Persistent currents through a quantum dot

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We study the persistent currents induced by the Aharonov-Bohm effect in a closed ring which either embeds or is directly side coupled to a quantum dot at Kondo resonance. We predict that in both cases, the persistent current is very sensitive to the ratio between the length of the ring and the size of the Kondo screening cloud which appears as a fundamental prediction of scaling theories of the Kondo effect. Persistent current measurements provide therefore an opportunity to detect this cloud which has so far never been observed experimentally.

I. INTRODUCTION

The Kondo effect has become one of the most studied paradigms in condensed matter theory in decades.\textsuperscript{11} The effect results from the interaction between conduction electrons and localized magnetic impurities. As a paradigm, the Kondo effect has played a considerable role in the development of renormalization group (RG) scaling ideas. The Kondo effect is associated with a large distance scale $\xi_K \approx \hbar v_F / T_K$, where $v_F$ is the Fermi velocity and $T_K$, the Kondo temperature, which also defines the energy scale of the problem: $T_K \sim D e^{-1/\lambda}$, where $D$ is the bandwidth and $\lambda$ the bare dimensionless Kondo coupling. The heuristic picture associated with this fundamental length scale is that a cloud of electrons with a size of order $\xi_K$ surrounds the impurity, forming a singlet with it. The remaining electrons outside the cloud do not feel the impurity spin any more but rather a scattering potential, caused by the complex formed between the cloud and the impurity, resulting in a $\pi/2$ phase shift at the Fermi energy. According to the renormalization group approach to the Kondo problem, all physical quantities should be expressed in terms of universal scaling functions of $r/\xi_K$, $r$ being the distance to impurity.\textsuperscript{2,3} Unfortunately, this large length scale of order 0.1 microns in metals has never been detected experimentally.

Quite recently, due to some experimental breakthroughs, various aspects of the Kondo effect have been measured in a semiconductor quantum dot coupled via weak tunnel junctions to leads and capacitively to gates. By tuning the gate voltage, the number of electrons inside the quantum dot can be adjusted due to the Coulomb blockade. In particular, when the number of electrons is odd, the quantum dot can act as a localized spin $S = 1/2$ impurity. Nevertheless, unlike magnetic impurities in metals the physical parameters of the quantum dots can be varied continuously. When the tunneling matrix elements between the leads and the quantum dot are small, the system can be essentially described by a Kondo model. It has been predicted theoretically for a Kondo impurity embedded between two leads, that the transmission probability should reach one at low temperature $T < T_K$ whereas it should be small at higher temperature.\textsuperscript{4,5} Such a manifestation of the Kondo effect has been confirmed experimentally in the last few years.\textsuperscript{6–9}

The Kondo temperatures in these experiments are generally considerably smaller than $1^0K$ and most importantly can be tuned via the gate voltage $V_g$. Therefore, the Kondo length scale is expected to be of order 1 micron or larger. This new experimental realization of the Kondo effect seems therefore to offer new opportunities to measure the screening cloud. In \cite{10}, we have emphasized that the transmission probability of a quantum dot, in the Kondo regime, and embedded in a closed ring, may be sensitive to the length of the ring versus the Kondo screening length. The idea is simply that a finite size should suppress the Kondo effect even if the temperature is much less than $T_K$. Such an idea, central to the interpretation of the Kondo problem, has been for example used by Nozières to interpret Wilson’s numerical renormalization group approach of the Kondo problem.\textsuperscript{11}

The purpose of this paper is to analyze in more detail the sensitivity of the transmission probability of a quantum dot to the Kondo length scale in two different geometries: one where the quantum dot is embedded in a closed mesoscopic circular ring and another one where the quantum dot is outside the ring and couples directly to it. We have shown schematically both devices in figure \textsuperscript{12–14}. In both geometries, the crucial point is that the screening cloud is “trapped” in the ring and cannot escape into macroscopic leads. A natural way to measure the transmission probability through a quantum dot
in a closed geometry would be by persistent current measurements. One might expect intuitively that the persistent current, as a function of the flux, $\Phi$ penetrating the ring, will be much different when the screening cloud is small compared to the circumference, $l$, of the ring than when it is much larger. Such persistent current experiments have been performed recently on micron sized rings not containing a quantum dot. We have shown that, when $\xi_K/l < 1$, the persistent current is that of a perfect ring with no impurity ($j \propto eV_F/l$) for the embedded quantum dot and is vanishing for the side coupled quantum dot. On the other hand, when $\xi_K/l > 1$, $jl$ becomes much smaller in the first geometry, vanishing as a power of the bare Kondo coupling, whereas it converges toward that of a perfect ring in the second geometry. We always assume in this article that $l \gg a$, with $a$ the lattice constant. In general, we expect $j(\Phi)$ to be a universal scaling function of $\xi_K/l$, in the usual scaling limit of the Kondo model (i.e. at small Kondo coupling and large ring size compared to the lattice constant).

The plan of the article is as follows. In section 2, we first introduce the two tight binding models we are to study and develop a continuum limit analysis of the models in an open geometry which clarifies the difference between the two models. In section 3, we calculate the persistent currents for the embedded quantum dot in the Kondo regime in both limits $\xi_K \gg l$ and $\xi_K \ll l$, and discuss the extension of these results for asymmetric tunneling amplitude between the quantum dots and the wires. A special emphasis on the role played by particle hole symmetry is also placed. We also study in this section how bulk interactions modifies the conductance of the quantum dot and hence the persistent current. In section 4, we present similar calculations for the side coupled quantum dot. Finally, in section 5, we discuss our results and the approximations we have used. Some technical details about our calculations have been relegated to three appendices.

![Diagram of Quantum Dot Models](image)

**FIG. 1.** Left: Embedded quantum dot. Right: side coupled quantum dot. $V_g$ is the gate voltage and $V_{R/L}$, $V$ control the tunneling amplitude between the quantum dot and the ring.

II. FORMULATION OF THE MODELS

The first model we are considering corresponds to a quantum dot embedded in a wire. In order to describe this situation, we begin with the simple tight-binding Anderson model:

$$H_1 = -t \sum_{j \leq -2} (c_j^\dagger c_{j+1} + \text{h.c.}) - t \sum_{j \geq 1} (c_j^\dagger c_{j+1} + \text{h.c.}) - t'[c_d^\dagger (c_{-1} + c_1) + \text{h.c.}] + \epsilon_d c_d^\dagger c_d + U n_{d\uparrow} n_{d\downarrow}. \quad (2.1)$$

Here sums over the electron spin index are implied. The quantum dot denoted by the subscript $d$ simply corresponds to the site 0 of the chain. $t'$, $\epsilon_d$ and $U$ are the tunneling matrix elements, gate voltage and charging energy of the dot, respectively. $n_{d\alpha}$ is the electron number at the origin for spin $\alpha$.

The second model we want to consider corresponds to a situation where the quantum dot is outside of the ring. It has been referred to as the side coupled quantum dot. This can be modeled by the following tight-binding Anderson model:

$$H_2 = -t \sum_{i = -\infty}^{+\infty} (c_{i+1}^\dagger c_i + \text{h.c.}) - t'[c_d^\dagger c_0 + c_0^\dagger c_d] + \epsilon_d c_d^\dagger c_d + U n_{d\uparrow} n_{d\downarrow}, \quad (2.2)$$

with similar notations and conventions. In this model the quantum dot couples directly to the site 0.
In order to distinguish both models, we will use in the following the notations 1 for the embedded quantum dot (EQD) and 2 for the side coupled quantum dot (SCQD). These two models have already received some attention in the past. The EQD has been first studied in [4] and [5]. In particular, Ng and Lee [5] have shown using Langreth formula that the conductance at zero temperature (and \( t' \neq 0 \)) reads:

\[
G_1 = \frac{2e^2}{h} \sin^2 \delta, \tag{2.3}
\]

with \( \delta \) the phase shift which has been approximated by:

\[
\delta = \frac{\pi}{2} \langle n_d \rangle, \tag{2.4}
\]

and \( \langle n_d \rangle \) the average occupation number of the quantum dot. Nevertheless, we would like to stress that the phase shift according to the generalization of the Friedel sum rule by Langreth is proportional to the number of electrons displaced by the impurity “among which are included not only the \( d \) electrons, but also some of the conduction electrons”. The approximation of Eq. (2.4) should be valid only when the reservoir bandwidth is much larger than the bare Kondo coupling (proportional to \( t'^2 \)) which implies that the contribution of the conduction electrons to the phase shift is negligible. We will come back to this point later (see section III D) with a special emphasis on the role played by particle-hole symmetry.

Kang et al. [14] have proved using similar reasoning that the conductance of the SCQD can be expressed as:

\[
G_2 = \frac{2e^2}{h} \cos^2 \delta, \tag{2.5}
\]

with similar notations.

In both models, the Kondo regime corresponds to the limit:

\[
t' \ll -\epsilon_d, U + \epsilon_d, \tag{2.6}
\]

(\( \epsilon_d < 0 \)). In this limit the dot essentially is locked into the singly occupied state and thus corresponds to an \( S = 1/2 \) spin impurity. In this regime, the EQD is predicted to exhibit perfect conductance according to Eq. (2.3) whereas the SCQD should give zero conductance according to (2.5). Note that at \( J = t' = 0 \), we obtain the opposite result.

In the sequel, we want to recover these results from the continuum limit analysis of both models in the Kondo regime. In order to compare both models, it is instructive to write down explicitly the corresponding Kondo models. In this regime, the tight-binding Anderson model (2.1) reads:

\[
H_1 = -t \sum_{j \leq -2} (c^\dagger_j c_{j+1} + h.c.) + -t \sum_{j \geq 1} (c^\dagger_{j-1} c_j + h.c.) + J (c^\dagger_{-1} + c^\dagger_1) \frac{\vec{S}}{2} (c_{-1} + c_1) \cdot \vec{S}. \tag{2.7}
\]

Here the Kondo coupling is:

\[
J = 2t'^2 \left( \frac{1}{-\epsilon_d} + \frac{1}{U + \epsilon_d} \right). \tag{2.8}
\]

In general a potential scattering term is also induced, except in the case of the symmetric Anderson model, \( U = -2\epsilon_d \). However, provided it is small, it has little effect on the transmission.

For the SCQD in the Kondo regime, the tight binding Anderson model (2.2) simply reduces to to the standard Kondo model:

\[
H_2 = -t \sum_{i=-\infty}^{+\infty} (c^\dagger_{i+1} c_i + h.c.) + J c^\dagger_0 \frac{\vec{S}}{2} c_0 \cdot \vec{S}. \tag{2.9}
\]

At first sight, the two Kondo Hamiltonians (2.7) and (2.9) just differ in slight details. Nevertheless, when the Kondo coupling constant is switched off (\( J = 0 \)), the first model reduces to two independent quantum wires and has therefore a transmittance \( T = 0 \) whereas the second one has a transmittance
Let us analyze the two models in the continuum limit. We can first linearize the spectrum around the Fermi points \( \pm k_F \) and introduce left \((L)\) and right \((R)\) moving chiral fields:

\[
\psi(x) \sim e^{-ik_F x} \psi_R(x) + e^{ik_F x} \psi_L(x),
\]

\(^{(2.10)}\)

\( \psi \) being the fermionic field associated with the operator \( c \) in the continuum limit. The next step is to introduce the even/odd basis defined as follow:

\[
\psi_{e/o}(x) = \frac{1}{\sqrt{2}}[\psi_L(x) \pm \psi_R(-x)].
\]

\(^{(2.11)}\)

Observe that these two fields are left movers. It may be advantageous in some situations to formulate the Kondo Hamiltonians as boundary problems.

For this purpose, we can fold the system by setting for \( x > 0 \):

\[
\psi_{L,e/o}(x) = \psi_{e/o}(x),
\]

\[
\psi_{R,e/o}(x) = \psi_{e/o}(-x).
\]

\(^{(2.12)}\)

All the fields are now defined on the infinite half line \( x > 0 \). In both models, the Hamiltonians \(^{(2.7)}\) and \(^{(2.9)}\) take the form:

\[
H = H^\text{odd}_0 + H^\text{even}_0 + H^\text{even}_{\text{int}},
\]

\(^{(2.13)}\)

the Hamiltonians of free fermions. Notice that the Kondo interactions couple only to the even sector which is the main advantage of this even/odd basis. In order to derive the continuum limit for the interacting parts of both Hamiltonians, we need to pay attention to the boundary conditions at the origin before including the interactions. For the EQD, we had free boundary conditions on site 1 and \(-1\) on the lattice implying therefore \( c_0 = 0 \). Consequently, we have \( \psi(0^+) = \psi(0^-) = 0 \) and therefore \( \psi_R(0^+) = -\psi_L(0^+) \) and \( \psi_R(0^-) = -\psi_L(0^-) \). Using \(^{(2.12)}\) and \(^{(2.11)}\), we can infer that \( \psi_{R,e}(0) = -\psi_{L,e}(0) \). Similarly, for the SCQD, we can prove that \( \psi_{R,e}(0) = \psi_{L,e}(0) \) using the continuity of \( \psi_{L/R}(x) \) at \( x = 0 \).

The boundary conditions can be summarized as follow:

\[
\text{EQD} : \quad \psi_{R,e}(0) = -\psi_{L,e}(0), \quad \psi_{L,e}(0) \approx \psi_{L,e}(0), \quad \psi_{R,e}(0) \approx \psi_{R,e}(0),
\]

\(^{(2.15)}\)

\[
\text{SCQD} : \quad \psi_{R,e}(0) = \psi_{L,e}(0).
\]

\(^{(2.16)}\)

The interacting part of the EQD reads:

\[
H^\text{even}_{\text{int}} = \frac{J}{2\pi} \int_0^\infty dx \left( \psi_{L,e/o}^\dagger(x) i \partial_x \psi_{L,e/o}(x) - (\psi_{R,e/o}^\dagger(x) i \partial_x \psi_{R,e/o}(x)) \right),
\]

\(^{(2.14)}\)

\[
= \frac{2J}{2\pi} (\psi_{L,e}^\dagger(1) e^{-ik_F} + \psi_{L,e}^\dagger(-1) e^{ik_F}) \frac{\sigma^y}{2} \cdot \hat{S}(\psi_{L,e}(1) e^{ik_F} + \psi_{L,e}(-1) e^{-ik_F}).
\]

\(^{(2.17)}\)

We then use the approximations \( \psi_e(1) = \psi_{L,e}(1) \approx \psi_{L,e}(0) \) and \( \psi_e(-1) = \psi_{R,e}(1) \approx \psi_{R,e}(0) \) and the boundary condition \(^{(2.15)}\). Finally, after these manipulations, the associated interacting parts of the Kondo Hamiltonians read:

\[
H^\text{even}_{\text{int},i} = v_F \lambda_i \psi_{L,e}^\dagger(0) \frac{\sigma^y}{2} \cdot \hat{S} \psi_{L,e}(0),
\]

\(^{(2.18)}\)

where the bare Kondo couplings are defined as follow:

\(^1\)we have used the notations \( 0^+ \) and \( 0^- \) to differentiate fermions to the right \((+)\) or left \((-)\) of the dot.
\[
\lambda_1 = \frac{4J \sin^2 k_F}{\pi v_F} = \frac{2J \sin k_F}{\pi t}, \quad (2.19)
\]
\[
\lambda_2 = \frac{J}{\pi v_F} = \frac{J}{2\pi t \sin k_F}. \quad (2.20)
\]

Note that the derivation of (2.21) comes simply from \(\psi_{L,\tau}(0) = \psi(0)/\sqrt{2}\) for the SCQD. Therefore, both Kondo Hamiltonians have almost the same continuum limit, the main difference being in the boundary conditions at the origin.

