We introduce a random differential operator, that we call CS\(_\tau\) operator, whose spectrum is given by the Sch\(_\tau\) point process introduced by Kritchevski, Valkó and Virág (2012) and whose eigenvectors match with the description provided by Rifkind and Virág (2018). This operator acts on \(\mathbb{R}^2\)-valued functions from the interval \([0, 1]\) and takes the form:

\[
2 \begin{pmatrix} 0 & -\partial_t \\ \partial_t & 0 \end{pmatrix} + \sqrt{\tau} \begin{pmatrix} dB + \frac{1}{\sqrt{2}} dW_1 & \frac{1}{\sqrt{2}} dW_2 \\ \frac{1}{\sqrt{2}} dW_2 & dB - \frac{1}{\sqrt{2}} dW_1 \end{pmatrix},
\]

where \(dB, dW_1\) and \(dW_2\) are independent white noises. Then, we investigate the high part of the spectrum of the Anderson Hamiltonian \(H_L := -\partial^2_t + dB\) on the segment \([0, L]\) with white noise potential \(dB\), when \(L \to \infty\). We show that the operator \(H_L\), recentred around energy levels \(E \sim L/\tau\) and unitarily transformed, converges in law as \(L \to \infty\) to CS\(_\tau\) in an appropriate sense. This allows to answer a conjecture of Rifkind and Virág on the behavior of the eigenvectors of \(H_L\). Our approach also explains how such an operator arises in the limit of \(H_L\). Finally we show that at higher energy levels, the Anderson Hamiltonian matches (asymptotically in \(L\)) with the unperturbed Laplacian \(-\partial^2_t\). In a companion paper, it is shown that at energy levels much smaller than \(L\), the spectrum is localized with Poisson statistics: the present paper therefore identifies the delocalized phase of the Anderson Hamiltonian.

1. Introduction. The original motivation of the present article was to study the asymptotic behavior as \(L \to \infty\) of the high part of the spectrum of the Anderson Hamiltonian

\[
\mathcal{H}_L := -\partial^2_t + dB, \quad t \in (0, L),
\]

endowed with Dirichlet b.c., where \(B\) is standard Brownian motion. It is known \([7]\) that \(\mathcal{H}_L\) is a self-adjoint operator with discrete spectrum, bounded below and of multiplicity one. In the sequel we denote by \((\lambda_k)_{k \geq 1}\) the increasing sequence of eigenvalues of \(\mathcal{H}_L\) and by \((\varphi_k)_{k \geq 1}\) the associated sequence of eigenvectors normalized in \(L^2(0, L)\).

By high part, we mean eigenvalues of order \(L\) or more. More precisely, we aimed at understanding the local statistics of \(\mathcal{H}_L\) near some energy level \(E = E(L)\) that goes to \(\infty\) with at least linear speed in \(L\). This question falls in the topic of Anderson localization for Schrödinger operators: it is expected that, the higher \(E\) lies within the spectrum the less localized the eigenvectors are. In the companion papers \([3, 4]\), we showed that around energies \(E \ll L\), the local statistics of the eigenvalues converge to a Poisson point process and the eigenvectors are localized.

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In the present article, we will show that the rest of the spectrum is delocalized. However two distinct regimes will arise:

1. Critical: \( E \sim L/\tau \) for some \( \tau \in (0, \infty) \),
2. Top: \( E \gg L \).

The most interesting case will be the Critical regime, so we now focus on it. At the end of the introduction, we will present our results in the other regime. Let us point out that the present results do not imply any delocalization for the infinite volume operator since we are looking at a region of the spectrum that goes to \(+\infty\) in the limit \( L \to \infty \). Actually, in a forthcoming paper, we show that the operator in infinite volume is completely localized [5].

There is a formal analogy between our Critical regime, and the behavior of critical models of \( N \times N \) tridiagonal matrices of the form “discrete Laplacian + diagonal noise” as studied by Kritchevski, Valkó and Virág in [11]. It was shown therein that the spectrum of these models, after appropriate recentring around some energy level that depends on a parameter \( \tau \in (0, \infty) \), converges in law as \( N \to \infty \) to a random point process that they called \( \text{Sch}_\tau \). The denomination “critical” for the underlying matrix models comes from the following fact: the family of point processes \( \text{Sch}_\tau \) that arises in the limit interpolates between the Poisson point process (\( \tau = \infty \), see [1] for a proof for a related model) and the picket fence \( 2\pi \mathbb{Z} (\tau = 0) \).

The point process \( \text{Sch}_\tau \), introduced in [11], admits a nice characterization in terms of a coupled system of SDEs. Let \( B \) and \( W \) be independent real and complex \(^1\) Brownian motions, and consider

\[
\begin{align*}
\frac{\lambda}{2} d\Theta_{\lambda}(t) &= \frac{\sqrt{\tau}}{2} dB(t) + \frac{\sqrt{\tau}}{2} \Re(e^{2i\Theta_{\lambda}(t)} dW(t)), \quad t \in [0,1], \quad \lambda \in \mathbb{R}, \quad (1)
\end{align*}
\]

where \( \Theta_{\lambda}(0) := 0 \) (note that this family \( \Theta_{\lambda} \) depends on \( \tau \)). The point process \( \text{Sch}_\tau \) can be defined as

\[
\text{Sch}_\tau := \{ \lambda \in \mathbb{R} : \Theta_{\lambda}(1) \in \pi \mathbb{Z} \}.
\]

Our first result shows not only convergence of the local statistics of \( \mathcal{H}_L \) near \( E \sim L/\tau \) towards \( \text{Sch}_\tau \), but also the joint convergence of the eigenvalues and eigenvectors (rescaled into probability measures) to an explicit \( 2d \)-point process whose first component is \( \text{Sch}_\tau \). To state the result, let us introduce

\[
\begin{align*}
d\rho_{\lambda}(t) &= \frac{\tau}{8} dt + \frac{\sqrt{\tau}}{2\sqrt{2}} \Re(e^{2i\Theta_{\lambda}(t)} dW(t)), \quad t \in [0,1], \quad \lambda \in \mathbb{R}, \quad (3)
\end{align*}
\]

with \( \rho_{\lambda}(0) = 0 \). Let us also set \( \ell_E := \{ L\sqrt{E} \}_{\pi} \) where \( \{ x \}_{\pi} \) is the unique value in \( [0, \pi) \) such that \( x = \{ x \}_{\pi} \) modulo \( \pi \).

**Theorem 1.1.** Fix \( \tau > 0 \) and assume that \( E = E(L) \sim L/\tau \). As \( L \to \infty \), the random point process

\[
\left\{ \frac{2\ell_E (L)}{(L/\sqrt{E})(\lambda - E)} + 2\ell_E, L \varphi_{\lambda}^2(Lt) dt, \lambda \text{ is an eigenvalue of } \mathcal{H}_L \right\}
\]

\[^1\]By complex Brownian motion, we mean that the real and imaginary parts are independent standard Brownian motions.
converges in law to
\[ \left\{ (\lambda, \frac{e^{2\rho_\lambda(t)}dt}{\int_{[0,1]} e^{2\rho_\lambda(t)}dt}), \lambda \in \text{Sch}_\tau \right\}. \]

In this statement, point processes are seen as elements of the set of locally finite measures on \( \mathbb{R} \times \mathcal{P}([0,1]) \) endowed with the vague topology, that is, the topology that makes continuous \( \mu \mapsto \langle \mu, f \rangle \) for all bounded and continuous maps \( f : \mathbb{R} \times \mathcal{P}([0,1]) \to \mathbb{R} \) that are compactly supported (in their first coordinate, since the second lives in the compact set \( \mathcal{P}([0,1]) \)).

Let us mention that Holcomb in [8] studied the transition between the large eigenvalues of the Stochastic Bessel operator and the Sine\( \beta \) point process, which corresponds to the eigenvalue point process convergence part of Theorem 1.1. It might be extended to cover the eigenvectors as well.

At this point, two natural questions arise:

1. Does there exist a self-adjoint operator whose spectrum is given by \( \text{Sch}_\tau \) and whose eigenvectors are (related to) the processes \( e^{2\rho_\lambda} \)?
2. Does the above convergence hold at the level of operators?

This topic is strongly related to the article [20] of Valkó and Virág. They have constructed a general framework of random operators corresponding to the limit of several famous random matrices models and notably the bulk, soft and hard edges of \( \beta \)-ensembles. In particular, they have introduced an operator whose spectrum corresponds to the translation invariant version of \( \text{Sch}_\tau \). These operators are differential random operators which fall into the scope of the theory of Dirac operators of Weidmann [21] and, more generally, into the scope of the theory of canonical systems of De Branges [2].

In the present article, we positively answer to the two questions stated above. We construct an operator whose spectrum is given by \( \text{Sch}_\tau \), and whose eigenvectors are the limits of the \( \text{properly transformed} \) eigenvectors of \( \mathcal{H}_L \). The rigorous definition of our operator relies on the theory of Dirac operators, however the differential expression of our operator is different from the generic expressions of the operators in [20], and rather matches with the form conjectured by Edelman-Sutton [6]: we believe that our approach sheds some light on the class of operators that should appear as scaling limits of tridiagonal matrices or continuum random Schrödinger operators. We will make a precise connection between our operator and its “natural” counterpart in the framework [20] of Valkó and Virág, see Theorem 1.4.

The Critical Schrödinger operator \( \mathcal{CS}_\tau \). Let us start by introducing the limiting operator and its first properties. We (formally) define the following operator on \( L^2([0,1], \mathbb{R}^2) \), that we call \( \mathcal{CS}_\tau \) for Critical Schrödinger:

\[ (4) \quad \mathcal{CS}_\tau := \begin{pmatrix} 0 & -\partial_t \\ \partial_t & 0 \end{pmatrix} + \sqrt{\tau} \begin{pmatrix} dB + \frac{1}{\sqrt{2}}dW_1 & \frac{1}{\sqrt{2}}dW_2 \\ \frac{1}{\sqrt{2}}dW_2 & dB - \frac{1}{\sqrt{2}}dW_1 \end{pmatrix}, \]

where \( B, W_1, W_2 \) are independent Brownian motions. The precise definition will be given in Section 2, let us only mention that we endow this operator with Dirichlet b.c., that is, any function \( f \in L^2([0,1], \mathbb{R}^2) \) lying in the domain is such that \( f(0) \) and \( f(1) \) are parallel to \( (0,1)^T \), where we denote by \( M^T \) the transpose of any matrix \( M \). Let us emphasize that the operator \( \mathcal{CS}_\tau \) acts on a space of \( \mathbb{R}^2 \)-valued functions while our initial operator \( \mathcal{H}_L \) acts...
on \( \mathbb{R} \)-valued functions: the reason for the “enlargement” of the underlying space will appear below. Note also that \( CS_\tau \) is of the form
\[
2 \begin{pmatrix} 0 & -\partial_t \\ \partial_t & 0 \end{pmatrix} + \text{“noise matrix”},
\]
a form that was conjectured by Edelman-Sutton [6] for the limit of certain tridiagonal ensembles.

**Theorem 1.2.** The operator \( CS_\tau \) is self-adjoint with discrete spectrum. Its collection of eigenvalues and normalized eigenvectors coincides in law with
\[
\left\{ (\lambda, \Psi_\lambda) : \Theta_\lambda(1) \in \pi \mathbb{Z} \right\},
\]
where \( \Psi_\lambda := e^{\rho_\lambda} \| e^{\rho_\lambda} \|_{L^2(0,1)} \left( \frac{\sin \Theta_\lambda}{\cos \Theta_\lambda} \right) \), where \( \Theta_\lambda \) and \( \rho_\lambda \) follow the diffusions (1) and (3).

Before we address our second question on the convergence at the operator level, let us provide some identities on the law of this point process. First of all, we recover the intensity measure of \( \text{Sch}_\tau \) obtained in Theorem 10 of [11], but with a different method: for any Borel set \( A \subset \mathbb{R} \)
\[
E \left[ \# \{ \lambda \in \text{Sch}_\tau \cap A \} \right] = \int_{\lambda \in A} \sum_{n \in \mathbb{Z}} g_{\lambda,\frac{\tau}{2}} (2n\pi) d\lambda,
\]
where \( g_{\lambda,\sigma^2} \) is the density of the real gaussian law \( N(\lambda, \sigma^2) \). Note that the above density is not translation invariant: it is only \( 2\pi \)-periodic. As \( \tau \downarrow 0 \), the intensity measure converges to \( \sum_{n \in \mathbb{Z}} \delta_{2n\pi} \), the intensity measure of the so-called picket fence. On the other hand, as \( \tau \uparrow \infty \), it “converges” to an infinite uniform measure on \( \mathbb{R} \). This is another hint that \( \text{Sch}_\tau \) is critical, in the sense that it interpolates between the localized and delocalized phases of Schrödinger operators.

Second, we show that the norm of the eigenvectors \( \Psi_\lambda \) follow a universal shape: for any given Borel set \( A \subset \mathbb{R} \) and any non-negative measurable map \( G \) on \( C([0,1], \mathbb{R}^2) \) it holds
\[
\frac{1}{E \left[ \# \lambda \in \text{Sch}_\tau \cap A \right]} E \left[ \sum_{\lambda \in \text{Sch}_\tau \cap A} G(|\Psi_\lambda|) \right] = E \left[ G(S/\|S\|_{L^2([0,1])}) \right],
\]
where the \textit{shape} \( S \) is the exponential of a two-sided real Brownian motion \( B_1 \) recentered at \( U \), an independent uniformly distributed r.v. over \([0,1]\), plus a drift:
\[
S(t) := e^{\frac{\tau}{2\sqrt{2}}} B_1(t-U) - \frac{\tau}{\sqrt{2}} t(U-U), \quad t \in [0,1].
\]

It is remarkable that this shape does not depend on \( \lambda \).

Identities (5) and (6) follow from the computation of the intensity measure of the point process of eigenvalues / eigenvectors, that is, the measure on \( \mathbb{R} \times C([0,1], \mathbb{R}^2) \) defined by
\[
\mu_\tau(A) := E \left[ \sum_{\lambda : \Theta_\lambda(1) \in \pi \mathbb{Z}} 1_A(\lambda, \Psi_\lambda(\cdot)) \right].
\]

**Theorem 1.3.** For any non-negative measurable map \( G \) on \( \mathbb{R} \times C([0,1], \mathbb{R}^2) \) we have
\[
\int G(\lambda, \Psi) d\mu_\tau(\lambda, \Psi) = \int \sum_{n \in \mathbb{Z}} E \left[ G(\lambda, X^{(n)}) \right] g_{\lambda,\frac{\tau}{2}} (2n\pi) d\lambda,
\]
where \( X^{(n)}(t) = S(t) \left( \begin{array}{c} \sin \beta^{(n\pi)}(t) \\ \cos \beta^{(n\pi)}(t) \end{array} \right) \) for any \( t \in [0, 1] \) and \( \beta^{(n\pi)} \) is a scaled Brownian bridge between \((0, 0)\) and \((1, n\pi)\):

\[
\beta^{(n\pi)}(t) = \frac{1}{2} \sqrt{\frac{3\pi}{2}} \left( B_2(t) - tB_2(1) \right) + n\pi t, \quad t \in [0, 1],
\]

associated to a Brownian motion \( B_2 \) independent from \( S \).

Lastly, we relate \( CS_\tau \) with the operators introduced by Valkó and Virág in [20]. The natural candidate in their framework would be the hyperbolic carousel operator \( \text{Sch}_\tau^* \) that they introduced: however, the set of eigenvalues of this operator is a translation invariant version of the \( \text{Sch}_\tau \) point process. A slight modification of the definition of \( \text{Sch}_\tau^* \) allows to get its non-translation invariant counterpart, that we denote \( \text{Sch}_\tau \), see Subsection 2.3.

**Theorem 1.4.** The operators \( \text{Sch}_\tau \) and \( CS_\tau \) are unitarily equivalent. The unitary transformation that we provide is an explicit function of the Brownian motions \( W_1, W_2, B \).

It turns out that the operator \( \text{Sch}_\tau \) only relies on the Brownian motions \( W_1 \) and \( W_2 \). The additional Brownian motion \( B \) is involved in the unitary transformation that maps it onto our operator \( CS_\tau \).

**Convergence of the eigenvectors.** We now address our second question on the convergence at the operator level. Given the statement of Theorem 1.1, one naturally starts from the centered operator \((L/\sqrt{E})(H_L - E) + 2\ell E\).

