Baryon fields with $U_L(3) \times U_R(3)$ chiral symmetry V:
Pion-Nucleon and Kaon-Nucleon $\Sigma$ Terms

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We have previously calculated the pion-nucleon $\Sigma_{\pi N}$ term in the chiral mixing approach with $u, d$ flavors only, and found the lower bound $\Sigma_{\pi N} \geq (1 + \frac{4}{9} \sin^2 \theta) \frac{1}{2} (m^b_u + m^b_d)$ where $m^b_u, m^b_d$ are the current quark masses, and $\theta$ is the mixing angle of the $[(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})]$ and the $[(1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)]$ chiral multiplets. This mixing angle can be calculated as $\sin^2 \theta = \frac{g_A^{(0)}}{g_A^{(3)}}$, where $g_A^{(0)}$ and $g_A^{(3)}$ are the flavor-singlet and the isovector axial couplings. With presently accepted values of current quark masses, this leads to $\Sigma_{\pi N} \geq 58.0 \pm 4.5$ MeV, which is in agreement with the values extracted from experiments, and substantially higher than most previous two-flavour calculations. The causes of this enhancement are: 1) the large, $(\frac{16}{9} \approx 5.3)$, purely $SU_L(2) \times SU_R(2)$ algebraic factor; 2) the admixture of the $[(1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)]$ chiral multiplet component in the nucleon, whose presence has been known for some time, but that had not been properly taken into account, yet. We have now extended these calculations of $\Sigma_{\pi N}$ to three light flavours, i.e., to $SU_L(3) \times SU_R(3)$ multiplet mixing. Phenomenology of chiral $SU_L(3) \times SU_R(3)$ multiplet mixing demands the presence of three chiral $SU_L(3) \times SU_R(3)$ multiplets, viz. $[(6, 3) \oplus (3, 6)]$, $[(3, 3) \oplus (3, 3)]$ and $[(3, 3) \oplus (3, 3)]$, in order to successfully reproduce the baryons’ flavor-octet and flavor-singlet axial currents, as well as the baryon anomalous magnetic moments. Here we use these previously obtained results, together with known constraints on the explicit chiral symmetry breaking in baryons to calculate the $\Sigma_{\pi N}$ term, but find little, or no change of $\Sigma_{\pi N}$ from the above successful two-flavor result. The physical significance of these results lies in the fact that they show no need for $q\bar{q}$ components, and in particular, no need for an $s\bar{s}$ component in the nucleon, in order to explain the large “observed” $\Sigma_{\pi N}$ value. We also predict the experimentally unknown kaon-nucleon sigma term $\Sigma_{K N}$.

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I. INTRODUCTION

For more than 35 years the deviation of the nucleon $\Sigma_{\pi N}$ term extracted from the measured $\pi N$ scattering partial wave analyses (in the following to be called “measured value”, for brevity) from the (naive quark model) value of 25 MeV was interpreted as an increase of Zweig-rule-breaking in the nucleon, or equivalently to an increased content of (unpolarized) $s\bar{s}$ pairs in the nucleon $[1, 2]$, defined as $y = \frac{2(N|\bar{s}s|N)}{(N|uu+ud|N)}$. Moreover, the anomalously small measured value of the flavor-singlet axial coupling $g_A^{(0)} = 0.33 \pm 0.06$ $[4, 5]$, or the older value $0.28 \pm 0.16$ $[3]$, as compared with the naive quark model prediction of $g_A^{(0)} = 1$, was long interpreted as evidence for an increased polarized $s\bar{s}$ content of the nucleon $[1]$. 

\[ \text{Keywords: baryon, chiral symmetry, pion-nucleon $\Sigma$ term} \]
Yet more recently, both of these conclusions and interpretations were checked directly in low-momentum transfer $Q$ parity-violating elastic electron scattering experiments, however, and were found to be incorrect \cite{11, 13}.

Whereas this situation is consistent with QCD, it seems in contradiction with earlier expectations, that were based on a combination of quark and chiral effective field theory models \cite{14, 16}. The question remains if one can explicitly construct an effective chiral field theory model for nucleons and mesons that is connected to the underlying quark structure of hadrons and reproduces these two “anomalous” results?

Gell-Mann and Lévy’s (GML) linear sigma model has been the principal example of an effective field theory model of strongly interacting nucleons and pions with spontaneously broken chiral symmetry ever since its inception more than 50 years ago \cite{17, 18}. It is well known that this linear sigma model does not always reproduce the correct phenomenology, e.g. a) the value of the isovector axial coupling strength $g_A^{(1)}$ equals unity in this model; b) the value of the isoscalar pion-nucleon scattering length is too large in this model.

Both of these shortcomings have been removed in an extended linear sigma model, proposed by Bjorken and Nauenberg \cite{19} and by Lee \cite{20}: a) The first one had been fixed by introducing an additional derivative-coupling term that is not renormalizable. b) Fred Myhrer and one of us (V.D.) showed in Ref. \cite{21} that consequently the phenomenology is considerably improved in the Bjorken-Nauenberg-Lee (BNL) extended linear sigma model, as compared to the original GML model; in particular, the value of the isoscalar pion-nucleon scattering length is reduced to its observed value. This improvement is directly related to the correct value of the (isovector) nucleon axial coupling constant $g_A^{(1)}$ in the BNL model, which in turn is a direct consequence of the new derivative coupling \cite{92}. This shows the phenomenological importance of having the correct value of the isovector axial coupling.

That is not the only axial coupling of the nucleon, however: there is also the isoscalar one $g_A^{(0)}$, whose measured value $g_A^{(0)} = 0.33 \pm 0.08$, or $0.28 \pm 0.16$ deviates even more from unity, which is the value that the naive non-relativistic quark model suggests and the GML model postulates. The BNL derivative coupling term does not fix the value of the isoscalar axial coupling strength $g_A^{(0)}$, however. Here one could continue with the BNL stratagem and introduce yet another derivative-coupling term to fix this problem, but clearly that would be \textit{ad hoc} and in no apparent way related to the underlying quark structure.

An alternative approach was attempted with the notion of chiral representation mixing, which is in fact older than the BNL model, but was able to reproduce realistic values of the isovector axial couplings \cite{22-24}. By choosing low dimensional representations, as corresponding to the ones of the nucleon’s three-quark interpolators at present days, that approach turned out to give some constraints on the values of the axial couplings $g_A^{(0,1)}$ without introducing derivative couplings of hadrons \cite{25-27}. In this sense it is rather different from the BNL model, and one should not be surprised if other predictions of the two models are different. What is perhaps not so well known is that there are two-flavor (linear realization) sigma model chiral Lagrangians based on the concept of chiral mixing, that reproduce the two-flavor chiral low-energy theorems \cite{20, 26-31}. Over the years, these Lagrangians have been extended to three flavors \cite{32-38} and adapted/fitted to the two axial couplings and other nucleon properties, such as the magnetic moments \cite{33, 40}.

An advantage of the chiral representation mixing is that the possible representations and their mixing may be inferred from the quark structure of the nucleon. For instance, in the Schwinger-Dyson-Faddeev-Bethe-Salpeter approach to QCD \cite{41}, different Dirac structures in the Faddeev-Bethe-Salpeter equation are sources of different chiral representation (or components in the Faddeev-Bethe-Salpeter amplitude), thus leading to a mixing of chiral representations in the physical nucleon wave function.

The purpose of the present paper is to coherently and systematically present the calculation of the pion-nucleon sigma term $\Sigma_{\pi N}$ and of the isoscalar axial coupling $g_A^{(0,1)}$ in the chiral-mixing linear sigma model. The isovector axial coupling has been studied by many authors as already mentioned above. The isoscalar axial coupling $g_A^{(0)}$ had been calculated in Refs. \cite{30, 42}, and in Refs. \cite{43, 44} we have
briefly presented our results for the pion-nucleon sigma term \( \Sigma_{\pi N} \) in the chiral mixing approach. The result of \( \Sigma_{\pi N} \) depends substantially on \( g_A^{(1)} \), in contrast to the BNL model one \([21]\), and agrees with the experimental value (almost embarrassingly) well. This phenomenological success has been a source of some open and more hidden criticism. We do not wish to over-emphasize this phenomenological agreement here, as it is subject to the time-dependent variation of the free parameters, more specifically, to the current quark masses, which were about 50\% larger 15 years ago - in this model, but rather we try and systematically explore the differences among various effective chiral models.

Moreover, we make a systematic exposition of our approach and we record this model's predictions of the kaon-nucleon sigma term, which have not been measured as yet, just in case some day they are measured if only on the lattice, and thus open ourselves to potential future criticism. In this way we explicitly show how to construct effective linear chiral model(s) of interacting nucleons and mesons based on the underlying quark structure, that does not need \( s\bar{s} \) content in the nucleon to reproduce the two crucial observables, the \( \Sigma_{\pi N} \) and the \( g_A^{(0)} \).

The crucial assumption here is the systematic implementation of chiral symmetry at all levels, i.e., at both the quark and the hadron levels. In Ref. \([21]\) we have shown how Weinberg’s “chiral boost” transformation of the BNL model leads to a nonlinear realization chiral Lagrangian, as corresponding to the leading order of the chiral perturbation theory Lagrangian. The same procedure can be applied to the chiral mixing Lagrangian, with the same result. For this to happen, two different linear Lagrangians lead to the same nonlinear one. That goes to show that the linear-to-nonlinear-realization mapping (Weinberg’s chiral boost) is of the many-to-one kind. Thus, it “hides” many details of the underlying dynamics at low values of momentum transfer as compared with \( f_\pi = 93 \text{ MeV} \) and \( m_\pi = 140 \text{ MeV} \), and emphasizes the dynamical aspects of chiral symmetry. In that sense, the nonlinear realization can be viewed as being “coarser” than the linear one. These dynamical details become increasingly visible as the momentum transfer is increased, however.

We believe that at least some of the generally valid chiral predictions of all chiral models are most economically obtained in the chiral-mixing model. In particular, we believe that the role of \( U_A(1) \) symmetry and its breaking in the baryon sector has been ignored thus far, and our study appears to be the first step in rectifying this lamentable situation.

Throughout this paper we shall use the first Born approximation at the tree level. In order to explore the various possibilities and to facilitate comparison with earlier studies of the Gell-Mann–Levy linear sigma model, we introduce three different chiral symmetry breaking [\( \chi \)SB] terms, as in Refs. \([43, 44]\).

This paper is the fifth one in a sequence of papers \([33–35, 47]\), consequently, we shall repeat here, for the sake of completeness and coherence of presentation, rather than merely cite, several (a bare minimum) of equations and tables that have already appeared in our previous papers.

The paper falls into six sections and two appendixes. In Section \( \text{II} \) we consider the chiral mixing phenomenology. Then in Section \( \text{III} \) which is devoted to a construction of a two- and three-flavor chiral Lagrangians that reproduce the chiral mixing phenomenology, we present the \( \chi \)SB terms and the canonical field variables, and show that the Noether charges close the chiral algebra although \( g_A \neq 1 \). In Section \( \text{IV} \) we examine the pion-nucleon sigma term \( \Sigma_{\pi N} \) - some of these results have been reported at conferences \([43, 44]\). In Section \( \text{V} \) we examine the kaon-nucleon \( \Sigma_{KN} \) term, and in Section \( \text{VI} \) we summarize the results.

### II. PHENOMENOLOGY OF CHIRAL MIXING

The basic premise of chiral mixing approach is that the chiral \( SU_L(3) \times SU_R(3) \) symmetry is spontaneously broken and therefore that the eigenstates do not form (ir)reducible representations of the chiral symmetry group \( SU_L(3) \times SU_R(3) \) \([93]\). Rather, the eigenstates are linear superpositions of several (ir)reducible representations of \( SU_L(3) \times SU_R(3) \). In general, such chiral representation mixing theories...
tend to be most powerful and predictive when only a few chiral multiplets are involved. As the number of admixed multiplets grows, this method becomes increasingly complicated and thus loses its predictive power.

Just which irreducible representations are being admixed, is a question that ultimately ought to be answered by QCD. In the absence of a QCD-based answer, the choice can be (severely) limited by the following mathematical and physical considerations.

A. Chiral representations

Group-theoretical considerations impose limitations on the allowed (ir)reducible representation of $SU_L(3) \otimes SU_R(3)$: any (ir)reducible representation of $SU_L(3) \otimes SU_R(3)$, that is described by two $SU(3)$ irreducible representations, $(G_L, G_R)$, leads to irreducible representations $G_F$ of $SU_F(3)$ as determined by the Clebsch-Gordan series of the tensor product: $G_F \in G_L \otimes G_R$. For example, $10 \oplus 8 = 6 \oplus 3 \in [(6, 3) \oplus (3, 6)]$, $1 \oplus 8 = 3 \oplus 3 \in [(3, 3) \oplus (3, 3)]$, and $8 = 8 \oplus 1 \in [(8, 1) \oplus (1, 8)]$.

If one demands that only the experimentally observed irreducible representations $G$ of $SU_F(3)$ appear in these Clebsch-Gordan series, then one is limited to the above three reducible chiral representations: Any other chiral representation, other than the (trivial) chiral-singlet one, $[(1, 1)]$, necessarily leads to $SU_F(3)$-exotics.

