ON SUMS OF POWERS OF ALMOST EQUAL PRIMES

ANGEL KUMCHEV AND HUAFENG LIU

Abstract. Let $k \geq 2$ and $s$ be positive integers, and let $n$ be a large positive integer subject to certain local conditions. We prove that if $s \geq k^2 + k + 1$ and $\theta > 31/40$, then $n$ can be expressed as a sum $p_1^k + \cdots + p_s^k$, where $p_1, \ldots, p_s$ are primes with $|p_j - (n/s)^{1/k}| \leq n^{\theta/k}$. This improves on earlier work by Wei and Wooley \cite{1} and by Huang \cite{8} who proved similar theorems when $\theta > 19/24$.

1. Introduction

The study of additive representations of integers as sums of powers of primes goes back to the work of Hua \cite{6,7}. In particular, Hua proved that when $k$ and $s$ are positive integers with $s > 2^k$, every sufficiently large natural number $n$ satisfying certain local solubility conditions can be represented as

$$n = p_1^k + \cdots + p_s^k,$$

where $p_1, \ldots, p_s$ are prime numbers. (Henceforth, the letter $p$, with or without subscripts, always denotes a prime number.) To describe the local conditions, we let $\tau = \tau(k, p)$ be the largest integer with $p^\tau \mid k$, and then define

$$K(k) = \prod_{(p-1)/k} p^{\gamma(k,p)}, \quad \gamma(k,p) = \begin{cases} \tau(k,p) + 2 & \text{when } p = 2, \tau > 0, \\ \tau(k,p) + 1 & \text{otherwise.} \end{cases}$$

One typically studies \((\ref{1.1})\) for $n$ restricted to the congruence class \(H_{k,s} = \{n \in \mathbb{N} : n \equiv s \pmod{K(k)}\}\).

In this paper, we are interested in the additive representations of the form \((\ref{1.1})\) with “almost equal” primes. Given a large integer $n \in H_{k,s}$, we ask whether it is possible to solve \((\ref{1.1})\) in primes subject to

$$|p_j - (n/s)^{1/k}| \leq H \quad (1 \leq j \leq s),$$

where $H = o(n^{1/k})$. There is a long list of results on sums of five or fewer almost equal squares ($k = 2$, $3 \leq s \leq 5$), beginning with the work of Liu and Zhan \cite{17} and culminating with the results of Kumchev and Li \cite{10} (see \cite{10} for a detailed history of that problem). In particular, Kumchev and Li showed that when $k = 2$ and $s = 5$ the problem has solutions with $H = n^{\theta/2}$ for any fixed $\theta > 8/9$. They were also the first to obtain results on sums of more than five almost equal squares, where the extra variables are used to reduce the admissible size of $H$. Let $\theta_{k,s}$ denote the least exponent $\theta$ such that \((\ref{1.1})\) and \((\ref{1.2})\) with $H = n^{\theta/k}$ can be solved for sufficiently large $n \in H_{k,s}$ whenever $\theta > \theta_{k,s}$. Kumchev and Li \cite{10} proved that $\theta_{2,5} \leq 19/24$ when $s \geq 17$. The lower bound on $s$ in this theorem was reduced to $s \geq 6$ in a recent paper by Wei and Wooley \cite{15}, in which those authors also established surprisingly strong results for higher values of $k$: they proved that if $s > 2k(k-1)$, one has

$$\theta_{k,s} \leq \begin{cases} 4/5 & \text{if } k = 3, \\ 5/6 & \text{if } k \geq 4. \end{cases}$$

(\ref{1.3})

Huang \cite{8} further reduced the bound \((\ref{1.3})\) to $\theta_{k,s} \leq 19/24$ for all $k \geq 3$ and $s > 2k(k-1)$.

The main goal of the present work is to establish the bound $\theta_{k,s} \leq 31/40$ for all $k \geq 2$. We also make use of a recent breakthrough by Bourgain, Demeter and Guth \cite{2} to reduce the lower bound on $s$ when $k \geq 4$. Our main result is as follows.

Date: March 28, 2018.

2010 Mathematics Subject Classification. 11P32; 11L20, 11N36, 11P05.
Theorem 1. Let \( k \geq 2, s \geq k^2 + k + 1, \) and \( \theta > 31/40. \) When \( n \in \mathcal{H}_{k,s} \) is sufficiently large, equation (1.1) has solutions in primes \( p_1, \ldots, p_s \) satisfying (1.2) with \( H = n^{\theta/k}. \)

Circle method experts will not be surprised that our methods lead also to improvements on the results established by Wei and Wooley [15] and by Huang [8] on solubility for “almost all” \( n \) and on the number of exceptions for representations by six almost equal squares. Indeed, by adapting the ideas in [15, §9], we obtain the following theorems.

Theorem 2. Let \( k \geq 2, s > k(k + 1)/2, \theta > 31/40, \) and \( N \to \infty. \) There is a fixed \( \delta > 0 \) such that equation (1.1) has solutions in primes \( p_1, \ldots, p_s \) satisfying (1.2) with \( H = n^{\theta/k} \) for all but \( O(N^{1-\delta}) \) integers \( n \leq N \) subject to \( n \in \mathcal{H}_{k,s} \) (and, when \( k = 3 \) and \( s = 7, \) also \( 9 \nmid n \)).

Theorem 3. Let \( \theta > 31/40, \) and \( N \to \infty. \) Let \( E_6(N; H) \) denote the number of integers \( n \equiv 6 \) (mod 24), with \( |n - N| \leq H N^{1/2}, \) such that equation (1.1) with \( k = 2 \) and \( s = 6 \) has solutions in primes \( p_1, \ldots, p_6 \) satisfying (1.2). There is a fixed \( \delta > 0 \) such that

\[
E_6(N; N^{\theta/2}) \ll N^{(1-\theta)/2-\delta}.
\]

Notation. Throughout the paper, the letter \( \epsilon \) denotes a sufficiently small positive real number. Any statement in which \( \epsilon \) occurs holds for each positive \( \epsilon, \) and any implied constant in such a statement is allowed to depend on \( \epsilon. \) The letter \( c \) denotes a constant that depends at most on \( k \) and \( s, \) but not necessarily the same in all occurrences. As usual in number theory, \( \mu(n), \Lambda(n), \phi(n), \) and \( \tau(n) \) denote, respectively, the Möbius function, von Mangoldt’s function, Euler’s totient function, and the number of divisors function. We write \( e(x) = \exp(2\pi ix) \) and \( (a, b) = \gcd(a, b) \), and we use \( m \sim M \) as an abbreviation for the condition \( M \leq m < 2M. \) If \( \chi \) denotes a Dirichlet character, we set \( \delta_{\chi} = 1 \) or 0 according as \( \chi \) is principal or not. The sums \( \sum_{\chi \mod q} \) and \( \sum_{\chi} \mod q \) denote summations over all the characters modulo \( q \) and over the primitive characters modulo \( q, \) respectively.

