Sharp lower bound on the curvatures of ASD connections over the cylinder

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Abstract. We prove a sharp lower bound on the curvatures of non-flat ASD connections over the cylinder.

1. Introduction.

The purpose of this note is to calculate explicitly a universal lower bound on the curvatures of non-flat ASD connections over the cylinder $\mathbb{R} \times S^3$.

First we fix our conventions. Let $S^3 = \{x^1_1 + x^2_2 + x^3_3 + x^4_4 = 1\} \subset \mathbb{R}^4$ be the unit 3-sphere equipped with the Riemannian metric induced by the Euclidean metric on $\mathbb{R}^4$. Set $X := \mathbb{R} \times S^3$. We give the standard metric on $\mathbb{R}$, and $X$ is equipped with the product metric.

Let $\mathbb{H}$ be the space of quaternions. Consider $SU(2) = \{x \in \mathbb{H} \mid |x| = 1\}$ with the Riemannian metric induced by the Euclidean metric on $\mathbb{H}$. (Hence it is isometric to $S^3$ above.) We naturally identify $su(2) := T_1SU(2)$ with the imaginary part $\text{Im}\mathbb{H} := \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$. Here $i$, $j$ and $k$ have length 1.

Let $E := X \times SU(2)$ be the product $SU(2)$-bundle. Let $A$ be a connection on $E$, and let $F_A$ be its curvature. $F_A$ is an $su(2)$-valued 2-form on $X$. Hence for each point $p \in X$ the curvature $F_A$ can be considered as a linear map

$$F_{A,p} : \Lambda^2(T_pX) \to su(2).$$

We denote by $|F_{A,p}|_{\text{op}}$ the operator norm of this linear map. The explicit formula is as follows: Let $x_1, x_2, x_3, x_4$ be the normal coordinate system on $X$ centered at $p$. Let $A = \sum_{i=1}^4 A_i dx_i$. Each $A_i$ is an $su(2)$-valued function. Then $F(A)_{ij} := F_A(\partial/\partial x_i, \partial/\partial x_j) = \partial_i A_j - \partial_j A_i + [A_i, A_j]$. Since $\partial/\partial x_i \wedge \partial/\partial x_j$ ($1 \leq i < j \leq 4$) form an orthonormal basis of $\Lambda^2(TX)$ at $p$, the norm $|F_{A,p}|_{\text{op}}$ is equal to

$$\sup \left\{ \left| \sum_{1 \leq i < j \leq 4} a_{ij} F(A)_{ij,p} \right| : a_{ij} \in \mathbb{R}, \sum_{1 \leq i < j \leq 4} a_{ij}^2 = 1 \right\}.$$ 

Let $\|F_A\|_{\text{op}}$ be the supremum of $|F_{A,p}|_{\text{op}}$ over $p \in X$. The main result is the following.

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Theorem 1.1. The minimum of $\|F_A\|_{op}$ over non-flat ASD connections $A$ on $E$ is equal to $1/\sqrt{2}$.

Note that we don’t assume $F_A \in L^2$ in this statement. As far as I know, this kind of explicit calculations have never been done in Yang-Mills theory. (See Remark 1.3 below.) The above minimum value $1/\sqrt{2}$ is attained by the following BPST instanton ([1]).

Example 1.2. We define an $SU(2)$ instanton $A$ on $\mathbb{R}^4$ by

$$A := \text{Im} \left( \frac{\bar{x} dx}{1 + |x|^2} \right), \quad (x = x_1 + x_2 i + x_3 j + x_4 k).$$

