Eigenvalue inequalities in terms of Schatten norm bounds on differences of semigroups, and application to Schrödinger operators

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Abstract

We develop a new method for obtaining bounds on the negative eigenvalues of self-adjoint operators $B$ in terms of a Schatten norm of the difference of the semigroups generated by $A$ and $B$, where $A$ is an operator with non-negative spectrum. Our method is based on the application of the Jensen identity of complex function theory to a suitably constructed holomorphic function, whose zeros are in one-to-one correspondence with the negative eigenvalues of $B$. Applying our abstract results, together with bounds on Schatten norms of semigroup differences obtained by Demuth and Van Casteren, to Schrödinger operators, we obtain inequalities on moments of the sequence of negative eigenvalues, which are different from the Lieb-Thirring inequalities.

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1 Introduction

Let $A$ be a self-adjoint operator on a complex Hilbert space, whose spectrum is non-negative. If $B$ is another self-adjoint operator, such that the difference $D_t = e^{-tB} - e^{-tA}$ of the semigroups corresponding to $A, B$ belongs to a Schatten ideal (trace class or Hilbert-Schmidt class), we will prove inequalities which provide bounds from above on the negative eigenvalues of $B$, in terms of Schatten norms of $D_t$. The usefulness of such results follows from the fact that for concrete operators, for example when $B$ is a Schrödinger operator $B = -\Delta + V$, and $A$ is the free Schrödinger operator $A = -\Delta$, it is known that, under appropriate conditions on the potential $V$, $D_t$ belongs to a Schatten ideal, and explicit bounds on the Schatten norm of $D_t$ are available [3]. Indeed such results are important in the study of the absolutely continuous spectrum of the perturbed operator $B$. The theorems proven here show that these bounds on the Schatten norms of $D_t$ can also be used in the study of the discrete spectrum of $B$.

The method used to prove our results is based on constructing a holomorphic function whose zeros are in one-to-one correspondence with the negative eigenvalues of $B$, and using complex function theory to bound these zeros. Specifically we will use the Jensen identity (see, e.g., [6], p. 307):

**Lemma 1** Let $\Omega_r$ be an open disk centered at 0 and with radius $r$. Let $h : U \rightarrow \mathbb{C}$ be a holomorphic function on the open set $U$, where $\overline{\Omega_r} \subset U$, and assume $h(0) = 1$. Then

$$\frac{1}{2\pi} \int_0^{2\pi} \log(|h(re^{i\theta})|)d\theta = \log \left( \prod_{z \in \overline{\Omega_r}, h(z) = 0} \frac{r}{|z|} \right) = \int_0^r \frac{n(u)}{u}du,$$

where $n(u)$ ($0 \leq u \leq r$) denotes the number of zeros of $h$ in $\overline{\Omega_u}$.

In Section 2 we prove general theorems which give bounds on the moments (sums of powers) of the sequence of negative eigenvalues of an operator $B$ in terms of the trace norm of the semigroup difference. In Section 3 we prove analogous bounds in terms of the Hilbert-Schmidt
norm of the semigroup difference. In Section 4 we apply the theorems of Section 3 to derive inequalities for the negative eigenvalues of Schrödinger operators under some conditions on the potential, which are different from the well-known Lieb-Thirring inequalities.

2 Eigenvalue inequalities in terms of trace-norm bounds on semigroup differences

In this section we will prove results under the assumption that \(A, B\) are selfadjoint operators, with the spectrum of \(A\) non-negative, and such that the difference of semigroups \(D_t = e^{-tB} - e^{-tA}\) is of trace class. This implies that the negative spectrum of \(B\), which we denote by

\[ \sigma^-(B) = \sigma(B) \cap (-\infty, 0), \]

consists only of eigenvalues, which can accumulate only at 0 (of course compactness of \(D_t\) is sufficient for this property). We shall denote by \(N(-s)\) the number of eigenvalues \(\lambda\) of \(B\) which satisfy \(\lambda < -s\).

We begin by proving identities expressing the moments of the negative eigenvalues of the operator \(B\) in terms of an integral. It should be noted that the identities hold also in the case that one side is infinite - which implies that the other side is infinite too.

**Theorem 1** Let \(A, B\) be self-adjoint in a complex Hilbert space \(\mathcal{H}\), with \(\sigma(A) \subset [0, \infty)\). Assume that \(D = e^{-B} - e^{-A}\) is of trace class. Then, for any \(\gamma > 1\), we have

\[
\sum_{\lambda \in \sigma^-(B)} |\lambda|^\gamma = \frac{\gamma(\gamma - 1)}{2\pi} \int_0^1 \frac{1}{r} |\log(r)|^{\gamma - 2} \int_0^{2\pi} \log \left( |\det(I - F(re^{i\theta}))| \right) d\theta dr,
\]

where \(F(z)\) is the operator-valued function defined by

\[
F(z) = z[I - ze^{-A}]^{-1} D,
\]
and for \( \gamma = 1 \) we have

\[
\sum_{\lambda \in \sigma^{-}(B)} |\lambda| = \lim_{r \to 1} \frac{1}{2\pi} \int_{0}^{2\pi} \log \left( |Det(I - F(re^{i\theta}))| \right) d\theta. \tag{3}
\]

**Proof:** We have, for all \( z \in \mathbb{C} \)

\[
I - ze^{-B} = I - ze^{-A} - zD, \tag{4}
\]

and if also \( |z| < 1 \), so that \( \|ze^{-A}\| < 1 \) then \( I - ze^{-A} \) is invertible, so that \( F(z) \) given by (2) is well defined, and we can write (4) as

\[
[I - ze^{-A}]^{-1}[I - ze^{-B}] = I - F(z).
\]

Thus we have the following equivalence for \( |z| < 1 \):

\[
\log(z) \in \sigma(B) \iff \frac{1}{z} \in \sigma(e^{-B}) \iff 1 \in \sigma(F(z)),
\]

so that

\[
\sigma^{-}(B) = \{ \log(z) \mid |z| < 1, \ 1 \in \sigma(F(z)) \}. \tag{5}
\]

Since we assume \( D \) is of trace class, then so is \( F(z) \). We note also that

\[
F(0) = 0. \tag{6}
\]

Since \( F(z) \) is a trace class operator, the determinant

\[
h(z) = Det(I - F(z))
\]

is well defined, and we have that \( h \) is holomorphic in the unit disk and

\[
h(z) = 0 \iff 1 \in \sigma(F(z)) \iff \log(z) \in \sigma^{-}(B).
\]

