CURVATURE OPERATOR OF HOLOMORPHIC VECTOR BUNDLES AND $L^2$-ESTIMATE CONDITION FOR $(n, q)$ AND $(p, n)$-FORMS

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ABSTRACT. We study the positivity properties of the curvature operator for holomorphic Hermitian vector bundles. The characterization of Nakano semi-positivity by $L^2$-estimate is already known. Applying our results, we give new characterizations of Nakano semi-negativity.

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1. INTRODUCTION

The aim of the present paper is to study the relation between the positivity properties of curvature operators for holomorphic Hermitian vector bundles and $L^2$-estimates by using the $(p, n)$-$L^2$-estimate condition (see Definition 1.4). This type of condition was firstly introduced in [H20], which was named as the twisted Hörmander condition. After that, in a paper [DNWZ20], Deng et al. generalized this concept as the optimal $L^p$-estimate condition and gave a new characterizations of Nakano semi-positivity by $L^2$-estimate. Here, Nakano positive is equivalent to the positivity of curvature operator for $(n, 1)$-forms. In this paper, by examining the properties of curvature operators, we extend this characterization in [DNWZ20] from $(n, 1)$-forms to $(n, q)$ and $(p, n)$-forms and obtain a characterizations of Nakano semi-negativity by $L^2$-estimates. Finally we obtain one definition of Nakano semi-negativity and dual Nakano semi-positivity for singular Hermitian metrics with $L^2$-estimates.

Let $(X, \omega)$ be a complex manifold of complex dimension $n$ equipped with a Hermitian metric $\omega$ and $(E, h)$ be a holomorphic Hermitian vector bundle of rank $r$ over $X$. Let $D = D' + \bar{\partial}$ be the Chern connection of $(E, h)$, and $\Theta_{E,h} = [D', \bar{\partial}] = D'\bar{\partial} + \bar{\partial}D'$.
be the Chern curvature tensor. Let \((U, (z_1, \cdots, z_n))\) be local coordinates. Denote by \((e_1, \cdots, e_r)\) an orthonormal frame of \(E\) over \(U \subset X\), and
\[
i \Theta_{E,h,x_0} = i \sum_{j,k,\lambda,\mu} c_{j,k\lambda\mu} dz_j \wedge d\bar{z}_k \otimes e^*_\lambda \otimes e_\mu, \quad c_{j,k\lambda\mu} = c_{k,j\mu\lambda}.
\]

To \(i \Theta_{E,h}\) corresponds a natural Hermitian form \(\theta_{E,h}\) on \(T_X \otimes E\) defined by
\[
\theta_{E,h}(u,u) = \sum c_{j,k\lambda\mu} u_j e_\lambda \otimes u_k \in T_{X,x_0} \otimes E_x,
\]
i.e. \(\theta_{E,h} = \sum c_{j,k\lambda\mu} (dz_j \otimes e^*_\lambda) \otimes (d\bar{z}_k \otimes e^*_\mu).

**Definition 1.1.** Let \(X\) be a complex manifold and \((E, h)\) be a holomorphic Hermitian vector bundle over \(X\).

- \((E, h)\) is said to be **Nakano positive** (resp. **Nakano semi-positive**) if \(\theta_{E,h}\) is positive (resp. semi-positive) definite as a Hermitian form on \(T_X \otimes E\), i.e. for any \(u \in T_X \otimes E, u \neq 0\), we have
  \[\theta_{E,h}(u,u) > 0 \quad (\text{resp.} \geq 0)\]
  We write \((E, h) >_{\text{Nak}} 0\) (resp. \(\geq_{\text{Nak}} 0\)) for Nakano positivity (resp. semi-negativity).

- \((E, h)\) is said to be **Griffiths positive** (resp. **Griffiths semi-positive**) if for any \(\xi \in T_{X,x}, \xi \neq 0\) and \(s \in E_x, s \neq 0\), we have
  \[\theta_{E,h}(\xi \otimes s, \xi \otimes s) > 0 \quad (\text{resp.} \geq 0)\]
  We write \((E, h) >_{\text{Griff}} 0\) (resp. \(\geq_{\text{Griff}} 0\)) for Griffiths positivity (resp. semi-negativity).

- **Nakano negative** (resp. **Nakano semi-negative**) and **Griffiths negative** (resp. **Griffiths semi-negative**) are similarly defined by replacing \(> 0\) (resp. \(\geq 0\)) by \(< 0\) (resp. \(\leq 0\)) in the above definitions respectively.

We now explain a few notions to state our results more precisely. Let \(\mathcal{E}^{p,q}(E)\) be the sheaf of germs of \(C^\infty\) sections of \(\Lambda^{p,q}T_X^* \otimes E\) and \(\mathcal{D}^{p,q}(E)\) be the space of \(C^\infty\) sections of \(\Lambda^{p,q}T_X^* \otimes E\) with compact support on \(X\). We say that \(\{(U_\alpha, \iota_\alpha)\}_\alpha\) is a local **Stein coordinate system** if any local coordinates \(\iota_\alpha : U_\alpha \to \iota(U_\alpha) \subset \mathbb{C}^n\) satisfies that \(\iota(U_\alpha)\) is Stein. By definition, every complex manifold always has a local Stein coordinate system.

**Definition 1.2.** (cf. [BP08], [Rau15] and [PT18]) Let \(X\) be a complex manifold and \(E\) be a holomorphic vector bundle over \(X\). We say that \(h\) is a **singular Hermitian metric** on \(E\) if \(h\) is a measurable map from the base manifold \(X\) to the space of non-negative Hermitian forms on the fibers satisfying \(0 < \det h < +\infty\) almost everywhere.

We already know that for a singular Hermitian metric, we cannot always define the curvature currents with measure coefficients (see [Rau15]). However, the following definition can be defined with the singular case by not using the curvature currents of a singular Hermitian metric directly.
Definition 1.3. Let $(X, \omega)$ be a Kähler manifold of dimension $n$ which admits a positive holomorphic Hermitian line bundle and $E$ be a holomorphic vector bundle over $X$ equipped with a (singular) Hermitian metric $h$. $(E, h)$ satisfies the $(n, q)$-$L^2$-estimate condition on $X$ for $q \geq 1$, if for any positive holomorphic Hermitian line bundle $(A, h_A)$ on $X$ and for any $f \in \mathcal{D}^{n,q}(X, E \otimes A)$ with $\overline{\partial} f = 0$, there is $u \in L^2_{n,q-1}(X, E \otimes A)$ satisfying $\overline{\partial} u = f$ and

$$\int_X |u|^2_{h \otimes h_A, \omega} dV \leq \int_X \langle [i\Theta_{A, h_A} \otimes \text{id}_E, \Lambda_{\omega}]^{-1} f, f \rangle_{h \otimes h_A, \omega} dV,$$

provided that the right hand side is finite.

And $(E, h)$ satisfies the $(n, q)$-$L^2$-estimate condition on $X$ if there exists a Kähler metric $\tilde{\omega}$ such that $(E, h)$ satisfies the $(n, q)$-$L^2$-estimate condition on $X$.

This definition is an extension of [DNWZ20, Definition 1.1] for $(n, 1)$-form to $(n, q)$-forms. Similarly, we define the following with respect to $(p, n)$-forms.

Definition 1.4. Let $(X, \omega)$ be a Kähler manifold of dimension $n$ which admits a positive holomorphic Hermitian line bundle and $E$ be a holomorphic vector bundle over $X$ equipped with a (singular) Hermitian metric $h$. $(E, h)$ satisfies the $(p, n)$-$L^2$-estimate condition on $X$ for $p \geq 0$, if for any positive holomorphic Hermitian line bundle $(A, h_A)$ on $X$ and for any $f \in \mathcal{D}^{p,n}(X, E \otimes A)$ with $\overline{\partial} f = 0$, there is $u \in L^2_{p,n-1}(X, E \otimes A)$ satisfying $\overline{\partial} u = f$ and

$$\int_X |u|^2_{h \otimes h_A, \omega} dV \leq \int_X \langle [i\Theta_{A, h_A} \otimes \text{id}_E, \Lambda_{\omega}]^{-1} f, f \rangle_{h \otimes h_A, \omega} dV,$$

provided that the right hand side is finite.

And $(E, h)$ satisfies the $(p, n)$-$L^2$-estimate condition on $X$ if there exists Kähler metric $\tilde{\omega}$ such that $(E, h)$ satisfies the $(p, n)$-$L^2$-estimate condition on $X$.

Let $(X, \omega)$ be a Hermitian manifold and $(E, h)$ be a holomorphic Hermitian vector bundle over $X$. We denote the curvature operator $[i\Theta_{E,h}, \Lambda_{\omega}]$ on $\Lambda^{p,q}T_X^* \otimes E$ by $A_{E,h,\omega}^{p,q}$. And the fact that the curvature operator $[i\Theta_{E,h}, \Lambda_{\omega}]$ is positive (resp. semi-positive) definite on $\Lambda^{p,q}T_X^* \otimes E$ is simply written as $A_{E,h,\omega}^{p,q} > 0$ (resp. $\geq 0$).

Let $(X, \omega)$ be a Kähler manifold of dimension $n$ which admits a positive holomorphic Hermitian line bundle. In [DNWZ20], by using fact that $(E, h)$ is Nakano semi-positive if and only if $A_{E,h,\omega}^{n,1} \geq 0$, they showed the following relationship between the $(n, 1)$-$L^2$-estimate condition and Nakano semi-positivity:

$(E, h)$ satisfies the $(n, 1)$-$L^2$-estimate condition $\implies$ $i\Theta_{E,h} \geq_{Nak} 0$, i.e. $A_{E,h,\omega}^{n,1} \geq 0$.

We have shown this result for the general case, the $(n, q)$ and $(p, n)$-$L^2$-estimate condition, focusing on $A_{E,h,\omega}^{n,q} \geq 0$ and $A_{E,h,\omega}^{p,n} \geq 0$ instead of $i\Theta_{E,h} \geq_{Nak} 0$.

Theorem 1.5. Let $(X, \omega)$ be a Kähler manifold of dimension $n$ which admits a positive holomorphic Hermitian line bundle and $(E, h)$ be a holomorphic Hermitian vector bundle over $X$ and $q$ be a positive integer. Then $(E, h)$ satisfies the $(n, q)$-$L^2$-estimate condition on $X$ if and only if $A_{E,h,\omega}^{n,q} \geq 0$.
**Theorem 1.6.** Let \((X, \omega)\) be a Kähler manifold of dimension \(n\) which admits a positive holomorphic Hermitian line bundle and \((E, h)\) be a holomorphic Hermitian vector bundle over \(X\) and \(p\) be a nonnegative integer. Then \((E, h)\) satisfies the \((p, n)\)-\(L^2\)-estimate condition on \(X\) if and only if \(A^{p,n}_{E,h,\omega} \geq 0\).

And, by studying the properties of the curvature operator, we obtain the following characterizations of Nakano semi-negativity using Theorem 1.6.

**Theorem 1.7.** Let \((X, \omega)\) be a Kähler manifold of dimension \(n\) which admits a positive holomorphic Hermitian line bundle and \((E, h)\) be a holomorphic Hermitian vector bundle over \(X\). Then \((E^*, h^*)\) satisfies the \((1,n)\)-\(L^2\)-estimate condition on \(X\) if and only if \((E, h)\) is Nakano semi-negative.

### 2. Properties of the curvature operator

In this section, we compute the value by the curvature operator in detail and study the properties related to the positivity of the curvature operator.

Let \((M, g)\) be an oriented Riemannian \(C^\infty\)-manifold with \(\dim \mathbb{R} M = m\). Let \(\xi_s\) be the interior product \(i_{\xi}\) for \(\xi \in T_M\). If \((\xi_1, \cdots , \xi_m)\) is an orthonormal basis of \((T_M, g)\) at \(x_0\), then for any ordered multi-index \(I = \{i_1, \cdots , i_p\}\) with \(i_1 < \cdots < i_p\) and \(|I| = p \in \{1, \cdots , m\}\), \(\bullet \cdot \bullet\) is the bilinear operator characterized by

\[
\xi_s \cdot (\xi_{i_1}^* \wedge \cdots \wedge \xi_{i_p}^*) = \begin{cases} 0 & \text{if } s \notin \{i_1, \cdots , i_p\} \\ (-1)^{k-1} \xi_{i_1}^* \wedge \cdots \wedge \xi_{i_k}^* \cdots \wedge \xi_{i_p}^* & \text{if } s = i_k. \end{cases}
\]

Therefore we introduce a symbol to represent this number \((-1)^{k-1}\) using \(s\) and \(I\).

**Definition 2.1.** Let \((M, g), (\xi_1, \cdots , \xi_m)\) and \(I\) be as in above. We define \(\varepsilon(s, I) \in \{-1, 0, 1\}\) as the number that satisfies \(\xi_s \cdot \xi_I^* = \varepsilon(s, I)\xi_I^*\), where if \(s \notin I\) then \(\varepsilon(s, I) = 0\) and if \(s \in I\) then \(\varepsilon(s, I) \in \{-1, 1\}\).

Let \((X, \omega)\) be a Hermitian manifold, \(\dim \mathbb{C} X = n\). If \((\partial / \partial z_1, \cdots , \partial / \partial z_n)\) is an orthonormal basis of \((T_X, \omega)\) at \(x_0\) then we define \(\varepsilon(s, I)\) in the same way as follows

\[
\frac{\partial}{\partial z_s} dz_I = \varepsilon(s, I) dz_I |_{x_0}. \quad \text{In particular, we have that } \frac{\partial}{\partial \bar{z}_s} dz_I = \varepsilon(s, I) d\bar{z}_I |_{x_0}.
\]

Using the symbols \(\varepsilon(s, I)\) in this definition, we obtain the following detailed calculation results.

