How to Solve Quantum Nonlinear Abelian Gauge Theory in Two Dimension in the Heisenberg Picture

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Abstract

The new method based on the operator formalism proposed by Abe and Nakanishi is applied to the quantum nonlinear abelian gauge theory in two dimension. The soluble models in this method are extended to wider class of quantum field theories. We obtain the exact solution in the canonical-quantization operator formalism in the Heisenberg picture. So this analysis might shed some light on the analysis of gravitational theory and non-polynomial field theories.
1 Introduction

Quantum Einstein gravity is non-renormalizable in the conventional perturbative approach. One candidate to overcome its difficulty is to modify the perturbative approach. In [1], Abe and Nakanishi have proposed the new method to solve quantum field theory in the operator formalism in the Heisenberg picture. The procedure is the following.

First we calculate equal-time commutation relations of the fundamental fields from the canonical commutation relations. From equal-time commutation relations and the equations of motion, we set up the Cauchy problems for two-dimensional commutation relations. By solving the Cauchy problems with operator coefficients, we obtain the two-dimensional commutation relations of fundamental fields. Finally, we construct full Wightman functions from the consistency with multiple commutation relations for fundamental fields and from the energy positivity requirement. This enables us to separate algebraic relations of operators and its realization or regularization on a state space.

So far, in the theories which have been solved exactly by this method there is at least one fundamental field, \( \phi(x) \), which commutes mutually:

\[
[\phi(x), \phi(y)] = 0, \tag{1}
\]

for arbitrary \( x \) and \( y \). This full commutativity is crucial in solvability of such models as two-dimensional quantum gravity in covariant gauge or light-cone gauge, BF theory and the one-loop model \[1, 2\]. In this paper, we extend the above method to a model with more general commutation relations, that is, the model in which there is no such a fundamental fields as \( \phi(x) \) satisfying (1).

The non-polynomiality is a common feature of the gravitational theories, which makes their analysis difficult. Moreover they are generally non-renormalizable. So it is unnatural to analyze it in the conventional perturbative approach based on the interaction picture. A natural method is to treat it in the Heisenberg picture.

In this paper, we consider a general \( U(1) \) gauge theory in two dimension as a toy model which can be solved in the Heisenberg operator formalism. Maxwell’s theory can be generalized to the nonlinear electromagnetic theory of a massless vector \( A_\mu \) with \( U(1) \) gauge
symmetry as
\[ S = \int d^2 x \mathcal{L}, \]
\[ \mathcal{L} = \Phi(F_{\mu\nu}F^{\mu\nu}) + \text{matters}, \] (2)
where \( \Phi(z) \) is a function of \( z \). The Born-Infeld theory is one example of nonlinear electromagnetic theory.

This Lagrangian density is generally non-polynomial in the fundamental fields, Thus the analysis of this theory might also give us a key to investigate the above problems.

In this paper, we exactly solve two-dimensional quantum nonlinear abelian gauge theory in the light-cone gauge.

We calculate all the exact multiple commutation relations and all n-point Wightman functions for the electromagnetic field.

2 Nonlinear Abelian Gauge Theory

The action of the nonlinear abelian gauge theory in two dimensions is written as
\[ S = \int d^2 x \mathcal{L}, \]
\[ \mathcal{L} = \Phi(F_{\mu\nu}F^{\mu\nu}), \] (3)
where \( \Phi(z) \) is a function of \( z \). The property of \( \Phi(z) \) is specified later. If \( \Phi(z) = \frac{1}{4}z \), we obtain the Maxwell theory.

\[ \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \] (4)
However the Lagrangian density generally becomes the infinite series of the fundamental fields.

