Towards a regularity theory for integral Menger curvature

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Abstract

We generalize the notion of integral Menger curvature introduced by Gonzalez and Maddocks [14] by decoupling the powers in the integrand. This leads to a new two-parameter family of knot energies intM^{p,q}.

We classify finite-energy curves in terms of Sobolev-Slobodecki˘ı spaces. Moreover, restricting to the range of parameters leading to a sub-critical Euler-Lagrange equation, we prove existence of minimizers within any knot class via a uniform bi-Lipschitz bound. Consequently, intM^{p,q} is a knot energy in the sense of O’Hara.

Restricting to the non-degenerate sub-critical case, a suitable decomposition of the first variation allows to establish a bootstrapping argument that leads to $C^\infty$-smoothness of critical points.

Contents

Introduction 2

1 Classification of finite-energy curves 6

2 Existence of minimizers within knot classes 11

3 Differentiability 15

4 Regularity of stationary points 19

A Product and chain rule 26

B Equivalence of fractional seminorms 27

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Introduction

Imagine a closed curve in the Euclidean space. Each triple of distinct points on the curve uniquely defines its circumcircle that passes through these three points. It degenerates to a line if and only if the points are collinear. The reciprocal of the circumcircle radius can be seen as some kind of approximate curvature. How much information on shape and regularity of the curve can be drawn from the $L^p$-norm of the latter quantity?

Motivated from applications in microbiology, Gonzalez and Maddocks [14] investigated this question for the case $p = \infty$. They were in search of a notion for the thickness of an embedded curve that, in contrast to other approaches, e.g. Litherland et al. [24], does not require initial regularity of the respective curves.

Thickness is influenced by both local and global properties of a curve and is additionally related to the regularity of the curve. In fact, the thickness of an arc-length parametrized curve is finite if and only if it is embedded and has a Lipschitz continuous tangent, i.e., it is $C^{1,1}$, see Gonzalez et al. [15]. Consequently, any curve of finite thickness parametrized by arc-length is bi-Lipschitz continuous with a bi-Lipschitz constant only depending on its thickness.

The latter is particularly interesting in the context of applications. Instead of trying to immediately determine the knot type of a given possibly quite entangled curve, one could first “simplify” it in order to obtain a nicely shaped curve, having large distances between distant strands. Such a deformation process could be defined by the gradient flow of a suitable functional which should prevent the curve from leaving the ambient knot class, preserving the bi-Lipschitz property.

This idea was formalized into the concept of knot energies by O’Hara [29, Def. 1.1]. A functional on a given space of knots is called a knot energy if it is bounded below and self-repulsive, i.e., it blows up on sequences of embedded curves converging to a non-embedded limit curve.

Among other functionals Gonzalez and Maddocks [14] also proposed to investigate the functional\[ M_p(\gamma) := \iint\int_{\mathbb{R}/\mathbb{Z}^3} \frac{|\gamma'(u_1)| |\gamma'(u_2)| |\gamma'(u_3)|}{R(\gamma(u_1), \gamma(u_2), \gamma(u_3))^p} \, du_1 \, du_2 \, du_3, \quad p \in (0, \infty), \]
which is called integral Menger curvature. Here $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ is an absolutely continuous curve and $R(x, y, z)$ denotes the circumcircle of the three points $x, y, z \in \mathbb{R}^n$ given by\[ R(x, y, z) := \frac{|y - z| |y - x| |z - x|}{2((y - x) \wedge (z - x))} = \frac{|y - z|}{2 \sin \alpha (y - x, z - x)}. \]

The functionals $M_p$ have been investigated by Strzelecki, Szumańska and von der Mosel in [35] wherein further references can be found. Their results cover the case $p > 3$ where $M_p$ is known to be a knot energy. Especially they have been able to show that finite energy of an arc-length parametrized curve implies $C^{1,1-3/p}$-regularity and its

\[\text{Named after Karl Menger, 1902 – 1985, US-Austrian mathematician, who used the circumcircle for generalizing geometric concepts to general metric spaces [27]. He worked on many fields including distance geometry, dimension theory, graph theory. Menger was a student of Hahn in Vienna where he received a professorship in 1927. Being member of the Vienna Circle, he was also interested in philosophy and social science. After emigrating to the USA in 1937, he obtained a position at Notre Dame, later at Chicago. See Kass [18] for further reading.}\]
image is $C^1$-homeomorphic to the circle. The regularity statement has been sharpened in [3].

The element $\mathcal{M}_2$ is referred to as total Menger curvature. Interestingly, it plays an important rôle in complex analysis, more precisely in the proof of Vitushkin’s conjecture, a partial solution to Painlevé’s problem which asks to determine removable sets. These are compact sets $K \subset \mathbb{C}$ such that for any open $U \subset \mathbb{C}$ containing $K$ and for any bounded analytic function $U \setminus K \to \mathbb{C}$, the latter can be extended to an analytic function on $U$. Vitushkin conjectured that a compact set $K$ with positive finite one-dimensional Hausdorff measure is removable if and only if it is purely unrectifiable, i.e. it intersects every rectifiable curve in a set of measure zero.

A central result in this context is the the curvature theorem of David and Léger [23] stating that one-dimensional Borel sets in $\mathbb{C}$ with finite total Menger curvature are 1-rectifiable. Mel’nikov and Verdera [25, 26, 39] discovered a connection between $L^2$-boundedness of the Cauchy integral operator on Lipschitz graphs and the Menger curvature. For further details regarding Vitushkin’s conjecture for removable sets we refer to Dudziak’s monography [12] and references therein.

Menger curvature for higher-dimensional objects has been discussed in [19, 21, 22, 4, 20, 34]. Further information on the context of the integral Menger curvature within the field of geometric knot theory and geometric curvature energies can be found in the recent surveys by Strzelecki and von der Mosel [38, 37].

In this article we make a first step towards the regularity theory of stationary points of integral Menger curvature. Regularity theory for minimizers of certain knot energies has been developed in [28, 13, 16, 32, 31, 8, 5]. A summary is given in [6, 7].

Unfortunately the Euler-Lagrange operator of $\mathcal{M}_p$ is not only non-local but also degenerate. In order to produce non-degenerate energies, we embed this family into the two-parameter family of generalized integral Menger curvature

$$\text{int} M^{(p,q)}(\gamma) := - \int \frac{|\gamma'(u_1)| |\gamma'(u_2)| |\gamma'(u_3)|}{R^{(p,q)}(\gamma(u_1), \gamma(u_2), \gamma(u_3))} \, du_1 \, du_2 \, du_3, \quad p, q > 0, \tag{0.2}$$

where

$$R^{(p,q)}(x, y, z) := \frac{|y - z|^p |y - x|^p |z - x|^p}{|y - x| \wedge |z - x|)^q} \sin \gamma(y - x, z - x), \quad x, y, z \in \mathbb{R}^n. \tag{0.3}$$

Note that the function $R^{(p,q)}$ is symmetric in all components. Of course, $\mathcal{M}_p = 2^p \text{int} M^{(p,p)}$.

The elements of this family are knot energies under certain conditions only. More precisely, we will see in Remark 1.2 that they are punishing self-intersections if and only if

$$p \geq \frac{2}{q} + 1. \tag{0.4}$$

On the other hand, these energies can only be finite on closed curves iff

$$p < q + \frac{2}{q}, \tag{0.5}$$

see Remark 1.3. The sub-critical range, i.e. the range of parameters that lead to a sub-critical Euler-Lagrange equation,\(^2\) is given by

$$p \in \left(\frac{2}{q} + 1, q + \frac{2}{q}\right) \quad (q > 1), \tag{0.6}$$

\(^2\)In contrast, the corresponding range is called super-critical by Strzelecki et al. [36] as it lies above the respective critical value for which the energy is scale-invariant.
and its non-degenerate part (leading to a non-degenerate equation) is
\[ p \in \left(\frac{2}{3}, \frac{4}{3}\right), \quad q = 2. \tag{0.7} \]
These areas are visualized in Figure 1.

Figure 1: The range of \( \text{int}M^{(p,q)} \). Above the green line, the integrand is not sufficiently singular to penalize self-intersections, thus \( \text{int}M^{(p,q)} \) is not a knot energy. On the other hand, below the red line, for \( q > 1 \), the integrand is so singular, that the integral is either equal to zero or infinite, so there are no finite-energy \( C^1 \)-knots at all. The hatched area reveals the strange behavior that there are no finite-energy \( C^1 \)-knots while it takes finite values on polygons.

In [3], a characterization of curves with finite \( M_p \) energy was given in terms of function spaces. Using this technique we infer

\section*{Theorem 1 (Classification of finite-energy curves).} Consider the sub-critical case (0.6) and let \( \gamma \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) be an injective curve parametrized by arc-length. Then \( \text{int}M^{(p,q)}(\gamma) < \infty \) if and only if \( \gamma \in W^{(3p-2)/q-1,q} \). Moreover, one then has, for constants \( C, \beta > 0 \) depending on \( p, q \) only,
\[ \|\gamma\|_{W^{(3p-2)/q-1,q}} \leq C \left( \text{int}M^{(p,q)}(\gamma) + \text{int}M^{(p,q)}(\gamma)^\beta \right). \tag{0.8} \]

We will use the last theorem to show

\section*{Theorem 2 (Existence of minimizers within knot classes).} In the sub-critical case (0.6), there is a minimizer of \( \text{int}M^{(p,q)} \) among all injective, regular curves \( \gamma \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) in any knot class.

To shorten notation we use the abbreviation
\[ \Delta_v \cdot := \bullet(u + v) - \bullet(u + w) \tag{0.9} \]
throughout this paper. Furthermore, we sometimes omit the argument of a function if it is precisely the variable \( u \), i.e. \( \gamma = \gamma(u) \) etc.

The first variation of \( \mathcal{M}_p \), \( p \geq 2 \), has been derived by Hermes [17, Thm. 2.33, Rem. 2.35]. Here we use a different approach to prove

**Theorem 3 (Differentiability).** In the sub-critical case \((0.6)\) the functional \( \operatorname{intM}^{(p,q)} \) is \( C^1 \) on the subspace of all regular embedded \( W^{(3p-2)/q-1,q} \)-curves. For any arc-length parameterized embedded \( \gamma \in W^{(3p-2)/q-1,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) and \( h \in W^{(3p-2)/q-1,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \), the first variation of \( \operatorname{intM}^{(p,q)} \) at \( \gamma \) in direction \( h \) amounts to

\[
\delta \operatorname{intM}^{(p,q)}(\gamma, h) = \iint_{(\mathbb{R}/\mathbb{Z})^2} 2q \left| \frac{\Delta_{\alpha,0} \gamma \wedge \Delta_{\alpha,0} h}{\left| \Delta_{\alpha,0} \gamma \right| \left| \Delta_{\alpha,0} h \right|} \right|^{q-2} \cdot \left( \frac{\Delta_{\alpha,0} \gamma \wedge \Delta_{\alpha,0} \gamma, \Delta_{\alpha,0} \gamma \wedge \Delta_{\alpha,0} h}{\left| \Delta_{\alpha,0} \gamma \right|^2} \cdot \left( \Delta_{\alpha,0} \gamma, \Delta_{\alpha,0} h \right) \right) - 3p \left| \frac{\Delta_{\alpha,0} \gamma \wedge \Delta_{\alpha,0} \gamma}{\left| \Delta_{\alpha,0} \gamma \right|^2} \cdot \left( \Delta_{\alpha,0} \gamma, \Delta_{\alpha,0} h \right) \right| \cdot \left( \Delta_{\alpha,0} \gamma, \Delta_{\alpha,0} h \right) \cdot \left( \gamma', h' \right) \right) dw \, du.
\]

Using this formula, we will see that stationary points of the energies \( \operatorname{intM}^{(p,q)} \) restricted to fixed length satisfy some kind of non-local uniformly elliptic pseudo-differential equation. If furthermore \( p \in \left( \frac{5}{3}, \frac{3}{2} \right) \), the non-linearity turns out to be sub-critical and we can finally use the Euler-Lagrange equation to prove the following main result of this article:

**Theorem 4 (Regularity of stationary points).** For \( p \in \left( \frac{5}{3}, \frac{3}{2} \right) \), let \( \gamma \in W^{3p/2-2-2,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) be a stationary point of \( \operatorname{intM}^{(p,q)} \) with respect to fixed length, injective and parametrized by arc-length. Then \( \gamma \in C^\infty \).

In a sense this concludes our study of the non-degenerate, subcritical cases of the most prominent knot energies for curves. Regularity theory for the non-degenerate subcritical case has already been performed for O’Hara’s energies [8] and for the generalized tangent-point energies [5]. The treatment of the critical case however turns out to be far more involved and has yet only be done for O’Hara’s knot energies [9].

