Abstract. The purpose of this paper is to give a new proof of results of Moscovici and Stanton on the orbital integrals associated with eta invariants on compact locally symmetric spaces. Moscovici and Stanton used methods of harmonic analysis on reductive groups. Here, we combine our approach to orbital integrals that uses the hypoelliptic Laplacian, with the introduction of a rotation on certain Clifford algebras. Probabilistic methods play an important role in establishing key estimates. In particular, we construct the proper Itô calculus associated with certain hypoelliptic diffusions.

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1. Introduction

The purpose of this paper is to give a new approach to the evaluation by Moscovici-Stanton [MoSt89] of the orbital integrals that appear in the eta invariant of a Dirac operator on a compact odd dimensional locally symmetric space. Moscovici and Stanton obtained their results by an appropriate use of Selberg trace formula. They were extending earlier results by Millson [Mi78] on compact manifolds of constant negative curvature. In a later paper [MoSt91], Moscovici and Stanton used similar methods to evaluate the integrand of the Ray-Singer analytic torsion for compact locally symmetric spaces. In both papers [MoSt89, MoSt91], Moscovici and Stanton built up on the evaluation of orbital integrals associated with heat kernels to give an explicit formula for the eta invariant and analytic torsion. Here, we will recover the results of Moscovici-Stanton on the orbital integrals in the case of eta invariants.

Our method uses the theory of the hypoelliptic Laplacian on symmetric spaces [B11], which we briefly review. The hypoelliptic Laplacian is a general construction that is valid on arbitrary Riemannian manifolds [B05, B08], in which a family of hypoelliptic operators acting on the total space of the tangent bundle, or of a larger vector bundle, is produced, that interpolates in the proper sense between a generalized Laplacian, and the generator of the geodesic flow. In [B11], a version of the hypoelliptic Laplacian was shown to exist on symmetric spaces, such that the semisimple orbital integrals associated with the heat kernel for the Casimir operator are preserved by the hypoelliptic deformation. By deforming all the way to the geodesic flow, a general explicit formula was given in [B11] for these orbital integrals.

In [B11, chapter 7], applications were given to the evaluation of the orbital integrals appearing in the evaluation of the Ray-Singer analytic torsion. We recovered this way some of the results of Moscovici-Stanton [MoSt91]. This is because, in the context of analytic torsion, the square of the relevant Dirac operator is just the Casimir operator. The same is true for the other index theoretic calculations of [B11, chapter 7] for even dimensional locally symmetric spaces, since the square of the Dirac operator differs from the Casimir operator by a covariantly constant matrix operator. Let us also point out the work by Shen [Sh16a, Sh16b] who was able to complete the proof by Moscovici-Stanton [MoSt91] of the Fried conjecture for the analytic torsion [F86] in the case of compact locally symmetric spaces. Shen used the explicit formulas established in [B11] for the orbital integrals for the heat kernel associated with the Casimir operator, and filled a gap in Moscovici-Stanton’s proof [MoSt91], also using arguments from representation theory.

The eta invariant is a spectral function of the Dirac operator, and not just of its square, and so the analysis in [B11] breaks down. Before we explain how to deal with this difficulty, let us describe the ideas of [B11] in more detail.

Let $G$ be a connected reductive Lie group, let $\theta$ be its Cartan involution, let $\mathfrak{g}$ be its Lie algebra, and let $K$ be the associated maximal compact subgroup of $G$ with Lie algebra $\mathfrak{k} \subset \mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{t}$ be the Cartan splitting of $\mathfrak{g}$. Let $X = G/K$ be the corresponding symmetric space. Let $B$ be a bilinear symmetric nondegenerate form on $\mathfrak{g}$, which is $G$-invariant and $\theta$-invariant, positive on $\mathfrak{p}$, and negative on $\mathfrak{t}$.

Let $C^\mathfrak{g}$ be the Casimir operator of $G$ associated with $B$, and let $F$ be the complex vector bundle on $X$ associated with an irreducible representation of $K$. Then $C^\mathfrak{g}$
descends to a second order elliptic operator $C^{g,X}$ acting on smooth sections of $F$ on $X$. Set $\mathcal{L}^{X} = \frac{1}{2}C^{g,X} + c$, where $c$ is an explicit constant.

Note that $TX$ is the vector bundle corresponding to the adjoint action of $K$ on $p$. Let $N$ be the vector bundle on $X$ associated with the adjoint action of $K$ on $\mathfrak{k}$. Let $\overline{\pi} : \overline{\mathcal{X}} \to X$ be the total space of $TX \oplus N$. The hypoelliptic deformation $\mathcal{L}^{X}_{b>0} |_{b>0}$ of $\mathcal{L}^{X}$ is obtained via a corresponding family of generalized Dirac operators $\mathcal{D}^{X}_{b} |_{b>0}$. The operators $\mathcal{D}^{X}_{b}, \mathcal{L}^{X}_{b}$ act on the smooth sections over $\overline{\mathcal{X}}$ of $\overline{\pi}^{*}(\Lambda^{\infty}(T^{*}X \oplus N^{*}) \otimes_{\mathbb{R}} F)$. Up to lower order terms, the operator $\mathcal{L}^{X}_{b}$ is a scaled sum of the harmonic oscillator along $TX \oplus N$, of the generator of the geodesic flow, and of a nonnegative scalar term of degree 4 in the variables in the fibre $TX \oplus N$.

In [B11], the fact that the semisimple orbital integrals for the heat kernel of $\mathcal{L}^{X}$ are invariant by the hypoelliptic deformation is shown to be a version of the McKean-Singer formula [MS67] for the Lefschetz supertrace associated with a classical Dirac operator. By making $b \to +\infty$, which forces the hypoelliptic Laplacian $\mathcal{L}^{X}_{b}$ to converge in the proper sense to the generator of the geodesic flow, we obtain an explicit geometric formula for the orbital integrals of the heat kernel of $\mathcal{L}^{X}$.

Let $c(\mathfrak{g}), \hat{c}(\mathfrak{g})$ be the Clifford algebras associated with $(\mathfrak{g}, B), (\mathfrak{g}, -B)$, and let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. In the construction of $\mathcal{D}^{X}_{b}$, one key idea in [B11] is to express the Casimir operator $\hat{c}(\mathfrak{g}) |_{\mathfrak{g}}$ up to a constant as minus the square of the Kostant Dirac operator $\hat{D}^{\mathfrak{g}} \in \hat{c}(\mathfrak{g}) \otimes U(\mathfrak{g})$ [K76, K97]. Ultimately, both Clifford algebras $c(\mathfrak{g}), \hat{c}(\mathfrak{g})$ are used in the constructions of [B11].

We assume $G$ to be simply connected. Let $TX$ be another copy of $TX$, and let $S^{TX}$ denote the vector bundle of $TX$ spinors. The classical Dirac operator $D^{X}$ on $X$ acts on $C^{\infty}(X, S^{TX} \otimes F)$. Let $\overline{p}$ be another copy of $p$. The Clifford algebra $c(\overline{p})$ descends to the Clifford algebra $c(TX)$, that is used in the construction of the Dirac operator $D^{X}$. The operator $D^{X,2}$ differs from $C^{g,X}$ by a constant tensor in which the Clifford algebra $c(\overline{p})$ does not appear. This is why in [B11, chapter 7], in which the index theory of Dirac operators is considered in relation with the trace formula, the odd part of the Clifford algebra $c(\overline{p})$ plays no role. These considerations suggest that to apply the methods of [B11] to the Dirac operator $D^{X}$ itself, we have to consider the three Clifford algebras $c(\mathfrak{g}), \hat{c}(\mathfrak{g}), c(\overline{p})$ together. In a formally similar context, three Clifford algebras were already considered in [B08] to construct a hypoelliptic version of the Dirac operator associated with an arbitrary Riemannian manifold.

The complex structure of $p \oplus \overline{p}$ will play an important role in our method to recover the results of Moscovi-Stanton [MoSt89]. What replaces the conservation of Lefschetz supertraces is a conservation principle for another class of orbital integrals. More precisely, we apply the odd superconnection formalism of Quillen [Q85] to a two parameter family of generalized Dirac operators $\mathcal{D}^{X}_{b,\vartheta}|_{b>0, \vartheta \in [0, \frac{\pi}{2}]}$ over $\overline{X}$. The corresponding family of hypoelliptic Laplacians is denoted $\mathcal{L}^{X}_{b,\vartheta}|_{b>0, \vartheta \in [0, \frac{\pi}{2}]}$. If $\vartheta = 0$, we recover the families $\mathcal{D}^{X}_{b}|_{b>0}, \mathcal{L}^{X}_{b}|_{b>0}$. If $\gamma \in G$ is semisimple and nonelliptic, we show that the integral of a 1-form $b$ constructed via orbital integrals associated with $\gamma$ on $[0, \frac{\pi}{2}]$ does not depend on $b > 0$. By making $b \to 0$, this quantity is shown to be an explicit multiple of the orbital integral $\text{Tr}\left[\mathcal{L}^{X}_{\gamma}(\frac{1}{\sqrt{2}} \exp (-sD^{X,2}/2) \right] * \frac{1}{\sqrt{2}}(t)$, where $*$ is the convolution of functions on $\mathbb{R}_{+}$. By making $b \to +\infty$, we express this quantity in geometric terms, by a localization procedure which is essentially
taken from our previous work [B11]. If \( \gamma = e^{a \kappa^{-1}}, a \in \mathfrak{p}, \kappa \in K, \text{Ad} (k) a = a \), like in [B11], the geometric expression involves an integral on \( \mathfrak{t} (\gamma) \), the \( \mathfrak{t} \) part of the Lie algebra \( \mathfrak{g} (\gamma) \) of the centralizer of \( \gamma \). Ultimately, we recover the vanishing results of Moscovici and Stanton [MoSt89], and also the explicit formulas they obtained in the case where the orbital integrals do not vanish.

In the paper, we tried to clearly separate the algebraic and geometric arguments from the analytic arguments. In particular, in sections 2–8, we construct the operators \( \mathcal{D}_{b, \vartheta}^X, \mathcal{L}_{b, \vartheta}^X \), key estimates on heat kernels for \( \mathcal{L}_{b, \vartheta}^X \) on \( \mathcal{X} \) are stated without proof, and are used to establish our main results in Theorems 8.3 and 8.6. The proof of these estimates is deferred to sections 9–12. Most of the estimates are obtained by properly adapting the estimates in [B11], except for a uniform estimate on solutions of a linear differential equation, that is established in Theorems 11.13 and 11.17. This estimate eluded us for some time, and largely explains the length of the paper, which should otherwise be a rather straightforward extension of [B11]. As in [B11], and for fundamental reasons, probabilistic techniques play an important role in the proofs.

This paper is organized as follows. In section 2, we briefly recall simple facts of linear algebra, that include Quillen’s superconnection formalism [Q85].

In section 3, we recall the construction of the hypoelliptic Laplacian \( \mathbb{L}_b^X \) of [B11], which up to an explicit constant, deforms \( C^{0,-\infty}/2 \). The operator \( \mathbb{L}_b^X \) is obtained via the construction of a Dirac like operator \( \mathcal{D}_b^X \), which itself deforms the operator 0.

In section 4, we construct the families of operators \( \mathcal{D}_{b, \vartheta}^X |_{(b, \vartheta) \in \mathbb{R}_+ \times [0, \frac{\pi}{2}]} \) and \( \mathcal{L}_{b, \vartheta}^X |_{(b, \vartheta) \in \mathbb{R}_+ \times [0, \frac{\pi}{2}]} \). Given \( \vartheta \in [0, \frac{\pi}{2}] \), the family \( \mathcal{D}_{b, \vartheta}^X |_{b > 0} \) is a deformation of \( \sin (\vartheta) iD^X \). For \( \vartheta > 0 \), the construction of \( \mathcal{D}_{b, \vartheta}^X \) now involves the Clifford algebra \( c (T/\mathbb{X}) \) explicitly.

In section 5, given a semisimple element \( \gamma \in G \), we construct a closed superconnection 1-form \( b \) on \( \mathbb{R}_+ \times [0, \frac{\pi}{2}] \), involving orbital integrals associated with the heat kernel for \( \mathcal{L}_{b, \vartheta}^X \), in which the operator \( \mathcal{D}_{b, \vartheta}^X \) also appears.

In section 6, we show that the integral of \( b \) on \( [0, \frac{\pi}{2}] \) does not depend on \( b > 0 \), and coincides with the integral on \( [0, \frac{\pi}{2}] \) of a 1-form \( a \) involving the orbital integrals \( \text{Tr}^{[\gamma]} \left[ D^X \exp \left( -sD^{X,2}/2 \right) \right] \).

In section 7, by making \( b \to +\infty \), we give an explicit geometric formula for the integral of \( a \). The geometric computations are closely related to our earlier work [B11].

In section 8, by working in more detail on the geometric formulas of section 7, we obtain our main results, i.e., we rederive the formulas of Moscovici-Stanton [MoSt89] for the orbital integrals \( \text{Tr}^{[\gamma]} \left[ D^X \exp \left( -tD^{X,2}/2 \right) \right] \).

In the sections that follow, we establish the estimates that are needed in the proofs of the previous results. The analysis of the operators \( \mathcal{L}_{b, \vartheta}^X \) involves an essentially different step from the analysis in [B11] when \( b > 0 \) is bounded, because we need to obtain uniform estimates in \( \theta \in [0, \frac{\pi}{2}] \) as \( b \to 0 \). When \( b \to +\infty \), the analysis is essentially the same as in [B11].

In section 9, we improve on the uniform estimates we had obtained in [B11, chapter 12–14] for the smooth kernels \( \exp (-tL_b^X) \), when \( b > 0, t > 0 \) remain uniformly bounded. In [B11], only the case where \( t \) remains bounded away from 0 was considered. The reason for doing this is that some of our estimates for the
kernels for $\exp\left(-\mathcal{L}_{b,\vartheta}^{X}\right)$ are equivalent to the just described estimates when $t > 0$ is small. Also we establish uniform estimates on the rate of escape from an open ball for the hypoelliptic diffusion associated with $\mathcal{L}_{b}^{X}|_{0<\theta\leq1}$. These estimates were not stated explicitly in \cite{B11}, and turn out to be useful in the sections that follow.

In section 10, we obtain uniform estimates for the heat kernels for a scalar version of $\mathcal{L}_{b,\vartheta}^{X}$ for $b > 0$ bounded and $\theta \in \left[0, \frac{\pi}{2}\right]$. These estimates are easy consequences of the results of section 9.

In section 11, we obtain the required estimates for the heat kernels $\exp\left(-\mathcal{L}_{b,\vartheta}^{X}\right)$ when $b > 0$ remains bounded and $\theta \in \left[0, \frac{\pi}{2}\right]$. This is the technically most difficult section of the paper. Indeed, passing from the estimates for the scalar version of $\mathcal{L}_{b,\vartheta}^{X}$ to the full operator $\mathcal{L}_{b,\vartheta}^{X}$ introduces new difficulties that did not appear in \cite{B11}, essentially because the exterior algebra $\Lambda (T^*X \oplus N^*)$ and the Clifford algebra $c (TX)$ are coupled in a nontrivial way for $\theta > 0$. Probabilistic techniques are especially useful there. While we described before the main new difficulty with respect to \cite{B11} to be the proof of uniform bounds on the solution of a family of differential equations, the way this control is obtained is via geometric considerations on the projection of this solution on the proper symmetric space.

Finally, in section 12, we obtain the required uniform estimates for the smooth kernels $\exp\left(-\mathcal{L}_{b,\vartheta}^{X}\right)$ for $b \geq 1$ and $\vartheta \in \left[0, \frac{\pi}{2}\right]$.

Let us also point out that in the very short subsection 9.7, we establish a generalized Itô formula for the hypoelliptic diffusion associated with the operator $\mathcal{L}_{b}^{X}$. This subsection can be read independently of the remainder of the paper. It makes clear that the proper Itô calculus for our hypoelliptic diffusion is deduced from the classical Itô calculus by a simple convolution. This new form of the Itô calculus plays a key role in the proof of our estimates, and should be of independent interest.

For an introduction to Brownian motion and the stochastic calculus, we refer to Ikeda-Watanabe \cite{IW89} and Le Gall \cite{LG16}.

In the whole paper, if $\mathcal{A}$ is a $\mathbb{Z}_2$-graded algebra, if $a, a' \in \mathcal{A}$, $[a, a']$ denotes the supercommutator of $a, a'$, so that

$$[a, a'] = aa' - (-1)^{\deg a \deg a'} a'a. \quad (1.1)$$

If $\mathcal{A}, \mathcal{A}'$ are $\mathbb{Z}_2$-graded algebras, $\mathcal{A} \hat{\otimes} \mathcal{A}'$ denotes the tensor product $\mathcal{A} \otimes \mathcal{A}'$ equipped with the induced $\mathbb{Z}_2$-graded structure.

Also, in our estimates, the constants $c > 0, C > 0$ may vary from line to line.
2. Linear algebra

The purpose of this section is to develop an algebraic formalism in a finite dimensional context that will be used in the next sections in infinite dimensions, to properly handle eta invariants on locally symmetric spaces.

This section is organized as follows. In subsection 2.1, if $E$ is a vector space, and $A = \text{End} (E)$, we introduce associated superconnections and $\sigma$-traces in the sense of Quillen [Q85]. Of special importance is the construction of a function on $A$ that transgresses a Quillen form of degree 1.

In subsection 2.2, we specialize the constructions of subsection 2.1 to the case where $A'$ is a subalgebra of $A$ and where $C$ is a central element in $A'$.

2.1. Superconnections and $\sigma$-traces. First, we describe Quillen’s odd superconnection formalism [Q85, §5]. Let $\sigma$ be the odd generator of the Clifford algebra $c(R)$, so that $\sigma^2 = 1$.

Let $E$ be a finite dimensional complex vector space. Put

\begin{equation}
A = \text{End} (E) .
\end{equation}

Set

\begin{equation}
A_\sigma = A \otimes_R c(R) .
\end{equation}

Then $A_\sigma$ is a $\mathbb{Z}_2$-graded algebra. The splitting of $A_\sigma$ into its even and odd parts is given by

\begin{equation}
A_\sigma = A \oplus \sigma A .
\end{equation}

If $\alpha \in A_\sigma$, then $\alpha = a + \sigma b, a, b \in A$. Put

\begin{equation}
\text{Tr}_\sigma [\alpha] = \text{Tr} [b] .
\end{equation}

Then $\text{Tr}_\sigma$ vanishes on the even part of $A_\sigma$, and also on supercommutators in $A_\sigma$.

Let $A^*$ denote the dual vector space to $A$. Put

\begin{equation}
B = \Lambda \cdot \hat{(A^*)} \otimes A_\sigma .
\end{equation}

Then $B$ is also a $\mathbb{Z}_2$-graded algebra. We extend $\text{Tr}_\sigma$ to a map from $B$ into $\Lambda \cdot (A^*)$, with the convention that if $\omega \in \Lambda \cdot (A^*), \alpha \in A_\sigma$,

\begin{equation}
\text{Tr}_\sigma [\omega \alpha] = \omega \text{Tr}_\sigma [\alpha] .
\end{equation}

Then $\text{Tr}_\sigma$ still vanishes on supercommutators in $B$.

We now view $A$ as a trivial vector bundle on the vector space $A$. Let $D$ denote the tautological section of $A$ over the vector space $A$. Let $d$ be the de Rham operator on $A$. Let $A$ be the superconnection over $A$,

\begin{equation}
A = d + D \sigma .
\end{equation}

Its curvature $A^2 \in B^{\text{even}}$ is given by

\begin{equation}
A^2 = D^2 + (dD) \sigma .
\end{equation}

It verifies the Bianchi identity

\begin{equation}
[A, A^2] = 0.
\end{equation}

Let $\varphi$ be a holomorphic function from $\mathbb{C}$ into itself. Then $\varphi$ extends to an analytic function from $B^{\text{even}}$ into itself. In particular, $\varphi (A^2)$ lies in $B^{\text{even}}$.

Set

\begin{equation}
\Phi = \text{Tr}_\sigma [\varphi (A^2)] .
\end{equation}
Then $\Phi$ is an odd form on $\mathcal{A}$. A result of Quillen [Q85] says that $\Phi$ is a closed form. Indeed, using the vanishing of $\text{Tr}_\sigma$ on supercommutators and Bianchi’s identity (2.9), we get

\begin{equation}
(2.11) \quad d\Phi = \text{Tr}_\sigma \left[ [d, \varphi (A^2)] \right] = \text{Tr}_\sigma \left[ [A, \varphi (A^2)] \right] = 0.
\end{equation}

We will be especially interested in the component $\Phi^{(1)}$ of degree 1 of $\Phi$, that is given by

\begin{equation}
(2.12) \quad \Phi^{(1)} = \text{Tr} \left[ \varphi' (D^2) dD \right].
\end{equation}

Let $F : \mathcal{A} \rightarrow \mathbb{C}$ be the smooth function that is given by

\begin{equation}
(2.13) \quad F(D) = \int_0^1 \text{Tr} \left[ D \varphi' (u^2 D^2) \right] du.
\end{equation}

Since $\Phi^{(1)}$ is closed, $F$ is the unique smooth function on $\mathcal{A}$ vanishing at 0 such that

\begin{equation}
(2.14) \quad dF = \Phi^{(1)}.
\end{equation}

For $t \geq 0$, set

\begin{equation}
(2.15) \quad \varphi_t (z) = \varphi (tz).
\end{equation}

The associated function $F_t$ is given by

\begin{equation}
(2.16) \quad F_t (D) = \sqrt{t} F \left( \sqrt{t} D \right).
\end{equation}

Equivalently,

\begin{equation}
(2.17) \quad F_t (D) = \sqrt{t} \int_0^t \frac{1}{2 \sqrt{u}} \text{Tr} \left[ D \varphi' (u D^2) \right] du.
\end{equation}

The function $t \in \mathbb{R}_+ \rightarrow F_t (D) \in \mathbb{C}$ is smooth, and moreover,

\begin{equation}
(2.18) \quad \frac{d}{dt} F_t (D) \big|_{t=0} = \varphi' (0) \text{Tr} [D].
\end{equation}

**Example 2.1.** Let log be the logarithm defined on \( \{ z \in \mathbb{C}^*, \text{Re} \, z \geq 0 \} \), with polar angle lying in \((-\pi/2, \pi/2]\). For \( z \in \mathbb{R}, \text{Re} \, z \geq 0 \), set

\begin{equation}
(2.19) \quad \varphi (z) = \log (1 + z).
\end{equation}

Here, we assume $E$ to be a Hermitian vector space. Let $\mathcal{A}_{\text{sa}}, \mathcal{A}_{\geq 0}$ be the vectors subspaces of $\mathcal{A}$ of self-adjoint elements, and self-adjoint nonnegative elements. If $B \in \mathcal{A}_{\geq 0}$, then

\begin{equation}
(2.20) \quad \text{Tr} [\varphi (B)] = \log \det (1 + B).
\end{equation}

Here, we restrict our construction to elements $D \in \mathcal{A}_{\text{sa}}$. We define the form $\Phi$ on $\mathcal{A}_{\text{sa}}$ as in (2.10). Then

\begin{equation}
(2.21) \quad \Phi^{(1)} = \text{Tr} \left[ (1 + D^2)^{-1} dD \right].
\end{equation}

As in Moscovici-Stanton [MoSt89, § 2], we introduce the Cayley transform

\begin{equation}
(2.22) \quad C(D) = \frac{1 + iD}{1 - iD}.
\end{equation}

Set

\begin{equation}
(2.23) \quad F(D) = \frac{1}{2i} \log \det [C(D)] = \text{Im} \log \det (1 + iD).
\end{equation}

Then $F$ is the real function on $\mathcal{A}_{\text{sa}}$ such that $F (0) = 0, dF = \Phi^{(1)}$. 

We denote by $\text{Tr}' \left[ \frac{D}{[D]} \right]$ the trace of $\frac{D}{[D]}$ in which the possible zero eigenvalues of $D$ have been eliminated. When $t \to +\infty$, then

$$\frac{F_t(D)}{\sqrt{t}} \to \frac{\pi}{2} \text{Tr}' \left[ \frac{D}{[D]} \right].$$

Another definition of $F(D)$ is by a formula like in (2.13),

$$F(D) = \int_0^1 \text{Tr} \left[ (1 + u^2 D^2)^{-1} D \right] du.$$

One can derive (2.24) from (2.25).

**Example 2.2.** Set

$$\varphi(z) = \exp(-z).$$

Then

$$\Phi^{(1)} = -\text{Tr} \left[ dd \exp(-D^2) \right].$$

Also

$$F_t(D) = -\sqrt{t} \int_0^t \frac{1}{2\sqrt{u}} \text{Tr} \left[ D \exp(-uD^2) \right] du.$$

If $D \in \mathcal{A}_{sa}$, when $t \to +\infty$,

$$\frac{F_t(D)}{\sqrt{t}} \to -\frac{\sqrt{\pi}}{2} \text{Tr}' \left[ \frac{D}{[D]} \right].$$

For $s \in \mathbb{C}, \Re s > 0$, if $B \in \mathcal{A}_{\geq 0}$, then

$$\frac{1}{\Gamma(s)} \int_0^{+\infty} u^{s-1} e^{-u} \varphi(uB) du = (1 + B)^{-s}.$$

By (2.30), we get

$$\frac{\partial}{\partial s} \left[ \frac{1}{\Gamma(s)} \int_0^{+\infty} u^{s-1} e^{-u} \varphi(uB) du \right] (0) = -\log(1 + B).$$

Note that (2.24) also follows from (2.29), (2.31).

### 2.2. A central element.

Let $\mathcal{A}'$ be a subalgebra of $\mathcal{A}$, let $C$ be a central element in $\mathcal{A}'$. We will restrict ourselves to forms on $\mathcal{A}'$. In particular $D$ now denotes the generic element of $\mathcal{A}'$, and $d$ is the de Rham operator on $\mathcal{A}'$. We still define the superconnection $A$ on $\mathcal{A}'$ by equation (2.7). Then

$$[A, C] = 0.$$

By (2.9), (2.32), we obtain the Bianchi identity

$$[A, A^2 - C] = 0.$$

Let $\varphi$ be a holomorphic function as in subsection 2.1. Set

$$\Phi_C = \text{Tr}_x \left[ \varphi (A^2 - C) \right].$$

By proceeding as in (2.11) and using (2.33), we get

$$d\Phi_C = \text{Tr}_x \left[ [d, \varphi (A^2 - C)] \right] = \text{Tr}_x \left[ A, \varphi (A^2 - C) \right] = 0,$$

i.e., $\Phi_C$ is a closed odd form on $\mathcal{A}'$. Also

$$\Phi_C^{(1)} = \text{Tr} \left[ \varphi' (D^2 - C) dD \right].$$
As in (2.13), for $D \in \mathcal{A}'$, set
\begin{equation}
F_C(D) = \int_0^1 \text{Tr} \left[ D \varphi' (u^2 D^2 - C) \right] du.
\end{equation}

Then $F_C$ is the unique function on $\mathcal{A}'$ that vanishes at 0 and is such that
\begin{equation}
dF_C = \Phi^{(1)}.
\end{equation}

More generally, the considerations of subsection 2.1 also apply to $F_C$. For $t \geq 0$, we denote by $F_{C,t}$ the function associated with $\varphi_t$.

**Example 2.3.** Set
\begin{equation}
\varphi(z) = \exp(z).
\end{equation}

Let $\Phi_{C,t}$ be associated with $\varphi_t$. Then
\begin{equation}
\Phi_{C,t}^{(1)} = t \text{Tr} \left[ dD \exp \left( t (D^2 - C) \right) \right].
\end{equation}

Also
\begin{equation}
F_{C,t}(D) = \sqrt{t} \int_0^t \frac{1}{2\sqrt{u}} \text{Tr} \left[ D \exp \left( uD^2 - tC \right) \right] du.
\end{equation}

The function $F_{C,t}(D)$ is a smooth function of $t \in \mathbb{R}_+$. Moreover,
\begin{equation}
\frac{d}{dt} F_{C,t}(D) \big|_{t=0} = \text{Tr}[D].
\end{equation}

We will now restrict the above forms to the $D \in \mathcal{A}'$ such that $D^2 = C$. Since $C$ is central, this set is invariant by conjugation by invertible elements in $\mathcal{A}'$. Then (2.41) takes the form
\begin{equation}
F_{C,t}(D) = \sqrt{t} \int_0^t \frac{1}{2\sqrt{u}} \text{Tr} \left[ D \exp \left( - (t-u) D^2 \right) \right] du.
\end{equation}

We can rewrite (2.43) in the form
\begin{equation}
F_{C,t}(D) = t \int_0^1 \frac{1}{2\sqrt{u}} \text{Tr} \left[ D \exp \left( -t (1-u) D^2 \right) \right] du.
\end{equation}

Let $\ast$ denote the convolution of distributions with support in $\mathbb{R}_+$. Then (2.43) can be written in the form
\begin{equation}
F_{C,t}(D) = \frac{\sqrt{t}}{2} \left( \text{Tr} \left[ D \exp \left( -sD^2 \right) \right] \ast \frac{1}{\sqrt{s}} \right)(t).
\end{equation}

We have the identity of distributions on $\mathbb{R}_+$,
\begin{equation}
\frac{1}{\sqrt{s}} \ast \frac{1}{\sqrt{s}} = \pi.
\end{equation}

Since $F_{C,t}(D)$ is a smooth function of $t \in \mathbb{R}_+$, $\frac{F_{C,t}(D)}{\sqrt{s}} \ast \frac{1}{\sqrt{s}}(t)$ is a smooth function of $t \in \mathbb{R}_+$.

**Proposition 2.4.** For $t > 0$, the following identity holds:
\begin{equation}
\text{Tr} \left[ D \exp \left( -tD^2 \right) \right] = \frac{2}{\pi} \frac{d}{dt} \left[ \frac{F_{C,t}(D)}{\sqrt{s}} \ast \frac{1}{\sqrt{s}} \right](t).
\end{equation}

**Proof.** This follows from (2.45), (2.46).
Assume now that $D$ is self-adjoint, and $C = D^2$ is central. We define $\text{Tr}' \left[ \frac{D}{|D|^{2\alpha}} \right]$ by taking the trace of $\frac{D}{|D|^{2\alpha}}$ in which the possible zero eigenvalues of $D$ have been eliminated. By (2.44), for $0 < \alpha < 1$, then

$$
(2.48) \quad \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-2} F_{C,t}(D) \, dt = \frac{\Gamma(1/2) \Gamma(1 - \alpha)}{\Gamma(\frac{3}{2} - \alpha)} \frac{1}{2} \text{Tr}' \left[ \frac{D}{|D|^{2\alpha}} \right].
$$

For $\alpha = 1/2$, we get

$$
(2.49) \quad \frac{1}{\sqrt{\pi}} \int_0^{+\infty} t^{-3/2} F_{C,t}(D) \, dt = \frac{\pi}{2} \text{Tr}' \left[ \frac{D}{|D|} \right].
$$

**Definition 2.5.** The self-adjoint operator $D$ is said to be fully symmetric if its spectrum, counted with multiplicity, is invariant under the map $\lambda \to -\lambda$.

**Proposition 2.6.** The self-adjoint operator $D$ is fully symmetric if and only if for any $t > 0$, $F_{C,t}(D) = 0$.

**Proof.** The operator $D$ is fully symmetric if and only if for any $s \geq 0$,

$$
(2.50) \quad \text{Tr} \left[ D \exp \left( -sD^2 \right) \right] = 0.
$$

By (2.45), (2.47), we get our proposition. $\square$
3. The hypoelliptic Laplacian on a symmetric space

Let \( G \) be a connected and simply connected reductive group, and let \( X = G/K \) be the symmetric space associated with \( G \). In this section, we recall the construction in \([B11]\) of the hypoelliptic Laplacian, a deformation \( \mathcal{L}_b^X \) of the operator \( C^0,\mathcal{X}/2 \) acting on \( C^\infty \left( X, S^{TX} \otimes F \right) \), where \( S^{TX} \) is the bundle of \( TX \) spinors, and \( F \) is a homogeneous vector bundle on \( X \). In section 4, we will add an extra deformation parameter \( \vartheta \in \left[ 0, \frac{\pi}{2} \right] \); the constructions in the present section corresponding to the case \( \vartheta = 0 \).

This section is organized as follows. In subsection 3.1, we introduce the connected reductive group \( G \), its Lie algebra \( \mathfrak{g} \), and the symmetric space \( X \).

In subsection 3.2, we lift the Cartan involution to homogeneous vector bundles on \( X \).

In subsection 3.3, we construct the Casimir operator \( C^g \).

In subsection 3.4, we introduce the exterior algebra \( \Lambda (\mathfrak{g}^*) \), and the Clifford algebras \( c (\mathfrak{g}) \), \( \tilde{\mathcal{C}} (\mathfrak{g}) \) associated with an invariant bilinear form \( \tilde{B} \) on \( \mathfrak{g} \).

In subsection 3.5, we describe a few properties of the symmetric algebra \( S (\mathfrak{g}^*) \), and the Bargmann isomorphism, that identifies a completion of \( S (\mathfrak{g}^*) \) with \( L_2 (\mathfrak{g}) \).

In subsection 3.6, when \( G \) is simply connected, if \( TX \) is another copy of \( TX \), we construct the bundle of spinors \( S^{TX} \) associated with \( TX \).

In subsection 3.7, we define the Dirac operator \( \hat{D}^X \) acting on \( C^\infty \left( X, S^{TX} \otimes F \right) \), and the elliptic operator \( \mathcal{L}_0^X \) acting on the same vector space, that differs by a constant from the action of \( \frac{1}{2} C^g \) on \( C^\infty \left( X, S^{TX} \otimes F \right) \), and by another constant from \( -\frac{1}{2} \hat{D}^{X,2} \).

In subsection 3.8, following \([B11]\), we use the Dirac operator of Kostant \( \hat{D}^g \) to define an operator \( \mathfrak{D}_b \) acting on \( C^\infty \left( G \times \mathfrak{g}, \Lambda (\mathfrak{g}^*) \right) \).

In subsection 3.9, we establish a simple compression property of the operator \( \mathfrak{D}_b \).

In subsection 3.10, we state a formula of \([B11]\) for \( \mathfrak{D}_b^2 \).

Let \( \pi : \tilde{X} \to X \) be the total space of the vector bundle \( TX \oplus N = G \times_K \mathfrak{g} \) on \( X \). In subsection 3.11, by quotienting the above constructions by \( K \), we descend the operator \( \mathfrak{D}_b \) to an operator \( \mathfrak{D}_b^X \) acting on \( C^\infty \left( \tilde{X}, \pi^* \left( \Lambda (T^*X \oplus N^*) \otimes S^{TX} \otimes F \right) \right) \).

Also we construct the hypoelliptic Laplacian \( \mathcal{L}_b^X \) that acts on the same vector space.

Finally, in subsection 3.12, we give an important formula established in \([B11]\) that explains why the family \( \mathcal{L}_b^X \) deforms \( \mathcal{L}_0^X \) as \( b \to 0 \).

In the constructions of the present section, the Dirac operator \( \hat{D}^X \) only appears through its square \( \hat{D}^{X,2} \) (that coincides, up to a constant, with the action of \( -\frac{1}{2} C^g \)), the family of operators \( \mathfrak{D}_b^X \) being a deformation of the 0 operator. In section 4, we will resurrect the operator \( \hat{D}^X \) through another family of operators, by introducing an extra parameter \( \vartheta \in \left[ 0, \frac{\pi}{2} \right] \).

### 3.1. A connected reductive group.

Let \( G \) be a real connected reductive group, and let \( \theta \) be its Cartan involution. Let \( K \subset G \) be the subgroup of \( G \) fixed by \( \theta \), so that \( K \) is a maximal compact subgroup. Then \( K \) is also connected.
Let $g, \mathfrak{k}$ be the Lie algebras of $G, K$. Then $\mathfrak{k}$ is the +1 eigenspace of $\theta$ in $g$. Let $p$ be the $-1$ eigenspace of $\theta$ so that

(3.1) $g = p \oplus \mathfrak{k}$.

Set

(3.2) $m = \dim p, \quad n = \dim \mathfrak{k}$.

Let $\theta^g$ be the $g$-valued canonical left-invariant 1-form on $g$, and let $\theta^g = \theta^p + \theta^\mathfrak{k}$ denote its splitting with respect to (3.1).

Let $\theta$ be a bilinear symmetric nondegenerate form on $g$ which is $G$-invariant, and also invariant under $\theta$, let $\varphi : g \to g^*$ denote the canonical isomorphism induced by $B$. Then (3.1) is still an orthogonal splitting with respect to $\langle \cdot, \cdot \rangle$. If $a \in g$, ad $(a) \in \text{End}(g)$ is antisymmetric with respect to $B$. If $a \in \mathfrak{k}$, ad $(a)$ exchanges $p$ and $k$, and it is symmetric with respect to $\langle \cdot, \cdot \rangle$.

Let $X = G/K$ be the symmetric space associated with $(G, K)$, and let $p : G \to X$ be the corresponding projection. Then $\theta^k$ defines a connection form on the $K$-bundle $p : G \to X$. The tangent bundle $TX$ to $X$ is given by

(3.3) $TX = G \times_K p$.

Then $TX$ comes equipped with a metric $g^{TX}$ induced by $B$, and with a Euclidean connection $\nabla^{TX}$, which coincides with the Levi-Civita connection of $TX$. Let $d$ denote the Riemannian distance on $X$.

Set

(3.4) $N = G \times_K \mathfrak{k}$.

The vector bundle $N$ is also equipped with a metric $g^N$ and with a Euclidean connection $\nabla^N$. Let $\nabla^{TX \oplus N}$ be the connection on $TX \oplus N$ that is induced by $\nabla^{TX}, \nabla^N$.

Clearly,

(3.5) $TX \oplus N = G \times_K g$.

The bilinear form $B$ descends to $TX \oplus N$. As explained in [B11, section 2.2], since $G$ also acts on $g$, the map $(g, a) \in G \times g \to \text{Ad}(g) a \in g$ identifies $TX \oplus N$ with the trivial vector bundle $g$ on $X$. Let $\nabla^{TX \oplus N}$ denote the corresponding flat connection on the bundle of Lie algebras $TX \oplus N$. By [B11, eq. (2.2.2)], we get

(3.6) $\nabla^{TX \oplus N} = \nabla^{TX \oplus N} + \text{ad}(\cdot)$.

Let $\nabla^{TX \oplus N}$ be the flat connection on $TX \oplus N$,

(3.7) $\nabla^{TX \oplus N} = \nabla^{TX \oplus N} - \text{ad}(\cdot)$.

The above flat connections preserve $B$.

Let $E$ be a finite dimensional real or complex vector space, let $\rho^E : K \to \text{Aut}(E)$ be a representation of $K$ on $E$, that preserves a scalar or Hermitian product. Let $F$ the vector bundle on $X$,

(3.8) $F = G \times_K E$. 
Then $F$ is a Euclidean or Hermitian vector bundle, that is canonically equipped with a metric preserving connection $\nabla^F$. If $a \in \mathbb{N}$, let $\rho^F (a) \in \text{End} (F)$ correspond to the action of $a$ via the representation $\rho^E$.

Let $R^F$ be the curvature of $\nabla^F$. If $e, f \in TX$, then
\[
R^F (e, f) = -\rho^F ([e, f]).
\]

### 3.2. The action of the Cartan involution.

The Cartan involution $\theta$ acts isometrically on $X$. It preserves the orientation of $X$ if $m$ is even, reverses the orientation if $m$ is odd.

The action of $\theta$ on $X$ lifts to two possible isometric involutions $\theta_{\pm}$ of $F$ that preserve $\nabla^F$. Namely, if $(g, f) \in G \times_k E$, then
\[
\theta_{\pm} (g, f) = (\theta g, \pm f).
\]

Then $\theta_- = -\theta_+$. If $E = p$, then $F = TX$, and $\theta_-$ corresponds to the derivative of the action of $\theta$ on $X$.

More generally, if $(g, h) \in G \times_k g = TX \oplus N$, one can also define a lift $\Theta$ of $\theta$ to $TX \oplus N$ by the formula
\[
(3.11) \quad \Theta (g, h) = (\theta g, \theta h).
\]

### 3.3. The Casimir operator.

In the sequel, we identify $g$ to the vector space of left-invariant vector fields on $G$. The enveloping algebra $U (g)$ will be identified with the algebra of left-invariant differential operators on $G$.

Let $C^g \in U (g)$ be the Casimir element of $G$. If $e_1, \ldots, e_m$ is a basis of $g$ and if $e^*_1, \ldots, e^*_{m+n}$ is the dual basis of $g$ with respect to $B$, then
\[
(3.12) \quad C^g = -\sum_{i=1}^{m+n} e^*_i e_i.
\]

Then $C^g$ lies in the centre of $U (g)$.

If we assume that $e_1, \ldots, e_m$ is an orthonormal basis of $p$ and $e_{m+1}, \ldots, e_{m+n}$ is an orthonormal basis of $\mathfrak{k}$, then
\[
(3.13) \quad e^*_i = e_i \text{ for } 1 \leq i \leq m, \quad -e_i \text{ for } m + 1 \leq i \leq m + n.
\]

In particular,
\[
(3.14) \quad C^g = -\sum_{i=1}^{m} e_i^2 + \sum_{i=m+1}^{m+n} e_i^2.
\]

The Casimir operator $C^\mathfrak{k}$ of $K$ will be calculated with respect to the bilinear form induced by $B$ on $\mathfrak{k}$, i.e.,
\[
(3.15) \quad C^\mathfrak{k} = \sum_{i=m+1}^{m+n} e_i^2.
\]

In the sequel, we use the notation
\[
(3.16) \quad C^{g,H} = -\sum_{i=1}^{m} e_i^2.
\]

By (3.14)–(3.16), we get
\[
(3.17) \quad C^g = C^{g,H} + C^\mathfrak{k}.
\]
Moreover,
\begin{equation}
[C^g,H,C^t] = 0.
\end{equation}

Let \( E,F \) be taken as in (3.8). Let \( C^{t,E} \in \text{End}(E) \) be the associated Casimir operator acting on \( E \),

\begin{equation}
C^{t,E} = \sum_{i=m+1}^{m+n} \rho^{E,2}(e_i).
\end{equation}

Then \( C^{t,E} \) commutes with the \( \rho^E(k), k \in K \). If \( \rho^E \) is irreducible, \( C^{t,E} \) is a constant. We denote by \( C^{t,F} \) the self-adjoint parallel endomorphism of \( F \) that corresponds to \( C^{t,E} \).

In particular \( C^{t,\xi} \in \text{End}(\xi), C^{t,p} \in \text{End}(p) \) are the Casimir operators associated with the actions of \( K \) on \( \xi, p \).

Let \( S^X \) be the scalar curvature of \( X \). By [B11, eq. (2.6.8)], we get
\begin{equation}
S^X = \text{Tr} p [C^{t,p}].
\end{equation}

3.4. **The algebras \( \Lambda^*(g^*) \) and \( c(g), \hat{c}(g) \).** Let \( \Lambda^*(g^*) \) be the exterior algebra of \( g^* \). Let \( N^\Lambda(g^*) \) denote the number operator of \( \Lambda(g^*) \). Let \( B^* \) be the bilinear symmetric form on \( \Lambda(g^*) \) that is induced by \( B \).

Let \( \kappa^\rho \in \Lambda^3(g^*) \) be such that if \( a,b,c \in g \),
\begin{equation}
\kappa^\rho(a,b,c) = B([a,b],c).
\end{equation}

We denote by \( \kappa^t \in \Lambda^3(t^*) \) the corresponding form associated with the restriction of \( B \) to \( t \). By [B11, eq. (2.6.7) and Proposition 2.6.1],
\begin{equation}
B^* (\kappa^\rho, \kappa^t) = \frac{1}{2} \text{Tr} p [C^{t,p}]^t + \frac{1}{6} \text{Tr} t [C^{t,t}]^t, \quad B^* (\kappa^t, \kappa^t) = \frac{1}{6} \text{Tr} t [C^{t,t}]^t.
\end{equation}

We follow [B11, sections 1.1 and 2.3]. Let \( c(g), \hat{c}(g) \) denote the Clifford algebras associated with \( (g,B), (g,-B) \). Then \( c(g), \hat{c}(g) \) are the algebras generated by \( 1 \in \mathbb{R}, e \in g \), and the commutation relations for \( e, f \in g \) given by \( ef + fe = -2B(e,f) \) if \( e \) and \( f \).

Recall that \( \varphi : g \to g^* \) is the canonical identification induced by \( B \). If \( a \in g \), let \( c(a), \hat{c}(a) \in \text{End}(\Lambda^*(g^*)) \) be given by
\begin{equation}
c(a) = \varphi a \wedge -i_a, \quad \hat{c}(a) = \varphi a \wedge +i_a.
\end{equation}

Then \( c(a) \) and \( \hat{c}(a) \) are odd operators, which are respectively antisymmetric and symmetric with respect to \( B^* \). If \( a,b \in g \), then
\begin{equation}
[c(a), c(b)] = -2B(a,b), \quad [\hat{c}(a), \hat{c}(b)] = 2B(a,b), \quad [c(a), \hat{c}(b)] = 0.
\end{equation}

By (3.24), \( \Lambda^*(g^*) \) is a \( c(g) \) and a \( \hat{c}(g) \) Clifford module. As explained in [B11, section 1.1], from the above we get canonical isomorphisms of \( \mathbb{Z}_2 \)-graded vector spaces,
\begin{equation}
c(g) \simeq \Lambda^*(g^*), \quad \hat{c}(g) \simeq \Lambda^*(g^*).
\end{equation}
The action of \( c(g) \) on \( \Lambda^*(g^*) \) corresponds to left multiplication on \( c(g) \), and the action of \( \hat{c}(g^*) \) to right multiplication on \( c(g) \) multiplied by \((-1)^{N^\Lambda(g^*)} \). By [B11, eq. (1.1.15)], we get
\begin{equation}
N^{\Lambda^*(g^*)} = \frac{1}{2} c(e_i^*) \hat{c}(e_i) + \frac{1}{2} (m+n).
\end{equation}
If \( a \in \mathfrak{g} \), we will denote by \( c(a), \tilde{c}(a) \) the corresponding elements in \( c(\mathfrak{g}), \tilde{c}(\mathfrak{g}) \). There will be no risk of confusion with the above definitions of \( c(a), \tilde{c}(a) \).

Let \( \mathcal{A}(\mathfrak{g}) \) be the Lie algebra of endomorphisms of \( \mathfrak{g} \) that are antisymmetric with respect to \( B \). Then \( \mathcal{A}(\mathfrak{g}) \) embeds as a Lie subalgebra of \( c(\mathfrak{g}) \) and \( \tilde{c}(g) \). Namely, let \( e_1, \ldots, e_{m+n} \) be a basis of \( \mathfrak{g} \), let \( e_1^*, \ldots, e_{m+n}^* \) be the corresponding dual basis of \( \mathfrak{g} \) with respect to \( B \). If \( \Lambda \in \mathcal{A}(\mathfrak{g}) \), as in [B11, eqs. (1.1.9) and (1.1.11)], we define \( \Lambda(A) \in \mathfrak{g}, \tilde{c}(A) \in \tilde{c}(\mathfrak{g}) \) by the formulas

\[
(3.32) \quad c(A) = \frac{1}{4} B(Ae_1^*, e_2^*) c(e_1) c(e_2), \quad \tilde{c}(A) = -\frac{1}{4} B(Ae_1^*, e_2^*) \tilde{c}(e_1) \tilde{c}(e_2).
\]

Then if \( e \in \mathfrak{g} \),

\[
(3.27) \quad [c(A), c(e)] = c(Ae), \quad [\tilde{c}(A), \tilde{c}(e)] = \tilde{c}(Ae).
\]

Let \( c(\kappa^\mathfrak{g}) \in c(\mathfrak{g}) \) be given by

\[
(3.28) \quad c(\kappa^\mathfrak{g}) = \frac{1}{6} \kappa^\mathfrak{g} (e_1^*, e_2^*, e_k^*) c(e_1) c(e_2) c(e_k),
\]

Then \( c(\kappa^\mathfrak{g}), \tilde{c}(\kappa^\mathfrak{g}) \) correspond to \( \kappa^\mathfrak{g}, -\kappa^\mathfrak{g} \) by the above canonical isomorphisms.

By (3.1), we get

\[
(3.30) \quad \Lambda^* (\mathfrak{g}^*) = \Lambda^* (\mathfrak{p}^*) \otimes \Lambda^* (\mathfrak{t}^*).
\]

By restricting \( B \) to \( \mathfrak{p}, \mathfrak{t} \), we obtain the Clifford algebras \( c(\mathfrak{p}), \tilde{c}(\mathfrak{p}), c(\mathfrak{t}), \tilde{c}(\mathfrak{t}) \). Since the splitting \( \mathfrak{g} = \mathfrak{p} \oplus \mathfrak{t} \) is orthogonal with respect to \( B \), we get

\[
(3.31) \quad c(\mathfrak{g}) = c(\mathfrak{p}) \hat{\otimes} c(\mathfrak{t}), \quad \tilde{c}(\mathfrak{g}) = \tilde{c}(\mathfrak{p}) \hat{\otimes} \tilde{c}(\mathfrak{t}).
\]

The algebras \( \Lambda^* (\mathfrak{p}^*), \Lambda^* (\mathfrak{t}^*) \) descend to the bundles of algebras \( \Lambda^* (T^*X), \Lambda^* (N^*) \), and \( \Lambda (\mathfrak{g}^*) \) descends to \( \Lambda (T^*X \oplus N^*) \). The vector bundle \( TX \oplus N \) is also equipped with the bilinear form \( B \). Let \( c(TX \oplus N), \tilde{c}(TX \oplus N) \) be the associated bundles of Clifford algebras. Let \( c(TX), c(N), \tilde{c}(TX), \tilde{c}(N) \) denote the bundles of algebras associated with the restriction of \( B \) to \( TX, N \). As in (3.31), we get

\[
(3.32) \quad c(TX \oplus N) = c(TX) \hat{\otimes} c(N), \quad \tilde{c}(TX \oplus N) = \tilde{c}(TX) \hat{\otimes} \tilde{c}(N).
\]

3.5. The symmetric algebra \( S (\mathfrak{g}^*) \). Let \( S (\mathfrak{g}^*) \) denote the symmetric algebra of \( \mathfrak{g}^* \). Equivalently \( S (\mathfrak{g}^*) \) is the polynomial algebra on \( \mathfrak{g} \). Let \( N^S(\mathfrak{g}^*) \) be the number operator of \( S (\mathfrak{g}^*) \). By (3.1), as in (3.30), we get

\[
(3.33) \quad S (\mathfrak{g}^*) = S (\mathfrak{p}^*) \otimes S (\mathfrak{t}^*).
\]

For the moment, we view \( \mathfrak{g} = \mathfrak{p} \oplus \mathfrak{t} \) as a Euclidean vector space. Then \( \langle \rangle \) induces a scalar product on \( S (\mathfrak{g}^*) \). Let \( \mathcal{S} (\mathfrak{g}^*) \) denote the Hilbert completion of \( S (\mathfrak{g}^*) \).

Let \( \Delta^\mathfrak{g} \) be the Laplacian on the Euclidean vector space \( \mathfrak{g} \). Let \( H^\mathfrak{g} \) be the harmonic oscillator on \( \mathfrak{g} \), so that if \( Y \) is the generic element of \( \mathfrak{g} \),

\[
(3.34) \quad H^\mathfrak{g} = \frac{1}{2} ( -\Delta^\mathfrak{g} + |Y|^2 - m - n ).
\]

Let \( L_2(\mathfrak{g}) \) denote the Hilbert space of square integrable functions on \( \mathfrak{g} \). As explained in [B11, section 1.4], we have the classical Bargmann isomorphism \( \mathcal{S} (\mathfrak{g}^*) \approx L_2(\mathfrak{g}) \). Under this isomorphism, \( N^S(\mathfrak{g}^*) \) corresponds to the harmonic oscillator \( H^\mathfrak{g} \).
Then \( S^* (p^*) , S^* (t^*) \) descends to \( S^* (T^* X) , S^* (N^*) \). Also \( H^q \) descend to the harmonic oscillator \( H^{TX \oplus N} \) along the fibres of \( TX \oplus N \). The Bargmann isomorphism identifies \( \mathcal{S} (T^* X \oplus N^*) \) with \( L_2 (TX \oplus N) \), and it maps \( N^S (T^* X \oplus N^*) \) to \( H^{TX \oplus N} \).

### 3.6. The spinors of \( \overline{T X} \)

We fix once and for all an orientation of \( p \), which in turn defines an orientation of \( X \).

Let \( \mathfrak{p} \) denote another copy of \( p \), which we equip with the orientation corresponding to the orientation of \( p \). We equip \( \mathfrak{p} \) with the scalar product corresponding to the scalar product of \( p \). If \( e \in p \), let \( \mathfrak{e} \) denote the corresponding element of \( \mathfrak{p} \).

Let \( c (\mathfrak{p}) \) be the Clifford algebra of \( \mathfrak{p} \). If \( e \in p \), \( c (\mathfrak{e}) \) acts as a skew-adjoint operator on \( \mathfrak{p} \).

Everything we did for \( c (\mathfrak{p}) \) remains valid for \( \hat{c} (\mathfrak{p}) \). We make the convention that if \( e \in p \), \( \hat{c} (\mathfrak{e}) \) acts on \( \mathfrak{p} \) like \( ic (\mathfrak{e}) \). In particular \( \hat{c} (\mathfrak{e}) \) acts on \( \mathfrak{p} \) as a self-adjoint operator. Using the same conventions as in (3.27), if \( A \in \text{End} (\mathfrak{p}) \) is antisymmetric,

\[
(3.35) \quad c (A) = \hat{c} (A)
\]

acts on \( \mathfrak{p} \). In this specific case, using the second notation instead of the first one will be a matter of convenience.

If \( m \) is even, \( \mathfrak{p} \) splits as \( \mathfrak{p} = \mathfrak{p} _+ \oplus \mathfrak{p} _- \), \( \mathfrak{p} _\pm \) being of dimension \( 2^{m/2-1} \). Moreover, we have an identification of \( \mathbb{Z}_2 \)-graded algebras,

\[
(3.36) \quad c (\mathfrak{p}) \otimes R \mathcal{C} \simeq \text{End} (\mathfrak{p} _+ \oplus \mathfrak{p} _-).
\]

If \( m \) is odd, then \( \mathfrak{p} \) has dimension \( 2^{(m-1)/2} \). Moreover, we have the identification of algebras

\[
(3.37) \quad c (\mathfrak{p}) \otimes R \mathcal{C} \simeq \text{End} (\mathfrak{p} _+ \oplus \mathfrak{p} _-).
\]

If we give its canonical orientation to \( \mathfrak{p} \), then

\[
(3.38) \quad \mathfrak{p} = \mathfrak{p} _+ \oplus R.
\]

The Lie group \( \text{Spin} (\mathfrak{p}) \) embeds in \( \text{even} (\mathfrak{p}) \), and acts unitarily on \( \mathfrak{p} \). In the sequel, we will assume that \( G \) is simply connected. Equivalently, we suppose that \( K \) is simply connected. Then the representation \( k \in K \rightarrow \text{Ad} (k) \in \text{SO} (\mathfrak{p}) \) lifts to a group homomorphism \( k \in K \rightarrow \sigma (k) \in \text{Spin} (\mathfrak{p}) \). If \( f \in t \), then

\[
(3.39) \quad \sigma (f) = \hat{c} (\text{ad} (f) |_{\mathfrak{p}}).
\]

By (3.39), we get

\[
(3.40) \quad C^t, \mathfrak{p} = \sum_{i=m+1}^{m+n} \hat{c} (\text{ad} (e_i) |_{\mathfrak{p}})^2.
\]

By [B11, eq. (7.8.6)], we have the identity

\[
(3.41) \quad C^t, \mathfrak{p} = \frac{1}{8} \text{Tr}_p [C^t, \mathfrak{p}].
\]

Ultimately, \( k \in K \rightarrow \sigma (k) \in \text{Spin} (\mathfrak{p}) \) defines a unitary representation of \( K \) into \( U (\mathfrak{p} \mathfrak{p}) \). If \( m \) is even, this representation preserves \( \mathfrak{p} \mathfrak{p} \).
Set
\begin{equation}
S^{TX} = G \times_K S^\mathcal{F}.
\end{equation}

Then $S^{TX}$ is a Hermitian vector bundle with connection on $X$. This bundle is just the bundle of spinors of $TX$ equipped with the connection $\nabla S^{TX}$ induced by $\nabla^{TX}$.

Let $\nabla S^{TX} \otimes F$ be the connection on $S^{TX} \otimes F$ that is induced by $\nabla S^{TX}$, $\nabla^F$.

As we saw in subsection 3.2, $\theta_{\pm}$ acts on $S^{TX} \otimes F$. The induced action of $\theta_{\pm}$ on $C^\infty (X, S^{TX} \otimes F)$ is given by
\begin{equation}
(3.43) \quad \theta_{\pm} s (x) = \theta_{\pm} s (\theta^{-1} x).
\end{equation}

3.7. The elliptic Dirac operator $\hat{D}^X$. In the sequel, $e_1, \ldots, e_m$ is an orthonormal basis of $p$, and $e_{m+1}, \ldots, e_{m+n}$ is an orthonormal basis of $\mathfrak{k}$. We will use the same notation for corresponding orthonormal bases of $TX$ and $N$.

**Definition 3.1.** Let $D \in c (\mathcal{F}) \otimes U (g)$, $\hat{D} \in \hat{c} (\mathcal{F}) \otimes U (\mathfrak{g})$ be given by
\begin{equation}
(3.44) \quad D = \sum_{i=1}^m c (\tau_i) e_i, \quad \hat{D} = \sum_{i=1}^m \hat{c} (\tau_i) e_i.
\end{equation}

Then $D, \hat{D}$ are $K$-invariant, so that $D, \hat{D}$ descend to Dirac operators $D^X, \hat{D}^X$ acting on $C^\infty (X, S^{TX} \otimes F)$. These operators are given by
\begin{equation}
(3.45) \quad D^X = \sum_{i=1}^m c (\tau_i) \nabla_{e_i}^{S^{TX} \otimes F}, \quad \hat{D}^X = \sum_{i=1}^m \hat{c} (\tau_i) \nabla_{e_i}^{S^{TX} \otimes F}.
\end{equation}

Then $D^X$ is formally self-adjoint, and $\hat{D}^X$ is formally skew-adjoint. Because of the conventions we made before, we have the identities,
\begin{equation}
(3.46) \quad \hat{D} = iD, \quad \hat{D}^X = iD^X.
\end{equation}

Also $D^X$ is a classical Dirac operator.

**Proposition 3.2.** The following identities hold:
\begin{equation}
(3.47) \quad \theta_{\pm} D^X \theta_{\pm}^{-1} = -D^X, \quad \theta_{\pm} \hat{D}^X \theta_{\pm}^{-1} = -\hat{D}^X.
\end{equation}

Proof. If $e \in TX$, using the considerations after (3.10), we have the identities,
\begin{equation}
(3.48) \quad \theta_{\pm} c (\tau) \theta_{\pm}^{-1} = c (\tau), \quad \theta_{\pm} \nabla_{e}^{S^{TX} \otimes F} \theta_{\pm}^{-1} = \nabla_{e}^{S^{TX} \otimes F}.
\end{equation}

By (3.45), (3.48), we get (3.47). \hfill \square

Let $C^g.X, C^g.H.X$ denote the action of $C^g, C^g.H$ on $C^\infty (X, S^{TX} \otimes F)$. By [B11, eqs. (2.12.17) and (7.2.6)] or by (3.17), $C^g.X$ splits as
\begin{equation}
(3.49) \quad C^g.X = C^g.H.X + C^k.S^{TX} \otimes F.
\end{equation}

If $\Delta^{X,H}$ denotes the Bochner Laplacian, then
\begin{equation}
(3.50) \quad C^g.H.X = -\Delta^{X,H}.
\end{equation}
Let $\mathcal{L}_0^X$ 
\footnote{In [B11], the operator $\mathcal{L}_0^X$ was instead denoted $\mathcal{L}^X$.} denote the operator that was defined in [B11, eq. (7.2.8)], i.e.,
\begin{equation}
(3.51) \quad \mathcal{L}_0^X = \frac{1}{2} C^\theta \cdot X + \frac{1}{8} B^* (\kappa^\theta, \kappa^\theta).
\end{equation}
Recall that by (3.35), if $f \in \mathfrak{t}$, $\hat{c} (\text{ad} (f) |_\mathfrak{p})$ acts on $\mathcal{S}^\mathfrak{p}$, and if $f \in \mathcal{N}$, $c (\text{ad} (f) |_{\mathcal{T}X})$ acts on $\mathcal{S}^{\mathcal{T}X}$.

**Proposition 3.3.** The following identity holds:
\begin{equation}
(3.52) \quad \mathcal{L}_0^X = \frac{1}{2} \left( -\Delta^{X,H} + \frac{1}{4} \text{Tr}^p [C^{t,p}] + 2 \sum_{i=m+1}^{m+n} \hat{c} (\text{ad} (e_i) |_{\mathcal{T}X}) \rho^F (e_i) \right) + \frac{1}{48} \text{Tr}^t [C^{t,t}] + \frac{1}{2} C^{t,F}.
\end{equation}

**Proof.** By (3.22), (3.49)–(3.51), we get
\begin{equation}
(3.53) \quad \mathcal{L}_0^X = \frac{1}{2} \left( -\Delta^{X,H} + C^{t,S^{\mathcal{T}X} \otimes F} + \frac{1}{8} \text{Tr}^p [C^{t,p}] \right) + \frac{1}{48} \text{Tr}^t [C^{t,t}].
\end{equation}
Moreover, using (3.41), we obtain
\begin{equation}
(3.54) \quad C^{t,S^{\mathcal{T}X} \otimes F} = \frac{1}{8} \text{Tr}^p [C^{t,p}] + C^{t,F} + 2 \sum_{i=m+1}^{m+n} \hat{c} (\text{ad} (e_i) |_{\mathcal{T}X}) \rho^F (e_i).
\end{equation}
By (3.53), (3.54), we get (3.52). \hfill \Box

By [B11, Theorem 7.2.1], we get
\begin{equation}
(3.55) \quad \frac{1}{2} \hat{D}^{X,2} = -\mathcal{L}_0^X + \frac{1}{8} B^* (\kappa^\mathfrak{t}, \kappa^\mathfrak{t}) + \frac{1}{2} C^{t,F}.
\end{equation}
Also Lichnerowicz’s formula asserts that
\begin{equation}
(3.56) \quad \hat{D}^{X,2} = \Delta^{X,H} - \frac{S^X}{4} + \frac{1}{2} \sum_{1 \leq i,j \leq m} \hat{c} (\tau_i) \hat{c} (\tau_j) R^F (e_i, e_j).
\end{equation}
As was explained in [B11, section 7.2], equations (3.55) and (3.56) are equivalent.

By (3.9), (3.20), (3.27), and (3.56), we get
\begin{equation}
(3.57) \quad \frac{1}{2} \hat{D}^{X,2} = \frac{1}{2} \Delta^{X,H} - \frac{1}{8} \text{Tr}^p [C^{t,p}] - \sum_{i=m+1}^{m+n} \hat{c} (\text{ad} (e_i) |_{\mathcal{T}X}) \rho^F (e_i).
\end{equation}

3.8. **The operator $\mathcal{D}_b$.** Set
\begin{equation}
(3.58) \quad \mathcal{A}^\theta = c (\mathfrak{g}) \otimes U (\mathfrak{g}), \quad \hat{\mathcal{A}}^\theta = \hat{c} (\mathfrak{g}) \otimes U (\mathfrak{g}).
\end{equation}
Then $\mathcal{A}^\theta, \hat{\mathcal{A}}^\theta$ are $\mathbb{Z}_2$-graded algebras. Moreover, $G$ acts on $\mathcal{A}^\theta, \hat{\mathcal{A}}^\theta$. Let $e_1, \ldots, e_{m+n}$ be a basis of $\mathfrak{g}$, and let $e_1^*, \ldots, e_{m+n}^*$ denote the dual basis of $\mathfrak{g}$ with respect to $B$.

Now, we introduce the Dirac operators of Kostant [K76, K97].

**Definition 3.4.** Let $D^\theta \in \mathcal{A}^\theta, \hat{D}^\theta \in \hat{\mathcal{A}}^\theta$ be the Dirac operators
\begin{equation}
(3.59) \quad D^\theta = \sum_{i=1}^{m+n} c (e_i^*) e_i + \frac{1}{2} c (\kappa^\theta), \quad \hat{D}^\theta = \sum_{i=1}^{m+n} \hat{c} (e_i^*) e_i + \frac{1}{2} \hat{c} (-\kappa^\theta).
\end{equation}
Now we recall a result of Kostant [K76, K97], [B11, Theorem 2.7.2].
Theorem 3.5. The following identities hold:

\[ D^g = C^g + \frac{1}{4} B^g (\kappa^g, \kappa^g), \quad \widehat{D}^g = -C^g - \frac{1}{4} B^g (\kappa^g, \kappa^g). \]

By (3.51), (3.60), \( \mathcal{L}_{\nu}^X \) is just the action of \( \frac{1}{2} D^g \) on \( C^\infty (X, S^TX \otimes F) \).

Let \( d^p, d^f \) be the de Rham operators on \( p, f \), and let \( d^{p*}, d^{f*} \) denote their formal \( L_2 \) adjoints with respect to the scalar products on \( p, f \). Let \( Y^p, Y^f \) be the tautological sections of \( p, f \) on \( p, f \). We identify \( Y^p, Y^f \) to the corresponding 1-forms via the scalar products of \( p, f \).

If \( f \in g \), let \( \nabla_e \) denote the corresponding differentiation on \( g \). Let \( e_1, \ldots, e_n \) be an orthonormal basis of \( p \), let \( e_{m+1}, \ldots, e_{m+n} \) be an orthonormal basis of \( f \). We use the notation in (3.23). Set

\[ \mathcal{D}^p = \sum_{i=1}^{m} c(e_i) \nabla_{e_i}, \quad \mathcal{E}^p = \widehat{c}(Y^p), \]
\[ \mathcal{D}^f = \sum_{i=m+1}^{m+n} c(e^*_i) \nabla_{e^*_i}, \quad \mathcal{E}^f = \widehat{c}(Y^f). \]

A trivial computation [B11, eqs. (2.8.6), (2.8.11)] shows that

\[ \mathcal{D}^p + \mathcal{E}^p = d^p + Y^p \wedge + d^{p*} + i_{Y^p}, \quad \mathcal{D}^f - \mathcal{E}^f = d^f + Y^f \wedge - d^{f*} - i_{Y^f}. \]

Let \( \Delta^p, \Delta^f \) denote the Laplacians on \( p, f \). By [B11, eqs. (2.8.8), (2.8.13)],

\[ \frac{1}{2} (\mathcal{D}^p + \mathcal{E}^p)^2 = \frac{1}{2} \left( -\Delta^p + |Y^p|^2 - m \right) + N^A(v^p), \]
\[ \frac{1}{2} (-i\mathcal{D}^f + i\mathcal{E}^f)^2 = \frac{1}{2} \left( -\Delta^f + |Y^f|^2 - n \right) + N^A(v^f). \]

If \( k \in K \), the action of \( k \) on \( C^\infty (G, \Lambda^*(g^*) \otimes S(g^*)) \) is given by

\[ k_s (g) = \rho^A(v^p \otimes S(v^f))(k) s (gk). \]

Also we have the identification

\[ C^\infty (G, \Lambda^*(g^*) \otimes C^\infty(g)) = C^\infty (G \times g, \Lambda^*(g^*)). \]

The action of \( K \) on (3.65) that corresponds to (3.64) is given by

\[ k_s (g, Y) = \rho^A(v^p) (k) s (gk, \text{Ad} (k^{-1}) Y). \]

Now we define the operator \( \mathcal{D}_b \) as in [B11, Definition 2.9.1].

Definition 3.6. For \( b > 0 \), let \( \mathcal{D}_b \in \text{End} (C^\infty (G \times g, \Lambda^*(g^*)) \) be given by

\[ \mathcal{D}_b = \widehat{D}^g + ic \left( [Y^p, Y^f] \right) + \frac{1}{b} (\mathcal{D}^p + \mathcal{E}^p - i\mathcal{D}^f + i\mathcal{E}^f). \]

Then \( \mathcal{D}_b \) commutes with \( K \).

3.9. The compression of the operator \( \mathcal{D}_b \). Let \( Y \) be the tautological section of \( g \) on \( g \), so that \( Y = Y^p + Y^f \in g \). We have the identity

\[ |Y|^2 = |Y^p|^2 + |Y^f|^2. \]

By [B11, section 1.6], the kernel \( H \subset \Lambda^*(g^*) \otimes L_2(g) \) of the operator \( \mathcal{D}^p + \mathcal{E}^p - i\mathcal{D}^f + i\mathcal{E}^f \) is 1-dimensional and spanned by \( \exp \left( -|Y|^2 / 2 \right) \). Let \( P \) denote the orthogonal projection operator on \( H \). Of course \( P \) acts on \( C^\infty (G, \Lambda^*(g^*) \otimes L_2(g)) \).
We identify \( C^\infty (G, \mathbf{R}) \) with a vector subspace of \( C^\infty (G \times \mathfrak{g}, \Lambda^* (\mathfrak{g}^*)) \) via the embedding \( s \mapsto s \exp \left(-|Y|^2/2\right)/\pi^{(m+n)/4} \).

Then we have the result in [B11, Proposition 2.10.1].

**Proposition 3.7.** The following identity holds:

\[
(3.69) \quad P \left( \tilde{D}^g + ic \left[ [Y^t, Y^p] \right] \right) P = 0.
\]

**3.10. A formula for \( \mathfrak{D}_g^2 \).** Now, we denote by \( \Delta^{g^\otimes \mathfrak{t}} \) the standard Euclidean Laplacian on the Euclidean vector space \( \mathfrak{g} = \mathfrak{p} \oplus \mathfrak{t} \).

We make the same assumptions on the basis \( e_1, \ldots, e_{m+n} \) of \( \mathfrak{g} \) as in (3.13). If \( V \in \mathfrak{t} \), \( \text{ad} \left( V \right) \mid _p \) acts as an antisymmetric endomorphism of \( \mathfrak{p} \), so that by (3.27),

\[
(3.70) \quad c \left( \text{ad} \left( V \right) \mid _p \right) = \frac{1}{4} \sum_{1 \leq i, j \leq m} \left[ [V, e_i], e_j \right] c \left( e_i \right) c \left( e_j \right).
\]

Also, if \( W \in \mathfrak{p} \), \( \text{ad} \left( W \right) \) exchanges \( \mathfrak{t} \) and \( \mathfrak{p} \) and is antisymmetric with respect to \( B \), i.e., it is symmetric with respect to the scalar product on \( \mathfrak{g} \). Moreover, by (3.27),

\[
(3.71) \quad c \left( \text{ad} \left( W \right) \right) = \frac{1}{2} \sum_{m+1 \leq i \leq m+n} \left[ [W, e_i^*], e_j \right] c \left( e_i \right) c \left( e_j \right).
\]

If \( a \in \mathfrak{g} \), let \( \nabla_a^V \) be the corresponding differentiation operator along \( \mathfrak{g} \). In particular, \( \nabla_{[Y^t, Y^p]}^V \) denotes the differentiation operator in the direction \( [Y^t, Y^p] \in \mathfrak{p} \). If \( Y \in \mathfrak{g} \), we denote by \( Y^p + iY^t \) the section of \( U (\mathfrak{g}) \otimes_{\mathbf{R}} \mathbf{C} \) associated to \( Y^p + iY^t \in \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C} \). Recall that \( N^\Lambda (\sigma^*) \) is the number operator of \( \Lambda^* (\mathfrak{g}^*) \).

