Diffusion in the Markovian limit of the spatio-temporal colored noise

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Abstract – We explore the diffusion process in the non-Markovian spatio-temporal noise. There is a non-trivial short-memory regime, i.e., the Markovian limit characterized by a scaling relation between the spatial and temporal correlation lengths. In this regime, a Fokker-Planck equation is derived by expanding the trajectory around the systematic motion and the non-Markovian nature amounts to the systematic reduction of the potential. For a system with the potential barrier, this fact leads to the renormalization of both the barrier height and collisional prefactor in the Kramers escape rate, with the resultant rate showing a maximum at some scaling limit.

Introduction. – It has been a long-standing problem to systematically treat the spatio-temporal colored noise in the thermal diffusion [1–3]. This is in contrast to the successful development of the noise-induced diffusion processes in the cases of additive noise [4], multiplicative noise [5], and temporally colored noise [6]. In the context of path coalescence, Deutsch [3] first examined the motion of a damped particle subjected to a force fluctuating in both space and time. Wilkinson, Mehlig and coworkers [7–9] pursued a similar subject, intensively explored the generalized Ornstein-Uhlenbeck process in momentum space, and showed the anomalous diffusion where the spatial memory led to the staggered ladder spectra [7]. A typical example of the spatio-temporal noise is given by the turbulent flow or a randomly moving gas which suspends small particles, and space coordinates often stand for collective or reaction coordinates and order parameters in general. Then small particles or molecules suspended by the turbulent flow can experience systematic forces. The above works [3,7–9], however, are limited to the diffusion with no systematic force. It is thus a natural challenge to consider the diffusion with a spatio-temporal correlated noise in the presence of the systematic force induced by a potential. The motivation of the present paper is to elucidate a scenario how the spatio-temporal correlated noise may radically affect the potential-induced motion with the purely-temporal noise. In this case, we shall see that there is a non-trivial Markovian limit characterized by the spatial and temporal correlation lengths, and the Fokker-Planck equation for the distribution of position, collective coordinates or the order parameter in general, is derived from the conservation of population. Then in order to numerically verify the validity of our analysis, we shall investigate the Kramers escape rate problem where the potential renormalization plays a crucial role. The new escape rate matches up to renewed attentions paid for the Kramers escape rate theory [10–12].

This letter is organized as follows: after a sketch of our model, the Markovian limit is discussed and then the Fokker-Planck equation is derived from the conservation of the population. Then, the analytical result is numerically verified by calculating the steady-state distribution and the Kramers escape rate with the use of stochastic simulation.

Model. – Our model is described by the overdamped Langevin equation with a colored noise

\[ \eta \ddot{x}(t) = -U'(x(t)) + f(x(t), t), \]
\[ \langle f(x, t) \rangle = 0, \langle f(x, t) f(x', t') \rangle = C(x - x', t - t'), \]

where \( \eta \) stands for the friction coefficient and \( U(x) \) is the potential. In our numerical analysis, \( U(x) \) consists of metastable wells and a potential barrier (see fig. 1). The bracket \( \langle \rangle \) stands for the average over the noise \( f(x, t) \).
and the initial distribution. The noise \( f(x,t) \) is assumed as a Gaussian process in accordance with the central limit theorem both in the spatial and temporal variables. An explicit expression of the noise \( f(x,t) \) is given in the last section concerning the stochastic simulation. As a typical form of the autocorrelation function, we consider the Gaussian memory

\[
C(x,t) = C_0 e^{-x^2/\xi^2 - t^2/\tau^2}, \tag{2}
\]

where \( \xi \) is the spatial correlation length and \( \tau \) is the correlation time, respectively. It is reasonable to assume that the usual overdamped Langevin equation is recovered as the case of position-independent noise by taking the limit of \( \xi \to \infty \) and the temporal Markovian limit \( \tau \to 0 \). For this purpose, motivated by the form of the autocorrelation function of the additive noise, we shall choose the constant \( C_0 \) as \( C_0 = 2\pi\theta/\tau \sqrt{\pi} \) where \( \theta \) is a parameter to control the noise strength, which is related but not necessarily equal to the temperature.