We briefly introduce Sobolev-Slobodeckiĭ spaces in the form we will use them in this text. Let \( f \in W^{1,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \). For \( s \in (0, 1) \) and \( q \in [1, \infty) \) we define the seminorm

\[
[f]_{W^{1+s,q}} := \left( \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{|f'(u + w) - f'(u)|^q}{|w|^{1+qs}} \, dw \, du \right)^{1/q}.
\]

On \( W^{1,q} \) this seminorm is equivalent to

\[
\|f\|_{W^{1+1,q}} := \left( \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/4}^{1/4} \frac{|f(u + w) - 2f(u) + f(u - w)|^q}{|w|^{1+qs(1+s)}} \, dw \, du \right)^{1/q},
\]

see Appendix B.
Now let $W^k(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, $k \in \mathbb{N}$, denote the usual Sobolev space (recall $W^0 := L^2$) and

$$W^{k+\frac{s}{\varrho}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) := \left\{ f \in W^k(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \mid \|f\|_{W^{k+\frac{s}{\varrho}}} < \infty \right\}$$

which we equip, depending on the situation, either with the norm $\|f\|_{W^k} + \|f^{(k-1)}\|_{W^{\frac{s}{\varrho}}}$ or with $\|f\|_{W^{k+\frac{s}{\varrho}}} + \|f^{(k-1)}\|_{W^{\frac{s}{\varrho}}}$ respectively.

Without further notice we will frequently use the embedding

$$W^{k+\frac{s}{\varrho}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \hookrightarrow C^{k-s+\frac{1}{\varrho}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n), \quad \varrho \in (1, \infty), \quad s \in (\varrho^{-1} - 1, 1). \quad (0.13)$$

We will denote by $C_{ia}$ resp. $W_{ia}$ injective (embedded) curves parametrized by arc-length. As usual, a curve is said to be regular if there is some $c > 0$ such that $|\gamma'| \geq c$ a.e. Constants may change from line to line.

1 Classification of finite-energy curves

Before we begin the discussion of the first variation, let us rewrite the integral Menger curvature using the symmetry and a suitable covering of the domain of integration $(\mathbb{R}/\mathbb{Z})^3$ by domains, on which it is easier to estimate the terms that will appear. The general idea here is quite similar to [3] and Hermes [17], but we will show that it is actually enough to integrate over a certain subdomain of $(\mathbb{R}/\mathbb{Z})^3$.

To this end we define the range of integration

$$D := \left\{ (v, w) \in (-\frac{1}{2}, 0) \times (0, \frac{1}{2}) \mid w \leq 1 + 2v, v \geq -1 + 2w \right\}, \quad (1.1)$$

which is depicted in Figure 2.

![Figure 2: The range of integration D](image)

**Lemma 1.1 (Domain decomposition).** Let $f \in L^1\left((\mathbb{R}/\mathbb{Z})^3\right)$ be symmetric in all components, i.e. $f = f \circ \sigma$ for all permutations $\sigma \in \Xi_3$. Then

$$\int_{(\mathbb{R}/\mathbb{Z})^3} f(u_1, u_2, u_3) \, du_1 \, du_2 \, du_3 = 6 \int_{(\mathbb{R}/\mathbb{Z})^2 \times D} f(u, u + v, u + w) \, dv \, dw \, du.$$
Proof. Let $P_\sigma \in \mathbb{R}^{3\times 3}$ denote the permutation matrix corresponding to $\sigma \in \Xi_3$. We first show that the images $\{ P_\sigma (\mathbb{R}/\mathbb{Z} \times D) \mid \sigma \in \Xi_3 \}$ cover $(\mathbb{R}/\mathbb{Z})^3$.

Consider $(u_1, u_2, u_3) \in (\mathbb{R}/\mathbb{Z})^3$. Then after a suitable permutation we can assume that

$$d_{\mathbb{R}/\mathbb{Z}}(u_1, u_3) = \max (d_{\mathbb{R}/\mathbb{Z}}(u_1, u_2), d_{\mathbb{R}/\mathbb{Z}}(u_2, u_3), d_{\mathbb{R}/\mathbb{Z}}(u_1, u_3))$$

where $d_{\mathbb{R}/\mathbb{Z}}(x, y) = \min \{ |x - y - k| : k \in \mathbb{Z} \} \in [0, 1/2]$ denotes the distance in $\mathbb{R}/\mathbb{Z}$. Hence, interchanging $u_1$ and $u_3$ if necessary, there are $(v, w) \in \left[ -1/2, 1/2 \right] \times \left[ -1/2, 1/2 \right]$ with

$$u_1 = u_2 + v, \quad u_3 = u_2 + w,$$

and

$$\max (-v, w) = \max (d_{\mathbb{R}/\mathbb{Z}}(u_1, u_2), d_{\mathbb{R}/\mathbb{Z}}(u_2, u_3)) \leq d_{\mathbb{R}/\mathbb{Z}}(u_1, u_3) = \min (w - v, 1 - (w - v)) \leq 1 - w + v,$$

so $(v, w) \in \overline{D}$. Since furthermore

$$\# \Xi_3 \cdot |\mathbb{R}/\mathbb{Z} \times D| = 6 \cdot \frac{1}{6} = \left| (\mathbb{R}/\mathbb{Z})^3 \right|$$

the sets $\{ P_\sigma (\mathbb{R}/\mathbb{Z} \times D) \mid \sigma \in \Xi_3 \}$ form up to sets of measure zero a disjoint partition of $(\mathbb{R}/\mathbb{Z})^3$.

Following Lemma 1.1 we derive using (0.9)

$$\text{int} M^{(p,q)}(\gamma) = 6 \iiint_{\mathbb{R}/\mathbb{Z} \times D} \frac{|A_{0,0} \gamma - \Delta_{0,0} \gamma| q}{|A_{0,0} \gamma||A_{0,0} \gamma|} \left| \gamma'(u) \right| |\gamma'(u + v)| |\gamma'(u + w)| \, dw \, dv \, du.$$  

(1.2)

Proof of Theorem 1. Recall that any embedded $W^{1+s}_{\text{loc}}$-curve, $s > \frac{1}{4}$, is bi-Lipschitz continuous [2, Lemma 2.1]. We obtain, using $|a \land b| = |a \land (a \pm b)| \leq |a| |a \pm b|$ for $a, b \in \mathbb{R}$,

$$\text{int} M^{(p,q)}(\gamma) \leq C \iint_{\mathbb{R}/\mathbb{Z} \times D} \left[ \int_{0}^{1} \left| \gamma'(u + \theta v) \right| \, dv \right]^{q} \, du \, \left[ \int_{0}^{1} \left| \gamma'(u + \theta w) \right| \, dw \right]^{p}$$

$$\leq C \iint_{\mathbb{R}/\mathbb{Z} \times D} \left[ \int_{0}^{1} \left| \gamma'(u + \theta v) - \gamma'(u + \theta w) \right| \, dv \right]^{q} \, du \, \left[ \int_{0}^{1} \left| \gamma'(u + \theta w) \right| \, dw \right]^{p}$$

$$\leq C \int_{0}^{1} \int_{\mathbb{R}/\mathbb{Z} \times D} \frac{\left| \gamma'(u + \theta v - w) - \gamma'(u + \theta w) \right|^{q}}{|v - w|^{p}} \, dv \, du \, \left[ \int_{0}^{1} \left| \gamma'(u + \theta w) \right| \, dw \right]^{p}$$

$$\leq C \int_{0}^{1} \int_{\mathbb{R}/\mathbb{Z} \times D} \frac{\left| \gamma'(u + \theta v - w) - \gamma'(u) \right|^{q}}{|v - w|^{p}} \, dv \, du \, \left[ \int_{0}^{1} \left| \gamma'(u) \right| \, dw \right]^{p}$$

where $C$ depends on $p, q$ and $\gamma$. Substituting $\Phi : (v, w) \mapsto (t, \tilde{w}) := \left( \frac{v}{v - w}, \theta(v - w) \right)$, $|\det \Phi(v, w)| = \frac{\theta}{|v - w|}$, $\Phi(D) \subset [0, 1] \times [-1, 0]$, we arrive at

$$\text{int} M^{(p,q)}(\gamma) \leq C \int_{0}^{1} \theta^{1-1} \iint_{\mathbb{R}/\mathbb{Z} \times D} \left[ \int_{0}^{1} \left| \frac{\gamma'(u + \tilde{w}) - \gamma'(u)}{|v - w|^{p}} \right| \, dv \right]^{q} \, du \, \left[ \int_{0}^{1} \left| \gamma'(u) \right| \, dw \right]^{p} \, d\theta$$

$$\leq C \int_{0}^{1} \theta^{3-2q} \, d\theta \int_{0}^{1} \frac{dr}{r} \int_{(1 - r)^{1-q}}^{1} \left( \int_{1}^{1} \left| \gamma'(u + \tilde{w}) - \gamma'(u) \right| \frac{du}{|\tilde{w}|^{p}} \right) \, dw \, \left[ \int_{0}^{1} \left| \gamma'(u) \right| \, dw \right]^{p}$$

$$\leq C \left( \frac{\gamma^{p}}{q^{p} + (1-q)q} + \frac{\gamma^{q}}{w^{q} + (1-q)q} \right) \leq C \frac{\gamma^{p}}{w^{q} + (1-q)q}.$$  

(1.3)
For the other implication, we first derive for given vectors \( a, b \in \mathbb{R}^n \), \(|a| = |b| = 1\), \((a, b) \geq 0\),

\[
|a \land b|^2 = |a|^2 |b|^2 - (a, b)^2 \geq 1 - (a, b) = \frac{1}{2} |a - b|^2.
\]  

(1.4)

By uniform continuity of \( \gamma' \) we may choose \( \delta = \delta(\gamma) \in (0, \frac{1}{2}) \) such that

\[
|\gamma'(u + v) - \gamma'(u + w)| \leq \frac{1}{10} \quad \text{for all } u \in \mathbb{R}/\mathbb{Z}, \quad v, w \in [-\delta, \delta].
\]  

(1.5)

In fact, we may choose \( \delta \) to be maximal, i.e. we assume that there are \( \bar{u} \in \mathbb{R}/\mathbb{Z}, \bar{v}, \bar{w} \in [-\delta, \delta] \) with

\[
|\gamma'(\bar{u} + \bar{v}) - \gamma'(\bar{u} + \bar{w})| = \frac{1}{10}.
\]  

(1.6)

We fix \( u_0 \in \mathbb{R}/\mathbb{Z} \). As \( \gamma \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) we may apply a suitable translation and rotation of the ambient space \( \mathbb{R}^n \) such that \( \gamma(u_0) = 0 \) and there is a function \( f \in C^1(\mathbb{R}, \mathbb{R}^{n-1}) \) with \( ||f||_{L^p} \leq 1 \) and \( f(0) = 0 \) such that \( \tilde{\gamma}(u) := (u, f(u)) \) satisfies \( \tilde{\gamma}(B_2(0)) \subset \gamma(\mathbb{R}/\mathbb{Z}) \). Then

\[
\frac{1}{2} |\Delta_{\delta, 0} \tilde{\gamma}| \leq |u| \leq |\Delta_{\delta, 0} \tilde{\gamma}|.
\]  

(1.7)

Arc-length parametrization of \( \gamma \) gives

\[
\text{intM}^{(p,q)}(\tilde{\gamma})
\]  

(1.8)
we arrive at

\[ \int_{|x|<\delta} \frac{|\nabla (u+v) - 2\gamma(u) + \tilde{\gamma}(u-v)|}{|u|^3} \, dv \, du - C \int_{|x|<\delta} \int_{|x|<\delta} \left| \frac{e^{-\alpha x}}{|x|^{3p-3q-1}} - \frac{e^{-\alpha y}}{|y|^{3p-3q-1}} \right|^q \, dv \, du. \]

By

\[ \left| \frac{u}{\Delta_{-0}\tilde{\gamma}} - \frac{v}{\Delta_{-0}\tilde{\gamma}} \right| \leq \left| (v, \triangle_{-0}\tilde{f}) + (-v, \triangle_{-0}\tilde{f}) \right| \leq \frac{\Delta_{-0}\tilde{\gamma}}{|\Delta_{-0}\tilde{\gamma}|} + \frac{\Delta_{-0}\tilde{\gamma}}{|\Delta_{-0}\tilde{\gamma}|}, \]

we may use (1.8) to absorb the last term which finally leads to

\[ \text{intM}(p,q)(\gamma) \geq c \int_{|x|<\delta} \int_{|x|<\delta} \left| \frac{\nabla (u+v) - 2\gamma(u) + \tilde{\gamma}(u-v)}{|u|^3} \right|^q \, dv \, du. \]

Since reparametrization to arc-length preserves regularity, we arrive at

\[ \text{intM}(p,q)(\gamma) \geq c \int_{|x|<\delta} \int_{|x|<\delta} \left| \frac{\nabla (u+v) - 2\gamma(u) + \gamma(u-v)}{|u|^3} \right|^q \, dv \, du. \]  \hspace{1cm} (1.9)

As \( u_0 \) was chosen arbitrarily, we obtain

\[ \|\gamma\|_{W^{3p-3p+2,\infty}} \leq C \left( \text{intM}(p,q)(\gamma) + \|\gamma\|_{L^\infty} \right) \delta^{-3p+1}\|w\|^\alpha \]  \hspace{1cm} (1.10)

uniformly on \( R/\mathbb{Z} \). Since the exponent \(-3p + 2q + 3\) is negative, we have to show that \( \delta \) is uniformly bounded away from zero in order to finish the proof. To this end we will establish the Morrey-type estimate

\[ \|\gamma' - w\|_{L^\infty} \leq C \text{intM}(p,q)(\gamma)^{1/q} w^\alpha \]  \hspace{1cm} (1.11)

where \( \alpha = 3(p-1)/q \). As \( \delta \) was chosen to be maximal with respect to (1.6), we arrive at

\[ \|\gamma' - w\|_{L^\infty} \leq C \text{intM}(p,q)(\gamma)^{1/q} \delta^{\alpha} \]

which, applied to (1.10), gives (0.8).

