INTEGRALITY AND GAUGE DEPENDENCE
OF HENNINGS TQFTS

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ABSTRACT. We provide a general construction of an integral TQFT over a Dedekind domain using the quantum double of a Hopf algebra which is projective as a module over the same domain. It specializes to the Hennings invariant on closed 3-manifolds.

Moreover, we show that TQFTs obtained from Hopf algebras that are related by a gauge transformation in the sense of Drinfeld are isomorphic, and provide an explicit formula for the natural isomorphism in terms of the twist element.

These two results are combined and applied to show that the Hennings invariant associated to quantum-$\mathfrak{sl}_2$ takes values in the cyclotomic integers. Using prior results of Chen et al we infer integrality also of the Witten-Reshetikhin-Turaev $SO(3)$ invariant for rational homology spheres.

As opposed to most other approaches the methods described in this article do not invoke or make reference to calculations of skeins, knots polynomials, or representation theory, but follow combinatorial constructions that use the elements and operations of the underlying Hopf algebras directly.

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CONTENTS

1. Introduction and Main Results 1
2. Topological and Algebraic Prerequisites 8
3. The Hennings TQFT Construction 15
4. Equivariance, Closed Surfaces, and Closed Manifolds 23
5. Gauge Transformations of Hopf algebras 30
6. Gauge Transformations for Hennings TQFTs 35
7. Integral TQFTs from Quantum Doubles 43
8. The Hennings TQFT for the quantum double of $B(\mathfrak{sl}_2)_\zeta$ 53
References 68

1. INTRODUCTION AND MAIN RESULTS

The mathematical axiomatization of 2+1-dimensional Topological Quantum Field Theory (TQFT) due to Atiyah [At89] has been refined in numerous ways since its
original formulation. Among the most interesting generalizations is the extension of the coefficient ring over which the TQFT is defined from the complex numbers to rings which possess a rich ideal structure suited to capture more subtle topological information.

A large and powerful family of TQFTs is based on the famous Witten-Reshetikhin-Turaev (WRT) construction \cite{RT91}, which are a-priori formulated over the complex numbers. It was soon realized that WRT invariants and TQFTs can essentially be defined over $\mathbb{D} = \mathbb{Z}[\zeta]$, the cyclotomic integers associated to a root of unity $\zeta$ (see \cite{Mu94,MR97,Le93,MW98,Gi04,CL05,Ha08}) and that the ideal and integrality structure of $\mathbb{Z}[\zeta]$ has useful topological applications as described, for example, in \cite{FK01,GKP02,CL04}.

In this article we establish analogous integrality results for the family of so called the Hennings TQFTs associated to a ribbon Hopf algebra $\mathcal{H}$, as developed and studied in \cite{He96,KR95,Ke95,Oh95}. Their construction starts, as for the WRT invariants, from surgery presentations but then uses elements of $\mathcal{H}$ directly in its computations, thus circumventing the use of the representation theory of $\mathcal{H}$ as in \cite{RT91} or the respective (essentially equivalent) combinatorial skein theory as in \cite{BHMV}.

The Hennings invariants (and TQFTs) behave manifestly differently than WRT invariants on 3-manifolds with non-trivial homology for non-semisimple Hopf algebras $\mathcal{H}$ but often coincide or are very similar in other cases. Indeed, in this article we will provide a precise relation between these two types of invariants in the special case of quantum $\mathfrak{sl}_2$. Let us now describe the main results of this article in more detail.

1.1. Some Basic Terminology. The primary cobordism category we consider is denoted $\text{Cob}^\bullet$ and has connected, compact, oriented surfaces with one boundary component as objects and classes of relative, 2-framed cobordisms as morphisms. The 2-framing may be more conveniently thought of as the signature of a bounding 4-manifold \cite{At90} or also the signature of a certain closure of the framed tangle representing the cobordism \cite{Ke99}. The respective category $\text{Cob}^\emptyset$ with closed surfaces is related by an obvious fill functor $\mathcal{F}_\emptyset : \text{Cob}^\bullet \to \text{Cob}^\emptyset$ which pastes a disk into the boundary component of a surface. See \cite{KL01} and Section 2 for more detailed definitions.

The target categories of the TQFT functors are defined with respect to a ribbon Hopf algebra $\mathcal{H}$ over a commutative ring $\mathbb{D}$, such that $\mathcal{H}$ is projective and finitely generated as a $\mathbb{D}$-module. We denote by $\text{proj}(\mathbb{D})$ the category of finitely generated
projective $\mathbb{D}$-modules, and by $\mathcal{K} \models \text{proj}(\mathbb{D})$ the category of $\mathcal{K}$-modules which are finitely generated and projective as $\mathbb{D}$-modules (but not necessarily as $\mathcal{K}$-modules). Besides the forgetful functor these categories are also related by the invariance functor

$$\text{Inv}_{\mathcal{K}} : \mathcal{K} \models \text{proj}(\mathbb{D}) \to \text{proj}(\mathbb{D}) : X \mapsto \text{Inv}_{\mathcal{K}}(X) = \text{Hom}_{\mathcal{K}}(1, X).$$

We also denote by $\mathcal{K} \models \text{free}(\mathbb{D})$ and $\text{free}(\mathbb{D})$ the respective subcategories of free $\mathbb{D}$-modules (for which the invariance functor is generally not well defined).

In this article we will focus on Hopf algebras given as the Drinfeld double $\mathcal{H} = \mathbb{D}(H)$ of a Hopf algebra $H$ \cite{Drinfeld, Kauffman} and assume that $\mathbb{D}$ is a Dedekind domain. In this case the technical assumptions on $H$ required for the TQFT construction reduce to only two basic ones, namely that $H$ as is projective and finitely generated as a $\mathbb{D}$-module as well as a double balancing condition which we describe next:

For a finite dimensional Hopf algebra $H$ over a field Sweedler shows in \cite{Sweedler} that the space of left cointegrals $\{ \Lambda : x\Lambda = \epsilon(x)\Lambda \ \forall x \in H \}$ is exactly one-dimensional. The analogous statement in the case when $\mathcal{H}$ is projective and finitely generated over a Dedekind ring is given in Proposition 28 in this article. An immediate implication is that there is a distinguished group-like element $\alpha \in H$ with $\Lambda y = \alpha(y)\Lambda \ \forall y \in H$ called comodulus. Analogously, we can associate to a right integral $\lambda \in H$ a distinguished group-like element $g \in H$, the modulus.

A well know result by Radford \cite{Radford} states that these elements implement the fourth order of the antipode by $S^4 = \text{ad}(g) \circ \text{ad}^*(\alpha)$ which readily extends to Hopf algebras with moduli over general rings. We call a Hopf algebra double balanced if $g$ and $\alpha$ have compatible square roots that correspondingly implement the square of the antipode. More precisely, we require that there are group-like elements $l \in H$ and $\beta \in H^*$ with

$$\beta^2 = \alpha, \quad l^2 = g, \quad \text{and} \quad S^2 = \text{ad}(l) \circ \text{ad}^*(\beta).$$

We also denote the numbers obtained by pairing the moduli as follows

$$\theta = \beta(l) \quad \text{with} \quad \theta^4 = \alpha(g),$$

which we will show to be a root of unity in $\mathbb{D}$ itself.

1.2. Statements of Main Results. With the definitions above we can now state the first of our main results.

**Theorem 1.** Suppose $H$ is a double balanced Hopf-algebra over a Dedekind domain $\mathbb{D}$ such that $H$ is finitely generated and projective as a $\mathbb{D}$-module.
(a) Then there are TQFT functors $\mathcal{V}_D^\emptyset$ and $\mathcal{V}_D^\bullet$ from cobordism categories to categories of $D$-modules as indicated in the horizontal arrows in Diagram (4) below. Moreover, this diagram of functors commutes.

$$\begin{array}{ccc}
\mathsf{Cob}^\bullet & \xrightarrow{\mathcal{V}_D^\bullet} & \mathcal{D}(H)\dagger \mathsf{proj}(D) \\
\mathsf{Cob}^\emptyset & \xrightarrow{\mathcal{V}_D^\emptyset} & \mathsf{proj}(D) \\
\mathcal{F}_\emptyset & \Downarrow & \mathcal{F}^\emptyset \\
\mathcal{F}_\emptyset & \Downarrow & \mathcal{F}^\emptyset \\
\mathcal{F}_\emptyset & \Downarrow & \mathcal{F}^\emptyset
\end{array}$$

(b) For $\theta$ as defined in (3) above and a closed 2-framed 3-manifold $M^*$, the value of $\mathcal{V}_D^\emptyset$ is related to the Hennings invariant $\varphi_D(M)$ for $D(H)$ as follows

$$\varphi_D(M) = \theta^{3\sigma(M^*)} \cdot \mathcal{V}_D^\emptyset(M^*),$$

where $M$ is the underlying 3-manifold of $M^*$ and $\sigma(M^*)$ is the signature corresponding to the 2-framing of $M^*$.

(c) If $H$ is a free $D$-module of finite rank then $\mathcal{V}_D^\bullet$ restricts to a TQFT functor

$$\mathcal{V}_D^\bullet : \mathsf{Cob}^\bullet \rightarrow \mathcal{D}(H)\dagger \mathsf{free}(D).$$

Theorem 1 is based on the more general Theorem 12, which asserts the existence of a TQFT functor, constructed explicitly in Section 2,

$$\mathcal{V}_D^\bullet : \mathsf{Cob}^\bullet \rightarrow \mathcal{H}\dagger \mathsf{proj}(D) \quad \text{(or } \mathcal{H}\dagger \mathsf{free}(D))$$

for a topogenic Hopf algebra, that is, a ribbon Hopf algebra $\mathcal{H}$ satisfying a list of technical assumptions described in detail in Section 2.2. The upshot of Theorem 1 is that all these technical assumptions are automatically fulfilled for the quantum double of a double balanced Hopf algebra.

In the second main result of this paper we describe the behavior of Hennings TQFT functors with respect (strict) gauge transformations of the coalgebra and quasi-triangular structure of $\mathcal{H}$. The notion of gauge transformations was introduced by Drinfeld in [Dr90] for quasi Hopf algebras, see also Section XV.3 of [Ka94]. Here an invertible element $F \in \mathcal{H} \otimes \mathcal{H}$ is used to define gauge transformed coproducts and R-matrices by $\Delta_F(x) = F \Delta(x) F^{-1}$ and $R_F = F_{21} R F^{-1}$. We denote by $\mathcal{H}_F$ the ribbon Hopf algebra with the so transformed coproduct and R-matrix.

In this article we confine ourselves to strict quasi-triangular Hopf algebras with trivial associators, which imposes an additional cocyle condition given in (52).
Theorem 2. Suppose $\mathcal{H}$ is a ribbon Hopf algebra fulfilling the prerequisites for the Hennings TQFT construction and $F \in \mathcal{H} \otimes \mathcal{H}$ fulfills the cocyle conditions from (52). Then there exists a natural isomorphism of TQFT functors
\[
\gamma_F : \mathcal{V}_\mathcal{H} \to \mathcal{V}_{\mathcal{H}F}.
\]
(8)

We will give an explicit formula for $\gamma_F$ in (94).

Of particular interest is the case when $H$ is the Borel subalgebra $B(\mathfrak{sl}_2)_\zeta$ of the quantum group $U_\zeta(\mathfrak{sl}_2)$ where $\zeta$ is a root of unity of odd order. Theorem 1 can now be used to infer integrality of the associated TQFT functor and Hennings invariant as stated in Corollary 34 below.

The Hennings invariant for $D(B(\mathfrak{sl}_2)_\zeta)$ is closely related to the WRT invariant $\tau_{SO^3}^\zeta$, which is also constructed via the same surgery presentations from categories obtained from $U_\zeta(\mathfrak{sl}_2)$. In order to state the precise relation we introduce the following semi-classical invariants. The first is
\[
h(M) = \begin{cases} 
|H_1(M,\mathbb{Z})| & \text{for } \beta_1(M) = 0 \\
0 & \text{for } \beta_1(M) > 0
\end{cases}.
\]
(9)
The second is the MOO invariant $\mathcal{Z}_\zeta(M)$ introduced in [MOO92] (see also Section 8.1), which is computed from only the linking matrix of a representing framed link.

Theorem 3. Let $M$ be a closed oriented 3-manifold and $\zeta$ be a root of unity of odd order $\ell$. Let $h(M)$ and $\mathcal{Z}_\zeta(M)$ be as above. Then the Hennings invariant $\varphi_{D(B(\mathfrak{sl}_2)_\zeta)}$ and the WRT $SO(3)$ invariant $\tau_{SO^3}^\zeta(M)$ are related as follows:
\[
\varphi_{D(B(\mathfrak{sl}_2)_\zeta)}(M) = h(M) \mathcal{Z}_\zeta(M) \tau_{SO^3}^\zeta(M).
\]
(10)

Theorem 3 is obtained from the more general Theorem 40 asserting an analogous factorization of TQFTs, which, in turn, is based on the almost factorization of $D(B(\mathfrak{sl}_2)_\zeta)$ into a version $U_\zeta$ of quantum-$\mathfrak{sl}_2$ and and algebra $\mathcal{A}$ underlying the MOO invariant. The subtlety that prevents a strict factorization is that $R$-matrix and coproduct of $D(B(\mathfrak{sl}_2)_\zeta)$ differs from that of $U_\zeta \otimes \mathcal{A}$ by a Drinfeld gauge twist as described above, see Proposition 36.

Thus Theorem 2 needs to be invoked and enters the proofs of Theorem 3 and Theorem 40. The gauge twist $F$ and isomorphism $\gamma_F$ can be defined over $\mathbb{Z}[\zeta, \frac{1}{\zeta}]$.

An immediate corollary and application of Theorem 3 is the rederivation of the integrality result given by Le in [Le08]. Unlike the original proof no reference to the colored Jones Polynomial or representations of quantum algebras is made here.
Corollary 4 ([Le08]). Suppose $\zeta$ is a root of unity of odd order $\ell > 1$ and $M$ is a rational homology sphere. If $h(M)$ and $\ell$ are coprime then $\tau_{\zeta}^{SO(3)}(M) \in \mathbb{Z}[\zeta]$.

Proof. Since $\varphi_{\mathcal{D}(B(sl_2)_{\zeta})}$ extends to an integral TQFT $\mathcal{V}^\bullet_{\mathcal{D}(B(sl_2)_{\zeta})}$ as in Theorem 1 we find $\varphi_{\mathcal{D}(B(sl_2)_{\zeta})}(M) \in \mathbb{Z}[\zeta]$. Given $(h(M), \ell) = 1$ we have by Lemma 31 that $\mathcal{L}_\zeta(M) = \pm 1$ so that Equation (10) implies $h(M) \tau_{\zeta}^{SO(3)}(M) \in \mathbb{Z}[\zeta]$. As remarked in the end of Section 1 of [KM91] we also have $\tau_{\zeta}^{SO(3)}(M) \in \mathbb{Z}[\zeta, \frac{1}{\ell}]$ and hence $\ell^m \tau_{\zeta}^{SO(3)}(M) \in \mathbb{Z}[\zeta]$ for some $m \in \mathbb{N}$. Using again $(h(M), \ell^m) = 1$ this immediately implies $\tau_{\zeta}^{SO(3)}(M) \in \mathbb{Z}[\zeta]$. \hfill $\square$

We note that the result in [Le08] has been generalized in [BL07]. In particular, the fact that $M$ is a rational homology sphere and that $(h(M), \ell) = 1$ are no longer required to ensure integrality. An argument similar to one used in the above proof can be found in [CYZ12].

1.3. Some Relations and Open Problems. Although our method of constructing integral TQFTs is rather different from approaches based on calculations of skeins and knots polynomials, it is interesting to observe that the same or similar technical constraints surface. For example, the combinatorial approach starting from WRT type TQFTs as proposed by Gilmer in [Gi04] finds Dedekind domains to be the natural class of ground rings to consider.

In other ways, the two types of TQFT constructions exhibit fundamentally different behaviors. For example, the Hennings setting used in this article naturally includes non-semisimple algebras and TQFTs, which are more often related to geometrically constructed TQFTs (for a basic example see [Ke03] and references therein). Non-semisimplicity, is, however, incompatible with the non-degeneracy conditions required in the construction of [Gi04], such as Axiom 3 of [Gi04], to infer integrality of TQFTs from previous integrality results on closed manifolds such as [Mu94, MR97, Le93]. (The latter results, in turn, also heavily rely on semisimplicity as they are using decompositions into simple objects).

Moreover, the two approaches imply entirely different strategies for finding bases in the cases where the $\mathbb{D}$-modules associated to surfaces are free. For the WRT-TQFT associated to $SO(3)$ bases are obtained by carefully and intricately chosen combinations of skeins as in [CM07]. There appear to be no obvious strategies for generalizations beyond the $SO(3)$-case. The Hennings TQFT associated to a general Hopf algebra $H$ assigns to a punctured connected surface a tensor product of copies
of $H$ and $H^*$. Thus if $H$ is freely and finitely generated as a module over $\mathcal{D}$ a basis for $H$ immediately implies a basis for the respective TQFT modules.

The transition from modules associated to punctured surfaces to those of closed surfaces tends to be more subtle since the invariance functor preserves projectiveness but not necessarily freeness. Particularly, the latter is identified with the subspace of $\mathcal{D}(H)^{\otimes g}$ invariant under the adjoint action of $\mathcal{D}(H)$. If $H = B(\mathfrak{g})_\zeta$ is a quantum Borel algebra associated to a Lie algebra $\mathfrak{g}$ one can expect to find bases for these subspaces via recursion relations on PBW-bases for $D(B(\mathfrak{g})_\zeta)$. In fact, for $\mathfrak{g} = \mathfrak{sl}_2$ computations suggest that bases can also be found for general roots of unity essentially generated by polynomial rings of Casimir-elements (see [Ke94] for the genus one case).

We finally note that freeness for the $\mathbb{Z}[\zeta]$-modules associated by the TQFT to closed surfaces is also implied for values of $\ell = 3, 4, 5, 7, 8, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21, 24, 25, 27, 28, 32, 33, 35, 36, 40, 44, 45, 48, 60, 84$ simply because in these cases $\mathbb{Z}[\zeta]$ has class number one [Wa97] and is thus a PID so that projective modules are automatically free.

1.4. **Summary of Contents.** This paper is organized as follows. Section 2 we review definitions of cobordism categories, their tangle presentations, and properties of Hopf algebras that will be required for the subsequent TQFT constructions. In Section 3 we describe in more formal terms the TQFT construction for surfaces with one boundary component that generalizes the Hennings invariant for closed 3-manifolds, introducing auxiliary categories of $\mathcal{H}$-labeled planar curves that will play an important technical role throughout the paper. Several functors involving these categories are constructed, assuming a list of properties of the underlying Hopf algebra, from which the TQFT is assembled. The extension of the TQFT functor to closed surfaces as well as the specialization to closed manifolds, resulting in the original Hennings invariant are discussed in Section 4.

Section 5 reviews Drinfeld’s notion of gauge transformations of quasi-triangular Hopf algebras in the strict case where the transformation element has to fulfill a cocycle condition. We extract canonical elements associated to a gauge transformation that are used to provide explicit formulae for the change of the antipode, balancing and ribbon elements as well as the integrals under the gauge transformations. This is applied in Section 6 to describe the effect of gauge transformations on the Hennings TQFT construction, resulting in the proof of Theorem 2 above together with an explicit formula for the natural isomorphism.
In Section 7 we develop the structure theory for quantum doubles of Hopf algebras over Dedekind domains as it pertains to the TQFT construction given in Section 3. This involves the existence of integrals, moduli, and ribbon elements, as well as their proper normalizations and projective phases. This analysis, summarized in Proposition 28, entails the proof of Theorem 1 given in Section 7.4.

These results are applied in Section 8 to the special case in which the Hopf algebra is the Borel subalgebra $B(\mathfrak{sl}_2)_{\zeta}$ of quantum-$\mathfrak{sl}_2$ at an odd root of unity. In Proposition 36 and Theorem 40 we provide the factorization up to gauge isomorphism of $\mathcal{D}(B(\mathfrak{sl}_2)_{\zeta})$ and the associated TQFT respectively. This implies the proof of Theorem 3 given in Section 8.4.

2. Topological and Algebraic Prerequisites

In this section we provide some technical background required for the construction of Hennings TQFTs. Particularly, we describe the definition of cobordism categories, their tangle presentations, and various relevant definitions and properties of Hopf algebras. In particular, we introduce the notion of topogenic Hopf algebra which absorbs the requirements for the TQFT construction given in the subsequent section.

2.1. The Cobordism Category. We summarize here the definition of the cobordism category $\text{Cob}^*$ from [KL01]. The set of objects $\text{Obj}(\text{Cob}^*)$ consists of compact, connected, oriented surfaces with one boundary component. In addition, each surface $\Sigma^* \in \text{Obj}(\text{Cob}^*)$ comes equipped with a fixed orientation preserving homeomorphism $\partial \Sigma^* \cong S^1$. We also assume that for each integer $g \in \{0, 1, 2, \ldots\}$ the set $\text{Obj}(\text{Cob}^*)$ contains exactly one surface of genus $g$. For two surfaces $\Sigma_1^*$ and $\Sigma_2^*$ consider the closed surface obtained by sewing a cylinder $C = S^1 \times [0, 1]$ between the two surfaces using the boundary homeomorphisms. We denote the resulting closed surface as follows:

$$\Sigma_1^* \# \Sigma_2^* := \begin{cases} -\Sigma_1^* & \text{if } \partial \Sigma_1^* \cong S^1 \times [0, 1], \\ \Sigma_2^* & \text{if } \partial \Sigma_2^* \cong S^1 \times [1, 0] \end{cases}$$

(11)

Here $(-\Sigma_1^*)$ denotes the surface with opposite orientation. Thus, using the isomorphisms $\partial \Sigma_1^* \cong S^1$ and $\partial C \cong -S^1 \cup S^1$ the combined surface $\Sigma_1^* \# \Sigma_2^*$ will admit an orientation compatible with its pieces $-\Sigma_1^*$ and $\Sigma_2^*$. See Figure 1 for two equivalent depictions of $\Sigma_1^* \# \Sigma_2^*$ in the case where $g_1 = 2$ and $g_2 = 3$.

A cobordism is represented by a compact oriented 3-manifold $M$ with corners, together with a homeomorphism $\xi : \Sigma_1^* \# \Sigma_2^* \cong \partial M$, mapping the $S^1$-strata in (11) to the 1-dimensional corners of $M$. As usual, we consider cobordisms $(M, \xi)$ and
INTEGRALITY AND GAUGE DEPENDENCE OF HENNINGS TQFTS

Figure 1. A morphism $M : \Sigma^*_1 \to \Sigma^*_2$ in $\text{Cob}^\bullet$

$(M', \xi')$ between the same surface to be equivalent if there is a homeomorphism $\eta : M \cong M'$ such that $\eta \circ \xi = \xi'$.

A morphism in $\text{Cob}^\bullet$ is now an equivalence class of cobordisms $[M, \xi]$ together with a 2-framing of $M$, or, equivalently, the signature of a 4-manifold bounding a standard closure of $M$. To simplify the notation we will denote $\text{For simplicity we will occasionally abuse notation and write } M \text{ for a morphism instead of } [M, \xi]$. Composition in $\text{Cob}^\bullet$ is defined by gluing over the respective surface pieces and rescaling of the cylindrical pieces. The composition is extended to include the signature information by gluing together representing 4-manifolds. See [KL01] for detailed definitions and constructions.

The objects of the category $\text{Cob}^\emptyset$ are the same surfaces from $\text{Obj}(\text{Cob}^\bullet)$ but with a standard disc $D^2$ glued in along each boundary $S^1 \cong \partial \Sigma^\bullet$ yielding a closed surface $\Sigma = \Sigma^\bullet \sqcup D^2$. Cobordisms are defined in exactly the same way as for $\text{Cob}^\bullet$. Moreover, given a cobordism $M : \Sigma^*_1 \to \Sigma^*_2$ in $\text{Cob}^\bullet$ we can obtain a cobordism $M^o : \Sigma_1 \to \Sigma_2$ in $\text{Cob}^\emptyset$ by gluing in a full cylinder $D^2 \times [0, 1]$ along the boundary piece $C = S^1 \times [0, 1] \subset \partial M$. This filling is consistent with the standard closure from which the signature extension is constructed (see [KL01]). Consequently, we have a
well defined surjective, cylinder-filling functor:

$$\mathcal{F}_\partial : \text{Cob}^* \longrightarrow \text{Cob}^0$$

(12)

2.2. Algebraic Prerequisites for Hennings TQFTs. Throughout this article $\mathcal{H}$ is a Hopf algebra over a commutative ring $\mathbb{D}$ with invertible antipode. We denote by $\mathcal{H}^* = \text{Hom}_\mathbb{D}(\mathcal{H}, \mathbb{D})$ the dual space of $\mathcal{H}$. An element $\lambda \in \mathcal{H}^*$ is called a right integral on $\mathcal{H}$ if

$$\lambda f = f(1)\lambda, \quad \forall f \in \mathcal{H}^*.$$  

(13)

An element $\Lambda \in \mathcal{H}$ is called a left (resp. right) cointegral in $\mathcal{H}$ if

$$x\Lambda = \epsilon(x)\Lambda \quad (\text{resp. } \Lambda x = \epsilon(x)\Lambda), \quad \forall x \in \mathcal{H},$$

where $\epsilon$ is the counit of $\mathcal{H}$. Note that the definition in (13) can be rewritten as

$$\sum \lambda(x')x'' = (\lambda \otimes \text{id})(\Delta(x)) = \lambda(x)1 \quad \forall x \in \mathcal{H},$$

where we use Sweedler’s notation for the coproduct

$$\Delta(x) = \sum x' \otimes x'', \quad \forall x \in \mathcal{H}.$$

Let us also review a few standard notations for actions of $\mathcal{H}$ and $\mathcal{H}^*$ on each other. $\mathcal{H}^*$ carries a natural $\mathcal{H}$-bimodule structure given by the formulas and notation

$$(f \leftarrow a)(b) = f(ab) = (b \rightarrow f)(a),$$

(14)

for all $f \in \mathcal{H}^*$, $a, b \in \mathcal{H}$. Similarly, $\mathcal{H}$ can be endowed with an $\mathcal{H}^*$-bimodule structure via the formulas

$$f \rightarrow a = \sum f(a'')a' \quad \text{and} \quad a \leftarrow f = \sum f(a')a'',$$

(15)

A Hopf algebra $\mathcal{H}$ is said to be unimodular if there exists a non-zero left cointegral for $\mathcal{H}$ that is also a right cointegral for $\mathcal{H}$. Given a right integral $\lambda \in \mathcal{H}^*$ and a two-sided cointegral $\Lambda \in \mathcal{H}$ with $\lambda(\Lambda) = 1$ define now maps

$$\beta : \mathcal{H} \rightarrow \mathcal{H}^* \quad \text{with} \quad \beta(a) = a \rightarrow \lambda$$

$$\overline{\beta} : \mathcal{H}^* \rightarrow \mathcal{H} \quad \text{with} \quad \overline{\beta}(f) = \Lambda \leftarrow f$$

(16)

It is well known that these are isomorphisms. Particularly, the following relation is easily verified.

$$\overline{\beta} \circ \beta = S$$

(17)

For $\mathcal{H}$ over a field it follows in [Ra98] that unimodularity implies (and in fact is equivalent) to the following identities.

$$S(\Lambda) = \Lambda \quad \text{and} \quad \lambda(xy) = \lambda(S^2(y)x).$$

(18)
The relations in (18) are essentially the duals of Proposition 5 and Lemma 3 in [Ra98] respectively using that the comodulus \( \alpha = \epsilon \) in the unimodular case. If \( H \) is finitely generated over some domain \( D \) these equations still hold in \( H \otimes K \) where \( K \) is the field of factions of \( D \). Thus equations (18) also hold in \( H \) provided that \( \Lambda \) and \( \lambda \) are also elements of \( H \) and \( H^* \) and provided that \( D \) is a domain.