For convergence purposes, one would like to deal with functions on \((0, 1)\) instead of \((0, L)\). Therefore we conjugate this operator with the rescaling map \( g \mapsto g(L \cdot) \) from \((0, L)\) to \((0, 1)\) and this yields the operator \( \mathcal{H}^{(E)} \) defined through

\[
\mathcal{H}^{(E)} f := \frac{1}{L\sqrt{E}} f'' + \sqrt{\frac{L}{E}} dB^{(L)} f + (2\ell E - L\sqrt{E}) f, \quad t \in [0, 1],
\]

where \( B^{(L)}(t) = L^{-1/2}B(tL) \) is again a standard Brownian motion. The eigenvalues and (normalized) eigenvectors \( (\lambda, \varphi^{(E)}_\lambda) \) of \( \mathcal{H}^{(E)} \) are in one-to-one correspondence with those of \( \mathcal{H}_L \) via

\[
\varphi^{(E)}_\lambda = \sqrt{L} \varphi_\mu(L \cdot), \quad \lambda = \frac{L}{\sqrt{E}} (\mu - E) + 2\ell E.
\]

Note that the domain of \( \mathcal{H}^{(E)} \) is a subset of \( L^2((0, 1), dt) \) which is nothing but the image of the domain of \( \mathcal{H}_L \) through the rescaling map.

Looking back at the statement of Theorem 1.1, we observe that the point process that appears in the limit is nothing but the following projection of the eigenvalues / eigenvectors of \( CS_\tau \)

\[
\left\{ (\lambda, |\Psi_\lambda|^2) : \Theta_\lambda(1) \in \pi\mathbb{Z} \right\}.
\]

\(^2\)The corresponding conjugation is not unitary, unless one defines \( \mathcal{H}^{(E)} \) on the Hilbert space \( L^2((0, 1), L dt) \).
A first naive guess would then be that the operator $\mathcal{H}^{(E)}$ converges to $\mathcal{C}_\tau$, and this would formally imply that the eigenvectors $\varphi^{(E)}_\lambda$ converge to the eigenvectors of $\mathcal{C}_\tau$. It turns out that in this regime of energy, the eigenvectors $\varphi^{(E)}_\lambda$ oscillate too much to converge as functions: indeed, from standard arguments of the theory of Sturm-Liouville operators, one can deduce that their numbers of zeros on $(0,1)$ is of order $L\sqrt{E}$.

One actually needs to remove these oscillations for the eigenvectors to converge. This can be done by considering the associated probability measures as we did in Theorem 1.1. However, in order to get convergence at the operator level, we need to remove these oscillations at the level of the functions. To do so, we successively apply two transformations:

**Step 1: From $\mathbb{R}$ to $\mathbb{R}^2$.** We consider the pair formed by the eigenvector and its derivative:

$$\left( \frac{\varphi_{\lambda}^{(E)}}{L\sqrt{E}}, \frac{(\varphi_{\lambda}^{(E)})'}{L\sqrt{E}} \right).$$

**Step 2: Unrotate.** Set $E' := L\sqrt{E} - \ell_E$ and introduce the (evolving) rotation matrix

$$Q_{E'} = Q_E(t) := \begin{pmatrix} \cos E't & -\sin E't \\ \sin E't & \cos E't \end{pmatrix}. \tag{9}$$

Then, we define

$$\Psi^{(E)}_{\lambda} := Q_{E'} \left( \frac{\varphi_{\lambda}^{(E)}}{L\sqrt{E}}, \frac{(\varphi_{\lambda}^{(E)})'}{L\sqrt{E}} \right). \tag{10}$$

**THEOREM 1.5 (Joint convergence of the eigenvalues and eigenvectors).** Fix $\tau > 0$ and consider $E = E(L) \sim L/\tau$. As $L \to \infty$, the point process on $\mathbb{R} \times C([0,1], \mathbb{R}^2)$:

$$\{ (\lambda, \psi^{(E)}_\lambda), \lambda \text{ eigenvalue of } \mathcal{H}^{(E)} \}$$

converges in law to the point process of eigenvalues/eigenvectors of $\mathcal{C}_\tau$, i.e.

$$\{ (\lambda, \Psi_{\lambda}), \lambda \text{ eigenvalue of } \mathcal{C}_\tau \}.$$

In this statement, the point processes are seen as elements of the set of measures on $\mathbb{R} \times C([0,1], \mathbb{R}^2)$ that are finite on $K \times C([0,1], \mathbb{R}^2)$ for any compact set $K \subset \mathbb{R}$, endowed with the smallest topology that makes continuous $\mu \mapsto \langle \mu, f \rangle$ for all bounded and continuous maps $f : \mathbb{R} \times C([0,1], \mathbb{R}^2) \to \mathbb{R}$ that are compactly supported in their first coordinate.

This result, combined with our description of the intensity measure of $\mathcal{C}_\tau$ given in Theorem 1.3, proves part (4) of [17, Conjecture 1.3] on the universal shape of a typical eigenvector associated to a “high eigenvalue” of $\mathcal{H}_L$. It is interesting to note that such a behavior can already be observed for the eigenvectors in the localized regime of $\mathcal{H}_L$ but for high enough energies (energies of order $1 \ll E \ll L$), see [4] for more details. It is conjectured in [17] that this shape should appear for various critical operators thus its denomination “universal”. It was also proved to arise in another random Schrödinger model recently, see [12].

---

3While $L\sqrt{E}$ is nothing but the order of magnitude of the oscillations of $\varphi_{\lambda}^{(E)}$, the correction $\ell_E$, which is of order 1, is more subtle: it is chosen in such a way that the rotation preserves the Dirichlet b.c.
Convergence at the operator level. Theorem 1.5 establishes a relationship between the eigenvalues/eigenvectors of $CS_\tau$ and those of $\mathcal{H}_L$. However it does not exactly answer our second question, and more importantly, it does not explain how the form (4) taken by $CS_\tau$ arises from $\mathcal{H}_L$. This is the purpose of our next result.

Given Theorem 1.5, our second question can be rephrased as follows: is there an operator associated to the point process $(\lambda, \Psi^{(E)}_\lambda)$ and does this operator converge to $CS_\tau$?

We will see later on that a.s. the space generated by the family of functions $\{\Psi^{(E)}_\lambda\}_\lambda$ is not dense in $L^2([0,1], \mathbb{R}^2)$, and therefore there is no self-adjoint operator on $L^2([0,1], \mathbb{R}^2)$ whose eigenvalues/eigenvectors are given by $\{\lambda, \Psi^{(E)}_\lambda\}_\lambda$. Consequently, “the” operator that we are looking for must live on a smaller space: this is not surprising since our original operator lives on $L^2([0,1], \mathbb{R})$.

We will construct in Subsection 3.1 an operator denoted $S^{(E)}_L$, which is unitarily equivalent to $\mathcal{H}^{(E)}$ and lives on a quotient space of $L^2([0,1], \mathbb{R}^2)$. The corresponding unitary map is the lift at the operator level of Steps 1 and 2 above. We will present at the end of the introduction how the form (4) taken by $CS_\tau$ arises from $\mathcal{H}^{(E)}$, see in particular Equation (13) below.

We now state our convergence result at the operator level: since there are some issues with the underlying spaces on which our operator acts, the precise statement will be given in Subsection 3.2.

**Theorem 1.6 (Strong-resolvent convergence).** Fix $\tau > 0$ and consider $E = E(L) \sim L/\tau$. As $L \to \infty$, the operator $S^{(E)}_L$ converges in law, in the strong resolvent sense, towards the operator $CS_\tau$. However it does not converge in law to $CS_\tau$ in the norm resolvent sense.

This answers our second question. Note that the strong resolvent convergence does not imply the convergence of the eigenvalues/eigenvectors, see e.g. [22]: in particular, Theorem 1.6 does not imply Theorem 1.5.

**Top of the spectrum.** Finally, let us examine the spectrum of $\mathcal{H}_L$ for energies $E$ that go to $\infty$ much faster than $L$. Note that now $L/E \to 0$ so that, formally, we are in the same situation as before but with $\tau = 0$. We keep the same definitions for $\mathcal{H}^{(E)}$, $\Psi^{(E)}_\lambda$ and $S^{(E)}_L$, in that case (see Subsection 3.1 for the precise definition of the latter). Our next result shows that all the previous results still hold but the limits are now deterministic: at first order, the influence of the white noise becomes negligible.

**Theorem 1.7.** Consider $E = E(L) \gg L$. As $L \to \infty$, the operator $S^{(E)}_L$ converges in probability, in the strong resolvent sense, towards the picket fence operator on $L^2$

$$F := 2 \begin{pmatrix} 0 & -\partial_t \\ \partial_t & 0 \end{pmatrix},$$

endowed with Dirichlet b.c. However the convergence does not hold in the norm resolvent sense.

Furthermore the point process:

$$\left\{ \left( \lambda, \frac{\Psi^{(E)}_\lambda}{\|\Psi^{(E)}_\lambda\|_{L^2([0,1], \mathbb{R}^2)}} \right) \right\}, \lambda \text{ eigenvalue of } \mathcal{H}^{(E)}$$
converges in probability to the eigenvalues/eigenvectors of $F$, which happen to be given by
\[
\left\{ \left( \lambda, \left( \begin{array}{c} \cos(\frac{\lambda}{2}) \\ \sin(\frac{\lambda}{2}) \end{array} \right) \right) \right\}, \lambda \in 2\pi\mathbb{Z}.
\]

**Remark 1.** This statement holds for the deterministic Laplacian operator $-\partial_t^2$ as long as $L\sqrt{E} \to \infty$ i.e. when the oscillations (total number of zeros) of the corresponding eigenfunctions go to infinity. The various behaviors that we observe for $H_L$ are due to the additional noise and are not present in the Laplacian.

**The operator $\mathcal{C}_\tau$ through SDEs.** The goal of this paragraph is to explain how we can guess the form of the operator $\mathcal{C}_\tau$ starting from the family of SDEs solved by the eigenvectors of $H^{(E)}$. Let us write the family of SDEs associated to the operator $H^{(E)}$:
\begin{equation}
\begin{aligned}
-\frac{1}{L\sqrt{E}}(u^{(E)}_\lambda)'' + \sqrt{\frac{L}{E}} u^{(E)}_\lambda dB^{(L)}(t) + (2\ell E - L\sqrt{E})u^{(E)}_\lambda(t) = \lambda u^{(E)}_\lambda(t), \quad t \in [0,1],
\end{aligned}
\end{equation}

with initial conditions $u^{(E)}_\lambda(0) = 0$ and $4(u^{(E)}_\lambda)'(0) = L\sqrt{E}$. By the Sturm-Liouville theory, the parameter $\lambda$ is an eigenvalue of $H^{(E)}$ if and only if $(u^{(E)}_\lambda(1), (u^{(E)}_\lambda)'(1))$ is parallel to $(0,1)$; and in that case, the associated normalized eigenvector $\varphi^{(E)}_\lambda$ is a multiple of $u^{(E)}_\lambda$.

Let us analyze how the SDE (11) is transformed through the two steps described above:

**Step 1: From $\mathbb{R}$ to $\mathbb{R}^2$.** Consider the matrix
\[
T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]
The above collection of SDEs (11) can be rewritten
\begin{equation}
\begin{aligned}
&\begin{pmatrix}
\sqrt{\frac{L}{E}} dB^{(L)}(t) + 2\ell E - L\sqrt{E} \\ -\partial_t
\end{pmatrix}
\begin{pmatrix}
u^{(E)}_\lambda \\ \frac{1}{L\sqrt{E}}(u^{(E)}_\lambda)' \end{pmatrix}
= \lambda T
\begin{pmatrix}
u^{(E)}_\lambda \\ \frac{1}{L\sqrt{E}}(u^{(E)}_\lambda)' \end{pmatrix}.
\end{aligned}
\end{equation}
The l.h.s. takes a form similar to (4): the main difference consists in the unbounded terms $L\sqrt{E}$ that still need to be “killed”, this will be the purpose of Step 2.

**Step 2: Unrotate.** We define
\[
y^{(E)}_\lambda := Q_{E'} \begin{pmatrix} u^{(E)}_\lambda \\ \frac{1}{L\sqrt{E}}(u^{(E)}_\lambda)' \end{pmatrix},
\]
where the rotating matrix $Q_{E'}$ was defined in (9). A simple computation shows that these processes solve
\begin{equation}
\begin{aligned}
&\begin{pmatrix}
0 -\partial_t \\ -\partial_t
\end{pmatrix}
\begin{pmatrix}
\sqrt{\frac{L}{E}} dB^{(L)} + \frac{1}{\sqrt{2}} dW^{(L)}_1 \\ \frac{1}{\sqrt{2}} dW^{(L)}_2
\end{pmatrix}
\begin{pmatrix}
u^{(E)}_\lambda \\ \frac{1}{\sqrt{2}}(u^{(E)}_\lambda)' \end{pmatrix}
+ \ell E(2R_{E'} - I)
\end{aligned}
\end{equation}

\[
= \lambda R_{E'} y^{(E)}_\lambda,
\]

\[4\] Any non-zero value for $(u^{(E)}_\lambda)'(0)$ would do, we choose $L\sqrt{E}$ for later convenience.
with initial condition $y_{X}^{(E)}(0) = (0, 1)^{\top}$. Here $I$ is the identity matrix, $R_{E'} := Q_{E'}TQ_{E'}^{T}$, and we have introduced the Brownian motions

$$(14) \quad W_{1}^{(L)}(t) := \sqrt{2} \int_{0}^{t} \cos(2E's)dB^{(L)}(s), \quad W_{2}^{(L)}(t) := \sqrt{2} \int_{0}^{t} \sin(2E's)dB^{(L)}(s).$$

Let us comment on Equation (13). First, the conjugation by the rotation matrix $Q_{E'}$ removed the unbounded terms $L\sqrt{E}$ from (12) so that all the terms appearing in this equation are bounded w.r.t. $L$. Moreover, we now see a clear resemblance between (4) and the l.h.s. of (13).

Interestingly, although we started from a single Brownian motion, the unbounded oscillations produce two additional, independent Brownian motions in the scaling limit: a phenomenon already observed by Kritchevski, Valkó and Virág in [11] for discrete 1-d Schrödinger operators and (at a larger scale) by Valkó and Virág in [19] for discrete Schrödinger operators on long boxes, that heuristically corresponds to dimension $1+\epsilon$, and that they called noise explosion in the latter paper (notice that in all cases, it leads to the delocalization of the spectrum). Second, by the Riemann-Lebesgue Lemma, the matrix $R_{E'}$ converges to $(1/2)I$ and thus the term whose prefactor is $\ell_{E}$ vanishes in the limit. Finally the prefactor 2 that appears in (4) is actually related to the r.h.s. of (13) where the term $R_{E'}$ converges towards $(1/2)I$.

The structure of the rest of the article is as follows. In Section 2, we properly define the Critical Schrödinger operator $CS_{\tau}$ using the theory of differential operators of Weidmann. Then we characterize its spectrum and compute its intensity measure, thus proving Theorems 1.2 and 1.3. We also link this operator to a Dirac operator of Valkó and Virág which proves Theorem 1.4. In Section 3, we provide the construction of the operator $S^{(E)}_{L}$ and then, we show the convergence at the operator level stated in Theorem 1.6. In Section 4, we exploit the systems of SDEs associated to the eigenvalues/eigenvectors of the operators at stake, and prove Theorems 1.1 and 1.5. We also prove a technical result stated in Section 3. Finally, in Section 5, we adapt the previous arguments in order to cover the top of the spectrum as in Theorem 1.7.

2. The Critical Schrödinger operator. The main objective of this section is to give a rigorous meaning to the operator $CS_{\tau}$ on $L^{2}([0, 1], \mathbb{R}^{2})$ formally defined by

$$CS_{\tau} := 2 \begin{pmatrix} 0 & -\partial_{t} \\ \partial_{t} & 0 \end{pmatrix} + \sqrt{\tau} \begin{pmatrix} dB + \frac{1}{\sqrt{2}}dW_{1} & \frac{1}{\sqrt{2}}dW_{2} \\ \frac{1}{\sqrt{2}}dW_{2} & dB - \frac{1}{\sqrt{2}}dW_{1} \end{pmatrix},$$

and endowed with Dirichlet b.c., that is, any $f$ in the domain is such that $f(0)$ and $f(1)$ are parallel to $(0, 1)^{\top}$. Here $B$, $W_{1}$ and $W_{2}$ are independent Brownian motions.

