The observed radial profiles of the X-ray emission from pulsar wind nebulae (PWNe) have been claimed to contradict the standard 1D steady model. However, the 1D model has not been tested to simultaneously reproduce the volume-integrated spectrum and the radial profile of the surface brightness. We revisit the 1D steady model and apply it to PWNe 3C 58 and G21.5–0.9. We find that the parameters of the pulsar wind, the radius of the termination shock $r_t$, and magnetization $\sigma$ greatly affect both the photon spectrum and radial profile of the emission. We have shown that the parameters constrained by the entire spectrum lead to an X-ray nebula smaller than the observed nebula. We have also tested the case that reproduces only the observations in X- and gamma-rays, ignoring the radio and optical components. In this case, there are parameter sets that reproduce both the spectrum and emission profile, but the advection time to the edge of the nebula becomes much smaller than the age. Our detailed discussion clarifies that the standard 1D steady model has severe difficulty to simultaneously reproduce both the volume-integrated spectrum and the surface brightness. This implies that the model should be improved by taking into account extra physical processes such as spatial diffusion of particles. Additionally, we calculate the surface brightness profile of the radio, optical, and TeV gamma-rays. The future observations in these wavelengths are also important to probe the spatial distributions of the relativistic plasma and the magnetic field of PWNe.

**Key words**: ISM: individual objects: (3C 58, G21.5–0.9) – magnetohydrodynamics – pulsars: general – radiation mechanisms: nonthermal – stars: winds, outflows

**1. Introduction**

Pulsar wind nebulae (PWNe) are extended sources around a rotation-powered pulsar. They show a broadband spectrum from radio to $\gamma$-rays so that they would contain very high energy nonthermal particles (Gaensler & Slane 2006; Kirk et al. 2009; Bühler & Blandford 2014). The central pulsar releases its rotational energy as the pulsar wind, which is a highly relativistic magnetized electron–positron outflow, and plays a role of the energy source of the PWN. The strong termination shock, which is formed by the interaction between the pulsar wind and an external supernova remnant (SNR) (or interstellar medium), has been assumed to produce the nonthermal electrons and positrons, and they emit the synchrotron radiation (Rees & Gunn 1974) and the inverse Compton (IC) emission (de Jager & Harding 1992).

PWNe are characterized by a center-filled morphology caused by confinement in an SNR, which is associated with the progenitor of the central pulsar. Most PWNe are detected as spatially extended sources in radio and X-rays. While the radio spectral index is almost spatially homogeneous (e.g., Bietenholz et al. 1997; Bietenholz & Bartel 2008), the X-ray spectral index increases with the distance from the pulsar (e.g., Bocchino & Bykov 2001; Slane et al. 2004; Schöck et al. 2010). Recently, PWNe have also been found to be very bright sources in TeV $\gamma$-rays (Kargaltsev et al. 2013). One-zone models of PWNe, which have been invented by Pacini & Salvati (1973) and have since been developed by several authors (e.g., Bednarek & Bartosik 2003; Chevalier 2005; Tanaka & Takahara 2010; Bucciantini et al. 2011; Vorster et al. 2013), can reproduce broadband spectra of the entire nebulae well. The one-zone models do not describe the spatial distribution of the emission (e.g., Amato et al. 2000), thus it is indispensable to invest a model that includes the spatial structure of nebulae.

The model of Kennel & Coroniti (1984a, 1984b, hereafter KC84s) has been established as a standard model of PWNe. The authors constructed a steady-state and 1D magnetohydrodynamic (MHD) model of the Crab Nebula. Assuming particle acceleration at the termination shock, they also calculated the evolution of nonthermal pairs along the flow and the synchrotron emission from advected particles. Adopting the model of KC84s, Atoyan & Aharonian (1996) succeeded in reproducing the entire photon spectrum, including the IC component. However, as Reynolds (2003) suggested, it is unclear whether KC84s can be applied to general PWNe other than the Crab Nebula. Furthermore, Slane et al. (2004) showed that the model of KC84s disagrees with the observed radial profile of the X-ray spectral index in 3C 58. They suggested that the radial profile of the X-ray spectral index in the model should change more rapidly, and the X-ray nebula size becomes more compact than the observation. Note that the entire photon spectrum was not taken into account as a model constraint in Slane et al. (2004).

This ignited the revision of KC84s. Tang & Chevalier (2012) introduced the effect of the spatial diffusion of the particles and reproduced the X-ray radial profile. Porth et al. (2016) supported this idea via 3D magnetohydrodynamic and test-particle simulations. In each of the studies, simultaneous verification of the entire spectrum and the spatial profile of the emission is not discussed, thus there is no consensus on the spatial structure in PWN models so far. In order to advance the study, it is essential to clarify controversial points in the simple steady 1D model before introducing nontrivial effects such as the particle diffusion.
We choose two objects, 3C 58 and G21.5−0.9, to examine the 1D steady model. Both of the PWNe show the feature that the extent of the X-ray emission is the same as the radio emission, in contrast to the Crab Nebula, in which the observed size shrinks with increasing frequency. Furthermore, the 1D steady model has never been applied to these two PWNe. Our purpose is to make the validity of the 1D steady model clear for general PWNe, so that these two PWNe are suitable targets for testing.

In this paper, we revisit the 1D steady model and calculate the photon spectrum and its radial profile numerically. In Section 2 we review the 1D steady model of PWNe based on KC84s. The parameter dependence of this model is investigated in Section 3. The application to the two observed sources (3C 58 and G21.5−0.9) is presented in Section 4. We discuss the 1D steady modeling in Section 5 and summarize our results in Section 6.

2. Model

In this paper, we adopt the 1D MHD model constructed by KC84s. We numerically solve the energy distribution of the electron–positron pair plasma along the outflow obtained by MHD equations. From the resulting spectral distribution of the pairs, we calculate the photon emission from these pairs, the resulting surface brightness, and the volume-integrated spectrum. In this section, we review the 1D steady model and present the method for calculating the radial evolution of the energy distribution of electrons and positrons.

Here, we consider that the PWN is a steady outflow and has a radius $R_{v}$ The relativistic magnetized wind emitted from the central pulsar forms a strong termination shock at a radius $r_{s}$. Assuming that the pre-shock plasma is cold, the wind property is regulated by three quantities, the number density in the comoving frame $n$, the bulk Lorentz factor $\gamma_{r}$, and the magnetic field in the laboratory frame $B$ at the shock. Almost all of the pulsar spin-down luminosity $L_{sd}$ is converted into the wind luminosity as

$$L_{sd} = 4\pi r_{s}^{2} n_{u} u_{u} \gamma_{u} m_{e} c^{5}(1 + \sigma),$$

where $u \equiv \sqrt{\gamma^{2} - 1}$, the subscript $u$ denotes values just upstream of the shock, and $\sigma$ is the ratio of the magnetic energy flux to the particle energy flux at the upstream of the shock,

$$\sigma \equiv \frac{B_{u}^{2}/4\pi}{n_{u} u_{u} \gamma_{u} m_{e} c^{2}}.$$ (2)

The magnetic field of the wind is dominated by the toroidal component (e.g., Goldreich & Julian 1969), and the upstream plasma is highly relativistic ($u_{u} / \gamma_{u} \approx 1$), which means that the downstream temperature is relativistic (adiabatic index 4/3). The Rankine–Hugoniot jump conditions provide the values in the downstream (KC84s) as

$$n_{d} = \frac{n_{u} u_{u}}{u_{d}},$$ (3)

$$u_{d}^{2} = 8\sigma^{2} + 10\sigma + 1 + \sqrt{64\sigma^{2}(\sigma + 1)^{2} + 20\sigma(\sigma + 1) + 1},$$ (4)

$$P_{d} = \frac{n_{d} m_{e}^{2} c^{2} u_{d}^{2}}{4\gamma_{d} u_{d}^{2}} \left[ 1 + \sigma \left( 1 - \frac{\gamma_{d}}{u_{d}} \right) \right],$$ (5)

$$B_{d} = B_{u} \frac{\gamma_{d}}{u_{d}},$$ (6)

where the subscript $d$ denotes the values just downstream of the shock, and $P$ is the thermal pressure. For $\gamma_{u} \gg 1$ and $\sigma \ll 1$, we obtain $u_{d} / \gamma_{d} \approx 1/3$, which coincides with the well-known result in relativistic hydrodynamics.

