Quadratic integrals of motion for the systems of identical particles-quantum case

Y. Brihaye
Faculte des Sciences,
Universite de Mons, 7000 Mons, Belgium
C. Gonera, P. Kosiński, P. Maślanka
Department of Theoretical Physics II
University of Łódź
Pomorska 149/153, 90 - 236 Łódź/Poland
S. Giller
Pedagogical University of Czestochowa,
Armii Krajowej 13/15, 42-200 Czestochowa Poland.

Abstract

The quantum dynamical systems of identical particles admitting an additional integral quadratic in momenta are considered. It is found that an appropriate ordering procedure exists which allows to convert the classical integrals into their quantum counterparts. The relation to the separation of variables in Schroedinger equation is discussed.

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1 Introduction

It has been shown by Braden [1] that assuming the permutational symmetry imposes severe restrictions on the form of potentials admitting polynomial in momenta integrals of motion. More precisely, he proved that the only system admitting third-order integral of motion with position-independent highest-order term is the celebrated CSM model.

In the previous paper [2] we studied classical systems of identical particles possessing quadratic integrals of motion. We obtained the complete classification of such systems. In the present paper we discuss their quantum versions.

2 The classical case.

The main result of ref. [2] is as follows: assume that
(i) the hamiltonian has a natural form

\[ H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + V(q_1, \ldots, q_N) \]  (1)

(ii) \( H \) is translationally invariant

\[ \{H, P\} = 0, \quad P = \sum_{i=1}^{N} p_i \]  (2)

(iii) \( H \) is invariant under the action of the group \( S_N \) of all permutations of canonical variables \( q_i, p_i, i = 1, \ldots, N \).

(iv) \( H \) admits at least one integral of motion quadratic in momenta and functionally independent of \( H \) and \( P \).

Then the general form of potential \( V \) reads

\[ V(q) = \tilde{V}(q) + U \left( \sum_{i, j=1}^{N} (q_i - q_j)^2 \right), \]  (3)

where \( \tilde{V} \) is a translationally invariant homogeneous function of degree-2 while \( U \) is an arbitrary differentiable function of one variable.

The form of additional integrals depends on the potential. If \( U \equiv 0 \) (or, more
generally, a constant) any quadratic integral is a linear combination of the following ones:

\begin{align*}
I_1 &= \left( \sum_{k=1}^{N} q_k p_k \right) P - 2QH \\
I_2 &= 2\left( \sum_{k=1}^{N} q_k^2 \right) H - \left( \sum_{k=1}^{N} q_k p_k \right)^2 \\
I_3 &= 2QP\left( \sum_{k=1}^{N} q_k p_k \right) - 2Q^2 H - \left( \sum_{k=1}^{N} q_k^2 \right) P^2
\end{align*}

where \( Q = \sum_{k=1}^{N} q_k \). The integrals \( I_1, I_2, I_3 \) obey the relation

\begin{equation}
I_1^2 + P^2 I_2 + 2HI_3 = 0
\end{equation}

For nontrivial \( U \) there exists one integral

\begin{align*}
I_4 &= 2 \left( \left( \sum_{k=1}^{N} q_k^2 \right) - Q^2 \right) \tilde{H} - N\left( \sum_{k=1}^{N} q_k p_k \right)^2 + \\
&\quad + 2QP\left( \sum_{k=1}^{N} q_k p_k \right) - P^2\left( \sum_{k=1}^{N} q_k^2 \right)
\end{align*}

where \( \tilde{H} \) is obtained from \( H \) by replacement \( V \to \tilde{V} \).

We have shown that the above integrals are related to the separation of variables in the H-J equation. More precisely, they appear from the separation of radial variables in Jacobi coordinates. Moreover, it has been explained in ref. [3] how the integrals are related to conformal \( sl(2, \mathbb{R}) \) symmetry.

3 The quantum case.

As it is seen from equations (4), (6) the form of integrals cannot be carried over directly to the quantum case: the ordering problem arises. However, due to natural form of Hamiltonian one can expect that the integrals related to separation of variables in H-J equation will emerge in the quantum case as the separation constants in the Schroedinger equation.

Assuming the form (3) of the potential one can find, by trial and error
method, the proper ordering for the quantum counterparts of $I_1 \div I_4$:

$$I_1 = \frac{1}{4} \left( \sum_{k=1}^{N} (q_k p_k + p_k q_k) \right) P + \frac{1}{4} P \left( \sum_{k=1}^{N} (q_k p_k + p_k q_k) \right) - QH - HQ$$

$$I_2 = \left( \sum_{k=1}^{N} q_k^2 \right) H + H \left( \sum_{k=1}^{N} q_k^2 \right) - \frac{1}{4} \left( \sum_{k=1}^{N} (q_k p_k + p_k q_k) \right)^2$$

$$I_3 = \frac{1}{4} \left[ \left( \sum_{k=1}^{N} (q_k p_k + p_k q_k) \right) (QP + PQ) + (QP + PQ) \left( \sum_{k=1}^{N} (q_k p_k + p_k q_k) \right) \right] +$$

