Finite pulse effects on $e^+e^-$ pair creation from strong electric fields

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Abstract

We investigate electron-positron pair creation from the vacuum in a pulsed electric background field. Employing the Sauter-type pulsed field $E(t) = E_0 \text{sech}^2(t/\tau)$ with height $E_0$ and width $\tau$, we demonstrate explicitly the interplay between the nonperturbative and perturbative aspects of pair creation in the background field. We analytically compute the number of produced pairs from the vacuum in the Sauter-type field, and the result reproduces Schwinger's nonperturbative formula in the long pulse limit (the constant field limit), while in the short pulse limit it coincides with the leading-order perturbative result. We show that two dimensionless parameters $\nu = |eE_0|\tau^2$ and $\gamma = |eE_0|\tau/m_e$ characterize the importance of multiple interactions with the fields and the transition from the perturbative to the nonperturbative regime. We also find that pair creation is enhanced compared to Schwinger's formula when the field strength is relatively weak $|eE_0|/m_e^2 \ll 1$ and the pulse duration is relatively short $m_e\tau \ll 1$, and reveal that the enhancement is predominantly described by the lowest order perturbation with a single photon.

1 INTRODUCTION

In the presence of extraordinarily strong gauge fields, we encounter essentially new phenomena that are not observed in the vacuum. Such phenomena are collectively called "strong-field physics," which has been attracting the attention of many researchers in various fields in physics [1]. There are intense laser facilities planned around the world; in addition, compact stars and relativistic nucleus-nucleus collisions offer unique opportunities to study the strong-field phenomena. Even for a system with a small coupling constant, the physics becomes nonperturbative because the smallness of the coupling constant is compensated by the strong background field strength, which requires some sort of resummations of higher-order contributions. Such nonperturbative nature of interactions between particles and fields is one of the outstanding properties of strong-field physics. At the same time, it is an important issue to understand the transition from the perturbative to the nonperturbative regime with increasing the strength of the fields and/or with changing other parameters. This is relevant also to experiments for realizing the setups for strong-field physics. The present paper is devoted to clarifying the interplay between perturbative and nonperturbative aspects of the phenomena under strong fields.

To be more specific, let us consider a system described by QED in a very strong electric field $|eE| \gtrsim m_e^2$ where $m_e$ is the electron mass [2]. In such a system, propagation of an electron significantly differs from that of the bare one because it receives large corrections from the strong field. From a naive dimensional argument, the propagation of an electron acquires corrections of the order of $O[(eE/m_e^2)^n]$ if it receives $n$ kicks from the strong field. Therefore, for $|eE| \gtrsim m_e^2$, the higher-order interactions are not suppressed and the propagation of an electron bears nonperturbative nature.
Typical examples of the nonperturbative phenomena include the Schwinger mechanism 3,5, i.e., spontaneous production of $e^+e^-$ pairs from the vacuum (see Ref. 6 for a review). Intuitively it is understood as the process where a virtual $e^+e^-$ pair produced as a vacuum fluctuation is kicked many times by the strong field so that they obtain enough energy to become a real pair. Formally, an imaginary part appears in the effective action of the background fields (the Euler-Heisenberg action 3) only after summing up the electron’s one-loop diagrams with infinitely many insertions of the external fields. The total number of created electrons in case of a constant electric background field with infinite duration is given by $N/(TV) \propto |eE|^2 \exp\{-\pi m_e^2/|eE|\}$ whose dependence on $eE$ clearly indicates nonperturbative nature of the Schwinger mechanism 5.

Another aspect of strong fields which becomes important in actual physical situations is the temporal dependence of the fields. As far as a strong electric field is concerned, it is not realistic to keep it for a long time compared to the typical time scale of the system. For example, electric fields produced in high-energy heavy-ion collisions decay quite fast. Those in intense lasers are also time dependent. Based on the intuitive picture that charged particles receive quantum corrections by kicks from the strong fields, one may expect that the number of kicks from the fields depends on the field lifetime. Indeed, perturbative calculation may be justified for a short-lived pulse field like a shock wave or a spike. Thus, we need to study carefully whether the process should be described in a perturbative or nonperturbative way for the pair creation in the presence of time-dependent strong fields. This is the problem we are going to address in the present paper.

Before going into the details, let us briefly explain here how the finite duration of the fields modifies our intuitive picture for the pair creation in the background field. In a static and homogeneous electric field, the system has only two dimensionful parameters, $eE$ and $m_e$. As mentioned above, a criterion for pair creation from the vacuum is then given by $|eE|/m_e^2 \gtrsim 1$. This is also understood in the following way. Pair creation will become possible if the work, $|eE|d$, done by the field on a virtual electron/positron for a distance $d$ is at least of the same order as the electron mass. Since this must happen within the lifetime of pair fluctuation, the distance $d$ is identified with the Compton length $d \sim 1/m_e$. However, when the external electric field is time dependent, we have another dimensionful parameter $\tau$, namely, a typical duration of the field. In the static field case, we care only the lifetime of the fluctuation, but now we need to deal with the two time scales: $\tau$ and $1/m_e$. We will find that two dimensionless parameters $\gamma = |eE_0|\tau/m_e$ and $\nu = |eE_0|\tau^2$ are relevant for this time dependent case, and can discuss the interplay between perturbative and nonperturbative physics with the values of $\gamma$ and $\nu$. In particular, we can study the interplay in detail by using the Sauter-type electric field $E(t) = E_0 \sech^2(t/\tau)$ which allows analytic calculation for the Schwinger mechanism 4.