When we further take into account the so-called “mirror” representations, in which the left- (L) and the right-handed (R) representations are interchanged, $(G_L \leftrightarrow G_R)$, in the chiral multiplet, then the number of allowed chiral multiplets is six. Mathematically, there is no difference between the “naive” (natural?) and “mirror” representations; physically, and historically, the “naive” ones were introduced first, mostly because there were no explicit examples how the “mirror” ones could arise in a three-quark system. That “objection” was finally raised by explicit examples of mirror (three-quark) interpolating fields in Refs. 22, 23, 47, 49. For octet baryons, this limits the permissible chiral multiplets to $[(6, 3) \oplus (3, 6)]$, and its “mirror” $[(3, 6) \oplus (6, 3)]$, to $[(3, 3) \oplus (3, 3)]$, and its “mirror” $[(3, 3) \oplus (3, 3)]$, and to $[(8, 1) \oplus (1, 8)]$, and its “mirror” $[(1, 8) \oplus (8, 1)]$. Of course, one may have other, “exotic” chiral multiplets that contain manifestly exotic flavor $SU_F(3)$ multiplets, but we exclude them per fiat, for lack of observed exotics.

Historically, after the observation, in Refs. 23, 24, 51, that several crucial SU(6) algebra results follow from its (smaller) SU(3) \otimes SU(3) sub-algebra, the notion of $SU(3) \otimes SU(3)$ representation mixing was proposed as an explanation of the nucleon’s (isovector) axial coupling $g_A^{(1)}$. The physical nature of this $SU(3) \otimes SU(3)$ sub-algebra was not immediately clear, however, as two options (the conventional chiral charge algebra, and the so-called “collinear” algebra) existed at the time.

Indeed, the “collinear” $SU(3) \otimes SU(3)$ algebra, which was generally assumed in the early work, holds only in a particular (the so-called $p_x \rightarrow \infty$) frame of reference, which appears to be in conflict with the general principles of special relativity. Moreover, Adler and Weisberger had also derived their sum rule(s) with the help of the $p_\infty$ frame. It was only after Weinberg’s clarification of the Adler-Weisberger sum rule as consisting of two independent statements (viz. a) the model-independent Goldberger-Miyazawa-Oehme sum rule for the $\pi N$ scattering lengths; and b) the chiral symmetry breaking-dependent predictions for the $\pi N$ scattering lengths) that this matter was settled in favor of chiral symmetry, and thus the way was paved for its later applications in QCD. Thus, only the chiral charge symmetry option leads to Lorentz-invariant quark interaction theories, such as QCD.

B. Chiral mixing

In one of the earliest physical applications of the chiral configuration mixing idea, Harari 22, Bincer 50, Gerstein and Lee 23, and Gatto et al. 57 used the mixing of three of the aforementioned six
chiral multiplets to fit the nucleon’s isovector axial coupling constant $g_A^{(1)}$ value at 1.267, and thus explain its being different from unity, as was seemingly demanded by the Gell-Mann–Lévy model. It turned out, however, that this application was not selective at all: all mixing scenarios could reproduce this value, so long as the $[(6, 3) \oplus (3, 6)]$ multiplet was involved: a) Gatto et al. - Harari scenario

$$|N(8)| = \sin \theta |(6, 3)\rangle + \cos \theta (\cos \varphi |(3, \bar{3})\rangle + \sin \varphi |(\bar{3}, 3)\rangle),$$

(1)

or b) Gerstein-Lee scenario

$$|N(8)| = \sin \theta |(6, 3)\rangle + \cos \theta (\cos \varphi |(3, \bar{3})\rangle + \sin \varphi |(8, 1)\rangle).$$

(2)

For other, more exotic scenarios, see Ref. 59. Simultaneously, or somewhat later, Refs. 24, 40, 52, 60, 61, used the same approach to saturate the electric dipole operator algebra and calculate the nucleon’s anomalous magnetic moments and charge radii. Moreover, other phenomenological applications of the current algebra e.g. to pion photoproduction can be found in Ref. 53. All of this was done in the framework of collinear $SU(3) \otimes SU(3)$ algebra, but algebraically these results must be the same as the chiral $SU(3) \otimes SU(3)$ algebra ones. The construction of corresponding chiral multiplets in the $SU(3) \otimes SU(3)$ chiral charge algebra is not as straightforward as in the collinear one, however (see our remarks about interpolators, below).

There is no guarantee that all six of the above chiral (not collinear) multiplets are allowed by the Pauli principle in the ground state of the nucleon, as composed of three Dirac quarks. The (formal) tool for this kind of study was provided around 1980, in the form of the so-called nucleon-three-quark interpolating fields.

Studies, in Refs. 25, 26, 47, 49, of local (S-wave, therefore ground state candidates), bi-local (P-wave and higher), and tri-local (D-wave and higher) three-quark interpolators have shown that only $[(\bar{3}, 3) \oplus (3, \bar{3})]$, and $[(8, 1) \oplus (1, 8)]$ are allowed in the local limit and that $[(6, 3) \oplus (3, 6)]$ appears as a spin 1/2 “complement” to the local Rarita-Schwinger spin 3/2 interpolator. Many other chiral multiplets appear in the non-local case, where the Pauli principle is less restrictive.

### C. Isoscalar axial coupling

The nucleon has also a flavor singlet axial coupling constant $g_A^{(0)}$, that was measured/extracted from spin-polarized lepton-nucleon DIS data after 1988 as $g_A^{(0)} = 0.28 \pm 0.16$, or the more recent value of 0.33 ± 0.03 ± 0.05, which is in the non-relativistic quark model predicted to be unity. Our studies of interpolating fields have shown that each $SU_L(3) \otimes SU_R(3)$ multiplet carries definite $U_A(1)$ transformation properties and the corresponding $U_A(1)$ charge, see Table I. Then, the next basic question becomes if the same set of chiral mixing angle(s) can simultaneously explain this anomalously low value. The answer, which is in the positive, manifestly depends on the $U_A(1)$ chiral transformation properties of the admixed nucleon fields, and leads to the so-called Harari scenario that mixes $[(6, 3) \oplus (3, 6)]$, with $[(\bar{3}, 3) \oplus (3, \bar{3})]$, and its “mirror” $[(3, \bar{3}) \oplus (\bar{3}, 3)]$ field. No admixture of $[(8, 1) \oplus (1, 8)]$, or its “mirror” $[(1, 8) \oplus (8, 1)]$ is preferred. This fact confirms the Gatto-Harari scenario, Eq. 11, and eliminates the Gerstein-Lee scenario, Eq. 2, from contention.

Moreover, we note that the above outlined program of fitting the hadron/nucleon observables in order to obtain chiral mixing angles is practically feasible only for the ground state(s): e.g. there is no hope of ever (sufficiently accurately) measuring the isovector axial coupling of the Δ resonance, except, perhaps, on the lattice. The same comments hold for the negative parity, and all of the higher-lying excited states. In this sense, the present scheme is of limited scope, but its potential to explain and illustrate the (fairly complex) QCD physics of baryons is undeniable.

The above no-$[(8, 1) \oplus (1, 8)]$, or $[(1, 8) \oplus (8, 1)]$ selection rule is in striking agreement with the results of so-called QCD sum rules and lattice QCD calculations 62, 63 that indicate only weak coupling of...
TABLE I: The Abelian and the non-Abelian axial charges (+ sign indicates “naive”, - sign “mirror” transformation properties) and the non-Abelian chiral multiplets of $J^\mu = \frac{1}{2}$, Lorentz representation $(\frac{1}{2}, 0)$ nucleon fields. The field denoted by 0 belongs to the $(1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)$ chiral multiplet and is the basic nucleon field that is mixed with various $(\frac{1}{2}, 0)$ nucleon fields.

| case | field | $g^{(0)}_A$ | $g^{(1)}_A$ | $SU_L(2) \times SU_R(2)$ | F | D | $SU_L(3) \times SU_R(3)$ |
|------|-------|-------------|-------------|------------------------|---|---|-------------------------|
| I    | $N_1 - N_2$ | -1 | +1 | $(\frac{3}{2}, 0) \oplus (0, \frac{3}{2})$ | 0 | +1 | $(3, 3) \oplus (3, 3)$ |
| II   | $N_1 + N_2$ | +3 | +1 | $(\frac{3}{2}, 0) \oplus (0, \frac{3}{2})$ | +1 | 0 | $(8, 1) \oplus (1, 8)$ |
| III  | $N_1 - N_2$ | +1 | -1 | $(0, \frac{3}{2}) \oplus (\frac{3}{2}, 0)$ | 0 | -1 | $(3, 3) \oplus (3, 3)$ |
| IV   | $N_1 + N_2$ | -3 | -1 | $(0, \frac{3}{2}) \oplus (\frac{3}{2}, 0)$ | -1 | 0 | $(1, 8) \oplus (8, 1)$ |
| 0    | $\partial_\mu (N^a_3 + \frac{1}{\sqrt{2}} N^a_8) + 1 + \frac{1}{\sqrt{2}} (1, \frac{3}{2}) \oplus (\frac{3}{2}, 1)$ | +1 | +1 | $(6, 3) \oplus (3, 6)$ |

the physical nucleon ground state to the $[(8, 1) \oplus (1, 8)]$, and/or its “mirror” $[(1, 8) \oplus (8, 1)]$ multiplet component. There is no dynamical, or symmetry-based explanation of this fact, as yet.

Specific dynamical models such as the Faddeev-Bethe-Salpeter-Schwinger-Dyson equation approach of Ref. [47], or the Faddeev-Salpeter equation approach of Ref. [65], ought to yield specific predictions for these mixing angles/parameters, and perhaps also to a dynamical explanation of empirical selection rules such as the above one.

Irrespective of specific dynamical model calculations, there ought to exist an effective chiral Lagrangian description of the corresponding hadron degrees of freedom. The task of constructing them was long drawn out: a) the first $(1, \frac{1}{2}) - (0, \frac{1}{2})$ chiral representation mixing Lagrangian with $SU_L(2) \otimes SU_R(2)$ chiral symmetry was presented by Hara [31]; b) the first “naive”-“mirror” $(\frac{3}{2}, 0) - (0, \frac{3}{2})$ chiral-mixing Lagrangian with $SU_L(2) \otimes SU_R(2)$ chiral symmetry was presented by Lee [20], and further extended by a number of researchers [29, 66–68], and most recently by Nagata et al. [26, 27, 30, 42]; c) the extension to $SU_L(3) \otimes SU_R(3)$ chiral symmetry has been accomplished in Refs. [22, 32–34, 37, 38, 42] and will be briefly reviewed in Sect. III.

D. Consistency of chiral algebra

In this paper we study the chiral properties of baryon states from the viewpoint of linear mixing of different chiral representations, which happens together with the chiral symmetry breaking. Before the chiral mixing, it has been proved in Ref. [32] that the axial charges of individual chiral multiplets close the $SU_L(3) \otimes SU_R(3)$ chiral algebra (see also Appendix B of this paper). Unfortunately, we do not know how to use the chiral algebra to exactly describe the chiral symmetry breaking process, but we can still show that the axial charges obey the same $SU_L(3) \otimes SU_R(3)$ chiral algebra even after the chiral mixing has occurred, so the chiral mixing process can be well described by using the chiral algebra. After we finetune the mixing parameters (angles) according to some experimental information, the obtained mixed nucleon state may be used to study the physical nucleon state appeared after the chiral symmetry breaking, and at the same time be described by the chiral algebra. Take the Gatto et al. - Harari scenario [22, 57] as an example:

$$\sin \theta |(6, 3)\rangle + \cos \theta (\cos \varphi |(3, 3)\rangle + \sin \varphi |(3, 3)\rangle) \xrightarrow{\text{chiral mixing}} |N(8)\rangle \xrightarrow{\text{chiral symmetry breaking}} |N_{\text{phy}}\rangle.$$ (3)

This will ensure that the chiral symmetry-breaking Dashen double commutator can be straightforwardly calculated. We note that in general the physical chiral charges have some singular components, related to the chiral non-invariant QCD vacuum, but these are the chiral charges after the chiral symmetry breaking, and we shall not study them in this paper.

