Evolution Equation for Generalized Parton Distributions

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Abstract

The extension of the method \textsuperscript{[1]} for solving the leading order evolution equation for Generalized Parton Distributions (GPDs) is presented. We obtain the solution of the evolution equation both for the flavor nonsinglet quark GPD and singlet quark and gluon GPDs. The properties of the solution and, in particular, the asymptotic form of GPDs in the small $x$ and $\xi$ region are discussed.

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I. INTRODUCTION

The outstanding problem in the theory of strong interactions is to understand how hadrons are built from quarks and gluons. The essential information on the internal structure of hadrons can be obtained from hard scattering processes. In such processes the duration of the interaction between a projectile and a fast moving hadron is small and the latter can be resolved into separate quarks and gluons (partons). Thus in hard scattering processes a hadron can be considered as a collection of non-interacting partons. This description of hadrons emerged from the study of deep inelastic scattering (DIS) and evolved into what is nowadays known as the parton model of hadrons and found its fundamental justification in QCD.

The key ingredient of the parton model are the parton densities which encode information on the distribution of partons with respect to longitudinal momentum. The recent trend in this field is to study processes which can provide additional information (e.g. distribution of partons in the transverse plane, their angular momentum) about the internal structure of hadrons. Such information is encoded in so-called Generalized Parton Distributions (GPDs) which were introduced in Refs. [2, 3, 4] in the analysis of Deeply Virtual Compton Scattering (DVCS). The GPD formalism is rather universal and applicable for the analysis of many different processes, ranging from completely inclusive to exclusive ones (for comprehensive reviews see Refs. [5, 6, 7]).

Since the GPDs are essentially nonperturbative quantities, no methods exist so far, which allow to obtain them directly from QCD. At present the only realistic way to get GPDs is to try to extract them from relevant experimental data [8, 9, 10] (which test only convolution of GPDs and specific kernels) and lattice data [11, 12, 13, 14] (which only can provide moments of GPDs). Probably only global fits will be selective enough to really determine more than the most dominant GPDs. Such fits naturally rely heavily on $Q^2$-evolution. This problem can be already treated within perturbative QCD. The equation governing the $Q^2$ dependence of GPDs – the evolution equation – is well known. Since the GPDs are related to matrix elements of certain operators, the evolution equation for GPDs follows from the renormalization group (RG) equation for the latter. In special cases the GPD evolution equation reduces to the famous DGLAP [16, 17, 18] and ERBL [19, 20] evolution equations. However, the general structure of solutions of the evolution equation for GPDs was poorly understood so far.

Several approaches [21, 22, 23, 24] were proposed in the past to get a solution for the general evolution equation for GPDs. All of them try to exploit the fact that Gegenbauer moments of GPDs have a simple scale dependence. However, the Gegenbauer polynomials do not form a complete set in the region where the GPDs are defined. This leads to the known problems when one tries to restore the GPDs by their Gegenbauer moments. The numerical algorithms based on these methods are rather cumbersome and ineffective (see for discussion Ref. [5]).

Recently, we presented a new method for solving the GPD evolution equation [1]. It is based on the correct incorporation of the symmetry properties of the evolution equation and provides a clear physical picture for the evolution. In Ref. [1] we considered the evolution equation for the gluon GPDs related to the matrix element of the twist two gluon operator and for simplicity neglected the effects of the mixing with quark-antiquark operators. In the present paper we give a detailed description of this approach for the example of the
flavor nonsinglet quark GPD and present the solution of the evolution equation for the flavor singlet GPD which takes into account mixing between quark and gluon operators. We would like to mention that an approach similar to ours was recently proposed in Ref. [25]. It will be briefly discussed in Sect. IV.

The paper is organized as follows. In the Sect. II we recall the definitions of GPDs and the corresponding evolution kernels. In Sect. III the symmetry properties of the evolution equations are discussed. In Sect. IV we consider in detail the evolution equation for the isovector quark GPD and construct its solution. In Sect. V we present the solution of the evolution equation for the singlet quark and gluon GPDs. Sect. VI contains the concluding remarks. In the Appendix some useful formulae are collected.

II. BACKGROUND

Throughout this paper we shall use the notations of Ref. [5]. The GPDs are usually defined in terms of matrix elements of certain non-local light-cone operators. We shall consider the isovector operator

$$O^a(z_1, z_2) = \bar{q}(z_1 n)\gamma^+ \tau^a q(z_2 n)$$

and quark and gluon isosinglet operators

$$Q(z_1, z_2) = \bar{q}(z_1 n)\gamma^+ q(z_2 n),$$

$$G(z_1, z_2) = G^{+\mu,i}(z_1 n)G^{+,i}_\mu(z_2 n).$$

Here $n$ is the light like vector, $n^2 = 0$, $\gamma^+ = n \cdot \gamma$ and $\tau^a$ ($a = 1, 2, 3$) are the Pauli matrices. In Eq. (2.2a) a summation over flavor indices is implied. The real coordinates $z_1, z_2$ specify the position of the operator on the light cone. As usual, we shall imply but do not display explicitly, the Wilson lines between the fields at points $z_1$ and $z_2$. Taking matrix elements of these operators between hadron states one obtains GPDs related to these operators. For example, the pion isovector quark GPD is defined as follows

$$i\epsilon^{abc} H(x, \xi, t) = \int \frac{dz}{2\pi} \epsilon^{iz2 \pi^+} \langle \pi^b(p')|O^c(-z, z)|\pi^a(p)\rangle,$$

where the kinematical variables are [5]

$$P = \frac{p + p'}{2}, \quad \xi = \frac{p^+ - p'^+}{p^+ + p'^+}, \quad t = (p - p')^2.$$

Henceforth we shall suppress the $t$–dependence of GPDs since it is irrelevant for the evolution. It is convenient to fix the normalization of the vector $n$ by the condition $(Pn) = P^+ = 1$. With such a normalization the matrix elements of the quark and gluon operators, Eqs. (2.1), (2.2a) and (2.2b), respectively, as well as the coordinates $z_1, z_2$ become dimensionless.

The isosinglet GPDs are defined in a similar way (see e.g. Ref. [5] for the definitions). We shall consider the nucleon GPDs

$$F^q(x, \xi) = \int \frac{dz}{2\pi} \epsilon^{iz2 \pi^+} \langle p'|Q(-z, z)|p\rangle,$$

$$F^g(x, \xi) = \int \frac{dz}{\pi} \epsilon^{iz2 \pi^+} \langle p'|G(-z, z)|p\rangle,$$
The functions $F^q(g)$ contain two different Lorentz structures and determine four scalar GPDs, $H^q, H^g$ and $E^q, E^g$. As the Lorentz structure of GPDs is irrelevant for the discussion of their evolution we shall treat the functions $F^q(g)$ as scalar functions. The GPDs $H, F^q, F^g$ have support in the $x$–interval $[-1, 1]$. The skewness parameter $\xi$ (Eq. (2.4)) is also restricted to the interval $[-1, 1]$ by kinematic, (in DVCS processes $\xi > 0$). It is standard to distinguish two different kinematical regions, $|\xi| < |x|$, the so-called DGLAP region, and $|\xi| > |x|$ – the ERBL or central region. In these regions GPDs describe different physical processes. In the central region a GPD describes the emission of a quark antiquark (gluon) pair from the initial hadron, while in the DGLAP region it describes the emission and absorption of a quark or an antiquark.

A. Evolution equations: Isovector case

All functions $H, F^q, F^g$ depend on the normalization scale $\mu$ at which the operators (2.1), (2.2a) and (2.2b) are defined. This dependence is governed by the renormalization group (RG) equation for the operators in question. We shall use the “coordinate space” version of this equation [26]. The RG equation for the operator $O^a$ at one loop order takes form

$$\left( \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right) O^a(z_1, z_2) = -\frac{\alpha_s}{\pi} [H O^a](z_1, z_2), \tag{2.6}$$

where $\alpha_s$ is the strong coupling constant and the operator $H$, which we shall refer to as Hamiltonian, is an integral operator. Eq. (2.6) represents a concise description of renormalization of the local composite operators. Indeed, expanding the l.h.s. and r.h.s of Eq. (2.6) in a Taylor series in $z_1, z_2$ one reproduces the RG equation for the local operators. The Hamiltonian $H$ (which encodes information on the mixing matrix) can be found in Ref. [26]. It is convenient to represent it in the following form

$$H = C_F \mathbb{H}, \tag{2.7}$$

$$\mathbb{H} = \mathbb{H}_1^q - \mathbb{H}_2^q + \mathbb{I}/2, \tag{2.8}$$

where $C_F = (N_c^2 - 1)/2N_c$ and the integral operators $\mathbb{H}_1^q, \mathbb{H}_2^q$ are defined as follows

$$[\mathbb{H}_1^q \varphi](z_1, z_2) = \int_0^1 d\alpha \frac{\bar{\alpha}}{\alpha} (2 \varphi(z_1, z_2) - \varphi(z_{12}^{\alpha}, z_2) - \varphi(z_1, z_{21}^{\alpha})), \tag{2.9a}$$

$$[\mathbb{H}_2^q \varphi](z_1, z_2) = \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \varphi(z_{12}^{\alpha}, z_{21}^{\beta}). \tag{2.9b}$$

We use here the standard notations $z_{ik}^{\alpha} = z_i \bar{\alpha} + z_k \alpha$, $\bar{\alpha} = 1 - \alpha$. As will be discussed later, both Hamiltonians are invariant under collinear conformal transformations and these two structures are the only structures which can appear in the Hamiltonian at one loop level (see Refs. [27, 28]). Also we shall see in which way these terms are responsible for the different evolution effects.

Let us now take the matrix element of both sides of Eq. (2.6). Then we get an equation for the function $\varphi^a(z_1, z_2) = \langle \text{out}|O^a(z_1, z_2)|\text{in}\rangle$, which is an ordinary function (not an operator) of two real variables $z_1, z_2$. Obviously it satisfies the same RG equation, (2.6), independently
of the choice of in and out states. Of course, the properties of the function $\varphi^a(z_1, z_2)$ depend strongly on the choice of the in and out states. Taking the matrix element between vacuum state and one hadron state gives the hadronic wave function, while taking the forward matrix element results in the parton densities. Usually one has to know all of these functions (wave functions, usual and generalized parton densities) not in coordinate space but in momentum space. In momentum space the RG equations look different for different physical situations. But it is useful to remember that all of them originate from one and the same RG equation for composite operators. In some sense the only difference between the DGLAP-, ERBL- and GPD-type evolution equation lies in the initial conditions for the evolution. The functions which one wants to evolve have different properties in all these cases. Therefore we prefer to work in the coordinate representation and only in the last step perform the transformation to momentum space.

For the isovector current we define the function $\varphi_\xi(z_1, z_2)$ by

$$i\epsilon^{abc}\varphi_\xi(z_1, z_2) = \langle \pi^b(p')|O^c(z_1, z_2)|\pi^a(p)\rangle.$$ (2.10)

It is related to the isovector GPD as follows

$$\varphi_\xi(z_1, z_2) = 2 e^{-i\xi(z_1+z_2)} \int dx e^{ix(z_1-z_2)} H(x, \xi).$$ (2.11)

The subscript $\xi$ of the function $\varphi$ indicates that this function has fixed total momentum. Under translation it transforms as $\varphi_\xi(z_1 + a, z_2 + a) = e^{-2i\xi a} \varphi_\xi(z_1, z_2)$. We also introduce a function given by the integral

$$\Phi_\xi(z) = \int dx e^{izx} H(x, \xi)$$ (2.12)

Provided that the GPD is a smooth function of $x$ the function $\Phi_\xi(z)$ (and $\varphi_\xi(z_1, z_2)$) vanishes fast for $z = z_1 - z_2 \to \pm\infty$. The function $\varphi_\xi(z_1, z_2)$, as a function of $z_1 + z_2$, is a plane wave. We shall present the solution of the evolution equation for the function $H(x, \xi)$, however, in order to deal with well defined expressions on intermediate steps it is convenient to regard the convolution of $\varphi_\xi(z_1, z_2)$ with some smooth function $\nu(\xi)$,

$$\varphi(z_1, z_2) = \int \frac{d\xi}{4\pi} \nu(\xi) \varphi_\xi(z_1, z_2).$$ (2.13)

One can always assume that this new function $\varphi(z_1, z_2)$ vanishes sufficiently fast as $z_1, z_2 \to \pm\infty$. Eq. (2.13) is a Fourier transform with respect to the total momentum $\xi$, therefore the function $\varphi(z_1, z_2)$ satisfies the same evolution equation (2.6). We postpone its solution till Sect. IV and discuss first the RG equation for the singlet quark and gluon GPDs.

**B. Evolution equations: Singlet case**

The quark and gluon operators (2.2a), (2.2b), mix under renormalization. However, the $C-$odd operator

$$Q^-(z_1, z_2) = Q(z_1, z_2) + Q(z_2, z_1)$$ (2.14)
can not mix with gluons and evolves autonomously according to Eq. (2.6), contrary to the $C$–even operator which we choose as

$$Q^+(z_1, z_2) = \frac{i}{2} (Q(z_1, z_2) - Q(z_2, z_1)) ,$$  \hspace{1cm} (2.15)

which mixes with the gluon operator (2.2b). To write down the RG equation in a compact form we introduce the vector notation $\mathcal{O} = \{\mathcal{O}^g, \mathcal{O}^q\}$ where $\mathcal{O}^g = G$ and $\mathcal{O}^q = Q^+$. Then one gets

$$\left( \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right) \mathcal{O}^i(z_1, z_2) = -\frac{\alpha_s}{\pi} [\mathcal{H}^{ik} \mathcal{O}^k](z_1, z_2) .$$  \hspace{1cm} (2.16)

The Hamiltonians $\mathcal{H}^{ik}$ can be found in Ref. [26]. Let us notice that we use here the conventional definition of the nonlocal gluon operator which differs by a sign from that used in Ref. [26]. It results in a change of sign for the off-diagonal kernels.