Integrals and cointegrals exist and are unique up to scalars if \( D \) is a principle ideal domain and \( H \) is a free \( D \)-module of finite rank [LS69]. Conversely, Sweedler also showed in [Sw69] that for \( H \) over field this implies that \( H \) is finite dimensional. In [Lo04] this implication is generalized, namely, that if \( D \) is an integral domain and \( H \) has an integral domain then \( H \) is finitely generated over \( D \). Thus we will always assume or imply that \( H \) is finitely generated over \( D \).

The next required ingredient for \( H \) is quasi-triangularity as defined by Drinfeld [Dr87], which stipulates the existence of an R-matrix \( R \in H^{\otimes 2} \) with functorial properties as follows.

\[
\Delta \otimes \text{id}(R) = R_{13}R_{23}, \quad \text{id} \otimes \Delta(R) = R_{13}R_{12}
\]

\[
R\Delta(x) = \Delta'(x)R, \quad \forall x \in H.
\]

Here \( \Delta' \) denotes the opposite coproduct. For other notations and more details see Section VIII.2 in [Ka94] or Section 10 in [Dr87].

A quasi-triangular Hopf algebra \( H \) is called \textit{ribbon} if it also contains an element \( r \in H \) with the following properties.

\[
r \text{ is central, \quad } S(r) = r, \quad \text{and} \quad R_{21}R = (r \otimes r)\Delta(r^{-1}).
\]

See also equation (2.48) in [Ke94]. Here \( R_{21} = \sum_j f_j \otimes e_j \) denotes the element \( R = \sum_i e_i \otimes f_i \) with transposed tensor factors. Note that \( r^2 \) is already determined by the quasi-triangular structure alone via \( m(\text{id} \otimes S)(R_{21}R) = r^2 \) so that the ribbon condition is really about the existence of compatible square roots.

The ribbon structure for a quasi-triangular Hopf algebra may, alternatively, be described as a \textit{balancing}. To this end consider the canonical element

\[
u = \sum_i S(f_i)e_i.
\]

It is well known [Dr89] that this element satisfies \( uxu^{-1} = S^2(x) \forall x \in H \) and that \( uS(u)^{-1} \) is group like. A \textit{balancing element} is defined to be a group like element for which the following hold.

\[
\kappa^2 = uS(u)^{-1} = S(u)^{-1}u, \quad S^2(x) = \kappa x \kappa^{-1}, \quad \forall x \in H.
\]
The existence of a balancing element is equivalent to the existence of a ribbon element $r$ and the two are related by
\[ r = u\kappa^{-1}. \] (23)

Yet another standard condition for the construction of TQFTs is modularity, which, in the original categorical framework, has been formulated as the invertibility of the so called “S-matrix”. In the setting of Hennings TQFTs this may be rephrased to require that the element
\[ M = R_{21} R = \sum_{ij} f_j e_i \otimes e_j f_i \] (24)
is (left) non-degenerate in the sense that the map
\[ \overline{M} : \mathcal{H}^* \to \mathcal{H} : l \mapsto l \otimes \text{id}(M) \] (25)
is an injection. It would, in indeed, be more accurate to speak of left modularity and consider as well the notion right modularity given by injectivity of $l \mapsto \text{id} \otimes l(M)$. These two conditions will turn out to be equivalent to each other and to a number of other conditions which will be discussed in greater detail in Lemma 6 below.

At this point let us formalize the the requirements on $\mathcal{H}$ needed for the construction of a Hennings TQFT from $\mathcal{H}$.

**Definition 5.** Let $\mathcal{H}$ be a quasi-triangular Hopf algebra, finitely generated over a commutative ground ring $\mathbb{D}$. We say $\mathcal{H}$ is topogenic if it satisfies the following additional conditions:

1. $\mathcal{H}$ is ribbon or, equivalently, balanced with elements $r$ and $\kappa$.
2. $\mathcal{H}$ is modular in the sense that (25) is an isomorphism.
3. There is a right integral $\lambda \in \mathcal{H}^*$ such that $\lambda(r)\lambda(r^{-1}) = 1$.
4. $\mathcal{H}$ admits a two-sided cointegral $\Lambda \in \mathcal{H}$ with $\lambda(\Lambda) = \lambda(S(\Lambda)) = 1$.

In Sections 3 and 4 we will show that any topogenic Hopf algebra gives rise to associated TQFT functors $\mathcal{V}^\bullet_\mathcal{H}$ and $\mathcal{V}^\emptyset_\mathcal{H}$, as described in Theorem 1, following the methods and constructions in [KL01] and [Ke03].

2.3. Tangle Presentations. The first key ingredient in the TQFT construction is a surgery presentation of cobordisms extending Kirby’s calculus of links for closed 3-manifolds. Instead of links we consider a category of admissible tangles $\mathcal{T}_{gl}$. Its set of objects is the set of non-negative integers $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$. For any pair $n, m \in \mathbb{Z}^+$ the set of morphisms $\text{Hom}_{\mathcal{T}_{gl}}(n, m) = \{ [T] : n \to m : T = \text{admissible} \}$ is given by equivalence classes of generic diagrams of admissible planar framed tangles in the
strip $\mathbb{R} \times [0,1]$. Each tangle consists of $n$ top components, $m$ bottom components, (each $\cong [0,1]$) and any number of closed components (each $\cong S^1$). The endpoints of the $j$-th top component connects the integers $2j - 1$ and $2j$ in the upper boundary $\mathbb{R} \times \{0\}$ of the strip, and the $k$-th bottom component connects $2k - 1$ and $2k$ in the lower boundary $\mathbb{R} \times \{0\}$. Figure 2 depicts an example of an admissible tangle $T : 2 \to 1$. The equivalences of tangle diagrams, defining the classes $[T] \in \text{Hom}_{\mathcal{T}gl}(n, m)$,

$$T : 2 \to 1 =$$

are given by isotopies of the diagrams in the plane and the usual Reidemeister moves for framed tangle diagrams. Consequently, we can think of the sets of morphisms in $\mathcal{T}gl$ also as isotopy classes of admissible, framed tangles in $\mathbb{R}^2 \times [0,1]$. As usual when drawing diagrams we assume that the framings of tangle components are given by the blackboard framing. Composition is defined by stacking diagrams on top of each other.

From any admissible tangle in $\mathbb{R}^2 \times [0,1]$ we obtain a cobordism by adding 1-handles to the $\mathbb{R}^2 \times \{1\}$ boundary of the 3-dimensional slice and continuing the bottom components through them. Moreover, we drill out holes along the bottom components of the tangle. This turns the bottom boundary of the slice into an open genus $m$ surface and the top part into a genus $n$ surface, which are canonically compactified as such that their boundary is a circle. The remaining closed as well as closed-off bottom components thus constitute a framed link $\mathcal{L}$. The cobordism is finally obtained by performing surgery in the usual fashion along $\mathcal{L}$.

This process thus describes a functor $\mathcal{H}^*$ from the category of tangles to the category cobordisms defined thus far, which assigns to each integer $n$ a surface $\Sigma_n^*$ of genus $n$, and to each tangle $T : n \to m$ a cobordism $M = \mathcal{H}^*(T) : \Sigma_n^* \to \Sigma_m^*$. This functor is surjective on morphisms – that is, any cobordism can be obtained by this type of surgery. However, generalizing the ordinary Kirby calculus for closed 3-manifolds, different tangles will yield the same cobordisms if and only if they are related by a sequence of the following moves $[\text{KL01}]$:

(T0) Isotopies
(T1) Addition and removal of a pair of isolated unknots with framings +1 and -1.
(T2) $O_2$-slides of any component over a closed component.
(T3) The $\sigma$-Move at any pair, see [KL01] or Section 3.4 below.

We denote by $\mathcal{T}C^*$ the category obtained from $\mathcal{T}gl$ by quotiening the morphism sets by the additional equivalences (T0)-(T3), and by $\mathcal{M}^*: \mathcal{T}gl \to \mathcal{T}C^*$ the functor that assigns a tangle its equivalence class with respect to these moves. We obtain the following commutative diagram.

\[
\begin{array}{ccc}
\mathcal{T}gl & \xrightarrow{\mathcal{K}^*} & \mathcal{Cob}^* \\
\downarrow{\mathcal{M}^*} & \cong & \\
\mathcal{T}C^* & \xrightarrow{\mathcal{K}} & \mathcal{Cob}^*
\end{array}
\]

(26)

It follows from the calculus described in Section 2.5.2.A of [KL01] that the functor $\mathcal{K}$ is indeed an isomorphism of categories. Let us add a few more comments on the moves above:

The $\sigma$-Move (T3) is given by replacing the tangle configuration $\Pi$ depicted in Figure 11 by two parallel strands connecting respective points at the top and bottom line.

Instead of (T1) the move described in [KL01] actually involve both the addition of Hopf links $0 \bigcirc \bigcirc 0$ and $0 \bigcirc \bigcirc 1$ for which one component is 0-framed and the other can have framing either 0 or 1. The diagrams $0 \bigcirc \bigcirc 1$ and $-1 \bigcirc \bigcirc 1$ differ only by a 2-handle slide. The calculation in Figure 3, which is essentially the same

\[
0 \bigcirc \bigcirc 0 \xrightarrow{O_2} 1 \bigcirc \bigcirc 1 \xrightarrow{O_2} 1 \bigcirc 0 \bigcirc 1
\]

Figure 3. Equivalence of Hopf link inclusions

for the corollary following Proposition 2 in [Ki78], shows that $0 \bigcirc \bigcirc 0$ and $0 \bigcirc \bigcirc 1$ are equivalent by 2-handle slides in the presence of an isolated $\bigcirc 1$. Thus, if addition or removal of the pair $1 \bigcirc \bigcirc -1$ is considered an equivalence besides the 2-handle slides it follows from $0 \bigcirc \bigcirc 0 \sim 0 \bigcirc \bigcirc 0 \bigcup 1 \bigcirc \bigcirc -1 \sim 0 \bigcirc \bigcirc 1 \bigcup 1 \bigcirc \bigcirc -1 \sim 1 \bigcirc \bigcirc -1 \bigcup 1 \bigcirc \bigcirc -1 \sim \emptyset$ that also the second Hopf link move is already implied. Thus (T1) suffices as an equivalence.
3. The Hennings TQFT Construction

The Reshetikhin-Turaev construction of 3-manifold invariants and TQFTs \cite{RT91} uses certain semisimple braided tensor categories. These categories are typically obtained as subquotients of representation categories of quantum groups or similar Hopf algebras.

Subsequently, Hennings formulated in \cite{He96} an invariant for closed 3-manifolds which entirely circumvents representation theory and, instead, computes the invariant directly from the elements of a quasi-triangular Hopf algebra. This simplified construction was further developed by Kauffman and Radford in \cite{KR95}, where also the role of the right integral is clarified.

The Hennings invariant has been extended to an algorithm for constructing TQFTs in \cite{Ke95}. Furthermore, the more abstract constructions in \cite{KL01} provide a unifying framework in which both the WRT and Hennings TQFTs occur as special cases.

In this chapter we will review and further develop the Hennings TQFT constructions described in \cite{KL01,Ke95,Ke03}. Particularly, it will be useful to break the construction of the TQFT functor $\mathcal{V}^\bullet_H$ into that of two functors that are composed over an intermediate category of $\mathcal{H}$-labeled planar curves $\mathcal{Dc}(\mathcal{H})$ as indicated in the following commutative diagram.

\[
\begin{array}{ccc}
\mathcal{Tgl} & \xrightarrow{Z_{\mathcal{H}}} & \mathcal{Dc}(\mathcal{H}) \\
\downarrow & & \downarrow \\
\mathcal{M}^\bullet & \xrightarrow{E_{\mathcal{H}}} & \mathcal{H} \equiv \mathfrak{m}(\mathcal{D}) \\
\downarrow & & \downarrow \\
\mathcal{Tc}^\bullet & \xrightarrow{\mathcal{V}_{\mathcal{H}} \circ \mathcal{H}} & \mathcal{Tc}^\bullet \\
\end{array}
\]  

(27)

Here we consider both the projective and free case by setting

\[
m(\mathcal{D}) = \begin{cases} 
\text{proj}(\mathcal{D}) & \text{whenever } \mathcal{H} \in \text{proj}(\mathcal{D}) \\
\text{free}(\mathcal{D}) & \text{whenever } \mathcal{H} \in \text{free}(\mathcal{D}) 
\end{cases}
\]  

(28)

We begin with the description of $\mathcal{Dc}(\mathcal{H})$ in Section 3.1 followed by the construction of the functors $Z_{\mathcal{H}}$ and $E_{\mathcal{H}}$ in Section 3.2. In Sections 3.3 and 3.4 we show that the composite $E_{\mathcal{H}} \circ Z_{\mathcal{H}}$ is invariant under the moves (T0)-(T3), which implies that that the functor factors into a functor $\mathcal{Tc}^\bullet \to \mathcal{H} \equiv \mathfrak{m}(\mathcal{D})$ as indicated. Composing this with the inverse of the presentation isomorphism from (26) yields the desired TQFT functor.
3.1. The Category of $\mathcal{H}$-labeled Planar Curves. We start with the definition of the category $\mathcal{Dc}(\mathcal{H})$ of planar $\mathcal{H}$-labeled planar curves. The objects of the category $\mathcal{Dc}(\mathcal{H})$ are integers as for $\mathcal{Tgl}$.

The morphisms of $\mathcal{Dc}(\mathcal{H})$ are equivalence classes of $\mathcal{H}$-labeled planar curves with transverse double points and the same component and boundary structure as required for admissible tangles in $\mathcal{Tgl}$.

Formally, we can define an $\mathcal{H}$-labeled planar curve as a pair $(D, a)$, where $D$ is a planar immersed curve in general position with $N$ ordered markings and $a$ is an element in $\mathcal{H} \otimes N$. If $a = \sum \nu a_{\nu}^1 \otimes \ldots \otimes a_{\nu}^N$ we also write $(D, a)$ as a formal sum, also with summation index $\nu$, of the same planar marked curve with elements in $\mathcal{H}$ associated to each marking. The label at the $j$-th marking of the $\nu$-th diagram would thus be $a_{\nu}^j$.

We consider $D$ to be in general position if it locally looks like either non-horizontal smooth intervals, such intervals with a marking, crossings without horizontal pieces, or non-degenerate maxima or minima. Moreover, we may require all markings, crossings, and extrema to occur at different heights.

The labeled curves $(D, a)$ are subject to equivalence relations as depicted in the Figure 4, Figure 5, and Figure 6 as well as their various reflections and general planar isotopies that preserve the extrema of the diagram. Here $\kappa$ is the balancing element from (22). The formal meaning of the equivalence on the right side of Figure 4, for example, is that $(D, a) \sim (D', a')$ where $D'$ is obtained from $D$ by combining two markings, and $a' \in \mathcal{H}^{\otimes (N-1)}$ is obtained by applying the multiplication map $m: \mathcal{H}^{\otimes 2} \rightarrow \mathcal{H}$ to the two tensor positions of $a \in \mathcal{H}^{\otimes N}$ (after suitable permutations of factors). The second equivalence $(D, a) \sim (D', a')$ indicates that $D'$ is obtained from $D$ by moving the marking over an extremum and $a' \in \mathcal{H}^{\otimes N}$ is found from $a \in \mathcal{H}^{\otimes N}$ by applying $S$ to the tensor position in $\mathcal{H}^{\otimes N}$ corresponding to this marking. The remaining equivalences are described analogously. We denote the equivalence class for $(D, a)$ by $[D, a] \in \text{Hom}_{\mathcal{Dc}(\mathcal{H})}(n, m)$.

Composition of two planar curves, $D_1 : n \rightarrow p$ and $D_2 : p \rightarrow m$, with $N_1$ and $N_2$ markings respectively is given by stacking the two diagram and shifting the the numbering of the marking in $D_2$ by $N_1$ denoted by $D_2^{*+N_1}$. The composition of $\mathcal{H}$-labeled diagrams is thus given by $(D_2, a_2) \circ (D_1, a_1) = (D_2^{*+N_1} \circ D_1, a_1 \otimes a_2)$, which is easily shown to be also well defined on the equivalence classes defining the morphisms in $\mathcal{Dc}(\mathcal{H})$. The tensor product of diagrams is defined analogously using juxtapositions.
3.2. Functors on $\mathcal{Dc}(\mathcal{H})$. The functor $\mathcal{Z}_\mathcal{H} : \mathcal{Tgl} \to \mathcal{Dc}(\mathcal{H})$ is identity on the set of objects. For a tangle $T$ representing a morphism in $\mathcal{Tgl}$ the diagram for $\mathcal{Z}_\mathcal{H}(T)$ is obtained by replacing positive or negative crossings of $T$ by a diagram with double point with a (sum of) labeled markings as depicted in Figure 7. The elements $e_i, f_i \in \mathcal{H}$ are the ones appearing in the expression for the universal $R$-matrix $R = \sum_i e_i \otimes f_i \in \mathcal{H} \otimes \mathcal{H}$, and $S$ is the antipode of $\mathcal{H}$, suppressing summation signs over diagrams depending on a summation index $i$.

More formally, a tangle diagram $T$ is assigned to an $\mathcal{H}$-labeled diagram $(D, a)$, where $D$ is obtained from $T$ by flattening each crossing and two adding markings just above or below the crossing depending on the orientation of the crossing in $T$. If $T$ has $p$ positive crossings (as defined by the left side of Figure 7) and $q$ negative crossings then $a = (R)^{\otimes p} \otimes (R^{-1})^{\otimes q} \in \mathcal{H}^{\otimes 2(p+q)}$, assuming an appropriate numbering of the markings and using $R^{-1} = S \otimes id(R)$.

![Figure 7. Definition of $\mathcal{Z}_\mathcal{H}$ at crossings](image-url)
The fact that $Z_{\mathcal{H}}$ is well defined on isotopy classes of framed tangle diagrams follows in standard fashion from the usual relations for $\mathcal{R}$ that can be derived from the axioms in \cite{He96}, see \cite{He96}.

The functor $\mathcal{E}_{\mathcal{H}} : \mathcal{D}c(\mathcal{H}) \to \mathcal{H} \vdash \mathfrak{m}(\mathbb{D})$ is constructed as follows. On the level objects we set

$$\mathcal{E}_{\mathcal{H}}(n) = \mathcal{H} \otimes n \in \mathcal{H} \vdash \mathfrak{m}(\mathbb{D}). \quad (29)$$

Here $\mathfrak{m}(\mathbb{D})$ is as in \cite{28} since the tensor product of projective $\mathbb{D}$-modules is again projective and similarly for free modules. The $\mathcal{H}$-action on $\mathcal{H} \otimes n$ is given by the $n$-fold tensor product of the adjoint action. This will be discussed in more detail in Section 4.

In order to define the linear map $\mathcal{E}_{\mathcal{H}}([D, a]) : \mathcal{H} \otimes n \to \mathcal{H} \otimes m$ for an $\mathcal{H}$-labeled planar curve $(D, a) : n \to m$ note that the moves in Figure 4 and Figure 5 can be used to remove all intersections from the planar diagram and move and combine the elements along a particular component to a single element on this component. We thus obtain an equivalent $\mathcal{H}$-labeled planar curve $(G, b)$, where $G$ consists of $n$ top arcs, $m$ bottom arcs, and some number $p$ of isolated circles. The $p + m + n$ components are disjoint and carry one marking as depicted in Figure 8 for each type. Thus, for an adequate numbering of these markings, we have $b \in \mathcal{H} \otimes p \otimes \mathcal{H} \otimes m \otimes \mathcal{H} \otimes n$.

Requiring that $\mathcal{H}$ is topogenic we may now assume the existence of a right integral $\lambda : \mathcal{H} \to \mathbb{D}$ with all of the listed properties listed in Definition 5. Considering the integral as a map $\lambda : \mathcal{H} \to \mathbb{D}$, the linear morphism assigned by $\mathcal{E}_{\mathcal{H}}$ is now given by the following formula:

$$\mathcal{E}_{\mathcal{H}}([D, a])(x_1 \otimes \ldots \otimes x_n) = \mathcal{E}_{\mathcal{H}}([G, b])(x_1 \otimes \ldots \otimes x_n) = \lambda \otimes \text{id} \otimes \lambda \otimes \lambda \otimes \lambda \otimes (1 \otimes p + m \otimes S(x_1) \otimes \ldots \otimes S(x_n)) b. \quad (30)$$

In the more combinatorial language indicated above this can be rephrased as an assignment of rank one linear maps to pure tensors as follows. We write the element $b = \sum_{\nu} r_1^{\nu} \otimes \ldots \otimes r_p^{\nu} \otimes s_1^{\nu} \otimes \ldots \otimes s_m^{\nu} \otimes t_1^{\nu} \otimes \ldots \otimes t_n^{\nu}$ so that the $\mathcal{H}$-labeled planar curve $(G, b)$ can be expressed by the union over indices $i, j, k$ and summation over the index $\nu$ of the pictures in Figure 8.

The diagrammatic rules to assign linear morphisms to building blocks of $G$ are now as indicated by the mappings in Figure 8. Specifically, we assign to a top arc with marking labeled by an element in $t \in \mathcal{H}$ the element $S^*(t \to \lambda) \in \text{Hom}_{\mathbb{D}}(\mathcal{H}, \mathbb{D}) = \mathcal{H}^*$, to a bottom arc the label $s \in \mathcal{H}$ considered as an element in $\text{Hom}_{\mathbb{D}}(\mathbb{D}, \mathcal{H}) = \mathcal{H}$, and to each isolated circle with label $r \in \mathcal{H}$ the element in $\lambda(r) \in \mathbb{D}$. 

INTEGRALITY AND GAUGE DEPENDENCE OF HENNINGS TQFTS

\[ \lambda(r_i^\nu), \quad s_j^\nu, \quad t_k^\nu \]

We thus take an ordered tensor product along the indices \( i, j, k \) of these factors we obtain, for fixed \( \nu \), an element in \( \text{Hom}_{\mathbb{D}}(\mathcal{H}, \mathbb{D})^\otimes n \otimes \mathbb{D}^\otimes p \otimes \mathbb{D}^\otimes m \subseteq \text{Hom}_{\mathbb{D}}(\mathcal{H}^\otimes n, \mathcal{H}^\otimes m) \). Summing over all indices \( \nu \) yields the desired morphism \( E_\mathcal{H}([D, a]) \) more formally described in (30).

We conclude this section with a minor but often useful extension of the tangle presentations of cobordisms described in Section 2.3 in which the notion of admissible tangles is generalized to include one more type. In addition to top, bottom, and closed components we also allow pairs of strands that connect a pair of markings \( \{2k - 1, 2k\} \) to another pair of markings \( \{2l - 1, 2l\} \). The surgery functor \( \mathcal{K}^* \) extends to such a configuration in the obvious manner. The 1-handle at the markings \( \{2k - 1, 2k\} \) is attached first and an arc along the core of the handle connecting the two markings is joined with the two strands. Along the resulting interval connecting the bottom markings \( \{2l - 1, 2l\} \) a hole is drilled out as for bottom components.

Although in this setting the category \( \mathcal{T}gl \) contains additional morphisms, dividing by the equivalences (T0)-(T3) above still yields the same category \( \mathcal{T}C^* \cong \mathcal{C}ob^* \).

The \( \mathcal{D}c(\mathcal{H}) \) are extended analogously by allowing pairs of strands with the same connectivities as admissible \( \mathcal{H} \)-labeled planar curves, allowing a corresponding extension of the functor \( Z_\mathcal{H} : \mathcal{T}gl \to \mathcal{D}c(\mathcal{H}) \).

The functor \( E_\mathcal{H} : \mathcal{D}c(\mathcal{H}) \to \mathcal{H} \sqsubset \mathbf{m}(\mathbb{D}) \) defined on the additional types of pairs of strands by assigning the map \( f : \mathcal{H} \to \mathcal{H} \), to be inserted in the respective factors, can be obtained by applying substitutions of Figure 7 and relations of Figures 4 and 5. The value \( f(x) \) is obtained by adding a cap as in the middle picture of Figure 8 but with \( x \) instead of \( b_k \) and collect elements in along the resulting bottom arc. The resulting rules are as follows:

\[ \begin{array}{cccc}
2k - 1 & 2k & \mapsto & x \mapsto axS(b) \\
2l - 1 & 2l
\end{array} \quad \begin{array}{cccc}
2k - 1 & 2k & \mapsto & x \mapsto \kappa^{-1}S(x) \\
2k - 1 & 2k
\end{array} \]
3.3. Invariance under T0-T2 Moves. In order to establish the functors in (27) one has to check that the composite \( E_H \circ Z_H \) indeed factors through \( \mathcal{M}^\bullet \). This means we need to verify that two morphisms given by admissible framed tangles related by the moves (T0)-(T3) above are mapped to the same linear map by \( E_H \circ Z_H \) given that \( H \) is a topogenic Hopf algebra.