The following identity was established in [B11, Theorem 2.11.1].

**Theorem 3.8.** The following identity holds:

\[
(3.72) \quad \frac{\mathfrak{D}_g^2}{2} = \frac{\tilde{D}^g - 2}{2} + \frac{1}{2} \left[ [Y^t, Y^p] \right]^2 + \frac{1}{2b^2} \left( -\Delta^{g^\otimes \mathfrak{t}} + |Y|^2 - m - n \right) + \frac{N^\Lambda (\sigma^*)}{b^2} + \frac{1}{b} \left( Y^p + iY^t - i\nabla_{[Y^t, Y^p]}^V + c \left( \text{ad} \left( Y^p + iY^t \right) \right) \right.
\]

\[
\left. + 2ic \left( \text{ad} \left( Y^t \right) \mid _p \right) - c \left( \text{ad} \left( Y^p \right) \right) \right).\]

**3.11. The operator \( \mathcal{L}_X^\Lambda \).** To make our notation simpler, from now on, and in the whole paper, if \( V \) is a real vector space and if \( W \) is a complex vector space, we will use the notation \( V \otimes W \) to denote the tensor product \( V \otimes_{\mathbf{R}} W \).

Observe that \( K \) acts on

\[
\left( C^\infty \left( G, \Lambda^* (\mathfrak{g}^*) \otimes S^* (\mathfrak{g}^*) \otimes S^\mathfrak{p} \otimes E \right) \right)
\]

by a formula similar to (3.64).

Let \( \hat{\mathfrak{X}} \) be the total space of \( TX \otimes N \) over \( X \), and let \( \hat{\pi} : \hat{\mathfrak{X}} \to X \) be the corresponding projection. By [B11, eq. (2.12.7)], since \( TX \otimes N \) can be identified with the trivial vector bundle \( \hat{\mathfrak{g}} \), then

\[
(3.73) \quad \hat{\mathfrak{X}} = X \times \hat{\mathfrak{g}},
\]
and \( \hat{\pi} : \hat{X} \to X \) is the projection \( X \times \mathfrak{g} \to X \). Let \( Y = Y^{TX} + Y^N, Y^{TX} \in TX, Y^N \in N \) be the canonical section of \( \hat{\pi}^* (TX \oplus N) \) over \( \hat{X} \). Then

\[
(3.74) \quad |Y|^2 = |Y^{TX}|^2 + |Y^N|^2.
\]

**Definition 3.9.** Let \( \nabla^{C^\infty} (TX \oplus N, \hat{\pi}^* (\Lambda (T^* X \oplus N^*) \otimes S^{TX} \otimes F)) \) be the connection on the vector bundle \( C^\infty (TX \oplus N, \hat{\pi}^* (\Lambda (T^* X \oplus N^*) \otimes S^{TX} \otimes F)) \) on \( X \) that is induced by the connection \( \theta^\pi \) on the \( K \)-bundle \( p : G \to X = G/K \).

Let \( \mathcal{H} \) be the vector space of smooth sections over \( X \) of the vector bundle \( C^\infty (TX \oplus N, \hat{\pi}^* (\Lambda (T^* X \oplus N^*) \otimes S^{TX} \otimes F)) \). Then

\[
(3.75) \quad \mathcal{H} = C^\infty (\hat{X}, \hat{\pi}^* (\Lambda (T^* X \oplus N^*) \otimes S^{TX} \otimes F)).
\]

Since \( \hat{D}^\theta, \mathcal{D}_b \) commute with the action of \( K \), they descend to operators \( \hat{D}^\theta X, \mathcal{D}^X_b \) acting on \( \mathcal{H} \). The operators \( \mathcal{D}^\theta, \mathcal{E}^\pi, \mathcal{D}^X, \mathcal{E}^X \) descend to operators \( \mathcal{D}^{TX}, \mathcal{E}^{TX}, \mathcal{D}^N, \mathcal{E}^N \) acting along the fibres of \( TX \oplus N \). Also the operator \( C^\theta \) descends to an operator \( C^\theta X \). By (3.60), we get

\[
(3.76) \quad \hat{D}^{\theta,X,2} = -C^{\theta,X} - \frac{1}{4} B^* (\kappa^\theta, \kappa^\theta).
\]

Put

\[
(3.77) \quad \mathcal{L}^X_b = -\frac{1}{2} \hat{D}^{\theta,X,2} + \frac{1}{2} \mathcal{D}^{X,2}.
\]

Let \( \Delta^{TX \oplus N} \) be the Laplacian acting along the fibres of the Euclidean vector bundle \( TX \oplus N \).

We will give an important formula that was established in [B11, Theorem 2.12.5 and eq. (2.13.5)]. Note that \( \hat{c} (\text{ad} (Y^N) |_{TX}) \) is a section of the Clifford algebra \( c(TX) \) and acts on \( S^{TX} \), while \( c (\text{ad} (Y^{TX})) \), \( \hat{c} (\text{ad} (Y^{TX})) \) lie in \( c(TX \oplus N), \hat{c}(TX \oplus N), \) and act on \( \Lambda (T^* X \oplus N^*) \).

**Theorem 3.10.** The following identities hold:

\[
\mathcal{D}^X_b = \hat{D}^{\theta,X} + i c ([Y^N, Y^{TX}]) + \frac{1}{b} (\mathcal{D}^{TX} + \mathcal{E}^{TX} - i \mathcal{D}^N + i \mathcal{E}^N),
\]

\[
(3.78) \quad \mathcal{L}^X_b = \frac{1}{2} \left[ |Y^N, Y^{TX}|^2 + \frac{1}{2b^2} \left( -\Delta^{TX \oplus N} + |Y|^2 - m - n \right) + \frac{N^2 (T^* X \oplus N^*)}{b^2} \right]
+ \frac{1}{b} \left( \nabla^{C^\infty} (TX \oplus N, \hat{\pi}^* (\Lambda (T^* X \oplus N^*) \otimes S^{TX} \otimes F)) + \hat{c} (\text{ad} (Y^{TX})) \right)
- c (\text{ad} (Y^{TX}) + i \theta \text{ad} (Y^N)) - i \hat{c} (\text{ad} (Y^N) |_{TX}) - i \rho^F (Y^N).
\]

In [B11, Theorem 2.13.2], it was shown that \( \frac{d}{dt} + \mathcal{L}^X_b \) is hypoelliptic, as a consequence of a result of Hörmander [Hö]. According to the terminology of [B11], \( \mathcal{L}^X_b \) is called a hypoelliptic Laplacian.
3.12. A formula relating $\mathcal{L}_b^X$ to $\mathcal{L}_0^X$. By (3.78), $\mathcal{L}_b^X$ can be written in the form

$$\mathcal{L}_b^X = \frac{\alpha}{b^2} + \frac{\beta}{b} + \gamma. \tag{3.79}$$

We still denote by $H$ the kernel of $\alpha$. Then $H$ is the module over $C^\infty(X, \mathbb{C})$ given by

$$H = \{\exp\left(-|Y|^2/2\right)\} \otimes S^{TX} \otimes F. \tag{3.80}$$

Let $H^\perp$ be the orthogonal space to $H$ in $L^2(\hat{X}, \hat{\pi}^* \left(\Lambda^{\cdot} (T^*X \oplus N^*) \otimes S^{TX} \otimes F\right))$.

We still denote by $P$ the orthogonal projection on $H$. Let $P^\perp$ be the orthogonal projection from $L^2(\hat{X}, \hat{\pi}^* \left(\Lambda^{\cdot} (T^*X \oplus N^*) \otimes S^{TX} \otimes F\right))$ on $H^\perp$. We embed $L^2(X, S^{TX} \otimes F)$ into $L^2(\hat{X}, \hat{\pi}^* \left(\Lambda^{\cdot} (T^*X \oplus N^*) \otimes S^{TX} \otimes F\right))$ via the isometric embedding $s \to \hat{\pi}^* s \exp\left(-|Y|^2/2\right)/\pi^{(m+n)/4}.$

Note that $\beta$ maps $H$ into $H^\perp$. Let $\alpha^{-1}$ be the inverse of $\alpha$ restricted to $H^\perp$.

The following result was established in [B11, Theorem 2.16.1].

**Theorem 3.11.** The following identity holds:

$$P \left(\gamma - \beta \alpha^{-1} \beta\right) P = \mathcal{L}_0^X. \tag{3.81}$$

**Remark 3.12.** In [B11], two proofs were given of (3.81). A first proof relies on Proposition 3.7. A second proof is based on explicit computations. In subsections 4.8 and 4.11, both arguments will be extended to the deformation $\mathcal{L}_{b,\vartheta}^X$ of $\mathcal{L}_b^X$.

In the above constructions, only the even part of $c(\tilde{p})$ or of $c(T\tilde{X})$ is involved. If $m$ is even, our operators preserve the splitting $S^{TX} = S_+^{TX} \oplus S_-^{TX}$. There is no way that by the above method, odd elements in $c(\tilde{p})$ or $c(T\tilde{X})$ would appear. Incidentally, equations (3.67), (3.69), and (3.78) suggest that $\mathcal{D}_b^X|_{b>0}$ is a deformation of the operator 0.
4. The hypoelliptic operators $\mathcal{L}_{b,\vartheta}^X$

In this section, we introduce a family of hypoelliptic operators $\mathcal{L}_{b,\vartheta}^X |_{(b,\vartheta)\in \mathbb{R}^*_+ \times [0,\frac{\pi}{2}]}$ acting on $C^\infty \left(\tilde{\mathcal{X}}, \pi^*(\Lambda^' (T^* X \oplus N^*)) \otimes \mathcal{S}^{TX} \otimes \mathcal{F}\right)$, that coincides with the family $\mathcal{L}_b^X |_{b>0}$ for $\vartheta = 0$. The construction of this new family is done through a new family of operators $\mathcal{D}_{b,\vartheta}^X |_{(b,\vartheta)\in \mathbb{R}^*_+ \times [0,\frac{\pi}{2}]}$, that coincides with the family $\mathcal{D}_b^X |_{b>0}$ for $\vartheta = 0$. While in Remark 3.12, one could argue that the family $\mathcal{D}_b^X |_{b>0}$ deforms the operator $0$ as $b \to 0$, here, given $\vartheta \in [0,\frac{\pi}{2}[$, $\mathcal{D}_{b,\vartheta}^X |_{b>0}$ deforms the classical Dirac operator $\sin (\vartheta) \hat{D}^X$ as $b \to 0$. It is in this way that the Dirac operator $\hat{D}^X$ enters the picture, and not only through its square as in section 3.

In this section, superconnections associated with the family $\mathcal{D}_{b,\vartheta}^X |_{(b,\vartheta)\in \mathbb{R}^*_+ \times [0,\frac{\pi}{2}]}$ do appear. Also we introduce the conjugate families $\mathcal{D}_{b,\vartheta}'^X, \mathcal{D}_{b,\vartheta}'^X$, the first family being more suitable when studying the limit $b \to 0$, and the second will be shown later to be more convenient when considering the limit $b \to +\infty$.

The organization of the section is closely related to the organization of section 3. In subsection 4.1, we introduce the family of operators $\hat{D}^X = \sin (\vartheta) \hat{D}^X, \vartheta \in [0,\frac{\pi}{2}[$, we construct a corresponding superconnection $A^X$, and a version $T^X$ of its curvature.

In subsection 4.2, if $\mathfrak{p}$ is another copy of $\mathfrak{p}$, we make $\text{SO} (2)$ act on $\mathfrak{p} \oplus \mathfrak{p}$.

In subsection 4.3, using the above action of $\text{SO} (2)$, we construct a deformation $\mathcal{D}_{b,\vartheta}$ of $\mathcal{D}_b$, and also the conjugate operators $\mathcal{D}_{b,\vartheta}', \mathcal{D}_{b,\vartheta}'$.

In subsection 4.4, we give a formula for a compression of $\mathcal{D}_{b,\vartheta}$.

In subsection 4.5, we give formulas for $\mathcal{D}_{b,\vartheta}^2, \mathcal{D}_{b,\vartheta}'^2$.

In subsection 4.6, we introduce superconnections $B, \mathcal{T}, B'$ on $\mathbb{R}^*_+ \times [0,\frac{\pi}{2}]$ that are associated with the above families, and we prove a compression identity.

In subsection 4.7, we descend the above operators to $X$. We obtain this way new operators $\mathcal{D}_{b,\vartheta}^X, \mathcal{D}_{b,\vartheta}'^X, \mathcal{D}_{b,\vartheta}'^X$ and new hypoelliptic Laplacians $\mathcal{L}_{b,\vartheta}^X, \mathcal{L}_{b,\vartheta}'^X, \mathcal{L}_{b,\vartheta}'^X$ that act on the same space as $\mathcal{D}_{b,\vartheta}^X, \mathcal{D}_{b,\vartheta}'^X, \mathcal{D}_{b,\vartheta}'^X$ and coincide with $\mathcal{D}_{b,\vartheta}^X, \mathcal{D}_{b,\vartheta}'^X, \mathcal{D}_{b,\vartheta}'^X$ for $\vartheta = 0$.

In subsection 4.8, we give a formula relating $\mathcal{T}_{b,\vartheta}^X$ to $\mathcal{L}_{b,\vartheta}^X$, that ultimately explains why $\mathcal{L}_{b,\vartheta}^X$ deforms $\mathcal{L}_{0,\vartheta}^X$. As in [B11, Theorem 2.16.1], the proof uses the compression identity of subsection 4.4.

In subsection 4.9, we descend the superconnections $B, \mathcal{T}, B'$ to superconnections $B^X, \mathcal{T}^X, B'^X$ over $\mathbb{R}^*_+ \times [0,\frac{\pi}{2}]$, and we introduce their proper curvatures $L^X, \mathcal{L}^X, L'^X$. We establish a corresponding compression identity, and we give a formula relating $\mathcal{L}^X$ to $T^X$, that extend in higher degree what we did in subsection 4.8.

In subsection 4.10, we establish a quartic compression identity on linear maps.

In subsection 4.11, as in [B11, section 2.16], using the compression identity of subsection 4.10, we give a direct computational proof of the formulas of subsection 4.8 and 4.9. One reason for giving this direct proof is that as in [B11], in section 11, and more specifically in subsection 11.11, we will use some of the intermediate identities established in this direct proof to study the behaviour of our hypoelliptic orbital integrals as $b \to 0$. 


Finally, in subsection 4.12, when the bilinear form $B$ on $\mathfrak{g}$ is replaced by $B/t$, we give a formula expressing the above operators associated with $B/t$ in terms of the original operators.

In this section, we make the same assumptions as in section 3, and we use the corresponding notation. In particular the connected reductive Lie group $G$ is assumed to be simply connected.

4.1. A deformation of $\hat{D}^X$. Here, as in [BF86, section 2], we will adopt the formalism of subsection 2.1 while omitting the Clifford variable $\sigma$ for simplicity. We will instead take into account the grading in the Clifford algebras $\hat{c}(\mathfrak{p})$. In the sequel, we form the $\mathbb{Z}_2$-graded tensor product $\Lambda^1(\mathfrak{r}^*) \hat{\otimes} \hat{c}(\mathfrak{p}) \otimes U(\mathfrak{g})$. In particular $c^{\text{odd}}(\mathfrak{p})$ anticommutes with $d\vartheta \in \Lambda^1(\mathfrak{r}^*)$.

Recall that the operator $\hat{D}$ was defined in (3.44). For $\vartheta \in [0, \frac{\pi}{2}]$, set

\begin{equation}
\hat{D}_\vartheta = \sin (\vartheta) \hat{D}.
\end{equation}

Then $\hat{D}_\vartheta \in \hat{c}^{\text{odd}}(\mathfrak{p}) \otimes U(\mathfrak{g})$.

Let $A$ be the superconnection over $[0, \frac{\pi}{2}]$, set

\begin{equation}
A = d\vartheta \frac{\partial}{\partial \vartheta} + \frac{\hat{D}_\vartheta}{\sqrt{2}}.
\end{equation}

The curvature $A^2$ of $A$ is given by

\begin{equation}
A^2 = \frac{1}{2} \sin^2 (\vartheta) \hat{D}^2 + \frac{d\vartheta}{\sqrt{2}} \cos (\vartheta) \hat{D}.
\end{equation}

Also $\hat{D}_\vartheta$ descends to an operator $\hat{D}_\vartheta^X$ acting on $C^\infty \left( \mathcal{X}, S^TX \otimes F \right)$ given by

\begin{equation}
\hat{D}_\vartheta^X = \sin (\vartheta) \hat{D}^X.
\end{equation}

The superconnection $A$ descends to a superconnection $A^X$ on the trivial vector bundle $C^\infty \left( \mathcal{X}, S^TX \otimes F \right)$ on $[0, \frac{\pi}{2}]$ given by

\begin{equation}
A^X = d\vartheta \frac{\partial}{\partial \vartheta} + \frac{\hat{D}_\vartheta^X}{\sqrt{2}}.
\end{equation}

The curvature $A^{X,2}$ of $A^X$ is given by

\begin{equation}
A^{X,2} = \frac{1}{2} \sin^2 (\vartheta) \hat{D}^{X,2} + \frac{d\vartheta}{\sqrt{2}} \cos (\vartheta) \hat{D}^X.
\end{equation}

Recall that the operator $\mathcal{L}_0^X$ was defined in (3.51).

**Definition 4.1.** Set

\begin{equation}
\mathcal{L}_{0, \vartheta}^X = \frac{1}{2} \sin^2 (\vartheta) \hat{D}^{X,2} + \mathcal{L}_0^X.
\end{equation}

Let $e_{m+1}, \ldots, e_{m+n}$ be an orthonormal basis of $N$. 

Proposition 4.2. The following identities hold:

\[
L_{\partial, \vartheta} = -\frac{1}{2} \cos^2(\vartheta) \hat{D}^{X,2} + \frac{1}{48} \text{Tr} \left[ C^{t,\vartheta} \right] + \frac{1}{2} C^{t,F},
\]

(4.8) \( L_{\partial, \vartheta} = \cos^2(\vartheta) L_{\partial}^X + \sin^2(\vartheta) \left( \frac{1}{8} B^*(\kappa^t, \kappa^t) + \frac{1}{2} C^{t,F} \right), \)

\[
L_{\partial, \vartheta} = \frac{1}{2} \cos^2(\vartheta) \left( -\Delta^{X,H} + \frac{1}{4} \text{Tr} \left[ C^{t,p} \right] + 2 \sum_{i=m+1}^{m+n} \hat{c}(\text{ad}(e_i)|_{TX}) \rho^F(e_i) \right)
\]

\[
+ \frac{1}{48} \text{Tr} \left[ C^{t,\vartheta} \right] + \frac{1}{2} C^{t,F}.
\]

Proof. By (3.55), (4.7), we get

\[
L_{\partial, \vartheta} = -\frac{1}{2} \cos^2(\vartheta) \hat{D}^{X,2} + \frac{1}{8} B^*(\kappa^t, \kappa^t) + \frac{1}{2} C^{t,F}
\]

\[
= \cos^2(\vartheta) L_{\partial}^X + \sin^2(\vartheta) \left( \frac{1}{8} B^*(\kappa^t, \kappa^t) + \frac{1}{2} C^{t,F} \right).
\]

By (3.22), (3.52), and (4.9), we get (4.8). \(\square\)

Definition 4.3. Set

\[
T^X = A^{X,2} + L_{\partial}^X.
\]

By (4.6), (4.7), and (4.10), we get

\[
T^X = L_{\partial, \vartheta} + \frac{d\vartheta}{\sqrt{2}} \cos(\vartheta) \hat{D}^X.
\]

Now we establish a version of Bianchi’s identity, similar to (2.33).

Proposition 4.4. The following identity holds:

\[
[A^X, T^X] = 0.
\]

Proof. As in (2.9), we have the classical Bianchi identity,

\[
[A^X, A^{X,2}] = 0.
\]

Moreover, \(C^g\) lies in the centre of \(U(g)\). By (3.51), we get an analogue of (2.32),

\[
[A^X, L_{\partial}^X] = 0.
\]

By (4.10), (4.13), and (4.14), we get (4.12). \(\square\)

Let \(e_1, \ldots, e_m\) be an orthonormal basis of \(TX\). At each \(x \in X\), we identify this frame to the orthonormal frame in \(TX\) defined on a neighbourhood of \(x\) obtained by parallel transport along geodesics centred at \(x\) with respect to the Levi-Civita connection. We have the identity

\[
\Delta^{X,H} = \sum_{i=1}^{m} \nabla_{e_i}^{TX} \otimes F;2.
\]

We have a version of an identity established in [BF86, Proposition 2.1].
Proposition 4.5. The following identity holds:

\[(4.16) \quad T^X = -\frac{1}{2} \sum_{i=1}^{m} \left( \cos (\vartheta) \nabla_{e_i} \circ \mathbb{R} - \frac{d\vartheta}{\sqrt{2}} \mathbb{C} (e_i) \right)^2 \]

\[
+ \frac{1}{2} \cos^2 (\vartheta) \left( \frac{1}{4} \text{Tr}^p [C^t \circ \mathbb{R}] + 2 \sum_{i=m+1}^{m+n} \mathbb{C} (\text{ad} (e_i) | T^X) \rho^F (e_i) \right) \]

\[
+ \frac{1}{48} \text{Tr}^t [C^t \circ \mathbb{R}] + \frac{1}{2} C^t \circ \mathbb{R}.
\]

Proof. This follows from the third identity in (4.8) and from (4.11). \qed

4.2. The action of SO(2) on \( p \oplus \bar{p} \). Let \( J \in \text{so}(2, \mathbb{R}) \) be given by

\[(4.17) \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\]

For \( \vartheta \in \mathbb{R} \), set

\[(4.18) \quad R_{\vartheta} = \exp (\vartheta J).
\]

Then \( R_{\vartheta} \) is the rotation of angle \( \vartheta \), i.e.,

\[(4.19) \quad R_{\vartheta} = \begin{bmatrix} \cos (\vartheta) & -\sin (\vartheta) \\ \sin (\vartheta) & \cos (\vartheta) \end{bmatrix}.
\]

We equip \( p \oplus \bar{p} \) with the direct sum of the obvious scalar products. Then \( J \) acts as an antisymmetric endomorphism of \( p \oplus \bar{p} \), and \( R_{\vartheta} \) acts as an isometry of \( p \oplus \bar{p} \).

Note that

\[(4.20) \quad g \oplus \bar{p} = p \oplus \bar{p} \oplus \mathfrak{t} \oplus \bar{p}.
\]

Let \( \beta \) be the bilinear symmetric form on \( g \oplus \bar{p} \) which coincides with \( B \) on \( g \) and with the scalar product on \( \bar{p} \), and is such that the splitting in (4.20) is orthogonal. We extend the action of \( J \) to \( g \oplus \bar{p} \), by making \( J \) act like 0 on \( \mathfrak{t} \). Then the action of \( R_{\vartheta} \) extends to \( g \oplus \bar{p} \). Also \( R_{\vartheta} \) preserves \( \beta \).

Let \( \mathbb{C} (g \oplus \bar{p}) \) be the Clifford algebra of \( (g \oplus \bar{p}, -\beta) \). Then

\[(4.21) \quad \mathbb{C} (g \oplus \bar{p}) = \mathbb{C} (g) \circ \mathbb{C} (\bar{p}).
\]

If \( f \in g \oplus \bar{p} \), let \( \mathbb{C} (f) \) denote the corresponding element in \( \mathbb{C} (g \oplus \bar{p}) \). By (4.21), \( \mathbb{C} (g \oplus \bar{p}) \) acts naturally on \( \Lambda^* (g^*) \otimes \bar{p} \).

Let \( \mathbb{C} (J) \in \mathbb{C} (g \oplus \bar{p}) \) correspond to \( J \). If \( e_1, \ldots, e_m \) is an orthonormal basis of \( p \), then

\[(4.22) \quad \mathbb{C} (J) = -\frac{1}{2} \sum_{i=1}^{m} \mathbb{C} (e_i) \mathbb{C} (e_i).
\]

For \( \vartheta \in \mathbb{R} \), set

\[(4.23) \quad \mathbb{R}_{\vartheta} = \exp (\vartheta \mathbb{C} (J)).
\]

Then

\[(4.24) \quad \mathbb{R}_{\vartheta} = \prod_{i=1}^{m} (\cos (\vartheta/2) - \sin (\vartheta/2) \mathbb{C} (e_i) \mathbb{C} (e_i)).
\]

For \( f \in g \oplus \bar{p} \), set

\[(4.25) \quad \mathbb{C}_{\vartheta} (f) = \mathbb{C} (R_{\vartheta} f).
\]
Then
\[ \hat{c}_\vartheta (f) = \hat{R}_\vartheta \hat{c}(f) \hat{R}_\vartheta^{-1}. \]

If \( A \in \text{End} (g \oplus \bar{p}) \) is antisymmetric with respect to \( \beta \), we denote by \( \hat{c}(A) \) the corresponding element in \( \hat{c}(g \oplus \bar{p}) \). Put
\[ \hat{c}_\vartheta (A) = \hat{R}_\vartheta \hat{c}(A) \hat{R}_\vartheta^{-1}. \]

Then
\[ \hat{c}_\vartheta (A) = \hat{c}(R_\vartheta A R_\vartheta^{-1}). \]

Recall that if \( e \in g \), \( \hat{c}(\text{ad} (e)) \in \hat{c}(g) \) is given by (3.27). If \( e \in g \), we denote by \( \text{ad} (e) |_g \) the endomorphism of \( g \oplus \bar{p} \), which is \( \text{ad} (e) \) on \( g \), and 0 on \( \bar{p} \). Then \( \text{ad} (e) |_g \) is antisymmetric with respect to \( \beta \), and \( \hat{c}(\text{ad} (e) |_g) = \hat{c}(\text{ad} (e)) \).

Set
\[ N^A (p^*_\vartheta) = \hat{R}_\vartheta N^A (p^*_\vartheta) \hat{R}_\vartheta^{-1} \]
and
\[ N^A (p^*_\vartheta) - \frac{m}{2} = \frac{1}{2} \sum_{i=1}^{m} c(e_i) \hat{c}_\vartheta (e_i). \]

By (3.26), we get
\[ N^A (p^*_\vartheta) = \cos (\vartheta) N^A (p^*) + \frac{1}{2} \sin (\vartheta) \sum_{i=1}^{m} c(e_i) \hat{c}(\pi_i) + \frac{m}{2} (1 - \cos (\vartheta)). \]

### 4.3. The deformation \( D_{b, \vartheta} \) of \( D_b \)

**Definition 4.6.** For \( \vartheta \in \left[ 0, \frac{\pi}{2} \right] \), set
\[ \hat{D}^g = \hat{R}_\vartheta \hat{D}^g \hat{R}_\vartheta^{-1}, \quad \hat{E}^p = \hat{R}_\vartheta \hat{E}^p \hat{R}_\vartheta^{-1}. \]

By (3.61), (4.25), we get
\[ \hat{E}^p = \hat{c}_\vartheta (Y^p) = \cos (\vartheta) \hat{c}(Y^p) + \sin (\vartheta) \hat{c}(\pi^p). \]

By (3.60), (3.61), and (4.32), we obtain
\[ \hat{D}^{p^2} = -C^g - \frac{1}{4} B^* (\kappa^g, \kappa^g), \quad \hat{E}^{p^2} = |Y^p|^2. \]

It is crucial to observe that \( \hat{D}^{p^2} \) does not depend on \( \vartheta \), and lies in the centre of \( U (g) \).

**Definition 4.7.** Set
\[ \hat{D}_{b, \vartheta} = \hat{D}^g + \cos (\vartheta) \text{i} c \left( [Y^t, Y^p] \right) + \frac{1}{b} \left( \hat{D}^p + \hat{E}^p - \text{i} \hat{D}^t + \cos (\vartheta) \text{i} \hat{E}^t \right), \]
\[ \hat{D}_{b, \vartheta} = \hat{D}^g + \cos (\vartheta) \text{i} c \left( [Y^t, Y^p] \right) + \frac{1}{b} \left( \hat{D}^p + \hat{E}^p - \text{i} \hat{D}^t + \cos (\vartheta) \text{i} \hat{E}^t \right). \]

The operators \( \hat{D}_{b, \vartheta}, \hat{D}_{b, \vartheta} \) lie in \( \text{End} \left( C^\infty (G \times g, \Lambda (g^*)) \otimes \hat{c}(\bar{p})^2 \right) \). Clearly,
\[ \hat{D}_{b, \vartheta} = \hat{R}_\vartheta^{-1} \hat{D}_{b, \vartheta} \hat{R}_\vartheta. \]

\(^2\text{If } m \text{ is even, we could have written instead } \text{End} \left( C^\infty \left( G \times g, \Lambda (g^*) \otimes S^F \right) \right). \text{ If } m \text{ is odd, one should use instead the identification in (3.36). This discrepancy is relevant since when } m \text{ is odd, the } \mathbb{Z}_2 \text{-grading of } c(\bar{p}) \text{ does not come from } S^F.\)
Comparing with (3.67), we get

\begin{equation}
D_{b,0} = D'_{b,0} = D_b.
\end{equation}

For \(a > 0\), set

\begin{equation}
K^t_a s (g, Y^p, Y^t) = a^{n/2} s (g, Y^p, aY^t).
\end{equation}

The factor \(a^{n/2}\) in (4.38) is introduced only to make \(K_a\) a \(L_2\) isometry.

**Definition 4.8.** For \(\vartheta \in [0, \frac{\pi}{2}\), set

\begin{equation}
\mathfrak{D}_{b,0} = K^t_{\pi/2(\vartheta)} D_{b,0} K^t_{\pi/2(\vartheta)}.
\end{equation}

By (4.35), (4.39), we get

\begin{equation}
\mathfrak{D}_{b,0} = \mathcal{D}_0 + \cos^{1/2}(\vartheta) ic \left( [Y^t, Y^p] \right) + \frac{1}{b} (\mathcal{D}^p + \mathcal{E}^p) + \frac{\cos^{1/2}(\vartheta)}{b} (-i\mathcal{D}^t + i\mathcal{E}^t).
\end{equation}

**Remark 4.9.** In our definition of \(\mathfrak{D}_{b,0}\), another possibility is to still use equation (4.35), with \(\cos(\vartheta)\) replaced by 1. This simplifies the algebraic computations which follow. However, this change would make the analysis more difficult.

4.4. **The compression of \(\mathfrak{D}_{b,0}\).** For \(\vartheta \in [0, \frac{\pi}{2}\), the results of section 3.9, the kernel \(H\) of the operator \(\mathcal{D}^p + \mathcal{E}^p + \cos^{1/2}(\vartheta) (-i\mathcal{D}^t + i\mathcal{E}^t)\) is 1-dimensional and is just \(\left\{ \exp \left( -|Y|^2/2 \right) \right\} \otimes \mathcal{S}^p\). We still denote by \(P\) the orthogonal projection on \(H\). Let \(H^\perp\) denote the orthogonal vector space to \(H\).

The operator \(\mathcal{D}_0\) defined in (4.1) acts on \(C^\infty(G, \mathcal{S}^p)\). As in section 3.9, the vector space \(C^\infty(G, \mathcal{S}^p)\) can be identified with \(H\), i.e., with a vector subspace of \(C^\infty(G \times \mathfrak{g}, \mathcal{L}^\perp(\mathfrak{g}^*) \otimes \mathcal{S}^p)\) via the embedding \(s \rightarrow s \exp \left( -|Y|^2/2 \right) / \pi^{(m+n)/4}\).

Now, we extend [B11, Proposition 2.10.1], that was stated before as Proposition 3.7.

**Proposition 4.10.** For \(\vartheta \in [0, \frac{\pi}{2}\), the following identity holds:

\begin{equation}
P \left( \mathcal{D}_0 + \cos^{1/2}(\vartheta) ic \left( [Y^t, Y^p] \right) \right) P = \mathcal{D}_0.
\end{equation}

**Proof.** In the proof, we will assume that \(e_1, \ldots, e_m\) is an orthonormal basis of \(\mathfrak{p}\), and \(e_{m+1}, \ldots, e_{m+n}\) is an orthonormal basis of \(\mathfrak{t}\). We proceed as in the proof of [B11, Proposition 2.10.1]. Recall that the kernel \(H\) is concentrated in degree 0 in \(\mathcal{L}^\perp(\mathfrak{g}^*)\), and that for \(e \in \mathfrak{g}\), \(c(e), \tilde{c}(e)\) act as odd operators on \(\mathcal{L}^\perp(\mathfrak{g}^*)\). It follows that

\begin{equation}
P \left( \sum_{i=1}^{m+n} \tilde{c}(R_0 e_i^*) e_i \right) P = \sin(\vartheta) \mathcal{D}.
\end{equation}

By [B11, eq. (2.7.4)], we get

\begin{equation}
\tilde{c}(-\kappa^\theta) = -2 \sum_{i=m+1}^{m+n} \tilde{c}(e_i) \tilde{c}(\text{ad}(e_i) \mid \mathfrak{p}) + \tilde{c}(-\kappa^t).
\end{equation}

By (4.43), we deduce easily that

\begin{equation}
P R_0 \tilde{c}(-\kappa^\theta) R_0^{-1} P = 0.
\end{equation}
Similarly,  
\begin{equation}
P_c \left( [Y^t, Y^p] \right) P = 0.
\end{equation}
By (3.59), (4.32), (4.42), (4.44), and (4.45), we get (4.41). The proof of our proposition is completed. \(\Box\)

4.5. A formula for \(D_{b,\vartheta}^2, D_{b,\vartheta}^2'.\) We use the same notation as in subsection 3.10.
We establish an extension of [B11, Theorem 2.11.1], stated here as Theorem 3.8.

**Theorem 4.11.** The following identities hold:
\begin{align*}
\mathcal{D}_{b,\vartheta}^2 &= \frac{\hat{D}_\vartheta^2}{2} + \frac{\cos^2 (\vartheta)}{2} \left| [Y^t, Y^p] \right|^2 + \frac{1}{2b^2} \left( -\Delta^{p\vartheta t} + |Y^p|^2 + \cos^2 (\vartheta) |Y^t|^2 \right) \\
&\quad - m - \cos (\vartheta) n + \frac{N^\Lambda (p^\vartheta r) + \cos (\vartheta) N^\Lambda (t^\vartheta r)}{b^2} \\
&\quad + \frac{\cos (\vartheta)}{b} \left( Y^p + iy^b - i\nabla_{[Y^t, Y^p]} + \hat{c}_\vartheta (\text{ad} (Y^p + iY^t) |_\vartheta) \right) \\
&\quad + 2ic (\text{ad} (Y^t) |_p) - c (\text{ad} (Y^p)) \right),
\end{align*}
\begin{align*}
\mathcal{D}_{b,\vartheta}'^2 &= \frac{\hat{D}_\vartheta^2}{2} + \frac{\cos^2 (\vartheta)}{2} \left| [Y^t, Y^p] \right|^2 + \frac{1}{2b^2} \left( -\Delta^{p\vartheta t} + |Y^p|^2 + \cos^2 (\vartheta) |Y^t|^2 \right) \\
&\quad - m - \cos (\vartheta) n + \frac{N^\Lambda (p^\vartheta r) + \cos (\vartheta) N^\Lambda (t^\vartheta r)}{b^2} \\
&\quad + \frac{\cos (\vartheta)}{b} \left( Y^p + iy^b - i\nabla_{[Y^t, Y^p]} + \hat{c} (\text{ad} (Y^p + iY^t) |_\vartheta) \right) \\
&\quad + 2ic (\text{ad} (Y^t) |_p) - c (\text{ad} (Y^p)) \right).
\end{align*}

**Proof.** The proof of the first identity in (3.72) is the same the proof of [B11, Theorem 2.11.1]. Combining this identity with (4.36), we get the second identity. The proof of our theorem is completed. \(\Box\)

**Remark 4.12.** A most important aspect of equation (4.46) is that the linear terms in the variable \(Y\) all have the same weight \(\cos(\vartheta)/b\). This would not be the case with the modification of \(\mathcal{D}_{b,\vartheta}^X\) that was suggested in Remark 4.9. This fact will play a most important role in the analysis.

4.6. The superconnections \(B, \overline{B}, B'.\) For \(a \in \mathbb{R}\), set
\begin{equation}
K_a f (Y) = a^{(m+n)/2} f (aY) .
\end{equation}
Let \((b, \vartheta)\) be the generic element of \(\mathbb{R}_+^* \times [0, \frac{\pi}{2}]\). Let \(d\mathbb{R}_+^* \times [0, \frac{\pi}{2}]\) denote the de Rham operator on \(\mathbb{R}_+^* \times [0, \frac{\pi}{2}]\).
Definition 4.13. Over $\mathbb{R}^*_+ \times \{0, \frac{x}{2}\}$, let $\nabla^{C^\infty}(G \times g, \Lambda \langle g^* \rangle \otimes S^F)$, $\nabla^{C^\infty}(G \times g, \Lambda \langle g^* \rangle \otimes S^F)'$ denote the flat connections on

$$C^\infty \left( G \times g, \Lambda \langle g^* \rangle \otimes S^F \right)$$

that are given by

$$\nabla^{C^\infty}(G \times g, \Lambda \langle g^* \rangle \otimes S^F) = K_b\tilde{\nabla}_0 dR^*_1 \times [0, \frac{x}{2}]; \tilde{K}_{b^{-1}},$$

(4.48)

$$\nabla^{C^\infty}(G \times g, \Lambda \langle g^* \rangle \otimes S^F)' = K_1^{t/\cos^{1/2}(\theta)} \nabla^{C^\infty}(G \times g, \Lambda \langle g^* \rangle \otimes S^F) K_{t/\cos^{1/2}(\theta)}^t,$$

$$\nabla^{C^\infty}(G \times g, \Lambda \langle g^* \rangle \otimes S^F)' = K_{b\cos(\theta)} dR^*_1 \times [0, \frac{x}{2}]; \tilde{K}_{b^{-1}}.$$!

Then

$$\nabla^{C^\infty}(G \times g, \Lambda \langle g^* \rangle \otimes S^F) = db \left( \frac{\partial}{\partial b} - \frac{1}{b} \nabla_Y - \frac{m+n}{2b} \right) + d\theta \left( \frac{\partial}{\partial \theta} - \tilde{c}(J) \right),$$

$$\nabla^{C^\infty}(G \times g, \Lambda \langle g^* \rangle \otimes S^F)' = db \left( \frac{\partial}{\partial b} - \frac{1}{b} \nabla_Y - \frac{m+n}{2b} \right)$$

$$+ d\theta \left( \frac{\partial}{\partial \theta} - \tilde{c}(J) - \frac{1}{2} \tan(\theta) \nabla_Y + \frac{1}{4} \tan(\theta) \right),$$

$$\nabla^{C^\infty}(G \times g, \Lambda \langle g^* \rangle \otimes S^F)' = db \left( \frac{\partial}{\partial b} - \frac{1}{b} \nabla_Y - \frac{m+n}{2b} \right)$$

$$+ d\theta \left( \frac{\partial}{\partial \theta} + \tan(\theta) \nabla_Y + \tan(\theta) \frac{m+n}{2} \right).$$

(4.49)

Proposition 4.14. The following identities hold:

$$\nabla^{C^\infty}(G \times g, \Lambda \langle g^* \rangle \otimes S^F) D_{b, \theta} = -\frac{2db}{b^2} \left( b \cos(\theta) ic \left( [Y^t, Y^p] \right) + E^p + \cos(\theta) iE^t \right)$$

$$- \frac{d\theta}{b} \left( b \sin(\theta) ic \left( [Y^t, Y^p] \right) + \tilde{c} \left( Y^p \right) + \sin(\theta) iE^t \right),$$

$$\nabla^{C^\infty}(G \times g, \Lambda \langle g^* \rangle \otimes S^F)' D_{b, \theta} = -\frac{2db}{b^2} \left( b \cos(\theta) ic \left( [Y^t, Y^p] \right) + E^p + \cos(\theta) iE^t \right)$$

$$- \frac{d\theta}{b} \left( b \frac{\sin(\theta)}{\cos^{1/2}(\theta)} ic \left( [Y^t, Y^p] \right) + \tilde{c} \left( Y^p \right) + \frac{\sin(\theta)}{\cos^{1/2}(\theta)} iE^t \right),$$

(4.50)

$$\nabla^{C^\infty}(G \times g, \Lambda \langle g^* \rangle \otimes S^F)' D_{b, \theta} = -\frac{2db}{b^2} \left( b \cos(\theta) ic \left( [Y^t, Y^p] \right) + E^p + \cos(\theta) iE^t \right)$$

$$+ \frac{d\theta}{b} \left( b \sin(\theta) ic \left( [Y^t, Y^p] \right) - \tan(\theta) \left( D^p - iD^t \right) - \frac{1}{\cos(\theta)} \tilde{c} \left( Y^p \right) \right).$$

Proof. Our proposition follows from (4.35), (4.39), (4.48), and (4.49). 

Definition 4.15. Let $B, \overline{B}, B'$ be the superconnections,

$$B = \nabla^{C^\infty}(G \times g, \Lambda \langle g^* \rangle \otimes S^F) + \frac{D_{b, \theta}}{\sqrt{2}},$$

(4.51)

$$\overline{B} = \nabla^{C^\infty}(G \times g, \Lambda \langle g^* \rangle \otimes S^F)' + \frac{D_{b, \theta}}{\sqrt{2}},$$

$$B' = \nabla^{C^\infty}(G \times g, \Lambda \langle g^* \rangle \otimes S^F)' + \frac{D_{b, \theta}}{\sqrt{2}}.$$
By (4.39), (4.48), we get
\[ \mathcal{B} = K^t_{\cos^{1/2} (\vartheta)} B K^t_{\cos^{1/2} (\vartheta)}. \]

Recall that the superconnection \( A \) was defined in (4.2). We will extend Proposition 4.10. We make temporarily \( db = 0 \).

**Proposition 4.16.** The following identity of superconnections on \( C^\infty (G, S \overline{\mathbb{F}}) \) over \([0, \pi/2] \) holds:
\[ P \left( \nabla \nabla^C_{\infty} (G \times \mathfrak{g}, \Lambda \cdot \Lambda (X^g, \mathcal{F})) |_{db = 0} + \frac{1}{\sqrt{2}} \left( \hat{D}^g_\vartheta + \cos^{1/2} (\vartheta) i c \left( \left[ Y^t, Y^p \right] \right) \right) \right) P = A. \]

**Proof.** In degree 0 in the variable \( d\vartheta \), equation (4.53) was already established in Proposition 4.10. To establish (4.53), we only need to show that we have the identity of connections on \( C^\infty (G, S \overline{\mathbb{F}}) \),
\[ P d\vartheta \left( \frac{\partial}{\partial \vartheta} - \hat{c} (J) - \frac{1}{2} \tan (\vartheta) \nabla Y^t - \frac{1}{4} \tanh (\vartheta) n \right) P = d\vartheta \left( \frac{\partial}{\partial \vartheta} \right). \]
By (4.22), we get
\[ P \hat{c} (J) P = 0. \]
Also by \([B11, \text{eq. (2.16.23)}]\), if \( u \in \mathfrak{k} \), we have the elementary identity
\[ P \left( u, Y^t \right)^2 P = \frac{1}{2} |u|^2. \]
From (4.56), it is easy to deduce that
\[ P \left( \nabla_{Y^t} + \frac{n}{2} \right) P = 0. \]
By (4.55), (4.57), we get (4.54). The proof of our proposition is completed. \( \Box \)

### 4.7. The operators \( \mathcal{L}_{\mathfrak{b}, \vartheta}, \mathcal{L}_{\mathfrak{X}, \mathfrak{b}, \vartheta}, \mathcal{L}_{\mathfrak{X}', \mathfrak{b}, \vartheta} \)

We use the same notation as in section 3. Then \( J \) and \( R_\vartheta \) still act on \( TX \oplus \mathcal{T}X \), so that \( \hat{c} (J) \) is a section of \( \hat{c} (TX \oplus N \oplus \mathcal{T}X) \).

We define \( \hat{R}_\vartheta \) as in (4.23). Recall that the vector space \( \mathcal{H} \) was defined in Definition 3.9. As in (4.38), if \( s \in \mathcal{H} \), set
\[ K^N_s (x, Y^T X, Y^N) = a^{n/2} s (x, Y^T X, a Y^N). \]
As in section 3.11, \( \hat{D}_\vartheta^s, \mathcal{D}_{\mathfrak{b}, \vartheta}, \mathcal{D}_{\mathfrak{X}, \vartheta}, \mathcal{D}_{\mathfrak{X}', \mathfrak{b}, \vartheta}, \mathcal{D}_{\mathfrak{X}'} \) descend to operators \( \hat{D}_\vartheta^s, \mathcal{D}_{\mathfrak{b}, \vartheta}, \mathcal{D}_{\mathfrak{X}, \vartheta}, \mathcal{D}_{\mathfrak{X}', \mathfrak{b}, \vartheta}, \mathcal{D}_{\mathfrak{X}'} \) acting on \( \mathcal{H} \). By (4.36), (4.39), we get
\[ \left( X \mathcal{D}_{\mathfrak{X}, \vartheta} = \hat{R}_\vartheta^{-1} \mathcal{D}_{\mathfrak{X}, \vartheta} \hat{R}_\vartheta, \quad \overline{\mathcal{D}_{\mathfrak{X}, \vartheta}} = K^N_{1/ \cos^{1/2} (\vartheta)} \mathcal{D}_{\mathfrak{b}, \vartheta} \right). \]

**Definition 4.17.** Put
\[ \mathcal{L}^X_{\mathfrak{b}, \vartheta} = -\frac{1}{2} \mathcal{D}^g_{\vartheta} X, 2 + \frac{1}{2} \mathcal{D}^X_{\mathfrak{b}, \vartheta}, \]
\[ \mathcal{L}^X_{\mathfrak{b}, \vartheta} = -\frac{1}{2} \mathcal{D}^g_{\vartheta} X, 2 + \frac{1}{2} \mathcal{D}^X_{\mathfrak{b}, \vartheta}, \]
\[ \mathcal{L}^{X'}_{\mathfrak{b}, \vartheta} = -\frac{1}{2} \mathcal{D}^g_{\vartheta} X, 2 + \frac{1}{2} \mathcal{D}^{X'}_{\mathfrak{b}, \vartheta}. \]
As observed after (4.34), we can rewrite the first equation in (4.60) in the form

\[(4.61)\]
\[L_{Xb,\vartheta} = -\frac{1}{2} \hat{D}_{g,X}^{b,\vartheta} + \frac{1}{2} \mathcal{L}_{b,\vartheta}.\]

By (4.32), (4.59), and (4.60), we get

\[(4.62)\]
\[L_{Xb,\vartheta} = \hat{R}_\vartheta^{-1} L_{Xb,\vartheta} \hat{R}_\vartheta, \quad \mathcal{E}_{Xb,\vartheta} = K_{1/\cos^{1/2} (\vartheta)} \mathcal{L}_{b,\vartheta} K_{\cos^{1/2} (\vartheta)}.\]

As before, we extend \(\text{ad} (Y_{TX})\) to an endomorphism \(\text{ad} (Y_{TX})\) of \(TX \oplus N\) that coincides with \(\text{ad} (Y_{TX})\) on \(TX \oplus N\), and vanishes on \(TX\). Similarly, we denote by \(\text{ad} (Y_{N|TX})\) the morphism of \(TX \oplus N \oplus TX\) that extends the action of \(\text{ad} (Y_{N})\) on \(TX\) by 0 on \(TX \oplus N\). As in (4.29), (4.32), put

\[(4.63)\]
\[N_{\alpha}^\Lambda (T^*X)^{\nu} = \hat{R}_\vartheta N_{\alpha}^\Lambda (T^*X) \hat{R}_\vartheta^{-1}, \quad \mathcal{E}^{TX} = \hat{R}_\vartheta \mathcal{E}^{TX} \hat{R}_\vartheta^{-1}.\]

Set

\[(4.64)\]
\[N_{\alpha}^\Lambda (T^*X \oplus N^*) = N_{\alpha}^\Lambda (T^*X) + \cos (\vartheta) N_{\alpha}^\Lambda (N^*), \quad N_{\alpha}^\Lambda (T^*X \oplus N^*)^{\nu} = N_{\alpha}^\Lambda (T^*X)^{\nu} + \cos (\vartheta) N_{\alpha}^\Lambda (N^*).\]

Then

\[(4.65)\]
\[N_{\alpha}^\Lambda (T^*X \oplus N^*)^{\nu} = \hat{R}_\vartheta N_{\alpha}^\Lambda (T^*X \oplus N^*) \hat{R}_\vartheta^{-1}.\]

By (4.31), (4.64), we obtain

\[(4.66)\]
\[N_{\alpha}^\Lambda (T^*X \oplus N^*)^{\nu} = \cos (\vartheta) N_{\alpha}^\Lambda (T^*X \oplus N^*)
\[- \frac{1}{2} \sin (\vartheta) \sum_{i=1}^m c(e_i) \hat{c}(e_i) + \frac{m}{2} (1 - \cos (\vartheta)).\]

Let \(\Delta^{TX}, \Delta^{N}\) denote the Laplacians acting along the fibres of the Euclidean vector bundles \(TX, N\). Now we extend [B11, Theorem 2.12.5 and eq. (2.13.5)], that was stated before as Theorem 3.10.

**Theorem 4.18.** The following identities hold:

\[(4.67)\]
\[\mathcal{D}_{b,\vartheta} = \hat{D}_{b}^{\vartheta,X} + \cos (\vartheta) ic \left( [Y_{N}, Y_{TX}] \right) + \frac{1}{b} \left( \mathcal{D}^{TX} + \mathcal{E}^{TX} - i \mathcal{D}^{N} + \cos (\vartheta) i \mathcal{E}^{N} \right),\]

\[\hat{\mathcal{D}}_{b,\vartheta} = \hat{D}_{b}^{\vartheta,X} + \cos^{1/2} (\vartheta) ic \left( [Y_{N}, Y_{TX}] \right) + \frac{1}{b} \left( \mathcal{D}^{TX} + \mathcal{E}^{TX} \right)
+ \frac{\cos^{1/2} (\vartheta)}{b} \left( -i \mathcal{D}^{N} + i \mathcal{E}^{N} \right),\]

\[\mathcal{D}_{b,\vartheta}^{TX} = \hat{D}_{b}^{\vartheta,X} + \cos (\vartheta) ic \left( [Y_{N}, Y_{TX}] \right) + \frac{1}{b} \left( \mathcal{D}^{TX} + \mathcal{E}^{TX} - i \mathcal{D}^{N} + \cos (\vartheta) i \mathcal{E}^{N} \right).\]
Moreover,
\[ L_{b,\vartheta}^X = \frac{\cos^2(\vartheta)}{2} \left[ |Y^N, Y^TX|^2 + \frac{1}{2b^2} \left( -\Delta^{TX} + |Y^TX|^2 + \cos^2(\vartheta) |Y^N|^2 - m \right) \right] - \cos(\vartheta) n + \frac{\cos(\vartheta)}{b^2} \left( \frac{1}{2b^2} \left( -\Delta^{TX} + |Y^TX|^2 + \cos^2(\vartheta) |Y^N|^2 - m \right) \right) \]

(4.68)

\[ V_{b,\vartheta}^X = \frac{\cos(\vartheta)}{2} \left[ |Y^N, Y^TX|^2 + \frac{1}{2b^2} \left( -\Delta^{TX} + |Y^TX|^2 + \cos^2(\vartheta) |Y^N|^2 - m \right) \right] - \cos(\vartheta) n + \frac{\cos(\vartheta)}{b^2} \left( \frac{1}{2b^2} \left( -\Delta^{TX} + |Y^TX|^2 + \cos^2(\vartheta) |Y^N|^2 - m \right) \right) \]

(4.69)

Proof. The identities in (4.67) follow from (4.35), (4.40). By proceeding as in the proof of [B11, Theorem 2.12.5], and using the second equation in (4.46), we get the third equation in (4.68). By conjugating this equation by \( \tilde{R}_{\vartheta} \), we get the first equation. By (4.62) and by the first identity in (4.68), we get the second identity. The proof of our theorem is completed. \( \square \)

Remark 4.19. The considerations of Remark 4.12 still apply here. Namely, in \( \Omega_{b,\vartheta}^X, \Phi_{b,\vartheta}^X \), the linear terms in \( Y^TX, Y^N \) all have the same factor \( \frac{\cos(\vartheta)}{b} \).

Now we follow [B11, Definition 2.4.1].

Definition 4.20. Let \( \nabla^\Lambda (T^*X \oplus N^*) \cdot f \cdot \hat{f} \) be the connection on \( \Lambda^* (T^*X \oplus N^*) \),

(4.69)

\[ \nabla^\Lambda (T^*X \oplus N^*) \cdot f \cdot \hat{f} = \nabla^\Lambda (T^*X \oplus N^*) - \cos(\vartheta) n + \frac{\cos(\vartheta)}{b} \left( \frac{1}{2b^2} \left( -\Delta^{TX} + |Y^TX|^2 + \cos^2(\vartheta) |Y^N|^2 - m \right) \right) \]

By [B11, Proposition 2.4.2], \( \nabla^\Lambda (T^*X \oplus N^*) \cdot f \cdot \hat{f} \) is a flat connection. Let

\[ \nabla^{C^\infty (T^*X \oplus N^*)} (T^*X \oplus N^*) \cdot f \cdot \hat{f} \]
be the connection on
\[ C^\infty \left( TX \oplus N, \hat{\pi}^* \left( \Lambda^1 (T^*X \oplus N^*) \otimes S^{TX} \otimes F \right) \right) \]
that is induced by \( \nabla^\Lambda_{(T^*X \oplus N^*)} f \cdot \hat{\jmath}_i, \nabla^{S^{TX}}, \nabla^F \).

We can rewrite the third equation in (4.68) in the form
\[
(4.70)
\mathcal{L}_{b, \vartheta}^X = \frac{\cos^2 (\vartheta)}{2} \left\| [Y^N, Y^{TX}] \right\|^2 + \frac{1}{2b^2} \left( -\Delta^{TX \oplus N} + |Y^{TX}|^2 + \cos^2 (\vartheta) |Y^N|^2 - m \right)
- \cos (\vartheta) n + \frac{N^{\Lambda (T^*X \oplus N^*)'}}{b^2} + \frac{\cos (\vartheta)}{b} \left( \nabla_{Y^{TX}} C^\infty \left( TX \oplus N, \hat{\pi}^* \left( \Lambda (T^*X \oplus N^*) \otimes S^{TX} \otimes F \right) \right), f \cdot \hat{\jmath} \right)
- ic \left( \text{ad} (Y^N) \right) - ic \left( \text{ad} (Y^N) |_{TX} \right) - i\rho^F (Y^N) \right) .
\]

4.8. A formula relating \( \mathcal{L}_{b, \vartheta}^X \) to \( \mathcal{L}_{0, \vartheta}^X \). We proceed as in subsection 3.12. By (4.68), we can write \( \mathcal{L}_{b, \vartheta}^X \) in the form
\[
(4.71)
\mathcal{L}_{b, \vartheta}^X = \frac{\alpha_{\vartheta}}{b^2} + \frac{\beta_{\vartheta}}{b} + \gamma_{\vartheta}.
\]

For \( \vartheta \in [0, \frac{\pi}{2}] \), the operator \( \alpha_{\vartheta} \) acting fibrewise has discrete spectrum. Its kernel is the vector space \( H \) in (3.80). We still define \( H^\perp \) as in subsection 3.12.

Note that \( \beta_{\vartheta} \) maps \( H \) into \( H^\perp \). Let \( \alpha_{\vartheta}^{-1} \) be the inverse of \( \alpha_{\vartheta} \) restricted to \( H^\perp \).

Recall that the operator \( \mathcal{L}_{0, \vartheta}^X \) was defined in Definition 4.1. Now we establish an extension of [B11, Theorem 2.16.1], which was stated before as Theorem 3.11.

**Theorem 4.21.** For \( \vartheta \in [0, \frac{\pi}{2}] \), the following identity holds:
\[
(4.72)
P \left( \gamma_{\vartheta} - \beta_{\vartheta} \alpha_{\vartheta}^{-1} \beta_{\vartheta} \right) P = \mathcal{L}_{0, \vartheta}^X .
\]

**Proof.** We proceed as in [B11, Theorem 2.16.1]. By (4.67), we can write \( \frac{1}{\sqrt{2}} \mathcal{L}_{b, \vartheta}^X \) in the form
\[
(4.73)
\frac{1}{\sqrt{2}} \mathcal{L}_{b, \vartheta}^X = E_{\vartheta} + \frac{F_{\vartheta}}{b} .
\]

Using (4.60), and comparing (4.71) and (4.73), we obtain
\[
(4.74)
\alpha_{\vartheta} = F_{\vartheta}^2 , \quad \beta_{\vartheta} = [E_{\vartheta}, F_{\vartheta}] , \quad \gamma_{\vartheta} = E_{\vartheta}^2 - \frac{1}{2} \hat{D}_{\vartheta}^{\gamma_b X^2} .
\]

By (4.74), we obtain
\[
(4.75)
P \left( \gamma_{\vartheta} - \beta_{\vartheta} \alpha_{\vartheta}^{-1} \beta_{\vartheta} \right) P = P \left( E_{\vartheta}^2 - E_{\vartheta} P^{b \perp} E_{\vartheta} - \frac{1}{2} \hat{D}_{\vartheta}^{\gamma_b X^2} \right) P .
\]

By equation (4.41) in Proposition 4.10, we get
\[
(4.76)
P \left( E_{\vartheta}^2 - E_{\vartheta} P^{b \perp} E_{\vartheta} \right) P = \left( PE_{\vartheta} P \right)^2 = \frac{1}{2} \sin^2 (\vartheta) \hat{D}_{\vartheta}^{X^2} .
\]

By (3.51), (3.60), (4.75), and (4.76), we obtain
\[
(4.77)
P \left( \gamma_{\vartheta} - \beta_{\vartheta} \alpha_{\vartheta}^{-1} \beta_{\vartheta} \right) P = \frac{1}{2} \sin^2 (\vartheta) \hat{D}_{\vartheta}^{X^2} + \mathcal{L}_{0, \vartheta}^X .
\]

By (4.7), (4.77), we get (4.72). The proof of our theorem is completed. \( \square \)
4.9. The superconnections $B^X, \overline{B}^X, B^{X'}$. The flat connections

$$\nabla^{C^\infty(G \times \mathfrak{g}, \mathfrak{X} \otimes S^\mathfrak{T})}, \nabla^{C^\infty(G \times \mathfrak{g}, \mathfrak{Y} \otimes S^\mathfrak{T})}, \nabla^{C^\infty(G \times \mathfrak{g}, A \otimes S^\mathfrak{T})}$$

induce flat connections $\nabla^H, \nabla^H, \nabla^{H'}$ on the vector bundle $\mathcal{H}$ over $\mathbb{R}^*_+ \times \mathbb{R}^*_+$. Let $dR^*_+ \times [0, \frac{\pi}{2}]$. Let $dR^*_+ \times [0, \frac{\pi}{2}]$ still denote the de Rham operator on $\mathbb{R}^*_+ \times \mathbb{R}^*_+$.

For $a \in \mathbb{R}$, we still define $K_a$ as in (4.47), i.e., if $s \in \mathcal{H}$, then

$$K_a s (x, Y) = a^{(m+n)/2} s (x,aY) .$$

By (4.48), we get

$$\nabla^H = K_b R_d dR^*_+ \times [0, \frac{\pi}{2}] [R_b^{-1} K_b^{-1}] ,$$

$$\nabla^H = K^N_1 \cos^{1/2} (\vartheta) \nabla^N K^{N}_1 \cos^{1/2} (\vartheta) ,$$

$$\nabla^{H'} = K_b \cos (\vartheta) dR^*_+ \times [0, \frac{\pi}{2}] [K_b^{-1}] .$$

By (4.49), we obtain

$$\nabla^H = db \left( \frac{\partial}{\partial b} \frac{1}{b} \nabla^Y - \frac{m+n}{2b} \right) + d\vartheta \left( \frac{\partial}{\partial \vartheta} - \hat{\vartheta} (J) \right) ,$$

$$\nabla^H = db \left( \frac{\partial}{\partial b} \frac{1}{b} \nabla^Y - \frac{m+n}{2b} \right) + d\vartheta \left( \frac{\partial}{\partial \vartheta} - \frac{1}{2} \tan (\vartheta) \nabla^V - \frac{1}{4} \tan (\vartheta) n \right) ,$$

$$\nabla^{H'} = db \left( \frac{\partial}{\partial b} \frac{1}{b} \nabla^Y - \frac{m+n}{2b} \right) + d\vartheta \left( \frac{\partial}{\partial \vartheta} + \tan (\vartheta) \nabla^V + \tan (\vartheta) \frac{m+n}{2} \right) ,$$

By (4.50), we get

$$\nabla^H \mathcal{D}^X_{b,\vartheta} = - \frac{2db}{b^2} \left( b \cos (\vartheta) ic \left( \left[ Y^N, Y^{TX} \right] \right) + \mathcal{E}^{TX} + \cos (\vartheta) i\mathcal{E}^N \right)$$

$$- \frac{d\vartheta}{b} \left( b \sin (\vartheta) ic \left( \left[ Y^N, Y^{TX} \right] \right) + \hat{\vartheta} \left( Y^{TX} \right) + \sin (\vartheta) i\mathcal{E}^N \right) ,$$

$$\nabla^{H'} \mathcal{D}^X_{b,\vartheta} = - \frac{2db}{b^2} \left( b \cos^{1/2} (\vartheta) ic \left( \left[ Y^N, Y^{TX} \right] \right) + \mathcal{E}^{TX} + \cos^{1/2} (\vartheta) i\mathcal{E}^N \right)$$

$$- \frac{d\vartheta}{b} \left( b \sin (\vartheta) ic \left( \left[ Y^N, Y^{TX} \right] \right) + \sin (\vartheta) \cos^{1/2} (\vartheta) i\mathcal{E}^N \right) ,$$

$$\nabla^H \mathcal{D}^X_{b,\vartheta} = - \frac{2db}{b^2} \left( b \cos (\vartheta) ic \left( \left[ Y^N, Y^{TX} \right] \right) + \mathcal{E}^{TX} + \cos (\vartheta) i\mathcal{E}^N \right)$$

$$+ \frac{d\vartheta}{b} \left( b \sin (\vartheta) ic \left( \left[ Y^N, Y^{TX} \right] \right) - \tan (\vartheta) \left( D^{TX} - i\mathcal{D}^N \right) - \frac{1}{\cos (\vartheta)} \hat{\vartheta} \left( Y^{TX} \right) \right) .$$
Note that the superconnections $B, \tilde{B}, B'$ in (4.51) descend to superconnections $B^X, \tilde{B}^X, B^{X'}$ on $\mathcal{H}$ given by
\[
B^X = \nabla^H + \frac{\mathcal{D}_{b,\theta}^X}{\sqrt{2}},
\]
\[
\tilde{B}^X = \nabla^H + \frac{\mathcal{D}_{b,\theta}^X}{\sqrt{2}},
\]
\[
B^{X'} = \nabla^{H'} + \frac{\mathcal{D}_{b,\theta}^{X'}}{\sqrt{2}}.
\]
(4.82)

By (4.52), we get
\[
B^X = K_N^1 \frac{1}{\cos \sqrt{2} \theta} B^X K_N^1 \cos \sqrt{2} \theta.
\]
Moreover, we have the identities
\[
B^{X,2} = \frac{1}{2} b_{X,2} + \frac{1}{\sqrt{2}} \nabla^H b_{X,2},
\]
\[
\tilde{B}^{X,2} = \frac{1}{2} b_{X,2} + \frac{1}{\sqrt{2}} \nabla^H b_{X,2},
\]
\[
B^{X',2} = \frac{1}{2} b_{X',2} + \frac{1}{\sqrt{2}} \nabla^{H'} b_{X',2}.
\]
(4.84)

**Definition 4.22.** Put
\[
L^X = -\frac{1}{2} \mathcal{D}^{X,2} + B^{X,2},
\]
\[
\tilde{L}^X = -\frac{1}{2} \mathcal{D}^{X,2} + \tilde{B}^{X,2},
\]
\[
L^{X'} = -\frac{1}{2} \mathcal{D}^{X',2} + B^{X',2}.
\]
(4.85)

By (4.34), (4.83), and (4.85), we obtain
\[
L^X = K_N^1 K_{\text{cos}^{1/2}(\theta)}^N,L^X K_{\text{cos}^{1/2}(\theta)}^N.
\]
(4.86)

By (4.34), (4.85), we get
\[
L^X = B^{X,2} + \frac{1}{8} \mathcal{D}_{b,\theta}^{X,2} + B^{X,2},
\]
\[
\tilde{L}^X = \tilde{B}^{X,2} + \frac{1}{8} \mathcal{D}_{b,\theta}^{X,2} + \tilde{B}^{X,2},
\]
\[
L^{X'} = B^{X',2} + \frac{1}{8} \mathcal{D}_{b,\theta}^{X',2} + B^{X',2}.
\]
(4.87)

By (4.60), (4.84), and (4.85), we obtain
\[
L^X = \mathcal{L}^X_{b,\theta} + \frac{1}{\sqrt{2}} \nabla^H \mathcal{D}^X_{b,\theta},
\]
\[
\tilde{L}^X = \tilde{\mathcal{L}}^X_{b,\theta} + \frac{1}{\sqrt{2}} \nabla^H \tilde{\mathcal{D}}^X_{b,\theta},
\]
\[
L^{X'} = \mathcal{L}^{X'}_{b,\theta} + \frac{1}{\sqrt{2}} \nabla^{H'} \mathcal{D}^{X'}_{b,\theta}.
\]
(4.88)

We will establish an analogue of the Bianchi identities in (2.33) and in Proposition 4.4.
Proposition 4.23. The following identities hold:
\[(4.89) \quad [B^X, L^X] = 0, \quad [B^X, \bar{L}^X] = 0, \quad [B^{X'}, L^X] = 0.\]

Proof. The classical Bianchi identity asserts that
\[(4.90) \quad [B^X, B^{X,2}] = 0.\]
Since \(C^g\) lies in the centre of \(U(g)\), by (4.34), we get
\[(4.91) \quad [B^X, C^{g, X}] = 0.\]
By (4.87), (4.90), and (4.91), we get the first identity in (4.89). The proof of the other identities is similar. \(\square\)

Recall that the superconnection \(A^X\) was defined in equation (4.5).

Proposition 4.24. The following identity holds:
\[(4.92) \quad P\left(\nabla^H|_{db=0} + \frac{1}{\sqrt{2}} \left(\hat{D}_{\partial}^{X} + \cos^{1/2}(\vartheta) \text{ic} ([Y^{N}, Y^{T}X])\right)\right) P = A^X.\]

Proof. Our proposition follows from Proposition 4.16. \(\square\)

By (4.68), (4.71), (4.81), and (4.88), we can write \(\bar{L}^X|_{db=0}\) in the form
\[(4.93) \quad \bar{L}^X|_{db=0} = \frac{\alpha_{\vartheta}}{b^2} + \frac{\beta_{\vartheta}}{b} + \gamma_{\vartheta},\]
and moreover,
\[(4.94) \quad \begin{align*}
\beta_{\vartheta} &= \beta_{\vartheta} - \frac{d\vartheta}{\sqrt{2}} \left(\hat{c}\left(Y^{TX}\right) + \frac{\sin (\vartheta)}{\cos^{1/2} (\vartheta)} iE^N\right), \\
\gamma_{\vartheta} &= \gamma_{\vartheta} - \frac{d\vartheta}{\sqrt{2} \cos^{1/2} (\vartheta)} \text{ic} ([Y^{N}, Y^{T}X]).
\end{align*}\]
Again, \(\beta_{\vartheta}\) maps \(H\) in \(H^\perp\).

Recall that \(T^X\) was defined in Definition 4.3 and is given by (4.11). Now, we extend Theorem 4.21.

Theorem 4.25. For \(\vartheta \in \left[0, \frac{\pi}{2}\right]\), the following identity holds:
\[(4.95) \quad P\left(\gamma_{\vartheta} - \frac{\beta_{\vartheta}}{b^2} \alpha_{\vartheta}^{-1} \beta_{\vartheta}\right) P = T^X.\]

Proof. By (4.73), (4.80), and (4.82), we get
\[(4.96) \quad B^X|_{db=0} = E_{\vartheta} + \frac{F_{\vartheta}}{b},\]
with
\[(4.97) \quad E_{\vartheta} = \nabla^H|_{db=0} + E_{\vartheta}.\]
Using (4.85), and comparing (4.93) and (4.96), we get
\[(4.98) \quad \begin{align*}
\alpha_{\vartheta} &= F^\vartheta_{\vartheta} , \\
\beta_{\vartheta} &= [E_{\vartheta}, F_{\vartheta}] , \\
\gamma_{\vartheta} &= E^\vartheta_{\vartheta} - \frac{1}{2} \hat{D}_{\vartheta}^{X,2}.
\end{align*}\]
By (4.98), we get
\[(4.99) \quad P\left(\gamma_{\vartheta} - \frac{\beta_{\vartheta}}{b^2} \alpha_{\vartheta}^{-1} \beta_{\vartheta}\right) P = P\left(\frac{E^\vartheta_{\vartheta}}{\vartheta} - E_{\vartheta} P_{\vartheta} \frac{1}{2} \hat{D}_{\vartheta}^{X,2}\right) P.\]
By equation (4.92) in Proposition 4.24, we obtain
\[(4.100)\quad P_\theta \left( E_\theta^2 - E_\theta P_\theta^\perp E_\theta \right) P_\theta = (P_\theta E_\theta P_\theta)^2 = A X^2.\]

By (3.51), (3.60), (4.10), (4.99), and (4.100), we get (4.95). □

4.10. A compression identity on linear maps.

**Definition 4.26.** Set
\[(4.101)\quad R_\theta (Y) = \cos (\phi) \left( \hat{c}_\theta \left( \mathrm{ad} \left( Y^TX \right) \right) - c \left( \mathrm{ad} \left( Y^TX \right) \right) \right)
- i \cos^{1/2} (\phi) \left( c \left( \mathrm{ad} \left( Y^N \right) \right) + \hat{c}_\theta \left( \mathrm{ad} \left( Y^N \right) \right) + \rho^F \left( Y^N \right) \right).\]

Then \( R_\theta (Y) \) splits as
\[(4.102)\quad R_\theta (Y) = R_\theta \left( Y^TX \right) + R_\theta \left( Y^N \right).\]

Let \( P \) be the orthogonal projection from \( \Lambda \cdot (T^* X \otimes N^*) \otimes S^TX \otimes F \) on \( S^TX \otimes F \).

