CONFORMAL DYNAMICS AT INFINITY FOR GROUPS WITH CONTRACTING ELEMENTS

WEN-YUAN YANG

Abstract. This paper develops a theory of conformal density at infinity for groups with contracting elements. We start by introducing a class of convergence boundary encompassing many known hyperbolic-like boundaries, on which a detailed study of conical points and Myrberg points is carried out. The basic theory of conformal density is then established, including the Sullivan shadow lemma and Hopf-Tsuji-Sullivan theorem. This gives a unification of the theory of conformal density on the Gromov and Floyd boundary for (relatively) hyperbolic groups, the visual boundary for rank-1 CAT(0) groups, and Thurston boundary for mapping class groups. Besides that, the conformal density on the horofunction boundary provides a new important example of our general theory. Applications include the identification of Poisson boundary of random walks, the co-growth problem of divergent groups, measure theoretical results for CAT(0) groups and mapping class groups.

Contents

1. Introduction 2
  1.1. Convergence compactification 2
  1.2. Conformal density at infinity 6
  1.3. Applications 3
  1.4. Historical remarks and further questions 8

Acknowledgments 10

2. Preliminary 10
  2.1. Notations 10
  2.2. Contracting subsets 10
  2.3. Contracting elements 12
  2.4. Admissible paths 13
  2.5. Extension lemma 14
  2.6. Projection complex 15
  2.7. Horofunction boundary 16

3. Convergence compactification 17
  3.1. Partitions 17
  3.2. Assumptions A & B 18
  3.3. Assumption C 20
  3.4. North-South dynamics of non-pinched contracting elements 21

4. Conical and Myrberg points 23
  4.1. Conical points 23
  4.2. Myrberg points 26
  4.3. Gromov boundary of projection complex 27

5. Convergence of horofunction boundary 27
  5.1. Verifying Assumptions 28
1. Introduction

Suppose that a group $G$ admits a proper and isometric action on a proper geodesic metric space $(Y,d)$. The group $G$ is assumed to be non-elementary: there is no finite index subgroup isomorphic to the integer group $\mathbb{Z}$ or the trivial group. This paper continues the investigation of such group actions with a contracting element from the point of dynamics at infinity. This provides a complementary view to several studies of growth problems carried out in [75, 76, 35, 27].

The contracting property captures the key feature of quasi-geodesics in Gromov hyperbolic spaces, rank-1 geodesics in CAT(0) spaces, and thick geodesics in Teichmüller spaces, and many others. In recent years, this notion and its variants have been proven fruitful in the setup of general metric spaces.

Let $X$ be a closed subset of $Y$, and $\pi_X : Y \to X$ be the shortest projection (set-valued) map. We say that $X$ is $C$-contracting for $C \geq 0$ if

$$\|\pi_X(B)\| \leq C$$

for any metric ball $B$ disjoint with $X$, where $\| \cdot \|$ denotes the diameter. An element of infinite order is called contracting, if it preserves a contracting quasi-geodesic.

The object of this paper is developing a general theory of the topological dynamics and measure theory on the boundary of groups with contracting elements, with applications to growth problems and measure theoretical results. Our results thus comprise the following three parts.

1.1. Convergence compactification. A bordification of a metric space $Y$ is a metrizable Hausdorff topological space $\overline{Y}$ in which $Y$ embeds as an open and dense subset, and the boundary is $\partial Y := \overline{Y} \setminus Y$. We say that a sequence of subsets $X_n$ is exiting if $d(o,X_n) \to \infty$ for some (thus any) basepoint $o \in Y$, and converges to a point $\xi \in \partial Y$ if any sequence of points $x_n \in X_n$ converges to $\xi$.

We are interested in a compactification of $Y$: $\overline{Y}$ and $\partial Y$ are compact, though the language of bordification is convenient, as $Y$ is bordified by any subset of the boundary.

We call $\overline{Y}$ a convergence bordification if the following two assumptions are true:

(A) Any contracting geodesic ray $X$ converges to a boundary point $\xi \in \partial Y$. Any sequence of $y_n \in Y$ with exiting projections $\pi_X(y_n)$ tends to the same $\xi$. 

References
(B) Let $X_n$ be any exiting sequence of $C$-contracting subsets. Then for any $x \in Y$, a subsequence of their cones
$$\Omega_x(X_n) := \{y \in Y : [x, y] \cap X_n \neq \emptyset\}$$
converges to a boundary point in $\partial Y$.

By definition, the convergence bordification persists under Hausdorff quotient, and one-point compactification is a convergence bordification. A convergence bordification is called nontrivial if

(C) the set $\mathcal{C}$ of non-pinched points $\xi \in \partial Y$ is non-empty: if $x_n, y_n \in Y$ converge to $\xi$, then $[x_n, y_n]$ is an exiting sequence.

(See Assump A, Assump B, Assump C in §3 for detailed discussions.)

---

Figure 1. Assumption A (left) and Assumption B (right)

Let us keep in mind the following examples, which motivate and satisfy the above assumptions:

**Examples.**

1. Gromov boundary $\partial G Y$ of a (possibly non-proper) Gromov hyperbolic space $Y$ gives a convergence bordification, where all boundary points are non-pinched.
2. Bowditch boundary $\partial B Y$ of a relatively hyperbolic group $Y$ gives a convergence compactification ([8, 30]). Non-pinched points are exactly conical points, while horocycles (i.e. pinched lines) exist for every parabolic points.
3. Floyd boundary $\partial F Y$ of any locally finite graph $Y$ forms a convergence compactification ([23, 30]). Any pinched point admits a horocycle, so non-pinched points contain at least all conical points.
   The same holds for the end boundary of a locally finite graph.
4. Visual boundary $\partial Vis Y$ of a proper CAT(0) space $Y$ is a convergence boundary, so that all boundary points are non-pinched. See Lemma 10.1.

As (Hausdorff) quotients of a convergence bordification are convergence, this amounts to endowing an $\text{Isom}(Y)$-invariant partition $[\cdot]$ on the original boundary $\partial Y$. Indeed, this gives the following quotient map by identifying points in the same $[\cdot]$-class
$$[\cdot] : \partial Y \rightarrow [\partial Y] := Y/\sim.$$  

However, we emphasize that the partition does not necessarily form a closed relation, so the quotient $[\partial Y]$ might be non-Hausdorff or even not $T_1$ ($[\cdot]$-classes is closed subsets).

With respect to such a partition $[\cdot]$, the bordification $\overline{Y}$ is *convergence* if the convergence to a point $\xi$ in the Assump A, Assump B and Assump C is relaxed to the convergence to a point in its equivalent class $[\xi]$. The above examples of convergence boundaries have been equipped with trivial partitions. See §3 for precise formulations.

Introducing the partition is also indispensable in obtaining a hyperbolic-like boundary, as certain natural subsets in the boundary are “inseparable” (from a view of negative curvature). The following examples illustrate some aspects of such consideration.

**Examples.** Let $\mathcal{T}$ be the Teichmüller space of a closed orientable surface of genus $g \geq 2$, endowed with the following two boundaries (recalled in §11.1 in details):

1. Thurston boundary $\partial Th \mathcal{T}$: it is the set $\mathcal{PMF}$ of projective measured foliations, which is a topological sphere of dimension $6g - 7$ ([20]). Kaimanovich-Masur ([40] considered a partition $[\cdot]$ on $\mathcal{PMF}$ induced by an intersection form. The restriction to the subset $\mathcal{MF}$ of the minimal measured foliations is closed. Its quotient, the (Hausdorff) ending lamination space, can be
identified with the Gromov boundary of the complex of curves on $\Sigma_g$ \[43\] [32]. Inside $\mathcal{MF}$, the partition is trivial exactly on the subset $\mathcal{UE}$ of uniquely ergodic points: their equivalent classes are singletons.

(2) Gardiner-Masur boundary $\partial_{GM} T$ \[24\]: it is homeomorphic to the horofunction boundary of $Y$ with respect to the Teichmüller metric \[48\]. It turned out that $\mathcal{MF}$ embeds as a proper subset (with a different subspace topology) in $\partial_{GM} T$. The relation between $\partial_{Th} T$ and $\partial_{GM} T$ has been carefully investigated in \[56\] [57] [58] [73].

Examples. The horofunction boundary $\partial_h Y$ was introduced by Gromov for any metric space $Y$. Two horofunctions are said to be equivalent if they differ by a bounded amount. This defines a finite difference relation on $\partial_h Y$, and the quotient $[\partial_h Y]$ is usually called reduced horofunction boundary. Here are two more examples we are interested in.

(1) Roller boundary of a locally finite CAT(0) cubical complex is homeomorphic to the horofunction boundary of $Y$ with combinatorial metric \[22\]. Every boundary point is represented by a geodesic ray and the finite difference of horofunctions is equivalent to the finite symmetric difference of boundary points viewed as ultrafilters. The quotient is homeomorphic to the combinatorial boundary studied in \[29\]. See Lemma 10.4.

(2) Martin boundary of a finitely supported random walk on a non-amenable group is homeomorphic to the horofunction boundary defined using (generally non-geodesic) Green metric. The points on which the finite difference relation restricts to be trivial are minimal harmonic functions. The resulting minimal Martin boundary is essential in representing all positive harmonic functions.

Our first result confirms that the above examples with the described partitions provide nontrivial convergence compactifications. The following is proved in \[85\].

**Theorem 1.1.** The horofunction boundary $\partial_h Y$ of a proper geodesic metric space $Y$ with contracting subsets is convergence, with finite difference relation, so that all boundary points are non-pinched.

As a corollary, Gardiner-Masur boundary and Roller boundary are examples of convergence boundary with finite difference relation, see Lemmas 10.4 and 11.4 for relevant properties. Thurston boundary is not a quotient of the Gardiner-Masur boundary, but is also convergence.

**Theorem 1.2** \[=11.3\]. The Thurston boundary $\partial_{Th} T$ is a convergence boundary, with respect to the above Kaimanovich-Masur partition, so that the set of non-pinched points contains $\mathcal{UE}$.

If $Y$ is a hyperbolic space, the finite difference relation can be shown to be closed, so the quotient is a Hausdorff space, homeomorphic to the Gromov boundary. In general, $[\partial_h Y]$ may be neither Hausdorff nor second countable, which cause the topological and measure theoretical issues. To remedy this situation, we shall consider the quotient of the so-called Myrberg points and their relative conical points. Both notions are analogous to and motivated by the corresponding ones in a convergence group action.

In the sequel, we always consider a non-elementary group $G < \text{Isom}(Y, d)$ so that

1. $G$ acts properly on $(Y, d)$ with a contracting element.

Define the limit set $\Lambda Go$ to be the set of accumulation points of a $G$-orbit $Go$ in $\partial Y$. It may depend on the choice of $o \in Y$ but the quotient $[AG] = [\Lambda Go]$ does not by Lemma 3.7. Moreover, under pleasant situations as in Lemma 3.2\[2\] including Roller and Gardiner-Masur boundaries, we can define a unique minimal $G$-invariant closed subset $\Lambda$ with $[\Lambda] = [AG]$ in boundary on which the minimal fixed point pairs of non-pinched contracting elements are doubly dense. Now, let

$$\Lambda^2 G := [AG] \times [AG] \setminus \{\xi = \eta\}$$

be the set of distinct pairs of $[\cdot]$-classes.

**Definition 1.3** (Myrberg points). A point $\xi \in C \subseteq \partial Y$ is called Myrberg point if for any $x \in Y$, the set of $G$-translates of the ordered pair $(x, \xi)$ is dense in $\Lambda^2 G$.

**Remark.** Introduced by Myrberg \[61\], this class of limit points was proved to be full Lebesgue measure in $S^1$ for finite co-volume Fuchsian groups. This was generalized later on to higher dimension \[11\] [71].
We prove that the quotient of Myrberg points by identifying two horofunctions with finite difference enjoys the Hausdorff and second countable properties. As the lack of second countability is a serious drawback in probability theory, this quotient provides an ideal candidate space in Theorem 1.11 on which the unique conformal density lives.

Proposition 1.4 (5.6). The quotient $[\Lambda_m G]$ of the Myrberg limit set in the horofunction boundary $\partial_h Y$ is a Hausdorff, second countable, topological space.

Moreover, $Y \cup [\Lambda_m G]$ forms a convergence bordification of $Y$, endowed with trivial partition, so that all boundary points are non-pinched.

Our notion of conical points is defined relative to a collection $\mathcal{F}$ of contracting subsets with bounded intersection. There are various (quantitatively) equivalent ways to define it. We give a quick definition in Introduction, and the other two elaborating ones are discussed in Section 4.

Let $F$ be a set of three (mutually) independent contracting elements $h_i$ ($i = 1, 2, 3$), which form a contracting system

$$\mathcal{F} = \{g \cdot \text{Ax}(h_i) : g \in G\}$$

where the axis $\text{Ax}(h_i)$ defined in (7) depends on the choice of a basepoint $o \in Y$.

One consequence of Assump A and Assump C together is as follows: the above projection map $\pi_X : Y \rightarrow X$ extends to the subset $L$ of non-pinched boundary points:

$$\pi_X : Y \setminus \Lambda \rightarrow X$$

which satisfies the bounded geodesic image property (for possibly larger $C$):

$$d_X(x, y) \geq C \Rightarrow [x, y] \cap N_C(X) = \emptyset$$

where $d_X(x, y) := \|\pi_X(x) \cup \pi_X(y)\|$. See Lemma 3.13.

Definition 1.5 (Conical points). A non-pinched point $\xi \in \mathcal{C}$ is called $L$-conical point for $L > 0$ relative to $\mathcal{F}$ if for some $x \in Y$, there exists a sequence of $X_n \in \mathcal{F}$ such that $d_{X_n}(x, \xi) \geq L$. Denote by $\Lambda^L \mathcal{F} (Go)$ the set of all such points.

An important ingredient in our argument is the axiomized construction of a projection complex $\mathcal{P}_K(\mathcal{F})$ from $\mathcal{F}$, developed by Bestvina-Bromberg-Fujiwara [6]. Loosely speaking, fixing a constant $K$, $\mathcal{P}_K(\mathcal{F})$ is a graph with vertex set $\mathcal{F}$ so that two vertices $U \neq V \in \mathcal{F}$ are adjacent if and only if they are visually $K$-small from any third party $W \in \mathcal{F} \setminus \{U, V\}$:

$$d_W(U, V) \leq K$$

(see §2.6 for more details). The basic fact is that for any $K \gg 0$, the graph $\mathcal{P}_K(\mathcal{F})$ is a quasi-tree (with universal hyperbolicity constant).

Our next result describes the above-defined conical points via the familiar Gromov boundary of the projection complex.

Theorem 1.6 (4.14). For any $L > 0$ there exists $K > 0$ such that the Gromov boundary of the projection complex $\mathcal{P}_K(\mathcal{F})$ admits a topological embedding onto the set of $L$-conical points relative to $\mathcal{F}$.

The following key fact relates conical points to Myrberg points.

Proposition 1.7 (4.13). The Myrberg limit set $\Lambda_m G$ is the (countable) intersection of all $(L, \mathcal{F})$-conical sets $\Lambda^L \mathcal{F} (Go)$, over all possible $\mathcal{F}$ given by (2) and all large integers $L \in \mathbb{N}$.

We now give an application of previous results to random walks on groups. A probability measure $\mu$ on $G$ is called irreducible if its support generates the group $G$ as a semi-group. A $\mu$-random walk on $G$ is the stochastic process $(w_n)$ defined as

$$w_n := g_1 g_2 \ldots g_n$$

where $(g_n)$ is a sequence of i.i.d. random variables valued in $G$ with law $\mu$. The entropy is defined by

$$H(\mu) := - \sum_{g \in \text{supp}(\mu)} \mu(g) \cdot |\log \mu(g)|.$$
By the work of Maher-Tiozzo [49], the following statement holds for the random walk for acylindrical actions on hyperbolic spaces. The previous two results transfer the results to the horofunction boundary. More details on the proof is given in [82].

**Theorem 1.8.** Let $G \simeq Y$ as in [4]. Consider an irreducible $\mu$-random walk driven by a probability measure $\mu$ on $G$ with finite logarithmic moment on $Y$:

$$
\sum_{g \in G} \mu(g) \cdot |\log d(o, go)| < \infty.
$$

Then almost every trajectory of $\mu$-random walk converges to the $[\cdot]$-class of a Myrberg point in $\partial hY$.

Moreover, if $H(\mu) < \infty$, then the reduced horofunction boundary $[\partial h Y]$ is the Poisson boundary, with harmonic measure supported on the quotient $[\Lambda_m G]$ of Myrberg limit set.

**Remark.** If $Y$ is a CAT(0) space with rank-1 elements, this completes the theorem of Karlsson-Margulis [42] by showing the positive drift. For the essential action on CAT(0) cube complex, our result recovers the main results [22, Thm 1.2] [21] that almost every trajectory tends to a point in the Roller boundary. See Lemma [10.4] for the relevant facts on Roller boundary.

1.2. **Conformal density at infinity.** A family of conformal measures on the limit set of Fuchsian groups were firstly constructed by Patterson [63] and studied extensively by Sullivan [69] in Kleinian groups. Patterson's construction is very robust and carries over verbatim to the general metric spaces with horofunction boundary instead of the limit set (see [10]). The second part of our study is to give a vast generalization of these works in groups with contracting elements, endowed with convergence compactification. Let us first introduce the setup for the Patterson's construction.

Fix a basepoint $o \in Y$. Consider the growth function of the ball of radius $R > 0$:

$$
N(o, R) := \{ v \in Go : d(o, v) \leq n \}.
$$

The critical exponent $\omega_T$ for a subset $\Gamma \subseteq G$:

$$
\omega_T = \limsup_{R \to \infty} \frac{\log \|N(o, R) \cap \Gamma o\|}{R},
$$

is independent of the choice of $o \in Y$, and intimately related to the Poincaré series

$$
\mathcal{P}_\Gamma(s, x, y) = \sum_{g \in \Gamma} e^{-sd(x, gy)}
$$

as $\mathcal{P}_\Gamma(s, x, y)$ diverges for $s < \omega_T$ and converges for $s > \omega_T$. Thus, the action $G \simeq Y$ as in [4] is called of divergent type (resp. convergent type) if $\mathcal{P}_G(s, x, y)$ is divergent (resp. convergent) at $s = \omega_G$.

Consider a convergence compactification $\overline{Y} = Y \cup \partial Y$, and denote by $\mathcal{M}_+ (\overline{Y})$ the set of positive Radon measures on $\overline{Y}$. If $G \simeq Y$ is of divergent type, the *Patterson-Sullivan measures* $\{\mu_x\}_{x \in Y}$ are accumulation points in $\mathcal{M}_+ (\overline{Y})$ of the following family of measures $\{\mu_x^s\}_{x \in Y}$ supported on $Go$

$$
\mu_x^s = \frac{1}{\mathcal{P}_G(s, o, o)} \sum_{g \in G} e^{-sd(x, go)} \cdot \text{Dirac}(go),
$$

as $s > \omega_G$ tends to $\omega_G$. Similar construction works for groups of convergent type via a Patterson's trick [63] on the Poincaré series. The $G$-equivariance of such measures $\mu_x$ on basestopes $x$ and their conformality on switching basestopes are formulated in a general notion of (quasi)-conformal density (see Definition [6.1]).

In [69], Sullivan proved a shadow lemma describing the local nature of Patterson’s measure on the limit set of Kleinian groups. We extend it to our setting. Define the usual shadow $\Pi_z(y, r)$ as the topological closure in $\partial Y$ of the cone:

$$
\Omega_z(y, r) = \{ z \in Y : [x, z] \cap B(y, r) \neq \emptyset \}
$$

Analogous to the partial cone and shadow in [77], we are actually working with a similar version of shadows via contracting segments. With a set $F$ of three independent contracting elements as in [2] and $r > 0$, a segment $\alpha$ in $Y$ contains an $(r, F)$-barrier at $go$ if for some $f \in F$, we have $go, gfo \in N_r([x, z])$. For $x \in Y, y \in Go$ and $r > 0$, the $(r, F)$-cone is defined as follows

$$
\Omega^F_z(y, r) := \{ z \in Y : y \text{ is an } (r, F) \text{-barrier for some geodesic } [x, z] \}$$
whose topological closure in $\partial Y$ gives the $(r, F)$-shadow $\Pi^x_F(y, r)$. See $\S 3$ for details.

Lemma 1.9 ($=6.3$). Let $G \simeq Y$ be as in $[4]$. Let $\{\mu_x\}_{x \in Y}$ be a $\omega$-dimensional $G$-quasi-equivariant, quasi-conformal density on $\partial Y$ for some $\omega > 0$, charging positive measure to the non-pinched points $C$. Then there exists $r_0 > 0$ such that

$$e^{-\omega(d(o, go))} \cdot \mu_o(\Pi^x_y(y, r) \cap C) \leq \mu_o(\Pi^x_y(y, r) \cap C) < e^{-\omega(d(o, go))}$$

for any $g \in G$ and $r \geq r_0$, where $<$, means the inequality holds up to a multiplicative constant depending on $r$ (or a universal constant if no $r$ is displayed).

Remark. By Theorem 1.1, we have $C = \partial_h Y$: the complete version of shadow lemma holds in the horofunction boundary. This is the main new instance, see $\S 1.4$ for a short history of it.

Recall that the usual version of conical points is defined via the usual shadow without involving $\mathcal{F}$:

$$\Lambda_c(G) := \bigcup_{r \geq 2} \bigcup_{y \in Go} \limsup_{y \rightarrow Go} \Pi^x_y(y, r).$$

The main result of this investigation is relating the divergence of Poincaré series to positive conformal measures on conical points. This type of results is usually refereed as Hopf-Tsuji-Sullivan theorem by researchers. Note here that the item on ergodicity of product measures is missing.

Theorem 1.10. Let $G \simeq Y$ be as in $[4]$. Let $\{\mu_x\}_{x \in Y}$ be a $\omega$-dimensional $G$-quasi-equivariant, quasi-conformal density on a convergence boundary $\partial Y$ for some $\omega > 0$. Assume that $\mu_o(C) > 0$. Then the following statements are equivalent: Let $\mathcal{F}$ be as in $[3]$ and any $L > 0$.

1. The Poincaré series $P_G(s, o, o)$ diverges at $s = \omega$;
2. The Myrberg limit set has the full $\mu_x$-measure in $C$;
3. The Myrberg limit set has the positive $\mu_x$-measure;
4. The set of conical points (or $(L, \mathcal{F})$-conical points) has positive $\mu_x$-measure;
5. The set of conical points (or $(L, \mathcal{F})$-conical points) has full $\mu_x$-measure in $C$.

If one of the above statements is true, then $\omega = \omega_G$.

Remark. The theorem is proved via the routine: (1) $\iff$ (4) $\implies$ (5) $\implies$ (3) $\implies$ (2) $\implies$ (4). We comment on the following directions while the others are trivial or classic.

- (1) $\implies$ (4): this is the most difficult direction, whose proof uses the projection complex to build a family of visual sphere decomposition of group elements. With light source at the basepoint $o \in Y$, each visual sphere is shadowed by the previous one and blocks the following one. Such a structure allows to re-arrange the series $P_G(s, o, o)$ as a geometric-like series. See $\S 7.2$ for more details.
- (5) $\implies$ (3): This follows from Proposition 1.7 as $\Lambda_m G$ is the countable intersection of $\mu_o$-full $(L, \mathcal{F})$-conical points, over all possible $\mathcal{F}$ in $[2]$.

Under any statement in Theorem 1.10 the conformal density turns out to be unique when passed to the quotient of Myrberg limit set (which is Hausdorff and second countable by Proposition 1.4 above).

Theorem 1.11 ($=8.4$). Let $G \simeq Y$ be as in $[4]$ of divergent type, compactified by the horofunction boundary $\partial_h Y$. Then the $\omega_G$-dimensional $G$-quasi-equivariant, quasi-conformal density on the quotient $[\Lambda_m G]$ of $\Lambda_m G \subseteq \partial_h Y$ is unique up to a bounded multiplicative constant, and ergodic.

By Rohklin’s theory, the quotient $[\Lambda_m G]$ with conformal measures are Lebesgue spaces ($15$, Lemma 15.4)], i.e. isomorphic to the interval with Lebesgue measure.

At last, we note the following corollary to the non-elementary action on a CAT(0) cube complex for further reference. Such an action is called essential if no half-space contains an orbit in its fixed finite neighborhood (See $\S 11$).

Theorem 1.12. Let $G \simeq Y$ in $[4]$ be an essential action of divergent type on a CAT(0) cube complex compactified by the Roller boundary $\partial_R Y$. Then the $\omega_G$-dimensional $G$-quasi-equivariant, quasi-conformal density on $\partial_R Y$ is unique up to a bounded multiplicative constant, and ergodic.
1.3. Applications. We now present various applications to the growth problems, and measure theoretical results in CAT(0) groups and mapping class groups.

We start with the following co-growth results. This extends the work of [53] about divergent group actions on hyperbolic spaces, and answers positively [2, Questions 4.1 and 4.2].

**Theorem 1.13** (9.1). Let $G \bowtie Y$ be as in (1). Suppose that the group $G$ is of divergent type. Then for any infinite normal subgroup $H$ of $G$, we have

$$\omega_H > \frac{\omega_G}{2}$$

**Remark.** In [2], Arzhantseva-Cashen proved the same result under the assumption that $G \bowtie Y$ has purely exponential growth. This already includes many interesting examples, but the divergent action is weaker than having purely exponential growth.

Contracting boundary for CAT(0) spaces was introduced by Charney-Sultan [12] as quasi-isometric invariant, and has attracted active research interests in recent years. It is observed that the contracting boundary is measurably negligible in harmonic measures. We derive the same result in conformal measures from a more general Theorem 10.5. Here let us state it in CAT(0) spaces.

The underlying set of the contracting boundary consists of the endpoints of contracting geodesic rays in the visual boundary.

**Theorem 1.14.** Let $G \bowtie Y$ in (1) be co-compact, where $Y$ is a proper CAT(0) space. Let $\{\mu_x\}_{x \in Y}$ be the $\omega_G$-dimensional conformal density on the visual boundary $\partial_{Vis} Y$. Then the underlying set of contracting boundary is $\mu_o$-null if and only if $G$ is not a hyperbolic group.

At last, for subgroups of mapping class groups, we can complete the Hopf-Tsuji-Sullivan Theorem 1.10 with ergodicity of product measures.

**Theorem 1.15** (§11.3). Consider any non-elementary subgroup $G$ of $\text{Mod}(\Sigma_g)$ $(g \geq 2)$. Let $\{\mu_x\}_{x \in Y}$ be a $\omega$-dimensional $G$-equivariant conformal density on $\partial_{Th} T = \mathcal{P.M.F}$. Then the following are equivalent:

1. The Poincaré series $P_G(s,o,o)$ of $G$ diverges at $\omega$;
2. The Myrberg set $\Lambda_m G$ has either full or positive $\mu_x$-measure;
3. The conical set $\Lambda_m G$ has either full or positive $\mu_x$-measure;
4. The diagonal action on $\mathcal{P.M.F} \times \mathcal{P.M.F}$ is ergodic with respect to the product measure $\mu_o \times \mu_o$.

**Remark.** If $G = \text{Mod}(\Sigma_g)$ the uniqueness of $\omega_G$-dimensional conformal density was proved [32] [46], and the ergodicity was known long before by work of Masur [51] and Veech [72]. The above statement, however, seems new for non-elementary proper subgroups.

1.4. Historical remarks and further questions. In a series of works [75, 76, 35, 27] without using ergodic theory, the elementary and geometric arguments are employed to establish coarse counting results for groups with contracting elements in general metric spaces. Analogous to Anosov shadowing/closing property, a geometric tool called extension lemma was used (recalled in Lemma 2.15): roughly speaking, any two geodesics can be connected by a short arc to form a quasi-geodesic. The short arc is provided by a contracting segment, whose existence facilities the extensive use of Gromov-hyperbolic geometry. On account of our previous works, the complementary view from the ergodic theory is anticipated, and actually forms the primary goal of the present paper, as forementioned.

The conformal density on the limit set of Fuchsian groups was famously constructed by Patterson [63], and further developed by Sullivan [69] in Kleinian groups and in a number of applications. Particularly, their work sets the right track for the further generalization of (quasi)-conformal density on boundaries of hyperbolic spaces [13], CAT(-1) spaces [10, 68], and CAT(0) spaces [66, 47]. The Sullivan Shadow Lemma [69] provides a very useful tool in applications, and thus is most desirable in any “good” theory of conformal density. Although the Sullivan’s proof carries over to Gromov hyperbolic spaces, its generalization in CAT(0) manifolds and spaces follows a different argument given by Knieper [45, 44, Remark on p.781]) and Link [47]. In Teichmüller space, the Shadow lemma is obtained in [70] with applications to fundamental inequality of random walks (see [25, Lemma 5.1] for a special case).

Attempting to formulate a unified framework for the (quasi)-conformal density in the above examples motivates the investigation of this paper.
In [77], the quasi-conformal density on Floyd and Bowditch boundary of relatively hyperbolic groups is used to study growth problems in word metrics. Like Teichmüller space, the Cayley graph of a relatively hyperbolic group is generally neither Gromov hyperbolic nor CAT(0) spaces. Moreover, the lack of Buseman cocyles at all boundary points forces us to define Buseman (quasi-)cocyles only at conical points. To handle non-conical points, we prove the constructed Patterson-Sullivan measures are fully supported on conical points. This strategy is axiomized in the present paper as the Assump D. As another instance, the conformal density obtained on the Thurston boundary [4] follows a similar route detailed in §11.2.

Concerning the convergence boundary, it would be interesting to compare with Karlsson’s theory of stars at infinity [41]. Defined using half-spaces, the stars describe some incidental relation on boundary, but do not form a partition in general. A nice feature of his theory allows to formulate an analogue of convergence group property. Some partial convergence property also exists on our conical points but is not included as we do not see an application.

**Further questions.** We discuss some questions we found interesting about the convergence boundary. The notion of contracting subsets is usually called strongly contracting in literature: there are many coexisting notions of contracting subsets (see [3]). We are working with proper geodesic metric spaces for simplicity, but the framework proposed here could be adapted to non-geodesic and non-symmetric metric spaces with strongly contracting elements. Here are two concrete examples in potential applications:

1. Martin boundary of groups with nontrivial Floyd boundary, that is the horofunction boundary of Green metric (see [28]).
2. Horofunction boundary of outer space for $\text{Out}(\mathbb{F}_n)$ with Lipschitz (asymmetric) metric.

On the other hand, a challenging question would be the following.

**Question 1.16.** Does there exist a “good” theory of the convergence boundary for groups with weakly contracting elements / Morse elements, and the corresponding conformal density at infinity?

Comparsion of different boundaries is an interesting research topic, and has been well-studied for convergence group actions, see the work of Floyd and Gerasimov [23, 30]. Recently, a surjective continuous map from Martin boundary to Floyd boundary was discovered in [28], where the Martin boundary can be seen as the horofunction boundary of the Green metric. As the dynamical notions of conical points and Myrberg points witness, the convergence boundary shows many similarities with convergence group actions. In view of these results, we are interested in understanding.

**Question 1.17.** Which group action on a convergence compactification $G \sim \overline{\mathbb{Y}}$ admit a quotient with a nontrivial convergence group action?

All known examples with this property have non-trivial Floyd boundary. It is easy to check that the horofunction boundary for word metric surjects to the Floyd boundary [1]. It is still open that whether such groups are necessarily relatively hyperbolic.

The last question concerns about the Myrberg limit set. We can prove that Myrberg limit set in Floyd boundary persists in quotients: there exists a homeomorphism from the Myrberg limit set in the Floyd boundary onto the Myrberg limit set in any of its quotient with a non-trivial convergence group action. This result seems providing a positive evidence to the following.

**Conjecture 1.18.** Suppose a group $G$ acts geometrically on two proper CAT(0) spaces $X$ and $Y$. Then there exists a $G$-equivariant homeomorphism between the corresponding Myrberg sets.

