0. Introduction

Let $\mathbb{C}$ be the field of complex numbers. Let $(C, p_1, \ldots, p_n)$ be a connected, reduced, projective, nodal curve over $\mathbb{C}$ with $n$ nonsingular marked points $(p_1, \ldots, p_n)$. Let $\omega_C$ be the dualizing sheaf of $C$. An algebraic map $\mu : (C, p_1, \ldots, p_n) \to \mathbb{P}^r$ is Kontsevich stable if $\omega_C(p_1 + \ldots + p_n) \otimes \mu^*(\mathcal{O}_{\mathbb{P}^r}(3))$ is ample on $C$. Let $\overline{M}_{g,n}(r,d)$ be the coarse moduli space of degree $d$, Kontsevich stable maps from $n$-pointed, genus $g$ curves to $\mathbb{P}^r$. In the genus zero case, $\overline{M}_{0,n}(r,d)$ is an irreducible, projective variety with finite quotient singularities. Only the following cases will be considered here:

$$d \geq 0, \quad g = 0, \quad r \geq 2.$$ 

The stack of Kontsevich stable maps was first defined in [K-M] and [K]. A treatment of the corresponding coarse moduli spaces can also be found in [P] and [Al].

The dimension of $\overline{M} = \overline{M}_{0,n}(r,d)$ is $m = rd + d + r + n - 3$. Let $Pic(\overline{M})$ be the Picard group of line bundles. Let $A_{m-1}(\overline{M})$ be the Chow group of Weil divisors modulo rational equivalence. Since $\overline{M}$ has finite quotient singularities, every Weil divisor is $\mathbb{Q}$-Cartier. Therefore, there is a canonical isomorphism:

$$Pic(\overline{M}) \otimes \mathbb{Q} \cong A_{m-1}(\overline{M}) \otimes \mathbb{Q}.$$ 

$Pic(\overline{M}) \otimes \mathbb{Q}$ is a finite dimensional vector space. An explicit set of generators is given below.

Let $P = \{1, 2, \ldots, n\}$ be the set of markings ($P$ may be the empty set). The $n$ markings of the moduli problem yield $n$ canonical line bundles
$L_i = \nu_i^*(O_{\mathbb{P}^r}(1))$ on $\overline{M}$ via the $n$ evaluation maps $\forall i \in P$, $\nu_i : \overline{M} \rightarrow \mathbb{P}^r$. The boundary of $\overline{M}$ is the locus corresponding to maps with reducible domain curves. Since the boundary is of pure codimension 1 in $\overline{M}$, each irreducible component is a Weil divisor. The irreducible components of the boundary are in bijective correspondence with data of weighted partitions $(A \cup B, d_A, d_B)$ where:

(i.) $A \cup B$ is a partition of $P$.
(ii.) $d_A + d_B = d$, $d_A > 0$, $d_B > 0$.
(iii.) If $d_A = 0$ (resp. $d_B = 0$), then $|A| \geq 2$ (resp. $|B| \geq 2$).

For example, if $P = \emptyset$, then $A = B = \emptyset$ and the boundary components correspond to positive partitions $d_A + d_B = d$. Let $\triangle$ be the set of components of the boundary.

In case $d \geq 1$, a Weil divisor is obtained on $\overline{M}$ by considering the locus of $\overline{M}$ corresponding to maps meeting a fixed $r - 2$ dimensional linear subspace of $\mathbb{P}^r$ (note $r \geq 2$). It is shown in [P] that this incidence Weil divisor is actually Cartier. Denote the corresponding line bundle on $\overline{M}$ by $H$. For convenience, let $H = 0$ in case $d = 0$.

**Proposition 1. Results on generation:**

(i.) If $d = 0$, $g = 0$, $r \geq 2$, $\{L_i\} \cup \triangle$ generate $\text{Pic}(\overline{M})$.
(ii.) If $d \geq 1$, $g = 0$, $r \geq 2$, $\{L_i\} \cup \triangle \cup \{H\}$ generate $\text{Pic}(\overline{M}) \otimes \mathbb{Q}$.

If $d = 0$, then (by stability) $n \geq 3$ and $\overline{M}_{0,n}(r,0) \cong \overline{M}_{0,n} \times \mathbb{P}^r$ where $\overline{M}_{0,n}$ is the Mumford-Knudsen space. In this case, $L_i$ is the pull-back of $O_{\mathbb{P}^r}(1)$ from the second factor. Therefore, part (i) is a consequence of the boundary generation of $\text{Pic}(\overline{M}_{0,n})$.

There is an intersection pairing $A_1(\overline{M}) \otimes \text{Pic}(\overline{M}) \rightarrow \mathbb{Z}$. Let $\text{Null} \subset \text{Pic}(\overline{M})$ be the null space with respect to the intersection pairing. Define

$$\text{Num}(\overline{M}) = \text{Pic}(\overline{M})/\text{Null}.$$  

By Proposition [P], the classes $\{L_i\} \cup \triangle \cup \{H\}$ generate $\text{Num}(\overline{M}) \otimes \mathbb{Q}$. The relations between these generators in $\text{Num}(\overline{M}) \otimes \mathbb{Q}$ can be algorithmically determined by calculating intersections with curves. It will be shown that all the relations in $\text{Num}(\overline{M}) \otimes \mathbb{Q}$ are obtained from linear equivalences in $\text{Pic}(\overline{M}) \otimes \mathbb{Q}$. 

2
Proposition 2. The canonical map $\text{Pic}(\overline{M}) \otimes \mathbb{Q} \to \text{Num}(\overline{M}) \otimes \mathbb{Q}$ is an isomorphism. The Picard numbers are:

- $(n = 0), \quad \dim_{\mathbb{Q}} \text{Pic}(\overline{M}) \otimes \mathbb{Q} = \left[ \frac{d}{2} \right] + 1.$
- $(n \geq 1), \quad \dim_{\mathbb{Q}} \text{Pic}(\overline{M}) \otimes \mathbb{Q} = (d + 1) \cdot 2^{n-1} - \left( \begin{array}{c} n \\ 2 \end{array} \right).$

The main result of this paper concerns the computations of top intersection products in $\text{Pic}(\overline{M}) \otimes \mathbb{Q}$.

Proposition 3. Let $d \geq 0, \ g = 0, \ r \geq 2$. There exists an explicit algorithm for calculating the top dimensional intersection products of the $\mathbb{Q}$-Cartier divisors $\{L_i\} \cup \Delta \cup \{H\}$ on $\overline{M}$.

Consider the space $R(d,r)$ of degree $d \geq 1$ rational curves in $\mathbb{P}^r$ ($r \geq 2$). The dimension of $R(d,r)$ is $rd + r + d - 3$. Classically, the characteristic numbers of $R(d,r)$ are the numbers of degree $d$ rational curves in $\mathbb{P}^r$ passing through $\alpha_i$ general linear spaces of codimension $i$ (for $2 \leq i \leq r$) and tangent to $\beta$ general hyperplanes where

$$(i - 1) \cdot \alpha_i + \beta = \dim R(d,r).$$

The characteristic numbers of rational curves excluding tangencies ($\beta = 0$) have been determined recursively by M. Kontsevich and Y. Manin in [K-M] (also by Y. Ruan and G. Tian in [R-T]). The divisor in $\overline{M}$ corresponding to the hyperplane tangency condition can be expressed as a linear combination of the classes $\{L_i\} \cup \Delta \cup \{H\}$. The characteristic numbers can then be expressed as top intersection products of $\{L_i\} \cup \Delta \cup \{H\}$ on suitably chosen Kontsevich spaces of maps $\overline{M}$. Therefore, all the characteristic numbers can be calculated by Proposition 3.

Proposition 4. There exists an explicit algorithm for calculating all the characteristic numbers of rational curves in projective space.

P. Di Francesco and C. Itzykson have modified the methods of [K-M] to determine some ($\beta \neq 0$) characteristic numbers for rational plane curves ([D-I]). Unfortunately, the relations they obtain from the WDVV associativity equations do not suffice to recursively determine all the characteristic numbers for rational plane curves from a finite set of data.

The structure of the paper is as follows. Propositions (1) and (2) are proven in section (1). In section (2), several geometric classes are
explicitly computed in $\text{Pic}(\overline{M}) \otimes \mathbb{Q}$. These classes will be used in the algorithms of Propositions (3) and (4). The algorithms are established in section (3). Section (4) is devoted to calculations of some characteristic numbers of rational curves for small values of $(d, r)$. As a final application, a new formula for cuspidal rational curves is derived in section (4.5).

The problem of calculating tangency characteristic numbers via Kontsevich’s moduli space was suggested to the author by W. Fulton. Conversations with W. Fulton on related topics have been of significant aid. Thanks are due to S. Kleiman and P. Aluffi for mathematical and historical remarks. The remarkable ideas in [K-M] have been a source of inspiration.

1. Pic($\overline{M}$) $\otimes \mathbb{Q}$ and Num($\overline{M}$) $\otimes \mathbb{Q}$

1.0. Summary. Propositions (3) and (4) are established in sections (1.1) and (1.2) respectively. Since these results are well known for $d = 0$,

$$\overline{M}_{0,n}(r, 0) \cong \overline{M}_{0,n} \times \mathbb{P}^r,$$

the conditions $d \geq 1$, $g = 0$, $r \geq 2$ are assumed throughout sections (1.1) and (1.2).

1.1. Generators. The proof of Proposition (1) is divided into four cases depending upon the number $n$ of marked point.

**Lemma 1.1.1.** If $n \geq 3$, then Pic($\overline{M}$) $\otimes \mathbb{Q}$ is generated by $\triangle \cup \{\mathcal{H}\}$.

**Proof.** Let $V = \bigoplus_0^r H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$. Let $U \subset \mathbb{P}(V)$ be the Zariski open set corresponding to a well defined (basepoint free) degree $d$ map from $\mathbb{P}^1$ to $\mathbb{P}^r$. The complement of $U$ in $\mathbb{P}(V)$ is of codimension at least $r \geq 2$. There is a universal map

$$\mathbb{P}^1 \times U \to \mathbb{P}^r.$$

Fix the first three marked points to be $0, 1, \infty \in \mathbb{P}^1$. Let

$$W = \mathbb{P}^1 \times \ldots \times \mathbb{P}^1 \setminus \{D_{i,j}, S_{0,i}, S_{1,i}, S_{\infty,i}\}$$

where the product is taken over $n - 3$ factors. $D_{i,j}$ is the large diagonal determined by factors $i$ and $j$. $S_{0,i}$ is the locus where the $i^{th}$ factor is
0 ∈ \mathbb{P}^1. S_{1,i}, S_{\infty,i} are defined similarly. It follows there is a universal family of Kontsevich stable degree \( d \) maps of \( n \)-pointed curves:

\[
\mathbb{P}^1 \times W \times U \to \mathbb{P}^r.
\]

The maps of the family are automorphism-free and distinct. By the universal property, there is an injection \( W \times U \to \overline{M} \). A tangent space calculation shows \( W \times U \) is an open set of \( \overline{M} \). The complement of \( W \times U \) is the boundary of \( \overline{M} \). Hence \( A_{m-1}(W \times U) \) is generated by \( \triangle \) and \( A_{m-1}(W \times U) \). Information about \( A_{m-1}(W \times U) \) is obtained from the open inclusion

\[
W \times U \subset \mathbb{P}^1 \times \ldots \times \mathbb{P}^1 \times \mathbb{P}(V).
\]

The Picard group of the right side of (1) is generated by the pull-backs of \( \mathcal{O}(1) \) from each factor. The pull-backs from the \( \mathbb{P}^1 \) factors are trivial on \( W \times U \) because of the removal of the loci \( S_{0,i} \). Hence, \( A_{m-1}(W \times U) \) is generated by \( \mathcal{O}_{\mathbb{P}(V)}(1) \). It is easily seen \( \mathcal{H} \) restricted to \( W \times U \) is the pull-back of a resultant hypersurface in \( \mathbb{P}(V) \). Therefore, \( \mathcal{H} \) restricted to \( W \times U \) is linearly equivalent to a multiple of \( \mathcal{O}_{\mathbb{P}(V)}(1) \).

