Moment transport equations for the primordial curvature perturbation

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Abstract. In a recent publication, we proposed that inflationary perturbation theory can be reformulated in terms of a probability transport equation, whose moments determine the correlation properties of the primordial curvature perturbation. In this paper we generalize this formulation to an arbitrary number of fields. We deduce ordinary differential equations for the evolution of the moments of $\zeta$ on superhorizon scales, which can be used to obtain an evolution equation for the dimensionless bispectrum, $f_{\text{NL}}$. Our equations are covariant in field space and allow identification of the source terms responsible for evolution of $f_{\text{NL}}$. In a model with $M$ scalar fields, the number of numerical integrations required to obtain solutions of these equations scales like $O(M^2)$. The performance of the moment transport algorithm means that numerical calculations with $M \gg 1$ fields are straightforward. We illustrate this performance with a numerical calculation of $f_{\text{NL}}$ in $N$flation models containing $M \sim 10^2$ fields, finding agreement with existing analytic calculations. We comment briefly on extensions of the method beyond the slow-roll approximation, or to calculate higher order parameters such as $g_{\text{NL}}$. 
1. Introduction

In a previous article [1] we introduced a new technique ("moment transport") designed to extract the statistics of primordial fluctuations generated by inflation. Our technique exploits the principle that one never uses a model of inflation to predict the properties of any particular universe. Instead, one determines averages over a large ensemble of universes. Observable quantities are predicted on the basis that our universe is a typical member of the ensemble. Therefore there is no need to study the evolution of fluctuations in any specific universe; it is sufficient to compute the evolution of the probability density function on the ensemble. Ref. [1] discussed the moment-transport method for a restricted range of models. In this paper we refine and extend our method to accommodate models containing many self-interacting scalar fields in a variety of gauges. We show that the moment-transport technique yields a system of coupled ordinary differential equations which can be used for rapid computation of observable quantities, such as the dimensionless bispectrum $f_{NL}$.

Our method is based on a transport equation. This is a partial differential equation which governs the evolution of a probability distribution. In Ref. [1], we showed how a transport equation for a bivariate distribution can be used to derive ordinary differential equations which describe the evolving moments of any nearly Gaussian distribution. Our method was based on diagonalization of the covariance matrix describing correlations between the two variables, a process analogous to Gram–Schmidt orthogonalization. In this paper our first objective is to introduce a more elegant method based on matrix decomposition. Such decompositions generalize immediately to an arbitrary number of scalar fields. Using the resulting equations we are able to evolve the joint probability distribution for any number of variables.

The moment transport equation approach can be thought of as a reformulation of cosmological perturbation theory on superhorizon scales with the additional virtue that, rather than simply evolving a perturbed quantity, such as $\zeta$, this approach evolves the statistical moments of a perturbed quantity. This is advantageous because these are the observationally relevant objects.

The equations presented in Ref. [1] were valid only in the spatially flat slicing. In this gauge, the joint probability distribution for a collection of light scalar fields at horizon-crossing is simple to calculate using the methods of quantum field theory [2]. Ref. [1] determined the statistics of observational quantities—such as the curvature perturbation, $\zeta$, in the uniform density gauge—by evolving this distribution to a later time and making an appropriate gauge transformation. In the present paper, our second objective is to generalize the moment-transport method to arbitrary slicings of spacetime. Once liberated from the restriction to spatially flat slicings we are free to calculate the horizon-crossing distribution of the curvature perturbation itself, and propagate this forward in time. Observational quantities may be read directly from this distribution at any desired time, with no need for an auxiliary gauge transformation.

Whichever gauge we pick, our predictions for observational quantities must agree.
Nevertheless, working directly in the uniform density gauge has certain advantages. First, slices of constant time have an intuitive and unambiguous meaning, being associated with hypersurfaces of constant Hubble parameter. Second, it is possible to write evolution equations for observable quantities such as $f_{\text{NL}}$. There is no need to keep track of intermediate, unphysical quantities with the associated risk that large cancellations occur when taking combinations designed to yield physical observables. Third, these evolution equations show explicitly which source terms are responsible for the growth and decay of the covariance and higher moments of $\zeta$. The possibility of cancellations obscures this interpretation for intermediate variables such as the field perturbations.

In this paper, we discuss two methods which may be used to extract the moment hierarchy associated with a general transport equation. In §2 we revisit the technique of Gauss–Hermite expansions, employed in Ref. [1], which explicitly invokes a perturbative expansion around a Gaussian distribution. Similar expressions were used by Contaldi & Magueijo to synthesize microwave background maps with a nongaussian component [2], and more recently have been applied to the distribution of collapsed structures; among others, see Refs. [3, 4, 5, 6]. Improving the discussion given in Ref. [1], we use a technique of matrix decomposition to find equations valid for an arbitrary number of fields. This method has several virtues. Most important, in the generalized version to be discussed in §2 it is explicitly tensorial. Therefore, combinatorial factors associated with multi-field models are built into the formalism and do not need to be addressed directly.

The Gauss–Hermite technique is an example of a cumulant expansion. Given a “kernel” distribution with a finite number of nonzero cumulants, up to some order $M$, we use it as a template for a general distribution with perturbatively small cumulants of order $> M$. For the Gauss–Hermite expansion the kernel is a Gaussian with $M = 2$. It has zero third- and higher-order cumulants. In the absence of special reasons to the contrary, the Gauss-Hermite method fails when a third- or higher cumulant of the probability density function of interest grows to the degree that it is not perturbatively small.

In §3 we give a different perspective on the results of §2, rederiving them using an alternative technique which does not rely on an expansion around an unperturbed kernel. Introducing generating functions for the moments and cumulants of the probability density function of interest, we demonstrate the surprising and remarkable fact that the same transport hierarchy applies for an arbitrary kernel function. The disadvantage of this method is a complicated treatment of the multi-field combinatorics. As well as a check on the correctness of the formulas derived in §3 the method of generating functions clarifies under which circumstances we can expect the transport hierarchy to apply. In particular, it allows a more refined discussion of the error involved in truncation of the hierarchy. In practical calculations the two methods are essentially equivalent because the transport hierarchy must be truncated by assuming that an infinite number of cumulants are perturbatively small.
In §4 we apply our general framework to the fluctuations generated by a model of inflation. This depends on the selection of a smoothing scale and a choice of gauge, which we discuss in §4.1. The smoothing scale is a measure of the spatial scale on which the probability density function measures correlations. In §4.2 we specialize to the uniform density gauge and obtain the moment transport equations on this slicing.

Throughout, we adopt units in which $c = \hbar = 1$ but explicitly retain the Planck mass, $M_P = (8\pi G)^{-1/2}$, because in some gauges we are obliged to mix dimensionless quantities—such as the accumulated e-folds, $N$, and its perturbation $\zeta$—with dimensionful ones, such as the field values $\phi_i$. We label the species of scalar fields by indices $\{i, j, \ldots\}$.

2. Method A: Gauss–Hermite expansions

In this section, we use the Gauss–Hermite method to derive the moment hierarchy associated with a transport equation for an arbitrary number of variables $x_i$, where $i = 1, \ldots, N$. In due course these variables will be associated with cosmological quantities such as field values or curvature, but at present we keep the discussion general. It will sometimes be convenient to adopt a matrix notation, in which the $x_i$ are treated as components of a vector $x$. In what follows we use matrix and component notation interchangeably.

2.1. The Gauss–Hermite expansion

We allow $x$ to have an arbitrary expectation value $X(t)$. Two-point correlations among the $x$ are expressed by the covariance matrix $\Sigma(t)$, which is defined to satisfy

$$\langle (x_i - X_i)(x_j - X_j) \rangle = \Sigma_{ij}.$$  \hspace{1cm} (1)

In addition there may be nontrivial third moments, $\alpha_{ijk}(t)$,

$$\langle (x_i - X_i)(x_j - X_j)(x_k - X_k) \rangle = \alpha_{ijk}.$$  \hspace{1cm} (2)

The covariance matrix and third-order moments are functions of time, as is the position of the centroid $X$. As in Eqs. (1)–(2), we will frequently suppress explicit time dependence to avoid unnecessary clutter.

Eq. (1) makes $\Sigma$ a real, symmetric positive-definite matrix. It therefore admits a decomposition of the form $\Sigma = AA^T$, where $A^T$ denotes the matrix transpose of $A$. A candidate decomposition can be obtained by setting $A = Q\lambda^{1/2}$ where $\lambda = Q^T\Sigma Q$ is the diagonal matrix of eigenvalues of $\Sigma$ and $Q$ is orthogonal. Other representations may exist. For our purpose it is sufficient to pick any one of these candidates. The matrix $A$ is only a tool with which to construct intermediate quantities and does not occur in the final transport equations, so this nonuniqueness does not lead to ambiguities. Given a choice of $A$, we may define standardized variables $z_i(x)$,

$$z_i = A^{-1}_{ij}(x_j - X_j).$$  \hspace{1cm} (3)
The $z_i$ have zero mean and orthonormal covariances. In this and subsequent expressions, the summation convention is applied to repeated indices.

