Abstract—Sampling from the lattice Gaussian distribution is an efficient way for solving the closest vector problem (CVP) in lattice decoding. In this paper, decoding by MCMC-based lattice Gaussian sampling is investigated in full details. First of all, the spectral gap of the transition matrix of the Markov chain induced by the independent Metropolis-Hastings-Klein (MHK) algorithm is derived, dictating an exponential convergence rate to the target lattice Gaussian distribution. Then, the decoding complexity of CVP is derived as $O(e^{O(\min\{\|b\|,1\})})$, where $d(\Lambda, c)$ represents the Euclidean distance between the query point $c$ and the lattice $\Lambda$, and $b$ is the $i$th Gram-Schmidt vector of the lattice basis $B$. Furthermore, the decoding radius from the perspective of bounded distance decoding (BDD) given a fixed number of Markov moves $t$ is also derived, revealing a flexible trade-off between the decoding performance and complexity. Finally, by taking advantages of $k$ trial samples from the proposal distribution, the independent multiple-try Metropolis-Klein (MTMK) algorithm is proposed to further enhance the exponential convergence rate. By adjusting $k$, the independent MTMK sampler enjoys a flexible decoding performance, where the independent MHK algorithm is just a case with $k = 1$. Additionally, the proposed decoding allows a fully parallel implementation, which is beneficial for the practical interest.

Keywords: Lattice decoding, lattice Gaussian sampling, Markov chain Monte Carlo, closest vector point, bounded distance decoding, large-scale MIMO detection.

I. INTRODUCTION

As a core problem of lattices, the closest vector problem (CVP) has drawn a lot of research attention in number theory, cryptography, and communications [1]–[6]. In communications, the CVP solvers have been widely applied in multiple-input multiple-output (MIMO) systems for signal detection. However, the optimal solution by enumeration, which corresponds to the sphere decoding (SD) in MIMO detection, tends to be unaffordable in practice due to its prohibitive complexity burden [7]. Even though lattice reduction was introduced to solve CVP approximately, the performance gap between maximum-likelihood (ML) detection and lattice-reduction-aided detection is still substantial especially in high-dimensional systems [8]–[11].

In [12], a sampling strategy was proposed by Klein, which performs lattice decoding by sampling from a discrete Gaussian distribution over lattices. In particular, the premise behind sampler decoding relies on the fact that the proposed lattice Gaussian distribution is centered at the query point $c$ while each lattice point entails a probability scaled by the Euclidean distance from it to $c$. In other words, the closer a lattice point is to $c$, the larger the sampling probability. Hence, if sampling from the underlying lattice Gaussian distribution can be carried out, the desired closest lattice point would conceivably be obtained with the largest sampling probability. Then, after multiple times independent samplings, the optimal solution of CVP will be encountered with a large enough probability [13], [14]. From this perspective, how to successfully collect samples from the lattice Gaussian distribution is the key. In fact, lattice Gaussian sampling is equivalent to CVP via a polynomial-time dimension-preserving reduction [15].

However, in sharp contrast to the continuous Gaussian density, it is by no means trivial even for sampling from a low-dimensional discrete Gaussian distribution, thus leaving a pressing challenge in lattice Gaussian sampling. For this reason, MCMC methods were introduced for lattice Gaussian sampling, which attempt to get samples from the target distribution through a Markov chain [16]. In principle, it randomly generates the next Markov state conditioned on the previous one, and after a burn-in time, which is normally measured by the mixing time, the Markov chain will step into a stationary distribution, where samples from the target distribution can be obtained thereafter [17]. In [18], it has been demonstrated that Gibbs sampling, which employs univariate conditional sampling to build a Markov chain, yields an ergodic Markov chain. Apart from Gibbs sampling, the Metropolis-Hastings (MH) algorithm is also capable of lattice Gaussian sampling. In [16], the independent Metropolis-Hastings Klein (MHK) algorithm is shown to be uniformly ergodic, which converges exponentially fast to the lattice Gaussian distribution. Meanwhile, the associated exponential convergence rate of the Markov chain is also accessible, resulting in a tractable estimation of the mixing time.

In this paper, lattice decoding by MCMC-based Gaussian sampling is studied. First of all, through the spectral gap of the transition matrix, the Markov chain induced by independent MHK algorithm is demonstrated to be uniformly ergodic. Then, we examine its decoding complexity of solving CVP. With a careful selection of the standard deviation during the sampling process, we show the decoding complexity $O(e^{O(\min\{\|b\|,1\})})$ for CVP, where $d(\Lambda, c)$ stands for the distance between the query point $c$ and the lattice $\Lambda$, i.e., $\|Bx - c\|$, and $\hat{b}$ is the $i$th Gram-Schmidt vector of the lattice basis $B$. From the perspective of bounded distance decoding (BDD), the trade-off between the decoding radius and number of Markov moves (corresponds to the incurred complexity) is also established. To further strengthen the convergence rate of the Markov chain, independent multiple-try Metropolis-Klein
(MTMK) is proposed, which fully exploits the trial samples generated from the proposal distribution. The uniform ergodicity is demonstrated and the enhanced exponential convergence rate is also given. Since independent MHK is only a special case of independent MTMK, a framework is established with respect to the number of trial samples.

The proposed decoding allows a parallel implementation and is easily adopted to MIMO detection to achieve near-optimal performance. With the development of millimeter wave in 5G, the demand for large-scale MIMO systems will increase in the next decade, which has triggered research activities towards low complexity decoding algorithms for large-scale MIMO detection. Therefore, there has been considerable interest of sampler decoding, and Gibbs sampling has been introduced into MIMO systems for the efficient decoding [19]–[24]. Nevertheless, the convergence rate of these Gibbs samplers has not been determined yet [25]. Note that different from the lattice Gaussian sampling algorithm in [26], [27] which has a clear lower bound $2^n$ on both the space and time complexity, only polynomial space complexity is required by both the independent MHK and MTMK sampling algorithms.

It should be noticed that the complexity of each Markov move is often insignificant, whereas the required number of Markov moves is more critical. Because of this, we are mostly concerned with the mixing time while the complexity is evaluated by the number of Markov moves. The rest of this paper is organized as follows. Section II introduces the lattice Gaussian distribution and briefly reviews the basics of MCMC. In Section III, we derive the spectral gap of the Markov chain associated with independent MHK algorithm, and show its uniform ergodicity as well as the convergence rate. Then, the decoding complexity of CVP by independent MHK algorithm is derived in Section IV, where the correct decoding radius by means of BDD is also given to establish the flexible trade-off. In Section V, independent MTMK algorithm is proposed to further strengthen the convergence performance. Simulation results with respect to MIMO detection are presented in Section VI. At the end, Section VII concludes the paper.

In practice, as an approximation solution for CVP, lattice reduction is often resorted to as a polynomial complexity decoder. A basis $B$ is said to be Lenstra-Lenstra-Lovász (LLL) reduced if it satisfies the following two conditions,

- $|\mu_{i,j}| \leq \frac{1}{2}$ for $\mu_{i,j} = \frac{\langle \hat{b}_i, b_j \rangle}{\langle b_i, b_j \rangle}$, $1 \leq j < i \leq n$;
- $\kappa \| \hat{b}_i \|^2 \leq \| \mu_{i,i+1} \hat{b}_i + \hat{b}_{i+1} \|^2$, for $1 \leq i < n$.

The LLL reduction based on the above criterions will terminate in a finite number of iterations for any given input basis $B$ [28]. The parameter $\frac{1}{2} \leq \kappa \leq 1$ controls both the convergence speed of the algorithm and the degree of orthogonality of the reduced basis [29].

