Vragov’s boundary value problem for an implicit equation of mixed type

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Abstract. We study a Vragov boundary value problem for a third-order implicit equation of mixed type with an arbitrary manifold of type switch. These Sobolev-type equations arise in many important applied problems. Given certain constraints on the coefficients and the right-hand side of the equation, we demonstrate, using nonstationary Galerkin method and regularization method, the unique regular solvability of the boundary value problem. We also obtain an error estimate for approximate solutions of the boundary value problem in terms of the regularization parameter and the eigenvalues of the Dirichlet spectral problem for the Laplace operator.

1. Introduction
Due to their importance in applications to gas dynamics, boundary value problems for nonclassical equations of mixed type were studied by F. Tricomi, S. Gellerstedt, M.A. Lavrentyev, A.V. Bitsadze, and others [1–11]. One can find examples [7–9] of implicit PDEs that arise in mathematical models of viscoelasticity, electrodynamics, semiconductor physics, polymer mechanics and other processes of modern physics. The equations of this kind are said to be of Sobolev type.

Note that the regular solvability of the first boundary value problem for equations of Sobolev type was studied [12–14].

Here, a Vragov [10] boundary value problem for a third-order implicit equation of mixed type will be studied. Given certain constraints on the coefficients and the right-hand side of the equation, the unique regular solvability of the boundary value problem will be demonstrated, using nonstationary Galerkin method and regularization method. Moreover, an error estimate for approximate solutions of the boundary value problem in terms of the regularization parameter and the eigenvalues of the Dirichlet spectral problem for the Laplace operator will be obtained.

2. Statement of the boundary value problem
Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\gamma$. Denote $Q = \Omega \times (0,T)$, $\Gamma = \gamma \times (0,T)$, $T > 0$, $\Omega_t = \Omega \times \{t\}$, $t \in [0,T]$.

In the cylindrical domain $Q$ consider the equation

$$Lu \equiv k(x,t)u_{tt} - \Delta u_t + a(x,t)u_t + c(x)u = f(x,t), \quad (x,t) \in Q. \quad (1)$$

The coefficients here are assumed to be smooth functions.
Note that the equation (1) belongs to the class of equations of mixed type with a non-fixed manifold where the type changes. Introduce the set
\[ P^\pm_0 = \{(x, 0) : k(x, 0) \gtrless 0, x \in \Omega \}, \quad P^{\mp}_T = \{(x, T) : k(x, T) \gtrless 0, x \in \Omega \}. \]

Boundary value problem. Find a solution of equation (1) in \( Q \), such that
\[ u|_{\Gamma} = 0, \quad u|_{t=0} = 0, \quad u|_{P^+_0} = 0, \quad u|_{P^-_T} = 0. \] (2)

3. Derivation of the a priori bound
In an anisotropic Sobolev space \( W^{m,s}_2(Q) \) introduce the norm
\[ \|u\|_{m,s}^2 = \int_Q \left[ \sum_{|\alpha| \leq m} (D^\alpha u)^2 + (D^s_t u)^2 \right] \, dQ, \quad u \in W^{m,s}_2(Q), \]
with \( \|u\|_{m,m} = \|u\|_m \) for \( u \in W^{m,m}_2(Q) = W^m_2(Q) \).
Denote
\[ (u, v)_0 = \int_{\Omega} u(x)v(x) \, dx, \quad \forall u, v \in L^2(\Omega) \]
the scalar product in the space \( L^2(\Omega) \) and \( (u, v) = \int_0^T (u, v)_0 dt \) for \( u, v \in L^2(Q) \), \( \|u\|^2 = (u, u) \).

Introduce the class of functions
\[ C_L = \{ u(x,t) : u \in W^2_2(Q), u_{x_i}x_j, t \in L^2(Q), i, j = 1, n \} \text{ and conditions (2) hold} \} . \]

**Lemma 1** Assume that \( c(x) \geq 0, \quad a - \frac{1}{2} k_t \geq \delta > 0. \)
Then the following inequality holds:
\[ (Lu, u_t) \geq C_1 \left[ \|u\|_1^2 + \int_Q \sum_{i=1}^n u_{tx}^2 \, dQ \right], \quad C_1 > 0 \]
for all functions \( u \in C_L \).