**Proposition 2.2.** Let \((X, \omega)\) be a Hermitian manifold and \((E, h)\) be a holomorphic Hermitian vector bundle over \(X\). Let \(x_0 \in X\) and \((z_1, \cdots , z_n)\) be local coordinates such that \((\partial / \partial z_1, \cdots , \partial / \partial z_n)\) is an orthonormal basis of \((T_X, \omega)\) at \(x_0\). Let \((e_1, \cdots , e_r)\) be an orthonormal basis of \(E_{x_0}\). We can write

\[
\omega_{x_0} = i \sum_{1 \leq j \leq n} d z_j \wedge d \bar{z}_j, \quad i \Theta_{E,h,x_0} = i \sum_{j,k,\lambda,\mu} c_{j,k,\lambda,\mu} d z_j \wedge d \bar{z}_k \otimes e_\lambda \otimes e_\mu.
\]

Let \(J, K, L\) and \(M\) be ordered multi-indices with \(|J| = |L| = p\) and \(|K| = |M| = q\). For any \((p, q)\)-form \(u = \sum_{|J|=p,|K|=q} u_{J,K,\lambda} dz_J \wedge d \bar{z}_K \otimes e_\lambda \in \Lambda^{p,q} T_{x_0}^* X \otimes E_{x_0}\), we have the
following

$$
[i \Theta_{E,h}, \Lambda_{\omega}] u = \left( \sum_{j \in J} + \sum_{j \in K} - \sum_{1 \leq j \leq n} \right) c_{jj} \lambda u_{J, K, \lambda} d z_j \wedge d \bar{z}_K \otimes e_{\mu}
+ \sum_{K \not\supset j \neq \bar{K}} c_{jj} \lambda u_{J, K, \lambda} \epsilon(j, K) d z_j \wedge d \bar{z}_k \wedge d \bar{z}_{K \setminus j} \otimes e_{\mu}
+ \sum_{J \not\supset k \neq j \neq \bar{J}} c_{jj} \lambda u_{J, K, \lambda} \epsilon(k, J) d z_j \wedge d z_{J \setminus k} \wedge d \bar{z}_K \otimes e_{\mu}, \quad \text{and}

\langle [i \Theta_{E,h}, \Lambda_{\omega}] u, u \rangle_{\omega} = \left( \sum_{j \in J} + \sum_{j \in K} - \sum_{1 \leq j \leq n} \right) c_{jj} \lambda u_{J, K, \lambda} \bar{u}_{J, K, \mu}
+ \sum_{j \neq k, K \setminus j = M \setminus k} c_{jj} \lambda u_{J, K, \lambda} \bar{u}_{J, M, \mu} \epsilon(j, K) \epsilon(k, M)
+ \sum_{j \neq k, L \setminus j = \bar{L} \setminus k} c_{jj} \lambda u_{L, K, \lambda} \bar{u}_{L, K, \mu} \epsilon(k, J) \epsilon(j, L).

Proof. As is well known, we have that

$$
\Lambda_{\omega} u = i(-1)^p \sum_{J, K, \lambda, s} u_{J, K, \lambda} \left( \frac{\partial}{\partial z_s} \right) d z_j \wedge \left( \frac{\partial}{\partial \bar{z}_s} \right) d \bar{z}_K \otimes e_{\lambda}.
$$

Then a simple computation gives

$$
i \Theta_{E,h} \wedge \Lambda_{\omega} u = \sum_{j} c_{jj} \lambda u_{J, K, \lambda} d z_j \wedge \left( \frac{\partial}{\partial z_s} \right) d z_j \wedge d \bar{z}_k \wedge \left( \frac{\partial}{\partial \bar{z}_s} \right) d \bar{z}_K \otimes e_{\mu},
\Lambda_{\omega} \wedge i \Theta_{E,h} u = \sum_{j} c_{jj} \lambda u_{J, K, \lambda} \left( \frac{\partial}{\partial z_s} \right) d z_j \wedge d z_j \wedge \left( \frac{\partial}{\partial \bar{z}_s} \right) d \bar{z}_k \wedge d \bar{z}_K \otimes e_{\mu}.
$$

For simplicity, we calculate only the terms in differential form as follows

$$
\sum_{j} \sum_{s \neq j} d z_j \wedge \left( \frac{\partial}{\partial z_s} \right) d z_j \wedge d \bar{z}_k \wedge \left( \frac{\partial}{\partial \bar{z}_s} \right) d \bar{z}_K
= \sum_{s \neq j} \left\{ \sum_{j \neq k} d z_j \wedge d \bar{z}_k \wedge \left( \frac{\partial}{\partial z_s} \right) d \bar{z}_K + \sum_{j \neq k} d z_j \wedge \left( \frac{\partial}{\partial z_s} \right) d z_j \wedge d \bar{z}_K + \sum_{s = k \neq j} d z_j \wedge d \bar{z}_k \wedge \left( \frac{\partial}{\partial \bar{z}_s} \right) d \bar{z}_K \right\},

+ \sum_{s = j} d z_j \wedge d \bar{z}_k + \sum_{s \neq j, k} d z_j \wedge \left( \frac{\partial}{\partial \bar{z}_s} \right) d z_j \wedge \left( \frac{\partial}{\partial \bar{z}_s} \right) d \bar{z}_K
= \sum_{j \neq k} \left\{ \sum_{s \neq j, k} d z_j \wedge \left( \frac{\partial}{\partial \bar{z}_s} \right) d \bar{z}_k \wedge d \bar{z}_K + \sum_{s = k \neq j} \left( \frac{\partial}{\partial \bar{z}_s} \right) d z_j \wedge d z_j \wedge d \bar{z}_K
+ \sum_{s = j} d z_j \wedge d \bar{z}_k + \sum_{s \neq j, k} \left( \frac{\partial}{\partial \bar{z}_s} \right) d z_j \wedge d z_j \wedge \left( \frac{\partial}{\partial \bar{z}_s} \right) d \bar{z}_K \right\}.
$$
Here if \( s \neq j, k \) then we get
\[
\left( \frac{\partial}{\partial z_s} \right) \left( d z_j \wedge d z_j \right) = -d z_j \wedge \left( \frac{\partial}{\partial z_s} \right) d z_j, \quad \left( \frac{\partial}{\partial z_s} \right) \left( d \overline{z}_k \wedge d \overline{z}_K \right) = -d \overline{z}_k \wedge \left( \frac{\partial}{\partial z_s} \right) d \overline{z}_K.
\]
From this, we have that
\[
\sum_{j \notin J, k \notin K} \sum_{s \neq j, k} \left( \frac{\partial}{\partial z_s} \right) d z_j \wedge d z_j \wedge d \overline{z}_k \wedge d \overline{z}_K = -d z_j \wedge \left( \frac{\partial}{\partial z_s} \right) d z_j \wedge d \overline{z}_k \wedge d \overline{z}_K
\]
\[
= \sum_{j \notin J, k \notin K} \sum_{s \neq j, k} d z_j \wedge \left( \frac{\partial}{\partial z_s} \right) d z_j \wedge d \overline{z}_k \wedge \left( \frac{\partial}{\partial z_s} \right) d \overline{z}_K
\]
\[
= \sum_{s \in \bar{J} \cap \bar{K}} \sum_{s \neq j, k} d z_j \wedge \left( \frac{\partial}{\partial z_s} \right) d z_j \wedge d \overline{z}_k \wedge \left( \frac{\partial}{\partial z_s} \right) d \overline{z}_K
\]
Hence the difference between the two equations is as follows
\[
\sum \left\{ d z_j \wedge \left( \frac{\partial}{\partial z_s} \right) d z_j \wedge d \overline{z}_k \wedge \left( \frac{\partial}{\partial z_s} \right) d \overline{z}_K \right\} = \left( \sum_{j=k=s \in \bar{J} \cap \bar{K}} \sum_{j=k=s \notin \bar{J} \cup \bar{K}} \right) d z_j \wedge d \overline{z}_K
\]
\[
+ \sum_{s \in \bar{J} \cap \bar{K}} \sum_{s \neq j, k} d z_j \wedge d \overline{z}_k \wedge \left( \frac{\partial}{\partial z_s} \right) d \overline{z}_K + \sum_{s \neq k} d z_j \wedge \left( \frac{\partial}{\partial z_s} \right) d z_j \wedge d \overline{z}_K
\]
\[
- \sum_{j \notin \bar{J}, k \notin \bar{K}} \sum_{s \neq j, k} d z_j \wedge \left( \frac{\partial}{\partial z_s} \right) d z_j \wedge d \overline{z}_k + \sum_{s \neq j, k} d z_j \wedge \left( \frac{\partial}{\partial z_s} \right) d z_j \wedge d \overline{z}_K
\]
\[
= \left( \sum_{j=k=s \in \bar{J} \cap \bar{K}} \sum_{j=k=s \notin \bar{J} \cup \bar{K}} \right) d z_j \wedge d \overline{z}_K + \left( \sum_{s \notin \bar{J}, k \notin \bar{K}} \sum_{j \notin \bar{J}, k \notin \bar{K}} \right) d z_j \wedge d \overline{z}_k \wedge \left( \frac{\partial}{\partial z_s} \right) d \overline{z}_K
\]
\[
+ \left( \sum_{k \in \bar{J} \cap \bar{K}} \sum_{j \notin \bar{J}, k \notin \bar{K}} \right) d z_j \wedge \left( \frac{\partial}{\partial z_k} \right) d \overline{z}_j \wedge d \overline{z}_K.
\]
We think about the conditions of sigma,
\[
\left( \sum_{j \in \bar{J} \cap \bar{K}} \sum_{j \notin \bar{J}, k \notin \bar{K}} \right) \sum_{j \notin \bar{J}, k \notin \bar{K}} d z_j \wedge \frac{\partial}{\partial z_s} d z_j = \sum_{j \in \bar{J} \cap \bar{K}} \sum_{j \notin \bar{J}} \frac{\partial}{\partial z_s} d z_j = \sum_{j \in \bar{J} \cap \bar{K}} \sum_{j \notin \bar{J}} \frac{\partial}{\partial z_s} d z_j = \sum_{j \in \bar{J} \cap \bar{K}} \frac{\partial}{\partial z_s} d z_j,
\]
\[
\left( \sum_{k \in \bar{J} \cap \bar{K}} \sum_{j \notin \bar{J}, k \notin \bar{K}} \right) \sum_{k \notin \bar{J}, j \notin \bar{K}} d z_j \wedge \frac{\partial}{\partial z_k} d \overline{z}_j = \sum_{k \in \bar{J} \cap \bar{K}} \sum_{k \notin \bar{J}} \frac{\partial}{\partial z_k} d \overline{z}_j = \sum_{k \in \bar{J} \cap \bar{K}} \sum_{k \notin \bar{J}} \frac{\partial}{\partial z_k} d \overline{z}_j = \sum_{k \in \bar{J} \cap \bar{K}} \frac{\partial}{\partial z_k} d \overline{z}_j.
\]
Therefore we have that

\[
\sum \left\{ dz_j \wedge \left( \frac{\partial}{\partial z_s} \right) \sum dz_k \wedge \left( \frac{\partial}{\partial z_s} \right) d\bar{z}_K - \left( \frac{\partial}{\partial z_s} \right) d\bar{z}_j \wedge d\bar{z}_j \right\}
\]

\[
= \left( \sum_{j=k \in J \setminus K} \sum_{j=k \notin J \cup K} \right) dz_j \wedge d\bar{z}_K
\]

\[
+ \sum_{K \ni j \notin K} dz_j \wedge d\bar{z}_k \wedge \left( \frac{\partial}{\partial z_j} \right) d\bar{z}_K + \sum_{J \ni k \notin J} dz_j \wedge \left( \frac{\partial}{\partial z_k} \right) d\bar{z}_j \wedge d\bar{z}_K
\]

From the above, we obtain the first claim as follows:

\[
[i \Theta_{E,h}, \Lambda] u = i \Theta_{E,h} \Lambda u - \Lambda u + i \Theta_{E,h} u
\]

\[
= \sum c_{jk\mu} u_{j,K,\lambda} \left\{ dz_j \wedge \left( \frac{\partial}{\partial z_s} \right) \sum dz_k \wedge \left( \frac{\partial}{\partial z_s} \right) d\bar{z}_K - \left( \frac{\partial}{\partial z_s} \right) d\bar{z}_j \wedge d\bar{z}_j \right\} \otimes e_{\mu}
\]

\[
= \left( \sum_{j \in J} \sum_{j \in K} \sum_{1 \leq j \leq n} \right) c_{jj\mu} u_{j,K,\lambda} d\bar{z}_j \wedge d\bar{z}_K \otimes e_{\mu}
\]

\[
+ \sum_{K \ni j \notin K} c_{jk\mu} u_{j,K,\lambda} \left( \frac{\partial}{\partial z_j} \right) d\bar{z}_j \wedge d\bar{z}_K \otimes e_{\mu}
\]

\[
+ \sum_{J \ni k \notin J} c_{jk\mu} u_{j,K,\lambda} \left( \frac{\partial}{\partial z_k} \right) \sum d\bar{z}_j \wedge d\bar{z}_j \wedge d\bar{z}_K \otimes e_{\mu}
\]

(2.2a)

\[
+ \sum_{K \ni j \notin K} c_{jk\mu} u_{j,K,\lambda} \varepsilon(j, K) d\bar{z}_j \wedge d\bar{z}_K \otimes e_{\mu}
\]

(2.2b)

\[
+ \sum_{J \ni k \notin J} c_{jk\mu} u_{j,K,\lambda} \varepsilon(k, J) d\bar{z}_j \wedge d\bar{z}_j \otimes d\bar{z}_K \otimes e_{\mu}
\]

(2.2c)

We calculate for the equation \( \langle [i \Theta_{E,h}, \Lambda] u, u \rangle \). First,

\[
\langle (2.2a), u \rangle = \left( \sum_{j \in J} \sum_{j \in K} \sum_{1 \leq j \leq n} c_{jj\mu} u_{j,K,\lambda} d\bar{z}_j \wedge d\bar{z}_K \otimes e_{\mu}
\]

\[
+ \sum_{|L|=p, |M|=q} u_{L,M,\tau} d\bar{z}_L \wedge d\bar{z}_M \otimes e_{\tau} \right\}_L
\]
We can write an orthonormal basis of \((X, \omega)\) at \((0,0)\) between the positivity and negativity of the curvature operator.

