On singular elliptic equations involving critical Sobolev exponent

K Tahri
Preparatory School in Economics, Business and Management Sciences
Department of Mathematics. B.P 1085 Bouhannak, Tlemcen. Algeria.
E-mail: tahri_kamel@yahoo.fr

Abstract. Given a $n$–dimensional compact Riemannian manifold $(M, g)$ with $n \geq 5$, we consider the following semi-linear elliptic equation:

$$P_g(u) := \Delta_g^2 u + \text{div}_g (a(x) \nabla_g u) + b(x) u = f(x) |u|^{N-2} u + \lambda h(x) |u|^{q-2} u$$

where the functions $a$, $b$ and $h$ are in suitable Lebesgue spaces, $2 < q < N$ and $\lambda > 0$ a real parameter, $f$ is a smooth positive function and the operator $P_g$ is coercive. Under some additional conditions, we obtain results concerning the existence of strong solutions of the above equation in $H^2_0(M)$.

1. Introduction

In 1983 Paneitz discovered a particular conformally fourth-order operator defined on 4–dimensional smooth Riemannian manifolds [1]. In 1987, Branson generalized the definition to higher dimensions in [2] as follows. Let $(M, g)$ be smooth, compact $n$–dimensional Riemannian manifold with $n \geq 5$, and $u \in C^4(M)$. The Paneitz-Branson operator $P^n_g$ is then defined via [2]:

$$P^n_g(u) = \Delta_g^2 u - \text{div}_g (a_n(x) du) + Q^n_g u$$

where

$$a_n(x) = \frac{(n-2)^2 + 4}{2(n-2)(n-1)} S_g g - \frac{4}{n-2} \text{Ric}_g,$$

$$Q^n_g = \frac{1}{(n-1)(n-4)} \Delta_g S_g + \frac{n^3 - 4n^2 + 16n - 16}{4(n-1)^2(n-2)^2(n-4)} S_g^2 - \frac{4}{(n-4)(n-2)^2} \text{[Ric}_g]_2^2,$$

being $\Delta_g$, $S_g$ and $\text{Ric}_g$ the Laplace-Beltrami operator, the scalar and the Ricci curvatures of $g$, respectively. The Paneitz-Branson operator enjoys interesting conformal properties that are very similar to those of the conformal Laplacian operator. Remark that if $\tilde{g} = \varphi^{\frac{4}{n-2}} g$ , with $\varphi$ a positive function of class $C^4(M)$, is a conformal metric to $g$, then for all $u \in C^4(M)$,

$$P^n_{\tilde{g}}(u \varphi) = \varphi^{\frac{n+4}{n-4}} P^n_g(u).$$

In particular, if $u \equiv 1$ then $P^n_g(\varphi) = Q^n_{\tilde{g}} \varphi^{\frac{n+4}{n-4}}$.

Many interesting results on Paneitz-Branson operator and related topics have been recently obtained by several authors, we refer the reader to Refs. [3]–[10]. Here we recall a few of these...
results that are pertinent to our investigation, see the list P1)-P3) below.

Let \((M, g)\) be an \(n\)-dimensional compact, smooth and oriented Riemannian manifold with \(n \geq 5\), \(H^2_n(M)\) be the standard Sobolev space consisting of function in \(L^2(M)\) whose derivatives up to the second order are in \(L^2(M)\), and let \(N = \frac{2n}{n-4}\) be the associated Sobolev critical exponent. Now, we define the best constant \(K_o\) of the embedding \(H^2_n(\mathbb{R}^n) \subset L^N(\mathbb{R}^n)\) given by

\[
\frac{1}{K_o} = \frac{n(n^2 - 4)(n-4)\omega_n^4}{16}
\]

where \(\omega_n\) is the volume of the unit Euclidean \(n\)-sphere \((S^n, h)\).

