From factorizable S-matrices to Conformal Invariance

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1 Introduction

Recently there has been spectacular progress in 1+1 dimensional quantum field theory\(^1\). Although one may argue about whether this field belongs to physics or mathematics, it is abundantly clear that a successful non-perturbative attack on 4-dimensional field theories will require a thorough understanding of the simpler counterpart in two dimensions. This is about all we will say to justify our living in so narrow an environment, except perhaps to proclaim our believe that if someone finds beautiful structures in mathematical physics, we may be sure that Nature will make use of it. And beautiful structures surely we will find. In the present lectures I'll try to make them as accessible as I can.

Trying to make progress in physics requires in general hard work. Yet this can be very rewarding, if miracles do happen along the way - and in exactly soluble models they do. After all some fine tuning of coupling constants has taken place in order to guarantee solubility. One way to ensure this is to impose factorizability on the \(S\)-matrix, i.e. we require the \(n\)-particle \(S\)-matrix to be a product of 2-particle ones, consequently not allowing particle production. This is guaranteed, if the model possesses an infinite number of conservation laws\(^0\).

In section 2, we will characterize our models by specifying their conservation laws and then explain the Bootstrap Principle to construct factorized \(S\)-matrices. This program is executed in section 3 for some models. In section 4 the Thermodynamic Bethe Ansatz will be used to connect these \(S\)-matrices to their ultra-violet limiting conformally invariant field theory. Section 5 contains a short review of conformal invariance, so that the unfamiliar reader doesn't have to go shopping somewhere else, although in practice he may want to look up ref.\(^0\). Finally in section 6 we will study perturbations of conformally invariant field theories, find out which of them are candidates for exhibiting an infinite number of conservation laws and thus make contact with section 2. For the benefit of the reader - hopefully -, I have tried to make these notes as self-contained as possible, at the risk of copying from too many places already well known material.

\(^1\)From a logical point of view, this last section should come first, but it has to contain older, more technical material for which excellent review articles exist\(^0\). Therefore I have placed it near the end, hoping to delay as much as possible the loss of my audience. In this spirit I tried to exhibit - as we say in portuguese - 'o caminho das pedras'.
2 1+1 dimensional factorizable S-matrix theory

2.1 Kinematics and scattering

We will consider scattering of $n$ particles $A_a$, $a = 1, 2\ldots, n$, whose masses are $m_a$. Their momenta satisfy the mass-shell condition

$$p_\mu p^\mu = p \bar{p} = m^2,$$

where the light-cone components of $p^\mu$ are $p = p^0 + p^1$ and $\bar{p} = p^0 - p^1$. The on-shell condition may be conveniently parametrized introducing the rapidity $\theta$:

$$(p^0, p^1) = (m \cosh \theta, m \sinh \theta).$$

In order to ensure exact integrability we will assume that the field theory giving rise to our $S$-matrix possess an infinite number of nontrivial, commutative integrals of motion.

Although this is a well suited requirement in $d = 2$ dimensions, for $d > 2$ a theorem by Coleman and Mandula \cite{0} states that in a non-trivial Poincaré invariant field theory the most general invariance group is a product of the Poincaré and an internal symmetry group. The proof assumes some analyticity in energy and momentum transfer (angle). This poses no problem in two dimensions, since the scattering angle is anyhow only either 0 or $\pi$.

These non-trivial, i.e. other than energy-momentum, conserved charges $Q_{\mu_1\ldots\mu_s}$ transform as $s$-th order tensors under the Lorentz group. In the light-cone representation we call them $P_s$, $s = s_1, s_2, \ldots$, where the label $s$ indicates the spin of $P_s$. Actually in two dimension we have no rotation group, so that spin refers to Lorentz-spin, specifying how $P_s$ behaves under a Lorentz transformation $L_\alpha : \theta \rightarrow \theta' = \theta + \alpha$. Thus

$$P_s \rightarrow P'_s = e^{s\alpha} P_s.$$

For example the momentum $p$ has spin one: $p \rightarrow p' = e^\alpha p$ and the parity transformed $\bar{p}$ has spin minus one. Therefore we have $P_1 = p$ and $P_{-1} = \bar{p}$. Since parity relates the integrals of motion $P_{-s}$ to $P_s$, we will consider only $s > 0$ as we will deal only with parity conserving theories.
$P_s$ acts on one-particle states as

$$P_s \mid A_a(p) > = \omega_s^a(p) \mid A_a(p) > .$$

(4)

Since $P_s$ carries spin $s$ the Lorentz-transformation property equ. (3) requires $\omega_s^a(p)$ to be of the form

$$\omega_s^a(p) = \kappa_s^a p^s = \kappa_s^a (m_a)^s e^{s\theta},$$

(5)

or

$$P_s | A_a(p) >= \gamma_s^a | A_a(p) >, \quad \gamma_s^a = \kappa_s^a (m_a)^s,$$

(6)

where $\kappa_s^a$ are real constants. For example $\kappa_1^a = 1$.

$P_s$ are integrals of local densities. Therefore their action on well separated multiparticle in or out states is the sum of the one-particle contributions. Consequently in a scattering process

$$p_1, \ldots, p_n \to p'_1, \ldots, p'_m$$

(7)

we have

$$\sum_{i=1}^{n} (p_i)^s = \sum_{i=1}^{m} (p'_i)^s.$$

(8)

At least one non-trivial conservation law with $s > 1$ implies in the above equation $n = m$, thus forbidding particle production and only time-delays and exchange of quantum numbers are allowed. Therefore after an eventual relabeling, we have $p_i = p'_i$ and a product of energy-momentum conserving $\delta$-functions can be factored out of both terms of the r.h.s. of $S = I + iT$.

When writing $S_{ab}(s)$ we will always assume these $\delta$-functions to have been factored out.

The $n$-particle $S$-matrix is then a product of $n(n-1)/2$ two-particle $S$-matrices. This decomposition can be effected in several ways and all of them must give the same result. The ensuing consistency conditions are called Yang-Baxter factorization equations.

Let us order our particles along the line as $p_1 > p_2 > \ldots > p_n$, so that in-states are arranged along the space-axis according to decreasing momenta and out-states the opposite. Thus in-states are going to scatter, whereas

\footnote{Except at most for a discrete set of momenta, which we exclude appealing to analyticity of the $S$-matrix.}
particles in \textit{out}-states are running away from each other. The action of the \textit{S}-matrix on a two-particle state is then defined as

\begin{equation}
|a(\theta_1), b(\theta_2)\rangle = S_{ab}(\theta_1 - \theta_2) |a'(\theta_2), b'(\theta_1)\rangle + S_{b'a}(\theta_1 - \theta_2) |b'(\theta_2), a'(\theta_1)\rangle. \tag{9}
\end{equation}

Note that the final states are ordered opposite to the initial states in rapidity space, so that the \textit{S}-matrix carries \textit{in}-states into \textit{out}-states. This ordering introduces at most a phase, which is irrelevant for two-body scattering, but when we write a many-body amplitude as a product of two-body amplitudes, this phase convention simplifies the factorization equations.

For simplicity we assume that in eq.\,(9) the second \textit{reflection} term vanishes and only the first \textit{transmission} term is present. Unitarity requires then the \textit{S}-matrix element $S_{ab}\equiv S_{ab}$ to be equal to $\exp i\delta(\theta)$ in the physical region. The Yang-Baxter factorization equations are now trivially satisfied for these diagonal \textit{S}-matrices. One way to ensure vanishing reflection is to require, that all particles $A_a$ have different masses, whenever $a \neq b$, i.e. there is no degeneracy and no internal symmetry present, (in particular particle and anti-particle are identical $A_a = A_{\overline{a}}$). In this case no exchange of quantum numbers is possible. Or, if this is not the case, some other mechanism, such as antiparticles being bound-states of particles [1], particular conservation laws [1] etc. has to be invoked to ensure vanishing reflection.

In order to pin down the possible functional dependence of $S(\theta)$, we have to discuss its analytical structure, which turns out to be very simple. In terms of the Mandelstam variable $s = (p_1 + p_2)^2\ S(s)$ has cuts along $s \geq (m_1 + m_2)^2$ and $s \leq (m_1 - m_2)^2$, which are required by two-particle unitarity. We suppose these to be the only cuts in the complex $s$-plane, since there is no production and anomalous thresholds turn into poles in $1 + 1$ dimensions [1]. The cuts are eliminated by the uniformizing variable $\theta$:

$$ s = s(\theta_1 - \theta_2) = m_1^2 + m_2^2 + 2m_1m_2 \cosh \theta_{12}, \tag{10} $$

where $\theta_{12} = |\theta_1 - \theta_2|$. The mapping from the $s$- to the $\theta$-plane is shown in figure 2.1.

Since poles in the $s$-plane generate poles in the $\theta$-plane, $S_{ab}(\theta)$ is a meromorphic function of $\theta$.

Real analyticity in the $s$-plane (Schwartz reflection principle), $S^*(s^*) = S(s)$, becomes $S^*(-\theta^*) = S(\theta)$ in the $\theta$-plane. In particular we see that
the bound-state region \((m_1 - m_2)^2 < s < (m_1 + m_2)^2\), where \(S(s)\) is real, is mapped into the segment \(0 < \Im m\theta < \pi\) of the imaginary \(\theta\)-axis. The unitarity condition \(S(s)S^\dagger(s) = S^\dagger(s)S(s) = I\) or \(S(\theta)S^\dagger(\theta) = S^\dagger(\theta)S(\theta) = I\), then becomes for real \(\theta\)

\[
S_{ab}(\theta)S_{ba}(-\theta) = 1. \quad (11)
\]

Parity invariance imposes

\[
S_{ab}(\theta) = S_{ba}(\theta) \quad (12)
\]

and charge-conjugation invariance implies

\[
S_{ab}(\theta) = S_{\overline{ab}}(\overline{\theta}), \quad (13)
\]

where \(\overline{b}\) indicates the antiparticle of \(b\). Finally crossing-symmetry

\[
S_{\overline{ab}}(s) = S_{\overline{mb}}(s) = S_{ab}(2m_1^2 + 2m_2^2 - s) \quad (14)
\]

becomes

\[
S_{\overline{mb}}(\theta) = S_{\overline{ab}}(\overline{\theta}) = S_{ab}(i\pi - \theta). \quad (15)
\]

It follows from equs. (11) and (14), that \(S(\theta)\) is a \(2\pi\text{-periodic}\) function of \(\theta\). We now exhibit the general solution \(S_{ab}(\theta)\) of these equations. Since we know that the scattering amplitudes are bounded functions of the momenta, one can show \([0]\) that this implies that any meromorphic, real analytic, \(2\pi\text{-periodic}\) function \(f(\theta)\) satisfying \(f(\theta)f(-\theta) = 1\), can be represented as

\[
f(\theta) = \prod_{\alpha \in \mathbb{A}} f_\alpha(\theta), \quad \text{with} \quad f_\alpha(\theta) = \frac{\sinh[1/2(\theta + i\pi\alpha)]}{\sinh[1/2(\theta - i\pi\alpha)]}, \quad (16)
\]

where \(\mathbb{A}\) is a set of complex numbers invariant under complex conjugation. If there are no unstable particles present, then all poles of \(f(\theta)\) occur on the imaginary \(\theta\)-axis, implying \(\alpha\) to be real and we may restrict it to \(-1 < \alpha \leq +1\). In this case \(f(\theta)\) has a simple pole of residue \(2i \sin \alpha \pi\) at \(\theta = i\alpha \pi\) and a simple zero at \(\theta = -i\alpha \pi\). Besides this we note the following useful properties:

\[
\begin{align*}
f_\alpha(\theta) &= f_{\alpha + 2}(\theta) = f_{-\alpha}(-\theta) \\
f_\alpha(\theta)f_{-\alpha}(\theta) &= f_\alpha(\theta)f_\alpha(-\theta) = 1 \\
f_\alpha(i\pi - \theta) &= -f_{-\alpha}(\theta) \quad (17)
\end{align*}
\]
Figure 1: Mapping from $s$- to $\theta$-plane, showing the position of cuts and bound-state poles.
Figure 2: Diagrams responsible for poles in the direct and crossed channel.

\[ f_\alpha(\theta - i\pi\beta)f_\alpha(\theta + i\pi\beta) = f_{\alpha+\beta}(\theta)f_{\alpha-\beta}(\theta) \]
\[ f_0(\theta) \equiv -f_1(\theta) \equiv 1. \]

If at least one of the particles \( A_a \) or \( A_b \) is self-conjugate, then crossing symmetry

\[ S_{ab}(\theta) = S_{ab}(i\pi - \theta) \] (18)

implies, that up to a sign \( S_{ab}(\theta) \) must be of the form

\[ F_\alpha(\theta) = f_\alpha(\theta)f_\alpha(i\pi - \theta) = \frac{\sinh\theta + i\sin \alpha \pi}{\sinh\theta - i\sin \alpha \pi} = \frac{\tanh[1/2(\theta + i\alpha \pi)]}{\tanh[1/2(\theta - i\alpha \pi)]}. \] (19)

The functions \( F_\alpha(\theta) \) satisfy:

\[ F_\alpha(\theta) = F_{\alpha+2}(\theta) = F_{1-\alpha}(\theta) = F_{-\alpha}(\theta) \]
\[ F_\alpha(\theta)F_{-\alpha}(\theta) = 1 \] (20)
\[ F_\alpha(\theta + i\pi\beta)f_\alpha(\theta - i\pi\beta) = F_{\alpha+\beta}(\theta)f_{\alpha-\beta}(\theta) \]
\[ F_0(\theta) \equiv 1. \]

For \( 0 < \alpha < 1/2 \), \( F_\alpha(\theta) \) has simple poles at \( i\alpha \pi \) and \( i(1-\alpha)\pi \) with residues \( 2i\tan \alpha \pi \) and \(-2i\tan \alpha \pi \), respectively, as well as zeroes at \(-i\alpha \pi \) and \(-i(1-\alpha)\pi \). \( F_{1/2}(\theta) \) has a double pole at \( i\pi/2 \) and a double zero at \(-i\pi/2 \).

Since bound-states will be all important in setting up the bootstrap principle, let us have a fast look at them, both in the s- and the \( \theta \)-plane. In the
s-plane \( S_{ab}(s) \) has bound-state poles for \( 0 < s = m_c^2 < (m_a + m_b)^2 \), whose residues are positive due to unitarity:

\[
S_{ab}(s) \sim (f_{ab}^c)^2/(s - m_c^2).
\] (21)

When we consider \( S_{ab}(\theta) \), we suppose always that the \( \delta \)-functions of the form \( \prod \delta(\theta_i - \theta_j) \) have been factored out. Therefore in going from \( S(s) \) to \( S(\theta) \), we have to take into account the Jacobian

\[
\frac{\partial(p_a, p_b)}{\partial(\theta_a, \theta_b)} = m_am_b \sinh(\theta_{ab}).
\] (22)

Therefore near the position of the pole in the \( \theta \)-plane given by

\[
m_c^2 = s(\theta = i\omega_{ab}^c) = m_a^2 + m_b^2 + 2m_am_b \cos \omega_{ab}^c,
\] (23)
i.e. near \( \theta = \omega_{ab}^c \), \( S_{ab}(\theta) \) behaves as

\[
S_{ab}(\theta) \sim iR_{ab}^c/(\theta - \omega_{ab}^c),
\] (24)

where \( R_{ba}^c = f_{ab}^c f_{ab}^\pi \). From crossing-symmetry we see, that there is also a pole with negative residue \( -R_{ba}^c \) at \( \theta = i\pi - \omega_{ab}^c \). This behavior is caused by the diagrams displayed in figure 2, which shows, that \( f_{abc} \equiv f_{ab}^c \) is a completely symmetric function of its indices. From the law of cosines applied to the triangle with sides \( m_a, m_b, m_c \), we obtain from equ. (23)

\[
u_{ab}^c + u_{\pi a}^b + u_{\pi c}^a = 2\pi.
\] (25)

Up to now we have listed the properties of \( S_{ab}(\theta) \) following from general principles. These are of course insufficient to construct the two-particle \( S \)-matrices. Since the factorization equations are void for diagonal \( S \)-matrices, we need another input. This is furnished by the Bootstrap Principle. Suppose our theory contains the set of particles \( A_1, \ldots, A_n \). Then we require that the bound-states of \( S_{ab}(\theta) \) with positive residue represent one of the particles \( A_a \) and vice-versa - any particle \( A_a \) occurs as a bound-state in some \( S \)-matrix element.