In both models the effective Kondo coupling renormalizes as:
\[
d\lambda_i/d \ln D = -\lambda_i^2 + O(\lambda_i^3) \quad i = 1, 2. \quad (2.21)
\]

\(D\) is the momentum space cutoff and effectively the bandwidth. In both models, we expect the effective dimensionless kondo couplings to renormalize to strong coupling namely \(\lambda_i \to \infty\). On the lattice, this corresponds to an electron near the origin forming a singlet with the impurity spin. The remaining electrons can propagate freely except that they cannot go into the same local orbital as the screening electron since this would break the singlet and cost a large energy of order \(J\). For the Hamiltonian \(H_2\), the screening electron resides at site 0. Thus, at \(J \to \infty\), the other electrons cannot cross the origin; they are trapped on either the positive or negative axis, corresponding to perfect reflectance. Formally, the low energy effective Hamiltonian is simply:
\[
H_{2,\text{low}} = -t \sum_{j \leq -2} (c_{j}^\dagger c_{j+1} + \text{h.c.}) - t \sum_{j \geq 1} (c_{j}^\dagger c_{j+1} + \text{h.c.}). \quad (2.22)
\]

Note that this effective Hamiltonian is similar to the embedded Kondo Hamiltonian (2.7) at \(J = 0\). On the other hand, for the Hamiltonian \(H_1\), the screening electron is in a symmetric orbital on sites 1 and \(-1\) \((\{c_1^\dagger\}^\dagger = (c_1^\dagger + c_{-1}^\dagger)/\sqrt{2}\)). The remaining electrons are now allowed to occupy sites 1 and \(-1\) provided that they only go into the anti-symmetric state: \(c_a^\dagger = (c_1^\dagger)^\dagger = (c_1^\dagger - c_{-1}^\dagger)/\sqrt{2}\). To obtain the low energy effective Hamiltonian in this case we must project the operators \(c_{\pm 1}\) into the anti-symmetric state:
\[
P_{c_{\pm 1}} = \pm \frac{c_a}{\sqrt{2}}. \quad (2.23)
\]

The effective low energy Hamiltonian then becomes:
\[
H_{1,\text{low}} = -t \sum_{j \leq -3} (c_{j}^\dagger c_{j+1} + \text{h.c.}) - t \sum_{j \geq 2} (c_{j}^\dagger c_{j+1} + \text{h.c.}) - \frac{t}{\sqrt{2}} (-c_{-2}^\dagger c_a + c_a^\dagger c_2 + \text{h.c.}). \quad (2.24)
\]

A simple calculation shows (see Appendix B) that this effective scattering Hamiltonian has a transmission probability:
\[
T(k) = \sin^2 k. \quad (2.25)
\]

Thus when \(k_F = \pi/2\), there is perfect transmission at the Fermi energy. Since the conductance is determined by \(T(k_F)\), this model exhibits perfect conductance at half-filling. This corresponds to a transmission resonance for non-interacting electrons; it occurs at half-filling since the model has particle-hole symmetry in that case. A major difference between the strong coupling limits of \(H_1\) and \(H_2\) is the sensitivity to particle-hole symmetry breaking of \(H_1\) but not of \(H_2\). While \(H_2\) exhibits zero conductance for any filling factor, \(H_1\) exhibits perfect conductance only at half-filling. In the limit of small bare Kondo coupling, the case relevant to experiments, particle-hole symmetry breaking effects are small in both Kondo models, of order \(J/D\). Similar terms are indeed neglected in both formula (2.3) and (2.5) when approximating \(\delta\) by \(\frac{\pi}{2}(n_d)\). We would like to emphasize that due to the fact that particle-hole symmetry breaking is an exactly marginal perturbation in renormalization group language, we can expect the conductance to be close to 1 for the Kondo Hamiltonian, \(H_1\), at weak coupling and any filling factor. The special role played by particle-hole symmetry is not apparent in the approximation leading to eqs. (2.3) and (2.5).
In our analysis, we have also supposed free electrons in the wire. The situation changes considerably if we include Coulomb interactions and has been extensively studied in ref. [23]. Nevertheless, the Kondo resonance can still be reached (and therefore the unitary limit in the embedded dot) but only for one special value of the gate voltage $V_g^*$ (and therefore $\epsilon_d^*$), which would make the observation of the Kondo resonance a pretty delicate task. We will come back to this point latter in section III E. The fact that the unitary limit has been successfully reached in [9] may be an indication that Coulomb interaction effects are very small in the leads.

A similar analysis of the strong coupling fixed point can be reproduced using the field theory language. The strong coupling regime $\lambda_i \to \infty$ corresponds to a $\pi/2$ phase shift in the even channel. In terms of right and left movers, it changes the boundary conditions at the origin as follow:

$$\psi_{L,e}(0) = -\psi_{R,e}(0), \quad \lambda_1 = 0, \quad \lambda_2 \to +\infty,$$

$$\psi_{L,e}(0) = \psi_{R,e}(0), \quad \lambda_2 = 0, \quad \lambda_1 \to +\infty.$$  

Therefore, the boundary conditions in the even channels of both models get completely interchanged as it is schematically depicted in figure 2.

![FIG. 2. Schematic flow for both models between the $|T| = 0$ fixed point and the $|T| = 1$ fixed point.](image)

The strong coupling fixed point of one model corresponds to the weak coupling fixed point of the other and vice versa. In this analysis, we have neglected all marginal operators which correspond to particle-hole symmetry breaking operators. The transmission probability depends on the scattering phase shift of both even and odd sector as follow:

$$|T| = \cos^2(\varphi_e - \varphi_o)$$  

where $\varphi_o = 0, \forall \lambda_i$ and $\varphi_e$ varies from $\pi/2 (\lambda_1 = 0, \lambda_2 = \infty)$ to $0 (\lambda_1 = +\infty, \lambda_2 = 0)$. From this analysis, we recover the results obtained from the explicit analysis of the low energy effective lattice Hamiltonian at strong coupling.

### III. PERSISTENT CURRENTS FOR THE EMBEDDED QUANTUM DOT

We consider in this section the persistent current in the transmission Kondo model (2.7). Thus we consider a closed ring of $l$ sites (including the impurity site, 0) and apply a magnetic flux:

$$\Phi = (e/\hbar)\alpha.$$  

(We work in units where $\hbar = 1$. $e > 0$ is the absolute value of the electron charge.) In the Anderson model of Eq. (2.1), this corresponds to modifying the hopping term between sites $j$ and $j + 1$ by a phase factor $e^{i\alpha}$ such that $\sum_j \alpha_j = \alpha$. Which links carry the phase can always be changed by a gauge transformation, $c_j \to e^{i\phi_j} c_j$. The total flux is, of course, gauge invariant. Choosing, for convenience, to put the phase factors on the 2 links connected to the impurity site, we obtain the Hamiltonian:

$$H = -t \sum_{j=1}^{l-2} (c_j^\dagger c_{j+1} + \text{h.c.}) - t'\left[\epsilon_d^1 (e^{-i\alpha/2} c_{l-1} + e^{i\alpha/2} c_1) + \text{h.c.}\right] + \epsilon_d c_d^\dagger c_d + U n_d^\dagger n_d.$$  

In the Kondo limit this becomes:
\[ H = -t \sum_{j=1}^{l-2} (c_j^\dagger c_{j+1} + h.c.) + J (e^{i\alpha/2} c_{l-1}^\dagger + e^{-i\alpha/2} c_l^\dagger) \frac{\Phi}{2} (e^{-i\alpha/2} c_{l-1} + e^{i\alpha/2} c_l) \cdot \vec{S}. \] (3.3)

The zero temperature persistent current is determined from the flux dependence of the ground state energy:

\[ j = -e \frac{dE_0}{d\Phi} = -e \frac{dE_0}{d\alpha}. \] (3.4)

We expect that the persistent current is given by universal scaling functions of the variables \( \alpha \) and \( l/\xi_K \) in the limit \( l, \xi_K \gg a \) (where \( a \) is the lattice constant, which we have set to 1):

\[ l_j = f(\alpha, \xi_K/l). \] (3.5)

These universal functions depend on the parity of the electron number, \( N \) and also, of course, on whether we consider the EQD or the SCQD. This behavior is a rather immediate consequence of Eq. (6.4), expressing the persistent current as the derivative of the groundstate energy with respect to the phase \( \alpha \) and the applicability of usual continuum field theory scaling arguments. Since \( \alpha \) is a phase, associated with the boundary conditions, we expect it to have zero anomalous dimension. The universal part of the ground state energy is expected to have the canonical dimension of \( 1/(\text{length}) \), and thus the result follows from standard finite size scaling hypotheses. \( l_j \) may be expressed as a universal function of \( \alpha \) and either the dimensionless renormalized Kondo coupling \( \lambda_{eff}(l) \) or equivalently of the ratio \( \xi_K/l \). A non-zero value of \( a/l \) leads to the appearance of various irrelevant operators in the effective Hamiltonian which produce corrections to \( l_j \), down by powers of \( a/l \). This scaling form implies that \( j \) can be calculated using renormalization group improved perturbation theory, at large \( \xi_K/l \) where \( \lambda_{eff}(l) \ll 1 \). It also implies that the current at small \( \xi_K/l \) will be a universal characteristic of the strong coupling RG fixed point.

We now want to calculate the persistent current in perturbation theory in \( J \). When \( J = 0 \) the impurity spin is decoupled from the rest of the chain which effectively consists of \( N - 1 \) electrons on \( l - 1 \) sites with open boundary conditions. The ground state at \( 0^{th} \) order in \( J \), consists of a product of the impurity state (which may be spin up or spin down) and the filled Fermi sea for the rest of the system. The \( 0^{th} \) order ground state energy is independent of flux. The single particle energy levels have energy \( E_n = -2t \cos k_n \), with \( k_n = \pi n/l, n = 1, 2, 3, \ldots, l - 1 \). The annihilation operator appearing in the interaction may be expanded in Fourier modes:

\[ \chi \equiv (e^{i\alpha/2} c_1 + e^{-i\alpha/2} c_{l-1}) = \sqrt{\frac{2}{l}} \sum_k [e^{i\alpha/2} \sin k + e^{-i\alpha/2} \sin (k - 1)] c_k. \] (3.6)

From this stage, we have to distinguish the cases \( N \) even and \( N \) odd. Here \( N \) is the total number of electrons, \textit{including the impurity spin}. Let us first consider the case of an even number of electrons.

**A. Perturbation theory for \( N \) even**

For even \( N \), the levels are doubly occupied for \( n \leq N/2 - 1 \). However, the state with \( k = \pi N/2 \) contains only one electron which may have spin up or down. Thus the ground state is 4-fold degenerate at \( J = 0 \). This degeneracy is split in first order in \( J \). It is convenient to decompose \( \chi \) into the term involving \( k = k_F = \pi N/2l \), which we label \( \chi_0 \), plus all other terms, which we label \( \chi' \).

\[ \chi_0 = \sqrt{\frac{2}{l}} \sin k_F [e^{i\alpha/2} - (-1)^{N/2} e^{-i\alpha/2}] c_{k_F}. \] (3.7)

Only \( \chi_0 \) contributes in first order perturbation theory. Keeping only this contribution, the Kondo interaction in Eq. (3.3) reduces to:

\[ H_{int} = \frac{8J}{l} \sin^2(k_F) \sin^2(\alpha/2) c_{k_F}^\dagger \frac{\Phi}{2} c_{k_F} \cdot \vec{S}. \] (3.8)

where
\[ \beta \equiv \alpha + k_F l = \alpha + \pi N / 2, \]  
\hspace{1cm} (3.9)

and hence:
\[ \sin^2 \beta / 2 = \left[ 1 - (-1)^{N / 2} \cos \alpha \right] / 2. \]  
\hspace{1cm} (3.10)

Diagonalizing \( H_{\text{int}} \) in the degenerate subspace picks out the singlet state, with the first order ground state energy:
\[ E_0 = \text{constant} - \frac{6 J}{l} \sin^2 k_F \sin^2 \beta / 2. \]  
\hspace{1cm} (3.11)

Thus the persistent current, to \( O(J) \) is given by:
\[ j_e(\alpha) = \frac{3 e J}{l} \sin^2 k_F \sin \beta. \]  
\hspace{1cm} (3.12)

We see from Eq. (3.9) that, at large \( l \), the current oscillates in \( N \) with period 4 and that the current for \( N / 2 \) odd is given by the current for \( N / 2 \) even with a shift of \( \alpha \) by \( \pi \):
\[ j_e,N + 2(\alpha) \approx j_e,N(\alpha + \pi). \]  
\hspace{1cm} (3.13)

Notice that Eq. (3.13) is indeed exact to all order of the perturbative theory ignoring corrections of \( O(a / l) \). Indeed, we can show that it is a property of the two different types of propagators involved in the calculation of the ground state at higher order of perturbation theory (see eqs. (3.18) and (A3)), therefore a property of the ground state energy itself, and thus a property of the persistent current.

To proceed to higher orders of perturbation in \( J \), it is convenient to include the \( \chi_0 \) part of the Kondo interaction, in Eq. (3.8) in the unperturbed Hamiltonian, \( H_0 \). This Hamiltonian can be diagonalized exactly. The ground state has all levels doubly occupied or else empty except for the \( k_F \) level. It has one electron in the \( k_F \)-level which forms a singlet with the impurity spin:
\[ |s> = \left( a_{k_F, \uparrow}^\dagger | \downarrow > - a_{k_F, \downarrow}^\dagger | \uparrow > \right) / \sqrt{2}. \]  
\hspace{1cm} (3.14)

(Here the double arrows label the spin state of the impurity.) The remaining perturbation, \( H_{\text{int}} \), has a term quadratic in \( \chi' \) and cross-terms containing one factor of \( \chi_0 \) and one factor of \( \chi' \). The second order correction to the ground state energy can be written in the standard form:
\[ E_0^{(2)} = - \frac{1}{2 l} \int_{-\infty}^{\infty} d\tau T <s|H_{\text{int}}(\tau)H_{\text{int}}(0)|s>. \]  
\hspace{1cm} (3.15)

The propagator for the field \( \chi' \) is, for \( \tau > 0 \):
\[ <\chi_{\alpha}^\dagger(\tau)\chi_{\beta}(0)> \equiv \delta_{\gamma \beta} G(\tau) = \frac{4 \delta_{\gamma \beta}}{l} \sum_{n=1}^{N/2-1} [1 - (-1)^n \cos \alpha] \sin^2(\pi n / l) e^{\xi_n \tau}, \]  
\hspace{1cm} (3.16)

where
\[ \xi_n \equiv -2 t \cos(\pi n / l) + 2 t \cos(\pi N / 2 l). \]  
\hspace{1cm} (3.17)

At large \( \tau \) the sum is dominated by terms near the Fermi surface, \( n = N / 2 - m \), with \( m << N \), so we may write:
\[ G(\tau) \approx \frac{4}{l} \sin^2 k_F \sum_{m=1}^{l-1} [1 - (-1)^{N/2-m} \cos \alpha] e^{-\nu_F \tau m / l} \]  
\[ = \frac{4}{l} \sin^2 k_F \left[ \frac{1}{e^{\nu_F \tau / l} - 1} + \frac{\cos \tilde{\alpha}}{e^{\nu_F \tau / l} + 1} \right], \hspace{1cm} (\tau > 0). \]  
\hspace{1cm} (3.18)

It can be easily seen that:
\[ T < \chi^I(\tau)\chi'(0) > = T < \chi'(\tau)\chi^I(0) > = G(\tau) = \epsilon(\tau)G(|\tau|), \]  
\[ (3.19) \]
where \( G(|\tau|) \) is given approximately by Eq. (3.18), and \( \epsilon(\tau) = 1 \) if \( \tau > 0 \) and \( \epsilon(\tau) = -1 \) if \( \tau < 0 \).

We have calculated in the appendix \[ \text{Appendix A1} \] \( j_c(\alpha) \) using (3.13) and the propagator (3.18) to second order in \( J \), for \( N \) even and large \( \tau \). We have ignored corrections of order \( O(a/l) \), with \( a \) the lattice spacing.

