ON STRUCTURE OF GRADED RESTRICTED SIMPLE LIE ALGEBRAS OF CARTAN TYPE AS MODULES OVER THE WITT ALGEBRA

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Abstract. Any graded restricted simple Lie algebra of Cartan type contains a subalgebra isomorphic to the Witt algebra over a field of prime characteristic. As some analogue of study on branching rules for restricted non-classical Lie algebras, it is shown that each graded restricted simple Lie algebra of Cartan type can be decomposed into a direct sum of restricted baby Verma modules and simple modules as an adjoint module over the Witt algebra. In particular, the composition factors are precisely determined.

1. Introduction

It is well known that in the late 1930’s E. Witt firstly found a non-classical simple Lie algebra over prime characteristic field which is called the Witt algebra $W(1)$. This contributed to the study of non-classical simple Lie algebras which were called Cartan-type Lie algebras later. A well-known classification result on simple modular Lie algebras asserts that each finite dimensional (restricted) simple Lie algebra over a field of prime characteristic $p > 5$ is of either classical type or Cartan type (cf. [1, 5]). There are four families of simple Lie algebras $X(n)$ of Cartan type $X$ for $X \in \{W, S, H, K\}$. They are subalgebras of derivation algebra of truncated polynomial algebras. Each simple Lie algebra of Cartan type contains a subalgebra isomorphic to the Witt algebra, which plays a similar role as the three dimensional simple Lie algebra $sl_2$ of type $A_1$ in classical simple Lie algebras. In view of this point, the Witt algebra is a "fundamental" non-classical simple Lie algebra.

The representation theory of the Witt algebra $W(1)$ was firstly studied by Ho-Jui Chang in the early 1940’s (cf. [2]). Its irreducible representations were completely determined. Based on this result, in the present paper we precisely determine the structure of graded restricted simple Lie algebras of Cartan type as adjoint modules over the Witt algebra. It is shown that any graded restricted simple Lie algebra of Cartan type can be decomposed as a direct sum of restricted baby Verma modules and simple modules over the Witt algebra. As a consequence, the composition factors are precisely determined. We hope the study on the decomposition of $X(n)$ as a $W(1)$-modules will provide some useful and interesting intrinsic observation on the structure of irreducible restricted $X(n)$-modules and branching rules in restricted representation category for $X(n)$, where $X \in \{W, S, H, K\}$.

This paper is organized as follows. In section 2, we introduce some basic concepts on restricted Lie algebras and their (restricted) representations, and the algebra structure on graded Lie algebras of Cartan type. In particular, we present precisely the embedding of the Witt algebra to the four families of Lie algebras of Cartan type. Moreover, the restricted representation theory of the Witt algebra is recalled. Section 3 is devoted to studying the decomposition of the Jacobson-Witt algebra.
as a module over the Witt algebra into a direct sum of submodules. In section 4, we first precisely give a basis for the special algebra. By using this basis, we decompose the special algebra into a direct sum of restricted baby Verma modules and simple modules over the Witt algebra. Sections 5 and 6 are devoted to determining the decomposition of the Hamiltonian algebra and the contact algebra as direct sums of restricted baby Verma modules and simple modules over the Witt algebra, respectively.

2. Preliminaries

Throughout this paper, $\mathbb{F}$ is assumed to be an algebraically closed field of prime characteristic $p > 2$. All modules (vector spaces) are over $\mathbb{F}$ and finite-dimensional. Set $I = \{0, 1, \ldots, p - 1\}$. For a finite set $S$, let $|S|$ denote the number of elements in $S$. For a Lie algebra $\mathfrak{g}$, let $U(\mathfrak{g})$ be its universal enveloping algebra, and $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. For a $\mathfrak{g}$-module $M$, let $[M]$ be the formal sum of all composition factors of $M$.

2.1. Restricted Lie algebras and their irreducible representations. Recall that a restricted Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ is a Lie algebra with a so-called restricted mapping $[p]: \mathfrak{g} \to \mathfrak{g}$ sending $x \mapsto x^{[p]}$ satisfying that $\text{ad} (x^{[p]}) = (\text{ad} x)^p$ and that $\xi : \mathfrak{g} \to Z(\mathfrak{g})$ sending $x \mapsto x^p - x^{[p]}$ is semi-linear.

For a restricted Lie algebra $(\mathfrak{g}, [p])$ and a simple $\mathfrak{g}$-module $M$, since $x^p - x^{[p]} \in Z(\mathfrak{g})$ for any $x \in \mathfrak{g}$, the element $x^p - x^{[p]}$ must act as a scalar, denoted by $\chi(x)^p$. The semilinearly of $\chi$ implies that $\chi \in \mathfrak{g}^*$. In general, a $\mathfrak{g}$-module $M$ is said to be $\chi$-reduced if $x^p \cdot v - x^{[p]} - v = \chi(x)^p v$ for all $x \in \mathfrak{g}, v \in M$. In particular, it is called a restricted module if $\chi = 0$. Let $U_\chi(\mathfrak{g})$ be the quotient of the universal enveloping algebra $U(\mathfrak{g})$ by the ideal generated by $\{x^p - x^{[p]} - \chi(x)^p \mid x \in \mathfrak{g}\}$ which is called a $\chi$-reduced enveloping algebra of $\mathfrak{g}$, i.e., $U_\chi(\mathfrak{g}) = U(\mathfrak{g})/(x^p - x^{[p]} - \chi(x)^p \mid x \in \mathfrak{g}_0)$. If $\chi = 0$, the algebra $U_0(\mathfrak{g})$ is called the restricted enveloping algebra and denoted by $u(\mathfrak{g})$ for brevity. All the $\chi$-reduced (resp. restricted) $\mathfrak{g}$-modules constitute a full subcategory of the $\mathfrak{g}$-module category, which coincides with the $U_\chi(\mathfrak{g})$ (resp. $u(\mathfrak{g})$)-module category. Each simple $\mathfrak{g}$-module is a $U_\chi(\mathfrak{g})$-module for a unique $\chi \in \mathfrak{g}^*$.

