Chow quotients of toric varieties as moduli of stable log maps

Qile Chen and Matthew Satriano
Chow quotients of toric varieties as moduli of stable log maps

Qile Chen and Matthew Satriano

Let $X$ be a projective normal toric variety and $T_0$ a rank-1 subtorus of the defining torus $T$ of $X$. We show that the normalization of the Chow quotient $X // T_0$, in the sense of Kapranov, Sturmfels, and Zelevinsky, coarsely represents the moduli space of stable log maps to $X$ with discrete data given by $T_0 \subset X$. We also obtain similar results when $T_0 \to T$ is a homomorphism that is not necessarily an embedding.

1. Introduction 2313
2. Log smoothness and irreducibility 2316
3. Tropical curves associated to stable log maps 2320
4. The Chow quotient as the coarse moduli space 2324
Appendix: Toric varieties have generalized Deligne–Faltings log structures 2326
Acknowledgments 2328
References 2328

1. Introduction

Throughout, we work over an algebraically closed field $k$ of characteristic 0.

Chow quotients of toric varieties were introduced by Kapranov, Sturmfels, and Zelevinsky in [Kapranov et al. 1991]. Given a projective normal toric variety $X$ and a subtorus $T_0$ of the defining torus $T$, the Chow quotient $X // T_0$ has the property that its normalization is the smallest toric variety that maps onto all GIT quotients of $X$ by $T_0$. We show in this paper that when $T_0$ has rank 1, the normalization of $X // T_0$ can be reinterpreted as the coarse moduli space of the stack of stable log maps introduced in [Chen 2011; Abramovich and Chen 2011] and independently in [Gross and Siebert 2013]. We also obtain similar results by replacing $T_0 \subset T$ with a homomorphism $T_0 \to T$ and the Chow quotient in the sense of [Kapranov et al. 1991] with that of [Kollár 1996].

MSC2010: primary 14H10; secondary 14N35.
Keywords: toric, Kontsevich, stable log map, Chow quotient.
Let $X$ be a normal toric variety of dimension $n$ with defining torus $T$. Denoting by $N \cong \mathbb{Z}^n$ the cocharacter lattice of $T$, we see that every point $v \in N$ corresponds to a morphism of multiplicative groups

$$\iota_v : T_0 := \mathbb{G}_m \rightarrow T.$$ (1-1)

It is convenient to view this map as the action of $T_0$ on the identity element $1 \in T$. Let $v = k\omega$ for some positive integer $k$ and primitive lattice point $\omega \in N$. Note that $\iota$ is an embedding if and only if $k = 1$.

We begin by introducing the Chow quotient $X \sslash T_0$. For every point $x \in T$, the closure $Z_x := \overline{T_0x}$ of the orbit of $x$ under $T_0$ with the reduced scheme structure is a subvariety of $X$. Thus, we obtain a Chow cycle $k \cdot [Z_x]$. For $x \in T$, the orbit closures $Z_x$ have the same dimension and homology class. Denoting by $T' := [T/T_0]$ the stack quotient, we therefore obtain a morphism from $T'$ to the Chow variety $C(X)$ of algebraic cycles of the given dimension and homology class. For the definition and construction of the Chow variety $C(X)$, we refer to [Kollár 1996, Chapter I]. Since the Chow variety is not actually a moduli space for cycles as above, one may initially be worried that we only obtain a map on the level of closed points. However, we will later see that there is a family of stable maps over $T'$ whose image is precisely the Chow cycle we obtained here; it then follows from [Kollár 1996, Chapter I, 3.17 and 3.21] that there is a natural map $T' \rightarrow C(X)$. We define the Chow quotient $X \sslash T_0$ to be the closure of the image $T'$ in $C(X)$ with the reduced scheme structure.

Note that when $k = 1$, $T'$ is a variety and $X \sslash T_0$ is the Chow quotient introduced by Kapranov et al. [1991]. In this case, it is a toric variety and the fan of its normalization is given explicitly in [Kapranov et al. 1991, §1].

As mentioned above, the goal of this paper is to relate $X \sslash T_0$ to moduli spaces of stable log maps. Notice that by compactifying $\iota$, we obtain a stable map $f_1 : \mathbb{P}^1 \rightarrow X$, where $\mathbb{P}^1$ is marked at the points $\{0, \infty\} \in \mathbb{P}^1 \setminus T_0$. By viewing $X$ as a log scheme with its canonical log structure $\mathcal{M}_X$ given by the boundary $X \setminus T$ and $\mathbb{P}^1$ as a log curve with log structure $\mathcal{M}_{\mathbb{P}^1}$ given by the two markings $\{0, \infty\}$, we obtain a stable log map

$$f_1 : (\mathbb{P}^1, \mathcal{M}_{\mathbb{P}^1}) \rightarrow (X, \mathcal{M}_X).$$

Let $\beta_0$ be the curve class of the stable map $f_1$, and let $c_0$ and $c_\infty$ be the contact orders of 0 and $\infty$ with respect to the toric boundary $X \setminus T$. Roughly speaking, $c_0$ and $c_\infty$ are functions that assign to the marked points their orders of tangency with the components of $X \setminus T$ (see [Abramovich et al. 2011] for more details). In the toric case, the contact orders can be explained as the slopes and weights of the unbounded edges of tropical curves associated to stable log maps; see Section 3.3. Let $\mathcal{H}_{\Gamma_0}(X)$ be the stack parametrizing stable log maps from rational curves with
two marked points to $X$ such that the curve class is $\beta_0$ and the marked points have contact orders given by $c_0$ and $c_\infty$; here the notation

$$\Gamma_0 := (0, \beta_0, 2, \{c_0, c_\infty\})$$

(1-2)

keeps track of the discrete data consisting of genus, curve class, number of marked points, and their tangency conditions. Our main result is:

**Theorem 1.1.** The normalization of $X \twoheadrightarrow T_0$ is the coarse moduli space of $\mathcal{H}_{\Gamma_0}(X)$.

**Remark 1.2.** In particular, we see that $\mathcal{H}_{\Gamma_0}(X)$ is irreducible.

**Remark 1.3.** In Proposition 2.3, we prove that for any $\Gamma = (0, \beta, 2, \{c_0, c_\infty\})$, either the stack $\mathcal{H}_{\Gamma}(X)$ is empty or $\Gamma = \Gamma_0$ for some $\Gamma_0$ as in (1-2). Thus, our discussion covers all two-pointed stable log maps to toric varieties.