The result reads:
\[ j_c(\alpha) = \frac{3\pi v_F\epsilon}{4l} [\sin \tilde{\alpha}(\lambda + \lambda^2 \ln(lc)) + (1/4 + \ln 2)\lambda^2 \sin 2\tilde{\alpha}] + O(\lambda^3) \quad (N \text{ even}) \]  
\[ (3.20) \]
where \( c \) is a constant of \( O(1) \) which we have not determined.

Let us now analyze the case \( N \) odd.

### B. Perturbation theory for \( N \) odd

In this case there is an exact two fold degeneracy of the ground state, which has \( S = 1/2 \). The unperturbed \((J = 0)\) ground state has all free electron levels doubly occupied for \( k < k_F \equiv \pi N/2 \) and empty for \( k > k_F \). (We choose \( k_F \) to lie exactly half-way between the highest filled and lowest empty levels.) Now first order perturbation theory vanishes since it necessarily gives states with a particle-hole excitation. Second order perturbation theory is now straightforward since the ground state is unique, once we specify a value of \( S^z_{\text{total}} = 1/2 \) for example. The exact time-ordered Green’s function is now, for \( \tau > 0 \):
\[ T < \chi^I_\alpha(\tau)\chi_\beta(0) > \equiv \delta_{\alpha\beta}G(\tau) = \frac{4\delta_{\alpha\beta}}{l} \sum_{n=1}^{(N-1)/2} [1 - (-1)^n \cos \alpha] \sin^2(\pi n/l)e^{\xi_n\tau}, \quad (\tau > 0). \]  
\[ (3.21) \]
where:
\[ \xi_n = -2t \cos \pi n/l + 2t \cos \pi N/2l. \]  
\[ (3.22) \]
Again, at large \( \tau \) this sum is dominated by states near the Fermi surface so we can approximate:
\[ G(\tau) \approx \frac{4}{l} \sin^2 k_F \sum_{m=0}^{l-1} [1 - (-1)^{(N-1)/2-m} \cos \alpha]e^{-v_F\tau \pi(m+1/2)/l} \]
\[ = \frac{2}{l} \sin^2 k_F \left[ \frac{1}{\sinh(\pi v_F \tau/2l)} - \frac{(-1)^{(N-1)/2} \cos \alpha}{\cosh(\pi v_F \tau/2l)} \right], \quad (\tau > 0). \]  
\[ (3.23) \]
For \( \tau < 0 \), we sum over the unoccupied levels, with \( n = (N + 1)/2 + m, \) \( m = 0, 1, 2, \ldots \) In this case:
\[ T < \chi^I_\alpha(\tau)\chi_\beta(0) > \equiv \delta_{\alpha\beta}G(\tau) = -\frac{4\delta_{\alpha\beta}}{l} \sum_{n=(N+1)/2}^{l-1} [1 - (-1)^n \cos \alpha] \sin^2(\pi n/l)e^{\xi_n\tau}, \quad (\tau < 0). \]  
\[ (3.24) \]
At large \( |\tau| \), this becomes:
\[ G(\tau) \approx -\frac{4}{l} \sin^2 k_F \sum_{m=0}^{\infty} [1 - (-1)^{(N+1)/2+m} \cos \alpha]e^{v_F\tau \pi(m+1/2)/l} \]
\[ = \frac{2}{l} \sin^2 k_F \left[ \frac{1}{\sinh(\pi v_F \tau/2l)} - \frac{(-1)^{(N-1)/2} \cos \alpha}{\cosh(\pi v_F \tau/2l)} \right], \quad (\tau < 0). \]  
\[ (3.25) \]
Thus \( G(\tau) \) is given by the same approximate expression, Eq. (3.23), for either sign of \( \tau \). As usual, we may write:
\[ T < \chi_\alpha(\tau)\chi^I_\beta(0) > = -\delta_{\alpha\beta}G(-\tau). \]  
\[ (3.26) \]
The second order term in the ground state energy is:
If we use our large \( \tau \) expression, Eq. (3.23) for \( G(\tau) \), then we get an \( \alpha \)-independent term whose integral diverges at \( \tau = 0 \). This is just an artifact of the large-\( \tau \) approximation to \( G(\tau) \) and in any event, this term doesn’t contribute to the persistent current. The other term is \( \alpha \)-dependent and finite:

\[
E_0^{(2)} = \text{constant} + \cos^2 \alpha \frac{3J^2}{4l^2} \int_{-\infty}^{\infty} d\tau \frac{1}{\cosh^2(\pi v_F \tau / 2l)} = \text{constant} + \cos^2 \alpha \frac{3J^2}{lv_F \pi} \sin^2 \alpha . \tag{3.28}
\]

Note that, in this case, the result does not depend on the parity of \((N-1)/2\) and is periodic in \( \alpha \) with period \( \pi \). We have generalized this result to third order in \( J \) in appendix A 2. Unfortunately, we have not been able to prove it to all orders of the perturbation theory. From the ground state energy, it is then straightforward to infer the persistent current:

\[
\bar{j}_0(\alpha) = \frac{3\pi v_F c}{16l} \sin 2\alpha \lambda^2 + 2 \lambda^3 \ln(\alpha c') + O(\lambda^4) \quad (N \text{ odd}), \tag{3.29}
\]

where \( c' \) is another undetermined constant.

Several remarks about the persistent currents expressions (3.21) and (3.22) are in order. First of all, we note a large parity effect— at small \( \lambda \) the persistent current is much larger for even \( N \) than for odd \( N \). Furthermore, as already noticed the periodicity is different in both cases.

We have chosen to present the calculations using the lattice propagator and then consider the limit \( l \gg a \), \( a \) the lattice spacing. We have also checked that we obtain the same results using directly the continuum limit representation of the Kondo Hamiltonian (3.3). The best strategy is first to unfold the chain with free boundary conditions in sites 1 and \( l-1 \) in order to work with left movers only (the size of the chain becomes so far 2\( l \)). In this representation, the linearized Hamiltonian (3.3) takes the form:

\[
H = \frac{v_F}{2\pi} \int_{-l}^{l} dx \psi_L^\dagger(i \partial_x \psi_L)
+ \frac{\lambda_1}{2} \left[ \psi_L^\dagger(0) e^{-i(\alpha-k_F l)/2} - \psi_L^\dagger(l) e^{i(\alpha-k_F l)/2} \right] \vec{\sigma} \cdot \vec{\sigma} \left[ \psi_L(0) e^{i(\alpha-k_F l)/2} - \psi_L(l) e^{-i(\alpha-k_F l)/2} \right]. \tag{3.30}
\]

From this point, the perturbative calculations become very similar to what was presented in the text and similar results are obtained using the propagators involving left movers.

C. Renormalization group arguments and the strong coupling limit

In this section, we want to comment on the \( l \)-dependence of the persistent currents (3.29) and (3.20). In both cases, to the order in \( \lambda \) that we have worked, the result has the form:

\[
\bar{j}_{e/o} = \frac{ev_F}{l} f_{e/o} \lambda_{eff}(l, \bar{\alpha}), \tag{3.31}
\]

where \( \lambda_{eff}(l) \) is the renormalized coupling constant at scale \( l \):

\[
\lambda_{eff}(l) = \lambda + \lambda^2 \ln l + \ldots \tag{3.32}
\]

If this result persists to all orders in perturbation theory, this would imply the scaling form for \( j \) we have anticipated on general arguments in Eq. (3.5) (which is valid in the limit of small bare coupling \( \lambda \)). In particular, this implies that, at \( l << \xi_K \), the perturbative result is valid. The finite size of the ring cuts off the infrared divergences of perturbation theory. On the other hand, when \( l \) is of order of \( \xi_K \) or greater, perturbation theory breaks down. It is well-known from numerical renormalization group, Bethe ansatz and other calculations that one may think of the effective coupling constant as renormalizing to \( \infty \) in the low energy effective Hamiltonian. Technically, this corresponds to a fixed point of the boundary renormalization group. Thus it is very reasonable to expect that the persistent current for \( l >> \xi_K \) can
be obtained by simply taking the limit \( J \to \infty \). This is expected to be valid because the persistent current is determined only by the low energy properties of the Hamiltonian and these are correctly represented by the infinite coupling fixed point.

It is a straightforward matter to calculate the persistent current for the tight-binding Hamiltonian of Eq. (2.7) at infinite \( J \). In this limit we may simply use the low energy effective Hamiltonian of Eq. (2.24). It is convenient to think of this as being a simple free electron model defined on \((l - 2)\) sites labeled \( a, 2, 3, \ldots, l - 2 \). (The impurity site is eliminated and the 2 neighboring sites, 1 and \( l - 1 \) have effectively collapsed to one site due to the projection onto the odd parity linear combination.) We can add a flux to the model by changing the phase of any hopping term. In particular, since the hopping between sites \( l - 2 \) and \( a \) already has a reversed sign, it is convenient to change the phase of that term, resulting in the model:

\[
H_{\text{low}} = -t \sum_{j \leq -3} (c_j^\dagger c_{j+1} + \text{h.c.}) - t \sum_{j \geq 2} (c_{j+1}^\dagger c_j + \text{h.c.}) - \frac{t}{\sqrt{2}} (e^{i(\alpha + \pi)} c_{-2}^\dagger c_a + c_a^\dagger c_2 + \text{h.c.}).
\]

(3.33)

We see that this is a standard potential scattering model with an effective flux of \( \alpha + \pi \). The \( \pi \) flux shift is a consequence of the projection onto the odd linear combination. As shown by Gogolin and Prokof'ev[2], the persistent current at zero temperature and large \( l \) is completely determined by \( T(k_F) \), the transmission probability at the Fermi surface, in such a potential scattering model (defined at \( \hbar = 0 \)). From Eq. (2.25), at half-filling, \( k_F = \pi/2, T(k_F) = 1 \). For this value of \( T(k_F) \), the persistent current is the same as for an ideal periodic ring. By summing over the energies of the filled Fermi sea, with momenta \( k = (2\pi n + \alpha)/l, n = 0, \pm 1, \pm 2, \ldots \), it can be shown that this gives, for \( M \) spinless fermions, an energy:

\[
E_0 = -2t \cos((\alpha)/l) \frac{\sin 2\pi(M + \frac{1}{2})/l}{\sin \pi/l} = \text{constant} + \frac{v_F}{2\pi l} [\alpha]^2 \quad (M \text{ odd})
\]

(3.34)

\[
E_0 = -2t \cos((\alpha - \pi)/l) \frac{\sin 2\pi(M)/l}{\sin \pi/l} = \text{constant} + \frac{v_F}{2\pi l} [\alpha - \pi]^2 \quad (M \text{ even}),
\]

(3.35)

where \( [\theta] \) equals the principal part of the angle \( \theta \), which lies between \( -\pi \) and \( \pi \) and jumps by \( -2\pi \) at \( \theta = (2n + 1)\pi \). ie.

\[
[\theta] = \theta, \quad ([\theta] < \pi)
\]

(3.36)

\[
= \theta - 2\pi, \quad (\pi < \theta < 3\pi),
\]


etc. Thus \( E_0(\alpha) \) has minima at \( \alpha = 2\pi n \), for \( M \) odd and at \( 2\pi(n + 1/2) \) for \( M \) even. When we include electron spin, for non-interacting electrons, we find a 4-fold periodicity in the total number of electrons, \( N \). If \( N \) is odd, then the numbers of electrons of opposite spin are necessarily of opposite parity, so the energy is the sum of the two terms in Eq. (3.34) and (3.35). On the other hand, if \( N \) is even, then the numbers of electrons of spin up and down are equal in the ground state, so the energy is twice Eq. (3.34) if \( N/2 \) is odd and twice Eq. (3.35) if \( N/2 \) is even.

To apply this free electron result to the large \( l \) (i.e. strong coupling) limit of the Kondo model, we should take into account the shift, \( \alpha \to \alpha + \pi \) in Eq. (3.33), and also the fact that two electrons are removed from the low energy subspace corresponding to the impurity spin and the screening electron. These two effects actually cancel. Writing the result in terms of \( \tilde{\alpha} \) defined in Eq. (3.3), we thus obtain the result for \( l >> \xi_K \):

\[
\tilde{j}_e(\alpha) = \frac{2ev_F}{\pi l} [\tilde{\alpha} - \pi], \quad (N \text{ even})
\]

(3.37)

\[
\tilde{j}_o(\alpha) = \frac{ev_F}{\pi l} ([\alpha] + [\alpha - \pi]) \quad (N \text{ odd}).
\]
FIG. 3. Persistent current of the EQD versus $\alpha/\pi$ for $N = 4p$ for $\xi_K/l \approx 50$ (solid line) and for $\xi_K/l \ll 1$ (dashed line). The persistent current for $N = 4p + 2$ is obtained by a simple translation of $\pi$. Note that $\lambda_{\text{eff}}(l) = 1/\ln(\xi_K/l)$.

FIG. 4. Persistent current of the EQD versus $\alpha/\pi$ for $N$ odd and for $\xi_K/l \approx 50$ (solid line) and for $\xi_K/l \ll 1$ (dashed line). $j_o$ has been multiplied by $\times 5$ for visibility.

D. Particle-hole symmetry breaking

A crucial aspect of Kondo physics is the exact marginality of particle-hole (P-H) symmetry breaking. By exact marginality we mean that the P-H symmetry breaking coupling constants grow neither larger nor smaller under renormalization to all orders in perturbation theory. (By contrast the Kondo coupling itself is marginally relevant, since the corresponding RG $\beta$-function vanishes in first order but is non-zero in second order.)

The model is easiest to analyze when it has exact P-H symmetry: for instance the Anderson model of
Eq. (2) with \( \epsilon_D = -U/2 \) at half-filling. Then various properties of the low energy physics follow exactly from symmetry. The average occupation number on every site \( <n_i> \) has exactly the P-H symmetric value 1. This implies that the formation of the Kondo screening cloud has no effect whatsoever on the charge density. The peak in the single electron density of states (Kondo resonance) occurs exactly at the Fermi energy, \( E = 0 \). The phase shift at the Fermi surface is exactly \( \pi/2 \), as follows, for instance, from Langreth’s application of the Friedel sum rule. Thus, the transmission probability at the Fermi surface is exactly 1 and the persistent current should have exactly the value for a perfect ring, Eq. (3.37) in the limit \( \xi_K/l \to 0 \).

Once P-H symmetry is broken, all these properties change somewhat. However, the important result is that the changes to these properties are small, of order the dimensionless bare Kondo coupling. This is a non-trivial fact given that these are all low energy properties and that the effective Kondo coupling at low energies is expected to renormalize to large values. A convenient way of seeing this is from taking the continuum limit of the Kondo model and then applying bosonization (either Abelian or non-Abelian) which leads immediately to spin-charge separation. One can then see that the Kondo interaction only involves the spin boson and the charge boson is apparently completely unaffected. If we start with an asymmetric Anderson model, \( \epsilon_d \neq -U/2 \), then the resulting large-\( U \) Kondo model contains a potential scattering term, of strength \( V \), which is typically of the same order of magnitude as the Kondo coupling \( J \). Under bosonization, this becomes \( V \partial_{\phi_c} \phi_c \) where \( \phi_c \) is the charge boson. Since this is linear in \( \phi_c \) it leaves the charge sector non-interacting and hence is strictly marginal. By factorizing the electron Green’s function into charge and spin factors, it can be seen that this has the effect of changing the phase shift at the Fermi surface by an amount of order \( V/D \). This is typically a small dimensionless number of order \( t^2/U t \sim J/t \sim \lambda_0 \). There is a corresponding change in the electron density in the vicinity of the impurity. This, in fact follows from the Friedel sum rule as pointed out by Langreth. The phase shift at the Fermi surface is related to the total charge displaced near the impurity. In the Kondo limit \( <n_d> \) is exactly 1 and the small change in the phase shift is associated with a small change in the charge density at the nearby sites, i.e. in the charge density of the conduction electrons, rather than the “d-electrons”. Even if \( \epsilon_d = -U/2 \) so that the Anderson model is nominally “symmetric”, a P-H symmetry breaking electron density, different than 1, has a similar effect. In the Kondo limit, an effective potential scattering term is generated at second order in the bare coupling \( \lambda \). However, we again expect no large renormalization of this potential scattering term for the reason discussed above. In these cases the transmission probability at the Fermi energy, will be slightly less than 1 and the persistent current will be slightly modified from that of Eq. (3.37) even in the limit \( \xi_K/l \ll 1 \). However, these modifications will be small, of order \( V/D \). Thus, as a function of \( \epsilon_d \), for other parameters fixed, the transmission probability at zero temperature is expected to have a broad plateau where it is only slightly less than 1 for a range of \( \epsilon_d \) of \( O(U) \). Changing the filling factor only slightly modifies the shape of this plateau.