There are canonical morphisms \( \overline{M}_{0,n}(r, d) \to \overline{M}_{0,n-1}(r, d) \) obtained by omitting the last marked point. Results for \( 0 \leq n \leq 2 \) are obtained via these morphisms.

**Lemma 1.1.2.** If \( n = 2 \), then \( \text{Pic}(\overline{M}) \otimes \mathbb{Q} \) is generated by \( \triangle \cup \{ \mathcal{L}_1, \mathcal{L}_2 \} \).

**Proof.** Let \( \overline{N} = \overline{M}_{0,3}(r, d) \) and \( \overline{M} = \overline{M}_{0,2}(r, d) \). Fix a hyperplane \( H_3 \subset \mathbb{P}^r \). Let \( X = \nu_3^{-1}(H_3) \) where \( \nu_3 \) is the third evaluation map, \( \nu_3 : \overline{N} \to \mathbb{P}^r \). There is a map \( \rho : X \to \overline{M} \) obtained by omitting the third point. The map \( \rho \) is surjective and generically finite. Let \( Z \subset \overline{M} \) be the open set corresponding to Kontsevich stable maps satisfying the following conditions:

(i.) The domain curve is \( \mathbb{P}^1 \).

(ii.) The images of the marked points \( \{1, 2\} \) do not lie in \( H_3 \).

It is clear the the complement of \( Z \) is the boundary union \( \nu_1^{-1}(H_3), \nu_2^{-1}(H_3) \). By the definition of \( Z \), \( \rho^{-1}(Z) \to Z \) is a finite, projective morphism. If \( A_{m-1}(\rho^{-1}(Z)) = 0 \), then \( A_{m-1}(Z) \) is torsion. To establish the Lemma, it therefore suffices to prove \( A_{m-1}(\rho^{-1}(Z)) = 0 \).
In the notation of the proof of Lemma (1.1.1), \( \rho^{-1}(Z) \subset U \subset M_{0,3}(r, d) \). In fact, the following is easily seen:

\[
\rho^{-1}(Z) = U \cap L_{\infty}(H_3) \setminus \{L_0(H_3), L_1(H_3)\}.
\]

\( L_\rho(H_3) \) is the hyperplane in \( U \) corresponding to maps sending the point \( p \in \mathbb{P}^1 \) to \( H_3 \). \( U \cap L_{\infty}(H_3) \) is an open set of \( L_{\infty}(H_3) \) with complement of codimension at least 2. Hence, \( A_{m-1}(U \cap L_{\infty}(H_3)) = \mathbb{Z} \) generated by the hyperplane class. Since \( \rho^{-1}(Z) \subset U \cap L_{\infty}(H_3) \) is the complement of hyperplanes, the desired conclusion \( A_{m-1}(\rho^{-1}(Z)) = 0 \) is obtained. \( \Box \)

**Lemma 1.1.3.** If \( n = 1 \), then \( \text{Pic}(\overline{M}) \otimes \mathbb{Q} \) is generated by \( \Delta \cup \{ \mathcal{L}_1, \mathcal{H} \} \).

**Proof.** Let \( \overline{N} = \overline{M}_{0,3}(r, d) \) and \( \overline{M} = \overline{M}_{0,1}(r, d) \). Fix two hyperplanes \( H_2, H_3 \subset \mathbb{P}^r \). Let \( X = \nu_2^{-1}(H_2) \cap \nu_3^{-1}(H_3) \) where \( \nu_2, \nu_3 \) are the second and third evaluation maps on \( \overline{N} \). There is a map \( \rho : X \to \overline{M} \) obtained by omitting the second and third points. The map \( \rho \) is surjective and generically finite. Let \( Z \subset \overline{M} \) be the open set corresponding to Kontsevich stable maps satisfying the following conditions:

(i.) The domain curve is \( \mathbb{P}^1 \).

(ii.) The image of the marked point \( \{1\} \) does not lie in \( H_2 \cup H_3 \).

(iii.) The map does not pass through the intersection \( H_2 \cap H_3 \).

The complement of \( Z \) is the boundary union \( \nu_1^{-1}(H_2), \nu_1^{-1}(H_3) \), and \( D_{2,3} \). \( D_{2,3} \) is the Cartier divisor of maps passing through \( H_2 \cap H_3 \). \( D_{2,3} \) is a divisor in the linear series of \( \mathcal{H} \). By the definition of \( Z \), \( \rho^{-1}(Z) \to Z \) is a finite, projective morphism. As before, it suffices to prove \( A_{m-1}(\rho^{-1}(Z)) = 0 \).

Let \( S \subset U \) be the union of the hyperplane sections \( \{L_0(H_2), L_0(H_3)\} \) with the resultant hypersurface of maps meeting \( H_2 \cap H_3 \). Conditions (i), (ii), and (iii) imply:

\[
\rho^{-1}(Z) = U \cap L_1(H_2) \cap L_{\infty}(H_3) \setminus S.
\]

As before, \( A_{m-1}(U \cap L_1(H_2) \cap L_{\infty}(H_3)) = \mathbb{Z} \) generated by the hyperplane class. \( S \) is a union of hyperplane classes and multiples of hyperplane classes. Hence, \( A_{m-1}(\rho^{-1}(Z)) = 0 \). \( \Box \)

**Lemma 1.1.4.** If \( n = 0 \), then \( \text{Pic}(\overline{M}) \otimes \mathbb{Q} \) is generated by \( \Delta \cup \{ \mathcal{H} \} \).

**Proof.** Let \( \overline{N} = \overline{M}_{0,3}(r, d) \) and \( \overline{M} = \overline{M}_{0,0}(r, d) \). Fix three general hyperplanes \( H_1, H_2, H_3 \subset \mathbb{P}^r \). Let \( X = \nu_1^{-1}(H_1) \cap \nu_2^{-1}(H_2) \cap \nu_3^{-1}(H_3) \)
where the $\nu_i$ are evaluation maps on $\overline{N}$. There is a map $\rho : X \to \overline{M}$ obtained by omitting the marked points. The map $\rho$ is surjective and generically finite. Let $Z \subset \overline{M}$ be the open set corresponding to Kontsevich stable maps satisfying the following conditions:

(i.) The domain curve is $\mathbb{P}^1$.
(ii.) The map does not pass through the intersections $H_1 \cap H_2$, $H_1 \cap H_3$, or $H_2 \cap H_3$.

The complement of $Z$ is the boundary union $D_{1,2}$, $D_{1,3}$, and $D_{2,3}$. By the definition of $Z$, $\rho^{-1}(Z) \to Z$ is a finite, projective morphism. As before, it suffices to prove $A_{m-1}(\rho^{-1}(Z)) = 0$.

Let $S \subset U$ be the union of the three resultant hypersurfaces of maps meeting $H_1 \cap H_2$, $H_1 \cap H_3$, and $H_2 \cap H_3$. Let $I \subset U$ be the hyperplane intersection defined by $I = U \cap L_0(H_1) \cap L_1(H_2) \cap L_\infty(H_3)$. Conditions (i) and (ii) imply:

$$\rho^{-1}(Z) = I \setminus S \cap I.$$  

Note $S \cap I$ contains the intersections of the following hyperlanes with $I$:

$$\{L_0(H_2), L_0(H_3), L_1(H_1), L_1(H_3), L_\infty(H_1), L_\infty(H_2)\}.$$  

As before, $A_{m-1}(U \cap I) = \mathbb{Z}$ generated by the hyperplane class. Since $S \cap I$ is a union of of hyperplane classes and multiples of hyperplane classes, $A_{m-1}(\rho^{-1}(Z)) = 0$.  

Lemmas (1.1.1) - (1.1.4) yield Proposition (I).

1.2. Relations. Curves in $\overline{M} = \overline{M}_{0,n}(r, d)$ are easily found. The following construction will be required for the calculations below. Let $C$ be a nonsingular, projective curve. Let $\pi : S = \mathbb{P}^1 \times C \to C$. Select $n$ sections $s_1, \ldots, s_n$ of $\pi$. A point $x \in S$ is an intersection point if two or more sections contain $x$. Let $\mathcal{N}$ be a line bundle on $S$ of type $(d, k)$ where $k$ is very large. Let $z_i \in H^0(S, \mathcal{N})$ $(0 \leq i \leq r)$ determine a rational map $\mu : S \to \mathbb{P}^r$ with simple base points. A point $y \in S$ is a simple base point of degree $1 \leq e \leq d$ if the blow-up of $S$ at $y$ resolves $\mu$ locally at $y$ and the resulting map is of degree $e$ on the exceptional divisor $E_y$. The set of special points of $S$ is the union of the intersection points and the simple base points. Three conditions are required:

(1.) There is at most one special point in each fiber of $\pi$.  


(2.) The sections through each intersection point $x$ have distinct tangent directions at $x$.

(3.) If $n$ or $n - 1$ sections pass through the point $x \in S$, then $x$ is not a simple base point of degree $d$. (If $n = 0$ or 1, there are no simple base points of degree $d$.)

Let $\overline{S}$ be the blow-up of $S$ at the special points. It is easily seen $\pi: \overline{S} \rightarrow \mathbb{P}^r$ is Kontsevich stable family of $n$-pointed, genus 0 curves over $C$. Condition (2) ensures the strict transforms of the sections are disjoint. Condition (3) implies Kontsevich stability. There is a canonical morphism $C \rightarrow \overline{M}$. Condition (1) implies $C$ intersects the boundary components transversally.

**Lemma 1.2.1.** Results on the span of $\{\mathcal{H}, \mathcal{L}_1, \mathcal{L}_2\}$:

(i.) The element $\mathcal{H}$ is not contained in the linear span of $\triangle$ in $\text{Pic}(\overline{M}) \otimes \mathbb{Q}$.

(ii.) If $n = 1$, $\{\mathcal{H}, \mathcal{L}_1\}$ are independent modulo $\triangle$.

(iii.) If $n = 2$, $\{\mathcal{L}_1, \mathcal{L}_2\}$ are independent modulo $\triangle$.

**Proof.** Consider $\pi: S = \mathbb{P}^1 \times C \rightarrow C$ with $n$ trivial sections. There are no intersection points. Let $\mathcal{N}, z_i \in H^0(S, \mathcal{N})$ be such that $\mu$ has no base points (note: since $r \geq 2$, this is easily accomplished). $\mathcal{N}$ has degree type $(d, k)$. For each component $K \in \triangle$, $C \cdot K = 0$. A simple calculation yields $C \cdot \mathcal{H} = \mathcal{N} \cdot \mathcal{N} = 2dk$. Hence $\mathcal{H}$ is not contained in the span of $\triangle$.

Consider $\pi: S = \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$. Let $s$ be the trivial section; let $s'$ be the diagonal section. Let $\mu: S \rightarrow \mathbb{P}^r$ be a base point free map of type $(d, k)$. The two sections $s, s'$ determine two maps $\tau, \tau': \mathbb{P}^1 \rightarrow \overline{M}_{0,1}(r, d)$. Intersection via $\tau$ yields:

$$\mathbb{P}^1 \cdot \mathcal{H} = 2dk, \quad \mathbb{P}^1 \cdot \mathcal{L}_1 = k.$$  

Intersection via $\tau'$ yields:

$$\mathbb{P}^1 \cdot \mathcal{H} = 2dk, \quad \mathbb{P}^1 \cdot \mathcal{L}_1 = d + k.$$  

In both cases $\mathbb{P}^1 \cdot K = 0$ for any $K \in \triangle$. Therefore $\{\mathcal{H}, \mathcal{L}_1\}$ are independent modulo $\triangle$ in $\text{Pic}(\overline{M}) \otimes \mathbb{Q}$ for $n = 1$.