As boundary conditions, we adopt Equations (3)–(6) at the radius $r = r_{s}$, and solve the steady-state and spherically symmetric MHD equations. Under the toroidal field approximation and the adiabatic assumption, the MHD equations are integrable. After some algebra with introducing $\delta \equiv u_{d} / (\sigma u_{u})$, we obtain (KC84s)

$$\sqrt{1 + u^{2}(r)} \left( \frac{(u_{d}^{2} / \sigma) - \frac{1}{2} \left( u(r) \right)^{2}}{u_{d}^{2} + \frac{1}{4}} + \frac{u_{d}}{u(r)} \right) = \gamma_{d} \left( \delta + \frac{(u_{d}^{2} / \sigma) - \frac{1}{2} \left( u(r) \right)^{2}}{u_{d}^{2} + \frac{1}{4}} + 1 \right),$$ (7)

from which we obtain the radial profile of the four velocity $u(r)$ (or equivalently $\gamma(r)$). In the strong shock approximation, $u_{d}$ is a function of only $\sigma$ as shown in Equation (4). If $\delta \ll 1$ is established, the above flow Equation (7) is depicted by only one parameter $\sigma$ independently of $n_{u}$ and $u_{u}$. We calculate the radial profile of $u(r)$ numerically, while KC84s neglected $\delta$ and adopted $\gamma_{d} \approx 1$ in the downstream. Then, the MHD conservation laws provide the other quantities as follows:

$$n_{tot}(r) = n_{d} u_{d} r_{s}^{2} / \left( u(r) r^{2} \right),$$ (8)

$$B(r) = B_{d} \frac{\gamma_{d} \left( u_{d} r_{s}^{2} / \gamma_{d} \right)}{u(r) r^{2}},$$ (9)

$$P(r) = P_{d} \left( u_{d} r_{s}^{2} / \left( u(r) r^{2} \right) \right)^{4/3},$$ (10)

where $n_{tot}(r)$ is the comoving number density in the wind.

In Figure 1 the radial profiles of $u(r)$ and $B(r)$ in our test calculations are shown for $\sigma = 10^{-6}$, $10^{-5}$, $10^{-4}$, $10^{-3}$, and $10^{-2}$. The results are almost the same as the behavior shown in KC84s because $\delta \ll 1$ for all the cases. For $\sigma \ll 1$, at a small radius, the pressure ratio $\beta_{pl} \equiv B^{2} / 8\pi P$ (at $r = r_{s}$) gradually increases with radius as $\beta_{pl} \propto r^{2}$. In that regime, the outflow behaves as $u \propto r^{-2}$ (equivalently $n_{tot} \propto r^{0}$), $B \propto r$ and $P \propto r^{0}$. At the radius

$$r \approx r_{eq} \equiv r_{s} / \sqrt{3\sigma},$$ (11)

$\beta_{pl}$ becomes unity, namely the magnetic pressure starts to dominate. Outside this radius, the radial four speed is approximately constant with $B \propto r^{-3}$ and $P \propto r^{-5/3}$. Thus, the magnetic field has a maximum value at $r \approx r_{eq}$ as shown in Figure 1.

The energy distribution of electron–positron pairs $n(E, r)$ is calculated consistently with the MHD model,

$$n_{tot}(r) = \int n(E, r) dE.$$ (12)
At the inner boundary \( r = r_s \), \( n(E , r_s) \) is assumed to have a broken power-law energy distribution at injection as follows:

\[
n(E , r_s) = \begin{cases} 
\frac{n_0}{E_b} \left( \frac{E}{E_b} \right)^{-p_1} & (E_{\text{min}} < E < E_b) \\
\frac{n_0}{E_b} \left( \frac{E}{E_b} \right)^{-p_2} & (E_b < E < E_{\text{max}}) 
\end{cases} ,
\]

(13)

where the parameters are the break energy \( E_b \), minimum energy \( E_{\text{min}} \), maximum energy \( E_{\text{max}} \), and two power-law indices \( p_1 \) and \( p_2 \) for low- and high-energy regions, respectively. The normalization \( n_0 = C_1 n_d \) is determined by Equation (12) as

\[
C_1 = \left[ \frac{1}{p_1 - 1} \left( \frac{E_{\text{min}}}{E_b} \right)^{-p_1} - 1 \right] \left[ \frac{1}{p_2 - 1} \left( 1 - \left( \frac{E_{\text{max}}}{E_b} \right)^{-p_2} \right) \right]^{-1}.
\]

(14)

While the origin of the radio spectral component may be different from that for the X-ray and optical components, as discussed in KC84s, in this paper, we use a broken power-law distribution that has been adopted by the one-zone studies (e.g., Tanaka & Takahara 2010).

The observed radio spectral index almost uniquely gives the index \( p_1 \), which is generally lower than 2 (e.g., Salter et al. 1989). In this case, the particles with energies \( \sim E_{\text{min}} \) dominate the particle number. For simplicity, we fix the minimum energy as \( E_{\text{min}} = 10 m_e c^2 \) and leave the pair density problem (see Tanaka & Takahara 2013a). The particles above \( E_b \) may be produced via the shock acceleration (Spitkovsky 2008). The maximum energy \( E_{\text{max}} \) is determined by the same method in KC84s; the energy at which a gyro radius is equal to the shock radius provides

\[
E_{\text{max}} = eB_u r_s = \sqrt{ \frac{e^2 L_{\text{sd}} \sigma}{c} }.
\]

(15)

The pressure obtained by \( n(E, r_s) \),

\[
P_d = \frac{1}{3} \int E n(E, r_s) dE,
\]

(16)

should satisfy Equation (5). We thus obtain

\[
n_d = \frac{3L_{\text{sd}}}{16\pi^2 c E_b (1 + \sigma)} \left( \frac{C_2}{C_1} \right) \left[ 1 + \sigma \left( 1 - \frac{\gamma_d}{u_d} \right) \right],
\]

(17)

where

\[
C_2 = \left[ \frac{1}{2 - p_1} \left( 1 - \left( \frac{E_{\text{min}}}{E_b} \right)^{-p_1} \right) \right] + \frac{1}{p_2 - 2} \left[ 1 - \left( \frac{E_{\text{max}}}{E_b} \right)^{-p_2} \right]^{-1}.
\]

(18)

We note that \( \gamma_d \) or \( u_d \) is already given as a function of only \( \sigma \) (see Equation (4)), so that the quantities in the upstream are written with the six parameters \( L_{\text{sd}}, \sigma, E_b, p_1, \) and \( p_2 \) as

\[
\gamma_d = \frac{4}{3} \frac{E_b}{m_e c^2} \frac{\gamma_d}{u_d} \left( \frac{C_2}{C_1} \right) \left[ 1 + \sigma \left( 1 - \frac{\gamma_d}{u_d} \right) \right]^{-1},
\]

(19)

\[
B_u = \left[ \frac{L_{\text{sd}}}{c r_s} \frac{\sigma}{1 + \sigma} \right]^{1/2},
\]

(20)

and

\[
n_u = \frac{9L_{\text{sd}}}{64\pi^2 c \gamma_d m_e c^2 (1 + \sigma)} \left( \frac{C_2 m_e c^2}{C_1 E_b} \right)^2 \times \left[ 1 + \sigma \left( 1 - \frac{\gamma_d}{u_d} \right) \right]^{-3/2}.
\]

(21)

From the above relations, the functional shape described in Equation (13) is also written with the six parameters.

In this 1D model, we have a unique parameter \( r_s \) that is not in the one-zone models, and \( r_s \) significantly affects the results, as we show in Section 3. The flow solution given by Equation (7) provides the advection time as

\[
t_{\text{adv}} = \int_{r_s}^{r_N} \frac{dr}{c u(r)}.
\]

(22)

While the age of the PWN is an important parameter in the one-zone time-dependent models such as Tanaka & Takahara (2010),
the parameter \( r_b \) in our steady model adjusts the advection time, which may be close to the age of the PWN.