2. OUTLINE OF THE PROOF

Let \( x = (n/s)^{1/k}, y = x^\theta, I = (x - y, x + y), \) and write

\[
R_{k,s}(n) = \sum_{\substack{n = p_1 + \cdots + p_s \\text{ \( p_i \in I \)}}} 1.
\]

Let \( 1_p \) denote the indicator function of the primes, and suppose that we have arithmetic functions \( \lambda^\pm \) such that, for \( m \in I, \)

\[
\lambda^- (m) \leq 1_p(m) \leq \lambda^+ (m).
\]

Then the vector sieve of Brüdern and Fouvry [3, Lemma 13] yields

\[
1_p(m_1) \cdots 1_p(m_5) \geq \sum_{i=1}^{5} \lambda^-(m_i) \prod_{j \neq i} \lambda^+(m_j) - 4 \lambda^+(m_1) \cdots \lambda^+(m_5).
\]

Thus, by the symmetry of the problem, we have

\[
R_{k,s}(n) \geq 5 R_{k,s}(n, \lambda^-) - 4 R_{k,s}(n, \lambda^+), \tag{2.3}
\]

where

\[
R_{k,s}(n, \lambda) = \sum_{\substack{n = p_1 + \cdots + p_s \\text{ \( p_i \in I \)}}} \lambda(m_1) \lambda^+(m_2) \cdots \lambda^+(m_5).
\]

To prove the theorem, we show that one can choose sieve functions \( \lambda^\pm \) satisfying (2.1) so that the right side of (2.3) is positive. Our choice of \( \lambda^\pm \) is borrowed from Baker, Harman and Pintz [11]—namely, \( \lambda^- \) and \( \lambda^+ \) are, respectively, the functions \( a_0 \) and \( a_1 \) constructed in §4 of that paper. In many ways, the functions \( \lambda^\pm \) imitate the indicator function \( 1_p \) of the primes \( p \in I. \) We will discuss the similarities in detail later (see [3] below) and will focus here on their most crucial property:
(A0) Let $A, B > 0$ be fixed (possibly large) numbers and let $x \to \infty$. If $\chi$ is a Dirichlet character modulo $q \leq (\log x)^B$ and $x^{11/20+\epsilon} \leq y \leq x \exp\left(-\log x^{1/3}\right)$, then one has

$$\sum_{|m-x|<y} \lambda^\pm(m)\chi(m) = \frac{2y}{\phi(q)\log x} \left(\delta_\chi \kappa^\pm + O\left((\log x)^{-A}\right)\right),$$

(2.4)

where $\kappa^\pm$ are absolute constants satisfying

$$\kappa^- > 0.99, \quad \kappa^+ < 1.01.$$  

(2.5)

We now sketch the application of the circle method to $R_{k,s}(n, \lambda)$. Let $\delta > 0$ be a fixed number, to be chosen later sufficiently small in terms of $k, s$ and $\theta$, and set

$$P = y^\delta, \quad Q = x^{k-2}y^2P^{-1}, \quad L = \log x.$$  

We write

$$M(q, a) = \{\alpha \in \mathbb{R} : |q\alpha - a| \leq Q^{-1}\},$$

and define the sets of major and minor arcs by

$$M = \bigcup_{1 \leq a \leq q \leq P \atop (a, q) = 1} M(q, a) \quad \text{and} \quad m = \left[Q^{-1}, 1 + Q^{-1}\right] \backslash M,$$

(2.7)

respectively. Further, for any Lebesgue measurable set $\mathcal{B}$, we write

$$R_{k,s}(n, \lambda; \mathcal{B}) = \int_{\mathcal{B}} f(\alpha, 1_p) s^{-5} f(\alpha, \lambda)f(\alpha, \lambda^+)^4 e(-n\alpha)\,d\alpha,$$

where

$$f(\alpha, \lambda) = \sum_{m \in \mathcal{L}} \lambda(m)e(mk\alpha).$$

(2.8)

By orthogonality and (2.7), we have

$$R_{k,s}(n, \lambda, \mathcal{B}) = R_{k,s}(n, \lambda; M) + R_{k,s}(n, \lambda; m).$$

(2.9)

In [4] we show that when $s \geq k^2 + k + 1$, $\delta \leq 1/(16k)$, and $\theta \leq 31/40$, one has

$$R_{k,s}(n, \lambda; m) \ll y^{s-1-\delta/3k}x^{1-k}.$$  

(2.10)

Then, in [5] we show that when $\delta \leq 2(\theta - 31/40)$, one has

$$R_{k,s}(n, \lambda^\pm; M) = \mathcal{C}(n)y^{s-1}x^{1-k}L^{-s}(\kappa^\pm \kappa^4 + O(L^{-1})),$$

(2.11)

where $1 \ll \mathcal{C}(n) \ll 1$ for sufficiently large $n \in \mathcal{H}_{k,s}$, and $\kappa^\pm$ are the constants from (2.4). Theorem [1] follows from (2.3), (2.5), and (2.9)–(2.11).

3. THE SIEVE WEIGHTS

As we said before, we use sieve weights $\lambda^\pm$ constructed by Baker, Harman and Pintz [1] to have properties [2] and (A0) above. We remark that (A0) is a short-interval version of the Siegel–Walfisz theorem: when the functions $\lambda^\pm$ are replaced by $1_p$, the asymptotic formula (2.4) with $\kappa = 1$ and $y \geq x^{7/12+\epsilon}$ is a well-known extension of a celebrated result of Huxley [2]. In this section, we record some additional properties of the weights $\lambda^\pm$ that we will need later in the paper:

(A1) The functions $\lambda^\pm(m)$ vanish if $m$ has a prime divisor $p < x^{1/10}$.

(A2) Let $\mathcal{S} = \{p^j : p \in \mathbb{P}, j \geq 2\}$. When $m \sim 2x/3$, one can express $\lambda^\pm(m)$ as a linear combination of a bounded function supported on $\mathcal{S}$ and of $O(L^c)$ triple convolutions of the form

$$\sum_{m \sim U, v \sim V} \xi_u \eta_v \zeta_w,$$

where $|\xi_u| \leq \tau(u)^c$, $|\eta_v| \leq \tau(v)^c$, $\max(U, V) < x^{11/20}$, and either $\zeta_w = 1$ for all $w$, or $|\zeta_w| \leq \tau(w)^c$ and $UV \gg x^{27/35}$. 

3
(A3) Let $A, B, \epsilon > 0$ be fixed, let $\chi$ be a Dirichlet character modulo $q \leq L^B$, and put $T_0 = \exp(L^{1/3})$ and $T_1 = x^{9/20-\epsilon}$. Then

$$\int_{T_0}^{T_1} \left| \sum_{m \sim 2\pi/3} \lambda^\pm(m) \chi(m) m^{-1/2-it} \right| dt \ll x^{1/2} L^{-A}.$$

Of the three properties above, (A3) is the easiest to justify, since it is a part of the proof of (A0) in [1]. Indeed, the method of Baker, Harman and Pintz reduces (2.3) to the classical Siegel–Walfisz theorem by decomposing $\lambda^\pm$ into a linear combination of $O(L^c)$ arithemetic functions for which (A3) holds and then applying [1] Lemma 11 to each of them. In order to justify that the functions $\lambda^\pm$ have also properties (A1) and (A2), we need to provide some details on their construction.

