Perturbative QCD and Tau Decay

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Sufficiently inclusive observables in the decay of the tau lepton can be calculated using the methods of perturbative QCD. These include the asymmetry parameter $A_\tau$ that determines the angular distribution of the total hadron momentum in the decay of a polarized tau. It should be possible to measure $A_\tau$ accurately using existing data from LEP. Reliable estimates of theoretical errors are essential in order to determine whether a given observable is sufficiently inclusive to be calculated using perturbative methods. The theoretical uncertainties due to higher orders in $\alpha_s$ can be estimated using recent calculations to all orders in the large-$\frac{33-2N_f}{N_f}$ limit. These estimates indicate that tau decay data can be used to determine $\alpha_s(M_Z)$ to a precision of 2% or better.

1. INTRODUCTION

What can perturbative QCD tell us about the decays of the tau lepton? If you ask this question to the typical man on the street, he will answer “Absolutely nothing!” Perturbative QCD tells us about the interactions of quarks and gluons with large momentum transfer. But the decay of the $\tau$ is dominated by decays into single particles and resonances: $\pi$, $\rho$, $a_1$, etc. The QCD interactions that bind quarks and gluons into these hadrons necessarily involve small momentum transfers and are therefore completely outside the domain of perturbation theory. Therefore one should not expect perturbative QCD to tell us anything about the decays of the $\tau$.

But suppose we ignore this objection, and proceed to calculate the decay rate of the tau into a neutrino plus quarks and gluons using perturbative QCD. The resulting expression for the decay rate is an expansion in powers of $\alpha_s$. Using the measured value of the decay rate, we can determine $\alpha_s$. We find that this value for the QCD coupling constant agrees amazingly well with the best determinations of $\alpha_s$. Is this just a remarkable coincidence? Or is it possible that sufficiently inclusive observables in tau decay really can be calculated using the methods of perturbative QCD?

2. QCD PREDICTION FOR $R_\tau$

The QCD prediction for the ratio $R_\tau$ of the hadronic and electronic branching fractions of the tau has the form

$$R_\tau = 3 \left[ |V_{ud}|^2 + |V_{us}|^2 \right] S_{EW} \times \left\{ 1 + \delta_{EW} + \delta_{pert} + \delta_{power} \right\}.$$  (1)

The electroweak corrections include a multiplicative factor $\delta_{EW} = 1.0194$ and a small additive correction $\delta_{EW} = 0.0010$ [2]. There is a perturbative QCD correction $\delta_{pert}$ that can be expressed as an expansion in powers of $\alpha_s$ at the scale $M_\tau$ (or any other scale $\mu$ of your choosing) [3]:

$$\delta_{pert} = \frac{\alpha_s(M_\tau)}{\pi} + 5.2 \left( \frac{\alpha_s(M_\tau)}{\pi} \right) \frac{m^2}{M^2} + 26.4 \left( \frac{\alpha_s(M_\tau)}{\pi} \right)^2 \frac{\langle \bar{\psi} \Gamma \psi \rangle}{M^4} + \ldots.$$  (2)

The remaining QCD corrections are the power corrections $\delta_{power}$, the most important of which are the following [3]:

$$\delta_{power} = -10 \frac{m^2}{M^2} + 32\pi^2 \frac{\langle \bar{m}_W^2 \rangle}{M^2}$$

$$+ \frac{11\pi^2}{4} \left( \frac{\alpha_s(M_\tau)}{\pi} \right)^2 \frac{\langle \bar{\psi} \Gamma \psi \rangle}{M^2}$$

$$+ \text{terms of the form } \frac{\langle \bar{\psi} \Gamma \psi \rangle}{M^2},$$  (3)

where $m^2$ is a weighted average of the running quark masses $m_u^2$, $m_d^2$, and $m_s^2$ evaluated at
the scale $M_{\tau}$ with weights $1/2$, $|V_{ud}|^2/2$, and $|V_{us}|^2/2$, respectively. Similarly, $\langle m\bar{\psi}\psi \rangle$ is the same weighted average of the quark condensate matrix elements $\langle m_u\bar{u}u \rangle$, $\langle m_d\bar{d}d \rangle$, and $\langle m_s\bar{s}s \rangle$. The matrix element $\langle 2\pi GG \rangle$ is called the gluon condensate.