The Hamiltonian $\mathcal{H}^{gg}$ has the following form

$$\mathcal{H}^{gg} = N_c (\mathbb{H}^g_1 - \mathbb{H}^g_2) + \left( \frac{7}{6} N_c + \frac{1}{3} n_f \right) \mathbb{I} .$$  \hspace{1cm} (2.17)

The integral operators $\mathbb{H}^g_1, \mathbb{H}^g_2$ are defined as follows

$$[\mathbb{H}^g_1 \varphi](z_1, z_2) = \int_0^1 d\alpha \frac{\bar{\alpha}^2}{\alpha} (2 \varphi(z_1, z_2) - \varphi(z_{12}^0, z_2) - \varphi(z_1, z_{21}^0)) ,$$

$$[\mathbb{H}^g_2 \varphi](z_1, z_2) = \int_0^1 d\alpha \int_0^{\bar{\alpha}} d\beta \bar{\alpha} \bar{\beta} \omega \left( \frac{\alpha \beta}{\alpha \bar{\beta}} \right) \varphi(z_{12}^\alpha, z_{21}^\beta) ,$$

where the weight function is $\omega(\tau) = 4(1 + 2\tau)$.

The Hamiltonian $\mathcal{H}^{qq}$ coincides with $\mathcal{H}$, (see Eqs. (2.7), (2.8))

$$\mathcal{H}^{qq} = \mathcal{H} .$$  \hspace{1cm} (2.19)

The off-diagonal Hamiltonians read [26]

$$\mathcal{H}^{gq} = C_F A^{-1} (\mathbb{I} + 2 \mathbb{H}^g_2) ,$$  \hspace{1cm} (2.20)

$$\mathcal{H}^{qg} = n_f A \mathbb{H}^g_2 ,$$  \hspace{1cm} (2.21)

where $\mathbb{H}^g_2$ is defined in Eq. (2.9b) and the Hamiltonian $\mathbb{H}^g_2$ has the form (2.18a) with the weight function $\omega(\tau) = 1 + 3\tau$. The operator $A$ is the operator of multiplication by $(z_1 - z_2)$

$$[A \varphi](z_1, z_2) = (z_1 - z_2) \varphi(z_1, z_2) .$$  \hspace{1cm} (2.22)

Further, in full analogy with the isovector case we introduce the functions

$$f_{\xi}^g(z_1, z_2) = \langle p' | \mathcal{O}^i(z_1, z_2) | p \rangle .$$  \hspace{1cm} (2.23)

They are related to the nucleon GPDs (2.5a), (2.5b) as follows

$$f_{\xi}^g(z_1, z_2) = e^{-i \xi (z_1 + z_2)} \int dx e^{ixz_{12}} F^g(x, \xi) ,$$  \hspace{1cm} (2.24a)

$$f_{\xi}^q(z_1, z_2) = e^{-i \xi (z_1 + z_2)} \int dx e^{ixz_{12}} F^{q, +}(x, \xi) ,$$  \hspace{1cm} (2.24b)
where \( z_{12} = z_1 - z_2 \) and
\[
\mathcal{F}^{q,+}(x, \xi) = i \left( \mathcal{F}^{q}(x, \xi) - \mathcal{F}^{q}(-x, \xi) \right).
\] (2.25)

For later convenience we introduce the notations
\[
\mathcal{F}_1(x, \xi) \equiv \mathcal{F}^{q}(x, \xi) - \mathcal{F}^{q}(-x, \xi),
\] (2.26a)
\[
\mathcal{F}_2(x, \xi) \equiv \mathcal{F}^{g}(x, \xi).
\] (2.26b)

Convoluting \( f_{\xi}^{i} \) with a smooth function \( \nu^{i}(\xi) \) one gets the function
\[
f^{i}(z_1, z_2) = \int \frac{d\xi}{2\pi} \nu^{i}(\xi) f_{\xi}^{i}(z_1, z_2)
\] (2.27)
which vanishes for \( z_1, z_2 \to \pm \infty \). Let us note also that the function \( f^{g}(z_1, z_2) (f^{q}(z_1, z_2)) \) is symmetric (antisymmetric) under permutation of variables, \( z_1 \leftrightarrow z_2 \) and this symmetry is preserved by the evolution.

C. Local operators

Let us discuss now the problems which arise in solving the evolution equation for GPDs. For the sake of simplicity we consider the isovector operator \( \mathcal{O}^{a}(z_1, z_2) \). As we said above the nonlocal operator \( \mathcal{O}^{a}(z_1, z_2) \) should be understood as the generating function for the local composite operators,
\[
\mathcal{O}^{a}(z_1, z_2) = \sum_{k,m=0}^{\infty} z_{12}^{k} \mathcal{D}_{k,m}^{a} \mathcal{O}^{a}_{k,m},
\] (2.28)
where \( \mathcal{O}^{a}_{k,m} \) is \((nD)^{k}q^{0}(0)\gamma^{+}\tau^{a}(nD)^{m}q(0)/k!m! \) and \( D_{\mu} \) is the covariant derivative. Inserting this expansion into Eq. \ref{eq:2.6} one gets an infinite set of equations describing the renormalization of the local operators. Solving these equations one finds a set of the multiplicatively renormalized local operators of twist two. They are
\[
\mathcal{O}_{N,k}^{a}(0) = (i\partial_{+})^{N+k}q^{0}(0)\gamma^{+}\tau^{a}C_{N}^{3/2} \left( \frac{\tilde{D}_{+} - \tilde{D}_{-}}{\tilde{\partial}_{+} + \tilde{\partial}_{-}} \right) q(0),
\] (2.29)
where \( C_{N}^{3/2} \) is a Gegenbauer polynomial. These operators enjoy an autonomous scale dependence to one-loop order
\[
\left( \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right) \mathcal{O}_{N,k}^{a}(0) = -\frac{\alpha_{s}}{2\pi} \gamma_{N} \mathcal{O}_{N,k}^{a}(0),
\] (2.30)
where \( \gamma_{N} \) is the anomalous dimension of the operator. One can rearrange the expansion \ref{eq:2.28} in the following way \ref{eq:2.28} \ref{eq:2.29}
\[
\mathcal{O}^{a}(z_1, z_2) = \sum_{N=0}^{\infty} B_{N} i^{N}(z_1 - z_2)^{N} \int_{0}^{1} du(u\bar{u})^{N+1} \mathcal{O}_{N}^{a}(z_{12}^{u}).
\] (2.31)
Here \( B_N = 2(2N + 3)/(N + 1)! \) and \( \mathcal{O}^2_N(z) \equiv \mathcal{O}^2_{N,0}(z) \). Inserting the expansion (2.31) into (2.10) one gets

\[
\varphi_\xi(z_1, z_2) = \sqrt{2\pi} e^{-i\xi(z_1 + z_2)} \sum_{N=0}^\infty i^N (2N + 3) c_N(\xi) \Psi_N(z_{12}), \tag{2.32}
\]

where \( z_{12} = z_1 - z_2 \) and

\[
\Psi_N(z_{12}) = (\xi z_{12})^{-3/2} J_{N+3/2}(\xi z_{12}), \tag{2.33a}
\]

\[
c_N(\xi) = \int_{-1}^1 dx \, C_N^{3/2} \left( \frac{x}{\xi} \right) H(x, \xi). \tag{2.33b}
\]

Here \( J_\nu(z) \) is the Bessel function. The coefficient \( c_N(\xi) \) is related to the matrix element of the local operator \( \mathcal{O}^c_N \) as follows

\[
ie^{abc} \xi^N c_N(\xi) = 2^{-N-1} \langle \pi^b(p')|\mathcal{O}^c_N(0)|\pi^a(p) \rangle. \tag{2.34}
\]

Taking into account the tensor structure of \( \mathcal{O}^c_N(0) \) one concludes that the coefficient \( \xi^N c_N(\xi) \) is a polynomial in \( \xi \) of degree \( N \). This is the so-called polynomiality condition. Since the operator \( \mathcal{O}^c_N(0) \), Eq. (2.29), is multiplicatively renormalized its matrix element depends on the scale in a simple manner

\[
c^\mu_\nu\xi(\xi) = c^\mu_\nu(\xi) L^{-\gamma/b_0}, \tag{2.35}
\]

where \( L = \alpha_s(\mu_1)/\alpha_s(\mu_2) \), and \( b_0 = \frac{\Pi}{2} N_c - \frac{2}{3} N_f \). Thus multiplying the coefficients \( c_N(\xi) \) in Eq. (2.32) by corresponding exponents one gets the solution of the evolution equation (2.6).

The representation (2.32) was obtained first in Ref. [21].

The series in (2.32) converges for any \( z_{12} \) but it converges non-uniformly with respect to \( z_{12} \) what causes some problems. Namely, in the Fourier transform of (2.32) it is not possible to interchange the integration and summation. Indeed, let us note that the Fourier transform of the function \( \Psi_N(z) \)

\[
i^N \int_{-\infty}^\infty dz e^{-iz\xi} \Psi_N(z) = \frac{\sqrt{2\pi} \xi^{-1}}{(N + 1)(N + 2)} \theta(x^2 - \xi^2) \left( 1 - \frac{x^2}{\xi^2} \right) C_N^{3/2} \left( \frac{x}{\xi} \right), \tag{2.36}
\]

has support in the central region \( |x| \leq |\xi| \) only, while the Fourier transform of \( \varphi_\xi(z_1, z_2) \) is nonzero in the whole \( x-\)region, \([-1, 1]\). In other words taking the Fourier transform of (2.32) one obtains a formal representation for the GPD \( H(x, \xi) \)

\[
H(x, \xi) = \frac{1}{\xi} \sum_{N=0}^\infty \omega_N c_N(\xi) \theta(x^2 - \xi^2) \left( 1 - \frac{x^2}{\xi^2} \right) C_N^{3/2} \left( \frac{x}{\xi} \right), \tag{2.37}
\]

where

\[
\omega_N = \frac{1}{\xi} \frac{2N + 3}{2(N + 1)(N + 2)},
\]

but this series diverges if \( H(x, \xi) \) is nonzero in the DGLAP region \( |x| > \xi \). Indeed, taking into account that for large \( N \) and \( t > 1 \), \( C_N^{3/2}(t) \sim (t + \sqrt{t^2 - 1})^N \) one concludes that the coefficients \( c_N(\xi) \) grow for large \( N \) as \( \xi^{-N} \) and one gets a divergence.
One can try to give sense to the series (2.37) by interpreting it as a distribution, i.e. one has at first to carry out a convolution with some test function and then to take the sum provided that the latter converges. However, it is clear that the distribution defined in a such way, \( \tilde{H}(x, \xi) \), is different from \( H(x, \xi) \), since the former has support in the central region only. The problem with the Fourier transform of Eq. (2.32) is related to the fact that the Gegenbauer polynomials \( C^{3/2}_N(x/\xi) \) are not orthogonal in the interval \([-1, 1]\) provided \(|\xi| < 1\). Nevertheless they form a complete set and the function \( H(x, \xi) \) can be restored by its Gegenbauer moments \( c_N(\xi) \), (see Eq. (2.33b)). The corresponding algorithm was suggested in Refs. [24, 30] and relies on the reexpansion of some system of orthogonal polynomials on the interval \([-1, 1]\), say Legendre polynomials, in terms of Gegenbauer polynomials, \( C^{3/2}_N(x/\xi) \),

\[
P_n(x) = \sum_{N=0}^{n} r^N_n(\xi) \xi^N C^{3/2}_N(x/\xi) .
\]

Obviously, one can also express the Legendre moments in terms of the Gegenbauer moments

\[
p_n(\xi) = \sum_{N=0}^{n} r^N_n(\xi) \xi^N c_N(\xi) . \tag{2.38}
\]

Then the series \( \sum_{n=0}^{\infty} (n + 1/2) p_n(\xi) P_n(x) \) converges to the function \( H(x, \xi) \). Since the scale dependence of the coefficients \( c_N(\xi) \) is known (Eq. (2.35)) one can calculate the coefficients \( p_n(\xi) \), and as a consequence the GPD \( H(x, \xi) \), at any scale. However, an attempt to apply this scheme in practice reveals the following problem – the coefficients \( r^N_n(\xi) \) grow fast with \( n \). The explicit expression for the coefficients \( r^N_n(\xi) \) can be found in [21, 24]. For the sake of clarity we consider the limiting case \( \xi = 0 \). In this limit one finds that \( \xi^N c_N(\xi) \to 2^N \bar{c}_N \), where

\[
\bar{c}_N = \frac{2\Gamma(N + 3/2)}{\sqrt{\pi N!}} \int_{-1}^{1} dx x^N H(x, 0) .
\]

The remaining dependence of the coefficients \( \bar{c}_N \) on \( N \) is then weak. The coefficients \( r^N_n \) is nonzero only if \( N \) and \( n \) have the same parity, (\( n - N = 2k \)). Denoting \( \bar{r}^N_n = 2^N r^N_n(0) \) one
The logarithm of the coefficients $|\tilde{r}_n^N|$ as function of $N$ for $n = 50$ is plotted in Fig. 1. One sees that in the sum

$$p_n = \sum_{N=0}^{n} \tilde{r}_n^N \tilde{c}_N$$

each term is large while the sum itself should be a number of order $O(1)$. Thus to calculate the coefficients $p_n$ with a reasonable accuracy one has to calculate the coefficients $\tilde{c}_N$ with “infinite” accuracy. It is clear that a numerical algorithm based on such a reexpansion sooner or later will have an accuracy problem. Thus, the methods for solving GPD evolution equations suggested in Refs. [21, 24, 30], although being mathematically correct, result in inefficient numerical algorithms. In what follows we suggest another method to solve the GPD evolution equation which relies heavily on the symmetry properties of the latter.