This expression is formal because the meaning of the product of the noise matrix with functions in the domain needs to be defined (this product cannot be interpreted in the Itô sense because elements in the domain are not adapted in general). To circumvent these issues, we extract from the formal expression above a condition on the eigenvalues/eigenvectors of the operator. More precisely, we start from the well-defined collection of SDEs

$$(15) \quad 2 \begin{pmatrix} 0 & -\partial_{t} \\ \partial_{t} & 0 \end{pmatrix} y_{\lambda} + \sqrt{\tau} \begin{pmatrix} dB + \frac{1}{\sqrt{2}}dW_{1} & \frac{1}{\sqrt{2}}dW_{2} \\ \frac{1}{\sqrt{2}}dW_{2} & dB - \frac{1}{\sqrt{2}}dW_{1} \end{pmatrix} y_{\lambda} = \lambda y_{\lambda}, \quad t \in [0, 1],$$
indexed by \( \lambda \in \mathbb{R} \), and initialized at \( y_\lambda(0) = (0, 1)^T \); and we aim at constructing an operator on \( L^2([0, 1], \mathbb{R}^2) \) whose eigenvalues / eigenvectors are precisely those pairs \( (\lambda, y_\lambda) \) for which \( y_\lambda(1) \) is parallel to \( (0, 1)^T \).

To carry out the construction, we extract from the collection of SDEs above a so-called Dirac operator, which, under some explicit conjugation, yields a self-adjoint operator that satisfies the desired property regarding its eigenvalues and eigenfunctions.

This section is organized as follows. The first subsection 2.1 recalls the basic material on Dirac operators following the book of Weidmann [21], and then presents a fairly general connection between systems of SDEs such as (15) (sometimes called Dirac equations) and Dirac operators. Subsection 2.2 then applies this material to \( \text{CS}_\tau \) and presents the proofs of Theorem 1.2 and 1.3. Finally, in subsection 2.3, we link the operator \( \text{CS}_\tau \) to a Dirac operator \( \text{Sch}_\tau \) of the same form as those appearing in [20] which proves Theorem 1.4.

Let us introduce some notations here. From now on, we write

\[
J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

Note the identity \( J^{-1} = -J \). Let us recall that we write \( M^T \) for the transpose of any matrix \( M \). Given two vectors \( u, v \in \mathbb{R}^2 \), we write \( u \parallel v \) to say that \( u \) is parallel to \( v \). Let \( S^+_2(\mathbb{R}) \) be the space of real-valued symmetric non-negative \( 2 \times 2 \) matrices and let \( R : [0, 1] \rightarrow S^+_2(\mathbb{R}) \) be a measurable function such that each entry of \( R \) is in \( L^1([0, 1], \mathbb{R}) \). The Hilbert space \( L^2_R([0, 1], \mathbb{R}^2) \), sometimes denoted \( L^2_R \) for short, is the Hilbert space of all measurable functions \( f : [0, 1] \rightarrow \mathbb{R}^2 \) such that

\[
\|f\|^2_{L^2_R} := \int_0^1 f^T R f < \infty.
\]

When \( R(t) \) is invertible for a.e. \( t \) (as it will be the case in this section), this is simply a twisted \( L^2([0, 1], \mathbb{R}^2) \) norm. In the complementary case (which will arise in Section 3), any element of \( L^2_R \) can be seen as an equivalence class of \( L^2([0, 1], \mathbb{R}^2) \) for the relation \( f \sim g \iff R f = R g \) almost everywhere.

2.1. From Dirac equations to Dirac operators.

2.1.1. Dirac operator. Suppose \( R \) is an integrable function from \([0, 1]\) into the space of \( 2 \times 2 \) positive symmetric matrices\(^5\) and introduce the differential operator:

\[
\mathcal{T} := R^{-1}(t) J \partial_t
\]

For any boundary conditions \( b_0, b_1 \in \mathbb{R}^2 \), let us introduce the domain\(^6\) of the Hilbert space \( L^2_R := L^2_R([0, 1], \mathbb{R}^2) \):

\[
\mathcal{D}_{b_0, b_1}(\mathcal{T}) := \left\{ f \in L^2_R : f \text{ A.C. on } (0, 1), R^{-1} J f' \in L^2_R, f(0) \parallel b_0, f(1) \parallel b_1 \right\}.
\]

It is proved in [21] that \( \mathcal{D}_{b_0, b_1}(\mathcal{T}) \) is dense in \( L^2_R \) and that the operator \( \mathcal{T} \) is a well-defined, self-adjoint operator on \( \mathcal{D}_{b_0, b_1}(\mathcal{T}) \).

\(^5\)To simplify the presentation, we have made several restrictive assumptions here. In the general theory, one works on an interval \([0, L]\) that can be unbounded and the matrix \( R \) is only assumed to be locally integrable on \([0, L]\).

\(^6\)The operator is limit circle at the two boundaries as \( R \) is integrable on the whole interval \([0, 1]\).
Note that \((\lambda, \phi)\) is an eigenvalue/eigenvector of \(T\) if and only if the solution \(v\) of
\[
(0 - \partial_t \begin{array}{c} 0 \\ \partial_t \end{array} )v(t) = z R(t)v(t), \quad t \in [0, 1], \quad z \in \mathbb{C},
\]
with \(z = \lambda\) satisfies the two b.c. and \(\phi\) is a multiple of \(v\).

**Remark 2.** If one parametrizes 
\[
R := \frac{X^TX}{\det X} \quad \text{for} \quad X = \begin{pmatrix} 1-x \\ 0 \\ 0 \\ y \end{pmatrix},
\]
with \(f := \det R\), the ratio \(q := v_1/v_2\) of the solution (19) satisfies:
\[
dq = \lambda f \frac{(q - x)^2 + y^2}{y} dt.
\]
Valkó and Virág give a nice geometric interpretation of this equation: the ratio \(q\) evolves as an hyperbolic carousel: at time \(t\), it rotates at speed \(\lambda f(t)\) around the point \(x(t) + iy(t)\) on the boundary of the upper half plane (it may explodes to \(+\infty\), in this case, it immediatly restarts at \(-\infty\)).

The resolvents of \(T\) admit explicit kernels [21, Th 7.8]. Fix a \(z \in \mathbb{C}\) that does not lie in the spectrum of \(T\). Let \(v_z\) be the solution of (19) that satisfies \(v_z(0) = b_0\). Let also \(\hat{v}_z\) be the solution of (19) that satisfies \(\hat{v}_z(1) \parallel b_1\) and \(v_z(0)^TJ\hat{v}_z(0) = 1\). Note that the last quantity is nothing but the Wronskian of \(v_z\) and \(\hat{v}_z\). Then for any \(f \in L^2_{\mathbb{R}}\) we have
\[
(T - z)^{-1} f(t) := \int_0^1 \begin{pmatrix} v_z(t) \hat{v}_z(s)^T \end{pmatrix} 1_{t \leq s} + \hat{v}_z(t)v_z(s)^T 1_{s < t} R(s)f(s).
\]
Our hypothesis imply that the functions \(v_z\) and \(\hat{v}_z\) are continuous on \([0, 1]\), and therefore bounded. This readily implies that the operator \((T - z)^{-1}\) is Hilbert-Schmidt so that \(T\) has discrete spectrum.

**Remark 3.** When \(R\) is non invertible, one can also define an operator associated to the so called canonical system
\[
0 - \partial_t \begin{array}{c} 0 \\ \partial_t \end{array} )v(t) = z R(t)v(t), \quad t \in [0, 1], \quad z \in \mathbb{C}.
\]
This is the theory of canonical systems introduced by De Branges in [2] (see e.g. [14, 15, 18] for reviews on the subject). Interestingly, Schrödinger operators can be transformed into a canonical system with a non invertible \(R\), see [15, 18, 20].

### 2.1.2. Dirac operator transformation associated to Dirac equations.

We would like to associate a self-adjoint operator to the following system of Dirac equations:
\[
J du_z(t) + dV(t) u_z(t) = zu_z(t)dt, \quad t \in [0, 1], \quad z \in \mathbb{C}.
\]
Here \(V\) is a \(2 \times 2\) “noise” matrix: its entries are Itô processes (they will be combinations of independent Brownian motions in the case of \(CS_\tau\)). Note that we have not set the initial condition yet.

**Remark 4.** Here we understand \(dV(t) u_z(t)\) in the Itô sense. It turns out that one can construct on a given probability space the solutions of the above SDE simultaneously for all \(z \in \mathbb{C}\) and all possible initial conditions. The solutions are continuous w.r.t. all parameters.
We will see in this paragraph that we can transform this system into a canonical system (19). When $dV$ is function-valued, this is already known (see e.g. Example 1 in [18]). Here $dV$ has the regularity of white noise and we thus need to adapt the arguments. The basic idea remains however the same.

One introduces the evolving matrix $M := \begin{pmatrix} u^{(N)}_1 & u^{(D)}_1 \\ u^{(N)}_2 & u^{(D)}_2 \end{pmatrix}$ where $u^{(N)} = (u^{(N)}_1, u^{(N)}_2)^T$, resp. $u^{(D)} = (u^{(D)}_1, u^{(D)}_2)^T$, is the solution of (22) with $z = 0$ and starting from $u^{(N)}(0) = (1,0)^T$, resp. $u^{(D)}(0) = (0,1)^T$. Of course the superscripts $N$ and $D$ refer to Neumann and Dirichlet. Note that $\det M$ remains however the same.

One introduces the evolving matrix $M := \begin{pmatrix} u^{(N)}_1 & u^{(D)}_1 \\ u^{(N)}_2 & u^{(D)}_2 \end{pmatrix}$ where $u^{(N)} = (u^{(N)}_1, u^{(N)}_2)^T$, resp. $u^{(D)} = (u^{(D)}_1, u^{(D)}_2)^T$, is the solution of (22) with $z = 0$ and starting from $u^{(N)}(0) = (1,0)^T$, resp. $u^{(D)}(0) = (0,1)^T$. Of course the superscripts $N$ and $D$ refer to Neumann and Dirichlet. Note that $\det M$ remains however the same.

One introduces the evolving matrix $M := \begin{pmatrix} u^{(N)}_1 & u^{(D)}_1 \\ u^{(N)}_2 & u^{(D)}_2 \end{pmatrix}$ where $u^{(N)} = (u^{(N)}_1, u^{(N)}_2)^T$, resp. $u^{(D)} = (u^{(D)}_1, u^{(D)}_2)^T$, is the solution of (22) with $z = 0$ and starting from $u^{(N)}(0) = (1,0)^T$, resp. $u^{(D)}(0) = (0,1)^T$. Of course the superscripts $N$ and $D$ refer to Neumann and Dirichlet. Note that $\det M$ remains however the same.

Coming back to the generic solution $u_z$ of the system (22), one considers the transformed process $v_z = M^{-1}u_z$. By computing $d(Mv_z)$, one deduces that

$$dv_z = -zM^{-1}JMv_zdt - M^{-1}d\langle M, v_z \rangle,$$

where for all $2 \times 2$ matrix $A$ and vector $x \in \mathbb{R}^2$ whose entries are Itô processes, we define their bracket through:

$$\langle A, x \rangle := \left( \langle A_{11}, x_1 \rangle + \langle A_{12}, x_2 \rangle \right) = \left( \langle A_{21}, x_1 \rangle + \langle A_{22}, x_2 \rangle \right).$$

Equation (24) shows that $v_z$ is differentiable and therefore $d\langle M, v_z \rangle$ vanishes. As a consequence

$$Jdv_z = -zJM^{-1}JMv_zdt = -\frac{z}{\det M}M^\top Mv_zdt,$$

where we used the identity $M^{-1} = -(\det M)^{-1}JM^\top J$ at the second line.

**Remark 5.** Note that $v_z$ is differentiable, while $u_z$ is Brownian-like. Our transformation removed the irregularity from the latter.

Denote by $R := M^\top M / \det M$. Almost surely the matrix $R$ is a positive definite symmetric matrix at all times, and is integrable as its entries are continuous. From the results on canonical systems recalled before, we can associate a self-adjoint operator to the system (25) by setting $\mathcal{T}_R := R^{-1}J\partial_t$ on $L^2(\mathbb{R})$, and by prescribing some boundary conditions $b_0$ and $b_1$. It acts on a domain $\mathcal{D}_{b_0,b_1}(\mathcal{T}_R)$ explicited in (18).

We finally associate to the system of equations (22) the following self-adjoint operator

$$\mathcal{S} := M\mathcal{T}_R M^{-1} = \det M (M^{-1})^\top J\partial_t M^{-1},$$

that acts on (recall that $M(0) = I$)

$$\mathcal{D}_{b_0,M(1)b_1}(\mathcal{S}) := \left\{ f \in L^2(\det M)^{-1} : M^{-1}f \in \mathcal{D}_{b_0,b_1}(\mathcal{T}_R) \right\}.$$
Since $S$ is a unitary\(^7\) transformation of $\mathcal{T}_R$, we deduce that it also has discrete spectrum. Furthermore, by conjugation we deduce the explicit expression of the kernel of its resolvents: for any $f \in L^2_{(\det M)^{-1} I}$ and any $z \in \mathbb{C}\setminus \mathbb{R}$

$$(S - z)^{-1} f(t) := \int_0^1 \left( u_z(t)\hat{u}_z(s)^\top 1_{t\leq s}(s) + \hat{u}_z(t)u_z(s)^\top 1_{s<t} \right) \frac{1}{\det M(s)} f(s),$$

where $u_z$ and $\hat{u}_z$ are the solutions of (22) that satisfy $u_z(0) = b_0$, $\hat{u}_z(1) \parallel b_1$ and $u_z(0)^\top J\hat{u}_z(0) = 1$.

In view of Equation (22), the operator $S$ can be written formally

$$S = \begin{pmatrix} 0 & -\partial_t \\ \partial_t & 0 \end{pmatrix} + dV,$$

with b.c. $b_0$ at 0 and $M(1)b_1$ at 1. In general, the elements of $\mathcal{D}(S)$ have Brownian like regularity but are not adapted (to the filtration of $V$); therefore one cannot apply Itô’s integration and the above expression for $S$ is only formal. On the other hand, the elements of $\mathcal{D}(\mathcal{T}_R)$ are absolutely continuous and the action of $\mathcal{T}_R$ given in (17) makes perfect sense. Let us mention that it would be possible to give a precise description of the action of $S$ on its domain using the theory of rough paths.

However, a rigorous connection with the formal equation of $S$ can be made at the level of the eigenvalues and eigenvectors:

**Lemma 2.1.** Almost surely for every $\lambda \in \mathbb{R}$, the pair $(\lambda, \varphi_\lambda)$ is an eigenvalue / eigenvector of $S$ if and only if the solution $u_\lambda$ of (22) that starts from $u_\lambda(0) = b_0$ is such that $u_\lambda(1)$ is parallel to $M(1)b_1$ and $\varphi_\lambda$ is a multiple of $u_\lambda$.

**Proof.** First note that almost surely for every $\lambda$, $(u_\lambda)$ is solution of (22) starting from $b_0$ at time 0 and is parallel to $M(1)b_1$ at time 1) is equivalent to $(v_\lambda$ is solution of (25) starting from $b_0$ at time 0 and is parallel to $b_1$ at time 1).

Second, almost surely for every $\lambda \in \mathbb{R}$, the pair $(\lambda, \varphi_\lambda)$ is an eigenvalue / eigenvector of $S$ if and only if $(\lambda, M^{-1}\varphi_\lambda)$ is an eigenvalue / eigenvector of $\mathcal{T}_R$. Then the equation $\mathcal{T}_R(M^{-1}\varphi_\lambda) = \lambda M^{-1}\varphi_\lambda$, together with the conditions that $M^{-1}\varphi_\lambda$ is parallel to $b_0$ at 0 and $b_1$ at 1, is equivalent to saying that $M^{-1}\varphi_\lambda$ is a multiple of $v_\lambda$ where $v_\lambda$ is the solution of (25) that starts from $b_0$ at time 0 and is parallel to $b_1$ at time 1. We thus conclude. \hfill $\square$

**2.2. Construction and properties of $\mathcal{C}S_{\tau}$.** Consider

$$(27) \quad dV := \frac{\sqrt{\tau}}{2} \begin{pmatrix} dB + \frac{1}{\sqrt{2}} dW_1 & \frac{1}{\sqrt{2}} dW_2 \\ \frac{1}{\sqrt{2}} dW_2 & dB - \frac{1}{\sqrt{2}} dW_1 \end{pmatrix},$$

where $B$, $W_1$ and $W_2$ are independent Brownian motions. Choose $b_0$ and $b_1$ in a such a way that $b_0 = M(1)b_1 = (0,1)^T$, where $M$ is defined in (23) (with $dV$ as above). A computation shows that $d\det M = 0$ so that $\det M \equiv 1$ and we thus set $R := M^T M$. We apply the general construction of the previous subsection and set

$$\mathcal{C}S_{\tau} := 2 M \mathcal{T}_R M^{-1}.$$ 

---

\(^7\)Note that $M$ is not a unitary matrix, but the transformation $f \in L^2_R \mapsto Mf \in L^2_{(\det M)^{-1} I}$ is indeed unitary.
A simple computation shows that
\[ dy_z(t) = -\frac{z}{2} J y_z dt + J dW(t) y_z(t), \quad t \in [0,1], \quad z \in \mathbb{C}, \]
with \( y_z(0) = (0,1)^T. \) We thus have the following corollary of Lemma 2.1.