A basic feature of the linear chiral realization is that the axial couplings are determined by the chiral representations. In Ref. [17], we found that for the nucleon octet, the three-quark chiral representations

\[ \sin \theta |(6, 3)\rangle + \cos \theta (\cos \varphi |(3, 3)\rangle + \sin \varphi |(3, 3)\rangle) \xrightarrow{\text{chiral mixing}} |N(8)\rangle \xrightarrow{\text{chiral symmetry breaking}} |N_{\text{phy}}\rangle. \]
of $SU_L(3) \times SU_R(3)$, $(8, 1) \oplus (1, 8)$, $(3, 3) \oplus (3, 3)$ and $(6, 3) \oplus (3, 6)$ provide the nucleon isovector axial coupling $g^{(3)}_A = 1, 1$ and $5/3$ respectively. Then in Ref. [33], we found that the mixing of chiral $(6, 3) \oplus (3, 6)$, $(3, 3) \oplus (3, 3)$, and $(3, 3) \oplus (3, 3)$ nucleons leads to the observed axial couplings (the case III-I in Ref. [33]):

$$
g^{(3)}_A = g^{(3)}_A (6, 3) \sin^2 \theta + \cos^2 \theta \left( g^{(3)}_A (3, 3) \cos^2 \varphi + g^{(3)}_A (3, \bar{3}) \sin^2 \varphi \right) = 1.267, \quad (4)$$

$$
g^{(0)}_A = g^{(0)}_A (6, 3) \sin^2 \theta + \cos^2 \theta \left( g^{(0)}_A (\bar{3}, 3) \cos^2 \varphi + g^{(0)}_A (\bar{3}, \bar{3}) \sin^2 \varphi \right) = 0.33 \pm 0.08, \quad (5)$$

where we used

$$
\langle N|Q^3_5|N\rangle = \langle N|Q^5_6(6, 3)|N\rangle \sin^2 \theta + \cos^2 \theta \left( \langle N|Q^5_6(3, 3)|N\rangle \cos^2 \varphi + \langle N|Q^5_5(3, 3)|N\rangle \sin^2 \varphi \right). \quad (6)
$$

Next we used Table II values of $g^{(3)}_A (6, 3) = \frac{5}{3} = -g^{(3)}_A (3, 6)$; $g^{(3)}_A (3, \bar{3}) = 1 = -g^{(3)}_A (\bar{3}, 3)$ and $g^{(0)}_A (3, \bar{3}) = -1 = -g^{(0)}_A (\bar{3}, 3)$, to find

$$
g^{(3)}_A = \frac{5}{3} \sin^2 \theta + \cos^2 \theta \cos 2\varphi = 1.267, \quad (7)$$

$$
g^{(0)}_A = \sin^2 \theta - \cos^2 \theta \cos 2\varphi = 0.33 \pm 0.08, \quad (8)$$

whose solutions are

$$\theta = 50.7^\circ \pm 1.8^\circ, \varphi = 66.1^\circ \pm 2.9^\circ. \quad (9)$$

Of course, this mixing appears to affect the $SU_L(3) \otimes SU_R(3)$ chiral algebra, as well, so we must first check that we did not spoil this algebra. The main “problematic” part of the $SU_L(3) \otimes SU_R(3)$ chiral algebra is the double-axial commutator

$$[Q^a_5, Q^b_5] = i f^{abc} Q^c. \quad (10)$$

We shall check this commutation rule in the nucleon sub-space of the full Hilbert space:

$$
\langle N|[Q^a_5, Q^b_5]|N\rangle = \langle N|[Q^a_5(6, 3), Q^b_5(6, 3)]|N\rangle \sin^2 \theta + \langle N|[Q^a_5(3, 3), Q^b_5(3, 3)]|N\rangle \cos^2 \theta \cos^2 \varphi + \langle N|[Q^a_5(3, \bar{3}), Q^b_5(3, \bar{3})]|N\rangle \cos^2 \theta \sin^2 \varphi. \quad (11)
$$

Next, we may use the commutators $[Q^a_5, Q^b_5] = i f^{abc} Q^c$ for the $(3, 3)$ and the $(6, 3)$ chiral multiplets worked out in Ref. [33] and listed in Appendix III which all lead to the same $SU(3)$ vector charges $Q^c$:

$$[Q^a_5(3, \bar{3}), Q^b_5(3, \bar{3})] = i f^{abc} Q^c(3, \bar{3}) = i f^{abc} Q^c, \quad (12)$$

$$[Q^a_5(6, 3), Q^b_5(6, 3)] = i f^{abc} Q^c(6, 3) = i f^{abc} Q^c. \quad (13)$$

Thus we find

$$\langle N|[Q^a_5, Q^b_5]|N\rangle = i f^{abc} \langle N|Q^c|N\rangle \left( \sin^2 \theta + \cos^2 \theta \left( \cos^2 \varphi + \sin^2 \varphi \right) \right) = i f^{abc} \langle N|Q^c|N\rangle, \quad (14)$$

which confirms the chiral charge $SU_L(3) \otimes SU_R(3)$ algebra. This ensures that the chiral symmetry-breaking Dashen double commutator can be safely/reliably calculated in the chiral mixing approach.
III. THE LINEAR SIGMA MODEL FOR CHIRAL MIXING

The next step is to try and reproduce this phenomenological mixing starting from a model interaction, rather than per flat. As the first step in that direction we must look for a dynamical source of chiral mixing. One, perhaps the simplest, such mechanism is the chirally symmetric non-derivative one-(σ, π)-meson interaction Lagrangian, which induces baryon masses via its σ-meson coupling. For this reason we need to know the form of the most general such Lagrangian(s); that problem was solved in Ref. [34] for three flavors and in Refs. [30, 42] for two flavors.

There is a significant difference between \( N_f=2 \) and \( N_f=3 \) chirally symmetric linear sigma models of chiral mixing, as only in the latter case there are strongly restrictive selection rules.

For example, most \( U_A(1) \) symmetry-breaking and \( SU_L(3) \times SU_R(3) \) chiral symmetry-conserving interactions are forbidden, see Tables III\( \text{III} \) taken from Ref. [44]. In particular only one \( SU_L(3) \times SU_R(3) \) symmetric, but \( U_A(1) \) symmetry-breaking interaction (the \([3, \bar{3}] \oplus (\bar{3}, 3)]-[ (8, 1) \oplus (1, 8)]\) and its Hermitian conjugate \([1, (8, 1)] [\text{mir}]-[(\bar{3}, 3) \oplus (3, \bar{3})] [\text{mir}]]\) is allowed. These results stand in marked contrast to the two-flavor case [30, 42], where all of the \( SU_L(2) \times SU_R(2) \) symmetric interactions have both a \( U_A(1) \) symmetry-conserving and a \( U_A(1) \) symmetry-breaking version. This is due to the fact that in the \( SU_L(2) \times SU_R(2) \) limit both the \([3, \bar{3}] \oplus (3, \bar{3})\) and the \([ (8, 1) \oplus (1, 8)]\) multiplet reduce to the same multiplet \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\), albeit with different \( U_A(1) \) symmetry properties.

Although, the \( SU_L(3) \times SU_R(3) \) symmetry is rather badly explicitly broken, we may expect that in the corresponding \( \pi - N \) sector, the \( SU_L(2) \times SU_R(2) \) symmetry may remain more-or-less conserved. So, although we shall be primarily interested in the pion-nucleon case, i.e., in \( N_f=2 \), we shall use the \( N_f=3 \) selection rules for guidance.

A. A brief Summary of \( N_f=3 \) Interactions

In this section, we introduce a shorthand notation

\[
\begin{align*}
N_{(8m)} & \sim [ (1, 8) \oplus (8, 1)] [\text{mir}], \\
N_{(9m)} & \sim [(3, 3) \oplus (\bar{3}, 3)] [\text{mir}], \\
N_{(18)} & \sim (6, 3) \oplus (3, 6), \\
N_{(10m)} & \sim [(1, 10) \oplus (10, 1)] [\text{mir}],
\end{align*}
\]

and similar for their mirror and naive representations. The scalar (σ) and pseudo-scalar (π) mesons are introduced and transformed under the chiral transformations as

\[
M = \sigma + i \gamma_5 \pi \sim (3, \bar{3}), \quad M^\dagger = \sigma - i \gamma_5 \pi \sim (\bar{3}, 3).
\]

Now the chiral structure of the Lagrangians for Yukawa type interactions is

\[
\bar{N}MN^\dagger + \bar{N}^\dagger M^\dagger N \sim \bar{N}_L M N_R^\dagger + \bar{N}_R^\dagger M^\dagger N_L,
\]

where \( N \) and \( N' \) may belong to different chiral representations. Our task is to form chiral singlet combinations for these interactions. For instance,

\[
\bar{N}_{(9m)} MN_{(18)} \sim (3, 3) \otimes (\bar{3}, \bar{3}) \otimes (3, 6) + (\bar{3}, \bar{3}) \otimes (3, 3) \otimes (6, 3)
\]

\[(18)\]

can make the \( SU_L(3) \times SU_R(3) \) chiral singlet \((3 \otimes 3 \otimes 3 \rightarrow 1, \bar{3} \otimes \bar{3} \otimes 6 \rightarrow 1)\). This corresponds to the cell at the 2nd row and 3rd column of Table II. Contrary, a combination like

\[
\bar{N}_{(8m)} MN_{(8m)} \sim (1, 8) \otimes (3, 3) \otimes (8, 1) + (8, 1) \otimes (\bar{3}, 3) \otimes (1, 8)
\]

\[(19)\]
Besides these, there is another single-term Lagrangian which is also chiral invariant:

\[ \mathcal{L} = g_B \bar{N}(8) M^\dagger T_{(B)}^a N(9) + h.c. \]

TABLE II: Allowed chiral invariant interaction Lagrangian with one pseudoscalar meson field, denoted by either \( M \) or \( M^\dagger \) as corresponding to Eqs. \((20)\) and \((21)\). The symbol – indicates that chiral invariant construction is not allowed. All cases are both \( SU_L(3) \times SU_R(3) \) and \( U_A(1) \) invariant except for the last (third) group where \( U_A(1) \) is broken.

| \( (1, 8) \oplus (8, 1) \) | \( (3, 3) \oplus (3, 3) \) | \( (6, 3) \oplus (3, 6) \) | \( (1, 10) \oplus (10, 1) \) |
|--------------------------|--------------------------|--------------------------|--------------------------|
| \( (1, 8) \oplus (8, 1) \) | \( M^\dagger \) | \( M \) | \( M \) |
| \( (3, 3) \oplus (3, 3) \) | \( M \) | \( M \) | \( M \) |
| \( (6, 3) \oplus (3, 6) \) | \( M^\dagger \) | \( M \) | \( M \) |
| \( (1, 10) \oplus (10, 1) \) | \( M \) | \( M \) | \( M \) |

We can also consider other possible combinations, all of which are listed in Table II.

The results are also expressed explicitly in the form of the Lagrangian which is given by

\[
\mathcal{L} = \begin{pmatrix}
N(8m) \\
N(9m) \\
N(18) \\
N(10m)
\end{pmatrix}
M
= 
\begin{pmatrix}
0_{8 \times 8} & 0_{8 \times 9} & 0_{8 \times 18} & 0_{8 \times 10} \\
0_{9 \times 8} & g(9) & g(9/18) & 0_{9 \times 10} \\
0_{18 \times 8} & g(9/18) & g(18/18) & 0_{18 \times 10} \\
0_{10 \times 8} & 0_{10 \times 9} & 0_{10 \times 18} & 0_{10 \times 10}
\end{pmatrix}
\begin{pmatrix}
N(8m) \\
N(9m) \\
N(18) \\
N(10m)
\end{pmatrix}.
\]

Here \( 0_{A \times B} \) is the null matrix of dimension \( A \times B \), and \( D^a(A) \). \( T_{(A)}^a \) are flavor transition matrix of dimension as indicated by their subscripts, which are defined in Ref. \[33\]. Similarly the mirror counterparts are given as:

\[
\mathcal{L}_{(m)} = \begin{pmatrix}
N(8) \\
N(9) \\
N(18m) \\
N(10)
\end{pmatrix}
M^\dagger
= 
\begin{pmatrix}
0_{8 \times 8} & 0_{8 \times 9} & 0_{8 \times 18} & 0_{8 \times 10} \\
0_{9 \times 8} & g'(9) & g'(9/18) & 0_{9 \times 10} \\
0_{18 \times 8} & g'(9/18) & g'(18/18) & 0_{18 \times 10} \\
0_{10 \times 8} & 0_{10 \times 9} & 0_{10 \times 18} & 0_{10 \times 10}
\end{pmatrix}
\begin{pmatrix}
N(8) \\
N(9) \\
N(18m) \\
N(10)
\end{pmatrix}.
\]

Besides these, there is another single-term Lagrangian which is also chiral invariant:

\[
\mathcal{L}(B) = g_B \bar{N}(8) M^\dagger T_{(B)}^a N(9) + h.c. 
\]
together with its mirror counterpart
\[
\mathcal{L}_{(Bm)} = g'_{(B)} \overline{N}^{(8m)} M T^a_{(B)} N^{(9)} + h.c.. \tag{23}
\]
These corresponds to the third (bottom) group of Table III.

We note that the Lagrangians \([20]\) and \([21]\) are also invariant under \(U_A(1)\) chiral transformation, while \([22]\) and \([24]\) are not. This is verified by counting the \(U_A(1)\) charge \(g_{A}^{(0)}\) in the interaction Lagrangian. Recall that the meson fields \(M\) and \(M^\dagger\) carry \(g_{A}^{(0)} = -2\) and \(+2\), respectively. Therefore, for the interaction \([18]\) as an example, by using the result of Table II we have the net \(U_A(1)\) charge as
\[
g_{A}^{(0)} = +1 - 2 + 1 = 0,
\]
where we have used the fact that the \(U_A(1)\) charge of the Dirac conjugate is the same as the original one because of the interchange of the left and right components.

These results stand in marked contrast to the two-flavor case \([30, 42]\). Namely, for \(SU_L(3) \times SU_R(3)\) chiral invariant Lagrangians with a certain representation structure as given in Table III (or to one term in Eqs. \([20] - [23]\)) are either \(U_A(1)\) symmetry-conserving or \(U_A(1)\) symmetry-breaking. In contrast, for \(SU_L(2) \times SU_R(2)\) chiral invariant Lagrangians with the same representation structure have both a \(U_A(1)\) symmetry-conserving and a \(U_A(1)\) symmetry-breaking version. This is due to the fact that the two \(SU_L(3) \times SU_R(3)\) representations, \((8, 1) \oplus (1, 8)\) and \((3, 3) \oplus (3, 3)\) reduce to the same \((\overline{3}, 0) \oplus (0, \overline{3})\) representation of \(SU_L(2) \times SU_R(2)\). Thus, generally speaking, the three-flavor chiral symmetry is more restrictive than the two-flavor one.