**Proof.** For a function \( u(x,t) \in C_L \)
\[ (Lu, u_t) = \int_Q \left[ \left( a - \frac{1}{2} k_t \right) u_t^2 + \sum_{i=1}^n u_{tx}^2 \right] \, dQ + I, \] (3)
where
\[ I = \frac{1}{2} \int_{P^+_T} ku_t^2 \, dx - \frac{1}{2} \int_{P^-_0} ku_t^2 \, dx + \frac{1}{2} \int_{\Omega_T} cu_t^2 \, dx \geq 0. \]

Due to inequalities
\[ \int_Q u_t^2 \, dQ \leq T^2 \int_Q u_t^2 \, dQ, \]
\[ \int_Q \sum_{i=1}^n u_{tx}^2 \, dQ \leq T^2 \int_Q \sum_{i=1}^n u_{tx}^2 \, dQ, \]
the statement of Lemma 1 follows from (3).

For \( \varepsilon > 0 \) define \( L_\varepsilon u = -\varepsilon u_{\varepsilon t} + Lu \). Assume the functions \( \varphi_k(x) \) are the solutions of a spectral problem

\[-\Delta \varphi = \lambda \varphi, \quad x \in \Omega, \quad \varphi|_\gamma = 0.\]

The functions \( \varphi_k(x) \) form an orthonormal basis in \( L_2(\Omega) \), and their corresponding eigenvalues satisfy the conditions \( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \) and \( \lambda_k \to +\infty \) for \( k \to \infty \).

Construct an approximate solution \( u^{N,\varepsilon}(x,t) \) of the boundary value problem (1)–(2) of the form

\[ u^{N,\varepsilon}(x,t) \equiv v(x,t) \equiv \sum_{k=1}^{N} c_k^{N,\varepsilon}(t)\varphi_k(x), \quad N \geq 1, \quad \varepsilon > 0, \]

where \( c_k^{N,\varepsilon}(t) \) are defined as the solution of the following boundary value problem for the third-order ordinary differential equation system:

\[ (L_\varepsilon u^{N,\varepsilon}, \varphi_l)_{0} = (f, \varphi_l)_{0}, \quad (4) \]

\[ c_l^{N,\varepsilon}|_{t=0} = 0, \quad D_t c_l^{N,\varepsilon}|_{t=0} = 0, \quad D_t^2 c_l^{N,\varepsilon}|_{t=T} = 0, \quad l = \overline{1,N}, \quad (5^1) \]

when \( k(x,0) > 0, \quad k(x,T) \geq 0, \)

\[ c_l^{N,\varepsilon}|_{t=0} = 0, \quad D_t c_l^{N,\varepsilon}|_{t=0} = 0, \quad D_t^2 c_l^{N,\varepsilon}|_{t=T} = 0, \quad l = \overline{1,N}, \quad (5^2) \]

when \( k(x,0) > 0, \quad k(x,T) < 0, \)

\[ c_l^{N,\varepsilon}|_{t=0} = 0, \quad D_t^2 c_l^{N,\varepsilon}|_{t=0} = 0, \quad D_t^2 c_l^{N,\varepsilon}|_{t=T} = 0, \quad l = \overline{1,N}, \quad (5^3) \]

when \( k(x,0) \leq 0, \quad k(x,T) < 0, \)

\[ c_l^{N,\varepsilon}|_{t=0} = 0, \quad D_t^2 c_l^{N,\varepsilon}|_{t=0} = 0, \quad D_t^2 c_l^{N,\varepsilon}|_{t=T} = 0, \quad l = \overline{1,N}, \quad (5^4) \]

when \( k(x,0) \leq 0, \quad k(x,T) \geq 0. \)

From Lemma 1 immediately follows

**Lemma 2** Assume the conditions of Lemma 1 hold, and \( f \in L_2(Q) \) satisfies either of the conditions \( k(x,0) > 0, \quad k(x,T) \geq 0, \) or \( k(x,0) > 0, \quad k(x,T) < 0, \) or \( k(x,0) \leq 0, \quad k(x,T) < 0, \) or \( k(x,0) \leq 0, \quad k(x,T) \geq 0. \) Then for an approximate solution of the boundary value problem (1)–(2) the following bound holds:

\[ \varepsilon \| v_{H\varepsilon}^2 \| + \| v \|^2 + \sum_{i=1}^{n} \| v_{t\varepsilon_i} \|^2 \leq C_2 \| f \|^2, \quad C_2 > 0. \]

