On discrete form of the mean value inequality for subharmonic functions

D S Telyakovskii

1National Research Nuclear University MEPhI (Moscow Engineering Physics Institute),
31 Kashirskoe shosse, 115409 Moscow, Russia
E-mail: dtelyakov@mail.ru

Abstract. We obtain a sufficient condition for the subharmonicity of a function \( u(x, y) = u(z) \), \( z \in G \subset \mathbb{R}^2 \), in which the mean value inequality has discrete form. Namely, it is assumed that for each point \( \zeta \in G \) there is a circle of arbitrarily small radius centered at \( \zeta \) and a set of nodes lying on this circle for which the value \( u(\zeta) \) does not exceed the arithmetic mean of the function values in the nodes of this set. A necessary and sufficient condition for the location of the nodes of the set is established when executed, the function \( u(z) \) satisfying at each point of \( G \) such discrete form of mean value inequality, and, additionally, some condition of summability and continuity in the directions is subharmonic in the domain \( G \).

1. Statement of the problem

There is obtained a sufficient condition for subharmonicity of functions \( u(x, y) = u(z) \), \( z \in G \subset \mathbb{R}^2 \), in which the mean value inequality is considered in discrete form.

Let points \( \zeta_1, \ldots, \zeta_n, n \geq 3 \) lie on a circle with the center at \( \zeta_0 \). We search conditions on the location of points \( \{ \zeta_j \} \), under which as the mean value inequality of can be considered an inequality

\[
 u(\zeta_0) \leq \frac{u(\zeta_1) + \cdots + u(\zeta_n)}{n} \tag{1}
 \]

between the value of the function at \( \zeta_0 \) and the arithmetic mean of its values in the set nodes \( \{ \zeta_j \} \).

In the book by I.I. Privalov [1], pp. 57–59, it was shown that continuous functions \( u(z) \) are subharmonic if at each point \( \zeta_0 \in G \) the inequality (1) is satisfied for all possible sets of points, lying at the vertices of regular \( n \)-gons inscribed in each circle of sufficiently small radius with center at \( \zeta_0 \). In the present work we generalized this sufficient condition for subharmonicity: the assumption of continuity of the function is significantly weakened and there is obtained a necessary and sufficient condition on location of the nodes of the set \( \{ \zeta_j \} \), under which the inequality (1) is a discrete form of the mean value inequality for subharmonic functions.

To formulate the condition of continuity in the sense, which is considered in this paper, definitions of an asterisk are required and \( h \)-regular functions [2].

Star \( S_\zeta = S(\zeta, \sigma_\zeta) \) with \( k \) rays \((k \geq 3)\) and center at point \( \zeta \) we will call the collection of non coinciding intervals \( s^{(j)}, j = 1, \ldots, k \) of length \( \sigma \), starting from point \( \zeta \). We will consider only those stars \( S_\zeta \), that inside of every half plane containing point \( \zeta \) at its bounding, contain at least one ray. It is clear that this condition is equivalent to the statement that point \( \zeta \) is inside...
of a polygon $P_2 = P(\zeta, \sigma)$ with vertexes at the end of the rays of star $S_\zeta$ and that also there can not be less than three rays for such star.

Let $h(t), t \geq 0$, be a modulus of continuity type function and and let function $u(z)$ be defined on some set $B_\zeta, z \in B_\zeta$, where point $\zeta$ is the limiting point. If for some value $L_\zeta > 0$ at every point $z \in B_\zeta$ the following inequality holds $|u(z) - u(\zeta)| \leq L_\zeta h(|z - \zeta|)$, then we say that function $u(z)$ is $h$-regular at point $\zeta$ with respect to set $B_\zeta$ with coefficient $L_\zeta$, or simply $h$-regular at $\zeta$ with respect to $B_\zeta$, if specific value of $L_\zeta$ is not important.

Denote by $C(\zeta, r)$ a circle with center at $\zeta$ and radius $r$.

The following theorem is proved.

**Theorem.** Let function $u(z)$ be defined in region $G \subset \mathbb{R}^2$ and for each point $\zeta_0 \in G$ there exist a circle $C(\zeta_0, \rho) \subset G$ of arbitrary small radius and a set of nodes on this circle $\zeta_j = \zeta_0 + \rho e^{i\varphi_j}, j = 1, \ldots, n, n = n(\zeta_0, \rho) \geq 3$, such that

1° for the set $\{\zeta_j\}$ the inequality (1) satisfies,

2° the point $\zeta_0$ is the center of gravity of the system of unit masses located at points $\{\zeta_j\}$ and systems of unit masses, located at $\{\zeta'_j := \zeta_0 + \rho e^{i\varphi_j}\}$.