The core idea behind the construction of $\lambda^\pm$ is explained in [1] pages 32–33, 41–42. It amounts to setting

$$\lambda^\pm(m) = 1_p(m) \pm \sum_{j=1}^{J^\pm} \lambda_j^\pm(m) \quad (3.1)$$

where $J^\pm = O(1)$ and the arithmetic functions $\lambda_j^\pm$ have the form

$$\lambda_j^\pm(m) = \sum_{\alpha = u_1, \ldots, u_{d+1}} \xi(u_1, \ldots, u_{d+1}) \quad (4 \leq d \leq 7),$$

with $\xi(u_1, \ldots, u_{d+1}) = 1$ or 0. The latter functions impose various restrictions on the sizes and arithmetic properties of $u_1, \ldots, u_{d+1}$ that amount to restricting the support of $\lambda_j^\pm$ to integers $m$ with very specific (undesirable) factorizations. Moreover:

(i) Only the cases $d = 4$ and $d = 6$ occur in the construction of $\lambda^-$, whereas only $d = 5$ and $d = 7$ occur in the construction of $\lambda^+$.

(ii) $\xi(u_1, \ldots, u_{d+1}) = 0$ if any of $u_1, \ldots, u_{d+1}$ has a prime divisor $< x^{1/10}$. Note that property (A1) is an immediate consequence of this observation.

(iii) When $d = 5$, $\lambda_j^\pm$ is supported on integers $m$ that have a divisor $u$ in the range $x^{0.46} \leq u \leq x^{1/2}$: see [1] p. 42).

(iv) When $d = 4$, $\lambda_j^-$ is supported on integers $m = n_1 n_2 n_3$, where $n_i = x^{\alpha_i}$ with $\alpha = (\alpha_1, \alpha_2)$ lying in one of the regions $\Gamma$, $\Delta_2, \Delta_3$, or $\Delta_4$ in [1] Diagram 1 on p. 33].

We now turn to property (A2). We note that when $\lambda_j^\pm$ is supported on integers $m = uv$, with $x^{9/20} \leq u \leq x^{11/20}$, it has property (A2). Thus, by (iii) above, property (A2) holds for all terms $\lambda_j^\pm$ with $d = 5$. Moreover, the same is true for $\lambda_j^-$ with $d = 4$ and $\alpha$ in one of the regions $\Delta_3$ or $\Delta_4$: we have $0.46 \leq \alpha_1 \leq 0.5$ when $\alpha \in \Delta_4$, and $0.46 \leq \alpha_1 + \alpha_2 \leq 0.54$ when $\alpha \in \Delta_3$.

We next consider the case $d \geq 6$ and suppose that the variables $u_i$ have been labelled so that $u_1 \geq u_2 \geq \cdots \geq u_{d+1}$. When $\lambda_j^\pm$ is supported on integers $m = u_1 \cdots u_{d+1}$ with $u_4 \cdots u_{d+1} \geq x^{11/20}$, we have

$$u_1 u_2 u_3 \ll x^{9/20} \quad \text{and} \quad u_4 \ll \sqrt[3]{u_1 u_2 u_3} \ll x^{3/20}.$$
we can verify that $\lambda_j^-$ has property (A2) by grouping the variables $u_1, \ldots, u_5$ into $u = u_1u_2u_3$, $v = u_4$, and $w = u_5$. Similarly, the functions $\lambda_j^-$ with $d = 4$ and $\alpha \in \Gamma$ are supported on integers $m = u_1 \cdots u_5$, where
\[
x^{0.32} \ll u_1u_2, u_3u_4 \ll x^{0.36}, \quad \text{and} \quad u_5 \ll x^{1/3}.
\]
(In this case, we have $u_1u_2 = x^{\alpha_1}$ and $u_5 = x^{\alpha_2}$.) If we assume that the variables are labelled so that $u_1 \leq u_2$ and $u_3 \leq u_4$, we have
\[
u_2u_4 \ll x^{0.72}/(u_1u_3) \ll x^{0.52}, \quad u_1u_5 \ll x^{0.18}x^{1/3} < x^{0.52}, \quad u_3 \ll x^{0.18}.
\]
Hence, we can once again verify that $\lambda_j^-$ has property (A2) by grouping the variables $u_1, \ldots, u_5$ into $u = u_2u_4$, $v = u_1u_3$, and $w = u_5$.

We have shown that each term $\lambda_j^\pm$ on the right side of (4.1) satisfies (A2). It remains to show that so does the indicator function $1_{\mathbb{P}}$. The proof of [1, Theorem 1] uses Heath-Brown’s identity to establish (A2) for von Mangoldt’s function. In the case of $1_{\mathbb{P}}$, we can use a variant of that argument based on Linnik’s identity instead of Heath-Brown’s.

### 4. The Minor Arcs

In this section, we establish inequality (2.10). Our main tools are Propositions 1 and 2 below.

**Proposition 1.** Suppose that $k \geq 2$, $s \geq k^2 + k$, and $y \geq x^{1/2}$. Then for any bounded arithmetic function $\lambda$, one has
\[
I_s(\lambda) := \int_0^1 |f(\alpha, \lambda)|^s \, d\alpha \ll y^{s-1}x^{1-k+\epsilon}.
\]  

**Proposition 2.** Let $k \geq 2$, $0 < \delta < 1/(16k)$, and $y \geq x^{31/40}$, and suppose that $\alpha \in \mathfrak{m}$. Then
\[
f(\alpha, 1_{\mathbb{P}}) \ll y^{1-\delta/(2k)+\epsilon}.
\]

It is straightforward to deduce (2.10) from these propositions. First, we remark that the functions $\lambda^\pm$ are bounded by construction—they are linear combinations of a bounded number of indicator functions. Thus, we may apply Proposition 1 to $\lambda = \lambda^\pm$. By Hölder’s inequality,
\[
|R_{k,s}(n, \lambda; \mathfrak{m})| \leq \left( \sup_{\alpha \in \mathfrak{m}} |f(\alpha, 1_{\mathbb{P}})| \right) I_{s-1}(\lambda)^uI_{s-1}(\lambda^+)^uI_{s-1}(1_{\mathbb{P}})^{1-5u},
\]
where $u = (s-1)^{-1}$. Thus, when $s \geq k^2 + k + 1$, we may use Propositions 1 and 2 to get
\[
R_{k,s}(n, \lambda; \mathfrak{m}) \ll y^{1-\delta/(2k)+\epsilon} y^{s-2}x^{1-k+\epsilon} \ll y^{s-1-\delta/(3k)}x^{1-k},
\]
provided that $\delta$ and $y$ satisfy the hypotheses of Proposition 2 and $\epsilon$ is chosen sufficiently small; this verifies (2.10). In the remainder of this section, we prove the propositions.