Thus

\[
\sigma^{-}(B) = \{ \log(z) \mid |z| < 1, \ h(z) = 0 \},
\]

so that, for all \( s > 0 \),

\[
N(-s) = n(e^{-s}), \tag{7}
\]
where $n(r)$ denotes the number of zeros of $h$ in $\Omega_r = \{ z \mid |z| < r \}$. By (6) we have

$$h(0) = \text{Det}(I - F(0)) = \text{Det}(I) = 1.$$  

Applying the Jensen identity, Lemma 1, we have, for any $0 < r < 1$,

$$\frac{1}{2\pi} \int_0^{2\pi} \log(|h(re^{i\theta})|) d\theta = \int_0^r \frac{n(u)}{u} du, \quad (8)$$

and making the substitution $u = e^{-s}$ in the integral on the right-hand side of (8) and using (7) we get

$$\frac{1}{2\pi} \int_0^{2\pi} \log(|h(re^{i\theta})|) d\theta = \int_{\log(\frac{1}{r})}^{\infty} N(-s) ds. \quad (9)$$

We now recall the well-known identity

$$\sum_{\lambda \in \sigma^{-}(B)} |\lambda|^\gamma = \gamma \int_0^\infty s^{\gamma-1} N(-s) ds. \quad (10)$$

Taking $\gamma = 1$, (10) becomes

$$\sum_{\lambda \in \sigma^{-}(B)} |\lambda| = \int_0^\infty N(-s) ds. \quad (11)$$

Taking the limit $r \to 1$ in (9), we have

$$\lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \log(|h(re^{i\theta})|) d\theta = \int_0^{\infty} N(-s) ds. \quad (12)$$

From (11) and (12), we conclude

$$\sum_{\lambda \in \sigma^{-}(B)} |\lambda| = \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \log(|h(re^{i\theta})|) d\theta,$$

so that we have (3).
We now assume that $\gamma > 1$. Multiplying (9) by $\frac{1}{r}|\log(r)|^{\gamma - 2}$ and integrating over $r \in [0, 1]$, we obtain

\[
\frac{1}{2\pi} \int_0^1 \int_0^{2\pi} \frac{1}{r} |\log(r)|^{\gamma - 2} \log(|h(re^{i\theta})|) d\theta dr = \int_0^1 \frac{1}{r} |\log(r)|^{\gamma - 2} \int_{\log(\frac{1}{r})}^{\infty} N(-s) ds dr = \int_0^\infty N(-s) \int_{e^{-s} r}^1 \frac{1}{r} |\log(r)|^{\gamma - 2} dr ds = \frac{1}{\gamma - 1} \int_0^\infty N(-s) s^{{\gamma - 1}} ds,
\]

which, together with (10), implies

\[
\sum_{\lambda \in \sigma^-(B)} |\lambda|^{\gamma} \leq \frac{\gamma(\gamma - 1)}{2\pi} \int_0^1 \int_0^{2\pi} \frac{1}{r} |\log(r)|^{\gamma - 2} \log(|h(re^{i\theta})|) d\theta dr,
\]

so that we have (11). ■

By bounding the function $h$ of Theorem 1 from above, we obtain bounds on the moments of the negative eigenvalues.

**Theorem 2** Let $A, B$ be self-adjoint in a complex Hilbert space $\mathcal{H}$, with $\sigma(A) \subset [0, \infty)$. Assume that $D = e^{-B} - e^{-A}$ is of trace class. Then for any $\gamma > 1$,

\[
\sum_{\lambda \in \sigma^-(B)} |\lambda|^{\gamma} \leq \frac{\gamma(\gamma - 1)}{2\pi} \int_0^1 |\log(r)|^{\gamma - 2} \int_0^{2\pi} ||[I - re^{i\theta} e^{-A}]^{-1} D||_{tr} d\theta dr,
\]

and for $\gamma = 1$ we have

\[
\sum_{\lambda \in \sigma^-(B)} |\lambda| \leq \limsup_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \|[I - re^{i\theta} e^{-A}]^{-1} D||_{tr} d\theta.
\]

**Proof:** We recall the general inequality for trace class operators $T$ (see, e.g., [7])

\[
|\text{Det}(I - T)| \leq e^{\|T\|_{tr}},
\]

(14)
which gives
\[ \log \left( |\text{Det}(I - F(re^{i\theta}))| \right) \leq \| F(re^{i\theta}) \|_{tr} = r \| [I - re^{i\theta}e^{-A}]^{-1}D \|_{tr}. \]

Substituting this inequality into (1), (3), we obtain the results. ■

Bounding the integral on the right-hand side of (13), we get the following theorem. Although we shall later prove a stronger result, Theorem 4, it is useful to present Theorem 3, whose proof is more straightforward, and for which the coefficient in the inequalities can be evaluated explicitly, in terms of Euler’s Γ-function and Riemann’s ζ-function.

**Theorem 3** Let \( A, B \) be self-adjoint in a complex Hilbert space \( \mathcal{H} \), with \( \sigma(A) \subset [0, \infty) \). Assume that, for some \( t > 0 \), \( D_t = e^{-tB} - e^{-tA} \) is of trace class.

Then, for any \( \gamma > 2 \), we have the inequality
\[
\sum_{\lambda \in \sigma^{-}(B)} |\lambda|^{\gamma} \leq \Gamma(\gamma + 1) \zeta(\gamma - 1) \frac{1}{t^{\gamma}} \| D_t \|_{tr},
\]
and the right-hand side is finite.

**Proof:** We note first that it suffices to prove (15) for \( t = 1 \), that is, setting \( D = D_1 = e^{-B} - e^{-A} \), to prove
\[
\sum_{\lambda \in \sigma^{-}(B)} |\lambda|^{\gamma} \leq \Gamma(\gamma + 1) \zeta(\gamma - 1) \| D \|_{tr},
\]
(16)
since (15) follows from (16) by replacing \( A, B \) by \( tA, tB \).

Since \( \sigma(A) \subset [0, \infty) \), we have \( \| e^{-A} \| \leq 1 \), so that, for \( |z| < 1 \),
\[
\| [I - z e^{-A}]^{-1} \| \leq \frac{1}{1 - |z|},
\]
hence
\[
\| F(re^{i\theta}) \|_{tr} = r \| [I - re^{i\theta}e^{-A}]^{-1}D \|_{tr} \leq r \| [I - re^{i\theta}e^{-A}]^{-1}D \|_{tr} \leq \| D \|_{tr} \frac{r}{1 - r}.
\]

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From the inequality (13) of Theorem 2 we thus have

\[
\sum_{\lambda \in \sigma(B)} |\lambda|^{\gamma} \leq \frac{\gamma(\gamma - 1)}{2\pi} \int_0^1 |\log(r)|^{\gamma-2} \int_0^{2\pi} \|[I - re^{i\theta}e^{-A}]^{-1}D\|_{tr} d\theta dr
\]

\[
\leq \gamma(\gamma - 1)\|D\|_{tr} \int_0^1 |\log(r)|^{\gamma-2} \frac{1}{1 - r} dr
\]

\[
= \gamma(\gamma - 1)\|D\|_{tr} \int_0^\infty \frac{x^{\gamma-2}}{e^x - 1} dx = \Gamma(\gamma + 1)\zeta(\gamma - 1)\|D\|_{tr}
\]

so we have (16). ■

A more refined estimate on the integral in (13) yields the following theorem, which is stronger than Theorem 3. We note that this theorem is valid for \(\gamma > 1\), rather than \(\gamma > 2\) as in Theorem 3. The value of the constant \(C_{tr}(\gamma)\) is given in the proof of the theorem, in terms of some integrals.

