Chain method for panchromatic colorings of hypergraphs

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Abstract. We deal with an extremal problem concerning panchromatic colorings of hypergraphs. A vertex r-coloring of a hypergraph $H$ is panchromatic if every edge meets every color. We prove that for every $r < \sqrt[3]{\frac{n}{100 \ln n}}$, every $n$-uniform hypergraph $H$ with $|E(H)| \leq \frac{1}{20n^2} \left( \frac{n}{\ln n} \right)^{\frac{r-1}{r}} (\frac{r}{r-1})^{n-1}$ has a panchromatic coloring with $r$ colors.

Keywords: panchromatic coloring, property B, proper coloring, uniform hypergraph.

1 Introduction and related work

We study colorings of uniform hypergraphs. Let us recall some definitions.

A vertex $r$-coloring of a hypergraph $H = (V, E)$ is a mapping from the vertex set $V$ to a set of $r$ colors. An $r$-coloring of $H$ is panchromatic if each edge has at least one vertex of each color.

The first sufficient condition on the existence of a panchromatic coloring of a hypergraph was obtained in 1975 by Erdős and Lováss [8]. They proved that if every edge of an $n$-uniform hypergraph intersects at most $\frac{r^{n-1}}{4(r-1)^n}$ other edges then the hypergraph has a panchromatic coloring with $r$ colors.

The next generalization of the problem was formulated in 2002 by Kostochka [11], who posed the following question: What is the minimum possible number of edges in an $n$-uniform hypergraph that does not admit a panchromatic coloring with $r$ colors? He denoted this number by $p(n, r)$.

Following closely behind this problem is a related one: a hypergraph $H = (V, E)$ has property $B$ if there is a coloring of $V$ by 2 colors so that no edge $f \in E$ is monochromatic. Erdős and Hajnal [7] (1961) proposed to find the value $m(n)$ equal to the minimum possible number of edges in a $n$-uniform hypergraph without property $B$. Erdős [6] (1963–1964) found bounds $\Omega (2^n) \leq m(n) = O (2^n n^2)$ and Radhakrishnan and Srinivasan [13] (2000) proved $m(n) \geq \Omega (2^n (n/\ln n)^{1/2})$.

Clearly, $m(n) = p(n, 2)$.

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We return to the panchromatic coloring. Kostochka [11] has found connections between $p(n, r)$ and minimum possible number of vertices in a $k$-partite graph with list chromatic number greater than $r$. Using results of Erdős, Rubin and Taylor [9] and also Alon’s result [2] Kostochka [11] proved the existence of constants $c_1$ and $c_2$ that for every large $n$ and fixed $r$:

$$\frac{e^{c_1 n}}{r} \leq p(n, r) \leq re^{c_2 r}.$$ (2)

In 2010, bounds (2) were considerably improved in the paper of Shabanov [15]:

$$p(n, r) \geq \frac{\sqrt{21} - 3}{4r} \left( \frac{n}{(r-1)^2 \ln n} \right)^{1/3} \left( \frac{r}{r-1} \right)^n, \quad \text{for all } r < n,$$

$$p(n, r) \leq \frac{1}{r} \left( \frac{r}{r-1} \right)^n \frac{e(\ln r) n^2}{2(r-1)} \varphi_1, \quad \text{when } r = o(\sqrt{n}),$$

$$p(n, r) \leq \frac{1}{r} \left( \frac{r}{r-1} \right)^n e(\ln r) n^{3/2} \varphi_2, \quad \text{when } n = o(r^2),$$

where $\varphi_1, \varphi_2$ some functions of $n$ and $r(n)$, tending to one at $n \to \infty$.

In 2012, Rozovskaya and Shabanov [14] improved Shabanov’s lower bound by proving that for $r < n/(2 \ln n)$:

$$\frac{1}{2r^2} \left( \frac{n}{\ln n} \right)^{1/2} \left( \frac{r}{r-1} \right)^n \leq p(n, r) \leq c_2 n^2 \left( \frac{r}{r-1} \right)^n \ln r.$$ (3)

Further research was conducted by Cherkashin [3] in 2018. In his work, Cherkashin introduced the auxiliary value $p'(n, r)$, which is numerically equal to the minimum number of edges in the class of $n$-uniform hypergraphs $H = (V, E)$, in which any subset of vertices $V' \subset V$ with $|V'| \geq \lceil \frac{r-1}{r} |V| \rceil$ must contain an edge. Analyzing the value $p'(n, r)$ and using Sidorenko’s [16] estimates on the Turan numbers, Cherkashin proved that for $n \geq 2, r \geq 2$

$$p(n, r) \leq c n^2 \frac{\ln r}{r} \left( \frac{r}{r-1} \right)^n.$$

Cherkashin also proved that for $r \leq c \frac{n}{\ln n}$

$$p(n, r) \geq c \max \left( \frac{n^{1/4}}{r \sqrt{r}}, \frac{1}{\sqrt{n}} \right) \left( \frac{r}{r-1} \right)^n.$$ (4)

And repeating the ideas of Gebauer [10] Cherkashin constructed an example of a hypergraph that has few edges and does not admit a panchromatic coloring in $r$ colors. The reader is referred to the survey [4] for the detailed history of panchromatic colorings.

