q-Analogues of some supercongruences related to Euler numbers

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ABSTRACT
Let \( E_n \) be the \( n \)th Euler number and \((a)_n = a(a + 1) \cdots (a + n - 1)\) the rising factorial. Let \( p > 3 \) be a prime. In 2012, Sun proved that

\[
\sum_{k=0}^{(p-1)/2} (-1)^k (4k + 1) \left(\frac{1}{2}\right)_k^3 \equiv p(-1)^{(p-1)/2} + p^3 E_{p-3} \mod p^4,
\]

which is a refinement of a famous supercongruence of Van Hamme. In 2016, Chen, Xie, and He established the following result:

\[
\sum_{k=0}^{p-1} (-1)^k (3k + 1) \left(\frac{1}{2}\right)_k^3 2^{3k} \equiv p(-1)^{(p-1)/2} + p^3 E_{p-3} \mod p^4,
\]

which was originally conjectured by Sun. In this paper, we give \( q \)-analogues of the above two supercongruences as well as another supercongruence related to Euler numbers by employing the \( q \)-WZ method. As a conclusion, we also provide a \( q \)-analogue of the following supercongruence of Sun:

\[
\sum_{k=0}^{(p-1)/2} \frac{(1/2)_k^2}{k!^2} \equiv (-1)^{(p-1)/2} + p^2 E_{p-3} \mod p^3.
\]

1. Introduction
In 1914, Ramanujan [30] gave a number of rapidly convergent series of \( 1/\pi \). Although the following series, due to Bauer [1], is not listed in [30], it gives an example of this kind:

\[
\sum_{k=0}^{\infty} (-1)^k (4k + 1) \left(\frac{1}{2}\right)_k^3 = \frac{2}{\pi},
\]
where \((a)_k = a(a+1)\cdots(a+k-1)\) denotes the rising factorial. Ramanujan’s formulas for \(1/\pi\) got widely admired in 1980s when they were discovered to offer efficient algorithms for calculating decimal digits of \(\pi\). See the monograph [2]. For a recent proof of Ramanujan’s series for \(1/\pi\), we refer the reader to Guillera [6].

In 1997, Van Hamme [35] developed interesting \(p\)-adic analogues of Ramanujan-type series. In particular, he conjectured the following supercongruence corresponding to (1):

\[
\sum_{k=0}^{(p-1)/2} (-1)^k (4k+1) \left(\frac{1}{2}\right)_k^3 \equiv p(-1)^{(p-1)/2} \pmod{p^3},
\]

where \(p\) is an odd prime. Note that we may calculate the sum in (2) for \(k\) up to \(p-1\), since the \(p\)-adic order of \((\frac{1}{2})_k/k!\) is 1 for \(k\) in the range \((p+1)/2 \leq k \leq p-1\). Congruences of this kind are called Ramanujan-type supercongruences. The congruence (2) was first proved by Mortenson [28] in 2008 using a \(6F_5\) transformation and the \(p\)-adic Gamma function and received a WZ (Wilf–Zeilberger [37,38]) proof by Zudilin [39] shortly afterwards.

In 2012, also employing the WZ method, Sun [33] gave the following refinement of (2):

\[
\sum_{k=0}^{m} (-1)^k (4k+1) \left(\frac{1}{2}\right)_k^3 \equiv p(-1)^{(p-1)/2} + p^3 E_{p-3} \pmod{p^4},
\]

where \(m = p-1\) or \((p-1)/2\), and \(E_{p-3}\) is the \((p-3)\)th Euler number, which may be given by

\[
\sum_{k=0}^{\infty} E_k \frac{x^k}{k!} = \frac{2}{e^x + e^{-x}}.
\]

In recent years, \(q\)-analogues (or rational function generalizations) of congruences and supercongruences have aroused the interest of many researchers (see [7,8,10–14,16–18,20–23,25,26,29,36,40]). For instance, the author in [7] gave the following \(q\)-analogue of (2): for any odd integer \(n > 1\),

\[
\sum_{k=0}^{(n-1)/2} (-1)^k q^k [4k+1] \left[\frac{(q; q^2)_k^3}{(q^2; q^2)_k^3}\right] \equiv [n]q^{(n-1)/4} (-1)^{(n-1)/2} \pmod{[n] \Phi_n(q)^2},
\]

