NORM ESTIMATES FOR SOLUTIONS OF ELLIPTIC BVPs OF THE DIRAC OPERATOR

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Abstract. We present norm estimates for solutions of first and second order elliptic BVPs of the Dirac operator \( D = \sum_{j=1}^{n} e_j \partial_{x_j} \) considered over bounded and smooth domain \( \Omega \) of \( \mathbb{R}^n \). The solutions whose norms to be estimated are in some Sobolev spaces \( W^{k,p}(\Omega) \) and the boundary conditions as traces of solutions and their derivatives are in some Slobodeckij spaces \( W^{\lambda,p}(\partial \Omega) \) where \( \lambda \) is some non integer but fractional number, for \( 1 \leq p < \infty \) and \( k \in \mathbb{Z} \).

1. Algebraic and Analytic Rudiments of \( Cl_n \)

Let \( \{e_j : j = 1, 2, ..., n\} \) be an orthonormal basis for \( \mathbb{R}^n \) that is equipped with an inner product so that

\[
e_i e_j + e_j e_i = -2\delta_{ij} e_0
\]

(1.1)

where \( \delta_{ij} \) is the Kronecker delta. The inner product satisfies an anti commutative relation

\[
x^2 = -\|x\|^2
\]

(1.2)

Therefore \( \mathbb{R}^n \) with these properties of base vectors generates a non commutative algebra called Clifford algebra denoted by \( Cl_n \).

The basis of \( Cl_n \) will then be

\[\{e_A : A \subset \{1 < 2 < 3 < ... < n\}\}\]

which implies:

\[\dim(Cl_n) = 2^n\]

The object \( e_0 \) used above is the identity element of the Clifford algebra \( Cl_n \).

Representation of elements of \( Cl_n \): every \( a \in Cl_n \) is represented by

\[
a = \sum e_A a_A
\]

(1.3)

where \( a_A \) is a real number.
Thus every \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) can be identified with \( \sum_{j=1}^{n} e_j x_j \) of \( Cl_n \) and therefore we have an embedding
\[
\mathbb{R}^n \hookrightarrow Cl_n
\]

We also define what is called a Clifford conjugate of
\[
a = \sum e_A a_A
\]
as
\[
\bar{a} = \sum e_A a_A
\]
where
\[
e_{i_1 \ldots i_r} = \begin{cases} (-1)^r e_{i_r} \ldots e_{i_1} & \text{for } i \neq j, \\ e_j & \text{for } i = j. \end{cases}
\]

For instance for \( i, j = 1, 2, \ldots, n \),
\[
e_j = -e_j, \quad e_j^2 = -1
\]
and for
\[
i \neq j : e_i e_j = (-1)^2 e_j e_i = e_j e_i
\]

**Definition 1.** We define the Clifford norm of
\[
a = \sum e_A a_A \in Cl_n
\]
by
\[
\|a\| = ((a\bar{a}))_0^{\frac{1}{2}} = \left( \sum_{A} a_A^2 \right)^{\frac{1}{2}}
\]
where \((a)_0\) is the real part of \(a\bar{a}\).

The norm \(\|\cdot\|\) satisfies the inequality:
\[
\|ab\| \leq c(n) \|a\| \|b\|
\]
with \(c(n)\) a dimensional constant.

Also each non zero element \( x \in \mathbb{R}^n \) has an inverse given by :
\[
x^{-1} = \frac{\bar{x}}{\|x\|^2}
\]

In the article it is always the case that \( 1 < p < \infty \) unless otherwise specified and \( \Omega \) is a bounded and smooth (at least with \( C^1 \)- boundary \( \partial \Omega \)) domain of \( \mathbb{R}^n \).