In the process of proving Theorem 1.1, we obtain an alternative description of $\mathcal{H}_{\Gamma_0}(X)$ that is more akin to the construction of the Chow quotient. As we saw above, $X \twoheadrightarrow T_0$ is defined as the closure of $T'$ in the Chow variety $C(X)$. Replacing $C(X)$ by other moduli spaces, we obtain alternate spaces analogous to $X \twoheadrightarrow T_0$. For each point $x \in T$, letting $T_0$ act on $x$ via the group morphism $\iota$ and taking the closure, we obtain a stable log map

$$f_x : (\mathbb{P}^1, \mathcal{M}_{\mathbb{P}^1}) \rightarrow (X, \mathcal{M}_X)$$

again with curve class $\beta_0$ and contact orders $c_0$ and $c_\infty$. Note that for any point $x' \in \overline{T_0x}$, the two stable log maps $f_x$ and $f_{x'}$ are canonically isomorphic. We thus obtain a family of stable log maps over the stack quotient $T'$. It is important to notice that the log structure on $T'$ is trivial (and is denoted by $\mathcal{O}^*$). The stack $\mathcal{H}_{\Gamma_0}(X)$ comes equipped with a log structure, and the above discussion defines a morphism of log stacks

$$(T', \mathcal{O}^*_T) \rightarrow (\mathcal{H}_{\Gamma_0}(X), \mathcal{M}_{\mathcal{H}_{\Gamma_0}(X)})$$

Forgetting the log structures, we obtain an immersion

$$T' \rightarrow \mathcal{M}_{0,2}(X, \beta_0),$$

where $\mathcal{M}_{0,2}(X, \beta_0)$ denotes the Kontsevich space of stable maps to $X$ with genus 0, curve class $\beta_0$, and two marked points. In analogy with the construction of the Chow variety, we let $\mathcal{M}$ denote the closure of $T'$ in $\mathcal{M}_{0,2}(X, \beta_0)$. Then we have:

**Theorem 1.4.** $\mathcal{H}_{\Gamma_0}(X)$ is the normalization of $\mathcal{M}$.

**Remark 1.5.** There is an analogous picture if one assumes that $X$ is an affine normal toric variety and replaces $\mathcal{M}_{0,2}(X, \beta_0)$ above by the toric Hilbert scheme, as defined in [Peeva and Stillman 2002]. That is, for all $x \in T$, the $Z_x$ are $T'$-invariant closed subschemes of $X$ that have the same discrete invariants. We therefore
obtain an immersion from $T'$ to an appropriate toric Hilbert scheme. The closure of $T'$ in this toric Hilbert scheme is called the main component. Olsson [2008, Theorem 1.7] shows that the normalization of the main component has a natural moduli interpretation in terms of log geometry. Theorem 1.4 above can therefore be viewed as an analogue of Olsson’s theorem, replacing his use of the toric Hilbert scheme by the Kontsevich space. That is, we show that the normalization of $\mathcal{M}$ carries a moduli interpretation in terms of stable log maps.

Recall that given any collection of discrete data $\Gamma = (g, \beta, n, \{c_i\})_{i=1}^n$, it is shown in [Chen 2011; Abramovich and Chen 2011; Gross and Siebert 2013] that there is a proper Deligne–Mumford stack $\mathcal{H}_{\Gamma}(X)$ that parametrizes stable log maps to $X$ from genus-$g$ curves with $n$ marked points having curve class $\beta$ and contact orders given by the $c_i$.\footnote{Strictly speaking, [Chen 2011; Abramovich and Chen 2011] only consider log schemes that are generalized Deligne–Faltings (see Definition A.1), so to apply their theory, one must first show that the natural log structure on $X$ satisfies this hypothesis. This is done in Proposition A.4, which we relegate to the Appendix since the theory developed in [Gross and Siebert 2013] is already known to apply to toric varieties.} We show in Proposition 2.1 that if $g = 0$, then $\mathcal{H}_{\Gamma}(X)$ is log smooth and in particular normal. This is a key ingredient in the proof of Theorem 1.4, which we give in Section 2. In Section 3, following [Nishinou and Siebert 2006; Gross and Siebert 2013], we explain the relationship between tropical curves and stable log maps to toric varieties. While the use of tropical curves is not strictly necessary for this paper, they serve as a convenient tool to study the boundary of $\mathcal{H}_{\Gamma}(X)$. Theorem 1.1 is then proved in Section 4.

**Remark 1.6.** One of the purposes of the theory of stable log maps is to define and compute Gromov–Witten invariants with tangency conditions. The authors plan to calculate the Gromov–Witten invariants in the case of this paper once the forthcoming paper [Abramovich et al. ≥ 2013] is ready to use; this latter paper will carefully treat the virtual cycle of the space of stable log maps as well as a version of the degeneration formula of Gromov–Witten invariants.

**Prerequisites.** We assume the reader is familiar with logarithmic geometry in the sense of Fontaine, Illusie, and Kato (see for example [Kato 1989] or [Ogus 2006]).

### 2. Log smoothness and irreducibility

Throughout this section, $X$ is a projective normal toric variety of dimension $d$ and $\Gamma$ is an arbitrary choice of discrete data $(0, \beta, n, \{c_i\})$. Let $T$ be the defining torus of $X$ and $M$ be the character lattice of $T$.

**Proposition 2.1.** $(\mathcal{H}_{\Gamma}(X), \mathcal{M}_{\mathcal{H}_{\Gamma}(X)})$ is log smooth over $(k, \mathcal{O}_k^*)$. Also, $\dim \mathcal{H}_{\Gamma}(X) = \dim X + n - 3$. 


Proof. The universal curve on $\mathcal{H}_\Gamma(X)$ induces a morphism of log stacks

$$\pi : (\mathcal{H}_\Gamma(X), \mathcal{M}_{\mathcal{H}_\Gamma(X)}) \to (\mathcal{M}_{0,n}, \mathcal{M}_{M_{0,n}}),$$

where $(\mathcal{M}_{g,n}, \mathcal{M}_{M_{g,n}})$ denotes the log stack of $(g, n)$-prestable curves; see [Kato 2000] and [Olsson 2007, Theorem 1.10] for the definition and construction of this log stack. Since $(\mathcal{M}_{g,n}, \mathcal{M}_{M_{g,n}})$ is log smooth over $(k, \mathcal{O}_k)$, it suffices to show that $\pi$ is log smooth. By [Olsson 2003, Theorem 4.6], this is equivalent to showing that the induced morphism

$$\pi' : \mathcal{H}_\Gamma(X) \to \text{Log}(\mathcal{M}_{0,n}, \mathcal{M}_{M_{0,n}})$$

of stacks is smooth, where $\text{Log}(S, \mathcal{M}_S)$ is the stack of log morphisms to a log scheme $(S, \mathcal{M}_S)$ as defined in the introduction of [loc. cit.].