Here we need to be familiar with the standard \(q\)-notation. The \(q\)-integer is defined by \([n] = 1 + q + \cdots + q^{n-1}\), and the \(q\)-shifted factorial is defined as \((a; q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})\) for \(n \geq 1\) and \((a; q)_0 = 1\). Moreover, let the \(n\)th cyclotomic polynomial \(\Phi_n(q)\) be

\[
\Phi_n(q) = \prod_{1 \leq k \leq n, \gcd(k,n)=1} (q - \zeta^k),
\]

where \(\zeta = e^{2\pi i/n}\) (an \(n\)th primitive root of unity). It is well known that \(\Phi_n(q)\) is an irreducible polynomial in the polynomial ring \(\mathbb{Z}[q]\).
Let $A(q)$ and $B(q)$ be two rational functions in $q$ and $P(q)$ a polynomial in $q$ with integer coefficients. We say that $A(q)$ is congruent to $B(q)$ modulo $P(q)$, denoted by $A(q) \equiv B(q) \pmod{P(q)}$, if $P(q)$ divides the reduced numerator of $A(q) - B(q)$ in $\mathbb{Z}[q]$.

In this paper, we shall give a $q$-analogue of (3).

**Theorem 1.1:** Let $n$ be a positive odd integer. Then

$$\sum_{k=0}^{N} (-1)^k q^{k^2} (4k + 1) \frac{q^{3k}}{(q^2; q^2)_k} \equiv (-1)^{(n-1)/2} q^{(1 - n^2)/4} \left( \frac{\binom{n}{2}}{n} + \frac{(n^2 - 1)(1 - q^2)}{24} [n]^3 \right)$$

$$+ [n]^3 \sum_{k=1}^{(n-1)/2} \frac{q^{k^2} (q^2; q^2)_k}{[2k][2k - 1][2k]^3} \pmod{[n] \Phi_n(q)^3},$$

(4)

where $N = (n - 1)/2$ or $n-1$.

Note that Sun [31, Equation (3.1)] proved that, for any odd prime $p$,

$$\sum_{k=1}^{(p-1)/2} \frac{4^k}{k(2k - 1)\binom{2k}{k}} \equiv 2E_{p-3} \pmod{p}.$$

Letting $n = p$ be a prime greater than 3 and taking $q \to 1$ in (4), we immediately get (3).

Still using the WZ method, Guillera and Zudilin [5] established the following supercongruence: for odd primes $p$,

$$\sum_{k=0}^{(p-1)/2} (-1)^k (3k + 1) \frac{\left(\frac{1}{2}\right)^3 k^{3k}}{k!^3} \equiv p(-1)^{(p-1)/2} \pmod{p^3}.$$  

(5)

Moreover, in 2016, Chen, Xie, and He [4] gave the following refinement of (5):

$$\sum_{k=0}^{p-1} (-1)^k (3k + 1) \frac{\left(\frac{1}{2}\right)^3 k^{3k}}{k!^3} \equiv p(-1)^{(p-1)/2} + p^3 E_{p-3} \pmod{p^4},$$

(6)

which was originally conjectured by Sun [32, Conjecture 5.1]. The author [10] established a $q$-analogue of (5):

$$\sum_{k=0}^{n-1} (-1)^k [3k + 1] \frac{q^{k^3}}{(q; q^2)_k} \equiv [n] q^{(n-1)^2/4} (-1)^{(n-1)/2} \pmod{[n] \Phi_n(q)^2}.$$  

(7)

We point out that a supercongruence for the left-hand side of (5) modulo $p^4$, also conjectured by Sun [32, Conjecture 5.1], was recently confirmed by Mao [27].

In this paper, we shall also establish a $q$-analogue of (6).
Let $n$ be a positive odd integer. Then

$$
\sum_{k=0}^{n-1} (-1)^k [3k + 1] \frac{(q^2)^3}{(q; q^2)_k^3} \equiv (-1)^{(n-1)/2} q^{(1-n^2)/4} \left( \frac{q^{(n)} [n]}{} + \frac{(n^2 - 1)(1 - q^2)}{24} [n]^3 \right)
$$

$$
+ [n]^3 \sum_{k=1}^{(n-1)/2} \frac{q^k (q^2; q^2)_k}{[2k][2k - 1](q; q^2)_k} \quad (\text{mod } [n] \Phi_n(q)^3).
$$

(8)

Recently, the author and Liu [15] established a supercongruence similar to (3): for any prime $p > 3$,

$$
\sum_{k=0}^{(p+1)/2} (-1)^k (4k - 1) \frac{(-1)^3}{k!^3} \equiv p(-1)^{(p+1)/2} + p^3(2 - E_{p-3}) \quad (\text{mod } p^4).
$$

(9)

Moreover, a partial $q$-analogue of the supercongruence (9) modulo $p^3$ is already known: for any positive odd integer $n$,

$$
\sum_{k=0}^{M} (-1)^k [4k - 1] \frac{(q^{-1}; q^2)_k^3}{(q^2; q^2)_k^3} q^{k^2+2k} \equiv [n](-q)^{(n-3)(n+1)/4} \quad (\text{mod } [n] \Phi_n(q)^2),
$$

(10)

where $M$ stands for $n-1$ or $(n + 1)/2$. See [17, Section 5] or [20, Theorem 4.9] with $r = -1, d = 2$ and $a = 1$.