2.2. Graded Lie algebras of Cartan type. Fix a positive integer $n$. Denote by $A(n)$ the index set $\{\alpha = (\alpha_1, \ldots, \alpha_n) \mid 0 \leq \alpha_i \leq p - 1, i = 1, 2, \cdots, n\}$, and denote $(p - 1, \ldots, p - 1)$ by $\tau$ for brevity. We have a truncated polynomial algebra $A(n)$ which is by definition a commutative associative algebra with a basis $\{x^\alpha \mid \alpha \in A(n)\}$, and multiplication subject to the following rule

$$x^\alpha x^\beta = x^{\alpha + \beta}, \quad \forall \alpha, \beta \in A(n),$$

additionally with

$$x^\alpha = 0 \text{ if } \alpha \notin A(n); \quad x_i := x^{\epsilon_i} \text{ for } \epsilon_i = (\delta_{1,i}, \ldots, \delta_{n,i}).$$

There is a natural graded structure on $A(n)$, and consequently a filtered structure there. The gradation and filtration of $A(n)$ induce the corresponding ones on the so-called Jacobson-Witt algebra $W(n)$, which is the derivation algebra of $A(n)$. Then $W(n)$ is free $A(n)$-module with a basis $\{\partial_1, \cdots, \partial_n\}$, where $\partial_i(x_j) = \delta_{ij}, 1 \leq i, j \leq n$. For more details, the readers are referred to the reference [4].

We can get other three series of subalgebras in $W(n)$, which are called (graded) Cartan type Lie algebras of series $S, H,$ and $K$ respectively, arising from the three exterior differentials $\omega_S, \omega_H, \omega_K$. Below, we recall the definitions, and cite some basic notations and facts we need later. The definitions here will be given by using some operators (cf. [6], Chapter 4), instead of using the original differential forms (cf. [1]).

Set $\bar{S}(n) = \{D \in W(n) \mid \text{div}(D) = 0\}$, where $\text{div}(\sum f_i \partial_i) = \sum \partial_i(f_i)$ for any $\sum f_i \partial_i \in W(n)$. Then by definition, the derived algebra of $\bar{S}(n)$ is called the special algebra $S(n)$, i.e. $S(n) = [\bar{S}(n), S(n)]$. 


The Hamiltonian algebra is by definition $H(2r) = \mathbb{F}\text{-span}\{D_H(x^\alpha) \mid 0 < \alpha < \tau\}$. Here $D_H$ is the Hamiltonian operator from $\mathfrak{A}(2r)$ to $W(2r)$ defined as follows:

$$
D_H : \mathfrak{A}(2r) \to W(2r)
$$

$$
f \mapsto D_H(f) = \sum_{i=1}^{2r} \sigma(i) \partial_i(f) \partial_{i'}
$$

where

$$
\sigma(i) := \begin{cases} 
1, & \text{if } 1 \leq i \leq r, \\
-1, & \text{if } r+1 \leq i \leq 2r,
\end{cases}
$$

and

$$
i' := \begin{cases} 
i + r, & \text{if } 1 \leq i \leq r, \\
i - r, & \text{if } r+1 \leq i \leq 2r.
\end{cases}
$$

Set $\bar{K}(2r+1) = \mathbb{F}\text{-span}\{D_K(x^\alpha) \mid \alpha \in A(2r+1)\}$, where the contact operator $D_K$ from $\mathfrak{A}(2r+1)$ to $W(2r+1)$ is defined as follows:

$$
D_K : \mathfrak{A}(2r+1) \to W(2r+1)
$$

$$
f \mapsto D_K(f) = \sum_{i=1}^{2r+1} f_i \partial_i
$$

where

$$
f_j = x_j \partial_{2r+1}(f) + \sigma(i') \partial_{i'}(f), \; j \leq 2r,
$$

$$
f_{2r+1} = 2f - \sum_{i=1}^{2r} \sigma(j') x_j f_{j'}.
$$

The contact algebra $K(2r+1)$ is the derived algebra of $\bar{K}(2r+1)$, i.e. $K(2r+1) = [\bar{K}(2r+1), \bar{K}(2r+1)]$. Both subalgebras $S(n)$ and $H(n)$ naturally inherit the graded and filtered structure of $W(n)$. But the contact algebra $K(2r+1)$ is not a graded subalgebra of $W(2r+1)$. One can define a new gradation on $K(2r+1)$ which is not inherited from the gradation of $W(2r+1)$. For that, define $||\alpha|| = \sum_{i=1}^{2r+1} \alpha_i + \alpha_{2r+1} - 2$ for $\alpha \in A(2r+1)$ and $K(2r+1)[i] = \mathbb{F}\text{-span}\{D_K(x^\alpha) \mid ||\alpha|| = i\}$. Then $K(2r+1) = \bigoplus_{i \geq -2} K(2r+1)[i]$ is a gradation of $K(2r+1)$. Associated with this gradation, one can also obtain the corresponding filtration

$$
K(2r+1) = K(2r+1)_{-2} \supset K(2r+1)_{-1} \supset \cdots
$$

where $K(2r+1)i = \bigoplus_{j \geq i} K(2r+1)[j]$.

Let $L = X(n), \; X \in \{W, S, H, K\}$. Then $L$ is a restricted Lie algebra with the restricted $[p]$-mapping given by taking the $p$-th power of derivations. Moreover, $L$ has a $\mathbb{Z}$-grading $L = \bigoplus_{i \geq -\delta_{XK}} L[i]$, associated with which there is a natural filtration $L = L_{-1} \supset L_{0} \supset \cdots$ with $L_j = \bigoplus_{i \geq j} L[i]$ for $j \geq -1 - \delta_{XK}$. 

2.3. Embeddings of the Witt algebra to restricted Lie algebras of Cartan type. Define the linear mappings from the Witt algebra to restricted Lie algebras of Cartan type as follows.

\[ \Theta_W : W(1) \rightarrow W(n) \]
\[ x_1^i \partial_1 \mapsto x_1^i \partial_1, \quad \forall 0 \leq i \leq p - 1, \]

\[ \Theta_S : W(1) \rightarrow S(n) \]
\[ x_1^i \partial_1 \mapsto D_{12}(x_1^i x_2) = x_1^i \partial_1 - ix_1^{i-1}x_2 \partial_2, \quad \forall 0 \leq i \leq p - 1, \]

\[ \Theta_H : W(1) \rightarrow H(2r) \]
\[ x_1^i \partial_1 \mapsto -D_H(x_1^i x_{r+1}) = x_1^i \partial_1 - ix_1^{i-1}x_{r+1} \partial_{r+1}, \quad \forall 0 \leq i \leq p - 1, \]

\[ \Theta_K : W(1) \rightarrow K(2r + 1) \]
\[ x_1^i \partial_1 \mapsto \frac{1}{2}D_K(x_{2r+1}^i), \quad \forall 0 \leq i \leq p - 1. \]

The following result asserts that the Witt algebra is a restricted subalgebra of any restricted Lie algebra of Cartan type. The proof follows from straightforward computation, we omit the details.