The QCD prediction (1) is based on the fact that the inclusive hadronic decay rate of the tau involves only 2 momentum scales: the mass $M_{\tau}$ and $\Lambda_{QCD}$, the scale associated with nonperturbative effects in QCD. One can imagine increasing $M_{\tau}$ while holding the QCD coupling constant, and therefore $\Lambda_{QCD}$, fixed. It is possible to systematically separate the effects of “hard” partons, whose momenta scale with $M_{\tau}$, from the effects of “soft” partons, whose momenta remain proportional to $\Lambda_{QCD}$ as $M_{\tau}$ increases. At very short distances of order $1/M_{W}$, the decay of the $\tau^-$ proceeds through the decay into a neutrino plus $d\bar{u}$ (or $s\bar{u}$). The QCD corrections involve both hard partons and soft partons. The perturbative correction $\delta_{\text{pert}}$ consists of corrections from the emission of hard partons and from the exchange of hard partons. It can be calculated as a perturbation series in $\alpha_s(M_{\tau})$. The first 3 terms in this series are known and they are given in (2).

What about the effects of soft partons? There are large corrections from the emission of soft gluons and large corrections from the exchange of soft gluons, but they cancel order-by-order in $\alpha_s$. In other words, the dominant effects of the soft gluons can be expressed as a unitary transformation that does not change the decay rate. At the perturbative level, this unitary transformation describes the evolution of an initial state consisting of hard partons emerging from the decay into final states consisting of hard partons and soft partons. At the nonperturbative level, the soft gluons have dramatic effects, binding the hard partons into final state hadrons. Nevertheless, the dominant effect of the soft gluons can still be expressed as a unitary transformation which evolves an initial state consisting of hard partons into final states consisting of hadrons. Since a unitary transformation preserves probabilities, it does not change the decay rate.

Now the effects of the soft partons are not exactly a unitary transformation. There are corrections which can be taken into account through power corrections $\delta_{\text{power}}$, like those given in (3). The term involving the gluon condensate $\langle 2\pi GG \rangle$ takes into account effects of soft gluons that can not be expressed as a unitary transformation. The perturbative corrections $\langle m\bar{\psi}\psi \rangle$ take into account corrections from soft quarks and antiquarks that are suppressed by the quark mass. The matrix elements of the form $\langle \psi\Gamma\bar{\psi}\Gamma\psi \rangle$ take into account corrections from soft quarks and antiquarks that would be present even if the quarks were massless.

The QCD predictions for all inclusive observables in tau decay have the same general form as the predictions for $R_{\tau}$ in (1). There is a free-quark prediction that corresponds to the decay into a neutrino plus $d\bar{u}$ or $s\bar{u}$. There are perturbative corrections that can be expressed as an expansion in $\alpha_s(M_{\tau})$ like that in (2). Finally, there are power corrections like those given in (3). They include corrections from running quark masses and soft parton corrections that are expressed in terms of vacuum matrix elements.

3. OBSERVABLES

Below, I enumerate the observables in tau decay that can be calculated using the methods of perturbative QCD.

3.1. Inclusive decay rate

The QCD prediction for the inclusive hadronic decay rate normalized to the electronic decay is given in (1). The ratio $R_{\tau}$ can be further resolved into the contributions from the weak vector current and the weak axial-vector current. It can also be resolved into “non-strange” contributions proportional to $|V_{ud}|^2$ and “strange” contributions proportional to $|V_{us}|^2$.

3.2. Invariant-mass distribution

Let $s$ be the square of the invariant mass of the hadrons in the decay of the $\tau$. The distribution $dR_{\tau}/ds$, or more accurately the moments of this distribution, can be calculated using perturbative methods. The $0^{th}$ moment is simply $R_{\tau}$ itself. The QCD predictions for the moments are given in Ref. [6]. These moments have been used to measure the matrix elements that appear in the power corrections (3).
3.3. Angular distribution

There is one inclusive observable in tau decay that has not yet been measured experimentally. This is a parameter $A_\tau$ that determines the angular distribution of the total hadron momentum in the decay of a polarized tau. Let $\theta$ be the angle in the tau rest frame between the total hadron momentum and the quantization axis for the spin of the $\tau$. If the tau is unpolarized, the angular distribution is uniform in $\cos \theta$. If the tau has polarization $P$, the angular distribution is

$$\frac{dR_\tau}{d\cos \theta} = \frac{1}{2} R_\tau (1 + A_\tau P \cos \theta). \quad (4)$$

The size of the asymmetry in $\cos \theta$ is determined by the parameter $A_\tau$. This parameter is known for various exclusive decay modes. It has the value $+1$ if the hadronic final state is a single $\pi$. The value is close to $-1$ for the transverse polarization modes of the $\rho$ and $a_1$, and close to $+1$ for the longitudinal polarization mode. The inclusive asymmetry parameter $A_\tau$ is the weighted average over all hadronic final states of the asymmetry parameter for each of the exclusive decay modes.