It remains to prove (1.11) which follows by standard arguments due to Campanato [11]. Let \( \gamma_{B_r(x)} \) denote the integral mean of \( \gamma \) over \( B_r(x) \). We calculate for \( x \in R/\mathbb{Z} \) and \( r \in (0,\delta) \)

\[ \frac{1}{2r} \int_{B_r(x)} |\gamma'(v) - \gamma_{B_r(x)}| \, dv \leq \frac{1}{4r^2} \int_{B_r(x)} \int_{B_r(x)} |\gamma'(v) - \gamma'(u)| \, du \, dv \]

\[ \leq \left( \frac{1}{4r^2} \int_{B_r(x)} \int_{B_r(x)} |\gamma'(v) - \gamma'(u)|^q \, du \, dv \right)^{1/q} \]

\[ \leq C \rho^\alpha \left( \int_{B_r(x)} \int_{B_r(x)} |\gamma'(v) - \gamma'(u)|^q \, du \, dv \right)^{1/q} \]

\[ = C \rho^\alpha \text{intM}(p,q)(\gamma)^{1/q}. \]

As (1.11) only involves the domain of \( \gamma \) up to a measure zero set, we may restrict to Lebesgue points. We choose two Lebesgue points \( u, v \in \mathbb{R}/\mathbb{Z} \) of \( \gamma' \) with \( r := |u-v| \in (0,\frac{\delta}{2}) \). Then

\[ |\gamma'(u) - \gamma'(v)| \leq \sum_{k=0}^{\infty} |\gamma_{B_{2^{-k}r}}(u) - \gamma_{B_{2^{-k}r}}(v)| + |\gamma_{B_{-2^{-k}r}}(u) - \gamma_{B_{-2^{-k}r}}(v)| + \sum_{k=0}^{\infty} |\gamma_{B_{2^{-k}r}}(u) - \gamma_{B_{2^{-k}r}}(v)|. \]
Since
\[ |\gamma_{B_0(u)} - \gamma_{B_0(v)}| \leq \frac{\int_{B_0(u)} |\gamma'(x)| \, dx + \int_{B_0(v)} |\gamma'(x)| \, dx}{|B_2(u) \cap B_2(v)|} \leq C|u - v|^q \int_{M}^{(p,q)}(\gamma)^{1/q} \]
as \( r = |u - v| \) and, for all \( y \in \mathbb{R}/\mathbb{Z}, R \in (0, \frac{\delta}{2}), \)
\[ |\gamma_{B_2(u)} - \gamma_{B_2(v)}| \leq \frac{\int_{B_2(u)} |\gamma'(x)| \, dx + \int_{B_2(v)} |\gamma'(x)| \, dx}{2R} \leq CR^q \int_{M}^{(p,q)}(\gamma)^{1/q}, \]
we deduce \( |\gamma'(u) - \gamma'(v)| \leq C \left( \sum_{k=0}^{\infty} 2^{-ka} + 1 + \sum_{k=0}^{\infty} 2^{-ka} \right) |u - v|^q \int_{M}^{(p,q)}(\gamma)^{1/q}. \) Thus \( |\gamma'(u) - \gamma'(v)| \leq C|u - v|^q \int_{M}^{(p,q)}(\gamma)^{1/q} \) for all Lebesgue points of \( \gamma' \) with \( |u - v| < \frac{\delta}{2}. \)
The case \( |u - v| \geq \frac{\delta}{2} \) follows by the triangle inequality.

Let us conclude this section by briefly commenting on the other ranges in the \((p,q)\)-domain, see Figure 1.

**Remark 1.2 (Non-repulsive energies for \( p < \frac{2}{3}q + 1 \)).** A bi-Lipschitz estimate is not guaranteed for injective curves if \( p < \frac{2}{3}q + 1 \). We briefly give the following example. Consider the curves \( u \mapsto (u, 0, 0) \) and \( u \mapsto (0, u, \delta) \) for \( u \in [-1, 1], \delta \in [0, 1]. \) The interaction of these strands leads to the \( \int_{M}^{(p,q)} \) value
\[ C \int_{-1}^{1} \int_{-1}^{1} \frac{(\delta^2 + u^2)^{q/2}}{|v - w|^{p/2} \left( \delta^2 + u^2 + v^2 + w^2 \right)^{p/2} \left( \delta^2 + u^2 + v^2 + w^2 \right)^{p/2} dw \, du. \]
Introducing polar coordinates \( u = r \cos \theta, v = r \sin \theta \cos \varphi, w = r \sin \theta \sin \varphi, \) the former quantity is bounded by
\[ C \int_{0}^{\sqrt{3}} \int_{0}^{\sqrt{3}} \frac{(\delta^2 + r^2 \cos^2 \theta)^{(q-p)/2}}{r^p q \sin^{q-1} \theta \left( \delta^2 + r^2 \right)^{p/2} \left( \delta^2 + r^2 \right)^{p/2}} \, d\theta \, dr \int_{0}^{\frac{2\pi}{\delta}} \frac{d\varphi}{|\cos \varphi - \sin \varphi|^{q-p}} \]
\[ \leq C \int_{0}^{\sqrt{3}} \int_{0}^{\sqrt{3}} \frac{(\delta^2 + r^2 \cos^2 \theta)^{(q-p)/2}}{r^p q \sin^{q-1} \theta \left( \delta^2 + r^2 \right)^{p/2} \left( \delta^2 + r^2 \right)^{p/2}} \, d\theta \, dr \int_{1}^{\frac{2\pi}{\delta}} \frac{d\varphi}{r^p q \sin^{q-1} \theta \left( \delta^2 + r^2 \right)^{p/2} \left( \delta^2 + r^2 \right)^{p/2}} \, d\varphi \]
\[ \leq C \int_{0}^{\sqrt{3}} \left( \delta + r \right)^{-3p+2q+2} + r^{-p+q+1} (\delta + r)^{-p} \int_{0}^{\sqrt{3}} (\delta^2 + \sigma^2)^{(q-p)/2} \, d\sigma \]
\[ \leq C \left( 1 - \delta^{-3p+2q+3} \right) \leq C. \]

Using Theorem 1 and the monotonicity of \( \int_{M}^{(p,q)} \) for fixed \( q, \) it is easy to produce a family of knots uniformly converging to a non-embedded curve without an energy blow-up as \( \delta \searrow 0, \) so these energies are not self-repulsive.
Remark 1.3 (Singular energies for \( p \geq q + \frac{2}{3}, q > 1 \)). For \( p \geq q + \frac{2}{3}, q > 1 \), and an absolutely continuous \( \gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R} \) we have \( \text{int}M^{p,q}(\gamma) \equiv \infty \) for all \( C^1 \)-curves \( \gamma \). To see this, note that we assumed \( p < \frac{2}{3}q + 1 \) in Theorem 1 mainly because neither (0.11) nor (0.12) is not defined for \( s \geq 1 \). For general \( p \geq \frac{2}{3}q + 1 \) we nevertheless still have

\[
\int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{|\gamma'(u+w) - \gamma'(u)|^q}{|w|^{3p-2q-1}} \, dw \, du \leq C \left( \text{int}M^{p,q}(\gamma) + \text{int}M^{p,q}(\gamma)^0 \right).
\]

Applying Brezis [10, Prop. 2], the function \( \gamma' \) is constant, hence \( \gamma \) lies on a straight line. Therefore, \( \gamma \) cannot be a closed \( C^1 \)-curve.

Remark 1.4 (Strange energies for \( p \in \{ q + \frac{2}{3}, \frac{2}{3}q + 1 \} \)). On \( p \in \{ q + \frac{2}{3}, \frac{2}{3}q + 1 \}, p, q > 0 \), see the hatched area in Figure 1, we find the strange behavior that there are no closed finite-energy \( C^1 \)-curves while self-intersections, and in particular corners, are not penalized. So piecewise linear curves (polygons) have finite energy.

The latter can be seen by adapting the calculation in Remark 1.2. For the former we recall that a closed arc-length parametrized \( C^2 \)-curve must have positive curvature \( |\gamma''| \) at some point \( u_0 \) and by continuity there are \( c, \delta > 0 \) with \( |\gamma''| \geq c > 0 \) on \([u_0-\delta, u_0+\delta]\). As \( \gamma'' \perp \gamma' \) we obtain \( |\gamma'' \wedge \gamma'| = |\gamma''| \geq c \). So \( \text{int}M^{p,q}(\gamma) \) is bounded by

\[
\int_{u_0-\delta}^{u_0+\delta} \int_{\frac{1}{3}|u|}^{\frac{2}{3}|u|} \int_{|v|}^{2|v|} \left| \frac{\Delta_x Y}{v} \wedge \frac{\Delta_{0,x} Y}{w} \right|^q \, dv \, du
\]

\[
= \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{w^2}{2} (1 - \vartheta)^2 \gamma'''(u + \vartheta_1 v) \, d\vartheta_1 \wedge (\gamma'(u) + \frac{w}{2} \gamma''(u))
\]

\[
+ \frac{2w}{2} \int_{[0,1]} (1 - \vartheta_2)^2 (1 - \vartheta_2)^2 \gamma'''(u + \vartheta_2 v) \wedge \gamma'''(u + \vartheta_2 v) \, d\vartheta_2 \, d\vartheta_1 \, d\vartheta_2
\]

\[
\geq \delta \left[ c - C \delta \| \gamma''' \|_{L^q} \left( \| \gamma'' \|_{L^q} + \| \gamma'' \|_{L^q} + 1 \right) \right] \int_{u_0-\delta}^{u_0+\delta} |v|^{-3p+3q+1} \, dv.
\]

Lessening \( \delta > 0 \), the square bracket is positive. This gives \( \text{int}M^{p,q}(\gamma) = \infty \).

2 Existence of minimizers within knot classes

The arguments here are quite similar as for the tangent-point energies [5], however, we provide full proofs for the readers' convenience.

Using Theorem 1 together with the Arzelà-Ascoli theorem, we see that sets of curves in \( C^1_{ia}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) with a uniform bound on the energy are sequentially compact in \( C^1 \). To this end we need the following result.

Proposition 2.1 (Uniform bi-Lipschitz estimate). For every \( M < \infty \) and (0.6) there is a constant \( C(M, p, q) > 0 \) such that every curve \( \gamma \in C^1_{ia}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) parametrized by arc-length with

\[
\text{int}M^{p,q}(\gamma) \leq M
\]
Figure 3: A pair of curves \((\gamma_1, \gamma_2) \in M_\mu\) defined in Lemma 2.2. Note that the arcs \(\gamma_1, \gamma_2\) cannot intersect each other.

**satisfies the bi-Lipschitz estimate**

\[
|u - v| \leq C(M, p, q) |\gamma(u) - \gamma(v)| \quad \text{for all } u, v \in \mathbb{R}/\mathbb{Z}. \tag{2.2}
\]

The proof is based on the following lemma. To be able to state it, we set for two arc-length parametrized curves \(\gamma_i : I_i \to \mathbb{R}, i = 1, 2, I_1, I_2\) open intervals,

\[
\text{int}M^{(p,q)}(\gamma_1, \gamma_2) := \text{int}M^{(p,q)}(\gamma_1) + \text{int}M^{(p,q)}(\gamma_2) + \\
\quad + \iiint_{I_1 \times I_2} |\gamma_1'(u_1)| |\gamma_1'(u_2)| |\gamma_2'(u_3)| \, du_1 \, du_2 \, du_3 \\
\quad + \iiint_{I_1 \times I_2} |\gamma_1'(u_1)| |\gamma_2'(u_2)| |\gamma_2'(u_3)| \, du_1 \, du_2 \, du_3.
\]

**Lemma 2.2.** Let \(\alpha \in (0, 1)\). For \(\mu > 0\) we let \(M_\mu\) denote the set of all pairs \((\gamma_1, \gamma_2)\) of curves \(\gamma_i \in C^1([-1, 1], \mathbb{R}^n)\) satisfying

(i) \(|\gamma_1(0) - \gamma_2(0)| = 1\),

(ii) \(\gamma_1'(0) \perp (\gamma_1(0) - \gamma_2(0)) \perp \gamma_2'(0)\),

(iii) \(\|\gamma_i'\|_{C^0} \leq \mu\), \(i = 1, 2\).