Suppose a particle \( A_c \) occurs as a bound-state in the scattering of \( A_aA_b \) at \( \theta_{ab} = \omega_{ab}^c \). Then scattering \( A_c \) with \( A_d \) must give the same result as the three-particle scattering

\[
S_{abd}(\theta_a, \theta_b, \theta_c) = S_{ab}(\theta_{ab})S_{ad}(\theta_{ad})S_{bd}(\theta_{bd})
\] (26)
at the relative rapidity $\theta_{ab} = \nu u_{ab}^c$ after dividing the $S$-matrix by the residue $\nu(f_{abc})^2$.

The kinematics goes as follows. Using equ.(25), we can write $\theta_{ab} = \nu u_{ab}^c = \nu(2\pi - u_{ca}^b - u_{bc}^a) = \nu(\pi_{ca}^b + \pi_{bc}^a)$. This suggests to apportion rapidities as:

$$\theta_a = \bar{\theta} + \nu \pi_{ca}^b, \quad \theta_b = \bar{\theta} - \nu \pi_{bc}^a,$$

where $\bar{\theta}$ is the center of mass rapidity of the $ab$-system. Hence we get

$$\theta_{ad} = \bar{\theta} - \theta_d + i\nu \pi_{ca}^b = \theta + i\nu \pi_{ca}^b,$$
$$\theta_{bd} = \bar{\theta} - \theta_d - i\nu \pi_{bc}^a = \theta + i\nu \pi_{bc}^a.$$

Thus the simplest equation compatible with the above requirement is the bootstrap equation

$$S_{cd}(\theta) = S_{ad}(\theta + i\pi_{ac}^b)S_{bd}(\theta - i\pi_{bc}^a).$$

(27)

It is this equation we will use as a consistency requirement to construct $S$-matrices in the next section.

The bootstrap equation generates a powerful constraint on the possible conservation laws $[S, P_s] = 0$. Namely, if we continue the two-particle state $|A_a(\theta_a)A_b(\theta_b)\rangle$ to $\theta_{ab} = \nu u_{ab}^c$ it will be dominated by $|A_c\rangle$, i.e.

$$\lim_{\epsilon \to 0} \epsilon \big| A_a(\theta + i\pi_{ac}^b - \epsilon)A_b(\theta - i\pi_{bc}^a + \epsilon)\big| = f_{ab} \big| A_c(\theta)\rangle.$$

(28)

This has to be compatible with our conservation laws $[S, P_s] = 0$. If we set $\theta_a = \theta + i\pi_{ac}^b - \epsilon/2$ and $\theta_b = \theta - i\pi_{bc}^a + \epsilon/2$, then the equation

$$P_s |A_a(\theta_a)A_b(\theta_b)\rangle >_{in} = (\gamma_s^a e^{s\theta_a} + \gamma_s^b e^{s\theta_b}) |A_a(\theta_a)A_b(\theta_b)\rangle >_{in},$$

(29)

when continued analytically to the pole, yields

$$P_s S|A_a(\theta_a)A_b(\theta_b)\rangle >_{in} = P_s f_{abc} S|A_c(\theta)\rangle = \frac{f_{abc}}{\epsilon} \gamma_s^c e^{s\theta} |A_c(\theta)\rangle > =$$
$$SP_s |A_a(\theta_a)A_b(\theta_b)\rangle >_{in} = S[\gamma_s^a e^{s\theta_a} + \gamma_s^b e^{s\theta_b}] |A_a(\theta_a)A_b(\theta_b)\rangle >_{in} =$$
$$[\gamma_s^a e^{s\theta_a} + \gamma_s^b e^{s\theta_b}] \frac{f_{abc}}{\epsilon} |A_c(\theta)\rangle >.$$

(30)

Therefore we get

$$\gamma_s^a e^{-i\pi_{ac}^b} + \gamma_s^b e^{i\pi_{bc}^a} = \gamma_s^c,$$

(31)

valid for all $f_{abc} \neq 0$. If all angles $u_{ab}^c$ are known, this equation can in general be solved for the $\gamma_s^a = k_s^a(m_a)^s \ldots$ only at certain values of $s$, determining in this way the possible conservation laws.

9
3 Factorized $S$-matrices

Let us use the prescription of the preceding section and construct some interesting $S$-matrices. Of course we still have to feed in information characterizing some particular model, such as some minimum particle content, symmetry properties, simplicity etc. In sections 4 to 6 we will link up with the underlying field theory.

3.1 The $S$-matrix of the Lee-Yang edge singularity

Consider the simplest possible model, where we have only one self-conjugate particle $A_1 = A_T$, which according to the bootstrap principle can be considered as an $A_1A_1$ bound-state of itself. This means that $f_{111} \neq 0$ and this will be referred to as $A_1$ having the $\phi^3$ property for obvious reasons. Thus setting $a = b = c = 1$ in equ.(31) and using $u_{11} = \frac{2\pi}{3}$ from equ.(25), we get

$$e^{-i\pi s/3} + e^{i\pi s/3} = 1.$$  \hspace{1cm} (32)

Rewriting this as $\cos(\pi s/3) = 1/2$, we see that this equation is satisfied only for $s = 6n \pm 1$, i.e. $s = 1, 5, 7, 11, 13, 17, 19, 23, 25, \ldots$.

Since we have a pole at $\theta = 2\pi i/3$ the minimal solution to the equ.(27) is

$$S_{11}(\theta) = F_{\frac{2}{3}}(\theta).$$ \hspace{1cm} (33)

The residue of this amplitude at $\theta = 2\pi i/3$ equals $+2i\sqrt{3}$. Thus, although the $S-$matrix satisfies the correct unitarity equation, the theory as a whole is not unitary, since the residue has got the wrong sign. The resolution of this problem goes noting that for the unitarity equation $SS^\dagger = 1$ to hold it is sufficient, that the \textit{in}- and \textit{out}-states form a complete set. In fact from $S|in> = |out>$ and $<in|S^\dagger =$ $<out|$ we get $<in|S^\dagger S|in>$ $= <out|out> = 1$. From the completeness of the \textit{in}-states we conclude that $SS^\dagger = 1$ and similarly for $S^\dagger S = I$. The hamiltonian doesn’t have to be hermitian [1].

We conclude, that a non-trivial unitary theory with a factorizing $S-$matrix has to contain at least two particles.

This model is only the first one of a whole series of non-unitary models, as we will show after discussing the $Z(N)$-models.

As we will see, the $S$-matrix equ.(33) belongs to the non-unitary, conformal field theory describing the Lee-Yang edge singularity [1], deformed by the only relevant operator it contains.
3.2 The Ising model in a magnetic field

Since one particle is not sufficient to produce a unitary theory, let us go one step further and consider an extra particle $A_2$. As above let us assume that $A_1$ has the $\phi^3$ property and also couples to the $A_2$ particle, i.e. assume that $f_{111}, f_{112}$ and $f_{221}$ do not vanish. We will furthermore assume all particles to be self-conjugate.

Using now equ.(31) with $a = b = 1; c = 2$ we get

$$\kappa_s^1 m_1^s e^{-is\pi^{12}} + \kappa_s^1 m_2^s e^{is\pi^{12}} = \kappa_s^2 m_2^s.$$  

(34)

Repeating the same for $a = b = 2; c = 1$ results in

$$x_1^s + x_1^{-s} = \left(\frac{m_2}{m_1}\right)^s \frac{\kappa_s^2}{\kappa_s^1}, \quad x_2^s + x_2^{-s} = \left(\frac{m_1}{m_2}\right)^s \frac{\kappa_s^1}{\kappa_s^2},$$

(35)

where

$$x_1 = e^{i\pi^{12}}; x_2 = e^{i\pi^{21}}.$$  

(36)

Eliminating the r.h.s. of these two equations, we finally obtain

$$(x_1^s + x_1^{-s})(x_2^s + x_2^{-s}) = 1.$$  

(37)

This is a very much overdetermined system of equations, if we have non-trivial conservation laws with $s > 1$. But, as we mentioned, miracles do happen and equs.(37) do have a solution for $s$ belonging to a subset of integers. For $s = 1$, we get:

$$x_1 = \exp(\pi i/5), \quad x_2 = \exp(2\pi i/5).$$

(38)

This solution can be obtained making the ansatz $x_2 = (x_1)^r$. $r = 1$ gives $m_1 = m_2$, which we don’t want, whereas the above solution corresponds to $r = 2$. For $s = 1$ we therefore obtain the golden mass ratio

$$\frac{m_2}{m_1} = 2 \cos \frac{\pi}{5} = \frac{\sqrt{5} + 1}{2} = 1.6180339 \ldots$$

(39)

With this information, we can write equ.(37) as

$$\cos(\frac{\pi}{5}) \cos(\frac{2\pi}{5}) = \cos(\frac{s\pi}{5}) \cos(\frac{2s\pi}{5}).$$

(40)
This equation is satisfied only for \( s \neq 0 \mod 5 \). Recalling that \( A_1 \) has the \( \phi^3 \) property, the allowed values for \( s \) are now
\[
s = 1, 7, 11, 13, 17, 19, 23, 29, 31, \ldots
\]
We may now start to use the bootstrap equation (27) to crank out the possible \( S \)-matrices of this model. Start with \( S_{11} \). It is supposed to have poles at \( \theta = \frac{2\pi}{3} \) and \( \theta = \frac{2\pi}{5} \) with positive residues corresponding to the particles \( A_1 \) and \( A_2 \) in the direct channel and corresponding poles with negative residues in the crossed channel. Using our functions \( F_\alpha(\theta) \) (which we abbreviate as \([\alpha]\) ) as building blocks, we choose the following 0’th order trial for \( S_{11} \):
\[
S^{(0)}_{11} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{5} \end{bmatrix}.
\] (42)
\( S^{(0)}_{11} \) has to satisfy the bootstrap equation (27), whose r.h.s. becomes:
\[
\begin{bmatrix} \frac{1}{3} \\ 1 \\ \frac{1}{15} \\ \frac{11}{15} \end{bmatrix}.
\] (43)
Here we used equ.(20) to maneuver the displaced \( \theta \)-dependence into the indices. Due to the presence of \([1/15]\) and \([11/15]\) , this cannot be equal to the l.h.s. of the bootstrap equation. Therefore we modify \( S^{(0)}_{11} \) to what will be our final version:
\[
S_{11}(\theta) = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{5} \\ \frac{1}{15} \end{bmatrix}.
\] (44)
It is now easily seen, that the bootstrap equation is satisfied at the expense of having introduced new poles into \( S_{11} \), which depending on the sign of the residues, will correspond to new particles. As a matter of fact the new pole at \( \theta = u_{11}^3 = \pi i/15 \) represents a new particle \( A_3 \).

Note that up to now we have from \( A_1 A_1 \to A_1 \), \( A_1 A_1 \to A_2 \), \( A_1 A_1 \to A_3 \) the following results:
\[
u^1_{11} = \frac{2\pi}{3}, \quad u^2_{11} = \frac{2\pi}{5}, \quad u^3_{11} = \frac{\pi}{15},
\]
\[u^1_{12} = \frac{4\pi}{5}, \quad u^2_{21} = \frac{2\pi}{5}.
\] (45)
Thus from equ.(28) we obtain
\[
\frac{m_3}{m_1} = 2 \cos \left( \frac{\pi}{30} \right) = 1.9890437 \ldots
\] (46)
Now we have to continue turning the crank, hoping that eventually no new particles will be necessary in order to satisfy the bootstrap equation equ.(27).

In the meanwhile, we use it with $a = b = d = 1, c = 2$:

$$S_{12}(\theta) = S_{11}(\theta - \pi l/5)S_{11}(\theta + \pi l/5).$$  \hfill (47)

This yields

$$S_{12}(\theta) = \left[\frac{4}{5}\right] \left[\frac{3}{5}\right] \left[\frac{4}{15}\right] \left[\frac{7}{15}\right].$$  \hfill (48)

We have a new pole with positive residue at $\theta = \nu u_{12}^4 = 4\pi l/15$, corresponding to a new particle $A_4$ with mass

$$m_4 \over m_1 = 4 \cos\left(\frac{\pi}{5}\right) \cos\left(\frac{7\pi}{30}\right) = 2.4048671 \ldots.$$  \hfill (49)

The amplitude $S_{22}$ can now be obtained from equ.(27) with $a = b = 1, c = d = 2$:

$$S_{22}(\theta) = S_{12}(\theta - \pi l/5)S_{12}(\theta + \pi l/5).$$  \hfill (50)

This yields

$$S_{22}(\theta) = S_{11}(\theta)S_{12}(\theta).$$  \hfill (51)

$S_{22}(\theta)$ exhibits two positive-residue poles at $\theta = \nu u_{22}^5 = 4\pi l/14$ and $\theta = \nu u_{22}^6 = \pi l/15$, representing new particles $A_5$ and $A_6$ with masses:

$$m_5 \over m_1 = 4 \cos\left(\frac{\pi}{5}\right) \cos\left(\frac{2\pi}{15}\right) = 2.957295 \ldots$$

$$m_6 \over m_1 = 4 \cos\left(\frac{\pi}{5}\right) \cos\left(\frac{\pi}{30}\right) = 3.2183404 \ldots.$$  \hfill (52)

We notice, that double poles occurring in, for example $S_{22}(\theta)$, correspond to anomalous thresholds \cite{1}.

Let us go on and construct $S_{13}(\theta) = S_{31}(\theta)$, putting $a = b = d = 1$ and $c = 3$ in equ.(31). We need $u_{13}$, which we lift from equ.(25), using $u_{11}^3 = \pi/15$ and $u_{ab}^c = u_{ba}^c$. We obtain

$$S_{13}(\theta) = S_{11}(\theta + \nu \pi/30)S_{11}(\theta - \nu \pi/30),$$  \hfill (53)

which yields

$$S_{13}(\theta) = \left[\frac{21}{30}\right] \left[\frac{19}{30}\right] \left[\frac{13}{30}\right] \left[\frac{11}{30}\right] \left[\frac{1}{10}\right] \left[\frac{1}{30}\right].$$
In a similar way we get:

\[
S_{23}(\theta) = \begin{bmatrix} \frac{1}{10} & \frac{1}{30} \\ \frac{11}{30} & \frac{13}{30} & \frac{21}{30} \end{bmatrix}^2.
\]

(54)

In a similar way we get:

\[
S_{23}(\theta) = \begin{bmatrix} \frac{1}{6} & \frac{19}{30} & \frac{3}{10} \\ \frac{7}{30} & \frac{13}{30} & \frac{1}{2} \end{bmatrix}^2,
\]

\[
S_{33}(\theta) = \begin{bmatrix} \frac{2}{3} & \frac{2}{15} & \frac{2}{10} \end{bmatrix} S_{11}(\theta) S_{22}(\theta),
\]

(55)

\[
S_{44}(\theta) = \begin{bmatrix} \frac{2}{15} & \frac{7}{15} & \frac{2}{3} \end{bmatrix}^2 S_{12}(\theta) S_{22}(\theta).
\]

We encounter two new particles: \(A_7\) as a pole at \(\theta = 2\pi i/15\) in \(S_{33}(\theta)\) and \(A_8\) as a pole at \(\theta = \pi i/15\) in \(S_{44}(\theta)\), whose masses are:

\[
\frac{m_7}{m_1} = 8 \cos^2 \left(\frac{\pi}{5}\right) \cos \left(\frac{7\pi}{30}\right) = 3.891156 \ldots
\]

\[
\frac{m_8}{m_1} = 8 \cos^2 \left(\frac{\pi}{5}\right) \cos \left(\frac{2\pi}{15}\right) = 4.783386 \ldots.
\]

(56)

At this point we may rest, since to our deep satisfaction we encounter no new poles and the bootstrap program closes with 8 of them. As we shall see these \(S\)-matrices describe the massive field theory, obtained by perturbing the critical \(T = T_c\) zero-field Ising model by a magnetic field. We also notice that an underlying structure pertaining to the exceptional Lie algebra \(E_8\) raises its delightful countenance: the conservation laws equ.\(\text{(41)}\) are labeled by spins, which are exactly the \textit{exponents} of the Lie algebra \(E_8\), repeated modulo the \textit{Coxeter} number of \(E_8\) \(\text{[4]}\). This is not completely unexpected, since the conformal field theory describing the critical Ising model can be obtained via the coset construction\(\text{[1]}\) \((E_8)_1 \otimes (E_8)_1 / (E_8)_2\), where the subscript denotes the \textit{level} of the Kac-Moody algebra.

