Abstract. The influence of oscillatory perturbations on autonomous strongly nonlinear systems in the plane is investigated. It is assumed that the intensity of perturbations decays with time, and their frequency increases according to a power law. The long-term behaviour of perturbed trajectories is discussed. It is shown that, depending on the structure and the parameters of perturbations, there are at least two different asymptotic regimes: a phase locking and a phase drifting. In the case of phase locking, resonant solutions with an unlimitedly growing energy occur. The stability and asymptotics at infinity of such solutions are investigated. The proposed analysis is based on a combination of the averaging technique and the method of Lyapunov functions.

Keywords: asymptotically autonomous system, chirped-frequency, damped perturbation, phase locking, stability, asymptotics

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1. Introduction

In this paper, the effect of oscillating perturbations on autonomous Hamiltonian systems is investigated. It is assumed that the intensity of perturbations decays with time and the limiting system describes strongly nonlinear oscillations. The conditions of existence and stability of resonant solutions with growing energy are discussed.

Qualitative properties of solutions for asymptotically autonomous systems have previously been studied in many papers. It is known, in particular, that if the limiting system is asymptotically stable, the trajectories of the perturbed system remain in some neighbourhood of a stable solution [1, 2]. See also [3], where the conditions were described under which decaying nonlinear disturbances do not violate the behaviour of solutions to linear autonomous oscillating systems. In the general case, the long-term behaviour of solutions to asymptotically autonomous systems can differ from the dynamics described by the corresponding autonomous systems [4]. It depends on the qualitative properties of solutions to the limiting system and the structure of decaying perturbations [5–9].

Asymptotically autonomous systems with oscillatory decreasing perturbations have been studied in several papers, where either linear equations were considered [10–14], or the behaviour of solutions in some vicinity of the equilibrium was discussed [15]. In the present paper, the effect of damped oscillatory disturbances on nonlinear systems far from equilibrium is investigated.

Note that the influence of small oscillatory perturbations on dynamical systems is well-studied problem [16–20]. In particular, chirped-frequency perturbations with a small parameter are effectively used to control the dynamics of nonlinear systems [21–25]. However, in this paper the presence of a small parameter is not assumed. We consider strongly nonlinear systems in the plane with chirped-frequency oscillatory perturbations vanishing at infinity in time. To the best of our knowledge, bifurcations in such systems have not been thoroughly investigated.

The paper is organized as follows. In section 2, the mathematical formulation of the problem is given and the class of decreasing perturbations is described. First we construct a change of variables
that simplifies the perturbed system in the leading asymptotic terms. The construction of this transformation is described in section 3. Depending on the structure of the simplified equations there are at least two asymptotic regimes for solutions of the perturbed system: a phase locking and a phase drifting. Such regimes are described in section 4. Nonlinear stability analysis of the phase locking is discussed in section 5. In section 6, the proposed theory is applied to the examples of non-autonomous systems with decaying oscillatory perturbations. The paper concludes with a brief discussion of the results obtained.

2. Problem statement

Consider the asymptotically autonomous system in the plane:

\[ \frac{dx}{dt} = \partial_y H(x,y) + t^{\frac{a}{q}} f(x,y,S(t),t), \quad \frac{dy}{dt} = -\partial_x H(x,y) + t^{\frac{b}{q}} g(x,y,S(t),t), \quad t > 0, \]

with \( S(t) = st^{1+b/q} \) and the parameters \( a, b, q \in \mathbb{Z}, s \in \mathbb{R} \) such that \( 1 \leq a, b \leq q, s > 0 \). It is assumed that the functions \( H(x,y), f(x,y,S,t) \) and \( g(x,y,S,t) \), defined for all \( (x,y,S) \) in \( \mathbb{R}^3, t > 0 \), are infinitely differentiable and \( 2\pi \)-periodic functions with respect to \( S \). The Hamiltonian \( H(x,y) \) of the corresponding limiting autonomous system

\[ \frac{dx}{dt} = \partial_y H(x,y), \quad \frac{dy}{dt} = -\partial_x H(x,y) \]

is assumed to have the following form

\[ H(x,y) = \frac{y^2}{2} + U(x), \quad U(x) = \frac{x^{2h}}{2h} + \sum_{i=1}^{2h-1} u_i x^{i} \]

with \( h \in \mathbb{Z}, h \geq 2 \) and \( u_i \) const. In this case, there exists \( E_0 > 0 \) such that for all \( E > E_0 \) the level lines \( \{ (x,y) \in \mathbb{R}^2 : H(x,y) = E \} \) are closed curves on the phase space \( (x,y) \) parameterized by the parameter \( E \) and do not contain any fixed points of system (2). Let \( x_-(E) < 0 < x_+(E) \) be the solutions of the equation \( U(x) = E \) with \( E > E_0 \). Then, to each closed curve there correspond a periodic solution \( x_0(t,E), y_0(t,E) \) of system (2) with a period

\[ T(E) \equiv \int_{x_-(E)}^{x_+(E)} \frac{\sqrt{2d\zeta}}{\sqrt{E - U(\zeta)}} = \kappa E^{1-2h} (1 + O(E^{-2h})), \quad E \to \infty, \quad \kappa = \sqrt{2(2h)} \int_{-1}^{1} \frac{d\zeta}{\sqrt{1 - \zeta^2}}. \]

The perturbations of the autonomous system (2) are described by the functions with power-law asymptotics:

\[ f(x,y,S,t) = \sum_{k=0}^{\infty} t^{-\frac{k}{2h}} f_k(x,y,S), \quad g(x,y,S,t) = \sum_{k=0}^{\infty} t^{-\frac{k}{2h}} g_k(x,y,S), \quad t \to \infty, \]

with

\[ f_k(x,y,S) = \sum_{i=0}^{l} \sum_{j=0}^{p} A_{k,i,j}(S)x^iy^j, \quad g_k(x,y,S) = \sum_{i=0}^{l} \sum_{j=0}^{p} B_{k,i,j}(S)x^iy^j, \]

where \( l, p \in \mathbb{Z}, 0 \leq l \leq p \leq 2h - 1 \) and the coefficients \( A_{k,i,j}(S), B_{k,i,j}(S) \) are \( 2\pi \)-periodic with respect to \( S \). It is assumed that \( A_{k,i,j}(S) \equiv B_{k,i,j}(S) \equiv 0 \) if \( i + j > p \) and \( A_{k,i,-1}(S) \equiv 0 \) for all \( k, i \geq 0 \). The parameter \( p \) is responsible for the maximum degree of the monomials \( x^iy^j \) in the perturbations with nonzero coefficients, while the parameter \( l \) corresponds to a maximum power of \( y \). It is also assumed that these parameters satisfy the following inequalities:

\[ -1 \leq \sigma < \frac{b}{q}, \quad \sigma := \frac{b}{q} \left( l - 1 + \frac{p - 1}{h - 1} \right) - \frac{a}{q}. \]

The role of this condition will be specified below.

Note that decreasing perturbations with power-law asymptotics appear, for example, in the study of Painlevé equations [26, 27], phase-locking phenomena [28–30], stochastic perturbations [31, 32], and in a wide range of other problems associated with nonlinear non-autonomous systems [33, 34].
In this paper, we investigate the existence and stability of resonant solutions with unboundedly growing energy for system (1) with decreasing oscillatory perturbations satisfying (3) and (4).

3. Change of variables

In this section, we construct the transformations of variables that simplify system (1).

3.1. Energy-angle variables. First, define auxiliary $2\pi$-periodic functions $X(\phi, E) = x_0(\phi/\omega(E), E)$ and $Y(\phi, E) = y_0(\phi/\omega(E), E)$ with $\omega(E) = 2\pi/T(E) > 0$ for all $E > E_0$. It follows easily that

$$\omega(E) \frac{\partial X}{\partial \phi} = \partial_Y H(X,Y), \quad \omega(E) \frac{\partial Y}{\partial \phi} = -\partial_X H(X,Y), \quad H(X(\phi, E), Y(\phi, E)) \equiv E.$$  

Application of the averaging method [16, 18] to system (6) yields the following asymptotic expansions as $E \to \infty$:

$$X(\phi, E) = E^{2h} \sum_{j=0}^{\infty} E^{-\frac{1}{2h}} X_j(\phi), \quad Y(\phi, E) = E^{2h} \sum_{j=0}^{\infty} E^{-\frac{1}{2h}} Y_j(\phi), \quad \omega(E) = E^{\frac{h-1}{2h}} \sum_{j=0}^{\infty} E^{-\frac{1}{2h}} \omega_j$$

with $2\pi$-periodic coefficients $X_j(\phi), Y_j(\phi)$ and $\omega_j = \text{const}$. Let $\xi(\alpha, J), \eta(\alpha, J)$ be a family of $2\pi$-periodic solutions of the system

$$\chi(J) \frac{\partial \xi}{\partial \alpha} = \eta, \quad \chi(J) \frac{\partial \eta}{\partial \alpha} = -\xi^{2h-1}, \quad \frac{\eta^2}{2} + \frac{\xi^{2h}}{2h} = J > 0, \quad \chi(J) = \frac{2\pi}{\kappa} J^{\frac{h-1}{2h}}.$$
Then, it is not hard to check that

\[\omega_0 = \chi(1), \quad \omega_1 = 0, \quad \omega_2 = u_{2h-2} \langle \partial_J (\xi^{2h-2})^{\chi} \rangle_{J=1} + \frac{u_{2h-1}^2}{2} \langle \xi^{4h-2} \partial_J^2 \chi \rangle_{J=1},\]

and

\[
\begin{align*}
\left(\begin{array}{c}
X_0 \\
Y_0
\end{array}\right) &= \left(\begin{array}{c}
\xi(\phi, 1) \\
\eta(\phi, 1)
\end{array}\right), \\
\left(\begin{array}{c}
X_1 \\
Y_1
\end{array}\right) &= \alpha_1(\phi) \left(\begin{array}{c}
\partial_\xi \xi(\phi, 1) \\
\partial_\eta \xi(\phi, 1)
\end{array}\right) + J_1(\phi) \left(\begin{array}{c}
\partial_\xi \xi(\phi, 1) \\
\partial_\eta \xi(\phi, 1)
\end{array}\right), \\
\left(\begin{array}{c}
X_2 \\
Y_2
\end{array}\right) &= \alpha_2(\phi) \left(\begin{array}{c}
\partial_\phi \xi(\phi, 1) \\
\partial_\phi \eta(\phi, 1)
\end{array}\right) + J_2(\phi) \left(\begin{array}{c}
\partial_\phi \xi(\phi, 1) \\
\partial_\phi \eta(\phi, 1)
\end{array}\right),
\end{align*}
\]

where

\[
\begin{align*}
\alpha_1(\phi) &= \alpha_1^0 + \omega_0 u_{2h-1} \int_0^\phi \left(\partial_J \langle \xi^{2h-1} \rangle \right)_{J=1} d\xi, \\
J_1(\phi) &= -u_{2h-1} \langle \xi^{2h-1} \rangle_{J=1}, \\
\alpha_2(\phi) &= \alpha_2^0 + \chi''(1) \frac{u_{2h-1}^2}{2} \int_0^\phi \left(\xi^{4h-2} - \langle \xi^{4h-2} \rangle \right)_{J=1} d\xi + \int_0^\phi \alpha'' \alpha_1 - \langle \alpha'' \alpha_1 \rangle d\xi \\
&\quad + \omega_0 u_{2h-2} \int_0^\phi \left(\partial_J \langle \xi^{2h-2} \rangle \right)_{J=1} d\xi, \\
J_2(\phi) &= - \left(u_{2h-2} \langle \xi^{2h-2} \rangle + u_{2h-1} J_1(\phi) \partial_J \langle \xi^{2h-1} \rangle \right)_{J=1}.
\end{align*}
\]

The parameters \(\alpha_1^0, \alpha_2^0\) are chosen such that

\[\langle \alpha_i(\phi) \rangle := \frac{1}{2\pi} \int_0^{2\pi} \alpha_i(\xi) d\xi = 0.\]

