An ideal proof for Fujisawa’s result and its generalization

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Abstract. — We give a generalization of Fujisawa’s theorem in [F]. Our proof of the generalized theorem is purely algebraic and it is simpler than his proof.

1 Introduction

This article is a sequel of my article [N3]. The principal aim in this article is to give an additional result to Fujisawa’s article [F] by giving an ideal proof for it. The proof for this result is partly useful as an ideal proof for the main theorem in [loc. cit.]. His proof is not purely algebraic; our proof is purely algebraic and simpler than his proof in [loc. cit.]. Because of the algebraicity of our proof, we can give a generalization of his result without difficulty. First let us recall his theorem.

Let $r$ be a fixed positive integer. Let $s_\mathbb{C}$ be a log analytic space whose underlying analytic space is $\text{Spec}(\mathbb{C}^{an})$ and whose log structure is associated to a morphism $\mathbb{N}^r \ni e_i \mapsto 0 \in \mathbb{C}$ of monoids. Here $e_i$ $(1 \leq i \leq r)$ is a canonical basis of $\mathbb{N}^r$. Let $M_{s_\mathbb{C}}$ be the log structure of $s_\mathbb{C}$. Set $\overline{M}_{s_\mathbb{C}} := M_{s_\mathbb{C}}/\mathbb{C}^*$. Let $t_i$ be the section of $\overline{M}_{s_\mathbb{C}}$ corresponding to $e_i$. Consider a section $d\log t_i \in \Omega^1_{s_\mathbb{C}/\mathbb{C}}$. This is independent of the choice of the lift of $t_i$ in $M_{s_\mathbb{C}}$. Let $f: X \rightarrow s_\mathbb{C}$ be a (not necessarily proper) log smooth morphism of reduced log analytic spaces. Let $\check{X}$ be the underlying analytic space of $X$. Assume that there exist local generators of the ideal sheaves of the irreducible components of $\check{X}$ which are the images of local sections of the log structure $M_X$ of $X$ by the structural morphism $\alpha_X: M_X \rightarrow \mathcal{O}_X$. Let $\hat{\Delta} := \{ x \in \mathbb{C} \mid |x| < 1 \}$ be the unit disk. Assume that the any irreducible component $\check{X}_\lambda$ ($\lambda \in \Lambda$) of $\check{X}$ is smooth over $\mathbb{C}$ and that $\check{X}$ fits locally into the following cartesian diagram

$$
\begin{array}{ccc}
\check{X} & \xrightarrow{c} & \hat{\Delta}^n \\
\downarrow & & \downarrow \\
\check{s}_\mathbb{C} & \xrightarrow{c} & \hat{\Delta}^r
\end{array}
$$

(1.0.1)

such that any $\check{X}_\lambda$ in this diagram is defined by equations $x_{i_1} = \cdots = x_{i_k} = 0$ for some $1 \leq i_1 < \cdots < i_k \leq n$, where $x_1, \ldots, x_n$ are the standard coordinates of $\hat{\Delta}^n$. If $X$ is locally a finitely many product of SNCL(=simple normal crossing log) analytic spaces over the standard log point of $\text{Spec}(\mathbb{C}^{an})$, then these assumptions are satisfied.

*2020 Mathematics subject classification number: 14F40.
Let $\Omega^{\bullet}_{X/sC}$ be the log de Rham complex of $X/sC$. Let $F$ be the log Hodge filtration on $\Omega^{\bullet}_{X/sC}$ as usual: $F^i\Omega^{\bullet}_{X/sC} := \Omega^{\leq i}_{X/sC}$. Let $\hat{X}^{(m)}$ (m $\in \mathbb{N}$) be the disjoint union of $(m+1)$-fold intersections of the different irreducible components of $\hat{X}$. Endow $\hat{X}^{(m)}$ with the inverse image of the log structure of $X$ by the natural morphism $a_{(m)}: \hat{X}^{(m)} \to \hat{X}$ and let $X^{(m)}$ be the resulting log analytic space. This space is a log analytic space over $sC$ and $\hat{sC}$. Let $f^{(m)}: X^{(m)} \to sC$ be the structural ideally log smooth morphism. Let $\Omega^{\bullet}_{X^{(m)}/sC}$ be the log de Rham complex of $X^{(m)}/sC$. By abuse of notation, denote by $d\log t_i \in \Omega^1_{X^{(m)}/sC}$ the image of $d\log t_i$ in $\Omega^1_{X^{(m)}/sC}$ by the morphism $f^{(m)*}: f^{(m)*}(\Omega^1_{sC}/sC) \to \Omega^1_{X^{(m)}/sC}$. Let $\Omega^{\bullet}_{X^{(m)}/sC}[u_1,\ldots,u_r]$ be the Hirsch extension of $\Omega^{\bullet}_{X^{(m)}/sC}$ by a morphism $\mathbb{C}^r \ni u \mapsto d\log t_i \in \Omega^1_{X^{(m)}/sC}$, where $u := (0,0,1,0,\ldots,0)$. (The notion of the Hirsch extension of a dga by a vector space appears in the definition of a minimal model of a dga in Sullivan’s theory.)

That is, $\Omega^{\bullet}_{X^{(m)}/sC}[u_1,\ldots,u_r]$ is a complex of $\mathbb{C}$-vector spaces on $X^{(m)}$ defined by $\Omega^{(m)}_{\hat{X}^{(m)}/sC}[u_1,\ldots,u_r] := \bigoplus_{\text{i.e.}} \otimes \Omega^i_{\hat{X}^{(m)}/sC}$ with boundary morphism induced by the derivative $d: \Omega^i_{\hat{X}^{(m)}/sC} \to \Omega^{i+1}_{\hat{X}^{(m)}/sC}$ and $u^e 1 \mapsto e u^e - d\log t_i \in \Omega^1_{X^{(m)}/sC}$. By abuse of notation, denote also by $d$ the boundary morphism $\Omega^i_{\hat{X}^{(m)}/sC}[u_1,\ldots,u_r] \to \Omega^{i+1}_{\hat{X}^{(m)}/sC}[u_1,\ldots,u_r]$. (In our mind, we consider $u$ as “$d\log t_i$” only in this introduction whose can be considered as a function of the universal cover of $\Delta \setminus \{O\}$.) Fix a total order on $\Lambda$. Let $H(X/sC)$ be the single complex of the double complex $\{a_{(m)^*}(\Omega^i_{X^{(m)}/sC}[u_1,\ldots,u_r])\}$ $(m,i \in \mathbb{N})$ with the following horizontal and vertical boundary morphisms as in [NI]:

\[
\begin{array}{cccccc}
\cdots & \longrightarrow & \cdots \\
\uparrow & & \uparrow \\
\cdots & \longrightarrow & \cdots \\
\uparrow & & \uparrow \\
\cdots & \longrightarrow & \cdots \\
\end{array}
\]

where $\rho^{(m)*}$ is the standard Čech morphism which will be recalled in the text. Set $\underline{\epsilon}^m := u_1^{e_1} \cdots u_r^{e_r}$ and $|\underline{\epsilon}| := \sum_{i=1}^r e_i$, where $\underline{\epsilon} := (e_1,\ldots,e_r)$. (The complex $H(X/sC)$ has been denoted by $K_C$ in [NI].) Set

\[
F^i(\Omega^{\bullet}_{X^{(m)}/sC}[u_1,\ldots,u_r]) := \bigoplus_{|\underline{\epsilon}| \leq i} \mathbb{C}[\underline{\epsilon}] \otimes \Omega^{\leq i-|\underline{\epsilon}|}_{X^{(m)}/sC}.
\]

This filtration indeed induces a filtration $F$ on $H(X/sC)$. Let $\Omega^{\bullet}_{X^{(m)}/sC}$ be the log de Rham complex of $X^{(m)}/sC$ and let $s(a_{\Lambda^*}(\Omega^{\bullet}_{X^{(m)}/sC}))$ be the single complex of the double complex $\{a_{\Lambda^*}(\Omega^i_{X^{(m)}/sC})\}_{m,i \in \mathbb{N}}$. The complex $s(\Omega^{\bullet}_{X^{(m)}/sC})$ has a natural log Hodge filtration $F$: $F^s(\Omega^{\bullet}_{X^{(m)}/sC}) := s(\Omega^{\leq i}_{X^{(m)}/sC})$. Let $(H(X/sC), F) \longrightarrow$
$(s(a^*(\Omega^*_{X^{(\bullet)}/sC}), F)$ be the natural morphism defined by the morphism $u_i \mapsto 0$ and the projection $\Omega^*_{X^{(\bullet)}/C} \rightarrow \Omega^*_{X^{(\bullet)}/sC}$. In his article [F] Fujisawa has proved that the natural morphism

$$
(\Omega^*_{X/sC}, F) \rightarrow (s(a^*(\Omega^*_{X^{(\bullet)}/sC}), F)
$$

is a filtered quasi-isomorphism. Consequently he has proved that there exists a filtered morphism

$$
(H(X/sC), F) \rightarrow (\Omega^*_{X/sC}, F)
$$

in the derived category $D^+(\mathbb{C}_X)$ of bounded below filtered complexes of sheaves of $\mathbb{C}$-vector spaces on $\hat{X}$. (Strictly speaking, this has not been stated in [loc. cit.] because the category $D^+(\mathbb{C}_X)$ has not appeared in [loc. cit.] (cf. [F, Corollary 5.7]).) Here note that the filtration $F$ on $H(X/sC)$ is not finite; we need a formalism of derived categories of filtered complexes in e.g., [NS], which is a special case of the formalism of derived categories of filtered complexes in [B]. Furthermore he has proved that the morphism (1.0.3) induces an isomorphism

$$
gr^i_F H(X/sC) \sim \rightarrow gr^i_F \Omega^*_{X/sC}.
$$

By this fact, we see that there exists a large number $N$ such that there exists an isomorphism

$$
H(X/sC)/F^N H(X/sC) \sim \rightarrow \Omega^*_{X/sC}
$$

if $\hat{X}$ is proper over $sC$. We think that the subsheaf $F^N H(X/sC)$ should be removed if possible. By using the isomorphism (1.0.3) and by constructing a $\mathbb{Q}$-structure $H_Q$ of $H(X/sC)$ with a weight filtration $W$ and by using his generalization on mixed Hodge complexes and a fundamental theorem on mixed Hodge structures, he has proved the following:

**Theorem 1.1 ([F, p. 94]).** Assume that the irreducible components of $\hat{X}$ are proper and Kähler. Then the morphism (1.0.5) induces the following isomorphism of cohomologies with Hodge filtrations:

$$
(H^q(X, H(X/sC), F) \sim \rightarrow (H^q(X, \Omega^*_{X/sC}, F).
$$

By using this theorem, the weight filtration on the $\mathbb{Q}$-structure $H_Q$ of $H(X/sC)$ constructed in [F] and the general theory of Hodge-Deligne theory, Fujisawa has proved that the log Hodge-de Rham spectral sequence

$$
E_1^{ij} = H^j(X, \Omega^i_{X/sC}) \Rightarrow H^{i+j}(X, \Omega^*_{X/sC})
$$

degenerates at $E_1$ under the assumption of (1.1).