In the $n = 2$ case, twisted families must be considered. Let $E(a, b)$ be the rank two bundle $\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ over $\mathbb{P}^1$. Let $S(a, b) = \mathbb{P}(E(a, b))$. Let

$$\mathcal{N} = \mathcal{O}_{\mathbb{P}(E)}(d) \otimes \pi^*(\mathcal{O}_{\mathbb{P}^1}(k)).$$
For large $k$, let $\mu : S(a, b) \to \mathbb{P}^r$ be a base point free map. The sub-bundles $\mathcal{O}(a)$, $\mathcal{O}(b)$ define sections $s_1$ and $s_2$. There is an induced map $\mathbb{P}^1 \to \overline{M}_{2,2}(r, d)$. A calculation yields:

$$\mathbb{P}^1 \cdot L_1 = -ad + k, \quad \mathbb{P}^1 \cdot L_2 = -bd + k.$$ 

As before $\mathbb{P}^1 \cdot K = 0$ for any $K \in \Delta$. It follows $\{L_1, L_2\}$ are independent modulo $\Delta$ in $\text{Pic}(\overline{M}) \otimes \mathbb{Q}$ for $n = 2$.

If $n \geq 1$, let $\Delta_i \subset \Delta$ be the subset of boundary components $(A \cup B, d_A, d_B)$ with marking partition $|A| + |B| = n$ equal to the partition $i + (n - i) = n$. There is a disjoint union

$$\Delta = \bigcup_{i=0}^{\lfloor \frac{n}{2} \rfloor} \Delta_i.$$ 

Let $\Delta' = \Delta \setminus (\Delta_0 \cup \Delta_1)$.

**Lemma 1.2.2.** Results on the span of $\Delta_0$, $\Delta_1$:

1. If $n = 0$, $\Delta_0 = \Delta$ is a set of linearly independent elements of $\text{Pic}(\overline{M}) \otimes \mathbb{Q}$.
2. If $n = 1$, $\Delta_0 = \Delta_1 = \Delta$ is a set of linearly independent elements of $\text{Pic}(\overline{M}) \otimes \mathbb{Q}$.
3. If $n \geq 2$, $\Delta_0 \cup \Delta_1$ is a set of linearly independent elements of $\text{Pic}(\overline{M}) \otimes \mathbb{Q}$. Moreover, the span of $\Delta_0 \cup \Delta_1$ does not intersect the span of $\Delta'$ in $\text{Pic}(\overline{M}) \otimes \mathbb{Q}$.

**Proof.** Let $\pi : S = \mathbb{P}^1 \times C \to C$ be as above with $n$ trivial sections. Let $\mathcal{N}$ be a line bundle on $S$ of degree type $(d, k)$. For large degrees $k$, the simple base points of $\mu$ of degree $1 \leq e \leq d$ can be selected arbitrarily satisfying conditions (1) and (3). For suitable choices of simple base points and base point degrees on $S$, the classes in assertions (i-iii) can be seen to be independent in $\text{Num}(\overline{M}) \otimes \mathbb{Q}$. Therefore, the classes are independent in $\text{Pic}(\overline{M}) \otimes \mathbb{Q}$. 

In case $n = 0$, $\Delta \cup \{\mathcal{H}\}$ is a basis of $\text{Pic}(\overline{M}) \otimes \mathbb{Q}$ via Lemmas (1.1.4), (1.2.1), and (1.2.2). For $1 \leq n \leq 3$, $\Delta_0 \cup \Delta_1 = \Delta$. Hence, the Lemmas show the generators of section (1.1) are also bases for $1 \leq n \leq 3$. The Picard numbers of Proposition (2) can be verified for $0 \leq n \leq 3$.

For $n \geq 4$, let $\overline{M}_{0,n}$ be the Mumford-Knudsen moduli space of $n$-pointed, genus 0 curves. The boundary components of $\overline{M}_{0,n}$ correspond
bijectively to partitions $A \cup B$ of $P = \{1, 2, \ldots, n\}$ such that $|A|, |B| \geq 2$. The boundary components generate $Pic(M_{0,n})$. The three boundary components of $\overline{M}_{0,4}$ are linearly equivalent. A four element subset $Q \subset P$ induces a natural map $M_{0,n} \to M_{0,Q}$. The pull-backs of the basic boundary linear equivalences on $M_{0,Q}$ induces boundary linear equivalences on $M_{0,n}$. The relations among the boundary components of $M_{0,n}$ are generated by these pull-back linear equivalences as $Q$ varies among all four element subsets of $P$. $Pic(M_{0,n})$ is a free group of rank $2^n - 1 - n$. Since there are $2^n - 2 - 2n$ boundary components of $\overline{M}_{0,n}$, it follows there are $\binom{n-1}{2} - 1$ independent relations among the boundary components. Finally, $Pic(M_{0,n}) \simeq Num(M_{0,n})$. See [Ke] for proofs of these results.

Let $n \geq 4$. There is canonical morphism $\eta : \overline{M} = M_{0,n}(r, d) \to \overline{M}_{0,n}$. The $\eta$ pull-back of a boundary component of $\overline{M}_{0,n}$ is a non-empty, multiplicity-free sum of boundary components $\triangle'$ of $M$:

$$\eta^{-1}((A \cup B)) = \sum_{d_A + d_B = d} (A \cup B, d_A, d_B).$$

**Lemma 1.2.3.** The relations among the boundary components $\triangle'$ in $Pic(M) \otimes \mathbb{Q}$ are the $\eta$ pull-backs of the relations among the boundary components of $\overline{M}_{0,n}$.

**Proof.** Let $\pi : S = \mathbb{P}^1 \times C \to C$ be a family with $n$ sections. Let $\mu : S \to \mathbb{P}^r$ be a rational maps with simple base points obtained from a line bundle of degree type $(d, k)$. Suppose the special points satisfy (1), (2), and

(3') An intersection point lies on at most $n - 2$ sections.

(4) Every simple base point is an intersection point.

Note condition (3') implies condition (3). For large $k$, the simple base points may be selected arbitrarily (with arbitrary degree) among the intersection points. Let $\overline{S}$ be the blow-up of $S$ at the special points; let $\lambda : C \to \overline{M}$ be the induced curve. By condition (3'), the family $\overline{S} \to C$ with the strict transforms of the sections is flat family of stable,
n-pointed, genus 0 curves. The induced morphism \( \gamma : C \to M_{0,n} \) is simply \( \gamma = \eta \circ \lambda \).

Suppose \( \sum_{K \in \Delta'} c_K K = 0 \) is a relation in \( \text{Pic}(M) \otimes \mathbb{Q} \). Let \( K = (A \cup B, d_A, d_B) \in \Delta' \).

Let \( (A \cup B) \) be corresponding boundary component of \( M_{0,n} \). The set theoretic intersection \( C \cdot K \) is the subset of \( C \cdot (A \cup B) \) with simple base points of the correct degree. Since the simple base points can be assigned arbitrary degrees, the coefficient \( c_K \) must depend only on the partition \( (A \cup B) \) and not on the weights \( d_A, d_B \). It now follows the relation \( \sum_{K \in \Delta'} c_K \cdot K = 0 \) must be the \( \eta \) pull-back of a boundary relation in \( M_{0,n} \). \( \square \)

In particular, it follows there are \( \binom{n-1}{2} - 1 \) independent relations among the boundary components \( \Delta' \). For \( n \geq 4 \),

\[
| \Delta | = d + dn + | \Delta' |
\]

\[
| \Delta' | = (d + 1) \cdot (2^{n-1} - 1 - n).
\]

By Lemmas (1.1.1), (1.2.1), (1.2.2), (1.2.3), the Picard number of \( M \) \( (n \geq 4) \) is:

\[
dim \text{Pic}(M) \otimes \mathbb{Q} = (d + 1) \cdot 2^{n-1} - \binom{n}{2}.
\]

All the numerical relations are obtained from linear equivalences. The proof of Proposition (2) is complete.

2. Computations in \( \text{Pic}(M) \otimes \mathbb{Q} \)

2.1. The Universal Curve and \( \pi_* (c_1(\omega_x)^2) \). Classes of certain canonical elements in \( \text{Pic}(M) \otimes \mathbb{Q} \) will be computed via intersections with curves. These computations will be used in the proof of Proposition (3). In order to use the coarse moduli space throughout, an automorphism result is required.

**Lemma 2.1.1.** Let \( d \geq 0, g = 0, r \geq 2 \). The locus of Kontsevich stable maps in \( \overline{M}_{0,n}(r, d) \) with nontrivial automorphisms is of codimension at least 2 except in one case: \( \overline{M}_{0,0}(2, 2) \).
Proof. The assertion follows from naive dimension estimates. If \( d = 0 \) or \( 1 \), the are no stable maps with nontrivial automorphisms. Let \( \overline{M} = \overline{M}_{0,n}(r,d), (d \geq 2, r \geq 2) \). Recall \( \dim \overline{M} = rd + d + r + n - 3 \). Certainly the generic elements of the boundary components are automorphism-free. Let \( A \subset \overline{M} \) be the locus of non-boundary, stable maps with nontrivial automorphisms. If a map \( \mu : \mathbb{P}^1 \to \mathbb{P}^r \) with \( n \) distinct marked points has an nontrivial automorphism, \( \mu \) must be a \( k \geq 2 \) to 1 map. For fixed \( 2 \leq k \leq d \), the map \( \mu \) moves in a family of dimension at most:

\[
(r + 1) \cdot \frac{d}{k} + 1 - 1 - 3 + 2 \cdot (k + 1) - 1 - 3 = (rd + d) \cdot \frac{1}{k} + r - 3 + 2k - 2.
\]

The \( n \) marked points must be fixed points of the nontrivial automorphism and hence move in a zero dimensional family for each \( \mu \). A calculation yields:

\[
\dim \overline{M} - \dim A \geq (rd + d) \cdot \left(1 - \frac{1}{k}\right) + n - 2k + 2
= rd + d + n + 2 - \frac{rd + d + 2k^2}{k}.
\]

A study of the function \( (rd + d + 2k^2)/k \) for \( 2 \leq k \leq d \) shows the maximum value must be attained at the end points \( k = 2, d \). If \( k = 2 \),

\[
rd + d - \frac{rd + d + 8}{2} = (r + 1)\frac{d}{2} - 4 \geq 0
\]

except when \( r = 2, d = 2 \). If \( k = d \),

\[
rd + d - \frac{rd + d + 2d^2}{d} = (r - 1)(d - 1) - 2 \geq 0
\]

except when \( r = 2, d = 2 \). \( A \) is of codimension at least 2 all cases except \( \overline{M}_{0,0}(2,2) \).

Since \( \overline{M}_{0,0}(2,2) \) is isomorphic to the space of complete conics, its intersection theory is well known. In the sequel, it will be assumed \((g,n,r,d) \neq (0,0,2,2)\). Let \( \overline{M}' \subset \overline{M} \) denote the automorphism-free locus. There is a universal Kontsevich stable family of maps over \( \overline{M}' \):

\[
\pi : U^* \to \overline{M}'
\]

with sections \( s_1, s_2, \ldots, s_n \) and a morphism

\[
\mu : U^* \to \mathbb{P}^r.
\]

See [P] for details. Let \( \omega_\pi \) be the relative dualizing sheaf of \( \pi \). Since the complement of \( \overline{M}' \) is of codimension at least 2 in \( \overline{M} \), the following
are well-defined elements of $Pic(M) \otimes \mathbb{Q}$:

$$\pi_*(c_1(\omega_\pi)^2), \pi_*(s_1^2). \quad (2)$$

Since $Pic(M) \otimes \mathbb{Q} \cong Num(M) \otimes \mathbb{Q}$, explicit expressions of the classes (2) in terms of the generators $\{L_i\} \cup \triangle \cup \{H\}$ can be found by calculating intersection products with curves in $M$. The methods of section (1.2) will be used to determine curves in $M$. First consider $\pi_*(c_1(\omega_\pi)^2)$:

**Lemma 2.1.2.** For $d \geq 0$, $g = 0$, $r \geq 2$, $\pi_*(c_1(\omega_\pi)^2) = -\sum_{K \in \triangle} K$ in $Pic(M) \otimes \mathbb{Q}$.