The functions \(X(\phi, E), Y(\phi, E)\) are used for rewriting system (1) in the energy-angle variables. In particular, the change of variables

\[x(t) = X(\phi(t), I(t)), \quad y(t) = Y(\phi(t), I(t))\]

transforms system (1) into the form:

\[
\begin{align*}
\frac{dI}{dt} &= t^{-\frac{\alpha}{2}} F(I, \phi, t), \\
\frac{d\phi}{dt} &= \omega(I) + t^{-\frac{\alpha}{2}} G(I, \phi, t),
\end{align*}
\]

where

\[
\begin{align*}
F(I, \phi, t) &\equiv f(X(\phi, I), Y(\phi, I), S(t), t) U'(X(\phi, I)) + g(X(\phi, I), Y(\phi, I), S(t), t) Y(\phi, I), \\
G(I, \phi, t) &\equiv \omega(I) \left(f(X(\phi, I), Y(\phi, I), S(t), t) \partial_E Y(\phi, I) - g(X(\phi, I), Y(\phi, I), S(t), t) \partial_E X(\phi, I)\right).
\end{align*}
\]

Moreover, it follows from (3) that

\[
F(I, \phi, t) = \sum_{k=0}^{\infty} \frac{t^{-\frac{\alpha}{2}}}{k!} F_k(I, \phi, S(t)), \quad G(I, \phi, t) = \sum_{k=0}^{\infty} \frac{t^{-\frac{\alpha}{2}}}{k!} G_k(I, \phi, S(t)), \quad t \to \infty,
\]

where the coefficients

\[
\begin{align*}
F_k(I, \phi, S) &\equiv f_k(X(\phi, I), Y(\phi, I), S) U'(X(\phi, I)) + g_k(X(\phi, I), Y(\phi, I), S) Y(\phi, I), \\
G_k(I, \phi, S) &\equiv \omega(I) \left(f_k(X(\phi, I), Y(\phi, I), S) \partial_E Y(\phi, I) - g_k(X(\phi, I), Y(\phi, I), S) \partial_E X(\phi, I)\right).
\end{align*}
\]
are 2π-periodic functions with respect to φ and S. Define
\[ F_k(I, \phi, S) \equiv I^{1-\frac{p-1+(i_1-i_2)(h-1)}{2n}} F_k(I, \phi, S), \quad F(I, \phi, t) \equiv I^{1-\frac{p-1+(i_1-i_2)(h-1)}{2n}} F(I, \phi, t), \]
\[ G_k(I, \phi, S) \equiv I^{1-\frac{p-1+(i_1-i_2)(h-1)}{2n}} G_k(I, \phi, S), \quad G(I, \phi, t) \equiv I^{1-\frac{p-1+(i_1-i_2)(h-1)}{2n}} G(I, \phi, t). \]

Then, taking into account (7), we have the following asymptotics:
\[ F_k(I, \phi, S) = \sum_{d=0}^{\infty} \hat{F}_{k,d}(\phi, S) I^{-\frac{d}{2h}}, \quad G_k(I, \phi, S) = \sum_{d=0}^{\infty} \hat{G}_{k,d}(\phi, S) I^{-\frac{d}{2h}}, \quad I \to \infty, \]
with 2π-periodic coefficients:
\[ \hat{F}_{k,d}(\phi, S) = \sum_{(i,j,i_1,i_2,j_2) \in \mathcal{X}_d} \delta_{j_2,0} (X_0(\phi))^{p-1-i} (Y_0(\phi))^{l-1-j} \times \left( (2h-i_2)u_{2h-i_2} A_{k,p-l+1,i-1,i_1} S X_{p-l-i+2h-i_2,i_1}(\phi) Y_{l-1-j,i_1} X_0(\phi) \right)^{2h-i_2} \]
\[ + \delta_{i_2,0} B_{k,p-l+i-1,i_1} S X_{p-l-i,i_1}(\phi) Y_{l-1-j,i_1} Y_{l-1-j,i_1} X_0(\phi) Y_{l-1-j,i_1} X_0(\phi) \]
\[ \equiv \sum_{(i,j,i_1,i_2,j_2) \in \mathcal{X}_d} \frac{\omega_{j_2}}{2h} (X_0(\phi))^{p-1-i} (Y_0(\phi))^{l-1-j} \times \left( (h-i_2) A_{k,p-l+i-1,i_1} S X_{p-l-i,i_1}(\phi) Y_{l-1-j,i_1} X_0(\phi) Y_{l-1-j,i_1} X_0(\phi) \right)^{2h-i_2} \]
\[ - (1-i_2) B_{k,p-l+i-1,i_1} S X_{p-l-i,i_1}(\phi) Y_{l-1-j,i_1} Y_{l-1-j,i_1} X_0(\phi) Y_{l-1-j,i_1} X_0(\phi) \],
where \( \delta_{i,0} \) is the Kronecker delta,
\[ \mathcal{X}_d = \{(i, j, i_1, i_2, j_2) \in \mathbb{Z}^6 : 0 \leq i+j \leq p, 0 \leq j \leq \ell, i_1, j_1, i_2, j_2 \geq 0, \]
\[ i + j + (h-1)j + i_1 + j_1 + i_2 + j_2 = d \}. \]

The functions \( \tilde{X}_{n,i}(\phi), \tilde{Y}_{n,i}(\phi) \) denote the coefficients of the asymptotic expansions:
\[ I^{-\frac{d}{2h}} \left( \frac{X(\varphi, I)}{X_0(\varphi)} \right)^n = \sum_{i=0}^{\infty} \tilde{X}_{n,i}(\phi) I^{-\frac{d}{2h}}, \quad I^{-\frac{d}{2h}} \left( \frac{Y(\varphi, I)}{Y_0(\varphi)} \right)^n = \sum_{i=0}^{\infty} \tilde{Y}_{n,i}(\phi) I^{-\frac{d}{2h}}, \quad I \to \infty. \]

For example, \( \tilde{X}_{0,0} = 1, \tilde{X}_{n,i} = 0, \tilde{X}_{n,0} = 1, \tilde{X}_{n,1} = n X_1 / X_0, \tilde{X}_{n,2} = n X_2 / X_0 + n(n-1)(X_1 / X_0)^2 / 2 \) for all \( n, i \neq 0 \). It is assumed that \( u_{2h} = 1/(2h), u_{2h-i_2} = 0 \) for all \( i_2 \geq 2h \) and \( A_{k,i_1-1} S \equiv 0 \) for all \( k, i_1 \geq 0 \).

From (6) it follows that
\[ \det \frac{\partial(X, Y)}{\partial(\varphi, E)} = \left| \begin{array}{cc} \partial_\phi X & \partial_\varphi X \\ \partial_\varphi Y & \partial_\varphi Y \end{array} \right| = \frac{1}{\omega(E)} > 0 \quad \forall E > E_0. \]

Hence, the transformation (10) is invertible for all \( I > E_0 \) and \( \phi \in \mathbb{R} \). Define \( D(E_0) = \{(x, y) \in \mathbb{R}^2 : H(x, y) > E_0\} \). Then, we have the following.

**Lemma 1.** For all \((x, y) \in D(E_0)\) and \( t > 0 \) system (1) can be transformed into (11) by the transformation (10).

### 3.2. Amplitude and phase difference.

It follows from (7) that \( \omega'(E) > 0 \) for all \( E \geq E_1 \) with some \( E_1 \geq E_0 \). Hence, there exists \( t_1 > 0 \) such that the equation
\[ \omega(I_s(t)) \equiv \varepsilon^{-1} S'(t) \]
has a smooth solution \( I_s(t) \geq E_1 \) defined for all \( t \geq t_1 \). Define
\[ z(t) \equiv c_\varepsilon e^{-\frac{\vartheta}{(h-1)q}} (I_s(t))^{\frac{1}{2h}}, \quad c_\varepsilon = \left( \frac{\omega(0)}{\vartheta} \right)^{\frac{1}{\varepsilon-1}}, \quad \vartheta = s \left( 1 + \frac{b}{q} \right) > 0. \]
Then, it can be easily seen that the function \( z(t) \) has power-law asymptotics:

\[
\begin{align*}
z(t) &= \sum_{k=0}^{\infty} z_k t^{-\frac{k\sigma}{M-N}}, \quad t \to \infty,
\end{align*}
\]

with constant coefficients \( z_k \). In particular, substituting (16) into (14), we obtain

\[
\begin{align*}
z_0 &= 1, \quad z_1 = 0, \quad z_2 = -\frac{\omega_2 c_\sigma^2}{(h-1)\omega_0}, \quad z_3 = -\frac{\omega_3 c_\sigma^3}{(h-1)\omega_0},
\end{align*}
\]

The solution \( I_\nu(t) \) of equation (14) is used in the following change of variables in system (11):

\[
I(t) = I_\nu(t)(1 + t^{-\mu}(\tau))^{\frac{2h}{\mu}}, \quad \phi(t) = \theta(\tau) + \sigma^{-1} S(t), \quad \tau = \frac{t^\nu}{\nu}
\]

with some constants \( \mu > 0, \nu > 0, \sigma \in \mathbb{Z}_+ \). Note that for all \( r_0 > 0 \) there exists \( t_2 = \max\{t_1, (2r_0)^{1/\mu}\} \) such that \( t^{-\mu}|r| < 1/2 \) for all \( |r| \leq r_0 \) and \( t \geq t_2 \). In this case, the mapping \((I, \phi, t) \mapsto (r, \theta, \tau)\) is invertible for all \( |r| \leq r_0, \theta \in \mathbb{R} \) and \( \tau \geq \tau_2 = t_2^\nu/\nu \).

We take

\[
\mu = \frac{1}{2} \left( \frac{\nu}{q} - \sigma \right), \quad \nu = 1 + 2\mu + \sigma, \quad M = 2\nu q(h-1), \quad N = 2\nu q(h-1).
\]

It follows from (4) that

\[
\mu > 0, \quad \nu - 2\mu \geq 0, \quad M, N \in \mathbb{Z}, \quad 1 \leq M \leq (h-1)(b+q), \quad 0 \leq N - 2M < 2(h-1)(b+q).
\]

Hence, substituting (17) into (11) yields the following asymptotically autonomous system:

\[
\frac{dr}{d\tau} = \tau^{-\frac{N-2M}{N}} A(r, \theta, \tau), \quad \frac{d\theta}{d\tau} = \tau^{-\frac{N-2M}{N}} B(r, \theta, \tau),
\]

with the right-hand sides

\[
A(r, \theta, \tau) \equiv F(r, \theta, \tau) + \tau^{-\frac{N-2M}{N}} P(r, \tau) + \tau^{-\frac{N-2M}{N}} M r, \quad B(r, \theta, \tau) \equiv Q(r, \tau) r + \tau^{-\frac{N-2M}{N}} G(r, \theta, \tau),
\]

where

\[
F(r, \theta, \tau) = \frac{c_{\nu} c^{\nu} \nu^{-\frac{\nu}{\mu}}}{2h z((\nu r)^{\frac{1}{\mu}})} c_{\nu} c^{\nu} \nu^{-\frac{\nu}{\mu}} (1 + (\nu r)^{-\frac{\nu}{\mu}} r)^{2h}, \theta + \sigma^{-1} \zeta(\tau), \zeta(\tau), (\nu r)^{\frac{1}{\mu}})
\]

\[
\times (c_{\nu} c^{\nu} \nu^{-\frac{\nu}{\mu}} (1 + (\nu r)^{-\frac{\nu}{\mu}} r)^{2h})^{p+(l-1)(h-1)},
\]

\[
P(r, \tau) = -\left(1 + (\nu r)^{-\frac{\nu}{\mu}} \right) r^{-\frac{\nu}{\mu}} \left( \frac{t I_\nu(t)}{2h I_\nu(t)} \right)_{l=(\nu r)^{1/\mu}},
\]

\[
Q(r, \tau) = \nu^{-1} \left[ \sigma (I_\nu((\nu r)^{\frac{1}{\mu}}) (1 + (\nu r)^{-\frac{\nu}{\mu}} r)^{2h}) - \omega (I_\nu((\nu r)^{\frac{1}{\mu}})) \right] r^{-\frac{\nu}{\mu}} \nu^{-1},
\]

\[
G(r, \theta, \tau) = \frac{c_{\nu} c^{\nu} \nu^{-\frac{\nu}{\mu}}}{2h z((\nu r)^{\frac{1}{\mu}})} c_{\nu} c^{\nu} \nu^{-\frac{\nu}{\mu}} (1 + (\nu r)^{-\frac{\nu}{\mu}} r)^{2h}, \theta + \sigma^{-1} \zeta(\tau), \zeta(\tau), (\nu r)^{\frac{1}{\mu}})
\]

\[
\times (c_{\nu} c^{\nu} \nu^{-\frac{\nu}{\mu}} (1 + (\nu r)^{-\frac{\nu}{\mu}} r)^{2h})^{p+(l-1)(h-1)},
\]

\[
\zeta(\tau) = S((\nu r)^{\frac{1}{\mu}}).
\]

Note that the condition (4) guarantees that the transformed system (19) is asymptotically autonomous and not trivial in the leading asymptotic terms.