Let \((X, \omega)\) be a Hermitian manifold and \((E, h)\) be a Hermitian vector bundle over \(X\). We denote the curvature operator \([i\Theta_{E,h}, \Lambda]_\omega\) on \(\Lambda^{p,q}T_X^* \otimes E\) by \(A_{E,h,\omega}^{p,q}\). For any ordered multi-index \(I\), we denote the ordered complementary multi-index of \(I\) by \(I^C\). Then \(\text{sgn}(I, I^C)\) is the signature of the permutation \((1, 2, \cdots, n) \to (I, I^C)\).

From Proposition 2.2, we obtain the following theorem, which represents the relation between the positivity and negativity of the curvature operator.

**Theorem 2.3.** Let \((X, \omega)\) be a Hermitian manifold and \((E, h)\) be a Hermitian vector bundle over \(X\). We have that

\[
A_{E,h,\omega}^{p,q} > 0 \quad (\text{resp. } \geq 0, < 0, \leq 0) \iff A_{E,h,\omega}^{n-q,n-p} < 0 \quad (\text{resp. } \leq 0, > 0, \geq 0).
\]

**Proof.** Let \(x_0 \in X\) and \((z_1, \ldots, z_n)\) be local coordinates such that \((\partial/\partial z_1, \ldots, \partial/\partial z_n)\) is an orthonormal basis of \((T_X, \omega)\) at \(x_0\). Let \((e_1, \ldots, e_r)\) be an orthonormal basis of \(E_{x_0}\).

We can write

\[
\omega_{x_0} = i \sum_{1 \leq j \leq n} dz_j \wedge d\bar{z}_j, \quad i\Theta_{E,h,x_0} = i \sum_{j,k,\lambda,\mu} c_{j,k,\lambda,\mu} dz_j \wedge d\bar{z}_k \otimes e^{\lambda}_\ast \otimes e_\mu.
\]
From Proposition 2.2 for any \((p, q)\)-form \(u = \sum_{|J| = p, |K| = q, \lambda} u_{J,K,\lambda} dz_J \wedge d\overline{z}_K \otimes e_\lambda \in \Lambda^{p,q} T^*_{x_0} \otimes E_{x_0}\) we have that

\[
\langle A_{E,h,\omega}^{p,q} u, u \rangle_\omega = \left( \sum_{j \in J} + \sum_{j \in K} - \sum_{1 \leq j \leq n} \right) c_{j,j,\lambda,\mu} u_{j,K,\lambda,\mu} \overline{u}_{j,K,\mu} \\
+ \sum_{j \neq k, K \setminus j = M \setminus k} c_{j,k,\lambda,\mu} u_{j,K,\lambda,\mu} \overline{u}_{j,M,\mu} \varepsilon(j, K) \varepsilon(k, M) \\
+ \sum_{j \neq k, L \setminus j = I \setminus k} c_{j,k,\lambda,\mu} u_{L,K,\lambda,\mu} \overline{u}_{j,K,\mu} \varepsilon(k, J) \varepsilon(j, L) .
\]

(2.3 I)

For the above \(u\), we take the \((n-q, n-p)\)-form

\[
\tilde{u} := \sum_{|J| = p, |K| = q, \lambda} \text{sgn}(J, J^C) \text{sgn}(K, K^C) (-1)^{|J|} (-1)^{|K|} u_{J,K,\lambda} d z_K \wedge d \overline{z}_K \otimes e_\lambda
\]

\[
= \sum_{|J| = p, |K| = q, \lambda} \alpha(J) \alpha(K) u_{J,K,\lambda} d z_K \wedge d \overline{z}_K \otimes e_\lambda,
\]

where \(\alpha(J) = \text{sgn}(J, J^C) (-1)^{|J|}\). Then from Proposition 2.2 we have that

\[
A_{E,h,\omega}^{n-q,n-p} \tilde{u} = \left( \sum_{j \in K^C} + \sum_{j \in J^C} - \sum_{1 \leq j \leq n} \right) c_{j,j,\lambda,\mu} \alpha(J) \alpha(K) u_{J,K,\lambda} d z_K \wedge d \overline{z}_K \otimes e_\mu
\]

(2.3 a)

\[
+ \sum_{J^C \ni j \neq k \notin J^C} c_{j,k,\lambda,\mu} \alpha(J) \alpha(K) u_{j,K,\lambda} \varepsilon(j, J^C) d z_K \wedge d \overline{z}_K \wedge d \overline{z}_{J^C} \otimes e_\mu
\]

(2.3 b)

\[
+ \sum_{K^C \ni k \neq j \notin K^C} c_{j,k,\lambda,\mu} \alpha(J) \alpha(K) u_{j,K,\lambda} \varepsilon(k, K^C) d z_j \wedge d \overline{z}_K \wedge d \overline{z}_{J^C} \otimes e_\mu
\]

(2.3 c)

We calculate for the equation \( \langle A_{E,h,\omega}^{n-q,n-p} \tilde{u}, \tilde{u} \rangle_\omega \). First,

\[
\langle 2.3 \text{ a} \rangle, \tilde{u} \rangle_\omega = \langle \left( \sum_{j \in K^C} + \sum_{j \in J^C} - \sum_{1 \leq j \leq n} \right) c_{j,j,\lambda,\mu} \alpha(J) \alpha(K) u_{J,K,\lambda} d z_K \wedge d \overline{z}_K \otimes e_\mu, \sum_{|L| = p, |M| = q, \tau} \alpha(L) \alpha(M) u_{L,M,\tau} d z_M \wedge d \overline{z}_M \otimes e_\tau \rangle
\]

\[
= \langle \left( \sum_{j \in K^C} + \sum_{j \in J^C} - \sum_{1 \leq j \leq n} \right) c_{j,j,\lambda,\mu} \alpha(J) \alpha(K) u_{J,K,\lambda} d z_K \wedge d \overline{z}_K, \sum_{|L| = p, |M| = q, \tau} \alpha(L) \alpha(M) u_{L,M,\tau} d z_M \wedge d \overline{z}_M \rangle
\]

\[
= \left( \sum_{j \in K^C} + \sum_{j \in J^C} - \sum_{1 \leq j \leq n} \right) c_{j,j,\lambda,\mu} \alpha(J)^2 \alpha(K)^2 u_{J,K,\lambda} \overline{u}_{J,K,\mu}
\]

\[
= \left( \sum_{1 \leq j \leq n} - \sum_{j \in K} + \sum_{j \in J} - \sum_{1 \leq j \leq n} \right) c_{j,j,\lambda,\mu} u_{J,K,\lambda} \overline{u}_{J,K,\mu}
\]

(2.3 II)

\[
= - \left( \sum_{j \in J} + \sum_{j \in K} - \sum_{1 \leq j \leq n} \right) c_{j,j,\lambda,\mu} u_{J,K,\lambda} \overline{u}_{J,K,\mu}
\]
\[ \langle 2.3 \rangle_b, \tilde{u} \rangle = \left\langle \sum_{J^C \ni j \neq k \notin J^C} c_{jk\lambda\mu} \alpha(J) \alpha(K) u_{j,K,\lambda} \varepsilon(j, J^C) dz_{K^C} \wedge d\tilde{z}_k \wedge d\tilde{z}_{J \setminus j} \otimes e_\mu, \right. \\
= \left. \sum_{\alpha(L) \alpha(M) u_{L,M,\tau} dz_{MC} \wedge d\tilde{z}_{LC} \otimes e_\tau} \right\rangle \\
= \left\langle \sum_{J^C \ni j \neq k \notin J^C} c_{jk\lambda\mu} \alpha(J) \alpha(K) u_{j,K,\lambda} \varepsilon(j, J^C) dz_{K^C} \wedge d\tilde{z}_k \wedge d\tilde{z}_{J \setminus j} \right. \\
\left. \sum_{\alpha(L) \alpha(M) u_{L,K,\lambda} dz_{K^C} \wedge d\tilde{z}_{LC}} \right\rangle \\
Here, in the same way as in Proposition 2.2, if \{k, J^C \setminus j\} = \{L^C\} as set then we get \\
J^C \setminus j = L^C \setminus k and \tilde{z}_{LC} = \varepsilon(k, L^C)dz_{k \setminus j}. Hence, we have that \\
\langle 2.3 \rangle_b, \tilde{u} \rangle = \left\langle \sum_{J^C \ni j \neq k \notin J^C} c_{jk\lambda\mu} \alpha(J) \alpha(K) u_{j,K,\lambda} \varepsilon(j, J^C) dz_{K^C} \wedge d\tilde{z}_k \wedge d\tilde{z}_{J \setminus j} \right. \\
\left. \sum_{\alpha(L) \alpha(M) u_{L,K,\lambda} dz_{K^C} \wedge d\tilde{z}_{LC}} \right\rangle \\
= \sum_{J^C \ni j \neq k \notin J^C, j \setminus L = L^C \setminus k} c_{jk\lambda\mu} \alpha(J) \alpha(L) u_{j,K,\lambda} \tilde{\nu}_{L,K,\mu} \varepsilon(j, J^C) \varepsilon(k, L^C) \\
= \sum_{j \notin k, J^C \setminus j = L^C \setminus k} c_{jk\lambda\mu} \alpha(J) \alpha(L) u_{j,K,\lambda} \tilde{\nu}_{L,K,\mu} \varepsilon(j, J^C) \varepsilon(k, L^C) \\
= \sum_{j \notin k, j \setminus k = L \setminus j} c_{jk\lambda\mu} \alpha(J) \alpha(L) u_{j,K,\lambda} \tilde{\nu}_{L,K,\mu} \varepsilon(j, J^C) \varepsilon(k, L^C) \\
Therefore we prove that if j \neq k and J \setminus k = L \setminus j, i.e. J^C \setminus j = L^C \setminus k then \\
-\varepsilon(k, J)\varepsilon(j, L) = \alpha(J) \alpha(L) \varepsilon(j, J^C) \varepsilon(k, L^C). \\
From j \notin J, i.e. j \in J^C, we have that \\
\frac{\partial}{\partial z_j} dz_N = \text{sgn}(J, J^C) \frac{\partial}{\partial z_j} (dz_j \wedge dz_{J^C}) \\
= \text{sgn}(J, J^C)(-1)^{|J|} dz_j \wedge \left( \frac{\partial}{\partial z_j} dz_{J^C} \right) \\
= \text{sgn}(J, J^C)(-1)^{|J|} \varepsilon(j, J^C) dz_j \wedge dz_{J \setminus j}. \\
Since k \in J, we get \\
\frac{\partial}{\partial z_k} \left( \frac{\partial}{\partial z_j} dz_N \right) = \text{sgn}(J, J^C)(-1)^{|J|} \varepsilon(j, J^C) \frac{\partial}{\partial z_k} \left( dz_j \wedge dz_{J \setminus j} \right) \\
= \text{sgn}(J, J^C)(-1)^{|J|} \varepsilon(j, J^C) \varepsilon(k, J) dz_{J \setminus k} \wedge dz_{J \setminus j}.
In a similar way to this equation, we get the following \\
\frac{\partial}{\partial z_j} \left( \frac{\partial}{\partial z_k} dz_N \right) = \text{sgn}(L, L^C)(-1)^{|L|} \varepsilon(k, L^C) \varepsilon(j, L) dz_{L \setminus j} \wedge dz_{L \setminus j}. \\
\text{CURVATURE OPERATOR AND L}^2\text{-ESTIMATE CONDITION}
From the assumption, we have that
\[
\text{sgn}(J, J^C)((-1)^{|J|}\varepsilon(j, J^C)\varepsilon(k, J)dz_{J\setminus k} \wedge dz_{J^C\setminus j}) = \frac{\partial}{\partial z_j} \left( \frac{\partial}{\partial z_j} dz_N \right)
\]
\[
= -\frac{\partial}{\partial z_j} \left( \frac{\partial}{\partial z_j} dz_N \right)
\]
\[
= -\text{sgn}(L, L^C)((-1)^{|\omega|}\varepsilon(k, L^C)\varepsilon(j, L)dz_{L\setminus j} \wedge dz_{L^C\setminus k})
\]
\[
= -\text{sgn}(L, L^C)((-1)^{|\omega|}\varepsilon(k, L^C)\varepsilon(j, L)dz_{J\setminus k} \wedge dz_{J^C\setminus j}.
\]