Besides the interaction Lagrangians \([20, 23]\), the so-called “naive”-“mirror” mass terms are also chiral invariant:
\[
\mathcal{L}_{(mass)} = m_{(8)} \overline{N}^{(8m)} \gamma_5 N^{(8)} + m_{(9)} \overline{N}^{(9m)} \gamma_5 N^{(9)} + m_{(18)} \overline{N}^{(18m)} \gamma_5 N^{(18)} + m_{(10)} \overline{N}^{(10m)} \gamma_5 N^{(10)} ,
\]
where \(m_{(8)}, \cdots, m_{(10)}\) are the mass parameters. The chiral structures of these terms are summarized in Table III.

**TABLE III:** Allowed chiral invariant mass terms as denoted by 1, while the symbol – indicates that chiral invariant construction is not allowed. All cases are both \(SU_L(3) \times SU_R(3)\) and \(U_A(1)\) invariant.

| \((SU_L(3), U_A(1))\) | \((8, 1) \oplus (1, 8)\) | \((3, 3) \oplus (3, 3)\) | \((3, 6) \oplus (6, 3)\) | \((10, 1) \oplus (1, 10)\) |
|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| \((1, 8) \oplus (8, 1)\) | 1                        | -                        | -                        | -                        |
| \((3, 3) \oplus (3, 3)\) | -                        | 1                        | -                        | -                        |
| \((6, 3) \oplus (3, 6)\) | -                        | -                        | 1                        | -                        |
| \((1, 10) \oplus (10, 1)\) | -                        | -                        | -                        | 1                        |

**B. Baryon Masses in the chiral limit**

Chiral symmetry is spontaneously broken through the “condensation” of the sigma field \(\sigma \rightarrow \sigma_0 = \langle \sigma \rangle_0 = f_\pi\), which leads to the dynamical generation of baryon masses, as can be seen from the linearized chiral invariant interaction Lagrangians Eqs. \([20]\), \([21]\), \([22]\) and \([23]\).

In this section, we study the masses of the octet baryons. There are altogether six types of octet baryon fields: \(N_+ (N^{(8)}), N_- (\text{contained in } N^{(9)}), N_\mu (\text{contained in } N^{(18)}), \) as well as their mirror fields \(N'_+ (N^{(8m)}), N'_- (\text{contained in } N^{(9m)}), N'_\mu (\text{contained in } N^{(18m)})\). The nucleon mass matrix is already in a
simple block-diagonal form when the nucleon fields form the following mass matrix:

\[
M = \frac{1}{\sqrt{6}} \begin{pmatrix}
0 & f_\gamma g_{(8/9)} & f_\gamma g_{(8/18)} & m_{(8)} \gamma_5 & f_\gamma g_B & 0 \\
-f_\gamma g_{(8/9)} & f_\gamma g_{(9/9)} & f_\gamma g_{(9/18)} & f_\gamma g_B & m_{(9)} \gamma_5 & 0 \\
f_\gamma g_{(8/18)} & f_\gamma g_{(9/18)} & f_\gamma g_{(9/18)} & f_\gamma g_B & m_{(18)} \gamma_5 & 0 \\
m_{(8)} \gamma_5 & f_\gamma g_B & 0 & m_{(8)} \gamma_5 & f_\gamma g_{(8/9)} & m_{(8)} \gamma_5 \\
f_\gamma g_B^* & m_{(9)} \gamma_5 & 0 & f_\gamma g_B^* & f_\gamma g_{(9/9)} & f_\gamma g_{(9/18)} \\
0 & 0 & m_{(18)} \gamma_5 & f_\gamma g_B^* & f_\gamma g_{(9/18)} & f_\gamma g_{(9/18)}
\end{pmatrix} N,
\]

where

\[
N = (N'_+, N'_-, N'_\mu, N_+, N_-, N'_\mu)^T.
\]

Since there are three nucleon fields as well as their mirror fields, there can be a nonzero phase angle. However, for simplicity, we assume all the axial couplings are real.

**C. Masses due to \([6, 3] \oplus (3, 6)] - [(3, 3] \oplus (3, 3)] - [(3, 3] \oplus (3, 3)] mixing**

As shown in Sec. I, the mixing of chiral \((6, 3] \oplus (3, 6)], (3, 3] \oplus (3, 3], and \((3, 3] \oplus (3, 3]) nucleons leads to the observed axial couplings (the case III-I in Ref. [33]). Accordingly, we investigate the following three nucleon chiral multiplets

\[
(B_2, \Delta) \in (6, 3] \oplus (3, 6], \\
(B_1, \Lambda_1) \in (3, 3] \oplus (3, 3] [\text{mir}], \\
(B_3, \Lambda_2) \in (3, 3] \oplus (3, 3],
\]

and one meson multiplet

\[
(\sigma, \pi) \in (3, \bar{3}] \oplus (\bar{3}, 3].
\]

Here all baryons have spin 1/2, while the isospin of \(B_1\) and \(B_2\) is 1/2 and that of \(\Delta\) is 3/2. The \(\Delta\) field is then represented by an isovector, Dirac-spinor field \(\Delta^i\), \((i = 1, 2, 3)\), which should not be confused with the spin-\(\frac{3}{2}\) \(\Delta(1232)\) resonance.

In writing down the Lagrangians Eqs. [20], we have implicitly assumed that the parities of \(B_1\), \(B_2\), \(\Lambda\) and \(\Delta\) are the same. In principle, they are arbitrary, except for the ground state nucleon, which must be even. For instance, if \(B_2\) has odd parity, the first term in the interaction Lagrangian Eq. [20] must include another \(\gamma_5\) matrix [67].

Having established the mixing interactions as well as the diagonal terms in Ref. [34], we calculated the masses of the baryon states, as functions of the pion decay constant \(f_\pi\) and the coupling constants \(g_1 \sim g_{(9/9)}, g_2 \sim g_{(18/18)}\) and \(g_3 \sim g_{(9/18)}\):

\[
\mathcal{L}(9) = -g_1 f_\pi (\bar{B}_1 B_1 - 2\bar{\Lambda}\Lambda) + \cdots,
\]

\[
\mathcal{L}(18) = -g_2 f_\pi (\bar{B}_2 B_2 - 2\bar{\Lambda}^i \Delta^i) + \cdots,
\]

\[
\mathcal{L}(9/18) = -g_3 f_\pi (\bar{B}_1 B_2) + \cdots,
\]

\[
\mathcal{L}'(9) = -g_4 f_\pi (\bar{B}_3 B_3 - 2\bar{\Lambda}_1 \Lambda_1) + \cdots,
\]

\[
\mathcal{L}'(9/9) = -g_5 f_\pi \bar{B}_1 B_3 - g_5 f_\pi \bar{\Lambda}_1 \Lambda_2 + \cdots.
\]
We note that $B_1$ and $B_3$ couple with each other through the naive combinations: $m_{(9)\overline{N}_{(9m)}\gamma N_{(9)}}$. Altogether we have

$$L = -f_\pi (B_1, B_3, B_2) \left( \begin{array}{ccc} g_1 & g_5 & g_3 \\ g_5 & g_4 & 0 \\ g_3 & 0 & g_2 \end{array} \right) \left( \begin{array}{c} B_1 \\ B_3 \\ B_2 \end{array} \right) - f_\pi (\overline{\Lambda}_1, \overline{\Lambda}_2) \left( \begin{array}{c} -2g_1 \\ g_5 \\ -2g_4 \end{array} \right) \left( \begin{array}{c} \Lambda_1 \\ \Lambda_2 \end{array} \right) + 2g_2 f_\pi \Delta^i \Delta^i.$$  

(31)

Let us now diagonalize the mass matrix and express the mixing angle in terms of diagonalized masses. We use the three nucleon candidates $N(940)$, $N(1440)$ and $N^*(1535)$ as well as the two mixing angles $\theta = 50.7^\circ$ and $\varphi = 66.1^\circ$, and finally find that there are two possibilities as shown in Table IV [34]. The odd-parity $\Delta$ option appears as the better one. Now, the first flavor-singlet $\Lambda$ lies at 1370 MeV, substantially closer to 1405 MeV; flavor-singlet $\Lambda$ lies at 1850 MeV, very close to the (three star PDG [69]) $P_{01}(1810)$ resonance. This is our best candidate in the $[(6, 3) \oplus (3, 6)] - [(3, 3) \oplus (3, 3)] - [(3, 3) \oplus (3, 3)]$ mixing scenario.

### Table IV: The values of the $\Delta$ and $\Lambda$ baryon masses predicted from the isovector axial coupling $g_A^{(1)} = g_A^{(1) \text{expt.}} = 1.267$ and $g_A^{(0) \text{mix.}} = 0.33 \pm 0.08$ due to $[(6, 3) \oplus (3, 6)] - [(3, 3) \oplus (3, 3)] - [(3, 3) \oplus (3, 3)]$ mixing.

| No. | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $\Lambda_1^+ \text{ (MeV)}$ | $\Lambda_2^+ \text{ (MeV)}$ | $\Delta^\nu \text{ (MeV)}$ |
|-----|-------|-------|-------|-------|-------|-----------------|-----------------|-----------------|
| 1   | -4.7  | 8.4   | -3.4  | 2.9   | 9.8   | 1370$^-$        | 1850$^+$        | 2170$^-$        |
| 2   | -7.2  | 4.6   | 7.9   | 9.1   | -4.2  | 1940$^+$        | 2430$^-$        | 1200$^-$        |

A comment about the comparatively high value of the $\Delta$ mass is in order: In the mid-1960-s Hara [31] noticed that the chiral transformation rules for a $(1, \frac{1}{2})$ multiplet impose a strict and seemingly improbable mass relation among its two members: $m_\Delta = 2m_N$. The mixing with the $(\frac{1}{2}, 0)$ multiplet only makes things worse, i.e., it makes the $\Delta$ even heavier. For this reason, the lowest-lying spin-1/2 $\Delta$ resonance cannot be a chiral partner of the lowest-lying nucleon $N(940)$, whereas, $\Delta(2150)$ seems to be a viable candidate for the $N(940)$’s chiral partner. Of course, $\Delta(2150)$ may contain components of (i.e. mix with) other high-lying resonances that do not significantly mix with $N(940)$.

### D. Chiral symmetry breaking

1. **Chiral symmetry breaking: bare quark masses**

In QCD one expects $\mathcal{H}_{\chi SB}$ to be determined solely by the current quark masses $m_{u}^0, m_{d}^0, m_{s}^0$ (modulo EM effects), i.e., $\mathcal{H}_{\chi SB} = \mathcal{H}_{\chi SB}^0$:

$$\mathcal{H}_{\chi SB}^0 = m_u^0 \bar{u}u + m_d^0 \bar{d}d + m_s^0 \bar{s}s + \sum_{a=0,3,8} n_a \bar{q} \gamma^a q,$$

(32)
The model is easily extended to broken SU(3) symmetry case: the explicit χSB “bare” nucleon mass and the corresponding χSB Hamiltonian density are

\[ H_{\chi SB}^{N(full)} = \sum_{i,j=1}^{3} \bar{N}_i M_{N,i}^0 N_j + \bar{Δ}_{(1,12)}^{N} M_{Δ(1,12)}^0 \bar{Δ}_{(1,12)}, \]

where \( i \) stands for the three chiral multiplets \((1, \frac{1}{2})\), \((\frac{1}{2}, 0)\) and \((0, \frac{1}{2})\). The off-diagonal components \((i \neq j)\) needs not be zero, but we find that only the diagonal components \((i = j)\) contribute to the \(Σ\) term, which will be calculated in the next section, Sec. [IV]. Hence, we only pay attention to the diagonal Hamiltonian

\[ H_{\chi SB} = \sum_{i=1}^{3} \bar{N}_i M_{N,i}^0 N_i + \bar{Δ}_{(1,12)}^{N} M_{Δ(1,12)}^0 \bar{Δ}_{(1,12)} \cdot \]

A priori, we do not know the values of the “current” nucleon masses, except for a lower limit - they cannot be smaller than three isospin-averaged current quark masses: \(M_{N,i}^0 \geq \frac{3}{2} (m_u^0 + m_d^0) \simeq 23 \text{ MeV} \) [71], or 14 MeV [72].

To see how this bound comes about, note that the isospin-averaged “bare” nucleon mass term,

\[ H_{\chi SB}(0) = M_{N,NN}^0, \quad (33) \]

where \(M_{N,NN}^0 = \frac{1}{2} (M_p^0 + M_n^0)\), can be readily expressed in terms of the current quark mass term, Eq. (32), with \(M_{N,NN}^0 = 3 \bar{m}_q^0 = \frac{3}{2} (m_u^0 + m_d^0)\).

It seems clear that the same “current” (or bare) nucleon mass \(M_{N,NN}^0\) ought to hold for any of the three chiral multiplets \((1, \frac{1}{2})\), \((\frac{1}{2}, 0)\) and \((0, \frac{1}{2})\), so long as they all correspond to three-quark interpolating fields. Of course, the same chiral multiplets may arise as five-quark interpolators, in which case their bare mass ought to be \(\frac{5}{2} (m_u^0 + m_d^0)\), i.e., larger than the above value \(\frac{3}{2} (m_u^0 + m_d^0)\). That explains the inequality in \(M_{N,NN}^0 \geq 3 \bar{m}_q^0 = \frac{3}{2} (m_u^0 + m_d^0)\).