A Clifford valued \((Cl_n\)-valued) function \( f \) defined on \( \Omega \) as
\[
f : \Omega \longrightarrow Cl_n
\]
has a representation
\[
f = \sum f_A e_A
\]
where \( f_A : \Omega \longrightarrow \mathbb{R} \) is a real valued component or section of \( f \).
Definition 2. For a function \( f \in C^1(\Omega) \cap C(\overline{\Omega}) \), we define the Dirac derivative of \( f \) by

\[
Df(x) = \sum_{j=1}^{n} e_j \partial x_j f(x)
\]

A function \( f : \Omega \rightarrow Cl_n \) is called left monogenic or left Clifford analytic over \( \Omega \) if

\[
Df(x) = 0, \quad \forall x \in \Omega
\]

and likewise it is called right monogenic over \( \Omega \) if

\[
f(x) D = \sum_{j=1}^{n} \partial x_j f(x) e_j = 0, \quad \forall x \in \Omega
\]

An example of both left and right monogenic function defined over \( \mathbb{R}^n \setminus \{0\} \) is given by

\[
\psi(x) = \frac{x}{\omega_n \|x\|^n}
\]

where \( \omega_n \) is the surface area of the unit sphere in \( \mathbb{R}^n \).

The function \( \psi \) is also a fundamental solution to the Dirac operator \( D \) and we define integral transforms as convolutions of \( \psi \) with functions of some function spaces below.

Definition 3. Let \( f \in C^1(\Omega, Cl_n) \cap C(\overline{\Omega}) \).

We define two integral transforms as follow:

\[
\zeta_{\Omega} f(x) = \int_{\Omega} \psi(y - x) f(y) d\Omega_y, \quad x \in \Omega
\]

\[
\xi_{\partial \Omega} f(x) = \int_{\partial \Omega} \psi(y - x) v(y) f(y) d\partial \Omega_y, \quad x \notin \partial \Omega
\]

The integral transform defined in (1.9) a domain integral is called the Theodorescu transform or the Cauchy transform. It is a convolution \( \psi * f \) over \( \Omega \). The integral transform defined in (1.10) is some times called the Feuter transform as a boundary integral which again is a convolution \( \psi * v f \) over \( \partial \Omega \). \( v(y) \) is a unit normal vector pointing outward at \( y \in \partial \Omega \).

2. SOBOLEV AND SLOBODECKIJ SPACES

Definition 4. For \( 1 < p < \infty \), \( k \in \mathbb{N} \cup \{0\} \) we define:

I: The Sobolev space \( W^{k,p}(\Omega) \) as

\[
W^{k,p}(\Omega) := \{ f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega), \| \alpha \| \leq k \}\]
with norm

\[ \|f\|_{W^{k,p}(\Omega)} = \left( \sum_{|\alpha| \leq k} \int_{\Omega} |D^\alpha f|^p dx \right)^{\frac{1}{p}} \]

II: The Slobodeckij spaces for \(0 < \lambda < 1\) as

\[ W^{\lambda,p}(\partial \Omega) := \{ f \in L^p(\partial \Omega) : \int_{\partial \Omega} \int_{\partial \Omega} \frac{|f(x) - f(y)|^p}{|x-y|^{n+\lambda p-1}} d\sigma_x d\sigma_y < \infty \} \]

and norm is defined by

\[ \|f\|_{W^{\lambda,p}(\partial \Omega)} = \left( \int_{\partial \Omega} \int_{\partial \Omega} \frac{|f(x) - f(y)|^p}{|x-y|^{n+\lambda p-1}} d\sigma_x d\sigma_y \right)^{\frac{1}{p}} \]

III: The Slobodeckij spaces for \(\lambda = [\lambda] + \{\lambda\}\) where \(0 < \{\lambda\} < 1\) :

\[ W^{\lambda,p}(\partial \Omega) := \{ f \in W^{[\lambda],p}(\partial \Omega) : \sum_{|\alpha| \leq [\lambda]} \int_{\partial \Omega} |Df|^p d\sigma_x + \sum_{|\alpha| = \{\lambda\}} \int_{\partial \Omega} \int_{\partial \Omega} \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x-y|^{n+[\lambda]p-1}} d\sigma_x d\sigma_y < \infty \} \]

and hence norm is given by

\[ \|f\|_{W^{\lambda,p}(\partial \Omega)} = \left( \sum_{|\alpha| \leq [\lambda]} \int_{\partial \Omega} |Df|^p d\sigma_x + \sum_{|\alpha| = \{\lambda\}} \int_{\partial \Omega} \int_{\partial \Omega} \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x-y|^{n+[\lambda]p-1}} d\sigma_x d\sigma_y \right)^{\frac{1}{p}} \]