The asymmetry parameter $A_\tau$ can be calculated using the methods of perturbative QCD with the same degree of rigor as $R_\tau$ itself [3]. The free-quark prediction for $A_\tau$ is $\frac{1}{3}$. The QCD corrections increase the prediction to $A_\tau = 0.415 \pm 0.022$. Thus the contributions of the $\pi$, $\rho$, $a_1$ and all the multihadronic modes must conspire to give a value that is about 25% larger than the free quark value.

An “experimental” value for $A_\tau$ has recently been obtained by Weinstein [3]. He generated 50,000 polarized tau decay events by running the Monte Carlo programs KORALB and TAUOLA and obtained the value $A_\tau = 0.385 \pm 0.007$. These programs use the known branching fractions for the various hadronic decay modes. They incorporate the correct asymmetry parameters for the $\pi$, $\rho$, and $a_1$ modes, but they are not tuned to give the correct asymmetries for the multihadronic modes. Thus a real experimental measurement of $A_\tau$ is needed in order to test the QCD prediction.

A good way to measure $A_\tau$ is to use the taus from $Z^0$ decay, because they are naturally polarized with polarization $P = -0.14$. Suppose one could assemble an unbiased sample of taus from $Z^0$ decay. If, for each tau decay, one measured the invariant mass $s$ of the hadrons and their total energy $E$ in the $Z^0$ rest frame, then $A_\tau$ could be determined by calculating the following average over the $\tau$ sample:

$$\langle \frac{1 - 2E/M_Z}{1 - s/M_Z^2} \rangle = \frac{1}{2} \left( 1 - \frac{1}{3} A_\tau P \right). \quad (5)$$

From simple statistics, one would need a sample of at least 2000 taus to distinguish the free quark prediction 0.508 for (5), which corresponds to $A_\tau = \frac{1}{3}$, from the unpolarized value of 0.5. One would need at least 30,000 taus to distinguish the QCD prediction 0.5097 $\pm$ 0.0005 from the free quark prediction. With a sample of 400,000 taus from $Z^0$ decay, it should be possible to measure $A_\tau$ rather accurately. Note that the prediction that the asymmetry parameter for all the exclusive modes must average out to a value near the free quark value $\frac{1}{3}$ is already rather remarkable and thus even a crude measurement of $A_\tau$ would be useful. It would be the first confirmation of a perturbative QCD prediction for spin-dependent observables in tau decay. An accurate measurement of $A_\tau$ in agreement with the QCD prediction would provide dramatic evidence that inclusive observables can be accurately calculated using the methods of perturbative QCD. It should lay to rest any questions of whether the accuracy of the value of $\alpha_s$ obtained from $\tau$ decay is merely fortuitous.

4. ERROR ESTIMATES

In order to make quantitative tests of the QCD predictions for the observables described in the last section, it is essential to have reliable estimates of the theoretical errors. In the proceedings of TAU 94, both Altarelli and Narison made attempts to estimate the theoretical errors in the determination of $\alpha_s(M_Z)$ from $\tau$ decay. Their estimates are given in Table 1. The bottom line is that Altarelli’s estimate of the theoretical error is larger than Narison’s by a factor of 3. This fac-
Table 1
Estimates of the theoretical errors (in units of $10^{-4}$) in the value of $\alpha_s(M_Z)$ determined from tau decay.

|                  | Altarelli | Narison |
|------------------|-----------|---------|
| running: $M_\tau \to M_z$ | 20        | 10      |
| freezing of $\alpha_s$ | 10        | –       |
| quark masses      | –         | 5       |
| condensates       | –         | 9       |
| higher orders in $\alpha_s$ | 65        | 14      |
| $\mu$ dependence  | –         | 9       |
| renormalization scheme | –       | 5       |
| "other theoretical errors" | 10       | –       |
| **Total Error**   | 70        | 23      |

4.1. Renormalization scale $\mu$

The ratio $R_\tau$, if calculated to all orders in $\alpha_s$, would be independent of the renormalization scale $\mu$. However, if the perturbation series is truncated at some order in $\alpha_s$, it will depend on $\mu$. The value of $\alpha_s$ determined by measuring $R_\tau$ will therefore depend on the choice for $\mu$.