If the joint probability distribution of $\mathbf{x}$ is close to Gaussian (an assumption which we relax in Section 3), it may be represented by a Gauss–Hermite expansion

$$P(\mathbf{x}) \, d^N x = P_g(\mathbf{z}) \left[ 1 + \frac{\alpha_{ijk}}{6} H_{ijk}(\mathbf{z}) \right] \, d^N x,$$

where $P_g$ is a normalized Gaussian kernel,

$$P_g(\mathbf{z}) = \frac{1}{(\det 2\pi \Sigma)^{1/2}} \exp \left( -\frac{\mathbf{z}^2}{2} \right).$$

Note that Eq. (4) should be considered a function of $\mathbf{x}$, via Eq. (3), although for convenience its right-hand side has been written in terms of $\mathbf{z}$. Also, its nongaussian part has been written using a set of third-order moments $\alpha_{ijk}$ associated with the $z_i$. These satisfy

$$\langle z_i z_j z_k \rangle = \alpha_{ijk}$$

and are related to the $\alpha_{ijk}$ by the rule

$$\alpha_{ijk} = A_{il} A_{jm} A_{kn} \alpha_{lmn}.$$ 

Thus, the $\alpha$ transform tensorially under the change of basis represented by $A_{ij}$. The basis functions $H_{ijk}$ which underlie the cumulant expansion are products of Hermite polynomials. The $n^{th}$ polynomial in the Hermite sequence is obtained from Rodrigues’ formula,

$$H_n(\mathbf{w}) = (-1)^n e^{\mathbf{w}^2/2} \frac{\partial^n}{\partial \mathbf{w}^n} e^{-\mathbf{w}^2/2}. $$

The $H_n$ satisfy an orthogonality relation,

$$\int_{-\infty}^{\infty} \frac{e^{-\mathbf{w}^2/2}}{\sqrt{2\pi}} H_n(\mathbf{w}) H_m(\mathbf{w}) \, d\mathbf{w} = n! \delta_{mn}.$$ 

The $H_{ijk}$ are defined by a generalized version of Rodrigues’ formula. They satisfy

$$H_{i_1 i_2 \cdots i_n} = (-1)^n \exp \left( \frac{z_j z_j}{2} \right) \frac{\partial^n}{\partial z_{i_1} \partial z_{i_2} \cdots \partial z_{i_n}} \exp \left( -\frac{z_k z_k}{2} \right).$$

Eqs. (8)–(9) make the $H_{i_1 \cdots i_n}$ orthogonal in the measure $\exp(-\mathbf{z}^2/2) \, d^N \mathbf{z}$.

Eq. (4) is a cumulant expansion in the sense discussed in §1. It is numerically close to a Gaussian for small $\alpha_{ijk}$. It has nonzero third moments, which are perturbatively small, but all higher cumulants are zero. Higher $n^{th}$ cumulants may be incorporated by including them in Eq. (4) as coefficients of $n^{th}$ order functions $H_{i_1 \cdots i_n}$. Nevertheless, Eq. (4) is not an asymptotic expansion in the $z_i$. Moreover, finite truncations may be negative for some values of $z_i$. For our purposes the cumulant expansion is a formal tool, and these mild pathologies are not a cause of serious difficulty because we do not make use of the probability distribution directly.

The $H_{i_1 \cdots i_n}$ obey certain important identities. By repeatedly commuting $z$ and $\partial/\partial z$ one can use the generalized form of Rodrigues’ identity to show

$$z_m H_{i_1 \cdots i_n} = H_{i_1 \cdots i_m} + \delta_{i_1 m} H_{i_2 \cdots i_n} + \cdots + \delta_{i_m m} H_{i_1 \cdots i_{n-1}}.$$ 

(10)
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Further, differentiating Eq. (9) and making use of (10) one can show
\[ \frac{\partial H_{i_1\cdots i_n}}{\partial z_m} = \delta_{i_1m}H_{i_2\cdots i_n} + \cdots + \delta_{i_nm}H_{i_1\cdots i_{n-1}}. \]  

(11)

Eqs. (10) and (11) play a significant role in extracting a moment hierarchy from the Gauss–Hermite expansion.

2.2. Transport of the probability density

Eq. (4) is time dependent, because of the explicit time dependence of \( \mathbf{X}(t), \Sigma(t), \) and \( \alpha_{ijk}(t) \). We assume that time evolution of \( \mathbf{x} \) is generated by a velocity field \( \mathbf{u}(t, \mathbf{x}) \), using the rule \( \dot{\mathbf{x}} = \mathbf{u} \). The vector \( \mathbf{u} \) depends on \( \mathbf{x} \), but may also depend explicitly on time. It is possible to interpret \( \mathbf{u} \) as a time-dependent Hamiltonian vector field whose integral curves are the allowed trajectories in phase space. As time evolves, the shape of the probability distribution is focused and sheared by the action of the velocity field. This geometrical evolution is described by the transport (or “continuity”) equation
\[ \frac{\partial P}{\partial t} + \frac{\partial (u_i P)}{\partial x_i} = 0. \]  

(12)

Eq. (12) accounts for changes in shape and profile of \( P \), but conserves the overall volume of the distribution. It is the zero-diffusion limit of a Chapman–Kolmogorov or Fokker–Planck equation. These equations occur in many areas in physics, including the heat equation and Schrödinger’s equation.

In principle Eq. (12) could be solved directly, but analytic progress is possible only for a limited choice of \( u_i \). Numerical approaches are rather involved. As an alternative to solving for \( P \) itself, Eq. (12) may be converted into a system of coupled equations for the \( n \)th moments of \( P \). The evolution equation for any moment will generally depend on all the others, but if increasingly high-order moments decrease in magnitude then it may be a reasonable approximation to truncate the coupled system at finite order. In what follows we will carry out this programme, assuming that it is only necessary to retain the moments \( X_i(t), \Sigma_{ij}(t) \) and \( \alpha_{ijk}(t) \).

In this section we are assuming that the third- and higher \( n \)th-order moments are perturbatively small. Therefore, probability is concentrated in the vicinity of the instantaneous centroid \( X_i(t) \). The influence of the velocity field in reshaping the distribution is greatest in this region. Near the centroid, we find
\[ u_i = u_{i0} + u_{ij}(x_j - X_j) + \frac{1}{2}u_{ijk}(x_j - X_j)(x_k - X_k) + \cdots \]
\[ = u_{i0} + u_{ij}A_{jk}z_k + \frac{1}{2}u_{ijk}A_{jm}A_{kn}z_mz_n + \cdots, \]  

(13)

where the coefficients \( u_{i0}, u_{ij} \) and \( u_{ijk} \) satisfy
\[ u_{i0}(t) = u_i(t)|_{\mathbf{x} = \mathbf{X}(t)}, \quad u_{ij}(t) = \frac{\partial u_i(t)}{\partial x_j}|_{\mathbf{x} = \mathbf{X}(t)}, \quad \text{and} \quad u_{ijk}(t) = \frac{\partial^2 u_i(t)}{\partial x_j \partial x_k}|_{\mathbf{x} = \mathbf{X}(t)}. \]  

(14)

A particle located at the centroid, \( x_i = X_i(t) \), would evolve according to \( dx_i/dt = u_{i0}(t) \). The presence of the higher terms in (13) reflects the way in which the wings of the
probability distribution, which are absent for a point particle, sample nearby parts of the velocity field.

Whether or not \(u_i\) depends explicitly on time, these coefficients will do so because they are evaluated at the time-dependent location \(x = X(t)\). Moreover, since we must truncate Eq. (13) at finite order to obtain a finite system of evolution equations, there will be an unaccounted remainder which makes the effective velocity field time dependent.

To proceed, we must determine \(\partial P/\partial t\) and \(\partial(u_i P)/\partial x_i\). According to Eq. (12) their sum must be equal to zero. Using Eqs. (10) and (11) to exchange factors of \(z\) or \(\partial/\partial z\) (as applied to the \(H_{i_1 \cdots i_n}\)) with sums of other \(H\)-functions, it can be cast as a Hermite tableau of the form

\[
P_g \left[ c_0 + c_i H_i + c_{ij} H_{ij} + c_{ijk} H_{ijk} + \cdots \right] = 0.
\]

(15)

The orthogonality of \(H_{i_1 \cdots i_n}\) in the measure \(P_g d^N z\) implies a hierarchy of equations,

\[
c_0 = c_i = c_{ij} = c_{ijk} = \cdots = 0,
\]

(16)

where brackets denote symmetrization of indices.