Let $i : \text{Spec } A \to \text{Spec } A'$ be a square zero thickening of Artin local rings, and let

$$\begin{array}{ccc}
\text{Spec } A & \longrightarrow & \mathcal{H}_\Gamma(X) \\
i & \downarrow & \downarrow \pi' \\
\text{Spec } A' & \longrightarrow & \text{Log}(\mathcal{M}_{0,n}, \mathcal{M}_{M_{0,n}})
\end{array}$$

be a commutative diagram. We may view this as a commutative diagram of log stacks by endowing the Artin local rings with the log structure pulled back from $\text{Log}(\mathcal{M}_{0,n}, \mathcal{M}_{M_{0,n}})$. Hence, the two vertical arrows are strict. Denote the induced log structures on $\text{Spec } A$ and $\text{Spec } A'$ by $\mathcal{M}_A$ and $\mathcal{M}_{A'}$, respectively. We therefore have a log smooth curve $h'$, a cartesian diagram

$$\begin{array}{ccc}
(C, \mathcal{M}_C) & \longrightarrow & (C', \mathcal{M}_{C'}) \\
h & \downarrow & \downarrow h' \\
(\text{Spec } A, \mathcal{M}_A) & \longrightarrow & (\text{Spec } A', \mathcal{M}_{A'})
\end{array}$$

and a minimal stable log map $f : (C, \mathcal{M}_C) \to (X, \mathcal{M}_X)$, which we must show deforms to a minimal stable log map $f' : (C', \mathcal{M}_{C'}) \to (X, \mathcal{M}_X)$. Since the minimality condition is open by [Chen 2011, Proposition 3.5.2], it suffices to show that $f$ deforms as a morphism of log schemes.

By standard arguments in deformation theory, it is enough to consider the case where the kernel $\mathcal{I}$ of $A' \to A$ is principal and killed by the maximal ideal $m$ of $A'$. Then the obstruction to deforming $f$ to a morphism of log schemes lies in

$$\text{Ext}^1(f_0^*\Omega^1_{(X, \mathcal{M}_X)/k, \mathcal{O}_C}, \mathcal{O}_C) \otimes_{\mathcal{O}_C} \mathcal{I},$$

where $f_0$ denotes the reduction of $f$ mod $m$ and $C_0$ denotes the fiber of $C$ over $A'/m = k$. By [Kato 1996, Example 5.6], $\Omega^1_{(X, \mathcal{M}_X)/k} \simeq \mathcal{O}_X \otimes_{\mathcal{O}_X} M$. Therefore,
\[
\text{Ext}^1(f_0^*\Omega^1_{(X,\mu_X)/k}, \mathcal{O}_{C_0}) = H^1(\mathcal{O}_{C_0}^d) = 0,
\]
where the last equality holds because \( C_0 \) is a curve of arithmetic genus 0. This shows that \( \mathcal{H}_\Gamma(X), \mathcal{M}_{\mathcal{H}_\Gamma(X)} \) is log smooth.

To prove the claim about the dimension of \( \mathcal{H}_\Gamma(X) \), note that
\[
\dim \text{Ext}^0(f_0^*\Omega^1_{(X,\mu_X)/k}, \mathcal{O}_{C_0}) = \dim H^0(\mathcal{O}_{C_0}^d) = d,
\]
and so \( \pi \) has relative dimension \( d \). Since \( \dim \mathcal{M}_{0,n} = n - 3 \), we see \( \dim \mathcal{H}_\Gamma(X) = d + n - 3 \).

Let \( \mathcal{H}_\Gamma^0(X) \) denote the nondegeneracy locus, that is, the locus of \( \mathcal{H}_\Gamma(X) \) where the log structure \( \mathcal{M}_{\mathcal{H}_\Gamma(X)} \) is trivial. By Proposition 2.1 and [Nizioł 2006, Proposition 2.6], \( \mathcal{H}_\Gamma^0(X) \) is an open dense subset of \( \mathcal{H}_\Gamma(X) \). Consider the Kontsevich moduli space of stable maps \( \mathcal{M}_{0,n}(X, \beta) \). The forgetful map
\[
\Phi : \mathcal{H}_\Gamma(X) \to \mathcal{M}_{0,n}(X, \beta)
\]
sending a stable log map to its underlying stable map induces a locally closed immersion
\[
\mathcal{H}_\Gamma^0(X) \to \mathcal{M}_{0,n}(X, \beta).
\]
Since the forgetful map does not change the underlying markings or the underlying maps, no stabilization of the underlying curve is needed here. Let \( \mathcal{M}_{\Gamma}(X) \) be the closure of \( \mathcal{H}_\Gamma^0(X) \) in \( \mathcal{M}_{0,n}(X, \beta) \). Then \( \Phi \) factors through a morphism
\[
\phi : \mathcal{H}_\Gamma(X) \to \mathcal{M}_{\Gamma}(X).
\]

**Lemma 2.2.** \( \phi \) is the normalization map.

**Proof.** By [Abramovich and Chen 2011, Corollary 3.10] and Proposition A.4, the morphism \( \Phi \) is representable and finite and so is \( \phi \). Since \( \mathcal{H}_\Gamma(X), \mathcal{M}_{\mathcal{H}_\Gamma(X)} \) is fs and log smooth over \( (k, \mathcal{O}_k^d) \) by Proposition 2.1, it follows that \( \mathcal{H}_\Gamma(X) \) is normal. Since \( \phi \) is an isomorphism over \( \mathcal{H}_\Gamma^0(X) \), it is birational, and so by Zariski’s main theorem, \( \phi \) is the normalization map. \( \square \)

Now we consider the case \( \Gamma = (0, \beta, 2, \{c_0, c_\infty\}) \), where \( \beta \) is an arbitrary curve class, and \( c_0 \) and \( c_\infty \) are two arbitrary contact orders along the two different markings. Note that both \( c_0 \) and \( c_\infty \) are nontrivial. Otherwise, there is a curve in toric variety intersect the boundary at only one point, which is impossible. Then we have the following result:

**Proposition 2.3.** (1) If \( \mathcal{H}_\Gamma(X) \neq \emptyset \), then \( \Gamma = \Gamma_0 \), for some \( \Gamma_0 \) as in (1-2), obtained from a group morphism (1-1).

(2) \( \mathcal{H}_{\Gamma_0}(X) \) is irreducible.
We see that for each $i$, as discussed in the introduction, we have an immersion which again follows from the above statement. □

$c$ where $U \subset X$

Lemma 2.4.

Since $\phi(0)$ induces a map $d$ dimension $/H5111$ has the same dimension as $Zariski’s$ main theorem shows that it is the normalization map.

Proof. As in the proof of Lemma 2.2, the equation $f*(e_i) = t^{c_i}a_i$ shows that $a_i = 1$. Note that such $f$ defines a group morphism $\iota_0$ as in (1-1). This implies that first statement.