The last aim of this paper is to give the following $q$-analogue of (9).

**Theorem 1.3:** Let $n$ be a positive odd integer. Then, modulo $[n] \Phi_n(q)^3$,

$$
\sum_{k=0}^{M} (-1)^k [4k - 1] \frac{(q^{-1}; q^2)_k^3}{(q^2; q^2)_k^3} q^{(k+1)^2} \equiv (-1)^{(n+1)/2} q^{(1-n^2)/4} \left( \frac{q^{(n)} [n]}{} + \frac{(n^2 - 1)(1 - q^2)}{24} [n]^3 \right)
$$

$$
+ [n]^3 q^2 + (-1)^{(n-1)/2} q^{(n-1)^2/4+1} - [n]^3 \sum_{k=1}^{(n-1)/2} \frac{q^{1-k} (q^2; q^2)_k}{[2k - 1](q; q^2)_k},
$$

(11)

where $M = (n + 1)/2$ or $n-1$.

Note that Sun [32] has proved that, for any odd prime $p$,

$$
\sum_{k=1}^{(p-1)/2} \frac{4^k}{(2k - 1)(2k)_k^3} \equiv E_{p-3} - 1 + (-1)^{(p-1)/2} \quad (\text{mod } p).
$$

Letting $n = p, M = (p + 1)/2$, and taking $q \to 1$ in (11), we are led to (9).
The remainder of the paper proceeds as follows. We prove Theorems 1.1–1.3 in Sections 2–4, respectively, by using the \( q \)-WZ method, together with a \( q \)-analogue of Wolstenholme’s congruence and a \( q \)-analogue of Morley’s congruence. In Section 5, we give some concluding remarks and two open problems. Particularly in Corollary 5.2, using a recent result of Wei [36], we shall deduce a \( q \)-analogue of another supercongruence of Sun from Theorem 1.1.

2. Proof of Theorem 1.1

Recall that the \( q \)-binomial coefficients \( \binom{M}{N} \) are defined by

\[
\binom{M}{N}_q = \begin{cases} 
(q; q)_M/(q; q)_N(q; q)_{M-N} & \text{if } 0 \leq N \leq M, \\
0 & \text{otherwise.}
\end{cases}
\]

We need the following \( q \)-analogue of Wolstenholme’s congruence (see [19, Lemma 3.1]).

**Lemma 2.1:** Let \( n \) be a positive integer. Then

\[
\binom{2n-1}{n-1}_q \equiv (-1)^{n-1}q^{(n)_2}/12 + (n^2 - 1)(1 - q^2)[n]^2 \pmod{\Phi_n(q^3)}.
\]

Moreover, a \( q \)-analogue of Morley’s congruence (see [24, (1.5)]) and a \( q \)-analogue of Fermat’s little theorem (see [19, Lemma 3.2]) will also be used in our proof.

**Lemma 2.2:** Let \( n \) be a positive odd integer. Then, modulo \( \Phi_n(q^3) \),

\[
\binom{n-1}{n-1/2}_{q^2} \equiv (-1)^{(n-1)/2}q^{(1-n^2)/4}\left((-q; q)_{n-1}^2 - \frac{(n^2 - 1)(1 - q^2)[n]^2}{24}\right).
\]

**Lemma 2.3:** Let \( n \) be a positive odd integer. Then

\[
(-q; q)_{n-1} \equiv 1 \pmod{\Phi_n(q)}.
\]  

**Proof of Theorem 1.1:** By [11, Theorem 6.1], modulo \([n]\Phi_n(q)(1 - aq^n)(a - q^n)\),

\[
\sum_{k=0}^{(n-1)/2} (aq; q^2)_k(q/a; q^2)_k(q/b; q^2)_k(q; q^2)_k b^k
\]

\[
\equiv \sum_{k=0}^{n-1} (aq; q^2)_k(q/a; q^2)_k(q/b; q^2)_k(q; q^2)_k b^k,
\]

where \( a \) and \( b \) are indeterminates. Letting \( b \to \infty \) and \( a = 1 \) in (13), we get

\[
\sum_{k=0}^{(n-1)/2} (-1)^k q^{k^2} (aq; q^2)_k^3 (aq^2; q^2)_k^3 \equiv \sum_{k=0}^{n-1} (-1)^k q^{k^2} (aq; q^2)_k^3 (aq^2; q^2)_k^3 \pmod{\Phi_n(q^4)}.
\]  

(14)
By [20, Theorem 4.1], both sides of (14) are congruent to 0 modulo \([n]\), and so (14) is also true modulo \([n]\Phi_n(q)^3\). Thus, to prove Theorem 1.1, it suffices to prove the \(N = (n - 1)/2\) case.