As the two CAT(0) actions share the same set of rank-1 elements, it still has some chance to have a positive answer. Compare with the famous examples of Croke-Kleiner [17] and the quasi-isometric invariant boundaries recently studied in [12, 14, 64].

At the completion of writing of this paper, the author received a preprint of Gekhtman-Qing-Rafi [26] which contains similar results as Theorems 1.8 and 1.14 for sublinearly Morse directions. On this regard, it would be interesting to compare sublinearly Morse boundary with Myrberg limit set.

Recently, R. Coulon independently developed a Patterson-Sullivan theory on the horofunction boundary in [10], in a large overlap with ours, including Shadow lemma 6.3, Theorems 1.10, 1.11 and 1.13.

\footnote{The author learnt this fact from A. Karlsson in 2010.}
However, our proof of Theorem 1.10 via the projection complex machinery are very different from his approach, and is actually a novelty of our study. The measurable quotient of horofunction boundary in Theorem 1.11 is a Lebesgue space, which is not known in [16].

**Organization of the paper.** The paper is organized into three parts. The first part from Sections 2 - 3 presents the basic theory of convergence boundary. After the preparatory section 2 we introduce the notion of convergence boundary in §3 and derive basic facts for later use. The study of conical points and Myrberg points is carried out in §4 with Proposition 1.7 and Theorem 1.6 proved among others. Theorem 1.11 about convergence of horofunction boundary is shown in §5.

The second part from Sections 6, §5 develops the theory of conformal density on convergence boundary. Shadow Lemma 6.3 and Principle 6.9 are established in §5 and the Hopf-Tsuji-Sullivan Theorem 1.10 in §6. Section §5 contains two different but related results: the unique conformal density Theorem 1.11 on divergent groups, and the reduced Myrberg limit set as Poisson boundary stated in Theorem 1.8.

The final sections 9 - 11 collect various applications of this study. In §9 we establish a co-growth property for divergent groups, answering questions in [2]. Among others, Theorem 1.14 is derived as a corollary of a general Theorem 10.5 in §10, and Theorem 1.15 for mapping class groups is proved in §11.

**Acknowledgments.** This work started in December 2015, where the author was visiting Université de Lille 1 under a 3-month CNRS research fellowship. He wishes to thank Professor Leonid Potyagailo for the invitation and the hospitality of Math department. During the visit, the proof of shadow lemma was obtained after discussions with Ilya Gekhtman and was explained to Fanny Kassel. The author thanks Ilya for his interests since then, and Rémi Coulon for related conversations in August 2018 and May 2019. Thanks also to Hideki Miyachi for sharing his prints, and Anthony Genevois, Weixu Su and Giulio Tiozzo for many helpful discussions. Part of the writing was completed during the visit to the Institute for Advanced Study in Mathematics in Zhejiang University.

2. **Preliminary**

2.1. **Notations.** Let $(Y,d)$ be a proper geodesic metric space. The shortest projection of a point $y \in Y$ to a closed subset $X \subseteq Y$ is given by

\[ \pi_X(y) = \{ x \in X : d(y,x) = d(y,X) \}, \]

and of a subset $A \subseteq Y$, $\pi_X(A) = \bigcup_{a \in A} \pi_X(a)$. Denote by $|A|$ the diameter of a set $A$.

Let $\alpha \in Y$ be a path with arc-length parametrization from the initial point $\alpha^-$ to the terminal point $\alpha^+$. $[x,y]_{\alpha}$ denotes the parametrized subpath of $\alpha$ going from $x$ to $y$, while $[x,y]$ is a choice of a geodesic between $x, y \in Y$.

A path $\alpha$ is called a $c$-quasi-geodesic for $c \geq 1$ if for any rectifiable subpath $\beta$,

\[ \ell(\beta) \leq c \cdot d(\beta^-,\beta^+) + c \]

where $\ell(\beta)$ denotes the length of $\beta$.

Denote by $\alpha \cdot \beta$ (or simply $\alpha \beta$) the concatenation of two paths $\alpha, \beta$ provided that $\alpha^+ = \beta^-$.

Let $\alpha: \mathbb{R} \to Y$ be an arc-length parameterized bi-infinite path. The restriction of $\alpha$ to $[a, +\infty)$ for $a \in \mathbb{R}$ is referred to as a positive ray, and its complement a negative ray. By abuse of language, we often denote them by $\alpha^+$ and $\alpha^-$ (in particular, when they represent boundary points to which the half rays converge).

Let $f, g$ be real-valued functions. Then $f <_c g$ means that there is a constant $C > 0$ depending on parameters $c_i$ such that $f < Cg$. The symbol $>_c$ is defined similarly, and $\simeq_c$ means both $<_c$ and $>_c$ are true. The constant $c_i$ will be omitted if it is a universal constant.

2.2. **Contracting subsets.**

**Definition 2.1** (Contracting subset). For given $C \geq 1$, a subset $U \in Y$ is called $C$-contracting if for any geodesic $\gamma$ with $d(\gamma,U) \geq C$, we have

\[ |\pi_U(\gamma)| \leq C. \]

A collection of $C$-contracting subsets is referred to as a $C$-contracting system.

Contracting property has several equivalent characterizations. When speaking about $C$-contracting property, the constant $C$ shall be assumed to satisfy the following three statements.
Lemma 2.2. Let $U$ be a contracting subset. Then there exists $C > 0$ such that

1. If $d(\gamma, U) \geq C$ for a geodesic $\gamma$, we have $\|\pi_U(\gamma)\| \leq C$.
2. If $\|\pi_U(\gamma)\| \geq C$ then $d(\pi_U(\gamma^-), \gamma) \leq C$, $d(\pi_U(\gamma^+), \gamma) \leq C$.
3. For a metric ball $B$ disjoint with $U$, we have $\|\pi_U(B)\| \leq C$.

We continue to list a few more consequences used throughout this paper.

A subset $U \subseteq Y$ is called $\sigma$-Morse for a function $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ if given $c \geq 1$, any $c$-quasi-geodesic with endpoints in $U$ lies in $N_{\sigma(c)}(U)$.

Lemma 2.3. Let $U \subseteq Y$ be a $C$-contracting subset for $C > 0$.

1. There exists $\sigma = \sigma(C)$ such that $U$ is $\sigma$-Morse.
2. For any $r > 0$, there exists $\hat{C} = \hat{C}(C, r)$ such that a subset $V \subseteq Y$ within $r$-Hausdorff distance to $U$ is $\hat{C}$-contracting.
3. Let $\gamma$ be a geodesic such that $N_r(U) \cap \gamma = \{\gamma^\pm\}$ for given $r \geq C$. Then
   $$\pi_U(\gamma^\pm) \subseteq B(\gamma^\pm, r + C).$$
4. For any geodesic $\gamma$, we have
   $$|d_U(\gamma^-, \gamma^+) - \|\gamma \cap N_C(U)\|| \leq 4C.$$
5. There exists $\hat{C} = \hat{C}(C)$ such that $d_U(y, z) \leq d(y, z) + \hat{C}$ for any $y, z \in Y$.

In this paper, we are interested in a contracting system $\mathcal{F}$ with $\tau$-bounded intersection property for a function $\tau : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ so that the following holds

$$\forall U, V \in \mathcal{F} : \|N_r(U) \cap N_r(V)\| \leq \tau(r)$$

for any $r \geq 0$. This is, in fact, equivalent to a bounded projection property of $\mathcal{F}$: there exists a constant $B > 0$ such that the following holds

$$\|\pi_U(V)\| \leq B$$

for any $U \neq V \in \mathcal{F}$.

We now state two elementary lemmas used later on. Consider a $C$-contracting subset $U \subseteq Y$ and $r \geq 100C$. Let $\alpha$ be a geodesic so that $\|N_r(U) \cap \alpha\| > 3r$. Consider the entry and exit points of $\alpha$ in $N_r(U)$: let $x, y \in \alpha \cap N_r(U)$ such that $d(x, y) = \|N_r(U) \cap \alpha\|$. Let $z \in N_r(U) \cap \alpha$.

Lemma 2.4. The entry and exit points of $\alpha$ in $N_C(U)$ are $(3r/2)$-close to $x$ and $y$ respectively. In particular,

$$\|N_C(U) \cap \alpha\| \geq \|N_r(U) \cap \alpha\| - 3r$$

$$d_U(\alpha^-, \alpha^+) \geq \|N_r(U) \cap \alpha\| - 4r.$$ 

Proof. By hypothesis, we have $d(x, y) > 3r$. First notice that $N_C(U) \cap \alpha \neq \emptyset$. Otherwise, if $N_C(U) \cap \alpha = \emptyset$, the contracting property gives

$$d(x, y) \leq d(x, U) + d(y, U) + \|\pi_U(\alpha)\| \leq 2r + C < 3r.$$ 

This is a contradiction.

Let us thus choose $u, v \in [x, y] \cap N_C(U)$ such that $d(u, v) = \|N_C(U) \cap \alpha\|$. Using again contracting property of $U$, we obtain

$$d(x, u) \leq d(x, U) + d(u, U) + \|\pi_U([x, u])\| \leq r + 2C \leq 3r/2.$$ 

Similarly, $d(v, y) \leq r + 2C \leq 3r/2$. Thus, $d(u, v) \geq d(x, y) - 3r$, so the conclusion follows. \hfill \Box

Lemma 2.5. Let $\beta$ be a geodesic such that $d_U(\alpha^-, \beta^-) \leq 10C$ and $d_U(\alpha^+, \beta^+) \leq 10C$. Then the enter and exit points of $\beta$ in $N_r(U)$ are $(4r)$-close to $x$ and $y$ respectively.
Proof. We first prove that $\beta \cap N_C(U) \neq \emptyset$. Indeed, assume to the contrary that $\beta \cap N_C(U) = \emptyset$, so $|\pi_U(\beta)| \leq C$ by the contracting property.

Since $[x, \alpha^-] \cap N_r(U) = \emptyset$ and $[\alpha^-, y] \cap N_r(U) = \emptyset$, the contracting property again shows $d_U(x, \alpha^-) \leq C$ and $d_U(\alpha^-, y) \leq C$. We compute

$$d(x, y) \leq d(x, U) + d_U(x, \alpha^-) + d_U(\alpha^-, \beta^-) + d_U(\beta^-, y) + d(y, U) \leq 14C + 2r < 3r.$$  

This contradicts the assumption that $d(x, y) > 3r$. Thus, $\beta \cap N_C(U) \neq \emptyset$ is proved, so we can choose $u, v \in \beta \cap N_C(U)$ such that $d(u, v) = |N_C(U) \cap \beta|$. Using the contracting property of $U$, we obtain

$$d(x, u) \leq d(x, U) + d_U(x, \alpha^-) + d_U(\alpha^-, \beta^-) + d_U(\beta^-, u) + d(u, U) \leq 2r + 12C \leq (5/2)r.$$  

Similarly, we have $d(v, y) \leq 2r + 12C \leq (5/2)r$.

By Lemma 2.4 the enter and exit points of $\beta$ in $N_r(U)$ are $(3r/2)$-close to $u$ and $v$ respectively, so are $(4r)$-close to the enter and exit points $x, y$ of $\alpha$ respectively. The proof of the lemma is proved. \qed

2.3. Contracting elements. A subgroup $H$ in $G$ is called contracting if for some (hence any) $o \in Y$, the subset $Ho$ is contracting in $Y$. Define the stabilizer $E(H)$ as follows:

$$E(H) = \{g \in G : \exists r > 0, gHo \leq N_r(Ho), Ho \leq N_r(gHo)\}.$$  

An infinite order element $h \in G$ is called contracting if the subgroup $\langle h \rangle$ is contracting. The set of contracting elements is preserved under conjugacy. We say that $h$ has the quasi-isometric image property if the orbital map

$$n \in \mathbb{Z} \rightarrow h^no \in Y$$  

is a quasi-isometric embedding. Under this assumption, we are able to precise the group $E(h) = E(\langle h \rangle)$.

**Lemma 2.6.** [75] Lemma 2.11] For a contracting element $h$ with quasi-isometric image, we have

$$E(h) = \{g \in G : \exists n \in \mathbb{N}_{>0}, (gh^n g^{-1} = h^n) \lor (gh^n g^{-1} = h^{-n})\}.$$  

Keeping in mind the basepoint $o \in Y$, the axis of $h$ is defined as the following quasi-geodesic

$$Ax(h) = \{fo : f \in E(h)\}.$$  

Notice that $Ax(h) = Ax(k)$ and $E(h) = E(k)$ for any contracting element $k \in E(h)$.

An element $g \in G$ preserves the orientation of a bi-infinite quasi-geodesic $\gamma$ if $\alpha$ and $g\alpha$ have finite Hausdorff distance for any half ray $\alpha$ of $\gamma$. Let $E^+(h)$ be the subgroup of $E(h)$ with possibly index 2 which elements preserve the orientation of their axis. Then we have

$$E^+(h) = \{g \in G : \exists n \in \mathbb{N}_{>0}, gh^n g^{-1} = h^n\}.$$  

and $E^+(h)$ contains all contracting elements in $E(h)$. In the sequel, unless explicitly mentioned, we always assume contracting elements to satisfy quasi-isometrically embedded image property [6].

Two contracting elements $h_1, h_2 \in G$ are called independent if the collection $\{gAx(h) : g \in G ; i = 1, 2\}$ is a contracting system with bounded intersection. Note that two conjugate contracting elements with disjoint fixed points are not independent in our sense. This is slightly different from the independence used by other researchers (cf. [19]).

**Definition 2.7.** Fix $r > 0$ and a set $F$ in $G$. A geodesic $\gamma$ contains an $(r, f)$-barrier for $f \in F$ if there exists an element $h \in G$ so that

$$\max\{d(h \cdot o, \gamma), d(h \cdot fo, \gamma)\} \leq r.$$  

By abuse of language, the point $ho$ or the axis $hAx(f)$ is called $(r, F)$-barrier of type $f$ on $\gamma$.

Recall that $|Fo|$ denotes the diameter of $Fo$. Sometimes, it is useful to denote $|Fo|_{\min} := \min\{d(o, fo) : f \in F\}$ the minimal length of elements in $F$. 
Lemma 2.8. For any \( r > 0 \) there exists \( \hat{r} = \hat{r}(r) > 0 \) with the following property.

Let \( f \) be a contracting element with \( d(o, fo) > 3r \). Suppose that a geodesic \( \alpha \) contains an \((r, f)\)-barrier \( X = gA(x, f) \). Let \( \beta \) be a geodesic such that \( d_X(\alpha^-, \beta^+) \leq 10C \) and \( d_X(\alpha^+, \beta^+) \leq 10C \). Then \( g \) is an \((\hat{r}, f)\)-barrier for \( \beta \).

Proof. Let \( x, y \in \alpha \cap N_r(X) \) be the enter and exit points. The \((r, f)\)-barrier \( go \) for \( \alpha \) implies that \( d(go, [x, y]) \leq r \) and \( d(gfo, [x, y]) \leq r \). By Lemma 2.5 the entry and exit points \( u, v \) of \( \beta \in N_r(X) \) are \((4r)\)-close to those \( x, y \) respectively.

As \( X \) is a contracting quasi-geodesic, any geodesic subsegment (say \([u,v]\)) in its \( r\)-neighborhood is also uniformly contracting by \([76, \text{Prop. 2.2}]\). By Morse property, there exists \( \tau = \tau(r) > 0 \) such that \([x,y]\) is contained in a \( \tau \)-neighborhood of \([u,v]\). Setting \( \hat{r} = 4r + \tau \) then completes the proof. \( \square \)

Let \( h \) be a contracting element and \( \gamma := Ax(h) \) be a \( C\)-contracting axis in \([7]\). Here we understand \( \gamma \) as a quasi-geodesic path for simplicity. By the Morse property, there exist constants \( r, L \) depending only on \( C \) such that for given \( f \in E^*(h) \), any geodesic segment in \( N_r(\gamma) \) of length \( L + d(o, fo) \) contains \([o, fo] \). We record this fact as the following.

Lemma 2.9. There exist \( r = r(C), L = L(C) > 0 \) with the following property.

Let \( x, y \in \alpha \cap N_r(\alpha) \) be the entry and exit points of \( \gamma \) in a geodesic segment \( \alpha \). If \( z \in [x, y] \gamma \) is a point such that \( d(z, x), d(z, y) \geq L + d(o, fo) \), then \( z \) is an \((r, F)\)-barrier for \( \alpha \), where \( F := \{f, f^{-1}\} \).

Lastly, the following result will be used later on.

Lemma 2.10. \([74, \text{Lemma 4.6}]\) Let \( G \sim Y \) be as in \([7]\) and \( \Gamma < G \) be an infinite normal subgroup. Then \( \Gamma \) must be non-elementary and contains infinitely many independent contracting elements.

2.4. Admissible paths. A class of admissible paths introduced in \([74]\) is of great importance in our study, which are quasi-geodesics with fellow travel property relative to a \( C\)-contracting system \( \mathcal{F} \) in \( Y \).

Definition 2.11 (Admissible Path). Given \( L, B \geq 0 \), a path \( \gamma \) is called \((L, B)\)-admissible relative to \( \mathcal{F} \), if the path \( \gamma = p_0 q_1 p_1 \ldots q_n p_n \) is a piece-wise geodesic path satisfying the Long Local and Bounded Projection properties:

(1) Each \( p_i \) for \( 0 \leq i \leq n \) has the two endpoints in \( X_i \in \mathcal{F} \) and length bigger than \( L \) unless \( i = 0 \) or \( i = n \).

(2) For each \( X_i \), we have

\[
\|\pi_{X_i}(q_{i+1})\| \leq B, \quad \|\pi_{X_i}(q_i)\| \leq B
\]

whenever the previous and next paths \( q_i, q_{i+1} \) are defined.

The indexed collection of \( X_i \in \mathcal{F} \) as above, denoted by \( \mathcal{F}(\gamma) \), will be referred to as the saturation of \( \gamma \).

We say that an \((L, B)\)-admissible ray \( \gamma \) is of infinite type if either its saturation \( \mathcal{F}(\gamma) \) contains infinitely many elements, or \( \gamma \) is eventually contained in a finite neighborhood of a contracting subset \( X \in \mathcal{F}(\gamma) \).

As the properties (LL) and (BP) are local conditions, we can manipulate admissible paths as follows:

Subpath. A subpath of an \((L, B)\)-admissible path is \((L, B)\)-admissible.

Concatenation. Let \( \alpha, \beta \) be two \((L, B)\)-admissible paths so that the last contracting subset associated to the geodesic \( p \) of \( \alpha \) is the same as the first contracting subset to the geodesic \( q \) of \( \beta \). If \( d(p^-, q^+) > L \), the path \( \gamma \) by concatenating paths

\[
\gamma := [\alpha^-, p^-]_\alpha \cdot [p^-, q^+] \cdot [q^+, \beta^+]_\beta
\]

is \((L, B)\)-admissible.

A sequence of points \( x_i \) in a path \( p \) is called linearly ordered if \( x_{i+1} \in [x_i, p^+]_p \) for each \( i \).

Definition 2.12 (Fellow travel). Let \( \gamma = p_0 q_1 p_1 \ldots q_n p_n \) be an \((L, B)\)-admissible path. We say \( \gamma \) has \( r\)-fellow travel property for some \( r > 0 \) if for any geodesic \( \alpha \) with the same endpoints as \( \gamma \), there exists a sequence of linearly ordered points \( z_i, w_i \) \((0 \leq i \leq n)\) on \( \alpha \) such that

\[
d(z_i, p^-_i) \leq r, \quad d(w_i, p^+_i) \leq r.
\]

Remark. In particular, \( |N_r(X_i) \cap \alpha| \geq L \) for each \( X_i \in \mathcal{F}(\gamma) \). This is basically amount to saying that \( X_i \) is an \((\hat{r}, F)\)-barrier for \( \alpha \). These two terminologies have their own convenience, even though.
The following result says that a local long admissible path enjoys the fellow travel property.

**Proposition 2.13.** [74] For any $B > 0$, there exist $L, r > 0$ depending only on $B, C$ such that any $(L, B)$-admissible path has $r$-fellow travel property.

The constant $B$ in an $(L, B)$-admissible path can be made uniform by the following operation.

**Truncation.** By the contracting property, a geodesic issuing from a $C$-contracting subset diverges in an orthogonal way from its $C$-neighborhood. At the expense of decreasing $L$, we can thus truncate these diverging parts to obtain the constant $B$ depending only on $\mathcal{F}$. We explain this procedure in a more general form.

**Lemma 2.14.** Assume that $\|F_0\|_{\min} > 3r$ for some $r > 0$. There exist $L = L(\|F_0\|_{\min}, r), B = B(\mathcal{F}) > 0$ with the following property.

Let $\alpha$ be a geodesic with a set of distinct $(r, F)$-barriers $X_\alpha \in \mathcal{F}$. Then there exists an $(L, B)$-admissible path $\hat{\alpha}$ with the same endpoints as $\alpha$ with saturation $\{X_\alpha\}$.

The path $\hat{\alpha}$ is called truncation of $\alpha$. If $\alpha$ is a fellow travel geodesic for the admissible path $\gamma$ in Proposition 2.13, then $\alpha$ has $(r, F)$-barriers $X \in \mathcal{F}(\gamma)$. The truncation $\hat{\alpha}$ will also be referred as truncation of $\gamma$ and has a uniform constant $B$.

**Proof.** By definition, $|\alpha \cap N_r(X_\alpha)| \geq \|F_0\|_{\min}$. Consider the entry and exit points $z'_i, w'_i \in \alpha$ in $N_C(X_i)$ by Lemma 2.4 such that $d(z'_i, w'_i) = |\alpha \cap N_C(X_i)| \geq \|F_0\|_{\min} - 3r$.

Let $z_i, w_i \in X_i$ such that $d(z_i, z'_i) = d(w_i, w'_i) = C$, where $z_0 := \alpha^-$ and $w_n := \alpha^+$. Since $\mathcal{F}$ has $\tau$-bounded intersection and then bounded projection, one deduces

$$\|\pi_{X_i}([w_i, z_{i+1}])\| \leq B$$

for some $B$ depending only on $\tau$ and $C$. Note that $d(z_i, w_i) \geq \|F_0\|_{\min} - 3r - 2C$. Denoting $\hat{p}_i = [z_i, w_i]$ and $\hat{q}_{i+1} = [w_i, z_{i+1}]$, the path

$$\hat{\alpha} := \hat{p}_0 \hat{q}_1 \hat{p}_1 \ldots \hat{q}_n \hat{p}_n$$

is an $(L, B)$-admissible path with the same endpoints as $\alpha$. $\blacksquare$

### 2.5. Extension lemma

Admissible paths have demonstrated strong fellow travel properties. We now introduce a basic tool to build admissible paths.

**Lemma 2.15 (Extension Lemma).** There exist $L, r, B > 0$ depending only on $C$ with the following property.

Choose any element $f_i \in \langle h_i \rangle$ for each $1 \leq i \leq 3$ to form the set $F$ satisfying $\|F_0\|_{\min} \geq L$. Let $g \in G$ and $\alpha$ be a geodesic with $\alpha^- = o$.

(1) There exists an element $f \in F$ such that the path

$$\gamma := [o, go] \cdot (g[o, f o]) \cdot (g f o)$$

is an $(L, B)$-admissible path relative to $\mathcal{F}$.

(2) The point $go$ is an $(r, f)$-barrier for any geodesic $[\gamma^-, \gamma^+]$.

**Remark.**

(1) In [75], the geodesic $\alpha = [o, ho]$ is chosen to be ending at $Go$ for some $h \in G$. However, the proof given there works for a geodesic starting from $o$ possibly ending at any point in $Y$.

(2) Since admissible paths are local conditions, we can concatenate the admissible paths for any number of elements provided $\alpha$ ending at $Go$. We refer the reader to [75] for precise formulation.

In defining conical limit points, we often need to specify the contracting system $\mathcal{F}$ and consider the admissible path relative to $\mathcal{F}$.

**Convention 2.16.** Let $(\mathcal{F}, F, r, \hat{r}, L, B)$ be given by Lemma 2.15 so that

(1) the constants $r, L, B$ further satisfy Lemmas 2.6 and 2.14.

(2) the constant $\hat{r}$ given by Lemma 2.6 depends only on $r$.

In particular, all the constants depend only on the contracting constant $C$ of $\mathcal{F}$. 
2.6. Projection complex. In this subsection, we briefly recall the work of Bestvina-Bromberg-Fujiwara \[6\] on constructing a quasi-tree of spaces.

**Definition 2.17** (Projection axioms). Let $\mathcal{F}$ be a collection of metric spaces equipped with (set-valued) projection maps $\\{\pi_U : \mathcal{F} \setminus \{U\} \to U \space{\cup} \mathcal{F} \}_{U \in \mathcal{F}}$. Denote $d_U(V,W) := \|\pi_U(V) \cup \pi_U(W)\|$ for $V \neq U \neq W \in \mathcal{F}$. The pair $(\mathcal{F}, \{\pi_U\}_{U \in \mathcal{F}})$ satisfies projection axioms for a constant $\kappa \geq 0$ if

1. $|\pi_U(V)| \leq \kappa$ when $U \neq V$.
2. if $U,V,W$ are distinct and $d_V(U,W) > \kappa$ then $d_U(V,W) \leq \kappa$.
3. the set $\{U \in \mathcal{F} : d_U(V,W) > \kappa\}$ is finite for $V \neq W$.

By definition, the following triangle inequality holds

\[ d_Y(V,W) \leq d_Y(V,U) + d_Y(U,W). \]

It is well-known that the projection axioms hold for a contracting system with bounded intersection (cf. \[74\] Appendix). 

**Lemma 2.18.** Let $h$ be any contracting element in a proper action of $G \sim Y$. Then the collection $\mathcal{F} = \{gAx(h) : g \in G\}$ with shortest projection maps $\pi_U(V)$ satisfies the projection axioms with constant $\kappa = \kappa(\mathcal{F}) > 0$.

In \[6\] Def. 3.1, a modified version of $d_U$ is introduced so that it is symmetric and agrees with the original $d_U$ up to an additive amount $2\kappa$. So, the axioms (1)-(3) are still true for $3\kappa$, and the triangle inequality in \[6\] holds up to a uniform error. In what follows, we actually need to work with this modified $d_U$ to define the complex projection, but for sake of simplicity, we stick on the above definition of $d_U$.

We consider the interval-like set for $K > 0$ and $V,W \in \mathcal{F}$ as follows

\[ \mathcal{F}_K(V,W) := \{U \in \mathcal{F} : d_U(V,W) > \kappa\}. \]

Denote $\mathcal{F}_K[V,W] := \mathcal{F}_K(V,W) \cup \{V,W\}$. It possesses a total order described in the next lemma.

**Lemma 2.19.** \[6\] Theorem 3.3.G There exist constants $D = D(\kappa), K = K(\kappa) > 0$ for the above $\mathcal{F}$ such that the set $\mathcal{F}_K[V,W]$ with order “$<$” is totally ordered with least element $V$ and great element $W$, such that given $U_0, U_1, U_2 \in \mathcal{F}_K[V,W]$, if $U_0 < U_1 < U_2$, then

\[ d_{U_1}(V,W) - D \leq d_{U_1}(U_0,U_2) \leq d_{U_1}(V,W), \]

and

\[ d_{U_1}(U_1,U_2) \leq D \quad \text{and} \quad d_{U_2}(U_0,U_1) \leq D. \]

We now give the definition of a projection complex.

**Definition 2.20.** The projection complex $\mathcal{P}_K(\mathcal{F})$ for $K$ satisfying Lemma 2.19 is a graph with the vertex set consisting of the elements in $\mathcal{F}$. Two vertices $u$ and $v$ are connected if $\mathcal{F}_K(U,V) = \emptyset$. We equip $\mathcal{P}_K(\mathcal{F})$ with a length metric $d_\mathcal{P}$ induced by assigning unit length to each edge.

The projection complex $\mathcal{P}_K(\mathcal{F})$ is connected since by \[6\] Proposition 3.7] the interval set $\mathcal{F}_K[U,V]$ gives a connected path between $U$ and $V$ in $\mathcal{P}_K(\mathcal{F})$: the consecutive elements directed by the total order are adjacent in $\mathcal{P}_K(\mathcal{F})$. The structural result about the projection complex is the following.

**Theorem 2.21.** \[6\] For $K > 0$ as in Lemma 2.19 the projection complex $\mathcal{P}_K(\mathcal{F})$ is a quasi-tree, on which $G$ acts co-boundedly.

For any two point $x \in X$ and $z \in Z$, we often need to lift a standard path to $Y$. It is a piecewise geodesic path (admissible path) as concatenation of the normal paths between two consecutive vertices and geodesics contained in vertices. This is explained by the following lemma proved in \[36\] Lemma 4.5]: we include the proof for completeness.
Lemma 2.22. For any $K > 0$, there exist a constant $L = L(K, \kappa) \geq 0$ with $L \to \infty$ as $K \to \infty$ and a uniform constant $B = B(\kappa) > 0$ with the following property.

For any two points $u \in U, v \in V$ there exists an $(L, B)$-admissible path $\gamma$ in $Y$ from $u$ to $v$ with saturation $F_{K}[U, V]$.

Proof. List $F_{K}[U, V] = \{ S_{0} = U, S_{1}, S_{2}, \ldots, S_{k}, S_{k+1} = V \}$ by the total order in Lemma 2.19. The admissible path is constructed by connecting projections between $S_{i}$ and $S_{i+1}$. Namely, choose a sequence of points $x_{i} \in \pi_{S_{i}}(S_{i+1}), y_{i} \in \pi_{S_{i+1}}(S_{i})$ for $0 \leq i \leq k$. We connect consecutively the points in $\{ u, x_{0}, y_{0}, \ldots, x_{k}, y_{k}, v \}$ to give a piecewise geodesic path $\gamma$. By bounded projection $|\pi_{S_{i}}(S_{i+1})| \leq B$, we have

$$d_{S_{i}}(x_{i-1}, y_{i-1}), d_{S_{i}}(x_{i}, y_{i}) \leq B.$$ 

By Lemma 2.19 we have

$$d_{S_{i}}(S_{i-1}, S_{i+1}) \geq d_{S_{i}}(U, V) - D \geq K - D,$$

for a constant $D = D(\kappa)$. Thus, $d(y_{i-1}, x_{i}) \geq d_{S_{i}}(S_{i-1}, S_{i+1}) - 2B \geq K - D - 2B := L(K, \kappa)$. So, $\gamma$ is an $(L, B)$-admissible path relative to $F_{K}[U, V]$. The lemma is then proved. \hfill $\Box$

In practice, we always assume that $K$ satisfies Proposition 2.13 so that the path $\gamma$ shall be a quasi-geodesic.

The following two results will be used later on.

Lemma 2.23. [6] Lemma 3.18] For any $K > 0$ there exists $\hat{K} > 0$ such that $F_{K}(X, Z)$ is contained in the geodesic from $X$ to $Z$ in $P_{K}(F)$.

A triangle formed by standard paths looks almost like a tripod.

Lemma 2.24. [7] Lemma 3.6] For every $U, V, W \in F$, the path $F_{K}[U, V]$ is contained in $F_{K}[U, W] \cup F_{K}[W, V]$ except for at most two consecutive vertices.

Quasi-tree of spaces. Fix a positive number $L$ so that $1/2K \leq L \leq 2K$. We now define a blowup version, $C(F)$, of the projection complex $P_{K}(F)$ by remembering the geometry of each $U \in F$. Namely, we replace each $U \in F$, a vertex in $P_{K}(F)$, by the corresponding subspace $U \subset Y$, and keep the adjacency relation in $P_{K}(F)$: if $U, V$ are adjacent in $P_{K}(F)$ (i.e. $d_{F}(U, V) = 1$), then we attach an edge of length $L$ from every point $u \in \pi_{U}V$ to $v \in \pi_{V}U$. This choice of $L$ by [6] Lemma 4.2 ensures that $U \subset Y$ are geodesically embedded in $C(F)$ (so the index $L$ is omitted here).