#### 4.1. Proof of Proposition 1

This is a variant of [15, Proposition 2.2], which we have extended in two ways. First, we have included the arbitrary coefficients $\lambda$. This is straightforward, due to the “maximal inequality”
\[
\int_0^1 |f(\alpha, \lambda)|^s \, d\alpha \ll y^{s-k^2-k} \int_0^1 |f(\alpha, 1)|^{k^2+k} \, d\alpha,
\]  

where $1$ is the constant function $1(n) = 1$ (compare this to [15, p. 1136]). Like Wei and Wooley, we estimate the right side of (4.2) by means of [5, Theorem 3] and standard bounds for Vinogradov’s mean-value integral. In particular, the recent work of Bourgain, Demeter and Guth [2] allows us to reduce the lower bound on $s$ to the one stated above. \[\square\]
4.2. Proof of Proposition [2] Although it looks somewhat different, Proposition [2] is merely a slight variation of the main theorem of Huang [8], and our proof follows closely Huang’s. We first obtain variants of some technical estimates from [8] by making some slight changes to Huang’s arguments.

**Lemma 1.** Let \( k \geq 2 \) be an integer and \( \rho \) be real, with \( 0 < \rho \leq t_k^{-1} \), where
\[
t_k = \begin{cases} 
2 & \text{if } k = 2, \\
k^2 - k + 1 & \text{if } k \geq 3.
\end{cases}
\]
Suppose also that \( y = x^\theta \), where
\[
\frac{1}{2 - t_k \rho} \leq \theta \leq 1.
\]
Then either
\[
\sum_{x < m \leq x + y} e(mk \alpha) \ll y^{1-\rho+\epsilon},
\]
or there exist integers \( a, q \) such that
\[
1 \leq q \leq y^{k^2}, \quad (a, q) = 1, \quad |qa - a| \leq x^{1-k} y^{k^2-1},
\]
and
\[
\sum_{x < m \leq x + y} e(mk \alpha) \ll y^{1-\rho+\epsilon} + \frac{y}{(q + yx^{k-1}|qa - a|)^{1/k}}.
\]

**Proof.** When \( k \geq 3 \), we follow the argument of Huang [8, Lemma 1] with \( \gamma = \rho^{-1}(t_k - 1) \). Within that argument, we apply the latest version of Vinogradov’s mean-value theorem due to Bourgain, Demeter and Guth [2] in place of the earlier version by Wooley [16] used by Huang. When \( k = 2 \), we follow the same argument with \( \gamma = (2\rho)^{-1} \) but observe that in this case the bound at the top of [8, p. 512] can be improved to
\[
\Delta \ll q^{1/2+\epsilon}(1 + x^2(qQ_0)^{-1})^{1/2} \ll P_0^{1/2+\epsilon} xy^{-1}.
\]
This slight improvement is possible, because in the quadratic case, Daemen’s proof of [3] (3.5) does not require the iterative process in [3] p. 78. Thus, we need not incur a loss of a factor of \( q^{-1/2} \) in the above bound which the iterative method causes when \( k \geq 3 \). \( \square \)

**Lemma 2** (Type II sum). Let \( k \geq 2 \) be an integer, let \( \rho \) be real, with \( 0 < \rho \leq \min \left\{ (4t_k)^{-1}, \frac{1}{20} \right\} \), and suppose that \( y = x^\theta \), where
\[
\frac{3}{4 - 4t_k \rho} \leq \theta \leq 1. \quad (4.3)
\]
Suppose also that \( \alpha \in \mathfrak{m} \) and that the coefficients \( \xi_u, \eta_v \) satisfy \( \xi_u \ll \tau(u)^c \) and \( \eta_v \ll \tau(v)^c \). Then
\[
\sum_{u \ll U} \sum_{u \in \mathbb{Z}} \xi_u \eta_v e(u^k v^k \alpha) \ll y^{1-\rho+\epsilon} + y^{1+\epsilon} P^{-1/(2k)},
\]
provided that
\[
xy^{-1+2\rho} \ll U \ll y^{-1-2\rho}. \quad (4.4)
\]

**Proof.** This is a version of [8, Proposition 2] that applies Lemma [1] above in place of [8, Lemma 1]. We have also altered slightly the choice of \( \nu \) in Huang’s argument by choosing it so that \( Y^\nu = y^{2\rho} L^{-1} \) as opposed to \( Y = x^{2\rho} L^{-1} \) (see [8, p. 515]). \( \square \)

**Lemma 3** (Type I sum). Let \( k \geq 2 \) be an integer, let \( \rho \) be real, with \( 0 < \rho \leq \min \left\{ (4t_k)^{-1}, \frac{1}{20} \right\} \), and suppose that \( y = x^\theta \), with \( \theta \) in the range \( [\frac{1}{2}, 1] \). Suppose also that \( \alpha \in \mathfrak{m} \) and that the coefficients \( \xi_u \) satisfy \( \xi_u \ll \tau(u)^c \). Then
\[
\sum_{u \ll U} \sum_{u \in \mathbb{Z}} \xi_u e(u^k \nu^k \alpha) \ll y^{1-\rho+\epsilon} + y^{1+\epsilon} P^{-1/(2k)},
\]
provided that
\[
U \ll y^{-1-2\rho}. \quad (4.5)
\]
Proof. This is a version of [8 Proposition 1]. Following the proof of that result, with our Lemma [1] in place of [8 Lemma 1] and with \( \nu \) chosen so that \( Y^\nu = y^{\alpha L^{-1}} \), one obtains the above bound when

\[
U \ll x^{-1} y^{2-\kappa \rho}, \quad U^{2k} \ll x^{k-1} y^{1-2k \rho}.
\]

On the other hand, when either of these inequalities fails, one has \( U \gg xy^{-1+2\rho} \) and the result follows from Lemma [2].

\[ \square \]

Proof of Proposition [3]. It suffices to bound \( f(\alpha, \Lambda) \), where \( \Lambda \) is von Mangoldt’s function. Let \( \rho = (31t_k)^{-1} \) and \( X = xy^{-1+2\rho} \). We note that this choice of \( \rho \) ensures that (4.3) holds for all \( \theta \geq 31/40 \) and that \( X \leq x^{9/40+(31\rho)/20} \leq x^{1/4} \). We may thus apply Vaughan’s identity for \( \Lambda \) (see [14, p. 28]) to decompose \( f(\alpha, \Lambda) \) into \( O(L) \) type I sums with \( U \leq X^2 \) and \( O(L) \) type II sums with \( X \leq U \leq xX^{-1} \). By the choice of \( X \) and \( \rho \), Lemma [2] can be applied to the arising type II sums. Moreover, since \( X^2 \leq xX^{-1} = y^{-1+2\rho} \), Lemma [3] can be applied to the type I sums. We conclude that when \( \alpha \in \mathfrak{M} \), one has

\[
f(\alpha, \Lambda) \ll y^{1-\rho + \epsilon} + y^{1-\delta/(2k) + \epsilon}.
\]

Since the hypothesis \( \delta < 1/(16k) \) ensures that \( \delta/(2k) < \rho \), this completes the proof. \[ \square \]