On the other hand, things are very different if one considers P-H symmetry breaking in the Kondo limit with a bare Kondo coupling which is not small. In this case the potential scattering term \( V/D \) becomes of order 1 and the P-H symmetry breaking effects are large. This is true regardless of whether or not \( \epsilon_d = -U/2 \). P-H symmetry breaking from the filling factor also has a large effect. Now the transmission probability is not, in general, close to 1 even when \( \xi_K/l \to 0 \). The modification of the charge density on the sites near the impurity is large, although \( <n_d> \) itself remains at exactly 1 in the Kondo limit. In the extreme limit where the Kondo coupling goes to \( \infty \), it is easy to calculate the transmission probability, and hence the persistent current, exactly. Using Eq. (2.25) and Gogolin and Proko’ev exact result, the persistent current at arbitrary filling becomes in this limit:

\[
 j_c(\alpha) = -\frac{2ev_F}{\pi l} \frac{\sqrt{T_F \sin \alpha}}{\sqrt{1 - T_F \cos^2 \alpha}} \left( \arccos(\sqrt{T_F \cos \alpha}) - \pi \delta_{N,4p} \right), \quad (N \text{ even}),
\]

\[
 j_o(\alpha) = -\frac{2ev_F}{\pi l} \frac{\sqrt{T_F \sin \alpha}}{\sqrt{1 - T_F \cos^2 \alpha}} \left( 2 \arccos(\sqrt{T_F \cos \alpha}) - \pi \right), \quad (N \text{ odd}),
\]

with \( T_F = T(k_F) = \sin^2(k_F) \). When \( k_F \neq \pi/2 \), these expressions differ considerably from (3.37).

### E. Inclusion of interactions in the ring

Electron-electron interactions in the ring have a dramatic effect on the conductance of the dot, and hence on the persistent current. This can be seen by considering the strong coupling effective Hamil-
tonian of Eq. (2.24) for the EQD case. Now consider the effect of P-H symmetry breaking. For instance, if we choose half-filling, then we could include P-H symmetry breaking by a local potential scattering term at the origin:

\[ H_{1,\text{low}} \rightarrow H_{1,\text{low}} + V c_a^\dagger c_a. \] (3.40)

The corresponding continuum limit Hamiltonian is just the free Dirac fermion Hamiltonian defined on \(-\infty < x < +\infty\), with a back scattering term, \(V[\psi_L^\dagger(0)\psi_R(0) + h. c.]\) added. If \(V\) is small, this has little effect, reducing the conductance by an amount of \(O(V^2)\). On the other hand, if we include electron-electron interactions in the ring, then the potential scattering has a much larger effect. We can simply use the analysis of Kane and Fisher \(23\) (see also Ref. \([24]\)). Upon bosonization, and introduction of spin and charge bosons, we parameterize the strength of the (screened, short range) interactions by a parameter, \(g_p\), related to the compactification radius of the charge boson by \(g_p = 1/\pi R_c^2\). For zero interactions, \(g_p = 2\). Repulsive interactions decrease \(g_p\). (We assume that the interactions preserve the full spin rotation symmetry so that the corresponding parameter for the spin boson is unchanged from its non-interacting value, \(g_s = 1/\pi R_s^2 = 2\).) The renormalization group scaling dimension of this backscattering term is:

\[ x = g_p/4 + 1/2. \] (3.41)

This boundary interaction is exactly marginal for zero bulk interactions, relevant for repulsive interactions and irrelevant for attractive interactions. Thus, with repulsive bulk interactions, this back scattering parameter, \(V\), renormalizes to large values at low energy scales corresponding to perfect reflection of spin and charge at the quantum dot. This implies that \(jl \rightarrow 0\) for large \(l\). The consistency of this assertion can be checked by considering the stability of the perfectly reflecting fixed point \(\text{At this fixed point, the ring is effectively severed at the quantum dot so that open boundary conditions can be applied to the fermion fields to the left and right of the dot. The leading irrelevant operator at this fixed point corresponds to a weak hopping process between the two ends of the severed chain, on either side of the quantum dot. This operator has scaling dimension:}\)

\[ x = 1/g_p + 1/2, \] (3.42)

which is \(> 1\) in the repulsive case. Hence this hopping process is irrelevant, for repulsive interactions, demonstrating the consistency of the assertion about the RG flow. We may think of this stable fixed point, where the Kondo coupling and the potential scattering, have both renormalized to \(\infty\), as corresponding to an electron in the even orbital at the original forming a spin singlet with the quantum dot with the odd orbital either empty or doubly occupied depending on the sign of \(V\). From this RG analysis of the perfectly reflecting fixed point we can deduce the scaling of the persistent current with length. The weak hopping amplitude scales with length as

\[ t_{eff}(l) \propto l^{1/2-1/g_p}. \] (3.43)

The conductance, being second order in \(t_{eff}\), scales with length as \(l^{1-2/g_p}\). On the other hand, \(jl\) scales as \(l_{eff}^{1/2}\), going as the square root of the conductance, or the transmission probability at the Fermi surface, \(T_F\). Thus, we expect

\[ j \propto l^{-(1/g_p+1/2)} \sin \alpha. \] (3.44)

This scales to zero faster than \(1/l\) for repulsive interactions. It is also interesting to consider the behavior for small \(V\), close to the resonance. It follows from Eq. (3.41) that the effective back-scattering potential at length \(l\) is given by:

\[ V_{eff}(l) \propto l^{1/2-g_p/4}. \] (3.45)

Thus the persistent current should depend on \(V\) and \(l\) as:

\[ jl = f(Vl^{1/2-g_p/4}, \alpha). \] (3.46)

(This result only holds in the case \(\xi_K \ll l\).) In particular, this implies that the width of the peak in \(j\), as a function of \(V\), scales as \(l^{-1/2+g_p/4}\). Thus, for weakly repulsive interactions, \(g_p \leq 2\), \(j\) will have a
broad maximum as a function of gate voltage for intermediate $l$ but this maximum will sharpen up with increasing ring length. The current at the maximum should be that of an ideal ring with no quantum dot.

We note that, provided $g_\rho$ is not too small, and assuming symmetric leads, there is only one relevant operator at the resonant fixed point. Thus, tuning one parameter, such as the gate voltage, should be sufficient to pass through the resonance. Strict P-H symmetry is not necessary to get perfect conductance; even away from half-filling there should be one value of gate voltage where perfect conductance occurs. With asymmetric leads $j_l$ will scale to 0 with $l$ for all gate voltages.

Further insight can be obtained by considering the weak Kondo coupling fixed point, with bulk interactions. Now the Kondo interaction breaks up into two parts which have different RG scaling dimension: the term that reflects an electron coming from the left or right and the term that transmits an electron. The folding transformation that we mentioned in Sec. II is not useful anymore because it would turn short range bulk interactions into infinite range interactions. We use subscripts $+$ and $-$ to label the fermion fields to the right or left of the dot. The open boundary conditions, at zero Kondo coupling, imply:

$$\psi_{L,+}(0) = -\psi_{R,+}(0). \quad (3.47)$$

Thus the two parts of the Kondo interaction are:

$$H_{int} = v_F \lambda_{++} \sum_\pm \psi_{L,\pm}^\dagger \vec{\sigma} \cdot \vec{S} + v_F \lambda_{+-} [\psi_{L,+}^\dagger \vec{\sigma} \psi_{L,-} + \psi_{L,-}^\dagger \vec{\sigma} \psi_{L,+}] \cdot \vec{S}. \quad (3.48)$$

Initially

$$\lambda_{++} = \lambda_{+-} = \lambda = \frac{2J \sin k_F}{\pi t}. \quad (3.49)$$

However, while $\lambda_{++}$ remains marginal, even in the presence of bulk interactions, $\lambda_{+-}$ does not. The marginality of $\lambda_{+-}$ follows since the interaction can be written entirely in terms of the spin bosons (or interaction parameter, $g_\sigma$) remains unchanged by the SU(2) invariant bulk interactions. On the other hand, the dimension of $\lambda_{+-}$ changes with bulk interactions. The corresponding operator has the same scaling dimension as a standard hopping operator between two severed chains, Eq. (3.42). Note that the spin-charge separation of the Kondo problem breaks down once bulk interactions are included. This results from the fact that we can’t transform to even and odd channels when bulk interactions are present. $\lambda_{+-}$ becomes irrelevant for repulsive interactions. We expect a flow from the $\lambda = 0$ fixed point either to the resonant fixed point if the backscattering term is tuned to 0, or otherwise to the charge and spin reflecting fixed point. The length scale at which the crossover to these other fixed points occurs is again given by $\xi_K \propto e^{-1/\lambda_{++}}$ as before. This length scale should be the appropriate one to render the scaling variable $V l^{1/2 - g_\rho/4}$, which controls the crossover between resonant and charge/spin reflecting fixed points, dimensionless. Thus we may write a general scaling form for the persistent current in the presence of bulk interactions:

$$j_l = f(\xi_K/l, (V/D)(l/\xi_K)^{1/2 - g_\rho/4}, \alpha). \quad (3.50)$$

Note in particular, that the width as a function of $V$ is given by $D(\xi_K/l)^{1/2 - g_\rho/4}$.

F. Asymmetric tunneling amplitudes

In this section, we would like to study how the persistent currents are modified if the tunneling amplitude between the wire and the dot are non-symmetric. The tight-binding Hamiltonian (3.2) is replaced by:

$$H = -t \sum_{j=1}^{l-2} (c_{j+1}^\dagger c_j + h.c.) - \epsilon_d c_d^\dagger c_d + t_R e^{i\alpha/2} c_1^\dagger c_{l-1} + h.c.) + \epsilon_d c_d^\dagger c_d + U n_d^\dagger n_d. \quad (3.51)$$
where we have introduced the left and right tunneling amplitudes \( t_L, t_R \). We have assumed no electronic interactions inside the ring. We refer to section III E for a discussion of this case, the idea being that one more relevant operator needs to be added to take into account the channel asymmetry in (3.48).

In the Kondo limit, the Hamiltonian (3.51) reads:

\[
H = \sum_{j=1}^{l-2} \left( c_j^\dagger c_{j+1} + h.c. \right) + 2J \left[ \frac{t_L}{\sqrt{t_L^2 + t_R^2}} e^{i\alpha/2} c_{j-1}^\dagger + \frac{t_R}{\sqrt{t_L^2 + t_R^2}} e^{-i\alpha/2} c_j^\dagger \right] \frac{\vec{\sigma}}{2} \cdot \vec{S} 
\]

(3.52)

where the Kondo coupling constant has been defined by \( \tilde{J} = (t_L^2 + t_R^2)(\frac{1}{e\xi_1} + \frac{1}{U+e\xi_2}) \). From (3.52), it is straightforward to reproduce our previous perturbative calculations. The persistent current (3.20) and (3.29) simply become:

\[
\tilde{j}_e(\alpha) = \frac{3\pi v_F e}{4l} \left[ \kappa \sin \tilde{\alpha}[\lambda + \lambda^2 \ln(l\epsilon)] + (1/4 + \ln 2)\kappa^2 \lambda^2 \sin 2\tilde{\alpha} \right] + O(\lambda^3, \kappa^3) \quad (N \text{ even}),
\]

(3.53)

\[
\tilde{j}_o(\alpha) = \frac{3\pi v_F e}{16l} \kappa^2 \sin 2\alpha[\lambda^2 + 2\lambda^3 \ln(l\epsilon')] + O(\lambda^4, \kappa^4) \quad (N \text{ odd}),
\]

(3.54)

where we have introduced the ratio:

\[
\kappa = \frac{2t_R t_L}{t_L^2 + t_R^2}.
\]

(3.55)

Notice first that the even and odd persistent currents are affected in a different manner. More importantly, the infrared logarithm divergences are only renormalizing the Kondo coupling as it should be. The persistent current is now a function of \( \tilde{\alpha}, \lambda_{e/o}, \kappa \). The universal scaling form for \( j \) (Eq. (3.31)) is modified to:

\[
\tilde{j}_{e/o}(\tilde{\alpha}, \lambda, l) = \frac{e v_F}{l} \tilde{g}_{e/o}(\xi_K/l, \tilde{\alpha}, \lambda)
\]

(3.56)

where the \( \tilde{g}_i \)'s are universal scaling functions. Our renormalization group arguments prove that the results obtained using perturbation theory are valid only when \( \xi_K \gg l \).

In order to analyse what happens in the limit \( \xi_K \ll l \), let us first observe that the asymmetric (infinite length) lattice Hamiltonian:

\[
H_{\text{asym}} = -t \sum_{j=-2}^{l-2} (c_j^\dagger c_{j+1} + h.c.) - t \sum_{j=1}^{l} (c_j^\dagger c_{j+1} + h.c.)
+ 2\tilde{J} \left[ \frac{t_L}{\sqrt{t_L^2 + t_R^2}} c_{j-1}^\dagger + \frac{t_R}{\sqrt{t_L^2 + t_R^2}} c_j^\dagger \right] \frac{\vec{\sigma}}{2} \cdot \vec{S} \left[ \frac{t_L}{\sqrt{t_L^2 + t_R^2}} c_{j-1} - \frac{t_R}{\sqrt{t_L^2 + t_R^2}} c_j \right]
\]

(3.57)

has exactly the same continuum limit (given by Eqs (2.13, 2.14, 2.18)) as the symmetric Kondo Hamiltonian defined in Eq. (2.7) (with \( J \to \tilde{J} \)), provided we generalize the even-odd basis in Eq. (2.11) to:

\[
\psi_e(x) = \frac{t_R}{\sqrt{t_L^2 + t_R^2}} \psi_L(x) + \frac{t_L}{\sqrt{t_L^2 + t_R^2}} \psi_R(-x),
\]

(3.58)

\[
\psi_o(x) = \frac{t_L}{\sqrt{t_L^2 + t_R^2}} \psi_L(x) - \frac{t_R}{\sqrt{t_L^2 + t_R^2}} \psi_R(-x).
\]

(3.59)

The generalized odd sector decouples from the generalized even sector which contains the Kondo interaction. We can therefore follow the same reasoning as in section 2. The effective Kondo coupling \( \tilde{\lambda}_1 = \frac{4J_1^2 \sin^2 \xi_1}{\pi v_F} \) renormalizes as in Eq. (2.21). This effective dimensionless Kondo coupling is driven to strong coupling namely \( \tilde{\lambda}_1 \to \infty \). On the lattice, it means that a screening electron forms a singlet with the impurity in the orbital defined by:
\[ c_e = \left( \frac{t_R}{\sqrt{t_R^2 + t_L^2}} c_1 + \frac{t_L}{\sqrt{t_R^2 + t_L^2}} c_{-1} \right). \]  

(3.60)

In order to obtain the low energy effective Hamiltonian, we project on the orthogonal state \( c_o \):

\[ P_{c_1} P = \frac{t_L}{\sqrt{t_R^2 + t_L^2}} c_o \]
\[ P_{c_{-1}} P = \frac{-t_R}{\sqrt{t_R^2 + t_L^2}} c_o. \]  

(3.61)

The low effective Hamiltonian then becomes:

\[ H_{\text{low}} = -t \sum_{j \leq -3} (c_j^\dagger c_{j+1} + \text{h.c.}) - t \sum_{j \geq 2} (c_j^\dagger c_{j+1} + \text{h.c.}) - t \left( \frac{t_L}{\sqrt{t_R^2 + t_L^2}} c_2^\dagger c_{i+2} - \frac{t_R}{\sqrt{t_R^2 + t_L^2}} c_0^\dagger c_{-2} + \text{h.c.} \right). \]  

(3.62)

It is straightforward to show that this Hamiltonian has the transmission probability:

\[ T(k) = \left( \frac{2t_R t_L}{t_R^2 + t_L^2} \right)^2 \sin^2 k. \]  

(3.63)

According to section [11] if \( t_L, t_R \) are small enough meaning the bare Kondo coupling small enough, particle-hole symmetry breaking effects are negligible. We can thus expect the transmission probability at the Fermi surface to be given by \( T(\epsilon_F) = \left( \frac{2t_R t_L}{t_R^2 + t_L^2} \right)^2 \). A similar result has been obtained in [13]. In order to calculate the persistent current, we can directly apply the result of Gogolin and Prokof’ev.[21]

The persistent current in the limit \( \xi_K \ll l \) is therefore given by Eq. (3.39) with \( T_F = T(\epsilon_F) \).