Let, further, for some function $h(t)$ of modulus of continuity type the function $u(z)$ is $h$-regular at each point $\zeta \in G$ with respect to some star $S_\zeta$ with center at point $\zeta$, and the angles between the rays of the star do not exceed $\alpha \pi$, $\alpha \geq 1/2$, and the function $(u^+(z))^{2\alpha}$ locally summable in $G$. Then the function $u(z)$ is subharmonic in the region $G$.

### 2. Proofs of Lemmas

To prove this theorem we need several lemmas.

**Lemma 1.** Assume that function $u(z)$ has a second Peano differential at point $z_0 = (x_0, y_0)$, i.e. for some numbers $p, q, r, s$ and $t$ holds

$$
u(z) = u(z_0) + \left[ p(x - x_0) + q(y - y_0) \right] + \frac{1}{2} \left[ r(x - x_0)^2 + 2s(x - x_0)(y - y_0) + t(y - y_0)^2 \right] +$$

$$+ \varepsilon(x, y) \left[ (x - x_0)^2 + (y - y_0)^2 \right] \quad \text{where} \quad \varepsilon(x, y) \to 0 \quad \text{when} \quad (x, y) \to (x_0, y_0). \quad (2)$$

Let $C(z_0, \rho)$ be a circle of sufficiently small radius. Then for any collection of nodes $z_1, \ldots, z_n, n = n(\rho) \geq 3$, of $C(z_0, \rho)$ that satisfies the condition 2° of the theorem, the following estimate holds

$$\frac{u(z_1) + \cdots + u(z_n)}{n} - u(z_0) = \frac{r + t}{4} \rho^2 + o(\rho^2) \quad \text{when} \quad \rho \to 0. \quad (3)$$

**Proof.** We represent coordinates of each point $z_j$ in the form $x_j = x_0 + \rho \cos \varphi_j, y_j = y_0 + \rho \sin \varphi_j, j = 1, \ldots, n$, and substitute them in the formula (2). We have

$$u(z_j) = u(z_0) + \rho [p \cos \varphi_j + q \sin \varphi_j] + \frac{1}{2} \rho^2 \left[r \cos^2 \varphi_j + 2s \cos \varphi_j \sin \varphi_j + t \sin^2 \varphi_j\right] + \varepsilon_j \rho^2 =$$

$$= u(z_0) + \rho [p \cos \varphi_j + q \sin \varphi_j] + \frac{1}{2} \rho^2 \left[\frac{1 + \cos 2\varphi_j}{2} + s \sin 2\varphi_j + t + \frac{1 - \cos 2\varphi_j}{2}\right] + \varepsilon_j \rho^2 =$$

$$= u(z_0) + \rho [p \cos \varphi_j + q \sin \varphi_j] + \frac{1}{2} \rho^2 \left[\frac{r + t}{2} + s \sin 2\varphi_j + \frac{r - t}{2} \cos 2\varphi_j\right] + \varepsilon_j \rho^2. \quad (4)$$

We substitute resulting expressions for $u(z_j)$ in the left part of the formula (3). We obtain

$$\frac{1}{n} \left[u(z_1) + \cdots + u(z_n)\right] - u(z_0) = \rho \left[\frac{p}{n} \sum_{j=1}^{n} \cos \varphi_j + \frac{q}{n} \sum_{j=1}^{n} \sin \varphi_j\right] +$$

$$+ \frac{1}{4} \rho^2 \left[\frac{r + t}{2} + s \sin 2\varphi_j + \frac{r - t}{2} \cos 2\varphi_j\right] + \varepsilon \rho^2. \quad (5)$$
\[ +\rho^2 \left[ \frac{r + t}{4} + \frac{r - t}{4n} \sum_{j=1}^{n} \cos 2\varphi_j + \frac{s}{4n} \sum_{j=1}^{n} \sin 2\varphi_j \right] + \rho^2 \frac{1}{n} \sum_{j=1}^{n} \varepsilon_j. \]