The \(c_0\) equation expresses overall conservation of probability, and is identically satisfied because Eq. (12) is conservative. The equation \(c_i = 0\) gives an evolution equation for the centroid,

\[
\frac{\partial X_i}{\partial t} = u_{i0} + \frac{1}{2} u_{imn} \Sigma_{mn} + \cdots,
\]

(17)

where ‘\(\cdots\)’ denotes omitted terms which are higher order in cumulant expansion. The equation \(c_{ij} = 0\) gives a similar evolution equation for the covariance matrix, \(\Sigma_{ij}\),

\[
\frac{\partial \Sigma_{ij}}{\partial t} = u_{im} \Sigma_{mj} + u_{jm} \Sigma_{mi} + \frac{1}{2} u_{imn} \alpha_{jmn} + \frac{1}{2} u_{jmn} \alpha_{imn} + \cdots.
\]

(18)

Finally, \(c_{ijk} = 0\) is equivalent to an evolution equation for the \(\alpha_{ijk}\)

\[
\frac{\partial \alpha_{ijk}}{\partial t} = u_{im} \alpha_{mjk} + u_{ins} \Sigma_{jsn} \Sigma_{kn} + (i \to j \to k) + \cdots,
\]

(19)

where \((i \to j \to k)\) denotes the preceding term with cyclic permutations of the indices. Eqs. (17)-(19) represent the first principal result of this paper. They agree with the corresponding evolution equations for first, second and third moments obtained in Ref. [1], but apply for an arbitrary number of scalar fields.

3. Method B: Generating functions

In this section we are going to rederive the results of §2 using a technique which is valid for a probability distribution in the neighbourhood of an arbitrary kernel distribution. It can accommodate an arbitrary number of fields, and applies to any order in the cumulant expansion. The method applied in Ref. [1] and §2 exploited orthogonality properties of Hermite polynomials and was therefore restricted to a Gaussian kernel. Here, we take a different approach. We introduce generating functions for the cumulants, and show that this method leads to an efficient derivation of the evolution equations.
We retain the notation of §2. In this section, we introduce the new method by using it to study a probability distribution with a single field. The extension to an arbitrary number of fields involves more complicated combinatorics, but ultimately yields expressions equivalent to Eqs. (17)–(19). We discuss the multiple-field case in Appendix A.

3.1. Generating functions

In the one-field case, we denote the single field by \( x \), and assume that the distribution of \( x \) is described by a time-dependent probability distribution of the form \( P(x, t) \, dx \). We denote the mean value of \( x \) by \( X(t) \), so that

\[
X(t) = \int x \, P(x, t) \, dx.
\]  

(20)

In §2 we restricted our attention to the third moment, \( \alpha \). In this section we would like to generalize the analysis to all orders in the moment expansion. For this reason we introduce moments \( \mu_n(t) \), \( n = 0, 1, 2, \ldots \) defined by

\[
\mu_n(t) = \int [x - X(t)]^n P(x, t) \, dx.
\]  

(21)

Since \( P \) is properly normalized we must have \( \mu_0(t) = 1 \), independent of time. Our definition of \( X \) implies \( \mu_1(t) = 0 \). Therefore the first nontrivial moment is the second, \( \mu_2(t) \). The entire infinite set of moments can be encoded using the “moment generating function” \( M(z, t) \), defined by

\[
M(z, t) \equiv \int e^{z(x-X)} P(x, t) \, dx = \sum_{n=0}^{\infty} \frac{z^n \mu_n(t)}{n!}.
\]  

(22)

This definition ensures that the \( n \)th moment can be recovered using the identity

\[
\frac{\partial^n}{\partial z^n} M(0, t) = \mu_n(t).
\]

In §2 and the foregoing discussion we have restricted our attention to the moments of \( P \). When discussing high orders in the moment expansion, it is sometimes more convenient to make use of the cumulants of \( P \) directly. We define the sequence of cumulants, \( \kappa_n \), via their generating function \( C(z, t) \). This satisfies

\[
C(z, t) \equiv \ln M(z, t) = \sum_{n=0}^{\infty} \frac{z^n \kappa_n(t)}{n!}.
\]  

(23)

Conversely, the moments are related to the cumulants through \( M(z, t) = e^{C(z, t)} \). Hence, knowledge of the moments is enough to determine all the cumulants, and vice-versa. For example, we always have \( \mu_1(t) = \kappa_1(t) = 0 \), \( \mu_2(t) = \kappa_2(t) \), and \( \mu_3(t) = \kappa_3(t) \), while

\[
\mu_4(t) = \kappa_4(t) + 3\kappa_2^2(t),
\]  

(24)

\[
\mu_5(t) = \kappa_5(t) + 10\kappa_2(t)\kappa_3(t),
\]  

(25)

\[
\mu_6(t) = \kappa_6(t) + 15\kappa_2(t)\kappa_4(t) + 10\kappa_3^2(t) + 15\kappa_2^3(t)
\]  

(26)

and so on \( ad \) \( infinitum \). To obtain the cumulant \( \kappa_n \) one need only know the moments up to order \( n \), and vice versa.
For $n > 3$ it is the $\kappa_n$ which are most useful for the characterization of primordial nongaussianity. In the language of field theory, the cumulants are the connected correlation functions of $x$, while the moments are the disconnected correlation functions. In particular, a pure Gaussian distribution has only one nonzero cumulant, $\kappa_2$. On the other hand, all of its even moments $\mu_0$, $\mu_2$, $\mu_4$, ..., are nonzero. Thus, beyond third order, the cumulants provide a more suitable measure of departures from Gaussianity.

3.2. Transport equation

The transport equation is the one-dimensional version of Eq. (12). Combining the transport equation with (20) yields

$$\frac{dX}{dt} = \int x \frac{\partial}{\partial t} P(x, t) \, dx = \int u(x) P(x, t) \, dx$$

(27)

As in §2, we assume that $x$ evolves under the influence of a velocity field which can be expanded around the instantaneous centroid $X(t)$,

$$u(x) = \sum_{n=0}^{\infty} \frac{u_n(t)}{n!} [x - X(t)]^n,$$

(28)

where, in comparison with Eq. (13), we have adopted a slightly different notation in which $u_n$ denotes the $n^{th}$ derivative $d^n u / dx^n$. Returning to (27) and applying (28) together with the definition of the moments, we find

$$\frac{dX}{dt} = \sum_{n=0}^{\infty} \frac{\mu_n(t)}{n!} u_n(t) = u_0(t) + \frac{1}{2} \mu_2(t) u_2(t) + \cdots$$

(29)

With the evolution of $X(t)$ in hand, we can derive the evolution of all other cumulants. Using (23), we conclude that the time derivatives $d\kappa_n / dt$ obey

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{d\kappa_n(t)}{dt} = \frac{1}{M(z, t)} \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{d\mu_n(t)}{dt},$$

(30)

from which it follows that $dC / dt$ is the generating function for the cumulant time derivatives. Following steps similar to those which led us to (27), it can be seen that the evolution equation for each moment can be written

$$\frac{d\mu_n(t)}{dt} = \sum_{k=0}^{n} \frac{n}{k!} \left[ \mu_{n+k-1}(t) - \mu_{n-1}(t) \mu_k(t) \right] u_k(t)$$

(31)

Inserting this expression in (30) yields a generating function for the time derivatives of each cumulant. The result can be written in terms of the moments. Finally, using the relationship between the moments and cumulants, the cumulant evolution equations can be written in terms of the cumulants alone. This gives a closed system of equations for the cumulants, with an infinite number of variables.

In practice, only a finite number of variables can be evolved and it is necessary to truncate both the series of cumulants and the expansion of $u$. If the Taylor expansion of the velocity field is truncated, then it is evident from (31) that the evolution equation
for each moment involves moments of higher order. Accordingly, the time-evolution of an individual cumulant always involves cumulants of higher order. Thus, even when the velocity field is truncated, the system of cumulant evolution equations does not close at any finite order. We must choose an order at which to approximate the full cumulant expansion. If desired, we can obtain an estimate of the error involved in truncation at this order by determining the degree to which time evolution sources higher cumulants, forcing them to become nonzero.

