Solution of boundary value problems for a system of polyharmonic equations of a special type with applications

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Abstract. The method of numerical solution of the system of polyharmonic equations of two types is proposed: 1) the right part is a given polyharmonic function; 2) the right part is also the desired polyharmonic function. The method is based on Green's integral identity and the boundary element method. The system of polyharmonic equations with boundary conditions is reduced to a system of integral identities, and then after the discretization to a system of linear algebraic equations. Numerical examples of solving boundary value problems on the plane are considered, including the problem of shell and plate theory.

1. Introduction
In the fundamental monograph by I N Vekua [1] a detailed study of linear differential equations of elliptic type [1] is given, which also provides information about the works of other authors. Also, the analytical solution of such equations using the apparatus of almost-periodic functions has been considered in [2]. It should be noted, however, that analytical solutions are expressed by complex integral representations via special functions or contour singular integrals, which presents some difficulties for numerical calculations. It is no accident that specific calculations were performed, mainly for special areas or specific analytical solutions. Therefore, it makes sense to apply direct numerical methods that would allow to calculate the desired function in an arbitrary single-sheet region with a closed continuous piecewise smooth and non-self-intersecting curve boundary.

In the work [3], the boundary value problem was reduced to a system of the linear equations by the method of direct numerical decomposition of the polyharmonic equation. As a result, a simple numerical algorithm for computing polyharmonic functions was developed. A series of numerical results confirms the efficiency and high accuracy of the calculations. The existence and uniqueness of the solution follows from the system of the linear algebraic equations.

The following is a generalization of numerical methods [3 - 4] to the linear differential equations of a special kind (an analogue of the Poisson problem with a known right-hand side or a system of polyharmonic equations). The method is based on Green's integral identity and multiple applications of the Laplace operator. The main attention is paid to the applied problems of the theory of plates and shells [5].

The following numerical algorithm for calculating the system is proposed:

\[
\begin{align*}
\Delta^n u(x, y) &= v(x, y), \\
\Delta^m v(x, y) &= 0.
\end{align*}
\]
The system of equations (1) is simplified, but the numerical algorithm is applicable to the system of the linear equations of arbitrary order.

Two cases are considered: 1) the function \( v(x, y) \) is a given \( m \)-harmonic function in the region \( D \) with a boundary \( C \); 2) the function \( v(x, y) \) is a solution of the \( m \)-harmonic equation. Referring to the results of I N Veleva, we leave aside the existence and uniqueness theorems of the solution, and focus on the construction of a numerical algorithm.

2. Approximation of the boundaries of the region and integral along the border

The boundary of the area is replaced by a polygon with \( N \) sides (elements), the corner points are called nodes. The number of nodes is equal \( N + 1 \). If there are corner points on the border, they are aligned with the nodes. The boundary conditions are satisfied in the mean (control) points of the elements.

We number the node and control points:

\[
Z_k = \frac{z_{k+1} + z_k}{2}, \quad h_k = |z_{k+1} - z_k|, \quad S_k = \sum_{m=1}^{k} h_m - \frac{h_k}{2}, \quad n_k = -i \frac{z_{k+1} - z_k}{h_k}. \tag{2}
\]

Consider two functions \( u(z) \) and \( g(z,z') \), \( z,z' \in C \). The first function is continuous at the boundary, while the second function can have an integrable function at the point \( z_k \). The integral of the product of two functions on the contour is determined by the mean formula that admits an error of order \( O\left(\frac{\max(h_m)}{m}\right) \):

\[
J(Z_k) = \oint_C u(s) \cdot g(Z_k, s) ds \approx \sum_{m=1}^{N} u(Z_m) \int_{h_m} g(Z_k, s) ds.
\tag{3}
\]

The approximate value of the integral can be written as a product of matrices

\[
J = MU\tag{4}
\]

The components of the matrix \( M \) are integrals over the elements \( h_m, (m = 1, N) \).