### 3.3 The \(Z(N)\)-models

This set of interesting models are the simplest generalization of the Ising model, which corresponds to \(N = 2\). Their symmetry is actually \(Z(2) \times Z(N)\), where the extra \(Z(2)\) factor stands for charge conjugation. They exhibit a very rich and interesting phase diagram\(\text{[1]}\), which includes special multicritical points, where exact solutions can be obtained, even off criticality. These points then become relevant for the \(S\)-matrix game.
On a lattice these models are defined by spins \( \sigma_i \) living on each site \( i \), which satisfy \( (\sigma_i)^N = 1 \) or equivalently \( \sigma_i^* = (\sigma_i)^{N-1} \). In the field-theoretic context this is translated into the property, that anti-particles are bound-states of \( N - 1 \) particles \([1]\). This property requires the reflection amplitude to vanish, since otherwise this would correspond to non-vanishing production in the crossed channel.

For the scattering of fundamental particle, we have then only two amplitudes

\[
< p_2, p_1 | S | p_1, p_2 > = S_{11}(\theta_{12})
\]
\[
< p_2, p_1 | S | p_1, p_2 > = S_{1\Pi}(\theta_{12}).
\]

Unitarity and crossing imply

\[
u(\theta)u(-\theta) = t(\theta)t(-\theta) = 1
\]
\[
u(\theta) = t(i\pi - \theta).
\]

We now make a minimality assumption and introduce a pole, corresponding to a two-particle bound-state at \( \theta_{12} = n\pi/2 = 2\pi i/N \) in \( u(\theta) \), the following mass spectrum is generated \([1]\):

\[
m_a = m \frac{\sin(\pi a/N)}{\sin(\pi/N)} , \quad a = 1, \ldots, N - 1.
\]

Here we have \( m_{N-1} = m \) or more generally, in agreement with charge-conjugation invariance, \( m_{N-a} = m_a \).

In the present model, as opposed to the Ising case with \( h \neq 0, T = T_c \), we know the mass spectrum and therefore don’t have to go through all the motions to find \( u_{ab}^c \). We immediately obtain the minimal \( S \)-matrix of the ‘fundamental’ particle \([1]\):

\[
S_{11}(\theta) = f_{\frac{\pi}{N}}(\theta).
\]

For \( N = 3 \) just apply the equ.(27). For larger \( N \) one has to fuse \( N - 1 \) particles to check, that anti-particles are bound-states of \( N - 1 \) particles \([1]\), which means that the following identity must hold :

\[
\prod_n u(\theta + n\pi/N) = t(\theta) = u(i\pi - \theta),
\]
where

$$n = \begin{cases} \pm 1, \pm 3, \ldots, \pm (N - 2) & \text{for } N \text{ odd} \\ 0, \pm 2, \pm 4, \ldots, \pm (N - 2) & \text{for } N \text{ even.} \end{cases}$$

For $N = 2$ this gives $S_{11}(\theta) = -1$, as it should for a free bosonic theory.

The complete two-particle $S$-matrix is obtained via the Bootstrap Principle as

$$S_{ab}(\theta) = f_{|a-b|}^{-1}(\theta) \left[ \prod_{k=1}^{\min(a,b)-1} f_{|a-b+2k|}^{-1}(\theta) \right]^2 f_{|a+b|}(\theta),$$

where $a, b = 1, 2, \ldots, N - 1$. The simple poles of $S_{ab}(\theta)$, which would violate the $Z(N)$-symmetry, if interpreted as particles, are doomed to exhibit residues with the wrong sign.

We can now check for which values of $s$ we encounter conservation laws. The mass spectrum together with equ. (23) yields

$$u_{ab}^c = \frac{\pi(a+b)}{N}, \quad c = a + b \pmod N.$$  \hspace{1cm} (63)

Using this information in equ. (31), we arrive at - recall $\pi = N - a$ -

$$\gamma_s^a e^{-is\pi b/N} + \gamma_s^b e^{is\pi a/N} = \gamma_s^{a+b}.$$  \hspace{1cm} (64)

Since the $\gamma_s^a$ are real, the imaginary part of the l.h.s. has to vanish, yielding the following condition:

$$\gamma_s^b \sin(s\pi b/N) = \gamma_s^a \sin(s\pi a/N)$$  \hspace{1cm} (65)

This therefore better be an $a, b$-independent constant, which we normalize to unity and get:

$$\gamma_s^a = \sin(s\pi a/N).$$  \hspace{1cm} (66)

The real part gives:

$$\gamma_s^a \cos(s\pi b/N) + \gamma_s^b \cos(s\pi a/N) = \gamma_s^{a+b} \sin[\pi(a+b)/N]$$  \hspace{1cm} (67)

and this is identically satisfied for $\gamma_s^a$ from equ.(63). All this is of course consistent only for $\sin(\pi sa/N) \neq 0$, or

$$s \not\equiv 0 \pmod N.$$  \hspace{1cm} (68)
and we get an infinite number of conservation laws for all these values of $s$.
Let us expose, that this set reveals an underlying structure corresponding to
the group $A_{N-1} = SU(N)$.

The relevant properties of simple Lie algebras can be encoded in Dynkin
diagrams. For $A_{N-1}$ or $SU(N)$ this looks like

1 2 3 4 N-1

The incidence matrix $I_{ab}$ of this graph is the $(N-1) \times (N-1)$ matrix

$$I = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
1 & 0 & 1 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}$$

The eigenvalues of $A$ are

$$i^{(s)} = 2 \cos \frac{s\pi}{N} = \frac{\sin \frac{2s\pi}{N}}{\sin \frac{s\pi}{N}}$$

for $s = 1, 2, \ldots, N-1$. The corresponding normalized eigenvectors are

$$\vec{\Gamma}_s = \Gamma^{(a)}_s = \sqrt{\frac{2}{N}} \sin \frac{sa\pi}{N}.$$  

The matrix $I$ has non-negative entries and one defines the Perron-Frobenius
vector as the unique eigenvector, all of whose components can be chosen to
be positive. The corresponding eigenvalue is not smaller in magnitude than
any other eigenvalue. In our case it is the vector $\vec{\Gamma}_1$.

At this point, we take a rest and compare our $Z(N)$ mass formula equ.(59)
and our result for $\gamma_s^a$ equ.(66) with the preceding algebraic constructs. We
immediately notice:

i) $\Gamma^{(a)}_1 = \sqrt{\frac{2}{N}} \sin \frac{a\pi}{N}$ and therefore the components of the Perron-Frobenius vector
   give - up to normalization - the mass spectrum.

ii) $\Gamma^{(a)}_s = \gamma^a_s$ or $\vec{\Gamma}_s = \vec{\gamma}_s$, where $\vec{\gamma}_s = \gamma^{1}_s, \ldots, \gamma^{N-1}_s$. The eigenvalues of the
    conserved charges $P_s$ are thence also given by group theory.
If we replace in the above formulas the number \( N \) by the Coxeter number \( h^A \) of the group \( A \), then these two results are valid for all the simple-laced groups \( A, D, E \). \( E_8 \) corresponds to the Ising model in a magnetic field and all the other cases have also been identified \[0, 0\].

We now want to reap some rewards by extending our solutions a bit. All the \( S \)-matrices obtained in this section are minimal in the sense, that they satisfy all the imposed constraints with a minimum number of poles. However there are models with the same symmetries, but which contain e.g. a free parameter. A very prominent set are the Toda field theories \[0\], which are exactly integrable and contain a coupling parameter \( \beta \), on which the \( S \)-matrices must depend. As first shown in ref. \[0\], one can find solutions of the relevant equations, which are products of two factors: one equals the minimum \( S_{ab}(\theta) \)-matrix and the other \( Z_{ab}(\theta) \) contains the \( \beta \) dependence. The mass-spectrum does not depend on \( \beta \) and the theory becomes a free one, as \( \beta \rightarrow 0 \). Therefore \( S_{ab}(\theta)Z_{ab}(\theta) \) goes to unity as \( \beta \rightarrow 0 \). If we build \( Z_{ab}(\theta) \) out of \( f_\alpha(\theta) \) with \( \alpha < 0 \), we introduce no new poles in the physical sheet and are able to cancel all poles of \( S_{ab}(\theta) \) as \( \beta \rightarrow 0 \). For the \( Z(N) \) Toda models the \( Z_{11}(\theta) \)-factor for example is

\[
Z_{11}(\theta) = f_\beta(\theta)f_{\frac{\pi}{2N-1}+\beta}(\theta).
\]

(72)

Factorized \( S \)-matrices describing Toda field theories with structure corresponding to all the simple Lie groups have been found recently \[0, 0\].

3.4 A non-unitary series \( A'(2N) \)

Finally we may generalize the non-unitary Lee-Yang edge singularity to a whole series of non-unitary models \[1\]. This is achieved, replacing the \( Z(N) \)-amplitudes \( f_\alpha(\theta) \) by \( F_\alpha(\theta) \) in eqn.(62), such that they reduce to the Lee-Yang case for \( N = 1 \). In this way, we certainly obtain minimal amplitudes for a set of \( N \) self-conjugate particles:

\[
S_{ab}(\theta) = F_{\frac{a-b}{2N-1}}(\theta) \left[ \prod_{k=1}^{\min(a,b)-1} F_{\frac{a-b+2k}{2N-1}}(\theta) \right]^2 F_{\frac{a+b}{2N-1}}(\theta),
\]

(73)

with \( 1 \leq a, b \leq N \). The mass-spectrum is given by

\[
m_a = \sin \left( \frac{\pi a}{2N-1} \right) / \sin \left( \frac{\pi}{2N-1} \right), \quad a = 1, 2, \ldots, N-1.
\]

(74)
As opposed to the $Z(N)$-models, here the absence $Z(N)$-invariance places no restrictions on simple poles to interpreted as particles. The whole series therefore violates unitarity. As we will see, these $S$-matrices belong to a series of non-unitary conformal theories with negative central charge, perturbed by a relevant operator.

The investigation of possible conserved charges $P_s$ runs parallel to our $Z(N)$ discussion, so that we don’t repeat it here. One finds, not surprisingly for us at this stage, that an infinite number of $P_s$ exists for

$$s = \text{odd} \neq 0 \mod (2N + 1).$$  \hspace{1cm} (75)

Here also exist non-minimal solutions and their fundamental $Z$-factor is given by

$$Z_{11}(\theta) = F_{-\beta}(\theta)F_{-\frac{\pi}{2N} + \theta}. \hspace{1cm} (76)$$

These non-minimal models are now unitary, due to the $Z$-factors, which change the sign of the relevant residues.

### 4 The Thermodynamic Bethe Ansatz

The Bethe Ansatz (BA) has been a powerful tool in analysing 2-dimensional field theory and statistical mechanical models. It has been used to diagonalize the hamiltonian of these models, to study their spectrum, finite-size corrections etc. Our purpose is to use the BA as a means to interpolate from the massive $S$-matrix theory to the ultra-violet, short-distance fixed point, in order to make contact with data from conformally invariant field theory.

One way to use the BA is to start with the unrenormalized Hamiltonian and to make an Ansatz for the wave functions of ‘pseudo-particles’ living on top of a pseudo-vacuum. Their spectrum is obtained by imposing e.g. periodic boundary conditions on the wave functions. This yields the so-called BA equations. Bound states correspond to complex solution in the rapidity of the BA equations.

These pseudo-particles have then to be used to fill up a Dirac sea in order to construct the physical vacuum. The excitation spectrum above this sea will correspond to the real particles of the theory, whose scattering is described by the (physical ) $S$-matrix.

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3See however reference [3].
In the Thermodynamic Bethe Ansatz (TBA) one deals with the real particles from the very beginning, the rapidities therefore being real numbers in the physical region. Bound states are treated on equal footing, which is also in accord with the bootstrap principle.

Suppose now we know the S-matrices of all the particles in the theory. We may then place them in a box, let them scatter till they are in equilibrium at a particular temperature $T$. As we shall see, it is reasonable to assume, that these equilibrium states will be described by BA wave functions, since there is no production in our models. As shown A. B. Zamolodchikov, we may now extract finite size effects and go in particular to the zero mass limit. In this way we obtain e.g. the conformal anomaly (or central charge) $c$ of the ultraviolet conformally invariant limit of our massive field theory, establishing a beautiful connection between factorizable S-matrices and conformal invariant field theories.

The importance of the central charge $c$, stems from the fact, that its knowledge is almost sufficient to pin down the conformal field theory we are talking about. As will be explained in section 5, it can be extracted from finite-size effects arising in a massive Euclidean field theory defined on a rectangle of size $L \times R$ with, say, periodic boundary conditions in the both directions - i.e. on a torus. In the following, we will emphasize periodicity in the $R$-(space-)direction and talk about a ‘vertical’ cylinder of radius $R$. If we slice the cylinder parallel to its base, we may define the transfer matrix $\hat{T}$ as the partition function of the small cylinder $\Delta L \times R$, where we fix the configurations of the fields on the neighboring bases. In order to write the partition function $Z(L, R)$ in terms of the transfer matrix $\hat{T}$ along the $L$-(time-)direction, we introduce a complete set of states $|n(L)\rangle$ for each slice at position $L$. Due to periodic boundary conditions in the vertical direction, the states on the bottom and top bases are the same. Labelling the slices by $1, 2, \ldots, L$, we get

$$Z(L, R) = \left( \prod_{j=1,2,\ldots,L} \sum_{n_j} \right) < n_1 | \hat{T} | n_2 > < n_2 | \hat{T} | n_3 > \ldots < n_{L-1} | \hat{T} | n_1 > =$$

$$\sum_{n_1} < n_1 | \hat{T}^L | n_1 > = Tr \hat{T}^L. \quad (77)$$

$^{4}$We always assume some UV cutoff, so that we can count configurations and take the limit at the end. We also choose $\Delta L = 1$. 

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The quantum hamiltonian of this theory is defined as $\hat{T} = \exp(-\hat{H})$. Since the hamiltonian $\hat{H}$ is related to the time-time component of the energy-momentum tensor, integrated over space, we have

$$\hat{H} = \frac{1}{2\pi} \int \hat{T}_{LL} dR,$$  \hspace{1cm} (78)

where a factor $1/2\pi$ has been extracted for later convenience.

If $E_0(R)$ is the lowest eigenvalue of $\hat{H}$ (the ground state energy of our field theory), then for $L \to \infty$ it will dominate the partition function:

$$Z(L, R) \simeq \exp(-LE_0(R)).$$  \hspace{1cm} (79)

The central charge $c$ is now obtained from the free energy per unit length - which equals the ground state energy $E_0$ of the two-dimensional conformal field theory -

$$F(R) = -\lim_{L \to \infty} \frac{1}{L} \ln Z(L, R) = E_0(R).$$

The TBA will extract $\tilde{c}$ and other interesting quantities, putting the system in a box at finite temperature $T$.

Take as ’box’ our cylinder. As $L \to \infty$, we may look at our system as a Euclidean quantum field theory (EQFT) living on an infinite one-dimensional space with periodic ’time’ in the $R$-direction. But this is exactly, what defines a finite temperature Gibbs state (remember $k = 1$!). Thus $Z(L, R)$ can be regarded as an EQFT at finite temperature $T = 1/R$ with free energy per unit length

$$f(T) = -kT \frac{1}{L} \ln Z(L, R) = -\frac{\ln Z(L, R)}{LR}.$$  \hspace{1cm} (80)

Therefore, comparing equus. (79) and (80), we finally get the relation

$$E_0(R) = Rf(R).$$  \hspace{1cm} (81)

This establishes the desired link between a quantity pertaining to a EQFT and a thermodynamic one accessible to the TBA using $S$-matrix elements as input.

\footnote{The period is $R$, the factor $2\pi$ being absorbed in a rescaling of the energy.}
As we will show in section 5, eqn.(156), the expansion of the free energy (of the UV limiting theory) in powers of $1/R$ is:

$$F(R) = f_0 R - \frac{\pi \tilde{c}}{6R} + O(1/R).$$

(82)

Here $f_0$ is the non-universal bulk free energy per unit area of the infinite plane, which is chosen to vanish. Due to periodic boundary conditions no term proportional to $R$ appears. $\tilde{c}$ is related to $c$ by $\tilde{c} = c - 12\Delta_0$, where $\Delta_0$ is the lowest critical dimension of the conformal field theory. For unitary theories $\Delta_0 = 0$ and $\tilde{c} = c$. In order to reach $c$, we still have to take an UV limit to get from our massive $S$-matrix theory to the conformally invariant fixed point. From eqn.(79) we see that $E_0(R)$ has the dimension length$^{-1}$ and scaling arguments then imply

$$E_0(R) = Rf(R) = \frac{1}{R}g(r) = -\frac{\pi \tilde{c}(r)}{6R},$$

(83)

where $r = R/\xi$. The correlation length $\xi$ is related to the mass of the lightest particle (in the thermodynamic limit, where Lorentz invariance holds and where the momentum $P = 2\pi n/R$, $n = integer$ becomes a continuous variable) as $\xi = m_0^{-1}$, so that $r = Rm_0$. In the UV limit, where $r << 1$, $\tilde{c}(r)$ becomes the central charge $c = \tilde{c}(0)$.