It is thus natural to consider the local case. Formally, the degree of an edge $A$ is the number of hyperedges intersecting $A$. Let $d(n, r)$ be the minimum possible value of the maximum edge
degree in an $n$-uniform hypergraph that does not admit panchromatic coloring with $r$ colors. Then, the Erdős and Lovász result (1) can be easily translated into following form:

\[ d(n, r) \geq \frac{r^{n-1}}{4(r - 1)^n}. \]  

(5)

However, the bound (5) appeared not to be sharp. The restriction on $d(n, r)$ have been improved by Rozovskaya and Shabanov [14]. In their work they achieved that

\[ d(n, r) > \sqrt{11 - \frac{3}{4(r - 1)}} \left( \frac{n}{\ln n} \right)^{1/2} \left( \frac{r}{r - 1} \right)^n, \quad \text{when } r \leq n/(2 \ln n). \]  

(6)

2 Our results

The main result of our paper improves the estimate (3) as follows.

**Theorem 1.** Suppose $r \leq \sqrt[3]{\frac{n}{100 \ln n}}$. Then we have

\[ p(n, r) \geq \frac{1}{20r^2} \left( \frac{n}{\ln n} \right)^{1/r} \left( \frac{r}{r - 1} \right)^n. \]  

(7)

**Corollary 1.** There is an absolute constant $C$ so that for every $n > 2$ and $\ln n < r < \sqrt[3]{\frac{n}{100 \ln n}}$

\[ p(n, r) \geq \frac{Cn}{r^2 \ln n} \cdot e^{\frac{n}{r^2} + \frac{n}{r^2}}. \]

We refine the bound (6) as follows.

**Theorem 2.** For every $2 < r < \sqrt[3]{\frac{n}{100 \ln n}}$

\[ d(n, r) \geq \frac{1}{40r^3} \left( \frac{n}{\ln n} \right)^{1/r} \left( \frac{r}{r - 1} \right)^n. \]  

(8)

2.1 Methods

In the work, we propose a new idea based on the Pluhar ordered chain method [12]. In the case of panchromatic coloring, the resulting structure is no longer a real ordered chain, but rather an intricate "snake ball". Nevertheless, with the help of probabilistic analysis, we managed to obtain a strong lower bound.

The rest of this paper is organised as follows. The next section describes a coloring algorithm. Section 4 is devoted to the detailed analysis of the algorithm. In Section 5 we collect some inequalities that will be subsequently useful. The last two sections contain proofs of Theorems 1 and 2.
3 The coloring algorithm

We may and will assume that $r \geq 3$, because case $r = 2$ corresponds to the case $m(n)$. Let $H = (V, E)$ be an $n$-uniform hypergraph with less than $\frac{1}{20r^2} \left( \frac{n}{\ln n} \right)^{r-1} \left( \frac{r}{r-1} \right)^n$ edges and let $r < \sqrt[3]{\frac{n}{100 \ln n}}$. We will show that $H$ has a panchromatic coloring with $r$ colors.

We define a special random order on the set $V$ of vertices of hypergraph $H$ using a mapping $\sigma : V \to [0, 1]$, where $\sigma(v), v \in V$ i.i.d. with uniform distribution on $[0, 1]$. The value $\sigma(v)$ we will call the weight of the vertex $v$. Reorder the vertices so that $\sigma(v_1) < \ldots < \sigma(v_{|V|})$. Put

$$p = \left( \frac{1}{r} \right) \cdot \frac{(r-1)^2 \ln \left( \frac{n}{\ln n} \right)}{n}.$$  \hspace{1cm} (9)

We divide the unit interval $[0, 1)$ into subintervals $\Delta_1, \Delta_2, \Delta_2, \ldots, \Delta_r$ as on the Figure 1, i.e.

$$\Delta_i = \left[ (i-1) \left( \frac{1-p}{r} \right) + \frac{p}{r-1} \right], \quad \delta_i = \left[ i \cdot \frac{1-p}{r} + (i-1) \cdot \frac{p}{r-1} \right], \quad i = 1, \ldots, r;$$

The length of each large subinterval $\Delta_i$ is equal to $\frac{1-p}{r}$ and every small subinterval $\delta_i$ has length equal to $\frac{p}{r-1}$. Since $p < \frac{1}{100r}$ under the given assumptions on $r$, we can see that the intervals $\Delta_1, \ldots, \Delta_r$ are each wider than the intervals $\delta_1, \ldots, \delta_{r-1}$. A vertex $v$ is said to belong to a subinterval $[c, d)$, if $\sigma(v) \in [c, d)$. We note that the same division of the segment $[0, 1]$ has already been used by the first author for proving some bounds on proper colorings [1].