P1) In 2002, F. Robert and P. Esposito in [10] considered the following equation

\[
\Delta^2_0 u + \text{div}_g (a(x)\nabla g u) + b(x)u = f(x)|u|^{N-2} u + h(x)|u|^{q-2} u
\]

where: i) \(a \in \Lambda^{1,\infty}_{(2,0)}(M)\) is a smooth symmetric \((2,0)\)-tensor field, ii) \(b, h, f\) are smooth functions in \(M\), with \(f\) positive, and iii) \(2 < q < N\). They established the following remarkable result:

**Theorem 1** Let \((M, g)\) be an \(n\)-dimensional compact Einsteinian manifold with \(n \geq 8\). Assume that \(P^m_n\) is coercive and let \(f \in C^\infty(M)\), \(f > 0\) such that there exists \(x_0 \in M\) with \(f(x_0) = \max_{x \in M} f(x), \Delta f(x_0) = 0\) and

\[
\frac{4(n^2 - 4n - 4)}{3(n+2)}|Wcg_f(x_0)|^2 + (n-6)(n-8)\frac{\Delta^2_0 f(x_0)}{f(x_0)} + 2(n-6)(n-8)\frac{(\nabla g f(x_0), Ric_g(x_0))}{f(x_0)} > 0.
\]

Then, there exists \(\tilde{g}\) conformal to \(g\) such that \(Q^2_{\tilde{g}}(x) = f(x)\).

P2) In 2010, M. Benalili in [5] considered the equation:

\[
\Delta^2_0 u + \text{div}_g (a(x)\nabla g u) + b(x)u = f(x)|u|^{N-2} u
\]

where \(f\) is a positive \(C^\infty\)-function on \(M\), \(a \in L^r(M)\) and \(b \in L^s(M)\), with \(r > \frac{n}{2}, s > \frac{n}{4}\). He established the following result:

**Theorem 2** Let \((M, g)\) is an \(n\)-dimensional compact manifold with \(n \geq 8\) and for \(2 < p < 5, \frac{3}{2} < s < 11\) or \(n = 7, \frac{7}{2} < p < 9, \frac{7}{4} < s < 9\) assume that there exists \(x_0 \in M\) such that \(f(x_0) = \max_{x \in M} f(x)\) and

\[
\frac{n^2 + 4n - 20}{6(n-6)(n^2 - 4)}S_g(x_0) + \frac{(n-4)}{2n(n-2)}\frac{\Delta_g f(x_0)}{f(x_0)} > 0.
\]

For \(n = 6, \frac{3}{2} < p < 2, 3 < s < 4\), assume that \(S_g(x_0) > 0\). Then, the equation (1) has a weak solution in \(H^2_n(M)\).

P3) Recently, M. Benalili and the author proved in [7], the following result:

**Theorem 3** Let \((M, g)\) be a compact manifold of dimension \(n \geq 6, a \in L^r(M), b \in L^s(M)\), with \(r > \frac{n}{2}, s > \frac{n}{4}\), \(0 < q < 2\) and \(f\) a positive \(C^\infty\)-function on \(M\). We suppose that \(P_g\) is coercive and the existence of a point \(x_0 \in M\) such \(f(x_0) = \max_{x \in M} f(x)\) and

\[
\begin{align*}
\frac{\Delta_g f(x_0)}{f(x_0)} < & \frac{1}{3} \left( \frac{(n-1)n(n^2 + 4n - 20)}{(n-6)(n-4)(n^2 - 4)} (1 + \|a\|_r + \|b\|_s)^{\frac{4}{n}} - \right) S_g(x_0) \quad &\text{if } n > 6 \\
S_g(x_0) > & 0 \quad &\text{if } n = 6
\end{align*}
\]
Throughout this section, we consider the energy functional $J$. Theorem 5 Let

$$\| \lambda \| \leq h \left( \frac{|N-2|}{2} \right) \Lambda^2 \left( \max \left( \left(1+\varepsilon \right) K_\alpha, A \right) \right)^{-\frac{1}{2}} \| h \|_{\alpha}^{-1}.$$

Our main results state as follows:

**Theorem 4** Let $(M, g)$ be an $n-$dimensional compact, smooth and oriented Riemannian manifold with $n > 6$ and $f$ a smooth positive function on $M$. Let $a \in L^r(M)$, $b \in L^s(M)$ and $h \in L^d(M)$, with $r > \frac{n}{2}$, $s > \frac{n}{4}$, $d > \frac{N}{N-q}$ and $2 < q < N$. We assume that the conditions $(h^1)$, $(h^2)$ and $(h^3)$ are satisfied and that there exists $x_0 \in M$ such that $f(x_0) = \max_{x \in M} f(x)$ and

$$\left( \frac{n(n-2\sqrt{6}+2)(n+2\sqrt{6}+2)-(n-6)(n-4)^3(n+2)}{3(n+2)(n-4)^2(n-6)(1+\|a\|_r+\|b\|_s)^\frac{4}{3}} S_g(x_0) - \left( \frac{n-4}{2} \right) \Delta f(x_0) \right) > 0.$$

Then, the equation (2) possesses a nontrivial solution in $H^2_{2}(M)$.

**Theorem 5** Let $(M, g)$ be a compact, smooth and oriented Riemannian manifold of dimension $n = 6$ under the same conditions of theorem 4 with

$$S_g(x_0) > 0$$

Then, the equation (2) possesses a nontrivial solution in $H^2_{2}(M)$.

### 2. Generic existence result

Throughout this section, we consider the energy functional $J_{\lambda}$, for each $u \in H^2_{2}(M)$,

$$J_{\lambda}(u) = \frac{1}{2} \int_M \left( \left( \Delta g \right) u^2 - a(x) |\nabla g u|^2 + b(x) u^2 \right) dv(g) - \frac{\lambda}{q} \int_M h(x) |u|^q dv(g) - \frac{1}{N} \int_M f(x) |u|^N dv(g)$$

First, we have the following lemma, whose proof is easy and can be found in [7].

**Lemma 6** $\| u \| = \left( \int_M \left( \left( \Delta g \right) u^2 - a(x) |\nabla g u|^2 + b(x) u^2 \right) dv(g) \right)^{\frac{1}{2}}$ is an equivalent norm of the usual one of $H^2_{2}(M)$ if only if the operator $P_g$ is coercive.
Proof. We follow closely the method used in [7].

Let \( J \in C^1(E, \mathbb{R}) \) where \( (E, \| \cdot \|) \) is a Banach space. We assume that:

(i) \( J(0) = 0 \).

(ii) \( \exists r, R > 0 \) such that \( J(u) \geq R > 0 \) for all \( u \in E \) such that \( \| u \| = r \).

(iii) \( \exists v \in E \) such that \( \lim \sup_{t \to +\infty} J(tv) < 0 \).

If

\[
 c = \min_{\eta \in \Gamma} \max_{t \in [0,1]} J(\eta(t))
\]

where \( \Gamma = \{ \eta \in C^1([0,1]; E) : \eta(0) = 0, \eta(1) = v \} \)

then there exists a sequence \((u_n)_n\) in \( E \) such that:

\[
 J(u_n) \to c \quad \text{and} \quad \nabla J(u_n) \to 0 \quad \text{in } E^*
\]

where \( E^* \) is the dual space of \( E \). Moreover, we have that: \( c \leq \sup_{t \geq 0} J(tv) \).

It is easily seen that \( J_\lambda \) is a \( C^1 \) functional and its Fréchet derivative is given by:

\[
 \langle \nabla J_\lambda(u), v \rangle = \int_M \left( \Delta_g u \Delta_g v - a(x)g(\nabla_g u, \nabla_g v) + b(x)uv \right) dv(g) + \nonumber \\
 -\lambda \int_M h(x) |u|^{q-2} uv dv(g) - \int_M f(x) |u|^{N-2} uv dv(g).
\]

Moreover, the functional \( J_\lambda \) verifies the Mountain-Pass conditions, namely:

**Lemma 8** Suppose that the conditions of (h\(^1\)), (h\(^2\)) and (h\(^3\)) of section 1 are satisfied. Then \( J_\lambda \) fulfills the following properties

1. There exist constants \( r, R > 0 \) such that \( J_\lambda(u) \geq R > 0 \), \( \| u \| = r \).

2. There exists \( v \in H^2_\lambda(M) \), with \( \| v \| > r \), such that \( J_\lambda(v) < 0 \).

**Lemma 9** Let \((M, g)\) be a \( n \)-dimensional compact, smooth and oriented Riemannian manifold with \( n \geq 5 \) and suppose that conditions (h\(^1\))-(h\(^2\)) are satisfied. Then each Palais-Smale sequence at level \( c_\lambda \) is bounded in \( H^2_\lambda(M) \).