**Proposition 4.27.** The following identity holds:
\[(4.103)\quad \mathbf{P} R_\theta \left( Y^TX \right) \mathbf{P} = 0,
\mathbf{P} R_\theta \left( Y^N \right) \mathbf{P} = - \cos^{5/2} (\phi) \hat{c} \left( \mathrm{ad} \left( Y^N \right) \right) - i \cos^{1/2} (\phi) \rho^F \left( Y^N \right).\]

**Proof.** If \( e_1, \ldots, e_m \) is an orthonormal basis of \( TX \), and if \( e_{m+1}, \ldots, e_{m+n} \) is an orthonormal basis of \( N \), by (3.27), we have the identities
\[(4.104)\quad c \left( \mathrm{ad} \left( Y^TX \right) \right) = - \frac{1}{2} \sum_{m+1 \leq i \leq m+n} \langle Y^TX, e_i \rangle \left( e^j - i_{e_i} \right) \left( e^j + i_{e_i} \right),
\hat{c} \left( \mathrm{ad} \left( Y^TX \right) \right) = \frac{1}{2} \sum_{1 \leq i \leq m} \langle Y^TX, e_i \rangle \left( e^j + i_{e_i} \right) \left( e^j - i_{e_i} \right).\]

By (4.104), we deduce that
\[(4.105)\quad \mathbf{P} c \left( \mathrm{ad} \left( Y^TX \right) \right) \mathbf{P} = 0, \quad \mathbf{P} \hat{c} \left( \mathrm{ad} \left( Y^TX \right) \right) \mathbf{P} = 0.\]

By (4.25), (4.104), we get
\[(4.106)\quad \hat{c}_\theta \left( \mathrm{ad} \left( Y^TX \right) \right) \mathbf{P} = \cos (\phi) \hat{c} \left( \mathrm{ad} \left( Y^TX \right) \right) + \sin (\phi) \frac{1}{2} \sum_{m+1 \leq i \leq m+n} \langle Y^TX, e_i \rangle \hat{c} (e_i) \left( e^j - i_{e_i} \right).\]

By (4.105), (4.106), we get
\[(4.107)\quad \mathbf{P} \hat{c}_\theta \left( \mathrm{ad} \left( Y^TX \right) \right) \mathbf{P} = 0.\]
Moreover, we have the identities

\[
\begin{align*}
&c(\theta \text{ad}(Y^N)) = -\frac{1}{4} \sum_{1 \leq i, j \leq m} \langle [Y^N, e_i], e_j \rangle (e^i - i e_i) (e^j - i e_j) \\
&- \frac{1}{4} \sum_{m+1 \leq i, j \leq m+n} \langle [Y^N, e_i], e_j \rangle (e^i + i e_i) (e^j + i e_j), \\
&\hat{c}(\text{ad}(Y^N)) = -\frac{1}{4} \sum_{1 \leq i, j \leq m} \langle [Y^N, e_i], e_j \rangle (e^i + i e_i) (e^j + i e_j) \\
&+ \frac{1}{4} \sum_{m+1 \leq i, j \leq m+n} \langle [Y^N, e_i], e_j \rangle (e^i - i e_i) (e^j - i e_j).
\end{align*}
\]  

(4.108)

Since \( \text{ad}(Y^N), \theta \text{ad}(Y^N) \) are antisymmetric, we get easily

\[
\begin{align*}
Pc(\theta \text{ad}(Y^N) |_{TX}) P &= 0, \\
P\hat{c}(\text{ad}(Y^N)) P &= 0.
\end{align*}
\]  

(4.109)

Also we have the identity

\[
\hat{c}(\text{ad}(Y^N) |_{TX}) = -\frac{1}{4} \sum_{1 \leq i, j \leq m} \langle [Y^N, e_i], e_j \rangle_{TX} \hat{c}(\overline{e}_i) \hat{c}(\overline{e}_j).
\]  

(4.110)

By (4.110), we deduce that

\[
\begin{align*}
\hat{c}_\phi(\text{ad}(Y^N) |_{TX}) &= \cos^2(\phi) \hat{c}(\text{ad}(Y^N) |_{TX}) + \sin^2(\phi) \hat{c}(\text{ad}(Y^N) |_{TX}) \\
&+ \frac{1}{2} \sin(\phi) \cos(\phi) \sum_{1 \leq i, j \leq m} \langle [Y^N, e_i], e_j \rangle_{TX} (e^i + i e_i) \hat{c}(\overline{e}_j).
\end{align*}
\]  

(4.111)

By (4.109), (4.111), we get

\[
P\hat{c}_\phi(\text{ad}(Y^N) |_{TX}) P = \cos^2(\phi) \hat{c}(\text{ad}(Y^N) |_{TX}).
\]  

(4.112)

By (4.101), (4.105), (4.107), (4.109), and (4.112), we get (4.103). \( \square \)

Now we extend [B11, Definition 2.16.2].

**Definition 4.28.** For \( 0 \leq \vartheta < \frac{\pi}{2} \), set

\[
S_\vartheta(Y) = P \left[ R_\vartheta(Y^{TX}) \left( 1 + N^{\vartheta \text{ad}(T^*X \oplus N^*)} \right) R_\vartheta(Y^{TX})^{-1} \\
+ R_\vartheta(Y^N) \left( \cos(\vartheta) + N^{\vartheta \text{ad}(T^*X \oplus N^*)} \right) R_\vartheta(Y^N) \right] P.
\]  

(4.113)

Then \( S_\vartheta(Y) \) lies in \( \text{End} \left( S^{TX} \otimes F \right) \). In the sequel, we use the notation

\[
x = \cos(\vartheta).
\]  

(4.114)

We give an extension of [B11, Proposition 2.16.3].
Proposition 4.29. The following identity holds:

\[ S_\theta (Y) = \frac{x^2}{4} + \frac{3}{4} \frac{\text{Tr} \left[ \text{ad}^2 (Y^T X) \right]}{x+2} |N| \mathbf{P} \]

\[ + \frac{1}{4} \left( \frac{x (x^2 - 2)^2}{x+2} - x^3 (x-1) \right) \frac{\text{Tr} \left[ -\text{ad}^2 (Y^N) \right]}{x+2} |T_X| \mathbf{P} \]

\[ + \frac{1}{24} \text{Tr} \left[ -\text{ad}^2 (Y^N) \right] |N| \mathbf{P} - (\rho^F (Y^N) + x^2 \hat{c} (\text{ad} (Y^N) \mid T_X))^2. \]

Proof. By (4.64), (4.101), (4.104), and (4.106), we get

\[ S_\theta (Y^T X) \left( 1 + N^\Lambda_{-\theta} (T^\star X \oplus N^\star) \right)^{-1} R_\theta (Y^T X) \mathbf{P} \]

\[ = \frac{x^2}{4} \sum_{1 \leq i,j \leq m, m+1 \leq i,j \leq m+n} \langle [Y^T X, e_i], e_j \rangle \langle [Y^T X, e_i'], e_j' \rangle \]

\[ \mathbf{P} \left( -\frac{(x+1)^2}{x+2} i_{e_i'} i_{e_j} e_i' e_j - \frac{-x^2 + 1}{x+1} \hat{c}(e_i') i_{e_j} \hat{c}(e_i) e_j \right) \mathbf{P}. \]

Equivalently,

\[ S_\theta (Y^N) \left( \cos (\theta) + N^\Lambda_{-\theta} (T^\star X \oplus N^\star) \right)^{-1} R_\theta (Y^N) \mathbf{P} \]

\[ = \frac{x^2}{4} + \frac{3}{4} \text{Tr} \left[ \text{ad}^2 (Y^T X) \right] |N| \mathbf{P}. \]

The variables \( \hat{c}(e_i'), \hat{c}(e_i) \) in the right-hand side of (4.116) disappear in (4.117) because in (4.116), a nonzero contribution is only possible if \( j = j' \). Ultimately only the case where \( i = i' \) contributes to the right-hand side of (4.117), which explains why no Clifford variable \( \hat{c} \) appears.

The same arguments as before combined with (4.108), (4.111) show that

\[ S_\theta (Y^N) \left( \cos (\theta) + N^\Lambda_{-\theta} (T^\star X \oplus N^\star) \right)^{-1} R_\theta (Y^N) \mathbf{P} \]

\[ = \frac{1}{4} \left( \frac{x (x^2 - 2)^2}{x+2} - \frac{x^3 (x-1)}{x+1} \right) \sum_{1 \leq i,j \leq m} \langle [Y^N, e_i], e_j \rangle^2 \mathbf{P} \]

\[ + \frac{1}{24} \sum_{m+1 \leq i,j \leq m+n} \langle [Y^N, e_i], e_j \rangle^2 - (\rho^F (Y^N) + x^2 \hat{c} (\text{ad} (Y^N) \mid T_X))^2. \]

Equivalently,

\[ S_\theta (Y^N) \left( \cos (\theta) + N^\Lambda_{-\theta} (T^\star X \oplus N^\star) \right)^{-1} R_\theta (Y^N) \mathbf{P} \]

\[ = \frac{1}{4} \left( \frac{x (x^2 - 2)^2}{x+2} - \frac{x^3 (x-1)}{x+1} \right) \text{Tr} \left[ -\text{ad}^2 (Y^N) \right] |T_X| \mathbf{P} \]

\[ + \frac{1}{24} \text{Tr} \left[ -\text{ad}^2 (Y^N) \right] |N| \mathbf{P} - (\rho^F (Y^N) + x^2 \hat{c} (\text{ad} (Y^N) \mid T_X))^2. \]

By (4.117), (4.119), we get (4.115). The proof of our proposition is completed. \( \square \)

Remark 4.30. By Proposition 4.29, \( S_\theta (Y) \) extends by continuity at \( \theta = \frac{\pi}{2} \).
Recall that the projector $P$ was defined in subsections 3.9 and 3.12.

**Definition 4.31.** Put

\[(4.120) \quad \delta \theta = PS_ \theta (Y) P.\]

Then $\delta \theta$ lies in $\text{End} \left( S^{TX} \otimes F \right)$.

Now we extend [B11, Proposition 2.16.5]. The absence of $S^{TX}$ in [B11] explains the fact that the results are different.

**Proposition 4.32.** The following identity holds:

\[(4.121) \quad \delta \theta = -\frac{1}{8} x (x + 1) \text{Tr}^p \left[ C^{t,p} \right] - \frac{1}{48} \text{Tr}^p \left[ C^{t,t} \right] - \frac{1}{2} C^{t,E} \]

\[\quad - x^2 \sum_{i=m+1}^{m+n} \hat{c} \left( \text{ad} \left( e_i \right) \right|_{\mathbb{T}X} \rho^F \left( e_i \right).\]

**Proof.** By [B11, eq. (2.16.23)] or by (4.56), if $u \in \mathfrak{g}$, we get

\[(4.122) \quad P \left( u, Y \right)^2 P = \frac{1}{2} |u|^2.\]

By (4.122), we deduce that

\[(4.123) \quad P \left( \text{Tr} \left[ \text{ad} \left( Y^{TX} \right) \right|_{\mathbb{N}} \right] P = -\frac{1}{2} \text{Tr}^p \left[ C^{t,p} \right],\]

\[P \left( \text{Tr} \left[ \text{ad} \left( Y^{Y^N} \right) \right|_{\mathbb{T}X} \right] P = -\frac{1}{2} \text{Tr}^p \left[ C^{t,t} \right],\]

\[P \left( \left( \rho^F \left( Y^{Y^N} \right) + x^2 \hat{c} \left( \text{ad} \left( Y^N \right) \right|_{\mathbb{T}X} \right)^2 \right) P \]

\[= -\frac{1}{2} C^{t,E} - \frac{x^4}{2} C^{t,S^F} - x^2 \sum_{i=m+1}^{m+n} \hat{c} \left( \text{ad} \left( e_i \right) \right|_{\mathbb{T}X} \rho^F \left( e_i \right).\]

By (4.115), (4.120), and (4.123), we get

\[(4.124) \quad \delta \theta = -\frac{1}{8} \left( \frac{2x^2 (x + 3) + x (x^2 - 2)^2}{2 (x + 2)} - x^3 (x - 1) \right) \text{Tr}^p \left[ C^{t,p} \right] - \frac{1}{48} \text{Tr}^p \left[ C^{t,t} \right]

\[\quad - \left( \frac{1}{2} C^{t,E} + \frac{x^4}{2} C^{t,S^F} + x^2 \sum_{i=m+1}^{m+n} \hat{c} \left( \text{ad} \left( e_i \right) \right|_{\mathbb{T}X} \rho^F \left( e_i \right).\right)\]

Also

\[(4.125) \quad \frac{2x^2 (x + 3) + x (x^2 - 2)^2}{2 (x + 2)} - x^3 (x - 1) = \frac{x}{2} \left( -x^3 + 2x + 2 \right).\]

By (3.41), (4.124), and (4.125), we get (4.121). \qed

**Remark 4.33.** If we make the modifications suggested in Remarks 4.9 and 4.12, in equation (4.121), $x (x + 1)$ should be replaced by $x^2 + 1$. 
Similarly, we have the identities

\[
\begin{align*}
(4.126) \quad T^\vartheta_\varphi (Y) &= PR^\vartheta_\varphi (Y) \left[ \left( 1 + N^A_{-\varphi} (TX \oplus N^*) \right)^{-1} R^\vartheta_\varphi (TY) \right. \\
& \quad \left. + \left( \cos (\vartheta) + N^A_{-\varphi} (TX \oplus N^*) \right)^{-1} R^\vartheta_\varphi (YN) \right] P.
\end{align*}
\]

Now we will establish an auxiliary identity that will be needed in section 11, in the proof of Theorem 11.19.

**Proposition 4.35.** The following identity holds:

\[
(4.127) \quad T^\vartheta_\varphi (Y) = S^\vartheta_\varphi (Y).
\]

**Proof.** By (4.113), it is enough to show that in (4.126), the contribution of the bilinear terms in \(YTX\), \(YN\) vanishes identically. By (4.101), (4.104), (4.106), (4.108), and (4.111), we get

\[
\begin{align*}
R^\vartheta_\varphi (YTX) P &= \cos (\vartheta) \sum_{1 \leq i \leq m \atop m + 1 \leq j \leq m + n} \langle [YTX, e_i], e_j \rangle \\
(4.128) \quad \left( (\cos (\vartheta) e^i \wedge + \sin (\vartheta) \tilde{\varphi} (\tilde{\varphi}_{\tilde{x}_i})) e^j + e^i \wedge e^j \right) P, \\
PR^\vartheta_\varphi (YN) &= iP \cos^{1/2} (\vartheta) \left( \sum_{1 \leq i, j \leq m} \langle [YN, e_i], e_j \rangle i_{e_i}, i_{e_j} \right. \\
& \quad + \sum_{m + 1 \leq i, j \leq m + n} \langle [YN, e_i], e_j \rangle i_{e_i}, i_{e_j} + \sum_{1 \leq i, j \leq m} \langle [YN, e_i], e_j \rangle \\
& \quad \left( - \sin (\vartheta) i_{e_i} \tilde{\varphi} (\tilde{\varphi}_{\tilde{x}_i}) \right) \left( - \sin (\vartheta) i_{e_j} \tilde{\varphi} (\tilde{\varphi}_{\tilde{x}_j}) \right) \right).
\end{align*}
\]

By (4.128), we deduce easily that

\[
(4.129) \quad PR^\vartheta_\varphi (YN) \left( 1 + N^A_{-\varphi} (TX \oplus N^*) \right)^{-1} R^\vartheta_\varphi (TY) P = 0.
\]

Similarly, we have the identities

\[
\begin{align*}
PR^\vartheta_\varphi (YTX) &= -P \cos (\vartheta) \sum_{1 \leq i \leq m \atop m + 1 \leq j \leq m + n} \langle [YTX, e_i], e_j \rangle \\
(4.130) \quad \left( (\cos (\vartheta) i_{e_i} + \sin (\vartheta) \tilde{\varphi} (\tilde{\varphi}_{\tilde{x}_i})) \right) i_{e_j} \tilde{\varphi} (\tilde{\varphi}_{\tilde{x}_j}), \\
R^\vartheta_\varphi (YN) P &= -i \cos^{1/2} (\vartheta) \left( - \sum_{1 \leq i, j \leq m} \langle [YN, e_i], e_j \rangle e^i \wedge e^j \right. \\
& \quad - \sum_{m + 1 \leq i, j \leq m + n} \langle [YN, e_i], e_j \rangle e^i \wedge e^j - \sum_{1 \leq i, j \leq m} \langle [YN, e_i], e_j \rangle \\
& \quad \left( - \sin (\vartheta) e^i + \cos (\vartheta) \tilde{\varphi} (\tilde{\varphi}_{\tilde{x}_i}) \right) \left( - \sin (\vartheta) e^j + \cos (\vartheta) \tilde{\varphi} (\tilde{\varphi}_{\tilde{x}_j}) \right) \right) P.
\end{align*}
\]
By (4.130), we also deduce that

\[ PR_0^T (Y^TX) \left( \cos (\vartheta) + N^\Lambda (T^*X \otimes N^*) \right)^{-1} R_0^0 (Y^N) P = 0. \]

By (4.129), (4.131), we get (4.127). □

**Definition 4.36.** Put

\[ S_\vartheta^0 (Y) = S_\vartheta^0 (Y) + \cos^4 (\vartheta) \hat{c} (\text{ad} (Y^N) |_{TX})^2. \]

By equation (4.115) in Proposition 4.29, $S_\vartheta^0 (Y)$ is a scalar operator, that depends quadratically on $Y$.

**Definition 4.37.** Let $\delta_\vartheta^0$ be the constant $\delta_\vartheta$ in (4.121) with $E$ the trivial representation. Set

\[ \delta_\vartheta^0 = \delta_\vartheta^0 + \cos^4 (\vartheta) \frac{1}{16} \text{Tr}^p [C^{t,p}]. \]

By (4.121), (4.133), if $x = \cos (\vartheta)$, we get

\[ \delta_\vartheta = \frac{1}{16} (x^4 - 2x^2 - 2x) \text{Tr}^p [C^{t,p}] - \frac{1}{48} \text{Tr}^t [C^{t,t}]. \]

**Proposition 4.38.** The following identity holds:

\[ P S_\vartheta^0 (Y) P = \delta_\vartheta^0. \]

**Proof.** Using (3.41), Proposition 4.32, the last equation in (4.123), (4.132), and (4.133), we get (4.135). □

### 4.11. A Computational Proof of Theorems 4.21 and 4.25

First, we give another proof of Theorem 4.21. By [B11, eq. (2.16.25)], we get

\[ P \frac{1}{2} ||Y^N, Y^TX||^2 P = -\frac{1}{8} \text{Tr}^p [C^{t,p}]. \]

By the second equation in (4.68), by (4.71), (4.113), (4.120), (4.122), and (4.136), we obtain

\[ P (\gamma_\vartheta - \beta_\vartheta \alpha_\vartheta^{-1} \beta_\vartheta) P = -\frac{x^2}{2} \Delta X,H - \delta_\vartheta - \frac{x}{8} \text{Tr}^p [C^{t,p}]. \]

By (4.121), we get

\[ \delta_\vartheta = \frac{1}{8} \text{Tr}^p [C^{t,p}] + \frac{1}{2} \text{Tr}^t [C^{t,t}] + x^2 \sum_{i=m+1}^{m+n} \hat{c} (\text{ad} (e_i) |_{TX}) \rho^F (e_i). \]

By (4.137), (4.138), we obtain

\[ P (\gamma_\vartheta - \beta_\vartheta \alpha_\vartheta^{-1} \beta_\vartheta) P = \frac{x^2}{2} \left( -\Delta X,H + \frac{1}{4} \text{Tr}^p [C^{t,p}] \right) \]

\[ + 2 \sum_{i=m+1}^{m+n} \hat{c} (\text{ad} (e_i) |_{TX}) \rho^F (e_i) \left( \frac{1}{48} \text{Tr}^t [C^{t,t}] + \frac{1}{2} C^{t,E} \right). \]

By (4.8), (4.139), we get (4.72). This completes the second proof of Theorem 4.21.
Now, we give another proof of Theorem 4.25. By Theorem 4.21, we only need to show the coincidence of the $d\vartheta$ components in (4.95). We use (4.94). We have the trivial identity

\[(4.140)\quad P_c \left( [Y^N, Y^{TX}] \right) P = 0.\]

By (4.11), (4.94), to establish (4.95), we only need to show that

\[(4.141)\quad P_\beta \alpha \frac{d\vartheta}{\sqrt{2}} \left( \hat{c} \left( Y^{TX} \right) + \frac{\sin (\vartheta)}{\cos^{1/2} (\vartheta)} i \mathcal{E}^N \right) P = \frac{d\vartheta}{\sqrt{2}} \cos (\vartheta) \hat{D}^N.\]

In (4.141), using (4.68), (4.128), and (4.130), an easy computation shows that we can replace $\beta$ by $\cos (\vartheta) \nabla \mathcal{C}_\infty \left( TX \oplus N, \hat{\pi}^* \left( \Lambda \cdot \left( T^* X \oplus N \right) \otimes S^{TX} \otimes F \right) \right)$, which combined with (4.122) gives (4.141). This completes the second proof of Theorem 4.25.

4.12. The scaling of the invariant form $B$. Given $t > 0$, we denote with an extra index $t$ the objects considered above that are associated with the form $B/t$ over $g$. We will establish an analogue of the results in [B11, section 2.14]. By (4.67), (4.80), as in [B11, eq. (2.14.3)], we get

\[(4.142)\quad K \sqrt{t} t^{-N^A \left( T^* X \otimes N^* \right) / 2} \mathcal{D}_{b, \vartheta, t}^{-N^A (T^* X \otimes N^*) / 2} K^{-1} = \sqrt{t} \mathcal{D}_{b, \vartheta}^{-N^A (T^* X \otimes N^*) / 2} = \nabla_{TX} \mathcal{H} \cdot \mathcal{D}_{b, \vartheta} \mathcal{H} \cdot \mathcal{D}_{b, \vartheta}.\]

By (4.142), we deduce that

\[(4.143)\quad K \sqrt{t} t^{-N^A \left( T^* X \otimes N^* \right) / 2} \mathcal{L}_{b, \vartheta, t}^{-N^A (T^* X \otimes N^*) / 2} K^{-1} = \sqrt{t} \mathcal{L}_{b, \vartheta}^{-N^A \left( T^* X \otimes N^* \right) / 2} = \mathcal{L}_{b, \vartheta} \cdot \mathcal{L}_{b, \vartheta}.\]

The above identities remain valid when replacing $\mathcal{D}_{b, \vartheta}, \nabla \mathcal{H}$ by $\mathcal{D}_{b, \vartheta}^X, \nabla^H$. 

5. A closed 1-form on $\mathbb{R}^*_+ \times [0, \frac{\pi}{2}]$

In this section, when $m = \dim \mathfrak{p}$ is odd, using hypoelliptic orbital integrals, we define a closed 1-form $\mathbf{b}$ on $\mathbb{R}^*_+ \times [0, \frac{\pi}{2}]$. This 1-form will play an important role in establishing our main result.

This section is organized as follows. In subsection 5.1, we define in our context the proper traces and supertraces.

In subsection 5.2, if $\gamma \in G$ is semisimple, we introduce orbital integrals associated with the family of elliptic operators $L_{\theta}^{X}$ considered in subsection 4.1. We show that we can limit ourselves to the case where $\gamma$ is nonelliptic. Also we introduce a 1-form $\mathbf{a}$ on $[0, \frac{\pi}{2}]$.

Finally, in subsection 5.3, if $\gamma \in G$ is semisimple, we introduce orbital integrals associated with the family of Dirac operators $L_{\theta}^{\mathfrak{v}}$ considered in subsection 4.7, and we construct the closed 1-form $\mathbf{b}$ on $\mathbb{R}^*_+ \times [0, \frac{\pi}{2}]$.

In this section, we make the same assumptions as in sections 3 and 4, and we use the corresponding notation. In particular, $G$ is still assumed to be simply connected.

5.1. Traces and supertraces. If $m = \dim \mathfrak{p}$ is even, $S^{\mathfrak{v}}$ is $\mathbb{Z}_2$-graded, and $\widehat{c}(\mathfrak{p}) \otimes_{\mathbb{R}} C = \text{End}(S^{\mathfrak{v}})$ is equipped with a corresponding supertrace. When combined with the usual trace on $\text{End}(E)$, we get a supertrace $\text{Tr}_{\mathfrak{s}} : \Lambda^{c}(\mathbb{R}^{2*}) \otimes \widehat{c}(\mathfrak{p}) \otimes \text{End}(E) \rightarrow \Lambda^{c}(\mathbb{R}^{2*}) \otimes_{\mathbb{R}} C$, with the convention that if $\eta \in \Lambda^{c}(\mathbb{R}^{2*})$, $a \in \Lambda^{c}(\mathbb{R}^{2*}) \otimes \widehat{c}(\mathfrak{p}) \otimes \text{End}(E)$,

$$\text{Tr}_{\mathfrak{s}}[\eta a] = \eta \text{Tr}_{\mathfrak{s}}[a].$$

Since $\Lambda^{c}(\mathfrak{g}^{*})$ is $\mathbb{Z}_2$-graded, $\text{End}(\Lambda^{c}(\mathfrak{g}^{*})) = c(\mathfrak{g}) \otimes \widehat{c}(\mathfrak{g})$ is equipped with a supertrace $\text{Tr}_{c}$, so that $\Lambda^{c}(\mathbb{R}^{2*}) \otimes \text{End}(\Lambda^{c}(\mathfrak{g}^{*})) \otimes \text{End}(E)$ is equipped with a supertrace $\text{Tr}_{c}$ with values in $\Lambda^{c}(\mathbb{R}^{2*}) \otimes_{\mathbb{R}} C$, that vanishes on supercommutators. When quotienting by $K$, it descends to a supertrace $\text{Tr}_{c}$ from

$$\Lambda^{c}(\mathbb{R}^{2*}) \otimes \text{End}(\Lambda^{c}(T^{*}X \oplus N^{*})) \otimes \widehat{c}(\mathfrak{p}) \otimes \text{End}(E)$$

into $\Lambda^{c}(\mathbb{R}^{2*}) \otimes_{\mathbb{R}} C$.

If $m = \dim \mathfrak{p}$ is odd, then $S^{\mathfrak{v}}$ is not $\mathbb{Z}_2$-graded, and by (3.37), $\widehat{c}(\mathfrak{p}) \otimes_{\mathbb{R}} C = \text{End}(S^{\mathfrak{v}}) \otimes \text{End}(S^{\mathfrak{v}})$. Instead of adopting the $\text{Tr}_{c}$ formalism of Quillen that was described in subsection 2.1, we will instead exploit the $\mathbb{Z}_2$-grading of $\widehat{c}(\mathfrak{p})$. We denote by $\text{Tr}^{\text{odd}}$ the linear map that vanishes on $\widehat{c}^{\text{even}}(\mathfrak{p})$ and coincides with $\text{Tr}^{S^{\mathfrak{v}}}$ on $c^{\text{odd}}(\mathfrak{p})$. Then $\text{Tr}^{\text{odd}}$ vanishes on supercommutators in $\widehat{c}(\mathfrak{p})$. We extend $\text{Tr}^{\text{odd}}$ to a map from $\Lambda^{c}(\mathbb{R}^{2*}) \otimes \widehat{c}(\mathfrak{p}) \otimes \text{End}(E)$ into $\Lambda^{c}(\mathbb{R}^{2*}) \otimes_{\mathbb{R}} C$, with the convention that if $\eta \in \Lambda^{c}(\mathbb{R}^{2*})$, $a \in \widehat{c}(\mathfrak{p}) \otimes_{\mathbb{R}} \text{End}(E)$, then

$$\text{Tr}^{\text{odd}}[\eta a] = \eta \text{Tr}^{\text{odd}}[a].$$

When combining $\text{Tr}^{\text{odd}}$ with the supertrace on $\text{End}(\Lambda^{c}(\mathfrak{g}^{*}))$, we get a linear map $\text{Tr}_{\mathfrak{s}}^{\text{odd}} : \Lambda^{c}(\mathbb{R}^{2*}) \otimes \text{End}(\Lambda^{c}(\mathfrak{g}^{*})) \otimes \widehat{c}(\mathfrak{p}) \otimes \text{End}(E) \rightarrow \Lambda^{c}(\mathbb{R}^{2*}) \otimes_{\mathbb{R}} C$. It still vanishes on supercommutators. It vanishes on the even part of $\text{End}(\Lambda^{c}(\mathfrak{g}^{*})) \otimes \widehat{c}(\mathfrak{p}) \otimes \text{End}(E)$ and is an ordinary supertrace on the odd part. If $\beta \in \left(\Lambda^{c}(\mathbb{R}^{2*}) \otimes \text{End}(\Lambda^{c}(\mathfrak{g}^{*})) \otimes \widehat{c}(\mathfrak{p}) \otimes \text{End}(E)\right)^{\text{even}}$,
then $\text{Tr}_{x,\text{odd}}[\beta]$ is a 1-form. When quotienting by $K$, this map descend to a map $\text{Tr}_{x,\text{odd}}$ from

$$\Lambda^1 \left( \mathbb{R}^{2*} \right) \otimes \text{End} \left( \Lambda^1 \left( T^*X \oplus N^* \right) \right) \otimes \text{c} \left( T^X \right) \otimes \text{End} \left( F \right)$$

with values in $\Lambda^1 \left( \mathbb{R}^{2*} \right) \otimes \mathbb{R} C$.

From now on, and in the whole paper, we assume that $m$ is odd.

5.2. Elliptic orbital integrals and the 1-form $a$. Let $\gamma \in G$. Let $d_\gamma : X \to \mathbb{R}_+$ be the displacement function associated with $\gamma$, i.e., if $x \in X$,

$$d_\gamma \left( x \right) = d \left( x, \gamma x \right).$$

Following [E96, 2.19.21], $\gamma$ is said to be semisimple if $d_\gamma$ reaches its minimum value on $X$. Semisimplicity is a property of the conjugacy class of $\gamma$ in $G$.

By [B11, Theorem 3.1.2], $\gamma$ is semisimple if and only if after conjugation, we can write $\gamma$ in the form

$$\gamma = e^a k^{-1}, \quad a \in \mathfrak{p},$$

$$k \in K, \quad \text{Ad} \left( k \right) a = a,$$

the factorization in (5.4) being unique. By [B11, Theorem 3.1.2],

$$\inf_{x \in X} d_\gamma \left( x \right) = \left| a \right|.$$

Recall that the vector space $\mathcal{H}$ was defined in Definition 3.9. The left action of $\gamma$ on $G$ descends to an action on $\mathcal{H}$.

Using the same arguments as in [B11, section 4.4], for $t > 0$, we can define the elliptic orbital integrals $\text{Tr}^{[\gamma]} \left[ D^X \exp \left( -tD^X,2/2 \right) \right]$. These integrals only depend on the conjugacy class of $\gamma$ in $G$.

Recall that since $\gamma$ is semisimple, $\theta \gamma$ is also semisimple.

**Proposition 5.1.** For $t > 0$, the following identity holds:

$$\text{Tr}^{[\theta \gamma]} \left[ D^X \exp \left( -tD^X,2/2 \right) \right] = -\text{Tr}^{[\gamma]} \left[ D^X \exp \left( -tD^X,2/2 \right) \right].$$

In particular, if $\gamma$ is elliptic, i.e., if $a = 0$, for $t > 0$, then

$$\text{Tr}^{[\gamma]} \left[ D^X \exp \left( -tD^X,2/2 \right) \right] = 0.$$

**Proof.** As we saw in subsection 3.7, $\theta_{\pm}$ acts on $C^\infty \left( X, S^{TX} \otimes F \right)$. Moreover, we have the identity of morphisms of $C^\infty \left( X, S^{TX} \otimes F \right)$,

$$\theta_{\pm} \gamma \theta_{\pm}^{-1} = \theta \left( \gamma \right).$$

Combining (3.47) and (5.8), we get

$$\theta_{\pm} \left[ \gamma D^X \exp \left( -tD^X,2/2 \right) \right] \theta_{\pm}^{-1} = -\theta \left( \gamma \right) D^X \exp \left( -tD^X,2/2 \right).$$

By (5.9), we get (5.6). If $\gamma$ is elliptic, after conjugation, we may assume that $\gamma \in K$, so that $\theta \gamma = \gamma$, and so from (5.6), we get (5.7). The proof of our proposition is completed.

In the sequel, we may and we will assume that $\gamma$ is nonelliptic, i.e., $a \neq 0$.

Recall that $T^X$ was defined in (4.10). A formula for $T^X$ is also given in (4.11).

**Definition 5.2.** Let $a$ be the 1-form on $\left[ 0, \frac{\pi}{2} \right]$, 

$$a = \text{Tr}^{[\gamma],\text{odd}} \left[ \exp \left( -T^X \right) \right].$$
By proceeding as in [B11, section 4.4], the orbital integral in (5.10) is well defined. Like any other 1-form on $[0, \frac{\pi}{2}]$, the 1-form $a$ is closed.

**Proposition 5.3.** The following identity holds:

$$a = -\Tr^t_{\gamma} \left[ \frac{\cos(\theta)}{\sqrt{2}} \tilde{D}^X \exp \left( -L_{b,\theta}^X \right) \right] d\theta.$$  

**Proof.** To establish (5.11), we use (4.11), (5.10), the fact that $\Tr_{\mathrm{odd}}$ vanishes on supercommutators, and also the arguments of [B11, chapter 4], and especially [B11, Theorem 4.3.4]. The easy details are left to the reader. \qed

5.3. Hypoelliptic orbital integrals and the 1-form $b$. Note that $L^X, \mathcal{L}^X, L^{X'}$ were defined in Definition 4.22, and are given by (4.88). They are both even in the proper algebra. It is now crucial to use the formalism of the second half of subsection 5.1. By [B11, chapter 4], the hypoelliptic orbital integrals

$$\Tr_s^{[\gamma], \mathrm{odd}} \left[ \exp \left( -L^X \right) \right], \Tr_s^{[\gamma], \mathrm{odd}} \left[ \exp \left( -L^X \right) \right], \Tr_s^{[\gamma], \mathrm{odd}} \left[ \exp \left( -L^{X'} \right) \right]$$

are well-defined. These are 1-forms on $R_+^* \times [0, \frac{\pi}{2}]$. Using the same arguments as in [B11, Theorem 4.3.4], and also (4.88), we get

$$\Tr_s^{[\gamma], \mathrm{odd}} \left[ \exp \left( -L^X \right) \right] = -\Tr_s^{[\gamma]} \left[ \frac{1}{\sqrt{2}} \nabla^H \mathcal{D}_{b,\theta}^X \exp \left( -L_{b,\theta}^X \right) \right],$$

(5.12) $$\Tr_s^{[\gamma], \mathrm{odd}} \left[ \exp \left( -L^{X'} \right) \right] = -\Tr_s^{[\gamma]} \left[ \frac{1}{\sqrt{2}} \nabla^H \mathcal{D}_{b,\theta}^{X'} \exp \left( -L_{b,\theta}^{X'} \right) \right].$$

In the right-hand side of (5.12), we have eliminated the mention odd, because the morphism that appears inside is indeed odd. Equivalently, the supertrace that appears in the right-hand side is an ordinary supertrace associated with the $Z_2$-grading of $\Lambda^\wedge (T^*X \oplus N^*)$.

**Theorem 5.4.** We have the identity of closed 1-forms on $R_+^* \times [0, \frac{\pi}{2}]$,

$$\Tr_s^{[\gamma], \mathrm{odd}} \left[ \exp \left( -L^X \right) \right] = \Tr_s^{[\gamma], \mathrm{odd}} \left[ \exp \left( -\mathcal{L}^X \right) \right] = \Tr_s^{[\gamma], \mathrm{odd}} \left[ \exp \left( -L^{X'} \right) \right].$$

**Proof.** Using equation (4.89) in Proposition 4.23, and proceeding as in [B11, proof of Theorem 4.6.1] and in (2.35), we get

$$dR_+^* \times [0, \frac{\pi}{2}] \left[ \Tr_s^{[\gamma], \mathrm{odd}} \left[ \exp \left( -L^X \right) \right] = \Tr_s^{[\gamma], \mathrm{odd}} \left[ \left[ B^X, \exp \left( -L^X \right) \right] \right] = 0,$$

i.e., the form $\Tr_s^{[\gamma], \mathrm{odd}} \left[ \exp \left( -L^X \right) \right]$ is closed. The same argument shows that the other forms in (5.13) are also closed. By (4.86), the first two forms in (5.13) are equal.

Set

$$\tilde{\nabla}^H = \tilde{R}^{-1}_\theta \nabla^H \tilde{R}_\theta, \quad \tilde{B}^X = \tilde{\nabla}^H + \frac{\mathcal{D}_{b,\theta}^{X'}}{\sqrt{2}}.$$

By (5.59), (4.82), and (5.15), we get

$$\tilde{B}^X = \tilde{R}^{-1}_\theta B^X \tilde{R}_\theta.$$
Set

\[ \widetilde{M}^X = -\frac{1}{2} \widehat{D}_{\nu}^{X, 2} + \widetilde{B}^{X, 2}. \]

By (4.85), (5.16), and (5.17), we get

\[ \widetilde{M}^X = \widehat{R}_0^{-1} L^X \widehat{R}_0. \]

By (5.18), we get the identity of closed 1-forms

\[ \text{Tr}_{\gamma}^{[\nu], \text{odd}} \left[ \exp \left( -L^X \right) \right] = \text{Tr}_{\gamma}^{[\nu], \text{odd}} \left[ \exp \left( -\widetilde{M}^X \right) \right]. \]

To establish the last identity in (5.13), we need to show that in (5.19), we can as well replace \( \nabla^H \) by \( \nabla^{H'} \) without changing the corresponding 1-form. We introduce an extra interpolation parameter \( \ell \in [0, 1] \) to interpolate linearly between \( \nabla^H \) and \( \nabla^{H'} \) through connections \( \nabla^{H'} \ell \) on \( \mathcal{H} \). Let \( \nabla^H = d\ell \frac{\partial}{\partial \ell} + \nabla^{H'} \ell \) denote the corresponding connection over \( \mathbb{R}^*_+ \times [0, \frac{\pi}{2}] \times [0, 1] \). By the same construction as before, we get an odd closed form \( \beta \) on \( \mathbb{R}^*_+ \times [0, \frac{\pi}{2}] \times [0, 1] \). Since \( \mathcal{D}^{X'}_{\nu, \theta} \) does not depend on \( \ell \), the component \( \beta^{(1)} \) of total degree 1 of \( \beta \) does not contain \( d\ell \). Since \( \beta^{(1)} \) is closed, as a 1-form on \( \mathbb{R}^*_+ \times [0, \frac{\pi}{2}] \), \( \beta^{(1)} \) does not depend on \( \ell \). Combining (5.19) with this result, we get the last identity in (5.13). The proof of our proposition is completed. \( \square \)

In the sequel, we denote by \( \mathfrak{b} \) the 1-form in (5.13).
6. A conserved quantity

In this section, we show that the integral of \( b \) on \([0, \frac{\pi}{2}]\) does not depend on \( b > 0 \) and coincides with the integral of \( a \).

This section is organized as follows. In subsection 6.1, we recall known estimates on the elliptic heat kernel over \( X \).

In subsection 6.2, we give without proof uniform estimates on the hypoelliptic heat kernel for \( L^X|_{dB=0} \) when \( b > 0 \) remains uniformly bounded, and also a convergence result of the hypoelliptic heat kernels to their elliptic counterpart when \( b \to 0 \). The proof of these results is deferred to section 11.

In subsection 6.3, we give a formula expressing \( \int_{\vartheta \leq \varphi \leq \pi/2} a \) in terms of the orbital integrals \( \text{Tr}_s[\gamma] D^X \exp(-sD^X,2/2) \).

In subsection 6.4, we state the conservation result that was mentioned before.

Finally, in subsection 6.5, we prove the identity of subsection 6.4 by integrating the closed 1-form \( b \) on a suitable contour, and by using the estimates of subsection 6.2.

We make the same assumptions as in section 5. In particular, we assume \( m \) to be odd, and also that \( \gamma \) is nonelliptic, i.e., \( a \neq 0 \).

6.1. Uniform estimates on the elliptic heat kernel. Recall that the operator \( T^X \) was defined in Definition 4.3. By Proposition 4.2 and by (4.11), for \( \vartheta \in [0, \frac{\pi}{2}] \), \( T^X \) is a second order elliptic operator.

**Definition 6.1.** For \( \vartheta \in [0, \frac{\pi}{2}] \), \( t > 0 \), let \( p^X_{\vartheta,t}(x,x') \) be the smooth kernel associated with the operator \( \exp(-tT^X) \) with respect to the volume \( dx' \). We use the notation

(6.1) \[ p^X_{\vartheta} = p^X_{\vartheta,1}. \]

Recall that \( d \) is the Riemannian distance on \( X \). By (4.8), (4.9), and (4.11), classical estimates on elliptic heat kernels show that given \( 0 < \epsilon \leq \frac{1}{2} < +\infty \), there exist \( C > 0, C' > 0 \) such that for \( \vartheta \in [0, \frac{\pi}{2}] \), \( \epsilon \leq t \leq M, x, x' \in X \), then

(6.2) \[ |p^X_{\vartheta,t}(x,x')| \leq C \cos^{m-1}(\vartheta) \exp\left(-C'\frac{d^2(x,x')}{\cos^2(\vartheta)}\right). \]

The uniformity of the estimate (6.2) on \( X \) comes from the fact that \( X \) is a symmetric space.

6.2. Uniform estimates on the hypoelliptic heat kernel for \( b \) small. Recall that the projector \( P \) was defined in Definition 4.26. Also \( L^X \) was defined in (4.86), and is given by (4.88).

**Definition 6.2.** For \( b > 0, \vartheta \in [0, \frac{\pi}{2}] \), \( t > 0 \), let \( q^X_{b,\vartheta,t}((x,Y),(x',Y')) \) be the smooth kernel associated with the operator \( \exp(-tL^X|_{dB=0}) \) with respect to the volume \( dx'dY' \). We use the notation

(6.3) \[ q^X_{b,\vartheta} = q^X_{b,\vartheta,1}. \]

Set

(6.4) \[ q^X_{b,\vartheta,t}((x,Y),(x',Y')) = P p^X_{\vartheta,t}(x,x') \pi^{-(m+n)/2} \exp\left(-\frac{1}{2} \left(|Y|^2 + |Y'|^2\right)\right) \cdot P. \]
For the proper functional analytic setting showing that the heat kernels $q^{X}_{b,\vartheta,t}$ are well defined, we refer to [B11, chapter 11].

Now we state an extension of [B11, Theorem 4.5.2].

**Theorem 6.3.** Given $0 < \epsilon \leq M < +\infty$, there exist $C > 0, C' > 0, k \in \mathbb{N}$ such that for $0 < b \leq M, \vartheta \in \left[0, \frac{\pi}{2}\right], \epsilon \leq t \leq M$, $(x, Y), (x', Y') \in \hat{X}$, then

$$
(6.5) \quad |q^{X}_{b,\vartheta,t}((x, Y), (x', Y'))| \leq C \cos^{-k}(\vartheta) \exp \left(-C' \left(\frac{d^2(x, x')}{\cos^2(\vartheta)} + |Y^TX|^2 + |Y'^TX|^2 \right) \right).
$$

Moreover, as $b \to 0$,

$$
(6.6) \quad q^{X}_{b,\vartheta,t}((x, Y), (x', Y')) \to q^{X}_{0,\vartheta,t}((x, Y), (x', Y')).
$$

**Proof.** Our theorem will be established in sections 10 and 11. \(\square\)

6.3. **Evaluation of the integral $\int_{0 \leq \vartheta \leq \frac{\pi}{2}} a$.** Recall that the displacement function $d_{\gamma}(x)$ was introduced in (5.3).

By (5.5), (6.2), there exist $C > 0, C' > 0$ such that

$$
(6.7) \quad |p^{X}_{\vartheta}(x, \gamma x)| \leq C \cos^{-m-1}(\vartheta) \exp \left(-C' \left(\frac{|a|^2}{\cos^2(\vartheta)} + d^2_{\gamma}(x) \right) \right).
$$

The 1-form $a$ on $\left[0, \frac{\pi}{2}\right]$ was defined in Definition 5.2. By proceeding as in [B11, Theorem 4.2.1], using (5.10), (6.7) and the fact that $|a| > 0$, for $\vartheta \in \left[0, \frac{\pi}{2}\right]$, we get

$$
(6.8) \quad |a| \leq C \exp \left(-C' \frac{|a|^2}{\cos^2(\vartheta)} \right).
$$

By (6.8), the integral $\int_{0 \leq \vartheta \leq \frac{\pi}{2}} a$ is well defined.

By the same arguments as before, for $s \in [0, 1]$, we get

$$
(6.9) \quad \left| \text{Tr}^{\gamma} \left[ \frac{D^X}{\sqrt{2}} \exp \left(-s \frac{D^X \cdot 2}{2} \right) \right] \right| \leq C \exp \left(-C' \frac{|a|^2}{s} \right).
$$

Therefore the integral $\int_{0}^{1} \frac{1}{2} \text{Tr}^{\gamma} \left[ \frac{D^X}{\sqrt{2}} \exp \left(-s \frac{D^X \cdot 2}{2} \right) \right] \frac{1}{\sqrt{1-s}} ds$ is well defined.

**Proposition 6.4.** The following identity holds:

$$
(6.10) \quad \int_{0 \leq \vartheta \leq \frac{\pi}{2}} a = -i \exp \left(-\frac{1}{48} \text{Tr}^{t} [C^{t,E}] - \frac{1}{2} C^{t,E} \right) \int_{0}^{1} \frac{1}{2} \text{Tr}^{\gamma} \left[ \frac{D^X}{\sqrt{2}} \exp \left(-s \frac{D^X \cdot 2}{2} \right) \right] \frac{1}{\sqrt{1-s}} ds.
$$

**Proof.** Using (3.46), the first identity in (4.8), and (5.11), and making the change of variables $s = \cos^2(\vartheta)$ in the integral of $a$, we get (6.10). \(\square\)

**Remark 6.5.** The right-hand side of equation (6.10) should be compared with equation (2.43) for $F^{t}_{C,t}(D)$. 
By (5.5) and by equation (6.5) in Theorem 6.3, given $M > 0$, there exist $C > 0, C' > 0$ such that if $0 < b \leq M, \vartheta \in [0, \pi/2]$, $(x, Y) \in \hat{X}$, we get

\begin{equation}
\left| \bar{\Pi}_{0, \vartheta}^{X} ((x, Y), \gamma (x, Y)) \right|
\leq C \cos^{-k} (\vartheta) \exp \left( -C' \left( \frac{|a|^2}{\cos^2 (\vartheta)} + d^2 (x) + |Y^T X|^2 + \cos (\vartheta) |Y^N|^2 \right) \right).
\end{equation}

Using (6.11) and proceeding as in [B11, section 4.3], for $0 < b \leq M$, we get

\begin{equation}
\left| \text{Tr}_{\pi}[^{\gamma}] \text{odd} \left[ \exp \left( -L^X |db = 0 \right) \right] \right| \leq C \cos^{-k-n/2} (\vartheta) \exp \left( -C' \frac{|a|^2}{\cos^2 (\vartheta)} \right).
\end{equation}

By (6.12), we deduce that for $B > 0$, the integral $\int_{0 \leq \vartheta \leq \pi/2} b$ is well defined.

6.4. A preserved quantity. Now we state the following key result, that replaces for us the conservation result of [B11, Theorem 4.6.1] in the context of more classical orbital integrals not involving the Dirac operator $D^X$.

**Theorem 6.6.** For any $B > 0$, the following identity holds:

\begin{equation}
\int_{0 \leq \vartheta \leq \pi} a = \int_{0 \leq \vartheta \leq \pi} b.
\end{equation}

**Proof.** Our theorem will be established in subsection 6.5. \hfill \Box

6.5. A proof of Theorem 6.6. Take $B_0, B, \epsilon$ such that $0 < B_0 < B, 0 < \epsilon < \pi/2$. Let $\Gamma = \Gamma_{B_0, B, \epsilon}$ be the oriented contour in $R^*_+ \times [0, \pi/2]$ shown in Figure 6.1. The contour $\Gamma$ is made of oriented segments $\Gamma_i, 1 \leq i \leq 4$.

By Theorem 5.4, the form $b$ is closed, and so

\begin{equation}
\int_{\Gamma} b = 0.
\end{equation}

For $1 \leq j \leq 4$, set

\begin{equation}
\Gamma_{i}^0 = \int_{\Gamma_{i}} b.
\end{equation}

By (6.14), we get

\begin{equation}
\sum_{i=1}^{4} \Gamma_{i}^0 = 0.
\end{equation}

We fix $B > 0$. We will study the terms $\Gamma_{i}^0, 1 \leq i \leq 4$ by making in succession $\epsilon \to 0, B_0 \to 0$.

1) The term $\Gamma_{1}^0$

We have the identity

\begin{equation}
\Gamma_{1}^0 = \int_{\Gamma_{1}} b.
\end{equation}

We can rewrite (6.17) in the form

\begin{equation}
\Gamma_{1}^0 = - \int_{0 \leq \vartheta \leq \pi/2 - \epsilon} \text{Tr}_{\pi}[^{\gamma}] \text{odd} \left[ \exp \left( -L^X |db = 0 \right) \right].
\end{equation}
Figure 6.1.

(1) \( \epsilon \to 0 \) By (6.12), as \( \epsilon \to 0 \), we get

\[
I_1^0 \to I_1^1 = -\int_{b=0}^{B_0} \text{Tr}_{s}[\gamma_{1}]^{\text{odd}} \left[ \exp \left( -L^X_{\mid db=0} \right) \right].
\]

(2) \( B_0 \to 0 \) We have the fundamental result.

**Theorem 6.7.** As \( B_0 \to 0 \),

\[
I_1^1 \to I_2^1 = -\int_{0 \leq \theta \leq \frac{\pi}{2}} a.
\]

**Proof.** By (6.6), (6.11), and by proceeding as in [B11, proof of Theorem 4.6.1], we obtain the pointwise convergence of 1-forms on \([0, \frac{\pi}{2}]\),

\[
\text{Tr}_{s}[\gamma_{1}]^{\text{odd}} \left[ \exp \left( -L^X_{\mid db=0} \right) \right] \to \text{Tr}[\gamma_{1}]^{\text{odd}} \left[ \exp \left( -T^X_{\mid db=0} \right) \right].
\]

Using the uniform bounds in (6.12), (6.21), and dominated convergence, we get (6.20). \( \square \)

2) The term \( I_2^0 \)

By (4.81), (5.12), we get

\[
I_2^0 = \frac{1}{\sqrt{2}} \int_{B_0} B_{\text{Tr}_{s}[\gamma_{1}]^{\text{odd}}} \left[ \left( \text{bic} \left( \left[ Y^N, Y^TX \right] \right) + \epsilon^T X + i\epsilon^N \right) \exp \left( -L^X_b \right) \right] \frac{2\,db}{b^2}.
\]

By equation (3.78), the term \(-\frac{i}{2}\epsilon \left( \text{ad} \left( Y^N \right) \right)_{TX}\) is the only term in \( L^N_b \) that contains Clifford variables in \( c \left( TX \right) \), and it lies in \( c^{\text{even}} \left( TX \right) \). Also as we saw in...
subsection 3.6, $K$ maps to $e^{even}(\mathfrak{p})$ via the spin representation. Therefore, the integrand in (6.22) vanishes identically, so that

(6.23) \[ I_2^0 = 0. \]

A related argument is that in the right-hand side of (6.22), only odd endomorphisms of $\Lambda (T^*X \oplus N^*) \otimes S^{TX} \otimes F$ appear, so that the corresponding supertrace vanishes.

3) The term $I_3^0$

By definition,

(6.24) \[ I_3^0 = \int_{\Gamma_3} b. \]

We can rewrite (6.24) in the form

(6.25) \[ I_3^0 = \int_{0 \leq \theta \leq \frac{\pi}{2} - \epsilon} \text{Tr}^{\gamma}_{odd} \left[ \exp \left( -\mathcal{L}_X \Big|_{db = 0} \right) \right]. \]

(1) $\epsilon \to 0$ By (6.12), as $\epsilon \to 0$,

(6.26) \[ I_3^0 \to I_3^1 = \int_{0 \leq \theta \leq \frac{\pi}{2}} \text{Tr}^{\gamma}_{odd} \left[ \exp \left( -\mathcal{L}_X \Big|_{db = 0} \right) \right]. \]

(2) As $B_0 \to 0$, $I_3^1$ remains constant and equal to $I_3^2$.

4) The term $I_4^0$

By definition, we get

(6.27) \[ I_4^0 = \int_{\Gamma_4} b. \]

**Proposition 6.8.** As $\epsilon \to 0$,

(6.28) \[ I_4^0 \to 0. \]

**Proof.** By (4.81), (5.12), we get

(6.29) \[ I_4^0 = -\frac{1}{\sqrt{2}} \int_{B_0} \text{Tr}_2 \left[ \left( b \sin^{1/2} (\epsilon) i c \left( [Y^N, Y^{TX}] \right) + E^{TX} + \sin^{1/2} (\epsilon) i E^N \right) \exp \left( -\mathcal{L}_X \Big|_{db = 0} \right) \right] \frac{2 db}{b^2}. \]

By making $d\theta = 0$ in (6.11) and using (6.29), we find that given $0 < B_0 < B < +\infty$, there exist $C > 0, C' > 0$ such that

(6.30) \[ \left| I_4^0 \right| \leq C \exp \left( -C' \frac{|a|^2}{\sin^2 (\epsilon)} \right), \]

which gives (6.28). \[ \square \]

We are now ready to prove Theorem 6.6. By taking the limit of (6.16) as $\epsilon \to 0$, $B_0 \to 0$, we get

(6.31) \[ \sum_{i=1}^{4} I_i^2 = 0, \]

which by (6.20), (6.23), (6.26), and (6.28) is just (6.13).
7. A geometric formula for $\int_{0 \leq \vartheta \leq \frac{\pi}{2}} a$

In this section, by making $B \to +\infty$ in Theorem 6.6, we give an explicit geometric formula for the elliptic orbital integral $\int_{0 \leq \vartheta \leq \frac{\pi}{2}} a$. This formula is in some sense the main result of this paper. It will be worked out in more detail in section 8, in order to make the proper comparison with the results of Moscovici-Stanton [MoSt89].

The proof of our main result relies on the results of [B11], and also on uniform estimates on the hypoelliptic heat kernels for $b > 0$ large, that will be established in section 12.

The structure of this section is strictly similar to the structure of [B11, chapter 9], where corresponding results are established for standard orbital integrals. The main new ingredients with respect to [B11] are the uniform upper bounds on certain integrands when $\vartheta \in [0, \frac{\pi}{2}]$.

This section is organized as follows. In subsection 7.1, we recall the results of [B11] that identify the minimizing set $X(\gamma)$ for the displacement function $d_\gamma$.

In subsection 7.2, we state our geometric formula for $\int_{0 \leq \vartheta \leq \frac{\pi}{2}} a$ in terms of objects previously obtained in [B11]. Among these objects, there is an important function $J_\gamma$ that will play an essential role in section 8. The remainder of this section is devoted to the proof of this result.

In subsection 7.3, we give a formula for the limit as $b \to +\infty$ of the 1-form $\text{Tr}_{\gamma, \text{odd}} \left[ \exp \left( -L_{X|_{db=0}} \right) \right]$. Our main result in subsection 7.2 is a trivial consequence of this convergence result. The subsections that follow are devoted to its proof.

In subsection 7.4, we give estimates for the smooth kernel for $\exp \left( -L_{X|_{db=0}} \right)$ in the range $b \geq 1$, $\vartheta \in [0, \frac{\pi}{2}]$ away from a submanifold of $\tilde{X}$ that fibres over $X(\gamma)$. The proof of these estimates is deferred to section 12.

In subsection 7.5, if $L_{X|_{db=0}}$ is a rescaled version of $L_{X|_{db=0}}$, we show that the orbital integral $\text{Tr}_{\gamma, \text{odd}} \left[ \exp \left( -L_{X|_{db=0}} \right) \right]$ localizes near a submanifold over $X(\gamma)$, and we suitably rescale coordinates on $\tilde{X}$.

In subsection 7.6, as in [B11, section 9.5], we introduce a conjugation on certain Clifford variables, which is equivalent to a suitable Getzler rescaling on the matrix part of the operator $L_{X|_{db=0}}$. From $L_{X|_{db=0}}$, we obtain an operator $L_{X|_{db=0}}$.

In subsection 7.7, we give a result on the limit as $b \to +\infty$ of a local supertrace of a rescaled heat kernel over a neighbourhood of $\tilde{\pi}^{-1}X(\gamma)$. The proof is deferred to subsections 7.9–7.12.

In subsection 7.8, we complete the proof of the result that was stated in subsection 7.3.

In subsection 7.9, we make the translation $Y^{TX} \to a^{TX} + Y^{TX}$ on the operator $\mathfrak{L}^{X|_{\vartheta}}_{a,b,0}$, and we obtain a new operator $\mathfrak{L}^{X|_{\vartheta}}_{a,b,0}$.

In subsection 7.10, as in [B11, section 9.9], we choose a coordinate system on $\tilde{X}$ based at $x_0 = p1$, and we trivialize our vector bundles. We obtain this way an operator $P^{X|_{\vartheta}}_{X|_{\vartheta}}$.

In subsection 7.11, we show that as $b \to +\infty$, $P^{X|_{\vartheta}}_{X|_{\vartheta}}$ converges in the proper sense to an operator $P^{X|_{\vartheta}}_{X|_{\vartheta}}$. The wonderful fact is that the dependence of this operator on $\vartheta$ is very mild, and that it differs very little from the operator $P^{X|_{\vartheta}}_{X|_{\vartheta}}$ already considered in [B11, section 9.10].
Finally, in section 7.12, we state a result of convergence of heat kernels, that implies the convergence results of section 7.7.

The techniques used in this chapter are variations on the techniques of [B11] in the range \( b \geq 1 \). The fact that \( \vartheta \) may approach \( \frac{\pi}{2} \) is handled using the fact that \( a \neq 0 \).

We make the same assumptions and we use the same notation as in section 6.

7.1. The geometry of the minimizing set. We follow [B11, chapter 3]. Let \( Z(\gamma) \subset G \) be the centralizer of \( \gamma \), and let \( \mathfrak{z}(\gamma) \) be its Lie algebra. Let \( Z(a) \subset G \) be the stabilizer of \( a \), and let \( \mathfrak{z}(a) \) be its Lie algebra. By [B11, Proposition 3.2.8], we have the identity
\[
Z(e^a) = Z(a).
\]

By [B11, eq. (3.1.2)], we have the identity
\[
\mathfrak{z}(a) = \ker \text{ad}(a).
\]

By [B11, eqs. (3.3.4) and (3.3.6)], we have
\[
Z(\gamma) = Z(e^a) \cap Z(k), \quad \mathfrak{z}(\gamma) = \mathfrak{z}(e^a) \cap \mathfrak{z}(k).
\]

Since \( \vartheta a = -a, \vartheta k = k \), by (7.3), \( \vartheta \) acts on \( Z(\gamma), Z(e^a), Z(k) \).

As in [B11, eq. (3.3.7)], put
\[
\mathfrak{p}(\gamma) = \mathfrak{z}(\gamma) \cap \mathfrak{p}, \quad \mathfrak{k}(\gamma) = \mathfrak{z}(\gamma) \cap \mathfrak{k}.
\]

By [B11, eq. (3.3.8)], we have the splitting
\[
\mathfrak{z}(\gamma) = \mathfrak{p}(\gamma) \oplus \mathfrak{k}(\gamma).
\]

Put
\[
r = \dim \mathfrak{z}(\gamma), \quad p = \dim \mathfrak{p}(\gamma), \quad q = \dim \mathfrak{k}(\gamma).
\]

By (7.5), we get
\[
r = p + q.
\]

Put
\[
K(\gamma) = K \cap Z(\gamma).
\]

Then \( K(\gamma) \) is a compact Lie subgroup of \( Z(\gamma) \) with Lie algebra \( \mathfrak{k}(\gamma) \).

Let \( Z^0(\gamma), K^0(\gamma) \) be the connected components of the identity in \( Z(\gamma), K(\gamma) \). The Cartan involution \( \vartheta \) acts on \( Z^0(\gamma) \). Then \( Z^0(\gamma) \) is a connected reductive group with maximal compact subgroup \( K^0(\gamma) \). Also the symmetric space \( Z^0(\gamma)/K^0(\gamma) \) embeds canonically in the symmetric space \( X = G/K \). The same considerations still hold when replacing \( \gamma \) by \( e^a \) or \( k \).

We denote by \( \mathfrak{z}^\perp(\gamma) \) the orthogonal to \( \mathfrak{z}(\gamma) \) with respect to \( \mathfrak{B} \). By (7.5), \( \mathfrak{z}^\perp(\gamma) \) splits as
\[
\mathfrak{z}^\perp(\gamma) = \mathfrak{p}^\perp(\gamma) \oplus \mathfrak{k}^\perp(\gamma).
\]

In the sequel, we use the notation
\[
Z_0 = Z(a), \quad K_0 = Z(a) \cap K, \quad z_0 = \mathfrak{z}(a).
\]

We denote by \( Z_0^0, K_0^0 \) the connected components of the identity in \( Z_0, K_0 \). Put
\[
p_0 = \ker \text{ad}(a) \cap \mathfrak{p}, \quad k_0 = \ker \text{ad}(a) \cap \mathfrak{k}.
By [B11, eq. (3.5.5)], we get
\[
(7.12) \quad \mathfrak{z}_0 = \mathfrak{p}_0 \oplus \mathfrak{k}_0.
\]
As was explained before, \( Z^0 \) equipped with the involution \( \theta \) is a connected reductive group with maximal compact subgroup \( K^0 \). Also (7.12) is the Cartan splitting of \( \mathfrak{z}_0 \) associated with \( \theta \). Observe that \( \mathfrak{p}_0 \) and \( \mathfrak{p} (k) \) intersect orthogonally along \( \mathfrak{p} (\gamma) \).

Let \( \mathfrak{z}_0^+ \) be the orthogonal vector space to \( \mathfrak{z}_0 \) in \( \mathfrak{g} \). Then \( \mathfrak{z}_0 \) splits as
\[
(7.13) \quad \mathfrak{z}_0^+ = \mathfrak{p}_0^+ \oplus \mathfrak{k}_0^+.
\]

Let \( \mathfrak{z}_0^+ (\gamma) \) be the orthogonal vector space to \( \mathfrak{z} (\gamma) \) in \( \mathfrak{z}_0 \) with respect to \( B \). Let \( \mathfrak{p}_0^+ (\gamma), \mathfrak{k}_0^+ (\gamma) \) be the orthogonal vector spaces to \( \mathfrak{p} (\gamma), \mathfrak{k} (\gamma) \) in \( \mathfrak{p}_0, \mathfrak{k}_0 \). By [B11, eq. (5.3.5)], we have the splitting
\[
(7.14) \quad \mathfrak{z}_0^+ (\gamma) = \mathfrak{p}_0^+ (\gamma) \oplus \mathfrak{k}_0^+ (\gamma).
\]

Recall that the displacement function \( d_\gamma \) on \( X \) was defined in (5.3). Let \( X (\gamma) \subset X \) be the minimizing set for \( d_\gamma \). By [BaGS85, p. 78], \( X (\gamma) \) is a closed convex subset of \( X \). By [B11, Theorem 3.3.1], \( X (\gamma) \) can be canonically identified with the symmetric space \( Z^0 (\gamma)/K^0 (\gamma) \). The embedding \( Z^0 (\gamma)/K^0 (\gamma) \subset G/K \) corresponds to the embedding \( X (\gamma) \subset X \). The same considerations apply to \( X (\epsilon^a), X (k) \).

By [B11, Theorem 3.3.1], we have the identity
\[
(7.15) \quad X (\gamma) = X (\epsilon^a) \cap X (k).
\]
Also the manifolds \( X (\epsilon^a) \) and \( X (k) \) intersect orthogonally along \( X (\gamma) \).

7.2. A fundamental identity. Note that if \( Y^*_0 \in \mathfrak{k} (\gamma) \), then \( \text{ad} (Y^*_0) \) acts on \( \mathfrak{z}_0^+ (\gamma) \) and preserves the splitting (7.14).

Set
\[
(7.16) \quad \hat{A} (x) = \frac{x/2}{\sinh (x/2)}.
\]
We identify \( \hat{A} \) with the corresponding multiplicative genus. If \( V \) is a finite dimensional Hermitian vector space and if \( B \in \text{End} (V) \) is self-adjoint, then \( \frac{B/2}{\sinh (B/2)} \) is a self-adjoint positive endomorphism. Set
\[
(7.17) \quad \hat{A} (B) = \det \frac{1}{2} \left[ \frac{B/2}{\sinh (B/2)} \right].
\]
In (7.17), the square root is taken to be the positive square root.

Now we follow [B11, Theorem 5.5.1].

Definition 7.1. Let \( J_\gamma (Y^*_0) \) be the function defined on \( \mathfrak{k} (\gamma) \) with values in \( \mathbb{C} \) given by
\[
(7.18) \quad J_\gamma (Y^*_0) = \frac{1}{\det (1 - \text{Ad} (\gamma))} \left[ \hat{A} \left( \text{ad} (Y^*_0) \right) \right]^{1/2} \frac{1}{\det (1 - \text{Ad} (k^{-1}))} \left[ \frac{\det (1 - \exp (-\text{ad} (Y^*_0))) \text{Ad} (k^{-1})}{\det (1 - \exp (-\text{ad} (Y^*_0)))} \text{Ad} (k^{-1}) \right]^{1/2}.
\]
As explained in [B11, section 5.5], the square root appearing in the right-hand side of (7.18) is unambiguously defined.

We now state the main result of this section. This result will be fully exploited in section 8.

**Theorem 7.2.** The following identity holds:

\[
\begin{align*}
&\left(i \exp \left(-\frac{1}{48} \text{Tr}^E \left[ C^{t,E} \right] - \frac{1}{2} C^{t,E} \right) \right)
&= \int_0^1 \frac{1}{2} \text{Tr}^\gamma \left[ \frac{D^X}{\sqrt{2}} \exp \left(-sD^X,2/2 \right) \right] \frac{1}{\sqrt{1-s}} ds \\
&= \int_0^\sqrt{\frac{\pi}{2}} \frac{\sqrt{\pi}}{2} \exp \left(-\frac{1}{2} \left( |a|^2 + |Y_0|^2 \right) \right) \\
&\quad \times J_\gamma (Y_0^t) \text{Tr}^S \left[ \frac{\tilde{c}(a)}{|a|} \text{Ad} (k^{-1}) \exp \left(-i\tilde{c} (\text{ad}(Y_0^t) |p) \right) \right] \\
&\quad \text{Tr}^E \left[ \rho^E (k^{-1}) \exp \left(-i\rho^E (Y_0^t) \right) \right] \frac{dY_0^t}{(2\pi)^{n/2}}.
\end{align*}
\]

**Proof.** Subsection 7.3 is devoted to the proof of our theorem.

### 7.3. The limit of the forms \( \text{Tr}_{s,\gamma}^{[\cdot],\text{odd}} \left[ \exp (-L^{X_t}|_{db=0}) \right] \) as \( b \to +\infty \)

In this subsection, we study the limit of the 1-form \( \text{Tr}_{s,\gamma}^{[\cdot],\text{odd}} \left[ \exp (-L^{X_t}|_{db=0}) \right] \) as \( b \to +\infty \).

**Theorem 7.3.** As \( b \to +\infty \), we have the pointwise convergence of 1-forms on \([0, \frac{\pi}{2}]\).

\[
\begin{align*}
\left| \text{Tr}_{s,\gamma}^{[\cdot],\text{odd}} \left[ \exp (-L^{X_t}|_{db=0}) \right] \right| &\to -d\vartheta (2\pi)^{-p/2} \exp \left(-\frac{|a|^2}{2} \cos^2 (\vartheta) \right) \\
&\int_{t(\gamma)} \frac{1}{\sqrt{2}} J_\gamma (Y_0^t) \text{Tr}^S \left[ \frac{\tilde{c}(a)}{\cos^2 (\vartheta)} \text{Ad} (k^{-1}) \exp \left(-i\tilde{c} (\text{ad}(Y_0^t) |p) \right) \right] \\
&\quad \text{Tr}^E \left[ \rho^E (k^{-1}) \exp \left(-i\rho^E (Y_0^t) \right) \right] \exp \left(-|Y_0|^2/2 \right) \frac{dY_0^t}{(2\pi)^{n/2}}.
\end{align*}
\]

Moreover, there exist \( C > 0, C' > 0 \) such that for \( b \geq 1, \vartheta \in \left[ 0, \frac{\pi}{2} \right] \),

\[
\begin{align*}
\left| \text{Tr}_{s,\gamma}^{[\cdot],\text{odd}} \left[ \exp (-L^{X_t}|_{db=0}) \right] \right| &\leq C \exp \left(-C' \frac{|a|^2}{\cos^2 (\vartheta)} \right).
\end{align*}
\]

**Proof.** The proof of our theorem is delayed to subsections 7.5–7.8.

**Remark 7.4.** We claim that Theorem 7.2 follows from Theorem 7.3. Indeed, let \( c \) be the 1-form in the right-hand side of (7.20). By equation (6.13) in Theorem 6.6, and by equations (7.20), (7.21) in Theorem 7.3, using dominated convergence, we get

\[
\begin{align*}
\int_{0 \leq \vartheta \leq \frac{\pi}{2}} \frac{a}{c} &= \int_{0 \leq \vartheta \leq \frac{\pi}{2}} \frac{a}{c}.
\end{align*}
\]
By making the change of variables $v = \tan(\vartheta)$, for $x > 0$, we get
\begin{equation}
\int_0^\infty \cos^2(\vartheta) \exp(-x^2/2 \cos^2(\vartheta)) \, d\vartheta = \exp(-x^2/2) \int_0^\infty \exp(-x^2v^2/2) \, dv.
\end{equation}

By (7.23), we obtain
\begin{equation}
\int_0^\infty \cos^2(\vartheta) \exp(-x^2/2 \cos^2(\vartheta)) \, d\vartheta = \frac{\sqrt{\pi}}{\sqrt{2x}} \exp(-x^2/2).
\end{equation}

By equation (6.10) in Proposition 6.4, and by equations (7.20), (7.22), and (7.24), we get (7.19), which completes the proof of Theorem 7.2.

7.4. Estimates on the heat kernel for $L^X|_{db=0}$ away from $\hat{i}_aN(k^{-1})$. As in [B11, section 3.4], we identify the total space of the normal bundle $N_{X(\gamma)/X}$ to the symmetric space $X$ via the normal geodesic coordinate based at $X(\gamma)$.

We proceed as in [B11, section 3.6]. Note that $\text{Ad}(k^{-1})$ acts on $N|_{X(\gamma)}$. Set
\begin{equation}
N(k^{-1}) = \{ Y^N \in N|_{X(\gamma)}, \text{Ad}(k^{-1}) Y^N = Y^N \}.
\end{equation}

Then $N(k^{-1})$ is the vector bundle on $X(\gamma)$ associated with the eigenspace of $\mathfrak{k}$ corresponding to the eigenvalue 1 of $\text{Ad}(k^{-1})$, i.e., with the Lie algebra $\mathfrak{k}(k^{-1})$ of the centralizer of $k^{-1}$ in $K$. Let $N(k^{-1})$ be the total space of $N(k^{-1})$.