The radial evolution of \( n(E, r) \) and photon emission are calculated with the numerical code used in Sasaki et al. (2015), which is based on the time-dependent code in Asano & Mészáros (2011) (see also Asano & Mészáros 2012). Taking into account the Klein-Nishina effect on the IC cooling, the code can follow the temporal evolution of the energy distribution along the stream. When we transform the elapsed time into radius as \( dr = cu(r)dt \), our calculation is practically equivalent to solving the steady transport equation (e.g., Ginzburg & Syrovatskii 1964; Parker 1965)

\[
\frac{\partial n(E, r)}{\partial r} = \frac{\partial}{\partial E} \left[ (\dot{E}_{\text{syn}} + \dot{E}_{\text{IC}})n(E, r) \right] + \frac{\partial}{\partial E} \left[ \frac{cE}{3r^2} \frac{d}{dr} (r^2 u(r)) \right] - \frac{c}{r^2} n(E, r) \frac{d}{dr} (r^2 u(r)),
\]

where \( \dot{E}_{\text{syn}} \) and \( \dot{E}_{\text{IC}} \) are the energy loss rates due to synchrotron radiation and IC scattering, respectively. The three terms on the right-hand side of Equation (23) represent the effects of radiative cooling, adiabatic approximation, and volume expansion, respectively.

Here, we have used the solution for \( u(r) \) with adiabatic approximation, namely the radiative cooling is assumed not to affect the dynamics of the flow. This approximation is valid when the cooling time for the particles with \( E \sim E_b \) is longer than the advection time (KC84s). Most of the results shown in this paper safely satisfy this condition.

The spectral emissivity \( j_v(r) \) per unit volume is obtained consistently with the energy distribution \( n(E, r) \), the magnetic field profile \( B(r) \), and the interstellar radiation field (ISRF) with the Klein-Nishina effect. The model of the ISRF is taken from GALPROP v54.1 (Vladimirov et al. 2011, and references therein), in which the results of Porter & Strong (2005) are adopted, as shown in Figure 2. The ISRF is assumed as uniform and isotropic in the PWN, and we neglect the synchrotron self-Compton, which significantly contributes only in limited cases like the Crab Nebula (Torres et al. 2013). We neglect the contribution of bremsstrahlung as well, because the density of pairs is low enough (Atoyan & Aharonian 1996).

When the emission from the pre-shock wind is neglected, the photon spectrum of the entire nebula \( F_v \) is given by

\[
F_v = \frac{1}{D^2} \int_0^{\infty} j_v(r) r^2 dr,
\]

where \( D \) is the distance to the PWN from us. The surface brightness \( B_v \) is also given by

\[
B_v(s) = 2 \int_{s_{\max}}^{s_{\max}} j_v(r) r dr \sqrt{r^2 - s^2},
\]

where \( s \) is the distance perpendicular to the line of sight from the central pulsar.

Our numerical code is checked by reproducing the result of Atoyan & Aharonian (1996) for the Crab Nebula. If we adopt the same assumption as Atoyan & Aharonian (1996), the resulting spectrum agrees with the observed spectrum. Note that our more conservative model leads to a slightly dimmer flux for the IC component, as shown in the Appendix.

![Figure 2. Spectra of the interstellar radiation field taken from GALPROP vs4.1. The black solid line is adopted for G21.5−0.9, which is located at \( R = 4 \) kpc and \( z = 0 \) kpc. The red dashed line is for 3C 58, which is located at \( R = 9.4 \) kpc and \( z = 0.5 \) kpc.](image)

### 3. Parameter Dependence in the 1D Steady Model

In this section, we discuss how the entire spectrum \( L_v = 4 \pi D^2 F_v \) and the surface brightness \( B_v \) depend on the parameters in this 1D steady model. There are some previous studies that discussed the parameter dependence of the 1D model. KC84s have already discussed how the parameters \( u_c \) and \( \sigma \) change the total synchrotron luminosity \( \int L_s d\nu \) (not the spectral distribution). While Schöck et al. (2010) have studied the X-ray spatial profile for different \( r_b \) assuming that the flow velocity decreases as a power law of \( r \) independently of \( \sigma \), we investigate the parameter dependence consistently with the MHD flow solution. We focus on the dependence on the parameters \( r_b \) and \( \sigma \), which largely affect the spatial structure of the emission. In this section, the nebula size \( r_b = 2.0 \) pc is fixed. The external photon field is taken from the model for G21.5−0.9 in Figure 2. For four parameters out of the six parameters in our model, we adopt a parameter set as \( L_{sd} = 10^{38} \) erg s\(^{-1}\), \( E_b = 10^5 m_e c^2 \), \( p_1 = 1.1 \), and \( p_2 = 2.5 \), and change \( r_b \) or \( \sigma \) below.

#### 3.1. Characteristic Frequencies and Energies of the Model

First, we introduce some typical particle energies and corresponding photon frequencies, and their dependence on the model parameters are discussed. In this subsection, we limit the discussion to the case of \( r_b < r_{eq} \) (i.e., \( r_b / r_b < (3\sigma)^{-1/2} \)). In this case, the magnetic field has a maximum value \( 3B_{sd} r_b / r_b \) at the edge of the nebula. The cooling effect for pairs with energies \( E \sim E_b \) is also found to be negligible. We find the first typical frequency, the intrinsic break frequency,

\[
\nu_b = \frac{3eB(r_b)}{4\pi m_e c^2} \left( \frac{E_b}{m_e c^2} \right)^2 \left( \frac{L_{sd}}{10^{38} \text{ erg s}^{-1}} \right)^{1/2} \left( \frac{\sigma}{10^{-4}} \right)^{1/2} \approx 4.7 \times 10^{12} \text{ Hz} \left( \frac{L_{sd}}{10^{38} \text{ erg s}^{-1}} \right)^{1/2} \left( \frac{\sigma}{10^{-4}} \right)^{1/2} \left( \frac{E_b}{10^5 m_e c^2} \right)^2 \left( \frac{r_b}{0.1 \text{ pc}} \right)^{-2} \left( \frac{r_{eq}}{2 \text{ pc}} \right)
\]

(26)

In the case of \( B \sim r \) or equivalently \( u \sim r^{-2} \), the energy of pairs injected with \( E = E_{\text{max}} \) decreases with \( r \) via the
The Astrophysical Journal, 838:142 (14pp), 2017 April 1

Shishizaki et al.

Figure 3. Test calculations to see the shock radius dependence. The parameters are \( L_{sd} = 10^{38} \text{erg s}^{-1} \), \( E_b = 10^5 m_e c^2 \), \( \sigma = 10^{-4} \), \( n_1 = 1.1 \), and \( n_2 = 2.5 \). (Left) The entire spectrum calculated for various values of \( r_s \) (see Table 1). (Right) The radial profile of the X-ray surface brightness for 0.5–10 keV range. The nebula radius \( r_N \) is 2 pc.

Table 1

| Obtained Parameters for the Test Calculation with \( \sigma = 10^{-4} \) Shown in Figure 3 |
|-----------------------------------------------|
| Radius of Termination Shock (pc) | 0.05 | 0.1 | 0.2 | 0.4 |
| Total number of pairs (10^{37}) | 9.5 | 3.4 | 0.97 | 0.25 |
| Advection time (year) | 6800 | 2400 | 690 | 180 |
| Total pressure at \( r = r_N \) (10^{-10} \text{dyne cm}^{-2}) | 36 | 16 | 5.4 | 1.5 |
| Maximum magnetic field (\( \mu G \)) | 250 | 100 | 32 | 8.7 |
| Averaged magnetic field (\( \mu G \)) | 240 | 88 | 26 | 6.8 |
| \( \rho_{eq}/\rho_N \) | 1.4 | 2.9 | 5.8 | 12 |

Note. See Figure 3 for the other parameters. In these parameter sets, the magnetic field always reaches its maximum at \( r = r_N \).

Above \( \nu_c \), the entire spectrum should show the softening behavior.