Invariance under (T0) is implied by isotopy invariance on tangles in \( \mathcal{T}_{gl} \). In order to see invariance under (T1) observe that \( E_H \circ Z_H (\bigcirc_{-1}) = \lambda(r) \). This follows from the application of relations in \( \mathcal{Dc}(H) \) as depicted in Figure 10. For the recombination of elements in the last step we compute

\[
\sum_i S(e_i)\kappa f_i = \sum_i S(e_i)S^2(f_i)\kappa = S(\sum_i S(f_i)e_i)\kappa = S(u)\kappa = u\kappa^{-1} = r.
\]

where we make use of the relations from (22).

![Figure 10. The ribbon element](image)

A similar calculation for \( \bigcirc_{+1} \) shows more generally that

\[
E_{\mathfrak{g}l} \circ Z_{\mathfrak{g}l} (\bigcirc_{+1}) = \lambda(r^{\pm 1}).
\]

Since all invariants factor over disjoint diagrams the addition of an isolated pair of \( \pm 1 \)-framed unknots \( \bigcirc_{-1} \sqcup \bigcirc_{+1} \) results in an extra factor \( \lambda(r)\lambda(r^{-1}) \). Property (3) of Definition 5, however, implies that this factor is one so that the assignment of linear morphisms is indeed invariant under the (T1) move.

Invariance under the (T2) move is a direct consequence of the defining properties of integrals in (13). See, for example, [KR95].

3.4. Modularity and T3 Move. The (T3) or \( \sigma \)-Move (see also Section 3.1.3 of [Ke99]) is closely related to the modularity condition (2) required in Definition 5 and defined via injectivity of the mapping in (25). We discuss next the equivalence of several conditions for modularity that arise in the context of the Hennings formalism. To this end we introduce in Figure 11 special morphisms in \( \mathcal{T}_{gl} \).

The tangles \( S_{\pm} \) and \( \Pi \) are admissible in the conventional sense and \( \Gamma_{\pm} \) are admissible in the extended sense explained at the end of Section 3.2 assuming suitable
labels at the top and bottom lines of the diagrams. The following relations are readily verified by composition of tangles and applications of isotopies.

\[ \Gamma_{-} = (\Gamma_{+})^{-1} \]
\[ \Gamma_{-} \circ S_{+} = S_{-} \]
\[ \Pi = S_{-} \circ S_{-} = S_{-} \circ S_{-} \]  (32)

For convenience let us denote by \( \mathcal{T} = \mathcal{E}_{\mathcal{H}} \circ \mathcal{Z}_{\mathcal{H}}(T) \in \text{Hom}_{D}(\mathcal{H}^{\otimes n}, \mathcal{H}^{\otimes m}) \) the image of a tangle under the composite functor. It follows readily from the rules laid out in Section 3.2 that \( \mathcal{S}_{+}(x) = \sum_{ij} \lambda(S(x)f_{j}e_{i})e_{j}f_{i} \), which can be expressed as

\[ \mathcal{S}_{+} = M \circ \beta^{*} \circ S . \]  (33)

with \( M \) as in (25) and \( \beta \) as in (16).

Note also that, by functoriality, the relations (32) also hold for the respective linear maps such as, for example, \( \Pi = \mathcal{S}_{-} \circ \mathcal{S}_{+} \). Indeed this morphism can be computed also by using the fact that doubling a strand is the same as taking a coproduct. That is, we have

\[ \Pi(x) = \sum \lambda(S(x)Q')Q'' , \]  (34)

where \( Q \) is the element defined by the last tangle in Figure 9 and give explicitly by

\[ Q = \text{id} \otimes (\lambda \circ S)(\mathcal{R}_{21}\mathcal{R}) = \sum_{ij} e_{i}f_{j} \cdot \lambda(S(f_{i}e_{j})) . \]  (35)

Using the rules depicted in Figure 9 we can also evaluate the morphisms for the diagrams for \( \Gamma_{\pm} \) as

\[ \Gamma_{-}(x) = u^{-1} \sum_{i} S(f_{i})S(x)e_{i} \quad \text{and} \quad \Gamma_{+}(x) = S(u) \sum_{i} S^{2}(e_{i})S(x)f_{i} . \]  (36)
The squares of these are given by the following well known identity.

\[ \Gamma_{\pm}^2(x) = ad(r^{\pm 1})(x) . \]  

We are now in a position to prove the following lemma relating conditions of modularity.

**Lemma 6.** Suppose \( \mathcal{H} \) is a quasi-triangular ribbon Hopf algebra with a right integral \( \lambda \in \mathcal{H}^* \) and a left integral \( \Lambda \in \mathcal{H} \) such that \( \lambda(\Lambda) = \lambda(S(\Lambda)) = 1 \). Then following are equivalent.

i) \( \mathcal{H} \) is left modular.

ii) \( \mathcal{H} \) is right modular.

iii) \( S_+ \) is injective.

iv) \( S_- \) is injective.

v) \( S_+ \) is invertible.

vi) \( S_- \) is invertible.

vii) \( \Pi \) is injective.

viii) \( \Pi \) is invertible.

ix) \( \Pi = id \).

x) \( Q = \Lambda \).

**Proof.** i)\(\Leftrightarrow\)ii): Note that by (20) and (24) we have \( M = (r \otimes r)\Delta(r^{-1}) \). If we denote \( \rho_L(l) = l \otimes id(\Delta(r^{-1})) \) the map in (25) can be written as \( l \mapsto r \cdot \rho_L(l \leftarrow r) \). Since \( r \) is invertible left modularity is thus equivalent to injectivity of \( \rho_L \). Similarly right modularity is equivalent to injectivity of \( l \mapsto \rho_R(l) = id \otimes l(\Delta(r^{-1})) \). Using \( S \otimes S(\Delta(r^{-1})) = \Delta'(S(r^{-1})) = \Delta'(r^{-1}) \) we obtain the identity \( S \circ \rho_L \circ S^* = \rho_R \) so that injectivity of \( \rho_L \) and \( \rho_R \) are obviously equivalent. The equivalence i)\(\Leftrightarrow\)iii) follows from (33) and invertibility of \( \beta \) given that \( \lambda(S(\Lambda)) = 1 \).

iii)\(\Leftrightarrow\)iv): Since \( Z_{\mathcal{H}} \) is already invariant under isotopy we have by (32) that \( \Gamma_{\pm} \) is invertible and by (32) that \( \Gamma_- \circ S_+ = S_- \) and \( \Gamma_+ \circ S_- = S_+ \). Given this equivalence we also have that both iii)\(\Rightarrow\)vii) and iv)\(\Rightarrow\)vii), since by (33) we have

\[ \Pi = S_+ \circ S_- = S_- \circ S_+ . \]  

vii)\(\Leftrightarrow\)viii) and vii)\(\Leftrightarrow\)ix): These equivalences follow from the tangle identity \( \Pi^2 = \Pi \), as computed, for example, in the proof of Lemma 4 of \([Kc03]\) using only isotopy (T0), handle slides over circles (T2) and the cancellation of \( \circ \otimes \circ \) implied by (T1). We already verified that the functor \( \mathcal{E}_{\mathcal{H}} \circ \circ_{\mathcal{H}} \) is already invariant also under these moves so that \( \Pi \) is also an idempotent, that is, \( \Pi^2 = \Pi \). The claimed equivalences are now immediate.

The implications ix)\(\Rightarrow\)v) and ix)\(\Rightarrow\)vi) are obvious from (38), and the implications v)\(\Rightarrow\)iii) and vi)\(\Rightarrow\)iv) are trivial. Hence i) through ix) are all equivalent and it remains to show ix)\(\Leftrightarrow\)x):
If \( \Lambda \) is a left integral with \( \lambda(S(\Lambda)) = 1 \) \( S(\Lambda) \) is a right integral so that by (34) \( \Pi(\Lambda) = \sum \lambda(S(\Lambda)Q')Q'' = \lambda(S(\Lambda))Q = Q \) so that \( \Pi = id \) implies \( Q = \Lambda \). Conversely, if \( Q = \Lambda \) is a left integral it follows by combining (17) and (18) that \( \Pi(x) = \sum \lambda(S(x)\Lambda')\Lambda'' = \lambda(\Lambda) \cdot x = x \). □

The conclusion from Lemma 6 most relevant to our construction is that modularity implies \( \Pi = id \), and hence also that \( \mathcal{E}_\mathcal{H} \circ \mathcal{Z}_\mathcal{H} \) is invariant under the required \( \sigma \)-move in (T3). Except for equivariance with respect to the \( \mathcal{H} \)-action we have proved the following.

**Corollary 7.** Give a topogenic Hopf algebra \( \mathcal{H} \) and functors \( \mathcal{E}_\mathcal{H} \) and \( \mathcal{Z}_\mathcal{H} \) as above.

Then the composite \( \mathcal{E}_\mathcal{H} \circ \mathcal{Z}_\mathcal{H} : \mathcal{Jgl} \rightarrow \mathbf{m}(\mathbb{D}) \) is invariant under equivalences (T0) through (T3) and hence factors in a functor \( \mathcal{V}_\mathcal{H}^* \circ \mathcal{K} : \mathcal{C}^* \rightarrow \mathbf{m}(\mathbb{D}) \) as indicated in Diagram 27.

In this corollary the target of functors is still only the category of free or projective \( \mathbb{D} \)-modules without consideration of the \( \mathcal{H} \)-action. The latter will be described in the following section, proving that the image of these functors lies indeed in \( \mathcal{H} = \mathbf{m}(\mathbb{D}) \).

### 4. Equivariance, Closed Surfaces, and Closed Manifolds

In this section we treat various aspects of equivariance of Hennings TQFTs with respect to the adjoint \( \mathcal{H} \)-action, starting with the outline of a diagrammatic proof of the same in Sections 4.1 and 4.2. In Sections 4.3 and 4.4 we discuss the invariance functor and use it to construct a TQFT functor for closed surfaces. Finally, we will define in Section 4.5 an invariant with values in the underlying ring \( \mathbb{D} \) for closed 3-manifolds without framings via the usual framing renormalization and identify it with Hennings’ original invariant.

#### 4.1. \( \mathcal{H} \)-Action and Equivariance in \( \mathcal{Dc}(\mathcal{H}) \).

The fact that for any admissible tangle class \( T : n \rightarrow m \) the morphism \( \mathcal{E}_\mathcal{H} \circ \mathcal{Z}_\mathcal{H}(T) \) intertwines the adjoint actions on the tensor products of \( \mathcal{H} \) can be inferred in an abstract manner from the construction in [KL01] that assigns a TQFT functor \( \mathcal{V}_\mathcal{C} \) to a suitable category \( \mathcal{C} \), and the fact that Hennings TQFT is equivalent to \( \mathcal{V}_\mathcal{C} \) given that \( \mathcal{C} = \mathcal{H} \mod \).

It is useful, however, to describe the \( \mathcal{H} \)-action and equivariance also concretely within the formalism of the combinatorial Hennings TQFT construction described thus far. For an element \( x \in \mathcal{H} \) and an integer \( g \) we assign a morphism in the extended version of \( \mathcal{Dc}(\mathcal{H}) \) by

\[
\Box^{(2g)}(x) = [\dot{1}, \ldots, \dot{2g}, \Delta^{(2g-1)}(x)] : g \rightarrow g,
\]  

(39)
where the underlying labeled planar curve is given by \(2g\) parallel strands, each with one marking in consecutive labels, and \(\Delta^{(2g-1)}(x) = \sum x^{(1)} \otimes \ldots \otimes x^{(2g)} \in \mathcal{H} \otimes \mathcal{H}\) is the \(2g\)-fold coproduct of \(x\). Figure 12 shows the respective picture of this morphism is the previously described diagrammatic language.

\[
\begin{array}{c}
\Delta^{(2g-1)}(x) \\
\end{array}
\]

\[
\begin{array}{c}
\Delta^{(2g-1)}(x) \\
\end{array}
\]

\[
\begin{array}{c}
\Delta^{(2g-1)}(x) \\
\end{array}
\]

Figure 12. Diagrammatic \(\mathcal{H}\)-action

Equivariance in the diagrammatic context is now given by the following statement.

**Lemma 8.** Suppose \(T : g \to h\) is a tangle in \(\mathcal{Tgl}\) and \(x \in \mathcal{H}\) then

\[
Z_{\mathcal{H}}^0(T) \square^{(2g)}(x) = \square^{(2h)}(x)Z_{\mathcal{H}}^0(T). \tag{40}
\]

**Proof.** Note that the categories \(\mathcal{Dc}(\mathcal{H})\) and \(\mathcal{Tgl}\) can be extended to a categories \(\mathcal{Dc}^0(\mathcal{H})\) and \(\mathcal{Tgl}^0\) respectively in which all of the connectivity requirements for admissible tangles or diagrams are dropped. The larger morphism sets of \(\mathcal{Tgl}^0\) have a simple set of generators given by over- and under-crossings as well as maxima and minima.

Using the same assignments given in Section 3.2 we obtain a functor \(Z_{\mathcal{H}}^0 : \mathcal{Tgl}^0 \to \mathcal{Dc}^0(\mathcal{H})\) such that \(Z_{\mathcal{H}}^0\) is given precisely as the restriction of \(Z_{\mathcal{H}}^0\) to \(\mathcal{Tgl}\). Moreover, the construction in (39) obviously generalizes to morphisms \(\square^{(N)}(x)\) in \(\mathcal{Dc}^0(\mathcal{H})\) with an odd number of parallel strands. For a generating tangle \(T\) of \(\mathcal{Tgl}^0\) with \(N\) top and \(M\) bottom end points we can now verify the intertwining relation

\[
Z_{\mathcal{H}}^0(T) \square^{(N)}(x) = \square^{(M)}(x)Z_{\mathcal{H}}^0(T). \tag{41}
\]

Particularly, \(T\) is a crossing (with possibly more parallel strands to the right and left) this relation follows directly from coassociativity as well as the second line in (19) after multiplying elements along strands. If \(T\) is a maximum or minimum (41) is a consequence of the antipode axiom and the relations in Figure 4.

Since every tangle is a composite of these generators (41) holds for all morphisms in \(\mathcal{Tgl}^0\). The claim now follows by restriction to \(\mathcal{Tgl}\). \(\square\)

An immediate consequence of (41) is that string links \(N = M = 1\) yield elements in the center of \(\mathcal{H}\), see for example [Re90, KRS98].
4.2. $\mathcal{H}$-Equivariance of the Hennings TQFT. As described at the end of Section 3.2 the fiber functor $\mathcal{E}_{\mathcal{H}}$ is defined on the extended version of $\mathcal{Dc}(\mathcal{H})$ and thus also on the morphisms $\Box(2^g)(x)$ for any $x \in \mathcal{H}$. The action of $\Box(2^g)(x)$ is readily found from the rule in Figure 9 to be just the $g$-fold tensor of the adjoint action of $\mathcal{H}$ on $\mathcal{H} \otimes x$.

$$\mathcal{E}_{\mathcal{H}}(\Box(2^g)(x))(b_1 \otimes \ldots \otimes b_g) = \sum x^{(1)} b_1 S(x^{(2)}) \otimes \ldots \otimes x^{(2g-1)} b_g S(x^{(2g)})$$

$$= \sum ad(x^{(1)})(b_1) \otimes \ldots \otimes ad(x^{(g)})(b_g) \quad (42)$$

Applying the functor $\mathcal{E}_{\mathcal{H}}$ to Lemma 8 and using (42) we thus arrive for any tangle $T : n \rightarrow m$ at the desired $\mathcal{H}$-equivariance relation

$$T \cdot ad(x)^{\otimes n} = ad(x)^{\otimes m} \cdot T, \quad (43)$$

where we denote with $T = \mathcal{E}_{\mathcal{H}} \circ Z_{\mathcal{H}}(T)$. This also implies that the target category of the functor $\mathcal{E}_{\mathcal{H}} \circ Z_{\mathcal{H}}$ is the one of $\mathcal{H}$-modules. Since $\mathcal{E}_{\mathcal{H}} \circ Z_{\mathcal{H}} = \mathcal{V}_{\mathcal{H}}^{\bullet} \circ \mathcal{K} \circ \mathcal{M}^{\bullet}$ and $\mathcal{H} \circ \mathcal{M}^{\bullet}$ is a full functor we can now refine the statement of Corollary 7 as follows.

**Corollary 9.** Suppose $\mathcal{H}$ is a topogenic Hopf algebra. With notations as above we have a TQFT functor

$$\mathcal{H}_{\mathcal{H}} : \mathcal{Cob} \rightarrow \mathcal{H} \models m(\mathbb{D}). \quad (44)$$

4.3. The $\tau$-Move for Closed Surfaces and the Invariance Functor. Let us now discuss the two main ingredients for extending the TQFT from Corollary 9 above to the category $\mathcal{Cob}^{\emptyset}$ of cobordisms between closed surfaces. Like $\mathcal{Cob}^{\bullet}$ the category $\mathcal{Cob}^{\emptyset}$ is represented by a tangle category $\mathcal{Tc}^{\emptyset}$ which is obtained from the category $\mathcal{Tgl}$ by dividing out relations (T0)-(T3) as for $\mathcal{Tc}^{\bullet}$ but in addition also the

(T4) $\tau$-Move

defined in [Ke99] (Figure 3.32, see also TS4 of Section 2.3.3 of [KL01]). This move allows passing a piece of a strand $\mathcal{L}$ through the entire collection of $2g$ parallel strands near the top (or bottom) end of the diagram as depicted in Figure 13.

![Figure 13. $\tau$-Move](image)
In order to describe the difference between these diagrams in the Hennings formalism we apply $Z_{\mathcal{H}}$ the left tangle piece with $4g$ crossings. Denoting by $M = \mathcal{R}'\mathcal{R} = \sum m_j \otimes n_j$ and repeatedly applying the relations in (19) we find that the resulting $\mathcal{H}$-labeled planar curve piece in $\mathcal{D}_c(\mathcal{H})$ can be written as

\[
\left( \bullet_1 \ldots \bullet_{2g} \bullet_{2g+1}, \Delta^{(2g-1)} \otimes id(M) \right).
\]

where the last strand with marking numbered $(2g+1)$ belongs to $\mathcal{L}$. That is, in diagrammatic terms we have a marking $\mathcal{L}$ labeled by elements $n_j$ and elements $\Delta^{(2g-1)}(m_j)$ distributed over the $2g$ parallel top strands with markings as in Figure 12. We use this observation now as follows.

**Lemma 10.** Suppose $v \in \mathcal{H}^{\otimes g}$ such that $ad(x)^{\otimes g}(v) = \epsilon(x)v$ for all $x \in \mathcal{H}$. Then $T \mapsto \varepsilon_{\mathcal{H}} \circ Z_{\mathcal{H}}(T).v$ is invariant under the $\tau$-Move given in Figure 13.

**Proof.** Let $T_l$ and $T_r$ be the tangles depicted on the left and right side of Figure 13, and let $(D, a)$ be the $\mathcal{H}$-labeled planar curve obtained from $T_r$ by application of the rules in Section 3.2. Moreover, let $D'$ be the diagram $D$ but with one additional marking on the $\mathcal{L}$ piece and numbering such that $[D, a] = [D', a \otimes 1]$.

The diagram assigned to $T_l$ consists of $4g$ additional crossing and thus $8g$ additional markings, of which each vertical strand carries two and the piece $\mathcal{L}$ has $4g$. Multiplying the elements along each of these strand pieces using equivalences in Figure 14 leaves $(2g+1)$ markings (one for each piece). As argued above the iteration of relations in (19) shows that the tensor assigned is as in (45). Writing this labeled curve piece as a sum $\sum_j \left( \bullet_1 \ldots \bullet_{2g} \bullet_{2g+1}, \Delta^{(2g-1)}(m_j) \otimes n_j \right)$, we can further move all of the first $2g$ markings along the parallel strands upwards so that they are separated by horizontal line from the rest of the diagram. For fixed $j$ the resulting diagram can consequently be expressed as a composition of the piece $\left( \bullet_1 \ldots \bullet_{2g}, \Delta^{(2g-1)}(m_j) \right)$ with the remainder of the diagram, which, in turn, represents the morphism from (45) for $x = m_j$. This implies the relation

\[
Z_{\mathcal{H}}(T_l) = \sum_j [D', a \otimes n_j] \circ \square^{(2g)}(m_j).
\]
For an element \( v \in \mathcal{H}^{\otimes g} \) as assumed in the lemma we thus compute
\[
\mathcal{E}_{\mathcal{H}}(Z_{\mathcal{H}}(T_l)).v = \sum_j E_{\mathcal{H}}([D', a \otimes n_j]) E_{\mathcal{H}}(\square^{(2g)}(m_j)).v
\]
by (42)
\[
\sum_j E_{\mathcal{H}}([D', a \otimes n_j]) . (\text{ad}(m_j)^{\otimes g}(v)) = \sum_j E_{\mathcal{H}}([D', a \otimes n_j]) . (\varepsilon(m_j)v) = E_{\mathcal{H}}([D', a \otimes 1]) . v = E_{\mathcal{H}}(Z_{\mathcal{H}}(T_r)).v
\]
where we also used \( \varepsilon \otimes id(M) = 1 \).

Thus the restrictions of the morphisms \( \mathcal{E}_{\mathcal{H}} \circ Z_{\mathcal{H}}(T) \) to the invariance subspace \( \text{Inv}_{\mathcal{H}}(\mathcal{H}^{\otimes g}) \), given by all elements \( v \in \mathcal{H}^{\otimes g} \) with \( \text{ad}(x)^{\otimes g}(v) = \varepsilon(x)v \) implements the \( \tau \)-Move. Regarding integrality this restriction will not necessarily preserve freeness of \( \mathbb{D} \)-modules but only projectiveness as described next.

**Lemma 11.** Suppose \( \mathbb{D} \) is Noetherian. Then the restriction to invariance subspaces \( \text{Inv}_{\mathcal{H}}(M) = \{ v \in M : x.v = \varepsilon(x)v \forall x \in \mathcal{H} \} \) of modules yields a functor
\[
\text{Inv}_{\mathcal{H}} : \mathcal{H} \models \text{proj}(\mathbb{D}) \rightarrow \text{proj}(\mathbb{D}) \quad M \mapsto \text{Inv}_{\mathcal{H}}(M).
\]

**Proof.** The definition of \( \text{Inv}_{\mathcal{H}} \) extends to morphisms since invariance spaces are mapped to each other by equivariance. Moreover, as a finitely generated module of a Noetherian ring \( \mathbb{D} \) also \( M \) is Noetherian so that the subspace \( \text{Inv}_{\mathcal{H}}(M) \) is Noetherian as well and thus finitely generated. Finally, the fact that finitely generated modules are projective precisely when they are torsion-free (e.g., §9 of [Ln94]) implies that \( \text{Inv}_{\mathcal{H}}(M) \) has to be projective as well since submodules of torsion free modules are obviously again torsion free.

4.4. Hennings TQFT for Closed Surfaces. In order to organize and summarize the TQFT constructions thus far let us consider the quotient functor \( \mathcal{F}_\tau : \mathcal{Cob}^* \rightarrow \mathcal{Cob}^\emptyset \), which quotients morphism sets by the additional \( \tau \)-Move, as well as the filling functor \( \mathcal{F}_\partial : \mathcal{Cob}^* \rightarrow \mathcal{Cob}^\emptyset \) that was introduced in (12). Observe that an isotopy class of a tangle in \( D_+^2 \times [0, 1] \) modulo the \( \tau \)-Move is the same as its isotopy class in \( S^2 \times [0, 1] \) after a complementary \( D_+^2 \times [0, 1] \) is glued in. The move really describes the tranverse passage of a piece of a strand \( \mathcal{L} \) through \( D_-^2 \times [0, 1] \).

Clearly we obtain the same cobordism in \( \mathcal{Cob}^\emptyset \) whether we first surger along the tangle and then glue in \( D_-^2 \times [0, 1] \) or whether we first add \( D_-^2 \times [0, 1] \) and then carry
out the surgery. Formally this can be expressed by the following relation of functors

\[ \mathcal{K}^0 \circ \mathcal{F}_r = \mathcal{F}_\partial \circ \mathcal{K}, \tag{48} \]

where \( \mathcal{K}^0 : \mathcal{TC}^\emptyset \to \mathcal{Cob}^\emptyset \) is the isomorphism functor providing a presentation of the category of cobordisms between closed surfaces analogous to \( \mathcal{K} \) defined in (26).

We summarize the existence and corresponding relations of TQFT functors in the following theorem.

**Theorem 12.** Suppose \( \mathcal{H} \) is a topogenic Hopf algebra over a Noetherian ring \( \mathbb{D} \), which is, as a \( \mathbb{D} \)-module, projective and finitely generated over \( \mathbb{D} \).

Then there is TQFT functor \( \mathcal{V}^\emptyset_{\mathcal{H}} : \mathcal{Cob}^\emptyset \to \text{proj}(\mathbb{D}) \) and a commutative diagram of functors as given in (49), where \( \mathcal{V}^\emptyset_{\mathcal{H}} \) is as in Corollary 9, \( \mathcal{F}_\partial \) as in (12), and \( \text{Inv}_{\mathcal{H}} \) as in Lemma 11.

\[
\begin{array}{ccc}
\mathcal{Cob}^\bullet & \xrightarrow{\mathcal{V}^\emptyset_{\mathcal{H}}} & \mathcal{H} \supset \text{proj}(\mathbb{D}) \\
\mathcal{F}_\partial & \downarrow & \text{Inv}_{\mathcal{H}} \\
\mathcal{Cob}^\emptyset & \xrightarrow{\mathcal{V}^\emptyset_{\mathcal{H}}} & \text{proj}(\mathbb{D})
\end{array}
\tag{49}
\]

**Proof.** Combining Lemma 10 and Corollary 7 we find that the composite \( \text{Inv}_{\mathcal{H}} \circ \mathcal{E}_{\mathcal{H}} \circ \mathcal{Z}_{\mathcal{H}} \) is invariant under moves (T0)-(T4). Consequently it factors through the functor \( \mathcal{M}^\emptyset : \mathcal{I}gl \to \mathcal{TC}^\emptyset \) quotiening out all these moves, which, in turn, is given by \( \mathcal{M}^\emptyset = \mathcal{F}_r \circ \mathcal{M}^\bullet \) with \( \mathcal{F}_r \) as in (18).