**Proof.** Let $\pi : S \to C$ be a projective bundle of rank 1 over a non-singular curve $C$. Let $\omega_\pi$ be the relative dualizing sheaf. A simple computation yields

$$\pi_*(c_1(\omega_\pi)^2) = 0$$

in $Num(C)$. Let $\rho : S \to S$ be the blow-up at $k$ points in distinct fibers of $\pi$. Let $\tilde{\pi} : \tilde{S} \to C$ be the composition.

$$\omega_{\tilde{\pi}} = \rho^*(\omega_\pi) + \sum_{i=1}^{k} E_i$$

where the $E_i$ are the exceptional divisors of $\rho$. Hence,

$$\pi_*(c_1(\omega_{\tilde{\pi}})^2) = -k$$

in $Num(C)$. By considering curves $C \to \overline{M}$ and the pull-back of $U^*$, it follows $\pi_*(c_1(\omega_\pi)^2) = -\sum_{K \in \triangle} K$ in $Num(M) \otimes \mathbb{Q}$. By Proposition (2), the Lemma is proven. \[\square\]

2.2. **The Class $\pi_*(s_1^2)$**. The determination of the class $\pi_*(s_1^2)$ is surprisingly different in the cases $d = 0$ and $d \geq 1$. If $d = 0$, it suffices to determine $\pi_*(s_1^2)$ for the universal family over $\overline{M}_{0,n}$. Let $\triangle$ be the set of boundary components of $\overline{M}_{0,n}$. There is a partition of $\triangle$ with respect to the first marking. For $2 \leq j \leq n-2$, let $\triangle^1_j \subset \triangle$ be defined by:

$$(A \cup B) \in \triangle^1_j \text{ if and only if } 1 \in A, \ |A| = j.$$ 

There is a disjoint union

$$\triangle = \bigcup_{j=2}^{n-2} \triangle^1_j.$$ 

Let $K^1_j = \sum_{K \in \triangle^1_j} K$. 

13
Lemma 2.2.1. The class $\pi_*(s_1^2)$ is expressed in $\text{Pic}(\overline{M}_{0,n}) \otimes \mathbb{Q}$ by:

$$\pi_*(s_1^2) = -\frac{1}{(n-1)} \sum_{j=2}^{n-2} \binom{n-j}{2} K_j^1. \quad (3)$$

Proof. The proof is by intersections with curves in $\overline{M}_{0,n}$. Let $S = \mathbb{P}^1 \times C$ be a family with $n$ sections $s_1, \ldots, s_n$. Let $s_1$ be of degree type $(1, q)$. For $2 \leq i \leq n$, let $s_i$ be of type $(1, p_i)$. As usual, assume the blow-up $\overline{S}$ up of $S$ at the intersection points yields a family of stable, $n$-pointed curves over $C$ with at most one exceptional divisor in each fiber. Let $\lambda : C \to \overline{M}_{0,n}$ be the induced map. It will be checked that the left and right sides of (3) have the same intersection with $C$.

A point of $C \cdot K_j^1$ can arise in exactly two cases. First an intersection point of $j$ sections including $s_1$ can be blown-up. Second, an intersection point of $n - j$ sections not including $s_1$ can be blown-up. Let

$$C \cdot K_j^1 = x_j + y_j$$

where $x_j, y_j$ are the number of instances of the first and second cases respectively. Let $\overline{s}_1$ be the strict transform of $s_1$ in $\overline{S}$. The intersection of $C$ with the left side of (3) is:

$$\pi_*(\overline{s}_1^2) = 2q - \sum_{j=2}^{n-2} x_j.$$

For $2 \leq i \leq n$, $s_i$ intersects $s_1$ in $q + p_i$ points. The following equation is easily obtained by analyzing intersection points contained in $s_1$:

$$(n-1)q + \sum_{i=2}^{n} p_i = \sum_{j=2}^{n-2} (j-1)x_j. \quad (4)$$

Similarly, the number of intersections of the sections $2 \leq i \leq n$ among themselves is $(n-2) \cdot \sum_{i=2}^{n} p_i$. Analysis of intersection points not contained in $s_1$ yields:

$$(n-2) \cdot \sum_{i=2}^{n} p_i = \sum_{j=2}^{n-2} \binom{j-1}{2} x_j + \binom{n-j}{2} y_j. \quad (5)$$
Via equations (4) and (5),
\[
\binom{n-1}{2} \cdot (2q - \sum_{j=2}^{n-2} x_j) = \sum_{j=2}^{n-2} \left( (n-2)(j-1) - \binom{j-1}{2} - \binom{n-1}{2} \right) x_j - \binom{n-j}{2} y_j
\]
\[
= -\sum_{j=2}^{n-2} \binom{n-j}{2} (x_j + y_j).
\]

The Lemma is proved. \(\square\)

Consider now the case \(d \geq 1\). Let \(\overline{M} = \overline{M}_{0,n(r,d)}\) where \(d \geq 1\), \(n \geq 1\). Let 1 be the first marking. There is another partition of \(\Delta\) with respect to the first marking depending upon the degree. For \(0 \leq j \leq d\), let \(\triangle^{1,j} \subset \Delta\) be defined by:

\[(A \cup B, d_A, d_B) \in \triangle^{1,j} \text{ if and only if } 1 \in A, \ d_A = j.\]

Note if \(n = 1\), then \(\triangle^{1,0}, \triangle^{1,d} = \emptyset\). If \(n = 2\), \(\triangle^{1,d} = \emptyset\). In all other cases \(\triangle^{1,j} \neq \emptyset\). There is a disjoint union

\[\Delta = \bigcup_{j=0}^{d} \triangle^{1,j}.\]

Let \(K^{1,j} = \sum_{K \in \triangle^{1,j}} K\). Let \(K^{1,j} = 0\) if \(\triangle^{1,j} = \emptyset\).

**Lemma 2.2.2.** In case \(d \geq 1\), The class \(\pi_{*}(s_1^2)\) is expressed in \(\text{Pic}(\overline{M}) \otimes \mathbb{Q}\) by:

\[
\pi_{*}(s_1^2) = -\frac{1}{d^2} H + \frac{2}{d} L_1 - \sum_{j=0}^{d} \frac{(d-j)^2}{d^2} K^{1,j}.
\]

**Proof.** The proof is by intersections with curves in \(\overline{M}\). Let \(\pi : S = \mathbb{P}^1 \times C \to \mathbb{P}^1\) be a family with \(n\) sections \(s_1, \ldots, s_n\). Let \(s_1\) be of degree type \((1, q)\). Let \(\mu : S- \to \mathbb{P}^r\) be a rational map with simple base points obtained from a line bundle of degree type \((d, k)\). Let conditions (1), (2), (3'), (4) of section (1.2) be satisfied. Let \(\overline{S} \to S\) be the blow-up at the special points. Let \(\lambda : C \to \overline{M}\) be the induced map. It will be checked that the left and right sides of (6) have the same intersection with \(C\).

A point of \(C \cdot K^{1,j}\) can arise in exactly two cases. First, a simple base point of degree \(j\) contained in \(s_1\) can be blown-up. Second, a simple base point of degree \(d-j\) not contained in \(s_1\) can be blown-up. Let

\[C \cdot K^{1,j} = x_j + y_j\]
where $x_j$, $y_j$ are the number of instances of the first and second cases respectively. Let $\pi_1$ be the strict transform of the section $s_1$ to $\overline{S}^2$. The intersection of $C$ with the left side of (3) is given by:

$$\pi_\ast(\pi_1^2) = 2q - \sum_{j=0}^{d} x_j.$$ 

A straightforward computation yields:

$$C \cdot H = 2dk - \sum_{j=0}^{d} j^2 x_j - \sum_{j=0}^{d} (d - j)^2 y_j,$$

$$C \cdot L_1 = dq + k - \sum_{j=0}^{d} j x_j.$$

The equality of the intersection of $C$ with the left and right sides of (6) is now a matter of simple algebra. \(\square\)

2.3. The Class $T$. Let $\overline{M} = \overline{M}_{0,n}(r,d)$, $d \geq 2$. Let $H \subset \mathbb{P}^r$ be a hyperplane. A tangency Weil divisor $T_H \subset \overline{M}$ is defined as follows. Let $W_H \subset \overline{M}$ be the open locus of maps $\mu : C \to \mathbb{P}^r$ where $\mu^{-1}(H)$ is a subscheme of $d$ reduced points of $C_{\text{nonsing}}$. Let $T_H$ be the complement of $W_H$.

It must be shown that $T_H$ is of pure codimension 1 in $\overline{M}$. Let $M_H \subset \overline{M}$ be the open locus of maps $\mu : C \to \mathbb{P}^r$ satisfying:

$$\forall x \in \mu^{-1}(H), \quad x \in C_{\text{nonsing}} \text{ and } d\mu_x \neq 0.$$ 

The intersection $T_H \cap M_H$ corresponds to geometric tangencies and is certainly of pure codimension 1 in $M_H$ ($d \geq 2$). The complement $\overline{M} \setminus M_H$ is of codimension 2 in $\overline{M}$. It is not hard to see the closure of $T_H \cap M_H$ in $\overline{M}$ contains the complement $\overline{M} \setminus M_H$. Therefore, $T_H$ is a Weil divisor.

Define for $0 \leq j \leq \lfloor \frac{d}{2} \rfloor$, $\Delta^j \subset \Delta$ as follows. A boundary component $(A \cup B, d_A, d_B) \in \Delta^j$ if and only if the degree partition $d_A + d_B = d$ equals the partition $j + (d - j) = d$. Let $K^j = \sum_{K \in \Delta^j} K$.

**Lemma 2.3.1.** The class of $T$ can be expressed in $\text{Pic}(\overline{M}) \otimes \mathbb{Q}$ by:

$$T = \frac{d-1}{d} H + \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} \frac{j(d-j)}{d} K^j.$$ 

(7)
Proof. Let $S$, $\mu$, $\lambda : C \to \overline{M}$ be exactly as in the proof of Lemma (2.2.2). It will be checked that the left and right sides of (7) have the same intersection with $C$.

As before, a point of the intersection $C \cdot K^j$ can arise in two cases. A simple point of degree $j$ or $d - j$ can be blown-up. Let $C \cdot K^j = x_j + y_j$ where $x_j$ and $y_j$ are the number instances of the first and second case respectively. Let $E_{x_j}$ be the union of the $x_j$ exceptional divisors in $\overline{S}$ obtained from the $x_j$ points of $C \cdot K^j$. Let $E_{y_j}$ be defined similarly.

First, the intersection $C \cdot T$ is calculated. A general element of $\overline{\mu}^*(\mathcal{O}_P(1))$ is a nonsingular curve $D$ in the linear series $(d,k) - \sum_j jE_{x_j} - \sum_j (d-j)E_{y_j}$. Adjunction yields:

$$2g_D - 2 = d(2g_C - 2) + 2dk - 2k - \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} j(j-1)x_j - \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} (d-j)(d-j-1)y_j.$$ 

Since $D$ is a $d$ sheeted cover of $C$, the Riemann-Hurwitz formula determines the ramifications:

$$C \cdot T = 2dk - 2k - \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} j(j-1)x_j + (d-j)(d-j-1)y_j.$$ 

$C \cdot H$ is simply $D^2$. Hence

$$C \cdot H = 2dk - \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} j^2x_j + (d-j)^2y_j.$$ 

Again, an algebraic computation yields the equality of the intersections of the left and right sides with $C$. \hfill \Box

3. Intersections of $\mathbb{Q}$-divisors

3.1. Intersections of the classes $\{L_i\} \cup \{H\}$. The top dimensional intersection products on $\overline{M} = \overline{M}_{0,n}(r,d)$ of the classes $\{L_i\}$ are algorithmically determined by the First Reconstruction Theorem [K-M]. These top classes are computed recursively in $d$ and $n$. The algorithm requires one initial value: the number of lines in $\mathbb{P}^r$ through two points. The top intersection products of $\{L_i\}$ are exactly the characteristic numbers $(\beta = 0)$ of rational curves in $\mathbb{P}^r$. 

