}$ denotes a local orthonormal basis of the horizontal space $\mathcal{H}$. For an orthonormal basis $\{e_0 = \xi, e_1, \ldots, e_n, e_{n+1} = \varphi e_1, \ldots, e_{2n} = \varphi e_n\}$ we denote $\varepsilon_i = \text{sign}(g(e_i, e_i)) = \pm 1$, where $\varepsilon_i = 1$ for $i = 0, 1, \ldots, n$ and $\varepsilon_i = -1$ for $i = n+1, \ldots, 2n$.

### 8.1. Almost contact complex Riemannian manifolds

Let $(\mathcal{M}, \varphi, \xi, \eta)$ be an almost contact manifold. An almost contact structure $(\varphi, \xi, \eta)$ on $\mathcal{M}$ is called normal and respectively $(\mathcal{M}, \varphi, \xi, \eta)$ is a normal almost contact manifold if the corresponding almost complex structure $\tilde{J}$ on $\mathcal{M}' = \mathcal{M} \times \mathbb{R}$ defined by

\begin{align}
\tilde{J}X &= \varphi X, \\
\tilde{J}\xi &= r \frac{d}{dr}, \\
\tilde{J}\frac{d}{dr} &= -\frac{1}{r}\xi
\end{align}

is integrable (i.e. $\mathcal{M}'$ is a complex manifold) \cite{129}.

Let the almost contact manifold $(\mathcal{M}, \varphi, \xi, \eta)$ be endowed with a $B$-metric as in \cite{4.2}. The manifold $(\mathcal{M}, \varphi, \xi, \eta, g)$ is known as an almost contact manifold with $B$-metric or an almost contact $B$-metric manifold \cite{39}. The manifold $(\mathcal{M}, \varphi, \xi, \eta, \tilde{g})$ is also an almost contact $B$-metric manifold. We will call these manifolds almost contact complex Riemannian manifolds.

#### 8.1.1. Relation with holomorphic complex Riemannian manifolds

Let us remark that the $2n$-dimensional distribution $\mathcal{H}$ is endowed with an almost complex structure $J = \varphi|_{\mathcal{H}}$ and a metric $h = g|_{\mathcal{H}}$, where $\varphi|_{\mathcal{H}}$
and $g|_{\mathcal{H}}$ are the restrictions of $\varphi$ and $g$ on $\mathcal{H}$, respectively, as well as the metric $h$ is compatible with $J$ as follows

\begin{equation}
(8.2) \quad h(JX, JY) = -h(X, Y), \quad \tilde{h}(X, Y) = h(X, JY).
\end{equation}

The distribution $\mathcal{H}$ can be considered as an $n$-dimensional complex Riemannian distribution with a complex Riemannian metric

\[ g^C = h + i \tilde{h} = g|_{\mathcal{H}} + i \tilde{g}|_{\mathcal{H}}. \]

We recall that a $2n$-dimensional manifold with almost complex structure $(J, h)$ endowed with a pseudo-Riemannian metric of signature $(n, n)$, satisfying (8.2), was discussed in the first three sections of the present work. When the almost complex structure $J$ is parallel with respect to the Levi-Civita connection $D$ of the metric $h$, i.e. $D J = 0$, then the manifold is known also as a holomorphic complex Riemannian manifold. In this case $J$ is integrable and the local components of the complex metric in a holomorphic coordinate system are holomorphic functions. A 4-dimensional example of a holomorphic complex Riemannian manifold has been given in [120]. Another approach to such manifolds has been used in [121]. In [147] has been proved that the 4-dimensional sphere of Kotel’nikov-Study carries a structure of a holomorphic complex Riemannian manifold.

### 8.1.2. The case of parallel structures

The simplest case of almost contact complex Riemannian manifolds is when the structures are $\nabla$-parallel, $\nabla \varphi = \nabla \xi = \nabla \eta = \nabla g = \nabla \tilde{g} = 0$, and it is determined by the condition $F(x, y, z) = 0$. In this case the distribution $\mathcal{H}$ is involutive. The corresponding integral submanifold is a totally geodesic submanifold which inherits a holomorphic complex Riemannian structure and the almost contact complex Riemannian manifold is locally a pseudo-Riemannian product of a holomorphic complex Riemannian manifold with a real interval.

### 8.2. Sasaki-like almost contact complex Riemannian manifolds

In this subsection we consider the complex Riemannian cone over an almost contact complex Riemannian manifold and we determine a Sasaki-like almost contact complex Riemannian manifold with the condition that its complex Riemannian cone is a holomorphic complex Riemannian manifold.
8.2.1. Holomorphic complex Riemannian cone

Let \((\mathcal{M}, \varphi, \xi, \eta, g)\) be an almost contact complex Riemannian manifold of dimension \(2n + 1\). We consider the cone over \(\mathcal{M}\), \(C(\mathcal{M}) = \mathcal{M} \times \mathbb{R}^−\), with the almost complex structure determined in (8.1) and the complex Riemannian metric defined by

\[
(8.3) \quad \tilde{g} \left( \left( x, \frac{a}{dr} \right), \left( y, \frac{b}{dr} \right) \right) = r^2 g(x, y) + \eta(x)\eta(y) - ab,
\]

where \(r\) is the coordinate on \(\mathbb{R}^−\) and \(a, b\) are \(C^∞\) functions on \(\mathcal{M} \times \mathbb{R}^−\).

Using the general Koszul formula (5.18), we calculate from (8.3) that the non-zero components of the Levi-Civita connection \(\tilde{\nabla}\) of the complex Riemannian metric \(\tilde{g}\) on \(C(\mathcal{M})\) are given by

\[
\tilde{g} \left( \tilde{\nabla}_X Y, Z \right) = r^2 g \left( \nabla_X Y, Z \right), \quad \tilde{g} \left( \tilde{\nabla}_X Y, \frac{d}{dr} \right) = -r g \left( X, Y \right),
\]

\[
\tilde{g} \left( \tilde{\nabla}_X Y, \xi \right) = r^2 g \left( \nabla_X Y, \xi \right) + \frac{1}{2} \left( r^2 - 1 \right) d\eta(X, Y),
\]

\[
\tilde{g} \left( \tilde{\nabla}_X \xi, Z \right) = r^2 g \left( \nabla_X \xi, Z \right) - \frac{1}{2} \left( r^2 - 1 \right) d\eta(X, Z),
\]

\[
\tilde{g} \left( \tilde{\nabla}_\xi Y, Z \right) = r^2 g \left( \nabla_\xi Y, Z \right) - \frac{1}{2} \left( r^2 - 1 \right) d\eta(Y, Z),
\]

\[
\tilde{g} \left( \tilde{\nabla}_\xi Y, \xi \right) = g \left( \nabla_\xi Y, \xi \right), \quad \tilde{g} \left( \tilde{\nabla}_\xi \xi, Z \right) = g \left( \nabla_\xi \xi, Z \right),
\]

\[
\tilde{g} \left( \tilde{\nabla}_X \frac{d}{dr}, Z \right) = r g \left( X, Z \right), \quad \tilde{g} \left( \tilde{\nabla}_{\frac{d}{dr}} Y, Z \right) = r g \left( Y, Z \right).
\]

Applying (8.1) we calculate from the formulas above that the non-zero components of the covariant derivative \(\tilde{\nabla}_{\tilde{J}}\) of the almost complex structure \(\tilde{J}\) are given by

\[
\tilde{g} \left( (\tilde{\nabla}_X \tilde{J}) Y, Z \right) = r^2 g \left( (\nabla_X \varphi) Y, Z \right),
\]

\[
\tilde{g} \left( (\tilde{\nabla}_X \tilde{J}) Y, \xi \right) = r^2 \left\{ g \left( (\nabla_X \varphi) Y, \xi \right) + g(X, Y) \right\} + \frac{1}{2} \left( r^2 - 1 \right) d\eta(X, \varphi Y),
\]

\[
\tilde{g} \left( (\tilde{\nabla}_X \tilde{J}) Y, \frac{d}{dr} \right) = -r \left\{ g \left( \nabla_X \xi, Y \right) + g(X, \varphi Y) \right\} + \frac{1}{2r} \left( r^2 - 1 \right) d\eta(X, Y),
\]

\[
\tilde{g} \left( (\tilde{\nabla}_X \tilde{J}) \xi, Z \right) = -\frac{1}{2r} \left( r^2 - 1 \right) d\eta(Y, X).
\]
\[ \tilde{g} \left( (\tilde{\nabla}_X \tilde{J}) \xi, Z \right) = -r^2 \{ g(\nabla_X \xi, \varphi Y) - g(X, Z) \} + \frac{1}{2} (r^2 - 1) \mathrm{d}\eta(X, \varphi Z), \]
\[ \tilde{g} \left( (\tilde{\nabla}_X \tilde{J}) \frac{d}{dr}, Z \right) = -r \{ g(\nabla_X \xi, Z) + g(X, \varphi Z) \} + \frac{1}{2} (r^2 - 1) \mathrm{d}\eta(X, Z), \]
\[ \tilde{g} \left( (\tilde{\nabla}_\xi \tilde{J}) Y, Z \right) = r^2 g ( (\nabla_\xi \varphi) Y, Z ) - \frac{1}{2} (r^2 - 1) \{ \mathrm{d}\eta(\varphi Y, Z) - \mathrm{d}\eta(Y, \varphi Z) \}, \]
\[ \tilde{g} \left( (\tilde{\nabla}_\xi \tilde{J}) Y, \xi \right) = -g(\nabla_\xi \xi, \varphi Y), \]
\[ \tilde{g} \left( (\tilde{\nabla}_\xi \tilde{J}) \xi, Z \right) = -g(\nabla_\xi \xi, \varphi Z), \]
\[ \tilde{g} \left( (\tilde{\nabla}_\xi \tilde{J}) Y, \frac{d}{dr} \right) = -\frac{1}{r} g(\nabla_\xi \xi, Y), \]
\[ \tilde{g} \left( (\tilde{\nabla}_\xi \tilde{J}) \frac{d}{dr}, Z \right) = -\frac{1}{r} g(\nabla_\xi \xi, Z). \] 

Consequently, we have

**Proposition 8.1.** The complex Riemannian cone \( C(\mathcal{M}) \) over an almost contact complex Riemannian manifold \((\mathcal{M}, \varphi, \xi, \eta, g)\) is a holomorphic complex Riemannian space if and only if the following conditions hold

\[ F(X, Y, Z) = F(\xi, Y, Z) = F(\xi, \xi, Z) = 0, \quad (8.4) \]
\[ F(X, Y, \xi) = -g(X, Y), \quad (8.5) \]

*Proof.* We obtain from the expressions above that the complex Riemannian cone \((\mathcal{C}(\mathcal{M}), \tilde{J}, \tilde{g})\) is a holomorphic Riemannian manifold (a Kähler manifold with Norden metric), i.e. \( \tilde{\nabla} \tilde{J} = 0 \), if and only if the almost contact complex Riemannian manifold \((\mathcal{M}, \varphi, \xi, \eta, g)\) satisfies the following conditions:

\[ F(X, Y, Z) = 0, \quad (8.6) \]
\[ F(X, Y, \xi) = -g(X, Y) - \frac{1}{2r^2} (r^2 - 1) \mathrm{d}\eta(X, \varphi Y), \]
\[ F(\xi, Y, Z) = \frac{1}{2r^2} (r^2 - 1) \{ \mathrm{d}\eta(\varphi Y, Z) - \mathrm{d}\eta(Y, \varphi Z) \}, \quad (8.7) \]
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(8.8) \[ F(\xi, \xi, Z) = 0, \quad \nabla_\xi \xi = 0. \]

The condition \( \tilde{\nabla} \tilde{J} = 0 \) implies the integrability of \( \tilde{J} \), hence the structure of \((M, \varphi, \xi, \eta, g)\) is normal.

Further, according to (8.6), we get

(8.9) \[ (\nabla_X \eta) Y = -g(X, \varphi Y) + \frac{1}{r^2} \left( r^2 - 1 \right) d\eta(X, Y), \]

yielding \( d\eta(X, Y) = \frac{1}{2r} \left( r^2 - 1 \right) d\eta(X, Y) \) since \( g(\cdot, \varphi \cdot) \) is symmetric.

The latter equality shows \( d\eta(X, Y) = 0 \) which together with (8.9) yields

(8.10) \[ (\nabla_X \eta) Y = -g(X, \varphi Y). \]

From (8.8) we get \( d\eta(\xi, X) = (\nabla_\xi \eta)(X) - (\nabla_X \eta)(\xi) = 0. \) Hence, we have \( d\eta = 0 \). We substitute \( d\eta = 0 \) into (8.6)-(8.7) to complete the proof of the proposition.

Definition 8.1. An almost contact complex Riemannian manifold \((M, \varphi, \xi, \eta, g)\) is said to be Sasaki-like if the structure tensors \( \varphi, \xi, \eta, g \) satisfy the equalities (8.4) and (8.5).

To characterize the Sasaki-like almost contact complex Riemannian manifolds by their structure tensors, we need the general result in Theorem 4.4.

The next result determines the Sasaki-like spaces by their structure tensors.

Theorem 8.2. Let \((M, \varphi, \xi, \eta, g)\) be an almost contact complex Riemannian manifold. The following conditions are equivalent:

(i) The manifold \((M, \varphi, \xi, \eta, g)\) is a Sasaki-like almost contact complex Riemannian manifold;

(ii) The covariant derivative \( \nabla \varphi \) satisfies the equality

(8.11) \[ (\nabla_x \varphi) y = -g(x, y)\xi - \eta(y)x + 2\eta(x)\eta(y)\xi; \]

(iii) The Nijenhuis tensors \( N \) and \( \tilde{N} \) satisfy the relations:

(8.12) \[ N = 0, \quad \tilde{N} = -4 \left( \tilde{g} - \eta \otimes \eta \right) \otimes \xi. \]

Proof. It is easy to check using (4.11) and (4.10) that (8.11) is equivalent to the system of the equations (8.4) and (8.5) which established the equivalence between (i) and (ii) in view of Proposition 8.1.
We substitute (8.11) consequently into (4.31) and (4.32) to get (8.12) which gives the implication (ii) ⇒ (iii).

Now, suppose (8.12) holds. Consequently, we obtain \( \hat{N}(\xi, y) = 0 \). Now, (8.11) follows with a substitution of the last equality together with (8.12) into (4.39) which completes the proof.

**Corollary 8.3.** Let \((\mathcal{M}, \varphi, \xi, \eta, g)\) be a Sasaki-like almost contact complex Riemannian manifold. Then we have

(i) the manifold \((\mathcal{M}, \varphi, \xi, \eta, g)\) is normal, the fundamental 1-form \(\eta\) is closed and the integral curves of \(\xi\) are geodesics;

(ii) the 1-forms \(\theta\) and \(\theta^*\) satisfy the equalities \(\theta = -2n \eta\) and \(\theta^* = 0\).

### 8.2.2. Examples

In this subsection we construct a number of examples of Sasaki-like almost contact complex Riemannian manifolds.

#### 8.2.2.1. Example 1

We consider the solvable Lie group \(G\) of dimension \(2n + 1\) with a basis of left-invariant vector fields \(\{e_0, \ldots, e_{2n}\}\) defined by the commutators

\[
[e_0, e_1] = e_{n+1}, \quad \ldots, \quad [e_0, e_n] = e_{2n},
\]

\[
[e_0, e_{n+1}] = -e_1, \quad \ldots, \quad [e_0, e_{2n}] = -e_n.
\]

We define an invariant almost contact complex Riemannian structure on \(G\) by

\[
g(e_0, e_0) = g(e_1, e_1) = \cdots = g(e_n, e_n) = 1
\]

\[
g(e_{n+1}, e_{n+1}) = \cdots = g(e_{2n}, e_{2n}) = -1,
\]

\[
g(e_i, e_j) = 0, \quad i, j \in \{0, 1, \ldots, 2n\}, \quad i \neq j,
\]

\[
\xi = e_0, \quad \varphi e_1 = e_{n+1}, \quad \ldots, \quad \varphi e_n = e_{2n}.
\]

Using the Koszul formula (5.18), we check that (8.4) and (8.5) are fulfilled, i.e. this is a Sasaki-like almost contact complex Riemannian structure.

Let \(e^0 = \eta, e^1, \ldots, e^{2n}\) be the corresponding dual 1-forms, \(e^i(e_j) = \delta^i_j\). From (8.13) and the formula for an arbitrary 1-form \(\alpha\)

\[
d\alpha(A, B) = A\alpha(B) - B\alpha(A) - \alpha([A, B]),
\]
it follows that the structure equations of the group are
\begin{equation}
\begin{aligned}
\de e^0 &= \de \eta = 0, \quad \de e^1 = e^0 \wedge e^{n+1}, \quad \ldots, \quad \de e^n = e^0 \wedge e^{2n}, \\
\de e^{n+1} &= -e^0 \wedge e^1, \quad \ldots, \quad \de e^{2n} = -e^0 \wedge e^n,
\end{aligned}
\end{equation}
and the Sasaki-like almost contact complex Riemannian structure has the form
\begin{equation}
\begin{aligned}
g &= \sum_{i=0}^{2n} \varepsilon_i (e^i)^2, \quad \varphi e^1 = e^{n+1}, \quad \ldots, \quad \varphi e^n = e^{2n}.
\end{aligned}
\end{equation}

The group $G$ is the following \textit{rank-1 solvable extension of the Abelian group} $\mathbb{R}^{2n}$
\begin{equation}
\begin{aligned}
e^0 &= \de t, \\
e^1 &= \cos t \de x^1 + \sin t \de x^{n+1}, \\
\ldots, \\
e^n &= \cos t \de x^n + \sin t \de x^{2n}, \\
e^{n+1} &= -\sin t \de x^1 + \cos t \de x^{n+1}, \\
\ldots, \\
e^{2n} &= -\sin t \de x^n + \cos t \de x^{2n}.
\end{aligned}
\end{equation}
Clearly, the 1-forms defined in (8.17) satisfy (8.15) and the Sasaki-like almost contact complex Riemannian metric has the form
\begin{equation}
\begin{aligned}
g &= \de t^2 + \cos 2t \left( \sum_{i=1}^{2n} \varepsilon_i (\de x^i)^2 \right) - \sin 2t \left( -2 \sum_{i=1}^n \de x^i \de x^{n+i} \right).
\end{aligned}
\end{equation}

It is known that the solvable Lie group $G$ admits a lattice $\Gamma$ such that the quotient space $G/\Gamma$ is compact (c.f. [144, Chapter 3]). The invariant Sasaki-like almost contact complex Riemannian structure $(\varphi, \xi, \eta, g)$ on $G$ descends to $G/\Gamma$ which supplies a compact Sasaki-like almost contact complex Riemannian manifold in any dimension.

It follows from (8.13), (8.16), (8.17) and (8.18) that the distribution $\mathcal{H} = \text{span}\{e_1, \ldots, e_{2n}\}$ is integrable and the corresponding integral submanifold can be considered as the holomorphic complex Riemannian flat space $\mathbb{R}^{2n} = \text{span}\{\de x^1, \ldots, \de x^{2n}\}$ with the following holomorphic complex Riemannian structure
\begin{equation}
\begin{aligned}
J\de x^1 &= \de x^{n+1}, \quad \ldots, \quad J\de x^n = \de x^{2n}; \\
h &= \sum_{i=1}^{2n} \varepsilon_i (\de x^i)^2, \quad \tilde{h} = -2 \sum_{i=1}^n \de x^i \de x^{n+i}.
\end{aligned}
\end{equation}
8.2.2.2. $S^1$-solvable extension

Inspired by Example 1 on page 114 we proposed the following more general construction. Let $(\mathcal{M}^{2n}, J, h, \tilde{h})$ be a $2n$-dimensional holomorphic complex Riemannian manifold, i.e. the almost complex structure $J$ acts as an anti-isometry on the neutral metric $h$, $h(JX, JY) = -h(X, Y)$ and it is parallel with respect to the Levi-Civita connection of $h$. In particular, the almost complex structure $J$ is integrable. The associated neutral metric $\tilde{h}$ is defined by $\tilde{h}(X, Y) = h(JX, Y)$ and it is also parallel with respect to the Levi-Civita connection of $h$.

We consider the product manifold $\mathcal{M}^{2n+1} = \mathbb{R}^+ \times \mathcal{M}^{2n}$. Let $dt$ be the coordinate 1-form on $\mathbb{R}^+$ and we define an almost contact complex Riemannian structure on $\mathcal{M}^{2n+1}$ as follows

$$\eta = dt, \quad \varphi|_H = J, \quad \eta \circ \varphi = 0,$$

$$(8.19)$$

$$g = dt^2 + \cos 2t \, h - \sin 2t \, \tilde{h}.$$

**Theorem 8.4.** Let $(\mathcal{M}^{2n}, J, h, \tilde{h})$ be a $2n$-dimensional holomorphic complex Riemannian manifold. Then the product manifold $\mathcal{M}^{2n+1} = \mathbb{R}^+ \times \mathcal{M}^{2n}$ equipped with the almost contact complex Riemannian structure defined in (8.19) is a Sasaki-like almost contact complex Riemannian manifold. If $\mathcal{M}^{2n}$ is compact and $S^1$ is an 1-dimensional sphere then $\mathcal{M}^{2n+1} = S^1 \times \mathcal{M}^{2n}$ with the structure (8.19) is a compact Sasaki-like almost contact complex Riemannian manifold.

**Proof.** It is easy to check using (5.18), (8.19) and the fact that the complex structure $J$ is parallel with respect to the Levi-Civita connection of $h$ that the structure defined in (8.19) satisfies (8.4) and (8.5) and thus $(\mathcal{M}, \varphi, \xi, \eta, g)$ is a Sasaki-like almost contact complex Riemannian manifold.

Now, suppose $\mathcal{M}^{2n}$ is a compact holomorphic complex Riemannian manifold. The equations (8.19) imply that the metric $g$ is periodic on $\mathbb{R}$ and therefore it descends to the compact manifold $\mathcal{M}^{2n+1} = S^1 \times \mathcal{M}^{2n}$. Thus we obtain a compact Sasaki-like almost contact complex Riemannian manifold.

We call the Sasaki-like almost contact complex Riemannian manifold constructed in Theorem 8.4 from a holomorphic complex Riemannian manifold an $S^1$-solvable extension of a holomorphic complex Riemannian manifold.
8.2.2.3. Example 2

Let us consider the Lie group $G^5$ of dimension 5 with a basis of left-invariant vector fields $\{e_0, \ldots, e_4\}$ defined by the commutators

$$
[e_0, e_1] = \lambda e_2 + e_3 + \mu e_4, \quad [e_0, e_2] = -\lambda e_1 - \mu e_3 + e_4,
$$
$$
[e_0, e_3] = -e_1 - \mu e_2 + \lambda e_4, \quad [e_0, e_4] = \mu e_1 - e_2 - \lambda e_3,
$$

where $\lambda, \mu \in \mathbb{R}$. Let $G^5$ be equipped with an invariant almost contact complex Riemannian structure as in (8.14) for $n = 2$. We calculate using (5.18) that the non-zero connection 1-forms of the Levi-Civita connection are the following

$$
\nabla_{e_0} e_1 = \lambda e_2 + \mu e_4, \quad \nabla_{e_1} e_0 = -e_3,
$$
$$
\nabla_{e_0} e_2 = -\lambda e_1 - \mu e_3, \quad \nabla_{e_2} e_0 = -e_4,
$$
$$
\nabla_{e_0} e_3 = -\mu e_2 + \lambda e_4, \quad \nabla_{e_3} e_0 = e_1,
$$
$$
\nabla_{e_0} e_4 = \mu e_1 - \lambda e_3, \quad \nabla_{e_4} e_0 = e_2,
$$
$$
\nabla_{e_1} e_3 = \nabla_{e_2} e_4 = \nabla_{e_3} e_1 = \nabla_{e_4} e_2 = -e_0.
$$

Similarly as in Example 1 we verify that the constructed manifold $(G^5, \varphi, \xi, \eta, g)$ is a Sasaki-like almost contact complex Riemannian manifold.

Take $\mu = 0$ and $\lambda \neq 0$. Then the structure equations of the group become

$$
d e^0 = d \eta = 0,
$$

(8.20) $$
de^1 = e^0 \wedge e^3 + \lambda e^0 \wedge e^2, \quad d e^2 = e^0 \wedge e^4 - \lambda e^0 \wedge e^1,
$$
$$
de^3 = -e^0 \wedge e^1 + \lambda e^0 \wedge e^4, \quad d e^4 = -e^0 \wedge e^2 - \lambda e^0 \wedge e^3.
$$

A basis of 1-forms satisfying (8.20) is given by $e^0 = dt$ and

$$
e^1 = \cos (1 - \lambda) t \, dx^1 - \cos (1 + \lambda) t \, dx^2 + \sin (1 - \lambda) t \, dx^3 - \sin (1 + \lambda) t \, dx^4,
$$
$$
e^2 = \sin (1 - \lambda) t \, dx^1 + \sin (1 + \lambda) t \, dx^2 - \cos (1 - \lambda) t \, dx^3 - \cos (1 + \lambda) t \, dx^4,
$$
$$
e^3 = -\sin (1 - \lambda) t \, dx^1 + \sin (1 + \lambda) t \, dx^2 + \cos (1 - \lambda) t \, dx^3 - \cos (1 + \lambda) t \, dx^4,
$$
$$
e^4 = \cos (1 - \lambda) t \, dx^1 + \cos (1 + \lambda) t \, dx^2 + \sin (1 - \lambda) t \, dx^3 + \sin (1 + \lambda) t \, dx^4.
$$
Then the Sasaki-like metric is of the form
\[
g = dt^2 - 4 \cos 2t \left( dx^1 dx^2 - dx^3 dx^4 \right) - 4 \sin 2t \left( dx^1 dx^4 + dx^2 dx^3 \right) .
\]
(8.21)

From (8.20) it follows that the distribution \( \mathcal{H} = \text{span}\{e_1, \ldots, e_4\} \) is integrable and the corresponding integral submanifold can be considered as the holomorphic complex Riemannian flat space \( \mathbb{R}^4 = \text{span}\{dx^1, \ldots, dx^4\} \) with the holomorphic complex Riemannian structure given by
\[
J dx^1 = dx^3, \quad J dx^2 = dx^4;
\]
\[
h = -4(dx^1 dx^2 - dx^3 dx^4), \quad \tilde{h} = 4(dx^1 dx^4 + dx^2 dx^3)
\]
and the Sasaki-like metric (8.21) takes the form
\[
g = dt^2 + \cos 2t \ h - \sin 2t \ \tilde{h}.
\]

8.3. Curvature properties

Let \( (\mathcal{M}, \varphi, \xi, \eta, g) \) be an almost contact complex Riemannian manifold. The curvature tensors of type \((1,3)\) and type \((0,4)\) are defined as usual (see Subsection 7.3, page 104). The Ricci tensor \( \text{Ric} \), the scalar curvature \( \text{Scal} \) and the \(*\)-scalar curvature \( \text{Scal}^* \) are the known traces of the curvature,
\[
\text{Ric}(x, y) = \sum_{i=0}^{2n} \varepsilon_i R(e_i, x, y, e_i),
\]
\[
\text{Scal} = \sum_{i=0}^{2n} \varepsilon_i \text{Ric}(e_i, e_i), \quad \text{Scal}^* = \sum_{i=0}^{2n} \varepsilon_i \text{Ric}(e_i, \varphi e_i).
\]

\[\text{Proposition 8.5.} \quad \text{On a Sasaki-like almost contact complex Riemannian manifold} \ (\mathcal{M}, \varphi, \xi, \eta, g) \ \text{the next formula holds}
\]
\[
R(x, y, \varphi z, u) - R(x, y, z, \varphi u) = [g(y, z) - 2\eta(y)\eta(z)] g(x, \varphi u) + [g(y, u) - 2\eta(y)\eta(u)] g(x, \varphi z) - [g(x, z) - 2\eta(x)\eta(z)] g(y, \varphi u) - [g(x, u) - 2\eta(x)\eta(u)] g(y, \varphi z).
\]
(8.22)

In particular, we have
\[
R(x, y)\xi = \eta(y)x - \eta(x)y,
\]
(8.23)
\[
[X, \xi] \in \mathcal{H}, \quad \nabla_\xi X = -\varphi X - [X, \xi] \in \mathcal{H};
\]
(8.24) \[ R(\xi, X)\xi = -X, \quad \text{Ric}(y, \xi) = 2n \ \eta(y), \quad \text{Ric}(\xi, \xi) = 2n. \]

**Proof.** The Ricci identity for \( \varphi \) reads

\[
R(x, y, \varphi z, u) - R(x, y, z, \varphi u) = g\left( (\nabla_x \nabla_y \varphi) z, u \right) - g\left( (\nabla_y \nabla_x \varphi) z, u \right).
\]

Applying (8.11) to the above equality and using (8.10), we obtain (8.22) by straightforward calculations. Set \( z = \xi \) into (8.22) and using (4.1), we get the first equality in (8.23). The rest follows from (8.10) and the condition \( d\eta = 0 \). The equalities (8.24) follow directly from the first equality in (8.23).

**8.3.1. The horizontal curvature**

From \( d\eta = 0 \) it follows locally \( \eta = dx \), \( \mathcal{H} \) is integrable and the manifold is locally the product \( \mathcal{M}^{2n+1} = \mathcal{M}^{2n} \times \mathbb{R} \) with \( T\mathcal{M}^{2n} = \mathcal{H} \). The submanifold \( (\mathcal{M}^{2n}, J = \varphi|_{\mathcal{H}}, h = g|_{\mathcal{H}}) \) is a holomorphic complex Riemannian manifold. Indeed, we obtain from (8.4) that

\[
h(D_X J)Y, Z) = F(X, Y, Z) = 0,
\]

where \( D \) is the Levi-Civita connection of \( h \).

We may consider \( \mathcal{M}^{2n} \) as a hypersurface of \( \mathcal{M} \) with the unit normal \( \xi = \frac{\partial}{\partial t} \). The equality (8.10) yields

\[
g(\nabla_X \xi, Y) = -g(\nabla_X Y, \xi) = -g(\varphi X, Y) = -\tilde{g}|_{\mathcal{H}}(X, Y), \quad \nabla_\xi \xi = 0.
\]

This means that the second fundamental form is equal to \( \tilde{g}|_{\mathcal{H}} = \tilde{h} \). The Gauss equation (see e.g. [61, Chapter VII, Proposition 4.1]) yields

\[
R(X, Y, Z, U) = R^h(X, Y, Z, U) + g(\varphi X, Z)g(\varphi Y, U)
\]

\[
- g(\varphi Y, Z)g(\varphi X, U),
\]

where \( R^h \) is the curvature tensor of the holomorphic complex Riemannian manifold \( (\mathcal{M}^{2n}, J, h) \).

For the horizontal Ricci tensor we obtain from (8.25) and (8.24) that

\[
\text{Ric}(Y, Z) = \sum_{i=1}^{2n} \varepsilon_i R(e_i, Y, Z, e_i) + R(\xi, Y, Z, \xi)
\]

\[
= \text{Ric}^h(Y, Z) + g(\varphi Y, \varphi Z) + g(Y, Z) = \text{Ric}^h(Y, Z),
\]

where \( \text{Ric}^h \) is the Ricci tensor of \( h = g|_{\mathcal{H}} \).
It follows from Proposition 8.5 that the curvature tensor in the direction of $\xi$ on a Sasaki-like almost contact complex Riemannian manifold is completely determined by $\eta, \varphi, g, \tilde{g}$. Indeed, using the properties of the Riemannian curvature, we derive from (8.23) that
\[
R(x, y, \xi, z) = R(\xi, z, x, y) = \eta(y)g(x, z) - \eta(x)g(y, z).
\]
Now, the equation (8.25) implies that the Riemannian curvature of a Sasaki-like almost contact complex Riemannian manifold is completely determined by the curvature of the underlying holomorphic complex Riemannian manifold $(\mathcal{M}^{2n}, J, h)$, where $TM^{2n} = \mathcal{H}$.

8.3.2. Example 3: $S^1$-solvable extension of the h-sphere

The next example illustrates Theorem 8.4. We consider $\mathbb{R}^{2n+2}$ for $n > 2$ as a flat holomorphic complex Riemannian manifold, i.e. $\mathbb{R}^{2n+2}$ is equipped with the canonical complex structure $J'$ and the canonical Norden metrics $h'$ and $\tilde{h}'$ defined by
\[
h'(x', y') = \sum_{i=1}^{n+1} (x^i y^i - x^{n+i+1} y^{n+i+1}),
\]
\[
\tilde{h}'(x', y') = -\sum_{i=1}^{n+1} (x^i y^{n+i+1} + x^{n+i+1} y^i)
\]
for the vectors $x' = (x^1, \ldots, x^{2n+2})$ and $y' = (y^1, \ldots, y^{2n+2})$ in $\mathbb{R}^{2n+2}$. Identifying the point $z' = (z^1, \ldots, z^{2n+2})$ in $\mathbb{R}^{2n+2}$ with the position vector $z'$, we consider the complex hypersurface $S^{2n}_h(z'_0; a, b)$ defined by the equations
\[
h'(z' - z'_0, z' - z'_0) = a, \quad \tilde{h}'(z' - z'_0, z' - z'_0) = b,
\]
where $(0, 0) \neq (a, b) \in \mathbb{R}^2$.

The codimension two submanifold $S^{2n}_h(z'_0; a, b)$ is $J'$-invariant and the restriction of $h'$ on $S^{2n}_h(z'_0; a, b)$ has rank $2n$ due to the condition $(a, b) \neq (0, 0)$. The holomorphic complex Riemannian structure on $\mathbb{R}^{2n+2}$ inherits a holomorphic complex Riemannian structure $(J'|_{S^{2n}_h}, h'|_{S^{2n}_h})$ on the complex hypersurface $S^{2n}_h(z'_0; a, b)$. The holomorphic complex Riemannian manifold $(S^{2n}_h(z'_0; a, b), J'|_{S^{2n}_h}, h'|_{S^{2n}_h})$ is sometimes called an h-sphere with center $z'_0$ and a pair of parameters $(a, b)$. The h-sphere $S^{2n}_h(z'_0; 1, 0)$ is the sphere of Kotel’nikov-Study [147]. The curvature of an h-sphere is
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given by the formula (35)

\begin{equation}
R'|_{S_{h}^{2n}} = \frac{1}{a^2 + b^2} \{a(\pi_1 - \pi_2) - b\pi_3\},
\end{equation}

where

\begin{align*}
\pi_1 &= \frac{1}{2} h'|_{S_{h}^{2n}} \otimes h'|_{S_{h}^{2n}}, \\
\pi_2 &= \frac{1}{2} \tilde{h}'|_{S_{h}^{2n}} \otimes \tilde{h}'|_{S_{h}^{2n}}, \\
\pi_3 &= -h'|_{S_{h}^{2n}} \otimes \tilde{h}'|_{S_{h}^{2n}}
\end{align*}

and \( \otimes \) stands for the Kulkarni-Nomizu product of two (0,2)-tensors (see (1.37)). Consequently, we have

\begin{align}
\text{Ric}'|_{S_{h}^{2n}} &= \frac{2(n-1)}{a^2 + b^2} (ah'|_{S_{h}^{2n}} + b\tilde{h}'|_{S_{h}^{2n}}), \\
\text{Scal}'|_{S_{h}^{2n}} &= \frac{4n(n-1)a}{a^2 + b^2}.
\end{align}

The product manifold \( \mathcal{M}^{2n+1} = \mathbb{R}^+ \times S_{h}^{2n}(z_0; a, b) \) equipped with the following almost contact complex Riemannian structure

\begin{align*}
\eta &= dt, \\
\varphi|_\mathcal{H} &= J'|_{S_{h}^{2n}}, \\
\eta \circ \varphi &= 0,
\end{align*}

\[ g = dt^2 + \cos 2t \ h'|_{S_{h}^{2n}} - \sin 2t \ \tilde{h}'|_{S_{h}^{2n}} \]

is a Sasaki-like almost contact complex Riemannian manifold according to Theorem 8.4.