The present paper is organized as follows: In the next section, we provide perturbative and nonperturbative formulations for the $e^+e^-$ pair creation in a time-dependent electric field without specifying any temporal profile. In section 3, we take the Sauter-type field as an example of pulsed electric fields and compute the number densities of produced electrons in both formulations. Then, we compare the two results. A summary is given in the last section.

2 PAIR CREATION IN TIME-DEPENDENT ELECTRIC FIELDS

The purpose of this section is to present a general expression for the number of electrons created from the vacuum in the presence of a time-dependent electric field. To this aim, we consider the following QED Lagrangian: $\mathcal{L} = \bar{\psi}(i\not\partial - m)\psi - e\bar{\psi}\not\!A\psi \equiv \mathcal{L}_0 + \mathcal{L}_{\text{BG}}$. Here $e > 0$ is the coupling constant, $\psi$ is the electron field, and $\not\!A^\mu$ is the background gauge field. We assume a background electric field $E$ directed to the $z$ axis, which is homogeneous in space but depends on time, and we set the background gauge field four-potential $\not\!A^\mu$ to

$$\not\!A^\mu(x) = (0, 0, 0, -\int_{-\infty}^{t} E(t')dt') . \quad (1)$$

Below we first derive a formula for the number of produced electrons in the lowest-order perturbation theory, and then briefly describe how to obtain the same quantity from the nonperturbative expression for the Schwinger mechanism that includes all order interactions with the background field.

*The parameter dependence of the pair creation in the Sauter-type electric field was studied previously by solving quantum kinetic equations numerically 7,10.*
2.1 Lowest order perturbation

We treat the interaction of the electron with the background $\mathcal{L}_{BG}$ as perturbation and compute an $S$-matrix element for the $e^+e^-$ pair creation from the vacuum, $S \equiv \langle e^- (p, s) e^+ (p', s') ; \text{in} | i \int d^4x \mathcal{L}_{BG} | \text{vac; in} \rangle = \langle e^- (p, s) e^+ (p', s') ; \text{in} | \text{T exp}[i \int d^4x \mathcal{L}_{BG}] | \text{vac; in} \rangle$, in the lowest order perturbation theory. The diagrammatic expression for the lowest-order contribution $S^{(1)}$ reads

$$S^{(1)} \equiv \langle e^- (p, s) e^+ (p', s') ; \text{in} | i \int d^4x \mathcal{L}_{BG} | \text{vac; in} \rangle$$

A straightforward calculation yields

$$S^{(1)} = -i \int d^4x (e^+ \bar{A}_3 (x^0)) \langle e^- (p, s) e^+ (p', s') ; \text{in} | \bar{\psi}(x) \gamma^3 \psi(x) | \text{vac; in} \rangle$$

$$= i \int d^4x \left( e^+ \bar{A}_3 (x^0) e^{2ip_0 x^0} \right) \left( \bar{u}(p, s) \gamma^3 v(-p, s') \right) \delta^3(p + p')$$

$$= \frac{e \bar{E}(2p_0)}{2p_0} \left( \bar{u}(p, s) \gamma^3 v(-p, s') \right) \delta^3(p + p').$$

Note that we have expanded the unperturbed operator $\psi$ as

$$\bar{\psi}(x) = \sum_s \int d^3p \left[ u(p, s) \frac{e^{-ip_0 x^0 - p \cdot x}}{\sqrt{(2\pi)^3}} + v(p, s) \frac{e^{ip_0 x^0 - p \cdot x}}{\sqrt{(2\pi)^3}} \right]$$

(4)

to get the second expression. Here $p_0 = \sqrt{m^2 + p^2}$, the annihilation operators $a(p, s)$ for an electron and $b(p, s)$ for a positron satisfy the following anticommutation relations: $\{a(p, s), a^\dagger(p', s')\} = \{b(p, s), b^\dagger(p', s')\} = \delta_{ss'} \delta^3(p - p')$, (otherwise) = 0, and the Dirac spinors $u(p, s), v(p, s)$ are normalized as $u^\dagger(p, s) u(p', s') = v^\dagger(p, s) v(p', s') = \delta_{ss'}$, $v^\dagger(p, s) u(p', s') = 0$. In the last line of Eq. (3) we have introduced the Fourier transform of the background electric field $\bar{E}(\omega) = \int dt \bar{E}(t) e^{i\omega t}$.