Since $h$ is contracting, the infinite cyclic subgroup $(h)$ is of finite index in $E(h)$ by Lemma 2.6 so $Ax(h) = E(h)o$ is quasi-isometric to a line $R$. Thus, $F$ consists of uniform quasi-lines. By [6] Theorem B], we have the following.

Theorem 2.25. [6] The quasi-tree of spaces $C(F)$ is a quasi-tree of infinite diameter, with each $U \in F$ totally geodesically embedded into $C(F)$. Moreover, the shortest projection from $U$ to $V$ in $C(F)$ agrees with the projection $\pi_{V}U$ up to a uniform finite Hausdorff distance.

2.7. Horofunction boundary. We recall the definition of horofunction boundary and setup the notations for further reference.

Fix a basepoint $o \in Y$. For each $y \in Y$, we define a Lipschitz map $b_{y}^{o} : Y \to \mathbb{R}$ by

$$b_{y}^{o}(x) = d(x, y) - d(o, y).$$

This family of 1-Lipschitz functions sits in the set of continuous functions on $Y$ vanishing at $o$. Endowed with the compact-open topology, the Arzela-Ascoli Lemma implies that the closure of $\{ b_{y}^{o} : y \in Y \}$ gives a compactification of $Y$. The complement, denote by $\partial_{o}Y$, of $Y$ is called the horofunction boundary.

A Buseman cocycle $B_{x} : Y \times Y \to \mathbb{R}$ (independent of $o$) is given by

$$B_{x}(x, x_{2}) = b_{x}^{o}(x_{1}) - b_{x}^{o}(x_{2}).$$

The topological type of horofunction boundary is independent of the choice of basepoints, since if $d(x, y_{n}) - d(o, y_{n})$ converges as $n \to \infty$, then so does $d(x, y_{n}) - d(o', y_{n})$. Moreover, the corresponding horofunctions differ by an additive amount:

$$b_{x}^{o}(\cdot) - b_{x'}^{o}(\cdot) = b_{x}^{o}(o').$$
so we will omit the upper index \( o \). Every isometry \( \phi \) of \( Y \) induces a homeomorphism on \( \partial h Y \):

\[
\forall y \in Y: \quad \phi(\xi) := b_\xi(\phi^{-1}(y)) = b_\xi(\phi^{-1}(o)).
\]

According to the context, both \( \xi \) and \( b_\xi \) are used to denote the boundary points.

**Finite difference relation.** Two horofunctions \( b_\xi, b_\eta \) have \( K \)-**finite difference** for \( K \geq 0 \) if the \( L^\infty \)-norm of their difference is \( K \)-bounded:

\[
\|b_\xi - b_\eta\|_\infty \leq K.
\]

The **locus** of \( b_\xi \) consists of horofunctions \( b_\eta \) so that \( b_\xi, b_\eta \) have \( K \)-finite difference for some \( K \) depending on \( \eta \). The loci \( \{b_\xi\} \) of horofunctions \( b_\xi \) form a **finite difference equivalence relation** \([\cdot]\) on \( \partial h Y \). The locus \([\Lambda]\) of a subset \( \Lambda \subseteq \partial h Y \) is the union of loci of all points in \( \Lambda \). We say that \( \Lambda \) is **saturated** if \([\Lambda]\) = \( \Lambda \).

The following fact follows directly by definition.

**Lemma 2.26.** Let \( x_n \in Y \rightarrow \xi \in \partial h Y \) and \( y_n \in Y \rightarrow \eta \in \partial h Y \) as \( n \rightarrow \infty \). If \( \sup_{n \geq 1} d(x_n, y_n) < \infty \), then \([\xi] = [\eta]\).

We say that the partition \([\cdot]\) restricted on a saturated subset \( \Lambda \) is called **K-finite** for \( K \geq 0 \) if any \([\cdot]\)-class in \( \Lambda \) consists of horofunctions with \( K \)-finite difference.

At last, we mention the following topological criterion to obtain second countable, Hausdorff quotient spaces.

**Lemma 2.27.** Let \( \Lambda \subseteq \partial h Y \) be a saturated subset so that the restricted relation \([\cdot]\) on \( \Lambda \) is **K-finite** for some \( K \geq 0 \). Then the quotient map

\[
[\cdot]: \quad \xi \in \Lambda \mapsto [\xi] \in [\Lambda]
\]

is a closed map from \( \Lambda \) to \([\Lambda]\) with compact fibers. Moreover, \([\Lambda]\) is Hausdorff and second countable with quotient topology.

**Proof.** A closed continuous surjective map \( f : M \rightarrow N \) between two topological spaces with compact fibers is called **perfect** in [10] Ex §31.7. A prefect map preserves the Hausdorff and second countable properties. Hence, we only need to prove that the quotient map \([\cdot]\) is a closed map.

Note first that the restricted \( K \)-finite difference relation \([\cdot]\) on \( \Lambda \) is closed. It suffices to recall that \( b_{\xi_n} \rightarrow b_\xi \) is amount to the uniform convergence of \( b_{\xi_n}(\cdot) \rightarrow b_\xi(\cdot) \) on any compact subset of \( Y \). Thus, \( |b_{\xi_n} - b_\eta|_\infty \leq K \) implies \( |b_\xi - b_\eta|_\infty \leq K \). Moreover, the \([\cdot]\)-class of a point in \( \Lambda \) is a closed subset, so is compact in \( \partial h Y \). Thus, the map \([\cdot]\) on \( \Lambda \) has compact fibers.

We now show the quotient map \([\cdot]\) is closed: the locus \([Z]\) (the preimage of the map) of a closed subset \( Z \subseteq \Lambda \) is closed in \( \Lambda \). Indeed, let \( w_n \in [Z] \) tend to a limit point \( \xi \in \Lambda \). Take a sequence \( z_n \in Z \) with \( w_n \in [z_n] \), and for \( \partial Y \) is compact, let \( \eta \in \partial Y \) be any accumulation point of \( z_n \). As the relation \([\cdot]\) on \( \Lambda \) is closed, we have \([\xi] = [\eta]\) and then \( \eta \in \Lambda \) for \( \Lambda \) is saturated. As \( Z \) is a closed subset, we have \( \eta \in \Lambda \), so \( \xi \in [Z] \). This implies that \([Z]\) is a closed subset in \( \Lambda \). Hence, the quotient map is closed, and the lemma is proved.

### 3. Convergence compactification

We develop a general theory of a convergence compactification.

#### 3.1. Partitions.** ** As in [1], let \((Y, d)\) be a proper metric space admitting an isometric action of a non-elementary countable group \( G \) with a contracting element. Consider a metrizable compactification \( \overline{Y} := \partial Y \cup Y \), so \( Y \) is open and dense in \( \overline{Y} \). We also assume that the action of \( \text{Isom}(Y) \) extends by homeomorphism to \( \partial Y \).

We equip \( \partial Y \) with a \( G \)-invariant partition \([\cdot]\): \([\xi] = [\eta]\) implies \([g\xi] = [g\eta]\) for any \( g \in G \). The **locus** \([Z]\) of a subset \( Z \subseteq \partial Y \) is the union of all \([\cdot]\)-classes of \( \xi \in Z \). We say that \( \xi \) is **minimal** if \([\xi]\) = \{\xi\}.

Partitions on the boundary might look unnatural. It is, however, amount to descending the action to the quotient \([\partial Y]\) of \( \partial Y \) induced by the relation \([\cdot]\) via the map:

\[
\xi \in \partial Y \quad \mapsto \quad [\xi] \in [\partial Y]
\]

so the open subsets of \([\partial Y]\) are precisely the images of saturated open sets \( U \in \partial Y \) with \( U = [U] \). In general, \([\partial Y]\) may not be Hausdorff or \( T_1 \).
Say that $[\cdot]$ is a closed partition if $x_n \to \xi \in \partial Y$ and $y_n \to \eta \in \partial Y$ are two sequences with $[x_n] = [y_n]$, then $[\xi] = [\eta]$. Equivalently, the relation $\{(\xi, \eta) : [\xi] = [\eta]\}$ is a closed subset in $\partial Y \times \partial Y$, so the quotient space $[\partial Y]$ is Hausdorff.

Lemma 3.1. Assume that the partition $[\cdot]$ is closed. For any given $Z \subseteq \partial Y$, we have $[\partial Z] \subseteq [\partial Z]$. In particular, the quotient map is a closed map: if $Z$ is a closed subset, then $[\partial Z]$ is closed.

Proof. Let $w_n \in [\partial Z]$ tend to $\xi \in \partial Y$. Then there exists $z_n \in Z$ with $z_n \in [w_n]$, and for $\partial Y$ is compact, let $\eta \in \partial Z$ be any accumulation point of $z_n$. As $[\cdot]$ is closed, we have $[\xi] = [\eta]$. Thus, we proved $\xi \in [\partial Z]$ and the proof is complete.

We say that $x_n$ tends (resp. accumulates) to $[\xi]$ if the limit point (resp. any accumulate point) is contained in $[\xi]$. This implies that $[x_n]$ tends or accumulates to $[\xi]$ in the quotient space $[\partial Y]$. So, an infinite ray $\gamma$ terminates at a point in $[\xi] \in \partial Y$ if any sequence of points on $\gamma$ accumulates in $[\xi]$.

Let $X$ be a subset in $Y$. The limit set of $X$, denoted by $\Lambda X$, is the topological closure of $X$ in $\partial Y$.

Convention 3.2. We write $\{\partial Z \subseteq \partial Y\}$ to emphasize $Z$ as a subset in $\partial Y$, and $\{\partial Z \subseteq [\partial Y]\}$ as a subset in $[\partial Y]$. In the latter case, the convergence to $[\xi]$ makes the usual sense in the quotient topology.

3.2. Assumptions A & B. The first two assumptions relate the contracting property of subsets in the space to the boundary.

Assump A. Any contracting geodesic ray $\gamma$ accumulates into a closed subset $[\xi]$ for some $\xi \in \partial Y$ such that any sequence of $y_n \in Y$ with exiting projections $\pi_{\gamma}(y_n)$ tends to $[\xi]$.

Lemma 3.3. Under Assump A, any contracting quasi-geodesic ray $\gamma$ and its finite neighborhood accumulates in only one $[\cdot]$-class, denoted by $[\gamma] \in [\partial Y]$.

Proof. By Morse property, a contracting quasi-geodesic ray is contained in a finite neighborhood of a contracting geodesic ray.

Let $\gamma : \mathbb{R} \to Y$ be a contracting quasi-geodesic, with the positive and negative half rays $\gamma^+, \gamma^-$. Following [31], $\gamma$ is called horocycle if $[\gamma^+] = [\gamma^-]$.

Definition 3.4. Let $h$ be a contracting element. The attracting set $[h^-]$ and repelling set $[h^+]$ in $\partial Y$ are the locus of all accumulation points of $\{h^n : n > 0\}$ and $\{h^{-n} : n > 0\}$ respectively. We also refer $[h^-],[h^+]$ as the fixed points of $h$. Write $[h^+] := h^+ \cup [h^-]$.

By Lemma 3.3 the sets $[h^-],[h^+]$ do not depend on $o \in Y$, and are represented by the negative/positive half rays $\gamma^-,\gamma^+$ of $\gamma := \text{Ax}(h)$ (in the obvious parametrization).

Our second assumption deals with an exiting sequence of contracting subsets.

Assump B. Let $\{X_n \subseteq Y : n \geq 1\}$ be an exiting sequence of $C$-contracting bi-infinite quasi-geodesics for some $C > 0$. Then for any given $o \in Y$, there exist a subsequence of $\{Y_n := \Omega_o(X_n) : n \geq 1\}$ (still denoted by $Y_n$) and $\xi \in \partial Y$ such that $Y_n$ accumulates into $[\xi]$; any convergent sequence of points $y_n \in Y_n$ tends to a point in $[\xi]$.

First of all, we clarify that the same convergence takes place under either enlarging $X_n$ by a fixed finite neighborhood or changing the basepoint.

Lemma 3.5. Let $Y_n = \Omega_o(X_n)$ be the subsequence accumulating into $[\xi]$ given by Assump B. Then

1. For any $r \geq 0$, $Z_n := \Omega_o(N_r(X_n))$ accumulates into the same $[\xi]$.
2. For any $x \in Y$, $Z_n := \Omega_x(X_n)$ accumulates into the same $[\xi]$.

Proof. (1). If $X_n$ is $C$-contracting, then $N_r(X_n)$ is $(2r + C)$-contracting for $r \geq 0$. We then apply Assump [3] to the exiting sequence $N_r(X_n)$ which thus has a subsequence accumulating into the same $[\cdot]$-class.

On the other hand, as $Y_n \subseteq Z_n$, accumulates into $[\xi]$, any convergent sequence $z_n \in Z_n$ necessarily has the limit point in $[\xi]$. Thus, $Z_n$ accumulates into $[\xi]$.

2. Choose $z_n \in Y_n$ such that $d_{X_n}(o, z_n) > 2C$ (which exists as $X_n$ is unbounded). As $X_n$ is exiting, we have $d_{X_n}(o, x) \leq C$ by contracting property. Thus, $d_{X_n}(x, z_n) > C$ so $[x, z] \cap N_C(X_n) \neq \emptyset$. That is to say, $z_n \in Y_n \cap \Omega_x(N_C(X_n))$. As $Y_n$ converges to $[\xi]$, any convergent subsequence of $\Omega_x(N_C(X_n))$ also has limit point in $[\xi]$. □
The following is immediate by $\text{Assump B}$, since the unbounded sequence of points on a geodesic ray $\gamma$ are eventually contained in $\Omega_n(X_n)$ for $n \gg 0$ where $o = \gamma_n$.

**Lemma 3.6.** If an exiting sequence of $C$-contracting bi-infinite quasi-geodesics $X_n$ intersects a geodesic ray $\gamma$, then $\gamma$ as well as $X_n$ accumulates in one $[\cdot]$-class, denoted by $[\gamma]$.

Consider a proper action $G \acts Y$ as in (1). If $G$ is an infinite group, the limit set $\Lambda G o$ of a $G$-orbit $Go$ may depend on $o \in Y$. Note that $[h^+]$ is the locus of the limit set of $(h)o$, independent of $o$.

**Lemma 3.7.** For any two points $o,o' \in Y$, we have $[\Lambda G o] = [\Lambda G o']$.

**Proof.** Fix a contracting element $h$ with $C$-contracting axis $Ax(h) = E(h) \cdot o$. Let $g_n o \to \xi \in \Lambda G o$. After passage to a subsequence, it suffices to prove that $g_n o' \to \eta \in [\xi]$. Indeed, consider the sequence $g_n o \in X_n \equiv g_n Ax(h)$. If $X := X_n$ is eventually the same, then we are done by Lemma 3.3. Otherwise, assume that $X_n$ is exiting. For $r = d(o,o')$, $N_r(X_n)$ is an exiting sequence of $(C+2r)$-contracting subsets containing $g_n o'$ and $X_n$. By $\text{Assump B}$ any convergent subsequence of $g_n o'$ has the limit point in $[\xi]$.

We now derive that the normal subgroup shares the limit set with the ambient group.

**Lemma 3.8.** Let $H < G$ be an infinite normal subgroup. Then $[\Lambda H o] = [\Lambda G o]$.

**Proof.** By Lemma 2.10, we can choose a set of independent contracting elements $F \subseteq H$. Let $g_n \in G$ such that $g_n o \to \xi \in \Lambda G o$. For each pair $(g_n,g_n')$, after passage to subsequences, there exists $f \in F$ such that $h_n := g_n f g_n^{-1} \in H$ satisfies Lemma 2.15 so $[o,h_n] \cap N_C(g_n Ax(f)) \neq \emptyset$. Applying $\text{Assump B}$ or $\text{Assump A}$ to $X_n = g_n Ax(f)$ according to whether $X_n$ is exiting or not, we see that $h_n o$ accumulates into $[\xi]$. We thus obtain $\Lambda G o \subseteq [\Lambda H o]$. The converse inclusion is obvious, so the lemma is proved.

The following lemma says that for any contracting element $h \in G$, the set of all $G$-translates of $[h^+]$ is dense in the limit set $[\Lambda G o]$ in the quotient $[\partial Y]$.

**Lemma 3.9.** Fix a contracting element $h \in G$ and a basepoint $o \in Y$. For any $\xi \in [h^+] \cap \Lambda G o$, we have $\overline{G \xi} \subseteq \Lambda G o \subseteq [\overline{G \xi}]$. In particular, $[\Lambda G o] = [\overline{G \xi}]$.

**Remark.** If the partition $[\cdot]$ is closed, we can obtain $[\overline{G \xi}] = [\overline{G \eta}]$ from $[\xi] = [\eta]$. In this case, the above conclusion works for any $\xi \in [h^+]$.

**Proof.** Let $\eta \in \Lambda G o$, and $x_n := g_n o \to \eta$ a sequence of points. Set $X_n := g_n Ax(h)$. Up to taking a subsequence of $X_n$, we consider two cases as follows.

**Case 1.** Assume that $d(o,X_n) \to \infty$ as $n \to \infty$. By $\text{Assump B}$ as $x_n \in X_n \to \eta$, any convergent sequence of $y_n \in X_n$ tends to $\eta' \in [\eta]$. Recall that $[\Lambda X_n] = g_m [h^+]$ for each $n \geq 1$: the point $g_m o \in \Lambda G o$ is an accumulation point of the set $X_n$. The compactification $Y \cup \partial Y$ is assumed to be metrizable, so let us fix a compatible metric $\delta$. Choose $y_n \in X_n$ so that $\delta(y_n,y_n') \leq 1/n$. Thus, $g_n o \to \eta' \in [\eta]$.

**Case 2.** There is a finite constant $M > 0$ such that $d(o,X_n) \leq M$ for all $n \geq 1$. By the proper action, the set of $G$-translated axis $X$ of $h$ with $d(o,X) \leq M$ is finite. Passing to a subsequence, let us assume that $X = X_n$ for all $n \geq 1$. Note that $X$ is a translated axis of $h$ on which $E(h)$ acts cocompactly. As a non-elementary $G$ contains infinitely many distinct conjugates of $h$, we can thus choose a sequence of their axes $Y_n$ (i.e. a copy of $X$) so that $\pi_X(Y_n)$ is uniformly close to $x_n$ independent of $n$. Pick up any sequence $y_n \in Y_n$. Since $d(o,x_n) \to \infty$ we have $y_n \to [\eta]$ by $\text{Assump A}$. We are therefore reduced to the setup of the Case 1, so the proof of the lemma is finished.

We now prove the double density of fixed points of non-pinched contracting elements. Recall that two contracting elements $h,k$ are independent if their $G$-translated axes have bounded projection / intersection.

**Lemma 3.10.** Let $h,k$ be two independent contracting elements. Then for any $n \gg 0$, the element $g_n = h^n k^n$ is contracting. Moreover, there exist $\xi_n \in [g_n]$ and $\eta_n \in [g_n]$ such that $\xi_n,\eta_n$ tend to $[h^+]$, $[k^+]$ respectively as $n \to \infty$. 
Proof. By assumption, $A_x(h)$ and $A_x(k)$ have $B$-bounded projection for some $B > 0$. Consider the admissible path relative to $\mathcal{K} = \{ gA_x(k) : g \in G \}$. For large enough $n > 0$, the element $g_n := h^n k^n$ is contracting; indeed, its orbital points form a bi-infinite $(L, B)$-admissible path $\gamma_n$ as follows:

$$\gamma_n = \bigcup_{i \in \mathbb{Z}} g_n^i([a, h_n^o] h_n^i [a, k_n^o])$$

where $L := d(o, k_n^o)$. By Proposition 2.9, $\gamma_n$ is a uniformly contracting quasi-geodesic independent of $n$, so by definition, $\gamma_n$ is a contracting element. The positive (resp. negative) ray $\gamma_n^+$ (resp. $\gamma_n^-$) denotes the half ray with positive (resp. negative) indices $i$ in $\gamma_n$. Let $\xi_n \in [g_n^+]$ be an accumulation point of $\gamma_n^+$. We now only prove that $\xi_n$ tends to $[h^+]$. The case for repelling points $\eta_n \to [k^-]$ is symmetric.

If denote $\alpha := \bigcup_{i \in \mathbb{Z}} h^i [o, h_0]$ and $\beta := \bigcup_{i \in \mathbb{Z}} k_i [o, k_0]$, the positive ray of $\alpha$ accumulates into $[h^+]$, and the negative ray of $\beta$ into $[k^-]$. Since $\gamma_n$ has the overlap $[a, h^n o]$ with $\alpha$, any sufficiently far point $y_n$ on the positive ray $\gamma_n^+$ projects to a fixed neighborhood of $h^n o$, whose radius depends only the contracting constant. Applying Assump C with $x_n := h^n o$ and $X := \alpha$, we obtain $y_n \to [h^+]$ for $x_n \to [h^+]$. Since $y_n$ can be chosen arbitrarily close to $\xi_n$, we conclude that the attracting fixed point $\xi_n$ tends to $[h^+]$. \qed

It is convenient to understand the next result in the quotient space $[\partial Y]$.

**Lemma 3.11.** Let $\Lambda^2 G$ be the set of distinct $[-]$-pairs in $[\Lambda G o]$ with quotient topology. Then the set of (ordered) fixed point pairs $([h^+], [h^-])$ of all non-pinched contracting elements $h \in G$ is dense in $\Lambda^2 G$.

**Proof.** Let $\xi, \eta \in \Lambda G o$ so that $[\xi] \neq [\eta]$. Consider any $[-]$-saturated open neighborhoods $[\xi] \subseteq U = [V]$ and $[\eta] \subseteq V = [V]$. By Lemma 3.9, there exist two contracting elements $h, k$ with $a \in [h^+] \cap \Lambda G o$ and $b \in [k^-] \cap \Lambda G o$ so that $[a] \subseteq U, [b] \subseteq V$. By Lemma 3.10, there exist $g_n = h^n k^n$ for $n \gg 0$ such that $[g_n^+] \subseteq U$ and $[g_n^-] \subseteq V$. This implies that $([g_n^+], [g_n^-])$ converges to $([\xi], [\eta])$ in the quotient topology. \qed

**3.3. Assumption C.** The third assumption demands the boundary to contain enough interesting points.

**Assump C.** Assume that the following set $C$ of points $\xi \in \partial Y$ is non-empty. If $x_n, y_n \in Y$ are two sequence of points converging to $[\xi]$, then $[x_n, y_n]$ is an exiting sequence of geodesic segments.

By definition, $C$ is Isom($Y$)-invariant and $C = [C]$ is saturated. A contracting element $h$ is called non-pinched if $[h^+] \cap [h^-]$ are contained in $C$; otherwise, it is pinched. It is necessary that $[h^+] \neq [h^-]$ for a non-pinched $h$. However, by definition, it is not clear whether a pinched element must have $[h^+] = [h^-]$.

As expected, two non-pinched contracting elements have either the same or disjoint fixed points. Note that non-pinched condition can be dropped in all the examples known to the author.

**Lemma 3.12.** Let $h, k$ be two contracting elements so that $A_x(h)$ and $A_x(k)$ have bounded intersection. If $h$ is non-pinched, then $[h^+] \cap [k^+] = \emptyset$.

**Proof.** Up to taking inverses, assume by contradiction that $\xi \in [h^+] \cap [k^+]$. Let $x_n \in A_x(h) \to \xi$ be a sequence of points. By $C$-contracting property, we have $[a, x_n]$ intersects the C-ball centered at $\pi_{A_x(k)}(x_n)$. Let $u_n \in [a, x_n]$ so that $d(u_n, A_x(k)) \leq C$. If $\{u_n\}$ is unbounded, the Morse property of $A_x(h)$ and $A_x(k)$ implies their infinite intersection, contradicting the hypothesis. Thus, $\{u_n\}$ is bounded. Let $y_n \in A_x(h)$ tending to $\xi$. The contracting property of $A_x(k)$ implies that $[x_n, y_n]$ intersects the $C$-neighborhood of $\pi_{A_x(k)}(x_n)$, which is contained in a fixed finite ball by boundedness of $\{u_n\}$. This contradicts Assump C for $[\xi] \subseteq C$. Thus the lemma is proved.

Under Assump C, we can further define the shortest projection of a boundary point $\xi \in C \setminus [\Lambda X]$ to a contracting subset $X$.

**Lemma 3.13.** Let $X$ be a $C$-contracting quasi-geodesic. For any $\xi \in C \setminus [\Lambda X]$, there exists a bounded set denoted by $\pi_X(\xi)$ of diameter at most $2C$ with the following property.

Let $y_n \to \xi$ be any sequence of points in $Y$. Then $\pi_X(y_n)$ is contained in $\pi_X(\xi)$ for all $n \gg 0$.

**Proof.** Fix any sequence $y_n \to \xi$. We claim that for some $N > 0$, the set

$$U := \{ \pi_X(y_n) : n \geq N \}$$

has diameter at most $C$. Indeed, if not, there exists two subsequences $\{x_n\}, \{z_n\}$ of points $\{y_n\}$ such that $d_X(x_n, z_n) > C$. By Lemma 2.2, $[x_n, z_n]$ intersects $N_C(\pi_X(x_n))$. Since $\xi$ is disjoint with $[\Lambda X]$,
all projection points \( \{ \pi_X(y_n) : n \geq 1 \} \) are contained in a fixed ball \( Z \) of finite radius by \textbf{Assump A}

Consequently, \([x_n, z_n]\) intersects \( N_{\lambda}(Z) \) and for \( x_n, z_n \rightarrow \xi \in C \), we get a contradiction with \textbf{Assump C}. Setting \( \pi_X(\xi) := N_{\lambda}(U) \) completes the proof. \( \Box \)

The following fact obtained in the above proof will be used later on.

**Corollary 3.14.** Let \( X \) be a \( C \)-contracting quasi-geodesic. Let \( x_n, y_n \in Y \) be two sequences of points tending to \( \xi \in C \setminus \{ \Lambda X \} \). Then \( d_X(x_n, y_n) \leq C \) for all \( n \gg 0 \).

In the next few lemmas, let us emphasize that given a quasi-geodesic \( \gamma \), we denote by \( \gamma^+ \) either a positive half ray of \( \gamma \) or the endpoint in \( \partial Y \) determined by \textbf{Assump A}. In the later use, we often write \([\gamma^+]\) for the \([-\)-class as the half ray \( \gamma^+ \) is only assumed to accumulate into the same \([-\)-class.

**Lemma 3.15.** Let \( \gamma \) be a geodesic ray ending at \( [\xi] \) for some \( \xi \in C \), and \( X \) a contracting quasi-geodesic. If \( \xi \in [\Lambda X] \), then \( \gamma \) is contained in a finite neighborhood of \( X \).

**Proof.** Without loss of generality, assume that \( \gamma \subset X \) by taking a finite neighborhood of \( X \). Let \( y_n \in \gamma \) tend to \([\xi]\). If \( \pi_X(y_n) \) is exiting then by \textbf{Assump A} any sequence of \( z_n \in \pi_X(y_n) \) tends to \([\xi]\). On the other hand, \( \gamma \) intersects \( C \)-ball of \( z_n \) by Lemma 2.2, and as \( y_n \) is unbounded and \( \gamma \subset X \), the Morse property of \( X \) implies that \( \gamma \) is contained in a finite neighborhood of \( X \).

So let us assume that \( \{ \pi_X(y_n) : n \geq 1 \} \) is contained in a compact set \( U \). If \( x_n \in X \rightarrow [\xi] \), the contracting property implies that \([x_n, y_n]\) intersects the \( C \)-neighborhood of \( \pi_X(y_n) \), and of the compact \( U \). This contradicts \( \xi \in C \) by \textbf{Assump C}. The proof of lemma is then completed. \( \Box \)

**Lemma 3.16.** Let \( \xi \in C \) and \( X \) a \( C \)-contracting quasi-geodesic such that \( \xi \notin [\Lambda X] \). Then for any \( \eta \in [\xi] \), we have \( d_X(\xi, \eta) \leq 6C \).

**Proof.** Assume by contradiction that \( d_X(\xi, \eta) > 6C \). If \( x_m \in Y \rightarrow \xi \) and \( z_m \in Y \rightarrow \eta \), by Lemma 3.13 we have

\[ \forall m \gg 1 : \quad d_X(x_m, z_m) \geq d_X(\xi, \eta) - 4C \geq 2C \]

so by Lemma 2.2 \([x_m, z_m]\) intersects the \( C \)-ball around \( \pi_X(\xi) \). As \( \pi_X(\xi) \) is a fixed subset of diameter at most \( 2C \), the non-exiting sequence of \([x_m, z_m]\) contradicts the assumption of \( \xi \in C \). So the proof is completed. \( \Box \)

### 3.4. North-South dynamics of non-pinched contracting elements

Let \( h \) be a non-pinched contracting element. We shall prove that the action of \( \{ h^t \} \) on the complement to \([h^+]\) in \( \partial Y \) has so-called North-South dynamics. Such type of dynamics can be found in many examples, but surprisingly, our proof is quite general and only uses the above three assumptions. Before stating the result, we first introduce the shortest projection for any boundary subset, which will be crucial later on in the proof.

**Lemma 3.17.** Let \( \gamma \) be a \( C \)-contracting bi-infinite quasi-geodesic and \( K \in \partial Y \setminus [\gamma^+] \) be a closed subset which may contain \([\gamma^-]\). Then there is a negative half ray \( \gamma^- \) depending on \( K \) such that for any \( \xi \in K \) and any \( y_n \in Y \rightarrow \xi \), we have \( \pi_{\gamma^-}(y_n) \subset \gamma^- \).

In particular, if \( K \cap [\gamma^+] = \emptyset \), then there exists a bounded subset \( Z \) of \( \gamma \) such that for any \( y_n \in Y \rightarrow \xi \in K \), \( \pi_{\gamma}(y_n) \) is contained in \( Z \).

**Proof.** We proceed by way of contradiction. Let us assume that for some \( \xi \in K \) and a sequence \( y_n \in Y \rightarrow \xi \), so that \( \{ \pi_{\gamma}(y_n) : n \geq 1 \} \) is not contained in any negative half way of \( \gamma \). By \textbf{Assump A} (a subsequence of) \( \pi_{\gamma}(y_n) \) is exiting and tends to \([\gamma^+]\). Let us fix a metric \( \delta \) compatible with the topology \( Y \cup \partial Y \). We thus have \( \delta(y_n, [\gamma^+]) \rightarrow 0 \) on one hand, and \( \delta(y_n, \xi) \rightarrow 0 \) on the other hand. This contradicts the assumption \( \delta(K, [\gamma^+]) > 0 \). \( \Box \)

As a consequence, we can define a projection \( \pi_{\gamma}(\xi) \) to \( \gamma \) for any boundary point \( \xi \in \partial Y \setminus [\gamma^+] \) as the bounded set \( Z \) for a sufficiently small compact neighborhood \( K \) of \( \xi \). Compared with Lemma 3.13 this definition works for any \( \xi \), but the diameter of \( \pi_{\gamma}(\xi) = Z \) is unknown to be uniform for \( \xi \in C \).

**Lemma 3.18.** Let \( \gamma \) be a \( C \)-contracting bi-infinite quasi-geodesic with \([\gamma^+]\) \( \subset C \). For any bounded subset \( Z \subseteq \gamma \), the preimage of \( Z \) under \( \pi_{\gamma} : Y \cup \partial Y \setminus [\gamma^+] \rightarrow \gamma \) has the closure disjoint with \([h^+]\).
Proof. Let $K$ be the closure of $\pi^{-1}_\gamma(Z)$ in $\partial Y \cup Y$. If $K \cap [\gamma^+] \neq \emptyset$, let $\xi_n \in \pi^{-1}_\gamma(Z) \to [\gamma^+]$ for definiteness. By definition of $\pi_\gamma$, choose $x_n \in Y$ in a small neighborhood of $\xi_n$ such that $x_n \to [\gamma^+]$, and $\pi_\gamma(x_n) \cap Z \neq \emptyset$. Choose $y_n \in \gamma$ so that $y_n \to [\gamma^+]$. By the $C$-contracting property of $\gamma$, each $[x_n, y_n]$ intersects the $C$-neighborhood of $\pi_\gamma(x_n)$, and thus of a compact subset as $\pi_\gamma(x_n) \cap Z \neq \emptyset$. This contradicts Assump C with the assumption $[\gamma^+] \in \mathcal{C}$.