**Lemma 1** [28]. During the execution of the LLL algorithm, $\max_{1 \leq i \leq n} \| \hat{b}_i \|$ is non-increasing while $\min_{1 \leq i \leq n} \| \hat{b}_i \|$ is non-decreasing.
Clearly, the LLL algorithm tends to reduce the interval $[\min_{1 \leq i \leq n} \| \mathbf{b}_i \|, \max_{1 \leq i \leq n} \| \mathbf{b}_i \|]$ where the lengths of GS vectors reside.

B. Lattice Gaussian Distribution

The Gaussian function centered at $c \in \mathbb{R}^n$ with standard deviation $\sigma > 0$ is defined as

$$
\rho_{\sigma,c}(z) = e^{-\frac{\|z-c\|^2}{2\sigma^2}},
$$

for all $z \in \mathbb{R}^n$. When $c$ or $\sigma$ are not specified, we assume that they are 0 and 1 respectively. Then, with respect to lattice $\Lambda$, the discrete Gaussian distribution over it is defined as

$$
D_{\Lambda,\sigma,c}(x) = \frac{\rho_{\sigma,c}(Bx)}{\rho_{\sigma,c}(\Lambda)} = \frac{e^{-\frac{\|Bx-c\|^2}{2\sigma^2}}}{\sum_{x \in \mathbb{Z}^n} e^{-\frac{\|Bx-c\|^2}{2\sigma^2}}},
$$

for all $x \in \mathbb{Z}^n$, where $\rho_{\sigma,c}(\Lambda) \triangleq \sum_{B \in \Lambda} \rho_{\sigma,c}(Bx)$ is just a scaling to yield a probability distribution. We remark that this definition differs slightly from the one in [4], where $\sigma$ is scaled by a constant factor $\sqrt{2\pi}$ (i.e., $s = \sqrt{2\pi}\sigma$). In fact, the discrete Gaussian resembles a continuous Gaussian distribution, but is only defined over a lattice. It has been shown that discrete and continuous Gaussian distributions share similar properties, if the flatness factor is small [30].

Intuitively, the CVP shown in (2) can be solved by the lattice Gaussian sampling. Since the distribution is centered at the query point $c$, the closest lattice point $Bx$ to $c$ is assigned with the largest sampling probability by default. Therefore, by multiple independent samplings, the solution of CVP is most likely to be returned. By adjusting the sample size, Meanwhile, the sample size can be sampling decoder enjoys a flexible trade-off between performance and complexity. In [12], Klein’s algorithm was introduced for approximate sampler decoding, which performs the sampling over a Gaussian-like distribution. It is shown in [12]–[14] that Klein’s algorithm is able to find the closest lattice point when it is very close to the input vector: this technique is known as bounded-distance decoding (BDD) in coding literature, which corresponds to a reduced variant of CVP.

In coding theory, the performance of BDD is captured by the correct decoding radius, which is defined as the radius of a sphere centered at the lattice point within which decoding is guaranteed to be correct. On the MIMO communications front, the transmitted signal can be successfully recovered when the noise norm is small compared to the correct decoding radius of the decoder. In other words, BDD actually serves as a suboptimal decoding strategy that enjoys lower complexity compared to ML decoding.

C. Classical MH Algorithms

The origins of Metropolis algorithm can be traced back to the celebrated work of [31] in 1950’s. In [32], the original Metropolis algorithm was successfully extended to a more general scheme known as the Metropolis-Hastings (MH) algorithm. In particular, let us consider a target invariant distribution $\pi$ together with a proposal distribution $q(x,y)$. Given the current state $x$ for Markov chain $X_t$, a state candidate $y$ for

Algorithm 1 Klein’s Algorithm

| Input: B, $\sigma$, c |
|----------------------|
| Output: Bx $\in \Lambda$ |

1. let $B = QR$ and $c' = Q^Tc$
2. for $i = n, \ldots , 1$ do
3. let $\sigma_i = \frac{\sigma}{\rho} \parallel x_i \parallel$ and $\tilde{x}_i = e^{-\frac{\sum_{j \neq i} r_j x_j}{\rho}}$
4. sample $x_i$ from $D_{z,\sigma_i,\tilde{x}_i}$
5. end for
6. return Bx

the next Markov move $X_{t+1}$ is generated from the proposal distribution $q(x,y)$. Then the acceptance ratio $\alpha$ is computed by

$$
\alpha = \min \left\{ \frac{\pi(y)q(y,x)}{\pi(x)q(x,y)} \right\},
$$

and $y$ will be accepted as the new state with probability $\alpha$. Otherwise, $x$ will be retained. In this way, a Markov chain $\{X_0, X_1, \ldots \}$ is established with the transition probability $P(x,y)$ as follows:

$$
P(x,y) = \begin{cases} q(x,y)\alpha & \text{if } y \neq x, \\ 1 - \sum_{z \neq x} q(x,z)\alpha & \text{if } y = x. \end{cases}
$$

It is interesting that in MH algorithms, the proposal distribution $q(x,y)$ can be any fixed distribution from which we can conveniently draw samples. Therefore, there is a large flexibility in the choice of $q(x,y)$ and it is a challenge to find a suitable one to obtain the satisfactory convergence. In fact, Gibbs sampling can be viewed as a special case of MH algorithm, whose proposal distribution is a univariate conditional distribution.

On the other hand, as an important parameter which measures the time required by a Markov chain to get close to its stationary distribution, the mixing time is defined as [17]

$$
t_{\text{mix}}(\epsilon) = \min \{ t : \max \parallel P^t(x,\cdot) - \pi(\cdot) \parallel_{TV} \leq \epsilon \},
$$

where $P^t(x,\cdot)$ denotes a row of the transition matrix $P$ for $t$ Markov moves and $\parallel \cdot \parallel_{TV}$ represents the total variation distance.

D. Independent MHK Algorithm

From the MCMC perspective, lattice Gaussian distribution can be viewed as a complex target distribution lacking of direct sampling methods. In order to obtain samples from $D_{\Lambda,\sigma,c}(x)$, the independent MHK sampling was proposed in [34]. Specifically, a state candidate $y$ for the next Markov move $X_{t+1}$ is generated by Klein’s algorithm shown in Algorithm 1 [12], via the following backward one-dimensional conditional sampling:

$$
P(y_i|\overline{y}[-i]) = \frac{P(y_i|\overline{y}[i+1], \ldots , y_n)}{\sum_{y_i \in \mathbb{Z}} e^{-\frac{\|y_i - Ry\|^2}{2\sigma^2}}}
$$

$1$Other measures by variance and exploration of the state space and so on also exist [33].
convergence analysis in MCMC [17]. Here, the proof of the following theorem was firstly given in [34].

\[ \rho \in B, \sigma, c, x_0, |\max(\epsilon)|; \]
\[ \text{Output: } x \sim D_{\Lambda, \sigma, e}; \]
1. let \( X_0 = x_0 \)
2. for \( t = 1, 2, \ldots \), do
3. let \( x \) denote the state of \( X_{t-1} \)
4. sample \( \gamma \) from the proposal distribution \( q(x, \gamma) \) in (10)
5. calculate the acceptance ratio \( \alpha(x, \gamma) \) in (11)
6. generate a sample \( u \) from the uniform density \( U[0, 1] \)
7. if \( u \leq \alpha(x, \gamma) \) then
8. let \( X_t = \gamma \)
9. else
10. \( X_t = x \)
11. end if
12. if \( t \geq |\max(\epsilon)| \) then
13. output \( x \)
14. end if
15. end for

Algorithm 2 Independent Metropolis-Hastings-Klein Sampling Algorithm

III. SPECTRAL GAP

In this section, the spectrum of the Markov chain induced by independent MHK algorithm is derived and analyzed. As a common way to evaluate the mixing time, spectral gap \( \gamma = 1 - |\tau_1| > 0 \) of the transition matrix is always preferred for convergence analysis in MCMC [17]. Here, \( \tau_1 \) represents the second largest eigenvalue in magnitude of the transition matrix \( P \) in the case of finite state space and becomes the spectral radius of \( P \) for a general state space Markov chain [35].