![Figure 1: Partition of [0, 1) into $\Delta_1, \delta_1, \Delta_2, \delta_2, \ldots, \Delta_5$ when $r = 5$.](image)

We color the vertices of hypergraph $H$ according to the following algorithm, which consists of two steps.

1. First, each $v \in \Delta_i$ is colored with color $i$ for every $i \in [r]$.

2. Then, moving with the growth of $\sigma$, we color a vertex $v \in \delta_i$ with color $i$ if there exists an edge $e, v \in e$ such that $e$ does not have color $i$ in the current coloring. Otherwise we color $v$ with color $i + 1$. 

4
4 Analysis of the algorithm

4.1 Short edge

We say that an edge $A$ is short if $A \cap (\Delta_i \cup \delta_i) = \emptyset$ or $A \cap (\Delta_{i+1} \cup \delta_i) = \emptyset$ for some $i \in [r-1]$. The probability of this event for fixed edge $A$ and fixed $i$ is at most $2 \left( 1 - \left( \frac{1-p}{r} + \frac{p}{r-1} \right) \right)^n$. Summing up this upper bound over all edges and $i \in [r-1]^n$ we get

$$2(r-1)|E| \left( 1 - \left( \frac{1-p}{r} + \frac{p}{r-1} \right) \right)^n \leq \frac{2(r-1)}{20r^2} \left( \frac{n}{\ln n} \right)^{\frac{r-1}{r}} \left( \frac{r}{r-1} \right)^n.$$

Hence, we conclude that the expected number of short edges is less than $1/10r$, hence with probability at least $1 - 1/10r$ there is no short edge.

4.2 Snake ball

Suppose our algorithm fails to produce a panchromatic $r$-coloring and there is no short edges. Let $A$ be an edge, which does not contain some color $i$.

Now we have two possibilities:

- $i < r$, in this situation edge $A$ is disjoint from the interval $\Delta_i \cup \delta_i$, which means that $A$ is short, a contradiction.

- $i = r$.

![Figure 2: Edges A and B in a snake ball.](image)

Edge $A$ is not short, so $A \cap (\delta_{r-1} \cup \Delta_r) \neq \emptyset$. Since $A$ does not contain color $r$ we have $A \cap \Delta_r = \emptyset$. Denote $v_A$ the last vertex of $A \cap \delta_{r-1}$. We note that $v_A$ could receive color $r - 1$ only if at the moment of coloring $v_A$ there was an edge $B$ without color $r - 1$ and $v_A$ was the first vertex of $B \cap \delta_{r-1}$. In this situation we say that the pair $(A, B)$ is conflicting in $\delta_{r-1}$ and the vertex $v_A$ is dangerous vertex in $\delta_{r-1}$.

Again, edge $B$ is not short and did not contain color $r - 1$ at the moment of coloring $v_A$, so $B \cap (\delta_{r-2} \cup \Delta_{r-1}) \neq \emptyset$ and $B \cap \Delta_{r-1} = \emptyset$. For $v_B$, the last vertex of $B \cap \delta_{r-2}$, there exists an edge $C$, which at the moment of coloring $v_B$ was without color $r - 2$ and $v_B$ was the first vertex of $C \cap \delta_{r-2}$. We get $(B, C)$ is conflicting pair in $\delta_{r-2}$ and $v_B$ is dangerous vertex in $\delta_{r-2}$.
Repeating the above arguments, we obtain a construction called *snake ball*. It is an edge sequence $H' = (C_1 = A, C_2 = B, \ldots, C_r)$ such that consecutive edges $(C_i, C_{i+1})$ form conflicting pairs in $\delta_{r-i}$.

Summarizing the above, we can say that

**Claim 1.** If for injective $\sigma : V \to [0; 1)$ there are neither snack balls nor short edges then Algorithm 1 produces a panchromatic $r$-coloring.

**Lemma 1.** Let $H' = (C_1, \ldots, C_r)$ be an ordered $r$-tuple of edges in the hypergraph $H$. Then the probability of the event that $H'$ forms a snake ball and all the edges $C_1, \ldots, C_r$ are not short does not exceed

$$
\left( \frac{p}{r-1} \right)^{r-1} \left( \frac{r-1}{r} \right)^{(n-2)r} \prod_{v \in H' : s(v) \geq 2} \left( \frac{1 - s(v) \frac{1-p}{r}}{1 - \left( \frac{1-p}{r} + \frac{2p}{r-1} \right)} \right)^{s(v)} \prod_{i=1}^{r-1} |C_i \cap C_{i+1}|,
$$

where $s(v)$ is the number of edges of $H'$ that contain vertex $v$.