Combining (12), (13) and (15), we obtain the asymptotic estimates as \( \tau \to \infty \):

\[
F = \sum_{K=0}^{\infty} F_K(r, \theta, \zeta) \tau^{-\frac{K}{N}}, \quad P = \sum_{K=0}^{\infty} P_K(r) \tau^{-\frac{K}{N}}, \quad Q = \sum_{K=0}^{\infty} Q_K(r) \tau^{-\frac{K}{N}}, \quad G = \sum_{K=0}^{\infty} G_K(r, \theta, \zeta) \tau^{-\frac{K}{N}}.
\]
for all \( |r| \leq r_0, \theta \in \mathbb{R} \), where

\[
F_K(r, \theta, \zeta) \equiv \nu \frac{K + M}{2h} \sum_{(k,d,i) \in \mathcal{Y}_K} \tilde{F}_{k,d,i}^\ell(\theta, \zeta)r^\ell, \quad \mathcal{P}_K(r) \equiv \nu \frac{K + M}{2h} \sum_{(k,d,i) \in \mathcal{Y}_K} \tilde{P}_{k,d,i}^\ell r^\ell, \\
Q_K(r) \equiv \nu \frac{K + M}{2h} \sum_{(k,d,i) \in \mathcal{Y}_K} \tilde{Q}_{k,d,i}^\ell r^\ell, \quad \mathcal{G}_K(r, \theta, \zeta) \equiv \nu \frac{K + M}{2h} \sum_{(k,d,i) \in \mathcal{Y}_K} \tilde{G}_{k,d,i}^\ell(\theta, \zeta)r^\ell,
\]

and

\[
\tilde{F}_{k,d,i}^\ell(\theta, \zeta) = \frac{1}{2h} \tilde{F}_{k,d}(\theta + x^{-1} \zeta)C_{p+(l-1)(h-1)-d,\ell}Z_{p+(l-1)(h-1)-d,i}c_{\alpha}^{-(p-1+(l-1)(h-1)-d)}, \\
\tilde{G}_{k,d,i}^\ell(\theta, \zeta) = \tilde{G}_{k,d}(\theta + x^{-1} \zeta)C_{p+(l-1)(h-1)-d,\ell}Z_{p+(l-1)(h-1)-d,i}c_{\alpha}^{-(p-1+(l-1)(h-1)-d)}, \\
\tilde{Q}_{k,d,i}^\ell = \omega_i \delta_{k,0} C_{h-1-i,\ell+1}Z_{h-1-i,\ell}c_{\alpha}^{-(h-1-i)}, \\
\tilde{P}_{k,d,i}^\ell = -\delta_{k,0} \frac{(2h-i)b}{2h(h-1)q} C_{1,\ell}Z_{2h,i}Z_{-2h,i}.
\]

Here \( \mathcal{Y}_K = \{(k,d,i) \in \mathbb{Z}^4 : k \geq 0, d \geq 0, i \geq 0, \ell \geq 0, 2(h-1)k + 2b(d+i) + M\ell = K \} \), and the parameters \( C_{n,i}, Z_{n,i} \) denote the coefficients of the asymptotic expansions

\[
(1 + t^{-\mu})^n = \sum_{i=0}^{\infty} C_{n,i} t^{-\mu i}, \quad (z(t))^n = \sum_{i=0}^{\infty} Z_{n,i} t^{-(n-1)q}, \quad t \to \infty.
\]

In particular, \( C_{0,0} = Z_{0,0} = 1, C_{0,i} = Z_{0,i} = 0, C_{n,0} = 1, C_{n,1} = n, C_{n,2} = n(n-1)/2, Z_{n,0} = 1, Z_{n,1} = 0, Z_{n,2} = n/2 \) for all \( n, i \neq 0 \).

Thus, the right-hand sides of system (19) have the following asymptotics:

\[
A(r, \theta, \tau) = \sum_{K=0}^{\infty} r^{-\frac{\mu}{2h}} A_K(r, \theta, \zeta(\tau)), \quad B(r, \theta, \tau) = \sum_{K=0}^{\infty} r^{-\frac{\mu}{2h}} B_K(r, \theta, \zeta(\tau))
\]
as \( \tau \to \infty \) uniformly for all \( |r| \leq r_0, \theta \in \mathbb{R} \), with

\[
A_K(r, \theta, \zeta) \equiv F_K(r, \theta, \zeta) + P_{K+2M-N}(r) + \delta_{K,N-M} \frac{M}{N} r, \\
B_K(r, \theta, \zeta) \equiv Q_K(r) + \mathcal{G}_{K-N}(r, \theta, \zeta).
\]

It is assumed that \( P_i \equiv G_i \equiv 0 \) if \( i < 0 \). It follows easily that \( A_K(r, \theta, \zeta) \) and \( B_K(r, \theta, \zeta) \) are \( 2\pi \)-periodic with respect to \( \theta \) and \( 2\pi \alpha \)-periodic with respect to \( \zeta \). In particular,

\[
\begin{pmatrix} A_K \\ B_K \end{pmatrix} \equiv \begin{pmatrix} A_0^0(\theta, \zeta) \\ Q_0^0(\theta, \zeta) \end{pmatrix}, \quad K \in [0, M), \\
\begin{pmatrix} A_K \\ B_K \end{pmatrix} \equiv \begin{pmatrix} A_1^1(\theta, \zeta)r + A_0^0(\theta, \zeta) \\ Q_1^1r + Q_0^0(\theta, \zeta) \end{pmatrix}, \quad K \in [M, 2M), \\
\begin{pmatrix} A_K \\ B_K \end{pmatrix} \equiv \begin{pmatrix} A_2^2(\theta, \zeta)r^2 + A_1^1(\theta, \zeta)r + A_0^0(\theta, \zeta) \\ Q_2^2r^3 + Q_1^1r^2 + Q_0^0(\theta, \zeta)r + G_0^0(\theta, \zeta) \end{pmatrix}, \quad K \in [2M, 3M),
\]

where

\[
A_1^1(\theta, \zeta) \equiv \nu \frac{K + M}{2h} \sum_{(k,d,i) \in \mathcal{Y}_K} \tilde{F}_{k,d,i}^\ell(\theta, \zeta) + \nu \frac{K + 2M - N}{2h} \sum_{(k,d,i) \in \mathcal{Y}_K} \tilde{F}_{k,d,i}^\ell + \delta_{k,1} \delta_{K,N-M} \frac{M}{N}, \\
G_1^1(\theta, \zeta) \equiv \nu \frac{K + M}{2h} \sum_{(k,d,i) \in \mathcal{Y}_K} \tilde{G}_{k,d,i}^\ell(\theta, \zeta), \quad Q_0^0 = \nu \frac{K + M}{2h} \sum_{(k,d,i) \in \mathcal{Y}_K} \tilde{G}_{k,d,i}^\ell.
\]

**Lemma 2.** Let assumption (4) hold. Then for all \( E \geq E_1, \phi \in \mathbb{R} \) and \( t \geq t_1 \) system (11) can be transformed into (19) by the transformation (17).
3.3. Averaging. Note that \( d\zeta/d\tau = \vartheta \). Hence, \( \zeta(\tau) \) changes rapidly in comparison to potential variations of \( r(\tau) \) and \( \Theta(\tau) \) for large values of \( \tau \). Further simplification of the system is associated with the averaging of the equations over \( \zeta \). This technique is usually used in perturbation theory (see, for example, [18, 35–37]).

Consider the following near-identity transformation:

\[
R_n(r, \theta, \tau) = r + \sum_{K=0}^{n} \tau^{-\frac{M+K}{N}} \rho_K(r, \theta, \zeta(\tau)), \quad \Psi_n(r, \theta, \tau) = \theta + \sum_{K=0}^{n} \tau^{-\frac{M+K}{N}} \psi_k(r, \theta, \zeta(\tau))
\]

with some integer \( n \geq 0 \). The coefficients \( \rho_K(r, \theta, \zeta) \), \( \psi_K(r, \theta, \zeta) \) are sought in such a way that the right-hand sides of the transformed equations in the variables \( R(\tau) = R_n(r(\tau), \theta(\tau), \tau) \) and \( \Psi(\tau) = \Psi_n(r(\tau), \theta(\tau), \tau) \) do not depend explicitly on \( \zeta \), at least in the first terms of the asymptotics:

\[
\frac{dR}{d\tau} = \sum_{K=0}^{n} \tau^{-\frac{M+K}{N}} \Lambda_K(R, \Psi) + \tilde{\Lambda}_n(R, \Psi, \tau), \quad \frac{d\Psi}{d\tau} = \sum_{K=0}^{n} \tau^{-\frac{M+K}{N}} \Omega_K(R, \Psi) + \tilde{\Omega}_n(R, \Psi, \tau),
\]

and the remainders \( \tilde{\Lambda}_n(R, \Psi, \tau), \tilde{\Omega}_n(R, \Psi, \tau) \) satisfy the estimates \( \tilde{\Lambda}_n(R, \Psi, \tau) = O(\tau^{-(M+n+1)/N}) \), \( \tilde{\Omega}_n(R, \Psi, \tau) = O(\tau^{-(M+n+1)/N}) \) as \( \tau \to \infty \). Calculating the total derivative of \( R_n(r, \theta, \tau) \) and \( \Psi_n(r, \theta, \tau) \) with respect to \( \tau \) along the trajectories of system (19) yields

\[
\frac{dR_n}{d\tau} \bigg|_{(19)} = \left( \tau^{-\frac{M}{N}} \left( A \partial_r + B \partial_\theta \right) + \partial_\tau \right) \left( R_n \right),
\]

\[
= \sum_{K=0}^{\infty} \tau^{-\frac{M+K}{N}} \left[ \partial_\tau \left( \rho_K \right) \left( \frac{\rho_j}{\psi_j} \right) + \left( \frac{A_K}{B_K} \right) + \frac{N-K}{N} \left( \frac{\rho_{K-M-N}}{\psi_{K-M-N}} \right) \right]
\]

\[
+ \sum_{K=0}^{\infty} \tau^{-\frac{2M+K}{N}} \sum_{i+j=K} \left( A_i \partial_r + B_i \partial_\theta \right) \left( \frac{\rho_j}{\psi_j} \right),
\]

where it is assumed that \( \rho_i \equiv \psi_i \equiv A_j \equiv B_j \equiv 0 \) if \( i, j < 0 \) or \( i > n \). Matching (23) with (22) gives the following chain of differential equations for determining \( \rho_K \) and \( \psi_K \):

\[
\partial_\tau \left( \frac{\rho_K}{\psi_K} \right) = \left( \frac{\Lambda_K(r, \theta) - A_K(r, \theta, \zeta) - \frac{A_K(r, \theta, \zeta)}{\Omega_K(r, \theta) - B_K(r, \theta, \zeta) - \frac{B_K(r, \theta, \zeta)}{\Omega_K(r, \theta)}} \right), \quad K \geq 0,
\]

where the functions \( A_K, B_K \) are expressed through \( \{\rho_i, \psi_i, A_i, \Omega_i\}_{j=0}^{K-M} \). In particular,