Analogous to the construction of \( \mathcal{V}^\bullet_{\mathcal{H}} \) and using the representation isomorphism \( \mathcal{H}^\emptyset \) we thus infer a TQFT functor \( \mathcal{V}^\emptyset_{\mathcal{H}} \) with \( \text{Inv}_{\mathcal{H}} \circ \mathcal{E}_{\mathcal{H}} \circ \mathcal{Z}_{\mathcal{H}} = \mathcal{V}^\emptyset_{\mathcal{H}} \circ \mathcal{K}^\emptyset \circ \mathcal{M}^\emptyset \). Substituting the above expression for \( \mathcal{M}^\emptyset \), and using \( \mathcal{E}_{\mathcal{H}} \circ \mathcal{Z}_{\mathcal{H}} = \mathcal{V}^\bullet_{\mathcal{H}} \circ \mathcal{K} \circ \mathcal{M}^\bullet \), which is implied by Corollary 7, we obtain \( \text{Inv}_{\mathcal{H}} \circ \mathcal{V}^\bullet_{\mathcal{H}} \circ \mathcal{K} \circ \mathcal{M}^\bullet = \mathcal{V}^\emptyset_{\mathcal{H}} \circ \mathcal{K}^\emptyset \circ \mathcal{F}_r \circ \mathcal{M}^\bullet \).

Since \( \mathcal{M}^\bullet \) is surjective we obtain \( \text{Inv}_{\mathcal{H}} \circ \mathcal{V}^\bullet_{\mathcal{H}} \circ \mathcal{K} = \mathcal{V}^\emptyset_{\mathcal{H}} \circ \mathcal{K}^\emptyset \circ \mathcal{F}_r = \mathcal{V}^\emptyset_{\mathcal{H}} \circ \mathcal{F}_\partial \circ \mathcal{K} \) using also (18). Given that \( \mathcal{K} \) is an isomorphism we conclude (19). \( \square \)

The problem remains how to obtain topogenic Hopf algebras that fulfill the prerequisites of Theorem 12 and Definition 5. We will show in Section 7 that a large naturally constructed family of topogenic Hopf algebras is given by quantum doubles of double balanced Hopf algebras.
4.5. The Hennings invariant for closed 3-manifolds. The TQFT functors can be used to associate an invariant for a closed, connected, compact, oriented 3-manifold $M$ as follows. We remove a 3-ball from $M$ with boundary $S^2 \cong D^2 \cup D^2$. This defines a relative cobordism $M^* : D^2 \to D^2$. It is easy to see that different choices of 3-balls and parametrizations of the $S^2$ boundary will yield equivalent cobordisms due to connectedness of $M$ and the fact that the oriented mapping class group of $S^2$ is trivial. If we endow $M^*$ with a 2-framing we obtain a morphism in $\mathbf{Cob}^*$ on which we can evaluate $\mathcal{Y}_{3\mathcal{K}}^*$. By construction we have $\mathcal{Y}_{3\mathcal{K}}^*(D^2) = \mathcal{K}^{\otimes 0} \cong \mathbb{D}$ so that $\text{End}(\mathcal{Y}_{3\mathcal{K}}^*(D^2)) = \{ x \cdot id : x \in \mathbb{D} \}$ is the free rank one $\mathbb{D}$-module with canonical generator given by the identity on $\mathcal{Y}_{3\mathcal{K}}^*(D^2)$. We thus have

$$\mathcal{Y}_{3\mathcal{K}}^*(M^*) = \widetilde{\varphi}_{3\mathcal{K}}(M^*) \cdot id \quad (50)$$

for a unique $\widetilde{\varphi}_{3\mathcal{K}}(M^*) \in \mathbb{D}$. Although this element of $\mathbb{D}$ does not depend on the choice of the removed ball and the parametrization of the bounding sphere, it still depends on the choice of the 2-framing on $M^*$ or signature of a 4-manifold. It does so, however, in an easily described manner.

The signature of a bounding 4-manifold is given by the signature $\sigma(L)$ of the linking matrix of a framed link $L$ representing $M^*$ or $M$ by surgery. It follows from elementary matrix algebra that $\sigma(L)$ is invariant under the moves (T0), (T1), and (T2) from Section 2.3 and, hence, depends only on $M^*$ so that we may write $\sigma(L) = \sigma(M^*)$.

For a given framed link $L$ in $S^3$ let us denote $L_\downarrow = L \sqcup \bigcirc_{-1}$ the link with an isolated, additional (-1)-framed unknot. The respectively represented morphisms $M^*$ and $M^*_\downarrow$ differ only by a shift in framing so that we have $M = M_\downarrow$ for the underlying (unframed) 3-manifolds. By definition, $\sigma(L_\downarrow) = \sigma(L) - 1$ so that also $\sigma(M^*_\downarrow) = \sigma(M^*) - 1$.

We also observe from the constructions in Section 3 that the evaluation of $\mathcal{E}_{3\mathcal{K}} \circ \mathcal{Z}_{3\mathcal{K}}$ on disjoint unions of links is multiplicative. The value of this functor on $\bigcirc_{-1}$ is given in (31) as $\lambda(r)$. It follows that $\mathcal{Y}_{3\mathcal{K}}^*(M^*_\downarrow) = \lambda(r) \mathcal{Y}_{3\mathcal{K}}^*(M^*)$ and the analogous relation for $\widetilde{\varphi}_{3\mathcal{K}}$. Combining these relations we find that $\varphi_{3\mathcal{K}}(M^*_\downarrow) = \varphi_{3\mathcal{K}}(M^*)$ where

$$\varphi_{3\mathcal{K}}(M) = \lambda(r)^{\sigma(M^*)} \cdot \widetilde{\varphi}_{3\mathcal{K}}(M^*) \quad . \quad (51)$$

Since $\varphi_{3\mathcal{K}}(M)$ is invariant under shifts of framings or signatures it does indeed depend only on the underlying topological 3-manifold. We will refer to $\varphi_{3\mathcal{K}}(M)$ as the Hennings invariant of $M$ for the algebra $\mathcal{K}$. Note that in this definition the normalization is such that $\varphi_{3\mathcal{K}}(S^3) = 1$, which also allows $\varphi_{3\mathcal{K}}(S^1 \times S^2) = \lambda(1)$ to be non-invertible [Ke98].
5. **Gauge Transformations of Hopf algebras**

The concept of gauge transformations of Hopf algebra structures naturally arose in Drinfeld’s discussion of *quasi Hopf algebras* in [Dr90]. Our investigation here of the effect of such transformations on the resulting TQFT constructions is motivated by the proof of Theorem 3 in Section 8 where we use the fact that the double of the quantum Borel algebra of $\mathfrak{sl}_2$ is equivalent to the tensor product of quantum $\mathfrak{sl}_2$ and a cyclic group algebra – but only up to a non-trivial gauge transformation.

For this purpose we will focus on the situation of gauge transformations between strictly associative Hopf algebras, which will require an additional cocycle condition but, in return, avoids the use of associators. A generalization of the following discussion to quasi Hopf algebras with non-trivial associators $\Phi \in \mathcal{H}^{\otimes 3}$ is expected to generalize to respective TQFT constructions for quasi Hopf algebra as in [Ge13].

### 5.1. Cocyle Condition, Special Elements, and Relations of Gauge Transformations.

For a Hopf algebra $\mathcal{H}$ we say that an element $F \in \mathcal{H} \otimes \mathcal{H}$ is a *cocycle* if it satisfies the conditions
\[
(1 \otimes F)(id \otimes \Delta)(F) = (F \otimes 1)(\Delta \otimes id)(F)
\]
\[
\epsilon \otimes id(F) = 1 = id \otimes \epsilon(F)
\]
(52)

We say that a cocycle $F \in \mathcal{H}^{\otimes 2}$ is a *gauge transformation* of Hopf algebras (as opposed to quasi Hopf algebras) if it is also has an inverse $F^{-1} \in \mathcal{H}^{\otimes 2}$. Note that one particular class of gauge transformations is given by *coboundaries* which are defined for any invertible element $c \in \mathcal{H}$ by $F_c = (c^{-1} \otimes c^{-1})\Delta(c)$.

Before discussing the modifications imposed on the Hopf algebra structure by a gauge transformation let us explore several useful relations implied by the cocycle condition. We use the following notations for the cocycle tensor expressions.

\[
F = \sum_i A_i \otimes B_i \quad \text{and} \quad F^{-1} = \sum_i C_i \otimes D_i.
\]

We also denote the maps $J = (id \otimes m)(\Delta \otimes S)$ and $\overline{J} = (m \otimes id)(S \otimes \Delta)$ from $\mathcal{H} \otimes \mathcal{H}$ to itself, where $m : \mathcal{H}^{\otimes 2} \to \mathcal{H}$ is the multiplication. For convenience we record the explicit action of $J$, $\overline{J}$, and their inverses on elements of $\mathcal{H}^{\otimes 2}$.

\[
J(a \otimes b) = \sum a' \otimes a'' S(b) \quad \overline{J}(a \otimes b) = \sum S(a)b' \otimes b''
\]
\[
J^{-1}(a \otimes b) = \sum a' \otimes S^{-1}(b)a'' \quad \overline{J}^{-1}(a \otimes b) = \sum b'S^{-1}(a) \otimes b''
\]
(53)

The two transformations are in fact conjugate by

\[
\overline{J} = J \circ J \circ J^{-1} \quad \text{with} \quad J := (S \otimes S) \circ \tau
\]
(54)
where $\tau(a \otimes b) = b \otimes a$. The following additional relations are readily verified as well.

$$\overline{J} \circ J = J \circ \overline{J}^{-1} \quad \text{and} \quad \overline{T}^2 \circ J = J \circ \overline{T}^2 \quad \text{with} \quad \overline{T}^2 = S^2 \otimes S^2. \quad (55)$$

We also associate the following special elements associated to a gauge transformation.

$$x_F = m(id \otimes S)(F) = \sum_i A_i S(B_i) \quad \text{and} \quad \overline{x}_F = m(S \otimes id)(F^{-1}) = \sum_i S(C_i) D_i. \quad (56)$$

They will play an important role in later formulae and duality consideration. For a coboundary $F_c$ is readily computed as $x_{F_c} = c^{-1} S(c)^{-1}$ and $\overline{x}_{F_c} = S(c)c$. Note also that if $F$ is a cocycle then also

$$F^\dagger := \overline{T}(F^{-1}) \quad (57)$$

is a gauge transformation for $\mathcal{H}$. This can be iterated to obtain more gauge transformations $F^{\dagger\dagger} = (S^2 \otimes S^2)(F)$, $F^{\dagger\dagger\dagger}$ etc. for the same $\mathcal{H}$. With the above convention we may now list several useful relations.

**Lemma 13.** Suppose $\mathcal{H}$ is a Hopf algebra and $F$ a gauge transformation with $x_F$ as above. Then the following hold.

$$\sum_i C_i x_F S(D_i) = 1 = \sum_i S(A_i) \overline{x}_F B_i \quad (58)$$

$$\overline{x}_F = x_F^{-1} \quad (59)$$

$$F^{-1} = \overline{T}(F)(1 \otimes x_F^2) = \overline{T}^{-1}(F)(S^{-1}(x_F^{-2}) \otimes 1) \quad (60)$$

$$F = (x_F \otimes 1) \overline{T}(F^{-1}) = (1 \otimes S^{-1}(x_F)) \overline{T}^{-1}(F^{-1}) \quad (61)$$

$$F^\dagger = S \otimes S(\tau(F^{-1})) = (x_F^{-1} \otimes x_F^{-2}) F \Delta(x_F) \quad (62)$$

**Proof.** The relations in (58) immediately follow by applying $m(id \otimes S)$ and $m(S \otimes id)$ to the equation $F^{-1}F = 1$ respectively. If we apply $m_{23}(id^{\otimes 2} \otimes S)$ and $m_{21}(S^{-1} \otimes id^{\otimes 2})$ to the cocycle equation (52) we find $1 \otimes x_F = F(\sum_i A'_i \otimes A''_i S(B_i))$ as well as $S^{-1}(x_F) \otimes 1 = F(\sum_i B'_i S^{-1}(A_i) \otimes B''_i)$ respectively. The latter imply $F^{-1}(1 \otimes x_F) = \overline{T}(F)$ and $F^{-1}(S^{-1}(x_F) \otimes 1) = \overline{T}^{-1}(F)$. Applying $m(S \otimes id)$ to both of these equations we obtain $\overline{x}_F x_F = 1$ and $x_F \overline{x}_F = 1$ respectively, which proves (59).

Given invertibility of $x_F$ the previous equations can be solved for $F^{-1}$ yielding (60). The second set of equations in (61) is obtained similarly by applying $m_{12}(S \otimes id^{\otimes 2})$ and $m_{32}(id^{\otimes 2} \otimes S^{-1})$ to the inverse of the cocycle equation given by

$$(id \otimes \Delta)(F^{-1})(1 \otimes F^{-1}) = (\Delta \otimes id)(F^{-1})(F^{-1} \otimes 1)$$
respectively. For the last identity we compute

\[
(1 \otimes x_F) T(F^{-1}) \xrightarrow{by (54)} T((S^{-1}(x_F) \otimes 1)F^{-1}) \xrightarrow{by (55)} T(J(T((S^{-1}(x_F) \otimes 1)F^{-1})))) \\
\xrightarrow{by (56)} T(J((x_F \otimes 1)F^{-1}))) \xrightarrow{by (61)} T(J(F)) \xrightarrow{by (60)} T(F^{-1}(1 \otimes x_F)) \\
\xrightarrow{by (53)} T(F^{-1}) \Delta(x_F) \xrightarrow{by (61)} (x_F^{-1} \otimes 1)F \Delta(x_F)
\]

which readily implies (62).

\[\square\]

5.2. Gauge Transformed Quasi-Triangular Structure. Let us now turn to defining the gauge transformed Hopf algebra structures. The following is well known and implied, for example, by specializing computations in [Dr90] to the strictly associative case with trivial associators.

**Lemma 14.** Suppose $H$ is a Hopf algebra and $F$ a gauge transformation of $H$ as defined above. Then a Hopf algebra $H_F$ can be defined with the same algebra structure as $H$ but with a Hopf algebra structure as given follows:

\[
\Delta_F(a) = F \Delta(a) F^{-1} \quad (63)
\]
\[
\epsilon_F = \epsilon \quad (64)
\]
\[
S_F(a) = x_F S(a) x_F^{-1} \quad (65)
\]

If $H$ is a quasi-triangular with R-matrix $R$. Then $H_F$ is also quasi-triangular with

\[
R_F = F_{21} R F^{-1} \quad (66)
\]

where we use the usual notation $F_{21} = \tau(F)$ with $\tau(a \otimes b) = b \otimes a$.

**Proof.** The counit and coassociativity axioms immediately follow from (52). Also $\Delta_F(ab) = \Delta_F(a) \Delta_F(b)$ is clear from (63). For the antipode we compute

\[
m(id \otimes S_F) \Delta_F(a) = \sum_{ij} A_i a' C_j x_F S(D_j) S(a'') S(B_i) x_F^{-1} = \\
= \sum_i A_i a' S(a'') S(B_i) x_F^{-1} = \epsilon(a) \sum_i A_i S(B_i) x_F^{-1} = \epsilon(a) x_F x_F^{-1} = \epsilon_F(a).
\]

The other antipode equation $m(S_F \otimes id) \Delta_F(a) = \epsilon_F(a)$ follows similarly from the second relation in (58). For the quasi-triangular structure the identity $\Delta'_F(a) R_F = R_F \Delta_F(a)$, where $\Delta'_F$ denotes the opposite coproduct, is immediate by conjugation.

Although already implied by computations in [Dr90] for quasi Hopf algebra let us also give a derivation of the remaining quasi-triangularity axioms as a warm up for later uses of the cocycle equation (58). To this end denote the left and right side of (58) by $\Upsilon_3 = F_{12}(\Delta \otimes id)(F) = F_{23}(id \otimes \Delta)(F) \in H^\otimes 3$. For $\pi$ a permutation
on $n$ letters let $\sigma_\pi$ denote the automorphism on $\mathcal{H}^{\otimes n}$ given by $\sigma_\pi(a_1 \otimes \ldots \otimes a_n) = a_{\pi(1)} \otimes \ldots \otimes a_{\pi(n)}$. The following two relations in $\mathcal{H}^{\otimes 3}$ are then easily verified.

$$(\mathcal{R}_F)_{12} \cdot \mathcal{T}_{[3]} = \sigma_{(12)}(\mathcal{T}_{[3]}) \cdot \mathcal{R}_{12} \quad \text{and} \quad (\mathcal{R}_F)_{23} \cdot \mathcal{T}_{[3]} = \sigma_{(23)}(\mathcal{T}_{[3]}) \cdot \mathcal{R}_{23}.$$ 

Combining these two identities we find

$$(\mathcal{R}_F)_{13} \cdot (\mathcal{R}_F)_{23} \cdot \mathcal{T}_{[3]} = \sigma_{(23)} \left( (\mathcal{R}_F)_{12} \cdot \mathcal{T}_{[3]} \right) \cdot \mathcal{R}_{23} = \sigma_{(123)} \left( \mathcal{T}_{[3]} \right) \cdot \mathcal{R}_{13} \cdot \mathcal{R}_{23}.$$ 

We also compute

$$(\Delta_F \otimes \text{id})(\mathcal{R}_F) \cdot \mathcal{T}_{[3]} = F_{12} \cdot (\Delta \otimes \text{id})(\mathcal{R}_F) \cdot (\Delta \otimes \text{id})(F) = F_{12}(\Delta \otimes \text{id})(F_{21}) \cdot (\Delta \otimes \text{id})(\mathcal{R}) = \sigma_{(123)} \left( \mathcal{T}_{[3]} \right) \cdot (\Delta \otimes \text{id})(\mathcal{R}).$$

Now since $(\Delta \otimes \text{id})(\mathcal{R}) = \mathcal{R}_{13} \cdot \mathcal{R}_{23}$ by assumption and $\mathcal{T}_{[3]}$ is invertible the identities above can be combined to yield the desired axiom $(\Delta_F \otimes \text{id})(\mathcal{R}_F) = (\mathcal{R}_F)_{13} \cdot (\mathcal{R}_F)_{23}$ for quasi-triangular Hopf algebras. The second axiom $(\text{id} \otimes \Delta_F)(\mathcal{R}_F) = (\mathcal{R}_F)_{13} \cdot (\mathcal{R}_F)_{12}$ follows entirely analogously.

5.3. Gauge Transformed Ribbon and Balancing Elements. We next discuss the effect of a gauge transformation on balancing and ribbon elements. To this end it is useful to introduce the following element.

$$z_F := x_F S(x_F)^{-1} \quad (67)$$

**Lemma 15.** Suppose $\mathcal{H}$ is balanced with ribbon element $r \in \mathcal{H}$ and balancing element $\kappa \in \mathcal{H}$, and $F$ is a gauge transformation as above. Then $\mathcal{H}_F$ is also balanced with elements as follows:

$$u_F = z_F u \quad r_F = r \quad \kappa_F = z_F \kappa $$ \quad (68)

**Proof.** Given that $\mathcal{R}_F = \sum_{ijk} B_i e_k C_j \otimes A_i f_k D_j$ we compute $u_F$ as follows.

$$u_F = \sum_{ijk} S_F(A_i f_k D_j) B_i e_k C_j \quad \text{by (65)} \quad \sum_{ijk} x_F S(D_j) S(f_k) S(A_i) x_F^{-1} B_i e_k C_j = \sum_{ijk} x_F S(D_j) S(f_k) e_k C_j = \sum_j x_F S(D_j) S^2(C_j) u = x_F S(\sum_j S(C_j) D_j) u \quad \text{by (69)} \quad x_F S(x_F^{-1}) u = z_F u.$$ 

We verify the ribbon element using the characterization in [18]. If we set $r_F = r$ centrality is obvious since multiplication is the same in $\mathcal{H}_F$. This also implies $S_F(r_F) = x_F S(r) x_F^{-1} = x_F r x_F^{-1} = r = r_F$. Moreover, we find

$$(\mathcal{R}_F)_{21} \mathcal{R}_F = F \mathcal{R}_{21} \mathcal{R} F^{-1} = F(r \otimes r) \Delta(r^{-1}) F^{-1} = (r \otimes r) \Delta_F(r^{-1}),$$
where we use again that \( r \) and hence \( r \otimes r \) is central. Finally, the equivalence of a ribbon structure and a balancing (see [Ke94]) implies that \( \kappa_F = u_F(r_F)^{-1} = z_F u^{-1} = z_F \kappa \) as desired. \( \square \)

Note that Lemma 15 implies that \( \kappa_F \) as given in (68) is group like with respect to \( \Delta_F \). This is indeed true even without an underlying quasi-triangular structure as verified in the following lemma.

**Lemma 16.** Suppose \( \mathcal{H} \) is a Hopf algebra with gauge transformation \( F \) and \( \kappa \) is a group like element with \( S^2(a) = \kappa a \kappa^{-1} \). Then \( \kappa_F = z_F \kappa \) is group like in \( \mathcal{H}_F \) and satisfies \( S^2_F(a) = \kappa_F a \kappa_F^{-1} \).

**Proof.** Iterating (65) we find that \( S^2_F(a) = z_F S^2(a) z_F^{-1} \) which immediately that \( S^2_F \) is given by conjugation with \( \kappa_F \). Moreover, using (62) we compute
\[
(\kappa \otimes \kappa) F \Delta(\kappa^{-1}) = (\kappa \otimes \kappa) F (\kappa^{-1} \otimes \kappa^{-1}) = S^2 \otimes S^2(F) = \mathcal{T}((\mathcal{T}(F^{-1})^{-1})
\]
\[
= \mathcal{T}(\Delta(x_F^{-1})F^{-1}(x_F \otimes x_F)) = (S(x_F) \otimes S(x_F))\mathcal{T}(F^{-1})\Delta(S(x_F)^{-1})
\]
\[
= (S(x_F)x_F^{-1} \otimes S(x_F)x_F^{-1})F \Delta(x_F) \Delta(S(x_F)^{-1}) = (z_F^{-1} \otimes z_F^{-1})F \Delta(z_F),
\]
which implies \( \kappa_F \otimes \kappa_F = (z_F \otimes z_F)(\kappa \otimes \kappa) = F \Delta(z_F) \Delta(\kappa) F^{-1} = \Delta_F(\kappa_F) \) as claimed. \( \square \)

5.4. **Gauge Transformed Integrals.** We finally consider the effect of gauge transformations on integrals of \( \mathcal{H} \). In our discussion we confine ourselves to the unimodular case (that is, when \( \Lambda \in \mathcal{H} \) is a two-sided integral) since this is the only relevant case for TQFT constructions. Generalizations to the non-unimodular case follow the same lines with additional elements such as \( (\alpha \otimes id)(F) \in \mathcal{H} \) where \( \alpha \) is the comodulus. Details are left to the interested reader.

In the unimodular case \( \Lambda_F = \Lambda \) is clearly also a two-sided integral in \( \mathcal{H}_F \) since the algebra structure remains the same. This simple observation allows us to determine \( \lambda_F \) using non-degenerate forms on \( \mathcal{H} \) and \( \mathcal{H}^* \) obtained from integrals as in [LS69].

**Lemma 17.** Let \( \mathcal{H} \) be a unimodular Hopf algebra with right integrals \( \lambda \in \mathcal{H}^* \) and \( \Lambda \in \mathcal{H} \) with \( \lambda(\Lambda) = 1 \). Then
\[
\lambda_F = \lambda \leftarrow z_F^{-1}
\]
is the unique right integral for \( \mathcal{H}_F \) with \( \lambda_F(\Lambda_F) = \lambda_F(\Lambda) = 1 \).

**Proof.** The strategy is to determine \( \lambda_F \) from (17) by computing \( \beta_F \) from formulae for \( \beta_F \) and \( S_F \) in \( \mathcal{H}_F \). In order to determine \( \beta_F \) observe that \( \mathcal{J}(a \otimes b) \Delta(\Lambda) = \mathcal{J}(a \otimes b \Lambda) = \epsilon(b)\mathcal{J}(a \otimes \Lambda) = \epsilon(b)(S(a) \otimes 1)\Delta(\Lambda) \) and, by similar calculation, \( \Delta(\Lambda)\mathcal{J}^{-1}(a \otimes b) = \)
\[ \Delta(\Lambda)e(b)\langle S^{-1}(a) \otimes 1 \rangle. \] This implies by (52) that \( \overline{\beta}(F^{-1})\Delta(\Lambda) = \Delta(\Lambda)\overline{\beta}^{-1}(F) \) and, hence,

\[
\Delta_F(\Lambda_F) = F\Delta(\Lambda)F^{-1} = (x_F \otimes 1)\overline{\beta}(F^{-1})\Delta(\Lambda)\overline{\beta}^{-1}(F)(S^{-1}(x_F^{-1}) \otimes 1)
\]

\[ = (x_F \otimes 1)\Delta(\Lambda)(S^{-1}(x_F^{-1}) \otimes 1) \]

The value for \( \overline{\beta}_F(f) \) is now obtained by applying \( f \otimes \text{id} \) to \( \Delta_F(\Lambda_F) \). Expressing the multiplication by the elements in the first tensor factor by the actions in (14) we find

\[
\overline{\beta}_F(f) = \overline{\beta}(S^{-1}(x_F^{-1}) \rightarrow f \leftarrow x_F) \implies \overline{\beta}^{-1}_F = S^{-1}(x_F) \rightarrow \overline{\beta}^{-1}(a) \leftarrow x_F^{-1}
\]

Now \( \lambda = \beta(1) = (\overline{\beta})^{-1}(S(1)) = (\overline{\beta})^{-1}(1) \) and similarly \( \lambda_F = \overline{\beta}^{-1}_F(1) \) so that

\[
\lambda_F = S^{-1}(x_F) \rightarrow \lambda \leftarrow x_F^{-1}.
\]

This can be rewritten as \( \lambda_F(a) = \lambda(x_F^{-1}aS^{-1}(x_F)) = \lambda(S(x_F)x_F^{-1}a) = \lambda(z_F^{-1}a) \) where we also used (13).

\[ \square \]

6. Gauge Transformations for Hennings TQFTs

In this section we will study how the TQFTs constructed in Section 3 behave under gauge transformations of the underlying Hopf algebras. Particularly, we will construct an explicit natural isomorphism between functors \( \mathcal{Y}_\mathcal{H}^* \) and \( \mathcal{Y}_\mathcal{F}^* \) for a given topogenic Hopf algebra and gauge transformation \( F \). The main goal of this section is thus to prove Theorem 2.