In order to carry out canonical quantization, we fix the gauge of \( U(1) \) symmetry. We take the light-cone gauge: \( A_- = 0 \), where
\[ A_\pm = A_0 \pm A_1. \] (5)
Then (4) is written as
\[ \mathcal{L} = \Phi(-2(\partial_- A_\pm)^2). \] (6)
where \( x^\pm = x^0 \pm x^1 \). The equations of motion are derived from (3) as follows:

\[
\partial_- \left( \frac{\partial \Phi}{\partial (-A_+)} \right) = 0,
\]

(7)
The canonical conjugate momentum of \( A_+ \) is

\[
\pi_{A_+} \equiv \frac{\partial L}{\partial (\partial_0 A_+)} = \frac{1}{2} \frac{\partial \Phi}{\partial (-A_+)}.
\]

(8)
If we define

\[
\Psi(\partial_- A_+) \equiv \frac{\partial \Phi}{\partial (-A_+)},
\]

(9)
then (8) is rewritten as

\[
\pi_{A_+} = \frac{1}{2} \Psi(\partial_- A_+).
\]

(10)
Let us analyze the solution of this theory. In order to quantize the theory, we set up the canonical commutation relations of the canonical quantities as follows:

\[
[\pi_{A_+}, A_+]_0 = -i\delta(x^1 - y^1),
\]

\[
[A_+, A_+]_0 = 0,
\]

\[
[\pi_{A_+}, \pi_{A_+}]_0 = 0,
\]

(11)
where \([, ]_0\) denotes the equal-time commutation relations at \( x^0 = y^0 \). We want to calculate the multiple commutation relations

\[
\cdots [A_+(x_1), A_+(x_2), A_+(x_3), \cdots, A_+(x_n)].
\]

(12)
In the precise Abe-Nakanishi method, we set up the Cauchy problems of (12) from (7), (10) and (11) and solve them directly. We can obtain the multiple commutation relations in the above method, but we use simpler method in this paper.

We can solve the equations of motion explicitly. Integrating (7), we can write

\[
\frac{\partial \Phi}{\partial (-A_+)} = \Psi(\partial_- A_+) = f(x^+),
\]

(13)
where \( f(x^+) \) is a hermitian operator depending on only \( x^+ \). Here we assume \( \Psi \) be invertible. Then (13) is rewritten as

\[
\partial_- A_+(x) = \Psi^{-1}(f(x^+)).
\]

(14)
Therefore we can solve $A_+(x)$ as
\[ A_+(x) = \Psi^{-1}(f(x^+))x^- + g(x^+), \tag{15} \]
where $g(x^+)$ is a hermitian operator depending only on $x^+$.

We can express $\pi_{A_+}$ in terms of $f(x^+)$ from (10) and (13):
\[ \pi_{A_+} = \frac{1}{2} f(x^+). \tag{16} \]

If we substitute (15) and (16) to (11), we obtain the equal-time commutation relations of $f$ and $g$. Since $f$ and $g$ depend on only $x^+$, two-dimensional commutation relations of $f$ and $g$ are calculated as follows:
\[ [f(x^+), f(y^+)] = [g(x^+), g(y^+)] = 0, \]
\[ [f(x^+), g(y^+)] = -2i\delta(x^+ - y^+). \tag{17} \]

As seen from $\partial_- f = \partial_- g = 0$, $f$ and $g$ are the currents which generate the residual gauge symmetries. From (17) and (15), we can derive the two-dimensional commutation relation of $A_+$ as follows:
\[ [A_+(x), A_+(y)] = -2i(\Psi^{-1}(f(x^+)))'(x^- - y^-)\delta(x^+ - y^+) \]
\[ = -2i(\Psi'(\partial_- A_+(x)))^{-1}(x^- - y^-)\delta(x^+ - y^+) \]
\[ = -2i \left( \frac{\partial^2 \Phi}{\partial (\partial_- A_+)^2} (x) \right)^{-1} (x^- - y^-)\delta(x^+ - y^+) \]
\[ = -\frac{i}{\pi} \left( \frac{\partial^2 \Phi}{\partial (\partial_- A_+)^2} (x) \right)^{-1} D(x - y), \tag{18} \]
where $'$ is the differentiation of a function and $D(x)$ is defined by
\[ D(x) = 2\pi x^- \delta(x^+). \tag{19} \]