We now illustrate this procedure in operation using a simple example. Suppose we wish to carry the velocity field expansion to third order in \( (x - X) \), so that

\[
u(x) = \nu_0(t) + [x - X(t)]u_1(t) + \frac{1}{2!}[x - X(t)]^2 u_2(t) + \frac{1}{3!}[x - X(t)]^3 u_3(t).
\]  

(32)

Assume further that the maximal nonzero cumulant is of order four. Applying the formulae above, we find

\[
\frac{dX}{dt} = \nu_0(t) + \frac{1}{2!}u_2(t)\kappa_2(t) + \frac{1}{3!}u_3(t)\kappa_3(t) + \frac{1}{4!}\left[3\kappa_2(t)^2 + \kappa_4(t)\right].
\]  

(33)

The cumulant evolution equations enforce \( d\kappa_0/dt = 0 \) and \( d\kappa_1/dt = 0 \). Furthermore, the cumulants \( \kappa_2, \kappa_3 \) and \( \kappa_4 \) evolve according to

\[
\frac{d\kappa_2}{dt} = 2u_1\kappa_2 + u_2\kappa_3 + u_3\left[\kappa_2^2 + \frac{1}{3}\kappa_4\right]
\]  

(34)

\[
\frac{d\kappa_3}{dt} = 3u_1\kappa_3 + u_2\left[3\kappa_2^2 + \frac{3}{2}\kappa_4\right] + \frac{9}{2}u_3\kappa_2\kappa_3
\]  

(35)

\[
\frac{d\kappa_4}{dt} = 4u_1\kappa_4 + u_2\left[12\kappa_2\kappa_3 + 2\kappa_5\right] + u_3\left[4\kappa_3^2 + 6\kappa_3^2 + 8\kappa_2\kappa_4\right],
\]  

(36)

where we have suppressed the time dependence of the \( \kappa_j \) and \( u_j \).

Under our assumptions, the evolution equation for \( \kappa_5 \) is

\[
\frac{d\kappa_5}{dt} = u_2\left[20\kappa_2\kappa_4 + 15\kappa_3^2\right] + u_3\left[30\kappa_2^2\kappa_3 + 25\kappa_3\kappa_4\right]
\]  

(37)

If a truncation to fourth-order quantities were self-consistent, this equation should vanish as a consequence of our assumption that \( \kappa_n = 0 \) for \( n \geq 5 \). That it does \textit{not} vanish is a measure of the error involved in our truncation. As explained above, a similar effect occurs no matter at which order the truncation is made. In the analysis of this section, we have made no assumption that either the moments or cumulants are small, or order themselves into an ultimately decreasing sequence. Eq. (37) demonstrates that the truncated hierarchy will be a useful predictive instrument only when a negligible variation in \( \kappa_5 \) is sourced over the time interval of interest. This will typically (but not absolutely) require the cumulants to fall in an ordered structure, for example \( |\kappa_2| > |\kappa_3| > |\kappa_4| \).

For example, we may suppose that \( |\kappa_n| = O(\delta^n) \) where \( \delta \ll 1 \) is a small positive number. Eq. (37) shows that source term for \( \kappa_5 \) is of order \( O(\delta^6) \), and therefore after a short time interval \( \Delta t \) we can expect \( \kappa_5 \sim (\Delta t)\delta^6 \). Truncation to fourth order, setting \( \kappa_5 \) and all higher cumulants to zero, may be acceptable approximation over sufficiently short times that \( \kappa_5 \) does not grow to the degree that it contaminates any observable Moment transport equations for the primordial curvature perturbation
of interest. How long this time can be is model dependent. A similar analysis can be given for all higher cumulants. Growing secular terms which eventually invalidate perturbation theory are a typical feature in studying the evolution of fluctuations. They are encountered directly in the Feynman amplitudes of quantum field theory, and in the most common implementations of the separate universe picture.

Eqs. (34)–(36) coincide with Eqs. (18) and (19) when specialized to the single-field case, but include the contribution of the third derivative of the velocity field, \( u_3 \). Eq. (36) gives, for the first time, the evolution of the kurtosis in a single-field setting.

In Appendix A we apply this method to the case of multiple fields, and again find agreement with the results of §2.1.

4. Inflationary perturbations

We now wish to apply the general framework assembled in §2–3 to the fluctuations which are generated and subsequently evolve during an inflationary era. To do this we identify the variables \( x_i \) of §2–3 with the light, scalar degrees of freedom which are excited at that time. We must also identify a choice of time variable, labelled \( t \) in §2–3, which corresponds to a choice of slicing in the spacetime picture.

4.1. Gauge choices

Consider a theory of \( M \) scalar fields coupled to a metric theory of gravity. The degrees of freedom in this system are the \( M \) fields themselves, \( \phi_i \) for \( i \in \{1, \ldots, M\} \), together with the lapse and shift functions, written \( N \) and \( N^a \), and the spatial 3-metric \( h_{ab} \),

\[
ds^2 = -N^2 \, dt^2 + h_{ab}(dx^a + N^a \, dt)(dx^b + N^b \, dt).
\]

In Einstein gravity the lapse and shift are not propagating fields, and are eliminated by constraint equations. The 3-metric \( h_{ab} \) encodes six independent degrees of freedom but in this paper we concentrate on the two spin-0 modes, of which only one is physical. We take this to be the volume modulus \( \text{det} \, h_{ab} \); the other can be absorbed by a spatial coordinate redefinition. We write \( \text{det} \, h_{ab} = e^{6(N-N_0)} \delta_{ab} \), and refer to \( N \) as the integrated e-foldings of expansion. The zero point \( N_0 \) is arbitrary. Note that the integrated number of e-folds is quite separate from the lapse function which occurs in Eq. (38), also traditionally denoted \( N \). In what follows we shall work in an unperturbed universe for which the lapse is unity. Therefore we have no need to refer to it explicitly, so that the appearance of \( N \) without qualification is unambiguous.

We now have \( M + 1 \) variables: the \( M \) fields, and the integrated expansion \( N \). Not all these are independent, and one of them can be eliminated by a suitable choice of time, \( t \). Whatever choice we make, surfaces of constant \( t \) foliate spacetime into spatial hypersurfaces which we refer to as a slicing. The transport equation, Eq. (12), evolves a probability distribution for the \( x_i \) from one slice in this foliation to the next. There are two slicings of particular importance for inflationary fluctuations.
Spatially flat slicing. In this gauge, spatial hypersurfaces are chosen so that the integrated number of e-foldings, $N$, is uniform across the slice. The scalar fields $\phi_i$ fluctuate from place to place. To apply the formalism of §2 we work on slices of uniform expansion $N$, and smooth the scalar fields on a comoving lengthscale $L$ to give an ensemble of $L$-sized regions.

These regions traverse a bundle of adjacent trajectories in field space, with some characteristic dispersion and higher-order moments which it is our intention to calculate. To make contact with observation the bundle should be chosen so that every trajectory reheats almost surely in our local vacuum. With this choice we can suppose the dispersion and higher-order moments of $L$-sized regions in our observable universe should be similar to those of the bundle. On scales comparable to the size of our observable patch we sample only a small number of trajectories, making the prospect of a mismatch with the bundle average (“cosmic variance”) more likely.

The centroid of the bundle follows a path $\Phi^L_i(t)$ in field space, where the superscript ‘$L$’ indicates dependence on the smoothing scale. The probability distribution we wish to calculate is a function of the scalar fields, $\phi_i = \Phi^L_i(t) + \delta \phi^L_i$. We take $x_i = \phi_i/M_P$ and $X_i(t) = \Phi^L_i(t)/M_P$. The probability density obtained in this way gives information about the distribution of field values on the scale $L$ only. To obtain a relation between bundles smoothed on different scales $L$ and $L'$, two separate distributions must be computed and their information combined. This would be necessary, for example, to obtain the spectral index.

We also require initial conditions, set at the time $t_L$ when the wavenumber corresponding to the smoothing scale $L$ crosses the horizon. We pick initial expectation values $\Phi^L_i(t_L)$ centred on the inflationary trajectory of interest. In the spatially flat slicing, the joint probability distribution of fluctuations in the scalar fields at time $t_L$ can be calculated directly \[7, 8, 9, 10, 11, 12\], with each field acquiring a variance of order $\langle \delta \phi^2_i/M_P^2 \rangle_L = \sigma^2_L$, where $\sigma_L = H_L/M_P \sim 10^{-5}$ and $H_L$ is the Hubble rate when the wavenumber $1/L$ crosses the horizon. The skewness of the bundle is negligible, with $\alpha^L_{ijk} \sim \sigma^4_L \approx 0 \[13, 14\]$. In addition, to leading order in $\epsilon = -\dot{H}/H^2$, the slow-roll condition makes the fields uncorrelated at horizon exit, with $\langle \delta \phi_i \delta \phi_j/M_P^2 \rangle_L \sim \epsilon L \sigma^2_L$ if $i \neq j$. Taking the initial dispersion to be given by $\sigma_L$ and setting $\alpha^L_{ijk}$ and any cross-correlations to be initially zero, we can evolve the $\sigma$ and $\alpha$ from horizon crossing until any desired future time, such as the end of inflation. At this point the field’s moments on the flat hypersurface can be used to calculate the moments of $\zeta$ using a gauge transformation. For two fields this procedure was performed in Ref. [1], and the technology developed in this paper makes this possible for an arbitrary number of fields.