The maximum particle energy decreases following Equation (27), while the magnetic field increases as \( B(r) = 3B_{bof}(r/r_N) \). The typical synchrotron frequency \( \propto B(r)E_{cut}(r)^2 \) peaks at

\[
r \simeq r_{pk} \equiv \left( \frac{5E_{bof}}{9E_{max}} \right)^{1/5}
\]

\[
\simeq 0.14 \text{pc} \left( \frac{L_{sd}}{10^{38} \text{erg s}^{-1}} \right)^{-3/10}
\]

\[
\times \left( \frac{\sigma}{10^{-4}} \right)^{3/10} \left( \frac{r_N}{0.1 \text{ pc}} \right)^{6/5},
\]

where we have assumed \( E_{max} \ll 5E_{bof} \). With \( r_{pk} \), the cutoff frequency in the synchrotron spectrum is obtained as

\[
\nu_{cut} = \frac{3eB(r_{pk})}{4\pi m_e c} \left( \frac{E_{cut}(r_{pk})}{m_e c^2} \right)^2
\]

\[
\simeq \frac{729e}{400\pi m_e c} \sqrt{\frac{L_{sd} \sigma}{c^2 r_{N}}} \left( \frac{5E_{bof}}{9E_{max}} \right)^{1/5} \left( \frac{E_{max}}{m_e c^2} \right)^2
\]

\[
\simeq 9.3 \times 10^{18} \text{Hz} \left( \frac{L_{sd}}{10^{38} \text{erg s}^{-1}} \right)^{11/10}
\]

\[
\times \left( \frac{\sigma}{10^{-4}} \right)^{11/10} \left( \frac{r_N}{0.1 \text{ pc}} \right)^{-4/5},
\]

above which the flux decreases exponentially.

3.2. Shock Radius Dependence

One-zone models do not include the shock radius as a parameter. The value \( r_s \) is a characteristic parameter in the 1D model. For the Crab Nebula (Schweizer et al. 2013), Vela (Helfand et al. 2001), and MSH 15-52 (Yatsu et al. 2009), a possible shock structure (inner ring) is detected with X-ray observations. In most PWNe, however, the shock radii are not well constrained observationally. In Figure 3, we show the \( r_s \)-dependences of the entire spectrum \( L_{\nu} \) and the X-ray surface brightness with \( \sigma = 10^{-4} \). The synchrotron component
dominates below $10^{20}$ Hz ($\sim$400 keV) and the IC component dominates in $\gamma$-rays. Note that in the case for $r_k = 0.05$ pc, $E_{\text{cut}}(r) < E_{\text{b}}$ beyond $r \gtrsim 1.5$ pc, i.e., the adiabatic approximation is invalid. In Table 1, for various $r_k$, we summarize the advection time, the volume-averaged magnetic field $B_{\text{av}}$ as,

$$\frac{B_{\text{av}}^2}{8\pi} = \int_{r}^{\infty} \frac{B(r)^2}{8\pi} 4\pi r^2 dr / \int_{r}^{\infty} 4\pi r^2 dr,$$  \hspace{1cm} (32)

and the maximum magnetic field $B_{\max} = B(r_N)$ because we have $E_{\text{eq}} > E_{\text{b}}$ from Equation (11) with $\sigma = 10^{-4}$.

First, as a benchmark case, we take up the case of $r_k = 0.1$ pc (the red solid line in Figure 3), which leads to $r_{\text{eq}} = 5.7$ pc. There are two breaks in the synchrotron spectrum: the intrinsic break at $\nu_b \sim 2.2 \times 10^{12}$ Hz corresponding to $E_b$ and the cooling break at $\nu_c \sim 3.6 \times 10^{14}$ Hz. Adopting the average magnetic field and the advection time, the cooling break energy of pairs is

$$E_c^{(\text{av})} \approx \frac{6\pi m_e^2 c^3}{\sigma T B_{\text{av}} I_{\text{adv}}} \approx 670 \text{ GeV} \left( \frac{B_{\text{av}}}{88 \mu G} \right)^{-2} \left( \frac{I_{\text{adv}}}{2400 \text{ year}} \right)^{-1}. $$  \hspace{1cm} (33)

The corresponding cooling break frequency is written as

$$\nu_c^{(\text{av})} = \frac{3eB_{\text{av}}}{4\pi m_e c} \left( \frac{E_c^{(\text{av})}}{m_e c^2} \right)^2 \approx 6.4 \times 10^{14} \text{ Hz} \left( \frac{B_{\text{av}}}{88 \mu G} \right)^{-3} \left( \frac{I_{\text{adv}}}{2400 \text{ year}} \right)^{-2}. $$  \hspace{1cm} (34)

The above value obtained with a one-zone-like treatment roughly agrees with our results.

The analytical descriptions of the spectral indices $\alpha$ ($F_\nu \propto \nu^{-\alpha}$) are $(p_1 - 1)/2$ below $\nu_b$, and $(p_2 - 1)/2$ for $\nu_b < \nu < \nu_c$. The calculated spectrum agrees with these values. In one-zone models, the index above $\nu_c$ steepened by 1/2. However, in 1D models following $B \propto r$ and $u \propto r^{-2}$, the spectral change $\Delta = (p + 7)/18$ is slightly different from 1/2 (Reynolds 2009), which also agrees with our result.

Let us consider the flux at $\nu = \nu_{\text{cut}}$. When the synchrotron cooling is efficient for particles of $E = E_{\text{max}}$, almost all of these energies are released by photon emission until $r = r_{pk}$. Since the energy density of pairs, which have an energy $E = E_{\text{max}}$ at

$$r = r_{pk}$$

is estimated as $L_{sd}(E_{\text{max}}/E_b)^{3-p_1}$, the synchrotron luminosity at $\nu \sim \nu_{\text{cut}}$ is calculated as $L_{sd}\nu_{\text{cut}} \sim 1.7 \times 10^{-2}L_{sd}$ for $\sigma = 10^{-4}$ and $p_2 = 2.5$, which seems consistent with the flux in Figure 3. The above estimate does not depend on $r_k$, which agrees with the results for $r_k \lesssim 0.2$ pc, where $\nu_c < \nu_{\text{cut}}$.

A particle of energy $E$ emits synchrotron photons of frequency $\nu \propto E^2B$, and power $p_{\nu\nu} \propto E^2B^2$, then the spectral emissivity $j_\nu$ is proportional to $n(E)E^2B^2(dE/\nu)$. When we can assume $\sigma \ll 1$ and $n_0 \propto \nu^0$, $j_\nu \propto n_0 E_0^{p-1} \nu^{-(p+1)/2}B(r)^{(p+1)/2}$ ($p$ is the index of the pair energy distribution) at the energy range where the cooling effect is negligible. The constant density implies that the magnetic field behaves as $B \sim B_0 r/r_k \propto L_{sd}^{1/2}r^{-2}\sigma^{1/2}r$. Since we can treat $C_2$ as a constant for $p_1 < 2 < p_2$ and $E_{\min} \ll E_b \ll E_{\text{max}}$, we obtain $n_0 \propto L_{sd} r_k^{-2}E_b^{-1}$. Finally, we obtain the entire specific luminosity ($L_{\nu_{\text{cut}}} \sim 4\pi \int_{r_k}^{\infty} dr r^2 j_\nu$) as

$$\nu L_{\nu_{\text{cut}}} \propto L_{sd}^{(p+5)/4} E_b^{p-2} \sigma^{(p+1)/4} r_k^{-(p+3)/2} r_N^{(p+7)/2} \nu^{-(p-3)/2},$$  \hspace{1cm} (35)

where $p$ is $p_1$ for $\nu < \nu_{\text{b}}$ and $p_2$ for $\nu_{\text{b}} < \nu < \nu_c$. At $\nu \sim \nu_b \propto L_{sd}^{1/2}2^{1/2}E_b r_k^{-2}r_N$ (Equation (26)),

$$\frac{L_{\nu_b} L_{\nu_{\text{cut}}}}{L_{sd}} \approx \left( \frac{L_{sd}}{10^{38} \text{ erg s}^{-1}} \right) \left( \frac{\sigma}{10^{-4}} \right) \left( \frac{E_b}{10^5 m_e c^2} \right)^{3-4} \left( \frac{r_k}{0.1 \text{ pc}} \right)^{-6} \left( \frac{r_N}{2 \text{ pc}} \right)^5. $$  \hspace{1cm} (36)
Here the absolute value, which is difficult to evaluate analytically, is provided by our numerical results. Above \( \nu = \nu_\star \propto L_{\text{sd}}^{-3/2} \sigma^{3/2} \gamma^2 r_N^{-3} \) (Equation (29)), using the above formula, the spectrum behaves as \( nL_{\nu} \propto \nu \gamma L_{\gamma} (\nu/\nu_\star)^{-5}(p_{1} - 2)/9 \) so that

\[
\nu L_{\nu} \propto L_{\text{sd}}^{(p_{2} + 4)/6} E_B^{p_{1} - 2} \sigma^{4(p_{1} - 2)/6} \nu^{-5(p_{1} - 2)/9},
\]

(37)

for \( \nu_\star < \nu < \nu_{\text{cut}} \). These formulae agree well with our numerical results.