**Theorem 4** Let \(A, B\) be self-adjoint in a complex Hilbert space \(\mathcal{H}\), with \(\sigma(A) \subset [0, \infty)\). Assume that, for some \(t > 0\), \(D_t = e^{-tB} - e^{-tA}\) is of trace class.

Then, for any \(\gamma > 1\), we have the inequality

\[
\sum_{\lambda \in \sigma(B)} |\lambda|^{\gamma} \leq C_{tr}(\gamma) \frac{1}{t^\gamma}\|D_t\|_{tr},
\]

where \(C_{tr}(\gamma)\) is a finite constant depending only on \(\gamma\).

**Proof**: As noted in the proof of Theorem 3 it suffices to prove (18) for \(t = 1\), that is, setting \(D = e^{-B} - e^{-A}\), to prove

\[
\sum_{\lambda \in \sigma(B)} |\lambda|^{\gamma} \leq C_{tr}(\gamma)\|D\|_{tr}.
\]
Since $\sigma(e^{-A}) \subset [0, 1]$, we have

$$|z||[I - z e^{-A}]^{-1}| = |[z^{-1} I - e^{-A}]^{-1}|$$

$$\leq \frac{1}{\min_{u \in [0,1]} |z^{-1} - u|} = \begin{cases} \frac{1}{|z^{-1} - 1|}, & \text{Re}(z^{-1}) \geq 1 \\ \frac{1}{|\text{Im}(z^{-1})|}, & 0 < \text{Re}(z^{-1}) < 1 \\ \frac{1}{|z^{-1}|}, & \text{Re}(z^{-1}) \leq 0 \end{cases} \quad (20)$$

so that

$$\|F(re^{i\theta})\|_{tr} = r \|[I - re^{i\theta} e^{-A}]^{-1}D\|_{tr} \leq r \|[I - re^{i\theta} e^{-A}]^{-1}\| \|D\|_{tr}$$

$$\leq r \|D\|_{tr} \begin{cases} \frac{1}{\sqrt{r^2 - 2r \cos(\theta) + 1}} & \cos(\theta) \geq r \\ \frac{1}{|\sin(\theta)|} & 0 < \cos(\theta) < r \\ \frac{1}{1} & \cos(\theta) \leq 0 \end{cases}$$

From the inequality (13) of Theorem 2 we thus have, for $\gamma > 1$,

$$\sum_{\lambda \in \sigma^{-}(B)} |\lambda|^\gamma \leq \frac{\gamma(\gamma - 1)}{2\pi} \int_0^1 \log(r) |r|^{-2} \int_0^{2\pi} \|I - re^{i\theta} e^{-A} \|_{tr} d\theta dr$$

$$\leq \frac{\gamma(\gamma - 1)}{\pi} \|D\|_{tr} \left[ \int_0^1 \log(r) |r|^{-2} \int_0^{\arccos(r)} \frac{1}{\sqrt{r^2 - 2r \cos(\theta) + 1}} d\theta dr \\
+ \int_0^1 \log(r) |r|^{-2} \int_0^{\frac{\pi}{2}} \frac{1}{|\sin(\theta)|} d\theta dr + \int_0^1 \int_0^{\frac{\pi}{2}} |\log(r)|^{-2} d\theta dr \right].$$

We estimate the integrals in (21) from above: making the substitution
\( s = \frac{1}{r}, \ y = \frac{1}{\cos(\theta)}, \) we have

\[
\begin{align*}
c_1(\gamma) &= \int_0^1 |\log(r)|^{\gamma - 2} \int_0^{\arccos(r)} \frac{1}{\sqrt{r^2 - 2r \cos(\theta) + 1}} d\theta dr \\
&= \int_1^\infty \frac{(\log(s))^{\gamma - 2}}{s \sqrt{s^2 + 1}} \int_1^s \frac{1}{\sqrt{y - \frac{2s}{s^2 + 1}}} \frac{1}{\sqrt{y^2 - 1}} dy ds \\
&\leq \int_1^\infty \frac{(\log(s))^{\gamma - 2}}{s \sqrt{s^2 + 1}} \int_1^s \frac{1}{\sqrt{y - \frac{2s}{s^2 + 1}}} \frac{1}{\sqrt{y - 1}} dy ds \\
&= \int_1^\infty \frac{(\log(s))^{\gamma - 2}}{s \sqrt{s^2 + 1}} \log \left( \frac{(s(s + 1) + \sqrt{s^2 + 1})^2}{s - 1} \right) ds
\end{align*}
\]

and since, for any \( \epsilon > 0, \) the integrand in the last integral is \( O((s - 1)^{\gamma - 2 - \epsilon}) \) as \( s \to 1+ \) and \( O(s^{\epsilon - 2}) \) as \( s \to \infty, \) this integral is finite whenever \( \gamma > 1, \) so \( c_1(\gamma) \) is finite.

\[
\begin{align*}
c_2(\gamma) &= \int_0^1 |\log(r)|^{\gamma - 2} \int_{\arccos(r)}^{\frac{\pi}{2}} \frac{1}{|\sin(\theta)|} d\theta dr \\
&= \frac{1}{2} \int_0^1 |\log(r)|^{\gamma - 2} \log \left( \frac{1 + r}{1 - r} \right) dr,
\end{align*}
\]

and since, for any \( \epsilon > 0, \) the integrand in the last integral is \( O(r^{1-\epsilon}) \) as \( r \to 0 \) and \( O((1 - r)^{\gamma - 2 - \epsilon}) \) as \( r \to 1, \) this integral is finite whenever \( \gamma > 1, \) so \( c_2(\gamma) \) is finite. Finally, we have

\[
\begin{align*}
c_3(\gamma) &= \int_0^1 \int_0^{\pi} |\log(r)|^{\gamma - 2} d\theta dr = \frac{\pi}{2} \int_0^\infty x^{\gamma - 2} e^{-x} dx = \frac{\pi}{2} \Gamma(\gamma - 1).
\end{align*}
\]

From (21) we thus obtain, for \( \gamma > 1, \)

\[
\sum_{\lambda \in \sigma^{-}(B)} |\lambda|^\gamma \leq \frac{1}{\pi} \gamma (\gamma - 1) [c_1(\gamma) + c_2(\gamma) + c_3(\gamma)] \|D\|_{tr}.
\]

so that we have (19), with

\[
C_\nu(\gamma) = \frac{1}{\pi} \gamma (\gamma - 1) [c_1(\gamma) + c_2(\gamma) + c_3(\gamma)].
\]
One could ask what is the best constant \( C_{tr}(\gamma) \) in inequality (18), that is, given \( \gamma > 1 \), what is the smallest number \( C_{tr}(\gamma) \) for which (18) will hold for any pair of selfadjoint operators with \( \sigma(A) \subset [0, \infty) \). We do not know how to answer this question, but we can give a simple lower bound for the possible values of \( C_{tr}(\gamma) \). We recall that the Lambert W-function is defined on \([-e^{-1}, \infty)\) as the inverse of the function \( f(x) = xe^x \).