Then there is a \(c = c(\alpha, \mu) > 0\) such that

\[
\text{int}M^{(p,q)}(\gamma_1, \gamma_2) \geq c \quad \text{for all } (\gamma_1, \gamma_2) \in M_\mu.
\]

**Proof.** It is easy to see that \(\text{int}M^{(p,q)}(\gamma_1, \gamma_2)\) is zero if and only if both \(\gamma_1\) and \(\gamma_2\) are part of one single straight line. We will show that \(\text{int}M^{(p,q)}(\gamma_1, \gamma_2)\) attains its minimum on \(M_\mu\). As \(M_\mu\) does not contain straight lines by (i), (ii), this minimum is strictly positive which thus proves the lemma.

Let \((\gamma_1^{(n)}, \gamma_2^{(n)})\) be a minimizing sequence in \(M_\mu\), i.e. we have

\[
\lim_{n \to \infty} \text{int}M^{(p,q)}(\gamma_1^{(n)}, \gamma_2^{(n)}) = \inf_{M_\mu} \text{int}M^{(p,q)}(\gamma_1, \gamma_2).
\]
Subtracting $\gamma_1(0)$ from both curves, i.e. setting
\[ \tilde{\gamma}_i^{(n)}(\tau) := \gamma_i^{(n)}(\tau) - \gamma_1(0), \quad i = 1, 2, \]
and using Arzelà-Ascoli we can pass to a subsequence such that
\[ \tilde{\gamma}_i^{(n)} \to \tilde{\gamma}_i \quad \text{in } C^1. \]
Furthermore, $(\tilde{\gamma}_1, \tilde{\gamma}_2) \in M_\mu$ since $M_\mu$ is closed under convergence in $C^1$. Since, by Fatou’s lemma, the functional $\text{int}M^{(p,q)}$ is lower semi-continuous with respect to $C^1$ convergence, we obtain
\[ \text{int}M^{(p,q)}(\tilde{\gamma}_1, \tilde{\gamma}_2) \leq \liminf_{n \to \infty} \text{int}M^{(p,q)}(\tilde{\gamma}_1^{(n)}, \tilde{\gamma}_2^{(n)}) = \lim_{n \to \infty} \text{int}M^{(p,q)}(\gamma_1^{(n)}, \gamma_2^{(n)}) = \inf_{\mu_0} \text{int}M^{(p,q)}(\cdot, \cdot). \]
\[ \square \]

Let us use this lemma to give the

**Proof of Proposition 2.1.** Applying Theorem 1 to (2.1) we obtain $\|y\|_{C^{0,\alpha}} \leq C(M)$ for $\alpha = 3^\frac{p-1}{q} - 2 \in (0, 1 - \frac{1}{q})$. As an immediate consequence there is a $\delta = \delta(\alpha, M) > 0$ such that
\[ |u - v| \leq 2|\gamma(u) - \gamma(v)| \quad (2.3) \]
for all $u, v \in \mathbb{R}/\mathbb{Z}$ with $|u - v| \leq \delta$. Let now
\[ S := \inf \left\{ |\gamma(u) - \gamma(v)| \left| u, v \in \mathbb{R}/\mathbb{Z}, |u - v| \geq \delta \right. \right\} \leq \frac{1}{2}. \]
We will complete the proof by estimating $S$ from below. Using the compactness of $\{u, v \in \mathbb{R}/\mathbb{Z}, |u - v| \geq \delta\}$, there are $s, t \in \mathbb{R}/\mathbb{Z}$ with $|s - t| \geq \delta$ and $|\gamma(s) - \gamma(t)| = S$. If now $|s - t| = \delta$ we obtain
\[ 2S = 2|\gamma(s) - \gamma(t)| \geq \delta \quad (2.3) \]
and hence
\[ |u - v| \leq \frac{S}{\delta} \leq \frac{|\gamma(u) - \gamma(v)|}{\delta(\alpha, M)} \]
for all $u, v \in \mathbb{R}/\mathbb{Z}$ with $|u - v| \geq \delta$. This proves the proposition in this case. If on the other hand $|s - t| > \delta$ then we infer using the minimality of $|\gamma(s) - \gamma(t)|$
\[ \gamma'(s) \perp (\gamma(s) - \gamma(t)) \perp \gamma'(t). \]
We define for $\tau \in [-1, 1]$ \[ \gamma_1(\tau) := \frac{1}{S} \gamma(s + S \tau) \quad \text{and} \quad \gamma_2(\tau) := \frac{1}{S} \gamma(t + S \tau). \]
Since $\|\gamma_1\|_{C^{0,\alpha}} \leq \|\gamma_2\|_{C^{0,\alpha}} \leq C(M)$ we may apply Lemma 2.2 which yields
\[ \text{int}M^{(p,q)}(\gamma_1, \gamma_2) \geq c(\alpha, M) > 0. \]
Together with $\text{int}M^{(p,q)}(\gamma_1, \gamma_2) \leq S^{3p-2q-3}\text{int}M^{(p,q)}(\gamma)$ this leads to
\[ S \geq \left( \frac{c(\alpha, M)}{\text{int}M^{(p,q)}(\gamma)} \right)^{\frac{1}{3p-2q-3}} \geq \left( \frac{c(\alpha, M)}{M} \right)^{\frac{1}{3p-2q-3}}. \]
Hence, $|u - v| \leq \frac{1}{2} \leq \frac{|\gamma(s) - \gamma(t)|}{2S} \leq C(M, p, q)|\gamma(u) - \gamma(v)|$ for all $u, v \in \mathbb{R}/\mathbb{Z}$ with $|u - v| \geq \delta$. \[ \square \]
We are now in the position to prove the compactness result which is crucial both to the existence of minimizers in any knot class and to the self-avoiding behavior of the energies.

**Proposition 2.3 (Sequential compactness).** For each $M < \infty$ the set

$$A_M := \{ \gamma \in C^1_{\text{ia}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \mid \text{int}M^{(p,q)}(\gamma) \leq M \}$$

is sequentially compact in $C^1$ up to translations.

**Proof.** By Theorem 1 there are $C(M) < \infty$ and $\alpha = \alpha(p,q) > 0$ such that $\|\gamma\|_{C^\alpha} \leq C(M)$ for all $\gamma \in A_M$ and hence

$$\|\tilde{\gamma}\|_{C^\alpha} \leq C(M) + 1$$

where $\tilde{\gamma}(u) := \gamma(u) - \gamma(0)$. By Proposition 2.1, the bi-Lipschitz estimate (2.2) holds.

Let now $\gamma_n \in A_M$. Then

$$\|\tilde{\gamma}_n\|_{C^\alpha} \leq C(M) + 1$$

and hence after passing to suitable subsequence we have $\tilde{\gamma}_n \to \gamma_0$ in $C^1$. Since $\gamma_n$ was parametrized by arc-length, $\gamma_0$ is still parametrized by arc-length and the bi-Lipschitz estimate carries over to $\gamma_0$. So, especially, $\gamma_0 \in C^1_{\text{ia}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. From lower semi-continuity with respect to $C^1$ convergence we infer

$$\text{int}M^{(p,q)}(\gamma_0) \leq \liminf_{n \to \infty} \text{int}M^{(p,q)}(\gamma_n) \leq M.$$ 

So $\gamma_0 \in A_M$. \qedsymbol

We may now pass to the

**Proof of Theorem 2.** Let $(\gamma_k)_{k \in \mathbb{N}} \in C^1_{\text{ia}}$ be a minimal sequence for $\text{int}M^{(p,q)}$ in a given knot class $K$, i.e. let

$$\lim_{k \to \infty} \text{int}M^{(p,q)}(\gamma_k) = \inf_{C^1_{\text{ia}} \cap K} \text{int}M^{(p,q)}.$$ 

After passing to a subsequence and suitable translations, we hence get by Proposition 2.3 a $\gamma_0 \in C^1_{\text{ia}}$ with $\gamma_k \to \gamma_0$ in $C^1$. As the intersection of every knot class with $C^1$ is an open set in $C^1$ [1, Cor. 1.5] (see [30] for an explicit construction), the curve $\gamma_0$ belongs to the same knot class as the elements of the minimal sequence $(\gamma_k)_{k \in \mathbb{N}}$. The lower semi-continuity of $\text{int}M^{(p,q)}$ furthermore implies

$$\inf_{C^1_{\text{ia}} \cap K} \text{int}M^{(p,q)}(\gamma_0) \leq \lim_{n \to \infty} \text{int}M^{(p,q)}(\gamma_n) = \inf_{C^1_{\text{ia}} \cap K} \text{int}M^{(p,q)}.$$ 

Hence, $\gamma_0$ is the minimizer we have been searching for. \qedsymbol

By the same reasoning one derives the existence of a global minimizer of $\text{int}M^{(p,q)}$.

Let us conclude this section by deriving that the generalized integral Menger curvature are in fact knot energies (in the sub-critical range).
Proposition 2.4 (int\(M^{(p,q)}\) is a strong knot energy [36, Cor. 2.3]). Let (0.6) hold.

(i) If \( (\gamma_k)_{k \in \mathbb{N}} \) is a sequence of embedded \( W^{(3p-2)/q-1,q} \)-curves uniformly converging to a non-injective curve \( \gamma_\infty \in C^{0,1} \) parametrized by arc-length then int\(M^{(p,q)}(\gamma_k) \to \infty. \)

(ii) For given \( E, L > 0 \) there are only finitely many knot types having a representative with int\(M^{(p,q)} < E \) and length \( = L. \)

Proof. The first statement immediately follows from the bi-Lipschitz estimate in Proposition 2.1, as a sequence with bounded energy would be sequentially compact in \( C^1_{\text{ia}} \) and thus cannot uniformly converge to a non-injective curve.

To show the second statement, let us assume that it was wrong, i.e., that there are curves \( (\gamma_k)_{k \in \mathbb{N}} \) of length \( L \), all belonging to different knot classes, with energy less than \( E. \) Of course we can assume that \( L = 1. \) After applying suitable transformations and passing to a subsequence, Proposition 2.3 guarantees the existence of \( \gamma_0 \in A_W \) with \( \gamma_k \to \gamma_0 \) in \( C^1. \) Again by [1, 30] this implies that almost all \( \gamma_k \) belong to the same knot class as \( \gamma_0, \) which is a contradiction. \( \square \)

3 Differentiability

Recall that we have for \( v, w \in D \)

\[ |v|, |w| \leq |v - w| \leq \frac{1}{3}. \]

Hence, we obtain for each curve \( \gamma \in C^{0,1}_{\text{ia}}(\mathbb{R}/\mathbb{Z}) \) with int\(M^{(p,q)}(\gamma) < \infty \) due to the bi-Lipschitz estimate

\[ |\gamma(u + v) - \gamma(u)| \approx |v|, \quad |\gamma(u + w) - \gamma(u)| \approx |w|, \quad |\gamma(u + v) - \gamma(u + w)| \approx |v - w| \]

for all \( (u, v, w) \in \mathbb{R}/\mathbb{Z} \times D. \) (Here \( a \approx b \) is an abbreviation for the existence of uniform constants \( 0 < c \leq C < \infty \) with \( cb \leq a \leqCb. \))

We derive the following form for the first variation which is at first site much more complicated than the formula derived by Hermes [17], but due to the special structure of \( D \) it is easier to do estimates using this formula. We abbreviate

\[ R^{(p,q)}(u_1, u_2, u_3) := R^{(p,q)}(\gamma(u_1), \gamma(u_2), \gamma(u_3)) \]

and we still use

\[ \Delta_{u,v} : = \bullet(u + v) - \bullet(u + w). \quad (0.9) \]

In contrast to O’Hara’s knot energies, we can use a rather direct argument to deduce that the integral Menger curvature is Gâteaux differentiable by investigating the integrand, i.e. by looking at the Lagrangian

\[ L(\gamma)(u, v, w) := \frac{|\Delta_{u,v} \gamma \wedge \Delta_{u,w} \gamma|^q}{|\Delta_{u,v} \gamma|^p |\Delta_{w,v} \gamma|^q |\Delta_{w,u} \gamma|^q} |\gamma'(u)||\gamma'(u + v)||\gamma'(u + w)|. \]