Let $\hat{i}_a$ be the embedding $(x, Y^N) \in N(k^{-1}) \to (x, aTX, Y^N) \in \hat{X}$. For the geometric interpretation of the set $\hat{i}_aN(k^{-1})$, we refer to [B11, Proposition 3.6.1].

For $a > 0$, set
\begin{equation}
K_{-a} s(x, Y) = a^{(m+n)/2} s(x, -aY).
\end{equation}

For $b > 0$, $\vartheta \in \left[0, \frac{\pi}{2}\right]$, by analogy with [B11, eq. (9.1.2)], where only the case $\vartheta = 0$ was considered, set
\begin{equation}
L^X_{b, \vartheta} = K_{-b/\cos(\vartheta)} L^X_{b, \vartheta} K_{-b/\cos(\vartheta)}, \quad L^X|_{db=0} = K_{-b/\cos(\vartheta)} L^X|_{db=0} K_{-b/\cos(\vartheta)}^{-1}
\end{equation}

By (4.70), we get
\begin{equation}
L^X_{b, \vartheta} = \frac{b^4}{2 \cos^2(\vartheta)} \left| [Y^N, Y^TX] \right|^2 + \frac{1}{2} \left( -\frac{\cos^2(\vartheta)}{b^4} \Delta_{TX \oplus N} + \frac{|Y^TX|^2}{\cos^2(\vartheta)} + |Y^N|^2 - \frac{1}{b^2} (m + \cos(\vartheta) n) \right) + \frac{N_{-\vartheta} \Lambda (T^* X \oplus N^*) \rho}{b^2}
\end{equation}
\begin{equation}
- \left( \nabla_{Y^TX} C_\infty (TX \oplus N, \hat{\mathbb{R}}^* (\Lambda (T^* X \oplus N^*) \oplus \hat{\mathbb{R}}^F \oplus \hat{F})), f \right) \cdot f
\end{equation}
\begin{equation}
- i c (\theta \text{ad}(Y^N)) - i \mathcal{C}(\text{ad}(Y^N)|_{TX}) - i \rho F(Y^N) \right).
\end{equation}
Also by (4.81), (4.88), and (7.27), we obtain

\[
\begin{align*}
(7.29) \quad & \mathcal{L}_{X_t} \big|_{db=0} = \mathcal{L}_{b,\theta} X_t + \frac{d\theta}{\sqrt{2b}} \left( \frac{b^3 \sin (\theta)}{\cos^2 (\theta)} \right) e^{\mathcal{L}} ([Y^N, Y^T X]) \\
& \quad + \frac{\sin (\theta)}{b} (D^T X - iT^N) + \frac{b}{\cos^2 (\theta)} \tilde{c} \left( Y^T X \right).
\end{align*}
\]

**Proposition 7.5.** The following identity holds:

\[
(7.30) \quad \text{Tr}_{\gamma} [\gamma]^{\text{odd}} \left[ \exp \left( -\mathcal{L}_{X_t} \right) \right]_{db=0} = -\frac{d\theta}{\sqrt{2b}} \text{Tr}_{\gamma} \left[ \left( \frac{b^3 \sin (\theta)}{\cos^2 (\theta)} \right) e^{\mathcal{L}} ([Y^N, Y^T X]) \\
+ \frac{\sin (\theta)}{b} (D^T X - iT^N) + \frac{b}{\cos^2 (\theta)} \tilde{c} \left( Y^T X \right) \right] \exp \left( -\mathcal{L}_{b,\theta} X_t \right).
\]

**Proof.** As we saw in subsection 5.1, \( \text{Tr}_{\gamma} [\gamma]^{\text{odd}} \) is a supertrace, i.e., it vanishes on supercommutators. Then (7.30) follows from (7.29). \( \square \)

**Definition 7.6.** Let \( g_{X_t,\theta}^{\gamma'} ((x,Y), (x',Y')) \) be the smooth kernel associated with the operator \( \exp \left( -t\mathcal{L}_{X_t}^{\gamma'} \right) \). When \( t = 1 \), we use the notation \( g_{X_t,\theta}^{\gamma} ((x,Y), (x',Y')) \).

First, we extend the estimates in [B11, Theorem 9.1.1] that are valid for \( \theta = 0 \).

**Theorem 7.7.** Given \( 0 < \epsilon \leq M < +\infty \), there exist \( C_{\epsilon, M} > 0, C'_{\epsilon, M} > 0 \) such that for \( b \geq 1, 0 \leq \theta < \frac{\pi}{2}, \epsilon \leq t \leq M, (x,Y), (x',Y') \in \tilde{X}, \)

\[
(7.31) \quad \left| g_{X_t,\theta}^{\gamma'} ((x,Y), (x',Y')) \right| \leq C_{\epsilon, M} \left( \frac{b}{\cos^{1/2} (\theta)} \right)^{4m+2n} \exp \left( -C'_{\epsilon, M} \left( \frac{d^2 (x,x')}{\cos^2 (\theta)} + |Y|^2 + |Y'|^2 \right) \right).
\]

Given \( \beta > 0, 0 < \epsilon \leq M < +\infty, \) there exist \( \eta_M > 0, C_{\epsilon, M} > 0, C''_{\gamma, M} > 0 \) such that for \( b \geq 1, 0 \leq \theta < \frac{\pi}{2}, \epsilon \leq t \leq M, (x,Y) \in \tilde{X}, \) if \( d(x, X(\gamma)) \geq \beta, \)

\[
(7.32) \quad \left| g_{X_t,\theta}^{\gamma} ((x,Y), \gamma(x,Y)) \right| \leq C_{\epsilon, M} \left( \frac{b}{\cos^{1/2} (\theta)} \right)^{4m+2n} \exp \left( -C'_{\epsilon, M} \left( \frac{d^2 (x) + |a|^2}{\cos^2 (\theta)} + |Y|^2 \right) \right) - \frac{C''_{\gamma, M}}{\eta_M} \exp \left( -2\eta_M \left| Y^T X \right| b^4/\cos^2 (\theta) \right).
\]

There exists \( \eta > 0 \) such that given \( \beta > 0, \mu > 0, \) there exist \( C > 0, C' > 0, C''_{\gamma, \beta, \mu} > 0 \) such that for \( b \geq 1, 0 \leq \theta < \frac{\pi}{2}, (x,Y) \in \tilde{X}, \) if \( d(x, X(\gamma)) \leq \beta, \left| Y^T X - a^T X \right| \geq \mu, \)

\[
(7.33) \quad \left| g_{X_t,\theta}^{\gamma} ((x,Y), \gamma(x,Y)) \right| \leq C \left( \frac{b}{\cos^{1/2} (\theta)} \right)^{4m+2n} \exp \left( -C' \left( \frac{|a|^2}{\cos^2 (\theta)} + |Y|^2 \right) \right) - \frac{C''_{\gamma, \beta, \mu}}{\eta} \exp \left( -2\eta \left| Y^T X \right| b^4/\cos^2 (\theta) \right).
\]
There exist \(c > 0, C > 0, C'_r > 0\) such that for \(b \geq 1, 0 \leq \vartheta < \frac{\pi}{2}, f \in \mathfrak{p}^\perp (\gamma)\), \(|f| \leq 1, Y \in (TX \oplus N)_x\), \(|Y^{TX} - a^{TX}| \leq 1\),

\[
\left| q^{X'}_{b, \vartheta} \left( (e^f p1, Y), \gamma (e^f p1, Y) \right) \right|
\leq c \left( \frac{b}{\cos^2 (\vartheta)} \right)^{4m+2n} \exp \left( -C \left( \frac{|a|^2}{\cos^2 (\vartheta)} + |Y^N|^2 \right) \right)
- C'_r \left( |f|^2 + |Y^{TX} - a^{TX}|^2 \right) b^4 / \cos^2 (\vartheta)
- C'_r \left( (\Ad (k^{-1}) - 1) Y^N \right) b^2 / \cos (\vartheta).
\]

The above inequalities remain valid when replacing \(q^{X'}_{b, \vartheta,t}\) or \(q^{X'}_{b, \vartheta}\) by \(\frac{\cos (\vartheta)}{b} \nabla_V q^{X'}_{b, \vartheta,t}\) or \(\frac{\cos (\vartheta)}{b} \nabla_V q^{X'}_{b, \vartheta}\).

**Proof.** The proof of our theorem is deferred to section 12. \qed

7.5. **A rescaling of the coordinates** \((f, Y)\). In the sequel, we use the notation

\[
q^X_{b, \vartheta} ((x, Y), (x', Y')) = -\frac{d\vartheta}{\sqrt{2b}} \left( \frac{b \sin \vartheta}{\cos^2 \vartheta} \right) (\left[ Y^N, V^X \right])
+ \frac{\sin \vartheta}{b} \left( D^TX - iD^N \right) + \frac{b}{\cos^2 \vartheta} \left( Y^{TX} \right) q^X_{b, \vartheta} ((x, Y), (x', Y')).
\]

We start proving Theorem 7.3. We proceed as in [B11, section 9.2]. When \(f \in \mathfrak{p}^\perp (\gamma)\), we identify \(e^f \in G\) and \(e^f p1 \in X\). By [B11, eq. (4.3.10)], we get

\[
\text{Tr}_x [\gamma, \text{odd}] \exp \left( -L^{X'} \right)_{\vert dh = 0} = \int [\mathbb{Z}^{-1}_{p^\perp (\gamma)}] \text{Tr}_x [\gamma, \text{odd}] q^X_{b, \vartheta} ((e^f, Y), \gamma (e^f, Y))) \right) r(f) dY df.
\]

As explained in detail in [B11, section 4.2], in the right-hand side of (7.36), \(\gamma\) is viewed as acting on \(X\), this action lifting to \(\Lambda (T^*X \oplus N^*) \otimes S^{TX} \otimes F\). Also \(r(f) > 0\) is a Jacobian. By [B11, eq. (3.436)], we have the estimate

\[
r(f) \leq C \exp (C' |f|)\)

By [B11, eq. (3.44)], there exists \(C_\gamma > 0\) such that

\[
d_\gamma (e^f) \geq |a| + C_\gamma |f|.
\]

By [B11, eq. (9.1.6)], given \(\eta > 0, C' > 0, C'' > 0, C''' > 0\), there exist \(c > 0, d > 0, e > 0\) such that for \(b \geq 1, Y^{TX} \in TX\), then

\[
C' |Y^{TX}|^2 + C'' \exp \left( -2\eta |Y^{TX}| \right) b^4 - C''' \log (b) \geq \frac{C'}{2} |Y^{TX}|^2
+ \frac{C''}{2} \exp \left( -2\eta |Y^{TX}| \right) b^4 - c \geq \frac{C' + e \log^2 (b)}{4} |Y^{TX}|^2 + e \log^2 (b) - d.
\]

Take \(\beta \in [0, 1]\). By equation (7.32) in Theorem 7.7, by the last part of this theorem, and by (7.35)–(7.39), there exist \(C > 0, C' > 0, C'' > 0\) such that for
(7.40) \[
\left| \int_{(f,Y) \in \mathbb{R}^{-1}_{\beta} \hat{p}^* (\gamma)} \Tr_{t_0} \left[ \gamma \Sigma_{t_0, \theta}^{X} \left( (e^f Y, \gamma (e^f Y)) \right) r (f) \right] dY df \right| \\
\leq C \exp \left( -C' |a|^2 / \cos^2 (\theta) - C'' \log^2 (b) \right). 
\]

By equation (7.33) in Theorem 7.7, by the last statement in this theorem, and by (7.39), given \( \beta > 0, \mu > 0 \), we may as well obtain a similar result for the integral

(7.41) \[
\Tr_{t_0} \left[ \gamma \Sigma_{t_0, \theta}^{X} \left( (e^f Y, \gamma (e^f Y)) \right) r (f) \right] 
\]

over the region considered in (7.33). In particular, as \( b \to +\infty \), both integrals tend to 0, and they can be estimated uniformly by an expression like the right-hand side of (7.21).

As in [B11, section 9.2], we trivialize \( TX, N \) by parallel transport with respect to the connections \( \nabla^TX, \nabla^N \) along the geodesics \( t \in \mathbb{R} \to e^{tf} p_1 \in X \). In this trivialization \( Y^N \in N \) splits as

(7.42) \[
Y^N = Y^t_0 + Y^N_{\perp}, \quad Y^t_0 \in \mathfrak{t} (\gamma), \quad Y^N_{\perp} \in \mathfrak{t}^\perp (\gamma).
\]

To control the behaviour of (7.36) as \( b \to +\infty \), given \( \beta > 0 \), we may as well consider the integral

(7.43) \[
\int_{Y^{TX} \in a^{TX} | \leq \beta} \Tr_{t_0} \left[ \gamma \Sigma_{t_0, \theta}^{X} \left( (e^f Y, \gamma (e^f Y)) \right) r (f) \right] dY^{TX} dY^N df \\
= (b^2 / \cos (\theta))^{-2m-n+r} \int_{Y^{TX} \leq \beta b^2 / \cos (\theta)} \Tr_{t_0} \left[ \gamma \Sigma_{t_0, \theta}^{X} \left( (e^{\cos (\theta) f / b^2} a^{TX}_{e^{\cos (\theta) f / b^2} + \cos (\theta) Y^{TX} / b^2, Y^t_0 + \cos (\theta) Y^N_{\perp} / b^2}) \right) \right. \\
\gamma \left( e^{\cos (\theta) f / b^2} a^{TX}_{e^{\cos (\theta) f / b^2} + \cos (\theta) Y^{TX} / b^2, Y^t_0 + \cos (\theta) Y^N_{\perp} / b^2}) \right) \right] \\
\left. r (\cos (\theta) f / b^2) \right) dY^{TX} dY^t_0 dY^N_{\perp} df.
\]

Using equation (7.34) in Theorem 7.7 and the last part of this theorem, and proceeding as in [B11, eqs. (9.2.8)–(9.2.12)], we find that for \( \beta > 0 \) small enough,
for \(|f| \leq \beta b^2 / \cos (\vartheta)\), \(|Y^{TX}| \leq \beta b^2 / \cos (\vartheta)\), then

\[
(7.44) \quad \left(\frac{b^2}{\cos (\vartheta)}\right)^{-2m-n} \left| g_{b,\vartheta}^{X_5} \left( e^{\cos (\vartheta) f / b^2} a^T_{\cos (\vartheta) f / b^2} + \cos (\vartheta) Y^{TX} / b^2, Y_0^\varphi + \cos (\vartheta) Y^{N,\perp} / b^2 \right) \right|
\]

By (7.45), we deduce that

\[
(a^T_{\cos (\vartheta) f / b^2} = a + O \left( |f|^2 / b^4 \right) ).
\]

By (7.45), we deduce that

\[
(7.46) \quad b^2 \left[ Y_0^\varphi + \cos (\vartheta) Y^{N,\perp} / b^2, a^T_{\cos (\vartheta) f / b^2} + \cos (\vartheta) Y^{TX} / b^2 \right] = \cos (\vartheta) \left[ Y_0^\varphi, Y^{TX} \right]
\]

\[
+ \cos (\vartheta) \left[ Y^{N,\perp}, a^T_{\cos (\vartheta) f / b^2} \right] + \cos^2 (\vartheta) \left[ \frac{Y^{N,\perp}}{b^2}, Y^{TX} \right] + |Y_0^\varphi| O \left( |f|^2 / b^2 \right).
\]

By (7.35), (7.44), and (7.46), in (7.44), we may as well replace \( g_{b,\vartheta}^{X_5} \) by \( g_{b,\vartheta}^{X_5} \).

As in [B11, section 9.2], we are still left with the diverging term \( \left( \frac{b^2}{\cos (\vartheta)} \right)^r \) in the right-hand side of (7.43). As in this reference, this term will be dealt with using local cancellation techniques.

7.6. A conjugation of the hypoelliptic Laplacian. We will proceed exactly as in [B11, sections 9.3–9.5].

Definition 7.8. Let \( \hat{\mathcal{A}}(\gamma) \) denote another copy of \( \mathcal{A}(\gamma) \), and let \( \hat{\mathcal{A}}(\gamma)^\ast \) denote the corresponding copy of the dual of \( \mathcal{A}(\gamma) \). If \( \alpha \in \Lambda \left( \hat{\mathcal{A}}(\gamma)^\ast \right) \), let \( \hat{\alpha} \) be the corresponding element in \( \Lambda \left( \hat{\mathcal{A}}(\gamma)^\ast \right) \). Also \( c(\mathcal{A}(\gamma)), \hat{c}(\mathcal{A}(\gamma)) \) denote the Clifford algebras of \( \mathcal{A}(\gamma), B_{\mathcal{A}(\gamma)} \), \( \hat{\mathcal{A}}(\gamma)^\ast \), \( \hat{\mathcal{A}}(\gamma), -B_{\hat{\mathcal{A}}(\gamma)} \).

Let \( e_1, \ldots, e_r \) be a basis of \( \mathcal{A}(\gamma) \), let \( e^1, \ldots, e^r \) be the corresponding dual basis of \( \mathcal{A}(\gamma)^\ast \).

As in [B11, eq. (9.3.4)], put

\[
(7.47) \quad \alpha = \sum_{i=1}^r c(e_i) \varepsilon^i, \quad \hat{\alpha} = \sum_{i=1}^r \hat{c}(e_i) \varepsilon^i.
\]

Then

\[
(7.48) \quad \alpha \in c(\mathcal{A}(\gamma)) \hat{\otimes} \Lambda \left( \hat{\mathcal{A}}(\gamma)^\ast \right), \quad \hat{\alpha} \in \hat{c}(\mathcal{A}(\gamma)) \hat{\otimes} \Lambda \left( \hat{\mathcal{A}}(\gamma)^\ast \right).
\]
Since \( \mathfrak{z} (\gamma) \subset \mathfrak{g} \), from (7.48), we get
\[
\alpha \in c (\mathfrak{g}) \hat{\otimes} \Lambda^1 \left( \mathfrak{z} (\gamma)^* \right), \quad \tilde{\alpha} \in \hat{c} (\mathfrak{g}) \otimes \Lambda^1 \left( \mathfrak{z} (\gamma)^* \right).
\]

We proceed as in [B11, sections 3.10, 9.3, and 9.4]. Under the identification \( TX \odot N = \mathfrak{g} \), we denote by \( (TX \odot N) (\gamma) \) the subvector bundle of \( TX \odot N \) corresponding to \( \mathfrak{z} (\gamma) \). Then \( (TX \odot N) (\gamma) \) is a flat vector subbundle of \( TX \odot N \) with respect to the connection \( \nabla^{TX \odot N, f} \) which is preserved by the left action of \( \gamma \). Let \( (TX \odot N) (\gamma) \) be the vector bundle on \( X \) corresponding to \( \mathfrak{z} (\gamma) \). Since \( (TX \odot N) (\gamma) \) is a subvector bundle of \( TX \odot N \), it inherits a scalar product. We equip \( (TX \odot N) (\gamma) \) with the corresponding scalar product.

Then \( \alpha \) can be viewed as a section of \( c (TX \odot N) \otimes \Lambda^1 (\mathfrak{z} (\gamma)^*) \), and \( \tilde{\alpha} \) as a section of \( \hat{c} (TX \odot N) \otimes \Lambda^1 (\mathfrak{z} (\gamma)^*) \). Moreover, \( \alpha \) and \( \tilde{\alpha} \) can also be considered as sections of \( \text{End} (\Lambda (TX \odot N^*)) \otimes \Lambda^1 (\mathfrak{z} (\gamma)^*) \).

From the flat connection \( \nabla^\Lambda (TX \odot N^*), f^*, \tilde{f} \) on \( \Lambda (TX \odot N^*) \) and from the trivial connection on \( \Lambda^1 (\mathfrak{z} (\gamma)^*) \), we obtain a flat connection on
\[
\Lambda^1 (TX \odot N^*) \otimes \Lambda (\mathfrak{z} (\gamma)^*),
\]
which is still denoted \( \nabla^\Lambda (TX \odot N^*), f^*, \tilde{f} \). By [B11, Proposition 9.4.1], we have
\[
\nabla^\Lambda (TX \odot N^*), f^*, \tilde{f} \Gamma = 0.
\]
By [B11, Proposition 9.3.2] and by (4.66), we get
\[
\exp \left( -\frac{b^2}{\cos (\vartheta)} \tilde{\alpha} \right) N_{-\vartheta}^* \exp \left( \frac{b^2}{\cos (\vartheta)} \tilde{\alpha} \right) = N_{-\vartheta}^* \exp \left( \frac{b^2}{\cos (\vartheta)} \tilde{\alpha} \right) + b^2 \alpha.
\]

As in [B11, Definition 9.5.1], set
\[
\mathcal{L}^\Lambda_{X, \vartheta} = \exp \left( -\frac{b^2}{\cos (\vartheta)} \tilde{\alpha} \right) \mathcal{L}^\Lambda_{X, \vartheta} \exp \left( \frac{b^2}{\cos (\vartheta)} \tilde{\alpha} \right),
\]
\[
\mathcal{L}^\Lambda_{X'} = \exp \left( -\frac{b^2}{\cos (\vartheta)} \tilde{\alpha} \right) \mathcal{L}^\Lambda_{X'} \exp \left( \frac{b^2}{\cos (\vartheta)} \tilde{\alpha} \right).
\]
Using (7.28), (7.29), and (7.50)–(7.52), we get
\[
\mathcal{L}^\Lambda_{X, \vartheta} = \mathcal{L}^\Lambda_{X'} + \alpha,
\]
\[
\mathcal{L}^\Lambda_{X'} = \mathcal{L}^\Lambda_{X, \vartheta} = \mathcal{L}^\Lambda_{X'} + \alpha.
\]
In (7.53), we used the fact that in the factor of \( d\vartheta \) in the right-hand side of (7.29), no term in \( \hat{c} (\mathfrak{g}) \) appears.

**Definition 7.9.** For \( t > 0 \), let \( q_{b, \vartheta, t}^\Lambda ((x, Y), (x', Y')) \) denote the smooth kernel associated with \( \exp \left( -t \mathcal{L}^\Lambda_{X, \vartheta} \right) \). Also we use the notation \( q_{b, \vartheta, \cdot}^\Lambda \) instead of \( q_{b, \vartheta, 1}^\Lambda \).

By (7.52), we get
\[
q_{b, \vartheta}^\Lambda ((x, Y), (x', Y')) = \exp \left( -\frac{b^2}{\cos (\vartheta)} \tilde{\alpha} \right) q_{b, \vartheta}^\Lambda ((x, Y), (x', Y')) \exp \left( \frac{b^2}{\cos (\vartheta)} \tilde{\alpha} \right).
\]

Put
\[
\mathcal{G} = \text{End} (\Lambda^1 (\mathfrak{g}^*)) \otimes \Lambda^1 (\mathfrak{z} (\gamma)^*).
\]

A basis \( e_1, \ldots, e_{m+n} \) of \( \mathfrak{g} \) is said to be unimodular if the determinant of \( B \) on this basis is equal to \((-1)^n\). Let \( e_1, \ldots, e_{m+n} \) be a unimodular basis of \( \mathfrak{g} \), such
that $e_1,\ldots,e_r$ is a basis of $\mathfrak{h}(\gamma)$. Let $\xi_1^\flat,\ldots,\xi_r^\flat$ be the basis of $\mathfrak{h}(\gamma)^*$ that is dual to $e_1,\ldots,e_r$. Let $\widehat{T}_r$ be the linear map from $\mathcal{G}$ into $\mathbb{R}$ that, up to permutation, vanishes on all the monomials in the $c(e_i),\hat{c}(e_i),1 \leq i \leq m+n,\xi^j,1 \leq j \leq r$ except on $c(e_1)\xi_1\ldots c(e_r)\xi^r c(e_{r+1})\hat{c}(e_{r+1})\ldots c(e_{m+n})\hat{c}(e_{m+n})$, and moreover,

\begin{equation}
\widehat{T}_r \left[ c(e_1)\xi_1\ldots c(e_r)\xi^r c(e_{r+1})\hat{c}(e_{r+1})\ldots c(e_{m+n})\hat{c}(e_{m+n}) \right] = (-1)^r (\xi^j)^{m+n-r} (-1)^{n-q}.
\end{equation}

If we assume that $e_{r+1},\ldots,e_{m+n}$ is a basis of $\mathfrak{h}(\gamma)$, and that $e_{r+1}^*,\ldots,e_{m+n}^*$ is the dual basis to $e_{r+1},\ldots,e_{m+n}$ with respect to $B_{\mathfrak{h}(\gamma)}$, then (7.56) can be replaced by

\begin{equation}
\widehat{T}_r \left[ c(e_1)\xi_1\ldots c(e_r)\xi^r c(e_{r+1})\hat{c}(e_{r+1})\ldots c(e_{m+n})\hat{c}(e_{m+n}) \right] = (-1)^r (\xi^j)^{m+n-r}.
\end{equation}

We can extend the map $\widehat{T}_r$ to a map $\widehat{T}_r^{-}\mathbb{C}$ that maps $\mathcal{G} \otimes c(TX) \otimes \text{End}(F)$ into $\mathbb{C}$.

Set

\begin{equation}
\varphi_{b,}\theta \left( ((x,Y),(x',Y')) \right) = -\frac{d\theta}{2b} \left( \frac{b^3 \sin (\theta)}{\cos^2 (\theta)} \right) i c \left( [Y^N,Y^{TX}] \right) + \frac{\sin (\theta)}{b} (D^{TX} - iD^N) + \frac{b}{\cos^2 (\theta)} \hat{c} \left( Y^{TX} \right) \varphi_{b,}\theta \left( ((x,Y),(x',Y')) \right).
\end{equation}

We recall a result in [B11, Proposition 9.5.4].

**Proposition 7.10.** For $b > 0$, the following identity holds:

\begin{equation}
\varphi_{b,}\theta \left( ((x,Y),(x',Y')) \right) = (b^2 / \cos (\theta))^{-\gamma} \varphi_{b,}\theta \left( ((x,Y),(x',Y')) \right).
\end{equation}

In the sequel, the norm of $\Lambda(T^*X \oplus N^*)$, $\Lambda(TX \oplus N)(\gamma)^*$, $S^{TX}$, and $F$ will be evaluated with respect to the norms of $\Lambda(T^*X \oplus N^*)$, $\Lambda(TX \oplus N)(\gamma)^*$, $S^{TX}$, and $F$.

Now, we give an extension of [B11, Theorem 9.5.6].

**Theorem 7.11.** Given $\beta > 0$, there exist $C > 0, C_\gamma > 0$ such that for $b \geq 1, \theta \in [0,\frac{\pi}{2}], f \in \mathfrak{p}(\gamma)$, $|f| \leq \beta b^2 / \cos (\theta)$, and $|Y^{TX}| \leq \beta b^2 / \cos (\theta)$,

\begin{equation}
\varphi_{b,}\theta \left( (e^\cos (\theta) f / b^2, a^{TX}_{e^\cos (\theta) f / b^2} + \cos (\theta) Y^{TX} / b^2, Y_0^l + \cos (\theta) Y^{N,\perp} / b^2) \right) \leq C \exp \left( -C' \left( |\theta|^2 / \cos^2 (\theta) + |Y_0^l|^2 \right) \right) - C'_\gamma \left( |f|^2 + |Y^{TX}|^2 \right) - C'_\gamma |(\text{Ad} (k^{-1}) - 1) Y^{N,\perp} - C'_\gamma |(|\theta|, Y^{N,\perp})|).
\end{equation}
The above inequality remains valid when replacing $q_{k,\theta}^{X'}$ by $\frac{\cos(\theta)}{\rho} \nabla^{Y} q_{k,\theta}^{X'}$.

Proof. The proof of our theorem will be given in section $12$. □

7.7. The limit of the rescaled heat kernel. By (7.43), (7.59), we get an analogue of [B11, eq. (9.6.1)]

\[
\int_{|f| \leq \beta} \int_{|Y^TX - a^TX| \leq \beta} \Tr_{n}^{\text{odd}} \left[ \gamma_{r}^{X,Y} \left( \left( (e^{f}, Y), \gamma (e^{f}, Y) \right) \right) \right] r (f) dY^TX dY^{N} df
\]

\[
= \int_{|f| \leq \beta} \int_{|Y^TX - a^TX| \leq \beta} \Tr_{n}^{\text{odd}} \left[ \gamma_{r}^{X,Y} \left( e^{\cos(\theta)f/b^2} \right) , a^{TX}_{e^{\cos(\theta)f/b^2} + \cos(\theta) Y^TX/b^2, Y_{0}^{r} + \cos(\theta) Y^{N,\perp}/b^2} \right) \right] r (\cos(\theta)f/b^2) dY^TX dY_{0}^{r} dY^{N,\perp} df.
\]

We consider the vector space

\[(7.62) \quad p \times g = p \times (p \oplus \mathfrak{t}).\]

Recall that $\Delta^{p \oplus \mathfrak{t}}$ is the standard Laplacian on $p \oplus \mathfrak{t}$. This operator acts along $p \oplus \mathfrak{t}$, i.e., on the second factor in the right-hand side of (7.62).

Let $\nabla^{H}$ denote differentiation in the variable $Y \in p$, and let $\nabla^{V}$ denote differentiation in the variable $Y^{0} \in g$.

Let $e_1, \ldots, e_p$ be an orthonormal basis of $p (\gamma)$, let $e_{p+1}, \ldots, e_r$ be an orthonormal basis of $\mathfrak{t} (\gamma)$. We denote with upper scripts the corresponding dual bases. Then $\xi^{1}, \ldots, \xi^{r}$ is a basis of $\mathfrak{g}^{\ast} (\gamma)$. Put

\[(7.63) \quad \alpha = \sum_{i=1}^{r} c (e_i) \xi^{i}.
\]

The notation in (7.63) is compatible with (7.47).

**Definition 7.12.** Given $Y_{0}^{r} \in \mathfrak{t} (\gamma)$, let $\mathcal{P}_{a,Y_{0}^{r}}$ be the differential operator on $p \times g$ that was defined in [B11, Definition 5.1.2], i.e.,

\[(7.64) \quad \mathcal{P}_{a,Y_{0}^{r}} = \frac{1}{2} \left[ [Y^{r}, a] + [Y_{0}^{r}, Y^{p}] \right]^{2} - \frac{1}{2} \Delta^{p \oplus \mathfrak{t}} + \alpha - \nabla^{H}_{Y^{p}}
\]

\[ - \nabla^{V}_{[a+Y_{0}^{r}, [a,Y^{p}]]} - \tilde{c} (\text{ad} (a)) + c (\text{ad} (a) + i \theta (\text{ad} (Y_{0}^{r}))).\]

Let $R_{Y_{0}^{r}} ((y,Y), (y', Y'))$ be the smooth kernel for $\exp \left( -\mathcal{P}_{a,Y_{0}^{r}} \right)$ with respect to $dy'dY'$.

We have the following convergence result that is an analogue of [B11, Theorem 9.6.1].
Theorem 7.13. For \( \vartheta \in \left[0, \frac{\pi}{2}\right] \), as \( b \to +\infty \), we have the convergence,

\[
\left( b^2 / \cos (\vartheta) \right)^{-2m-n} \widehat{\mathrm{Tr}}_s^{\text{odd}} \left[ \gamma_{b, \vartheta} X f, T^X \right] \rightarrow -d \vartheta \sqrt{2} \exp \left( -|a|^2 / 2 \cos^2 (\vartheta) \right) 
\]

\[
\gamma \left( e^{\cos (\vartheta) f/b^2}, a T^X e^{\cos (\vartheta) f/b^2} + \cos (\vartheta) Y^{X,\perp} / b^2 \right) 
\]

\[
\gamma \left( e^{\cos (\vartheta) f/b^2}, a T^X e^{\cos (\vartheta) f/b^2} + \cos (\vartheta) Y^{X,\perp} / b^2 \right) 
\]

\[
\rightarrow -d \vartheta \sqrt{2} \exp \left( -|a|^2 / 2 \cos^2 (\vartheta) \right) 
\]

\[
\widehat{\mathrm{Tr}}_s \left[ \Ad (k^{-1}) R_{Y^0} ((f, Y), \Ad (k^{-1}) (f, Y)) \right] 
\]

\[
\mathrm{Tr}^{Sp} \left[ \hat{c} (a) \cos^2 (\vartheta) \Ad (k^{-1}) \exp \left( -i \hat{c} (\ad (Y^0_0) |_p) \right) \right] 
\]

\[
\mathrm{Tr}^E \left[ \rho^E (k^{-1}) \exp \left( -i \rho^E (Y^0_0) \right) \right] \exp \left( -|Y^0_0|^2 / 2 \right) 
\]

Proof. The proof of this result will be given in subsections 7.9–7.12.

7.8. A proof of Theorem 7.3. By (7.40), and by the considerations following (7.41), we may as well study the behaviour of the integral in (7.61). By equation (7.60) in Theorem 7.11 and by the last statement in this theorem, we get the uniform bound (7.21). Using dominated convergence, we deduce from Theorems 7.11 and 7.13 that as \( b \to 0 \),

\[
\mathrm{Tr}_{\gamma, \text{odd}} \left[ \exp \left( -L^{X,f}_{|db=0} \right) \right] 
\]

\[
\rightarrow -d \vartheta \sqrt{2} \exp \left( -|a|^2 / 2 \cos^2 (\vartheta) \right) \int_{\mathrm{Sp}^+ (\gamma) \times (p \in \mathrm{Sp}^+ (\gamma) \times \mathbb{R})} (y, Y, Y^0_0) \mathrm{Tr}_s \left[ \Ad (k^{-1}) R_{Y^0} ((y, Y), \Ad (k^{-1}) (y, Y)) \right] 
\]

\[
\mathrm{Tr}^{Sp} \left[ \hat{c} (a) \cos^2 (\vartheta) \Ad (k^{-1}) \exp \left( -i \hat{c} (\ad (Y^0_0) |_p) \right) \right] 
\]

\[
\mathrm{Tr}^E \left[ \rho^E (k^{-1}) \exp \left( -i \rho^E (Y^0_0) \right) \right] \exp \left( -|Y^0_0|^2 / 2 \right) dY^0 dY^0_0. 
\]
Theorem 7.14. If $\theta \in \left[0, \frac{\pi}{2}\right]$, the following identity holds:

$$\exp \left(-\frac{|a|^2}{2 \cos^2 (\theta)} \right) \int_{\Theta \times TX N} \left( y, Y^*_0 \right) \left( y, Y^*_0 \right) \left( y, Y^*_0 \right) \left( y, Y^*_0 \right) \left( y, Y^*_0 \right)$$

$$\left[ \frac{\partial}{\partial \phi} - \frac{1}{2} \left[ \frac{\partial}{\partial \phi} \right]^2 \right] \left[ (y, Y), \left( y, Y \right) \left( y, Y \right) \left( y, Y \right) \left( y, Y \right) \left( y, Y \right) \right]$$

$$\left\{ \begin{array}{l}
\text{Tr} \left[ \text{Ad} \left( k^{-1} \right) R_{Y^*_0} \left( y, Y \right) \right] \\
\text{Tr} \left[ \text{Tr} \left( y, Y \right) \cdot \left( y, Y \right) \cdot \left( y, Y \right) \cdot \left( y, Y \right) \cdot \left( y, Y \right) \right] \\
\text{Tr} \left[ \text{Tr} \left( y, Y \right) \cdot \left( y, Y \right) \cdot \left( y, Y \right) \cdot \left( y, Y \right) \cdot \left( y, Y \right) \right] \\
\text{Tr} \left[ \text{Tr} \left( y, Y \right) \cdot \left( y, Y \right) \cdot \left( y, Y \right) \cdot \left( y, Y \right) \cdot \left( y, Y \right) \right] \\
\text{Tr} \left[ \text{Tr} \left( y, Y \right) \cdot \left( y, Y \right) \cdot \left( y, Y \right) \cdot \left( y, Y \right) \cdot \left( y, Y \right) \right] \\
\end{array} \right.$$

$$= \left( 2\pi \right)^{-r/2} \exp \left(-\frac{|a|^2}{2 \cos^2 (\theta)} \right) \int_{\Theta} (E) \left( Y^*_0 \right) \left( Y^*_0 \right) \left( Y^*_0 \right) \left( Y^*_0 \right) \left( Y^*_0 \right)$$

Equation (7.20) in Theorem 7.3 now follows from (7.66), (7.67). This completes the proof of Theorem 7.3.

7.9. A translation of the variable $Y^{TX}$. We begin the proof of Theorem 7.13. We proceed as in [B11, section 9.8].

Recall that $a \in p$. As explained in [B11, section 2.17], there are sections $a^{TX}, a^N$ of $TX, N$ that are such that via the identification $TX \otimes N = g$, we have the identity $a = a^{TX} + a^N$. The section $a^{TX}$ is just the vector field associated with $a \in g$ via the left action of $G$ on $X$. By [B11, eq. (2.17.10)], if $A \in TX$, then

$$\left[ A, a^N \right] = 0, \quad \left[ A, a^{TX} \right] = 0.$$

If $e \in TX \otimes N$, $\nabla_e^L$ denotes differentiation along $e$. Let $L_a$ be the Lie derivative operator associated with $a \in p$. Then $L_a$ acts on $\mathcal{H}$. By [B11, eq. (2.18.1)], this action is given by

$$L_a = \nabla^\infty_{a^{TX}} + \left[ A, a^N \right] = 0, \quad \nabla_a^N a^N + \left[ A, a^{TX} \right] = 0.$$

In (7.69), the fibrewise Lie derivative operator $L^V_{[a^N, Y]}$ is given by

$$L^V_{[a^N, Y]} = \nabla^V_{[a^N, Y]} - \left( e + \frac{\partial}{\partial \phi} \right) (a^N).$$

Now we follow [B11, Definition 9.8.1].

Definition 7.15. Set

$$T_a s \left( x, Y^{TX}, Y^N \right) = s \left( x, a^{TX} + Y^{TX}, Y^N \right).$$

By [B11, eq. (9.8.7)] or by using (7.69), (7.70), we get

$$T_a L_a T_a^{-1} = L_a.$$
As in [B11, eq. (9.8.2)], set

\[ N_{x}^{a,b,\vartheta} = T_{a} \Sigma_{b,\vartheta} T_{a}^{-1}, \quad N_{x}^{a} = T_{a} \Sigma_{x} T_{a}^{-1}. \]  

(7.74)

Now we establish an extension of [B11, Proposition 9.8.2].

Proposition 7.16. The following identity holds:

\[
\mathcal{N}_{a}^{x_{r}} = \frac{b^4}{2 \cos^2(\vartheta)} \left| [Y^{\perp}, a^{T}X + Y^{T}X] \right|^2 + \frac{1}{2} \left( -\frac{\cos^2(\vartheta)}{b^4} \Delta_{X^{r}_{\perp}} \right.
\]
\[
+ \frac{1}{\cos^2(\vartheta)} \left| a^{T}X + Y^{T}X \right|^2 + \left| Y^{\perp} \right|^2 - \frac{1}{b^2} (m + \cos(\vartheta)n) \right)
\]
\[
+ \frac{N^{r_{X}^{T}=N,\vartheta}}{b^2} + \alpha - \left( \nabla_{a,T_{X}^{r}+Y^{T}X}^{L^{r_{X}^{T}=N,\vartheta}} \right) f_{s,\tilde{f}}
\]
\[
- i\vartheta \left( \theta \text{ad}(Y^{N}) \right) - i\vartheta \left( \text{ad}(Y^{N}) \right) + \nabla_{[\hat{X},a,T_{X}^{r}+Y^{T}X]}^{L^{r_{X}^{T}=N,\vartheta}}
\]
\[
+ \frac{d\vartheta}{\sqrt{2}b} \left( \frac{b^3 \sin(\vartheta)}{\cos^2(\vartheta)} \right) \left( \frac{\vartheta}{\vartheta} \right) \left( [Y^{\perp}, a^{T}X + Y^{T}X] \right)
\]
\[
+ \frac{\sin(\vartheta)}{b} \left( \Delta^{T}X - \Delta^{Y^{N}} \right) + \frac{b}{\cos^2(\vartheta)} \left( \frac{\vartheta}{\vartheta} \right) \left( \Delta^{T}X + \Delta^{Y^{T}X} \right)
\].

Proof. Using (7.28), (7.29), (7.53), and (7.68), we get (7.75). \qed

Put

\[ \mathfrak{D}_{a,b,\vartheta}^{x_{r}} = \mathfrak{N}_{a,b,\vartheta}^{x_{r}} + L_{a}, \quad \mathfrak{D}_{a}^{x_{r}} = \mathfrak{N}_{a}^{x_{r}} + L_{a}. \]  

(7.76)

By (7.52), (7.72)–(7.74), and (7.76), we get the analogue of [B11, eq. (9.8.8)],

\[
\mathfrak{D}_{a,b,\vartheta}^{x_{r}} = T_{a} \exp \left( -\frac{b^2}{\cos(\vartheta)} \hat{\alpha} \right) \left( \mathfrak{L}_{b,\vartheta}^{x_{r}} + L_{a} \right) \exp \left( \frac{b^2}{\cos(\vartheta)} \hat{\alpha} \right) T_{a}^{-1},
\]
\[
\mathfrak{D}_{a}^{x_{r}} = T_{a} \exp \left( -\frac{b^2}{\cos(\vartheta)} \hat{\alpha} \right) \left( \mathfrak{L}_{b=0}^{x_{r}} + L_{a} \right) \exp \left( \frac{b^2}{\cos(\vartheta)} \hat{\alpha} \right) T_{a}^{-1}.
\]

(7.77)

Now we extend [B11, Proposition 9.8.3].
Proposition 7.17. The following identity holds:

\[
(7.78) \quad \mathcal{D}_a^{X^t} = \frac{b^k}{2 \cos^2 (\vartheta)} \left[ |Y^N, a^{TX} + Y^{TX}|^2 + \frac{1}{2} \left( - \frac{\cos^2 (\vartheta)}{b^4} \Delta_{TX \oplus N} \right. \right. \\
+ \left. \left. \frac{1}{\cos^2 (\vartheta)} \left| a^{TX} + Y^{TX} \right|^2 + \left| Y^N \right|^2 - \frac{1}{b^2} (m + \cos (\vartheta) n) \right) \right] \\
+ \frac{N_a^{T \oplus N} Y}{b^2} + \alpha \left( \nabla_{Y^{TX}, a, TX + 2Y^{TX} + Y_N} \left[ \right] + \frac{d\vartheta}{2 \sqrt{b}} \left( \frac{b^3 \sin (\vartheta)}{\cos^2 (\vartheta)} i c \left( \left[ Y^N, a^{TX} + Y^{TX} \right] \right) \right. \right. \\
\left. \left. + \frac{\sin (\vartheta)}{b} \left( \bar{D}^T \mathcal{X} - i \mathcal{D} \mathcal{N} \right) + \frac{b}{\cos^2 (\vartheta)} \hat{c} \left( \pi^T X + Y^{TX} \right) \right) \right).
\]

Proof. By (4.69), (7.69), (7.70), (7.75), and (7.76), we get (7.78). \qed

Remark 7.18. Most of the terms in equation (7.78) for \( \mathcal{D}_a^{X^t} |_{d\vartheta = 0} = \mathcal{D}_a^{X^t} |_{d\vartheta = 0} \) can be obtained from the corresponding terms in [B11, eq. (9.8.5)] for \( \mathcal{D}_a^{X^t} |_{d\vartheta = 0} \) by replacing \( b \) by \( b/\cos^{1/2} (\vartheta) \). Since \( b/\cos^{1/2} (\vartheta) \geq b, \) when \( b \to +\infty, \) the presence of \( \vartheta \) improves the situation with respect to [B11]. The only term for which this is not the case is \( \frac{1}{2 \cos^2 (\vartheta)} |a^{TX} + Y^{TX}|^2. \) However, because this term is positive, the fact that it is larger than when \( \vartheta = 0 \) will work again in our favour.

7.10. A coordinate system and a trivialization of vector bundles near \( X (\gamma) \). If \( x_0 = p1 \in X (\gamma) \), we take the same coordinate system on \( X \) near \( x_0 \) and the same trivialization of the vector bundles as in [B11, section 9.9]. In particular, \( S_{\mathcal{X}_N} \) is treated exactly like \( F \).

We proceed as in [B11, Definitions 9.9.1 and 9.10.1]. Let \( \nabla^H \) denote differentiation in the coordinate \( y \in \mathfrak{p} \), and \( \nabla^V \) denote differentiation in \( Y \in \mathfrak{g} \). Also \( Y \in \mathfrak{g} \) splits as \( Y = Y^p + Y^\xi, Y^p \in \mathfrak{p}, Y^\xi \in \mathfrak{e} \).

Definition 7.19. If \( Y_0^e \in \mathfrak{e} (\gamma) \), set

\[
(7.79) \quad H_{b, \vartheta, Y_0} s (y, Y) = s \left( \cos (\vartheta) y/b^2, Y_0^e + \cos (\vartheta) Y/b^2 \right).
\]

---

Footnote 3: In [B11, Definition 9.9.1], \( Y_0^e \in \mathfrak{e} (\gamma) \) should be corrected to \( Y_0^e \in \mathfrak{e} (\gamma) \).
In (7.83), \( \delta \) operator was defined in [B11, eq. (4.1.11)].

Set Definition 7.21.

Put (7.80)

\[
P_{a,b,\vartheta,Y_0}^{X_t} = H_{b,\vartheta,Y_0}^{-1} \mathcal{D}_{a,b,\vartheta} H_{b,\vartheta,Y_0}^{-1},
\]

(7.81)

\[
\mathcal{E}_{a,b,\vartheta,Y_0}^{X_t} = H_{b,\vartheta,Y_0}^{-1} \frac{d\vartheta}{\sqrt{2b}} \left( \frac{b^3 \sin(\vartheta)}{\cos^2(\vartheta)} ic \left( \left[ Y^N, a^TX + Y^TX \right] \right) 
+ \frac{\sin(\vartheta)}{b} \left( D^T + iD^N \right) + \frac{b}{\cos^2(\vartheta)} \hat{c} \left( a^TX + Y^TX \right) \right) H_{b,\vartheta,Y_0}^{-1},
\]

Put (7.82)

\[
P_{a,Y_0}^{X_t} = P_{a,b,\vartheta,Y_0}^{X_t} + \mathcal{E}_{a,b,\vartheta,Y_0}^{X_t}.
\]

**Definition 7.20.** Let \( p_{a,b,\vartheta,Y_0}^{X_t} ((y, Y), (y', Y')) \) be the smooth kernel for the operator \( \exp(-P_{a,b,\vartheta,Y_0}^{X_t}) \) with respect to \( dy'dY' \). Set

(7.82)

\[
\tau_{a,b,\vartheta,Y_0}^{X_t} ((y, Y), (y', Y')) = -\mathcal{E}_{a,b,\vartheta,Y_0}^{X_t} P_{a,b,\vartheta,Y_0}^{X_t} ((y, Y), (y', Y')).
\]

By proceeding as in [B11, eq. (9.9.11)], we get

(7.83)

\[
(\frac{b^2}{\cos(\vartheta)})^{-2m-n} \gamma^{X_t}_{a,\vartheta} \left( \left( e^{\cos(\vartheta)f/b^2} a^TX + \cos(\vartheta)Y^TX/b^2, Y_0^t + \cos(\vartheta)Y^N, 1/b^2 \right), \right.
\]

\[
\left. \gamma \left( e^{\cos(\vartheta)f/b^2} a^TX + \cos(\vartheta)Y^TX/b^2, Y_0^t + \cos(\vartheta)Y^N, 1/b^2 \right) \right) \delta (\cos(\vartheta)f/b^2)
\]

\[
= \text{Ad}(k^{-1}) |_{\Lambda(g')} \otimes \rho^{\mathfrak{g}}(k^{-1}) \otimes \rho^{\mathfrak{e}}(k^{-1}) \tau_{a,b,\vartheta,Y_0}^{X_t} ((f, Y), \text{Ad}(k^{-1}) (f, Y)).
\]

In (7.83), \( \delta \) is the Jacobian of the geodesic exponential map based at \( x_0 = p1 \). That function was defined in [B11, eq. (4.1.11)].

**7.11. The asymptotics of the operator \( P_{a,Y_0}^{X_t} \), as \( b \to +\infty \).** Recall that the operator \( P_{a,Y_0}^{X_t} \) was defined in Definition 7.12.

**Definition 7.21.** Set

(7.84)

\[
Q_{a,\infty,\vartheta,Y_0^t}^{X_t} = P_{a,Y_0^t} + \frac{1}{2} \left( \frac{|a|^2}{\cos^2(\vartheta)} + |Y_0^t|^2 \right) + i\hat{c} \left( \text{ad}(Y_0^t) |_{\mathfrak{p}} \right) + i\rho^{\mathfrak{e}}(Y_0^t),
\]

(7.85)

\[
Q_{a,\infty,Y_0^t}^{X_t} = Q_{a,\infty,\vartheta,Y_0^t}^{X_t} + \mathcal{E}_{a,\infty,\vartheta,Y_0^t}^{X_t}.
\]
Let $p_{a,\infty,\vartheta,Y_0^t}^{X_t}(y,Y),(y',Y'))$ be the smooth kernel for $\exp\left(-\mathcal{P}_{a,\infty,\vartheta,Y_0^t}^{X_t}\right)$ with respect to $dy'dY'$. Set
\begin{equation}
(7.86) \quad r_{a,\infty,\vartheta,Y_0^t}^{X_t}(y,Y),(y',Y')) = -\mathcal{S}_{a,\infty,\vartheta,Y_0^t} p_{a,\infty,\vartheta,Y_0^t}^{X_t}(y,Y),(y',Y')) .
\end{equation}

By (7.84), we get
\begin{equation}
(7.87) \quad p_{a,\infty,\vartheta,Y_0^t}^{X_t}(y,Y),(y',Y')) = \exp\left(-\frac{1}{2} \left( \frac{|a|^2}{\cos^2(\vartheta)} + |Y_0^t|^2 \right) \right)
R_{Y_0^t}(y,Y),(y',Y')) exp\left(-i\vartheta (\text{ad} (Y_0^t))_{\vartheta} \right). 
\end{equation}

For $\vartheta = 0$, $\mathcal{P}_{a,\infty,\vartheta,Y_0^t}^{X_t}$ coincides with the operator $\mathcal{P}_{a,0,\infty,Y_0^t}^{X_t}$ in [B11, eqs. (9.10.1), (9.10.2)].

By [B11, eq. (9.10.4)], we have
\begin{equation}
(7.88) \quad a_{T,X_0}^{TX} = a + O\left(|y|^2/b^4\right), \quad a_{y/b^2}^{N} = \frac{|a|}{b^2} + O\left(|y|^3/b^6\right).
\end{equation}

Using (7.64), (7.78), (7.80), (7.88), and proceeding as in the proof of [B11, Theorem 9.10.2], we find that as $b \to +\infty$, 
\begin{equation}
(7.89) \quad \mathcal{P}_{a,Y_0^t}^{X_t} \to \mathcal{P}_{a,\infty,Y_0^t}^{X_t}.
\end{equation}

The convergence in (7.89) just means that the coefficients of the operators together with their derivatives of arbitrary order converge uniformly on compact sets.

By Remark 7.18, since with respect to [B11], for most terms in (7.78), the estimate of the difference in (7.89) is better than in [B11], i.e., where positive powers of $1/b$ appear in [B11], they are replaced here by positive powers of $\cos^{1/2}(\vartheta)/b$, which are smaller. The only exception comes from the term $\frac{1}{\cos(\vartheta)} |a^{TX} + Y^{TX}|^2$. However, because of (7.88), proceeding as in [B11, eq. (9.10.6)], we get
\begin{equation}
(7.90) \quad \frac{1}{\cos^2(\vartheta)} \left|a^{TX} + \cos(\vartheta) Y^{TX} / b^2\right|_{\cos(\vartheta)y/b^2}^2 = \frac{|a|^2}{\cos^2(\vartheta)}
+ \frac{1}{\cos^2(\vartheta)} \left(O\left(\cos^4(\vartheta) |y|^4 / b^8 + \cos^2(\vartheta) |Y^{TX}|^2 / b^4\right)\right)
+ \frac{1}{\cos^2(\vartheta)} \left(O\left(|a| \left(\cos^2(\vartheta) |y|^2 / b^4 + \cos(\vartheta) |Y^{TX}| / b^2\right)\right)\right).
\end{equation}

In (7.90), among the small terms as $b \to +\infty$, the only potentially annoying term, not uniform in $\vartheta$ is given by $\frac{1}{\cos(\vartheta)} O\left(\cos(\vartheta) |Y^{TX}| / b^2\right)$. But dominated convergence allows us to ignore these questions of uniformity.

7.12. A proof of Theorem 7.13. Now, we establish an analogue of [B11, Theorem 9.11.1].

**Theorem 7.22.** As $b \to +\infty$,
\begin{equation}
(7.91) \quad r_{a,b,\vartheta,Y_0^t}^{X_t}(y,Y),(y',Y')) \to r_{a,\infty,\vartheta,Y_0^t}^{X_t}(y,Y),(y',Y')).
\end{equation}

**Proof.** Given $\vartheta \in \left[0,\frac{\pi}{2}\right]$, the structure of the operator $\mathcal{P}_{a,Y_0^t}^{X_t}$ is very similar to the structure of the operator $\mathcal{P}_{a,0,b,Y_0^t}^{X_t}$ that was considered in [B11, section 9.10 and 15.10]. In view of (7.89) and of the considerations that follow, the proof of (7.91) is exactly the same as the proof of [B11, Theorem 9.11.1].
We are now ready to prove Theorem 7.13. Indeed by (7.83) and by equation (7.91) in Theorem 7.22, as \( b \to +\infty \),

\[
(b^2 / \cos (\vartheta))^{-2m-n} \widetilde{T}_{\varphi} \text{odd} \left[ \gamma \tau^{X'}_{b,0} \left( e^{\cos(\vartheta)f/b^2} a^{TX}_{e^{\cos(\vartheta)f/b^2}} \cos (\vartheta) Y^{TX} / b^2, Y_0^t \right)
\]

\[
+ \cos (\vartheta) Y^{N.\perp} / b^2 \right) \), \gamma \left( e^{\cos(\vartheta)f/b^2} a^{TX}_{e^{\cos(\vartheta)f/b^2}} + \cos (\vartheta) Y^{TX} / b^2,
\]

\[
Y_0^t + \cos (\vartheta) Y^{N.\perp} / b^2 \right] \to \widetilde{T}_{\varphi} \text{odd} \left[ \Ad (k^{-1}) |_{\Lambda (\mathfrak{g}^*)} \otimes \rho^{S_{\mathfrak{g}}} (k^{-1}) \otimes \rho^{E} (k^{-1})
\]

\[
\tau^{X'}_{a,\infty,\vartheta,Y_0^t} \left( (f, Y) , \Ad (k^{-1}) (f, Y) \right) \right].
\]

By (7.64), (7.84), the operator \( \Psi^{X'}_{a,\infty,\vartheta,Y_0^t} \) is even in every possible way, including in the Clifford variables in \( c(\mathfrak{g}) \). By (7.83), (7.86), (7.87), and by equation (7.91) in Theorem 7.22, we get equation (7.65) in Theorem 7.13. In that equation, we replaced \( S_{\mathfrak{g}} \) by \( S_{\mathfrak{p}} \), since the distinction has now become irrelevant. This concludes the proof of Theorem 7.13.
8. AN EXPlicit FORMULA FOR THE ODD ORBITAL INtegrALS

In this section, as an application of the results of section 7, we give a simple formula for the orbital integrals \( \text{Tr}^{[\gamma]} [D^X \exp \left( -tD^{X,2}/2 \right)] \). In particular, we recover all the results by Moscovici-Stanton [MoSt89] on the explicit evaluation of such orbital integrals.

This section is organized as follows. In subsection 8.1, using Theorem 7.2, we give a formula for \( \text{Tr}^{[\gamma]} [D^X \exp \left( -tD^{X,2}/2 \right)] \).

In subsection 8.2, we find conditions under which the above orbital integrals vanish identically. Our conditions are exactly the ones in Moscovici-Stanton [MoSt89].

In subsection 8.3, we establish a simple convolution identity.

Finally, in subsection 8.4, when the orbital integrals do not vanish, we give an explicit formula for these orbital integrals in terms of characteristic forms on \( X(\gamma) \). We recover this way the explicit geometric formulas of Moscovici-Stanton [MoSt89].

We make the same assumptions and we use the same notation as in sections 6 and 7. In particular \( m \) is still assumed to be odd, and \( \gamma \) to be semisimple and nonelliptic.

8.1. A reformulation of Theorem 7.2. Let \( \Delta^{s(\gamma)} \) be the usual Laplacian on \( \mathfrak{z}(\gamma) = \mathfrak{p}(\gamma) \oplus \mathfrak{k}(\gamma) \) with respect to the scalar product induced by the scalar product of \( \mathfrak{g} \). For \( t > 0 \), let \( \exp \left( t\Delta^{s(\gamma)}/2 \right) \) be the corresponding heat operator, and let \( \exp \left( t\Delta^{s(\gamma)}/2 \right) \left( (y, Y_k^s), (y', Y_k'^s) \right) \) be the associated Gaussian heat kernel with respect to \( dy'dY_k^s \).

Let \( (y, Y_k^s) \) be the generic element in \( \mathfrak{z}(\gamma) = \mathfrak{p}(\gamma) \oplus \mathfrak{k}(\gamma) \). Observe that

\[
\text{(8.1)} \quad J_{\gamma} (Y_k^s) \text{Tr}^\mathfrak{p} \left[ \frac{c(a)}{|a|} \text{Ad} \left( k^{-1} \right) \exp \left( -ic \left( \text{ad} \left( Y_k^s \right) \right) \right) \right] \text{Tr}^\mathfrak{E} \left[ \frac{\rho^E}{k-1} \exp \left( -i\rho^E \left( Y_k^s \right) \right) \right] \delta_y = a
\]

is a distribution on \( \mathfrak{z}(\gamma) \). For \( t > 0 \), the heat operator \( \exp \left( t\Delta^{s(\gamma)}/2 \right) \) can be applied to this distribution and we obtain this way a smooth function on \( \mathfrak{z}(\gamma) \).

Theorem 8.1. For any \( t > 0 \), the following identity holds:

\[
\text{(8.2)} \quad \text{Tr}^{[\gamma]} \left[ D^X \exp \left( -sD^{X,2}/2 \right) \right] \star \frac{1}{\sqrt{s}} (t) = \sqrt{2\pi} \exp \left( \frac{t}{48} \text{Tr}^\mathfrak{p} \left[ C^{E,\mathfrak{p}} \right] + \frac{t}{2} C^{E,\mathfrak{p}} \right) \int_{\mathfrak{z}(\gamma)} \exp \left( -\frac{1}{2t} \left( |a|^2 + |Y_k^s|^2 \right) \right) J_{\gamma} (Y_k^s) \text{Tr}^\mathfrak{p} \left[ \frac{c(a)}{|a|} \text{Ad} \left( k^{-1} \right) \exp \left( -ic \left( \text{ad} \left( Y_k^s \right) \right) \right) \right] \text{Tr}^\mathfrak{E} \left[ \frac{\rho^E}{k-1} \exp \left( -i\rho^E \left( Y_k^s \right) \right) \right] \frac{dY_k^s}{(2\pi t)^{n/2}}.
\]
Equivalently, for any \( t > 0 \),

\[
(8.3) \quad \cT \left[ D^X \exp \left( -sD^X, 2 \right) \right] * \frac{1}{\sqrt{s}} (t) = \sqrt{2\pi} \exp \left( \frac{t}{48} \cT \left[ C^{t,E} \right] + \frac{t}{2} C^{t,E} \right) \\
\exp \left( \frac{t}{2} \Delta s (\gamma) \right) \left( J_\gamma \left( Y_0^t \right) \cT \left[ \frac{c(a)}{|a|} \Ad (k^{-1}) \exp (-ic (\Ad (Y_0^t) |p)) \right] \right) \\
C^{t,E} \left[ \rho^E \left( k^{-1} \right) \exp (-i\rho^E (Y_0^t)) \right] \delta_{Y_0^t = a} (0) .
\]

Proof. Recall that if \( e \in \mathfrak{p} \), the action of \( \hat{c}(e) \) on \( S^p \) is given by \( \hat{c}(e) = ic(e) \). When \( t = 1 \), by equation (7.19) in Theorem 7.2, we get (8.2). For an arbitrary \( t > 0 \), (8.2) is just the same equation with \( t = 1 \) when \( B \) is replaced by \( B/t \). By (8.2), we get (8.3).

From now on, we assume that the representation \( \rho^E : K \to U(E) \) is irreducible. In particular \( C^{t,E} \) is scalar. Let \( T \) be a maximal torus in \( K \), and let \( \mathfrak{t} \) be its Lie algebra. Let \( R_\pm \) be an associated positive root system, and let \( \rho \) be the half-sum of the positive roots. Let \( \lambda \in \mathfrak{t}^* \) be the nonnegative weight that defines the irreducible representation \( \rho^E \).

By [B11, eq. (7.2.15) and Proposition 7.5.2], we get

\[
(8.4) \quad \frac{1}{48} \cT \left[ C^{t,E} \right] + \frac{1}{2} C^{t,E} = -2\pi^2 |\rho + \lambda|^2 .
\]

Theorem 8.2. For any \( t > 0 \), the following identity holds:

\[
(8.5) \quad \cT \left[ D^X \exp \left( -tD^X, 2 \right) \right] = \sqrt{2\pi} \left( \frac{d}{dt} \right) \left( -2\pi^2 |\rho + \lambda|^2 \right) \\
\exp \left( \frac{t}{2} \Delta s (\gamma) \right) \left( J_\gamma \left( Y_0^t \right) \cT \left[ \frac{c(a)}{|a|} \Ad (k^{-1}) \exp (-ic (\Ad (Y_0^t) |p)) \right] \right) \\
C^{t,E} \left[ \rho^E \left( k^{-1} \right) \exp (-i\rho^E (Y_0^t)) \right] \delta_{Y_0^t = a} (0) * \frac{1}{\sqrt{s}} (t) .
\]

Proof. By (2.46) and (8.3), we get (8.5).

8.2. The vanishing of the orbital integrals. Since \( \mathfrak{p} \) is odd dimensional and since \( \Ad (k) \) preserves the orientation of \( \mathfrak{p} \), \( \dim \mathfrak{p} (k) \) is odd. Since \( a \in \mathfrak{p} (k) \), then \( \dim \mathfrak{p} (k) \geq 1 \).

Now we make the same discussion as in [B11, section 7.9]. Set

\[
(8.6) \quad \mathfrak{b} = \{ e \in \mathfrak{p}, [e, t] = 0 \} .
\]

Since \( \mathfrak{p} \) is odd dimensional, \( \mathfrak{b} \) is also odd dimensional.

Put

\[
(8.7) \quad \mathfrak{h} = \mathfrak{b} \oplus t .
\]

By [Kn86, p. 129], \( \mathfrak{h} \) is a Cartan subalgebra of \( \mathfrak{g} \). Also \( \dim \mathfrak{t} \) is the complex rank of \( K \), and \( \dim \mathfrak{h} \) is the complex rank of \( G \).

By (7.3), we get

\[
(8.8) \quad K (\gamma) \subset Z (a) \cap Z (k) .
\]
Recall that $K^0(\gamma) \subset K(\gamma)$ is the connected component of the identity. Let $T(\gamma) \subset K^0(\gamma)$ be a maximal torus in $K^0(\gamma)$, and let $t(\gamma) \subset \mathfrak{t}(\gamma)$ be its Lie algebra. Since $T(\gamma)$ is commutative, and since by (8.8), $k$ commutes with $T(\gamma)$, we may and we will assume that $T(\gamma) \subset T, k \in T$. In particular $t(\gamma) \subset t$.

Set
\begin{equation}
(8.9) \quad b(\gamma) = \{ e \in p(k), [e, t(\gamma)] = 0 \}.
\end{equation}

Since $p(k)$ is odd dimensional, $b(\gamma)$ is also odd odd dimensional, so that
\begin{equation}
(8.10) \quad \dim b(\gamma) \geq 1.
\end{equation}

Also $a \in b(\gamma)$.

Moreover, since $t(\gamma) \subset t$, and $k \in T$,
\begin{equation}
(8.11) \quad b \subset b(\gamma).
\end{equation}

Here, we will recover a first result of Moscovici-Stanton \cite{MoSt89}.

**Theorem 8.3.** If $\dim b(\gamma) \geq 3$, for any $t > 0$,
\begin{equation}
(8.12) \quad \text{Tr}^{[\gamma]} [D^X \exp (-tD^{X,2}/2)] = 0.
\end{equation}

In particular, this is the case if $\dim b \geq 3$.

**Proof.** By (8.5), to establish (8.12), we only need to show that if $Y^t_0 \in \mathfrak{t}(\gamma)$,
\begin{equation}
(8.13) \quad \text{Tr}^{\mathfrak{p}} \left[ \frac{c(a)}{|a|} \text{Ad} (k^{-1}) \exp \left(-ic \left( \text{ad} (Y^t_0) \right) |p\right) \right] = 0.
\end{equation}

Using the adjoint action of $K^0(\gamma)$ on $\mathfrak{t}(\gamma)$, we may as well assume that $Y^t_0 \in \mathfrak{t}(\gamma)$. Then $\text{Ad} (k^{-1})$ acts like the identity on $b(\gamma)$, and $\text{ad} (Y^t_0)$ vanishes on $b(\gamma)$. Therefore $b(\gamma)$ lies in the eigenspace for the action on $p$ of $\text{Ad} (k^{-1}) \exp (-\text{ad} (Y^t_0))$ associated with the eigenvalue 1. If $\dim b(\gamma) \geq 3$, this eigenspace is of dimension $\geq 3$.

Let $\{a\}^\perp$ be the orthogonal space to $a$ in $p$. By the above, the eigenspace for the action of $\text{Ad} (k^{-1}) \exp (-\text{ad} (Y^t_0))$ on $\{a\}^\perp$ for the eigenvalue 1 is of dimension $\geq 2$. Let $e$ be a unit vector in this eigenspace. We have the identity
\begin{equation}
(8.14) \quad c(e), \frac{c(a)}{|a|} \text{Ad} (k^{-1}) \exp (-c \left( \text{ad} (Y^t_0) \right) |p\right) = 0.
\end{equation}

By (8.14), we get
\begin{equation}
(8.15) \quad \frac{c(a)}{|a|} \text{Ad} (k^{-1}) \exp (-c \left( \text{ad} (Y^t_0) \right) |p\right)) = \frac{1}{2} \left[ c(e), \frac{c(a)}{|a|} \text{Ad} (k^{-1}) \exp (-c \left( \text{ad} (Y^t_0) \right) |p\right) c(e) \right].
\end{equation}

Since $\text{Ad} (k^{-1}) \exp (-c \left( \text{ad} (Y^t_0) \right) |p\right) \in e^{even}(p)$, the right-hand side of (8.15) is a commutator. Since $\text{Tr}^{\mathfrak{p}}$ vanishes on commutators, by (8.15), we get
\begin{equation}
(8.16) \quad \text{Tr}^{\mathfrak{p}} \left[ \frac{c(a)}{|a|} \text{Ad} (k^{-1}) \exp (-c \left( \text{ad} (Y^t_0) \right) |p\right) = 0.
\end{equation}

By analyticity, from (8.16), we get (8.13). The proof of our theorem is completed.
Remark 8.4. Now we reproduce the content of [B11, Remark 7.9.2]. For \( p, q \in \mathbb{N} \), let \( \text{SO}^0 (p, q) \) be the connected component of the identity in the real group \( \text{SO} (p, q) \). By [H78, Table V p. 518] and [Kn86, Table C1 p. 713, and Table C2 p. 714], among the noncompact simple connected complex groups such that \( m \) is odd and \( \dim b = 1 \), there is only \( \text{SL}_2 (\mathbb{C}) \), and among the noncompact simple real connected groups with the same property, there are only \( \text{SL}_3 (\mathbb{R}), \text{SL}_4 (\mathbb{R}), \text{SL}_2 (\mathbb{H}) \), and \( \text{SO}^0 (p, q) \) with \( pq \) odd > 1. Also by [H78, pp. 519, 520], \( \text{sl}_2 (\mathbb{C}) = \text{so} (3, 1), \text{sl}_4 (\mathbb{R}) = \text{so} (3, 3), \) \( \text{and} \text{sl}_2 (\mathbb{H}) = \text{so} (5, 1) \). Therefore the above list can be reduced to \( \text{SL}_3 (\mathbb{R}) \) and \( \text{SO}^0 (p, q) \) with \( pq \) odd > 1. This is exactly the list given by [MoSt89] that implies the vanishing of the odd traces in (8.12).

8.3. A convolution identity. For \( x \geq 0 \), set

\[
(8.17) \quad \phi (x) = \int_x^{+\infty} \exp (-\lambda) \lambda^{-1/2} d\lambda = 2 \int_{\sqrt{x}}^{+\infty} \exp (-\lambda^2) d\lambda.
\]

Proposition 8.5. For \( x > 0, t > 0 \),

\[
(8.18) \quad \frac{1}{\sqrt{s}} \exp (-x/s) * \frac{1}{\sqrt{s}} (t) = \sqrt{\pi} \phi (x/t).
\]

Proof. Clearly,

\[
(8.19) \quad \frac{1}{\sqrt{s}} \exp (-x/s) * \frac{1}{\sqrt{s}} (t) = \int_1^{+\infty} \exp \left( \frac{x s}{t} \right) \frac{1}{s^{1/2} - 1} ds = 2 \exp (-x/t) \int_0^{+\infty} \exp (-x s^2/t) (1 + s^2)^{-1} ds.
\]

Moreover,

\[
(8.20) \quad 2 \int_0^{+\infty} \exp (-x s^2/t) (1 + s^2)^{-1} ds = 2 \int_{\mathbb{R}_+^2} \exp (-x s^2/t - (1 + s^2) \lambda) d\lambda d\lambda\]

\[
= \sqrt{\pi} \int_0^{+\infty} \exp (-\lambda (\lambda + x/t)^{-1/2} d\lambda = \sqrt{\pi} \exp (x/t) \int_{x/t}^{+\infty} \exp (-\lambda) \lambda^{-1/2} d\lambda.
\]

By (8.19), (8.20), we get (8.18). \( \square \)

8.4. The case where \( \dim b (\gamma) = 1 \). In the sequel, we assume that \( \dim b (\gamma) = 1 \).

By (8.11), \( b = b (\gamma) \) is 1-dimensional and generated by \( a \). Since \( \text{ad} (t) \) vanishes on \( a \), and \( k \in T \), we get

\[
(8.21) \quad t (\gamma) = t, \quad T (\gamma) = T.
\]

Note that \( \text{ad} (a) \) is an invertible endomorphism of \( \mathfrak{g}_0^1 \) that exchanges \( \mathfrak{p}_0^1 \) and \( \mathfrak{t}_0^1 \) and commutes with \( \text{Ad} (k^{-1}) \).