† Note that $\Phi^L_i(t)$ need not be an integral curve of the velocity field, and therefore may not constitute an allowed inflationary trajectory.

‡ The conclusion that the $\delta \phi_i$ are virtually uncorrelated at horizon exit implies that the inflationary trajectories—described by the integral curves of the velocity field $u_i$ in this gauge—are effectively straight lines for a few e-folds around horizon-crossing, up to corrections of $O(\epsilon)$. This applies in canonical inflationary models, but in more general examples this may not occur [14]. In such cases the theory becomes more complicated and the formalism of this paper will no longer apply.
Uniform density slicing. Alternatively, we may choose our spatial hypersurfaces so that the Hubble rate, $H$, is uniform over the slice. In Einstein gravity the Friedmann constraint enforces $3M_P^2H^2 = \rho$, so slices of uniform $H$ are also slices of uniform density. On this slicing the integrated number of e-foldings, $N$, will typically vary from place to place. The counting of independent fluctuations is the same as in the spatially flat slicing, because the Friedmann constraint makes one field a function of all the others and the Hubble rate, $H$. Without loss of generality we can suppose that $\phi_M = \phi_M(\phi_1, \ldots, \phi_{M-1}, H)$. For notational convenience we define $s_i = \phi_i/M_P$. To apply the framework of §§2–3 we must set the time variable, $t$, to equal $H/M_P$ and choose the variables $x_i$ whose distribution we wish to calculate to be $\{s_1, \ldots, s_{M-1}, N\}$.

It is still necessary to choose a smoothing scale, $L$. The centroid of the bundle is characterized by the expectation values of the first $M-1$ scalar fields, $\{\Phi^L_1(t), \ldots, \Phi^L_{M-1}(t)\}$ together with the mean integrated expansion experienced by trajectories within the bundle, $\bar{N}_L(t)$. We write $N = \bar{N}_L(t) + \zeta_L$, where $\zeta_L$ is the uniform density gauge curvature perturbation smoothed on scale $L$. On the large scales we are considering, the uniform density gauge and comoving gauge coincide which makes $\zeta_L$ numerically equal to the comoving curvature perturbation, $R_L$.

4.2. Moment transport in the uniform density slicing

To implement moment transport in the uniform density slicing, we will require initial conditions for the dispersion and higher moments of the fluctuations $\{s_1, \ldots, s_{M-1}, \zeta\}$. These have not yet been calculated directly, but can be obtained from the joint probability distribution in the spatially flat slicing [2] in conjunction with gauge transformations relating super-horizon quantities in the uniform density gauge to those in spatially flat gauge. These transformations can be obtained using conventional cosmological perturbation theory [15], or using the separate universe assumption, which was the approach taken in Ref. [1]. Using the separate universe assumption, one writes $\zeta$ using the ‘$\delta N$’ formula [16]

$$
\zeta(t) = N(t, \phi_* + \delta \phi) - N(t, \phi_*) = N, i \delta \phi_* + \frac{1}{2} N, ij \delta \phi_* \delta \phi_j + \cdots \tag{39}
$$

where $N(t, \phi_*)$ measures the e-foldings between a spatially flat slice on which the fields take prescribed values $\phi_*$ and a subsequent uniform density slice at time $t$. We are considering the special case of flat and uniform density slices which coincide on average, and are therefore only perturbatively separated. The coefficients satisfy $N, i = \partial N/\partial \phi_*$ with similar definitions for $N, ij$ and higher derivatives.

Initial conditions for isocurvature fields. The fluctuations $\{s_1, \ldots, s_{M-1}\}$ can be calculated using an analogous formula, as described in Ref. [1],

$$
\phi_i M_P = \frac{\partial \phi^c_i}{\partial \phi_*} \delta \phi_j + \frac{1}{2} \frac{\partial^2 \phi^c_i}{\partial \phi_j \partial \phi_k} \delta \phi_j \delta \phi_k + \cdots \tag{40}
$$

The superscript ‘$c$’ denotes scalar fields evaluated on a comoving (uniform density) spatial slice, in the same way that ‘$*$’ denotes fields evaluated on spatially flat slices.
Eq. (40) gives each field $s_i$ a dispersion $\sigma^L_i$, approximately satisfying
\[ \sigma^L_i \approx \frac{1}{4\pi M^2 \rho} \sum_j \frac{\partial \phi^c_j}{\partial\phi_{j*}} \frac{\partial \phi^c_i}{\partial\phi_{i*}} \] (no sum on $i$) \hspace{1cm} (41)
together with negligible third moments and cross-correlations. By a suitable choice of origin we can arrange that $N = 0$ on the initial spatially flat slice, and therefore $\bar{N}$ satisfies
\[ \bar{N}_L = \frac{1}{2} N_{ij} \langle \delta \phi_i \delta \phi_j* \rangle_L = \frac{1}{8\pi^2 M^2 \rho} \sum_i M^2 \rho N_{ii} \] \hspace{1cm} (42)
on the initial uniform density slice. Since this is very small it is a reasonable approximation to take $\bar{N}_L \approx 0$. Eq. (39) generates nonzero correlations between $\zeta$ and the $s_i$, allowing us to calculate the covariances $\langle \zeta s_i \rangle_L$. It can also be used to obtain the $\zeta$-dispersion, $\langle \zeta^2 \rangle_L$. We find
\[ \langle \zeta s_i \rangle_L \approx \frac{M^2 \rho H^2}{4\pi^2 M^2 \rho} \sum_i N_{ii} \frac{\partial \phi^c_i}{\partial\phi_{i*}} \] \hspace{1cm} (43)
and
\[ \langle \zeta^2 \rangle_L \approx \frac{1}{4\pi^2 M^2 \rho} \sum_i N_{ii} N_{ii}. \] \hspace{1cm} (44)

**Expressions for velocity field.** With this choice of variables and the assumption that third-order moments are zero initially, the initial conditions, Eqs. (17)–(19), can be used to compute the covariance matrix and third-order moments at any later time, provided expressions can be found for the velocity potential and its derivatives. In the uniform density gauge, the allowed trajectories are integral curves of
\[ u_i = M^2 \rho \frac{ds_i}{dH} \quad \text{and} \quad u_N = M^2 \rho \frac{dN}{dH}. \] \hspace{1cm} (45)
Applying the slow-roll approximation allows us to write the velocity potential as a function of $s_i$ and $H$, and we find
\[ u_i = -\frac{M^2 \rho H^2}{2\epsilon} \] \hspace{1cm} (46)
and
\[ u_N = -\frac{M^2 \rho H^2}{H\epsilon}. \] \hspace{1cm} (47)
In order to write these formulae we have defined a set of partial slow-roll parameters, $\epsilon_i$, which satisfy
\[ \epsilon_i = \frac{1}{2M^2 \rho} \frac{\dot{\phi}^2_i}{H^2}, \] \hspace{1cm} (48)
§ It is clearly possible to extend the approach we have described to calculate the initial third order moments in the uniform density gauge, given the known initial conditions in the flat gauge. This requires the use of $N_{ij}$ and $\partial^2 \phi^c_i/\partial\phi_{j*}$. The assumption that these are zero, however, introduces an error only of the same order as that already present from the typical approximation of taking the initial third moments to be zero in the flat gauge.
where $\dot{\phi} = \partial V / \partial \phi_i / 3H$. To leading order in the slow-roll approximation, $\epsilon = \sum_i \epsilon_i$. Note that both $u_i$ and $u_N$ are independent of $N$, but depend explicitly on the time variable $H$. Therefore, all $N$-derivatives of the velocity field vanish identically. Derivatives of Eqs. (16) and (17) with respect to the $s_i$ can be found after using the Friedmann constraint to eliminate the variations $\partial \phi_M / \partial \phi_i$ and $\partial \phi_M / \partial H$.