In the definitions of the Slobodeckij spaces and associated norms, the irregularity exponent \(n + \{\lambda\}p - 1\) is due to the fact that the dimension of \(\partial \Omega\) is \(n - 1\) and \(d\sigma\) is a hypersurface measure on \(\partial \Omega\).

Slobodeckij spaces as subspaces of Sobolev spaces but with fractional exponents are analogues of the Hölder spaces in classical spaces of continuous functions.

3. SOME PROPERTIES AND RELATIONS BETWEEN \(D, \zeta, \tau\) AND \(\xi_{\partial \Omega}\)

**Proposition 1.** \(D : W^{k,p}(\Omega, C_{\partial \Omega}) \to W^{k-1,p}(\Omega, C_{\partial \Omega})\) is continuous with

\[ \|Df\|_{W^{k-1,p}(\Omega, C_{\partial \Omega})} \leq \gamma \|f\|_{W^{k,p}(\Omega, C_{\partial \Omega})} \]

for \(\gamma = \gamma (n, p, \Omega)\) a positive constant.

**Proof.** Let \(f \in W^{k,p}(\Omega, C_{\partial \Omega})\). We need to show that

\[ \|Df\|_{W^{k-1,p}(\Omega, C_{\partial \Omega})} \leq c \|f\|_{W^{k,p}(\Omega, C_{\partial \Omega})} \]

for some positive constant \(c\).

\[ f \in W^{k,p}(\Omega, C_{\partial \Omega}) \implies \|f\|_{W^{k,p}(\Omega, C_{\partial \Omega})} = \left( \sum_{|\alpha| \leq k} \int_{\Omega} |D^\alpha f|^p dx \right)^{\frac{1}{p}} < \infty \]
But then
\[
\|Df\|_{W^{k-1,p}(\Omega,CL_n)} = \left( \sum_{\|\alpha\| \leq k-1} \int_{\Omega} |D^\alpha f|^p \, dx \right)^{\frac{1}{p}}
\]
\[
\leq \left( \sum_{\|\alpha\| \leq k-1} \int_{\Omega} |D^\alpha f|^p \, dx + \sum_{\|\alpha\| = k-1} \int_{\Omega} |D^\alpha f|^p \, dx \right)^{\frac{1}{p}}
\]
\[
= \left( \sum_{\|\alpha\| \leq k} |D^\alpha f|^p \, dx \right)^{\frac{1}{p}}
\]
\[
= \|f\|_{W^{k,p}(\Omega,CL_n)}
\]

Therefore for \( c = 1 \), the proposition is proved. \( \square \)

**Proposition 2.** \( D : L^p(\Omega) \rightarrow W^{k,p}(\Omega) \) is continuous for \( 1 < p < \infty \).

**Proof.** Let \( f \in L^p(\Omega) \). Then

\[
\|Df\|_{W^{k-1,p}(\Omega)} = \sup \{ \frac{|\langle Df, v \rangle|}{\|v\|_{W^{1,q}_0(\Omega)}} : v \neq 0, v \in W^{1,q}_0(\Omega) \}
\]

for \( p^{-1} + q^{-1} = 1 \).