The proof of the following theorem was firstly given in [34] using the coupling technique. Here, we reformulate it in terms of spectral gap and provide a more straightforward proof by following [36].

**Theorem 1.** Given the invariant lattice Gaussian distribution \( D_{\Lambda, \sigma, e} \), the Markov chain induced by independent MHK sampling converges exponentially fast to the stationary distribution in total variational distance:

\[ ||P^t(x, \cdot) - D_{\Lambda, \sigma, e}(\cdot)||_{TV} \leq (1 - \delta)^t, \]

where

\[ \delta = \frac{\rho(\Lambda)}{\prod_{i=1}^n \rho_i(\Lambda)} \]

denotes a lower bound on the spectral gap \( \gamma \).

**Proof.** From (10) and (11), the transition probability \( P(x, y) \) of each Markov move in the independent MHK sampling is given by

\[ P(x, y) = \begin{cases} \min \left\{ q(y), \pi(y) q(x) \frac{\pi(y) q(x)}{\pi(x)} \right\} & \text{if } y \neq x, \\ 1 - \min_{z \neq x} \left\{ q(z) \frac{\pi(z) q(x)}{\pi(x)} \right\} & \text{if } y = x. \end{cases} \]

For notational simplicity, we define the importance weight \( w(x) \) as

\[ w(x) = \frac{\pi(x)}{q(x)}. \]

Then the transition probability can be rewritten as

\[ P(x, y) = \begin{cases} q(y) \cdot \min \left\{ 1, \frac{w(y)}{w(x)} \right\} & \text{if } y \neq x, \\ q(x) + \sum_{z \neq x} q(z) \cdot \max \left\{ 0, 1 - \frac{w(z)}{w(x)} \right\} & \text{if } y = x. \end{cases} \]

Without loss of generality, we label the countably infinite state space \( \Omega = \mathbb{Z}^n \) as \( \Omega = \{x_1, x_2, \ldots, x_\infty\} \) and assume that these states are sorted according to their importance weights, namely,

\[ w(x_1) \geq w(x_2) \geq \cdots \geq w(x_\infty). \]

From (16) and (17), the transition matrix \( P \) of the Markov
chain can be exactly expressed as
\[
P = \begin{bmatrix}
q(x_1) + \eta_1 & \frac{\pi(x_2)}{w(x_1)} & \frac{\pi(x_3)}{w(x_1)} & \cdots & \frac{\pi(x_n)}{w(x_1)} \\
q(x_1) & q(x_2) + \eta_2 & \frac{\pi(x_3)}{w(x_2)} & \cdots & \frac{\pi(x_n)}{w(x_2)} \\
q(x_1) & q(x_2) & q(x_3) + \eta_3 & \cdots & \frac{\pi(x_n)}{w(x_3)} \\
q(x_1) & q(x_2) & q(x_3) & \cdots & \frac{\pi(x_n)}{w(x_3)} \\
q(x_1) & q(x_2) & q(x_3) & \cdots & q(x_\infty)
\end{bmatrix}
\]

where
\[
\eta_j = \sum_{i=1}^{\infty} \left( q(x_i) - \frac{\pi(x_i)}{w(x_i)} \right)
\]

stands for the probability of being rejected in the decision stage with the current state \(x_j\) for \(X_t\).

Let \(q = [q(x_1), q(x_2), \ldots]^T\) denote the vector of proposal probabilities. Then by decomposition, it follows that
\[
P = G + eq^T,
\]

where \(e = [1, 1, \ldots]^T\) and \(G\) is an upper triangular matrix of the form
\[
G = \begin{bmatrix}
\eta_1 & \pi(x_2)/w(x_1) - q(x_2) & \cdots & \cdots & \cdots \\
0 & \eta_2 & \pi(x_3)/w(x_2) - q(x_3) & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix}
\]

It is well-known that for a Markov chain, the largest eigenvalue of the transition matrix \(P\) always equals 1. Here, as \(e\) is a common right eigenvector for both \(P\) and \(P - G\), it naturally corresponds to the largest eigenvalue 1. Meanwhile, since the rank of \(P - G\) is 1, the other eigenvalues of \(G\) are exactly the same as those of \(P\).

Thanks to the ascending order in (17), it is easy to verify that the spectral radius \(\tau_1\) exactly characterizes by \(\eta_1\) as
\[
\tau_1 = \eta_1
\]

and
\[
1 > |\eta_1| \geq |\eta_2| \geq \cdots > 0,
\]

thereby raising the interest of identifying the value of \(\tau_1\).

Therefore, according to (18) and (20), we can easily get that
\[
\tau_1 = \sum_{i=1}^{\infty} \left( q(x_i) - \frac{\pi(x_i)}{w(x_i)} \right) = \sum_{i=1}^{\infty} q(x_i) - \frac{1}{w(x_1)} \sum_{i=1}^{\infty} \frac{\pi(x_i)}{w(x_1)} = 1 - \frac{1}{w(x_1)} = 1 - \frac{q(x_1)}{\pi(x_1)}.
\]

In other words, the spectral gap \(1 - \tau_1\) is exactly captured by the ratio \(q(x_1)/\pi(x_1)\). Next, we invoke the following lemma to lower bound the ratio \(q(x)/\pi(x)\) for \(x \in \mathbb{Z}^n\).

**Lemma 2 (16).** In the independent MHK algorithm, there exists \(\delta > 0\) such that
\[
\frac{q(x)}{\pi(x)} \geq \delta
\]

for all \(x \in \mathbb{Z}^n\), where
\[
\delta = \frac{\rho_{\infty} c(\Lambda)}{\prod_{i=1}^{n} \rho_{\infty} c(i)}.
\]

Now, assume two independent Markov chains \(\{X_0, X_1, \ldots\}\) and \(\{Y_0, Y_1, \ldots\}\) proceed in the same update rule, where \(\{X_t\}\) starts from some initial distribution \(\pi_0\) and \(\{Y_t\}\) is supposed to start from the stationary distribution \(\pi\). Without loss of generality, assume \(X_t = x\) and \(Y_t = y\), then a state candidate \(z\) generated by both Markov chains at step \(t + 1\) will be accepted simultaneously can be expressed as
\[
P(X_{t+1} = Y_{t+1} | X_t = x, Y_t = y) = \sum_{Z} q(z) \min \left\{ \frac{\omega(z)}{\omega(x)}, \frac{\omega(y)}{\omega(z)} \right\}
\]

\[
= \sum_{Z} \pi(z) \min \left\{ \frac{1}{\omega(z)}, \frac{1}{\omega(y)} \right\} \geq \frac{1}{\omega_{\text{max}}(x)} \geq \delta,
\]

where \(\omega_{\text{max}}(x) = \omega(x_1)\) stands for the largest importance ratio in (17).

