Phase and Scaling Properties of Determinants Arising in Topological Field Theories

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Abstract

In topological field theory determinants of maps with negative as well as positive eigenvalues arise. We give a generalisation of the zeta-regularisation technique to derive expressions for the phase and scaling-dependence of these determinants. For theories on odd-dimensional manifolds a simple formula for the scaling dependence is obtained in terms of the dimensions of cohomology spaces. This enables a non-perturbative feature of Chern-Simons gauge theory to be reproduced by semiclassical methods.

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Topological field theories (TFTs) are of interest because they provide examples of quantum field theories which are exactly solvable and because they provide a new way of looking at topological invariants of manifolds [2], [3]. A particular TFT, the Chern-Simons gauge theory on 3-dimensional manifolds, has led to new invariants [4], [5]. (For a review of TFTs see [1]).

In topological field theory, given a topological action functional \( S(\omega) \) for fields \( \omega \) on a manifold \( M \), an object of interest is the partition function

\[
Z(\beta) = \int_\Gamma D\omega e^{-\beta S(\omega)} \tag{1}
\]

where the formal integration is over the infinite-dimensional vector space \( \Gamma \) of fields \( \omega \). We have included in (1) a scaling parameter \( \beta \) which we allow to be complex-valued. (Typically \( \beta \) is either real or purely imaginary; it is often taken to be a constant equal to 1 or \(-i\)). For the cases we consider in this paper the manifold \( M \) is required to be compact, without boundary and oriented (e.g. a sphere of arbitrary dimension).

For a wide class of TFTs where the action \( S(\omega) \) is quadratic (see (16) below for a specific example) the partition function can be formally evaluated by the method of A. Schwarz [2], [6]. This leads to an expression for (1) consisting of a product of determinants of certain maps associated with \( S(\omega) \). One of these determinants is

\[
det(\beta \tilde{T})^{-1/2} \tag{2}
\]

where \( \tilde{T} \) is obtained by discarding the zero-modes of the self-adjoint map \( T \) on \( \Gamma \) given by

\[
S(\omega) = \langle \omega, T \omega \rangle. \tag{3}
\]

The inner product \( \langle \cdot, \cdot \rangle \) in \( \Gamma \) used to obtain \( T \) from \( S(\omega) \) in (3) is constructed from a Euclidean metric on \( M \) (as in [3, p.437]). The other determinants in the expression for the partition function appear because of the zero-modes of \( T \). They are all real-valued and do not involve the parameter \( \beta \). Hence the phase of the determinant should really be \( \det(\beta \frac{1}{\pi} \tilde{T})^{-1/2} \) since \( \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \). However the numerical factor \( 1/\pi \) in the determinant is usually considered to be irrelevant and discarded.

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partition function (1) and its dependence on the scaling parameter $\beta$ are determined solely by the determinant (2). The determinants in the expression for the partition function are determinants of maps on infinite-dimensional vector spaces and must therefore be regularised in order to obtain a finite expression. This is done using the zeta-regularisation technique.

In this paper we consider a subtlety in the zeta-regularisation of the determinant (2). The zeta-regularisation technique requires the map to be positive, i.e. all its eigenvalues must be positive. But the action functional $S(\omega)$ of the TFT typically takes negative as well as positive values, so from (3) it follows that $\tilde{T}$ has negative as well as positive eigenvalues. This problem was sidestepped by Schwarz (and in most subsequent work on TFTs) by replacing $\tilde{T}$ in (2) by the positive map $|\tilde{T}|$. This map is defined in the following way: Take a basis $\{\omega_j\}$ for $\Gamma$ of eigenvectors of $T$ with eigenvalues $\{\lambda_j\}$, then $|T|$ is defined by setting $|T|\omega_j = |\lambda_j|\omega_j$ and $|\tilde{T}|$ is obtained from $|T|$ by discarding the zero-modes.