**Corollary 1.** Almost surely for every \( \lambda \in \mathbb{R}, \) the pair \( (\lambda, \Psi(\lambda)) \) is an eigenvalue / eigenvector of \( \text{CS}_\tau \) if and only if \( \dot{y}_\lambda(1) \) is parallel to \( (0,1)^T \) and \( \Psi(\lambda) \) is a multiple of \( y_\lambda. \)

For any \( z \in \mathbb{C} \setminus \mathbb{R}, \) the resolvent writes:
\[ (\text{CS}_\tau - z)^{-1} f(t) := \frac{1}{2} \int_0^1 \left( y_z(t) \dot{y}_z(s)(s) + \dot{y}_z(t) y_z(s)(s) 1_{s < t} \right) f(s), \]
where \( f \in L^2, \) \( y_z \) and \( \dot{y}_z \) are solutions of (28) such that \( y_z(0) = (0,1)^T, \) \( \dot{y}_z(1) \parallel (0,1)^T \) and \( \dot{y}_z(0) \parallel J \dot{y}_z(0) = 1. \) Note that \( \dot{y}_z \) can be constructed by setting
\[ \dot{y}_z := v_z - \alpha y_z, \quad \alpha := \begin{pmatrix} 1,0 \\ 1,0 \end{pmatrix} \]
where \( v_z \) is the solution of (28) that starts from \( (1,0)^T \) at time 0.

Let us now introduce the polar coordinates, also called Prüfer coordinates, associated to \( y_\lambda \) for any \( \lambda \in \mathbb{R} \) through the relation
\[ (y_\lambda)_2 + i(y_\lambda)_1 := \Gamma_\lambda e^{i\Theta_\lambda}. \]
A simple computation shows that
\[ d\Theta_\lambda(t) = \frac{\lambda}{2} dt + \frac{\sqrt{\tau}}{2} dB(t) + \frac{\sqrt{\tau}}{2\sqrt{2}} \text{Re}(e^{2i\Theta_\lambda(t)} dW(t)), \]
\[ d\ln \Gamma_\lambda(t) = \frac{\tau}{8} dt + \frac{\sqrt{\tau}}{2\sqrt{2}} \text{Im}(e^{2i\Theta_\lambda(t)} dW(t)), \]
where \( \mathcal{W} = (\mathcal{W}_1 + i \mathcal{W}_2) \) is a complex Brownian motion.

**Proof of Theorem 1.2.** It is a consequence of the material above, noticing that \( \Theta_\lambda(1) \in \pi \mathbb{Z} \) if and only if \( y_\lambda(1) \) is parallel to \( (0,1)^T \). \( \square \)

**Remark 6.** We could have endowed the operator with other b.c. For instance, let \( \text{CS}_\tau,\ell \) be defined similarly as \( \text{CS}_\tau \) except that we impose \( M(1)b_1 = (\sin \ell, \cos \ell)^T \) for some \( \ell \in [0,\pi]. \) The eigenvalues of \( \text{CS}_\tau,\ell \) are those \( \lambda \) for which \( \Theta_\lambda(1) \) equals \( \ell \) modulo \( \pi. \) We have the following scaling property of the family \( \text{CS}_\tau,\ell: \)
\[ Q_{-\ell}(\text{CS}_\tau + 2\ell) Q_{-\ell}^{-1} = \text{CS}_\tau,\ell, \]
where \( \begin{pmatrix} \cos(-\ell t) & \sin(-\ell t) \\ \sin(-\ell t) & \cos(-\ell t) \end{pmatrix}. \)
Consequently the set of eigenvalues of \( \text{CS}_\tau,\ell \) coincides in law with the point process \( \text{Sch}_\tau + 2\ell. \) It is then easy to deduce that the point process \( \text{Sch}_\tau \) is invariant in law under translation by integer multiples of \( 2\pi. \)

\(^8\)We stuck to standard definitions of Dirac operators and Dirac equations, while the defining expression of \( \text{CS}_\tau, \) and therefore of the associated SDEs (15) have an additional factor 2.
We turn to the computation of the intensity measure of the point process of eigenvalues / eigenvectors of $\mathcal{C}_S$. We start with some preliminary results. We start with some properties of the phase function, which were also established in [11] and [17]:

**Lemma 2.2.** The phase $\Theta_\lambda$ is differentiable with respect to $\lambda$ its derivative $\lambda \mapsto \Theta_\lambda(t)$ writes:

$$\partial_\lambda \Theta_\lambda(t) = \frac{1}{2} \int_0^t \exp \left(2 \ln \Gamma_\lambda(u) - 2 \ln \Gamma_\lambda(t)\right) du.$$ 

It implies in particular that almost surely for any $t \in [0, 1]$, the phase $\lambda \mapsto \Theta_\lambda(t)$ is increasing.

**Proof of Lemma 2.2.** It is standard to check that $\Theta_\lambda$ is differentiable (in fact real-analytic) with respect to $\lambda$, see e.g. [13, Chap. V, Theorem 40] (see also Theorem 24 of [11]). Its derivative $\lambda \mapsto \Theta_\lambda(t)$ satisfies the following SDE

$$d(\partial_\lambda \Theta_\lambda) = \frac{1}{2} dt - \frac{\sqrt{\pi}}{\sqrt{2}} (\partial_\lambda \Theta_\lambda) \Im(e^{2i\theta_\lambda} dW(t)).$$

An application of Itô’s formula yields (32). We thus deduce that almost surely, for all $t \in [0, 1]$ and all $\lambda \in \mathbb{R}$, we have $\partial_\lambda \Theta_\lambda(t) > 0$, which is enough to conclude. \qed

Note however that $t \mapsto |\Theta_\lambda(t)/\pi|$ is not non-decreasing, while the phase associated to 1-d Schrödinger operators satisfies this property, often called the Sturm-Liouville property.

Second, we show that the number of points in $\text{Sch}_\tau$ that fall in any given compact set has finite expectation.

**Lemma 2.3.** For any $\mu < \lambda$, we have $\mathbb{E}[\#\{\text{Sch}_\tau \cap [\mu, \lambda]\}] < \infty$.

**Proof.** Thanks to the characterization of the $\text{Sch}_\tau$ point process (2) and the monotonicity property of $\lambda \mapsto \Theta_\lambda(1)$ seen in Lemma 2.2, we have:

$$\#\{\text{Sch}_\tau \cap [\mu, \lambda]\} \leq \frac{\Theta_\lambda(1) - \Theta_\mu(1)}{\pi} + 1,$$

hence it suffices to bound the expectation of $\Theta_\lambda(1) - \Theta_\mu(1)$. For any $t \in [0, 1]$, we have

$$\Theta_\lambda(t) - \Theta_\mu(t) = \frac{\lambda - \mu}{2} t + \frac{\sqrt{\pi}}{2\sqrt{2}} \int_0^t \Im\left((e^{2i\Theta_\lambda(s)} - e^{2i\Theta_\mu(s)}) dW(s)\right).$$

It implies:

$$\mathbb{E}[\Theta_\lambda(t) - \Theta_\mu(t)] = \frac{\lambda - \mu}{2} t,$$

which proves the result. \qed

Third we compute a change of measure, which is essentially the same as in [17, Proof of Lemma 3.6].

**Lemma 2.4.** Fix $u \in [0, 1]$ and let $B_1$ be a real Brownian motion on $[0, 1]$ starting from 0. Set $f^u(t) := (u - |u - t|)/2$ and $Y(t) := t^\frac{\tau}{8} + \frac{\sqrt{\pi}}{2\sqrt{2}} B_1(t)$ for $t \in [0, 1]$. Then for any bounded measurable map $G$ on $C([0, 1], \mathbb{R})$ we have

$$\mathbb{E}\left[G(Y)e^{(u-1)\frac{\tau}{8} + \frac{\sqrt{\pi}}{2\sqrt{2}} (B_1(u) - B_1(1))}\right] = \mathbb{E}\left[G\left(\frac{\tau}{4} f^u + \frac{\sqrt{\pi}}{2\sqrt{2}} B_1\right)\right].$$
The identity
\[ \frac{\tau}{4} f^u(t) + \frac{\sqrt{\tau}}{2\sqrt{2}} B_1(t) = Y(t), \quad t \in [0, u] . \]
onumber
On the other hand, Girsanov’s Theorem [16, Th VIII.1.7] shows that under $\mathbb{Q}$, the process $(B_1(t) - B_1(u) + \sqrt{\frac{\tau}{2}} (t - u))_{u \leq 1}$ is a Brownian motion starting from 0 at time $u$. Consequently, under $\mathbb{Q}$ the process $(\tilde{Y}(t))_{u \leq 1}$ has the same law as the process
\[ \frac{\tau}{4} (f^u(t) - f^u(u)) + \frac{\sqrt{\tau}}{2\sqrt{2}} (B_1(t) - B_1(u)), \quad u \leq t \leq 1 , \]
onumber
under $\mathbb{P}$. This completes the proof. \hfill \Box

We now proceed with the computation of the intensity measure.

**PROOF OF THEOREM 1.3.** Assume that
\begin{equation}
\mathbb{E} \left[ \sum_{\lambda : \Theta_\lambda(1) \in \pi \mathbb{Z}} G(\lambda, \ln \Gamma_\lambda, \Theta_\lambda) \right] = \int \sum_{n \in \mathbb{Z}} \mathbb{E} \left[ G(\lambda, \sqrt{\frac{\tau}{2}} B_1 + \frac{\tau f^U}{4}, \beta^{(n\pi)}) \right] g_{\lambda, \frac{\tau}{2}} (2n\pi) d\lambda .
\end{equation}

The identity
\[ \Psi_\lambda = \frac{\Gamma_\lambda}{\| \Gamma_\lambda \|_{L^2([0,1], \mathbb{R})}} \left( \frac{\sin \Theta_\lambda}{\cos \Theta_\lambda} \right) , \]
shows that $\Psi_\lambda$ is the image through a continuous map of $(\ln \Gamma_\lambda, \Theta_\lambda)$. From (33) we thus deduce that
\[ \mathbb{E} \left[ \sum_{\lambda : \Theta_\lambda(1) \in \pi \mathbb{Z}} G(\lambda, \Psi_\lambda) \right] = \int \sum_{n \in \mathbb{Z}} \mathbb{E} \left[ G(\lambda, \frac{e^Z}{\| e^Z \|_{L^2([0,1])}} (\frac{\sin \beta^{(n\pi)}}{\cos \beta^{(n\pi)})}) \right] g_{\lambda, \frac{\tau}{2}} (2n\pi) d\lambda , \]
where $Z := \frac{\sqrt{\tau}}{2\sqrt{2}} B_1 + \frac{\tau f^U}{4}$. Theorem 1.3 then follows from the equality in law
\[ \left( \frac{e^Z(t)}{\| e^Z \|_{L^2([0,1])}}, t \in [0, 1] \right) \overset{(d)}{=} \left( \frac{e^{\frac{\sqrt{\tau}}{2\sqrt{2}} B_1(t-U) - \frac{\tau}{2}|t-U|}}{\| e^{\frac{\sqrt{\tau}}{2\sqrt{2}} B_1(t-U) - \frac{\tau}{2}|t-U|} \|_{L^2([0,1])}}, t \in [0, 1] \right) . \]

We are left with the proof of (33). Recall that the process $(\Theta_\lambda(t), \Gamma_\lambda(t); t \in [0, 1], \lambda \in \mathbb{R})$ is continuous in both variables and satisfies the SDEs (31). We have seen in Lemma 2.2 that almost surely $\lambda \mapsto \Theta_\lambda(1)$ is a $C^1$-diffeomorphism. In the sequel, we denote by $\theta \mapsto \lambda(\theta)$ its inverse.
By standard approximation arguments, it suffices to take $G$ non-negative, bounded and continuous, and such that $G(\lambda, \cdot) = 0$ whenever $\lambda \notin [-a, a]$ for some $a > 0$. Note that

$$
\sum_{\lambda, \Theta_{\lambda}(1) \in \pi Z} G(\lambda, \ln \Gamma_{\lambda}, \Theta_{\lambda}) = \sum_{\theta \in \pi Z} G(\lambda(\theta), \ln \Gamma_{\lambda(\theta)}, \Theta_{\lambda(\theta)}) .
$$

By continuity, the right hand side is the almost sure limit as $\epsilon \downarrow 0$ of

$$
X_\epsilon := \frac{1}{2\epsilon} \int_{\theta \in [-\epsilon, \epsilon] + \pi Z} G(\lambda(\theta), \ln \Gamma_{\lambda(\theta)}, \Theta_{\lambda(\theta)}) d\theta
$$

$$
= \frac{1}{2\epsilon} \int_{\lambda \in \mathbb{R}} 1\{\Theta_{\lambda} \in [-\epsilon, \epsilon] + \pi Z\} G(\lambda, \ln \Gamma_{\lambda}, \Theta_{\lambda}) \partial \Theta_{\lambda}(1) d\lambda .
$$

Recall that $G$ is compactly supported in its first variable. Provided that $\epsilon < \pi$, we see that almost surely $|X_\epsilon| \leq \|G\|_\infty (\#\{\lambda \in [-a, a]: \Theta_{\lambda}(1) \in \pi Z\} + 2)$. The latter r.v. has finite expectation by Lemma 2.3. The Dominated Convergence and Fubini Theorems thus yield

$$
\begin{aligned}
\mathbb{E}[\sum_{\lambda, \Theta_{\lambda}(1) \in \pi Z} G(\lambda, \ln \Gamma_{\lambda}, \Theta_{\lambda})]
&= \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_{\lambda \in \mathbb{R}} \mathbb{E}[1\{\Theta_{\lambda} \in [-\epsilon, \epsilon] + \pi Z\} G(\lambda, \ln \Gamma_{\lambda}, \Theta_{\lambda}) \partial \Theta_{\lambda}(1)] d\lambda \\
\end{aligned}
$$

(34)

Fix $\lambda \in \mathbb{R}$ and recall the definition of the SDEs (31) and formula (32) of Lemma 2.2. Observe that $\int_0^t \mathcal{Y}(\epsilon^2(\Theta_{\lambda}, d\mathcal{W})$ and $\int_0^t \mathcal{Y}(\epsilon^2(\Theta_{\lambda}, d\mathcal{W})$ are independent Brownian motions. Since $\mathcal{W}$ is independent of $\mathcal{B}$, we deduce that the process $\Theta_{\lambda}$ is independent of $(\ln \Gamma_{\lambda}, \partial \Theta_{\lambda}(\lambda))$. We now provide some identities in law on the process $\Theta_{\lambda}$. First, the r.v. $\Theta_{\lambda}(1)$ has a Gaussian law, centered at $\lambda/2$ with variance $3\pi/8$. Moreover the process $\Theta_{\lambda}$ conditioned to $\{\Theta_{\lambda}(1) = x\}$ has the same law as $\beta^{(x)}$ where, for some Brownian motion $\mathcal{B}_2$,

$$
d\beta_{\Theta}^{(x)}(t) = x dt + \sqrt{\frac{3\pi}{8}}(d\mathcal{B}_2(t) - \mathcal{B}_2(1) dt) .
$$

The process $\beta_{\Theta}^{(x)}$ is a scaled Brownian bridge from $(0, 0)$ to $(1, x)$ (note that the law of $\beta^{(x)}$ no longer depends on $\lambda$).