### 4.3. Numerical performance of the moment transport method

The imminent arrival of high-quality microwave background data from the Planck satellite implies that it will soon be necessary to obtain accurate estimates of the nonlinearity parameters in a wide range of inflationary models. Although analytic formulas for $f_{NL}$ exist, they are available only for certain forms of the potential and even when their use is possible they rapidly become unwieldy in the limit of a large number of fields. For these reasons we expect numerical methods for computing $f_{NL}$ to become of increasing importance. In $\S 5$ below, we discuss a numerical implementation of the moment transport method using the uniform density slicing. Here, we briefly comment on the computational efficiency of the moment transport algorithm in comparison with alternative approaches.

In a system with $M$ fields, the moment transport method requires a solution of Eqs. (17)–(19). These comprise $M$ unique equations for the centroid, $M(M + 1)/2$ for the covariance matrix and $M(M + 1)(M + 2)/6$ for the 3-point functions. Therefore, for large $M$ we must solve a coupled system of $O(M^3)$ equations. In comparison with popular alternative methods based on the $\delta N$ formula [17, 18, 19, 16], we argue that the transport method is computationally simpler.

To make use of the $\delta N$ formula requires calculation of the derivatives $\partial N / \partial \phi_i$, $\partial^2 N / \partial \phi_i \partial \phi_j$, $\ldots$, and so on. Therefore a direct implementation of the $\delta N$ formula with $M$ fields requires numerical evolution of the background field equations for many initial conditions, from which the necessary derivatives may be extracted.

How many evolutions of the background equations are required? This will determine the computational efficiency. To calculate $f_{NL}$, we require derivatives up to second order but not higher. The first-order derivatives may be obtained by taking finite differences between inflationary trajectories with initial conditions separated by a small distance $\delta$, giving derivatives up to an error of order $\delta^2$. This requires of order $M + 1$ evolutions of the $M$ equations, or the solution to $\sim M^2$ ordinary differential equations. Extending this argument to the second order derivatives shows that to determine $f_{NL}$ we must solve $\sim M^3$ ordinary differential equations. For higher moments the same counting applies, so that an evaluation of the trispectrum will typically require the solution to $\sim M^4$ ordinary differential equations, or more generally $\sim M^n$ for the amplitude of $n$-point correlations.

We conclude that, to compute $f_{NL}$, both the transport method and a direct $\delta N$ require the solution of $O(M^3)$ ordinary differential equations. Asymptotically, their

\[ || \text{In practice, a more accurate discretization scheme may be required.} \]
relative efficiency depends on details of the algorithm. However, the method of moment transport is especially simple. Numerical $\delta N$ requires a discretization scheme to compute the partial derivatives. The final accuracy can depend on our choice of discretization. Also, as we will discuss below, $\delta N$ requires the e-folds of expansion to be determined very accurately on successive time slices. In comparison, the transport method requires only the solution to a set of coupled ordinary differential equations. This allows off-the-shelf differential equation solvers to be brought to bear on the problem immediately.

In a direct $\delta N$ algorithm, very high accuracy is required when evolving the background field equations because at the end of the calculation we must take finite differences to construct the derivatives of $N$. In simulations with up to $M = 5$ fields we have found this process to be sensitive to small numerical inaccuracies; the moment transport algorithm produces accurate results with larger numerical tolerances. This observation can be explained in a simple way. Suppose we wish to compute $\delta N$ to fractional accuracy $f$, where

$$f = \frac{\text{Error}(\delta N)}{\delta N} \quad (49)$$

On the one hand, if we use the naïve $\delta N$ algorithm and compute $N$ using an integration routine which operates at fractional accuracy $f_1$, then the absolute error in $N$ will be $f_1 N$. Therefore, the absolute error in $\delta N$ is also $f_1 N$, and to achieve the target precision (19) we must choose $f_1 \sim f \delta N / N$. On the other hand, the moment transport approach essentially integrates $\delta N$ directly. Using an integration routine with fractional accuracy $f_2$ we will evaluate $\delta N$ with the same fractional accuracy, so $f_2 \sim f$. Since $\delta N / N \ll 1$, we have $f_1 \ll f_2$. We conclude that the moment transport algorithm can operate with much lower numerical tolerances than the naïve $\delta N$ approach.

Finally, we note that direct implementation of $\delta N$ is not the only possibility. Yokoyama, Suyama & Tanaka [20, 21] suggested an approach which is broadly similar to that employed in Ref. [1], reviewed in Ref. [22]. In this approach, one seeks to calculate the field perturbations on a uniform curvature hypersurface at the time of interest, as functions of their initial values, using the $\delta N$ formula to effect the final gauge transformation. One important difference between the formulation of this paper (and Ref. [1]) and that of Yokoyama et al. is that the authors of Refs. [20, 21] continue to work in terms of the field perturbations, rather than evolving the moments of the distribution. We believe this makes our formulation simpler to implement in practice.

5. Numerical examples

In this section we present results obtained from the moment transport method, implemented on a desktop computer. This demonstrates that it is possible to compute $f_{NL}$ in systems with at least $M \sim 10^2$ fields using commodity hardware. The exact time required for a simulation is highly model dependent, but a simple $M = 10$ example may only take a few seconds, while a $M = 100$ calculation can be carried out in less than an hour. Our numerical code is implemented using Matlab, and evolves the uniform
density gauge transport equations, Eqs. (17)–(19). Using this code we are able to obtain numerical solutions for the potential \[ V = \sum_i \frac{1}{2} m_i^2 \phi_i^2. \] (50)

This describes a number of uncoupled fields with quadratic potentials. It is also the small-field approximation to a collection of uncoupled axions, which have trigonometric potentials; this latter case is often known as Nflation. The variables evolved in this gauge and their initial conditions are determined by applying the discussion of §4.2.

In the special case where all \( m_i \) are equal there is an \( O(M) \) symmetry, and the fields roll radially to the origin. Where this symmetry exists we have verified that our code reproduces the expected single-field result of constant \( \zeta \) and negligible \( f_{\text{NL}} \), for up to \( 10^3 \) fields.

Where the \( m_i \) are different, it is known from analytic calculation that this potential does not give rise to a large nongaussianity \[27, 28\]. Our results confirm this conclusion, but the simplicity of the model and the existence of analytic predictions makes it a useful test of our method. In Figs. 1–3 we show illustrative results. Consider first the case of a small number of fields. Rigopoulos, Shellard & van Tent \[29, 30\] analysed a two-field example with mass ratio \( m_1 = 9m_2 \). Vernizzi & Wands later studied the same model using a combination of analytic and numerical methods \[27\] and concluded that \( f_{\text{NL}} \) at the end of inflation was very small, of order \( 10^{-2} \).

We depict the evolution of \( f_{\text{NL}} \) in Fig. 1. For comparison, we plot a calculation of \( f_{\text{NL}} \), for the same model and initial conditions, using a numerical slow-roll implementation of the \( \delta N \) formula, finding complete agreement. In this and subsequent figures the horizontal axis is labelled by \( N \), the number of e-folds which have elapsed since horizon-crossing of the wavenumber of interest. Because our moment-transport and \( \delta N \) calculation are performed using the slow-roll equations of motion, we do not need to allow a ‘lead time’ for the simulation to converge to the slow-roll attractor.

The qualitative behaviour of \( f_{\text{NL}} \) in this model was discussed at the end of §4.2. Our evolution agrees with this discussion, and also the explicit calculations of Vernizzi & Wands \[27\] and the moment transport method using the spatially flat slicing \[1\]. The most prominent feature is a well-documented spike, which occurs when the heavier field reaches the vicinity of its minimum and decouples from the dynamics.

In Fig. 2, we increase the number of fields to 10. For simplicity we take \( \phi_i = 5 \) for all fields and distribute the masses logarithmically, in such a way that \( m_i = 2m_{i-1} \). As successive fields evolve to their minima, a wavelike structure is produced in \( f_{\text{NL}} \). Its value at the end of inflation is again of order \( 10^{-2} \). Kim & Liddle \[26\] argued that, at the end of inflation in a model of the form \(50\), the bispectrum imprinted for a mode of wavenumber \( k \) could be parametrized by \( (6/5)f_{\text{NL}} \approx 1/2N_*.1 \) In this estimate, \( N_* \)

† We have neglected a quantum-mechanical contribution generated by interference among field modes at horizon crossing. This contribution is not determined by the moment transport method, and its contribution is not represented in Figs. 1–3. It is known to be small \[31, 27\] and may be neglected when \( |f_{\text{NL}}| \gtrsim 1 \).
Figure 1. Evolution of $f_{NL}$ (solid red line) calculated using the moment transport equations for a two-field inflation model, working in the quadratic approximation. The green dashed line shows $f_{NL}$ calculated for the same model using the $\delta N$ formula. Initial conditions are described in the text. In this and subsequent figures the horizon axis is labelled by $N$, the number of e-folds which have elapsed since horizon-crossing of the wavenumber of interest.