In this article we prove that the filtered morphism

$$
(H(X/sC), F) \rightarrow (\Omega^*_{X/sC}, F)
$$

is indeed a filtered isomorphism in the filtered derived category $D^+(\mathbb{C}_X)$ without assuming that the irreducible components of $\hat{X}$ are proper nor Kähler (we assume only that the irreducible components of $\hat{X}$ are smooth) and without using any result on
(generalized) mixed Hodge complex in [F]. (The theory of generalized mixed Hodge complex becomes unnecessary.) Consequently we give a much simpler and algebraic proof of the isomorphism (1.1.1) without the assumption in (1.1). In fact, we give the isomorphism at the level of the filtered complexes. Moreover we give a generalization of this result for certain coefficients as follows:

\textbf{Theorem 1.2.} Let \( X/\mathcal{C} \) be as before \( (1.1) \). Let \( \mathcal{E} \) be a locally free coherent \( \mathcal{O}_X \)-module and let \( \nabla: \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathcal{C}} \) be a locally nilpotent integrable connection on \( Y \) with respect to \( s_{\mathcal{C}} \) (see the text for the definition of this notion). Let \( \nabla: \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathcal{C}} \) be the induced connection. Endow \( \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_{X/\mathcal{C}} \) with a filtration \( F \) on \( \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_{X/\mathcal{C}} \) as follows: \( F^i(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_{X/\mathcal{C}}) := \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_{X/\mathcal{C}}^{i+1} \). Set

\[ H(X/\mathcal{C}, \mathcal{E}) := s(\mathcal{E} \otimes_{\mathcal{O}_X} u_e(\mathcal{E}) \Omega^*_{X/\mathcal{C}}^{[u_1, \ldots, u_r]}) \]

with already essentially defined boundary morphism and

\[ F^i(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_{X/\mathcal{C}}^{[u_1, \ldots, u_r]}) := \bigoplus_{\mathbb{C} \in \mathbb{N}_r} \mathbb{C} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_{X/\mathcal{C}}^{[u_1, \ldots, u_r]} \].

Then there exists the following filtered isomorphism

\begin{equation}
(1.2.1) \quad (H(X/\mathcal{C}, \mathcal{E}), F) \sim (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_{X/\mathcal{C}}, F).
\end{equation}

Moreover this isomorphism is contravariantly functorial with respect to certain morphisms of \( X/\mathcal{C} \)'s.

This theorem is a special case of the main result (1.7) below. As a corollary we obtain the following without assuming that \( \hat{X} \) is proper over \( \mathbb{C} \) nor without assuming that the irreducible components of \( \hat{X} \) are Kähler:

\textbf{Corollary 1.3.} There exists the following contravariantly functorial isomorphism of filtered complexes with respect to certain morphisms of \( X/\mathcal{C} \)'s and \( (\mathcal{E}, \nabla) \):

\begin{equation}
(1.3.1) \quad R^\Gamma(X, (H(X/\mathcal{C}, \mathcal{E}), F)) \sim R^\Gamma(X, (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_{X/\mathcal{C}}, F)).
\end{equation}

Consequently there exists the following isomorphism

\begin{equation}
(1.3.2) \quad (H^q(X, (H(X/\mathcal{C}, \mathcal{E}), F)) \sim (H^q(X, \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_{X/\mathcal{C}}), F) \quad (q \in \mathbb{Z}).
\end{equation}

(1.3.2) is a generalization of (1.1.1). The example of \( \mathcal{E} \) is obtained by a local system on \( \hat{X} \). More generally \( \mathcal{E} \) is obtained by a locally unipotent local system on \( X^{log} \) by the log Riemann-Hilbert correspondence established in [KN]. By using the isomorphism (1.3.2), it may be possible to prove that the following log Hodge-de Rham spectral sequence

\begin{equation}
(1.3.3) \quad E_1^{i,j} = H^j(X, \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathcal{C}}) \Rightarrow H^{i+j}(X, \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_{X/\mathcal{C}})
\end{equation}

degenerates at \( E_1 \) under the assumption of (1.1).

To prove the theorem (1.2), we use theory of (PD-)Hirsch extensions in [N3] and a key theorem obtained in [loc. cit.].

See [N3] for the log crystalline analogue of (1.2) for the case \( r = 1 \). In [loc. cit] we have had to prove the log crystalline analogue of the isomorphism

\begin{equation}
(1.3.4) \quad H(X/\mathcal{C}, \mathcal{E}) \sim \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^*_{X/\mathcal{C}}
\end{equation}
for the case $r = 1$ in order to prove the compatibility of the weight filtration on the log crystalline cohomology sheaf of a proper SNCL scheme in characteristic $p > 0$ with the cup product of the log crystalline cohomology sheaf when $(\mathcal{E}, \nabla)$ has no log poles. Because there is no good analogue of the Hodge filtration on this cohomology sheaf, we have had to give a purely algebraic proof for the log crystalline analogue of the isomorphism $\{1, 2, 3\}$.

The contents of this article is as follows.

In §2 we recall results on (PD-)Hirsch extensions in [N3].

In §3 we recall a result in [N3] and we give a key theorem in this article. It is an analytic version of a generalization of a result in [N3], which plays a starting theorem in this article.

In §4 we give the main result in this article, which is a generalization of (1.2).

2 Recall on the Hirsch extension

In this section we recall results on the (PD-)Hirsch extension in [N3].

Let $n$ be a positive integer. Let $A$ be a commutative ring with unit element. Let $M$ be a projective $A$-module. Let $\Gamma_A(M)$ be the PD-algebra generated by $M$. Let $B_i$ ($1 \leq i \leq n$) be an $A$-algebra. Let $\Omega^\bullet_A$ be a dga over $A$ such that each $\Omega^q_i$ ($q \in \mathbb{Z}$) is a $B_i$-module. Let $E_i$ be a $B_i$-module. Let $\varphi_i: M \to \text{Ker}(d: \Omega^1_i \to \Omega^2_i)$ be a morphism of $A$-modules. Let $(\bigoplus_{i=1}^n (E_i \otimes B_i, \Omega^\bullet_i), d)$ be a complex of $A$-modules. Here $d$ is not necessarily the direct sum of certain boundary morphisms of $E_i \otimes B_i, \Omega^\bullet_i$. Then the following $A$-linear morphism

$$d_H := d_{H,(\varphi_1, \ldots, \varphi_r)}: \Gamma_A(M) \otimes_A \left(\bigoplus_{i=1}^n (E_i \otimes B_i, \Omega^\bullet_i)\right) \to \Gamma_A(M) \otimes_A \left(\bigoplus_{i=1}^n (E_i \otimes B_i, \Omega^\bullet_i^{r+1})\right)$$

defined by the following formula

$$d_H \left(\sum_{i=1}^n \sum_{j(i)_1, \ldots, j(i)_r \in \mathbb{N}} m_{i}^{j(i)_1} \cdots m_{r}^{j(i)_r} \otimes e_i \otimes \omega_i\right)$$

$$:= \sum_{i=1}^n \sum_{j(i)_1, \ldots, j(i)_r \in \mathbb{N}} \sum_{k=1}^n m_{i}^{j(i)_1} \cdots m_{i}^{j(i)_{k-1}} \cdots m_{i}^{j(i)_r} \otimes e_i \otimes \varphi_M(m_k) \wedge \omega_i$$

$$+ \sum_{i=1}^n \sum_{j(i)_1, \ldots, j(i)_r \in \mathbb{N}, m_1, \ldots, m_r \in M, e_i \in E_i, \omega_i \in \Omega^\bullet_i} m_{i}^{j(i)_1} \cdots m_{i}^{j(i)_r} \otimes d(e_i \otimes \omega_i)$$

makes $\Gamma_A(M) \otimes_A (\bigoplus_{i=1}^n (E_i \otimes B_i, \Omega^\bullet_i))$ a complex of $A$-modules. We call the following natural injective morphism

$$\bigoplus_{i=1}^n E_i \otimes B_i, \Omega^\bullet_i \to \Gamma_A(M) \otimes_A \left(\bigoplus_{i=1}^n (E_i \otimes B_i, \Omega^\bullet_i)\right)$$

of $A$-modules the PD-Hirsch extension of $(\bigoplus_{i=1}^n (E_i \otimes B_i, \Omega^\bullet_i), d)$ by $(M, (\varphi_1, \ldots, \varphi_r))$. We denote it by $(\bigoplus_{i=1}^n (E_i \otimes B_i, \Omega^\bullet_i))(M) = (\bigoplus_{i=1}^n (E_i \otimes B_i, \Omega^\bullet_i))(\varphi_1, \ldots, \varphi_r)$.
Let $A \rightarrow A'$ and $B \rightarrow B'$ be morphisms of commutative rings with unit elements. Assume that $B'$ is an $A'$-algebra and that the following diagram

$$
\begin{array}{ccc}
A & \longrightarrow & A' \\
\downarrow & & \downarrow \\
B & \longrightarrow & B'
\end{array}
$$

is commutative. Let $\Omega^\bullet$ (resp. $\Omega'^\bullet$) be a dga over $A$ (resp. $A'$) such that each $\Omega^q$ (resp. $\Omega'^q$) ($q \in \mathbb{Z}$) is a $B$-module (resp. $B'$-module). Let $\varphi: M \rightarrow \text{Ker}(\Omega^1 \rightarrow \Omega^2)$ (resp. $\varphi': M \rightarrow \text{Ker}(\Omega'^1 \rightarrow \Omega'^2)$) be a morphism of $A$-modules (resp. $A'$-modules). Let $h: \Omega^\bullet \rightarrow \Omega'^\bullet$ be a morphism of complexes of $A$-modules fitting into the following commutative diagram

$$
\begin{array}{ccc}
M & \longrightarrow & M \\
\varphi \downarrow & & \varphi' \\
\Omega^1 & \longrightarrow & \Omega^1.
\end{array}
$$