### IV. PERSISTENT CURRENTS FOR THE SIDE COUPLED QUANTUM DOT

We now want to consider the persistent current of a closed ring with Aharonov-Bohm flux and a side coupled quantum dot. As in the previous section, we suppose the ring has \( l \) sites. The Hamiltonian reads:

\[ H = -t \sum_{i=0}^{l-1} (c_i^\dagger c_{i+1} + \text{h.c.}) + J \frac{\Phi}{2} \hat{S}_0. \]  

(4.1)

There is no explicit flux dependence in the above Hamiltonian since we can choose to gauge it away. Obviously, the flux dependence is encoded into the non trivial boundary condition \( c_{j+l} = e^{i\alpha} c_j \) and in particular \( c_l = e^{i\alpha} c_0 \), with \( \Phi = (c/e)\alpha \). Due to this \( \alpha \)-dependent boundary condition, the possible values of the momentum are

\[ k_n = 2\pi n/l + \alpha/l, \quad n = -(l - 2)/2, \ldots, l/2 \quad (l \text{ even}). \]

When \( J = 0 \), the persistent current is the one of free fermions given in Eq. (3.37) except that \( N \) has to be changed to \( N - 1 \). Note that the persistent current is an odd function of \( \alpha \) and is \( 2\pi \) periodic for \( N \) odd and \( \pi \) periodic for \( N \) even. We will prove in the following that these relations persist to all order of the perturbation scheme in the Kondo coupling \( J \). Let us first distinguish the two cases \( N \) even and \( N \) odd.

#### A. \( N \) even

We first assume that \( \alpha \in [0, \pi] \). The ground state is well defined and has all levels empty or doubly occupied except the Fermi level at \( k_F \) defined by
\[
\begin{align*}
  k_F &= \frac{2\pi N}{4l} - \frac{\alpha}{l}, \quad N/2 \text{ even} \\
  k_F &= \frac{2\pi N}{4l} - \frac{(\pi - \alpha)}{l}, \quad N/2 \text{ odd}
\end{align*}
\] (4.2)

which contains one electron forming a singlet with the impurity. Note that \( k_F = k_F(\alpha) \). The interaction is the standard Kondo interaction between the quantum dot and the site 0 of the ring. There is a contribution at first order which does not depend on \( \alpha \):

\[
E_0^{(1)} = -\frac{3J}{4l}.
\] (4.3)

The second order contributions can be obtained according to Eq. (3.15). Following the procedure developed in appendix A, we first expand \( \psi(0) \) in Fourier modes and separate in the time ordered Green function the \( k_F \) term (which we call \( \psi_0 \)) from the rest of the series (which we call \( \psi' \)). Due to \( \alpha \) dependence of \( k_F \), the left and right branches gives different contributions to the propagator.

Let us first suppose that \( N/2 \) is even. We will show how the case \( N/2 \) odd is related to the case \( N/2 \) even later. We have depicted a schematic level diagram in figure 5 for \( \alpha \in [0, \pi] \) and \( N = 12 \). The lowest energy level has momentum \( k = \alpha/l \) corresponding to \( n = 0 \). For \( N/2 \) even, the Fermi level is reached by one electron with spin up or down belonging to the left branch (corresponding to \( n = -3 \) on the figure).

\[
\begin{align*}
  \xi^L_m &= -2t \cos[2\pi(N/4 - m)/l - \alpha/l] + 2t \cos[2\pi N/4l - \alpha/l], \\
  \xi^R_m &= -2t \cos[2\pi(N/4 - m)/l + \alpha/l] + 2t \cos[2\pi N/4l - \alpha/l].
\end{align*}
\] (4.6)

We assume that at large \( |\tau| \), the sum is dominated by excitations near the Fermi surface. We can therefore use the approximation \( \xi^L_m \approx -2\pi v_F m/l \) and \( \xi^R_m \approx -2\pi v_F (m - \alpha/\pi) \), such that the Green functions can be written as:

\[
\begin{align*}
  G(\tau) &= \frac{1}{l} \sum_{|k| < k_F} e^{i(E_k - E_F)\tau} \quad \text{if } \tau > 0, \\
  &= \frac{1}{l} \sum_{m=1}^{l/2} (\xi^L_m - \xi^R_m),
\end{align*}
\] (4.5)

From this diagram, it is straightforward to compute the propagator for the field \( \psi' \) which are taken at site 0:

\[
\mathcal{T}(\psi^\dagger_\alpha(\tau)\psi^\prime_\beta(0)) = \delta_{\alpha,\beta} G(\tau),
\] (4.4)
\[ G(\tau) \approx \frac{1}{\mathcal{T}} \left( \frac{e^{-\frac{2\pi v_F}{\tau}}}{1 - e^{-\frac{2\pi v_F}{\tau}}} + \frac{e^{-\frac{2\pi v_F}{\tau}(1 - \alpha/\pi)}}{1 - e^{-\frac{2\pi v_F}{\tau}} \tau} \right) \equiv G_1(\tau) \text{ if } \tau > 0. \] (4.8)

Similar calculations for \( \tau < 0 \) leads to:

\[ G(\tau) = -\frac{1}{\mathcal{T}} \sum_{|k| > k_F} e^{(E_k - E_F)\tau} \text{ if } \tau > 0 \]

\[ \approx -\frac{1}{\mathcal{T}} \left( \frac{e^{\frac{2\pi v_F}{\tau}}}{1 - e^{\frac{2\pi v_F}{\tau}}} + \frac{e^{2\pi v_F\tau\alpha/\pi}}{1 - e^{2\pi v_F\tau}} \right) \equiv -G_2(\tau) \text{ if } \tau < 0. \] (4.9)

The first part of the propagator comes from the left excitations whereas the second part comes from the right excitations (see figure [3]).

We now want to emphasize that the only \( \alpha \) dependence of the ground state energy comes from this propagator. Thus, properties of this propagator extend to properties of the ground state and therefore of the persistent current. First, notice that from time reversal symmetry it results that the ground state energy is an even function of \( \alpha \) and therefore the persistent current an odd function of \( \alpha \). It is straightforward to show that the case \( N/2 \) odd is obtained from the case \( N/2 \) even by changing \( \alpha \to \pi - \alpha \) in (4.8) and (4.9). Note that it also a property of the Fermi momenta defined in (4.2). Using this property and time reversal invariance, we can prove that \( E_{4p+2}^{(0)}(\alpha) = E_{4p}^{(0)}(\pi + \alpha) \). We can go one step further. Indeed, it is worth noticing that under \( \alpha \to \alpha + \pi \) the propagators (4.8) and (4.9) exchange with each other: \( G_1(\tau) \to G_2(-\tau), \ G_2(\tau) \to G_1(\tau) \). \( G_1 \) is associated with particle excitations and \( G_2 \) with hole excitations. When calculating the ground state energy in perturbation theory, each order of the perturbation theory appears as a time integral of a function which should be symmetric under interchanging \( G_1(\tau) \) with \( G_2(-\tau) \) due to particle-hole symmetry. This proves the \( \pi \) periodicity of \( E_{4p}^{(0)}(\alpha) \) and also that:

\[ E_{4p+2}^{(0)}(\alpha) = E_{4p}^{(0)}(\alpha). \] (4.10)

We also need the correlator for \( \psi_0 \):

\[ \mathcal{T} \langle \sigma | \psi_{0,\gamma}^{\dagger}(\tau) \psi_{0,\nu}(0) | \epsilon \rangle = \frac{1}{\mathcal{T}} [\delta_{\sigma \gamma} \delta_{\nu \epsilon} - \theta(\tau) \delta_{\gamma \nu} \delta_{\sigma \epsilon}], \]

\[ \mathcal{T} \langle \sigma | \psi_{0,\nu}(\tau) \psi_{0,\gamma}^{\dagger}(0) | \epsilon \rangle = \frac{1}{\mathcal{T}} [-\delta_{\sigma \gamma} \delta_{\nu \epsilon} + \theta(\tau) \delta_{\gamma \nu} \delta_{\sigma \epsilon}], \] (4.11)

where \( |\epsilon\rangle \) indicates an electron with spin \( \epsilon \) lying at the Fermi level. We have performed the calculations in the basis defined by:

\[ |A\rangle, |B\rangle = |\uparrow, \downarrow\rangle, |\downarrow, \uparrow\rangle, \]

for convenience. The double arrows denote the spin state of the impurity. There are three contributions to the ground state energy at second order. The first contribution comes from the correlators involving only \( \psi' \) and reads:

\[ -\frac{J^2 l}{4\pi v_F} \int_0^\infty du \ G_1(u) G_2(-u) \mathcal{T} \langle \sigma^a \sigma^b \rangle \left[ \langle A| S^a(u) S^b(0)|B \rangle + \langle A| S^b(0) S^a(-u)|B \rangle \right], \] (4.12)

where we have made the change of variable \( \tau = 2\pi v_F u/l \). We need to add also the cross terms (containing two \( \psi' \) and two \( \psi_0 \)). The full result for

\[ E_{AB}^{(2)}(\alpha) = -\frac{1}{2} \int d\tau \langle A| H_{\text{int}}(\tau) H_{\text{int}}(0)|B \rangle, \] (4.13)

reads:
\[ E_{AB}^{(2)}(\alpha) = -\frac{J^2 l}{4\pi v_F} \left[ \int_0^\infty du \ (G_1(u)G_2(-u) + \frac{1}{l}G_1(u)) \ Tr \left( \frac{\sigma^a \sigma^b}{2} \right) \ [\langle A|S^a(u)S^b(0)|B \rangle + \langle A|S^b(0)S^a(-u)|B \rangle] \right. 

- \frac{1}{l} \int_0^\infty du \ \left( G_1(u)\langle A|c_{k_F}^\dagger \left( \frac{\sigma^b \sigma^a}{2} \right) c_{k_F}S^a(u)S^b(0)|B \rangle - G_2(-u)\langle A|c_{k_F}^\dagger \left( \frac{\sigma^b \sigma^a}{2} \right) c_{k_F}S^b(0)S^a(-u)|B \rangle \right) \]

+ \frac{1}{l} \int_0^\infty du \ \left( G_2(-u)\langle A|c_{k_F}^\dagger \left( \frac{\sigma^a \sigma^b}{2} \right) c_{k_F}S^a(u)S^b(0)|B \rangle - G_1(u)\langle A|c_{k_F}^\dagger \left( \frac{\sigma^a \sigma^b}{2} \right) c_{k_F}S^b(0)S^a(-u)|B \rangle \right), \tag{4.14} \]

where we have also used \( \tau = 2\pi v_F u/l \). Using the identity:

\[ \mathcal{T}\langle A|S^a(u)S^b(0)|B \rangle = \frac{1}{4} \delta^{ab} \delta_{AB} + \frac{i}{2} \epsilon^{abc} \langle A|S^c|B \rangle, \]

and gathering all contributions, we finally obtain the second order correction to the ground state energy:

\[ E_0^{(2)}(\alpha) = -\frac{J^2 l}{4\pi v_F} \int_0^\infty du \ \left[ \frac{3}{4} G_1(u)G_2(-u) + \frac{9}{8l} (G_1(u) + G_2(-u)) \right]. \tag{4.15} \]

Note that this expression is symmetric under \( G_1(u) \leftrightarrow G_2(-u) \) as it should be. It is also worth noticing that had we just focussed on the leading logarithm divergences in (4.15), we would find \( E_0^{(2)}(\alpha) \approx -\frac{3J^2 \ln(lc)}{4\pi v_F} = -\frac{3\pi v_F \lambda^2 \ln(lc)}{4l} \) which would renormalize perfectly the first order contribution to the ground state energy given in (4.13) (it also provides a non trivial check of the calculations). Nevertheless, in order to calculate the persistent current, we are only interested in the \( \alpha \) dependent terms in (4.13):

\[ E^{(2)}(\alpha) \approx -\frac{J^2 l}{4\pi v_F} \int_0^\infty du \ \left[ \frac{3}{4} \left( e^{(-2u+u_0)/\pi} + e^{-u(1+\alpha)/\pi} \right) \right. \]

\[ + \left. \frac{9}{8} \left( e^{-u_0/\pi} + e^{-u(1-\alpha)/\pi} \right) \right]. \tag{4.16} \]

This expression contains some UV divergences which are removed when we consider the persistent current defined by \( (j^c)^{(2)}(\alpha) = -\frac{e v_F \lambda^2}{4l} E^{(2)}(\alpha) \):

\[ (j^c)^{(2)}_1(\alpha) = \frac{e v_F \lambda^2}{4l} \int_0^\infty du \ \left[ \frac{3}{4} \left( e^{(-2u+u_0)/\pi} - e^{-u(1+\alpha)/\pi} \right) \right. \]

\[ - \left. \frac{9}{8} \left( e^{-u_0/\pi} - e^{-u(1-\alpha)/\pi} \right) \right]. \tag{4.17} \]

The subscript 1 is for latter convenience. This expression is completely antisymmetric under \( \alpha \to \pi - \alpha \) as it should be (since \( E_0(\alpha) = E_0(\pi - \alpha) \)). In order to check explicitly the renormalization of the persistent current with the Kondo coupling, we would have to go the third order in perturbation. We have not performed the third order calculation but we expect the dimensionless Kondo coupling constant \( \lambda = \frac{J}{\pi v_F} \) to renormalize as \( \lambda_{\text{eff}}(l) = \lambda + \lambda^2 \ln l + \ldots \) as in the previous section (see also further). Note that the \( \alpha \) dependence of \( k_F \) gives corrections smaller by a factor of \( a/l \). This expression contains some severe infrared divergences when \( \alpha \to 0 \) or \( \alpha \to \pi \). For example when \( \alpha \to 0 \), the level at \( k_F \) and the next higher level (corresponding to \( n = 3 \) in Fig. 3) become degenerate, which is not taken into account in our initial choice of the ground state. For small values of \( \alpha \), we can evaluate the integral approximately by

\[ (j^c)_1^{(2)}(\alpha) \approx -\frac{9e v_F \lambda_{\text{eff}}^2}{32l} \frac{\pi^2}{\alpha^2}. \tag{4.18} \]

Physically, this corresponds to truncating the summation over intermediate states occurring in second order perturbation to the first exited state only which approaches the ground state as \( \alpha \to 0 \) and therefore gives the most important contribution to the persistent current. For perturbation theory to make sense we are restricted to \( (j_1^{(2)}(\alpha) \ll (j^c)^{(0)}(\alpha)) \) namely

\[ \alpha \gg \pi \lambda_{\text{eff}} \frac{3}{\sqrt{32}}. \tag{4.19} \]
When this condition is not satisfied, this approach fails. Note that this result is expected. Indeed, the persistent current is discontinuous at \( \alpha = 0 \) for \( \lambda = 0 \) which precludes a naive perturbative analysis near this singular point which corresponds simply to a level crossing. To overcome this difficulty occurring for \( \alpha \ll \pi \) or \( |\alpha - \pi| \ll \pi \), we need to do perturbation theory around the correct ground state which is built in this case from the two levels close to the Fermi surface which are mixed by the perturbation. In the sequel we consider the case \( \alpha \ll \pi \), which extends trivially to \( |\alpha - \pi| \ll \pi \) by symmetry around \( \alpha = \pi/2 \). These two levels have momenta \( k_{j/2} = \pm \frac{2N\pi}{3l} + \alpha/l \). They correspond on the figure 6 to the levels labeled by \( n = -3 \) and \( n = 3 \). In fact, the strategy we follow is analogous to a second order degenerate perturbation theory. When \( \alpha \ll \pi \), the ground state is built with all levels with \( |k| < k_1 \) full and one electron lying on one of the two almost degenerate levels forming a singlet with the impurity (defining therefore two possible states).