We are now ready to prove the North-South dynamics of contracting elements.

Lemma 3.19 (NS dynamics). Let $h$ be a non-pinched contracting element. Then the action of $\langle h \rangle$ on $\partial Y \setminus [h^+]$ has the North-South dynamics: for any two open sets $[h^+] \subseteq U$ and $[h^-] \subseteq V$ in $\partial Y$, there exists an integer $n > 0$ such that $h^n(\partial Y \setminus V) \subseteq U$ and $h^n(\partial Y \setminus U) \subseteq V$.

Proof. Since $h$ is a contracting element, the path $\gamma := \cup_{i \in \mathbb{Z}} h^i[o, ho]$ is a contracting quasi-geodesic. Let $K$ be the closed set of points $x \in Y$ such that $[o, ho] \cap \pi_\gamma(x) \neq \emptyset$, so
\[ \bigcup_{g \in \mathcal{G}} gK = Y. \]
Denote by $\tilde{K}$ the topological closure of $K$ in $Y \cup \partial Y$, and $\partial K := \partial Y \cap \tilde{K}$. By definition, $[h^+] = [\gamma^+]$. By Lemma 3.18 we have $\tilde{K} \cap [h^+] = \emptyset$ for $[h^+] \in \mathcal{C}$.

Let $\pi_\gamma : \partial Y \setminus [h^+] \to \gamma$ be the shortest projection map given by Lemma 3.17. Observe that
\[ \partial Y \cup Y = \left( \bigcup_{g \in \mathcal{G}} g\tilde{K} \right) \cup [h^+]. \]
Indeed, if $\xi \in \partial Y \setminus [h^+]$, any sequence $x_n \in Y \cup \partial Y \to \gamma$ has bounded projection to $\gamma$ by Assump A. Then $\{x_n\}$ are contained in finitely many copies of $K$, so the inclusion $\subseteq$ follows. The other direction follows by definition $\tilde{K}$ and $\cup_{g \in \mathcal{G} \cap \mathcal{G} K = Y}$.

By a similar argument, Lemma 3.17 also implies that $\pi_\gamma(\tilde{K})$ lies in a finite neighborhood of $[o, ho]$. Consequently, the projection map $\pi_\gamma$ coarsely commutes two actions of $\langle h \rangle$ on $Y \cup \partial Y \setminus [h^+]$ and on $\gamma$: for any $h \in \mathcal{G}$ and $\eta \in \partial Y \setminus [h^+]$,
\[ \pi_\gamma(h\eta) = h\pi_\gamma(\eta) \]
where $\approx$ means one of the two sets being contained into a uniform neighborhood of the other.

As $\langle h \rangle$ acts by translation on $\gamma$, we see from (11) that $\langle h \rangle$ acts properly on $Y \cup \partial Y \setminus [h^+]$: for any compact subset $L \subseteq \partial Y \setminus [h^+]$, the set $\{g \in \mathcal{G} : gL \cap \tilde{L} \neq \emptyset\}$ is finite. Moreover, $[h^iL : i \in \mathbb{Z}]$ is a locally finite family of closed subsets, so the union of any subfamily is a closed subset in $Y \cup \partial Y \setminus [h^+]$.

If $W \subseteq \partial Y \cup U$ is a compact neighborhood of $\tilde{K}$ disjoint with $[h^+]$, then $U_N := \partial Y \setminus (\cup_{i \leq N} h^iU \cup [h^-])$ for $N \geq 1$ forms a neighborhood basis for $[h^+]$. Indeed, let $[h^+] \subseteq U$ be any open subset. By Lemma 3.17 applied to $U^c$, there exists $N > 0$ such that $U^c \subseteq \cup_{i \leq N} h^iW \cup [h^-]$. That is to say, $U_N \subseteq U$.

Similarly, $V_M := \partial Y \setminus (\cup_{i \geq M} h^iW \cup [h^-])$ for $M > 0$ is a neighborhood basis for $[h^-]$. Taking large enough $n > N + M$, we obtain $h^n(\partial Y \setminus V_M) \subseteq U_N$, so the North-South dynamics follows.

The following result is implicit in the proof.

Corollary 3.20. Let $h \in \mathcal{G}$ be a contracting element with axis $X = Ax(h)$. For any $L \geq 0$ there exist open neighborhoods $[h^+] \subseteq V$ and $[h^-] \subseteq U$ in $Y \cup \partial Y$ with the following property. For any $x \in V \setminus [h^-]$ and $y \in U \setminus [h^+]$ we have
\[ d_Y(x, y) \geq L. \]
Conversely, if $d_Y(x_n, y_n) \to \infty$ for $x_n, y_n \in Y \cup \partial Y \setminus [AX]$, we have $(x_n, y_n)$ converges to $([h^-], [h^+])$.

Proof. Keep the notions as in the proof of Lemma 3.19. As $\tilde{K} \subseteq W$ is a compact set disjoint with $[h^+]$, $\pi_\gamma(\tilde{K}) \subseteq \pi_\gamma(W)$ is bounded by Lemma 3.17. By (11), we can choose $U = U_N$ and $V = V_M$ for large $M, N \gg L$ so that $d_Y(x, y) > L$ for $x \in U, y \in V$. The converse direction also follows by the North-South dynamics.

Lemma 3.21. Let $F$ be given by Lemma 2.15. For any $f \in F$ and $r > 0$, there exist open neighborhoods $U \subseteq V$ of the attracting fixed point $[f^+]$ so that for any $g \in \mathcal{G}$, we have
\[ gU \subseteq \Pi^*_g(go, r) \subseteq \Pi_*^g(go, r) \subseteq gV. \]
Proof. Set $X := \text{Ax}(f)$. Choose a shortest projection point $w \in \pi_{gX}(o)$ from $o$ to $gX$, so $d(o, go) \geq d(o, w)$ for $go \in gX$. Let $U$ be an open neighborhood of $[f^+]$ by Corollary 3.20 such that
\[
\forall z \in U \cap Y : \quad d_X(g^{-1}o, z) = d_{gX}(o, gz) \gg d(o, fo)
\]
This is to say, the distance from $go \in gX$ to the exit point of $[o, gz]$ at $N_C(gX)$ is sufficiently larger than $d(o, fo)$. For the constant $r$ given by Lemma 2.9 $go$ is an $(r, f)$-barrier for $[o, gz]$. Thus, the open set $gU \cap \partial_Y$ is contained in $\Pi_f^R(go, r)$.

The definition of $z \in \Pi_f^R(go, r)$ implies $|N_r(gX) \cap [o, z]| \geq |Fo|_{\text{min}}$, so by Lemma 2.4 we have $d_{gX}(o, z) \geq |Fo|_{\text{min}} - 4r$. For large enough $|F|$ $\gg 0$, Corollary 3.20 gives an open neighborhood $V$ of $[f^+]$ such that $z \in gV$. That is, we proved that $\Pi_f^R(go, r) \subseteq gV$.

By Lemma 3.7 the limit set $\Lambda Go$ is well-defined up to taking the locus. One desirable property for limit sets is the topological minimality and the uniqueness. We describe a typical situation where this happens. Recall that a point in $\partial Y$ is called minimal if its $[-]$-locus contains only one point.

**Lemma 3.22.** Assume that a non-pinched contracting element $h \in G$ be admits the North-South dynamics with minimal fixed points. Let $\Lambda$ be the closure of the fixed points of all conjugates of $h$. Then

1. $\Lambda$ is the unique, minimal, $G$-invariant closed subset.
2. $[\Lambda] = [\Lambda Go]$ for any $o \in Y$.
3. The set of the fixed point pairs of all non-pinched contracting element with minimal fixed points in $G$ is dense in the distinct pairs of $\Lambda \times \Lambda$.

**Remark.** This is a generalization of the known result on CAT(0) groups with rank-1 elements ([3], Theorem III.3.4 [34]) and dynamically irreducible subgroups of mapping class groups ([54]). The applications to Roller and Gardiner-Masur boundary in Lemmas 10.4 and 11.4 seem new.

**Proof.** We only need to prove the assertion (1). The assertion $[\Lambda] = [\Lambda Go]$ follows by Lemma 3.7. Assuming assertion (1), the double density follows the same proof of Lemma 3.11 (without taking care of $[-]$-classes).

Note that conjugates of $h$ are also non-pinched with minimal fixed points. A non-elementary group $G$ thus contains infinitely many conjugates of $h$ with pairwise bounded intersection, so the set $Z$ of their fixed points are infinite by Lemma 3.12. Let $A$ be any $G$-invariant closed subset in $\partial Y$. First of all, $A$ contains at least three points. Otherwise, if $G$ fixes a point or a pair of points, this contradicts the North-South dynamics of conjugates of $h$, as the set $Z$ of their fixed points is infinite. As $[h^+] = \{h^+\}$ is minimal, we can choose $\xi \in A \setminus h^+$. By Lemma 3.19, $h^n\xi \in A$ converges to $h^+$. This implies $h^+ \in \Lambda$, since $A$ is closed and $G$-invariant. Thus, $A$ contains $Z$ and $\Lambda \subseteq A$ is proved.

4. Conical and Myrberg points

The notion of conical and Myrberg points shall be central in the study of conformal density in Section 4 on convergence boundaries. Two definitions of conical points are presented, which are quantitatively equivalent.

4.1. Conical points. Fix a basepoint $o \in Y$, so everything we talk about below depends on $o$. First of all, define the usual cone and shadow:
\[
\Omega_f(y, r) := \{ z \in Y : \exists [x, z] \cap B(y, r) \neq \emptyset \}
\]
and $\Pi_f(y, r) \subseteq \partial Y$ be the topological closure in $\partial Y$ of $\Omega_f(y, r)$.

**Standing Assumption.** In the sequel, unless explicitly mentioned, we always assume that the choice of $F, F$ and the constants $r, \bar{r}, L, B$ conform with Convention 2.16. They depend on the contracting constant $C$ of $F$ and $|F|_{\text{min}} \geq L$.

We are actually working with a technically involving but useful notion of partial cones, which are defined relative to $F$.

**Definition 4.1** (Partial cone and shadow). For $x, y, y \in Go$, the $(r, F)$-cone $\Omega^f(x, y, r)$ is the set of elements $z \in Y$ such that $y$ is a $(r, F)$-barrier for some geodesic $[x, z]$.

The $(r, F)$-shadow $\Pi^f(y, r) \subseteq \partial Y$ is the topological closure in $\partial Y$ of the cone $\Omega^f(y, r)$. 


We give the first definition of a conical point. Recall that $\mathcal{C} \subseteq \partial Y$ is the set of non-pinched boundary points in Assump C

**Definition 4.2.** A point $\xi \in \mathcal{C}$ is called a $(r, F)$-conical point if for some $x \in G_o$, the point $\xi$ lies in infinitely many $(r, F)$-shadows $\Pi^F_x(y_n, r)$ for $y_n \in G_o$. We denote by $\Lambda^F_x(G)$ the set of $(r, F)$-conical points.

**Remark.** If a geodesic ray $\gamma$ starting at $x \in G_o$ contains infinitely many $(r, F)$-barriers in $y_n \in G_o$, then by definition, any accumulation points of $\gamma$ are $(r, F)$-conical.

By definition, $\Lambda^F_x(G)$ is a $G$-invariant set, and depends on the choice of $o \in Y$. It is the limit superior of the family of partial shadows:

\[
\Lambda^F_x(G) = \bigcup_{x \in G_o} \left( \limsup_{y \in G_o} \Pi^F_x(y, r) \right).
\]

It is clear that $\Lambda^F_x(G) \subseteq \Lambda_x(G)$ where $\Lambda_x(G)$ is given in (5).

We clarify that the definition in Introduction is equivalent to the above one.

**Lemma 4.3.** Let $r, F$ be as above. There exist $L_1, L_2 > 0$ depending on $r, F$ such that an $L_1$-conical point relative to $\mathcal{F}$ is $(r, F)$-conical, and an $(r, F)$-conical point is $L_2$-conical.

**Proof.** Let $\xi$ be an $L_1$-conical point in Definition 1.5 so there exists $X_n \in \mathcal{F}$ such that

\[
d_{X_n}(x, \xi) \geq L_1
\]

Consider a convergent sequence $z_m \to \xi$. By Lemma 3.13 we have $d_{X_n}(x, z_m) \geq L_1 - 2C$ for a fixed $n \geq 1$ and any $m \gg 0$. Choosing $L_1 > \|F_0\|$ by Lemma 2.9, we have $[x, z_m]$ contains an $(r, f)$-barrier at $y_n \in \pi_{X_n}(x)$. As $z_m \to \xi$, we have $\xi \in \limsup_{y \in G_o} \Pi^F_x(y, r)$. Thus, $\xi$ is $(r, F)$-conical.

Conversely, if $\xi \in \Pi^F_x(y_n, r)$, choose $z_m \in \Omega^F_x(y_n, r)$ tends to $\xi$. There exists an $(r, F)$-barrier $y_n \in X_n \in \mathcal{F}$ so we have

\[
[[x, z_m] \cap N_r(X_n)] \geq \|F_0\|_{\text{min}}
\]

By Lemma 2.4 we have $d_{X_n}(x, z_m) \geq L_2 := \|F_0\|_{\text{min}} - 4r$. Thus, $\xi$ is an $L_2$-conical point.

A point $\xi \in \partial Y$ is called visual: for any $x \in Y$, there exists a geodesic ray starting at $x$ ending at $[\xi]$. We first prove that conical points are visual.

**Lemma 4.4.** Let $\xi \in \Lambda^F_x(G)$. For any $x \in G_o$, there exists a geodesic ray $\gamma$ from $x$ and accumulating in $[\xi]$ such that $\gamma$ contains infinitely many distinct $(\hat{r}, F)$-barriers $y_n \in G_o$, and $y_n \to [\xi]$.

**Proof.** By definition, for each $n \geq 1$, there exists a sequence of points $z_{n, m} \in Y$ in the cone $\Omega^F_x(y_n, r)$ for some $x \in G_o$ such that $z_{n, m}$ tends to $\xi$ as $m \to \infty$. We shall prove the conclusion for this $x$, while the same is true with $x \in G_o$ holds by Lemma 2.8.

Given $n \geq 1$, consider geodesic segments $\alpha_m = [x, z_{n, m}]$ which contains $(r, f_n)$-barriers $y_n$ for $f_n \in F$. Passing to a subsequence as $m \to \infty$, as $Y$ is a proper metric space, we obtain a limiting geodesic ray denoted by $\gamma_n$ so that $\gamma_n$ contains $(r, f_n)$-barriers $y_n$ for $f_n \in F$. Moreover, $y_n$ is an $(\hat{r}, f_n)$-barrier for $\gamma_n$ by Lemma 2.8. And the same statement is true with $x \in G_o$.

Passing to a subsequence of $\gamma_n$ further produces a limiting geodesic ray $\gamma$, with infinitely many distinct $(\hat{r}, F)$-barriers $y_n$. We obtain $y_n \to [\xi]$ from Lemma 3.6.

As a direct corollary, we can fix the light source $x$ in (12) for convenience.

**Corollary 4.5.** For any $r > 0$ there exists $\hat{r} > 0$ such that the following

\[
\Lambda^F_x(G) \subseteq \limsup_{y \in G_o} \Pi^F_x(y, \hat{r})
\]

holds for any $x \in G_o$.

Furthermore, conical points are visual from each other.

**Lemma 4.6.** For any $[\xi] \neq [\eta]$ in $\Lambda^F_x(G)$, there exists a bi-infinite geodesic so that two half rays have infinitely many $(\hat{r}, F)$-barriers ending at $[\xi], [\eta]$ respectively.
Proof. By Lemma 4.8 let \( \alpha, \beta \) be \((L, B)\)-admissible rays of infinite type ending at \([\xi], [\eta]\) respectively. Let \( \mathcal{F}(\alpha) = \{X_n\} \) and \( \mathcal{F}(\beta) = \{Y_n\} \) be the corresponding saturations of \( \alpha \) and \( \beta \). If \( \mathcal{F}(\alpha), \mathcal{F}(\beta) \) are both finite, then \( \alpha, \beta \) are both eventually contained in a finite neighborhood of contracting quasi-geodesics. The proof is well-known in the study of contracting boundary (cf. [12, 14]) and left to the interested reader.

Let us thus assume that \( \mathcal{F}(\alpha) \) is infinite. By Lemma 3.15 we have \( \xi \notin [\Lambda X_m] \) for each fixed \( X_m \).

Take two unbounded sequences \( x_n \in \alpha \cap N_r(X_n) \) and \( y_n \in \beta \cap N_r(\gamma_n) \), and connect them by geodesics \( \gamma_n := [x_n, y_n] \). We claim that \( d(o, \gamma_n) \) are uniformly bounded for all \( n \gg 0 \). Indeed, suppose to the contrary that \([x_n, y_n] \) are exiting. For \( \xi \notin [\Lambda X_m] \), Corollary 3.14 shows

\[ \forall n \gg m : \quad d_{X_m}(x_n, y_n) \leq C \]

By Proposition 2.13 \( \alpha \) has \( r \)-fellow travel property for \( X_m \) and then by Lemma 2.4, we have

\[ d_{X_m}(o, x_n) \geq [([o, x_n] \cap N_r(X_m)] - 4r \geq L - 4r \]

which yields by the triangle inequality \( d_{X_m} \):

\[ d_{X_m}(o, y_n) > L - C - 4r \]

By Lemma 2.4 if \( L > C - 7r \), we have \( \beta \) intersects \( N_r(X_m) \). By Lemma 3.6 as \( x_n \) accumulates into \([\xi]\), we obtain that \( \beta \) does so as well. This leads to a contradiction with \( \alpha \neq \beta \). The claim follows.

As \( Y \) is proper, a subsequence of \( \gamma_n \) converges to a bi-infinite geodesic \( \gamma \), which intersects \( N_r(\gamma_n) \) and \( N_r(Y_m) \) for all \( m \gg 0 \). Thus, the two half rays of \( \gamma \) accumulate into \([\xi] \neq [\eta]\) respectively. This shows that two conical points are visual to each other, completing the proof of lemma.

We give the second formulation of conical points. The constant \( B \) in the following definition is given by Convention 2.16 and depends only on \( \mathcal{F} \).

Definition 4.7. A point \( \xi \in \mathcal{F} \) is called an \((L, \mathcal{F})\)-conical point if there exists an \((L, B)\)-admissible ray of infinite type accumulating in \([\xi]\).

By definition, the set of \((L, \mathcal{F})\)-conical points is \( G \)-invariant.

Lemma 4.8. Let \( r > 4C, F \notin G \) be as above. There exist \( L_1, L_2 > 0 \) such that the locus of an \((L_1, \mathcal{F})\)-conical point \( \xi \in \partial Y \) contains an \((r, F)\)-conical point, and the locus of an \((r, F)\)-conical point contains \((L_2, \mathcal{F})\)-conical point \( \xi \in \partial Y \).

Proof. The second half of the statement already follows from Lemma 4.4 if \( \xi \) is an \((r, F)\)-conical point, then there exists a geodesic ray \( \gamma \) with infinitely many \((r, F)\)-barriers, which accumulates in \([\xi]\). By Lemma 2.14 the truncation of \( \gamma \) relative to barriers is an \((L_2, B)\)-admissible ray of infinite type, for some \( L_2 = L_2([\Lambda F_{\text{min}}]) \). By definition, \( \xi \) is a \((L_2, \mathcal{F})\)-conical point.

Let \( L_1 = \|F_0\| + L_0 + 6r > 0 \), where \( L_0 \) is given by Lemma 2.9. Let us now assume that \( \xi \) is an \((L_1, \mathcal{F})\)-conical point. If \( \gamma \) is an \((L_1, B)\)-admissible ray of infinite type ending in \([\xi]\), then any subpath of \( \gamma \) is \((L_1, B)\)-admissible, so has the \( r \)-fellow travel property by Proposition 2.13. Connecting \( o \) to an unbounded sequence of points on \( \gamma \), we can obtain as in the proof of Lemma 4.6 a limiting geodesic ray \( \alpha \) from \( o \), so that there exists a sequence of \( X_i \in \mathcal{F}(\gamma) \) satisfying \( \|\alpha \cap N_r(X_i)\| > L_1 - 2r \). From Lemma 3.6 we know that \( \alpha \) accumulates into \([\xi]\). By Lemma 2.4 we have

\[ d_{X_i}(\alpha^+, \alpha^+) > \|\alpha \cap N_r(X_i)\| - 4r > L_1 - 6r \geq \|F_0\| + L_0. \]

By Lemma 2.9 \( \alpha \) contains infinitely many \((r, F)\)-barriers, so by definition, any accumulation point of \( \alpha \) is an \((r, F)\)-conical point.

We next show that our notion of conical points satisfies the dynamical property of conical points in the theory of convergence groups.

Lemma 4.9. Let \( \xi \in \Lambda^F G \) be a conical point. Then there exists a sequence of elements \( g_n \in G \) and a pair of \([a] \neq [b] \in \partial Y \) such that the following holds

\[ g_n(\xi, x) \rightarrow ([a], [b]), \]

for any point \( x \in Y \).
exists a sequence of elements \( \alpha, \beta \) converging respectively to two geodesic rays \( A \).

Lemma 4.12. If \( k \) is a projection to the axis of \( \gamma \) in line with the convergence property of conical points in Lemma 4.9. Then the locus of fixed points of a contracting element cannot contain a Myrberg point. \( \boxtimes \)

4.2. Myrberg points. We now formulate a more intrinsic subclass of conical points, Myrberg points, without involving the reference to \( \mathcal{F} \). It has several equivalent characterizations (cf. [21]). We give one in line with the convergence property of conical points in Lemma 4.9.

Definition 4.10. A visual point \( \xi \in \mathcal{C} \) is called Myrberg point if for any pair of \( [a] \neq [b] \subseteq [\Lambda Go] \) there exists a sequence of elements \( g_n \in G \) such that the following holds for any \( x \in Y \). Denote by \( \Lambda_n G \) the set of Myrberg points.

Lemma 4.11. The locus of fixed points of a contracting element cannot contain a Myrberg point.

Proof. Let \( h \) be a contracting element. As \( G \) is assumed to be non-elementary, there exists a contracting element \( k \in G \) independent with \( h \). That is to say, any \( G \)-translated axis of \( h \) has uniform bounded projection to the axis of \( k \). This implies that the set of pairs \( G(x, [h^+]) \) cannot accumulate to the pair \( ([k^-], [k^+] \) by Corollary 3.20. \( \boxtimes \)

Lemma 4.12. A point \( \xi \in \mathcal{C} \) is a Myrberg point if and only if the following holds.

Let \( h \in G \) be any non-pinched contracting element and \( f \in E(h) \) be a contracting element. Any geodesic ray \( \gamma \) ending at \( [\xi] \) contains infinitely many distinct \((r, f)\)-barriers. Here \( r \) depends only on the contracting constant of \( Ax(h) \).

Proof. “\( \Rightarrow \)” Let \( \gamma \) be a geodesic ray starting at \( x \in Y \) and ending at a Myrberg point \([\xi]\). Fix a non-pinched contracting element \( h \). Applying the definition of Myrberg points to \([a] := [h^-], [b] := [h^+] \), we have \( g_n(\xi, x) \to ([a], [b]) \) for a sequence of \( g_n \in G \). By Lemma 3.13 we have a well-defined projection map \( \pi_X : Y \cup \mathcal{C} \setminus [\Lambda X] \to X \) where \( X = Ax(h) \).

Set \( X_n := g_n^{-1} X \). As \( g_n(\xi, x) \to ([a], [b]) \), by Corollary 3.20 there exist \( n_0 > 0 \) such that for any \( n \geq n_0 \),

\[
d_{X_n}(x, \xi) \gg d(o, fo)\]

By Lemma 2.9 \( \gamma \) contains an \((r, f)\)-barrier \( g_n^{-1} X \). So the direction “\( \Rightarrow \)” follows.

“\( \Leftarrow \)” By Lemma 3.11 the set of the pairs \(([h^-], [h^+] \) for all non-pinched contracting elements \( h \in G \) is dense in the distinct \([\cdot]\)-pairs in \([\Lambda Go] \). It suffices to verify the definition of Myrberg point for the pair \(([h^-], [h^+] \)). This however follows from Corollary 3.20. \( \boxtimes \)

The following corollary summarizes the relation between conical points and Myrberg points.

Corollary 4.13. We have

\[
\Lambda_n G = \bigcap F \Lambda_F (G)
\]

where the intersection is taken over the sets \( F \subseteq G \) of three mutually independent non-pinched contracting elements in \( G \).
4.3. Gromov boundary of projection complex. It turns out that the conical points are closely related to the Gromov boundary of the projection complex $\mathcal{P}_K(\mathcal{F})$ for $K \geq 0$.

**Lemma 4.14.** For any $r, F$ satisfying Convention 2.16 there exists $K > 0$ such that the following holds.

There is a continuous embedding of the Gromov boundary $\partial G\mathcal{P}_K(\mathcal{F})$ of $\mathcal{P}_K(\mathcal{F})$ into $[\Lambda^F_r(G)]$ of $(r, F)$-conical points in the quotient space $[\partial Y]$.

**Remark.** By Lemma 2.23 one can prove that the $(r, F)$-conical points $[\Lambda^F_r(G)]$ are embedded into the Gromov boundary $\partial G\mathcal{P}_K(\mathcal{F})$ for some (possibly different) $K$, provided that $\|F_0\|_{\text{min}} \gg K$. This is not required in this paper, so we omit it.

**Proof.** Fix a basepoint $o \in \mathcal{P}_K(\mathcal{F})$ (we omit the bar in this proof). By definition, the Gromov boundary of $\mathcal{P}_K(\mathcal{F})$ consists of equivalent classes of unbounded sequences $\{u_n\}$ satisfying $\{u_n, u_m\}_o \to \infty$ as $n, m \to \infty$.

Two sequences $\{u_n\}, \{v_n\}$ are equivalent if $\{u_n, v_n\}_o \to \infty$.

Let $\alpha_n$ be the standard path from $o$ to $u_n$. By the tripod-like property in Lemma 2.22 as Gromov products $\{u_n, u_m\}_o \to \infty$, one can extract a subsequence of $\alpha_n$ such that it converges to a quasi-geodesic ray $\alpha$ in a sense that restricting on any bounded subset, $\alpha_n$ and $\alpha$ coincide. Thus, any subpath of $\alpha$ are standard paths. We call $\alpha$ **standard ray**, which converges to the boundary point $\xi$.

By Lemma 2.22 there exist $L = L(K), B = B(\mathcal{F})$ so that we can lift $\alpha$ to get an $(L, B)$-admissible quasi-geodesic ray $\gamma$. Moreover, $L \to \infty$ as $K \to \infty$. For given $F$, we can choose $K$ large enough so that an $(L, \mathcal{F})$-conical point is an $(r, F)$-conical point by Lemma 4.13. So $\gamma$ determines a unique $(r, F)$-conical point $[\gamma] \in [\partial Y]$. We set $\Phi(\xi) := [\gamma]$, and shall prove the continuity of $\Phi$ below.

Let $\xi_n \to \xi$ in the Gromov boundary. As in previous paragraph, for each $\xi_n$, there exist a standard ray $\beta_n$ ending at $\xi_n$, and its lift $(L, B)$-admissible path $\gamma_n$ in $Y$ ending at $\Phi(\xi_n)$. After passage of subsequence, $\alpha_n$ converges in the above sense to a standard ray ending at $\xi$. Let $X_n \in \mathcal{F}$ be the vertex space corresponding to the last vertex of the overlap $\alpha_n \cap \alpha$.

Let $v_n$ be the exit point of $\gamma_n$ at $N_r(X_n)$ for $r \geq 0$ given by Proposition 2.13. For any sufficiently deep $z_n \in [v_n, \Phi(\xi_n)]$, we have $\pi_{X_n}(z_n)$ is $(r + C)$-close to $v_n$ by Lemma 2.3. So the triangle inequality gives

$$d_{X_n}(o, z_n) \geq d_{X_n}(o, v_n) - r - C$$

As $d_{X_n}(o, v_n) \geq [\gamma_n \cap N_r(X_n)] - 4r \geq L - 4r$ by Lemma 2.4 we have

$$d_{X_n}(o, z_n) \geq L - 5r - C$$

Setting $L > 5r + 2C$, Assum B implies that $z_n$ converges to $\Phi(\xi)$.

Fix a metric $\delta$ on the metrizable boundary $\partial Y$, and choose $z_n$ so that $\delta(z_n, \Phi(\xi_n)) \leq 1/n$. We see then $\Phi(\xi_n)$ converges to $\Phi(\xi)$. The proof is complete. $\square$

For completeness, we state a similar result for the quasi-tree of spaces $\mathcal{C}(\mathcal{F})$. Recall that $\mathcal{C}(\mathcal{F})$ recovers the collapsed vertex spaces in $\mathcal{F}$, the set of axes of contracting elements.

**Corollary 4.15.** Under the assumption of Lemma 4.14 there is a continuous map $\Phi$ from the Gromov boundary $\partial G\mathcal{C}(\mathcal{F})$ of the quasi-tree of spaces $\mathcal{C}(\mathcal{F})$ into $[\partial Y]$ so that

1. the image is the set $[\Lambda^F_r(G)]$ of $(r, F)$-conical points,
2. the map $\Phi$ fails to be injective only at fixed points of pinched contracting elements.

**Proof.** By the construction of quasi-tree of spaces, there are two types of such a sequence. Either the corresponding vertices $\tilde{u}_n$ are unbounded in the original projection complex $\mathcal{P}_K(\mathcal{F})$ or $u_n$ are eventually contained in one vertex space, i.e.: an axis $U$ in $\mathcal{F}$. The former case has been dealt in Lemma 4.14.

In the latter case, $U$ is a contracting quasi-geodesic in $Y$. Depending on whether $U$ is pinched in $\overline{Y}$, one defines $\Phi(\xi)$ to be either $[U^-] = [U^+]$ or the corresponding $[U^-]$ or $[U^+]$. The pinched points $[U^-] = [U^+]$ are exactly where the injectivity of $\Phi$ fails.

The continuity follows a similar argument as in Lemma 4.14 and is left to the interested reader. $\square$

5. Convergence of horofunction boundary

In this section, we verify that the horofunction boundary with finite difference relation is a convergence boundary, i.e. satisfying Assum A, Assum B and Assum C.
The limit set. The limit set $\Lambda Go$ of a group $G$ is defined to be the topological closure of a $G$-orbit $Go$ in $\partial_h Y$. We have $[\Lambda Go] = [\Lambda Go']$ by Lemma 2.26 and denote $[\Lambda G] = [\Lambda Go]$.

5.1. Verifying Assump A. This subsection is devoted to the proof of Theorem 1.1. We start by verifying Assump A for the horofunction boundary.

**Lemma 5.1.** Let $X \subseteq Y$ be a $C$-contracting subset. If $y_n \in Y \rightarrow \eta \in \partial_h Y$ and $x_n \rightarrow \xi \in \partial_h Y$ for a sequence $x_n \in \pi_X(y_n)$, then

$$\|b_\xi - b_\eta\|_\infty \leq 4C$$

**Proof.** The proof is to estimate that for any $z \in Y$, the difference

$$\lim_{n \to \infty} (b_{x_n}(z) - b_{y_n}(z)) = \lim_{n \to \infty} (d(z, x_n) - d(o, x_n) + d(o, y_n) - d(z, y_n))$$

is uniformly bounded. As $x_n \in \pi_X(y_n)$ exits every compact set, the $C$-contracting property of $X$ thus implies for all $n \gg 0$:

$$d(x_n, [y_n, o]) \leq C, \quad d(x_n, [y_n, z]) \leq C.$$

Combining these estimates, we get for $n \gg 0$,

$$|d(z, x_n) - d(o, x_n) + d(o, y_n) - d(z, y_n)| \leq 4C$$

completing the proof.