Thus, we have shown the following
\[
\text{sgn}(J, J^C)((-1)^{|\omega|}\varepsilon(j, J^C)\varepsilon(k, J) = -\text{sgn}(L, L^C)((-1)^{|\omega|}\varepsilon(k, L)\varepsilon(j, L))
\]
\[
\leftrightarrow -\varepsilon(k, J)\varepsilon(j, L) = \text{sgn}(J, J^C)((-1)^{|\omega|}\varepsilon(k, L^C)\varepsilon(j, L^C))
\]
\[
= \alpha(J)\alpha(L)\varepsilon(j, J^C)\varepsilon(k, L^C).
\]

Hence, we have that
\[
\langle (2.3b), \tilde{u} \rangle_{\omega} = \sum_{j \neq k, J=L\setminus j} c_{jk\lambda\mu} \alpha(J) \alpha(L) u_{J,K;\lambda} \bar{u}_{L,K;\mu} \varepsilon(j, J^C)\varepsilon(k, L^C)
\]
\[
= -\sum_{j \neq k, J=L\setminus j} c_{jk\lambda\mu} u_{J,K;\lambda} \bar{u}_{L,K;\mu} \varepsilon(k, J)\varepsilon(j, L).
\]

In a similar way to this equation, we get
\[
\langle (2.3c), \tilde{u} \rangle_{\omega} = -\sum_{j \neq k, K=M\setminus k} c_{jk\lambda\mu} u_{J,K;\lambda} \bar{u}_{J,M;\mu} \varepsilon(j, K)\varepsilon(k, M).
\]

By the above conditions (2.3I)-(2.3IV), we have that for any \((p, q)\)-form \(u \in \Lambda^p q T^*_{X,x_0} \otimes E_{x_0}\), there exists a \((n-q, n-p)\)-form \(\tilde{u} \in \Lambda^{n-q, n-p} T^*_{X,x_0} \otimes E_{x_0}\) such that
\[
\langle A_{E,h,\omega}^{p,q} u, u \rangle_{\omega} = -\langle A_{E,h,\omega}^{n-q, n-p} \tilde{u}, \tilde{u} \rangle_{\omega}.
\]
Therefore we obtain that \(A_{E,h,\omega}^{n-q, n-p} > 0 \implies A_{E,h,\omega}^{p,q} < 0\). And the other claim can also be shown in the same way.

We consider the relationship between the curvature operator and the Hodge-star operator. Let \((X, \omega)\) be a Hermitian manifold and \(E\) be a holomorphic vector bundle over \(X\) equipped with a smooth Hermitian metric \(h\). We denote by \(L^2_{p,q}(X, E, h, \omega)\) the Hilbert space of \(E\)-valued \((p, q)\)-forms \(u\) which satisfy
\[
||u||_{h,\omega}^2 = \int_X |u|^2_{h,\omega} dV_\omega < +\infty.
\]
Let \(*_E\) be the Hodge-star operator \(L^2_{p,q}(X, E, h, \omega) \to L^2_{n-p,n-q}(X, E^*, h^*, \omega)\) as in [CS12], [Dem-book] and \(\bar{\partial}_E\) be the Hilbert space adjoint to \(\bar{\partial}_E : L^2_{p,q}(X, E, h, \omega) \to L^2_{p,q+1}(X, E, h, \omega)\). Let \(i\Theta_{E,h}\) be the Chern curvature tensor of \((E, h)\) and \(\Lambda_\omega\) be the adjoint of multiplication of \(\omega\).
Lemma 2.4. Let \((X, \omega)\) be a Hermitian manifold and \((E, h)\) be a holomorphic Hermitian vector bundle over \(X\). Let \(x_0 \in X\) and \((U, (z_1, \ldots, z_n))\) be local coordinates such that \((\partial/\partial z_1, \ldots, \partial/\partial z_n)\) is an orthonormal basis of \((T_X, \omega)\) at \(x_0\). Then for any \((p, q)\)-form \(u = \sum_{|J|=p, |K|=q, \lambda} u_{J,K, \lambda} dz_J \wedge d\overline{z}_K \otimes e_\lambda \in \Lambda^{p,q} T^*_{X,x_0} \otimes E_{x_0}\), we have that

\[
*_{E} u = \sum_{|J|=p, |K|=q, \lambda} \overline{\pi}_{J,K, \lambda} C_{J,K} dz_J \wedge d\overline{z}_K \otimes e_\lambda^* \in \Lambda^{n-q,n-p} T^*_{X,x_0} \otimes E^*_{x_0},
\]

where \(C_{J,K} = i^{n^2} (-1)^{q(n-p)} \text{sgn}(J, J^C) \text{sgn}(K, K^C)\).

Proof. For this lemma, it suffices to show the following

\[
*_{E}(dz_J \wedge d\overline{z}_K \otimes e_\lambda) = i^{n^2} (-1)^{q(n-p)} \text{sgn}(J, J^C) \text{sgn}(K, K^C) dz_J \wedge d\overline{z}_K \otimes e_\lambda^*.
\]

Here the Hodge-star operator \(\ast_E : \mathcal{E}^{p,q}(X, E, h, \omega) \to \mathcal{E}^{n-p,n-q}(X, E^*, h^*, \omega)\) defined by

\[
\langle \alpha, \beta \rangle_{h, \omega} dV_\omega = \alpha \wedge \ast_E \beta.
\]

Therefore we define the constant number \(C_{J,K}\) by

\[
*_{E}(dz_J \wedge d\overline{z}_K \otimes e_\lambda) = C_{J,K} dz_J \wedge d\overline{z}_K \otimes e_\lambda^*.
\]

From these definitions, we have that

\[
i^{n^2} dz_N \wedge d\overline{z}_N = idz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge idz_n \wedge d\overline{z}_n = \frac{\omega^n}{n!} = dV_\omega = \langle dz_J \wedge d\overline{z}_K \otimes e_\lambda, dz_J \wedge d\overline{z}_K \otimes e_\lambda \rangle_{h, \omega} dV_\omega
\]

\[
= dz_J \wedge d\overline{z}_K \otimes e_\lambda \wedge \ast_E (dz_J \wedge d\overline{z}_K \otimes e_\lambda)
\]

\[
= C_{J,K} dz_J \wedge d\overline{z}_K \wedge d\overline{z}_K \wedge d\overline{z}_K
\]

\[
= C_{J,K} (-1)^{q(n-p)} dz_J \wedge dz_J \wedge d\overline{z}_K \wedge d\overline{z}_K
\]

\[
= C_{J,K} (-1)^{q(n-p)} \text{sgn}(J, J^C) \text{sgn}(K, K^C) dz_N \wedge d\overline{z}_N.
\]

Hence, we obtain that

\[
i^{n^2} = C_{J,K} (-1)^{q(n-p)} \text{sgn}(J, J^C) \text{sgn}(K, K^C),
\]

i.e.

\[
C_{J,K} = i^{n^2} (-1)^{q(n-p)} \text{sgn}(J, J^C) \text{sgn}(K, K^C).
\]

From the above, this proof is completed. \(\square\)

Using Proposition 2.2 and Lemma 2.4 we obtain the following theorem, which shows the relation between the curvature operator and the Hodge-star operator.

Theorem 2.5. Let \((X, \omega)\) be a Hermitian manifold and \((E, h)\) be a holomorphic Hermitian vector bundle over \(X\). Then we have that

\[
*_{E}[i\Theta_{E,h}, \Lambda]\omega = [i\Theta_{E^*,h^*}, \Lambda \omega] *_{E}, \text{ i.e. } *_{E} A^{p,q}_{E,h,\omega} = A^{n-p,n-q}_{E^*,h^*,\omega} *_{E},
\]

and for any \((p, q)\)-form \(u \in \mathcal{E}^{p,q}(X, E)\) we have that

\[
\langle [i\Theta_{E,h}, \Lambda \omega] u, u \rangle_{h, \omega} = \langle [i\Theta_{E^*,h^*}, \Lambda \omega] *_{E} u, *_{E} u \rangle_{h^*, \omega}, \text{ and }
\]

\[
\langle [i\Theta_{E,h}, \Lambda \omega]^{-1} u, u \rangle_{h, \omega} = \langle [i\Theta_{E^*,h^*}, \Lambda \omega]^{-1} *_{E} u, *_{E} u \rangle_{h^*, \omega}.
\]
Furthermore, it follows that
\[ A_{E,h,\omega}^{p,q} > 0 \quad (\text{resp. } \geq 0, < 0, \leq 0) \iff A_{E^*,h^*,\omega}^{n-p,n-q} > 0 \quad (\text{resp. } \geq 0, < 0, \leq 0). \]

Proof. Let \( x_0 \in X \) and \((z_1, \ldots, z_n)\) be local coordinates such that \((\partial/\partial z_1, \ldots, \partial/\partial z_n)\) is an orthonormal basis of \((T_{x_0}^\ast X, \omega)\). Let \((e_1, \ldots, e_r)\) be an orthonormal basis of \( E_{x_0}^\ast \). We can write \( \omega_{x_0} = i \sum_{1 \leq j \leq n} dz_j \wedge d\bar{z}_j \) and \nabla \text{and} \nabla^\ast_i \text{with respect to}\nabla \text{and} \nabla^\ast_i \text{with respect to}
\[ i\Theta_{E,h,x_0} = \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} dz_j \wedge d\bar{z}_k \otimes e_\lambda \otimes e_\mu, \]
\[ i\Theta_{E^*,h^*,x_0} = -i \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} dz_j \wedge d\bar{z}_k \otimes (e_\lambda)^* \otimes e_\mu \]
\[ = -i \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} dz_j \wedge d\bar{z}_k \otimes (e_\mu)^* \otimes e_\lambda. \]

Let \( u = \sum_{|J|=p,|K|=q} u_{J,K}\lambda \, dz_J \wedge d\bar{z}_K \otimes e_\lambda \in \Lambda^{p,q} T_{x_0}^\ast X \otimes E_{x_0} \). By Lemma 2.4, we have
\[ *_E u = \sum_{|J|=p,|K|=q} \bar{u}_{J,K}\lambda \, C_{J,K} \, dz_J \wedge d\bar{z}_ IS_{X,x_0} \otimes E_{x_0}. \]

Then since \( \bar{c}_{jk\lambda\mu} = c_{kj\mu\lambda} \) and Proposition 2.2, we have that
\[ *_E [i\Theta_{E,h}, \Lambda_{\omega}] u = *_E \left\{ \left( \sum_{j \notin J} + \sum_{j \in K} - \sum_{1 \leq j \leq n} \right) c_{jj\lambda\mu} u_{J,K}\lambda \, dz_J \wedge d\bar{z}_K \otimes e_\mu \right\} \]
\[ = \left( \sum_{j \notin J} + \sum_{j \in K} - \sum_{1 \leq j \leq n} \right) \bar{c}_{jj\lambda\mu} \bar{u}_{J,K}\lambda \, C_{J,K} \, dz_J \wedge d\bar{z}_K \otimes e_\mu \]
\[ + \sum_{K \ni j \neq j} c_{jk\lambda\mu} u_{J,K}\lambda \, dz_J \wedge d\bar{z}_K \otimes e_\mu \]
\[ + \sum_{J \ni k \neq j} \bar{c}_{jk\lambda\mu} \bar{u}_{J,K}\lambda \, C_{J,K} \, dz_J \wedge d\bar{z}_K \otimes e_\mu \]
\[ = \left( \sum_{j \notin J} + \sum_{j \in K} - \sum_{1 \leq j \leq n} \right) c_{jj\lambda\mu} u_{J,K}\lambda \, dz_J \wedge d\bar{z}_K \otimes e_\mu \]
\[ + \sum_{K \ni j \neq j} c_{jk\lambda\mu} u_{J,K}\lambda \, dz_J \wedge d\bar{z}_K \otimes e_\mu \]
\[ + \sum_{J \ni k \neq j} \bar{c}_{jk\lambda\mu} \bar{u}_{J,K}\lambda \, C_{J,K} \, dz_J \wedge d\bar{z}_K \otimes e_\mu \]
\[ \quad \left( \text{resp. } \geq 0, < 0, \leq 0 \right). \]
(2.5 a) \[ = \left( \sum_{j \in J} + \sum_{j \in K} - \sum_{1 \leq j \leq n} \right) c_{j,k} \bar{u}_{j,K} \lambda C_{j,K} dz_{j,C} \land d\bar{z}_{K,C} \otimes e_{\mu} \]

(2.5 b) \[ + \sum_{K \neq K, j \neq K} c_{j,k} \bar{u}_{j,K} \lambda C_{j,(k,K \setminus j)} \varepsilon(k, K) dz_{j,C} \land d\bar{z}_{(j,K \setminus k),j} \otimes e_{\mu} \]

(2.5 c) \[ + \sum_{j \neq j, k \neq j} c_{j,k} \bar{u}_{j,K} \lambda C_{j,(k,J \setminus j)} \varepsilon(j, J) dz_{(k,J \setminus j),j} \land d\bar{z}_{K,C} \otimes e_{\mu}, \text{ and} \]

\[ \left[ i \Theta_{E^*, h^*}, \Lambda_\omega \right] * u = \left[ i \Theta_{E^*, h^*}, \Lambda_\omega \right] \sum_{|J| = p, |K| = q} \bar{u}_{j,K} \lambda C_{j,K} dz_{j,C} \land d\bar{z}_{K,C} \otimes e_{\lambda} \]