Desintegrating the expectation appearing on the r.h.s. of (34) according to the law of $\Theta_{\lambda}(1)$ we get

$$
\begin{aligned}
\mathbb{E}[1\{\Theta_{\lambda}(1) \in [-\epsilon, \epsilon] + \pi Z\} G(\lambda, \ln \Gamma_{\lambda}, \Theta_{\lambda}) \partial \Theta_{\lambda}(1)]
&= \int_{x \in \mathbb{R}} 1\{x \in [-\epsilon, \epsilon] + \pi Z\} \mathbb{E}\left[G(\lambda, \ln \Gamma_{\lambda}, \beta^{(x)} \partial \Theta_{\lambda}(1))\right] g_{\lambda/2 \pi}(x) dx .
\end{aligned}
$$

Note that $g_{\lambda/2 \pi}(x) = 2g_{\lambda/2 \pi}(2x)$ Applying again the Dominated Convergence Theorem, we obtain

$$
\begin{aligned}
\mathbb{E}[\sum_{\lambda, \Theta_{\lambda}(1) \in \pi Z} G(\lambda, \ln \Gamma_{\lambda}, \Theta_{\lambda})]
&= \int_{\lambda \in \mathbb{R}} \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_{x \in [-\epsilon, \epsilon] + \pi Z} 2 \mathbb{E}\left[G(\lambda, \ln \Gamma_{\lambda}, \beta^{(x)} \partial \Theta_{\lambda}(1))\right] g_{\lambda/2 \pi}(2x) dx d\lambda \\
&= \int_{\lambda \in \mathbb{R}} \sum_{n \in \mathbb{Z}} 2 \mathbb{E}\left[G(\lambda, \ln \Gamma_{\lambda}, \beta^{(n \pi)} \partial \Theta_{\lambda}(1))\right] g_{\lambda/2 \pi}(2n \pi) d\lambda .
\end{aligned}
$$
It remains to compute the expectation that appears on the r.h.s. Recall that $\beta^{(n\pi)}$ is independent of the pair $(\ln \Gamma_\lambda, \partial_\lambda \Theta_\lambda)$. We already saw that $B_1(t) := \int_0^t \Re(e^{2i\Theta_\lambda} dW)$ is a standard Brownian motion. Consequently

$$\ln \Gamma_\lambda(t) = t \frac{\tau}{8} + \frac{\sqrt{\tau}}{2\sqrt{2}} B_1(t) =: Y(t).$$

Using the explicit expression (32) of $\partial_\lambda \Theta_\lambda$ in terms of $\ln \Gamma_\lambda$, we obtain

$$2 \mathbb{E} \left[ G(\lambda, \ln \Gamma_\lambda, \beta^{(n\pi)}) \partial_\lambda \Theta_\lambda(1) \right] = \int_0^1 \mathbb{E} \left[ G(\lambda, Y, \beta^{(n\pi)}) e^{(u-1)\frac{\tau}{4} + \sqrt{\tau} \sqrt{2} (B_1(u) - B_1(1))} \right] du.$$

Lemma 2.4 allows to conclude.

2.3. CS as a transformation of $\text{Sch}_\tau$. In [20], Valkó and Virág introduced Dirac operators $R^{-1} J \partial_t$ corresponding to the limit of many famous matrix ensembles. Writing the matrix $R$ as $R = \frac{f}{\det X} X^T X$ for $X := \begin{pmatrix} 1-x & 0 \\ 0 & y \end{pmatrix}$, those operators are encoded by a speed function $f$, a path $x + iy$ in the upper half plane and some boundary conditions (i.e. two directions in the limit circle case). The authors defined in particular the operator $\text{Sch}_\tau^*$ which corresponds to the translation invariant (in law) version of the bulk limit of discrete 1d Schrödinger operator (see [11] and Section 9.5 of [20] for more details).

Following their approach, we would like to define a Dirac operator $\text{Sch}_\tau$ whose eigenvalue point process corresponds to $\text{Sch}_\tau^*$. Necessarily, this operator cannot be translation invariant. We will see that it is constructed similarly to $\text{Sch}_\tau^*$ except that we have to change in a non-trivial way the right boundary condition.

Let us set $X := \begin{pmatrix} 1-x & 0 \\ 0 & y \end{pmatrix}$ satisfying

$$dX = \sqrt{\frac{\tau}{2}} \begin{pmatrix} 0 & dB_2 \\ 0 & dB_1 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{where } B_1 \text{ and } B_2 \text{ are independent Brownian motions. Thus the path } x + iy \text{ evolves in the upper half plane as a hyperbolic Brownian path of variance } \tau/2.$$

We set $\tilde{R} := X^T X / \det X$ ($f := 1$ in this case).

Since $\tilde{R}$ is invertible at all times, we can apply the theory of Dirac operators recalled in Subsection 2.1 and define $\text{Sch}_{\tau, \ell} := 2\tilde{R}^{-1} J \partial_t$ on the domain (18) with boundary conditions $b_0 := (0, 1)^T$ and $b_1 := (\sin \ell, \cos \ell)^T$. It turns out that for the eigenvalues of $\text{Sch}_{\tau, \ell}$ to be given by the $\text{Sch}_\tau$ point process, one needs to choose carefully the boundary condition $\ell$. More precisely, let $\theta := \sqrt{\frac{\tau}{2}} (B_2 - \sqrt{2} B)$ where $B$ is a Brownian motion independent of $B_1, B_2$ and consider the random variable $^9$

$$\tan L = y(1) \tan \theta(1) + x(1).$$

Then we set $\text{Sch}_\tau := \text{Sch}_{\tau, L}$. A tedious computation shows that its eigenvalues are distributed as the $\text{Sch}_\tau$ point process. However, this property can also be obtained from the statement of Theorem 1.4 that we now prove.

Note that we define $\theta$ for all times $t \in [0, 1]$ but we only need its value at $t = 1$ for the definition of $L$. The evolution of $\theta$ will be useful below for the definition of the unitary transformation.
PROOF OF THEOREM 1.4. Recall that the operator $\mathcal{C}_\tau$ was defined as $M(2R^{-1}J\partial_t)M^{-1}$ endowed with the b.c. $b_0 = M(1)b_1 = (0,1)^\top$, where the matrix $M$ satisfies (23) with $V$ defined by (27), and the matrix $\tilde{R} := M^\top M$.

Recall the process $\theta$ introduced right above, and set

$$Q := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \tilde{M} := \frac{1}{\sqrt{\det X}} Q X.$$  

We claim that there exists a coupling of $B, B_1, B_2$ and $\mathcal{B}, \mathcal{W}_1, \mathcal{W}_2$ such that almost surely $M = \tilde{M}$.

We postpone the proof of this claim, and carry on the proof the theorem. Since at all times $Q$ is a rotation matrix, we deduce that

$$\tilde{R} = \frac{1}{\det X} X^\top X = (\tilde{M})^\top \tilde{M} = M^\top M = R.$$  

Regarding the boundary conditions, let us check that $\tilde{b}_1$ and $b_1$ are parallel. To that end, it suffices to check that $X(1)\tilde{b}_1$ is parallel to $Q^{-1}(1)(0,1)^\top = (\sin \theta(1), \cos \theta(1))^\top$. We compute

$$X(1)\tilde{b}_1 = \begin{pmatrix} \sin L - x(1) \cos L \\ y(1) \cos L \end{pmatrix},$$  

and the latter is parallel to $(\sin \theta(1), \cos \theta(1))^\top$ if and only if

$$\frac{1}{y(1)} \tan L - \frac{x(1)}{y(1)} = \tan \theta(1),$$  

and this identity is granted by definition of $L$.

Consequently, we have shown that $\mathcal{S}_\tau$ coincides with $2R^{-1}J\partial_t$ and therefore

$$\mathcal{C}_\tau = M \mathcal{S}_\tau M^{-1}.$$  

We are left with the proof of the claim. Let $T := X/\sqrt{\det X}$. One can check that

$$dT = \left(-\sqrt{\frac{\pi}{8}} dB_1 + \frac{3}{16} \tau, \sqrt{\frac{\pi}{8}} dB_2 - \frac{1}{16} \tau \right) T.$$  

Take now $Q$ rotation matrix of angle $\frac{\sqrt{\pi}}{2\sqrt{2}}(B_2 + \sqrt{2}B)$ and compute the matrix SDE of the product $\tilde{M} = QT$. We have

$$d\tilde{M} = dQ T + Q dT + d\langle Q,T \rangle$$  

where

$$d\langle Q,T \rangle := \left(d\langle Q_{11}, T_{11} \rangle + d\langle Q_{12}, T_{21} \rangle d\langle Q_{11}, T_{12} \rangle + d\langle Q_{12}, T_{22} \rangle d\langle Q_{12}, T_{21} \rangle d\langle Q_{11}, T_{12} \rangle + d\langle Q_{22}, T_{22} \rangle d\langle Q_{22}, T_{21} \rangle d\langle Q_{21}, T_{12} \rangle + d\langle Q_{22}, T_{22} \rangle d\langle Q_{22}, T_{21} \rangle d\langle Q_{21}, T_{12} \rangle \right).$$  

Using $dQ = (Jd\theta - \frac{1}{2}d\langle \theta \rangle I)Q = Q(Jd\theta - \frac{1}{2}d\langle \theta \rangle I)$, a computation gives

$$d\tilde{M} = Jd\tilde{V} \tilde{M},$$  

with

$$d\tilde{V} = \frac{\sqrt{\pi}}{2} \left( dB + \frac{1}{\sqrt{2}} \Re(e^{2i\theta}(dB_2 + idB_1)) + \frac{1}{\sqrt{2}} \Im(e^{2i\theta}(dB_2 + idB_1)) \right).$$  

Observe that $\int_0^t \Re(e^{2i\theta}(dB_2 + idB_1))$, $\int_0^t \Im(e^{2i\theta}(dB_2 + idB_1))$ and $B$ are three independent Brownian motions of variance 1. We get the desired coupling by taking

$$B = B, \quad \mathcal{W}_1 := \int_0^t \Re(e^{2i\theta}(dB_2 + idB_1)), \quad \mathcal{W}_2 := \int_0^t \Im(e^{2i\theta}(dB_2 + idB_1)).$$  

□
Using Remark 2, it makes explicit the hyperbolic carousel interpretation for the (non-translation invariant) Schrödinger point process (which was not detailed in [11] and [20]).

It also shows that the Dirac operator constructed in Subsection 2.2 acts as the Dirac operator of Valkó and Virág Sch$^*$. The only difference lies in the domain of definition: the right boundary condition is non-trivial, it depends on the two Brownian motions $B_1$ and $B_2$ and an additional independent random variable.

3. Convergence of the operators. We start this section with a detailed presentation of the unitary map that allows to construct $S^{(E)}_L$ from $H^{(E)}_L$. Then, we deal with the operator convergence of $S^{(E)}_L$ towards $CS_\tau$ and thus prove Theorem 1.6. In the first two subsections, no assumption is made on the value of $E$, while in the subsequent subsections we always assume that $E \sim L/\tau$ for some fixed $\tau > 0$.

3.1. The unitary transformation. The operator $H_L$ is a generalized Sturm-Liouville operator. Its domain is made of (random) $H^1([0, L])$-functions that satisfy Dirichlet b.c.: namely,

$$D(H_L) := \left\{ f \in L^2([0, L]) : f(0) = f(L) = 0, \ f \ \text{A.C.}, \ f' - Bf \ \text{A.C.}, \right.$$

$$\left. \text{and } -(f' - Bf)' - Bf' \in L^2([0, L]) \right\}.$$

Recall from the introduction that we consider the recentered operator $(L/\sqrt{E})(H_L - E) + 2f_E$, and that we conjugate it with the rescaling map $g \mapsto g(L \cdot)$ that goes from $L^2([0, L], \mathbb{R})$ into $L^2([0, 1], \mathbb{R})$. This yields the operator

$$H^{(E)}_L f := -\frac{1}{L/\sqrt{E}} f'' + \sqrt{\frac{L}{E}} dB^{(L)} f + (2f_E - L\sqrt{E})f, \quad x \in [0, 1],$$

whose domain is the image through the rescaling map of the domain of $H_L$. In particular any element $f$ of this domain belongs to $H^1([0, 1])$ and satisfies Dirichlet b.c.: $(f(0), f'(0))$ and $(f(1), f'(1))$ are parallel to $(0, 1)$.

Recall from (16) the definition of the Hilbert space $L^2_T([0, 1], \mathbb{R}^2)$. Recall in particular that for non-invertible $R$, it is a space of equivalence classes on $L^2_T([0, 1], \mathbb{R}^2)$ for the relation $f \sim g \iff Rf = Rg$ almost everywhere. It implies that $f \in L^2_T$ can have distinct representatives in $L^2_T$.

We now present the transformation corresponding to Step 1 at the operator level.

Step 1: From $\mathbb{R}$ to $\mathbb{R}^2$. Although we considered a lift of our system from $\mathbb{R}$ to $\mathbb{R}^2$, it is not canonically associated to a densely defined operator on $L^2((0, 1), \mathbb{R}^2)$ and one needs to work on a “smaller” space.

Recall from the introduction the matrix $T := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. We will consider the Hilbert space $L^2_T([0, 1], \mathbb{R}^2)$. Note that $f = (f_1, f_2)^T$ and $g = (g_1, g_2)^T$ represent the same element in $L^2_T$ iff a.e. $f_1 = g_1$.

Let us introduce the map $\iota : L^2((0, 1), \mathbb{R}) \to L^2_T((0, 1), \mathbb{R}^2)$ that sends any $f \in L^2((0, 1), \mathbb{R})$ on its canonical equivalence class in $L^2_T((0, 1), \mathbb{R}^2)$ (the first coordinate of any representative of this equivalence class coincides with $f$ a.e.). Since $\iota$ is unitary, the operator

$$H^{(E)}_L := \iota H^{(E)}_L \iota^{-1}.$$

is a self-adjoint operator on $L^2_T((0, 1), \mathbb{R}^2)$ with domain $D(H^{(E)}_L) := \iota(D(H^{(E)}_L))$. It is equivalent to $H^{(E)}_L$: the eigenvalues and normalized eigenvectors of $H^{(E)}_L$ are given by
we deduce that it is a multiple of the equivalence class of $\pi\varphi^{(E)}_{\lambda}$ where $(\lambda, \varphi^{(E)}_{\lambda})$ are the eigenvalues and normalized eigenvectors of $H^{(E)}$.

Any element $f$ of $L^2_T$ is an equivalence class of $L^2_T$; its second coordinate $f_2$ is “arbitrary”. However, in our context there is a convenient representative given by
\[ Pf := (f_1, \frac{1}{L\sqrt{E}} f'_1)\top. \]

Note that $Pf \in L^2_T$ provided $f_1 \in H^1((0,1), \mathbb{R})$; this holds in particular for any $f \in \mathcal{D}(H^{(E)}_L)$. Thus, any $f \in \mathcal{D}(H^{(E)}_L)$ satisfies Dirichlet b.c. in the following sense: $Pf(0)$ and $Pf(1)$ are parallel to $(0,1)^\top$.

Our next lemma establishes the relationship between $H^{(E)}_L$ and the collection of SDEs (12).

**Lemma 3.1.** Almost surely the eigenvalues of $H^{(E)}_L$ are those $\lambda$ for which the solution $(u^{(E)}_{\lambda},(u^{(E)}_{\lambda})')$ of (12) is parallel to $(0,1)^\top$ at time 1. In that case the associated eigenvector $\varphi^{(E)}_{\lambda}$ is a multiple of the equivalence class associated to $u^{(E)}_{\lambda}$ and we have
\[ P\varphi^{(E)}_{\lambda} = (\varphi^{(E)}_{\lambda}, \frac{1}{L\sqrt{E}} (\varphi^{(E)}_{\lambda})')\top. \]

**Proof.** $\lambda$ is an eigenvalue of $H^{(E)}_L$ if and only if $\lambda$ is an eigenvalue of $H^{(E)}_L$ if and only if the solution $u^{(E)}_{\lambda}$ of (11) vanishes at time 1. The latter is equivalent with: the solution $(u^{(E)}_{\lambda},(u^{(E)}_{\lambda})')$ of (12) is parallel to $(0,1)^\top$ at time 1. The theory of Sturm-Liouville operators shows that the eigenvector $\varphi^{(E)}_{\lambda}$ of $H^{(E)}$ is then a multiple of $u^{(E)}_{\lambda}$. Since $\varphi^{(E)}_{\lambda} = \iota\varphi^{(E)}_{\lambda}$ we deduce that it is a multiple of the equivalence class of $(u^{(E)}_{\lambda},(u^{(E)}_{\lambda})')$. This implies that $P\varphi^{(E)}_{\lambda} = (\varphi^{(E)}_{\lambda}, \frac{1}{L\sqrt{E}} (\varphi^{(E)}_{\lambda})')\top$. \hfill \square

**Step 2: Urotate.** Recall $Q_{E'}$ defined in (9) and note that $Q_{E'}^{-1} = Q_{E'}^\top$. We set $R_{E'} := Q_{E'}TQ_{E'}^\top$. The evolving rotation matrix $Q_{E'}$ can be viewed\(^{10}\) as a unitary map from $L^2_T$ into $L^2_{R_{E'}}$. Indeed, for any $f \in L^2_T$ and for any representative $g \in L^2_T$ of $f$ we define $Q_{E'} f$ as the equivalence class of $Q_{E'} g$ in $L^2_{R_{E'}}$. It is simple to check that this definition is independent of the choice of the representative $g$ and that it gives a unitary map.

We then introduce the operator
\[ S^{(E)}_L := Q_{E'} H^{(E)}_L Q_{E'}^{-1}, \]
acting on the domain $\mathcal{D}(S^{(E)}_L) := Q_{E'} \mathcal{D}(H^{(E)}_L) \subset L^2_{R_{E'}}([0,1], \mathbb{R}^2)$. Similarly to Step 1, there is a canonical choice of representative in $L^2_T$ for the elements $f$ in $L^2_{R_{E'}}$ given by $P_{E'} f$ where $P_{E'} := Q_{E'} P Q_{E'}^{-1}$.