For \( \gamma, h \in W^{(3p-2)/q-1,q} \) as in the statement of the lemma and \( \gamma_r := \gamma + rh \) one calculates

\[ \delta L(\gamma; h)(u, v, w) := \left. \frac{\partial}{\partial r} (L_{\gamma_r}(u, v, w)) \right|_{r=0} \]

15
\[
\begin{align*}
&= \left\{ q \left| \frac{\Delta_{w_0} \gamma \wedge \Delta_{v_0} \gamma}{|\Delta_{w_0} \gamma|^p |\Delta_{v_0} \gamma|^p |\Delta_{w_0} \gamma|^p} \right|^2 \right. \\
&\quad - p \frac{|\Delta_{w_0} \gamma \wedge \Delta_{v_0} \gamma|^p |\Delta_{w_0} \gamma|^p |\Delta_{v_0} \gamma|^p}{|\Delta_{w_0} \gamma|^p |\Delta_{v_0} \gamma|^p |\Delta_{w_0} \gamma|^p} \left. \right. \delta_{w_0} \gamma, \delta_{w_0} h \\
&\quad - p \frac{|\Delta_{w_0} \gamma \wedge \Delta_{v_0} \gamma|^p |\Delta_{w_0} \gamma|^p |\Delta_{v_0} \gamma|^p}{|\Delta_{w_0} \gamma|^p |\Delta_{v_0} \gamma|^p |\Delta_{w_0} \gamma|^p} \left. \right. \delta_{v_0} \gamma, \delta_{v_0} h \\
&\quad - p \frac{|\Delta_{w_0} \gamma \wedge \Delta_{v_0} \gamma|^p |\Delta_{w_0} \gamma|^p |\Delta_{v_0} \gamma|^p}{|\Delta_{w_0} \gamma|^p |\Delta_{v_0} \gamma|^p |\Delta_{w_0} \gamma|^p} \left. \right. \delta_{w_0} \gamma, \delta_{v_0} h \\
&\quad + R^{p,q}(u, u + w, u + v) \left( \gamma'(u) \frac{h'(u)}{|\gamma'(u)|} \right) + R^{p,q}(u, u + w, u + v) \left( \gamma'(u + v) \frac{h'(u + v)}{|\gamma'(u + v)|} \right) + R^{p,q}(u, u + w, u + v) \left( \gamma'(u + w) \frac{h'(u + w)}{|\gamma'(u + w)|} \right)
\end{align*}
\]

For future reference, we denote the seven terms one obtains from this formula after factoring out the outermost bracket by \(\delta L_1, \ldots, \delta L_7\).

**Lemma 3.1.** Let (0.6) hold and \(\gamma \in W^{(3p-2)/q-1,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)\) be an injective regular curve. Then \(\text{intM}^{p,q}\) is Gâteaux differentiable in \(\gamma\) and the first variation in direction \(h \in W^{(3p-2)/q-1,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)\) is given by

\[
\delta \text{intM}^{p,q}(\gamma; h) = 6 \int_{\mathbb{R}/\mathbb{Z} \times D} \delta L(\gamma; h)(u, v, w) \, dw \, dv \, du \tag{3.1}
\]

and (0.10) holds.

**Proof.** Let \(U\) be a neighborhood of \(\gamma\) in \(W^{(3p-2)/q-1,q} \subset C^{(3p-3)/q-1} \subset C^1\) consisting only of regular curves with

\[
\inf_{\gamma \in U \cap \mathbb{R}/\mathbb{Z}} |\gamma'(u)| =: M_1 > 0
\]

and

\[
\sup_{\gamma \in U \cap \mathbb{R}/\mathbb{Z}} \frac{|\gamma(u) - \gamma(v)|}{|u - v|} =: M_2 < \infty.
\]

Using

\[
\begin{align*}
\left( \frac{\Delta_{w_0} \gamma}{w} \wedge \frac{\Delta_{v_0} \gamma}{v} + \frac{\Delta_{w_0} h}{w} \wedge \frac{\Delta_{v_0} h}{v} \right) = \left( \frac{\Delta_{w_0} \gamma}{w} - \frac{\Delta_{v_0} \gamma}{v} \right) \wedge \frac{\Delta_{w_0} h}{w} + \frac{\Delta_{w_0} \gamma}{w} \wedge \left( \frac{\Delta_{v_0} h}{v} - \frac{\Delta_{v_0} \gamma}{v} \right) + \left( \frac{\Delta_{w_0} \gamma}{w} - \frac{\Delta_{v_0} \gamma}{v} \right) \wedge \frac{\Delta_{v_0} h}{v},
\end{align*}
\]

and

\[
R^{p,q}(u, v, w) = \left| \frac{\Delta_{w_0} \gamma}{w} - \frac{\Delta_{v_0} \gamma}{v} \right|^p \wedge \frac{\Delta_{v_0} \gamma}{v} \wedge \frac{\Delta_{w_0} \gamma}{w} \wedge \frac{\Delta_{v_0} h}{v} \wedge \frac{\Delta_{v_0} \gamma}{v} \wedge \frac{\Delta_{v_0} \gamma}{v}
\]

16
together with the bi-Lipschitz estimate we infer for $\tilde{\gamma} \in U$

$$|\delta L(\tilde{\gamma}; h)(u, v, w)| \leq C \left( \frac{\Delta u \delta y}{w} - \frac{\Delta u \delta y q}{q} \|h\|_{L^p} + \frac{\Delta u \delta h}{w} - \frac{\Delta u \delta h q}{q} \right) \frac{|u|^{p-\eta} |v|^{p-\eta} |w - u|^p}{|u|^{p-\eta} |v|^{p-\eta} |w - u|^p}.$$

(3.2)

So for $0 < |\tau| \leq 1$ so small that $\gamma_\tau \in U$

$$\left| \frac{\partial}{\partial \tau} L(\gamma_\tau)(u, v, w) \right| \leq C \left( \frac{\Delta u \delta y}{w} - \frac{\Delta u \delta y q}{q} \|h\|_{L^p} + \frac{\Delta u \delta h}{w} - \frac{\Delta u \delta h q}{q} \right) \frac{|u|^{p-\eta} |v|^{p-\eta} |w - u|^p}{|u|^{p-\eta} |v|^{p-\eta} |w - u|^p} \triangleq g(u, v, w)$$

(3.3)

where $g$ does not depend on $\tau$ and $C$ depends on $M_1, M_2, p, q$ only.

For $f \in \mathcal{W}^{(3p-2)/q-1,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, we have

$$\int_{\mathbb{R}/\mathbb{Z} \times D} \frac{|\Delta u \delta f|}{|u|^{p-\eta} |v|^{p-\eta} |w - u|^p} \, dw \, du = \int_{\mathbb{R}/\mathbb{Z} \times D} \frac{\|f(u + \theta w) - f(u + \theta v)\|_q^q}{|u|^{p-\eta} |v|^{p-\eta} |w - v|^p} \, dw \, dv \, d\theta$$

(3.4)

$$\leq C \int_0^1 \int_{\mathbb{R}/\mathbb{Z} \times D} \frac{\|f(u + \theta (w - v)) - f(u)^q}{|u|^{p-\eta} |v|^{p-\eta} |w - v|^p} \, dw \, dv \, d\theta \leq C \|f\|_{\mathcal{W}^{(3p-2)/q-1,q}}^q \cdot$$

Hence,

$$\int_{\mathbb{R}/\mathbb{Z} \times D} |g(u, v, w)| \, dw \, dv \, du \leq C \left[ \gamma_{1/y}^{\|y\|_{\mathcal{W}^{(3p-2)/q-1,q}}} \|h\|_{L^p}^{\|h\|_{L^p}} \left( 1 + \|h\|_{L^p} \right) \right] + C \left( \int_{\mathbb{R}/\mathbb{Z} \times D} \left( \frac{|\Delta u \delta y}{w} - \frac{\Delta u \delta h}{w} \right) \frac{|u|^{p-\eta} |v|^{p-\eta} |w - u|^p}{|u|^{p-\eta} |v|^{p-\eta} |w - u|^p} \, dw \, dv \right)^{1/2} \leq C \left( \gamma_{1/y}^{\|y\|_{\mathcal{W}^{(3p-2)/q-1,q}}} + C \left[ \gamma_{1/y}^{\|y\|_{\mathcal{W}^{(3p-2)/q-1,q}}} \|h\|_{L^p} \right] \right)$$

(3.5)

So $\delta L(\gamma_\tau; h)$ has a uniform $L^1$-majorant for $\tau$ sufficiently small. Therefore, by Lebesgue’s theorem of dominated convergence, we finally can use the fundamental theorem of calculus to write for $\tau$ small

$$\lim_{\tau \to 0} \int_{\mathbb{R}/\mathbb{Z} \times D} \left( \frac{\Delta u \delta y}{w} - \frac{\Delta u \delta h}{w} \right) \left( 1 + \|h\|_{L^p} \right) \, dw \, dv \, du = 6 \int_{\mathbb{R}/\mathbb{Z} \times D} \int_0^1 \delta L(\gamma_\tau; h)(u, v, w) \, ds \, dw \, du \xrightarrow{\tau \to 0} 6 \int_{\mathbb{R}/\mathbb{Z} \times D} \delta L(\gamma; h)(u, v, w) \, du \, dv \, dw.$$
Consequently, the first variation exists and has the form (3.1).

Using once more the symmetry of the integrand, we can bring this into the form (0.10) as follows. Due to the symmetry of the integrand we have \( L \circ P_\sigma = L \) for any permutation matrix \( P_\sigma \in \mathbb{S}_3 \). So we obtain

\[
6 \int_{\mathbb{R}/\mathbb{Z} \times D} \delta L(\gamma; h) = \sum_{\sigma \in \mathbb{S}_3} \int_{\mathbb{R}/\mathbb{Z} \times D} \delta L(\gamma; h) = \int_{\mathbb{R}/\mathbb{Z}^3} \delta L(\gamma; h).
\]

The symmetry of \( R^{p,q} \) now leads to the desired.

Furthermore, by (3.2), the first variation defines a bounded linear operator on \( W^{(3p-2)/q-1,q} \). Hence \( \text{int}M^{p,q} \) is Gâteaux differentiable.

In fact, we can even show that \( \text{int}M^{p,q} \) is \( C^1 \), though we will not use this fact in the rest of this article.

**Lemma 3.2.** The functional \( \text{int}M^{p,q} \) is \( C^1 \) on the subspace of all regular embedded \( W^{(3p-2)/q-1,q} \)-curves.

**Proof.** We will prove by contradiction that \( \text{int}M^{p,q} \) is a continuous map from embedded regular \( W^{(3p-2)/q-1,q} \)-curves to \( (W^{(3p-2)/q-1,q})^* \). So let us assume that \( \text{int}M^{p,q} \) was not continuous in \( \gamma_0 \). Consequently, there are some \( \varepsilon_0 > 0 \) and sequences \( (\gamma_k)_{k \in \mathbb{N}}, (h_k)_{k \in \mathbb{N}} \subseteq W^{(3p-2)/q-1,q} \), \( \gamma_k \to \gamma_0 \) in \( W^{(3p-2)/q-1,q} \), \( \|h_k\|_{W^{(3p-2)/q-1,q}} \leq 1 \), with

\[
|\text{int}M^{p,q}(\gamma_k; h_k) - \text{int}M^{p,q}(\gamma; h_0)| \geq \varepsilon_0. \tag{3.6}
\]

As in the proof of Lemma 3.1, we can exploit the embedding \( W^{(3p-2)/q-1,q} \hookrightarrow C^1 \) and the openness of the set of regular embedded curves in \( C^1 \), to find an open neighborhood \( U \) of \( \gamma_0 \) consisting only of embedded curves, such that

\[
\inf_{\tilde{\gamma} \in U, \tilde{\gamma} \neq \gamma_0} |\tilde{\gamma}(u)| =: M_1 > 0
\]

and

\[
\sup_{\tilde{\gamma} \in U} \|\tilde{\gamma}\|_{W^{(3p-2)/q-1,q}} + \sup_{\tilde{\gamma} \in U, \tilde{\gamma} \neq \gamma_0} \frac{|\tilde{\gamma}(u) - \tilde{\gamma}(v)|}{|u - v|} =: M_2 < \infty.
\]