**Lemma 2.1.** Keep notations as above. Then \( \Theta_X \) is an injective restricted Lie algebra homomorphism from the Witt algebra \( W(1) \) to the restricted Lie algebra \( X(n) \) of Cartan type \( X \) for \( X \in \{ W, S, H, K \} \).

**Remark 2.2.** Thanks to Lemma 2.1, any Lie algebra of Cartan type is a restricted module over the Witt algebra under the adjoint action. We will determine this module structure in the sequel sections.

2.4. Restricted representations of the Witt algebra \( W(1) \). In this subsection, we always assume that \( \mathfrak{g} = W(1) \) is the Witt algebra over \( \mathbb{F} \). We will recall the known classification results on simple \( \mathfrak{g} \)-modules given by Chang in [2]. Recall that \( \mathfrak{g} \) has a natural \( \mathbb{Z} \)-gradation \( \mathfrak{g} = \bigoplus_{i=-1}^{p-1} \mathfrak{g}[i] \), associated with which there is a filtration \( \mathfrak{g} = \mathfrak{g}_{-1} \supset \mathfrak{g}_0 \cdots \supset \mathfrak{g}_{p-2} \supset 0 \), where \( \mathfrak{g}_i = \bigoplus_{j \geq i} \mathfrak{g}[j] \) for \( -1 \leq i \leq p - 2 \). For any \( \lambda \in I \), let \( \mathbb{F}_\lambda \) be the one dimensional restricted \( \mathfrak{g}_0 \)-module given by multiplication by the scalar \( \lambda \). Then we can regard \( \mathbb{F}_\lambda \) as a restricted \( \mathfrak{g}_0 \)-module with trivial action by \( \mathfrak{g}_1 \). Let \( V(\lambda) = u(\mathfrak{g}) \otimes_{u(\mathfrak{g}_0)} \mathbb{F}_\lambda \) which is called a restricted baby Verma \( \mathfrak{g} \)-module. Each restricted baby Verma \( \mathfrak{g} \)-module \( V(\lambda) \) has a unique simple quotient denoted by \( L(\lambda) \) for \( \lambda \in I \). Then the set \( \{ L(\lambda) \mid \lambda \in I \} \) exhausts all non-isomorphic irreducible restricted \( \mathfrak{g} \)-modules. Moreover, \( L(\lambda) = V(\lambda) \) if and only if \( 0 < \lambda < p - 1 \). While both \( V(0) \) and \( V(p - 1) \) have two composition factors \( L(0) \) and \( L(p - 1) \). The natural module \( A(1) \) is isomorphic to \( V(p - 1) \), while the adjoint module \( W(1) \) is isomorphic to \( L(p - 2) \).

3. Structure of the Jacobson-Witt algebras as modules over the Witt algebra

In this section, we study the structure of the Jacobson-Witt algebra \( W(n) \) as a module over the Witt algebra \( W(1) \). For that, denote \( x^\mathbf{i} = x_2^{i_2} \cdots x_n^{i_n} \) for any \( \mathbf{i} = (i_2, \cdots, i_n) \in I^{n-1} \). Then

\[ W(n) = \bigoplus_{j=1}^{n} \bigoplus_{\mathbf{i} \in I^{n-1}} A(1)x^\mathbf{i} \partial_j. \]
Moreover, each summand in \((3.1)\) is a \(W(1)\)-module. More precisely, for each \(i \in I^{n-1}\), we have the following isomorphism as modules over the Witt algebra \(W(1)\),
\[
A(1)x_i^j \partial_j \cong \begin{cases} 
W(1), & \text{if } j = 1, \\
A(1), & \text{if } 2 \leq j \leq n.
\end{cases}
\]

Consequently, we have

**Theorem 3.1.** As a module over the Witt algebra \(W(1)\), we have
\[
W(n) \cong V(p-1)^{\oplus (n-1)p^{n-1}} \oplus L(p-2)^{\oplus p^{n-1}}.
\]

Hence,
\[
[W(n)] = (n-1)p^{n-1} ([L(0)] + [L(p-1)]) + p^{n-1}[L(p-2)].
\]

**Proof.** Since \(A(1) \cong V(p-1)\) and \(W(1) \cong L(p-2)\) as \(W(1)\)-modules, \((3.2)\) follows. Furthermore, note that the restricted baby Verma module \(V(p-1)\) has two composition factors \(L(0)\) and \(L(p-1)\). This together with \((3.2)\) yields \((3.3)\). \(\square\)

4. Structure of the special algebras as modules over the Witt algebra

In this section, we study the structure of the special algebra \(S(n)\) as a module over the Witt algebra \(W(1)\). For that, for each \(a = (a_1, \cdots, a_n) \in I^n\), we define
\[
x^a = x_1^{a_1} \cdots x_n^{a_n}, \quad \Omega(a) = \{i \mid a_i \neq p-1\} \quad \text{and} \quad \ell(a) = |\Omega(a)|.
\]

In the following, if \(\Omega(a) = \{i_1, \cdots, i_s\}\), we always assume that \(1 \leq i_1 < \cdots < i_s \leq n\).

4.1. **Basis of \(S(n)\).** In this subsection, we give a basis of \(S(n)\). This may be known for experts. However, we do not find it in literature.

Set
\[
\mathfrak{B}_1 = \{x^a_i \mid a \in I^n, 1 \leq i \leq n, \text{ and } a_i = 0, a_j \neq p-1 \text{ for some } j \neq i\},
\]
and
\[
\mathfrak{B}_2 = \{D_{ij,i_{j+1}}(x^a_i x_j x_{i_{j+1}}) \mid a \in I^n \text{ with } \Omega(a) = \{i_1, \cdots, i_s\}, 1 \leq j \leq s-1\}.
\]

Let \(V_1\) be the linear subspace of \(S(n)\) spanned by elements in \(\mathfrak{B}_1\), and let \(V_2\) be the linear subspace of \(S(n)\) spanned by \(D_{ij}(x^a)\) for \(a \in I^n\) with \(a_i \neq 0\) and \(a_j \neq 0\), \(1 \leq i < j \leq n\). Note that \(D_{ij}(x^a) \in V_2\) if and only if \(x^a = x^a_i x_j x_{i_{j+1}}\) for some \(a \in I^n\) such that \(a_i, a_j \neq p-1\).

We have the following basic observation.