6.1. Gauge Elements for Higher Tensor Products. The first ingredient in the construction of an isomorphisms between these functors is the extension of \( F \) as an intertwiner of coalgebra structures to higher tensor powers as follows. We start by defining the operation

\[
\nabla : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H} : x \mapsto F\Delta(x) \quad \text{and} \quad \overline{\nabla} : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H} : x \mapsto \Delta(x)F^{-1}
\]

\[
\nabla_i^{(n)} = id^\otimes(i-1) \otimes \nabla \otimes \text{id}^\otimes(n-i) : \mathcal{H}^\otimes n \rightarrow \mathcal{H}^\otimes(n+1)
\]

and analogously \( \overline{\nabla}_i^{(n)} : \mathcal{H}^\otimes n \rightarrow \mathcal{H}^\otimes(n+1) \). Moreover, for a sequence of integer indices \( (i_1, \ldots, i_n) \) with \( 1 \leq i_k \leq k \) for all \( k \) we define

\[
\tau_{[i+1]} := \nabla_{i_n}^{(n)} \circ \nabla_{i_{(n-1)}}^{(n-1)} \circ \ldots \circ \nabla_{i_1}^{(1)}(1) \in \mathcal{H}^\otimes(n+1)
\]

\[
\overline{\tau}_{[i+1]} := \overline{\nabla}_{i_n}^{(n)} \circ \overline{\nabla}_{i_{(n-1)}}^{(n-1)} \circ \ldots \circ \overline{\nabla}_{i_1}^{(1)}(1) \in \mathcal{H}^\otimes(n+1)
\]

For example, we have \( \tau_{[1]} = F \), and \( \overline{\tau}_{[1]} = F^{-1} \). For \( n = 2 \) we obtain the two sides of the cocycle condition (or its inverse) depending on whether we choose \( i_2 = 1 \) or
Similarly, we have the following identities.

\[ i_2 = 2. \] That is, \( T_{[2]} = (F \otimes 1)(\Delta \otimes id)(F) = (1 \otimes F)(id \otimes \Delta)(F) = \) and \( \overline{T}_{[2]} = (\Delta \otimes id)(F^{-1})(F^{-1} \otimes 1) = (id \otimes \Delta)(F^{-1})(1 \otimes F^{-1}) \) are inverses and independent of

the choice of \( i_2 \). This observation and other properties of \( F \) are generalized to \( T_{[n]} \) next.

**Lemma 18.** For \( T_{[n]} \) and \( \overline{T}_{[n]} \) as above and for any \( f \in \mathcal{H}^* \), \( x \in \mathcal{H} \) and \( i = 1, \ldots, n \) we have the following identities.

\[
T_{[n]} \text{ does not depend on the choice of indices } (i_1, \ldots, i_n) \text{ but only on } n \quad (73)
\]

\[
id^{\otimes i-1} \otimes \epsilon \otimes id^{\otimes (n-i)}(T_{[n]}) = T_{[n-1]} \quad (74)
\]

\[
T_{[n]}^{-1} = \overline{T}_{[n]} \quad (75)
\]

\[
T_{[n]} \Delta^{(n)}(x) = \Delta_T^{(n)}(x)T_{[n]} \quad (76)
\]

\[
(1^{\otimes i-1} \otimes R_F \otimes 1^{\otimes (n-i)})T_{[n+1]} = \sigma_{(i,i+1)}(T_{[n+1]}) (1^{\otimes i-1} \otimes R \otimes 1^{\otimes (n-i)}) \quad (77)
\]

\[
(id^{\otimes i-1} \otimes f \circ m \circ (id \otimes S)) \otimes 1^{\otimes (n-i)})(T_{[n+1]}) = f(x_F)T_{[n+1]} \quad (78)
\]

\[
(id^{\otimes i-1} \otimes (f \circ m \circ (S \otimes id)) \otimes 1^{\otimes (n-i)})(\overline{T}_{[n+1]}) = f(x_F^{-1})\overline{T}_{[n+1]} \quad (79)
\]

**Proof.** For the statement in (73) observe that (52) and coassociativity imply that \( \nabla_1^{(3)} \circ \nabla = \nabla_2^{(3)} \circ \nabla \) and hence \( \nabla_i^{(n+1)} \circ \nabla_i^{(n)} = \nabla_{i+1}^{(n+1)} \circ \nabla_i^{(n)} \). Assuming the statement in (73) is true for \( n = m - 1 \) we may set \( i_{m-1} \) to be \( i_m \). We find

\[
T_{[m+1]} = \nabla_i^{(m)}(T_{[m]}) = \nabla_i^{(m)}(\nabla_i^{(m-1)}(T_{[m-1]})) = \nabla_i^{(m+1)}(\nabla_i^{(m-1)}(T_{[m-1]})) = \nabla_i^{(m+1)}(T_{[m]}). \]

That is, \( i_m \) may be replaced by \( i_m + 1 \) implying independence of the choice of \( i_m \). (73) thus follows by induction in \( m \).

Identity (74) follows immediately from \( (\epsilon \otimes id)(\nabla(x)) = (id \otimes \epsilon)(\nabla(x)) = x \).

For (75) note that \( \nabla(x)\overline{\nabla(y)} = \Delta_T(xy) \) so that, again by induction in \( n \), we find

\[
T_{[n+1]} \overline{T}_{[n+1]} = \nabla_1(T_{[n]})\overline{\nabla_1(T_{[n]})} = (\Delta_T \otimes id^{\otimes n-1})(T_{[n]}\overline{T}_{[n]}) = (\Delta_T \otimes id^{\otimes n-1})(1) = 1. \]

Similarly, \( \overline{\nabla(x)\nabla(y)} = \Delta(xy) \) implies \( \overline{T}_{[n+1]}T_{[n+1]} = 1 \). The induction step for proving (76) is as follows.

\[
T_{[n+1]} \Delta^{(n+1)}(x) = (F \otimes 1^{\otimes n-1})(\Delta \otimes id^{\otimes n-1})(T_{[n]} \Delta^{(n)}(x)) \]

\[
= (F \otimes 1^{\otimes n-1})(\Delta \otimes id^{\otimes n-1})(\Delta_T^{(n)}(x)T_{[n]}) \]

\[
= (\Delta_F \otimes id^{\otimes n-1})(\Delta_T^{(n)}(x))(F \otimes 1^{\otimes n-1})(\Delta \otimes id^{\otimes n-1})(T_{[n]}) \]

\[
= \Delta_T^{(n+1)}(x)T_{[n+1]}.
\]
For (77) substitute \( \Upsilon_{n+1} = \nabla_i(\Upsilon_n) \) and observe that \( \mathcal{R}_F \nabla(x) = F_{21} \mathcal{R}_\Delta(x) = \sigma_{(12)}(F)\sigma_{(12)}(\Delta(x)) \mathcal{R} = \sigma_{(12)}(\nabla(x)) \mathcal{R} \). Substituting also \( \Upsilon_{n+1} = \nabla_i(\Upsilon_n) \) but using

\[
(f \circ m \circ (id \otimes S)) (\nabla(x)) = f(\sum_i A_i x' S(B_i x'')) = f(\sum_i A_i x' S(B_i)) = \\
= \epsilon(x) f(\sum_i A_i S(B_i)) = \epsilon(x) f(x_F)
\]

we prove (78) using also (74). Equation (79) follows analogously. \( \square \)

6.2. **Gauge Transformations of Planar Diagrams.** Next relate the algebraic formalism of gauge transformations above to the combinatorial calculus introduced in Section 3.1 for categories of \( \mathcal{H} \)-labeled planar diagrams. Consider the following map of decorated diagrams which inserts into a given diagram an additional element \( x_F^{\pm 1} \) near each maximum and minimum as indicated in Figure 14 but leaves the diagram unchanged otherwise.

\[\begin{array}{cc}
\begin{array}{c}
\text{Figure 14. Extrema insertions}
\end{array}
\end{array}\]

**Lemma 19.** The map of decorated diagrams defined in Figure 14 factors into a well defined functor

\[ I_F : \mathcal{Dc}(\mathcal{H}_F) \rightarrow \mathcal{Dc}(\mathcal{H}). \]  

**Proof.** We need to check that equivalence classes of diagrams in \( \mathcal{Dc}(\mathcal{H}_F) \) are mapped to classes for \( \mathcal{Dc}(\mathcal{H}) \). The fact is clear for the planar second and third Reidemeister moves since these do not involve extrema. Similarly, moving an extremum through a crossing follows since the extra \( x_F^{\pm 1} \) can be moved through the crossing as in Figure 4.

Cancellation of a maximum and a minimum is seen by canceling the \( x_F \) assigned to the maximum with the \( x_F^{-1} \) added near the minimum. It remains to verify the moves in Figures 4 and Figure 5. The first and third picture in Figure 4 as well as the first in Figure 5 are immediate. For the second equivalence in Figures 4 we apply \( I_F \) to either side of the equation (in \( \mathcal{Dc}(\mathcal{H}_F) \)) which yields the diagrams in \( \mathcal{Dc}(\mathcal{H}) \) as depicted in Figure 15. Clearly, by (65), the resulting diagrams are the same.

For the second picture in Figure 5 we proceed similarly by applying \( I_F \) to both sides of the equivalence as indicated in Figure 16. The equality of the resulting diagrams in \( \mathcal{Dc}(\mathcal{H}) \) is implied by (67), (68) and (22), as well as \( x_F^{-1} \kappa_F = x_F^{-1} z_F \kappa = S(x_F^{-1}) \kappa = \kappa S^{-1}(x_F^{-1}). \) \( \square \)
We observe next that the elements $\Upsilon_{[n]} \in \mathcal{H} \otimes g$ introduced in (72) can be thought of as a collection of morphisms in the categories of decorated diagrams by distributing the tensor factors of $\Upsilon_{[n]} = \sum \Upsilon_{[1]}^{(1)} \otimes \ldots \otimes \Upsilon_{[n]}^{(n)}$ over $n$ parallel strands exactly in the way we did for $\Delta^{(2g-1)}(x)$ in (12).

In the category $\mathcal{Dc}^0(\mathcal{H})$ of decorated diagrams without connectivity constraints (see beginning of Section 4) this yields a collection $\Upsilon^0_{[\ast]} = \{ \Upsilon_{[n]}^0 : n \to n \}$ of isomorphisms in $\mathcal{Dc}^0(\mathcal{H})$. In the (sub) category $\mathcal{Dc}(\mathcal{H})$ we thus obtain a collection $\Upsilon_{[\ast]} = \{ \Upsilon_{[2k]} : k \to k \}$ of isomorphisms.

In the next lemma we will interpret $\Upsilon^0_{[\ast]}$ and $\Upsilon_{[\ast]}$ as natural transformations between functors with target categories $\mathcal{Dc}^0(\mathcal{H})$ and $\mathcal{Dc}(\mathcal{H})$ respectively.

**Lemma 20.** With conventions as above we have the following natural isomorphism of functors from $\mathcal{Tgl}$ to $\mathcal{Dc}(\mathcal{H})$:

$$\Upsilon_{[\ast]} : \mathcal{Z}_{\partial(\mathcal{H})} \rightarrow \mathcal{J}_F \circ \mathcal{Z}_{\partial(\mathcal{H})}.$$  

(81)

*Proof.* We will in fact prove that the isomorphism $\Upsilon^0_{[\ast]} : \mathcal{Z}_{\partial(\mathcal{H})} \rightarrow \mathcal{J}_F \circ \mathcal{Z}_{\partial(\mathcal{H})}$ is defined on general tangles without connectivity constraints, which obviously implies Lemma 20. Explicitly we need to show that for every tangle $T$ with $N$ end points at the top and
end points at the bottom we have
\[ \Upsilon_{n+1}^0 \cdot \Theta_{y_g}(T) \cdot (\Upsilon_{n+1}^0)^{-1} = \mathcal{I}_F \left( \Theta_{y_g}(T) \right) . \] (82)

By functoriality it suffices to prove (82) for the generators of \( \mathcal{I} gl \). The three types of generators are a single maximum, a single minimum, and a single positive crossing, each having some number of parallel strands to the right and left. In the following picture we evaluate the left hand side of (82) by making use of the identities in Lemma 18.

In the evaluation of a crossing in Figure 17 we first apply the rule from Figure 7 assigning the factors of \( \mathcal{R} \in \mathcal{H}_{\otimes 2} \) to the two markings and then conjugate the diagram by \( \Upsilon_{n+1}^0 \) assuming we have \( n+1 \) strands. The equality between the last two diagrams follows by multiplying them both by \( \Upsilon_{n+1}^0 \) and a crossing from the bottom. Given that conjugation of \( \Upsilon_{n+1}^0 \) by crossings is the same as transposition of factors the resulting equality is the same as (77).

Since there are no extrema involved in a crossing the resulting diagram in \( \mathcal{D} c^0(\mathcal{H}_F) \) is the same as evaluating \( \mathcal{I}_F \circ \Theta_{y_g}(T) \) on a single crossing so that (82) holds for this case.

The proof for maximum proceeds similarly as depicted in Figure 18. For the last identity we can multiply the diagram from the top with \( \Upsilon_{n-1}^0 \) and evaluate the combined decoration along the arc with an arbitrary linear form \( f \in \mathcal{H}^* \). The identity to prove is then the same as in (78) where the map \( m \circ (id \otimes S) \) accounts for combining the \( i \)-th and \( i+1 \)-st factors of \( \Upsilon_{n+1}^0 \) along the arc using the rules in (4).

The resulting diagram in \( \mathcal{D} c^0(\mathcal{H}_F) \) is again clearly the same as evaluating \( \mathcal{I}_F \circ \Theta_{y_g}(T) \) on a single maximum. The case for a minimum follows analogously from (78). Thus (82) holds on all generators of the tangle category.
6.3. Gauge Transformations of Modular Evaluations. The relation between the evaluation functors $E_{\mathbf{H}}$ and $E_{\mathbf{H}_F}$ also involves a natural transformation, which we defined next. Let us denote right multiplication by $x_F$ as

$$\rho_F : \mathbf{H} \to \mathbf{H} : z \mapsto z x_F. \quad (83)$$

In order to describe equivariance of a tensor power $\rho_F^{\otimes g}$ that maps $\mathcal{H}^{\otimes g}$ to itself (as a $\mathfrak{D}$-module) we need to identify the latter space as an $\mathcal{H}$-module. One $\mathcal{H}$-action is given by the usual $g$-fold tensor product of adjoint action on $\mathcal{H}$ with respect to the transformed coalgebra structure given by $\Delta_F$ and $S_F$. We denote the respective $\mathcal{H}$-module by $\mathcal{H}_F^{\otimes g}$.

Another $\mathcal{H}$-action is defined similar to the adjoint action using the transformed coproduct $\Delta_F$ but the original antipode $S$. That is, if $\Delta_F^{2g-1}(x) = \sum x_F^{(1)} \otimes \ldots \otimes x_F^{(2g)}$ then $x.(b_1 \otimes \ldots \otimes b_g) = \sum x_F^{(1)} b_1 S(x_F^{(2)}) \otimes \ldots \otimes x_F^{(2g-1)} b_g S(x_F^{(2g)}).$ It is obvious that this also defines an $\mathcal{H}$-module which we denote by $\widetilde{\mathcal{H}}^{\otimes g}$. We also readily verify that the morphisms in

$$\rho_F^* = \{ \rho_F^{\otimes g} : \mathcal{H}_F^{\otimes g} \to \widetilde{\mathcal{H}}^{\otimes g} \}. \quad (84)$$

indeed commute with the $\mathcal{H}$-action on the so defined modules. Particularly, with $\Delta_F^{2g-1}(x) = \sum x_F^{(1)} \otimes \ldots \otimes x_F^{(2g)}$ as before we have

$$x.(\rho_F^{\otimes g}(b_1 \otimes \ldots \otimes b_g)) = x.(b_1 x_F \otimes \ldots \otimes b_g x_F)$$

$$= \sum x_F^{(1)} b_1 x_F S(x_F^{(2)}) \otimes \ldots \otimes x_F^{(2g-1)} b_g x_F S(x_F^{(2g)}) \quad \text{by (83)}$$

$$= \rho_F^{\otimes g} \left( \sum ad_F(x_F^{(1)})(b_1) \otimes \ldots \otimes ad_F(x_F^{(g)})(b_g) \right) \quad \text{(85)}$$

$$= \rho_F^{\otimes g}(ad_F^{\otimes g}(x)(b_1 \otimes \ldots \otimes b_g))$$

The maps in (84) are thus morphisms in $\mathcal{H} \models m(\mathfrak{D})$. In the next lemma we will interpret the collection $\rho_F^*$ of these isomorphisms as a natural transformation.
Lemma 21. With conventions as above we have the following natural isomorphism of functors:
\[ \rho^\circ_F : \mathcal{E}_{\mathcal{H}_F} \xrightarrow{\circ} \mathcal{E}_{\mathcal{H}} \circ \mathcal{I}_F. \]  
(86)

Proof. Expressing naturality of (86) more explicitly we have to show for every class of functors:
\[ \rho^\circ_F : \mathcal{E}_{\mathcal{H}_F} \circ \mathcal{I}_F = \mathcal{E}_{\mathcal{H}_F} \circ (\mathcal{I}_F(D)). \]  
(87)

As before we can use functoriality to reduce the proof to generators of \( \mathcal{Dc}(\mathcal{H}_F) \). Moreover, all functors and natural transformations in (86) also respect the tensor products of the five types of diagrams depicted in Figures 8 and 9. It thus suffices to prove (87) for each of these.

We start by evaluating \( \mathcal{E}_{\mathcal{H}} \circ \mathcal{I}_F \) on the second picture in Figure 8. The functor \( \mathcal{I}_F \) inserts a \( \mathbf{x}_F \) directly below the \( \mathbf{t}_k \) which combines to a single decoration \( s^\nu_j \mathbf{x}_F \) in place of \( s^\nu_j \). \( \mathcal{E}_{\mathcal{H}} \) assigns to this the map \( v^\nu_j : \mathbf{D} \to \mathbf{H} : 1 \mapsto s^\nu_j \mathbf{x}_F \).

Next we evaluate \( \mathcal{E}_{\mathcal{H}} \circ \mathcal{I}_F \) on the third picture in the same figure. \( \mathcal{I}_F \) inserts a \( \mathbf{x}_F^{-1} \) directly below the \( \mathbf{t}_k \) which combines to \( \mathbf{x}_F^{-1} \mathbf{t}_k^\nu \) in place of \( \mathbf{t}_k \). \( \mathcal{E}_{\mathcal{H}} \) assigns to this the form \( t_k^\nu : \mathbf{H} \to \mathbf{D} : x \mapsto \lambda(S(x)\mathbf{x}_F^{-1} \mathbf{t}_k) \). If we apply \( \mathcal{E}_{\mathcal{H}_F} \) to the same picture we obtain the form \( m_k^\nu : \mathbf{H}_F \to \mathbf{D} : x \mapsto \lambda \mathbf{F}(S_F(x)\mathbf{t}_k) \). We verify (87) by computation:
\[ \rho^{\circ 0}_F \circ m_k^\nu \circ (\rho^{\circ 1}_F)^{-1}(x) = m_k^\nu(\rho^{\circ 1}_F(x)) = \lambda_S(x\mathbf{x}_F^{-1} \mathbf{t}_k^\nu) \]
by (85)
\[ = \lambda((S(x)\mathbf{x}_F^{-1})^{-1} \mathbf{x}_F S(x) \mathbf{x}_F^{-1} \mathbf{x}_F^{-1} \mathbf{t}_k^\nu) \]
\[ = \lambda(S(x)\mathbf{x}_F^{-1} \mathbf{t}_k) = t_k^\nu(x) \]
(88)

The first picture is a consequence of the second and third by composing them (setting \( s^\nu_j = 1 \) and \( t_k^\nu = r_k^\nu \)).

In Figure 9 we start with the left picture. Since no extrema are involved \( \mathcal{E}_{\mathcal{H}} \circ \mathcal{I}_F \) assigns to this diagram the morphism \( f : \mathbf{H} \to \mathbf{H} : x \mapsto axS(b) \). Similarly, \( \mathcal{E}_{\mathcal{H}_F} \) assigns to it the map \( g : \mathbf{H} \to \mathbf{H} : x \mapsto axS_F(b) \). We compute \( \rho^{\circ 1}_F \circ g \circ (\rho^{\circ 1}_F)^{-1}(x) = \rho_F(g(\rho^{\circ 1}_F(x))) = (a(x\mathbf{x}_F^{-1} S_F(b)) \mathbf{x}_F = ax\mathbf{x}_F^{-1} S(b) \mathbf{x}_F \mathbf{x}_F = f(x). \)
Finally, $\mathcal{E}_{2q} \circ \mathcal{I}_F$ assigns to the picture on the right of Figure 9 the map $q : \mathcal{H} \to \mathcal{H} : x \mapsto \kappa^{-1}S(x)$ and $\mathcal{E}_{3q}$ assigns to it $p : \mathcal{H} \to \mathcal{H} : x \mapsto \kappa^{-1}S_F(x)$. As before verification of (87) is done by computing $\rho_F^{\otimes 1} \circ p \circ (\rho_F^{\otimes 1})^{-1}(x) = \rho_F(p(\rho_F^{-1}(x))) = \kappa^{-1}S_F(x\cdot x_F^{-1})x_F \kappa^{-1}z_F^{-1}x_F S(x\cdot x_F^{-1}) = \kappa^{-1}z_F^{-1}x_F S(x\cdot x_F^{-1})S(x) = q(x)$. \hfill \Box

### 6.4. Construction of a Natural Isomorphism and Proof of Theorem 2

In order to construct the natural transformation from Theorem 2 we also need to consider the evaluation of the transformations in Lemma 20

$$\mathcal{T}_g = \mathcal{E}_{2q}(\mathcal{T}_{[2q]}) : \mathcal{H}^{\otimes g} \to \tilde{\mathcal{H}}^{\otimes g} \tag{89}$$

Denoting $\mathcal{T}_{[2q]} = \sum \mathcal{T}_{[2q]}^{(1)} \otimes \ldots \otimes \mathcal{T}_{[2q]}^{(2g)}$ the explicit form of $\mathcal{T}_g$ is readily derived from Figure 9 as

$$\mathcal{T}_g(b_1 \otimes \ldots \otimes b_g) = \sum \mathcal{T}_{[2q]}^{(1)}b_1 S(\mathcal{T}_{[2q]}^{(2)}) \otimes \ldots \otimes \mathcal{T}_{[2q]}^{(2g-1)}b_g S(\mathcal{T}_{[2q]}^{(2g)}) \tag{90}$$

If we consider $\mathcal{H}^{\otimes g}$ to be a $\mathcal{H}$-module equipped with the tensor product of the regular adjoint action and $\tilde{\mathcal{H}}^{\otimes g}$ with the action defined for (84) above the maps $\mathcal{T}_g$ also commute with the actions of $\mathcal{H}$. This follows from the following calculation.

\begin{align*}
\notag x. (\mathcal{T}_g(b_1 \otimes \ldots \otimes b_g)) &= \\
\notag & = x. \left( \sum \mathcal{T}_{[2q]}^{(1)}b_1 S(\mathcal{T}_{[2q]}^{(2)}) \otimes \ldots \otimes \mathcal{T}_{[2q]}^{(2g-1)}b_g S(\mathcal{T}_{[2q]}^{(2g)}) \right) \\
\notag & = \sum x_F^{(1)} \mathcal{T}_{[2q]}^{(1)}b_1 S(\mathcal{T}_{[2q]}^{(2)}) S(x_F^{(2)}) \otimes \ldots \otimes x_F^{(2g-1)} \mathcal{T}_{[2q]}^{(2g-1)}b_g S(\mathcal{T}_{[2q]}^{(2g)}) S(x_F^{(2g)}) \\
\notag & = \sum x_F^{(1)} \mathcal{T}_{[2q]}^{(1)}b_1 S(x_F^{(2)} \mathcal{T}_{[2q]}^{(2)}) \otimes \ldots \otimes x_F^{(2g-1)} \mathcal{T}_{[2q]}^{(2g-1)}b_g S(x_F^{(2g)} \mathcal{T}_{[2q]}^{(2g)}) \\
\notag \text{by (76)} & = \sum \mathcal{T}_{[2q]}^{(1)} x^{(1)} b_1 S(\mathcal{T}_{[2q]}^{(2)} S(x^{(2)}) \otimes \ldots \otimes \mathcal{T}_{[2q]}^{(2g-1)} x^{(2g-1)} b_g S(\mathcal{T}_{[2q]}^{(2g)} S(x^{(2g)})) \\
\notag & = \mathcal{T}_g \left( \sum x^{(1)} b_1 S(\mathcal{T}_{[2q]}^{(2)} S(x^{(2)}) \otimes \ldots \otimes x^{(2g-1)} b_g S(\mathcal{T}_{[2q]}^{(2g)} S(x^{(2g)})) \right) \\
\notag & = \mathcal{T}_g \left( \sum \ldots \right) \\
\notag & = \mathcal{T}_g \left( ad^{\otimes g}(x)(b_1 \otimes \ldots \otimes b_g) \right)
\end{align*}

Proof of Theorem 2: Applying the functor $\mathcal{E}_{3q}$ to Equation (81) in Lemma 20 we obtain the natural transformation $\mathcal{E}_{3q} \mathcal{T}_{[q]} : \mathcal{E}_{3q} \circ \mathcal{Z}_{\mathcal{F}} \to \mathcal{E}_{3q} \circ \mathcal{I}_F \circ \mathcal{Z}_{\mathcal{F}}$, where the isomorphisms of $\mathcal{E}_{3q} \mathcal{T}_{[q]}$ are given by the $\mathcal{T}_g$ from (89). Similarly, (21) implies a natural isomorphism $\rho_F^{\otimes 1} \mathcal{Z}_{\mathcal{F}} : \mathcal{E}_{3q} \circ \mathcal{Z}_{\mathcal{F}} \to \mathcal{E}_{3q} \circ \mathcal{I}_F \circ \mathcal{Z}_{\mathcal{F}}$ which combines to

$$\tilde{\gamma}_F := (\rho_F^{\otimes 1} \mathcal{Z}_{\mathcal{F}})^{-1} \circ (\mathcal{E}_{3q} \mathcal{T}_{[q]}) : \mathcal{E}_{3q} \circ \mathcal{Z}_{\mathcal{F}} \to \mathcal{E}_{3q} \circ \mathcal{Z}_{\mathcal{F}}. \tag{92}$$
Given that for objects $Z_{\alpha_F}(g) = g$, the morphisms for (92) are given by

$$\tilde{\gamma}_F : H^g \rightarrow \tilde{H}^g \rightarrow \rho_F^{\otimes g} \rightarrow \tilde{H}^g \rightarrow \tilde{H}^g.$$

As shown in (91) and (85) above both $\Upsilon_g$ and $\rho^{\otimes g}$ commute with the respective actions of $H$ so that each $\tilde{\gamma}_F(g)$ is a morphism in $H = m(D)$.