It is known that the first time that both chains accept a same state refers to the coupling time, where these two chains are deemed as identical thereafter (17). According to *coupling inequality* (17), the variation distance \(||TV\||\) between distributions formed by Markov chains \(\{X_t\}\) and \(\{Y_t\}\) is upper bounded by the probability that they are not coupled, then we can obtain that
\[
||P^t(x, \cdot) - \pi(\cdot)||_{TV} \leq P(X_t \neq Y_t)
\]

\[
= \prod_{i=1}^{t} P(X_i \neq Y_i | X_{i-1} \neq Y_{i-1}) \cdot P(X_0 \neq Y_0)
\]

\[
\leq \prod_{i=1}^{t} P(X_i \neq Y_i | X_{i-1} \neq Y_{i-1})
\]

\[
= \prod_{i=1}^{t} \left[ 1 - P(X_i = Y_i | X_{i-1} \neq Y_{i-1}) \right]
\]

\[
= \left[ 1 - P(X_i = Y_i | X_{i-1} \neq Y_{i-1}) \right]^{t} \leq (1 - \delta)^{t},
\]

completing the proof.

Note that the rate of convergence derived here is the same as that given in (16), which suggests the analysis based on the coupling technique is tight enough. More details on the exponential decay coefficient \(\delta\) can be found in our prior work (16).

**IV. Complexity of Lattice Decoding**

In this section, we apply the independent MHK sampling to CVP and BDD and examine their performance and complex-
A. The Complexity of Solving CVP

In MCMC, because samples from the stationary distribution tend to be correlated with each other, we leave a gap, which is the mixing time $t_{\text{mix}}$, to pick up the desired independent samples. Put it another way, this is essentially equivalent to operating the Markov chains in parallel to guarantee i.i.d. samples. Therefore, without loss of generality, we define the complexity of solving CVP by MCMC as follows\(^2\).

**Definition 1.** Let the mixing time $t_{\text{mix}}$ serve as the gap for i.i.d. samples from the stationary distribution. The complexity (i.e., the number of Markov moves $t$) of solving CVP by MCMC is

$$C_{\text{CVP}} \triangleq \frac{t_{\text{mix}}}{D_{\Lambda, \sigma, c}(x)}.$$  

(27)

Obviously, given the value $\delta < 1$, the mixing time of the Markov chain can be calculated by (8) and (12), that is

$$t_{\text{mix}}(\epsilon) = \frac{\ln \epsilon}{\ln(1 - \delta)} < (-\ln \epsilon) \cdot \left(\frac{1}{\delta}\right), \quad \epsilon < 1$$  

(28)

where we use the bound $\ln(1 - \delta) < -\delta$ for $0 < \delta < 1$. Therefore, the mixing time is proportional to $1/\delta$, and becomes $O(1)$ if $\delta \to 1$.

Then, with the mixing time given in (28), $C_{\text{CVP}}$ can be upper bounded by

$$C_{\text{CVP}} \leq \log \left(\frac{1}{\epsilon}\right) \cdot \frac{1}{\delta} \cdot \frac{\rho_{\sigma, c}(A)}{\rho_{\sigma, c}(Bx)}$$

$$\leq \log \left(\frac{1}{\epsilon}\right) \cdot \frac{1}{\delta} \cdot \prod_{i=1}^{n} \frac{\rho_{\sigma, c}(Z)}{\rho_{\sigma, c}(Bx)}$$

$$\leq \log \left(\frac{1}{\epsilon}\right) \cdot \prod_{i=1}^{n} \frac{\rho_{\sigma, c}(Z)}{\rho_{\sigma, c}(Bx)} \cdot C,$$  

(29)

where

$$C = \prod_{i=1}^{n} \frac{\rho_{\sigma, c}(Z)}{\rho_{\sigma, c}(Bx)}.$$  

(30)

**Theorem 2.** The decoding complexity of solving CVP by independent MHK algorithm is

$$C_{\text{CVP}} \leq \log \left(\frac{1}{\epsilon}\right) \cdot \frac{d^{2}(\Lambda, c)}{d}\left(\frac{\epsilon}{\min f_{\sigma}||b||}\right).$$  

(31)

where $d(\Lambda, c) = ||Bx - c||$ denotes the Euclidean distance between the query point $c$ and the lattice $\Lambda = Bx$.

**Proof.** To start with, let us firstly focus on the numerator $\prod_{i=1}^{n} \rho_{\sigma, c}(Z)$ from (30).

$$\prod_{i=1}^{n} \rho_{\sigma, c}(Z) = \prod_{i=1}^{n} \sum_{x_i \in \mathbb{Z}} e^{-\frac{1}{2\rho_{\sigma, c}^{2}} ||x_i||^{2}}$$

$$= \prod_{i=1}^{n} \sum_{x_i \in \mathbb{Z}} e^{-\frac{1}{2\rho_{\sigma, c}^{2}} ||x_i||^{2}}$$

(32)

It may be possible to reduce the gap to obtain lower decoding complexity, which is one of our future work.

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**Algorithm 3** Independent Metropolis-Hastings-Klein Sampling Decoder

**Input:** $B, \sigma, c, x_0, \epsilon$

**Output:** $x \sim D_{\Lambda, \sigma, c}, \epsilon$

1: let $x_{\text{cvp}} = x^0$ and $X_0 = x_0$
2: for $i = 1, \ldots, t$
3: let $x$ denote the state of $X_i$
4: sample $y$ from the proposal distribution $q(x, y)$ in (10)
5: calculate the acceptance ratio $\alpha(x, y)$ in (11)
6: generate a sample $u$ from the uniform density $U[0, 1]$
7: if $u \leq \alpha(x, y)$ then
8: let $X_i = y$ and $\hat{x} = y$
9: if $||c - B\hat{x}|| < ||c - Bx_{\text{cvp}}||$ then
10: update $x_{\text{cvp}} = \hat{x}$
11: end if
12: else
13: $X_i = x$
14: end if
15: end for
16: output $x_{\text{cvp}} = \hat{x}$

---

(33) Here, (a) follows the inequality from [1, Lemma 1.4]

$$\sum_{x \in \Lambda} e^{-\pi s^{-1} x^2} \leq s^\Phi \cdot \sum_{x \in \Lambda} e^{-\pi x^2}, \quad \text{for } s \geq 1$$

and (b) applies the Jacobi theta function $\vartheta_3$ [38]

$$\vartheta_3(\tau) = \sum_{n=-\infty}^{+\infty} e^{-\pi n^2},$$

(34)

which means

$$2\pi \alpha \sigma^2 \geq 1,$$

(35)

and the equality holds if and only if

$$\alpha = \frac{1}{\sqrt{2\pi \sigma}}.$$  

(36)

By substituting (32) to (30), the complexity $C$ is upper bounded as

$$C \leq \left(\frac{\sqrt{2\pi \sigma}}{2\pi \sigma^2}\right)^n \prod_{i=1}^{n} \vartheta_3(\alpha ||b_i||^2)$$

$$\leq \sum_{i=1}^{n} \frac{\vartheta_3(\alpha ||b_i||^2)}{e^{-\pi \alpha^2 ||b||^2}}.$$  

(37)

for $2\pi \alpha \sigma^2 \geq 1$.