Now, we can compute the number density of electrons $d^3N/dp^3$ created from the vacuum in the lowest-order perturbation theory,

$$\frac{1}{V} \frac{d^3N}{dp^3} = \frac{1}{V} \sum_s \int d^3p' |S^{(1)}|^2 = \frac{1}{(2\pi)^3} \left( 1 - \frac{p^2}{p_0^2} \right) \frac{|e \bar{E}(2p_0)|^2}{4p_0^2},$$

(5)

where integration over the positron momentum $p'$ gives a volume factor $V = (2\pi)^3 \delta^3(0)$. The physical meaning of the formula is evident. For an electric field oscillating in time $E(x^0) = E_0 \cos \omega x^0$, $\bar{E}(2p_0)$ is proportional to $\delta(2p_0 + \omega) + \delta(2p_0 - \omega)$. For the on-shell electron energy $p_0 = m$, the number of produced electrons vanishes if $|\omega| < 2m$, which means that the pair creation does not occur when the energy supplied by a single photon is below this threshold. This is certainly true for a constant electric field $\omega \rightarrow 0$, no matter how strong the background electric field is (within the perturbation theory). For a general time-dependent background field the number of produced electrons is nonvanishing even for a single photon as long as the background electric field has a nonzero Fourier spectrum $\bar{E}(\omega)$ above the threshold $|\omega| \geq 2m$.

The total number of produced electrons, $N$, is obtained after integration over $p$:

$$\frac{N}{V} = \frac{1}{(4\pi)^2} \int_{2m}^{\infty} d\omega \sqrt{1 - \frac{4m^2}{\omega^2}} \left( 2 + \frac{4m^2}{\omega^2} \right) |e \bar{E}(\omega)|^2.$$  

(6)

The $\omega$ integral is not possible in general unless we specify the background electric field $\bar{E}$. 
2.2 Nonperturbative evaluation — Schwinger mechanism

Creation of $e^+ e^-$ pairs from the vacuum is possible in the presence of an electric field as a nonperturbative process, which is characterized by the critical field strength $E_c = m^2/e$. Schwinger [5] computed formulas for the number of created pairs and for the vacuum persistent probability in case of the electric fields which is homogeneous in space and constant in time. It is also known that we can equally formulate the case with temporal dependence. Here, we briefly explain such a case. Notice that this is a nonperturbative calculation because we include the interaction of electrons or positrons with the electric fields up to infinite order.

We employ a formalism based on the canonical quantization in the presence of external fields. Namely, we expand the field operator $\psi$ by the exact mode functions of an electron $\psi_{ps}^{\text{in}}$ (as = in/out) and a positron $\psi_{ps}^{\text{out}}$ (as = in/out) under the background field $A^\mu$ as

$$\psi(x) = \sum_s \int d^3p \left[ \psi_{ps}^{\text{in}}(x) a_s(x, p) + \psi_{ps}^{\text{out}}(x) b_s(x, p) \right],$$

$$[i\partial_s - eA^\mu - m] \psi_{ps}^{\text{in/out}}(x) = 0 \quad (\text{as} = \text{in/out}).$$

Noting that any linear combinations of $\psi_{ps}^{\text{in/out}}(x)$ satisfy the equation of motion (8), we identify the electron (positron) mode as the positive (negative) frequency mode in the asymptotic in- and out-region, respectively. The important point is that the mode function of the in-state $\psi_{ps}^{\text{in}}$ and that of the out-state $\psi_{ps}^{\text{out}}$ do not coincide with each other in the presence of the background field and the electron (positron) mode function of the in-state $\psi_{ps}^{\text{in}}$ (as $\psi_{ps}^{\text{in}}$) becomes a linear combination of the mode functions of the out-state:

$$\begin{pmatrix} \alpha_p \\ \beta_p \end{pmatrix} = \begin{pmatrix} a_p & -\beta_p \\ \beta_p & a_p \end{pmatrix} \begin{pmatrix} \alpha_p \\ \beta_p \end{pmatrix}_{\text{in}},$$

The Bogoliubov coefficients, $\alpha_p$ and $\beta_p$, satisfy the relation $|\alpha_p|^2 + |\beta_p|^2 = 1$. One may simply understand this fact [9] in analogy with an one-dimensional barrier scattering problem where we always have a mixture of in-coming wave and its reflection on one side of the barrier, but the other side consists of out-going wave only. This difference between in- and out-state mode functions results in the difference between the in- and out-state annihilation operators, $a^\gamma(p, s)$ and $b^\gamma(p, s)$. From the orthonormality of the mode functions, one can obtain

$$a_{\text{out}}^\gamma(p, s) = \int d^3x \left( \psi_{ps}^{\text{out}}(x) \right)^\dagger \psi(x) = \alpha_p a_{\text{in}}^\gamma(p, s) + \beta_p b_{\text{in}}^\gamma(-p, s),$$

$$b_{\text{out}}^\gamma(-p, s) = \int d^3x \left( -\psi_{ps}^{\text{out}}(x) \right)^\dagger \psi(x) = -\beta_p a_{\text{in}}^\gamma(p, s) + \alpha_p b_{\text{in}}^\gamma(-p, s).$$