Let $E$ (resp. $E'$) be a $B$-module (resp. $B'$-module). Let $g: E \rightarrow E'$ be a morphism of $B$-modules. (We can consider $E'$ as a $B'$-module.) Let $(E \otimes_B \Omega^\bullet, d)$ (resp. $(E' \otimes_{B'} \Omega'^\bullet, d')$) be a complex of $A$-modules (resp. a complex of $A'$-modules). Assume that $f := g \otimes h: E \otimes_B \Omega^\bullet \rightarrow E' \otimes_{B'} \Omega'^\bullet$ is a morphism of complexes of $A$-modules. Then we have the following morphism

$$
f_{(M)}: (E \otimes_B \Omega^\bullet)(M) \longrightarrow (E' \otimes_{B'} \Omega'^\bullet)(M).
$$

of complexes. Consider the mapping cone $\text{MC}(f_{(M)})$ of the morphism $f_{(M)}$. Then

$$
\text{MC}(f_{(M)})^q = \{ \Gamma_A(M) \otimes_A (E \otimes_B \Omega^{q+1}) \} \oplus \{ \Gamma_A(M) \otimes_A (E' \otimes_{B'} \Omega'^q) \} = \Gamma_A(M) \otimes_A (E \otimes_B \Omega^{q+1} \oplus E' \otimes_{B'} \Omega'^q) = \Gamma_A(M) \otimes_A \text{MC}(f)^q.
$$

The following is the commutativity of the operation of the mapping cone and that of the PD-Hirsch extension.

**Lemma 2.1 ([N3 (3.11)])**. The following diagram is commutative:

$$
\begin{array}{ccc}
\text{MC}(f_{(M)})^q & \longrightarrow & \Gamma_A(M) \otimes_A \text{MC}(f)^q \\
\downarrow_{d_{\text{MC}(f_{(M)})^q}(\varphi,\varphi')} & & \downarrow_{d_{\Gamma_A(M) \otimes_A \text{MC}(f)^q}(\varphi,\varphi')} \\
\text{MC}(f_{(M)})^{q+1} & \longrightarrow & \Gamma_A(M) \otimes_A \text{MC}(f)^{q+1}.
\end{array}
$$

(Note that the delicate sign appears before $\varphi$).

We have also obtained the following:

**Proposition 2.2 ([N3 (3.12)])**. Let the notations be as above. Assume that $M$ is a direct summand of a free $A$-module of countable rank. If $(\bigoplus_{i=1}^n (E_i \otimes_B \Omega^\bullet_i), d)$ is acyclic, then $(\bigoplus_{i=1}^n (E_i \otimes_B \Omega'^\bullet_i))(M)$ is acyclic.

By using (2.2), we obtain the following result which we need in this article:

**Corollary 2.3 ([N3 (3.13)])**. Let $f: E \otimes_B \Omega^\bullet \longrightarrow E' \otimes_{B'} \Omega'^\bullet$ be a quasi-isomorphism of $A$-modules. Assume that $M$ is a direct summand of a free $A$-module of countable rank. Then the morphism $f_{(M)}: (E \otimes_B \Omega^\bullet)(M) \longrightarrow (E' \otimes_{B'} \Omega'^\bullet)(M)$ is a quasi-isomorphism of $A$-modules.
Remark 2.4. In the later sections, we consider the Hirsch extension in characteristic 0. Hence the notion of the PD-Hirsch extension and that of the Hirsch extension is the same. In the later sections we denote \( \langle M \rangle \) by \([M]\).

3 The key result

In this section we give a key result (3.8) below in this article.

Let \( r \) be a positive integer. Let \( S \) be a log analytic family over \( \mathbb{C} \) of log points of virtual dimension \( r \), that is, zariski locally on \( S \), the log structure \( M_S \) of \( S \) is isomorphic to \( N^r \oplus \mathcal{O}_S^* \ni ((n_1, \ldots, n_r), u) \mapsto 0^i u \in \mathcal{O}_S \) (cf. [N2 §2]), where \( 0^n = 0 \) when \( n \neq 0 \) and \( 0^0 = 1 \). Set \( \overline{M}_S := M_S/\mathcal{O}_S^* \). Let \( g: Y \rightarrow S \) be a log smooth morphism of log analytic spaces over \( \mathbb{C} \).

Locally on \( S \), there exists a family \( \{t_i\}_{i=1}^r \) of local sections of \( M_S \) giving a local basis of \( \overline{M}_S \). Let \( M_i \) be the ideal sheaf of \( M_S \) generated by a lift of \( t_i \) in \( M_S \). For all \( 1 \leq i \leq r \), the submonoid sheaf \( \bigoplus_{j=1}^i M_j \) of \( M_S \) and \( S \) defines a family of log points of virtual dimension \( i \). Let \( S_i := S(M_1, \ldots, M_i) = (\tilde{S}, (\bigoplus_{j=1}^i M_j \rightarrow O_S)^n) \) be the resulting local log analytic space. Set \( S_0 := S \). Then we have the following sequence of families of log points of virtual dimensions locally:

\[ S = S_r \rightarrow S_{r-1} \rightarrow S_{r-2} \rightarrow \cdots \rightarrow S_1 \rightarrow S_0 = \tilde{S}. \]

Let \( \tilde{t}_i \) be a lift of \( t_i \) in \( M_S \). The one-form \( d \log \tilde{t}_i \in \Omega^1_{S/\tilde{S}} \) is independent of the choice of the lift \( \tilde{t}_i \). Hence we denote \( d \log \tilde{t}_i \) by \( d \log t_i \). Denote also by \( d \log t_i \in \Omega^1_{Y/\tilde{S}} \) the image of \( d \log t_i \in \Omega^1_{S/\tilde{S}} \) in \( \Omega^1_{Y/\tilde{S}} \).

Let \( \mathcal{F} \) be a (not necessarily coherent) locally free \( \mathcal{O}_Y \)-module and let

\[ \nabla: \mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^1_{Y/\tilde{S}} \]

be an integrable connection. Then we have the complex \( \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/\tilde{S}} \) of \( g^{-1}(\mathcal{O}_S) \)-modules.

Lemma 3.1. (1) The sheaf \( \Omega^i_{Y/\tilde{S}} \) \((i \in \mathbb{N})\) is a locally free \( \mathcal{O}_Y \)-module.

(2) Locally on \( S \), the following sequence

\[ 0 \rightarrow \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/\tilde{S}}[-1] \xrightarrow{id \mathcal{F} \otimes (d \log t_i \wedge \cdot)} \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S_i} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S_{i-1}} \rightarrow 0 \quad (1 \leq i \leq r) \]

is exact. (Note that the morphism

\[ id \mathcal{F} \otimes (d \log t_i \wedge \cdot): \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/\tilde{S}}[-1] \rightarrow \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S_{i-1}} \]

is indeed a morphism of complexes of \( g^{-1}(\mathcal{O}_S) \)-modules.)

Proof. (1), (2): First note that the connection (3.0.2) induces the local integrable connection \( \mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^1_{Y/\tilde{S}} \) for any \( 0 \leq i \leq r \). Consequently we indeed have the local log de Rham complex \( \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/S_i} \). Consider the following exact sequence (cf. [K] (3.12)):

\[ f^* (\Omega^1_{S/\tilde{S}}) \rightarrow \Omega^1_{Y/\tilde{S}} \rightarrow \Omega^1_{Y/S} \rightarrow 0. \]
Because $Y/S$ is log smooth, the following sequence is exact and locally split (cf. [loc. cit.]):

\[
(3.1.3) \quad 0 \rightarrow f^*(\Omega^1_{S/\bar{S}}) \rightarrow \Omega^1_{Y/\bar{S}} \rightarrow \Omega^1_{Y/S} \rightarrow 0.
\]

Hence we obtain (1).

Since \( \{d \log t_i\}_{i=1}^r \) is a basis of \( \Omega^1_{S/\bar{S}} \), we see that \( \{d \log t_i\}_{i=1}^r \) can be a part of a basis of \( \Omega^1_{Y/\bar{S}} \) by (3.1.3). Hence the following sequence

\[
(3.1.4) \quad 0 \rightarrow \mathcal{O}_Y \xrightarrow{d \log t_i \wedge} \Omega^1_{Y/S, i-1} \rightarrow \Omega^1_{Y/S, i} \rightarrow 0 \quad (1 \leq i \leq r)
\]

is a locally split exact sequence. Now it is clear that the sequence (3.1.1) is exact. \( \square \)

**Remark 3.2.** In the exact sequence in [K] (3.12)], \( \omega^1_{Y/S} \) (resp. \( \omega^1_{X/S} \)) should be replaced by \( \omega^1_{X/S} \) (resp. \( \omega^1_{Y/S} \)).