After passing to a subsequence we may assume \( (\gamma_k)_{k \in \mathbb{N}} \subseteq U \) and \( h_k \to h_0 \in W^{(3p-2)/q-1,q} \) in \( C^1 \) due to the compactness of the embedding \( W^{(3p-2)/q-1,q} \hookrightarrow C^1 \), which then also gives

\[
\delta L(\gamma_k; h_k) - \delta L(\gamma_0; h_0) \to 0
\]

pointwise almost everywhere on \( \mathbb{R}/\mathbb{Z} \times D \) and hence in measure, i.e., for all \( \varepsilon > 0 \) we have

\[
\lim_{k \to \infty} \mathcal{L}^1(A_{\varepsilon,k}) = 0, \tag{3.7}
\]

where \( \mathcal{L}^1 \) denotes the Lebesgue measure and

\[
A_{\varepsilon,k} := \left\{ (u,v,w) \in \mathbb{R}/\mathbb{Z} \times D \left| |\delta L(\gamma_k; h_k)(u,v,w) - \delta L(\gamma_0; h_0)(u,v,w)| \geq \varepsilon \right. \right\}.
\]
For all $\varepsilon > 0$ we can deduce from (3.2) using Young’s inequality that there is a $C_\varepsilon > 0$ with

$$|\delta L(\gamma_k; h_k)(u, v, w)| \leq C_\varepsilon \left[ \frac{\delta u_k^2}{p} + \frac{\delta v_k^2}{q} + \frac{\delta w_k^2}{r} \right] + \varepsilon \left[ \frac{\delta u_k^2}{p} + \frac{\delta v_k^2}{q} + \frac{\delta w_k^2}{r} \right]$$

(3.8)

for all $k \in \mathbb{N} \cup \{0\}$. Since the first summand converges in $L^1$ as $k \to \infty$, it is uniformly integrable, so there is a $\delta_\varepsilon > 0$ such that $L^1(E) \leq \delta_\varepsilon$ for any measurable subset $E \subset \mathbb{R}/\mathbb{Z} \times D$ implies

$$\int \int \int_E g_k^{(1)} \leq \frac{\varepsilon}{C_\varepsilon} \quad \text{for all } k \in \mathbb{N} \cup \{0\}.$$  \hspace{1cm} (3.9)

Furthermore we infer from (1.3)

$$\int \int \int_{\mathbb{R}/\mathbb{Z} \times D} g_k^{(2)} \, du \, dv \, dw \leq C \|h_k\|_{W^{(3p-2)/q-1, q}} \leq C.$$  \hspace{1cm} (3.10)

By (3.7) there is some $k_0 = k_0(\varepsilon) \in \mathbb{N}$ with

$$L^1(A_{\varepsilon, k}) \leq \delta_\varepsilon \quad \text{for all } k \geq k_0$$

which yields for $k \geq k_0$ and $B_{\varepsilon, k} := \mathbb{R}/\mathbb{Z} \times D \setminus A_{\varepsilon, k}$

$$\frac{1}{|\delta int M^{(p,q)}(\gamma_k; h_k)|} - \frac{1}{|\delta int M^{(p,q)}(\gamma_0; h_0)|} \leq \int \int \int_{A_{\varepsilon, k}} |\delta L(\gamma_k; h_k) - \delta L(\gamma_0; h_0)| + \int \int \int_{B_{\varepsilon, k}} |\delta L(\gamma_k; h_k) - \delta L(\gamma_0; h_0)|$$

$$\leq \int \int \int_{A_{\varepsilon, k}} \left( C \epsilon g_k^{(1)} + \epsilon g_k^{(2)} \right) + \int \int \int_{B_{\varepsilon, k}} \left( C \epsilon g_k^{(1)} + \epsilon g_k^{(2)} \right) + L^1(B_{\varepsilon, k})$$

$$\leq C \varepsilon.$$  \hspace{1cm} (3.11)

Hence,

$$e_k \leq \left| \frac{1}{|\delta int M^{(p,q)}(\gamma_k; h_k)|} - \frac{1}{|\delta int M^{(p,q)}(\gamma; h_k)|} \right|$$

$$\leq \left| \delta int M^{(p,q)}(\gamma_k; h_k) - \delta int M^{(p,q)}(\gamma; h_k) \right| + \left| \delta int M^{(p,q)}(\gamma; h_k) - \delta int M^{(p,q)}(\gamma; h) \right|$$

$$\leq C \epsilon + C \|h_k - h\|_{W^{(3p-2)/q-1, q}}$$

for all $\varepsilon > 0$ and $k \geq k_0(\varepsilon)$ which leads to a contradiction.  \hspace{1cm} \Box

4 Regularity of stationary points

For the rest of this section, let us restrict to the case that $\gamma$ is parametrized by arc-length. Then we get using Lemma 3.1

$$\delta int M^{(p,q)}(\gamma; h) := 6q \tilde{Q}^{(p,q)}(\gamma, h) + 6R_1^{(p,q)}(\gamma h)$$

where

$$\tilde{Q}^{(p,q)}(\gamma, h) := \int \int \int_{\mathbb{R}/\mathbb{Z} \times D} \left| \Delta_{w,0}\gamma \wedge \Delta_{v,0}\gamma \right|_{w-2} \frac{\left( \Delta_{w,0}\gamma \wedge \Delta_{v,0}\gamma, \Delta_{w,0}\gamma \wedge \Delta_{u,0}\gamma \wedge \Delta_{v,0}\gamma \wedge \Delta_{w,0}\gamma \right)}{|\Delta_{w,0}\gamma|^p|\Delta_{v,0}\gamma|^p|\Delta_{u,0}\gamma|^p} \, du \, dv \, dw$$

19
and

\[
R_1(\gamma, h) := \iint_{\mathbb{R}^2 \times D} \left( -p \frac{\langle \Delta_{\omega,0}\gamma \wedge \Delta_{\omega,0}\gamma \rangle}{|\Delta_{\omega,0}\gamma|^{p-2} |\Delta_{\omega,0}\gamma|} \cdot \langle \Delta_{\omega,0}\gamma, \Delta_{\omega,0}h \rangle \\ - p \frac{\langle \Delta_{\omega,0}\gamma \wedge \Delta_{\omega,0}\gamma \rangle}{|\Delta_{\omega,0}\gamma|^{p-2} |\Delta_{\omega,0}\gamma|} \cdot \langle \Delta_{\omega,0}\gamma, \Delta_{\omega,0}h \rangle \\ - p \frac{\langle \Delta_{\omega,0}\gamma \wedge \Delta_{\omega,0}\gamma \rangle}{|\Delta_{\omega,0}\gamma|^{p-2} |\Delta_{\omega,0}\gamma|} \cdot \langle \Delta_{\omega,0}\gamma, \Delta_{\omega,0}h \rangle \\ + R^\beta(a, u + w, u + v) \langle \gamma'(u), h'(u) \rangle \\ + R^\beta(a, u + w, u + v) \langle \gamma'(u + v), h'(u + v) \rangle \right) \, dw \, dv \, du.
\]

For \( q = 2 \) will see that \( \hat{Q}^\rho := \hat{Q}^{h,2} \) contains the highest order term of the Euler-Lagrange operator. To see this we use

\[
\langle a \wedge b, c \wedge d \rangle = \det \begin{pmatrix}
(a, c) & (a, d) \\
(b, c) & (b, d)
\end{pmatrix}
\]
to get

\[
\langle a \wedge b, a \wedge c \rangle = \langle a, a \rangle \langle c, b \rangle - \langle a, c \rangle \langle a, b \rangle = |a|^2 \langle P^+_a b, c \rangle.
\]

Hence,

\[
\hat{Q}^\rho(\gamma; h) = \iint_{\mathbb{R}^2 \times D} \left( \frac{P^+_a b}{|v - w|^p |v|^p} \frac{\langle \Delta_{a,0} \gamma, \Delta_{a,0}h \rangle}{|\Delta_{a,0} \gamma|^{p-2} |\Delta_{a,0} \gamma|} + \frac{P^+_a b}{|v - w|^p |v|^p} \frac{\langle \Delta_{a,0} \gamma, \Delta_{a,0}h \rangle}{|\Delta_{a,0} \gamma|^{p-2} |\Delta_{a,0} \gamma|} \right) \, dw \, dv \, du \\
+ \iint_{\mathbb{R}^2 \times D} \left( \frac{P^+_a b}{|v - w|^p |v|^p} \frac{\langle \Delta_{a,0} \gamma, \Delta_{a,0}h \rangle}{|\Delta_{a,0} \gamma|^{p-2} |\Delta_{a,0} \gamma|} \right) \, dw \, dv \, du + R_2(\gamma; h)
\]
Using
\[ Q^{(p)}(\gamma; h) : = \iiint_{\mathbb{R}^2 \times D} \left( \frac{\partial_u \gamma}{\nu} - \frac{\partial_u \phi}{\nu} - \frac{\partial_u \phi}{\nu} - \frac{\partial_u \phi}{\nu} \right) \, dw \, dv \, du \]
we hence get
\[ \delta M^{p,2} = 12 \left( Q^{(p)} + \frac{1}{2} R_1 + R_2 + R_3 \right). \tag{4.2} \]

**Proposition 4.1.** The functional \( Q^{(p)} \) is bilinear on \( \left( W^{3,p/2-2,2} \right)^2 \), more precisely
\[ Q^{(p)}(f, g) = \sum_{k \in \mathbb{Z}} \langle \hat{f}_k, \hat{g}_k \rangle_{\mathbb{C}^d} \quad \text{where} \quad \hat{\varrho}_k = c|k|^{p-4} + o\left(|k|^{p-4}\right) \quad \text{as} \quad |k| \to \infty \]
and \( c > 0 \). Here \( \hat{\varrho}_k \) denotes the \( k \)-th Fourier coefficient
\[ \hat{f}_k := \int_0^1 f(x)e^{-2\pi ikx} \, dx. \]

**Proof.** Testing with the basis \( e_l \cdot e^{2\pi ikx} \) of \( L^2 \), \( l = 1, 2, 3 \), \( k \in \mathbb{Z} \), where \( e_1, e_2, e_3 \) is the standard basis of \( \mathbb{R}^3 \), we get
\[ Q^{(p)}(f, g) = \sum_{k \in \mathbb{Z}} \langle \hat{f}_k, \hat{g}_k \rangle_{\mathbb{C}^d} \hat{\varrho}_k \]
where
\[ \hat{\varrho}_k := \iint_{D_k} \frac{|e^{2\pi ikx} - e^{2\pi ikx}|^2}{|e^{p-2}| |p-2| |v - u|^p} \, dw \, dv. \]
A simple substitution leads to
\[ \hat{\varrho}_k = |k|^{p-4} \iint_{D_1} \frac{|e^{2\pi ikx} - e^{2\pi ikx}|^2}{|e^{p-2}| |p-2| |v - u|^p} \, dw \, dv \]
where \( D_k := k \cdot D \). We use the fundamental theorem of calculus and Jensen’s inequality to get
\[ \frac{|e^{2\pi ikx} - e^{2\pi ikx}|^2}{|e^{p-2}| |p-2| |v - u|^p} \leq 4\pi^2 \int_0^1 \frac{|e^{2\pi ikx} - e^{2\pi ikx}|^2}{|e^{p-2}| |p-2| |v - u|^p} \, d\theta \]
\[ = 4\pi^2 \int_0^1 \frac{\frac{1}{|e^{p-2}| |p-2| |v - u|^p}}{d\theta} \]
\[ \leq C \int_0^1 \frac{1}{|e^{p-2}| |p-2| |v - u|^p} \, d\theta \]
Thus, we have shown that

\[
\gamma(\Delta) = 0
\]

and

\[
\Gamma \leq 0
\]

The term \( R = \langle \Delta, \gamma, \Delta, \rangle \) is an analytic function, \( s \in [0, \theta) \) for \( j = 1, \ldots, K \).

\[
g^\delta(u, v, w; s_1, \ldots, s_{K-2}) = G^\delta \left( \frac{\Delta_{u,v} \gamma}{|u|}, \frac{\Delta_{u,w} \gamma}{|w|} - \frac{\Delta_{v,w} \gamma}{v - w} \right) \Gamma(u, v, w, s_1, s_2) \cdot
\]

\[
\left( \bigotimes_{i=1}^{K_1} \gamma'(u + s_i) \right) \otimes \left( \bigotimes_{j=1}^{K_2} \gamma'(u + s_j, w) \right) \otimes \left( \bigotimes_{j=K_2+1}^{K-2} \gamma'(u + v + s_j(w - v)) \right)
\]

and \( \Gamma(u, v, w, s_1, s_2) \) is a term of one of the four types

\[
\frac{\gamma'(u + s_1 w) - \gamma'(u + s_1 v)}{|v - w|} \otimes \frac{\gamma'((u + s_1 w) - \gamma'(u + s_2 v))}{|v - w|} \quad \frac{|\gamma'(u + s_1 w) - \gamma'(u + s_2 w)|^2}{|v - w|} \quad \frac{|\gamma'(u + s_1 w) - \gamma'(u + s_1 v)|^2}{|v - w|} \quad \frac{|\gamma'(u + v + s_1(w - v)) - \gamma'(u + v + s_2(w - v))|^2}{|v - w|}
\]

\[
\frac{\left| \Delta_{u,v} \gamma \wedge \Delta_{u,v} h \right|^2}{\left| \Delta_{u,v} \gamma \right|^p \left| \Delta_{u,v} h \right|^p} \cdot \langle \Delta_{u,v} \gamma, \Delta_{u,v} h \rangle = G^{\delta^p} \left( \frac{\Delta_{u,v} \gamma}{|u|}, \frac{\Delta_{u,w} \gamma}{|w|} - \frac{\Delta_{v,w} \gamma}{v - w} \right) \frac{\left| \Delta_{u,v} \gamma \wedge \Delta_{u,v} h \right|^2}{|v - w|^{p+2}|u - w|^p} \cdot \langle \Delta_{u,v} \gamma, \Delta_{u,v} h \rangle
\]
where $G^\alpha(z_1, z_2, z_3) = z_1^{-\alpha} z_2^{-\beta} z_3^{-\gamma}$ together with

\[
\frac{\Delta_{w,0} y}{w} \wedge \frac{\Delta_{v,0} y}{v} = \left( \frac{\Delta_{w,0} y}{w} - \frac{\Delta_{v,0} y}{v} \right) \wedge \frac{\Delta_{w,0} y}{v} = \int_0^1 \int_0^1 (y'(u + s_1 w) - y'(u + s_1 v)) \wedge (u + s_1 v) \, ds_1 \, ds_3
\]

we see that the first term of $R^1$ is of type 1. Similarly, one gets that all the terms of $R^1$ are of type 1.