For simplicity’s sake, we shall assume, as a first approximation, that all three chiral components have the same “current” nucleon mass \(M_{N,NN}^0 = M_{N,NN}^0 (6,3) = m_{N,NN}^{(1,\frac{1}{2})} = m_{N,NN}^{(\frac{1}{2},0)} = M_{N,NN}^0 (3,3) = m_{N,NN}^{(\frac{1}{2},0)} = M_{N,NN}^0 (3,3) = m_{N,NN}^{(0,\frac{1}{2})} = \frac{3}{2} (m_u^0 + m_d^0)\). In principle, the nucleon bare mass value may differ from one chiral multiplet to another, albeit not much, e.g. in the three-flavor case it may contain different \(F\) and \(D\) components, due to different \(F\) and \(D\) structures of the chiral multiplets, see below. This difference may be important in the three-flavor extension(s) of the model, but not in the two-flavor case.

The model is easily extended to broken SU(3) symmetry case: the explicit χSB “bare” nucleon mass and the corresponding diagonal χSB Hamiltonian density are

\[ H_{\chi SB} = \sum_{i=1}^{3} B_i M_{B_i}^0 B_i + \bar{Δ}_{(6,3)} M_{Δ(6,3)}^0 \bar{Δ}_{(6,3)}, \quad (34) \]
where \( i \) stands for the three chiral multiplets \((6, 3), (3, 3)\) and \((3, \overline{3})\), and the nucleon-octet mass matrix \(M_{B_i}^0\) in the “physical” basis reads

\[
M_{B_i}^0 = m_3 (d_3 + f_3) - \frac{3}{2\sqrt{3}} d_8 (M_A - M_\Sigma) + 2 f_8 m_8 + \sqrt{6} m_0 U, \tag{35}
\]

\[
M_{B_i}^0 = \text{diagonal} \left( \begin{array}{c}
\frac{3(M_A - M_\Sigma)}{2} + m_d + 2m_u,
\frac{3M_A - M_\Sigma}{2} + 2m_d + m_u,
\frac{3M_A - M_\Sigma + 3m_d + 2m_s + m_u}{2},
\frac{3M_A - 3M_\Sigma + 2m_d + 8m_s + 2m_u}{2},
\end{array} \right) + \text{off–diagonal}. \tag{36}
\]

The off-diagonal term in Eq. \(36\), given by

\[
\text{off–diagonal} = \left( \frac{m_u - m_d}{2\sqrt{3}} \right) \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

determines the \( \Lambda - \Sigma_0 \) mixing and mass splitting. As we shall not concern ourselves with hyperons in this paper, this term is of no interest here.

**IV. THE PION-NUCLEON \(\Sigma_{\pi N}\) TERM**

**A. Introduction: \(\Sigma_{\pi N}\) at quark level**

The \(\Sigma\) operator, defined as the double commutator

\[
\Sigma = \frac{1}{3} \delta^{ab} [Q_5^a, [Q_5^b, \mathcal{H}_{\chi SB}(0)]] , \tag{37}
\]

was introduced by Dashen as a measure of explicit chiral \(SU_L(2) \times SU_R(2)\) symmetry breaking. It is sensitive to the flavor indices of the axial charges \(Q_5^a\) and the form of the \(SU_L(3) \times SU_R(3)\) chiral symmetry breaking Hamiltonian density \(\mathcal{H}_{\chi SB}\): Other choices of summed over indices \(a, b\) probe different parts of symmetry breaking Hamiltonian. Its nucleon matrix element is the pion-nucleon \(\Sigma_{\pi N}\) term

\[
\Sigma_{\pi N} = \frac{1}{3} \delta^{ab} \langle N | [Q_5^a, [Q_5^b, \mathcal{H}_{\chi SB}(0)]] | N \rangle , \tag{38}
\]

which is of importance for the determination of the flavour content, in particular of the \(s\bar{s}\) content of the nucleon \(14\). In QCD one expects \(\mathcal{H}_{\chi SB}\) to be determined solely by the current quark masses \(m_u^0, m_d^0, m_s^0\) (modulo EM effects) Eq. \(33\). Then, the axial charges \(Q_5^a\) are then also constructed from the quark fields:

\[
Q_5^a = \int dx q^i(x) \gamma_5 \frac{1}{2} \lambda^a q(x) , \tag{39}
\]
which leads, after some basic algebraic manipulations, to

$$\Sigma_{\pi N} = \frac{m_u^0 + m_d^0}{2} \langle N|\bar{u}u + \bar{d}d|N\rangle + m_s^0 \langle N|\bar{s}s|N\rangle,$$  

(40)

and thus the value of the \(\pi N \Sigma_{\pi N}\) term that is given by the sum of the current quark masses in the nucleon. (This is reflected in a nonzero “bare”, or “current” nucleon mass on the hadronic level.) Assuming the nucleon contains no, or little strange quark component, i.e., \(\langle N|\bar{s}s|N\rangle \sim 0\), one has

$$\Sigma_{\pi N} = \frac{m_u^0 + m_d^0}{2} \langle N|\bar{u}u + \bar{d}d|N\rangle.$$  

(41)

The matrix element \(\langle N|\bar{u}u + \bar{d}d|N\rangle\) counts the number of \(u\) and \(d\) quarks and/or antiquarks in the nucleon, so that \(\Sigma_{\pi N} \sim \frac{3}{2}(m_u^0 + m_d^0) = 3\bar{m}^0\varepsilon 23\) MeV (with the current quark mass estimates that were valid at the time in PDG1998 \[72\]); these have dropped in the meantime significantly down to roughly \(3\bar{m}^0 \sim 14\) MeV in PDG2012 \[72\]). At the same time the nucleon mass shift due to the \(SU(3)\)-breaking Hamiltonian was evaluated at the baryonic level, assuming the contribution of strange quark to be zero, as

$$\Sigma_{\pi N} = \frac{3\bar{m}^0}{m_u^0 - m_d^0} (M_\Xi - M_\Lambda) \simeq 26\ \text{MeV},$$  

(42)

where \(M_\Xi, M_\Lambda\) are the hyperon ground state masses. As these two essentially independent estimates yielded basically one and the same number, any deviation of \(\Sigma_{\pi N}\) from the value of 25 MeV seemed to indicate some \(\bar{s}s\) content in the nucleon (this agreement between these two methods has disappeared with time, however: with the PDG2012 \[72\] values one finds \(\Sigma_{\pi N} = \frac{3\bar{m}^0}{m_u^0 - m_d^0}(M_\Xi - M_\Lambda) \simeq 22\) MeV vs. \(\Sigma_{\pi N} \sim \frac{3}{2}(m_u^0 + m_d^0) \simeq 13\) MeV). But, all estimates of \(\Sigma_{\pi N}\) from the \(\pi N\) scattering data yielded substantially larger values, ranging from 55 MeV to 80 MeV \[72,80\]. Consequently, the importance of the \(\Sigma_{\pi N}\) term cannot be exaggerated for the \(\bar{s}s\) content of the nucleon. These arguments go back to 1976 \[1\], and have, by now, found their way into textbooks on particle physics \[14,15\].

In the meantime there has been a large number of attempts at a theoretical explanation, most of which rely on the enlarged \(\bar{s}s\) content of the nucleon. More recently a number of lattice calculations with (almost physical) pions have also reached an enlarged value of \(\Sigma_{\pi N}\) \[81–84\]. But, there have also been many experimental searches for the \(\bar{s}s\) contributions to the nucleon observables, none of which produced a significant result (meaning larger than 1% of the \(\bar{u}u\) and \(\bar{d}d\) contributions; otherwise they are compatible with isospin violating corrections) thus making \(\bar{s}s\) effectively negligible \[11,13\]. Thus the enigma deepens: how is it possible to have such a large \(\Sigma\) term without \(\bar{s}s\) content?

**B. \(\Sigma_{\pi N}\) at the baryon level**

The results obtained at the quark level are not always the same as those obtained at the hadronic level, however. The purpose of this study is to lay bare the dependence of the \(\Sigma_{\pi N}\) term on the mixing of chiral multiplets, i.e., on the isovector axial coupling \(g_A^{(3)}\), and the flavor-singlet axial coupling \(g_A^{(0)}\).

The axial current coupling constants of the baryon flavor octet are well known \[58\]. The zeroth (time-like) components of these axial currents are generators of the \(SU_L(3) \times SU_R(3)\) chiral symmetry of QCD. The general flavor \(SU_F(3)\) symmetric form of the nucleon axial current contains two free parameters, called the \(F\) and \(D\) couplings, that are empirically determined as \(F=0.459 \pm 0.008\) and \(D=0.798 \pm 0.008\) \[58\]. The nucleon also has a flavor singlet axial coupling \(g_A^{(0)}\), that has been estimated from spin-polarized lepton-nucleon DIS data as \(g_A^{(0)} = 0.28 \pm 0.16\) \[2\], or more recently as \(0.33 \pm 0.03 \pm 0.05\) \[4,16\], subject to certain assumptions about hyperon decays (the axial \(F\) and \(D\) values).

In the chiral mixing approach the value \(g_A \neq 1\) is achieved naturally by way of mixing different chiral multiplets, without derivative couplings. We shall display the \(\Sigma_{\pi N}\) term’s dependence on \(g_A^{(0)}\) and show
that a large value of $\Sigma_{\pi N}$ is easily obtained even with the present day (significantly smaller) current quark masses and with a vanishing $s\bar{s}$ component of the nucleon.

First, we discuss the nucleon $\Sigma_{\pi N}$ term as obtained from the $\Sigma$ double commutator. We adopt different chiral symmetry breaking $[\chi_{SB}]$ terms, in accord with Refs. [45, 46]. Then we evaluate the $\Sigma_{\pi N}$ term in this approach.

1. Chiral $SU_L(3) \times SU_R(3)$ Commutators

We note that the matrix calculation $[Q^a_{5}, N]$ is equivalent (up to a multiplicative factor) to the $SU(3)_A$ chiral transformation which we have found in our previous papers: Eqs. (11) and (13) in Ref. [33], lead to

$$[Q^{a}_{5}, N_{(6,3)}] = \gamma_5 \left( \left( D^{a}_{(8)} + \frac{2}{3} P^{a}_{(8)} \right) N_{(6,3)} + \frac{2}{\sqrt{3}} T^{a}_{(8/10)} \Delta_{(6,3)} \right),$$

$$[Q^{a}_{5}, \Delta_{(6,3)}] = \gamma_5 \left( \frac{2}{\sqrt{3}} T^{a}_{(8/10)} N_{(6,3)} + \frac{1}{3} F^{a}_{(10)} \Delta_{(6,3)} \right),$$

$$[Q^{a}_{5}, N_{(3,3)}] = \gamma_5 D^{a}_{(3)} N_{(3,3)},$$

$$[Q^{a}_{5}, \Delta_{(3,3)}] = -\gamma_5 D^{a}_{(3)} N_{(3,3)}.$$

These $SU(3)$-spurion matrices $T^{a}$ (sometimes we use $T^{a}_{(10/8)}$) and $F^{a}_{(10)}$ have the following properties

$$F^{a}_{(10)} = -i f^{abc} T^{b}_{10/8} T^{c}_{10/8},$$

$$T^{a}_{10/8} T^{a}_{10/8} = \frac{5}{2} \times 1_{8 \times 8},$$

$$T^{a}_{10/8} T^{a}_{10/8} = 2 \times 1_{10 \times 10}.$$ (45)

The octet generators $\left( D^{a}_{(8)} + \frac{2}{3} P^{a}_{(8)} \right)$, the transition matrices $T^{a}_{(10/8)}$ and the decuplet generators $F^{a}_{(10)}$ are listed in Appendices A1., A.2 and A.3, respectively, of Ref. [33].