This can be illustrated by considering the perturbation series truncated after the order $-\alpha_s$ term. If we choose the renormalization scale to be $\mu = M_\tau$, the prediction for $R_\tau$ (ignoring electroweak and power corrections and setting $|V_{ud}|^2 + |V_{us}|^2 = 1$ for simplicity) is

$$R_\tau = 3 \left( 1 + \frac{\alpha_s(M_\tau)}{\pi} \right).$$

If we choose the scale to be some fraction $x$ of the tau mass $\mu = xM_\tau$, then the prediction is

$$R_\tau = 3 \left( 1 + \frac{\alpha_s(xM_\tau)}{\pi} \right).$$

Having determined $\alpha_s(xM_\tau)$ from the measured value of $R_\tau$, we can then determine $\alpha_s(M_\tau)$ by using the renormalization group:

$$\alpha_s(M_\tau) = \frac{\alpha_s(xM_\tau)}{1 + \frac{9}{2}\alpha_s(xM_\tau) \log(1/x)}.$$  \hspace{1cm} (8)

The resulting numerical value of $\alpha_s(M_\tau)$ will differ from that obtained directly from (6). The difference between (6) and the combination of (7) and (8) amounts to summing certain terms of the form $\alpha^n_s \log^n(M_\tau/\mu)$ to all orders in $n$.

The dependence on the renormalization scale $\mu$ decreases if the perturbation series is calculated to higher order in $\alpha_s$. However, as long as the perturbation series is truncated, there will always be some dependence on $\mu$.

4.2. Truncation of the perturbation series

In (2), we have truncated the perturbation expansion for $R_\tau$ in powers of $\alpha_s(M_\tau)$, but this is not the only way to truncate the perturbation series. The ratio $R_\tau$ can be expressed in the form of a contour integral

$$R_\tau = \frac{12\pi^2}{2\pi i} \int_{|t|=M_\tau^2} \frac{dt}{t} \times \left( 1 - 2 \frac{t}{M_\tau^2} + 2 \frac{t^3}{M_\tau^6} - \frac{t^4}{M_\tau^8} \right) D(\alpha_s(-t)),$$

where the function $D(\alpha_s)$ has a perturbation expansion in powers of $\alpha_s$:

$$D(\alpha_s) = \frac{1}{4\pi^2} \left( 1 + \frac{\alpha_s}{\pi} + 1.6 \left( \frac{\alpha_s}{\pi} \right)^2 + 6.4 \left( \frac{\alpha_s}{\pi} \right)^3 \right).$$ \hspace{1cm} (10)
The contour integral in (9) runs counterclockwise around the circle |t| = M_{\tau}^2, beginning at t = M_{\tau}^2 + i\epsilon and ending at t = M_{\tau}^2 - i\epsilon. The function α_s(-t) in (9) is the analytic continuation of the running coupling constant to the complex t-plane. It is defined by the boundary condition α_s(-t) = α_s(M_{\tau}^2) at t = -M_{\tau}^2 and by the differential equation t(d/dt)α_s(-t) = β(α_s(-t)), where β(α_s) is the beta function for QCD: β(α_s) = -(9/4\pi)α_s^2 + \ldots.

One way to truncate the perturbation series is to first carry out the contour integral and then truncate the expansion for R_\tau. This gives the usual perturbation series for R_\tau given in (2). An alternative possibility is to truncate the expansion for D(α_s) and then carry out the contour integral analytically. This has been advocated in particular by deLiberder and Pich [3]. The resulting expression for R_\tau is

\[ R_\tau = 3 \left\{ 1 + \frac{α_s(-t)}{π} + 1.6 \left( \frac{α_s(-t)}{π} \right)^2 \right. \]
\[ \left. + 6.4 \left( \frac{α_s(-t)}{π} \right)^3 \right\}, \quad (11) \]

where the coupling constant with the bar over it represents the following weighted average of its values along the contour in the complex plane:

\[ \bar{α}_s^n(-t) = \frac{1}{2\pi} \int_0^{2\pi} dθ \left( 1 - 2e^{iθ} + 2e^{3iθ} - e^{4iθ} \right) \times α_s^n(-M_{\tau}^2e^{iθ}). \quad (12) \]