By the conformal map

$$\mathbb{R} \times S^3 \to \mathbb{R}^4 \setminus \{0\}, \quad (t, \theta) \mapsto e^t \theta,$$

the connection $A$ is transformed into an ASD connection $A'$ on $E$ over $\mathbb{R} \times S^3$. Then

$$|F_{A',(t,\theta)}|_{op} = \frac{2\sqrt{2}}{(e^t + e^{-t})^2}.$$

Hence

$$\|F_{A'}\|_{op} = \frac{1}{\sqrt{2}}.$$

Remark 1.3. The essential point of the statement of Theorem 1.1 is the explicitness of $1/\sqrt{2}$. Indeed the following general statement is easy to prove: Let $Y$ be a closed Riemannian 3-fold, and assume that all flat $SU(2)$ connections $\rho$ on $Y$ satisfy the non-degeneracy condition $H_1^{\rho} = 0$. (See [2, p. 25, Definition 2.4]. $S^3$ satisfies this condition. More generally lens spaces $S^3/\mathbb{Z}_p$ satisfy it.) Then the infimum of $\|F_{\rho}\|_{op}$ over non-flat $SU(2)$ ASD connections $A$ on $\mathbb{R} \times Y$ is positive. The proof is just a direct application of [2, p. 81, Proposition 4.4]. But it is difficult to determine the value of $\inf \|F_{\rho}\|_{op}$ explicitly.

Theorem 1.1 is a Yang-Mills analogy of the classical result of Lehto [7, Theorem 1] in complex analysis. (The formulation below is due to Eremenko [4, Theorem 3.2]. See also Lehto-Virtanen [8, Theorem 1].) Consider $C^* := \mathbb{C} \setminus \{0\}$ with the length element $|dz|/|z|$. We give a metric on $CP^1 = \mathbb{C} \cup \{\infty\}$ by (naturally) identifying it with the unit 2-sphere $\{x_1^2 + x_2^2 + x_3^2 = 1\}$. For a map $f : C^* \to CP^1$ we denote its Lipschitz constant by Lip$(f)$. Then Lehto [7, Theorem 1] proved that the minimum of Lip$(f)$ over non-constant holomorphic maps $f : C^* \to CP^1$ is equal to 1. The function $f(z) = z$ attains the minimum. Eremenko [4, Section 3] discussed the relation between this result of Lehto and a quantitative homotopy argument of Gromov [6, Chapter 2, 2.12. Proposition]. Our proof of Theorem 1.1 is inspired by this idea.
2. Preliminaries: Connections over $S^3$.

In this section we study the method of choosing good gauges for some connections over $S^3$. The argument below is a careful study of \cite[p. 146–148]{5}. Set $N := (1, 0, 0, 0) \in S^3$ and $S := (-1, 0, 0, 0) \in S^3$. Let $P := S^3 \times SU(2)$ be the product $SU(2)$-bundle over $S^3$. For a connection $B$ on $P$ we define the operator norm $\|F_B\|_{op}$ in the same way as in Section 1.

Let $v_1, v_2 \in T_NS^3$ be two unit tangent vectors at $N$. $(|v_1| = |v_2| = 1)$ Let $\exp_N : T_NS^3 \to S^3$ be the exponential map at $N$. Since $|v_1| = |v_2| = 1$, we have $\exp_N(\pi v_1) = \exp_N(\pi v_2) = S$. We define a loop $l : [0, 2\pi] \to S^3$ by

$$l(t) := \begin{cases} 
\exp_N(tv_1) & (0 \leq t \leq \pi) \\
\exp_N((2\pi - t)v_2) & (\pi \leq t \leq 2\pi).
\end{cases}$$

**Lemma 2.1.** Let $B$ be a connection on $P$. Let $\text{Hol}(B) \in SU(2)$ be the holonomy of $B$ along the loop $l$. Then

$$d(\text{Hol}(B), 1) \leq 2\pi \|F_B\|_{op}. \quad \text{Here } d(\cdot, \cdot) \text{ is the distance on } SU(2) \text{ defined by the Riemannian metric.}$$

**Proof.** This follows from the standard fact that curvature is an infinitesimal holonomy \cite[p. 36]{3}. $(2\pi$ is half the area of the unit 2-sphere.) The explicit proof is as follows: Take a unit tangent vector $v_3 \in T_NS^3$ orthogonal to $v_1$ such that there is $\alpha \in [0, \pi]$ satisfying $v_2 = v_1 \cos \alpha + v_3 \sin \alpha$. Consider (the spherical polar coordinate of the totally geodesic $S^2 \subset S^3$ tangent to $v_1$ and $v_3$):

$$\Phi : [0, \alpha] \times [0, \pi] \to S^3, \quad (\theta_1, \theta_2) \mapsto \exp_N\{\theta_2(v_1 \cos \theta_1 + v_3 \sin \theta_1)\}.$$  