**Proposition 1** If \( \gamma > 1 \) and \( C_{tr}(\gamma) \) is a constant for which Theorem 4 holds, then

\[
C_{tr}(\gamma) \geq -W(-\gamma e^{-\gamma})(\gamma + W(-\gamma e^{-\gamma}))^{\gamma - 1}.
\]  

(22)

**Proof**: If the inequality (18) holds then in particular it must hold when \( A \) and \( B \) are \( 1 \times 1 \) matrices. Thus let \( A = 0, B = -b, (b > 0), t = 1 \). Then the moment of the negative eigenvalues of \( B \) of order \( \gamma \) is simply \( b^\gamma \), and \( D_1 = e^{-B} - e^{-A} = e^b - 1 \). Thus inequality (18) becomes in this case

\[
b^\gamma \leq C_{tr}(\gamma)(e^b - 1).
\]

Since this must hold for all \( b > 0 \), we have

\[
C_{tr}(\gamma) \geq \sup_{b>0} \frac{b^\gamma}{e^b - 1}.
\]  

(23)

By differentiating the function of \( b \) on the right-hand side of (23) we find its maximum on \([0, \infty)\) to be given by the expression on the right-hand of (22). [11]

As an example, we take \( \gamma = 2 \). From (22) we obtain \( C_{tr}(2) \geq 0.647.. \). Theorem 4 gives (evaluating the integrals numerically) \( C_{tr}(2) \leq 2.5.. \)

Using the above argument one can see that for \( \gamma < 1 \), Theorem 4 cannot be true. Indeed, if \( \gamma < 1 \), then the expression on the right-hand side of (23) goes to \(+\infty\) as \( b \to 0 \), so that the supremum is infinite.

We remark that the inequalities for the moments of eigenvalues derived here imply inequalities for the number of eigenvalues less than a given
negative number $-s$ ($s > 0$), which we denote by $N(-s)$. Indeed since

$$\sum_{\lambda \in \sigma^{-}(B)} |\lambda|^\gamma \geq \sum_{\lambda \in \sigma(B) \cap (-\infty, -s)} |\lambda|^\gamma \geq \sum_{\lambda \in \sigma(B) \cap (-\infty, -s)} s^\gamma = s^\gamma N(-s),$$

we have, from (18), assuming that $D_t$ is trace-class for all $t > 0$,

$$N(-s) \leq \frac{1}{s^\gamma} \sum_{\lambda \in \sigma(B) \cap (-\infty, 0)} |\lambda|^\gamma \leq \inf_{t>0,\gamma>1} \frac{C_{tr}(\gamma)}{(st)^\gamma} \|D_t\|_{tr}.$$

3 Eigenvalue inequalities in terms of Hilbert-Schmidt norm bounds on semigroup differences

In this section we prove theorems analogous to those in the previous section, for the case in which the semigroup difference is Hilbert-Schmidt rather than trace class. The proofs are similar, the difference being that we have to get around the fact that the determinant is not defined for a general Hilbert-Schmidt perturbation of the identity. In the applications to Schrödinger operators, it is easier to verify that the semigroup difference is Hilbert-Schmidt than to verify that it is trace class, so the theorems of this section will be used in these applications, to be presented in Section 4.

The following theorem is the Hilbert-Schmidt analog of Theorem 1. It should be noted, however, that unlike in Theorem 1 here we have only inequalities rather than identities.

**Theorem 5** Let $A, B$ be self-adjoint in a complex Hilbert space $\mathcal{H}$, with $\sigma(A) \subset [0, \infty)$. Assume that $D = e^{-B} - e^{-A}$ is Hilbert-Schmidt. Then we have, for any $\gamma > 1$,

$$\sum_{\lambda \in \sigma^{-}(B)} |\lambda|^\gamma \leq \frac{\gamma^{(\gamma - 1)}}{2\pi} \int_{0}^{1} \frac{1}{r} \log(r)|^{\gamma-2} \int_{0}^{2\pi} \log \left(|\text{Det}(I - (F(re^{i\theta}))^2)|\right) d\theta dr.$$
where $F(z)$ is the operator-valued function defined by

$$F(z) = z[I - z e^{-A}]^{-1}D,$$

and for $\gamma = 1$

$$\sum_{\lambda \in \sigma^-(B)} |\lambda| \leq \lim_{r \to 1} \frac{1}{2\pi} \int_{0}^{2\pi} \log \left(|\text{Det}(I - (F(re^{i\theta}))^2)|\right) d\theta. \quad (25)$$

**Proof**: Like in the proof of Theorem 1, we have

$$\sigma^-(B) = \{ \log(z) \mid |z| < 1, \ 1 \in \sigma(F(z)) \}.$$ 

Since we assume $D$ is Hilbert-Schmidt, then so is $F(z)$, and this implies that $(F(z))^2$ is trace class, so we can define the holomorphic function

$$h(z) = \text{Det}(I - (F(z))^2),$$

and we have

$$1 \in \sigma(F(z)) \Rightarrow 1 \in \sigma((F(z))^2) \Leftrightarrow h(z) = 0,$$

and thus

$$\sigma^-(B) \subset \{ \log(z) \mid |z| < 1, \ h(z) = 0 \}. \quad (26)$$

Since (26) is an inclusion rather than an equality as in (5), (7) is replaced by the inequality

$$N(-s) \leq n(e^{-s}),$$

Since $F(0) = 0$ we have $h(0) = 1$. Applying the Jensen identity, as in the proof of Theorem 1, we get the results. $\blacksquare$

The next theorem is the Hilbert-Schmidt analog of Theorem 2.

**Theorem 6** Let $A, B$ be self-adjoint in a complex Hilbert space $\mathcal{H}$, with $\sigma(A) \subset [0, \infty)$. Assume that $D = e^{-B} - e^{-A}$ is Hilbert-Schmidt. Then, for any $\gamma > 1$, we have the inequality

$$\sum_{\lambda \in \sigma^-(B)} |\lambda|^\gamma \leq \frac{\gamma(\gamma - 1)}{2\pi} \int_{0}^{1} r |\log(r)|^{\gamma - 2} \int_{0}^{2\pi} ||[I - r e^{i\theta} e^{-A}]^{-1}D||_H^2 d\theta dr. \quad (27)$$
and for $\gamma = 1$ we have
\[
\sum_{\lambda \in \sigma^-(B)} |\lambda| \leq \limsup_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \|[I - re^{i\theta}e^{-A}]^{-1}D\|_{HS} d\theta.
\]

Proof: Using (14), we have
\[
\log \left( |\text{Det}(I - (F(z))^2)| \right) \leq \|F(z)^2\|_{tr}, \quad (28)
\]
and since, for any Hilbert-Schmidt operator $T$ we have $\|T^2\|_{tr} \leq \|T\|_{HS}^2$, we get
\[
\|F(z)^2\|_{tr} \leq \|F(z)\|_{HS}^2. \quad (29)
\]
From (28) and (29), together with (24), (25), we obtain the results. 