\[
\begin{align*}
\left( \frac{A_K}{B_K} \right) &\equiv \left( \begin{array}{c} 0 \\ 0 \end{array} \right), & K &\in [0, M), \\
\left( \frac{A_K}{B_K} \right) &\equiv \sum_{i+j=K-M} \left( A_i \partial_r + B_i \partial_\theta \right) \left( \frac{\rho_j}{\psi_j} \right) - \left( \rho_i \partial_r + \psi_i \partial_\theta \right) \left( \frac{A_j}{\Omega_j} \right), & K &\in [M, 2M), \\
\left( \frac{A_K}{B_K} \right) &\equiv \sum_{i+j=K-M} \left( A_i \partial_r + B_i \partial_\theta \right) \left( \frac{\rho_j}{\psi_j} \right) - \left( \rho_i \partial_r + \psi_i \partial_\theta \right) \left( \frac{A_j}{\Omega_j} \right) - \frac{1}{2} \sum_{i+j+k=K-2M} \left( \rho_i \rho_j \partial_r^2 + (\rho_i \psi_j + \rho_j \psi_i) \partial_r \partial_\theta + \psi_i \psi_j \partial_\theta^2 \right) \left( \frac{A_k}{\Omega_k} \right), & K &\in [2M, 3M),
\end{align*}
\]

etc. Define

\[
\Lambda_K(r, \theta, \zeta) \equiv \left\langle A_K(r, \theta, \zeta) + \frac{A_K(r, \theta, \zeta)}{\Omega_K(r, \theta)} \right\rangle_{\zeta}, \quad \Omega_K(r, \theta, \zeta) \equiv \left\langle B_K(r, \theta, \zeta) + \frac{B_K(r, \theta, \zeta)}{\Omega_K(r, \theta)} \right\rangle_{\zeta},
\]

where

\[
\left\langle \mathcal{C}(r, \theta, \zeta) \right\rangle_{\zeta} \equiv \frac{1}{2\pi} \int_{0}^{2\pi} \mathcal{C}(r, \theta, \zeta) d\zeta = \frac{1}{2\pi r} \int_{0}^{2\pi} \mathcal{C}(r, \theta, \zeta) d\zeta.
\]
Then, for all \( K \geq 0 \) the right-hand sides of system (24) are \( 2\pi \)-periodic with respect to \( \zeta \) with zero average. Integrating (24) yields

\[
(26) \quad \begin{pmatrix} \rho_K(r, \theta, \zeta) \\ \psi_K(r, \theta, \zeta) \end{pmatrix} = -\frac{1}{\nu} \int_0^\zeta \left( \left\{ A_K(r, \theta, \zeta) + \mathfrak{A}_K(r, \theta, \zeta) \right\}_\zeta + \left\{ \hat{\rho}_K(r, \theta) \right\}_\zeta \right) d\zeta,
\]

where \( \{ \mathcal{C} \}_\zeta := \mathcal{C} - \langle \mathcal{C} \rangle_\zeta \), and the functions \( \hat{\rho}_K(r, \theta) \), \( \hat{\psi}_K(r, \theta) \) are chosen such that \( \langle \rho_K \rangle_\zeta \equiv \langle \psi_K \rangle_\zeta \equiv 0 \). Thus, the functions \( \rho_K(r, \theta, \zeta) \), \( \psi_K(r, \theta, \zeta) \) are smooth and periodic with respect to \( \theta \) and \( \zeta \).

From (20), (24) and (26) it follows that

\[
\begin{aligned}
\left( \begin{array}{c}
\Lambda_K \\
\Omega_K
\end{array} \right) &\equiv \left( \begin{array}{c}
\Lambda_K^0(\theta) \\
\Omega_K^0(\theta)
\end{array} \right) \\
\rho_K &\equiv \left( \begin{array}{c}
\rho_K^0(\theta, \zeta) \\
\psi_K^0(\theta, \zeta)
\end{array} \right)
\end{aligned},
\]

where \( \Lambda_K^0(\theta) \equiv \langle A_K^0(\theta, \zeta) \rangle_\zeta, \Omega_K^0(\theta) \equiv \langle G_K^0(\theta, \zeta) \rangle_\zeta, \rho_K^0(\theta, \zeta) \equiv \langle \rho_K^0(\theta, \zeta) \rangle_\zeta, \psi_K^0(\theta, \zeta) \equiv \langle \psi_K^0(\theta, \zeta) \rangle_\zeta \).

In this case, it can easily be checked that

\[
\begin{aligned}
\tilde{\lambda}_n(R, \Psi, \tau) &\equiv \frac{d}{d\tau} R_n(r, \theta, \tau) \bigg|_{(24)} - \sum_{K=0}^\infty \tau^{-\frac{M+K}{M}} \Lambda_K(R, \Psi) = O\left( \tau^{-\frac{M+K+1}{M}} \right), \\
\tilde{\gamma}_n(R, \Psi, \tau) &\equiv \frac{d}{d\tau} \Psi_n(r, \theta, \tau) \bigg|_{(24)} - \sum_{K=0}^\infty \tau^{-\frac{M+K}{M}} \Omega_K(R, \Psi) = O\left( \tau^{-\frac{M+K+1}{M}} \right)
\end{aligned}
\]

as \( \tau \to \infty \) uniformly for all \( |R| \leq d_0 \) and \( \Psi \in \mathbb{R} \) with some \( d_0 = \text{const} > 0 \).

It follows from (21) that for all \( r_0 > 0 \) and \( \varepsilon \in (0, r_0) \) there exists \( \tau_0 \geq \tau_2 \) such that

\[
\begin{aligned}
|R_n(r, \theta, \tau) - r| &\leq \varepsilon, \\
|\partial_r R_n(r, \theta, \tau) - 1| &\leq \varepsilon, \\
|\partial_{\theta} R_n(r, \theta, \tau)| &\leq \varepsilon,
\end{aligned}
\]

\[
\begin{aligned}
|\Psi_n(r, \theta, \tau) - \theta| &\leq \varepsilon, \\
|\partial_r \Psi_n(r, \theta, \tau)| &\leq \varepsilon, \\
|\partial_{\theta} \Psi_n(r, \theta, \tau) - 1| &\leq \varepsilon
\end{aligned}
\]

for all \( |r| \leq r_0, \theta \in \mathbb{R} \) and \( \tau \geq \tau_0 \). Hence, the mapping \((r, \theta) \mapsto (R, \Psi)\) is invertible for all \( |R| \leq d_0, \Psi \in \mathbb{R} \) and \( \tau \geq \tau_0 \) with \( d_0 = \tau_0 - \varepsilon > 0 \).

**Lemma 3.** For all \( r_0 > 0 \) there exists \( \tau_0 > 0 \) such that for all \( |r| \leq r_0, \theta \in \mathbb{R} \) and \( \tau \geq \tau_0 \) system (19) can be transformed into (22) by the transformations (10), (17) and (21).

Thus, combining Lemmas 1, 2 and 3, we obtain the following.

**Theorem 1.** Let assumptions (3), (4) hold. Then there exists \( t_0 > 0 \) such that for all \( (x, y) \in D(E_1) \) and \( t > t_0 \) system (1) can be transformed into (22) by the transformations (10), (17) and (21).
4. Asymptotic regimes

Let $0 \leq L \leq \min \{N - M, 2M - 1\}$ be a whole number such that

\begin{equation}
\Lambda_K(p, \varphi) \equiv 0 \quad \forall K < L, \quad \Lambda_L(p, \varphi) \neq 0.
\end{equation}

Then, we take sufficiently large $n \geq L$ and consider a system obtained from (22) by the truncation of the remainders $\tilde{\Lambda}_n$ and $\tilde{\Omega}_n$:

\begin{equation}
\frac{dp}{d\tau} = \sum_{K=L}^{n} \tau^{-\frac{M+K}{N}} \Lambda_K(p, \varphi), \quad \frac{d\varphi}{d\tau} = \sum_{K=0}^{n} \tau^{-\frac{M+K}{N}} \Omega_K(p, \varphi), \quad \tau > \tau_0.
\end{equation}

The functions $\Lambda_K(p, \varphi), \Omega_K(p, \varphi)$ are defined by (25). In particular, for all $K \in [0, M]$ we have $\Omega_K(p, \varphi) \equiv Q^0_K p$, $\Omega_{M+K}(p, \varphi) = \Omega^0_{M+K}(\varphi) + O(p)$, $\Lambda_L(p, \varphi) = \Lambda_L^0(\varphi) + O(p)$ as $p \to 0$ uniformly for all $\varphi \in \mathbb{R}$.

Let us show that system (28) admits at least two different asymptotic regimes depending on the properties of the function $\Lambda_L(p, \varphi)$. The first regime is associated with solutions such that $\varphi(\tau) \to \text{const}$ as $\tau \to \infty$. Another mode is characterized by an unlimitedly growing phase difference.

Consider the following cases:

\begin{equation}
\exists \varphi_0 \in \mathbb{R} : \quad \Lambda_L^0(\varphi_0) = 0, \quad \lambda_L := \partial \Lambda_L^0(\varphi_0) \neq 0;
\end{equation}

\begin{equation}
\Lambda_L(p, \varphi) \neq 0 \quad \forall (p, \varphi) \in \mathbb{R}^2.
\end{equation}

The degenerate case with $\Lambda_L^0(\varphi) \equiv 0$ is not discussed in this paper.

We have the following.

**Lemma 4.** Let assumptions (4), (27) and (29) hold. Then system (28) has a particular solution $p_*(\tau)$, $\varphi_*(\tau)$ with asymptotic expansion in the form

\begin{equation}
p(\tau) = \sum_{K=0}^{\infty} \varphi_K \tau^{-\frac{M+K}{N}}, \quad \varphi(\tau) = \varphi_0 + \sum_{K=1}^{\infty} \varphi_k \tau^{-\frac{K}{N}}, \quad \tau \to \infty,
\end{equation}

where $\varphi_K, \varphi_K \equiv \text{const}.$

**Proof.** Substituting these series in system (28) and grouping the terms of the same power of $\tau$ yields $\varphi_0 = -Q^0_M(\varphi_0)/Q^0_M$ with $Q^0_M = \omega_0(h - 1) > 0$, and the chain of linear equations for the coefficients $\varphi_K, \varphi_K, K \geq 1$:

\begin{equation}
Q^0_M \varphi_0 + \left( \partial \varphi \Omega^0_M(\varphi_0) + \delta_{N, 2M} \frac{K}{N} \right) \varphi_K = \mathcal{R}_K, \quad \lambda_L \varphi_K = \mathcal{G}_K,
\end{equation}

where the functions $\mathcal{R}_K$ and $\mathcal{G}_K$ are expressed through $\varphi_0, \varphi_0, \ldots, \varphi_{K-1}, \varphi_{K-1}$. For instance, if $M > 3$, we have

$\mathcal{R}_1 = -Q^0_M \varphi_0 - \Omega^0_{M+1}(\varphi_0),$

$\mathcal{G}_1 = -\Lambda^0_{L+1}(0, \varphi_0) - \delta_{N, L+1} \frac{M}{N} \varphi_0,$

$\mathcal{R}_2 = -Q^0_M \varphi_0 - \Omega^0_{M+2}(\varphi_0) - \left( \partial \varphi \Omega^0_M(\varphi_0) + \delta_{N, 2M+1} \frac{1}{N} \right) \varphi_1 - \partial \varphi \Omega^0_M(\varphi_0) \frac{\varphi_1^2}{2},$

$\mathcal{G}_2 = -\Lambda^0_{L+2}(0, \varphi_0) - \partial \varphi \Lambda_{L+1}(0, \varphi_0) \varphi_1 - \partial \varphi \Lambda_L(0, \varphi_0) \frac{\varphi_1^2}{2} - \delta_{N, L+2} \frac{M+1}{N} \varphi_1 - \delta_{N, L+2} \frac{M}{N} \varphi_0,$

$\mathcal{R}_3 = -Q^0_M \varphi_0 - Q^0_M \varphi_2 - \Omega^0_{M+3}(\varphi_0) - \partial \varphi \Omega^0_M(\varphi_0) \varphi_2 - \partial \varphi \Omega^0_M(\varphi_0) \frac{\varphi_2^2}{2} - \partial \varphi \Omega^0_M(\varphi_0) \varphi_1 \varphi_2 - \partial \varphi \Omega^0_M(\varphi_0) \frac{\varphi_1^2}{6} - \left( \partial \varphi \Omega^0_{M+1}(\varphi_0) + \delta_{N, 2M+2} \frac{1}{N} \right) \varphi_1 - \partial \varphi \Omega^0_{M+1}(\varphi_0) \varphi_1 \varphi_2 - \partial \varphi \Lambda_L(0, \varphi_0) \varphi_1 \varphi_2 - \partial \varphi \Lambda_L(0, \varphi_0) \frac{\varphi_1^2}{6} - \delta_{N, L+1} \frac{M+2}{N} \varphi_2 - \delta_{N, L+2} \frac{M+1}{N} \varphi_1 - \delta_{N, L+2} \frac{M}{N} \varphi_0.$
Note that system (32) is solvable whenever $\lambda_L \neq 0$. The existence of a particular solution of system (28) with power-law asymptotics at infinity follows from [34, 38, 39].