The horizontal metrics on \( \mathcal{M}^{2n+1} = \mathbb{R}^+ \times S_{h}^{2n}(z_0; a, b) \) are

\begin{align}
\begin{split}
h &= g|_\mathcal{H} = \cos 2t \ h'|_{S_{h}^{2n}} - \sin 2t \ \tilde{h}'|_{S_{h}^{2n}}, \\
\tilde{h} &= \tilde{g}|_\mathcal{H} = \sin 2t \ h'|_{S_{h}^{2n}} + \cos 2t \ \tilde{h}'|_{S_{h}^{2n}}.
\end{split}
\end{align}

The Levi-Civita connection \( \nabla' \) of the metric \( h'|_{S_{h}^{2n}} \) coincides with the Levi-Civita connection of \( \tilde{h}'|_{S_{h}^{2n}} \) since \( \nabla'J' = 0 \). Using this fact, the Koszul formula (5.18) together with (8.29) gives for the Levi-Civita connection \( D \) of \( h \) the expression

\[ h(D_X Y, Z) = \cos 2t \ h'|_{S_{h}^{2n}} (\nabla'_X Y, Z) - \sin 2t \ h'|_{S_{h}^{2n}} (\nabla'_X Y, JZ) \]

which implies \( D_X Y = \nabla'_X Y \). The latter equality together with (8.29) yields for the curvature of \( h \) the formula

\[ R^h = \cos 2t \ R'|_{S_{h}^{2n}} - \sin 2t \ \tilde{R}'|_{S_{h}^{2n}}, \]

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where \( \tilde{R}'|_{S^{2n}_h} := J'R'|_{S^{2n}_h} \). The above equality together with (8.27) implies

\[
R^h = \frac{1}{a^2 + b^2} \left\{ \cos 2t[a(\pi_1 - \pi_2) - b\pi_3] - \sin 2t[-a\pi_3 - b(\pi_1 - \pi_2)] \right\}
\]

(8.30)

\[
= \frac{1}{a^2 + b^2} \left\{ (a \cos 2t + b \sin 2t)(\pi_1 - \pi_2) - (b \cos 2t - a \sin 2t)\pi_3 \right\}.
\]

Taking into account (8.25), (8.29) and (8.30), we obtain that the horizontal curvature \( R|_H \) of the Sasaki-like almost contact complex Riemannian manifold \( M^{2n+1} = \mathbb{R}^+ \times S^{2n}_h(z'_0; a, b) \) is given by the formula

\[
R|_H = R^h - (\sin 2t)^2 \pi_1 - (\cos 2t)^2 \pi_2 + \sin 2t \cos 2t \pi_3
\]

\[
= \left\{ \frac{1}{a^2 + b^2} (a \cos 2t + b \sin 2t) - (\sin 2t)^2 \right\} \pi_1
\]

\[
- \left\{ \frac{1}{a^2 + b^2} (a \cos 2t + b \sin 2t) + (\cos 2t)^2 \right\} \pi_2
\]

\[
- \left\{ \frac{1}{a^2 + b^2} (b \cos 2t - a \sin 2t) - \sin 2t \cos 2t \right\} \pi_3.
\]

For the horizontal Ricci tensor we obtain from (8.26), (8.28) and (8.29) the formula

\[
\text{Ric}|_H = \text{Ric}^h = \frac{2(n - 1)}{a^2 + b^2} \left( ah'|_{S^{2n}_h} + b\tilde{h}'|_{S^{2n}_h} \right)
\]

\[
= \frac{2(n - 1)}{a^2 + b^2} \left[ (a \cos 2t - b \sin 2t)h + (b \cos 2t + a \sin 2t)\tilde{h} \right].
\]

### 8.4. Contact conformal (homothetic) transformations

In this subsection we investigate when the Sasaki-like condition is preserved under contact conformal transformations. We recall that a general contact conformal transformation of an almost contact complex Riemannian manifold \( (M, \varphi, \xi, \eta, g) \) is defined by (5.17) [78, 95, 96]. If the functions \( u, v, w \) are constant we have a contact homothetic transformation.

A relation between the tensors \( \overline{F} \) and \( F \) is given in [78], see also (5.20),
\[ 2F(x, y, z) = 2e^{2u} \cos 2v F(x, y, z) \]
\[ + 2e^{2u} \eta(x) \left[ \eta(y) dw(\varphi z) + \eta(z) dw(\varphi y) \right] \]
\[ + e^{2u} \sin 2v \left[ F(\varphi y, z, x) - F(y, \varphi z, x) \right. \]
\[ \left. + F(\varphi z, y, x) - F(z, \varphi y, x) \right] \]
\[ + F(x, \varphi y, \xi) \eta(z) + F(x, \varphi z, \xi) \eta(y) \]
\[ + (e^2 - e^{2u} \cos 2v) \left\{ \left[ F(x, y, \xi) + F(\varphi y, \varphi x, \xi) \right] \eta(z) \right. \]
\[ \left. + \left[ F(x, z, \xi) + F(\varphi z, \varphi x, \xi) \right] \eta(y) \right\} \]
\[ + (F(y, z, \xi) + F(\varphi z, \varphi y, \xi)) \eta(x) \]
\[ + \left[ F(z, y, \xi) + F(\varphi y, \varphi z, \xi) \right] \eta(x) \}
\[ - 2e^{2u} \left\{ \left[ \cos 2v [du(\varphi z) + dv(z)] \right. \right. \]
\[ \left. + \sin 2v [du(z) - dv(\varphi z)] \right] g(x, \varphi y) \]
\[ + \left[ \cos 2v [du(\varphi y) + dv(y)] \right. \]
\[ + \sin 2v [du(y) - dv(\varphi y)] \right] g(\varphi x, \varphi z) \]
\[ + \left[ \cos 2v [du(z) - dv(\varphi z)] \right. \]
\[ - \sin 2v [du(\varphi z) + dv(z)] \right] g(x, \varphi y) \]
\[ + \left[ \cos 2v [du(y) - dv(\varphi y)] \right. \]
\[ - \sin 2v [du(\varphi y) + dv(y)] \right] g(x, \varphi z) \} \]
\[ (8.31a) \]

The Sasaki-like condition \((8.11)\) also reads as
\[ F(x, y, z) = g(\varphi x, \varphi y) \eta(z) + g(\varphi x, \varphi z) \eta(y). \]
\[ (8.32) \]

We obtain the Sasaki-like condition for the metric \(\overline{g}\) substituting \((5.17)\) into \((8.32)\) which yields
\[ \overline{F}(x, y, z) = e^{u+2u} \left\{ \cos 2v \left[ \eta(z)g(\varphi x, \varphi y) + \eta(y)g(\varphi x, \varphi z) \right] \right. \]
\[ - \sin 2v \left[ \eta(z)g(x, \varphi y) + \eta(y)g(x, \varphi z) \right] \} \cdot \]
\[ (8.33) \]

We substitute \((8.32)\) into \((8.31)\) to get the following expression.
\[
\overline{F}(x, y, z) = e^{2w} \eta(x) \left\{ \eta(y) dw(\varphi z) + \eta(z) dw(\varphi y) \right\} \\
+ e^{2u} \left\{ \cos 2v [\eta(z) g(\varphi x, \varphi y) + \eta(y) g(\varphi x, \varphi z)] \\
- \sin 2v [\eta(z) g(x, \varphi y) + \eta(y) g(x, \varphi z)] \\
- \{ \cos 2v [du(\varphi z) + dv(z)] \\
+ \sin 2v [du(z) - dv(\varphi z)] \} g(\varphi z, \varphi y) \right\}
\]

(8.34)

\[
- \{ \cos 2v [du(\varphi y) + dv(y)] \\
+ \sin 2v [du(y) - dv(\varphi y)] \} g(x, \varphi z) \\
- \{ \cos 2v [du(z) - dv(\varphi z)] \\
- \sin 2v [du(\varphi z) + dv(z)] \} g(x, \varphi y) \\
- \{ \cos 2v [du(y) - dv(\varphi y)] \\
- \sin 2v [du(\varphi y) + dv(y)] \} g(x, \varphi z) \right\}.
\]

The equalities \((8.34)\) and \((8.33)\) imply

\[
(1 - e^w) e^{2u} \left\{ \cos 2v [\eta(z) g(\varphi x, \varphi y) + \eta(y) g(\varphi x, \varphi z)] \\
- \sin 2v [\eta(z) g(x, \varphi y) + \eta(y) g(x, \varphi z)] \right\} \\
- e^{2u} \left\{ \cos 2v [du(\varphi z) + dv(z)] \\
+ \sin 2v [du(z) - dv(\varphi z)] \} g(\varphi x, \varphi y) \\
+ \{ \cos 2v [du(\varphi y) + dv(y)] \\
+ \sin 2v [du(y) - dv(\varphi y)] \} g(\varphi x, \varphi z) \\
+ \{ \cos 2v [du(z) - dv(\varphi z)] \\
- \sin 2v [du(\varphi z) + dv(z)] \} g(x, \varphi y) \right\}
\]

(8.35a)
\begin{align}
+ \{ & \cos 2v [du(y) - dv(\varphi y)] \\
& - \sin 2v [du(\varphi y) + dv(y)] \} g(x, \varphi z) \\
+ e^{2w} & \eta(x) \{ \eta(y)dw(\varphi z) + \eta(z)dw(\varphi y) \} = 0.
\end{align}

We set \( x = y = \xi \) into (8.35) to get

\begin{equation}
(8.36)
dw(\varphi z) = 0.
\end{equation}

Now, using (8.36) we write (8.35) in the form

\begin{equation}
(8.37)
A(z)g(\varphi x, \varphi y) + B(z)g(x, \varphi y) + A(y)g(\varphi x, \varphi z) + B(y)g(x, \varphi z) = 0,
\end{equation}

where the 1-forms \( A \) and \( B \) are defined by

\begin{equation}
(8.38)
A(z) = \cos 2v [(e^w - 1)\eta(z) + du(\varphi z) + dv(z)] \\
+ \sin 2v [du(z) - dv(\varphi z)],
\end{equation}

\begin{equation}
B(z) = \sin 2v [(e^w - 1)\eta(z) + du(\varphi z) + dv(z)] \\
- \cos 2v [du(z) - dv(\varphi z)].
\end{equation}

We take the trace of (8.37) with respect to \( x = e_i, \ z = e_i \) and \( y = e_i, \ z = e_i \) to get

\begin{equation}
(8.39)
-(2n + 1)A(z) + \eta(z)A(\xi) + B(\varphi z) = 0,
\end{equation}

\begin{equation}
A(z) - \eta(z)A(\xi) - B(\varphi z) = 0.
\end{equation}

We derive from (8.39) that \( A(z) = 0 \). Similarly, we obtain \( B(z) = 0 \).

Now, (8.38) imply

\begin{equation}
\begin{aligned}
& \cos 2v [du(\varphi z) + dv(z)] \\
+ & \sin 2v [du(z) - dv(\varphi z)] = (1 - e^w) \cos 2v \eta(z), \\
& \sin 2v [du(\varphi z) + dv(z)] \\
- & \cos 2v [du(z) - dv(\varphi z)] = (1 - e^w) \sin 2v \eta(z).
\end{aligned}
\end{equation}

Then we derive

**Proposition 8.6.** Let \( (\mathcal{M}, \varphi, \xi, \eta, g) \) be a Sasaki-like almost contact complex Riemannian manifold. Then the structure \( (\varphi, \xi, \eta, g) \) defined by (5.17) is Sasaki-like if and only if the smooth functions \( u, v, w \) satisfy the following conditions

\begin{equation}
(8.41)
dw \circ \varphi = 0, \quad du - dv \circ \varphi = 0, \quad du \circ \varphi + dv = (1 - e^w)\eta.
\end{equation}
In particular
\[ du(\xi) = 0, \quad dv(\xi) = 1 - e^w. \]
In the case \( w = 0 \) the global smooth functions \( u \) and \( v \) does not depend on \( \xi \) and they are locally defined on the complex submanifold \( \mathcal{M}^{2n} \), \( T\mathcal{M}^{2n} = \mathcal{H} \), as well as the complex valued function \( u + \sqrt{-1} v \) is a holomorphic function on \( \mathcal{M}^{2n} \).

Proof. Solve the linear system (8.40) to get the second and the third equality into (8.41). Now, (8.36) completes the proof of (8.41). \( \square \)

### 8.4.1. Contact homothetic transformations

Let us consider contact homothetic transformations of an almost contact complex Riemannian manifold \( (\mathcal{M}, \varphi, \xi, \eta, g) \). Since the functions \( u, v, w \) are constant, it follows from (5.17) using the Koszul formula (5.18) that the Levi-Civita connections \( \nabla \) and \( \nabla \) of the metrics \( \overline{g} \) and \( g \), respectively, are connected by the formula

\begin{equation}
\nabla_x y = \nabla_x y + e^{2(u-w)} \sin 2v \; g(\varphi x, \varphi y) \xi \\
- \left( e^{-2w} - e^{2(u-w)} \cos 2v \right) g(x, \varphi y) \xi.
\end{equation}

(8.42)

For the corresponding curvature tensors \( \overline{R} \) and \( R \) we obtain from (8.42) that

\begin{equation}
\overline{R}(x, y) z = R(x, y) z \\
+ e^{2(u-w)} \sin 2v \{ g(y, \varphi z) \eta(x) \xi - g(\varphi y, \varphi z) \varphi x \\
- g(x, \varphi z) \eta(y) \xi + g(\varphi x, \varphi z) \varphi y \} \\
+ \left( e^{-2w} - e^{2(u-w)} \cos 2v \right) \{ g(\varphi y, \varphi z) \eta(x) \xi + g(y, \varphi z) \varphi x \\
- g(\varphi x, \varphi z) \eta(y) \xi - g(x, \varphi z) \varphi y \}.
\end{equation}

(8.43)

We have

**Proposition 8.7.** The Ricci tensor of an almost contact complex Riemannian manifold is invariant under a contact homothetic transformation,

\begin{equation}
\overline{\text{Ric}} = \text{Ric}.
\end{equation}

(8.44)

Consequently, we obtain the following relations:
Scal = $e^{-2u} \cos 2v \, \text{Scal} - e^{-2u} \sin 2v \, \text{Scal}^*$

+ \left( e^{-2w} - e^{-2u} \cos 2v \right) \text{Ric}(\xi, \xi),

(8.45)

\text{Scal}^* = e^{-2u} \sin 2v \, \text{Scal} + e^{-2u} \cos 2v \, \text{Scal}^*

- e^{-2u} \sin 2v \, \text{Ric}(\xi, \xi).

In particular, the scalar curvatures of a Sasaki-like almost contact complex Riemannian manifold changes under a contact homothetic transformation with $w = 0$ as follows

\text{Scal} = e^{-2u} \cos 2v \, \text{Scal} - e^{-2u} \sin 2v \, \text{Scal}^*

+ 2n \left( 1 - e^{-2u} \cos 2v \right),

(8.46)

\text{Scal}^* = e^{-2u} \sin 2v \, \text{Scal} + e^{-2u} \cos 2v \, \text{Scal}^*

- 2n \, e^{-2u} \sin 2v.

Proof. Taking the trace of (8.43) we get $\text{Ric} = \text{Ric}$.

We consider the basis $\{\overline{\xi} = \xi, \overline{\xi}_1, \ldots, \overline{\xi}_n, \overline{\xi}_{n+1} = \varphi \overline{\xi}_1, \ldots, \overline{\xi}_{2n} = \varphi \overline{\xi}_n\}$, where

$\overline{\xi}_i = e^{-u} \{\cos v \, e_i - \sin v \, \varphi e_i\}, \quad i = 1, \ldots, n.$

It is easy to check that this basis is orthonormal for the metric $\overline{g}$. Then (8.44) gives

$$\overline{\text{Scal}} = \sum_{i=0}^{2n} \overline{\varepsilon}_i \text{Ric}(\overline{\xi}_i, \overline{\xi}_i), \quad \overline{\text{Scal}}^* = \sum_{i=0}^{2n} \overline{\varepsilon}_i \text{Ric}(\overline{\xi}_i, \varphi \overline{\xi}_i),$$

which yield the formulas for the scalar curvatures.

The formulas (8.46) follow from (8.45) and (8.24). \qed

Consequently, we have

Proposition 8.8. A Sasaki-like almost contact complex Riemannian manifold $(\mathcal{M}, \varphi, \xi, \eta, g)$ is Einstein if and only if the underlying holomorphic complex Riemannian manifold $(\mathcal{M}^{2n}, TM^{2n} = \mathcal{H}, J, h)$ is an Einstein manifold with scalar curvature not depending on the vertical direction $\xi$.

Proof. We compare (8.24) with (8.26) to see that $(\mathcal{M}, \varphi, \xi, \eta, g)$ is an Einstein manifold if and only if $\mathcal{M}^{2n}$ is an Einstein manifold with Einstein constant equal to $2n$, i.e. $\text{Ric}^h = 2ng$. 

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Further, we consider a contact homothetic transformation with \( w = v = 0 \) and we get that \( (\mathcal{M}, \varphi, \xi, \eta, \overline{\mathcal{G}} = e^{2u}g + (1 - e^{2u})\eta \otimes \eta) \) is again Sasaki-like due to Proposition 8.6. Applying Proposition 8.7 and (8.26), we get the following sequence of equalities

\[
\overline{\operatorname{Ric}} = \overline{\operatorname{Ric}}|_{\mathcal{H}} = \operatorname{Ric}|_{\mathcal{H}} = \operatorname{Ric} = \frac{\operatorname{Scal}}{2n}g|_{\mathcal{H}} = \frac{e^{-2u}\operatorname{Scal}}{2n}g|_{\mathcal{H}},
\]

which yield \( \overline{\operatorname{Scal}} = e^{-2u}\operatorname{Scal} = 4n^2 \) by choosing the constant \( u \) to be equal to \( e^{-2u} = \frac{4n^2}{\operatorname{Scal}} \), i.e. the Einstein constant of the complex holomorphic Einstein manifold \( \mathcal{M}^{2n} \) can always be made equal to \( 4n^2 \) which completes the proof.

Suppose we have a Sasaki-like almost contact complex Riemannian manifold which is Einstein, i.e. \( \operatorname{Ric} = 2ng \), and we make a contact homothetic transformation

\[
\overline{\eta} = \eta, \quad \overline{\xi} = \xi, \quad \overline{g}(x, y) = c g(x, y) + d g(x, \varphi y) + (1 - c)\eta(x)\eta(y),
\]

where \( c, d \) are constants. According to Proposition 8.7 and using (8.47), we obtain that

\[
\overline{\operatorname{Ric}}(x, y) = \operatorname{Ric}(x, y) = 2ng(x, y)
\]

\[
= \frac{2n}{c^2 + d^2} \{c \overline{g}(x, y) - d \overline{g}(x, \varphi y) + (c^2 + d^2 - c)\eta(x)\eta(y)\}.
\]

We may call a Sasaki-like space whose Ricci tensor satisfies (8.48) an \( \eta \)-complex-Einstein Sasaki-like manifold and if the constant \( d \) vanishes, \( d = 0 \), we have \( \eta \)-Einstein Sasaki-like space. Thus, we have shown

**Proposition 8.9.** Any \( \eta \)-complex-Einstein Sasaki-like space is contact homothetic to an Einstein Sasaki-like space.
Chapter II.

On manifolds with almost hypercomplex structures and almost contact 3-structures, equipped with metrics of Hermitian-Norden type
§9. Almost hypercomplex manifolds with Hermitian-Norden metrics

In the present section, we give some facts about the almost hypercomplex manifolds with Hermitian-Norden metrics known from [2, 46, 47, 82].

Let us recall the notion of almost hypercomplex structure $H$ on a manifold $\mathcal{M}^{4n}$. This structure is a triad of anticommuting almost complex structures such that each of them is a composition of two other structures [2, 134].

We equip an almost hypercomplex structure $H$ with a metric structure, generated by a pseudo-Riemannian metric $g$ of neutral signature [46, 47]. In our case, one (resp., the other two) of the almost complex structures of $H$ acts as an isometry (resp., act as anti-isometries) with respect to $g$ in each tangent fibre. Thus, there exist three $(0,2)$-tensors associated by $H$ to the metric $g$: a Kähler form and two metrics of the same type. The metric $g$ is Hermitian with respect to one of almost complex structures of $H$ and $g$ is a Norden metric regarding the other two almost complex structures of $H$. For this reason we call the derived almost hypercomplex structure an almost hypercomplex structure with Hermitian-Norden metrics.

Let $(\mathcal{M}, H)$ be an almost hypercomplex manifold, i.e. $\mathcal{M}$ is a $4n$-dimensional differentiable manifold and $H = (J_1, J_2, J_3)$ is a triad of almost complex structures on $\mathcal{M}$ with the following properties for all cyclic permutations $(\alpha, \beta, \gamma)$ of $(1, 2, 3)$:

\[
(9.1) \quad J_\alpha = J_\beta \circ J_\gamma = -J_\gamma \circ J_\beta, \quad J_\alpha^2 = -I,
\]

where $I$ denotes the identity.

Let $g$ be a neutral metric on $(\mathcal{M}, H)$ with the properties

\[
(9.2) \quad g(\cdot, \cdot) = \varepsilon_\alpha g(J_\alpha \cdot, J_\alpha \cdot),
\]
where

\[ \varepsilon_\alpha = \begin{cases} 1, & \alpha = 1; \\ -1, & \alpha = 2; 3. \end{cases} \]  

Further, the index \( \alpha \) runs over the range \( \{1, 2, 3\} \) unless otherwise stated.

The associated (Kähler) 2-form \( \tilde{g}_1 \) and the associated neutral metrics \( \tilde{g}_2 \) and \( \tilde{g}_3 \) are determined by

\[ \tilde{g}_\alpha (\cdot, \cdot) = g(J_\alpha \cdot, \cdot) = -\varepsilon_\alpha g(\cdot, J_\alpha \cdot). \]  

Let us note that \( J_1 \) (resp., \( J_3 \), \( J_2 \)) acts as an isometry with respect to \( g \) (resp., \( \tilde{g}_2 \), \( \tilde{g}_3 \)) and moreover \( J_2 \) and \( J_3 \) (resp., \( J_1 \) and \( J_2 \), \( J_1 \) and \( J_3 \)) act as anti-isometries with respect to \( g \) (resp., \( \tilde{g}_2 \), \( \tilde{g}_3 \)). On the other hand, a quaternionic inner product \( \langle \cdot, \cdot \rangle \) is generated in a natural way by the bilinear forms \( g, \tilde{g}_1, \tilde{g}_2, \tilde{g}_3 \) by the following decomposition:

\[ \langle \cdot, \cdot \rangle = -g + ig_1 + j\tilde{g}_2 + k\tilde{g}_3. \]

We call the structure \((H, G) = (J_1, J_2, J_3; g, \tilde{g}_1, \tilde{g}_2, \tilde{g}_3)\) on \( \mathcal{M}^{4n} \) an almost hypercomplex structure with Hermitian-Norden metrics. We call the manifold \((\mathcal{M}, H, G)\) an almost hypercomplex manifold with Hermitian-Norden metrics. These manifolds are introduced and studied in [47, 81, 103, 46, 82, 97, 117].

According to [47], the fundamental tensors of such a manifold are the following three \((0, 3)\)-tensors

\[ F_\alpha (x, y, z) = g((D_x J_\alpha) y, z) = (D_x \tilde{g}_\alpha) (y, z), \]

where \( D \) is the Levi-Civita connection generated by \( g \). These tensors have the following basic properties caused by the structures

\[ F_\alpha (x, y, z) = -\varepsilon_\alpha F_\alpha (x, z, y) = -\varepsilon_\alpha F_\alpha (x, J_\alpha y, J_\alpha z). \]

The following relations between the tensors \( F_\alpha \) are valid

\[ F_1 (x, y, z) = F_2 (x, J_3 y, z) + F_3 (x, y, J_2 z), \]

\[ F_2 (x, y, z) = F_3 (x, J_1 y, z) + F_1 (x, y, J_3 z), \]

\[ F_3 (x, y, z) = F_1 (x, J_2 y, z) - F_2 (x, y, J_1 z). \]

The corresponding Lee forms \( \theta_\alpha \) are defined by

\[ \theta_\alpha (\cdot) = g^{ij} F_\alpha (e_i, e_j, \cdot) \]

for an arbitrary basis \( \{e_1, e_2, \ldots, e_{4n}\} \) of \( T_p \mathcal{M}, p \in \mathcal{M} \).
In [47], we study the so-called hyper-Kähler manifolds with Hermitian-Norden metrics, i.e. the almost hypercomplex manifold with Hermitian-Norden metrics in the class $\mathcal{K}$, where $D J_\alpha = 0$ for all $\alpha$. A sufficient condition for $(\mathcal{M}, H, G)$ to be in $\mathcal{K}$ is this manifold to be of Kähler-type with respect to two of the three complex structures of $H$ [46].

As $g$ is an indefinite metric, there exist isotropic vectors $x$ on $\mathcal{M}$, i.e. $g(x, x) = 0$, $x \neq 0$. In [46], we define the invariant square norm
\[
\|D J_\alpha\|^2 = g^{ij} g^{kl} g((D_i J_\alpha) e_k, (D_j J_\alpha) e_l),
\]
where $\{e_1, e_2, \ldots, e_{4n}\}$ is an arbitrary basis of the tangent space $T_p \mathcal{M}$ at an arbitrary point $p \in \mathcal{M}$. We say that an almost hypercomplex manifold with Hermitian-Norden metrics is an isotropic hyper-Kähler manifold with Hermitian-Norden metrics if $\|D J_\alpha\|^2 = 0$ for each $J_\alpha$ of $H$. Clearly, if the manifold is a hyper-Kähler manifold with Hermitian-Norden metrics, then it is an isotropic hyper-Kähler manifold with Hermitian-Norden metrics. The inverse statement does not hold.

Let us note that according to (9.2) the manifold $(\mathcal{M}, J_1, g)$ is almost Hermitian whereas the manifolds $(\mathcal{M}, J_\alpha, g)$, $\alpha = 2, 3$, are almost Norden manifolds. The basic classes of the mentioned two types of manifolds are given in [44] by A. Gray, L.M. Hervella and in [34] by G. Ganchev, A. Borisov, respectively. They are determined for dimension $4n$ as follows:

a) for $\alpha = 1$
\[
\mathcal{W}_1(J_1) : F_1(x, y, z) = -F_1(y, x, z);
\]
\[
\mathcal{W}_2(J_1) : \bigotimes_{x,y,z} \{F_1(x, y, z)\} = 0;
\]
\[
\mathcal{W}_3(J_1) : F_1(x, y, z) = F_1(J_1 x, J_1 y, z), \quad \theta_1 = 0;
\]
\[
\mathcal{W}_4(J_1) : F_1(x, y, z) = \frac{1}{4n - 2} \{g(x, y)\theta_1(z) - g(x, z)\theta_1(y) - g(x, J_1 y)\theta_1(J_1 z) + g(x, J_1 z)\theta_1(J_1 y)\}
\]

b) for $\alpha = 2$ or $3$
\[
\mathcal{W}_1(J_\alpha) : F_\alpha(x, y, z) = \frac{1}{4n} \{g(x, y)\theta_\alpha(z) + g(x, z)\theta_\alpha(y) + g(x, J_\alpha y)\theta_\alpha(J_\alpha z)\}
\]
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\[ + g(x, J_\alpha z) \theta_\alpha(J_\alpha y) \}\];

\[(9.11b)\]

\[ \mathcal{W}_2(J_\alpha) : \mathcal{G}_{x,y,z} \{ F_\alpha(x,y,J_\alpha z) \} = 0, \quad \theta_\alpha = 0; \]

\[ \mathcal{W}_3(J_\alpha) : \mathcal{G}_{x,y,z} \{ F_\alpha(x,y,z) \} = 0. \]

The special class \( \mathcal{W}_0(J_\alpha) : F_\alpha = 0 \) of the Kähler-type manifolds belongs to any other class within the corresponding classification.

In the 4-dimensional case, the four basic classes of almost Hermitian manifolds with respect to \( J_1 \) are restricted to two: \( \mathcal{W}_2(J_1) \), the class of the almost Kähler manifolds, and \( \mathcal{W}_4(J_1) \), the class of the Hermitian manifolds.

It is known that the class of complex manifolds with Hermitian metric for \( J_1 \) is \( (\mathcal{W}_3 \oplus \mathcal{W}_4)(J_1) \) and the class of complex manifolds with Norden metric for \( J_\alpha (\alpha = 2, 3) \) is \( (\mathcal{W}_1 \oplus \mathcal{W}_2)(J_\alpha) \).

By definition, an almost hypercomplex structure \( H = (J_\alpha) \) is a hypercomplex structure if the Nijenhuis tensors \([J_\alpha, J_\alpha] \), given by

\[(9.12)\]

\[ [J_\alpha, J_\alpha](\cdot, \cdot) = [J_\alpha, J_\alpha]\cdot - J_\alpha [J_\alpha, \cdot] - J_\alpha [\cdot, J_\alpha] - [\cdot, \cdot], \]

vanish on \( \mathcal{X}(\mathcal{M}) \) for each \( \alpha \). Moreover, it is known that \( H \) is hypercomplex if and only if two of \([J_\alpha, J_\alpha]\) vanish.

Then the class \( \mathcal{HC} \) of hypercomplex manifolds with Hermitian-Norden metrics is

\[ (\mathcal{W}_3 \oplus \mathcal{W}_4)(J_1) \cap (\mathcal{W}_1 \oplus \mathcal{W}_2)(J_2) \cap (\mathcal{W}_1 \oplus \mathcal{W}_2)(J_3), \]

which for the 4-dimensional case is restricted to

\[ \mathcal{W}_4(J_1) \cap (\mathcal{W}_1 \oplus \mathcal{W}_2)(J_2) \cap (\mathcal{W}_1 \oplus \mathcal{W}_2)(J_3). \]

Let \( R \) be the curvature tensor of the Levi-Civita connection \( D \), generated by \( g \). Obviously, \( R \) is a Kähler-type tensor on an arbitrary hyper-Kähler manifold with Hermitian-Norden metrics, i.e.

\[(9.13)\]

\[ R(x, y, z, w) = \varepsilon_\alpha R(x, y, J_\alpha z, J_\alpha w) = \varepsilon_\alpha R(J_\alpha x, J_\alpha y, z, w). \]

A basic property of the hyper-Kähler manifolds with Hermitian-Norden metrics is given in [47] by the following

**Theorem 9.1** ([47]). Each hyper-Kähler manifold with Hermitian-Norden metrics is a flat pseudo-Riemannian manifold of signature \((2n, 2n)\).

In [82], it is proved the following more general property.
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Theorem 9.2 ([82]). Each Kähler-type tensor on an almost hypercomplex manifold with Hermitian-Norden metrics is zero.
§10. Hypercomplex structures with Hermitian-Norden metrics on 4-dimensional Lie algebras

In the present section, integrable hypercomplex structures with Hermitian-Norden metrics on Lie groups of dimension 4 are considered. The corresponding five types of invariant hypercomplex structures with hyper-Hermitian metric are constructed here. The different cases regarding the signature of the basic pseudo-Riemannian metric are considered. The main results of this section are published in [86].

The study in the present section is inspired by the work of M.L. Barberis [6] where invariant hypercomplex structures \( H \) on 4-dimensional real Lie groups are classified. In that case the corresponding metric is positive definite and Hermitian with respect to the triad of complex structures of \( H \). Our main goal is to classify 4-dimensional real Lie algebras which admit hypercomplex structures with Hermitian-Norden metrics.

Let us remark that in [132] and [126] are classified the invariant complex structures on 4-dimensional solvable simply-connected real Lie groups where the dimension of commutators is less than three and equal to three, respectively.

A hypercomplex structure is called Abelian, if \( [J_{\alpha}, J_{\alpha}] = [\cdot, \cdot] \) is valid for arbitrary vector fields and for all \( \alpha \) [8]. Abelian hypercomplex structures are considered in [7, 27] and they can occur only on solvable Lie algebras [30]. It is clear that the condition

\[
[J, J](x, y) = [Jx, Jy] - J[Jx, y] - J[x, Jy] - [x, y] = 0
\]

can be rewritten as

\[
[Jx, Jy] - [x, y] = J([Jx, y] + [x, Jy])
\]

for all vector fields \( x, y \). Thus, Abelian complex structures and therefore Abelian hypercomplex structure are integrable.
In the present section we construct different types of hypercomplex structures on Lie algebras following the Barberis classification. The basic problem here is the existence and the geometric characteristics of hypercomplex structures with Hermitian-Norden metrics on 4-dimensional Lie algebras according to the Barberis classification. The main results of this section are the construction of the different types of the considered structures and their characterization.

A hypercomplex structure on a Lie group is said to be invariant if left translations by elements of the Lie group are holomorphic with respect to $J_\alpha$ for all $\alpha$. Obviously, a hypercomplex structure on the corresponding Lie algebra induces an invariant hypercomplex structure on the Lie group by left translations.

Let $\mathcal{L}$ be a simply connected 4-dimensional real Lie group admitting an invariant hypercomplex structure. A left invariant metric on $\mathcal{L}$ is called invariant hyper-Hermitian if it is hyper-Hermitian with respect to some invariant hypercomplex structure on $\mathcal{L}$. It is known that all such metrics on given $\mathcal{L}$ are equivalent up to homotheties.