We introduce two rational functions in \(q\):

\[
F(m, k) = (-1)^{m+k} q^{(m-k)k/2} [4m + 1] (q; q^2)_m (q; q^2)_{m+k} (q^2; q^2)_m (q; q^2)_{m-k} (q; q^2)_k^2,
\]

\[
G(m, k) = (-1)^{m+k} q^{(m-k)k/2} (q; q^2)_m (q; q^2)_{m+k-1} (1 - q)(q^2; q^2)_{m-1} (q; q^2)_{m-k} (q; q^2)_k^2,
\]

where we assume that \(1/(q^2; q^2)_M = 0\) for negative integers \(M\). As mentioned in [7], the functions \(F(m, k)\) and \(G(m, k)\) form a \(q\)-WZ pair. Namely, they satisfy the following equality

\[
F(m, k - 1) - F(m, k) = G(m + 1, k) - G(m, k).
\]

(15)

Summing (15) over \(m\) from 0 to \((n - 1)/2\), we get

\[
\sum_{m=0}^{(n-1)/2} F(m, k - 1) - \sum_{m=0}^{(n-1)/2} F(m, k) = G\left(\frac{n + 1}{2}, k\right) - G(0, k) = G\left(\frac{n + 1}{2}, k\right).
\]

(16)

Summing (16) further over \(k\) from 1 to \((n - 1)/2\) and noticing that \(F(m, (n - 1)/2) = 0\) for \(m < (n - 1)/2\), we obtain

\[
\sum_{m=0}^{(n-1)/2} F(m, 0) - F\left(\frac{n - 1}{2}, \frac{n - 1}{2}\right) = \sum_{k=1}^{(n-1)/2} G\left(\frac{n + 1}{2}, k\right).
\]

(17)

Note that, for \(k = 1, 2, \ldots, (n - 1)/2\), we have

\[
G\left(\frac{n + 1}{2}, k\right) = (-1)^{(n+1)/2+k} q^{((n+1)/2-2-k)^2} (q; q^2)_{(n+1)/2} (q; q^2)_{(n-1)/2+k} (1 - q)(q^2; q^2)_{(n-1)/2-k} (q; q^2)_k^2
\]

\[
= (-1)^{(n+1)/2+k} q^{((n+1)/2-2-k)^2} \frac{(1 - q^n)(q; q^2)^3_{(n-1)/2} (q^{n+2}; q^2)_{k-1}}{(1 - q)(q^2; q^2)_{(n-1)/2-k} (q^2; q^2)_{(n-1)/2-k} (q; q^2)_k^2}.
\]

(18)

Since \(q^n \equiv 1\ (\text{mod } \Phi_n(q))\), there hold

\[
(q^2; q^2)_{(n+1)/2-k} = \frac{(q^2; q^2)_{(n-1)/2}}{(q^{n+3-2k}; q^2)_{k-1}}
\]

\[
\equiv \frac{(q^2; q^2)_{(n+1)/2}}{(q^{3-2k}; q^2)_{k-1}} = (-1)^{k-1} q^{(k-1)^2} \frac{(q^2; q^2)_{(n-1)/2}}{(q; q^2)_{k-1}} \ (\text{mod } \Phi_n(q)),
\]

(19)
and
\[
\frac{(q; q^2)_{(n-1)/2}}{(q^2; q^2)_{(n-1)/2}} = \prod_{j=1}^{(n-1)/2} \frac{1 - q^{2j-1}}{1 - q^{n-2j+1}}
\equiv \prod_{j=1}^{(n-1)/2} \frac{1 - q^{2j-1}}{1 - q^{1-2j}} = (-1)^{(n-1)/2} q^{(n-1)^2/4} \pmod{\Phi_n(q)}. \tag{20}
\]

Employing the above two $q$-congruences, we deduce from (18) that, for $1 \leq k \leq (n - 1)/2$,
\[
G\left(\frac{n + 1}{2}, k\right) = \frac{q^k(1 - q^n)^3(q^2; q^2)_{k-1}}{(1 - q)(1 - q^{2k-1})(q; q^2)_k} = \frac{q^k[n]^3(q^2; q^2)_k}{[2k][2k - 1](q; q^2)_k} \pmod{\Phi_n(q)^4}. \tag{21}
\]

Since $(q^2; q^2)_k/(q; q^2)_k = (-q; q^2)^2/[2k]_k$ and the $q$-binomial coefficient can be written as a product of different cyclotomic polynomials (see [3, Lemma 1]), we see that the right-hand side of (21) is congruent to 0 modulo $[n]$, and so is (18). Namely, the $q$-congruence (21) holds modulo $[n]\Phi_n(q)^3$.