Let $Q$ be the pull-back of the bundle $P$ by $\Phi$. Since $\Phi([0, \alpha] \times \{0\}) = \{N\}$ and $\Phi([0, \alpha] \times \{\pi\}) = \{S\}$, $Q$ admits a trivialization under which the pull-back connection $\Phi^*B$ is expressed as $\Phi^*B = B_1 d\theta_1 + B_2 d\theta_2$ with $B_1 = 0$ on $[0, \alpha] \times \{0, \pi\}$.

We take a smooth map $g : [0, \alpha] \times [0, \pi] \to SU(2)$ satisfying

$$g(\theta_1, 0) = 1 \quad (\forall \theta_1 \in [0, \alpha]), \quad (\partial_2 + B_2)g = 0.$$  

We have $\text{Hol}(B) = g(\alpha, \pi)^{-1}g(0, \pi)$. Then $F_{\Phi^*B}(\partial_1, \partial_2)g = [\partial_1 + B_1, \partial_2 + B_2]g = -(\partial_2 + B_2)(\partial_1 + B_1)g$. From $B_1 = 0$ on $[0, \alpha] \times \{0, \pi\}$ and Kato’s inequality $|\partial_2(\partial_1 + B_1)g| \leq |(\partial_2 + B_2)(\partial_1 + B_1)g| = |F_{\Phi^*B}(\partial_1, \partial_2)g|$,  

$$|\partial_1 g(\theta_1, \pi)| = |(\partial_1 + B_1)g(\theta_1, \pi) - (\partial_1 + B_1)g(\theta_1, 0)|$$

$$\leq \int_{\{\theta_1\} \times [0, \pi]} |\partial_2(\partial_1 + B_1)g| d\theta_2 \leq \int_{\{\theta_1\} \times [0, \pi]} |F_{\Phi^*B}(\partial_1, \partial_2)| d\theta_2.$$
Then
\[ d(\text{Hol}_1(B), 1) = d(g(0, \pi), g(\alpha, \pi)) \leq \int_{[0,\alpha] \times [0,\pi]} |F_{\Phi^*_B}(\partial_1, \partial_2)| d\theta_1 d\theta_2. \]

The injectivity radius of \( F \) is a positive constant depending on \( \alpha \). From \( 0 \leq \alpha \leq \pi \),
\[ d(\text{Hol}_1(B), 1) \leq \|F_B\|_{\text{op}} \int_{[0,\alpha] \times [0,\pi]} \sin \theta_2 d\theta_1 d\theta_2 = 2\alpha \|F_B\|_{\text{op}} \leq 2\pi \|F_B\|_{\text{op}}. \]

Let \( \tau < 1/2 \). Let \( B \) be a connection on \( P \) satisfying \( \|F_B\|_{\text{op}} \leq \tau \). We construct a good connection matrix of \( B \).

Fix \( v \in T_N S^3 \). By the parallel translation along the geodesic \( \exp_N(tv) \) \((0 \leq t \leq \pi)\) we identify the fiber \( P_S \) with the fiber \( P_N \). Let \( g_N \) and \( g_S \) be the exponential gauges (see [5, p. 146] or [3, p. 54]) centered at \( N \) and \( S \) respectively:
\[ g_N : P|_{S^3 \setminus \{S\}} \to (S^3 \setminus \{S\}) \times P_N, \quad g_S : P|_{S^3 \setminus \{N\}} \to (S^3 \setminus \{N\}) \times P_N. \]
(In the definition of \( g_S \) we identify \( P_S \) with \( P_N \) as in the above.) By Lemma 2.1, for \( x \in S^3 \setminus \{N, S\} \),
\[ d(g_N(x), g_S(x)) \leq 2\pi \|F_B\|_{\text{op}} \leq 2\pi \tau < \pi. \]