**Proof.** The proof follows from the coerciveness of the operator \( P_\lambda \), the Sobolev’s inequality and the condition (h\(^2\)).

**Theorem 10** Let \((M, g)\) is an \( n \)-dimensional compact, smooth and oriented Riemannian manifold with \( n \geq 5 \). Let \((u_n)_m\) be a Palais-Šmalse sequence at level \( c_\lambda \). Assume that conditions (h\(^1\))-(h\(^2\)) and (h\(^3\)) are satisfied and that

\[
 c_\lambda < \frac{1}{(1 + \varepsilon)^{\frac{n}{2}}} \frac{n}{K_0} \max_{x \in M} f(x).
\]

Then, there is a subsequence of \((u_m)_m\) converging strongly in \( H^2_\lambda(M) \).

**Proof.** We follow closely the method used in [7].
3. The sharp case
Let \( P \in M \), we define the distance function \( \rho \) on \( M \) by
\[
\rho_P(Q) = \begin{cases} 
\delta(M) & \text{if } d(P, Q) \geq i_g(M) \\
\frac{d(P, Q)}{i_g(M)} & \text{if } d(P, Q) < i_g(M)
\end{cases}
\]
and \( i_g(M) \) is the injectivity radius of \( M \). Furthermore, we define the space \( L^p(M, \rho^\gamma) \) as follows.

**Definition 11** Let \((M, g)\) be a compact \( 5 \leq n \)-dimensional Riemannian manifold. We consider the space \( L^p(M, \rho^\gamma) \) where \( 1 \leq p \leq +\infty \) of measurable functions \( u \) on \( M \) such that \( \rho^\gamma |u|^p \) is integrable, i.e.
\[
\|u\|_{p, \rho^\gamma}^p := \int_M \rho^\gamma |u|^p \, dv < +\infty
\]

Now, we use the following Hardy-Sobolev inequalities proven in [5] (the Hardy-Sobolev inequalities for the singular Yamabe equation was proven in [9]).

**Theorem 12** [5] Let \((M, g)\) be a compact \( 5 \leq n \)-dimensional Riemannian manifold and \( p, q \) and \( \gamma \) three real numbers satisfying \( \frac{n}{q} = \frac{n}{p} - \frac{n}{2} - 2 \) and \( 2 \leq p \leq \frac{2n}{n-2} \).

For any \( \epsilon > 0 \), there is a constant \( A(\epsilon, q, \gamma) \) such that
\[
\forall u \in H^2(M) : \|u\|_{p, \rho^\gamma}^2 \leq (1 + \epsilon) K(n, 2, \gamma)^2 \|\Delta_g u\|_{L^2}^2 + A(\epsilon, q, \gamma) \|u\|_2^2
\]

In particular: \( K(n, 2, 0)^2 = K_0 \) is the optimal constant of Sobolev inequality.

**Theorem 13** [5] Let \((M, g)\) be a compact \( 5 \leq n \)-dimensional Riemannian manifold and \( p, q \) and \( \gamma \) three real numbers satisfying: \( 1 \leq q \leq p \leq \frac{n}{n-2} \) and \( \gamma < 0 \).

- If \( \frac{n}{q} = n \left( \frac{1}{q} - \frac{1}{p} \right) - 2 \), then the imbedding \( H^2(M) \subset L^p(M, \rho^\gamma) \) is continuous.
- If \( \frac{n}{q} > n \left( \frac{1}{q} - \frac{1}{p} \right) - 2 \), then the imbedding \( H^2(M) \subset L^p(M, \rho^\gamma) \) is compact.