Let \( \{t_1, \ldots, t_r\} \) be a set of local generators of \( \mathcal{M}_S \). Set

\[
U_S(t_1, \ldots, t_r) := \bigoplus_{i=1}^r \mathcal{O}_S t_i
\]

(a free \( \mathcal{O}_S \)-module with basis \( t_1, \ldots, t_r \)).

In the following we denote \( t_i \) in \( U_S \) by \( u_i \). Let \( \Gamma_{\mathcal{O}_S}(U_S) := \text{Sym}_{\mathcal{O}_S}(U_S) \) be the symmetric algebra of \( U_S \) over \( \mathcal{O}_S \). Consider the following Hirsch extension

\[
(3.2.2) \quad \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/\bar{S}}[U_S] := \Gamma_{\mathcal{O}_S}(U_S) \otimes_{\mathcal{O}_S} \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/\bar{S}}
\]

of \( \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/\bar{S}} \) by the morphism

\[
(3.2.3) \quad d \log : g^{-1}(U_S) \ni u_i \mapsto d \log t_i \in \text{Ker}(\Omega^1_{Y/\bar{S}} \rightarrow \Omega^2_{Y/\bar{S}})
\]

([K] §3). We have omitted to write \( g^{-1} \) for \( g^{-1}(\text{Sym}_{\mathcal{O}_S}(U_S)) \otimes_{\mathcal{O}_S}(\mathcal{O}_S) \) in (3.2.2). Note that \( d \log (at_i) = d \log t_i \) for \( a \in \mathcal{O}_S \). (It may be better to denote \( \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/\bar{S}}[U_S] \) by \( \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^*_{Y/\bar{S}} \) as in [K] and the previous section. However see (3.2.5) below.)

It is easy to check that the morphism \( g^{-1}(U_S) \rightarrow \Omega^1_{Y/\bar{S}} \) is well-defined (cf. [N2] §2).

The boundary morphism

\[
\nabla : \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^i_{Y/\bar{S}}[U_S] \rightarrow \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^{i+1}_{Y/\bar{S}}[U_S] \quad (i \in \mathbb{Z}_{\geq 0})
\]

is defined by the following:

\[
(3.2.4) \quad \nabla(m_1^{[e_1]} \cdots m_r^{[e_r]} \otimes \omega) = \sum_{j=1}^r m_1^{[e_1]} \cdots m_{j-1}^{[e_{j-1}]} \cdots m_r^{[e_r]} d \log m_j \wedge \omega + m_1^{[e_1]} \cdots m_r^{[e_r]} \otimes \nabla(\omega)
\]

\[
( \omega \in \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^i_{Y/\bar{S}}, m_1, \ldots, m_r \in U_S, e_1, \ldots, e_r \in \mathbb{Z}_{\geq 1}, m_i^{[e_i]} = ((e_i)!)^{-1}m_i^{e_i} ).\]

We can easily see that \( \nabla^2 = 0 \). We have the following equality

\[
(3.2.5) \quad \Gamma_{\mathcal{O}_S}(U_S) = \mathcal{O}_S(\mathcal{M}_S)
\]
as sheaves of commutative rings on $S$.

The projection $\text{Sym}_{\mathcal{O}_S}(U_S) \longrightarrow \text{Sym}_{\mathcal{O}_S}(U_S) = \mathcal{O}_S$ induces the following natural morphism of complexes:

$$(3.2.6) \quad \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega_{Y/S}^\bullet [U_S] \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega_{Y/S}^\bullet.$$

The following is a simpler analytic version of [N3 (4.3)]:

**Definition 3.3** (cf. [N3 (4.3)]). (1) Take a local basis of $\Omega_{Y/S}^1$ containing \{ $d \log t_1, \ldots, d \log t_r$ \}.

Let $(d \log t_i)^* : \Omega_{Y/S}^1 \longrightarrow \mathcal{O}_Y$ be the local morphism defined by the local dual basis of $d \log t_i$. We say that the connection $(\mathcal{F}, \nabla)$ has no poles along $S$ if the composite morphism $\mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega_{Y/S}^1 \longrightarrow \mathcal{F}$ vanishes for $1 \leq i \leq r$.

(2) (cf. [KN p. 163]) If there exists locally a finite increasing filtration \{$\mathcal{F}_i$\}_{i \in \mathbb{Z}} on $\mathcal{F}$ such that $\text{gr}_i \mathcal{F} := \mathcal{F}_i / \mathcal{F}_{i-1}$ is a locally free $\mathcal{O}_Y$-module and $\nabla$ induces a connection $\mathcal{F}_1 \longrightarrow \mathcal{F}_1 \otimes_{\mathcal{O}_Y} \Omega_{Y/S}^1$, whose induced connection $\text{gr}_i \mathcal{F} \longrightarrow (\text{gr}_i \mathcal{F}) \otimes_{\mathcal{O}_Y} \Omega_{Y/S}^1$ has no poles along $M_S$, then we say that $(\mathcal{F}, \nabla)$ a locally nilpotent integrable connection on $Y$ with respect to $S$.

It is obvious that the notion “no poles along $M_S$” is independent of the choice of the local sections $t_1, \ldots, t_r$. It is also obvious that, if the connection 3.0.2 factors through $\mathcal{F} \otimes_{\mathcal{O}_Y} \Omega_{Y/S}^1$, then $(\mathcal{F}, \nabla)$ has no poles along $S$. Especially $(\mathcal{O}_Y^{\oplus n}, d^{\oplus n})$ \((n \in \mathbb{Z}_{\geq 1})\) has no poles along $M_S$. Here $d_Y : \mathcal{O}_Y \longrightarrow \Omega_{Y/S}^1$ is the usual derivative.

We need the following results (3.4) and (3.5) below. The analytic version of the first result in [N3] is as follows. This plays the most important role in this article:

**Theorem 3.4** (cf. [N3 (4.7)]). Assume that $r = 1$. Let $(\mathcal{F}, \nabla)$ be a locally nilpotent integrable connection on $Y$ with respect to $S$. Then the morphism $(3.2.6)$ is a quasi-isomorphism.

*Proof.* Though the same proof as that of [N3 (4.6)] works in the analytic case, we give the proof here for the completeness of this article.

This is a local problem. We may assume that $U_S = \mathcal{O}_S u_1$. Denote $t_1$ and $u_1$ by $t$ and $u$, respectively. Assume that there exists a sub free $\mathcal{O}_Y$-module $\mathcal{F}_0$ of $\mathcal{F}$ such that $\mathcal{F}_0$ and $\text{gr}_1 \mathcal{F} := \mathcal{F} / \mathcal{F}_0$ have no poles along $S$ and $\text{gr}_1 \mathcal{F}$ is a free $\mathcal{O}_Y$-module. Let $\iota : \mathcal{F}_0 \otimes_{\mathcal{O}_Y} \Omega_{Y/S}^\bullet \hookrightarrow \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega_{Y/S}^\bullet$ be the natural inclusion. For an injective morphism $f : \mathcal{G}^\bullet \longrightarrow \mathcal{H}^\bullet$ of complexes of abelian sheaves on $\mathcal{O}_Y$, let $\text{MC}(f)$ be the mapping cone of $f$. Then we have the natural quasi-morphism $\text{MC}(\iota) \longrightarrow \text{gr}_1 \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega_{Y/S}^\bullet$. By (2.3) we have a quasi-isomorphism

$$\text{MC}(\iota)[U_S] \longrightarrow \text{gr}_1 \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega_{Y/S}^\bullet [U_S].$$

Hence, by (2.1), we obtain the following isomorphism

$$\text{MC}(\iota)[U_S] : \mathcal{F}_0 \otimes_{\mathcal{O}_Y} \Omega_{Y/S}^\bullet [U_S] \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega_{Y/S}^\bullet [U_S] \sim \text{gr}_1 \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega_{Y/S}^\bullet [U_S].$$


Take a local section \( \omega \). Hence, if the morphism

\[
E = F_0 \otimes \Omega^\bullet_{Y/S} \rightarrow \mathcal{E} \otimes \Omega^\bullet_{Y/S}.
\]

is an isomorphism for \( \mathcal{E} = F_0 \) and \( \text{gr}_1 F \), then the morphism

\[
F \otimes \Omega^\bullet_{Y/S} \rightarrow \mathcal{E} \otimes \Omega^\bullet_{Y/S}
\]

is an isomorphism. Consequently we may assume that \( F \) has no poles along \( S \). By (3.1.1) the following sequence is exact:

\[
0 \longrightarrow F_0 \otimes \Omega^\bullet_{Y/S} \rightarrow \mathcal{E} \otimes \Omega^\bullet_{Y/S} \rightarrow \mathcal{E} \otimes \Omega^\bullet_{Y/S} \rightarrow 0.
\]

Here we have denoted \( \text{id}_F \otimes (d \log t \wedge) \) only by \( d \log t \wedge \). We claim that the following sequence

\[
0 \longrightarrow \mathcal{H}^{q-1}(F \otimes \Omega^\bullet_{Y/S}) \rightarrow \mathcal{H}^q(F \otimes \Omega^\bullet_{Y/S}) \rightarrow 0
\]

is exact. And we have only to prove that the following inclusion holds:

\[
\nabla(F \otimes \Omega^\bullet_{Y/S}) \cap (d \log t \wedge (F \otimes \Omega^\bullet_{Y/S})) \subset d \log t \wedge \nabla(F \otimes \Omega^\bullet_{Y/S}).
\]