(2.5 a') \[ = - \left( \sum_{j \in J} + \sum_{j \in K} - \sum_{1 \leq j \leq n} \right) c_{j,k} \bar{u}_{j,K} \lambda C_{j,K} dz_{j,C} \land d\bar{z}_{K,C} \otimes e_{\mu} \]

(2.5 b') \[ - \sum_{K \neq K, j \neq K} c_{j,k} \bar{u}_{j,K} \lambda C_{j,(k,C)} \varepsilon(j, K^C) dz_{j,C} \land d\bar{z}_{K,K \cap j} \otimes e_{\mu} \]

(2.5 c') \[ - \sum_{j \neq j, k \neq j} c_{j,k} \bar{u}_{j,K} \lambda C_{j,(j,J \setminus j)} \varepsilon(j, J^C) dz_{j,J \setminus j,k} \land d\bar{z}_{K,C} \otimes e_{\mu} \]

From the condition
\[ \sum_{j \in J} + \sum_{j \in K} - \sum_{1 \leq j \leq n} = \sum_{j \in J} - \sum_{j \in K} + \sum_{1 \leq j \leq n} = - \left( \sum_{j \in J} + \sum_{j \in K} - \sum_{1 \leq j \leq n} \right), \]
we have the equation (2.5 a) = (2.5 a'). We prove that if \( j \neq k \) then
\[-\text{sgn}(K, K^C) \varepsilon(j, K^C) dz_{j,K \setminus j} \land d\bar{z}_{K \setminus j} = \text{sgn}((j, K \setminus k), (j, K \setminus k)^C) \varepsilon(k, K) d\bar{z}_{(j,K \setminus k),k} \]
to show that (2.5 b) = (2.5 b') and (2.5 c) = (2.5 c'). From definition, we have that
\[ \varepsilon(s, K) dz_s \land d\bar{z}_{K \setminus j} = d\bar{z}_K \]
and that \( \text{sgn}(K, K^C) d\bar{z}_{K,C} \) is characterized by \( \text{sgn}(K, K^C) d\bar{z}_K \land d\bar{z}_{K,C} = d\bar{z}_N \). Then the above equation are proven from the following equation:
\[-\text{sgn}(K, K^C) \varepsilon(j, K^C) \varepsilon(k, K) dz_{j,K \setminus k} \land d\bar{z}_{k,K \cap j} \]
\[ = -\text{sgn}(K, K^C) \varepsilon(j, K^C) \varepsilon(k, K) dz_{j,K \setminus k} \land d\bar{z}_{k,K \cap j} \land d\bar{z}_k \land d\bar{z}_{K \cap j} \]
\[ = \text{sgn}(K, K^C) \varepsilon(j, K^C) \varepsilon(k, K) dz_{j,K \setminus k} \land d\bar{z}_{k,K \cap j} \land d\bar{z}_j \land d\bar{z}_{K \cap j} \]
\[ = d\bar{z}_N. \]

Therefore we get the following equation
\[ C_{J,(j,K \setminus k)} \varepsilon((k, K) dz_{j,C} \land d\bar{z}_{(j,K \setminus k),k} \]
\[ = i^{n^2} (-1)^{q(n-p)} \text{sgn}(J, K^C) \text{sgn}((k, j \setminus K), (k, j \setminus K)^C) \varepsilon(k, K) dz_{j,C} \land d\bar{z}_{(j,K \setminus k),k} \]
\[ = -i^{n^2} (-1)^{q(n-p)} \text{sgn}(J, K^C) \varepsilon((j, K^C) dz_{j,C} \land d\bar{z}_{k,K \cap j} \]
\[ = -C_{J,K} \varepsilon((j, K^C) dz_{j,C} \land d\bar{z}_{k,K \cap j}. \]

Thus from this equation and the fact that \( K \ni k \neq j \notin K \iff K^C \ni j \neq k \notin K^C \), we have that (2.5 b) = (2.5 b'). And the other equation (2.5 c) = (2.5 c') can also be shown in the same way. Hence, from the above we have that
\[ *_E \left[ i \Theta_{E^*, h^*}, \Lambda_\omega \right] = *_E \left[ i \Theta_{E^*, h^*}, \Lambda_\omega \right]. \]
For any \((p, q)\)-form \(u \in \mathcal{E}^{p,q}(X, E)\), we have that \(\langle [i\Theta_{E,h}, \Lambda_\omega] u, u \rangle_{h,\omega} = \langle [i\Theta_{E,h}^*, \Lambda_\omega]^* E u, *E u \rangle_{h^*,\omega}\) from the following calculation:

\[
\langle [i\Theta_{E,h}, \Lambda_\omega] u, u \rangle_{h,\omega} dV_\omega = \frac{u, [i\Theta_{E,h}, \Lambda_\omega] u}{h,\omega} dV_\omega = u \wedge [i\Theta_{E,h}, \Lambda_\omega] u = (-1)^{(n+p)(n-p)} [i\Theta_{E,h}^*, \Lambda_\omega]^* E u \wedge u
\]

\[
= (-1)^{(p+q)(n-p+n-q)} [i\Theta_{E,h}^*, \Lambda_\omega]^* E u \wedge *E = -1 *E u
\]

\[
= (-1)^{(p+q)^2} (-1)^{n-p+n-q} [i\Theta_{E,h}^*, \Lambda_\omega]^* E u \wedge *E
\]

\[
= \langle [i\Theta_{E,h}^*, \Lambda_\omega]^* E u, *E u \rangle_{h^*,\omega} dV_\omega.
\]

And, we obtain that \(\langle [i\Theta_{E,h}, \Lambda_\omega]^{-1} u, u \rangle_{h,\omega} = \langle [i\Theta_{E,h}^*, \Lambda_\omega]^{-1} E u, *E u \rangle_{h^*,\omega}\) as follows:

Let \(v := [i\Theta_{E,h}, \Lambda_\omega]^{-1} u\) then \(u = [i\Theta_{E,h}, \Lambda_\omega] v\). Therefore we get

\[
[i\Theta_{E,h}^*, \Lambda_\omega]^* E v = *E [i\Theta_{E,h}, \Lambda_\omega] v = *E u,
\]

\[
[i\Theta_{E,h}^*, \Lambda_\omega]^{-1} E u = [i\Theta_{E,h}, \Lambda_\omega]^{-1} E [i\Theta_{E,h}, \Lambda_\omega] v = [i\Theta_{E,h}^*, \Lambda_\omega]^{-1} [i\Theta_{E,h}^*, \Lambda_\omega]^* E v = *E v.
\]

Hence, we have that

\[
\langle [i\Theta_{E,h}, \Lambda_\omega]^{-1} u, u \rangle_{h,\omega} = \langle v, [i\Theta_{E,h}, \Lambda_\omega] v \rangle_{h,\omega} = \langle [i\Theta_{E,h}, \Lambda_\omega]^{-1} v, v \rangle_{h,\omega} = \langle [i\Theta_{E,h}^*, \Lambda_\omega]^{-1} E v, v \rangle_{h^*,\omega} = \langle [i\Theta_{E,h}^*, \Lambda_\omega]^{-1} E u, E u \rangle_{h^*,\omega}.
\]

From the above, this proof is completed. \(\square\)

For the \(L^2\)-estimate that is of \(\overline{\partial}\)-type, we obtain the \(\overline{\partial}^*\)-type \(L^2\)-estimate in the same way from equation \(\overline{\partial}^* \circ \overline{\partial} = 0\). By using Theorem 2.5 we obtain a more direct relationship, as follows.

**Corollary 2.6.** Let \((X, \omega)\) be a Hermitian manifold and let \((E, h)\) be a holomorphic Hermitian vector bundle over \(X\). If \(\omega\) is complete on \(X\) and the curvature operator \([i\Theta_{E,h}, \Lambda_\omega]\) is positive definite on \(\Lambda^{p,q} T^*_X \otimes E\), i.e. \(A^{p,q}_{E,h,\omega} > 0\), for some \(q \geq 1\), then for any form \(g \in L^2_{p,q}(X, E, h, \omega)\) satisfying \(\overline{\partial}_E g = 0\) and \(\int_X \langle [i\Theta_{E,h}, \Lambda_\omega]^{-1} g, g \rangle_{h,\omega} dV_\omega < +\infty\), there exists \(f \in L^2_{p,q+1}(X, E, h, \omega)\) such that \(\overline{\partial}^*_E f = g\) and

\[
||f||^2_{h,\omega} \leq \int_X \langle [i\Theta_{E,h}, \Lambda_\omega]^{-1} g, g \rangle_{h,\omega} dV_\omega.
\]

In particular, we have that \(*E g \in L^2_{n-p,n-q}(X, E^*, h^*, \omega)\) satisfying \(\overline{\partial}^*_E *E g = 0\) and

\[
\int_X \langle [i\Theta_{E,h}^*, \Lambda_\omega]^{-1} *E g, *E g \rangle_{h^*,\omega} dV_\omega = \int_X \langle [i\Theta_{E,h}, \Lambda_\omega]^{-1} g, g \rangle_{h,\omega} dV_\omega < +\infty,
\]

and that \(*E f \in L^2_{n-p,n-q}(X, E^*, h^*, \omega)\) satisfying \((-1)^{p+q+1} \overline{\partial}^*_E *E f = *E g\) and

\[
||*E f||^2_{h^*,\omega} = ||f||^2_{h,\omega} \leq \int_X \langle [i\Theta_{E,h}, \Lambda_\omega]^{-1} g, g \rangle_{h,\omega} dV_\omega
\]

\[
= \int_X \langle [i\Theta_{E,h}^*, \Lambda_\omega]^{-1} *E g, *E g \rangle_{h^*,\omega} dV_\omega.
\]
In other words, when the solution to the $\overline{\partial}$-type $L^2$-estimate for $(p,q)$-form $g$ is $f$, the solution to the $L^2$-estimate for $*_{E^*}g$ can be given by $(-1)^{p+q+1} *_{E^*}f$.

**Proof.** From the formula $\overline{\partial}_E = - *_{E^*} \overline{\partial}_{E^*} *_{E}$ (cf. [CS12]), we get $\overline{\partial}_{E^*} *_{E} g = 0$. Since $*_{E^*} *_{E} |_{E^p,q(E,X,E,h,\omega)} = (-1)^{p+q}$ (cf. [CS12]), we have that

$$ *_{E^*}g = *_{E^*} \overline{\partial}_E f = - *_{E^*} \overline{\partial}_{E^*} *_{E} f = -(-1)^{n-p-q} \overline{\partial}_{E^*} *_{E} f = (-1)^{p+q+1} \overline{\partial}_{E^*} *_{E} f. $$

From the above and Theorem 2.5, this proof is completed. □

3. The $(n,q)$ and $(p,n)$-$L^2$-estimate condition and semi-positivity of the curvature operator

In this section, we prove Theorem 1.5 and 1.6 which is an extension of [DNWZ20, Theorem 1.1] by using the following proposition.

**Proposition 3.1.** (cf. [DNWZ20, Proposition 2.1]) Let $X$ be a Kähler manifold, which admits a positive holomorphic Hermitian line bundle, and $(A, h_A)$ be a positive holomorphic Hermitian line bundle over $X$. Let $(U, (z_1, \cdots, z_n))$ be a local coordinate on $X$, such that $A|_U$ is trivial, and $B \subset \subset U$ be a coordinate ball. Then for any smooth strictly plurisubharmonic function $\psi$ on $U$, there is a positive integer $m$, and a Hermitian metric $h_m$ on the line bundle $A^\otimes m$ such that $h_m = e^{-\psi m}$ on $U$ with $\psi_m|_B = \psi$, where $\psi_m$ is a smooth strictly plurisubharmonic function.

**Proof.** Assume that $h_A|_U = e^{-\phi}$ for some smooth strictly plurisubharmonic function $\phi$ on $U$. We may assume that $\phi > 0$. We may assume that $B := B_1$ is the unit ball, and the ball $B_{1+\delta}$ with radius $1+3\delta$ is also contained in $U$, for $0 < \delta << 1$. Let $\chi$ be a cut-off function on $U$ such that $\chi|_{B_{1+\delta}} = 1$ and $\chi|_{U \setminus B_{1+2\delta}} = 0$. Let $\phi_m := m\phi + \chi \log(||z||^2 - 1)$ on $U \setminus B_1$, where $m >> 1$ is an integer such that $\psi_m$ is strictly plurisubharmonic on $U \setminus B_1$ and $\phi_m > \psi$ on $\partial B_{1+\delta}$.

Now we define a function $\psi_m$ on $U$ as follows:

$$ \psi_m(z) = \begin{cases} \phi_m, & \text{outside } B_{1+\delta}; \\ \max\{\phi_m, \psi\}, & \text{on } B_{1+\delta} \setminus B_1; \\ \psi, & \text{on } B_1. \end{cases} $$

Then for $0 < \varepsilon << 1$, $\psi_m$ is strictly plurisubharmonic on $U$, $\psi_m|_B = \psi$, and equals to $m\psi$ on $U \setminus B_{1+2\delta}$. So $\psi_m$ gives a Hermitian metric on $A^\otimes m|_U$ which coincides with $h^\otimes m$ on $U \setminus B_{1+2\delta}$. □

**Theorem 3.2.** (cf. [Wat22, Theorem 3.7]) Let $(X, \tilde{\omega})$ be a complete Kähler manifold, $\omega$ be another Kähler metric which is not necessarily complete and $(E, h)$ be a holomorphic vector bundle which satisfies $A_{E,h,\omega}^{p,n} \geq 0$. Then for any $\overline{\partial}$-closed $f \in L^2_{p,n}(X, E, h, \omega)$ there exists $u \in L^2_{p,n-1}(X, E, h, \omega)$ satisfies $\overline{\partial}u = f$ and

$$ \int_X |u|^2_{h,\omega}dV_\omega \leq \int_X \langle (A_{E,h,\omega}^{p,n})^{-1}f, f \rangle_{h,\omega}dV_\omega, $$

where we assume that the right-hand side is finite.
Using Proposition 3.1 and Theorem 3.2, we obtain the following characterization of $A^p_{E,h,\omega} \geq 0$ by $L^2$-estimate. Moreover, by this theorem, we obtain the characterization of Nakano semi-negativity by $L^2$-estimate in the next section.