Using Eqs. (10) and (11), we can construct a nonperturbative formula to compute the number of electrons created via the Schwinger mechanism:

$$\frac{1}{V} \frac{d^3N}{dp^3} = \frac{1}{V} \langle \text{vac}; in | a_{\text{out}}(p, s) a_{\text{out}}^\dagger(p, s) | \text{vac}; in \rangle = \frac{1}{(2\pi)^3} |\beta_p|^2.$$ 

Thus, the problem now is reduced to computing the coefficient $\beta_p$ or solving the Dirac equation [8]. Notice that we have not specified the time dependence of the background field $A^\mu$ so far and thus Eq. (12) is a general formula. However, since there are only a few cases where analytic solutions for the Dirac equation [8] are available, we are usually forced to use numerical methods to evaluate Eq. (12).

When the background electric field is constant and homogeneous, one can easily solve the Dirac equation [8] [11]. The total number of produced electrons per unit volume and time is given by $N/(TV) \propto |eE|^2 \exp\{-\pi m^2/|eE|\}$, which clearly shows the nonperturbative nature of the formula (12).
3 PULSED ELECTRIC FIELD: SAUTER-TYPE BACKGROUND FIELD

In this section, we consider a special case where the background electric field is applied as a pulse in time. In particular, we work with a Sauter-type pulse field[13] with height $E_0$ and width $\tau$:

$$\tilde{A}_3(t) = E_0 \tau \tanh(t/\tau) \quad \text{or} \quad E(t) = E_0 \text{sech}^2(t/\tau).$$  \hspace{1cm} (13)

As mentioned in the Introduction, the advantage of the Sauter-type background field is that the analytic solution for the Dirac equation is known so that we can explicitly compute the particle number created via the Schwinger mechanism [9,12,13]. Thus, by comparing this nonperturbative result with that of our perturbative computation obtained in the Sauter-type background field, we can discuss which picture, perturbative or nonperturbative, is appropriate for studying the pair production in a strong field with finite duration.

3.1 Perturbative result

We substitute the Sauter-type background field [13] into Eqs. (5) and (6) to get the electron number density $d^3 N/dp^3$ and the total electron number $N$, respectively, in the lowest-order perturbation theory. By using

$$\tilde{E}(\omega) = i\omega \tilde{A}_3(\omega) = \frac{i\pi E_0 \tau \omega}{\sinh \frac{\pi\tau}{2}},$$  \hspace{1cm} (14)

we find that the electron number density $d^3 N/dp^3$ is given by

$$\frac{1}{V} \frac{d^3 N}{dp^3} = \frac{1}{(2\pi)^3} \left(1 - \frac{p^2}{p_0^2}\right) \left|\frac{eE_0}{m}\right|^2 \frac{(\pi p_0 \tau)^4}{\pi^2 \sinh[\pi p_0 \tau]^2}. $$  \hspace{1cm} (15)

We obtain a nonvanishing result because the Fourier spectrum of the Sauter-type field $\tilde{E}(\omega)$ is nonzero at any value of $\omega$, in particular in the region $\omega \geq 2m$. Also, the total electron number $N$ is given by

$$\frac{N}{V} = m^3 \left|\frac{eE_0}{m}\right|^2 f(\pi m\tau).$$  \hspace{1cm} (16)

Here we have introduced the function $f$ by

$$f(x) = \frac{x^4}{2\pi^4} \int_1^\infty d\omega \omega^2 \sqrt{1 - \frac{1}{\omega^2}} \frac{1}{3} \left(2 + \frac{1}{\omega^2}\right) \frac{1}{\sinh[\omega x]^2},$$  \hspace{1cm} (17)

which behaves asymptotically as (see the Appendix)

$$f(x) \sim \begin{cases} \frac{x}{18\pi^2} & \text{for} \quad x \lesssim 1, \\ \frac{x^5/2}{2\pi^7/2} \left(1 + \frac{7}{16} \frac{1}{x}\right) e^{-2x} & \text{for} \quad x \gtrsim 1. \end{cases}$$  \hspace{1cm} (18)

Figure 1 shows the momentum ($p$) dependence of the electron number density $(1/V)d^3 N/dp^3$ [13] for $|eE_0|/m^2 = 10$ and $m\tau = 0.01$. We see that the peak is located at $p = 0$, which reflects the fact that the energy threshold for creating one $e^+e^-$ pair $E_{\text{thres}} = 2p_0$ takes its minimum $E_{\text{thres}} = 2m$ at $p = 0$. We also find that the distribution decays exponentially for large $p_\perp$. Actually, one can check this by taking the limit of $p_\perp \gg p_z, m$:

$$\frac{1}{V} \frac{d^3 N}{dp^3} \propto p_z, m \quad \frac{1}{2\pi} \left|\frac{eE_0}{m}\right|^2 (m\tau)^4 e^{-2\pi|p_\perp|\tau}. $$  \hspace{1cm} (19)

\hspace{1cm} \text{[Originally, Sauter [14] considered the Dirac equation in an inhomogeneous potential $V(z) = V_0 \text{sech}^2(z/d)$. Since the problem is essentially reduced to solving a differential equation in one dimension, we can equally solve the potential with the similar functional dependence on time.]}
Figure 1: (color online). The momentum ($p$) dependence of the number density of electrons ($1/V) d^3 N/dp^3$). Parameters are set to $|eE_0|/m^2 = 10$ and $m\tau = 0.01$.