Let \( \{a\}^\perp \subset \mathfrak{p} \) be the orthogonal vector space to \( a \) in \( \mathfrak{p} \). We have the orthogonal splitting

\[
(8.22) \quad \{a\}^\perp = \{a\}^1 \cap \mathfrak{p}_0 \oplus \mathfrak{p}_0^1.
\]

\(^{4}\)I am indebted to Yves Benoist for providing the above information.
Then \( t \) preserves \( \{a\}^\perp \). Since \( t \subset \mathfrak{t}_0 \), \( t \) preserves the splitting (8.22). Since \( \mathfrak{b} \) is reduced to \( \{a\} \), \( \{a\}^\perp \cap \mathfrak{p}_0 \) and \( \mathfrak{p}_0^\perp \) are even dimensional, and preserved by \( T \). Since \( k \in T \), the action of \( \text{Ad}(k) \) on these two vector spaces preserves their orientation. In particular, the eigenspaces of \( k \) that are associated with the eigenvalue 1 are even dimensional.

As we saw in subsection 7.1, \( \mathfrak{p}_0 \) and \( \mathfrak{p}(k) \) intersect orthogonally along \( \mathfrak{p}(\gamma) \).

Since \( \{a\}^\perp \cap \mathfrak{p}_0 \) is even dimensional, \( \mathfrak{p}_0 \) is odd dimensional. Since \( T = T(\gamma) \), then \( T \) preserves \( \mathfrak{p}_0 \). In particular \( k \in T \) acts like an oriented isomorphism of \( \mathfrak{p}_0 \). Since \( \mathfrak{p}(\gamma) \) is the part of \( \mathfrak{p}_0 \) that is fixed by \( \text{Ad}(k) \), \( \mathfrak{p}(\gamma) \) is also odd dimensional, and \( \mathfrak{p}_0^\perp(\gamma) \) is even dimensional. In the same way, \( \text{Ad}(k) \) acts like an oriented isomorphism of \( \mathfrak{p}_0^\perp \). Since \( \mathfrak{p}_0^\perp \) is even dimensional, \( \mathfrak{p}_0^\perp \cap \mathfrak{p}(k) \), which is the vector subspace of \( \mathfrak{p}_0^\perp \) fixed by \( \text{Ad}(k) \), is even dimensional. Since \( \mathfrak{p}_0 \) and \( \mathfrak{p}(k) \) intersect orthogonally along \( \mathfrak{p}(\gamma) \), \( \mathfrak{p}_0^\perp \cap \mathfrak{p}(k) \) is just the orthogonal space to \( \mathfrak{p}(\gamma) \) in \( \mathfrak{p}(k) \).

We orient \( \{a\}^\perp \) so that when completing an oriented basis of \( \{a\}^\perp \) by \( a \), we obtain an oriented basis of \( \mathfrak{p} \). We orient the vector spaces in the right-hand side of (8.22), so that (8.22) is an identity of oriented vector spaces. The orientation of these two vector spaces is noncanonical. Since \( \text{ad}(a) \) induces an isomorphism from \( \mathfrak{p}_0^\perp \) into \( \mathfrak{t}_0^\perp \), we equip \( \mathfrak{t}_0^\perp \) with the corresponding orientation. Similarly, since \( \{a\}^\perp \cap \mathfrak{p}_0 \) is oriented, we orient \( \mathfrak{p}_0 \) by the procedure that was outlined before.

Since \( k \) acts as an oriented isometry of \( \mathfrak{p} \), \( \mathfrak{p}(k) \) is odd dimensional, and \( \mathfrak{p}^\perp(k) \) is even dimensional. Since \( k \in T \), there is \( t_0 \in \mathfrak{t} \) such that

\[
(8.23) \quad k = e^{t_0}.
\]

In particular \( \text{ad}(t_0) \) acts as an invertible endomorphism of \( \mathfrak{p}^\perp(k) \), so that \( \mathfrak{p}^\perp(k) \) is canonically oriented (the orientation depending on the choice of \( t_0 ) \). It follows that the orientation line \( o(\mathfrak{p}) \) of \( \mathfrak{p} \) is just the orientation line \( o(\mathfrak{p}(k)) \) of \( \mathfrak{p}(k) \), i.e.,

\[
(8.24) \quad o(\mathfrak{p}) = o(\mathfrak{p}(k)).
\]

Other orientations lines will be denoted in the same way. By (8.24), we get

\[
(8.25) \quad o(\mathfrak{p}) = o(\mathfrak{p}(\gamma)) \otimes o(\mathfrak{p}_0^\perp \cap \mathfrak{p}(k)).
\]

Since the simply connected group \( K \) acts on \( \mathfrak{S}^\mathfrak{p} \), the action of \( k \) on \( \mathfrak{S}^\mathfrak{p} \) is unambiguously determined by (3.70), (8.23).

Let \( S^{(a)}^\perp \) be the spinors associated with the oriented Euclidean vector space \( \{a\}^\perp \). Then \( S^{(a)}^\perp \) is a \( \mathbb{Z}_2 \)-graded vector space. Let \( \tau = \pm 1 \) be the endomorphism defining the \( \mathbb{Z}_2 \)-grading. Then

\[
(8.26) \quad \mathfrak{S}^\mathfrak{p} = S^{(a)}\mathfrak{S}.
\]

Moreover, \( c^{(a)} \) acts on \( S^{(a)}^\perp \) like \(-i\tau \). Let \( S^{(a)}^\perp \cap \mathfrak{p}_0, \mathfrak{S}^\mathfrak{p}^\perp_0 \) be the spinors associated with the oriented Euclidean vector spaces \( \{a\}^\perp \cap \mathfrak{p}_0, \mathfrak{p}_0^\perp \). Again, these vector spaces are \( \mathbb{Z}_2 \)-graded. Moreover, because of the splitting (8.22), we get

\[
(8.27) \quad S^{(a)}^\perp = S^{(a)}^\perp \cap \mathfrak{p}_0 \otimes \mathfrak{S}^\mathfrak{p}^\perp_0.
\]

The simply connected group \( K \) acts on \( \mathfrak{S}^\mathfrak{p} = S^{(a)}\mathfrak{S} \). Recall that \( K^0_0 \) is the connected component of the identity in \( K_0 = Z(a) \cap K \). Since \( K^0_0 \subset K \), \( K^0_0 \) also acts on \( S^{(a)}^+ \). However, \( K^0_0 \) is not necessarily simply connected. While \( K^0_0 \) preserves the splitting (8.22), the action of \( K^0_0 \) does not necessarily lift to an action
of $K_0^0$ on $S^{(a)} \otimes p_0$, $S^{p_0}$, the possible lift having a $\pm 1$ ambiguity. However, because of (8.27), the ambiguity is the same when acting on both vector spaces.

Recall that (7.1) holds. By the results of [B11] that were explained in subsection 7.1, $X (e^a)$ can be identified with the symmetric space associated with $Z_0^0$, so that $X (e^a) = Z_0^0 / K_0^0$. Then $TX (e^a)$ is the vector bundle associated with the action of $K_0^0$ on $p_0$, and the normal bundle $N_{X (e^a) / X}$ is associated with the action of $K_0^0$ on $p_0 \perp$. In particular $X (e^a)$ is odd dimensional. Moreover, $k^{-1}$ acts on $X (e^a)$, and its fixed point set is given by $X (\gamma)$. Also $N_{X (\gamma) / X (e^a)}$ is just the vector bundle on $X (\gamma)$ associated with the action of $K_0^0 (\gamma)$ on $p_0^0 (\gamma)$.

Then $k^{-1}$ acts naturally on $TX (e^a) \mid_{X (\gamma)}$. Also $TX (\gamma)$ is the eigenbundle of this action associated with the eigenvalue 1. The distinct angles $\pm \theta_1, \ldots, \pm \theta_s$, $0 < \theta_i \leq \pi$ of the action of $k^{-1}$ on $N_{X (\gamma) / X (e^a)}$ are exactly the nonzero angles of the action of $\text{Ad} (k^{-1})$ on $p_0^0 (\gamma)$. Let $N_{X (\gamma) / X (e^a)} \mid_{X (\gamma)} \theta_i, 1 \leq i \leq s$ be the part of $N_{X (\gamma) / X (e^a)}$ on which $\text{Ad} (k^{-1})$ acts by a rotation of angle $\theta_i$.

We will consider characteristic forms of homogeneous vector bundles on $X (\gamma)$. Since these vector bundles are equipped with canonical connections, when noting their corresponding characteristic forms, we will not note the connection forms explicitly.

If $\theta \in \mathbb{R} \setminus 2\pi \mathbb{Z}$, set

$$\hat{A}^\theta (x) = \frac{1}{2 \sinh \left( \frac{x + i \theta}{2} \right)}.$$ (8.28)

Given $\theta$, we identify $\hat{A}^\theta (x)$ with the corresponding multiplicative genus. We define the following closed form on $X (\gamma)$,

$$\hat{A}^{k^{-1}} (TX (e^a) \mid_{X (\gamma)}) = \hat{A} (TX (\gamma)) \prod_{i=1}^s \hat{A}^{\theta_i} (N_{X (\gamma) / X (e^a)} \mid_{X (\gamma)}),$$ (8.29)

As usual in such formulas, there is a $\pm 1$ sign ambiguity in the right-hand side of (8.29). However, because the action of $k \in T$ lifts to $S^p$, the ambiguity disappears when considering instead $\left( \hat{A}^{k^{-1}} \mid_{p_0^0} (0) \right)^{-1} \hat{A}^{k^{-1}} (TX (e^a))$

Similarly, the group $Z (a)$ acts on $E$ via the representation $\rho^E$. The corresponding vector bundle on $X (e^a)$ is just the restriction of $E$ to $X (e^a)$. Also $k^{-1}$ acts on the left on this vector bundle. Recall that $R^E$ is the curvature of $\nabla^E$. Let $\text{ch}^{k^{-1}} (F)$ denote the Chern character form on $X (\gamma)$ that is given by

$$\text{ch}^{k^{-1}} (F \mid_{X (\gamma)}) = \text{Tr} \left[ \rho^E (k^{-1}) \exp \left( -\frac{R^E \mid_{X (\gamma)}}{2i\pi} \right) \right].$$ (8.30)

The closed forms in (8.29), (8.30) on $X (\gamma)$ are exactly the ones that appear in the Lefschetz fixed point formula of Atiyah-Bott [AB67, AB68] when considering the action of $k^{-1}$ on $X (e^a)$. Note that there are questions of signs to be taken care of, because of the need to distinguish between $\theta_1$ and $-\theta_1$. We refer to the above references for more detail.

Let $N_{X (e^a) / X} (k)$ be the subvector bundle of $N_{X (e^a) / X} \mid_{X (\gamma)}$ that is fixed by $k$. This vector bundle is associated with the action of $K_0^0 (\gamma)$ on $p_0^0 \cap p (k)$. Since $p_0^0 \cap p (k)$ is even dimensional, $N_{X (e^a) / X} (k)$ is an even dimensional vector bundle. Since
$X(\gamma)$ and $X(e^a)$ intersect orthogonally, $N_{X(e^a)/X}(k)$ is also the normal bundle $N_{X(\gamma)/X(k)}$.

Let $e(N_{X(\gamma)/X(k)})$ denote the Euler form of $N_{X(\gamma)/X(k)}$ on $X(\gamma)$. The form $e(N_{X(\gamma)/X(k)})$ is a section of $\Lambda^{\dim p^+ \cap p^+ (T^*X(\gamma))} \otimes o(N_{X(\gamma)/X(k)})$. By (8.25), the form $e(N_{X(\gamma)/X(k)})$ can be considered as a section of $\Lambda^{\dim p^+ \cap p^+ (T^*X(\gamma))} \otimes o(TX(\gamma)) \otimes o(TX)$. Also $o(TX(\gamma)) \otimes o(TX) = o(p(\gamma)) \otimes o(p)$.

Also $K_0^0$ preserves the splitting $\xi = t_0 \oplus t_0^\perp$. Let $N_0, N_0^\perp$ denote the corresponding vector bundles on $X(e^a)$.

The Lie group $Z_0$ acts on $z(a) = z_0$, and so it acts on $z_0^\perp = p_0^\perp \oplus t_0^\perp$. By proceeding as in [B11, eq. (7.7.5)], we find that on $X(e^a)$, $N_{X(e^a)/X} \oplus N_0^\perp$ is equipped with a Euclidean connection preserving the splitting, and also with a flat connection. Also $k^{-1}$ acts on the restriction of these vector bundles to $X(\gamma)$. Our characteristic forms will be computed using the relevant Euclidean connections. By proceeding as in [B11, eq. (7.7.5)], we get the identity of forms on $X(\gamma)$.

\begin{equation}
(8.31) \quad \hat{A}^{k^{-1}}(N_{X(e^a)/X}) \hat{A}^{k^{-1}}(N_0^\perp) = A^{k^{-1}}{z_0^\perp}(0).
\end{equation}

Also $ad(a)$ is a parallel isomorphism from $N_{X(e^a)/X}$ into $N_0^\perp$ with respect to their canonical Euclidean connections. It follows that on $X(\gamma)$, we have the identity of differential forms

\begin{equation}
(8.32) \quad \hat{A}^{k^{-1}}(N_{X(e^a)/X}) = \hat{A}^{k^{-1}}(N_0^\perp).
\end{equation}

By (8.31), (8.32), we get

\begin{equation}
(8.33) \quad \hat{A}^{k^{-1}}(N_{X(e^a)/X}) = A^{k^{-1}}{z_0^\perp}(0).
\end{equation}

Let $\eta|_{X(\gamma)}$ be the canonical section of norm 1 in $\Lambda^p(T^*X(\gamma)) \otimes o(TX(\gamma))$. Equivalently, $\eta|_{X(\gamma)}$ is the volume form on $X(\gamma)$. If $\alpha \in \Lambda^p(T^*X(\gamma)) \otimes o(TX(\gamma)) \otimes o(TX)$, and if $\alpha(p)$ denotes its component of top degree $p$, let $\alpha^\max$ be the section of $o(TX)$ given by

\begin{equation}
(8.34) \quad \alpha^\max = \frac{\alpha(p)}{\eta|_{X(\gamma)}}.
\end{equation}

We will also view $\alpha^\max$ as a section of $o(p)$.

Let $a^*$ be the 1-form on $X(\gamma)$ which is dual to $a^{TX}$. Put

\begin{equation}
(8.35) \quad a = \frac{a^*}{|a|}.
\end{equation}

As we saw in subsection 7.1, $Z_0^0$ is a reductive group. By the above, we know that $k \in Z_0^0$. To the couple $(Z_0(a), k^{-1})$, we will apply the constructions we made before for $(G, \gamma)$. The analogue of $\xi(\gamma)$ is still equal to $\xi(\gamma)$. Let $J_{k^{-1}}(Y_0^\gamma), Y_0^\gamma \in \xi(\gamma)$ be the function defined in [B11, Theorem 5.5.1] and in Definition 7.1 which is associated with the group $Z_0^0$ and with $k^{-1} \in Z_0^0$. By [B11, Theorem 5.5.1] or by (7.18), for $Y_0^\gamma \in \xi(\gamma)$, we get

\begin{equation}
(8.36) \quad J_{\gamma}(Y_0^\gamma) = \frac{J'_{k^{-1}}(Y_0^\gamma)}{\left|\det (1 - \Ad(\gamma))\right|_{k_0^\perp}^{1/2}}.
\end{equation}
We will now recover the explicit formula by Moscovici-Stanton [MoSt89, Theorem 5.10] for the orbital integrals $\text{Tr}^{[\gamma]} [D^X \exp (-t D^X, 2/2)]$. Since they depend on the choice of an orientation of $p$, they are just sections of $o(p)$.

**Theorem 8.6.** When $\dim b(\gamma) = 1$, for $t > 0$, the following identity of sections of $o(p)$ holds:

$$
\text{Tr}^{[\gamma]} [D^X \exp (-t D^X, 2/2)] = -i \frac{(-1)^{\dim p^0_+}/2}{\left| \det (1 - \text{Ad} (\gamma))_{|b^0_+} \right|^{1/2}} \max \left[ \hat{A}^{-1} (T X (e^{\gamma})) e (N_{X(\gamma)/X(k)}) \text{ch} (F) \frac{a}{\sqrt{2\pi}} \right]
$$

$$
= -i \frac{\sqrt{\pi} d t}{\sqrt{\pi} dt} \frac{1}{\sqrt{s}} \exp \left( -\frac{|a|^2}{2s} \right) \max \left[ \hat{A}^{-1} (T X (e^{\gamma})) e (N_{X(\gamma)/X(k)}) \text{ch} (F) \frac{a^*}{\sqrt{2\pi}} \right] t^{-3/2} \exp \left( -\frac{|a|^2}{2t} \right).
$$

**Proof.** By the considerations we made after equation (8.26), we get

$$
\text{Tr}^{\gamma} \left[ \frac{c(a)}{|a|} \text{Ad} (k^{-1}) \exp (-ic (\text{ad} (Y^*_0) \mid p)) \right] = -i \text{Tr}^{S^{(a)^-}} \left[ \text{Ad} (k^{-1}) \exp (-ic (\text{ad} (Y^*_0) \mid p)) \right].
$$

In the right-hand side of (8.38), the supertrace is associated with the $Z_2$-grading $S^{(a)^-} = S^{(a)^+} \oplus S^{(a)^-}$. In general $\text{ad} (a)$ does not induce an isometry from $p_0^+$ into $t_0^-$. However, it intertwines the action of $\text{Ad} (k^{-1}) \exp (-ic (Y^*_0))$ on these two vector spaces. Let $S^{t_0^-}$ denote the spinors associated with $t_0^-$. By (8.38), we get

$$
\text{Tr}^{\gamma} \left[ \frac{c(a)}{|a|} \text{Ad} (k^{-1}) \exp (-ic (\text{ad} (Y^*_0) \mid p)) \right] = -i \text{Tr}^{S^{(a)^+ \cap p_0}} \left[ \text{Ad} (k^{-1}) \exp \left( -ic \left( \text{ad} (Y^*_0) \mid (a)^+ \cap p_0 \right) \right) \right]
$$

$$
= \text{Tr}^{S^{t_0^-}} \left[ \text{Ad} (k^{-1}) \exp \left( -ic \left( \text{ad} (Y^*_0) \mid t_0^- \right) \right) \right].
$$
By (8.36), (8.39), we get

$$\exp \left( \frac{1}{2} \Delta^{s(\gamma)} \right) \left( J_{\gamma} \left( Y_{0}^{t} \right) \text{Tr}^{S_{0}} \left[ \frac{c(a)}{|a|} \text{Ad} \left( k^{-1} \right) \exp \left( -ic \left( \text{ad} \left( Y_{0}^{t} \right) \right) p \right) \right] \right)$$

$$= \text{Tr}^{E} \left[ \rho^{E} \left( k^{-1} \right) \exp \left( -i\rho^{E} \left( Y_{0}^{t} \right) \right) \right] \delta_{Y_{0}^{t} = a} (0)$$

$$= -i \exp \left( \frac{-|a|^2/2}{(2\pi)^{p/2}} \right) \exp \left( \frac{1}{2} \Delta^{k(\gamma)} \right)$$

$$\left( J_{k-1} \left( Y_{0}^{t} \right) \text{Tr}^{S_{0}} \left[ \text{Ad} \left( k^{-1} \right) \exp \left( -ic \left( \text{ad} \left( Y_{0}^{t} \right) \right) \right) \right] \right)$$

$$\text{Tr}^{S_{0}} \left[ \text{Ad} \left( k^{-1} \right) \exp \left( -ic \left( \text{ad} \left( Y_{0}^{t} \right) \right) \right) \right] \text{Tr}^{E} \left[ \rho^{E} \left( k^{-1} \right) \exp \left( -i\rho^{E} \left( Y_{0}^{t} \right) \right) \right].$$

Let \( \Lambda \subset \mathfrak{t} \) be the coroot lattice associated with \( K \), so that \( T = \mathfrak{t}/\Lambda \). Recall that \( R_{+} \subset \mathfrak{t}^{*} \) is a positive root system associated with the group \( K \). Let \( R_{+} ^{0} (a) \subset R_{+} \) be a positive subroot system associated with the group \( K_{0}^{0} = K^{0} (a) \). Let \( W, W_{0} \) be the Weyl groups of \( K, K_{0}^{0} \). Then \( W_{0} \subset W \).

Let \( \sigma (t), \sigma_{0} (t), t \in T \) be the denominators of Weyl’s character formulas for the groups \( K, K_{0}^{0} \). Then

$$\sigma (t) = \prod_{\alpha \in R_{+}} \left( e^{\pi \langle \alpha, t \rangle} - e^{-i\pi \langle \alpha, t \rangle} \right), \quad \sigma_{0} (t) = \prod_{\alpha \in R_{+}^{0}, a} \left( e^{i\pi \langle \alpha, t \rangle} - e^{-i\pi \langle \alpha, t \rangle} \right).$$

Since \( K \) is simply connected, the function \( \sigma (t) \) is well defined on \( T \). Since \( K_{0}^{0} (a) \) is not necessarily simply connected, the function \( \sigma_{0} (t) \) is only defined up to sign on \( T \). Moreover, for \( t \neq 0 \), we have the identity up to sign

$$\text{Tr}^{S_{0}^{+}} \left[ \text{Ad} \left( t^{-1} \right) \right] = \frac{\sigma}{\sigma_{0}} (t).$$

Note here again that by nature, both sides of (8.42) are only defined up to sign.

Let \( \chi_{\lambda} \) be the character of the representation of the irreducible representation \( \rho^{E} \) of \( K \) with highest weight \( \lambda \). By Weyl’s character formula, for \( t \in T \), we get

$$\frac{\sigma}{\sigma_{0}} (t) \chi_{\lambda} (t) = \frac{1}{\sigma_{0} (t)} \sum_{w \in W} \epsilon_{w} \exp \left( 2i\pi \langle \rho + \lambda, wt \rangle \right),$$

the two sides being well-defined up to sign. Note that (8.43) is \( W_{0} \)-invariant. In the sequel, we denote by \( \chi_{\lambda}^{0} \) the associated function on \( K_{0}^{0} \). By (8.42), (8.43), for \( t \in T \), we get

$$\chi_{\lambda}^{0} (t) = (-1)^{|R_{+} \setminus R_{+}^{0}|} \text{Tr}^{S_{0}^{+}} \left[ \text{Ad} \left( t \right) \right] \chi_{\lambda} (t).$$

Again, both sides in (8.44) have the same sign ambiguity.
By (8.43), we get

\[
\begin{align*}
     & \frac{1}{(2\pi)^{P/2}} \exp \left( \Delta_1^{(\gamma)} / 2 \right) \left( J_{k-1} \left( Y_0^* \right) \right) \\
     & \Tr_s S^{(\gamma) \cap r_{p_0}} \left[ \Ad (k^{-1}) \exp \left( -ic \left( \ad \left( Y_0^* \right) |_{(a) \cap p_0} \right) \right) \right] \\
     & \Tr_s \left[ \Ad (k^{-1}) \exp \left( -ic \left( \ad \left( Y_0^* \right) | \{a \} \cap p_0 \right) \right) \right] \Tr E \left[ \rho^E \left( k^{-1} \right) \exp \left( -i\rho^E \left( Y_0^* \right) \right) \right] (0) \\
     & = \frac{(-1)^{|R_+ \setminus R_{+,0}|}}{(2\pi)^{P/2}} \exp \left( \Delta_1^{(\gamma)} / 2 \right) \left( J_{k-1} \left( Y_0^* \right) \right) \\
     & \Tr_s S^{(\gamma) \cap r_{p_0}} \left[ \Ad (k^{-1}) \exp \left( -ic \left( \ad \left( Y_0^* \right) |_{(a) \cap p_0} \right) \right) \right] \chi^0_\lambda \left( k^{-1} e^{-iY_0^*} \right) (0).
\end{align*}
\]

Again, the last two terms in the right-hand side of (8.45) suffer from a ±1 ambiguity, but their product is unambiguously defined. We denote by \( L \) the expression in the right-hand side (8.45).

Let \( \Omega^{(\gamma)} \) be the curvature of the canonical connection on the \( K^0(\gamma) \) principal bundle \( Z^0(\gamma) \to X(\gamma) \). Then \( \Omega^{(\gamma)} \) is a \( \mathfrak{t}(\gamma) \)-valued 2-form. By proceeding as in [B11, eqs. (7.7.7)–(7.7.13)], we deduce from (8.43), (8.45) that

\[
(8.46) \quad L = \frac{\exp \left( 2\pi^2 |\rho + \lambda|^2 \right)}{(2\pi)^{P/2}} \left[ \hat{A}^{k^{-1}} \left( i\ad \left( \Omega^{(\gamma)} \right) |_{p_0} \right) \chi^0_\lambda \left( k^{-1} \exp \left( i\Omega^{(\gamma)} \right) \right) \right] a_{\text{max}}.
\]

The reason why \( a \) appears is because when following the methods of [B11], we should obtain forms of maximal degree on the orthogonal space to \( a \) in \( TX(\gamma) \). When multiplying by \( a \), we obtain the corresponding term of maximal degree on \( TX(\gamma) \).

By the argument that was given after (8.38), and by (8.44), we can rewrite (8.46) in the form

\[
(8.47) \quad L = \frac{\exp \left( 2\pi^2 |\rho + \lambda|^2 \right)}{(2\pi)^{P/2}} \left[ \hat{A}^{k^{-1}} \left( i\ad \left( \Omega^{(\gamma)} \right) |_{p_0} \right) \\
\Tr_s S^{(\gamma) \cap r_{p_0}} \left[ \Ad (k^{-1}) \exp \left( ic \left( \ad \left( \Omega^{(\gamma)} \right) | \{p \} \cap p_0 \right) \right) \right] \\
\Tr E \left[ \rho^E \left( k^{-1} \right) \exp \left( i\rho^E \left( \Omega^{(\gamma)} \right) \right) \right] a_{\text{max}}.
\]
By proceeding as in [B11, eq. (7.7.7)], we get

\[(8.48) \quad \text{Tr}_{x^{\mathfrak{p}}} \left[ \text{Ad} \left( k^{-1} \right) \exp \left( i c \left( \text{ad} \left( \Omega^\gamma \right) \right) \right) \right] = \text{Pf} \left[ -\text{ad} \left( \Omega^\gamma \right) \right] \tilde{A}^{-1} \left( i \text{ad} \left( \Omega^\gamma \right) \right) \left( \tilde{A}^{-1} \left( i \text{ad} \left( \Omega^\gamma \right) \right) \right)^{-1} \left( 0 \right)^{1}.
\]

By (8.47), (8.48), we obtain

\[(8.49) \quad L = (-1)^{\dim \mathfrak{p}^0/2} \exp \left( 2\pi^2 |\rho + \lambda|^2 \right) \left[ \tilde{A}^{-1} \left( TX (e^0) \right) e \left( N_{X(\gamma)/X(k)} \right) \right]^{-1} \left( N_{X(e^0)/X(\gamma)} \right) \ch^{-1} \left( F \right) \frac{a}{\sqrt{2\pi}} \max.
\]

For \( s > 0 \), let \( L_s \) be the obvious analogue of \( L \), when replacing \( B \) by \( B/s \). We can use equation (8.50) to evaluate \( L_s \). Recall that to properly use equation (8.50), we need to modify our definition of \( \alpha^{\max} \). Ultimately, we find that

\[(8.51) \quad L_s = s^{(p-1)/2} (-1)^{\dim \mathfrak{p}^0/2} \exp \left( 2\pi^2 s |\rho + \lambda|^2 \right) \left( \tilde{A}^{-1} \left( TX (e^0) \right) e \left( N_{X(\gamma)/X(k)} \right) \right]^{-1} \left( N_{X(e^0)/X(\gamma)} \right) \ch^{-1} \left( F \right) \frac{a}{\sqrt{2\pi}} \max.
\]

By (8.5), (8.36), (8.40), (8.45), and (8.51), we get the first identity in (8.37). By (8.17), (8.18), and the identity just proved, we get the second identity in (8.37). The proof of our theorem is completed.

Remark 8.7. We will check that the two terms in the right-hand side of (8.37) lie in \( \mathfrak{o}(\mathfrak{p}) \), i.e., they behave properly under change of the orientation of \( \mathfrak{p} \). Indeed note that \( e \left( N_{X(\gamma)/X(k)} \right) \) takes its values in \( \mathfrak{o} \left( N_{X(\gamma)/X(k)} \right) \), which is modelled on the orientation line \( \mathfrak{o} \left( \mathfrak{p}_0 \cap \mathfrak{p}(k) \right) \). Therefore, the right-hand side of (8.37) takes its values in \( \mathfrak{o}(\mathfrak{p}(\gamma)) \otimes \mathfrak{o}(\mathfrak{p}_0 \cap \mathfrak{p}(k)) \). Using (8.25), we find that the right-hand side of (8.37) is indeed a section of \( \mathfrak{o}(\mathfrak{p}) \).
9. Scalar hypoelliptic operators and their corresponding diffusions

The purpose of this section is to establish uniform estimates on the heat kernel
\( r_{b,t}^X((x,Y),(x',Y')) \) of a scalar version \( A^X_b \) of the operator \( L^X_b \). These estimates were established in [B11, chapters 12 and 13] in the range \( 0 < b \leq b_0, \epsilon \leq t \leq M \) with \( 0 < \epsilon \leq M < +\infty \). For later purposes, we need to extend such estimates in the range \( 0 < b, \tau b^2 \leq t \leq M \), with \( \tau > 0, M > 0 \).

The techniques we use to establish our estimates are exactly the ones in [B11]. They combine the Malliavin calculus with uniform estimates on the rate of escape of an open ball of the associated hypoelliptic diffusions, a result of independent interest, and a slight reinforcement of the results in [B11].

This section is organized as follows. In subsection 9.1, if \( E \) is a Euclidean vector space, we recall the definition of the harmonic oscillator, we give Mehler formula for its heat kernel, and we describe its probabilistic interpretation in terms of the Ornstein-Uhlenbeck process \( Y \).

In subsection 9.2, we obtain a uniform bound on \( \sup_{0 \leq s \leq t} |bY_s/b^2| \) that will be used in section 11.

In subsection 9.3, we define the scalar hypoelliptic Laplacian \( A^E_b \) on \( E \times E \), and we give the formula for its heat kernel that was obtained in [B11].

In subsection 9.4, if \( X = G/K \), we define the scalar elliptic heat kernel, which we relate to classical Brownian motion on \( X \). Also we state Itô’s formula, and we recall a well-known result on the rate of escape of Brownian motion from a ball.

In subsection 9.5, we introduce the hypoelliptic scalar operator \( A^X_b \) on the total space \( X \) of \( TX \).

In subsection 9.6, we construct the corresponding hypoelliptic heat operators, and the associated hypoelliptic diffusions on \( X \).

In subsection 9.7, we give a generalized formula of Itô for the hypoelliptic diffusion. This formula was already established in [B11]. However, we present it here as a suitable convolution of the classical Itô formula for the Brownian motion on \( X \). In the next sections, we will use this convolution formula. This approximate Itô formula is of independent interest.

In subsection 9.8, we establish the uniform rate of escape of the projection on \( X \) of our hypoelliptic diffusions. We need to establish these estimates with the proper Gaussian weight on the coordinate \( Y \in TX \). The techniques we rely on are the ones in [B11].

In subsection 9.9, we recall the result established in [B11] stating that as \( b \to 0 \), the projection on \( X \) of the hypoelliptic diffusion converges in probability law to Brownian motion on \( X \).

Finally, in subsection 9.10, we establish the suitable uniform upper bound for the hypoelliptic heat kernel \( r_{b,t}^X((x,Y),(x',Y')) \) on \( X \).

9.1. Harmonic oscillator and Mehler formula. Let \( E \) still be a Euclidean vector space of dimension \( m \), and let \( Y \) be the generic element of \( E \). As in [B11, Proposition 10.3.1], put

\[
H^E_t(Y,Y') = \frac{1}{2} \left( \tanh \left( \frac{t}{2} \right) \left( |Y|^2 + |Y'|^2 \right) + \frac{1}{\sinh(t)} |Y' - Y|^2 \right).
\]

Let \( \Delta^E \) be the Laplacian on \( E \). Let \( O^E \) be the harmonic oscillator on \( E \),

\[
O^E = \frac{1}{2} \left( -\Delta^E + |Y|^2 - m \right).
\]
Given $t > 0$, let $h_E^t (Y, Y')$ be the smooth kernel associated with $\exp (-tO_E)$ with respect to $dY'$. By Mehler’s formula [GIJ87], [B11, eq. (10.4.2)], we get

\begin{equation}
(9.3) \quad h_E^t (Y, Y') = \left( \frac{e^t}{2\pi \sinh (t)} \right)^{m/2} \exp \left( -\frac{1}{2} \tanh (t/2) \left( |Y|^2 + |Y'|^2 \right) \right).
\end{equation}

By (9.3), we deduce that

\begin{equation}
(9.4) \quad h_E^t (Y, Y') \leq \left( \frac{e^t}{2\pi \sinh (t)} \right)^{m/2} \exp \left( -\frac{1}{2} \tanh (t/2) \left( |Y|^2 + |Y'|^2 \right) \right).
\end{equation}

As $t \to +\infty$, we get

\begin{equation}
(9.5) \quad h_E^t (Y, Y') \to \left( \frac{1}{\pi} \right)^{m/2} \exp \left( -\frac{1}{2} \tanh (t/2) \left( |Y|^2 + |Y'|^2 \right) \right).
\end{equation}

From (9.3), we deduce the identity in [B11, eq. (10.7.12)],

\begin{equation}
(9.6) \quad \int_E h_E^t (Y, Y') \, dY' = \left( \frac{e^t}{\cosh (t)} \right)^{m/2} \exp \left( -\frac{1}{2} \tanh (t/2) |Y|^2 \right).
\end{equation}

Set

\begin{equation}
(9.7) \quad P_E = \exp \left( |Y|^2/2 \right) O_E \exp \left( -|Y|^2/2 \right).
\end{equation}

Let $\nabla_Y^{E,V}$ denote the radial vector field on $E$. Then

\begin{equation}
(9.8) \quad P_E = \frac{1}{2} \left( -\Delta_E + 2\nabla_Y^{E,V} \right).
\end{equation}

Let $w_E$ denote the Brownian motion in $E$ with $w_0^E = 0$. Given $Y \in E$, consider the stochastic differential equation

\begin{equation}
(9.9) \quad \dot{Y} = -Y + \dot{w}_E, \quad Y_0 = Y.
\end{equation}

Then $Y$ is given by

\begin{equation}
(9.10) \quad Y_t = e^{-t} Y + \int_0^t e^{-(t-s)} \, dw_s^E.
\end{equation}

The process $Y$ is called an Ornstein-Uhlenbeck process. Let $Q$ be the probability law of $Y$ on $C(\mathbb{R}_+, E)$ in (9.10), and let $E^Q$ be the corresponding expectation operator.

**Proposition 9.1.** For $t > 0$, if $f \in C^{\infty,c}(E, \mathbb{R})$, then

\begin{equation}
(9.11) \quad \exp (-tO_E) f (Y) = \exp \left( -|Y|^2/2 \right) E^Q \left[ \exp \left( |Y_t|^2/2 \right) f (Y_t) \right],
\end{equation}

\begin{equation}
\exp (-tQ^E) f (Y) = E^Q [f (Y_t)].
\end{equation}

**Proof.** This is a trivial consequence of Itô’s formula. 

Instead of (9.9), we consider the equation

\begin{equation}
(9.12) \quad \dot{Y} = \dot{w}_E, \quad Y_0 = Y,
\end{equation}

so that

\begin{equation}
(9.13) \quad Y_t = Y + w_t^E.
\end{equation}

In (9.13), $Y$ is just a Brownian motion. We denote by $P$ the probability law of $Y$ on $C(\mathbb{R}_+, E)$ and by $E^P$ the corresponding expectation operator.
Proposition 9.2. For \( t > 0 \), if \( f \in C^{\infty, c} (E, \mathbb{R}) \), then

\[
(9.14) \quad \exp (-tO^E) f (Y) = E_P \left[ \exp \left( \frac{mt}{2} - \frac{1}{2} \int_0^t |Y_s|^2 \, ds \right) f (Y_t) \right].
\]

Proof. Equation (9.14) follows from (9.2) and from the formula of Feynman-Kac. \( \square \)

Remark 9.3. The first equation in (9.11) and (9.14) are equivalent. Indeed an elementary version of a formula of Girsanov shows that if \( Q^t, P^t \) are the probability laws of \( Y \cdot \) on \( C ([0, t], E) \) in (9.9), (9.12), then

\[
(9.15) \quad \frac{dQ^t}{dP^t} = \exp \left( -\int_0^t \langle Y_s, \delta w^E_s \rangle - \frac{1}{2} \int_0^t |Y_s|^2 \, ds \right).
\]

Using (9.12) and Itô’s formula, we get

\[
(9.16) \quad \frac{1}{2} |Y_t|^2 = \frac{1}{2} |Y|^2 + \frac{mt}{2} + \int_0^t \langle Y_s, \delta w^E_s \rangle.
\]

By (9.16), we can rewrite (9.15) in the form

\[
(9.17) \quad \frac{dQ^t}{dP^t} = \exp \left( \frac{mt}{2} - \frac{1}{2} \int_0^t |Y_s|^2 \, ds + \frac{1}{2} |Y|^2 - \frac{1}{2} |Y_t|^2 \right).
\]

By (9.17), the two versions of \( \exp (-tO^E) f (Y) \) in (9.11), (9.14) are indeed equivalent.

Let \( Y \) be as in (9.9).

Proposition 9.4. Given \( \beta \geq 0 \), the following identity holds:

\[
(9.18) \quad E_P \left[ \exp \left( -\frac{\beta^2}{2} \int_0^t |Y_s|^2 \, ds \right) \right] = \frac{1}{cosh (\beta t)} \exp \left( -\frac{1}{2} \tanh (\beta t) |Y|^2 \right).
\]

Proof. For \( a > 0 \), set

\[
(9.19) \quad K_{as} (Y) = s (aY).
\]

For \( \beta > 0 \), we get

\[
(9.20) \quad K_{\sqrt{\beta}O^E} K_{\sqrt{\beta}}^{-1} = \frac{1}{2} \left( -\Delta^E + \beta^2 |Y|^2 - m\beta \right).
\]

By (9.14), (9.20), we deduce that

\[
(9.21) \quad E_P \left[ \exp \left( -\frac{\beta^2}{2} \int_0^t |Y_s|^2 \, ds \right) \right] = \exp (-m\beta t/2) \int_E h^E_{\beta t} \left( \sqrt{\beta}Y, Y' \right) dY'.
\]

By (9.6), (9.21), we get (9.18). \( \square \)

For \( 0 \leq \beta \leq 1 \), let \( \rho_\beta \geq 0 \) be defined by

\[
(9.22) \quad \rho_\beta^2 = 1 - \beta^2.
\]

Let \( Y \) be as in (9.9).
**Proposition 9.5.** The following identity holds:

\[
E^Q \left[ \exp \left( \frac{\beta^2}{2} \int_0^t |Y_s|^2 \, ds \right) \right] = \left[ \frac{\exp(t)}{\cosh(\rho_\beta t) + \frac{\sinh(\rho_\beta t)}{\rho_\beta}} \right]^{m/2} \exp \left( \frac{\beta^2}{2} \frac{\tanh(\rho_\beta t)}{\rho_\beta + \tanh(\rho_\beta t)} |Y|^2 / 2 \right).
\]

In particular, we have

\[
E^Q \left[ \exp \left( \frac{\beta^2}{2} \int_0^t |Y_s|^2 \, ds \right) \right] \leq \exp \left( \frac{\beta^2}{2} \left( mt + |Y|^2 \right) \right).
\]

**Proof.** Equation (9.23) was established in [B11, eq. (13.2.54)]. By (9.23), we get

\[
E^Q \left[ \exp \left( \frac{\beta^2}{2} \int_0^t |Y_s|^2 \, ds \right) \right] \leq \exp \left( \frac{\beta^2}{2} \left( m (1 - \rho_\beta) t / 2 + \beta^2 |Y|^2 / 2 \right) \right).
\]

Since \(1 - \rho_\beta \leq 1 - \rho_\beta^2 = \beta^2\), from (9.25), we get (9.24). The proof of our proposition is completed.

\[\square\]

**Remark 9.6.** Equation (9.23) can be extended by analyticity to

\[
\beta^2 \leq 1 + \left( \frac{\pi}{2t} \right)^2.
\]

**Proposition 9.7.** For \(c \geq 0, t \geq 0\), for \(0 < \beta \leq 1\), then

\[
E^Q \left[ \exp \left( c \int_0^t |Y_s| \, ds \right) \right] \leq \exp \left( c^2 t / 2 \beta^2 + \beta^2 mt / 2 + c \left( 1 - e^{-t} \right) |Y| \right).
\]

**Proof.** Let \(Y_b\) be \(Y\) in (9.10) with \(Y_0 = 0\). We can rewrite (9.10) in the form

\[
Y_t = e^{-t}Y + \hat{Y}_t.
\]

By (9.28), we get

\[
\int_0^t |Y_s| \, ds \leq \int_0^t |Y_s| + (1 - e^{-t}) |Y|.
\]

Also

\[
c \int_0^t |Y_s| \, ds \leq \frac{\beta^2}{2} \int_0^t |Y_s|^2 \, ds + \frac{c^2 t}{2 \beta^2}.
\]

By (9.24), (9.29), (9.30), we get (9.27).

\[\square\]

### 9.2. A uniform estimate on \(bY_{b/\beta}\).

We still consider equation (9.9) and the corresponding probability law \(Q\). Given \(b > 0\), set

\[
Y_{b/\beta} = Y_{b/\beta},
\]

There is a Brownian motion, which is still denoted \(w^E\), such that

\[
dY_{b/\beta} = -\frac{Y_{b/\beta}}{b^2} + \frac{\hat{w}^E}{b}.
\]

By (9.32), we deduce that

\[
Y_{b/\beta} = e^{-t/b^2}Y + \int_0^t e^{-(t-s)/b^2} \frac{d\hat{w}^E_s}{b}.
\]
By (9.33), we deduce that

\[(9.34)\quad Y_{b,t} = e^{-t/b^2} Y + \frac{w_t}{b} - \int_0^t \frac{\exp \left( -(t-s)/b^2 \right) w_s}{b} \, ds.\]

By (9.34), we deduce that

\[(9.35)\quad \sup_{0 \leq s \leq t} |bY_{b,s}| \leq |bY| + \left( 2 - e^{-t/b^2} \right) \sup_{0 \leq s \leq t} |w_s|.\]

By [B11, Proposition 14.10.1], for \( \alpha > 1/2 \), as \( b \to 0 \), \(|\log (b)| - \alpha Y_{b,\cdot} - \cdot \) converges uniformly to 0 over compact subsets in probability. Equation (9.35) is not strong enough to produce this result.

9.3. The hypoelliptic Laplacian on \( E \times E \). The generic element of \( E \times E \) will be denoted \((x, Y)\). As in [B11, Proposition 10.3.2], for \( b > 0, t > 0, x, x', Y, Y' \in E \), set

\[(9.36)\quad H_{b,t} ((x, Y), (x', Y')) = \frac{b^2}{2} \left( \tanh \left( t/2b^2 \right) \left( |Y|^2 + |Y'|^2 \right) + \frac{|Y' - Y|^2}{\sinh \left( t/2b^2 \right)} \right) + \frac{1}{2} \left( \frac{t}{t - 2b^2 \tanh \left( t/2b^2 \right)} \right) |x' - x - b^2 \tanh \left( t/2b^2 \right) (Y + Y')|^2.\]

Put

\[(9.37)\quad \beta = b/\sqrt{t}.\]

By (9.36), we get

\[(9.38)\quad H_{b,t} ((x, Y/b), (x', Y'/b)) = H_{\beta,1} \left( (x/\sqrt{t}, Y/\beta) , (x'/\sqrt{t}, Y'/\beta) \right).\]

Also

\[(9.39)\quad H_{\beta,1} \left( (x/\sqrt{t}, Y/\beta) , (x'/\sqrt{t}, Y'/\beta) \right) = \frac{1}{2} \left( \tanh \left( 1/2\beta^2 \right) \left( |Y|^2 + |Y'|^2 \right) + \frac{|Y' - Y|^2}{\sinh \left( 1/\beta^2 \right)} \right) + \frac{1}{2} \left( \frac{1}{1 - 2\beta^2 \tanh \left( 1/2\beta^2 \right)} \right) \frac{|x' - x - \beta \tanh \left( 1/2\beta^2 \right) (Y + Y')|^2}{\sqrt{t}}.\]

We give the following extension of [B11, eq. (10.3.52)].

**Proposition 9.8.** There exists \( C > 0 \) such that for \( b > 0, t > 0 \),

\[(9.40)\quad H_{b,t} ((x, Y/b), (x', Y'/b)) \geq C \left( \frac{|x' - x|^2}{t} + \left( 1 - e^{-t/b^2} \right) \left( |Y|^2 + |Y'|^2 \right) \right).\]

In particular if \( \tau > 0 \) is such that \( t \geq \tau b^2 \), then

\[(9.41)\quad H_{b,t} ((x, Y/b), (x', Y'/b)) \geq C \left( \frac{|x' - x|^2}{t} + \left( 1 - e^{-\tau} \right) \left( |Y|^2 + |Y'|^2 \right) \right).\]
Proof. By (9.38), (9.39), if \( \beta \tanh (1/2\beta^2) (|Y| + |Y'|) \leq |x' - x|/2\sqrt{t} \), (9.40) holds true. If \( \beta \tanh (1/2\beta^2) (|Y| + |Y'|) > |x' - x|/2\sqrt{t} \), then

\[
(9.42) \quad \tanh (1/2\beta^2) \left( |Y|^2 + |Y'|^2 \right) \geq C \frac{1}{\beta^2} \tanh (1/2\beta^2) \frac{|x' - x|^2}{t}.
\]

Since for \( x \geq 0 \), \( \tanh (x) \leq x \), from (9.42), we get

\[
(9.43) \quad \tanh (1/2\beta^2) \left( |Y|^2 + |Y'|^2 \right) \geq 2C \frac{|x' - x|^2}{t},
\]

so that (9.40) still holds. From (9.40), we get (9.41). The proof of our proposition is completed.

We will consider hypoelliptic differential operators on \( E \times E \). Recall that the generic element of \( E \times E \) is denoted \((x, Y)\). The harmonic oscillator \( \mathcal{O}_E \), will act on the second factor \( E \). We denote by \( \nabla_Y^E \) the vector field on \( E \times E \) that differentiates along the first copy \( E \) in the direction \( Y \).

**Definition 9.9.** Given \( b > 0 \), let \( \mathcal{A}_b^E \) be the differential operator

\[
(9.44) \quad \mathcal{A}_b^E = \frac{\mathcal{O}_b^E}{b^2} - \frac{\nabla_Y^E b^2}{b}.
\]

Then \( \frac{\partial}{\partial t} + \mathcal{A}_b^E \) is a hypoelliptic operator on \( E \times E \).

**Definition 9.10.** For \( b > 0, t > 0 \), let \( r_{b,t}^E ((x, Y), (x', Y')) \) be the smooth kernel for the operator \( \exp (-t \mathcal{A}_b^E) \) with respect to the volume \( dx'dY' \).

Here is a result established in [B11, Proposition 10.5.1].

**Proposition 9.11.** For \( b > 0, t > 0 \), the following identity holds:

\[
(9.45) \quad r_{b,t}^E ((x, Y), (x', Y')) = \left[ \frac{e^{t/b^2}}{4\pi^2 \sinh (t/b^2) (t - 2b^2 \tanh (t/2b^2))} \right]^{m/2} \exp \left( -H_{b,t} (((x, Y), (x', Y')))/b \right).
\]

**Remark 9.12.** By (9.40), since for \( y \geq 0 \), the function \( \tanh (y)/y \) is decreasing, given \( \tau > 0 \), there exist \( C_\tau > 0, C' > 0 \) such that if \( b > 0, t \geq \tau b^2 \), then

\[
(9.46) \quad r_{b,t}^E ((x, Y), (x', Y')) \leq \frac{C_\tau}{b^{m/2}} \exp \left( -C' \left( \frac{|x' - x|^2}{t} + (1 - e^{-\tau}) \left( |Y|^2 + |Y'|^2 \right) \right) \right).
\]

9.4. **Elliptic heat kernel and Brownian motion.** Let \( \Delta^X \) be the Laplace-Beltrami operator on \( X \). For \( t > 0 \), let \( p_t (x, x') \) be the smooth kernel associated with \( \exp (t\Delta^X/2) \) with respect to the volume \( dx' \). Classically, given \( M \geq 0 \), there exist \( C > 0, C' > 0 \) such that for \( 0 < t \leq M \), \( x, x' \in X \),

\[
(9.47) \quad |p_t (x, x')| \leq C t^{-m/2} \exp \left( -C' d^2 (x, x')/t \right).
\]

The uniformity of the constants is a consequence of the fact that \( X \) is a symmetric space.

Set \( x_0 = p_1 \). Let \( P_{x_0} \) be the probability law on \( C (\mathbb{R}_+, X) \) of the Brownian motion in \( X \) starting at \( x_0 \). There is a Brownian motion \( w^{TX} \) with values in \( T_{x_0}X \) such that

\[
(9.48) \quad \dot{x} = w^{TX}.
\]
In (9.48), \( \dot{w}^{TX} \) denotes the differential of \( w^{TX} \) in the sense of Stratonovitch \(^5\). The precise meaning of (9.48) is that \( \dot{x} \) is the parallel transport along \( x \) of \( \dot{w}^{TX} \in T_{x}_{0}X \) with respect to the Levi-Civita connection. The theory of stochastic differential equations gives an unambiguous meaning to (9.48). Let \( E \) be the corresponding expectation operator.

We denote by \( C^{\infty,c}(X, \mathbb{R}) \) the vector space of smooth real functions with compact support. The same notation will be used when \( X \) is replaced by any other space.

If \( f \in C^{\infty,c}(X, \mathbb{R}) \), the basic relation between the heat operator and Brownian motion is that for \( t > 0 \),
\[
(9.49) \quad \exp \left( t \Delta^{X}/2 \right) f(x_{0}) = E \left[ f(x_{t}) \right] .
\]

Let \( f : X \to \mathbb{R} \) be a smooth function. Let \( \delta w^{TX} \) be the Itô differential of \( w^{TX} \). By Itô’s formula, from (9.48), we get
\[
(9.50) \quad f(x_{t}) = f(x_{0}) + \int_{0}^{t} \frac{1}{2} \Delta^{X} f(x_{s}) \, ds + \int_{0}^{t} \langle \nabla f(x_{s}), \delta w^{TX}_{s} \rangle .
\]

For later purposes, we first reprove a well-known result on the process \( x \).

**Proposition 9.13.** Given \( M > 0 \), there exist \( C > 0, C' > 0 \) such that for \( 0 < t \leq M, r > 0 \),
\[
(9.51) \quad P_{x_{0}} \left[ \sup_{0 \leq s \leq t} d(x_{0}, x_{s}) \geq r \right] \leq C \exp \left( -C' r^{2}/t \right).
\]

**Proof.** Let \( k : \mathbb{R}_{+} \to \mathbb{R}_{+} \) be a smooth increasing function such that
\[
(9.52) \quad k(u) = u^{2} \quad \text{for} \quad u \leq 1/2, \quad u \quad \text{for} \quad u \geq 1.
\]

We fix \( \epsilon > 0 \). For \( z \in X \), set
\[
(9.53) \quad f(z) = \epsilon k \left( \frac{d(x_{0}, z)}{\epsilon} \right).
\]

Then \( f \) is a smooth function on \( X \). Moreover, \( d(x_{0}, z) \geq \epsilon \) if and only if \( f(z) \geq \epsilon \).

We use (9.50) with the previous choice of \( f \). By [B11, Proposition 13.1.2], we get
\[
(9.54) \quad \Delta^{X} f \leq C.
\]

By (9.50), (9.54), for \( r \geq \epsilon \), we get
\[
(9.55) \quad P_{x_{0}} \left[ \sup_{0 \leq s \leq t} d(x_{0}, x_{s}) \geq r \right] \leq P_{x_{0}} \left[ \sup_{0 \leq s \leq t} \int_{0}^{t} \langle \nabla f(x_{s}), \delta w^{TX}_{s} \rangle \geq r - Ct/2 \right].
\]

Since \( |\nabla d(x_{0}, x)| = 1 \), \( \nabla f \) is uniformly bounded. By [StrV79, eq. (2.1) in Theorem 4.2.1], we get
\[
(9.56) \quad P_{x_{0}} \left[ \sup_{0 \leq s \leq t} \int_{0}^{t} \langle \nabla f(x_{s}), \delta w^{TX}_{s} \rangle \geq r - Ct/2 \right] \leq c \exp \left( -c' (r - Ct/2)^{2}/t \right).
\]

\(^5\)Brownian motion is nowhere differentiable. Still, there is an efficient calculus along its trajectories, the Itô calculus. Because it is based on a mean-variance description of local variations, it is not coordinate invariant. The main advantage of the calculus of Stratonovitch is that it is invariant under change of coordinates. The two calculi can be deduced from each other. We refer to Ikeda-Watanabe [IW89] and Le Gall [LG16] for more details.
Also for $0 < t \leq M$, we get
\[
(r - Ct/2)^2/t \geq r^2/t - Cr \geq r^2/2t - C^2t/2 \geq r^2/2t - C^2M/2.
\]
By (9.55)–(9.57), when $r \geq \epsilon$, we get (9.51).

Using the exponential map, we can identify a neighbourhood of $x_0 = p1$ in $X$ with a neighbourhood of 0 in $p$. To handle the case where $r > 0$ is small, we can instead take smooth functions $f_1, \ldots, f_m$ on $X$ with compact support, which coincide with coordinates $x^1, \ldots, x^m$ near $p1 \simeq 0$. By proceeding as before with the functions $f_1, \ldots, f_m$ instead of $f$, we obtain (9.51) in the case of a small $r > 0$.

The proof of our proposition is completed. \(\Box\)

9.5. The scalar analogues of the operator $L^X_b$. Let $\pi : \mathcal{X} \to X$ be the total space of $TX = G \times_K p$, and let $Y^{TX}$ be the tautological section of $\pi^*TX$ on $\mathcal{X}$. We denote by $\nabla_{Y^{TX}}$ the generator of the geodesic flow on $\mathcal{X}$. Let $\nabla^V$ denotes differentiation along the fibre $TX$. Then $\nabla_{Y^{TX}}^V$ is the fibrewise radial vector field. As in [B11, eqs. (11.1.1) and (11.1.2)], let $A^X_b, B^X_b$ be the scalar differential operators on $X$,
\[
A^X_b = \frac{1}{2b^2} \left( -\Delta^{TX} + |Y^{TX}|^2 - m \right) - \frac{1}{b} \nabla_{Y^{TX}},
\]
\[
B^X_b = \frac{1}{2b^2} \left( -\Delta^{TX} + 2\nabla^V_{Y^{TX}} \right) - \frac{1}{b} \nabla_{Y^{TX}}.
\]
Then
\[
B^X_b = \exp \left( \frac{|Y^{TX}|^2}{2} \right) A^X_b \exp \left( -\frac{|Y^{TX}|^2}{2} \right).
\]

By Hörmander [H6], the operators $\frac{\partial}{\partial t} + A^X_b, \frac{\partial}{\partial t} + B^X_b$ are hypoelliptic.

By [B11, section 11.5], for $t > 0$, the heat operators $\exp \left( -tA^X_b \right), \exp \left( -tB^X_b \right)$ are unambiguously defined. The difficulty in defining them properly is that $X$ is noncompact.

9.6. Hypoelliptic heat operators and probability. Recall that $x_0 = p1$, and that $TX_{x_0} \simeq p$. Let $w^p$ be a Brownian motion with values in $p$ such that $w^p_0 = 0$, and let $w^{TX}$ denote the corresponding Brownian motion in $TX$. Let $Q$ be the probability law of $w^p$ on $C(R_+, p)$, and let $E^Q$ denote the corresponding expectation operator.

Now we follow [BL08, section 14.2], and [B11, section 12.2]. Fix $Y^p \in p$. Consider the stochastic differential equation on $\mathcal{X}$,
\[
\dot{x} = \frac{Y^{TX}}{b}, \quad \dot{Y}^{TX} = -\frac{Y^{TX}}{b^2} + \frac{\dot{w}^{TX}}{b},
\]
\[
x_0 = p1, \quad Y^{TX}_0 = Y^p.
\]
In (9.60), $Y^{TX}$ denotes the covariant derivative of $Y^{TX}$ with respect to $\nabla^{TX}$. The first line of equation (9.60) can be rewritten in the form
\[
b^2 \ddot{x} + \dot{x} = \ddot{w}^{TX}, \quad Y^{TX} = b\dot{x}.
\]
Instead of (9.60), as in [B11, eq. (12.2.8)], we may instead consider the horizontal lift \( g \) of \( x \) in \( G \), i.e., consider the system
\[
\begin{align*}
\dot{g} &= \frac{Y_p}{b}, \\
\dot{Y}_p &= -\frac{Y_p}{b^2} + \frac{\dot{w}}{b}, \\
g_0 &= 1, \\
Y_p^0 &= Y_p,
\end{align*}
\]
so that
\[
x = \pi g.
\]

**Proposition 9.14.** If \( F \in C_{\infty,c}(X, \mathbb{R}) \), for \( t > 0 \), then
\[
\exp(-tA^X_b) F (x_0, Y_{TX}^0) = \exp\left(-\frac{|Y_{TX}|^2}{2}\right)
\]
\[
E^Q \left[ \exp\left(\frac{|Y_{TX}|^2}{2}\right) F (x_t, Y_{TX}^t) \right],
\]
\[
\exp(-tB^Y_b) F (x_0, Y_{TX}^0) = E^Q \left[ F (x_t, Y_{TX}^t) \right].
\]

**Proof.** The second identity in (9.64) was established in [B11, Theorem 12.2.1] using the Itô calculus. Using (9.59), we also obtain the first identity. The proof of our proposition is completed.

We give another construction of the semigroup \( \exp(-tA^X_b) \). Instead of (9.60), we consider the stochastic differential equation
\[
\begin{align*}
\dot{x} &= \frac{Y_{TX}}{b}, \\
\dot{Y}_{TX} &= \frac{\dot{w}_{TX}}{b}, \\
x_0 &= p1, \\
Y_{TX}^0 &= Y_p.
\end{align*}
\]
By (9.65), we get
\[
b^2 \ddot{x} = \dot{w}_{TX}.
\]

Instead of (9.62), we consider the system
\[
\begin{align*}
\dot{g} &= \frac{Y_p}{b}, \\
\dot{Y}_p &= \frac{\dot{w}}{b}, \\
g_0 &= 1, \\
Y_p^0 &= Y_p,
\end{align*}
\]
so that (9.63) still holds. To distinguish the systems (9.60), (9.62), and (9.65), (9.67), for the last two equations, we denote the associated expectation operator by \( E^P \).

**Proposition 9.15.** If \( F \in C_{\infty,c}(X, \mathbb{R}) \), then
\[
\exp(-tA^X_b) F (x_0, Y_{TX}^0) = E^P \left[ \exp\left(\frac{mt}{2b^2} - \frac{1}{2b^2} \int_0^t |Y_{TX}(s)|^2 \, ds\right) F (x_t, Y_{TX}^t) \right].
\]

**Proof.** Equation (9.68) was established in [B11, eq. (13.2.12)] using the Itô calculus.

**Remark 9.16.** As explained in Remark 9.3, equations (9.64) and (9.68) can be deduced from each other. The first equation will be useful when \( b \to 0 \), the second equation when \( b \to +\infty \).
9.7. A generalized Itô formula. From now on, we assume that (9.60), (9.62) hold.

In the sequel, $dw^{TX}$ denotes the Stratonovitch differential of $w^{TX}$. As before, $\delta w^{TX}$ is our notation for its Itô differential. First, we give formula established in [B11, eq. (12.3.19)].

**Proposition 9.17.** Let $f : X \to \mathbb{R}$ be a smooth function. Then

$$
(9.69) \quad f(x_t) + b \nabla_{Y^{TX}} f(x_t) = f(x_0) + b \nabla_{Y^{TX}} f(x_0) + \int_0^t \nabla_{Y^{TX}} \nabla_{Y^{TX}} f(x_s) \, ds + \int_0^t \nabla_{\delta w^{TX}} f(x_s) \, ds.
$$

**Proof.** Equation (9.69) follows from an easy application of Itô’s formula to the process $f(x_t) + b \nabla_{Y^{TX}} f(x_t)$. □

Now we establish our generalized Itô formula.

**Theorem 9.18.** Set

$$
(9.70) \quad A^f_t = \int_0^t \nabla_{Y^{TX}} \nabla_{Y^{TX}} f(x_s) \, ds + \int_0^t \nabla_{\delta w^{TX}} f(x_s) \, ds.
$$

The following identity holds:

$$
(9.71) \quad \left( b^2 \frac{d}{dt} + 1 \right) f(x_t) = \left( b^2 \frac{d}{dt} + 1 \right) f(x_0) + A^f_t.
$$

Moreover,

$$
(9.72) \quad f(x_t) = f(x_0) + b^2 \frac{d}{dt} f(x_t) \bigg|_{t=0} \left( 1 - e^{-t/b^2} \right) + \int_0^t \frac{e^{-(t-s)/b^2}}{b^2} A^f_s \, ds.
$$

**Proof.** From (9.60), (9.69), we get (9.71). By integrating the differential equation (9.71), we get (9.72). The proof of our theorem is completed. □

**Remark 9.19.** The standard form of Itô’s formula (9.50) should be compared with its approximate version (9.72). As was shown in [B11, Theorem 12.8.1], as $b \to 0$, the probability law of $x$ in (9.60) converges to the probability law of $x$ in (9.48). Equation (9.69) plays a key role in the proof of this result given in [B11].

9.8. A uniform estimate on the rate of escape of the process $x$. We still consider the probability measure $Q$ in subsection 9.6, and the corresponding stochastic differential equation in (9.60). The corresponding process $x$ depends on the parameter $b > 0$. We will improve on the results obtained in [B11, section 13.2].

**Theorem 9.20.** Given $M > 0$, there exist $C > 0, C' > 0$ such that for $b > 0, \epsilon \leq t \leq M, r > 0, then

$$
(9.73) \quad Q \left[ \sup_{0 \leq s \leq t} d(x_0, x_s) \geq r \right] \leq C \exp \left( -C' r^2 / t + \frac{1}{2} |Y_0^{TX}|^2 \right).
$$

Given $M > 0$, there exist $c > 0, C > 0, C' > 0$ such that for $b > 0, 0 < t \leq M, r > 0, then

$$
(9.74) \quad \exp \left( -|Y_0^{TX}|^2 / 2 \right) E^Q \left[ 1_{\sup_{0 \leq s \leq t} d(x_0, x_s) \geq r} \exp \left( |Y_t^{TX}|^2 / 2 \right) \right] \leq C \exp \left( -C' \left( r^2 / t + \left( 1 - e^{-ct/b^2} \right) |Y_0^{TX}|^2 \right) \right).
$$
Given $M > 0, \tau > 0$, there exist $C > 0, C' > 0$ such that for $b > 0, \tau b^2 \leq t \leq M, r > 0$, then

\begin{equation}
\text{(9.75)} \quad \exp \left( - \frac{1}{2} \frac{\tau^2}{\tau b^2} \right) E^Q \left[ \sup_{0 \leq s \leq t} d(x_0, x_s) \geq r \right] \leq C \exp \left( - C' \frac{r^2}{t} + \frac{1}{2} \frac{\tau^2}{\tau b^2} \right).
\end{equation}

**Proof.** We temporarily assume that equation (9.60) is replaced by (9.65). Let $P$ denote the corresponding probability measure. Using Girsanov’s transformation as in [B11, proof of Theorem 13.2.2]. We take

\begin{equation}
\text{(9.76)} \quad Q \left[ \sup_{0 \leq s \leq t} d(x_0, x_s) \geq r \right] = E^P \left[ \exp \left( - \frac{1}{b} \int_0^t \langle Y^T X_s, \delta w^T X_s \rangle \right) - 1 \frac{1}{2b^2} \right] \left( \sup_{0 \leq s \leq t} d(x_0, x_s) \geq r \right).
\end{equation}

By Itô’s formula, as in (9.16), we get

\begin{equation}
\text{(9.77)} \quad \frac{1}{2} \left( |Y^T X_t|^2 - |Y^T X_0|^2 \right) = \frac{m t}{2b^2} + \frac{1}{b} \int_0^t \langle Y^T X_s, \delta w^T X_s \rangleds.
\end{equation}

Using (9.77), equation (9.76) can be rewritten in the form

\begin{equation}
\text{(9.78)} \quad Q \left[ \sup_{0 \leq s \leq t} d(x_0, x_s) \geq r \right] = E^P \left[ \exp \left( - \frac{1}{b} \int_0^t |Y^T X_s|^2 ds \right) \right.
\end{equation}

\begin{equation}
\left. + \frac{mt}{2b^2} \right] \left( \sup_{0 \leq s \leq t} d(x_0, x_s) \geq r \right).
\end{equation}

By equation (9.65), we get

\begin{equation}
\text{(9.79)} \quad \frac{1}{2b^2} \int_0^t |Y^T X_s|^2 ds \geq \frac{1}{2} \sup_{0 \leq s \leq t} d^2(x_0, x_s).
\end{equation}

From (9.79), we obtain

\begin{equation}
\text{(9.80)} \quad \frac{1}{2b^2} \int_0^t |Y^T X_s|^2 ds \geq \frac{1}{2t} \sup_{0 \leq s \leq t} d^2(x_0, x_s).
\end{equation}

By (9.78), (9.80), we get

\begin{equation}
\text{(9.81)} \quad Q \left[ \sup_{0 \leq s \leq t} d(x_0, x_s) \geq r \right] \leq \exp \left( \frac{1}{2} \frac{|Y^T X_0|^2}{|Y^T X_0|^2} \right) \exp \left( - \frac{r^2}{2t} + \frac{mt}{2b^2} \right).
\end{equation}

By (9.81), if $\tau > 0$ is fixed, and if $0 < t \leq \tau b^2$, equation (9.73) holds. This is the case if $b > 0$ is bounded away from 0, and if $0 < t \leq M$.

In the sequel, we may as well assume that there is $b_0 > 0$ small enough so that $0 < b \leq b_0$ and that $b^2 \leq t \leq M$. Even in this case, the estimate (9.81) will still be used.

We proceed as in [B11, proof of Theorem 13.2.2]. We fix $\epsilon > 0$. We take $f : X \to \mathbb{R}$ as in (9.53). We define $A^f$ as in (70). Since $f(x_0) = 0, \frac{df}{dt}(x_t)|_{t=0} = 0$, by equation (9.72) in Theorem 9.18, we get

\begin{equation}
\text{(9.82)} \quad f(x_t) = \frac{1}{b^2} \int_0^t e^{-t-s/b^2} A^f ds.
\end{equation}
By (9.82), we deduce that
\begin{equation}
\sup_{0 \leq s \leq t} f(x_s) \leq \sup_{0 \leq s \leq t} A'_f.
\end{equation}

For \( r \geq \epsilon \), \( \sup_{0 \leq s \leq t} d(x_0, x_s) \geq r \) if and only if \( \sup_{0 \leq s \leq t} f(x_s) \geq r \). By (9.83), this is the case only if \( \sup_{0 \leq s \leq t} A'_f \geq r \). If \( \sup_{0 \leq s \leq t} A'_f \geq r \), at least one of the sup of the two terms in the right-hand side of (9.70) is larger than \( r/2 \).

By [B11, eq. (13.1.17)], there is \( C > 0 \) such that
\begin{equation}
\nabla^T X \cdot \nabla f \leq \frac{C}{2}.
\end{equation}

By (9.84), for \( 0 \leq s \leq t \), we get
\begin{equation}
\int_0^s \nabla^T X \nabla_{Y^T X} f(x_u) \, du \leq \frac{C}{2} \int_0^t |Y^T X|^2 \, du.
\end{equation}

By (9.85), using Chebyshev’s inequality, for any \( \alpha > 0 \), we get
\begin{equation}
Q \left[ \frac{1}{2b^2} \int_0^t |Y^T X|^2 \, du \right] \leq \exp \left( -\alpha r/2b^2 \right)
E^Q \left[ \exp \left( \frac{\alpha C}{2b^2} \int_0^t |Y^T X|^2 \, du \right) \right].
\end{equation}

In the sequel, we choose \( \alpha \) given by
\begin{equation}
\alpha = \frac{1}{C}.
\end{equation}

Using equation (9.23) in Proposition 9.5 with \( \beta = 1 \), and \( t \) replaced by \( t/b^2 \), we get
\begin{equation}
E^Q \left[ \exp \left( \frac{1}{2b^2} \int_0^t |Y^T X|^2 \, du \right) \right] = \exp \left( \frac{mt}{2b^2} \right)
\left[ \frac{1}{1 + \frac{t}{1 + \frac{t}{2}} \frac{1}{1 + \frac{t}{2}}} \right] \exp \left( \frac{t/b^2}{1 + \frac{t}{2}} \frac{1}{2} |Y^T X|^2 \right).
\end{equation}

By (9.86)–(9.88), we obtain
\begin{equation}
Q \left[ \int_0^t \nabla^T X \nabla_{Y^T X} f(x_u) \, du \right] \leq \exp \left( -\frac{1}{b^2} \left( \frac{r}{2C} - \frac{mt}{2} \right) + \frac{1}{2} |Y^T X|^2 \right).
\end{equation}

Since \( \nabla f \) is uniformly bounded, by [StrV79, eq. (2.1) in Theorem 4.2.1], we get
\begin{equation}
Q \left[ \sup_{0 \leq s \leq t} \int_0^s \nabla_{\delta w_u} f(x_u) \geq r/2 \right] \leq C \exp \left( -C'r^2/t \right).
\end{equation}

By (9.89), (9.90), if \( r \geq \epsilon \), then
\begin{equation}
Q \left[ \sup_{0 \leq s \leq t} d(x_0, x_s) \geq r \right] \leq \exp \left( -\frac{1}{b^2} \left( \frac{r}{2C} - \frac{mt}{2} \right) + \frac{1}{2} |Y^T X|^2 \right) + C \exp \left( -C'r^2/t \right).
\end{equation}
Combining (9.81) and (9.91), if \( r \geq \epsilon \), then

\[
(9.92) \quad Q \left[ \sup_{0 \leq s \leq t} d(x_0, x_s) \geq r \right] \leq \exp \left( -\frac{1}{b^2} \left( \frac{r}{4C} - \frac{mt}{2} \right) - \frac{r^2}{4t} + \frac{1}{2} |Y_0^{TX}|^2 \right) + C \exp \left( -C' r^2 / t \right). 
\]

If \( t \leq r/2mc \), by (9.92), we get

\[
(9.93) \quad Q \left[ \sup_{0 \leq s \leq t} d(x_0, x_s) \geq r \right] \leq \exp \left( -r^2/4t + \frac{1}{2} |Y_0^{TX}|^2 \right) + C \exp \left( -C' r^2 / t \right),
\]

which is compatible with (9.73). If \( t > r/2mc \), then

\[
(9.94) \quad \frac{r^2}{t} \leq 4m^2 C^2 t.
\]

If \( M > 0 \) is given, and if \( \frac{r}{2mc} < t \leq M \), by (9.94), we get

\[
(9.95) \quad 1 \leq \exp \left( 4m^2 C^2 M - r^2 / t \right).
\]

By (9.95), equation (9.73) still holds, which completes the proof of (9.73) when \( r \geq \epsilon \).