The following corollary is proved in the proof.

**Corollary 5.2.** Let $\gamma$ be a $C$-contracting ray. For any $\xi, \eta \in [\gamma]$, we have

$$\|b_\xi - b_\eta\|_\infty \leq 4C$$

**Proof.** Let $y_n \in Y \rightarrow \zeta \in [\gamma]$. By Lemma 5.1, it suffices to prove that $\pi_\gamma(y_n)$ is an exiting sequence. Indeed, if not, $\{\pi_\gamma(y_n) : n \geq 1\}$ is contained in a $R$-ball at $\gamma^-$ for $R > 0$. By $C$-contracting property, as $z \in \gamma$ tends to infinity, the geodesic $[y_n, z]$ intersects $B(\gamma^-, R + C)$. Estimating the value of $b_{y_n}(\cdot)$ at $z \in \gamma$ shows $b_{y_n}(z) \rightarrow +\infty$. However, the horofunction $b_\gamma$ associated to $\gamma$ takes negative values as $z \in \gamma$ tends to infinity. This contradiction completes the proof.

Let $h$ be a contracting element on $Y$, so that $(h)o$ is a contracting quasi-geodesic. The repelling and attracting fixed points $[h^-], [h^+]$ in $\partial_h Y$ do not depend on $o \in Y$, and any two horofunctions in $[h^+]$ (or $[h^-]$) has uniform difference by Corollary 5.2.

We now prove Assump B for the horofunction boundary.

**Lemma 5.3.** Let $X_n \subseteq Y$ be an exiting sequence of $C$-contracting subsets. Fix a basepoint $o \in Y$. Let $x_n \in \pi_{X_n}(o)$ such that $x_n \rightarrow \xi \in \partial_h Y$. If $y_n \in Y \rightarrow \eta \in \partial_h Y$ so that $N_C(X_n) \cap [o, y_n] \neq \emptyset$ for $n \geq 1$, then

$$\|b_\xi - b_\eta\|_\infty \leq 20C.$$

**Proof.** Fix $z \in Y$ and $z_n \in \pi_{X_n}(z)$. Since the ball at $o$ with radius $d(o, z)$ misses $X_n$ for $d(o, X_n) \rightarrow \infty$, the $C$-contracting property of Lemma 2.2 implies $d(z_n, x_n) \leq d_X(z, o) \leq C$, so for $n \gg 0$,

$$d_{X_n}(z_n, y_n) \geq d(z_n, w_n) \geq 2C$$

Let $w_n \in [o, y_n] \cap N_C(X_n)$ be the exit point. Using again the contracting property,

$$d(x_n, [z, w_n]) \leq C, \quad d(x_n, [o, w_n]) \leq C$$

$$d(w_n, [z, y_n]) \leq C, \quad d(w_n, [o, y_n]) \leq C$$

Accordingly, we obtain

$$|b_{w_n}(z) - b_{z_n}(z)| \leq 10C, \quad |b_{y_n}(z) - b_{w_n}(z)| \leq 10C$$

completing the proof.

We prove now that all boundary points in the horofunction boundary $\partial_h Y$ are non-pinched. As a corollary, every contracting elements are non-pinched in the horofunction boundary.

**Lemma 5.4.** For any $\xi \in \partial_h Y$, there exist no sequences $x_n, y_n \in Y$ such that $x_n, y_n \rightarrow [\xi]$ and $[x_n, y_n]$ intersects a fixed ball for any $n \geq 1$. 


Proof. For concreteness, assume that \( x_n \to \xi, y_n \to \eta \) such that \([x_n] = [y_n]\). The finite difference relation implies that there exists \( K \geq 10 \) such that \(|B_{\xi}(z, w) - B_{\eta}(z, w)| \leq K\) for any \( z, w \in Y\). Arguing by contradiction, as \( Y \) is proper, assume that \([x_n, y_n]\) converges locally uniformly to a bi-infinite geodesic \( \alpha \). Choose \( z, w \in \alpha \) with distance at least \( 2K\) such that \( z, w \in N_1([x_n, y_n]) \) and \( z \) is closer to \( x_n \) than \( w \).

By direct computation,
\[
|B_{x_n}(z, w) + d(z, w)| \leq 2, \quad |B_{y_n}(z, w) - d(z, w)| \leq 2.
\]

For \( x_n, y_n \to \xi \), we have \( B_{x_n}(z, w), B_{y_n}(z, w) \to B_{\xi}(z, w) \) as \( n \to \infty \). With \(|B_{\xi}(z, w) - B_{\eta}(z, w)| \leq K\), this contradicts the assumption of \( d(z, w) > 2K\).

5.2. Horofunctions associated to conical points. We are not presumably working with the horofunction boundary in this subsection. Still, the goal is to define for conical points in any nontrivial convergence boundary \( \overline{Y} = Y \cup \partial Y \).

Lemma 5.5. Let \( \xi \in \Lambda^F(\hat{G}) \). Consider two sequences \( x_n \in Y \to \xi \in [\xi], z_n \in Y \to \eta \in [\eta] \) in \( \partial Y \), and simultaneously, \( x_n \to b_\xi, z_n \to b_\eta \) in the horofunction boundary \( \partial \overline{Y} \). Then
\[
|b_\xi - b_\eta|_{\infty} \leq 20C.
\]
Moreover, the locus of an \((r, F)\)-conical point \( \xi \) consists of \((\hat{r}, F)\)-conical points.

As a consequence, we can associate to any \( \xi \in \Lambda^F(\hat{G}) \) a Busemann quasi-cocycle
\[
B_{\xi}(x, y) = \limsup_{y \to \xi} (d(x, z_n) - d(y, z_n))
\]
where the convergence \( z \to \xi \) takes place in \( \overline{Y} = Y \cup \partial Y \).

Proof. By Lemma 4.4 let \( \gamma \) be a geodesic ray with infinitely many distinct \((\hat{r}, F)\)-barriers \( y_n := g_n o \in X_n \in \mathcal{F}\) and ending at \([\xi]\). For given \( X_n \) and \( m \gg n \), we have
\[
d_{X_n}(o, y_m) \geq \|N_r(X_n) \cap \gamma\| - 4r \geq \|F o\|_{\min} - 4r > 10C,
\]
By Lemma 5.3 we thus see \( |b_\xi - b_\eta|_{\infty} \leq 20C\).

We close this section with the following property of Myrberg set in the horofunction boundary.

Lemma 5.6. The Myrberg limit set \( \Lambda_m G \subseteq \partial Y \) is saturated in any nontrivial convergence boundary \( \partial Y \), and the finite difference relation \([\cdot]\) of the horofunction boundary restricted on \( \Lambda_m G \subseteq \partial Y \) is \( K\)-finite for some uniform constant \( K > 0 \).

As a corollary of Lemma 2.27 the quotient \([\Lambda_m G]\) for \( \Lambda_m G \subseteq \partial Y \) is a Hausdorff, second countable topological space. The same would be true in a general convergence boundary, if we knew that the partition restricted on \( \Lambda_m G \) is a closed partition.

Proof. As Myrberg points are conical, the finite difference relation on \( \Lambda_m G \) is \( K\)-finite for some uniform \( K \) by Lemma 5.5. By Lemma 4.12 a point \( \xi \in \Lambda_m G \) is Myrberg if and only if for any non-pinched contracting element \( h \) and for any \( L \gg 10C\), there exists \( g \in G \) such that
\[
d_X(o, \xi) \geq L
\]
for \( X = gAx(h) \). It remains to show that \( \eta \in [\xi] \) is Myrberg. By Lemma 3.16 we have \( d_X(\xi, \eta) \leq 6C \), so by the triangle inequality for \( d_X \),
\[
d_X(o, \eta) \geq L - 6C.
\]
As $h$ and $L$ is arbitrary, it is proved that $\eta$ is Myrberg as well. \hfill \Box

6. Conformal density on the convergence boundary

This section develops a general theory of quasi-conformal density on a convergence compactification $\overline{Y} = Y \cup \partial Y$, under an additional Assump D.

6.1. Quasiconformal density. Buseman (quasi-)cocycles are indispensable to formulate (quasi-)conformal density on $\partial Y$. As shown in Lemma 5.3, conical points in any convergence boundary can be associated with Buseman quasi-cocycles. Let $C$ be the $G$-invariant subset of non-pinched points in Assump C. Our last assumption requests this property holds for the set $C$ of non-pinched points in Assump C.

Given $z \in Y$, let $B_z(x,y) := d(x,z) - d(y,z)$ for $x, y \in Y$.

**Assump D.** There exists a family of Buseman quasi-cocycles based at points in $C$

$$\{B_\xi : Y \times Y \to \mathbb{R}\}_{\xi \in C}$$

so that for any $x, y \in Y$, we have

$$\limsup_{z \xrightarrow{\xi} x} |B_\xi(x,y) - B_z(x,y)| \leq \epsilon,$$

where $\epsilon \geq 0$ is a universal constant not depending on $x, y$ and $\xi$. If $\epsilon = 0$, we say that Buseman cocycles extend continuously to $C$.

Let $\mathcal{M}^+(\overline{Y})$ be the set of finite positive Borel measures on $\overline{Y} := \partial Y \cup Y$, on which $G$ acts by push-forward:

$$g_*\mu(A) = \mu(g^{-1}A)$$

for any Borel set $A$. Thus, $\mu_{g\cdot x} = g_*\mu_x$; equivalently, $\mu_{g\cdot x}(gA) = \mu_x(A)$.

**Definition 6.1.** Let $\omega \in [0, \infty[$. Under Assump D, a map $\mu : Y \to \mathcal{M}^+(\overline{Y})$

$$x \mapsto \mu_x$$

is a $\omega$-dimensional $G$-quasi-equivariant quasi-conformal density if for any $g, h \in G$ and any $x \in Y$, we have

$$\forall \xi \in \partial Y : \frac{d\mu_x}{d\mu_h}(\xi) \in \left[\frac{1}{\lambda}, \lambda\right],$$

$$\forall \xi \in C : \frac{1}{\lambda}e^{-\omega B_\xi(gx,hx)} \leq \frac{d\mu_{gx}}{d\mu_{hx}}(\xi) \leq \lambda e^{-\omega B_\xi(gx,hx)}$$

for a universal constant $\lambda \geq 1$. We normalize $\mu_o$ to be a probability measure: its mass $\|\mu_o\| = \mu_o(\overline{Y}) = 1$.

If $\lambda = 1$ for (14), the map $\mu : Y \to \mathcal{M}^+(\overline{Y})$ is $G$-equivariant. If both $\lambda = 1$, we call $\mu$ a conformal density.

**Remark.** We clarify by examples the advantage of the conformality (15) restricted on a smaller subset $C$ of $\partial Y$.

(1) First of all, it seems implausible to define Buseman cocyles at pinched points, e.g. if the pinched point is the two endpoints of a bi-infinite geodesic (i.e.: a horocycle). For example, the horocycles exist in the Cayley graph of relatively hyperbolic groups, so that in [77], we can only define Buseman (quasi-)cocyles at conical points in Floyd boundary.

(2) Secondly, if the action $G \acts Y$ is of divergent type, one may prove via other means that the quasi-conformal measures are fully supported on the set $C$ (in view of Theorem 1.10).

As finitely generated groups are of divergent type for the action on the Cayley graph, we then prove in [77] the atomless of the Patterson-Sullivan measures on Bowditch boundary via an argument in [18]. As there are only countably many non-conical points, the conformality is recovered on almost every boundary points.

As another example, the conformal density constructed from Thurston measures in [3] (not using Patterson’s construction) is supported on uniquely ergodic points in Thurston boundary (this follows by Masur-Veech’s result [51, 72] on Teichmüller geodesic flow). It turned out that the horofunction could be defined on those points (this is nontrivial as Thurston boundary is not the horofunction boundary of Teichmüller metric), on which the conformality suffices in application.
For simplicity, we write $\mu_g = \mu_o$ if the basepoint $o \in U$ does not matter in context. In particular, $\mu_1$ denotes the measure $\mu_o$ where 1 is the identity of $G$.

**Patterson-Sullivan measures.** Let $(Y, d)$ be a proper geodesic space. Choose a basepoint $o \in Y$. The Poincaré series for the action of $G \sim Y$

$$P_G(s, x, y) = \sum_{g \in G} e^{-sd(x, gy)}, \ s \geq 0$$

diverges at $0 \leq s < \omega_G$ and converges at $s > \omega_G$. The action of $G$ on $Y$ is called divergent if $P_G(s, x, y)$ diverges at the critical exponent $\omega_G$. Otherwise, $G$ is called convergent.

We start to construct a family of measures $\{\mu_{s,y}^G\}_{s,y \in Y}$ supported on $Gy$ for any given $s > \omega_G$. Assume that $P_G(s, x, y)$ diverges at $s = \omega_G$. Set

$$\mu_{s,y}^G = \frac{1}{P_G(s, o, y)} \sum_{g \in G} e^{-sd(x, gy)}. \ \text{Dirac}(gy),$$

where $s > \omega_G$. Note that $\mu_{s,y}^G$ is a probability measure supported on $Gy$. If $P_G(s, x, y)$ is convergent at $s = \omega_G$, the Poincaré series in (16) needs to be replaced by a modified series as in (63).

Fix $y \in Y$. Choose $s_i \to \omega_G$ such that $\mu_{s_i,y}^G$ are convergent in $M(\Lambda G)$. The Patterson-Sullivan measures $\mu_{\omega_G}^G = \lim_{s \to \omega_G} \mu_{s,y}^G$ are the limit measures. Note that $\mu_1(\Lambda G) = 1$. In what follows, we usually write $\mu_{s} = \mu_{s,y}^G$ for $x \in Y$.

**Lemma 6.2.** Let $G$ act properly on a proper geodesic space $(Y, d)$ with a contracting element, and compactified by a nontrivial convergence boundary $\partial Y$. Then under Assump D for any fixed $y \in Y$, the family of Patterson-Sullivan measures $\{\mu_y^G\}_{y \in Y}$ on $\partial Y$ is a $\omega_G$-dimensional $G$-equivariant quasi-conformal density supported on the limit set $\Lambda Gy$.

Moreover, if $\partial Y$ is the horofunction boundary, then it is a conformal density.

**Proof.** As the orbit $Gy$ is discrete in $Y$, the measure $\mu_y^G$ is supported on the closure of $Gy$. The proof for quasi-conformal density follows the same argument in [13]. If $\partial Y$ is the horofunction boundary, the Buseman cocycle extends continuously to $\partial Y$ so the constant $\epsilon = 0$ in Assump D and we obtain a conformal density: $\lambda = 1$. See [10] for more details. \qed

In the sequel, we write PS-measures as shorthand for Patterson-Sullivan measures.

### 6.2. Shadow Lemma

We assume the set $\mathcal{C}$ in Assump C is measurably significant: $\mu_1(\mathcal{C}) > 0$. The goal of this subsection is to prove the shadow lemma.

The partial shadows $\Pi_o^F(go, r)$ and cones $\Omega_o^F(go, r)$ given in Definition 4.1 depend on the choice of a contracting system $\mathcal{F}$ as in [2]. Without index $F$, $\Pi_o(go, r)$ denotes the usual shadow.

**Lemma 6.3 (Shadow Lemma).** Let $\{\mu_g\}_{g \in G}$ be a $\omega$-dimensional $G$-quasi-equivariant quasi-conformal density on $\partial Y$ for some $\omega > 0$. Assume that $\mu_1(\mathcal{C}) > 0$. Then there exists $r_0 > 0$ such that

$$e^{-\omega d(o, go)} \prec_\lambda \mu_1(\Pi_o^F(go, r) \cap \mathcal{C}) \leq \mu_1(\Pi_o(go, r) \cap \mathcal{C}) \prec_{\lambda, \epsilon, r} e^{-\omega d(o, go)}$$

for any $g \in G$ and $r \geq r_0$, where $\epsilon, \lambda$ are given respectively in Assump D and Definition 6.1.

**Proof.** We start proving the lower bound which is the key part of the proof.

1. **Lower Bound.** Fix a Borel subset $U$ in $\mathcal{C}$ such that $\mu_1(U) > 0$ (e.g.: let $U = \mathcal{C}$). For any given $\xi \in U$, there exists a sequence of points $z_n \in Y$ such that $z_n \to \xi$. By Lemma 2.15, there exist $f_n \in F$ and $r > 0$ such that $g$ is an $(r, f_n)$-barrier for any geodesic $[o, gfz_n]$. Since $F$ consists of three elements, we assume that $f := f_n$ are all equal, up to taking a subsequence of $z_n$. This implies that $gfz_n \in \Omega_o^F(go, r)$ tends to $gf\xi$, so by definition,

$$gf\xi \in \Pi_o^F(go, r) \cap \mathcal{C}.$$ 

Consequently, we decompose the set $U$ into a disjoint union of three sets $U_1, U_2, U_3$: for each $U_i$, there exists $f_i \in F$ such that $gf_iU_i \subseteq \Pi_o^F(go, r)$.

Since $B_{\xi}(\cdot, \cdot)$ is Lipschitz, we obtain a constant $\theta = \theta(|F_0|) > 0$ from [13] such that for each $1 \leq i \leq 3$, we have

$$\mu_1(f_iU_i) \geq \theta \cdot \mu_1(f_iU_i) \geq \theta \cdot \lambda^{-1} \cdot \mu_1(U_i)$$
where the universal constant \( \lambda \) comes from the almost \( G \)-equivariance \((14)\). Thus,

\[
\sum_{1 \leq i \leq 3} \mu_1(f_i U_i) \geq \theta \lambda^{-1} \cdot \mu_1(U)
\]

so

\[
\mu_1(\gamma_1 \Pi^F(g_0, r_0) \cap C) \geq \theta \lambda^{-1} \cdot \mu_1(U).
\]

Using again \((14)\) and \((15)\), we have

\[
\mu_1(\Pi^F_0(g_0, r_0) \cap C) \geq \lambda^{-1} \cdot \mu_{g^{-1}}(\gamma_1 \Pi^F_0(g_0, r_0) \cap C)
\]

\[
\geq \lambda^{-2} \cdot e^{-\omega d(o, go)} \cdot \mu_1(\gamma_1 \Pi^F_0(g_0, r_0))
\]

where the last line uses \( B_\xi(g^{-1} o, o) \leq d(o, go) \). Setting \( M := \theta \cdot \lambda^{-3} \cdot \mu_1(U) \), we get the lower bound:

\[
\mu_1(\Pi_0^F(g_0, r_0) \cap C) \geq \frac{1}{2} \cdot e^{-\omega d(o, go)}.
\]

We remark here that by the same proof, the subset \( C \) could be replaced with any \( G \)-invariant subset of positive \( \mu_1 \)-measure. We restrict to the \( \mathcal{C} \) satisfying [Assump D] only for the upper bound.

2. Upper Bound. Fix \( r \geq r_0 \). Given \( \xi \in g^{-1} \Pi_0(g_0, r) \cap C \), there is a sequence of \( z_n \in Y \) tending to \( \xi \) such that \( \gamma_n \cap B(o, r) \neq \emptyset \) for \( \gamma_n := [g^{-1} o, z_n] \). By [Assump D] we obtain

\[
|B_\xi(g^{-1} o, o) - d(g^{-1} o, o)| \leq 2r + \epsilon.
\]

As \( \mu_1(g^{-1} \Pi_0(g_0, r)) \leq \| \mu_1 \| = 1 \), the upper bound is given as follows:

\[
\mu_1(\Pi_0(g_0, r) \cap C) \leq \lambda \mu_{g^{-1}}(g^{-1} \Pi_0(g_0, r) \cap C)
\]

\[
\leq \lambda e^{2r + \epsilon} \int g^{-1} \Pi_0(g_0, r) e^{-\omega B_\xi(g^{-1} o, o)} d\mu_1(\xi)
\]

\[
\leq \lambda e^{2r + \epsilon} \cdot e^{-\omega d(o, go)}
\]

The proof of lemma is complete. \( \square \)

Remark (Alternative proof). We sketch another proof for the lower bound which was carried out in Teichmüller space \([70]\). In the proof of Lemma \(3.19\) as \((f) \cdot \partial K = \partial Y \setminus [f^+]\), the set \( \partial K \cap \mathcal{C} \) has positive \( \mu_1 \)-measure. By Lemma \(3.21\) there exists an open set \( U \) of \([f^+]\) of positive \( \mu_1 \)-measure so that

\( U \subseteq \Pi_{g^{-1} o}^F(o, r) \), and then the remaining proof proceeds as above.

Standing Assumption. From now on, we assume further that the constant \( r > 0 \) in Convention \(2.16\) satisfies the Shadow Lemma \(6.3\).

As all boundary points in the horofunction boundary are non-pinched by Lemma \(5.4\) we obtain a full version of shadow lemma.

Lemma 6.4. Let \( \{ g \}_{g \in G} \) be a \( \omega \)-dimensional \( G \)-equivariant conformal density for some \( \omega > 0 \) on the horofunction boundary \( \partial_h Y \). Then there exists \( r_0 > 0 \) such that

\[
\exp(-\omega \cdot d(o, go)) \prec \mu_1(\Pi_0^F(g_0, r)) \leq \mu_1(\Pi_0(g_0, r)) \prec \exp(-\omega \cdot d(o, go))
\]

for any \( go \in G_o \) and \( r \geq r_0 \).

Corollary 6.5. Under the assumption as Lemma \(6.4\), \( \mu_1 \) has no atoms at conical points in \( \partial_h Y \).

Proof. Recall that any conical point \( \xi \in \Lambda_0^F(G) \) is contained in the \((r, F)\)-shadows of an unbounded sequence of orbital points in \( G_o \). By Lemma \(3.21\) the \((r, F)\)-shadows are sandwiched between open sets, so by increasing \( r \), we can assume that \( \xi \) is contained in a sequence of open sets whose \( \mu_1 \)-measure tends to \( 0 \). This shows that \( \xi \) is not atom.

By the existence of \( \omega_G \)-dimensional conformal density in Lemma \(6.2\), we obtain a upper bound on the growth of balls first proved in \([75]\), by other methods without using PS-measures.
Proposition 6.6. Let \( \{ \mu_g \}_{g \in G} \) be a \( \omega \)-dimensional \( G \)-quasi-equivariant, quasi-conformal density on \( \partial \Omega \) for \( \omega > 0 \) charging positive measure on \( \mathcal{C} \). Then for any \( n \geq 0 \),
\[
|N(o, n)| < \exp(\omega n).
\]
In particular, \( \omega \geq \omega_G \) and \( |N(o, n)| < \exp(\omega_G n) \).

To prove this result, we need the following technical result, which shall be also used in later on. Recall that \( \Pi^{\mathcal{F}}_o(v, r) \) denotes the union of \([\cdot]\)-classes of \( \xi \in \Pi^{\mathcal{F}}_o(v, r) \).

Lemma 6.7. There exist \( \hat{r}, R > 0 \) with the following property. Assume that \( |F_0|_{\min} \gg 0 \). Let \( v, w \in Go \) such that the intersection
\[
\xi \in [\Pi^{\mathcal{F}}_o(v, \hat{r})] \cap [\Pi^{\mathcal{F}}_o(w, \hat{r})] \cap \mathcal{C}
\]
is non-empty. If \( d(o, v) \leq d(o, w) \), then \( d(v, [o, w]) \leq R \) and \( \Pi^{\mathcal{F}}_o(v, R) \subseteq \Pi^{\mathcal{F}}_o(v, \hat{r}) \).

Proof. Write \( v = go, w = g'o \). By definition, there exist \( x_m \in \Omega^{\mathcal{F}}_o(go, r), y_m \in \Omega^{\mathcal{F}}_o(g'o, r) \) such that \( x_m \) and \( y_m \) converge to \( [\xi] \). Let \( X = gAx(f) \) and \( Y = g'Ax(f') \) be the corresponding \((r, F)\)-barriers satisfying (18)
\[
go, go \in N_r((o, x_m)), \quad g'o, go \in N_r((o, y_m))
\]
Case 1. Assume that \( X = Y \). For \( go, g'o \in X = Y, [o, go] \) and \([o, g'o] \) intersect the \( C \)-ball at \( \pi_X(o) \) by Lemma 2.2. As \( d(o, go) \leq d(o, g'o) \) is assumed, the Morse quasi-geodesic \( X \) implies the existence of a constant \( R \) independent of \( go, g'o \) such that \( d(go, [o, g'o]) \leq R \).

Case 2. Assume now that \( X \neq Y \in \mathcal{F} \). As \( \mathcal{F} \) has bounded intersection, we have \([AX] \cap [AY] = \emptyset \). For concreteness, assume that \( \xi \in [AX] \). By Corollary 6.14 we have \( d_X(x_m, y_m) \leq C \) for all \( m \gg 0 \). By triangle inequality for \( d_X \), we have \( d_X(o, y_m) \leq d_X(o, x_m) - C \), so Lemma 2.3 shows
\[
||N_C(X) \cap [o, y_m]|| \geq d_X(o, x_m) - 5C \geq L - 4r - 5C > 0.
\]
By Lemma 2.2, the entry points of \([o, x_m]\) and of \([o, y_m]\) into \( N_C(X) \) have a distance at most \( 2C \). As \( ||N_C(X) \cap N_C(Y)|| \leq B \) for some \( B = B(C) \), we see that \([o, y_m]\) exits \( N_C(X) \) within at most \( B \) distance to the entry point in \( N_C(Y) \). By (18), we have \( d(go, [o, x_m]), d(g'o, [o, y_m]) \leq r \). Thus, there exists a constant \( R \) such that \( d(go, [o, g'o]) \leq R \).

Apply Lemma 2.8 to the geodesic \( \alpha := [o, x_m] \) and \( \beta := [o, y_m] \), where \( \alpha \) contains \((r, f)\)-barrier \( go \in X \). Recalling \( d_X(x_m, y_m) \leq C \) as above, we have \( go \) is an \((\hat{r}, f)\)-barrier for \([o, y_m]\) with \( m \gg 0 \). This implies by definition that \( \Pi^{\mathcal{F}}_o(w, r) \subseteq \Pi^{\mathcal{F}}_o(v, \hat{r}) \). \( \square \)

Consider the annulus of radius \( n \) centered at \( o \in \Omega \) with width \( \Delta \geq 1 \):
\[
A(o, n, \Delta) = \{ v \in Go : |d(o, v) - n| \leq \Delta \}
\]

Lemma 6.8. For given \( \Delta > 0 \), there exists \( N = N(\Delta) \) such that for every \( n \geq 1 \), any \( \xi \in \mathcal{C} \) is contained in at most \( N \) shadows \( \Pi^{\mathcal{F}}_o(v, r) \) where \( v \in A(o, n, \Delta) \).

Proof. Indeed, let \( v, w \in A(o, n, \Delta) \) such that \( \xi \in \Pi^{\mathcal{F}}_o(v, R) \cap \Pi^{\mathcal{F}}_o(w, R) \). Lemma 6.7 gives an constant \( R > 0 \) so that \( d(v, [o, w]) \leq R \). Taking into account of the relation
\[
d(o, v), d(o, w) \in [n - \Delta, n + \Delta],
\]
there exists \( D = D(\Delta, R) \) such that \( d(v, w) \leq D \). Therefore it suffices to set \( N = |N(o, D)| \) to complete the proof. \( \square \)

All the ingredients are prepared for the proof of Proposition 6.6.

Proof of Proposition 6.6 By Lemma 6.8 that every point \( \xi \in \mathcal{C} \) is contained in a uniformly finite number of shadows. Hence,
\[
\sum_{v \in A(o, n, \Delta)} \mu_1(\Pi^{\mathcal{F}}_o(v, r) \cap \mathcal{C}) < \mu_1(\mathcal{C})
\]
which by the shadow lemma gives \( |A(o, n, \Delta)| \leq C R^n \), and thus the desired estimates on \( |N(o, n)| \). \( \square \)
6.3. **Shadow principle.** Let $N(G)$ be the normalizer of $G$ in Isom(Y). The following result called Shadow Principle by Roblin [67] is proved in CAT(-1) spaces. The shadow lemma is a special case where $x, y \in Go$ and $\| \mu_x \| = \| \mu_y \|$.

**Lemma 6.9** (Shadow Principle). Let $\{ \mu_x \}_{x \in Y}$ be a $\omega$-dimensional $G$-quasi-equivariant quasi-conformal density supported on $\partial Y$ with $\mu_x(C) > 0$ for some $\omega > 0$. Let $Z = \Gamma o$ for $\Gamma = N(G)$. Then there exists $r_0 > 0$ such that
\[
\mu_y(C) e^{-\omega d(x, y)} < \lambda \mu_x(\Pi_x^y(y, r) \cap C) \leq \mu_x(\Pi_x^y(y, r) \cap C) < \lambda^{-1} \mu_y(C) e^{-\omega d(x, y)}
\]
for any $x, y \in Z$ and $r \geq r_0$, where $\lambda, \omega$ are given respectively in Assump D and Definition 6.1.

**Proof.** Write explicitly $x = g_1 o$ and $y = g_2 o$ for $g_1, g_2 \in \Gamma$. Set $U = C$. Repeating the proof for the lower bound of Lemma 6.3 with $g := g_1^{-1} g_2$, we decompose the set $U$ into a disjoint union of three sets $U_1, U_2, U_3$; for each $U_i$, there exists $f_i \in F$ such that
\[
(20) \quad g_2 f_i U_i \subseteq \Pi_{g_1 o}^F(g_2 o, r) \cap C.
\]
By quasi-conformality (15) for $d(o, f_i o) \leq L$, there exists $\theta = \theta(L) > 0$ such that for each $1 \leq i \leq 3$,
\[
(21) \quad \mu_{g_1 o}(g_2 f_i U_i) \geq \theta \cdot \mu_{g_2 f_i o}(g_2 f_i U_i).
\]
By the $G$-quasi-equivariance in (14), we apply the element $g_2 f_i^{-1} g_2^{-1} \in G$ to obtain
\[
(22) \quad \mu_{g_2 f_i o}(g_2 f_i U_i) \geq \lambda^{-1} \cdot \mu_{g_2 o}(g_2 U_i).
\]
$(g_2 f \in \Gamma \setminus G$ cannot be cancelled directly for $\mu_x$ is not known $\Gamma$-equivariant). By (21) and (22),
\[
\sum_{1 \leq i \leq 3} \mu_{g_2 o}(g_2 f_i U_i) \geq \lambda^{-1} \cdot \mu_{g_2 o}(g_2 U).
\]
As $U = C$ is $\Gamma$-invariant and so $g_2 U = U$, we obtain from (20):
\[
\mu_{g_2 o}(\Pi_{g_1 o}^F(g_1 o, g_2 o)) \geq \lambda^{-1} \cdot \mu_{g_1 o}(g_2 U) \geq \lambda^{-1} \theta \cdot \mu_{g_2 o}(U).
\]
By quasi-conformality (15) again,
\[
\mu_{g_1 o}(\Pi_{g_1 o}^F(g_2 o, r) \cap C) \geq \lambda^{-2} \cdot \mu_{g_1 o}(g_2 o) \cdot \mu_{g_2 o}(\Pi_{g_1 o}^F(g_2 o, r)) \geq \lambda^{-1} \theta \cdot \mu_{g_2 o}(U) \cdot e^{-\omega d(g_1 o, g_2 o)}.
\]
where $M := \lambda^{-3} \cdot \theta$. The upper bound proceeds exactly as Lemma 6.3 so we omit it. \qed

7. **HOPF-TSUJI-SULLIVAN THEOREM**

This section is devoted to the proof of Theorem 1.10. Let $G \varsubsetneq Y$ as in 1, equipped with a convergence boundary $\partial Y$. Let $\{ \mu_x \}_{x \in Y}$ be any $\omega$-dimensional $G$-equivariant quasi-conformal density charging positive measure on $C$, where $C$ satisfies Assump C and Assump D. The proof is based on the construction of a family of visual spheres described as follows. A similar strategy was used by Takia [71] for real hyperbolic spaces whose arguments are hard to be implemented in general metric spaces. We are eventually turned to the technique of projection complex.