**Lemma 5.** Let assumptions (4), (27) and (30) hold. Then the solutions of system (28) exit from any bounded domain in a finite time.

**Proof.** Since $\Lambda_L(\varrho, \varphi) \neq 0$ for all $(\varrho, \varphi) \in \mathbb{R}^2$, it follows that for all $\delta > 0$ there exist $C_1, C_2, C_3 > 0$ and $\tau_1 \geq \tau_0$ such that

\[
\left| \frac{d\varrho}{dt} \right| \geq \tau^{-\frac{M+L}{N}} C_1, \quad \left| \frac{d\varphi}{dt} \right| \geq \tau^{-\frac{M}{N}} \left( C_2|\varrho| - C_3 \right)
\]

for all $|\varrho| \leq \delta$, $\varphi \in \mathbb{R}$ and $\tau \geq \tau_1$. Integrating the last inequalities with respect to $\tau$ in the case $L < N - M$ yields

\[
|\varrho(\tau) - \varrho(\tau_1)| \geq \tilde{C}_1 \left( \frac{N-M-L}{N} - \frac{\tau_1 N-M}{N} \right), \quad |\varphi(\tau) - \varphi(\tau_1)| \geq \tilde{C}_2 \left( \frac{2N-2M-L}{N} - \frac{\tau_1 2N-2M-L}{N} \right) - \tilde{C}_3 \left( \frac{N-M}{N} - \frac{\tau_1 N}{N} \right)
\]

as $\tau \geq \tau_1$ with positive constants $\tilde{C}_1 = N C_1/(N-M-L)$, $\tilde{C}_2 = N \tilde{C}_1 C_2/(2N-2M-L)$, $\tilde{C}_3 = N(C_3 + C_2|\varrho(\tau_1)|) + \tilde{C}_1 \tilde{C}_2 \tau_1^{(N-M-L)/N}/(N-M)$. Similar estimates hold in the case $L = N - M$. Hence, there exists $\tau_2 \geq \tau_1$ such that $|\varrho(\tau_2)| + |\varphi(\tau_2)| \geq \delta$. \hfill $\Box$

Let us remark that the case of (29) corresponds to a phase locking regime, when the phase $\phi(t)$ for solutions of the perturbed system (1) is synchronized with the perturbations, $\phi(t) - \varphi^{-1} S(t) = O(1)$ as $t \to \infty$, and the energy $E(t)$ increases significantly. The stability of such solutions is discussed in the next section.

In the case of (30), the phase $\phi(t)$ of solutions can significantly differ from that of the perturbations and the power-mode growth of the energy $E(t)$ does not occur. Such solutions correspond to a phase drifting. The qualitative analysis of this case requires special attention and is not discussed in this paper.

5. **Stability of phase locking**

Let $\varrho_*(\tau)$, $\varphi_*(\tau)$ be a solution of system (28) with asymptotics (31). First, the stability of this solution in the truncated system is investigated. Then, we discuss its persistence in the full system (22).

Substitution $\varrho(t) = \varrho_*(\tau) + \hat{\varrho}(\tau)$, $\varphi(t) = \varphi_*(\tau) + \hat{\varphi}(\tau)$ into (28) gives the following system with a fixed point at $(0,0)$:

\[
\tau \frac{d\hat{\varrho}}{d\tau} = \Lambda(\hat{\varrho}, \hat{\varphi}, \tau), \quad \tau \frac{d\hat{\varphi}}{d\tau} = \Omega(\hat{\varrho}, \hat{\varphi}, \tau),
\]

where

\[
\Lambda(\hat{\varrho}, \hat{\varphi}, \tau) = \sum_{K=L}^{n} \tau^{-\frac{K}{N}} \left( \Lambda_K(\varrho_* + \hat{\varrho}, \varphi_* + \hat{\varphi}) - \Lambda_K(\varrho_*, \varphi_*) \right)
\]

\[
= \tau^{-\frac{L}{N}} \left( \lambda_{L} \hat{\varphi} + \partial_\varphi \Lambda_L(0, \varphi_0) \hat{\varrho} + O(\Delta^2) \right) \left( 1 + O(\tau^{-\frac{1}{N}}) \right),
\]

\[
\Omega(\hat{\varrho}, \hat{\varphi}, \tau) = \sum_{K=0}^{n} \tau^{-\frac{K}{N}} \left( \Omega_K(\varrho_* + \hat{\varrho}, \varphi_* + \hat{\varphi}) - \Omega_K(\varrho_*, \varphi_*) \right)
\]

\[
= \left( \partial_\varrho \Omega_0(\varrho_0, \varphi_0) \hat{\varphi} + O(\Delta^2) \right) \left( 1 + O(\tau^{-\frac{1}{N}}) \right)
\]

as $\tau \to \infty$ and $\Delta = \sqrt{\hat{\varrho}^2 + \hat{\varphi}^2} \to 0$. Note that if $L < M$, then $\partial_\varrho \Lambda_L(\hat{\varrho}, \hat{\varphi}) \equiv 0$. 


5.1. Linear analysis. Consider the linearized system:

$$\tau \frac{d}{d\tau} \left( \hat{\rho} \hat{\varphi} \right) = a(\tau) \left( \hat{\rho} \hat{\varphi} \right), \quad a(\tau) = \left( \begin{array}{cc} \partial_\psi \Lambda(0, 0, \tau) & \partial_\theta \Lambda(0, 0, \tau) \\ \partial_\psi \Omega(0, 0, \tau) & \partial_\theta \Omega(0, 0, \tau) \end{array} \right).$$

The roots of the characteristic equation \(|a(t) - \epsilon I| = 0\) have the form:

$$e_{\pm}(\tau) = \frac{1}{2} \text{tr} a(\tau) \pm \frac{1}{2} \sqrt{(\text{tr} a(\tau))^2 - 4 \det a(\tau)}.$$

Taking into account (31) and the structure of the functions \(\Lambda_K, \Omega_K\), we obtain

$$\text{tr} a(\tau) = O(\tau^{-\frac{M}{2}}), \quad \det a(\tau) = -\lambda_0 L_0 \tau^2 (1 + O(\tau^{-\frac{N}{2} + 1})), \quad \tau \to \infty.$$ 

Since \(0 \leq L \leq \min\{N - M, 2M - 1\}\), we see that if \(\lambda_L > 0\), both eigenvalues \(e_+(\tau), e_-(\tau)\) are real and have different signs:

$$e_{\pm}(\tau) = 2\tau^{-\frac{M}{2}} \sqrt{\lambda_0 \lambda_0 \tau^2 (1 + O(\tau^{-\frac{N}{2}}))}, \quad \tau \to \infty.$$ 

In this case, the fixed point \((0, 0)\) of the linearized system is a saddle in the asymptotic limit, and the particular solution \(\varphi_*(\tau), \varphi_*(\tau)\) of system (28) is unstable.

Lemma 6. Let assumptions (27) and (29) hold with \(\lambda_L > 0\). Then the solution \(\varphi_*(\tau), \varphi_*(\tau)\) of system (28) with asymptotics (31) is unstable.

In the opposite case, when \(\lambda_L < 0\), the eigenvalues are complex:

$$e_{\pm}(\tau) = 2i\tau^{-\frac{M}{2}} \sqrt{|\lambda_0| \tau^2 (1 + O(\tau^{-\frac{N}{2}}))}, \quad \Re e_{\pm}(\tau) = O(\tau^{-\frac{N}{2}}), \quad \tau \to \infty.$$ 

Hence, the fixed point \((0, 0)\) is a centre in the asymptotic limit, and the linear analysis fails to determine the stability of the solution \(\varphi_*(\tau), \varphi_*(\tau)\) in the full nonlinear system (see, for example, [40]).

5.2. Nonlinear analysis. Assume that there exists \(D \geq M\) such that

$$\partial_\psi \Lambda_K(\theta, \varphi) + \partial_\theta \Omega_K(\theta, \varphi) \equiv 0 \quad \forall K < D,$$

$$\gamma_D := \partial_\psi \Lambda_D(0, \varphi_0) + \partial_\theta \Omega_D(0, \varphi_0) \neq 0.$$ 

We choose \(N \geq L + D\) in (28), then we have the following.

Lemma 7. Let assumptions (4), (27), (29), (34) hold with \(\lambda_L < 0\). Then then the solution \(\varphi_*(\tau), \varphi_*(\tau)\) of system (28) with asymptotics (31) is

- exponentially stable if \(\gamma_D < 0\) and \(M + D < N\);
- polynomially stable if \(\gamma_D + \frac{N}{2} < 0\) and \(M + D = N\);
- stable if \(\gamma_D < 0\), \(L = 0\), and \(M + D > N\);
- unstable if \(\gamma_D > 0\).

Proof. The proof is based on the construction of suitable Lyapunov function for system (33). We first note that system (33) can be written as

$$\tau \frac{d}{d\tau} \hat{\theta} = -\partial_\theta \Theta(\hat{\theta}, \hat{\varphi}, \tau), \quad \tau \frac{d}{d\tau} \hat{\varphi} = \partial_\psi \Theta(\hat{\theta}, \hat{\varphi}, \tau) + \Upsilon(\hat{\theta}, \hat{\varphi}, \tau),$$

with

$$\Theta(\hat{\theta}, \hat{\varphi}, \tau) = \int_0^\hat{\theta} \Omega(r, 0, \tau) dr - \int_0^{\hat{\varphi}} \Lambda(\hat{\theta}, \theta, \tau) d\theta, \quad \Upsilon(\hat{\theta}, \hat{\varphi}, \tau) = \int_0^{\hat{\varphi}} \left( \partial_\psi \Lambda(\hat{\theta}, \theta, \tau) + \partial_\theta \Omega(\hat{\theta}, \theta, \tau) \right) d\theta.$$ 

It can easily be checked that \(\Theta(\hat{\theta}, \hat{\varphi}, \tau) = \Theta_L(\hat{\theta}, \hat{\varphi}, \tau) + O(\Delta^2 \tau^{-\frac{M}{2}})\), where

$$\Theta_L(\hat{\theta}, \hat{\varphi}, \tau) = \sum_{K=0}^L \tau^{-\frac{M}{2}} \int_0^\hat{\theta} \left( \Omega_K(r, \varphi_0) - \Omega_K(0, \varphi_0) \right) dr - \tau^{-\frac{M}{2}} \int_0^{\hat{\varphi}} \left( \Lambda_L(\hat{\theta}, \theta + \varphi_0) - \Lambda_L(0, \varphi_0) \right) d\theta$$

$$= Q_0 \frac{\hat{\varphi}^2}{2} \left( 1 + O(\tau^{-\frac{N}{2}}) \right) - \tau^{-\frac{M}{2}} \left( Q_0 \frac{\hat{\varphi}^2}{2} + \partial_\psi \Lambda_L(0, \varphi_0) \hat{\varphi} + \varphi_0 \Delta^2 \right).$$
as $\Delta \to 0$ and $\tau \to \infty$. From (34) it follows that

$$\mathbf{T}(\hat{\theta}, \hat{\varphi}, \tau) \equiv \sum_{K=M}^{n} \tau^{-\frac{K}{2}} \int_{0}^{\tau} \left( \partial_{\theta} \Lambda_{K}(\hat{\theta} + \hat{\varphi}, \hat{\varphi} + \theta) + \partial_{\varphi} \Omega_{K}(\hat{\theta} + \hat{\varphi}, \hat{\varphi} + \theta) \right) d\theta$$

$$= \tau^{-\frac{D}{2}} \hat{\varphi} (\gamma_{D} + \mathcal{O}(\Delta) + \mathcal{O}(\tau^{-\frac{1}{2}})), \quad \Delta \to 0, \quad \tau \to \infty.$$

Here, the asymptotic estimates are uniform with respect to $(\hat{\theta}, \hat{\varphi}, \tau)$ in the domain $\{ (\hat{\theta}, \hat{\varphi}, \tau) \in \mathbb{R}^{3} : \Delta \leq \Delta_{*}, \tau \geq \tau_{0} \}$ with some constants $\Delta_{*} > 0$ and $\tau_{0} > 0$.