But

\[
|\langle Df, v \rangle| = |\langle f, Dv \rangle| \leq \|f\|_{L^p(\Omega)} \|Dv\|_{L^q(\Omega)} \leq \|f\|_{L^p(\Omega)} \|v\|_{W^{1,q}_0(\Omega)}
\]

Thus by the Cauchy-Schwartz inequality we have

\[
\frac{|\langle Df, v \rangle|}{\|v\|_{W^{1,q}_0(\Omega)}} \leq \frac{\|f\|_{L^p(\Omega)} \|v\|_{W^{1,q}_0(\Omega)}}{\|v\|_{W^{1,q}_0(\Omega)}} = \|f\|_{L^p(\Omega)}
\]

Therefore

\[
\|Df\|_{W^{k-1,p}(\Omega)} = \sup \{ \frac{|\langle Df, v \rangle|}{\|v\|_{W^{1,q}_0(\Omega)}} : v \neq 0, v \in W^{1,q}_0(\Omega) \}
\]
\[
\leq \sup \{ \frac{\|f\|_{L^p(\Omega)} \|v\|_{W^{1,q}_0(\Omega)}}{\|v\|_{W^{1,q}_0(\Omega)}} : v \neq 0, v \in W^{1,q}_0(\Omega) \}
\]
\[
= \|f\|_{L^p(\Omega)}
\]

\( \square \)
Proposition 3. (Mapping properties) [4, 9]

Let \( k \in \mathbb{N} \cup \{0\} \) and \( 1 < p < \infty \). Then there are positive constants \( \beta = \beta(n, p, \Omega) \), \( \theta = \theta(n, p, \Omega) \) and \( \delta = \delta(n, p, \Omega) \) such that

\[
\zeta_{\partial \Omega} : W^{k,p}(\Omega, Cl_n) \to W^{k+1,p}(\Omega, Cl_n)
\]

with

\[
\|\zeta_{\partial \Omega}f\|_{W^{k+1,p}(\Omega, Cl_n)} \leq \beta\|f\|_{W^{k,p}(\Omega, Cl_n)}
\]

and

\[
\xi_{\partial \Omega} : W^{\lambda,p}(\partial \Omega, Cl_n) \to W^{\lambda+\frac{1}{p},p}(\Omega, Cl_n)
\]

with

\[
\|\xi_{\partial \Omega}f\|_{W^{\lambda+\frac{1}{p},p}(\Omega, Cl_n)} \leq \theta\|f\|_{W^{\lambda,p}(\partial \Omega, Cl_n)}
\]

is the trace operator with

\[
\sum_{\|\alpha\| \leq \lambda + \frac{1}{p}} \int_{\Omega} |D^\alpha f|^p dx + \sum_{\|\alpha\| = \lambda + \frac{1}{p}} \int_{\partial \Omega} \int_{\partial \Omega} |D^\alpha f(x) - D^\alpha f(y)|^p \frac{dxdy}{|x-y|^n+(\lambda+\frac{1}{p})^p} \leq \delta^p \left( \sum_{\|\alpha\| \leq \lambda} \int_{\partial \Omega} |D^\alpha f|^p dx + \sum_{\|\alpha\| = \lambda} \int_{\partial \Omega} \int_{\partial \Omega} |D^\alpha f(x) - D^\alpha f(y)|^p \frac{d\sigma_x d\sigma_y}{|x-y|^n-1+(\lambda+\frac{1}{p})^p} \right)
\]

Proposition 4. The composition \( \xi_{\partial \Omega} \circ \tau \) preserves regularity of a function in a Sobolev space.

Proof. Indeed, \( \tau \) makes a function to lose a regularity fractional exponent of \( \frac{1}{p} \) when taken along the boundary of the domain. But the boundary or Feuter integral \( \xi_{\partial \Omega} \) augments the regularity exponent of a function defined on the boundary by an exponent of \( \frac{1}{p} \).

Therefore the composition operator \( \xi_{\partial \Omega} \circ \tau \) preserves or fixes the regularity exponent of a function in a Sobolev space.

\[\square\]

Proposition 5. (Borel-Pompeiu)

Let \( f \in W^{k,p}(\Omega, Cl_n) \). Then

\[ f = \xi_{\partial \Omega} \tau f + \zeta_{\Omega} Df \]

Corollary 1. (i) If \( f \in W^{k,p}_{0} (\Omega, Cl_n) \), then

\[ f = \zeta_{\Omega} Df \]

That is \( D \) is a right inverse for \( \zeta_{\Omega} \) and \( \zeta_{\Omega} \) is a left inverse for \( D \) over traceless spaces.
(ii) If $f$ is monogenic function over $\Omega$, then
\[ f = \xi_{\partial \Omega} \tau f \]
Therefore monogenic functions are always Cauchy transforms of their traces over the boundary.