Before we present the proof of this lemma, we introduce some facts and give the basic scheme of the proof. Note that if $v \in C_i$ then $\sigma(v) \notin \Delta_{r-i+1}$. Furthermore, for each $v$ its weight $\sigma(v)$ belongs to the subintervals of total length at most

$$
1 - s(v) \frac{1-p}{r}. \quad (10)
$$

The scheme of the proof is following:

- fix vertex $v_j \in C_j \cap C_{j+1}$ and its weight $\sigma(v_j)$ for all $j = 1, \ldots, r-1$. Assuming that $v_j$ is the dangerous vertex in $\delta_{r-j}$ calculate conditional probability given weights of dangerous vertices.

- sum up (integrate) the previous probability over all possible values of weights, using that $\sigma(v_j) \in \delta_{r-j}$, as this is needed for $H'$ to be a snake ball.

- Finally, sum over all choices of $v_1, \ldots, v_{r-1}$.

**Proof.** Fix dangerous vertex $v_j \in C_j \cap C_{j+1}$ for each $j = 1, \ldots, r-1$. Put $[\alpha_j, \beta_j] = \delta_j$, $\beta_j - \alpha_j = p/(r-1)$ and $y_j = \beta_j - \alpha_j - \sigma(v_j)$. Recall that $0 \leq y_j \leq p/(r-1)$.

Fix for a moment variables $y_1, \ldots, y_{r-1}$. Then, for $v \in C_i$ with $s(v) = 1$ its weight $\sigma(v)$ belongs to the subinterval of total length at most

$$
1 - \left( \frac{1-p}{r} + y_{i+1} + \frac{p}{r-1} - y_i \right) \quad \text{if} \quad i \in [2, r-1].
$$

And similarly, $1 - \left( \frac{1-p}{r} + y_1 \right)$ for $i = 1$ and $1 - \left( \frac{1-p}{r} + \frac{p}{r-1} - y_{r-1} \right)$ for $i = r$.

Now we are ready to give an upper bound for the probability of the event that “$H'$ forms a snake ball”, conditional on the value taken by $y_1, \ldots, y_{r-1}$:
Here we estimated as if all the rest of the vertices have \( s(v) = 1 \) (factors \((11)\) and factor \((12)\)), and then using \((10)\), edited for vertices with \( s(v) > 1 \) by multiplying by \( 1 - s(v) \frac{1-p}{r} \) and divided by \( (1 - \frac{1-p}{r} + \frac{2p}{r-1}) \). The factor \( 1 - \frac{1-p}{r} + \frac{2p}{r-1} \) is obviously no more than any factor for \( s(v) = 1 \), so we get a correct upper bound.

Taking out factor \((r-1)/r)^{(n-2)r+2}\) in the above equation and using estimate \((1 + y)^n \leq \exp\{ys\}\), we get the following upper bound on product of \((11)\) and \((12)\):

\[
\left(\frac{r-1}{r}\right)^{r(n-2)+2} \exp\left(\frac{(n-1)p - (n-2)p}{r-1} - \frac{p}{(r-1)^2} - \frac{ry_1 + ry_r-1}{r-1} \right) \leq \left(\frac{r-1}{r}\right)^{r(n-2)+2} \exp\left(\frac{p(r-2)}{(r-1)^2} + \frac{rp}{(r-1)^2} \right) = \left(\frac{r-1}{r}\right)^{r(n-2)+2} \exp\left(\frac{2p}{r-1} \right) < \left(\frac{r-1}{r}\right)^{r(n-2)}.
\]

To obtain the final estimate, we have to integrate over the weights \( y_1, y_2, \ldots y_{r-1} \) (factor \((p/(r-1))^{r-1}\)) and sum up over all possible choices for the \( v_1, \ldots, v_{r-1} \) (factor \( \prod_{i=1}^{r-1} |C_i \cap C_{i+1}| \)).

\[\Box\]

## 5 Auxiliary calculations

Under the assumptions of Theorem 1 we will formulate and prove three auxiliary lemmas needed to prove Theorem 1. In particular, in Lemma 2, we replace product of pairwise intersections on their sum \( \sum_{i<j} |C_i \cap C_j| \) and in Lemma 4, we will use double-counting for estimating the sum \( \sum_{i<j} |C_i \cap C_j| \), which can be large with \( n \), by special bounded terms.