**Theorem 3.3.** (Theorem 1.6) Let $(X, \omega)$ be a Kähler manifold of dimension $n$ which admits a positive holomorphic Hermitian line bundle and $(E, h)$ be a holomorphic Hermitian vector bundle over $X$ and $p$ be a nonnegative integer. Then $(E, h)$ satisfies the $(p,n)-L^2$-estimate condition on $X$ if and only if $A^p_{E,h,\omega} \geq 0$.

**Proof.** First, we show $A^p_{E,h,\omega} \geq 0$ if $(E, h)$ satisfies the $(p,n)-L^2$-estimate condition on $X$. Let $(A, h_A)$ be a positive holomorphic Hermitian line bundle over $X$. Let $(U, (z_1, \cdots, z_n))$ be a local coordinate on $X$, such that $A|_U$ is trivial, and $B \subset U$ be a coordinate ball. From Proposition 3.1 for any smooth strictly plurisubharmonic function $\psi$ on $U$, there is a positive integer $l$, and a Hermitian metric $h_l$ on the line bundle $A^\otimes l$, such that $h_l = e^{-\psi}$ on $U$ with $\tilde{\psi}|_B = \psi$, where $\tilde{\psi}$ is a smooth strictly plurisubharmonic function. By assumption, for any $f \in \mathcal{D}^{p,q}(X, E \otimes A^\otimes l)$ such that $\bar{\partial}f = 0$ and $\text{supp } f \subset U$, there is $u \in \mathcal{L}^2_{p,q}(X, E \otimes A^\otimes l)$ such that $\bar{\partial}u = f$ and

$$\|u\|^2_{h_l,\omega} = \int_X |u|^2_{h_l,\omega} d\omega \leq \int_X \langle B^{-1}_l f, f \rangle_{h,\omega} e^{\tilde{\psi}} d\omega,$$

where $B^{-1}_l := [i\bar{\partial}\tilde{\psi} \otimes \text{id}_E, \Lambda_\omega]$. Here, from $\text{supp } f \subset U$ and $A|_U$ is trivial, we have that $f \in \mathcal{D}^{p,q}(U, E \otimes A^\otimes l) \subset \mathcal{D}^{p,q}(X, E)$ and

$$J = \int_X \langle B^{-1}_l f, f \rangle_{h,\omega} e^{\tilde{\psi}} d\omega = \int_X \langle [i\Theta_{A,h_l} \otimes \text{id}_E, \Lambda_\omega]^{-1} f, f \rangle_{h,\omega} d\omega.$$

From above estimate and Bochner-Kodaira-Nakano identity, for any $\alpha \in \mathcal{D}^{p,q}(X, E)$ such that $\text{supp } \alpha \subset U$, we have that

$$\left| \int_X \langle f, \alpha \rangle_{h,\omega} e^{\tilde{\psi}} d\omega \right|^2 = \langle \langle f, \alpha \rangle \rangle_{\tilde{\psi}} = \langle \langle \bar{\partial}u, \alpha \rangle \rangle_{\tilde{\psi}} = \langle \langle u, \bar{\partial} \alpha \rangle \rangle_{\tilde{\psi}} \leq \|u\|^2_{h_l,\omega} \|\bar{\partial} \alpha\|^2_{\tilde{\psi}}$$

$$\leq \int_X \langle B^{-1}_l f, f \rangle_{h,\omega} e^{\tilde{\psi}} d\omega \times (\|D\alpha\|^2_{\tilde{\psi}} + \|D^*\alpha\|^2_{\tilde{\psi}})$$

$$\leq \int_X \langle B^{-1}_l f, f \rangle_{h,\omega} e^{\tilde{\psi}} d\omega \\ \times (\langle \langle i\Theta_{E,h_l} + i\bar{\partial}\tilde{\psi} \otimes \text{id}_E, \Lambda_\omega \rangle \alpha, \alpha \rangle_{\tilde{\psi}} + \|D^*\alpha\|^2_{\tilde{\psi}}),$$

where $D'$ is the (1,0) part of the Chern connection on $E \otimes A^\otimes l$ with respect to the metric $h \otimes h_l$. In particular, $D'|_{U}$ is also the (1,0) part of the Chern connection on $E \otimes A^\otimes l|_{U} = E|_{U}$ with respect to the metric $h \otimes h_l|_{U} = he^{-\tilde{\psi}}$.

Let $\alpha = B^{-1}_l f$, i.e. $f = B^{-1}_l \alpha$. Then the above inequality becomes

$$\langle \langle \alpha, B^{-1}_l \alpha \rangle \rangle_{\tilde{\psi}} \leq \langle \langle \alpha, B^{-1}_l \alpha \rangle \rangle_{\tilde{\psi}} (\langle \langle \bar{\partial} \alpha, \alpha \rangle \rangle_{\tilde{\psi}} + \|D\alpha\|^2_{\tilde{\psi}} + \|D^*\alpha\|^2_{\tilde{\psi}})$$

$$= \langle \langle \alpha, B^{-1}_l \alpha \rangle \rangle_{\tilde{\psi}} (\langle \langle \bar{\partial} \alpha, \alpha \rangle \rangle_{\tilde{\psi}} + \|B^{-1}_l \alpha\|^2_{\tilde{\psi}} + \|D\alpha\|^2_{\tilde{\psi}} + \|D^*\alpha\|^2_{\tilde{\psi}}).$$
Therefore we get

\[(\ast) \quad \langle \langle [i \Theta_{E,h}, \Lambda_\omega] \alpha, \alpha \rangle \rangle \psi + \|D' \alpha \|^2_\psi + \|D^* \alpha \|^2_\psi \geq 0.\]

Using this formula (\ast), we show the theorem by contradiction.

Suppose that $A^{p,n}_{E,h,\omega}$ is not semi-positive on $X$. Then there is $x_0 \in X$ and $\xi_0 \in A^{p,n}T_{X,x_0} \otimes E_{x_0}$ such that $|\xi_0| = 1$ and $\langle [i \Theta_{E,h}, \Lambda_\omega] \xi_0, \xi_0 \rangle_{h,\omega} = -2c$ for some $c > 0$.

For any small number $\varepsilon > 0$, let $M_\varepsilon$ is a regularized max function (see [Demailly, Chapter I, Section 5]). Here, the function $M_\varepsilon$ possesses the following properties (see the proof of [Watanabe, Proposition 3.2]):

(a) $M_\varepsilon(x,y)$ is non-decreasing in all variables, smooth and convex on $\mathbb{R}^2$,
(b) $\max \{x,y\} \leq M_\varepsilon(x,y) \leq \max \{x,y\} + \varepsilon$ and
(c) $M_\varepsilon(x,y) = \max \{x,y\}$ on $\{(x,y) \in \mathbb{R}^2 \mid |x - y| \geq 2\varepsilon\}$.

Let $(U, (z_1, \cdots, z_n))$ be a holomorphic coordinate in $X$ centered at $x_0$ such that $\omega = i \sum dz_j \wedge d\bar{z}_j + O(|z|^2)$. For any small number $R > 0$, we define $B_R := \{z \in U \mid |z| < R\}$ such that $B_{2R} \subset U$. Let $\varphi = |z|^2 - R^2$ and $\varphi_\varepsilon = |z|^2 - (R^2 + \varepsilon^2)$, where $\varepsilon > 0$ is enough small with respect to $R$. Then we define the smooth strictly plurisubharmonic function $\psi_\varepsilon$ on $B_{2R}$ by $\psi_\varepsilon := M_{\varepsilon/2}(\varphi, m\varphi_\varepsilon)$. Since the above conditions (b) and (c), for any $m \geq 4$ we have that $\psi_\varepsilon = \max \{\varphi, m\varphi_\varepsilon\}$ on $(B_{R+2\varepsilon} \setminus B_R)^c$, i.e.

$$
\psi_\varepsilon|_{B_R} = \varphi < 0, \quad \psi_\varepsilon|_{B_{2R} \setminus B_{R+2\varepsilon}} = m\varphi_\varepsilon > 0,
$$

and that $\max \{\varphi, m\varphi_\varepsilon\} \leq \psi_\varepsilon \leq \max \{\varphi, m\varphi_\varepsilon\} + \varepsilon$.

There exists a holomorphic frame $(e_1, \cdots, e_r)$ of $E$ on $U$ such that $h = I + O(|z|^2)$ at $z = 0$ (see [Weil, Chapter III, Lemma 2.3]). Then we have that

$$
D'_{h} = \partial + h^{-1} \partial h = \partial + (I + O(|z|^2))O(|z|) = \partial + O(|z|),
$$

and

$$
D'_{e^{-m\varphi_\varepsilon}} = \partial + e^{m\varphi_\varepsilon} \partial e^{-m\varphi_\varepsilon} = \partial - \partial m\varphi = \partial - m \sum \bar{z}_j dz_j.
$$

From Proposition [3.1] we can construct a Hermitian metric $h_\lambda$ on the line bundle $A^{\otimes l}$ such that $h_\lambda = e^{-\psi_\varepsilon}$ on $U$ with $\psi_\varepsilon|_{B_{2R}} = \psi_\varepsilon|_{B_{R+2\varepsilon}}$. Hence, we get

$$
D'|_{B_R} = D'_{he^{-\varphi}} = \partial - \sum \bar{z}_j dz_j + O(|z|),
$$

$$
D'|_{B_{2R} \setminus B_{R+2\varepsilon}} = D'_{he^{-m\varphi_\varepsilon}} = \partial - m \sum \bar{z}_j dz_j + O(|z|),
$$

where $O(|z|)$ is independent of $m$. From the fact

$$
\omega = i \sum dz_j \wedge d\bar{z}_j + O(|z|^2) = i \partial \bar{\partial} \varphi + O(|z|^2),
$$

$$
B_\varphi = [i \partial \bar{\partial} \varphi \otimes \text{id}_E, \Lambda_\omega] = [\omega \otimes \text{id}_E + O(|z|^2), \Lambda_\omega]
$$

$$
= p \cdot \text{id}_E + O(|z|^2)
$$

on $A^{p,n}T_X \otimes E$, we have that

$$
B_{\psi_\varepsilon}|_{B_R} = B_\varphi = p \cdot \text{id}_E + O(|z|^2),
$$

$$
B_{\psi_\varepsilon}|_{B_{2R} \setminus B_{R+2\varepsilon}} = B_{m\varphi_\varepsilon} = [m i \partial \bar{\partial} \varphi_\varepsilon \otimes \text{id}_E, \Lambda_\omega] = mp \cdot \text{id}_E + O(|z|^2)
$$

on $A^{p,n}T_X \otimes E$. 

Let $\xi = \sum \xi_{J,K,\lambda} dz_J \wedge d\bar{z}_K \otimes e_\lambda \in \mathcal{E}^{p,n}(U, E)$, with constant coefficients such that $\xi(x_0) = \xi_0$. We may assume
\[
\langle [i\Theta_{E,h}, \Lambda_\omega] \xi, \xi \rangle_{h,\omega} < -c
\]
on $U$. For any small number $R > 0$.