### 4. Elliptic First and Second Order BVPs

**Proposition 6.** Let $f \in W^{k-1,p}(\Omega, C^{n})$ for $k \geq 1$. Then the first order elliptic BVP:

\[
\begin{cases}
Du = f & \text{in } \Omega \\
\tau u = g & \text{on } \partial \Omega
\end{cases}
\]

has a solution $u \in W^{k,p}(\Omega, C^{n})$ given by
\[ u(x) = \xi_{\partial \Omega} g + \zeta_{\Omega} f \]

**Proof.** The proof follows from the Borel-Pompeiu relation. As to where exactly $u$ and $g$ belong, we make the argument: $f$ is in $W^{k-1,p}(\Omega, C^{n})$ and hence from the mapping property of $D$, we have $u$ to be a function in $W^{k,p}(\Omega, C^{n})$.

Also from the mapping property of the trace operator $\tau$ we have
\[ \tau u = u|_{\partial \Omega} = g \in W^{k-\frac{1}{p},p}(\partial \Omega, C^{n}) \]

\[ \Box \]

**Proposition 7.** The solution $u \in W^{k,p}(\Omega, C^{n})$ has a norm estimate:

\[
\|u\|_{W^{k,p}(\Omega, C^{n})} \leq \gamma_1 \left( \sum_{\|\alpha\| \leq k-1} \int_{\partial \Omega} |D^\alpha g|^p d\sigma x + \sum_{\|\alpha\| = k-1} \int_{\partial \Omega} \int_{\partial \Omega} \frac{|D^\alpha g(x) - D^\alpha g(y)|^p}{|x-y|^{n+p-2}} d\sigma x d\sigma y \right)^\frac{1}{p} + \gamma_2 \left( \sum_{\|\alpha\| = k-1} \int_{\partial \Omega} |f|^p dx \right)^\frac{1}{p}
\]

where $\gamma_1, \gamma_2$ are constants the depend on $p, n$ and $\Omega$.

**Proof.** First let us determine regularity exponents of $g \in W^{k-\frac{1}{p},p}(\partial \Omega, C^{n})$.

For the regularity index $k - \frac{1}{p}$ the integer part is
\[ \lfloor k - \frac{1}{p} \rfloor = k - 1 \]
and the fractional part is
\[
\{ k - \frac{1}{p} \} = 1 - \frac{1}{p}
\]
Besides \( \text{dim} (\partial \Omega) = n - 1 \). From the mapping properties of \( D, \zeta_{\partial \Omega}, \tau \) and \( \xi_{\partial \Omega} \), we have
\[
u \in W^{k,p}(\Omega, C_{1n})
\]
and
\[
\tau \nu = g \in W^{k-\frac{1}{p}p}(\partial \Omega, C_{1n})
\]
Therefore the solution \( \nu \) given by:
\[
\nu(x) = \xi_{\partial \Omega} g + \zeta_{\Omega} f
\]
has norm estimate
\[
\| \nu \|_{W^{k,p}(\Omega, C_{1n})} = \| \xi_{\partial \Omega} g + \zeta_{\Omega} f \|_{W^{k,p}(\Omega, C_{1n})}
\]
\[
\leq \| \xi_{\partial \Omega} g \|_{W^{k,p}(\Omega, C_{1n})} + \| \zeta_{\Omega} f \|_{W^{k,p}(\Omega, C_{1n})}
\]
\[
\leq \gamma_1 \| g \|_{W^{k-\frac{1}{p}p}(\partial \Omega, C_{1n})} + \gamma_2 \| f \|_{W^{k-\frac{1}{p}p}(\partial \Omega, C_{1n})}
\]
\[
= \gamma_1 \left( \sum_{|\alpha| \leq k-1} \int_{\partial \Omega} |D^\alpha g|^p d\sigma + \sum_{|\alpha| = k-1} \int_{\partial \Omega} \int_{\partial \Omega} \frac{|D^\alpha g(x) - D^\alpha g(y)|^p}{|x-y|^{n-1+(k-\frac{1}{p})p}} d\sigma_x d\sigma_y \right)^{\frac{1}{p}}
\]
\[
+ \gamma_2 \left( \sum_{|\alpha| = k-1} \int_{\partial \Omega} |f|^p d\sigma \right)^{\frac{1}{p}}
\]
\[
= \gamma_1 \left( \sum_{|\alpha| \leq k-1} \int_{\partial \Omega} |D^\alpha g|^p d\sigma + \sum_{|\alpha| = k-1} \int_{\partial \Omega} \int_{\partial \Omega} \frac{|D^\alpha g(x) - D^\alpha g(y)|^p}{|x-y|^{n+p-2}} d\sigma_x d\sigma_y \right)^{\frac{1}{p}}
\]
\[
+ \gamma_2 \left( \sum_{|\alpha| = k-1} \int_{\partial \Omega} |f|^p d\sigma \right)^{\frac{1}{p}}
\]
The constants \( \gamma_1 \) and \( \gamma_2 \) are from the mapping properties of \( \xi_{\partial \Omega}, \zeta_{\Omega} \) and \( \tau \). \( \square \)