In addition, by Lemmas 2.1–2.3,
\[
F\left(\frac{n - 1}{2}, \frac{n - 1}{2}\right) = \frac{[n]}{(-q; q^2)_{n-1}^2} \left[\frac{2n - 1}{n - 1}\right]^{n-1/2} q^2
\equiv (-1)^{(n-1)/2} q^{(1-n^2)/4} \left\{ q([n]) + \frac{(n^2 - 1)(1 - q^2)}{24} \left(2 - \frac{q^2([n])}{(-q; q^2)_{n-1}^2}\right) [n]^3 \right\}
\equiv (-1)^{(n-1)/2} q^{(1-n^2)/4} \left\{ q([n]) + \frac{(n^2 - 1)(1 - q^2)}{24} [n]^3 \right\} \pmod{[n]\Phi_n(q)^3}, \tag{22}
\]
where we have used $q([n]) \equiv 1 \pmod{\Phi_n(q)}$ for odd $n$ in the last step.

Substituting (22) and the modulus $[n]\Phi_n(q)^3$ case of (21) into (17), we are led to (4) in the case where $N$ is equal to $(n - 1)/2$. This completes the proof of the theorem. \hfill \blacksquare

### 3. Proof of Theorem 1.2

The author [10] employed the following functions
\[
F(m, k) = (-1)^m [3m - 2k + 1] \left[\frac{2m - 2k}{m}\right] \frac{(q; q^2)_m(q; q^2)_{m-k}}{(q; q)_m(q^2; q^2)_{m-k}}
\]
\[
G(m, k) = (-1)^{m+1} [m] \left[\frac{2m - 2k}{m-1}\right] \frac{(q; q^2)_m(q; q^2)_{m-k}q^{m+1-2k}}{(q; q)_m(q^2; q^2)_{m-k}}
\]
to establish (7). It is not difficult to verify that $F(m, k)$ and $G(m, k)$ satisfy the relation
\[
F(m, k - 1) - F(m, k) = G(m + 1, k) - G(m, k). \tag{23}
\]
That is, they form a $q$-WZ pair.

Since (8) is clearly true for $n = 1$, we now assume that $n \geq 3$. Summing (23) over $m = 0, 1, \ldots, n - 1$, we obtain

$$\sum_{m=0}^{n-1} F(m, k - 1) - \sum_{m=0}^{n-1} F(m, k) = G(n, k). \quad (24)$$

Summing (24) further over $k = 1, \ldots, n - 1$ and noticing that $F(m, n - 1) = 0$ for $m \leq n - 1$, we arrive at

$$\sum_{m=0}^{n-1} F(m, 0) = \sum_{k=1}^{n-1} G(n, k) = \sum_{k=1}^{(n+1)/2} G(n, k). \quad (25)$$

In view of

$$\left[ \begin{array}{c} 2m \\ m \end{array} \right] = \frac{(q; q^2)_m (-q; q^2)_m}{(q^2; q^2)_m},$$

the identity (25) can be written as

$$\sum_{m=0}^{n-1} (-1)^n [3m + 1] \frac{(q; q^2)_m^3}{(q; q^2)_m} = \frac{[n][2n-1]}{(-q; q)_n-1} \sum_{k=1}^{(n+1)/2} \left[ \begin{array}{c} 2n - 2k \\ n - 1 \end{array} \right] \frac{(q; q^2)_{n-k}q^{n+1-2k}}{(q^2; q^2)_{n-k}}. \quad (26)$$

For $1 \leq k \leq (n - 1)/2$, we have

$$\frac{(q; q^2)_{n-k}}{(q^2; q^2)_{n-k}} = \frac{(1 - q^n)(q; q^2)_{(n-1)/2}(q^{n+2}; q^2)_{(n-1)/2-k}}{(1 - q^{n-1})(q^2; q^2)_{(n-3)/2}(q^{n+1}; q^2)_{(n+1)/2-k}}$$

$$= \frac{(1 - q^n)(q^{n+2-2k}; q^2)_{k-1}}{(1 - q^{n-1})(q^{n+1-2k}; q^2)_{k-1}}$$

$$= \frac{(1 - q^n)(q^{2-2k}; q^2)_{k-1}}{(1 - q^{1})(q^{1-2k}; q^2)_{k-1}} = \frac{q^k(1 - q^n)(q^{2}; q^2)_{k-1}}{(q^2; q^2)_k} \pmod{\Phi_n(q)^2},$$

and

$$\left[ \begin{array}{c} 2n - 2k + 1 \\ n \end{array} \right] = \prod_{k=1}^{n-2k+1} \frac{1 - q^{n+j}}{1 - q^j} \equiv 1 \pmod{\Phi_n(q)}.$$