To prove the second statement, it is enough to show that $\mathcal{H}^0_\Gamma(X)$ is irreducible, which again follows from the above statement.

Now observe that the point $1 \in T \subset U$ is given by $e_i = 1$ for all $i$. Since $f(1) = 1$, the equation $f*(e_i) = t^{c_i}a_i$ shows that $a_i = 1$. Note that such $f$ defines a group morphism $\iota_0$ as in (1-1). This implies that first statement.

To prove the second statement, it is enough to show that $\mathcal{H}^0_\Gamma(X)$ is irreducible, which again follows from the above statement.

Now we set $\Gamma = \Gamma_0$ as in (1-2) and use the setting and notation of the introduction. As discussed in the introduction, we have an immersion $T' \to \mathcal{H}^0_\Gamma(X)$. Let $X_\Gamma$ be the closure of $T'$ in $\mathcal{H}^0_\Gamma(X)$. The forgetful morphism $\Phi$ then induces a map $\phi' : X_\Gamma \to \mathcal{M}$.

Since $\mathcal{H}^0_\Gamma(X)$ is irreducible, Theorem 1.4 follows from the next lemma.

**Lemma 2.4.** $X_\Gamma$ is an open substack of $\mathcal{H}^0_\Gamma(X)$, and so $\phi'$ is the normalization map.

Proof. As in the proof of Lemma 2.2, $\phi'$ is representable and finite. If $X_\Gamma$ is an open substack of $\mathcal{H}^0_\Gamma(X)$, it is then normal. Since $\phi'$ is an isomorphism over $T'$, Zariski’s main theorem shows that it is the normalization map.

To show that $X_\Gamma$ is open in $\mathcal{H}^0_\Gamma(X)$, it suffices to prove that $X^0_\Gamma := X_\Gamma \cap \mathcal{H}^0_\Gamma(X)$ has the same dimension as $\mathcal{H}^0_\Gamma(X)$. Since $T'$ is dense in $X_\Gamma$, we see that $X_\Gamma$ has dimension $d - 1$. On the other hand, the map $\pi : \mathcal{H}^0_\Gamma(X), \mathcal{M}(\mathcal{H}^0_\Gamma(X)) \to (\mathcal{M}_{0,2}, \mathcal{M}_{0,2})$ in the proof of Proposition 2.1 induces a map $\mathcal{H}^0_\Gamma(X) \to \mathcal{M}_{0,2}$. 

where \( \mathcal{M}_{0,2}^\circ \) denotes the open substack of \( \mathcal{M}_{0,2} \) with smooth fiber curves. By Proposition 2.1, we see that \( \mathcal{H}_T^\circ(X) \) has dimension \( d - 1 \).

### 3. Tropical curves associated to stable log maps

The goal of this section is to prove Proposition 3.8. Following [Nishinou and Siebert 2006; Gross and Siebert 2013], we explain the connection between tropical curves and stable log maps to toric varieties.

#### 3.1. Review of tropical curves.

Let \( G \) be the geometric realization of a weighted, connected finite graph with weight function \( \omega \). That is, \( G \) is the CW complex associated to a finite connected graph with vertex set \( G^{[0]} \) and edge set \( G^{[1]} \), and

\[
\omega : G^{[1]} \to \mathbb{N}
\]

is a function. Here we allow \( G \) to have divalent vertices. Given an edge \( l \in G^{[1]} \), we denote its set of adjacent vertices by \( \partial l \). If \( l \) is a loop, then we require \( \omega(l) = 0 \).

Let \( G^{[0]} \subset G^{[0]}_\infty \) be the set of one-valent vertices, and let

\[
G := G \setminus G^{[0]}_\infty.
\]

Let \( G^{[1]}_\infty \) be the set of noncompact edges in \( G \), which we refer to as unbounded edges. A flag of \( G \) is a pair \((v, l)\) where \( l \) is an edge and \( v \in \partial l \). We let \( FG \) be the set of flags of \( G \), and for each vertex \( v \), we let

\[
FG(v) := \{(v, l) \in FG\}.
\]

Let \( N \) be a lattice and \( M = N^\vee \). We let \( N_\mathbb{Q} := N \otimes \mathbb{Z} \mathbb{Q} \) and \( N_\mathbb{R} := N \otimes \mathbb{Z} \mathbb{R} \).

**Definition 3.2.** A parametrized tropical curve in \( N_\mathbb{Q} \) is a proper map \( \varphi : G \to N_\mathbb{R} \) of topological spaces satisfying the following conditions:

1. For every edge \( l \) of \( G \), the restriction \( \varphi|_l \) acts as dilation by a factor \( \omega(l) \) with image \( \varphi(l) \) contained in an affine line with rational slope. If \( \omega(l) = 0 \), then \( \varphi(l) \) is a point.

2. For every vertex \( v \) of \( G \), we have \( \varphi(v) \in N_\mathbb{Q} \).

3. For each \((v, l) \in FG(v)\), let \( u_{v,l} \) be a primitive integral vector emanating from \( \varphi(v) \) along the direction of \( h(l) \). Then

\[
\epsilon_v := \sum_{(v,l) \in FG(v)} \omega(l)u_{v,l} = 0,
\]

which we refer to as the balancing condition.

An isomorphism of tropical curves \( \varphi : G \to N_\mathbb{R} \) and \( \varphi' : G' \to N_\mathbb{R} \) is a homeomorphism \( \Phi : G \to G' \) compatible with the weights of the edges such that \( \varphi = \varphi' \circ \Phi \).

A tropical curve is an isomorphism class of parametrized tropical curves.
3.3. Tropical curves from nondegenerate stable log maps. Let \((X, \mathcal{M}_X)\) be a toric variety with its standard log structure, and let \(T \subset X\) be its defining torus. We denote by \(N\) the lattice of one-parameter subgroups of \(T\). Let \(f : (C, \mathcal{M}_C) \to (X, \mathcal{M}_X)\) be a stable log map over \((S, \mathcal{M}_S)\) with \(S\) a geometric point. Further assume that \(f\) is nondegenerate; that is, the log structure \(\mathcal{M}_S\) is trivial.

In this subsection, we show how to assign a tropical curve \(\mathrm{Trop}(f) : G \to N\) to any such nondegenerate stable log map \(f\). Note that in this case, the points on the source curve with nontrivial log structures are marked points or nodal points.

To begin, let \(G\) be the graph with a single vertex \(v\), which we think of as being associated to the unique component of \(C\), and with one unbounded edge for each marked point of \(C\). We let \(\mathrm{Trop}(f)(v) = 0\).

Let \(l\) be an edge corresponding to a marked point \(p\) of \(C\). If \(p\) has trivial contact orders, then we set \(\omega(l) = 0\) and let \(\mathrm{Trop}(f)\) contract \(l\) to 0. Otherwise, the contact order is equivalent to giving a nontrivial map \(c_l : \mathcal{M}_{X, f(p)} \to \mathcal{M}_C, p = \mathbb{N}\).