Consider the combination

$$(36) \quad V(\hat{\theta}, \hat{\varphi}, \tau) = \Theta(\hat{\theta}, \hat{\varphi}, \tau) + \tau^{-\frac{D}{2}} \gamma_{D} \hat{\varphi} + \tau^{-\frac{D+L}{2N}} \gamma_{D} \partial_{\theta} \Lambda_{L}(0, \hat{\varphi}, \varphi) \frac{3\hat{\varphi}^{2}}{4Q_{0}}$$

as a Lyapunov function candidate for system (35). It is easily shown that for all $\epsilon \in (0, 1)$ there exist $\Delta_{1} > 0$ and $\tau_{1} \geq \tau_{0}$ such that

$$\Delta_{1} \leq \Delta \leq \Delta_{*}, \tau \geq \tau_{1} \quad \text{and} \quad \Delta \to 0, \quad \tau \to \infty.$$
as \( \tau \geq \tau_2 \). Therefore, for all \( \Delta_3 > 0 \) there exists \( \tau_3 > \tau_2 \) such that \( W_0(\hat{\theta}(\tau), \hat{\varphi}(\tau), \tau_2) \geq \Delta_3^2 \) as \( \tau \geq \tau_3 \). It follows that the fixed point \((0, 0)\) of system (35) is unstable.

Returning to the variables \( \varrho(\tau), \varphi(\tau) \), we obtain the result of the Lemma. \( \square \)

Note that the constructed Lyapunov function does not allow to prove the stability of the particular solution \( \varrho_*(\tau), \varphi_*(\tau) \) in the case of \( \gamma_D < 0, M + D \geq N, L > 0 \). Let us show that in this case there is at least a stability on a finite but asymptotically long time interval.

**Lemma 8.** Let assumptions (4), (27), (29), (34) hold with \( \lambda_L < 0 \). If \( \gamma_D < 0, M + D \geq N \) and \( L > 0 \), then the particular solution \( \varrho_*(\tau), \varphi_*(\tau) \) of system (28) with asymptotics (31) is stable on a finite but asymptotically long time interval.

**Proof.** From (37) and (38) it follows that for all \( \delta \in (0, \Delta_2) \) there exists

\[
\Delta_\delta = \delta \left( \frac{1 - \varepsilon}{1 + \varepsilon} \right)^\frac{1}{2} \frac{\tau_2}{\tau_2 - \tau}\n
\]
such that

\[
\sup_{(\hat{\varrho}, \hat{\varphi}) : W_0(\hat{\varrho}, \hat{\varphi}, \tau_2) \leq \Delta_\delta} V(\hat{\varrho}, \hat{\varphi}, \tau) \leq (1 + \varepsilon) \Delta_\delta^2 < (1 - \varepsilon) \delta^2 \frac{\tau_2}{\tau_2 - \tau} \leq \inf_{(\hat{\varrho}, \hat{\varphi}) : W_0(\hat{\varrho}, \hat{\varphi}, \tau_2) = \delta^2} V(\hat{\varrho}, \hat{\varphi}, \tau)
\]
for all \( 1 \leq \tau/\tau_2 \leq \Delta_\delta \) with \( \Gamma_\delta = (\Delta_2/\delta)^2N/L \). Combining this with (40), we see that any solution of system (35) with initial data \( \{(\hat{\varrho}, \hat{\varphi}) : W_0(\hat{\varrho}, \hat{\varphi}, \tau_2) \leq \Delta_\delta^2\} \) at \( \tau = \tau_3 \) satisfies the inequality \( W_0(\hat{\varrho}(\tau), \hat{\varphi}(\tau), \tau/\tau_2) < \delta^2 \) as \( 1 \leq \tau/\tau_2 \leq \Delta_\delta \), and \( \Gamma_\delta \to \infty \) as \( \delta \to 0 \). Thus, the fixed point \((0, 0)\) of system (35) and the particular solution \( \varrho_*(\tau), \varphi_*(\tau) \) of system (28) are stable on a finite but asymptotically long time interval. \( \square \)

5.3. Persistence of phase locking. Let us show that if the particular solution \( \varrho_*(\tau), \varphi_*(\tau) \) is stable, the phase locking regime occurs in the full system (22). We have the following.

**Lemma 9.** Let assumptions (4), (27), (29), (34) hold with \( \lambda_L < 0 \). If any of the following conditions holds:

- \( \gamma_D < 0 \) and \( M + D < N \);
- \( \gamma_D + \frac{L}{\tau} < 0 \) and \( M + D = N \);
- \( \gamma_D < 0 \), \( L = 0 \), and \( M + D > N \),

then for all \( \epsilon > 0 \) there exists \( \delta > 0 \) and \( \tau_s > 0 \) such that \( \forall (R_\epsilon, \Psi_\epsilon) : |R_\epsilon - \varrho_*(\tau_s)| < \delta, |\Psi_\epsilon - \varphi_*(\tau_s)| < \delta \) the solution \( R(\tau), \Psi(\tau) \) of system (22) with initial data \( R(\tau_s) = \varrho_*(\tau_s), \Psi(\tau_s) = \varphi_*(\tau_s) \) satisfies the inequality: \( |R(\tau) - \varrho_*(\tau)| + |\Psi(\tau) - \varphi_*(\tau)| < \epsilon \) for all \( \tau > \tau_s \).

**Proof.** Substituting \( R(\tau) = \varrho_*(\tau) + \hat{R}(\tau), \Psi(\tau) = \varphi_*(\tau) + \hat{\Psi}(\tau) \) into (22) yields

\[
\tau^M \frac{d\hat{R}}{dt} = -\partial_\varphi \Theta(\hat{R}, \hat{\Psi}, \tau) + \tilde{\Lambda}_n(\hat{R}, \hat{\Psi}, \tau), \quad \tau^M \frac{d\hat{\Psi}}{dt} = \partial_\varrho \Theta(\hat{R}, \hat{\Psi}, \tau) + \gamma(\hat{R}, \hat{\Psi}, \tau) + \tilde{\Omega}_n(\hat{R}, \hat{\Psi}, \tau),
\]

where \( \tilde{\Lambda}_n = \tau^M \Lambda_n(\varrho_*(\tau) + \hat{R}, \varphi_*(\tau) + \hat{\Psi}, \tau), \tilde{\Omega}_n = \tau^M \Omega_n(\varrho_*(\tau) + \hat{R}, \varphi_*(\tau) + \hat{\Psi}, \tau) \). It follows that

\[
\tilde{\Lambda}_n(R, \Psi, \tau) = O(\tau^{-\frac{M+1}{2}}), \quad \tilde{\Omega}_n(R, \Psi, \tau) = O(\tau^{-\frac{M+1}{2}}), \quad \tau \to \infty
\]

uniformly for all \((\hat{R}, \hat{\Psi}) \in \mathbb{R}^2\) such that \( W_0(\hat{R}, \hat{\Psi}) \leq \Delta_2^2 \) with some \( \Delta_1 = \text{const} > 0 \). Note that the functions \( \tilde{\Lambda}_n \) and \( \tilde{\Omega}_n \) play the role of persistent disturbances of system (35). Let us show that the particular solution \( \varrho_*(\tau), \varphi_*(\tau) \) of system (35) is stable with respect to these perturbations.

Using \( V(\hat{R}, \hat{\Psi}, \tau) \) defined by (36) as a Lyapunov function candidate for system (42), we obtain

\[
\frac{dV}{dt} \Big|_{(42)} = -\frac{dV}{dt} \Big|_{(35)} + \tau^{-\frac{M}{2}} \left( \partial_\varrho V(\hat{R}, \hat{\Psi}, \tau) \tilde{\Lambda}_n(\hat{R}, \hat{\Psi}, \tau) + \partial_\varphi V(\hat{R}, \hat{\Psi}, \tau) \tilde{\Omega}_n(\hat{R}, \hat{\Psi}, \tau) \right).
\]

It follows from (43) that there exist \( C_s > 0 \) and \( \tau_1 \geq \max\{\tau_0, 1\} \) such that

\[
\partial_\varrho V \tilde{\Lambda}_n + \partial_\varphi V \tilde{\Omega}_n \leq C_s \tau^{-\frac{M}{2}} \left( Q_0^0 ||\hat{R}|| + \tau^{-\frac{M}{2}} |\lambda_L| ||\hat{\Psi}|| \right)
\]
as $\tau \geq \tau_1$ and for all $(\hat{R}, \hat{\Psi}) \in \mathbb{R}^2$ such that $W_0(\hat{R}, \hat{\Psi}, \tau_1) \leq \Delta_1^2$. Since $n \geq L + D$, we have

$$\left| \frac{\partial V}{\partial \tau} \right| \leq \frac{1}{2D} \left( \frac{\tau^2}{D} + \frac{\tau^2}{D} \right),$$

(45)

$$\partial_R V_{\hat{\alpha}_n} + \partial_q V_{\hat{\om}_{\eta}} \leq \tau^{-\frac{D+1}{\Delta_2}} W_0(\hat{R}, \hat{\Psi}, \tau) \frac{C_n}{\delta} \left( 1 + \frac{1}{Q_0} + \frac{1}{|\eta|} \right)$$

as $\tau \geq \tau_1$ and for all $(\hat{R}, \hat{\Psi}) \in \mathbb{R}^2$ such that $\delta^2 \leq W_0(\hat{R}, \hat{\Psi}, \tau_1) \leq \Delta_1^2$ with some $\delta = \text{const} > 0$. Thus, it follows from (39), (44) and (45) that for all $\varepsilon \in (0, 1)$ there exists $\delta < \Delta_2 \leq \Delta_1$ and $\tau_2 \geq \tau_1$ such that

$$\frac{dW}{d\tau} \leq -\tau^{-\frac{M+D}{\gamma_0}} V(\hat{R}, \hat{\Psi}, \tau) \leq 0,$$

(46)

as $\tau \geq \tau_2$. By choosing $\varepsilon \in (0, 1)$ small enough, we can ensure that $\gamma_0 (1-\varepsilon)/(1+\varepsilon) + L/N < 0$. Hence, for all $\varepsilon \in (0, \Delta_2)$ there exist $\delta = \varepsilon \sqrt{(1-\varepsilon)/(1+\varepsilon)} < \varepsilon$ such that any solution of system (42) starting from $(\hat{R}, \hat{\Psi}) : W_0(\hat{R}, \hat{\Psi}, \tau_2) \leq \Delta_2^2$ at $\tau_\ast > \tau_2$ cannot leave the domain $(\hat{R}, \hat{\Psi}) : W_0(\hat{R}, \hat{\Psi}, \tau_2) \leq \varepsilon^2$ as $\tau > \tau_\ast$. Similar estimates hold in the case $M + D \neq N$. Returning to the variables $(R, \Psi)$, we obtain the result of the Lemma.