**Lemma 2.** Let \( H' = (C_1, \ldots, C_r) \) be an ordered \( r \)-tuple of edges in the hypergraph \( H \). Then

\[
\sum_{\pi \in S_r} |C_{i_1} \cap C_{i_2}||C_{i_2} \cap C_{i_3}| \cdots |C_{i_{r-1}} \cap C_{i_r}| \leq \left( \frac{2 \sum_{i<j} |C_i \cap C_j| + r}{r} \right)^r, \quad (14)
\]

where \( S_r \) denotes all permutations \( \pi = (i_1, \ldots, i_r) \) of \( (1, 2, \ldots, r) \).
Proof. Denote the cardinality of the edge intersection $|C_i \cap C_j|$ by $x_{i,j}$. Then, we have to prove that

$$\sum_{\pi \in S_r} x_{i_1,i_2} x_{i_2,i_3} \cdots x_{i_{r-1},i_r} \leq \left( \frac{2 \sum_{i<j} x_{i,j} + r}{r} \right)^r.$$

First, we will show that

$$\sum_{\pi \in S_r} x_{i_1,i_2} x_{i_2,i_3} \cdots x_{i_{r-1},i_r} \leq (x_{1,2} + \ldots + x_{1,r} + 1) \cdots (x_{r,1} + \ldots + x_{r,r-1} + 1). \quad (15)$$

Let us call $(x_{i,1} + \ldots + x_{i,r} + 1)$ from (15) the \textit{bracket number} $i$. We define a mapping $f$ between elements from the left-hand side of (15) and ordered sets that are obtained after performing the multiplication in (15).

Let $f : x_{i_1,i_2} x_{i_2,i_3} \cdots x_{i_{r-1},i_r} \mapsto x_{i_1,t_1} x_{i_2,t_2} \cdots x_{r,t_r}$, where $x_{i_1,t_1} x_{i_2,t_2} \cdots x_{r,t_r}$ is the product of the following $r$ elements: $x_{i_{r-1},i_r}$ from the bracket number $i_{r-1}$, $x_{i_{r-2},i_{r-1}}$ from the bracket number $i_{r-2}$ and so forth, finally we take the factor 1 from the unused bracket. For example,

$$x_{5,6} x_{6,1} x_{1,4} x_{4,3} x_{3,2}$$

is mapped to $x_{1,4} \cdot 1 \cdot x_{3,2} \cdot x_{4,3} \cdot x_{5,6} \cdot x_{6,1}$.

We note that $f$ is an injection. Indeed, for each $x_{i_1,t_1} x_{i_2,t_2} \cdots x_{r,t_r}$ there exists at most one sequence $x_{i_1,i_2} x_{i_2,i_3} \cdots x_{i_{r-1},i_r}$, with $i_1 \neq i_2 \ldots \neq i_r$, such as $f(x_{i_1,i_2} x_{i_2,i_3} \cdots x_{i_{r-1},i_r}) = x_{i_1,t_1} x_{i_2,t_2} \cdots x_{r,t_r}$.

So, since $f$ does not change the product and $f$ is an injection we get that the right-hand side of (15) is not less than the left-hand side.

Finally, by the inequality on the arithmetic-geometric means and by $x_{i,j} = x_{j,i}$

$$(x_{1,2} + \ldots + x_{1,r} + 1) \cdots (x_{r,1} + \ldots + x_{r,r-1} + 1) \leq \left( \frac{2 \sum_{i<j} x_{i,j} + r}{r} \right)^r.$$

\[\square\]

\textbf{Lemma 3.} For all $s \in \{2, \ldots, r-1\}$

$$\frac{(1 - \frac{s(1-p)}{r})}{(1 - (\frac{1-p}{r} + \frac{2p}{r-1}))} \leq e^{-\frac{s^2}{20r^2}}. \quad (16)$$

\textbf{Proof.} First prove the case $s \geq 3$.

$$\frac{(1 - \frac{s(1-p)}{r})^s}{(1 - (\frac{1-p}{r} + \frac{2p}{r-1}))^s} = \frac{(1 - \frac{s(1-p)}{r})}{(1 - (\frac{1-p}{r}))^s} \frac{(1 - \frac{2pr}{(r-1)(r-1+p)})}{(1 - \frac{2pr}{(r-1)^2})^s} \leq \frac{(1 - \frac{s(1-p)}{r})}{(1 - \frac{1-p}{r-1+p})^s} \frac{(1 + \frac{1-p}{r-1+p})^s}{(1 - \frac{2pr}{(r-1)^2})^s}. \quad (17)$$
Now we deal with factors in (17) separately:

\[
\left(1 + \frac{1 - p}{r - 1 + p}\right)^s \leq \left(1 + \frac{1 - p}{r - 1}\right)^s = \text{Apply Taylor’s formula with Lagrange Remainder} = \]

\[
1 + \frac{s(1 - p)}{r - 1} + \frac{s(s - 1)(1 - p)^2}{2(r - 1)^2} + \frac{s(s - 1)(s - 2)(1 - p)^3(1 + \theta \cdot \frac{1 - p}{r - 1})^{s - 3}}{6(r - 1)^3} \leq
\]

bound \((s - 1)/(r - 1)\) by \(s/r\), \((s - 1)(s - 2)/(r - 1)^2\) by \(s^2/r^2\) and \((1 + \theta/(r - 1))^{s - 3}\) by \(e\).