Let $\mathfrak{l}$ denote the corresponding Lie algebra of $\mathcal{L}$. Then it is known the following

**Theorem 10.1** ([1]). *The only 4-dimensional Lie algebras admitting an integrable hypercomplex structure are the following types:*

$(\text{hc1})$ $\mathfrak{l}$ is Abelian;

$(\text{hc2})$ $\mathfrak{l} \cong \mathbb{R} \oplus \mathfrak{so}(3)$;

$(\text{hc3})$ $\mathfrak{l} \cong \mathfrak{aff}(\mathbb{C})$;

$(\text{hc4})$ $\mathfrak{l}$ is the solvable Lie algebra corresponding to $\mathbb{R}H^4$;

$(\text{hc5})$ $\mathfrak{l}$ is the solvable Lie algebra corresponding to $\mathbb{C}H^2$, where $\mathbb{R} \oplus \mathfrak{so}(3)$ is the Lie algebra of the Lie groups $U(2)$ and $S^3 \times S^1$; $\mathfrak{aff}(\mathbb{C})$ is the Lie algebra of the affine motion group of $\mathbb{C}$, the unique 4-dimensional Lie algebra carrying an Abelian hypercomplex structure; $\mathbb{R}H^4$ is the real hyperbolic space; $\mathbb{C}H^2$ is the complex hyperbolic space.

Let $\{e_1, e_2, e_3, e_4\}$ be a basis of a 4-dimensional real Lie algebra $\mathfrak{l}$ with center $\mathfrak{z}$ and derived Lie algebra $\mathfrak{l}' = [\mathfrak{l}, \mathfrak{l}]$. A standard hypercomplex structure on $\mathfrak{l}$ is defined as in [134]:

\begin{align*}
J_1 e_1 &= e_2, & J_1 e_2 &= -e_1, & J_1 e_3 &= -e_4, & J_1 e_4 &= e_3; \\
J_2 e_1 &= e_3, & J_2 e_2 &= e_4, & J_2 e_3 &= -e_1, & J_2 e_4 &= -e_2; \\
J_3 e_1 &= -e_4, & J_3 e_2 &= e_3, & J_3 e_3 &= -e_2, & J_3 e_4 &= e_1.
\end{align*}

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Let us introduced a pseudo-Euclidian metric $g$ of neutral signature as follows
\begin{equation}
  g(x, y) = x^1y^1 + x^2y^2 - x^3y^3 - x^4y^4,
\end{equation}
where $x(x^1, x^2, x^3, x^4), y(y^1, y^2, y^3, y^4) \in \mathbb{I}$. This metric satisfies (9.2) and (9.4). Then the metric $g$ generates an almost hypercomplex structure with Hermitian-Norden metrics on $\mathbb{I}$.

Let us consider the different cases of Theorem 10.1.

10.1. Hypercomplex structure of type (hc1)

Obviously, in this case the considered manifold belongs to $\mathcal{K}$, the class of hyper-Kähler manifolds with Hermitian-Norden metrics.

10.2. Hypercomplex structure of type (hc2)

Let $\mathbb{I}$ be not solvable and let us determine it by
\begin{equation}
  [e_2, e_4] = e_3, \quad [e_4, e_3] = e_2, \quad [e_3, e_2] = e_4.
\end{equation}
In this consideration the $(+)$-unit $e_1 \in \mathbb{R}$, i.e. $g(e_1, e_1) = 1$, is orthogonal to $\mathbb{I}'$ with respect to $g$.

Then we compute the covariant derivatives in the basis with respect to the Levi-Civita connection $D$ of $g$ and the nontrivial ones are
\begin{equation}
  D_{e_2} e_3 = -\frac{3}{2}e_4, \quad D_{e_3} e_2 = -\frac{1}{2}e_4, \quad D_{e_4} e_2 = \frac{1}{2}e_3, \\
  D_{e_2} e_4 = \frac{3}{2}e_3, \quad D_{e_3} e_4 = -\frac{1}{2}e_2, \quad D_{e_4} e_3 = \frac{1}{2}e_2.
\end{equation}

By virtue of (10.4), (10.1) and (9.5), we obtain components $(F_\alpha)_{ijk} = F_\alpha(e_i, e_j, e_k), i, j, k \in \{1, 2, 3, 4\}$, as follows:
\begin{equation}
  (F_1)_{314} = - (F_1)_{323} = (F_1)_{332} = - (F_1)_{341} = \\
  = - (F_1)_{413} = - (F_1)_{424} = (F_1)_{431} = (F_1)_{442} = \frac{1}{2}, \\
  (F_2)_{214} = - (F_2)_{223} = - (F_2)_{232} = (F_2)_{241} = \frac{3}{2}, \\
  (F_2)_{412} = (F_2)_{421} = (F_2)_{434} = (F_2)_{443} = \frac{1}{2}, \\
  (F_3)_{213} = (F_3)_{224} = (F_3)_{231} = (F_3)_{242} = \frac{3}{2}, \\
  (F_3)_{312} = (F_3)_{321} = - (F_3)_{334} = - (F_3)_{343} = \frac{1}{2},
\end{equation}
and the others are zero. The only non-zero components $(\theta_i) = (\theta_i)(e_i)$, $i = 1, 2, 3, 4$, of the corresponding Lee forms are

\[(\theta_1)_2 = -1, \quad (\theta_2)_3 = -2, \quad (\theta_3)_4 = 2.\]

Using the results in (10.5), (10.6) and the classification conditions (9.10), (9.11) for dimension 4, we obtain

Proposition 10.2. The hypercomplex manifold with Hermitian-Norden metrics on a 4-dimensional Lie algebra, determined by (10.3), belongs to the largest class of the considered manifolds, i.e. $HC$, as well as this manifold does not belong to neither $W_1$ nor $W_2$ for $J_2$ and $J_3$.

The other possibility is the signature of $g$ on $\mathbb{R}$ to be $(-)$, e.g. $e_3 \in \mathbb{R}$, where $g(e_3, e_3) = -1$. By similar computations we establish the same class in the statement of Proposition 10.2.

10.3. Hypercomplex structure of type (hc3)

We analyze separately the cases of signature $(1,1)$, $(0,2)$ and $(2,0)$ of $g$ on $\mathfrak{l}'$.

10.3.1. Case 1

Firstly, we consider $g$ of signature $(1,1)$ on $\mathfrak{l}'$.

Let us determine $\mathfrak{l}$ by

\[(e_2, e_3) = [e_1, e_4] = e_2, \quad [e_2, e_1] = [e_4, e_3] = e_4.\]

Then we compute covariant derivatives and the nontrivial ones are

\[(\theta_1)_2 = -1, \quad (\theta_2)_3 = -2, \quad (\theta_3)_4 = 2.\]

Using the results in (10.5), (10.6) and the classification conditions (9.10), (9.11) for dimension 4, we obtain

Proposition 10.2. The hypercomplex manifold with Hermitian-Norden metrics on a 4-dimensional Lie algebra, determined by (10.3), belongs to the largest class of the considered manifolds, i.e. $HC$, as well as this manifold does not belong to neither $W_1$ nor $W_2$ for $J_2$ and $J_3$.

The other possibility is the signature of $g$ on $\mathbb{R}$ to be $(-)$, e.g. $e_3 \in \mathbb{R}$, where $g(e_3, e_3) = -1$. By similar computations we establish the same class in the statement of Proposition 10.2.

10.3. Hypercomplex structure of type (hc3)

We analyze separately the cases of signature $(1,1)$, $(0,2)$ and $(2,0)$ of $g$ on $\mathfrak{l}'$.

10.3.1. Case 1

Firstly, we consider $g$ of signature $(1,1)$ on $\mathfrak{l}'$.

Let us determine $\mathfrak{l}$ by

\[(e_2, e_3) = [e_1, e_4] = e_2, \quad [e_2, e_1] = [e_4, e_3] = e_4.\]

Then we compute covariant derivatives and the nontrivial ones are

\[(\theta_1)_2 = -1, \quad (\theta_2)_3 = -2, \quad (\theta_3)_4 = 2.\]

Using the results in (10.5), (10.6) and the classification conditions (9.10), (9.11) for dimension 4, we obtain

Proposition 10.2. The hypercomplex manifold with Hermitian-Norden metrics on a 4-dimensional Lie algebra, determined by (10.3), belongs to the largest class of the considered manifolds, i.e. $HC$, as well as this manifold does not belong to neither $W_1$ nor $W_2$ for $J_2$ and $J_3$.

The other possibility is the signature of $g$ on $\mathbb{R}$ to be $(-)$, e.g. $e_3 \in \mathbb{R}$, where $g(e_3, e_3) = -1$. By similar computations we establish the same class in the statement of Proposition 10.2.

10.3. Hypercomplex structure of type (hc3)

We analyze separately the cases of signature $(1,1)$, $(0,2)$ and $(2,0)$ of $g$ on $\mathfrak{l}'$.

10.3.1. Case 1

Firstly, we consider $g$ of signature $(1,1)$ on $\mathfrak{l}'$.

Let us determine $\mathfrak{l}$ by

\[(e_2, e_3) = [e_1, e_4] = e_2, \quad [e_2, e_1] = [e_4, e_3] = e_4.\]

Then we compute covariant derivatives and the nontrivial ones are

\[(\theta_1)_2 = -1, \quad (\theta_2)_3 = -2, \quad (\theta_3)_4 = 2.\]

Using the results in (10.5), (10.6) and the classification conditions (9.10), (9.11) for dimension 4, we obtain

Proposition 10.2. The hypercomplex manifold with Hermitian-Norden metrics on a 4-dimensional Lie algebra, determined by (10.3), belongs to the largest class of the considered manifolds, i.e. $HC$, as well as this manifold does not belong to neither $W_1$ nor $W_2$ for $J_2$ and $J_3$.

The other possibility is the signature of $g$ on $\mathbb{R}$ to be $(-)$, e.g. $e_3 \in \mathbb{R}$, where $g(e_3, e_3) = -1$. By similar computations we establish the same class in the statement of Proposition 10.2.
and the others are zero. The only non-zero components of the corresponding Lee forms are

\[(\theta^2)_1 = (\theta^3)_2 = 4.\]

Using that \(F_1 = 0\), the results in \((10.9)\), \((10.10)\) and the classification conditions \((9.10)\), \((9.11)\), we obtain

**Proposition 10.3.** The hypercomplex manifold with Hermitian-Norden metrics on a 4-dimensional Lie algebra, determined by \((10.7)\), belongs to the subclass of the Kähler manifolds with respect to \(J_1\) of the largest class of the considered manifolds, i.e.

\[\mathcal{W}_0(J_1) \cap (\mathcal{W}_1 \oplus \mathcal{W}_2) (J_2) \cap (\mathcal{W}_1 \oplus \mathcal{W}_2) (J_3),\]

as well as this manifold does not belong to neither \(\mathcal{W}_1\) nor \(\mathcal{W}_2\) for \(J_2\) and \(J_3\).

**10.3.2. Case 2**

Secondly, we consider \(g\) of signature \((2,0)\) on \(\mathfrak{l}'\). The case for signature \((0,2)\) is similar.

Let us determine \(\mathfrak{l}\) by

\[(10.11) \quad [e_1, e_3] = [e_4, e_2] = e_1, \quad [e_1, e_4] = [e_2, e_3] = e_2.\]

Then we compute covariant derivatives and the nontrivial ones are

\[(10.12) \quad D_{e_1} e_1 = D_{e_2} e_2 = e_3, \quad D_{e_2} e_3 = -D_{e_4} e_1 = e_2, \quad D_{e_1} e_3 = D_{e_4} e_2 = e_1.\]

By virtue of \((10.11)\), \((10.1)\) and \((9.5)\), we obtain the following components of \(F_\alpha:\)

\[(F_1)_{114} = -(F_1)_{123} = (F_1)_{132} = -(F_1)_{141} \]
\[= (F_1)_{213} = (F_1)_{224} = -(F_1)_{231} = -(F_1)_{242} = -1,\]
\[(F_2)_{111} = (F_2)_{133} = 2,\]
\[(F_2)_{212} = (F_2)_{221} = (F_2)_{234} = (F_2)_{243} = -1,\]
\[(F_3)_{222} = (F_3)_{233} = 2,\]
\[-(F_3)_{112} = -(F_3)_{121} = (F_3)_{134} = (F_3)_{143} \]
\[= (F_3)_{413} = (F_3)_{424} = (F_3)_{431} = (F_3)_{442} = -1\]
and the others are zero. The only non-zero components of the corresponding Lee forms are

\[(\theta_1)_4 = -2, \quad (\theta_2)_1 = (\theta_3)_2 = 4.\]

Using the results in (10.13), (10.14) and the classification conditions (9.10), (9.11), we obtain that the considered manifold belongs to the class

\[\mathcal{W} = \mathcal{W}_4(J_1) \cap \mathcal{W}_1(J_2) \cap \mathcal{W}_1(J_3).\]

Remark that, according to [47], necessary and sufficient conditions a 4-dimensional almost hypercomplex manifold with Hermitian-Norden metrics to be in the class \(\mathcal{W}\) are:

\[(\theta_2) \circ J_2 = (\theta_3) \circ J_3 = -2 (\theta_1) \circ J_1.\]

These conditions are satisfied bearing in mind (10.14).

Let us consider the class \(\mathcal{W}^0 = \{\mathcal{W} \mid d(\theta_1 \circ J_1) = 0\}\), which is the class of the (locally) conformally equivalent \(\mathcal{K}\)-manifolds, where a conformal transformation of the metric is given by \(\tilde{g} = e^{2u}g\) for a differentiable function \(u\) on the manifold.

Using (10.14) and (10.15), we establish that the considered manifold belongs to the subclass \(\mathcal{W}^0\).

**Proposition 10.4.** The hypercomplex manifold with Hermitian-Norden metrics on a 4-dimensional Lie algebra, determined by (10.11), belongs to the class \(\mathcal{W}^0\) of the (locally) conformally equivalent \(\mathcal{K}\)-manifolds.

### 10.4. Hypercomplex structure of type (hc4)

In this case, \(\mathfrak{I}\) is solvable and the derived Lie algebra \(\mathfrak{I}'\) is 3-dimensional and Abelian.

**10.4.1. Case 1**

Firstly, we fix \(e_1 \in \mathfrak{I}\) with \(g(e_1, e_1) = 1\) as an element orthogonal to \(\mathfrak{I}'\) with respect to \(g\). Therefore \(\mathfrak{I}\) is determined by

\[(e_1, e_2] = e_2, \quad [e_1, e_3] = e_3, \quad [e_1, e_4] = e_4.\]

Then we compute covariant derivatives and the nontrivial ones are

\[D_{e_2} e_1 = -e_2, \quad D_{e_3} e_1 = -e_3, \quad D_{e_4} e_1 = -e_4, \quad D_{e_2} e_2 = -D_{e_3} e_3 = -D_{e_4} e_4 = e_1.\]
By similar computation as in the previous cases, the components \((F_\alpha)_{ijk}\) are as follows:

\[
(F_1)_{143} = -(F_1)_{323} = (F_1)_{332} = -(F_1)_{341} = -(F_1)_{413} = -(F_1)_{424} = (F_1)_{431} = (F_1)_{442} = 1,
\]

\[
(F_2)_{311} = (F_2)_{333} = -2,
\]

\[
(F_2)_{214} = -(F_2)_{223} = -(F_2)_{232} = (F_2)_{241} = (F_2)_{412} = (F_2)_{421} = (F_2)_{434} = (F_2)_{443} = -1,
\]

\[
(F_3)_{411} = (F_3)_{444} = 2,
\]

\[
(F_3)_{213} = (F_3)_{224} = (F_3)_{231} = (F_3)_{242} = (F_3)_{312} = (F_3)_{332} = -(F_3)_{334} = -(F_3)_{343} = -1
\]

and the others are zero. The only non-zero components of the corresponding Lee forms are

\[
(\theta_1)_2 = -(\theta_2)_3 = (\theta_3)_4 = -2.
\]

The results in \((10.18), (10.19)\) and the classification conditions \((9.10), (9.11)\) imply

**Proposition 10.5.** The hypercomplex manifold with Hermitian-Norden metrics on a 4-dimensional Lie algebra, determined by \((10.16)\), belongs to the largest class of the considered manifolds, i.e. \(\mathcal{HC}\), as well as this manifold does not belong to neither \(\mathcal{W}_1\) nor \(\mathcal{W}_2\) for \(J_2\) and \(J_3\).

### 10.4.2. Case 2

Secondly, we choose \(e_4 \in \mathfrak{l}\) with \(g(e_4, e_4) = -1\) as an element orthogonal to \(\mathfrak{l}'\) with respect to \(g\). Therefore, in this case \(\mathfrak{l}\) is determined by

\[
[e_4, e_1] = e_1, \quad [e_4, e_2] = e_2, \quad [e_4, e_3] = e_3.
\]

Therefore, the nontrivial covariant derivatives are

\[
D_{e_1} e_1 = D_{e_2} e_2 = -D_{e_3} e_3 = -e_4,
\]

\[
D_{e_1} e_4 = -e_1, \quad D_{e_2} e_4 = -e_2, \quad D_{e_3} e_4 = -e_3.
\]

In a similar way we obtain:

\[
(F_1)_{113} = (F_1)_{124} = -(F_1)_{131} = -(F_1)_{142} = (F_1)_{214} = (F_1)_{223} = -(F_1)_{232} = (F_1)_{241} = -1,
\]

\[
(F_2)_{222} = (F_2)_{244} = -2,
\]

\[
(F_3)_{314} = -F_3_{323} = (F_3)_{332} = -(F_3)_{341} = -(F_3)_{413} = -(F_3)_{424} = (F_3)_{431} = (F_3)_{442} = 1,
\]

\[
(F_4)_{311} = (F_4)_{333} = -2,
\]

\[
(F_4)_{214} = -(F_4)_{223} = -(F_4)_{232} = (F_4)_{241} = (F_4)_{412} = (F_4)_{421} = (F_4)_{434} = (F_4)_{443} = -1,
\]

\[
(F_5)_{411} = (F_5)_{444} = 2,
\]

\[
(F_5)_{213} = (F_5)_{224} = (F_5)_{231} = (F_5)_{242} = (F_5)_{312} = (F_5)_{332} = -(F_5)_{334} = -(F_5)_{343} = -1
\]

and the others are zero. The only non-zero components of the corresponding Lee forms are

\[
(\theta_1)_2 = -(\theta_2)_3 = (\theta_3)_4 = -2.
\]

The results in \((10.18), (10.19)\) and the classification conditions \((9.10), (9.11)\) imply

**Proposition 10.5.** The hypercomplex manifold with Hermitian-Norden metrics on a 4-dimensional Lie algebra, determined by \((10.16)\), belongs to the largest class of the considered manifolds, i.e. \(\mathcal{HC}\), as well as this manifold does not belong to neither \(\mathcal{W}_1\) nor \(\mathcal{W}_2\) for \(J_2\) and \(J_3\).
\[(F_2)_{112} = (F_2)_{121} = (F_2)_{134} = (F_2)_{143} = (F_2)_{314} = -(F_2)_{323} = -(F_2)_{332} = (F_2)_{341} = -1,\]

\[(F_3)_{111} = (F_3)_{144} = 2,\]

\[-(F_3)_{212} = -(F_3)_{221} = (F_3)_{234} = (F_3)_{243} = (F_3)_{313} = (F_3)_{324} = (F_3)_{331} = (F_3)_{342} = -1\]

and the others are zero. The only non-zero components of the corresponding Lee forms are

\[(\theta_1)_3 = -2, \quad (\theta_2)_2 = -(\theta_3)_1 = -4.\]

Then, analogously to Case 2 on page 139, we obtain the following

**Proposition 10.6.** The hypercomplex manifold with Hermitian-Norden metrics on a 4-dimensional Lie algebra, determined by \[(10.20),\]

belongs to the class \(W^0\) of the (locally) conformally equivalent \(K\)-manifolds.

### 10.5. Hypercomplex structure of type (hc5)

In this case, \(\mathfrak{l}\) is solvable and \(\mathfrak{l}'\) is a 3-dimensional Heisenberg algebra.

#### 10.5.1. Case 1

Firstly, we fix \(e_1 \in \mathfrak{l}\) with \(g(e_1, e_1) = 1\) as an element orthogonal to \(\mathfrak{l}'\) with respect to \(g\). Then \(\mathfrak{l}\) is determined by

\[
[e_1, e_2] = e_2, \quad [e_1, e_3] = \frac{1}{2} e_3, \quad [e_1, e_4] = \frac{1}{2} e_4, \quad [e_3, e_4] = \frac{1}{2} e_2.
\]

Then we compute covariant derivatives and the nontrivial ones are

\[
D_{e_2} e_2 = -2 D_{e_3} e_3 = -2 D_{e_4} e_4 = e_1,
\]

\[
-D_{e_2} e_1 = 4 D_{e_3} e_4 = -4 D_{e_4} e_3 = e_2,
\]

\[
-4 D_{e_2} e_4 = -2 D_{e_3} e_1 = -4 D_{e_4} e_2 = e_3,
\]

\[
4 D_{e_2} e_3 = 4 D_{e_3} e_2 = -2 D_{e_4} e_1 = e_4.
\]

Analogously of the previous cases, we obtain the components \((F_{\alpha})_{ijk}\) as follows:

\[(F_1)_{314} = -(F_1)_{323} = (F_1)_{332} = -(F_1)_{341} = - (F_1)_{413} = -(F_1)_{424} = (F_1)_{431} = (F_1)_{442} = \frac{1}{4},\]

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(\(F_2\))_{214} = -(\(F_2\))_{223} = -(\(F_2\))_{232} = (\(F_2\))_{241} = -\frac{5}{4},
(\(F_2\))_{311} = -2(\(F_2\))_{322} = (\(F_2\))_{333} = -2(\(F_2\))_{344} = -1,
(\(F_2\))_{412} = (\(F_2\))_{421} = (\(F_2\))_{434} = (\(F_2\))_{443} = -\frac{3}{4},
(\(F_3\))_{213} = (\(F_3\))_{224} = (\(F_3\))_{231} = (\(F_3\))_{242} = -\frac{5}{4},
(\(F_3\))_{312} = (\(F_3\))_{321} = -(\(F_3\))_{334} = -(\(F_3\))_{343} = -\frac{3}{4},
(\(F_3\))_{411} = -2(\(F_3\))_{422} = -2(\(F_3\))_{433} = (\(F_3\))_{444} = 1

and the others are zero. The only non-zero components of the corresponding Lee forms are

\[(\theta_1)_2 = -\frac{1}{2}, \quad (\theta_2)_3 = - (\theta_3)_4 = 3.\]

The results in (10.26), (10.27) and the classification conditions (9.10), (9.11) imply

**Proposition 10.7.** The hypercomplex manifold with Hermitian-Norden metrics on a 4-dimensional Lie algebra, determined by (10.24), belongs to the largest class of the considered manifolds, i.e. \(\mathcal{HC}\), as well as this manifold does not belong to neither \(\mathcal{W}_1\) nor \(\mathcal{W}_2\) for \(J_2\) and \(J_3\).

**10.5.2. Case 2**

The other possibility is to choose \(e_4 \in \mathfrak{l}\) with \(g(e_4, e_4) = -1\) as an element orthogonal to \(\mathfrak{l}'\) with respect to \(g\). We rearrange the basis in (10.24) and then \(\mathfrak{l}\) is determined by

\[
[e_1, e_2] = -\frac{1}{2}e_3, \quad [e_1, e_4] = -\frac{1}{2}e_1,
\]
\[
[e_2, e_4] = -\frac{1}{2}e_2, \quad [e_3, e_4] = -e_3.
\]

By similar computations we establish the same statement as of Proposition 10.7 for the Heisenberg algebra introduced by (10.28).

\[
\ast \ast \ast
\]
§11. Tangent bundles with complete lift of the base metric and almost hypercomplex structure with Hermitian-Norden metrics

In the present section, the tangent bundle of an almost Norden manifold and the complete lift of the Norden metric are considered as a $4n$-dimensional manifold. It is equipped with an almost hypercomplex Hermitian-Norden structure. It is characterized geometrically. The case when the base manifold is an h-sphere is considered.

The main results of this section are published in [87].

The investigation of the tangent bundle $T\mathcal{M}$ of a manifold $\mathcal{M}$ helps us to study the manifold $\mathcal{M}$. Moreover, $T\mathcal{M}$ has own structure closely related to the structure of $\mathcal{M}$, which implies mutually related geometric properties.

In this section we consider the following situation: it is given a base almost Norden manifold and we study its tangent bundle equipped with a metric, which is the complete lift of the base metric. Thus, we get a manifold with an almost hypercomplex structure and Hermitian-Norden metrics which we characterize.

Similar investigations are made in [81]. There, it is used the diagonal lift of the base metric (known as a Sasaki metric) on the tangent bundle. The almost hypercomplex structure with Hermitian-Norden metrics is generated in the same manner.

Our goal is to determine an almost hypercomplex structure with Hermitian-Norden metrics $(H,G)$ on $T\mathcal{M}$ when the base manifold $\mathcal{M}$ has an almost Norden structure $(J,\tilde{g},\tilde{g})$.

We use the horizontal and vertical lifts of the vector fields on $\mathcal{M}$ to get the corresponding components of the considered tensor fields on $T\mathcal{M}$. 

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These components are sufficient to describe the characteristic tensor fields on $T\mathcal{M}$ in general.

11.1. Almost hypercomplex structure on the tangent bundle

It is well known [152], for any affine connection on $\mathcal{M}$, the induced horizontal and vertical distributions of $T\mathcal{M}$ are mutually complementary. Then we define tensor fields $J_1$, $J_2$ and $J_3$ on $T\mathcal{M}$ by their action over the horizontal and vertical lifts of an arbitrary vector field on $\mathcal{M}$:

$$
\begin{align*}
J_1 : & \begin{cases} 
X' \to -(JX)' \\
X'' \to (JX)''
\end{cases} \\
J_2 : & \begin{cases} 
X' \to X'' \\
X'' \to -X'
\end{cases} \\
J_3 : & \begin{cases} 
X' \to (JX)'' \\
X'' \to (JX)'
\end{cases}
\end{align*}
$$

(11.1)

where $J$ is the given almost complex structure on $\mathcal{M}$; moreover, we denote the horizontal and vertical lifts by $X', X'' \in \mathfrak{X}(T\mathcal{M})$ of any $X \in \mathfrak{X}(\mathcal{M})$ at $u \in T_p\mathcal{M}$, $p \in \mathcal{M}$, where an affine connection $D$ on $\mathcal{M}$ is used.

By direct computations we get the following

**Proposition 11.1.** There exists an almost hypercomplex structure $H$, defined by (11.1) on $T\mathcal{M}$ over an almost complex manifold $(\mathcal{M}, J)$ with an affine connection $D$. The constructed $4n$-dimensional manifold is an almost hypercomplex manifold $(T\mathcal{M}, H)$.

Following (9.12), let $[J_\alpha, J_\alpha]$ denote the Nijenhuis tensor of $J_\alpha$ for each $\alpha$ and $\hat{X}, \hat{Y} \in \mathfrak{X}(T\mathcal{M})$, i.e.

$$
[\alpha, \alpha](\hat{X}, \hat{Y}) = [J_\alpha \hat{X}, J_\alpha \hat{Y}] - J_\alpha[J_\alpha \hat{X}, \hat{Y}] - J_\alpha[\hat{X}, J_\alpha \hat{Y}] - [\hat{X}, \hat{Y}].
$$

(11.2)

If $D$ is torsion-free and its curvature tensor is denoted by $R$, then we have the following (see also [152])

$$
\begin{align*}
[X', Y'] &= [X,Y]' - \{R(X,Y)u\}'' \\
[X', Y''] &= (DX)'', \\
[X'', Y'] &= -(DY)'', \\
[X'', Y''] &= 0.
\end{align*}
$$

(11.3)

Using (11.1), (11.2) and (11.3), we get
Proposition 11.2. Let \((\mathcal{M}, J)\) be an almost complex manifold with Nijenhuis tensor \([J, J]\), a torsion-free affine connection \(D\) and its curvature tensor \(R\). Then the Nijenhuis tensors of the structure \(H\) on \(T\mathcal{M}\) for the corresponding horizontal and vertical lifts have the following form

\[
\begin{align*}
[J_1, J_1](X', Y') &= ([J, J](X, Y))' \\
&\quad - (R(JX, JY)u + JR(JX, Y)u \\
&\quad + JR(X, JY)u - R(X, Y)u)'',
\end{align*}
\]

\[
\begin{align*}
[J_1, J_1](X', Y'') &= -((D_{JX} J)(Y) - (D_X J)(JY))'', \\
[J_1, J_1](X'', Y') &= -((D_Y J)(JX) - (D_{JY} J)(X))'', \\
[J_1, J_1](X'', Y'') &= 0;
\end{align*}
\]

\[
\begin{align*}
[J_2, J_2](X', Y') &= -[J_2, J_2](X'', Y'') = (R(X, Y)u)'', \\
[J_2, J_2](X', Y'') &= [J_2, J_2](X'', Y') = (R(X, Y)u)', \\
[J_3, J_3](X', Y') &= -(J(D_X J)(Y) - J(D_Y J)(X))' \\
&\quad + (R(X, Y)u)'', \\
[J_3, J_3](X', Y'') &= -(J(D_X J)(Y) + (D_{JY} J)(X))'' \\
&\quad + (JR(X, JY)u)', \\
[J_3, J_3](X'', Y') &= ((D_{JX} J)(Y) + J(D_Y J)(X))'' \\
&\quad + (JR(JX, Y)u)', \\
[J_3, J_3](X'', Y'') &= ((D_{JX} J)(Y) - (D_{JY} J)(X))' \\
&\quad - (R(JX, JY)u)''.
\end{align*}
\]

The last equalities for \([J_\alpha, J_\alpha]\) imply the following necessary and sufficient conditions for the integrability of \(J_\alpha\) and \(H\).

Theorem 11.3. Let \(T\mathcal{M}\) be the tangent bundle manifold equipped with an almost hypercomplex structure \(H = (J_1, J_2, J_3)\) defined as in (11.1). Let also \(\mathcal{M}\) be its base manifold with the almost complex structure \(J\). We additionally assume that the affine connection used for \(H\) be torsion-free. Then the following relations hold:

(i) \((T\mathcal{M}, J_\alpha)\) for \(\alpha = 1\) or 3 is complex if and only if \(\mathcal{M}\) is flat and \(J\) is parallel;

(ii) \((T\mathcal{M}, J_2)\) is complex if and only if \(\mathcal{M}\) is flat;
(iii) \((T\mathcal{M}, H)\) is hypercomplex if and only if \(\mathcal{M}\) is flat and \(J\) is parallel.

**Remark 11.1.** The assertion (ii) above is a corollary of the theorem of Dombrowski in [26], where the structure \(J_2\) is defined and studied.

**Corollary 11.4.** (i) \((T\mathcal{M}, J_1)\) is complex if and only if \((T\mathcal{M}, J_3)\) is complex.

(ii) If \((T\mathcal{M}, J_1)\) or \((T\mathcal{M}, J_3)\) is complex then \((T\mathcal{M}, H)\) is hypercomplex.

### 11.2. Complete lift of the base metric on the tangent bundle

Let us introduce a metric \(\hat{g}\) on \(T\mathcal{M}\), which is the complete lift of the base metric \(g\) on \(\mathcal{M}\), by

\[
\hat{g}(X', Y') = \hat{g}(X'', Y'') = 0,
\hat{g}(X', Y'') = \hat{g}(X'', Y') = g(X, Y).
\]

It is known that \(\hat{g}\), associated with a (pseudo-)Riemannian metric \(g\), is a pseudo-Riemannian metric on \(T\mathcal{M}\) of signature \((m, m)\), where \(m = \dim \mathcal{M}\). The metric \(\hat{g}\) coincides with the horizontal lift of \(g\) with respect to its Levi-Civita connection [151]. This metric is introduced by Yano and Kobayashi as \((T\mathcal{M}, \hat{g})\) has zero scalar curvature and it is an Einstein space if and only if \(\mathcal{M}\) is Ricci-flat [152].

As it is known from [154], whenever \(D\) is the Levi-Civita connection of \(\mathcal{M}\) with respect to the pseudo-Riemannian metric \(g\), then the complete lift of \(D\) is the Levi-Civita connection of \(T\mathcal{M}\) generated by \(\hat{g}\). Since \(\hat{D}\) is the Levi-Civita connection of \(\hat{g}\) on \(T\mathcal{M}\) and the same holds for \(D\) on \((\mathcal{M}, g)\), then using the Koszul formula (5.18) we obtain the covariant derivatives of the horizontal and vertical lifts of the vector fields on \(T\mathcal{M}\) at \(u \in T_p\mathcal{M}\) as follows (see also [152]):

\[
\begin{align*}
\hat{D}_X Y' &= (D_X Y)' + (R(u, X)Y)'', \\
\hat{D}_X Y'' &= (D_X Y)'', \\
\hat{D}_X'' Y' &= 0, \\
\hat{D}_X'' Y'' &= 0.
\end{align*}
\]

After that, we calculate the components of the curvature tensor \(\hat{R}\) of \(\hat{D}\) with respect to the horizontal and vertical lifts of the vector fields on \(\mathcal{M}\). We obtain the following non-zero components for the curvature tensors \(R\) and \(\hat{R}\) as well as the Ricci tensors \(\rho\) and \(\hat{\rho}\), corresponding to the metrics \(g\) and \(\hat{g}\) on \(\mathcal{M}\) and \(T\mathcal{M}\), respectively (see also [152]):

\[
\begin{align*}
\hat{R}(X', Y') Z' &= \{R(X, Y)Z\}' + \{(D_u R)(X, Y)Z\}'', \\
\hat{R}(X'', Y'') Z'' &= \{R(X, Y)Z\}''.
\end{align*}
\]
\( \hat{R}(X', Y')Z'' = \hat{R}(X', Y'')Z' = \{R(X, Y)Z\}'' ; \)
\( \hat{\rho}(Y', Z') = 2\rho(Y, Z). \)

Hence, we get

Corollary 11.5. (i) \((TM, \hat{g})\) is flat if and only if \((M, g)\) is flat.

(ii) \((TM, \hat{g})\) is Ricci flat if and only if \((M, g)\) is Ricci flat.

(iii) \((TM, \hat{g})\) is scalar flat.

Remark 11.2. The results in (11.5), (11.6) and Corollary 11.5 are confirmed also by [152], where \(g\) is a Riemannian metric.

11.3. Tangent bundle with almost hypercomplex structure and Hermitian-Norden metrics

Suppose that \((M, J, g, \tilde{g})\) is an almost complex manifold with Norden metric \(g\) and its associated Norden metric \(\tilde{g}\). Suppose also that \((TM, H)\) is its almost hypercomplex tangent bundle with the Hermitian-Norden metric structure \(\hat{G} = (\hat{g}, \hat{g}_1, \hat{g}_2, \hat{g}_3)\) derived (as in (9.4)) from the metric \(\hat{g}\) on \(TM\), the complete lift of \(g\). The generated 4n-dimensional manifold is denoted by \((TM, H, \hat{G})\).