It is natural to choose $\alpha$ to minimize the above upper bound. However, once $2\pi \alpha \sigma^2 > 1$, then the term $(\sqrt{2\pi \sigma})^n$ may become overwhelming. Therefore, we choose

$$\alpha = \frac{1}{2\pi \sigma^2}$$

(38)

and the upper bound (37) can be rewritten as

$$C \leq \prod_{i=1}^{n} \frac{\vartheta_3(\alpha ||b_i||^2/2\pi \sigma^2)}{e^{-\pi \alpha^2 ||b||^2}}.$$  

(39)
Now, let us recall some facts about Jacobi theta function \( \vartheta_3(\tau) \). From \( (34) \), it is obvious that \( \vartheta_3(\tau) \) is monotonically decreasing with \( \tau \), and particularly
\[
\lim_{\tau \to -\infty} \inf \vartheta_3(\tau) = 1.
\] (40)
By simple calculation, we can get that
\[
\vartheta_3(2) = \sum_{n=-\infty}^{+\infty} e^{-2\pi n^2} = \frac{\sqrt{6\pi + 4\sqrt{2}\pi}}{2\Gamma\left(\frac{3}{4}\right)} = 1.0039,
\] (41)
where \( \Gamma(\cdot) \) stands for the Gamma function. Clearly, if
\[
\min_{n \leq i \leq n} ||\mathbf{b}_i||^2 \geq 2
\] (42)
then it turns out that the following term
\[
\prod_{i=1}^{n} \vartheta_3(||\mathbf{b}_i||^2/2\pi\sigma^2) \leq \vartheta_3^n(2) = 1.0039^n
\] (43)
will be rather small even for values of \( n \) up to hundreds (e.g., 1.0039\(10^4 \approx 1.4759 \)). The key point here is that the pre-exponential factor is extremely close to 1. For better accuracy, \( \vartheta_3(3) = 1.0002 \) (or \( \vartheta_3(4) \), etc.) can be applied so that 1.0002\(10^4 \approx 1.2214 \). More options about \( \vartheta_3 \) can be found in Table I.

Therefore, let \( \sigma \) satisfy the condition given in (42), that is
\[
\sigma \leq \min_{1 \leq i \leq n} ||\mathbf{b}_i||/(2\sqrt{\pi}),
\] (44)
then we have
\[
C \leq 1.0039^n \cdot e^{-\frac{\pi}{2\sigma^2}||\mathbf{B}x-c||^2} \approx e^{-\frac{\pi}{2\sigma^2}||\mathbf{B}x-c||^2}.
\] (45)
Consequently, with \( \sigma = \min_{1 \leq i \leq n} ||\mathbf{b}_i||/(2\sqrt{\pi}) \), the following result is finally achieved
\[
C_{CVP} \leq \log \left( \frac{1}{c} \right) \cdot e^{-\frac{\pi}{2\sigma^2}||\mathbf{B}x-c||^2}
\] (46)
\[
= \log \left( \frac{1}{c} \right) \cdot e^{-\frac{2\sigma^2}{\pi}||\mathbf{B}x-c||^2}
\]
\[
= O(e^{d(\Lambda,c)/\min_i||\mathbf{b}_i||}),
\] (47)
completing the proof.

Here, let us highlight the significance of lattice reduction technique \( [28] \). As shown in Lemma 1, through the swap step, LLL reduction is able to significantly improve \( \min_i ||\mathbf{b}_i|| \) while reduce \( \max_i ||\mathbf{b}_i|| \) \( [28] \). Therefore, \( \min_i ||\mathbf{b}_i|| \) is always increasing all along the execution of the reduction, which will significantly decrease the complexity shown above.

**Remark 1.** In fact, such an analysis also holds for Klein’s algorithm, where the probability of sampling \( x \) follows a Gaussian-like distribution \( [12] \)
\[
P(x) \geq \frac{e^{-\frac{\pi}{2\sigma^2}||\mathbf{B}x-c||^2}}{\prod_{i=1}^{n} \vartheta_3(||\mathbf{b}_i||^2/2\pi\sigma^2)}.
\] (48)
Specifically, Klein chose \( \sigma = \min_i ||\mathbf{b}_i||/\sqrt{2\log n} \), which corresponds to \( O(n\sigma^2/\Lambda,c)/\min_i ||\mathbf{b}_i|| \) in terms of complexity. Here, we have shown that the decoding complexity can be

**B. Trade-off Between Decoding Radius and Markov Moves**

In coding theory, \( \eta \)-BDD (with \( \eta \leq 1/2 \)) is a special instance of the CVP where the noise norm i.e., the distance from the query point \( c \) to the lattice \( \Lambda \), namely, \( d(\Lambda,c) \), is less than the correct decoding radius \( R = \eta \cdot \lambda_1 \), where \( \lambda_1 \) is the first successive minimal distance of the lattice \( [39] \). Although for some sophisticated lattice codes BDD algorithms could achieve optimal \( \eta \approx 1/2 \) in polynomial time, polynomial complexity algorithms only solve \( \eta \)-BDD for much smaller values of \( \eta \) in general.

In fact, from \( (31) \), if \( d(\Lambda,c) \) is much larger than \( \min_i ||\mathbf{b}_i|| \), then the complexity will become intensively overwhelming, making BDD with \( d(\Lambda,c) \leq R \) more meaningful on the analysis front. Furthermore, of particular interest is the decoding performance with a given number of Markov moves \( t \gg 1 \). Since the decoding complexity of CVP shown above is determined by \( d(\Lambda,c) \), the following analysis is carried out from the perspective of BDD.

According to (46), if \( C_{BDD} \) is a given number of Markov moves \( t \), we have
\[
\log \left( \frac{1}{c} \right) \cdot e^{-\frac{\pi}{2\sigma^2}||\mathbf{B}x-c||^2} \geq C_{BDD} \triangleq t.
\] (49)
Then it follows
\[
||\mathbf{B}x-c|| \geq \sigma \sqrt{2 \log \frac{t}{a}},
\] (50)
where \( a = \log \left( \frac{1}{c} \right) \) is a constant.

Typically, the sampling decoder will output the closest vector point \( \mathbf{B}x \) if the distance from \( c \) to lattice \( \Lambda \) is less than the right-hand-side (RHS) of (50). In this regard, the RHS of (50) can be interpreted as the correct decoding radius of BDD, that is
\[
R = ||\mathbf{B}x-c|| \triangleq \sigma \sqrt{2 \log \frac{t}{a}}.
\] (51)
It should be pointed that the above derivation is based on the lower bound in (50), a larger size of \( R \) is possible in practice. Obviously, the correct decoding radius \( R \) is decided by \( \sigma \)
and $t$. When $\sigma = \min_i ||\hat{b}_i||/(2\sqrt{\pi})$, we obtain

$$R = \sqrt{\frac{1}{2\pi} \cdot \ln \frac{t}{a} \cdot \min_i ||\hat{b}_i||}. \quad (52)$$

Clearly, $R$ increases monotonically with $t$, implying a flexible trade-off between the decoding performance and the number of Markov moves. In addition, the significance of lattice reduction can be seen due to an increased value of $\min_i ||\hat{b}_i||$.

V. INDEPENDENT MULTIPLE-TRY METROPOLIS-KLEIN ALGORITHM

In this section, the independent multiple-try Metropolis-Klein (MTMK) algorithm is proposed to enhance the Markov mixing. We firstly prove its validity and then show its uniform ergodicity with an improved convergence rate.

A. Multiple-Try Metropolis Method

Rather than directly generating the state candidate $y$ from the proposal distribution $q(x, y)$, the multiple-try Metropolis (MTM) method selects $y$ among a set of i.i.d. trial samples from $q(x, y)$, which significantly expands the searching region of proposals [40]. In particular, the MTM method consists of the following steps:

1) Given the current state $X_t = x$, draw $k$ i.i.d. state candidates $y_1, \ldots, y_k$ from the proposal distribution $q(x, y)$.

2) Select $y = y_c$ among $\{y_1, \ldots, y_k\}$ with probability proportional to the weight

$$\omega(y_1, x) = \pi(y_1)q(y_1, x)\lambda(y_1, x), \quad i = 1, \ldots, k, \quad (53)$$

where $\lambda(y, x)$ defined initially is a nonnegative symmetric function about $y$ and $x$.

3) Draw $k-1$ i.i.d. reference candidates $x_1, \ldots, x_{k-1}$ from the proposal distribution $q(y, x)$ and let $x_0 = x$.