17
Top dimensional intersections of the classes \( \{L_i\} \cup \{H\} \) are also characteristic numbers of rational curves in \( \mathbb{P}^r \). Each factor of \( H \) is a codimension-2 characteristic condition. For example, if \( \overline{M} = \overline{M}_{0,0}(2,3) \), then \( H^8 \) equals the number of rational plane cubics through 8 general points. If \( \overline{M} = \overline{M}_{0,2}(3,4) \), then \( c(L_1)^3 \cdot c(L_2)^3 \cdot H^{12} \) equals the number of rational space quartics passing through 2 general points and meeting 12 general lines.

### 3.2. Boundary Components

Let \( \bar{K} = (A \cup B, d_A, d_B) \) be a boundary component of \( \overline{M}_{0,n}(r,d) \). Let \( \overline{M}_A = \overline{M}_{0,|A|+1}(r,d_A) \) and \( \overline{M}_B = \overline{M}_{0,|B|+1}(r,d_B) \). Let the additional markings be \( p_A \) and \( p_B \) respectively. Let \( e_A : \overline{M}_A \to \mathbb{P}^r \) and \( e_B : \overline{M}_B \to \mathbb{P}^r \) be the evaluation maps obtained from the markings \( p_A \) and \( p_B \). Let \( \tau_A, \tau_B \) be the projections from \( \overline{M}_A \times \overline{M}_B \) to the first and second factors respectively. Let \( \tilde{K} = \overline{M}_A \times_{\mathbb{P}^r} \overline{M}_B \) be the fiber product with respect to the evaluation maps \( e_A, e_B \). \( \tilde{K} \subset \overline{M}_A \times_{\mathbb{C}} \overline{M}_B \) is the closed subvariety \((e_A \times_{\mathbb{C}} e_B)^{-1}(D)\) where \( D \subset \mathbb{P}^r \times \mathbb{P}^r \) is the diagonal. \( \tilde{K} \) is easily seen to be an irreducible, normal, projective variety with finite quotient singularities. These results follow, for example, from the local construction given in [P]. The class of \( \tilde{K} \) in \( \overline{M}_A \times \overline{M}_B \) can be computed by the pull-back of the Künneth decomposition of the diagonal in \( \mathbb{P}^r \times \mathbb{P}^r \):

\[
[\tilde{K}] = \sum_{i=0}^r \tau_A^*(c_1(L_A)^i) \cdot \tau_B^*(c_1(L_B)^{r-i})
\]

where \( L_A, L_B \) are the line bundles on \( \overline{M}_A, \overline{M}_B \) induced by the marking \( p_A, p_B \).

There is a natural map \( \psi : \tilde{K} \to K \). The set theoretic description of \( \psi \) is clear: \( \psi([\mu_A], [\mu_B]) \) is the moduli point of the map obtained by gluing maps \( \mu_A, \mu_B \) along the markings \( p_A, p_B \). It is not hard to define \( \psi \) algebraically. \( \psi \) is a birational morphism except when \( n = 0 \) and \( d_A = d_B = d/2 \). In the latter case, \( \psi \) is generically 2-1.

The pull-backs of the classes \( \{L_i\} \cup \{L\} \) on \( \overline{M} \) to \( \tilde{K} \) are determined in the following manner. Let \( \mathcal{H}_A, \mathcal{H}_B \) be the codimension-2 plane incidence classes on \( \overline{M}_A, \overline{M}_B \). Clearly,

\[
\psi^*(\mathcal{H}) = (\tau_A^*(\mathcal{H}_A) + \tau_B^*(\mathcal{H}_B)) \big|_{\tilde{K}}.
\]

Let \( P \) be the marking set of \( \overline{M} \). For each \( i \in P \), \( i \) is either in \( A \) or \( B \). It follows

\[
\psi^*(L_i) = \tau_A^*(L_i) \big|_{\tilde{K}}, \quad \psi^*(L_i) = \tau_B^*(L_i) \big|_{\tilde{K}}.
\]
in case \( i \in A, i \in B \) respectively.

Let \( T = (A' \cup B', d_{A'}, d_{B'}) \) be a boundary component of \( \overline{M} \) not equal to \( K \). \( T \) intersects \( K \) exactly when one of the following two conditions hold:

(i.) There exists a subset \( C \subset A \) and an integer \( d_C \) such that
\[
((A \setminus C) \cup (B \cup C), d_A - d_C, d_B + d_C) = T.
\]

(ii.) There exists a subset \( C \subset B \) and an integer \( d_C \) such that
\[
((A \cup C) \cup (B \setminus C), d_A + d_C, d_B - d_C) = T.
\]

\[
\psi^*(T) = \sum_{C, d_C} \tau_A^*(A \cup (C \cup \{p_A\}), d_A, d_C) \big|_{\tilde{K}} + \sum_{C, d_C} \tau_B^*(B \cup (C \cup \{p_B\}), d_B, d_C) \big|_{\tilde{K}}. \tag{11}
\]

The sums on the right are taken over subsets \( C \) and degrees \( d_c \) that satisfy (i) and (ii) above respectively. The main point is distinct boundary divisors have transverse (if nonempty) intersections in the stack \( \overline{M}_{0,n}(r, d) \). This can be seen as a property inherited from the Mumford-Knudsen space \( \overline{M}_{0,n} \) by the local construction given in [P]. Since the automorphism loci of \( \overline{M}_{0,n}(r, d) \) and the boundary component \( (A \cup B, d_A, d_B) \) are of codimension at least two in \( \overline{M}_{0,n} \), \( (A \cup B, d_A, d_B) \) respectively, the transverse intersection property descends to the coarse moduli space.

Let \( \omega_{\pi A}, \omega_{\pi B} \) denote the relative dualizing sheaves of the universal families over \( \overline{M}_A^*, \overline{M}_B^* \) respectively. There are two universal curves over \( \tilde{K}^* = \tilde{K} \cap (\overline{M}_A \times \overline{M}_B) \) obtained via pull-back of the universal families \( U_A^* \) and \( U_B^* \). These universal curves glue on the sections \( s_{pA} \) and \( s_{pB} \) to form a universal family
\[
\tilde{\pi} : U_{\tilde{K}^*}^* \to \tilde{K}^*
\]
of maps for the moduli problem of \( \overline{M} \). It follows,
\[
\omega_{U_{\tilde{K}^*}} \big|_{\tau_A^*(U_A^*)} = \tau_A^*(\omega_{\pi A}) + s_{pA},
\]
\[
\omega_{U_{\tilde{K}^*}} \big|_{\tau_B^*(U_B^*)} = \tau_B^*(\omega_{\pi B}) + s_{pB}.
\]

Hence
\[
\psi^*(\pi_*(c_1(\omega_{\pi}^2))) = \tau_A^*(\pi_*(c_1(\omega_{\pi A}) + s_{pA})^2) + \tau_B^*(\pi_*(c_1(\omega_{\pi B}) + s_{pB})^2)).
\]
A normal bundle calculation yields $c_1(\omega_{\pi A}) \cdot s_{pA} = -s_{pA}^2$. Hence,

$$(c_1(\omega_{\pi A}) + s_{pA})^2 = c_1(\omega_{\pi A})^2 - s_{pA}^2$$

(similarly for $B$). Recall $\pi_*(c_1(\omega_\pi)^2) = -\sum_{T \in \triangle} T$. Finally,

$$-\psi^*(K) = \sum_{T \in \triangle, T \neq K} \psi^*(T) + \tau_A^*(\pi_*(c_1(\omega_{\pi A})^2 - s_{pA}^2))$$

$$+ \tau_B^*(\pi_{\pi B}*(c_1(\omega_{\pi B})^2 - s_{pB}^2)).$$

Lemmas (2.1.2), (2.2.1), and (2.2.2) express $\pi_{\pi A}^*(c_1(\omega_{\pi A})^2)$ and $\pi_{\pi A}^*(s_{pA}^2)$ explicitly in terms of the standard classes $\{L_i\} \cup \triangle \cup \{H\}$ on $\overline{M}_A$ (similarly for $\overline{M}_B$). Via equations (9) - (12), the $\psi$ pull-back of every standard class $\{L_i\} \cup \triangle \cup \{H\}$ on $\overline{M}$ has now been expressed as the restriction to $\tilde{K}$ of a linear combination of the $\tau_A$ and $\tau_B$ pull-backs of standard classes on $\overline{M}_A$ and $\overline{M}_B$.

3.3. The Algorithm. The inductive algorithm for computing top intersection products is now clear. All top monomials in the elements $\{L_i\} \cup \{H\}$ are known by the First Reconstruction theorem. If a monomial product on $\overline{M}$ includes a boundary class $K$, the intersection is carried out on $\tilde{K}$. By the above formulas (8) - (12), the desired monomial can be expressed as a sum of top products of standard classes on $\overline{M}_A$ and $\overline{M}_B$. Since $\overline{M}_A$ is of lesser degree or of lesser marking number than $\overline{M}$ (similarly for $\overline{M}_B$), the inductive process terminates.

3.4. Characteristic Numbers. Lemma (2.3.1) expresses the hyperplane tangency condition in terms of the standard classes. Hence all top products of the classes $\{L_i\} \cup \{H, T\}$ can be effectively computed by the above algorithm.

It remains to check the top intersections of $\{L_i\} \cup \{H, T\}$ are the characteristic numbers of rational curves. Let

$$c_1(L_1)^{l_1} \cdots c_1(L_n)^{l_n} \cdot H^\alpha \cdot T^\beta$$

be a top product on $\overline{M} = \overline{M}_{0,n}(r,d)$. Since the $L_i$ are pull-backs of $O_{\mathbb{P}^r}(1)$ via the evaluation maps, codimension $l_i$ linear spaces of $\mathbb{P}^r$ determine representatives of $c_1(L_i)^{l_i}$. The cycle $H^\alpha$ is determined by $\alpha$ codimension 2 linear spaces in $\mathbb{P}^r$. Finally, the cycle $T^\beta$ is determined by $\beta$ hyperplanes in $\mathbb{P}^r$. When $\beta = 0$, it is assumed $d \geq 2$. The first step is to show for general choices of all the linear spaces of $\mathbb{P}^r$ in question, the intersection cycle (13) in $\overline{M}$ is at most 0 dimensional.
and corresponds (set theoretically) to the correct geometric locus. The second step is to show the intersection cycle is multiplicity free.

Let $P^r$ be the parameter space of hyperplanes in $\mathbb{P}^r$. Defined the universal tangency subvariety

$$T_{\text{univ}} \subset \overline{M} \times \mathbb{P}^r$$

as follows. Let $W_{\text{univ}} \subset \overline{M} \times \mathbb{P}^r$ be the open locus of pairs $(\mu : C \to \mathbb{P}^r, H)$ where $\mu^{-1}(H)$ is a subscheme of $d$ reduced points of $C_{\text{nonsing}}$. Let $T_{\text{univ}}$ be the complement of $W_{\text{univ}}$. Let $T_H$ be the the fiber of $T_{\text{univ}}$ over the parameter point of the hyperplane $H$. $T_H$ is exactly the tangency Weil divisor defined in section (2). Similarly, let $H_{\text{univ}} \subset \overline{M} \times G(P^r-2, \mathbb{P}^r)$ be the universal codimension 2 plane incidence subvariety. The fiber of $H_{\text{univ}}$ over the parameter point of the codimension 2 plane $P$ is $H_P$. Let

$$I_{\text{univ}} \subset \overline{M} \times G(r-l_1, r) \times \cdots \times G(r-l_n, r) \times G(r-2, r) \times \cdots \times G(r-2, r) \times \mathbb{P}^r \times \cdots \times \mathbb{P}^r$$

be the universal intersection cycle (13) defined by the universal divisors $T_{\text{univ}}, H_{\text{univ}}$ and the evaluation maps. $I_{\text{univ}}$ is closed subvariety.