Bearing in mind (11.1), we verify immediately that \(\hat{g}\) satisfies (9.2) and therefore it is valid the following

Theorem 11.6. The tangent bundle \(TM\) equipped with the almost hypercomplex structure \(H\) and the metric \(\hat{g}\), defined by (11.1) and (11.4), respectively, is an almost hypercomplex manifold with Hermitian-Norden metrics \((TM, H, \hat{G})\).

We use (11.5) and (11.1) in order to characterize the fundamental tensors \(F_\alpha\) with respect to \(\hat{g}\) and \(\hat{D}\) at each \(u \in T_pM\) on \((TM, H, \hat{G})\). Then we obtain the following

Proposition 11.7. The nonzero components of \(F_\alpha\) with respect to the horizontal and vertical lifts of the vector fields depend on the fundamental tensor \(F\) and the curvature tensor \(R\) of \((M, J, g, \tilde{g})\) in the following way:

\[
\begin{align*}
F_1(X', Y', Z') &= -R(u, X, JY, Z) - R(u, X, Y, JZ), \\
F_1(X', Y', Z'') &= -F_1(X', Y'', Z') = -F(X, Y, Z); \\
F_2(X', Y', Z'') &= -F_2(X', Y'', Z') = R(u, X, Y, Z); \\
F_3(X', Y', Z') &= F_3(X', Y'', Z'') = F(X, Y, Z), \\
\end{align*}
\]

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\[ F_3(X', Y', Z'') = -R(u, X, Y, JZ), \]
\[ F_3(X', Y'', Z') = R(u, X, JY, Z). \]

We use the components \((11.7)\) of \(F_\alpha\) and bearing in mind the definition conditions \((9.10)\) and \((9.11)\) of the basic classes as well as their direct sums in the corresponding classifications from \([44]\) and \([34]\), we obtain

**Proposition 11.8.** (i) The almost Hermitian manifold \((T_M, J_1, \hat{g})\) belongs to the class

\[
\{(W_1 \oplus W_2 \oplus W_3 \oplus W_4) \setminus (W_1 \cup (W_3 \oplus W_4))\} \cup W_0,
\]
where \(W_1 \oplus W_2 \oplus W_3 \oplus W_4, W_1, W_3 \oplus W_4\) and \(W_0\) stand for the classes of the almost Hermitian, nearly Kähler, Hermitian and Kähler manifolds, respectively. For the 4-dimensional case, the class of \((T_M, J_1, \hat{g})\) is restricted to the class \(W_2\) of the almost Kähler manifolds.

(ii) The almost Norden manifold \((T_M, J_\alpha, \hat{g}), (\alpha = 2, 3)\), belongs to the class

\[
\{(W_1 \oplus W_2 \oplus W_3) \setminus ((W_1 \oplus W_2) \cup (W_1 \oplus W_3))\} \cup W_0.
\]

The corresponding Lee forms are determined by \((9.8)\). Hence, we can compute them with respect to an adapted frame.

Let \(\{\hat{e}_A\} = \{e'_i, e''_i\}\) be the adapted frame at each point of \(T_M\) derived by the orthonormal basis \(\{e_i\}\) of signature \((n, n)\) at each point of \(M\). The indices \(i, j, \ldots\) run over the range \(\{1, 2, \ldots, 2n\}\), while the indices \(A, B, \ldots\) belong to the range \(\{1, 2, \ldots, 4n\}\). We use summation convention for these indices.

For example, we compute as follows

\[
\theta_3(Z') = \hat{g}^{AB} F_3(\hat{e}_A, \hat{e}_B, Z') = \hat{g}^{ij} \{F_3(e'_i, e''_j, Z') + F_3(e''_i, e'_j, Z')\}
\]
\[
= \hat{g}^{ij} F_3(e'_i, e''_j, Z') = \hat{g}^{ij} R(u, e_i, Je_j, Z) = \rho^*(u, Z).
\]

Analogously, we have

\[
\theta_3(Z'') = g^{ij} F_3(e'_i, e''_j, Z'') = g^{ij} F(e_i, e_j, Z) = \theta(Z).
\]

Thus, we obtain the following nonzero components of the Lee forms \(\theta_\alpha\)

\[
\theta_1(Z') = \theta_3(Z'') = \theta(Z),
\]
\[
\theta_2(Z') = -\rho(u, Z),
\]
\[
\theta_3(Z') = \rho^*(u, Z),
\]
where $\rho$ and $\rho^*$ are the Ricci tensor and its associated Ricci tensor stemming from $g$ and $J$. Therefore, we obtain

**Proposition 11.9.** The following necessary and sufficient conditions are valid:

(i) $\theta_1 = 0$ if and only if $\theta = 0$;

(ii) $\theta_2 = 0$ if and only if $\rho = 0$;

(iii) $\theta_3 = 0$ if and only if $\theta = 0$ and $\rho^* = 0$.

**Remark 11.3.** Let us recall that the vanishing condition of the corresponding Lee form determines the class $(W_1 \oplus W_2 \oplus W_3)(J_1)$ of the semi-Kähler manifolds among the almost Hermitian manifolds and the class $(W_2 \oplus W_3)(J_\alpha), (\alpha = 2, 3)$ among the almost Norden manifolds, respectively.

Bearing in mind Proposition 11.7 and Theorem 11.3, it is easy to conclude the following

**Proposition 11.10.** The following necessary and sufficient conditions are valid:

(i) $(TM, J_\alpha, \widehat{g})$ for $\alpha = 1$ or $3$ has a parallel complex structure $J_\alpha$ if and only if $(M, J, g, \widehat{g})$ is flat and $J$ is parallel.

(ii) $(TM, J_2, \widehat{g})$ has a parallel complex structure $J_2$ if and only if $(M, J, g, \widehat{g})$ is flat.

(iii) $(TM, H, G)$ is a hyper-Kähler manifold with Hermitian-Norden metrics if and only if $(M, J, g, \widehat{g})$ is flat and $J$ is parallel.

**Corollary 11.11.** (i) $J_1$ is parallel if and only if $J_3$ is parallel.

(ii) If $J_1$ or $J_3$ is parallel then $(TM, H, \widehat{G})$ is a hyper-Kähler manifold with Hermitian-Norden metrics.

**Corollary 11.12.** (i) If the manifold $(TM, J_\alpha, \widehat{g})$ for any $\alpha$ is a complex manifold then it is of Kähler type.

(ii) If the manifold $(TM, H, \widehat{G})$ is hypercomplex then it is a hyper-Kähler manifold with Hermitian-Norden metrics.

**Corollary 11.13.** (i) If $(M, J, g, \widehat{g})$ is flat, then $TM$ has parallel $J_2$.

(ii) If $(M, J, g, \widehat{g})$ is flat and its Lee form $\theta$ is zero, i.e. $(M, J, g, \widehat{g}) \in (W_2 \oplus W_3)$, then

$$(TM, H, \widehat{G}) \in (W_1 \oplus W_2 \oplus W_3)(J_1) \cup W_0(J_2) \cup (W_2 \oplus W_3)(J_3);$$
(iii) If $(\mathcal{M}, J, g, \tilde{g})$ is a Kähler-Norden manifold, then $(TM, J_1, \tilde{g}) \in \mathcal{W}_2(J_1)$.

**Remark 11.4.** The corresponding result in the Riemannian case (see [143]) reads as follows. The manifold $(TM, J_1, \tilde{g})$ is almost Kähler (i.e. symplectic) for any Riemannian metric $g$ on the base manifold when the connection used to define the horizontal lifts is the Levi-Civita connection.

### 11.3.1. Tangent bundle of an h-sphere

Let $(\mathcal{M}, J, g, \tilde{g})$ be a Kähler-Norden manifold, $\dim \mathcal{M} = 2n \geq 4$. Let $x, y, z, w$ be arbitrary vectors in $T_p\mathcal{M}$, $p \in \mathcal{M}$. The curvature tensor $R$ is of Kähler type in this case. This implies that the associated tensor $R^*$ of type $(0, 4)$ defined by the equality $R^*(x, y, z, w) = R(x, y, z, Jw)$ has the property $R^*(x, y, z, w) = -R^*(x, y, w, z)$ [35]. Therefore $R^*$ has the properties of a curvature tensor, i.e. it is a curvature-like tensor. The following tensors are essential curvature-like $(0, 4)$-tensors:

$$\pi_1 = -\frac{1}{2}g \otimes g, \quad \pi_2 = -\frac{1}{2}\tilde{g} \otimes \tilde{g}, \quad \pi_3 = g \otimes \tilde{g}.$$  

Every non-degenerate 2-plane $\beta$ with respect to $g$ in $T_p\mathcal{M}$, $p \in \mathcal{M}$, has the following two sectional curvatures

$$k(\beta; p) = \frac{R(x, y, y, x)}{\pi_1(x, y, y, x)}, \quad k^*(\beta; p) = \frac{R^*(x, y, y, x)}{\pi_1(x, y, y, x)},$$

where $\{x, y\}$ is a basis of $\beta$.

A 2-plane $\beta$ is said to be holomorphic (resp., totally real) if $\beta = J\beta$ (resp., $\beta \perp J\beta \neq \beta$) with respect to $g$ and $J$.

The orthonormal $J$-basis $\{e_i, \bar{e}_i\}$, where $i \in \{1, 2, \ldots, n\}$, $\bar{i} = n + i$, $e_i = Je_i$ of $T_p\mathcal{M}$ with respect to $g$ generates an orthogonal basis of isotropic vectors $\{e'_i, e''_i, e'''_i\}$ of $T_u(TM)$ with respect to $\tilde{g}$. Then, the basis $\{\xi_i, \bar{\xi}_i, \eta_i, \bar{\eta}_i\}$, where

$$\xi_i = \frac{1}{\sqrt{2}}(e'_i + e''_i), \quad \bar{\xi}_i = \frac{1}{\sqrt{2}}(e'_i + e''_i),$$

$$\eta_i = \frac{1}{\sqrt{2}}(e'_i - e''_i), \quad \bar{\eta}_i = \frac{1}{\sqrt{2}}(e'_i - e''_i),$$

is orthonormal with respect to $\tilde{g}$. Moreover, we have

$$g(\xi_i, \xi_i) = -g(\xi_i, \bar{\xi}_i) = -g(\eta_i, \eta_i) = g(\bar{\eta}_i, \bar{\eta}_i) = 1,$$
if \( g(e_i, e_i) = -g(e_i, e_i) = 1 \) are valid. Then, using (11.1), we obtain
\[
J_1 \xi_i = \eta_i, \quad J_1 \eta_i = \xi_i, \quad J_2 \eta_i = \xi_i,
\]
\[
J_2 \eta_i = \xi_i, \quad J_3 \xi_i = \xi_i, \quad J_3 \eta_i = \eta_i,
\]
i.e. the basis \( \{ \xi_i, \eta_i, \eta_i \} \) is an adapted \( H \)-basis.

Thus, the following basic 2-planes in \( T_u(TM) \) are special with respect to \( H \) \((i \neq j)\):

a) \( J_\alpha \)-totally-real 2-planes \((\alpha = 1, 2, 3)\):
\[
\{ \xi_i, \xi_j \}, \quad \{ \xi_i, \eta_j \}, \quad \{ \xi_i, \eta_j \}, \quad \{ \xi_i, \xi_j \}, \quad \{ \xi_i, \eta_j \}, \quad \{ \eta_i, \eta_j \};
\]
b) \( J_1 \)-holomorphic and \( J_\alpha \)-totally-real 2-planes \((\alpha = 2, 3)\):
\[
\{ \xi_i, \eta_i \}, \quad \{ \eta_i, \xi_i \};
\]
c) \( J_2 \)-holomorphic and \( J_\alpha \)-totally-real 2-planes \((\alpha = 1, 3)\):
\[
\{ \eta_i, \xi_i \}, \quad \{ \eta_i, \xi_i \};
\]
d) \( J_3 \)-holomorphic and \( J_\alpha \)-totally-real 2-planes \((\alpha = 1, 2)\):
\[
\{ \xi_i, \xi_j \}, \quad \{ \eta_i, \eta_j \}.
\]

The sectional curvatures \( \hat{k} \) of these 2-planes and the sectional curvatures \( k_{ij}, k_{ij}, k_{ij} \) and \( k_{ii} \) of the special basic 2-planes in \( T_pM \): \( J \)-totally-real 2-planes \( \{ e_i, e_j \}, \{ e_i, e_j \}, \{ e_i, e_j \} \) \((i \neq j)\) and \( J \)-holomorphic 2-planes \( \{ e_i, e_i \} \), respectively, are related as follows:
\[
\hat{k}(\xi_i, \xi_j) = \frac{1}{4} (D_u k)_{ij} + k_{ij}, \quad \hat{k}(\xi_i, \xi_j) = \frac{1}{4} (D_u k)_{ij} + k_{ij},
\]
\[
\hat{k}(\xi_i, \eta_j) = -\frac{1}{4} (D_u k)_{ij}, \quad \hat{k}(\xi_i, \eta_j) = -\frac{1}{4} (D_u k)_{ij}
\]
\[
\hat{k}(\xi_i, \xi_j) = \frac{1}{4} (D_u k)_{ij} + k_{ij}, \quad \hat{k}(\xi_i, \eta_j) = -\frac{1}{4} (D_u k)_{ij},
\]
\[
\hat{k}(\xi_i, \eta_j) = -\frac{1}{4} (D_u k)_{ij}, \quad \hat{k}(\eta_i, \eta_j) = \frac{1}{4} (D_u k)_{ij} - k_{ij},
\]
\[
\hat{k}(\eta_i, \eta_j) = \frac{1}{4} (D_u k)_{ij} - k_{ij}, \quad \hat{k}(\eta_i, \eta_j) = \frac{1}{4} (D_u k)_{ij} - k_{ij}.
\]

Therefore, we obtain

**Proposition 11.14.** The manifold \((TM, H, \hat{G})\) for an arbitrary almost Norden manifold \((M, J, g)\) has equal sectional curvatures of the \( J_1 \)-holomorphic 2-planes and vanishing sectional curvatures of the \( J_2 \)-holomorphic 2-planes.
Identifying the point $z = (x^1, ..., x^{n+1}; y^1, ..., y^{n+1})$ in $\mathbb{R}^{2n+2}$ with the position vector $z$, we consider an $h$-sphere with center $z_0$ and a pair of parameters $(a, b)$. The $h$-sphere $S^{2n}(z_0; a, b)$ is the holomorphic hypersurface of the Kähler-Norden manifold $\mathbb{R}^{2n+2}$ equipped with the canonical complex structure and the canonical Norden metrics $h$ and $\tilde{h}$. Then $S^{2n}(z_0; a, b)$ is defined by

$$h(z - z_0, z - z_0) = a, \quad \tilde{h}(z - z_0, z - z_0) = b,$$

where $(0, 0) \neq (a, b) \in \mathbb{R}^2$. It was also considered in Subsection 8.3.2. Each $S^{2n}$, $n \geq 2$, has vanishing holomorphic sectional curvatures and constant totally real sectional curvatures $\nu = \frac{a}{a^2 + b^2}, \nu^* = -\frac{b}{a^2 + b^2}$.

Then, according to [15], the curvature tensor of the $h$-sphere is

$$R = \nu(\pi_1 - \pi_2) + \nu^*\pi_3$$

and therefore $D R = 0$. Moreover, we have

$$\rho = 2(n - 1)(\nu g - \nu^*\tilde{g}),$$

$$\rho^* = 2(n - 1)(\nu^*\tilde{g} + \nu g),$$

$$\tau = 4n(n - 1)\nu,$$

$$\tau^* = 4n(n - 1)\nu^*,$$

where $\rho^* = \rho(R^*), \tau^* = \tau(R^*)$. Because of the form of $\rho$, $S^{2n}$ is called almost Einstein.

Let us consider $(TS, H, \hat{G})$, the tangent bundle with almost hypercomplex structure with Hermitian-Norden metrics of the $h$-sphere $(S, J, g)$ with parameters $(a, b)$, as its base Kähler-Norden manifold. Then, bearing in mind Proposition 11.7, Proposition 11.8 and Corollary 11.13 (iii), we get that $(TS, J_1, \hat{g}) \in W_2(J_1)$ and $(TS, J_\alpha, \hat{g})$ ($\alpha = 2, 3$) belongs to $(W_1 \oplus W_2 \oplus W_3)(J_\alpha) \setminus (W_i \oplus W_j)(J_\alpha)$, where $i \neq j \in \{1, 2, 3\}$. Moreover, from (11.6) and (11.8) we get the components of the curvature tensor $\hat{R}$ of $(TS, H, \hat{G})$. Then we have

$$\hat{k}(\xi_i, \xi_j) = \hat{k}(\xi_i, \bar{\xi}_j) = \hat{k}(\xi_i, \xi_j)$$

$$= -\hat{k}(\eta_i, \eta_j) = -\hat{k}(\bar{\eta}_i, \eta_j) = -\hat{k}(\eta_i, \eta_j) = \frac{a}{a^2 + b^2},$$

$$\hat{k}(\xi_i, \eta_j) = \hat{k}(\xi_i, \eta_j) = \hat{k}(\xi_i, \eta_j) = \hat{k}(\xi_i, \eta_j) = 0.$$
Corollary 11.15. The manifold \((TS, H, \hat{G})\) of an arbitrary \(h\)-sphere \((S, J, g)\) has constant sectional curvatures of the \(J_\alpha\)-totally-real 2-planes and vanishing sectional curvatures of the \(J_\alpha\)-holomorphic 2-planes \((\alpha = 1, 2, 3)\).
§12. Associated Nijenhuis tensors on almost hypercomplex manifolds with Hermitian-Norden metrics

In the present section, it is introduced an associated Nijenhuis tensor of endomorphisms in the tangent bundle of an almost hypercomplex manifold with Hermitian-Norden metrics. There are studied relations between the six associated Nijenhuis tensors of an almost hypercomplex structure as well as their vanishing. It is given a geometric interpretation of the associated Nijenhuis tensors for an almost hypercomplex structure and Hermitian-Norden metrics. Finally, an example of a 4-dimensional manifold of the considered type with vanishing associated Nijenhuis tensors is given.

The main results of this section are published in [91].

The vanishing of the Nijenhuis tensors as conditions for integrability of the manifold are long-known [150]. The goal here is to introduce an appropriate tensor on an almost hypercomplex manifold and to establish that its vanishing is a necessary and sufficient condition for existing of an affine connection with totally skew-symmetric torsion preserving the almost hypercomplex structure and Hermitian-Norden metrics.

The present section is organised as follows. In Subsection 12.1, for endomorphisms in the tangent bundle, it is introduced a symmetric tensor, which is associated to the Nijenhuis tensor. In Subsection 12.2, it is deduced a series of relations between the six associated Nijenhuis tensors of an almost hypercomplex structure as well as it is proved that if two of these six tensors vanish, then the others also vanish. In Subsection 12.3, it is proved the main theorem for the geometric interpretation of the associated Nijenhuis tensors for an almost hypercomplex structure and Hermitian-Norden metrics. In Subsection 12.4, it is given an example of a 4-dimensional manifold of the considered type with vanishing associated Nijenhuis tensors.
12.1. The associated Nijenhuis tensors of endomorphisms

Let us consider a differentiable manifold $\mathcal{M}$ and let the symmetric braces $\{\cdot, \cdot\}$ be defined by (1.19) for all differentiable vector fields on $\mathcal{M}$.

As it is well known [60], the Nijenhuis tensor $[J, K]$ for arbitrary tensor fields $J$ and $K$ of type (1,1) on $\mathcal{M}$ is a tensor field of type (1,2) defined by

\[(12.1)\]
\[
2[J, K](x, y) = (JK + KJ)[x, y] \\
+ [Jx, Ky] - J[Kx, y] - J[x, Ky] \\
+ [Kx, Jy] - K[Jx, y] - K[x, Jy].
\]

As a consequence, the Nijenhuis tensor $[J, J]$ for an arbitrary tensor field $J$ is determined by the following equality

\[(12.2)\]
\[
[J, J](x, y) = [Jx, Jy] + J^2[x, y] - J[Jx, y] - J[x, Jy].
\]

If $J$ is an almost complex structure, then the Nijenhuis tensor of $J$ is determined by the latter formula but with $J^2 = -I$, where $I$ stands for the identity, i.e. (1.18) is valid.

Besides that, by (1.20) it is defined the associated Nijenhuis tensor $\{J, J\}$ of the almost complex structure $J$ and the pseudo-Riemannian metric $g$.

Bearing in mind (12.1), we give the following

**Definition 12.1.** The (1,2)-tensor $\{J, K\}$ for (1,1)-tensors $J$ and $K$, defined by

\[(12.3)\]
\[
2\{J, K\}(x, y) = (JK + KJ)\{x, y\} \\
+ \{Jx, Ky\} - J\{Kx, y\} - J\{x, Ky\} \\
+ \{Kx, Jy\} - K\{Jx, y\} - K\{x, Jy\},
\]

is called an associated Nijenhuis tensor of two endomorphisms on a pseudo-Riemannian manifold.

Obviously, it is symmetric with respect to tensor (1,1)-fields, i.e.

\[(12.4)\]
\[
\{J, K\}(x, y) = \{K, J\}(x, y).
\]

Moreover, it is symmetric with respect to vector fields, i.e.

\[
\{J, K\}(x, y) = \{J, K\}(y, x).
\]

In the case when some of the tensor fields in $\{J, K\}$ is the identity $I$ (let us suppose $K = I$), then (12.3) yields that the corresponding associated
Nijenhuis tensor vanishes, i.e.

\[(12.5) \quad \{J, I\}(x, y) = 0.\]

For the case of two identical tensor (1,1)-fields, \[(12.3)\] implies

\[(12.6) \quad \{J, J\}(x, y) = J^2\{x, y\} + \{Jx, Jy\} - J\{x, Jy\} - J\{x, y\}.\]

Let \(L\) be also a tensor field of type (1,1) and \(S\) be a tensor field of type (1,2). Further, we use the following notation of Frölicher-Nijenhuis from \[33\]

\[(12.7) \quad (S \bar{\wedge} L)(x, y) = S(Lx, y) + S(x, Ly),\]

\[(12.8) \quad (L \bar{\wedge} S)(x, y) = L(S(x, y)).\]

According to \[150\], the following properties of the latter notation are valid

\[(12.9) \quad (S \bar{\wedge} J) \bar{\wedge} K - (S \bar{\wedge} K) \bar{\wedge} J = S \bar{\wedge} JK - S \bar{\wedge} KJ,\]

\[(12.10) \quad (J \bar{\wedge} S) \bar{\wedge} K = J \bar{\wedge} (S \bar{\wedge} K).\]

**Lemma 12.1.** For arbitrary endomorphisms \(J, K\) and \(L\) in \(\mathfrak{X}(\mathcal{M})\) and the relevant associated Nijenhuis tensors, the following identity holds

\[(12.11) \quad \{J, KL\} + \{K, JL\} = \{J, K\} L + J \bar{\wedge} \{K, L\} + K \bar{\wedge} \{J, L\}.\]

**Proof.** Using \[(12.3), (12.7)\) and \[(12.8)\], we get consequently

\[
\begin{aligned}
&\{J, K\} L(x, y) + (J \bar{\wedge} \{K, L\})(x, y) + (K \bar{\wedge} \{J, L\})(x, y) \\
&= \{J, K\}(Lx, y) + \{J, K\}(x, Ly) + J(\{K, L\}(x, y)) \\
&\quad + \{J, L\}(x, y) \\
&= \frac{1}{2} \left[ (JK + KJ)\{Lx, y\} + JKLx, Ky + KKLx, Jy \right. \\
&\quad - J\{Klx, y\} - K\{JLx, y\} - J\{Lx, Ky\} - K\{Lx, Jy\} \\
&\quad + (JK + KJ)x, Ly + \{Jx, KLy\} + \{Kx, JLy\} \\
&\quad - J\{Kx, Ly\} - K\{Jx, Ly\} - \{Jx, KLy\} - K\{x, JLy\} \\
&\quad + J(KL + LK)x, y + \{Kx, Ly\} + J\{Lx, Ky\} \\
&\quad - JK\{Lx, y\} - JL\{Kx, y\} - JK\{x, Ly\} - JL\{x, Ky\} \\
&\quad + K(JL + LJ)x, y + K\{Jx, Ly\} + K\{Lx, Jy\} \\
&\quad - KJ\{Lx, y\} - KL\{Jx, y\} - KJ\{x, Ly\} - KL\{x, Jy\}\right].
\end{aligned}
\]
\[ \frac{1}{2}\left[ \{JLx, Ky\} + \{KLx, Jy\} - J\{KLx, y\} - K\{JLx, y\} \\
+ \{Jx, KLy\} + \{Kx, JLy\} - J\{x, KLy\} - K\{Jx, JLy\} \\
+ J(KL + LK)\{x, y\} - JL\{Kx, y\} - JL\{x, Ky\} \\
+ K(JL + LJ)\{x, y\} - KL\{Jx, y\} - KL\{x, Jy\}\right] \\
= \{J, KL\}(x, y) + \{K, JL\}(x, y). \]

Therefore, (12.11) is proved.

12.2. The associated Nijenhuis tensors of the almost hypercomplex structure

Let \( \mathcal{M} \) be a differentiable manifold of dimension \( 4n \). It admits an almost hypercomplex structure \((J_1, J_2, J_3)\), i.e. a triad of almost complex structures satisfying the properties (9.1).

12.2.1. Relations between the associated Nijenhuis tensors

The presence of three almost complex structures, which form an almost hypercomplex structure, implies the existence of six associated Nijenhuis tensors: three for each almost complex structure and three for each pair of almost complex structures, namely

\( \{J_1, J_1\}, \{J_2, J_2\}, \{J_3, J_3\}, \{J_1, J_2\}, \{J_2, J_3\}, \{J_3, J_1\} \).

They are called the associated Nijenhuis tensors of the almost hypercomplex structure \((J_1, J_2, J_3)\) on \((\mathcal{M}, g)\).

In the present section we examine some relations between them, which we use later.

Lemma 12.2. The following relations between the associated Nijenhuis tensors of an almost hypercomplex manifold are valid:

\[ \{J_3, J_1\} = \frac{1}{2}\{J_1, J_1\} \bar{\wedge} J_2 + J_1 \bar{\wedge} \{J_1, J_2\}, \]
\[ \{J_3, J_1\} = -\{J_1, J_2\} \bar{\wedge} J_1 - J_1 \bar{\wedge} \{J_1, J_2\} - J_2 \bar{\wedge} \{J_1, J_1\}, \]
\[ J_2 \bar{\wedge} \{J_1, J_1\} + \frac{1}{2}\{J_1, J_1\} \bar{\wedge} J_2 \]
\[ + 2J_1 \bar{\wedge} \{J_1, J_2\} + \{J_1, J_2\} \bar{\wedge} J_1 = 0, \]
\[ \{J_2, J_3\} = -\frac{1}{2}\{J_2, J_2\} \bar{\wedge} J_1 - J_2 \bar{\wedge} \{J_1, J_2\}, \]

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(12.16) \[ \{J_2, J_3\} = J_1 \bar{\wedge} \{J_2, J_2\} + \{J_1, J_2\} \bar{\wedge} J_2 + J_2 \bar{\wedge} \{J_1, J_2\}, \]

(12.17) \[ J_1 \bar{\wedge} \{J_2, J_2\} + \frac{1}{2} \{J_2, J_2\} \bar{\wedge} J_1 \]

\[ + \{J_1, J_2\} \bar{\wedge} J_2 + 2J_2 \bar{\wedge} \{J_1, J_2\} = 0, \]

(12.18) \[ \{J_3, J_3\} - \{J_1, J_1\} = \{J_3, J_1\} \bar{\wedge} J_2 + J_3 \bar{\wedge} \{J_1, J_2\} \]

\[ + J_1 \bar{\wedge} \{J_2, J_3\}, \]

(12.19) \[ \{J_3, J_3\} = \frac{1}{2} \{\{J_1, J_1\} + \{J_3, J_1\} \bar{\wedge} J_2 - J_2 \bar{\wedge} \{J_3, J_1\} \]

\[ - \{J_2, J_3\} \bar{\wedge} J_1 + J_1 \bar{\wedge} \{J_2, J_3\}, \]

(12.20) \[ \{J_1, J_1\} - \{J_2, J_2\} + \{J_3, J_1\} \bar{\wedge} J_2 + J_2 \bar{\wedge} \{J_3, J_1\} \]

\[ + 2J_3 \bar{\wedge} \{J_1, J_2\} + \{J_2, J_3\} \bar{\wedge} J_1 + J_1 \bar{\wedge} \{J_2, J_3\} = 0, \]

(12.21) \[ \{J_2, J_2\} \bar{\wedge} J_2 = -2J_2 \bar{\wedge} \{J_2, J_2\}. \]

**Proof.** The identity (12.11) for \( J = K = J_1 \) and \( L = J_2 \) has the form in (12.12), because of \( J_1J_2 = J_3 \) from (9.1) and (12.4).

On the other hand, (12.11) for \( J = J_2 \) and \( K = L = J_1 \) implies

\[ \{J_2, J_1J_1\} + \{J_1, J_2J_1\} = \{J_2, J_1\} \bar{\wedge} J_1 + J_2 \bar{\wedge} \{J_1, J_1\} + J_1 \bar{\wedge} \{J_2, J_1\}. \]

Next, applying \( J_1^2 = -I \), \( J_2J_1 = -J_3 \), which are corollaries of (9.1), as well as (12.4) and (12.5), we obtain (12.13).

By virtue of (12.12) and (12.13) we get (12.14).

Next, setting \( J = J_1 \) and \( K = J_2 \) in (12.11), we find

\[ \{J_2, J_2J_1\} + \{J_2, J_2J_2\} = \{J_2, J_2\} \bar{\wedge} J_1 + J_2 \bar{\wedge} \{J_2, J_1\} + J_2 \bar{\wedge} \{J_2, J_1\}, \]

which is equivalent to (12.15), taking into account (12.4) and \( J_2J_1 = -J_3 \).

On the other hand, setting \( J = J_1 \) and \( K = L = J_2 \) in (12.11), we find

\[ \{J_1, J_2J_2\} + \{J_2, J_1J_2\} = \{J_1, J_2\} \bar{\wedge} J_2 + J_1 \bar{\wedge} \{J_2, J_2\} + J_2 \bar{\wedge} \{J_1, J_2\} \]

and applying (12.5) and \( J_1J_2 = J_3 \), we have (12.16).

By virtue of (12.15) and (12.16) we get (12.17).

Now, setting \( J = J_3 \), \( K = J_1 \) and \( L = J_2 \) in (12.11) and applying \( J_1J_2 = J_3 \), \( J_3J_2 = -J_1 \) and (12.4), we have (12.18).

The equality (12.18) and the resulting equality from it by the substitutions \( J_3, J_1, J_2 \) for \( J_1, J_2, J_3 \), respectively, imply (12.19) and (12.20).

At the end, the identity (12.11) for \( J = K = L = J_2 \) implies (12.21) because of (12.5). \[\square\]
12.2.2. Vanishing of the associated Nijenhuis tensors

Now, we will study at least how many associated Nijenhuis tensors (and which) must be vanished to become all associated Nijenhuis tensors zeros on an almost hypercomplex manifold.

As proof steps of the relevant main theorem, we will prove a series of lemmas.

Lemma 12.3. If \( \{J_1, J_1\} \) and \( \{J_2, J_2\} \) vanish, then \( \{J_1, J_2\}, \{J_2, J_3\}, \{J_3, J_1\} \) and \( \{J_3, J_3\} \) vanish.

Proof. The formulae (12.12) and (12.14), because of \( \{J_1, J_1\} = 0 \), imply respectively

\[
\{J_3, J_1\} = J_1 \bar{\wedge} \{J_1, J_2\},
\]

(12.22)

\[
\{J_1, J_2\} \bar{\wedge} J_1 = -2J_1 \bar{\wedge} \{J_1, J_2\}.
\]

(12.23)

Similarly, since \( \{J_2, J_2\} = 0 \), the equalities (12.15) and (12.17) take the corresponding form

\[
\{J_2, J_3\} = -J_2 \bar{\wedge} \{J_1, J_2\},
\]

(12.24)

\[
\{J_1, J_2\} \bar{\wedge} J_2 = -2J_2 \bar{\wedge} \{J_1, J_2\}.
\]

(12.25)

We set \( \{J_1, J_1\} = 0, \{J_2, J_2\} = 0, \) (12.22) and (12.24) in (12.20) and we obtain

\[
2J_3 \bar{\wedge} \{J_1, J_2\} + (J_1 \bar{\wedge} \{J_1, J_2\}) \bar{\wedge} J_2 + J_2 \bar{\wedge} (J_1 \bar{\wedge} \{J_1, J_2\}) - (J_2 \bar{\wedge} \{J_1, J_2\}) \bar{\wedge} J_1 - J_1 \bar{\wedge} (J_2 \bar{\wedge} \{J_1, J_2\}) = 0.
\]

The latter equality is equivalent to

\[
(J_1 \bar{\wedge} \{J_1, J_2\}) \bar{\wedge} J_2 = (J_2 \bar{\wedge} \{J_1, J_2\}) \bar{\wedge} J_1,
\]

(12.26)

because of the following corollaries of (12.8) and the identities \( J_3 = J_1J_2 = -J_2J_1 \) from (9.1)

\[
J_2 \bar{\wedge} (J_1 \bar{\wedge} \{J_1, J_2\}) = -J_1 \bar{\wedge} (J_2 \bar{\wedge} \{J_1, J_2\})
\]

(12.27)

According to (12.10), (12.23), (12.25) and (12.27), the equality (12.26) yields

\[
0 = (J_1 \bar{\wedge} \{J_1, J_2\}) \bar{\wedge} J_2 - (J_2 \bar{\wedge} \{J_1, J_2\}) \bar{\wedge} J_1
\]

\[
= J_1 \bar{\wedge} (\{J_1, J_2\} \bar{\wedge} J_2) - J_2 \bar{\wedge} (\{J_1, J_2\} \bar{\wedge} J_1)
\]

\[
= -2J_1 \bar{\wedge} (J_2 \bar{\wedge} \{J_1, J_2\}) + 2J_2 \bar{\wedge} (J_1 \bar{\wedge} \{J_1, J_2\})
\]

\[
= -4J_3 \bar{\wedge} \{J_1, J_2\},
\]

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i.e. it is valid
\[(12.28) \quad J_3 \bar{\wedge} \{J_1, J_2\} = 0.\]

Therefore, because of \(J_3^2 = -I\), we get
\[\{J_1, J_2\} = 0.\]

Next, \textbf{(12.22)} and \textbf{(12.24)} imply \(\{J_3, J_1\} = 0\) and \(\{J_2, J_3\} = 0\), respectively. Finally, since \(\{J_1, J_1\}, \{J_2, J_2\}, \{J_2, J_3\}\) and \(\{J_3, J_1\}\) vanish, the relation \textbf{(12.19)} yields \(\{J_3, J_3\} = 0\). \(\square\)

**Lemma 12.4.** If \(\{J_1, J_1\}\) and \(\{J_1, J_2\}\) vanish, then \(\{J_2, J_2\}, \{J_3, J_3\}, \{J_2, J_3\}\) and \(\{J_3, J_1\}\) vanish.