Let us now set up the TBA equations, considering a EQFT at temperature $T = 1/R$ on a periodic one-dimensional space of length $L$. For simplicity, let us start considering a system of $N$ identical particles of mass $m$ at equilibrium at temperature $T = 1/R$. Suppose we take particle $a$ on a round-trip along the circle of length $L$. On its way it will scatter against all other particles it encounters. Let us look at this process in some detail. If our system contains $N$ particles at positions $x_j$, there are $N!$ regions in configuration space, where all particles are well separated - $|x_i - x_j| >> \xi$ - and their mutual interaction can be neglected. In each of these regions, we can describe the system by an $N$-particle wave function. Due to a scattering process - made up of two-particle scatterings - , the system will pass from one region to another. The wave function of the final region is now obtained by multiplying the wave function of the initial region by the relevant $S$-matrix ($exp(-i\delta(\theta)$). On it’s trip ‘around the world’ all the phase shifts will add

Typically it will decay as $exp(-x/\xi)$. 

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up and due to periodic boundary conditions the final wave function will be identical to the one we started with. This has to be true for anyone of the \(N\) particles. Thus we get:

\[
e^{\sum_j i(p_i+\delta_{ij})(x_i+L)} = e^{ip_i x_i}
\]

or

\[
e^{iLm \sinh(\theta_i)} \prod_{j \neq i}^N S(\theta_i - \theta_j) = 1, \ i = 1, 2, \ldots, N, \quad (84)
\]

or taking the logarithm

\[
Lm \sinh \theta_i + \sum_{j \neq i}^N \delta(\theta_i - \theta_j) = 2\pi J_i, \ i = 1, 2, \ldots, N, \quad (85)
\]

where \(J_i\) are integers to be determined by equilibrium conditions. The above Bethe Ansatz equations (BAE) reduce to the well known equations in the free case \(\delta_{ij}(\theta) = 0\), where they too determine the possible values of the momenta \(p_i = m_i \sinh(\theta_i) = 2\pi J_i/L\). In our non-trivial, interacting situation they are a set of \(n\) coupled transcendental equation for \(\theta_{ij}\). The energy and momentum are given by

\[
E = \sum_{j=1}^N m \cosh \theta_j, \quad P = \sum_{j=1}^N m \sinh \theta_j, . \quad (86)
\]

Notice that factorization of the \(S\)-matrix once more has given us an unexpected surprise: how to extract off-shell information from on-shell data.

In the following we will assume our Bethe wave functions to obey an exclusion principle, i.e. they are of fermionic type \(\mathbb{I}, \mathbb{I}\). All our \(S\)-matrices are compatible with this condition. This seems to be a necessary requirement for the construction of a physical vacuum in the usual Bethe-Ansatz \(\mathbb{I}\), although it is not necessary for the implementation of the TBA \(\mathbb{I}\).

The wave function of identical bosons or fermions has in his case to be anti-symmetric under their interchange. This implies the following constraint

\[\text{If the } S\text{-matrix is not diagonal, the wave functions will have internal indices, on which the S-matrix acts. In this case the following equations turn into matrix equations.}\]

\[\text{In another step to implement this program, one obtains the form-factors from the on-shell information } \mathbb{I}.\]
on the $S$-matrix. From the unitarity condition equ.(11), we have $S_{aa}(0) = \pm 1$. If $S_{aa}(0) = -1$, the wave function is anti-symmetric under interchange. Therefore, in order to enforce an exclusion principle, we require $S_{aa}(0) = -1$ for bosons. For fermion the situation is reversed and we require $S_{aa}(0) = +1$ for fermions.

4.1 The TBA equations in the thermodynamic limit

To obtain information about the spectrum of our theory, we have to investigate the roots of the BAE, which is a complicated system of transcendental equations. But this system simplifies in the thermodynamic limit, where $L \to \infty$ together with the number of particles, which also grows $\sim L$. The difference between adjacent solutions of equ.(85) goes to zero as $\theta_i - \theta_j \sim 1/m_0 L$. It is then reasonable to define a density $J(\theta)$ giving the number of real roots in the rapidity interval $\theta$ and $\theta + \Delta \theta$.

Suppose now that $\{\theta_1, \theta_2, \ldots, \theta_n\}$ is a self-consistent solution of equ.(85). Consider then the function

$$J(\theta) = \frac{mL}{2\pi} \sinh \theta + \sum_j^N \delta(\theta - \theta_j),$$

(87)

which - as will be checked later on - is a monotonically increasing function of $\theta$. Whenever $J(\theta)$ passes through one of the integers $J_i$, the corresponding $\theta$ equals $\theta_i$. However there may be integers $J(\theta)$ for which the corresponding $\theta$ is not in the set $\theta_1, \theta_2, \ldots, \theta_n$. Such a $\theta$ will be called a hole. Thus a natural definition of the density of roots and holes is

$$\rho(\theta) = \frac{1}{L} \frac{dJ(\theta)}{d\theta}.$$  (88)

A linear integral equation for $J(\theta)$ can now be obtained in the limit $L \to \infty$, by first differentiating equ.(87)

$$\frac{1}{L} \frac{dJ(\theta)}{d\theta} = \rho(\theta) = \frac{m}{2\pi} \cosh \theta + \sum_i^N \varphi(\theta - \theta_j),$$

(89)

---

9By this we mean that the $\theta_j$, besides obeying the BAE, also satisfy the equilibrium conditions at temperature $T$ to be set up below.
where $\varphi(\theta)$ is defined as

$$\varphi(\theta) \equiv \frac{d}{d\theta} \delta(\theta) = -i \frac{d}{d\theta} \ln S(\theta).$$  \hfill (90)

We now make the replacement (valid for sufficiently large $L$)

$$\sum_{j=1}^{n} f(\theta_j) = \int d\theta \rho(\theta) f(\theta) - \sum_{j=1}^{l} f(\theta_j)$$  \hfill (91)

$$= \int d\theta \rho_r(\theta) f(\theta),$$

where $\theta_1, \ldots, \theta_l$ are the hole rapidities and we introduced the density of roots $\rho_r(\theta)$. Clearly the hole density is: $\rho_h = \rho - \rho_r$.

This transforms equ.(89) into

$$\rho(\theta) = \frac{m}{2\pi} \cosh \theta + \varphi * \rho_r(\theta),$$  \hfill (92)

where we have introduced the convolution

$$(f * \rho)(\theta) = \int_{-\infty}^{+\infty} \frac{d\theta'}{2\pi} f(\theta - \theta')\rho(\theta').$$  \hfill (93)

In general we may have more than one type of particles. Therefore introduce densities for each type, labeled by $a = 1, \ldots, n$ : $\rho^{(a)}(\theta)$, etc. The integral equation for $\rho^{(a)}(\theta) = \rho_r^{(a)} + \rho_h^{(a)}$ becomes then:

$$\rho^{(a)}(\theta) = \frac{m_a}{2\pi} \cosh \theta + \sum_{b=1}^{n} \varphi_{ab} * \rho^{(b)}_r(\theta), \quad a = 1, \ldots, n.$$  \hfill (94)

As advertised, the very hard problem of solving $N$ coupled transcendental equations has been converted to the solution of a simple linear integral equation for the density $\rho^{(a)}_r$. However we still need a relation between $\rho^{(a)}_r$ and $\rho^{(a)}_h$.

In order to obtain the necessary additional information, we impose the condition of thermodynamic equilibrium. Thus we will calculate the free energy per unit length $f = e - Ts$, whose minimization will provide what we want.
It is easy to compute the entropy of the distribution of particles and holes on the circle \( L \). We have a large number of roots \( L \rho^a(\theta) \Delta \theta \) and holes \( \rho^h(\theta) \Delta \theta \) in the interval \( \Delta \theta \). The number of ways to distribute them in this interval - according to our exclusion principle - is

\[
\frac{[L(\rho^a(\theta) + \rho^h(\theta)) \Delta \theta]!}{[L \rho^a(\theta) \Delta \theta]! [L \rho^h(\theta) \Delta \theta]!}.
\]

(95)

Taking the logarithm, we obtain the entropy per unit length:

\[
s[\rho, \rho] = \sum_{a=1}^{n} s_a[\rho, \rho]
\]

= \sum_{a=1}^{n} \int_{-\infty}^{+\infty} d\theta \left[ (\rho^a(\theta) + \rho^h(\theta)) \ln(\rho^a(\theta) + \rho^h(\theta)) - \rho^a(\theta) \ln \rho^a(\theta) - \rho^h(\theta) \ln \rho^h(\theta) \right].
\]

(96)

The energy per unit length is obviously given by

\[
e[\rho, \rho] = \sum_{a=1}^{n} \int_{-\infty}^{+\infty} d\theta \rho^a(\theta) m_a \cosh \theta.
\]

(97)

Let us introduce the function \( \epsilon_a(\theta) \) by

\[
\frac{\rho^a(\theta)}{\rho^h(\theta)} = 1 + e^{\epsilon_a(\theta)}
\]

(98)

and also

\[
L_a(\theta) = \ln(1 + e^{-\epsilon_a(\theta)}).
\]

(99)

Using this the entropy may easily be expressed as

\[
s[\rho, \epsilon_a] = \sum_{a=1}^{n} \int_{-\infty}^{+\infty} d\theta \rho^a(\theta)[\epsilon_a(\theta) + (1 + e^{\epsilon_a(\theta)}) \ln(1 + e^{-\epsilon_a(\theta)})].
\]

(100)

The equilibrium distribution at temperature \( T \) is now given by minimizing the free energy \( f = e - Ts \) with respect to \( \rho^a(\theta) \), subject to the periodic boundary condition equ.(94) relating \( \rho^a(\theta) \) to \( \rho^a(\theta) \).

Thus using

\[
\frac{\delta s}{\delta \rho} = \frac{\delta s}{\delta \rho} + \frac{\delta s}{\delta \rho} \frac{\delta \rho}{\rho} \ln \frac{\rho}{\rho} - \frac{\delta \rho}{\rho} \ln \frac{\rho}{\rho - \rho}
\]

(101)
\[
\frac{\delta \rho(\theta)}{\delta \rho_r(\theta')} = \varphi(\theta - \theta'),
\]
coming from the periodic boundary condition, we get finally get the following extremum condition:
\[
\frac{m_a}{T} \cosh \theta = \epsilon_a(\theta) + \sum_{b=1}^{n} \varphi_{ab} * L_b(\theta).
\]
Note that this condition depends only on \(\epsilon_a(\theta)\). The solution of this equation for \(\epsilon_a(\theta)\) will always be real. From equ.(\ref{eq:98}) we see that this implies \(\rho^\rho(\theta) > 0\), since \(\rho_r^\rho(\theta)\) is positive by definition. This in turn results - see equ.(\ref{eq:88}) - in \(J^\rho(\theta)\) being a monotonically increasing function of \(\theta\), as announced.

The equilibrium free energy can now be obtained by eliminating \(\rho_h\), using the periodic boundary condition equ.(\ref{eq:94}) and \(\rho_h/\rho_r = e^{\epsilon(\theta)}\) in the expression for \(s[\rho_r, \rho_h]\). Using the extremum condition equ.(\ref{eq:103}) then yields the free energy:
\[
f(\theta) = -T \sum_{a=1}^{n} \int_{-\infty}^{+\infty} \frac{d\theta}{2\pi} m_a \cosh \theta L_a(\theta).
\]
Usually numerical work will now be involved in order to compute this equilibrium free energy. But a number like the conformal anomaly can be extracted analytically.

To accomplish this we have to go to the UV limit, where the masses may be neglected : \(r = m/T \to 0\). Let us first take the limit of the extremum condition equ.(\ref{eq:103})
\[
\tilde{m}_a r \cosh \theta = \tilde{\epsilon}_a(\theta) + \sum_{b=1}^{n} \varphi_{ab} * \tilde{L}_b(\theta),
\]
where we have introduced dimensionless mass ratios \(\tilde{m}_a = m_a/m_0 = m_a \xi\). As \(r \to 0\), \(\theta\) has to go to \(\infty\) for the l.h.s. to give a finite contribution, in which case it behaves as
\[
\tilde{m}_a r \cosh \theta \sim \tilde{m}_a \frac{r}{2} e^{\theta} = \tilde{m}_a \exp(\theta - \ln \frac{2}{r}).
\]
We therefore make the shift \(\theta \to \theta + \ln(2/r)\) to get the following equation:
\[
e^{\theta} = \tilde{\epsilon}_a(\theta) + \sum_{b=1}^{n} (\varphi_{ab} * \tilde{L}_b)(\theta),
\]
whose solution provides the \( r \)-independent functions

\[
\tilde{\epsilon}_a(\theta) \equiv \epsilon(\theta + \ln(2/r)) \tag{107}
\]

\[
\tilde{L}_b(\theta) \equiv L_b(\theta + \ln(2/r)).
\]

Solving numerically the extremum condition for \( \epsilon_a(\theta) \), one realizes that this function equals the constant \( \epsilon_a \) in the range \(-\ln(2/r) < \theta < +\ln(2/r)\) and eventually grows exponentially, so that \( \tilde{\epsilon}_a(\theta) = \epsilon_a \), except at \( \infty \), for \( r \rightarrow 0 \). Similarly \( \tilde{L}_b(\theta) \) interpolates between \( \epsilon_a \) at \( \theta = 0 \) and \( -\) through a double exponential decay - 0 for \( \theta \neq 0 \).

If we now use equ.(83) and take the limit \( r \rightarrow 0 \) in equ.(105), we get a compact formula for \( \tilde{c}(0) \):

\[
\tilde{c}(0) = \frac{3}{\pi^2} \sum_{a=1}^{n} \hat{m}_a \int_{-\infty}^{+\infty} d\theta \tilde{L}_a(\theta) e^\theta, \tag{108}
\]

which is very convenient for a numerical solution.

A perhaps more explicit formula for \( \tilde{c}(0) \) may also be obtained in terms of Roger’s dilogarthmic function \[10\]. It is more convenient to use the entropy equ.(100) and to get the \( \tilde{c}(0) \) from

\[
S = \frac{\pi \tilde{c}(r)}{3} T L + O(T^2), \tag{109}
\]

which is equivalent to equ.(82) \[10\].

Again the entropy \( s[\rho_r, \epsilon_a] \) vanishes in the limit \( r \rightarrow 0 \), as long as the limits of the integral over \( \theta \) are finite. Now check the behavior of the integrand. On one hand taking the derivative with respect to \( \theta \) of the extremum condition equ.(103) we get

\[
\frac{d\epsilon_a(\theta)}{d\theta} \simeq \frac{m_a}{T} \sinh \theta \simeq \frac{m_a}{2T} e^{\mid\theta\mid} \text{sgn} \theta, \quad \mid\theta\mid \rightarrow \infty. \tag{110}
\]

On the other hand, since \( \varphi(\theta) = 0 + O(e^{-\theta}) \) as \( \theta \rightarrow \infty \), we get for the asymptotic behavior of \( \rho^{(a)} \) from the BAE equ.(114) :

\[
\rho^{(a)} \simeq \frac{m_a}{4\pi} e^{\mid\theta\mid} \tag{111}
\]

\[10\] We have \( f(T) = e - Ts = \ldots = T^2(\pi \tilde{c}/6) \). But \( s = -\partial f/\partial T = T(\pi \tilde{c}/6) + O(T^2) \) and equ.(108) follows.