**Lemma 4.1.** Keep notations as above. Then the following statements hold.

1. \(S(n) = V_1 \oplus V_2\).
2. \(\mathfrak{B}_1\) is a basis of \(V_1\).
3. \(\dim(V_1) = n(p^{n-1} - 1), \dim(V_2) = np^{n-1}(p-1) + 1\).

**Proof.** (1) It’s obvious that \(V_1 \cap V_2 = \{0\}\). Moreover, if \(a \in I^n\) with \(a_i = 0\) and \(a_j \neq 0\), then \(D_{ij}(x^a) = a_j x^a_i x_j \partial_i \in V_1\). Hence, (1) follows.

(2) is obvious.

(3) The first assertion for \(\dim(V_1)\) follows from (2). Furthermore, it follows from (1) and \(\frac{\ell}{2}\). Theorem 3.7, Chapter 4] that
\[
\dim(V_2) = \dim(S(n)) - \dim(V_1) = (n-1)(p^n - 1) - n(p^{n-1} - 1) = np^{n-1}(p-1) + 1.
\]

\(\square\)

The following result is crucial to our final determination of a basis of \(S(n)\).
Lemma 4.2. Suppose $\ell(\mathfrak{a}) = s$ and $\Omega(\mathfrak{a}) = \{i_1, \cdots, i_s\}$. Then for any $k, l \in \Omega(\mathfrak{a})$,
\[ D_{kl}(x^a x_k x_l) \in \text{span}_F \{ D_{ij,i_{j+1}}(x^a x_j x_{i_{j+1}}) \mid 1 \leq j \leq s - 1 \}. \]

Proof. Without loss of generality, we can suppose that $k = i_1$, $l = i_s$. It is readily shown that \{ $D_{ij,i_{j+1}}(x^a x_j x_{i_{j+1}})$ \mid 1 \leq j \leq s - 1 \} are linear independent. Recall that $D_{kl}(x^a x_k x_l) = x^a((a_l + 1)x_k \partial_k - (a_k + 1)x_l \partial_l)$. Since
\[
\begin{vmatrix}
  a_{i_2} + 1 & 0 & \cdots & 0 & a_{i_s} + 1 \\
  -(a_{i_1} + 1) & \ddots & \ddots & \vdots & 0 \\
  0 & \ddots & \ddots & \ddots & \vdots \\
  0 & \cdots & a_{i_s} + 1 & 0 & \\
  0 & \cdots & -(a_{i_{s-1}} + 1) & -(a_{i_1} + 1) & \\
\end{vmatrix} = 0,
\]
\[ D_{kl}(x^a x_k x_l) \in \text{span}_F \{ D_{ij,i_{j+1}}(x^a x_j x_{i_{j+1}}) \mid 1 \leq j \leq s - 1 \}. \]

We need the following combination formulas for later use.

Lemma 4.3. The following equalities hold.
\[
\sum_{s=0}^{n} s C_n^s (p - 1)^s = n(p - 1)p^{n-1},
\]
\[
\sum_{s=2}^{n} C_n^s (p - 1)^{s-2}(s - 1) = \sum_{i=1}^{n-1} i p^{i-1}.
\]

Proof. Take derivations on both sides of the following equations respectively,
\[
\sum_{s=0}^{n} C_n^s (x - 1)^s = x^n,
\]
\[
\sum_{s=2}^{n} C_n^s (x - 1)^{s-1} = \frac{1}{x-1}(x^n - n(x-1) - 1) = \sum_{i=0}^{n-1} x^i - n,
\]
and put $x = p$. Then the desired equalities follows. \qed

As a consequence of Lemma 4.2 and Lemma 4.3, we have

Corollary 4.4. The subspace $V_2$ has a basis $\mathfrak{B}_2$. Consequently, $S(n)$ has a basis $\mathfrak{B}_1 \cup \mathfrak{B}_2$.

Proof. By Lemma 4.2,
\[
V_2 = \sum_{D \in \mathfrak{B}_2} F D = \sum_{s=2}^{n} \sum_{\Omega(\mathfrak{a}) \subseteq \{i_1, \cdots, i_s\}} \sum_{j=1}^{s-1} F D_{ij,i_{j+1}}(x^a x_j x_{i_{j+1}}).
\]
Moreover, it follows from Lemma 4.1(3) and Lemma 4.3 that
\[
|\mathfrak{B}_2| = \sum_{s=2}^{n} C_n^s (p - 1)^s (s - 1) = (p - 1)^2 \sum_{s=2}^{n} C_n^s (p - 1)^{s-2}(s - 1) = (p - 1)^2 \sum_{i=1}^{n-1} i p^{i-1} = \dim(V_2).
\]
Hence, $\mathfrak{B}_2$ is a basis of $V_2$. This together with Lemma 4.1(2) yields the second assertion. \qed
4.2. The structure of $S(n)$ as a module over the Witt algebra. Recall the Lie algebra embedding $\Theta_S : W(1) \hookrightarrow S(n)$ given by $\Theta_S(x_i^1 \partial_1) = D_{12}(x_i^1 x_2^1) = x_i^1 \partial_1 - ix_i^1 x_2^1 \partial_2$, $i \in I$. In this subsection, we study the structure of the special algebra $S(n)$ as a module over the Witt algebra $W(1)$. For that, for each $l = (l_2, \ldots, l_n) \in I^{n-1}$, we denote $x_l := x_2^{l_2} \cdots x_n^{l_n}$.

For any $2 \leq i \leq n$ and $l \in I^{n-1}$ with $l_i = 0$, let

$$N_i := \text{span}_F \{x_i^1 x_2^{p-1} \cdots x_i^{p-1} x_{i+1}^{q-1} \cdots x_n^{p-1} \partial_i \mid 0 \leq t \leq p - 2 \}$$

and $N_{l_i} := \text{span}_F \{x_i^1 x_l \partial_l \mid l \in I \}$ if $l_j \neq p - 1$ for some $j \neq i$. Then both $N_i$ and $N_{l_i}$ are submodules over the Witt algebra. More precisely, we have

**Lemma 4.5.** Keep notations as above. Then as modules over the Witt algebra $W(1)$, we have

1. For each $2 \leq i \leq n$, $N_i \cong L(p - 1)$.
2. Suppose $l \in I^{n-1}$ with $l_2 = 0$ and $l_j \neq p - 1$ for some $j \neq 2$. Then $N_{l_2} \cong V(0)$.
3. For each $3 \leq i \leq n$, suppose $l \in I^{n-1}$ with $l_i = 0$ and $l_j \neq p - 1$ for some $j \neq i$. Then $N_{l_i} \cong V(p - 1 - l_2)$.