Figure 2 shows the $\tau$ dependence of the total electron number $N/V$. As is seen in Fig. 2, $N$ increases monotonically for small $m\tau$ while it decreases exponentially for large $m\tau$. This tendency can be roughly understood as follows: Since the threshold energy of the pair creation is $E_{\text{thres}} \sim 2m$, the background field must supply energy $\Omega$ larger than $2m$ for the pair creation to occur. In our computation based on the lowest-order perturbation theory, such energy $\Omega$ is supplied by a single (virtual) photon from the background electric field $E$. Since the typical energy $\omega$ of a photon which forms the Sauter-type background field $E$ is $\omega \sim 1/\tau$ (see Eq. (14) and the left panel of Fig. 3), we find $\Omega \sim (\text{number of photons}) \times (\text{typical photon energy}) \sim 1 \times \omega \sim 1/\tau$. Thus, $m\tau \lesssim 1/2$ is required for the pair creation in the lowest order perturbation theory. The upper limit $m\tau = 1/2$ is shown as a vertical line in Fig. 2. Pair creation from a single photon occurs when the pulse duration $\tau$ is short enough. On the other hand, as shown in the right panel of Fig. 3, the strength of the Fourier component $\tilde{E}(\omega)$ decreases with decreasing $\tau$. This essentially explains the decrease of electron number density for $\tau \to 0$ as seen in Fig. 2.

### 3.2 Nonperturbative result

One can obtain analytic solutions to the Dirac equation in the presence of the Sauter-type background field [12–14] (see also Ref. 9 for discussion in $SU(2)$ case), which enables us to compute the number of produced electrons [12]. After some calculations, one finds

$$
\frac{(2\pi)^3}{V} \frac{d^3 N}{dp^3} = \frac{\sinh \left( \frac{\pi}{\tau} \left( 2eE_0\tau + p_0^{(-)} - p_0^{(+)} \right) \right)}{\sinh \left( \pi \tau p_0^{(-)} \right)} \frac{\sinh \left( \frac{\pi}{\tau} \left( 2eE_0\tau - p_0^{(-)} + p_0^{(+)} \right) \right)}{\sinh \left( \pi \tau p_0^{(+)} \right)},
$$

(20)

where $p_0^{(\pm)} = \sqrt{m^2 + p_\perp^2} + (p_z \pm eE_0\tau)^2$ are the energy of the electron (positron) with the transverse momentum $p_\perp$ and the canonical longitudinal momentum $p_z$. Note that the corresponding electron mode originally has the kinetic longitudinal momentum $p_z - eE_0\tau$ in the infinite past and $p_z + eE_0\tau$ in the infinite future. We stress that this result is clearly nonperturbative with respect to the coupling constant $e$, while the lowest order perturbation gave the result proportional to $e^2$ [see Eq. (15)]. The total number of produced electrons, $N/V$, is obtained after integration over the momentum $p$. 


3.3 Comparison of the perturbative and nonperturbative results

We compare the nonperturbative result (20) with the perturbative one (15) and (16) in Figs. 4, 5, and 6. Figure 4 shows the comparison of the momentum ($p_z, p_\perp$) dependence of the number density of electrons $d^3N/dp^3$. The peak strength of the field is taken as $|eE_0|/m^2 = 1$. Three lines are for different values of $\tau$. We have shown only relatively short pulse cases: $m\tau = 0.1, 0.3$, and 0.6. We immediately observe that the nonperturbative result (20) and the perturbative result (15) coincide with each other for the short pulse. The deviation becomes larger as $\tau$ increases, which can be explicitly seen in Fig. 5. There, $\tau$ dependence is shown for different values of the peak strength: The left panel is for the subcritical field strength $|eE_0|/m^2 \ll 1$, perturbative result dominates the nonperturbative result even when the pulse is not very short $m\tau \sim 1$. On the other hand, for supercritical field strength $|eE_0|/m^2 \gg 1$, perturbative description is applicable only for finite pulses.