4) Accept $y = y_c$ as the state of $X_{t+1}$, i.e., $X_{t+1} = y$ with probability

$$\alpha_{\text{MTM}} = \min \left\{ 1, \frac{\omega(y_1, x) + \ldots + \omega(y_k, x)}{\omega(x_1, y) + \ldots + \omega(x_k, y)} \right\}. \quad (54)$$

otherwise, with probability $1 - \alpha_{\text{MTM}}$, let $X_{t+1} = X_t = x$.

Obviously, MTM is developed based on the formula of MH algorithms. By exploring the searching region more thoroughly, a convergence improvement can be achieved. Based on a number of trial samples generated from the proposal distribution, the Markov chain enjoys a large step-size jump within every single move without lowering the acceptance rate. As the MCMC sampler is able to explore the state space more efficiently, the convergence rate improves, thus leading to faster mixing. It should be noticed that the $k-1$ reference samples $x_i$'s are involved only for the validity of MTM by satisfying the detailed balance condition shown below [40]

$$\pi(x)P(x, y) = \pi(y)P(y, x). \quad (55)$$

Clearly, the efficiency of MTM method relies on the number of trial samples $k$ and the traditional MH sampling behaves as a special case with $k = 1$. Similar to MH sampling, there is a large flexibility in the choice of the proposal distribution $q(x, y)$ in MTM [41]. Actually, it is even possible to use different proposal distributions to generate trial samples without altering the ergodicity of the Markov chain [42]. Meanwhile, the nonnegative symmetric function $\lambda(x, y)$ in [53] is also flexible, where the only requirement is that $\lambda(x, y) > 0$ whenever $q(x, y) > 0$.

B. The Proposed Algorithm

With the great flexibility offered by $q(x, y)$ and $\lambda(x, y)$, we now propose the independent multiple-try Metropolis-Klein (MTMK) algorithm, which is described by the following steps:

1) Given the current state $X_t = x$, use Klein’s algorithm to draw $k$ i.i.d. state candidates $y_1, \ldots, y_k$ from the independent proposal distribution in [10]

$$q(x, y) = \prod_{i=1}^{n} P(y_{n+1-i} | \sum_{j=1,j \neq i}^{n} y_j) = q(y). \quad (56)$$

2) Let $\lambda(x, y) = [q(x, y)q(y, x)]^{-1} = [q(y)q(x)]^{-1}$. Then select $y = y_c$ among $\{y_1, \ldots, y_k\}$ with probability proportional to the importance weight

$$\omega(y_i, x) = \frac{\pi(y_i)}{q(y_i)} = \omega(y_i), \quad i = 1, \ldots, k. \quad (57)$$

3) Accept $y = y_c$ as the state of $X_{t+1}$ with acceptance rate

$$\alpha_{\text{MTM}} = \min \left\{ 1, \frac{\omega(y_c) + \sum_{j=1, j \neq c}^{k} \omega(y_j)}{\omega(x) + \sum_{j=1, j \neq c}^{k} \omega(y_j)} \right\}. \quad (58)$$

otherwise, with probability $1 - \alpha_{\text{MTM}}$, let $X_{t+1} = X_t = x$.

In the proposed algorithm, the basic formulation of MTM is modified in three aspects. First, Klein’s algorithm is applied to generate trial state candidates from the independent proposal distribution $q(x, y) = q(y)$. Then, by setting $\lambda(x, y) = [q(x, y)q(y, x)]^{-1}$, $\omega(x, y)$ becomes the importance weight of

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**Algorithm 4 Independent Multiple-try Metropolis-Klein Sampling Decoder**

**Input:** $B, \sigma, c, x_0, t$

**Output:** $x \sim D_{X, \sigma, c}$

1. let $x_{\text{cvp}} = x^0$ and $x_0 = x_0$

2. for $i = 1, \ldots, t$

3. let $x$ denote the state of $X_{t-1}$

4. sample $k$ trial samples $y_1, \ldots, y_k$ from $q(x, y)$ in [56]

5. select $y = y_c$ from $y_1, \ldots, y_k$ based on $\omega(y_i)$ in [57]

6. calculate the acceptance ratio $\alpha(x, y)$ in [58]

7. generate a sample $u$ from the uniform density $U[0, 1]$

8. if $u < \alpha(x, y)$ then

9. let $X_i = y$ and $\hat{x} = y$

10. if $||c - B\hat{x}|| < ||c - Bx_{\text{cvp}}||$ then

11. update $x_{\text{cvp}} = \hat{x}$

12. end if

13. else

14. $X_i = x$

15. end if

16. end for

17. output $x_{\text{cvp}} = \hat{x}$
\[ p(y_c|x, c) = \sum_{y_1, \ldots, 1 \in \mathbb{Z}^n} \sum_{y_{c+1} \in \mathbb{Z}^n} \left\{ \prod_{j=1}^{k} q(x, y_j) \right\} \cdot \frac{\omega(y_c)}{\sum_{i=1}^{k} \omega(y_i)} \cdot \min \left\{ 1, \frac{\omega(y_c)}{\omega(x) + \sum_{j=1, j \neq c}^{k} \omega(y_j)} \right\} \]

\[ = q(y_c) \cdot \omega(y_c) \cdot \sum_{y_1, \ldots, 1 \in \mathbb{Z}^n} \sum_{y_{c+1} \in \mathbb{Z}^n} \left\{ \prod_{j=1}^{k} q(y_j) \right\} \cdot \min \left\{ 1, \frac{1}{\omega(y_c) + \sum_{j=1, j \neq c}^{k} \omega(y_j)} \right\} \cdot \omega(x) + \sum_{j=1, j \neq c}^{k} \omega(y_j) \right\} \]

\[ = \pi(y_c) \cdot \min \left\{ \sum_{y_1, \ldots, 1 \in \mathbb{Z}^n} \sum_{y_{c+1} \in \mathbb{Z}^n} \left\{ \prod_{j=1}^{k} q(y_j) \right\} \cdot \omega(x) + \sum_{j=1, j \neq c}^{k} \omega(y_j) \right\} \]

\[ \text{(61)} \]

\( x \) that we have defined in (15). Finally and interestingly, thanks to the independent proposals, the generation of reference samples \( x_i \)'s can be removed without changing the ergodicity of the chain.

Specifically, the \( k - 1 \) reference samples \( x_i \)'s in MTM are only drawn to fulfill the detailed balance condition. However, in the case of independent proposals, because both the trial samples \( y_i \)'s and the reference samples \( x_i \)'s are generated independently from the identical distribution \( q(\cdot) \), the generation of reference samples can be greatly simplified by just setting \( x_i = y_i \) for \( i = 1, \ldots, c-1, c+1, \ldots, k \) and \( x_c = x \). Actually, with the same arguments, the trial samples generated in the previous Markov moves can also be used by \( x_i \). [43]

It is well known that a Markov chain which is irreducible and aperiodic will be ergodic if the detailed balance condition is satisfied [17]. Since irreducible and aperiodic are easy to verify, we show the validity of the proposed algorithm by demonstrating the detailed balance condition.

**Theorem 3.** Given the target lattice Gaussian distribution \( D_{\Lambda, \sigma, x} \), the Markov chain induced by the independent MTMK algorithm is ergodic.