**Proof.** (1) For each $2 \leq i \leq n$, let $\Delta = (p - 1, \ldots, p - 1) - (p - 1) \epsilon_i \in I^{n-1}$. Then

$$[\Theta_S(x_i^1 \partial_1), x_i^1 x_l \partial_l] = (t + r)x_i^1 x_l^{t-1} \partial_l, \quad \forall r \in I, \quad 0 \leq t \leq p - 2.$$ 

Therefore, $N_i$ is a simple $W(1)$-module with a maximal vector $x_1^{p-2} x_l \partial_l$ of weight $p - 1$. Hence, $N_i \cong L(p - 1)$.

(2) Since $l \in I^{n-1}$ with $l_2 = 0, l_i \neq p - 1$ for some $i \neq 2$, we have

$$[\Theta_S(x_i^1 \partial_1), x_i^1 x_2 \partial_2] = (t + r)x_i^1 x_2^{t-1} \partial_2, \quad \forall r \in I.$$ 

This implies that $N_{l_2}$, as a $W(1)$-module, has a maximal vector $x_1^{p-1} x_2 \partial_2$ of weight 0. Consequently, $N_{l_2} \cong V(0)$.

(3) For each $i \geq 3$, if $l \in I^{n-1}$ with $l_i = 0, l_j \neq p - 1$ for some $j \neq i$, then

$$[\Theta_S(x_i^1 \partial_1), x_i^1 x_l \partial_l] = (t - r l_2)x_i^1 x_l^{t-1} \partial_l, \quad \forall r, t \in I.$$ 

This implies that $N_{l_i}$, as a $W(1)$-module, has a maximal vector $x_1^{p-1} x_i \partial_i$ of weight $p - 1 - l_2$. Consequently, $N_{l_i} \cong V(p - 1 - l_2)$.

For each $1 \leq i < j \leq n$ and $\Delta \in I^{n-1}$, let $M_{\Delta, i,j} := \text{span}_F \{D_{ij}(x_i^1 x_l) \mid t \in I \}$.

**Lemma 4.6.** Suppose $l = (l_2, \ldots, l_n) \in I^{n-1}$ with $\ell(l) = s$ and $\Omega(l) = \{i_1, \ldots, i_s\} \subseteq \{2, \ldots, n\}$. Then as modules over the Witt algebra $W(1)$, we have

1. For each $1 \leq j \leq s - 1$, $M_{(l)^+\epsilon_{i_j}+\epsilon_{i_{j+1}}i_{j+1}} \cong V(p - 1 - l_2)$.
2. If $i_1 = 2$, then $M_{(l)^{+\epsilon_2,12}} \cong V(p - 2 - l_2)$.
3. If $i_1 > 2$, then $M_{(l)^{+\epsilon_{i_1}i_{i_1}}} \cong V(p - 1)$.

**Proof.** (1) For each $1 \leq j \leq s - 1$ and $r, t \in I$,

$$[\Theta_S(x_i^1 \partial_1), D_{ij}(x_i^1 x_l \partial_l)] = (t - r l_2)D_{ij}(x_i^1 x_l^{t-1} x_i^1 x_l \partial_l).$$

This implies that $M_{(l)^{+\epsilon_{i_j}+\epsilon_{i_{j+1}}i_{j+1}}}$, as a $W(1)$-module, has a maximal vector $D_{ij}(x_i^1 x_l x_i^1 x_l)$ of weight $p - 1 - l_2$. Hence, $M_{(l)^{+\epsilon_{i_j}+\epsilon_{i_{j+1}}i_{j+1}}} \cong V(p - 1 - l_2)$.

(2) For each $r, t \in I$, we have

$$[\Theta_S(x_i^1 \partial_1), D_{12}(x_i^1 x_{l+\epsilon_2})] = (t - r l_2 - r)D_{ij}(x_i^1 x_l^{t-1} x_{l+\epsilon_2}).$$

This implies that $M_{(l)^{+\epsilon_2,12}}$, as a $W(1)$-module, has a maximal vector $D_{12}(x_i^1 x_l^{t+1} x_{l+\epsilon_2})$ of weight $p - 2 - l_2$. Hence, $M_{(l)^{+\epsilon_2,12}} \cong V(p - 2 - l_2)$. 

\[ \]
(3) Since \( i_1 > 2 \), \( l_2 = p - 1 \). For each \( r, t \in I \), we have
\[
[\Theta_S(x_r^i \partial_1), D_{l_1 r}(x_1^i x_r^j \epsilon_i \epsilon_j)] = t D_{l_1 r}(x_1^r x_r^{j-1} x_r^{j+1} \epsilon_i \epsilon_j).
\]
This implies that \( M_{l_1 + \epsilon_i, \epsilon_j} \), as a \( W(1) \)-module, has a maximal vector \( D_{l_2}(x_1^{p-1} x_1^{j+1} \epsilon_i \epsilon_j) \) of weight \( p - 1 \). Hence, \( M_{l_1 + \epsilon_i, \epsilon_j} \cong V(p - 1) \).

We are now in the position to present the following main result on the decomposition of \( S(n) \) as a direct sum of \( W(1) \)-modules.

**Theorem 4.7.** As a \( W(1) \)-module, we have
\[
S(n) \cong V(0)^{(n-1)(p^{n-2} - 1) + 1} \oplus V(p - 1)^{(n-1)p^{n-2} - 1)} \\
\oplus \left( \bigoplus_{i=1}^{p-2} L(i)^{(n-1)p^{n-2} - 1} \right) \oplus L(p - 1)^{(n-1)p^{n-2} - 1)}.
\]