Set \( \omega_1 := d \log t \). Let us take a local basis \( \{\omega_1, \ldots, \omega_m\} \) \( m \in \mathbb{N} \) of \( \Omega^1_{Y/S} \). Take a local section

\[
f := \sum_{1 \leq i_1 < \cdots < i_{q-1} \leq m} f_{i_1 \cdots i_{q-1}} \omega_{i_1} \wedge \cdots \wedge \omega_{i_{q-1}} \quad (f_{i_1 \cdots i_{q-1}} \in F)
\]

of \( F \otimes \Omega^\bullet_{Y/S} \). Decompose \( f \) as follows:

\[
f = \sum_{2 \leq i_2 < \cdots < i_{q-1} \leq m} f_{i_2 \cdots i_{q-1}} \omega_{i_2} \wedge \cdots \wedge \omega_{i_{q-1}} + \sum_{2 \leq i_1 < \cdots < i_{q-1} \leq m} f_{i_1 \cdots i_{q-1}} \omega_{i_1} \wedge \cdots \wedge \omega_{i_{q-1}}.
\]

Then

\[
\nabla(f) = \sum_{2 \leq i_2 < \cdots < i_{q-1} \leq m} \nabla(f _{i_2 \cdots i_{q-1}}) \wedge \omega_1 \wedge \cdots \wedge \omega_{i_{q-1}} - \sum_{2 \leq i_2 < \cdots < i_{q-1} \leq m} f_{i_2 \cdots i_{q-1}} \omega_1 \wedge d(\omega_{i_2} \wedge \cdots \wedge \omega_{i_{q-1}}) + \sum_{2 \leq i_1 < \cdots < i_{q-1} \leq m} \nabla(f_{i_1 \cdots i_{q-1}}) \wedge \omega_1 \wedge \cdots \wedge \omega_{i_{q-1}}.
\]
Because $\mathcal{F}$ has no poles along $M_S$, the third term in (3.4.5) can be expressed as follows:

\[
(3.4.6) \quad \sum_{2 \leq i_1 < \cdots < i_q \leq m} g_{i_1 \cdots i_q} \omega_{i_1} \wedge \cdots \wedge \omega_{i_q}
\]

for some $g_{i_1 \cdots i_q} \in \mathcal{F}$. Assume that $\nabla(f) \in \omega_1 \wedge (\mathcal{F} \otimes \mathcal{O}_Y \Omega^{q-1}{Y/S})$. Then (3.4.6) vanishes. Hence

\[
\nabla(f) = -\omega_1 \wedge (\sum_{2 \leq i_2 < \cdots < i_q \leq m} \nabla(f_{1 \cdots i_q}) \omega_{i_1} \wedge \cdots \wedge \omega_{i_q}) + \sum_{2 \leq i_2 < \cdots < i_q \leq m} f_{1 \cdots i_q} \wedge d(\omega_{i_1} \wedge \cdots \wedge \omega_{i_q - 1})) - \omega_1 \wedge \nabla(\sum_{2 \leq i_2 < \cdots < i_q \leq m} f_{1 \cdots i_q} \omega_{i_1} \wedge \cdots \wedge \omega_{i_q - 1})
\]

Thus we have proved that the inclusion (3.4.4) holds.

By (3.4.3) the following sequence is exact:

\[
(3.4.7) \quad 0 \longrightarrow H^0(\mathcal{F} \otimes \mathcal{O}_Y \Omega^* Y/S) \xrightarrow{d \log t \wedge} H^1(\mathcal{F} \otimes \mathcal{O}_Y \Omega^* Y/S) \longrightarrow \cdots \xrightarrow{d \log t \wedge} H^q(\mathcal{F} \otimes \mathcal{O}_Y \Omega^* Y/S) \xrightarrow{d \log t \wedge} H^{q+1}(\mathcal{F} \otimes \mathcal{O}_Y \Omega^* Y/S) \longrightarrow 0.
\]

The complex $\mathcal{F} \otimes \mathcal{O}_Y \Omega^* Y/S[S] = \Gamma_{\mathcal{O}_S}(S) \otimes \mathcal{F} \otimes \mathcal{O}_Y \Omega^* Y/S$ is the single complex of the following double complex:

\[
\begin{array}{cccccc}
\vdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \\
\text{id} \otimes \nabla & \longrightarrow & \text{id} \otimes \nabla & \longrightarrow & \text{id} \otimes \nabla & \\
\mathcal{O}_S u^2 \otimes \mathcal{O}_S \mathcal{F} & \xrightarrow{(?)^t \log t \wedge} & \mathcal{O}_S u \otimes \mathcal{O}_S \mathcal{F} \otimes \mathcal{O}_Y \Omega^1 Y/S & \xrightarrow{(?)^t \log t \wedge} & \mathcal{F} \otimes \mathcal{O}_Y \Omega^2 Y/S & \\
\text{id} \otimes \nabla & \longrightarrow & \text{id} \otimes \nabla & \longrightarrow & \text{id} \otimes \nabla & \\
\mathcal{O}_S u \otimes \mathcal{O}_S \mathcal{F} & \xrightarrow{(?)^t \log t \wedge} & \mathcal{F} \otimes \mathcal{O}_Y \Omega^1 Y/S & \\
\nabla & \longrightarrow & \mathcal{F} & \\
\end{array}
\]

where the horizontal arrow $(?)^t \log t \wedge$ is defined by $u^{[i]} \otimes f \otimes \omega \longrightarrow u^{[i-1]} \otimes f \otimes (d \log t \wedge \omega)$. For $i, j \in \mathbb{N}$, set

\[
d'' := \text{id} \otimes \nabla : \mathcal{O}_S u^i \otimes \mathcal{O}_S \mathcal{F} \otimes \mathcal{O}_Y \Omega^{j-i} Y/S \longrightarrow \mathcal{O}_S u^i \otimes \mathcal{O}_S \mathcal{F} \otimes \mathcal{O}_Y \Omega^{j-i+1} Y/S.
\]

Let "$\mathcal{H}^q$" be the cohomology with respect to $d''$. Consider the filtration $G$ by columns of (3.4.8) for $\mathcal{F} \otimes \mathcal{O}_Y \Omega^* Y/S[S]$: \n
\[
G^i(\mathcal{F} \otimes \mathcal{O}_Y \Omega^* Y/S[S]) := \bigoplus_{j \geq i} \mathcal{O}_S u^j \otimes \mathcal{O}_S \mathcal{F} \otimes \mathcal{O}_Y \Omega^* Y/S.
\]
By the same proof as that of (3.4) again, we see that the following morphism is a quasi-isomorphism. Analogously, we have the following quasi-isomorphism

\[(3.5.2)\]

\[E^p_1 = H^{p+q}(gr^p_G (F \otimes \Omega^\bullet_{Y/S}[U_S])) \Rightarrow H^{p+q}(F \otimes \Omega^\bullet_{Y/S}[U_S]).\]

Here

\[E^p_1 = H^q(O_{S}F \otimes \Omega^\bullet_{Y/S}) = H^q(F \otimes \Omega^\bullet_{Y/S}).\]

The exact sequence (3.4.7) tells us that \(E_2^p = H^q(F \otimes \Omega^\bullet_{Y/S})\) and \(E_1^p = 0\) for \(p \neq 0\). Hence the spectral sequence (3.4.9) is regular and bounded below. Hence, by the cohomological version of \[W, \text{Complete Convergence Theorem 5.5.10}\], (3.4.9) is convergent and

\[H^q(F \otimes \Omega^\bullet_{Y/S}[U_S]) = H^q(F \otimes \Omega^\bullet_{Y/S}).\]

We complete the proof of this theorem.

The following is a key result in this article:

**Theorem 3.5.** Let \((F, \nabla)\) be a locally nilpotent integrable connection on \(Y\) with respect to \(S\). Then the morphism (3.2.6) is a quasi-isomorphism.

**Proof.** This is a local problem. Hence we may assume that there exists the sequence (3.0.1). Consider the exact sequence (3.1.1) for the case \(i = r\). By the same proof as that of (3.2.4), the following sequence

\[(3.5.1)\]

\[0 \to H^{q-1}(F \otimes \Omega^\bullet_{Y/S}) \xrightarrow{d \log t^\wedge} H^q(F \otimes \Omega^\bullet_{Y/S_{r-1}}) \to H^q(F \otimes \Omega^\bullet_{Y/S}) \to 0 \quad (q \in \mathbb{N})\]

obtained by (3.1.1) is exact. Let

\[F \otimes \Omega^\bullet_{Y/S_{i-1}}[u_i, \ldots, u_j] := \text{Sym}_{\Omega^\bullet_{Y/S}} \left( \bigoplus_{k=i}^j \Omega^\bullet_{S} u_k \right) \otimes S F \otimes \Omega^\bullet_{Y/S_{i-1}} \quad (1 \leq i \leq j \leq r)\]

be the Hirsch extension of \(F \otimes \Omega^\bullet_{Y/S_i}\) by the following morphism

\[\bigoplus_{k=i}^j \Omega^\bullet_{S} u_k \ni u_k \mapsto d \log t_k \in \Omega^1_{Y/S_{i-1}}.\]

By the same proof as that of (3.2.4) again, we see that the following morphism

\[(3.5.2)\]

\[F \otimes \Omega^\bullet_{Y/S_{r-1}}[u_r] \to F \otimes \Omega^\bullet_{Y/S}\]

is a quasi-isomorphism. Analogously, we have the following quasi-isomorphism

\[(3.5.3)\]

\[F \otimes \Omega^\bullet_{Y/S_{r-2}}[u_{r-1}] \to F \otimes \Omega^\bullet_{Y/S_{r-1}}.\]

By (3.5.2), (3.5.3) and (3.2.5) we see that the following morphism

\[(F \otimes \Omega^\bullet_{Y/S_{r-2}}[u_{r-1}])[u_r] \to F \otimes \Omega^\bullet_{Y/S}\]

is a quasi-isomorphism. Since the source of this quasi-isomorphism is equal to \(F \otimes \Omega^\bullet_{Y/S_{r-2}}[u_{r-1}, u_r]\), we have the following quasi-isomorphism

\[F \otimes \Omega^\bullet_{Y/S_{r-2}}[u_{r-1}, u_r] \to F \otimes \Omega^\bullet_{Y/S}.\]

Continuing this process, we see that the morphism (3.2.6) is a quasi-isomorphism. \(\square\)
Let $F$ be the filtration on $\mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^i_{Y/S}$ defined by the following formula:

$$F^i(\mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^i_{Y/S}) := \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^i_{Y/S}.$$

Let $\Gamma_{\mathcal{O}_S,n}(U_S)$ be the degree $n$-part of $\Gamma_{\mathcal{O}_S}(U_S)$. Let $F$ be the filtration on $\mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^i_{Y/S}$ defined by the following formula:

$$F^i(\mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^i_{Y/S}[U_S]) := \bigoplus_{n \geq 0} \Gamma_{\mathcal{O}_{S,n}}(U_S) \otimes_{\mathcal{O}_S} \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^i_{Y/S}.$$

It is clear that $F$ indeed gives the filtration on the complex $\mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^i_{Y/S}[U_S]$ by (3.2.4).