7.1. **Family of visual spheres.** Recall $(\mathcal{F}, F, r, \hat{r}, L, B)$ are given in Convention 2.16 where $\mathcal{F}$ is the collection of $G$-translated $C$-contracting $F$-axes. Moreover, the constant $r$ satisfies Shadow Lemma 6.3.

**Convention 7.1.** In the sequel, $U \in \mathcal{F}$ denotes either the subset $U$ in $Y$ or in the quasi-tree $\mathcal{C}(\mathcal{F})$, so the points $u \in U$ are understood accordingly. The corresponding vertex in $\mathcal{P}_K(\mathcal{F})$ is denoted by $\hat{u}$.

Choose $K \gg 0$ such that for any $U \neq V \in \{ \text{Ax}(f) : f \in F \}$, we have $\mathcal{F}_K(U, V) = \emptyset$. Thus, $\{ \text{Ax}(f) : f \in F \}$ form a clique (three vertices pairwise adjacent) in $\mathcal{P}_K(\mathcal{F})$. Fix $O \in \{ \text{Ax}(f) : f \in F \}$, and a base vertex $\hat{o}$ in $\mathcal{P}_K(\mathcal{F})$. As $o \in O$, this is consistent to the above convention.
Visual sphere. For an integer $n \geq 0$, consider the sphere of radius $n$ centered at $\bar{v}$:

$$\hat{S}_n := \{ \bar{u} \in \mathcal{P}_K(\mathcal{F}) : d(\bar{v}, \bar{u}) = n \}$$

The $n$-th visual sphere of orbital points in $Go$ is defined as

$$T_n := \bigcup_{\bar{v} \in \hat{S}_n} V.$$ 

That is, $T_n$ is the union of all $V \in F$ corresponding to $\bar{v} \in \hat{S}_n$. If $go \in U$, we say that $U$ or $\bar{u}$ are an associated (non-unique) vertex space or a vertex.

Observe that the union $\bigcup_{n=0}^\infty T_n$ covers $Go$ with multiplicity between 1 and 3. Indeed, the vertex spaces $V$ are not necessarily disjoint, and any element of $Go$ is possibly contained in at most three $V \in F$, as $V = qE(f) \cdot o$ and $f \in F$ consists of three elements.

We now introduce some auxiliary sets. For any $V \in F$ and $L, \Delta > 0$, denote

$$V_L := \{ v \in V : d_V(o, v) > L \}$$

$$T_n'(L) := \bigcup_{\bar{v} \in \hat{S}_n} V_L$$

(where $V$ is the axis corresponding to the vertex $\bar{v}$.) By definition, $T_n'(L)$ forms an $L$-net in $T_n$: any point in $T_n$ has a distance less than $L$ to $T_n'$.

Let us fix $\Delta := \min\{d(o, go) : g \in E(f), f \in F\}$, so for any $L > 0$, the set

$$V(L, \Delta) := \{ v \in V : |d_V(o, v) - L| \leq \Delta \}$$

is always non-empty, and has at most $|A(o, L, \Delta)|$ elements.

The union of $V(L, \Delta)$ over $\bar{v} \in \hat{S}_n$ is definitely not a net of $T_n$, however, on which the Poincaré series takes a major proportion.

**Lemma 7.2.** For any $\omega > 0$ and $L > 0$, there exists $\theta = \theta(L, \omega) > 0$ such that for any $V \in F$, we have

$$\sum_{v \in V} e^{-\omega d(o, v)} \leq \theta \sum_{v \in V(L, \Delta)} e^{-\omega d(o, v)}.$$

**Proof.** Write $V = qE(f) \cdot o$ for some $g \in G$ and $f \in F$. Let $u \in V$ such that $d(o, u) = d(o, V)$, so $|d(u, v) - L| \leq \Delta$ for $v \in V(L, \Delta)$. The $C$-contracting property further implies for any $v \in V$:

$$|d(o, v) - d(o, u) - d(u, v)| \leq 4C.$$

Thus,

$$\sum_{v \in V} e^{-\omega d(o, v)} < C, \Delta e^{-\omega d(o, u)} \sum_{n \geq 1} |A(u, n, \Delta) \cap Ax(f)| e^{-\omega n}.$$

As the maximal elementary group $E(f)$ is virtually $\mathbb{Z}$ and $Ax(f) = E(f) \cdot o$ is a quasi-geodesic, we see $|A(u, n, \Delta) \cap Ax(f)|$ is a linear function of $n$. For any given $L > 0$, we ignore the contribution from $V_{L+\Delta}$ in the series on the right-hand side, so that

$$\sum_{v \in V} e^{-\omega d(o, v)} \leq \theta \left( \sum_{v \in V(L, \Delta)} e^{-\omega d(o, v)} \right)$$

where the implied constant $\theta$ depends only on $L, C, \Delta$ and $E(f)$. Since there are finitely many $E(f)$ for $f \in F$, the conclusion follows. \hfill $\square$

The following lemma is analogous to Lemma 6.8 but the proof is much more involved.

**Lemma 7.3.** There exists $L \gg 0$ such that for any $\xi \in C$ and $n > 0$, the set of orbital points

$$T_n(\xi) := \{ v \in T_n'(L) : \xi \in \Pi^F_o(v, r) \}$$

is contained in at most one axis in $F$.

**Proof.** Let $go, ho \in T_n'(L)$ so that $\xi \in \Pi^F_o(go, r) \cap \Pi^F_o(ho, r)$. By definition of $T_n'(L)$, let $\bar{u}, \bar{v} \in \hat{S}_n$ be associated vertices with $go \in U$ and $ho \in V$ satisfying

$$d_U(o, go) \geq L, \quad d_V(o, ho) \geq L.$$ 

We shall prove that $U = V$. Assume to the contrary that $U \neq V$, and $d(o, U) \leq d(o, V)$ for concreteness.
As \( \xi \in \Pi^F_o(g_o, r) \cap \Pi^F(o, ho, r) \), there exist two sequences of points \( \{x_m\} \subseteq \Omega^F_o(g_o, r), \{y_m\} \subseteq \Omega^F_ho(ho, r) \) both tending to \( \xi \) as \( m \to \infty \), such that \( d(go, [o, x_m]) \leq r, d(ho, [o, y_m]) \leq r \). By the contracting property, the exit point of \([o, x_m]\) at \( N_o(U) \) is \( 2C \)-close to \( \pi_U(x_m) \), and has a distance to \( o \) bigger than \( d(o, go) \) for \( go \in U \). Thus, there exists a constant \( M = M(r) > 0 \) such that

\[
d_U(o, x_m) \geq d_U(o, go) - M \geq L - M.
\]

The same reasoning shows \( d_U(o, y_m) \geq L - M \). As \( U \neq V \) is assumed, we have \( AU \cap AV = \emptyset \). For concreteness, assume that \( \xi \notin AU \); the case \( \xi \notin AV \) is similar.

By Assumption \( C \), we have \( d([o, x_m], [o, y_m]) \to \infty \) for \( \xi \in C \). By Lemma \( 3.14 \) as \( \xi \notin AU \), we have \( d_U(x_m, y_m) \leq C \) for \( m \gg 0 \). The triangle property for \( d_U \) then gives

\[
d_U(o, y_m) \geq d_U(o, x_m) - d_U(x_m, y_m) \geq L - C - M.
\]

If we choose \( L > K + 2M + C \), where \( K \) is given by Lemma \( 2.23 \). Thus, \( d_U(o, ho) \geq K \), so that \( \bar{u} \) appears in any geodesic \([\bar{u}, \bar{\bar{u}}] \). This is a contradiction, as \( d(\bar{u}, \bar{\bar{u}}) = d(\bar{u}, \bar{\bar{u}}) = n \).

In conclusion, the set \( T_n(\xi) \) is contained in one axis \( U = V \). In particular, \( T_n(\xi) = V_L \).

**Lemma 7.4.** There exists a uniform constant \( \theta > 0 \) independent of \( n \geq 1 \) such that

\[
\sum_{go \in T_n} e^{-\omega(d(go, go))} \leq \theta.
\]

**Proof.** Fix a constant \( L > 0 \) satisfying Lemma \( 7.3 \) so if \( T_n(\xi) \) for \( \xi \in C \) is non-empty, then it is contained in one \( V \in \mathcal{F} \). Precisely, \( T_n(\xi) \subseteq V_L \). By Lemma \( 7.2 \) we have

\[
\sum_{v \in T_n} e^{-\omega(d(o, v))} \leq \theta \sum_{u \in S_n} \left( \sum_{u \subseteq V(L, \Delta)} e^{-\omega(d(o, u))} \right)
\]

By Lemma \( 6.7 \) there exists \( N = N(L) \) such that any point \( \xi \in C \) are shadowed by at most \( N \) elements in \( V(L, \Delta) \), we have

\[
\sum_{u \in S_n} \sum_{u \subseteq V(L, \Delta)} \mu_1(\Pi^F_o(u, r)) \leq N \cdot \mu_o(C)
\]

The conclusion then follows from the shadow lemma \( 6.3 \).

**Lemma 7.5.** For any integer \( n \geq 1 \) and for any \( s > \omega_G \), we have

\[
\mathcal{P}_{T_n}(s, o, o) \succ \mathcal{P}_{T_{n+1}}(s, o, o)
\]

and

\[
\mathcal{P}_{T_{n+1}}(s, o, o) \succ \mathcal{P}_{T_n}(s, o, o) \cdot \mathcal{P}_{T_m}(s, o, o)
\]

where the implied constants do not depend on \( m, n \) and \( s \).

**Proof.** Recall the definition

\[
\mathcal{P}_{T_n}(s, o, o) = \sum_{v \in T_n} e^{-\omega d(o, v)}
\]

(1). We start by proving the direction of \( 25 \): \( \mathcal{P}_{T_n}(s, o, o) \prec \mathcal{P}_{T_{n+1}}(s, o, o) \). The other direction follows from \( 26 \), as proven below in (2).

Let \( K = K(K) \geq B \) be given by Lemma \( 2.23 \) where \( B \) is the bounded projection constant of \( \mathcal{F} \). Consider the \( (2K)-net \) \( T'_n := T_n(2K) \) in \( T_n \), so that

\[
\mathcal{P}_{T_n}(s, o, o) \simeq_K \mathcal{P}_{T'}(s, o, o).
\]

The goal of the proof is then to prove for any \( go \in T'_n \) there exists \( f' \in F \) such that \( gf'o \in T_{n+1} \). As \( d(o, fo) \) is a fixed number, this implies \( \mathcal{P}_{T_n}(s, o, o) \prec \mathcal{P}_{T_{n+1}}(s, o, o) \).
Let \( go \in T_n'(2\hat{K}) \), so there exist an associated vertex \( \bar{u} \in \tilde{S}_n \) of type \( f \in F \) and the vertex space \( U = gAx(f) \) such that \( d_U(o, go) > 2\hat{K} \). By Lemma 2.13 there exists \( f' \neq f \in F \) such that \( gAf' \) labels a quasi-geodesic. The axis \( V := gAx(f') \) is adjacent to \( U \) in \( P_K(\mathcal{F}) \), so \( d(\bar{u}, \bar{v}) = 1 \), then \( |d_P(\bar{v}, \bar{o}) - n| \leq 1 \). We claim that \( \bar{v} \in \Tilde{S}_{n+1} \). To this end, we need to exclude the possibility of \( d(\bar{v}, \bar{o}) \leq n \).

Since \( Ax(f) \) and \( Ax(f') \) have \( B \)-bounded intersection, \( \{go, gAf'\} \subseteq V \) has \( B \)-bounded projection to \( U = gAx(f) \). This implies \( d_U(O, V) \geq d(o, go) - B \geq 2\hat{K} - B \geq \hat{K} \). By Lemma 2.23 \( \bar{u} \) is contained in any geodesic from \( \bar{o} \) to \( \bar{v} \) in \( P_K(\mathcal{F}) \). If \( d(\bar{o}, \bar{v}) \leq n \) was true, we would obtain a contradiction: \( d(\bar{u}, \bar{o}) < n \). Hence, \( d(\bar{u}, \bar{v}) = n + 1 \) and \( \bar{v} \in \Tilde{S}_{n+1} \). In other words, we proved that for any \( g \in T_n' \) there exists \( f' \in F \) such that \( gAf' \) is of convergent type. Denote \( \Lambda : \text{covers a product of at most three elements so that} \)

\[
\begin{align*}
\Lambda_n &:= \bigcup_{i \geq 1} \{\Pi^{F}(go, r) : g \in \cup_{k \geq n} T_k\}, \\
\text{We have } \lim_{n \to \infty} \Lambda_n &\subseteq \Lambda_n^F(G) \text{ by } (12). \text{ By hypothesis, } \lim_{n \to \infty} \mu_1(\Lambda_n) = 0. \text{ Using Lemma } 6.3 \text{ there exist } n_0 > 0 \text{ and } 0 < q < 1 \text{ such that } \theta q \leq 1 \text{ and} \\
\sum_{g \in T_{k_0}} e^{-\omega d(a, go)} &< q \\
\text{ where } \theta > 0 \text{ is the implied constant given by Lemma } 7.5. \text{ For each } k \geq 0 \text{ and } 1 \leq i \leq n_0, \text{ we obtain from } (25) \text{ that} \\
\sum_{g \in T_{k_0}} \left( \sum_{v \in T_{k_0}} e^{-\omega d(a, v)} \right) &< \sum_{k \geq 1} \left( \theta \sum_{v \in T_{k_0}} e^{-\omega d(a, v)} \right)^{k} \leq \sum_{k \geq 1} (\theta q)^{k} < \infty \text{ and} \\
\sum_{v \in T_{k_0+1}} e^{-\omega d(a, v)} &< C \sum_{v \in T_{k_0}} e^{-\omega d(a, v)} \\
\text{ By } (27), \text{ we obtain that } \mathcal{P}_G(\omega, o, o) \text{ is convergent. The same argument works for the usual conical points } \Lambda_n(G) \text{ in } (5) \text{ without index } F: \mu_1(\Lambda_n(G)) = 0 \text{ implies } \mathcal{P}_G(\omega, o, o) < \infty. \]
Step 2: (4) \implies (1). If \( G \) is of convergent type, then the set \( \Lambda^G_\omega(G) \) is \( \mu_1 \)-null. Indeed, assume that \( \mu_1(\Lambda^G_\omega(G)) > 0 \) and we are going to prove that \( \mathcal{P}_G(\omega, o, o) \) is convergent. By Corollary 4.3, we could fix the light source at \( o \) and increase \( r \), so that the following family of sets

\[
\Lambda_n := \bigcup_{k \geq n} \left( \bigcup_{v \in A(o, k, \Delta)} \Pi^G_\nu(v, r) \right)
\]

tends to \( \Lambda^G_\omega(G) \), where \( A(o, n, \Delta) \) is defined in [19]. Thus, \( \mu_1(\Lambda_n) \geq \frac{1}{2} \mu_1(\Lambda^G_\omega(G)) \) for \( n \gg 0 \). By Lemma 6.3, we obtain a uniform lower bound of the partial sum of Poincaré series \( \mathcal{P}_G(\omega, o, o) \) for any \( n \gg 0 \)

\[
\mathcal{P}_G(\omega, o, o) \ni \frac{1}{n} \sum_{k=1}^{n} \left( \sum_{v \in A(o, k, \Delta)} e^{-\omega n} \right) > \mu_1(\Lambda_n)
\]

contradicting to the convergence of \( \mathcal{P}_G(\omega, o, o) \). Again, the same argument shows that \( \Lambda_c(G) \) is \( \mu_1 \)-null.

Step 3: (4) \implies (5). If the set \( A := \Lambda^G_\omega(G) \) or \( A := \Lambda_c(G) \) is \( \mu_1 \)-positive then it is \( \mu_1 \)-full in \( C \): \( \mu_1(A) = \mu_1(C) \). Suppose to the contrary that the complement \( C \setminus A \) is \( \mu_1 \)-positive. The restriction of \( \{\mu_x\}_{x \in Y} \) on \( C \setminus A \) gives a conformal density \( \{\nu_x\}_{x \in Y} \) of the same dimension \( \omega \), so that \( \nu_1(\Lambda^G_\omega(G)) = 0 \).

By Step 1, we have that \( G \) is of convergent type: this in turn contradicts the Step 2.

Step 4: (5) \implies (3). Applying \( "(1) \implies (4)" \), we have that for every \( \mathcal{F} \) in \( (2) \), the set \( \Lambda^G_\omega(G) \) is \( \mu_1 \)-full. By Corollary 4.13, their countable intersection gives the Myrberg limit set.

Finally, the direction \( "(2) \implies (4)" \) follows as \( \Lambda^G_\omega(G) \in \Lambda_m G \), and \( "(3) \implies (2)" \) is trivial. The proof of Theorem 1.10 is completed.

8. Conformal and harmonic measures on Myrberg limit set

This section focuses on conformal and harmonic measures on the reduced horofunction boundary. First of all, we prove the uniqueness and ergodicity of conformal density, provided that \( G \) is of divergent type: Theorem 1.11. Secondly, we explain the proof of Theorem 1.8 concerning the Poisson boundary.

8.1. Conformal density on the reduced Myrberg limit set. Let \( \{\mu_x\}_{x \in Y} \) be a \( \omega \)-dimensional \( G \)-quasi-equivariant quasi-conformal density on \( \partial_h Y \) with \( \lambda \geq 1 \) given in Definition 6.1. As \( G \sim Y \) is of divergent type, \( \mu_x \) is supported on \( \Lambda_m G \).

By Theorem 1.10, we have \( \mu_1 \) is supported on the Myrberg limit set \( \Lambda_m G \). By Lemma 5.6 and 2.27, the finite difference relation restricted on \( \Lambda_m G \) induces a closed surjective map from \( \Lambda_m G \) to \( [\Lambda_m G] \) with compact fibers. This implies that \( [\Lambda_m G] \) is Hausdorff and second countable.

**Push forward conformal density.** Push forward \( \{\mu_x\}_{x \in Y} \) to \( [\Lambda_m G] \) via the quotient map. We obtain a \( \omega \)-dimensional \( G \)-quasi-equivariant quasi-conformal density \( \{[\mu_x]\}_{x \in Y} \) on \( [\Lambda_m G] \), where \( [\mu_x] \) denotes the image measure of \( \mu_x \). The constant \( \lambda' \) now depends on \( \lambda \) and \( C \), as any two horofunctions in a \([\cdot]\)-class of Myrberg points has \( (2C) \)-difference by Lemma 5.3. Here the contracting constant \( C \) could taken to be the infimum over all \( \mathcal{F} \) in \( (2) \).

The proof consists in releasing conformal measures as a Hausdorff-Caratheodory measure on \( [\Lambda_m G] \). To that end, we first state a variant of Vitali covering lemma in [77, Appendix A].

Let \( B \in \mathcal{B} \) be a family of subsets in a topological space \( X \) with extensions \( B \subseteq \hat{B} \) of each \( B \). Then \( \mathcal{B} \) is called **hierarchically locally finite** if there exist a positive integer \( N > 0 \) and a disjoint decomposition \( \mathcal{B} = \bigcup_{n \geq 0} \mathcal{B}_n \) for \( n \geq 0 \) where each \( \mathcal{B}_n \) is possibly empty for certain \( n \) such that the following two properties are true:

1. Given \( B \in \mathcal{B}_n \) for \( n \geq 0 \), the set \( N(B) \) of \( A \in \mathcal{B}_{n-1} \cup \mathcal{B}_n \cup \mathcal{B}_{n+1} \) with \( A \cap B \neq \emptyset \) has cardinality at most \( N \), where \( \mathcal{B}_{-1} := \emptyset \).

2. If \( A \in \mathcal{B}_n \) intersects \( B \in \mathcal{B}_m \) for \( n \geq m - 2 \), then \( A \) is contained in the extension \( \hat{B} \) of \( B \).

By definition, any subfamily of \( \mathcal{B} \) is hierarchically locally finite.

**Lemma 8.1.** [77, Lemma A.1] Let \( \mathcal{B} \) be a hierarchically locally finite family of subsets in a space \( X \). Then there exists a sub-family \( \mathcal{B}' \subseteq \mathcal{B} \) of pairwise disjoint subsets such that

\[
\bigcup_{B \in \mathcal{B}} B \subseteq \bigcup_{A \in \mathcal{B}'} \left( \bigcup_{B \subseteq \mathcal{N}(A)} \hat{B} \right).
\]
Construction of hierarchically locally finite family. Define the family $B$ of sets $B(v)$ and their extension $\hat{B}(v)$ as follows

$$B(v) := [\Pi^F_v(v, r)], \quad \hat{B}(v) := [\Pi^F_v(v, \hat{r})]$$

for every $v \in \mathcal{O}$. By abuse of language, we define diameter $\|B(v)\| := e^{-d(o, v)}$, even though there may not exist a metric on $\partial Y$ inducing it. A subfamily of $\mathcal{E} \subseteq \mathcal{B}$ is called an $\epsilon$-covering, if $\|B\| \leq \epsilon$ for every $B \in \mathcal{B}'$.

Lemma 8.2. For any $v \in \mathcal{O}$, we have

$$\mu_1(B(v)) \sim \|B(v)\|^\omega.$$  

Proof. The lower bound on $\mu_1(B(v))$ follows directly from the shadow lemma 6.4. For the upper bound, $B(v)$ contains the locus of every points in $\Pi^F_o(v, r)$. Since $G$ is of divergent type, the measure $\mu_1$ is supported on $\Lambda^F_o$, so by Lemma 5.5, any two horofunctions in the same locus of points in $\Pi^F_o(v, r)$ have uniform difference. Thus, the proof of Lemma 6.3 for the upper bound works to give an uniform coefficient before $e^{-d(o, v)}$.

We define a Hausdorff-Caratheodory measure $\mathcal{H}^\omega$ on $[\Lambda_mG]$ relative to $\mathcal{B}$ and the gauge function $r^\omega$. Namely, for any $\epsilon > 0$,

$$\mathcal{H}^\omega(A) = \inf \left\{ \sum_{B \in \mathcal{B}'} \|B\|^\omega : A \subseteq \bigcup_{B \in \mathcal{B}'} B, \|B\| \leq \epsilon \right\}$$

where the infimum is taken over $\epsilon$-cover $\mathcal{B}'$ over $A$. Then

$$\mathcal{H}^\omega(A) := \sup_{\epsilon > 0} \mathcal{H}^\omega(A).$$

The Hausdorff measure in the usual sense is taking $\mathcal{B}$ as the set of all metric balls in $\Lambda G$.

Fix a large $L > 0$, and subdivide $\mathcal{B}$ into a sequence of (possibly empty) sub-families $\mathcal{B}_n$ for $n \geq 0$:

$$\mathcal{B}_n := \{ B(v) \in \mathcal{B} : nL \leq d(o, v) < (n + 1)L \}.$$

Lemma 8.3. There exists a uniform constant $L > 0$ such that $\mathcal{B}$ defined as above is a hierarchically locally finite family over $[\Lambda_mG]$.

Proof. We verify the two conditions in order. Let $B(v) = [\Pi^F_o(v, r)] \in \mathcal{B}_n$. Applying Lemma 6.8 to the annulus $A(o, n, 2L)$, there are at most $N = N(L)$ sets $B(w) \in \mathcal{B}_{n-1} \cup \mathcal{B}_n \cup \mathcal{B}_{n+1}$ intersecting $B(v)$

Let $B(w) \in \mathcal{B}_n$ for $m < n$ intersecting $B(v)$, so $d(v, w) \leq d(o, v) - d(o, w) \geq L$. We have $B(v) \subseteq \hat{B}(w) = [\Pi^F_o(w, \hat{r})]$ by Lemma 6.7. Both conditions are thus verified and the lemma is proved.

We are now ready to prove Theorem 1.11.

Lemma 8.4. Suppose that $G \rightharpoonup \gamma$ as in 1 is of divergent type, where $\gamma$ is compactified by the horofunction boundary $\partial \gamma Y$. Let $\{[\mu_z]\}_{z \in \gamma}$ be a $\omega G$-dimensional $G$-quasi-equivariant quasi-conformal density on the quotient $[\Lambda_mG]$. Then we have

$$\mathcal{H}^\omega(A) \sim [\mu_z](A)$$

for any Borel subset $A \subseteq [\Lambda_mG]$, where the implied constants depend on the constant $\lambda$ in Definition 6.7.

In particular, $[\mu_z]$ is unique in the following sense: if $[\mu, [\mu']]$ are two such quasi-conformal densities, then the Radon-Nikodym derivative $d[\mu] / d[\mu']$ is bounded from above and below by a constant depending only on $\lambda$.

Proof. Take an $\epsilon$-covering $\mathcal{B}' \subseteq \mathcal{B}$ of $A$, so that $[\mu_1(A)] \leq \sum_{B \in \mathcal{B}'} [\mu_1](B)$. Letting $\epsilon \to 0$, we obtain from 29 that $[\mu_1](A) < \mathcal{H}^\omega(A)$.

To establish the other inequality, for any $0 < \epsilon < \epsilon_0$, let $\mathcal{B}_1 \subseteq \mathcal{B}$ be an $\epsilon$-covering of $A$. By Lemma 8.1 there exist a disjoint sub-family $\mathcal{B}_2$ of $\mathcal{B}_1$ and an integer $N > 1$ such that

$$\mathcal{H}^\omega(A) < \sum_{B \in \mathcal{B}_2} N \cdot \|B\|^\omega < [\mu_1](U),$$

Letting $\epsilon \to 0$ yields $\mathcal{H}^\omega(A) < [\mu_1](A)$.
As in the usual Hausdorff measure, we define the Hausdorff dimension $\text{HDim}(A)$ of a subset $A \subseteq [\Lambda_m G]$:

$$\text{HDim}(A) := \sup\{\omega \geq 0 : \mathcal{H}^\omega(A) = \infty\} = \inf\{\omega \geq 0 : \mathcal{H}^\omega(A) = 0\}$$

where $\sup\emptyset := \infty$ and $\inf\emptyset := 0$ by convention. The following corollary is immediate.

**Corollary 8.5.** The Hausdorff dimension of $[\Lambda_m G]$ equals $\omega_G$.

**Lemma 8.6.** Let $\{[\mu_x]\}_{x \in Y}$ be a $\omega_G$-dimensional $G$-equivariant quasi-conformal density on $[\Lambda_m G]$. Then $\{[\mu_x]\}_{x \in Y}$ is ergodic with respect to the action of $G$ on $[\Lambda_m G]$.

**Proof.** Let $A$ be a $G$-invariant Borel subset in $[\Lambda_m G]$ such that $[\mu_x](A) > 0$. Restricting $[\mu_x]$ on $A$ gives rise to a $\omega_G$-dimensional conformal density $\nu_x := [\mu_x]|_A$ on $[\Lambda_m G]$. By Lemma 8.4 for any subset $X \subseteq [\Lambda_m G]$, we have $\nu_x(X) < \mathcal{H}^\omega(A \cap X)$. Thus, $[\mu_x](\Lambda_m G \setminus A) = 0$. \qed

### 8.2. Myrberg limit set as Poisson boundary

We now present the proof of Theorem 1.8 concerning the random walks on $G$.

By [7] Theorem 5.10, the action of $G$ on the projection complex $\mathcal{P}_K(\mathcal{F})$ is acylindrical (see [27] Thm 6.3 for a detailed proof). By the construction of projection complex, the orbital map

$$g_0 \in Y \mapsto g_0 \in \mathcal{P}_K(\mathcal{F})$$

is clearly a Lipschitz map. The finite logarithmic moment of $\mu$-random walk on $Y$ then implies the finite logarithmic moment of the induced random walk on $\mathcal{P}_K(\mathcal{F})$. [49] Theorem 1.5 then says that almost every trajectory on $\mathcal{P}_K(\mathcal{F})$ tends to a point in the Gromov boundary of $\mathcal{P}_K(\mathcal{F})$. If $H(\mu) < \infty$, the Gromov boundary endowed with hitting measure $\mu$ is the Poisson boundary. Via Lemma 4.14 we obtain that almost every trajectory on $Y$ tends to the locus of a conical point in $\partial Y$. As the countable intersection of the embeddings of those Gromov boundaries is contained in the Myrberg limit set by Corollary 6.13, we prove that almost every trajectory in $Y$ hits at the locus of a Myrberg point.

By Lemmas 5.6 and 2.27, the finite difference relation $[\cdot]$ restricted on $\Lambda_m G$ induces a closed map from $\Lambda_m G$ to $[\Lambda_m G]$ with compact fibers. This implies that $[\Lambda_m G]$ is Hausdorff and second countable. As the hitting measure $\nu$ is supported on $\Lambda_m G$, the partition $[\cdot]$ thus forms a measurable partition in sense of Rokhlin: there is a countable coarser partition separating almost every two $[\cdot]$-classes ([13] Def. 15.1]).

By Rokhlin’s theory, the measurable quotient space $((\partial Y), [\nu])$ by the above discussion is isomorphic to the Poisson boundary with hitting measure. The proof of Theorem 1.8 is therefore completed.

### 9. Cogrowth for groups of divergent type

From this section, we begin to present the applications of results established in previous sections. The first application is a general result about co-growth of normal subgroups.

**Theorem 9.1.** Assume that $G \varsubsetneq Y$ is a non-elementary group of divergent type with a contracting element. Then for any non-elementary normal subgroup $H$ of $G$, we have

$$\omega_H > \frac{\omega_G}{2}.$$ 

The remainder of this section is devoted to the proof, which are subdivided into the following two cases.

#### Case 1: Normal subgroups of divergent type

Assume that $H \varsubsetneq Y$ is of divergent type. We are going to prove that $\omega_H = \omega_G$. The proof presented here is inspired by the one in [55].

Let $\{\mu_x\}_{x \in Y}$ be a PS measure on the horofunction boundary $\partial H Y$, which is an accumulation point of $\{\mu_x^\omega\}_{x \in \mathcal{S}}$ in [10] supported on $Ho$ for a sequence $s \varpropto \omega_H$. This is a $\omega_H$-dimensional $H$-equivariant conformal density by Lemma 6.2. That is, $\lambda = 1$ and $\epsilon = 0$ as in Definition 6.1 and Assump D.

The goal of proof is providing a constant $M > 0$ such that

$$\forall \ x \in Go : \quad ||\mu_x|| \leq M$$

Once this is proved, the shadow principal 6.5 implies that

$$\mu_1([\Pi^{\infty}_r(go, r)]) \leq e^{-\omega_H d_o(go)}$$
for any \( g_0 \in G_0 \). The same argument as in Proposition 6.6 proves \( \omega_H \geq \omega_G \). Thus the remainder of the proof is to prove (32). For this purpose, for given \( g_0 \), we consider another PS-measure \( \{ \mu_{x, s}^{g_0} \}_{s \in Y} \) on \( \partial_h Y \) as an accumulation point of \( \{ \mu_{x, s}^{g_0} \}_{s \in Y} \) supported on \( Hg_0 \) for a sequence \( s \searrow \omega_H \).