**Proof.** Set
\[
V_{11} := \text{span} \{x^{2, i} \partial_1 \mid a \in I^n \text{ with } a_1 = 0 \text{ and } a_j \neq p - 1 \text{ for some } j \neq 1\},
\]
\[
V_{12} := \text{span} \{x^{2, i} \partial_1 \mid 2 \leq i \leq n, a \in I^n \text{ with } a_i = 0 \text{ and } a_j \neq p - 1 \text{ for some } j \neq i\}.
\]
Then \( V_1 = V_{11} \oplus V_{12} \) as a vector space. Moreover, both \( V_{12} \) and \( V_{11} + V_2 \) are \( W(1) \)-modules, and \( S(n) = V_{12} \oplus (V_{11} + V_2) \). It follows from Lemma 4.3 that \( V_{12} = U_1 \oplus U_2 \oplus U_3 \oplus U_4 \), where
\[
U_1 = \bigoplus_{i=2}^{n} N_i \cong L(p - 1)^{(n-1)}, \quad U_2 = \bigoplus_{l \in I^{n-1}, l_2 = 0} N_{L_2} \cong V(0)^{(p^{n-2} - 1)},
\]
\[
U_3 = \bigoplus_{i=3}^{n} \bigoplus_{j=0}^{p-2} N_{L_i} \cong \bigoplus_{i=3}^{n} \bigoplus_{j=0}^{p-2} V(p - 1 - j) \cong \bigoplus_{i=1}^{p-1} V(i)^{(n-2)p^{n-3}},
\]
\[
U_4 = \bigoplus_{i=3}^{n} \bigoplus_{l \in I^{n-1}} N_{L_i} \cong \bigoplus_{i=3}^{n} V(0) \cong V(0)^{(n-2)(p^{n-3} - 1)}.
\]
Since \( V(i) = L(i) \) for \( 1 \leq i \leq p - 1 \) (see §2.4), it follows that
\[
V_{12} \cong V(0)^{(n-1)(p^{n-2} - 1) + 1} \oplus \left( \bigoplus_{i=1}^{p-2} L(i)^{(n-1)p^{n-2} - 1} \right) \\
\oplus L(p - 1)^{(n-1)p^{n-2} - 1)} \oplus V(p - 1)^{(n-1)p^{n-2} - 1)}.
\]

It follows from Corollary 4.4 and Lemma 4.6 that \( V_{11} + V_2 = W_1 \oplus W_2 \oplus W_3 \oplus W_4 \), where
\[
W_1 = \bigoplus_{s=2}^{n-1} \bigoplus_{l \in I^{n-1}, l_1 = 2} M_{l_1 + \epsilon_i + j, \epsilon_j + 1, \epsilon_j + j, \epsilon_j + 1} \quad W_2 = \bigoplus_{s=2}^{n-1} \bigoplus_{l \in I^{n-1}, l_1 = 2} M_{l_1 + \epsilon_i + j, \epsilon_j + 1, \epsilon_j + j, \epsilon_j + 1},
\]
\[
W_3 = \bigoplus_{s=1}^{n-1} \bigoplus_{l \in I^{n-1}, l_1 = 2} M_{l_1 + \epsilon_i + l_1, \epsilon_i + l_1} \quad W_4 = \bigoplus_{s=1}^{n-1} \bigoplus_{l \in I^{n-1}, l_1 = 2} M_{l_1 + \epsilon_i + l_1, \epsilon_i + l_1}.
\]
Thanks to Lemma 4.3, we have
\[ \sum_{s=2}^{n-1} (s-1)C_{n-2}^{s-1}(p-1)^{s-1} = (n-2)(p-1)p^{n-3}, \]
and
\[ \sum_{s=2}^{n-2} (s-1)C_{n-2}^s(p-1)^s = (n-2)(p-1)p^{n-3} - (p^{n-2} - 1). \]
Moreover, it follows from Lemma 4.5 that
\[ W_1 \cong \bigoplus_{i=1}^{p-1} \bigoplus_{s=2}^{n-1} V(i) \oplus (s-1)C_{n-2}^{s-1}(p-1)^{s-1} = \bigoplus_{i=1}^{p-1} V(i) \oplus (n-2)(p-1)p^{n-3}, \]
\[ W_2 \cong \bigoplus_{s=2}^{n-2} V(0) \oplus (s-1)C_{n-2}^{s-1}(p-1)^s = V(0) \oplus ((n-2)(p-1)p^{n-3} - p^{n-2} + 1), \]
\[ W_3 \cong \bigoplus_{i=0}^{p-2} V(i) \oplus (\bigoplus_{s=1}^{n-1} C_{n-2}^{s-1}(p-1)^s) = \bigoplus_{i=0}^{p-2} V(i) \oplus p^{n-2}, \]
\[ W_4 \cong \bigoplus_{s=1}^{n-2} V(p-1) \oplus C_{n-2}^{s-1}(p-1)^s = V(p-1) \oplus (p^{n-2} - 1). \]
Since \( V(i) = L(i) \) for \( 1 \leq i \leq p-1 \) (see [2.4]), it follows that
\[ V_{11} + V_2 \cong V(0) \oplus ((n-2)(p-1)p^{n-3} + 1) \oplus \left( \bigoplus_{i=1}^{p-2} L(i) \oplus ((n-2)(p-1)p^{n-3} + p^{n-2}) \right) \]
\[ \oplus V(p-1) \oplus ((n-2)(p-1)p^{n-3} + p^{n-2} - 1). \]
Now the desired assertion follows from (4.1) and (4.2). \( \square \)

As a consequence of Theorem 4.7 we further have

**Corollary 4.8.** As a module over the Witt algebra \( W(1) \),
\[ [S(n)] = (2(n-1)p^{n-2} - n + 1)[L(0)] + \sum_{i=1}^{p-2} (n-1)p^{n-2}[L(i)] + 2(n-1)p^{n-2}[L(p-1)]. \]

**Proof.** Since \( [V(0)] = [V(p-1)] = [L(0)] + [L(p-1)] \), the assertion follows directly from Theorem 4.7 \( \square \)

5. Structure of the Hamiltonian algebras as modules over the Witt algebra

Recall the Lie algebra embedding \( \Theta_H : W(1) \hookrightarrow H(2^r) \) given by \( \Theta_H(x_1^j \partial_1) = -D_H(x_1^j x_{r+1}) = x_1^j \partial_1 - ix_1^{j-1} x_{r+1} \partial_{r+1}, i \in I \). In this section, we study the structure of the Hamiltonian algebra \( H(2^r) \) as a module over the Witt algebra \( W(1) \).

We first investigate the case \( H(2) \). Set
\[ H_j = \begin{cases} \text{span}_F \{D_H(x_1^j x_2^i) \mid 1 \leq i \leq p-1 \}, & \text{if } j = 0, \\ \text{span}_F \{D_H(x_1^j x_2^i) \mid i \in I \}, & \text{if } 1 \leq j \leq p-2, \\ \text{span}_F \{D_H(x_1^j x_2^i) \mid 0 \leq i \leq p-2 \}, & \text{if } j = p-1. \end{cases} \]

Since
\[ [\Theta_H(x_1^j \partial_1), D_H(x_1^j x_2^i)] = (i - sj)D_H(x_1^{s+i-1} x_2^j), \forall i, j, s \in I, \]
(5.1)
each $H_j$ is a $W(1)$-module for any $j \in I$. Moreover, we have the following decomposition of $H(2)$ as a $W(1)$-module.