2. Chiral $SU_L(2) \times SU_R(2)$ Commutators

For the chiral $SU_L(2) \times SU_R(2)$ subgroup, i.e. for $a = 1, 2, 3$, this leads to

$$[Q^{a}_{5}, N_{(1, \frac{1}{2})}] = \gamma_5 \left( \frac{5}{3} \frac{r^{a}}{2} N_{(1, \frac{1}{2})} + \frac{2}{\sqrt{3}} T^{a}_{(3/2)} \Delta_{(1, \frac{1}{2})} \right),$$

$$[Q^{a}_{5}, \Delta_{(1, \frac{1}{2})}] = \gamma_5 \left( \frac{2}{\sqrt{3}} T^{a}_{(3/2)} N_{(1, \frac{1}{2})} + \frac{1}{3} F^{a}_{(3/2)} \Delta_{(1, \frac{1}{2})} \right),$$

$$[Q^{a}_{5}, N_{(0, \frac{1}{2})}] = \gamma_5 \frac{r^{a}}{2} N_{(0, \frac{1}{2})},$$

$$[Q^{a}_{5}, \Delta_{(0, \frac{1}{2})}] = -\gamma_5 \frac{r^{a}}{2} N_{(0, \frac{1}{2})},$$

where $a = 1, 2, 3, t^{i}_{(\frac{3}{2})}$ are the isospin-$\frac{3}{2}$ generators of the SU(2) group and $T^{i}$ are the so-called iso-spurion ($4 \times 2$) matrices, see Appendix B of Ref. [27].
Consequently,

\[
[Q_5^a, N_{(1, \frac{1}{2})} N_{(1, \frac{1}{2})}] = \frac{5}{3} \tilde{N}_{(1, \frac{1}{2})} \gamma^a \gamma^5 N_{(1, \frac{1}{2})}
\]

\[
+ \frac{2}{\sqrt{3}} \left( \tilde{N}_{(1, \frac{1}{2})} \gamma^a T^a \Delta_{(1, \frac{1}{2})} + \Delta_{(1, \frac{1}{2})} \gamma^5 T^a \tilde{N}_{(1, \frac{1}{2})} \right),
\]  

\[
(47)
\]

\[
[Q_5^a, \tilde{N}_{(\frac{1}{2}, 0)} N_{(\frac{1}{2}, 0)}] = \tilde{N}_{(\frac{1}{2}, 0)} \gamma^a N_{(\frac{1}{2}, 0)},
\]

\[
[Q_5^a, \tilde{N}_{(0, \frac{1}{2})} N_{(0, \frac{1}{2})}] = -\tilde{N}_{(0, \frac{1}{2})} \gamma^a N_{(0, \frac{1}{2})},
\]

and similarly for the \(\Delta\)-field commutator

\[
[Q_5^a, \tilde{\Delta}_{(1, \frac{1}{2})} \Delta_{(1, \frac{1}{2})}] = \frac{2}{3} \tilde{\Delta}_{(1, \frac{1}{2})} \gamma^a \gamma^5 t^a_{(3/2)} T^a \tilde{\Delta}_{(1, \frac{1}{2})}
\]

\[
+ \frac{2}{\sqrt{3}} \left( \tilde{\Delta}_{(1, \frac{1}{2})} \gamma^a T^a \Delta_{(1, \frac{1}{2})} + \tilde{\Delta}_{(1, \frac{1}{2})} \gamma^5 T^a \tilde{N}_{(1, \frac{1}{2})} \right).
\]  

\[
(48)
\]

3. Chiral \(SU_L(2) \times SU_R(2)\) Double Commutators

Here we have

\[
[Q_5^b, [Q_5^a, N_{(1, \frac{1}{2})} N_{(1, \frac{1}{2})}]] = \left( \frac{5}{3} \right)^2 \delta^{ab} \tilde{N}_{(1, \frac{1}{2})} N_{(1, \frac{1}{2})}
\]

\[
+ \left( \frac{5}{3} \right) \frac{1}{\sqrt{3}} \left( \tilde{N}_{(1, \frac{1}{2})} \gamma^a T^a \Delta_{(1, \frac{1}{2})} + \tilde{\Delta}_{(1, \frac{1}{2})} \gamma^5 T^a \tilde{N}_{(1, \frac{1}{2})} \right)
\]

\[
+ [Q_5^b, 2 \sqrt{3} \left( \tilde{N}_{(1, \frac{1}{2})} \gamma^a T^a \Delta_{(1, \frac{1}{2})} + \tilde{\Delta}_{(1, \frac{1}{2})} \gamma^5 T^a \tilde{N}_{(1, \frac{1}{2})} \right)]
\]

\[
= \left( \frac{25 + 16}{9} \right) \delta^{ab} \tilde{N}_{(1, \frac{1}{2})} N_{(1, \frac{1}{2})}
\]

\[
+ \left( \frac{4}{3} \right) \tilde{\Delta}_{(1, \frac{1}{2})} \left( \frac{3}{2} \delta^{ab} - \frac{1}{3} \left\{ t^a_{(3/2)}, t^b_{(3/2)} \right\} \right) \Delta_{(1, \frac{1}{2})} + \ldots,\]

\[
(49)
\]

where \(\ldots\) stand for the off-diagonal terms, such as \(\tilde{N}_{(1, \frac{1}{2})} \ldots \Delta_{(1, \frac{1}{2})}\), and their Hermitian conjugates. Similarly for the \(\Delta\)-field double commutator

\[
[Q_5^b, [Q_5^a, \tilde{\Delta}_{(1, \frac{1}{2})} \Delta_{(1, \frac{1}{2})}]] = \left( \frac{16}{9} \right) \delta^{ab} \tilde{N}_{(1, \frac{1}{2})} N_{(1, \frac{1}{2})} + 2 \delta^{ab} \tilde{\Delta}_{(1, \frac{1}{2})} \Delta_{(1, \frac{1}{2})}
\]

\[
- \left( \frac{2}{9} \right) \tilde{\Delta}_{(1, \frac{1}{2})} \left\{ t^a_{(3/2)}, t^b_{(3/2)} \right\} \Delta_{(1, \frac{1}{2})} + \ldots,\]

\[
(50)
\]

where \(\ldots\) again stand for the off-diagonal terms. The \(\left(\frac{1}{2}, 0\right)\) and \(\left(0, \frac{1}{2}\right)\) chiral multiplets double commutators are much simpler

\[
[Q_5^b, [Q_5^a, \tilde{N}_{(\frac{1}{2}, 0)} N_{(\frac{1}{2}, 0)}]] = \delta^{ab} \tilde{N}_{(\frac{1}{2}, 0)} N_{(\frac{1}{2}, 0)},\]

\[
(51)
\]

\[
[Q_5^b, [Q_5^a, \tilde{N}_{(0, \frac{1}{2})} N_{(0, \frac{1}{2})}]] = \delta^{ab} \tilde{N}_{(0, \frac{1}{2})} N_{(0, \frac{1}{2})}.
\]

\[
(52)
\]

Finally, we contract these Eqs. \([19], [50], [51], [52]\) with \(\frac{1}{3} \delta^{ab}\) (where summation over repeated indices is understood) to find:

\[
\frac{1}{3} \delta^{ab} [Q_5^b, [Q_5^a, N_{(1, \frac{1}{2})} N_{(1, \frac{1}{2})}]] = \left( \frac{41}{9} \right) \tilde{N}_{(1, \frac{1}{2})} N_{(1, \frac{1}{2})} + \left( \frac{8}{9} \right) \tilde{\Delta}_{(1, \frac{1}{2})} \Delta_{(1, \frac{1}{2})} + \ldots,\]

\[
(53)
\]

and similarly

\[
\frac{1}{3} \delta^{ab} [Q_5^b, [Q_5^a, \tilde{\Delta}_{(1, \frac{1}{2})} \Delta_{(1, \frac{1}{2})}]] = \left( \frac{16}{9} \right) \tilde{N}_{(1, \frac{1}{2})} N_{(1, \frac{1}{2})} + \left( \frac{13}{9} \right) \tilde{\Delta}_{(1, \frac{1}{2})} \Delta_{(1, \frac{1}{2})} + \ldots,\]

\[
(54)
\]
where we have used the identity \( t^a_{(3/2)} t^a_{(3/2)} = \frac{15}{4} \mathbf{1}_{4 \times 4} \).

\[
\frac{1}{3} \delta^{ab} [Q_5^b, [Q_5^a, \bar{N}_{(\frac{1}{2}, 0)} N_{(\frac{1}{2}, 0)}]] = \bar{N}_{(\frac{1}{2}, 0)} N_{(\frac{1}{2}, 0)},
\]  

\[
\frac{1}{3} \delta^{ab} [Q_5^b, [Q_5^a, \bar{N}_{(0, \frac{1}{2})} N_{(0, \frac{1}{2})}]] = \bar{N}_{(0, \frac{1}{2})} N_{(0, \frac{1}{2})}.
\]  

4. Chirally Mixed Double Commutators

As shown in our previous papers, the physical nucleon field contains several chiral multiplet components:

\[
|N\rangle = \sin \theta |(6, 3)\rangle + \cos \theta (\cos \varphi |(3, \bar{3})\rangle + \sin \varphi |(\bar{3}, 3)\rangle).
\]  

Use the identities

\[
N_{(\frac{1}{2}, \frac{1}{2})} |N(p)\rangle = N_{(\frac{1}{2}, \frac{1}{2})} (\sin \theta |(6, 3)\rangle + \cos \theta (\cos \varphi |(3, \bar{3})\rangle + \sin \varphi |(\bar{3}, 3)\rangle))
\]  

\[
= u(p)_{(\frac{1}{2}, \frac{1}{2})} \sin \theta,
\]  

\[
N_{(\frac{1}{2}, 0)} |N(p)\rangle = N_{(\frac{1}{2}, 0)} (\sin \theta |(6, 3)\rangle + \cos \theta (\cos \varphi |(3, \bar{3})\rangle + \sin \varphi |(\bar{3}, 3)\rangle))
\]  

\[
= u(p)_{(\frac{1}{2}, 0)} \cos \theta \cos \varphi,
\]  

\[
N_{(0, \frac{1}{2})} |N(p)\rangle = N_{(0, \frac{1}{2})} (\sin \theta |(6, 3)\rangle + \cos \theta (\cos \varphi |(3, \bar{3})\rangle + \sin \varphi |(\bar{3}, 3)\rangle))
\]  

\[
= u(p)_{(0, \frac{1}{2})} \cos \theta \sin \varphi,
\]  

and the Dirac conjugate

\[
\langle N(p) | \bar{N}_{(\frac{1}{2}, \frac{1}{2})} = (\sin \theta |(6, 3)\rangle + \cos \theta (\cos \varphi |(3, \bar{3})\rangle + \sin \varphi |(\bar{3}, 3)\rangle) \bar{N}_{(\frac{1}{2}, \frac{1}{2})}
\]  

\[
= \bar{u}(p)_{(\frac{1}{2}, \frac{1}{2})} \sin \theta,
\]  

\[
\langle N(p) | \bar{N}_{(\frac{1}{2}, 0)} = (\sin \theta |(6, 3)\rangle + \cos \theta (\cos \varphi |(3, \bar{3})\rangle + \sin \varphi |(\bar{3}, 3)\rangle) \bar{N}_{(\frac{1}{2}, 0)}
\]  

\[
= \bar{u}(p)_{(\frac{1}{2}, 0)} \cos \theta \cos \varphi,
\]  

\[
\langle N(p) | \bar{N}_{(0, \frac{1}{2})} = (\sin \theta |(6, 3)\rangle + \cos \theta (\cos \varphi |(3, \bar{3})\rangle + \sin \varphi |(\bar{3}, 3)\rangle) \bar{N}_{(0, \frac{1}{2})}
\]  

\[
= \bar{u}(p)_{(0, \frac{1}{2})} \cos \theta \sin \varphi,
\]  

which leads to

\[
\langle N(p) | \bar{N}_{(\frac{1}{2}, \frac{1}{2})} N_{(\frac{1}{2}, \frac{1}{2})} | N(p)\rangle = \bar{u}(p)_{(\frac{1}{2}, \frac{1}{2})} u(p)_{(\frac{1}{2}, \frac{1}{2})} \sin^2 \theta = \frac{E_p}{m} \sin^2 \theta,
\]  

\[
\langle N(p) | \bar{N}_{(\frac{1}{2}, 0)} N_{(\frac{1}{2}, 0)} | N(p)\rangle = \bar{u}(p)_{(\frac{1}{2}, 0)} u(p)_{(\frac{1}{2}, 0)} \cos^2 \theta \cos^2 \varphi = \frac{E_p}{m} \cos^2 \theta \cos^2 \varphi,
\]  

\[
\langle N(p) | \bar{N}_{(0, \frac{1}{2})} N_{(0, \frac{1}{2})} | N(p)\rangle = \bar{u}(p)_{(0, \frac{1}{2})} u(p)_{(0, \frac{1}{2})} \sin^2 \theta \sin^2 \varphi = \frac{E_p}{m} \sin^2 \theta \sin^2 \varphi,
\]  

which in the \( p \to 0 \) limit implies

\[
\lim_{p \to 0} \langle N(p) | \bar{N}_{(\frac{1}{2}, \frac{1}{2})} N_{(\frac{1}{2}, \frac{1}{2})} | N(p)\rangle = \sin^2 \theta,
\]  

\[
\lim_{p \to 0} \langle N(p) | \bar{N}_{(\frac{1}{2}, 0)} N_{(\frac{1}{2}, 0)} | N(p)\rangle = \cos^2 \theta \cos^2 \varphi,
\]  

\[
\lim_{p \to 0} \langle N(p) | \bar{N}_{(0, \frac{1}{2})} N_{(0, \frac{1}{2})} | N(p)\rangle = \cos^2 \theta \sin^2 \varphi.
\]  

Take the definition

\[
\Sigma_{\pi N} = \frac{1}{3} \delta^{ab} \langle N_\alpha | [Q_5^a, Q_5^b, H_{\chi SB}(0)] | N_\alpha \rangle,
\]
where \( \alpha \) is the chiral representation, and evaluate it using Eqs. \([53]\), \([54]\), \([55]\), \([56]\)

\[
\Sigma^{(1, \frac{1}{2})}_{\pi N} = \left( \frac{41}{9} m_N^{(1, \frac{1}{2})} + \frac{16}{9} m_\Delta^{(1, \frac{1}{2})} \right),
\]

\[
\Sigma^{(0, 0)}_{\pi N} = m_N^{(0, 0)},
\]

\[
\Sigma^{(0, \frac{1}{2})}_{\pi N} = m_N^{(0, \frac{1}{2})},
\]

thus find

\[
\Sigma_{\pi N} = \frac{1}{3} \delta_{\alpha b} \langle N||Q_a^b, [Q_b^b, H_{\chi SB}(0)]||N \rangle
\]

\[
= \sin^2 \theta \Sigma^{(1, \frac{1}{2})}_{\pi N} + \cos^2 \theta \left( \cos^2 \varphi \Sigma^{(0, 0)}_{\pi N} + \sin^2 \varphi \Sigma^{(0, \frac{1}{2})}_{\pi N} \right)
\]

\[
= \sin^2 \theta \left( \frac{41}{9} m_N^{(1, \frac{1}{2})} + \frac{16}{9} m_\Delta^{(1, \frac{1}{2})} \right) + \cos^2 \theta \left( \cos^2 \varphi m_N^{(0, 0)} + \sin^2 \varphi m_N^{(0, \frac{1}{2})} \right),
\]

which is our basic result here.