Note that the filtration $F$ is separated, but not finite. The morphism (3.2.6) induces the following filtered morphism

$$\text{(3.5.4)} \quad (\mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^i_{Y/S}[U_S], \mathcal{F}) \longrightarrow (\mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^i_{Y/S}, \mathcal{F}).$$

The following is a generalization of [F] (5.3):

**Proposition 3.6 (cf. [F] (5.3)).** The following morphism

$$\text{(3.6.1)} \quad \text{gr}_Y^{\bullet}(\mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^i_{Y/S}[U_S]) \longrightarrow \text{gr}_Y^{\bullet}(\mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^i_{Y/S})$$

induced by (3.5.4) is a quasi-isomorphism.

**Proof.** (The following proof is essentially the same as that of [F] (5.3),.) First recall the definition of the Koszul complex.

For a ringed topos $(\mathcal{T}, \mathcal{A})$ and a morphism $\psi: \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ of $\mathcal{A}$-modules and a nonnegative integer $n$, $\text{Kos}(\psi, n)^q := \Gamma_{\mathcal{A}, n-q}(\mathcal{E}_1) \otimes_{\mathcal{A}} \mathcal{E}_2$ for $q \geq 0$ with the following boundary morphism

$$\text{Kos}(\psi, n)^q \ni e_1^{[i_1]} \cdots e_k^{[i_k]} \otimes f \longmapsto \sum_{j=1}^k e_1^{[i_1]} \cdots e_j^{[i_j-1]} \cdots e_k^{[i_k]} \otimes \psi(e_j) \wedge f \in \text{Kos}(\psi, n)^{q+1},$$

where $e_1, \ldots, e_k$ and $f$ are local sections of $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively, and $i_1, \ldots, i_k$ are positive integers such that $\sum_{j=1}^k i_j = n - q$. Here $\Gamma_{\mathcal{A}}(\mathcal{E}_1) = \bigoplus_{m=0}^{\infty} \Gamma_{\mathcal{A}, m}(\mathcal{E}_1)$ is the graded PD-algebra over $\mathcal{A}$ generated by $\mathcal{E}_1$.

Consider the following injective morphism

$$\varphi: \mathcal{O}_Y \otimes_{\mathcal{O}_S} U_S \ni b \otimes \sum_{i=1}^{r} a_i u_i \longmapsto b \otimes \sum_{i=1}^{r} a_i g^i(d \log t_i) \in \Omega^1_{Y/S}.$$  

This morphism is independent of the choice of $\{t_i\}_{i=1}^r$ and this is well-defined. Note that we do not consider the following morphism

$$\text{id}_\mathcal{F} \otimes_{\mathcal{O}_S} \varphi: \mathcal{F} \otimes_{\mathcal{O}_S} U_S \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S}.$$  

The following sequence

$$\text{(3.6.2)} \quad 0 \longrightarrow \mathcal{O}_Y \otimes_{\mathcal{O}_S} U_S \xrightarrow{\varphi} \Omega^1_{Y/S} \longrightarrow \Omega^1_{Y/S} \longrightarrow 0$$

is exact. Hence $\text{Coker}(\varphi)$ is a flat $\mathcal{O}_Y$-module since $Y/S$ is log smooth.
We obtain the following equalities:

\[
\text{gr}^i_F(F \otimes_{O_Y} \Omega^k_{Y/S}[U_S]) = \bigoplus_{j \geq 0} \Gamma_{O_S,j}(U_S) \otimes_{O_S} F \otimes_{O_Y} \text{gr}^{i-j}_F \Omega^k_{Y/S} = \Gamma_{O_S,i-k}(U_S) \otimes_{O_S} \bigwedge^k_{Y/S} \Omega^1_{Y/S} = F \otimes_{O_Y} \bigwedge^k_{Y/S} \Omega^1_{Y/S} = F \otimes_{O_Y} \text{Kos}(\varphi, i)^k.
\]

Since the morphism

\[
\text{gr}^i_F(F \otimes_{O_Y} \Omega^k_{Y/S}[U_S]) \longrightarrow \text{gr}^i_F(F \otimes_{O_Y} \Omega^k_{Y/S}[U_S])
\]

obtained by the second term on the right hand side of (3.6.3) is 0, we see that \( \text{gr}^i_F(F \otimes_{O_Y} \Omega^k_{Y/S}[U_S]) \) is equal to \( F \otimes_{O_Y} \text{Kos}(\varphi, i)^k \) by (3.6.3). Because \( \text{Coker}(\varphi) \) is a flat \( O_Y \)-module, we have the following isomorphism

\[
\Gamma_{O_Y,i-q}(\text{Ker}(\varphi)) \otimes_{O_Y} \bigwedge^q \text{Coker}(\varphi) \overset{\sim}{\longrightarrow} \mathcal{H}^q(\text{Kos}(\varphi, i))
\]

by \([1, \text{Proposition 4.4.1.6}]\). Hence

(3.6.4) \( F \otimes_{O_Y} \bigwedge^i \text{Coker}(\varphi) \overset{\sim}{\longrightarrow} F \otimes_{O_Y} \mathcal{H}^i(\text{Kos}(\varphi, i)) = \mathcal{H}^i(F \otimes_{O_Y} \text{Kos}(\varphi, i)) \)

and

(3.6.5) \( \mathcal{H}^q(F \otimes_{O_Y} \text{Kos}(\varphi, i)) = F \otimes_{O_Y} \mathcal{H}^q(\text{Kos}(\varphi, i)) = 0 \quad (q \neq i). \)

On the other hand, the target of the morphism (3.6.1) is equal to \( F \otimes_{O_Y} \Omega^k_{Y/S} = F \otimes_{O_{O_Y}} \bigwedge^i \text{Coker}(\varphi) \). Hence the equalities (3.6.4) and (3.6.5) tell us that the morphism (3.6.1) is a quasi-isomorphism. We can complete the proof of (3.6).

**Remark 3.7.** Because the filtration \( F \) on \( F \otimes_{O_Y} \Omega^k_{Y/S}[U_S] \) is not finite, \( 3.6 \) does not imply \( 3.5 \).

**Corollary 3.8.** The filtered morphism (3.5.4) is a filtered quasi-isomorphism in the sense of [NS].

**Proof.** This follows from (3.5) and (3.6).

4 Main Result

In this section we give the main result in this article.

Let the notations be as in the previous section. Let \( f : X \longrightarrow S \) be a log smooth morphism of analytic spaces.

In the following we assume that there exists a set \( \{ \tilde{X}_\lambda \}_{\lambda \in \Lambda} \) of smooth closed analytic spaces of \( \tilde{X} \) such that \( \bigcup_{\lambda \in \Lambda} \tilde{X}_\lambda = \tilde{X} \) and such that there exists no empty
open subset $\hat{V}$ of $\hat{X}$ such that $\hat{X}_\lambda|_{\hat{V}} = \hat{X}_{\lambda'}|_{\hat{V}}$ for $\lambda \neq \lambda'$ except in the case where both hand sides are empty sets.

Set $\Delta^\circ_S := \Delta^\circ \times \hat{S}$. Then the standard coordinates of $\Delta^\circ$ define an fs log structure on $\Delta^\circ_S$. Let $\Delta^\circ_S$ be the resulting log analytic space. Locally on $S$, we have the following natural exact closed immersion

$$S \hookrightarrow \Delta^\circ_S$$

of log analytic spaces. This immersion fits into the following commutative diagram

$$\begin{array}{ccc}
S & \hookrightarrow & \Delta^\circ_S \\
\downarrow & & \downarrow \\
\hat{S} & = & \hat{S},
\end{array}$$

where the vertical morphisms are natural morphisms.

By replacing $\Delta^\circ_S$ in [N2 (1.1)] by $\Delta^\circ_S$, we have a well-defined log analytic space $\overline{S}$ with an exact closed immersion $S \hookrightarrow \overline{S}$ over $\hat{S}$. Locally on $S$, $\overline{S}$ is isomorphic to a polydisk $\Delta^\circ_S$. We also assume that, locally on $X$, the morphism $\tilde{f}$ fits into the following cartesian diagram

$$\begin{array}{ccc}
\tilde{X} & \hookrightarrow & \Delta^n_S \\
\downarrow \tilde{f} & & \downarrow \\
\hat{S} & \hookrightarrow & \overline{S}
\end{array}$$

such that $\tilde{X}_\lambda$ is empty or defined by equations $x_{i_1} = \cdots = x_{i_k} = 0$ for some $1 \leq i_1 < \cdots < i_k \leq n$, where $x_1, \ldots, x_n$ are the standard coordinates of $\Delta^n_S$ in this cartesian diagram. We also assume that $x_{i_1}, \ldots, x_{i_k}$ of the images of local sections of the log structure of $X$ by the structural morphism.