It is straightforward to calculate multiple commutation relations of $A_+$ as follows:
\[ [\cdots [A_+(x_1), A_+(x_2)], A_+(x_3)], \cdots, A_+(x_n)] \]
\[ = (-2i)^{n-1} S_n \left[ \Psi^{-1}(z)^{(n-1)} \bigg|_{z=f(x_i)} \right] \]
\[ \times \delta(x_1^+ - x_2^+)\delta(x_2^+ - x_3^+) \cdots \delta(x_{n-1}^+ - x_n^+), \]
\[ = \left( -\frac{i}{\pi} \right)^{n-1} S_n \left[ \Psi^{-1}(z)^{(n-1)} \bigg|_{z=f(x_i)} \right] \]
\[ \times D(x_1 - x_2)\partial_-^{x_2} D(x_2 - x_3) \cdots \partial_-^{x_{n-1}} D(x_{n-1} - x_n), \tag{20} \]
where $S_n$ is an arbitrary symmetrization of the arguments $x_1^+,\cdots,x_n^+$ in $\Psi^{-1}$. Symmetrization is not necessary in the operator solutions, but it is necessary when we construct the Wightman functions. These solutions (20) also can be obtained by solving the operator Cauchy problems with respect to the multiple commutation relations.

We can prove that it is sufficient to obtain the above commutation relations in order to construct all the commutation relations of the fundamental fields. (20) is the exact operator solution of the quantum nonlinear abelian gauge theory in the Heisenberg picture.

3 The Wightman Functions

Next, we construct the Wightman functions in this theory. We set the vacuum expectation values of $f$ and $g$ as

$$\langle f(x^+) \rangle = f(x^+),$$
$$\langle g(x^+) \rangle = g(x^+),$$

(21)

where $f$ and $g$ are arbitrary real c-number functions. If these one-point functions are non-vanishing, Lorentz invariance is broken, but we dare to include nonzero expectation values to consider general situations. From

$$[f(x), f(y)] = 0,$$

(22)

and the energy positivity requirement we can trivially calculate truncated $n$-point Wightman functions of $f$. For example, the non-truncated two-point Wightman function of $f(x)$ is given by the product of two one-point functions:

$$\langle f(x)f(y) \rangle = \langle f(x) \rangle \langle f(y) \rangle = f(x)f(y).$$

(23)

In order to obtain the one-point function of $A_+$, we take the following generalized normal-product rule:

The $n$-point Wightman function $W(x_1,\cdots,x_n)$ with $x_i^\mu = x_1^\mu = \cdots = x_j^\mu (i < j)$ is defined from $W(x_1,\cdots,x_n)$ by setting $x_i^\mu = x_1^\mu = \cdots = x_j^\mu$ and by
deleting the resulting divergent terms in such a way that it be independent of the
ordering of $i, i + 1, \ldots, j$.

Since the truncated $n$-point Wightman functions of $f$ is trivial, we obtain the one-point
function of $A_+$ from the above rule as follows:

$$
\langle A_+(x) \rangle = \Psi^{-1}(f(x))x^- + g(x), \quad (24)
$$

Hence (18) and the energy positivity requirement lead the two-point Wightman function of
$A_+$ to

$$
\langle A_+(x_1)A_+(x_2) \rangle_T = -\frac{1}{\pi} \left\langle S_2 \left[ \left( \frac{\partial^2 \Phi}{\partial (\partial_+ A_+(x_i))^2} \right)^{-1} \right] D^{(+))(x_1 - x_2) \right\rangle
$$

$$
= -\frac{1}{\pi} S_2 [ (\Psi^{-1}(f(x_i))') D^{(+))(x_1 - x_2)], \quad (25)
$$