The cooling frequency strongly depends on \( r_\star \). The frequencies \( \nu_\star \) and \( \nu_c \) have similar values for \( r_\star = 0.05 \) pc, while \( \nu_\star \) and \( \nu_{\text{cut}} \) merge for \( r_\star = 0.2 \) pc. For \( r_\star = 0.4 \) pc, \( \nu_c \) becomes higher than \( \nu_{\text{cut}} \). Namely, radiative and adiabatic cooling effects are both negligible in this case. This leads to a flux at \( \nu = \nu_{\text{cut}} \) for \( r_\star = 0.4 \) pc that is lower than the fluxes for smaller \( r_\star \).

The value of \( E_{\text{cut}}(r_N) \) increases with \( r_\star \) because the synchrotron cooling becomes less effective. Reflecting this, the IC spectra show a soft-to-hard evolution with \( \nu_\star \). This is the high-energy cutoff of the IC component determined by the maximum energy of pairs. Electron–positron pairs expand only a small fraction of their energy in the IC emission. Since the \( L_{\text{sd}} \) and \( E_B \) are common for the examples in Figure 3, the IC flux in the low-energy range (10^20–10^22 Hz) is basically proportional to the total number of corresponding low-energy particles in the nebula. As shown in the left panel in Figure 1, a flow with a low ratio of \( r_N/r_\star \) reaches the edge of the nebula before significant deceleration. Consequently, the advection time becomes shorter for a smaller \( r_N/r_\star \) as shown in Table 1. If we can neglect the cooling effect, the total particle number \( \propto L_{\text{sd}} E_B^{p_{1} - 2} \nu^{-5} \) decreases with \( r_\star \). This effect is seen as the flux growth with increasing \( t_{\text{adv}} \) below \( \sim 10^{22} \) Hz. For \( r_\star = 0.05 \) pc, the synchrotron cooling is crucial (\( E_{\text{cut}}(0.7 r_N) \approx E_B \)), which practically reduces the particle number above \( E_B \) in the nebula. Therefore, the IC flux in this case does not follow the aforementioned trend of the flux growth.

The surface brightness profile in X-rays (see the right panel of Figure 3) is regulated by the synchrotron cooling. For a smaller \( r_\star \), the stronger magnetic field results in a compact X-ray profile. The X-ray extent is proportional to \( r_\star^{10/9} \) for \( r_\star \leq 0.2 \) pc as explained as follows. When the cooling effect is significant, \( E_{\text{cut}} \propto E_{\text{cut}}(r_N/r_\star)^{5/2} \gamma^2 r_\star^{-5} \), while the magnetic field behaves as \( B \propto \sigma^{1/2} r_\star^{-2} r \). For a given frequency \( \nu \propto B_\star^2 \), the maximum radius to emit photons of \( \nu \) is proportional to \( \sigma^{-1/9} r_\star^{10/9} \). This supports the X-ray size growth with \( r_\star^{10/9} \) for the case where the cooling effect is significant. For \( r_\star = 0.4 \) pc, the synchrotron cooling has a negligible effect for the X-ray profile. In order to reconcile the fact that the X-ray extent is comparable to the radio nebula, a large \( r_\star \) is preferable, although the synchrotron component becomes very hard and dim in this case.

As shown in Table 1, the total pressure \( \rho_\text{tot} = 4\rho u^2 + P + B^2/8\pi \) at the outer boundary, which may balance the pressure outside the nebula, decreases with \( r_\star \) by roughly an order of magnitude. Since the uncertainty of the current observation of the outside pressure is larger than this variance, it may be difficult to constrain the value of \( r_\star \) directly.

### 3.3. \( \sigma \) Dependence

Figure 4 shows the \( \sigma \) dependences of the entire spectrum and the X-ray surface brightness. While the shock radius is fixed to \( r_\star = 0.1 \) pc and the other parameters are the same as those in the previous subsection, the value of \( \sigma \) changes from 10^{-6} to
to $10^{-2}$. As is shown in the figure, a complicated behavior appears in the spectral shape with increasing $\sigma$. The change of $\sigma$ modifies the profiles of the emission through two processes: the strength of the magnetic field (see Equation (20)) and the deceleration profile as shown in Figure 1. The magnetic field strength affects the typical synchrotron frequency and the cooling efficiency. The flow velocity profile adjusts the radius $r_{eq}$, where the magnetic field becomes maximum, and the advection time, which controls the total energy in the nebula and the cooling efficiency (the ratio of cooling time to advection time).

For $\sigma \lesssim 10^{-4}$, the radius $r_{eq}$ is outside $r_N$ (see Table 2, note that $r_N/r_c$ corresponds to 20 in Figure 1), so that the behaviors of the characteristic frequencies are well explained by Equations (26), (29), and (31) as $\nu_b \propto \sigma^{1/2}$, $\nu_c \propto \sigma^{-3/2}$, and $\nu_{cut} \propto \sigma^{11/10}$. In the case of $\sigma = 10^{-6}$, the frequency $\nu_c$ is much higher than $\nu_{cut}$, so that the power-law portion for $\nu_c < \nu < \nu_{cut}$ is absent. Below $\nu_{cut}$, the formulae of Equations (35) and (36) well represent the spectral behavior for $\sigma \lesssim 10^{-4}$. In the frequency range of $\nu_c < \nu < \nu_{cut}$, the spectrum practically follows Equation (37) for the cases of $\sigma = 10^{-3}$ and $10^{-4}$. As $\sigma$ increases, the resulting stronger magnetic field leads to a large efficiency of the energy release during the advection time. The peak of the synchrotron flux at $\nu = \nu_c$ grows with $\sigma^{p-2}$ (see Equations (29) and (35)) for $\sigma \lesssim 10^{-4}$, accompanying the shift of $\nu_c$ to a lower frequency.

The spectral behavior deviates from the above trends for $\sigma \gtrsim 10^{-3}$. This is because the radial evolution of the magnetic field can no longer be approximated by $B \propto r$. As shown in Table 2, the radius $r_{eq}$ is inside the nebula radius $r_N$ in this parameter region, unlike in the discussion in Section 3.1. The magnetic pressure prevents the flow from deceleration at $r > r_{eq}$, and the adiabatic cooling starts to play its role. The decline of the advection time (see Table 2) leads to the reduction of the total energy in the nebula. As a result, above $\sigma = 10^{-4}$, the synchrotron peak flux begins to decrease and the cooling frequency begins to increase. Therefore, we cannot decrease $\nu_c$ to extremely low values in the 1D steady model, differently from the one-zone models.

In the case of $r_{eq} < r_N$, the contribution to the entire spectrum is dominated by the emission from nonthermal pairs at $r < r_{eq}$. At $r > r_{eq}$, electrons/positrons lose their energies via adiabatic cooling as $E \propto (r/r_{eq})^{-2/3}$ and the magnetic field decays as $\propto r^{-1}$. Thus, the energy loss rate rapidly decreases as $r^{-10/3}$ so that the emission beyond $r = r_{eq}$ is almost negligible. From Equation (27), assuming $E_{\text{max}} \gg E_{\text{hot}}$, we obtain $E_{\text{cut}}(r_{eq}) \approx 5E_{\text{hot}}(r_{eq})^{\gamma} \approx 5E_{\text{hot}}(3\sigma)^{1/2}$. In this case, the cooling frequency for large enough $\sigma$ that $r_{eq}$ is smaller than $r_N$ may be estimated with $E_{\text{cut}}$ at $r = r_{eq}$ as

$$\nu_{\text{cut}}(\sigma) = \frac{3eB(r_{eq})}{4\pi m_e c} \left(\frac{E_{\text{cut}}(r_{eq})}{m_e c^2}\right)^2 \approx 5.9 \times 10^{16} \left(\frac{L_{\text{rad}}}{10^{38} \text{ erg s}^{-1}}\right)^{-2/3} \left(\frac{\sigma}{10^{-2}}\right)^{1/2} \left(\frac{r_N}{0.1 \text{ pc}}\right).$$

Although the estimate of the value of $\nu_{\text{cut}}(\sigma)$ almost agrees with our numerical results, the dependence on $\sigma$ is slightly too strong. This is because the case for $\sigma = 10^{-2}$, $10^{-3}$ is a marginal situation ($r_{eq} \sim r_N$). Nevertheless, it is certain that the cooling frequency for $\sigma > 10^{-4}$ ($r_{eq} < r_N$) becomes high with increasing $\sigma$.