*Proof.* Setting \(J = K = J_2\) and \(L = J_3\) in \textbf{(12.11)}, using \(J_2J_3 = J_1\) and \textbf{(12.4)}, we obtain
\[\{J_1, J_2\} = \frac{1}{2}\{J_2, J_2\} \bar{\wedge} J_3 + J_2 \bar{\wedge} \{J_2, J_3\}.\]

Since \(\{J_1, J_2\} = 0\), we get
\[(12.29) \quad \{J_2, J_2\} \bar{\wedge} J_3 = -2J_2 \bar{\wedge} \{J_2, J_3\}.\]

The condition \(\{J_1, J_2\} = 0\) and \textbf{(12.16)} imply
\[\{J_2, J_3\} = J_1 \bar{\wedge} \{J_2, J_2\}.\]

The latter equality and \textbf{(12.29)}, using \(J_2J_1 = -J_3\), yield
\[\{J_2, J_2\} \bar{\wedge} J_3 = -2J_2 \bar{\wedge} \left(J_1 \bar{\wedge} \{J_2, J_2\}\right) = 2J_3 \bar{\wedge} \{J_2, J_2\},\]

that is
\[(12.30) \quad \{J_2, J_2\} \bar{\wedge} J_3 = 2J_3 \bar{\wedge} \{J_2, J_2\}.\]

On the other hand, setting \(S = \{J_2, J_2\}, J = J_1\) and \(K = J_2\) in \textbf{(12.9)} and using \(J_1J_2 = -J_2J_1 = J_3\), we obtain
\[(12.31) \quad 2\{J_2, J_2\} \bar{\wedge} J_3 = \left(\{J_2, J_2\} \bar{\wedge} J_1\right) \bar{\wedge} J_2 - \left(\{J_2, J_2\} \bar{\wedge} J_2\right) \bar{\wedge} J_1.\]

However, since \(\{J_1, J_2\} = 0\), then \textbf{(12.17)} and \textbf{(12.18)} imply
\[(12.32) \quad \{J_2, J_2\} \bar{\wedge} J_1 = -2J_1 \bar{\wedge} \{J_2, J_2\}.\]

Now, substituting \textbf{(12.21)} and \textbf{(12.32)} into \textbf{(12.31)}, we obtain
\[\{J_2, J_2\} \bar{\wedge} J_3 = -\left(J_1 \bar{\wedge} \{J_2, J_2\}\right) \bar{\wedge} J_2 + \left(J_2 \bar{\wedge} \{J_2, J_2\}\right) \bar{\wedge} J_1\]
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and applying (12.10), we have

\[ \{ J_2, J_2 \} \bar{\wedge} J_3 = -J_1 \bar{\wedge} (\{ J_2, J_2 \} \bar{\wedge} J_2) + J_2 \bar{\wedge} (\{ J_2, J_2 \} \bar{\wedge} J_1). \]

In the latter equality, applying (12.21), (12.32) and \( J_1 J_2 = -J_2 J_1 = J_3 \), we get

\[ \{ J_2, J_2 \} \bar{\wedge} J_3 = 2J_1 \bar{\wedge} (J_2 \bar{\wedge} \{ J_2, J_2 \}) - 2J_2 \bar{\wedge} (J_1 \bar{\wedge} \{ J_2, J_2 \}) \]

\[ = 4J_3 \bar{\wedge} \{ J_2, J_2 \}, \]

that is

(12.33) \[ \{ J_2, J_2 \} \bar{\wedge} J_3 = 4J_3 \bar{\wedge} \{ J_2, J_2 \}. \]

Comparing (12.30) and (12.33), we conclude that

\[ J_3 \bar{\wedge} \{ J_2, J_2 \} = 0, \]

which is equivalent to

\[ \{ J_2, J_2 \} = 0 \]

by virtue of \( J_3^2 = -I \). This completes the proof of the first assertion in the lemma.

Combining it with Lemma 12.3, we establish the truthfulness of the whole lemma.

\[ \square \]

Lemma 12.5. If \( \{ J_1, J_2 \} \) and \( \{ J_3, J_1 \} \) vanish, then \( \{ J_1, J_1 \} \), \( \{ J_2, J_2 \} \), \( \{ J_3, J_3 \} \) and \( \{ J_2, J_3 \} \) vanish.

*Proof.* From (12.13) and the vanishing of \( \{ J_1, J_2 \} \) and \( \{ J_3, J_1 \} \), we get

\[ J_2 \bar{\wedge} \{ J_1, J_1 \} = 0, \]

which is equivalent to

\[ \{ J_1, J_1 \} = 0. \]

Now, combining the latter assertion and Lemma 12.4, we have the validity of the present lemma.

\[ \square \]

Lemma 12.6. If \( \{ J_1, J_1 \} \) and \( \{ J_2, J_3 \} \) vanish, then \( \{ J_2, J_2 \} \), \( \{ J_3, J_3 \} \), \( \{ J_1, J_2 \} \) and \( \{ J_3, J_1 \} \) vanish.

*Proof.* Firstly, from (12.12) and \( \{ J_1, J_1 \} = \{ J_2, J_3 \} = 0 \), we obtain

(12.34) \[ \{ J_3, J_1 \} = J_1 \bar{\wedge} \{ J_1, J_2 \}. \]

Secondly, setting \( J = K = J_2 \) and \( L = J_3 \) in (12.11) and using the equalities \( J_2 J_3 = J_1 \), \( \{ J_1, J_1 \} = \{ J_2, J_3 \} = 0 \) and (12.4), we get

(12.35) \[ 2\{ J_1, J_2 \} = \{ J_2, J_2 \} \bar{\wedge} J_3. \]
On the other hand, setting $J = L = J_2$ and $K = J_3$ in (12.11), using the assumptions, as well as $J_2^2 = -I$, $J_3 J_2 = -J_1$ and (12.5), we find

$$\{J_1, J_2\} = -J_3 \wedge \{J_2, J_2\},$$

which, because of $J_3^2 = -I$, is equivalent to

$$\{J_2, J_2\} = J_3 \wedge \{J_1, J_2\}. \tag{12.36}$$

Moreover, the formula (12.20) and $\{J_1, J_1\} = \{J_2, J_3\} = 0$ yield

$$2 J_3 \wedge \{J_1, J_2\} + \{J_3, J_1\} \wedge J_2 + J_2 \wedge \{J_3, J_1\} - \{J_2, J_2\} = 0.$$  

The latter equality, (12.34) and (12.36), using $J_2 J_1 = -J_3$, (12.8) and (12.10), imply

$$J_1 \wedge (\{J_1, J_2\} \wedge J_2) = 0$$

and therefore

$$\{J_1, J_2\} \wedge J_2 = 0. \tag{12.37}$$

By virtue of (12.36), (12.37) and (12.10), we get

$$\{J_2, J_2\} \wedge J_2 = 0. \tag{12.38}$$

On the other hand, (12.21) and (12.38) imply

$$J_2 \wedge \{J_2, J_2\} = 0,$$

which is equivalent to

$$\{J_2, J_2\} = 0$$

and the first assertion in the present lemma is proved.

Combining it with Lemma 12.3, we obtain the validity of the rest equalities.

Now, we are ready to prove the main theorem in the present section.

**Theorem 12.7.** If two of the six associated Nijenhuis tensors

$$\{J_1, J_1\}, \quad \{J_2, J_2\}, \quad \{J_3, J_3\}, \quad \{J_1, J_2\}, \quad \{J_2, J_3\}, \quad \{J_3, J_1\}$$

vanish, then the others also vanish.

**Proof.** The truthfulness of this theorem follows from Lemma 12.3, Lemma 12.4, Lemma 12.5 and Lemma 12.6.
12.3. Natural connections with totally skew-symmetric torsion on almost hypercomplex manifolds with Hermitian-Norden metrics

Let $g$ be a pseudo-Riemannian metric on an almost hypercomplex manifold $(\mathcal{M}, J_1, J_2, J_3)$ defined by (9.2). Then, we call that the almost hypercomplex manifold is equipped with Hermitian-Norden metrics. Namely, the metric $g$ is Hermitian for $\alpha = 1$, whereas $g$ is a Norden metric in the cases $\alpha = 2$ and $\alpha = 3$ [46, 97].

Let us consider $(\mathcal{M}, J_1, g)$ belonging to $G_1 = (\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4)(J_1)$ (the class of cocalibrated manifolds with Hermitian metric), according to the classification (9.10) from [44]. Moreover, let $(\mathcal{M}, J_\alpha, g)$, $(\alpha = 2; 3)$ belong to $\mathcal{W}_3(J_\alpha)$ (the class of quasi Kähler manifolds with Norden metric), according to the classification (9.11) from [34]. The mentioned classes are determined in terms of the fundamental tensors $F_\alpha$, defined by (9.5), as follows:

\begin{align}
G_1(J_1) : & \quad F_1(x, x, z) = F_1(J_1x, J_1x, z), \\
(12.39) & \\
\mathcal{W}_3(J_\alpha) : & \quad F_\alpha(x, y, z) + F_\alpha(y, z, x) + F_\alpha(z, x, y) = 0
\end{align}

for $\alpha = 2; 3$.

**Remark 12.2.** It is known from [44] that the class $G_1$ of almost Hermitian manifolds $(\mathcal{M}, J_1, g)$ exists in general form when the dimension of $\mathcal{M}$ is at least 6. At dimension 4, $G_1$ is restricted to its subclass $\mathcal{W}_4$, the class of locally conformally equivalent manifolds to Kähler manifolds with Norden metric. According to [34], the lowest dimension for almost Norden manifolds in the class $\mathcal{W}_3$ is 4. Thus, the almost hypercomplex manifold with Hermitian-Norden metrics belonging to the classes $G_1(J_1)$, $\mathcal{W}_3(J_2)$, $\mathcal{W}_3(J_3)$ exists in general form when $\dim \mathcal{M} = 4n \geq 8$ holds.

Let the corresponding tensors of type $(0,3)$ with respect to $g$ of the pair of Nijenhuis tensors be denoted by

\[
[J_\alpha, J_\alpha](x, y, z) = g([J_\alpha, J_\alpha](x, y), z), \\
\{J_\alpha, J_\alpha\}(x, y, z) = g(\{J_\alpha, J_\alpha\}(x, y), z).
\]

These tensors can be expressed by $F_\alpha$, as follows:

\begin{align}
(12.41) & \\
[J_\alpha, J_\alpha](x, y, z) = F_\alpha(J_\alpha x, y, z) + \varepsilon_\alpha F_\alpha(x, y, J_\alpha z) \\
& \quad - F_\alpha(J_\alpha y, x, z) - \varepsilon_\alpha F_\alpha(y, x, J_\alpha z),
\end{align}
\[
\{J_\alpha, J_\alpha\}(x, y, z) = F_\alpha(J_\alpha x, y, z) + \varepsilon_\alpha F_\alpha(x, y, J_\alpha z) \\
+ F_\alpha(J_\alpha y, x, z) + \varepsilon_\alpha F_\alpha(y, x, J_\alpha z).
\]

In the case \(\alpha = 2; 3\), the latter formulae are given in [34], as \(\{J_\alpha, J_\alpha\}\) coincides with the tensor \(\tilde{N}\) introduced there by the second equality in (1.21).

In [44], it is given an equivalent definition of \(G_1\) by the condition the Nijenhuis tensor \([J_1, J_1](x, y, z)\) to be a 3-form, i.e.
\[
(12.43) \quad G_1(J_1) : [J_1, J_1](x, y, z) = -[J_1, J_1](x, z, y).
\]

**Proposition 12.8.** For the Nijenhuis tensor and the associated Nijenhuis tensor of \((\mathcal{M}, J_1, g)\) we have:

(i) the following relation
\[
(12.44) \quad \{J_1, J_1\}(x, y, z) = [J_1, J_1](z, x, y) + [J_1, J_1](z, y, x);
\]

(ii) \(\{J_1, J_1\}\) vanishes if and only if \([J_1, J_1]\) is a 3-form.

**Proof.** We compute the right hand side of (12.44) using (12.41). Applying (9.6) and their consequence
\[
(12.45) \quad F_1(x, y, J_1 z) = F_1(x, J_1 y, z),
\]
we obtain
\[
[J_1, J_1](z, x, y) + [J_1, J_1](z, y, x) = -F_1(J_1 x, z, y) - F_1(x, z, J_1 y) \\
- F_1(J_1 y, z, x) - F_1(y, z, J_1 x).
\]

Using again (12.45) and the first equality in (9.6), we establish that the right hand side of the latter equality is equal to \(\{J_1, J_1\}(x, y, z)\), according to (12.42) for \(\alpha = 1\).

The identity in (ii) follows immediately from (i).

The assertion (ii) of Proposition 12.8 and (12.43) imply the following

**Proposition 12.9.** The manifolds in the class \(G_1(J_1)\) are characterized by the condition \(\{J_1, J_1\} = 0\).

For almost Norden manifolds it is known from [34] that the manifolds in the class \(\mathcal{W}_3(J_2)\) (respectively, \(\mathcal{W}_3(J_3)\)) are characterized by the condition \(\{J_2, J_2\} = 0\) (respectively, \(\{J_3, J_3\} = 0\)).

From Theorem 12.7 we have immediately that if two of associated Nijenhuis tensors \(\{J_1, J_1\}, \{J_2, J_2\}, \{J_3, J_3\}\) vanish, then the third one also vanishes. Thus, we establish the truthfulness of the following
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**Theorem 12.10.** If an almost hypercomplex manifold with Hermitian-Norden metrics belongs to two of the classes \( \mathcal{G}_1(J_1), \mathcal{W}_3(J_2), \mathcal{W}_3(J_3) \) with respect to the corresponding almost complex structures, then it belongs also to the third class.

In [97], it is proved that if \( \mathcal{M} \) is in \( \mathcal{W}_3(J_2) \) and \( \mathcal{W}_3(J_3) \), then it belongs to \( \mathcal{G}_1(J_1) \).

For the almost Hermitian manifold \( (\mathcal{M}, J_1, g) \) we apply Theorem 10.1 in [31]. Then there exists an affine connection \( D^1 \) with totally skew-symmetric torsion \( T_1 \) preserving \( J_1 \) and \( g \) if and only if \( (\mathcal{M}, J_1, g) \) belongs to \( \mathcal{G}_1(J_1) \). In this case \( D^1 \) is unique and determined by

\[
T_1(x, y, z) = d\tilde{g}_1(J_1x, J_1y, J_1z) + [J_1, J_1](x, y, z),
\]

where \( \tilde{g}_1 \) is the Kähler form determined by (9.4).

Using properties (9.6) and (12.41) for \( \alpha = 1 \), the relation \( d\tilde{g}_1 = \mathfrak{S}F_1 \), where \( \mathfrak{S} \) is the cyclic sum by three arguments, we get the expression of \( T_1 \) in terms of \( F_1 \) as follows

\[
T_1(x, y, z) = F_1(x, y, J_1z) - F_1(y, x, J_1z) - F_1(J_1z, x, y).
\]

On the other side, for the case \( \alpha = 2 \) or \( \alpha = 3 \) we dispose with an almost Norden manifold \( (\mathcal{M}, J_\alpha, g) \). Then, according to Theorem 3.1 in [112], there exists an affine connection \( D^\alpha \) with totally skew-symmetric torsion \( T_\alpha \) preserving \( J_\alpha \) and \( g \) if and only if \( (\mathcal{M}, J_\alpha, g) \) belongs to \( \mathcal{W}_3(J_\alpha) \). In this case \( D^\alpha \) is unique and determined, according to (3.13), by the following expression of its torsion

\[
T_\alpha(x, y, z) = -\frac{1}{2} \mathfrak{S}_{x,y,z} F_\alpha(x, y, J_\alpha z), \quad \alpha = 2, 3.
\]

By virtue of Theorem 12.10 and the comments above, we obtain the validity of the following

**Theorem 12.11.** Let \( (\mathcal{M}, H, G) \) be an almost hypercomplex manifold with Hermitian-Norden metrics. Then it admits an affine connection \( D^* \) with totally skew-symmetric torsion preserving the structure \( (H, G) \) if and only if two of the three associated Nijenhuis tensors \( \{J_\alpha, J_\alpha\} \) (\( \alpha = 1, 2, 3 \)) vanish and the equalities \( T_1 = T_2 = T_3 \) are valid, bearing in mind (12.47) and (12.48). If \( D^* \) exists, it is unique and determined by its torsion \( T^* = T_1 = T_2 = T_3 \).
12.4. A 4-dimensional example

In [46], it is considered a connected Lie group \( L \) with a corresponding Lie algebra \( \mathfrak{l} \), determined by the following conditions for the global basis of left invariant vector fields \( \{x_1, x_2, x_3, x_4\} \):

\[
\begin{align*}
[x_1, x_3] &= \lambda_2 x_2 + \lambda_4 x_4, \\
[x_2, x_4] &= \lambda_1 x_1 + \lambda_3 x_3, \\
[x_3, x_2] &= \lambda_2 x_1 + \lambda_3 x_4, \\
[x_4, x_3] &= \lambda_4 x_1 - \lambda_3 x_2, \\
[x_4, x_1] &= \lambda_1 x_2 + \lambda_4 x_3, \\
[x_1, x_2] &= \lambda_2 x_3 - \lambda_1 x_4,
\end{align*}
\]

(12.49)

where \( \lambda_i \in \mathbb{R} \) (\( i = 1, 2, 3, 4 \)) and \( (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \neq (0, 0, 0, 0) \). The pseudo-Riemannian metric \( g \) is defined by

\[
\begin{align*}
g(x_1, x_1) &= g(x_2, x_2) = -g(x_3, x_3) = -g(x_4, x_4) = 1, \\
g(x_i, x_j) &= 0, \quad i \neq j.
\end{align*}
\]

(12.50)

There, it is introduced an almost hypercomplex structure \( (J_1, J_2, J_3) \) on \( L \) as follows:

\[
\begin{align*}
J_1 x_1 &= x_2, & J_2 x_1 &= x_3, & J_3 x_1 &= -x_4, \\
J_1 x_2 &= -x_1, & J_2 x_2 &= x_4, & J_3 x_2 &= x_3, \\
J_1 x_3 &= -x_4, & J_2 x_3 &= -x_1, & J_3 x_3 &= -x_2, \\
J_1 x_4 &= x_3, & J_2 x_4 &= -x_2, & J_3 x_4 &= x_1
\end{align*}
\]

(12.51)

and then (9.1) are valid.

In [48], it is constructed the manifold \( (L, J_2, g) \) as an example of a 4-dimensional quasi-Kähler manifold with Norden metric. The conditions (12.50) and (12.51) imply the properties (9.2) and therefore the introduced structure on \( L \) is an almost hypercomplex structure with Hermitian-Norden metrics. Hence, it follows that the constructed manifold is an almost hypercomplex manifold with Hermitian-Norden metrics. Moreover, in [46], it is shown that this manifold belongs to basic classes \( \mathcal{W}_4(J_1), \mathcal{W}_3(J_2), \mathcal{W}_3(J_3) \) with respect to the corresponding almost complex structures. These conclusions are made using the following nonzero basic components of \( F_\alpha \):

\[
\begin{align*}
\frac{1}{2} \lambda_1 &= (F_1)_{124} = -(F_1)_{132} = (F_1)_{132} = -(F_1)_{141} \\
&= (F_1)_{213} = (F_1)_{224} = -(F_1)_{231} = -(F_1)_{242}, \\
\frac{1}{2} \lambda_2 &= -(F_1)_{113} = -(F_1)_{124} = (F_1)_{131} = (F_1)_{142} \\
&= (F_1)_{214} = -(F_1)_{223} = (F_1)_{232} = -(F_1)_{241},
\end{align*}
\]
\[ \frac{1}{2}\lambda_3 = -(F_1)_{314} = (F_1)_{323} = -(F_1)_{332} = (F_1)_{341} = (F_1)_{413} = (F_1)_{424} = -(F_1)_{431} = -(F_1)_{442}, \]
\[ \frac{1}{2}\lambda_4 = -(F_1)_{313} = -(F_1)_{324} = (F_1)_{331} = (F_1)_{342} = -(F_1)_{414} = (F_1)_{423} = -(F_1)_{432} = (F_1)_{441}, \]
\[ \frac{1}{2}\lambda_1 = -\frac{1}{2}(F_2)_{122} = -\frac{1}{2}(F_2)_{144} = (F_2)_{212} = (F_2)_{221} = (F_2)_{234} = (F_2)_{243} = -(F_2)_{414} = -(F_2)_{423} = -(F_2)_{432} = (F_2)_{441}, \]
\[ \frac{1}{2}\lambda_2 = (F_2)_{112} = (F_2)_{121} = (F_2)_{134} = (F_2)_{143} = -\frac{1}{2}(F_2)_{211} = -(F_2)_{233} = -(F_2)_{314} = (F_2)_{323} = (F_2)_{332} = -(F_2)_{341}, \]
\[ \frac{1}{2}\lambda_3 = (F_2)_{214} = -(F_2)_{223} = -(F_2)_{232} = (F_2)_{241} = \frac{1}{2}(F_2)_{322} = \frac{1}{2}(F_2)_{344} = -(F_2)_{412} = -(F_2)_{421} = -(F_2)_{434} = -(F_2)_{443}, \]
\[ \frac{1}{2}\lambda_4 = -(F_2)_{114} = (F_2)_{123} = (F_2)_{132} = -(F_2)_{141} = -(F_2)_{312} = -(F_2)_{321} = -(F_2)_{334} = -(F_2)_{343} = \frac{1}{2}(F_2)_{411} = \frac{1}{2}(F_2)_{433}; \]
\[ \frac{1}{2}\lambda_1 = (F_3)_{112} = (F_3)_{121} = -(F_3)_{134} = -(F_3)_{143} = -\frac{1}{2}(F_3)_{211} = -\frac{1}{2}(F_3)_{244} = (F_3)_{413} = (F_3)_{431} = (F_3)_{424} = (F_3)_{442}, \]
\[ \frac{1}{2}\lambda_2 = \frac{1}{2}(F_3)_{122} = \frac{1}{2}(F_3)_{133} = -(F_3)_{212} = -(F_3)_{221} = (F_3)_{234} = (F_3)_{243} = -(F_3)_{313} = -(F_3)_{331} = -(F_3)_{324} = -(F_3)_{342}, \]
\[ \frac{1}{2}\lambda_3 = (F_3)_{213} = (F_3)_{231} = (F_3)_{224} = (F_3)_{242} = -(F_3)_{212} = -(F_3)_{321} = -(F_3)_{343} = (F_3)_{334} = -(F_3)_{422} = -\frac{1}{2}(F_3)_{422} = -\frac{1}{2}(F_3)_{443}, \]
\[ \frac{1}{2}\lambda_4 = -(F_3)_{113} = -(F_3)_{124} = -(F_3)_{131} = -(F_3)_{142} = \frac{1}{2}(F_3)_{311} = \frac{1}{2}(F_3)_{344} = (F_3)_{412} = (F_3)_{421} = -(F_3)_{434} = -(F_3)_{443}. \]

Bearing in mind the discussions in the previous two subsections, this is an example of a 4-dimensional manifold with vanishing associated Nijenhuis tensors for the almost hypercomplex structure and there exist affine connections \( D^\alpha \) with totally skew-symmetric torsion \( T_\alpha \) preserving \( J_\alpha \) and \( g \). Using (12.47), (12.48) and the components above, we compute the basic components of \( T_\alpha \). They are determined by the following nonzero components for \( \alpha = 1, 2, 3 \):

\[ (T_\alpha)_{123} = \lambda_2, \quad (T_\alpha)_{124} = -\lambda_1, \quad (T_\alpha)_{134} = \lambda_4, \quad (T_\alpha)_{234} = -\lambda_3. \]

Therefore the connections \( D^1, D^2 \) and \( D^3 \) coincide. Then, according to Theorem [12.11] \( (\mathcal{L}, H, G) \) admits a unique affine connection \( D^* \) with totally skew-symmetric torsion \( T^* \) preserving the structure \( (H, G) \) and it
is determined by $T^*$ with nonzero components

$$T^*_{123} = \lambda_2, \quad T^*_{124} = -\lambda_1, \quad T^*_{134} = \lambda_4, \quad T^*_{234} = -\lambda_3.$$
§13. Quaternionic Kähler manifolds with Hermitian-Norden metrics

In the present section, almost hypercomplex manifolds with Hermitian-Norden metrics and more specially the corresponding quaternionic Kähler manifolds are considered. Some necessary and sufficient conditions for the studied manifolds to be isotropic hyper-Kählerian and flat are found. It is proved that the quaternionic Kähler manifolds with the considered metric structure are Einstein for dimension at least 8. The class of the non-hyper-Kähler quaternionic Kähler manifold of the considered type is determined.

The main results of this section are published in [84].

The basic problem of this section is the existence and the geometric characteristics of the quaternionic Kähler manifolds with Hermitian-Norden metrics. The main results here is that every quaternionic Kähler manifold with Hermitian-Norden metrics is Einstein for dimension at least 8 and it is not flat hyper-Kählerian only when belongs to the general class $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ or the class $\mathcal{W}_1 \oplus \mathcal{W}_3$, where the manifold is Ricci-symmetric.

The present section is organised as follows. In Subsection 13.1 we introduce the corresponding quaternionic Kähler manifold of an almost hypercomplex manifold with Hermitian-Norden metrics. We establish that the quaternionic Kähler manifolds with Hermitian-Norden metrics are Einstein for dimension $4n \geq 8$. For comparison, it is known that the quaternionic Kähler manifolds with hyper-Hermitian metric structure are Einstein for all dimensions [1]. In Subsection 13.2 we consider the location of the quaternionic Kähler manifolds with Hermitian-Norden metrics in the classification of the corresponding almost hypercomplex manifolds with respect to the covariant derivatives of the almost complex structures. We get only one class (except the general one) of the considered classification where these manifolds are non-hyper-Kählerian and consequently non-flat always. In Subsection 13.3 we characterize the non-hyper-Kähler
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§13. Quaternionic Kähler manifolds with Hermitian-Norden metrics

Quaternionic Kähler manifolds with Hermitian-Norden metrics obtained in the previous subsection.

13.1. Quaternionic Kähler manifolds with Hermitian-Norden metrics

Let us consider again only an almost hypercomplex manifold \((\mathcal{M}, H)\). The endomorphism \(Q = \lambda_1 J_1 + \lambda_2 J_2 + \lambda_3 J_3\), \(\lambda_i \in \mathbb{R}\), is called a quaternionic structure on \((\mathcal{M}, H)\) with an admissible basis \(H\). Let \(D\) be the Levi-Civita connection of a pseudo-Riemannian metric \(g\) on \(\mathcal{M}\). A quaternionic structure with the condition \(DQ = 0\) is called a quaternionic Kähler structure on \((\mathcal{M}, H)\). An almost hypercomplex manifold with quaternionic Kähler structure is determined by

\[
(D_J^\alpha) y = \omega_\gamma(x) J_\beta y - \omega_\beta(x) J_\gamma y
\]

for all cyclic permutations \((\alpha, \beta, \gamma)\) of \((1, 2, 3)\), where \(\omega_\alpha\) are local 1-forms associated to \(H = (J_\alpha)\), \((\alpha = 1, 2, 3)\). \([127, 2]\)

Next, we equip the quaternionic Kähler manifold with a structure of Hermitian-Norden metrics \(G = (g, g_1, g_2, g_3)\), determined by \((9.2)\) and \((9.4)\), and we obtain a quaternionic Kähler manifold with Hermitian-Norden metrics.

Bearing in mind \((13.1)\) and \((9.9)\), for a quaternionic Kähler manifold with Hermitian-Norden metrics we obtain the following form of the square norm of \(D_J^\alpha\):

\[
\|D_J^\alpha\|^2 = 4n \left\{ \varepsilon_\beta \omega_\gamma (\omega_{\beta}^\gamma) + \varepsilon_\gamma \omega_\beta (\omega_{\beta}^\gamma) \right\},
\]

where \(\omega_1^\alpha, \omega_2^\alpha, \omega_3^\alpha\) are the corresponding vectors of \(\omega_1, \omega_2, \omega_3\) regarding \(g\), respectively. The coefficients \(\varepsilon_1, \varepsilon_2\) and \(\varepsilon_3\) are defined in \((9.3)\).

Therefore, we have immediately the following

**Proposition 13.1.** A quaternionic Kähler manifold with Hermitian-Norden metrics is an isotropic hyper-Kähler manifold with Hermitian-Norden metrics if and only if the corresponding vectors of the 1-forms \(\omega_1, \omega_2\) and \(\omega_3\) with respect to \(g\) are isotropic vectors regarding \(g\).

Bearing in mind \((13.1)\), we obtain the following property of the curvature tensor \(R\) of \(D\) for all cyclic permutations \((\alpha, \beta, \gamma)\) of \((1, 2, 3)\):

\[
R(x, y) J_\alpha z = J_\alpha R(x, y) z - \Psi_\beta (x, y) J_\gamma z + \Psi_\gamma (x, y) J_\beta z,
\]

where

\[
\Psi_\beta (x, y) = d\omega_\beta (x, y) + \omega_\gamma (x) \omega_\alpha (y) - \omega_\alpha (x) \omega_\gamma (y)
\]
are 2-forms associated to the local 1-forms $\omega_1, \omega_2, \omega_3$. Therefore, we have

$$R(x, y, z, w) - \varepsilon_\alpha R(x, y, J_\alpha z, J_\alpha w) = \Psi_\beta(x, y)\tilde{g}_\beta(z, w) + \Psi_\gamma(x, y)\tilde{g}_\gamma(z, w).$$

(13.5)

According to the antisymmetry of $R$ by the third and the forth entries, we establish that $\Psi_2 = \Psi_3 = 0$, i.e.

**Lemma 13.2.** The local 1-forms $\omega_1, \omega_2$ and $\omega_3$, determining a quaternionic Kähler manifold with Hermitian-Norden metrics, satisfy the following identities

$$d\omega_2(x, y) = -\omega_3(x)\omega_1(y) + \omega_1(x)\omega_3(y),$$

(13.6)

$$d\omega_3(x, y) = -\omega_1(x)\omega_2(y) + \omega_2(x)\omega_1(y).$$

Then, according to (13.6), equations (13.5) take the form

$$R(x, y, J_1 z, J_1 w) = R(x, y, z, w),$$

(13.7)

$$R(x, y, J_2 z, J_2 w) = R(x, y, J_3 z, J_3 w)$$

(13.8)

$$= -R(x, y, z, w) + \Psi_1(x, y)\tilde{g}_1(z, w).$$

Bearing in mind (13.7), (13.8) and (9.13), we have immediately

**Lemma 13.3.** The curvature tensor $R$ of a quaternionic Kähler manifold with Hermitian-Norden metrics is of Kähler-type if and only if $\Psi_1 = 0$, i.e. the following condition is valid

$$d\omega_1(x, y) = -\omega_2(x)\omega_3(y) + \omega_3(x)\omega_2(y).$$

(13.9)

According to Lemma 13.3 and Theorem 9.1, we have

**Proposition 13.4.** The necessary and sufficient condition an arbitrary quaternionic Kähler manifold with Hermitian-Norden metrics to be flat is condition (13.9).

**Lemma 13.5.** The Ricci tensor $\rho$ and the 2-form $\Psi_1$, defined by (13.4), have the following relation on any quaternionic Kähler manifold with Hermitian-Norden metrics:

$$\rho(x, y) = n\Psi_1(J_1 x, y).$$

(13.10)

*Proof.* From (13.8) for $z \to e_i, w \to J_1 e_j$ by contraction with $g^{ij}$ we have

$$-g^{ij}R(x, y, J_2 e_i, J_3 e_j) = g^{ij}R(x, y, J_3 e_i, J_2 e_j)$$

(13.11)

$$= -g^{ij}R(x, y, e_i, J_1 e_j) + 4n\Psi_1(x, y).$$
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Bearing in mind the antisymmetry on the second pair arguments of $R$ and $J_1 = J_2 J_3$, we get

\[-g^{ij} R(x, y, J_2 e_i, J_3 e_j) = g^{ij} R(x, y, J_3 e_i, J_2 e_j) = g^{ij} R(x, y, e_i, J_1 e_j)\]

and therefore from (13.11) we have

(13.12) \[ g^{ij} R(x, y, e_i, J_1 e_j) = 2n \Psi_1(x, y). \]

After that, from (13.12), applying the properties of $R$, (9.2) for $\alpha = 1$ and (13.7), we obtain consequently

\[2n \Psi_1(x, y) = g^{ij} R(x, y, e_i, J_1 e_j) = g^{ij} \left\{-R(x, e_i, J_1 e_j, y) - R(x, J_1 e_j, y, e_i)\right\} = g^{ij} R(x, e_i, y, J_1 e_j) + g^{ij} R(x, e_j, y, J_1 e_i) = 2g^{ij} R(x, e_i, y, J_1 e_j) = 2g^{ij} R(e_i, x, J_1 y, e_j) = 2\rho(x, J_1 y),\]

i.e. we get

(13.13) \[ \Psi_1(x, y) = \frac{1}{n} \rho(x, J_1 y). \]

Because of the symmetry of $\rho$ and the antisymmetry of $\Psi_1$ we have the property

(13.14) \[ \Psi_1(x, J_1 y) = -\Psi_1(J_1 x, y) \]

and therefore

(13.15) \[ \rho(J_1 x, J_1 y) = \rho(x, y). \]

Hence, from (13.13), (13.14) and (13.15), we obtain (13.10). \(\square\)

**Proposition 13.6.** A quaternionic Kähler manifold with Hermitian-Norden metrics is Ricci-flat if and only if it is flat.

**Proof.** Using Lemma 13.5 property (13.8) takes the form

\[ R(x, y, J_2 z, J_2 w) = R(x, y, J_3 z, J_3 w) \]

(13.16) \[ = -R(x, y, z, w) - \frac{1}{n} \rho(J_1 x, y) g(J_1 z, w). \]

Then, according to (13.16), (9.13) and Theorem 9.2, we obtain the equivalence in the statement. \(\square\)
Theorem 13.7. Quaternionic Kähler manifolds with Hermitian-Norden metrics are Einstein for dimension $4n \geq 8$.