It follows that

$$\sum_{k=1}^{(n+1)/2} \left[ \begin{array}{c} 2n - 2k \\ n - 1 \end{array} \right] \frac{(q; q^2)_{n-k}q^{n+1-2k}}{(q^2; q^2)_{n-k}}$$

$$= \frac{(q; q^2)_{(n-1)/2}}{(q^2; q^2)_{(n-1)/2}} + \sum_{k=1}^{(n-1)/2} \left[ \begin{array}{c} 2n - 2k \\ n - 1 \end{array} \right] \frac{(q; q^2)_{n-k}q^{n+1-2k}}{(q^2; q^2)_{n-k}}$$
\[ 
\begin{align*}
\sum_{n=0}^{\frac{n-1}{2}} \frac{1}{q^2 (-q; q)_{n-1}} + \sum_{k=1}^{\frac{(n-1)/2}{n}} \left[ \frac{2n-2k+1}{n} \right] (1-q^n)(q; q^2)_{n-k} q^{n+1-2k} / (1-q^{2n-2k+1})(q; q^2)_{n-k} \\
\equiv (-1)^{(n-1)/2} q^{(1-n^2)/4} \left( (-q; q)_{n-1} - \frac{(n^2 - 1)(1-q)^2 [n]^2}{24} \right) \\
+ [n]^2 \sum_{k=1}^{(n-1)/2} \frac{q^k (q^2; q^2)_k}{[2k][2k-1](q; q^2)_k} (\text{mod } \Phi_n(q)^3),
\end{align*}
\]

where we have used Lemmas 2.2 and 2.3 in the last step.

By Lemmas 2.1 and 2.3, we have
\[ 
\frac{[2n-1]}{(-q; q)_{n-1}} \equiv (-1)^{n-1} q^{2} (-q; q)_{n-1} + \frac{(n^2 - 1)(1-q)^2 [n]^2}{12} (\text{mod } \Phi_n(q)^3).
\]

Substituting (27) and (28) into (26) and making some simplifications, we immediately obtain (8).

### 4. Proof of Theorem 1.3

This time we need the following two rational functions in \( q \):
\[ 
F(m, k) = (-1)^{m+k} q^{(m-k+1)^2} \frac{[4m - 1](q^{-1}; q^2)_m (q^{-1}; q^2)_{m+k}}{(q^2; q^2)_m (q^2; q^2)_{m-k} (q^{-1}; q^2)_k},
\]
\[ 
G(m, k) = (-1)^{m+k} q^{(m-k+1)^2} \frac{(q^{-1}; q^2)_m (q^{-1}; q^2)_{m+k-1}}{(1-q)(q^2; q^2)_{m-1} (q^2; q^2)_{m-k} (q^{-1}; q^2)_k}.
\]

It is easy to check that the functions \( F(m, k) \) and \( G(m, k) \) form a \( q \)-WZ pair (in fact, they are the \((d, r) = (2, -1)\) case of the \( q \)-WZ pair in [9]). Namely, they meet the following relation:
\[ 
F(m, k - 1) - F(m, k) = G(m + 1, k) - G(m, k).
\]

Summing (29) over \( m \) from 0 to \((n + 1)/2\), we get
\[ 
\sum_{m=0}^{(n+1)/2} F(m, k - 1) - \sum_{m=0}^{(n+1)/2} F(m, k) = G \left( \frac{n + 3}{2}, k \right) - G(0, k) = G \left( \frac{n + 3}{2}, k \right).
\]

Summing (30) further over \( k \) from 1 to \((n + 1)/2\) and noticing that \( F(m, (n + 1)/2) = 0 \) for \( m < (n + 1)/2 \), we obtain
\[ 
\sum_{m=0}^{(n+1)/2} F(m, 0) = F \left( \frac{n + 1}{2}, \frac{n + 1}{2} \right) + \sum_{k=1}^{(n+1)/2} G \left( \frac{n + 3}{2}, k \right).
\]

By (22) and the \( q \)-congruence \([2n + 1]/[n + 1]^2 \equiv 1 - q[n]^2 \) (mod \( \Phi_n(q)^3 \)), we have
\[ 
F \left( \frac{n + 1}{2}, \frac{n + 1}{2} \right).
\]
This means that
\[
\sum_{k=1}^{(n+1)/2} G\left(\frac{n+3}{2}, k\right) \equiv q^2[n]^3 - [n]^3 \sum_{k=1}^{(n-1)/2} \frac{q^{1-k}(q^2; q^2)_k}{[2k-1](q; q^2)_k} \pmod{[n] \Phi_n(q)^3}. \tag{33}
\]
Substituting (32) and (33) into (31), we immediately obtain (11) for \( M = (n + 1)/2 \).