### III. SYMMETRY PROPERTIES

As it is well known the classical QCD Lagrangian is invariant under conformal transformations. This symmetry, however, does not survive in the full quantum theory due to the renormalization effects. Nevertheless, at one-loop level the counterterms inherit the symmetry of the classical Lagrangian (for a review see e.g. Ref. [31]). This means that if two local (gauge invariant) operators are related to each other by a symmetry transformation, $O^2 = \delta_\epsilon O^1$, their one loop counterterms are related by the same symmetry transformation, $\Delta O^2 = \delta_\epsilon \Delta O^1$. If one uses the formulation in terms of nonlocal operators this statement is translated into a statement about the invariance of the evolution kernel (Hamiltonian) with respect to the symmetry (conformal) transformations.

We consider operators of a special type — the symmetric traceless operators of twist two (Eqs. (2.1), (2.2)). This set of operators is closed under renormalization. The restriction of the full conformal group to this class of operators gives the so-called collinear conformal group which is the ordinary $SL(2,\mathbb{R})$ group. The generators of the $SL(2,\mathbb{R})$ group, $S^+, S^-, S^0$, satisfy the commutation relations

$$[S^+, S^-] = 2S^0, \quad [S^0, S^\pm] = \pm S^\pm$$

and can be chosen as

$$S^+ = z^2 \partial_z + 2sz, \quad S^- = -\partial_z, \quad S^0 = z\partial_z + s.$$  (3.2)

Here the parameter $s$, the conformal spin, specifies the representation of the $SL(2,\mathbb{R})$ group. Under an infinitesimal $SL(2,\mathbb{R})$ transformation a nonlocal operator transforms as follows

$$\delta_\epsilon \mathcal{O}(z_1, z_2) = \epsilon_\alpha (S^\alpha_1 + S^\alpha_2) \mathcal{O}(z_1, z_2),$$  (3.3)

where the spin operators should be taken in the representation corresponding to the conformal spin $s = 1$ for the quark operator (both singlet and nonsinglet), Eqs. (2.1), (2.2a), and
to the conformal spin $s = 3/2$ for the gluon operator, Eq. (2.2b). For later convenience we introduce the following notations for the two-particle generators

\[ S^\alpha_q = (S^\alpha_1 + S^\alpha_2)_{s=1}, \quad \text{(3.4a)} \]
\[ S^\alpha_g = (S^\alpha_1 + S^\alpha_2)_{s=3/2}. \quad \text{(3.4b)} \]

Since, as was discussed above, the conformal symmetry survives at one loop level one gets

\[ [\mathcal{H}, S^\alpha_q] = 0 \quad \text{(3.5)} \]

for the Hamiltonian (2.7). For the singlet case the Hamiltonian has the matrix form (see Eqs. (2.16)) therefore the symmetry relation takes form

\[ \sum_{k=1,2} [\mathcal{H}^{ik}, S^\alpha_{kj}] = 0, \quad \text{(3.6)} \]

where the matrix $\mathbf{S}$ is defined as

\[ S^\alpha = \begin{pmatrix} S^\alpha_g & 0 \\ 0 & S^\alpha_q \end{pmatrix}. \quad \text{(3.7)} \]

For the individual components Eq. (3.6) reads

\[ [\mathcal{H}^{gg}, S^\alpha_g] = 0, \quad [\mathcal{H}^{qq}, S^\alpha_q] = 0, \quad \text{(3.8a)} \]
\[ S^\alpha_g \mathcal{H}^{gg} = \mathcal{H}^{gg} S^\alpha_g, \quad S^\alpha_q \mathcal{H}^{qq} = \mathcal{H}^{qq} S^\alpha_q. \quad \text{(3.8b)} \]

To solve the evolution equations it is helpful to take this symmetry into account.

Let us note that the transformations (3.3) are the infinitesimal form of the finite \((SL(2,R))\) transformations

\[ \mathcal{O}(z_1, z_2) \rightarrow (cz_1 + d)^{-2s}(cz_2 + d)^{-2s}\mathcal{O}(z'_1, z'_2), \quad \text{(3.9)} \]

where $z'_k = (az_k + b)/(cz_k + d)$, $ad - bc = 1$, and the spin $s = 1(3/2)$ for the quark (gluon) operators. To proceed we recall some facts about the representations of the \(SL(2,R)\) group.

A. \(SL(2,R)\) group

The \(SL(2,R)\) group is the group of the real unimodular $2 \times 2$ matrices,

\[ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ab - cd = 1. \quad \text{(3.10)} \]

A detailed description of the representations of \(SL(2,R)\) group can be found in [32, 33]. Since the \(SL(2,R)\) group is a noncompact group its unitary representations are infinite-dimensional. They can be organized into three series, namely the discrete series, the principal and supplementary continuous series [32]. The last one will not appear in our analysis.

The representation of the unitary principal continuous series is determined by two parameters - the conformal spin $s$, which takes values $s = 1/2 + i\rho$, where $\rho$ is a real number;
and by a discrete parameter, the so-called signature $\epsilon = 0, 1/2$. We need to consider the representations with signature $\epsilon = 0$ only. In this case, the representation, which is denoted by $T^\rho$, can be realized by unitary operators

$$
T(g^{-1})\psi(z) = \frac{1}{|cz + d|^{2s}}\psi(z'), \quad z' = \frac{az + b}{cz + d},
$$

acting on the Hilbert space of functions of a real variable equipped with the scalar product

$$
\langle \psi_1 | \psi_2 \rangle = \int_{-\infty}^{\infty} dz \, \overline{\psi_1(z)} \psi_2(z).
$$

The generators in this representation have the form with conformal spin $s = 1/2 + i\rho$.

The representations of the discrete series are characterized by the integer or half-integer conformal spin $s$, $s = 1/2, 1, 3/2, \ldots$ and by the transformation law

$$
T(g^{-1})\psi(z) = \frac{1}{(cz + d)^{2s}}\psi(z').
$$

They fall into two classes usually denoted by $D^\pm_s$. The difference between them lies in the space of functions they are defined on. The representations $D^+_s(D^-_s)$ are defined on the functions of the real variable which can be continued analytically to the upper(lower) complex half-plane. The scalar product in these cases reads

$$
\langle \psi_1 | \psi_2 \rangle^\pm = \frac{2s - 1}{\pi} \int_{\pm \text{Im} z \geq 0} d^2z (2\text{Im} z)^{2s-2} \overline{\psi_1(z)} \psi_2(z).
$$

It is clear that functions from the representation space of $D^\pm_s$ can be represented in the form of a Fourier integral

$$
\psi^\pm(x) = \int_0^\infty dp \, e^{\pm ipx} \psi^\pm(p).
$$

In the momentum representation the scalar product reads

$$
\langle \psi_1 | \psi_2 \rangle^\pm = \Gamma(2s) \int_0^\infty dp \, p^{-1-2s} \overline{\psi_1(p)} \psi_2(p).
$$

One sees that in order to have a finite norm $\psi(p)$ has to vanish at $p = 0$.

The tensor product of two unitary representations can always be decomposed into irreducible components. It will become clear later that we are interested in the tensor product decomposition of the representations of the discrete series, $D^\pm_s$. This decomposition depends strongly on whether one multiplies the representations of the same type or not. Namely, the decomposition of the tensor product $D^+_s \otimes D^+_s$ (or $D^-_s \otimes D^-_s$) contains only representations of the same type

$$
D^\pm_s \otimes D^\pm_s = \sum_{n=0}^{\infty} \oplus D^\pm_{n+2s}.
$$

In contrast, the tensor product $D^+_s \otimes D^-_s$ (or $D^-_s \otimes D^+_s$) can be decomposed into direct integral of the representations of the unitary principal continuous series

$$
D^\pm_s \otimes D^\pm_s = \int_0^\infty d\rho \oplus T^\rho.
$$
Now let us try to understand the group theoretical properties of GPDs in more details. To be concrete we consider the quark GPD. In coordinate representation the GPD \( \varphi(z_1, z_2) \) inherits the symmetry properties of the operator \( \mathcal{O}(z_1, z_2) \), Eq. (3.9). Since the conformal spin \( s = 1 \), this transformation corresponds to the tensor product of the discrete series representations of the \( SL(2, R) \) group. Let us represent the quark and antiquark fields by

\[
q(zn) = q^+(zn) + q^-(zn),
\]
\[
\bar{q}(zn) = \bar{q}^+(zn) + \bar{q}^-(zn),
\]

(3.19)

where the components \( q^+ \) and \( \bar{q}^+ \) contain the creation operators and thus, only positive Fourier harmonics, \( e^{i(pn)z}, p > 0 \), while the components \( q^- \) and \( \bar{q}^- \) contain the annihilation operators (negative Fourier harmonics, \( e^{-i(pn)z}, p > 0 \)). Then, it is clear that e.g. the matrix element \( \langle p' | \bar{q}^- (z_1 n) \gamma^+ q^- (z_2 n) | p \rangle \) as a function of two real variables, \( z_1, z_2 \), admits an analytic continuation to the lower half-plane for each argument. Therefore, the GPD \( \varphi(z_1, z_2) \) can be represented as the sum of four functions

\[
\varphi(z_1, z_2) = \varphi^{++}(z_1, z_2) + \varphi^{+-}(z_1, z_2) + \varphi^{-+}(z_1, z_2) + \varphi^{--}(z_1, z_2),
\]

(3.20)

where \( \varphi^{\alpha\beta} \sim \langle p' | q^+ (z_1 n) \gamma^+ q^- (z_2 n) | p \rangle \) and \( \alpha(\beta) = \pm \). Each of the functions \( \varphi^{\alpha\beta}(z_1, z_2) \) transforms according to the tensor product of the representations \( D^\alpha_s \otimes D^\beta_s \), with \( s = 1 \). We notice here that the functions \( \varphi^{\pm+}(z_1, z_2) \) being transformed to the momentum representation have support in the central region, while the functions \( \varphi^{\pm-}(z_1, z_2) \) in the DGLAP one. Since the form of the tensor product decomposition \( D^\alpha \otimes D^\beta \) depends on whether \( \alpha = \beta \) or \( \alpha \neq \beta \) one has to decompose the functions \( \varphi^{\alpha\beta} \) over different set of functions in these two cases.

Let us note that our conclusion is in agreement with the solutions of the evolution equation in the two limiting cases, \( \xi = 1 \) and \( \xi = 0 \). In the first one, \( \xi = 1 \), when the DGLAP region shrinks to zero, to solve the evolution equation one has to use the expansion in Gegenbauer polynomials, in agreement with the decomposition (3.17). In the case of forward scattering, \( \xi = 0 \), one should use a Mellin transform to solve the DGLAP evolution equation. This, in turn, agrees with the decomposition (3.18).

All this shows that the two kinematical regions, "DGLAP" (\( |x| > |\xi| \)) and "central" (\( |x| < |\xi| \)) are quite different, both from the physical and mathematical point of view and, therefore, should be treated in a different way.

IV. QUARK GPD

In this section we consider in detail the case of the quark GPD (2.3). We shall construct the solution of the evolution equation for \( H(x, \xi) \). As it was explained earlier it is convenient to start analysis from the evolution equation in the coordinate representation

\[
\left( \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right) \varphi(z_1, z_2) = -\frac{\alpha_s}{\pi} [\mathcal{H} \varphi](z_1, z_2),
\]

(4.1)

where the Hamiltonian \( \mathcal{H} \) and function \( \varphi(z_1, z_2) \) are defined in Sect. II.A.

We remind that the conventional strategy to solve equations of this type is to expand the function \( \varphi(z_1, z_2) \) in eigenfunctions of the Hamiltonian \( \mathcal{H} \), \( \varphi(z_1, z_2) \sim \sum_E c_E \psi_E \). If such an expansion is found, the dependence of the expansion coefficients \( c_E \) on the scale \( \mu \) can be
easily restored and, provided that the corresponding sum can be effectively calculated, this determines \( \varphi \) at any scale \( \mu \). We shall try to follow this scheme.

We start the discussion with the following remark. In order to talk rigorously about expansion in eigenfunctions of the Hamiltonian \( H \), one should specify the Hilbert space in which the eigenvalue problem for the operator has to be solved. In other words one has to define the scalar product on the space of functions \( \varphi(z_1, z_2) \). It is clear that this construction (scalar product, Hilbert space) is external with respect to the problem. Indeed, Eq. (4.1) is an integro-differential equation which knows nothing about the Hilbert space. However, to solve this equation it is convenient to introduce the structure of a Hilbert space, i.e. a scalar product, and its choice is almost completely in our hands.

The only condition which restricts the choice of the scalar product is the requirement that the function \( \varphi(z_1, z_2) \) belongs to the Hilbert space, i.e. it has to be normalizable with respect to the chosen scalar product. This is the place where physics imposes some constraints. The other requirements (such as the \( SL(2, R) \) invariance in the case under consideration) are useful but not mandatory.