Note that we have a surjective cospecialization map of groups

\[ M := N^\vee \to \mathcal{M}^{\mathrm{gp}}_{X, f(p)} \]

corresponding to the specialization of the generic point of \(T\) to \(f(p)\). Composing with \(c_l^{\mathrm{gp}}\), we obtain a map

\[ \mu_l : M \to \mathbb{Z}, \]

which defines an element \(\mu_l \in N\). Let \(u_l\) be the primitive vector with slope given by \(\mu_l \in N\). We define \(\omega(l)\) to be the positive integer such that \(\mu_l = \omega(l)u_l\) and define the image \(\mathrm{Trop}(f)(l)\) to be the unbounded ray emanating from 0 along the direction of \(u_l\). This defines our desired map \(\mathrm{Trop}(f) : G \to N\) up to reparametrization.

**Proposition 3.4.** \(\mathrm{Trop}(f) : G \to N\) defines a tropical curve.

**Proof.** It remains to check that the balancing condition holds. That is, we must show \(\epsilon_v = 0\). Note that every \(m \in M\) defines a rational function on \(C\) and that the degree of the associated Cartier divisor is \(0 = \epsilon_v(m)\). Therefore, \(\epsilon_v \in N = M^\vee\) is 0. \(\square\)

3.5. Tropical curves from stable log maps over the standard log point. Suppose \((X, \mathcal{M}_X)\) is a toric variety with its standard log structure, and let \(T \subset X\) be its defining torus. Fix discrete data \(\Gamma = (g, \beta, n, \{c_i\})\), and let \(f : (C, \mathcal{M}_C) \to (X, \mathcal{M}_X)\) be a stable log map with discrete data \(\Gamma\) over the standard log point \((S, \mathcal{M}_S)\); that is, \(S\) is a geometric point and \(\mathcal{M}_S\) is the log structure associated to the map \(\mathbb{N} \to \mathcal{O}_S\) sending 1 to 0. This is equivalent to giving a (not necessarily strict) log map

\[ (S, \mathcal{M}_S) \to (\mathcal{K}_\Gamma(X), \mathcal{M}_\mathcal{K}_\Gamma(X)) \],
and the stable log map $f$ is obtained by pulling back the universal stable log map over $(\mathcal{M}_\Gamma(X), \mathcal{M}_\Gamma(X))$. In this subsection, we associate a tropical curve

$$\text{Trop}(f) : G \to N_{\mathbb{R}}$$

to $f$ by modifying the construction given in [Gross and Siebert 2013, §1.3].

We define $G$ to be the dual graph of $C$ where we attach an unbounded edge for each marked point. Given a vertex $v$, let $t$ be the generic point of the corresponding component of $C$. We therefore have a morphism

$$\overline{\mathcal{M}}_{X, f(t)} \to \overline{\mathcal{M}}_{C, t} = \mathbb{N}$$

of monoids. Taking the associated groups and composing with the cospecialization map $M \to \overline{\mathcal{M}}_{X, f(t)}^\text{gp}$ yields a map

$$\tau_v : M \to \mathbb{Z}$$

and hence a point in $N$. We define $\text{Trop}(f)(v) = \tau_v$.

Let $l$ be an edge of $G$. If $\partial l = \{v, v'\}$ and $v \neq v'$, then we define the image of $l$ under $\text{Trop}(f)$ to be the line segment joining $\tau_v$ and $\tau_{v'}$. In this case, $\tau_{v'} - \tau_v = e_l \mu_l$, where $e_l \in \overline{\mathcal{M}}_S = \mathbb{N}$ is the section that smooths the node corresponding to $l$, and $\mu_l$ is an element of $N$. We define $\omega(l)$ to be the positive integer such that $\mu_l = \omega(l) u_l$, where $u_l$ is a primitive integral vector.

Suppose now that $l$ is an unbounded edge corresponding to a marked point $p$. If $p$ has trivial contact orders, then we set $\omega(l) = 0$ and let $\text{Trop}(f)$ contract $l$ to $\tau_v$, where $\partial l = \{v\}$. Otherwise, the contact orders of $p$ define a nontrivial map

$$e_l : \overline{\mathcal{M}}_{X, f(p)} \to \overline{\mathcal{M}}_{C, p} = \mathbb{N} \oplus \overline{\mathcal{M}}_S \to \mathbb{N},$$

where the last map is the projection. Again taking the associated groups and composing with the cospecialization map $M \to \overline{\mathcal{M}}_{X, f(p)}^\text{gp}$, we obtain

$$\mu_l : M \to \mathbb{Z}.$$  

We define $\omega(l)$ to be the positive integer such that $\mu_l = \omega(l) u_l$, where $u_l \in N$ is a primitive integral vector, and we let $\text{Trop}(f)(l)$ be the unbounded ray emanating from $\tau_v$ in the direction of $u_l$.

**Proposition 3.6.** $\text{Trop}(f) : G \to N_{\mathbb{R}}$ defines a tropical curve.

**Proof.** We must check that the balancing condition holds for each vertex $v$ of $G$. As in the proof of Proposition 3.4, every $m \in M$ defines a rational function on the irreducible component of $C$ corresponding to $v$. The associated Cartier divisor has degree $0 = \epsilon_v(m)$, and so $\epsilon_v = 0$; see [Gross and Siebert 2013, Proposition 1.14]. □
we produce an infinite sequence of distinct edges $l$ with the map
where the last map is the projection. Taking associated groups and precomposing $Trop$ points. Hence, $C$ is a contradiction.

Since $\mu$ is unbounded, then a segment or ray parallel to $\mu$ have nontrivial contact orders. The balancing condition for $Trop$ the dual graphs of $C$ is an embedding whose image is a line. Moreover, $C$ is a chain of $\mathbb{P}^1$s and $f$ does not contract any components of $C$.

**Proposition 3.8.** If the discrete data $\Gamma$ is given by $g = 0$, $n = 2$, and $\beta \neq 0$, then $Trop(f)$ is an embedding whose image is a line. Moreover, $C$ is a chain of $\mathbb{P}^1$s and $f$ does not contract any components of $C$.