5. THE MAJOR ARCS

In this section, we establish [2,11]. First, we need to introduce some notation. We write

\[
S(q,a) = \sum_{1 \leq h \leq q} e(ah^k/q), \quad v(\beta; s) = \int_I u^{s-1} e(u^k \beta) \, du,
\]

and define the singular series \( \mathfrak{S}(n) \) and the singular integral \( \mathfrak{I}(n) \) by

\[
\mathfrak{S}(n) = \sum_{q=1}^\infty \phi(q)^{s-1} \sum_{1 \leq a \leq q} S(q,a)^s e(-an/q), \quad \mathfrak{I}(n) = \int_{\mathbb{R}} v(\beta; 1)^s e(-n \beta) \, d\beta.
\]

If \( \lambda \) denotes one of the functions \( \lambda^\pm \) and \( \kappa \) the respective constant \( \kappa_\pm \), we define a function \( f^*(\alpha, \lambda) \) on the major arcs \( \mathfrak{M} \) by setting

\[
f^*(\alpha, \lambda) = \kappa \phi(q)^{-1} S(q,a) v(\beta; 1) L^{-1} \quad \text{if } \alpha \in \mathfrak{M}(q,a).
\]

This is the “major arc approximation” to \( f(\alpha, \lambda) \). We also define a major arc approximation to \( f(\alpha, 1^\pm) \) by

\[
f^*(\alpha) = \phi(q)^{-1} S(q,a) v(\beta; 1) L^{-1} \quad \text{if } \alpha \in \mathfrak{M}(q,a).
\]

Finally, we adopt the convention that for any arithmetic function \( \lambda \), there is an associated Dirichlet polynomial \( F(s, \lambda) \), given by

\[
F(s, \lambda) = \sum_{m \sim 2x^{1/3}} \lambda(m) m^{-s}.
\]

5.1. SOME TECHNICAL ESTIMATES.

Lemma 4. Let \( x^{1/20} \leq y \leq x \) and suppose that \( P, Q \) satisfy

\[
PQ \leq y x^{k-1}, \quad Q \geq x^{k-9/20}.
\]

Suppose also that \( g \) is a positive integer, \( \nu \geq 1 \), and \( \lambda \) is a bounded arithmetic function satisfying hypothesis (A2) above. Then

\[
\sum_{r \leq P} [g, r]^{-\nu} \sum_{x \mod r} \left( \int_{1/(rQ)}^{1/(rQ)} |f(\beta, \lambda \chi)|^2 d\beta \right)^{1/2} \ll g^{-\nu + \epsilon} y^{1/2} x^{(1-k)/2} L^c.
\]

(5.1)

Proof. When \( k = 2 \) and \( \nu = 1 - \epsilon \), this is [10] Lemma 4.5. The proof for general \( k \geq 2 \) and \( \nu \geq 1 \) uses the same argument with some obvious changes: e.g., \( T_1 = \Delta x^k \) and \( H \ll \Delta^{-1} x^{-k} \) in place of the respective statements in [10] p. 618. \[ \square \]
Letting $b$ and the trivial estimate by the first-derivative test for exponential integrals (see [13, Lemma 4.5]). Combining this bound with (5.7) if we change $u$ to $u_1$, where $|u_1 - u| < 1/2$, the left side will change by $O(1)$ and the integral on the right side will change by $O(T)$. Hence, the integral representation (5.2) can be extended to all $u \in I$. The conclusion of the lemma then follows by partial summation.

**Lemma 5.** Let $x$ be a large integer, and suppose that $y, b, T$ are reals with: $y = o(x), \|y\| = 1/2, 0 < b \leq 1,$ and $1 \leq T \leq x^{1/2}$. Suppose also that $\lambda$ is a bounded arithmetic function. Then

$$f(\beta, \lambda) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} F(s, \lambda) v(\beta; s) ds + O((1 + yx^{b-1}/|\beta|)xLT^{-1}).$$

**Proof.** For any $u \in I$ with $\|u\| = 1/2$, Perron’s formula (see [12 Corollary 5.3]) gives

$$\sum_{x-y < m \leq u} \lambda(m) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} F(s, \lambda) \frac{u^s - (x-y)^s}{s} ds + O(xLT^{-1}). \tag{5.2}$$

If we change $u$ in (5.2) to $u_1$, where $|u_1 - u| < 1/2$, the left side will change by $O(1)$ and the integral on the right side will change by $O(T)$. Hence, the integral representation (5.2) can be extended to all $u \in I$. The conclusion of the lemma then follows by partial summation.

**Lemma 6.** Under the assumptions of Lemma 5, we have

$$\sum_{r \leq P} [g, r]^{-\nu} \sum_{\chi \pmod{r}} \max_{|\beta| < 1/(rQ)} |f(\beta, \lambda)| \ll g^{-\nu + \epsilon} yL^\epsilon. \tag{5.3}$$

Furthermore, for any given $A > 0$, there is a $B = B(A, \nu) > 0$ such that

$$\sum_{r \leq P} r^{-\nu} \sum_{\chi \pmod{r}} \max_{|\beta| < 1/(rQ)} |f(\beta, \lambda)| \ll yL^{-A}. \tag{5.4}$$

**Proof.** Let $1 \leq R_0 \leq P$. By a simple splitting argument,

$$\sum_{0 \leq r \leq P} [g, r]^{-\nu} \sum_{\chi \pmod{r}} \max_{|\beta| < 1/(rQ)} |f(\beta, \lambda)| \ll (gR)^{-\nu} L \sum_{d|g} d^{\nu} S(R, d), \tag{5.5}$$

where $R_0 \leq R \leq P$ and

$$S(R, d) = \sum_{d|g} d^{\nu} S(R, d),$$

where $S(R, d)$ is merely the linear combination of triple convolutions of the kind described in (A2). We may also assume that $x \in \mathbb{Z}$ and $\|y\| = 1/2$.

Let $0 < b \leq 1, |\beta| \leq (RQ)^{-1}, T_3 = 3k\pi x^{b}Q^{-1},$ and the integral on the right side of $\|y\| = 1/2$. Thus, we may assume that $\lambda$ is merely the linear combination of triple convolutions of the kind described in (A2). We may also assume that $x \in \mathbb{Z}$ and $\|y\| = 1/2$.