The following theorem is the Hilbert-Schmidt analog of Theorem 4.

**Theorem 7** Let $A, B$ be self-adjoint in a complex Hilbert space $\mathcal{H}$, with $\sigma(A) \subset [0, \infty)$. Assume that, for some $t > 0$, $D_t = e^{-tB} - e^{-tA}$ is Hilbert-Schmidt.

Then, for every $\gamma > 2$, we have the inequality
\[
\sum_{\lambda \in \sigma^-(B)} |\lambda|^\gamma \leq C_{HS}(\gamma) \frac{1}{t^\gamma} \|D_t\|_{HS}^2, \quad (30)
\]
where $C_{HS}(\gamma)$ is a finite constant depending only on $\gamma$.

Proof: We first note that it suffices to prove (30) with $t = 1$, that is, setting $D = D_1 = e^{-B} - e^{-A}$, to prove
\[
\sum_{\lambda \in \sigma^-(B)} |\lambda|^\gamma \leq C_{HS}(\gamma) \|D\|_{HS}^2, \quad (31)
\]
since (30) follows from (31) by replacing $A, B$ by $tA, tB$.

Using the inequality (20), we have
\[
\|D\|_{HS} \leq \left\{ \begin{array}{ll}
\frac{1}{r^2 - 2r \cos(\theta) + 1} & \cos(\theta) \geq r \\
\frac{1}{(\sin(\theta))^2 - 1} & 0 < \cos(\theta) < r \\
1 & \cos(\theta) \leq 0
\end{array} \right.
\]
for $0 < r < 1$.
Therefore from inequality (27) of Theorem 6
\[
\sum_{\lambda \in \sigma(B)} |\lambda|^\gamma 
\leq \frac{\gamma(\gamma - 1)}{2\pi} \int_0^1 r |\log(r)|^{\gamma - 2} \int_0^{2\pi} |I - re^{i\theta}e^{-\lambda}|^{-1} D^2_{HS} d\theta dr
\]
\[
\leq \|D_t\|^2_{HS} \frac{\gamma(\gamma - 1)}{\pi} \left[ \int_0^1 |\log(r)|^{\gamma - 2} \int_0^{\arccos(r)} \frac{r}{r^2 - 2r \cos(\theta) + 1} d\theta dr 
+ \int_0^1 r |\log(r)|^{\gamma - 2} \int_0^{\frac{\pi}{2}} \frac{1}{(\sin(\theta))^2} d\theta dr + \int_0^1 \int_{\frac{\pi}{2}}^\pi r |\log(r)|^{\gamma - 2} d\theta dr \right].
\]
To verify that the above integrals are indeed finite for \(\gamma > 2\), we estimate from above:
\[
c_4(\gamma) = \int_0^1 |\log(r)|^{\gamma - 2} \int_0^{\arccos(r)} \frac{r}{r^2 - 2r \cos(\theta) + 1} d\theta dr
\]
\[
= \int_0^1 |\log(r)|^{\gamma - 2} \frac{2r}{1 - r^2} \arctan\left( \frac{1 + r}{1 - r} \right) dr,
\]
and since, for any \(\epsilon > 0\) the integrand is \(O(r^{1-\epsilon})\) as \(r \to 0\), and \(O((1 - r)^{\gamma - 3})\) as \(r \to 1\), the integral is finite when \(\gamma > 2\).
\[
c_5(\gamma) = \int_0^1 r |\log(r)|^{\gamma - 2} \int_0^{\frac{\pi}{2}} \frac{1}{(\sin(\theta))^2} d\theta dr
\]
\[
= \int_0^1 |\log(r)|^{\gamma - 2} \frac{r^2}{\sqrt{1 - r^2}} dr,
\]
and since, for any \(\epsilon > 0\) the integrand is \(O(r^{2-\epsilon})\) as \(r \to 0\), and \(O((1 - r)^{\gamma - \frac{5}{2}})\) as \(r \to 1\), the integral is finite when \(\gamma > \frac{3}{2}\). Finally,
\[
c_6(\gamma) = \int_0^1 \int_{\frac{\pi}{2}}^\pi r |\log(r)|^{\gamma - 2} d\theta dr = \frac{\pi}{2} \int_0^\infty e^{-2x} x^{\gamma - 2} dx = \pi 2^{-\gamma} \Gamma(\gamma - 1),
\]
finite for any \(\gamma > 1\). From (27) we thus have, for \(\gamma > 2\),
\[
\sum_{\lambda \in \sigma(B)} |\lambda|^\gamma \leq \frac{1}{\pi} \gamma(\gamma - 1) [c_4(\gamma) + c_5(\gamma) + c_6(\gamma)] \|D\|^2_{HS}.
\]
so that (31) holds, with \( C_{HS}(\gamma) = \frac{1}{\pi} \gamma (\gamma - 1) [c_4(\gamma) + c_5(\gamma) + c_6(\gamma)] \).

An argument involving one-dimensional operators, like in the end of the previous section, shows that Theorem 7 is not true if \( \gamma < 2 \).

### 4 Application to Schrödinger operators

We now apply our general results to the study of the discrete spectrum of Schrödinger operators \(-\Delta + V\). Recall that the potential \( V : \mathbb{R}^d \to \mathbb{R} \) is said to belong to the class \( K(\mathbb{R}^d) \) if

\[
\lim_{t \to 0} \sup_{x \in \mathbb{R}^d} \int_0^t (e^{\eta \Delta} |V|)(x) \, d\eta = 0.
\]

\( V \) is said to belong to class \( K^{loc}(\mathbb{R}^d) \) if \( \chi_Q V \in K(\mathbb{R}^d) \) for any ball \( Q \subset \mathbb{R}^d \), where \( \chi_Q \) denotes the characteristic function of \( Q \). \( V \) is said to be a Kato potential if \( V_- = \min(V, 0) \in K(\mathbb{R}^d) \) and \( V_+ = \max(V, 0) \in K^{loc}(\mathbb{R}^d) \).