\footnote{We tentatively use the words “supercritical” and “subcritical” for the cases $|eE_0|/m^2 > 1$ and $|eE_0|/m^2 < 1$, respectively, but precisely speaking, the condition $|eE_0|/m^2 = 1$ (valid for a constant electric field) does not play the same role for finite pulses.}
We note that the peak structure in the short pulse region $m\tau \ll 1$. We will clarify the reason for this behavior in the later discussion. The important point here is that for any field strength $|eE_0|/m^2$ there surely exists a region (short pulse region) where pair creation can be understood as a purely perturbative phenomenon. We can also observe that there is a clear deviation between the two in the long pulse limit where the nonperturbative result approaches Schwinger’s result (horizontal lines). In particular, the deviation is larger for supercritical field strength $|eE_0|/m^2 \gtrsim 1$. This can be understood as follows. Notice first that the perturbative result always approaches 0 in the long pulse limit $m\tau \gg 1$ because the typical energies of a (virtual) photon which forms the Sauter-type field $\omega \sim 1/\tau \to 0$ for large $\tau$ and thus not enough to create a pair. On the other hand, Schwinger’s formula valid in the long pulse region says that pair creation for subcritical field strength is exponentially suppressed $d^3N/d\mathbf{p}^3 \propto \exp(-\pi m^2/|eE_0|)$. Therefore, the deviation between the two is almost negligible for weak field strength $|eE_0|/m^2 \lesssim 1$, while it increases with increasing peak strength $|eE_0|/m^2$ in the supercritical regime $|eE_0|/m^2 \gtrsim 1$.

The same tendency is found in the comparison of the total number of electrons $N$ as shown in Fig. 6. We note that the peak structure in the short pulse region is reproduced by the perturbative result quite well. Although for small $p_z$ the perturbative value of the density $d^3N/d\mathbf{p}^3$ is somewhat larger than the nonperturbative one, while it becomes smaller for large $p_z$ (see the left panel of Fig. 4), these differences cancel out with each other in integration over $p_z$. Thus we have a nice agreement in the total number $N$.
as displayed in Fig. 6.

Figure 6 also shows an interesting behavior. For relatively short pulses \( m \tau \lesssim 1 \) with subcritical field strength \( |eE_0|/m^2 \lesssim 1 \) (left panel), the results of the Sauter-type field are enhanced as compared to Schwinger’s value (horizontal lines). Since the pair creation in this region is dominated by the perturbative contribution, this enhancement should be understood as a purely perturbative effect. It shows up because Schwinger’s nonperturbative result \( d^3N/dp^3 \propto \exp\{-\pi m^2/|eE_0|\} \) is exponentially small for subcritical field strength \( |eE_0|/m^2 \lesssim 1 \), while the perturbative result is only power-suppressed as \( d^3N/dp^3 \propto |eE_0/m^2|^2 \) (see Eq. (5)). By using the perturbative formula for \( d^3N/dp^3 \), we immediately find that the peak position \( \tau_{\text{peak}} \) is given by \( 2 = (\pi p_0 \tau_{\text{peak}}) \coth(\pi p_0 \tau_{\text{peak}}) \) or \( p_0 \tau_{\text{peak}} \sim 0.61 \), which does not depend on the field strength \( |eE_0| \) as is seen in Fig. 5. Accordingly, the peak value is given by \( (1/V)d^3N/dp^3 \sim (5.0 \times 10^{-4}) \times (1 - p_0^2/p_0^2)|eE_0/p_0^2|^2 \).

The peak value, \( \max \left[ d^3N/dp^3 \right] \), at \( p = 0 \) is displayed in Fig. 7 as a function of the field strength \( |eE_0|/m^2 \), together with Schwinger’s value and the peak value of the perturbative contribution. The extrapolation of Schwinger’s value to the weak field case \( |eE_0|/m^2 \lesssim 1 \) underestimates the pair creation in the Sauter-type pulsed field; the pair creation from the vacuum in the region \( |eE_0|/m^2 \lesssim 1, m \tau \lesssim 1 \) is actually more abundant than Schwinger’s value, owing to the perturbative contribution with a single photon. Indeed, the compact formula for the perturbative peak nicely describes the enhancement for the subcritical fields, which is explicitly depicted with a dashed line in Fig. 7. Similar behavior was found in Refs. [9][10], who however regarded this peak as a result of nonperturbative physics.

Now we return to the question: To what extent are we able to say a pulse is short? To answer this, we expand the nonperturbative result [20] by the pulse duration \( \tau \). More precisely, we expand [20] by the following two dimensionless parameters,

\[
\nu \equiv \frac{|eE_0| \tau^2}{m}, \quad \gamma \equiv \frac{|eE_0| \tau}{m},
\]

because there are two dimensionful quantities \( |eE_0|, m \) in addition to \( \tau \). The result is

\[
\text{Eq. (20)} \to \left\{ \begin{array}{ll}
\frac{1}{(2\pi)^2} \left( 1 - \frac{p_0^2}{p_0^2} \right) \left( \frac{|eE_0|}{p_0^2} \right)^2 \frac{(\pi p_0 \tau)^4}{\pi^2 |\sinh(\pi p_0 \tau)|^2} & (\nu, \gamma \ll 1) \\
\exp \left[ -\pi \frac{(m^2 + p_0^2)}{|eE_0|} \right] & (\nu, \gamma \gg 1).
\end{array} \right.
\]
Figure 7: (color online). The peak value $\max_{\vec{D}} [d^3N/\partial \vec{p}^3]$ of the nonperturbative result (20) (solid line) as a function of the field strength $|eE_0|/m^2$. For comparison, Schwinger’s value and an estimate $(1/V)d^3N/\partial \vec{p}^3|_{\text{peak}} \sim (5.0 \times 10^{-4}) \times (1 - p^2_z/p_0^2)|eE_0/p_0^2|^2$ obtained with the perturbative result (15) are shown in dotted and dashed lines, respectively. Parameters are set to $p_z/m = p_\perp/m = 0$.