Let $0 < b \leq 1, |\beta| \leq (RQ)^{-1}, T_3 = 3k\pi x^{b}Q^{-1},$ and $T_0 = T_1/\pi. Then, by Lemma 5 with $T = T_1$,

$$f(\beta, \lambda) = \frac{1}{2\pi i} \int_{b-it_1}^{b+iT_1} F(s, \lambda) v(\beta; s) ds + O(yR^{-1}L). \tag{5.6}$$

Letting $b \downarrow 0$ in (5.6), we obtain

$$f(\beta, \lambda) = \frac{1}{2\pi i} \int_{-T_1}^{T_1} F(it, \lambda) v(\beta; it) dt + O(yR^{-1}L). \tag{5.7}$$

When $|\beta| \leq (RQ)^{-1}$ and $|t| \geq T_0$, we have

$$v(\beta; it) \ll |t|^{-1},$$

by the first-derivative test for exponential integrals (see [13 Lemma 4.5]). Combining this bound with (5.7) and the trivial estimate $|v(\beta; it)| \ll yx^{-1},$ we find that

$$f(\beta, \lambda) \ll yx^{-1} \int_{-T_0}^{T_0} |F(it, \lambda)| dt + \int_{T_0 \leq |t| < T_1} |F(it, \lambda)| \frac{dt}{|t|} + yR^{-1}L.$$
where

$$S_1(R, d; T) = \sum_{r \sim R} \sum_{x \bmod r} \int_{-T}^T |F(it, \lambda x)| \, dt.$$ 

Since $\lambda$ is assumed to be a linear combination of convolutions of the type in (A2), we may apply [4, Theorem 2.1] to obtain the bound

$$S_1(R, d; T) \ll (x + (R^2T/d)x^{11/20})L^c.$$

Combining this bound, (5.5) and (5.8), we conclude that the left side of (5.3) is

$$\ll g^{-\nu}y\left(1 + x^{k-9/20}Q^{-1} + x^{1-k}y^{-1}PQ + Px^{11/20}y^{-1}\right)L^c.$$

This establishes the first claim of the lemma.

When $g = 1$, the above argument yields the bound

$$\ll yR_0^{-\nu}(1 + x^{k-9/20}Q^{-1} + x^{1-k}y^{-1}PQ + Px^{11/20}y^{-1})L^c$$

for the left side of (5.4). When $R_0 = L^B$ for a sufficiently large $B > 0$, this establishes the second claim of the lemma. 

Lemma 7. Let $x^{11/20+2\epsilon} \leq y \leq x^{1-\epsilon}$ and suppose that $P, Q$ satisfy

$$PQ \leq yx^{k-1}, \quad Q \geq x^{k-9/20+2\epsilon}. \quad (5.9)$$

Suppose also that $\nu > 1$ and $\lambda$ is a bounded arithmetic function that satisfies hypotheses (A0), (A2) and (A3) above. Then, for any given $A > 0$,

$$\sum_{r \leq B} \sum_{\chi \bmod r} \max_{|\beta| \leq 1/(rQ)} |f(\beta, \lambda x) - \rho_xv(\beta; 1)| \ll yL^{-A}, \quad (5.10)$$

where $\rho_x = \delta_x\kappa L^{-1}$, $\kappa$ being the constant in hypothesis (A0) for $\lambda$.

Proof. By the second part of Lemma 4, it suffices to show that

$$\max_{|\beta| \leq 1/Q} |f(\beta, \lambda x) - \rho_xv(\beta; 1)| \ll yL^{-B-A} \quad (5.11)$$

for all primitive characters $\chi$ with moduli $r \leq L^B$, where $B = B(A, \nu)$ is the number that appears in (5.4). Let $\chi$ be such a character and suppose that $|\beta| \leq Q^{-1}$. By Lemma 4 with $b = 1/2$ and $T = T_1 = x^{9/20-\epsilon}$,

$$f(\beta, \lambda x) = \frac{1}{2\pi i} \int_{1/2 - i\tau}^{1/2 + i\tau} F(s, \lambda x)v(\beta; s) \, ds + O(yx^{-\epsilon/2} + xy^{k-9/20+\epsilon}Q^{-1}L). \quad (5.12)$$

Since $v(\beta; 1/2 + it) \ll yx^{-1/2}$, we deduce from (5.12) and hypothesis (A3) that

$$f(\beta, \lambda x) = \frac{1}{2\pi i} \int_{1/2 - i\tau}^{1/2 + i\tau} F(s, \lambda x)v(\beta; s) \, ds + O(yL^{-B-A}),$$

where $T_0 = \exp(L^{1/3})$. Note that when $\text{Re}(s) = 1/2$,

$$v(\beta; s) - x^{s-1}v(\beta; 1) \ll (|s| + 1)y^{2}x^{-3/2}.$$

Hence,

$$f(\beta, \lambda x) = \frac{v(\beta; 1)}{2\pi i} \int_{1/2 - iT_0}^{1/2 + iT_0} F(s, \lambda x)x^{s-1} \, ds + O(yL^{-B-A}). \quad (5.13)$$

When $\beta = 0$, we can evaluate the left side of (5.13) directly by means of hypothesis (A0). Thus,

$$\frac{1}{2\pi i} \int_{1/2 - iT_0}^{1/2 + iT_0} F(s, \lambda x)x^{s-1} \, ds = \rho_x + O(L^{-B-A}). \quad (5.14)$$

The desired inequality (5.11) follows from (5.13) and (5.14). □
Lemma 8. Let $x^{7/12+2\epsilon} \leq y \leq x^{1-\epsilon}$ and suppose that $P, Q$ satisfy

$$PQ \leq yx^{k-1}, \quad Q \geq x^{k-5/12+\epsilon}. $$

Suppose also that $\nu > 1$. Then, for any given $A > 0$,

$$\sum_{\nu \leq (r/\nu)^{1/5}} \sum_{|\beta| \leq 1/(rQ)} |f(\beta, 1)\chi) - \delta_x L^{-1} \nu(\beta; 1)| \ll yL^{-A}. \quad (5.15)$$

Proof. This is a slight variation of [10, Lemma 4.7]. We use the same argument, but we alter slightly the choice of $T$ in [10, p. 620]: instead of $T = (x/y)^2 x^{3\epsilon}$, we choose

$$T = x^\nu \max \{xy^{-1}, x^k Q^{-1}\}, $$

which suffices to complete the proof.

\[\square\]

5.2. The asymptotic formula for $R_{k,s}(n, \lambda; \mathcal{M})$. We have

$$R_{k,s}(n, \lambda; \mathcal{M}) = \sum_{p_1, \ldots, p_t \in \mathcal{M}} \int_{\mathfrak{M}} f(\alpha, \lambda) f(\alpha, \lambda^+) e(-n_p \alpha) \, d\alpha, \quad (5.16)$$

where $t = s - 5$ and $n_p = n - p_1^k - \cdots - p_t^k$. We now proceed to show that, for any fixed $A > 0$, one has

$$\int_{\mathfrak{M}} (f(\alpha, \lambda) f(\alpha, \lambda^+) - f^*(\alpha, \lambda) f^*(\alpha, \lambda^+)) e(-n_p \alpha) \, d\alpha \ll y^4 x^{1-k} L^{-A}. \quad (5.17)$$