**Lemma 3** Assume that \( c(x) \geq 0, \quad a - \frac{1}{2}|k| \geq \delta > 0, \quad f \in W_{L}^{0,1}(Q) \) and either of the four conditions \( k(x,0) > 0, \quad k(x,T) \geq 0, \quad f(x,0) = 0, \) or \( k(x,0) > 0, \quad k(x,T) < 0, \quad f(x,0) = 0, \quad f(x,T) = 0, \) or \( k(x,0) \leq 0, \quad k(x,T) < 0, \quad f(x,0) = 0, \) or \( k(x,0) \leq 0, \quad k(x,T) \geq 0 \) holds. Then for an approximate solution of the boundary value problem (1)–(2) the following bound holds:

\[ \varepsilon \| v_{H\varepsilon}^2 \| + \int_Q \left[ v_{H\varepsilon}^2 + \sum_{i=1}^{n} v_{t\varepsilon_i}^2 \right] dQ \leq C_3 \| f \|^2_{0,1}, \quad C_3 > 0. \]
Proof. From (4) it is not hard to obtain an equality
\[
\varepsilon \|v_{tt}^2\| + \int_Q \left[ \left( a + \frac{1}{2} k_t \right) v_{tt}^2 + \sum_{i=1}^{n} v_{ttx_i}^2 + (a_t + c) v_t v_{tt} \right] dQ + J = (f_t, v_t),
\]
(6)
where
\[
J \equiv -\frac{1}{2} \int_{\Omega_T} k v_{tt}^2 dx + \frac{1}{2} \int k_{tt} v_{tt}^2 dx - \int_{\Omega_T} c v_t v_{tt} dx.
\]

Note that if either of boundary value conditions (5\(^1\)) or (5\(^4\)) holds then \(J \geq 0\). If the boundary value condition (5\(^2\)) or (5\(^3\)) is satisfied then \(-k(x,T) \geq k_0 > 0\), and the second term in the expression above for \(J\) is nonnegative. The well-known trace theorem then yields the inequality
\[
\int_{\Omega_T} v^2(x,T) dx \leq C_4 \|v_t\|_2^2, \quad C_4 > 0.
\]

Using this inequality, together with the bound of Lemma 2, and applying the Cauchy inequality to the equality (6), yields the required a priori bound of Lemma 3.

Lemma 4 Under the conditions of Lemma 3, the approximate solutions of the boundary value problem (1)–(2) satisfy the bound
\[
\|\Delta v_t\| \leq C_5 \|f\|_{0,1}, \quad C_5 > 0.
\]

Proof. Due to the boundary conditions (5\(^p\)), \(p = 1,4\), the following equality holds:
\[
\varepsilon \int_Q \sum_{i=1}^{n} v_{ttx_i} dQ + \int_Q \left[ (\Delta v_t)^2 - (kv_{tt} + av_t + cv)\Delta v_t \right] dQ = -(f, \Delta v),
\]
which, together with the Cauchy inequality and Lemmas 2, 3, yields the bound of Lemma 4.

4. Regular solvability and the error estimate

Note that Lemma 4 implies the a priori bound
\[
\|\Delta v_t\| \leq C_5 T \|f\|_{0,1}.
\]
Then from Lemmas 2, 3, 4 follows the a priori bound
\[
\|u_{N,\varepsilon}\|_2 + \|\Delta u_{N,\varepsilon}\| \leq C_6 \|f\|_{0,1}, \quad C_6 > 0.
\]
(7)
The bound (7) provides the basis for the proof of the following theorem.

Theorem 1 Assume all conditions of Lemma 3 are satisfied. Then the boundary value problem (1)–(2) has a unique solution \(u(x,t)\) in \(C_L\), and the following inequality holds:
\[
\|u\|_2 + \|\Delta u_t\|_2 \leq C_7 \|f\|_{0,1}. \quad C_7 > 0.
\]
(8)

Using inequalities (7)–(8), as well as Lemma 1, in a fashion analogous to that of the works [15,16], the error estimate for nonstationary Galerkin method is obtained.

Theorem 2 Assume all conditions of Lemma 3 are satisfied. Then the following error estimate for nonstationary Galerkin method holds:
\[
\|u - u_{N,\varepsilon}\|_1 + \|u_{t} - u_{t_{N,\varepsilon}}\|_{1,0} \leq C_8 \|f\|_{0,1}(\sqrt{\varepsilon} + \lambda_{N+1}^{-1/2}), \quad C_8 > 0,
\]
where \(u(x,t)\) is the exact solution of the boundary value problem (1)–(2).
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