In the frequency range of $\nu_{\text{cut}}(\sigma) < \nu < \nu_{\text{cut}}$ for a larger $\sigma$, the spectrum is harder than the analytical estimate $\alpha = (5p_e - 1)/9$ for $r_{eq} < r_N$. On the other hand, since $r_{ph} \sim r_c$, the frequency $\nu_{\text{cut}}$ agrees well with the analytical estimate of Equation (31) even for a larger $\sigma$. For $\sigma \gtrsim 10^{-4}$, as we have mentioned in the previous subsection, the luminosity around $\nu_{\text{cut}}$ is almost independent of $\sigma$, while the peak luminosity at $\nu = \nu_c$ decreases with increasing $\sigma$ following the decline of the total energy in the nebula. These complicated effects cause the spectral hardening between $\nu_c$ and $\nu_{\text{cut}}$.

The peak flux of the IC component declines monotonically with increasing $\sigma$. Below the spectral break frequency $10^{23}$ Hz, which corresponds to the photon energy emitted by particles of $E_b$ interacting with dust photons, all the model curves for $\sigma \lesssim 10^{-4}$ almost overlap each other. In this range of $\sigma$, the flow profiles are almost the same, so that IC emission processes, which do not directly depend on the magnetic field, are common. On the other hand, above $10^{28}$ Hz, the softening of the photon spectrum with increasing $\sigma$ is seen. The softening is caused by the decline of the cutoff energy $E_{\text{cut}}(r_N)$ due to synchrotron cooling, although $E_{\text{max}}$ is higher for a larger $\sigma$.
Above $\sigma = 10^{-4}$, the short advection time leads to the reduction of the IC flux. The drop of the cooling efficiency due to the short advection time causes the spectral hardening of the IC component. While the flux decrease of the synchrotron component due to the reduction of the advection time is mitigated by the magnetic field growth, the IC component more rapidly falls with $\sigma$ than the synchrotron component.

In the right panel of Figure 4, the X-ray surface brightness profile is shown. The size of the X-ray nebula contracts with increasing $\sigma$. The dependence of $\sigma^{-1/6}$ obtained in the previous subsection is a reasonable approximation.

4. Application to Observed Sources

Here, we apply our model to the observed sources 3C 58 and G21.5−0.9, for which significant data sets are available to constrain the model parameters. Moreover, in these two PWNe, the extent of the X-ray emission is close to the radio emission. We examine both the entire spectra and spatial profiles for those objects.

The images of 3C 58 were obtained in the radio wavelength (e.g., Reynolds & Aller 1988) and X-ray band (e.g. Slane et al. 2004). The radial profile of the photon index in the X-ray band was also obtained. The extents of the radio and X-ray images are similar as $\sim 5' \times 9'$. From the distance $D \approx 2$ kpc (Kothes 2013), we adopt $N_e = 2$ pc (Tanaka & Takahara 2013b).

The spin period and its time derivative for the central pulsar of 3C 58 are $1.94 \times 10^{-13}$ s s$^{-1}$ (Livingstone et al. 2009), respectively. The spin-down luminosity is estimated as $2.7 \times 10^{37}$ erg s$^{-1}$, assuming $10^{45}$ g cm$^{-2}$ for the moment of inertia of the pulsar.

G21.5−0.9 shows spherical structures in the radio (Bietenholz & Bartel 2008) and X-ray (Matheson & Safi-Harb 2005, 2010; Camilo et al. 2006) images. The radio and X-ray sizes of the PWN ($\sim 40''$ in radius) are almost the same again. Adopting the distance $D = 4.8$ kpc (Tian & Leahy 2008), we have $N_e = 0.9$ pc. PSR J1833-1034, the central object of G21.5−0.9, has a spin period 61.9 ms (Gupta et al. 2005) and its derivative $2.02 \times 10^{-13}$ s s$^{-1}$ (Roy et al. 2012), from which we obtain the spin-down luminosity $L_{\text{sd}} = 3.5 \times 10^{37}$ erg s$^{-1}$.

The parameters we used to fit the spectra for the two PWNe are summarized in Table 3. See Figure 2 for the ISRFs taken from GALPROP v54.1 for the two cases. First, we discuss the parameter sets denoted with “broadband” in the Table 3 (hereafter we call them broadband model). The resulting radial profiles of $u(r)$ and $B(r)$ are shown in Figure 5. In the two PWNe, $N_e$ is outside the nebula radius $r_N$ in our parameter sets. The particle spectra in Figure 6 show the evolution of $E_{\text{cut}}$ as discussed in Section 3.1. The volume-averaged spectra (dashed lines) $\tilde{n}(E)$ are well expressed by broken power-laws. The differences in particle spectral index above the cooling break energy are 0.67 for both cases. Given the particle energy $E$, Equation (27) implies that the maximum radius to which these particles survive is $r_N \propto E^{-1/5}$. The differences in the index seem consistent with a naive estimate $\tilde{n}(E) \propto E^{-\alpha} r_N^\alpha$.

Figure 7 shows the volume-integrated photon spectra for the two PWNe. Our models roughly reproduce the entire structures of the spectra. In 3C 58, the data points obtained with Fermi (Abdo et al. 2013) may contain large systematic errors due to the emission from the central pulsar so that we treat these data as upper limits. The model spectrum in the X-ray range is apparently softer than the X-ray data in 3C 58 due to the cooling effect. As discussed in Section 3.3, $\nu_c$ can be higher than the X-ray energy range by adopting a larger $\sigma$ or conversely lower $\sigma$. In such cases, the X-ray model spectrum may be as hard as the observed spectrum. However, when we adopt a lower $\sigma$ to increase $\nu_c$ to above the X-ray frequency, the synchrotron component does not extend to the X-ray energy, as Equation (31) indicates. Neither do we find a consistent high-$\nu_c$ model with a very large $\sigma$ or a slightly large $r_N$ for which the radio and X-ray fluxes are hard to reproduce simultaneously. One may assume that a smaller $\nu_c$ can agree with the observed X-ray spectral index, even if $\nu_c$ is below the X-ray frequency. The extrapolation from the X-ray data requires $\nu_c < 10^{14}$ Hz to increase $\nu_{\text{rb}}$ to above the radio data points. Such a low $\nu_c$ is hard to be realized in this model (see the $\nu_c$-turnover in Figure 4); the radio or X-ray flux becomes inconsistent for such extreme parameter sets. Therefore, our model spectra cannot be reconciled with the X-ray spectral index.
A similar problem to the case in 3C 58 arises in the X-ray spectrum of G21.5–0.9. When we fit a model with $p_s = 2$, which yields a flat spectrum ($\nu L_{\nu} \propto \nu^0$) above $\nu_c$, a much lower $\sigma$ is required to adjust the X-ray flux. For such a low $\sigma$, $\nu_{\text{cut}}$, becomes lower than the X-ray band.

In these two objects, we are forced to have $\nu_c$ below the X-ray band. As a result, the X-ray spectra show softer shapes than the observed spectra. The X-ray extents are more compact than the radio images (Figure 8). The radial profiles of photon indices in 0.5–10.0 keV range (Figure 9) also deviate from the observed data. However, the discrepancy in 3C 58 is not as prominent as the model curve by Slane et al. (2004) based on Reynolds (2003). Note that the radial profiles of photon indices in optical (3944–4952 Å) and radio (4.75 GHz) band for two objects do not depend on angular distance from the pulsar. Since the $\nu_c$ is higher than the frequencies of these bands in the parameters of the broadband models, the pairs can emit radio and optical photons without the cooling effect all over the nebula, thus the emission has the same spectral index in each radial position.