We consider the following equation:
\[
\Delta^2 u + div_g \left( \frac{a(x)}{\rho^\sigma} \nabla_g u \right) + \frac{b(x)}{\rho^\mu} u = f(x) |u|^{N-2} u + \lambda \frac{h(x)}{\rho^\beta} |u|^{q-2} u \tag{3}
\]
where \( a, b \) and \( h \) are three smooth functions and the distance function defined before in section 1, \( 2 < q < N \) and \( \lambda > 0 \) a real parameter. The energy functional \( J_\lambda : H^2_2(M) \to \mathbb{R} \) associated to equation (3) is defined as:
\[
J_\lambda(u) = \frac{1}{2} \int_M \left( (\Delta_g u)^2 - \frac{a(x)}{\rho^\sigma} |\nabla_g u|^2 + \frac{b(x)}{\rho^\mu} u^2 \right) \, dv(g) + \frac{\lambda}{q} \int_M \frac{h(x)}{\rho^\beta} |u|^q \, dv(g) - \frac{1}{N} \int_M f(x) |u|^N \, dv(g),
\]
where \( u \in H^2_2(M) \) and it is well-known that the critical points of \( J_\lambda \) are the weak solutions of (3).

**Theorem 14** Let \( 0 < \sigma < \frac{n}{2} < 2, \ 0 < \mu < \frac{n}{2} < 4 \) and \( 0 < \beta < \frac{N}{\alpha} < N - q \). We suppose that the conditions \((h^1), (h^2)\) and \((b^3)\) are satisfied and
\[
\sup_{u \in H^2_2(M)} J^{\sigma, \mu, \beta}_{\sigma, \mu, \beta} (u) < \frac{2}{nK_0^{\frac{q}{2}}(f(x_0))^{\frac{2}{q}}}.
\]

Then, the equation (3) has a non trivial solution \( u_{\sigma, \mu, \beta} \in H^2_2(M) \).

**Proof.** The result follows in that if we put \( \tilde{a} = \frac{a(x)}{\rho^\sigma}, \tilde{b}(x) = \frac{b(x)}{\rho^\mu} \) and \( \tilde{h}(x) = \frac{h(x)}{\rho^\beta} \), then \( \tilde{a} \in L^r(M), \tilde{b} \in L^s(M) \) and \( \tilde{h} \in L^d(M), \) with \( r > \frac{n}{2}, s > \frac{n}{2} \) and \( d > \frac{N}{N-q} \).
4. Critical cases

Strategies developed in [7] and [8] enable us to derive another result, that refers to the critical cases when \( \sigma = 2, \mu = 4, \) and \( \beta = \frac{n(q-2)}{2} - 2q. \)

**Theorem 15** Let \( (M,g) \) be an \( n \)-dimensional compact, smooth and oriented Riemannian manifold with \( n \geq 5 \) and suppose that the conditions \( (h^1), (h^2) \) and \( (h^3) \) are satisfied. In addition, let \( (u_m)_m := (u_{\sigma, \mu, \beta})_m \) be a sequence in \( H^2_2(M) \) such that:

\[
\begin{cases}
J^\sigma_{\lambda, \mu, \beta}(u_m) \to c^\sigma_{\lambda, \mu, \beta} \\
\nabla J^\sigma_{\lambda, \mu, \beta}(u_m) \to 0
\end{cases}
\text{ for all } n \in \mathbb{N} \quad \text{weakly in } H^2_2(M)
\]

with \( c^\sigma_{\lambda, \mu, \beta} < \frac{2}{nK^2_0(f(x_0))^{\frac{n-2}{4}}} \) \( (4) \)

and

\[
1 + a^- \max(K(n,2,\sigma); A(\epsilon,\sigma)) + b^- \max(K(n,2,\mu); A(\epsilon,\mu)) > 0. \quad (5)
\]

Then, the equation

\[
\Delta^2 u + \text{div}_g\left(\frac{a(x)}{\rho^2} \nabla_g u\right) + \frac{b(x)}{\rho^2} u = f(x) |u|^{N-2} u + \lambda \frac{h(x)}{\rho^2} |u|^{q-2} u
\]

has a nontrivial solution \( u_{\sigma, \mu, \beta} \in H^2_2(M) \).