We choose the following \( SL(2, R) \) invariant scalar product

\[
\langle \varphi_1 | \varphi_2 \rangle = \int dz_1 dz_2 (z_1 - z_2)^2 \varphi_1(z_1, z_2) \varphi_2(z_1, z_2).
\]

(4.2)

Obviously, it is invariant under \( SL(2, R) \) transformations

\[
\varphi_k(z_1, z_2) \rightarrow (cz_1 + d)^{-2s}(cz_2 + d)^{-2s} \varphi_k(z'_1, z'_2),
\]

(4.3)

with \( s = 1 \) and \( z' = (az + b)/(cz + d) \). Going to the momentum representation

\[
\varphi(z_1, z_2) = \int \frac{d\xi dx}{2\pi} e^{i(-\xi(z_1 + z_2) + x(z_1 - z_2))} \varphi(x, \xi)
\]

(4.4)

one gets for the norm of the function \( \varphi \)

\[
||\varphi||^2 = \frac{1}{2} \int d\xi dx |\partial_x \varphi(x, \xi)|^2.
\]

(4.5)

Then, taking into account Eqs. (4.11), (4.14) one concludes that the physical GPD has a finite norm (4.5) if the integral \( \int dx |\partial_x H(x, \xi)|^2 \) is finite. Since the GPD \( H(x, \xi) \), as a function of \( x \), has support on the interval \([-1, 1]\) this implies that it should vanish faster than \((1 \pm x)^{1/2}\) at the endpoints and be continuous inside this interval. These requirements are in agreement with the standard assumptions on the properties of quark GPDs (see e.g. Refs. [34, 35]).

Let us remark here that the other possible choice for the invariant scalar product

\[
||\varphi||^2 \sim \int d\xi dx \frac{|\varphi(x, \xi)|^2}{|x^2 - \xi^2|},
\]

(4.6)

is ruled out. Indeed, in this case the function \( \varphi(x, \xi) \) in order to be normalizable has to vanish at \( x = \pm \xi \), what is physically unaccepteable.

Next, it is useful to introduce a new function \( \psi(z_1, z_2) \) related to \( \varphi(z_1, z_2) \) by the simple relation

\[
\psi(z_1, z_2) = [A \varphi](z_1, z_2) = (z_1 - z_2) \varphi(z_1, z_2).
\]

(4.7)
One can easily check that if the function \( \varphi \) transforms according to (4.3) the function \( \psi \) obeys the same transformation law with \( s = 1/2 \). The map \( A : \varphi \to \psi \) is a one to one isometric map of the Hilbert space defined by the scalar product (4.2) to the standard, \( L^2(R \times R) \) Hilbert space,

\[
\langle \psi_1 | \psi_2 \rangle = \int dz_1 dz_2 \overline{\psi_1(z_1, z_2)} \psi_2(z_1, z_2) .
\tag{4.8}
\]

The transformation of the Hilbert space \( L^2(R \times R) \)

\[
\psi(z_1, z_2) \rightarrow (c z_1 + d)^{-1} (c z_2 + d)^{-1} \psi(z'_1, z'_2) ,
\tag{4.9}
\]
determines the unitary representation of the \( SL(2, R) \) group which is equivalent to the tensor product of the representations

\[
D^\uparrow_{1/2} = \left( D^+_{1/2} \oplus D^-_{1/2} \right) \otimes \left( D^+_{1/2} \oplus D^-_{1/2} \right) .
\tag{4.10}
\]

According to the Eqs. (3.17) and (3.18) the decomposition of this representations into irreducible ones has form

\[
D^\uparrow_{1/2} = \sum_{n=0}^\infty D^\pm_{n+1} + 2 \int_0^\infty d\rho \otimes T^\rho .
\tag{4.11}
\]

### A. Eigenvalue problem

We remind that our purpose is to construct the expansion of the GPD \( \varphi(z_1, z_2) \) in terms of eigenfunctions of the Hamiltonian (2.7). Unfortunately, one can check that the Hamiltonian \( \mathcal{H} \) (Eq. (2.7)) is non-hermitian with respect to the scalar product (4.2). It is possible to find its eigenfunctions, (it will be done later) but they do not form a complete set and, in general, it is not clear how to construct the expansion over this set. Nevertheless such an expansion can be constructed. To do so we use the following trick. As was explained before, the Hamiltonian commutes with the generators of the \( SL(2, R) \) transformations (see Eq. (3.5)), and as a consequence it commutes with the Casimir operator of the \( SL(2, R) \) group, \([\mathcal{H}, \mathbf{J}^2] = 0\),

\[
\mathbf{J}^2 = (\vec{S}_1 + \vec{S}_2)^2 = S^+_1 S^-_1 + S^+_2 S^-_2 + (S^0_1 - I) ,
\tag{4.12}
\]

where \( S^0_{12} = S^\alpha_1 + S^\alpha_2 \) and \( I \) is the identity operator. Taking into account the explicit form of the generators (3.2) one finds that in the case \( s_1 = s_2 = s \) the Casimir operator takes form

\[
\mathbf{J}^2 = -z^2_{12} - 2s \partial_1 \partial_2 z^2_{12} ,
\tag{4.13}
\]

where \( z_{12} \equiv z_1 - z_2 \). For \( s = 1 \) this operator is a self-adjoint operator with respect to the scalar product (4.2) (and for \( s = 1/2 \) with respect to the scalar product (4.8)). So one can expand any function over the set of eigenfunctions of the Casimir operator. Nevertheless, the operators \( \mathcal{H} \) and \( \mathbf{J}^2 \), despite the fact that they commute, have different eigenfunctions. However, one of the crucial points of our approach is that, as we shall show, the expansion in eigenfunctions of the Casimir operator can be transformed into an expansion in eigenfunctions of the Hamiltonian.
So as a first step we find the eigenfunctions of the Casimir operator $J^2$ for the conformal spin $s = 1/2$. This problem is equivalent to the decomposition of the representation of the $SL(2, R)$ group into irreducible ones and its solution is well known [36]. The Casimir operator has eigenfunctions of both the discrete and continuous spectrum. The eigenvalues of the Casimir operators are usually written in the form $j(j - 1)$. The parameter $j$ is called conformal spin and takes integer values for the eigenfunctions of the discrete spectrum and is $j = 1/2 + i\rho$, with $\rho$ being real and positive, for the eigenfunctions of the continuous spectrum.

For our purpose it is convenient to choose the eigenfunctions in the form $\psi(z_1, z_2) \sim e^{-i\xi(z_1 + z_2)}\Psi(z_{12})$. Inserting this function into the equation

$$J^2 \psi(z_1, z_2) = j(j - 1)\psi(z_1, z_2) \quad (4.14)$$

one finds that the function $z_{12}^{1/2}\Psi(z_{12})$ satisfies the equation for Bessel functions, $J_{\pm(j - 1/2)}(|\xi|z_{12})$. It is therefore convenient to define the function

$$\Psi_j^\xi(z) = e^{-\frac{i\pi}{2}(j-1/2)} z^{-1/2} J_{j-1/2}(|\xi|z) \quad (4.15)$$

For integer $j$ it is a single valued function of $z$ in the whole complex plane. For non-integer $j$ it has a cut from 0 to $-\infty$.

The eigenfunctions of the discrete spectrum, $P_j^\xi(z_1, z_2)$, $j = n + 1$, have the form

$$P_j^\xi(z_1, z_2) = e^{-i\xi(z_1 + z_2)}\Psi_j^\xi(z_{12}) \quad (4.16)$$

Calculating the scalar product one finds

$$\langle j', \xi'|j, \xi \rangle = 2\pi\delta(\xi - \xi') \frac{\delta_{jj'}}{2j - 1} \quad (4.17)$$

where we used the standard notation for the scalar product of $P_j^\xi$ and $P_{j'}^\xi$. The subspace spanned by the function $P_j^\xi(z_1, z_2)$, $\xi > 0$ ($\xi < 0$) corresponds to the subspace $D^-_j$ ($D^+_j$) in the tensor product decomposition [4.11].

The eigenfunctions of the continuous spectrum, $P_{j}^{\xi, \pm}(z_1, z_2)$, $j = 1/2 + i\rho$, (there are two eigenfunctions corresponding to each value of $j$) have the form

$$P_{j}^{\xi, \pm}(z_1, z_2) = e^{-i\xi(z_1 + z_2)}\Psi_{j}^{\xi, \pm}(z_{12}) \quad (4.18)$$

The functions $\Psi_{j}^{\xi, \pm}(z_{12})$ are defined as follows

$$\Psi_{j}^{\xi, \pm}(z) = \frac{1}{2\cos \pi j} \left[ \Psi_{j}^{\xi}(z_+) - \Psi_{j-1}^{\xi}(z_-) \right] \quad (4.19)$$

where $z_\pm = \pm z + i0$. Again, the subspaces spanned by the eigenfunctions $P_{j}^{\xi, \pm}(z_1, z_2)$ correspond to two copies of the subspaces $T^\rho$ in the tensor product decomposition [4.11]. We notice also that these two functions differ only by permutation of their arguments,

$$P_{j}^{\xi, +}(z_1, z_2) = P_{j}^{\xi, -}(z_2, z_1).$$
They are also invariant under the interchange $j \rightarrow 1 - j$, $P_{j,x}^{\xi}(z_1, z_2) = P_{1-j,x}^{\xi}(z_1, z_2)$. Taking into account that the function $\Psi_j^{\xi}(z)$ is proportional to a Hankel function

$$\Psi_j^{\xi}(z) \sim (z + i0)^{-1/2} H_{j-1/2}^{(1)}(|\xi|(z + i0))$$

one can check that $P_{j,x}^{\xi}(z_1, z_2)$ is an analytic function of $z_1$ in the upper and of $z_2$ in the lower complex half-plane. Next we conclude that from the properties of Bessel functions follows that the functions (4.16), (4.18) of the Casimir operator $J^2$ form a complete basis of the Hilbert space $L^2(R \times R)$. Thus any function from $L^2(R \times R)$ can be expanded as follows

$$\psi(z_1, z_2) = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \left\{ \sum_{j=1}^{\infty} \omega(j) a_\xi(j) P_{j,x}^{\xi}(z_1, z_2) - i \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} dj \omega^c(j) a_\xi^{\pm}(j) P_{j,x}^{\xi}(z_1, z_2) \right\}. \tag{4.21}$$

Here $\omega(j) = 2j - 1$, $\omega^c(j) = (j - 1/2) \cot \pi j$ and the expansion coefficients $a_\xi(j)$ and $a_\xi^{\pm}(j)$ are given by the scalar products

$$a_\xi(j) = \langle j, \xi | \psi \rangle, \quad a_\xi^{\pm}(j) = \langle j, \xi, \pm | \psi \rangle. \tag{4.22}$$

It follows from the properties of the eigenfunctions that the coefficients $a_\xi^{\pm}(j)$ are entire functions of the conformal spin $j$ in the whole complex plane satisfying the symmetry relation

$$a_\xi^{\pm}(j) = a_\xi^{\pm}(1 - j). \tag{4.23}$$

Using the explicit expressions for the eigenfunctions (4.16), (4.18) one can represent the expansion coefficients (4.22) in the form

$$a_\xi(j) = \kappa(-1)^{j-1} \int_{-|\xi|}^{|\xi|} dx P_{j-1,x} \left( \frac{x}{|\xi|} \right) \psi(x, \xi),$$

$$a_\xi^{\pm}(j) = \kappa \int_{|\xi|}^1 dx P_{j-1,x} \left( \frac{x}{|\xi|} \right) \psi(\pm x, \xi), \tag{4.23}$$

where $P_j(x)$ are the Legendre functions and

$$\kappa = \frac{1}{2} e^{i\pi/4} \sqrt{2\pi/|\xi|}. \tag{4.24}$$

$\psi(x, \xi)$ is the Fourier transform of $\psi(z_1, z_2)$ as in (2.11). It can be shown that if the function $\psi(x, \xi)$ behaves like $(1 - x)^{\alpha}$ for $x \rightarrow 1$, then the coefficients $a_\xi^{\pm}(j)$ vanish as $j^{-3/2-\alpha}$ when $j$ goes to infinity along the imaginary axis.