Substituting the functor composites in (92) using (27) we thus have the natural isomorphism $\tilde{\gamma}_F : \Upsilon_{\alpha_F} \circ H \circ \mathcal{M} \rightarrow \Upsilon_{\alpha_F} \circ H \circ \mathcal{M}$. Clearly, the functor $H \circ \mathcal{M}$ is a one-to-one correspondence on objects (with $H \circ \mathcal{M}(g) = g$) and maps morphism spaces surjectively onto each other. This ensures that $\tilde{\gamma}_F$ indeed gives rise to a natural isomorphism $\gamma_F : \Upsilon_{\alpha_F} \rightarrow \Upsilon_{\alpha_F}$ defined by the same set of morphisms given in (93). □

The explicit form of the morphisms of the natural transformation is readily worked out from (89) and (83) to be

$$\gamma_F(g)(b_1 \otimes \ldots \otimes b_g) = \sum \Upsilon_{\alpha_F}^{(1)} b_1 S(\Upsilon_{\alpha_F}^{(2)} x_F^{-1} \otimes \ldots \otimes \Upsilon_{\alpha_F}^{(g-1)} b_g S(\Upsilon_{\alpha_F}^{(g)}) x_F^{-1}.$$

Finally, let us note that the assignment of natural transformations $F \mapsto \gamma_F$ is well behaved under compositions of gauge transformations. More precisely, if $G$ is a cocycle with respect to $\Delta_F$ then $G \cdot F$ is a cocycle with respect to $\Delta$ and $(H_F)_G = H_{G \cdot F}$. In this case we have that

$$\gamma_{G \cdot F} = \gamma_G \cdot \gamma_F.$$

7. Integral TQFTs from Quantum Doubles

In this section we specialize the previous TQFT constructions to the case in which the underlying algebra is the Drinfeld quantum double $H = D(H)$ of a Hopf algebra $H$ over $D$. The conditions that ensure integrality of the resulting TQFT on $\text{Cog}^\emptyset$ and $\text{Cog}$ as postulated for the general case in Theorem 12 will reduce to only a few very mild assumptions on the algebra $H$. In the course of this section we will also generalize various facts about integrals on and in Hopf algebras over commutative rings instead of fields. Our findings will cumulate in the proof of Theorem 1.

7.1. Hopf Algebras over Dedekind Domains. In this section we establish and collect several basic but crucial facts about integrals and moduli of a Hopf algebra $H$ over a Dedekind domain $D$ which is finitely generated and projective as a $D$-module as assumed in Theorem 1.
Recall that an element $x$ is said to be group-like if $\Delta(x) = x \otimes x$. Denote the set of group-like elements of $H$ by $G(H)$ which is clearly a group itself. The dual $H^* = \text{Hom}_D(H, \mathbb{D})$, with respect to the ground ring $\mathbb{D}$, is also a Hopf algebra whose structural maps are induced by dualizing. The respective set $G(H^*)$ is thus the group of multiplicative forms on $H$ with values in $\mathbb{D}$. We start with the following technical lemma.

**Lemma 22.** Let $H$ be a Hopf algebra over a domain $\mathbb{D}$ that is finitely generated as a $\mathbb{D}$-module. For any elements $f \in G(H^*)$ and $x \in G(H)$, $f(x)$ is a root of unity.

This readily follows, for example, from Theorem 2.1.2 in [Ab77] asserting that the elements of $G(\mathcal{P})$ are linearly independent if $\mathcal{P}$ is over a field. Applying this to $\mathcal{P} = H \otimes \mathbb{F}$, where $\mathbb{F}$ denotes the field of fractions of $\mathbb{D}$, and using that $\mathcal{P}$ is, by assumption, finite dimensional implies that $G(\mathcal{P})$ is finite. Thus also $G(H)$ is finite implying the lemma. This observation can be extended to more general rings although we will concern ourselves here only with domains.

The next proposition provides the required technical properties for a Hopf algebra over a Dedekind domain $\mathbb{D}$, including the existence and normalization of left and right integrals and cointegrals as defined in Section 2.2. This main assertion below is proved in [LS69] when $\mathbb{D}$ is a principal ideal domain. The mere existence of integrals for general commutative rings is also established by Lomp in [Lo04]. We do, however, require also the normalization relation in (96) and the existence of moduli (97) for our constructions for which we need to make more assumptions on $\mathbb{D}$.

**Proposition 23.** Let $\mathbb{D}$ be a Dedekind domain and $H$ be a Hopf algebra over $\mathbb{D}$, which is finitely generated and projective as a $\mathbb{D}$-module. Then there exists a left cointegral $\Lambda$ and a right integral $\lambda$ for $H$ such that every left cointegral (resp. right integral) for $H$ is a multiple of $\Lambda$ (resp. $\lambda$) and

$$\lambda(\Lambda) = \lambda(S(\Lambda)) = 1. \quad (96)$$

The elements $\Lambda$ and $\lambda$ induce distinguished group-like elements $\alpha \in G(H^*)$ and $g \in G(H)$ respectively, called moduli, such that

$$\Lambda x = \alpha(x)\Lambda, \quad \text{and} \quad f\lambda = f(g)\lambda, \quad \forall x \in H, \ f \in H^*. \quad (97)$$

**Proof.** We start by considering the set $P$ of left cointegrals for $H$ defined as

$$P = \{ p \in H \mid qp = \epsilon(q)p, \ \forall q \in H \}. \quad (96)$$

The space of left integrals $Q$ in $H^*$ is defined analogously.
According to Lemma 1.1 of [Lr72], we have \( H \cong H^* \otimes P \) as \( D \)-modules for a Dedekind domain \( D \). Since \( H \) is projective and finitely generated over \( D \) we also know the same to be true for \( H^* \) so that \( H^* \cong H^{**} \otimes Q \). Since \( H \) is assumed to be finitely generated and projective we have \( H^{**} = H \), which allows us to conclude
\[
H \cong H \otimes P \otimes Q . \tag{98}
\]

In order to characterize the \( D \)-module structure of \( P \) note first that by results in [LS69] spaces of integrals for finite dimensional Hopf algebras are one-dimensional so that \( F \otimes P \cong F \) where \( F \) is again the field of fractions of \( D \). This, in turn, is equivalent to saying that \( P \) is isomorphic, as a \( D \)-module, to a fractional ideal of \( D \). Since we assumed that \( D \) is a Dedekind domain it thus follows that \( P \) is indeed isomorphic, as a \( D \)-module, to an ideal in \( F \).

Next we note that since \( H \) is assumed to be projective it is also a flat module so that, for example, by Proposition 3.7 in Chapter XVI of [Ln02] we have that for any ideal \( J \subset D \) that the natural \( H \otimes_D J \rightarrow HJ \) is an isomorphism. Together with (98) and interpreting \( P \) and \( Q \) as ideals this thus implies
\[
H \cong HPQ . \tag{99}
\]

By Proposition 27 in [Ln94] there exist unique ideals \( I_i \), called the elementary divisors, \( 1 \leq i \leq s \) such that \( I_i | I_{i+1} \). Inserting this decomposition into (99) yields
\[
H \cong \bigoplus_{i=1}^{s} I_i \cong \bigoplus_{i=1}^{s} I_i PQ .
\]

By the uniqueness of the elementary divisors we must have \( P \cong Q \cong D \). That is, both \( P \) and \( Q \) viewed as modules are free of rank one over \( D \).

Let \( \Lambda \) be now a left cointegral that is a generator of \( P = D\Lambda \). Then by Corollary 1.2 of [Lr72]
\[
\phi : H^* \rightarrow H : f \mapsto \phi(f) = \Lambda \leftarrow f \tag{100}
\]
is a \( D \)-module isomorphism. By Proposition 3 in [Ra90], \( \lambda := \phi^{-1}(1) \) is a right integral for \( H \) and \( \lambda(\Lambda) = \lambda(S(\Lambda)) = 1 \).

The space of right integrals \( Q' = S(Q) \) is similarly free of rank one over \( D \) so that \( \iota_{\Lambda} : Q' \rightarrow D : f \mapsto f(\Lambda) \) is zero or injective. Since \( \iota_{\Lambda}(\lambda) = 1 \) the latter is the case and \( q = \iota_{\Lambda}(q)\lambda \) for all \( q \in Q' \). Thus \( \lambda \) is indeed a generator of \( Q' \).

Given the fact that the integral spaces are free modules the existence of the group-like moduli follows now from the same arguments as in the field case. Particularly,
for any \( x \in H \) it is obvious that \( \Lambda x \) is also an a left integral so that it needs to be a multiple of \( \Lambda \). Hence there is an \( \alpha \in H^* \) such that
\[
\Lambda x = \alpha(x)\Lambda, \quad \forall x \in H.
\]

It follows from standard calculations that \( \alpha \) is indeed group-like. Similarly \( \lambda \) induces the distinguished group-like element \( g \in H \) concluding the proof. \( \square \)

An important consequence for the construction of doubles is the following:

**Corollary 24.** Let \( H \) be as in the proposition above. Then the antipode \( S \) of \( H \) is invertible.

**Proof.** As before let \( \mathbb{F} \) be the field of fractions of \( \mathbb{D} \) and denote \( \mathbb{H} = H \otimes \mathbb{F} \). In \([RG98]\) Radford proves that
\[
S^4(x) = g(\alpha \rightarrow x \leftarrow \alpha^{-1})g^{-1}, \quad \forall x \in \mathbb{H},
\]
which implies that the same formula holds on \( H \) since by Proposition 23 the actions of \( \alpha \in H^* \) and \( g \in H \) involved in (101) map \( H \) to itself. Therefore \( S^4 \) and whence \( S \) are invertible on \( H \). \( \square \)

### 7.2. Quantum Doubles for Projective Hopf Algebras

In this section we describe the construction of a quantum double \( \mathcal{D}(H) \) of a Hopf algebra \( H \) over a unital commutative ring \( \mathbb{D} \) instead of a field. As usual we will denote the multiplication of \( H \) by \( \mu : H \otimes H \to H \) with unit \( 1 \in H \), its comultiplication by \( \Delta : H \to H \otimes H \) with counit \( \epsilon : H \to \mathbb{D} \) and its antipode by \( S : H \to H \), all of which are \( \mathbb{D} \)-module morphisms.

In order to be able to handle duals we assume from now on that \( H \) is projective and finitely generated as a \( \mathbb{D} \)-module. Equivalently, this means that there is a \( \mathbb{D} \)-module, \( Q \), such that \( H \oplus Q = \mathbb{D}^n \) for some \( n \in \mathbb{N} \). Note that for the dual space \( H^* = \text{Hom}_{\mathbb{D}}(H, \mathbb{D}) \) this implies that \( H^* \oplus Q^* \cong \mathbb{D}^n \) so that \( H^* \) is also projective and finitely generated. Moreover, the condition implies that the canonical map
\[
\eta : \text{Hom}_{\mathbb{D}}(H, M) \otimes_{\mathbb{D}} N \to \text{Hom}_{\mathbb{D}}(H, M \otimes_{\mathbb{D}} N)
\]
given by \( \eta(f \otimes y)(x) = f(x) \otimes y \) is an isomorphism. See, for example, Exercise 6 on Page 155 in Section 3.10 of \([Ja89]\). Specializing (102) to \( M = \mathbb{D} \) and \( N = H^* \) yields \( H^* \otimes_{\mathbb{D}} H^* \cong \text{Hom}_{\mathbb{D}}(H, H^*) = \text{Hom}_{\mathbb{D}}(H, \text{Hom}_{\mathbb{D}}(H, \mathbb{D})) \). Using the adjointness relation \( \text{Hom}_{\mathbb{D}}(H, \text{Hom}_{\mathbb{D}}(H, \mathbb{D})) \cong \text{Hom}_{\mathbb{D}}(H \otimes_{\mathbb{D}} H, \mathbb{D}) \) (see, for example, Proposition 3.8 in Section 3.8 of \([Ja89]\)) this yields that the canonical map
\[
\tau : H^* \otimes_{\mathbb{D}} H^* \to (H \otimes_{\mathbb{D}} H)^* \quad \text{with} \quad \tau(l \otimes m)(x \otimes y) = l(x) \cdot m(y)
\]
is an isomorphism for $H$ as above.

The relevant implication of (103) is that we can define a coalgebra structure also on $H^*$ with coproduct $\tau^{-1} \circ \mu^* : H^* \to H^* \otimes H^*$. It is readily verified that this makes $H^*$ into a Hopf algebra with product $\Delta^* \circ \tau : H^* \otimes H^* \to H^*$ and antipode $S^* : H^* \to H^*$.

As a coalgebra the quantum double of $H$ is defined as $D(H) = H^{\text{cop}} \otimes H$, where $H^{\text{cop}}$ is identical to $H^*$ except that the opposite comultiplication is used. As an algebra $D(H)$ is defined as a bi-crossed product for which the multiplication is given as follows with notation as in (14) and (15).

$$(p \otimes x)(q \otimes y) = \sum_{(q)} pq'' \otimes (S^*(q') - x \leftarrow q''')y.$$ 

(104)

See, for example, Section IX.4 of [Ka94]. Note that formula (104) requires no assumptions on the ground ring $\mathbb{D}$ so that we have a well defined bi-algebra structure on $D(H)$.

The requirement for $H$ to be projective and finitely generated allows us to apply the Dual Basis Lemma as in Proposition 3.11 and the following corollary in Section 3.10 of [Ja89]. It asserts that there is a finite collection of pairs $(h_i, h^i)$ with $h_i \in H$ and $h^i \in H^*$ such that

$$x = \sum_i h^i(x) h_i \quad \forall x \in H.$$ 

(105)

Note that (102) yields in the case of $M = \mathbb{D}$ and $N = H$ an isomorphism $\eta : H^* \otimes H \to \text{End}(H)$ so that the dual bases can also be defined as $\eta^{-1}(id_H) = \sum_i h^i \otimes h_i$. The existence of dual bases allows us to define a quasi-triangular structure. Particularly, we can use the tensor $\sum_i h^i \otimes h_i$ to define a canonical universal $R$-matrix for $D(H)$ by

$$R = \sum (\epsilon \otimes h_i) \otimes (h^i \otimes 1).$$ 

(106)

The dual basis relation in (105) indeed suffices to verify the axioms for $R$-matrices as introduced by [Dr87] following standard calculations, for example, as in Section IX of [Ka94]. We summarize the above discussion in the following lemma.

**Lemma 25.** Suppose $\mathbb{D}$ is a unital commutative ring and $H$ a Hopf algebra over $\mathbb{D}$ which is finitely generated and projective as a $\mathbb{D}$-module. Then the quantum double $D(H)$ is well-defined as a bi-algebra with $R$-matrix as in (106).

The missing ingredient in order to establish $D(H)$ has a Hopf algebra is the existence of an antipode, for which we need to make further assumptions on the ring $\mathbb{D}$. 


Lemma 26. Suppose $\mathbb{D}$ is a Dedekind domain and $H$ is a Hopf algebra that is finitely generated and projective over $\mathbb{D}$. Then $\mathcal{D}(H)$ with the above is well-defined as a Hopf algebra.

Proof. Since $\mathbb{D}$ is also assumed to be Dedekind we can now invoke Corollary 24 that ensures the existence of an inverse $S^{-1}$ of the antipode $S$ of $H$. This allows us to define an antipode on $\mathcal{D}(H)$ consistent with this bi-algebra structure by the following composition of isomorphisms:

$$S_\mathcal{D} : \mathcal{D}(H) = H^* \otimes H \xrightarrow{S^{-1} \otimes S} H^* \otimes H \xrightarrow{\sigma_{(12)}} H \otimes H^* \cdot \mathcal{D}(H) \quad (107)$$

As before $\sigma_{(12)}$ denotes the transposition of tensor factors, and the last arrow denotes the bi-crossed product map onto $\mathcal{D}(H)$ given by $x \otimes l \mapsto (e \otimes x)(l \otimes 1)$ using the specialization of (104) above. The verification of the antipode axiom for $S_\mathcal{D}$ as given in (107) is exactly the same well known calculation for Hopf algebras over fields (see for example [KR93]).

We conclude this section with a couple of useful formulae assuming invertibility of $S$. The first is a simple consequence of the fact that squares of antipodes are homomorphisms of $H$.

$$S_\mathcal{D}^2 = S^{*-2} \otimes S^2. \quad (108)$$

Moreover, the second equivalent version of the bi-crossed product can now also also be written as follows:

$$(p \otimes x)(q \otimes y) = \sum_{(x)} p(x' \mapsto q \gets S^{-1}(x'')) \otimes x'' y \quad (109)$$

7.3. Ribbon Elements and Integrals for Doubles over Dedekind Domains.
After having extended the quantum double construction $\mathcal{D}(H)$ to rings in the previous section our next task is to establish the existence of a ribbon structure and integrals for $\mathcal{D}(H)$.

Throughout this section and for the remainder of this article we will now assume that $\mathbb{D}$ be a Dedekind domain and that $H$ is a Hopf algebra over $\mathbb{D}$, which is finitely generated and projective as a $\mathbb{D}$-module.

Lemma 27. With $H$ and $\mathbb{D}$ as assumed the quantum double $\mathcal{D}(H)$ is a ribbon Hopf algebra if and only if the distinguished group-like elements (moduli) $\alpha$ and $g$ have square roots, denoted $\beta$ and $l$ respectively, such that

$$S^2(x) = l(\beta \mapsto x \gets \beta^{-1})l^{-1}, \quad \forall x \in H. \quad (110)$$
In this case a ribbon element is given by
\[ r = \sum S_D(f_i)e_i(\beta^{-1} \otimes l^{-1}), \quad (111) \]
where \( S_D \) is the antipode of \( D(H) \) and \( R = \sum e_i \otimes f_i \) is the canonical universal R-matrix of \( D(H) \).

Proof. Then main observation is that Drinfeld’s formula in Proposition 6.1 in \([Dr89]\) for a canonical group element of a quasi-triangular Hopf algebra that implements \( S^4 \) in the case of a double \( D(H) \) also holds when the Hopf algebra \( H \) is defined over a ring as above. In our notation and conventions it is expressed as
\[ uS(u)^{-1} = \alpha \otimes g \in D(H), \quad (112) \]
where \( u \) is as in (21) and the moduli \( \alpha \) and \( g \) are as in Section 7.1. The proof in \([Dr89]\) relies, besides Radford’s formula as given in (101) for \( D \), only on computations using Hopf algebra operations that easily extend to general commutative rings.

The first condition for a balancing element \( \kappa \) as stated in (22) thus becomes \( \kappa^2 = \alpha \otimes g \). Note that for a Hopf algebra \( A \) over \( D \) that is projective and finitely generated as a \( D \)-module we still have \( G(A^*) = \text{Alg}_D(A, D) \). Using this and the duality properties given in Section 7.2 it follows that the canonical map \( G(H^*) \times G(H) \rightarrow G(D(H)) \) is a group isomorphism by adapting the proof of this statement for Hopf algebras over fields given by Radford in Proposition 9 of \([Ra93]\).

The existence of a group like element \( \kappa \) that fulfills the first identity in (22) is thus equivalent to finding group like square roots \( \beta \) and \( l \) for \( \alpha \) and \( g \) respectively so that \( \kappa = \beta \otimes l \in D(H) \). With this the second condition in (22) combined with the general formula in (108) turns out to equivalent to the formula in (110) by a straightforward calculation as given in \([KR93]\), for which the assumption that \( H \) is over a field is not required.

As noted in Section 2.2 the existence of a balancing element is equivalent to the existence of a ribbon element via the relation in (23). In our case the latter readily translates into Formula (111) above.

We shall call a Hopf algebra \( H \) with integrals and moduli double balanced if the conditions in Lemma 27 are fulfilled. That is, if the moduli have group like square roots such that (110) holds or, equivalently, if its double \( D(H) \) is balanced or ribbon.

We note that a double balancing of \( H \) is already implied by a double balancing of the fraction field completion \( \overline{H} = H \otimes F \) if we are given that \( G(H) = G(\overline{H}) \), which appears to be the case in many situations. In this case the balancing prerequisite of Theorem \( \overline{I} \) can be weakened to finding the balancing of a Hopf algebra over a field.
Proposition 28. Let $D$ and $H$ be as assumed. Moreover, let $\lambda \in H^*$ and $\Lambda \in H$ be integrals with normalizations as in (96) and assume $D(H)$ is balanced with balancing elements given by $(\beta, g) \in G(H^*) \times G(H)$ as in Lemma 27.

Then $\lambda_D \in D(H)^*$ defined by
\[
\lambda_D(f \otimes x) = \beta(l)^{-2} f(\Lambda) \lambda(x), \quad \forall f \in H^*, \ x \in H
\] (113)

is a generator of the ideal of right integrals of $D(H)$ with
\[
\lambda_D(r) = \beta(l)^3 \quad \text{and} \quad \lambda_D(r^{-1}) = \beta(l)^{-3}.
\] (114)

Moreover, the element $\Lambda_D \in D(H)$ defined by
\[
\Lambda_D = \beta(l)^2 S^*(\lambda) \otimes S(\Lambda)
\] (115)
is a two-sided cointegral of $D(H)$ which generates the ideal of integrals for $D(H)$ and for which
\[
\lambda_D(\Lambda_D) = 1 \quad \text{and} \quad S_D(\Lambda_D) = \Lambda_D.
\] (116)

Proof. In order to show that $\lambda_D$ is a right integral for $D(H)$ we need to verify that for every $f \in H^*$ and $x \in H$,\[
\sum (f \otimes x)' \lambda_D(f \otimes x) = (\epsilon \otimes 1)\lambda_D(f \otimes x).
\] (117)

Note that by Lemma 22, $\beta(l)$ is invertible. Using that $D(H) = H^{*\text{cop}} \otimes H$ as a co-algebra as well as the integral and cointegral equations for $\lambda$ and $\Lambda$ we compute that
\[
\text{LHS of (117)} = \sum (f' \otimes x'') \lambda_D(f'' \otimes x') = \beta(l)^{-2} \sum (f' \otimes x'') f''(\Lambda) \lambda(x')
\]
\[
= \beta(l)^{-2} \sum f''(\Lambda) f' \otimes \lambda(x') x'' = \beta(l)^{-2} \sum f''(\Lambda) f' \otimes \lambda(x) \cdot 1
\]
\[
= \beta(l)^{-2} f(\Lambda) \epsilon \otimes \lambda(x) \cdot 1 = \beta(l)^{-2} f(\Lambda) \lambda(x) (\epsilon \otimes 1)
\]
\[
= \text{RHS of (117)}.
\]

To prove that $\Lambda_D$ as in (115) defines a cointegral we recall from Proposition 5 in [Ke94] that if $\mu \in H^*$ and $\Lambda \in H$ are left (co)integrals then
\[
\Lambda_D^\lambda := \mu \otimes S^{-1}(\Lambda) \in H^* \otimes H = D(H)
\] (118)
is a two-sided cointegral. Although generally assumes $H$ to be over a field, the proof of Proposition 5 only depends on calculations with Hopf algebra operations and thus extends verbatim to Hopf algebras over rings.

Given that $\lambda$ is a right integral we can define a left integral as $\mu = S^*\lambda$. Now by Proposition 5 in [Ke94] we also have $S_D(A_D^b) = A_D^b$ and, using the expression in (108), also $A_D^b = S_D^2(\Lambda_D^b) = S^*(\lambda) \otimes S(\Lambda)$. Clearly the multiple $\Lambda_D$ of this expression is then also an $S_D$-invariant two-sided integral.

The normalization condition in (116) is a straightforward calculation from the formulae in (113) and (115) as $\lambda_D(\Lambda_D) = (\beta(l)^2\lambda(S(\Lambda))(\beta(l)^{-2}\lambda(S(\Lambda)) = 1$ using (96). This in turn readily implies that both $\lambda_D$ and $\Lambda_D$ are generators using arguments similar to those in the proof of Proposition 23 above.

The formulae for the integral evaluations in (114) are obtained by the following calculations. One can rewrite $r$ using the canonical universal $R$-matrix $R$ in (106) as follows.

$$r = \sum S_D(h^i \otimes 1)(\epsilon \otimes h_i)(\beta^{-1} \otimes l^{-1})$$

$$= \sum (S^{*-1}(h^i) \otimes 1)(\epsilon \otimes h_i)(\beta^{-1} \otimes l^{-1})$$

by (104) $$= \sum S^{*-1}(h^i)\beta^{-1} \otimes (\beta \rightarrow h_i \leftarrow \beta^{-1})l^{-1}$$

by (110) $$= \sum S^{*-1}(h^i)\beta^{-1} \otimes l^{-1}S^2(h_i).$$

Hence

$$\beta(l)^2\lambda_D(r) = \sum (S^{*-1}(h^i)\beta^{-1}(\Lambda) \cdot \lambda(l^{-1}S^2(h_i))$$

since $l \in G(H)$

$$= \sum S^{*-1}(h^i)\beta^{-1}(\Lambda') \cdot \lambda(S^2(l^{-1}h_i))$$

$$= \sum h^i(S^{-1}(\Lambda'))\beta^{-1}(\Lambda'') \cdot S^2(\lambda)(l^{-1}h_i)$$

by (105) $$= \sum h^i(S^{-1}(\Lambda'))(S^2(\lambda) \leftarrow l^{-1})(h_i)\beta^{-1}(\Lambda'')$$

since $l \in G(H)$

$$= \sum S^2(\lambda)(l^{-1}S^{-1}(\Lambda'))\beta^{-1}(\Lambda'')$$

since $\beta \in G(H^*)$

$$= \sum S^*(\lambda)(\Lambda'l)\beta^{-1}(\Lambda''l)\beta^{-1}(l^{-1})$$

$$= (S^*(\lambda)\beta^{-1})(\Lambda)l\beta(l)$$
= (S^*(\lambda)\beta^{-1})(\Lambda)\alpha(l)\beta(l)
= S^*(\beta(\lambda)(\Lambda)\alpha(l)\beta(l)
= \beta(g)S^*(\lambda)(\Lambda)\beta(l^2)\beta(l)
= \beta(l^5)\lambda(S(\Lambda))
= \beta(l)^5.