**Proof.** To start with, let us specify the transition probability \( P(x, y) \) of the underlying Markov chain. For ease of presentation, we only consider the case of \( x \neq y \), since the case \( x = y \) is trivial. The transition probability \( P(x, y) \) can be expressed as

\[ P(x, y = y_c) = \sum_{i=1}^{k} p(y_c|x, c = i). \]  

Here, \( p(y_c|x, c = i) \) represents the probability of accepting \( y = y_c \) as the new state of \( X_{t+1} \) given the previous one \( X_t = x \) when the \( c \)th candidate among \( y_i \)'s is selected. Moreover, as \( y_i \) is exchangeable and independent, it follows that \( p(y_i|x, i) = p(y_j|x, j) \) by symmetry, namely,

\[ P(x, y = y_c) = k \cdot p(y_c|x, c). \]  

(60)

In contrast to MH algorithms, the generation of the state candidate \( y = y_c \) for Markov move \( X_{t+1} \) in MTM actually follows a distribution formed by \( q(x, y) \) and \( \omega(y, x) \) together [40]. More precisely, \( p(y_c|x, c) \) can be further expressed as \( \omega(y_c) \sum_j \omega(x) + \sum_{j=1, j \neq c}^{k} \omega(y_j) \), where the terms inside the sum correspond to \( q(x, y) \), \( \omega(y, x) \) and \( \alpha \) respectively.

From (61), it is straightforward to verify the term \( \pi(x) \cdot p(y_c|x, c) \) is symmetric in \( x \) and \( y_c \), namely

\[ \pi(x) \cdot p(y_c|x, c) = \pi(y_c) \cdot p(x|y_c, c). \]  

Then, by simple substitution, the detailed balance condition is satisfied as

\[ \pi(x) \cdot P(x, y = y_c) = \pi(y) \cdot p(y = y_c, x), \]  

completing the proof.

However, ergodicity only implies asymptotic convergence, and it is desirable to further specify the convergence rate, so as to estimate the mixing time.

**C. Convergence Analysis**

We then arrive at the following theorem to show the uniform ergodicity of the proposed algorithm.

**Theorem 4.** Given the invariant lattice Gaussian distribution \( D_{\Lambda, \sigma, x} \), the Markov chain induced by the independent MTMK sampling algorithm converges exponentially fast to the stationary distribution:

\[ \| P^t(x, \cdot) - D_{\Lambda, \sigma, x}(\cdot) \|_{TV} \leq (1 - \delta_{\text{MTM}})^t \]

with

\[ \delta_{\text{MTM}} = \frac{k}{k - 1 + \frac{1}{\gamma}}. \]

(65)

The proof of Theorem 4 is provided in Appendix A.

Clearly, from (64), it can be observed that with the increase of the trial sample size \( k \), the exponential decay coefficient \( \delta_{\text{MTM}} = \frac{k}{k - 1 + \frac{1}{\gamma}} \) will approach 1 asymptotically. In other words, with a sufficiently large \( k \), sampling from the target distribution can even be realized efficiently. On the other hand, it should be noticed that the independent MHK algorithm is just a case of the independent MTMK with \( k = 1 \). More importantly, the generation of \( k \) trial samples at each Markov move not only allows fully parallel implementation, but also can be carried out by a preprocessing stage, which is beneficial for the practical interest.

Now, given \( \delta_{\text{MTM}} = \frac{k}{k - 1 + \frac{1}{\gamma}} \), the mixing time of the underlying Markov chain can be estimated. Specifically, according to (8) and (64), we obtain

\[ t_{\text{mix}}^{\text{MTM}}(\epsilon) = \frac{\ln \epsilon}{\ln(1 - \delta_{\text{MTM}})} \leq \log \left( \frac{1}{\epsilon} \right) \cdot \left( \frac{1}{\delta_{\text{MTM}}} \right) \]
where we again use the bound $\ln(1-\alpha) \approx -\alpha$ for $0 < \alpha < 1$ in (d). Clearly, the mixing time is proportional to $1/\epsilon$, and becomes $O(1)$ if $k\delta \to 1$. Overall, compared with the mixing time given in (28), the mixing time of the independent MTMK is significantly reduced by a factor of $k$. Since the independent MTMK inherits all the formulations of the independent MHK, we have

$$C_{\text{CVT}}^{\text{MTM}} = \frac{t_{\text{mix}}^\text{MTM}(\epsilon)}{D_{\Lambda,\sigma,c}(x)}$$

$$= \frac{1}{k} \cdot \log \left( \frac{1}{\epsilon} \right) \cdot C$$

$$= O\left(\frac{\|Bx-e\|^2}{\|B\|} \right)$$

for $\sigma = \min_i \|B_i\|/(2\sqrt{n})$.

Compared to (31), the impact of $\log(1/\epsilon)$ can be reduced by the usage of $k$ trial samples. Meanwhile, following the afore-mentioned derivation, the correct decoding radius of the independent MTMK algorithm can be easily obtained as

$$R^\text{MTM} = \sqrt{\frac{1}{2\pi} \cdot \ln \frac{kt}{\alpha} \cdot \min_{1 \leq i \leq n} \|B_i\|}$$

with $\sigma = \min_i \|B_i\|/(2\sqrt{n})$.

Remark 2. Although independent MTMK algorithm is capable to reduce the mixing time of the Markov chain, its complexity incurred in each Markov move also increases accordingly by the multiple recalls of the trial samples. Therefore, parallel implementation or preprocessing stage is highly desired to ease the complexity burden.

Moreover, it is possible to have a varying $k$ at each Markov move, thereby resulting in an adaptive independent MTMK algorithm as

$$\|P^t(x, \cdot) - D_{\Lambda,\sigma,c}(\cdot)\|_{TV} \leq \prod_{i=1}^t (1 - \delta^i_{\text{MTM}}),$$

where $\delta^i_{\text{MTM}} = k_i \frac{k_i - 1}{k_i + 1}$ and $k_i$ denotes the size of trial samples at each Markov move [44].

VI. EXPERIMENTS OF MIMO DETECTION

In this section, performance of the MCMC sampler decoding algorithms are evaluated through MIMO detection. Specifically, we present simulation results for an $n \times n$ MIMO system with a square channel matrix containing i.i.d. Gaussian entries. Here, the $i$th entry of the transmitted signal $x$, denoted as $x_i$, is a modulation symbol taken independently from an $M$-QAM constellation $\mathcal{X}$ with Gray mapping. Meanwhile, we assume a flat fading environment, where the channel matrix $H$ contains uncorrelated complex Gaussian fading gains with unit variance and remains constant over each frame duration. Let $E_b$ represents the average power per bit at the receiver, then $E_b/N_0 = n/(\log_2(M))\sigma_w^2$ holds where $M$ is the modulation level and $\sigma_w^2$ is the noise power. Then, we can construct the system model as

$$c = Hx + w.$$  

(70)

Typically, in the case of AWGN $w$ with zero mean and variance $\sigma_w^2$ in MIMO detection, it follows that from (47)

$$C \approx O(e^{n\sigma_w^2/\min_i \|B_i\|})$$

(71)

as $\|Bx-e\|^2 \approx n \sigma_w^2$ statistically by the law of large numbers. Therefore, in MIMO detection, the decoding complexity $C$
account for decoding. This corresponds to a lattice decoding
the samples generated by MCMC algorithms are taken into
n
8

in a

x

high SNRs [45].

noise variance \( \sigma_w^2 \) is different from the standard deviation \( \sigma \) of the lattice Gaussian distribution\(^3\) where the latter is carefully designed for the efficient sampler decoding with \( \sigma = \min_i \| \hat{b}_i \|/(2\sqrt{\pi}) \).