In the following we get rid of the Fourier integral over $\xi$ and construct the expansion for the function $\psi(z)$ which is defined by

$$\psi(z_1, z_2) = \int \frac{d\xi}{2\pi} e^{-i\xi(z_1 + z_2)} \psi(z_1 - z_2). \tag{4.25}$$
and using Eq. (4.19) one derives

\[ \psi(\xi) = \sum_{j=1}^{\infty} \omega(j) a_\xi(j) \Psi_\xi^j(z) - \frac{i}{2} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \frac{dj}{\sin\pi j} \omega(j) a_\xi^\pm(j) \Psi_\xi^j(z_{\pm}). \] (4.26)

Let us discuss the properties of the expansion (4.26). First, we note that when \( j \) goes to infinity along the imaginary axis the functions \( \Psi_\xi^j(z_{\pm}) \) behave as

\[ \Psi_\xi^j(z_{\pm}) \sim 2^{-1/2} e^{-i\pi j(j-1/2)} \frac{\xi^{j-1}}{\Gamma(j+1/2)} \frac{z_{\pm}^{j-1}}{2} (1 + O(1/j)) \] (4.27)

so that the integrals in (4.26) converge. Moreover, shifting the integration contour to the line \( \text{Re} j = 1/2 \). In order to study its small \( z \) behavior let us shift the contour of integration to the line \( \text{Re} j = 3/2 \). Since the integrand has a pole at \( j = 1 \) due to \( \sin\pi j \) one should evaluate the residue at this point. Adding the contributions from residues and the first term in the sum one gets

\[ \left[ a_\xi(1) + a_\xi^+(1) + a_\xi^-(1) \right] \Psi_\xi^j(z). \]

The term in the square brackets is zero, what is easy to see using Eq. (4.23) and taking into account that \( \psi(\xi) = \sum_{j=2}^{\infty} \omega(j) a_\xi(j) \Psi_\xi^j(z) \). Thus one gets the following expansion for the function \( \varphi_\xi(z) = \psi_\xi(z)/z \)

\[ \varphi_\xi(z) = z^{-1} \left\{ \sum_{j=2}^{\infty} \omega(j) a_\xi(j) \Psi_\xi^j(z) - \frac{i}{2} \int_{C} \frac{dj}{\sin\pi j} \omega(j) a_\xi^\pm(j) \Psi_\xi^j(z_{\pm}) \right\}, \] (4.28)

where the integration follows the line \( \text{Re} j = k, \quad 1 < k < 2 \). We shall show below that this is an expansion in eigenfunctions of the Hamiltonian. But before we want to notice that due to the asymptotics (4.27) the integration contour in (4.28) can be closed in the right half-plane. Calculating the integrals in (4.28) by residues one finds

\[ \varphi_\xi(z) = z^{-1} \sum_{j=2}^{\infty} \omega(j) A_\xi(j) \Psi_\xi^j(z), \] (4.29)

where

\[ A_\xi(j) = i \kappa(-1)^{j-1} \int_{-1}^{1} dx P_{j-1} \left( \frac{x}{|\xi|} \right) \partial_x \varphi_\xi(x) = i \kappa(-1)^{j} \int_{-1}^{1} dx C_{j-2}^{3/2} \left( \frac{x}{|\xi|} \right) \varphi_\xi(x). \] (4.30)

The representation (4.29) coincides with (2.32), which was obtained in Ref. [21]. Next, the sum in (4.28) can be rewritten as an integral over \( j \) resulting in a representation similar to that derived in Ref. [26]

\[ \varphi_\xi(z) = -\frac{i}{2z} \int_{C_\alpha} \frac{dj}{\sin\pi j} \omega(j) A_\xi^\pm(j) \Psi_\xi^j(z_{\pm}), \] (4.31)
where
\[ A_\xi^\pm(j) = i\kappa \int_{-|\xi|}^{|\xi|} dx P_{j-1} \left( \frac{x}{|\xi|} \right) \left[ \partial_x \varphi_\xi^\pm(\pm x) \right]. \quad (4.32) \]

The integration contour \( C_a \) is shown in Fig. 2. Deriving Eq. (4.31) we represented the function \( \varphi_\xi(x) \) as sum of two functions \( \varphi_\xi^+(x) \) and \( \varphi_\xi^-(x) \), such that \( \varphi_\xi^+(x) = 0 \) at \( x < -|\xi| \), and \( \varphi_\xi^-(x) = 0 \) at \( x > |\xi| \). We also notice that the contour can not be deformed to be parallel to the imaginary axis, since in this case the integral over \( j \) starts to diverge.

All three expansions (4.28), (4.29), (4.31) are equivalent, the difference appears when one wants to make a Fourier transform. One can interchange the Fourier transformation with integration (summation) over \( j \) only in the representation (4.28).

Let us now show that the expansion (4.28) runs over the eigenfunctions of the Hamiltonian. First of all, we note that the Hamiltonian (2.7) is not a hermitian operator with respect to the scalar product (4.2). Then the commutativity, \([\mathbb{H}, J^2] = 0\), does not imply that the operators share the same eigenfunctions. Indeed, one can easily check that the eigenfunctions of the continuous spectrum (4.18) of the Casimir operator are not the eigenfunctions of the Hamiltonian (2.7).

Nevertheless, one can find functions which diagonalize the Hamiltonian. (Of course, they are not mutually orthogonal and do not form a basis of the Hilbert space.) To solve the equation
\[ J^2 \varphi_j(z_1, z_2) \equiv -\partial_1 \partial_2 z_1^2 \varphi_j(z_1, z_2) = j(j-1)\varphi_j(z_1, z_2) \]
we use the ansatz \( \varphi_j(z_1, z_2) = e^{-\kappa(z_1+z_2)}\Phi_j^\xi(z_{12}) \), where \( j \) is an arbitrary complex number. This results in a second order differential equation for the function \( \Phi_j^\xi(z) \). It has two independent solutions \( \Phi_j^\xi(z) = z^{-1}\Psi_j^\xi(z) \) and \( \Phi_{1-j}^\xi(z) = z^{-1}\Psi_{1-j}^\xi(z) \). Due to commutativity of the integral operator \( \mathbb{H} \) and the differential operator \( J^2 \) one concludes that
\[ [\mathbb{H} \varphi_j](z_1, z_2) = A(j) \varphi_j(z_1, z_2) + B(j) \varphi_{1-j}(z_1, z_2), \quad (4.33) \]
whenever the action of the Hamiltonian \( \mathbb{H} \) on the function \( \varphi_j(z_1, z_2) \) is well defined. Substituting \( \varphi_j(z_1, z_2) \) in the form \( e^{-\kappa(z_1+z_2)}\Phi_j^\xi(z_{12}) \) into (2.9) one finds that the integrals converge when \( \text{Re } j > 1 \). Next, let us notice that when the variables \( \alpha, \beta \) run over the integration region, the argument of the function \( \Phi_j^\xi \) varies from 0 to \( z \). Thus to fix the coefficients \( A(j) \) and \( B(j) \) it is sufficient to study the asymptotics \( z \to 0 \) of the r.h.s and l.h.s. of (4.33).
In this case one can substitute $\Phi_j^\xi(z)$ by its leading term $\sim z^{j-2}$ and gets $B(j) = 0$ and $A(j) \equiv E(j)$ with
\[
E(j) = 2 [\psi(j) - \psi(2)] - \frac{1}{j(j-1)} + \frac{1}{2}, \tag{4.34}
\]
Furthermore, we have to fix the ambiguity in the definition of the function $\Phi_j^\xi(z_{12})$ related to the existence of a branching point at $z_{12} = 0$. To this end we note that the Hamiltonian \((4.33)\) transforms the functions with support in the region $z_{12} > 0$ ($z_{12} < 0$) into functions with the same support property. Thus, we conclude, that both functions, $\theta(\pm z_{12}) e^{-i\xi(z_{12})} \Phi_j^\xi(\pm z_{12})$, (or their linear combinations such as, $e^{-i\xi(z_{12})} \Phi_j^\xi(\pm z_{12} + i0)$) are eigenfunctions of the Hamiltonian corresponding to the same eigenvalues, $E(j)$. We notice also that since the integral $\int dzz^2 |\Psi_j^\xi(\pm z)|^2$ is finite for Re $j > 1/2$, the eigenvalue $E(j)$ belongs to the discrete spectrum of the operator $\mathbb{H}$.
Thus we have shown that for any complex $j$ such that Re $j > 1$ the function
\[
\varphi_j^\pm(z_1, z_2) = e^{-i\xi(z_1 + z_2)} z_{12}^{-1} \Phi_j^\xi(\pm z_{12} + i0) \tag{4.35}
\]
is an eigenfunction of the discrete spectrum of the Hamiltonian $\mathbb{H}$ with the eigenvalue given by \((4.34)\).

B. Solution of the evolution equation

Since we have shown that the expansion in the Eq. \((4.28)\) contains eigenfunctions of the Hamiltonian one can easily write down the solution to the evolution equation \((4.30)\) in the LO approximation
\[
\varphi_j^\nu'(z) = z^{-1} \left\{ \sum_{j=2}^\infty \omega(j) a_j^\nu(j) L^{-\gamma(j)} \Psi_j^\xi(z) \right. \]
\[
- \sum_{\alpha=\pm} \frac{i}{2} \int_{\frac{3+\infty}{2}}^{\frac{3-\infty}{2}} \frac{dj}{\sin \pi j} \omega(j) a_j^\nu(\bar{j}) L^{-\gamma(j)} \Psi_j^\xi(\bar{z}) \left. \right\}, \tag{4.36}
\]
where $\omega(j) = 2j - 1$,
\[
L = \frac{\alpha(\mu)}{a(\mu')} \quad \text{and} \quad \gamma(j) = 2 \frac{C_F}{b_0} E(j). \tag{4.37}
\]
After Fourier transform Eq. \((4.36)\) can be cast into the form
\[
\varphi_j^\nu(x) = \sum_{j=2}^\infty c_j^\nu(j) L^{-\gamma(j)} P_j^{(1)} \left( \frac{x}{|\xi|} \right) + \sum_{\alpha=\pm} \int_C \frac{dj}{2\pi i} c_j^\nu(\bar{j}) L^{-\gamma(\bar{j})} P_j^{(1)} \left( \frac{a\alpha}{|\xi|} \right), \tag{4.38}
\]
where the integration follows the line Re $j = k, 1 < k < 2$. The expansion coefficients are given by the following expressions
\[
c_j^\nu(j) = v_j \int_{-|\xi|}^{|\xi|} dx P_{j-1} \left( \frac{x}{|\xi|} \right) \partial_x \varphi_j^\nu(x), \tag{4.39a}
\]
\[
c_j^\nu,\pm(j) = v_j \int_{|\xi|}^{1} dx P_{j-1} \left( \frac{x}{|\xi|} \right) [\partial_x \varphi_j^\nu](\pm x), \tag{4.39b}
\]
where \( P_{j-1}(x) \) are the Legendre functions of the first kind and \( v_j = (2j - 1)/2 \). Notice, that the coefficients \( c_{\xi}^{\mu,\pm}(j) \) are antisymmetric under the interchange \( j \to 1 - j \), \( c_{\xi}^{\mu,\pm}(j) = -c_{\xi}^{\mu,\pm}(1 - j) \). We also want to stress here that the coefficients \( c_{\xi}^{\mu,\pm}(j) \) are entire functions of \( j \) in the whole complex plane.

The functions \( p_j^{(1)}(x) \), \( q_j^{(1)}(x) \) are expressed in terms of the Legendre functions of the first and second kind \[^{37}\]. We give here the expressions for the functions \( p_j^{(m)}(x) \), \( q_j^{(m)}(x) \) for general integer \( m \) since they appear in the solution of the evolution equation in the singlet sector. These functions (up to some normalization factors) are the Fourier transform of the functions \( z^{-m} \Psi_j^\xi(z) \) \( (p_j^{(m)}(x)) \) and \( z^{-m} \Psi_j^\xi(z + i0) \) \( (q_j^{(m)}(x)) \). The function \( p_j^{(m)}(x) \) is defined as follows

\[
p_j^{(m)}(x) = (-1)^m \theta(1 - x^2)(1 - x^2)^{m/2} P_{j-m}^{-m}(x) = r_j^m \theta(1 - x^2)(1 - x^2)^m C_{j-m-1}^{m+1/2}(x),
\]

where

\[
r_j^m = (-1)^m 2^{-m} \frac{\Gamma(2m + 1) \Gamma(j - m)}{\Gamma(m + 1) \Gamma(j + m)}.
\]

The function \( q_j^{(m)}(x) = 0 \) for \( x < -1 \), while for \( x > -1 \) it is given by

\[
q_j^{(m)}(x) = (x^2 - 1)^{m/2} Q_{j-1}^{-m}(x), \quad x > 1
\]

\[
q_j^{(m)}(x) = \frac{\pi}{2 \sin \pi j} (1 - x^2)^{m/2} P_{j-1}^{-m}(-x), \quad |x| < 1.
\]

The function \( q_j^{(1)}(x) \) is continuous at the point \( x = 1 \),

\[
q_j^{(1)}(x)|_{x=1} = -\frac{1}{j(j-1)},
\]

but its first derivative has a logarithmic singularity at this point

\[
\frac{d}{dx} q_j^{(1)}(1 + x) \sim -\frac{1}{2} \log x + \ldots.
\]

The formula \(4.38\) represents the solution to the evolution equation for the pion isovector quark GPD, \( H(x, \xi) \equiv \varphi_\xi(x) \). It can be used both for numerical and analytical study of the evolution.

Let us discuss the solution \(4.38\) in more details. First of all we notice that the integrals over \( j \) in \(4.38\) vanish whenever \( |x| > 1 \). Second, at the input scale \( \mu' = \mu \) \( (L = 1) \) the Eq. \(4.38\) can be represented as an expansion in eigenfunctions of the Casimir operator. To this end we shift the integration contour to the line \( \text{Re} j = 1/2 \). Since the function \( q_j^{(1)}(x) \) for \( |x| < 1 \) has a pole at \( j = 1 \) one should calculate the residue at the point \( j = 1 \). Taking into account that

\[
c_{\xi}^{\mu,\pm}(j)|_{j=1} = \mp \frac{1}{2} \varphi_\xi^{\mu}(\pm|\xi|)
\]

and \( (1 - x^2)^{1/2} P_{0}^{-1}(-x) = (1 + x) \) one finds that the contribution from residues is

\[
\Delta^{\mu}(x, \xi) = \frac{1}{2} \theta(|\xi| - |x|) \left[ \left( 1 + \frac{x}{|\xi|} \right) \varphi_\xi^{\mu}(|\xi|) + \left( 1 - \frac{x}{|\xi|} \right) \varphi_\xi^{\mu}(-|\xi|) \right].
\]
Using the antisymmetry of the expansion coefficients, \( c_{\xi}^{\mu a}(j) \), under \( j \to 1 - j \) and the relation \((A.6)\) between the Legendre functions one can rewrite Eq. \((4.38)\) in the form

\[
\varphi_{\xi}^{\mu}(x) = \Delta^{\mu}(x, \xi) + \sum_{j=2}^{\infty} c_{\xi}^{\mu a}(j) P_j^{(1)} \left( \frac{x}{|\xi|} \right) + \sum_{a=\pm} \frac{a}{2i} \int_{1/2-i\infty}^{1/2+i\infty} dj \cot \pi j \, c_{\xi}^{\mu a}(j) P_j^{(1)} \left( \frac{ax}{|\xi|} \right),
\]

(4.47)

where

\[
P_j^{(m)}(x) = \theta(x - 1) (x^2 - 1)^{m/2} P_{j-1}^{m}(x).
\]

(4.48)

Thus, one sees that the contribution of the integrals in Eq. \((4.38)\) at \( L = 1 \) in the region \( |x| < |\xi| \) is given by the term \( \Delta^{\mu}_\xi(x) \), which is determined completely by the values of the function at the points \( x = \pm \xi \). Obviously, the first two terms in \((4.47)\) have support in the ERBL region, while the integrals live in the DGLAP region. The term \( \Delta^{\mu}_\xi(x) \) can be restored from the knowledge of the function in the DGLAP region.