The polyharmonic equation \( \Delta^n u = 0 \) can be represented as a system of linear equations [3]:

\[
eu_j + \sum_{p=0}^{n-1-j} \oint_{C} u_{j+p} H_p ds - \sum_{p=0}^{n-1-j} \oint_{C} q_{j+p} G_p ds = 0, \quad (j = 0, n-1), \tag{5}
\]

where

\[
u_j = \Delta' u, \quad q_j = \frac{\partial u}{\partial n}, \quad \Delta^p G_p (r) = G(r), \quad H_p = \frac{\partial G_p}{\partial n}.
\tag{6}
\]

As the contour is smooth at the control point \( Z_k \), the multiplier \( \varepsilon = 1/2 \); at the inner point \( \varepsilon = 1 \).

The third equation in (6) is an ordinary differential equation of the form

\[
\left( \frac{1}{r} \frac{d}{dr} \frac{dG_j(r)}{dr} \right) = G(r).
\]

The solution is a function

\[
G_j = \frac{1}{2\pi} \frac{r^{2j}}{(j!)^2} \left( \ln \frac{1}{r} + \sum_{m=1}^{k} \frac{1}{m} \right).
\]
3. Discretization of the equation

Using an approximate representation of (3) or (4), we can write the equation (5) in the form of a linear system of the discrete values of the function \( u_j(s) \) and the normal derivative \( q_j(Z_k) = \mathbf{n}_k \cdot \Delta u_j(Z_k) \)

\[
\frac{1}{2} u_j(Z_k) + \sum_{p=0}^{n-1} \sum_{m=1}^{N} \left[ u_{j+p}(Z_m) \int_{h_k} H_p(Z_k,s) ds - q_{j+p}(Z_m) \int_{h_k} G_p(Z_k,s) ds \right] = 0
\]

or in the form of \( n \) matrix equations

\[
\frac{1}{2} \mathbf{U}_j + \sum_{p=0}^{n-1} \left( \mathbf{H}_p \mathbf{U}_{j+p} - \mathbf{G}_p \mathbf{Q}_{j+p} \right) = 0, \quad \left( j = 0, n-1 \right).
\]

Equality (7) represents \( n \) equations with respect to \( N \) components \( U_{j,k} \) and \( Q_{j,k} \), \( (k = 1, N) \) which are elements of matrices and are defined as \( U_{j,k} = u_j(Z_k) \), \( Q_{j,k} = q_j(Z_k) \).

4. Boundary value problems

**Task 1.** Assume \( v(z) \), the continuous and differentiable in self-contained area \(\overline{D} \), is the known polyharmonic function of an order of \( m \). Then the second equation of system (1) is satisfied identically. Applying \( m \) of times the operator of Laplace, we receive one polyharmonic equation of the order of \( m+n \):

\[
\Delta^{m+n} u = 0
\]

According to the formula (7), the equation (8) is written as a system of matrix equations

\[
\frac{1}{2} \mathbf{U}_j + \sum_{p=0}^{n+m-1} \left( \mathbf{H}_p \mathbf{U}_{j+p} - \mathbf{G}_p \mathbf{Q}_{j+p} \right) = 0, \quad \left( j = 0, n+m-1 \right).
\]

The components of the matrices and vectors are the values of the corresponding functions in \( N \) control points.

**Task 2.** The function \( v(z) \) is unknown, like the function \( u(z) \). It is determined from the second equation of system (1) with \( m \) boundary conditions. If we represent this equation in the form of system (9), then to solve it we need to set in addition \( mN \) values at control points.

Applying the Laplace operator \( m \) times to the first equation of the system, we again come to the polyharmonic equation of order of \( (m+n) \), which can also be represented as a system of equations (9). Thus, from the system of equations (9) it is possible to find simultaneously the values of the required functions \( u_j \) and \( v_j \) in the control points. The value of the function \( v_j \) in an arbitrary interior point of the domain is from the equality (6) provided \( \varepsilon = 1 \).