The injectivity radius of \( SU(2) = S^3 \) is \( \pi \) (this is a crucial point of the argument). Hence there uniquely exists \( u(x) \in \text{ad}P_N(\cong su(2)) \) satisfying
\[ |u(x)| \leq 2\pi \|F_B\|_{\text{op}}, \quad g_S(x) = e^{u(x)} g_N(x). \]

We take and fix a cut-off function \( \varphi : S^3 \to [0, 1] \) such that \( \varphi(x_1, x_2, x_3, x_4) \) is equal to 0 over \( \{x_1 > 1/2\} \) and equal to 1 over \( \{x_1 < -1/2\} \). We can define a bundle trivialization \( g \) of \( P \) all over \( S^3 \) by \( g := e^{\varphi u} g_N \). Then the connection matrix \( g(B) \) satisfies
\[ |g(B)| \leq C_{\tau} \|F_B\|_{\text{op}}. \]

Here \( C_{\tau} \) is a positive constant depending on \( \tau \).

3. Proof of Theorem 1.1.

In this section we denote by \( t \) the standard coordinate of \( \mathbb{R} \). Let \( A \) be an ASD connection on \( E \) satisfying \( \|F_A\|_{\text{op}} < 1/\sqrt{2} \). We will prove that \( A \) must be flat. Set \( \tau := \|F_A\|_{\text{op}}/\sqrt{2} < 1/2 \).

The ASD equation implies that \( F_A \) has the following form:
\[ F_A = -dt \wedge (\star_3 F(A|_{\{t\} \times S^3})) + F(A|_{\{t\} \times S^3}), \]

where \( A|_{\{t\} \times S^3} \) is the restriction of \( A \) to \( \{t\} \times S^3 \) and \( \star_3 \) is the Hodge star on \( \{t\} \times S^3 \). Hence

\[ |F_{A,(t,\theta)}|_{op} = \sqrt{2}|F(A|_{\{t\} \times S^3})\theta|_{op}. \]

Therefore

\[ \|F(A|_{\{t\} \times S^3})\|_{op} \leq \frac{1}{2} (\forall t \in \mathbb{R}). \]

Thus we can apply the construction of Section 2 to \( A|_{\{t\} \times S^3} \).

Fix a bundle trivialization of \( E \) over \( \mathbb{R} \times \{N\} \). (Any choice will do.) Then the construction in Section 2 gives a bundle trivialization \( g \) of \( E \) over \( X \) satisfying

\[ |g(A)|_{\{t\} \times S^3} \leq C_\tau \|F(A|_{\{t\} \times S^3})\|_{op} \ (\forall t \in \mathbb{R}). \]

Set \( A' := g(A) \). We consider the Chern-Simons functional

\[ cs(A') := \text{tr} \left( A' \wedge F_{A'} - \frac{1}{3} A'^3 \right). \]

For \( R > 0 \)

\[ \int_{[-R,R] \times S^3} |F_A|^2 d\text{vol} = \int_{\{R\} \times S^3} cs(A') - \int_{\{-R\} \times S^3} cs(A') \quad \text{(because } A \text{ is ASD)} \]

\[ \leq \text{const} \_\tau \left( \|F(A|_{\{R\} \times S^3})\|_{op} + \|F(A|_{\{-R\} \times S^3})\|_{op} \right). \quad (1) \]

Here we have used \( |A'||_{\{\pm R\} \times S^3} \leq C_\tau \|F(A|_{\{\pm R\} \times S^3})\|_{op} \) and \( \|F(A|_{\{\pm R\} \times S^3})\|_{op} \leq \tau \).

Let \( R \rightarrow +\infty \). Then we get

\[ \int_X |F_A|^2 d\text{vol} < +\infty. \]

This implies that the curvature \( F_A \) has an exponential decay at the ends (see [2, Theorem 4.2]). In particular

\[ \|F(A|_{\{\pm R\} \times S^3})\|_{op} \rightarrow 0 \quad (R \rightarrow +\infty). \]

By the above (1)

\[ \int_X |F_A|^2 d\text{vol} = 0. \]

This shows \( F_A \equiv 0 \). So \( A \) is flat.
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