**Lemma 5.1.** As a $W(1)$-module, $H(2)$ is completely reducible and

$$H(2) \cong \left( \bigoplus_{i=1}^{p-2} L(i) \right) \oplus L(p-1)^{\oplus 2}.$$

In particular,

$$[H(2)] = \sum_{j=1}^{p-2} [L(j)] + 2[L(p-1)]. \quad (5.2)$$

**Proof.** For any $1 \leq j \leq p-2$, it follows from (5.1) that $H_j$ has a maximal vector $x_1^{p-1}x_j^j$ of weight $p-1-j$. Since $H_j$ is $p$-dimensional, $H_j \cong V(p-1-j) \cong L(p-1-j)$ as $W(1)$-modules. Furthermore, it again follows from (5.1) that $H_0$ has a maximal vector $x_1^{p-1}$ of weight $p-1$, and $H_{p-1}$ has a maximal vector $x_1^{p-2}x_2^{p-1}$ of weight $p-1$. And dim $H_0 = \dim H_{p-1} = p-1$, it follows that $H_0 \cong H_{p-1} \cong L(p-1)$ as $W(1)$-modules. Since $H(2) = \oplus_{j=0}^{p-1} H_j$, the desired assertion follows. 

**Remark 5.2.** (5.2) was obtained in [3, Lemma 2.6].

In general, for $r > 1$, $j \in I$, \( \underline{1} = (l_2, \cdots, l_r, l_{r+2}, \cdots, l_{2r}) \in I^{2r-2} \), let $x_{\underline{1}} := x_2^{l_2} \cdots x_r^{l_r} x_{r+2}^{l_{r+2}} \cdots x_{2r}^{l_{2r}}$,

$$H_{j, \underline{1}} := \text{span}_F \{ D_H(x_1^i, x_2^j, x_{\underline{1}}) \mid i \in I \text{ such that } (i, j, \underline{1}) \neq (0, \cdots, 0), (p-1, \cdots, p-1) \},$$

and

$$H_{\underline{1}} := \text{span}_F \{ D_H(x_1^i, x_2^j, x_{\underline{1}}) \mid i, k \in I \text{ such that } (i, k, \underline{1}) \neq (0, \cdots, 0), (p-1, \cdots, p-1) \}.$$

Since

$$[\Theta_H(x_1^i, \partial_1), D_H(x_1^i, x_2^j, x_{\underline{1}})] = (i-sj)D_H(x_1^{s+i-1}, x_2^j, x_{\underline{1}}), \quad \forall i, j, s \in I, \underline{1} \in I^{n-2}, \quad (5.3)$$

both $H_{t, \underline{1}}$ and $H_{\underline{1}}$ are $W(1)$-modules for any $t \in I, \underline{1} \in I^{2r-2}$. The following result describes the $W(1)$-module structure on $H_{\underline{1}}$.

**Lemma 5.3.** Keep notations as above, then the following decompositions hold as $W(1)$-modules.

1. If $\underline{1} = (0, \cdots, 0)$, then

$$H_{\underline{1}} \cong V(0) \oplus \left( \bigoplus_{i=1}^{p-1} L(i) \right).$$

2. If $\underline{1} = (p-1, \cdots, p-1)$, then

$$H_{\underline{1}} \cong \left( \bigoplus_{i=1}^{p-1} L(i) \right) \oplus V(p-1).$$

3. If $\underline{1} \neq (0, \cdots, 0)$ and $(p-1, \cdots, p-1)$, then

$$H_{\underline{1}} \cong V(0) \oplus \left( \bigoplus_{i=1}^{p-2} L(i) \right) \oplus V(p-1).$$

**Proof.** Note that $H_{\underline{1}} = \bigoplus_{j \in I} H_{j, \underline{1}}$. We need to determine the $W(1)$-module structure of $H_{j, \underline{1}}$ for any $t \in I$. We just show the assertion for the case $\underline{1} = (0, \cdots, 0)$. Similar arguments yield the assertion for the other cases.

In the following, we always assume that $\underline{1} = (0, \cdots, 0)$. For each $0 \leq j \leq p-1$, it follows from (5.3) that $H_{j, \underline{1}}$, as a $W(1)$-module, contains a maximal vector $D_H(x_1^{p-1} x_j^j, x_{\underline{1}})$ of weight $p-1-j$. However, in the following, we always assume that $\underline{1} = (0, \cdots, 0)$. For each $0 \leq j \leq p-1$, it follows from (5.3) that $H_{j, \underline{1}}$, as a $W(1)$-module, contains a maximal vector $D_H(x_1^{p-1} x_j^j, x_{\underline{1}})$ of weight $p-1-j$. 


Since \( \dim H_{j,l} = p \) for \( 1 \leq j \leq p - 1 \), \( H_{j,l} \cong V(p - 1 - j) \) as a \( W(1) \)-module. While \( \dim H_{0,l} = p - 1 \), we have \( H_{0,l} \cong L(p - 1) \). Hence,

\[
H_l = \bigoplus_{j \in I} H_{j,l} \cong L(p - 1) \oplus \left( \bigoplus_{j=1}^{p-1} V(p - 1 - j) \right) \cong V(0) \oplus \left( \bigoplus_{i=1}^{p-1} L(i) \right).
\]

We are now in the position to present the following main result on the structure of the Hamiltonian algebra \( H(2r) \) as a module over the Witt algebra \( W(1) \).

**Theorem 5.4.** As a module over the Witt algebra \( W(1) \), we have

\[
H(2r) \cong V(0)^{\oplus (p^{2r-2} - 1)} \oplus \left( \bigoplus_{i=1}^{p-2} L(i)^{\oplus p^{2r-2}} \right) \oplus V(p - 1)^{\oplus (p^{2r-2} - 1)} \oplus L(p - 1)^{\oplus 2}.
\]

In particular,

\[
[H(2r)] = (2p^{2r-2} - 2)[L(0)] + p^{2r-2} \sum_{j=1}^{p-2} [L(j)] + 2p^{2r-2}[L(p - 1)].
\]

**Proof.** For \( r = 1 \), the assertion follows from Lemma 5.1. While for \( r > 1 \), \( H(2r) = \bigoplus_{l \in I^{2r-2}} H_l \), then the desired assertion follows directly from Lemma 5.3.