By the min-max principle, the eigenvalues of \(-\Delta + V_-\) are smaller then or equal to the corresponding eigenvalues of \(-\Delta + V\), and therefore we have

\[
\sum_{\lambda \in \sigma_{-}(\Delta + V_-)} |\lambda|^{\gamma} \leq \sum_{\lambda \in \sigma_{-}(\Delta + V)} |\lambda|^{\gamma}, \quad (32)
\]

so that to bound the left-hand side of (32) it suffices to bound the right-hand side. We shall therefore take \( A = H_0 = -\Delta \), \( B = H_0 + V_- \), so that

\[
D_t = e^{-t(H_0 + V_-)} - e^{-tH_0}.
\]

We quote the following bounds for the Hilbert-Schmidt norm of \( D_t \) \((\text{3}, \text{Theorem 5.7})\)

**Lemma 2** Assuming \( V_- \in K(\mathbb{R}^d) \), we have

\[
\|D_t\|_{HS}^2 \leq 2t \int_{\mathbb{R}^d} e^{-2t(H_0 + V_-)}(x, x) |V_- (x)| \, dx.
\]
Lemma 3 Assuming $V_\tau \in K(\mathbb{R}^d)$, we have

$$
\|D_t\|_{HS}^2 \leq t^2 \int_{\mathbb{R}^d} e^{-2t(H_0 + V_\tau)}(x,x)|V_\tau(x)|^2 dx.
$$

We also quote the following inequality (see [3], p. 66, in the proof of Theorem 2.9):

Lemma 4 Assuming $V_\tau \in K(\mathbb{R}^d)$, we have

$$
e^{-t(H_0 + V_\tau)}(x,y) \leq \|e^{-t(H_0 + 2V_\tau)}\|_{L^1,L^\infty}^{\frac{1}{2}}(e^{-tH_0}(x,y))^{\frac{1}{2}}.
$$

Since $e^{-tH_0}(x,x) = \frac{1}{(4\pi t)^{\frac{d}{2}}}$, Lemmas [2] and [4] imply

$$
\|D_t\|_{HS}^2 \leq \frac{2t}{(8\pi t)^{\frac{d}{4}}} \|e^{-2t(H_0 + 2V_\tau)}\|_{L^1,L^\infty}^{\frac{1}{2}} \|V_\tau\|_{L^1}, \quad (33)
$$

$$
\|D_t\|_{HS}^2 \leq \frac{t^2}{(8\pi t)^{\frac{d}{4}}} \|e^{-2t(H_0 + 2V_\tau)}\|_{L^1,L^\infty}^{\frac{1}{2}} \|V_\tau\|_{L^2}^2. \quad (34)
$$

From (33) and Theorem 7 we have

**Theorem 8** Let $V$ be a Kato potential, and assume also $V_\tau \in L^1(\mathbb{R}^d)$.

We have the following inequality for any $\gamma > 2$,

$$
\sum_{\lambda \in \sigma(-\Delta + V)} |\lambda|^\gamma \leq \frac{2C_{HS}(\gamma)}{(8\pi)^{\frac{d}{4}}} \|V_\tau\|_{L^1} \inf_{t > 0} \frac{\|e^{-2t(H_0 + 2V_\tau)}\|_{L^1,L^\infty}^{\frac{1}{2}}}{t^{\gamma + \frac{d}{4} - 1}}.
$$

Similarly, from (34) and Theorem 7 we have

**Theorem 9** Let $V$ be a Kato potential, and assume also $V_\tau \in L^2(\mathbb{R}^d)$.

We have the following inequality for any $\gamma > 2$,

$$
\sum_{\lambda \in \sigma(-\Delta + V)} |\lambda|^\gamma \leq \frac{C_{HS}(\gamma)}{(8\pi)^{\frac{d}{4}}} \|V_\tau\|_{L^2}^2 \inf_{t > 0} \frac{\|e^{-2t(H_0 + 2V_\tau)}\|_{L^1,L^\infty}^{\frac{1}{2}}}{t^{\gamma + \frac{d}{4} - 2}}.
$$
In order to make the bounds given by Theorems 8, 9 more explicit we are going to bound \( \| e^{-2t(H_0+2V_-)} \|_{L^1,L^\infty} \) in terms of the quantity (\( c > 0 \))

\[
\beta(c) = \| (c - \Delta)^{-1} V_- \|_{L^\infty}.
\]  
(35)

We note that (see, e.g., \[2\], Lemma 4.2.4) \( V_- \in K(\mathbb{R}^d) \) implies that

\[
\lim_{c \to \infty} \beta(c) = 0.
\]  
(36)

From \[3\], Proposition 2.2, we have

**Lemma 5** Assume \( V \) is a Kato potential. Then, for any \( c > 0 \) for which \( \beta(c) < 1 \), we have

\[
\| e^{-t(H_0+V_-)} \|_{L^\infty,L^\infty} \leq \frac{e^{ct}}{1 - \beta(c)}.
\]

**Lemma 6** Let \( V \) be a Kato potential. If \( c > 0 \) is such that

\[
4\beta(c) < 1,
\]  
(37)

then

\[
\| e^{-2t(H_0+2V_-)} \|_{L^1,L^\infty} \leq \frac{1}{(4\pi t)^{\frac{d}{2}}} \frac{e^{ct}}{1 - 4\beta(c)}.
\]

**Proof:** We have (as in \[3\], proof of Theorem 2.9):

\[
\| e^{-2t(H_0+2V_-)} \|_{L^1,L^\infty} \leq \| e^{-t(H_0+2V_-)} \|_{L^1,L^2} \| e^{-t(H_0+2V_-)} \|_{L^2,L^\infty}
\]

\[
= \| e^{-t(H_0+2V_-)} \|_{L^2,L^\infty}^2 \leq \| e^{-t(H_0+4V_-)} \|_{L^\infty,L^\infty} \| e^{-tH_0} \|_{L^1,L^\infty}
\]

\[
= \frac{1}{(4\pi t)^{\frac{d}{2}}} \| e^{-t(H_0+4V_-)} \|_{L^\infty,L^\infty}.
\]

Using Lemma 5, we get the result. ■

Using Lemma 6, Theorem 8 implies, for \( c \) satisfying (37),

\[
\sum_{\lambda \in \sigma(-\Delta + V)} |\lambda|^\gamma \leq \frac{2^{\frac{d}{2}+1}}{(8\pi)^{\frac{d}{2}}} C_{HS}(\gamma) \frac{1}{t^{\gamma + \frac{d}{2} - 1}} \frac{e^{\frac{1}{2}ct}}{[1 - 4\beta(c)]^{\frac{1}{2}}} \| V_- \|_{L^1}.
\]  
(38)
We can now minimize the expression on the right-hand side of (38) over \( t \). Since
\[
\min_{t>0} e^{\frac{1}{2}ct} t^{\gamma + \frac{4}{d} - 1} = \left( \frac{ec}{2\gamma + d - 2} \right)^{\gamma + \frac{4}{d} - 1}
\]
we obtain

**Theorem 10** Let \( V \) be a Kato potential, and assume also \( V_- \in L^1(\mathbb{R}^d) \). If \( c > 0 \) is such that \( 4\beta(c) < 1 \), then, for any \( \gamma > 2 \),
\[
\sum_{\lambda \in \sigma(-\Delta + V)} |\lambda|^{\gamma} \leq \frac{2^{\frac{d}{2}}}{(8\pi)^{\frac{d}{2}}} C_{HS}(\gamma) \left( \frac{ec}{2\gamma + d - 2} \right)^{\gamma + \frac{4}{d} - 1} \frac{1}{[1 - 4\beta(c)]^\frac{1}{2}} \| V_- \|_{L^1}.
\]

Similarly, from Theorem 9 we obtain

**Theorem 11** Let \( V \) be a Kato potential, and assume also \( V_- \in L^2(\mathbb{R}^d) \). If \( c > 0 \) is such that \( 4\beta(c) < 1 \), then, for any \( \gamma > 2 \),
\[
\sum_{\lambda \in \sigma(-\Delta + V)} |\lambda|^{\gamma} \leq \frac{2^{\frac{d}{2}}}{(8\pi)^{\frac{d}{2}}} C_{HS}(\gamma) \left( \frac{ec}{2\gamma + d - 4} \right)^{\gamma + \frac{d}{2} - 2} \frac{1}{[1 - 4\beta(c)]^\frac{1}{2}} \| V_- \|_{L^2}.
\]

We note that (36) assures us that there always exists \( c > 0 \) with \( 4\beta(c) < 1 \), so that Theorems 10, 11 apply.