The advection time of G21.5–0.9 agrees well with the age of 870 years (Bietenholz & Bartel 2008). However, Wang et al. (2006) argued that this object is associated with the BC48 guest star and its age is thus about 2000 years. In this case, $t_{\text{adv}}$ in the broadband model becomes less than half of the age. In 3C 58, if this object is associated with SN 1181 (Stephenson 1971), $t_{\text{adv}}$ is about twice larger than the age. The characteristic ages of these objects are 5370 year for 3C 58 and 4850 year for G21.5–0.9, respectively. We note that the characteristic age tends to be older than the actual pulsar age, especially for young pulsars. Meanwhile, the previous one-zone time-dependent models have obtained the ages. For 3C 58, Bucciantini et al. (2011), Torres et al. (2013) and Tanaka & Takahara (2013b) obtained $t_{\text{age}} \sim 2000$ year, which is comparable with $t_{\text{adv}}$ in our broadband model. However, Bucciantini et al. (2011) and Torres et al. (2014) adopted a different value of the distance to the object, and these three studies did not include the data of Aleksić et al. (2014). A direct comparison of the age with our advection times does not seem very meaningful. On the other hand, for G21.5–0.9, $t_{\text{age}}$ was estimated to be 870 years (Vorster et al. 2013; Torres et al. 2014) or 1000 years (Tanaka & Takahara 2011) in one-zone models. These values are close to our estimate. These studies adopted the same condition for the distance and the observed flux, so that the coincidence of the age and $t_{\text{adv}}$ supports our 1D model.

Next, we discuss the case where the radio/IR/optical emission can be treated separately as an additional component. In the broken power-law spectrum for the pair injection, the low-energy portion dominates the number. In the case of the Crab Nebula, the particle number required to reconcile the radio flux is much larger than the theoretically expected value (Tanaka & Takahara 2010, 2011). Atoyan & Aharonian (1996) and Olmi et al. (2015) treated the low-energy component as a different component from the wind particles in their calculations. Thus, as an alternative model, we assume that the low-energy particle component responsible for the radio/IR emission has a different origin from the high-energy component. In the alternative models, we incorporate only the high-energy particles above $E_b$ and neglect the emission below the optical band.

The blue dashed lines in Figures 7–9 show the alternative models, whose model parameters are summarized in Table 3. We adopt a slightly large $r_s$ and large $\sigma$, which lead to X-ray extents consistent with observation as shown in Figure 8. The difference in X-ray profiles in the two PWNe is attributed to the effect of the adiabatic cooling in G21.5–0.9, as shown in Figure 6. Since we have adopted a $\sigma$ large enough to establish $r_{\text{eq}} < r_s$ for G21.5–0.9, a signature of adiabatic cooling appears. Although these models seem to reproduce the observed X-ray surface brightness and $X-\gamma$ fluxes, the resulting advection times become very short (see Table 3).

5. Discussion

As shown in Section 4, we have fitted the entire spectra of 3C 58 and G21.5–0.9. The obtained $\sigma$ by fitting the entire spectrum of nebula in the broadband models is about 10 times smaller than the conceivable value in the Crab Nebula and the $r_s$ we obtained is similar to the value of the Crab Nebula. Our 1D model has difficulties to reproduce both the hard spectra and the size of PWNe in X-rays. As discussed in
Section 4, our model struggles to avoid the spectral softening due to the cooling effect in X-ray range. As a result, the X-ray nebula size becomes more compact than the observed extents. The 1D model should be improved by introducing possible physical processes, such as the spatial diffusion of high-energy particles, reacceleration by turbulences, and amplification/dissipation of the magnetic field.

The 1D model tends to lead to a lower $\sigma$ than the values derived from one-zone models. One-zone time-dependent models resulted in $\sigma \sim 0.03$–0.5 (Bucciantini et al. 2011; Tanaka & Takahara 2013b; Torres et al. 2013) and 0.01–0.2 (Tanaka & Takahara 2011; Vorster et al. 2013; Torres et al. 2014) for 3C 58 and G21.5–0.9, respectively, while the value in our paper is $\sigma \sim 10^{-4}$. Since the magnetic field increases with radius in the 1D model, the average magnetic field $B_{av}$ is consistent with the previous one-zone models (see also Equations (33) and (34)). On the other hand, $\nu_{cut}$ is determined by the magnetic field near the shock rather than $B_{av}$, in contrast to the cooling break $\nu_c$. Therefore, the one-zone model overestimates the maximum synchrotron frequency about $B_{av}/B_0$ times higher than the 1D model. Note that hard X-ray observations around $\nu_{cut}$ are also interesting to investigate how the maximum energy of nonthermal pairs is determined. Although $E_{max}$ is constrained by the size of $r$, in our case, as shown in Equation (15), $\nu_{cut} \sim 100$ MeV in the Crab Nebula implies that $E_{max}$ is determined by the balance of the acceleration and cooling times (e.g., de Jager et al. 1996).

The fitted shock radius $r_s$ of 3C 58 is twice as large as that of G21.5–0.9. The total pressures at $r = r_s$ in our models are $p_{tot,3C 58} \sim 3.7 \times 10^{-10}$ erg cm$^{-3}$ and $p_{tot,G21.5} \sim 2.1 \times 10^{-9}$ erg cm$^{-3}$. From the pressure balance, the large $p_{tot}$ implies the large plasma pressure of the surrounding remnants of their supernovae. Slane et al. (2004) obtained $kT_e \sim 0.23$ keV and $p_{SNR} \sim 0.38$ cc$^{-1}$ for 3C 58 and then its pressure $\sim 1.4 \times 10^{-10}$ erg cm$^{-3}$. Matheson & Saff-Harb (2010) obtained $kT_e \sim 0.3$ keV and $p_{SNR} \sim 0.63$ cm$^{-3}$ for G21.5–0.9 and then its pressure $\sim 3.0 \times 10^{-10}$ erg cm$^{-3}$. While the pressure values may be not so robust, this indicates that the surrounding SNR of G21.5–0.9 has a higher pressure than 3C 58. In addition, the fact that the bright shell-like SNR is clearly seen in G21.5–0.9 also supports a higher pressure for G21.5–0.9 than that for 3C 58.

In the broadband models, although we reproduce the flux levels of the entire spectrum, the X-ray spectral indices

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**Figure 8.** Radial profiles of the surface brightness in various frequencies for 3C 58 (left panel) and G21.5–0.9 (right panel). The X-ray data points are taken from Slane et al. (2004) and Matheson & Saff-Harb (2005) for 3C 58 and G21.5–0.9, respectively. The thick lines are the X-ray surface brightnesses for the broadband (red solid) and alternative (blue dashed) models.

**Figure 9.** Radial profiles of photon indices in 0.5–10.0 keV range for 3C 58 (left panel) and G21.5–0.9 (right panel) in the broadband (red solid) and alternative (blue dashed) models. The model and data points in Slane et al. (2004) are also plotted for 3C 58. The data points for G21.5–0.9 are taken from Matheson & Saff-Harb (2005).
disagree with the observations. We studied alternative models, in which the emission in radio and optical is assumed to be different from the direct emission from the pulsar wind (Atoyan & Aharonian 1996; Olmi et al. 2014, 2015). As shown in Table 3, the obtained $\sigma$ in the alternative models tends to be larger than the value in the broadband models. This tendency is similar to some one-zone models (Torres et al. 2013; Vorster et al. 2013). The time-dependent model of Torres et al. (2013) introduced an energy density of the ISRF higher by an order of magnitude larger (i.e., the lager magnetic field strength) than ours in order to reproduce the X-ray spectral index of 3C 58. In the model of Vorster et al. (2013), the X-ray spectrum of G21.5–0.9 was also reproduced by a strong magnetic field ($230 \mu G$) and a hard spectral index ($p_b = 2.0$). Note that their predicted GeV flux seems above the Fermi upper limit (Ackermann et al. 2011). The cooling break was set well below keV range in these models, while it is hard to set $\nu_{c}$ low enough in our model because the high $\sigma$ leads to a shorter advection time than the cooling timescale of low-energy particles, as discussed in Section 3.3. The temporal evolution of the magnetic field in one-zone models causes the gradual hardening of the particle spectrum (see also Tanaka & Takahara 2010), which is favorable to fit the X-ray data differently from our steady model.