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and consequently
\[
\rho_r^{(a)} \approx \frac{m_a}{4\pi} \frac{e^{\mid \theta \mid}}{1 + e^{\epsilon_a(\theta)}}
\]
\[
\approx \frac{T}{2\pi} \frac{d\epsilon_a(\theta)}{d\theta} (1 + e^{\epsilon_a(\theta)})^{-1} (sgn\theta). \quad (112)
\]
Substituting this into equ.(100) for the entropy, we get
\[
\lim_{r \to 0} s[\rho_r] = \frac{T}{2\pi} \sum_{a=1}^{n} \int_{-\infty}^{+\infty} d\theta \frac{d\epsilon_a(\theta)}{d\theta} (sgn\theta) \ln(1 + e^{\epsilon_a(\theta)}) - \frac{\epsilon_a(\theta)}{1 + e^{-\epsilon_a(\theta)}}. \quad (113)
\]
Here we change variables \(\theta \to \epsilon_a\) and use \(\varphi_{ab}(\theta) = \varphi_{ab}(-\theta)\), which follows from the unitarity of the S-matrix, to show that \(\epsilon_a(\theta)\) is even in \(\theta\). Finally changing variables again from \(\epsilon_a\) to \(f(\epsilon_a) = 1/(1 + e^{\epsilon_a(\theta)})\), we get:
\[
\lim_{r \to 0} s[\rho_r] = \frac{T}{2\pi} \sum_{a=1}^{n} \int_{f(\epsilon_a(\infty))}^{f(\epsilon_a(0))} dy \left[ \frac{\ln y}{1 - y} + \frac{\ln(1 - y)}{y} \right]. \quad (114)
\]
The upper limit of the integral equals 0, since \(\epsilon(\theta) \to \infty\) there. The lower limit may be obtained from the extremum condition equ.(103), setting there \(\theta \to 0\) and \(r \to 0\):
\[
0 = \epsilon_a(0) + \sum_{b=1}^{n} \int_{-\infty}^{+\infty} d\theta' \frac{d\varphi_{ab}(\theta - \theta')}{2\pi} \varphi_{ab}(\theta - \theta') L_b(\theta'). \quad (115)
\]
However \(\varphi_{ab}(\theta)\) decreases exponentially off the origin and we get an equation for \(\epsilon_a \equiv \epsilon_a(0)\)
\[
e^\epsilon_a = \prod_{b=1}^{n} (1 + e^{-\epsilon_b}) N_{ab} \quad a = 1, \ldots, n, \quad (116)
\]
where the symmetric matrix \(N_{ab}\) is:
\[
N_{ab} \equiv -\int_{-\infty}^{+\infty} d\theta \frac{d\varphi_{ab}(\theta)}{2\pi} = -\frac{1}{2\pi} [\delta_{ab}(+\infty) - \delta_{ab}(-\infty)]. \quad (117)
\]
Introducing Rogers’ dilogarithmic function \(L(x)\) as
\[
L(x) = -\frac{1}{2} \int_{0}^{x} dy \left[ \frac{\ln y}{1 - y} + \frac{\ln(1 - y)}{y} \right], \quad (118)
\]
29
we may finally express \( \tilde{c} \), using the finite-size scaling formula equ.(109), as

\[
\tilde{c} = \sum_{a=1}^{n} \tilde{c}_a(\epsilon_a), \quad \text{where}
\]

\[
\tilde{c}(\epsilon) = \frac{6}{\pi^2} L \left( \frac{1}{1 + e^\epsilon} \right) = \frac{6}{\pi^2} L \left( e^{-\epsilon} \right)
\]

(119)

Each particle species contributes with \( \tilde{c}_a(\epsilon_a) \) to the total central charge. The function \( \tilde{c}_a(\epsilon_a) \) is strictly monotonically decreasing, approaching 0 as \( \epsilon \to \infty \) and \( \tilde{c}_a(0) = 1 \). In order to obtain \( \tilde{c} \) the non-linear equation (116) has to be solved.

4.2 Computation of the central charge

Let us try things out in some simple cases, although there exist solutions for whole sets of models.

1) Ising model \(( h = 0, T \neq T_c )\)

The Ising model ( or \( Z(2) \) model ) is a free fermion theory with \( \varphi_{ab} = \text{const} \) and \( N_{ab} = 0 \). We may start directly from equ.(104) for the free energy. Thus we have to compute

\[
E_0(r) = R f(R) = -\frac{\pi}{6R} c =
\]

\[
-\frac{1}{R} \int_{-\infty}^{+\infty} \frac{d\theta}{2\pi} r \cosh \theta \ln(1 + e^{-\epsilon(\theta)})
\]

in the limit \( r \to 0 \). With \( \epsilon(\theta) = r \cosh \theta \approx r e^{\theta}/2 \), we get

\[
r \int_{-\infty}^{+\infty} \frac{d\theta}{2\pi} \cosh \theta \ln(1 + e^{-\epsilon(\theta)}) \approx 2 \int_{0}^{\infty} \frac{d\theta}{2\pi} \epsilon(\theta)' \ln(1 + e^{-\epsilon(\theta)}) \approx \int_{0}^{\infty} d\epsilon \ln(1 + e^{-\epsilon}) = \frac{\pi}{12}
\]

With this result we obtain, as expected \( c = 1/2 \). As a matter of fact, we may obtain an expansion of \( RE_0(R) \) in powers \( r \) and the result compares well with numerical finite-size calculations.

2) Lee-Yang edge singularity

In section 2 we found the \( S \)-matrix

\[
S_{11}(\theta) = F_{2/3}(\theta).
\]

(120)
From the properties equ.(20), we have

\[ F_{\alpha}(\theta) = f_{\alpha}(\theta)f_{\alpha}(1\pi - \theta) = f_{\alpha}(\theta)(-1)f_{1-\alpha}(\theta). \]

Since all our S-matrices are products of the \( f_{\alpha}(\theta) \)'s, their contributions to the phase shifts and the matrix \( N_{ab} \) of equ.(117) has the structure

\[ N_{ab} = \sum_{j} N[f_{\alpha}] \]

and \( c \) can be extracted form equ.(119) to give \( c = 2/5 \). If we take the value of \( \Delta_0 \) from conformal field theory: \( \Delta_0 = -2/5 \), we get for the central charge \( c = -22/5 \). Notice the negative value for \( c \), as appropriate for non-unitary models.

3) \( Z(N) \)-models

Now let us look at \( Z(N) \)- (also called \( A_{N-1} \)) models in some detail. The complete two-particle S-matrix was written in equ.(62):

\[ S_{ab}(\theta) = \frac{|a - b|}{N} \prod_{k=1}^{\min(a,b)-1} \left\{ \frac{|a - b| + 2k}{N} \right\}^{2} \frac{((a + b)/N)}{N}, \]

where \( a, b = 1, 2, \ldots, N - 1 \) and we used the shorthand \( \{\alpha\} \equiv f_{\alpha}(\theta) \).

Let us check \( Z(3) \). We have : \( N_{11} = N_{22} = N[f_{2/3}] = 1/3 \) and \( N_{12} = N_{21} = N[f_{1/3}] + N[f_{3/3}] = 2/3 \). Therefore we have to solve the equ.(116):

\[ e^\epsilon_1 = x_1 = (1 + e^{-\epsilon_1})^{2/3}(1 + e^{-\epsilon_2})^{2/3} \]

\[ e^\epsilon_2 = x_2 = (1 + e^{-\epsilon_1})^{2/3}(1 + e^{-\epsilon_2})^{2/3}. \]

Their solution is \( x_1 = x_2 = 2 \cos(\pi/5) \). If we now use the sum-rule [1]

\[ \sum_{k=2}^{n+1} L \left( \frac{\sin^2(\pi/n + 3)}{\sin^2(\pi k/n + 3)} \right) = L(1) \frac{2n}{n + 3} = \frac{\pi^2}{6} \frac{2n}{n + 3}, \]

31
we obtain $c$ from equ.(119) as

$$c = \frac{6}{\pi^2} \sum_{a=1}^{2} L(e^{-\epsilon_a}) = \frac{6}{\pi^2} \frac{4}{6} = \frac{1}{5},$$  

(126)

which is the expected value $c(Z(3)) = 4/5$.

For general $N$ the solution of equs. (116) is

$$e^{\epsilon_a} = \frac{\sin \frac{\alpha \pi}{N+2} \sin \frac{(a+2)\pi}{N+2}}{\sin^2 \frac{\pi}{N+2}}. \quad (127)$$

and the same sum rule plugged into equ.(119) yields :

$$c(Z(N)) = \frac{2(N-1)}{N + 2}, \quad (128)$$

which is the central charge of the $Z(N)$ parafermion models.

The non-minimal models describing Toda field theories have, as shown in section 3, a free UV limit and therefore their value of $c$ is simply the number of bosonic fields, as can be explicitly verified [5].

For further applications of this scheme see e.g. ref.[6].

3) The non-unitary series

Following the reasoning exhibited for the Lee-Yang edge singularity and the $Z(N)$ models, we obtain:

$$e^{\epsilon_a} (A'(2N)) = e^{\epsilon_a} (Z(2N)). \quad (129)$$

From the same sum-rule used in the $Z(N)$ case we get:

$$\tilde{c} (A'(2N)) = \frac{1}{2} \tilde{c} (Z(N)) = \frac{2N}{2N + 3}. \quad (130)$$

Borrowing $\Delta_0 = -\frac{2(N-1)(6N-1)}{2N+1}$, we get for the central charge:

$$c(A'(2N)) = -\frac{2(N-1)(6N-1)}{2N + 1}. \quad (131)$$

At this point we have completed the first of our connections between a massive $S$-matrix theory and it’s UV-limiting fixed point data. It gives us confidence, that the plausibility assumptions made in the previous section, are indeed correct. In order to further clarify this point, we will dig into this UV theory in more detail in the next section.
5 Lightning Overview of Conformal Invariance

The purpose of this section is to provide a minimum of vocabulary for the newcomer or a reminder for the rusty practitioner, but it is not a substitute for the excellent review articles and books on the subject [1, 2].

5.1 Conformal transformations

Conformal transformations are coordinate transformations $x_\mu \to y_\mu(x)$, which preserve the angle between two arbitrary vectors at any point, although their length may change:

$$dy_\mu dy^\mu = \left(\frac{\partial y_\rho}{\partial x_\mu}\right) \left(\frac{\partial y^\rho}{\partial x'^\nu}\right) dx_\mu dx'^\nu \equiv \rho(x) dx_\mu dx^\mu. \quad (132)$$

A special case are dilatations, which scale all lengths up by a factor $\rho(x) = \lambda$. Under mild assumptions - for $D=2$ a discrete spectrum of dimensions - a scale invariant quantum field theory is also conformally invariant [1]. Therefore in Statistical Mechanics the long distance behavior at a critical point, which is scale invariant due to a diverging correlation length, is described by a conformally invariant Euclidean field theory [1].

Although for dimensions $D > 2$, the conformal group is finite-dimensional, for $D = 2$ it involves an infinite number of parameters, which entails powerful restrictions on the theory.

Consider then a Euclidean Field theory in the 2-dimensional space of coordinates $x_1, x_2$ [1]. We, very conveniently, introduce complex (light-cone) coordinates

$$z = x^1 + i x^2; \quad \bar{z} = x^1 - i x^2. \quad (133)$$

Then any transformation

$$z \to z' = f(z); \quad \bar{z} \to \bar{z}' = \bar{f}(\bar{z}), \quad (134)$$

where $f(z)$ and $\bar{f}(\bar{z})$ are differentiable functions, are conformal transformations satisfying equ. (132). We will consider $z$ and $\bar{z}$ to be independent complex variables. However, since physical correlation functions live in the real

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11The Minkowskian space-time would have coordinates $x^1$ and $x^2 = it$. 

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space $R^2$, we eventually have to take into account that $\overline{z}$ is the complex conjugate of $z$: $\overline{z} = z^*$, when constructing physical correlation functions.

Under the finite conformal transformation $z \rightarrow z' = f(z)$, $\overline{z} \rightarrow \overline{z}' = \overline{f(\overline{z})}$ the Jacobian is

$$\frac{\partial(z', \overline{z}')}{\partial(z, \overline{z}')} = f'(z) \overline{f'(\overline{z})}.$$  \hspace{1cm} (135)

Tensor fields will transform with some powers of $f'$ and $\overline{f'}$. We will define **conformal fields of weight** $(h, \overline{h})$ to transform as

$$\phi(z, \overline{z}) \rightarrow \phi'(z, \overline{z}) = f'(z)^h \overline{f'(\overline{z})}^\overline{h} \phi(f(z), \overline{f(\overline{z})}).$$  \hspace{1cm} (136)

The scaling dimension is $\Delta = h + \overline{h}$, whereas the spin is $s = h - \overline{h}$. We also want to consider $f(z)$ and $\overline{f(\overline{z})}$ to be an arbitrary, but infinitesimal change of coordinates:

$$\delta z = \epsilon(z), \delta \overline{z} = \overline{\epsilon(\overline{z})}.$$  \hspace{1cm} (137)

and fields transforming this way are called **primary**. All fields we will deal with are primary, except the energy-momentum tensor.

### 5.2 The energy-momentum tensor

The energy-momentum tensor $T^{\mu\nu}$ is the generator of space-time symmetries and as such is symmetric and conserved: $T^{\mu\nu} = T^{\nu\mu}$, $\partial^\mu T^{\mu\nu} = 0$. Due to these properties it is a prominent object in the study of conformal transformations in field theory. If we consider the infinitesimal coordinate transformation - not necessarily conformal -

$$x^\mu \rightarrow x^\mu + \epsilon^\mu(x),$$  \hspace{1cm} (138)

then we may define $T^{\mu\nu}$ as inducing the following change in the action:

$$\delta S = -\frac{1}{2\pi} \int T_{\mu\nu} \partial^\mu \epsilon^\nu \, d^2 x.$$  \hspace{1cm} (139)

---

12. In doing this we want to stay within the conformal plane. Thus by **finite** we mean **global** transformations of the form $z' = \frac{az + b}{cz + d}$ with $ad - bc = 1$, which are the only one-to-one mappings of the complex plane onto itself.

13. Let us mention a delicate point here. Infinitesimal, everywhere analytic functions do not exist, since they are bound to develop singularities somewhere, if they are nontrivial. Thus we consider a bounded domain, which excludes the singularities.

14. For example, in the case of a free massless boson with Lagrangian density $L = \frac{1}{2}(\partial_\mu \phi)^2$, we get with this definition $T_{\mu\nu} = -\partial_\mu \phi \partial_\nu \phi + \frac{\delta_{\mu\nu}}{2} \partial_\alpha \phi \partial^\alpha \phi$. 

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In terms of correlation functions this is equivalent to the identity

\[\sum_{j=1}^{n} \phi_{1}(x_{1}) \ldots \delta_{e} \phi_{j}(x_{j}) \ldots \phi_{n}(x_{n}) = - \int \frac{d^{2}x}{2\pi} \partial_{\mu} \epsilon_{\alpha}(x) < T^{\mu\nu}(x) \phi_{1}(x_{1}) \ldots \phi_{n}(x_{n}) >,\]  

(140)

where \(\delta_{e} \phi(x)\) is the variation of \(\phi(x)\) under the transformation equ.(138). This is a very useful definition of \(T_{\mu\nu}\), if we don’t know the Lagrangian of the theory we are studying. If we are after exactly integrable models it is often more profitable to shift the emphasis to other types of structures like symmetry properties, operator product expansions, fusion rules etc.; the corresponding Lagrangian being determined a posteriori.

If we introduce the fields \(T = \left( T^{11} - T^{22} + 2iT^{12} \right) / 2\) and \(\overline{T} = \left( T^{11} - T^{22} - 2iT^{12} \right) / 2\), then the conservation equations \(T^{\mu\nu} = T^{\nu\mu}, \partial_{\mu}T^{\mu\nu} = 0\) become

\[\partial_{z}T = -\partial_{\overline{z}}\Theta / 2; \partial_{\overline{z}}\overline{T} = -\partial_{z}\Theta / 2;\]  

(141)

where \(\Theta = T^{11} + T^{22}\) is the trace of the energy-momentum tensor. In a conformally invariant theory this trace vanishes and \(T\) and \(\overline{T}\) are analytic functions of \(z\) and \(\overline{z}\) respectively:

\[T = T(z), \overline{T} = \overline{T}(\overline{z}).\]  

(142)

The 2-dimensional problem is thus reduced to effectively two one-dimensional problems and it is this feature, which makes conformally invariant models exactly soluble. Equs.(142) mean that the correlation functions

\[< T(z) \phi_{1}(z_{1}, \overline{z}_{1}) \ldots \phi_{n}(z_{n}, \overline{z}_{n}) >\]  

(143)

\(^{15}\)Fast derivation! Define correlation functions by the functional integral (or Gibbs average on a lattice in statistical mechanics):

\[< \phi(x) \ldots > = \frac{\int D\phi \phi(x) \ldots e^{-S[\phi]}}{\int D\phi e^{-S[\phi]}}.\]

In the numerator of the functional integral on the r.h.s. change the dummy variable \(\phi(x) \rightarrow \phi(x) + \delta_{e} \phi(x)\). Expand everything to first order in \(\delta_{e} \phi(x)\), getting two terms: one from \(\phi(x)\) and another from the variation of the action \(S[\phi + \delta_{e} \phi] = S[\phi] + \int d^{2}x/(2\pi)T^{\mu\nu} \partial_{\mu} \epsilon_{\alpha}(x)\). The zeroth order term cancels the l.h.s. and we get the above Ward identity.
are single-valued analytic functions with singularities (at most poles of finite order) only at the points $z_1, z_2, \ldots, z_n$ and similarly for $T(z)$.