Hence \lambda_D(r) = \beta(l)^3 as asserted. The equation \lambda_D(r^{-1}) = \beta(l)^{-3} follows from a similar but simpler calculation starting from the following known identity for the inverse of the ribbon element.

\[ r^{-1} = \sum (h^i \otimes 1)S^2_D(\epsilon \otimes h_i)(\beta \otimes l). \]

This concludes the proof of the proposition. \(\square\)

Finally, combining Lemma 22 with (114) we observe the following:

**Corollary 29.** Let \( H \) and \( D \) be as above and \( \lambda_D \in D(H)^* \) and \( r \in D(H) \) the integral and ribbon element of \( D(H) \) as in Proposition 28.

Then \( \lambda_D(r^{\pm 1}) \) is a root of unity.

### 7.4. Proof of Theorem 1

This section now combines the special algebraic properties of quantum doubles over Dedekind domains as laid out in the preceding sections with the general TQFT constructions from Section 3. The main observation is the following characterization of topogenic doubles.

**Lemma 30.** Suppose \( H \) is a double balanced Hopf algebra over a Dedekind domain \( D \) which is projective and finitely generated as a \( D \)-module.

Then its double \( D(H) \) is a topogenic Hopf algebra.

**Proof.** We will verify each of the four conditions from Definition 5. The ribbon or balancing property (1) follows immediately from Lemma 27.

In order to prove the modularity condition (2) of Definition 5 we note that by standard arguments \( \tau^\#: H^* \otimes H \to (H^* \otimes H)^* = D(H) \) given by \( \tau^\#(l \otimes x)(k \otimes y) = k(x)l(y) \) is an isomorphism if \( H \) is projective and finitely generated over \( D \). We can derive an explicit expression for the composition of \( \tau \) with the map in (25) using duality relation in (105), the form of \( \mathcal{R} \) in (106), as well as the bi-crossed product in
as follows:
\[
\overline{M}(\tau^#(l \otimes x)) = \tau^#(l \otimes x) \otimes \text{id}(R_2, R)
\]
\[
= \tau^#(l \otimes x) \sum_{ij} (h^j \otimes 1)(\epsilon \otimes h_i)(\epsilon \otimes h_j)(h^i \otimes 1)
\]
\[
= \sum_{ij} \tau^#(l \otimes x)(h^j \otimes h_i)(\epsilon \otimes h_j)(h^i \otimes 1)
\]
\[
= \sum_{ij} h^j(x)l(h_i)(\epsilon \otimes h_j)(h^i \otimes 1)
\]
\[
= (\epsilon \otimes x)(l \otimes 1) = \sum_{(x)} x' \mapsto l \leftarrow S^{-1}(x'') \otimes x''
\]

It is now easily verified from Hopf algebra axioms that the map
\[
H^* \otimes H \to H^* \otimes H : k \otimes y \mapsto \sum_{(y)} S^{-1}(y') \leftarrow k \leftarrow y''' \otimes y''
\]
is a two-sided inverse for \(\overline{M} \circ \tau^#\). Thus also \(\overline{M}\) is invertible as required in (2) of Definition 5.

The normalization required in (3) of Definition 5 is immediate from (114) in Proposition 28. Finally, condition (4) is also implied by Proposition 28 using both parts of (116). \(\square\)

Part (a) of Theorem 1 now follows by specialization of Corollary 9 and Theorem 12 to the case where \(H = D(H)\) and using Lemma 30 above. The specialization to the Hennings invariant \(\varphi_{D(H)}\) in Part (b) of Theorem 1 follows from (51) and (114) which determine the signature phase as \(\lambda_D(r^{\pm 1}) = \beta(l)^{\pm 3}\).

Finally, to see Part (c) we note that by (29) the space associated to a surface of genus \(n\) (and one boundary component) is given by \(D(H)^{\otimes n}\). As a \(D\)-module this is a tensor product of copies of \(H\) and \(H^*\). Thus if \(H\) is a free \(D\)-module is also \(H^*\) and hence also \(D(H)^{\otimes n}\).

This completes the proof of Theorem 1.

8. The Hennings TQFT for the quantum double of \(B(\mathfrak{sl}_2)\)\(\zeta\)

The previously developed techniques and results, summarized in Theorem 1 and Theorem 2, are applied in this section to the study of example \(H = B(\mathfrak{sl}_2)\zeta\), the quantum Borel algebra at a root of unity \(\zeta\). The main results are Theorem 40 below and the proof of Theorem 3.
The first example given by Drinfeld in [Dr87] for his quantum double construction is that for the quantum universal enveloping algebra $U_{\hbar} \mathfrak{b}$ where $\mathfrak{b}$ is the Borel algebra associated to a simple Lie algebra $\mathfrak{g}$. He shows in Section 13 that $\mathcal{D}(U_{\hbar} \mathfrak{b}) = U_{\hbar} \mathfrak{g} \otimes U_{\hbar} \mathfrak{h}$, where $\mathfrak{h}$ is a second copy of the Cartan algebra of $\mathfrak{g}$.

In Proposition 36 of Section 8.3 we will establish an analogous factorization in the case of the algebra $B(\mathfrak{sl}_2)_\zeta$ which is of finite rank over $\mathbb{Z}[\zeta]$. The generating set of the dual algebra is essentially the opposite Borel part of the divided powers introduced by Lusztig in [Lu93]. The analogous product relation will hold naïvely only on the level of associate algebras but requires an additional gauge twist to yield a factorization of quasi-triangular Hopf algebras.

In Section 8.4 this is used to establish the respective factorization of TQFTs and Hennings invariants and infer the formula in Theorem 3. In Section 8.1 we discuss the Hennings invariant associated to the Cartan algebra, and discuss in detail the double construction over $B(\mathfrak{sl}_2)_\zeta$ and choices of generators in Section 8.2.

Throughout this section we assume that $\ell$ is an odd integer and $\zeta$ is a primitive $\ell$-th root of unity. In numerous calculation we will need the multiplicative inverse of 2 in $\mathbb{Z}/\ell$, for which we thus introduce the following notation:

$$1/2 = \frac{\ell + 1}{2}. \quad (121)$$

### 8.1. The MOO invariant.

The MOO invariant was introduced by Murakami, Ohtsuki and Okada in [MOO92]. Its construction generalizes Kirby and Melvin’s formula for the WRT $SU(2)$ invariant at the third root of unity in [KM91]. The construction of the MOO invariant as described in Section 7 of [MOO92] follows the standard WRT process starting from a particular ribbon Hopf algebra, which is defined as follows.

For $\ell$ an odd integer, let $\mathcal{A} := \mathcal{A}_\ell$ be the $\mathbb{Q}[\zeta, \sqrt{\zeta-1}]$-algebra generated by $z$ with the relation $z^\ell = 1$. Then $\mathcal{A}$ is a Hopf algebra with

$$S(z) = z^{-1}, \quad \Delta(z) = z \otimes z, \quad \epsilon(z) = 1.$$ 

$\mathcal{A}$ in endowed with a ribbon Hopf algebra structure as follows: The universal $R$-matrix is given by

$$R_{\mathcal{A}} = \frac{1}{\ell} \sum_{i,j=0}^{\ell-1} \zeta^{-2ij} z^i \otimes z^j, \quad (122)$$

where $\zeta$ is again a primitive $\ell$-th root of unity. We compute for the canonical element from (21) that

$$u_{\mathcal{A}} = \frac{1}{\ell} \sum_{i,j=0}^{\ell-1} \zeta^{-2ij} z^{i-j} = \frac{\gamma}{\ell} \sum_{n=0}^{\ell-1} \zeta^{1/2n^2} z^n. \quad (123)$$
where $\frac{1}{2}$ as in (121) and we denote the Gauss sum

$$
\gamma_\ell = \sum_{m=0}^{\ell-1} \zeta^{-\frac{1}{2}m^2} = \varepsilon_\ell \sqrt{\ell} \left( \frac{p}{\ell} \right), \quad \text{so that } |\gamma_\ell|^2 = \ell .
$$

Here $\varepsilon_\ell = 1$ if $\ell \equiv 1 \mod 4$ and $\varepsilon_\ell = \sqrt{-1}$ if $\ell \equiv 3 \mod 4$. Moreover, $p$ is defined by $\zeta^{-\frac{1}{2}} = e^{2\pi \sqrt{-1} \frac{p}{\ell}}$ and $(\cdot)$ is the Jacobi symbol. Since $S(u_{\mathcal{A}}) = u_{\mathcal{A}}$ and $S^2 = id$ for the above definitions the balancing and ribbon structure for $\mathcal{A}$ are trivial in the sense that

$$
r_{\mathcal{A}} = u_{\mathcal{A}} \quad \text{and} \quad \kappa_{\mathcal{A}} = 1 .
$$

Since $\mathcal{A}$ is semisimple it follows from Lemma 1 in [Ke95] that $\mathcal{Z}_\zeta$ can also be computed using the Hennings algorithm as described in Section 4.5 of this article. That is, we have

$$
\mathcal{Z}_\zeta = \varphi_{\mathcal{A}},
$$

Let $\lambda_{\mathcal{A}}$ be the element in $\mathcal{A}^*$ defined by

$$
\lambda_{\mathcal{A}}(z_a) = \sqrt{\ell} \delta_{a,0} .
$$

Note that (124) implies that $\sqrt{\ell} \in \mathbb{Q}[\zeta, \sqrt{-1}]$. It is easy to check that $\lambda_{\mathcal{A}}$ is a right integral for $\mathcal{A}$ and $\lambda_{\mathcal{A}}(r_{\mathcal{A}})\lambda_{\mathcal{A}}(r_{\mathcal{A}}^{-1}) = 1$.

The results in [MOO92] imply an explicit formula for the values of $\mathcal{Z}_\zeta$. If the order of the first homology $h(M)$ as defined in (9) is coprime to $\ell$ they simplify to Jacobi symbols as follows.

**Lemma 31 ([MOO92]).** Suppose $M$ is a rational homology sphere, and let $\ell$ be an odd integer with $(h(M), \ell) = 1$. Then $\mathcal{Z}_\zeta(M) = \left( \frac{h(M)}{\ell} \right)$ where $(\cdot)$ is the Jacobi symbol.

**Proof.** Suppose $\ell = p_1^{n_1} \cdots p_k^{n_k}$ is the prime factorization of $\ell$ with each $n_j > 0$. It follows by iteration of Proposition 2.3 of [MOO92] that

$$
\mathcal{Z}_\zeta(M) = \mathcal{Z}_{\zeta_1}(M) \mathcal{Z}_{\zeta_2}(M) \cdots \mathcal{Z}_{\zeta_k}(M) ,
$$

where $\zeta_j$ is a primitive $p_j^{n_j}$-th root of unity. Since $M$ is a homology sphere with $(h(M), \ell) = 1$ we have $p_j \nmid |H_1(M, \mathbb{Z})| < \infty$ so that $H_1(M, \mathbb{Z})$ cannot have any $p_j$-torsion or free parts. Consequently, $H_1(M, \mathbb{Z}/p_j) = H_1(M, \mathbb{Z}) \otimes \mathbb{Z}/p_j = 0$ with $j = 1, \ldots, k$.

Corollary 4.8 of [MOO92] now asserts that $\mathcal{Z}_{\zeta_j}(M) = \left( \frac{h(M)}{p_j} \right)^{n_j}$ where $(\cdot)$ is the Legendre symbol. Combined with (128) this yields

$$
\mathcal{Z}_\zeta(M) = \left( \frac{h(M)}{p_1} \right)^{n_1} \cdots \left( \frac{h(M)}{p_k} \right)^{n_k} = \left( \frac{h(M)}{\ell} \right)
$$

as claimed. □
8.2. The Borel subalgebra and its quantum double. The quantum double described here is the same as the one in \[Ke98\] except that ours has a different ground ring. In order to simplify notation we will use \(B = B(\mathfrak{sl}_2)_\zeta\) to denote the Borel subalgebra of quantum \(\mathfrak{sl}_2\) at the root of unity \(\zeta\). It is defined as the \(\mathbb{Z}[\zeta]\)-algebra generated by \(e\) and \(k\) with relations

\[
k^\ell = 1, \quad e^\ell = 0, \quad \text{and} \quad kek^{-1} = \zeta e.
\]

(129)

It is a Hopf algebra with structural maps:

\[
\Delta(k) = k \otimes k, \quad S(k) = k^{-1}, \quad \epsilon(k) = 1,
\]
\[
\Delta(e) = e \otimes 1 + k^2 \otimes e, \quad S(e) = -k^{-2}e, \quad \epsilon(e) = 0.
\]

(130)

Obviously \(B\) is a free \(\mathbb{Z}[\zeta]\)-module with basis \(\{e^i k^j \mid 0 \leq i, j \leq \ell - 1\}\). A left cointegral for \(B\) is

\[
\Lambda = \left(\sum_{j=0}^{\ell-1} k^j\right)e^{\ell-1},
\]

(131)

and a right integral \(\lambda\) for \(B^*\) is given by

\[
\lambda(e^n k^j) = \delta_{n,\ell-1}\delta_{j,0}.
\]

(132)

These are readily checked to fulfill the normalizations in \((96)\) that are required in Proposition 28. The moduli \(\alpha\) and \(g\) defined in Proposition 23 are

\[
\alpha(e^i k^j) = \delta_{i,0}\zeta^j \quad \text{and} \quad g = k^{-2}.
\]

(133)

They have group like square roots

\[
\beta(e^i k^j) = \delta_{i,0}\zeta^{\frac{(i+1)}{2}} \quad \text{and} \quad l = k^{-1}.
\]

(134)

One can easily check that \((110)\) holds in \(B\) for these choices. Proposition 28 thus implies that the quantum double \(D(B)\) is a ribbon Hopf algebra.

We next describe this ribbon algebra in terms of generators and relations starting with explicit formulae for the dual algebra \(B^*\). They will involve the so called q-number expressions in \(\mathbb{Z}[\zeta]\) denoted as follows:

\[
[i] = \frac{\zeta^i - \zeta^{-i}}{\zeta - \zeta^{-1}}, \quad [n]! = \prod_{i=1}^{n} [i] \quad \text{and} \quad \left[\begin{array}{c} a \\ b \end{array}\right] = \frac{[a]!}{[a-b]![b]!}.
\]

(135)

For \(k, s \in \{0, 1, \ldots, \ell - 1\}\) we define special elements \(f^{(k)}\) and \(\omega_s\) with in \(B^*\) by

\[
f^{(k)}(e^n k^j) = \delta_{n,k} \quad \text{and} \quad \omega_s(e^n k^j) = \delta_{n,0}\delta_{s,j}.
\]

(136)
We note that $\alpha$ and the $\omega_j$ are related by transformations

$$\alpha^k = \sum_{j=0}^{l-1} \zeta^{kj} \omega_j \quad \text{and} \quad \omega_j = \frac{1}{l} \sum_{i=0}^{l-1} \zeta^{-ij} \alpha^i,$$  \hspace{1cm} (137)

where the second relation is to be used with caution as it is strictly only defined over $\mathbb{Q}(\zeta)$. The next lemma describes the dual algebra $B^*$ and follows readily from the relations for $B$ above.

**Lemma 32.** Let $B^* = \text{Hom}_{\mathbb{Z}[\zeta]}(B, \mathbb{Z}[\zeta])$ be the algebra over $\mathbb{Z}[\zeta]$ dual to $B$ with coproduct denoted by $\Delta_*$. Then $\{f^{(n)}\omega_j\}_{0 \leq n, j \leq \ell - 1}$ is a basis of $B^*$ dual to $\{e^n k^j\}_{0 \leq n, j \leq \ell - 1}$.

Moreover, $B^*$ is isomorphic to the bi-algebra over $\mathbb{Z}[\zeta]$ given by generators $\{f^{(n)}\}$ and $\{\omega_j\}$ subject to the relations

$$f^{(n)} f^{(m)} = \zeta^{mn} \left[\begin{array}{c} n + m \\ n \end{array}\right] f^{(n+m)} , \quad \omega_j f^{(n)} = f^{(n)} \omega_j , \quad \text{and} \quad \omega_i \omega_j = \delta_{i,j} \omega_j \hspace{1cm} (138)$$

as well as co-relations

$$\Delta_* (f^{(n)}) = \sum_{q=0}^{n} f^{(n-q)} \alpha^q \otimes f^{(q)} \quad \text{and} \quad \Delta_* (\omega_j) = \sum_{s=0}^{l-1} \omega_{j-s} \otimes \omega_s . \hspace{1cm} (139)$$

Note that the first relation in (138) implies that $f^{(n)} f^{(m)} = 0$ whenever $n + m \geq l$. We also imply $\sum_i \omega_i = \alpha^0 = \epsilon = 1$. The evaluation of the bi-crossing formulae in (104) and (109) on these generators yields the relations in $\mathcal{D}(B)$ as follows

$$kf^{(n)} = \zeta^{-n} f^{(n)} k , \quad e \omega_j = \omega_{j-2} e , \quad k \omega_j = \omega_j k$$

$$\quad \text{and} \quad e f^{(n)} = f^{(n)} e + f^{(n-1)} (\alpha - \zeta^{-2(n-1)} k^2) .$$

(140)

Give that a set of free generators over $\mathbb{Z}[\zeta]$ for $\mathcal{D}(B) = B^* \otimes B$ is readily given by such generators over $B$ and $B^*$ and that the above relations can be used to write any expression in terms of these we make the following observation.

**Lemma 33.** The double $\mathcal{D}(B)$ is freely generated as a $\mathbb{Z}[\zeta]$-module by the basis

$$\{f^{(m)} \omega_i \otimes e^n k^j \mid 0 \leq m, n, i, j \leq \ell - 1\} . \hspace{1cm} (141)$$

It is, as an algebra over $\mathbb{Z}[\zeta]$, isomorphic to the algebra defined in terms of generators $\{e, k^{\pm 1}, \omega_j, f^{(n)}\}_{0 \leq j, n \leq \ell - 1}$ and relations (129), (138), and (140). The coalgebra structure is given by (139) and $\Delta_*^{\text{opp}}$ as in (139) for these generators.

From now on we will use the fact that for $f \in B^*$ and $x \in B$ we have $(f \otimes 1)(e \otimes x) = f \otimes x$ in $\mathcal{D}(B)$ to omit the tensor symbol and simply write $fx$ for the same expression. We next list the remaining ingredients of $\mathcal{D}(B)$ relevant to the TQFT
construction. The antipode $S_{D(B)}$ of this double is identical to the one in (130) for the generators of $B$ and on the remaining generators it is given by

$$S_{D(B)}(f(n)) = (-1)^n \zeta^{-n(n-1)} f(n) \alpha^{-n} \quad \text{and} \quad S_{D(B)}(\omega_j) = \omega_{-j}.$$  \hspace{1cm} (142)

The quasi-triangular structure of a double is given by the canonical universal $R$-matrix as in (106). For $D(B)$ this can be factored as follows.

$$R_{D(B)} = \sum_{0 \leq m, i \leq \ell-1} e^m k^i \otimes f^{(m)} \omega_i = \left( \sum_{m=0}^{\ell-1} e^m \otimes f^{(m)} \right) D_{D(B)},$$

where we denote

$$D_{D(B)} = \sum_{i=0}^{\ell-1} k^i \otimes \omega_i$$

which is sometimes called the diagonal part of $R_{D(B)}$. The special group like element defined in (22) is readily found from the group like square roots given in (134) to be

$$\kappa_{D(B)} = \beta \cdot l = \alpha^{1/2} k^{-1},$$

and the root of unity defined by the contraction of these elements is

$$\theta = \beta(l) = \zeta^{-1/2}.$$  \hspace{1cm} (146)

In order to determine the special integrals from Proposition 28 note first that we can express the integral from (132) as $\lambda = f^{(l-1)} \omega_0$ so that also $S^*(\lambda) = S_{D(B)}(\lambda) = \zeta^2 f^{(l-1)} \omega_2$. Using also (131) this implies

$$\Lambda_{D(B)} = \zeta f^{(l-1)} \omega_2 \left( \sum_{i=0}^{l-1} \zeta^i k^i \right) e^{l-1} = \zeta \omega_0 \left( \sum_{i=0}^{l-1} k^i \right) f^{(l-1)} e^{l-1}.$$  \hspace{1cm} (147)

Combining (113) of Proposition 28 with (146), (132), (131), as well as

$$f^{(m)} \omega_i(A) = f^{(m)} \omega_i(k^i e^{l-1}) = f^{(m)} \omega_i(\zeta^{-i} e^{l-1} k^i) = \zeta^{-i} \delta_{m, \ell-1}$$

we obtain the following formula for the normalized right integral $D(B)$ of Proposition 28

$$\lambda_{D(B)}(f^{(m)} \omega_i e^n k^j) = \zeta^{1-i} \delta_{m, \ell-1} \delta_{n, \ell-1} \delta_{j, 0},$$  \hspace{1cm} (148)

Furthermore, we compute the ribbon element from (143), (21), (145), (23), and the relations of $D(B)$:

$$r_{D(B)} = \sum_{n,j=0}^{l-1} (-1)^n \zeta^{1/2 j + n(n+j+1)} f^{(n)} \omega_{-j-2n} e^n k^{j+1}$$

$$\hspace{1cm} (149)$$
The evaluation of the integral on the ribbon element is given by (114) in Proposition 28 using (146) but can also be obtained by applying (148) to (149) directly.

\[ \lambda_{\mathcal{D}(B)}(r^\pm_1) = \theta^{\pm} = \zeta^{\pm \frac{(\ell-3)}{2}} \]  

(150)

Since \( B = B(\mathfrak{sl}_2)_\zeta \) is a free module over the Dedekind domain \( \mathbb{D} = \mathbb{Z}[\zeta] \) and double balanced by (134) we can apply Theorem 1 to construct TQFTs.

Corollary 34. With \( B = B(\mathfrak{sl}_2)_\zeta \) as above there is a TQFT functor

\[ \mathcal{F}^{\bullet}_{\mathcal{D}(B(\mathfrak{sl}_2)_\zeta)} : \text{Cob}^\bullet \to \mathcal{D}(B) \cong \text{free}(\mathbb{Z}[\zeta]) \]  

(151)

that assigns to a surface of genus \( n \) the module \( \mathcal{D}(B)^{\otimes n} \). Its evaluation on the framed \( S^3 \) represented by \( \bigotimes_{-1} \) (or framing anomaly) is given by \( \theta = \zeta^{\frac{1}{2}} \). In particular the associated Hennings invariant is integral in the sense that

\[ \mathcal{F}_{\mathcal{D}(B(\mathfrak{sl}_2)_\zeta)} \in \mathbb{Z}[\zeta]. \]  

(152)

The bi-algebra \( B^{\ast \cop} \subset \mathcal{D}(B) \) with \( B^{\ast} \) as described in Lemma 32 above has a simpler subalgebra \( B^{\dagger} \) that is generated by the group like element \( \alpha \) from (133) as well as \( f = f^{(1)} \). Their powers are related to the original generators through (137) as well as

\[ f^n = \zeta^{\frac{n(n-1)}{2}} [n]! f^{(n)}. \]  

(153)

The relations and co-relations for \( \mathcal{D}(B) \) imply the following for these generators:

\[ \alpha f\alpha^{-1} = \zeta^2 f, \quad f^\ell = 0, \quad ef - fe = \alpha - k^2, \]
\[ \alpha e\alpha^{-1} = \zeta^{-2} e, \quad \alpha^\ell = 1, \]
\[ kfk^{-1} = \zeta^{-1} f, \quad \alpha k = k\alpha \]  

(154)

The relations in (154) imply that the bicrossing closes in the subalgebra so that \( \mathcal{D}(B)^\dagger = B^{\dagger} \otimes B \subseteq B^{\ast \cop} \otimes B = \mathcal{D}(B) \) is indeed a subalgebra over \( \mathbb{Z}[\zeta] \) with an analogous PBW type basis \( \{ f^m \alpha^e k^j \} \). It follows that \( \mathcal{D}(B) \) may be equivalently defined in terms of generators \( \{ f, \alpha, e, k \} \) and relations (129) and (154).

Note that the transformation formulae in (133) and (153) also imply that the field completion as the same, that is, \( B^{\dagger} \otimes \mathbb{Q}(\zeta) = B \otimes \mathbb{Q}(\zeta) \) as well as \( \mathcal{D}(B)^\dagger \otimes \mathbb{Q}(\zeta) = \mathcal{D}(B) \otimes \mathbb{Q}(\zeta) \). Finally, \( B^{\dagger} \) inherits a well defined Hopf algebra structure given by

\[ \Delta(\alpha) = \alpha \otimes \alpha, \quad \Delta(f) = f \otimes \alpha + 1 \otimes f, \]
\[ S(\alpha) = \alpha^{-1}, \quad S(f) = -f\alpha^{-1}. \]  

(155)

This identifies \( \mathcal{D}(B)^\dagger \) as a Hopf algebra. Note, however, that the quasi-triangular structure will generally not extend over \( \mathbb{Z}[\zeta] \) to this subalgebra so that one has to consider the field completions.
8.3. Rational Factorization of a Gauge Twisted $\mathcal{D}(B)$.

Theorem 3 is based on the fact that the Hennings invariant for $\mathcal{D}(B(\mathfrak{sl}_2)_\zeta)$ can be written as the product of the Hennings invariant for the standard quantum-$\mathfrak{sl}_2$ and the MOO invariant, which is also the Hennings invariants for the algebra $\mathcal{A}$ as discussed in Section 8.1.