Recall that $E'$ is taken equal to $L\sqrt{E} - \ell_E$: the order 1 correction $\ell_E = \{L\sqrt{E}\}$ has been chosen in such a way that any element $f \in \mathcal{D}(S^{(E)}_L)$ satisfies Dirichlet b.c., that is, $P_{E'} f(0)$ and $P_{E'} f(1)$ are parallel to $(0,1)^\top$.

Note that the eigenvalues and normalized eigenvectors of $S^{(E)}_L$ are given by $(\lambda, \psi^{(E)}_{\lambda} = Q_{E'} \iota \varphi^{(E)}_{\lambda})$. Our next result connects $S^{(E)}_L$ with the collection of processes $(y^{(E)}_{2,\lambda}, z \in \mathbb{C})$

\(^{10}\)Note that there is a slight abuse of notation here: we denote by $Q_{E'}$ both the operator that acts on $L^2_T$ and the one that acts on $L^2_T((0,1), \mathbb{R}^2)$.\]
(this is the same equation as in (13), except it is written here for a complex $z$)

$$
\left( \begin{array}{cc} 0 & -\partial_t \\ \partial_t & 0 \end{array} \right) + \frac{1}{2} \sqrt{L/E} \left( dB^{(L)} + \frac{1}{\sqrt{2}} dW^{(L)}_1 - \frac{1}{\sqrt{2}} dW^{(L)}_2 \right) + \ell_E (2R_{E^*} - I) y^{(E)}_z = z R_{E^*} y^{(E)}_z, \\
$$

with initial condition $y^{(E)}_z(0) = (0, 1)^\top$.

**Lemma 3.2.** The eigenvalues of $\mathcal{S}_L^{(E)}$ are those $\lambda \in \mathbb{R}$ for which the solution $y^{(E)}_\lambda$ of (37) is parallel to $(0, 1)^\top$ at time 1. The corresponding normalized eigenvector $\psi^{(E)}_\lambda$ is then a multiple of the equivalence class in $L^2_{R_{E^*}}$ of $y^{(E)}_\lambda$, and we have (recall (10))

$$
P_{E^*} \psi^{(E)}_\lambda = Q_{E^*} \left( \frac{\varphi^{(E)}_\lambda}{L^{1/2}_E (\varphi^{(E)}_\lambda)^\top} \right) = \psi^{(E)}_\lambda.
$$

**Proof.** By definition of $\mathcal{S}_L^{(E)}$, $\lambda$ is an eigenvalue of $\mathcal{S}_L^{(E)}$ if and only if $\lambda$ is an eigenvalue of $\mathcal{H}_L^{(E)}$. By the previous lemma, this is equivalent with: $(u^{(E)}_\lambda, (u^{(E)}_\lambda)^\top)$ of (12) is parallel to $(0, 1)$ at time 1. But since $y^{(E)}_\lambda = Q_{E^*} (u^{(E)}_\lambda, 1/L^{1/2}_E (u^{(E)}_\lambda)^\top)^\top$, we deduce that it is in turn equivalent to: $y^{(E)}_\lambda$ of (37) is parallel to $(0, 1)^\top$ at time 1. In that case, since $\psi^{(E)}_\lambda = Q_{E^*} \varphi^{(E)}_\lambda$ and given the previous lemma, we deduce that $\psi^{(E)}_\lambda$ is a multiple of the equivalence class of $y^{(E)}_\lambda$. The last identity then easily follows.

**Remark 7.** One can show that for any given $f$ in the domain of $\mathcal{S}_L^{(E)}$, any representative $g_0 \in L^2_{\overline{\mathbb{R}_E}}$ of the function $g := \mathcal{S}_L^{(E)} f \in L^2_{R_{E^*}}$ satisfies the equation:

$$
\left( \begin{array}{cc} 0 & -\partial_t \\ \partial_t & 0 \end{array} \right) + \sqrt{L/E} \left( dB^{(L)} + \frac{1}{\sqrt{2}} dW^{(L)}_1 - \frac{1}{\sqrt{2}} dW^{(L)}_2 \right) + 2\ell_E (2R_{E^*} - I) P_{E^*} f = 2 R_{E^*} g_0,
$$

where the equality holds in $L^2_{\overline{\mathbb{R}_E}}$. Note the similarity with (4).

**Remark 8.** In the introduction, we mentioned that the space generated by the family $\{\psi^{(E)}_\lambda\}_{\lambda}$ is not dense in $L^2_{\overline{\mathbb{R}_E}}$. Indeed, if we set $f := Q_{E^*} (0, 1)^\top$, we observe that

$$
\langle f, \psi^{(E)}_\lambda \rangle_{L^2_{\overline{\mathbb{R}_E}}} = \langle (0, 1)^\top, (\varphi^{(E)}_\lambda, 1/L^{1/2}_E (\varphi^{(E)}_\lambda)^\top)^\top \rangle_{L^2_{\overline{\mathbb{R}_E}}} = \int \frac{1}{L^{1/2}_E} (\varphi^{(E)}_\lambda)^\top = 0.
$$

3.2. The resolvents and the precise statement of Theorem 1.6. Our goal is now to prove convergence of the resolvents of $\mathcal{S}_L^{(E)}$ to those of $\mathcal{C}_\varphi$. While the resolvents of the latter are operators on $L^2_{\overline{\mathbb{R}_E}}$, the resolvents of $\mathcal{S}_L^{(E)}$ are defined on the quotient space $L^2_{R_{E^*}}$: our first task is to extend these resolvents into well-defined, bounded operators on $L^2_{\overline{\mathbb{R}_E}}$.

Of course, such an extension is far from being unique. We will see that the extension that we opt for is related to the functions $\{\psi^{(E)}_\lambda\}_{\lambda}$: in light of the statement of Theorem 1.5, this justifies a posteriori our choice.
Fix $\in \mathbf{C}\setminus \mathbf{R}$. For any $g \in L^2_j$, let $\hat{g}$ be its equivalence class in $L^2_{R_E}$; and set

$$(S_L^{(E)} - z)^{-1}g := P_{E'}(S_L^{(E)} - z)^{-1}\hat{g}.$$ 

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node (L2j) at (0,0) {$L_j$};
  \node (L2I) at (3,0) {$L_j$};
  \node (L2RE) at (0,3) {$L^2_{R_E}$};
  \node (L2RE') at (3,3) {$L^2_{R_E}$};
  \draw[->] (L2j) -- (L2RE) node[midway,above] {$(S_L^{(E)} - z)^{-1}$};
  \draw[->] (L2RE) -- (L2RE') node[midway,above] {$(S_L^{(E)} - z)^{-1}$};
  \draw[->] (L2j) -- (L2I) node[midway,above] {$(S_L^{(E)} - z)^{-1}$};
  \draw[->] (L2I) -- (L2RE') node[midway,above] {$(S_L^{(E)} - z)^{-1}$};
\end{tikzpicture}
\caption{Extension of $(S_L^{(E)} - z)^{-1}$.}
\end{figure}

**Remark 9.** The overline in the notation $(\overline{S_L^{(E)} - z})^{-1}$ should not be confused with the complex conjugate. We use it to indicate that $(\overline{S_L^{(E)} - z})^{-1}$ is an “extension” of the initial operator $(S_L^{(E)} - z)^{-1}$ to the space $L^2_j$.

This definition is illustrated on Figure 1. We have composed $(S_L^{(E)} - z)^{-1}$ to the right with the (canonical) projection from $L^2_j$ to $L^2_{R_E}$; this is clearly the only reasonable operation to apply here. On the other hand, we have composed it to the left with the densely defined injection $P_{E'}$ from $L^2_{R_E}$ into $L^2_j$ and this may seem arbitrary. However note that for any eigenvalue $\lambda$ of $S_L^{(E)}$ we find

$$(\overline{S_L^{(E)} - z})^{-1}\Psi^{(E)}_\lambda = (\lambda - z)^{-1}\Psi^{(E)}_\lambda,$$

thus justifying a posteriori the composition with $P_{E'}$.

Our next proposition provides explicitly the kernel of $(\overline{S_L^{(E)} - z})^{-1}$. Let $v_z^{(E)}$ be the solution of (37) starting from $v_z^{(E)}(0) = (1,0)^\top$. Note that $y_z^{(E)}$ and $v_z^{(E)}$ are respectively the Dirichlet and Neumann solutions of (37), and that these functions live in $L^2([0,1], \mathbf{C}^2)$. We then set

$$\hat{y}_z^{(E)} = v_z^{(E)} - \alpha^{(E)} y_z^{(E)}, \quad \alpha^{(E)} := \begin{pmatrix} 1,0 \\ 1,0 \end{pmatrix} v_z^{(E)}(1).$$

Note that $\hat{y}_z^{(E)}$ is a solution of (37) which is parallel to $(0,1)^\top$ at time 1 (but not at time 0), and whose Wronskian with $y_z^{(E)}$ equals 1.

**Proposition 1.** For any $g \in L^2_j$, we have

$$(\overline{S_L^{(E)} - z})^{-1}g(t) = \int \left(\hat{y}_z^{(E)}(t)y_z^{(E)}(s)^\top 1_{s\leq t} + y_z^{(E)}(t)\hat{y}_z^{(E)}(s)^\top 1_{s>t}\right)R_{E'}(s)\, g(s)\, ds.$$

As a consequence, there is a constant $C > 0$ independent of $E$ s.t. the operator norm satisfies

$$\|((S_L^{(E)} - z)^{-1})\|_\infty \leq C\|y_z^{(E)}\|_\infty \|\hat{y}_z^{(E)}\|_\infty.$$
By the theory of Sturm-Liouville operators, we have an explicit integral form for the resolvent
$$Q(E)$$
for
$$E$$
follows.
Now observe that we have
$$\lambda = Q(t)u(t)$$
for
$$t \leq \tau$$
and
$$u(t)\sim\lambda$$
for
$$t > \tau$$.
As
$$L \sim (0,1)$$
the bound on the operator norm
$$\|L\|$$
is bounded by
$$1$$.
Furthermore from the identities
$$P_{E'}=Q(E)PQ^{-1}$$
and
$$S_{E'}(E) = Q(E)LH_{E'}^{-1}Q_{E'}^{-1}$$,
we obtain
$$P_{E'}(S_{E'}(E) - z)^{-1} = Q(E)LH_{E'}^{-1}Q_{E'}^{-1}.$$}
Now observe that we have
$$u(E)\lambda^{-1}Q_{E'}^{-1}g = (u(E)\frac{1}{L\sqrt{E}}(u(E))^{\prime})^{-1}TQ_{E'}Q_{E'}^{-1}g = (y(E))^{\prime}R_{E'}g,$$
and similarly
$$\lambda^{-1}Q_{E'}^{-1}g = (\frac{1}{L}\sqrt{E}(\frac{\lambda}{\sqrt{E}})^{\prime})^{-1}TQ_{E'}Q_{E'}^{-1}g = (\hat{y}(E))^{\prime}R_{E'}g.$$}
Furthermore
$$Q_{E'}Pr\lambda^{-1}Q_{E'}^{-1}g = \hat{y}(E), \quad Q_{E'}P\lambda t\lambda^{-1}Q_{E'}^{-1}g = y(E).$$
Putting everything together, we deduce the asserted expression for
$$P_{E'}(S_{E'}(E) - z)^{-1}g.$$ Finally, since the entries of the matrix
$$R_{E'}$$
are all bounded by
$$1$$, the bound on the operator norm follows.

Let
$$A_n, A$$
be random bounded operators on
$$L_2$$.
Recall that
$$A_n \rightarrow A$$
in law for the strong operator topology if the finite-dimensional marginals of the process
$$(A_n f, f \in L_2)$$
converge in law to those of
$$(Af, f \in L_2)$$.
Furthermore
$$A_n \rightarrow A$$
in law for the norm operator topology if the process
$$(A_n f, f \in L_2)$$
converges in law to the topology of uniform convergence on bounded sets to
$$(Af, f \in L_2)$$. The precise statement of Theorem 1.6 is then:

**Theorem (Strong-resolvent convergence: precise statement).** Fix
$$\tau > 0$$
and consider
$$E = E(L) \sim L/\tau.$$ As
$$L \rightarrow \infty$$
and for any given
$$z \in C \setminus \mathbb{R},$$
the operator
$$(S_{E'}(E) - z)^{-1}$$
converges in law towards
$$(CS_{E'} - z)^{-1}$$
for the strong operator topology. However it does not converge in law to
$$(CS_{E'} - z)^{-1}$$
for the norm operator topology.
3.3. **Strong resolvent convergence.** Recall from (40) and (29) the expressions of the resolvents at stake. These resolvents depend respectively on the pair of processes \((y_z^{(E)}, \hat{y}_z^{(E)})\), and \((y_z, \hat{y}_z)\). The main technical step consists in showing convergence in law of the former towards the latter.

Recall from (39) and (30) that \(\hat{y}_z^{(E)} = v_z^{(E)} - \alpha^{(E)} y_z^{(E)}\) and \(\hat{y}_z^{(E)} = v_z - \alpha y_z\). Note that almost surely \(\alpha^{(E)}\) is neither 0 nor \(\infty\). Indeed, suppose for instance that with positive probability \(\alpha^{(E)} = \infty\), then it means that \(y_z^{(E)}\) satisfies Dirichlet b.c. at 0 and 1 so that \(z\) is a non-real eigenvalue of the self-adjoint operator \(S_L^{(E)}\), thus yielding a contradiction. The reasoning is the same for \(\alpha^{(E)} = 0\) (with a contradiction with Neumann b.c.). Similarly almost surely \(\alpha\) is neither 0 nor \(\infty\).

Our main technical step is the following result, whose proof is postponed to the next section.

**Proposition 2.** For any \(z \in \mathbb{C}\setminus\mathbb{R}\), the process \((y_z^{(E)}, v_z^{(E)})\) converges in law to \((y_z, v_z)\) for the topology of uniform convergence.

We also need to control some oscillating terms. This is the content of the following lemma, which is also a key ingredient in the proof of Proposition 2. Its proof is also postponed to Section 4.

**Lemma 3.3.** Let \(h : [0, 1] \to \mathbb{R}\) be a smooth function and let \(f\) be either \(\sin(2E^i\cdot)\) or \(\cos(2E^i\cdot)\). Then for any \(i \in \{1, 2\}\) we have as \(L \to \infty\)

\[
\mathbb{E}\left[\sup_{0 \leq t \leq 1} \left| \int_0^t f(s)(y_z^{(E)})_i(s)h(s)ds \right| \right] \to 0.
\]

With these results at hand, we can proceed with the proof of the first part of the theorem.

**Proof of Theorem 1.6 - Strong resolvent convergence.** To prove strong resolvent convergence, it suffices to show that for any fixed \(g_1, \ldots, g_n \in L^2_1\), the vector \((S_L^{(E)} - z)^{-1}g_i\) converges in law to \((S_{\tau} - z)^{-1}g_i\). Note that the resolvents at stake are measurable functions of the processes \((y_z^{(E)}, v_z^{(E)})\) and \((y_z, v_z)\). We thus combine Proposition 2 and Skorohod’s Representation Theorem, and work under a coupling for which \((y_z^{(E)}, v_z^{(E)})\) converges almost surely to \((y_z, v_z)\). It now suffices to prove that for any function \(g \in L^2_1\), \((S_L^{(E)} - z)^{-1}g\) converges in probability to \((S_{\tau} - z)^{-1}g\).

Recall that the norm of the operator \((S_L^{(E)} - z)^{-1}\) is bounded by a constant times \(\|y_z^{(E)}\|_{\infty}\|\hat{y}_z^{(E)}\|_{\infty}\), and similarly for the norm of \((S_{\tau} - z)^{-1}\). From the almost sure uniform convergence of \((y_z^{(E)}, v_z^{(E)})\) towards \((y_z, v_z)\), we deduce that almost surely the norms of \((S_L^{(E)} - z)^{-1}\) and \((S_{\tau} - z)^{-1}\) are uniformly bounded. Since smooth functions are dense in \(L^2_1\), we can restrict ourselves to considering smooth functions \(g : [0, 1] \to \mathbb{R}^2\) in the sequel.