At the scale \( \mu' \) the function \( \varphi_{\xi}(x) \) is given by the expression \((4.38)\). To rewrite it in the form \((4.47)\) one has to find the expansion coefficients \( c_{\xi}^{\mu a}(j) \) and \( c_{\xi}^{\mu a,\pm}(j) \) at the scale \( \mu' \). To this end one should insert the expansion \((4.38)\) into \((4.39)\) and evaluate the corresponding integrals. This results in the following expression for the expansion coefficients at scale \( \mu' \)

\[
c_{\xi}^{\mu^+}(j) = \frac{v_j}{\pi i} \int_{C_{11}} dj' \frac{L^{-\gamma(j')} \, c_{\xi}^{\mu^+}(j')}{(j' - j)(j' + j - 1)},
\]

(4.49a)

\[
c_{\xi}^{\mu^+}(j) = \frac{v_j}{\pi i} \int_{C_{11}} dj' \frac{L^{-\gamma(j')} \, c_{\xi}^{\mu^+}(j') + (-1)^{j-1} c_{\xi}^{\mu^-}(j')}{(j' - j)(j' + j - 1)},
\]

(4.49b)

where the integration contour in both cases follows the line parallel to the imaginary axis such that \( 1 < \Re j' < j \) (contour \( C_1 \)), and \( \max\{1, \Re j, 1 - \Re j\} < \Re j' \) (contour \( C_{11} \)). We notice that the integration contours can not be closed in the right half-plane because the integrals over the large semicircle do not vanish. Next, let us check that at \( \mu' = \mu \ (L = 1) \) the r.h.s reproduces the l.h.s. To this end let us shift the integration contour to the line \( \Re j' = 1/2 \). Then the r.h.s of Eq. \((4.49)\) will be given by the sum of the residue at the point \( j' = j \) and the integral over the line \( \Re j' = 1/2 \). The integral vanishes due to antisymmetry of the integrand under \( j \leftrightarrow 1 - j \) while the residue gives the coefficient \( c_{\xi}^{\mu^+}(j) \).

One sees that the coefficients \( c_{\xi}^{\mu a}(j) \), \( c_{\xi}^{\mu a,\pm}(j) \) mix under evolution \[51\]. Therefore, their scale dependence is not independent. Nevertheless, it is straightforward to check that the transformation \( \mu \to \mu' = f(\mu', \mu) \) generated by Eqs. \((4.49)\) possesses the necessary group property, \( f(\mu', \mu') = f(\mu'^+, \mu'^-, \mu^a) \).

The coefficients \( c_{\xi}^{\mu a}(j) \) get contributions from \( c_{\xi}^{\mu a,\pm}(j) \). This effect can be interpreted as migration of partons from the DGLAP region to the ERBL region \[4, 15\].

Further, let us note also that it follows from the Eq. \((4.49)\) that for integer \( j \geq 2 \)

\[
\left[ c_{\xi}^{\mu}(j) + c_{\xi}^{\mu^+,}(j) + (-1)^{j-1} c_{\xi}^{\mu^-}(j) \right] = \left[ c_{\xi}^{\mu}(j) + c_{\xi}^{\mu^+}(j) + (-1)^{j-1} c_{\xi}^{\mu^-}(j) \right] L^{-\gamma(j)}. \quad (4.50)
\]

It is not surprising that the combination in the brackets is nothing but the Gegenbauer moment of the GPD

\[
C_{j-2}^\mu(\xi) = -v_j \int_{-1}^{1} \frac{dx}{|\xi|} C_{j-2}^{3/2} \left( \frac{x}{|\xi|} \right) \varphi_{\xi}^{\mu}(x).
\]

(4.51)
FIG. 3: Example for the evolution of the function $\varphi_{\xi=0.4}(x) = (1 - x^2)^{2.7}$ (curve (1)) and its derivative (curve (2)) to the scale $L = 1.5$, curves (3) and (4), respectively.

For the physical GPD, $H(x, \xi)$, moments $\xi^N C_\mu^N(\xi)$ have to be polynomials in $\xi$ of degree $N$. This is the so-called polynomiality conditions \[3\]. It means that the knowledge of the GPD in the DGLAP region allows to restore the GPD in the central region, up to terms which satisfy the polynomiality conditions.

Next, all singularities of the integrand in the integral in Eq. (4.49a) lie to the left of the integration line. This allows to shift it arbitrarily far to the right. Since the anomalous dimension $\gamma(j)$ behaves as $\log j$ for large $j$ we conclude that the coefficients $c_{\xi}^{\mu, \pm}(j)$ vanish faster than any power of $1/L$ for $L \to \infty$. As far as the function $\varphi_{\xi}(x)$ in the DGLAP region $|x| \geq |\xi|$ is expressed solely in terms of the coefficients $c_{\xi}^{\mu, \pm}(j)$ this implies also that the function in this region vanishes faster any power of $1/L$ for $L \to \infty$. Thus the asymptotic expansion for the function $\varphi_{\xi}^{\mu'}(x)$ for $\mu' \to \infty$ (large $L$) has the form

$$
\left[ \varphi_{\xi}^{\mu'}(x) - \sum_{j=2}^{n} C_{j-2}^{\mu} L^{-\gamma(j)} p_{j}^{(1)} \left( \frac{x}{|\xi|} \right) \right] \sim O \left( L^{-\gamma(n+1)} \right) .
$$

(4.52)

Since the anomalous dimension $\gamma(j)$ vanishes for $j = 2$ the function $\varphi_{\xi}^{\mu'}(x)$ tends to its asymptotic form, $p_{2}^{(1)}(x/|\xi|) \sim (1 - x^2/\xi^2)$, (see Refs. [15, 24]), for $\mu' \to \infty$. We stress that the expansion (4.52) is an asymptotic expansion and that the series in (4.52) diverges for any $x$.

Let us analyze now the behavior of the function $\varphi_{\xi}^{\mu'}(x)$ at the point $x = \pm \xi$. As follows from the properties of the functions $q_{j}^{(1)}(x)$ the GPD $\varphi_{\xi}^{\mu'}(x)$ is continuous at the points $x = \pm \xi$ and has the following values (see Eqs. (4.45) and (4.49a))

$$
\varphi_{\xi}(\pm |\xi|) = \pm \frac{1}{\pi i} \int_{-\infty}^{\infty} d\nu \frac{c_{\nu, \pm}(\nu')}{j'(j'-1)} L^{-\gamma(j')} ,
$$

(4.53)
where the $\kappa > 1$. Taking into account the Eqs. (A.1) and (A.7) one finds that the first
derivative of the function $\varphi^\mu_\xi(x)$ has at least a logarithmic singularity at $x = \pm \xi$ even if thefunction $\varphi(x)$ is a smooth function at the input scale $\mu$ (see Fig. 3)

$$\frac{d}{dx} \varphi^\mu_\xi(x) \sim \alpha_\pm |\xi|^{-1} \log |x/|\xi| + 1| .$$  \hspace{1cm} (4.54)

Here

$$\alpha_\pm = -\frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} dj' c_{\mu, \pm}(j') L^{-\gamma(j')} .$$  \hspace{1cm} (4.55)

If the integral in (4.55) diverges it means that the derivative of $\varphi^\mu_\xi(x)$ has a more singular
behaviour at $x = \pm \xi$. It is clear that the convergence properties of the integral in (4.55) is
determined by the behavior of the coefficients $c_{\mu, \pm}(j)$ for $j \to \pm i\infty$. The latter depends on
the smoothness of the function $\varphi(x)$ at the scale $\mu$, i.e. on the behavior of the function at
the points $x = \pm \xi$ and $\xi = \pm 1$.

The amplitude of a scattering process (DVCS, light meson production, etc.) is given by
the convolution of a GPD with a hard scattering amplitude. The typical integral occurring
in such a convolution at lowest order in $\alpha_s$ is

$$\mathcal{J}(\mu', \xi) = \int_{-\infty}^{\infty} dx \frac{\varphi^\mu_\xi(x)}{\xi - x - i0} ,$$  \hspace{1cm} (4.56)

where due to the support properties of the GPD the integration is restricted to the interval
$[-1, 1]$. We assume here that $\xi > 0$. Substituting $\varphi(x)$ by the expansion (4.38) and changing
the order of summation (integration) over $j$ and $x$ one derives with the help of the Eqs. (A.4)

$$\mathcal{J}(\mu', \xi) = -2 \sum_{j=2} c^\mu_j \frac{c^\mu_\xi}{j(j-1)} L^{-\gamma(j)} + \frac{1}{\pi i} \int_C \frac{dj}{\sin \pi j} \frac{\left[e^{-\pi j} c^\mu_\xi(j) - c^\mu_j(j)\right]}{j(j-1)} L^{-\gamma(j)} ,$$  \hspace{1cm} (4.57)

where the integration follows the line $\text{Re} j = \kappa, 1 < \kappa < 2$.

Let us compare the approach presented here with the one developed in [25]. One can see
that the only difference between Eq. (81) in Ref. [25] and our representation of the GPD, i.e.
Eq. (4.38) at $L = 1$, (or analogously Eq. (4) in Ref. [1]) is, that the sum over discrete $j$ in
Eq. (4.38) is rewritten as an integral over $j$. To this end one has to construct the analytical
continuation of the coefficients $c_j(j)$ with suitable analytic properties. This has been done
in [25]. However, Eq. (69) in Ref. [25], which provides such an analytical continuation,
involves the function $\varphi_\xi(x)$ (or $w(x, \eta)$ in the notations of Ref. [25]) outside the region where
it is originally defined. Thus, such reformulation is possible only under certain additional
assumptions on the analytic structure of GPDs.

C. Small $\xi$ asymptotics

In this subsection we want to discuss the structure of the solution (4.38) in the limit
$\xi \to 0$. (It is clear that when $\xi \to 1$ one recovers the conventional ERBL evolution [19, 20].)
Obviously, the form of the function in the small \( \xi \) region can not differ strongly from the input function at \( L \sim 1 \). However, one may hope that the form of the evolved function for large scales, \( L \gg 1 \), depends weakly on the initial profile.

The sum in Eq. (4.38) contributes only to the central region. To estimate this sum one has to have some information on the expansion coefficients \( c_\xi(j) \). If the coefficients \( c_\xi(j) \) are not singular as \( \xi \to 0 \), then at large \( L \) one can omit all the terms in sum except the first one. This type of behavior holds for some models of GPDs [35, 38] but the opposite situation is not excluded. Thus, we restrict ourself to the analysis of the small \( \xi \) behavior of the integrals in Eq. (4.38).

First of all we shall show, that in the limit \( \xi \to 0 \) and \( x \) fixed, Eq. (4.38) takes the standard DGLAP form. Till the end of this section we shall assume that \( \xi > 0 \). Let us consider the integral in (4.38) corresponding to the term “\( a = + \)”. For \( \xi \to 0 \) and \( x \) fixed, the argument of the function \( q_\xi(1) \) goes to infinity and so we can replace it by its asymptotic value

\[
q_\xi^{(1)}(x/\xi) = -x^{1-j} \frac{\sqrt{\pi} \Gamma(j-1/2)}{2 \Gamma(j+1)} \left( \frac{\xi}{2} \right)^{j-1} (1 + \mathcal{O}(\xi^2)) .
\]  

(4.58)

The contour of the integration over \( j \) follows the line \( \text{Re} j = \kappa, \kappa > 1 \). The function \( q_\xi(1) \) vanishes as \( \xi^{\kappa-1} \) for \( \xi \to 0 \) and the parameter \( \kappa \) can be chosen sufficiently large. In its turn the coefficient \( c_\xi^{+}(j) \) becomes singular at \( \xi \to 0 \). Indeed, using Eq. (A.10) one can derive the following representation

\[
c_\xi^{+}(j) = -\frac{v_j}{4\pi^{3/2}i} \int_C ds \varphi_\xi(s) \left( \frac{s}{2} \right)^{-s} \frac{\Gamma(s+j/2) \Gamma(s-j+1/2)}{\Gamma(s)} ,
\]  

(4.59)

where the integration goes from \(-i\infty\) to \(+i\infty\) staying to the right of all singularities of the integrand and

\[
\varphi_\xi(s) = \int_0^1 dx x^{s+1} \varphi_\xi(x) .
\]  

(4.60)

If \( \text{Re} s \) is sufficiently large then \( \varphi_\xi(s) \to \varphi_\xi=0(s) \) for \( \xi \to 0 \) even if the function \( \varphi_\xi(x) \) is singular at \( x = \xi = 0 \). The leading \( \xi \to 0 \) asymptotics of \( c_\xi^{+}(j) \) is determined by the rightmost singularity of the integrand in (4.59) which is situated at \( s = j-1 \). Thus one gets

\[
c_\xi^{+}(j) = -\left( \frac{\xi}{2} \right)^{1-j} \frac{\Gamma(j+1/2)}{\sqrt{\pi} \Gamma(j-1)} \varphi_\xi(j-1) (1 + \mathcal{O}(\xi^2)) .
\]  