**Remark 4.1** (cf. [F (4.3)]). In the following we do not use the following fact.

Let $Y/S$ be as in the previous section. Locally on $Y$, there exists a log smooth lift $\overline{Y}$ into a log analytic space over $\overline{S}$ of $Y$ fitting into the following cartesian diagram

$$\begin{array}{ccc}
Y & \hookrightarrow & \overline{Y} \\
\downarrow & & \downarrow \\
S & \hookrightarrow & \overline{S}
\end{array}$$

(cf. [N2 (1.8)]). Hence the condition of the existence of $(U, M_U)$ in [F (4.3)] is automatically satisfied (if $Y/S$ is log smooth). In the case above, $M_U$ is fine.

For a subset $\underline{\lambda} = \{\lambda_1, \ldots, \lambda_m\}$ of $\Lambda$, set $\hat{X}_{\underline{\lambda}} := \hat{X}_{\lambda_1} \cap \cdots \cap \hat{X}_{\lambda_m}$ and let $\tilde{\iota}_{\underline{\lambda}} : \hat{X}_{\underline{\lambda}} \rightarrow \hat{X}$ be the natural closed immersion. Endow $\hat{X}_{\underline{\lambda}}$ with the inverse image of the log structure of $X$ by the morphism $\tilde{\iota}_{\underline{\lambda}}$. Let $\hat{X}_{\underline{\lambda}}$ be the resulting log analytic space. Then the structural morphism $X_{\underline{\lambda}} \rightarrow S$ is ideally log smooth. Here we say that a morphism $Y \rightarrow S$ of fine log analytic spaces is ideally log smooth if, locally on $Y$, there exists an exact closed immersion $Y \hookrightarrow Z$ over $S$ into a log smooth analytic
space such that the ideal of definition of this exact closed immersion is defined by the image of an ideal of the log structure of \( Z \). Let \( \iota_\lambda : X_\lambda \rightarrow X \) be the resulting exact closed immersion. Let \( \iota_\lambda : X_\lambda \rightarrow X \) be the resulting exact closed immersion. Let \( \Omega^\bullet_{X_\lambda/S} \) and \( \Omega^\bullet_{X_\lambda/S} \) be the log de Rham complex of \( X_\lambda \) and \( X_\lambda \), respectively. It is convenient to define \( X_\emptyset = X \) and \( \iota_\emptyset = \text{id} \). In [F] Fujisawa has essentially proved the following two results:

**Proposition 4.2 (cf. [F (4.13)])**. For a subset \( \Lambda \) of \( \Lambda \), \( \Omega^i_{X_\lambda/S} = \iota_\lambda^*(\Omega^i_{X/S}) \) \( (i \in \mathbb{N}) \) and \( \Omega^i_{X_\lambda/S} = \iota_\lambda^*(\Omega^i_{X/S}) \) \( (i \in \mathbb{N}) \). Consequently \( \Omega^i_{X_\lambda/S} \) and \( \Omega^i_{X_\lambda/S} \) are locally free \( \mathcal{O}_{X_\lambda} \)-modules.

**Proof.** Let \( I_\lambda \) be the defining ideal sheaf of \( X_\lambda \) of \( X \). Set \( T := \mathcal{O}_X \). Then we have the following exact sequence

\[
I_\lambda/IP_\lambda^2 \rightarrow \mathcal{O}_{X_\lambda} \otimes \mathcal{O}_X \Omega^1_{X/T} \rightarrow \Omega^1_{X_\lambda/T} \rightarrow 0
\]

by [KN (3.6) (1)]. By the assumption, the left morphism is zero as in [loc. cit. (3.6) (2)].

**Proposition 4.3 (cf. [F (4.16)])**. Set \( X^{(n)} := \bigsqcup_{\lambda = n+1} X_\lambda \) and let \( a^{(n)} : X^{(n)} \rightarrow X \) be the natural morphism. Fix a total order on \( \Lambda \). For a subset \( \Lambda = \{\lambda_0, \ldots, \lambda_n\} \) \( (\lambda_0 < \cdots < \lambda_n) \) of \( \Lambda \), set \( \Lambda_2 := \{\lambda_0, \ldots, \lambda_j, \lambda_{j+1}, \ldots, \lambda_n\} \). Let \( \iota_{\Lambda_2} : X_{\Lambda_2} \rightarrow X_{\Lambda} \) be the natural inclusion. Let

\[
\rho^{n-1} : a^{(n-1)}(\mathcal{O}_{X^{(n-1)}}) \rightarrow a^{(n)}(\mathcal{O}_{X^{(n)}})
\]

be the morphism

\[
\sum_{\lambda = n+1} (-1)^{|\lambda_2|} \iota_{\lambda_2}^* : a^{(n-1)}(\bigoplus_{\lambda = n} \mathcal{O}_{X_{\lambda}}) \rightarrow a^{(n)}(\bigoplus_{\lambda = n+1} \mathcal{O}_{X_{\lambda}}).
\]

Then the following sequence

\[
0 \rightarrow \mathcal{O}_X \rightarrow a^{(0)}(\mathcal{O}_{X^{(0)}}) \xrightarrow{\rho^0} a^{(1)}(\mathcal{O}_{X^{(1)}}) \xrightarrow{\rho^1} a^{(2)}(\mathcal{O}_{X^{(2)}}) \xrightarrow{\rho^2} \cdots
\]

is exact.

**Proof.** The exactness of (4.3.2) is a local question. We have only to take the tensorization \( \mathcal{O}_X \otimes \mathcal{O}_\mathcal{C} \) in the proof of [F (4.16)].

**Remark 4.4.** If \( \overset{\circ}{X} \) is a locally product of SNC analytic spaces, we can prove (4.3) in a very quick way. Indeed this is well-known in the case where \( \overset{\circ}{X} \) is an SNC(=simple normal crossing) analytic space (cf. [St, p. 115]). In the general case, because \( \overset{\circ}{X} \) is a locally product of SNC analytic spaces, this follows from the case of SNC analytic spaces.

In the rest of this article, we fix a total order on \( \Lambda \). Let \( \mathcal{E} \) be a (not necessarily coherent) locally free \( \mathcal{O}_X \)-module and let

\[
\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{O}_X \Omega^1_{X/S}
\]
be an integrable connection. This connection induces the following integrable connection

\[(4.4.2)\quad \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}\]
and we have the log de Rham complex \(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}\). We also have the log de Rham complex \(\mathcal{E} \otimes_{\mathcal{O}_X} a^*_n(\Omega^\bullet_{X(n)/S})\) for each \(n \in \mathbb{N}\). Let \(T\) be \(\tilde{S}\) or \(S\). Then we have a morphism

\[
\rho^{n-1,j}: \mathcal{E} \otimes_{\mathcal{O}_X} a^*_n(\Omega^j_{X(n-1)/T}) \to \mathcal{E} \otimes_{\mathcal{O}_X} a^*_n(\Omega^j_{X(n)/T})
\]
as in (4.3.1).

**Corollary 4.5.** Let \(\mathcal{E} \otimes_{\mathcal{O}_X} a^*_n(\Omega^\bullet_{X(n)/T})\) be the double complex whose \((i,j)\)-component \(K^{ij}\) is \(\mathcal{E} \otimes_{\mathcal{O}_X} a^*_n(\Omega^j_{X(n)/T})\) and whose horizontal boundary morphism is \(\{\rho^j: K^{ij} \to K^{i+1,j}\}\) and whose vertical morphism is \((-1)^i\nabla\). Let \(s(\mathcal{E} \otimes_{\mathcal{O}_X} a^*_n(\Omega^\bullet_{X(n)/T}))\) be the single complex of \(\mathcal{E} \otimes_{\mathcal{O}_X} a^*_n(\Omega^\bullet_{X(n)/T})\). Endow \(s(\mathcal{E} \otimes_{\mathcal{O}_X} a^*_n(\Omega^\bullet_{X(n)/T}))\) with the Hodge filtration \(F\) defined by the following formula:

\[(4.5.1)\quad F^i s(\mathcal{E} \otimes_{\mathcal{O}_X} a^*_n(\Omega^\bullet_{X(n)/T})) := s(\mathcal{E} \otimes_{\mathcal{O}_X} a^*_n(\Omega^{\geq i}_{X(n)/T})), \quad (i \in \mathbb{N}).\]

Then the natural filtered morphism

\[(4.5.2)\quad (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/T}, F) \to (s(\mathcal{E} \otimes_{\mathcal{O}_X} a^*_n(\Omega^\bullet_{X(n)/T})), F)\]
is a filtered quasi-isomorphism.

**Proof.** This immediately follows from (4.2) and (4.3).

Consider the Hirsch extension \(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}[U_S]\) of \(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}\) as in the previous section. We also have the Hirsch extension \(\mathcal{E} \otimes_{\mathcal{O}_X} a^*_n(\Omega^\bullet_{X(n)/S})[U_S]\) of \(\mathcal{E} \otimes_{\mathcal{O}_X} a^*_n(\Omega^\bullet_{X(n)/S})\) as in the previous section. We obtain the double complex \(\mathcal{E} \otimes_{\mathcal{O}_X} a^*_n(\Omega^\bullet_{X(n)/S})[U_S]\) and the single complex \(s(\mathcal{E} \otimes_{\mathcal{O}_X} a^*_n(\Omega^\bullet_{X(n)/S})[U_S])\) of it.