The first order correction in the Kondo coupling \( J \) mixes these two states. It is straightforward to show that at this order the contribution of these two levels to the ground state energy is found by diagonalizing the \( 2 \times 2 \) matrix

\[
\langle H_{12} \rangle = \left( \begin{array}{cc} \epsilon_1 - \frac{3J}{4l} & \frac{-3J}{4l} \\ \frac{-3J}{4l} & \epsilon_2 - \frac{3J}{4l} \end{array} \right),
\]

written in the basis:

\[
\frac{|\uparrow; \alpha; \downarrow \rangle - |\downarrow; \alpha; \uparrow \rangle}{\sqrt{2}} ; \quad \frac{|\alpha, \uparrow; \downarrow \rangle - |\alpha, \downarrow; \uparrow \rangle}{\sqrt{2}},
\]

where \( |\uparrow; \alpha; \downarrow \rangle = c_{1, \uparrow}| \downarrow \rangle, |\alpha, \uparrow; \downarrow \rangle = c_{2, \uparrow}| \downarrow \rangle \), etc. Here the double arrows denote the spin state of the impurity.

The other levels are giving second order contributions in \( J \) in (4.20). We have explicitly shown in appendix C that the Kondo coupling constant in Eq. (4.20) gets perfectly renormalized at second order, namely \( J/l \) has to be replaced by \( \pi v_F(\lambda + \lambda^2 \ln(|\epsilon|))/l = \pi v_F \lambda_{eff}(l)/l \). This result is not surprising since we expect the persistent current to be a universal function of \( \lambda_{eff} \) and \( \alpha \) as in Eq. (3.3).

Therefore the final result for the ground state energy can be cast in the form:

\[
E^{(0)} = -\frac{3\pi v_F \lambda_{eff}}{4l} + E^{free}(\alpha) + \frac{\epsilon_2 - \epsilon_1}{2} - \sqrt{(\epsilon_1 - \epsilon_2)^2 - \frac{9\pi^2 v_F^2 \lambda^2_{eff}}{16l^2}}.
\]

\[
E^{free}(\alpha) + 2t \sin\left[\frac{2\pi N}{4l}\right] \sin[\alpha/l] - \sqrt{4t^2 \sin^2\left[\frac{2\pi N}{4l}\right] \sin^2[\alpha/l] + \frac{9\pi^2 v_F^2 \lambda^2_{eff}}{16l^2}},
\]

where \( E^{free}(\alpha) \) is the ground state energy of the free case for \( N \) even. This is easily deduced from Eq. (3.37) by replacing \( N \) by \( (N - 1) \):

\[
E^{free}(\alpha) = constant + \frac{v_F}{2\pi l} \left( \alpha^2 + (\alpha - \pi)^2 \right).
\]

For \( \lambda_{eff} = 0 \), the free case is recovered as it should be. The singularity in \( \alpha = 0 \) is smoothed by the square root. The persistent current becomes so far in the large \( l \) limit:

\[
j_2^\alpha(\alpha) = -\frac{ev_F}{l} \left( \frac{2\alpha}{\pi} - 1 + \cos[\alpha/l] - \frac{1}{2} \frac{\sin[2\alpha/l]}{\sin^2[\alpha/l] + \frac{9\pi^2 v_F^2 \lambda^2_{eff}}{16l^2}} \right),
\]

\[
\approx -\frac{ev_F}{l} \left( \frac{2\alpha}{\pi} - \frac{2}{\sqrt{(2 \pi l)^2 + (\frac{3\lambda_{eff}}{2})^2}} \right).
\]
FIG. 6. Persistent current of the SCQD for $N$ even calculated for $\xi_K/l \approx 50$ (solid line) compared to the $J = 0$ case (dashed line).

Notice that for $\lambda_{\text{eff}} \neq 0$, the persistent current is continuous in $0$ and $j_e(0) = 0$. The same analysis can be reproduced near $\alpha = \pi$. Indeed, by symmetry it is enough to replace in (4.25) $\alpha/\pi \rightarrow 1 - \alpha/\pi$ and to change the overall sign. It is clear that the result (4.25) is valid only for small value of $\alpha \ll \pi$ and for $\lambda_{\text{eff}} \ll 1$ where the ground state can be regarded as a superposition of states belonging to the two almost degenerate levels. For larger value of $\alpha \gg \lambda_{\text{eff}}$ but still small compared to $\pi$, we may evaluate the validity of this approach by developing (4.25) in powers of $\lambda_{\text{eff}}/\alpha$:

$$j_e^2(\alpha) \approx (j_e^r)^{(0)}(\alpha) - \frac{9ev_F\lambda_{\text{eff}}^2}{32l} \frac{\pi^2}{\alpha^2} \approx (j_e^r)^{(0)}(\alpha) + (j_e^r)^{(2)}(\alpha).$$

(4.26)

In this regime, we recover the small $\alpha$ limit of (4.17). Thus Eqs (4.17) and (4.25) agree in the limit $\lambda_{\text{eff}} \ll \alpha \ll \pi$. Therefore, together, they cover all the $\alpha$ range at small $\lambda_{\text{eff}}$. We want to emphasize that these arguments are valid for small value of $\lambda_{\text{eff}} \ll 1$ meaning $l \ll \xi_K$ where perturbation makes sense.

We have plotted the persistent current from Eq. (4.17) and (4.25) (more exactly $j_e^r/l ev_F$) in figure 6 for $\xi_K/l \approx 50$ ($\lambda_{\text{eff}} \approx 0.25$). We notice that the amplitude of the persistent current is already strongly renormalized for this value of $\lambda_{\text{eff}}$. Moreover the singularities at $\alpha = n\pi$ have been completely wiped out by the Kondo coupling.

In the opposite limit $l \gg \xi_K$, where perturbation breaks down, we may expect the persistent current to be given by the $J \rightarrow \infty$ limit according to standard renormalization group arguments (see the discussion in section III). In this limit, we have shown in section 2 that the transmission $T$ tends toward 0. According to the argument given by Gogolin and Prokof‘ev [2], we thus expect:

$$j_e^r \rightarrow 0 \quad \text{for} \quad l \gg \xi_K,$$

(4.27)

the picture being simply that the ring is “cut” into two pieces which prevents non zero persistent currents.

**B. N odd**

The ground state is defined by all states under the Fermi level ($k_F = \pi(N-1)/4l$) being doubly occupied and all states above the Fermi level being empty. The number of electrons inside the ring is even therefore the system, including the impurity, is in a doublet state. Consequently, there is no contribution to the ground state at first order in $J$. In the following, we assume $N = 4p + 1$ and $0 < \alpha < \pi$. We will show later how the case $N = 4p + 3$ is related to that case $N = 4p + 1$. We have depicted a schematic diagram
of the different levels for $N = 13$ in figure 7. The first level has a momentum $\alpha/l$ corresponding to $n = 0$ and the last filled level has momentum $k = (2\pi p - \alpha)/l$ (labeled by $-3$ in the figure 7).

![Level diagram](image)

**FIG. 7.** Level diagram for $N = 13$ and $\alpha \in [0, \pi]$. The filled circles are for full levels whereas empty circles are for empty levels. The Fermi level $k_F$ lays half between the highest filled level corresponding to $n = -3$ and the lowest empty level corresponding to $n = 4$.

The second order correction to the ground state can be written as

$$E^{(2)}_0 = -\frac{1}{2} \int_{-\infty}^{+\infty} d\tau \, \langle 0 | H_{\text{int}}(\tau) H_{\text{int}}(0) | 0 \rangle. \quad (4.28)$$

The propagator reads:

$$\mathcal{T}(\psi^\dagger_\alpha(\tau) \psi_\beta(0)) = \delta_{\alpha,\beta} G(\tau) = \frac{\delta_{\alpha,\beta}}{l} \sum_n e^{\xi_n \tau} \quad (4.29)$$

where

$$\xi_n = -2t \cos\left[\frac{2\pi}{l} n + \frac{\alpha}{l}\right] + 2t \cos k_F = -2t \cos\left[\frac{2\pi}{l} n + \frac{\alpha}{l}\right] + 2t \cos\left[\frac{2\pi(N-1)}{4l}\right] \quad (4.30)$$

As for $N$ even, at large $\tau$, the sum is dominated by terms near the Fermi surface corresponding to $n = \frac{N-1}{4} - m$, with $m \ll N$. Due to the $\alpha$ dependence of the quantum numbers, there is an asymmetry between left and right movers. By help of the level diagram, it can be easily shown that

$$G(\tau) \approx \frac{1}{l} \sum_{m=1}^{1/2} e^{-\frac{2\pi v_F}{l} \tau (m-\alpha/2\pi)} + \frac{1}{l} \sum_{m=0}^{1/2} e^{-\frac{2\pi v_F}{l} \tau (m+\alpha/2\pi)}$$

$$\approx \frac{1}{l} \left( e^{-\frac{2\pi v_F}{l} \tau (1-\alpha/2\pi)} + e^{-\frac{2\pi v_F}{l} \tau \alpha/l} \right) \quad (4.31)$$

This expression is valid whatever the sign of $\tau$. Following similar arguments as for $N$ even, properties of the propagator translate into properties of the ground state energy. We can show explicitly that $E_0(\alpha) = E_0(\alpha + 2\pi)$ and especially $E_0(\alpha, 4p + 3) \equiv E_0(\alpha + \pi, 4p + 1)$. Therefore the case $N = 4p + 3$ is simply related to the case $N = 4p + 1$ by a translation of $\pi$. The second order term in the ground state energy reads therefore:

$$E^{(2)}_0 = -\frac{J^2}{2} \sum_{a,b} \text{tr} \left( \sigma^a \sigma^b \right) \langle S^a S^b \rangle \int_{-\infty}^{\infty} G(\tau) [-G(-\tau)] \quad (4.32)$$
After plugging the time ordered Green function (4.31) in this expression we get, retaining only $\alpha$ dependent terms:

$$E_0^{(2)} = \frac{3J^2}{8l(2\pi v_F)^2} \int_0^\infty du \, \frac{1 + \cosh u(1 - \alpha/\pi)}{1 - \cosh u}. \quad (4.33)$$

This result can be generalized to the third order in $J$. Following similar calculations presented in appendix A2, we can show that:

$$E_0^{(3)} = -\frac{3J^3}{16l(2\pi v_F)^2} 4 \ln(lc) \int_{-\infty}^{+\infty} dv G(v)[G(-v)]. \quad (4.34)$$

We define the Kondo dimensionless coupling constant $\lambda$ as $J = \pi v_F \lambda$. Gathering all terms, we find

$$E_0 \approx -\frac{3\pi v_F}{16l}[\lambda^2 + 2 \ln(lc) \lambda^3] \int_0^\infty du \, \frac{1 + \cosh u(1 - \alpha/\pi)}{\cosh u - 1}. \quad (4.35)$$

The correction to the persistent current finally takes the form:

$$(j_o^{(2)})_1^2(\alpha) = \frac{-3\pi v_F \lambda_{eff}^2}{16l} \int_0^\infty du \, u \sinh u(1 - \alpha/\pi) \cosh u - 1, \quad (4.36)$$

where we have defined as usual $\lambda_{eff} = \lambda + \lambda^2 \ln(lc)$ the renormalized labeled Kondo coupling constant. Exactly as in the case $N$ even, this expression diverges when $\alpha \to 0$, i.e. when the two levels close to the Fermi surface become degenerate. In figure 7, it corresponds to the levels labeled by $n = -3$ and $n = 3$.

For small value of $\alpha$, we can evaluate the integral approximately by

$$(j_o^{(2)})_1^2(\alpha) \approx -\frac{3\pi v_F \lambda_{eff}^2 \pi^2}{16l \alpha^2}. \quad (4.37)$$

Therefore, perturbation theory applies when $(j_o^{(2)})_1^2(\alpha) \ll (j_o^{(0)})_1^2(\alpha)$ namely

$$\alpha \gg \frac{\pi \lambda_{eff}}{\sqrt{3}} \frac{\sqrt{3}}{2\sqrt{8}}. \quad (4.38)$$

For smaller value of $\alpha \ll \pi$ the Kondo coupling mixes strongly the states belonging to the two levels close to the Fermi surface. We need therefore to perform our perturbative calculations around a new ground state built from these two almost degenerate levels which we denote by 1 and 2. They have energy close to the Fermi surface. We need therefore to perform our perturbative calculations around a new
different states with $N$ even. This expression diverges when $\alpha \to 0$, i.e. when the two levels close to the Fermi surface become degenerate. In figure 7, it corresponds to the levels labeled by $n = -3$ and $n = 3$.

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For smaller value of $\alpha \ll \pi$ the Kondo coupling mixes strongly the states belonging to the two levels close to the Fermi surface. We need therefore to perform our perturbative calculations around a new ground state built from these two almost degenerate levels which we denote by 1 and 2. They have energy $\epsilon_{1/2} = -2t \cos(k_F \mp \alpha/l)$ respectively, where we have defined $k_F = \frac{2\pi N - 1}{N}$. We can now repeat the same analysis as for $N$ even.

The ground state has all levels with $|k| < k_1$ full and two electrons in the levels 1, 2. We have four possible states with $S_{tot} = S^z_{tot} = 1/2$:

$$\sqrt{\frac{3}{2}} \langle \uparrow, \downarrow; \uparrow \rangle + \langle \downarrow, \uparrow; \uparrow \rangle - 2 \langle \uparrow, \downarrow; \downarrow \rangle; \quad \langle \uparrow, \downarrow; \alpha; \uparrow \rangle \quad ; \quad \langle \alpha, \uparrow; \uparrow \rangle \quad ; \quad \langle \uparrow, \downarrow; \uparrow \rangle - \langle \downarrow, \uparrow; \uparrow \rangle \quad . \quad (4.39)$$

The states are defined according to $|\alpha, \beta, \uparrow \rangle = c_{1,1}^\dagger c_{2,\beta}^\dagger \uparrow \rangle$, $|\uparrow, \downarrow, \alpha; \uparrow \rangle = c_{1,\uparrow}^\dagger c_{1,\downarrow}^\dagger \uparrow \rangle$, etc. The first order contributions in $J$ mixes these states. The associated matrix taking into account the contribution of these two levels to the ground state energy at first order in $J$ reads:

$$J_{12} = \begin{pmatrix}
\epsilon_1 + \epsilon_2 - \frac{J}{4l} & -\frac{J\sqrt{\pi}}{4l} & 0 \\
-\frac{J\sqrt{\pi}}{4l} & 2\epsilon_1 & 0 \\
\frac{J\sqrt{\pi}}{4l} & 0 & 2\epsilon_2 \\
0 & 0 & \epsilon_1 + \epsilon_2
\end{pmatrix}. \quad (4.40)$$
We first note that the fourth state does not mix at this order and the matrix is effectively $3 \times 3$. We want now to include the effects of the higher levels which are giving second order contributions in $J$ in Eq. (4.40). We have shown in appendix C2 using second order degenerate perturbation theory that the Kondo coupling constant in the Eq. (4.40) gets perfectly renormalized namely $J/l$ to be replaced by $\pi v_F (\lambda + \lambda^2 \ln|\epsilon|)/l = \pi v_F \lambda_{eff}/l$.