(4.61)

Therefore, for \( \xi = 0 \) the solution (4.38) \((x > 0)\) takes the DGLAP form

\[
\varphi_{\xi=0}(x) = \frac{1}{2\pi i} \int_C dj x^{-j} \varphi_{\xi=0}(j) L^{-\gamma(j+1)} .
\]  

(4.62)

The integrals in (4.38) depend on the three parameters \( x, \xi \) and \( L \). Eq. (4.62) was obtained in the limit \( \xi \to 0 \), \( x \) and \( L \) fixed. Let us now keep the ratio \( x/\xi = r > 1 \) fixed (i.e. we shall consider the DGLAP region) and evaluate the integral

\[
\mathcal{I}(\xi, L, r) = \frac{1}{\pi i} \int_C dj c_\xi^{+}(j) L^{-\gamma(j)} q_\xi^{(1)}(r) .
\]  

(4.63)
for large $1/\xi$ and $L$. The answer depends on the relative size of $L$ and $\xi$. Let’s start with the situation when $\log L \gg \log 1/\xi$. In this case taking into account Eq. (4.61) one can evaluate the integral by the saddle point method. The position of the saddle point is determined from the equation

$$1 - \gamma'(j_*) \sigma = 0, \quad \sigma = \log L / \log (2/\xi). \quad (4.64)$$

For large $\sigma$, (which of course corresponds to an unphysically high scale $\mu'$) $j_* = 4C_F/b_0 \sigma$ and the integral (4.63) scales with $L$ as

$$I \sim L^{-4C_F/b_0 j_*-1} q^{(1)}_{j_*}(r).$$

Note that we restricted our consideration to the DGLAP region, $x/\xi > 1$, only because the integrand in this region has no singularities in the half-plane $\text{Re} \, j > 1$, so that that we can freely deform the integration contour. In the ERBL region the integrand has poles, (see Eq. (4.42)), so when deforming contour one has to take into account the corresponding residues. These residues together with the terms in the sum in (4.38) result in the asymptotic expansion (4.52). Consequently, the number of terms in the asymptotic expansion (4.52) is controlled by the position of the saddle point $j_* \sim \log L / \log (2/\xi)$. This means that the “time” $\tau = \log L$ which is necessary for the function to reach the asymptotic regime is proportional to $\log 2/\xi$.

Now let us decrease $L$ or $\xi$. Then $\sigma$ and $j_*$ decrease as well. When $\sigma \ll 1$ (i.e. when the function is far from its asymptotic form) one finds

$$j_* \simeq 1 + \sqrt{2C_F/b_0 \sigma} \quad (4.65)$$

and recovers the well known “double scaling” behaviour [40, 41]

$$I \sim \frac{\log^{1/4} L}{\log^{3/4}(2/\xi)} e^{\sqrt{8C_F/b_0 \log L \log 2/\xi} q^{(1)}_{j_*}(r)}. \quad (4.66)$$

Let us now consider the situation when the function $\varphi_{\xi=0}(x)$ is enhanced at small $x$. Recent fits of parton densities (see Ref. [39]) shows that in the $x$ region, $10^{-4} \leq x \leq 10^{-2}$, they display a power-like behavior $\varphi_{\xi=0}(x) \sim x^{-\alpha}$. In this case the Mellin transform of the function $\varphi_{\xi=0}(x)$ has a pole at $s = \alpha$. (In the case $\alpha > 0$ the first pole of $\varphi_{\xi=0}(j - 1)$ lays to the right of the first pole of the anomalous dimension $\gamma(j)$.) At the same time the function $\varphi_{\xi}(x)$ is a regular function of $x$ for $\xi \neq 0$, so that the first pole of its Mellin transform lies at $s = 0$. It can be seen that near the point $s = \alpha$ the Mellin transform $\varphi_{\xi}(s)$ takes the following form $\varphi_{\xi}(s) \sim (1 - \xi^{-s-\alpha} f(\xi, s))/(s - \alpha)$, where $f(\xi, \alpha) = 1$. One sees that the numerator contains two terms with different $\xi$ dependence. Thus the integrand in (4.63) near the point $j = 1 + \alpha$ can be split up into two terms, which are proportional to $\xi^{1-j}$ and $\xi^{-\alpha}$, respectively. Each of these terms has poles, which, of course, cancel in the sum. Calculating the asymptotics of the first term ($\xi^{1-j}$) one finds that in the case $j_* < 1 + \alpha$ it is necessary to take into account the contribution due the pole at $j = 1 + \alpha$ which is

$$I^{\text{pole}} \sim \xi^{-\alpha} q^{(1)}_{1+\alpha} \left(\frac{x}{|\xi|}\right) L^{-\gamma(1+\alpha)}. \quad (4.67)$$
For $0 < \alpha < 1$ the anomalous dimension $\gamma(1 + \alpha)$ is negative so that the evolution results in an enhancement of this contributions. The second term has the same $\xi$ dependence as the pole contribution, $\xi^{-\alpha}$. However, this term vanishes faster with $L$ than the first one. So one can expect that the pole contribution (1.67) will dominate in the limit $L \to \infty$, $\log L/ \log 1/\xi$ fixed. Our result (1.67) agrees with the result of Shuvaev et al. [42]. (One can easily check that the integral (22) in Ref. [42] gives rise to the function $q_{2+1,\xi}^{(\nu+1)}(x/\xi)$.)

V. SINGLET CASE

In the singlet case one is dealing with the evolution of the quark, $F^{q,+}(x, \xi)$, and gluon, $F^{g}(x, \xi)$, GPDs which are related to matrix elements of the $C$–even quark and gluon operators, (see Eqs. (2.5), (2.21)). We shall proceed along the same lines as in the case of the isovector quark GPD. Similarly, we start with the formulation of the problem in coordinate space. The only difference with respect to the previous case is that we have now two functions $f^q(z_1, z_2)$, $f^g(z_1, z_2)$ (see Eqs. (2.21)) instead of one ($\varphi(\zeta_1, \zeta_2)$). For the sake of brevity we shall denote by $f(z_1, z_2)$ the two component function $f(z_1, z_2) = (f^q(z_1, z_2), f^g(z_1, z_2)) \equiv (f(1)(z_1, z_2), f(2)(z_1, z_2))$. Next, let us introduce a function $\psi$, $\psi = (\psi^q, \psi^g) \equiv (\psi^{(1)}, \psi^{(2)})$, as follows

$$\psi^{(k)}(z_1, z_2) = (z_1 - z_2)^k f^{(k)}(z_1, z_2), \quad (5.1a)$$

where $k = 1, 2$. The functions $\psi^{q,g}(\zeta)$ transform under $SL(2, R)$ transformations according to Eq. (4.3) with $s = 1/2$ while the functions $f^{q,g}(\zeta)$ obey the same transformation with $s = 1$ and $s = 3/2$, respectively. We define the scalar product on the space of functions $f$ as follows

$$\langle f_1 | f_2 \rangle \equiv \langle \psi_1 | \psi_2 \rangle = \langle \psi_1^q | \psi_2^q \rangle + \langle \psi_1^g | \psi_2^g \rangle, \quad (5.2)$$

where the scalar product $\langle \psi_1^q | \psi_2^q \rangle$ is the standard scalar product on the space $L^2(R \times R)$, (see Eq. (4.3)).

Since the functions $\psi^{q,g}(\zeta)$ obey the $SL(2, R)$ transformations (4.3) with conformal spin $s = 1/2$ they can be expanded in eigenfunctions of the Casimir operator according to the Eq. (4.21). Again, we shall omit the Fourier integral over $\xi$ and write down all expansions for the function $f_\xi(z)$ ($\psi_\xi(z)$)

$$f(z_1, z_2) = \int \frac{d\xi}{2\pi} e^{-i\xi(z_1 + z_2)} f_\xi(z_1 - z_2). \quad (5.3)$$

Obviously, an expansion for the function $\psi^\xi_\xi(x)$ has the form (4.26). Then we shift the integration contour from the line $\Re j = 1/2$ to $\Re j = 5/2$. Doing so one has to take into account the residues at the points $j = 1, 2$ which cancel identically with the first two terms in the sum over $j$. To see that the sum of the residues at $j = 2$ for the function $\psi_\xi^q(z)$ vanishes one should remember that this function is symmetric with respect to $z \to -z$. Thus the expansion for the function $f_\xi(z)$ can be written in the following form

$$f^{(k)}(z) = \sum_{j=3}^\infty \omega(j) a^{(k)}_j(z) \psi_j^{(k)}(z) - \frac{i}{2} \int_{\infty}^{\pm\infty} \frac{dj}{\sin \pi j} \omega(j) a^{(k),\pm}_j(z) \psi_j^{(k)}(z), \quad (5.4)$$

27
where $2 < \kappa < 3$, $k = 1, 2$, $z_\pm = \pm z + i0$ and the coefficients $a^{(k)}_\xi(j), a^{(k)\pm}_\xi(j)$ are given by Eq. (4.23) after the substitution $\psi \rightarrow \psi^{(k)}$. The functions $\Psi^{(k)}_j(z_\pm)$ are defined as follows

$$\Psi^{(k)}_j(z_\pm) = z^{-k}\Psi_j(z_\pm),$$

(and similar for $\Psi^{(k)}_j(z)$) where $\Psi_j(z)$ is defined in Eq. (4.13). Now, using the same arguments as in Sect. IV one can show that the Hamiltonians describing the renormalization of the singlet operators Eqs. (2.17)–(2.21) act on the functions $\Phi^{(k)}_j(z_1, z_2) = e^{-i(\gamma_1 + \gamma_2)}\Psi^{(k)}_j(z_{12})$ as follows

$$\mathcal{H}^{ik}_j \Phi^{(k)}_j(z_1, z_2) = E^{ik}_j(j_1, j_2).$$

(5.5)

Note that in the formula above no summation over repeated indices $i, k$ is implied. The matrix of the ”anomalous dimensions” $E^{ik}$ has the well known form

$$E^{11}_j \equiv E^{11}_j = C_F \left(2\left[\psi(j) - \psi(2)\right] - \frac{1}{j(j-1)} + \frac{1}{2}\right),$$

(5.6a)

$$E^{12}_j = n_f \frac{j^2 - j + 2}{(j+1)(j(j-1)(j-2))},$$

(5.6b)

$$E^{21}_j \equiv E^{21}_j = C_F \frac{j^2 - j + 2}{j(j-1)},$$

(5.6c)

$$E^{22}_j \equiv E^{22}_j = 2N_c \left[\psi(3) - \psi(3) - \frac{1}{j(j+1)} - \frac{1}{(j-1)(j-2)}\right] + \frac{7}{6}N_c + \frac{1}{3}n_f.$$ (5.6d)

The solution of the evolution equation (2.16) for the function $f_\xi(z)$ can be written in the form

$$f^{(k)\mu'}_\xi(z) = \sum_{j=3}^{\infty} \omega(j) \left(L^{-\gamma(j)}a^{\mu}(j)\right)_k \Psi^{(k)}_j(z)$$

$$- \frac{i}{2} \int_{c} \frac{dj}{\sin \pi j} \omega(j) \left(L^{-\gamma(j)}a^{\mu\pm}(j)\right)_k \Psi^{(k)}_j(z_\pm).$$

(5.7)

Here, $\gamma(j)$ is the matrix of the anomalous dimensions, $\gamma(j) = 2E(j)/b_0$, $a^{\mu}(j)$ is a two-dimensional vector, $a^{\mu}(j) = (a^{(1)}_\xi(j), a^{(2)}_\xi(j))$, and the integration contour is the same as in Eq. (5.2).