\[\text{□}\]

Define $\gamma_D := \gamma_D + \delta_{M+D,N}L/N$. Then we have the following:

**Corollary 1.** Let assumptions (4), (27), (29), (34) hold with $\lambda_L < 0$. If $\gamma_D < 0$ and $M + D \leq N$, then for all $\zeta \in (0, 1)$ there exists $\Delta_2 > 0$ and $\tau_\ast > 0$ such that $\forall (R_\ast, \Psi_\ast) : |R_\ast - \Psi_\ast(\tau_\ast)| < \delta$, $|\Psi_\ast - \varphi_\ast(\tau_\ast)| < \Delta_2$, the solution $R(\tau)$, $\Psi(\tau)$ of system (22) with initial data $(\hat{R}_0, \hat{\Psi}_0)$ has the following estimates as $\tau \to \infty$:

$$R(\tau) = \delta_{M+D,N}O\left(\tau^{-\frac{1}{\gamma_D}}\right) + O\left(\tau^{-\frac{D}{\gamma_D}}\right), \quad \Psi(\tau) = \varphi_0 + \delta_{M+D,N}O\left(\tau^{-\frac{1}{\gamma_D}}\right) + O\left(\tau^{-\frac{D}{\gamma_D}}\right).$$

(48)

**Proof.** Let $M + D = N$. Then, by taking $\varepsilon = (1-\zeta)|\gamma_D|/(2|\gamma_D| - |\gamma_D|)$ in (47), we see that $W_0(\hat{R}(\tau), \hat{\Psi}(\tau), \tau_2) = \mathcal{O}(\tau^{-\frac{1}{\gamma_D}})$ at $\tau \to \infty$ for solutions of system (42) with initial data from $(\hat{R}, \hat{\Psi}) : W_0(\hat{R}, \hat{\Psi}, \tau_2) \leq \Delta_2^2$ with some $0 < \Delta_2 \leq \Delta_2$. Similarly, if $M + D < N$, then from (46) it follows that $W_0(\hat{R}, \hat{\Psi}, \tau_2)$ has exponentially decaying bound on the trajectories. Returning to the variables $(R, \Psi)$ and taking into account (31), we obtain the corresponding asymptotic estimates.

\[\text{□}\]

Combining this with Theorem 1, we obtain the following:

**Theorem 2.** Let assumptions (3), (4), (27), (29), (34) hold with $\lambda_L < 0$ and some $\omega \in \mathbb{Z}_+$. If $\gamma_D < 0$ and $M + D \leq N$, then for all $\zeta \in (0, 1)$ there exist $t_\ast > 0$ and $\Delta_2 \subset \mathcal{D}(E_0)$ such that for all $(x_\ast, y_\ast) \in \Delta_2$, the solution $x(t)$, $y(t)$ of system (1) with initial data $x(t_\ast) = x_\ast$, $y(t_\ast) = y_\ast$ has the following estimates as $t \to \infty$:

$$x(t) = t^{\frac{b}{(\nu-\omega-\mu)}(2h)} e^\frac{c}{\kappa} x_0(\omega^{-1} S(t) + \Psi(\tau)) \left( 1 + t^{-\mu} R(\tau) \right)^\frac{\nu}{2} \left( 1 + \mathcal{O}(t^{-\frac{b}{(\nu-\omega-\mu)}}) \right),$$

$$y(t) = t^{\frac{h}{(\nu-\omega-\mu)} \sqrt{2}} e^{-b} x_0(\omega^{-1} S(t) + \Psi(\tau)) \left( 1 + t^{-\mu} R(\tau) \right)^\frac{\nu}{2} \left( 1 + \mathcal{O}(t^{-\frac{b}{(\nu-\omega-\mu)}}) \right),$$

where $R(\tau)$, $\Psi(\tau)$ have asymptotics (48), $\tau = t^\nu/\nu$, $\mu = (b/q - \sigma)/2 > 0$, $\nu = 1 + b/q$, and $X_0(\phi)$, $Y_0(\phi)$ is a $2\pi$-periodic solution of the system

$$\frac{2\pi}{\kappa} \partial_\phi X_0 = Y_0, \quad \frac{2\pi}{\kappa} \partial_\phi Y_0 = -X_0^{2h-1}, \quad \frac{X_0^{2b}}{2h} + \frac{Y_0}{2} = 1.$$
6. Examples

1. Consider again equation (5) that satisfies the assumptions (3) and (4) with \( h = 2, a = b = 1, q = 3, l = p = 0, \) and \( \sigma = -1. \) From (18) it follows that \( \mu = 2/3, \nu = 4/3, M = 4, N = 8. \) In this case, we have the following [41]:

\[
\kappa = 2\sqrt{2}K\left(\frac{1}{2}\right), \quad X_0(\phi) = \sqrt{2}cn\left(\frac{\kappa \phi}{\pi \sqrt{2}}, \frac{1}{2}\right), \quad Y_0(\phi) = \frac{2\pi}{\kappa}d_\phi X_0(\phi),
\]

where \( K(k) \) is the complete elliptic integral of the first kind, \( cn(t; k) \) is the Jacobi elliptic function [42]. Moreover, the \( 2\pi \)-periodic function \( X_0(\phi) \) admits the Fourier expansion:

\[
X_0(\phi) = \sum_{j=1}^{\infty} x_j \cos((2j-1)\phi), \quad x_j = \frac{4\pi\sqrt{2}}{\kappa} \sech\left((2j-1)\frac{\pi}{2}\right).
\]

It is not hard to check that the corresponding averaged system (22) takes the form

\[
\frac{dR}{d\tau} = \tau^{-\frac{1}{2}} \sum_{K=0}^{4} \tau^{-\frac{K}{2}} A_K(R, \Psi) + O(\tau^{-\frac{5}{2}}), \quad \frac{d\Psi}{d\tau} = \tau^{-\frac{1}{2}} \sum_{K=0}^{4} \tau^{-\frac{K}{2}} \Omega_K(R, \Psi) + O(\tau^{-\frac{5}{2}}), \quad \tau \to \infty,
\]

with \( \tau = 3t^{4/3}/4, \) \( \Lambda_1 = \Lambda_3 \equiv \Omega_1 \equiv \Omega_2 \equiv \Omega_3 \equiv 0, \)

\[
\Lambda_0 \equiv \nu^{-\frac{1}{4}} \left( c_4^2 \frac{B}{4} \langle Y_0(\Psi + \chi^{-1}\zeta) \cos \zeta \rangle_{\chi} - \frac{1}{3} \right),
\]

\[
\Lambda_2 \equiv \nu^{-\frac{1}{4}} \frac{c_4^2 B}{4} \langle Y_1(\Psi + \chi^{-1}\zeta) \cos \zeta \rangle_{\chi},
\]

\[
\Lambda_4 \equiv -\nu^{-1} \left( c_4^2 \frac{B}{4} \langle Y_0(\Psi + \chi^{-1}\zeta) \cos \zeta \rangle_{\chi} - \frac{1}{3} \right) + \nu^{-1} \frac{2\omega_2 c_4^2}{3\omega_0}
\]

\[
+ \nu^{-1} \frac{c_4^2 B}{4} \left( \langle Y_2(\Psi + \chi^{-1}\zeta) \cos \zeta \rangle_{\chi} + \frac{2\omega_2}{\omega_0} \langle Y_0(\Psi + \chi^{-1}\zeta) \cos \zeta \rangle_{\chi} \right),
\]

\[
\Omega_0 \equiv \omega_0 \nu^{-\frac{1}{4}} c_4^{-1} R,
\]

\[
\Omega_4 \equiv \nu^{-1} \left( R \omega_2 c_4 (\omega_2 - 1) - \frac{\omega_0 c_4^2 B}{4} \langle Y_0(\Psi + \chi^{-1}\zeta) \cos \zeta \rangle_{\chi} \right).
\]

The parameters \( \omega_0 \) and \( 2\pi \)-periodic functions \( X_k(\phi), Y_k(\phi) \) are defined in (8), (9) with \( h = 2. \) It is easily shown that \( \langle Z(\Psi + \chi^{-1}\zeta) \cos \zeta \rangle_{\chi} \equiv \langle Z(\chi^{-1} \zeta) \cos \zeta \rangle_{\chi} \cos(\chi \Psi) + \langle Z(\chi^{-1} \zeta) \sin \zeta \rangle_{\chi} \sin(\chi \Psi), \) for any continuous \( 2\pi \)-periodic function \( Z(\phi). \)

Consider resonant solutions with \( \chi = 2m - 1, m \in \mathbb{Z}_+. \) In this case, \( \langle X_0(\Psi + \chi^{-1}\zeta) \cos \zeta \rangle_{\chi} \equiv \langle x_m/2 \rangle \cos(\chi \Psi) \) and \( \langle Y_0(\Psi + \chi^{-1}\zeta) \cos \zeta \rangle_{\chi} \equiv -\langle \pi \chi x_m/\kappa \rangle \sin(\chi \Psi). \) Hence, the condition (27) holds with \( L = 0 \) and

\[
\Lambda_0(R, \Psi) \equiv -\nu^{-\frac{1}{4}} \frac{\pi B \chi x_m c_4^2}{4K} \left( \sin(\chi \Psi) + \frac{s^2}{B d_\chi} \right), \quad d_\chi := \frac{27\pi^4 \chi^3 \sech^2 \left( \frac{\pi x_m}{2} \right)}{2\sqrt{2} \kappa^3}, \quad c_\chi = \frac{3\pi}{2\kappa s}.
\]

We see that if \( s^2/|B| < d_\chi, \) there exists \( \varphi_0 \) such that \( \Lambda_0(0, \varphi_0) = 0 \) and \( \Lambda_0 = \partial_\Psi \Lambda_0(0, \varphi_0) < 0 \) (see Fig. 2, a). Moreover, it can easily be checked that the conditions (29), (34) are satisfied with \( D = 4 \) and \( \gamma_4 = \partial_R \Lambda_4(0, \varphi_0) + \partial_\Psi \Omega_4(0, \varphi_0) = -1/4. \) It follows that equation (5) satisfies the assumptions of Theorem 2 with \( \gamma_\varphi = \gamma_D < 0, \) \( M = D = N. \) Hence, the phase locking regime with \( \chi = 2m - 1 \) is stable, and the resonant solutions of equation (5) have the following asymptotics:

\[
x(t) = t^\frac{1}{4} \sqrt{2} c_\chi^{-1} cn\left( \frac{\kappa \phi(t)}{\pi \sqrt{2}}, \frac{1}{2} \right) \left( 1 + O(t^{-\frac{1}{2}}) \right),
\]

\[
I(t) = c_\chi^{-4} t^\frac{3}{4} \left( 1 + O(t^{-\frac{1}{4}}) \right), \quad \phi(t) = \chi^{-1} S(t) + \varphi_0 + O(t^{-\frac{1}{4}}), \quad t \to \infty.
\]
It is easy to verify that equation (30) with $a = 2$, $U(x) \equiv x^4/4 - x^2/2$, and satisfies (3) with $l = 0$, $p = 1$, $f \equiv 0$, $g \equiv B_{0,1,0}(S)x$, $B_{0,1,0}(S) \equiv \cos S$. If $a + b \leq q$, then the condition (4) is satisfied with $\sigma = -(a + b)/q$. Consider the case $q = 3$, $a = 2$, $b = 1$. From (18) it follows that $\mu = 2/3$, $\nu = 4/3$, $M = 4$, $N = 8$, and the corresponding averaged system (22) takes the form

\[
\frac{dR}{d\tau} = \tau^{-\frac{1}{2}} \sum_{K=0}^{4} \tau^{-\frac{K}{2}} \Lambda_K(R, \Psi) + O(\tau^{-\frac{9}{8}}), \quad \frac{d\Psi}{d\tau} = \tau^{-\frac{1}{2}} \sum_{K=0}^{4} \tau^{-\frac{K}{2}} \Omega_K(R, \Psi) + O(\tau^{-\frac{9}{8}})
\]
as $\tau \to \infty$, with $\tau = (3/4)t^{4/3}$, $\Lambda_1 \equiv \Lambda_3 \equiv \Omega_1 \equiv \Omega_2 \equiv \Omega_3 \equiv 0$, 