Assuming that all three chiral components have the same “current” nucleon mass \( M_N^{(0)} = M_{(6,3)}^{(6,3)} = m_N^{(1, \frac{1}{2})} = m_\Delta^{(1, \frac{1}{2})} = M_{(3,3)}^{(3,3)} = m_N^{(0, 0)} = m_N^{(0, \frac{1}{2})} \) one finds finally

\[
\Sigma_{\pi N} = \left( \frac{57}{9} \sin^2 \theta + \cos^2 \theta \right) M_N^{(0)}
\]

\[
= \left( 1 + \frac{16}{3} \sin^2 \theta \right) M_N^{(0)}.
\]

Note that the factor \( 1 + \frac{16}{3} \sin^2 \theta \) in front of the current nucleon mass is always larger than unity (for real values of the mixing angle \( \theta \)).

C. Comparison with experiment

As most “measurements” of \( \Sigma_{\pi N} \) have yielded values ranging from 55 MeV to 75 MeV \([95]\), that are substantially larger than the naively expected 25 MeV, it has consequently appeared that the \( s \bar{s} \) content of the nucleon must be (very) large.

The nucleon current mass is \( M_N^{(0)} = 3 m_q^0 = \frac{3}{2} (m_u + m_d) \simeq 14.4 \text{ MeV} \), i.e. \( \frac{1}{2} (m_u^0 + m_d^0) \simeq 4.79 \text{ MeV} \) in PDG2012 \([72]\). We note that here \( m_u^0 = 2.3 \times 1.35 \text{ MeV} \) and \( m_d^0 = 4.8 \times 1.35 \text{ MeV} \), where 1.35 is the rescaling factor due to the change of the energy scale from 2 GeV down to 1 GeV \([72]\), yielding \( \frac{1}{2} (m_u^0 + m_d^0) \simeq 4.79 \text{ MeV} \), substantially lower than 7.6 MeV in PDG1998 \([71]\).

The constraint on the mixing angle \( \theta \) by the experimental values of the axial couplings has been discussed in Sec. \([11]\) which gives \( \theta = 50.7^\circ \) and \( \varphi = 66.1^\circ \). Inserting these values into Eq. \([73]\), one finds \( \Sigma_{\pi N} = 60.3 \text{ MeV} \).

The \( \Sigma \) operator, Eq. \([77]\), is often identified with the chiral symmetry breaking (\( \chi \text{SB} \)) Hamiltonian itself. In Eq. \([77]\) the nucleon \( \Sigma \) term is a measure of the \( \chi \text{SB} \) in the nucleon. In such a case it equals the shift of the nucleon mass \( \delta M \) due to the \( \chi \text{SB} \) terms in the Hamiltonian. This reasoning underlies the standard interpretation of the nucleon \( \Sigma \) term as being a measure of the strangeness content of the nucleon Ref. \([3]\).

A large value of \( \Sigma_{\pi N} \), such as 65 MeV, has often been interpreted as a sign of a substantial \( s \bar{s} \) content of the nucleon. We have shown that in the chiral-mixing approx large values of \( \Sigma_{\pi N} \) can be obtained without any strangeness degrees of freedom in the nucleon as a natural consequence of the rather substantial chiral \([6, 3) \oplus (3, 6)] \) multiplet component in the nucleon field.
V. THE KAON-NUCLEON $\Sigma_{KN}$ TERM

A. SU(3) Quark level

To calculate the kaon-nucleon $\Sigma_{KN}$ term, we use the $\Sigma^{ab}$ operator defined as the double commutator

$$\Sigma^{ab} = [Q^a_5, [Q^b_5, H_{\chi_{SB}}]],$$

(74)

double commutator of the axial charges $Q^a_5$ and the chiral symmetry breaking Hamiltonian $H_{\chi_{SB}}$. It was introduced by Dashen [74] as a way of separating out the explicit chiral $SU_L(3) \times SU_R(3)$ symmetry breaking part $H_{\chi_{SB}}$ from the total Hamiltonian.

Its (diagonal) nucleon matrix element $\Sigma_{KN} = \frac{1}{4} \sum_{a=4}^{7} \langle N | \Sigma^{aa} | N \rangle$ is due to the (explicit) chiral symmetry breaking current quark masses [74, 75]. Then the kaon-nucleon $\Sigma$-terms are

$$\Sigma^{44} = \begin{pmatrix} 2 (m_s + m_u) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 (m_s + m_u) \end{pmatrix},$$

$$\Sigma^{55} = \begin{pmatrix} 2 (m_s + m_u) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 (m_s + m_u) \end{pmatrix},$$

$$\Sigma^{66} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 (m_d + m_s) & 0 \\ 0 & 0 & 2 (m_d + m_s) \end{pmatrix},$$

$$\Sigma^{77} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 (m_d + m_s) & 0 \\ 0 & 0 & 2 (m_d + m_s) \end{pmatrix}.$$ 

Summing them up and dividing by 4, we find

$$\Sigma_{KN} = \frac{1}{4} \sum_{a=4}^{7} \Sigma^{aa} = \left( \begin{array}{ccc} m_s + m_u & 0 & 0 \\ 0 & m_d + m_s & 0 \\ 0 & 0 & m_d + 2m_s + m_u \end{array} \right).$$

If we assume that $m_u = m_d$, then

$$\Sigma_{KN} = (m_s + m_{u/d}).$$

B. SU(3) Hadron level

Double commutator of the axial charges $Q^a_5$ for $a = 4, 5, 6, 7$ and the current/bare nucleon mass Hamiltonian $H_{\chi_{SB}}$ (also) gives the kaon $\Sigma$ term operator

$$\Sigma_K = \frac{1}{4} \sum_{a=4}^{7} [Q^a_5, [Q^a_5, H_{\chi_{SB}}]].$$

Kaon-nucleon $\Sigma_{KN}$ term - matrix element $\Sigma_{KN} = \frac{1}{4} \sum_{a=4,5,6,7} \langle N | [Q^a_5, [Q^a_5, H_{\chi_{SB}}(0)]] | N \rangle = \langle N | \Sigma_K | N \rangle$ - is also a chiral mixture:

$$\Sigma_{KN} = \sin^2 \theta \Sigma_{KN(6,3)} + \cos^2 \theta \left( \cos^2 \varphi \Sigma_{KN(\bar{3},3)} + \sin^2 \varphi \Sigma_{KN(\bar{3},3)} \right).$$
Thus, we need three double commutators:

1. The $(6,3)$ and $(3,6)$ chiral multiplets

\[
\frac{1}{4} \sum_{i=4}^{7} ([Q_5^i, [Q_5^i, \tilde{N}_{(6,3)}]N_{N_{(6,3)}]}]
= \frac{1}{4} \tilde{N}_{(6,3)} \left( \begin{array}{cc}
\frac{20}{9} m_u + \frac{41}{9} m_d + \frac{5}{3} m_s \\
0
\end{array} \right) \left( \begin{array}{cc}
\frac{41}{9} m_u + \frac{70}{9} m_d + \frac{5}{3} m_s
\end{array} \right) N_{N_{(6,3)}},
\]

\[
= \frac{1}{4} \tilde{N}_{(6,3)} \left( \begin{array}{cc}
\frac{20}{9} m_u + \frac{41}{9} m_d + \frac{5}{3} m_s \\
0
\end{array} \right) \left( \begin{array}{cc}
\frac{41}{9} m_u + \frac{70}{9} m_d + \frac{5}{3} m_s + \frac{4}{3} m_s
\end{array} \right) N_{N_{(6,3)}},
\]

thus leading to
\[
\Sigma_{KN_{(6,3)}} = \frac{1}{4} (10 m_u + 5 m_d + 3 m_s).
\]

2. The $(\bar{3},3)$ and $(3,\bar{3})$ chiral multiplets

\[
\frac{1}{4} \sum_{i=4}^{7} ([Q_5^i, [Q_5^i, \tilde{N}_{(3,3)}]M_{N_{N_{(3,3)}}]}]
= \frac{1}{4} \tilde{N}_{(3,3)} \left( \begin{array}{cc}
\frac{4}{9} m_u + \frac{14}{9} m_d + \frac{5}{3} m_s \\
0
\end{array} \right) \left( \begin{array}{cc}
\frac{14}{9} m_u + \frac{28}{9} m_d + \frac{5}{3} m_s
\end{array} \right) N_{N_{(3,3)}},
\]

\[
= \frac{1}{4} \tilde{N}_{(3,3)} \left( \begin{array}{cc}
\frac{4}{9} m_u + \frac{14}{9} m_d + \frac{5}{3} m_s \\
0
\end{array} \right) \left( \begin{array}{cc}
\frac{14}{9} m_u + \frac{28}{9} m_d + \frac{5}{3} m_s + \frac{4}{3} m_s
\end{array} \right) N_{N_{(3,3)}},
\]

thus leading to
\[
\Sigma_{KN_{(3,3)}} = \frac{1}{4} (10 m_u + 5 m_d + 3 m_s).
\]

3. The $(8,1)$ and $(1,8)$ chiral multiplets

\[
\frac{1}{4} \sum_{i=4}^{7} ([Q_5^i, [Q_5^i, \tilde{N}_{(8,1)}]M_{N_{N_{(8,1)}}]}]
= \frac{1}{4} \tilde{N}_{(8,1)} \left( \begin{array}{cc}
10 m_u + 5 m_d + 3 m_s \\
0
\end{array} \right) \left( \begin{array}{cc}
5 m_u + 10 m_d + 3 m_s
\end{array} \right) N_{N_{(8,1)}},
\]

thus leading to
\[
\Sigma_{KN_{(8,1)}} = \frac{1}{4} (10 m_u + 5 m_d + 3 m_s).
\]
C. Numerical results

One can see that the kaon-nucleon $\Sigma_{KN}$ terms are identical in these three chiral multiplets, $\Sigma_{KN(6,3)} = \Sigma_{KN(3,3)} = \Sigma_{KN(8,1)}$, so that the $\Sigma_{KN}$ term of their admixture also equals the same number:

$$
\Sigma_{KN} = \Sigma_{KN(6,3)} = \Sigma_{KN(3,3)} = \Sigma_{KN(8,1)} = \frac{1}{4} (10m_u + 5m_d + 3m_s).
$$

The 2012 edition of the Particle Data Group, Ref. [72] has $m_0^u = 2.3 \times 1.35 \text{ MeV}$ and $m_0^d = 4.8 \times 1.35 \text{ MeV}$, i.e., $\frac{1}{2} (m_0^u + m_0^d) = 4.79 \text{ MeV}$ and $m_0^s = (93.5 \pm 2.5) \times 1.35 = (126.225 \pm 3.375) \text{ MeV}$, yielding

$$
\Sigma_{KN} = 111 \text{ MeV}.
$$

Note that these values are substantially lower than before, see e.g. the PDG1998 values, Ref. [71].

One would like to compare this value with the “experimental” one. In this case, the status of “experimental” $\Sigma_{KN}$ is even worse than that of “experimental” $\Sigma_{\pi N}$: the kaon-nucleon scattering data are nowhere near of pion-nucleon ones in terms of overall quality, abundance, kinematic range, precision and accuracy.

Only some very old “experimental” estimates are available: 1) $\Sigma_{KN} \simeq 170 \text{ MeV}$ from 1970, Ref. [85], 2) $\Sigma_{KN} = -370 \pm 110 \text{ MeV}$ from 1972, Ref. [86], 3) $\Sigma_{KN} \simeq 170 \text{ MeV}$, from 1973, Ref. [87], 4) $\Sigma_{KN} = 540 \pm 160 \text{ MeV}$, from 1973, Ref. [88], 4) $\Sigma_{KN} = 246 \text{ MeV}$, from 1976, Ref. [89]. The most recent reviews, Refs. [3, 90] have calculated $\Sigma_{KN}$ at zero strangeness content $y_N = 0$ of the nucleon as being 170 MeV, but with the 1987 values of current quark masses. Their formulae translate to the value of 110 MeV with the 2012 values of masses, Ref. [72]. This is (very) close to our predicted value (111 MeV) of the same quantity.

VI. SUMMARY AND CONCLUSIONS

In this paper we have calculated the pion-nucleon $\Sigma_{\pi N}$ term in the chiral mixing approach, first with two light ($u,d$) flavors, and then we extended it to the case with three light ($u,d,s$) flavors, i.e., to $SU_L(3) \times SU_R(3)$ multiplet mixing, which we then used to calculate the kaon-nucleon sigma term $\Sigma_{KN}$. We based our calculations on the chiral mixing formalism and the phenomenology developed previously in Refs. [33, 34, 43, 44, 47].

The physical significance of our present work is that it shows that there is no need to introduce $s\bar{s}$ components in addition to the three-quark “core”, so as to agree with the observed values of the pion-nucleon $\Sigma$ term, the baryon axial couplings, and the nucleon magnetic moments: the phenomenologically necessary $[(6,3) \oplus (3,6)]$ chiral component and the $[(3,3) \oplus (3,3)]$ “mirror” component exist as bi-local three-quark fields, Ref. [39, 48]. Thus, we have shown that there is no need for “meson cloud”, or (non-exotic) “pentaquark” components in the Fock expansion of the baryon wave function, to explain (at least) the axial currents, magnetic moments and the pion-nucleon $\Sigma$ term, contrary to established opinion, Ref. [2]. This goes to show that the algebraic complexity of three Dirac quark fields is such that it can mimick the presence of $q\bar{q}$ pairs, at least in certain observables. For us this was a surprise.