In the first step, slightly more than the dimensionality of the general intersection cycle will be established. A map $\mu : C \to \mathbb{P}^r$ is simply tangent to a hyperplane $H$ if

(i.) $\mu^{-1}(H) \subset C_{\text{nonsing}}$.  
(ii.) As a subscheme, $\mu^{-1}(H)$ consists of 1 double and $d - 2$ reduced points.

A map $\mu : C \to \mathbb{P}^r$ has simple intersection with a codimension 2 plane $P$ if

(i.) $\mu^{-1}(P)$ consists of 1 point $x \in C_{\text{nonsing}}$.  
(ii.) $Im(d\mu(x))$ and the tangent space of $P$ span maximal rank.

**Lemma 3.4.1.** For general choices of linear spaces

$$L_1, \ldots, L_n, \ P_1, \ldots, P_\alpha, \ H_1, \ldots, H_\beta$$

the intersection cycle (13) is at most 0 dimensional and set theoretically corresponds to maps $\mu : C \to \mathbb{P}^r$ satisfying:
(1.) $C \cong \mathbb{P}^1$, $\mu$ is an immersion/embedding ($r = 2 / r \geq 3$).
(2.) $\forall k$, $\mu$ is simply tangent to the hyperplanes $H_k$.
(3.) $\forall j$, $\mu$ intersects the linear spaces $P_j$ simply.
(4.) $\forall i$, the $\mu$-image of the $i^{th}$ marked point lies in $L_i$.

**Proof.** The intersection cycle $I$ determined by the linear spaces (14) is the fiber of $I_{\text{univ}}$ over the parameter points of the linear spaces. $\dim(I) \leq 0$ is an open condition in the parameter space. It is first checked that general choice of the linear spaces (14) yields an intersection cycle of dimension at most 0.

Let $[\mu] \in \overline{M}$ be the moduli point of a map $\mu : C \to \mathbb{P}^r$. By Bertini’s Theorem, the general hyperplane $H$ is transverse to $\mu$. Therefore, the general tangency divisor $T_H$ satisfies $[\mu] \notin T_H$. Similarly, the general incidence divisor $H_P$ satisfies $[\mu] \notin H_P$. By choosing at each stage tangency and incidence divisors that reduce the dimension of every component of the intersection, it follows

$$H_{P_1} \cap \ldots \cap H_{P_\alpha} \cap T_{H_1} \cap \ldots \cap T_{H_\beta}$$

has codimension at least $\alpha + \beta$. Since the remaining intersections are obtained from basepoint free linear series, the general intersection cycle has dimension at most 0.

If the general parameter point yields an empty cycle $I$, there is nothing more to prove. Let $W$ be the open set of the parameter space where $\dim(I) = 0$. The conditions (1-3) on $I$ determine open sets $W_1, W_2, W_3 \subset W$. Condition (iv) is automatic. It suffices to show $W_i$ is nonempty for $1 \leq i \leq 3$.

The subset $Y \subset \overline{M}$ of maps that are not immersion/embedding ($r = 2 / r \geq 3$) is of codimension at least 1. Hence, by the dimension reduction argument above, $Y \cap I = \emptyset$ for a general parameter point. Therefore, $W_1 \neq \emptyset$.

Let $W_{2,k}, W_{3,j} \subset W$, be the set of parameter points that satisfy condition (2), (3) for the hyperplane $H_k$, linear space $P_j$ respectively. Since $W_2 = \bigcap_{k=1}^{\beta} W_{2,k}$ and $W_3 = \bigcap_{j=1}^{\alpha} W_{3,j}$, it suffices to show $W_{2,k}, W_{3,j} \neq \emptyset$. Let $H_k$ be any hyperplane. The locus of moduli points $[\mu] \in T_{H_k}$ that are not simply tangent is of codimension at least 2 in $\overline{M}$. By the dimension reduction argument, $W_{2,k} \neq \emptyset$. Similarly, the locus of moduli points $[\mu] \in H_{P_j}$ that do not intersect simply is of codimension at least 2 in $\overline{M}$. As before $W_{3,j} \neq \emptyset$. \qed
It must now be shown that the intersection cycle (13) is reduced for
general linear spaces. This transversality is established by Kleiman’s
Bertini Theorem. Unfortunately, since the divisors \( T_H, H_P \) need not move \textit{linearly}, Bertini’s Theorem can not be directly applied to \( \overline{M} \).
Instead, an auxiliary construction is undertaken. Kleiman’s Bertini
Theorem is applied to the universal curve over \( \overline{M} \). It will be shown that suitable transversality on the universal curve implies transversality on \( \overline{M} \).

Let \( \overline{M}^0 \subset \overline{M} \) be the open set of immersed/embedded \((r = 2, r \geq 3)\) maps with irreducible domains. Since (for general linear spaces) the in-
tersection cycle (13) lies in \( \overline{M}^0 \), transversality need only be established in \( \overline{M}^0 \). Note \( \overline{M}^0 \) is in the automorphism-free locus. Let \( U \to \overline{M}^0 \) be the universal curve. Let \( \mu : U \to \mathbb{P}^r \) be the universal map. \( U, \overline{M}^0 \) are nonsingular. Let \( \mathbb{P}T \) be the projective tangent bundle of \( \mathbb{P}^r \). Since each point of \( \overline{M}^0 \) corresponds to an immersion/embedding, there is a natural algebraic map \( \nu : U \to \mathbb{P}T \) given by the differential of \( \mu \). The map \( \nu \) is a lifting of \( \mu \).

By projectivizing tangent spaces, the hyperplanes \( H_1, \ldots, H_\beta \) define nonsingular, codimension 2 subvarieties of \( \mathbb{P}T \):
\[
\mathbb{P}H_1, \ldots, \mathbb{P}H_\beta
\]
Let \( U_1, \ldots, U_\beta \) be \( \beta \) copies of the universal curve \( U \). Let \( U'_1, \ldots, U'_\alpha \) be \( \alpha \) more copies of \( U \). Define the product:
\[
X \cong U'_1 \times _{\overline{M}^0} \ldots \times _{\overline{M}^0} U'_\alpha \times _{\overline{M}^0} U_1 \times _{\overline{M}^0} \ldots \times _{\overline{M}^0} U_\beta.
\]
Let \( \mu'_j : X \to \mathbb{P}^r, \nu_k : X \to \mathbb{P}T \) be the maps obtained by projection onto \( U'_j, U_k \) and composition with \( \mu, \nu \) respectively.

Kleiman’s Bertini Theorem may now be applied. The group \( GL_{r+1}(\mathbb{C}) \) acts transitively on \( \mathbb{P}^r, \mathbb{P}T \). Hence, the general intersection
\[
\mu'_1^{-1}(P_1) \cap \ldots \cap \mu'_\alpha^{-1}(P_\alpha) \cap \nu_1^{-1}(\mathbb{P}H_1) \cap \ldots \cap \nu_\beta^{-1}(\mathbb{P}H_\beta) \subset X
\]
is nonsingular and of the correct codimension (if nonempty).

It remains to obtain the corresponding result on \( \overline{M} \). Consider the nonsingular, codimension 2 subvariety \( \mu^{-1}(P_j) \subset U \). The projection \( \mu^{-1}(P_j) \to \mathcal{H}_{P_j} \cap \overline{M}^0 \) is étale and 1-1 over the locus of maps meeting \( P_j \) simply. Similarly, the projection \( \nu^{-1}(\mathbb{P}H_k) \to \mathcal{T}_{H_k} \cap \overline{M}^0 \) is étale and 1-1 over the the locus of maps simply tangent to \( H_k \). From Lemma
Below, the projection
\[ \mu_1^{-1}(P_1) \cap \ldots \cap \mu_\alpha^{-1}(P_\alpha) \cap \nu_1^{-1}(\mathbb{P}H_1) \cap \ldots \cap \nu_\beta^{-1}(\mathbb{P}H_\beta) \rightarrow \]
\[ \mathcal{H}_{P_1} \cap \ldots \cap \mathcal{H}_{P_\alpha} \cap \mathcal{T}_{H_1} \cap \ldots \cap \mathcal{T}_{H_\beta} \cap \overline{M}^0 \]
is étale and 1-1 over the locus of points in \( \mathcal{H}_{P_1} \cap \ldots \cap \mathcal{H}_{P_\alpha} \cap \mathcal{T}_{H_1} \cap \ldots \cap \mathcal{T}_{H_\beta} \cap \overline{M}^0 \) corresponding to simple intersection and tangency. It has therefore been proved, for general linear spaces, the locus of \( \mathcal{H}_{P_1} \cap \ldots \cap \mathcal{H}_{P_\alpha} \cap \mathcal{T}_{H_1} \cap \ldots \cap \mathcal{T}_{H_\beta} \cap \overline{M}^0 \) corresponding to simple intersection and tangency is nonsingular and of the correct codimension (if nonempty). It was shown above, for general linear spaces, the intersection cycle involves only maps that have simple intersection and tangency with the \( P_j, H_k \). Since the intersections \( c_1(\mathcal{L}_i)^k \) are obtained from basepoint free linear series on \( M \), the further intersections yield a reduced intersection cycle by Bertini’s Theorem.

**Lemma 3.4.2.** Let \( M \) be a nonsingular base. Let \( \pi : U \rightarrow M \) be smooth map of relative dimension 1. Let \( D_1, D_2, \ldots, D_l \subset U \) be nonsingular, codimension 2 subvarieties such that \( D_i \) is étale and 1-1 over \( \pi(D_i) \). Let \( X \cong U_1 \times_M \ldots \times_M U_l \) be the fiber product of copies of \( U \). Let \( \rho_i : X \rightarrow U_i \) be the projection. Then
\[ \rho_1^{-1}(D_1) \cap \ldots \cap \rho_l^{-1}(D_l) \subset X \]
is étale and 1-1 over the intersection \( \pi(D_1) \cap \ldots \cap \pi(D_l) \subset M \).

**Proof.** The issue is local on \( M \). Let \( m \in M \) be in the intersection of the \( \pi(D_i) \). Choose local defining equations \((f_i)\) of \( \pi(D_i) \) near \( m \). Let \( u_i \in D_i \) be points over \( m \). Locally (in the analytic topology) at \( u_i, U_i \) is an open set of the trivial product \( C_i \times M \) and \( D_i \) is the intersection of \((f_i)\) with a section \((z_i)\) of this product \((z_i)\) is the coordinate on \( C_i \). It now follows local equations for for \( \rho_1^{-1}D_1 \cap \ldots \cap \rho_l^{-1}D_l \) at \((u_1, \ldots, u_l)\) are \((z_1, \ldots, z_l, f_1, \ldots, f_l)\) in \( C_1 \times \ldots \times C_l \times M \) which is certainly étale over \((f_1, \ldots, f_l) \subset M \).

All the characteristic numbers of rational curves in projective space can be algorithmically computed. For example, the number of twisted cubics in \( \mathbb{P}^3 \) through 2 points, 6 lines, and tangent to 2 planes can be expressed as
\[ c_1(\mathcal{L}_1)^3 \cdot c_1(\mathcal{L}_2)^3 \cdot c_1(\mathcal{L}_3)^2 \cdot \ldots \cdot c_1(\mathcal{L}_8)^2 \cdot \mathcal{T}^2 \]
on \( \overline{M}_{0,8}(3, 3) \) or
\[ c_1(\mathcal{L}_1)^3 \cdot c_1(\mathcal{L}_2)^3 \cdot \mathcal{H}^6 \cdot \mathcal{T}^2 \]
4. Examples

4.1. Conics in \( \mathbb{P}^2 \) and \( \mathbb{P}^3 \). Since the Hilbert schemes of lines and conics are Grassmannians and projective bundles over Grassmannians, the \( \beta = 0 \) characteristic numbers of rational curves in degrees 1 and 2 can be calculated directly via intersection theory on these Hilbert schemes. The tangency characteristic numbers for conics classically required the beautiful space of complete conics. \( \overline{M}_{0,0}(2,2) \) is the space of complete conics. A new calculation of the characteristic numbers for plane conics is obtained by considering the pointed space \( \overline{M}_{0,1}(2,2) \).