From Proposition 2, we deduce that the coefficient \(\alpha^{(E)}\) converges almost surely to \(\alpha\), and that the pair \((y_z^{(E)}, \hat{y}_z^{(E)})\) converges almost surely to the pair \((y_z, \hat{y}_z)\).
Let us rewrite the resolvents in the following way

\[
(S^{(E)}_L - z)^{-1} g(t) = \hat{y}_z^{(E)}(t) u^{(E)}(t) + y_z^{(E)}(t) \hat{u}^{(E)}(t), \quad t \in [0, 1],
\]

with

\[
\begin{align*}
u^{(E)}(t) & := \int_0^t y_z^{(E)}(s)^T R_E'(s) g(s) ds, \\
\hat{u}^{(E)}(t) & := \int_t^1 \hat{y}_z^{(E)}(s)^T R_E'(s) g(s) ds.
\end{align*}
\]

Similarly

\[
(CS_T - z)^{-1} g(t) = \hat{y}_z(t) u(t) + y_z(t) \hat{u}(t), \quad t \in [0, 1],
\]

with

\[
\begin{align*}
u(t) & := \frac{1}{2} \int_0^t y_z(s)^T g(s) ds, \\
\hat{u}(t) & := \frac{1}{2} \int_t^1 \hat{y}_z(s)^T g(s) ds.
\end{align*}
\]

Then we have

\[
\left\| (S^{(E)}_L - z)^{-1} g - (CS_T - z)^{-1} g \right\|_{L_2^2} \leq \left\| \hat{y}_z^{(E)} u^{(E)} - \hat{y}_z u \right\|_{L_2^2} + \left\| y_z^{(E)} \hat{u}^{(E)} - y_z \hat{u} \right\|_{L_2^2}.
\]

The arguments to bound the two terms on the r.h.s. are the same, so we provide the details only for the first. We write

\[
\left\| \hat{y}_z^{(E)} u^{(E)} - \hat{y}_z u \right\|_{L_2^2} \leq \left\| \hat{y}_z^{(E)} - \hat{y}_z \right\|_{L_2^2} \| u^{(E)} \|_{L_2^2} + \| y_z^{(E)} \hat{u}^{(E)} - y_z \hat{u} \|_{L_2^2}.
\]

The a.s. convergence of \( \hat{y}_z^{(E)} \) to \( \hat{y}_z \) ensures that the first term on the r.h.s. goes to 0 almost surely. Regarding the second term, we have

\[
\| y_z^{(E)} \hat{u}^{(E)} - y_z \hat{u} \|_{L_2^2} \leq \sup_{t \in [0, 1]} | y_z^{(E)} | \sup_{t \in [0, 1]} | u^{(E)}(t) - u(t) |.
\]

and it remains to show that \( \sup_{t \in [0, 1]} | u^{(E)}(t) - u(t) | \) goes to 0 in probability. Observe that

\[
u^{(E)}(t) - u(t) = \int_0^t y_z^{(E)}(s)^T (R_E'(s) - (1/2)I) g(s) ds + \frac{1}{2} \int_0^t (y_z(s) - y_z^{(E)})^T g(s) ds.
\]

From the almost sure convergence of \( y_z^{(E)} \) to \( y_z \), we deduce that the second term goes to 0 almost surely. Note that

\[
R_E'(s) - (1/2)I = \frac{1}{2} \begin{pmatrix} \cos 2E's & \sin 2E's \\ \sin 2E's & -\cos 2E's \end{pmatrix},
\]

so that the first term is a linear combination of expressions of the form

\[
\int_0^t f(s)(y_z^{(E)} i_1(s) g_j(s)) ds,
\]

where \( f(s) \) is either \( \cos 2E's \) or \( \sin 2E's \) and \( i, j \in \{1, 2\} \). Since \( E' \to \infty \) as \( L \to \infty \), the Riemann-Lebesgue Lemma should imply that this term goes to 0 almost surely as \( L \to \infty \): however, we are not exactly within the scope of this lemma since the (random) function \( y_z^{(E)} \) depends on \( L \). We thus rely on Lemma 3.3 stated above, and this suffices to conclude. \( \square \)
3.4. Absence of norm resolvent convergence. Set
\[ g_E(t) := (\sin^2 E't, -\sin E't \cos E't)^\top, \quad t \in [0, 1]. \]

It is easy to check that
\[ \|g_E\|_{L^2_t} \to \frac{1}{2}, \quad L \to \infty, \]
and therefore \( (g_E)_{L \geq 1} \) remains in a bounded set of \( L^2_t \).

Note that \( R_E^* g_E = 0 \) so that \( (S_L^{(E)} - z)^{-1} g_E = 0 \). To conclude, it suffices to show that as \( L \to \infty \), with positive probability \( (CS_\tau - z)^{-1} g_E \) does not converge to 0 in \( L^2_t \).

Recall that
\[ (CS_\tau - z)^{-1} g_E(t) = \hat{y}_z(t) u_E(t) + y_z(t) \hat{u}_E(t), \quad t \in [0, 1], \]
with
\[ u_E(t) := \frac{1}{2} \int_0^t y_z(s)^\top g_E(s) ds, \quad \hat{u}_E(t) := \frac{1}{2} \int_t^1 \hat{y}_z(s)^\top g_E(s) ds. \]

By the Riemann-Lebesgue Lemma, almost surely \( u_E, \hat{u}_E \) converge pointwise to \( u, \hat{u} \) where
\[ u(t) = \frac{1}{4} \int_0^t (y_z(s))_1 ds, \quad \hat{u}(t) = \frac{1}{4} \int_t^1 (\hat{y}_z(s))_1 ds. \]

Note that \( |u_E|_\infty \) and \( |\hat{u}_E|_\infty \) are almost surely bounded by \( (|y_z|_\infty + |\hat{y}_z|_\infty) \). Therefore by the Dominated Convergence Theorem, almost surely \( (CS_\tau - z)^{-1} g_E \) converges in \( L^2_t \) to
\[ t \mapsto \hat{y}_z(t) u(t) + y_z(t) \hat{u}(t). \]

Since \( y_z(t) \) and \( \hat{y}_z(t) \) are linearly independent for all \( t \in [0, 1] \), we deduce that the \( L^2_t \)-norm of the latter vanishes if and only if \( u(t) = \hat{u}(t) = 0 \) for almost every \( t \in [0, 1] \). The latter property would imply that \( (y_z)_1 \) and \( (\hat{y}_z)_1 \) are identically 0, which is not true almost surely. Consequently, almost surely \( (CS_\tau - z)^{-1} g_E \) converges in \( L^2_t \) to a non-degenerate limit, thus concluding the proof of Theorem 1.6.

4. Convergence of the SDEs. In this section, we prove Theorems 1.1 and 1.5. The arguments are relatively elementary: we show convergence of the system of SDEs associated with the operator \( S_L^{(E)} \) towards its counterpart for \( CS_\tau \). At the end of the section, we present the proof of Proposition 2 and Lemma 3.3, since the arguments are small modifications of the previous ones. Until the end of the section we always assume that \( E \sim L/\tau \) for some \( \tau > 0 \).

We consider the solutions \( y_\lambda^{(E)} \) of (37) starting from \( (0, 1)^\top \) at time 0. For \( \lambda \in \mathbb{R} \), it is convenient to consider the associated polar coordinates, also called Prüfer coordinates, implicitly defined by:
\[ (y_\lambda^{(E)})_1 = r_\lambda^{(E)}(t) \sin \theta_\lambda^{(E)}(t), \quad (y_\lambda^{(E)})_2 = r_\lambda^{(E)}(t) \cos \theta_\lambda^{(E)}(t). \]

Set \( W^{(L)} := W^{(L)}_1 + i W^{(L)}_2 := \sqrt{2} \int_0^t e^{2 E's} dB^{(L)}(s) \). Tidious applications of Itô’s formula (see also Remark 10) show that the equations for \( r_\lambda^{(E)} \) and \( \theta_\lambda^{(E)} \) are
\[ \begin{align*}
d\theta_\lambda^{(E)}(t) &= \frac{\lambda}{2} dt - \frac{\sqrt{L/E}}{2} dB^{(L)}(t) + \frac{\sqrt{L/E}}{2 \sqrt{2}} \Re(e^{i \theta_\lambda^{(E)}} dW^{(L)}(t)) + \mathcal{E}(\theta_\lambda^{(E)})(t) dt, \\
d \ln r_\lambda^{(E)}(t) &= \frac{L/E}{8} dt + \frac{\sqrt{L/E}}{2 \sqrt{2}} \Im(e^{i \theta_\lambda^{(E)}} dW^{(L)}(t)) + \mathcal{E}(\ln r_\lambda^{(E)})(t) dt,
\end{align*} \]
where the terms $\mathcal{E}(\cdot)$, which will be proven to be negligible in the limit $L \to \infty$, are given by

$$
\mathcal{E}(\theta^{(E)}_\lambda)(t) := \frac{2L - \lambda}{2} \cos(2\theta^{(E)}_\lambda(t) + 2E't) + \frac{L}{4E} \sin(2\theta^{(E)}_\lambda(t) + 2E't) - \frac{L}{8E} \sin(4\theta^{(E)}_\lambda(t) + 4E't),
$$

$$
\mathcal{E}(\ln r^{(E)}_\lambda)(t) := \frac{2L - \lambda}{2} \sin(2\theta^{(E)}_\lambda(t) + 2E't) - \frac{L}{4E} \cos(2\theta^{(E)}_\lambda(t) + 2E't) + \frac{L}{8E} \cos(4\theta^{(E)}_\lambda(t) + 4E't).
$$

In the above equations, the initial conditions are taken to be $\ln r^{(E)}_\lambda(0) = 0$ and $\theta^{(E)}_\lambda(0) = 0$.

**Remark 10.** Recall $u^{(E)}_\lambda$ from (11) and note that $u^{(E)}_\lambda = r^{(E)}_\lambda \sin(\theta^{(E)}_\lambda + E't)$ and $\frac{1}{L\sqrt{E}}(u^{(E)}_\lambda)' = r^{(E)}_\lambda \cos(\theta^{(E)}_\lambda + E't).$ Since the evolution equation of $u^{(E)}_\lambda$ is simpler than that of $y^{(E)}_\lambda$, one may prefer to apply Itô’s formula at this level, namely

$$
\ln r^{(E)}_\lambda = \frac{1}{2} \ln((u^{(E)}_\lambda)^2 + \frac{1}{L\sqrt{E}}(u^{(E)}_\lambda)')^2, \quad \theta^{(E)}_\lambda + E't = \arccotan \left( \frac{(u^{(E)}_\lambda)'}{L\sqrt{E}u^{(E)}_\lambda} \right).
$$

Let $\Theta_\lambda, \Gamma_\lambda$ be the solutions of (31) starting from $\Theta_\lambda(0) = 0$ and $\ln \Gamma_\lambda(0) = 0$. Observe the similarity between the SDEs solved by $(\Theta_\lambda, \Gamma_\lambda)$ and $(\theta^{(E)}_\lambda, r^{(E)}_\lambda)$.

**Proposition 3.** Fix $\tau > 0$ and consider $E = E(L) \sim L/\tau$. The collection, indexed by $L > 1$, of continuous processes $(\theta^{(E)}_\lambda(t), r^{(E)}_\lambda(t); t \in [0, 1], \lambda \in \mathbb{R})$ converges in law to $(\Theta_\lambda(t), \Gamma_\lambda(t); t \in [0, 1], \lambda \in \mathbb{R})$, for the topology of uniform convergence on compact sets of $[0, 1] \times \mathbb{R}$.

With this proposition at hand, we can proceed with the proof of the theorems.

**Proofs of Theorems 1.1 and 1.5.** Thanks to Lemma 2.2, we know that almost surely $\mathbb{R} \ni \lambda \mapsto \Theta_\lambda(1)$ is a continuous, increasing bijection from $\mathbb{R}$ to $\mathbb{R}$. The very same arguments ensure that this property also holds for $\mathbb{R} \ni \lambda \mapsto \theta^{(E)}_\lambda(1)$.

The convergence in law stated in Proposition 3 thus implies that the ordered sequence of hitting “times” of $\pi \mathbb{Z}$ by $\lambda \mapsto \theta^{(E)}_\lambda(1)$ converges to the corresponding sequence associated to $\lambda \mapsto \Theta_\lambda(1)$.

Note that $y^{(E)}_\lambda$, resp. $y_\lambda$, is a continuous function of $(\theta^{(E)}_\lambda, r^{(E)}_\lambda)$, resp. $(\Theta_\lambda, \Gamma_\lambda)$. We deduce that the point process

$$
\left\{ \left( \lambda, \frac{y^{(E)}_\lambda}{\|y^{(E)}_\lambda\|_{L^2}} \right) : \theta^{(E)}_\lambda(1) \in \pi \mathbb{Z} \right\},
$$

converges in law to the point process

$$
\left\{ \left( \lambda, \frac{y_\lambda}{\|y_\lambda\|_{L^2}} \right) : \Theta_\lambda(1) \in \pi \mathbb{Z} \right\}.
$$

This is exactly the convergence stated in Theorem 1.5. Since the point processes involved in the convergence stated in Theorem 1.1 are continuous projections of the above point processes, Theorem 1.1 follows. 

Remark 11. As already mentioned, the convergence of the eigenvalues of Theorems 1.1 and 1.5 is the continuous analog of the result of [11]. We believe that in Corollary 4 in [11], there should be no constant $\pi$ i.e. that the correct statement for the convergence of the eigenvalues of the discrete model (using their notations) is:

$$\Lambda_n - \arg(z^{2n+2}) \to \text{Schur}.$$  

The next three subsections are devoted to the proof of Proposition 3, while the last subsection provides the arguments for the proof of Proposition 2.

4.1. Tightness. Suppose we can show that for any $p \geq 2$, there exists a constant $C > 0$ such that for all $L > 1$, for all $\mu < \lambda$ and all $0 \leq s \leq t \leq 1$

$$E[|\theta^{(E)}_{\mu}(t) - \theta^{(E)}_{\lambda}(s)|^{2p}] \leq C|t - s|^p, \quad E[|\ln r^{(E)}_{\mu}(t) - \ln r^{(E)}_{\lambda}(s)|^{2p}] \leq C|t - s|^p,$$

and

$$E[|\theta^{(E)}_{\lambda}(t) - \theta^{(E)}_{\mu}(t)|^{2p}] \leq C|\lambda - \mu|^p, \quad E[|\ln r^{(E)}_{\lambda}(t) - \ln r^{(E)}_{\mu}(t)|^{2p}] \leq C|\lambda - \mu|^p.$$  

Then, by Kolmogorov-Centsov’s Theorem [9, Th 2.23 & Th 14.9], we deduce that there exists a constant $\beta > 0$ such that for any $p \geq 1$ and any compact set $K \subset \mathbb{R}$

$$\sup_{L \geq 1} E \left[ \sup_{0 \leq s \leq t \leq 1} \sup_{\lambda, \mu \in K} \left( \frac{|\theta^{(E)}_{\mu}(t) - \theta^{(E)}_{\lambda}(s)|}{(t - s) + |\lambda - \mu|^{\beta}} \right)^p \right] < \infty,$$

and similarly for $\ln r^{(E)}_{\lambda}$. Since in addition $\theta^{(E)}_{\lambda}(0) = \ln r^{(E)}_{\lambda}(0) = 0$, we deduce that the collection of processes is tight.

It remains to prove the above bounds. The increments in $t$ are easy to control: since the drift and diffusion coefficients of the SDE are bounded by some constant (uniformly over all parameters), we get the desired bound using the triangle inequality (to control separately the terms coming from the drift and the martingale) and the Burkholder-Davis-Gundy inequality (to control the martingale term). Note that the bound of the drift term is of order $|t - s|^{2p}$ while the bound of the martingale term is only of order $|t - s|^p$.

On the other hand, the increments in $\lambda$ require some work: fix $p \geq 2$ and let us start with $\theta^{(E)}_{\lambda}$. Since the coefficients of the SDE are Lipschitz in $\theta^{(E)}_{\lambda}$, we deduce that there exists $C = C(p) > 0$ such that for all $\mu \leq \lambda$ and for all $t \in [0, 1]$ we have

$$E[|\theta^{(E)}_{\lambda}(t) - \theta^{(E)}_{\mu}(t)|^{2p}] \leq C \left( |\lambda - \mu|^p + \int_0^t E[|\theta^{(E)}_{\lambda}(s) - \theta^{(E)}_{\mu}(s)|] ds + E[|M_t|^{2p}] \right),$$

where

$$M_t := \frac{\sqrt{2n}L/L}{2} \int_0^t \left( \cos(2\theta^{(E)}_{\lambda}(s) + 2E^s) - \cos(2\theta^{(E)}_{\mu}(s) + 2E^s) \right) dB^{(L)}(s).$$

Combining the Burkholder-Davis-Gundy inequality and the Jensen inequality, there exists a constant $C' > 0$ such that for all $t \in [0, 1]$,

$$E[|M_t|^{2p}] \leq C' \int_0^t E[|\theta^{(E)}_{\lambda}(s) - \theta^{(E)}_{\mu}(s)|^{2p}] ds.$$

The desired bound on $E[|\theta^{(E)}_{\lambda}(t) - \theta^{(E)}_{\mu}(t)|^{2p}]$ then follows from Grönwall’s lemma.