Let $\alpha \in \mathfrak{M}(q, a)$ and write $\beta = \alpha - a/q$. Since $q \leq P$, property (A1) ensures that the function $\lambda$ is supported on integers $m$ with $(m, q) = 1$. Hence, by the orthogonality of the characters modulo $q$, we have

$$f(\alpha, \lambda) = \sum_{\chi \mod q} \lambda(\chi) e(a\chi^k/q) \sum_{\chi \mod q} \lambda(m) e(m^k \beta)$$

$$= \phi(q)^{-1} \sum_{\chi \mod q} S(\chi, a) f(\beta, \lambda \chi),$$

where

$$S(\chi, a) = \sum_{h=1}^q \chi(h)e(a\chi^k/q).$$

Hence,

$$f(\alpha, \lambda) = f^*(\alpha, \lambda) + \Delta(\alpha, \lambda), \quad (5.18)$$

where

$$\Delta(\alpha, \lambda) = \phi(q)^{-1} \sum_{\chi \mod q} S(\chi, a) W(\beta, \lambda \chi),$$

$$W(\beta, \lambda \chi) = f(\beta, \lambda \chi - \rho \chi), \quad \rho \chi = \delta \lambda \chi L^{-1}.$$  

Using (5.18), we can express the integral in (5.17) as the linear combination of integrals of the form

$$\int_{\mathfrak{M}} f^*(\alpha, \lambda)^n \Delta(\alpha, \lambda)(1-a) f^*(\alpha, \lambda)^b \Delta(\alpha, \lambda)^{4-b} e(-n_p \alpha) \, d\alpha, \quad (5.19)$$

where $a \in \{0, 1\}$, $b \in \{0, 1, \ldots, 4\}$ and $a + b < 5$. The estimation of all those integrals follows the same pattern, so we shall focus on the most troublesome among them, namely,

$$\int_{\mathfrak{M}} \Delta(\alpha, \lambda) \Delta(\alpha, \lambda)^{4} e(-n_p \alpha) \, d\alpha. \quad (5.20)$$

We can rewrite (5.20) as the multiple sum

$$\sum_{q \in P} \sum_{\chi_1 \mod q} \cdots \sum_{\chi_5 \mod q} B(q; \chi_1, \ldots, \chi_5) J(q; \chi_1, \ldots, \chi_5), \quad (5.21)$$

where

$$B(q; \chi_1, \ldots, \chi_5) = \phi(q)^{-1} \sum_{\chi \mod q} S(\chi, a) W(\beta, \lambda \chi),$$

$$J(q; \chi_1, \ldots, \chi_5) = f(\beta, \lambda \chi - \rho \chi).$$

This completes the proof of Lemma 8.
where
\[ B(q; \chi_1, \ldots, \chi_5) = \phi(q)^{-5} \sum_{1 \leq a \leq q} \sum_{(a,q)=1} S(\chi_1, a) \cdots S(\chi_5, a) e(-an_p/q), \]
\[ J(q; \chi_1, \ldots, \chi_5) = \sum_{1 \leq a \leq q} \sum_{(a,q)=1} S(\chi_1, a) \cdots S(\chi_5, a) e(-an_p/q). \]

First, we reduce (5.21) to a sum over primitive characters. If \( \chi \) is a Dirichlet character modulo \( q \) that is induced by a primitive character \( \chi^* \) modulo \( r, r | q \), then by property (A1), \( \lambda^\pm \chi = \lambda^\pm \chi^* \). Thus,
\[ W(\beta, \lambda^\pm \chi) = W(\beta, \lambda^\pm \chi^*). \]  
(5.22)

Let \( \chi_1^* \) modulo \( r_1 \), \( r_i | q \), be the primitive character inducing \( \chi_i \) and set \( q_0 = [r_1, \ldots, r_5] \). By (5.22), we have
\[ J(q; \chi_1, \ldots, \chi_5) = J(q; \chi_1^*, \ldots, \chi_5^*). \]

Therefore, the sum (5.21) does not exceed
\[ \sum_{r_1 \leq P \chi_1 \mod r_1} \cdots \sum_{r_5 \leq P \chi_5 \mod r_5} \sum_{P \chi_1}^{*} \sum_{P \chi_5}^{*} J_0(\chi_1, \ldots, \chi_5) B_0(\chi_1, \ldots, \chi_5), \]
where
\[ B_0(\chi_1, \ldots, \chi_5) = \sum_{q \leq P} |B(q; \chi_1, \ldots, \chi_5)|, \]
\[ J_0(\chi_1, \ldots, \chi_5) = \sum_{1 \leq a \leq q_0} \sum_{(a,q_0)=1} S(\chi_1, a) \cdots S(\chi_5, a) e(-an_p/q_0). \]

Recalling the bound (see [15, Lemma 6.1])
\[ B_0(\chi_1, \ldots, \chi_5) \leq q_0^{-3/2+\varepsilon} L^c, \]
we conclude that the sum (5.21) is
\[ \ll L^c \sum_{r_1 \leq P \chi_1 \mod r_1} \cdots \sum_{r_5 \leq P \chi_5 \mod r_5} q_0^{-3/2+\varepsilon} V(\lambda \chi_1) V(\lambda^+ \chi_2) V(\lambda^+ \chi_3) W(\lambda^+ \chi_4) W(\lambda^+ \chi_5), \]
(5.23)

where for a character \( \chi \) modulo \( r \), we write
\[ V(\lambda \chi) = \max_{|\beta| \leq 1/(rQ)} |W(\beta, \lambda \chi)|, \]
\[ W(\lambda \chi) = \left( \int_{-1/(rQ)}^{1/(rQ)} |W(\beta, \lambda \chi)|^2 d\beta \right)^{1/2}. \]

Next, we proceed to estimate the sum in (5.23) by Lemmas 3, 6, and 7, which we will denote by \( \Sigma \). When \( y = x^\theta \) with \( \theta > 31/40 \) and \( \delta \leq 2(\theta - 31/40) \), the definitions of \( P \) and \( Q \) (recall (2.6)) ensure that they satisfy inequalities (5.9). Since the sieve functions \( \lambda^\pm \) have properties (A0)–(A3), this means that all the hypotheses of the lemmas are in place.

To begin the estimation of \( \Sigma \), we note that Lemma 4 yields
\[ \sum_{r \leq P \chi \mod r} \sum_{r_5 \leq P \chi_5 \mod r} [g, r]^{-\nu} W(\lambda^+ \chi) \ll g^{-\nu+\varepsilon} y^{1/2} x^{(1-k)/2} L^c + g^{-\nu} I_0^{1/2}, \]
(5.24)

where
\[ I_0 = \int_{-1/Q}^{1/Q} |v(\beta; 1)|^2 d\beta \ll \int_{T}^{\infty} \frac{du_1 du_2}{2|Q + |u_1^k - u_2^k|}| \]
\[ \ll y x^{1-k} + y L Q L^{-1} \ll y x^{1-k}. \]
(5.25)
Finally, we apply Lemma 7 to the last sum and conclude that
\[ A \text{ for any fixed } A > 0. \]
we obtain
\[ \sum_{P \chi \mod r} \sum_{r \leq P} g^{\nu} V(\lambda^+ \chi) \ll g^{\nu + \varepsilon} y \Lambda. \]  
(5.26)

Applying (5.24) to the summations over \( r_5 \) and \( r_4 \) in \( \Sigma \) and then (5.26) to the summations over \( r_3 \) and \( r_2 \), we obtain
\[ \Sigma \ll y^3 x^{1-k} L^c \sum_{r \leq P \chi \mod r} \sum_{r \leq P} g^{\nu} V(\lambda \chi). \]

Finally, we apply Lemma 7 to the last sum and conclude that
\[ \Sigma \ll y^4 x^{1-k} L^{-A} \]
for any fixed \( A > 0 \). This inequality and its variants for other integrals of the form (5.19) establish (5.17).