For a particular case (with $\beta = i$) E. Witten has shown in [3, §2] how the zeta-regularisation technique can be generalised to evaluate (4) (see also [4, §7.2]). He found that a complex phase factor appears, determined by $\eta(0 ; |T|)$, where $\eta(s ; T)$ is the eta-function of $T$. In this paper we evaluate the determinant (2) (with arbitrary $\beta \in \mathbb{C}$) for all the above-mentioned TFTs considered by Schwarz. This is done using a straightforward generalisation of the usual zeta-regularisation technique and analytic continuation in $\beta$, and generalises the calculation of Witten mentioned above. The following expression is obtained: Let $\mathbb{C}_+$ and $\mathbb{C}_-$ denote the upper and lower halfplanes of $\mathbb{C}$ respectively, then for $\beta = |\beta|e^{i\theta} \in \mathbb{C}_\pm$ with $\theta \in [-\pi, \pi]$ we find

$$
det(\beta\tilde{T})^{-1/2} = e^{-\frac{i\pi}{4}((\frac{2\pi}{\pi}+1)\zeta \pm \eta)} |\beta|^{-\zeta/2} \det(|\tilde{T}|)^{-1/2}
$$

(4)

where $\zeta$ and $\eta$ are the analytic continuations to $s = 0$ of the zeta-function $\zeta(s ; |T|)$ and eta-function $\eta(s ; T)$ respectively (defined as in (7) and (10) below) and $\det(|\tilde{T}|)^{-1/2}$ is defined by the usual zeta-regularisation technique. In particular, for $\lambda \in \mathbb{R}_+$ we get

$$
det(\lambda\tilde{T})^{-1/2} = e^{\pm \frac{i\pi}{4}(\zeta - \eta)} \lambda^{-\zeta/2} \det(|\tilde{T}|)^{-1/2}
$$

(5)
and
\[ det(i\lambda \tilde{T})^{-1/2} = e^{-\frac{i\pi}{4}} \lambda^{-\zeta/2} det(|\tilde{T}|)^{-1/2}. \] (6)

Note that for \( \beta \in \mathbb{R} \) there is a phase ambiguity in (4) (analogous to the ambiguity in \( \sqrt{-1} = \pm i \)) while there is no ambiguity for \( \beta \in \mathbb{C} - \mathbb{R} \) (e.g. when \( \beta \) is purely imaginary). It is not immediately obvious that the phase and scaling factors in (4)–(6) are finite, since this requires the zeta-function \( \zeta(s; |T|) \) and eta-function \( \eta(s; T) \) to have analytic continuations regular at \( s = 0 \). If \( T \) were elliptic then this would follow from standard results in mathematics; however for the cases arising in TFTs the map \( T \) in (3) is not elliptic. We will nevertheless show below that \( \zeta(s; |T|) \) and \( \eta(s; T) \) do in fact have analytic continuations regular at \( s = 0 \), so the expressions (4)–(6) are finite. (We do not claim that this is a new mathematical result, but for the sake of completeness we give a simple derivation). We also derive a simple formula for \( \zeta = \zeta(0; |T|) \) in terms of the dimensions of certain cohomology spaces when \( M \) has odd dimension ((23) below). This leads to a simple expression for the scaling dependence of (4)–(6).

Determinants of the form (2) are also relevant for TFTs where the action \( S(\omega) \) contains higher order terms as well as the quadratic term. In this case determinants of the form (2) appear in the semiclassical approximation for the partition function of the theory. A particular TFT with non-quadratic action is the Chern-Simons gauge theory on 3-dimensional manifolds (given by (28) below), which was shown to be solvable by E. Witten in [5]. We will discuss below how the dependence of the semiclassical approximation on the parameter \( k \) in this theory can be obtained from our calculation of (2). Because it is a solvable theory for a field with self-interactions the Chern-Simons gauge theory provides a “mathematical laboratory” in which predictions of perturbation theory can be tested. A basic prediction of perturbative quantum field theory is that the semiclassical approximation should coincide with the non-perturbative expression for the partition function in the limit where the parameter \( k \) of the theory becomes large. The large \( k \) limit of the partition function, with gauge group \( SU(2) \), has been explicitly calculated by Witten’s non-perturbative method for a large number of 3-manifolds in a program initiated by
D. Freed and R. Gompf \cite{8,9}. They found that the $k$–dependence of the partition function in this limit is given by a simple expression ((32) below). Subsequent work by L. Jeffrey \cite{10} and L. Rozansky \cite{11} has verified this expression for large classes of 3-manifolds. The expression we obtain below for the $k$–dependence of the semiclassical approximation turns out to be identical to this non-perturbative expression. Thus we reproduce a non-perturbative feature of the Chern-Simons gauge theory from perturbation theory.