The effect of this prescription for calculating R_\tau is more easily illustrated using the simpler example of the ratio R_{e^+e^-}. It can also be expressed as a contour integral involving the same function D(α_s) as in (9):

\[ R_{e^+e^-}(\sqrt{s}) = 12π^2 \frac{1}{2\pi i} \int_{|t|=s} \frac{dt}{t} D(α_s(-t)). \quad (13) \]

We choose the center-of-mass energy √s of the e^+e^- to be below charm threshold, so that the number of light quark flavors is 3, just as in tau decay. If we integrate and then truncate R_{e^+e^-}, then at one loop, we obtain

\[ R_{e^+e^-}(\sqrt{s}) = 3 \left\{ 1 + \frac{α_s(\sqrt{s})}{π} \right\}. \quad (14) \]

If we truncate D(α_s) and then integrate, we obtain

\[ R_{e^+e^-}(\sqrt{s}) = 3 \left\{ 1 + \frac{α_s(-t)}{π} \right\}, \quad (15) \]

where \( α_s(-t) = α_s(-se^{iθ}) \) averaged over the angles θ. Expressed in terms of \( α_s(\sqrt{s}) \), this average is \( \frac{4}{9} \arctan(\sqrt{s}/α_s(\sqrt{s})) \). Its expansion in powers of \( α_s(\sqrt{s}) \) is

\[ α_s(-t) = α_s(\sqrt{s}) \left\{ 1 - \frac{27π^2}{16} \left( \frac{α_s(\sqrt{s})}{π} \right)^2 \right. \]
\[ + \frac{6561π^4}{1280} \left( \frac{α_s(\sqrt{s})}{π} \right)^4 + \ldots \}. \quad (16) \]

Thus the expression (15) for R_{e^+e^-} differs from (14) by the resummation of terms of the form \( π^{2n}(α_s/π)^{2n} \) to all orders in n.

Similarly, the perturbative expansion (11) for R_\tau differs from the one given in (2) by the resummation of terms of the form \( π^{2n}(α_s/π)^{2n} \) to all orders in n. DeLiberder and Pich give a number of arguments why this resummation should be preferred, but I don’t find them convincing. One argument is that it greatly decreases the dependence of R_\tau on the renormalization scale µ. This is true, but there is an equally convincing argument against the resummation, and that is that the power series expansion for D(α_s) is more severely divergent than that for R(α_s).

4.3. Divergence of the perturbation series

It has been known for a long time that most perturbation expansions in field theory, including QED and QCD, are actually divergent series. Suppose you were able to calculate the correction of order \( α_s^n \) for any value of n. For small values of n, you might find that adding more and more correction terms gives a better and better approximation. But if you continued adding higher and higher orders in α_s, you would eventually find that the approximation would get worse and worse. The order in α_s at which the series begins to diverge depends on the value of α_s, decreasing roughly as 1/α_s. Thus the divergence of the perturbation series is a much more important issue for tau decay than it is for applications of perturbative QCD at higher energy, where the running coupling constant is smaller.
There has been a dramatic development in the last five years that has made the problem of the divergence of the perturbation series much more concrete. A subset of the Feynman diagrams for $R_\tau$ and $R_{\tau+e^-}$ have been calculated to all orders in $\alpha_s$ \cite{3}. The diagrams are those of order $\alpha_s^n$ with the maximum number $n - 1$ of quark loops. The complete perturbation series for $R_\tau$ has the form

$$R_\tau = 3 \left\{ 1 + \frac{\alpha_s(M_\tau)}{\pi} + \sum_{n=2}^{\infty} r_n b^n(M_\tau) \right\}.$$

The coefficients $r_n$ are polynomials in the number $N_f$ of light quark flavors, or equivalently, in $33 - 2N_f$:

$$r_n = r_n^{(0)} + r_n^{(1)}(33 - 2N_f) + \ldots + r_n^{(n-1)}(33 - 2N_f)^{n-1}.$$  \hspace{1cm} (18)

It is the numbers $r_n^{(n-1)}$ that have been computed to all orders in $n$.