Defining $E' = E - \epsilon_1 - \epsilon_2 - \sum_{|k|<k_1} \epsilon_k$, the ground state energy is obtained by finding the minimum root of the cubic equation:

\[(E')^3 + \frac{\pi v_F \lambda_{eff}}{l}(E')^2 - E' \left( (\epsilon_1 - \epsilon_2)^2 + \frac{3\pi^2 v_F^2 \lambda_{eff}^2}{4l^2} \right) - \frac{\pi v_F \lambda_{eff}}{l} (\epsilon_1 - \epsilon_2)^2 = 0.\]  

(4.41)

After the change of variable $E' = \frac{\pi v_F}{l} (X - \frac{1}{2})$, and using $\epsilon_2 - \epsilon_1 \approx 2 v_F \alpha/l$, the equation can be cast in the reduced form:

\[(X)^3 - X\left[\frac{13}{12} \lambda_{eff}^2 + a^2\right] - \frac{2}{3} \alpha^2 \lambda_{eff} + \frac{35}{108} \lambda_{eff}^3 = 0,\]  

(4.42)

where $a = 2\alpha/\pi$.

The smallest root is found to be:

\[X_{min} = 2\sqrt{Q} \cos\left[\frac{\theta}{3} + \frac{2\pi}{3}\right] \quad \text{with} \quad \theta = \arccos\left[\frac{-R}{Q^2}\right],\]  

(4.43)

where $Q, R$ are defined by:

\[Q = \frac{1}{3}\left(\frac{13}{12} \lambda_{eff}^2 + a^2\right), \quad R = -\frac{1}{2}\left(-\frac{2}{3} a^2 \lambda_{eff} + \frac{35}{108} \lambda_{eff}^3\right).\]  

(4.44)

Therefore, the ground state energy reads:

\[E^{(0)} = E^{free} + \epsilon_2 - \epsilon_1 + \frac{\pi v_F}{l} X_{min}(2\alpha/\pi, \lambda_{eff}),\]  

(4.45)

where $E^{free}$ is the ground state energy for $J = 0$.

The persistent current can be then readily deduced as:

\[j_2^{(a)} \approx \frac{2ev_F}{l} \left(\frac{\alpha}{\pi} + X_{min}(a = 2\alpha/\pi, \lambda_{eff})\right),\]  

(4.46)

with $\alpha = \alpha$ if $(N-1)/2$ is even and $\alpha = \alpha - \pi$ if $(N-1)/2$ is odd. Moreover

\[X_{min}' = \frac{2a}{3\sqrt{Q}} \cos\left[\frac{\theta}{3} + \frac{2\pi}{3}\right] - \frac{2\sqrt{Q}}{3} \sin\left[\frac{\theta}{3}\right] + \frac{2\pi}{3} |\theta'(a)|,\]  

(4.47)

and

\[\theta'(a) = \frac{a \lambda_{eff}}{3} \left(\frac{2a}{3} \lambda_{eff}^2 - \frac{1}{4} a^2\right) \frac{Q^{5/2}}{\sqrt{1 - \frac{a^2}{Q^2}}}.\]  

(4.48)

Notice first that for $\lambda = 0$, the free case is recovered as it should be. To analyze the range of validity of the formula (4.46), it is interesting to expand (4.46) in $\lambda_{eff}$. It is straightforward to show that:

\[j_2^{(a)} = (j_2^{(0)})^{(0)} - \frac{3ev_F}{16l} \frac{\lambda_{eff}^2 \pi^2}{\alpha^2} \approx (j_2^{(0)})^{(0)} + (j_2^{(0)})^{(2)},\]  

(4.49)

which corresponds the small $\alpha$ limit of the persistent current calculated with standard perturbation theory (see Eq. (4.36)). Therefore Eqs (4.36) and (4.46) together cover all $\alpha$ for small $\lambda_{eff}$. From both expressions (4.36) and (4.46), we have plotted the persistent current for $(N-1)/2$ even on figure 8 for $\xi_K/l \approx 50$ ($\lambda_{eff} \approx 0.25$).
FIG. 8. Persistent current for \( N = 4p + 1 \) calculated for \( \xi_K/l \approx 50 \) (solid line) versus the \( J = 0 \) case (dashed line). The \( N = 4p + 3 \) case is obtained by a translation of \( \pi \) of the horizontal axis.

Notice that the corrections are weaker for \( N \) odd that for \( N \) even.

In the opposite limit \( \xi_K \ll l \), perturbation theory does not apply. However we can use the same reasoning as for \( N \) even in order to prove that

\[
\lim_{\xi_K \to 0} j = 0 \quad \text{for} \quad \xi_K \ll l, \quad (4.50)
\]

therefore no persistent current should be observed.

V. DISCUSSION AND CONCLUSION

In the two previous sections, we have calculated the persistent current in the two limits \( \xi_K/l \gg 1 \) and \( \xi_K \ll 1 \) for both the EQD and SCQD. Our results are summarized in figures 3, 4, 6, 8. In the embedded case, our results are actually very much different that those obtained in \[14\] by solving some approximate self-consistent equations. In particular, the current was predicted to be small for \( N \) odd for large \( l/\xi_K \) but not for small \( l/\xi_K \), the opposite of our result. Furthermore the \( \pi \) periodicity in \( \alpha \) that we find for odd \( N \) was not obtained even in the limit of small \( l/\xi_K \) where our perturbative calculations are robust. This variational approach seems to give the inverse behavior. We have obtained similar disagreement for the SCQD with ref. \[15\] and \[16\]. We refer to our previous paper \[10\] for a discussion of the validity of their results.

In this article, we have analyzed the validity and robustness of our results against various perturbations which may occur experimentally. Let us summarize our results and also discuss some other potential experimental limitations like disorder.

We have first analyzed how the results are modified if we consider electrons interactions inside the ring. From the theoretical point of view, particle-hole symmetry breaking terms are relevant for repulsive interactions. As a consequence, the gate voltage controlling \( \epsilon_d \) has to be tuned to resonance in order to reach a perfect transmittance. Therefore, it may imply experimentally that it might be difficult to reach the unitary limit and thus to observe the “saw-tooth” form of the persistent current which is expected for \( \xi_K \ll l \).

We have also treated the case of asymmetric tunneling amplitudes \( t_L \neq t_R \) between the EQD and the wires. This affects especially the strong coupling regime \( l \gg \xi_k \) because the transmission probability is no longer 1 and therefore the persistent current loses its “saw-tooth” shape. Such effect could disguise the cross-over between the regimes \( l \ll \xi_K \) to \( l \gg \xi_K \). From the experimental point of view, it would be important to tune the voltage gates \( V_l \) and \( V_R \) (see figure \[1\]) associated with \( t_L \) and \( t_R \) as close as possible.
to each other in order to observe such cross-over. Moreover, in the presence of repulsive interactions, the operator corresponding to the tunneling asymmetry becomes relevant at resonance. It would therefore imply experimentally one more parameter to tune.

We have supposed the wire to be relatively clean such that we can neglect the effect of non-magnetic impurities. One may wonder how our results are modified when we have both the quantum dot and just one non-magnetic impurity with a potential scattering $V$ localized at a distance $r$ from the quantum dot. In the regime $r < l \ll \xi_K$, we can easily generalize our perturbative calculations for both the EQD and SCQD. We expect the persistent current to be given by a scaling function of $\alpha, \xi_K/l, r/l, V$ in this limit:

$$jl = g(\alpha, \xi_k/l, r/l, V),$$

(5.1)

where $g$ is a universal function depending on the system we consider (EQD versus SCQD) but also on the parity of the system. We have explicitly checked for the EQD that the first correction to the persistent current is $O(JV^2)$ for $N$ even and $O(J^2V^2)$ for $N$ odd. In the other limit, $l > r \gg \xi_K$, it is reasonable to expect that the persistent current can be obtained by simply taking the limit $J \rightarrow \infty$. We can then easily calculate the transmission probability of the associated non interacting model. At half filling, the expression reduces to the one of the non-magnetic impurity: $T = v_F^2/(v_F^2 + V^2)$ (away from half filling, the expression is more complicated and has some $r$ dependence). Plugging this result into (3.39), provides the expression for the persistent current. The outcome is a decrease in the amplitude of the persistent current but especially the loss of the “saw-tooth” form here too. Such a result might also disguise a clear cross-over from $\xi_K \gg l$ to $\xi_K \ll l$ if $V$ is strong enough. From the experimental point of view, it seems quite important that the wires should be relatively clean. The fact that the unitary limit has been reached in [9] may indicate indeed that the wires are clean enough such that these effects are negligible.

We have assumed only one electronic channel but despite the narrowness of present quantum wires, several electronic channels can be activated. However, it is reasonable to expect that one channel will have a stronger tunneling amplitude to the dot than the others. The RG equations, to third order, for the multi-channel Kondo problem are:

$$d\lambda_i/d\ln l = \lambda_i^2 - (1/2)\lambda_i\sum_j \lambda_j^2.$$  

(5.2)

We see that if all but one of the couplings, $\lambda_1$, are small, while $\lambda_1$ is larger and positive, then we may approximate the equation for the small couplings by only the second term in Eq. (5.2), keeping only the $\lambda_i\lambda_j^2$ term. This equation then predicts that all the small couplings shrink. Meanwhile, the larger coupling grows. Therefore, the impurity spin is screened by an electron from the most strongly coupled channel and the other channels decouple at low energies. Thus, for $l \gg \xi_K$, we expect the single channel result obtained for the embedded quantum dot to still apply. Yet, we expect the results for the side coupled quantum dot to be fairly different. Indeed, if just one channel is screened by the spin impurity, the other channels do not feel the effects of the quantum dot in the scaling limit. The larger the number of channels, the less the transmission is affected by the dot. Therefore, if there are several active channel in a side coupled dot experiment, the observation of the cross-over between large $\xi_K/l$ to small $\xi_K/l$ could be severely concealed. In this respect, the embedded quantum dot would be a better candidate to detect the screening cloud through persistent current measurements.

In this paper we have shown that the persistent current in a mesoscopic ring coupled to a quantum dot is a highly sensitive function of the ratio of the ring circumference to the size of the Kondo screening cloud. On the other hand, electron interactions in the ring, asymmetric tunnelling amplitudes, disorder and the presence of several channels can all serve to mask this sensitivity. Thus, while the screening cloud size is an important length scale in quantum dot physics, its separation from various other effects is likely to be a challenging experimental problem.

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APPENDIX A: PERTURBATION THEORY FOR THE EMBEDDED QUANTUM DOT

1. \( N \) even

In this appendix, we calculate the ground state energy at second order in perturbation theory for \( N \) even. The correction to the ground state energy is given by:

\[
E_0^{(2)} = -\frac{J^2}{2!} \int_{-\infty}^{+\infty} d\tau \mathcal{T} |\chi_1^\dagger(\tau)\frac{\partial}{2}\mathbf{S} \chi(\tau)\chi_0^\dagger(0)| \frac{\partial}{2}\mathbf{S} \chi(0)|s>,
\]

where \( \chi \) has been defined in Eq. (3.6). The first contribution to \( E_0^{(2)} \), coming from the term involving 2 factors of \( \chi \) in \( H_{int} \), reads:

\[
-\frac{1}{2!} J^2 \text{tr} \left( \frac{\sigma^a}{2} \frac{\sigma^b}{2} \right) <s|S^a S^b|s> \int_{-\infty}^{+\infty} d\tau G(\tau)^2,
\]

where \( G(\tau) \) has been defined in Eq. (3.18). To evaluate the contribution of the cross-terms between \( \chi_0 \) and \( \chi \) we need:

\[
\mathcal{T} <|\chi_0^\dagger(\tau)\chi_0^\dagger(0)|\beta> = \frac{4}{l} \sin^2 k_F (1 - \cos \hat{\alpha}) [\delta_{\gamma\delta} \delta_{\beta\nu} - \theta(-\tau) \delta_{\nu\delta} \delta_{\beta\gamma}]
\]

\[
\mathcal{T} <|\chi_0(\tau)\chi_0(0)|\beta> = \frac{4}{l} \sin^2 k_F (1 - \cos \hat{\alpha}) [-\delta_{\gamma\delta} \delta_{\beta\nu} + \theta(\tau) \delta_{\nu\delta} \delta_{\beta\gamma}],
\]

where \( |\gamma\rangle \) labels the state with one electron of spin \( \gamma \) at the Fermi level. The full result for the ground state energy to second order in \( J \) can be written:

\[
E_0^{(2)} = -\frac{J^2}{2!} \int_{-\infty}^{+\infty} d\tau \sum_{a,b} \left\{ \text{tr} \left[ \frac{\sigma^a}{2} \frac{\sigma^b}{2} \right] <s|S^a S^b|s> \left[ G^2(\tau) + \frac{4}{l} \sin^2 k_F (1 - \cos \hat{\alpha}) G(\tau) \epsilon(\tau) \right] \ight. \\
+ \frac{4}{l} \sin^2 k_F (1 - \cos \hat{\alpha}) <s|c_{kF}^\dagger \left[ \frac{\sigma^a}{2} \frac{\sigma^b}{2} \right] c_{kF} \mathcal{T} |S^a(\tau)S^b(0)|s> G(\tau) \right\}.
\]

Here we have reinserted the creation operator at the Fermi surface, \( c_{kF} \) (at \( \tau = 0 \)) in order to facilitate comparison with the first order term. Now using the impurity spin Green’s function:

\[
\mathcal{T} <S^a(\tau)S^b(0)> = \frac{1}{4} \delta^{ab} + \frac{i}{2} \epsilon(\tau) \epsilon^{abc} S^c,
\]

this can be simplified to:

\[
E_0^{(2)} = -\frac{J^2}{2!} \int_{-\infty}^{+\infty} d\tau \left\{ \frac{3}{8} \left[ G^2(\tau) + \frac{4}{l} \sin^2 k_F (1 - \cos \hat{\alpha}) G(\tau) \epsilon(\tau) \right] \ight. \\
- \frac{4}{l} \sin^2 k_F (1 - \cos \hat{\alpha}) <s|c_{kF}^\dagger \left[ \frac{\sigma^a}{2} \frac{\sigma^b}{2} \right] c_{kF} \mathcal{T}|s> G(\tau) \epsilon(\tau) \right\}.
\]

We now wish to examine the behavior of these various terms at large \( l \). In the limit \( l >> v_F |\tau| \),

\[
G(\tau) \approx \frac{4 \sin^2 k_F}{\pi v_F^2}. 
\]

To estimate the large \( -l \) behavior of \( \int d\tau \epsilon(\tau) G(\tau) \) we may use this expression for \( \tau_0 \approx |\tau| < l/v_F \) where \( v_F \tau_0 \) is of order the lattice constant (1 in our units). At larger \( \tau \), \( G(\tau) \) decays to zero exponentially, as we see from Eq. (3.18). Thus,
\[\int_{-\infty}^{\infty} d\tau \epsilon(\tau) G(\tau) \to \frac{8\sin^2 k_F}{\pi v_F} \ln lc_1 + \cos \hat{\alpha} \ln 2, \tag{A8}\]

where \(c_1\) is a constant of \(O(1)\), independent of \(l\) at large \(l\). Now consider:

\[\int_{-\infty}^{\infty} G(\tau)^2 \approx \frac{16 \sin^4 k_F}{l^2} \int d\tau \left[ \frac{1}{\left( e^{\pi v_F |\tau|/l} - 1 \right)^2} + \frac{2 \cos \hat{\alpha}}{e^{2\pi v_F |\tau|/l} - 1} + \frac{\cos^2 \hat{\alpha}}{e^{2\pi v_F |\tau|/l} + 1} \right]. \tag{A9}\]

The integral of the first term is divergent at \(\tau = 0\). This just reflects our use of the large \(|\tau|\) form of \(G(\tau)\). Using the exact expression would give an integral of \(O(l^2)\), corresponding to a term of \(O(1)\) in the ground state energy. This is independent of \(\alpha\) however, and so does not contribute to the persistent current, using Eq. (3.4). Thus we have:

\[\int_{-\infty}^{\infty} G(\tau)^2 \to \frac{16 \sin^4 k_F}{l} \left[ 2 \cos \hat{\alpha} \left( \frac{\ln l}{\pi v_F} + \text{finite} \right) + \frac{\cos^2 \hat{\alpha}}{\pi v_F} (-1 + 2 \ln 2) \right] + \text{constant}. \tag{A10}\]

Important, the \(\ln l\) terms in the term proportional to \(\text{tr}\sigma^a\sigma^b\), in Eq. (A4), cancel. Now evaluating the various terms in Eq. (A6), we have:

\[E_0^{(2)} \to -\frac{J^2 \sin^4 k_F}{2l^2 \pi v_F} \left\{ \cos \hat{\alpha} \times \text{finite} - 6 \cos^2 \hat{\alpha} \right\} - (1 - \cos \hat{\alpha})2(\ln lc_1 + \ln 2 \cos \hat{\alpha}) \langle s|c^\dagger_{k_F} \frac{\hat{\sigma}}{2} c_{k_F} \cdot \tilde{S}|s \rangle + \text{constant}. \tag{A11}\]

Combining this with the term of \(O(J)\), gives:

\[E_0 = \frac{3\pi v_F}{4l} \left[ \cos \hat{\alpha} [\lambda + \lambda^2 \ln (lc)] + \cos^2 \hat{\alpha} (1/4 + \ln 2)\lambda^2 \right] + \text{constant}, \tag{A12}\]

where \(c\) is a dimensionless constant which we have not calculated explicitly, and \(\lambda\) is the dimensionless Kondo coupling defined in Eq. (2.11).