For the term $R^3$ we use

\[
\left\langle P_{\Delta,\gamma} \right| \left( \Delta_{w,0} \gamma, \Delta_{0,0} h \right) = \frac{1}{|\Delta_{w,0} \gamma|^2} \left\langle \left( \left| \Delta_{w,0} \gamma \right|, \left| \Delta_{0,0} \gamma \right| \right), \left( \left| \Delta_{w,0} \gamma \right|, \left| \Delta_{0,0} \gamma \right| \right) \right\rangle
\]

\[
= \frac{v^2 w^2}{|\Delta_{0,0} \gamma|^2} \int_0^1 \cdots \int_0^1 y'(u + s_1 v) \wedge y'(u + s_2 w) \wedge y'(u + s_3 v) \wedge \delta'(u + s_4 w) \, ds_1 \, ds_2 \, ds_3 \, ds_4
\]

together with the fact that for $w \in \mathbb{R}, u \in \mathbb{R}/\mathbb{Z}$

\[
\frac{1}{|\Delta_{w,0} \gamma|^2} = 2G^{(\alpha)} \left( \frac{|\Delta_{w,0} \gamma|}{|w|} \right) \frac{1}{|w|^2} \left( \frac{|\Delta_{w,0} \gamma|}{|w|} \right)
\]

\[
= G^{(\alpha)} \left( \frac{|\Delta_{w,0} \gamma|}{|w|} \right) \int_0^1 \frac{y'(u + s_1 v) - y'(u + s_2 w)}{|w|^2} \, ds_1 \, ds_2
\]

where $G^{(\alpha)}(z) = \frac{1}{z^2} \cdot \frac{1}{1 - e^{-z \alpha}}$, $z^{-\alpha}$ is analytic on $(0, \infty)$ for $\alpha > 0$. Both equations together

\[
\frac{1}{|\Delta_{w,0} \gamma|^2} \frac{\Delta_{w,0} \gamma}{v} \frac{\Delta_{v,0} \gamma}{w} = \frac{1}{|v - w|^p |w^{p-2} |w|^p}
\]

\[
= \left\langle \frac{\Delta_{w,0} \gamma}{w}, \frac{\Delta_{v,0} \gamma}{v} \right\rangle \frac{1}{|v - w|^p |w^{p-2} |w|^p} - \left\langle \frac{\Delta_{w,0} \gamma}{w}, \frac{\Delta_{v,0} \gamma}{v} \right\rangle \frac{1}{|w|^p}
\]

\[
+ \left\langle \frac{\Delta_{w,0} \gamma}{w}, \frac{\Delta_{v,0} \gamma}{v} \right\rangle \frac{1}{|w|^p} - \frac{1}{|v - w|^p |w^{p-2} |w|^p}
\]

\[
+ \left\langle \frac{\Delta_{w,0} \gamma}{w}, \frac{\Delta_{v,0} \gamma}{v} \right\rangle \frac{1}{|v - w|^p |w^{p-2} |w|^p} - \frac{1}{|v - w|^p |w^{p-2} |w|^p}
\]

show that the first term of $R_2$ is the sum of terms of type 2 to 4. Similarly for the second term in $R_2$.

Let us turn to the last term, $R_3$. Again, we restrict to the first term, the second is parallel. We obtain

\[
\left\langle P_{\Delta,\gamma} \right| \frac{\Delta_{w,0} \gamma}{w} = \frac{\Delta_{w,0} \gamma}{v} \frac{\Delta_{v,0} \gamma}{w} \frac{\Delta_{v,0} h}{w} = \left\langle \frac{\Delta_{w,0} \gamma}{w}, \frac{\Delta_{v,0} \gamma}{v} \right\rangle \frac{\Delta_{v,0} h}{w}
\]

\[
= \left\langle \frac{\Delta_{w,0} \gamma}{v} \right\rangle \frac{1}{|v - w|^p |w^{p-2} |w|^p} - \left\langle \frac{\Delta_{w,0} \gamma}{v} \right\rangle \frac{1}{|v - w|^p |w^{p-2} |w|^p}
\]
Recall that the argument of regularity theorem which is deferred to the end of this section. This statement together with Proposition 4.1 immediately leads to the proof of the follows from the succeeding auxiliary result.

\[ \frac{\Delta_{u,0}Y}{v} \leq \frac{(\Delta_{u,0}Y - \Delta_{u,0}Z)}{v} - \frac{1}{2} \left( \frac{\Delta_{u,0}Y}{v} - \frac{\Delta_{u,0}Z}{v} \right)^{-2} \left( \frac{\Delta_{u,0}Y}{v} - \frac{\Delta_{u,0}Z}{v} \right) \]

which gives rise to type 4.

Our next task is to show that \( R^p \) is in fact a lower-order term. More precisely, we have

**Proposition 4.3 (Regularity of the remainder term).** If \( \gamma \in W^{(3p-4)/2+\sigma, 2} \) for some \( \sigma \geq 0 \) then \( R^p(\gamma, \cdot) \in \left(W^{3/2-\epsilon, 2}\right)^\ast \) for any \( \epsilon > 0 \).

This statement together with Proposition 4.1 immediately leads to the proof of the regularity theorem which is deferred to the end of this section.

To prove Proposition 4.3, we first note that, by partial integration, the terms of \( R^{(p)}(\gamma, h) \) may be transformed into

\[
\int_\mathbb{R} \int D \int_{\mathbb{R}^n} \left( (-\Delta)^{\sigma/2} g^p(u, v, w, s_1, \ldots, s_{K-2}) \right)(u) \otimes \left( (-\Delta)^{\sigma/2} h^p \right)(u + s_{K-1}u + s_K w) \, du \, dw \, \theta_1 \cdots \theta_K
\]

\[
\leq \int_\mathbb{R} \int D \int_{\mathbb{R}^n} \left\| g^p(u, v, w, \ldots) \right\|_{0,1,1} \, du \, dw \, \theta_1 \cdots \theta_K \left\| (-\Delta)^{\sigma/2} h^p \right\|_{3/2+\epsilon, 2} 
\]

where \( \tilde{\sigma} \in \mathbb{R}, \epsilon > 0 \) can be chosen arbitrarily, and \((-\Delta)^{\sigma/2}\) denotes the fractional Laplacian. We let \( \tilde{\sigma} := 0 \) if \( \sigma = 0 \) and \( \tilde{\sigma} := \sigma - \frac{\epsilon}{2} \) otherwise. Now the claim directly follows from the succeeding auxiliary result.

**Lemma 4.4 (Regularity of the remainder integrand).** Let \( \gamma \in W^{(3p-4)/2+\sigma, 2} \).

- If \( \sigma = 0 \) then \( g^p \in L^1(\mathbb{R}^n) \) and
- If \( \sigma > 0 \) then \( g^p(u, v, w, \ldots) \in L^1(D, W^{\sigma, 1}(\mathbb{R}^n)) \) for any \( \tilde{\sigma} < \sigma \).

The respective norms are bounded independently of \( s_1, \ldots, s_K \).

**Proof.** Recall that the argument of \( G^{(p)} \) is compact and bounded away from zero. Using arc-length parametrization, we immediately obtain for the first type

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |g^{(p)}(u, v, w)| \, dw \, du \leq C \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|g^(u+s_1 w) - g^(u+s_1 v)|^2}{|u|^p-2|u|^p-2|w| - |v|^p} \, dw \, du
\]

\[
= C \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|g^(u+s_1 (w-v)) - g^(u)|^2}{|u|^p-2|u|^p-2|w| - |v|^p} \, dw \, du
\]

\[
\leq C \|g\|^2_{W^{(3p-4)/2, 2}}.
\]
For a term of the second type we get
\[
\int_{\mathbb{R}^3} |g^p(u, v, w)| \, dv \, dw \, du \leq C \int_{\mathbb{R}^3} \frac{|y'(u + s_1 u) - y'(u + s_2 u)|^2}{|w|^p - |v|^p} \, dw \, dv \, du
\]
and of course the same estimate is true for a term of the third kind. For term of type
4, we get along the same lines
\[
\int_{\mathbb{R}^3} |g^p(u, v, w)| \, dv \, dw \, du \leq C \int_{\mathbb{R}^3} \frac{|y'(u + v + s_1 (w - v) - y'(u + v + s_2 (w - v))|^2}{|w|^p - |v|^p} \, dw \, dv \, du
\]
and of course the same estimate is true for a term of the third kind. For term of type
four, we get along the same lines
\[
\int_{\mathbb{R}^3} |g^p(u, v, w)| \, dv \, dw \, du \leq C \int_{\mathbb{R}^3} \frac{|y'(u + v + s_1 (w - v) - y'(u + v + s_2 (w - v))|^2}{|w|^p - |v|^p} \, dw \, dv \, du
\]
and of course the same estimate is true for a term of the third kind. For term of type
four, we get along the same lines
\[
\int_{\mathbb{R}^3} |g^p(u, v, w)| \, dv \, dw \, du \leq C \int_{\mathbb{R}^3} \frac{|y'(u + v + s_1 (w - v) - y'(u + v + s_2 (w - v))|^2}{|w|^p - |v|^p} \, dw \, dv \, du
\]
and of course the same estimate is true for a term of the third kind. For term of type
four, we get along the same lines
\[
\int_{\mathbb{R}^3} |g^p(u, v, w)| \, dv \, dw \, du \leq C \int_{\mathbb{R}^3} \frac{|y'(u + v + s_1 (w - v) - y'(u + v + s_2 (w - v))|^2}{|w|^p - |v|^p} \, dw \, dv \, du
\]
and of course the same estimate is true for a term of the third kind. For term of type
four, we get along the same lines
\[
\int_{\mathbb{R}^3} |g^p(u, v, w)| \, dv \, dw \, du \leq C \int_{\mathbb{R}^3} \frac{|y'(u + v + s_1 (w - v) - y'(u + v + s_2 (w - v))|^2}{|w|^p - |v|^p} \, dw \, dv \, du
\]
and of course the same estimate is true for a term of the third kind. For term of type
four, we get along the same lines
\[
\int_{\mathbb{R}^3} |g^p(u, v, w)| \, dv \, dw \, du \leq C \int_{\mathbb{R}^3} \frac{|y'(u + v + s_1 (w - v) - y'(u + v + s_2 (w - v))|^2}{|w|^p - |v|^p} \, dw \, dv \, du
\]
Together this leads to
\[ \|g^p(\cdot, v, w)\|_{W^{s,r}} \leq C \frac{\|\gamma'(\cdot + s_1(w - v)) - \gamma'(\cdot)\|_{W^{s,2}}^2}{|v|^{p-2} |w - v|^p} \]

and finally
\[
\iint_{D} \|g^p(\cdot, w)\|_{W^{s,r}} \, dv \, dw \leq C \iint_{D} \frac{\|\gamma'(\cdot + s_1(w - v)) - \gamma'(\cdot)\|_{W^{s,2}}^2}{|v|^{p-2} |w - v|^p} \, dv \, dw
\]  
\[
\overset{(1.3)}{\leq} C \|\gamma\|_{W^{(3p-4)/2,2}}. \]