The dependence on \( V_- \) in Theorems 10, 11 is both through its \( L^1 \)-norm and through the quantity \( \beta(c) \). The quantity \( \beta(c) \) can be written more explicitly by using the integral representation of \((c - \Delta)^{-1}\),
\[
((c - \Delta)^{-1}V_-)(x) = c \frac{d-2}{2} \int_{\mathbb{R}^d} G(c^\frac{1}{2}(x - y)) V_-(y) dy,
\]
where
\[
G(x) = \frac{1}{(2\pi)\frac{d}{2}} K_{\frac{d}{2} - 1}(|x|) \frac{1}{|x|^{\frac{d}{2} - 1}},
\]
in which \( K_{\frac{d}{2} - 1} \) is the modified Bessel function of the third kind (see, e.g., [1]). Thus
\[
\beta(c) = c \frac{d-2}{2} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} G(c^\frac{1}{2}(x - y)) |V_-(y)| dy = \frac{1}{c} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} G(x - y) |V_-(c^{-\frac{1}{2}}y)| dy.
\]

(39)
We now introduce an apparently new norm on potentials, which is natural in this context, in terms of which we can derive some useful inequalities from Theorems 10, 11. For $\alpha > 0$, we say that a measurable function $W : \mathbb{R}^d \to \mathbb{R}$ belongs to $K^\alpha(\mathbb{R}^d)$ if $\|W\|_{K^\alpha} < \infty$, where

$$
\|W\|_{K^\alpha} = \sup_{c > 0} c^\alpha \| (c - \Delta)^{-1} W \|_{L^\infty} = \sup_{x \in \mathbb{R}^d, c > 0} c^\alpha \int_{\mathbb{R}^d} G(x - y) |W(c^{-\frac{1}{2}} y)| dy.
$$

(40)

$K^\alpha(\mathbb{R}^d)$ is a normed space with the above norm, and we have $K^\alpha(\mathbb{R}^d) \subset K(\mathbb{R}^d)$ for all $\alpha > 0$. By the definition of the $K^\alpha$-norm and by (39) we have, when $V_- \in K^\alpha(\mathbb{R}^d)$

$$
\beta(c) \leq \|V_-\|_{K^\alpha} c^{-\alpha}, \quad \forall c > 0.
$$

(41)

To see that $K^\alpha(\mathbb{R}^d)$ is a sufficiently large class of functions, we note that

**Lemma 7** If $d \geq 3$ and $p > \frac{d}{2}$ then $L^p(\mathbb{R}^d) \subset K^\alpha(\mathbb{R}^d)$, where $\alpha = 1 - \frac{d}{2p}$, and we have, for all $W \in L^p(\mathbb{R}^d)$,

$$
\|W\|_{K^\alpha} \leq C_{d,p} \|W\|_{L^p},
$$

(42)

where

$$
C_{d,p} = \left( \int_{\mathbb{R}^d} |G(x)| \frac{p}{p-1} dx \right)^{\frac{p-1}{p}}.
$$

(43)

**Proof:** Using Hölder’s inequality we have

$$
\int_{\mathbb{R}^d} G(x - y) |W(c^{-\frac{1}{2}} y)| dy \leq C_{d,p} c^{\frac{d}{2p}} \|W\|_{L^p} = C_{d,p} c^{1-\alpha} \|W\|_{L^p},
$$

which, using (40), implies (42). We note that the fact that $C_{d,p}$ is finite follows from the condition $p > \frac{d}{2}$, which implies $\frac{p}{p-1} < \frac{d}{d-2}$. ■

Another fact, which shows that $K^\alpha(\mathbb{R}^d)$ contains functions which are not in any $L^p(\mathbb{R}^d)$ is
Lemma 8 If $W$ is measurable and
\[ |W(x)| \leq \frac{A}{|x|^\eta}, \quad \forall x \in \mathbb{R}^d, \]
where $\eta \in (0, 2)$, then $W \in K^{2-\eta}(\mathbb{R}^d)$, and
\[ ||W||_{K^{2-\eta}} \leq \frac{1}{\pi^{\frac{d}{2}}} \frac{1}{2\eta+1} \Gamma\left(1 - \frac{\eta}{2}\right) \Gamma\left(\frac{d-\eta}{2}\right) A \]

Proof: We have
\[
\int_{\mathbb{R}^d} G(x-y)|W(c^{-\frac{1}{2}}y)|dy \leq A c^{\frac{d}{2}} \int_{\mathbb{R}^d} G(x-y)|y|^{-\eta}dy \\
\leq A c^{\frac{d}{2}} \int_{\mathbb{R}^d} G(y)|y|^{-\eta}dy = \frac{1}{\pi^{\frac{d}{2}}} \frac{1}{2\eta+1} \Gamma\left(1 - \frac{\eta}{2}\right) \Gamma\left(\frac{d-\eta}{2}\right) A c^{\eta-1},
\]
where the second inequality follows from the fact that both $G(x)$ and $|x|^{-\eta}$ are radially symmetric functions which are decreasing in $|x|$, so that their convolution is maximized at the origin. ■

We now derive eigenvalue inequalities using the norms $||V_-||_{K^\alpha}$. From Theorem 10 and (41) we have
\[
\sum_{\lambda \in \sigma^{-(\Delta+V)}} |\lambda|^\gamma \leq C_{HS}(\gamma) \frac{2^{\frac{d}{2}+1}}{(8\pi)^{\frac{d}{2}}} \frac{\left(\frac{\epsilon_c}{\frac{\epsilon_c}{2\gamma+d-2}}\right)^{\frac{\gamma+d-1}{2}}}{\left[1 - 4||V_-||_{K^\alpha c^{-\alpha}}\right]^\frac{1}{2}} ||V_-||_{L^1}. \tag{44}
\]