2. \(N\) odd

Let us write in this case an expression for the term in the ground state energy of third order in \(J\):

\[\beta E_0^{(3)} = \frac{J^3}{3!} \sum_{a,b,c} \text{tr} \left( \frac{\sigma^a \sigma^b \sigma^c}{2} \right) \mathcal{T} < S^a(\tau_1) S^b(\tau_2) S^c(\tau_3) > \cdot \int d\tau_1 d\tau_2 d\tau_3 G(\tau_2 - \tau_1) G(\tau_3 - \tau_2) G(\tau_1 - \tau_3). \tag{A13}\]

The factor of 2 arises from the two ways of ordering the three vertices. Here \(\beta\) is the inverse temperature which must be taken to \(\infty\). The integrals run between \(\pm \beta/2\). We now use:

\[\mathcal{T} < S^a(\tau_1) S^b(\tau_2) S^c(\tau_3) > = \frac{i}{8} \epsilon^{abc} \epsilon(\tau_1, \tau_2, \tau_3), \tag{A14}\]

where \(\epsilon(\tau_1, \tau_2, \tau_3) = 1\) if \(\tau_1 > \tau_2 > \tau_3\) and is completely antisymmetric in its arguments. Using Eq. (2.23) for \(G(\tau)\), we now get four terms proportional to \(\cos^n \alpha\) with \(n = 0, 1, 2, 3\). The \(n = 0\) term does not contribute to the persistent current so we ignore it. The \(n = 1\) and \(n = 3\) terms vanish because the integrands are antisymmetric. This leaves only the \(\cos^2 \alpha\) term, which is:

\[E_0^{(3)} \approx -\frac{3J^3 \cos^2 \alpha}{2l^3} \sin^6 k_F \int_{-\infty}^{\infty} d\tau_1 d\tau_2 \frac{\epsilon(\tau_1, \tau_2, 0)}{\cosh(\pi \tau_1 v_F/2l) \cosh(\pi \tau_2 v_F/2l) \sinh[\pi (\tau_2 - \tau_1)v_F/2l]} \tag{A15}\]

This integral is actually divergent at \(\tau_1 = \tau_2\), signaling the breakdown of the large \(\tau\) approximation to \(G(\tau)\). Integrating over \(v_F |\tau_1 - \tau_2| \leq l\) gives a contribution to the energy proportional to \(\ln l\). We may set
\( \tau_2 \) equal to \( \tau_1 \) inside the argument of the (non-singular) \( \cosh \) function, so that the logarithmic term can be obtained as:

\[
E^{(3)}_0 \approx \frac{3J^3 \cos^2 \alpha}{2l^3} \sin^6 kF \int d\tau_1 \frac{1}{\cosh^2(\pi v_F \tau_1/2l)} \int d\tau \frac{\epsilon(\tau)2l}{v_F(\tau)\pi},
\]

where the \( \tau \) integral should be taken over \( \tau_0 < |\tau| < l/v_F \), where \( \tau_0 v_F \) is a distance of order a lattice spacing (one in our units). Thus we obtain:

\[
E^{(3)}_0 = \frac{24J^3 \cos^2 \alpha}{(v_F\pi)^2} \sin^6 kF \ln(lc') + \text{constant},
\]

(A17)

where \( c' \) is another constant of \( O(1) \) which we have not determined. Combining this with the second order result, of Eq. (3.28) we obtain:

\[
E_0 = \cos^2 \alpha \frac{3\pi v_F}{16l} \left[ \lambda^2 + 2\lambda \ln(|c|) \right] + \text{constant}.
\]

(A18)

**APPENDIX B: \( J \rightarrow \infty \) TRANSMISSION PROBABILITY IN THE EMBEDDED QUANTUM DOT**

We can easily calculate the exact transmission coefficient for the model with \( \alpha + \pi = 0 \) in the Eq. (3.33). We simply solve the lattice Schrödinger equation:

\[
E \phi_j = -t(\phi_{j+1} + \phi_{j-1}), \quad (j = 3, 4, \ldots l - 3)
\]

\[
E \phi_2 = -(t/\sqrt{2})\phi_a - t\phi_3
\]

\[
E \phi_{l-2} = -t\phi_{l-3} - (t/\sqrt{2})\phi_a
\]

\[
E \phi_a = -(t/\sqrt{2})(\phi_2 + \phi_{l-2}).
\]

(B1)

The phase shift results from the reduction of the hopping matrix element by a factor of \( 1/\sqrt{2} \) at the site \( a \), again a consequence of the projection onto the odd state. We can write down the general solution of Eq. (B1), by inspection:

\[
\phi_j = A \epsilon(j) \sin(k(|j| - 1) + B \sin k|j|), \quad (|j| \geq 2),
\]

\[
\phi_a = B \sqrt{2} \sin k,
\]

(B2)

with \( \epsilon(j) = 1 \) if \( j > 0 \) and \(-1 \) if \( j < 0 \). In Eq. (B2), we label sites to the left of the impurity by negative integers, so \( \phi_j \) is the wave-function at site \( l + j \) with \( j = -2, -3, \ldots \). The allowed values of \( k \) depend on \( l \) but we don’t need to consider them explicitly since we just need the transmission probability. For suitably chosen \( A \) and \( B \) we can obtain a scattering solution of the form:

\[
\phi_j = e^{ikj} + \sqrt{Re^{i\delta_j}} e^{-ikj}, \quad (j \leq -2)
\]

\[
\phi_j = \sqrt{T} e^{i\delta_j} e^{ikj}, \quad (j \geq 2),
\]

with

\[
T(k) = \sin^2 k.
\]

(B3)

**APPENDIX C: PERTURBATION THEORY FOR THE SIDE COUPLED QUANTUM DOT**

1. Degenerate perturbation theory for \( N \) even

When \( \alpha \ll \pi \), we need to perform our perturbative calculations around a different ground state which is composed of the two levels near \( k_M = \pi N/4l \) (we have used \( k_M \) which is different from \( k_F \) defined in
we suppose that the states \( k \) momenta \( k \).

Notice that the log(\( J \)) terms in the first line of Eq. (C5) cancel. We find that the leading behavior at large \( l \) is:

\[
E_{pq}^{(2)} = -
\frac{3J^2}{4\pi v_F l} \ln l = -
\frac{3\pi v_F \lambda^2}{4l} \ln l,
\]

which is independent of \( p, q \) and renormalizes perfectly the matrix elements of Eq. (4.20). We have therefore proven that in the range \( \alpha \ll \pi \), the main effect of the higher levels is to renormalize the Kondo coupling, the infrared divergences being cut off by the size of the ring.

**2. Degenerate perturbation theory for N odd**

When \( \alpha \ll \pi \), the two levels near \( k_F = \pi(N - 1)/4l \) become almost degenerate (see Fig. 7). We want now to include the effects of the other levels in the Eq. (4.40). They are contributing to the kinetic energy but are also giving second order contributions in \( J \) in Eq. (4.40). We follow exactly the same scheme as for N even. Namely, we assume that the two levels 1 and 2 close to the Fermi surface are almost degenerate and have energy \( \epsilon_{1/2} = -2t \cos[k_F] \). By analogy with second order degenerate perturbation theory, we need to compute:

\[
E_{ij}^{(2)} = -
\frac{1}{2} \int d\tau \langle i | T \psi_{\text{int}}(\tau) \psi_{\text{int}}(0) | j \rangle.
\]
where $1 \leq i, j \leq 4$ correspond to the four states defined in $[4.33]$ which are taken to have the same ground state energy $\epsilon_i \approx \epsilon_F$ for $[C7]$ to make sense. It seems more convenient to first compute the quantities $A_{kl}^{(2)} = \frac{-J^2}{2} \int d\tau \left( \langle S_k | H_{\text{int}}(\tau) H_{\text{int}}(0) | S_l \rangle \right)$ where the states $|S_k\rangle$ define the basis

$$|S_1\rangle = |\uparrow, \uparrow; \psi\rangle ; \quad |S_2\rangle = |\uparrow, \downarrow; \not\psi\rangle ; \quad |S_3\rangle = |\downarrow, \uparrow; \not\psi\rangle ; \quad |S_4\rangle = |\downarrow, \downarrow; \not\psi\rangle ; \quad |S_5\rangle = |0, \uparrow, \not\psi\rangle$$  

As usual, we separate in the Fourier de-exponentiate of $\psi_i = \frac{1}{\sqrt{N}} \sum_k e^{ik\varphi} \psi_k$ the modes involving $k_{1/2} = k_F \mp \alpha/l$ from other modes, which we label by $\psi'$. The propagator involving the prime fields only is easily performed:

$$T \langle \psi'^\dagger(\tau) \psi'(0) \rangle = \frac{\varepsilon(\tau)}{l} \frac{2 \cosh(v_F \tau \alpha/l)}{\varepsilon(\tau)^2} = G(\tau) .$$  

(C8)

Let us define the electronic states $|r_i\rangle$ and $|r_j\rangle$ as follow:

$$|r_i\rangle = \psi^\dagger_{p_1, \gamma_1} \psi^\dagger_{p_2, \gamma_2} |0\rangle ; \quad \langle r_j| = \langle 0| \psi_{q_2, \gamma_1} \psi_{q_1, \beta_1} ,$$

where the momenta $p_1, p_2, q_1, q_2$ take the values $k_1$ or $k_2$ and $\gamma_1, \gamma_2, \beta_1, \beta_2$ are the associated spins (the vacuum $|0\rangle$ means the Fermi sea i.e which has levels with $|k| < k_f$ full). Let us also define the impurity state $|u_i\rangle$ such that

$$|S_i\rangle = |r_i\rangle \otimes |u_i\rangle$$

The other propagators we need are the following:

$$T \langle r_i | \psi^\dagger_{k, \epsilon}(\tau) \psi_{k, \nu}(0) | r_j \rangle = T \langle 0 | \psi_{q_2, \beta_2} \psi_{q_1, \beta_1} \psi^\dagger_{k, \epsilon}(\tau) \psi_{k, \nu}(0) \psi^\dagger_{p_1, \gamma_1} \psi^\dagger_{p_2, \gamma_2} | 0 \rangle$$

$$= \frac{1}{l} \left[ \delta_{k'p_1} \delta_{\nu \gamma_1} (\delta_{kq_2} \delta_{\beta_2, \beta_1} - \delta_{kq_2} \delta_{\beta_2, \beta_1}) + \delta_{k'p_2} \delta_{\gamma_2} (\delta_{kq_2} \delta_{\beta_2, \beta_1} - \delta_{kq_2} \delta_{\beta_2, \beta_1}) - \Theta(\tau) \delta_{kk'} \delta_{\epsilon \nu} \delta_{\alpha \beta} \right]$$  

(C9)

$$T \langle r_i | \psi_{k, \epsilon}(\tau) \psi^\dagger_{k, \nu}(0) | r_j \rangle = T \langle 0 | \psi_{q_2, \beta_2} \psi_{q_1, \beta_1} \psi_{k, \epsilon}(\tau) \psi^\dagger_{k, \nu}(0) \psi^\dagger_{p_1, \gamma_1} \psi^\dagger_{p_2, \gamma_2} | 0 \rangle$$

$$= \frac{1}{l} \left[ -\delta_{k'p_1} \delta_{\gamma_1} (\delta_{kq_2} \delta_{\beta_2, \beta_1} - \delta_{kq_2} \delta_{\beta_2, \beta_1}) - \delta_{k'p_2} \delta_{\gamma_2} (\delta_{kq_2} \delta_{\beta_2, \beta_1} - \delta_{kq_2} \delta_{\beta_2, \beta_1}) + \Theta(\tau) \delta_{kk'} \delta_{\epsilon \nu} \delta_{\alpha \beta} \right]$$  

(C10)

where $k, k' = k_1, k_2$. Using the expressions of various propagators, it can be shown after some algebra that

$$A_{S_i, S_j}^{(2)} = -\frac{J^2}{2} \int d\tau \sum_{a,b} \left\{ tr \frac{a^n b^b}{2} < u_i | S_a S_b | u_j \rangle \left[ G^2(\tau) + \frac{2}{l} G(\tau) e(\tau) \right] \right.$$

$$+ \frac{1}{l} G(\tau) \left[ \delta_{p_2 q_1} \delta_{\gamma_2 \beta_2} \left[ \frac{a^n b^b}{2} \right]_{\beta_2 \gamma_2} + \delta_{p_1 q_1} \delta_{\gamma_1 \beta_1} \left[ \frac{a^n b^b}{2} \right]_{\beta_1 \gamma_1} \right] - \delta_{p_1 q_1} \delta_{\gamma_1 \beta_1} \left[ \frac{a^n b^b}{2} \right]_{\beta_1 \gamma_1}$$

$$\left. - \delta_{p_2 q_1} \delta_{\gamma_2 \beta_2} \left[ \frac{a^n b^b}{2} \right]_{\beta_2 \gamma_2} \right\} T \langle u_i | S_a S_b(0) | u_j \rangle$$

(C11)

We are only focusing on terms leading to infrared divergences which will renormalize the Kondo coupling constant. Notice that the first line of (C11) do not give any logarithm contributions. The leading contribution at large $l$ can be then computed straightforwardly in all cases. We can establish

$$A_{11}^{(2)} = -\frac{J^2}{2 \pi v_F^2} \ln l c = -\frac{\pi v_F^2 \lambda^2}{4l} \ln l c ,$$

(C12)

$$A_{12}^{(2)} = A_{13}^{(2)} = \frac{J^2}{2 \pi v_F^2} \ln l c = \frac{\pi v_F^2 \lambda^2}{2l} \ln l c ,$$

(C13)

$$A_{14}^{(2)} = -A_{15}^{(2)} = \frac{J^2}{2 \pi v_F^2} \ln l c = -\frac{\pi v_F^2 \lambda^2}{2l} \ln l c ,$$

(C14)

$$A_{24}^{(2)} = A_{34}^{(2)} = -A_{25}^{(2)} = -A_{35}^{(2)} = -\frac{J^2}{4 \pi v_F^2} \ln l c = -\frac{\pi v_F^2 \lambda^2}{4l} \ln l c .$$

(C15)
All other matrix elements do not give any logarithm divergences. From these results, it is then easy to compute the matrix elements defined in (27) and to show that the Kondo coupling constant in the Eq. (4.40) gets perfectly renormalized namely $J/l$ has to be replaced by $\pi v_f (\lambda + \lambda^2 \ln|l_c|)/l$.

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