Hence, the numerator of (17) does not exceed

\[
\left(1 - \frac{s(1 - p)}{r}\right) \left(1 + \frac{s(1 - p)}{r - 1} + \frac{s^2(1 - p)}{2r(r - 1)} + \frac{s^3(1 - p)^2}{2r^2(r - 1)}\right) \leq 1 - \frac{s^2(1 - p)}{r(r - 1)} \left(1 - p - 1/2\right) + \frac{s(1 - p)}{r(r - 1)} (1/2 - 1/s - p) > 1 - \frac{s^2(1/6 - p)(1 - p)}{r^2} < 1 - \frac{s^2}{7r^2} \leq \exp \left\{-\frac{s^2}{7r^2}\right\}.
\]

Using bounds \(1/(1 - x) < 1 + 2x\) for \(x < 1/2\) and estimating \(pr < 1/100\), which follows from restrictions on \(r\), we finally get

\[
\frac{(1 - s^{1 - p}/r)}{(1 - (1 - p) + 2p\cdot r^{-1})^s} \leq \exp \left\{-\frac{s^2}{7r^2}\right\} \left(1 - \frac{2pr}{(r - 1)^2}\right)^s < \exp \left\{-\frac{s^2}{7r^2}\right\} \left(1 + \frac{4pr}{(r - 1)^2}\right)^s \leq \exp \left\{-\frac{s^2}{7r^2}\right\} \leq \exp \left\{-\frac{s^2}{7r^2}\right\}.
\]

Consider the case \(s = 2\).

\[
\frac{1 - 2(1 - p)/r}{(1 - (1 - p) + 2p\cdot r^{-1})} \leq \frac{1 - 2(1 - p)/r}{1 - \frac{2(1 - p)}{r} - \frac{4p}{r^2} + \frac{1}{r^2}} \leq \frac{1 - 2(1 - p)/r}{1 - \frac{2(1 - p)}{r} - \frac{1}{r^2} + \frac{1}{2r^2}} = 1 - \frac{1/4r^2}{1 - 2\cdot \frac{1 - 1/4r^2}{r^2}} \leq 1 - 1/4r^2 \leq \exp \{-1/4r^2\} < \exp \{-1/5r^2\},
\]

where we used that \(4p/(r - 1) < 8p/r = 8pr/r^2 < 8/100r^2 < 1/4r^2\).

\[\square\]

\textbf{Lemma 4.}

\[
\left(\prod_{v \in H:s(v) \geq 2} \left(1 - s(v)\frac{1 - p}{r}\right)\right) \cdot \sum_{\sigma \in S_r} |C_{i_1} \cap C_{i_2} \cap \ldots \cap C_{i_r}| \leq 20^r e^{-r + 1}
\]

\[
\tag{18}
\]

\textbf{Proof.} By Lemmas 2 and 3 the left hand side of (18) does not exceed

\[
\exp \left\{-\sum_{v \in H:s(v) \geq 2} s^2(v) \cdot \left(\frac{2 \sum_{i < j} |C_i \cap C_j| + r^2}{20r^2}\right)^r\right\}.
\]

9
Now we will use the following double-counting: \( \sum_{i<j} |C_i \cap C_j| \) is equal to \( \sum_{v \in H': s(v) \geq r} \left( \frac{s(v)}{2} \right) < 1/2 \sum_{v \in H': s(v) \geq r} s^2(v) \). Hence,

\[
\exp \left\{ - \sum_{v \in H': s(v) \geq r} \frac{s^2(v)}{20r^2} \right\} \leq \exp \left\{ - \sum_{v \in H': s(v) \geq r} \frac{s^2(v)}{20r^2} \right\} \cdot r^r.
\]

\[
\left( \sum_{v \in H': s(v) \geq r} s^2(v) \right) \leq r^r e^{-t/20} (t+1)^r \leq \frac{20r^2}{e^{t-1}},
\]

where we used \( t = \sum_{v \in H': s(v) \geq r} s^2(v) / r^2 \) and observed that the expression \( ((t+1)^r e^{-t/20}) \) is maximized when \( t = 20r - 1 \). \( \square \)

6 Proof of Theorem 1

We want to show that there is a positive probability that no edge is short and no tuple of edges forms a snake ball.