Sums \( \rho \sum \cos \varphi_j \) and \( \rho \sum \cos 2\varphi_j \) are equal to the sums of moments of vertical forces applied to the points of systems \( \{z_j\} \) and \( \{z_j'\} \) \( (z_j' := (x_0 + \rho \cos 2\varphi_j, y_0 + \rho \sin 2\varphi_j), j = 1, \ldots, n) \), respectively relative to the point \( z_0 \) and \( \rho \sum \sin \varphi_j \) and \( \rho \sum \sin 2\varphi_j \) — to the sums of moments of horizontal forces applied to the points of these systems, relative to \( z_0 \). The set of points \( z_1, \ldots, z_n \) is selected on the circle \( C(z_0, \rho) \) so that the center of gravity of the systems \( \{z_j\} \) and \( \{z_j'\} \) is at \( z_0 \). So all these four the amounts vanish and therefore,

\[ \frac{1}{n} (u(z_1) + \cdots + u(z_n)) - u(z_0) = \rho^2 \frac{r + t}{4} + \rho^2 \frac{1}{n} \sum_{j=1}^{n} \varepsilon_j. \]  

(4)

According to the statement of Lemma for the function \( u(z) \) at point \( z_0 \) holds the estimate (2), so for sufficiently small radii \( \rho \) modules all the values \( \varepsilon_j \) do not exceed arbitrary predetermined positive \( \varepsilon \). In this case, modulus of the sum \( |\sum \varepsilon_j| \) on the right side of formula (4) is not greater than \( n\varepsilon \) and hence (3) is evaluated. Lemma 1 is proved.

Note that for second-order polynomials the equality (3) is exact, that is, it does not contain the term \( o(\rho^2) \).

**Lemma 2.** Let function \( u(z) \) be continuous in the domain \( G \subset \mathbb{R}^2 \) and in any neighborhood of each point \( \zeta_0 \in G \) there exists a set of \( n \) nodes \( \{\zeta_j\} \) (number of nodes in sets for different points \( \zeta_0 \in G \) and in different sets for the same point can be different, but in any case of \( n \geq 3 \) lying on a circle with center at \( \zeta_0 \), for which the conditions \( 1^\circ \) and \( 2^\circ \) of the Theorem hold. Then the function \( u(z) \) is subharmonic in the region \( G \).

Proof of this lemma follows the approach similar to the theorem 2, I.I. Privalov [3]. Proof of the Privalov’s theorem is also provided in the footnote in the work of I.G. Petrovskii [4], pg. 107.

**Remark 1.** The condition \( 1^\circ \) of the theorem can be loosened somewhat by assuming that for set \( \{\zeta_j\} \) of circle points \( C(\zeta_0, \rho) \) completed not the inequality (1), but the next one for the negative part of the difference between the arithmetic mean of the values of the function \( u(z) \) at nodes of the collection \( \{\zeta_j\} \) and its value at \( \zeta_0 \)

\[ \frac{u(\zeta_1) + \cdots + u(\zeta_n)}{n} - u(\zeta_0) = o(\rho^2) \quad \text{when} \quad \rho \to 0. \]  

(5)

It is sufficient to prove the Lemma 2 under this assumption. We shall prove this version of Lemma 2.

**Proof.** Suppose the Lemma is not true and the function \( u(z) \) is subharmonic not everywhere in the domain \( G \). Violation of subharmonicity means that there exist a closed disk \( D \subset G \) and function \( v(z) \), continuous at \( D \) and harmonic in the interior of \( D \), such that the inequality \( v(z) \geq u(z) \) holds on \( \partial D \), but in some points of the interior of \( D \) is executed the opposite inequality \( v(z) < u(z) \). The function \( u(z) \) is continuous, so as a function of \( v(z) \) we can take the solution of the Dirichlet problem in the disk \( D \) with values \( u(z) \) on \( \partial D \).

On the disk \( D \) we set \( w(z) := u(z) - v(z) \). The function \( w(z) \) is continuous on the closed disk \( D \), equals zero on its boundary \( \partial D \) and is positive at some interior points of \( D \).

Let \( z_0 \) be an arbitrary inner point of the disk \( D \). According to the statement of the Lemma inside \( D \) there exist circles of arbitrarily small radius with center at \( z_0 \) and sets of points \( \{z_1, \ldots, z_n\} \) of these circles, such that the function \( u(z) \) satisfies the conditions of \( 1^\circ \) and \( 2^\circ \) of
the Theorem. Consider the expression included in the inequality (1) for the function \( w(z) \) on such sets of nodes \( \{ z_j \} \). We have

\[
\frac{w(z_1) + \cdots + w(z_n)}{n} - w(z_0) = \left[ \frac{u(z_1) + \cdots + u(z_n)}{n} - u(z_0) \right] + \left[ \frac{v(z_1) + \cdots + v(z_n)}{n} - v(z_0) \right].
\] (6)

According to the Lemma condition on the set \( \{ z_j \} \) the negative part of the value in the first pair of brackets on the right side of the formula (6) satisfies the estimate of \( o(\rho^2) \) as \( \rho \to 0 \) (we take the radii of the circles \( C(z_0, \rho) \) on which collections \( \{ z_j \} \) lie). The function \( v(z) \) is harmonic in a sufficiently small neighborhood of the point \( z_0 \). So according to the Lemma 1 the value in the second pair of brackets on the right side of the formula (6) also satisfies the estimate \( o(\rho^2) \) as \( \rho \to 0 \). Therefore, the negative part of (6) satisfies (5) on some sequence \( \rho_n \to 0 \).