By taking the Fourier transform Eq. (5.7) can be brought into the form

$$\hat{F}^{\mu}(x, \xi) = \sum_{j=3}^{\infty} \hat{p}_j \left(\frac{x}{|\xi|}\right) L^{-\gamma(j)}c^{\mu}_\xi(j) + \sum_{a=\pm} \int_{c} \frac{dj}{\pi i} \hat{q}_a \left(\frac{ax}{|\xi|}\right) L^{-\gamma(j)}c^{\mu,a}_\xi(j).$$

(5.8)

The quark and gluon GPDs $F^{\mu}_{ij}(x, \xi)$ are defined in Eq. (2.26). The matrix functions $\hat{p}_j(x)$ and $\hat{q}_j(x)$ are given by the following expressions

$$\hat{p}_j(x) = \left(\begin{array}{cc} p^{(1)}_j(x) & 0 \\ 0 & \xi p^{(2)}_j(x) \end{array}\right),$$

(5.9)

$$\hat{q}_j^{\pm}(x) = \left(\begin{array}{cc} \pm q^{(1)}_j(x) & 0 \\ 0 & \xi q^{(2)}_j(x) \end{array}\right).$$

(5.10)
and the functions \( p_j^{(m)} \), \( q_j^{(m)} \) are defined by Eqs. (1.40), (1.41). The expansion coefficients, \( \bar{c} = (c^{(1)}, c^{(2)}) \) have the following form

\[
\begin{align*}
c^{(k),\mu}(j) &= v_j \int_{-|\xi|}^{|\xi|} dx P_{j-1} \left( \frac{x}{|\xi|} \right) \left[ \frac{d^k}{dx^k} F^\mu_k \right] (x, \xi), & (5.11a) \\
c^{(k),\mu,\pm}(j) &= v_j \int_{|\xi|}^1 dx P_{j-1} \left( \frac{x}{|\xi|} \right) \left[ \frac{d^k}{dx^k} F^\mu_k \right] (\pm x, \xi). & (5.11b)
\end{align*}
\]

Note, that the gluon GPD \( F_2(x, \xi) \equiv F_g(x, \xi) \) is a symmetric function of \( x \), while the quark GPD \( F_1(x, \xi) \equiv F_q(x, \xi) - F_q(-x, \xi) \) is an antisymmetric one. Therefore, all integer moments with even \( j \) (Eq. (5.11a)) vanish and \( c^{\mu,\pm}(j) = c^{\mu,-}(j) \). The matrix exponent \( L^{-\gamma(j)} \) can be written in the form

\[
L^{-\gamma(j)} = L^{-\gamma_+(j)} A_+(j) + L^{-\gamma_-(j)} A_-(j),
\]

where

\[
\begin{align*}
\gamma_\pm(j) &= \frac{\gamma_{11}(j) + \gamma_{22}(j)}{2} \mp \Delta(j), & (5.13) \\
\Delta(j) &= \sqrt{\left(\frac{\gamma_{11}(j) - \gamma_{22}(j)}{4}\right)^2 + \gamma_{12}(j)\gamma_{21}(j)}. & (5.14)
\end{align*}
\]

The projectors \( A_\pm(j) \), \( A_\pm^2(j) = A_\pm(j) \), \( A_+(j)A_-(j) = 0 \) are given by the following expression

\[
A_\pm(j) = \frac{1}{2} \left[ \hat{I} \mp \frac{1}{\Delta(j)} \left( \frac{\gamma_{11}(j) - \gamma_{22}(j)}{\gamma_{21}(j)} \right) \frac{\gamma_{12}(j) - \gamma_{21}(j)}{\gamma_{22}(j) - \gamma_{11}(j)} \right].
\]

Let us note that although each term on the r.h.s of (5.12) has branching points, their sum is an analytic function. For large \( j \), \( \gamma_+(j) \simeq \gamma_{11}(j) \equiv \gamma_q(j) \) and \( \gamma_-(j) \simeq \gamma_{22}(j) \equiv \gamma_g(j) \) so we shall refer to \( \gamma_+(j)(\gamma_-(j)) \) as the quark (gluon) anomalous dimension. Let us notice that due to mixing the quark (but not gluon) anomalous dimension, \( \gamma_+(j) \), has a pole at \( j = 2 \).

The formulae (5.8), (5.11) represent the solution of the evolution equation for the quark and gluon GPDs.

Having set \( L = 1 \) in Eq. (5.8) one derives a representation analogous to that in the isovector case, Eq. (4.17).

\[
\bar{F}^\mu(x, \xi) = \bar{A}^\mu(x, \xi) + \sum_{j=3}^\infty \bar{P}_j \left( \frac{x}{|\xi|} \right) c^{\mu,\pm}(j) + \sum_{a=\pm} \frac{1}{2} \int_C \frac{dx}{x} \cot x \pi j \hat{P}^a_j \left( \frac{ax}{|\xi|} \right) \bar{c}^{\mu,a}(j),
\]

(5.16)

where the integration follows the line \( \text{Re} j = 1/2 \). The function \( \hat{P}_j \) is defined as

\[
\hat{P}_j^\pm(x) = \begin{pmatrix} \pm P^{(1)}_j(x) & 0 \\ 0 & \mp P^{(2)}_j(x) \end{pmatrix},
\]

(5.17)

where

\[
P^{(k)}_j(x) = \theta(x-1) (x^2 - 1)^{k/2} P^{-k}_{j-1}(x).
\]

(5.18)
The vector $\vec{\Delta}^{\mu}(x, \xi)$ is given by the following expression

$$\Delta^{\mu}_1(x, \xi) = \theta(|\xi| - |x|) \frac{x}{|\xi|} F^{\mu}_1(|\xi|, \xi), \quad (5.19)$$

$$\Delta^{\mu}_2(x, \xi) = \theta(|\xi| - |x|) \left( F^{\mu}_2(|\xi|, \xi) + \frac{x^2 - \xi^2}{2|\xi|} \left[ \frac{d}{dx} F^{\mu}_2 \right] (|\xi|, \xi) \right). \quad (5.20)$$

The expansion coefficients at the scale $\mu'$ are given by the same expressions as in the non-singlet case, Eqn. (4.49), which should now be understood as vector equations.

The anomalous dimension $\gamma_+(j)$ vanishes at $j = 3$, so that for $\mu' \to \infty$ the GPD takes the well known asymptotic form [6, 35]

$$F^{as}_1 = n_f \frac{x}{\xi^3} \left( 1 - \frac{x^2}{\xi^2} \right) D, \quad (5.21)$$

$$F^{as}_2 = C_F \frac{1}{\xi} \left( 1 - \frac{x^2}{\xi^2} \right)^2 D, \quad (5.22)$$

where

$$D = \frac{15}{4(n_f + 4C_F)} \int_{-1}^{1} dx \left[ x F_1(x, \xi) + F_2(x, \xi) \right]. \quad (5.23)$$

Similarly to the case of the isovector GPD one can analyze the small $\xi$ behavior of the quark and gluon GPDs. In the situation when the GPDs are the smooth functions at the input scale their form after evolution in the small $\xi$ region is determined by the rightmost pole of the anomalous dimension $\gamma_+(j)$, which is located at $j = 2$. The corresponding asymptotics have the “double scaling” form which results from evaluation of the integral in Eq. (5.8) by the saddle point method. We again consider the integral corresponding to the term with $a = +$ in Eq. (5.8). The expansion coefficients (5.11) can be expressed as follows

$$\vec{c}^{\mu,+}_j \approx \left( \frac{\xi}{2} \right)^{1-j} \vec{m}(j), \quad (5.24)$$

$$\vec{m}(j) = \frac{\Gamma(j + 1/2)}{\sqrt{\pi} \Gamma(j - 1)} \left( \frac{1}{(j - 2)} F_1(j - 2) \right), \quad (5.25)$$

where

$$\bar{F}(s) = \int_{0}^{1} dx x^{s-1} \bar{F}^{\mu}(x, \xi). \quad (5.26)$$

Inserting (5.24) into Eq. (5.8) one finds that the saddle point is determined by the equation

$$1 - \gamma_-(j_*) \sigma = 0, \quad \sigma = \log L \log 2/\xi. \quad (5.27)$$

For small $\sigma$ one finds $j = 2 + \epsilon$ with

$$\epsilon = 2 \sqrt{\frac{N_c}{b_0}} \sigma (1 + O(\sigma)). \quad (5.28)$$
Then one finds for the integral \( \tilde{I} \) (which corresponds to the term \( a = + \) in Eq. (5.8) ) in the “double scaling” approximation

\[
\tilde{I}(x, \xi) \simeq \left( \frac{\pi^2 N_c}{4b_0} \right)^{1/4} \frac{\log^{1/4} L}{\log^{3/4} 2/\xi} e^{4\sqrt{N_c/b_0} \log L \log 2/\xi} - \frac{2}{\xi} \int_0^1 r \, dr \, \hat{q}_{2+\epsilon}(r) \, A_+(2 + \epsilon) \, \tilde{m}(2 + \epsilon).
\]  

(5.29)

Here \( r = x/\xi \) and we assume that \( \xi > 0 \). Due to symmetry properties taking into account the term \( a = - \) results in the replacement

\[
\hat{q}_{2+\epsilon}(r) \rightarrow \hat{q}_{2+\epsilon}(r) + \hat{q}_{2-\epsilon}(-r)
\]

in the above expression.

VI. SUMMARY

We developed a method for solving the leading order evolution equations for Generalized Parton Distributions. The form of the solution, Eqs. (4.38) and (5.8), is fixed completely by the symmetry properties of the evolution kernels. We have shown that the GPD should be treated differently in the ERBL and DGLAP regions, which reflects both the mathematical structure of the equation and its physical content.

The Eqs. (4.38) and (5.8) can be used for both analytical and numerical studies of the GPD’s evolution. It is important to stress here that all quantities entering these equations are unambiguously defined. We demonstrated that in the limits \( \xi \rightarrow 1 \) and \( \xi \rightarrow 0 \) the solutions (4.38) and (5.8) of the GPD evolution equations take the form of the solutions of the ERBL and DGLAP evolution equations, respectively. Furthermore, using the representations (4.38) and (5.8) one can easily obtain the asymptotic form of the GPDs in different regimes (large \( L \), small \( \xi \), and so on).

It would be quite interesting to apply the developed approach to the analysis of the analytic structure of the anomalous dimensions of twist-3 operators. It is well known that the knowledge of the energies of the reggeon bound states allows to extract anomalous dimensions of the operators at the singular point of in the \( j_- \) plane. This correspondence has been checked for the twist two-operators. Recently, methods for the calculation of multi-reggeon bound states were developed and the predictions for the anomalous dimensions of higher twist operators at the singular points were obtained. However, so far it is not known how to solve the problem of the analytical continuation of the anomalous dimensions of higher twist operators. We hope that our approach can be generalized to higher twist operators, at least for the class of the twist-3 operators for which the evolution equation are known to be integrable.

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APPENDIX A: AUXILIARY FORMULAE

In this Appendix we collected formulae which were useful in our analysis. We use the definition of the Bessel and Legendre functions of Ref. [37]. It follows from the properties of the Legendre functions that the functions $p_j^{(m)}(x), q_j^{(m)}(x)$ defined in (4.40), (4.41) satisfy the following relations

\[ \frac{d}{dx} p_j^{(m)}(x) = p_j^{(m-1)}(x), \quad (A.1a) \]
\[ \frac{d}{dx} q_j^{(m)}(x) = q_j^{(m-1)}(x). \quad (A.1b) \]

The function $q_j^{(m)}(x)$ can be represented in the form

\[ q_j^{(m)}(x) = \frac{e^{i\pi j} Q_j^{(m)}(x + i0) - e^{-i\pi j} Q_j^{(m)}(x - i0)}{2i \sin \pi j}, \quad (A.2) \]

where

\[ Q_j^{(m)}(z) = (z^2 - 1)^{m/2} Q_{j-1}^{-m}(z). \quad (A.3) \]

We give here some integrals involving the functions $p_j^{(m)}(x)$ and $q_j^{(m)}(x)$

\[ \int_{-1}^{1} \frac{dx}{1 - x} p_j^{(m)}(x) = (-1)^m 2^m \Gamma(m) \frac{\Gamma(j - m)}{\Gamma(j + m)}, \quad (A.4a) \]
\[ \int_{-1}^{\infty} \frac{dx}{x + 1} q_j^{(m)}(x) = \frac{2^m \Gamma(m) \Gamma(j - m)}{2 \sin \pi j \Gamma(j + m)} \quad (A.4b) \]
and

\[ \int_{-1}^{\infty} \frac{dx}{x - 1 \pm i0} q_j^{(m)}(x) = (-1)^m 2^m \Gamma(m) \frac{e^{\mp i\pi j} \Gamma(j - m)}{2 \sin \pi j \Gamma(j + m)}. \quad (A.4c) \]

The following integrals involving the Legendre functions were useful

\[ \int_{-1}^{1} dx P_\nu(x) P_m(x) = \frac{2}{\pi} \frac{(-1)^m \sin \pi \nu}{(\nu - m)(\nu + m + 1)}, \quad (A.5a) \]
\[ \int_{1}^{\infty} dx P_\nu(x) Q_\lambda(x) = \frac{1}{(\nu - \lambda)(\nu + \lambda + 1)}, \quad (A.5b) \]

where $m$ is integer. Next, we remind the following relation

\[ P_\nu(z) = \frac{1}{\pi} \tan \pi \nu [Q_\nu(z) - Q_{-\nu-1}(z)] \quad (A.6) \]

between the Legendre functions of the first and second kind. Next, for $z \to 1$, the Legendre function $Q_\nu(z)$ has the following behavior

\[ Q_\nu(z) = -\frac{1}{2} \log(z - 1) + O(1). \quad (A.7) \]
It follows from the Eqs. (A.5b) and (A.6) that
\[
\int_1^\infty dx P_{-1/2+i\rho}(x)P_{-1/2+i\rho'}(x) = \rho^{-1}\coth \pi\rho \left[ \delta(\rho - \rho') + \delta(\rho + \rho') \right]. \tag{A.8}
\]

The Fourier transform of the function \( \Psi_{j}^{(m)}(z) \) (see Eq. (4.15)) is
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dz}{z^m} e^{-izx} \Psi_{j}^{(m)}(z + i0) = \frac{1}{i^m} e^{-i\pi/4 |\xi|} e^{-i\pi/2} \sin \pi j \frac{\Gamma(m-1/2)}{\sqrt{2\pi}} \frac{\Gamma(1-s)}{\Gamma \left( \frac{3-s+\nu}{2} \right)} \tag{A.9}
\]
where \( \text{Re} \, j > m \). We also remind that
\[
\int_{1}^{\infty} dx x^{-s} P_\nu(x) = 2^{s-2} \frac{\Gamma \left( \frac{s+\nu}{2} \right) \Gamma \left( \frac{s-\nu-1}{2} \right)}{\pi^{1/2} \Gamma(s)} \tag{A.10}
\]
\[
\int_{0}^{1} dx x^{-s} P_\nu(x) = 2^{s-1} \frac{\pi^{1/2} \Gamma(1-s)}{\Gamma \left( \frac{2-s-\nu}{2} \right) \Gamma \left( \frac{3-s+\nu}{2} \right)} \tag{A.11}
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