To see this integrate the r.h.s. of equ. (140) over a domain excluding small regions around the points $z_j$ and choose for $\epsilon(z), \bar{\epsilon}(\bar{z})$ functions vanishing sufficiently fast at infinity to allow partial integrations. In this domain $\partial_\mu T^{\mu\nu}$ will be zero and only the surface terms around the points $x_j$ survive. Putting the origin at one such point, and choosing for the excluded region a small circle of radius $\rho$, yields the surface term $\int d\sigma_\mu T^{\mu\nu} \epsilon_\nu$. With

$$d\sigma_\mu = d^2x \frac{\partial F(x)}{\partial x^\mu} \delta(F(x) - \rho), \quad F(x) = x_1^2 + x_2^2$$

we get

$$\int d\sigma_\mu = \oint_\rho d\theta x_\mu.$$ Transforming from rectangular to complex coordinates, we obtain for the r.h.s. of equ. (140) the result:

$$\int d\sigma_\mu T^{\mu\nu} \epsilon_\nu = \frac{1}{2\pi i} \left\{ \oint d\bar{z} T(z) \epsilon(z) + \oint d\bar{z} \bar{T}(\bar{z}) \bar{\epsilon}(\bar{z}) \right\}. \quad (144)$$

Thus the Ward identity for $T(z)$ (and a similar one for $\bar{T}(\bar{z})$) becomes:

$$\sum_{j=1}^n < \phi_1(z_1, \bar{z}_1) \ldots \delta \phi_j(z_j, \bar{z}_j) \ldots > =$$

$$\sum_{j=1}^n \left( \epsilon(z_j) \partial_{z_j} + h_j \epsilon'(z_j) \right) < \phi_1(z_1, \bar{z}_1) \ldots \phi_n(z_n, \bar{z}_n) > =$$

$$\oint \frac{dz}{2\pi i} \epsilon(z) < T(z) \phi_1(z_1, \bar{z}_1) \ldots \phi_n(z_n, \bar{z}_n) >,$$

where $\mathcal{C}$ encircles the points $z_1, \ldots, z_n$ once in the positive sense. This statement is equivalent to - as can be seen multiplying equ. (140) by $\epsilon(z)$ and integrating along $\mathcal{C}$ -

$$< T(z) \phi_1(z_1, \bar{z}_1) \ldots \phi_n(z_n, \bar{z}_n) > =$$

$$\sum_{j=1}^n \left( \frac{h_j}{(z - z_j)^2} + \frac{1}{z - z_j} \partial_{z_j} \right) < \phi_1(z_1, \bar{z}_1) \ldots \phi_n(z_n, \bar{z}_n) > \quad (146)$$

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and a similar equation for $T(z)$. These relations express the transformation properties of primary fields under conformal transformations.

Surprisingly the energy-momentum tensor itself is not a primary field. As a matter of fact, in a non-trivial theory the two-point function $< T(z)T(0) >$ cannot vanish and since the dimension of $T(z)$ is $h = 2^{16}$, scale invariance implies

$$< T(z)T(0) > = \frac{c/2}{z^4}. \tag{147}$$

The dimensionless constant $c$ is called central charge or conformal anomaly and it’s knowledge characterizes to a large extent a conformal model. $c$ has been normalized such that the free scalar massless boson has $c = 1$. The analog of equ.(146) for $T(z)$ itself now becomes

$$T(z_1)T(z_2) = \frac{c/2}{(z_1 - z_2)^4} + \frac{2}{(z_1 - z_2)^2}T(z_2) + \frac{1}{(z_1 - z_2)} \partial_1 T(z_2) + \ldots \tag{148}$$

Equ.(148) translates into the following transformation law for $T(z)$:

$$\delta T(z) = \frac{1}{12} c \epsilon'''(z) + 2c'(z) + c(z) \partial_z T(z). \tag{149}$$

It is not completely trivial to integrate this equation to obtain the transformation law for finite conformal transformations. We state only the result

$$T(z) = (f'(z'))^2 T'(z') + \frac{c}{12} S(f, z), \tag{150}$$

where the Schwartzian derivative is

$$S(f, z) = \frac{\partial_z f \partial^2 f - \frac{3}{2} (\partial^2 f)^2}{(\partial f)^2}.$$  

This is a very useful result, for it permits to connect informations pertaining to different geometries. For arbitrary $f(z)$, not of the form $f(z) = 16^{16}$

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16 This can already be seen from equ.(146). Or : the generator of translations by a vector $a_\mu$ is $exp(ia_\mu P^\mu)$. Therefore $P^\mu = \int T^\mu d\sigma$ has dimension one and $T^\mu$ has dimension two.

17 This expression is to be understood as occurring inside a correlation function. The ellipsis stands for terms, which are regular as $z_1 \rightarrow z_2$, generated when $T(z_1)$ hits other fields of the correlation function. The reader, who is uneasy with this kind of operator statements should take refuge in Furlan et al. Furlan et al.
\[(az + b)/(cz + d)\], the conformal plane will be mapped into a different geometrical domain. We now broaden the concept of conformal invariance to include invariance under this active transformation. Let us, for example, conformally map the complex \(z\)-plane (without the origin) onto a periodic horizontal strip of width \(R\) in the \(w = u + \nu\)-plane by \(z = \exp(2\pi wR)\) and analogously for \(\bar{z}\). We now postulate that objects in the \(z\)-plane go, via this mapping, over to the corresponding objects in the strip. Applying this to the energy-momentum tensor, we get

\[T_{\text{strip}}(u) = \left(\frac{2\pi}{R}\right)^2 [T_{\text{plane}}(z)z^2 - c/24]. \tag{151}\]

In particular, if we now take the expectation value of \(T(z)\), since \(<T_{\text{plane}}(z)>\) has been renormalized to zero, we find

\[<T_{\text{strip}}(u)> = \frac{c}{24} \frac{2\pi^2}{R}. \tag{152}\]

This formula allows us to measure the central charge \(c\) exploring finite size effects in the statistical mechanical version of our Euclidean quantum field theory, where \(c\) is obtained via the free energy.

In fact the variation of the free energy is given by a formula analogous to equ.(140) with no fields on the l.h.s.:

\[\delta \ln Z = \int_\mathcal{D} \frac{d^2u}{2\pi} <T_{\mu\nu}(u, \bar{\pi})> \partial_\mu \epsilon_\nu, \tag{153}\]

where the integration domain \(\mathcal{D}\) is the infinite strip. Choose a transversal dilatation by \(\delta \eta\) of the strip: \(\epsilon_1 = 0, \epsilon_2 = u_1 \delta \eta\). We get \(T_{\mu\nu} \partial_\mu \epsilon_\nu = (T(u) + \mathcal{T}(\bar{\pi})) \delta \eta\). The accessible statistical mechanical observable is the free energy per unit (vertical) length with suitable boundary conditions in the vertical direction

\[F(R) = -\lim_{M \to \infty} \frac{1}{M} \ln Z(R, M). \tag{154}\]

Under the transversal dilatation \(F(L)\) changes by \(\delta F(L) = R \delta \eta dF(R)/dR\) so that from equ.(153) follows, that

\[dF/dR = (2\pi/R^2)(c/12). \tag{155}\]

Using \(F(\infty) = 0\), we get

\[F(R) = -c \frac{\pi}{6} \frac{1}{R}. \tag{156}\]
This equation holds true, if all operators have positive dimensions, as is the case of unitary models. As we shall see, for non-unitary models, we have at least one operator with negative dimension $\Delta_0 < 0$. Then equ.(156) has to be modified to $c \rightarrow \tilde{c} = c - 12\Delta_0$. This equation is then used to extract the value of $c$ by numerical finite-size studies or, as we have done, using the Thermodynamic Bethe Ansatz.

5.3 The representation space of the Virasoro algebra

In order to construct a Hilbert space, in which the conformal fields may act as operators, we have to go from correlation functions to an operator formalism. Remember that the charges, which via equal time commutators, generate the variations of the fields under some transformations, are written as integrals over space-like surfaces. Looking at equ.(144), we see that this integral is in our case $\int dz T(z)$. This is made explicit adopting the radial quantization in the complex plane. One considers a Minkowski space $\sigma^0, \sigma^1$ with a periodic space coordinate $\sigma^1$. This amounts to quantize the theory on a cylinder of radius $R$, which then acts as an infra-red cutoff for the massless fields. These separate into left- and right-moving fields living on the light-cones $\sigma^0 \pm \sigma^1$. With the map $z = e^{\sigma^0 + i\sigma^1}$, which we have already used, the cylinder is then mapped onto the complex $z$-plane and analogously for right-movers. The space-like ’equal time’ slices are concentric circles around the origin, whereas the ’time’ direction is radially outward, $t = -\infty$ corresponding to $z = 0$ and $t = +\infty$ to the point $z = \infty$ of the complex plane. Time translations $\sigma^0 \rightarrow \sigma^0 + \tau$ on the cylinder are mapped into dilatations $z \rightarrow ze^\tau$ (and $\bar{z} \rightarrow \bar{z}e^\tau$) in the plane. Therefore, what we call Hamiltonian on the cylinder, is the generator of dilatations (= scale transformations) in $z$ and $\bar{z}$.

Remembering that Euclidean correlation functions correspond to time-ordered Green-functions, we can consider operators $\hat{\phi}(z)$ defined such that

$$< \phi_1(z_1) \ldots \phi_n(z_n) > =< 0|\mathcal{R} (\hat{\phi}_1(z_1) \ldots \hat{\phi}_n(z_n)) |0 >,$$

where we introduced the radial ordering operation $\mathcal{R}$:

$$\mathcal{R}(\hat{\phi}_1(z_1)\hat{\phi}_2(z_2)) = \begin{cases} \hat{\phi}_1(z_1)\hat{\phi}_2(z_2) & |z_1| > |z_2| \\ \hat{\phi}_2(z_2)\hat{\phi}_1(z_1) & |z_2| > |z_1|. \end{cases}$$

(158)
With this understanding, we can write equ.(143) as

\[
\delta \hat{\phi}(z, \bar{z}) = \frac{1}{2\pi i} \left( \oint_{|w|>|z|} - \oint_{|w|<|z|} \right) dw \epsilon(w) R(\hat{T}(w)\hat{\phi}(z, \bar{z}), (159)
\]

the difference of the two line integrals giving the integral around the point \(z\). This is now an operator statement.

For the particular case of \(T(z)\) itself, we get

\[
\delta \hat{T}(z) = \frac{1}{12} \epsilon''' + (2\epsilon'(z) + \epsilon(z)\frac{d}{dz})\hat{T}(z) = \left\{ \oint_{|w|>|z|} - \oint_{|w|<|z|} \right\} \epsilon(w) R(\hat{T}(z)\hat{T}(w)). (160)
\]

We now expand \(\hat{T}(z)\) and \(\hat{T}(\tau)\) in a Laurent series:

\[
\hat{T}(z) = \sum_{n=-\infty}^{+\infty} \frac{L_n}{z^{n+2}}, \quad \hat{T}(\tau) = \sum_{n=-\infty}^{+\infty} \frac{T_n}{\tau^{n+2}}, (161)
\]

where the factor \(z^{-2}\) has been introduced so that the operators \(L_n\) have dimension \(n\). If this expansion, together with a similar one for \(\epsilon(z)\), is now inserted into equ.(160), we obtain the infinite-dimensional Virasoro algebras:

\[
[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}cn(n^2 - 1)\delta_{n+m,0}
\]

\[
[L_n, T_m] = (n-m)T_{n+m} + \frac{1}{12}cn(n^2 - 1)\delta_{n+m,0}
\]

\[
[T_n, T_m] = 0. (162)
\]

The space in which the representation of the Virasoro algebra acts is then constructed as follows. We want to build up a Hilbert space in which the primary field \(\phi_h(z, \bar{z})\) with conformal weight \((h, \bar{h})\) may act. Define the vector

\[
|h> \equiv \lim_{z \to 0} \phi_h(z, \bar{z})|0> . (163)
\]

---

\[18\] This is what we mean by the equal-time commutator \([\int dx A(x), B(y)]\). For euclidean Green functions, the time evolution converges only for \(\Delta \tau > 0\) in \(\exp(-\Delta \tau \hat{H})\). Therefore the \(\hat{R}\)-operation does give sense to equ.(159).

\[19\] We may expand be around any point, not necessarily around the origin as done here.
From equ.(146) it follows, that

\[ [L_n, \phi(z)] = \oint \frac{dw}{2\pi i} w^{n+1} T(w) \phi(z) = h(n+1)z^n \phi(z) + z^{n+1} \partial \phi(z), \tag{164} \]

consequently \( [L_0, \phi(0)] = h\phi(0) \) and \([L_n, \phi(0)] = 0, n > 0 \). The state \(|h\rangle\) therefore satisfies

\[
L_n|h\rangle = T_n|h\rangle = 0 \quad \text{for } n > 0 \tag{165}
\]

\[
L_0|h\rangle = h|h\rangle, \quad T_0|h\rangle = 0.
\]

From the commutation relations of the \(L_n\)'s, we see that \(L_n\) decreases the dimension by \(n\) units, so that the above equations guarantee, that the spectrum of dimensions is bounded from below.

Notice that \(L_{-n}, \bar{T}_{-n}\) for \(n > 0\) create new states and therefore act like creation operators, whereas \(L_n, \bar{L}_n\) are destruction operators. The space is therefore built up by all vectors of the form

\[
L_{-n_1} \ldots L_{-n_N} \bar{T}_{-m_1} \ldots \bar{T}_{-m_M} |h\rangle \quad \text{with } n_j, m_k > 0. \tag{166}
\]

These vectors are also generated acting with descendant fields on the vacuum. These fields are the regular terms in equ.(146) as \(z \to w\) in the following equation, which we now write out explicitly:

\[
T(z)\phi(w, \bar{w}) \equiv \sum_{n\geq 0} (z-w)^{n-2} \bar{T}_{-n} \phi(w, \bar{w})
\]

\[
= \frac{1}{(z-w)^2} \bar{L}_0 \phi + \frac{1}{z-w} \bar{L}_{-1} \phi + \bar{L}_{-2} \phi + (z-w) \bar{L}_{-3} \phi + \ldots. \tag{167}
\]

The descendant fields are

\[
\phi^{(-n)}(w, \bar{w}) = \bar{T}_{-n} \phi(w, \bar{w}) = \oint \frac{dz}{2\pi i} \frac{1}{(z-w)^n} T(z) \phi(w, \bar{w}). \tag{168}
\]

Comparing with the l.h.s. of equ.(146), we note that \(\phi^{(0)} = \bar{L}_0 \phi = h\phi\) and \(\phi^{(-1)} = \bar{L}_{-1} \phi = \partial_z \phi\). The most important descendant field - and therefore not a primary field ! - is the energy-momentum tensor, which is a level two descendant of the identity:

\[
(\bar{L}_{-2} I)(w) = \oint \frac{dz}{2\pi i} \frac{1}{z-w} T(z) I = T(w). \tag{169}
\]

\footnote{This fact allows to write down differential equations satisfied by correlation functions.}
From equ.(168) and the definition of $L_n$ as
\[ L_n \phi(0) = \oint \frac{dz}{2\pi i} z^{n+1} T(z) \phi(0), \]
we see that the representation space is generated by the descendant fields as:
\[ L^{-n}|h> = L^{-n}(\phi(0)|0> = (L^{-n}\phi)|0> = \phi^{(-n)}(0)|0>. \]  
(170)

This representation is irreducible, unless there exists a null vector
\[ |\chi_{h,N}> = \sum a_k...L_{k_1}...L_{k_n}|h> \]
with $\sum k_j = N$, satisfying
\[ L_n|\chi_{h,N}> = 0, n > 0 \quad L_0|\chi_{h,N}> = (h + N)|\chi_{h,N}> \]  
(171)
for some positive integer $N$.

For example for $N = 2$, which is the first non-trivial case, these conditions are
\[ L_{+1}(L_{-2} + aL^2_{-1})\phi(z,\bar{z}) = 0 \]  
(172)
and conditions for $n \geq 3$ follow automatically from the Virasoro algebra. Now we move the destruction operators $L_{+1}, L_{+2}$ to the right till they give zero due to equ.(171). The resulting two equations are
\[ a = -\frac{3}{2(2h + 1)}, \quad c = \frac{2h(5 - 8h)}{2h + 1}. \]  
(173)

We will use analogous equations for level 3 degeneracy in order to obtain a first nontrivial conservation law in the next chapter.

Since the subspace generated by applying $L_n, L_m$’s with $n, m < 0$ is invariant, it has to be factored out in order to get an irreducible representation. This factor space is called a degenerate irreducible representation space (or modul ) generated by the degenerate field $\phi(z)$ and $N$ is it’s level.