Let \( \overline{M} = \overline{M}_{0,1}(2,2) \). \( \text{Pic}(\overline{M}) \otimes \mathbb{Q} \) is freely generated by \( \mathcal{H}, \mathcal{L}_1 \), and the unique boundary component \( \mathcal{K} \) corresponding to the partition \( (\{1\} \cup \emptyset, 1+1 = 2) \). The top intersection numbers are (\( \text{dim}\overline{M}_{0,1}(2,2) = 6 \)):

\[
\begin{array}{ccccccc}
\mathcal{H}^6 & 0 & \mathcal{H}^5K & 0 & \mathcal{H}^4K^2 & 0 \\
\mathcal{H}^5\mathcal{L}_1 & +2 & \mathcal{H}^4\mathcal{K}\mathcal{L}_1 & +6 & \mathcal{H}^3K^2\mathcal{L}_1 & +18 \\
\mathcal{H}^4\mathcal{L}_1^2 & +1 & \mathcal{H}^3K\mathcal{L}_1^2 & +3 & \mathcal{H}^2K^2\mathcal{L}_1^2 & +9 \\
\mathcal{H}^3K^3 & 0 & \mathcal{H}^2K^4 & 0 & \mathcal{H}K^5 & 0 & K^6 & 0 \\
\mathcal{H}^2K^3\mathcal{L}_1 & -10 & \mathcal{H}K^4\mathcal{L}_1 & -30 & \mathcal{K}\mathcal{L}_1 & +102 \\
\mathcal{H}K^3\mathcal{L}_1^2 & -5 & K^4\mathcal{L}_1^2 & -15 & & & & & & \\
\end{array}
\]

Note \( \mathcal{L}_1^3 = 0 \). The line tangency class \( \mathcal{T} = \frac{1}{2}(\mathcal{H} + \mathcal{K}) \) is determined by Lemma (2.3.1). The characteristic number of plane conics through \( \alpha \) points and tangent to \( \beta \) lines is \( \frac{1}{2}\mathcal{H}^{\alpha}\mathcal{T}^{\beta}\mathcal{L}_1 \):

\[
\begin{align*}
(1/2) \cdot \mathcal{H}^5\mathcal{L}_1 & \quad 1 \\
(1/2) \cdot \mathcal{H}^4\mathcal{T}\mathcal{L}_1 & \quad 2 \\
(1/2) \cdot \mathcal{H}^3\mathcal{T}^2\mathcal{L}_1 & \quad 4 \\
(1/2) \cdot \mathcal{H}^2\mathcal{T}^3\mathcal{L}_1 & \quad 4 \\
(1/2) \cdot \mathcal{H}\mathcal{T}^4\mathcal{L}_1 & \quad 2 \\
(1/2) \cdot \mathcal{T}^5\mathcal{L}_1 & \quad 1 \\
\end{align*}
\]

The class of maps tangent to a fixed conic can be easily calculated by the methods of Lemma (2.3.1). Let \( \mathcal{C} \in \text{Pic}(\overline{M}) \otimes \mathbb{Q} \) denote this conic tangency class. \( \mathcal{C} = 3\mathcal{H} + \mathcal{K} \). The number of plane conics tangent to 5 fixed conics is therefore \( \frac{1}{2}\mathcal{C}^5\mathcal{L}_1 = 3264 \).
For $r \geq 3$, $\mathcal{M}_{0,0}(r, 2)$ differs from the space of complete conics and the algorithm described above yields a new computation of the characteristic numbers in these cases. Let $\mathcal{M} = \mathcal{M}_{0,0}(3, 2)$. $Pic(\mathcal{M}) \otimes \mathbb{Q}$ is freely generated by $\mathcal{H}$ and the unique boundary component $K$ corresponding to the degree partition $1 + 1 = 2$. $\tilde{K} \subset \mathcal{M}_{0,1}(3, 1) \times \mathcal{M}_{0,1}(3, 1)$. Since $\mathcal{M}_{0,1}(3, 1)$ has no boundary, all top intersections are known. Using the formulas of section (3), the answers for the top intersections of $\mathcal{H}$ and $K$ on $\mathcal{M}_{0,0}(3, 2)$ ($\dim \mathcal{M}_{0,0}(3, 2) = 8$) are:

\[
\begin{align*}
\mathcal{H}^8 & \quad +92 \\
\mathcal{H}^7K & \quad +140 \\
\mathcal{H}^6K^2 & \quad +140 \\
\mathcal{H}^5K^3 & \quad -100 \\
\mathcal{H}^4K^4 & \quad -68 \\
\mathcal{H}^3K^5 & \quad +172 \\
\mathcal{H}^2K^6 & \quad -20 \\
\mathcal{H}K^7 & \quad -580 \\
K^8 & \quad +1820
\end{align*}
\]

By Lemma $(2.3.1)$, $\mathcal{T} = \frac{1}{2}(\mathcal{H} + K)$. The characteristic number of space conics through $\alpha$ lines and tangent to $\beta$ planes is $\mathcal{H}^a\mathcal{T}^3$:

\[
\begin{align*}
\mathcal{H}^8 & \quad 92 \\
\mathcal{H}^7\mathcal{T} & \quad 116 \\
\mathcal{H}^6\mathcal{T}^2 & \quad 128 \\
\mathcal{H}^5\mathcal{T}^3 & \quad 104 \\
\mathcal{H}^4\mathcal{T}^4 & \quad 64 \\
\mathcal{H}^3\mathcal{T}^5 & \quad 32 \\
\mathcal{H}^2\mathcal{T}^6 & \quad 16 \\
\mathcal{H}\mathcal{T}^7 & \quad 8 \\
\mathcal{T}^8 & \quad 4
\end{align*}
\]

These characteristic numbers (with complete proofs) were known classically.

4.2. Rational Plane Cubics. Let $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{0,0}(2, 3)$. $Pic(\overline{\mathcal{M}}) \otimes \mathbb{Q}$ is freely generated by $\mathcal{H}$ and the unique boundary component $K$ corresponds to the degree partition $1 + 2 = 3$. The algorithm described above yields the top intersections of $\mathcal{H}$ and $K$ inductively. Since $\overline{K} \subset \overline{\mathcal{M}}_{0,1}(2, 1) \times \overline{\mathcal{M}}_{0,1}(2, 2)$, first the top intersections on these Kontsevich spaces must be computed. $\overline{\mathcal{M}}_{0,1}(2, 1)$ has no boundary, hence all top products are known. There is a unique boundary component $B$ of $\overline{\mathcal{M}}_{0,1}(2, 2)$. $\overline{B} \subset \overline{\mathcal{M}}_{0,2}(2, 1) \times \overline{\mathcal{M}}_{0,1}(2, 1)$. Thus the top products on
$\overline{M}_{0,2}(2,1)$ must be computed. Finally, the unique boundary component of $\overline{M}_{0,2}(2,1)$ requires knowledge of the top products on $\overline{M}_{0,3}(2,0)$ and $\overline{M}_{0,1}(2,1)$ which are known. The answers for the top intersections of $\mathcal{H}$ and $K$ on $\overline{M}_{0,0}(2,3)$ ($\dim \overline{M}_{0,0}(2,3) = 8$) are:

\begin{align*}
\mathcal{H}^8 & \quad +12 \\
\mathcal{H}^7 K & \quad +42 \\
\mathcal{H}^6 K^2 & \quad +129 \\
\mathcal{H}^5 K^3 & \quad +285 \\
\mathcal{H}^4 K^4 & \quad +336 \\
\mathcal{H}^3 K^5 & \quad -(2541/4) \\
\mathcal{H}^2 K^6 & \quad -(8259/16) \\
\mathcal{H} K^7 & \quad +(19641/8) \\
K^8 & \quad -(44835/16)
\end{align*}

Note since $K$ is $\mathbb{Q}$-Cartier, the intersections $\mathcal{H}^i \cdot K^j$ need not be integers. By Lemma (2.3.1), $T = \frac{2}{3}(H + K)$. The characteristic number of plane cubics through $\alpha$ points and tangent to $\beta$ lines is $\mathcal{H}^\alpha T^\beta$:

\begin{align*}
\mathcal{H}^8 & \quad 12 \\
\mathcal{H}^7 \cdot T & \quad 36 \\
\mathcal{H}^6 \cdot T^2 & \quad 100 \\
\mathcal{H}^5 \cdot T^3 & \quad 240 \\
\mathcal{H}^4 \cdot T^4 & \quad 480 \\
\mathcal{H}^3 \cdot T^5 & \quad 712 \\
\mathcal{H}^2 \cdot T^6 & \quad 756 \\
\mathcal{H} \cdot T^7 & \quad 600 \\
T^8 & \quad 400
\end{align*}

These characteristic numbers have been calculated by H. Zeuthen, S. Maillard, H. Schubert, G. Sacchiero, S. Kleiman, S. Speiser, and P. Aluffi. ([S], [Sa], [K-S], [A]). Complete proofs appear in [Sa], [K-S], and [A].

4.3. **Twisted Cubics in $\mathbb{P}^3$.** In case $\overline{M} = \overline{M}_{0,0}(3,3)$, $\text{Pic}(\overline{M}) \otimes \mathbb{Q}$ is still generated freely by $\mathcal{H}$, $K$. A similar analysis yields the top
intersections \( \dim(M) = 12 \):

\[
\begin{align*}
\mathcal{H}^{12} & +80160 \\
\mathcal{H}^{11}K & +121440 \\
\mathcal{H}^{10}K^2 & +148920 \\
\mathcal{H}^{9}K^3 & +112080 \\
\mathcal{H}^{8}K^4 & -7824 \\
\mathcal{H}^{7}K^5 & -104100 \\
\mathcal{H}^{6}K^6 & +35880 \\
\mathcal{H}^{5}K^7 & +(190095/2) \\
\mathcal{H}^{4}K^8 & -(222855/2) \\
\mathcal{H}^{3}K^9 & -(674007/16) \\
\mathcal{H}^{2}K^{10} & +(10112745/32) \\
\mathcal{H}K^{11} & -(5995065/8) \\
K^{12} & +(58086435/32)
\end{align*}
\]

The hyperplane tangency class is again \( \mathcal{T} = \frac{2}{3}(\mathcal{H} + K) \). The characteristic number of twisted cubics through \( \alpha \) lines and tangent to \( \beta \) planes is \( \mathcal{H}^\alpha \mathcal{T}^\beta \):

\[
\begin{align*}
\mathcal{H}^{12} & 80160 \\
\mathcal{H}^{11}\mathcal{T} & 134400 \\
\mathcal{H}^{10}\mathcal{T}^2 & 209760 \\
\mathcal{H}^{9}\mathcal{T}^3 & 297280 \\
\mathcal{H}^{8}\mathcal{T}^4 & 375296 \\
\mathcal{H}^{7}\mathcal{T}^5 & 415360 \\
\mathcal{H}^{6}\mathcal{T}^6 & 401920 \\
\mathcal{H}^{5}\mathcal{T}^7 & 343360 \\
\mathcal{H}^{4}\mathcal{T}^8 & 264320 \\
\mathcal{H}^{3}\mathcal{T}^9 & 188256 \\
\mathcal{H}^{2}\mathcal{T}^{10} & 128160 \\
\mathcal{H}\mathcal{T}^{11} & 85440 \\
\mathcal{T}^{12} & 56960
\end{align*}
\]

These characteristic numbers have been calculated by H. Schubert and others ([S], [K-S-X]). Complete proofs appear in [K-S-X].