Before evaluating (2) we briefly recall the usual zeta-regularisation technique. The zeta-function of a positive selfadjoint linear map $A$ is defined by

$$\zeta(s ; A) = \sum_j \frac{1}{\lambda_j^s}, \quad s \in \mathbb{C}$$

where $\{\lambda_j\}$ are the non-zero eigenvalues of $A$ (so $\lambda_j > 0$ for all $\lambda_j$ in (7)) with each eigenvalue appearing the same number of times as its multiplicity. With $\tilde{A}$ obtained from $A$ by discarding the zero-modes we can formally write

$$\det(\tilde{A}) = \prod_j \lambda_j = e^{-\zeta'(0 ; A)}.$$  \hspace{1cm} (8)

When $A$ acts on an infinite-dimensional vectorspace $\zeta(s ; A)$ is divergent around $s = 0$. However in many cases of interest it turns out that $\zeta(s ; A)$ is well-defined for $\text{Re}(s) >> 0$ and extends by analytic continuation to a meromorphic function on $\mathbb{C}$ which is regular at $s = 0$. Then we can use the analytic continuation of $\zeta(s ; A)$ to give well-defined meaning to the r.h.s. of (8) and use this to define $\det(\tilde{A})$ in (8). For $\beta \in \mathbb{R}_+$ we then obtain a well-defined expression for $\det(\beta \tilde{A})$ by replacing $\tilde{A}$ by $\beta \tilde{A}$ in (8). This leads to

$$\det(\beta \tilde{A}) = \beta^{\zeta(0 ; A)} e^{-\zeta'(0 ; A)}.$$ \hspace{1cm} (9)

Using (9) we can define $\det(\beta \tilde{A})$ for arbitrary $\beta \in \mathbb{C}$ via analytic continuation in $\beta$. To do this we must fix a convention for defining $z^a$ for $z \in \mathbb{C}$ and $a \in \mathbb{R}$. The natural way to do this is to write $z = |z| e^{i\theta}$ with $\theta \in [-\pi, \pi]$ and set $z^a = |z|^a e^{ia \theta}$. This is well-defined for all $a \in \mathbb{R}$ provided $z \not\in \mathbb{R}_-$; if $z \in \mathbb{R}_-$ then there is a phase ambiguity. With this convention (8) is defined for all $\beta \in \mathbb{C}$ up to a phase ambiguity.
for $\beta \in \mathbb{R}_-$. Finally, recall that the eta-function of a selfadjoint linear map $B$ (which may have both positive and negative eigenvalues) is defined by

$$
\eta(s; B) = \sum_k \frac{1}{(\lambda_k^+)^s} - \sum_l \frac{1}{(-\lambda_l^-)^s}
$$

(10)

where $\{\lambda_k^+\}$ and $\{\lambda_l^-\}$ are the strictly positive- and strictly negative eigenvalues of $B$ respectively. In many cases of interest it turns out that $\eta(s; B)$ is well-defined for $\text{Re}(s) >> 0$ and extends by analytic continuation to a meromorphic function on $\mathbb{C}$ which is regular at $s = 0$.

We shall now evaluate the determinant (2). Formally we have

$$
\det(\beta \tilde{T})^{-1/2} = (\det(\beta T_+) \det(\beta T_-))^{-1/2}
$$

(11)

where $T_+$ and $T_-$ are obtained from $T$ by restricting to the strictly positive- and strictly negative modes respectively. Note that $-T_-$ is positive (i.e. has positive eigenvalues) and that

$$
\zeta(s; |T|) = \zeta(s; T_+) + \zeta(s; -T_-)
$$

(12)

$$
\eta(s; T) = \zeta(s; T_+) - \zeta(s; -T_-)
$$

(13)

From (14), using (8), (9) and (12) we get

$$
\det(\beta \tilde{T})^{-1/2} = \det(\beta T_+)^{-1/2} \det((-\beta) (-T_-))^{-1/2}
$$

$$
= \beta^{-\zeta(0; T_+)/2} (-\beta)^{-\zeta(0; -T_-)/2} e^{(\zeta'(0; T_+) + \zeta'(0; -T_-))/2}
$$

$$
= \beta^{-\zeta(0; T_+)/2} (-\beta)^{-\zeta(0; -T_-)/2} \det(|\tilde{T}|)^{-1/2}
$$