These degenerate fields are very important, because for them the operator algebra
\[ \Phi_l \Phi_k = \sum_n c_{lkn} \Phi_n \]  
(174)
\[ ^{21}\text{This equation is written in an abbreviated notation only.} \]
closes with a finite number of terms. The corresponding central charge can be labeled by two positive integers $p, p'$ with no common divisor:

$$c(p, p') = 1 - \frac{6(p - p')}{pp'}.$$  \hfill (175)

For each value of $c(p, p')$ there are $(p - 1)(p' - 1)/2$ primary fields with dimensions

$$h_{r,s} = h_{p' - r, p - s} = \frac{(rp - sp')^2 - (p - p')^2}{4pp'}$$ \hfill (176)

$$1 \leq r \leq p' - 1, \quad 1 \leq s \leq p - 1.$$

For $p' - p \geq 2$, there will exist negative weights corresponding to growing correlation function in non-unitary theories. For $c < 1$ all unitary theories are special cases of the above with $p' - p = 1$ or:

$$c(m) = 1 - \frac{6}{m(m + 1)}, \quad m = 3, 4, \ldots$$ \hfill (177)

For each value of $c(m)$ there are again $m(m - 1)/2$ allowed values of $h$:

$$h_{r,s}(m) = h_{m-r,m+1-s} = \frac{[(m + 1)r - ms]^2 - 1}{4m(m + 1)}$$ \hfill (178)

with the integers $r, s$ satisfying $1 \leq r \leq m - 1, 1 \leq s \leq m$.

We also know that the representation with highest weight $h_{rs}$ is degenerate at level $rs$. Besides this the correlation functions of these fields satisfy linear differential equations of order $rs$.

We may now clarify the shift $c \to \tilde{c}$ for non-unitary models. Recalling equ.(78) of section 4, the quantum hamiltonian $\hat{H}$ is given by:

$$\hat{H} = \frac{1}{2\pi} \int_0^R T_{uu}(v)dv = \frac{1}{2\pi} \int_0^R (T(v) + \overline{T}(v))dv = \frac{2\pi}{R} (L_0 + \overline{L}_0) - \frac{\pi c}{6R}.$$  \hfill (179)

As advertised, we see that time evolution on the cylinder, corresponds to dilatations on the plane. From equ.(78), it follows that states with higher eigenvalues of $L_0$ and $\overline{L}_0$, contribute exponentially smaller corrections to $Z(L, R)$. However in non-unitary theories some operators may have negative dimensions. Suppose there is one such operator with dimension $\Delta_0 < 0$. In this case a factor $exp(-2\pi \Delta_0/R)$ has to be kept in $Z(L, R)$, effectively shifting $c$ to $\tilde{c} = c - 12\Delta_0$. 

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5.4 Characters of the Virasoro algebra

For Zamolochikov’s counting argument, which permits a partial identification of the infinite number of conservation laws surviving perturbations breaking conformal invariance, we need to determine the dimensional decomposition of the representations of the Virasoro algebras. In the representation space of highest weight $h$ the characters $\chi_h(q)$ are the generating functions for the number of linearly independent vectors at level $n$, therefore having eigenvalues $h + n$ of $L_0$. For $q, |q| < 1$ we define

$$\chi_h(q) \equiv q^{-c/24} T \left. L_0 \right|_h = q^{h-c/24} \sum_{n=0}^{\infty} d_h(n) q^n. \quad (180)$$

Here $d_h(n)$ counts the degeneracy of the states in the representation at level $n$. An analogous definition holds for the right Virasoro algebra.

If there are no null states in the representation of weight $h$, the states at level $n$ are of the form

$$L_{-n_1} L_{-n_2} \ldots L_{-n_k} |h > \sum_{i} n_i = n. \quad (181)$$

In this case $d_h(n)$ equals $p(n)$ - the number of partitions of the integer $n$. Euler’s generating function for these parts gives

$$\sum_{n=0}^{\infty} q^n p(n) = \prod_{n=1}^{\infty} (1 - q^n)^{-1} = \frac{1}{q^{-1/24} \eta(q)} \quad (182)$$

and

$$\chi_h(q) = q^{(h+(1-c)/24)} \eta(q)^{-1}. \quad (183)$$

Here we have introduced Dedekind’s $\eta$-function, central in the study of elliptic functions. The $c$-dependent factor has introduced, so that $\chi_{rs}(q)$ has nice modular transformation properties.

Let us compute the characters for the degenerate, unitary series with $c < 1$. The allowed weights are given by equ. (178) and are degenerate at level $rs$. Thus in counting the states we have to subtract the null state at level $rs$ and all its descendants, getting

$$\chi_{rs}(q) = q^{(1-c)/24} \eta(q)^{-1} (q^{h_{rs}} - q^{h_{rs}+rs} + \ldots). \quad (184)$$
But the null state has weight $h_{r,s} + rs = h_{m+r,m+1-s}$ and therefore we have to subtract in turn it (and all its descendants) at level $(m+r)(m+1-s)$. Correcting the above formula for $\chi_{r,s}$, we get

$$\chi_{r,s}(q) = q^{(1-c)/24} \eta(q)^{-1} (q^{h_{r,s}} - q^{h_{r,-s}} (1 - q^{(m+r)(m+1-s)} + \ldots))$$

$$= q^{(1-c)/24} \eta(q)^{-1} (q^{h_{r,s}} - q^{h_{r,-s}} + q^{h_{2m+r,s}} - \ldots).$$

(185)

Repeating this process yields finally the correct expression for the character:

$$\chi_{r,s}(q) = q^{(1-c)/24} \eta(q)^{-1} \sum_{k=-\infty}^{\infty} (q^{h_{2mk+r,s}} - q^{h_{2mk+r,-s}}).$$

(186)

6 Deformations of Conformal Invariant Field Theories

In this section we will study relevant perturbations of Conformal Invariant Field Theories (CFT), i.e. perturbations which drive the system away from its UV critical unstable fixed point. An example would be the perturbation of a statistical mechanical system at its critical point by a magnetic field, the raising of the temperature off $T = T_c$ etc. The system will then flow away from its UV fixed point and may either end up at another critical conformally invariant fixed point or develop a finite correlation length, i.e. the theory becomes massive.

In a CFT conservation laws are trivially satisfied by fields, which depend only on either of the light cone coordinates $z$ or $\overline{z}$, for example the energy-momentum tensor and its regularized powers (in most of the models, we are going to look at, there are also other conserved currents). If an infinite number of local integrals of motion survives the perturbation of the CFT, then we know that the massive theory is described by factorized $S$-matrices, of which some examples have been studied in chapter [4]. There we made some assumptions, which resulted in the existence of an infinite number of conserved charges $P_s$, where $s$ belonged to a certain subset of the natural numbers $\mathbb{N}$. Suppose we perturb the critical action $S_*$ by a relevant scalar

\[22\] Recall that $P_{-s}$ is related to $P_{+s}$ by parity, which we always assume to be a symmetry, allowing us to restrict ourselves to $s > 0$.  

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operator:

\[ S = S^* - \lambda \int \phi(z, \bar{z}) \, d^2 z, \quad (187) \]

where the weight of \( \phi \) is \((h, h)\), so that the dimension of \( \lambda \) is \((1 - h, 1 - h)\). For this to be a relevant perturbation, we need \( y = 2(1 - h) > 0 \) or \( h < 1 \).

Charges, whose conservation is preserved by the perturbation, now become

\[ P_s = \oint [ T_{s+1} \, d z + \Theta_{s-1} \, d \bar{z} ], \quad (188) \]

where \( T_s \) and \( \Theta_s \) are local fields of spin \( s \), satisfying the continuity equation

\[ \partial \bar{z} T_{s+1} = \partial_z \Theta_{s-1}. \quad (189) \]

If \( S^* \) described a free field theory, such a perturbation would be called super-renormalizable and adding the finite number of terms with dimensions smaller or equal than that of \( \phi \) one obtains a finite theory without UV-divergencies. In particular this doesn’t change the structure of the UV fixed point and the fields continue to have the same dimensions at short distances, although the large distance behaviour of the perturbed theory is very much different. We will assume that this is also true for our case, where we perturb around a theory, which is not free.

Thus \( S^* \) describes an UV-finite ( renormalized ) theory, which contains \( \phi \) as one of its operators and we assume, that we know all its correlation functions ( as well as those of all other operators ). Our primary aim is to check whether there exist currents \( J(z) \), whose conservation survives the perturbation, i.e. whether it satisfies equ.(189).

The perturbed correlation functions of a particular operator \( J(z) \) are now given by

\[ < J(z, \bar{z}) \ldots > = < J(z) \ldots >_{S^*} + \lambda \int d^2 z_1 < J(z) \phi(z_1, \bar{z}_1) \ldots >_{S^*} + O(\lambda^2). \quad (190) \]

If this integral were finite, it would be independent of \( \bar{z} \). Therefore any \( \bar{z} \)-dependence can come only from possible singular points \( z \to z_1 \). In their neighborhood, we can use the short distance expansion ( SDE ):

\[ J(z) \phi(z_j, \bar{z}_j) = \sum_k \frac{a_k}{[z - z_j]^\Delta_j + \Delta - \Delta_k} \phi_k(z_j, \bar{z}_j), \quad (191) \]
where $\Delta = 2h$ and $\Delta_J$ and $\Delta_k$ are the scaling dimensions of $J$ and $\phi_k$. Since only non-integrable singularities will contribute, this requires $\Delta_J + \Delta - \Delta_k \geq 2$. In a unitary theory all dimensions are $> 0$ and therefore only a finite number of operators $\phi_k$ will contribute to first order in $\lambda$.

Let us look at the specific example of the energy-momentum tensor. In this case the SDE is

$$T(z)\phi(z_1, \bar{z}_1) = \frac{h}{(z - z_1)^2} \phi(z_1, \bar{z}_1) + \frac{1}{z - z_1} \partial_1 \phi(z_1, \bar{z}_1).$$  \hspace{1cm} (192)

Since we only want to check whether the current $T(z, \bar{z})$ is still conserved and to avoid bothering with infra-red singularities, let us only calculate the $\partial_{\bar{z}}$ derivative of $T(z, \bar{z})$. For this purpose we use the equation

$$\partial_{\bar{z}}(z - \xi)^{-m-1} = 2\pi i \frac{(-1)^m}{m!} \partial_{\bar{z}}^m \delta^{(2)}(z - \xi).$$  \hspace{1cm} (193)

This can easily be proved, either remembering that the logarithm is the 2-dimensional Green function $\Delta \ln r = -2\pi \delta^{(2)}(\bar{r})$, i.e.

$$\partial_{\bar{z}} \partial_z \ln[(z - z')(\bar{z} - \bar{z}')] = -2\pi i \delta^{(2)}(z - z')$$

and taking derivatives thereof, or proceeding as follows. Let us regulate the UV divergence by a cut-off $a$, inserting the step function $H(x)$, which vanishes for $x < 0$. Then

$$\int d^2z \partial_{\bar{z}} \frac{H[z\bar{z} - a^2]}{z^{m+1}} f(z, \bar{z}) = \int d^2z \frac{z \delta(z\bar{z} - a^2)}{z^{m+1}} f(z, \bar{z}) = i \int d\varphi \frac{f(\alpha e^{i\varphi}, \alpha e^{-i\varphi})}{a^m e^{im\varphi}} = 2\pi i \frac{(-1)^m}{m!} \int d^2z \partial_{\bar{z}}^m \delta^{(2)}(z) f(z, \bar{z}).$$  \hspace{1cm} (194)

Hence $T(z, \bar{z})$ satisfies

$$\partial_{\bar{z}} \int d^2z_1 < T(z) \phi(z, \bar{z}) \ldots > = \int d^2z_1 \partial_{\bar{z}} < \left(\frac{h}{(z - z_1)^2} \phi(z_1, \bar{z}_1) + \frac{1}{z - z_1} \partial_1 \phi(z_1, \bar{z}_1)\right) \ldots > = \int d^2z_1 \left(h 2\pi \partial_{\bar{z}} \delta^{(2)}(z - z_1) + 2\pi \delta^{(2)}(z - z_1) \partial_1\right) < \phi(z_1, \bar{z}_1) \ldots >.$$  \hspace{1cm} (195)
We therefore immediately get the conservation law for the energy-momentum tensor - as expected, since the energy-momentum tensor must remain conserved - :

\[ \partial_z T + \partial_z \Theta = 0, \quad (196) \]

where

\[ \Theta = \pi \lambda (1 - h) \phi(z, \bar{z}). \quad (197) \]

We see, that the term in the SDE equ.(192), relevant for the conservation of the current \( J(z, \bar{z}) \) is picked out by :

\[ \partial_z J(z, \bar{z}) = \lambda \pi \oint_{C_z} \frac{dz_1}{2\pi i} J(z) \phi(z_1, \bar{z}_1). \quad (198) \]

Remembering the discussion on radial quantization leading to equ.(159), we realize that the r.h.s. of this equation is a commutator and we get the following suggestive form :

\[ \partial_z J(z, \bar{z}) = [J(z, \bar{z}), H_{\text{int}}(\bar{z})], \quad (199) \]

where

\[ H_{\text{int}}(\bar{z}) = \lambda \int dz_1 \phi(z_1, \bar{z}). \]

It remains to be seen, if the r.h.s. of equ.(198) can be expressed as \( \partial_z \) of some operator. If yes, the conservation law will continue to hold to first order in \( \lambda \).

To provide an example of how this can be checked, let us take the energy-momentum tensor and it’s powers. Define \( \Lambda \) to be the irreducible Virasoro modul with highest weight \( h = 0 \), to which the energy-momentum tensor belongs. Introduce Virasoro generators as in equ.(161), but via an expansion around an arbitrary point \( \zeta \):

\[ T(z) = \sum_{n=-\infty}^{+\infty} \frac{L_n}{(\zeta - z)^{n+2}}. \quad (200) \]

Then, using the commutation relation

\[ [L_{-1}, \phi(\zeta, \bar{\zeta})] = \partial_\zeta \phi(\zeta, \bar{\zeta}), \quad (201) \]

it immediately follows for the operator \( \partial_\zeta \) in equ.(198) - with \( J(z) \) replaced by \( T(z) \) -

\[ \partial_\zeta L_{-1} \Lambda = L_{-1} \partial_\zeta \Lambda. \quad (202) \]
Now we streamline our algebra following reference [0]. Introduce a set of operators $D_n$, $n = 0, \pm 1, \pm 2, \ldots$ as

$$D_n \Lambda(z, \overline{z}) = \oint_z \frac{d\zeta}{2\pi i} \phi(\zeta, \overline{z})(\zeta - z)^n \Lambda(z), \quad (203)$$

i.e. $D_n$ projects out the term proportional to $(\zeta - z)^{n+1}$ in the SDE of $\phi$ and $\Lambda$. The following equations show, why the $D_n$ are of good use to compute $\partial_\overline{z}$:

$$\partial_\overline{z} = D_0. \quad (204)$$

Also

$$D_{-n-1}I = \oint_d \frac{d\zeta}{2\pi i} (\zeta - z)^{-n-1} \phi(\zeta, \overline{z}), \quad n \geq 0,$$

and using the residue theorem for the $(n+1)$-th order pole:

$$D_{-n-1}I = \frac{1}{n!} \partial^2_\overline{z} \phi(z, \overline{z}). \quad (205)$$

Since $T(z)$ is generated from the unity $I$ by applying operators $L_{-n}$, we need the commutation relations between $L_{-n}$ and $D_m$. First remember

$$[L_{-n}, \phi(\zeta, \overline{\zeta})] = \{(\zeta - z)^{n+1} \partial_\zeta + (n + 1)h(\zeta - z)^n\} \phi(\zeta, \overline{\zeta}). \quad (206)$$

Now compute the commutator of $[L_n, D_m]$:

$$L_n D_m \Lambda(z, \overline{z}) = \oint_z \frac{d\zeta}{2\pi i} L_n \phi(\zeta, \overline{z}) =$$

$$(\zeta - z)^m \Lambda(z) = \oint_z \frac{d\zeta}{2\pi i} ([L_n, \phi(\zeta, \overline{z})] + \phi(\zeta, \overline{z})L_n) (\zeta - z)^m \Lambda(z).$$

Therefore

$$[L_n, D_m] \Lambda(z, \overline{z}) = \oint_z \frac{d\zeta}{2\pi i} [L_n, \phi(\zeta, \overline{z})](\zeta - z)^m \Lambda(z) =$$

$$\oint \frac{d\zeta}{2\pi i} \left((\zeta - z)^{n+m+1} \partial_\zeta + (n + 1)h(\zeta - z)^{n+m}\right) \phi(\zeta, \overline{z}) \Lambda(z) =$$

$$\{-(1-h)(n+1)\} \oint \frac{d\zeta}{2\pi i} \phi(\zeta, \overline{z})(\zeta - z)^{n+m} \Lambda(z).$$
Hence we get the commutator

\[ [L_n, D_m] = -\{(1 - h)(n + 1) + m\}D_{n+m}. \]  

(207)

A trivial application is

\[ \partial_z T(z, \overline{z}) = \lambda D_0 L_{-2} I = \lambda(h - 1)D_{-2} I = \lambda(h - 1)L_{-1}\phi(z, \overline{z}) \]  

(208)

reproducing equ. (196).