**Proof.** Since $\mathcal{X}_\Gamma(X)$ is log smooth by Proposition 2.1, there exists a stable log map $h : (\mathcal{E}, \mathcal{M}_\mathcal{E}) \to (X, \mathcal{M}_X)$ over $(R, \mathcal{M}_R)$ as in Remark 3.7. Let $p, p' : \text{Spec } R \to \mathcal{E}$ be the two marked sections, and let $l_0, l'_0, l_\eta$, and $l'_\eta$ be the corresponding edges of the dual graphs of $C$ and $\mathcal{E}_\eta$. Since $\beta \neq 0$, the two marked points $p_\eta$ and $p'_\eta$ of $\mathcal{E}_\eta$ have nontrivial contact orders. The balancing condition for $Trop(h_\eta)$ then shows $\mu_{l_\eta} = -\mu_{l'_\eta} \neq 0$. By Remark 3.7, we therefore have $\mu_{l'_0} = -\mu_{l_0} \neq 0$. In particular, $Trop(f)$ maps $l_0$ and $l'_0$ to unbounded rays.

We next show that if $l$ is an edge of $G$, then $Trop(f)(l)$ is a point or it is a line segment or ray parallel to $\mu_{l_0}$. Suppose $Trop(f)(l)$ is not a point. If $Trop(f)(l)$ is unbounded, then $l$ is $l_0$ or $l'_0$, and so $Trop(f)(l)$ is parallel to $\mu_{l_0}$. Otherwise, $Trop(f)(l)$ is a line segment and $\partial l = \{v, v_1\}$ with $v \neq v_1$. If $Trop(f)(l)$ is not parallel to $\mu_{l_0}$, then the balancing condition shows that there is an edge $l_1 \neq l$ such that $v_1 \in \partial l_1$ and $Trop(f)(l_1)$ is not parallel to $\mu_{l_0}$. Hence, $l_1$ is a line segment with endpoints $v_1$ and $v_2$. Again, the balancing condition shows that there is an edge $l_2$ containing $v_2$ such that $Trop(f)(l_2)$ is a line segment which is not parallel to $\mu_{l_0}$. Since $C$ has genus 0, we see $l, l_1$, and $l_2$ are distinct. Continuing in this manner, we produce an infinite sequence of distinct edges $l_i$ of the dual graph of $C$. This is a contradiction.

Lastly, we show that every irreducible component $A$ of $C$ has exactly two special points. Hence, $C$ is a chain of $\mathbb{P}^1$s, $f$ does not contract any component of $C$,
and Trop(f)(G) is a line parallel to µ_{i_0}. Suppose A is a component with at least three special points, and let v be the vertex of G corresponding to A. Then G \ v is a disjoint union of nonempty trees T_1, T_2, ..., T_m with m ≥ 3. Without loss of generality, T_1 only contains bounded edges. The argument in the preceding paragraph then shows that Trop(f) maps every edge of T_1 to a single point. If \( C_1 \) denotes the subcurve of \( C \) corresponding to \( T_1 \), then we see that every special point of \( C_1 \) has a trivial contact order, and so f contracts \( C_1 \). Since \( T_1 \) is a tree, \( C_1 \) contains components with only two special points. This contradicts the stability of f. □

4. The Chow quotient as the coarse moduli space

Throughout this section, we let \( \Gamma = \Gamma_0 \) and \( C(X) \) denote the Chow variety as in the introduction. Let \( K \) be the normalization of \( X // T_0 \). Since the stack \( \mathcal{H}_\Gamma(X) \) is normal, it follows from [Kollár 1996, Chapter I, 3.17 and 3.21] that there is a map

\[
F : \mathcal{H}_\Gamma(X) \to C(X)
\]

sending a stable log map \( f : (C, M_C) \to (X, M_X) \) to the image cycle \( f_*[C] \). Since \( \mathcal{H}_\Gamma(X) \) is irreducible by Theorem 1.4, \( F \) factors as

\[
\mathcal{H}_\Gamma(X) \xrightarrow{F'} X // T_0 \xrightarrow{i} C(X),
\]

where \( i \) is the natural inclusion. Since \( F \) is an isomorphism over \( T' \) and \( \mathcal{H}_\Gamma(X) \) is normal, by Proposition 2.1, we obtain an induced morphism

\[
G : \mathcal{H}_\Gamma(X) \to K.
\]

To prove Theorem 1.1, we show:

**Proposition 4.1.** \( G \) is a coarse space morphism.

**Proof.** Note that both \( \mathcal{H}_\Gamma(X) \) and \( K \) are normal and proper, and \( G \) is bijective on the level of closed points over \( T' \). To show that \( K \) is the coarse moduli space of \( \mathcal{H}_\Gamma(X) \), by Zariski’s main theorem, it suffices to show \( G \) is quasifinite. To do so, it is enough to show \( F' \) is quasifinite at the level of closed points. That is, we show that if \( x \in X // T_0 \) is a closed point and \( E_x \) denotes the corresponding cycle of \( X \), then there are finitely many stable log maps whose image cycles are given by \( E_x \). Let

\[
E_x = \sum a_i Z_i,
\]

where the \( a_i \) are positive integers and the \( Z_i \) are reduced irreducible closed sub-schemes of \( X \). Let \( \widetilde{Z}_i \) be the normalization of \( Z_i \). Since \( E_x \) is of dimension 1, we have \( \widetilde{Z}_i \simeq \mathbb{P}^1 \).

We claim that if \( f : (C, M_C) \to (X, M_X) \) is a stable log map that defines a closed point of \( \mathcal{H}_\Gamma(X) \) such that the image cycle of \( f \) is \( E_x \), then \( f \) can only be
ramified at the special points of $C$. Given this claim, $F'$ is quasifinite. Indeed, since Proposition 3.8 shows that no component of $C$ is contracted under $f$, the number of irreducible components of $C$ is bounded by $\sum a_i$. For each irreducible component $A$ of $C$, the restriction $f|_A$ factors as

$$A \to \tilde{Z}_i \to X$$

for some $i$. Since the first map $A \to \tilde{Z}_i$ can only be ramified at the two fixed special points, it is determined by the degree of $f|_A$. This implies that there are only finitely many choices for the underlying map $C \to X$. Since the forgetful morphism $\Phi : \mathcal{H}_\Gamma(X) \to \mathcal{M}_{0,2}(X, \beta)$ is finite, there are finitely many choices for the stable log map $f$.

It remains to prove the claim. By Proposition 2.3, $\mathcal{H}_\Gamma(X)$ is irreducible and $T'$ is dense, so there exists a toric morphism $\mathbb{A}^1 \to \mathcal{H}_\Gamma(X)$ such that the fiber over $0 \in \mathbb{A}^1$ is our given stable log map $f : (C, M_C) \to (X, M_X)$ whose image cycle is $E_X$. Let $R$ denote the complete local ring $\mathcal{O}_{\mathbb{A}^1,0}$ and let

$$\begin{array}{ccc}
\mathcal{C} & \overset{\mathcal{C}}{\overset{h}{\longrightarrow}} & X \\
\Spec R & \downarrow & \\
\end{array}$$

be the associated underlying stable map. Let $\eta \in \Spec R$ be the generic point.