Notice that the asymptotic forms (24) exactly reproduce the perturbative result (15) for $\nu, \gamma \ll 1$ and the nonperturbative expression for the Schwinger mechanism in a constant electric background field [5] for $\nu, \gamma \gg 1$. Thus, we conclude that pulses, such that the condition $\nu, \gamma \ll 1$ i.e., $m\tau \ll \sqrt{m^2/|eE_0|}$, $m^2/|eE_0|$ is satisfied, are so short that pair creation becomes purely perturbative, where the lowest order perturbation theory works very nicely. On the other hand, pulses, such that the condition $\nu, \gamma \gg 1$ i.e., $m\tau \gg \sqrt{m^2/|eE_0|}$, $m^2/|eE_0|$ is satisfied, are so long that pair creation becomes nonperturbative, where perturbation theory completely breaks down. We can also say that for middle pulses, such that neither condition $\nu, \gamma \gg 1$ nor $\nu, \gamma \ll 1$ is satisfied, perturbation theory is still applicable; however, the lowest-order perturbation theory does not work because higher-order corrections $\mathcal{O}((eE)^n)$ ($n > 1$) become important. We summarize our picture in Fig. 8. These considerations clearly show that in order to investigate the nonperturbative nature of the Schwinger mechanism we must require not only the strength $|eE_0|/m^2 \gtrsim 1$ but also a sufficient duration $m\tau \gg \sqrt{m^2/|eE_0|}$, $m^2/|eE_0|$; otherwise pair creation from the vacuum can be understood simply as a perturbative phenomenon.

![Figure 8](image_url)

Figure 8: Sketch of the appropriate picture for pair creation from the vacuum for various pulses with height $E_0$ and width $\tau$. $m$ is the electron mass.

The discussion given above is a natural result if we consider the meaning of the dimensionless parameters $\nu, \gamma$. Recall the fact that the work $W$ done by a pulsed electric background field with height $E_0$ and width $\tau$ is given by $W \sim |eE_0|\tau$ and that the typical energy $\omega$ of a photon that forms the pulsed background field is given by $\omega \sim 1/\tau$. Then, we can understand the physical meaning of $\nu, \gamma$ as follows:
\( \nu \sim W/\omega \) is the number of (virtual) photons of the background field involved in a scattering process. \( \gamma \sim W/m \sim \nu \omega /m \) is the work done by the background field scaled by the typical energy scale of the system \( m \). Keeping these in mind, we can interpret that the perturbative condition \( \nu, \gamma \ll 1 \) corresponds to the case where both the number of photons involved in a scattering process \( \nu \) and its correction to the system \( \gamma \) are very small. This is obviously a natural criterion for the lowest order perturbation theory to work. We can also interpret the nonperturbative condition \( \nu, \gamma \gg 1 \) in the same way.

It is interesting to compare our discussion with Ref. \[15\], which claims that the Keldysh parameter \( \gamma_K = |eE|/(\omega m) \), where \( \omega \) is the typical frequency of the background field, discriminates whether the system is perturbative or nonperturbative. Note that their discussion is limited to the case where (i) an oscillating electric field background \( E(t) = E_0 \cos \omega t \), (ii) \( \omega \) is sufficiently small compared to the electron mass \( \omega /m \ll 1 \), and (iii) the background field is sufficiently weak \( |eE_0|/m^2 \ll 1 \). If we assume that the typical frequency \( \omega \) of a pulsed background field is given by the inverse of the pulse duration \( \omega \sim 1/\tau \), we find that our discussion obtained in a pulsed background field (see Fig. 3) agrees with Ref. \[15\] as long as the limitation (iii) is satisfied. In such a condition, \( \gamma \) determines the “perturbativeness” of the system in our discussion and is equivalent to the Keldysh parameter because \( \gamma = |eE_0|/(\omega m) = \gamma_K \).

4 SUMMARY AND DISCUSSION

We have explicitly demonstrated that, by using the Sauter-type electric field, an interplay between perturbative and nonperturbative effects for the \( e^+e^- \) pair creation from the time-dependent electric field is controlled by two dimensionless parameters \( \gamma = |eE_0|\tau/m \) and \( \nu = |eE_0|\tau^2 \). Perturbative pair creation occurs when \( \gamma, \nu \ll 1 \) is satisfied, while nonperturbative pair creation (the Schwinger mechanism) occurs when \( \gamma, \nu \gg 1 \) is satisfied. In particular, an enhancement of the electron number density seen when the pulse duration and the field strength is relatively short \( m\tau \lesssim 1 \) and weak \( |eE_0|/m^2 \lesssim 1 \), respectively, can be understood as the lowest order perturbative process with a single photon.