As \( H \sim Y \) is of divergent type, the Patterson’s construction gives

\[
\forall x \in Y: \quad \| \mu_{x, s}^{g_0} \| = \lim_{s \to \omega^+} \frac{\mathcal{P}_H(s, x, g_0)}{\mathcal{P}_H(s, o, g_0)}, \quad \| \mu_{x, s}^{g_0} \| = \lim_{s \to \omega^+} \frac{\mathcal{P}_H(s, x, o)}{\mathcal{P}_H(s, o, o)}.
\]

From the definition of Poincaré series (4), we have

\[
\forall s > 0: \quad \frac{\mathcal{P}_H(s, g_0, g_0)}{\mathcal{P}_H(s, o, g_0)} = \left[ \frac{\mathcal{P}_H(s, g_0, o)}{\mathcal{P}_H(s, o, o)} \right]^{-1}
\]

For given \( g_0 \in G_0 \), let us take the same sequence of \( s \) (depending on \( g_0 \)) so that \( \{ \mu_{x, s}^{g_0} \}_{s \in Y} \) converge to \( \{ \mu_{x, s}^{g_0} \}_{s \in Y} \) and \( \{ \mu_{x, s}^{g_0} \}_{s \in Y} \) respectively, so

\[
\| \mu_{x, s}^{g_0} \| = \| \mu_{x, s}^{g_0} \|^{-1}
\]

Push forward the limiting measures \( \{ \mu_{x}^{g_0} \}_{s \in Y} \) and \( \{ \mu_{x}^{g_0} \}_{s \in Y} \) to the conformal densities on the quotient \([A_\infty G]\), which we keep by the same notation. The constant \( \lambda \) in Definition 6.1 is universal for any \( \{ \mu_{x, s}^{g_0} \}_{s \in Y} \) with \( g_0 \in G_0 \), as the difference of two horofunctions in the same locus of a Myrberg point is universal by Lemma 5.3. Thus, Lemma 8.4 gives a constant \( M = M(\lambda) \) independent of \( g_0 \) such that for any \( g_0 \in G_0 \):

\[
M^{-1} \mu_{x, s}^{g_0} \leq \| \mu_{x, s}^{g_0} \| \leq M \| \mu_{x, s}^{g_0} \|
\]

so the relation (33) implies \( \| \mu_{x, s}^{g_0} \|^2 \leq M \). It still remains to note that \( \{ \mu_{x, s} \} \) and \( \{ \mu_{x, s} \} \) may be different limit points of \( \{ \mu_{x, s} \}_{s \in Y} \), where \( \{ \mu_{x, s} \} \) is the PS measure fixed at the beginning of the proof. However, applying Lemma 8.4 again gives \( \| \mu_{x, s} \| \leq \mu_{x, s} \) for any \( x \in Y \) and thus for \( x = g_0 \). As desired in (32), we obtain a upper bound on \( \| \mu_{x, s} \| \) depending only on \( \lambda, M \). The proof of the Case 1 is completed.

**Case 2: Normal subgroups of convergent type.** Assume that \( H \sim Y \) is of convergent type. It suffices to prove

\[
\omega(H) > \frac{\omega(G)}{2}.
\]

Indeed, if denote \( A_n = A(o, n, \Delta) \) defined in (19), we have

\[
\mathcal{P}_G(\omega, o, o) \geq \Delta \sum_{n=0}^{\infty} |A_n| e^{-n\omega}.
\]

A simple exercise shows that for any \( \Gamma \subseteq G \), we have

\[
\limsup_{n \to \infty} \frac{\log n}{n} = \omega(\Gamma).
\]

Let \( r, F > 0 \) given by Lemma 2.15 and \( R = |Fo| + 4r + 4\Delta \). Let \( B_n \) be a maximal \( R \)-separated subset of \( A_n \):

1. For any \( v, w \in B_n \) we have \( d(v, w) > R \)
2. For any \( w \in A_n \) there exists \( v \in B_n \) such that \( d(v, w) \leq R \).

The idea of the proof is to produce at least \( |B_n| \) elements in \( H \cap A(o, 2n, R) \):

\[
|B_n| \leq |H \cap A(o, 2n, R)|
\]

Indeed, this follows from [73] Lemma 2.19. As it is short, let us give the proof at the convenience of reader.

Apply Lemma 2.15 to the pair \( (g, g^{-1}) \) for each \( g \in B_n \). There exists \( f \in F \) such that the admissible path labeled by \( gfg^{-1} \) \( r \)-fellow travels a geodesic \( \gamma = [o, gfg^{-1}o] \) so \( d(\gamma, \gamma) \leq r \) and

\[
|d(o, gfg^{-1}o) - 2d(o, go)| \leq |Fo| + 4r.
\]
If \(fg^{-1}o = g'f'g^{-1}o\) for two \(g, g' \in B_n\), then \(d(go, \gamma), d(g'o, \gamma) \leq r\). As \(go, g' \in A_n\) lies in the same annulus of width \(\Delta\), we see that \(d(go, g') \leq 2r + 2\Delta\). This contradicts the item (1). Hence, (35) is proved and by (34),

\[
\omega_H \geq \frac{\omega_G}{2}.
\]

By the item (2), \(|A_n| \leq |B_n|\). If \(\omega_H = \frac{\omega_G}{2}\), then

\[
\mathcal{P}_G(\omega_G, o, o) \times_R \sum_{n=1}^{\infty} |B_n|e^{-\omega_G n} \leq \sum_{n=1}^{\infty} |A(0, 2n, R) \cap H|e^{-\omega_H n} < \infty
\]

where the last inequality follows from the convergence of \(\mathcal{P}_H(\omega_H, o, o)\) at \(\omega_H\). This contradicts the divergent action of \(G \sim Y\). Hence, the proof of the second case is completed.

10. CAT(0) GROUPS WITH RANK-1 ELEMENTS

We present applications to the Myrberg limit set and the contracting boundary for CAT(0) spaces.

10.1. Myrberg limit set of the visual boundary.

**Lemma 10.1.** Let \(Y\) be a proper CAT(0) space. Then the visual boundary \(\partial_{vis} Y\) is a convergence boundary with respect to the trivial partition, so that all boundary points are non-pinched.

**Proof.** It is well-known that the visual boundary is homeomorphic to horofunction boundary \([7]\). By Theorem \([14]\) it suffices to prove that this finite difference partition is trivial: every point is minimal.

Indeed, let \(\xi, \eta \in \partial_{vis} Y\) be represented by two geodesic rays \(\alpha, \beta\) from \(o \in Y\), so that \(|b_\xi - b_\eta| \leq K\) for some \(K > 0\). For any \(x \in \alpha\) we have \(b_\xi(x, n) = d(o, x)\) so \(b_\xi(x) - d(o, x) \leq K\). Taking a sequence \(y_n \in \beta\) so that \(b_\eta(x) = d(x, y_n) - d(o, y_n)\) tends to \(b_\xi(x)\), we have \(d(x, y_n) - d(o, y_n) \leq K\). Considering the Euclidean comparison triangle for triangle \(o, x, y_n\), we see that \(d(x, y_n) \leq K\) for any \(x \in \alpha\). This shows that \(\xi = \eta\).

**□**

**Lemma 10.2.** \([5]\) Let \(\gamma : \mathbb{R} \to Y\) be a geodesic in a proper CAT(0) space \(Y\) without bounding a flat strip of width \(R > 0\). Then there exist neighborhoods \(U_{\gamma^+}\) of \(\gamma^+\) and \(V_{\gamma^-}\) of \(\gamma^-\) in \(Y\) such that any \(\xi \in U_{\gamma^+}\) and any \(\eta \in V_{\gamma^-}\), there exists a geodesic from \(\xi\) to \(\eta\) within \(R\)-distance to \(\gamma(0)\) and any such geodesic does not bound a \((2R)\)-flat strip.

We now give the main result of this subsection, recovering \([17]\) Lemma 7.1] via completely different methods (also compare with \([60]\) Theorem 2]). Such results are crucial in applying Hopf’s argument in proving ergodicity of geodesic flows (see \([17]\) Section 7) for details).

**Lemma 10.3.** Assume that \(G \sim Y\) in \([4]\) contains a rank-1 element with zero width axis. Then

1. any bi-infinite geodesic \(\gamma\) with one endpoint at a Myrberg point has zero width.
2. any two geodesic rays \(\beta, \gamma\) ending at a Myrberg point are asymptotic: there exists \(a \in \mathbb{R}\) such that \(d(\beta(t), \gamma(t+a)) \to 0, \ t \to \infty\).

**Proof.** Let \(h\) be a rank-1 element with a geodesic axis \(\alpha\) with zero width: for any \(R > 0\) there is no flat strip of width \(R\) with one boundary component \(\alpha\). Fix \(o \in \alpha\).

(1). We are going to prove that \(\gamma\) has zero width. Suppose to the contrary that \(\gamma\) bounds an flat strip of width \(R\) for some \(R > 0\). By Corollary \([13]\) \(\gamma\) contains infinitely many \((r, f_n)\)-barriers \(g_n o\), where \(f_n \in (h)\) is any sequence of elements with unbounded length. Thus,

\[
|N_r(g_n o) \cap \gamma| \to \infty
\]

Thus, there exists a segment \(p_n\) of \(g_n o\) such that \(p_n \subseteq N_r(\gamma)\) and \(\ell(p_n) \to +\infty\).

For this \(R/2\), let \(U, V\) be the open neighborhoods of \(\alpha^-, \alpha^+\) satisfying Lemma \([12]\). As \(\ell(p_n) \to +\infty\), the cone topology on the visual boundary of \(Y\) implies that \(g_n^{-1} \gamma^- \in U, g_n^{-1} \gamma^+ \in V\) for \(n \gg 0\). By Lemma \([12]\) the geodesic \(g_n^{-1} \gamma\) does not bound an flat strip of width \(R\). This contradicts the hypothesis, so we proved that \(\gamma\) has zero width.

(2). As the CAT(0) distance has convexity property, it suffices to find two unbounded sequence of points \(x_n \in \beta, y_n \in \gamma\) such that \(d(x_n, y_n) \to 0\) as \(n \to \infty\). By the same argument as above, we have \([36]\) holds for \(\gamma\) and \(\beta\), so for any \(R > 0\), Lemma \([12]\) implies that \(\gamma, \beta\) are \(R/2\)-close to \(g_n o(0)\) for any \(n \gg 0\). Thus we obtain \(x_n \in \beta, y_n \in \gamma\) so that \(d(x_n, y_n) \leq R\). Letting \(R \to 0\) completes the proof. \(\square\)
10.2. **Roller boundary for CAT(0) cube complexes.** Let $Y$ be a proper CAT(0) cube complex, which is a CAT(0) space built out of unit Euclidean $n$-cubes for $n \geq 1$ by side isometric gluing. The collection of hyperplanes or called walls as connected components of mid-cubes endows $Y$ with a wall structure: every wall separates the space into two connected components called half-spaces. An orientation of walls picks up exactly one half-space for each wall, and is called consistent if any two such chosen half-spaces intersect non-trivially. For every vertex $x$ of $Y^0$, the set $U_x$ of half-spaces containing $x$ forms a consistent orientation. The Roller compactification $\overline{Y}$ of $Y$ is thus the closure of the vertex set $Y^0$ in all the consistent orientations of walls, and Roller boundary is $\partial_R Y := \overline{Y} \setminus Y^0$.

Assume that $Y$ is irreducible: it cannot be written as a nontrivial product. If $G$ acts essentially on $Y$ without fixed point in the visual boundary, this is equivalent to the existence of strongly separated of two half-spaces: no wall transverses both of them \cite{11}. Following \cite{21}, a boundary point is regular if the orientation of walls contains an infinite sequence of a descending chain of pairwise strongly separated half-spaces. We can equip $\partial_R Y$ with finite symmetric difference partition $[\cdot]$; two orientations are equivalent if their symmetric difference is finite. It is proved in \cite{22} Lemma 5.2 that the regular points are $[\cdot]$-minimal: the $[\cdot]$-class is a singleton.

Rather than CAT(0) metric, we are interested in the combinatorial metric $d(x, y) := \frac{1}{2} |U_x \Delta U_y|$ on $Y^0$, where $\Delta$ denotes the symmetric difference of sets. A unpublished result of Bader-Guralnik says that the Roller boundary is homeomorphic to the horofunction boundary of $(Y^0, d)$ \cite{22 Prop. 6.20}).

**Lemma 10.4.** The Roller boundary $\partial_R Y$ is a convergence boundary, with finite symmetric difference partition $[\cdot]$ so that $[\cdot]$ coincides with the partition of finite difference of horofunctions.

Moreover, assume that a non-elementary group $G$ acts essentially on $Y$. Then the following hold.

1. Every contracting element satisfies the North-South dynamics with minimal fixed points.
2. $G$ admits a unique invariant closed subset, where the fixed point pairs of contracting elements are dense in the distinct pairs.
3. The Myrberg points of $G$ are regular, and then minimal.

**Proof.** By \cite{29 Prop A.2}, every point in $\partial_R Y$ is represented by a geodesic ray, i.e.: via the identification with the half-spaces into which $\partial_R Y$ eventually enters. We first prove that if two horofunctions $b_\xi, b_\eta \in \partial_R Y$ have finite difference, then $\xi$ and $\eta$ are in the same symmetric difference $[\cdot]$-class.

Let $\alpha, \beta$ be two geodesic rays ending at $\xi, \eta$ as in the proof of Lemma 10.1. We obtain as there $2(o, y_n, x) = d(o, x) + d(o, y_n) - d(x, y_n) \leq K$ for $x \in o, y_n \in \beta$. As half-spaces are combinatorially convex, $\alpha$ and $\beta$ with same initial point share the same set of half-spaces containing them. Thus, it suffices to consider the difference of the half-spaces whose bounding walls are dull to edges of $\alpha$ or $\beta$. Recall that the median $m$ of $o, x, y_n$ is the unique common point on all three sides of a geodesic triangle with these vertices. We thus have $(o, y_n, x) = d(x, m) \leq K/2$. Upon replacing the initial subpaths, we can assume that $m \in \alpha$ and $m \in \beta$, so the walls intersecting $[o, x]_\alpha$ but not $[o, m]_\beta$ is at most $K/2$ half-spaces, that are due to the geodesic $[x, m]$. As $x \rightarrow \xi$, a limiting argument shows that the symmetric difference of $\xi$ and $\eta$ is at most $K$ half-spaces.

Conversely, if $\xi, \eta$ has finite symmetric difference on $K$ walls, the similar reasoning as above produces two unbounded sequence of points $x_n \in \alpha, m_n \in \beta$ with $d(x_n, m_n) \leq K/2$. Thus, $\|b_\xi - b_\eta\|_\infty \leq K/2$.

To complete the proof, it remains to show that a Myrberg point is regular. Then the statements (1) (2) are consequences of Lemmas 3.19, 3.22.

Let $\gamma$ be a geodesic ray ending at a Myrberg point $\xi$. By \cite{29 Theorem 3.9}, for any contracting isometry $h$, the combinatorial geodesic axis contains two strongly separated half-spaces with arbitrarily large distance. As each wall separates $Y$ into two convex components, for any two pairs $(x, y)$ and $(z, w)$ with $d(x, z), d(y, w) \leq r$, the walls due to $[x, y]$ differ by at most $2r$ from the ones to $[z, w]$. By Corollary 4.13 for any large $L > 0$, $\gamma$ contains a sequence of disjoint sub-segments of length $L$ within the $r$-neighborhood of a translated contracting axis of $h$. Hence, $\gamma$ has to be crossed by infinitely many pairs of strongly separated walls. By strong separateness, the half-spaces bounding by those walls and into which $\gamma$ eventually enters form a descending chain, so $\xi$ is a regular point by definition. Thus the lemma is proved. \(\square\)
10.3. Conformal measure of contracting boundary. We shall prove a general result for any convergence boundary $\partial Y$. Recall that a point $\xi$ in $\partial Y$ is called visual if there exists a geodesic ray $\gamma$ from any point $\gamma \in Y$ ending at $[\xi]$. If in addition, $\gamma$ is contracting, then $\xi \in \partial Y$ is called a contracting point.

Theorem 10.5. Suppose that $G \asym Y$ as in \cite{7} is a co-compact action. Let $\{\mu_x\}_{x \in Y}$ be a $\omega_G$-dimensional quasi-conformal density on a convergence boundary $\partial Y$. Then the set of contracting points in $\partial Y$ is $\mu_1$-null if and only if $G$ is not a hyperbolic group.

Proof. Fix a basepoint $o \in Y$. As the action $G \asym Y$ is co-compact, let $M > 0$ such that $N_M(Go) = Y$. In the proof, we need the following result which is of independent interest.

Let $\Lambda \subseteq \partial Y$ be a set of contracting points. Denote by $\text{Hull}_R(\Lambda)$ the set of orbital points $v \in Go$ such that $v$ is $R$-closed to a geodesic from $o$ to $\xi \in \Lambda$ for $R \geq 0$.

Lemma 10.6. Suppose $\Lambda \subseteq \partial Y$ has positive $\mu_1$-measure. Then for any $R \gg M$, there exists $c = c(\Lambda, R) > 0$ such that

$$|\text{Hull}_R(\Lambda) \cap N(o, n)| \geq c \cdot e^{\omega_G n},$$

for any $n > 0$.\hfill $\square$

Proof. Denote $A_n = \text{Hull}_R(\Lambda) \cap A(o, n, \Delta)$. Set $R > \max\{r_0, M\}$, where $r_0 > 0$ is given by Lemma 6.3. By definition, we have

$$\Lambda \subseteq \bigcup_{v \in A_n} [\Pi^o_v(R)],$$

which yields

$$\mu_1(\Lambda) \leq \sum_{h \in A_n} \mu_1([\Pi_o(v, R)]).$$

A co-compact action must be divergent type, so by Theorem 1.10, $\mu_1$ is supported on Myrberg limit set. By Lemma 6.3 any two horofunctions in the locus of a Myrberg point have uniform bounded difference. The proof for the upper bound in Lemma 10.6 gives

$$\mu_1([\Pi_o(v, R)]) < R e^{\omega_G d(o, v)}$$

Hence, there exists a constant $c > 0$ such that

$$|A_n| \geq c \cdot e^{\omega_G n}$$

completing the proof. We are now ready to give the proof.

Proof of Theorem 10.4. If $G$ is a hyperbolic group, then the co-compact action implies that $Y$ is hyperbolic and the action $G \asym Y$ is of divergent type. By Lemma 4.4, any $(r, F)$-conical point is visual and thus contracting, because every geodesic ray in a hyperbolic space is contracting. By Theorem 1.10 the set of contracting points is $\mu$-full, hence the "$\Rightarrow$" direction follows.

Let us prove the other direction by assuming that $G$ is not hyperbolic. Let $\Lambda$ denote the set of contracting points, and assume $\mu(\Lambda) > 0$ for a contradiction. Let $\Lambda_C(x)$ be the set of $\xi \in \Lambda$ so that there exists a $C$-contracting geodesic ray at $\xi$ starting at $x \in Y$. Then $\Lambda$ admits the following countable union:

$$\Lambda = \bigcup_{c \in \mathbb{N}} \left( \bigcup_{x \in Y} \Lambda_C(x) \right)$$

where $\Lambda(gx) = g\Lambda(x)$ for any $g \in G, x \in Y$. Since the conformal measures are almost $G$-equivariant and the $M$-neighborhood of a $G$-orbit $Go$ covers $Y$, there exists $C > 0$ such that $\mu_1(\Lambda_C(o)) > 0$ for some (or any) $o \in Y$. Let $U = \text{Hull}_R(\Lambda_C(o))$ be the geodesic hull of $\Lambda_C(o)$ for some $R \gg 0$. Then by Lemma 10.6 $U \subseteq Go$ has the growth rate $\omega_G$.

On the other hand, recall that by \cite{72} Prop 2.2, there exists a constant $D = D(C, R)$ such that any geodesic segment in a $R$-neighborhood of a $C$-contracting geodesic is $D$-contracting. It is well-known that if every geodesic segment in $Y$ is uniformly contracting then $Y$ is Gromov-hyperbolic. Thus there exists a segment $\alpha$, so that $\alpha$ is not $D$-contracting. As $G$ acts co-compactly on $Y$, we can assume without generality that $\alpha = [o, ho]$ for a basepoint $o \in Y$ and $h \in G$. 
Any segment with two points in $U$ is $C$-contracting, so cannot contain $\alpha$ in its neighborhood. That is to say, any $[a, b]$ for $v \in U$ does not contain any $(\epsilon, h)$-barrier. By [24, Theorem C], the set $U$ is growth tight: $\omega_U < \omega_G$. This is a contradiction. Hence, it is proved that $\mu_1(\Lambda) = 0$. $\square$

11. Conformal dynamics at infinity

11.1. Convergence of Thurston boundary. Let $\mathcal{S}$ be the set of (isotopy classes of) essential simple closed curves on $\Sigma_g$. Given $x \in \mathcal{T}$, each element $c \in \mathcal{S}$ can be assigned either the length $\ell_x(c)$ of a closed geodesic representative in hyperbolic geometry or the extremal length $\text{Ext}_x(c)$ in complex structure.

Denote by $\mathbb{R}_{\geq 0}$ the set of positive functions on $\Sigma$. We then obtain the embedding of $Y$ into $\mathbb{R}_{\geq 0}$ in two ways

$$x \in \mathcal{T} \mapsto \left(\ell_x(c)\right)_{c \in \mathcal{S}}, \quad x \in \mathcal{T} \mapsto \left(\sqrt{\text{Ext}_x(c)}\right)_{c \in \mathcal{S}}$$

whose topological closures in the projective space $\mathbb{P}\mathbb{R}_{\geq 0}$ give the corresponding Thurston and Gardiner-Masur compactifications of $\mathcal{T}$ ([20, 24]). We first describe the Thurston boundary $\partial_{\text{Th}} \mathcal{T}$ in more details.

**Thurston boundary.** Let $\mathcal{MF}$ be the set of measured foliations on $\Sigma_g$ with a natural topology, so that $\mathcal{MF}$ admits a topological embedding into $\mathbb{R}_{\geq 0}$, by taking the intersection number $I(\xi, c)$ of a measured foliation $\xi$ with a curve $c$ in $\mathcal{S}$. As the weighted multi-curves form a dense subset in $\mathcal{MF}$, their geometric intersection number extends continuously to a bi-linear intersection form

$$I(\cdot, \cdot) : \mathcal{MF} \times \mathcal{MF} \to \mathbb{R}_{\geq 0}.$$ 

This allows to understand an essential simple closed curves $c \in \mathcal{S}$ as a measured foliation, where the transversal measure is given by the geometric intersection number.

Let $\mathcal{MF} \to \mathcal{PMF}$ be the projection to the set $\mathcal{PMF}$ of projective measured foliations by positive reals scaling. Thurston proved that $\mathcal{PMF}$ is homeomorphic to the sphere of dimension $6g - 7$, and compactifies $\mathcal{T}$ in the projective space $\mathbb{P}\mathbb{R}_{\geq 0}$. Then $\partial_{\text{Th}} \mathcal{T} := \mathcal{PMF}$ is the so-called Thurston boundary.

By abuse of language, recalling $I(\cdot, \cdot)$ is bi-linear, we shall write $I(\xi, \eta) = 0$ for two projective measured foliations $\xi, \eta \in \mathcal{PMF}$ to mean $I(\xi, \eta) = 0$ for any of their lifts $\xi, \eta \in \mathcal{MF}$.

**Gardiner-Masur boundary.** In [24], Gardiner-Masur proved that $\partial_{\text{GM}} \mathcal{T}$ contains $\mathcal{PMF}$ as a proper subset. In [57, Corollary 1], Miyachi proved that the identification between the Teichmüller space $\mathcal{T}$ extends continuously

$$(37) \quad \mathcal{T} \cup \mathcal{U} \subseteq \mathcal{T} \cup \partial_{\text{Th}} \mathcal{T} \quad \quad \mathcal{T} \cup \mathcal{V} \subseteq \mathcal{T} \cup \partial_{\text{GM}} \mathcal{T}$$

to a homeomorphism between the subsets $\mathcal{U} \subseteq \mathcal{V}$ in both boundaries. Furthermore, the following result will be crucially used later on.

**Lemma 11.1.** [59, Corollary 1] For any $o \in \mathcal{T}$, there exists a unique continuous extension of the Gromov product

$$(x, y) \in \mathcal{T} \times \mathcal{T} \quad \quad \rightarrow \quad \quad \langle x, y \rangle_o = \frac{d(x, o) + d(y, o) - d(x, y)}{2} \in \mathbb{R}_{\geq 0}$$

to the Gardiner-Masur boundary $\mathcal{PMF} \subseteq \partial_{\text{GM}} \mathcal{T}$ so that for any $\xi, \eta \in \mathcal{PMF}$

$$(38) \quad e^{-2\langle \xi, \eta \rangle_o} = \frac{I(\xi, \eta)}{\text{Ext}_o(\xi)^{1/2}\text{Ext}_o(\eta)^{1/2}}$$

where $\xi, \eta$ are any lifts of $\xi, \eta$ in $\mathcal{MF}$. 

Remark. Thanks to (37), the equality (38) holds for uniquely ergodic points $\xi, \eta \in \partial_T \mathcal{T}$. We now explain it also holds for $\xi, \eta \in \mathcal{S} \subseteq \mathcal{P \mathcal{M} \mathcal{F}} = \partial_T \mathcal{T}$. By Hubbard-Masur theorem, for any basepoint $o \in \mathcal{T}$ and for any $c \in \mathcal{S} \subseteq \mathcal{P \mathcal{M} \mathcal{F}}$, there exists a quadratic differential $q$ on the Riemann surface $o$ whose vertical foliation realizes $c$. Recall that the Teichmüller geodesic ray issuing from $o$ has the limit point $c$ in either Thurston or Gardiner-Masur boundary (cf. [52 Theorem], [57 Theorem 2]). Let $x_n, y_n$ be on the geodesic rays ending at $\xi \in \partial_T \mathcal{T}$ and $\eta \in \partial_T \mathcal{T}$ so that $x_n \to \xi$ and $y_n \to \eta$. Thus, (38) holds for $\xi, \eta \in \mathcal{S} \subseteq \partial_T \mathcal{T}$.

**Kaimanovich-Masur partition.** Following [40], we define a partition on $\mathcal{MIN}$ as follows. Let $\mathcal{MIN}$ be the subset of projective minimal foliations $\xi \in \mathcal{P \mathcal{M} \mathcal{F}}$ so that $I(\xi,c) > 0$ for any $c \in \mathcal{S}$. The $[\cdot]$-class of $\xi \in \mathcal{MIN}$ defined as

$$[\xi] := \{\eta \in \mathcal{MIN} : I(\xi,\eta) = 0\}$$

forms a partition of $\mathcal{MIN}$, as $I(\xi,\eta) = 0$ for $\xi \in \mathcal{MIN}$ implies $\eta \in \mathcal{MIN}$ (65). The remaining non-minimal foliations $\mathcal{F} \setminus \mathcal{MIN}$ are partitioned into countably many classes

$$[\xi] := \{\eta : \forall c \in \mathcal{S}, I(\eta,c) = 0 \iff I(\xi,c) = 0\}$$

according to whether they share the same disjoint set of curves in $\mathcal{S}$. It is proved in [40] Lemma 1.1.2] that the partition $[\cdot]$, generated by a countable partition, is measurable in the sense of Rokhlin.

The following useful fact is proved in [40] Lemma 1.4.2] where $\xi$ lies in $\mathcal{UE}$ and the conclusion follows as $y_n \to \xi$. We provide some explanation on the changes.

**Lemma 11.2.** Assume that $x_n \in \mathcal{T}$ tends to $\xi \in \mathcal{MIN}$. If a sequence $y_n \in \mathcal{T}$ satisfies

$$d(o,y_n) - d(x_n,y_n) \to +\infty$$

then $y_n \to [\xi]$.

**Proof.** In the proof [40] Lemma 1.4.1], a sequence of $\beta_n \in \mathcal{S}$ is produced such that $\text{Ext}_{x_n}(\beta_n)$ is uniformly bounded and

$$I(\beta_n,F_n) \to 0$$

where $F_n$ is the vertical measured foliation of the terminal quadratic differential of the Teichmüller map from $x_0$ to $x_n$. Taking convergence in $\mathcal{P \mathcal{M} \mathcal{F}}$ and passing to a subsequence, assume that

$$\beta_n \mathcal{P \mathcal{M} \mathcal{F}} \Rightarrow F, \quad [F_n] \mathcal{P \mathcal{M} \mathcal{F}} \Rightarrow F_+$$

for some $F_\mathcal{F}, F_\mathcal{G} \in \mathcal{MIN}$. Then $I(F,F_\mathcal{G}) = 0 = I(F_\mathcal{G},\xi)$ by [41] Lemma 43]. As all the foliations in $[\xi]$ for $\xi \in \mathcal{MIN}$ is minimal (65), we obtain $F \in [\xi]$.

The remaining proof from page 245 of [40] Lemma 1.4.1] shows that $y_n$ tends to $F \in \mathcal{P \mathcal{M} \mathcal{F}}$. This is what we wanted. \hfill \Box

We are now ready to prove the main result, Theorem 1.2 of this subsection.

**Proposition 11.3.** The Thurston boundary with the above partition $[\cdot]$ is a convergence boundary of Teichmüller space, so that $\mathcal{UE}$ consists of non-pinched points.

**Proof.** It is well-known that a contracting geodesic ray $X$ tends to a unique ergodic point (see [11]). Any sequence $y_n$ with an exiting $x_n \in \pi_X(y_n)$ satisfies the condition of Lemma 11.2 so Assump A follows.

The non-pinched points in Assump C contain $\mathcal{UE}$, which follows by Lemma 11.1 and (37).

Let us prove now Assump B. Indeed, after passage to subsequence, let $X_n$ be an exiting sequence of $C$-contracting subsets so that $x_n \in \pi_X(o)$ tends to $\xi \in \mathcal{P \mathcal{M} \mathcal{F}}$. Take any convergent sequence of points $y_n \in \Pi_\alpha(X_n)$ to some $\eta \in \mathcal{P \mathcal{M} \mathcal{F}}$ by compactness. Recall that $\Pi_\alpha(X_n)$ is the set of points $y \in \mathcal{T}$ with $[o,y] \cap X_n \neq \emptyset$. The contracting property in Lemma 2.2 implies $d(\pi_X(o),[o,y_n]) \leq C$, so we have

$$d(o,y_n) - d(o,x_n) - d(x_n,y_n) \leq 2C$$

(39)

If $\xi \in \mathcal{MIN}$, then $\eta \in [\xi]$ follows by Lemma 11.2 as $d(o,x_n) \to \infty$.

To finish the proof of Assump B it thus remains to consider the case $\xi \in \mathcal{P \mathcal{M} \mathcal{F}} \setminus \mathcal{MIN}$, and prove $[\xi] = [\eta]$; they has the same set of disjoint curves in $\mathcal{S}$. That is to say, $I(\xi,c) = 0$ for $c \in \mathcal{S}$ is equivalent to $I(\eta,c) = 0$. The two directions are symmetric. We only prove that if $I(\xi,c) = 0$ then $I(\eta,c) = 0$. 


Let $z_n$ be on a geodesic ray ending at $c \in \mathcal{P}_\infty \mathcal{F} = \partial_{th} \mathcal{T}$, so $z_n \to c$ also takes place in $\partial_{GM} \mathcal{T}$. By Lemma 11.1 as $I(\xi, c) = 0$, we have $(x_n, z_n)_o \to \infty$. By definition of Gromov product and (39),

$$
2(y_n, z_n)_o = d(y_n, o) + d(z_n, o) - d(y_n, z_n)
\geq d(y_n, x_n) + d(o, x_n) + d(z_n, o) - d(y_n, z_n) - 2C
\geq d(o, x_n) + d(z_n, o) - d(x_n, z_n) - 2C = 2(x_n, z_n)_o - 2C
$$

so by Lemma 11.1 with the Remark after it, we have $I(\eta, c) = 0$.

In [54], McCarthy-Papadopoulos defined the limit set $\Lambda$ in Thurston boundary for non-elementary subgroups of mapping class groups, as the closure of fixed points of pseudo-Anosov elements. They further showed that $\Lambda Go$ is contained in the intersection locus of $\Lambda$ (those measured foliations with zero intersection with the ones in $\Lambda$). We showed $[\Lambda] = [\Lambda Go]$ in Lemma 3.22. It would be interesting to examine the exact relation between the Kaimanovich-Masur partition [$\cdot$] and the intersection locus.

In [48, Liu-Su proved that Gardiner-Masur boundary is the horofunction boundary of Teichmüller space (with Teichmüller metric). We list a few consequence of the general theory, in view of the corresponding results on Thurston boundary [54].