where

$$
D^{(+))(x)} = \frac{x^-}{x^+ - i0^+} \quad (26)
$$

There are plural symmetrization methods of the arguments $x^+_1, \ldots, x^+_n$ in $S_2[(\Psi^{-1}(f(x_i))]$. For example,

$$
S_2[(\Psi^{-1}(f(x_i))') = \frac{1}{2} [ (\Psi^{-1}(f(x_1))' + (\Psi^{-1}(f(x_2)'))', \quad (27)
$$

or

$$
S_2 [(\Psi^{-1}(f(x_i))'] = \left( \Psi^{-1}(f(x_1))' (\Psi^{-1}(f(x_2))', \quad (28)
$$

are some symmetrizations. We find any symmetrized solution in (27) is consistent with (24).
Therefore there are plural Wightman functions consistent with (18) and the energy positivity
requirement. In order to determine them uniquely, we need other assumptions.

From (24) and the energy positivity requirement, we can calculate the $n$-point Wightman
functions of $A_+$,

$$
\langle A_+(x_1)A_+(x_2) \cdots A_+(x_n) \rangle_T = \frac{1}{n!} \sum_{p(i_1, \ldots, i_n)} S_n \left[ \left( \frac{d}{dZ} \right)^{n-1} \Psi^{-1}(Z) \right]_{Z=f(x_i)}
$$

$$
\times \left( -\frac{1}{\pi} \right)^{n-1} D^{(+)i_1}(x_{i_1} - x_{i_2})\partial_{x_2}^{i_2} D^{(+)i_2}(x_{i_2} - x_{i_3}) \cdots \partial_{x_{i_{n-1}}}^{i_{n-1}} D^{(+)i_{n-1}}(x_{i_{n-1}} - x_{i_n}), \quad (29)
$$
where
\[
D_{<}^{(+)}(x_i - x_j) = \begin{cases} 
D^{(+)}(x_i - x_j), & \text{if } i < j \\
D^{(+)}(x_j - x_i), & \text{if } i > j 
\end{cases}
\]  
(30)
and \(P(i_1, \ldots, i_n)\) is a permutation of \((1, \ldots, n)\). The symmetrization \(S_n\) is arbitrary same as (23). Therefore there are plural solutions consistent with (20).

Here, we give some remark on the expression for the \(n\)-point functions. In (20), we can identically replace the arguments of delta functions. So there is apparent ambiguity at the representations of multiple commutation relations by \(D^{(+)}(x)\). For example, we can have another expression for them as follows:

\[
\begin{align*}
& \cdots [A_+(x_1), A_+(x_2)], A_+(x_3)], \cdots, A_+(x_n)] \\
& = (-2i)^{n-1} S_n \left[ \Psi^{-1}(z)^{(n-1)} \bigg|_{z=f(x)} (x_1^--x_2^-) \right. \\
& \times \delta(x_1^+-x_2^+)\delta(x_1^+-x_3^+)\cdots \delta(x_1^+-x_n^+), \\
& = \left( -\frac{i}{\pi} \right)^{n-1} S_n \left[ \Psi^{-1}(z)^{(n-1)} \bigg|_{z=f(x)} \right. \\
& \times D(x_1 - x_2) \partial_{x_1} \cdots \partial_{x_1} D(x_1 - x_n). \\
\end{align*}
\]  
(31)

Of course, (31) is equal to (20). From (31), we can derive the different expression for of \(n\)-point Wightman functions:

\[
\langle A_+(x_1)A_+(x_2)\cdots A_+(x_n) \rangle_T = \frac{1}{(n-2)!n} \sum_{P(i_1, \ldots, i_n)} S_n \left[ \left( \frac{d}{dz} \right)^{n-1} \Psi^{-1}(z) \bigg|_{z=f(x)} \right. \\
& \times \left( -\frac{1}{\pi} \right)^{n-1} D_{<}^{(+)}(x_{i_1} - x_{i_2}) \partial_{x_{i_1}} D_{<}^{(+)}(x_{i_1} - x_{i_3}) \cdots \partial_{x_{i_1}} D_{<}^{(+)}(x_{i_1} - x_{i_n}),
\]  
(32)