Proof. By virtue of (13.7), (13.16) and (13.15) we obtain the following properties

\[(13.17) \quad R(J_1x, J_1y, J_1z, J_1w) = R(x, y, z, w), \]
\[R(J_2x, J_2y, J_2z, J_2w) = R(J_3x, J_3y, J_3z, J_3w)\]
\[= R(x, y, z, w)\]

\[(13.18) \quad \frac{1}{n}g(x, J_1y)\rho(J_1z, w) + \frac{1}{n}\rho(J_2x, J_3y)g(J_1z, w).\]

Hence, for the Ricci tensor we have (13.15) and

\[(13.19) \quad (n^2 - 1)\rho(J_2y, J_2z) = (n^2 - 1)\rho(J_3y, J_3z)\]
\[= -(n^2 - 1)\rho(y, z).\]

Then for $n > 1$ the Ricci tensor is hybrid with respect to $J_2$ and $J_3$, i.e.

\[\rho(J_2y, J_2z) = \rho(J_3y, J_3z) = -\rho(y, z).\]

Conditions (13.7), (13.8) and (13.10) imply for $n > 1$ the following

\[A(x, z) = -\frac{2}{n}\rho(x, x)g(z, z) = -\frac{2}{n}g(x, x)\rho(z, z),\]

where

\[A(x, z) = R(x, J_1x, z, J_1z) - R(x, J_1x, J_2z, J_3z)\]
\[- R(J_2x, J_3x, z, J_1z) + R(J_2x, J_3x, J_2z, J_3z)\]

Then for arbitrary non-isotropic vectors we have $\rho = \lambda g$, $\lambda \in \mathbb{R}$. □

By Theorem 13.7, identity (13.16) implies the following corollary for $n \geq 2$:

\[(13.20) \quad R(x, y, J_2z, J_2w) = R(x, y, J_3z, J_3w)\]
\[= -R(x, y, z, w) - \frac{\tau}{4n^2}g(J_1x, y)g(J_1z, w).\]

Therefore from (13.20), using (9.13) and Theorem 9.2 we obtain the following
Proposition 13.8. A quaternionic Kähler manifold with Hermitian-Norden metrics of dimension $4n \geq 8$ is scalar flat if and only if it is flat.

Proposition 13.9. A quaternionic Kähler manifold with Hermitian-Norden metrics of dimension $4n \geq 8$ is determined by the local 1-forms satisfying conditions (13.6) and

$$d\omega_1(x, y) = -\omega_2(x)\omega_3(y) + \omega_3(x)\omega_2(y) - \frac{\tau}{4n^2}g(J_1x, y).$$

13.2. Quaternionic Kähler manifolds with Hermitian-Norden metrics in a classification of almost hypercomplex manifolds with Hermitian-Norden metrics

Firstly, let us consider the case when $H$ is (integrable) hypercomplex structure, i.e. when $[J_\alpha, J_\alpha]$ vanishes for each $\alpha = 1, 2, 3$.

Taking into account (1.21) and (13.1), for the quaternionic Kähler manifolds we have

$$[J_\alpha, J_\alpha](x, y) = -[\omega_\gamma(x) + \omega_\beta(J_\alpha x)] J_\gamma y - [\omega_\beta(x) - \omega_\gamma(J_\alpha x)] J_\beta y + [\omega_\gamma(y) + \omega_\beta(J_\alpha y)] J_\gamma x + [\omega_\beta(y) - \omega_\gamma(J_\alpha y)] J_\beta x,$$

$$\{J_\alpha, J_\alpha\}(x, y) = -[\omega_\gamma(x) + \omega_\beta(J_\alpha x)] J_\gamma y - [\omega_\beta(x) - \omega_\gamma(J_\alpha x)] J_\beta y - [\omega_\gamma(y) + \omega_\beta(J_\alpha y)] J_\gamma x - [\omega_\beta(y) - \omega_\gamma(J_\alpha y)] J_\beta x.$$ (13.21)

The latter equations imply immediately the next two lemmas.

Lemma 13.10. The tensors $[J_\alpha, J_\alpha]$ and $\{J_\alpha, J_\alpha\}$ ($\alpha = 1, 2, 3$) vanish if and only if the following equality is valid for any fixed cyclic permutation $(\alpha, \beta, \gamma)$ of $(1, 2, 3)$:

$$\omega_\gamma = -\omega_\beta \circ J_\alpha.$$

Lemma 13.11. The tensors $[J_\alpha, J_\alpha]$ and $\{J_\alpha, J_\alpha\}$ ($\alpha = 1, 2, 3$) vanish if and only if the following equalities hold for all cyclic permutations $(\alpha, \beta, \gamma)$ of $(1, 2, 3)$:

$$\omega_\alpha = \omega_\beta \circ J_\gamma = -\omega_\gamma \circ J_\beta.$$ (13.22)
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§13. Quaternionic Kähler manifolds with Hermitian-Norden metrics

Now, according to (13.1) and (9.4), the fundamental tensors and their corresponding Lee forms of the derived quaternionic Kähler manifold with Hermitian-Norden metrics, defined by (9.5) and (9.8), have the form

\[
F_\alpha(x, y, z) = \omega_\gamma(x)g(J_\beta y, z) - \omega_\beta(x)g(J_\gamma y, z),
\]

(13.23)

\[
\theta_\alpha(z) = -\varepsilon_\beta \omega_\gamma(J_\beta z) + \varepsilon_\gamma \omega_\beta(J_\gamma z).
\]

(13.24)

Proposition 13.12. If a quaternionic Kähler manifold with Hermitian-Norden metrics \((\mathcal{M}, H, G)\) is integrable, then it is a hyper-Kähler manifold with Hermitian-Norden metrics, i.e. the following implication is valid

\[
(\mathcal{M}, H, G) \in (\mathcal{W}_3 \oplus \mathcal{W}_4)(J_1) \cap (\mathcal{W}_1 \oplus \mathcal{W}_2)(J_2) \cap (\mathcal{W}_1 \oplus \mathcal{W}_2)(J_3)
\]

\[
\Rightarrow \quad (\mathcal{M}, H, G) \in \mathcal{K}.
\]

Proof. Let \((\mathcal{M}, H, G)\) be an integrable hypercomplex manifold with Hermitian-Norden metrics, i.e. \((\mathcal{M}, H, G)\) belongs to the class \(\mathcal{W}_3 \oplus \mathcal{W}_4\) with respect to \(J_1\) in (9.10) and \((\mathcal{M}, H, G)\) is an element of \(\mathcal{W}_1 \oplus \mathcal{W}_2\) regarding \(J_2\) and \(J_3\), according to (9.11).

Therefore \([J_1, J_1] = [J_2, J_2] = [J_3, J_3] = 0\) hold and then, according to Lemma 13.11, conditions (13.22) are valid. Hence, according to \(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = -1\) and \(\varepsilon_1\varepsilon_2\varepsilon_3 = 1\), relation (13.24) takes the form

\[
\theta_\alpha = -(1 + \varepsilon_\alpha)\omega_\alpha,
\]

which implies

\[
\theta_1 = -2\omega_1, \quad \theta_2 = \theta_3 = 0.
\]

On the other hand, \([J_\alpha, J_\alpha] = \{J_\alpha, J_\alpha\} = 0\) and (1.21) imply

\[
(D_x J_\alpha) y = (D_{J_\alpha x} J_\alpha) J_\alpha y
\]

and finally the fact that the manifold is hyper-Kählerian with Hermitian-Norden metrics.

\[\square\]

Proposition 13.13. If an almost hypercomplex manifold with Hermitian-Norden metrics \((\mathcal{M}, H, G)\) with vanishing Lee forms \(\theta_2\) and \(\theta_3\) is quaternionic Kählerian, then it is a hyper-Kähler manifold with Hermitian-Norden metrics, i.e. the following implication is valid

\[
(\mathcal{M}, H, G) \in (\mathcal{W}_2 \oplus \mathcal{W}_3)(J_2) \cap (\mathcal{W}_2 \oplus \mathcal{W}_3)(J_3) \quad \Rightarrow \quad (\mathcal{M}, H, G) \in \mathcal{K}.
\]

Proof. Since \(\theta_2 = \theta_3 = 0\), we have \([J_2, J_2] = [J_3, J_3] = 0\), because of (13.24) and (13.21). Consequently, \([J_1, J_1]\) vanishes, too. Then, according to Proposition 13.12 and conditions (9.11), the considered manifold belongs to the class \(\mathcal{K}\).

\[\square\]
Let us remark for $\alpha = 2$ or $3$, using (9.11), that an almost complex manifold with Norden metric belongs to $W_1 \oplus W_3$ regarding $J_\alpha$ if and only if the following property holds

\[(13.25) \quad \mathcal{S}_{x,y,z} F_\alpha(x,y,z) = \frac{1}{2n} \mathcal{S}_{x,y,z} \{ g(x,y)\theta_\alpha(z) + g(J_\alpha x, y)\theta_\alpha(J_\alpha z) \} \]

**Proposition 13.14.** Let $(\mathcal{M}, H, G)$ be an almost hypercomplex manifold with Hermitian-Norden metrics belonging to the class $W_1 \oplus W_3$ with respect to $J_2$ and $J_3$. If $(\mathcal{M}, H, G)$ is quaternionic Kählerian, then it is a Kähler manifold with respect to $J_1$, i.e. the following implication is valid

\[
(\mathcal{M}, H, G) \in (W_1 \oplus W_3)(J_2) \cap (W_1 \oplus W_3)(J_3) \Rightarrow (\mathcal{M}, H, G) \in W_0(J_1).
\]

Moreover, we have

\[(13.26) \quad (D_x J_1)y = 0, \quad (D_x J_2)y = \omega_1(x)J_3y, \quad \omega_1(x) = -\theta_2(J_3x), \quad (D_x J_3)y = -\omega_1(x)J_2y, \quad \omega_1(x) = \theta_3(J_2x).\]

**Proof.** From (13.25) for $\alpha = 2$ and $3$ we obtain

\[
\theta_2 = \omega_1 \circ J_3, \quad \theta_3 = -\omega_1 \circ J_2.
\]

Then, according to (13.24), we get

\[(13.27) \quad \omega_2 = \omega_3 = 0\]

and therefore we obtain (13.26). \qed

From Proposition 13.13 and Proposition 13.14 we have directly

**Corollary 13.15.** Let $(\mathcal{M}, H, G)$ be an almost hypercomplex manifold with Hermitian-Norden metrics, belonging to the class $W_3$ with respect to $J_2$ and $J_3$. If $(\mathcal{M}, H, G)$ is quaternionic Kählerian, then it is a hyper-Kähler manifold with Hermitian-Norden metrics, i.e. the following implication is valid

\[
(\mathcal{M}, H, G) \in W_3(J_2) \cap W_3(J_3) \Rightarrow (\mathcal{M}, H, G) \in \mathcal{K}.
\]

Bearing in mind Propositions 13.12, 13.14, Corollary 13.15 and Theorem 9.1, we give the following
**Theorem 13.16.** Let a quaternionic Kähler manifold with Hermitian-Norden metrics \((\mathcal{M}, H, G)\) be in some of the classes \(\mathcal{W}_1 \oplus \mathcal{W}_2\) (and in particular \(\mathcal{W}_0, \mathcal{W}_1\) and \(\mathcal{W}_2\)), \(\mathcal{W}_2 \oplus \mathcal{W}_3\) and \(\mathcal{W}_3\) with respect to both of the structures \(J_2\) and \(J_3\). Then \((\mathcal{M}, H, G)\) is a flat hyper-Kähler manifold with Hermitian-Norden metrics. The unique class (except the class without conditions for \(D J_2\) and \(D J_3\)), where \((\mathcal{M}, H, G)\) is not flat hyper-Kählerian, is \(\mathcal{W}_1 \oplus \mathcal{W}_3\) and its manifolds are determined by (13.26).

**13.3. Non-hyper-Kähler quaternionic Kähler manifolds with Hermitian-Norden metrics**

In this subsection we characterize the manifold satisfying the conditions of Proposition 13.14. It is a non-hyper-Kähler quaternionic Kähler manifold with Hermitian-Norden metrics.

We apply (13.27) to (13.2) and obtain the square norms of the non-zero quantities \(D J_2\) and \(D J_3\) in the considered case as follows:

\[\|D J_2\|^2 = \|D J_3\|^2 = -4n \omega_1(\omega_1^\#),\]

where \(\omega_1^\#\) is the corresponding vector to \(\omega_1\) with respect to \(g\).

**Corollary 13.17.** Let \((\mathcal{M}, H, G)\) be a quaternionic Kähler manifold with Hermitian-Norden metrics, determined by a local 1-form \(\omega_1\) in (13.26). It is an isotropic hyper-Kähler manifold with Hermitian-Norden metrics if and only if \(\omega_1^\#\) is an isotropic vector regarding \(g\).

Using (13.27) and Proposition 13.9, for the considered manifolds here we have

**Proposition 13.18.** Let \(\omega_1\) be the local 1-form of a quaternionic Kähler manifold with Hermitian-Norden metrics of dimension \(4n \geq 8\), determined by (13.26). Then \(\omega_1\) satisfies the following condition

\[
d\omega_1(x, y) = -\frac{\tau}{4n^2} \tilde{g}_1(x, y).
\]

According to (13.26) we have \(F_1(x, y, z) = (D_x \tilde{g}_1)(y, z) = 0\) and then we obtain \(d \tilde{g}_1(x, y, z) = \mathcal{G}_{x,y,z} \{ F_1(x, y, z) \} = 0\). Hence we establish that \(\tau = \text{const}\), using (13.28), i.e. \((\mathcal{M}, H, G)\) determined by (13.26) has a constant scalar curvature. Then, bearing in mind Theorem 13.7, we get the following

**Proposition 13.19.** Quaternionic Kähler manifolds with Hermitian-Norden metrics determined by (13.26) for dimension \(4n \geq 8\) are Ricci-symmetric, i.e. \(D \rho = 0\).
As in Proposition 13.1 for an arbitrary quaternionic Kähler manifold with Hermitian-Norden metrics, in the following proposition we give a necessary and sufficient condition for the considered manifold in this subsection to be flat.

**Proposition 13.20.** Let \((\mathcal{M}, H, G)\) be a quaternionic Kähler manifold with Hermitian-Norden metrics, determined by a non-zero local 1-form \(\omega_1\) in (13.26). Then \((\mathcal{M}, H, G)\) is flat non-hyper-Kählerian if and only if \(\omega_1\) is closed.

*Proof.* Since \(\mathcal{M}\) is a Kähler manifold with respect to \(J_1\), then \(R\) is of Kähler-type with respect to \(J_1\).

Bearing in mind (13.27) and (13.4), we have that \(\Psi_1 = d\omega_1\) in the considered case. Then identity (13.8) takes the following form

\[
R(x, y, J_2z, J_2w) = R(x, y, J_3z, J_3w) = -R(x, y, z, w) + d\omega_1(x, y)g(J_1z, w).
\]

It is clear that \(R\) is a Kähler-type tensor with respect to \(H = (J_\alpha)\) if and only if \(\omega_1\) is closed. Hence, according to Theorem 9.2, we obtain the statement. 

**Corollary 13.21.** Let \((\mathcal{M}, H, G)\) be a quaternionic Kähler manifold with Hermitian-Norden metrics, determined by a non-zero local 1-form \(\omega_1\) in (13.26). Then \((\mathcal{M}, H, G)\) is flat non-hyper-Kählerian if and only if the following identity is valid for \(\alpha = 2\) or \(\alpha = 3\):

\[
d\theta_\alpha(x, y) + d\theta_\alpha(J_1x, J_1y) - d\theta_\alpha(J_2x, J_2y) - d\theta_\alpha(J_3x, J_3y) = 0.
\]

*Proof.* It follows directly from Proposition 13.20 and the relations

\[
\omega_1 = -\theta_2 \circ J_3 = \theta_3 \circ J_2
\]

in (13.26).  

\begin{center}
\textbf{...}
\end{center}
§14. Manifolds with almost contact 3-structure and metrics of Hermitian-Norden type

In the present section, it is introduced a differentiable manifold with almost contact 3-structure which consists of an almost contact metric structure and two almost contact B-metric structures. The corresponding classifications are discussed. The product of this manifold and a real line is an almost hypercomplex manifold with Hermitian-Norden metrics. It is proved that the introduced manifold of cosymplectic type is flat. Some examples of the studied manifolds are given.

The main results of this section are published in [92].

Our goal here is to consider a \((4n + 3)\)-dimensional manifold with almost contact 3-structure and to introduce a pseudo-Riemannian metric on it having another kind of compatibility with the triad of almost contact structures. The product of this manifold of new type and a real line is a \((4n + 4)\)-dimensional manifold which admits an almost hypercomplex structure with Hermitian-Norden metrics.

The purpose of this development is to launch a study of the manifolds with almost contact 3-structure and metrics of a Hermitian-Norden type.

14.1. Almost contact metric manifolds

Let \(\mathcal{M}\) be an odd-dimensional smooth manifold which is compatible with an almost contact structure \((\varphi_1, \xi_1, \eta_1)\), i.e. \(\varphi_1\) is an endomorphism of the tangent bundle, \(\xi_1\) is a Reeb vector field and \(\eta_1\) is its dual contact 1-form satisfying the identities (4.1), i.e.

\[
(\varphi_1)^2 = -I + \xi_1 \otimes \eta_1, \quad \varphi_1 \xi_1 = 0, \quad \eta_1 \circ \varphi_1 = 0, \quad \eta_1(\xi_1) = 1,
\]

where \(I\) is the identity in the Lie algebra \(\mathfrak{X}(\mathcal{M})\) and \(o\) is the zero element of \(\mathfrak{X}(\mathcal{M})\). Moreover, let \(g\) be a pseudo-Riemannian metric on \(\mathcal{M}\) which
is compatible with \((\varphi_1, \xi_1, \eta_1)\) as follows:

\[
(14.2) \quad g(\xi_1, \xi_1) = -\varepsilon, \quad \eta_1(x) = -\varepsilon g(\xi_1, x), \\
g(\varphi_1 x, \varphi_1 y) = g(x, y) + \varepsilon \eta_1(x) \eta_1(y),
\]

where \(\varepsilon = +1\) or \(\varepsilon = -1\). Then \((\varphi_1, \xi_1, \eta_1, g)\) is called an almost contact metric structure on \(M\).

Usually, one can assume that \(\varepsilon = -1\) in (14.2) without loss of generality. This is conditioned since if we put \(\varphi_1 = \varphi_1, \xi_1 = -\xi_1, \eta_1 = -\eta_1, g = -g\), then \((\varphi_1, \xi_1, \eta_1, g)\) for \(\varepsilon = -\varepsilon\) is also an almost contact metric structure on \(M\) \([138]\). Here, we pay attention of the case when \(\varepsilon = +1\) which is in relation with our topic. We call the corresponding \((\varphi_1, \xi_1, \eta_1, g)\)-structure an almost contact metric structure and \(g\) — a compatible metric on \(M\).

Since \(g\) is a Hermitian metric with respect to the almost complex structure \(\varphi_1|_{\mathcal{H}_1}\) on the contact distribution \(\mathcal{H}_1 = \ker(\eta_1)\), any metric with properties (14.2) can be considered as an odd-dimensional counterpart of the corresponding pseudo-Riemannian Hermitian metric, or this compatible metric is a pseudo-Riemannian metric of Hermitian type on an odd-dimensional differentiable manifold.

A classification of the almost contact metric manifolds is given by V. Alexiev and G. Ganchev in \([4]\). There, it is considered the vector space of the tensors of type \((0, 3)\) defined by \(F_1(x, y, z) = g((\nabla_x \varphi_1)y, z)\), where \(\nabla\) is the Levi-Civita connection generated by \(g\). They have the following basic properties

\[
F_1(x, y, z) = -F_1(x, z, y) \\
= -F_1(x, \varphi_1 y, \varphi_1 z) + F_1(x, \xi_1, z) \eta_1(y) \\
+ F_1(x, y, \xi_1) \eta_1(z).
\]

Bearing in mind (14.2), we establish that the covariant derivatives of the structure tensors with respect to \(\nabla\) are related by

\[
(14.4) \quad F_1(x, \varphi_1 y, \xi_1) = -((\nabla_x \eta_1)(y) = g(\nabla_x \xi_1, y).
\]

In \([4]\), this vector space is decomposed in 12 orthogonal and invariant subspaces with respect to the action of the structure group \(U(m) \times 1, m = \frac{1}{2}(\dim M - 1)\), where \(U\) is the unitary group. In such a way, it is obtained a classification of 12 basic classes with respect to \(F_1\), which we denote by \(P_i\) \((i = 1, 2, \ldots, 12)\) in the present work. Bearing in mind the
above remarks we can use the same classification for almost contact metric manifolds. The basic classes for \( \dim M = 4n + 3 \) can be determined as follows:

\[
\begin{align*}
\mathcal{P}_1 : \quad & F_1(x, y, z) = \eta_1(x) \{ \eta_1(y) \omega_1(z) - \eta_1(z) \omega_1(y) \} ; \\
\mathcal{P}_2 : \quad & F_1(x, y, z) = \frac{\theta_1(\xi_1)}{2(2n + 1)} \{ g(x, y) \eta_1(z) - g(x, z) \eta_1(y) \} ; \\
\mathcal{P}_3 : \quad & F_1(x, y, z) = \frac{\theta^*_1(\xi_1)}{2(2n + 1)} \{ g(\varphi_1 x, y) \eta_1(z) \\ & \quad - g(\varphi_1 x, z) \eta_1(y) \} ; \\
\mathcal{P}_4 : \quad & F_1(x, y, z) = F_1(x, y, \xi_1) \eta_1(z) - F_1(x, z, \xi_1) \eta_1(y), \\
& F_1(x, y, \xi_1) = F_1(y, x, \xi_1) = F_1(\varphi_1 x, \varphi_1 y, \xi_1), \\
& \theta_1 = 0; \\
\mathcal{P}_5 : \quad & F_1(x, y, z) = F_1(x, y, \xi_1) \eta_1(z) - F_1(x, z, \xi_1) \eta_1(y), \\
& F_1(x, y, \xi_1) = -F_1(y, x, \xi_1) = F_1(\varphi_1 x, \varphi_1 y, \xi_1), \\
& \theta^*_1 = 0; \\
\mathcal{P}_6 : \quad & F_1(x, y, z) = F_1(x, y, \xi_1) \eta_1(z) - F_1(x, z, \xi_1) \eta_1(y), \\
& F_1(x, y, \xi_1) = F_1(y, x, \xi_1) = -F_1(\varphi_1 x, \varphi_1 y, \xi_1); \\
\mathcal{P}_7 : \quad & F_1(x, y, z) = F_1(x, y, \xi_1) \eta_1(z) - F_1(x, z, \xi_1) \eta_1(y), \\
& F_1(x, y, \xi_1) = -F_1(y, x, \xi_1) = -F_1(\varphi_1 x, \varphi_1 y, \xi_1); \\
\mathcal{P}_8 : \quad & F_1(x, y, z) = -\eta_1(x) F_1(\xi_1, \varphi_1 y, \varphi_1 z); \\
\mathcal{P}_9 : \quad & F_1(x, y, z) = \frac{1}{4n} \{ g(\varphi_1 x, y) \theta(\varphi_1 z) - g(\varphi_1 x, z) \theta(\varphi_1 y) \\ & \quad - g(\varphi_1 x, \varphi_1 y) \theta(\varphi_1^2 z) \\ & \quad + g(\varphi_1 x, \varphi_1 z) \theta(\varphi_1^2 y) \}; \\
\mathcal{P}_{10} : \quad & F_1(\xi_1, y, z) = F_1(x, y, \xi_1) = 0, \\
& F_1(x, y, z) = F_1(\varphi_1 x, \varphi_1 y, z), \quad \theta_1 = 0; \\
\mathcal{P}_{11} : \quad & F_1(\xi_1, y, z) = F_1(x, y, \xi_1) = 0, \\
& F_1(x, y, z) = -F_1(y, x, z); \\
\mathcal{P}_{12} : \quad & F_1(\xi_1, y, z) = F_1(x, y, \xi_1) = 0, \\
& F_1(x, y, z) + F_1(y, z, x) + F_1(z, x, y) = 0.
\end{align*}
\] (14.5)

The class \( \mathcal{P}_0 \) of cosymplectic metric manifolds is determined by \( F_1 = 0 \)
and it is contained in any other class $\mathcal{P}_i$. There are $2^{12}$ classes at all.

Besides definitional conditions of the basic classes $\mathcal{P}_i$, $i \in \{1, 2, \ldots, 12\}$, in [4], there are given the corresponding component $P^i$ of $F_1$ for every class $\mathcal{P}_i$. An almost contact metric manifold $(\mathcal{M}, \varphi_1, \xi_1, \eta_1, g)$ belongs to some of the basic classes $\mathcal{P}_i$ or their direct sum $\mathcal{P}_i \oplus \mathcal{P}_j \oplus \cdots$ for $i, j \in \{1, 2, \ldots, 12\}$, $i \neq j$, if and only if the fundamental tensor $F_1$ on the manifold has the following form $F_1 = P^i$ or $F_1 = P^i + P^j + \ldots$, respectively.

Some of the classes of almost contact metric manifolds are discovered before the complete classification and they are known by special names. For example, $\mathcal{P}_6 \oplus \mathcal{P}_{11}$ is the class of almost cosymplectic metric manifolds, $\mathcal{P}_2 \oplus \mathcal{P}_4$ is the class of quasi-Sasakian metric manifolds, $\mathcal{P}_3 \oplus \mathcal{P}_5$ is the class of quasi-Kenmotsu metric manifolds, $\mathcal{P}_3 \oplus \mathcal{P}_6 \oplus \mathcal{P}_{11}$ is the class of almost $\beta$-Kenmotsu metric manifolds, $\mathcal{P}_6$ is the class of $\beta$-Kenmotsu metric manifolds and so on.

### 14.2. Almost contact B-metric manifolds

Let $\mathcal{M}$ be equipped with another almost contact structure $(\varphi_2, \xi_2, \eta_2)$ and the metric $g$ is a B-metric with respect to $(\varphi_2, \xi_2, \eta_2)$, i.e the relations (4.1) and (4.2) are satisfied for the structure $(\varphi_2, \xi_2, \eta_2, g)$ and we have

\begin{align}
(14.6) \\
g(\xi_2, \xi_2) &= 1, \quad \eta_2(x) = g(\xi_2, x), \\
g(\varphi_2 x, \varphi_2 y) &= -g(x, y) + \eta_2(x)\eta_2(y).
\end{align}

Then $(\varphi_2, \xi_2, \eta_2, g)$ is an almost contact B-metric structure on $\mathcal{M}$, according to §4.

A classification of the almost contact B-metric manifolds, given by G. Ganchev, V. Mihova and K. Gribachev, is presented in (4.14) following [39]. In the present case, the fundamental tensor is $F_2$ defined by $F_2(x, y, z) = g((\nabla_x \varphi_2) y, z)$. Its properties (4.10) and (4.11) take the following form

\begin{align}
(14.7) \\
F_2(x, y, z) &= F_2(x, z, y) \\
&= F_2(x, \varphi_2 y, \varphi_2 z) + F_2(x, \xi_2, z) \eta_2(y) \\
&\quad + F_2(x, y, \xi_2) \eta_2(z),
\end{align}

\begin{align}
(14.8) \\
F_2(x, \varphi_2 y, \xi_2) &= (\nabla_x \eta_2) (y) = g(\nabla_x \xi_2, y).
\end{align}

The vector space of tensors having the properties of $F_2$ is decomposed in 11 orthogonal and invariant subspaces with respect to the ac-
tion of the structure group \((GL(m; \mathbb{C}) \cap O(m, m)) \times 1, m = \frac{1}{2}(\dim M - 1)\), where \(O(m, m)\) is the pseudo-orthogonal group of neutral signature. Thus, the obtained basic classes \(F_i\) \((i = 1, 2, \ldots, 11)\) with respect to \(F_2\) for \(\dim M = 4n + 3\) can be determined as follows:

\[
\mathcal{F}_1 : \quad F_2(x, y, z) = \frac{1}{2(2n + 1)} \{g(x, \varphi_2 y)\theta_2(\varphi_2 z) + g(x, \varphi_2 z)\theta_2(\varphi_2 y) + g(\varphi_2 x, \varphi_2 y)\theta_2(\varphi_2^2 z) + g(\varphi_2 x, \varphi_2 z)\theta_2(\varphi_2^2 y)\};
\]

\[
\mathcal{F}_2 : \quad F_2(\xi_2, y, z) = F_2(x, y, \xi_2) = 0,
\]

\[
F_2(x, y, \varphi_2 z) + F_2(y, z, \varphi_2 x) + F_2(z, x, \varphi_2 y) = 0,
\]

\[
\theta_2 = 0;
\]

\[
\mathcal{F}_3 : \quad F_2(\xi_2, y, z) = F_2(x, y, \xi_2) = 0,
\]

\[
F_2(x, y, z) + F_2(y, z, x) + F_2(z, x, y) = 0;
\]

\[
\mathcal{F}_4 : \quad F_2(x, y, z) = -\frac{\theta_2(\xi_2)}{2(2n + 1)} \{g(\varphi_2 x, \varphi_2 y)\eta_2(z) + g(\varphi_2 x, \varphi_2 z)\eta_2(y)\};
\]

\[
\mathcal{F}_5 : \quad F_2(x, y, z) = -\frac{\theta_2^*(\xi_2)}{2(2n + 1)} \{g(x, \varphi_2 y)\eta_2(z) + g(x, \varphi_2 z)\eta_2(y)\};
\]

\[
\mathcal{F}_6 : \quad F_2(x, y, z) = F_2(x, y, \xi_2)\eta_2(z) + F_2(x, z, \xi_2)\eta_2(y),
\]

\[
F_2(x, y, \xi_2) = F_2(y, x, \xi_2) = -F_2(\varphi_2 x, \varphi_2 y, \xi_2),
\]

\[
\theta_2 = \theta_2^* = 0;
\]

\[
\mathcal{F}_7 : \quad F_2(x, y, z) = F_2(x, y, \xi_2)\eta_2(z) + F_2(x, z, \xi_2)\eta_2(y),
\]

\[
F_2(x, y, \xi_2) = -F_2(y, x, \xi_2) = -F_2(\varphi_2 x, \varphi_2 y, \xi_2);
\]

\[
\mathcal{F}_8 : \quad F_2(x, y, z) = F_2(x, y, \xi_2)\eta_2(z) + F_2(x, z, \xi_2)\eta_2(y),
\]

\[
F_2(x, y, \xi_2) = F_2(y, x, \xi_2) = F_2(\varphi_2 x, \varphi_2 y, \xi_2);
\]

\[
\mathcal{F}_9 : \quad F_2(x, y, z) = F_2(x, y, \xi_2)\eta_2(z) + F_2(x, z, \xi_2)\eta_2(y),
\]

\[
F_2(x, y, \xi_2) = -F_2(y, x, \xi_2) = F_2(\varphi_2 x, \varphi_2 y, \xi_2);
\]
\[ F_{10} : \quad F_2(x, y, z) = F_2(\xi_2, \varphi_2 y, \varphi_2 z)\eta_2(x); \]
\[ F_{11} : \quad F_2(x, y, z) = \eta_2(x) \{ \eta_2(y) \omega(z) + \eta_2(z) \omega(y) \}. \]

Obviously, the class of cosymplectic B-metric manifolds \( F_0 \) is determined by the condition \( F_2 = 0 \).

### 14.3. Almost contact 3-structure with metrics of Hermitian-Norden type

Let \( (\mathcal{M}, \varphi_\alpha, \xi_\alpha, \eta_\alpha), (\alpha = 1, 2, 3) \) be a manifold with an almost contact 3-structure, i.e. \( \mathcal{M} \) is a \((4n + 3)\)-dimensional differentiable manifold with three almost contact structures \((\varphi_\alpha, \xi_\alpha, \eta_\alpha), (\alpha = 1, 2, 3)\) consisting of endomorphisms \( \varphi_\alpha \) of the tangent bundle, Reeb vector fields \( \xi_\alpha \) and their dual contact 1-forms \( \eta_\alpha \) satisfying the following identities:

\[ \varphi_\alpha \circ \varphi_\beta = -\delta_{\alpha\beta} I + \xi_\alpha \otimes \eta_\beta + \epsilon_{\alpha\beta\gamma} \varphi_\gamma, \]
\[ \varphi_\alpha \xi_\beta = \epsilon_{\alpha\beta\gamma} \xi_\gamma, \quad \eta_\alpha \circ \varphi_\beta = \epsilon_{\alpha\beta\gamma} \eta_\gamma, \quad \eta_\alpha (\xi_\beta) = \delta_{\alpha\beta}, \]

where \( \alpha, \beta, \gamma \in \{1, 2, 3\} \), \( I \) is the identity on the algebra \( \mathcal{X}(\mathcal{M}) \), \( \delta_{\alpha\beta} \) is the Kronecker delta, \( \epsilon_{\alpha\beta\gamma} \) is the Levi-Civita symbol, i.e. either the sign of the permutation \((\alpha, \beta, \gamma)\) of \((1, 2, 3)\) or \(0\) if any index is repeated \([146, 62]\).

Further, the indices \( \alpha, \beta \) run over the range \( \{1, 2, 3\} \) unless otherwise stated.

In the present subsection we discuss the coherence of compatible metrics and B-metrics in an almost contact 3-structure. In \([62]\), it is considered the case of a Riemannian metric which is compatible by equations \((14.2)\) for the three almost contact structures.

Suppose that \( \mathcal{M} \) admits two almost contact structures \((\varphi_\alpha, \xi_\alpha, \eta_\alpha), (\alpha = 2, 3)\). If a pseudo-Riemannian metric \( g \) is a B-metric for both structures, then the property in the first line of \((14.10)\) implies the properties in the second line of the same equations.

In \([62]\) for the case of Riemannian metrics (positive definite), it is proved that if the almost contact 3-structure admits two almost contact metric structures, then the third one is of the same type. We consider the relevant cases for our investigations in the following

**Theorem 14.1.** Let \( \mathcal{M} \) admit an almost contact 3-structure \((\varphi_\alpha, \xi_\alpha, \eta_\alpha)\), and a pseudo-Riemannian metric \( g \). If one of the three structures \((\varphi_\alpha, \xi_\alpha, \eta_\alpha, g)\) is an almost contact B-metric structure, then the other two ones are an almost contact metric structure and an almost contact B-metric structure.
Proof. First we establish on $\mathcal{M}$ that if the pseudo-Riemannian metric $g$ and two of the almost contact structures generate:

(i) two almost contact metric structures, then the third one is an almost contact metric structure;

(ii) two almost contact B-metric structures, then the third one is an almost contact metric structure;

(iii) an almost contact metric structure and an almost contact B-metric structure, then the third one is an almost contact B-metric structure.