**Proof.** We follow closely the method used in [7] and [8]. First by using the condition \( (5) \) we obtain, as in [7], that the sequence \( (\Lambda_{\alpha, \mu})_{\alpha, \mu} \) of constants of coerciveness of the operator

\[
u \to \Delta^2 u + \text{div}_g\left(\frac{a(x)}{\rho^2} \nabla_g u\right) + \frac{b(x)}{\rho^2} u
\]

is bounded below by a constant \( \Lambda > 0 \) as \( (\alpha, \mu) \to (2^-,4^-) \).

Let \( (u_m)_m \subset H^2_2(M) \), such that :

\[
J^\sigma_{\lambda, \mu, \beta}(u_m) = c^\sigma_{\lambda, \mu, \beta} + o(1) \quad \text{and} \quad \nabla J^\sigma_{\lambda, \mu, \beta}(u_m) = o(1) \quad \text{in } (H^2_2(M))^*
\]

Then we have:

\[
J^\sigma_{\lambda, \mu, \beta}(u_m) - \frac{1}{N} \left(J^\sigma_{\lambda, \mu, \beta}(u_m), u_m\right) = \left(\frac{1}{2} - \frac{1}{N}\right) \|u_m\|^2 - \lambda \left(\frac{1}{q} - \frac{1}{N}\right) \int_M h(x) |u_m|^q dv(g)
\]

By Hölder and Sobolev inequalities, we get that

\[
J^\sigma_{\lambda, \mu, \beta}(u_m) - \frac{1}{N} \left(J^\sigma_{\lambda, \mu, \beta}(u_m), u_m\right) = c^\sigma_{\lambda, \mu, \beta} + o(1)
\]

and

\[
c^\sigma_{\lambda, \mu, \beta} + o(1) \geq \left(\frac{1}{2} - \frac{1}{N}\right) \|u_m\|^2 - \left(\frac{1}{q} - \frac{1}{N}\right) \left(\max((1 + \varepsilon)K_0, A_\varepsilon)\right)^\frac{q}{2} \|h\|_\alpha \|u_m\|^q_{H^2_2(M)}
\]

In addition the hypothesis \( (h^1) \) and \( (h^2) \) are satisfied and if we have \( \|u_m\| \geq 1 \), then we obtain

\[
\|u_m\| \leq \left[\left(\frac{N-2}{2} - \lambda \frac{N-q}{q} \Lambda^\frac{q}{2} (\max((1 + \varepsilon)K_0, A_\varepsilon)) \|h\|_\alpha\right)^{-\frac{1}{2}} \right]^{\frac{1}{q}} \text{ o}(1)
\]

Then \( (u_m)_m \) is bounded in \( H^2_2(M) \). The rest of the proof is the same as in Theorem 10. \( \square \)
Concluding remark. To prove main Theorems given in the Introduction, let \( \delta \in \left( \frac{i^*}2, 2 \right) \) and \( \eta \in C^\infty(M) \) such that:

\[
\eta(x) = \begin{cases} 
1 & \text{if } x \in B(x_0, \delta) \\
0 & \text{if } x \in M - B(x_0, 2\delta)
\end{cases}
\]

For \( \epsilon > 0 \), we define the radial function \( u_\epsilon \) by:

\[
u_\epsilon(x) := \frac{\eta(x)}{(\epsilon^2 + (\xi\rho)^2)^{\frac{n-4}{4}}} \quad \text{with} \quad \xi = (1 + \|a\|_r + \|b\|_s)\frac{1}{n} \tag{6}\]

We next point out that, by resorting to the strategy outlined in \[7, 8\], the function given by (6) can be proved to verify condition (4) of the generic theorem. This step completes our discussion on the solutions of Equation (2).

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