5. Boundary conditions

In both cases for solving the equation (8) it is necessary to set \((m+n)\) boundary conditions. They can be of the following type:

1) \( u|_{c} = u_0(s) \), \( \Delta u|_{c} = u_1(s) \), \ldots \( \Delta^{m+n-1} u|_{c} = u_{m+n-1}(s) \), \( s \) - arc coordinate on the boundary;

2) \( \frac{\partial u}{\partial n}|_{c} = q_0(s) \), \( \frac{\partial \Delta u}{\partial n}|_{c} = q_1(s) \), \ldots \( \frac{\partial \Delta^{m+n-1} u}{\partial n}|_{c} = q_{m+n-1}(s) \);
3) the mixed conditions, when on one part of the border functions \( u_j \) and on other function \( q_j \) are set. The given boundary conditions are similar to the Dirichlet and Neumann conditions.

In the transition to the system of equations (9) will be determined \((n + m) \cdot N\) values \( U_{j,k} \) or \( Q_{j,k} \), or partially \( U_{j,k} \) and \( Q_{j,k} \). For the problem of the 1st type, it is enough to set \( nN \) values in the control points, the additional \( mN \) conditions can be found from the given function \( v(x,y) \) in the region \( D \) and, therefore, on the border \( C \):

\[
U_{n+l} = \Delta^{n+l} v, \quad q_{n+l} = \frac{\partial (\Delta^l v)}{\partial n}, \quad l = 0, m - 1.
\]

For the problem of the 2nd type \( nN \) values at control points are determined from the boundary conditions for the first equation of the system (1), and the remaining \( mN \) values from the boundary conditions for the second equation of this system.

6. Numerical examples

6.1. Example 1

To compare a numerical solution with an analytic function, we consider the polyharmonic function

\[
u(x,y) = x^3 \left( x^2 - 5y^2 \right),
\]

which is the solution of the equation

\[
\Delta u = 10x^3 - 30xy^2. \tag{10}
\]

Let’s solve the equation (10) numerically in a circular ring if the Dirichlet conditions are prescribed on the boundary:

\[
C_1 : \begin{cases} x = a \cos s, \\ y = -a \sin s; \\ s \in [0, 2\pi), \ b > a. 
\end{cases}
\]

Let’s move on to the equation \( \Delta^2 u = 0 \) with boundary conditions

\[
u|_{C_1} = a^5 \cos^5 s - 5 \cdot a^3 \sin^2 s \cdot \cos^3 s, \quad \Delta u|_{C_1} = 10a^3 \cos^3 s - 30 \cdot a^3 \sin^2 s \cdot \cos s.
\]

On the basis of the method described above, we proceed to the system of the linear matrix equations

\[
\begin{align*}
(0.5E + H_0)U_1 - G_o Q_1 &= 0, \\
(0.5E + H_0)U_0 - G_o Q_0 + H_i U_1 - G_i Q_1 &= 0
\end{align*}
\]

with respect to the unknown vectors \( Q_0, Q_1 \). The solution of the equation (11) has the form

\[
Q_1 = (G_o)^{-1} (0.5E + H_0)U_1,
\]

\[
Q_0 = (G_o)^{-1} \left[ (0.5E + H_0)U_0 + H_i U_1 - G_i (G_o)^{-1} (0.5E + H_0)U_1 \right].
\]
The value of the desired function $u$ at any point in the area is determined by the formula (6) at values $\varepsilon = 1$ and $n + m = 2$.

The numerical solution is found for a circular ring with the internal and external radiiuses $a=5$ and $b=4$. As a comparison, it is represented on a contour $x = 4.5\cos s$, $y = 4.5\sin s$, $s \in [0, 2\pi)$. together with the analytical solution (fig.1, the numerical solution, the analytical solution). The number of the elements on each contour is $N = 50$.