Set

\[H(X/S, \mathcal{E}) := s(\mathcal{E} \otimes_{\mathcal{O}_X} a^*_n(\Omega^\bullet_{X(n)/S})[U_S]).\]

Endow \(H(X/S, \mathcal{E})\) with the Hodge filtration defined by the following formula

\[F^i H(X/S, \mathcal{E}) := s(\bigoplus_{n \geq 0} \Gamma_{O_{S}, n}(U_S) \otimes_{O_S} \mathcal{E} \otimes_{\mathcal{O}_X} a^*_n(\Omega^{\geq i-n}_{X(n)/S})).\]

**Corollary 4.6.** The following filtered morphism

\[(4.6.1)\quad (\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S}[U_S], F) \to (H(X/S, \mathcal{E}), F)\]
is a filtered quasi-isomorphism.
Proof. By (4.5) and (2.3) we see that the following morphism

\[ E \otimes_{O_X} \Omega^\bullet_{X/S}[U_S] \to H(X/S, \mathcal{E}) \]

is a quasi-isomorphism. We have the following equalities:

\[
\text{gr}_i F(E \otimes_{O_X} \Omega^\bullet_{X/S}[U_S]) := (\cdots \to \text{gr}_i F(s(\bigoplus_{n \geq 0} \Gamma_{O_S,n}(U_S) \otimes_{O_S} \mathcal{E} \otimes_{O_X} \Omega^k_{X/S}))) \to \cdots)
\]

and

\[
\text{gr}_i H(X/S, \mathcal{E}) := (\cdots \to s(\Gamma_{O_S,j-k}(U_S) \otimes_{O_S} \mathcal{E} \otimes_{O_X} \Omega^k_{X/S}))) \to \cdots).
\]

Hence the natural morphism

\[ \text{gr}_i F(E \otimes_{O_X} \Omega^\bullet_{X/S}[U_S]) \to \text{gr}_i H(X/S, \mathcal{E}) \]

is an isomorphism by the proof of (4.5).

The following is the main result of this article:

**Theorem 4.7.** There exists the following filtered isomorphism

\[(4.7.1) \quad (H(X/S, \mathcal{E}), F) \sim \to (E \otimes_{O_X} \Omega^\bullet_{X/S}, F)
\]

in the derived category \(D^+ F(f^{-1}(O_S))\) of bounded below filtered complexes of \(f^{-1}(O_S)\)-modules on \(X\).

Proof. By (3.8) and (4.6) we have the following filtered quasi-isomorphisms:

\[(4.7.2) \quad (E \otimes_{O_X} \Omega^\bullet_{X/S}, F) \sim \to (E \otimes_{O_X} \Omega^\bullet_{X/S}[U_S], F) \sim \to (H(X/S, \mathcal{E}), F).
\]

Hence we have the filtered isomorphism (4.7.1).

**Remark 4.8.** By (3.8), (3.9) and (4.6) we have the following commutative diagram

\[(4.8.1) \quad \begin{array}{ccc}
(E \otimes_{O_X} \Omega^\bullet_{X/S}[U_S], F) & \sim \to & (H(X/S, \mathcal{E}), F) \\
\approx & \downarrow & \\
(E \otimes_{O_X} \Omega^\bullet_{X/S}, F) & \sim \to & (s(E \otimes_{O_X} s^*(\Omega^\bullet_{X/S})), F).
\end{array}
\]

Hence the right vertical morphism in (4.8.1) is also a quasi-isomorphism.

The following is a generalization of (1.3):

**Corollary 4.9.** There exists a filtered isomorphism

\[(4.9.1) \quad Rf_*((H(X/S, \mathcal{E}), F)) \sim \to Rf_*((E \otimes_{O_X} \Omega^\bullet_{X/S}, F)).
\]
Lastly we consider the contravariant functoriality of the isomorphism (4.7.1) whose formulation is not obvious.

Let $S'$ be an analytic family over $\mathbb{C}$ of log points of virtual dimension $r'$ and let $v: S \rightarrow S'$ be a morphism of log analytic spaces. This morphism induces a morphism $v^*: \overline{M}_S \rightarrow v_*(\overline{M}_S)$. Locally on $S$, this morphism is equal to a morphism $v^*: \mathbb{N}^r \rightarrow \mathbb{N}$. Set $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}^r$ for $s = r$ or $r' (1 \leq i \leq r)$. Let $A = (a_{ij})_{1 \leq i \leq r, 1 \leq j \leq r'} \in M_{r'}(\mathbb{N})$ be the representing matrix of $v^*$:

$$v^*(e_1, \ldots, e_{r'}) = (e_1, \ldots, e_r)A.$$ 

Let $t_1, \ldots, t_r$ and $t'_1, \ldots, t'_{r'}$ be local sections of $M_S$ and $M_{S'}$ whose images in $\overline{M}_S \sim \mathbb{N}^r$ and $\overline{M}_{S'} \sim \mathbb{N}^{r'}$ are $e_1, \ldots, e_r$ and $e_1, \ldots, e_{r'}$, respectively. Let $\mathbf{t}_i$ and $\mathbf{t}'_i$ be the images of $t_i$ and $t'_i$ in $\overline{M}_S$ and $\overline{M}_{S'}$, respectively. Then there exists a local section $b_j \in \mathcal{O}_S^r (1 \leq j \leq r')$ such that

$$v^*(t'_j) = b_j t_1^{a_{1j}} \cdots t_r^{a_{rj}}.$$ 

Let $u_1, \ldots, u_r$ and $u'_1, \ldots, u'_{r'}$ be the corresponding local sections to $\mathbf{t}_1, \ldots, \mathbf{t}_r$ and $\mathbf{t}'_1, \ldots, \mathbf{t}'_{r'}$ of $U_S$ and $U_{S'}$, respectively. Then we define an $\mathcal{O}_{S'}$-linear morphism $v^*: \Gamma_{\mathcal{O}_{S'}}(U_{S'}) \rightarrow v_*(\Gamma_{\mathcal{O}_S}(U_S))$ by the following formula:

$$v^*(u'_j) = a_{1j} u_1 + \cdots + a_{rj} u_r.$$ 

and by $\text{Sym}(v^*)$ for the $v^*$ in (4.9.3). Since $\text{Aut}(\mathbb{N}^r) = \mathfrak{S}_r$ ([11, p. 47]), the morphism $v^*: \Gamma_{\mathcal{O}_{S'}}(U_{S'}) \rightarrow v_*(\Gamma_{\mathcal{O}_S}(U_S))$ is independent of the choice of the local isomorphisms $\overline{M}_{S'} \sim \mathbb{N}^{r'}$ and $\overline{M}_S \sim \mathbb{N}^r$. In conclusion, we obtain the following well-defined morphism

$$v^*: \Gamma_{\mathcal{O}_{S'}}(U_{S'}) \rightarrow v_*(\Gamma_{\mathcal{O}_S}(U_S))$$

of sheaves of commutative rings of unit elements on $S'$. This morphism satisfies the usual transitive relation

$$(v \circ v')^* = v'^* v^*.$$ 

**Remark 4.10.** Note that the morphism (4.9.4) is different from the natural morphism

$$v^*: M_{S'} \rightarrow v_*(M_S)$$

induced by $v^*: M_{S'} \rightarrow v_*(M_S)$. It is clear that $U_{\overline{S}}$ is more important than $\overline{M}_S$ in this article.

Let $f': X' \rightarrow S'$ be an analogous morphism of log analytic spaces to $f: X \rightarrow S$. Let $X'_{\lambda}$'s be analogous log analytic spaces to $X_{\lambda}$'s for $X'$. Assume that we are given a commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{g} & X' \\
\downarrow f & & \downarrow f' \\
S & \xrightarrow{v} & S'
\end{array}$$

such that, for any smooth component $\tilde{X}_{\lambda}$ of $\tilde{X}$ over $\tilde{S}$, there exists a unique smooth component $X'_{\lambda'}$ of $X'$ over $\tilde{S}'$ such that $g$ induces a morphism $\tilde{X}_{\lambda} \rightarrow X'_{\lambda'}$.  

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**Proposition 4.11** (Contravariant functoriality). Assume that we are given a commutative diagram

\[
\begin{array}{ccc}
E' & \xrightarrow{\nabla'} & E' \otimes_{O_{X'}} \Omega^1_{X'/\mathcal{S}'} \\
\downarrow & & \downarrow \\
g_*(E) & \xrightarrow{g_*(\nabla)} & g_*(E \otimes_{O_X} \Omega^1_{X/\mathcal{S}}),
\end{array}
\]

where \((E', \nabla')\) is an analogous connection to \((E, \nabla)\) on \(X'\). Then the filtered morphism \((4.7.1)\) is contravariantly functorial for the commutative diagram \((4.10.2)\) and \((4.11.1)\). This contravariance satisfies the usual transitive relation \(\((g \circ h)^* = h^* g^*\)\).

**Proof.** By the assumption and the existence of the morphism \((4.9.4)\), we have the pull-back

\[g^*: (H(X/S, E), F) \rightarrow Rg_*(((H(X'/S', E'), F)\]

fitting into the following commutative diagram

\[
\begin{array}{ccc}
(E' \otimes_{O_{X'}} \Omega^1_{X'/\mathcal{S}'}, F) & \xleftarrow{\sim} & (E' \otimes_{O_{X'}} \Omega^1_{X'/\mathcal{S}'}, [U_{\mathcal{S}'}], F) \\
\downarrow & & \downarrow \quad \quad \downarrow \\
Rg_*(E \otimes_{O_X} \Omega^1_{X/\mathcal{S}}) & \xrightarrow{\sim} & Rg_*(E \otimes_{O_X} \Omega^1_{X/\mathcal{S}}), F) \\
\end{array}
\]

The transitive relation is now obvious.

**Corollary 4.12.** This isomorphism \((4.9.1)\) is contravariantly functorial with respect to \(g\) in \((4.4)\) and the commutative diagrams \((4.10.2)\) and \((4.11.1)\). This functoriality satisfies the transitive relation.

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