To prove (11) is also true for \( M = n - 1 \), we only need to prove that
\[
\sum_{k=(n+3)/2}^{n-1} (-1)^k [4k - 1] \frac{(q^{-1}; q^2)_k}{(q^2; q^2)_k} (q^{k+1})^2 \equiv 0 \pmod{[n] \Phi_n(q)^3}.
\]
By (10), the above congruence holds modulo \([n]\). It remains to prove the modulus \( \Phi_n(q)^4 \) case. This can be easily done by showing that the \(((n + 3)/2 + k)\)-th and \((n - 1 - k)\)-th summands cancel each other modulo \( \Phi_n(q)^4 \). (Note that each summand is congruent to 0 modulo \( \Phi_n(q)^3 \).)
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5. Concluding remarks and open problems

From (3) and (6), one sees that, for any prime \( p > 3 \),

\[
\sum_{k=0}^{m} (-1)^k (4k + 1) \left( \frac{1}{2} \right)_k^3 \equiv \sum_{k=0}^{p-1} (-1)^k (3k + 1) \left( \frac{1}{2} \right)_k^3 2^{3k} \pmod{p^4}, \tag{34}
\]

where \( m = p-1 \) or \( (p - 1)/2 \). Further, combining (4) and (8), we have the following \( q \)-analogue of (34): for any positive odd integer \( n \),

\[
\sum_{k=0}^{N} (-1)^k q^k [4k + 1] \frac{(q; q^2)_k}{(q^2; q^2)_k} = \sum_{k=0}^{n-1} (-1)^k [3k + 1] \frac{(q; q^2)_k}{(q; q^2)} (\mod{[n] \Phi_n(q)^3}), \tag{35}
\]

where \( N = (n - 1)/2 \) or \( n-1 \).

We now provide a conjectural parametric generalization of (35).

**Conjecture 5.1:** Let \( n \) be a positive odd integer. Then, modulo \([n] \Phi_n(q)(1 - aq^n)(a - q^n)\),

\[
\sum_{k=0}^{N} (-1)^k q^k [4k + 1] \frac{(aq; q^2)_k(q/a; q^2)_k(q; q^2)_k}{(aq^2; q^2)_k(q^2/a; q^2)_k(q^2; q^2)_k} = \sum_{k=0}^{n-1} (-1)^k [3k + 1] \frac{(aq; q^2)_k(q/a; q^2)_k(q; q^2)_k}{(aq; q)_k(q/a; q)_k(q; q)_k}, \tag{36}
\]

where \( N = (n - 1)/2 \) or \( n-1 \).

Using the ‘creative microscoping’ method introduced in [20] and the Chinese remainder theorem for relatively prime polynomials, the author [11, Theorem 5.3] has shown that the left-hand side of (36) is congruent to

\[
(-1)^{(n-1)/2} q^{(n-1)^2/4} [n] + (-1)^{(n-1)/2} q^{(n-1)^2/4} \frac{(1 - aq^n)(a - q^n)}{(1 - a)^2} [n]
\]

\[
- \frac{(1 - aq^n)(a - q^n)}{(1 - a)^2} [n] \sum_{k=0}^{(n-1)/2} \frac{(q; q^2)_k}{(aq^2; q^2)_k(q^2/a; q^2)_k}
\]

modulo \([n] \Phi_n(q)(1 - aq^n)(a - q^n)\). But it seems rather difficult to prove that the right-hand side of (36) is also congruent to the above expression, though a three-parametric generalization of (7) was already proved by the author and Schlosser [17, Theorem 6.1] (see also [20, Conjecture 4.6]).