The aim of this section is to prove this fact by establishing a respective factorization for the associated Hopf algebras. More precisely, our goal is to identify the double $\mathcal{D}(B)$ as a product of the standard quantum-$\mathfrak{sl}_2$ and the group algebra of the cyclic group of order $\ell$. This will not only imply the identities between invariants but also factorizations of the associated Hennings TQFTs.

The factorization of Hopf algebras will, however, not hold in the naïve sense. The first caveat is that the ground ring needs to be extended since, in particular, $U_\zeta$ is not naturally defined over $\mathbb{Z}[\zeta]$. The second subtlety is that the factorization is only true up to a gauge twist transformation of the coalgebra structure, for which we developed the general theory in Section 6.

The sought identities of invariants will not depend on these modification since an equality in an extended ring will obviously imply equality in the original ring and since the associated gauge transformations of TQFTs are trivial on genus zero surfaces.

We begin by defining the following change of generators for $\mathcal{D}_e(B) = \mathcal{D}(B) \otimes \mathbb{Q}(\zeta)$:

$$E := \frac{\alpha^{-1/2} k^{-1} e}{\zeta - \zeta^{-1}}, \quad F := -f, \quad K := \alpha^{-1/2} k, \quad Z := \alpha^{1/2} k.$$  \hfill (156)

The inverse relations are as follows:

$$e = (\zeta - \zeta^{-1}) E Z, \quad f = -F, \quad k = K^{1/2} Z^{1/2}, \quad \alpha = K^{-1} Z.$$  \hfill (157)

The relations (129) and (154) for $\mathcal{D}_e(B)$ are reexpressed in the new generators by the following two sets of relations:

$$Z^\ell = 1, \quad ZK = KZ, \quad ZE = EZ, \quadZF = FZ$$  \hfill (158)

and

$$KEK^{-1} = \zeta^2 E, \quad K^\ell = 1,$$

$$KFK^{-1} = \zeta^{-2} F, \quad (\zeta - \zeta^{-1})(EF - FE) = K - K^{-1}.$$  \hfill (159)

Note, relations (158) imply that $Z$ generates a central subalgebra $\mathcal{A} \cong \mathbb{Q}(\zeta)[\mathbb{Z}/\ell\mathbb{Z}]$ in $\mathcal{D}_e(B)$. Moreover, the relations in (159) show that the set $\{E, F, K\}$ generates a
subalgebra $U_\zeta$ isomorphic to the standard quantum $\mathfrak{sl}_2$ over $\mathbb{Q}(\zeta)$. Thus as algebras we have indeed a factorization

$$\mathcal{D}_e(B) \cong U_\zeta \otimes \mathcal{A}.$$  \hfill (160)

The factorization as algebras indicated above, however, does not extend to a factorization of Hopf-algebras. Particularly, the co-relations (130) and (155) for $\mathcal{D}_e(B)$ yield mixed terms as follows:

$$\begin{align*}
\Delta(Z) &= Z \otimes Z, & \epsilon(Z) &= 1, & S(Z) &= Z^{-1}, \\
\Delta(K) &= K \otimes K, & \epsilon(K) &= 1, & S(K) &= K^{-1}, \\
\Delta(E) &= E \otimes Z^{-1} + K \otimes E, & \epsilon(E) &= 0, & S(E) &= -ZK^{-1}E, \\
\Delta(F) &= F \otimes K^{-1}Z + 1 \otimes F, & \epsilon(F) &= 0, & S(F) &= -FKZ^{-1}.
\end{align*}$$

Similarly, we can express the canonical R-matrix from (161) and (144) in terms of the new generators. The resulting expression is not a product of R-matrices but contains mixed terms of $K$ and $Z$ generators.

$$\mathcal{R}_{\mathcal{D}(B)} = \left( \sum_{m=0}^{\ell-1} \tau_m E^m Z^m \otimes F^m \right) D_{\mathcal{D}(B)},$$  \hfill (161)

with diagonal part

$$D_{\mathcal{D}(B)} = \frac{1}{\ell} \sum_{i,j=0}^{\ell-1} \zeta^{2ij} K^i Z^i \otimes K^j Z^{-j}$$  \hfill (162)

and coefficients

$$\tau_m = \frac{(\zeta^{-1} - \zeta)^m}{\zeta^{\frac{m(m+1)}{2}} [m]!} = (-1)^m \frac{\zeta^{m(1-m)}}{[m]!}.$$  \hfill (163)

It is useful to introduce the following idempotents of $\mathcal{A}$.

$$P_i = \frac{1}{\ell} \sum_{i=0}^{\ell-1} \zeta^{-2ij} Z^j$$

so that

$$\Delta(P_i) = \sum_{s=0}^{\ell-1} P_s \otimes P_{j-s}.$$  \hfill (164)

With these conventions now we define the relevant gauge transformation.

**Lemma 35.** The element $F \in \mathcal{D}(B)^{\otimes 2}$ given by

$$F = \frac{1}{\ell} \sum_{i,j=0}^{\ell-1} \zeta^{-2ij} K^i \otimes Z^j = \sum_{i=0}^{\ell-1} K^i \otimes P_i$$  \hfill (165)

is a gauge transformation.
Proof. It is readily computed that each side of (52) equals \( \sum_{m,j} K^m \otimes K^j P_{m-j} \otimes P_j \). Moreover, the counit condition is easily verified and the element has an inverse

\[
F^{-1} = \frac{1}{\ell} \sum_{i,j=0}^{\ell-1} \zeta^{2ij} K^i \otimes Z^j = \sum_{i=0}^{\ell-1} K^{-i} \otimes P_\ell.
\]  

(166)

The element \( F \) can thus be used to gauge transform the coalgebra structure of \( D(B) \). In order to compute the coproduct and R-matrix of \( D_e(B)_F \) it is useful to record the following, easily verified identities:

\[
F(E \otimes 1)F^{-1} = E \otimes Z \quad \text{and} \quad F(F \otimes 1)F^{-1} = F \otimes Z^{-1}.
\]  

(167)

Moreover, we have commutation relations

\[
[F, 1 \otimes E] = [F, 1 \otimes F] = [F, g \otimes h] = 0 \quad \text{with} \ g, h \in \{ K^i Z^j \}.
\]  

(168)

Using equations (167) and (168) we find for the gauge transformed coproduct \( \Delta_F(X) = F \Delta(X) F^{-1} \) by straightforward computation that

\[
\Delta_F(E) = E \otimes 1 + K \otimes E \quad \Delta_F(K) = K \otimes K \quad \Delta_F(F) = F \otimes K^{-1} + 1 \otimes F \quad \Delta_F(Z) = Z \otimes Z.
\]  

(169)

From the relations in (167) and (168) we also compute for the gauge twisted R-matrix in the sense of (66) for this example:

\[
(R_{D(B)})_F = F_{21} R_{D(B)} F^{-1} = R_{U_\zeta} \cdot R_{af}.
\]  

(170)

Here \( R_{af} \) is as in (122) with \( z \) substituted by \( Z \), and \( R_{U_\zeta} \) is defined as

\[
R_{U_\zeta} = \left( \sum_{m=0}^{\ell-1} \tau_m E^m \otimes F^m \right) D_{U_\zeta},
\]  

(171)

where

\[
D_{U_\zeta} = \frac{1}{\ell} \sum_{0 \leq i,j \leq \ell-1} \zeta^{2ij} K^i \otimes K^j.
\]  

(172)

The main steps in the calculation for (170) is the identity \( F_{21}(E^m Z^m \otimes F^m) = (E^m \otimes F^m) F_{21} \) readily implied by (167) and (168) as well as \( F_{21} F^{-1} D_{D(B)} = D_{U_\zeta} R_{af} \), which is an exercise in resummations. We summarize our findings as follows.
Proposition 36. We have the factorization of quasi-triangular Hopf algebras given by the canonical isomorphism
\[ \varpi : (D_e(B))_F \rightarrow U_\zeta \otimes A. \] (173)
which assigns a PBW basis element \( F^aE^bK^cZ^d \) to \( F^aE^bK^c \otimes Z^d \). It is an isomorphism of quasi-triangular algebras in the sense that
\[ \varpi \otimes 2((R_{D_e(B)})_F) = \varpi \otimes 2((R_{D_e(B)})_F) = (R_{U_\zeta})_{13}(R_A)_{24} = (U_\zeta \otimes A)^{\otimes 2}, \] (174)
where the R-matrix of \( U_\zeta \) is given by \( R_{U_\zeta} \) and the one of \( A \) is given by \( R_A \) as above.

Proof. The factorization as associative algebras was already noted in (160). The coproduct in (169) clearly restricts to coproducts on \( U_\zeta \) and \( A \). Moreover, \( R_{U_\zeta} \) and \( R_A \) are readily verified to be the R-matrices of these respective algebras and (170) show that their product is the R-matrix of \( (R_{D(B)})_F \).

In the remainder of this section let us compute the other special elements associated to this gauge transformation as defined in Section 6. To begin with the elements defined in (56) are given by
\[ x_F = \frac{1}{\ell} \sum_{i,j=0}^{\ell-1} \zeta^{2ij}K^iZ^j = \sum_{j=0}^{\ell-1} K^{-j}P_j \quad \text{and} \quad x_F^{-1} = \frac{1}{\ell} \sum_{i,j=0}^{\ell-1} \zeta^{-2ij}K^iZ^j = \sum_{j=0}^{\ell-1} K^jP_j \] (175)
where we use (59) as well as (166). From (175) we see that \( S(x_F) = x_F \) so that
\[ z_F = 1 \] (176)
for the element defined in (67). By Lemma 15 and Lemma 17 this implies that all elements related to the ribbon structure and integrals remain unchanged under this gauge twist. Particularly, we have
\[ u_F = u, \quad \kappa_F = \kappa, \quad r_F = r, \quad \Lambda_F = \Lambda, \quad \text{and} \quad \lambda_F = \lambda. \] (177)

Combining (177) with Proposition 36 implies that in the basis chosen in (156) and (157) the special elements of the untwisted \( D_e(B) \) indeed factor accordingly into elements in or on \( U_\zeta \) and \( A \). For later use let us list here in more detail the explicit formulae for these factorized elements, starting with the canonical element \( u \) as defined in (21). Particularly, we find
\[ u_{D(H)} = u_{U_\zeta}u_{A} \quad \text{with} \]
\[ u_{U_\zeta} = \left( \frac{\tau}{\ell} \sum_{n=0}^{\ell-1} \zeta^{-1/2m^2}K^n \right) \left( \sum_{m=0}^{\ell-1} (-1)^m \zeta^m \tau_m F^m F^m K^m \right). \] (178)
Here $\gamma_\ell$ is as in (124) and $u_{\mathcal{A}}$ as in (123) with $z$ replaced by $Z$. Note further that (125) and (145) imply
\[
\kappa_D(H) = \kappa_U \zeta = K^{-1} \quad \text{and} \quad \kappa_{\mathcal{A}} = 1.
\]
Consequently we also have
\[
r_D(H) = r_{\mathcal{A}} r_U \zeta \quad \text{with} \quad r_{\mathcal{A}} = u_{\mathcal{A}} \quad \text{and} \quad r_U \zeta = u_U \zeta K.
\]
Also the integral from (148) is reexpressed in the basis\{\(F_a E_b K_c Z_d\}\} for $D_e(B)$ as defined in (156) and (157) in the following factorizable form.
\[
\lambda_D(B)(F^a E^b K^c Z^d) = \delta_{a,\ell-1} \delta_{b,\ell-1} \delta_{c,\ell-1} \delta_{d,0} \frac{[\ell - 1]! \ell}{(\zeta - \zeta^{-1})^{\ell-1}}
\]
\[
= \lambda_U \zeta(F^a E^b K^c) \cdot \lambda_{\mathcal{A}}(Z^d).
\]
Here $\lambda_{\mathcal{A}}$ is as in (127) and $\lambda_{\mathcal{U}}$ is the normalized right integral for $U \zeta$ given by
\[
\lambda_U \zeta(F^a E^b K^c) = \delta_{a,\ell-1} \delta_{b,\ell-1} \delta_{c,\ell-1} \delta_{d,0} \frac{[\ell - 1]! \sqrt{\ell}}{(\zeta - \zeta^{-1})^{\ell-1}}.
\]
As already mentioned in Section 8.1, if $\ell \equiv 3 \mod 4$ we actually need to extend the ground ring further to $Q(\zeta)[\sqrt{-1}]$ in order to ensure that $\sqrt{\ell}$ lies in that ring so that indeed $\lambda_{\mathcal{A}} \in \mathcal{A}^*$ and $\lambda_{\mathcal{U}} \in U^*_\zeta$. This technicality can also be circumvented at the level of TQFTs by restricting to the index 2 subcategory of evenly 2-framed cobordisms as defined in Lemma 10 of [Ke03].

8.4. Factorization of TQFTs and Proof of Theorem 3

We begin with a fairly straightforward observation about the factorization of a general Hennings TQFT if the underlying quasi-triangular Hopf algebra is the direct product of two quasi-triangular Hopf algebras.

**Lemma 37.** For two quasi-triangular ribbon Hopf algebras $H$ and $K$ satisfy the prerequisites of Theorem 12 over the same domain $D$ then so does $H \otimes_D K$ for the canonical product ribbon structure. Moreover, there is a canonical natural isomorphism of TQFT functors
\[
f : \mathcal{V}^\bullet_{H \otimes_K} \rightarrow \mathcal{V}^\bullet_H \otimes \mathcal{V}^\bullet_K,
\]
given by permutation of tensor factors.

**Proof.** The fact that the product of two topogenic Hopf algebras is again topogenic is obvious from the product form of the various special elements. Also tensor products of projective finitely generated modules are again projective and finitely generated.
For a particular genus $g$ the natural isomorphism in (183) is given by the obvious permutation of tensor factors $(\mathcal{H} \otimes \mathcal{K})^{\otimes g} \cong (\mathcal{K}^{\otimes g}) \otimes (\mathcal{H}^{\otimes g})$ with

$$ f_g ((h_1 \otimes k_1) \otimes \ldots \otimes (h_g \otimes k_g)) = (h_1 \otimes \ldots \otimes h_g) \otimes (k_1 \otimes \ldots \otimes k_g). \quad (184) $$

For an $\mathcal{H} \otimes \mathcal{K}$-labeled planar curve $(D, c)$ in the sense of Section 3.1, where $D$ has $N$ markings we similarly identify $c \in (\mathcal{H} \otimes \mathcal{K})^{\otimes N}$ with an element $c' = \sum \mu h_\mu \otimes k_\mu$ with $h_\mu \in \mathcal{H}^{\otimes N}$ and $k_\mu \in \mathcal{K}^{\otimes N}$. It follows from the constructions in Section 3.2 that $\mathcal{E}_{\mathcal{H} \otimes \mathcal{K}}([D, c])$ is the same morphism as $\sum \mu \mathcal{E}_{\mathcal{H}}([D, h_\mu]) \otimes \mathcal{E}_{\mathcal{K}}([D, k_\mu])$ conjugated by the respective permutations of tensor factors $f_n$ and $f_m$.

We note that the images of $Z_{\mathcal{H} \otimes \mathcal{K}} : \mathcal{D}^g \rightarrow \mathcal{D}^g_{\mathcal{H} \otimes \mathcal{K}}$ have all pure tensors in the sense that if $Z_{\mathcal{H} \otimes \mathcal{K}}(T) = [D, c]$ for a tangle $T : n \rightarrow m$ we have that the respective element $c' = h \otimes k$ with $h \in \mathcal{H}^{\otimes N}$ and $k \in \mathcal{K}^{\otimes N}$. Moreover, we have $Z_{\mathcal{H}}(T) = [D, h]$ and $Z_{\mathcal{K}}(T) = [D, k]$. This follows from the fact that a tangle $T$ can be broken into crossings, maxima, and minima, and the $R$-matrix, integrals, co-integrals, and balancing elements assigned to these pictures for $\mathcal{H} \otimes \mathcal{K}$ are the pure tensors of the respective elements in the assignments for $\mathcal{H}$ and $\mathcal{K}$.

Combining the properties of $\mathcal{E}_{\mathcal{H} \otimes \mathcal{K}}$ and $Z_{\mathcal{H} \otimes \mathcal{K}}$ above we thus find that their composite maps a tangle $T$ to the tensor product of the morphism $\mathcal{E}_{\mathcal{H}} \circ Z_{\mathcal{H}}(T)$ and $\mathcal{E}_{\mathcal{K}} \circ Z_{\mathcal{K}}(T)$ conjugated by $f_n$ and $f_m$. The respective statement for the TQFT functors defined on cobordisms instead of the representing tangles is immediate. □

In Section 6 we constructed the natural isomorphism between Hennings TQFTs whose underlying Hopf algebras are related by gauge twisting. For the particular gauge transformation $F$ from (165) this isomorphism takes on a special form which we compute next.

**Lemma 38.** For the gauge transformation $F$ as given in (165) the natural transformation $\mathcal{T}_g$ from (89) is given by factor-wise left multiplication by $x_F$ as in (175). That is, we have

$$ \mathcal{T}_g (a_g \otimes \ldots \otimes a_1) = x_F a_g \otimes \ldots \otimes x_F a_1. \quad (185) $$

**Proof.** We start by computing the special elements $\mathcal{T}_{[a]} \in \mathcal{D}(H)^{\otimes n}$ defined in (172). It follows by induction that

$$ \mathcal{T}_{[n+1]} = \sum_{j_1, \ldots, j_n=0}^{\ell-1} K_{j_n} \otimes P_{j_n-j_{n-1}} K_{j_{n-1}} \otimes \ldots \otimes P_{j_2-j_1} K_{j_1} \otimes P_{j_1} \quad (186) $$

where we use the iteration $\mathcal{T}_{[n+1]} = \nabla_1 (\mathcal{T}_{[n]})$ as defined in (171) and the identity $\nabla(K^j) = F(K^j \otimes K^j) = \sum_j K^{j^*} \otimes P_{j^*-j} K^j$. The action of $\mathcal{T}_g$ is now obtained from
\( \Upsilon_{[2g]} \) via the expression from (90). We compute

\[
\Upsilon_g(a_g \otimes \ldots \otimes a_1) = \sum_{j_1, \ldots, j_{2g-1}} K^{j_{2g-1}} a_g S(P_{j_{2g-1}-j_{2g-2}} K^{j_{2g-2}}) \otimes \ldots \otimes P_{j_{2k}-j_{2k-1}} K^{j_{2k-1}} a_k S(P_{j_{2k-1}-j_{2k-2}} K^{j_{2k-2}}) \otimes \ldots \otimes P_{j_{2-j_1}} K^{j_1} a_1 S(P_{ji})
\]

\[
= \sum_{j_1, \ldots, j_{2g-1}} P_{j_{2g-2}-j_{2g-1}} K^{j_{2g-1}} a_g K^{-j_{2g-2}} \otimes \ldots \otimes P_{j_{2k}-j_{2k-1}} K^{j_{2k-1}} a_k K^{-j_{2k-2}} \otimes \ldots \otimes P_{j_{2-j_1}} P_{-j_1} K^{j_1} a_1
\]

(187)

where we used that \( S(P_j) = P_{-j} \) and that the \( P_j \) are central. Since \( P_a P_b = \delta_{a,b} P_a \) we only consider terms for which \( j_{2k} - j_{2k-1} = j_{2k-2} - j_{2k-1} \) and \( j_2 - j_1 = -j_1 \). That is, we have contributions only when \( 0 = j_2 = \ldots = j_{2k-2} = j_{2k} = \ldots = j_{2g-2} \) are all zero. Relabeling the remaining summation indices as \( n_k = j_{2k-1} \) we obtain

\[
\Upsilon_g(a_g \otimes \ldots \otimes a_1) = \sum_{n_1, \ldots, n_g} P_{-n_g} K^{n_g} a_g \otimes \ldots \otimes P_{-n_k} K^{n_k} a_k \otimes \ldots \otimes P_{-n_1} K^{n_1} a_1
\]

(188)

Clearly, the summation can now be distributed over the factors to yield the desired form (185) using the expression in (175) for \( x_F \). \( \square \)

We note that the natural isomorphism is \( \gamma_F \) from Theorem 2 as a map on \( \mathcal{H}^{\otimes g} \) is by (93) actually the composite of \( \Upsilon_g \) as computed above and \((\rho^{\otimes g}_F)^{-1}\) which multiplies \( x_F^{-1} \) from the right to each tensor factor, see also (94). Thus, for the gauge transformation from (165) the natural isomorphism \( \gamma_F : \mathcal{V}^\bullet_{\mathcal{D}_e(B)} \to \mathcal{V}^\bullet_{\mathcal{D}_e(B)} \) of TQFTs is given by

\[
(\gamma_F)_g = \chi^{\otimes g}_F \quad \text{where} \quad \chi_F(x) = x_F x F^{-1}.
\]

(189)

The latter inner automorphism can be reexpressed by multiplication of an element by \( Z^{-d} \) where \( d \) is the \( K \)-degree of the element. More precisely, the following can be verified by a straightforward computation.

**Lemma 39.** Suppose for \( X \in \mathcal{D}(B) \) we have \( K X K^{-1} = \zeta^{2d} \). Then

\[
\chi_F(X) = X Z^{-d}.
\]

(190)

For example, we have \( \chi_F(E) = EZ^{-1}, \chi_F(F) = FZ, \) and that \( \chi_F \) is identity on \( K \) and \( Z \). In applications it is useful to think of (190) as the definition of \( \chi_F \). We
denote the composite with the isomorphism from (173) as follows:

\[
\hat{\chi}_F : \mathcal{D}_e(B) \xrightarrow{\chi_F} \mathcal{D}_e(B)_F \xrightarrow{\varpi} U_\zeta \otimes \mathcal{A}
\]  

(191)

Finally we note that there is an obvious natural isomorphism
\[
\varpi : \mathcal{V} \mathcal{D}_e(B) \rightarrow \mathcal{V} U_\zeta \otimes \mathcal{A}
\]
given by \(\varpi = \varpi \otimes g\). Composing the natural isomorphisms \(\gamma_F\) from (189), \(\varpi\), and \(f\) described in Lemma 37 we obtain the following isomorphism of TQFTs:

**Theorem 40.** Let \(l \geq 3\) be an odd integer as before. If \(\ell \equiv 1 \mod 4\) or if we restrict TQFT-functors to evenly framed cobordisms assume \(\mathbb{D} = \mathbb{Q}(\zeta)\). If \(\ell \equiv 3 \mod 4\) and all framings of cobordisms are included let \(\mathbb{D} = \mathbb{Q}(\zeta)[\sqrt{-1}]\).

Then there is a natural isomorphism of TQFT functors over \(\mathbb{D}\) given by

\[
\eta : \mathcal{V}_D(B) \xrightarrow{\mathbf{\ast}} \mathcal{V}_{U_\zeta} \otimes \mathcal{A} \]

(192)

with \(\eta_g = f_g \circ (\hat{\chi}_F)^{\otimes g} : \mathcal{D}_e(B)^{\otimes g} \rightarrow (U_\zeta)^{\otimes g} \otimes \mathcal{A}^{\otimes g}\).

As an application of Theorem 40 we consider the special case \(g = 0\) which leads to the respective invariants for closed 3-manifolds as described in Section 4.5. In all cases the associated module is free of rank one. It is obvious that in this case \(\eta_0 = id\) so that we have strict equality \(\mathcal{V}_D(B)(M^*) = \mathcal{V}_{U_\zeta}(M^*)\mathcal{V}_{\mathcal{A}}(M^*)\) for \(M^* : D^2 \rightarrow D^2\) as in Section 4.5. It follows immediately from the factorizations in (180) and (181) that the extra factor \(\lambda(r)\) occurring in (51) also factors as

\[
\lambda_D(B)(r_{D(H)}) = \lambda_{U_\zeta}(r_{U_\zeta}) \lambda_{\mathcal{A}}(r_{\mathcal{A}}).
\]

(193)

Combining our results with the definitions in (51) we thus find the following factorization of the associated Hennings invariants.

**Proposition 41.** Let \(M\) be a closed oriented 3-manifold and \(\zeta\) be a root of unity of odd order. Then

\[
\varphi_{D(B(sl_2\zeta))}(M) = \varphi_{U_\zeta}(M)\varphi_{\mathcal{A}}(M).
\]

(194)

**Proof.** Following the description of the construction of Hennings invariants in Section 4.5 we first construct a cobordisms \(M^* : D^2 \rightarrow D^2\) for \(M\) (with choice of some even framing). We note that in the case of \(g = 0\) the associated modules are all free of rank one. Thus, the isomorphism

\[
\text{Ad}(\eta_0) : \text{End}(\mathcal{V}_{D(B)}(D^2)) \rightarrow \text{End}(\mathcal{V}_{U_\zeta}(D^2)) \otimes \text{End}(\mathcal{V}_{\mathcal{A}}(D^2))
\]
is between rank one spaces. Indeed it is the canonical one mapping identities to each other. Since, by Theorem 40, it also maps the values of $M^*$ assigned by the TQFTs to each other it follows from (50) that $	ilde{\varphi}_{D(B)}(M^*) = \tilde{\varphi}_{U_\zeta}(M^*) \tilde{\varphi}_{A}(M^*)$.

Given (193) above we see thus that all terms for $D(B)$ in (51) factor into the respective terms for $U_\zeta$ and $A$, implying (194). □

Note that by choosing even framings for $M^*$ we avoid working over ring extensions by $\sqrt{-1}$ and all invariants are in $\mathbb{Q}(\zeta)$. In fact, other results imply that they have values in $\mathbb{Z}[\zeta]$. The remaining ingredient in the proof of Theorem 3 is the following relation established and proved in [CKS07] for a closed, connected, compact, oriented 3-manifold $M$.

$$\varphi_{U_\zeta}(M) = h(M)\tau_\zeta(M).$$

(195)

Here $\tau_\zeta(M)$ denotes the Witten-Reshetikhin-Turaev $SO(3)$ invariant associated to the root of unity $\zeta$, and $h(M)$ is the order of first homology group as defined in (9).

Combining (195), (194), and (126) we infer (10) and thus Theorem 3.

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INTEGRALITY AND GAUGE DEPENDENCE OF HENNINGS TQFTS

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