We turn to $\ln r^{(E)}_{\lambda}$. The strategy is the same, the only difference is that the coefficients of the SDE do not depend on $\ln r^{(E)}_{\mu}$ but on $\theta^{(E)}_{\mu}$. Since we already established bounds on the increments of the latter, one can easily conclude.
4.2. Control of the error terms. Before we identify the limit of any converging subsequence, let us control the error terms appearing in the SDEs (41).

**Lemma 4.1.** For any \( \lambda \in \mathbb{R} \), the following convergences hold in probability as \( L \to \infty \)

\[
\sup_{0 \leq t \leq 1} \left| \int_0^t \mathcal{E}(\theta^{(E)}_\lambda)(s) \, ds \right| \to 0 , \quad \sup_{0 \leq t \leq 1} \left| \int_0^t \mathcal{E}(\ln r^{(E)}_\lambda)(s) \, ds \right| \to 0 .
\]

**Proof.** Given the terms that appear in \( \mathcal{E} \), it suffices to show that for any functions \( f, g \) of the form \( \cos(a \cdot), \sin(a \cdot) \), with \( a \in \{1, 2, 4\} \), we have the following convergence in probability as \( L \to \infty \)

\[
\sup_{0 \leq t \leq 1} \left| \int_0^t f(\theta^{(E)}_\lambda(s)) g(E's) \, ds \right| \to 0 .
\]

This is not a direct consequence of the Riemann-Lebesgue Lemma since \( \theta^{(E)}_\lambda \) depends on \( L \). Without loss of generality, we can take \( f = \sin(\cdot) \) and \( g = \cos(\cdot) \). By Itô’s formula we find

\[
\int_0^t \sin(\theta^{(E)}_\lambda(s)) \cos(E's) \, ds = \frac{1}{E'} \sin(\theta^{(E)}_\lambda(t)) \sin(E't) - \frac{1}{E'} \int_0^t \sin(E's) \left( \cos(\theta^{(E)}_\lambda(s)) d\theta^{(E)}_\lambda(s) - \frac{1}{2} \sin(\theta^{(E)}_\lambda(s)) d(\theta^{(E)}_\lambda) \right)
\]

Recall that \( E' \to \infty \) as \( L \to \infty \). Obviously, the first term on the r.h.s. goes to 0 uniformly over \( t \in [0, 1] \). Regarding the second term, it can be split into martingale and non-martingale terms. The non-martingale terms go to 0 in probability uniformly over \( t \in [0, 1] \) since all the terms appearing inside the integral are uniformly bounded by some deterministic constant.

The martingale term is given by

\[
N_t := -\frac{1}{E'} \int_0^t \sin(E's) \cos(\theta^{(E)}_\lambda(s)) \frac{\sqrt{L/E}}{2\sqrt{2}} \Re(e^{2i\theta^{(E)}_\lambda} \, dW^{(L)}(s)) .
\]

The Burkholder-Davis-Gundy inequality ensures that there exists a constant \( C > 0 \) such that

\[
\mathbb{E}[\sup_{0 \leq t \leq 1} N_t^2] \leq C \mathbb{E}[\langle N \rangle_1] .
\]

Since the r.h.s. is of order \((1/E')^2\), we deduce that \( \sup_{0 \leq t \leq 1} |N_t| \) goes to 0 in probability, as required. \( \square \)

4.3. Identification of the limit. Fix \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \). We will identify the law of any converging subsequence of \( (\theta^{(E)}_{\lambda_i}, \ln r^{(E)}_{\lambda_i})_{1 \leq i \leq n} \) through the following standard martingale problem, whose proof can be found in [10, Prop 5.4.6]

**Proposition 4.** Let \( (\theta_{\lambda_i}, \ln r_{\lambda_i})_{1 \leq i \leq n} \) be a continuous process on \([0, 1]\) and let \( \mathcal{F} \) be the associated filtration. Assume that

\[
M_{\lambda_i}(t) = \theta_{\lambda_i}(t) - \frac{\lambda_i}{2} t , \quad N_{\mu_i}(t) = \ln r_{\mu_i}(t) - \frac{\mu_i}{8} t .
\]

together with

\[
M_{\lambda_i}(t) M_{\lambda_j}(t) - \frac{\tau}{4} t - \frac{\tau}{8} \int_0^t \cos(2\theta_{\lambda_i} - 2\theta_{\lambda_j}) \, ds ,
\]

\[
N_{\lambda_i}(t) N_{\lambda_j}(t) - \frac{\tau}{8} \int_0^t \cos(2\theta_{\lambda_i} - 2\theta_{\lambda_j}) \, ds ,
\]

(44)

\[
M_{\lambda_i}(t) N_{\lambda_j}(t) - \frac{\tau}{8} \int_0^t \sin(2\theta_{\lambda_i} - 2\theta_{\lambda_j}) \, ds ,
\]
are $\mathcal{F}$-martingales. Then $(\theta_{\lambda_i}, \ln r_{\lambda_i})_{1 \leq i \leq n}$ coincides in law with the unique solution of the SDEs (31) associated to the parameters $\lambda_1, \ldots, \lambda_n$.

Recall the SDEs (41) and define the martingales
\begin{align}
M^{(E)}_{\lambda_i}(t) &= \theta^{(E)}_{\lambda_i}(t) - \frac{\lambda_i}{2} t - \int_0^t \mathcal{E}(\theta^{(E)}_{\lambda_i})(s) ds , \\
N^{(E)}_{\lambda_i}(t) &= \ln r^{(E)}_{\lambda_i}(t) - \frac{L/E}{8} t - \int_0^t \mathcal{E}(\ln r^{(E)}_{\lambda_i})(s) ds .
\end{align}
From the moment bounds established for tightness, we easily deduce that all moments of these martingales are bounded uniformly over $L > 1$.

Let $(\theta_{\lambda_i}, \ln r_{\lambda_i})_{1 \leq i \leq n}$ be the limit of a converging subsequence of $(\theta^{(E)}_{\lambda_i}, \ln r^{(E)}_{\lambda_i})_{1 \leq i \leq n}$; for simplicity we keep the same notation for the subsequence. We naturally define
\begin{align}
M_{\lambda_i}(t) &= \theta_{\lambda_i}(t) - \frac{\lambda_i}{2} t , \\
N_{\lambda_i}(t) &= \ln r_{\lambda_i}(t) - \frac{\tau}{8} t .
\end{align}
Thanks to Lemma 4.1, we can pass to the limit on (45) and we obtain (46). Given the aforementioned moment bounds, we also deduce that $M_{\lambda_i}, N_{\lambda_i}$ are martingales (in the natural filtration associated to the processes at stake). We now identify their brackets.

Again from (41) we see that the processes
\begin{align}
M^{(E)}_{\lambda_i}(t)M^{(E)}_{\lambda_j}(t) - \frac{L/E}{4} \int_0^t (-1 + \cos(2\theta^{(E)}_{\lambda_i}(s) + 2E's))(-1 + \cos(2\theta^{(E)}_{\lambda_j}(s) + 2E's))ds , \\
N^{(E)}_{\lambda_i}(t)N^{(E)}_{\lambda_j}(t) - \frac{L/E}{4} \int_0^t \sin(2\theta^{(E)}_{\lambda_i}(s) + 2E's)\sin(2\theta^{(E)}_{\lambda_j}(s) + 2E's)ds , \\
M^{(E)}_{\lambda_i}(t)N^{(E)}_{\lambda_j}(t) - \frac{L/E}{4} \int_0^t (-1 + \cos(2\theta^{(E)}_{\lambda_i}(s) + 2E's))\sin(2\theta^{(E)}_{\lambda_j}(s) + 2E's)ds ,
\end{align}
are martingales. We aim at passing to the limit on (47). We can compute the limits of the three integrals therein: by expanding the cos and sin functions, the oscillating terms in $E'$ will vanish thanks to the Riemann-Lebesgue-type argument of the previous subsection, and the remaining terms match with the ones in the integrals that appear in (44). Combining this with the aforementioned moment bounds, we deduce that the processes of (44) are also martingales. We can then apply the martingale problem recalled above and this completes the proof of Proposition 3.

4.4. Proof of Proposition 2. The proof is very close to the proof of Proposition 3. The main difference is that we consider solutions of (37) with a non-real parameter $z$ so that the polar representation used previously does not hold anymore: hence we work directly at the level of the SDEs (37) to prove the convergences. Note that we prove the convergence for a fixed $z$, although the arguments could be adapted to get the local uniform convergence of the family.
The SDE (37) solved by $y_z^{(E)}$ can be written in the following way:

$$
dy_z^{(E)}(t) = \left( \frac{z}{2} J^{-1} dt - \frac{\sqrt{L/E}}{2} J^{-1} \left( dB(t) + \frac{1}{\sqrt{2}} dW_1(t) \frac{1}{\sqrt{2}} dW_2(t) \right) y_z^{(E)}(t) \right) + \mathcal{E}(t) y_z^{(E)}(t) dt ,
$$

where

$$
\mathcal{E}(t) := \frac{z - 2\ell_E}{2} J^{-1}(2R_{E'} - I) , \quad J^{-1}(2R_{E'} - I) = \begin{pmatrix} \sin 2E' t & -\cos 2E' t \\ -\cos 2E' t & \sin 2E' t \end{pmatrix}.
$$

The putative limit satisfies

$$
dy_z(t) = \left( \frac{z}{2} J^{-1} dt - \frac{\sqrt{\tau}}{2} J^{-1} \left( dB(t) + \frac{1}{\sqrt{2}} dW_1(t) \frac{1}{\sqrt{2}} dW_2(t) \right) y_z(t) \right) .
$$

The processes $y_z^{(E)}$ and $v_z^{(E)}$, resp. $y_z$ and $v_z$, satisfy the same equations, the only difference lies in the initial conditions:

$$
y_z^{(E)}(0) = y_z(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad v_z^{(E)}(0) = v_z(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} .
$$

Consequently, the proof of the tightness relies on exactly the same arguments for $y_z^{(E)}$ and $v_z^{(E)}$, and we restrict ourselves to presenting the details for the former. We start with some a priori bounds.

**Lemma 4.2.** For any $p \geq 1$, there exists $C > 0$ such that

$$
\sup_{L > 1} \sup_{t \in [0,1]} \mathbb{E}[|y_z^{(E)}(t)|^p] < \infty , \quad \sup_{t \in [0,1]} \mathbb{E}[|y_z(t)|^p] < \infty .
$$

**Proof.** Fix $p \geq 2$. From the integral form of the SDE above, applying successively the Burkholder-Davis-Gundy inequality and the Jensen inequality we get the existence of some deterministic constants $C, C' > 0$ (depending on $z$ and $p$) such that uniformly over all $L$ and $t$:

$$
\mathbb{E}[|y_z^{(E)}(t)|^p] \leq 1 + C \left( \int_0^t \mathbb{E}[|y_z^{(E)}(s)|^p] ds + \mathbb{E}[\left( \int_0^t |y_z^{(E)}(s)|^2 ds \right)^{p/2}] \right)
$$

$$
\leq 1 + C' \int_0^t \mathbb{E}[|y_z^{(E)}(s)|^p] ds .
$$

Grönwall’s Lemma then yields the desired bound. The proof is the same for $y_z$.

We now control the oscillations thanks to Lemma 3.3, which was stated in Subsection 3.3 and also used for the proof of the strong resolvent convergence.

**Proof of Lemma 3.3.** The arguments are essentially the same as those of the proof of Lemma 4.1. Take $f = \cos(2E')$ and $i = 1$ without loss of generality. By Itô’s formula, we have

$$
\int_0^t \cos(2E's)(y_z^{(E)})_1(s)h(s)ds = \frac{1}{2E'} (y_z^{(E)})_1(t)h(t) \sin(2E't)
$$

$$
- \frac{1}{2E'} \int_0^t \sin(2E's) \left( h'(s)(y_z^{(E)})_1(s) + h(s)d(y_z^{(E)})_1(s) \right) ds .
$$
Since $E' \to \infty$ and given the bounds of Lemma 4.2 it is easy to check that the expectation of all terms goes to 0, except for the martingale term produced by $d(y_z^{(E)})_1(s)$ which requires some additional work. This martingale term is given by

$$N_t := \frac{\sqrt{L/E}}{4E'} \int_0^t \sin(2E's)h(s)\left(\frac{1}{\sqrt{2}}(y_z^{(E)}(s))_1dW_2^{(L)}(s) + (y_z^{(E)}(s))_2 dB^{(L)}(s) - \frac{1}{\sqrt{2}}dW_1^{(L)}(s)\right) ds.$$  

By the Burkholder-Davis-Gundy inequality there exists $C > 0$ such that

$$\mathbb{E}\left[\sup_{0 \leq t \leq 1} N_t^2\right] \leq \frac{C}{(E')^2} \mathbb{E}\left[\int_0^t h(s)^2 |y_z^{(E)}(s)|^2 ds\right],$$

which, in view of Lemma 4.2, goes to 0 as $L \to \infty$.\hfill $\square$

As a consequence of Lemma 3.3, we deduce that $\sup_{0 \leq t \leq 1} |\int_0^t \mathcal{E}(s)y_z^{(E)}(s) ds|$ goes to 0 in probability as $L \to \infty$.

Let us now prove tightness of $(y_z^{(E)})_{L \geq 1}$ (once again, the arguments are exactly the same for $(v_z^{(E)})_{L \geq 1}$). Fix $p \geq 1$. By the triangle and the Burkholder-Davis-Gundy inequalities at the first line, the H"{o}lder inequality at the second line and Lemma 4.2 at the third line, there exist some constants $C, C' > 0$ such that for all $L > 1$ and all $0 \leq s \leq t \leq 1$

$$\mathbb{E}[|y_z^{(E)}(t) - y_z^{(E)}(s)|^{2p}] \leq C\mathbb{E}\left[\left(\int_s^t |y_z^{(E)}(r)| dr\right)^{2p}\right] + \mathbb{E}\left[\left(\int_s^t |y_z^{(E)}(r)|^2 dr\right)^p\right]$$

$$\leq C\int_s^t \mathbb{E}\left[|y_z^{(E)}(r)|^2 dr\right] \times \left((t-s)^{2p-1} + (t-s)^{p-1}\right)$$

$$\leq C'|t-s|^p.$$  

By Kolmogorov-Centsov’s Theorem [9, Th 2.23 & Th 14.9], we deduce that there exists a constant $\beta > 0$ such that for any $p \geq 1$

$$\sup_{L \geq 1} \mathbb{E}\left[\sup_{0 \leq s \leq t \leq 1} \left(\frac{|y_z^{(E)}(t) - y_z^{(E)}(s)|}{|t-s|^\beta}\right)^p\right] < \infty,$$

and tightness follows.

The identification of the limit of any converging subsequence of $(y_z^{(E)}, v_z^{(E)})_{L \geq 1}$ can be carried out with a martingale problem as in Subsection 4.3 and with the help of Lemma 3.3: the arguments being virtually the same, we do not provide the details.

5. Top of the spectrum. In this section, we assume that $E = E(L) \gg L$ and we explain how the previous arguments can be adapted to establish Theorem 1.7. First of all, since the limiting objects appearing in that statement are all deterministic the asserted convergences in probability are granted provided that convergence in law holds, and this is what we are going to prove.

Let $\mathcal{D}(F)$ be the closure in $H^1((0,1), \mathbb{R}^2)$ of all smooth functions $f : [0,1] \to \mathbb{R}^2$ such that $f_1(0) = f_1(1) = 0$. One can check that the operator $F$ on $\mathcal{D}(F)$ is self-adjoint. Following the same steps as in Section 2 we see that the SDEs associated to the operator $F$ are trivial:

$$dy_z(t) = -\frac{z}{2} J y_z(t) dt, \quad t \in [0,1], \quad z \in \mathbb{C}, \quad y_z(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
and for $\lambda \in \mathbb{R}$

$$
d\Theta_\lambda(t) = \frac{\lambda}{2} dt, \quad \Theta_\lambda(0) = 0,$$

$$
d\ln \Gamma_\lambda(t) = 0 dt, \quad \Gamma_\lambda(0) = 1.
$$

Their solutions are given by $y_\lambda(t) = (\sin(\lambda t/2), \cos(\lambda t/2))$, $\Theta_\lambda(t) = \lambda t/2$ and $\Gamma_\lambda(t) = 1$.

The proofs of the convergences are exactly the same as those presented in Sections 3 and 4: the only difference is that many terms, that had non-trivial contributions in the limit in the regime $E \sim L/\tau$, now vanish in the limit since $L/E \to 0$.

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