The present formalism and phenomenology can be used to attack other outstanding issues of baryon chiral dynamics: the hyperon radiative decays, for example, have been a long-standing unsolved problem.
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Appendix A: Isospin-$\frac{3}{2}$ generators

From Ref. [27] we have

$$\delta^a_{\frac{5}{2}} \phi^6_{\frac{3}{2}} = i \gamma_5 a_3 \phi^6_{\frac{3}{2}},$$

$$\delta^a_{\frac{3}{2}} \left( \begin{array}{c} \phi^6_{\frac{3}{2}} \\ \phi^6_{\frac{3}{2}} \end{array} \right) = i \gamma_5 a_3 \left( \begin{array}{c} \frac{5}{3} \\ 1 \end{array} \right) \left( \begin{array}{c} \phi^6_{\frac{3}{2}} \\ \phi^6_{\frac{3}{2}} \end{array} \right),$$

(A1)

(A2)

with the familiar ("SU$_F$(6)" value $\frac{5}{3}$ for its "nucleon" component $N^\mu$. In order to read off the value of $g_A$, it is convenient to express this as

$$\delta^a_{\frac{5}{2}} \Delta^\mu = i \gamma_5 \left( \frac{1}{3} t^i_{(\frac{3}{2})} \cdot a \Delta^\mu + \frac{2}{\sqrt{3}} a \cdot T^i N^\mu \right),$$

(A3)

where $t^i_{(\frac{3}{2})}$ are the isospin-$\frac{3}{2}$ generators of the SU(2) group and $T^i$ are the so-called iso-spurion (4 × 2) matrices, see Appendix B of Ref. [27]. The $t^i_{(\frac{3}{2})}$ are defined as

$$t^1_{(\frac{3}{2})} = \left( \begin{array}{ccc} 0 & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 0 & 1 \\ 0 & 1 & 0 \end{array} \right),$$

$$t^2_{(\frac{3}{2})} = i \left( \begin{array}{ccc} 0 & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 0 & 1 \\ 0 & 1 & 0 \end{array} \right),$$

$$t^3_{(\frac{3}{2})} = \left( \begin{array}{ccc} \frac{3}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{3}{2} \end{array} \right),$$

(A4)

which leads to the conventional normalization of the SU(2) Casimir operator. The $T^i$ are defined by

$$T^1 = \left( \begin{array}{ccc} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} \end{array} \right),$$

$$T^2 = i \left( \begin{array}{ccc} -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{6}} \end{array} \right),$$

$$T^3 = \left( \begin{array}{ccc} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{array} \right),$$

(A5)
with the properties
\[ i t \frac{\partial}{\partial x} \cdot a = \frac{3}{2} T^{i \dagger} (i \tau \cdot a \delta^{jk}) T^k = -\frac{3}{2} T^{i \dagger} (\epsilon^{ijk} a^j) T^k, \]
\[ T^i T^{k \dagger} = P^{ik}. \]  
\[ \text{(A6)} \]

**Appendix B: Closure of the chiral SU(3) × SU(3) algebra**

The SU(3) vector charges \( Q^a = \int dx J^a_0 (t, x) \) defined as
\[ -2 b \cdot J_\mu = \sum_i \frac{\partial L}{\partial \partial \mu B_i} \delta^5 B_i, \]  
\[ \text{(B1)} \]

Together with the axial charges \( Q^a_5 = \int dx J^{a \mu}_5 (t, x) \) defined as
\[ -2 a \cdot J_{\mu 5} = \sum_i \frac{\partial L}{\partial \partial \mu B_i} \delta^5 B_i, \]  
\[ \text{(B2)} \]

ought to close the chiral algebra
\[ [Q^a, Q^b] = if^{abc} Q^c, \]  
\[ \text{(B3)} \]
\[ [Q^a_5, Q^b] = if^{abc} Q^c_5, \]  
\[ \text{(B4)} \]
\[ [Q^a_5, Q^b_5] = if^{abc} Q^c, \]  
\[ \text{(B5)} \]

where \( f^{abc} \) are the SU(3) structure constants. Eqs. (B3) and (B4) usually hold automatically, as a consequence of the canonical (anti)commutation relations between Dirac baryon fields \( B_i \), whereas Eq. (B5) is not trivial for the chiral multiplets that are different from the \([(8, 1) \oplus (1, 8)]\), because of the (nominally) fractional axial charges and the presence of the off-diagonal components. When taking a matrix element of Eq. (B5) by baryon states in a certain chiral representation, the axial charge mixes different flavor states within the same chiral representation. This is an algebraic version of the Adler-Weisburger sum rule \[91\]. In the following we shall check and confirm the validity of Eq. (B5) in the three multiplets of SU(3)_L × SU(3)_R.

1. **Closure of the Chiral SU(3)_L × SU(3)_R Algebra in the (8, 1) ⊕ (1, 8) Multiplet**

Due to the absence of fractional coefficients in the \((8, 1) \oplus (1, 8)\) multiplet’s axial charge \( Q^a_5 = \int dx J^{a \mu}_5 (t, x) \) defined by the current given in
\[ J^{a \mu}_5 = \overline{N} \gamma_\mu \gamma_5 F^{a (8)} N, \]  
\[ \text{(B6)} \]

the vector charge \( Q^a = \int dx J^a_0 (t, x) \) defined by the current given in
\[ J^a_\mu = \overline{N} \gamma_\mu F^{a (8)} N, \]  
\[ \text{(B7)} \]

and the axial charge close the chiral algebra defined by Eqs. (B3), (B4) and (B5). The same comments holds for the \((10, 1) \oplus (1, 10)\) chiral multiplet for the same reasons as in the example shown above.

2. **Closure of the Chiral SU(3)_L × SU(3)_R Algebra in the (3, 3) ⊕ (\overline{3}, 3) Multiplet**

The vector charge \( Q^a = \int dx J^a_0 (t, x) \) defined by the current given in
\[ J^a_\mu = \overline{N} \gamma_\mu F^{a (8)} N, \]  
\[ \text{(B8)} \]
together with the axial charge $Q_5 = \int dx J_{5\mu}^a(t, x)$ defined by the current given in
\[
J_{5\mu}^a = \nabla_\mu \gamma_5 \left( D^a N + \frac{2}{3} T_{1/8}^{ai} \Lambda_i \right) + \nabla_1 \gamma_5 \left( T_{1/8}^a N \right),
\]
(E9)
ought to close the chiral algebra defined by Eqs. (B3), (B4) and (B5). Eqs. (B3) and (B4) hold here, whereas Eq. (B5) is the non-trivial one: the diagonal $D$ charge of $N (Q_{5D}^a(N))$ axial charge,
\[
Q_{5D}^a(N) = \int dx \left( \bar{\gamma}_5 \gamma_5 D^a N \right),
\]
(E10)
whereas Eq. (B5) is once again the non-trivial one: the fractions $\Sigma_a$ and $\Delta$ axial charges, respectively, and the diagonal $D$ matrices
\[
\int dx \left( T_{1/8}^a T_{1/8}^{ai} \right) + \frac{2}{3} \left( T_{1/8}^a T_{1/8}^{ai} \right).
\]
(B15)
Therefore, the chiral algebra Eqs. (B3), (B4) and (B5) close.

3. Closure of the Chiral $SU_L(3) \times SU_R(3)$ Algebra in the $(3,6) \oplus (6,3)$ Multiplet

The vector charge $Q^a = \int dx J_{0\mu}^a(t, x)$ defined by the current in
\[
J_{0\mu}^a = \left( \nabla_\mu \gamma_5^a \left( F_{(8)}^a \right) N \right) + \left( \Sigma_\mu \gamma_5^a F_{(10)}^a \right) \Delta,
\]
(B16)
together with the axial charge $Q_5 = \int dx J_{5\mu}^a(t, x)$ defined by the current in
\[
J_{5\mu}^a = \nabla_\mu \gamma_5 \left( D^a + \frac{2}{3} F_{(8)}^a N + \frac{2}{3} T^a \Delta \right) + \nabla_1 \gamma_5 \left( \frac{2}{3} T^a N + \frac{1}{3} F_{(10)}^a \Delta \right),
\]
(B17)
ought to close the chiral algebra defined by Eqs. (B3), (B4) and (B5). Eqs. (B3) and (B4) hold here, whereas Eq. (B5) is once again the non-trivial one: the fractions $\Sigma_a$ and $\Delta$ in the diagonal $F$ charge of $N (Q_{5F}^a(N))$ and $\Delta$ axial charges, respectively, and the diagonal $D$ charge of $N (Q_{5D}^a(N))$:
\[
Q_{5F}^a(N) = \frac{2}{3} \int dx \left( \bar{\gamma}_5 \gamma_5^a F_{(8)}^a N \right),
\]
(B18)
\[
Q_{5F}^a(\Delta) = \frac{1}{3} \int dx \left( \bar{\Sigma} \gamma_5^a F_{(10)}^a \Delta \right),
\]
(B19)
\[
Q_{5D}^a(N) = \int dx \left( \bar{\gamma}_5 \gamma_5^a D^a N \right),
\]
(B20)
lead to

\[
[Q^a_{5D} (N), Q^b_{5D} (N)] = \int dx \left( \overline{N} \gamma_0 \left( \left( D^a + \frac{2}{3} F^a_8 \right) \left( D^b + \frac{2}{3} F^b_8 \right) \right) - \left( D^b + \frac{2}{3} F^b_8 \right) \left( D^a + \frac{2}{3} F^a_8 \right) \right) N , \tag{B21}
\]

\[
[Q^a_{5F} (\Delta), Q^b_{5F} (\Delta)] = i f^{abc} \frac{1}{9} Q^c (\Delta) , \tag{B22}
\]

lead to “only” part of the \( N \) and \( \Delta \) vector charges, respectively, on the right-hand-side of Eq. (B22) and (B22).

Once again, it turns out that the off-diagonal terms in the axial charge

\[
Q^a_5 (N, \Delta) = \int dx \left( \frac{2}{\sqrt{3}} \left( \overline{N} \gamma_0 \gamma_5 T^a_{10/8} \Delta + \overline{\Delta} \gamma_0 \gamma_5 T^a_{1/8} N \right) \right) , \tag{B23}
\]

play a crucial role in the closure of the chiral algebra Eq. (B23). The additional terms in the commutator add up to

\[
\frac{1}{3} \int dx \left( \overline{N} \gamma_0 \left( T^a_{10/8} T^b_{1/8} - T^b_{10/8} T^a_{1/8} \right) N + \overline{N} \gamma_0 \left( T^a_{10/8} T^b_{10/8} - T^b_{10/8} T^a_{10/8} \right) \Delta \right) , \tag{B24}
\]

which provide the “missing” factors due to the following properties of the off-diagonal flavor operators \( T^i \) and \( D^i \) matrices

\[
i f^{ijk} (F^i_8) = \left( \left( D^i + \frac{2}{3} F^i_8 \right) \left( D^j + \frac{2}{3} F^j_8 \right) - \left( D^j + \frac{2}{3} F^j_8 \right) \left( D^i + \frac{2}{3} F^i_8 \right) \right) + \frac{4}{3} \left( T^i_{10/8} T^j_{10/8} - T^j_{10/8} T^i_{10/8} \right) , \tag{B25}
\]

\[
i \frac{2}{3} f^{ijk} (F^k_{10}) = \left( T^i_{10/8} T^j_{10/8} - T^j_{10/8} T^i_{10/8} \right) . \tag{B26}
\]

Therefore, the chiral algebra Eqs. (B23), (B24) and (B25) closes in spite, or perhaps because of the apparent fractional axial charges \((\frac{2}{3} \) and \(\frac{1}{3})\).

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This BNL extended linear sigma model allows one to study the dependence of the pN scattering lengths, \(a_{\pi N}\), and of the nucleon sigma term \(\Sigma_{\pi N}\). It is well known that \(a_{\pi N}^{(+)\,s}\) depends crucially on the value of \(g_A^{(+)\,s}\), whereas the \(\Sigma_{\pi N}\) dependence on \(g_A^{(+)\,s}\) was not known (and may be model-dependent). We displayed this dependence of \(a_{\pi N}^{(+)\,s}\) and showed that a large value of \(\Sigma_{\pi N}\) could easily be obtained without recourse to any \(s\bar{s}\) component of the nucleon with the values of bare parameters available at the time (which have changed drastically in the meantime, however). We also reproduced the then new, tiny experimental value of the isoscalar pN scattering length \(a_{\pi N}^{(+)\,s}\).

Chiral symmetry does not require irreducible representations for parity-conserving interactions.

That aspect of the problem could be safely neglected in the collinear approach, which allows arbitrary values of the orbital angular momentum \(L\) and restricts only its z-projection \(L_z\).

Phenomenologically, the sigma term is related to the pN scattering amplitude at a certain non-physical kinematical point, (at the so-called Cheng-Dashen point \(t = +2m_{\pi}^2\)). Its extracted value is typically in the range \(\Sigma_{\pi N}^{CD} = 70-90\) MeV. After the corrections for the finite value of \(t\) are taken into account, which roughly amount to -15 MeV, one obtains \(\Sigma_{\pi N} = 55-75\) MeV.

For normalization and notational conventions see Ref. [21].