Given the coefficients $r_n^{(n-1)}$ defined in (18), we can calculate a subset of the higher order corrections to $R_\tau$ to all orders in $\alpha_s$. I will denote the resulting expression by $\hat{R}_\tau$ and refer to it as the large -- $(33 - 2N_f)$ limit:

$$\hat{R}_\tau = 3 \left\{ 1 + \frac{\alpha_s(M_\tau)}{\pi} + \sum_{n=2}^{\infty} \hat{r}_n b^n(M_\tau) \right\},$$  \hspace{1cm} (19)

where $\hat{r}_n = r_n^{(n-1)}(33 - 2N_f)^{n-1}$. The first few terms in the expansion for $\hat{R}_\tau$ are

$$\hat{R}_\tau = 3 \left\{ 1 + \frac{\alpha_s(M_\tau)}{\pi} + 5.1 \left( \frac{\alpha_s(M_\tau)}{\pi} \right)^2 \right. \right.$$  

$$\left. + 28.8 \left( \frac{\alpha_s(M_\tau)}{\pi} \right)^3 + 156.7 \left( \frac{\alpha_s(M_\tau)}{\pi} \right)^4 \right.$$  

$$\left. + 900.8 \left( \frac{\alpha_s(M_\tau)}{\pi} \right)^5 + \ldots \right\}.$$  \hspace{1cm} (20)

This expansion can be compared with the exact expansion for the corrections to $R_\tau$, which is given in (2). Note the remarkable agreement between the coefficients of $\alpha_s^2$ and $\alpha_s^3$ in the two expansions. This gives us some confidence that the large -- $(33 - 2N_f)$ limit can predict correctly the sign and order of magnitude of the coefficient of the $\alpha_s^3$ term.

Since we know the expansion (19) for $R_\tau$ in the large -- $(33 - 2N_f)$ limit to all orders in $\alpha_s$, we can use it to study the behavior of the perturbation expansion \cite{3}. One finds that (19) is a divergent series with coefficients $\hat{r}_n$ that grow asymptotically like $n!$:

$$\hat{r}_n \to \frac{2}{15\pi} e^{-5\frac{3}{9}} \left( -\frac{9}{4\pi} \right)^n n!.$$  \hspace{1cm} (21)

Thus the sum in (19) if taken literally is meaningless. However, there is a standard procedure for recovering an analytic function from the divergent power series generated by its Taylor expansion. It is called Borel resummation. From the series (19) for $R_\tau$, one constructs the Borel transform $\hat{B}(b)$ by dividing each coefficient by $(n - 1)!$:

$$\hat{B}(b) = \sum_{n=1}^{\infty} \frac{\hat{r}_n}{(n - 1)!} b^{n-1}.$$  \hspace{1cm} (22)

The power series for $\hat{B}(b)$ is less divergent than that for $R_\tau$, and actually converges for $|b| < 4\pi/9$.

An analytic expression for $\hat{B}(b)$ is known. It has poles at integer multiples of $4\pi/9$. There are poles on the negative real axis at $b = n(4\pi/9)$, $n = -1, -2, \ldots$ that are called ultraviolet renormalons. There are also poles on the positive real axis at $b = n(4\pi/9)$, $n = 2, 3, \ldots$ that are called infrared renormalons. Given the analytic expression for the function $\hat{B}(b)$ defined by (22), we can recover the desired function $R_\tau$ by computing the inverse Borel transform:

$$\hat{R}_\tau = 3 \left\{ 1 + \int_0^{\infty} db \exp \left( \frac{-b}{\alpha_s(M_\tau)} \right) \hat{B}(b) \right\}. \hspace{1cm} (23)$$

The infrared renormalon poles on the integration contour can be handled using a principal value prescription.

Now the divergence of the perturbation series for $R_\tau$ is dominated by the singularity of $\hat{B}(b)$ that is closest to the origin. This is the first ultraviolet renormalon, which is a pole at $b = -4\pi/9$. The behavior of $\hat{B}(b)$ near this pole is

$$\hat{B}(b) \to \frac{2}{15\pi} e^{-5\frac{3}{9}} \frac{1}{1 + \frac{b}{4\pi}}.$$  \hspace{1cm} (24)

The remainder is analytic inside the circle $|b| < 8\pi/9$. The pole term can be expanded as a power
series in $b$, but the series has a radius of convergence of only $4\pi/9$. Thus perturbation theory gives a convergent approximation to the function $B(b)$ in the integrand of (23) only in the interval $0 < b < 4\pi/9$. Outside that interval, the pole term given in (24) cannot be approximated by a truncated perturbation series. The integral of the pole term from $4\pi/9$ to infinity therefore gives a “renormalon correction” to $R_\tau/3$:

$$\delta_{\text{ren}} = \frac{2}{15\pi} e^{-5/3} \int_{4\pi/9}^{\infty} \frac{db}{db} \exp \left( -\frac{b}{\alpha_s(M_\tau)} \right) \times \frac{1}{1 + \frac{3}{4\pi} b}$$