**Proposition 8.** Let \( f \in W^{k,p}(\Omega, C_{1n}) \). Then the second order elliptic BVP

\[
(4.2) \begin{cases}
-\Delta u = f & \text{in } \Omega \\
\tau Du = g_1 & \text{on } \partial \Omega \\
\tau u = g_2 & \text{on } \partial \Omega
\end{cases}
\]

has a solution given by
\[
u = \xi_{\partial \Omega} (g_2) + \zeta_{\Omega} \xi_{\partial \Omega} (g_1) + \zeta_{\Omega} \circ \zeta_{\Omega} (f)
\]
in \( W^{k+2,p}(\Omega) \) with
\[
g_1 \in W^{k+1-\frac{1}{p}p} (\partial \Omega), \quad g_2 \in W^{k+2-\frac{1}{p}p} (\partial \Omega)
\]
Proof. As \( f \in W^{k,p}(\Omega, Cl_n) \), the solution \( u \) is in the Sobolev space \( W^{k+2,p}(\Omega) \) and hence

\[
\tau u = g_2 \in W^{k+2-\frac{1}{p},p}(\partial \Omega)
\]

But then \( Du \) is in \( W^{k+1,p}(\Omega) \) and hence

\[
\tau Du = g_1
\]

is in the Slobodeckij space \( W^{k+1-\frac{1}{p},p}(\partial \Omega) \).

The solution \( u \) of the BVP is obtained by repeated application of the Borel-Pompeiu formula by writing the Laplacian \( \Delta \) as \(-D^2\).

Now let us first determine the integer and fractional parts of indices \( k+2-\frac{1}{p} \) and \( k+1-\frac{1}{p} \) as

\[
[k+2-\frac{1}{p}] = k+1, \quad \{k+2-\frac{1}{p}\} = 1-\frac{1}{p}
\]
\[
[k+1-\frac{1}{p}] = k, \quad \{k+1-\frac{1}{p}\} = 1-\frac{1}{p}
\]