Having established (5.17), we can combine it with (5.16) to get
\[ R_{k,s}(n, \lambda; \mathfrak{M}) = \int_{\mathfrak{M}} f(\alpha, \mathfrak{M}) \frac{f^*(\alpha, \lambda) f^*(\alpha, \lambda^+)}{\alpha} d\alpha + O(y^{s-1} x^{1-k} L^{-A}). \]

We now define a new, slimmer set of major arcs \( \mathfrak{M}_0 \), given by (2.7) with \( Q_0 = x^{k-1} y P^{-1} \) in place of \( Q \). From the bound
\[ f^*(\alpha, \lambda^\pm) \ll y^{\frac{1}{2} + \varepsilon} (1 + y x^{k-1} |\alpha - a/q|)^{-1} \quad \text{if } \alpha \in \mathfrak{M}(q, a), \]
we find that
\[ \int_{\mathfrak{M} \cap \mathfrak{M}_0} |f(\alpha, \mathfrak{M}) - f^*(\alpha, \lambda) f^*(\alpha, \lambda^+)| d\alpha \ll \sum_{1 \leq \alpha \leq Q} \int_{|\beta| \geq 1/(Q_0)} \frac{y^{\frac{1}{2} + \varepsilon} x^{k-1} L^{-A}}{(1 + y x^{k-1} |\beta|)^{\frac{3}{2}}} d\beta. \]

Hence, for any fixed \( A > 0 \), we have
\[ R_{k,s}(n, \lambda; \mathfrak{M}) = \int_{\mathfrak{M}_0} f(\alpha, \mathfrak{M}) \frac{f^*(\alpha, \lambda) f^*(\alpha, \lambda^+)}{\alpha} d\alpha + O(y^{s-1} x^{1-k} L^{-A}). \]

Finally, we have
\[ \int_{\mathfrak{M}_0} (f(\alpha, \mathfrak{M}) - f^*(\alpha))^2 d\alpha \ll y^{s-1} x^{1-k} L^{-A}. \]

The proof of this inequality is similar to the proof of (5.17), except that we do not need to use Lemma 3 and (5.25) can be used instead. We use Lemma 8 instead of Lemma 7. We remark that during the process, we need to verify the hypotheses \( Q \geq x^{k-9/20} \) and \( Q \geq x^{k-5/12 + \varepsilon} \) of those lemmas for \( Q = Q_0 \); with our choice of \( Q_0 \), those hypotheses are satisfied when \( y \geq x^{7/12 + \delta} \).

By (5.27) and (5.28), we have
\[ R_{k,s}(n, \lambda; \mathfrak{M}) = \kappa \int_{\mathfrak{M}_0} f^*(\alpha) e(-n\alpha) d\alpha + O(y^{s-1} x^{1-k} L^{-A}). \]

The evaluation of the last integral uses standard major arc techniques (e.g., see Wei and Wooley [15, pp. 1150–1151]), so we can omit it and report that
\[ \int_{\mathfrak{M}_0} f^*(\alpha) e(-n\alpha) d\alpha = \mathcal{E}(n) \mathcal{J}(n) L^{-s} + O(y^{s-1} x^{1-k} P^{-1}). \]

We note that \( \mathcal{E}(n) \) is the standard singular series in the Waring–Goldbach problem for \( s \) \( k \) th powers. In particular, it is known that \( 1 \ll \mathcal{E}(n) \ll 1 \) when \( n \in \mathcal{H}_{k,s} \). Since the inequality
\[ y^{s-1} x^{1-k} \ll \mathcal{J}(n) \ll y^{s-1} x^{1-k} \]
is also standard (compare to [15, (6.5)]), we conclude that (2.11) holds with
\[ \mathcal{C}(n) = \mathcal{E}(n) \mathcal{J}(n) y^{1-s} x^{k-1}. \]
Acknowledgments. The second author would like to thank Professor Jianya Liu for his constant encouragement. He also wants to thank the China Scholarship Council (CSC) for supporting his studies in the United States and the Department of Mathematics at Towson University for the hospitality and the excellent conditions.

References

[1] R. C. Baker, G. Harman, and J. Pintz. The exceptional set for Goldbach’s problem in short intervals. In “Sieve Methods, Exponential Sums, and Their Applications in Number Theory”, pp. 1–54. Cambridge University Press, 1997.
[2] J. Bourgain, C. Demeter, and L. Guth. Proof of the main conjecture in Vinogradov’s mean value theorem for degrees higher than three. Ann. of Math. (2) 184 (2016), 633–682.
[3] J. Brüdern and E. Fouvry. Lagrange’s four squares theorem with almost prime variables. J. reine angew. Math. 454 (1994), 59–96.
[4] S. K. K. Choi and A. V. Kumchev. Mean values of Dirichlet polynomials and applications to linear equations with prime variables. Acta Arith. 123 (2006), 125–142.
[5] D. Daemen. The asymptotic formula for localized solutions in Waring’s problem and approximations to Weyl sums. Bull. London Math. Soc. 42 (2010), 75–82.
[6] L. K. Hua. Some results in additive prime number theory. Quart. J. Math. (Oxford) 9 (1938), 68–80.
[7] L. K. Hua. Additive Theory of Prime Numbers. American Mathematical Society, 1965.
[8] B. R. Huang. Exponential sums over primes in short intervals and an application to the Waring–Goldbach problem. Mathematika 62 (2016), 508–523.
[9] M. N. Huxley. On the difference between consecutive primes. Invent. Math. 15 (1972), 164–170.
[10] A. V. Kumchev and T. Y. Li. Sums of almost equal squares of primes. J. Number Theory 132 (2012), 608–636.
[11] J. Y. Liu and T. Zhan. On sums of five almost equal prime squares. Acta Arith. 77 (1996), 369–383.
[12] H. L. Montgomery and R. C. Vaughan. Multiplicative Number Theory: I. Classical Theory. Cambridge University Press, 2007.
[13] E. C. Titchmarsh. The Theory of the Riemann Zeta-Function, 2nd ed., revised by D. R. Heath-Brown. Oxford University Press, 1986.
[14] R. C. Vaughan. The Hardy–Littlewood Method, 2nd ed. Cambridge University Press, 1997.
[15] B. Wei and T. D. Wooley. On sums of powers of almost equal primes. Proc. London Math. Soc. (3) 111 (2015), 1130–1162.
[16] T. D. Wooley. The cubic case of the main conjecture in Vinogradov’s mean value theorem. Adv. Math. 294 (2016), 532–561.

Department of Mathematics, Towson University, Towson, MD 21252, USA
E-mail address: akumchev@towson.edu

School of Mathematics, Shandong University, Jinan, Shandong 250100, China
E-mail address: hfliu_ed@hotmail.com