We will now establish (9.73) for \( r > 0 \) is small. As before, we will assume that \( b^2 \leq t \leq M \). We take \( f_1, \ldots, f_m \) as in the proof of Proposition 9.13. We still consider equation (9.72) with \( f = \pm f_i, 1 \leq i \leq m \). With respect to what we did before, we have the extra term \( b^2 \frac{d}{dt} f \left( x_i \right) \big|_{t=0} \left( 1 - e^{-t/b^2} \right) \). Note that

\[
(9.96) \quad \left| b^2 \frac{d}{dt} f \left( x_i \right) \right|_{t=0} \left( 1 - e^{-t/b^2} \right) \leq C'' b \left( 1 - e^{-t/b^2} \right) |Y_0^{TX}| \leq C'' b |Y_0^{TX}|.
\]

By proceeding as before, for \( r > 0 \) small, we get the obvious analogue of (9.91), where \( r \) is replaced by \( (r - C'' b |Y_0^{TX}|)_{+} \). More precisely, for \( r > 0 \) small, we obtain

\[
(9.97) \quad Q \left[ \sup_{0 \leq s \leq t} d(x_0, x_s) \geq r \right] \leq \exp \left( -\frac{1}{b^2} \left( \frac{(r - C'' b |Y_0^{TX}|)_{+}}{2C} - \frac{mt}{2} \right) + \frac{1}{2} |Y_0^{TX}|^2 \right) + C \exp \left( -C' \left( r - C'' b |Y_0^{TX}| \right)^2 / t \right).
\]

By combining (9.81) and (9.97), we get

\[
(9.98) \quad Q \left[ \sup_{0 \leq s \leq t} d(x_0, x_s) \geq r \right] \leq \exp \left( -\frac{1}{b^2} \left( \frac{(r - C'' b |Y_0^{TX}|)_{+}}{4C} - \frac{mt}{2} \right) \right. \\
\left. - \frac{r^2}{4t} + \frac{1}{2} |Y_0^{TX}|^2 \right) + C \exp \left( -C' \left( r - C'' b |Y_0^{TX}| \right)^2 / t \right).
\]

We claim that if \( t \geq b^2 \), then

\[
(9.99) \quad C' \left( r - C'' b |Y_0^{TX}| \right)^2 / t + \frac{1}{2} |Y_0^{TX}|^2 \geq C'' r^2 / t.
\]

Indeed this is the case if \( C'' b |Y_0^{TX}| \leq r/2 \). If \( C'' b |Y_0^{TX}| > r/2 \), then

\[
(9.100) \quad \frac{1}{2} |Y_0^{TX}|^2 \geq r^2 / 8C''^2 b^2 \geq r^2 / 8C''^2 t.
\]
By (9.99), we conclude that the second term in (9.97) can be dominated by the right-hand side of (9.73). If \( t \leq (r - C'')b Y_0^{TX} \)/2mC, this also the case for the first term.

If \( t > (r - C''b Y_0^{TX})/2mC \), then
\[
(r - C''b Y_0^{TX})^2/t \leq 4m^2C^2t.
\]
By (9.106), (9.107), we get
\[
(9.101)
1 \leq \exp \left(\left(4m^2C^2 M - (r - C''b Y_0^{TX})^2/t\right)\right).
\]
By (9.99), (9.102), we obtain
\[
(9.103)
1 \leq \exp \left(4m^2C^2 M - C''r^2/t + \frac{1}{2} |Y_0^{TX}|^2\right),
\]
so that (9.73) also holds in this case. This concludes the proof of (9.73).

Now we establish (9.74). By equation (9.6) with \( t \) replaced by \( t/b^2 \) and by (9.11), we get
\[
(9.104) \quad \exp \left(-|Y_0^{TX}|^2/2\right) E^Q \left[\exp \left(|Y_t^{TX}|^2/2\right)\right] = \left(\frac{e^{t/b^2}}{\cosh(t/b^2)}\right)^{m/2} \exp \left(-\frac{1}{2} \tanh(t/b^2) |Y_0^{TX}|^2\right).
\]
Therefore when \( r^2/t \) remains uniformly bounded, (9.74) holds. In the sequel, we may take \( r^2/t \) as large as needed.

The same argument as in (9.78) shows that
\[
(9.105) \quad \exp \left(-|Y_0^{TX}|^2/2\right) E^Q \left[\mathbf{1}_{\sup_{0 \leq s \leq t} d(x_0, x_s) \geq r} \exp \left(|Y_t^{TX}|^2/2\right)\right] = E^P \left[\exp \left(-\frac{1}{2b^2} \int_0^t |Y_s^{TX}|^2 \, ds + \frac{mt}{2b^2}\right) \mathbf{1}_{\sup_{0 \leq s \leq t} d(x_0, x_s) \geq r}\right].
\]
By (9.80), (9.105), we get
\[
(9.106) \quad \exp \left(-|Y_0^{TX}|^2/2\right) E^Q \left[\mathbf{1}_{\sup_{0 \leq s \leq t} d(x_0, x_s) \geq r} \exp \left(|Y_t^{TX}|^2/2\right)\right] \leq \exp \left(-r^2/4t + \frac{mt}{2b^2}\right) E^P \left[\exp \left(-\frac{1}{4b^2} \int_0^t |Y_s^{TX}|^2 \, ds\right)\right].
\]
By equation (9.18) in Proposition 9.4, we get
\[
(9.107) \quad E^P \left[\exp \left(-\frac{1}{4b^2} \int_0^t |Y_s^{TX}|^2 \, ds\right)\right] = \left(\cosh\left(t/\sqrt{2}b^2\right)\right)^{-m/2} \exp \left(-\frac{1}{2\sqrt{2}} \tanh\left(t/\sqrt{2}b^2\right) |Y_0^{TX}|^2\right).
\]
By (9.106), (9.107), we get
\[
(9.108) \quad \exp \left(-|Y_0^{TX}|^2/2\right) E^Q \left[\mathbf{1}_{\sup_{0 \leq s \leq t} d(x_0, x_s) \geq r} \exp \left(|Y_t^{TX}|^2/2\right)\right] \leq \exp \left(-r^2/4t + mt/2b^2 - \frac{1}{2\sqrt{2}} \tanh\left(t/\sqrt{2}b^2\right) |Y_0^{TX}|^2\right).
\]
By (9.108), if \( \tau > 0 \) is fixed, if \( 0 < t \leq \tau b^2 \), equation (9.108) is compatible with (9.74). In the sequel, we may as well assume that \( b^2 \leq t \) and that \( \tau^2 / t \) is large.

In the sequel, Hölder norms are calculated with respect to the probability measure \( Q \). By [B11, eq. (10.7.1)], for \( 1 \leq \theta < 2 \), we get

\[
\| \exp \left( Y_t^{T^X} \right)^2 / 2 \|_\theta \leq \frac{1}{(1 - \theta/2)^{m/2\theta}} \exp \left( \frac{1}{2} \left( 1 - e^{-2t/b^2} \right) \frac{|Y_0^{T^X}|^2}{1 - \theta e^{-t/b^2} \sinh (t/b^2)} \right).
\]

By (9.109), we deduce that

\[
\exp \left( -\frac{1}{2} |Y_0^{T^X}|^2 \right) \| \exp \left( |Y_t^{T^X}|^2 / 2 \right) \|_\theta \leq \frac{1}{(1 - \theta/2)^{m/2\theta}} \exp \left( - \left( 1 - e^{-2t/b^2} \right) \left( 1 - \theta/2 \right) \frac{|Y_0^{T^X}|^2}{2} \right).
\]

By (9.110), we get

\[
\exp \left( -\frac{1}{2} |Y_0^{T^X}|^2 \right) \| \exp \left( |Y_t^{T^X}|^2 / 2 \right) \|_\theta \leq \frac{1}{(1 - \theta/2)^{m/2\theta}} \exp \left( - \left( 1 - e^{-2t/b^2} \right) \left( 1 - \theta/2 \right) \frac{|Y_0^{T^X}|^2}{2} \right).
\]

We take \( r \geq \epsilon \). Let \( f \) be as in (9.53). We still use (9.83) and the arguments that follow. Using Hölder’s inequality, we get

\[
E^Q \left[ \sup_{0 \leq s \leq t} \int_0^s \nabla_y T^X \nabla_y T^X f(x_u) du \left( |Y_t^{T^X}|^2 / 2 \right) \right] \leq \| \exp \left( |Y_t^{T^X}|^2 / 2 \right) \|_\theta \left( Q \left[ \sup_{0 \leq s \leq t} \int_0^s \nabla_y T^X \nabla_y T^X f(x_u) du \geq r / 2 \right] \right)^{(\theta - 1)/\theta}.
\]

Let \( \beta, 0 < \beta < 1 \). We will use (9.86) with \( \alpha = \beta^2 / C \). By equation (9.23) in Proposition 9.5 and by (9.111), (9.112), we obtain

\[
\exp \left( -|Y_0^{T^X}|^2 / 2 \right) E^Q \left[ \sup_{0 \leq s \leq t} \int_0^s \nabla_y T^X \nabla_y T^X f(x_u) du \geq r / 2 \right] \exp \left( |Y_t^{T^X}|^2 / 2 \right) \leq \frac{1}{(1 - \theta/2)^{m/2\theta}} \exp \left( - \left( 1 - e^{-2t/b^2} \right) \left( 1 - \theta/2 \right) \frac{|Y_0^{T^X}|^2}{2} \right) \exp \left( -\beta^2 r / 2C b^2 + m (1 - \rho_\beta) t / 2b^2 \right) \exp \left( \beta^2 \frac{\tanh (\rho_\beta t / b^2)}{\rho_\beta + \tanh (\rho_\beta t / b^2)} \frac{|Y_0^{T^X}|^2}{2} \right)^{(\theta - 1)/\theta}.
\]
Observe that since \( \rho_\beta < 1 \), we have

\[
(9.114) \quad \left( 1 - e^{-2t/b^2} \right) (1 - \theta/2) - \frac{\theta - 1}{\theta} \beta^2 \frac{\tanh (\rho_\beta t/b^2)}{\rho_\beta + \tanh (\rho_\beta t/b^2)} \geq \left( 1 - \theta/2 - \frac{\theta - 1}{\theta} \beta^2 \right) \left( 1 - e^{-2\rho_\beta t/b^2} \right).
\]

Given \( \beta \), by taking \( \theta \) close enough to 1 in (9.114), there is \( c > 0 \) such that

\[
(9.115) \quad \left( 1 - e^{-2t/b^2} \right) (1 - \theta/2) - \frac{\theta - 1}{\theta} \beta^2 \frac{\tanh (\rho_\beta t/b^2)}{\rho_\beta + \tanh (\rho_\beta t/b^2)} \geq c \left( 1 - e^{-2\rho_\beta t/b^2} \right).
\]

By (9.113), (9.115), we get

\[
(9.116) \quad \exp \left( - |Y_0^{TX}|^2 / 2 \right) E^Q \left[ \sup_{0 \leq s \leq t} \int_0^s \nabla Y_0^{TX} \nabla_{Y_i} f(x_u) du \geq r/2 \exp \left( |Y_t^{TX}|^2 / 2 \right) \right]
\]

\[
\leq C \left[ \exp \left( - \beta^2 r/2C^2 b^2 + m (1 - \rho_\beta) t/2b^2 \right) \right]^{(\theta - 1)/\theta}
\]

\[
\exp \left( - c \left( 1 - e^{-2\rho_\beta t/b^2} \right) |Y_0^{TX}|^2 / 2 \right).
\]

Observe that

\[
(9.117) \quad \beta^2 r/4C - m (1 - \rho_\beta) t/2 = \sqrt{t} \left( \beta^2 \frac{r}{4C \sqrt{t}} - m (1 - \rho_\beta) \sqrt{t}/2 \right).
\]

Given \( M > 0 \), from (9.117), we deduce that if \( r^2/t \) is large enough and \( 0 < t \leq M \), (9.117) is nonnegative. By (9.116), we get

\[
(9.118) \quad \exp \left( - |Y_0^{TX}|^2 / 2 \right) E^Q \left[ \sup_{0 \leq s \leq t} \int_0^s \nabla Y_0^{TX} \nabla_{Y_i} f(x_u) du \geq r/2 \exp \left( |Y_t^{TX}|^2 / 2 \right) \right]
\]

\[
\leq C \exp \left( -C'r^2/5 + \left( 1 - e^{-2\rho_\beta t/b^2} \right) |Y_0^{TX}|^2 / 2 \right).
\]

By Hölder’s inequality, for \( \theta \in [1, +\infty] \), we get

\[
(9.119) \quad E^Q \left[ \sup_{0 \leq s \leq t} \int_0^s \nabla Y_0^{TX} \nabla_{Y_i} f(x_u) du \geq r/2 \exp \left( |Y_t^{TX}|^2 / 2 \right) \right]
\]

\[
\leq \left\| \exp \left( |Y_t^{TX}|^2 / 2 \right) \right\|_\theta \left( Q \left[ \sup_{0 \leq s \leq t} \int_0^s \nabla_{Y_i} f(x_u) du \geq r \right] \right)^{(\theta - 1)/\theta}.
\]

By (9.90), (9.111), and (9.119), we get

\[
(9.120) \quad \exp \left( - |Y_0^{TX}|^2 / 2 \right) E^Q \left[ \sup_{0 \leq s \leq t} \int_0^s \nabla Y_0^{TX} \nabla_{Y_i} f(x_u) du \geq r/2 \exp \left( |Y_t^{TX}|^2 / 2 \right) \right]
\]

\[
\leq C \exp \left( -C'r^2/5 + \left( 1 - e^{-2\rho_\beta t/b^2} \right) |Y_0^{TX}|^2 / 2 \right).
\]

Equation (9.120) is compatible with (9.74).
By (9.118), (9.120), we find that under the above conditions,
\begin{align}
\exp \left( - \frac{\|Y_{0}^{TX} \|^2}{2} / 2 \right) E^{Q} \left[ 1_{\sup_{0 \leq s \leq t} d(x_{0},x_{s}) \geq r} \exp \left( \frac{\|Y_{t}^{TX} \|^2}{2} / 2 \right) \right] \\
\leq C \exp \left( -C' \left( \frac{r}{b^2} + \left( 1 - e^{-2\beta t/b^2} \right) \frac{\|Y_{0}^{TX} \|^2}{2} \right) \right) \\
+ C \exp \left( -C' \left( \frac{r^2}{t} + \left( 1 - e^{-2t/b^2} \right) \frac{\|Y_{0}^{TX} \|^2}{2} \right) \right). \end{align}

We will now combine the estimates (9.108) and (9.121). We proceed as in (9.117). Observe that
\begin{align}
C' r - mt/2 = \sqrt{t} \left( C' r / \sqrt{t} - m \sqrt{t} / 2 \right). \end{align}

Given $M > 0$, we deduce that if $0 < t \leq M$, if $r / \sqrt{t}$ is large enough, (9.122) is positive. By (9.108), (9.121), for $r / \sqrt{t}$ large enough, we get the full estimate (9.74). This completes the proof of (9.74) when $r \geq \epsilon$.

Let us now extend (9.121) with $r > 0$ small, while $r^2 / t$ can be taken to be large and $t \geq b^2$. We take again $f_{1}, \ldots, f_{m}$ as in the proof of Proposition 9.13. We still consider equation (9.72) with $f = \pm f_{i}, 1 \leq i \leq m$. With respect to what we did before, we have the extra term $b^2 d / t f(x_{i}) |_{t=0} \left( 1 - e^{-t/b^2} \right)$. We still use the estimate (9.96). By proceeding the way we did before, the estimate (9.121) is still valid when replacing $r$ by $(r - Cb |Y_{0}^{TX}|)_{+}$ in the right-hand side. In particular, for $b > 0, 0 < t \leq M, r > 0$, we have
\begin{align}
\exp \left( - \frac{\|Y_{0}^{TX} \|^2}{2} / 2 \right) E^{Q} \left[ 1_{\sup_{0 \leq s \leq t} d(x_{0},x_{s}) \geq r} \exp \left( \frac{\|Y_{t}^{TX} \|^2}{2} / 2 \right) \right] \\
\leq \exp \left( -C' \left( (r - Cb |Y_{0}^{TX}|)_{+} / b^2 + |Y_{0}^{TX} |^2 \right) \right) \\
+ C \exp \left( -C' \left( (r - Cb |Y_{0}^{TX}|)_{+} / t + \left( 1 - e^{-2t/b^2} \right) \frac{\|Y_{0}^{TX} \|^2}{2} \right) \right). \end{align}

We claim that if $t \geq b^2$, \begin{align}
(r - Cb |Y_{0}^{TX}|)_{+} / t + |Y_{0}^{TX} |^2 \geq C'' \left( r^2 / t + \left( 1 - e^{-t/b^2} \right) \frac{\|Y_{0}^{TX} \|^2}{2} \right). \end{align}
Indeed (9.124) holds if $Cb |Y_{0}^{TX}| \leq r/2$. If $Cb |Y_{0}^{TX}| > r/2$, since $t \geq b^2$, then
\begin{align}
|Y_{0}^{TX} |^2 \geq r^2 / 4C^2b^2 \geq r^2 / 4C^2t, \end{align}
so that we still get (9.124). By (9.124), the second term in the right-hand side of (9.123) is compatible with (9.74). Consider the first term in the right-hand side of (9.123). By (9.125), only the case $Cb |Y_{0}^{TX}| \leq r/2$ should be considered. In this case, we get
\begin{align}
\exp \left( -C' \left( (r - Cb |Y_{0}^{TX}|)_{+} / b^2 + |Y_{0}^{TX} |^2 \right) \right) \leq \exp \left( -C' \left( r^2 / 2b^2 + \left( 1 - e^{-2t/b^2} \right) \frac{\|Y_{0}^{TX} \|^2}{2} \right) \right). \end{align}
Using (9.108), (9.122), and (9.126), we find that the first term in the right-hand side of (9.123) is also compatible with (9.74), when taking into account the fact that $t \geq b^2$.

Finally, (9.75) follows from (9.74). The proof of our theorem is completed. \qed
9.9. Convergence in probability of $x$. Let $(\Omega, \mathcal{F}, \Pi)$ be a probability space, and let $(F, \delta)$ be a metric space. Let $X_n|_{n \in \mathbb{N}}$ be a family of random variables defined on $\Omega$ with values in $F$. If $X_\infty$ is another such random variable, we say that as $n \to +\infty$, $X_n$ converges to $X_\infty$ in probability if for any $\epsilon > 0$, as $n \to +\infty$, $\Pi[\delta(X_n, X_\infty) \geq \epsilon]$ tends to 0.

For greater clarity, we will now write the dependence of $x, g$ on $b > 0$ in equations (9.60), (9.62) explicitly, i.e., $x, g$ will now be denoted $x_b, g_b$.

**Definition 9.21.** Let $x_{0,}$ be the solution of the stochastic differential equation

\begin{align}
(9.127) \quad \dot{x}_{0,} &= \dot{w}^TX, \\
\quad x_{0,0} &= p1.
\end{align}

Let $g_{0,}$ denote the horizontal lift of $x_{0,},$ so that $g_{0,}$ is the solution of

\begin{align}
(9.128) \quad \dot{g}_{0,} &= \dot{w}^p, \\
\quad g_{0,0} &= 1.
\end{align}

Note that

\begin{align}
(9.129) \quad x_{0,} &= pg_{0,}.
\end{align}

Observe that $x_{b,}, g_{b,}$ in (9.60) and $x_{0,}, g_{0,}$ in (9.127) have been constructed on the same probability space.

We will improve on a result established in [B11, proof of Theorem 12.8.1], where a weaker result of convergence in probability law was established. This stronger form of convergence will not be used in the paper.

**Theorem 9.22.** For any $M > 0,$ as $b \to 0,$ $(x_{b,}, g_{b,})$ converges to $(x_{0,}, g_{0,})$ uniformly over $[0, M]$ in probability.

**Proof.** By [B11, proof of Theorem 12.8.1 and Remark 12.8.2], as $b \to 0,$ the probability law of $x_{b,}$ on $C([0, M], X)$ converges to the probability law of $x_{0,}$.

Set

\begin{align}
(9.130) \quad H_{b, t} &= \int_0^t \frac{Y_s^TX}{b} ds, \\
\quad H_{0, t} &= w_t^p.
\end{align}

The same argument shows that as $b \to 0,$ the probability law of $(x_{b,}, H_{b,})$ on $C([0, M], X \times p)$ converges to the probability law of $(x_{0,}, H_{0,})$. It is easy to deduce from this that the probability law of $(x_{b,}, p^b)$ converges to the probability law of $(x_{0,}, p^0)$. An elementary argument detailed in [B81, p. 48] allows us to obtain our theorem for $x_{b,}$. The same argument can be used also for $g_{b,}$. The proof of our theorem is completed. \hfill \Box

9.10. A uniform estimate on the heat kernel for $A_b^X$. By the results of [B11, section 11.5], for $t > 0,$ the operators $\exp(-tA_b^X), \exp(-tB_b^X)$ are well-defined, and there are associated smooth kernels

\begin{align}
r_{b, t}^X((x, Y^TX), (x', Y^{TX'})), \quad s_{b, t}^X((x, Y^TX), (x', Y^{TX'}))
\end{align}

with respect to the volume $dx'dY^{TX'}$. By Proposition 9.14, we get

\begin{align}
(9.131) \quad r_{b, t}^X \geq 0, \\
\quad s_{b, t}^X \geq 0.
\end{align}

Recall that the smooth kernel $p_t(x, x')$ was defined in subsection 9.4.

**Definition 9.23.** For $t > 0,$ $(x, Y^TX), (x', Y^{TX'}) \in X$, put

\begin{align}
(9.132) \quad r_{0, t}^X((x, Y^TX), (x', Y^{TX'})) = p_t(x, x') \pi^{-m/2} \exp\left(-\frac{1}{2} \left(|Y^TX|^2 + |Y^{TX'}|^2\right)\right).
\end{align}
We explain results established in [B11, Theorems 12.8.1 and 13.2.4], where we extend the range of parameters.

**Theorem 9.24.** Given \( \tau > 0, M > 0 \), there exist \( k \in \mathbb{N}, C > 0, C' > 0 \) such that if \( b > 0, \tau b^2 \leq t \leq M, (x, Y^{TX}) \), \((x', Y^{TX'}) \) \( \in \mathcal{X}, \)

\[
r_{b,t}^X ((x, Y^{TX}), (x', Y^{TX'})) \leq C \exp \left( \frac{C'(d^2(x, x') + |Y^{TX}|^2 + |Y^{TX'}|^2)}{t} \right).
\]

Given \( t > 0 \), as \( b \to 0 \), we have the pointwise convergence

\[
r_{b,t}^X ((x, Y^{TX}), (x', Y^{TX'})) \to r_{0,t}^X ((x, Y^{TX}), (x', Y^{TX'})).
\]

**Proof.** Equation (9.134) was proved in [B11, Theorem 12.8.1]. Equation (9.133) was established in [B11, Theorem 13.2.4] when \( 0 < b \leq b_0, \epsilon \leq t \leq M \). Also observe that if \( X \) is a Euclidean vector space \( E \), equation (9.133) was already established in (9.46).

We will explain how to extend the arguments of [B11] to the general case. We consider the Euclidean vector space \( E = \mathfrak{p} \). For \( t > 0 \), let \( L_{2,t} \) be the vector space of \( L_2 \) functions from \([0, t] \) with values in \( \mathfrak{p} \) with respect to the Lebesgue measure. For \( t = 1 \), we write \( L_2 \) instead of \( L_{2,1} \).

We proceed as in [B11, section 12.5]. For \( b > 0, t > 0, e \in \mathfrak{p}, f \in \mathfrak{p}, v \in L_{2,t} \), as in [B11, eq. (12.5.2)], consider the differential equation on the interval \([0, t]\) on the function \( J^{TX} : [0, t] \to \mathfrak{p}, \)

\[
b^2 J^{TX} + J^{TX} = v, \quad J^{TX}_0 = 0, \quad J^{TX}_t = e, \quad J^{TX}_t = f/b.
\]

For \( s \in [0, 1] \), set

\[
K^{TX}_s = J^{TX}_s, \quad w_s = tv_{st}.
\]

If \( \beta = b/\sqrt{t} \), we have

\[
\beta^2 \dot{K}^{TX} + \dot{K}^{TX} = w, \quad K^{TX}_0 = 0, \quad K^{TX}_1 = e, \quad K^{TX}_1 = \sqrt{t} f/\beta.
\]

We take in (9.137) the \( w \) minimizing \( |w|_{L_2}^2 \) so that the given boundary conditions are verified. By the considerations after [B11, eq. (10.3.47)], as long as \( e, f \) remain uniformly bounded, and \( \beta > 0, t > 0 \) remain uniformly bounded, \( |w|_{L_2}^2 \) remains uniformly bounded, and the functions \( K^{TX}, \beta \dot{K}^{TX} \) also remain uniformly bounded.

We can now transfer these results to corresponding results on equation (9.135). Given \( b, e, f, \) let \( v \in L_{2,t} \) minimize \( |v|_{L_2}^2 \), so that the boundary conditions in (9.135) are verified. By (9.136), we get

\[
|v|_{L_2,t}^2 = \frac{|w|_{L_2}^2}{t}.
\]

Under the same conditions as before, \( t|v|_{L_2,t}^2 \) remains uniformly bounded, and the functions \( J^{TX}, b\sqrt{t} J^{TX} \) remain uniformly bounded.

In general, to estimate the kernels \( r_{b,t}^X, s_{b,t}^X \), we proceed exactly as in [B11, chapter 12], where \( t > 0 \) was fixed. The main point is to keep track of the dependence on \( t \) of the estimates in [B11]. Let us first consider [B11, Theorems 12.6.1]. Then
the above uniform estimates on $t |v|^2$, $J$, $b\sqrt{t}J$, show that in the estimate [B11, eq. (12.6.2)] in [B11, Theorem 12.6.1], we have an extra constant $1/\sqrt{t}$ in the right-hand side. Namely, given $c' > 0$, $M > 0$, $p \in [1, +\infty]$, there is $C > 0$ such that for $0 < b \leq M$, $0 < t \leq M$, $\beta \leq c'$, \footnote{We refer to [B11] for more details on $M_t$. Let us just mention that an integration by parts formula coming from the Malliavin calculus was established in [B11, Theorem 12.5.1], in which the random tensor $M_t$ appears. Estimating $\|M_t\|_p$ is used in [B11] to control the kernel $r_{b,t}$ for bounded $b$ and fixed $t > 0$.}

\begin{equation}
(9.139) \quad \|M_t\|_p \leq \frac{C}{\sqrt{t}} \left(1 + |Y|^2 \right).
\end{equation}

We can still use the formula of integration by parts in [B11, Theorem 12.7.1]. We claim that given $\tau > 0$, $M > 0$, there exist $\theta, 1 < \theta < 2$ and $C_\theta > 0$ such that for $0 < b \leq M$, $\tau b^2 \leq t \leq M$, we have the obvious analogue of [B11, eq. (12.7.2)],

\begin{equation}
(9.140) \quad \left\| \exp \left(\frac{|Y|^2}{2} \right) \right\|_p \leq C_\theta \exp \left(\frac{|Y|^2}{2\theta} \right).
\end{equation}

This is just a consequence of (9.109), of the fact that if $1 < \theta \leq \frac{e^t}{\cosh(t)}$, then

\begin{equation}
(9.141) \quad \frac{e^{-2t}}{1 - \theta e^{-t} \sinh(t)} \leq \frac{1}{\theta},
\end{equation}

and of the fact that $\frac{e^t}{\cosh(t)}$ is an increasing function of $t$.

By proceeding as in [B11, Theorem 12.7.4], we find that given $c' > 0$, $M > 0$, $k \in \mathbb{N}$, there exists $k' \in \mathbb{N}$ such that if $0 < b \leq M$, $0 < t \leq M$, $0 < \beta \leq c'$, the covariant derivatives of $t^{k'} r_{b,t}^X ((x, Y^{TX}), (x', Y^{TX'}))$ of order $\leq k$ are uniformly bounded and uniformly rapidly decreasing when $|Y^{TX}|$ or $|Y^{TX'}|$ tend to $+\infty$.

Now we use the semigroup identity

\begin{equation}
(9.142) \quad r_{b,t}^X ((x, Y^{TX}), (x', Y^{TX'})) = \int_X r_{b,t/2}^X ((x, Y^{TX}), (z, Z^{TX})) r_{b,t/2}^X ((z, Z^{TX}), (x', Y^{TX'})) dzdZ^{TX}.
\end{equation}

We proceed as in [B11, section 13.2]. First, using the above uniform bounds, we deduce from (9.142) that

\begin{equation}
(9.143) \quad r_{b,t}^X ((x, Y^{TX}), (x', Y^{TX'})) \leq C t^{-k'} \int_X r_{b,t/2}^X ((x, Y^{TX}), (z, Z^{TX})) dzdZ^{TX}.
\end{equation}

Recall that the heat kernel $h_t^p (Y, Y')$ is given by (9.3). As explained in [B11, eq. (13.2.5)], we have the identity

\begin{equation}
(9.144) \quad \int_X r_{b,t/2}^X ((x, Y^{TX}), (z, Z^{TX})) dzdZ^{TX} = \int_p h_{t/2b^2}^p (Y^{TX}, Z^{TX}) dZ^{TX}.
\end{equation}

By (9.6), (9.144), we get

\begin{equation}
(9.145) \quad \int_X r_{b,t/2}^X ((x, Y^{TX}), (z, Z^{TX})) dzdZ^{TX}
= \left(\frac{e^{t/2b^2}}{\cosh (t/2b^2)} \right)^m \frac{m}{2} \exp \left( -\frac{\tanh (t/2b^2)}{2} |Y^{TX}|^2 \right).
\end{equation}
By combining (9.143) and (9.145), we deduce that given \( \tau > 0 \), there exist \( C > 0, C' > 0 \) such that if \( t \geq \tau b^2 \), then

\[
(9.146) \quad r_{b,t}^X \left( (x, Y^{TX}), (x', Y^{TX'}) \right) \leq C t^{-k'} \exp \left( -C' |Y^{TX}|^2 \right).
\]

The \( L_2 \) formal adjoint of \( A^X_{b,t} \) is the operator \( A^X_{b,t} \) deduced from \( A^X_b \) by making the change of variables \( Y^{TX} \to -Y^{TX} \). By interchanging the roles of \( Y^{TX} \) and \( Y^{TX'} \) in (9.146), under the same conditions, we get

\[
(9.147) \quad r_{b,t}^X \left( (x, Y^{TX}), (x', Y^{TX'}) \right) \leq C t^{-k'} \exp \left( -C' |Y^{TX'}|^2 \right).
\]

In the integral in (9.142), either \( d(x, z) \geq d(x, x')/2 \) or \( d(x', z) \geq d(x, x')/2 \). From the uniform bounds on \( r_{b,t}^X \) that were described before equation (9.142), we deduce from (9.142) that under the conditions of our theorem,

\[
(9.148) \quad r_{b,t}^X \left( (x, Y^{TX}), (x', Y^{TX'}) \right)
\leq C t^{-k'} \int_{d(x, z) \geq d(x, x')/2} r_{b,t/2}^X \left( (x, Y^{TX}), (z, Z^{TX}) \right) dzdZ^{TX}
\leq C t^{-k'} \int_{d(x', z) \geq d(x, x')/2} r_{b,t/2}^X \left( (z, Z^{TX}), (x', Y^{TX'}) \right) dzdZ^{TX}.
\]

By equation (9.64) in Proposition 9.14, we get

\[
(9.149) \quad \int_{d(x, z) \geq d(x, x')/2} r_{b,t/2}^X \left( (x, Y^{TX}), (z, Z^{TX}) \right) dzdZ^{TX}
= \exp \left( -|Y^{TX}|^2/2 \right) E^Q \left[ 1_{d(x, z) \geq d(x, x')/2} \exp \left( \left| Y_{t/2}^{TX} \right|^2 / 2 \right) \right].
\]

From (9.149), we deduce that

\[
(9.150) \quad \int_{d(x, z) \geq d(x, x')/2} r_{b,t/2}^X \left( (x, Y^{TX}), (z, Z^{TX}) \right) dzdZ^{TX}
\leq \exp \left( -|Y^{TX}|^2/2 \right) E^Q \left[ 1_{s \leq M \geq d(x, x')/2} \exp \left( \left| Y_{t/2}^{TX} \right|^2 / 2 \right) \right].
\]

By equation (9.75) in Theorem 9.20 and by (9.150), if \( 0 < \tau b^2 \leq t \leq M \), we get

\[
(9.151) \quad \int_{d(x, z) \geq d(x, x')/2} r_{b,t}^X \left( (x, Y^{TX}), (z, Z^{TX}) \right) dzdZ^{TX}
\leq C \exp \left( -C' \left( d^2 (x, x') / t + |Y^{TX}|^2 \right) \right).
\]

Still using the fact that the \( L_2 \) adjoint of \( A^X \) is just \( A^X_{b,-} \), we also get the estimate

\[
(9.152) \quad \int_{d(x', z) \geq d(x, x')/2} r_{b,t}^X \left( (z, Z^{TX}), (x', Y^{TX'}) \right) dzdZ^{TX}
\leq C \exp \left( -C' \left( d^2 (x, x') / t + |Y^{TX'}|^2 \right) \right).
\]
By (9.148), (9.151), and (9.152), we get

\[ r^{X}_{b,t}((x,Y^{TX},(x',Y'^{TX}))) \leq Ct^{-k'} \left( \exp \left( -C'd^2(x,x')/t + |Y^{TX}|^2 \right) \right) \]
\[ + Ct^{-k'} \exp \left( -C' \left( d^2(x,x')/t + |Y'^{TX}|^2 \right) \right). \]

By combining (9.146), (9.147), and (9.153), we get (9.133). The proof of our theorem is completed.
10. Uniform estimates for \( b \) small: the scalar case

The purpose of this section is to establish uniform estimates for the heat kernels of a scalar version \( A_{X,b,\vartheta} \) of the operator \( L_{X,b,\vartheta} \). These estimates correspond to the estimates stated in Theorem 6.3 for the nonscalar heat kernels \( q_{X,b,\vartheta,t} \). They are established for bounded values of \( b > 0 \). The main difficulty in the proof of these estimates is that when \( \vartheta \) gets close to \( \frac{\pi}{2} \), our scalar operators become singular. Our estimates will be essentially consequences of results of [B11] that were properly extended in section 9.

As will be shown in section 11 and in dramatic contrast with [B11], this analogy is no longer valid for the kernels \( q_{X,b,\vartheta,t} \) themselves, because the matrix terms in \( L_{X,b,\vartheta} \) are different from the ones in \( L_{X,b} \).

This section is organized as follows. In subsection 10.1, we introduce a scalar analogue \( A_{X,b,\vartheta} \) of \( L_{X,b,\vartheta} \) over \( \hat{X} \).

In subsection 10.2, we show that the estimates on the heat kernel \( r_{X,b,\vartheta,t} \) for \( A_{X,b,\vartheta} \) are trivial consequences of the results of section 9.

Finally, in subsection 10.3, we introduce the scalar operator \( A_{X,b,\vartheta} \) on \( \hat{X} \), and we establish the required estimates on its heat kernel.

We use the same notation as in section 9.

10.1. The scalar analogues of the operator \( L_{X,b,\vartheta} \) on \( \hat{X} \). By analogy with equation (4.68) for \( L_{X,b,\vartheta} \) and with equation (9.58) for \( A_{X,b}, B_{X,b} \), given \( b > 0, \vartheta \in [0, \frac{\pi}{2}] \), let \( A_{X,b,\vartheta}, B_{X,b,\vartheta} \) be the scalar differential operators on \( \hat{X} \),

\[
A_{X,b,\vartheta} = \frac{1}{2b^2} \left( -\Delta_{TX} + |Y_{TX}|^2 - m \right) - \frac{\cos(\vartheta)}{b} \nabla_{Y_{TX}},
\]

\[
B_{X,b,\vartheta} = \frac{1}{2b^2} \left( -\Delta_{TX} + 2\nabla_{Y_{TX}}^V \right) - \frac{\cos(\vartheta)}{b} \nabla_{Y_{TX}}.
\]

Then

\[
B_{X,b,\vartheta} = \exp \left( \frac{|Y_{TX}|^2}{2} \right) A_{X,b,\vartheta} \exp \left( -\frac{|Y_{TX}|^2}{2} \right).
\]

Comparing with (9.58), we get

\[
A_{X,b,\vartheta} = \cos^2(\vartheta) A_{\cos(\vartheta)b}, \quad B_{X,b,\vartheta} = \cos^2(\vartheta) B_{\cos(\vartheta)b}.
\]

For \( \vartheta = 0 \), our operators coincide with \( A_{X,b}, B_{X,b} \).

10.2. A uniform estimate on the heat kernel for \( A_{X,b,\vartheta} \). Recall that the kernel \( p_t(x,x') \) was defined in subsection 9.4, and the kernel \( r_{X,b,t}((x,Y_{TX}),(x',Y_{TX'})) \) was defined in subsection 9.10.

Definition 10.1. For \( t > 0 \), let \( r_{X,b,\vartheta,t}((x,Y_{TX}),(x',Y_{TX'})) \) be the smooth kernel associated with the operator \( \exp \left( -tA_{X,b,\vartheta} \right) \) with respect to the volume \( dx'dY_{TX'} \).

Set

\[
r_{X,b,\vartheta,t}((x,Y_{TX}),(x',Y_{TX'})) = p_{\cos^2(\vartheta)t}(x,x') \pi^{-m/2} \exp \left( -\frac{1}{2} \left( |Y_{TX}|^2 + |Y_{TX'}|^2 \right) \right).
\]
By (9.132), (10.3), and (10.4), for \( b \geq 0, \vartheta \in \left[ 0, \frac{\pi}{2} \right], t > 0 \), we get

\[
(10.5) \quad r^X_{b, \vartheta, t} = r^X_{b, \cos(\vartheta), t}. 
\]

Now we establish an analogue of Theorem 6.3 for the kernel \( r^X_{b, \vartheta, t} \), which is also an extension of Theorem 9.24 for the kernel \( r^X_{b, t} \).

**Theorem 10.2.** Given \( 0 < \epsilon \leq M < +\infty \), there exist \( C > 0, C' > 0, k \in \mathbb{N} \) such that for \( 0 < b \leq M, 0 \leq \vartheta < \frac{\pi}{2}, \epsilon \leq t \leq M, (x, Y^{TX}), (x', Y^{TX'}) \in X', \)

\[
(10.6) \quad r^X_{b, \vartheta, t} (\langle x, Y^{TX} \rangle, \langle x', Y^{TX'} \rangle) \leq \frac{C}{\cos^k (\vartheta)} \exp \left( -C' \left( \frac{d^2 (x, x')}{\cos^2 (\vartheta)} + |Y^{TX}|^2 + |Y^{TX'}|^2 \right) \right). 
\]

Given \( \vartheta \in \left[ 0, \frac{\pi}{2} \right], t > 0, as b \to 0 \), we have the pointwise convergence,

\[
(10.7) \quad r^X_{b, \vartheta, t} (\langle x, Y^{TX} \rangle, \langle x', Y^{TX'} \rangle) \to r^X_{0, \vartheta, t} (\langle x, Y^{TX} \rangle, \langle x', Y^{TX'} \rangle). 
\]

**Proof.** Our theorem is a consequence of Theorem 9.24 and of equation (10.5). □

### 10.3. A uniform estimate on a scalar heat kernel over \( \tilde{X} \).

Recall that \( \tilde{\pi} : \tilde{X} \to X \) is the total space of \( TX \oplus N \), and that \( Y = Y^{TX} \oplus Y^N \) is the canonical section of \( \tilde{\pi}^* (TX \oplus N) \). Let \( \mathfrak{A}^X_{b, \vartheta}, \mathfrak{B}^X_{b, \vartheta} \) be the scalar differential operators on \( \tilde{X} \),

\[
\mathfrak{A}^X_{b, \vartheta} = \frac{1}{2b^2} \left( -\Delta^X + |Y^{TX}|^2 - m \right) + \frac{\cos (\vartheta)}{2b^2} \left( -\Delta^N + |Y^N|^2 - n \right) 
\]

\[
(10.8) \quad \mathfrak{B}^X_{b, \vartheta} = \frac{1}{2b^2} \left( -\Delta^X + 2\nabla_{Y^{TX}}^V \right) + \frac{\cos (\vartheta)}{2b^2} \left( -\Delta^N + 2\nabla_{Y^N}^V \right) - \frac{\cos (\vartheta)}{b} \nabla_{Y^{TX}}. 
\]

Then

\[
(10.9) \quad \mathfrak{B}^X_{b, \vartheta} = \exp \left( \frac{|Y|^2}{2} \right) \mathfrak{A}^X_{b, \vartheta} \exp \left( -\frac{|Y|^2}{2} \right). 
\]

Note that \( \mathfrak{A}^X_{0, \vartheta}, \mathfrak{B}^X_{0, \vartheta} \) are just the operators \( \mathfrak{A}^X_{b, \vartheta}, \mathfrak{B}^X_{b, \vartheta} \) defined in [B11, eqs. (11.6.1) and (11.6.2)].

For \( t > 0 \), let \( r^X_{b, \vartheta, t} (\langle x, Y \rangle, \langle x', Y' \rangle) \), \( b, \vartheta, t \) \( \langle x, Y \rangle, \langle x', Y' \rangle \) be the smooth kernels associated with the operators \( \exp (-t\mathfrak{A}^X_{b, \vartheta}), \exp (-t\mathfrak{B}^X_{b, \vartheta}) \). The existence of such kernels follows from [B11, section 11.6].

Set

\[
(10.10) \quad r^X_{0, \vartheta, t} (\langle x, Y \rangle, \langle x', Y' \rangle) = p_{\cos^2 (\vartheta), t} (x, x') \pi^{-(m+n)/2} \exp \left( -\frac{1}{2} \left( |Y|^2 + |Y'|^2 \right) \right). 
\]

Now we establish an analogue of Theorem 6.3 for the kernel \( r^X_{b, \vartheta, t} \). These results extend corresponding results established in [B11, Theorems 12.10.2 and 13.3.1].
Theorem 10.3. Given $0 < \epsilon \leq M < +\infty$, there exist $C > 0, C' > 0, k \in \mathbb{N}$ such that for $0 < b \leq M, \vartheta \in \left[0, \frac{\pi}{2}\right]$, $\epsilon \leq t \leq M$, $(x, Y), (x', Y') \in \mathcal{X}$,

\begin{equation}
\tau^X_{b, \vartheta, k}((x, Y), (x', Y')) \leq \frac{C}{\cos^k(\vartheta)} \exp \left( -C' \left( \frac{d^2(x, x')}{\cos^2(\vartheta)} + \left| Y^T X \right|^2 + \left| Y^T X' \right|^2 \right) \right.
- \frac{1}{4} \left( 1 - e^{-\cos(\vartheta)\epsilon/b^2} \right) \left( \left| Y^N \right|^2 + \left| Y^N' \right|^2 \right).
\end{equation}

There exist $C > 0, C' > 0, k \in \mathbb{N}$ such that under the above conditions,

\begin{equation}
\tau^X_{b, \vartheta, k}((x, Y), (x', Y')) \leq \frac{C}{\cos^k(\vartheta)} \exp \left( -C' \left( \frac{d^2(x, x')}{\cos^2(\vartheta)} + \left| Y^T X \right|^2 + \left| Y^T X' \right|^2 \right) \right.
- \left. \left( \cos(\vartheta) \left| Y^N \right|^2 + \left| Y^N' \right|^2 \right) \right).
\end{equation}

Given $t > 0, \vartheta \in \left[0, \frac{\pi}{2}\right]$, as $b \to 0$, we have the pointwise convergence,

\begin{equation}
\tau^X_{b, \vartheta, k}((x, Y), (x', Y')) \to \tau^X_{0, \vartheta, k}((x, Y), (x', Y')).
\end{equation}

Proof. Take $x_0 \in X$. Let $w = (w^{TX}, w^N)$ be a Brownian motion in $(TX \oplus N)_{x_0}$, and let $E$ be the corresponding expectation operator. Consider the stochastic differential equation

\begin{equation}
\dot{x} = \frac{\cos(\vartheta)}{b} Y^T X, \quad Y^T X = \frac{1}{b} \dot{w}^T X, \quad Y^N = \frac{\cos^{1/2}(\vartheta)}{b} \dot{w}^N,
(x, Y)_0 = (x_0, Y_0).
\end{equation}

In (10.14), $Y^T X, Y^N$ denote the covariant derivatives along the path $x$, with respect to the connections $\nabla^T X, \nabla^N$. As explained in [B11, eq. (13.2.12)], an application of It and Feynman Kac formulas shows that if $F \in C^{\infty, \epsilon}(X, \mathbb{R})$, then

\begin{equation}
\exp \left(-t \mathfrak{A}^X_{b, \vartheta}\right) F(x_0, Y_0) = \exp \left( (m + \cos(\vartheta) n) \frac{t}{2b^2} \right) E \left[ \exp \left( -\frac{1}{2b^2} \int_0^t \left( \left| Y^T X \right|^2 + \cos(\vartheta) \left| Y^N \right|^2 \right) ds \right) F(x_t, Y_t) \right].
\end{equation}

We will now be more precise on the proper interpretation of (10.14), (10.15). Indeed let $Y^N_s \in N_{x_0}$ be the path corresponding to $Y^N$, so that if $\tau^0_s$ is the parallel transport from $N_{x_0}$ to $N_x$ along $x$, and if $\tau^0_s$ denotes its inverse, we have

\begin{equation}
y^N_s = \tau^0_{s\tau^N_{s}}.
\end{equation}

Then we rewrite (10.15) in the form

\begin{equation}
\exp \left(-t \mathfrak{A}^X_{b, \vartheta}\right) F(x_0, Y_0) = \exp \left( (m + \cos(\vartheta) n) \frac{t}{2b^2} \right) E \left[ \exp \left( -\frac{1}{2b^2} \int_0^t \left( \left| Y^T X \right|^2 + \cos(\vartheta) \left| Y^N \right|^2 \right) ds \right) F(x_t, \tau^0_{t\tau^N_{t}}) \right].
\end{equation}
We will use the fact that \( x \) and \( Y_N \) are independent processes. Recall that the heat kernel \( h_t^N (Y, Y') \) is given by (9.3). By (10.17), we get

\[
\exp \left( -t \mathfrak{A}_b^X \right) F (x_0, Y_0) = \exp \left( \frac{mt}{2b^2} \right) \left[ \exp \left( -\frac{1}{2b^2} \int_0^t |Y^T X|^2 \, ds \right) \right] \int_{N_{x_0}} h_{\cos (\theta/t/2b)}^N (Y_0^N, Y^N) F (x_t, Y^T X, Y^N) \, dY^N.
\]

Since \( \tau^0_s \) is an isometry, we can rewrite (10.18) in the form

\[
\exp \left( -t \mathfrak{A}_b^X \right) F (x_0, Y_0) = \exp \left( \frac{mt}{2b^2} \right) \left[ \exp \left( -\frac{1}{2b^2} \int_0^t |Y^T X|^2 \, ds \right) \right] \int_{N_{x_0}} h_{\cos (\theta/t/2b)}^N (\tau_0^0 Y_0^N, Y^N) F (x_t, Y^T X, Y^N) \, dY^N.
\]

Assume that \( F \) is nonnegative. Using (10.19), we obtain

\[
\exp \left( -t \mathfrak{A}_b^X \right) F (x_0, Y_0) \leq \exp \left( \frac{mt}{2b^2} \right) \exp \left( -\frac{1}{2} \tanh \left( \frac{1}{2} \frac{t}{b^2} \right) \left( \frac{1}{2} |Y^N_0|^2 + |Y^N|^2 \right) \right) F (x_t, Y^T X, Y^N) \, dY^N.
\]

By (10.20), we deduce that

\[
\tau_{b, \theta}^X \left( (x, Y), (x', Y') \right) \leq \tau_{b, \theta}^X \left( (x, Y^T X), (x', Y^T X') \right) \left( \frac{e^{\cos (\theta/t)^2/2b^2}}{2\pi \sinh (\cos (\theta/t)^2/2b^2)} \right)^{n/2} \left( \frac{1}{2} \frac{t}{b^2} \tanh \left( \frac{1}{2} \frac{t}{b^2} \right) \left( |Y^N_0|^2 + |Y^N|^2 \right) \right).
\]

For \( x \geq 0 \), we get

\[
\tanh (x) = \frac{1 - e^{-2x}}{1 + e^{-2x}} \geq \frac{1}{2} \left( 1 - e^{-2x} \right).
\]

For \( 0 < b \leq M, t \geq \epsilon \), then \( t/b^2 \geq \epsilon/M^2 \), and so

\[
\frac{e^{\cos (\theta/t)^2/2b^2}}{2\pi \sinh (\cos (\theta/t)^2/2b^2)} \leq \frac{e^{\cos (\theta)^2/2M^2}}{2\pi \cos (\theta)} \leq \frac{C}{\cos (\theta)}.
\]

By combining equation (10.6) in Theorem 10.2 and (10.21)–(10.23), we get (10.11), (10.12).

By equation (9.134) in Theorem 9.24, by (10.10) and (10.21), we get

\[
\limsup_{b \to 0} \tau_{b, \theta, t}^X \left( (x, Y), (x', Y') \right) \leq \tau_{0, \theta, t}^X \left( (x, Y), (x', Y') \right).
\]
Also by (9.4), (9.134), and (10.18), as $b \to 0$,

\begin{equation}
\exp\left(-t\mathfrak{A}_{b,\vartheta}^X\right) F(x, Y) \to \int_{\tilde{\mathcal{X}}} \mathfrak{r}_{0,\vartheta,t}^X ((x, Y), (x', Y')) F(x', Y') \, dx'dY'.
\end{equation}

The above is not enough to obtain the pointwise convergence in (10.13). However, given $\vartheta \in [0, \frac{\pi}{2}]$, as explained in [B11, Theorems 12.10.1 and 12.10.2], we can still use the Malliavin calculus, and show that given $(x, Y) \in \tilde{\mathcal{X}}$, $\mathfrak{r}_{b,\vartheta}^X ((x, Y), (x', Y'))$ and its derivatives in $(x', Y')$ of arbitrary order are uniformly bounded on any compact set. By combining this with equation (10.25), we get (10.13). The proof of our theorem is completed. \qed
11. The hypoelliptic heat kernel $q^{X}_{b,ϑ,t}$ for small $b$

The purpose of this section is to prove Theorem 6.3. More precisely, we establish uniform estimates on the smooth kernel $q^{X}_{b,ϑ,t}((x,Y),(x',Y'))$ when $0 < b \leq 1$, and we prove that as $b \to 0$, $q^{X}_{b,ϑ,t}$ converges to $q^{X}_{0,ϑ,t}$. A scalar version of our estimates was established in section 10. While in [B11], when $ϑ = 0$, the corresponding estimates for $q^{X}_{b,ϑ,t}$ could be derived relatively easily from the estimates on the scalar version of this heat kernel, this is not the case here, because of the matrix structure of our operator. At a technical level, while the solutions of certain linear stochastic differential equations could be controlled in [B11] using a version of Gronwall’s lemma, this is no longer possible. This alone explains the length of this section.

This section is organized as follows. In subsection 11.1, we give a probabilistic expression for the semigroup $\exp(-tT^{X})$.

In subsection 11.2, we give a crucial identity on the operator $\tilde{M}^{X}_{b,ϑ}$, which is conjugate to the operator $\tilde{L}^{X}_{b,ϑ}$. In subsection 11.3, we make the innocuous change of coordinates $Y \to -Y$.

In subsection 11.4, we give a probabilistic construction of the hypoelliptic heat equation semigroup associated with $\tilde{L}^{X}_{b,ϑ}$. The solution $U^{0}_{b,ϑ,t}$ of a linear differential stochastic equation appears, whose uniform estimate in the proper $L_{p}$ space turns out to be the main difficulty in this section.

In subsection 11.5, we give a crude and insufficient estimate on $U^{0}_{b,ϑ,t}$. In subsection 11.6, we obtain a uniform $L_{p}$ estimate on the solution $E^{b,ϑ,·}_{b,ϑ}$ of another stochastic differential equation.

In subsection 11.7, we introduce a stochastic process $H^{b,ϑ,t}$, which will be used in subsection 11.8 to estimate $U^{0}_{b,ϑ,t}$. The process $H^{b,ϑ,t}$ can itself be easily estimated.

In subsection 11.8, we express $U^{0}_{b,ϑ,t}$ as an infinite series in which the process $H^{b,ϑ,·}_{b,ϑ}$ appears.

In subsection 11.9, we give a uniform estimate on the $L_{p}$ norm of $|U^{0}_{b,ϑ,t}|$ for bounded $t$. To obtain this estimate, we estimate the various terms of the series obtained in subsection 11.8. The generalized Itô formula of subsection 9.7 and the Girsanov transformation play an essential role in the proof of the estimate.

In subsection 11.10, using the estimate of subsection 11.9, we obtain a uniform estimate on the $L_{p}$ norm of $\sup_{0 \leq t \leq M} |U^{0}_{b,ϑ,t}|$.

In subsection 11.11, we evaluate the limit as $b \to 0$ of $U^{0}_{b,ϑ,·}$. In subsection 11.12, we also handle the $dϑ$ component of our operators.

Finally, in subsection 11.13, we establish Theorem 6.3. The uniform estimates on the kernel $q^{X}_{b,ϑ,t}$ are proved using the Malliavin calculus and the estimates of the previous subsections. The convergence of the heat kernels as $b \to 0$ in a weak sense is proved using the results of the previous subsections. The convergence of the heat kernels themselves is proved by combining these results.

We use the conventions and notation of the previous sections.

11.1. A probabilistic expression for $\exp(-t\tilde{L}^{X}_{0,ϑ})$. Recall that the elliptic operator $\tilde{L}^{X}_{0,ϑ}$ on $X$ was defined in Definition 4.1.

We use the same notation as in subsection 9.6. Let $w^{p}$ be a Brownian motion with values in $p$ such that $w^{p}_{0} = 0$, and let $w^{TX}$ denote the corresponding Brownian
motion in $T_{x_0}X$. Let $E$ denote the corresponding expectation operator. Consider the stochastic differential equation in $X$,

\begin{equation}
\dot{x} = \cos(\vartheta) \dot{w}^{TX}, \quad x_0 = p_1G,
\end{equation}

and also its horizontal lift in $G$,

\begin{equation}
\dot{g} = \cos(\vartheta) \dot{w}^{p}, \quad g_0 = 1,
\end{equation}

so that

\begin{equation}
x = pg.
\end{equation}

Equation (9.48) is a special case of (11.2) for $\vartheta = 0$.

Let $\tau^{0}$ denote the parallel transport operator from $x_0$ to $x_t$ along the curve $x$. with respect to the connection $\nabla^{S^{TX} \otimes F}$, and let $\tau_{0}$ be its inverse. Since $g$ is the horizontal lift of $x$, $\tau^{0}$ can easily be obtained from $g_t$.

We give an extension of [B11, Proposition 14.1.1].

**Proposition 11.1.** Let $u \in C^\infty_c \left( X, S^{TX} \otimes F \right)$. For $t \geq 0$, the following identity holds:

\begin{equation}
\exp \left( \frac{t}{2} \nabla^{S^{TX} \otimes F} \right) u (x_0) = \exp \left( -t \nabla \frac{1}{2} \left[ C^t,F \right] - \frac{1}{2} \left[ C^t,F \right] \right) E \left[ \tau^t_{0} u (x_t) \right].
\end{equation}

**Proof.** Our theorem follows from the third expression for $\nabla\frac{1}{2} \left[ C^t,F \right]$ in equation (4.8) and from Itô’s formula. \hfill \square

Recall that the elliptic operator $T^X$ was defined in Definition 4.3. Let us show how to modify equation (11.4) in order to give a formula for the action of $\exp \left( -tT^X \right)$. With the notation in (11.1)–(11.2), we introduce the stochastic differential equation

\begin{equation}
dA = A \left( -\cos^2(\vartheta) \sum_{i=m+1}^{m+n} \widehat{c} (\text{ad} (e_i) |_{TX}) \rho^F (e_i) \right) dt - \frac{d\vartheta}{\sqrt{2}} \widehat{c} (dw^{TX}) dt,
\end{equation}

$A_0 = 1$.

Using equation (4.16) for $T^X$, for $t \geq 0$, we get

\begin{equation}
\exp \left( -tT^X \right) u (x_0) = \exp \left( \left( -\cos^2(\vartheta) \sum_{i=m+1}^{m+n} \widehat{c} (\text{ad} (e_i) |_{TX}) \rho^F (e_i) \right) t - \frac{1}{48} \nabla\frac{1}{2} \left[ C^t,F \right] \right) E \left[ A_t \tau^t_{0} u (x_t) \right].
\end{equation}

11.2. **An identity of partial differential operators.** Recall that $\mathcal{L}^X_{b,\vartheta}$ is given by equation (4.68), and $\mathcal{L}^X$ by (4.88). Set

\begin{align}
\mathcal{M}^X_{b,\vartheta} &= \exp \left( \frac{|Y|^2}{2} \right) \mathcal{L}^X_{b,\vartheta} \exp \left( -|Y|^2/2 \right), \\
\mathcal{M}^X &= \exp \left( \frac{|Y|^2}{2} \right) \mathcal{L}^X \exp \left( -|Y|^2/2 \right).
\end{align}
By (4.68), we get
\[(11.8) \quad \mathcal{M}^X_{b,0} = \frac{\cos (\vartheta)}{2} \left[ |Y^N, Y^TX|^2 + \frac{1}{2b^2} (-\Delta^T X + 2\nabla_Y^N) + \frac{N^{\lambda (T^* X \oplus N^*)}_{- \theta}}{b^2} \right] \]
\[+ \frac{\cos (\vartheta)}{2b^2} (-\Delta^N + 2\nabla_Y^N) \quad \text{by (4.68), we get} \]
\[+ \frac{\cos (\vartheta)}{b} \left( \nabla_{Y^TX}^{\infty} (T^X \oplus N^* \oplus S^\infty \otimes F) \right) + \tilde{c}_\vartheta (\text{ad} (Y^TX)|_{T^X \oplus N}) \]
\[- c (\text{ad} (Y^TX)) - \frac{i \cos^{1/2} (\vartheta)}{b} (c (\theta \text{ad} (Y^N)) + \tilde{c}_\vartheta (\text{ad} (Y^N)|_{T^X}) + \rho^F (Y^N)) . \]

By (4.81), (4.88), and (11.7), we have the identities
\[(11.9) \quad \mathcal{T}^X_{|db=0} = \mathcal{T}^X_{b,0} \]
\[= \frac{d\vartheta}{\sqrt{2b}} \left( b \frac{\sin (\vartheta)}{\cos^{1/2} (\vartheta)} \right) (|Y^N, Y^TX|^2 + \frac{\sin (\vartheta)}{\cos^{1/2} (\vartheta)} \mathcal{E}^N) \]
\[+ \frac{\cos (\vartheta)}{b} \left( \nabla_{Y^TX}^{\infty} (T^X \oplus N^* \oplus S^\infty \otimes F) \right) + \tilde{c}_\vartheta (\text{ad} (Y^TX)|_{T^X \oplus N}) \] \[- c (\text{ad} (Y^TX)) - \frac{i \cos^{1/2} (\vartheta)}{b} (c (\theta \text{ad} (Y^N)) + \tilde{c}_\vartheta (\text{ad} (Y^N)|_{T^X}) + \rho^F (Y^N)) . \]

Set
\[\mathcal{R}_\vartheta (Y^TX) = \cos (\vartheta) \left( \nabla_{Y^TX}^{\infty} (T^X \oplus N^* \oplus S^\infty \otimes F) \right) \]
\[+ \tilde{c}_\vartheta (\text{ad} (Y^TX)|_{T^X \oplus N}) - c (\text{ad} (Y^TX)) \] \[= \mathcal{R}_\vartheta (Y^N) \quad \text{by (4.68), we get} \]
\[= \mathcal{R}_\vartheta (Y^N) \quad \text{by (4.81), (4.88), and (11.7), we have the identities} \]
\[\mathcal{R}_\vartheta (Y^TX) = \cos (\vartheta) \left( \nabla_{Y^TX}^{\infty} (T^X \oplus N^* \oplus S^\infty \otimes F) \right) + \mathcal{R}_\vartheta (Y^TX) \]
\[\mathcal{R}_\vartheta (Y^N) = R_{\vartheta} (Y^N) \]
\[\mathcal{R}_\vartheta (Y) = \cos (\vartheta) \left( \nabla_{Y^TX}^{\infty} (T^X \oplus N^* \oplus S^\infty \otimes F) \right) + \mathcal{R}_\vartheta (Y) . \]

With the notation in (4.101), (4.102), we have the identities
\[(11.10) \quad \mathcal{R}_\vartheta (Y^N) = -i \cos^{1/2} (\vartheta) (c (\theta \text{ad} (Y^N)) + \tilde{c}_\vartheta (\text{ad} (Y^N)|_{T^X}) + \rho^F (Y^N)) , \]
\[\mathcal{R}_\vartheta (Y) = \mathcal{R}_\vartheta (Y^TX) = R_{\vartheta} (Y^N) \quad \text{by (11.8), we get} \]
\[\mathcal{R}_\vartheta (Y) = \cos (\vartheta) \left( \nabla_{Y^TX}^{\infty} (T^X \oplus N^* \oplus S^\infty \otimes F) \right) \]
\[+ \mathcal{R}_\vartheta (Y) . \]

Now we establish an extension of [B11, Proposition 14.2.1].
Proposition 11.2. For $0 \leq \vartheta < \frac{\pi}{2}$, if $s \in C^\infty \left( X, \Lambda^1 (T^*X \oplus N^*) \otimes S^{TX} \otimes F \right)$, then

\begin{equation}
(11.13) \quad \left( \mathcal{M}_{b,\vartheta}^X - \frac{\cos (\vartheta)}{2} \left[ [Y^N, Y^{TX}] \right]^2 \right) \nonumber \\
- b \left( 1 + N_{-\vartheta}^X (T^*X \oplus N^*) \right)^{-1} \mathcal{R}_\vartheta (Y^{TX}) \nonumber \\

= \frac{N_{-\vartheta}^X (T^*X \oplus N^*)}{b^2} \pi^* s - \mathcal{R}_\vartheta (Y) \left( 1 + N_{-\vartheta}^X (T^*X \oplus N^*) \right)^{-1} \mathcal{R}_\vartheta (Y^{TX}) \nonumber \\

= \left( \cos (\vartheta) + N_{-\vartheta}^X (T^*X \oplus N^*) \right)^{-1} \mathcal{R}_\vartheta (Y^{TX}) \nonumber \\

\end{equation}

Proof. We proceed as in [B11]. Note that the eigenspaces of the fibrewise operators

b,\vartheta, X

are spanned by sections that are linear in $X$. Let $w$ be Brownian motions on $TX$. Instead of [B11, eq. (14.4.1)], we consider the differential equation for $(w, Y) \in Y^{TX}, Y^N$.

11.3. Changing $Y$ in $-Y$. We follow [B11, section 14.3].

Definition 11.3. Let $I$ be the map $s (x, Y) \to s (x, -Y)$. Set

\begin{equation}
(11.14) \quad \mathcal{T}_{b,\vartheta,}^X = \mathcal{T}_{b,\vartheta,}^X I^{-1}, \quad \mathcal{M}_{b,\vartheta,}^X = \mathcal{M}_{b,\vartheta,}^X I^{-1}, \nonumber \\
\mathcal{T}_-^X = I \mathcal{T}_-^X I^{-1}, \quad \mathcal{M}_-^X = I \mathcal{M}_-^X I^{-1}. \nonumber 
\end{equation}

11.4. A probabilistic construction of the hypoelliptic semigroups. As in [B11, section 14.6], we denote by $K_C$ the complexification of the compact Lie group $K$. The splitting of the Lie algebra $\mathfrak{k}_C$ that corresponds to the splitting $g = \mathfrak{p} \oplus \mathfrak{k}$ is just $\mathfrak{k}_C = i \mathfrak{t} \oplus \mathfrak{k}$. Let $X_{K_C} = K_C/K$ denote the corresponding symmetric space. We use the same notation on $X_{K_C}$ as we used before for the symmetric space $X$.

In particular $p$ is the projection $K_C \to X_{K_C}$, and $d$ still denotes the Riemannian distance on $X_{K_C}$. Also the action of $K$ on $S^F$ extends to $K_C$.

Let $w^\mathfrak{p}, w^\mathfrak{k}$ be Brownian motions in $p, k$ such that $w^\mathfrak{p}_0 = 0, w^\mathfrak{k}_0 = 0$. We denote by $w^X, w^N$ the corresponding processes with values in $T_{x_0}X, N_{x_0}$. Let $E^p$ be the corresponding expectation operator.

Instead of [B11, eq. (14.4.1)], we consider the differential equation for $(x, y) \in X \times X_{K_C}, Y = (Y^{TX}, Y^N) \in (TX \oplus N)_{x_0},$

\begin{equation}
(11.15) \quad \dot{x} = \frac{\cos (\vartheta)}{b} Y^{TX}, \quad \dot{y} = -i \frac{\cos^{1/2} (\vartheta)}{b} Y^N, \nonumber \\
\dot{Y}^{TX} = \frac{w^{TX}}{b}, \quad \dot{Y}^N = \cos^{1/2} (\vartheta) \frac{w^N}{b}, \nonumber \\
x_0 = p1_G, \quad y_0 = p1_{K_C}, \quad Y_0 = Y. \nonumber 
\end{equation}

By (11.15), we get

\begin{equation}
(11.16) \quad b^2 \dot{x} = \cos (\vartheta) w^{TX}, \quad b^2 \dot{y} = -i \cos (\vartheta) w^N. \nonumber 
\end{equation}
Instead of [B11, eqs. (14.6.4) and (14.6.5)], we also consider the associated equations on \((g, h) \in G \times K_C,\)

\[
\begin{align*}
\dot{g} &= \frac{\cos(\vartheta)}{b} Y^p, \\
\dot{h} &= -i \frac{\cos^{1/2}(\vartheta)}{b} Y^t, \\
\dot{Y}^p &= \frac{\dot{w}^p}{b}, \\
\dot{Y}^t &= \cos^{1/2}(\vartheta) \frac{\dot{w}^t}{b}, \\
g_0 &= 1_{G}, \\
h_0 &= 1_{K_C}, \\
Y_0 &= Y.
\end{align*}
\]

In (11.17), \((Y^p, Y^t) \in \mathfrak{p} \oplus \mathfrak{k}\) corresponds to \((Y^{TX}, Y^N) \in TX \oplus N.\) Then

\[
\begin{align*}
x. &= pg., \\
y. &= ph.. 
\end{align*}
\]

We still denote by \(\tau^0_t\) the parallel transport along \(x.\) from \(x_0\) to \(x_t\) and by \(\tau^{-1}_t\) its inverse.

Instead of [B11, eq. (14.4.2)], we consider the differential equation,

\[
\frac{dU_{b, \vartheta}}{dt} = U_{b, \vartheta} \left[ -\frac{N^2(X_{\vartheta^*}X^{(N^*)})}{b^2} + R_\vartheta(Y) \right], \quad U_{b, \vartheta, 0} = 1.
\]

Then we have the extension of [B11, Theorem 14.4.1].

**Theorem 11.4.** If \(s \in C^{\infty, c} \left( \mathfrak{X}, \pi^* \left( \Lambda^1 (T^{*}X \oplus N^*) \otimes S^{TX} \otimes F \right) \right),\) then

\[
\exp \left( -it \mathfrak{L}_{b, \vartheta, \alpha, \beta} \right) s(x_0, Y) = E^P \left[ \exp \left( \frac{m + \cos(\vartheta) n}{2b^2} t \right) \right.
\]

\[
\left. - \cos(\vartheta) \int_0^t \left[ \left[ Y^N, Y^{TX} \right] \right]^2 ds - \frac{1}{2b^2} \int_0^t \left( \left| Y^{TX} \right|^2 + \cos(\vartheta) \left| Y^N \right|^2 \right) ds \right] \nonumber
\]

\[
\left. U_{b, \vartheta, 1} \tau^0_t s(x_t, Y_t) \right].
\]

**Proof.** Equation (11.20) is a consequence of (4.68), (4.101), (11.14), and of Itô’s formula. \(\square\)

Instead of [B11, eq. (14.8.8)], given \(Y = (Y^{TX}, Y^N) \in (TX \oplus N)_{x_0},\) we consider now the differential equation,

\[
\begin{align*}
\dot{x} &= \frac{\cos(\vartheta)}{b} Y^{TX}, \\
\dot{y} &= -i \frac{\cos^{1/2}(\vartheta)}{b} Y^N, \\
\dot{Y}^{TX} &= -\frac{Y^{TX}}{b^2} + \frac{\dot{w}^{TX}}{b}, \\
\dot{Y}^N &= -\frac{\cos(\vartheta)}{b^2} Y^N + \frac{\cos^{1/2}(\vartheta)}{b} \dot{w}^N, \\
x_0 &= p1_G, \\
y_0 &= 1_{K_C}, \\
Y_0 &= Y.
\end{align*}
\]

Also we define \((g, h)\) as in (11.17), so that (11.18) holds. We still define \(U_{b, \vartheta, .}\) as in (11.19). By (11.21), we get

\[
\begin{align*}
b^2 \ddot{x} + \dot{x} &= \cos(\vartheta) \dot{w}^{TX}, \\
b^2 \ddot{y} + \cos(\vartheta) \dot{y} &= -i \cos(\vartheta) \dot{w}^N.
\end{align*}
\]

We now denote by \(E^Q\) the expectation operator. Then we have the following extension of [B11, eq. (14.9.2)].
Theorem 11.5. If $s \in C^{\infty,c} \left( \tilde{\mathcal{X}}, \tilde{\pi}^* \left( \Lambda^* (X^* \oplus N^*) \otimes \mathcal{S}^{TX} \otimes F \right) \right)$, then

\begin{equation}
(11.23) \quad \exp \left( -t \mathcal{L}^X_{b,\vartheta,-} \right) \cdot s (x_0, Y) = \mathcal{E}_Q^t \left[ \exp \left( -\frac{\cos (\vartheta)}{2} \int_0^t \left| [Y^N, Y^{TX}] \right|^2 ds \right) U_{b,\vartheta,t} \tau_{b,t}^0 s (x_t, Y_t) \right].
\end{equation}

Proof. This follows from (4.101), (11.8), (11.14), and from Itô's formula. \qed

We will show how to modify (11.20), (11.23) when replacing $\mathcal{L}^X_{b,\vartheta,-}, \mathcal{M}^X_{b,\vartheta,-}$ by $\mathcal{L}^X_{-b=0}, \mathcal{M}^X_{-b=0}$. Set

\begin{equation}
(11.24) \quad R_\vartheta (Y) = R_\vartheta (Y) - \frac{d\vartheta}{\sqrt{2}} \left( \tilde{\mathcal{C}} \left( Y^{TX} \right) + \frac{\sin (\vartheta)}{\cos^{1/2} (\vartheta)} i \mathcal{E}^N \right).
\end{equation}

Let $U_{b,\vartheta,-}$ be the solution of the differential equation,

\begin{equation}
(11.25) \quad \frac{dU_{b,\vartheta}}{dt} = U_{b,\vartheta} \left[ -\frac{N^X (Y^{TX} \otimes N^*)}{b^2} + R_\vartheta (Y) \right] b + \frac{d\vartheta}{\sqrt{2} \cos^{1/2} (\vartheta)} i \mathcal{E} \left( [Y^N, Y^{TX}] \right),
\end{equation}

$U_{b,\vartheta,0} = 1$.