A less non-trivial calculation is to check the conservation of higher powers of the momentum. Let us define a regularized square of \( T T \equiv T^2 \) as

\[ T^4(z) \equiv (L_{-2} L_{-2} I)(z) = \oint_z d\zeta (\zeta - z)^{-1} T(\zeta) T(z). \]  

(209)

Now check it’s conservation :

\[ \partial_z T_4 = \lambda D_0 L_{-2} L_{-2} I = \lambda(h - 1)(D_{-2} L_{-2} + L_{-2} D_{-2}) I = \lambda(h - 1)(2L_{-2} L_{-1} + \frac{h - 3}{6} L_{-1}^3) \phi. \]  

(210)

Therefore, due to the first term above, the r.h.s. is in general not a derivative of \( z \). This is to be expected, since the existence or not of this conservation law is a dynamical question. Let us take as perturbation one of the fields \( \phi_{1,3} \) of the unitary models with \( c < 1 \) as an example. It is degenerate at level 3 and repeating the steps that led to equ. (172) for the present case, we would get the following null-vector equation:

\[ \left( L_{-3} - \frac{2}{h + 2} L_{-1} L_{-2} + \frac{1}{(h + 1)(h + 2)} L_{-1}^3 \right) \phi_{1,3}(z) = 0. \]  

(211)

Hence the term containing \( L_{-2} \) in equ. (210) can be eliminated in favor of the derivative \( L_{-1} : L_{-2} L_{-1} = L_{-1} L_{-2} - L_{-3} \) and using for \( L_{-3} \) the null-vector equation, we get

\[ \partial_z T_4(z, \overline{z}) = \partial_z \Theta_2(z, \overline{z}), \]  

(212)

with

\[ \Theta_2 = \frac{\lambda}{h + 1} \left( 2hL_{-2} + \frac{(h - 2)(h - 1)(h + 3)}{6(h + 1)} L_{-1}^3 \right) \phi_{1,3}. \]

Finally let us check that, similarly to the first order perturbation, the perturbation expansion to n-th order, which usually contains an infinite number
of terms, is here also drastically truncated. The n-th order term will have
the form
\[ \lambda^n \int d^2 z_1 \ldots d^2 z_n < J(z) \prod_{j=1}^{n} \phi(z_j, \overline{z}_j) \ldots > S^* \]  
(213)

We easily see, that the condition for non-integrable singularities is now
\[ \Delta_J + n \Delta - \Delta_k > 2n \] or \[ \Delta_J - (2 - \Delta) n - \Delta_k \geq 0 \]. Since \( \Delta < 2 \), this condition
will eventually be violated and the perturbation expansion has to stop with a finite number of terms.

6.1 Counting Arguments

In principle we may now take specific models and start looking for surviving
conservation laws. It turns out [1], that at the expense of some formalism
this job can be significantly simplified. Let us then present, what is called
Zamoldchikov's counting argument. It will enable us to find out which of
the \( P_s \) are still conserved, at least for small \( s \), without having to compute
explicitly the term \( \Theta_s \) in equ.(189). Once we know the conservation laws, we
may link up with chapter 2 and lift a candidate \( S \)-matrix for the model in
question.

Let us talk about the energy-momentum tensor \( T \), since \( T \) and its regular-
ized powers will provide the conservation of the momentum and it's powers.
The same type of reasoning also applies to other conserved currents.

Recall that \( \Lambda \) was defined to be the irreducible Virasoro modul with
highest weight \( h = 0 \), to which the energy-momentum tensor belongs. That
is, \( \Lambda \) is the infinite-dimensional space spanned by all the fields of the form
\( L_{-n_1} L_{-n_2} \ldots L_{-n_k} I \), where \( n_i \) are positive integers or zero. \( \Lambda \) may be decom-
posed as
\[ \Lambda = \bigoplus_{s=0}^{\infty} \Lambda_s, \]  
(214)

where the fields belonging to \( \Lambda_s \) satisfy \( \sum_{i=1}^{k} n_i = s \). From the commutation
relations for the \( L_n \) we easily see that the \( \Lambda_s \) are eigenspaces of \( L_0 \):
\[ L_0 \Lambda_s = s \Lambda_s. \]  
(215)

Thus all fields belonging to \( \Lambda_s \) have conformal weight \((s, 0)\) and therefore
dimension and spin equal to \( s \). Besides this all these fields depend only on \( z \)
and are thus analytic, satisfying $\partial_{z}\Lambda = 0$ : they all give rise to integrals of motion. However for our counting argument it is important to exclude fields, which are derivatives of others. These would lead situations like $\partial_{z}\partial_{z}T = \partial_{z}R$, which do not correspond to conservation laws at all. They are contained in $L_{-1}\Lambda$. Therefore let us define a new space $\hat{\Lambda}$, where these fields are divided out : $\hat{\Lambda} = \Lambda / L_{-1}\Lambda$. This space also has the decomposition

$$\hat{\Lambda} = \bigoplus_{s=0}^{\infty} \hat{\Lambda}_{s}, \quad L_{0}\hat{\Lambda}_{s} = s\hat{\Lambda}_{s}. \quad (216)$$

We can now take advantage of character formulas to obtain the dimensions of $\hat{\Lambda}_{s}$:

$$\sum_{s=0}^{\infty} q^{s} \dim \hat{\Lambda}_{s} = (1 - q)\chi_{0}(q) + q, \quad (217)$$

where $\chi_{0}(q)$ is the character of $\Lambda$ defined as

$$\chi_{0}(q) = \sum_{s=0}^{\infty} q^{s} \dim \Lambda_{s}. \quad (218)$$

Equ.(217) can easily be shown as follows. First note that

$$\dim ((L_{-1}\Lambda)_{s}) = \begin{cases} \dim \Lambda_{s-1}, & \text{for } s > 1 \\ 0, & s = 1, \end{cases}$$

since $L_{-1}I = 0$. Therefore:

$$\sum_{s=0}^{\infty} q^{s} \dim ((L_{-1}\Lambda)_{s}) = \sum_{s=2}^{\infty} q^{s} \dim \Lambda_{s-1} = q \sum_{s=2}^{\infty} q^{s-1} \dim \Lambda_{s-1} = q \sum_{s=0}^{\infty} q^{s} \dim \Lambda_{s} - q = q \chi_{0}(q) - q,$$

and equ.(217) follows. If $c$ doesn’t belong to the degenerate set given by equ.(175) or equ.(177), the dimension of $\Lambda_{s}$, according to its definition, equals $p(s)$ - the number of partitions of the integer $s$. Equ.(182) gives then

$$\sum_{s=0}^{\infty} q^{s} p(s) = \prod_{n=1}^{\infty} (1 - q^{n})^{-1} = \chi_{0}(q), \quad (219)$$
so that finally we get \[23\]

\[
\sum_{s=0}^{\infty} q^s \text{dim}(\hat{\Lambda}_s) = 1 + q^2 + q^4 + 2q^6 + \ldots.
\]

Now we have to ask, which of these conserved currents survive perturbation? Suppose we add a relevant perturbation \(\lambda \phi(z, \bar{z})\). \(\phi(z, \bar{z})\) is the highest weight vector of the irreducible module \(\Phi \otimes \overline{\Phi}\), generated by all the fields of the form \(L_{-n_1} \ldots L_{-n_k} \overline{T}_{-m_1} \ldots \overline{T}_{-m_l} \phi\), where \(n_i, m_j\) are positive integers or zero. As \(\Lambda, \Phi, \Phi\) may be decomposed as

\[\Phi = \bigoplus_{s=0}^{\infty} \Phi_s, \quad L_0 \Phi_s = (\Delta + s)\Phi_s, \quad \overline{T}_0 \Phi_s = \Delta \Phi_s. \tag{221}\]

Now the fields \(T^{(s)}_s \in \hat{\Lambda}_s\) do not satisfy \(\partial \phi T^{(s)}_s = 0\), but

\[\partial \phi T^{(s)}_s = \lambda \Phi^{(s)}_{s-1} + O(\lambda). \tag{222}\]

Here \(\Phi^{(s)}_{s-1}\) are local fields belonging to \(\Phi_{s-1}\), and for simplicity assume, that only the first order term in \(\lambda\) contributes \[24\]. We use this equation to define the linear operator

\[\partial \phi : \hat{\Lambda}_s \to \Phi_{s-1}. \tag{223}\]

Let us consider the space \(\hat{\Phi}\), where we factored out the derivatives : \(\hat{\Phi}_s = \Phi_s / L_{-1} \Phi_{s-1}\). Consider now the mapping \(\mathcal{M}_s\) from \(\Lambda_s \to \hat{\Phi}_{s-1}\). \(\mathcal{M}_s\) is implemented by \(\mathcal{M}_s = \Pi_s D_{0,s}\), where \(\Pi_s\) is the projector from \(\Phi_s\) to \(\Phi_{s-1}\) and \(D_{0,s}\) is \(\partial \phi\) restricted to \(\Lambda_s\). Since in \(\hat{\Phi}\) we factored out the derivatives, all the fields satisfying

\[\partial \phi T^{(s+1)}_s = \partial \phi \Theta^{(s+1)}_{s-1}, \tag{224}\]

are mapped into the null element of \(\hat{\Phi}_{s-1}\); i.e. all elements \(T^{(s+1)}_s\) satisfying eqn.\[224\] belong to the kernel of \(\mathcal{M}_s\). If \(\text{dim} \hat{\Lambda}_{s+1} > \text{dim} \hat{\Phi}_s\), then \(\text{Ker} \mathcal{M}_s \neq 0\) and we have conserved charges surviving the perturbation.

The dimension of \(\hat{\Phi}_s\) can be computed as that of \(\hat{\Lambda}_s\) and we obtain:

\[
\sum_{s=0}^{\infty} q^{\Delta+s} \text{dim} \hat{\Phi}_s = (1 - q) \chi \Delta(q), \tag{225}\]

\[23\]For the degenerate cases, we have to subtract invariant subspaces and use eqn.\[186\]. Even then eqn.\[217\] is still valid for \(s < m(m - 1)\).

\[24\]See ref.\[5\] for a discussion when this is not true.
where $\chi_\Delta(q)$ is the character of the modul with highest weight $(h, h)$. 

Let us apply this scheme to specific cases.

1) Lee-Yang edge singularity and the associated non-unitary series

The Lee-Yang singularity describes the critical behavior of an Ising model in a purely imaginary magnetic field $ih$. Above a critical value $h_c$ the zeroes of the partition function condense on the imaginary $h$-axis and the following Ginzburg-Landau lagrangian describes this behavior:

$$L = \int d^2x \left[ \frac{1}{2} (\partial_\mu \varphi(x))^2 - \nu (h - h_c) \varphi(x) - \nu \varphi(x)^3 \right]. \quad (226)$$

This is clearly a non-unitary theory and at the critical point $h = h_c$, $\varphi(x)$ is the only relevant operator. In ref.[0] it was shown that there exists only one model with these properties and no extra symmetries. With the aid of equs.(175) and eq.(176) one easily identifies it to have $p = 2$ and $p' = 5$ with central charge $c = c(2, 5) = -22/5$. It contains only two primary fields: the identity and the field $\varphi = \varphi_{(1,2)} = \varphi_{(1,3)}$ with weights $(0, 0)$ and $(-1/5, -1/5)$ respectively. The negative values reflect lack of unitarity here.

We now check for conservation laws computing the dimensions of the spaces $\hat{\Lambda}_s$ and $\hat{\Phi}_s$, using eq.(217) and eq.(225). One finds, that the dimension of $\hat{\Lambda}_s$ exceeds the one of $\hat{\Phi}_s$ by one unit for

$$s = 1, 5, 7.$$ \hspace{1cm} (227)

These are the first 3 conservation laws found in section 2. This reasoning has been extended[1] to the whole series of non-unitary models $p = 2, p' = 2N + 3$ with

$$c(2, 2N + 3) = -\frac{2N(6N + 5)}{2N + 3} \quad (228)$$

perturbed by the field $\varphi_{(1,3)}$.

2) Ising model.

It’s conformal anomaly is $c = c(m = 3) = 1/2$. This model contains three spinless primary fields: $I = \varphi_{(1,1)}, \sigma = \varphi_{(1,2)}$ and $\epsilon = \varphi_{(1,3)}$ with

---

Notice that the identity is the only primary field satisfying $L_{-1} I = 0$, which accounts for the difference between equs.(217) and (225).
conformal weights \((0, 0), (1/16, 1/16)\) and \((1/2, 1/2)\), which are identified with the identity operator, the spin density (magnetization) and the energy density respectively. All the other local fields in this model are obtained by applying the left and right Virasoro generators \(L_n\) and \(\bar{L}_n\) with \(n < 0\) to these primary fields.

Suppose we perturb this model by a magnetic field, which breaks the \(Z_2\) invariance and couples to the magnetization \(\sigma(x) = \varphi_{(1,2)}(x)\). The total action is

\[
S_{1/2}^{(1,2)} = S^* + h \int d^2 x \sigma(x).
\]

The dimensions of the spaces \(\hat{\Lambda}_s\) and \(\hat{\Phi}_s\) can be calculated, using equ.(217) and equ.(225). For the characters, we use equ.(186). The relevant formulas are

\[
\chi_0(q) = \chi_{1,1}(q) = \frac{1}{2} \left\{ \prod_{n=1}^{\infty} (1 + q^{n+1/2}) + \prod_{n=1}^{\infty} (1 - q^{n+1/2}) \right\},
\]

\[
\chi_{1/16}(q) = \chi_{1,2}(q) = q^{1/16} \prod_{n=0}^{\infty} (1 - q^{2n+1})^{-1} = q^{1/16} \prod_{n=1}^{\infty} (1 + q^n).
\]

Computing now the dimensions, we check that the dimension of \(\hat{\Lambda}_s\) exceeds the one of \(\hat{\Phi}_s\) by one unit for

\[
s = 1, 7, 11, 13, 19.
\]

For larger values of \(s\) the dimension of \(\hat{\Phi}_s\) is greater or equal and nothing can be concluded. However we obtained five nontrivial conservation laws for the Ising model in a magnetic field and one conjectures the infinite set, corresponding to all integers \(s\), relatively prime to 30 as found in section 2.

3) \(Z(N)\)-models.

Using again the by now familiar scheme let us consider the \(Z(3)\) model perturbed by the field \(\varphi_{(1,2)}\), which is the most relevant thermal operator. We get conservation laws for \(s = 1, 2, 4, 5, 7, 8\), which are the first six integers of the infinite series \(s \neq 0, \text{mod}(3)\).

This is a particular case of the following general results obtainable using the machinery developed in the lectures.
The UV limiting CFT of the $Z(N)$ models exhibit primary fields of integer spin $s = 3, 4, \ldots, N$, which depend only on the variable $z$ - and analogously for $\bar{z}$. These are therefore conserved currents and the Virasoso algebra is enlarged to a $W(N)$ algebra, containing the energy-momentum tensor and those currents [0]. This additional symmetry is also suggested by the coset construction [1], from which the $Z(N)$ models may be obtained, using the $A_{N-1}$ Lie algebra. The central charge of the $W(N)$-models depends on two integers:

$$c(p, N) = (N - 1) \left( 1 - \frac{N(N + 1)}{p(p + 1)} \right).$$

(233)

For $p = N + 1$ we obtain our $Z(N)$ series. The generalization of the $\varphi_{(1,2)} \in Z(3)$ is a primary field of weight $(\frac{2}{h^A+2}, \frac{2}{h^A+2})$, which preserves the global symmetry and is the most relevant thermal operator. Here $h^A = N$ is the so-called dual Coxeter number of the Lie group $A_{N-1}$. Perturbing the UV limit with this operator, we obtain our massive $Z(N)$ models with conservation laws $P_s$, where $s$ takes the exponents of $A_{N-1}$ (which are all integers $1, 2, \ldots, N$), repeated modulo the Coxeter number $h^G$, i.e. modulo $N$.

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