$$-Q^2H - HQ - \frac{1}{2} \left( \sum_{k=1}^{N} q_k^2 \right) P^2 - \frac{1}{2} P^2 \left( \sum_{k=1}^{N} q_k^2 \right)$$

$$I_4 = \left( N \left( \sum_{k=1}^{N} q_k^2 \right) - Q^2 \right) \tilde{H} + \tilde{H} \left( N \left( \sum_{k=1}^{N} q_k^2 \right) - Q^2 \right) - \frac{1}{4} N \left( \sum_{k=1}^{N} (q_k p_k + p_k q_k) \right)^2 +$$

$$+ \frac{1}{4} \left[ \left( \sum_{k=1}^{N} (q_k p_k + p_k q_k) \right) (QP + PQ) + (QP + PQ) \left( \sum_{k=1}^{N} (q_k p_k + p_k q_k) \right) \right]$$

$$- \frac{1}{2} \left( \sum_{k=1}^{N} q_k^2 \right) P^2 - \frac{1}{2} P^2 \left( \sum_{k=1}^{N} q_k^2 \right).$$

One can check explicitly that the above expressions are hermitean and commute with the hamiltonian (again, for $I_1$, $I_2$, $I_3$ being the constants of motion, one has to assume $V = \tilde{V}$).

4 Separation of variables.

As in the classical case, one can relate the quadratic integrals to the separation of variables; the relevant equation is now the Schroedinger equation. We separate the center-of-mass motion and introduce the polar variables in the space of Jacobi coordinates. Then the integral $I_4$ arises from the separation of radial variable.

Indeed, let us define

$$\tilde{q}_i = q_i - \frac{1}{N} Q$$

$$\tilde{p}_i = p_i - \frac{1}{N} P$$

(8)
\[\rho = \sqrt{\sum_{k=1}^{N} \tilde{q}_k^2}\]
\[p_{\rho} = \frac{1}{2} \sum_{k=1}^{N} \left( \frac{\tilde{p}_k}{\rho} \tilde{p}_k + \tilde{p}_k \tilde{p}_k / \rho \right)\]

Then the hamiltonian can be written as
\[H = \frac{1}{2N} \tilde{P}^2 + \frac{M^2}{2\rho^2} + \frac{\hbar^2 (N-2)(N-4)}{8\rho^2} + \frac{F}{\rho^2} + U(\rho^2) \quad (9)\]
with \(M^2\) being the square of angular momentum in the center-of-mass system i.e.
\[M^2 = \frac{1}{2} \sum_{i,j=1}^{N} (\tilde{q}_i \tilde{p}_j - \tilde{q}_j \tilde{p}_i)^2 \quad (10)\]

We have also used the general form of potentials \(\tilde{F}\) which implies that \(F\) is a function of angular variables only. From equation \((9)\) we conclude that
\[\rho^2 \left( \tilde{H} - \frac{1}{2N} \tilde{P}^2 - \frac{1}{2}\tilde{p}_{\rho}^2 - \frac{\hbar^2 (N-2)(N-4)}{8\rho^2} \right) \quad (11)\]
is the operator responsible for separating the radial variable

Now, one can check that the expression \((11)\) equals to
\[\frac{1}{2N} I_4 = \frac{\hbar^2 (N-2)(N-4)}{8} \quad (12)\]

5 \textit{sl}(2, \mathbf{R}) \textit{symmetry}

The results obtained for \(U \equiv 0\) can be understood from the point of view of \textit{sl}(2, \mathbf{R}) symmetry \(\tilde{F}\). Defining \(X = \sum_{k=1}^{N} q_k^2\) and \(Y = \frac{1}{2} \sum_{k=1}^{N} (q_k p_k + p_k q_k)\) one easily checks that, together with \(H\), the above operators span \textit{sl}(2, \mathbf{R}) algebra:
\[\begin{align*}
[Y, H] &= 2i\hbar H \\
[Y, X] &= -2i\hbar X \\
[H, Y] &= 2i\hbar Y
\end{align*}\]
The Casimir operator of this algebra is of course an integral of motion. One readily finds it is equal to $I_2$:

$$I_2 = XH + HX - Y^2 \tag{14}$$

In order to explain the meaning of $I_1$ we note the following. Any integral of motion $I$ which is homogeneous function of natural degree $n$ provides a highest-weight vector for some representation in the adjoint action of $sl(2, \mathbb{R})$, $[H, I] = 0$, $[Y, I] = i\hbar n I$. Therefore, the next-to-highest vector $J = \frac{1}{\hbar}[X, I]$ evolves linearly in time:

$$\dot{J} = \frac{1}{\hbar}[J, H] = -2nI \tag{15}$$

Now, if $I'$ is a second integral of degree $n'$ it follows from eq.\,(15) that $n'I'J - nIJ'$ is also an integral of motion. Applying this reasoning to the triple $(H, Y, X)$ and the doublet $(P, Q)$ one obtains $I_1$.

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