On the other hand, soft-output decoding in MIMO bit-interleaver coded modulation (BICM) system is also available from the samples generated by MCMC. Specifically, the sample candidates can be used to approximate the log-likelihood ratio (LLR), as in [46]. For bit \( b_i \in \{0, 1\} \), the approximated LLR is computed as

\[
L(b_i|c) = \log \frac{\sum_{x: b_i = 1} \exp \left( -\frac{1}{2\sigma_x^2} \| c - Hx \|^2 \right)}{\sum_{x: b_i = 0} \exp \left( -\frac{1}{2\sigma_x^2} \| c - Hx \|^2 \right)},
\]

where \( b_i \) is the \( i \)th information bit associated with sample \( x \). The notation \( x : b_i = \mu \) means the set of all vectors \( x \) for which \( x : b_i = \mu \).

Fig. 2 shows the bit error rate (BER) of MCMC decoding in a \( 8 \times 8 \) uncoded MIMO system with 16-QAM, where all the samples generated by MCMC algorithms are taken into account for decoding. This corresponds to a lattice decoding scenario with dimension \( n = 16 \). The performance of zero-forcing (ZF) and maximum-likelihood (ML) decoding are shown as benchmarks. For a fair comparison, a sequential Gibbs sampling is applied here, which performs the one-dimensional conditional sampling over \( x_i \) in a backwards order\(^4\), thus completing a full iteration [19]. This corresponds to one Markov move in the independent MHK and MTMK algorithms, which also update \( n \) components of \( x \) in one iteration.

As expected, with \( t = 50 \) Markov moves (i.e., iterations), independent MHK outperforms Gibbs sampling. As \( \sigma \) has a vital impact on the sampling algorithms, Gibbs sampling is illustrated by tuning \( \sigma \) with different values. Note that the detection performance may be affected due to the finite constellation. Furthermore, as shown in (52), under the help of LLL reduction, the decoding radius of the independent MHK sampling is significantly strengthened by a larger size of \( \min_i \| \hat{b}_i \| \), thereby leading to a much better decoding performance. Additionally, compared to independent MHK, further decoding gain can be obtained by the independent MTMK algorithm, where cases with \( k = 5 \) and \( k = 10 \) trial samples are illustrated respectively.

On the other hand, in Fig. 3, the bit error rates (BER) of MCMC sampling detectors are evaluated against the number of Markov moves (i.e., iterations) in a \( 8 \times 8 \) uncoded MIMO system with 16-QAM. The SNR is fixed as \( E_b/N_0 = 15 \) dB. Clearly, the decoding performance of all the MCMC detectors improve with the number of Markov moves. Meanwhile, under the same number of Markov moves, substantial performance gain is obtained by LLL reduction. With an increasing number of trial samples, better decoding performance can be obtained by the independent MTMK algorithm due to a larger decoding radius shown in (68).

VII. CONCLUSION AND DISCUSSION

In this paper, we have studied the MCMC sampling technique for lattice decoding. Based on the spectrum of the transition matrix, the exponential convergence rate of the Markov chain induced by independent MHK algorithm has been derived. With a tractable mixing time, an analysis about the decoding performance and complexity of MCMC was carried out.

We have proven that the decoding complexity of CVP by independent MHK algorithm is \( O(\exp(\lambda |e|)/\min_l \| \hat{b}_i \|) \). Fur-
therm因此，从BDD点的视角，交易的离散-正确解码半径和马尔可夫移动的有效性是建立的。我们还表明，LLR降低对解码有积极影响，降低了解码半径和减少了计算复杂性。调整样本大小，独立MTMK可享受其有效性和推导出精确的收敛率。我们将进一步增强收敛。我们首先展示了工作。

此外，从BDD的角度来看，独立MTMK算法在实践中具有前景。同时，因为这个推导独立MTMK算法和

\[ B = \sum_{y_1, \ldots, y_n} \frac{1}{\omega(x) + \omega(\Xi)} \sum_{j=1, j \neq c}^{k} \omega(y_j) \quad (70) \]

与此同时，因为每个k个试验样本从概率分布q(·)是独立的，每个集合Ω是定义的，包含了k-1个试验样本y_j, j \neq c.

然后我们可以表示A和B为

\[ A = \sum_{\Xi} Q(\Xi) \cdot \frac{1}{\omega(y_c) + \omega(\Xi)} = \sum_{\Xi} Q(\Xi) \cdot F_A(\Xi) \quad (75) \]

和

\[ B = \sum_{\Xi} Q(\Xi) \cdot \frac{1}{\omega(x) + \omega(\Xi)} = \sum_{\Xi} Q(\Xi) \cdot F_B(\Xi). \quad (76) \]

这里，Q(\Xi) = \prod_{j=1, j \neq c}^{k} q(y_j) 表示一个概率分布，包含了所有的q(y_j), j \neq c进入计算。另一方面，F_A(\Xi)和F_B(\Xi)表示关于Ω的函数

\[ F_A(\Xi) = \frac{1}{\omega(y_c) + \omega(\Xi)} \quad (77) \]

和

\[ F_B(\Xi) = \frac{1}{\omega(x) + \omega(\Xi)} \quad (78) \]

其中

\[ \omega(\Xi) = \sum_{j=1, j \neq c}^{k} \omega(y_j). \quad (79) \]

现在，让我们聚焦在函数A上，我们得到

\[ A = \sum_{\Xi} Q(\Xi) \cdot F_A(\Xi) \]

\[ = \mathbb{E}_Q(\Xi) | F_A(\Xi) \]

\[ \geq \frac{1}{\mathbb{E}_Q(\Xi) [\omega(y_c) + \omega(\Xi)]} \]

\[ = \frac{1}{\omega(y_c) + \mathbb{E}_Q(\Xi) [\omega(\Xi)]} \]

\[ \frac{1}{k - 1 + \omega(y_c)}. \quad (80) \]

这里，E_u(x)[v(x)]代表函数v(x)的期望，而x是从分布u(x)抽样的，(e)来自Jensen’s inequality的多变量情况。此外，由于k-1个独立样本从q(·)，(f)遵循了下述推导

\[ \mathbb{E}_Q(\Xi) [\omega(\Xi)] = (k-1) \cdot \mathbb{E}_Q(y_j) [\omega(y_j)] \]

\[ \geq (k-1) \cdot \sum_{y_j \in \mathbb{Z}^n} q(y_j) \cdot \omega(y_j) \]

\[ \geq (k-1) \cdot \sum_{y_j \in \mathbb{Z}^n} \pi(y_j) \]

\[ \geq k - 1. \quad (81) \]

所以，我们可以通过(80)和(82)得到

\[ B \geq \frac{1}{k - 1 + \omega(x)}. \quad (82) \]

因此，从(80)和(82)得到

\[ P(x, y = y_c) = k \cdot p(y_c | x, c) \]

\[ \geq k \cdot \pi(y_c) \cdot \min \{ A, B \} \]

\[ \geq \pi(y_c) \cdot \min \left\{ \frac{k}{k - 1 + \omega(y_c)}, \frac{k}{k - 1 + \omega(x)} \right\} \]

\[ \geq \pi(y_c) \cdot \frac{k}{k - 1 + \omega_{\text{max}}(x)} \]

\[ \geq \delta_{\text{MTM}} \cdot \pi(y_c), \quad (83) \]

其中 \[ \omega_{\text{max}}(x) \triangleq \sup_{x \in \mathbb{Z}^n} \omega(x) = \sup_{x \in \mathbb{Z}^n} \frac{\pi(x)}{q(x)} \]

\[ \leq \frac{1}{\delta}. \quad (84) \]

对于x \in \mathbb{Z}^n 来自Lemma 2。从(83)，它很容易看出，所有马尔可夫转移都有一个共同的组成部分大小 \delta_{\text{MTM}}。然后，当技巧可以应用于证明均匀等价性时，这被简要地忽略。更多的细节可以在[16]中找到。

\[ \text{REFERENCES} \]

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