(14)

For $\beta = |\beta| e^{i\theta} \in \mathbb{C}_+$ with $\theta \in [-\pi, \pi]$ we have $-\beta = |\beta| e^{i(\theta \mp \pi)}$ with $\theta \mp \pi \in [-\pi, \pi]$ and a simple calculation using (12) and (13) shows

$$
\beta^{-\zeta(0; T_+)/2} (-\beta)^{-\zeta(0; -T_-)/2} = e^{-\frac{\pi i}{4}((\theta \mp \pi) \zeta(0; |T|) \pm \eta(0; T))}.
$$

(15)

Substituting this in (14) gives (4).

As pointed out previously, for the expression (4) to have well-defined meaning $\zeta(s; |T|)$ and $\eta(s; T)$ must be regular at $s = 0$. We will now show that this is the
case for the cases of interest in TFT. In doing so we derive a simple formula for \( \zeta(0; |T|) \) when \( M \) has odd dimension. For the sake of concreteness we will work with a specific topological action functional

\[
S(\omega) = \int_M \omega \wedge d_m \omega .
\]  

The fields \( \omega \) are the real-valued differential forms on \( M \) of degree \( m \) and \( d_q \) denotes the exterior derivative on \( q \)-forms. \( M \) is required to have odd dimension \( n = 2m + 1 \) and we assume that \( m \) is odd, since for \( m \) even (16) is identically zero. The quadratic action functionals in other TFTs are generalisations of (16) and it is easily checked that the following arguments continue to hold for these. A choice of metric on \( M \) enables us to construct an inner product in the space of differential forms in the usual way (as in [6, p.437]) and with this we can write

\[
S(\omega) = \langle \omega , T \omega \rangle , \quad T = * d_m
\]

where \(*\) is the Hodge star-map (as in [6, p.437]). We denote the space of \( q \)-forms on \( M \) by \( \Omega^q(M) \) and define the Laplace-operator on \( \Omega^q(M) \) by

\[
\Delta_q = d_q^* d_q + d_{q-1}^* d_{q-1}^- \quad , \quad q = 0, 1, \ldots , n
\]

(18) (with \( d_{-1} = d_n = 0 \)). We will derive a relationship between the zeta-function of \(|T|\) and the zeta-functions of \( \Delta_0, \Delta_1, \ldots , \Delta_m \). To do this we will use the following simple observation: Consider linear maps \( A \) and \( B \) on a vector space, satisfying \( AB = BA = 0 \). Then if \( \{ \lambda_j \} \) denotes the collection of non-zero eigenvalues of \( A + B \) (with each eigenvalue appearing the same number of times as its multiplicity) we have

\[
\{ \lambda_j \} = \{ \lambda'_k \} \cup \{ \lambda''_l \}
\]

(19) where \( \{ \lambda'_k \} \) and \( \{ \lambda''_l \} \) are the non-zero eigenvalues of \( A \) and \( B \) respectively. (This is an elementary fact in linear algebra which is easily verified). Setting \( A = d_q^* d_q \) and \( B = d_{q-1}^* d_{q-1}^- \) the property \( AB = BA = 0 \) follows from \( d_q d_{q-1} = 0 \), and it follows from (18) and (19) that

\[
\zeta(s; \Delta_q) = \zeta(s; d_q^* d_q) + \zeta(s; d_{q-1}^* d_{q-1}^-)
\]

(20)
where we have used the simple fact that for any linear map $C$ the maps $C^*C$ and $CC^*$ have the same non-zero eigenvalues. A simple induction argument based on (20) and starting with $\zeta(s; d_m^*d_m) = \zeta(s; \Delta_m) - \zeta(s; d_{m-1}^*d_{m-1})$ shows that

$$\zeta(s; d_m^*d_m) = (-1)^m \sum_{q=0}^m (-1)^q \zeta(s; \Delta_q). \quad (21)$$

The map $T$ in (17) has the property $T^2 = d_m^*d_m$ and from the definition (7) we see that $\zeta(s; T^2) = \zeta(2s; |T|)$. It follows from (21) that

$$\zeta(s; |T|) = (-1)^m \sum_{q=0}^m (-1)^q \zeta(\frac{s}{2}; \Delta_q). \quad (22)$$