Choose $\chi \in \mathcal{D}(B_{2R}, \mathbb{R}_{\geq 0})$ such that $\chi|_{B_{R+2\varepsilon}} = 1$. Let
\[
v = \frac{1}{n} \sum_{J,N,\lambda,j} (-1)^{|J|} \varepsilon(j, N) \bar{z}_j \xi_{J,N,\lambda}(z) dz_J \wedge d\bar{z}_{N\setminus j} \otimes e_\lambda \in D^{p,n-1}(X, E),\]
then from $(-1)^{|J|} \varepsilon(j, N) d\bar{z}_j \wedge dz_J \wedge d\bar{z}_{N\setminus j} = dz_J \wedge d\bar{z}_N$, we have that
\[
\overline{\partial} v|_{B_{R+2\varepsilon}} = \frac{1}{n} \partial \sum_{J,N,\lambda,j} (-1)^{|J|} \varepsilon(j, N) \bar{z}_j \xi_{J,N,\lambda} d z_J \wedge d\bar{z}_{N\setminus j} \otimes e_\lambda
\]
\[
= \frac{1}{n} \sum_{J,N,\lambda,j} (-1)^{|J|} \varepsilon(j, N) \xi_{J,N,\lambda} \sum_{l=1}^n \frac{\partial}{\partial z_l} \bar{z}_j d z_l \wedge dz_J \wedge d\bar{z}_{N\setminus j} \otimes e_\lambda
\]
\[
= \frac{1}{n} \sum_{J,N,\lambda} \sum_{j \in N} \xi_{J,N,\lambda} d z_J \wedge d\bar{z}_N \otimes e_\lambda
\]
\[
= \xi,
\]
where if $j \notin N$ then $\varepsilon(j, N) = 0$. Therefore let $f := \overline{\partial} v \in D^{p,n}(U, E) = D^{p,n}(U, E \otimes A^{\otimes l})$ then $\overline{\partial} f = 0$ and $f = \xi$ with constant coefficients on $B_{R+2\varepsilon}$. We define $\alpha_m = B^{-1}_{\psi_m} f \in D^{p,n}(U, E \otimes A^{\otimes l})$. From $i[\Lambda_\omega, \overline{\partial}] = D^* \xi$ (cf. [Dem10, Chapter4]), we get
\[
D' \alpha_m = D' B^{-1}_{\psi_m} f = D' \left( \frac{1}{p} \xi + O(|z|^2) \right)
\]
\[
= -\frac{1}{p} \xi \wedge \sum \bar{z}_j dz_j + O(|z|^3),
\]
\[
D^* \alpha_m = D^* B^{-1}_{\psi_m} f = D^* \left( \frac{1}{p} \xi + O(|z|^2) \right) = O(|z|^2)
\]
on $B_R$. And we get
\[
D' \alpha_m = D' B^{-1}_{\psi_m} f = \frac{1}{mp} D' f + O(|z|^2)
\]
\[
= \frac{1}{mp} \left( \partial f - f \wedge \sum \bar{z}_j dz_j \right) + O(|z|^3),
\]
\[
D^* \alpha_m = D^* B^{-1}_{\psi_m} f = \frac{1}{mp} D^* f + O(|z|^2)
\]
on $B_{2R} \setminus B_{R+2\varepsilon}$. Since $D'\alpha_m(0) = 0$ and $\alpha_m$ is constant coefficients on $B_R$, so after shrinking $R$, we can get
\[
|D'\alpha_m|_{h,\omega}^2 = \left(\frac{1}{p}\right)^2 |\xi \wedge \sum \overline{z_j} d\omega_j|_{h,\omega}^2 \\
\leq \left(\frac{1}{p}\right)^2 |z^2| |\xi|_{h,\omega}^2 \leq \frac{1}{4}c,
\]
\[
|D^*\alpha_m|_{h,\omega}^2 = |O(|z|)^2|_{h,\omega}^2 \leq \frac{1}{4}c
\]
on $B_R$. From $f$ and $\alpha_m$ have compact support in $B_{2R}$, there is a constant $C$, such that
\[
|\langle [i\Theta_{E,h}, \Lambda]\alpha_m, \alpha_m \rangle_{h,\omega}| = \left(\frac{1}{mp}\right)^2 |\langle [i\Theta_{E,h}, \Lambda], f, f \rangle_{h,\omega} + O(|z|^2)| \leq \frac{C}{m^2},
\]
\[
|D'\alpha_m|_{h,\omega}^2 = \left(\frac{1}{mp}\right)^2 |D'f|_{h,\omega}^2 + O(|z|^2)
\]
\[
= \left(\frac{1}{mp}\right)^2 (|\partial f|^2_{h,\omega} + m^2 |f \wedge \sum z_j d\omega_j|_{h,\omega}^2) + O(|z|^2)
\]
\[
\leq \frac{1}{p^2} \left(\frac{1}{m^2} |\partial f|^2_{h,\omega} + |z^2| |f|_{h,\omega}^2\right) + O(|z|^2) \leq C \left(\frac{1}{m^2} + |z|^2\right),
\]
\[
|D^*\alpha_m|_{h,\omega}^2 = \left(\frac{1}{mp}\right)^2 |D^*f|_{h,\omega}^2 + O(|z|^2) \leq \frac{C}{m^2}
\]
on $B_{2R} \setminus B_{R+2\varepsilon}$ and that
\[
|\langle [i\Theta_{E,h}, \Lambda]\alpha_m, \alpha_m \rangle_{h,\omega}| \leq C, \ |D'\alpha_m|_{h,\omega} \leq C, \ |D^*\alpha_m|_{h,\omega} \leq C
\]
on $B_{R+2\varepsilon} \setminus B_R$ for any $m \geq 4$.

Then we consider the left-hand side of (*) with $\alpha$ and $\psi$ replaced by $\alpha_m$ and $\psi_m$ defined as above.
\[
\langle [i\Theta_{E,h}, \Lambda]\alpha_m, \alpha_m \rangle_{h,\omega} \tilde{\psi}_m + ||D'\alpha_m||_{\psi_m}^2 + ||D^*\alpha_m||_{\tilde{\psi}_m}^2
\]
\[
= \int_{B_R} \langle [i\Theta_{E,h}, \Lambda]\alpha_m, \alpha_m \rangle_{h,\omega} e^{-\varphi} dV_\omega + \int_{B_R} (|D'\alpha_m|_{h,\omega}^2 + |D^*\alpha_m|_{h,\omega}^2) e^{-\varphi} dV_\omega
\]
\[
+ \int_{B_{R+2\varepsilon} \setminus B_R} \langle [i\Theta_{E,h}, \Lambda]\alpha_m, \alpha_m \rangle_{h,\omega} e^{-\psi_m} dV_\omega + \int_{B_{R+2\varepsilon} \setminus B_R} (|D'\alpha_m|_{h,\omega}^2 + |D^*\alpha_m|_{h,\omega}^2) e^{-\psi_m} dV_\omega
\]
\[
+ \int_{B_{2R} \setminus B_{R+2\varepsilon}} \langle [i\Theta_{E,h}, \Lambda]\alpha_m, \alpha_m \rangle_{h,\omega} e^{-m\varphi_\varepsilon} dV_\omega + \int_{B_{2R} \setminus B_{R+2\varepsilon}} (|D'\alpha_m|_{h,\omega}^2 + |D^*\alpha_m|_{h,\omega}^2) e^{-m\varphi_\varepsilon} dV_\omega
\]
\[
\leq -\frac{c}{2} \int_{B_R} e^{-\varphi} dV_\omega + 3C \int_{B_{R+2\varepsilon} \setminus B_R} e^{-\psi_m} dV_\omega + C \int_{B_{2R} \setminus B_{R+2\varepsilon}} \left(\frac{3}{m^2} + |z|^2\right) e^{-m\varphi_\varepsilon} dV_\omega
\]
\[
\leq -\frac{c}{2} \text{Vol}(B_R) + C \left(3\text{Vol}(B_{R+2\varepsilon} \setminus B_R) + \int_{B_{2R} \setminus B_{R+2\varepsilon}} \left(\frac{3}{m^2} + |z|^2\right) e^{-m\varphi_\varepsilon} dV_\omega\right),
\]
where $\varphi < 0$ on $B_R$ and $\psi_m, \varphi_\varepsilon > 0$ on $B_{2R} \setminus B_R$. Since $\lim_{m \to +\infty} m\varphi_\varepsilon(z) = +\infty$ for $z \in B_{2R} \setminus B_R$, if $4 \leq m \to +\infty$ then we get
\[
0 \leq \int_{B_{2R} \setminus B_{R+2\varepsilon}} \left(\frac{3}{m^2} + |z|^2\right) e^{-m\varphi_\varepsilon} dV_\omega \leq (1 + 4R^2) \int_{B_{2R} \setminus B_{R+2\varepsilon}} e^{-m\varphi_\varepsilon} dV_\omega \to 0.
Here, if $1/2 > \varepsilon \to 0$ then
\[
\text{Vol}(B_{R+2\varepsilon} \setminus B_R) = \text{Vol}(B_{R+2\varepsilon}) - \text{Vol}(B_R) = C_n((R + 2\varepsilon)^{2n} - R^{2n})
\]
\[
= \varepsilon C_n \sum_{k=1}^{2n} \binom{n}{k} R^{2n-k}(2\varepsilon)^{k-1} \leq 2\varepsilon C_n \sum_{k=1}^{2n} \binom{n}{k} R^{2n-k} \to 0,
\]
where $C_n$ is a constant depending only on $n$.

Therefore we obtain that
\[
\langle [i\Theta_{E,h}, \Lambda_\omega] \alpha_m, \alpha_m \rangle_{\psi_m} + \|D'\alpha_m\|_{\psi_m}^2 + \|D^*\alpha_m\|_{\psi_m}^2 < 0
\]
for $m \gg 4$ and $1/2 > \varepsilon > 0$, which contradicts to the inequality (*). Hence, we have that $A_{E,h,\omega}^{p,n} \geq 0$.

Finally, we show that $(E,h)$ satisfies the $(p,n)$-$L^2_\omega$-estimate condition on $X$ if $A_{E,h,\omega}^{p,n} \geq 0$. As above, we have that $[i\Theta_{A,h_A} \otimes \text{id}_E, \Lambda_\omega] > 0$ on $A_{p,n}^n T_X^* \otimes E \otimes A$. Then by the inequality
\[
A_{E,h,\omega}^{p,n} = [i\Theta_{E,h}, \Lambda_\omega] + [i\Theta_{A,h_A} \otimes \text{id}_E, \Lambda_\omega]
\]
\[
= A_{E,h,\omega}^{p,n} + [i\Theta_{A,h_A} \otimes \text{id}_E, \Lambda_\omega] \geq [i\Theta_{A,h_A} \otimes \text{id}_E, \Lambda_\omega] > 0
\]
and Theorem 3.2 for any $f \in D_{p,n}^p(X, E \otimes A)$ with $\bar{\partial} f = 0$, there is $u \in L_{p,n-1}^2(X, E \otimes A)$ satisfying $\bar{\partial} u = f$ and
\[
\int_X |u|_{h_{A,h_A}}^2 dV_\omega \leq \int_X \langle (A_{E,h,\omega}^{p,n})^{-1} f, f \rangle_{h_{A,h_A}} dV_\omega < +\infty.
\]
Since $A_{E,h,\omega}^{p,n} \geq 0$, we have the inequality
\[
\langle (A_{E,h,\omega}^{p,n})^{-1} f, f \rangle_{h_{A,h_A}} \leq \langle [i\Theta_{A,h_A} \otimes \text{id}_E, \Lambda_\omega]^{-1} f, f \rangle_{h_{A,h_A}}.
\]
Hence, $(E,h)$ satisfies the $(p,n)$-$L^2_\omega$-estimate condition.

Using the similar proof technique as Theorem 1.6 and [DNWZ20, Theorem 1.1], Theorem 1.5 is proved using the following theorem. In the case of $(n,q)$-forms, i.e. Theorem 1.5, the proof itself is easier than in $(p,n)$-forms because $D'\alpha_m$ vanishes.

**Theorem 3.4.** (cf. [Dem-book, ChapterVIII, Theorem 6.1]) Let $(X, \omega)$ be a complete Kähler manifold, $\omega$ be another Kähler metric which is not necessarily complete and $(E,h)$ be a holomorphic Hermitian vector bundle which satisfies $A_{E,h,\omega}^{n,q} \geq 0$. Then for any $\bar{\partial}$-closed $f \in L^2_{n,q} (X, E, h, \omega)$ there exists $u \in L^2_{n,q-1} (X, E, h, \omega)$ satisfies $\bar{\partial} u = f$ and
\[
\int_X |u|_{h_{h,\omega}}^2 dV_\omega \leq \int_X \langle (A_{E,h,\omega}^{n,q})^{-1} f, f \rangle_{h_{h,\omega}} dV_\omega,
\]
where we assume that the right-hand side is finite.

From Theorem 1.5 and Theorem 2.5, we obtain the following corollary.

**Corollary 3.5.** Let $(X, \omega)$ be a Kähler manifold of dimension $n$ which admits a positive holomorphic Hermitian line bundle and $(E,h)$ be a holomorphic Hermitian vector bundle over $X$. Let $p$ and $q$ be positive integers with $q \leq n-1$. Then we have the following
\* \((E^*, h^*)\) satisfies the \((n, n - q)\)-\(L^2\)-estimate condition if and only if \(A_{E,h,\omega}^{0,q} \geq 0\).

\* \((E^*, h^*)\) satisfies the \((n - p, n)\)-\(L^2\)-estimate condition if and only if \(A_{E,h,\omega}^{p,0} \geq 0\).

Here, the characterization of semi-negative curvature operator, i.e.

\[ A_{E,h,\omega}^{n,q} \leq 0, \quad A_{E,h,\omega}^{0,q} \leq 0 \text{ for } q \geq 0, \quad A_{E,h,\omega}^{p,n} \leq 0 \text{ for } p \geq 1 \text{ and } A_{E,h,\omega}^{p,0} \leq 0 \text{ for } n - 1 \geq p, \]

by \(L^2\)-estimates can be obtained immediately by using Theorem 2.3, Theorem 1.5 and Corollary 3.5.

4. Characterizations of Nakano semi-negativity

In this section, we obtain characterizations of Nakano semi-negativity by \(L^2\)-estimate, i.e. Theorem 1.6, from Theorem 1.5 and properties of the curvature operator. Let \(X\) be a Hermitian manifold and \((E, h)\) be a holomorphic Hermitian vector bundle. We denote the condition that there exists a Hermitian metric \(\omega\) on \(X\) such that \(A_{E,h,\omega}^{p,q} = [i\Theta_{E,h}, \Lambda_{\omega}] > 0\) on \(X\) by \(A_{E,h}^{n,1} > 0\) on \(X\).

\textbf{Lemma 4.1.} (cf. [DNWZ20, Lemma 2.5]) Let \(X\) be a complex manifold and \((E, h)\) be a holomorphic Hermitian vector bundle over \(X\). Then \((E, h)\) is Nakano positive if and only if for any local coordinates \(U\), \(A_{E,h}^{n,1} > 0\) on \(U\). In particular, if \(X\) is a Hermitian manifold then for any Hermitian metric \(\omega\) on \(X\) we have that

\[(E, h) >_{\text{Nak}} 0 \iff A_{E,h}^{n,1} > 0 \text{ on } \forall U \iff A_{E,h,\omega}^{n,1} > 0,\]

where \(U\) is any local coordinates.