**Figure 1.** Solution of the Dirichlet problem for the biharmonic equation

6.2. **Example 2**

Consider a thin plate with a thickness of 0.003 m rectangular plate with sides $a=1$ m and $b=1$ m in the undeformed state on the $Oxy$ plane, the edges of which are rigidly fixed. A dimensionless linear external load

$$q(x, y) = 5.94 \times 10^6 (1 + x^2 + y^2) \text{N}\text{m}^{-3}$$

is distributed on the plate surface. It is required to calculate a function of a bend.

In the plane theory of elasticity, the study of the plate bending is related to the solution of the differential equation by Sophie Germain

$$\Delta^2 u = q$$

The equation will be solved in the field $D = \{(x, y) | -0.5 \leq x \leq 0.5, -0.5 \leq y \leq 0.5\}$, if the boundary conditions on the contour $C$ of the plate have the form:

$$u |_c = 0, \quad \frac{\partial u}{\partial n} |_c = 0.$$

Applying the above algorithm, we proceed to the system of the matrix equations

$$
\begin{align*}
(0.5E + H_0)U_1 - G_0Q_1 - G_1Q_2 &= -H_1U_2, \\
H_1U_1 - G_1Q_1 - G_2Q_2 &= -H_2U_2, \\
G_0Q_2 &= (0.5E + H_0)U_2.
\end{align*}
$$

The shape of the plate deflection is shown in figure 2.

On figure 3 the form of the deformed elliptic aluminum plate with a thickness of 0.003 and semiaxes of 1 m and 0.75 m is represented at
\[ q(x, y) = 5.94 \cdot 10^6 (1 + x) \text{ N}\times\text{m}^{-3}. \]

**Figure 2.** The shape of the deformed plate

**Figure 3.** The shape of the deformed elliptic plate

### 6.3. Example 3

It is required to solve a system of differential equations

\[
\begin{cases}
\Delta u(x, y) = v(x, y), \\
\Delta^2 v(x, y) = 0
\end{cases}
\]

(12)

in a closed region, if the boundary of the region that represents the circle of the unit radius has conditions

\[
\begin{align*}
u|_c &= \frac{1}{2} \sin 2s \left(1 - \frac{1}{2} \sin^2 2s\right), \\
v|_c &= 10 \sin 2s, \\
\Delta v|_c &= 120 \sin 2s, \quad s \in [0, 2\pi).
\end{align*}
\]

Analytical solution of the system (12):

\[
\begin{align*}
u &= x^3 y + x y^3, \\
v &= 20(x^3 y + x y^3).
\end{align*}
\]

For the numerical solution it is necessary to proceed to a third-order polyharmonic equation with the boundary conditions

\[
\begin{align*}
u|_c &= \frac{1}{2} \sin 2s \left(1 - \frac{1}{2} \sin^2 2s\right), \\
\Delta u|_c &= 10 \sin 2s, \\
\Delta^2 u|_c &= 120 \sin 2s, \quad s \in [0, 2\pi).
\end{align*}
\]

The solution of this equation is reduced to the solution of a system of the linear algebraic equations:

\[
\begin{align*}
G_0 Q_0 + G_1 Q_1 + G_2 Q_2 &= (0.5E + H_0) U_0 + H_1 U_1 + H_2 U_2, \\
G_0 Q_1 + G_1 Q_2 &= (0.5E + H_0) U_1 + H_1 U_2, \\
G_0 Q_2 &= (0.5E + H_0) U_2,
\end{align*}
\]

which can be reduced to a simple matrix equation
As a comparison, the analytical and numerical (number of elements $N=50$) solutions are built on a circle $x=0.75\cos s, y=0.75\sin s, s \in [0, 2\pi)$. The results of constructing the function $u$ are presented in Fig. 4, functions $v$ in Fig. 5 (____ the numerical solution, —— the analytical solution).

![Figure 4. The function $u$](image)

![Figure 5. The function $v$](image)

The presented numerical algorithm can be applied to a wide class of such systems of equations. The effectiveness of the method is confirmed by comparing the numerical results and analytical solutions.

References

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