6. **Structure of the contact algebras as modules over the Witt algebra**

Recall the Lie algebra embedding \( \Theta_K : W(1) \hookrightarrow K(2r+1) \) given by \( \Theta_K(x^r_1 \partial_i) = \frac{1}{2}D_K(x^r_{2r+1}) = \sum_{j=1}^{2r} x^{r-1}_{2r+1} x_j \partial_j + x^r_{2r+1} \partial_{2r+1} \), \( i \in I \). In this section, we study the structure of the contact algebra \( K(2r+1) \) as a module over the Witt algebra \( W(1) \).

For \( i \in I \), set

\[
\Gamma_i := \left\{ \ell = (l_1, \cdots, l_{2r}) \in I^{2r} \bigg| \frac{1}{2} \left( \sum_{j=1}^{2r} l_j - 4 \right) \equiv i \pmod{p} \right\}.
\]

For arbitrary \( i \in I \) and \( (l_1, \cdots, l_{2r-1}) \in I^{2r-1} \), there is a unique \( l_{2r} \in I \) such that

\[
l_{2r} \equiv 2i + 4 - \sum_{j=1}^{2r-1} l_j \pmod{p}.
\]

Hence, \( |\Gamma_i| = p^{2r-1} \) for any \( i \in I \).

For any \( \ell = (l_1, \cdots, l_{2r}) \in I^{2r} \), we denote \( x^\ell := x^{l_1}_{2r+1} \cdots x^{l_{2r}}_{2r+1} \) and

\[
K_\ell := \begin{cases} \text{span}_F \{ x^\ell x^t_{2r+1} \mid 0 \leq t \leq p - 2 \}, & \text{if } 2r + 4 \equiv 0 \pmod{p} \text{ and } \ell = \tau \equiv (p - 1) \epsilon_{2r+1}, \\ \text{span}_F \{ x^\ell \partial_{2r+1} \mid 0 \leq t \leq p - 1 \}, & \text{otherwise}. \end{cases}
\]

Since

\[
[\Theta_K(x^r_1 \partial_i), x^\ell x^t_{2r+1}] = \frac{1}{2} \left( \sum_{j=1}^{2r} l_j - 2 \right) + 2t x^\ell x^{t+i-1}_{2r+1}, \quad \forall i, t \in I, \ell \in I^{2r},
\]

\( K_\ell \) is a \( W(1) \)-module for any \( \ell \in I^{2r} \). More precisely, we have

**Lemma 6.1.** Keep notations as above, then for any \( \ell = (l_1, \cdots, l_{2r}) \in I^{2r} \), as a \( W(1) \)-module

\[
K_\ell \cong \begin{cases} L(p - 1), & \text{if } 2r + 4 \equiv 0 \pmod{p} \text{ and } \ell = \tau \equiv (p - 1) \epsilon_{2r+1}, \\ V\left( \frac{1}{2} \left( \sum_{j=1}^{2r} l_j \right) - 2 \right), & \text{otherwise}. \end{cases}
\]
Proof. If $2r + 4 \equiv 0 \pmod{p}$ and $\ell = \tau - (p - 1)e_{2r+1}$, it follows from (6.1) that $x^{\ell}e_{2r+1}^{p^2-2}$ is a maximal vector of weight $p - 1$ in $K_{\ell}$, so that $K_{\ell}$, as a $W(1)$-module, is a homomorphic image of $V(p - 1)$. Furthermore, since $\dim K_{\ell} = p - 1$, it follows that $K_{\ell} \cong L(p - 1)$.

If $2r + 4 \not\equiv 0 \pmod{p}$ or $\ell = \tau - (p - 1)e_{2r+1}$, it follows from (6.1) that $x^{\ell}e_{2r+1}^{p^2-1}$ is a maximal vector of weight $\frac{1}{2}((\sum_{j=1}^{2r} l_j) - 2)$ in $K_{\ell}$. This together with the dimension of $K_{\ell}$ yields that $K_{\ell} \cong V(\tau)$.

As a consequence of Lemma 6.1, we have the following main result on the decomposition of the contact algebra $K(2r + 1)$ as a module over the Witt algebra $W(1)$.

**Theorem 6.2.** As a $W(1)$-module, we have

$$K(2r+1) \cong \begin{cases} V(0)^{\oplus(p^{2r-1}-1)} \oplus V(p-1)^{\oplus p^{2r-1}} \oplus \left( \bigoplus_{i=1}^{p-2} L(i)^{\oplus p^{2r-1}} \right) \oplus L(p-1), & \text{if } 2r + 4 \equiv 0 \pmod{p}, \\ V(0)^{\oplus p^{2r-1}} \oplus V(p-1)^{\oplus p^{2r-1}} \oplus \left( \bigoplus_{i=1}^{p-2} L(i)^{\oplus p^{2r-1}} \right), & \text{if } 2r + 4 \not\equiv 0 \pmod{p}. \end{cases}$$

Consequently,

$$[K(2r+1)] = \begin{cases} (2p^{2r-1}-1)[L(0)] + \sum_{i=1}^{p-2} p^{2r-1}[L(i)] + 2p^{2r-1}[L(p-1)], & \text{if } 2r + 4 \equiv 0 \pmod{p}, \\ 2p^{2r-1}[L(0)] + \sum_{i=1}^{p-2} p^{2r-1}[L(i)] + 2p^{2r-1}[L(p-1)], & \text{if } 2r + 4 \not\equiv 0 \pmod{p}. \end{cases}$$

Proof. Note that $K(2r+1) = \bigoplus_{\ell \in I^2} K_{\ell}$. It follows from Lemma 6.1 that

$$K(2r+1) \cong \begin{cases} V(0)^{\oplus(|I_0|-1)} \oplus V(p-1)^{\oplus|I_{p-1}|} \oplus \left( \bigoplus_{i=1}^{p-2} L(i)^{\oplus|I_i|} \right) \oplus L(p-1), & \text{if } 2r + 4 \equiv 0 \pmod{p}, \\ V(0)^{\oplus|I_0|} \oplus V(p-1)^{\oplus|I_{p-1}|} \oplus \left( \bigoplus_{i=1}^{p-2} L(i)^{\oplus|I_i|} \right), & \text{if } 2r + 4 \not\equiv 0 \pmod{p}. \end{cases}$$

Since $|I_i| = p^{2r-1}$ for any $i \in I$, the first assertion holds. Moreover, since

$$[V(0)] = [V(p-1)] = [L(0)] + [L(p-1)],$$

the second assertion follows.

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