In 2011, Sun [32] studied many interesting supercongruences related to Euler numbers. In particular, Sun [32, Theorems 1.1 and 1.2] proved that, for any prime \( p > 3 \),

\[
\sum_{k=0}^{p-1} \frac{1}{2^k} \binom{2k}{k} \equiv (-1)^{(p-1)/2} - p^2 E_{p-3} \pmod{p^3}, \tag{37}
\]
Hence, the

\[
\sum_{k=0}^{p-1} \frac{1}{16k} \binom{2k}{k}^2 \equiv (-1)^{(p-1)/2} + p^2 E_{p-3} \pmod{p^3},
\]

(38)

\[
\sum_{k=0}^{p-1} \frac{1}{16k} \binom{2k}{k}^2 \equiv (-1)^{(p-1)/2} - p^2 E_{p-3} \pmod{p^3}.
\]

(39)

Recently, Wei [36, Theorem 1.1 with \(c = q, d \to \infty\)] gave the following result: for any positive odd integer \(n\), and \(N = (n - 1)/2\) or \(n-1\),

\[
\sum_{k=0}^{N} (-1)^k q^k [4k + 1] \frac{(q; q^2)_k^3}{(q^2; q^2)_k^3} \equiv q^{(1-n)/2} \left( [n] + \frac{(n^2 - 1)(1 - q)^2}{24} \right) \times \sum_{k=0}^{(n-1)/2} \frac{(q; q^2)_k^2}{(q^2; q^2)_k^2} q^{2k} \pmod{[n] \Phi_n(q)^3},
\]

(40)

which is clearly a \(q\)-analogue of the relation between (3) and (38): for any prime \(p > 3\),

\[
\sum_{k=0}^{m} (-1)^k (4k + 1) \frac{1}{k!} \equiv p \sum_{k=0}^{(p-1)/2} \frac{1}{16k} \binom{2k}{k}^2 \pmod{p^4},
\]

where \(m = p-1\) or \((p - 1)/2\).

Note that the author, Pan and Zhang [16] gave the following \(q\)-supercongruence:

\[
\sum_{k=0}^{(n-1)/2} \frac{(q; q^2)_k^2}{(q^2; q^2)_k^2} q^{2k} \equiv (-1)^{(n-1)/2} q^{(n^2-1)/4} \pmod{\Phi_n(q)^2}.
\]

(41)

Hence, the \(q\)-supercongruence (40) may be written as

\[
q^{(n-1)/2} \sum_{k=0}^{N} (-1)^k q^k [4k + 1] \frac{(q; q^2)_k^3}{(q^2; q^2)_k^3}
\]

\[
\equiv [n] \sum_{k=0}^{(n-1)/2} \frac{(q; q^2)_k^2}{(q^2; q^2)_k^2} q^{2k}
\]

\[
+ \frac{(n^2 - 1)(1 - q)^2}{24} [n]^3 (-1)^{(n-1)/2} q^{(n^2-1)/4} \pmod{[n] \Phi_n(q)^3}.
\]

(42)

Combining (4) and (42), we immediately obtain a \(q\)-analogue of (38).

**Corollary 5.2:** Let \(n\) be a positive odd integer. Then

\[
\sum_{k=0}^{(n-1)/2} \frac{(q; q^2)_k^2}{(q^2; q^2)_k^2} q^{2k} \equiv (-1)^{(n-1)/2} q^{(n^2-1)/4}
\]

\[
+ (-1)^{(n-1)/2} \frac{(n^2 - 1)(1 - q)^2}{24} [n]^2 \left( q^{-(n-1)/4} - q^{(n^2-1)/4} \right)
\]
\[ q^{(n-1)/2} \left\lfloor \frac{n-1}{2} \right\rfloor^2 \sum_{k=1}^{(n-1)/2} \frac{q^k (q^2; q^2)_k}{[2k][2k - 1](q; q^2)_k} \quad (\text{mod } \Phi_n(q)^3). \]

However, to the best of the author’s knowledge, no \(q\)-analogues of (37) and (39), even conjectural, are known in the literature. It follows from (37) and (39) that, for any odd prime \(p\),

\[
\sum_{k=0}^{p-1} \frac{1}{2^k} \binom{2k}{k} \equiv \sum_{k=0}^{p-1} \frac{1}{16^k} \binom{2k}{k}^2 \quad (\text{mod } p^3). \quad (43)
\]

On the other hand, for odd \(n\), the author [8] proved that

\[
\sum_{k=0}^{n-1} q^k \frac{2k}{(-q; q)_k} \equiv (-1)^{(n-1)/2} q^{(n^2-1)/4} \quad (\text{mod } \Phi_n(q)^2), \quad (44)
\]
confirming a conjecture of Tauraso [34].

In light of (41) and (44), we believe that the following \(q\)-analogue of (43) is true.

**Conjecture 5.3:** Let \(n\) be a positive odd integer. Then

\[
\sum_{k=0}^{n-1} q^k \frac{2k}{(-q; q)_k} \equiv \sum_{k=0}^{n-1} \frac{(q; q^2)_k^2}{(q^2; q^2)_k^2} q^{2k} \quad (\text{mod } \Phi_n(q)^3).
\]

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