We first handle the case when $X$ is smooth. Let $\mathcal{C}^0$ be the open subset of $\mathcal{C}$ obtained by removing the special points. Note that $\mathcal{C}^0$ is normal, and $h|_{\mathcal{C}^0}$ is quasifinite by Proposition 3.8. By the purity of the branch locus theorem [Altman and Kleiman 1971, p. 461], if $h|_{\mathcal{C}^0}$ is ramified, then the ramification locus $D$ is pure of codimension 1. Since $h|_{\mathcal{C}^0}$ is not everywhere ramified over the central fiber, $D$ must intersect the generic fiber. However, $h|_{\mathcal{C}^0}$ is unramified over the generic fiber, so we conclude that $D$ is empty.

We now consider the case when $X$ is singular. Let $p : \tilde{X} \to X$ be a toric resolution. We may replace $R$ by a ramified extension as this does not affect the set of closed points. By the properness of $\mathcal{H}_\Gamma(X)$, we can assume we have a stable log map $\tilde{h} : (\mathcal{C}, M_{\mathcal{C}}) \to (\tilde{X}, M_{\tilde{X}})$ and a commutative diagram of the underlying maps

$$\begin{array}{ccc}
\mathcal{C} & \overset{\mathcal{C}}{\overset{\tilde{h}}{\longrightarrow}} & \tilde{X} \\
\mathcal{C} & \overset{\mathcal{C}}{\overset{h}{\longrightarrow}} & X \\
\end{array}$$

over $R$. Here $h$ is the underlying map of the stable log map to $X$, which can be also obtained by taking the stabilization of the prestable map $p \circ \tilde{h}$. The previous
paragraph shows that $\tilde{h}$ only ramifies at the special points. Since Proposition 3.8 shows that $\tilde{c}$ and $\tilde{e}$ are both chains of $\mathbb{P}^1$s, we see that $h$ only ramifies at the special points as well. □

Appendix: Toric varieties have generalized Deligne–Faltings log structures

The theory of moduli spaces of stable log maps $\mathcal{M}_\Gamma(Y, \mathcal{M}_Y)$ is developed in [Chen 2011; Abramovich and Chen 2011] and [Gross and Siebert 2013] for different classes of log schemes $(Y, \mathcal{M}_Y)$. In [Chen 2011; Abramovich and Chen 2011], Abramovich and the first author consider log schemes that are generalized Deligne–Faltings (see Definition A.1); Gross and Siebert [2013] consider log schemes that are quasigenerated Zariski. It is shown in [Abramovich and Chen 2011, Proposition 4.8] that when $(Y, \mathcal{M}_Y)$ is both generalized Deligne–Faltings and quasigenerated Zariski, the Abramovich–Chen and Gross–Siebert constructions are identical. Gross and Siebert show that the standard log structure $\mathcal{M}_X$ on a normal toric variety $X$ is always quasigenerated Zariski. Here we show that if $X$ is also projective, then $\mathcal{M}_X$ is generalized Deligne–Faltings. Therefore, the two theories agree for projective normal toric varieties.

**Definition A.1.** A log structure $\mathcal{M}_Y$ on a scheme $Y$ is called generalized Deligne–Faltings if there exists a fine saturated sharp monoid $P$ and a morphism $P \to \mathcal{M}_Y$ that locally lifts to a chart $P \to \mathcal{M}_Y$.

**Remark A.2.** Given a fine saturated sharp monoid $P$, let $A_P = \text{Spec} \ k[P]$ with its standard log structure $\mathcal{M}_{A_P}$. Then there is a natural action of $T_P := \text{Spec} \ k[P^{gp}]$ on $(A_P, \mathcal{M}_{A_P})$ induced by the morphism $P \to P \oplus P^{gp}$ sending $p$ to $(p, p)$. The log structure $\mathcal{M}_{A_P}$ descends to yield a log structure $\mathcal{M}_{[A_P/T_P]}$ on the quotient stack $[A_P/T_P]$. By [Olsson 2003, Remark 5.15], a log scheme $(Y, \mathcal{M}_Y)$ is generalized Deligne–Faltings if and only if there exists a strict morphism

$$(Y, \mathcal{M}_Y) \to ([A_P/T_P], \mathcal{M}_{[A_P/T_P]})$$

for some fine saturated sharp monoid $P$.

Let $X$ be a projective normal toric variety, and let $\mathcal{M}_X$ be its standard log structure. Let $Q \subset \mathbb{R}^n$ be a polytope associated to a sufficiently positive projective embedding of $X$. Placing $Q$ at height 1 in $\mathbb{R}^n \times \mathbb{R}$ and letting $P$ be the monoid of lattice points in the cone over $Q$, we have $X = \text{Proj} \ k[P]$. Note that $P$ is fine, saturated, and sharp. Let $(A_P, \mathcal{M}_{A_P})$ be as in Remark A.2, let $U$ be the compliment of the closed subscheme of $A_P$ defined by the irrelevant ideal of $k[P]$, and let $\mathcal{M}_U = \mathcal{M}_{A_P} |_U$. The function $\text{deg} : P \to \mathbb{Z}$ sending an element to its height induces a $\mathbb{G}_m$-action on $(A_P, \mathcal{M}_{A_P})$. Hence, $\mathcal{M}_U$ descends to yield a log structure $\mathcal{M}_P$ on $X$.

**Lemma A.3.** $\mathcal{M}_P$ is generalized Deligne–Faltings.
Proof. We have a cartesian diagram

\[
\begin{array}{ccc}
(U, \mathcal{M}_U) & \longrightarrow & (A_P, \mathcal{M}_P) \\
\downarrow & & \downarrow \\
(X, \mathcal{M}_P) & \longrightarrow & ([A_P/\mathbb{G}_m], \mathcal{M}_{[A_P/\mathbb{G}_m]})
\end{array}
\]

where all morphisms are strict and the vertical morphisms are smooth covers. Note that the $\mathbb{G}_m$-action on $(A_P, \mathcal{M}_P)$ is induced from the morphism $\sigma : P \to P \oplus \mathbb{Z}$ defined by $p \mapsto (p, \deg p)$. Since $\sigma$ factors as $P \to P \oplus P^{gp} \to P \oplus \mathbb{Z}$, we see that there is a strict smooth cover

\[
([A_P/\mathbb{G}_m], \mathcal{M}_{[A_P/\mathbb{G}_m]}) \to ([A_P/T_P], \mathcal{M}_{[A_P/T_P]}).
\]

Hence, Remark A.2 shows that $\mathcal{M}_P$ is generalized Deligne–Faltings. 