The asymptotic value of the integral for small values of $\alpha_s(M_\tau)$ is

$$\delta_{\text{ren}} = \frac{1}{15} e^{-5/3} \frac{\alpha_s(M_\tau)}{\pi} \exp \left( -\frac{4\pi}{9\alpha_s(M_\tau)} \right).$$

Finally, using the expression $2\pi/(9 \log(M_\tau/\Lambda))$ for the running coupling constant $\alpha_s(M_\tau)$, the leading renormalon correction reduces to

$$\delta_{\text{ren}} = \frac{1}{15} e^{-5/3} \frac{\alpha_s(M_\tau)}{\pi} \frac{\Lambda^2}{M_\tau^2},$$

where $\Lambda$ is the renormalization group invariant scale parameter in the $\overline{MS}$ scheme.

The analysis above was carried out for the standard truncation of the perturbation series for $R_\tau$. With deLiberder-Pich resummation, one must apply Borel resummation to the function $D(\alpha_s)$ in the integrand of (9). The Borel transform of this function has a double pole at $b = -4\pi/9$. The renormalon correction analogous to (25) therefore has two terms corresponding to the double pole and a single pole. After integrating over $t$, it reduces to

$$\delta_{\text{ren}} = \frac{2}{9\pi} e^{-5/3} \int_{4\pi/9}^{\infty} \frac{db}{db} \exp \left( -\frac{b}{\alpha_s(M_\tau)} \right) \times \left[ \frac{2}{(1 + \frac{3}{4\pi} b)^2} + \frac{5}{1 + \frac{3}{4\pi} b} \right] \times \frac{12 \sin(\frac{\pi}{b})}{b(1 - \frac{9}{4\pi} b)(3 - \frac{9}{4\pi} b)(4 - \frac{9}{4\pi} b)}. \quad (28)$$

The asymptotic value of the integral for small values of $\alpha_s(M_\tau)$ is

$$\delta_{\text{ren}} = \frac{4}{3} e^{-5/3} \frac{\alpha_s(M_\tau)}{\pi} \frac{\Lambda^2}{M_\tau^2}. \quad (29)$$

Note that this is larger by a factor of 20 than the corresponding correction (27) for the standard truncation. This simply reflects the fact that the perturbation series for $D(\alpha_s)$ diverges more severely than that for $R_\tau$.

4.4. Estimates of the perturbative error

We are now in a position to assess the error estimates of Altarelli and Narison. The perturbation series for $R_\tau$ can be written

$$R_\tau = 3 \left\{ 1 + \sum_{n=1}^{3} r_n \alpha_s^n(M_\tau) + 4 \alpha_s^4(M_\tau) + \sum_{n=5}^{3} r_n \alpha_s^n(M_\tau) \right\}. \quad (30)$$

The first 3 correction terms in the expansion are given in (2). We have separated the unknown $\alpha_s^4$ term from the sum of all the higher-order corrections.

Narison’s error estimate is based on the assumption that the error from the $\alpha_s^4$ term in (30) dominates over that from the sum of all higher orders. He used the values of the lower-order coefficients $r_1$, $r_2$, and $r_3$ to guess a reasonable range for the values of $r_4$. The resulting estimate of the error from higher orders is

$$\left( \Delta_{\text{pert}} \right)_{\text{Narison}} = \pm 50 \left( \frac{\alpha_s(M_\tau)}{\pi} \right)^4. \quad (31)$$

Translating this into an error on $\alpha_s(M_Z)$, we obtain the value $\pm 14 \times 10^{-4}$ given in Table 1.