Since energy positivity conditions may give us sufficient constraints to determine the \(n\)-point Wightman functions uniquely from \(n\)-ple commutation relations except for symmetrization ambiguity for the arguments \(x_1^+, \ldots, x_n^+\) in \(\Psi^{-1}\), (29) and (32) should be equivalent. Indeed we can confirm (29) and (32) are identical by using the explicit expression for \(D^{(+)}(x)\) given by (26).

The exact Wightman functions may break the equations of motion or the Ward-Takahashi identities in some theories. This 'anomaly' arises from regularization of divergences of Wight-
man functions at the same spacetime points. We have defined the rule to construct the Wightman functions with the same spacetime points at the sentence after (23). If we define the Wightman functions for the composite fields according to it, we find that we can subtract divergences so as to be consistent with the equations of motion and the Ward-Takahashi identities in this model as follows.

We find that if a truncated Wightman function includes \( \partial_{-} x_k A_+ (x_k) \), it does not depend on \( x_k^- \) from (29). And since \( \partial_- A_+ \) commute mutually in two dimension and the truncated \( n \)-point functions of \( \partial_- A_+ \) are zero, any Wightman function which includes a product of \( \partial_- A_+ \)'s at the same spacetime point is non-singular. Thus

\[
\langle \Psi (\partial_- A_+ (x_1)) A_+ (x_2) \cdots A_+ (x_n) \rangle, \tag{33}
\]
is obtained consistently, and does not depend on \( x_1^- \). Therefore

\[
\langle \partial_{-} x_1 \Psi (\partial_- A_+ (x_1)) A_+ (x_2) \cdots A_+ (x_n) \rangle = 0, \tag{34}
\]
and we can confirm the Wightman functions are consistent with the equation of motion.

4 The Born-Infeld Theory

The action (3) reduces to the Born-Infeld theory in two dimension when we set

\[
\Phi (F^{\mu \nu} F_{\mu \nu}) = \frac{1}{2\kappa} \left[ -\sqrt{1 + \kappa F^{\mu \nu} F_{\mu \nu}} + 1 \right], \tag{35}
\]
where \( \kappa \) is a coupling constant. Then \( \Psi \) is written

\[
\Psi (\partial_- A_+) = \frac{\partial_- A_+}{\sqrt{1 - 2\kappa (\partial_- A_+)^2}}, \tag{36}
\]
and \( \Psi^{-1} \) is calculated as

\[
\Psi^{-1} (f(x^+)) = \frac{f(x^+)}{\sqrt{1 + 2\kappa f(x^+)^2}}, \tag{37}
\]

Expanding the Lagrangian density in power of \( \kappa \), we obtain the Maxwell theory at zeroth order:

\[
\mathcal{L} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \cdots. \tag{38}
\]
However the Lagrangian density becomes the infinite series of the fundamental fields.

We can calculate a solution by assuming
\[
\sqrt{1 - 2\kappa (\partial \cdot A_+)} \neq 0,
\tag{39}
\]
if we substitute (35) and (36) into the results in the section 2.

From (18), we can derive the two-dimensional commutation relation of \( A_+ \) as follows:
\[
[A_+(x), A_+(y)] = -2i \frac{1}{(1 + 2\kappa f(x)^2)^2} (x^- - y^-) \delta(x^+ - y^+)
\]
\[
= -2i [1 - 2\kappa (\partial_+ A_+(x))]^2 (x^- - y^-) \delta(x^+ - y^+)
\]
\[
= -\frac{i}{\pi} S(x,y) \left[ [1 - 2\kappa (\partial_+ A_+(x))]^2 \right] D(x - y).
\tag{40}
\]