Denote \( \sum^* \) the sum over all \( r \)-sets \( J \subseteq (1, 2, \ldots, |E|) \), \( \sum^o \) the sum over all ordered \( r \)-tuples \( (j_1, \ldots, j_r) \), with \( \{j_1, \ldots, j_r\} \) forming such a \( J \) and \( \sum_{\pi \in S_r} \) denote the sum over all permutations \( \pi = (i_1, \ldots, i_r) \) of \( (1, 2, \ldots, r) \).

In Section 4.1 we already proved that the expected number of short edges does not exceed \( 1/(10r) \). The expected number of snake ball can be upper bounded as follows:

\[
\sum^o \mathbb{P} ((C_{j_1}, \ldots, C_{j_r}) \text{ forms a snake ball}) = \sum^* \sum_{\pi \in S_r} \mathbb{P} ((C_{i_1}, \ldots, C_{i_r}) \text{ forms a snake ball}).
\]

On the other hand,

\[
\sum_{\pi \in S_r} \mathbb{P} ((C_{i_1}, \ldots, C_{i_r}) \text{ forms a snake ball}) \leq \sum_{\pi \in S_r} \left( \frac{p}{r-1} \right)^{r-1} \left( \frac{r-1}{r} \right)^{n-2r} \prod_{v \in H': s(v) \geq r} \frac{(1 - s(v) \frac{1-p}{r})}{\left( (1 - (\frac{1-p}{r} + \frac{2p}{r-1}) s(v) \right) |C_{i_1} \cap C_{i_2}| \ldots |C_{i_{r-1}} \cap C_{i_r}|} \right.
\]

\[
= \left( \frac{p}{r-1} \right)^{r-1} \left( \frac{r-1}{r} \right)^{n-2r} \prod_{v \in H': s(v) \geq r} \frac{(1 - s(v) \frac{1-p}{r})}{(1 - (\frac{1-p}{r} + \frac{2p}{r-1}) s(v) \right) \sum_{\pi \in S_r} |C_{i_1} \cap C_{i_2}| \ldots |C_{i_{r-1}} \cap C_{i_r}| \right. \]

\[
\leq \left( \frac{p}{r-1} \right)^{r-1} \left( \frac{r-1}{r} \right)^{n-2r} 20r^2 e^{r-1} \leq \left( \frac{(r-1)^2 \ln(n \ln n)}{rn} \right)^{r-1}, \left( \frac{r-1}{r} \right)^{(n-2)r} \cdot 20r^2 e^{r-1},
\]

where for the first inequality we used Lemma 1 and for the second Lemma 4 and in the final
inequality we took $p$ from 9. Finally,

$$
\sum_{\pi \in S_r} \prod_{i=1}^{s} \mathbb{P}(C_i, ..., C_r \text{ forms a snake ball}) \leq
$$

$$
\left(\frac{|E|}{r}\right) \cdot \left(\frac{(r - 1)^2 \ln\left(\frac{n}{rn}\right)}{\frac{rr_n}{r!}}\right)^{r - 1} \cdot \left(\frac{r - 1}{r}\right)^{(n - 2)r} \cdot \frac{20r^2 r}{e^{r - 1}} \leq
$$

$$
\left(\frac{1}{20r^2} \left(\frac{n}{\ln n}\right)^{r - 1} \left(\frac{r}{r - 1}\right)^n\right)^r \cdot \left(\frac{(r - 1)^2 \ln\left(\frac{n}{rn}\right)}{\frac{rr_n}{r!}}\right)^{r - 1} \cdot \left(\frac{r - 1}{r}\right)^{(n - 2)r} \cdot \frac{20r^2 r}{e^{r - 1}} \leq \frac{1}{r} \left(\frac{r}{r - 1}\right)^2.
$$

Since $1 - \frac{1}{10r} - \frac{1}{r} \left(\frac{r}{r - 1}\right)^2 > 0$, with positive probability the Algorithm creates a panchromatic coloring with $r$ colors, which proves Theorem 1.

**Corollary 2.** There is an absolute constant $c$ so that for every $n > 2$ and $\ln n < r < \sqrt[3]{\frac{n}{100 \ln n}}$

$$
p(n, r) \geq c \frac{n}{r^2 (\ln n)} \cdot e^{\frac{2}{r} + \frac{\ln r}{2r^2}}.
$$

**Proof.** By applying Taylor’s formula with Peano remainder, we obtain

$$
\left(1 + \frac{1}{r - 1}\right) e^{-\frac{1}{r - 1}} = 1 + \frac{1}{3r^3} + O\left(\frac{1}{r^4}\right).
$$

Thus, $\left(1 + \frac{1}{r - 1}\right) > e^{\frac{2}{r} + \frac{\ln r}{2r^2}}$. Finally, we use $\left(\frac{n}{\ln n}\right)^{-\frac{1}{r}} > \frac{1}{e}$ when $r > \ln n$ and Theorem 1. \qed

## 7 Local variant: proof of Theorem 2

A useful parameter of $H$ is its maximal edge degree

$$
D := D(H) = \max_{e \in E(H)} |\{e' \in E(H) : e \cap e' \neq \emptyset\}|.
$$

We show that for $3 < r < \sqrt[3]{\frac{n}{100 \ln n}}$ every $n$-uniform hypergraph with $D \leq \frac{1}{40r^2} \left(\frac{n}{\ln n}\right)^{r - 1} \left(\frac{r}{r - 1}\right)^n$ has a panchromatic coloring with $r$ colors, which implies Theorem 2.