Altarelli’s estimate was based on the assumption that the error from the sum of all higher order terms in (30) dominates over that from the $\alpha_s^4$ term. As shown in the previous subsection, there is a renormalon contribution to that sum that is proportional to $\Lambda^2/M_\tau^2$. Altarelli’s error estimate is

$$\left( \Delta_{\text{pert}} \right)_{\text{Altarelli}} = \pm \frac{1}{4} \frac{\Lambda^2}{M_\tau^2}. \quad (32)$$

Translating this into an error on $\alpha_s(M_Z)$, we obtain the value $\pm 65 \times 10^{-4}$ given in Table 1. There is no apparent calculation underlying the coefficient 1/4 in (32). It seems to be pure guesswork on the part of Altarelli.
Table 2
Estimates of the theoretical error in $\alpha_s(M_\tau)$ (in units of $10^{-4}$) from higher orders in $\alpha_s(M_\tau)$.

| Error Source                                      | Error from $n = 4$ term | Error from sum of $n \geq 5$ terms |
|---------------------------------------------------|-------------------------|-------------------------------------|
| large–$(33 - 2N_f)$ limit (truncation in $\alpha_s(M_\tau)$) | 40                      | 0.2                                 |
| large–$(33 - 2N_f)$ limit (deLiberder-Pich resummation) | 4                       | 4                                   |
| Altarelli                                         | –                       | 65                                  |
| Narison                                           | 14                      | –                                   |

One of the problems with both of these error estimates is that they require guessing the magnitude of unknown coefficients: $r_4$ in the case of Narison, the coefficient of $\Lambda^2/M_\tau^2$ from the renormalon correction in the case of Altarelli. One way to avoid such guesswork is to use the explicit calculations to all orders in the large–$(33 - 2N_f)$ to estimate the errors. As an estimate of the error from the $\alpha_s^4$ term in (30), we can use the magnitude of the $\alpha_s^4$ term in the large–$(33 - 2N_f)$ limit:

$$\Delta_{\text{pert}} = \pm |\hat{r}_4| \alpha_s^4(M_\tau).$$

(33)

As an estimate of the error from the sum of all higher order terms in (30), we can use the magnitude of the sum of all higher order terms in the large–$(33 - 2N_f)$ limit:

$$\Delta_{\text{pert}} = \pm \left| \sum_{n=5}^{\infty} \hat{r}_n \alpha_s^n(M_\tau) \right|.$$  

(34)

The divergent series on the right side of (34) is defined by Borel resummation. A simpler estimate, which can be compared directly with that of Altarelli, is to pick out the leading renormalon correction, which is given in (27) for the standard truncation in $\alpha_s(M_\tau)$ and in (29) for deLiberder-Pich resummation.

The error estimates based on the large–$(33 - 2N_f)$ limit are given in Table 2. The standard truncation gives a much larger error from the $\alpha_s^4$ term and a much smaller error from the sum of all higher-order terms. With deLiberder-Pich resummation, there is a better balance between these two sources of uncertainty. Adding the two errors, we take the total error in $\alpha_s(M_Z)$ from higher orders in $\alpha_s$ to be $8 \times 10^{-4}$. The error estimates of Altarelli and Narison are also given in Table 2 for comparison. The error estimate from the large–$(33 - 2N_f)$ limit is consistent with that of Narison, but much smaller than that of Altarelli. The explicit calculations in the large–$(33 - 2N_f)$ limit indicate that the coefficient of the $\Lambda^2/M_\tau^2$ correction arising from the first ultraviolet renormalon is an order of magnitude smaller than assumed by Altarelli.

5. CONCLUSIONS

Sufficiently inclusive observables in tau decay can be calculated using the methods of perturbative QCD. A particularly interesting observable that has yet to be measured is the asymmetry parameter $A_\tau$ that describes the angular distribution of the total hadron momentum in the decay of a polarized tau. Using existing data from LEP, it should be possible to measure this parameter with sufficient precision to discriminate between the free-quark prediction of $A_\tau = 0.013$ and the QCD prediction $A_\tau = 0.41 \pm 0.02$. Such a measurement would be the first test of QCD predictions for spin-dependent observables in tau decay and it would demonstrate conclusively that methods based on perturbative QCD give accurate predictions for sufficiently inclusive observables.

In order to determine whether a given observable in tau decay is sufficiently inclusive to calculate using perturbative methods, it is essential to have reliable estimates of the theoretical errors. One can use recent calculations to all orders in $\alpha_s$ in the large–$(33 - 2N_f)$ limit to estimate the errors from the higher-order terms in the perturbation series. The resulting error esti-
mate on $\alpha_s(M_Z)$ is consistent with the 1994 estimate of Narison, but much smaller than the error estimate of Altarelli. These explicit calculations indicate that Altarelli overestimated by an order of magnitude the size of $\Lambda^2/M_Z^2$ corrections associated with the first ultraviolet renormalon. The bottom line is that $\alpha_s(M_Z)$ can indeed be determined from $\tau$-decay data with a precision of 2% (or perhaps even better).

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