Figure 2. Evolution of $f_{NL}$ calculated using the moment transport equations for a 10-field inflation model in the quadratic approximation. The initial conditions and distribution of masses are described in the main text.
Figure 3. Evolution of $f_{NL}$ calculated using the moment transport equations for a 100-field inflation model in the quadratic approximation. The initial conditions and distribution of masses are described in the main text.

is the number of e-folds to the end of inflation from the field values at horizon exit of mode $k$, dropping any correction from the end of inflation. Within this approximation, the result is independent of the masses, initial conditions, or number of fields. In this model, we find $N_\ast \approx 63$ yielding a value for $f_{NL}$ in good agreement with the formula of Kim & Liddle.

In order to demonstrate the ability of our algorithm to deal with a large number of fields, we give two examples with $M = 10^2$ fields. First, we retain the potential (50) and set $\phi_i = 1.5M_P$ for each field. We distribute masses such that $m_i = m_{i-1} + 0.1m_1$. The evolution of $f_{NL}$ in this model is shown in Fig. 3. At the end of inflation, $f_{NL}$ is marginally smaller than in the 10-field case. This suppression can be thought of as a consequence of the central limit theorem, which requires that $\zeta$ becomes Gaussian if it receives comparable contributions from a large number of field perturbations with finite variance. To achieve a large nongaussianity as $M$ becomes large, one must arrange that $\zeta$ be dominated by only a few fields [32], as we will discuss below.

These plots show that the horizon-crossing approximation is valid, in these models, to within roughly 5% and 15%, in the 10- and 100-field cases respectively. In the latter case, this discrepancy can likely be ascribed to the relatively large fraction of fields still in motion at the end of inflation. As described above, the typical effect causes $f_{NL}$ to decrease as a consequence of the central limit theorem. As the horizon-crossing approximation begins to fail the value of $f_{NL}$ ceases to be universal and acquires a dependence on the details of the model.
Second, we study $f_{\text{NL}}$ in an inflation model which retains the full trigonometric form of each potential.\footnote{We carry out this calculation in the spatially flat slicing, which validates our $M$-field formulae in this gauge.}

\begin{equation}
V = \sum_i \Lambda_i^4 \left(1 - \cos \frac{2\pi \phi_i}{f_i}\right).
\end{equation}

(51)

It has recently been shown that these models have a phenomenology quite different from Eq. (50) if the initial conditions populate regions in field space close to the maximum of the potential, where the quadratic approximation is poor.\footnote{This illustrates a subtle feature of the ‘horizon-crossing’ approximation, used in Refs. \cite{26,32}. If all trajectories converge to an attractor, then the statistics of the curvature perturbation are determined only by the fields’ values at the time of horizon crossing. In the horizon-crossing approximation, only this contribution is kept. This does not imply that the final $f_{\text{NL}}$ has any relation to its value actually at the time of horizon crossing, and indeed it will typically be quite different. In the present model $f_{\text{NL}} \approx 0$ for almost the whole history of inflation, only achieving its ‘horizon-crossing’ value as the inflationary phase comes to an end.}

We choose the $f_i$ to have a common value, $f$. Where $N$ fields explore the hilltop region and contribute roughly equally to the curvature perturbation, Kim et al.\footnote{\cite{32}} estimate that it is possible to achieve $f_{\text{NL}}$ of order

\begin{equation}
f_{\text{NL}} \approx \frac{5\pi^2}{3N} \left(\frac{M_P}{f}\right)^2.
\end{equation}

(52)

This can easily be of order $1 - 10$ for $f \approx M_P$ and only a single field in the vicinity of the hilltop. Therefore, Eq. (51) constitutes an excellent test that our methods are effective in models for which $f_{\text{NL}}$ is not negligible.

We work with $M = 10^2$ fields as before, and set $f = 5M_P$. We fix initial conditions so that $\phi_i = 1.25M_P$ for all but one field, and assume this remaining field is very close to its potential maximum, with $\phi = 2.49M_P$. Under these conditions Eq. (52) yields $f_{\text{NL}} \approx 0.66$. We plot the evolution of $f_{\text{NL}}$ in this model in Fig. 4. Initially, the field closest to the hilltop is held in place by the large Hubble friction generated by all the other fields. This is the classical assisted inflation mechanism.\footnote{\cite{33}} Our choice of $f_i$ and initial conditions implies an $O(M - 1)$ symmetry among the remaining fields, which roll radially away from the hilltop. In a more general model, the fields furthest from the hilltop would be sequentially ejected into their minima, where they decouple from the dynamics. During this phase $f_{\text{NL}}$ is constant and practically zero.

Eventually, the Hubble friction decreases sufficiently to allow the field closest to the hilltop to roll. While this field is near the maximum of the potential it can support a few e-foldings of inflation, but it is rapidly ejected from the vicinity of the hilltop and accelerated expansion ceases. During this process, $f_{\text{NL}}$ suddenly receives a large contribution from the latent nongaussianity which was imprinted in the fluctuations of this field around the time of horizon crossing.\footnote{This contribution to $f_{\text{NL}}$ is proportional to the curvature, or $\eta$-parameter, of the cosine potential near its hilltop region. As the single hilltop field joins and then decouples from the dynamics, there is a transient spike} This contribution to $f_{\text{NL}}$ is proportional to the curvature, or $\eta$-parameter, of the cosine potential near its hilltop region. As the single hilltop field joins and then decouples from the dynamics, there is a transient spike
where $f_{NL}$ grows rapidly, and oscillates in sign. At the end of inflation, it settles down to a time-independent value $f_{NL} \approx 0.65$, which is an accurate match to the prediction of Kim et al. [32], obtained using the horizon-crossing approximation.

6. Conclusions

In this paper we have refined and extended the “moment transport” method, introduced in Ref. [1] to compute the bispectrum nongaussianity parameter $f_{NL}$ in a two-field model of inflation. The formulation given in this paper contains three significant improvements.

First, the results of Ref. [1] were valid only for a two-field model. We have extended the evolution equations quoted in that paper to an arbitrary number of fields, confirming the conjecture made there that these equations were field-space covariant. This is essential if our formalism is to be applied to models with a large field content, such as Nflation. It is also a practical requirement in many quasi-realistic models of inflation—perhaps deriving from supergravity, string compactifications, or the scalar fields available within the MSSM—which typically contain a more modest number of fields of order $M \sim 10$.

Second, we have shown how to write the evolution equations for the curvature perturbation $\zeta$ directly. As an ancillary benefit this leads to a simple formula for the time evolution of the $f_{NL}$ parameter. Third, discussed in §§3 and Appendix A, we have developed an entirely different method of deriving the moment transport hierarchy, making use of cumulant expansion in the distribution of field values. The original version of the moment transport method made the simplifying assumption that the field
distribution was nearly Gaussian, by expanding the true distribution as a perturbative series around a Gaussian “kernel” distribution. Our new technique works with the probability distribution directly and does not require a Gaussian kernel, providing an alternate derivation of the moment transport system and extending its range of validity.

Both versions of the moment transport method share the advantage that they are simpler to implement numerically, and may have accuracy and performance advantages over a direct $\delta N$ algorithm. Similar properties may be shared by the approach of Yokoyama et al. [20, 21]. This is because they require only the solution of a set of coupled ordinary differential equations. Furthermore, error propagation in the moment transport system can be controlled more straightforwardly, since one evolves the nongaussian moments directly and need not compute differences of large quantities, as in the $\delta N$ framework. The $\zeta$ version of the moment transport procedure presented here also highlights the source terms which are responsible for the growth and decay of the moments of $\zeta$, and of $f_{NL}$. We hope that this will eventually clarify the origin of nongaussianity on a model-by-model basis.

Our method is limited by the requirement that the cumulant expansion remains valid at all intermediate times. We may expect a breakdown when the third-order cumulants become comparable to the variance, so as a conservative estimate we expect the expansion to be valid provided $f_{NL} < 10^4$ throughout the evolution. A similar limitation is shared by any approach which uses the evolution equations of perturbation theory [20, 21], but is avoided by direct numerical $\delta N$.

In our method, there is no obstruction to dropping the inflationary slow-roll assumption, except at horizon crossing where it is needed to fix the initial conditions for the moment transport equations. We have made use of the slow-roll approximation in our numerical computations in Ref. [1] and [5]. Relaxing this assumption would entail additional initial conditions and evolution equations for the field velocities. This can be accomplished quite naturally in the moment transport framework, since the second-order differential equations describing the non-slow-roll dynamics can always be expressed as a larger set of first-order differential equations. This could be achieved by doubling the variables $x_i$ of [2] and identifying the new variables as field velocities. The equations of motion would reduce to a velocity field on this doubled space, and our formalism would go through as before.