This shows that $\zeta(s; |T|)$ is well-defined for $\text{Re}(s) >> 0$ with analytic continuation regular at $s = 0$, since the zeta-functions of the Laplace-operators $\Delta_q$ are known to have these properties (see e.g. [12, ch.28]). When $\text{dim}M$ is odd we have $\zeta(0; \Delta_q) = -\text{dim}H^q(d)$ (see [12, ch.28]), where $H^q(d) = \ker(d_q)/\text{Im}(d_{q-1})$ is the $q$’th cohomology space of $d$. It follows from (22) that in this case

$$\zeta(0; |T|) = (-1)^{m+1} \sum_{q=0}^m (-1)^q \text{dim}H^q(d). \quad (23)$$

We now consider the eta-function $\eta(s; T)$. A standard result in elliptic operator theory states that the eta-function of an elliptic selfadjoint map is regular at $s = 0$. (This is due to M. Atiyah, V. Patodi and I. Singer [13] in the case where $\text{dim}M$ is odd, and P. Gilkey [14] when $\text{dim}M$ is even). The map $T$ in (17) is selfadjoint but not elliptic. However we can construct an elliptic selfadjoint map $D$ such that $\eta(s; D) = \eta(s; T)$, from which it follows that $\eta(s; T)$ is regular at $s = 0$. For $q = 0, 1, \ldots, m$ we extend $d_q$ to a map on $\oplus_{q=0}^m \Omega^q(M)$ by setting $d_q = 0$ on $\Omega^p(M)$ for $p \neq q$. We define the map $\tilde{D}$ on $\oplus_{q=0}^m \Omega^q(M)$ by $\tilde{D} = \sum_{q=0}^m (d_q + d_q^*)$ and set $D = T + \tilde{D}$, with $T$ as in (17). $D$ is clearly selfadjoint and a simple calculation using the property $d_q d_{q-1} = 0$ shows that $D^2 = \sum_{q=0}^m \Delta_q$, which is elliptic, so $D$ is elliptic. It is immediate from the definitions of $\tilde{D}$ and $T$ that $T\tilde{D} = \tilde{D}T = 0$ and it follows from (19) that

$$\eta(s; D) = \eta(s; T) + \eta(s; \tilde{D}). \quad (24)$$
To show \( \eta(s; D) = \eta(s; T) \) we must show that \( \eta(s; \widetilde{D}) = 0 \). We consider the eigenvalue equation \( \widetilde{D}\omega = \lambda\omega \) with \( \omega = \bigoplus_{q=0}^{m} \omega_q \in \bigoplus_{q=0}^{m} \Omega^q(M) \). This is equivalent to the collection of equations

\[
d_q \omega_q + d_{q+1}^* \omega_{q+2} = \lambda \omega_{q+1} , \quad q = 0, 1, \ldots, m - 1
\]

(with \( \omega_{m+1} = 0 \)). If \( \omega \) is a solution to (25) then we set \( \omega' = \bigoplus_{q=0}^{m} \omega_q' \) with \( \omega_q' = (-1)^q \omega_q \). Then

\[
d_q \omega_q' + d_{q+1}^* \omega_{q+2}' = (-1)^q (d_q \omega_q + d_{q+1}^* \omega_{q+2}) = (-1)^q \lambda \omega_{q+1} = -\lambda \omega_{q+1}'
\]

and it follows from (23) that \( \widetilde{D}\omega' = -\lambda \omega' \). This shows that there is a one-to-one correspondence \( \omega \leftrightarrow \omega' \) between eigenvectors for \( \widetilde{D} \) with eigenvalue \( \lambda \) and eigenvectors with eigenvalue \(-\lambda \), and it follows from the definition (11) that \( \eta(s; \widetilde{D}) = 0 \) as claimed. (The statement \( \eta(s; T) = \eta(s; D) \) is similar to [15, proposition(4.20)]).