By (11.9), when replacing $\mathcal{L}^X_{b,\vartheta,-}$ by $\mathcal{L}^X_{-b=0, \mathcal{M}^X_{b,\vartheta,-}}$ by $\mathcal{M}^X_{-b=0}$ in (11.20), (11.23), one should simply replace $U_{b,\vartheta,t}$ by $U_{b,\vartheta,t}$.

Recall that $R_\vartheta (Y)$ is given by (4.101). In Definition 4.34, $R_{\vartheta}^0 (Y)$ was defined to be $R_\vartheta (Y)$ when $E = C$ is the trivial representation of $K$. By (4.101), we get

\begin{equation}
(11.26) \quad R_{\vartheta}^0 (Y) = \cos (\vartheta) \left( \tilde{\mathcal{C}}_\vartheta \left( \mathrm{ad} (Y^{TX}) \right) + c \left( \mathrm{ad} (Y^{TX}) \right) \right),
\end{equation}

By (11.26), we get

\begin{equation}
(11.27) \quad |R_{\vartheta}^0 (Y)| \leq C' \left( \left| \cos (\vartheta) \right| |Y^{TX}| + \cos^{1/2} (\vartheta) |Y^N| \right).
\end{equation}

Let $U_{b,\vartheta,-}$ be $U_{b,\vartheta,-}$ in this special case. More precisely, $U_{b,\vartheta,-}$ is the solution of the differential equation,

\begin{equation}
(11.28) \quad \frac{dU_{b,\vartheta}^0}{ds} = U_{b,\vartheta}^0 \left[ -\frac{N^X (Y^{TX} \otimes N^*)}{b^2} + R_{\vartheta}^0 (Y) \right] b, \quad U_{b,\vartheta,0}^0 = 1.
\end{equation}

Let $E_{b,\vartheta,-}$ be the solution of the differential equation,

\begin{equation}
(11.29) \quad \frac{dE_{b,\vartheta}^0}{ds} = E_{b,\vartheta}^0 \left[ -i \frac{\cos^{1/2} (\vartheta)}{b} \rho^E (Y^N) \right], \quad E_{b,\vartheta,0}^0 = 1.
\end{equation}

Then we have the obvious identity

\begin{equation}
(11.30) \quad U_{b,\vartheta,-} = U_{b,\vartheta,-}^0 \otimes E_{b,\vartheta,-}.
\end{equation}

Put

\begin{equation}
(11.31) \quad R^1 (Y) = \tilde{\mathcal{C}} \left( \mathrm{ad} (Y^{TX}) \right) - c \left( \mathrm{ad} (Y^{TX}) \right) - i \left( c (\theta \mathrm{ad} (Y^N)) \right).
\end{equation}

By (11.26), $R_0^0 (Y)$ splits as the sum of two commuting pieces,

\begin{equation}
(11.32) \quad R_0^0 (Y) = R^1 (Y) - i \tilde{\mathcal{C}} \left( \mathrm{ad} (Y^N) \right).
\end{equation}
We make temporarily $\vartheta = 0$ in (11.15), (11.21). Let $V_{b,\vartheta}, W_{b,\vartheta}$ be the solutions of the differential equations,

\begin{equation}
\frac{dV_b}{ds} = V_b \left[ -\frac{N^A(T^*X \oplus N^*)}{b^2} + \frac{R^1(Y)}{b} \right], \quad V_{b,0} = 1,
\end{equation}

\begin{equation}
\frac{dW_b}{ds} = W_b \left[ -\frac{i}{b} \hat{c} \left( \text{ad} \left( Y^N \right) \right|_{T^*X} \right], \quad W_{b,0} = 1.
\end{equation}

As in (11.30), $U^0_{b,\vartheta} = U^0_{b,0,\vartheta}$ can be written in the form,

\begin{equation}
U^0_{b,\vartheta} = V_{b,\vartheta} \otimes W_{b,\vartheta}.
\end{equation}

The factorization (11.34) was already obtained in [B11, eq. (14.4.6)]. Using the fact that $R^1(Y)$ maps forms of degree zero to forms of positive degree, a coarse estimate on $V_{b,\vartheta}$ could be obtained from the differential equation (11.33) using Gronwall’s lemma. This coarse estimate played a critical role in obtaining the behaviour of $V_{b,\vartheta}$ as $b \to 0$. The behaviour of $W_{b,\vartheta}$ as $b \to 0$ was obtained via the analysis of the hypoelliptic Laplacian on the complexification $K_C$ of $K$.

In the sequel, the main difficulty is that for $\vartheta > 0$, there is no factorization of $U^0_{b,\vartheta}$ similar to (11.34), to which the above methods could be applied.

For $\vartheta = \frac{\pi}{2}$, in (11.15) or in (11.21), we have

\begin{equation}
Y^N_t = Y^N_0.
\end{equation}

By (4.64), (4.101), and (11.19), we get

\begin{equation}
U_{b,\vartheta,t} = \exp \left( -t N^A(T^*X)/b^2 \right).
\end{equation}

Let $P^\Lambda(N^*)$ be the orthogonal projection from $\Lambda(T^*X \oplus N^*)$ on $\Lambda(N^*)$. By (11.36), as $b \to 0$, for $t > 0$,

\begin{equation}
U^0_{b,\vartheta,t} \to P^\Lambda(N^*),
\end{equation}

and the convergence is uniform when $t$ stays away from 0.

When $E = C$, is the trivial representation, let $U^0_{b,\vartheta}, \vartheta$ be $U'_{b,\vartheta}, \vartheta$ in this special case. Instead of (11.30), we have the identity

\begin{equation}
U'_{b,\vartheta,t} = U^0_{b,\vartheta,t} \otimes E_{b,\vartheta,t}.
\end{equation}

11.5. **A crude estimate on $U^0_{b,\vartheta,t}, E_{b,\vartheta,t}$.** We will now establish a crude estimate on $U^0_{b,\vartheta,t}, E_{b,\vartheta,t}$.

**Proposition 11.6.** There exists $C > 0$ such that for $b > 0, \vartheta \in [0, \frac{\pi}{2}], t \geq 0$, then

\begin{equation}
|U^0_{b,\vartheta,t}| \leq \exp \left( \frac{C}{b} \left( \cos(\vartheta) \int_0^t |Y^T_X| ds + \cos^{1/2}(\vartheta) \int_0^t |Y^N_s| ds \right) \right),
\end{equation}

\begin{equation}
|E_{b,\vartheta,t}| \leq \exp \left( C \frac{\cos^{1/2}(\vartheta)}{b} \int_0^t |Y^N_s| ds \right).
\end{equation}

Moreover, there exists $C > 0$ such that for $b > 0, \vartheta \in [0, \frac{\pi}{2}], t > 0$,

\begin{equation}
|U^0_{b,\vartheta,t}| \leq \exp \left( \frac{C}{b} \left( \cos(\vartheta) \int_0^t |Y^T_X|^2 ds + \cos^{1/2}(\vartheta) \int_0^t |Y^N_s| ds \right) \right).
\end{equation}
By (11.43), using Gronwall’s lemma, we get the first inequality in (11.39). The proof of the second inequality in (11.39) is similar.

By equation (4.103) in Proposition 4.27, we get
\[ \mathbf{P} R_{d}^{0} (Y^{TX}) \mathbf{P} = 0. \]  

By (4.64), (11.27), and (11.44), we get
\[ \langle R_{d}^{0} (Y^{TX}) f, f \rangle \leq C \cos^{1/2} (\theta) |Y^{TX}| \left| \sqrt{N_{-\vartheta}^{\Lambda (T^{*} X \oplus N^{*})}} f \right| |f|. \]

By (11.45), we deduce that
\[ \left\langle -\frac{N_{-\vartheta}^{\Lambda (T^{*} X \oplus N^{*})}}{b^{2}} + \frac{R_{d}^{0} (Y^{TX})}{b} \right\rangle f, f \leq C \cos (\theta) |Y^{TX}|^{2} |f|^{2}. \]

Using (11.27), (11.42), (11.46), and proceeding as before, we get (11.40). The proof of our proposition is completed.

**Remark 11.7.** Under \( Q \), our \( Y^{TX} \) has the same probability law as \( Y_{/b^{2}} \) where \( Y \) is taken as in (9.9) with \( E = TX \), and our \( Y^{N} \) has the same probability law as \( Y_{\cos (\vartheta),/b^{2}} \), and \( Y \) is taken as in (9.9) with \( E = N \). Using (9.27) and (11.39), given \( p \geq 1 \), there is a constant \( C > 0 \) such that for \( 0 \leq \beta \leq 1 \), we get
\[ E^{Q} \sup_{0 \leq s \leq t} \left| U_{b,\vartheta,s}^{0} \right|^{p} \leq \exp \left( C^{2} t / \beta^{2} + \frac{1}{2} \frac{(m + \cos (\vartheta) n) \beta^{2}}{b^{2}} t \right) \]
\[ + Cb \left( 1 - e^{-t/b^{2}} \right) \cos (\vartheta) |Y_{0}^{TX}| + C \frac{b^{2}}{\cos^{1/2} (\vartheta)} \left( 1 - e^{-t \cos (\vartheta)/b^{2}} \right) |Y_{0}^{N}|. \]

Since \( 1 - e^{-x} \leq x \), from (11.47), we get
\[ E^{Q} \sup_{0 \leq s \leq t} \left| U_{b,\vartheta,s}^{0} \right|^{p} \leq \exp \left( C^{2} t / \beta^{2} + \frac{1}{2} \frac{(m + \cos (\vartheta) n) \beta^{2}}{b^{2}} t \right) \]
\[ + C \frac{t^{2}}{b^{2}} \cos (\vartheta) |Y_{0}^{TX}| + C t^{\cos^{1/2} (\vartheta)} |Y_{0}^{N}|. \]
In particular, when \( b \) stays away from 0, when \( |Y_0|, t \geq 0 \) are uniformly bounded, (11.48) remains uniformly bounded.

Also by (9.24), (9.27), and (11.40), given \( p \geq 1 \), there is \( C > 0 \) such that for \( b \leq C^{-1/2} \), we get

\[
E Q \left[ \sup_{0 \leq s \leq t} |U^0_{b,\varrho,s}|^p \right] \leq \exp \left( \frac{C}{2} \cos(\varrho) mt + \frac{1}{2} \cos(\varrho) Cb^2 |Y_0^{TX}|^2 + \frac{C^2t}{2} + \frac{1}{2} \frac{\cos(\varrho)}{b^2} nt + \frac{b \cos(\varrho)}{\cos^{1/2}(\varrho)} \left( 1 - e^{-t \cos(\varrho)/b^2} \right) |Y_0^N| \right).
\]

Using again the fact that \( 1 - e^{-x} \leq x \), from (11.49), we deduce that if \( b \leq C^{-1/2} \), then

\[
E Q \left[ \sup_{0 \leq s \leq t} |U^0_{b,\varrho,s}|^p \right] \leq \exp \left( \frac{C}{2} \cos(\varrho) mt + \frac{1}{2} \cos(\varrho) |Y_0^{TX}|^2 + \frac{C^2t}{2} + \frac{1}{2} \frac{\cos(\varrho)}{b^2} nt + \frac{C \cos^{1/2}(\varrho)}{b} t |Y_0^N| \right).
\]

The estimates in (11.39), (11.40), (11.47)–(11.50) deteriorate as \( b \to 0 \). As we will see in subsection 11.6, a stronger estimate can be found for \( E_{b,\varrho,.} \). The problem is much more serious for \( U^0_{b,\varrho,.} \).

### 11.6. A uniform estimate on \( E_{b,\varrho,.} \)

We assume equation (11.21) to be in force. In equation (11.29) for \( E_{b,\varrho,.}, Y^N \) in (11.21) also depends on \( b, \varrho \). In the sequel, we use the notation

\[
E_{b,.} = E_{b,0,.}
\]

In the definition of \( E_{b,0,.} \), \( \varrho \) is made equal to 0 also in the definition of \( Y^N \).

Note that

\[
E_{b,\varrho,.} = E_{b/\cos^{1/2}(\varrho),.}
\]

Let \( E_{0,.} \) be the solution of the stochastic differential equation

\[
dE_0 = -E_0 \rho E \left( idw^t \right), \quad E_{0,0} = 1.
\]

Observe that both \( E_{b,.} \) and \( E_{0,.} \) are constructed on the same probability space for \( w^t \).

**Theorem 11.8.** Given \( M \geq 0, p \geq 1 \), there exist \( C_p > 0, C' > 0 \) such that for \( b > 0, \varrho \in [0, \frac{\pi}{2}] \),

\[
E^Q \left[ \sup_{0 \leq t \leq M} |E_{b,\varrho,t}|^p \right] \leq C_p \exp \left( \frac{1}{2} |Y_0^N|^2 \right).
\]

Given \( b_0 > 0, 0 < \epsilon \leq M < +\infty, p \geq 1 \), there exist \( C_{b_0,M,p} > 0, C'_{b_0,\epsilon,M} > 0 \) such that for \( 0 < b \leq b_0, \varrho \in [0, \frac{\pi}{2}], \epsilon \leq t \leq M \), then

\[
\exp \left( -\frac{|Y_0^N|^2}{2} \right) E^Q \left[ \sup_{0 \leq t \leq M} |E_{b,\varrho,t}|^p \exp \left( \frac{|Y_0^N|^2}{2} \right) \right] \leq C_{b_0,M,p} \exp \left( -C'_{b_0,\epsilon,M} \cos(\varrho) |Y_0^N|^2 \right).
\]
Finally, given $M > 0$, as $b \to 0$, with respect to $Q$, $E_{b, \cdot}$ converges uniformly in probability to $E_{0, \cdot}$ on $[0, M]$.

Proof. As in [B11, eq. (14.6.3)], if $h \in K_C$ then
\begin{equation}
|\rho^E(h)| \leq e^{Cd(p_1, ph)}, \quad |\rho^E(h^{-1})| \leq e^{Cd(p_1, ph)}
\end{equation}
Let $Y^t \approx Y^N$ be as in (11.21) with $b > 0, \vartheta = 0$. Let $k_{b, \cdot} \in K_C$ be the solution of the differential equation,
\begin{equation}
\dot{k}_{b} = -iY^t_{b}, \quad k_{b, 0} = 1.
\end{equation}
Let $k_{0, \cdot} \in K_C$ be the solution of the stochastic differential equation,
\begin{equation}
\dot{k}_{0} = -i\dot{w}^t, \quad k_{0, 0} = 1.
\end{equation}
By (11.29), (11.53), (11.57), and (11.58), we get
\begin{equation}
E_{b, \cdot} = \rho^E k_{b, \cdot}, \quad E_{0, \cdot} = \rho^E k_{0, \cdot}.
\end{equation}
Equation (11.54) follows from equations (9.73) in Theorem 9.20, and from (11.52), (11.56), and (11.59).

11.7. The process $H_{b, \vartheta, \cdot}$. When $\vartheta > 0$, even when $F$ is trivial, it is no longer true that $R_0^\vartheta (Y)$ maps forms of degree 0 into forms of positive degree. More precisely, the spin representation $S^p_T$ is now coupled to the forms in $\Lambda (p^* \otimes t^*)$, so that the product technique used in [B11] does not apply any more. We will be forced to suitably combine the two sorts of techniques used in [B11].

We use the same notation as in Definition 4.26. Here $P$ denotes the orthogonal projection from $\Lambda (T^*X \otimes N^*) \otimes S^{TX}$ on $S^{TX}$. Set
\begin{equation}
P^\perp = 1 - P.
\end{equation}

Definition 11.9. Put
\begin{equation}
R_{0}^{\vartheta, \perp} (Y) = R_{0}^{\vartheta} (Y) + \cos \frac{5}{2} (\vartheta) \tilde{c} (\text{iad} (Y^N) |_{TX}) .
\end{equation}
By Proposition 4.27, we get
\begin{equation}
P R_{0}^{\vartheta, \perp} (Y) P = 0.
\end{equation}
We still consider the stochastic differential equation (11.21), i.e., we work under the probability measure $Q$. Let $H_{b, \vartheta, \cdot}$ be the solution of the differential equation,
\begin{equation}
\frac{dH_{b, \vartheta}}{ds} = H_{b, \vartheta} \cos \frac{5}{2} (\vartheta) \tilde{c} (\text{iad} (Y^N) |_{TX}), \quad H_{b, \vartheta, 0} = 1 .
\end{equation}
By proceeding as in the proof of Proposition 11.6, we have the trivial bound,
\begin{equation}
|H_{b, \vartheta, t}| \leq \exp \left( C \frac{\cos \frac{5}{2} (\vartheta)}{b} \int_{0}^{t} |Y_{s}^N| ds \right).
\end{equation}
Definition 11.10. Let \( h_{b, \vartheta,} \in K_C \) be the solution of the differential equation,

\[
(11.66) \quad \dot{h}_{b, \vartheta} = \frac{\cos^{5/2}(\vartheta)}{b} i Y^t, \quad h_{b, \vartheta, 0} = 1.
\]

Set

\[
(11.67) \quad z_{b, \vartheta,} = ph_{b, \vartheta,}.
\]

Then \( H_{b, \vartheta,} \) is the image of \( h_{b, \vartheta,} \) by the spin representation. By \([B11, \text{eq. } (14.6.3)]\), as in (11.56), we get

\[
(11.68) \quad |H_{b, \vartheta,}| \leq e^{Cd(p_1, z_{b, \vartheta,})}, \quad \left|H_{b, \vartheta}^{-1}\right| \leq e^{Cd(p_1, z_{b, \vartheta,})}.
\]

By (11.66), we get

\[
(11.69) \quad \dot{z}_{b, \vartheta} = -\frac{\cos^{5/2}(\vartheta)}{b} i Y^N, \quad z_{b, \vartheta, 0} = p1 K_C.
\]

By (11.21), (11.69), \( z_{b, \vartheta,} \) is a solution of the differential equation,

\[
(11.70) \quad b^2 \dot{z}_{b, \vartheta} + \cos(\vartheta) \dot{z}_{b, \vartheta} = -\cos^3(\vartheta) i \dot{w}^N, \quad z_{b, \vartheta, 0} = p1 K_C, \quad \dot{z}_{b, \vartheta, 0} = -\frac{\cos^{5/2}(\vartheta)}{b} i Y_0^N.
\]

We denote by \( X_{K_C} \) the analogue of \( X \) for \( K_C \). Namely, \( X_{K_C} \) is the total space of \( TX_{K_C} \). We define the scalar differential operators \( A_{K_C}^{X, K_C}, B_{K_C}^{X, K_C} \) acting on \( X_{K_C} \) as in (9.58). One verifies easily that the operator on \( X_{K_C} \) associated with the process \( (z_{b, \vartheta,}, Y^N) \) is given by \( \cos^4(\vartheta) B_{\cos^{3/2}(\vartheta) b}^{X, K_C} \).

To make our notation more transparent, we will briefly note the dependence of \( Y^N \) on \( b, \vartheta \) explicitly. Put

\[
(11.71) \quad z_{b, \vartheta,} = z_{b, 0, \vartheta}, \quad Y^N_{b, \vartheta,} = Y^N_{b, 0, \vartheta}.
\]

By the above considerations, it follows that the probability law of \( (z_{b, \vartheta,}, Y^N_{b, \vartheta,}) \) is the same as the probability law of \( (z_{b, 0, \vartheta,} Y^N_{b, 0, \vartheta,}) \).

We omit again the subscript \( b, \vartheta \) in \( Y^N \). We proceed as in Theorem 9.18. Let \( f : X_{K_C} \rightarrow \mathbb{R} \) be a smooth function. Set

\[
(11.72) \quad B^f_t = \cos^2(\vartheta) \int_0^t \nabla_{Y^N} f(z_{b, \vartheta, s}) \, ds + \int_0^t \nabla_{-i\omega} f(z_{b, \vartheta, s}) \, ds.
\]

By (11.21), (11.69), we obtain an analogue of equation (9.71),

\[
(11.73) \quad \left( b^2 \frac{d}{dt} \cos(\vartheta) \right) f(z_{b, \vartheta, t}) = \left( b^2 \frac{d}{dt} \cos(\vartheta) \right) f(z_{b, \vartheta, t}) |_{t=0} + \cos^3(\vartheta) B^f_t.
\]

From (11.73), we get the analogue of (9.72),

\[
(11.74) \quad f(z_{b, \vartheta, t}) = f(z_{b, \vartheta, 0}) + \frac{b^2}{\cos(\vartheta)} \frac{d}{dt} f(z_{b, \vartheta, t}) |_{t=0} \left( 1 - e^{-\cos(\vartheta) t/b^2} \right) + \frac{1}{b^2} \int_0^t e^{-\cos(\vartheta)(t-s)/b^2} \cos^3(\vartheta) B^f_s \, ds.
\]
Let $H_{0,\vartheta,}$ be the solution of the stochastic differential equation
\begin{equation}
(11.75) \quad dH_{0,\vartheta} = H_{0,\vartheta} \cos^2(\vartheta) \tilde{c} (-\text{iad} (dw^j) |\pi^X\rangle), \quad H_{0,\vartheta,0} = 1.
\end{equation}

Let $h_{0,\vartheta,} \in K_C$ be the solution of
\begin{equation}
(11.76) \quad \dot{h}_{0,\vartheta} = -\cos^2(\vartheta) i w^s, \quad h_{0,\vartheta,0} = p_{1K_C}.
\end{equation}

Then $H_{0,\vartheta,}$ is the image of $h_{0,\vartheta,}$ via the spin representation. If $z_{0,\vartheta} = ph_{0,\vartheta,},$ then
\begin{equation}
(11.77) \quad \dot{z}_{0,\vartheta} = -\cos^2(\vartheta) i w^s, \quad z_{0,\vartheta,0} = p_{1K_C}.
\end{equation}

**Theorem 11.11.** Given $M > 0, p \geq 1,$ there exist $C_p > 0, C^\prime$ such that for $b > 0, \vartheta \in \left[0, \frac{\pi}{2}\right].$

\begin{equation}
(11.78) \quad E^Q \left[ \sup_{0 \leq t \leq M} |H_{b,\vartheta,t}|^p \right] \leq C_p \exp \left( |Y_0^N|^2 / 2 \right),
\end{equation}

\begin{equation}
E^Q \left[ \sup_{0 \leq t \leq M} |H_{b,\vartheta,t}^{-1}|^p \right] \leq C_p \exp \left( |Y_0^N|^2 / 2 \right).
\end{equation}

Given $b_0 > 0, 0 < \epsilon \leq M, p \geq 1,$ there exist $C_{p,M} > 0, C'_{b_0,\epsilon,M} > 0$ such that for $0 < b \leq b_0, \vartheta \in \left[0, \frac{\pi}{2}\right], \epsilon \leq t \leq M,$ then

\begin{equation}
(11.79) \quad \exp \left( -|Y_0^N|^2 / 2 \right) E^Q \left[ \sup_{0 \leq t \leq M} |H_{b,\vartheta,t}|^p \exp \left( |Y_t^N|^2 / 2 \right) \right] \leq C_{p,M} \exp \left( -C'_{b_0,\epsilon,M} \cos(\vartheta) |Y_0^N|^2 \right),
\end{equation}

\begin{equation}
\exp \left( -|Y_0^N|^2 / 2 \right) E^Q \left[ \sup_{0 \leq t \leq M} |H_{b,\vartheta,t}^{-1}|^p \exp \left( |Y_t^N|^2 / 2 \right) \right] \leq C_{b_0,p,M} \exp \left( -C'_{b_0,\epsilon,M} \cos(\vartheta) |Y_0^N|^2 \right).
\end{equation}

Given $\vartheta \in \left[0, \frac{\pi}{2}\right], as b \rightarrow 0, H_{b,\vartheta,}$ converges to $H_{0,\vartheta,}$ uniformly on $[0, M]$ in probability.

**Proof.** Using the considerations we made after (11.70), (11.71), equation (11.78) follows from equation (9.73) in Theorem 9.20 and from (11.68). Also note that
\begin{equation}
(11.80) \quad \cos^3(\vartheta) / \left( \cos^{3/2}(\vartheta) b \right)^2 = \cos(\vartheta) / b^2.
\end{equation}

Equation (11.79) now follows from equation (9.74) in Theorem 9.20 and from (11.60), (11.68) and (11.80). The last part of our theorem is a consequence of Theorem 9.22.

11.8. **The process** $U^0_{b,\vartheta,}$ **as an infinite series.** Set
\begin{equation}
(11.81) \quad K_{b,\vartheta,s} = H_{b,\vartheta,s} \exp \left( -s N^{-1}_\vartheta (T' X \otimes N^\prime) / b^2 \right).
\end{equation}

Then $K_{b,\vartheta,}$ commutes with $P$ and $P^\perp$. 

Proposition 11.12. The following identity holds:

$$U_{b,\vartheta,t}^{0} = \sum_{k=0}^{\infty} \int_{0 \leq s_{1} \leq \ldots \leq s_{k} \leq t} K_{b,\vartheta,s_{1}} \frac{R_{\vartheta}^{0,\perp}(Y_{s_{1}})}{b} K_{b,\vartheta,s_{2}}^{1} \frac{R_{\vartheta}^{0,\perp}(Y_{s_{2}})}{b} K_{b,\vartheta,s_{k}}^{1} \ldots K_{b,\vartheta,s_{k}} K_{b,\vartheta,t} ds_{1} \ldots ds_{k},$$

and the series in (11.82) is uniformly convergent over compact sets in \( \mathbb{R}_{+} \).

Proof. Put

$$L_{b,\vartheta,\cdot} = U_{b,\vartheta,\cdot}^{0} K_{b,\vartheta,\cdot}^{-1}.$$ 

By (11.28), (11.62), (11.64), and (11.83), \( L_{b,\vartheta,\cdot} \) is the solution of the differential equation,

$$\frac{dL_{b,\vartheta}}{ds} = L_{b,\vartheta} \frac{R_{\vartheta}^{0,\perp}(Y)}{b} K_{b,\vartheta}^{1}, \quad L_{b,\vartheta,0} = 1,$$

We can express \( L_{b,\vartheta,\cdot} \) as the sum of iterated integrals which converges uniformly over compact subsets of \( \mathbb{R}_{+} \). Using (11.83), we get (11.82). \( \square \)

11.9. A uniform estimate on \( \sup_{0 \leq s \leq t} E^{Q} \left[ \left| U_{b,\vartheta,s}^{0} \right|^{p} \right] \). Now we establish the following key estimate.

Theorem 11.13. Given \( p \geq 1, M > 0 \), there exist \( C > 0, C' > 0 \) such that for \( 0 < b \leq 1, \theta \in \left[ 0, \frac{\pi}{2} \right], \) then

$$\sup_{0 \leq t \leq M} E^{Q} \left[ \left| U_{b,\vartheta,t}^{0} \right|^{p} \right] \leq C \exp \left( C' \left( \cos(\vartheta) |Y_{0}^{TX}|^{2} + |Y_{0}^{N}|^{2} \right) \right).$$

Given \( p \geq 1, 0 < \epsilon \leq M < +\infty \), there exist \( C > 0, C' > 0 \) such that for \( 0 < b \leq 1, \theta \in \left[ 0, \frac{\pi}{2} \right], \epsilon \leq t \leq M, \)

$$\exp \left( -|Y_{0}|^{2}/2 \right) E^{Q} \left[ \left| U_{b,\vartheta,t}^{0} \right|^{p} \exp \left( \left| Y_{t}^{2}/2 \right| \right) \right] \leq C \exp \left( -C' \left( |Y_{0}^{TX}|^{2} + \cos(\vartheta) |Y_{0}^{N}|^{2} \right) \right).$$

Proof. As we saw in Remark 11.7, equation (11.85) is trivial when \( b \) stays away from 0. In the sequel, we may as well assume that \( b_{0} > 0 \) is small enough and that \( 0 < b \leq b_{0}. \)

Note that

$$R_{\vartheta}^{0,\perp}(Y) = (P + P^{\perp}) R_{\vartheta}^{0,\perp}(Y) (P + P^{\perp}).$$

By (11.63), (11.87), we get

$$R_{\vartheta}^{0,\perp}(Y) = P R_{\vartheta}^{0,\perp}(Y) P^{\perp} + P^{\perp} R_{\vartheta}^{0,\perp}(Y) P + P^{\perp} R_{\vartheta}^{0,\perp}(Y) P^{\perp}.$$ 

Consider the splitting

$$\Lambda^{c}(TX^{*} \oplus N^{*}) \oplus \Lg = \Lg(TX) \oplus \left( \Lambda^{(>0)}(T^{*}X \oplus N^{*}) \oplus \Lg(TX) \right).$$

As a subalgebra of \( \text{End}(\Lambda^{c}(TX^{*})), \) \( \Lg(TX) \) inherits a corresponding norm, and \( \text{End}(\Lambda^{c}(T^{*}X \oplus N^{*})) \) inherits a norm from the Euclidean norm of \( \Lambda^{c}(T^{*}X \oplus N^{*}). \)
We will write elements of \( \text{End}(\Lambda^* (T^* X \oplus N^*)) \) as \((2, 2)\) matrices with respect to the splitting (11.89). In particular the norm of such an element is not a nonnegative number but a \((2, 2)\) matrix of nonnegative numbers. Such a norm is still such that

\[
|AB| \leq |A| |B|.
\]

The inequality just means that each of the 4 coefficients in the left-hand side is dominated by the corresponding coefficient in the right-hand side.

For \(0 \leq s_1 \leq \ldots \leq s_k \leq t\), set

\[
R_{s_1, \ldots, s_k, t} = |H_{b, \vartheta, s_1}| \left| \exp \left( -s_1 N_{\vartheta}^\Lambda (T^* X \oplus N^*) / b^2 \right) \right| \left| R_{\vartheta}^{0,1} (Y_{s_1}) \right| b
\]

\[
|H_{b, \vartheta, s_1}^{-1} H_{b, \vartheta, s_2}| \left| \exp \left( - (s_2 - s_1) N_{\vartheta}^\Lambda (T^* X \oplus N^*) / b^2 \right) \right| \left| R_{\vartheta}^{0,1} (Y_{s_2}) \right| b \ldots
\]

\[
\ldots |H_{b, \vartheta, s_{k-1}}^{-1} H_{b, \vartheta, s_k}| \left| \exp \left( - (s_k - s_{k-1}) N_{\vartheta}^\Lambda (T^* X \oplus N^*) / b^2 \right) \right| \left| R_{\vartheta}^{0,1} (Y_{s_k}) \right| b
\]

\[
|H_{b, \vartheta, s_j}^{1} H_{b, \vartheta, t}| \left| \exp \left( - (t - s_k) N_{\vartheta}^\Lambda (T^* X \oplus N^*) / b^2 \right) \right| b
\]

By (11.81), (11.82), and (11.91), we get

\[
|U_{b, \vartheta, t}^0| \leq \sum_{k=0}^{\infty} \int_{0 \leq s_1 \leq \ldots \leq s_k \leq t} R_{s_1, \ldots, s_k, t} ds_1 ds_2 \ldots ds_k.
\]

We fix temporarily \(s_1 \geq 0\). By [B11, eq. (14.6.3)] or by (11.68), there exists \(C > 0\) such that for \(s_1 \leq s_2\),

\[
|H_{b, \vartheta, s_1}^{-1} H_{b, \vartheta, s_2}| \leq \exp (Cd (\vartheta, s_1, \vartheta, s_2)).
\]

Let \(k : \mathbf{R}_+ \rightarrow \mathbf{R}_+\) be a smooth increasing function such that

\[
k(u) = 0 \text{ for } u \leq 1/2,
\]

\[
= u \text{ for } u \geq 1.
\]

For \(y, z \in X_{K_C}\), set

\[
f(y, z) = k (d(y, z)).
\]

Then \(f(y, z)\) is a smooth function such that

\[
d(y, z) \leq f(y, z) + 1.
\]

Moreover, if \(d(y, z) \leq 1/2\), then \(f(y, z) = 0\). In the sequel \(\nabla f(y, z)\) denotes the gradient of \(f\) with respect to the second variable \(z\). Note that \(f\) and \(\nabla f\) vanish on the diagonal.

By (11.93), (11.96), for \(0 \leq s_1 \leq s_2\), we get

\[
|H_{b, \vartheta, s_1}^{-1} H_{b, \vartheta, s_2}| \leq \exp (C + Cf (\vartheta, s_1, \vartheta, s_2)).
\]

Now we use the formulas in (11.72)–(11.74). As in (11.72), for \(s \geq s_1\), set

\[
B_{s_1, s}^f = \cos^2 (\vartheta) \int_{s_1}^{s} \nabla f (\nu_{iY_{\alpha}^N}) \nabla f (\vartheta, s, \vartheta, u) du + \int_{s_1}^{s} \nabla \omega_{iY_{\alpha}^N} f (\vartheta, s, \vartheta, u).
\]
By (11.74), since \( f(z_b, \vartheta, s_1, \cdot) \) vanishes near \( z_b, \vartheta, s_1 \), we get

\[
(11.99) \quad f(z_b, \vartheta, s_1, z_b, \vartheta, s_2) = \frac{1}{b^2} \int_{s_1}^{s_2} e^{-\cos(\vartheta)(s_2 - s)/b^2} \cos^3(\vartheta) B^f_{s_1, s} \, ds.
\]

By [B11, Proposition 13.1.2], \( f(z_1, z_2) \) and its first and second covariant derivatives in the variable \( z_2 \) are uniformly bounded. In particular,

\[
(11.100) \quad |\nabla_{-Y} f(z_1, z_2)| \leq c |Y^N|, \quad \nabla_{iY}^{TX} \nabla_{iY} f(z_1, z_2) \leq c |Y^N|^2.
\]

For \( s \geq s_1 \), put

\[
(11.101) \quad M^f_{s_1, s} = \int_{s_1}^{s} \nabla_{-i\delta w} f(z_b, \vartheta, s_1, z_b, \vartheta, u) \, ds.
\]

By (11.98), (11.100), and (11.101), we get

\[
(11.102) \quad B^f_{s_1, s} \leq c \cos^2(\vartheta) \int_{s_1}^{s} |Y^N|^2 \, ds + M^f_{s_1, s}.
\]

By (11.99), (11.102), we get

\[
(11.103) \quad f(z_b, \vartheta, s_1, z_b, \vartheta, s_2) \leq c \cos^4(\vartheta) \int_{s_1}^{s_2} |Y^N_s|^2 \, ds + \frac{1}{b^2} \int_{s_1}^{s_2} e^{-\cos(\vartheta)(s_2 - s)/b^2} \cos^3(\vartheta) M^f_{s_1, s} \, ds.
\]

Also using (11.101) and integration by parts, we obtain

\[
(11.104) \quad \frac{1}{b^2} \int_{s_1}^{s_2} e^{-\cos(\vartheta)(s_2 - s)/b^2} \cos^3(\vartheta) M^f_{s_1, s} \, ds
\]

\[
= \int_{s_1}^{s_2} \cos^2(\vartheta) \left(1 - e^{-\cos(\vartheta)(s_2 - s)/b^2}\right) \nabla_{-i\delta w} f(z_b, \vartheta, s_1, z_b, \vartheta, s) \, ds.
\]

By (11.97), (11.103), and (11.104), we obtain

\[
(11.105) \quad |H_{b, \vartheta, s_1} H_{b, \vartheta, s_2}| \leq \exp \left(C + cC \cos^4(\vartheta) \int_{s_1}^{s_2} |Y^N_s|^2 \, ds\right)
\]

\[
\exp \left(C \int_{s_1}^{s_2} \cos^2(\vartheta) \left(1 - e^{-\cos(\vartheta)(s_2 - s)/b^2}\right) \nabla_{-i\delta w} f(z_b, \vartheta, s_1, z_b, \vartheta, s) \, ds\right).
\]

We fix \( s_1, s_2 \) with \( 0 \leq s_1 \leq s_2 \). For \( s \geq s_1 \), set

\[
(11.106) \quad N^f_{s_1, s_2, s} = \exp \left(C \int_{s_1}^{s} \cos^2(\vartheta) \left(1 - e^{-\cos(\vartheta)(s_2 - u)/b^2}\right) \nabla_{-i\delta w} f(z_b, \vartheta, s_1, z_b, \vartheta, u) \, du\right)
\]

\[
- \frac{C^2}{2} \int_{s_1}^{s} \cos^4(\vartheta) \left(1 - e^{-\cos(\vartheta)(s_2 - u)/b^2}\right)^2 |\nabla f(z_b, \vartheta, s_1, z_b, \vartheta, u)|^2 \, du.
\]

Then \( N^f_{s_1, s_2, s} \) is the unique solution of the Itô stochastic differential equation,

\[
(11.107) \quad N^f_{s_1, s_2, s} = 1 + \int_{s_1}^{s} N^f_{s_1, s_2, u} C \cos^2(\vartheta) \left(1 - e^{-\cos(\vartheta)(s_2 - u)/b^2}\right)
\]

\[
\nabla_{-i\delta w} f(z_b, \vartheta, s_1, z_b, \vartheta, u) \, du.
\]
In particular $N^f_{s_1, s_2}.$ is a martingale. In the sequel, we will write $N^f_{s_1, s_2}$ instead of $N^f_{s_1, s_2, s_2}.$ By (11.100), (11.105), and (11.106), we get

\[(11.108) \quad \left| H^{-1}_{b, \vartheta, s_1} H_{b, \vartheta, s_2} \right| \leq \exp \left( C + cC \cos^4 (\vartheta) \int_{s_1}^{s_2} |Y^N_s|^2 ds + \frac{1}{2} C^2 c^2 (s_2 - s_1) \right) N^f_{s_1, s_2}.
\]

For $s_1 \leq s \leq s_2$, set

\[(11.109) \quad \overline{w}^N_s = u^N_s - \int_{s_1}^{s} C \cos^2 (\vartheta) \left( 1 - e^{-\cos (\vartheta) (s - u)/b^2} \right) \nabla_{-i} f (z_{b, \vartheta, s_1}, z_{b, \vartheta, u}) du.
\]

Using the properties of the Girsanov transformation, we know that with respect to the probability measure $dQ' = N^f_{s_1, s_2} dQ$, $\overline{w}^N$ is a Brownian motion on $[s_1, s_2]$. By (11.21), for $s_1 \leq s \leq s_2$, we get

\[(11.110) \quad Y^N_s = e^{-\cos (\vartheta) (s - s_1)/b^2} Y^N_{s_1} + \frac{\cos^{1/2} (\vartheta)}{b} \int_{s_1}^{s} e^{-\cos (\vartheta) (s - u)/b^2} du \overline{w}^N_u.
\]

By (11.109), we can rewrite (11.110) in the form,

\[(11.111) \quad Y^N_s = e^{-\cos (\vartheta) (s - s_1)/b^2} Y^N_{s_1} + \frac{\cos^{1/2} (\vartheta)}{b} \int_{s_1}^{s} e^{-\cos (\vartheta) (s - u)/b^2} C \cos^2 (\vartheta) \left( 1 - e^{-\cos (\vartheta) (s - u)/b^2} \right) \nabla_{-i} f (z_{b, \vartheta, s_1}, z_{b, \vartheta, u}) du.
\]

By (11.100), for $s_1 \leq s \leq s_2$, we get

\[(11.112) \quad \left| \frac{\cos^{1/2} (\vartheta)}{b} \int_{s_1}^{s} e^{-\cos (\vartheta) (s - u)/b^2} C \cos^2 (\vartheta) \left( 1 - e^{-\cos (\vartheta) (s - u)/b^2} \right) \nabla_{-i} f (z_{b, \vartheta, s_1}, z_{b, \vartheta, u}) du \right| \leq cC \cos^{3/2} (\vartheta) b.
\]

Set

\[(11.113) \quad \overline{Y}^N_s = e^{-\cos (\vartheta) (s - s_1)/b^2} Y^N_{s_1} + \frac{\cos^{1/2} (\vartheta)}{b} \int_{s_1}^{s} e^{-\cos (\vartheta) (s - u)/b^2} du \overline{w}^N_u.
\]

Under $Q'$, the process $\overline{Y}^N$ on $[s_1, s_2]$ has the same probability law as $Y^N$ under $Q$. By (11.111), (11.113), we get

\[(11.114) \quad |Y^N_s| \leq |\overline{Y}^N_s| + cC \cos^{3/2} (\vartheta) b.
\]
By (11.91), (11.108), given \( k \in \mathbb{N} \), with the convention that \( s_0 = 0, s_{k+1} = t \), we get

\[
(11.115) \quad R_{s_1, \ldots, s_k, t} \leq \exp \left( (k + 1) C + cC \cos^4(\theta) \int_0^t |Y_s^N|^2 \, ds + \frac{1}{2} C^2 t^2 \right) \prod_{i=0}^k N^f_{s_i, s_{i+1}}
\]

Under the probability law (11.116), define

\[
\exp \left( -s_1 N^A(T^*X \oplus N^*) / b^2 \right) \left| R_{\theta, s_1}^0 \right| \left| \exp \left( -(s_2 - s_1) N^A(T^*X \oplus N^*) / b^2 \right) \right| \prod_{i=0}^k N^f_{s_i, s_{i+1}}
\]

Again \( \prod_{i=0}^k N^f_{s_i, s_{i+1}} \) is a Girsanov exponential. Instead of (11.109), on \([0, t]\), we define \( \overline{w}^N \) by the formula

\[
(11.116) \quad \overline{w}^N_s = w^N_s - \int_0^s \sum_{i=0}^k 1_{s_i \leq u \leq s_{i+1}} C \cos^2(\theta) \left( 1 - e^{-\cos(\theta)(s_{i+1}-u)/b^2} \right) \nabla_{-i} f(z_b, \vartheta, s_i, z_b, \vartheta, u).
\]

Under the probability law \( dQ_{s_1, \ldots, s_k, t} = \prod_{i=0}^k N^f_{s_i, s_{i+1}} dQ \), on \([0, t]\), \( \overline{w}^N \) is a Brownian motion. Instead of (11.111), we now have

\[
(11.117) \quad Y^N_s = e^{-\cos(\theta)s/b^2} Y^N_0 + \frac{\cos^{1/2}(\theta)}{b} \int_0^s e^{-\cos(\theta)(s-u)/b^2} \overline{w}^N_u + \frac{\cos^{1/2}(\theta)}{b} \int_0^s e^{-\cos(\theta)(s-u)/b^2} \nabla_{-i} f(z_b, \vartheta, s_i, z_b, \vartheta, u).
\]

Under \( Q_{s_1, \ldots, s_k, t} \), on \([0, t]\), \( \overline{Y}^N_s \) as the same probability law as \( Y^N_s \) under \( Q \).

In the sequel, we denote by \( \overline{Y} \) the process whose \( TX \) component coincides with \( Y^N \), and whose \( N \) component coincides with \( \overline{Y}^N_s \). It is still true that under \( Q_{s_1, \ldots, s_k, t} \), the probability law of \( \overline{Y} \) on \([0, t]\) is the same as the probability law of \( Y \) under \( Q \).

Instead of (11.113), set

\[
(11.118) \quad \overline{Y}^N_s = e^{-\cos(\theta)s/b^2} Y^N_0 + \frac{\cos^{1/2}(\theta)}{b} \int_0^s e^{-\cos(\theta)(s-u)/b^2} \overline{w}^N_u.
\]

Under \( Q_{s_1, \ldots, s_k, t} \), on \([0, t]\), \( \overline{Y}^N_s \) as the same probability law as \( Y^N \) under \( Q \).

Instead of (11.112), for \( 0 \leq s \leq t \), we obtain

\[
(11.119) \quad \left| \frac{\cos^{1/2}(\theta)}{b} \int_0^s e^{-\cos(\theta)(s-u)/b^2} \left( \sum_{i=0}^k 1_{s_i \leq u \leq s_{i+1}} C \cos^2(\theta) \right) \left( 1 - e^{-\cos(\theta)(s_{i+1}-u)/b^2} \right) \nabla_{-i} f(z_b, \vartheta, s_i, z_b, \vartheta, u) \right| du \leq cC \cos^{3/2}(\theta) b.
\]
By (11.117)–(11.119), as in (11.114), we get

\[(11.120) \quad |Y_s^N| \leq |\overline{Y}_s^N| + cC \cos^{3/2} (\vartheta) b.\]

By (11.120), we get

\[(11.121) \quad \int_0^t |Y_s^N|^2 \, ds \leq 2 \int_0^t |\overline{Y}_s^N|^2 \, ds + 2c^2C^2 \cos^3 (\vartheta) b^2 t.\]

Let \(E^{Q,s_1,\ldots,s_k,t}\) be the expectation operator with respect to \(Q_{s_1,\ldots,s_k,t}\). By (11.115), we obtain

\[(11.122) \quad E^Q [R_{s_1,\ldots,s_k,t}] \leq \exp \left( (k + 1) C + \frac{1}{2} C^2 c^2 t \right) \exp \left( cC \cos^4 (\vartheta) \int_0^t |Y_s^N|^2 \, ds \right) \left| \exp \left( -s_1 N_{-\vartheta}^{(\mathcal{T}^*X\oplus N^*)} / b^2 \right) \right| \left| \frac{R_{\vartheta}^{0,\perp} (Y_{s_1})}{b} \right| \left| \exp \left( - (s_2 - s_1) N_{-\vartheta}^{(\mathcal{T}^*X\oplus N^*)} / b^2 \right) \right| \left| \frac{R_{\vartheta}^{0,\perp} (Y_{s_2})}{b} \right| \ldots \left| \frac{R_{\vartheta}^{0,\perp} (Y_{s_k})}{b} \right| \left| \exp \left( - (t - s_k) N_{-\vartheta}^{(\mathcal{T}^*X\oplus N^*)} / b^2 \right) \right| .\]

Let \(K\) be the \((2,2)\) matrix

\[(11.123) \quad K = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} .\]

For simplicity, we will write \(\mathbb{R}^2\) in the form

\[(11.124) \quad \mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}^{0,\perp},\]

so that \(K\) acts on \(\mathbb{R} \oplus \mathbb{R}^{0,\perp}\).

By (11.27), (11.63), and (11.120), we get

\[(11.125) \quad \left| \frac{R_{\vartheta}^{0,\perp} (Y_s)}{b} \right| \leq \left| \frac{R_{\vartheta}^{0,\perp} (\overline{Y}_s)}{b} \right| + cCC' \cos^2 (\vartheta) bK.\]

Using (11.121), (11.122), (11.125), and the fact that under \(Q_{s_1,\ldots,s_k,t}\), the probability law of \(\overline{Y}\) is the same as the probability law of \(Y\) under \(Q\), we get

\[(11.126) \quad E^Q [R_{s_1,\ldots,s_k,t}] \leq \exp \left( (k + 1) C + \frac{1}{2} C^2 c^2 t + 2c^3 C^3 \cos^7 (\vartheta) b^2 t \right) \exp \left( 2cC \cos^4 (\vartheta) \int_0^t |Y_s^N|^2 \, ds \right) \left| \exp \left( -s_1 N_{-\vartheta}^{(\mathcal{T}^*X\oplus N^*)} / b^2 \right) \right| \left| \frac{R_{\vartheta}^{0,\perp} (Y_{s_1})}{b} \right| + cCC' \cos^2 (\vartheta) K \left| \exp \left( - (s_2 - s_1) N_{-\vartheta}^{(\mathcal{T}^*X\oplus N^*)} / b^2 \right) \right| \left| \frac{R_{\vartheta}^{0,\perp} (Y_{s_2})}{b} \right| + cCC' \cos^2 (\vartheta) K \left| \frac{R_{\vartheta}^{0,\perp} (Y_{s_k})}{b} \right| \ldots \left| \frac{R_{\vartheta}^{0,\perp} (Y_{s_k})}{b} \right| + cCC' \cos^2 (\vartheta) K \left| \exp \left( - (t - s_k) N_{-\vartheta}^{(\mathcal{T}^*X\oplus N^*)} / b^2 \right) \right| .\]
By (11.92), (11.126), we get

\[ E^Q \left[ |U_{b,\vartheta, t}^0| \right] \leq \sum_{k=0}^{+\infty} \exp \left( (k+1)C + \frac{1}{2}C^2C^2t + 2c^3C^3 \cos^7(\vartheta) b^2 t \right) \]

(11.127)

\[
E^Q \left[ \int_{\vartheta s_1 \leq t} \exp \left( 2C \cos^4(\vartheta) \int_0^t |Y_s^N|^2 \, ds \right) \right] \exp \left( -s_1 N_{-\vartheta}^{(T^*X \otimes N^*)} / b^2 \right) \]

\[
\left( R_{\vartheta}^{0, \perp} (Y_{s_1}) + cCC' \cos^2(\vartheta) K \right) \exp \left( - (s_2 - s_1) N_{-\vartheta}^{(T^*X \otimes N^*)} / b^2 \right) \]

\[
\left( R_{\vartheta}^{0, \perp} (Y_{s_2}) + cCC' \cos^2(\vartheta) K \right) \ldots \left( R_{\vartheta}^{0, \perp} (Y_{s_k}) + cCC' \cos^2(\vartheta) K \right) \]

\[
\exp \left( - (t - s_k) N_{-\vartheta}^{(T^*X \otimes N^*)} / b^2 \right) \, ds_1 \ldots ds_k .
\]

Let \( V_{b,\vartheta} \) be the solution of the differential equation,

\[ \frac{dV_{b,\vartheta}}{ds} = V_{b,\vartheta} \left[ - \frac{N_{-\vartheta}^{(T^*X \otimes N^*)}}{b^2} \right] + cC R_{\vartheta}^{0, \perp} (Y_s) + cCC' e^C \cos^2(\vartheta) K , \]

\[ V_{b,\vartheta,0} = 1 . \]

In (11.128), \( V_{b,\vartheta} \) is a process of (2, 2) matrices. One verifies easily that equation (11.127) can be rewritten in the form,

\[ E^Q \left[ |U_{b,\vartheta, t}^0| \right] \leq \exp \left( C + \frac{1}{2}C^2c^2t + 2c^3C^3 \cos^7(\vartheta) t \right) \]

\[ E^Q \left[ \exp \left( 2C \cos^4(\vartheta) \int_0^t |Y_s^N|^2 \, ds \right) V_{b,\vartheta, t} \right] . \]

We will estimate \( V_{b,\vartheta} \) by the method used in [B11, Theorem 14.5.2] and in the proof of Proposition 11.6. Put

\[ M_{b,\vartheta} = - \left( N_{-\vartheta}^{(T^*X \otimes N^*)} \right) / b^2 + cC R_{\vartheta}^{0, \perp} (Y) + cCC' e^C \cos^2(\vartheta) K . \]

Then \( M_{b,\vartheta} \) is a self-adjoint matrix. Equation (11.128) can be written in the form,

\[ \frac{dV_{b,\vartheta}}{ds} = V_{b,\vartheta, s} M_{b,\vartheta} , \quad V_{b,\vartheta,0} = 1 . \]

Since the component of \( |R_{\vartheta}^{0, \perp} (Y)| \) mapping \( \mathbf{R} \) into \( \mathbf{R} \) vanishes identically, using (11.27), if \( f \in \mathbf{R}^2 \), then

(11.132)

\[
\left| \left( R_{\vartheta}^{0, \perp} (Y), f \right) \right| \leq C' \left( \cos^{3/2}(\vartheta) |Y^T X| + |Y^N| \right) |f| \sqrt{\left| N_{-\vartheta}^{(T^*X \otimes N^*)} \right|} .
\]
By (11.130)–(11.133), we obtain

\[ \frac{1}{2} \left( \frac{\eta}{N^2} \right) \left( |N_{\alpha}^{\lambda} (T^X \oplus N^\perp) | f, f \right) + \frac{2}{\eta} \left( \cos (\vartheta) |Y^TX|^2 + |Y^N|^2 \right) |f|^2. \]

By (11.130)–(11.133), we obtain

\[ (M_{b,0} f, f) \leq \frac{1}{2} \left( \frac{\eta}{N^2} \right) \left( |N_{\alpha}^{\lambda} (T^X \oplus N^\perp) | f, f \right) + \frac{2}{\eta} \left( \cos (\vartheta) |Y^TX|^2 + |Y^N|^2 \right) |f|^2. \]

By taking \( \eta > 0 \) small enough, we deduce from (11.134) that there exists \( C' > 0 \) such that

\[ (M_{b,0} f, f) \leq \frac{C'}{2} \left( \cos (\vartheta) |Y^TX|^2 + |Y^N|^2 \right) |f|^2. \]

By (11.131), we get

\[ \frac{d}{ds} |V_{b,0,s} f|^2 \leq 2 (M_{b,0,s} V_{b,0,s}^*, X_{b,0,s} f). \]

Using (11.135), (11.136), we get

\[ \frac{d}{ds} |V_{b,0,s} f|^2 \leq \frac{C''}{2} \left( \cos (\vartheta) |Y^TX|^2 + |Y^N|^2 \right) |V_{b,0,s} f|^2. \]

By Gronwall’s lemma, from (11.137), we obtain

\[ |V_{b,0,s} f|^2 \leq \exp \left( \frac{C''}{2} \int_0^t \left( \cos (\vartheta) |Y^TX|^2 + |Y^N|^2 \right) ds + \frac{C''}{2} \cos (\vartheta) t \right). \]

Equation (11.138), is equivalent to

\[ |V_{b,0,s} f|^2 \leq \exp \left( \frac{C''}{2} \int_0^t \left( \cos (\vartheta) |Y^TX|^2 + |Y^N|^2 \right) ds + \frac{C''}{2} \cos (\vartheta) t \right). \]

By (11.129), (11.139), we get

\[ E^Q \left[ |U_{b,0,t}^0|^2 \right] \leq e^C E^Q \left[ \exp \left( \frac{C'''}{2} \int_0^t \left( \cos (\vartheta) |Y^TX|^2 + |Y^N|^2 \right) ds + C'''t \right) \right]. \]

By (9.24), (11.140), and using the same arguments as in Remark 11.7, given \( c_0, 0 < c_0 \leq 1 \), if \( b > 0 \) is such that \( C'''b^2 / \cos (\vartheta) \leq c_0 \), then

\[ E^Q \left[ |U_{b,0,t}^0|^2 \right] \leq e^C \exp \left( \frac{C'''}{2} \cos (\vartheta) \left( mt + b^2 |Y^0|^2 \right) \right) + C'' \left( \frac{n}{2} + 1 \right) t + \frac{c_0}{2} |Y^0|^2. \]
Also by (11.50), if \( b \leq C^{-1/2}, C^{m}b^{2}/\cos(\vartheta) > 1 \), then

\[
E^{Q} \left[ [U^0_{b,\vartheta,t}] \right] \leq \exp \left( \frac{1}{2} C \cos(\vartheta) nt + \frac{1}{2} \cos(\vartheta) Y_0^{TX} + \frac{C^2 t}{2} \right.
\]

\[
+ \frac{1}{2} C^{m} nt + C C^{m1/2} t |Y_0^{N}| \right).
\]

By (11.141), (11.142), we get (11.85) when \( p = 1 \). To obtain (11.85) for arbitrary \( p \geq 1 \), we may limit ourselves to the case where \( p \in \mathbb{N}^{*} \). However, it is easy to verify that the \( p^{th} \) power of the right-hand side of (11.92) has a similar expansion. The arguments that were given before then extend easily to the case of a general \( p \).

Let us now establish (11.86). By Hölder’s inequality\(^\text{7}\), for \( 1 < \theta < 2 \), we get

\[
\exp \left( - |Y_0|^2 / 2 \right) E^{Q} \left[ [U^0_{b,\vartheta,t}]^p \exp \left( |Y_t|^2 / 2 \right) \right] \leq \left\| [U^0_{b,\vartheta,t}]^p \right\| \exp \left( - |Y_0|^2 / 2 \right) \left\| \exp \left( |Y_t|^2 / 2 \right) \right\|_{\theta}.
\]

By (9.111), (11.47), (11.60), and (11.143), for \( 0 < b \leq 1, t \geq \epsilon \)

\[
\exp \left( - |Y_0|^2 / 2 \right) E^{Q} \left[ [U^0_{b,\vartheta,t}]^p \exp \left( |Y_t|^2 / 2 \right) \right] \leq C \exp \left( C^2 t + \frac{1}{2} \frac{(m + \cos(\vartheta)n)}{b^2} t \right.
\]

\[
+ C \cos(\vartheta) b |Y_0^{TX}| + C \frac{b}{\cos^{1/2}(\vartheta)} \left( 1 - e^{-t \cos(\vartheta)/b^2} \right) |Y_0^{N}| \exp \left( -C^* \left( |Y_0^{TX}|^2 + \cos(\vartheta) |Y_0^{N}|^2 \right) \right).
\]

By (11.144), if \( b_0 \) such that \( 0 < b_0 \leq 1 \) is given, if \( b_0 \leq b \leq 1, \epsilon \leq t \leq M \), then

\[
\exp \left( - |Y_0|^2 / 2 \right) E^{Q} \left[ [U^0_{b,\vartheta,t}]^p \exp \left( |Y_t|^2 / 2 \right) \right] \leq
\]

\[
C \exp \left( C \cos(\vartheta) b |Y_0^{TX}| + C \frac{\cos^{1/2}(\vartheta)}{b} |Y_0^{N}| \right)
\]

\[
\exp \left( -C^* \left( |Y_0^{TX}|^2 + \cos(\vartheta) |Y_0^{N}|^2 \right) \right).
\]

By (11.145), we deduce that (11.86) holds in the above range of parameters. In the sequel, we may as well take \( b \) to be arbitrarily small.

By (9.111), (11.49), and (11.143), for \( b > 0 \) small enough, and \( \epsilon \leq t \leq M \), we get

\[
\exp \left( - |Y_0|^2 / 2 \right) E^{Q} \left[ [U^0_{b,\vartheta,t}]^p \exp \left( |Y_t|^2 / 2 \right) \right] \leq C \exp \left( \frac{1}{2} \frac{\cos(\vartheta)}{b^2} nM + C \frac{b}{\cos^{1/2}(\vartheta)} \left( 1 - e^{-t \cos(\vartheta)/b^2} \right) |Y_0^{N}| \right.
\]

\[
\exp \left( -C^* \left( |Y_0^{TX}|^2 + \left( 1 - e^{-2t \cos(\vartheta)/b^2} \right) |Y_0^{N}|^2 \right) \right).
\]

\(^7\)As before, the Hölder norms are calculated with respect to \( Q \).
Take \( c_0 > 0 \). If \( b > 0 \) is small enough, if \( b^2 / \cos (\vartheta) \geq c_0, \epsilon \leq t \leq M \), then

\[
(11.147) \quad C' \left( 1 - e^{-2t \cos (\vartheta)/b^2} \right) |Y_0^N|^2 - C \frac{b}{\cos^{1/2} (\vartheta)} \left( 1 - e^{-t \cos (\vartheta)/b^2} \right) |Y_0^N| \\
\geq c' \cos (\vartheta) \frac{b}{b^2} |Y_0^N|^2 - C \cos^{1/2} (\vartheta) \frac{b}{b^2} |Y_0^N|^2 \geq \frac{c'}{2} \cos (\vartheta) |Y_0^N|^2 - C'.
\]

By (11.147), if we have also \( 0 < b \leq 1 \), we get

\[
(11.148) \quad C' \left( 1 - e^{-2t \cos (\vartheta)/b^2} \right) |Y_0^N|^2 - C \frac{b}{\cos^{1/2} (\vartheta)} \left( 1 - e^{-t \cos (\vartheta)/b^2} \right) |Y_0^N| \\
\geq c' \cos (\vartheta) |Y_0^N|^2 - C'.
\]

By (11.146), (11.148), we deduce that for \( 0 < b \leq 1 \), for \( b \) small enough, \( b^2 / \cos (\vartheta) \geq c_0, \epsilon \leq t \leq M \), then (11.86) still holds.

By the above, in the sequel, we can take \( b > 0 \) and \( b^2 / \cos (\vartheta) \) as small as needed.

By (9.111), by (11.141) which we use with an arbitrary power instead of with the power 1, and (11.143), given \( c_0 \in [0, 1] \), for \( 0 < b \leq 1, \epsilon \leq t \leq M, C' b^2 / \cos (\vartheta) \leq c_0 \), we get

\[
(11.149) \quad \exp \left( - |Y_0^N|^2 / 2 \right) E^Q \left[ |U_{b, \vartheta, t}^0|^p \exp \left( |Y_0^N|^2 / 2 \right) \right] \\
\leq C \exp \left( - C \left( |Y_0^{TX}|^2 + |Y_0^N|^2 \right) \right) \\
\exp \left( \frac{C' b^2}{2} \cos (\vartheta) |Y_0^{TX}|^2 + \frac{c_0}{2} |Y_0^N|^2 \right).
\]

Since we can take \( b, c_0 \) to be arbitrarily small, from (11.149), we still get (11.86).

The proof of our theorem is completed. \( \square \)

11.10. **A uniform estimate on** \( \sup_{0 \leq t \leq M} \left< \left< U_{b, \vartheta, t}^0 \right> \right> _p \).

**Theorem 11.14.** Given \( p > 2, M > 0 \), there exist \( C_{p, M} > 0, C' > 0 \) such that for \( 0 < b \leq 1, \vartheta \in [0, \frac{\pi}{2}] \),

\[
(11.150) \quad \left< \sup_{0 \leq t \leq M} \left< \left| \left( U_{b, \vartheta, t}^0 - \exp \left( - t N^A_{-\vartheta} (T^* X^{\perp N^+}) / b^2 \right) \right) P \right| \right> \right> _p \\
\leq C_{p, M} \exp \left( C' \left( \cos (\vartheta) |Y_0^{TX}|^2 + |Y_0^N|^2 \right) \right) \\
\inf \left( \frac{b}{\cos 1/2 (\vartheta)} \right)^{(p-2)/p} \cos 1/2 (\vartheta) / b.
\]

In particular, given \( Y_0 \), for \( 0 < b \leq 1, \vartheta \in [0, \frac{\pi}{2}] \), \( 0 \leq t \leq M \), the left-hand side of (11.150) is uniformly bounded.

Given \( M > 0, \vartheta \in [0, \frac{\pi}{2}] \), as \( b \to 0 \), \( \left< U_{b, \vartheta}^0 - \exp \left( - t N^A_{-\vartheta} (T^* X^{\perp N^+}) / b^2 \right) \right> P \)
converges uniformly to 0 on \([0, M]\) in probability. In particular, given \( 0 < \epsilon \leq M < +\infty, \vartheta \in [0, \frac{\pi}{2}] \), as \( b \to 0 \), \( U_{b, \vartheta}^0, P \) converges uniformly to 0 on \([\epsilon, M]\) in probability.
Proof. We proceed as in the proof of [B11, Proposition 14.10.2]. By (11.28), we get

\begin{equation}
U_{b,\vartheta,t}^0 = \exp \left( -tN_{-\vartheta}^\Lambda (T^*X \oplus N^*) / b^2 \right) + \int_0^t U_{b,\vartheta,s}^0 \frac{R_0^b (Y_s)}{b} \exp \left( - (t - s) N_{-\vartheta}^\Lambda (T^*X \oplus N^*) / b^2 \right) ds.
\end{equation}

Clearly,

\begin{equation}
\left| \int_0^t U_{b,\vartheta,s}^0 \frac{R_0^b (Y_s)}{b} \exp \left( - (t - s) N_{-\vartheta}^\Lambda (T^*X \oplus N^*) / b^2 \right) ds \right| \leq \int_0^t \left| U_{b,\vartheta,s}^0 \frac{R_0^b (Y_s)}{b} \right| \exp \left( - (t - s) N_{-\vartheta}^\Lambda (T^*X \oplus N^*) / b^2 \right) ds.
\end{equation}

Take \( p > 2 \), and let \( q \) be such that \( \frac{1}{p} + \frac{1}{q} = 1 \). By (11.27) and by Hölder’s inequality, for \( 0 \leq t \leq M \), we get

\begin{equation}
\left| \int_0^t U_{b,\vartheta,s}^0 \frac{R_0^b (Y_s)}{b} \exp \left( - (t - s) N_{-\vartheta}^\Lambda (T^*X \oplus N^*) / b^2 \right) ds \right| \leq C b^{(p-2)/p} \left[ \int_0^M \left| U_{b,\vartheta,s}^0 \right| ^{2p} ds \right] ^{1/2p} \left( \cos (\vartheta) \left[ \int_0^M \left| Y_s^{-T,X} \right| ^{2p} ds \right] ^{1/2p} + \cos^{1/2} (\vartheta) \left[ \int_0^M \left| Y_s^{-N} \right| ^{2p} ds \right] ^{1/2p} \right) \left| 1 - \exp \left( -qMN_{-\vartheta}^\Lambda (T^*X \oplus N^*) / b^2 \right) \right| ^{1/2}.
\end{equation}

By Cauchy-Schwarz, we have

\begin{equation}
E^Q \left[ \left[ \int_0^M \left| U_{b,\vartheta,s}^0 \right| ^{2p} ds \right] ^{1/2} \left[ \int_0^M \left| Y_s^{-T,X} \right| ^{2p} ds \right] ^{1/2} \right] \leq \left\{ E^Q \left[ \int_0^M \left| U_{b,\vartheta,s}^0 \right| ^{2p} ds \right] \right\} ^{1/2} \\left\{ E^Q \left[ \int_0^M \left| Y_s^{-T,X} \right| ^{2p} ds \right] \right\} ^{1/2}.
\end{equation}

In (11.154), we can replace \( Y_{-T,X} \) by \( Y_{-N} \).
By (11.151)–(11.154), we obtain

\[(11.155) \quad \left\| \sup_{0 \leq t \leq M} \left( U_{b, \vartheta, t}^0 - \exp \left( -t N^\Lambda_{-\vartheta} (T^* X \oplus N^*) / b^2 \right) \right) \right\|_p \leq Cb^{(p-2)/p} \left( E^Q \left[ \int_0^M |U_{b, \vartheta, s}^0|^{2p} \, ds \right] \right)^{1/2p} \cos^{1/2} (\vartheta) \]

\[ \left( \cos^{1/2} (\vartheta) \right) \left( E^Q \left[ \int_0^M |Y_t^{TX} |^{2p} \, ds \right] \right)^{1/2p} + \left( E^Q \left[ \int_0^M |Y_t^{N} |^{2p} \, ds \right] \right)^{1/2p} \]

\[ \left| 1 - \exp \left( -qMN^\Lambda_{-\vartheta} (T^* X \oplus N^*) / b^2 \right) \right| \cdot qN^\Lambda_{-\vartheta} (T^* X \oplus N^*) \] \( \leq C \inf \left( \frac{1}{\cos (\vartheta)}, \frac{1}{b^2} \right). \]

By [B11, Proposition 10.8.2], we get

\[(11.156) \quad E^Q \left[ |Y_t^{TX}|^{2p} \right] \leq C_p \left( 1 + |Y_0^{TX}|^{2p} \right), \quad E^Q \left[ |Y_t^{N}|^{2p} \right] \leq C_p \left( 1 + |Y_0^{N}|^{2p} \right). \]

For \( a > 0, x \geq 0 \), we have the inequality,

\[(11.157) \quad \frac{1 - e^{-x/a}}{x} \leq \inf \left( \frac{1}{a}, \frac{1}{x} \right). \]

From (11.157), we obtain

\[(11.158) \quad \left| 1 - \exp \left( -qMN^\Lambda_{-\vartheta} (T^* X \oplus N^*) / b^2 \right) \right| \cdot qN^\Lambda_{-\vartheta} (T^* X \oplus N^*) \right\| \leq C \inf \left( \frac{1}{\cos (\vartheta)}, \frac{1}{b^2} \right). \]

Using equation (11.85) in Theorem 11.13 and (11.155)–(11.158), for \( 0 \leq t \leq M \), we get (11.150). The last part of our theorem follows from (11.150). The proof of our theorem is completed.

Now we get an extension of [B11, Proposition 14.10.3], which is also a path integral version of Proposition 11.2.
Proposition 11.15. For $0 \leq \vartheta < \frac{\pi}{2}$, the following identity holds:

\begin{equation}
(11.159) \quad U_{b,\vartheta,t}^0 \left( 1 + b \left( 1 + N_{-\vartheta}^\Lambda (T^*X \oplus N^*) \right)^{-1} R_0^0 \left( Y_t^{TX} \right) \\
+ b \left( \cos (\vartheta) + N_{-\vartheta}^\Lambda (T^*X \oplus N^*) \right)^{-1} R_0^0 \left( Y_t^N \right) \right)
\end{equation}

$$= 1 + b \left( 1 + N_{-\vartheta}^\Lambda (T^*X \oplus N^*) \right)^{-1} R_0^0 \left( Y_0^{TX} \right) + b \left( \cos (\vartheta) + N_{-\vartheta}^\Lambda (T^*X \oplus N^*) \right)^{-1} R_0^0 \left( Y_0^N \right)$$

$$+ \int_0^t U_{b,\vartheta,s}^0 \left( \frac{N_{-\vartheta}^\Lambda (T^*X \oplus N^*)}{b^2} + R_0^0 \left( Y_s \right) \left( 1 + N_{-\vartheta}^\Lambda (T^*X \oplus N^*) \right)^{-1} R_0^0 \left( Y_s^{TX} \right) \right) ds$$

$$+ \int_0^t U_{b,\vartheta,s}^0 \left( 1 + N_{-\vartheta}^\Lambda (T^*X \oplus N^*) \right)^{-1} R_0^0 \left( \delta w^{TX} \right)$$

$$+ \left( \cos (\vartheta) + N_{-\vartheta}^\Lambda (T^*X \oplus N^*) \right)^{-1} \cos^{3/2} (\vartheta) R_0^0 \left( \delta w^N \right).$$

Proof. By (11.21), (11.28), we get (11.159). \hfill \Box

Remark 11.16. In the right hand-side of (11.159), we could have replaced the Itô integrals by more classical Stratonovitch integrals. However, in the proof of Theorem 11.17, the estimates will be established using the fact that Itô integrals appear in (11.159).

Now we will establish a crucial estimate.

Theorem 11.17. For any $p > 2, M > 0$, there exist $C_{p,M} > 0, C' > 0$ such that for $0 < b \leq 1, \vartheta \in [0, \frac{\pi}{2}[$, then

\begin{equation}
(11.160) \quad \left\| \sup_{0 \leq t \leq M} \left| U_{b,\vartheta,t}^0 \right| \right\|_p \leq C_{p,M} \exp \left( C' \left( \cos (\vartheta) \left| Y_0^{TX} \right|^2 + \left| Y_0^N \right|^2 \right) \right).
\end{equation}

Proof. If we just consider $U_{b,\vartheta}^0 \perp P$, our estimate follows from Theorem 11.14. We should then prove the corresponding estimate for $U_{b,\vartheta}^0 \perp P$. 