The function \( w(z) \) is continuous in the disk \( D \), positive in some internal points of \( D \) and is identically zero on \( \partial D \). Hence, the maximum on \( D \) of the function \( w(z) \) is positive and this maximum is located strictly inside \( D \). Let \( \zeta_0 \) be the maximum point: \( \max_{z \in D} w(z) = w(\zeta_0) \). We set

\[
\tilde{w}(z) := w(z) + \frac{w(\zeta_0)}{2 \text{diam}^2 D} |z - \zeta_0|^2.
\]

According to the definition of the function \( \tilde{w}(z) \) the inequality \( \tilde{w}(z) \geq w(z) \) holds everywhere in the disk \( D \), so

\[
\max_{z \in D} \tilde{w}(z) \geq \max_{z \in D} w(z) = w(\zeta_0).
\]

There holds the identically \( w(z) \equiv 0 \) on the boundary \( \partial D \) so we have

\[
\tilde{w}(z) := w(z) + \frac{w(\zeta_0)}{2 \text{diam}^2 D} |z - \zeta_0|^2 < \frac{w(\zeta_0)}{2 \text{diam}^2 D} \text{diam}^2 D = \frac{w(\zeta_0)}{2}
\]

Therefore, the maximum of the function \( \tilde{w}(z) \) on the disk \( D \) is achieved strictly inside \( D \). Let us denote \( z_0 \) a point of maximum: \( \max_{z \in D} \tilde{w}(z) = \tilde{w}(z_0) \).

The expression \( |z - \zeta_0|^2 = (x-\xi_0)^2 + (y-\eta)^2 \) is a second order polynomial. Then by Lemma 1 and a remark after it, there exist a circle \( C(z_0, \rho) \) of arbitrarily small radius and a set of nodes \( \{ z_j \} \subset C(z_0, \rho) \) such that

\[
\frac{\tilde{w}(z_1) + \cdots + \tilde{w}(z_n)}{n} - \tilde{w}(z_0) = \left[ \frac{w(z_1) + \cdots + w(z_n)}{n} - w(z_0) \right] + \frac{w(\zeta_0)}{4 \text{diam}^2 D} \rho^2,
\] (7)

and the negative part of the expression in brackets in (7) satisfies the estimate of \( o(\rho^2) \) as \( \rho \to 0 \) (estimate (5)). Since \( w(\zeta_0) > 0 \), for sufficiently small \( \rho \) the value (7) is positiv. It is clear that in this case the value of the function \( \tilde{w}(z) \) at least one of the nodes of the set \( \{ z_j \} \) exceeds value \( \tilde{w}(z_0) \). This contradicts the assumption that the function \( \tilde{w}(z) \) has maximum in the disk \( D \) at \( z_0 \).

Thus, the assumption that the function \( u(z) \) is subharmonic is not everywhere in the region \( G \) leads to a contradiction. Hence, the function \( u(z) \) subharmonic. Lemma 2 is proved.

Further, the proof of the Theorem from the present work follows the proof of the Theorem by author of [2] on the harmonicity of functions satisfying discrete Laplace equation. So why we don’t bring it.

The author was partially supported by the Program of competitiveness increase of the National Research Nuclear University MEPhI (Moscow Engineering Physics Institute); contract No 02.a03.21.0005, 27.08.2013.
References

[1] I.I. Privalov, Subharmonic Functions (ONTI, Moscow–Leningrad, 1937), Vol. 2 [in Russian].

[2] D.S. Telyakovskii. A sufficient condition for the harmonicity of a function of two variables satisfying the
Laplace difference equation, Tr. Inst. Mat. Mekh. UrO RAN, 2016, 22:4, 269–283, [in Russian].

[3] I. Priwaloff. Sur les fonctions harmoniques. — Mat. Sb., 1925, 23:3, 464–471.

[4] I.G. Petrovskii, Perron’s method for solving the Dirichlet problem, Uspekhi Mat. Nauk, 1941, 8, 107–114
[Selected Works, Gordon and Breach Publishers, Amsterdam, 1996, Vol. 2, 20–29].