Finally, as promised, we apply our results to the semiclassical approximation for the partition function of the Chern-Simons gauge theory on 3-manifolds. The partition function of this theory is

\[
Z(k) = \int \mathcal{D}A \, e^{ikS(A)} , \quad k \in \mathbb{Z}
\]

where

\[
S(A) = \frac{1}{4\pi} \int_M Tr(A\wedge dA + \frac{2}{3} A\wedge A\wedge A).
\]

The gauge fields \( A \) are the 1-forms on \( M \) with values in the Lie algebra of the gauge group \( SU(N) \). The parameter \( k \) is required to be integer-valued, then the integrand in (27) is gauge-invariant. An expression for the semiclassical approximation for (27) can be obtained from the invariant integration method of A. Schwarz [16, §5]. (We emphasise that Schwarz’s method is ideally suited for evaluating the semiclassical approximation for (27). This method leads to the appearance of inverse volume factors \( V(H_A)^{-1} \) in the integrand of the expression (24) below for the semiclassical approximation (see [13, §5, formula(1)]), where \( H_A \) is the subgroup of gauge transformations which leaves the gauge field \( A \) unchanged. These factors are necessary to reproduce
the numerical factors appearing in the large \( k \) limit of the non-perturbative expression for the partition function and have not been obtained in a self-contained way in other evaluations of the semiclassical approximation for the Chern-Simons partition function\(^4\). We will be discussing this in more detail in a future paper; see also \([17]\). The expression obtained from Schwarz’s method for the semiclassical approximation for (27) has the form

\[
Z_{sc}(k) = \int_{\mathcal{M}^F} DA e^{ikS(A)} \mu(k; A)
\]

(29)

where \( \mathcal{M}^F \) is the moduli space of flat gauge fields modulo gauge transformations. (The flat gauge fields are the solutions to the field equations corresponding to \((28)\)). The integrand \( e^{ikS(A)} \mu(k; A) \) is gauge-invariant and is therefore a well-defined function on \( \mathcal{M}^F \). The quantity \( \mu(k; A) \) is given by \([16, \S 5, \text{formula(1)}]\) and its dependence on \( k \) enters through the determinant

\[
\text{det}(ick\tilde{T}_A)^{-1/2}, \quad T_A = *d_A
\]

(30)

where \( c \) is a numerical constant (involving \( \pi \) ) and \( d_A \) is the flat covariant derivative on the Lie algebra-valued \( q \)-forms obtained from \( d_q \) by “twisting” by the flat gauge field \( A \). (See \([18, \S 15.2]\) for the definition of this). The results above concerning the map \( T \) in (17) generalise for the map \( T_A \) in (30). Since in the present case \( \dim M = 3 \), \( m = 1 \) and it follows from (6) and (23) that the \( k \)-dependence of the determinant in (30) is given by

\[
k^{-\zeta(\dim H^0(d^A) + \dim H^1(d^A))/2}. \quad (31)
\]

It follows that in the limit of large \( k \) the \( k \)-dependence of the semiclassical approximation (29) (ignoring phase factors) is given by

\[
k^{\left( \max_A \left\{ \frac{-\dim H^0(d^A)}{2 + \dim H^1(d^A)/2} \right\} \right)}
\]

(32)

where the maximum is taken over the flat gauge fields. This is precisely the \( k \) dependence \([1, \text{formula(1.37)}]\) of the large \( k \) limit of the partition function (27) obtained

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\(^4\)These volume factors were put in by hand in the expression for the semiclassical approximation given by L. Rozansky in \([11]\) and shown to lead to agreement with the large \( k \) limit of the non-perturbative expression for the partition function for large classes of 3-manifolds.
We illustrate this with a specific example. When $M$ is the 3-sphere the expression for the partition function obtained from Witten’s non-perturbative method [3, §4] with gauge group $SU(2)$ is

$$Z(k) = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}\right) \sim \sqrt{2} \pi k^{-3/2} \quad \text{for} \quad k \to \infty.$$  \hspace{1cm} (33)

Since $\pi_1(S^3)$ is trivial the only flat gauge field on the 3-sphere up to gauge equivalence is the trivial field $A=0$, and in this case we have $\dim H^0(dA) = \dim (su(2)) \dim H^0(S^3) = 3$ and $\dim H^1(dA) = \dim (su(2)) \dim H^1(S^3) = 0$. It follows from (31) that the $k-$dependence of the semiclassical approximation in this case is $\sim k^{-3/2}$, in agreement with the large $k$ limit of (33).

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