Therefore from the properties of the mappings studied above, we have a norm estimate of the solution \( u \) in \( W^{k+2,p}(\Omega) \) in terms of norms of \( f, g_1 \) and \( g_2 \) as follow:

\[
\|u\|_{W^{k+2,p}(\Omega)} = \|\xi_{\partial \Omega} (g_2) + \zeta_{\Omega} \xi_{\partial \Omega} (g_1) + \zeta_{\Omega} \circ \zeta_{\Omega} (f)\|_{W^{k+2,p}(\Omega)}
\]

\[
\leq \gamma_1 \left( \sum_{\|\alpha\| \leq k} \int_{\partial \Omega} |D^\alpha g_2|^p d\sigma_x + \sum_{\|\alpha\| = k+1} \int_{\partial \Omega} \int_{\partial \Omega} \frac{|D^\alpha g_2(x) - D^\alpha g_2(y)|^p}{|x-y|^{n+p-2}} d\sigma_x d\sigma_y \right)^{\frac{1}{p}}
\]

\[
+ \gamma_2 \left( \sum_{\|\alpha\| \leq k} \int_{\partial \Omega} |D^\alpha g_1|^p d\sigma_x + \sum_{\|\alpha\| = k} \int_{\partial \Omega} \int_{\partial \Omega} \frac{|D^\alpha g_1(x) - D^\alpha g_1(y)|^p}{|x-y|^{n+p-2}} d\sigma_x d\sigma_y \right)^{\frac{1}{p}}
\]

\[
+ \gamma_3 \left( \sum_{\|\alpha\| \leq k} \int_{\partial \Omega} |D^\alpha f|^p dx \right)^{\frac{1}{p}}
\]

for some positive constants \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) that depend on \( p, n, \Omega \) \( \square \)

**Proposition 9.** For the BVP \( \text{[4.1]} \) there exist positive constants \( c, \gamma_1 \) and \( \gamma_2 \) such that the solution \( u \in W^{k,2n}(\Omega) \) satisfies the norm estimate:
\[
c^{-1} \left( \sup_{x, y \in \Omega, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\lambda} + \|u\|_{C(\Omega)} \right) \\
\leq \gamma_1 \left( \sum_{\|\alpha\| \leq k-1} \int_{\partial \Omega} |D^\alpha g|^{2n} d\sigma x + \sum_{\|\alpha\| = k-1} \int_{\partial \Omega} \int_{\partial \Omega} \frac{|D^\alpha g(x) - D^\alpha g(y)|^{2n}}{|x - y|^{n+p-2}} d\sigma x d\sigma y \right)^{\frac{1}{2}} \\
+ \gamma_2 \left( \sum_{\|\alpha\| = k-1} \int_{\partial \Omega} |f|^{2n} d\sigma x \right)^{\frac{1}{2}} 
\]

**Proof.** From the Sobolev embedding theorems, if \( p > n \), then \( W^{k,p}(\Omega) \hookrightarrow C^{0,\lambda}(\Omega) \) for \( 0 < \lambda \leq 1 - \frac{n}{p} \).

But then for \( p = 2n \), we have \( 0 < \lambda \leq \frac{1}{2} \) and therefore the solution \( u \) which is in \( W^{k,2n}(\Omega) \) is contained in Hölder spaces \( C^{0,\lambda}(\Omega) \).

Thus \( \exists c = c(p,n,\Omega) > 0 \) such that

\[
c^{-1} \|u\|_{C^{0,\lambda}(\Omega)} \leq \|u\|_{W^{k,2n}(\Omega)} 
\]

That is

\[
c^{-1} \left( \sup_{x, y \in \Omega, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\lambda} + \|u\|_{C(\Omega)} \right) \\
\leq \|u\|_{W^{k,2n}(\Omega)} \\
\leq \gamma_1 \left( \sum_{\|\alpha\| \leq k-1} \int_{\partial \Omega} |D^\alpha g|^{2n} d\sigma x + \sum_{\|\alpha\| = k-1} \int_{\partial \Omega} \int_{\partial \Omega} \frac{|D^\alpha g(x) - D^\alpha g(y)|^{2n}}{|x - y|^{n+p-2}} d\sigma x d\sigma y \right)^{\frac{1}{2}} \\
+ \gamma_2 \left( \sum_{\|\alpha\| = k-1} \int_{\partial \Omega} |f|^{2n} d\sigma x \right)^{\frac{1}{2}} 
\]

Choosing \( \lambda = \frac{1}{2} \), we have the required result. \( \square \)

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