Note that $\mathcal{M}_P|_T = \mathcal{O}_T^*$, where $T$ is the torus of $X$. We therefore obtain a map

\[
\psi : \mathcal{M}_P \to \log^* \mathcal{O}_T^* =: \mathcal{M}_X.
\]

**Proposition A.4.** $\psi$ is an isomorphism, and so $(X, \mathcal{M}_X)$ is generalized Deligne–Faltings.

**Proof.** To show $\psi$ is an isomorphism, it is enough to look Zariski locally on $X$. Note that $X$ has an open cover by the $X_v := \text{Spec } k[Q_v]$, where $v$ is a vertex of the polytope $Q$ and $Q_v$ is the monoid of lattice points in the cone over $Q - v := \{q - v \mid q \in Q \subset \mathbb{R}^n\}$. Let $P_v$ be the submonoid of $P^{gp}$ generated by $P$ and $-v$. Then we have a cartesian diagram

\[
\begin{array}{ccc}
A_{P_v} & \xrightarrow{i} & U \\
\downarrow \pi & & \downarrow \\
X_v & \longrightarrow & X
\end{array}
\]

where $\pi$ is induced from the map $Q_v \to P_v$ embedding $Q_v$ at height 0 in $P_v$ and where the composite of $i$ and $U \to A_P$ is induced from the inclusion $P \to P_v$. Hence,

\[
\mathcal{M}_{Q_v} = (\mathcal{M}_{P_v})_{\mathbb{G}_m},
\]

and so $\psi$ is an isomorphism over $X_v$. 

Acknowledgments

We would like to thank Dan Abramovich, Dustin Cartwright, Anton Geraschenko, Noah Giansiracusa, and Martin Olsson. We also thank the anonymous referee for helpful comments. Chen was partially supported by the Simons Foundation. Satriano was partially supported by NSF grant DMS-0943832 and an NSF postdoctoral fellowship (DMS-1103788).

References

[Abramovich and Chen 2011] D. Abramovich and Q. Chen, “Stable logarithmic maps to Deligne–Faltings pairs, II”, preprint, 2011. arXiv 1102.4531v2

[Abramovich et al. 2011] D. Abramovich, Q. Chen, W. D. Gillam, and S. Marcus, “The evaluation space of logarithmic stable maps”, preprint, 2011. arXiv 1012.5416v1

[Abramovich et al. ≥2013] D. Abramovich, Q. Chen, M. Gross, and B. Siebert, in preparation.

[Altman and Kleiman 1971] A. Altman and S. L. Kleiman, “On the purity of the branch locus”, Compos. Math. 23 (1971), 461–465. MR 46 #7233 Zbl 0242.14001

[Chen 2011] Q. Chen, “Stable logarithmic maps to Deligne–Faltings pairs, I”, preprint, 2011. arXiv 1008.3090v4

[Gross and Siebert 2013] M. Gross and B. Siebert, “Logarithmic Gromov–Witten invariants”, J. Amer. Math. Soc. 26:2 (2013), 451–510. MR 3011419 Zbl 06168513

[Kapranov et al. 1991] M. M. Kapranov, B. Sturmfels, and A. V. Zelevinsky, “Quotients of toric varieties”, Math. Ann. 290:4 (1991), 643–655. MR 92g:14050 Zbl 0762.14023

[Kato 1989] K. Kato, “Logarithmic structures of Fontaine–Illusie”, pp. 191–224 in Algebraic analysis, geometry, and number theory (Baltimore, 1988), edited by J.-I. Igusa, Johns Hopkins Univ. Press, Baltimore, MD, 1989. MR 99b:14020 Zbl 0776.14004

[Kato 1996] F. Kato, “Log smooth deformation theory”, Tohoku Math. J. (2) 48:3 (1996), 317–354. MR 99a:14012 Zbl 0876.14007

[Kato 2000] F. Kato, “Log smooth deformation and moduli of log smooth curves”, Internat. J. Math. 11:2 (2000), 215–232. MR 2001d:14016 Zbl 1100.14011

[Kollár 1996] J. Kollár, Rational curves on algebraic varieties, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 32, Springer, Berlin, 1996. MR 98c:14001 Zbl 0833161

[Nishinou and Siebert 2006] T. Nishinou and B. Siebert, “Toric degenerations of toric varieties and tropical curves”, Duke Math. J. 135:1 (2006), 1–51. MR 2007h:14083 Zbl 1105.14073

[Nizioł 2006] W. Nizioł, “Toric singularities: log-blow-ups and global resolutions”, J. Algebraic Geom. 15:1 (2006), 1–29. MR 2006i:14015 Zbl 1100.14011

[Ogus 2006] A. Ogus, “Lectures on logarithmic algebraic geometry”, preprint, 2006, available at http://math.berkeley.edu/~ogus/preprints/log_book/logbook.pdf.

[Olsson 2003] M. C. Olsson, “Logarithmic geometry and algebraic stacks”, Ann. Sci. École Norm. Sup. (4) 36:5 (2003), 747–791. MR 2004k:14018 Zbl 1069.14022

[Olsson 2007] M. C. Olsson, “(Log) twisted curves”, Compos. Math. 143:2 (2007), 476–494. MR 2008d:14021 Zbl 1138.14017

[Olsson 2008] M. Olsson, “Logarithmic interpretation of the main component in toric Hilbert schemes”, pp. 231–252 in Curves and abelian varieties, edited by V. Alexeev et al., Contemp. Math. 465, Amer. Math. Soc., Providence, RI, 2008. MR 2009j:14007 Zbl 1153.14007
Chow quotients of toric varieties as moduli of stable log maps

[Peeva and Stillman 2002] I. Peeva and M. Stillman, “Toric Hilbert schemes”, *Duke Math. J.* **111**:3 (2002), 419–449. MR 2003m:14008 Zbl 1067.14005

Communicated by Ravi Vakil
Received 2012-10-22 Revised 2013-02-04 Accepted 2013-03-12

q_chen@math.columbia.edu Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027, United States
satriano@umich.edu Department of Mathematics, University of Michigan, 2074 East Hall, Ann Arbor, MI 48109, United States
Multiplicities associated to graded families of ideals
STEVEN DALE CUTKOSKY

Normal coverings of linear groups
JOHN R. BRITNELL and ATTILA MARÓTI

Modularity of the concave composition generating function
GEORGE E. ANDREWS, ROBERT C. RHoades and SANDER P. Zwegers

Moduli of elliptic curves via twisted stable maps
ANDREW NILES

Regular permutation groups of order \(mp\) and Hopf Galois structures
TIMOTHY KOHL

Further evidence for conjectures in block theory
BENJAMIN SAMBALE

Network parametrizations for the Grassmannian
KELLI TALASKA and LAUREN WILLIAMS

Chow quotients of toric varieties as moduli of stable log maps
QILE CHEN and MATTHEW SATRIANO

Vinberg’s representations and arithmetic invariant theory
JACK A. THORNE