Moreover, we obtain the claim replaced positive with semi-positive or negative, semi-negative respectively.

From Theorem 2.3, Theorem 2.5 and Lemma 4.1 we obtain the following corollary.

\textbf{Corollary 4.2.} Let \(X\) be a complex manifold and \((E, h)\) be a holomorphic Hermitian vector bundle over \(X\). Then we have that

\[(E, h) >_{\text{Nak}} 0 \iff A_{E,h}^{n,1} > 0 \text{ on } U \iff A_{E,h,\omega}^{0,n-1} > 0 \text{ on } U\]

\[\iff A_{E,h}^{n-1,0} < 0 \text{ on } U \iff A_{E,h,\omega}^{1,n} < 0 \text{ on } U, \text{ and}\]

\[(E, h) <_{\text{Nak}} 0 \iff A_{E,h}^{n,1} < 0 \text{ on } U \iff A_{E,h,\omega}^{0,n-1} < 0 \text{ on } U\]

\[\iff A_{E,h}^{n-1,0} > 0 \text{ on } U \iff A_{E,h,\omega}^{1,n} > 0 \text{ on } U,\]

where \(U\) is any local coordinates. In particular if \(X\) is a Hermitian manifold then \(U\) can be changed to \(X\).

Hence, from Theorem 1.6, Lemma 4.1 and Corollary 4.2 we have the following theorem. This type of theorem for Nakano semi-positive was first shown in [DNWZ20].

\textbf{Theorem 4.3.} (= Theorem 1.7) Let \((X, \omega)\) be a Kähler manifold of dimension \(n\) which admits a positive holomorphic Hermitian line bundle and \((E, h)\) be a holomorphic Hermitian vector bundle over \(X\). Then \((E^*, h^*)\) satisfies the \((1, n)\)-\(L^2\)-estimate condition on \(X\) if and only if \((E, h)\) is Nakano semi-negative.

We introduce another notion about Nakano-type positivity.
Definition 4.4. (cf. [LSY13], [Dem20]) Let $X$ be a complex manifold of complex dimension $n$ and $(E, h)$ be a holomorphic Hermitian vector bundle of rank $r$ over $X$. $(E, h)$ is said to be dual Nakano positive (resp. dual Nakano semi-positive) if $(E^*, h^*)$ is Nakano negative (resp. Nakano semi-negative).

From definitions, we see immediately that if $(E, h)$ is Nakano positive or dual Nakano positive then $(E, h)$ is Griffiths positive. And there is an example of dual Nakano positive as follows. Let $h_{FS}$ be the Fubini-Study metric on $T_P^n$, then $(T_P^n, h_{FS})$ is dual Nakano positive and Nakano semi-positive (cf. [LSY13, Corollary 7.3]). $(T_P^n, h_{FS})$ is easily shown to be ample, but it is not Nakano positive. In fact, if $(T_P^n, h_{FS})$ is Nakano positive then from the Nakano vanishing theorem (see [Nak55]), we have that

$$H^{n-1,n-1}(P^n, C) = H^{n-1}(P^n, K_{P^n} \otimes T_{P^n}) = 0.$$ 

However, this contradicts $H^{n-1,n-1}(P^n, C) = C$.

In the same way as in Theorem 1.7, we obtain the following corollary.

Corollary 4.5. Let $(X, \omega)$ be a Kähler manifold of dimension $n$ which admits a positive holomorphic Hermitian line bundle and $(E, h)$ be a holomorphic Hermitian vector bundle over $X$. Then $(E, h)$ satisfies the $(1, n)$-$L^2$-estimate condition if and only if $(E, h)$ is dual Nakano semi-positive.

Finally, we obtain the following characterizations of Nakano semi-negativity, semi-positivity and dual Nakano semi-positive by $L^2$-estimates for smooth Hermitian metrics on a general complex manifold.

Corollary 4.6. Let $X$ be a complex manifold of dimension $n$ and $(E, h)$ be a holomorphic Hermitian vector bundle over $X$. For any local Stein coordinate system $\{(U_\alpha, \iota_\alpha)\}_\alpha$, we have the following

- $(E, h)$ is Nakano semi-positive if and only if $(E, h)$ satisfies the $(n, 1)$-$L^2$-estimate condition on any $U_\alpha$.
- $(E, h)$ is Nakano semi-negative if and only if $(E^*, h^*)$ satisfies the $(1, n)$-$L^2$-estimate condition on any $U_\alpha$.
- $(E, h)$ is dual Nakano semi-positive if and only if $(E, h)$ satisfies the $(1, n)$-$L^2$-estimate condition on any $U_\alpha$.

Proof. From Steinness of $U_\alpha$, the set $U_\alpha$ has a complete Kähler metric $\hat{\omega}$. Here, the Kähler manifold $(U_\alpha, \hat{\omega})$ admits a positive holomorphic Hermitian line bundle. Since Theorem 1.5 and 1.6 and Corollary 4.1, this proof is completed.

5. Applications

In this section, as applications of main theorems, we prove that the $(n, q)$ and $(p, n)$-$L^2$-estimate condition is preserved with respect to a increasing sequence. This phenomenon is first mentioned in [Ina21] as an extension of the properties seen in plurisubharmonic functions. After that, it is extended to the case of Nakano semi-positivity in [Ina22].
Proposition 5.1. Let \((X, \omega)\) be a Kähler manifold of dimension \(n\) which admits a positive holomorphic Hermitian line bundle and \(E\) be a holomorphic vector bundle over \(X\) equipped with a singular Hermitian metric \(h\) and let \(p\) be a positive integer. Assume that there exists a sequence of smooth Hermitian metrics \(\{h_{\nu}\}_{\nu \in \mathbb{N}}\) increasing to \(h\) pointwise such that \(A^{p,n}_{E,h_{\nu},\omega} \geq 0\). Then \((E, h)\) satisfies the \((p, n)\)-\(L^2\)-estimate condition on \(X\).

Proof. For any positive holomorphic Hermitian line bundle \((A, h_A)\) on \(X\), for any \(f \in \mathcal{D}^{p,n}(X, E \otimes A, h \otimes h_A, \omega)\) with \(\overline{\partial} f = 0\) we have that \(f \in \mathcal{D}^{p,n}(X, E \otimes A, h_{\nu} \otimes h_A, \omega)\). Since \((E, h_{\nu})\) satisfies the \((p, n)\)-\(L^2\)-estimate condition on \(X\), we get a solution \(u_{\nu}\) of \(\overline{\partial} u_{\nu} = f\) satisfying

\[
\int_X |u_{\nu}|_{h_{\nu} \otimes h_A, \omega}^2 dV_\omega \leq \int_X \langle [i \Theta_{A,h_A} \otimes \text{id}_E, \Lambda_{\omega}]^{-1} f, f \rangle_{h_{\nu} \otimes h_A, \omega} dV_\omega \leq \int_X \langle [i \Theta_{A,h_A} \otimes \text{id}_E, \Lambda_{\omega}]^{-1} f, f \rangle_{h \otimes h_A, \omega} dV_\omega < +\infty
\]

for each \(\nu \in \mathbb{N}\). Here, the right-hand side of the inequality above has an upper bound independent of \(\nu\). Then \(\{u_{\nu}\}_{\nu \geq j}\) forms a bounded sequence in \(L^2_{p,n-1}(X, E \otimes A, h_j \otimes h_A, \omega)\) due to the monotonicity of \(\{h_{\nu}\}\). Therefore we can get a weakly convergent subsequence \(\{u_{\nu_k}\}_{k \in \mathbb{N}}\) by using a diagonal argument and the monotonicity of \(\{h_{\nu}\}\). We have that \(\{u_{\nu_k}\}_{k \in \mathbb{N}}\) weakly converges in \(L^2_{p,n-1}(X, E \otimes A, h_{\nu} \otimes h_A, \omega)\) for every \(\nu\). Hence, the weak limit denoted by \(u_\infty\) satisfies \(\overline{\partial} u_\infty = f\) and

\[
\int_X |u_\infty|_{h \otimes h_A, \omega}^2 dV_\omega \leq \int_X \langle [i \Theta_{A,h_A} \otimes \text{id}_E, \Lambda_{\omega}]^{-1} f, f \rangle_{h \otimes h_A, \omega} dV_\omega
\]

due to the monotone convergence theorem. From the above, we have that \((E, h)\) satisfies the \((p, n)\)-\(L^2\)-estimate condition on \(X\).

From Theorem \ref{thm:5.1} and Corollary \ref{cor:5.2} and Corollary \ref{cor:5.3} and \ref{cor:5.4} it is natural to define semi-positivity of curvature operators, Nakano semi-positive, semi-negative and dual Nakano semi-positive by extending from smooth Hermitian metrics to singular Hermitian metrics as follows.

Definition 5.2. Let \(X\) be a Kähler manifold equipped with a complete Kähler metric and \(E\) be a holomorphic vector bundle over \(X\) equipped with a singular Hermitian metric \(h\). For any positive integers \(p\) and \(q\) and any Kähler metric \(\omega\), a curvature operator

- \(A^{p,q}_{E,h,\omega}\) is said to be \textit{semi-positive in the sense of singular} if \((E, h)\) satisfies the \((n, q)\)-\(L^2\)-estimate condition on \(X\).
- \(A^{0,q}_{E,h,\omega}\) is said to be \textit{semi-positive in the sense of singular} if \((E^*, h^*)\) satisfies the \((n, n-q)\)-\(L^2\)-estimate condition on \(X\), where \(q \leq n-1\).
- \(A^{p,n}_{E,h,\omega}\) is said to be \textit{semi-positive in the sense of singular} if \((E, h)\) satisfies the \((p, n)\)-\(L^2\)-estimate condition on \(X\).
- \(A^{p,0}_{E,h,\omega}\) is said to be \textit{semi-positive in the sense of singular} if \((E^*, h^*)\) satisfies the \((n-p, n)\)-\(L^2\)-estimate condition on \(X\).
**Definition 5.3.** Let $X$ be a complex manifold and $E$ be a holomorphic vector bundle over $X$ equipped with a singular Hermitian metric $h$. For any local Stein coordinate system $\{(U_\alpha, \omega_\alpha)\}_\alpha$, we define the following.

- $(E, h)$ is said to be Nakano semi-positive in the sense of singular if $(E, h)$ satisfies the $(n, 1)$-$L^2$-estimate condition on any $U_\alpha$.
- $(E, h)$ is said to be Nakano semi-negative in the sense of singular if $(E^*, h^*)$ satisfies the $(1, n)$-$L^2$-estimate condition on any $U_\alpha$.
- $(E, h)$ is said to be dual Nakano semi-positive in the sense of singular if $(E, h)$ satisfies the $(1, n)$-$L^2$-estimate condition on any $U_\alpha$.

Since Proposition 5.1 and the proof of [Ina22, Proposition 6.1], we obtain the following corollaries.

**Corollary 5.4.** Let $X$ be a Kähler manifold equipped with a complete Kähler metric and $E$ be a holomorphic vector bundle over $X$ equipped with a (singular) Hermitian metric $h$. Let $p$ and $q$ be positive integers and $\omega$ be a Kähler metric on $X$. Assume that there exists a sequence of smooth Hermitian metrics $\{h_\nu\}_{\nu \in \mathbb{N}}$ such that $A_{E,h_\nu,\omega}^p \geq 0 \ (\text{resp. } A_{E,h_\nu,\omega}^q \geq 0, \ A_{E,h_\nu,\omega}^0 \geq 0)$ for each $\nu$.

If $\{h_\nu\}_{\nu \in \mathbb{N}}$ increases to $h$ pointwise then $h$ is semi-positive in the sense of singular as in Definition 5.2.

If $\{h_\nu\}_{\nu \in \mathbb{N}}$ decreases to $h$ pointwise then $h$ is semi-negative in the sense of singular as in Definition 5.2.

**Corollary 5.5.** (cf. [Ina22, Proposition 6.1]) Let $X$ be a complex manifold and $E$ be a holomorphic vector bundle over $X$ equipped with a singular Hermitian metric $h$. Assume that there exists a sequence of smooth Nakano semi-positive metrics $\{h_\nu\}_{\nu \in \mathbb{N}}$ increasing to $h$ pointwise. Then $h$ is Nakano semi-positive in the sense of singular as in Definition 5.3.

**Corollary 5.6.** Let $X$ be a complex manifold and $E$ be a holomorphic vector bundle over $X$ equipped with a singular Hermitian metric $h$. Assume that there exists a sequence of smooth Nakano semi-negative metrics $\{h_\nu\}_{\nu \in \mathbb{N}}$ decreasing to $h$ pointwise. Then $h$ is Nakano semi-negative in the sense of singular as in Definition 5.3.

**Corollary 5.7.** Let $X$ be a complex manifold and $E$ be a holomorphic vector bundle over $X$ equipped with a singular Hermitian metric $h$. Assume that there exists a sequence of smooth dual Nakano semi-positive metrics $\{h_\nu\}_{\nu \in \mathbb{N}}$ increasing to $h$ pointwise. Then $h$ is dual Nakano semi-positive in the sense of singular as in Definition 5.3.

**Acknowledgement.** I would like to thank my supervisor Professor Shigeharu Takayama for guidance and helpful advice. I would also like to thank Professor Takahiro Inayama for useful advice on applications.

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