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Appendix A. Multiple field evolution equations

The derivation of the evolution equations for multiple fields follows essentially the same steps as the one-field case, but with more complicated combinatorics. We assume we
have $D$ fields, with probability distribution

$$P(x_1, x_2, ... x_D, t) \, dx_1 dx_2 ... dx_D = P(x, t) \, d^Dx$$

We define the mean position of the distribution $X_j$ by

$$X_j = \int x_j P(x, t) \, d^Dx$$

and denote the moments by

$$\mu_{n_1 n_2 ... n_D}(t) = \int (x_1 - X_1(t))^{n_1} (x_2 - X_2(t))^{n_2} ... (x_D - X_D(t))^{n_D} P(x, t) \, d^Dx$$

The rank of a given moment $\mu_{n_1 n_2 ... n_D}(t)$ is $n_1 + n_2 + \cdots + n_D$. The moment generating function $M$ now has $D$ dummy variables $z_j$, and is given by

$$M(z_1, z_2, ... z_D, t) = \int \exp \left[ \sum_{j=1}^{D} z_j (x_j - X_j(t)) \right] P(x, t) \, d^Dx$$

$$= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_D=0}^{\infty} \frac{z_1^{n_1} z_2^{n_2} ... z_D^{n_D}}{n_1! n_2! ... n_D!} \mu_{n_1 n_2 ... n_D}(t)$$

The cumulants are defined by the cumulant generating function

$$C(z_1, z_2, ... z_D, t) = \ln M(z_1, z_2, ... z_D, t)$$

$$= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_D=0}^{\infty} \frac{z_1^{n_1} z_2^{n_2} ... z_D^{n_D}}{n_1! n_2! ... n_D!} \kappa_{n_1 n_2 ... n_D}(t)$$

The cumulants of zeroth, first, second, and third rank are identical to their corresponding cumulants. At higher order, we have for example (when $D = 2$)

$$\mu_{40} = \kappa_{40} + 3\kappa_{20}$$

$$\mu_{31} = \kappa_{31} + 3\kappa_{11}\kappa_{20}$$

$$\mu_{22} = \kappa_{22} + 2\kappa_{11}^2 + \kappa_{02}\kappa_{20}$$

$$\mu_{13} = \kappa_{13} + 3\kappa_{11}\kappa_{02}$$

$$\mu_{04} = \kappa_{04} + 3\kappa_{02}^2$$

The equation for the cumulant time derivatives is then analogous to the one-field case. For example, using the multi-field probability conservation equation

$$\frac{\partial P(x, t)}{\partial t} + \sum_{j=1}^{D} \frac{\partial}{\partial x_j} [u_j(x) P(x, t)] = 0$$

we find

$$\frac{dX_j}{dt} = \int x_j P(x) \, d^Dx = \int u_j P(x) \, d^Dx$$

If we define the velocity field expansion in the natural way by

$$u_j(x) = \sum_{n_1, n_2, ... n_D = 0}^{\infty} \frac{u_j(n_1 n_2 ... n_D)}{n_1! n_2! ... n_D!} (x_1 - X_1)^{n_1} (x_2 - X_2)^{n_2} ... (x_D - X_D)^{n_D}$$

(A.14)
then we find
\[
\frac{dX_j}{dt} = \sum_{n_1, n_2 \ldots n_D=0}^{\infty} \frac{w_{j|n_1 n_2 \ldots n_D} \mu_{n_1 n_2 \ldots n_D}}{n_1!n_2! \ldots n_D!} \tag{A.16}
\]
which is clearly the multi-field analogue of (29).

Just as in the single-field case, we can use (A.16) to derive the equations of motion for the cumulants. As in the single-field case, we have
\[
\sum_{n_1, n_2 \ldots n_D=0}^{\infty} \frac{z_{n_1}^{n_1} z_{n_2}^{n_2} \ldots z_{n_D}^{n_D}}{n_1!n_2! \ldots n_D!} \frac{d\mu_{n_1 n_2 \ldots n_D}}{dt} = \frac{1}{M(z_1, z_2, \ldots z_D, t)} \sum_{n_1, n_2 \ldots n_D=0}^{\infty} \frac{z_{n_1}^{n_1} z_{n_2}^{n_2} \ldots z_{n_D}^{n_D}}{n_1!n_2! \ldots n_D!} \frac{d\mu_{n_1 n_2 \ldots n_D}}{dt} \tag{A.17}
\]
Following a derivation which entirely parallels the single-field case, we arrive at the expression
\[
\frac{d\mu_{n_1 n_2 \ldots n_D}}{dt} = \sum_{n_1, n_2 \ldots n_D=0}^{\infty} \sum_{j=1}^{D} \frac{n_j(A_j + B_j)}{m_1!m_2! \ldots m_D!} u_{j|m_1 m_2 \ldots m_D} \tag{A.19}
\]
where
\[
A_j = \mu_{n_1+m_1 n_2+m_2 \ldots n_j+m_j-1 \ldots n_D+n_D} \tag{A.20}
\]
and
\[
B_j = -\mu_{n_1 n_2 \ldots n_j-1 \ldots n_D} \mu_{m_1 m_2 \ldots m_D} \tag{A.21}
\]
These equations, combined with the expressions for the moments in terms of the cumulants, enables the system of equations for the cumulants to be derived.

The algebra involved in deriving the evolution equations for the cumulants is straightforward, but tedious. Fortunately the required manipulations are entirely mechanical and can be implemented in a computer algebra system, such as \textit{Mathematica}. As an example, if we expand to the third cumulant and to quadratic order in the velocity field, we find
\[
\frac{d\kappa_{20}}{dt} = 2u_{1|01} \kappa_{11} + 2u_{1|10} \kappa_{20} + u_{1|02} \kappa_{12} + 2u_{1|11} \kappa_{21} + u_{1|20} \kappa_{30} \tag{A.22}
\]
\[
\frac{d\kappa_{11}}{dt} = u_{1|01} \kappa_{02} + u_{1|10} \kappa_{11} + u_{2|01} \kappa_{11} + u_{2|10} \kappa_{20} + u_{1|11} \kappa_{12} + \frac{1}{2} u_{1|02} \kappa_{03} + \frac{1}{2} u_{2|02} \kappa_{12} + u_{2|11} \kappa_{21} + \frac{1}{2} u_{1|20} \kappa_{21} + \frac{1}{2} u_{2|20} \kappa_{30} \tag{A.23}
\]
\[
\frac{d\kappa_{02}}{dt} = 2u_{2|01} \kappa_{02} + u_{2|10} \kappa_{11} + u_{2|02} \kappa_{03} + 2u_{2|11} \kappa_{12} + u_{2|20} \kappa_{21} \tag{A.24}
\]
and
\[
\frac{d\kappa_{30}}{dt} = 3u_{1|02} \kappa_{11} + 6u_{1|11} \kappa_{11} \kappa_{20} + 3u_{1|20} \kappa_{20} + 3u_{1|01} \kappa_{21} + 3u_{1|10} \kappa_{30} \tag{A.25}
\]
\[
\frac{d\kappa_{21}}{dt} = 2u_{1|02} \kappa_{02} \kappa_{11} + 2u_{1|11} \kappa_{11}^2 + u_{2|02} \kappa_{11}^2 + 2u_{1|01} \kappa_{12} + 2u_{1|11} \kappa_{02} \kappa_{20} + 2u_{1|11} \kappa_{11} \kappa_{20} + u_{2|20} \kappa_{20}^2 + 2u_{1|10} \kappa_{21}
\]
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\[ \frac{d\kappa_{12}}{dt} = u_{1|02}\kappa_{02}^2 + u_{1|01}\kappa_{03} + 2u_{1|11}\kappa_{02}\kappa_{11} + 2u_{2|02}\kappa_{02}\kappa_{11} + u_{1|20}\kappa_{11}^2 + 2u_{2|11}\kappa_{11}\kappa_{11} + 2u_{2|12}\kappa_{12} \quad (A.26) \]

\[ \frac{d\kappa_{03}}{dt} = 3u_{2|02}\kappa_{02}^2 + 3u_{2|01}\kappa_{03} + 6u_{2|11}\kappa_{02}\kappa_{11} + 3u_{2|20}\kappa_{11}^2 + 3u_{2|10}\kappa_{12} \quad (A.27) \]

These equations can be generalized to any number of fields, or any order in the cumulant expansion, by using the expressions given above.

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