For spatially correlated noise with \( \xi < +\infty \), the magnitude of the noise should be affected by the particle fluctuation, and the relation between the noise strength \( \theta \) and the physical temperature constitutes a non-trivial open problem.

We remark that essentially the same result is obtained for the other form of the memory, \( C(x,t) = C_0 \sin(x/\xi) e^{-x^2/\xi^2} \) which behaves as Gaussian at the origin, and converges to the Dirac-delta. Note that Sinai’s random walk in random environment leads to the anomalously slow diffusion [13], while in our case the symmetric nature of the random force may enhance the diffusion.

**Fokker-Planck equation.** — The Fokker-Planck equation is obtained for a short-correlation regime from the conservation of probability [14,15]. The stochastic Liouville equation for the probability distribution \( p(x,t) \) of each realization of noise is given as

\[
\frac{\partial}{\partial t} p(x,t) = -L p(x,t) - \frac{1}{\eta} \frac{\partial}{\partial x} f(x,t) p(x,t), \tag{3}
\]

where \( L p(x,t) = -\frac{\partial}{\partial x} \frac{1}{\eta} U'(x) p(x,t) \) is the deterministic Liouville operator. With the use of the identity [14,15]

\[
p(x,t) = e^{-tL} p(x,0) - \frac{1}{\eta} \int_0^t dse^{-sL} \frac{\partial}{\partial x} f(x,s) p(x,s), \tag{4}
\]

eq (3) is rewritten in the form of a diffusion equation as

\[
\frac{\partial}{\partial t} p(x,t) = -L p(x,t) - \frac{1}{\eta} \frac{\partial}{\partial x} f(x,t) e^{-tL} p(x,0) + \frac{1}{\eta^2} \frac{\partial}{\partial x} f(x,t) \int_0^t dse^{-sL} \frac{\partial}{\partial x} f(x,s) p(x,s). \tag{5}
\]

By averaging over the noise, the population distribution \( P(x,t) \equiv \langle p(x,t) \rangle \) obeys the diffusion equation

\[
\frac{\partial}{\partial t} P(x,t) = \frac{1}{\eta} \frac{\partial}{\partial x} U'(x) P(x,t) \]

\[
+ \frac{1}{\eta^2} \frac{\partial}{\partial x} \int_0^t ds f(x,t) \frac{\partial}{\partial x} f(y(x),s) p(y(x),s)) \tag{6}
\]

Here, \( y(x) \equiv e^{(t-s)L} x = x + \int_0^{t-s} ds' U'(x(s'))/\eta \) is the time-reversed deterministic trajectory which starts at \( x \).

The last term in eq. (6) is composed of three different terms: a) the correlation between noise terms which contributes to the drift velocity, b) the diffusion term \( \frac{1}{\eta^2} \frac{\partial}{\partial x} \int_0^t ds f(x,t) \frac{\partial}{\partial x} f(y(x),s) \rangle \) and c) the correlation term between noise \( f \) and the population \( P \). Each contribution is evaluated as follows.

Firstly, the correlation between noise terms is evaluated as follows. There is a contribution to the drift velocity

\[
\int_0^t ds \left\langle f(x,t) \left( \frac{\partial}{\partial x} f(y(x),s) \right) \right\rangle = \int_0^t ds C_0 \frac{\partial}{\partial x} U'(x) e^{-xy(x)/\xi^2 - (t-s)^2/\tau^2} \approx \int_0^t ds C_0 \frac{\partial}{\partial x} U'(x) e^{-xy(x)/\xi^2 - (t-s)^2/\tau^2} \approx
\]

\[
- \int_0^t ds C_0 \left( 1 + O(\tau) / \eta \right) \frac{2U'(x)}{\eta \xi^2} e^{-xy(x)/\xi^2 - (t-s)^2/\tau^2} \approx
\]

\[
- U'(0) \frac{2\theta}{\sqrt{\pi} \tau U'(x)^2 + 2\xi^2} \tag{7}
\]