**Lemma 11.4.** $\partial_{GM} \mathcal{T}$ is a convergence boundary with respect to finite difference relation $[\cdot]$. Moreover,

1. The finite difference relation $[\cdot]$ restricts on $\mathcal{U} \mathcal{D} \subseteq \partial_{GM} \mathcal{T}$ as a trivial partition.
2. All conical points are uniquely ergodic points.
3. Every pseudo-Anosov elements are non-pinched contracting elements with minimal fixed points and have the North-South dynamics on $\partial_{GM} \mathcal{T}$.
4. Every non-elementary subgroup $G < \text{Mod}(\Sigma_g)$ has a unique minimal $G$-invariant closed subset $\Lambda$, so that $[\Lambda Go] = [\Lambda]$ for any $o \in \mathcal{T}$.

**Proof.** The assertions (3) and (4) are corollaries of Lemmas 3.19 and 3.22 provided that (1) and (2) are proved.

1. Let $[\xi] = [\eta]$ with $\xi \in \mathcal{U} \mathcal{D}$, so $|b_\xi - b_\eta| \leq K < \infty$. Take a geodesic ray $\gamma = [o, \xi]$ and $x_n \in \gamma$ tends to $\xi$. Let $y_n \to \eta$. As $|b_\xi(x_n) - b_\eta(x_n)| = |d(o, x_n) + d(o, y_n) - d(x_n, y_n)| \leq K$, then $y_n \to \xi$ by Lemma 11.2 so $\xi = \eta$ is proved.

2. Projecting to the quotient $\mathcal{T}/\text{Mod}(\Sigma_g)$, any geodesic ray ending at a $(r, F)$-conical point $\xi \in \partial_{GM} \mathcal{T}$ is recurrent into the $r$-neighborhood of closed geodesics corresponding to pseudo-Anosov elements in $F$. Thus, through [37], $\xi$ is uniquely ergodic by a criterion of Masur [30].

### 11.2. Conformal density on Thurston and Gardiner-Masur boundary.

We first clarify the relation of (formal) conformal density defined in [11] on $\partial_{th} \mathcal{T}$ with the ones obtained from the Patterson’s construction.

Let $\mu_{th}$ be the Thurston measure on $\mathcal{MF}$. According to [4], a family of conformal measures are constructed via $\mu_{th}$ as follows. Let $U$ be a Borel subset in $\mathcal{P}_\infty \mathcal{F}$. Define for any $x \in \mathcal{T}$,

$$
\lambda_x(U) := \mu_{th}(\{\xi \in \mathcal{MF} : \hat{\xi} \in U, \text{Ext}_x(\xi) \leq 1\}).
$$

where $\hat{\xi}$ denotes the image of $\xi$ in $\mathcal{MF}$.

By definition, the family of $\text{Mod}(\Sigma_g)$-equivariant measures $\{\lambda_x\}_{x \in \mathcal{T}}$ are mutually absolutely continuous, and are related as follows

$$
\forall \xi \in \mathcal{P}_\infty \mathcal{F} : \frac{d\lambda_x(\xi)}{d\lambda_y} = e^{-\omega_G B_\xi(x, y)}
$$

where $\omega_G = 6g - 6$ and the cocycle $B_\xi : \mathcal{T} \times \mathcal{T} \to \mathbb{R}$ is given by

$$
\forall x, y \in \mathcal{T}, \quad B_\xi(x, y) := \frac{1}{2} \cdot \log \frac{\text{Ext}_x(\hat{\xi})}{\text{Ext}_y(\hat{\xi})}.
$$

where $\hat{\xi} \in \mathcal{MF}$ is any lift of $\xi$. By [72, Theorem 1], one can verify that for $\xi \in \mathcal{U} \mathcal{D}$, $B_\xi(\cdot, \cdot)$ is indeed the Buseman cocycle at $\xi$ in the horofunction boundary $\partial_{GM} \mathcal{T}$ (see [70] for details). By [37], the Buseman cocycles extend continuously to $\mathcal{U} \mathcal{D} \subseteq \partial_{th} \mathcal{T}$, so Assump D is satisfied with $\epsilon = 0$. 

It is well-known that \( \lambda_x \) is supported on \( \mathcal{U} \) (\([51],[72]\)). In terms of Definition 6.1, \( \{\lambda_x\}_{x \in T} \) is a \( (6g-6) \)-dimensional \( \text{Mod}(\Sigma_g) \)-equivariant conformal density on \( \partial_{\text{Th}} T \). The shadow lemma 6.3 and principle 6.9 thus hold in this context, where the former has been already obtained in \([70]\) via a different method.

We emphasize the above construction is not obtained via the Patterson’s construction. However, the action \( \text{Mod}(\Sigma_g) \sim T \) has purely exponential growth (\([41]\) Thm 1.2)), hence it is of divergent type. By Theorem 1.11 there exists a unique \( (6g-6) \)-dimensional conformal density on \( \partial_{\text{Th}} T \), which was first proved in \([33],[40]\). Thus, the conformal density obtained by Patterson’s construction coincides with \( \lambda_x \).

We record the following result for further counting applications.

**Lemma 11.5.** Let \( G < \text{Mod}(\Sigma_g) \) be a non-elementary subgroup. Assume that the limit set \( \Lambda G \) of \( G \) is a proper subset of \( \mathcal{P} \mathcal{M} \mathcal{F} \). Then its Poincaré series \( P_G(s,o,o) \) is convergent at \( 6g-6 \).

In particular, the Thurston measure of the conical limit set \( \Lambda_c(G) \) defined in \([5]\) is zero.

**Remark.** Analogous to Ahlfors area theorem in Kleinian groups, it would be interesting to study which subgroups of \( \text{Mod}(\Sigma_g) \) have limit set of zero Lebesgue measure. If \( G \) is a convex-cocompact subgroup, then the conical limit set \( \Lambda_c(G) \) is the whole limit set, so we recover \([59]\) Corollary 4.8. The limit set of geometrically finite Kleinian groups consists of conical points and countable many parabolic points. If Lebesgue measure has no charge on conical limit points, then the whole limit set is null. We expect the “in particular” statement helpful to solve the zero measure problem for certain limit sets.

**Proof.** A non-elementary subgroup contains at least two independent pseudo-Anosov elements, so is sufficiently large in terms of \([54]\). With respect to Kaimanovich-Masur \([\cdot]\)-partition, any pseudo-Anosov element has minimal fixed points \( \{h^\pm\} = \{\xi\} \) for some \( \xi \in \mathcal{U} \). By \([54]\), \( G \) possesses a unique \( G \)-invariant minimal closed subset \( \Lambda \subseteq \mathcal{P} \mathcal{M} \mathcal{F} \) defined as the closure of all fixed points of pseudo-Anosov elements (see also Lemma 3.22). Let \( Z(\Lambda) := \{\eta \in \mathcal{P} \mathcal{M} \mathcal{F} : I(\xi,\eta) = 0 \text{ for some } \xi \in \Lambda\} \). Then \( \Lambda G_0 \subseteq Z(\Lambda) \) by \([51]\) Prop 8.1.

By \([54]\) Thm 6.16, if \( \Lambda \neq \mathcal{P} \mathcal{M} \mathcal{F} \), then \( G \) acts properly discontinuously on the complement \( \Omega := \mathcal{P} \mathcal{M} \mathcal{F} \setminus Z(\Lambda) \). By Lemma 3.11 the fixed point pairs of all pseudo-Anosov elements are dense in distinct pairs of \( \mathcal{P} \mathcal{M} \mathcal{F} \). Choose a pseudo-Anosov element \( h \in \text{Mod}(\Sigma_g) \) and a closed neighborhood \( U \) of \( h^+ \) such that \( U \) is contained in \( \Omega \). By the proper action of \( G \sim \Omega \), the set \( A := \{1 \neq g \in G : U \cap gU \neq \emptyset\} \) is finite. As \( G < \text{Mod}(\Sigma_g) \) is residually finite, let us choose a finite index subgroup \( \hat{G} < G \) such that \( \hat{G} \cap A = \emptyset \). Thus, any nontrivial element \( g \in \hat{G} \) sends \( U \) into \( V := \mathcal{P} \mathcal{M} \mathcal{F} \setminus U \).

By the North-South dynamics, there exists a high power of \( h \) (with the same notation) such that \( h^n(U) \subseteq V \) for any \( n \neq 0 \). Thus, \( \langle h \rangle \) and \( \hat{G} \) are ping-pong players on \( (U,V) \), so generate a free product \( \langle h \rangle * \hat{G} \). By \([75]\) Lemma 2.23, the Poincaré series of \( G \) converges at \( \omega_{\text{Mod}(\Sigma_g)} = 6g-6 \). If \( G \sim T \) is of divergent type, we have \( \omega_G = \omega_{\hat{G}} < 6g-6 \).

By Theorem 1.15 we have the Thurston measure of the conical limit set of \( G \) is null.

For further reference, we record the following corollary of Theorem 1.11 for Gardiner-Masur boundary.

**Lemma 11.6.** \( \partial_{\text{GM}} T \) admits a unique \( \text{Mod}(\Sigma_g) \)-equivariant conformal density of dimension \( 6g-6 \), supported on the Myrberg set consisting of uniquely ergodic points.

11.3. **Teichmüller geodesic flow on covers of moduli space.** Let \( G \) be a non-elementary subgroup of \( \text{Mod}(\Sigma_g) \), so that \( \mathcal{M} := T/G \) is a (branched) cover of the moduli space of \( \Sigma_g \). Let \( \{\mu_x\}_{x \in T} \) be a \( \omega_G \)-dimensional conformal density on \( \partial_{\text{Th}} T = \mathcal{P} \mathcal{M} \mathcal{F} \) in Theorem 1.15.

By Theorem 1.10 it suffices to prove the direction “(3) \iff (4)” of Theorem 1.15. This requires to run the well-known Hopf’s argument which almost follows verbatim the case for \( \text{CAT}(1) \) spaces \([37],[62],[10]\). What follows only sets up the background and points out the necessary ingredients.

Let \( \pi : \mathcal{Q}T \to T \) be the vector bundle of quadratic differentials over Teichmüller space. We first explain a \( G \)-invariant geodesic flow on the sub-bundle \( \mathcal{Q}^1 T \) of unit area quadratic differentials, and then construct a flow invariant measure called Bowen-Margulis-Sullivan measure. This construction is due to Sullivan in hyperbolic spaces.

Teichmüller geodesics \( \gamma \) are uniquely determined by a pair of projective classes \( (\xi,\eta) \) of transversal measured foliations \( (\xi,\eta) \), and vice versa. Such pairs \( (\xi,\eta) \) form exactly the complement of the following
big diagonal

\[ \Delta := \{ (\xi, \eta) \in \mathcal{M} \times \mathcal{M} : \exists c \in \mathcal{P}, I(c, \xi) + I(c, \eta) = 0 \} \]

According to Hubbard-Masur theorem \[^{38}\], the transversal pair \((\xi, \eta)\) is realized as the corresponding vertical and horizontal foliations \((q^+, q^-)\) of a unique quadratic differential \(q \in \mathcal{Q}T\). According to \[^{40}\], we consider the following \(G\)-equivariant homeomorphism

\[
\mathcal{Q}T \rightarrow \mathcal{M} \times \mathcal{M} \setminus \Delta
\]

\[
q \mapsto (q^+, q^-).
\]

**Remark.** By abuse of language, we write \(\gamma = (\hat{\xi}, \hat{\eta}) = (q^+, q^-)\) and \(\gamma^+ = \hat{\xi}, \gamma^- = \hat{\eta}\). Indeed, the two half rays of \(\gamma\) do not necessarily converge to \(\hat{\xi}, \hat{\eta}\) in \(\mathcal{T} \cup \partial_{\text{Th}} \mathcal{T}\), which however do converge for \(\xi, \eta \in \mathcal{U} \mathcal{E}\).

The geodesic flow \((\mathcal{G}^t)_{t \in \mathbb{R}}\) on \(\mathcal{Q}^1T\) is defined as \(\mathcal{G}^t(q) := (e^{t}q^+, e^{-t}q^-)\) for any \(q \in \mathcal{Q}^1T\) and \(t \in \mathbb{R}\). The flow line \(t \mapsto q_t := \mathcal{G}^t(q)\) gives the lift of the Teichmüller geodesic \(\gamma = (q_0^+, q_0^-)\). Endow \(\mathcal{Q}^1T\) with a \(\text{Mod}(\Sigma_g)\)-invariant distance \(d_Q\) as follows:

\[
d_Q(q, q') := \int_{-\infty}^{\infty} \frac{d\tau(\pi(q_t), \pi(q'_t))}{2e^{\tau}} dt
\]

which descends to the quotient metric still denoted by \(d_Q\) on \(\mathcal{Q}^1\mathcal{M} := \mathcal{Q}^1T/\mathcal{G}\).

**Hopf parameterization.** Set \(\partial^2 \mathcal{T} := \left( (\mathcal{P} \mathcal{M} \times \mathcal{P} \mathcal{M}) \times \Delta \right)\). Fix a basepoint \(o \in \mathcal{T}\), the Hopf parameterization of \(\mathcal{Q}^1T\) is given by the following \(G\)-equivariant homeomorphism

\[
\mathcal{Q}^1T \rightarrow \partial^2 \mathcal{T} \times \mathbb{R}
\]

\[
q \mapsto (\xi, \eta, B_\xi(o, \pi(q)))
\]

where \(B_\xi(\cdot, \cdot)\) is the cocycle in \[^{40}\]. The \(G\)-action on the target is given by

\[
g(q^+, q^-, s) = (gq^+, gq^-, s + B_\xi(g^{-1}o, o)).
\]

As a consequence, the geodesic flow \((\mathcal{G}^t)_{t \in \mathbb{R}}\) on \(\mathcal{Q}^1T\) is conjugate to the additive action on the time factor of \(\partial^2 \mathcal{T} \times \mathbb{R}\) as follows:

\[
(\xi, \eta, s) \mapsto (\xi, \eta, t + s).
\]

**BMS measures.** Following \[^{4}\] Section 2.3.1, define

\[
\beta(x, \xi, \eta) := [e^{B_\xi(x, p) + B_\xi(x, p)}]^{6g-6} = \left[ \frac{\sqrt{\text{Ext}_x(\hat{\xi}) \text{Ext}_x(\hat{\eta})}}{I(\xi, \eta)} \right]^{6g-6}
\]

for any \(p \in [\xi, \eta]\), where \(\hat{\xi}, \hat{\eta} \in \mathcal{M}\) are lifts of \(\xi, \eta\), and \(\text{Ext}_p(\hat{\xi}) \text{Ext}_p(\eta) = I(\hat{\xi}, \hat{\eta})^2\) by \[^{24}\] Thm 5.1]. Define the Bowen-Margulis-Sullivan measure \(m\) on \(\partial^2 \mathcal{T} \times \mathbb{R}\) for any \(x \in \mathcal{T}\),

\[
m := \beta(x, \xi, \eta)^{-1} \cdot \mu_x \times \mu_x \times \text{Leb}
\]

which is independent of \(x\). As \(m\) is \(G\)-invariant, it descends to the flow invariant measure \(m\) on \(\mathcal{Q}^1\mathcal{M}\).

**Complete conservativity \(\iff\) Full measure of conical points.** The geodesic flow \((\mathcal{Q}^1\mathcal{M}, \mathcal{G}^t, m)\) is a measure preserving dynamical system. It is called *completely conservative* if there is no \(m\)-positive wandering set \(\Omega\): \(\{G^n\Omega : n \in \mathbb{Z}\}\) is pairwise disjoint. By Poincaré recurrence theorem, almost every point \(q \in \mathcal{Q}^1\mathcal{M}\) is recurrent: \(G^nq \rightarrow q\) for an unbounded sequence \(n > 0\). Let \(\gamma\) be a Teichmüller geodesic ray ending at (the projective class of) \(q^+\). As \(\mathcal{Q}^1\mathcal{M}\) is a sphere bundle over \(\mathcal{T}/\mathcal{G}\), we see that \(\gamma\) returns infinitely often to a compact subset of \(\mathcal{T}/\mathcal{G}\). In other words, \(q^+ \in \Lambda_\circ(G)\) is a conical point as defined in \[^{3}\]. By the dichotomy of Theorem \[^{1,10}\], the \(\mu_x\)-measure of conical points is either null or full, so \(\mu_x\) is fully supported on the set of conical points and thus on the Myrberg limit set. By Hopf decomposition, \(\mathcal{Q}^1\mathcal{M}\) consists of the conservative and dissipative sets, uniquely up to \(m\)-null sets. The flow lines in the dissipative set exit every compact sets, so the above argument is reversible.
Complete conservativity $\iff$ Ergodicity via Hopf argument. If $(G_t)_{t \in \mathbb{T}}$ on $\mathbb{Q}^1 \mathcal{T}$ is ergodic without atoms, then it is completely conservative. The converse direction requires to run the Hopf argument, and to that end, we need a positive $m$-integrable function $\Phi(x)$ on $\mathbb{Q}^1 \mathcal{M}$. If $m(\mathbb{Q}^1 \mathcal{M})$ is finite, we can choose $\Phi = 1$. If $G = \text{Mod}(\Sigma_g)$, $m$ is the Masur-Veech measure which is finite. The proof of ergodicity via Hopf argument is given in [71] Theorem 4.

In general, define $\Phi(x) := d_Q(x, o)$ for some $o \in \mathbb{Q}^1 \mathcal{M}$. The same argument as in [62] Lemma 8.3.1 proves that $\Phi(x)$ is $m$-integrable function and for some $c > 0$ and all $x, y \in \mathbb{Q}^1 \mathcal{M}$ with $d_Q(x, y) < 1$

$$\frac{|\Phi(x) - \Phi(y)|}{\Phi(y)} < c \cdot d_Q(x, y)$$

Recall that $\mu_x$ is supported on the Myrberg limit set $\Lambda_m G$, which consists of uniquely ergodic points. By [50], any two geodesic rays $\alpha, \beta$ ending at a unique ergodic point are asymptotic: there exists $a \in \mathbb{R}$ such that $d_T(\alpha(t + a), \beta(a)) \to 0$ and thus for two $p, q \in \mathbb{Q}^1 \mathcal{M}$ with the same vertical foliation,

$$d_Q(\mathbb{G}^{t+a}(p), \mathbb{G}^{t}(q)) \to 0, \quad t \to \infty.$$  

With these ingredients, the ergodicity of the geodesic flow and thus product measures $\mu_x \times \mu_x$ follows verbatim the Hopf’s argument (see [62] Thm 8.3.2 for details). So Theorem 1.15 is proved.

References

1. S. Agard, A geometric proof of Mostow’s rigidity theorem for groups of divergence type, Acta Math. 151 (1983), no. 3-4, 231–252. MR 723011

2. G. Arzhantseva and C. Cashen, Cogrowth for group actions with strongly contracting elements, Ergodic Theory Dynam. Systems 40 (2020), no. 7, 1738–1754. MR 4108903

3. G. Arzhantseva, C. Cashen, D. Gruber, and D. Hume, Characterizations of Morse quasi-geodesics via superlinear divergence and sublinear contraction, Doc. Math. 22 (2017), 1193–1224. MR 3690269

4. J. Athreya, A. Bufetov, A. Eskin, and M. Mirzakhani, Lattice point asymptotics and volume growth on Teichmüller space, Duke Math. J. 161 (2012), no. 6, 1055–1111.

5. W. Ballmann, Lectures on spaces of nonpositive curvature, Birkhäuser, Basel, 1995.

6. M. Bestvina, K. Bromberg, and K. Fujiwara, Constructing group actions on quasi-trees and applications to mapping class groups, Publications mathématiques de l’IHÉS 122 (2015), no. 1, 1–64, arXiv:1006.1939.

7. M. Bestvina, K. Bromberg, K. Fujiwara, and A. Sisto, Acylindrical actions on projection complex, Enseign. Math. 65 (2019), no. 1-2, 1–32. MR 4057354

8. B. Bowditch, Relatively hyperbolic groups, Int. J. Algebra Comput. (2012), no. 22, p1250016.

9. M. Britzton and A. Haefliger, Metric spaces of non-positive curvature, vol. 319, Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 1999.

10. M. Burger and S. Mozes, CAT(-1)-spaces, divergence groups and their commensurators, Journal of the American Mathematical Society (1996), no. 1, 57–93.

11. P. Caprace and M. Sageev, Rank rigidity for CAT(0) cube complexes, Geom. Funct. Anal. 21 (2011), 851–891.

12. R. Charney and H. Sultan, Contracting boundaries of CAT(0) spaces, J Topology 8 (2015), no. 1, 93–117.

13. M. Coornaert, Mesures de Patterson-Sullivan sure le bord d’un espace hyperbolique au sens de Gromov, Pac. J. Math. 193 (1999), no. 2, 241–270.

14. M. Cordes, Morse boundaries of proper geodesic metric spaces, Groups Geom. Dyn. 11 (2017), no. 4, 1281–1306. MR 3737283

15. Y. Coudene, Ergodic theory and dynamical systems, Universitext, Springer-Verlag London, Ltd., London; EDP Sciences, [Les Ulis], 2016. Translated from the 2013 French original [ MR3184308] by Reinie Erné. MR 3586310

16. R. Coulon, Patterson-Sullivan theory for groups with a strongly contracting element, preprint, arXiv:2206.07361.

17. C. Croke and B. Kleiner, Spaces with nonpositive curvature and their ideal boundaries, Topology 39 (2000), no. 3, 549–556. MR 1749008

18. F. Dal’bo, P. Otal, and M. Peigné, Séries de Poincaré des groupes géométriquement finis, Israel Journal of math. 118 (2000), no. 3, 109–124.

19. B. Farb and D. Margalit, A primer on mapping class groups, Princeton Mathematical Series, vol. 49, Princeton University Press, 2012.

20. A. Fathi, F. Laudenbach, and V. Poénaru, Travaux de Thurston sur les surfaces, vol. 66-67, Astérisque, Soc. Math. France, Paris, 1979.

21. T. Fernós, The Furstenberg-Poisson boundary and CAT(0) cube complexes, Ergodic Theory Dynam. Systems 38 (2018), no. 6, 2180–2223. MR 3833446

22. T. Fernós, J. Lécureux, and F. Mathéus, Random walks and boundaries of CAT(0) cubical complexes, Comment. Math. Helv. 93 (2018), no. 2, 291–333. MR 3811753

23. W. Floyd, Group completions and limit sets of Kleinian groups, Inventiones Math. 57 (1980), 205–218.
24. F. Gardiner and H. Masur, *Extremal length geometry of Teichmüller space*, Complex Variables Theory Appl. 16 (1991), no. 2-3, 209–237. MR 1099913

25. I. Gekhtman, *Stable type of the mapping class group*, arXiv: 1310.5364.

26. I. Gekhtman, Y.-L. Qi, and R. Rafi, *Genericity of sublinearly morse directions in CAT(0) spaces and the Teichmüller space*, preprint, arXiv:2208.04778.

27. I. Gekhtman and W. Y. Yang, *Counting conjugacy classes in groups with contracting elements*, J. Topol. 15 (2022), no. 2, 620–665. MR 4441600

28. Y. Gekhtman, V. Gerasimov, L. Potyagailo, and W. Y. Yang, *Martin boundary covers Floyd boundary*, Invent. Math. 223 (2021), no. 2, 759–809. MR 4209863

29. A. Genevois, *Contracting isometries of CAT(0) cube complexes and acylindrical hyperbolicity of diagram groups*, Algebr. Geom. Topol. 20 (2020), no. 1, 49–134. MR 4071367

30. V. Gerasimov, *Floyd maps for relatively hyperbolic groups*, Geom. Funct. Anal. (2012), no. 22, 1361 – 1399.

31. V. Gerasimov and L. Potyagailo, *Quasi-isometries and Floyd boundaries of relatively hyperbolic groups*, J. Eur. Math. Soc. 15 (2013), 2115 – 2137.

32. U. Hamenstädt, *Train tracks and the Gromov boundary of the complex of curves*, Spaces of Kleinian groups, London Math. Soc. Lecture Note Ser., vol. 329, Cambridge Univ. Press, Cambridge, 2006, pp. 187–207. MR 2258749

33. ——, *Invariant Radon measures on measured lamination space*, Invent. Math. 176 (2009), no. 2, 223–273. MR 2495764

34. ——, *Rank-one isometries of proper CAT(0)-spaces*, Discrete groups and geometric structures, Contemp. Math., vol. 501, Amer. Math. Soc., Providence, RI, 2009, pp. 43–59. MR 2581914

35. S.Z. Han and W. Y. Yang, *Generic free subgroups and statistical hyperbolicity*, Algebr. Geom. Topol. 20 (2020), no. 1, 101–140. MR 4235749

36. Junwu He, Jinsong Liu, and W. Y. Yang, *Large quotients of group actions with a contracting element*, Proceedings of the International Consortium of Chinese Mathematicians 2017, Int. Press, Boston, MA, [2020] ©2020, pp. 319–338. MR 4251117

37. E. Hopf, *Ergodic theory and the geodesic flow on surfaces of constant negative curvature*, Bull. Amer. Math. Soc. 77 (1971), 863–877. MR 284564

38. J. Hubbard and H. Masur, *Quadratic differentials and foliations*, Acta Math. 142 (1979), no. 3-4, 221–274. MR 523212

39. R. Kent IV and C. Leininger, *Subgroups of the mapping class group from the geometrical viewpoint*, In the tradition of Ahlfors–Bers IV, Contemp. Math., 432, Amer. Math. Soc., Providence, RI, 2007.

40. V. Kaimanovich and H. Masur, *The poisson boundary of the mapping class group*, Geom. Topol. 9 (2005), 2359–2394. MR 2209375

41. A. Karlsson and G. Margulis, *A multiplicative ergodic theorem and nonpositively curved spaces*, Comm. Math. Phys. 208 (1999), no. 1, 107–123. MR 1729880

42. E. Klarreich, *The boundary at infinity of the curve complex and the relative Teichmüller space*, https://arxiv.org/abs/1803.10339, preprint, 1999.

43. G. Kneiper, *On the asymptotic geometry of nonpositively curved manifolds*, GAFA 7 (1997), 755–782.

44. ——, *The uniqueness of the measure of maximal entropy for geodesic flows on rank 1 manifolds*, Annals of Mathematics 148 (1998), no. 1, 291–314.

45. E. Lindenstrauss and M. Mirzakhani, *Ergodic theory of the space of measured laminations*, Int. Math. Res. Not. IMRN (2008), no. 4, Art. ID rnn126, 49. MR 2424174

46. G. Link, *Hopf-Tsuji-Sullivan dichotomy for quotients of Hadamard spaces with a rank one isometry*, Discrete Contin. Dyn. Syst. 38 (2018), no. 11, 5577–5613. MR 3917781

47. L.X. Liu and W.X. Su, *The horofunction compactification of the Teichmüller metric*, Train tracks and the Gromov boundary of the complex of curves, Spaces of Kleinian groups, London Math. Soc. Lecture Note Ser., vol. 329, Cambridge Univ. Press, Cambridge, 2006, pp. 283–304. MR 3009545

48. H. Miyachi, *Ergodic theory and the geodesic flow on surfaces of constant negative curvature*, Bull. Amer. Math. Soc. 77 (1971), 863–877. MR 284564

49. R. Kaimanovich and H. Masur, *Invariant Radon measures on measured lamination space*, Invent. Math. 176 (2009), no. 2, 223–273. MR 2495764

50. H. Masur, *Uniquely ergodic quadratic differentials*, Comment. Math. Helv. 55 (1980), no. 2, 255–266. MR 576605

51. ——, *Interval exchange transformations and measured foliations*, Ann. of Math. (2) 115 (1982), no. 1, 169–200. MR 640418

52. ——, *Two dimensions of Teichmüller space*, Duke Math. J. 49 (1982), no. 1, 183–190. MR 650376

53. K. Matsuzaki, Y. Yabuki, and J. Jaerisch, *Normalizer, divergence type, and Patterson measure for discrete groups of the Gromov hyperbolic space*, Groups Geom. Dyn. 14 (2020), no. 2, 369–411. MR 4118622

54. J. McCarthy and A. Papadopoulos, *Dynamics on Thurston’s sphere of projective measured foliations*, Comment. Math. Helvetici 66 (1991), 133–166.

55. Y. Minsky, *Quasi-projections in Teichmüller space*, J. Reine Angew. Math. 473 (1996), 121–136.

56. H. Miyachi, *Teichmüller rays and the Gardiner-Masur boundary of Teichmüller space*, Geom. Dedicata 137 (2008), 113–141. MR 2449148

57. ——, *Teichmüller rays and the Gardiner-Masur boundary of Teichmüller space II*, Geom. Dedicata 162 (2013), 283–304. MR 3095645

58. ——, *Extremal length boundary of the Teichmüller space contains non-Busemann points*, Trans. Amer. Math. Soc. 366 (2014), no. 10, 5409–5430. MR 3240928
59. Unification of extremal length geometry on Teichmüller space via intersection number, Math. Z. 278 (2014), no. 3-4, 1065–1095. MR 3278905
60. J. Munkres, Topology, Prentice Hall, Inc., Upper Saddle River, NJ, 2000. Second edition of [ MR0464128]. MR 3728284
61. F. J. Myrberg, Ein Approximationsatz für die Fuchsschen Gruppen, Acta Math. 57 (1931), no. 1, 389–409. MR 1555338
62. P. Nicholls, The ergodic theory of discrete groups, London Mathematical Society Lecture Note Series, vol. 143, Cambridge University Press, Cambridge, 1989. MR 1041575
63. S. Patterson, The limit set of a Fuchsian group, Acta Mathematica (1976), no. 1, 241–273.
64. Y. L. Qing and K. Rafi, Sublinearly Morse boundary I: CAT(0) spaces, Adv. Math. 404 (2022), Paper No. 108442, 51. MR 4423805
65. M. Rees, An alternative approach to the ergodic theory of measured foliations on surfaces, Ergodic Theory Dynam. Systems 1 (1981), no. 4, 461–488 (1982). MR 662738
66. R. Ricks, Flat strips, Bowen-Margulis measures, and mixing of the geodesic flow for rank one CAT(0) spaces, Ergodic Theory Dynam. Systems 37 (2017), no. 3, 939–970. MR 3628926
67. T. Roblin, Sur la fonction orbitale des groupes discrets en courbure négative, Ann. Inst. Fourier 52 (2002), 145–151.
68. Ergodité et équidistribution en courbure négative, no. 95, Mémoires de la SMF, 2003.
69. D. Sullivan, The density at infinity of a discrete group of hyperbolic motions, Publ. Math. IHES (1979), 171–202.
70. G. Tiozzo and W. Y. Yang, Fundamental inequality of random walks on Teichmüller space, preliminary version, 2021.
71. P. Tukia, The Poincaré series and the conformal measure of conical and Myrberg limit points, Journal d’Analyse Mathématique 62 (1994), 241–259.
72. W. Veech, Gauss measures for transformations on the space of interval exchange maps, Ann. of Math. (2) 115 (1982), no. 1, 201–242. MR 644019
73. C. Walsh, The asymptotic geometry of the Teichmüller metric, Geom. Dedicata 200 (2019), 115–152. MR 3956189
74. W. Y. Yang, Growth tightness for groups with contracting elements, Math. Proc. Cambridge Philos. Soc 157 (2014), 297 – 319.
75. Statistically convex-cocompact actions of groups with contracting elements, Int. Math. Res. Not. IMRN (2019), no. 23, 7259–7323. MR 4039013
76. Genericity of contracting elements in groups, Math. Ann. 376 (2020), no. 3-4, 823–861. MR 4081104
77. Patterson-Sullivan measures and growth of relatively hyperbolic groups, Peking Math. J. 5 (2022), no. 1, 153–212. MR 4389490

BEIJING INTERNATIONAL CENTER FOR MATHEMATICAL RESEARCH, PEKING UNIVERSITY, BEIJING 100871, CHINA P.R.
Email address: wyang@math.pku.edu.cn