\[
\begin{align*}
\Lambda_0 &\equiv \nu^{-\frac{1}{2}} \left( \frac{c_x B}{4} \langle X_0(\Psi + \zeta^{-1})Y_0(\Psi + \zeta^{-1})\cos \zeta \rangle_{\zeta} - \frac{1}{3} \right), \\
\Lambda_2 &\equiv \nu^{-\frac{3}{2}} \frac{c_x^2 B^2}{4} \sum_{i+j=1} \langle X_i(\Psi + \zeta^{-1})Y_j(\Psi + \zeta^{-1})\cos \zeta \rangle_{\zeta}, \\
\Lambda_4 &\equiv \nu^{-\frac{1}{2}} \frac{c_x^2 B^2}{4} \sum_{i+j=2} \langle X_i(\Psi + \zeta^{-1})Y_j(\Psi + \zeta^{-1})\cos \zeta \rangle_{\zeta} + \nu^{-1} \frac{R}{3} + \nu^{-\frac{1}{2}} \frac{c_x^2 B^2}{\omega_0} \left( \frac{c_x B}{4} \langle X_0(\Psi + \zeta^{-1})Y_0(\Psi + \zeta^{-1})\cos \zeta \rangle_{\zeta} + \frac{2}{3} \right), \\
\Omega_0 &\equiv \omega_0 \nu^{-\frac{3}{2}} c_x^{-1} R, \\
\Omega_4 &\equiv \nu^{-1} \left( \omega_0 \bar{z}_2 c_x^{-1} - \omega_2 c_x \right) R - \frac{\omega_0 c_x B}{4} \langle X_0^2(\Psi + \zeta^{-1})\cos \zeta \rangle_{\zeta}.
\end{align*}
\]

Consider a phase locking with $\zeta = 2$. In this case, system (51) satisfies the condition (27) with $L = 0$ and

\[
\Lambda_0(R, \Psi) \equiv \nu^{-\frac{1}{2}} \frac{c_x B a_{11}}{4} \left( \sin(2\Psi) + \frac{s}{B d_2} \right),
\]

where

\[
a_{11} = \langle X_0(\zeta)Y_0(\zeta)\sin(2\zeta) \rangle_{\zeta} = \frac{\pi x_2}{2} - \frac{\pi}{\kappa} \sum_{j=1}^{\infty} x_j x_{j+1}, \quad d_2 := -\frac{9\pi a_{11}}{4\kappa} > 0, \quad c_2 = \frac{3\pi}{\kappa s}.
\]

It follows that if $s/|B| < d_2$, there exists $\varphi_0$ such that $\Lambda_0(0, \varphi_0) = 0$ and $\lambda_0 = \partial_\Psi \Lambda_0(0, \varphi_0) < 0$ (see Fig. 3, a). In particular,

\[
\begin{align*}
\varphi_0 &\in \left\{- \frac{1}{2} \arcsin \left( \frac{s}{B d_2} \right) + \pi k, \quad k \in \mathbb{Z} \right\} \quad \text{if} \quad B > B_2; \\
\varphi_0 &\in \left\{ \frac{\pi}{2} + \frac{1}{2} \arcsin \left( \frac{s}{B d_2} \right) + \pi k, \quad k \in \mathbb{Z} \right\} \quad \text{if} \quad B < -B_2,
\end{align*}
\]

where $B_2 = s/d_2$. Hence, the condition (29) holds. Since

\[
\partial_\Psi \langle X_0^2(\Psi + \zeta^{-1})\cos \zeta \rangle_{\zeta} = \frac{2}{\omega_0} \langle X_0(\Psi + \zeta^{-1})Y_0(\Psi + \zeta^{-1})\cos \zeta \rangle_{\zeta} = \frac{\kappa a_{11}}{\pi} \sin(2\Psi),
\]

we have $\gamma_4 = \partial_R \Lambda_4(0, \varphi_0) + \partial_\Psi \Omega_4(0, \varphi_0) = -1/4$. Consequently, the condition (34) is satisfied with $D = 4$ and $\gamma_4 \neq 0$. Therefore, equation (50) satisfies the assumptions of Theorem 2 with $\tilde{\gamma}_D = \gamma_D < 0$, $M + D = N$. Hence, the phase locking regime is stable and the solutions of (50) corresponding to parametric resonance have asymptotics (49) with $\zeta = 2$ (see Fig. 3, b, c).

3. Finally, consider a system with a nonlinear parametric perturbation and a weak nonlinear damping:

\[
\frac{d^2x}{dt^2} - x + \left(1 - Bt^{-\frac{1}{q}} \cos S(t)\right)x^3 + Ct^{-\frac{n+1}{q}}x^\frac{q}{2} \frac{dx}{dt} = 0, \quad S(t) = st^{1+\frac{b}{a}},
\]

where $B, C = \text{const}, C > 0$. It is readily seen that (50) takes the form (1) with $h = 2, U(x) \equiv x^4/4 - x^2/2$, and satisfies (3) with $l = 1, p = 3, f \equiv 0, g \equiv B_{0,3,0}(S)x^3 + t^{-1/q}B_{1,2,1}(S)x^2 y, B_{0,3,0}(S) \equiv B \cos S, B_{1,2,1}(S) \equiv -C$. Note that if $b < a$, then the condition (4) holds with $\sigma = (2b - a)/q$. We take $a = q = 3$ and $b = 1$. Then $\sigma = -2/3, \mu = 1/3, \nu = 4/3, M = 2, N = 8$, and the corresponding averaged system (22) takes the following form:

\[
\begin{align*}
\frac{dR}{d\tau} &= \tau^{-\frac{1}{q}} \sum_{K=0}^{4} \tau^{-\frac{K}{q}} \Lambda_K(R, \Psi) + O(\tau^{-1}), \\
\frac{d\Psi}{d\tau} &= \tau^{-\frac{1}{q}} \sum_{K=0}^{4} \tau^{-\frac{K}{q}} \Omega_K(R, \Psi) + O(\tau^{-1}), \quad \tau \to \infty,
\end{align*}
\]
with \( \tau = (3/4)t^{4/3} \), \( \Lambda_0 \equiv \Lambda_1 \equiv \Lambda_3 \equiv \Omega_1 \equiv \Omega_2 \equiv \Omega_3 \equiv 0 \),

\[
\begin{align*}
\Lambda_2 &\equiv -\nu^{-4}\frac{C}{4\nu\omega^2} \langle X^0_0(\Psi + \nu^{-1}\zeta)Y^2_0(\Psi + \nu^{-1}\zeta) \rangle_{\nu\zeta} + \nu^{-4}\frac{B}{4\nu\omega^2} \langle X^3_0(\Psi + \nu^{-1}\zeta)Y_0(\Psi + \nu^{-1}\zeta) \cos \zeta \rangle_{\nu\zeta}, \\
\Lambda_4 &\equiv -\nu^{-4}\frac{3CR}{4\nu\omega^2} \langle X^2_0(\Psi + \nu^{-1}\zeta)Y^2_0(\Psi + \nu^{-1}\zeta) \rangle_{\nu\zeta} + \nu^{-4}\frac{BR}{2\nu\omega^2} \langle X^3_0(\Psi + \nu^{-1}\zeta)Y_0(\Psi + \nu^{-1}\zeta) \cos \zeta \rangle_{\nu\zeta} \\
&\quad - \nu^{-4}\frac{C}{2\nu\omega^2} \langle X_0Y_0(X_1Y_0 + X_0Y_1) \rangle_{\nu\zeta} + \nu^{-4}\frac{B}{4} \langle X^2_0(Y_0X_1 + 3X_1Y_0) \cos \zeta \rangle_{\nu\zeta} - \nu^{-4}\frac{1}{3}, \\
\Omega_0 &\equiv \omega_0\nu^{-4}c_{\nu}^{-1}R, \\
\Omega_4 &\equiv \nu^{-4}\left(-2\nu\omega_0R - \frac{\omega_0B}{4\nu\omega^2} \langle X^4_0(\Psi + \nu^{-1}\zeta) \cos \zeta \rangle_{\nu\zeta}\right).
\end{align*}
\]

It is easy to verify that for resonant solutions with \( \nu = 2 \), the condition (27) is satisfied with \( L = 2 \) and

\[
\Lambda_2(R, \Psi) \equiv \nu^{-4} \frac{B_0a_{31}k\zeta}{12\pi} \left( \sin(2\Psi) + \frac{sC}{Bd_2} \right), \quad d_2 := \frac{3\pi a_{31}}{\nu^2v_{22}} > 0,
\]

where \( v_{22} \equiv \langle X^2_0(\Psi)Y^2_0(\zeta) \rangle_{\zeta} > 0 \), \( a_{31} \equiv \langle X^3_0(\Psi)Y_0(\zeta) \sin 2\zeta \rangle_{\zeta} \approx \nu^2 a_{11}/2 < 0 \). Hence, if \( sC/|B| < d_2 \), the condition (29) holds (see Fig. 4, a): there exists \( \varphi_0 \) such that \( \Lambda_2(0, \varphi_0) = 0 \) and \( \lambda_2 = \partial_{\Psi}\Lambda_2(0, \varphi_0) < 0 \). In this case, we have

\[
\begin{align*}
\varphi_0 &\in \left\{ -\frac{1}{2} \arcsin \left( \frac{sC}{Bd_2} \right) + \pi k; \quad k \in \mathbb{Z} \right\} \quad \text{if} \quad B > B_2; \\
\varphi_0 &\in \left\{ \frac{\pi}{2} + \frac{1}{2} \arcsin \left( \frac{sC}{Bd_2} \right) + \pi k; \quad k \in \mathbb{Z} \right\} \quad \text{if} \quad B < -B_2,
\end{align*}
\]
Figure 4. (a) Partition of the parameter plane \((B, s)\) for equation (50) with \(\kappa = 2\) and different values of the parameter \(C\). (b), (c) The evolution of \(I(t) = H(x(t), \dot{x}(t))\) and \(\bar{\theta}(t) = \phi(t) - S(t)/2\), \(\tan \bar{\theta}(t) = -\dot{x}(t)/x(t)\) for solutions of (52) with \(a = q = 3\), \(b = 1\), \(s = C = 1\), \((B_2 \approx 0.71)\) and different values of the parameter \(B\). (b) The gray dashed curve corresponds to \(c_{-2}^{-4}t^4\), \(c_{-2}^{-4} \approx 0.095\). (c) The gray dashed lines correspond to \(\bar{\theta} = \varphi_0\).

where \(B_2 = sC/d_2\). Moreover, it can easily be checked that the assumption (34) is satisfied with \(D = 2\) and

\[
\gamma_4 \equiv \partial_R \Lambda_4(0, \varphi_0) + \partial_{\varphi} \Omega_4(0, \varphi_0) = -5 Cv_{22} \nu^{-\frac{3}{2}} \left(\frac{\nu s}{\nu_0} S\right) \frac{2}{\nu_0} < 0.
\]

Thus, by applying Theorem 2 with \(\dot{\gamma}_D = \gamma_D < 0\) and \(M + D < N\), we see that the phase locking regime is stable and the resonant solutions of (52) have asymptotics (49) with \(\kappa = 2\) (see Fig. 4, b, c).

7. Conclusion

Thus, we have shown that decreasing chirped-frequency oscillatory perturbations of strongly nonlinear Hamiltonian systems in the plane can lead to the appearance of at least two different asymptotic regimes away from the equilibrium: a phase locking and a phase drifting. In the case of phase locking the energy of system can increase significantly and the phase of system is synchronised with the phase of the perturbation. We have described the conditions that guarantee the existence and stability of resonant solutions with growing energy. A violation of these conditions can lead to a phase drifting. Numerical examples show that in this case the energy of the perturbed system remains bounded. Note that such solutions have not been investigated in detail in this paper. This will be discussed elsewhere.

The results obtained show that it is possible to use vanishing in time perturbations for capture and holding of strongly nonlinear systems at resonance.

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