Let us recall Lovász Local Lemma, which shows a useful sufficient condition for simultaneously avoiding a set $A_1, A_2, \ldots, A_N$ of “bad” events:

**Lemma 5** (The Local Lemma; General Case, [8]). Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. A directed graph $\overrightarrow{D} = (V, E)$ on the set of vertices $V = \{1, 2, \ldots, n\}$ is a dependency digraph for the events $A_1, \ldots, A_n$ if for each $i$, $1 \leq i \leq n$, the event $A_i$ is mutually independent of all the events $\{A_j : (i, j) \notin E\}$. Suppose that $\overrightarrow{D} = (V, E)$ is a dependency digraph
for the above events and suppose there are real numbers \( x_1, \ldots, x_n \) such that \( 0 \leq x_i < 1 \) and 
\[ \mathbb{P}[A_i] \leq x_i \prod_{(i,j) \in E} (1 - x_j) \] 
for all \( 1 \leq i \leq n \). Then
\[ \mathbb{P} \left[ \bigwedge_{i=1}^{n} \overline{A_i} \right] \geq \prod_{i=1}^{n} (1 - x_i). \]

In particular, with positive probability, no event \( A_i \) holds.

To prove Theorem 2 we will use the following generalization of Lemma 5.

**Lemma 6.** If all events have probability \( \mathbb{P}(A_i) \leq \frac{1}{2} \), and for all \( i \)
\[ \sum_{j: (i,j) \in E} \mathbb{P}(A_j) \leq \frac{1}{4}, \]  
(19)
then there is a positive probability that no \( A_i \) holds.

For the sake of completeness, we give the proof of Lemma 6 here.

**Proof.** Put \( x_i = 2 \mathbb{P}(A_i) \). Then, for all \( i \)
\[ x_i \prod_{(i,j) \in E} (1 - x_j) = 2 \mathbb{P}(A_i) \prod_{(i,j) \in E} (1 - 2 \mathbb{P}(A_j)) \geq \mathbb{P}(A_i). \]

\[ \square \]

In our case the set of bad events has two types: short edges and snake balls. Let \( Q(C) \) be the event “edge \( C \) is short” and \( W(C_1, \ldots, C_r) \) be the event “(\( C_1, \ldots, C_r \)) forms a snake ball and all the edges \( C_1, \ldots, C_r \) are not short”. Note that \( Q(C) \) depends on at most \( D + 1 \) events \( Q(C') \) and at most \( 2r(D + 1)D^{r-1} \) events \( W(C_1, \ldots, C_r) \). Similarly, \( W(C_1, \ldots, C_r) \) depends at most on \( r(D + 1) \) events \( Q(C') \) and at most on \( 2r^2(D + 1)D^{r-1} \) events \( W(C_1', \ldots, C_r') \). Hence, using bounds from Sections 4.1 and 6 we get the following upper bounds:

1. if \( A_i = W(C_1, \ldots, C_r) \):
\[ \sum_{j: (i,j) \in E} \mathbb{P}(A_j) \leq r(D + 1) \cdot 2(r - 1) \left( 1 - \left( \frac{1 - p}{r} + \frac{p}{r - 1} \right) \right)^n + \]
\[ + 2r^2(D + 1)D^{r-1} \cdot \left( \frac{r - 1}{r} \right)^{(n-2)r} \left( \frac{p}{r - 1} \right)^{r-1} \frac{20^r r^2 e^{r-1}}{e^{r-1}} < \frac{2 r^2}{40 r^3} + \frac{2 r^2}{r^2 e^{r-1}} < \frac{1}{4}. \]

2. if \( A_i = Q(C) \):
\[ \sum_{j: (i,j) \in E} \mathbb{P}(A_j) \leq (D + 1) \cdot 2(r - 1) \left( 1 - \left( \frac{1 - p}{r} + \frac{p}{r - 1} \right) \right)^n + \]
\[ + 2r(D + 1)D^{r-1} \cdot \left( \frac{r - 1}{r} \right)^{(n-2)r} \left( \frac{p}{r - 1} \right)^{r-1} \frac{20^r r^2 e^{r-1}}{e^{r-1}} < \frac{1}{4}. \]

In both cases inequality (19) holds, completing the proof of Theorem 2.
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