If \( \kappa \to 0 \), the above expression goes to results of the Maxwell theory as it should be. We derive multiple commutation relations of \( A_+ \) from (20):
\[
\cdots, [A_+(x_1), A_+(x_2)], \cdots, A_+(x_n)]
\]
\[
= (-2i)^{n-1} S_n \left[ \left( \frac{d}{dz} \right)^{n-1} \left( \frac{z}{\sqrt{1 + 2\kappa z^2}} \right) \right]_{z = f(x_1)} (x_1 - x_2)
\]
\[
\times \delta(x_1^+ - x_2^+) \delta(x_2^+ - x_3^+) \cdots \delta(x_n^+ - x_1^+)
\]
\[
= \left( \frac{\pi}{i} \right)^{n-1} S_n \left[ \left( \frac{d}{dz} \right)^{n-1} \left( \frac{z}{\sqrt{1 + 2\kappa z^2}} \right) \right]_{z = f(x_1)}
\]
\[
\times D(x_1 - x_2) \partial z_1^+ D(x_2 - x_3) \cdots \partial z_n^+ D(x_n - x_1).
\tag{41}
\]

From (24), we obtain the one-point function of \( A_+ \) as follows:
\[
\langle \partial_- A_+ (x) \rangle = \frac{f(x)}{\sqrt{1 + 2\kappa f(x)^2}},
\]
\[
\langle A_+ (x) \rangle = \frac{f(x)}{\sqrt{1 + 2\kappa f(x)^2}} x^- + g(x),
\tag{42}
\]

From (29), we can calculate the \( n \)-point Wightman functions of \( A_+ \),
\[
\langle A_+(x_1) A_+(x_2) \cdots A_+(x_n) \rangle_T = \frac{1}{n} \sum_{P(i_1, \ldots, i_n)} S_n \left[ \left( \frac{d}{dZ} \right)^{n-1} \left( \frac{Z}{\sqrt{1 + 2\kappa Z^2}} \right) \right]_{Z = f(x_i)}
\]
\[
\times \left( \frac{1}{\pi} \right)^{n-1} D^{(+)}_<(x_{i_1} - x_{i_2}) \partial x_{i_2}^+ D^{(+)}_<(x_{i_2} - x_{i_3}) \cdots \partial x_{i_n-1}^+ D^{(+)}_<(x_{i_{n-1}} - x_{i_n}).
\tag{43}
\]
5 Conclusion and Discussion

We have exactly solved the quantum nonlinear abelian gauge theory in two dimension in the light-cone gauge by the canonical-quantization operator formalism in the Heisenberg picture. For the solvability of this theory, we have assumed $\Psi$ be invertible. We have calculated the exact multiple commutation relations and $n$-point Wightman functions. The generalization to the nonlinear non-abelian gauge theory is trivial.

In this model, two dimensional commutation relation (18) is

$$[A_+(x), A_+(y)] \neq 0.$$  

(44)

So we have obtained a new model which can be solved by the method proposed by Abe and Nakanishi.

There are symmetrization ambiguities in the Wightman functions (25) and (29). There are plural Wightman functions consistent with multiple commutation relations and energy positivity. Therefore the consistency with multiple commutation relations and energy positivity is insufficient to determine the Wightman functions uniquely in this model. Moreover it is important to clarify the relation between our results and the analysis in conventional perturbative approach.

In the covariant gauge, we encounter more general commutation relations than in this paper. Since it is an example with new commutation relations, it is interesting to analyze it by the operator formalism in the Heisenberg picture. We should develop general mathematical techniques of the Cauchy problems involving noncommutative quantities in order to analyze a model with more general commutation relations.

The regularization procedures in our method is a generalization of traditional one, for example, the normal ordering of free fields. And ours are applicable to not only free fields but general Heisenberg fields. Therefore it is interesting to apply to the gravitational theory, which is essentially non-polynomial and non-renormalizable. This method for solving quantum theory will be useful to treat non-polynomial quantum field theories.
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