This means a systematic renormalization of the potential. We approximated the trajectory as \( y(x) \approx x + U'(x)(t-s)/\eta \), because it appears in the exponentially decaying factor as \( t-s \) gets larger and, in the Markovian limit, only the short time integral does contribute to the correlation function. A similar approximation was used in [1] by assuming the weak spatial disorder, and in [7,8] for the calculation of the
diffusion coefficient for the underdamped case. Similarly, the diffusion coefficient is given as
\[
\int_0^t ds(f(x, t)f(y(x), s)) = \int_0^t ds C_0 e^{-(x-y)^2/\xi^2 - (t-s)^2/\tau^2} \approx \int_0^\infty ds C_0 e^{-U'(x)^2(t-s)^2/\xi^2 \eta^2 - (t-s)/\tau^2} = \frac{\theta}{\eta \sqrt{(\tau U'(x))^2 + (\xi \eta)^2}}.
\]  
(8)

In order to explore the role of spatial correlation, we shall consider a Markovian limit where both the drift velocity and diffusion coefficient are well defined: keeping the dimensionless parameter
\[
\kappa = 2\tau/\sqrt{\pi \xi^2 \eta}
\]  
(9)
as a constant, both the temporal and spatial correlations go to zero, i.e., \(\tau, \xi \to 0\). The systematic renormalization of the potential (7) converges to \(-\kappa U'(x)\), and the diffusion coefficient becomes
\[
\frac{\theta}{\eta \sqrt{\tau \kappa \sqrt{\pi / 2 \eta^2 + 1}}}
\]  
in the limit. It is instructive to derive the diffusion constant in a more general way. We assume that the kernel \(K(x, t)\) rapidly vanishes as \(t/\tau \to \infty\). Then the time integral of the noise correlation function is calculated as
\[
\frac{1}{\eta^2} \int_0^t ds C(x-y(x), t-s) \approx \frac{1}{\eta^2} \int_0^t ds C\left(\frac{\xi U'(x) t - s}{\tau}, \frac{t - s}{\tau}\right) \approx \frac{1}{\eta^2} \int_0^t ds C\left(0, \frac{t - s}{\tau}\right) = \frac{\theta}{\eta},
\]  
(10)

where we employed the same approximation, i.e., the expansion around the deterministic trajectory \(y(x)\). Also, to evaluate the \(x-y(x)\) in the first line of eq. (10) we used that \(\tau_x = \xi \cdot \sqrt{\pi \eta \kappa / 2\theta} \to 0\) vanishes in the Markovian limit. With the use of this relation, the spatial dependence of the autocorrelation function disappears, since the spatial displacement becomes negligible in the unit of correlation length \(\xi\) even when \(t-s = o(\tau)\). On the other hand, the temporal dependence remains finite for \(t-s = o(\tau)\). The integral gives the diffusion coefficient.

The truncation of the hierarchy of the stochastic Liouville equation should be corrected by the correlation between the noise \(f\) and the population \(p\). The correlation term yields a series of \(\kappa\), \(\sum_n c_n(\tau)\kappa^n\), which seems to diverge for considerably large \(\kappa\). Therefore, we confine ourselves to the case \(0 \leq \kappa < 1\), where the correlation between the noise \(f\) and \(p\) is negligible. In the appendix, we shall briefly explore the correction due to the correlation between the noise and the population.