4.4. **Rational Plane Quartics.** Let \( \overline{M} = \overline{M}_{0,0}(2, 4) \). \( \text{Pic}(\overline{M}) \otimes \mathbb{Q} \) is freely generated by \( \mathcal{H} \) and the boundary components \( J, K \) corresponding to the degree partitions \( 2 + 2 = 4, 1 + 3 = 4 \). The top intersection
numbers are \((\text{dim} \overline{M}_{0,0}(2, 4) = 11)\):

\[
\begin{align*}
\mathcal{H}^{11} & = +620 \\
\mathcal{H}^{10}K & = +1620 \\
\mathcal{H}^9K^2 & = +3564 \\
\mathcal{H}^8K^3 & = +4052 \\
\mathcal{H}^7K^4 & = -8340 \\
\mathcal{H}^6K^5 & = -48300 \\
\mathcal{H}^5K^6 & = +1260 \\
\mathcal{H}^4K^7 & = +153300 \\
\mathcal{H}^3K^8 & = -(338620/3) \\
\mathcal{H}^2K^9 & = -(13690660/27) \\
\mathcal{H}K^{10} & = +1620 \\
K^{11} & = -(278947820/81)
\end{align*}
\]

\[
\begin{align*}
\mathcal{H}^8J^3 & = -364 \\
\mathcal{H}^7J^3K^2 & = -1260 \\
\mathcal{H}^6J^3K^3 & = -3852 \\
\mathcal{H}^5J^3K^4 & = -8836 \\
\mathcal{H}^4J^3K^5 & = +4980 \\
\mathcal{H}^3J^3K^6 & = +16356 \\
\mathcal{H}^2J^3K^7 & = -22060 \\
\mathcal{H}J^3K^8 & = -46452 \\
J^3K^9 & = +255444
\end{align*}
\]

\[
\begin{align*}
\mathcal{H}^5J^6 & = +(2419/8) \\
\mathcal{H}^4J^6K & = -(4743/8) \\
\mathcal{H}^3J^6K^2 & = -(18549/8) \\
\mathcal{H}^2J^6K^3 & = -(3455/8) \\
\mathcal{H}J^6K^4 & = +(39075/8) \\
J^6K^5 & = -(56631/8)
\end{align*}
\]

\[
\begin{align*}
\mathcal{H}^2J^9 & = +(4375/8) \\
\mathcal{H}J^9K & = +189 \\
J^9K^2 & = -189
\end{align*}
\]

The line tangency class is \(\mathcal{T} = \frac{3}{4}\mathcal{H} + J + \frac{3}{4}K\). The characteristic number of rational plane quartics through \(\alpha\) points and tangent to \(\beta\) lines is

\(29\)
The characteristic numbers of rational plane quartics have been calculated by H. Zeuthen in [Z].

4.5. Cuspidal Rational Plane Curves. For \( d \geq 1 \), let \( N_d \) be the number of irreducible, nodal rational plane curves passing through \( 3d-1 \) general points in \( \mathbb{P}^2 \). \( N_d \) is a \( \beta = 0 \) characteristic number. The numbers \( N_d \) satisfy a beautiful recursion relation ([K-M]):

\[
N_1 = 1
\]

\[
\forall d \geq 2, \quad N_d = \sum_{i+j=d, i,j>0} N_i N_j i^2 j \left( j \frac{3d-4}{3i-2} - i \frac{3d-4}{3i-1} \right)
\]

The first few \( N_d \)'s are:

\[
N_1 = 1, \quad N_2 = 1, \quad N_3 = 12, \quad N_4 = 620, \quad N_5 = 87304, \quad N_6 = 26312976, \ldots
\]

As a final application, the enumerative geometry of cuspidal rational plane curves is considered. A rational plane curve, \( C \), is 1-cuspidal if the singularities of \( C \) consist of nodes and exactly 1 cusp. For \( d \geq 3 \), let \( C_d \) be the number of irreducible, 1-cuspidal rational plane curves passing through \( 3d-2 \) general points in \( \mathbb{P}^2 \).

**Proposition 5.** The numbers \( C_d \) can be expressed in terms of the \( N_d \):

\[
\forall d \geq 3, \quad C_d = \frac{3d-3}{d} N_d + \frac{1}{2d} \sum_{i=1}^{d-1} \left( \frac{3d-2}{3i-1} \right) N_i N_{d-i} (3i^2(d-i)^2 - 2di(d-i))
\]

The first few \( C_d \)'s are:

\[
C_3 = 24, \quad C_4 = 2304, \quad C_5 = 435168, \quad C_6 = 156153600, \ldots
\]
$C_3$ is the degree of the locus of cuspidal cubics. $C_4$ has been computed by H. Zeuthen ([Z]). The 1-cuspidal numbers $C_d$ are evaluated by intersecting divisors on $\overline{M}_{0,0}(2, d)$.

Let $d \geq 3$. Let $M_{0,0}(2, d)$ be $\overline{M}_{0,0}(2, d)$ minus the boundary. Let $Z \subset M_{0,0}(2, d)$ be the subvariety of maps that are not immersions. It is easily seen $Z$ is of pure codimension 1 and the generic element of every component corresponds to a 1-cuspidal rational plane curve. Let $Z$ be the Weil divisor obtained by the closure of $Z$ in $\overline{M}_{0,0}(2, d)$. By the dimension reduction argument of section (3.4), the intersection cycle on $\overline{M}_{0,0}(2, d)$

$$Z \cap \mathcal{H}^{3d-2}$$

determined by general points $P_1, \ldots, P_{3d-2}$ is of dimension (at most) 0 and lies in $Z$. A simple modification of the corresponding argument in section (3.4) can be applied to show ([13]) is reduced for general choices of $P_j$. Hence $C_d = Z \cdot \mathcal{H}^{3d-2}$.

The boundary of $\overline{M}_{0,0}(2, d)$ simply consists of the $[\frac{d}{2}]$ Weil divisors $K^i$ ($1 \leq i \leq \frac{d}{2}$). Recall $K^i$ is the boundary component corresponding to the degree partition $i + (d - i) = d$. By Lemmas ([1.2.1]-[1.2.2]), the elements $\{\mathcal{H}\} \cup \{K^i\}$ span a basis of $Pic(\overline{M}_{0,0}(2, d)) \otimes \mathbb{Q}$.

**Lemma 4.5.1.** The class of $Z$ in $Pic(\overline{M}_{0,0}(2, d)) \otimes \mathbb{Q}$ is determined by ($d \geq 3$):

$$Z = \frac{3d - 3}{d} \mathcal{H} + \sum_{i=1}^{\frac{d}{2}} \frac{3i(d - i) - 2d}{d} K^i. \quad (16)$$

**Proof.** Let $S$, $\mu$, $\overline{S}$, $\lambda : C \to \overline{M}_{0,0}(2, d)$ be exactly as in the proof of Lemma (2.3.1). It will be checked that the left and right sides of (16) have the same intersection with $C$. As before, a point of the intersection $C \cdot K^i$ can arise in two cases. A simple point of degree $i$ or $d - i$ can be blown-up. Let

$$C \cdot K^i = x_i + y_i$$

where $x_i$ and $y_i$ are the number instances of the first and second case respectively. Let $E_{x_i}$ be the union of the $x_i$ exceptional divisors in $\overline{S}$ obtained from the $x_i$ points of $C \cdot K^i$. Let $E_{y_i}$ be defined similarly.

First, the intersection $C \cdot Z$ is calculated. Consider $\overline{\mu} : \overline{S} \to \mathbb{P}^2$. $\overline{\mu}^*(\mathcal{O}_F(1))$ is the element $(d, k) - \sum_{x_i} i E_{x_i} - \sum_{y_i}(d - i)E_{y_i}$. The differential
map yields an injection of sheaves:

$$0 \to T_S^\mu \to \mu^*(T_{P^2}) \to Q \to 0.$$  \hspace{1cm} (17)

For general maps $\mu, Q$ is line bundle supported on a nonsingular curve $D$. The restriction of the sequence (17) to $D$ yields an exact sequence of bundles on $D$:

$$0 \to L \to T_S^\mu|_D \to \mu^*(T_{P^2})|_D \to Q|_D \to 0$$  \hspace{1cm} (18)

where $L$ is a line bundle on $D$. Finally, there is exact sequence of bundles on $D$ obtained from the projection $\pi : S \to C$:

$$0 \to V \to T_S^\mu|_D \to \pi^*(T_C) \to 0$$  \hspace{1cm} (19)

where $V$ is a line bundle on $D$. Maps in the family $\pi$ have zero differential exactly at the points of intersection $P(V) \cdot P(L) \subset P(T_S^\mu|_D)$.

Hence

$$C \cdot Z = P(V) \cdot P(L).$$

A lengthy, routine exercise in Chern classes and exact sequences now yields:

$$C \cdot Z = (6d-6)k + \sum_{i=1}^{\lfloor \frac{d}{2} \rfloor} (-3i^2 + 3i - 2)x_i + \sum_{i=1}^{\lfloor \frac{d}{2} \rfloor} (-3(d-i)^2 + 3(d-i) - 2)y_i.$$

Algebraic manipulation and the relation

$$C \cdot H = 2dk - \sum_{i=1}^{\lfloor \frac{d}{2} \rfloor} i^2 x_i - \sum_{i=1}^{\lfloor \frac{d}{2} \rfloor} (d-i)^2 y_i$$

yields the result. \hfill \square

It remains to compute $Z \cdot H^{3d-2}$. By Lemma (4.5.1), it suffices to determine the products $K^i \cdot H^{3d-2}$. If $i \neq d/2$, the result

$$K^i \cdot H^{3d-2} = \binom{3d-2}{3i-1} i(d-i) N_i N_{d-i}$$

is obtained from a simple geometric argument. In case $i = d/2$, division by 2 is required to account for symmetry:

$$K^{d/2} \cdot H^{3d-2} = \frac{1}{2} \binom{3d-2}{3d/2-1} \left( \frac{d}{2} \right)^2 N_{d/2}^{d/2}.$$

Evaluation of $Z \cdot H^{3d-2}$ yields the formula for $C_d$. The proof of Proposition (5) is complete.
References

[Al] V. Alexeev, Moduli Spaces $M_{g,n}(W)$ For Surfaces, preprint 1994.
[A] P. Aluffi, The Enumerative Geometry of Plane Cubics II: Nodal and Cuspidal Cubics, Math. Ann., 289 (1991), 543-572.
[D-I] P. Di Francesco and C. Itzykson, Quantum Intersection Rings, preprint 1994.
[Ke] S. Keel, Intersection Theory of Moduli Spaces of Stable $n$-Pointed Curves of Genus 0, Trans. AMS, 330 (1992), 545-574.
[K] M. Kontsevich, Enumeration of Rational Curves Via Torus Actions, preprint 1994.
[K-M] M. Kontsevich and Y. Manin, Gromov-Witten Classes, Quantum Cohomology, and Enumerative Geometry, preprint 1994.
[K-S] S. Kleiman, R. Speiser, Enumerative Geometry of Nodal Plane Curves, Algebraic Geometry - Sundance 1986, Lecture Notes in Mathematics 1311, Springer-Verlag: Berlin, 1988.
[K-S-X] S. Kleiman, S. Stromme, S. Xambo, Sketch of a Verification of Schubert’s Numbers 5, 819, 539, 753, 680 of Twisted Cubics, Lecture Notes in Mathematics 1266, Springer-Verlag: Berlin, 1987.
[P] R. Pandharipande, Notes on Kontsevich’s Compactification of the Moduli Space of Maps, Unpublished Course Notes, Winter 1995.
[R-T] Y. Ruan and G. Tian, A Mathematical Theory of Quantum Cohomology, preprint 1994 (to appear in JDG).
[Sa] G. Saccheiro, Numeri Caratteristici delle Cubishe Piane Nodali, preprint 1985.
[S] H. Schubert, Kalkül der Abzählenden Geometrie, B. G. Teubner: Leipzig, 1879.
[Z] H. Zeuthen, Abmindelige Egenskaber…, Danske Videnkabernes Selskabs Skrifter-Natur og Math, 10 (1873).

Department of Math, University of Chicago, rahul@math.uchicago.edu