Now, we argue for the case (ii). Let $(\varphi_2, \xi_2, \eta_2, g)$ be an almost contact B-metric structure, i.e. (14.6) holds. Moreover, let $(\varphi_3, \xi_3, \eta_3, g)$ be also an almost contact B-metric structure, i.e. the following properties are valid

\begin{align}
    g(\xi_3, \xi_3) &= 1, \quad \eta_3(x) = g(\xi_3, x), \\
    g(\varphi_3 x, \varphi_3 y) &= -g(x, y) + \eta_3(x)\eta_3(y).
\end{align}

Then, by virtue of the relations

$\varphi_1 = \varphi_2 \circ \varphi_3 - \xi_2 \otimes \eta_3, \quad \eta_2 \circ \varphi_2 = 0, \quad \eta_2 \circ \varphi_3 = \eta_1,$

which are consequences of (14.10), using (14.6) and (14.11), we obtain

\begin{align}
    g(\varphi_1 x, \varphi_1 y) &= g(\varphi_2(\varphi_3 x) - \eta_3(x)\xi_2, \varphi_2(\varphi_3 y) - \eta_3(y)\xi_2) \\
    &= g(x, y) + \eta_1(x)\eta_1(y).
\end{align}

Therefore, comparing with (14.2), the metric $g$ is a compatible metric with respect to the almost contact structure $(\varphi_1, \xi_1, \eta_1)$.

The verifications of the other cases are similar.

Since any compatible metric and any B-metric on an almost contact manifold $\mathcal{M}$ are metrics corresponding to a Hermitian metric and a Norden metric on the corresponding almost complex manifold $\mathcal{M} \times \mathbb{R}$ (or on the corresponding contact distribution $\mathcal{H} = \ker(\eta)$), respectively, we said that the compatible metric and the B-metric are metrics of Hermitian type and Norden type on $\mathcal{M}$, respectively. Then, we give the following

**Definition 14.1.** We call a pseudo-Riemannian metric $g$ a *metric of Hermitian-Norden type* on a manifold with almost contact 3-structure $(\mathcal{M}, \varphi_\alpha, \xi_\alpha, \eta_\alpha)$, if it satisfies the identities

\begin{align}
    g(\varphi_\alpha x, \varphi_\alpha y) &= \varepsilon_\alpha g(x, y) + \eta_\alpha(x)\eta_\alpha(y)
\end{align}
for some cyclic permutation $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ of $(1, -1, -1)$. Then, we call $(\varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$ an *almost contact 3-structure with metrics of Hermitian-Norden type*.

Let us suppose for the sake of definiteness that the coefficients $\varepsilon_\alpha$ have values as in (9.3), i.e. $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, -1, -1)$.

As a sequel of (14.12) we have the following properties:

\[(14.13)\]
$$\eta_\alpha = -\varepsilon_\alpha \xi_\alpha g,$$

\[(14.14)\]
$$g(\varphi_\alpha x, y) = -\varepsilon_\alpha g(\varphi_\alpha y, x),$$

\[(14.15)\]
$$F_\alpha(x, \varphi_\alpha y, \xi_\alpha) = -\varepsilon_\alpha (\nabla_x \eta_\alpha)(y) = g(\nabla_x \xi_\alpha, y),$$

\[(14.16)\]
$$F_\alpha(x, \varphi_\alpha y, \xi_\alpha) = -\varepsilon_\alpha \left( (\nabla_x \eta_\alpha)(y) + (\nabla_y \eta_\alpha)(x) \right).$$

Bearing in mind (14.14), we deduce the following. In the case $\alpha = 1$, the associated tensor field of type $(0, 2)$ is a 2-form. Let us denote it by $\tilde{g}_1$, i.e. $\tilde{g}_1(x, y) = g(\varphi_1 x, y)$. It is actually opposite to $\Phi(x, y) = g(x, \varphi_1 y)$, known as the *fundamental 2-form* of the almost contact metric structure. In other two cases $\alpha = 2$ and $\alpha = 3$, the tensor $(0, 2)$-field $g(\varphi_\alpha x, y)$ is symmetric as well as $\eta_\alpha \otimes \eta_\alpha$. Then, we define the following fundamental tensor $(0, 2)$-fields by

\[(14.17)\]
$$\tilde{g}_\alpha(x, y) = g(\varphi_\alpha x, y) + \eta_\alpha(x)\eta_\alpha(y), \quad \alpha = 2, 3.$$ 

Then $\tilde{g}_2$ and $\tilde{g}_3$ satisfy condition (14.12) and they are also metrics of Hermitian-Norden type, which we call *associated metrics* to $g$ with respect to $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)$ for $\alpha = 2$ and $\alpha = 3$, respectively.

Bearing in mind the structure groups of the almost contact 3-structures with compatible metric (62) and the almost hypercomplex manifolds with Hermitian-Norden metrics (46), we can conclude the following. The structure group of the manifolds with almost contact 3-structure and metrics of Hermitian-Norden type is $(\mathcal{GL}(n, \mathbb{H}) \cap O(2n, 2n)) \times O(2, 1)$, where $\mathcal{GL}(n, \mathbb{H})$ is the group of invertible quaternionic $(n \times n)$-matrices and $O(p, q)$ is the pseudo-orthogonal group of signature $(p, q)$ for natural numbers $p$ and $q$.

The fundamental tensors of a manifold with almost contact 3-structure and metrics of Hermitian-Norden type are the three $(0, 3)$-tensors determined by

\[(14.18)\]
$$F_\alpha(x, y, z) = g((\nabla_x \varphi_\alpha)y, z).$$
They have the following basic properties as a generalization of (14.3) and (14.7)

\[ F_\alpha(x, y, z) = -\varepsilon_\alpha F_\alpha(x, z, y) \]

\[ = -\varepsilon_\alpha F_\alpha(x, \varphi_\alpha y, \varphi_\alpha z) + F_\alpha(x, \xi_\alpha, z) \eta_\alpha(y) + F_\alpha(x, y, \xi_\alpha) \eta_\alpha(z). \]

The following associated 1-forms, defined as traces of \( F_\alpha \), are known as their Lee forms:

\[ \theta_\alpha(z) = g^{ij} F_\alpha(e_i, e_j, z), \]

\[ \theta^*_\alpha(z) = g^{ij} F_\alpha(e_i, \varphi_\alpha e_j, z), \]

\[ \omega_\alpha(z) = F_\alpha(\xi_\alpha, \xi_\alpha, z), \]

where \( g^{ij} \) are the components of the inverse matrix of the metric \( g \) with respect to an arbitrary basis of the type \( \{e_1, e_2, \ldots, e_{4n+2}, \xi_\alpha\} \).

The simplest case of the manifolds with almost contact 3-structure and metrics of Hermitian-Norden type is when the structures are \( \nabla \)-parallel, i.e. \( \nabla \varphi_\alpha = \nabla \xi_\alpha = \nabla \eta_\alpha = \nabla g = \nabla \tilde{g}_\alpha = 0 \), and it is determined by the condition \( F_\alpha = 0 \). We call these structures cosymplectic 3-structure with metrics of Hermitian-Norden type.

14.4. Relation with pseudo-Riemannian manifolds equipped with almost complex or almost hypercomplex structures

We can consider each of the three \((4n + 2)\)-dimensional distributions \( \mathcal{H}_\alpha = \ker(\eta_\alpha) \), equipped with a corresponding pair of an almost complex structure \( J_\alpha = \varphi_\alpha|\mathcal{H}_\alpha \) and a metric \( h_\alpha = g|\mathcal{H}_\alpha \), where \( \varphi_\alpha|\mathcal{H}_\alpha, g|\mathcal{H}_\alpha \) are the restrictions of \( \varphi_\alpha, g \) on \( \mathcal{H}_\alpha \), respectively, and the metrics \( h_\alpha \) are compatible with \( J_\alpha \) as follows

\[ h_\alpha(J_\alpha X, J_\alpha Y) = \varepsilon_\alpha h_\alpha(X, Y), \]

\[ \tilde{h}_\alpha(X, Y) := h_\alpha(J_\alpha X, Y) = -\varepsilon_\alpha h_\alpha(X, J_\alpha Y). \]

for arbitrary \( X, Y \in \mathfrak{X}(\mathcal{H}_\alpha) \). Obviously, in the cases \( \alpha = 2 \) and \( \alpha = 3 \) the metrics \( h_\alpha \) and their associated \((0, 2)\)-tensors \( \tilde{h}_\alpha \) are Norden metrics, whereas for \( \alpha = 1 \) the structure \((J_1, h_1)\) is an almost Hermitian pseudo-Riemannian structure with Kähler form \( \Omega = -\tilde{h}_1 \). In such a way, any of the distributions \( \mathcal{H}_\alpha \) for \( \alpha = 2 \) or \( \alpha = 3 \) can be considered as a \((2n + 1)\)-dimensional complex Riemannian distribution with a complex Riemannian metric \( g^C_\alpha = h_\alpha + \tilde{h}_\alpha \sqrt{-1} = g|\mathcal{H}_\alpha + \tilde{g}|\mathcal{H}_\alpha \sqrt{-1} \). In another
point of view, the distribution $\mathcal{H}_\alpha$ for $\alpha = 2$ or $\alpha = 3$ is a $(4n + 2)$-dimensional almost complex distribution with a Norden metric $\tilde{h}_\alpha$ and its associated Norden metric $\tilde{h}_\alpha$. Moreover, the $4n$-dimensional distribution $\mathcal{H} = \mathcal{H}_1 \cap \mathcal{H}_2 \cap \mathcal{H}_3$ has an almost hypercomplex structure $(J_1, J_2, J_3)$, i.e. $J_\alpha^2 = -I$, $J_3 = J_1 J_2 = -J_2 J_1$, $J_\alpha = \varphi_\alpha \vert_\mathcal{H}$, with a pseudo-Riemannian metric $h = g \vert_\mathcal{H}$ which is Hermitian with respect to $J_1$ and a Norden metric with respect to $J_2$ and $J_3$ since $h(J_\alpha X, J_\alpha Y) = \varepsilon_\alpha h(X, Y)$.

Let the vector $4n$-tuple $$(e_1, \ldots, e_n; J_1 e_1, \ldots, J_1 e_n; J_2 e_1, \ldots, J_2 e_n; J_3 e_1, \ldots, J_3 e_n)$$ be an adapted basis (or a $J_\alpha$-basis) of the almost hypercomplex structure. Then, according to (14.12), the basis

$$(e_1, \ldots, e_n; \varphi_1 e_1, \ldots, \varphi_1 e_n; \varphi_2 e_1, \ldots, \varphi_2 e_n; \varphi_3 e_1, \ldots, \varphi_3 e_n; \xi_1, \xi_2, \xi_3)$$

is an an adapted basis (or a $\varphi_\alpha$-basis) for the almost contact 3-structure and it is orthonormal with respect to $g$, i.e.

$$g(e_i, e_i) = \varepsilon_\alpha g(\varphi_\alpha e_i, \varphi_\alpha e_i) = -\varepsilon_\alpha g(\xi_\alpha, \xi_\alpha) = 1,$$

$$g(e_i, e_j) = g(e_i, \varphi_\alpha e_i) = g(e_i, \varphi_\alpha e_j) = g(e_i, \xi_\alpha) = g(\varphi_\beta e_i, \xi_\alpha) = 0$$

for arbitrary $i \neq j \in \{1, 2, \ldots, n\}$.

It is well known that an even-dimensional manifold endowed with almost complex structure $J$ and a compatible Riemannian metric $h$, i.e. $h(J \cdot, J \cdot) = h(\cdot, \cdot)$, is an almost Hermitian manifold. There are considered also almost pseudo-Hermitian manifolds, i.e. the case when $h$ is a pseudo-Riemannian metric with the same compatibility (cf. [106, 107]). We recall that this manifold equipped with a pseudo-Riemannian metric of neutral signature satisfying the identity $h(J \cdot, J \cdot) = -h(\cdot, \cdot)$ is known as an almost complex manifold with Norden metric (see §1). In the case when the almost complex structure $J$ is parallel with respect to the Levi-Civita connection $D$ of the metric $h$, i.e. $D J = 0$, then the manifold is known as a Kähler-Norden manifold or a holomorphic complex Riemannian manifold. Then the almost complex structure $J$ is integrable and the local components of the complex metric in holomorphic coordinate system are holomorphic functions.

From another point of view, the almost hypercomplex structure $(J_1, J_2, J_3)$ and the metric $h$ generate two almost complex structures with Norden
metrics (e.g., for $\alpha = 2, 3$) and one almost complex structure with Hermitian pseudo-Riemannian metric (e.g., for $\alpha = 1$) because of (14.21), i.e. an almost hypercomplex structure with Hermitian-Norden metrics (see §9).

The manifolds with almost contact 3-structure and metrics of Hermitian-Norden type can be considered as real hypersurfaces of an almost hypercomplex manifold with Hermitian-Norden metrics.

In case of cosymplectic manifolds with metrics of Hermitian-Norden type, the distribution $\mathcal{H}$ is involutive. The corresponding integral submanifold is a totally geodesic submanifold which inherits a holomorphic hypercomplex Riemannian structure and the manifold with almost contact 3-structure and metrics of Hermitian-Norden type is locally a pseudo-Riemannian product of a holomorphic hypercomplex Riemannian manifold with a 3-dimensional Lorentzian real space.

14.5. Curvature properties of manifolds with almost contact 3-structure and metrics of Hermitian-Norden type

Let us recall that a tensor $L$ of type $(0, 4)$ with the properties (1.35) is called a curvature-like tensor. We say that a curvature-like tensor $L$ is a Kähler-like tensor on a manifold with almost contact 3-structure and metrics of Hermitian-Norden type when $L$ satisfies the properties:

\begin{equation}
(14.24) \quad L(x, y, z, w) = \varepsilon_\alpha L(x, y, \varphi_\alpha z, \varphi_\alpha w).
\end{equation}

Kähler-like tensors on almost contact manifolds with B-metric are considered in [102].

Using (14.10) and (14.12), we obtain that for a Kähler-like tensor $L$ the following properties are valid

\begin{equation}
(14.25) \quad L(x, y, z, w) = \varepsilon_\alpha L(x, \varphi_\alpha y, \varphi_\alpha z, w) = \varepsilon_\alpha L(\varphi_\alpha x, \varphi_\alpha y, z, w)
\end{equation}

\begin{align*}
L(\xi_\alpha, y, z, w) &= L(x, \xi_\alpha, z, w) = L(x, y, \xi_\alpha, w) = L(x, y, z, \xi_\alpha) = 0.
\end{align*}

The latter properties show that if $L$ is a Kähler-like tensor on a manifold with almost contact 3-structure and metrics of Hermitian-Norden type then $L$ is a Kähler-like tensor on $(\mathcal{H}, J_\alpha = \varphi_\alpha|_H, h = g|_H)$ which is a manifold with almost hypercomplex structure with Hermitian-Norden metrics. It is known from [82] that every Kähler-like tensor vanishes on an almost hypercomplex manifold with Hermitian-Norden metrics. Therefore, it is valid the following
**Proposition 14.2.** Every Kähler-like tensor vanishes on a manifold with almost contact 3-structure and metrics of Hermitian-Norden type.

Let $R$ be the curvature tensor of the Levi-Civita connection $\nabla$ generated by $g$.

According to [47], every hyper-Kähler manifold with Hermitian-Norden metrics is flat. Since $R$ is a Kähler-like tensor on every manifold with cosymplectic 3-structure with metrics of Hermitian-Norden type, i.e. $\nabla \varphi_\alpha$ vanishes, then applying Proposition 14.2 we obtain

**Proposition 14.3.** Every manifold with cosymplectic 3-structure with metrics of Hermitian-Norden type is flat.

### 14.6. Examples of manifolds with almost contact 3-structure and metrics of Hermitian-Norden type

#### 14.6.1. A real vector space with contact 3-structure with metrics of Hermitian-Norden type

Let $V$ be a real $(4n + 3)$-dimensional vector space and a (local) basis of $V$ is denoted by $\left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial u^i}, \frac{\partial}{\partial v^i}, \frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial c} \right\}$, $(i = 1, 2, \ldots, n)$ or

\[
\left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}, \frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^n}, \frac{\partial}{\partial v^1}, \ldots, \frac{\partial}{\partial v^n} \right\}.
\]

Any vector $z$ of $V$ can be represented in the mentioned basis as follows

\[
z = x^i \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial y^i} + u^i \frac{\partial}{\partial u^i} + v^i \frac{\partial}{\partial v^i} + a \frac{\partial}{\partial a} + b \frac{\partial}{\partial b} + c \frac{\partial}{\partial c}.
\]

A standard contact 3-structure in $V$ is defined as follows:

\[
\begin{align*}
\varphi^1 & = \frac{\partial}{\partial y^i} & \varphi^2 & = \frac{\partial}{\partial u^i} & \varphi^3 & = -\frac{\partial}{\partial v^i}, \\
\varphi^1 & = -\frac{\partial}{\partial x^i} & \varphi^2 & = \frac{\partial}{\partial v^i} & \varphi^3 & = \frac{\partial}{\partial u^i}, \\
\varphi^1 & = -\frac{\partial}{\partial u^i} & \varphi^2 & = -\frac{\partial}{\partial v^i} & \varphi^3 & = \frac{\partial}{\partial x^i}, \\
\varphi^1 & = \frac{\partial}{\partial v^i} & \varphi^2 & = \frac{\partial}{\partial x^i} & \varphi^3 & = -\frac{\partial}{\partial y^i}.
\end{align*}
\]
\[
\varphi_1 \frac{\partial}{\partial a} = 0, \quad \varphi_2 \frac{\partial}{\partial a} = -\frac{\partial}{\partial c}, \quad \varphi_3 \frac{\partial}{\partial a} = \frac{\partial}{\partial b},\]
(14.27b)
\[
\varphi_1 \frac{\partial}{\partial b} = \frac{\partial}{\partial c}, \quad \varphi_2 \frac{\partial}{\partial b} = 0, \quad \varphi_3 \frac{\partial}{\partial b} = -\frac{\partial}{\partial a},
\]
\[
\varphi_1 \frac{\partial}{\partial c} = -\frac{\partial}{\partial b}, \quad \varphi_2 \frac{\partial}{\partial c} = \frac{\partial}{\partial a}, \quad \varphi_3 \frac{\partial}{\partial c} = 0,
\]
\[
\xi_1 = \frac{\partial}{\partial a}, \quad \xi_2 = \frac{\partial}{\partial b}, \quad \xi_3 = \frac{\partial}{\partial c},
\]
\[
\eta_1 = da, \quad \eta_2 = db, \quad \eta_3 = dc.
\]

We check immediately that the properties (14.10) hold.

If \(z \in V\), i.e. \(z(x^i; y^i; u^i; v^i; a, b, c)\) then according to (14.27) we have
\[
\varphi_1 z(-y^i; x^i; v^i; -u^i; 0, -c, b), \quad \eta_1(z) = a,
\]
(14.28)
\[
\varphi_2 z(-u^i; -v^i; x^i; y^i; c, 0, -a), \quad \eta_2(z) = b,
\]
\[
\varphi_3 z(v^i; -u^i; y^i; -x^i; -b, a, 0), \quad \eta_3(z) = c.
\]

**Definition 14.2.** The structure \((\varphi_\alpha, \xi_\alpha, \eta_\alpha)\) on \(V\) is called a contact 3-structure on \(V\).

Let us introduce a pseudo-Euclidean metric \(g\) of signature \((2n+2, 2n+1)\) as follows
\[
g(z, z') = \sum_{i=1}^{n} (x^i x'^i + y^i y'^i - u^i u'^i - v^i v'^i) - aa' + bb' + cc',
\]
where \(z(x^i; y^i; u^i; v^i; a, b, c), z'(x'^i; y'^i; u'^i; v'^i; a', b', c') \in V, (i = 1, 2, \ldots, n)\). This metric satisfies the following properties
\[
g(\varphi_\alpha z, \varphi_\alpha z') = \varepsilon_\alpha g(z, z') + \eta_\alpha(z)\eta_\alpha(z'),
\]
which is actually (14.12).

We check immediately that \(\nabla \varphi_\alpha\) vanishes for \(\nabla\), the Levi-Civita connection of \(g\). Therefore we get the following

**Proposition 14.4.** The space \((V, \varphi_\alpha, \xi_\alpha, \eta_\alpha, g)\) is a manifold with cosymplectic 3-structure and metrics of Hermitian-Norden type.

14.6.2. A time-like sphere with almost contact 3-structure and metrics of Hermitian-Norden type

It is known that any real hypersurface of an almost hypercomplex manifold carries in a natural way an almost contact 3-structure.
In a similar way it can be shown that on every real nonisotropic hypersurface of an almost hypercomplex manifold with Hermitian-Norden metrics there arises an almost contact 3-structure with metrics of Hermitian-Norden type.

Let us consider

$$\mathbb{R}^{4n+4} = \{ (x^i, y^i; u^i; v^i) \mid x^i, y^i, u^i, v^i \in \mathbb{R}, \ i \in \{1, 2, \ldots, n+1\}\},$$

a vector space of dimension $4n+4$ with an almost hypercomplex structure ($J_1, J_2, J_3$) determined as follows [17]

$$J_1 z(-y^i; x^i; v^i; -u^i), \quad J_2 z(-u^i; -v^i; x^i; y^i), \quad J_3 z(v^i; -u^i; y^i; -x^i)$$

for an arbitrary vector $z(x^i; y^i; u^i; v^i)$. This space is equipped with a pseudo-Euclidean metric of neutral signature, i.e. $(2n+2, 2n+2)$, by

$$g(z, z') = \sum_{i=1}^{n+1} (x^i x'^i + y^i y'^i - u^i u'^i - v^i v'^i)$$

for arbitrary $z(x^i; y^i; u^i; v^i), z'(x'^i; y'^i; u'^i; v'^i) \in \mathbb{R}^{4n+4}$.

Identifying an arbitrary point $p \in \mathbb{R}^{4n+4}$ with its position vector $z$, we study the following hypersurface of $\mathbb{R}^{4n+4}$. Let $S : g(z, z) = -1$ be the unit time-like sphere of $g$ in $\mathbb{R}^{4n+4}$. Then $z$ coincides with the unit normal $U$ to the tangent space $T_p S$ at $p \in S$.

We determine the Reeb vector fields by the equalities

$$\xi_\alpha = \lambda_\alpha U + \mu_\alpha J_\alpha U,$$

such that $g(U, \xi_\alpha) = 0$ and $g(\xi_\alpha, \xi_\alpha) = -\varepsilon_\alpha$ are valid.

We substitute $g(U, J_\alpha U) = \tan \psi_\alpha$ for $\psi_\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then we obtain

$$\xi_\alpha = \sin \psi_\alpha U + \cos \psi_\alpha J_\alpha U.$$

Since $g(U, J_1 U) = 0$ then $\psi_1 = 0$ and therefore $\xi_1 = J_1 U$. Because of $g(U, \xi_\alpha) = 0$ we have that $\xi_\alpha$ are in $T_p S$. The conditions $g(\xi_\alpha, \xi_\beta) = 0$ for $\alpha \neq \beta$ are equivalent to $\psi_2 = \psi_3 = 0$. Therefore we obtain the following equality for all $\alpha$

$$\xi_\alpha = J_\alpha U. \tag{14.31}$$

Using the latter equality and $J_\alpha J_\alpha U = -U$, we obtain that $J_\alpha \xi_\alpha = -U$.

We define the structure endomorphisms $\varphi_\alpha$ and the contact 1-forms $\eta_\alpha$ in $T_p S$ by the following orthonormal decomposition of $J_\alpha x$ for arbitrary $x \in T_p S$

$$J_\alpha x = \varphi_\alpha x - \eta_\alpha(x) U, \tag{14.32}$$
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i.e. $\varphi_\alpha x$ is the tangent component of $J_\alpha x$ and $-\eta_\alpha (x)U$ is the corresponding normal component. By direct computation (14.32) implies (14.10). Then, using (14.31), we obtain (14.12) and (14.13). Thus, we equip the unit time-like sphere $S$ in $\mathbb{R}^{4n+4}$ with an almost contact 3-structure with metrics of Hermitian-Norden type $(\varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$.

Let $\nabla$ and $\nabla$ be the Levi-Civita connections of the metric $g$ in $\mathbb{R}^{4n+4}$ and $S$, respectively. Since $\nabla$ is flat, the formulae of Gauss and Weingarten have the form

\begin{equation}
\nabla_x y = \nabla_x y + g(x, y)U, \quad \nabla_x U = x.
\end{equation}

Therefore one can obtain by (14.31), (14.32) and (14.33) that

\[ \nabla_x \xi_\alpha = \varphi_\alpha x, \quad F_{\alpha}(x, y, \xi_\alpha) = -g(\varphi_\alpha x, \varphi_\alpha y). \]

Then, for the Lee forms we have

\[ \theta_\alpha(\xi_\alpha) = 4n + 2, \quad \theta_*^\alpha(\xi_\alpha) = 0, \quad \omega_\alpha = 0. \]

Finally, we get

\begin{equation}
F_{\alpha}(x, y, z) = -g(\varphi_\alpha x, \varphi_\alpha y)\eta_\alpha(z) + \varepsilon_\alpha g(\varphi_\alpha x, \varphi_\alpha z)\eta_\alpha(y).
\end{equation}

In the case $\alpha = 1$, the equality (14.34) takes the form

\begin{equation}
F_1(x, y, z) = g(x, y)\eta_1(z) - g(x, z)\eta_1(y).
\end{equation}

i.e. by virtue of (14.12), (14.13) and (14.18), we have

\[ (\nabla_x \varphi_1)y = g(x, y)\xi_1 - g(\xi_1, y)x. \]

According to [12, Theorem 6.3], the latter equality is a necessary and sufficient condition for a Sasakian manifold.

Similarly, in the case $\alpha = 2$ or $\alpha = 3$, from (14.34), according to Theorem 8.2, we get a necessary and sufficient condition for a Sasaki-like almost contact complex Riemannian manifold.

We recall that a Sasakian manifold (respectively, a Sasaki-like almost contact complex Riemannian manifold) is defined as an almost contact metric manifold (respectively, an almost contact B-metric manifold) which complex cone is a Kähler manifold (respectively, a Kähler-Norden manifold) (cf. [12] and §8).

Thus, we obtain the following

**Proposition 14.5.** The manifold $(S, \varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is:

(i) a Sasakian manifold for $\alpha = 1$;
(ii) a Sasaki-like almost contact complex Riemannian manifold for $\alpha = 2, 3$.

In view of (14.5), (14.9), (14.34) and (14.35), we obtain that $(\mathcal{S}, \varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$ belongs to the class $\mathcal{P}_2$ of almost contact metric manifolds for $\alpha = 1$ and to the class $\mathcal{F}_4$ of almost contact B-metric manifolds for $\alpha = 2, 3$.

In [39], it is considered a unit time-like sphere with almost contact B-metric structure and it is proved that it belongs to the class $\mathcal{F}_4 \oplus \mathcal{F}_5$, the analogue of trans-Sasakian manifold of type $(\alpha, \beta)$.

***
§15. Associated Nijenhuis tensors on manifolds with almost contact 3-structure and metrics of Hermitian-Norden type

In the present section, it is considered a differentiable manifold equipped with a pseudo-Riemannian metric and an almost contact 3-structure so that one almost contact metric structure and two almost contact B-metric structures are generated. There are introduced associated Nijenhuis tensors for the studied structures. The vanishing of the Nijenhuis tensors and their associated tensors is considered. It is given a geometric interpretation of the vanishing of associated Nijenhuis tensors for the studied structures as a necessary and sufficient condition for existence of affine connections with totally skew-symmetric torsions preserving the structure. An example of a 7-dimensional manifold with connections of the considered type is given.

The main results of this section are published in [93] and [94].

In §1, §4 and §12 were defined and then studied associated Nijenhuis tensors on almost complex manifold with Norden metric, almost contact manifold with B-metric and almost hypercomplex manifold with Hermitian-Norden metrics, respectively. The goal of the present section is to introduce an appropriate associated Nijenhuis tensor on a manifold with almost contact 3-structure and metrics of Hermitian-Norden type which will be used in studying of the considered manifold.

As it is known, for each \( \alpha \in \{1, 2, 3\} \) the Nijenhuis tensor \( N_\alpha \) of an almost contact manifold \( (\mathcal{M}, \varphi_\alpha, \xi_\alpha, \eta_\alpha) \) is defined as in (4.25). Moreover, if two of almost contact structures in an almost contact 3-structure are normal, then the third one is also normal [62, 150, 153].
15.1. Associated Nijenhuis tensors of an almost contact 3-structure with a pseudo-Riemannian metric

Let us consider the symmetric braces \{·, ·\} on \(\mathfrak{X}(\mathcal{M})\) introduced by (1.19) for a pseudo-Riemannian metric \(g\), as well as the tensors \{\varphi_1, \varphi_1\} and \(\mathfrak{L}_{\xi_1}g\) determined by (4.27) and (14.16), respectively.

**Definition 15.1.** The symmetric (1,2)-tensor \(\widehat{N}_1\), defined by

\[
\widehat{N}_1 = \{\varphi_1, \varphi_1\} - \xi_1 \otimes \mathfrak{L}_{\xi_1}g,
\]

is called the associated Nijenhuis tensor of the almost contact metric structure \((\varphi_1, \xi_1, \eta_1, g)\).

The corresponding tensors of type \((0, 3)\) for \(N_1\) and \(\widehat{N}_1\) are given by

\[
N_1(x, y, z) = g(N_1(x, y), z) \quad \text{and} \quad \widehat{N}_1(x, y, z) = g(\widehat{N}_1(x, y), z),
\]

respectively.

By direct consequences of the definitions, we get that \(N_1, \widehat{N}_1\) and \(\mathfrak{L}_{\xi_1}g\) are expressed in terms of \(F_1\) as follows:

\[
N_1(x, y, z) = F_1(\varphi_1 x, y, z) - F_1(\varphi_1 y, x, z)
\]

\[
+ F_1(x, y, \varphi_1 z) - F_1(y, x, \varphi_1 z)
\]

\[
+ F_1(x, \varphi_1 y, \xi_1) \eta_1(z) - F_1(y, \varphi_1 x, \xi_1) \eta_1(z),
\]

\[
\widehat{N}_1(x, y, z) = F_1(\varphi_1 x, y, z) + F_1(\varphi_1 y, x, z)
\]

\[
+ F_1(x, y, \varphi_1 z) + F_1(y, x, \varphi_1 z)
\]

\[
+ F_1(x, \varphi_1 y, \xi_1) \eta_1(z) + F_1(y, \varphi_1 x, \xi_1) \eta_1(z),
\]

\[
(\mathfrak{L}_{\xi_1}g)(x, y) = F_1(x, \varphi_1 y, \xi_1) + F_1(y, \varphi_1 x, \xi_1).
\]

According to (4.28), the associated Nijenhuis tensor \(\widehat{N}_2\) for the almost contact B-metric structure \((\varphi_2, \xi_2, \eta_2, g)\) is defined by

\[
\widehat{N}_2 = \{\varphi_2, \varphi_2\} + \xi_2 \otimes \mathfrak{L}_{\xi_2}g.
\]

**Proposition 15.1.** For the almost contact B-metric manifold \((\mathcal{M}, \varphi_2, \xi_2, \eta_2, g)\), the vanishing of \(\widehat{N}_2\) implies that \(\xi_2\) is Killing.

**Proof.** The formula for \(F_2\) in terms of \(N_2\) and \(\widehat{N}_2\) is known from (4.39) of Theorem 4.1, whereas the expression of \(\widehat{N}_2\) by \(F_2\) follows from (4.32). By these relations, (4.35), (14.15) and (14.16), we obtain
\[(\mathfrak{L}_{\xi_2}g)(x, y) = -\frac{1}{2}\left\{\hat{N}_2(\varphi_2x, \varphi_2y, \xi_2) + \hat{N}_2(\xi_2, \varphi_2x, \varphi_2y) + \hat{N}_2(\xi_2, \varphi_2y, \xi_2) + \eta_2(x)\hat{N}_2(\xi_2, \xi_2, y) + \eta_2(y)\hat{N}_2(\xi_2, \xi_2, x)\right\} ,\]

which yields the statement.

Let us remark that a similar statement of Proposition 15.1 for an almost contact metric manifold is not true whereas the corresponding proposition for the almost contact B-metric structure \((\varphi_3, \xi_3, \eta_3, g)\) and \(\hat{N}_3\) defined as in (15.5) holds.

Let \(M, \dim M = 4n + 3\), be equipped with an almost contact 3-structure \((\varphi_\alpha, \xi_\alpha, \eta_\alpha)\) and then we consider the product \(M \times \mathbb{R}\). Let \(X\) be a vector field on \(M \times \mathbb{R}\) which is presented by a pair \(\left(x, a \frac{d}{dt}\right)\), where \(x\) is a tangent vector field on \(M\), \(t\) is the coordinate on \(\mathbb{R}\) and \(a\) is a differentiable function on \(M \times \mathbb{R}\) [12, Sect. 6.1]. The almost complex structures \(J_\alpha\) are defined on the manifold \(M \times \mathbb{R}\) by

\[
(15.6) \quad J_\alpha X = J_\alpha \left(x, a \frac{d}{dt}\right) = \left(\varphi_\alpha x - a \xi_\alpha, \eta_\alpha(x) \frac{d}{dt}\right).
\]

In such a way, an almost hypercomplex structure on \(M \times \mathbb{R}\) is defined in [153] when \(M\) has an almost contact 3-structure.

Moreover, we equip \(M \times \mathbb{R}\) with the product metric \(h = g - dt^2\). By virtue of (15.6), (14.12) and its consequence \(g(\xi_\alpha, \xi_\alpha) = -\varepsilon_\alpha\), we obtain

\[
h(J_\alpha X, J_\alpha Y) = \varepsilon_\alpha h(X, Y)
\]

for arbitrary

\[
X = \left(x, a \frac{d}{dt}\right), \quad Y = \left(y, b \frac{d}{dt}\right) \in \mathfrak{X}(M \times \mathbb{R}),
\]

i.e. the manifold \(M \times \mathbb{R}\) admits \((J_\alpha, h)\), an almost hypercomplex structure with Hermitian-Norden metrics.

We introduce the braces \(\{X, Y\}\) on \(M \times \mathbb{R}\) defined by

\[
(15.7) \quad \{X, Y\} = \left(\{x, y\}, (x(b) + y(a)) \frac{d}{dt}\right).
\]
where \( \{x, y\} \) are given in (1.19). Obviously, the braces are symmetric, i.e. \( \{X, Y\} = \{Y, X\} \).