We now wish to minimize the right-hand side of (44) with respect to $c$. We compute
\[
\min_{c>(4||V_-||_{K^\alpha})^\frac{1}{\delta}} \left\{ \frac{c^{\gamma+d-1}}{\left[1 - 4||V_-||_{K^\alpha c^{-\alpha}}\right]^\frac{1}{2}} \right\} = \frac{2^\delta (2\delta + 1)^{\delta+\frac{1}{2}}}{\delta^\delta} ||V_-||_{K^\alpha}^{\delta},
\]
where
\[ \delta = \frac{1}{\alpha} \left( \gamma + \frac{d}{2} - 1 \right). \]
Thus from (44) we get
Theorem 12 Let $V$ be a Kato potential, and assume also $V_\gamma \in L^1(\mathbb{R}^d) \cap K^\alpha(\mathbb{R}^d)$, where $\alpha > 0$. Then, for any $\gamma > 2$,

$$\sum_{\lambda \in \sigma^-(\Delta + V)} |\lambda|^\gamma \leq \kappa \|V_\gamma\|_{L^1} \|V\|_{K^\alpha}^\delta,$$

where the constants are given by

$$\delta = \delta_{d,\alpha,\gamma} = \frac{1}{\alpha} \left( \gamma + \frac{d}{2} - 1 \right),$$

$$\kappa = \kappa_{d,\alpha,\gamma} = C_{HS}(\gamma) \frac{2^{\frac{d}{2}+1}}{(8\pi)^{\frac{d}{2}}} \frac{2^\delta (2\delta + 1)^{\delta + \frac{1}{2}}}{\delta^\delta} \left( \frac{e}{2\delta\alpha} \right)^{\delta\alpha}.$$

Similarly, using Theorem 11 we obtain

Theorem 13 Let $V$ be a Kato potential, and assume also $V_\gamma \in L^2(\mathbb{R}^d) \cap K^\alpha(\mathbb{R}^d)$, where $\alpha > 0$. Then, for any $\gamma > 2$,

$$\sum_{\lambda \in \sigma^-(\Delta + V)} |\lambda|^\gamma \leq \kappa \|V_\gamma\|_{L^2} \|V\|_{K^\alpha}^\delta,$$

where the constants are given by

$$\delta = \delta_{d,p,\gamma} = \frac{1}{\alpha} \left( \gamma + \frac{d}{2} - 2 \right),$$

$$\kappa = \kappa_{d,\alpha,\gamma} = C_{HS}(\gamma) \frac{2^{\frac{d}{2}+1}}{(8\pi)^{\frac{d}{2}}} \frac{2^\delta (2\delta + 1)^{\delta + \frac{1}{2}}}{\delta^\delta} \left( \frac{e}{2\delta\alpha} \right)^{\delta\alpha}.$$

We particularize to the case in which $d \geq 3$, $V_\gamma \in L^p(\mathbb{R}^d)$, $p > \frac{d}{2}$. Using Lemma 7, Theorem 12,13 imply

Corollary 1 Assume $d \geq 3$. Let $V$ be a Kato potential, and assume also $V_\gamma \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, where $p > \frac{d}{2}$. Then, for any $\gamma > 2$,

$$\sum_{\lambda \in \sigma^-(\Delta + V)} |\lambda|^\gamma \leq \kappa \|V_\gamma\|_{L^1} \|V\|_{L^p}^\delta,$$  \hfill (45)
where the constants are given by

\[
\delta = \delta_{d,p,\gamma} = \frac{\gamma + \frac{d}{2} - 1}{1 - \frac{d}{2p}},
\]

\[
\kappa = \kappa_{d,p,\gamma} = C_{HS}(\gamma)(2C_{d,p})^\delta \frac{2^{\frac{d}{2} + 1}}{(8\pi)\frac{d}{2}} \frac{(2\delta + 1)^{\delta + \frac{1}{2}}}{\delta^\delta} \left(\frac{e}{2\delta(1 - \frac{d}{2p})}\right)^{\delta(1 - \frac{d}{2p})},
\]

with \(C_{d,p}\) given by (43).

**Corollary 2** Assume \(d \geq 3\). Let \(V\) be a Kato potential, and assume also \(V_\in L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)\), where \(p > \frac{d}{2}\). Then, for any \(\gamma > 2\),

\[
\sum_{\lambda \in \sigma(-\Delta + V)} |\lambda|^\gamma \leq \kappa \|V_\in -\|_{L^2} \|V_\in \|^2_{L_p},
\]

where the constants are given by

\[
\delta = \delta_{d,p,\gamma} = \frac{\gamma + \frac{d}{2} - 2}{1 - \frac{d}{2p}},
\]

\[
\kappa = \kappa_{d,p,\gamma} = C_{HS}(\gamma) \left(2C_{d,p}\right)^\delta \frac{2^{\frac{d}{2}}}{(8\pi)\frac{d}{2}} \frac{(2\delta + 1)^{\delta + \frac{1}{2}}}{\delta^\delta} \left(\frac{e}{2\delta(1 - \frac{d}{2p})}\right)^{\delta(1 - \frac{d}{2p})},
\]

with \(C_{d,p}\) given by (43).

It is interesting to compare the inequalities given by Corollaries 1, 2 with a different bound on the moments of eigenvalues, given by the Lieb-Thirring inequalities [4, 5]. These state that

\[
\sum_{\lambda \in \sigma(-\Delta + V)} |\lambda|^\gamma \leq C_{d,\gamma} \|V_\in -\|_{L^\gamma} \|V_\in \|_{L^\gamma + \frac{d}{2}},
\]

holds for any \(\gamma \geq 0\) when \(d \geq 3\), for any \(\gamma > 0\) when \(d = 2\), and for any \(\gamma \geq \frac{1}{2}\) when \(d = 1\).

Let us compare the bounds given by the inequalities when both of them are valid. The following argument shows that our inequality (45) and
the Lieb-Thirring inequality are independent, in the sense that neither of them is stronger than the other: fixing \( \gamma > 2, p > \frac{d}{2} \), if we take some potential \( W \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \cap L^{\gamma + \frac{d}{2}}(\mathbb{R}^d) \), and define the family \( V_\mu \) (\( \mu > 0 \)) by
\[
V_\mu(x) = \mu^{-\frac{d}{\gamma + \frac{d}{2}}} W(\mu x)
\]
then, for any \( r > 0 \),
\[
\|V_\mu - \|_{L^r} = \mu^{-\frac{d}{\gamma + \frac{d}{2}} - \frac{d}{r}} \|W - \|_{L^r},
\]
hence
\[
\|V_\mu - \|_{L^{\gamma + \frac{d}{2}}} = \|W - \|_{L^{\gamma + \frac{d}{2}}},
\]
\[
\|V_\mu - \|_{L^1} \|V_\mu - \|_{L^p} = \mu^{-\frac{2d\delta}{(2\gamma + d)p}} \|W - \|_{L^1} \|W - \|_{L^p},
\]
where \( \delta \) is defined by (46). Thus the right-hand side of the inequality (45) is arbitrarily small for \( \mu \) large and arbitrarily large for \( \mu \) small, while the right-hand side of (48) does not depend on \( \mu \), so that our inequalities are sometimes weaker and sometimes stronger than the Lieb-Thirring inequalities - depending on the potential \( V \). In particular (45) is better than the bound given by the Lieb-Thirring inequality when \( \mu \) is large. A similar conclusion holds with respect to the inequality (47) of Corollary 2.

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