In the above-mentioned short-correlation regime, the Fokker-Planck equation becomes
\[
\frac{\partial}{\partial t} P(x, t) = \frac{1}{\eta} \frac{\partial}{\partial x} \left(U'(x)(1-\kappa) + \theta \frac{\partial}{\partial x}\right) P(x, t).
\]  
(11)

It is our main result that the spatial correlation amounts to the systematic reduction of the drift velocity. As described in the previous paragraph, our Fokker-Planck equation is available for \(\kappa \ll 1\). On the other hand, for the wide range of \(0 \leq \kappa < 1\), our stochastic simulation shows that the Fokker-Planck description is quite accurate. It is also remarked that the diffusion coefficient is constant, since the noise is invariant under the uniform spatial translation. The steady-state distribution is now given as
\[
P^{st}(x) = N e^{-U(x)(1-\kappa)/\theta},
\]  
(12)

which has the form of a canonical distribution with the renormalized potential \(U(x)(1-\kappa)\) (fig. 3). Here, the natural boundary condition is assumed, and \(N\) is the normalization constant. The usual Langevin dynamics driven by the white noise is recovered in the case of long-enough spatial correlation length which is expressed as \(\kappa = 0\). Mathematically, this condition is satisfied when the ratio \(\tau / \xi^2 \to 0\), which only requires that \(\tau\) should vanish faster than \(\xi^2\). Intuitively, however, the spatial correlation length can be seen as infinite when it exceeds the typical length of the particles’ displacement within the relaxation time \(\tau\).

To see the role of the potential renormalization, let us investigate the escape rate problem characterized by the potential in fig. 1. For the diffusion process governed by the Fokker-Planck equation (11), the mean-first-passage time is now straightforwardly calculated, and its inverse gives the escape rate. We assume the initial probability distribution \(P(x, t = 0) = \delta(x-x_0)\) confined in a metastable region bordered by the free energy barrier at \(x = x_{\text{max}}\) and the infinite wall at \(x = -\infty\). The mean-first-passage time \(\tau(x_0)\) from the initial position \(x_0\) is given by the standard procedure [4] as a quadratic integral
\[
\tau(x_0) = \frac{\eta}{\theta} \int_{x_0}^{x_{\text{max}}} d\xi e^{-U(\xi) / \theta} \int_0^\infty d\tau e^{-U(\xi) / \theta},
\]  
(13)

where the dependence on the initial point may not be important at a weak noise strength. For a high-enough barrier, the escape rate is evaluated by the steepest descent approximation at the maximum \(x_{\text{max}}\) and the minimum \(x_{\text{min}}\):
\[
\Gamma \equiv \tau(x_0)^{-1} = \frac{1 - \kappa}{2\pi \eta} \sqrt{U''(x_{\text{min}})U''(x_{\text{max}})} e^{-\Delta U (1-\kappa)},
\]  
(14)

where \(\Delta U\) is the barrier height.

**Stochastic simulation.** – To confirm the validity of eq. (14), we numerically solved the non-Markovian Langevin equation (1) in the special case of the Landau
potential $U = x^4/4 - x^2/2$, under sufficiently short correlations of noise. The spatio-temporal correlated noise can be given in a product form $f(x, t) = g_1(x, \xi)g_2(t, \tau)$ where the noise $g_{i=1,2}(s, a)$ have the Gaussian correlation $\langle g_i(s, a)g_j(s', a) \rangle \propto \delta_{ij}e^{-\frac{(s-s')^2}{\alpha^2}}$ with the correlation $a$. In order to construct the colored noise $g_i(s, a)$, we considered an assembly of harmonic oscillators which mimics the Langevin force

$$g_i(s, a) = \sum_k \left( x_k \cos \omega_k s + p_k \sin \omega_k s \right),$$  

where the coefficients $x_k$ and $p_k$ are the Gaussian stochastic variables with zero means and the variances

$$\langle x_k^2 \rangle = \theta_i / m \omega_k^2, \quad \langle p_k^2 \rangle = \theta_i / m,$$  