Since a larger $\sigma$ is required in the alternative models, the resulting short advection time prevents high-energy particles from cooling before reaching the edge of the nebula. However, such a short advection time may contradict the age of the PWNe. To validate $t_{age} \gg t_{adv}$, the efficient particle escape at the nebula surface should be required. Although the large amount of the escaped high-energy particles should emit photons outside the PWNe, such a signature outside PWNe has not been claimed. For example, the model of Holler et al. (2012), in which the radial velocity profile is artificially tuned, also implies $\sim 100$ years for the advection time in G0.9+0.1, although the age is more than kyr. We should carefully note the advection time in modeling the outflow property (see Equation (22)). Even for the models including the effect of the spatial diffusion (Tang & Chevalier 2012; Porth et al. 2016), a short diffusion timescale may be required to reproduce the X-ray surface brightness. This leads to the same problem in the case of short advection times. If the outer supernova ejecta efficiently confines the PWN, the fast outflow implied in the high $\sigma$ model should be decelerated near the edge of the PWN, and should induce turbulence inside the PWN. As a result, the wind material may be efficiently mixed inside the PWN (e.g., Porth et al. 2014). In this case, the one-zone approximation may be rather adequate.

As shown in Figure 8, we calculated the radial radio profiles in the broadband model. They appear to be an almost uniform profile in radius and seem to agree with the observational facts (e.g., Bietenholz & Bartel 2008; Bietenholz et al. 2013). Additionally, the radiation in $\nu < \nu_{c}$ ($\sim 10^{15}$ Hz) shows a similar profile as radio profile. This is because the radial evolution of the density is common for the non-cooled particles. If a clear distinction of surface brightness between the radio and optical is detected, this would be the strong basis of the hypothesis that the radio emission of PWNe has a different origin from the optical and X-rays. We also calculate the radial profile of the surface brightness in 0.8–2.0 and 8–10 TeV. The extents of 3C 58 and G21.5–0.9 in 0.8–2 TeV are larger than X-rays in the broadband models because the photon of $\sim 1$ TeV is emitted by nonthermal pairs with lower energies than the energy of particles emitting the synchrotron radiation at $\nu \sim \nu_{c}$ (see de Jager & Djannati-Ataï 2009). Since the extent of 3C 58 is about 200", CTA will be able to present the spatially resolved $\gamma$-ray map of 3C 58 in $\sim 1$ TeV.

6. Conclusion

We have revisited the 1D steady model and applied it to the PWNe, in order to find a parameter set consistent with both the entire photon spectrum and the surface brightness profile. It is still controversial whether the simple 1D model reproduces observed properties of PWNe other than the Crab Nebula. As we have shown in Section 3, the entire photon spectrum and the surface brightness profile both largely depend on the parameters, the uncertain shock radius $r_{s}$, and the magnetization parameter $\sigma$. The flux of the IC component becomes dim with increasing $\sigma$. In contrast, the synchrotron component is not a monotonic function of $\sigma$. For the dependence on $r_{s}$, the synchrotron component becomes dim with increasing $r_{s}$, the IC component shows complicated behaviors. The X-ray size of a PWN becomes large with increasing $r_{s}$ and decreasing $\sigma$.

We have fitted the entire spectrum of two observed sources, 3C 58 and G21.5–0.9. Calculating the radial profile of the surface brightness for these models, we showed that the resulting X-ray extents are significantly smaller than the observed sizes. Furthermore, we have performed another parameter set called “alternative” model, where we treated the radio and optical emissions as extra components. The alternative models successfully reproduce the observed X-ray surface brightness and the X-ray and $\gamma$-ray fluxes. However, these models imply a too short advection time. In summary, the 1D model constructed by KC84s has severe difficulties in reproducing both the spectrum and the spatial emission profile of PWNe consistently. The model should be improved by taking some possible physical processes into consideration, such as spatial diffusion of nonthermal particles and reacceleration by turbulence.

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Appendix

In order to check our model, we calculate the entire spectrum and the X-ray profile of the Crab Nebula and compare the result with Atoyan & Aharonian (1996) (hereafter AA96). We treat the SSC approximately by assuming that the synchrotron photons exist in the nebula homogeneously. The other differences of our model from AA96 are as follows. In our model, the low-energy particles responsible for the radio emission are supplied by the pulsar wind, while AA96 considered them as another component. In AA96, the maximum energy of pairs was treated as a parameter of the model (but see Equation (15) for our model). Finally, AA96 introduced the correction factor $\kappa$, which is a parameter to
adjust the ratio of the synchrotron flux to the SSC flux. The parameter $\kappa$ (see, how AA96 adopted $\kappa \sim 0.5$) may represent the effects of deviation from the spherical symmetry or inhomogeneity inside the PWN.

In Figure 10, the entire photon spectrum and the X-ray radial profile for the Crab Nebula are shown. All the parameters to calculate the spectrum for the Crab Nebula are the same as in AA96 without the “obtained parameter,” and they are summarized in Table 4. The flux of IC becomes slightly lower than the value observed and calculated by AA96. The difference in the SSC flux is due to the additional parameters $\kappa$ and $E_{\text{max}}$ in AA96. Even if we take the smaller $\sigma$ to enhance the SSC flux, the maximum energy of the synchrotron emission becomes much lower than the cutoff energy of the observed spectrum (see Equation (31)). Within our conservative model assumption for the maximum energy of particles, we are not able to fit the spectrum around the cutoff of the spectrum.

The right panel of Figure 10 shows the radial profile of the surface brightness in the 3.0–5.0 keV range. The extent of the X-ray nebula calculated by our model is comparable with observations. Comparing this with the cases of 3C 58 and G21.5–0.9, we do not have so strong a motivation to improve the 1D steady model in the case of the Crab Nebula.

### Table 4

Parameters in the Calculations for the Crab Nebula

| Given Parameters | Symbol | Crab |
|------------------|--------|------|
| Spin-down luminosity (erg s$^{-1}$) | $L_{\text{sd}}$ | $5 \times 10^{38}$ |
| Distance (kpc) | $D$ | 2.0 |
| Radius of the nebula (pc) | $r_\text{N}$ | 1.8 |

| Fitting Parameters | $E_{\text{b}}$ | $2.5 \times 10^{11}$ |
|--------------------|-----------------|----------------------|
| Break energy (eV) | $E_{\text{b}}$ | $2.5 \times 10^{11}$ |
| Low-energy power-law index | $p_1$ | 1.6 |
| High-energy power-law index | $p_2$ | 2.4 |
| Radius of the termination shock (pc) | $r_\text{tr}$ | 0.1 |
| Magnetization parameter | $\sigma$ | $5.0 \times 10^{-3}$ |

| Obtained Parameters | $\gamma_\text{u}$ | $3.2 \times 10^{3}$ |
|--------------------|-------------------|---------------------|
| Initial bulk Lorentz factor | $\gamma_\text{u}$ | $3.2 \times 10^{3}$ |
| Pre-shock density (cm$^{-3}$) | $n_\text{u}$ | $1.7 \times 10^{-9}$ |
| Pre-shock magnetic field ($\mu G$) | $B_\text{u}$ | 30 |
| Maximum Energy (eV) | $E_{\text{max}}$ | $2.7 \times 10^{15}$ |
| Advection time (year) | $t_{\text{adv}}$ | 380 |
| Averaged magnetic field ($\mu G$) | $B_{\text{av}}$ | 234 |
| Total pressure at $r = r_\text{N}$ (erg cm$^{-3}$) | $p_\text{tot}$ | $2.0 \times 10^{-9}$ |
| $\rho_{\text{eq}/rN}$ | $...$ | 0.45 |

Note.

$^a$ The parameters denoted “given parameter” and “fitting parameter” are adopted at the same value as AA96.

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