Throughout this paper, we considered the case where the background field is described by the Sauter-type pulse in order to explicitly perform an analytic calculation. However, we stress that our qualitative discussion should be valid for more general pulse fields smoothly characterized by its height \( E_0 \) and width \( \tau \). It is also interesting to note that, though we focused on the pulse in time in this paper, our analysis implies that the finite space extension would also affect the interplay between perturbative and nonperturbative aspects of the phenomena under strong fields.

Our analysis is instructive when we consider the effects of time-dependent strong fields in actual physical situations. Let us briefly discuss the case in high-energy heavy-ion collisions as an example. It is estimated that a very strong field \( |eE| \gg m_e^2 \) is generated in heavy-ion collisions operated in the Relativistic Heavy Ion Collider (RHIC) at BNL and the Large Hadron Collider (LHC) at CERN. (a) In noncentral collisions such that two nuclei can touch each other, numerical simulations \[10,17\] have shown that its strength is of the order of \( |eE| \sim 1 \times m_e^2 \sim (1 \times 10^5) \times m_e^2 \) for RHIC and \( |eE| \sim 10 \times m_e^2 \sim (1 \times 10^6) \times m_e^2 \) for LHC, where \( m_e \) is the pion mass. (b) In the ultraperipheral collisions where two nuclei do not touch each other, the electric field is still very strong and is estimated as \( |eE| \sim Z \alpha_{EM} \gamma_L/b^2 \), where \( b \) is the impact parameter and \( \gamma_L \) is the Lorentz factor. With modest parameters, the strength of the field reaches \( |eE| \sim 60 \times m_e^2 \) for RHIC (\( Z = 79, \gamma_L \sim 100, b \sim 1/m_e \)) and \( |eE| \sim (2 \times 10^5) \times m_e^2 \) for LHC (\( Z = 82, \gamma_L \sim 3000, b \sim 1/m_e \)). At first sight, it might be natural to expect there exists nonperturbative strong field effects such as the Schwinger mechanism because the field is extremely strong \( |eE|/m_e^2 \gg 1 \). However, from the analysis of the present paper, we have learned that we need to be careful about the finite lifetime of the strong fields. Indeed, the duration \( \tau \) of the strong field is extremely short when compared to the typical energy scale of the system \( m_e \): \( m_e \tau \ll 1 \). For instance, a rough estimate yields \( \tau \sim 0.1 \text{fm} \sim 3 \times 10^{-4} \text{m}_e^{-1} \) (RHIC, LHC) for case (a) and \( \tau \sim b/\gamma_L \sim 1 \times 10^{-2} \times \text{m}_e^{-1} \) (RHIC; \( \gamma_L \sim 100, b \sim 1/m_e \)) and \( \sim 3 \times 10^{-4} \text{m}_e^{-1} \) (LHC; \( \gamma_L \sim 3000, b \sim 1/m_e \)) for case (b). With such short durations, the important parameter \( \gamma \) can be large \( \gamma \gtrsim 1 \); however, \( \nu \) is always so small \( \nu \ll 1 \) that the pair creation in these processes is no longer nonperturbative and the perturbative treatment would be sufficient. However, as suggested in Ref. \[17\] for case (a), the matter created in the collisions could let the electric field survive longer than the naive estimation. If this is the case, there is a possibility that pair creation could be nonperturbative.
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APPENDIX: ASYMPTOTIC EXPRESSION OF \( f(x) \)

Let us find an asymptotic expression of \( f(x) \) for \( x \lesssim 1 \) and \( x \gtrsim 1 \).

For small \( x \), we change the variable \( \omega \) by \( \xi = \omega x \) to obtain

\[
\begin{align*}
f(x) &= \frac{x}{6\pi^4} \int_x^\infty d\xi \xi^2 \sqrt{1 - \frac{x^2}{\xi^2}} \left( 2 + \frac{x^2}{\xi^2} \right) \frac{1}{\sinh^2 \xi} \\
&\sim \frac{x}{6\pi^4} \int_0^\infty d\xi \frac{2}{\sinh^2 \xi} \\
&= \frac{x}{18\pi^2}.
\end{align*}
\]

At large \( x \), \( \sinh(\omega x) \sim \exp(\omega x)/2 \) and only \( \omega = 1 + \epsilon \sim 1 \) contributes to the integral. Thus we find

\[
\begin{align*}
f(x) &\sim \frac{2x^4}{3\pi^4} \int_1^\infty d\omega \omega^2 \sqrt{1 - \frac{1}{\omega^2}} \left( 2 + \frac{1}{\omega^2} \right) e^{-2\omega x} \\
&= \frac{2x^4}{3\pi^4} \int_0^\infty d\epsilon \left( 3\sqrt{2}\epsilon^{1/2} + \frac{7}{2\sqrt{2}} \epsilon^{3/2} \right) e^{-2(1+\epsilon)x} \\
&\sim \frac{x^{5/2}}{2\pi^{7/2}} \left( 1 + \frac{7}{16} \right) e^{-2x}.
\end{align*}
\]

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