with the mass $m = 1$ and the noise strength $\theta_i$. The density of the states $G_i(\omega)$ is chosen as $G_i(\omega) \propto \omega^2 e^{-\frac{\omega^2}{\alpha^2}}$. Then the desired Gaussian correlation with correlation length $a$ is achieved. Note that the stochastic process $f(x(t), t)$ is not uniquely determined from the variance, and the product form is one of the possible choices which guarantee the Gaussian memory (2). We fixed the parameters $\tau = 0.01$, $\eta = 1$, $\theta = 0.1$, and changed the correlation length $\xi$. The correlation time $\tau$ is much shorter than the quantity $\frac{\eta^2 L_0^4}{m \pi^2}$ (see (A.2)) so that the correlation between external noise $f$ and implicit noise contained in population $p$ is negligible. Here $L_0$ is the typical length scale accompanied by the probability distribution. For each parameters, more than 200 trajectories are simulated so that the mean-first-passage time well converges. The time step $\delta t$ of the stochastic simulations is around 0.09 which is longer than $\tau$, but sufficiently short so that the discretization of the equation makes sense.

Note that the time step larger than $\tau$ can consistently verify the form of the renormalized potential i)–iii), and is thus useful.

i) We assume that there exists a renormalized potential with which a Fokker-Planck equation approximates the spatio-temporal correlated diffusion process. The form of the renormalized potential is to be determined.

ii) The Fokker-Planck equation has a self-similarity, i.e. if $\hat{x}(n\delta t)$ is the solution of the discretized Markovian Langevin equation corresponding to the Fokker-Planck equation, $\sqrt{n}x(n\delta t)$ is also the solution with the scaled time step by an arbitrary factor $a$, $\delta t \rightarrow a\delta t$. Therefore our simulation is directly connected to the simulations with much shorter time step.

iii) So-obtained numerical results are consistent with our theoretical renormalized potential.

The $\kappa$-dependence of the numerical escape rate $\Gamma$ is compared with both the theoretical prediction (14) and the traditional Kramers formula in fig. 2. We find a very nice agreement between the theoretical and numerical results. In fig. 3, we also examined the steady-state distribution for parameters near the maximum of the escape rate of fig. 2 ($\tau = 0.01$, $\xi = 0.04750$, $\eta = 1$, and $\theta = 0.1$). In this way, the quasi-stable state becomes unstable due to the spatial correlation and leads to the renormalization of both the barrier height and the collisional prefactor in the Kramers escape rate. Noting that the normalization factor $(1 - \kappa)$ is positive, the escape rate shows a maximum at a weak but finite noise strength.

**Summary.** – In summary, stimulated by the work on the generalized Ornstein-Uhlenbeck process in momentum space [7], we investigated the diffusion process with the spatially and temporally correlated noise for the strong-damping regime. A Fokker-Planck equation is derived in a kind of Markovian limit that is characterized by a dimensionless parameter $\kappa = 2\pi \theta / \sqrt{\pi \xi^2 \eta}$. The drift term is renormalized by the spatial correlation. In contrast to the case of multiplicative noise, the diffusion coefficient is constant, since the noise is invariant under the uniform spatial translation. Intuitive understanding of the
renormalization of the potential should be as follows: the spatial correlation yields domains of the spatial coherence of the random noise with the typical size $\xi$. The systematic force pushes the particle out of the domain of coherence which amounts to an extra relaxation of the random noise. Thus the systematic force $-U'(x)$ accompanied by the spatial relaxation is equivalently replaced by the weaker force but without the spatial relaxation of the noise. We think that the basic scenario is ubiquitous at least qualitatively. The role of the potential renormalization is most clearly seen in the escape rate formula. The consequent escape rate has a maximum at an optimal $\kappa$, clearly seen in the escape rate formula. This correction term behaves as

$$\frac{\eta^2 U'^2}{2 L_0^2} \approx \frac{\xi^2 \eta^2}{\tau} \approx \frac{2U'^2 \tau}{\pi L_0^2 \kappa^2}. \quad (A.2)$$

Here $L_0$ is a typical spatial length scale accompanied by the potential $U$. Similarly, the $2n$-th-order correction with respect to noise $f$ contains terms of order $\kappa^n$, since each spatial derivative $\partial_x^n$ produces a factor $\kappa$. The total correction is thus expressed as a series of $\kappa$, which seems to diverge for large $\kappa$.

Our numerical simulation shows that the series seems to converge for the wide range of $\kappa \in [0, 1]$.

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