**Proof of Theorem 4.** We start with the Euler-Lagrange Equation
\[
\delta \int_{M} (p, q)(\gamma, h) + \lambda \langle \gamma', h' \rangle_{L^2} = 0 \quad (4.4)
\]
for any \( h \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) where \( \lambda \in \mathbb{R} \) is a Lagrange parameter stemming from the side condition (fixed length). Using (4.2) this reads
\[
12Q^p(\gamma, h) + \lambda \langle \gamma', h' \rangle_{L^2} + 12R^p(\gamma, h) = 0. \quad (4.5)
\]
Since first variation of the length functional satisfies
\[
\langle \gamma', h' \rangle_{L^2} = \sum_{k \in \mathbb{Z}} |2\pi k|^2 \langle \hat{\gamma}_k, \hat{h}_k \rangle_{\ell^2},
\]
we get using Proposition 4.1 that there is a \( \tilde{c} > 0 \) such that
\[
12Q^p(\gamma, h) + \lambda \langle \gamma', h' \rangle_{L^2} = \sum_{k \in \mathbb{Z}} \tilde{c} \langle \hat{\gamma}_k, \hat{h}_k \rangle_{\ell^2}, \quad (4.6)
\]
where
\[
\tilde{c} = \tilde{c} |k|^{3p-4} + o(|k|^{p-1}) \quad \text{as } |k| \to \infty.
\]
Assuming that \( \gamma \in W^{(3p-4)/2+\sigma,2} \) for some \( \sigma \geq 0 \), we infer
\[
12Q^p(\gamma, \cdot) + \lambda \langle \gamma', \cdot \rangle_{L^2} \in \left(W^{3/2-\sigma+2,2}\right)^* \]
applying Proposition 4.3 to (4.5). Equation (4.6) implies
\[
\langle \tilde{c} |k|^{-3+\sigma+\varepsilon} \hat{\gamma}_k \rangle_{\ell^2} \in \ell^2.
\]
Recalling that \( \tilde{c} |k|^{-3+p+4} \) converges to a positive constant as \( |k| \to \infty \), we are led to
\[
\gamma \in W^{3p-4 + \sigma + \frac{3p-7}{2} - \varepsilon}. \]
Choosing \( \varepsilon := \frac{3p-7}{2} > 0 \), we gain a positive amount of regularity that does not depend on \( \sigma \). So by a simple iteration we get \( \gamma \in W^{s,2} \) for all \( s \geq 0 \). \( \square \)

### A Product and chain rule

As in [8], we make use of the following results which we briefly state for the readers’ convenience.
We first prove the equivalence of the two norms for smooth Slobodecki˘ı spaces we used in this article. We give a straightforward proof of the equivalence of two seminorms on the Sobolev-B Equivalence of fractional seminorms

We also refer to Runst and Sickel [33, Lem. 5.3.7 theorem of calculus and the triangle inequality tell us covered by [33, Thm. 5.3.6 one mainly has to treat

where k is the smallest integer greater than or equal to s.

**Lemma A.1 (Product rule).** Let \( q_1, \ldots, q_k \in (1, \infty) \) with \( \sum_{i=1}^{k} \frac{1}{q_i} = \frac{1}{r} \in (1, \infty) \) and s > 0. Then, for \( f_i \in W^{s,r}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \), \( i = 1, \ldots, k \),

\[
\left\| \prod_{i=1}^{k} f_i \right\|_{W^{s,r}} \leq C_{k,s} \prod_{i=1}^{k} \| f_i \|_{W^{s,r}}.
\]

**Lemma A.2 (Chain rule).** Let \( f \in W^{s,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \), s > 0, \( p \in (1, \infty) \). If \( \psi \in C^\infty(\mathbb{R}) \) is globally Lipschitz continuous and \( \psi \) and all its derivatives vanish at 0 then \( \psi \circ f \in W^{s,p} \) and

\[
\| \psi \circ f \|_{W^{s,p}} \leq C \| \psi \|_C \| f \|_{W^{s,p}}
\]

where k is the smallest integer greater than or equal to s.

### B Equivalence of fractional seminorms

We give a straightforward proof of the equivalence of two seminorms on the Sobolev-Slobodeckii spaces we used in this article.

**Lemma B.1.** For \( s \in (0,1) \), \( p \in [1, \infty) \) the seminorms

\[
[f]_{W^{1+s,p}} := \left( \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/4}^{1/4} \frac{|f(u+w) - f(u)|^p}{|u|^{1+sp}} \, dw \, du \right)^{1/p},
\]

\[
\| f \|_{W^{1+s,p}} := \left( \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/4}^{1/4} \frac{|f(u+w) - 2f(u) + f(u-w)|^p}{|u|^{1+(1+sp)}} \, dw \, du \right)^{1/p}
\]

are equivalent on \( W^{1,p} \).

**Proof.** We first prove the equivalence of the two norms for smooth \( f \). The fundamental theorem of calculus and the triangle inequality tell us

\[
\| f \|_{W^{1+s,p}} \geq \left( \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/4}^{1/4} \frac{|f(u+w) - 2f(u) + f(u-w)|^p}{|u|^{1+(1+sp)}} \, dw \, du \right)^{1/p}
\]

\[
= \left( \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/4}^{1/4} \left( \frac{\int_{0}^{1} f(u + \tau w) - f'(u) \, d\tau |w|^p}{|u|^{1+sp}} \right) \, dw \, du \right)^{1/p}
\]

\[
\leq \left( \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/4}^{1/4} \left( \frac{\int_{0}^{1} f(u + \tau w) - f'(u) \, d\tau |w|^p}{|u|^{1+sp}} \right) \, dw \, du \right)^{1/p}
\]

\[
+ \left( \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/4}^{1/4} \left( \frac{\int_{0}^{1} f'(u) - f'(u - \tau w) \, d\tau |w|^p}{|u|^{1+sp}} \right) \, dw \, du \right)^{1/p}
\]

27
To get an estimate in the other direction, we calculate for \( \varepsilon > 1 \)

\[
\phi_{1/4} \geq -\int_{-\tau/4}^{\tau/4} \frac{f'(u + \tau w) - f'(u)}{|\tilde{u}|^{1+p}} \, d\tilde{w} \, dr \, du \leq \frac{1}{1 + sp} [f]_{W^{1+s,p}}.
\]

Using Fubini’s theorem and substituting \( \tilde{w} = \tau w \), we can estimate this further by

\[
2 \left( \int_{\mathbb{R}/Z} \int_{-1/4}^{1/4} \int_{0}^{1} \frac{|f'(u + \tau w) - f'(u, w)|^{p}}{|\tilde{u}|^{1+p}} \, d\tilde{w} \, dr \, du \right)^{1/p} \leq \frac{2}{1 + sp} [f]_{W^{1+s,p}}.
\]

Hence,

\[
\|f\|_{W^{1+s,p}} \leq \frac{2}{1 + sp} [f]_{W^{1+s,p}}.
\]

To get an estimate in the other direction, we calculate for \( \varepsilon > 0 \)

\[
\|f\|_{W^{1+s,p}} = \left( \int_{\mathbb{R}/Z} \int_{-1/4}^{1/4} \frac{|f(u + w) - 2f(u) + f(u - w)|^{p}}{|u|^{1+s+p}} \, dw \, du \right)^{1/p}.
\]

Substituting \( \tilde{w} = \varepsilon w \) and \( \tilde{u} = (1 - \varepsilon)w \), we get

\[
I_2 = \left( \int_{\mathbb{R}/Z} (1 - \varepsilon)^{(1+s)p} \int_{1-\varepsilon}^{1} \int_{0}^{1} \frac{|f'(u + \tau w) - f'(u, w)|^{p}}{|\tilde{u}|^{1+p}} \, d\tilde{w} \, dr \, du \right)^{1/p}
\]

\[
I_1 \geq \left( \int_{\mathbb{R}/Z} \int_{-1/4}^{1/4} \int_{0}^{1} \frac{|f'(u + w) - f'(u, w)|^{p}}{|u|^{1+s+p}} \, dw \, du \right)^{1/p}.
\]

For \( I_1 \) we observe

\[
= \left( \int_{\mathbb{R}/Z} \int_{-1/4}^{1/4} \int_{0}^{1} \frac{|f'(u + \tau w) - f'(u + w)|^{p}}{|u|^{1+s+p}} \, d\tilde{w} \, dr \, du \right)^{1/p}.
\]

\[
- \left( \int_{\mathbb{R}/Z} \int_{-1/4}^{1/4} \int_{0}^{1} \frac{|f'(u - \tau w) - f'(u + w)|^{p}}{|u|^{1+s+p}} \, d\tilde{w} \, dr \, du \right)^{1/p}.
\]

\[
= \varepsilon \left( \int_{\mathbb{R}/Z} \int_{-1/4}^{1/4} \frac{|f'(u + w) - f'(u, w)|^{p}}{|u|^{1+s+p}} \, dw \, du \right)^{1/p}.
\]

\[
- 2 \left( \int_{\mathbb{R}/Z} \int_{-1/4}^{1/4} \int_{0}^{1} \frac{|f'(u + \tau w) - f'(u, w)|^{p}}{|u|^{1+s+p}} \, dw \, du \right)^{1/p}.
\]
To bound the first integral from below, we calculate

$$\left[ f \right]_{\mathcal{W}^{1+s,p}} = \left( \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{|f'(u + w) - f'(u)|^p}{|w|^{1+s}} \, dw \, du \right)^{1/p}$$

$$= 2^{-s} \left( \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/4}^{1/4} \frac{|f'(u + w) - f'(u - w)|^p}{|w|^{1+s}} \, dw \, du \right)^{1/p}.$$ 

The second integral can be estimated further

$$\int_{\mathbb{R}/\mathbb{Z}} \int_{-1/4}^{1/4} \frac{1}{|w|^{1+s}} \left[ f'(u + \tau w) - f'(u + w) \right] \, d\tau \, dw \, du \leq \epsilon^{p-1} \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/4}^{1/4} \int_{1-\epsilon}^{1+\epsilon} \frac{|f'(u + \tau w) - f'(u + w)|^p}{|w|^{1+s}} \, d\tau \, dw \, du$$

$$= \epsilon^{p-1} \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/4}^{1/4} \int_{1-\epsilon}^{1+\epsilon} \frac{|f'(u + (\tau - 1)w) - f'(u)|^p}{|w|^{1+s}} \, d\tau \, dw \, du$$

$$= \epsilon^{p-1} \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/4}^{1/4} \int_{0}^{1} \frac{|f'(u + \tau w) - f'(u)|^p}{|w|^{1+s}} \, d\tau \, dw \, du$$

$$\leq \frac{\epsilon^{p-1}}{sp + 1} \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/4}^{1/4} \frac{|f'(u + w) - f'(u)|^p}{|w|^{1+s}} \, dw \, du.$$

So we finally arrive at

$$\| f \|_{\mathcal{W}^{1+s,p}} \geq 2^s \epsilon \left[ f \right]_{\mathcal{W}^{1+s,p}} - 2 \frac{\epsilon^{s+1}}{\sqrt{s}p + 1} \left[ f \right]_{\mathcal{W}^{1+s,p}} - (1 - \epsilon)^{1+s} \| f \|_{\mathcal{W}^{1+s,p}}.$$ 

For $\epsilon = 2^{1-2s}$ this leads to

$$\| f \|_{\mathcal{W}^{1+s,p}} \geq 2^{1-2s} \left[ f \right]_{\mathcal{W}^{1+s,p}}.$$ 

To get the statement for $f \in \mathcal{W}^{1,p}$ we use a standard mollifier $\phi \in C^\infty(\mathbb{R})$ with $\phi \geq 0$, bounded support and

$$\int_{\mathbb{R}} \phi \, dx = 1.$$ 

We set $\phi_\epsilon(x) := \frac{1}{\epsilon} \phi(\frac{x}{\epsilon})$ and $f_\epsilon = f * \phi_\epsilon$.

Then $f_\epsilon$ converges to $f$ in $\mathcal{W}^{1,p}$ and we can hence chose a sequence $\epsilon_k \to 0$ such that $f_k := f_{\epsilon_k}$ converge pointwise almost everywhere to $f$ and $f'_{\epsilon_k}$ to $f'$.

Using Hölder’s inequality, we see that

$$\left[ f_\epsilon \right]_{\mathcal{W}^{1+s,p}}^p = \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{|f'(u + w - z) - f'(u - z)|^p}{|w|^{1+s}} \, dw \, du$$

$$\leq \left( \int_{\mathbb{R}} \phi_\epsilon(z) \, dz \right)^{p-1} \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{|f'(u + w - z) - f'(u - z)|^p \phi_\epsilon(z)}{|w|^{1+s}} \, dw \, du \, dz$$

$$= \int_{\mathbb{R}} \phi_\epsilon(z) \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{|f'(u + w) - f'(u)|^p}{|w|^{1+s}} \, dw \, du \, dz.$$
for all $\varepsilon > 0$. Similarly,

$$\|f_\varepsilon\|_{W^{1,p}} \leq \|f\|_{W^{1,p}}.$$ 

Hence Fatou’s lemma tells us that

$$\|f\|_{W^{1,p}} \leq \liminf_{k \to \infty} \|f_k\|_{W^{1,p}} \leq C \liminf_{k \to \infty} \|f_k\|_{W^{1,p}} \leq C \|f\|_{W^{1,p}}.$$ 

This completes the proof of the lemma. \qed

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