By equation (11.159) in Proposition 11.15, we get an analogue of [B11, eq. (14.10.32)],

\[(11.161) \quad U_{b,ϑ,t}^0 \left( 1 + b \left( 1 + N_{-0}^{\Lambda(T^*X⊕N^*)} \right)^{-1} R_{ϑ}^0 (Y^TX) + b \left( \cos (ϑ) + N_{-0}^{\Lambda(T^*X⊕N^*)} \right)^{-1} R_{ϑ}^0 (Y^N) \right) P \]

\[= \left( 1 + b \left( 1 + N_{-0}^{\Lambda(T^*X⊕N^*)} \right)^{-1} R_{ϑ}^0 (Y^TX) + b \left( \cos (ϑ) + N_{-0}^{\Lambda(T^*X⊕N^*)} \right)^{-1} R_{ϑ}^0 (Y^N) \right) P \]

\[+ \int_0^t U_{b,ϑ,s}^0 R_{ϑ}^0 (Y_s) \left[ \left( 1 + N_{-0}^{\Lambda(T^*X⊕N^*)} \right)^{-1} R_{ϑ}^0 (Y^TX) + \left( \cos (ϑ) + N_{-0}^{\Lambda(T^*X⊕N^*)} \right)^{-1} R_{ϑ}^0 (Y^N) \right] ds \]

\[+ \int_0^t U_{b,ϑ,s}^0 \left[ \left( 1 + N_{-0}^{\Lambda(T^*X⊕N^*)} \right)^{-1} R_{ϑ}^0 (\delta w^T) + \left( \cos (ϑ) + N_{-0}^{\Lambda(T^*X⊕N^*)} \right)^{-1} \cos^{1/2} (ϑ) R_{ϑ}^0 (\delta w^N) \right] P. \]

By (11.27) and by Hölder’s inequality, for \( p > 1 \), we get

\[(11.162) \quad \int_0^M \left| U_{b,ϑ,s}^0 R_{ϑ}^0 (Y_s) \left[ \left( 1 + N_{-0}^{\Lambda(T^*X⊕N^*)} \right)^{-1} R_{ϑ}^0 (Y^TX) + \left( \cos (ϑ) + N_{-0}^{\Lambda(T^*X⊕N^*)} \right)^{-1} R_{ϑ}^0 (Y^N) \right] P \right| ds \leq C_{M,p} \left[ \int_0^M |U_{b,ϑ,s}^0|^p \left( \cos^p (ϑ) |Y^TX|^{2p} + |Y^N|^{2p} \right) ds \right]^{1/p}. \]

In (11.162), equation (11.27) has been used in particular to overcome the potentially singular term \( \left( \cos (ϑ) + N_{-0}^{\Lambda(T^*X⊕N^*)} \right)^{-1} \). By (11.162), and by Cauchy-Schwarz,
we obtain

\[
\begin{align*}
(11.163) \quad E^Q \left[ \int_0^M & \left| U_{b, \vartheta, s}^0 R^0_\vartheta \left( Y_s^{TX} \right) \right| \right. \\
+ & \left( \cos (\vartheta) + N^A_{-\vartheta} \left( T^* X \oplus N^* \right) \right)^{-1} R^0_\vartheta \left( Y_s^N \right) \left( 1 + N^A_{-\vartheta} \left( T^* X \oplus N^* \right) \right)^{-1} R^0_\vartheta \left( Y_s^{TX} \right) \left. \right| ds \right]^p \\
\leq C_{M,p} \left\{ E^Q \left[ \int_0^M \left| U_{b, \vartheta, s}^0 \right|^2 ds \right] \right\}^{1/2} \\
& \left\{ E^Q \left[ \int_0^M \left( \cos^2 (\vartheta) \left| Y^{TX} \right|^4 + \left| Y^N \right|^4 \right) ds \right] \right\}^{1/2} .
\end{align*}
\]

Using equation (11.85) in Theorem 11.13, (11.156), and (11.163), we find that the estimate in (11.163) is compatible with (11.160).

The contribution of the last stochastic integral \( I_t \) in the right-hand side of (11.161) can be estimated using an inequality by Burkholder-Davis-Gundy [BuG70], which asserts that if \( f \cdot \delta w \) is an adapted stochastic process such that

\[
\int_0^M |f| ds < +\infty \quad \text{a.s., for } p > 1
\]

(11.164)

\[
\left[ E^Q \left[ \sup_{0 \leq t \leq M} \left| \int_0^t \langle f_s, \delta w_s \rangle \right|^p \right] \right]^{1/p} \leq C_p \left[ E^Q \left[ \left( \int_0^M |f_s|^2 ds \right)^{p/2} \right] \right]^{1/p} .
\]

By (11.27), (11.164), for \( p > 1 \), we get

(11.165)

\[
\left[ E^Q \left[ \sup_{0 \leq t \leq M} |I_t|^p \right] \right]^{1/p} \leq C_p E^Q \left[ \left( \int_0^M \left| U_{b, \vartheta, s}^0 \right|^2 ds \right)^{p/2} \right]^{1/p} .
\]

By Hölder’s inequality, for \( p > 2 \), we get

(11.166)

\[
\left( \int_0^M \left| U_{b, \vartheta, s}^0 \right|^2 ds \right)^{p/2} \leq M^{(p-2)/2} \int_0^M \left| U_{b, \vartheta, s}^0 \right|^p ds .
\]

Using equation (11.85) in Theorem 11.13, (11.165), and (11.166), the contribution of \( I_t \) to the estimation of the right-hand side of (11.161) is still compatible with (11.160).

The most annoying term in (11.161) is the contribution of

\[
(11.167) \quad A_{b, \vartheta, t} = U_{b, \vartheta, t}^0 b \left( 1 + N^A_{-\vartheta} \left( T^* X \oplus N^* \right) \right)^{-1} R^0_\vartheta \left( Y_t^{TX} \right) \\
+ \left( \cos (\vartheta) + N^A_{-\vartheta} \left( T^* X \oplus N^* \right) \right)^{-1} R^0_\vartheta \left( Y_t^N \right) - b \left( 1 + N^A_{-\vartheta} \left( T^* X \oplus N^* \right) \right)^{-1} R^0_\vartheta \left( Y_0^{TX} \right) \\
+ \left( \cos (\vartheta) + N^A_{-\vartheta} \left( T^* X \oplus N^* \right) \right)^{-1} R^0_\vartheta \left( Y_0^N \right) \right) \right) .
\]
Set

\[
B_{b,\vartheta,t} = U_{b,\vartheta,t}^0 P b \left( \left( 1 + N_{-\vartheta}^\Lambda (T^*X \oplus N^*) \right)^{-1} R_{0}^0 (Y^T X) \right)
+ \left( \cos (\vartheta) + N_{-\vartheta}^\Lambda (T^*X \oplus N^*) \right)^{-1} R_{0}^0 (Y^T N) \right) P
- P b \left( \left( 1 + N_{-\vartheta}^\Lambda (T^*X \oplus N^*) \right)^{-1} R_{0}^0 (Y^T X) \right)
+ \left( \cos (\vartheta) + N_{-\vartheta}^\Lambda (T^*X \oplus N^*) \right)^{-1} R_{0}^0 (Y^T N) \right) P,
\]

(11.168)

\[
B_{b,\vartheta,t}^\perp = U_{b,\vartheta,t}^0 P^\perp b \left( \left( 1 + N_{-\vartheta}^\Lambda (T^*X \oplus N^*) \right)^{-1} R_{0}^0 (Y^T X) \right)
+ \left( \cos (\vartheta) + N_{-\vartheta}^\Lambda (T^*X \oplus N^*) \right)^{-1} R_{0}^0 (Y^T N) \right) P
- P^\perp b \left( \left( 1 + N_{-\vartheta}^\Lambda (T^*X \oplus N^*) \right)^{-1} R_{0}^0 (Y^T X) \right)
+ \left( \cos (\vartheta) + N_{-\vartheta}^\Lambda (T^*X \oplus N^*) \right)^{-1} R_{0}^0 (Y^T N) \right) P.
\]

Then

(11.169)

\[
A_{b,\vartheta,t} = B_{b,\vartheta,t} + B_{b,\vartheta,t}^\perp.
\]

First we will control \(B_{b,\vartheta,t}^\perp\), and later \(B_{b,\vartheta,t}\).

• Control of \(B_{b,\vartheta,t}^\perp\).

By (11.27), we get

(11.170)

\[
b \left| \left( 1 + N_{-\vartheta}^\Lambda (T^*X \oplus N^*) \right)^{-1} R_{0}^0 (Y^T X) + \left( \cos (\vartheta) + N_{-\vartheta}^\Lambda (T^*X \oplus N^*) \right)^{-1} R_{0}^0 (Y^T N) \right| \leq C \left( b \cos (\vartheta) |Y^T X| + \frac{b}{\cos^{1/2} (\vartheta)} |Y^T N| \right).
\]
By Theorem 11.14 and by (11.170), for $p > 2$, we get

\begin{align}
(11.171) \quad \left\| \sup_{0 \leq t \leq M} \left( U_{b, \theta, t}^0 - \exp \left( -t N_{-\theta}^\Lambda \left( T^* X \oplus N \right) / b^2 \right) \right) P^\perp \right\|_p & \\
& \quad \leq C_{2p, M} \exp \left( C \left( \cos (\vartheta) \sqrt{Y^{TX}_0} + \sqrt{Y^{N}_0} \right) \right) \\
& \quad \cdot \left( b \cos (\vartheta) \left\| \sup_{0 \leq t \leq M} |Y^{TX}_t| \right\|_{2p} + \inf \left( \left( b / \cos^{1/2} (\vartheta) \right)^{(2p-1)/p}, 1 \right) \right) \left\| \sup_{0 \leq t \leq M} |Y^{N}_t| \right\|_{2p}.
\end{align}

By (9.35), for $0 < b \leq 1$, we get

\begin{align}
(11.172) \quad \left\| \sup_{0 \leq t \leq M} |bY^{TX}_t| \right\|_{2p} & \leq C \left( 1 + |Y^{TX}_0| \right).
\end{align}

The same argument shows that if $b / \cos^{1/2} (\vartheta) \leq 1$, then

\begin{align}
(11.173) \quad \frac{b}{\cos^{1/2} (\vartheta)} \left\| \sup_{0 \leq t \leq M} |Y^{N}_t| \right\|_{2p} & \leq C \left( 1 + |Y^{N}_0| \right).
\end{align}

Since $p > 2$, then $(2p - 2) / p > 1$, so that in (11.173), $b / \cos^{1/2} (\vartheta)$ can be replaced by $\left( b / \cos^{1/2} (\vartheta) \right)^{(2p-1)/p}$. If $b / \cos^{1/2} (\vartheta) > 1$, by (9.35), we get

\begin{align}
(11.174) \quad \left\| \sup_{0 \leq t \leq M} |Y^{N}_t| \right\|_{2p} & \leq C \left( 1 + |Y^{N}_0| \right).
\end{align}

By (11.172)–(11.174), we can give an upper bound for the left-hand side of (11.171) which is compatible with (11.160).
By (11.170), we get

(11.175) \[ \sup_{0 \leq t \leq M} \left\| \left( \exp \left( -tN^{\Lambda_{-\vartheta}}(T^*X \oplus N) / b^2 \right) - 1 \right) \mathbf{P}^{1} \right\| \]

\[ b \left( 1 + N^{\Lambda_{-\vartheta}}(T^*X \oplus N^*) \right)^{-1} \mathbf{R}^0_\vartheta (Y^T_X - Y^T_0) \]

\[ + \left( \cos (\vartheta) + N^{\Lambda_{-\vartheta}}(T^*X \oplus N^*) \right)^{-1} \mathbf{R}^0_\vartheta (Y^N - Y^N_0) \right) \mathbf{P} \right\|_p \]

\[ \leq C b \cos (\vartheta) \sup_{0 \leq t \leq M} \left\| Y^T_t \right\|_p + \left( 1 - \exp \left( -\cos (\vartheta) nM / b^2 \right) \right) \frac{b}{\cos^{1/2} (\vartheta)} \]

\[ + b \cos^{1/2} (\vartheta) \sup_{0 \leq t \leq M} \left\| Y^N_t \right\|_p \right) |f|. \]

The last two terms in the right-hand side of (11.175) correspond to the cases where \( N^{\Lambda_{-\vartheta}}(T^*X) = 0 \) and \( N^{\Lambda_{-\vartheta}}(T^*X) > 0 \). If \( b / \cos^{1/2} (\vartheta) \leq 1 \), by (11.172), (11.173), we still get an upper bound for the left-hand side of (11.175) that is compatible with (11.160). If \( 0 < b \leq 1 \), using (11.172), (11.174), and (11.176), we still obtain a good upper bound for the left-hand side of (11.175).

By (11.27), we get

(11.176) \[ \left( 1 - \exp \left( -\cos (\vartheta) nM / b^2 \right) \right) \frac{b}{\cos^{1/2} (\vartheta)} \leq nM \frac{\cos^{1/2} (\vartheta)}{b} \leq nM. \]

Using (11.172), (11.174), and (11.176), we still obtain a good upper bound for the left-hand side of (11.175).

By (11.172), the first term in the right-hand side of (11.177) can be dominated by the right-hand side of (11.160). If \( b / \cos^{1/2} (\vartheta) \leq 1 \), using (11.173), we can dominate properly the contribution of the second term in the right-hand side of (11.177). Let \( Y^{N,0} \) be the solution of (11.21) with \( Y^{N,0}_0 = 0 \). Then

(11.178) \[ Y^N_t - Y^N_0 = \left( e^{-\cos (\vartheta) t / b^2} - 1 \right) Y^N_0 + Y^{N,0}_t. \]

Then

(11.179) \[ \frac{b}{\cos^{1/2} (\vartheta)} \sup_{0 \leq t \leq M} \left\| \left( Y^N_t - Y^N_0 \right) \right\|_p \leq \frac{b}{\cos^{1/2} (\vartheta)} \left( 1 - e^{-\cos (\vartheta) M / b^2} \right) \left| Y^N_0 \right| \]

\[ + \frac{b}{\cos^{1/2} (\vartheta)} \sup_{0 \leq t \leq M} \left\| Y^{N,0}_t \right\|_p. \]
Also

\[ (11.180) \quad \frac{b}{\cos^{1/2}(\vartheta)} \left( 1 - e^{-\cos(\vartheta)M/b^2} \right) \leq \frac{\cos^{1/2}(\vartheta)}{b} M. \]

If \( b/\cos^{1/2}(\vartheta) > 1 \), the right-hand side of (11.180) is dominated by \( M \). By (9.35), when \( b/\cos^{1/2}(\vartheta) > 1 \), we get

\[ (11.181) \quad \frac{b}{\cos^{1/2}(\vartheta)} \left\| \sup_{0 \leq t \leq M} |Y_t^N,0| \right\|_p \leq C. \]

By (11.177)–(11.181), we obtain an upper bound for (11.177) that is compatible with (11.160).

By (11.171)–(11.181), we find that

\[ \left\| \sup_{0 \leq t \leq M} |B_{b,\vartheta,t} \perp b,\vartheta| \right\|_p \text{ is dominated by an} \]

expression like the right-hand side of (11.160).

\begin{itemize}
  \item \textit{Control of } \( B_{b,\vartheta,t} \).
  \end{itemize}

Set

\[ (11.182) \quad C_{b,\vartheta,t} = U_{0}^{0}(b,\vartheta,t)P + B_{b,\vartheta,t}. \]

It follows from equation (11.161) and from the above that

\[ (11.183) \quad \left\| \sup_{0 \leq t \leq M} |C_{b,\vartheta,t}| \right\|_p \leq C_{p,M} \exp \left( C' \left( \cos(\vartheta) |Y_{0}^{TX} |^2 + |Y_{0}^{N} |^2 \right) \right). \]

Note that

\[ (11.184) \quad B_{b,\vartheta,t} = U_{0}^{0}(b,\vartheta,t)P \left( R_{0}^{0}(Y_{t}^{TX}) + \frac{R_{0}^{0}(Y_{t}^{N})}{\cos(\vartheta)} \right) P \]

\[ \quad - P \left( R_{0}^{0}(Y_{t}^{TX}) + \frac{R_{0}^{0}(Y_{0}^{N})}{\cos(\vartheta)} \right) P. \]

Using equation (4.103) in Proposition 4.27, we can rewrite (11.184) in the form,

\[ (11.185) \quad B_{b,\vartheta,t} = U_{0}^{0}(b,\vartheta,t)P \frac{b}{\cos(\vartheta)} R_{0}^{0}(Y_{t}) P - \frac{b}{\cos(\vartheta)} P R_{0}^{0}(Y_{0}) P. \]

We rewrite (11.185) in the form

\[ (11.186) \quad B_{b,\vartheta,t} = \frac{b}{\cos(\vartheta)} U_{0}^{0}(b,\vartheta,t)R_{0}^{0}(Y_{t}) P - U_{0}^{0}(b,\vartheta,t)P \frac{b}{\cos(\vartheta)} R_{0}^{0}(Y_{t}) P - \frac{b}{\cos(\vartheta)} P R_{0}^{0}(Y_{0}) P. \]

By combining (12.28) and (11.186), we obtain

\[ (11.187) \quad B_{b,\vartheta,t} = \frac{b^2}{\cos(\vartheta)} \frac{d}{dt} U_{0}^{0}(b,\vartheta,t) P - U_{0}^{0}(b,\vartheta,t)P \frac{b}{\cos(\vartheta)} R_{0}^{0}(Y_{t}) P - \frac{b}{\cos(\vartheta)} P R_{0}^{0}(Y_{0}) P. \]

By (11.182), (11.187), we get

\[ (11.188) \quad \left( \frac{b^2}{\cos(\vartheta)} \frac{d}{dt} + 1 \right) U_{0}^{0}(b,\vartheta,t) P = C_{b,\vartheta,t} + U_{0}^{0}(b,\vartheta,t)P \frac{b}{\cos(\vartheta)} R_{0}^{0}(Y_{t}) P + \frac{b}{\cos(\vartheta)} P R_{0}^{0}(Y_{0}) P. \]
By (11.188), we obtain

\begin{equation}
U_{b,\vartheta,t}^0 P = \exp \left( -\cos (\vartheta) t / b^2 \right) P + \int_0^t \exp \left( -(t - s) \cos (\vartheta) / b^2 \right) \frac{\cos (\vartheta)}{b^2} \left( C_{b,\vartheta,s} + U_{b,\vartheta,s}^0 \frac{b}{\cos (\vartheta)} R_{b,\vartheta}^0 (Y_s) P + \frac{b}{\cos (\vartheta)} P R_{b,\vartheta}^0 (Y_t) P \right) ds.
\end{equation}

For 0 \leq t \leq M, we have the obvious inequalities,

\begin{equation}
\left\| \int_0^t \exp \left( -(t - s) \cos (\vartheta) / b^2 \right) \frac{\cos (\vartheta)}{b^2} C_{b,\vartheta,s} ds \right\|_p \leq \sup_{0 \leq s \leq M} |C_{b,\vartheta,t}|.
\end{equation}

By (11.183), (11.190), we get

\begin{equation}
\left\| \sup_{0 \leq s \leq M} \left\| \int_0^t \exp \left( -(t - s) \cos (\vartheta) / b^2 \right) \frac{\cos (\vartheta)}{b^2} C_{b,\vartheta,s} ds \right\|_p \leq C_{p,M} \exp \left( C' \left( \cos (\vartheta) |Y_0^{TX}|^2 + |Y_0^N|^2 \right) \right).\end{equation}

For 0 \leq t \leq M, by (11.27), we have the inequality,

\begin{equation}
\left\| \sup_{0 \leq s \leq M} \left\| \int_0^t \exp \left( -(t - s) \cos (\vartheta) / b^2 \right) \frac{\cos (\vartheta)}{b^2} U_{b,\vartheta,s}^0 \frac{b}{\cos (\vartheta)} R_{b,\vartheta}^0 (Y_s) P ds \right\|_p \leq \sup_{0 \leq s \leq M} \left| bY^T_s \right| + 2 \sup_{0 \leq s \leq M} \left| w_s^{TX} \right| + 2 \sup_{0 \leq s \leq M} \left| w_s^N \right|.
\end{equation}

Using the notation in (9.35), we get

\begin{equation}
\sup_{0 \leq s \leq M} \left| bY^T_s \right| \leq \left| bY^T_0 \right| + 2 \sup_{0 \leq s \leq M} \left| w_s^{TX} \right|,
\end{equation}

\begin{equation}
\sup_{0 \leq s \leq M} \left| \frac{b}{\cos^{1/2}(\vartheta)} Y_s^N \right| \leq \left| \frac{b}{\cos^{1/2}(\vartheta)} Y_0^N \right| + 2 \sup_{0 \leq s \leq M} \left| w_s^N \right|.
\end{equation}

By (11.193), we obtain

\begin{equation}
\left( 1 - e^{-M \cos(\vartheta)/b^2} \right) \sup_{0 \leq s \leq M} \left| bY^T_s \right| \leq C \cos^{1/2}(\vartheta) \left| Y_0^T \right| + 2 \sup_{0 \leq s \leq M} \left| w_s^{TX} \right|,
\end{equation}

\begin{equation}
\left( 1 - e^{-M \cos(\vartheta)/b^2} \right) \sup_{0 \leq s \leq M} \left| \frac{b}{\cos^{1/2}(\vartheta)} Y_s^N \right| \leq C \left| Y_0^N \right| + 2 \sup_{0 \leq s \leq M} \left| w_s^N \right|.
\end{equation}

Using Theorem 11.14, and equations (11.192), (11.194), we obtain

\begin{equation}
\left\| \sup_{0 \leq t \leq M} \left\| \int_0^t \exp \left( -(t - s) \cos (\vartheta) / b^2 \right) \frac{\cos (\vartheta)}{b^2} U_{b,\vartheta,s}^0 \frac{b}{\cos (\vartheta)} R_{b,\vartheta}^0 (Y_s) P ds \right\|_p \leq C_{p,M} \exp \left( C' \left( \cos (\vartheta) |Y_0^{TX}|^2 + |Y_0^N|^2 \right) \right).\end{equation}
By Proposition 4.27, we get

\begin{equation}
\int_0^t \exp \left(\frac{-(t-s)}{b^2} \cos (\vartheta) b^2 \cos (\vartheta) \right) \frac{b}{b^2} \cos (\vartheta) \mathbf{P} R^0_{t, \vartheta}(Y_0) \mathbf{P} ds = \left(1 - e^{-t \cos (\vartheta)/b^2}\right) b \cos^{3/2} (\vartheta) \tilde{c} \left(-i \text{ad} (Y_0^N) \mid \mathcal{T}_X\right) \mathbf{P}.
\end{equation}

By (11.196), we deduce that for \(0 < b \leq 1\),

\begin{equation}
\left| \int_0^t \exp \left(-\left(\frac{(t-s)}{b^2} \cos (\vartheta) b^2 \cos (\vartheta) \right) \frac{b}{b^2} \cos (\vartheta) \mathbf{P} R^0_{t, \vartheta}(Y_0) \mathbf{P} ds \right| \leq C |Y_0^N|.
\end{equation}

By (11.197), we deduce that

\begin{equation}
\sup_{0 \leq t \leq M} \left| U^0_{b, \vartheta, t} \mathbf{P} \right|_p \leq C_{p, M} \exp \left(C' \left(\cos (\vartheta) |Y_0^{TX}|^2 + |Y_0^N|^2\right)\right).
\end{equation}

By (11.182), (11.183), and (11.198), \(\left| \sup_{0 \leq t \leq M} |B_{b, \vartheta, t} \mathbf{P}|_p \right) \) is dominated by an expression like the right-hand side of (11.160).

This concludes the proof of equation (11.160), and of our theorem. \(\square\)

11.1. The limit of \(U_{b, \vartheta}^0\), as \(b \to 0\). Recall that \(\mathcal{S}_0^0(Y)\) was defined in Definition 4.35. By equation (4.115) in Proposition 4.29, we get

\begin{equation}
\left| \mathcal{S}_0^0(Y) \right| \leq C \left(\cos^2 (\vartheta) |Y^{TX}|^2 + |Y^N|^2\right).
\end{equation}

Recall that \(H_{b, \vartheta, t}\) was defined in equation (11.64). Using (11.21), (11.64), we conclude that

\begin{equation}
\frac{b^2}{\cos (\vartheta)} \dot{H}_{b, \vartheta} + \dot{H}_{b, \vartheta} = H_{b, \vartheta} \left(-\cos^4 (\vartheta) \tilde{c} \left(\text{ad} (Y^N) \mid \mathcal{T}_X\right)^2 + \cos^2 (\vartheta) \tilde{c} \left(\text{ad} (-i \dot{w}^N) \mid \mathcal{T}_X\right)\right).
\end{equation}

Equations (11.70) and (11.200) are intimately related.

By (11.64), we get

\begin{equation}
\frac{dH_{b, \vartheta}^{-1}}{ds} = \frac{\cos^5/2 (\vartheta)}{b} \tilde{c} \left(\text{ad} (Y^N) \mid \mathcal{T}_X\right) H_{b, \vartheta}^{-1}, \quad H_{b, \vartheta, 0}^{-1} = 1.
\end{equation}

By (11.21), (11.201), we conclude that

\begin{equation}
\frac{b^2}{\cos (\vartheta)} \dot{H}_{b, \vartheta}^{-1} + \dot{H}_{b, \vartheta}^{-1} = \left(\tilde{c} \left(\text{ad} (Y^N) \mid \mathcal{T}_X\right)^2 - \cos^2 (\vartheta) \tilde{c} \left(\text{ad} (-i \dot{w}^N) \mid \mathcal{T}_X\right)\right) H_{b, \vartheta}^{-1}.
\end{equation}

Definition 11.18. Put

\begin{equation}
L_{b, \vartheta, t} = \exp \left(\int_0^t \mathcal{S}_0^0(Y) ds\right) H_{b, \vartheta, t}.
\end{equation}

By equation (9.24) in Proposition 9.5, in which \(\beta\) is replaced by \(\beta b\), and \(t\) by \(t/b^2\), given \(\beta > 0\), if \(b > 0\) is small enough so that \(\beta b \leq 1\), then

\begin{equation}
E^2 \left[\exp \left(\frac{\beta^2}{2} \int_0^t |Y_{b, \vartheta}^{TX}|^2 ds\right)\right] \leq \exp \left(\frac{\beta^2}{2} \left(mt + b^2 |Y_0^{TX}|^2\right)\right).
\end{equation}
Under the above conditions, by (11.204), we get

\[(11.205)\quad E^Q \left[ \exp \left( \frac{\beta^2}{2} \int_0^t |Y_s^T X|^2 \, ds \right) \right] \leq \exp \left( \frac{1}{2} \left( \beta^2 mt + |Y_0^T X|^2 \right) \right).\]

Similarly, given \( \vartheta \in [0, \frac{\pi}{2}], \beta > 0 \), for \( b > 0 \) small enough so that \( \beta b / \cos^{1/2}(\vartheta) \leq 1 \), then

\[(11.206)\quad E^Q \left[ \exp \left( \frac{\beta^2}{2} \int_0^t |Y_s^N|^2 \, ds \right) \right] \leq \exp \left( \frac{1}{2} \left( \beta^2 mt + |Y_0^N|^2 \right) \right).\]

Under the above conditions, by (11.206), we deduce that

\[(11.207)\quad E^Q \left[ \exp \left( \frac{\beta^2}{2} \int_0^t |Y_s^N|^2 \, ds \right) \right] \leq \exp \left( \frac{1}{2} \left( \beta^2 mt + |Y_0^N|^2 \right) \right).\]

We fix \( \vartheta \in [0, \frac{\pi}{2}], Y_0^T X, Y_0^N \). By Theorem 11.11, and by (11.199), (11.203), (11.205), and (11.207), we conclude that given \( p \geq 1 \), for \( b > 0 \) small enough, \( \sup_{0 \leq t \leq M} |L_{b, \vartheta, t}| \) and \( \sup_{0 \leq t \leq M} |L_{b, \vartheta, t}^{-1}| \) are uniformly bounded in \( L_p \).

Recall that the constant \( \delta_0^\vartheta \) was defined in (4.133), and is given by (4.134). Also \( H_{0, \vartheta, \cdot} \) was defined in equation (11.75).

Now we establish an extension of [B11, Theorem 14.10.6].

**Theorem 11.19.** Given \( \vartheta \in [0, \frac{\pi}{2}], M > 0 \), as \( b \to 0 \), \( U_{b, \vartheta, \cdot}^0 P \) converges uniformly on \( [0, M] \) to \( \exp (\delta_0^\vartheta) H_{0, \vartheta, \cdot} P \) in probability.

**Proof.** In the proof, \( \vartheta \in [0, \frac{\pi}{2}] \) will be fixed. We start from equation (11.161), that will be combined with equations (11.201) and (11.202). Let \( A_{b, \vartheta, \cdot} \) be the stochastic process,

\[(11.208)\quad A_{b, \vartheta, \cdot} = U_{b, \vartheta, \cdot}^0 \left( 1 + b \left( 1 + N_{-\vartheta}^\Lambda (T^* X \oplus N^*) \right) R_0^\vartheta (Y^T X) \right. \]

\[+ b \left( \cos (\vartheta) + N_{-\vartheta}^\Lambda (T^* X \oplus N^*) \right)^{-1} R_0^\vartheta (Y^N) + b \cos^{3/2} (\vartheta) \hat{\sigma} (iad \, (Y^N | TX)) \left( L_{b, \vartheta, \cdot}^{-1} P \right).\]
Using (11.21), (11.28), (11.161), and (11.201), we get

\[ A_{b,\vartheta,t} = A_{b,\vartheta,0} + \int_0^t U_{b,\vartheta,0}^0 \left[ -\mathcal{S}_\varrho (Y) + R_\varrho^0 (Y) \left( 1 + N_{-\varrho}^\Lambda (T^*X \otimes N^*) \right)^{-1} R_\varrho^0 \left( Y^{TX} \right) \\
+ \left( \cos (\vartheta) + N_{-\varrho}^\Lambda (T^*X \otimes N^*) \right)^{-1} R_\varrho^0 (Y^{-}) \right] ds + \left( 1 + N_{-\varrho}^\Lambda (T^*X \otimes N^*) \right)^{-1} R_\varrho^0 (dw^{TX}) \]

\[ + \left( \left( 1 + N_{-\varrho}^\Lambda (T^*X \otimes N^*) \right)^{-1} R_\varrho^0 (Y^{TX}) + \left( \cos (\vartheta) + N_{-\varrho}^\Lambda (T^*X \otimes N^*) \right)^{-1} R_\varrho^0 (Y^{-}) \right) \right] ds + \cos^2 (\vartheta) \hat{\alpha} (iad (dw^N) |_{TX}) \]

\[ + \cos^2 (\vartheta) \hat{\alpha} (iad (dw^N) |_{TX}) \right] L_{b,\vartheta}^{-1} \mathbf{P}. \]

In (11.209), the stochastic integrals with respect to \( w^{TX}, w^N \) are standard Itô integrals, i.e., \( dw^{TX}, dw^N \) can be replaced by \( \delta w^{TX}, \delta w^N \).

We denote by \( B_{b,\vartheta,t}, B_{b,\vartheta,t}^\perp \) be obtained from \( A_{b,\vartheta,t} - A_{b,\vartheta,0} \) in (11.209) by replacing \( U_{b,\vartheta,0}^0, U_{b,\vartheta,0}^0 \mathbf{P}, U_{b,\vartheta,0}^0 \mathbf{P}^\perp \), so that

\[ A_{b,\vartheta,t} = A_{b,\vartheta,0} + B_{b,\vartheta,t} + B_{b,\vartheta,t}^\perp. \]

By equation (4.103) in Proposition 4.27, by equation (4.127) in Proposition 4.35, by (4.132), and (11.209), we get

\[ B_{b,\vartheta,t} = \int_0^t U_{b,\vartheta,0}^0 \left[ -\cos^4 (\vartheta) \hat{\alpha} (iad (Y^{-}) |_{TX})^2 ds - \cos^2 (\vartheta) \hat{\alpha} (iad (dw^N) |_{TX}) \right] + \cos^4 (\vartheta) \hat{\alpha} (iad (Y^{-}) |_{TX})^2 ds + \cos^2 (\vartheta) \hat{\alpha} (iad (dw^N) |_{TX}) \right] L_{b,\vartheta}^{-1} \]

By (11.211), we deduce that

\[ B_{b,\vartheta,t} = 0. \]

By equation (11.150) in Theorem 11.14, and by the considerations following equation (11.207), by proceeding as in [B11, Proposition 14.10.4], as \( b \to 0 \), we have the uniform convergence over \( [0,M] \) in probability,

\[ B_{b,\vartheta}^\perp \to 0. \]

By (11.210), (11.212), and (11.213), we get the uniform convergence over \([0,M]\) in probability,

\[ A_{b,\vartheta} - A_{b,\vartheta,0} \to 0. \]
By [B11, Proposition 14.10.1], as $b \to 0$, $bY$ converges uniformly on $[0, M]$ to $0$ in probability. By Theorem 11.17 and by the considerations we made after (11.207) on the process $L_{b, \vartheta}$, we deduce from (11.208), (11.214) that we have the uniform convergence over $[0, M]$ in probability,

\[(11.215)\]

\[U_{b, \vartheta}^{-1}P \to P.\]

By the considerations we made after (11.207), we deduce from (11.215) that as $b \to 0$, we have the uniform convergence in probability over $[0, M]$,

\[(11.216)\]

\[U_{b, \vartheta}^{-1}P - L_{b, \vartheta}, P \to 0.\]

By [B11, Proposition 14.10.1] and by (4.135), as $b \to 0$, the process $\int_0^s \delta(\vartheta) \, Y \, ds$ converges uniformly on $[0, M]$ to the process $\delta_\vartheta^t$ in probability. Using Theorem 11.11, we conclude that as $b \to 0$, the process $L_{b, \vartheta}$ converges to $\exp(\delta_\vartheta^t)H_{0, \vartheta}$ uniformly on $[0, M]$ in probability. Our theorem follows from (11.216) and from this last result.

We now give an extension of [B11, Theorem 14.10.7].

**Theorem 11.20.** Given $\vartheta \in \left[0, \frac{\pi}{2}\right]$, $0 < \epsilon \leq M$, as $b \to 0$, $U_{b, \vartheta}^{\epsilon}$ converges uniformly on $[\epsilon, M]$ to $\exp(\delta_\vartheta^t)H_{0, \vartheta}$ in probability.

**Proof.** This is a consequence of Theorems 11.14 and 11.19. \(\square\)

11.12. The case of $U_{b, \vartheta}$... There is an obvious analogue of Proposition 11.2 when replacing $M_{b, \vartheta}$ by $X_{b, \vartheta}|_{\vartheta=0}$. Recall that $U_{b, \vartheta}$ was defined in (11.25). Let $U_{b, \vartheta}$ be $U_{b, \vartheta}$, when $\rho^{\infty}$ is the trivial representation. Proposition 11.6 has a trivial extension to $U_{b, \vartheta}$. The presence of $\int_0^s \delta(\vartheta) \, ds$ introduces an extra factor $\frac{1}{\cos(\vartheta)}$ in whatever estimate we make of $U_{b, \vartheta}$, with respect to the estimates we made for $U_{b, \vartheta}$. Indeed the term $-\frac{d\delta}{\sqrt{2}} \left(\hat{c}(Y^{TX}) + \frac{\sin(\vartheta)}{\cos^{3/2}(\vartheta)}i\epsilon N\right)$ just introduces a factor $1/\cos^{1/2}(\vartheta)$. The presence of $\exp\left(-\frac{\cos(\vartheta)}{2} \int_0^s |[Y^{TX}]|^2 \, ds\right)$ in the right-hand side of (11.23) allows us to dominate the contribution of $\frac{d\delta}{\sqrt{2}} \cos^{3/2}(\vartheta) i\epsilon \left([Y^{TX}]ight)$ with an extra factor $1/\cos(\vartheta)$. Equation (11.64) for $H_{b, \vartheta}$ should be replaced by

\[(11.217)\]

\[\frac{dH_{b, \vartheta}}{ds} = H_{b, \vartheta} \frac{1}{b} \left(\cos^{3/2}(\vartheta) \, \hat{c}(\text{iad}(Y^{TX})|_{TX}) - \frac{d\vartheta}{\sqrt{2}} \hat{c}(Y^{TX})\right), \quad H_{b, \vartheta, 0} = 1.\]

Equation (11.217) can be easily integrated by the formula

\[(11.218)\]

\[H_{b, \vartheta, s} = H_{b, \vartheta, s} - \frac{1}{b} \int_0^s H_{b, \vartheta, u} \frac{d\vartheta}{\sqrt{2}} \hat{c}(Y^{TX}) H_{b, \vartheta, u} du H_{b, \vartheta, s}.\]

Using (11.21), (11.64), equation (11.218) can be rewritten in the form,

\[(11.219)\]

\[H_{b, \vartheta, s} = H_{b, \vartheta, s} + \text{dadb} H_{b, \vartheta, s} \hat{c}(Y^{TX}_s) - d\vartheta \hat{c}(Y^{TX}_0) H_{b, \vartheta, s}\]

\[+ d\vartheta \int_0^s H_{b, \vartheta, u} \cos^{3/2}(\vartheta) \, \hat{c}(\text{iad}(Y^{TX}_u, Y^{TX}_u)) H_{b, \vartheta, u} du H_{b, \vartheta, s}\]

\[\quad - d\vartheta \int_0^s H_{b, \vartheta, u} \hat{c}(\text{dW}^{TX}_u) H_{b, \vartheta, u} H_{b, \vartheta, s}.\]
By equations (9.35) and (11.219), we can estimate \( H_{b,\vartheta}^0 \), using the previous estimates on \( H_{b,\vartheta} \).

Let \( H_{0,\vartheta}^0 \) be the solution of the stochastic differential equation,

\[
dH_{0,\vartheta}^0 = H_{0,\vartheta}^0 \left( \cos^2 (\vartheta) \hat{\varphi} \right) \left( \frac{-d\vartheta}{\sqrt{2}} \right),
\]

\( H_{0,\vartheta}^0 = 1 \).

The obvious extensions of Theorem 11.11, of Proposition 11.12, of Theorems 11.13 and 11.14, of Proposition 11.15, and of Theorem 11.17 still hold.

Now we state an extension of Theorems 11.19 and 11.20.

**Theorem 11.21.** Given \( \vartheta \in [0, \frac{\pi}{2}] \), \( M > 0 \), as \( b \to 0 \), \( U_{0,\vartheta}^0 \) converges uniformly on \([0, M]\) to \( \exp (\delta_{0,\vartheta}^0) H_{0,\vartheta}^0 \) in probability.

**Proof.** The proof is essentially the same as the proof of Theorems 11.19 and 11.20. \( \Box \)

11.13. **The final steps in the proof of Theorem 6.3.**

Recall that the smooth kernels \( \tilde{\eta}_{b,\vartheta,t}^X, \tilde{\eta}_{0,\vartheta,t}^X \) were defined in Definition 6.2. We will give an extension of [B11, Theorem 14.11.2].

**Theorem 11.22.** Let \( s \in C^{\infty, \epsilon} \left( \tilde{\mathcal{X}}, \tilde{\pi}^* \left( \Lambda (T^* X \oplus N^*) \otimes S^{TX} \otimes F \right) \right) \). Given \( \vartheta \in [0, \frac{\pi}{2}] \), \( t > 0 \), as \( b \to 0 \),

\[
\int_{\tilde{\mathcal{X}}} \tilde{\eta}_{b,\vartheta,t}^X ((x, Y), (x', Y')) s (x', Y') \, dx' \, dY' \to \int_{\tilde{\mathcal{X}}} \tilde{\eta}_{0,\vartheta,t}^X ((x, Y), (x', Y')) s (x', Y') \, dx' \, dY'.
\]

**Proof.** First, we prove equation (11.221) when making \( d\vartheta = 0 \). In the sequel, we fix \( \vartheta \in [0, \frac{\pi}{2}] \), \( t > 0 \). Set \( x = \cos (\vartheta) \). By Proposition 11.1, by equations (11.17), (11.14), by Theorem 11.5, and by (11.30), (11.52), we only need to prove that given \( \vartheta \in [0, \frac{\pi}{2}] \), \( t > 0 \), as \( b \to 0 \),

\[
E^Q \left[ \exp \left( \frac{x^2}{2} \int_0^t \left[ \left| Y_b^N, Y_b^TX \right| \right]^2 \, ds \right) U_{b,\vartheta,t}^0 \otimes E_{b,\vartheta,t}^{\cos^1/2(\vartheta),t} \tilde{\eta}_{b,t}^s (x_b, Y_b) \right] 
\]

\[
\to \exp \left( -x^2 \left( \frac{1}{4} \text{Tr}^X \left[ C_{t,p} \right] + 2 \sum_{i=m+1}^{m+n} \hat{\varphi} (\text{ad} (e_i) \text{Tr}^X) \rho^F (e_i) \right) t \right)
\]

\[
- \left( \frac{1}{48} \text{Tr}^X \left[ C_{t,F} \right] + \frac{1}{2} C_{t,F} \right) t \right)
\]

\[
E^Q \left[ \frac{x^2}{2} \right. \int_{(TX \oplus N)_{x_0,t}} \left. \text{Ps} (x_{0,t}, Y) \exp \left( -|Y|^2 \right) \frac{dY}{\pi^{m+n/2}} \right],
\]

where in the left-hand side, \( (x_b, Y_b) \) verifies (11.21), and where in the right-hand side, \( x_0 \), verifies (11.1). The dependence of these processes on \( \vartheta \) will not be denoted explicitly.
By the ergodic theory arguments explained in [B11, Proposition 14.10.1], for any 
$M > 0$, the process $t \in \mathbb{R}_+ \to \int_0^t \left| \left[ Y_{b,s}, Y_{b,s}^{TX} \right] \right|^2 ds$ converges uniformly on $[0, M]$ to 
t $t \in \mathbb{R}_+ \to - \frac{3}{4} \text{Tr}^p \left[ C^p \right] t$ in probability. By Theorem 11.17, given $p > 2$, for $b > 0$
small enough, $\sup_{0 \leq t \leq M} \left| U^0_{b,0,t} \right|$ remains uniformly bounded in all the $L_p, 1 \leq p < +\infty$. By Theorem 11.8, $\sup_{0 \leq t \leq M} \left| E_{b/ \text{cos } 2(\theta), t} \right|$ remains also uniformly bounded in all the $L_p, 1 \leq p < +\infty$.

Combining Theorems 11.8, 11.11, and 11.20 with the above considerations, we find that as $b \to 0$, then

$$
(11.223) \quad E^Q \left[ \exp \left( -\frac{X}{2} \int_0^t \left| \left[ Y_{b,s}^N, Y_{b,s}^{TX} \right] \right|^2 ds \right) U^0_{b,0,t} \otimes E_{b/ \text{cos } 2(\theta), t} r^t_0 s (x_{b,t}, Y_t) \right] 
- \exp \left( \left( \frac{X}{8} \text{Tr}^p \left[ C^p \right] + \delta_0^q \right) t \right) E^Q \left[ H_{b,0,t} \otimes E_{b/ \text{cos } 2(\theta), t} r^t_0 \mathbb{P}s (x_{b,t}, Y_t) \right] \to 0.
$$

We are now ready to use the full strength of [B11, Theorem 12.8.1 and Remark 12.8.2], and more specifically [B11, eq. (12.8.47)], from which we deduce that as $b \to 0$,

$$
(11.224) \quad E^Q \left[ H_{b,0,t} \otimes E_{b/ \text{cos } 2(\theta), t} r^t_0 \mathbb{P}s (x_{b,t}, Y_t) \right] 
\to E^Q \left[ H_{0,0,t} \otimes E_{0,0} r^t_0 \int_{(TX \otimes N)_{x_0,t}} \mathbb{P}s (x_{0,t}, Y) \exp \left( -|Y|^2 \right) \frac{dY}{\pi^{m+n/2}} \right].
$$

Note that $x_{0,\cdot}$, which only depends on $w^{TX}$, and $H_{0,0} \otimes E_{0,\cdot}$, which only depends on $w^N$, are independent, so that

$$
(11.225) \quad E^Q \left[ H_{0,0,t} \otimes E_{0,t} \int_{(TX \otimes N)_{x_0,t}} \mathbb{P}s (x_{0,t}, Y) \exp \left( -|Y|^2 \right) dY \right] 
= E^Q \left[ H_{0,0,t} \otimes E_{0,t} \right] E^Q \left[ \int_{(TX \otimes N)_{x_0,t}} \mathbb{P}s (x_{0,t}, Y) \exp \left( -|Y|^2 \right) dY \right].
$$

From the stochastic differential equations (11.53), (11.75), using Itô’s formula, we get

$$
(11.226) \quad d (H_{0,0} \otimes E_0) = (H_{0,0} \otimes E_0) \frac{1}{2} \left( -x^4 \sum_{m+1}^{m+n} \bar{c} (\text{ad} (e_i) | p)^2 - \sum_{m+1}^{m+n} \rho^E (e_i)^2 
- 2x^2 \sum_{m+1}^{m+n} \bar{c} (\text{ad} (e_i) | p) \rho^E (e_i) \right) + (H_{0,0} \otimes E_0) \left( -x^2 \bar{c} (i \text{ad} (\delta w^t)) - \rho^E (i \delta w^t) \right).
$$

By (11.226), we deduce that

$$
(11.227) \quad dE^Q \left[ (H_{0,0} \otimes E_0) \right] = E^Q \left[ (H_{0,0} \otimes E_0) \right] \frac{1}{2} \left( -x^4 \sum_{m+1}^{m+n} \bar{c} (\text{ad} (e_i) | p)^2 - \sum_{m+1}^{m+n} \rho^E (e_i)^2 
- 2x^2 \sum_{m+1}^{m+n} \bar{c} (\text{ad} (e_i) | p) \rho^E (e_i) \right).
$$
By (3.40), (3.41), and (11.227), we conclude that

\[(11.228) \quad E^Q [ (H_{0,\vartheta,t} \otimes E_{0,t}) ]
= \exp \left( -\frac{1}{2} \left( \frac{x^4}{8} \text{Tr}^p \left[ C^t p \right] + C^t E + 2x^2 \sum_{m+1}^{m+n} \tilde{c} (\text{ad} (e_i) | \pi) \rho^E (e_i) \right) t \right). \]

By (4.134), (11.223)–(11.225), and (11.228), we get (11.222).

Now we will consider the \(d\vartheta \) component of (11.221). Using (11.6), (11.23), the considerations that follow (11.25), and (11.38), to establish (11.222), we have to prove that as \( b \rightarrow 0 \),

\[(11.229) \quad E^Q \left[ \exp \left( \frac{x}{2} \int_0^t \left[ Y_b^N, Y_b^T X \right] \| ds \right) U_{b,\vartheta,t} \otimes E_{b/\cos^{1/2} (\vartheta)},t \tau_0^s (x_{b,t}, Y_{b,t}) \]
\rightarrow \exp \left( -\left( \frac{x^2}{8} \text{Tr}^p \left[ C^t p \right] + \frac{1}{48} \text{Tr}^f \left[ C^t f \right] + \frac{1}{2} C^t F \right) t \right)
\]

\[E^Q \left[ A_t \tau_0^s \int_{(TX \otimes N)_{x_{0,t}}} P^s (x_{0,t}, Y) \exp \left( -\| Y \|^2 \right) \frac{dY}{\pi^{(m+n)/2}} \right]. \]

By Theorem 11.21, in the extension of (11.223), we have to replace \( H_{b,\vartheta,t} \) by \( H'_{b,\vartheta,t} \). In (11.224) we should replace \( H_{0,\vartheta,t} \) by \( H'_{0,\vartheta,t} \).

Let \( E^Q w^p \) denote conditional expectation with respect to \( w^p \). Set

\[(11.230) \quad D_{\vartheta,} = E^Q,w^p [ H'_{0,\vartheta,.} \otimes E_{0,.} ]. \]

Since \( x_{0,.} \) depends only on \( w^p \), instead of (11.225), we get

\[(11.231) \quad E^Q \left[ H'_{0,\vartheta,t} \otimes E_{0,t} \int_{(TX \otimes N)_{x_{0,t}}} P^s \tau_0^s (x_{0,t}, Y) \exp \left( -\| Y \|^2 \right) dY \right]
= E^Q \left[ D_{\vartheta,t} \int_{(TX \otimes N)_{x_{0,t}}} P^s \tau_0^s (x_{0,t}, Y) \exp \left( -\| Y \|^2 \right) dY \right]. \]

By (11.53), (11.220), instead of (11.226), we get

\[(11.232) \quad d (H'_{0,\vartheta} \otimes E_0) = \left( H'_{0,\vartheta} \otimes E_0 \right) \frac{1}{2} \left( -x^4 \sum_{m+1}^{m+n} \tilde{c} (\text{ad} (e_i) | \pi) \rho^E (e_i)^2 \right.
- 2x^2 \sum_{m+1}^{m+n} \tilde{c} (\text{ad} (e_i) | \pi) \rho^E (e_i) 
+ \left( H'_{0,\vartheta} \otimes E_0 \right) \left( -x^2 \tilde{c} (\text{ad} (i\delta w^t)) - \rho^E (i\delta w^t) - \frac{d\vartheta}{\sqrt{2}} \tilde{c} \left( dw^p \right) \right). \]

By (11.230), we get

\[(11.233) \quad dD_{\vartheta} = D_{\vartheta} \frac{1}{2} \left( -x^4 \sum_{m+1}^{m+n} \tilde{c} (\text{ad} (e_i) | \pi) \rho^E (e_i)^2 \right.
- 2x^2 \sum_{m+1}^{m+n} \tilde{c} (\text{ad} (e_i) | \pi) \rho^E (e_i) 
- \left. \frac{d\vartheta}{\sqrt{2}} \tilde{c} \left( dw^p \right) \right). \]
Using the same arguments as in (11.228), and comparing with (11.5), we deduce from (11.233) that

\[
D_{\vartheta,t} = \exp \left( -\frac{t}{2} \left( \frac{x^4}{8} \text{Tr} [C^t,p] + C^t,E \right) \right) A_t.
\]

By (4.134), by the analogue of (11.223), (11.224), and using (11.231), (11.234), we get (11.229). The proof of our theorem is completed. \(\square\)

Given what we did before, we are now ready to indicate the main steps of the proof of Theorem 6.3, by properly referring to \([B11, \text{chapters 12–14}]\) when necessary.

In Theorem 10.3, we gave a complete proof of the corresponding estimates for the scalar heat kernel \(r_{X,b,\vartheta,t}^X\) over \(\tilde{X}\) and of the corresponding convergence result as \(b \to 0\).

For the moment, we make \(d\vartheta = 0\).

As in \([B11]\), the idea is to proceed in two steps:

1. In a first step, we obtain rough estimates on \(q_{X,b,\vartheta,t}^X\) and its derivatives in the considered range of parameters, by using the proper version of the Malliavin calculus, and we prove the convergence in (6.6).
2. In a second step, using the uniform bounds (11.55) in Theorem 11.8, (11.86) in Theorem 11.13, and the uniform bounds of the first step, we obtain the uniform estimates in (6.5).

The main difficulty is the presence of \(U_{b,\vartheta,t} = U_{0,\vartheta,t} \otimes E_{b,\vartheta,t}\) in the right-hand side of (11.20), (11.23).

By (11.7), (11.14), we get

\[
\begin{align*}
L_{X,b,\vartheta,-} &= \exp \left( -|Y|^2 / 2 \right) \mathcal{M}_{b,\vartheta,-} \exp \left( |Y|^2 / 2 \right), \\
\exp \left( -t L_{X,b,\vartheta,-} \right) s(x_0,Y_0) &= \exp \left( -|Y|^2 / 2 \right) \\
E^Q \left[ \exp \left( -\cos(\vartheta) \frac{1}{2} \int_0^t \left[ |Y^N,Y^T,X| \right]^2 ds \right) U_{b,\vartheta,t} \tau_b s(x_t,Y_t) \exp \left( |Y|^2 / 2 \right) \right].
\end{align*}
\]

An important observation is that contrary to what we did in \([B11, \text{sections 14.7 and 14.8}]\) for the kernel \(\tilde{q}_{b,t}^X\), it is not possible to deduce the uniform pointwise estimate (6.5) from the corresponding estimate (10.12) for the scalar kernel \(q_{b,\vartheta,t}^X\).

In \([B11]\), a simple proof was possible because one could obtain a simple pointwise estimate for \(U_{b,\vartheta,t}^0\), which is not the case here when \(\vartheta > 0\). So we start from (11.236), apply the Malliavin calculus, and use also the uniform bound (11.55) in Theorem 11.8 for \(E_{b,\vartheta,t}\), and the estimate (11.86) in Theorem 11.13 for \(U_{b,\vartheta,t}^0\). This last estimate, that provides the required uniformity in \(\vartheta \in \left[ 0, \frac{\pi}{2} \right]\) plays a critical role. Ultimately, given \(0 < \epsilon \leq M < +\infty\), we obtain the existence of \(C > 0, k \in \mathbb{N}\) such that for \(0 < b \leq 1, \vartheta \in \left[ 0, \frac{\pi}{2} \right], \epsilon \leq t \leq M\),

\[
\left| \tilde{q}_{b,\vartheta,t}^X ((x,Y), (x',Y')) \right| \leq C \cos^{-k}(\vartheta),
\]

Also, we obtain corresponding bounds for the derivatives of arbitrary order of \(\tilde{q}_{b,\vartheta,t}^X ((x,Y), (x',Y'))\) in \((x',Y')\).
In a second step, we use the semigroup property of \( \exp\left(-tL_{b,\vartheta,t}^X\right) \), and we get an analogue of (9.142). From this analogue, we obtain

\[
(11.238) \quad \left| \frac{\partial}{\partial t} \varphi_{b,\vartheta,t}^X ((x,Y),(x',Y')) \right| \leq \int_{\hat{X}} \left| \varphi_{b,\vartheta,t/2}^X ((x,Y),(z,Z)) \right| \left| \frac{\partial}{\partial t} \varphi_{b,\vartheta,t/2}^X ((z,Z),(x',Y')) \right| dzdZ.
\]

By (11.237), (11.238), we get the obvious extension of (9.143),

\[
(11.239) \quad \left| \varphi_{b,\vartheta,t}^X ((x,Y),(x',Y')) \right| \leq C \cos^{-k} (\vartheta) \int_{\hat{X}} \left| \varphi_{b,\vartheta,t/2}^X ((x,Y),(z,Z)) \right| dzdZ.
\]

By (11.236), we get

\[
(11.240) \quad \int_{\hat{X}} \left| \varphi_{b,\vartheta,t/2}^X ((x,Y),(z,Z)) \right| dzdZ \leq C \exp \left(-|Y|^2 / 2\right) E^{Q} \left[ U_{b,\vartheta,t} \exp \left(|Y|^2 / 2\right) \right].
\]

Using again the uniform bounds (11.55), (11.86), we deduce from (11.239), (11.240) that

\[
(11.241) \quad \left| \varphi_{b,\vartheta,t}^X ((x,Y),(x',Y')) \right| \leq C \cos^{-k} (\vartheta) \exp \left(-C' \left(|Y|^2 X + \cos (\vartheta) |Y| N^2 \right) \right).
\]

The \( L_2 \) formal adjoint of \( L_{b,\vartheta,t}^X \) has the same structure as \( L_{b,\vartheta,t}^X \). By exchanging the roles of \((x,Y)\) and \((x',Y')\), we obtain the analogue of (11.241),

\[
(11.242) \quad \left| \varphi_{b,\vartheta,t}^X ((x,Y),(x',Y')) \right| \leq C \cos^{-k} (\vartheta) \exp \left(-C' \left(|Y|^2 X + \cos (\vartheta) |Y| N^2 \right) \right).
\]

From (11.237), (11.238), we get the obvious analogue of (9.148),

\[
(11.243) \quad \left| \varphi_{b,\vartheta,t}^X ((x,Y),(x',Y')) \right|
\leq C \cos^{-k} (\vartheta) \int_{d(x,z) \geq d(x,x')/2} \left| \varphi_{b,\vartheta,t/2}^X ((x,Y),(z,Z)) \right| dzdZ
\]

\[+ C \cos^{-k} (\vartheta) \int_{d(x',z) \geq d(x,x')/2} \left| \varphi_{b,\vartheta,t/2}^X ((z,Z),(x',Y^X_t)) \right| dzdZ.
\]

By (11.236), instead of (9.149), we get

\[
(11.244) \quad \int_{d(x,z) \geq d(x,x')/2} \left| \varphi_{b,\vartheta,t/2}^X ((x,Y),(z,Z)) \right| dzdZ
\leq \exp \left(-|Y|^2 / 2\right) E^{Q} \left[ 1_{\sup_{0 \leq s \leq t/2} d(x,x_s) \geq d(x,x')/2} U_{b,\vartheta,t} \exp \left(|Y|^2 / 2\right) \right].
\]

It is now crucial to observe that in (11.21), \((x,Y^X_t)\) has the same probability law as \( \left|x \cos (\vartheta) \right| Y_{\cos (\vartheta)}^{Xt} \) in (9.60), when replacing \( b \) by \( b \cos (\vartheta) \). Similarly, \( Y^N_t \) in (11.21) has the same probability law as \( Y_{\cos (\vartheta)}^{\vartheta/b^2} \) in (9.9) (with \( E = N \)). In equations (10.3) and (10.18), we already used a similar argument. By equation (9.6) (used with \( t \) replaced by \( \cos (\vartheta) t/b^2 \)), and by equation (9.75) in Theorem 9.20, if
0 < b ≤ 1, ϑ ∈ [0, π/2], ϵ ≤ t ≤ M, we get
\begin{equation}
\exp\left(-|Y|^{2}/2\right) E^{Q}\left[\mathbf{1}_{\sup_{b_{t}≤t≤t^{1/2}} d(x,x') ≥ d(x,x')/2} \exp\left(\left|Y_{t/2}\right|^{2}\right)\right]
\leq C \exp\left(-c' \left(\frac{d^{2}(x,x')}{\cos^{2}(\vartheta)} + |Y^{TX}|^{2} + \cos(\vartheta)|Y|^{N}|^{2}\right)\right).
\end{equation}

By using equation (11.86) in Theorem 11.13, (11.245), and Hölder’s inequality, in the considered range of parameters, we get
\begin{equation}
\exp\left(-|Y|^{2}/2\right) E^{Q}\left[\mathbf{1}_{\sup_{b_{t}≤t≤t^{1/2}} d(x,x) ≥ d(x,x')/2} |U_{b,\vartheta,t}| \exp\left(\left|Y_{t/2}\right|^{2}\right)\right]
\leq C \exp\left(-c' \left(\frac{d^{2}(x,x')}{\cos^{2}(\vartheta)} + |Y^{TX}|^{2} + \cos(\vartheta)|Y|^{N}|^{2}\right)\right).
\end{equation}

By (11.244), (11.246), we obtain
\begin{equation}
\int_{(x,z)∈\bar{X}} \left|\tilde{q}_{b,\vartheta,t/2}^{X}((x,Y'),(z,Z))\right| dzdZ
\leq C \exp\left(-c' \left(\frac{d^{2}(x,x')}{\cos^{2}(\vartheta)} + |Y^{TX}|^{2} + \cos(\vartheta)|Y|^{N}|^{2}\right)\right).
\end{equation}

Still using the properties of the formal adjoint of \(\tilde{T}_{b,\vartheta,\cdot,\cdot}^{X}\), the same arguments as before give
\begin{equation}
\int_{(x,z)∈\bar{X}} \left|\tilde{q}_{b,\vartheta,t/2}^{X}((z,Z),(x',Y'))\right| dzdZ
\leq C \exp\left(-c' \left(\frac{d^{2}(x,x')}{\cos^{2}(\vartheta)} + |Y^{TX}|^{2} + \cos(\vartheta)|Y|^{N}|^{2}\right)\right).
\end{equation}

By (11.243), (11.247), and (11.248), we get
\begin{equation}
\left|\tilde{q}_{b,\vartheta,t}^{X}((x,Y),(x',Y'))\right| \leq C \cos^{-k}(\vartheta) \exp\left(-c' \frac{d^{2}(x,x')}{\cos^{2}(\vartheta)}\right).
\end{equation}

By combining (11.241), (11.242), and (11.249), we get the estimate (6.5), at least when making \(d\bar{\vartheta} = 0\). The proof of this estimate when \(d\bar{\vartheta}\) is not equal to 0 is exactly the same, given the estimates we obtained on \(U_{b,\bar{\vartheta},\cdot,\cdot}^{0}\).

Let us now establish the convergence result in equation (6.6). As was explained before, the methods of the Malliavin calculus show that given \(0 < b ≤ 1, \vartheta ∈ [0, \pi/2]\), \((x,Y) \in \bar{X}, \tilde{q}_{b,\vartheta,t}^{X}((x,Y),(x',Y'))\) and its derivatives of arbitrary order in \((x',Y')\) are uniformly bounded over compact subsets of \(\bar{X}\). Note here that we can also obtain corresponding uniform bounds for these derivatives that are similar to (6.5), but this plays no role in the sequel. Using Theorem 11.22, we get the convergence in (6.6). More precisely, the above convergence is uniform over compact subsets, and derivatives of arbitrary order converge as well.

This completes the proof of Theorem 6.3.
12. Estimates on the hypoelliptic heat kernel for \( b \) large

The purpose of this section is to establish Theorems 7.7 and 7.11, i.e., to obtain uniform estimates on the hypoelliptic heat kernels \( q_t^{X'} \) and \( q_t^{X_0} \) for large values of \( b \).

When \( \vartheta = 0 \), Theorem 7.7 coincides with [B11, Theorem 9.1.1]. We will briefly explain how to adapt the techniques used in the proof of [B11, Theorem 9.1.1] in order to establish Theorem 7.7.

The operator \( L_{b,\vartheta}^{X',t} \) is given by (7.28). The operators \( L_{b,0}^{X'} \) and \( L_{b/cos^{1/2}(\vartheta),0}^{X'} \) can be easily compared. The only substantial difference is that in the first operator, we have the term \( \frac{1}{2\cos^2(\vartheta)} |Y^{TX}|^2 \), while in the second operator, the smaller term \( \frac{1}{2} |Y^{TX}|^2 \) appears. The operator \( L_{b,0}^{X'} \) and its heat kernel were studied in detail in [B11, Theorem 9.1.1 and chapter 15] when \( b \to +\infty \). Here, the difference between \( L_{b,\vartheta}^{X',t} \) and \( L_{b/cos^{1/2}(\vartheta),0}^{X'} \) will play in our favour.

Put

\[
L_{b,\vartheta}^{X',t} = \frac{b^4}{2\cos^2(\vartheta)} \left[ |Y^N, Y^{TX}|^2 + \frac{1}{2} \left( -\frac{\cos^2(\vartheta)}{b^2} \Delta_{TX \oplus N} + \frac{1}{2} |Y^{TX}|^2 + |Y^N|^2 \right) \right.

- \frac{1}{b^2} (m + \cos(\vartheta) n) + \frac{N_{\vartheta}^A (T^{TX \oplus N} \varphi)}{b^2}

- \left( \nabla_{Y^{TX}} C^\infty (TX \oplus N \hat{\varphi} \Lambda (T^{TX \oplus N} \hat{\varphi}) \otimes T_{\vartheta} \otimes F) \right) f, \tilde{f}

- \frac{c (i\vartheta \text{ad} (Y^N)) - i \hat{c} (\text{ad} (Y^N) |_{TX}) - i \rho F (Y^N)}{b^2} \left].
\]

Up to terms that are irrelevant in the range of parameters \( b \geq 1, \vartheta \in [0, \frac{\pi}{2}] \), the operator \( L_{b,\vartheta}^{X',t} \) is now easily comparable to the operator \( L_{b,0}^{X',t} \).

For \( t > 0 \), let \( q_t^{X'}, ((x, Y), (x', Y')) \) be the smooth kernel associated with the operator \( \exp (-t L_{b,\vartheta}^{X',t}) \). We use the notation \( q_t^{X',0} = q_t^{X',1} \). It is easy to see that the estimates in [B11, Theorem 9.1.1] for \( q_t^{X',t} \) remain valid for \( q_t^{X',b,\vartheta} \) in the range \( b \geq 1, \vartheta \in [0, \frac{\pi}{2}] \).

Comparing (7.28) and (12.1), we get

\[
L_{b,\vartheta}^{X',t} = L_{b,\vartheta}^{X',t} + \frac{1}{2} \left( \frac{1}{\cos^2(\vartheta)} - \frac{1}{2} \right) |Y^{TX}|^2.
\]

Also

\[
\frac{1}{\cos^2(\vartheta)} - \frac{1}{2} \geq \frac{1}{2\cos^2(\vartheta)}.
\]

If \( \vartheta \in [0, \frac{\pi}{2}] \), let \( s \in [0, 1] \to x_s \in X \) be a \( C^1 \) path such that

\[
\dot{x}_s = Y_s^{TX}.
\]
If \( x_0 = x, x_t = x' \), using (12.3), we get

\[
\frac{1}{2} \left( \frac{1}{\cos^2(\vartheta)} - \frac{1}{2} \right) \int_0^t |Y^{-T}X|^2 \, ds \geq \frac{d^2(x, x')}{4 \cos^2(\vartheta) t}.
\]

From the probabilistic representation of the heat kernels \( q^{X'}_{b, \vartheta, t} \), using equation (12.5), up to fixed constants, in the range of parameters described before, an upper bound for \( |q^{X'}_{b, \vartheta, t}((x, Y), (x', Y'))| \) is the product of \( \exp \left( -\frac{d^2(x, x')}{4 \cos^2(\vartheta) t} \right) \) by the upper bounds for \( |\tilde{q}^{X'}_{b/\cos^{1/2}(\vartheta), t}((x, Y), (x', Y'))| \) that were obtained in [B11, Theorem 9.1.1]. Since \( d_\gamma(x) \geq |a| \), from these estimates, we get Theorem 7.7 except for the very last statement concerning vertical derivatives. However, the same arguments as in [B11, Theorem 12.11.2] permits us also to properly control the vertical derivatives.

By exactly the same arguments, we can derive Theorem 7.11 from [B11, Theorem 9.5.6]. More precisely, we obtain exactly the same estimates as in [B11] for the rescaled heat kernel associated with \( L^{X'}_{b, \vartheta} \). Equations (12.4), (12.5) are responsible for the appearance of \( |a|^2 / \cos^2(\vartheta) \) in the right-hand side of (7.60). As to replacing \( q^{X'}_{b, \vartheta} \) by \( \frac{\cos(\vartheta)}{a^2} \nabla^V q^{X'}_{b, \vartheta} \) in (7.60), this can be done by exactly the same arguments.
References

[AB67] M. F. Atiyah and R. Bott. A Lefschetz fixed point formula for elliptic complexes. I. Ann. of Math. (2), 86:374–407, 1967.

[AB68] M. F. Atiyah and R. Bott. A Lefschetz fixed point formula for elliptic complexes. II. Applications. Ann. of Math. (2), 88:451–491, 1968.

[BaGS85] W. Ballmann, M. Gromov, and V. Schroeder. Manifolds of nonpositive curvature, volume 61 of Progress in Mathematics. Birkhäuser Boston Inc., Boston, MA, 1985.

[B81] J.-M. Bismut. Mécánique aléatoire, volume 866 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1981. With an English summary.

[B05] J.-M. Bismut. The hypoelliptic Laplacian on the cotangent bundle. J. Amer. Math. Soc., 18(2):379–476 (electronic), 2005.

[B08] J.-M. Bismut. The hypoelliptic Dirac operator. In Geometry and dynamics of groups and spaces, volume 265 of Progr. Math., pages 113–246. Birkhäuser, Basel, 2008.

[B11] J.-M. Bismut. Hypoelliptic Laplacian and orbital integrals, volume 177 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2011.

[BF86] J.-M. Bismut and D.S. Freed. The analysis of elliptic families. II. Dirac operators, eta invariants, and the holonomy theorem. Comm. Math. Phys., 107(1):103–163, 1986.

[BL08] J.-M. Bismut and G. Lebeau. The hypoelliptic Laplacian and Ray-Singer metrics, volume 167 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2008.

[BuG70] D. L. Burkholder and R. F. Gundy. Extrapolation and interpolation of quasi-linear operators on martingales. Acta Math., 124:249–304, 1970.

[E96] P. B. Eberlein. Geometry of nonpositively curved manifolds. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1996.

[F86] D. Fried. Analytic torsion and closed geodesics on hyperbolic manifolds. Invent. Math., 84(3):523–540, 1986.

[GIJ87] J. Glimm and A. Jaffe. Quantum physics. Springer-Verlag, New York, second edition, 1987. A functional integral point of view.

[H78] S. Helgason. Differential geometry, Lie groups, and symmetric spaces, volume 80 of Pure and Applied Mathematics. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978.

[Hö6] L. Hörmander. Hypoelliptic second order differential equations. Acta Math., 119:147–171, 1967.

[IW89] N. Ikeda and S. Watanabe. Stochastic differential equations and diffusion processes, volume 24 of North-Holland Mathematical Library. North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, second edition, 1989.

[Kn86] A. W. Knapp. Representation theory of semisimple groups, volume 36 of Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1986. An overview based on examples.

[K76] B. Kostant. On Macdonald’s η-function formula, the Laplacian and generalized exponents. Advances in Math., 20(2):179–212, 1976.

[K97] B. Kostant. Clifford algebra analogue of the Hopf-Koszul-Samelson theorem, the ρ-decomposition C(g) = End Vρ ⊗ C(P), and the g-module structure of ∧ g. Adv. Math., 125(2):275–350, 1997.

[LG16] J.-F. Le Gall. Brownian motion, martingales, and stochastic calculus, volume 274 of Graduate Texts in Mathematics. Springer, [Cham], French edition, 2016.

[MS67] H. P. McKean, Jr. and I. M. Singer. Curvature and the eigenvalues of the Laplacian. J. Differential Geometry, 1(1):43–69, 1966.

[MI78] J. J. Millson. Closed geodesics and the η-invariant. Ann. of Math. (2), 108(1):1–39, 1978.

[MoSt89] H. Moscovici and R. J. Stanton. Eta invariants of Dirac operators on locally symmetric manifolds. Invent. Math., 95(3):629–666, 1989.

[MoSt91] H. Moscovici and R. J. Stanton. R-torsion and zeta functions for locally symmetric manifolds. Invent. Math., 105(1):185–216, 1991.

[Q85] D. Quillen. Superconnections and the Chern character. Topology, 24(1):89–95, 1985.

[Sh16a] S. Shen. Analytic torsion, dynamical zeta functions and orbital integrals. C. R. Math. Acad. Sci. Paris, 354(4):433–436, 2016.
[Sh16b] S. Shen. Analytic torsion, dynamical zeta functions, and the Fried conjecture. *Analysis & PDE*, 11(1):1–74, 2018.

[StrV79] D. W. Stroock and S. R. S. Varadhan. *Multidimensional diffusion processes*, volume 233 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1979.

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