It is known from (12.1) (see also [60]), the Nijenhuis tensor of two endomorphisms \( J_\alpha \) and \( J_\beta \) has the following form:

\[
2[J_\alpha, J_\beta](X, Y) = [J_\alpha X, J_\beta Y] - J_\alpha [J_\beta X, Y] - J_\alpha [X, J_\beta Y]
+ [J_\beta X, J_\alpha Y] - J_\beta [J_\alpha X, Y] - J_\beta [X, J_\alpha Y]
+ (J_\alpha J_\beta + J_\beta J_\alpha)[X, Y].
\]

Then, bearing in mind (12.2), the Nijenhuis tensor of an almost complex structure \( J_\alpha \equiv J_\beta \) is presented by

\[
[J_\alpha, J_\alpha](X, Y) = [J_\alpha X, J_\alpha Y] - J_\alpha [J_\alpha X, Y] - J_\alpha [X, J_\alpha Y] - [X, Y].
\]

Analogously of the last two equalities, using the braces (15.7) instead of the Lie brackets, we define consequently the associated Nijenhuis tensors in the two respective cases as follows:

\[
2\{J_\alpha, J_\beta\}(X, Y) = \{J_\alpha X, J_\beta Y\} - J_\alpha \{J_\beta X, Y\} - J_\alpha \{X, J_\beta Y\}
+ \{J_\beta X, J_\alpha Y\} - J_\beta \{J_\alpha X, Y\} - J_\beta \{X, J_\alpha Y\}
+ (J_\alpha J_\beta + J_\beta J_\alpha)\{X, Y\},
\]

\[
\{J_\alpha, J_\alpha\}(X, Y) = \{J_\alpha X, J_\alpha Y\} - J_\alpha \{J_\alpha X, Y\} - J_\alpha \{X, J_\alpha Y\}
- \{X, Y\}.
\]

Recalling [44], \( G_1 \)-manifolds are almost Hermitian manifolds whose corresponding Nijenhuis (0,3)-tensor by the Hermitian metric is a 3-form (see (12.43)). This condition is equivalent to the vanishing of the associated Nijenhuis tensor, according to Proposition [12.9].

As it is known from [34], the class of the quasi-Kähler manifolds with Norden metric is the only basic class of almost Norden manifolds with non-integrable almost complex structure, because the corresponding Nijenhuis tensor is non-zero there. Moreover, this class is determined by the condition that the associated Nijenhuis tensor is zero.

According to Theorem [12.7], if two of its six associated Nijenhuis tensors for the almost hypercomplex structure vanish, then the others also vanish.

We seek to express in terms of the structure tensors of \( (\varphi_\alpha, \xi_\alpha, \eta_\alpha, g) \) on \( \mathcal{M} \) a necessary and sufficient condition for vanishing of \( \{J_\alpha, J_\alpha\} \) on \( \mathcal{M} \times \mathbb{R} \).
For the structure \((\varphi_\alpha, \xi_\alpha, \eta_\alpha, g)\) let us define the following four tensors of type \((1,2), (0,2), (1,1), (0,1)\), respectively:

\[
\begin{align*}
\widehat{N}_\alpha^{(1)}(x, y) &= \{\varphi_\alpha, \varphi_\alpha\}(x, y) - \varepsilon_\alpha (\mathfrak{L}_{\xi_\alpha} g)(x, y) \xi_\alpha, \\
\widehat{N}_\alpha^{(2)}(x, y) &= -\varepsilon_\alpha (\mathfrak{L}_{\xi_\alpha} g)(\varphi_\alpha x, y) - \varepsilon_\alpha (\mathfrak{L}_{\xi_\alpha} g)(x, \varphi_\alpha y), \\
\widehat{N}_\alpha^{(3)}(x) &= \{\varphi_\alpha, \varphi_\alpha\}(\varphi_\alpha x, \xi_\alpha) + (\mathfrak{L}_{\xi_\alpha} \eta_\alpha)(\varphi_\alpha x) \xi_\alpha \\
&\quad + 2\eta_\alpha(x) \varphi_\alpha \nabla_{\xi_\alpha} \xi_\alpha, \\
\widehat{N}_\alpha^{(4)}(x) &= - (\mathfrak{L}_{\xi_\alpha} \eta_\alpha)(x).
\end{align*}
\]

(15.9)

**Proposition 15.2.** The associated Nijenhuis tensor \(\{J_\alpha, J_\alpha\}\) of an almost complex structure \(J_\alpha\) for \((\mathcal{M} \times \mathbb{R}, J_\alpha, h)\) vanishes if and only if the four tensors \(\widehat{N}_\alpha^{(1)}, \widehat{N}_\alpha^{(2)}, \widehat{N}_\alpha^{(3)}, \widehat{N}_\alpha^{(4)}\) for the structure \((\varphi_\alpha, \xi_\alpha, \eta_\alpha, g)\) vanish.

**Proof.** First of all we need of the following relations

(15.10) \((\mathfrak{L}_{\xi_\alpha} g)(\xi_\alpha, x) = -\varepsilon_\alpha (\mathfrak{L}_{\xi_\alpha} \eta_\alpha)(x) = g(\nabla_{\xi_\alpha} \xi_\alpha, x)\).

These equalities follow by virtue of \((14.13), (14.15), (14.16)\).

Since each \(\{J_\alpha, J_\alpha\}\) is a tensor of type \((1,2)\), it suffices to compute the tensors

\[
\{J_\alpha, J_\alpha\}(X_0, Y_0), \quad \{J_\alpha, J_\alpha\}(X_0, Z_0),
\]

where \(X_0 = \left(x, 0 \frac{d}{dt}\right), Y_0 = \left(y, 0 \frac{d}{dt}\right), Z_0 = \left(o, \frac{d}{dt}\right)\) and \(o\) is the zero element of \(\mathfrak{X}(\mathcal{M})\). Taking into account \((4.27), (15.6), (15.7), (15.8)\), we get the equalities:

\[
\begin{align*}
\{J_\alpha, J_\alpha\}(X_0, Y_0) &= \left(\widehat{N}_\alpha^{(1)}(x, y), \widehat{N}_\alpha^{(2)}(x, y) \frac{d}{dt}\right), \\
\{J_\alpha, J_\alpha\}(X_0, Z_0) &= \left(\widehat{N}_\alpha^{(3)}x, \widehat{N}_\alpha^{(4)}(x) \frac{d}{dt}\right),
\end{align*}
\]

which show the correctness of the statement. \(\square\)

**Proposition 15.3.** For an almost contact structure \((\varphi_\alpha, \xi_\alpha, \eta_\alpha)\) and a pseudo-Riemannian metric \(g\), the vanishing of \(\widehat{N}_\alpha^{(1)}\) implies the vanishing of \(\widehat{N}_\alpha^{(2)}, \widehat{N}_\alpha^{(3)}\) and \(\widehat{N}_\alpha^{(4)}\).

**Proof.** We set \(y = \xi_\alpha\) in \(\widehat{N}_\alpha^{(1)}(x, y) = 0\) and apply \(\eta_\alpha\). Then, using \((4.27)\) and \((14.10)\), we obtain \((\mathfrak{L}_{\xi_\alpha} g)(x, \xi_\alpha) = 0\) and thus \(\widehat{N}_\alpha^{(4)} = 0\), according to \((15.10)\).
Therefore, from the form of $\hat{N}_\alpha^{(1)}$ in (15.9), we get $\{\varphi_\alpha, \varphi_\alpha\}(\varphi_\alpha x, \xi_\alpha) = 0$. On the other hand, bearing in mind (15.10), we have that the vanishing of $(\mathcal{L}_{\xi_\alpha} g)(x, \xi_\alpha)$ is equivalent to the vanishing of $(\mathcal{L}_{\xi_\alpha} \eta_\alpha)(x)$ and $\nabla_{\xi_\alpha} \xi_\alpha$. Thus, we obtain $\hat{N}_\alpha^{(3)} = 0$.

Finally, applying $\eta_\alpha$ to $\hat{N}_\alpha^{(1)}(\varphi_\alpha x, y) = 0$ and using (4.27), we have

$$\eta_\alpha(\{\varphi_\alpha^2 x, \varphi_\alpha y\}) - \varepsilon_\alpha (\mathcal{L}_{\xi_\alpha} g)(\varphi_\alpha x, y) = 0.$$  

The first term in the latter equality can be expressed in the following form $-\varepsilon_\alpha (\mathcal{L}_{\xi_\alpha} g)(x, \varphi_\alpha y)$, using that $\mathcal{L}_{\xi_\alpha} \eta_\alpha$ vanishes. In such a way, we obtain that $\hat{N}_\alpha^{(2)}(x, y) = 0$.

**Proposition 15.4.** For an almost contact structure $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)$ and a pseudo-Riemannian metric $g$, where $\xi_\alpha$ is Killing, the following assertions are valid:

(i) $\hat{N}_\alpha^{(1)}$ vanishes if and only if $\{\varphi_\alpha, \varphi_\alpha\}$ vanishes;

(ii) $\hat{N}_\alpha^{(2)}$ vanishes;

(iii) $\hat{N}_\alpha^{(3)}$ vanishes if and only if $\xi_\alpha \mathcal{J} \{\varphi_\alpha, \varphi_\alpha\}$ vanishes;

(iv) $\hat{N}_\alpha^{(4)}$ vanishes.

**Proof.** Taking into account that $\mathcal{L}_{\xi_\alpha} g$ vanishes, we have $\hat{N}_\alpha^{(1)} = \{\varphi_\alpha, \varphi_\alpha\}$ and $\hat{N}_\alpha^{(2)} = 0$, i.e. (ii). Further, we obtain $\hat{N}_\alpha^{(4)} = 0$, i.e. (iv), and $\hat{N}_\alpha^{(3)} x = \{\varphi_\alpha, \varphi_\alpha\}(\varphi_\alpha x, \xi_\alpha)$, according to (15.10). Then, (i) is obvious whereas (iii) holds, bearing in mind the assumption for $\xi_\alpha$.

**Definition 15.2.** Let $(\mathcal{M}, \varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a manifold with almost contact 3-structure and metrics of Hermitian-Norden type. The symmetric $(1, 2)$-tensors defined by

$$\hat{N}_\alpha = \{\varphi_\alpha, \varphi_\alpha\} - \varepsilon_\alpha \xi_\alpha \otimes \mathcal{L}_{\xi_\alpha} g$$

we call associated Nijenhuis tensors on $(\mathcal{M}, \varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$.

The corresponding $(0, 3)$-tensors are denoted by

$$\hat{N}_\alpha(x, y, z) = g(\hat{N}_\alpha(x, y), z),$$

$$\{\varphi_\alpha, \varphi_\alpha\}(x, y, z) = g(\{\varphi_\alpha, \varphi_\alpha\}(x, y), z).$$

Then, taking into account (14.13) and (15.11), we obtain

$$\hat{N}_\alpha(x, y, z) = \{\varphi_\alpha, \varphi_\alpha\}(x, y, z) + (\mathcal{L}_{\xi_\alpha} g)(x, y) \eta_\alpha(z).$$
Theorem 15.5. Let \((M, \varphi_\alpha, \xi_\alpha, \eta_\alpha, g)\) be a manifold with almost contact 3-structure and metrics of Hermitian-Norden type. For any \(\alpha\), the associated Nijenhuis tensor \(\{J_\alpha, J_\alpha\}\) of the almost complex structure \(J_\alpha\) on \((M \times \mathbb{R}, J_\alpha, h)\) vanishes if and only if the associated Nijenhuis tensor \(\hat{N}_\alpha\) of the structure \((\varphi_\alpha, \xi_\alpha, \eta_\alpha, g)\) vanishes.

Proof. The statement follows from Proposition 15.2 and Proposition 15.3, bearing in mind (15.9) and (15.11).

Theorem 15.6. Let \((M, \varphi_\alpha, \xi_\alpha, \eta_\alpha, g)\) be a manifold with almost contact 3-structure and metrics of Hermitian-Norden type. If two of the associated Nijenhuis tensors \(\hat{N}_\alpha\) vanish, the third also vanishes.

Proof. It follows by virtue of Theorem 12.7 and Theorem 15.5.

15.2. Natural connections with totally skew-symmetric torsion on a manifold with almost contact 3-structure and metrics of Hermitian-Norden type

An affine connection \(\nabla^*\) is said to be a natural connection for \((\varphi_\alpha, \xi_\alpha, \eta_\alpha, g)\), if it preserves the structure, i.e. \(\nabla^* \varphi_\alpha = \nabla^* \xi_\alpha = \nabla^* \eta_\alpha = \nabla^* g = 0\).

Theorem 15.7. Let \((M, \varphi_1, \xi_1, \eta_1, g)\) be a pseudo-Riemannian manifold with an almost contact metric structure. The following statements are equivalent:

(i) The manifold belongs to the class \(\mathcal{P}_2 \oplus \mathcal{P}_4 \oplus \mathcal{P}_9 \oplus \mathcal{P}_10 \oplus \mathcal{P}_11\) determined by

\[
F_1(\varphi_1 x, y, z) + F_1(\varphi_1 y, x, z) + F_1(x, y, \varphi_1 z) + F_1(y, x, \varphi_1 z) = 0.
\]

(ii) The associated Nijenhuis tensor \(\hat{N}_1\) vanishes and \(\xi_1\) is a Killing vector field;

(iii) The tensor \(\{\varphi_1, \varphi_1\}\) vanishes and \(\xi_1\) is a Killing vector field;

(iv) The Nijenhuis tensor \(N_1\) is a 3-form and \(\xi_1\) is a Killing vector field;

(v) There exists a natural connection \(\nabla^1\) with totally skew-symmetric torsion for the structure \((\varphi_1, \xi_1, \eta_1, g)\) and this connection is unique and determined by its torsion.
\[ T_1(x, y, z) = F_1(x, y, \varphi_1 z) - F_1(y, x, \varphi_1 z) - F_1(\varphi_1 z, x, y) \]
\[ + 2F_1(x, \varphi_1 y, \xi_1) \eta_1(z). \]

**Proof.** Using (15.3) and (15.4), we have that the vanishing of \( \hat{N}_1 \) and \( \mathcal{L}_{\xi_1} g \) implies the identity (15.12). Vice versa, setting \( x = y = \xi_1 \) in (15.12), we have \( F_1(\xi_1, \xi_1, z) = 0 \). If we put \( x = \varphi_1 x, y = \varphi_1 y, z = \xi_1 \) in (15.12) and use the latter vanishing, we obtain \( \mathcal{L}_{\xi_1} g = 0 \) and therefore \( \hat{N}_1 = 0 \). The determination of the class in (i) by (15.12) becomes under the definitions of the basic classes by (14.5) and the form of the corresponding components \( P^i(F_1) \), given in [4]. So, the equivalence between (i) and (ii) is valid.

Now, we need to prove the following relation
\[ \hat{N}_1(x, y, z) = N_1(z, x, y) + N_1(z, y, x). \]
We calculate the right hand side of (15.14) using (15.2). By (9.6) we obtain
\[ N_1(z, x, y) + N_1(z, y, x) = -F_1(\varphi_1 x, z, y) - F_1(\varphi_1 y, z, x) \]
\[ - F_1(x, z, \varphi_1 y) - F_1(y, z, \varphi_1 x) \]
\[ - F_1(x, \varphi_1 z, \xi_1) \eta_1(y) - F_1(y, \varphi_1 z, \xi_1) \eta_1(x) \]
and then we establish that the right hand side of the latter equality is equal to \( \hat{N}_1(x, y, z) \), according to (15.3). Therefore, (15.14) is valid.

The relation (15.14) implies the equivalence between (ii) and (iv), whereas the equivalence between (iv) and (v) is given in Theorem 8.2 of [31]. The equivalence between (ii) and (iii) follows from (15.1).

For the connection \( \nabla^1 \) from (v) we have
\[ g(\nabla^1_{x y} z) = g(\nabla_{x y} z) + \frac{1}{2} T_1(x, y, z). \]
According to Theorem 8.2 in [31], its torsion \( T_1 \) is determined in our notations by
\[ T_1 = -\eta_1 \wedge d\eta_1 + d^{\varphi_1} \Phi + N_1 - \eta_1 \wedge (\xi_1 \lrcorner \mathcal{N}_1), \]
where it is used the notation \( d^{\varphi_1} \Phi(x, y, z) = -d\Phi(\varphi_1 x, \varphi_1 y, \varphi_1 z) \) for the fundamental 2-form \( \Phi \) of the almost contact metric structure, i.e. \( \Phi(x, y) = g(x, \varphi_1 y) \). Since \( \eta_1 \wedge d\eta_1 = \mathcal{G}\{\eta_1 \otimes d\eta_1\} \) holds and because of (14.12), (9.6) and the fact that \( \xi_1 \) is Killing, it is valid
\[ (\eta_1 \wedge d\eta_1)(x, y, z) = -2 \mathcal{G}\{\eta_1(x)F_1(y, \varphi_1 z, \xi_1)\}. \]
Moreover, from the equalities $d\Phi(x, y, z) = -\bigotimes_{x,y,z} F_1(x, y, z)$ and (9.6), we get

$$d\varphi_1 \Phi(x, y, z) = -\bigotimes_{x,y,z} \left\{ F_1(\varphi_1 x, y, z) + 2F_1(x, \varphi_1 y, \xi_1)\eta_1(z) \right\}. \tag{15.18}$$

So, applying (15.17), (15.18), (15.2) and (9.6) to the equality (15.16), we obtain an expression of $T_1$ in terms of $F_1$ given in (15.13).

**Theorem 15.8.** The following statements for an almost contact B-metric manifold $(M, \varphi_2, \xi_2, \eta_2, g)$ are equivalent:

(i) It belongs to the class $F_3 \oplus F_7$, which is characterised by the conditions: the cyclic sum of $F_2$ by the three arguments vanishes and $\xi_2$ is Killing;

(ii) It has a vanishing associated Nijenhuis tensor $\hat{N}_2$;

(iii) It has a vanishing tensor $\{\varphi_2, \varphi_2\}$ and $\xi_2$ is Killing;

(iv) It admits the existence of a unique natural connection $\nabla^2$ with totally skew-symmetric torsion determined by

$$T_2(x, y, z) = -\frac{1}{2} \bigotimes_{x,y,z} \left\{ F_2(x, y, \varphi_2 z) - 3\eta_2(x)F_2(y, \varphi_2 z, \xi_2) \right\}. \tag{15.19}$$

**Proof.** The equivalence of (i), (ii) and (iv) is known from [83] and Corollary 5.11. Bearing in mind Proposition 15.1 and the definition of $\hat{N}_2$, we obtain the equivalence of (ii) and (iii).

For the natural connection $\nabla^2$ with totally skew-symmetric torsion $T_2$ for the structure $(\varphi_2, \xi_2, \eta_2, g)$ we have

$$g \left( \nabla^2 y, z \right) = g(\nabla y, z) + \frac{1}{2} T_2(x, y, z), \tag{15.20}$$

where $T_2$ is determined by $T_2 = \eta_2 \wedge d\eta_2 + \frac{1}{4} \bigotimes N_2$ and it is expressed in terms of $F_2$ by (15.19).

Using Theorem 15.6, Theorem 15.7 and Theorem 15.8, we get

**Theorem 15.9.** The existence of unique natural connections with totally skew-symmetric torsion for two of the three structures of an almost contact 3-structure with metrics of Hermitian-Norden type implies an existence of a unique natural connection with totally skew-symmetric torsion for the remaining third structure.

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Corollary 15.10. Let \((M, \varphi_\alpha, \xi_\alpha, \eta_\alpha, g)\) be a manifold with almost contact 3-structure and metrics of Hermitian-Norden type. If the manifold belongs to two of the following three classes for the corresponding structure, then the manifold belongs to the remaining third class for the corresponding structure: \(P_2 \oplus P_4 \oplus P_9 \oplus P_{10} \oplus P_{11}\) for \(\alpha = 1\); \(F_3 \oplus F_7\) for \(\alpha = 2\) and \(F_3 \oplus F_7\) for \(\alpha = 3\).

Now, we are interested on conditions for coincidence of these three natural connections \(\nabla^\alpha\) with totally skew-symmetric torsion for the particular almost contact structures with the metric \(g\). Then we shall say that it exists a natural connection with totally skew-symmetric torsion for the almost contact 3-structure with metrics of Hermitian-Norden type.

Theorem 15.11. Let \((M, \varphi_\alpha, \xi_\alpha, \eta_\alpha, g)\) be a manifold with almost contact 3-structure and metrics of Hermitian-Norden type. Then the manifold admits an affine connection \(\nabla^*\) with totally skew-symmetric torsion preserving the structure \((\varphi_\alpha, \xi_\alpha, \eta_\alpha, g)\) if and only if the associated Nijenhuis tensors \(\hat{N}_\alpha\) vanish, \(\xi_1\) is Killing and the equalities \(T_1 = T_2 = T_3\) are valid, bearing in mind (15.13) and (15.19). If \(\nabla^*\) exists, it is unique and determined by its torsion \(T^* = T_1 = T_2 = T_3\).

Proof. According to Theorem 15.7 and Theorem 15.8 since \(\hat{N}_\alpha = \mathcal{L}_{\xi_i}g = 0\) are valid then there exist the natural connections \(\nabla^\alpha\) with totally skew-symmetric torsion \(T_\alpha\) for the structures \((\varphi_\alpha, \xi_\alpha, \eta_\alpha, g)\), \((\alpha = 1, 2, 3)\). Bearing in mind (15.13), (15.15), (15.20) and (15.19), the coincidence of \(\nabla^1\), \(\nabla^2\) and \(\nabla^3\) is equivalent to the conditions to equalise of their torsions. ∎

15.3. A 7-dimensional Lie group as a manifold with almost contact 3-structure and metrics of Hermitian-Norden type

Let \(L\) be a 7-dimensional real connected Lie group, and \(l\) be its Lie algebra with a basis \(\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}\). Now we introduce an almost contact 3-structure and metrics of Hermitian-Norden type \((\varphi_\alpha, \xi_\alpha, \eta_\alpha, g)\) by a standard way as follows

\[
\begin{align*}
\varphi_1 e_1 &= e_2, & \varphi_1 e_3 &= e_4, & \varphi_1 e_6 &= e_7, & \varphi_1 e_5 &= o, \\
\varphi_2 e_1 &= e_3, & \varphi_2 e_4 &= e_2, & \varphi_2 e_7 &= e_5, & \varphi_2 e_6 &= o, \\
\varphi_3 e_1 &= e_4, & \varphi_3 e_2 &= e_3, & \varphi_3 e_5 &= e_6, & \varphi_3 e_7 &= o, \\
\xi_1 &= e_5, & \xi_2 &= e_6, & \xi_3 &= e_7, \\
\eta_1 &= e^5, & \eta_2 &= e^6, & \eta_3 &= e^7, & e^i(e_j) = \delta^i_j,
\end{align*}
\]
(15.21b) \[ g(e_i, e_j) = 0, \; i \neq j; \]
\[ g(e_i, e_i) = -g(e_j, e_j) = 1, \; i = 1, 2, 6, 7; \; j = 3, 4, 5. \]

where \( o \) denotes the zero vector in \( T_pL \) at \( p \in L \) and bearing in mind that \( \varphi^2_\alpha = -I \).

Let us consider \( (L, \varphi_\alpha, \xi_\alpha, \eta_\alpha, g) \) with the Lie algebra \( \mathfrak{l} \) determined by the following nonzero commutators for \( \lambda \in \mathbb{R} \setminus \{0\} \)
\[ [e_1, e_2] = [e_3, e_4] = \lambda e_7. \]

By the Koszul equality (5.18), we compute the components of \( \nabla \) with respect to the basis and the nonzero ones of them are:
\[ \nabla_{e_1} e_2 = -\nabla_{e_2} e_1 = \nabla_{e_3} e_4 = -\nabla_{e_4} e_3 = \frac{1}{2}\lambda e_7, \]
\[ \nabla_{e_1} e_7 = \nabla_{e_7} e_1 = -\frac{1}{2}\lambda e_2, \quad \nabla_{e_2} e_7 = \nabla_{e_7} e_2 = \frac{1}{2}\lambda e_1, \]
\[ \nabla_{e_3} e_7 = \nabla_{e_7} e_3 = \frac{1}{2}\lambda e_4, \quad \nabla_{e_4} e_7 = \nabla_{e_7} e_4 = -\frac{1}{2}\lambda e_3. \]

**Proposition 15.12.** Let \( (L, \varphi_\alpha, \xi_\alpha, \eta_\alpha, g) \) be the Lie group \( L \) with almost contact 3-structure and metrics of Hermitian-Norden type depending on \( \lambda \in \mathbb{R} \setminus \{0\} \). Then this manifold belongs to the following basic classes, according to the corresponding classification in (14.5) and (14.9):

(i) \( \mathcal{P}_{10} \) with respect to \( (\varphi_1, \xi_1, \eta_1, g) \);
(ii) \( \mathcal{F}_3 \) with respect to \( (\varphi_2, \xi_2, \eta_2, g) \);
(iii) \( \mathcal{F}_7 \) with respect to \( (\varphi_3, \xi_3, \eta_3, g) \).

**Proof.** Using (15.21) and (15.22), we obtain the property \( (F_\alpha)_{ijk} = -(F_\alpha)_{jik} \)
for \( i \neq j \) and the following values of the basic components
\[ (F_\alpha)_{ijk} = F_\alpha(e_i, e_j, e_k) \]
of \( F_\alpha \):
\[ \frac{1}{2}\lambda = (F_1)_{117} = (F_1)_{126} = (F_1)_{227} = (F_1)_{337} \]
\[ = (F_2)_{346} = (F_1)_{447} = (F_2)_{125} = (F_2)_{147} \]
\[ = (F_3)_{237} = (F_3)_{345} = -(F_3)_{137} = (F_3)_{247} \]
and the others are zero. From here, applying the classification conditions for the relevant classification in [4] or [39], we have the classes in the statement, respectively. \( \Box \)
Bearing in mind Proposition 15.12, we deduce that the conditions (i) of Theorem 15.7 and Theorem 15.8 are fulfilled and hence there exist natural connections $\nabla^\alpha$ for the corresponding structure $(\varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$ on $L$. We get the basic components of their torsions $T_\alpha$ by direct computations from (15.15), (15.13), (15.20), (15.19) and (15.23) as follows:

$$(T_1)_{127} = (T_1)_{347} = (T_3)_{127} = (T_3)_{347} = -\lambda,$$

$$(T_2)_{127} = -(T_2)_{145} = -(T_2)_{235} = (T_2)_{347} = -\frac{1}{2}\lambda$$

and the others are zero.

Obviously, $\nabla^1$ and $\nabla^3$ coincides but $\nabla^2$ differs from them. The condition for equality of the three torsions in Theorem 15.11 is not fulfilled and therefore it does not exist a unique connection with totally skew-symmetric torsion preserving the almost contact 3-structure and the metrics of Hermitian-Norden type on $(L, \varphi_\alpha, \xi_\alpha, \eta_\alpha, g)$.

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Conclusion

Contributions of the Dissertation

The present dissertation contains recent author’s investigations on differential geometry of smooth manifolds equipped with some tensor structures (almost complex structures, almost contact structures, almost hypercomplex structures and almost contact 3-structures) and metrics of Norden type.

According to the author, the main contributions of the dissertation are the following:

1. There have been introduced and studied the twin interchange of the pair of Norden metrics (the basic one and its associated metric) on almost complex manifolds as well as there have been found invariant and anti-invariant geometric objects and characteristics under this interchange. (§2)

2. It has been developed further the study on the basic natural connections: the B-connection, the canonical connection and the KT-connection on almost complex manifolds with Norden metric. It has been characterized all basic classes of the considered manifolds with respect to the torsions of the canonical connection, the Nijenhuis tensor and its associated one. (§3)

3. It has been introduced an associated Nijenhuis tensor on an almost contact manifold with B-metric having important geometrical characteristics. Moreover, these manifolds have been classified with respect to the Nijenhuis tensor and its associated one. (§4)

4. It has been introduced and studied a $\varphi$-canonical connection on the almost contact manifolds with B-metric and it has been found the relation between this connection and other two important natural connections on these manifolds – the $\varphi$B-connection and the $\varphi$KT-connection.
Conclusion. Contributions of the Dissertation

It has been established that the torsion of the $\varphi$-canonical connection is invariant with respect to a subgroup of the general conformal transformations of the almost contact $B$-metric structure. Thereby, the basic classes of the considered manifolds have been characterized in terms of the torsion of the $\varphi$-canonical connection. (§5)

5. There have been classified all affine connections on an almost contact manifold with $B$-metric with respect to the properties of their torsions regarding the manifold’s structure. Three studied natural connections have been characterized regarding this classification. (§6)

6. There have been introduced and studied a pair of associated Schouten-van Kampen affine connections adapted to the contact distribution and the almost contact $B$-metric structure generated by the pair of associated $B$-metrics and their Levi-Civita connections. By means of the constructed non-symmetric connections, there have been characterized the basic classes of the almost contact $B$-metric manifolds. (§7)

7. There have been introduced and studied Sasaki-like almost contact complex Riemannian manifolds. In addition, it has been presented a canonical construction (called an $S^1$-solvable extension) producing such a manifold from a holomorphic complex Riemannian manifold. (§8)

8. There have been studied integrable hypercomplex structures with Hermitian-Norden metrics on 4-dimensional Lie groups by constructing the five types corresponding of invariant hypercomplex structures with hyper-Hermitian metric. (§10)

9. It have been studied the tangent bundle of an almost complex manifold with Norden metric and the complete lift of the Norden metric as an almost hypercomplex manifold with Hermitian-Norden metrics. (§11)

10. There have been introduced and studied the associated Nijenhuis tensors of an almost hypercomplex manifold with Hermitian-Norden metrics. It has been found a geometric interpretation of the vanishing of these tensors as a necessary and sufficient condition for existence of affine connections with totally skew-symmetric torsions preserving the manifold’s structure. (§12)

11. There have been introduced and studied quaternionic Kähler manifolds corresponding to almost hypercomplex manifolds with Hermitian-Norden metrics. (§13)
12. There have been introduced manifolds with almost contact 3-structure and metrics of Hermitian-Norden type as well as corresponding associated Nijenhuis tensors. It has been found a geometric interpretation of the vanishing of these tensors as a necessary and sufficient condition for existence of affine connections with totally skew-symmetric torsions preserving the manifold’s structure. (§14, §15)

13. There have been constructed and studied a variety of explicit examples of manifolds equipped with studied structures: almost complex structure with Norden metric, almost contact structure with B-metric, almost hypercomplex structure with Hermitian-Norden metrics and almost contact 3-structure with metrics of Hermitian-Norden type.

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Main results of the present dissertation are published in the following papers and preprints:

1. [84] M. Manev. *Quaternionic Kähler manifolds with Hermitian and Norden metrics*. Journal of Geometry, vol. 103, no. 3 (2012), 519–530; ISSN:0047-2468, DOI:10.1007/s00022-012-0139-x, MCQ(2012): 0.16, SJR(2012):0.278.

2. [99] M. Manev, M. Ivanova. *Canonical-type connection on almost contact manifolds with B-metric*. Annals of Global Analysis and Geometry, vol. 43, no. 4 (2013), 397–408; ISSN:0232-704X, DOI:10.1007/s10455-012-9351-z, IF(2013):0.794, SJR(2013):1.248.

3. [100] M. Manev, M. Ivanova. *A classification of the torsion tensors on almost contact manifolds with B-metric*. Central European Journal of Mathematics, vol. 12, no. 10 (2014), 1416–1432; ISSN:1895-1074, DOI:10.2478/s11533-014-0422-1, IF(2014):0.578, MCQ(2014):0.39, SJR(2014):0.610.

4. [86] M. Manev. *Hypercomplex structures with Hermitian-Norden metrics on four-dimensional Lie algebras*. Journal of Geometry, vol. 105, no. 1 (2014), 21–31; ISSN:0047-2468, DOI:10.1007/s00022-013-0188-9, MCQ(2014):0.26, SJR(2014):0.345.

5. [87] M. Manev. *Tangent bundles with complete lift of the base metric and almost hypercomplex Hermitian-Norden structure*. Comptes rendus de l’Academie bulgare des Sciences, vol. 67, no. 3 (2014), 313–322; ISSN:1310-1331, IF(2014):0.284, SJR(2014):0.205.

6. [88] M. Manev. *On canonical-type connections on almost contact complex Riemannian manifolds*. Filomat, vol. 29, no. 3 (2015), 411–425; ISSN:0354-5180, DOI:10.2298/FIL1503411M, IF(2015):0.603, SJR(2015):0.487.

*The number in square brackets is from the general list of Bibliography.*
7. [90] M. Manev. *Pair of associated Schouten-van Kampen connections adapted to an almost contact B-metric structure*. Filomat, vol. 29, no. 10 (2015), 2437–2446; ISSN:0354-5180, IF(2015):0.603, SJR (2015):0.487.

8. [58] S. Ivanov, H. Manev, M. Manev. *Sasaki-like almost contact complex Riemannian manifolds*. Journal of Geometry and Physics, vol. 105 (2016), 136–148; ISSN:0393-0440, DOI:10.1016/j.geomphys.2016.05.009, IF(2015):0.752, SJR(2015):0.705.

9. [89] M. Manev. *Invariant tensors under the twin interchange of Norden metrics on almost complex manifolds*. Results in Mathematics, vol. 70, no. 1 (2016), 109–126; ISSN:1422-6383, DOI:10.1007/s00025-015-0464-0, IF(2015):0.768, MCQ(2015):0.43, SJR(2015):0.636.

10. [91] M. Manev. *Associated Nijenhuis tensors on manifolds with almost hypercomplex structures and metrics of Hermitian-Norden type*. Results in Mathematics, vol. 71 (2017), ISSN:1422-6383, DOI:10.1007/s00025-016-0624-x, IF(2015):0.768, MCQ(2015):0.43, SJR(2015):0.636.

11. [92] M. Manev. *Manifolds with almost contact 3-structure and metrics of Hermitian-Norden type*. Journal of Geometry, (accepted, 4.05.2017), DOI:10.1007/s00022-017-0386-y, MCQ(2015):0.27, SJR(2015):0.272.

12. [93] M. Manev. *Associated Nijenhuis tensors on manifolds with almost contact 3-structure and metrics of Hermitian-Norden type*. Comptes rendus de l’Academie bulgare des Sciences (accepted, 28.02.2017), IF(2015):0.233.

13. [94] M. Manev. *Natural connections with totally skew-symmetric torsion on manifolds with almost contact 3-structure and metrics of Hermitian-Norden type*. arXiv:1604.02039 (part 2).

***
Conclusion. Declaration of Originality

Declaration of Originality

by
PROF. DR. MANCHO HRISTOV MANEV
Department of Algebra and Geometry
Faculty of Mathematics and Informatics
Paisii Hilendarski University of Plovdiv

In connection with the conducting of the procedure for award of the scientific degree of Doctor of Science in Mathematics from Paisii Hilendarski University of Plovdiv and the protection of the presented by me dissertation, I declare:

The results and the contributions of the scientific studies presented in my dissertation on the topic

ON GEOMETRY OF MANIFOLDS WITH SOME TENSOR STRUCTURES AND METRICS OF NORDEN TYPE

are original and not taken from research and publications in which I do not participate.

1.02.2017
Plovdiv

Signature:
(Prof. Dr. Mancho Manev)

***
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