SHARP ASYMPTOTIC ESTIMATES FOR A CLASS OF LITTLEWOOD-PALAELY OPERATORS

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Abstract. It is well-known that Littlewood-Paley operators formed with respect to lacunary sets of finite order are bounded on \( L^p(\mathbb{R}) \) for all \( 1 < p < \infty \). In this note it is shown that

\[
S \rightarrow L^p(\mathbb{R}) \sim (p - 1)^{-2} \quad (p \to 1^+),
\]

where \( S \) denotes the classical Littlewood-Paley operator formed with respect to the second order lacunary set \( E_2 = \{ \pm (2^k - 2^l) : k, l \in \mathbb{Z} \text{ with } k > l \} \).

1. Introduction

If \( I \) is a collection of mutually disjoint intervals in the real line, then the corresponding Littlewood-Paley operator \( S_I \) is given by

\[
S_I(f) := \left( \sum_{j \in I} |P_I(f)|^2 \right)^{1/2}.
\]

Here \( P_I \) denotes the Fourier multiplier operator with symbol \( \chi_I \). It is well-known that if we consider the collection \( I_E := \{[2^j, 2^{j+1}) \}_{j \in \mathbb{Z}} \cup \{-[2^j+1, -2^j) \}_{j \in \mathbb{Z}} \), then \( S_I \) is an \( L^p \)-bounded operator for all \( p \in (1, \infty) \) and moreover, for each \( p \in (1, \infty) \) there exist positive constants \( A_{I_E, p} \) and \( B_{I_E, p} \) such that

\[
A_{I_E, p} \|f\|_{L^p(\mathbb{R})} \leq \|S_{I_E}(f)\|_{L^p(\mathbb{R})} \leq B_{I_E, p} \|f\|_{L^p(\mathbb{R})} \quad (1 < p < \infty).
\]

See e.g. Chapter IV in [24]. In [3], L. Carleson proved that if we consider the collection \( I_0 := \{[2^j, 2^{j+1}) \}_{j \in \mathbb{Z}} \cup \{-[2^j+1, -2^j) \}_{j \in \mathbb{Z}} \), then \( S_{I_0} \) is \( L^p \)-bounded for \( p \in (1, \infty) \) and is not bounded on \( L^p(\mathbb{R}) \) when \( p \in (1, 2) \). A different proof of Carleson’s result was given by A. Córdoba in [8]. In [21], J. L. Rubio de Francia extended the aforementioned result by showing that for any collection \( I \) in \( \mathbb{R} \) of mutually disjoint intervals the corresponding Littlewood-Paley operator \( S_I \) is bounded on \( L^p(\mathbb{R}) \) for all \( p \in [2, \infty) \). For alternative proofs and extensions of Rubio de Francia’s theorem in the one-dimensional case, see J. Bourgain [2], P. Sjölin [22], and S. V. Kislyakov and D. V. Parilov [16]. To the best of our knowledge, the problem of characterising all admissible collections \( I \) in \( \mathbb{R} \) of mutually disjoint intervals such that \( S_I \) is \( L^p \)-bounded for all \( p \in (1, \infty) \) seems to be still open; see the paper [11] of K. E. Hare and I. Klemes for a relevant conjecture as well as for some related partial results in the periodic setting. See also [12] and [13]. For more details on topics related to Rubio de Francia’s theorem, see M. Lacey’s paper [17] and the references therein.

In 1939, J. Marcinkiewicz in his classical paper [20] showed that the periodic Littlewood-Paley operator formed with respect to the second order lacunary set

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\{\pm(2^k + 2^l) : k, l \in \mathbb{N}_0 \text{ with } k > l\} \text{ is bounded on } L^p(\mathbb{T}) \text{ for all } p \in (1, \infty); \text{ see } [20, \text{ Théorème 8}]. \text{ In 1981, in } [22], \text{ P. Sjögren and Sjölin developed a systematic method of successively constructing collections } \mathcal{I} \text{ of intervals in } \mathbb{R} \text{ such that the corresponding Littlewood-Paley operator } S_{\mathcal{I}} \text{ is bounded on } L^p(\mathbb{R}) \text{ for all } p \in (1, \infty). \text{ To be more specific, following } [22], \text{ if } E, E' \text{ are two closed null sets in } \mathbb{R}, \text{ then } E' \text{ is said to be a \textit{successor} of } E \text{ whenever there exists a constant } c_{E, E'} > 0 \text{ such that for every } x, y \in E \text{ with } x \neq y \text{ one has } |x - y| \geq c_{E, E'} \text{dist}(x, E). \text{ It was shown by Sjögren and Sjölin that if a set } E' \text{ is a successor of a closed null set } E, \text{ then } S_{\mathcal{I}_{E'}} \text{ is } L^p \text{-bounded for some } p \in (1, \infty), \text{ then } S_{\mathcal{I}_{E'}} \text{ is also bounded on } L^p(\mathbb{R}); \text{ see } [22, \text{ Theorem 1.2}]. \text{ In particular, since the second order lacunary set } E_2 := \{\pm(2^k - 2^l) : k, l \in \mathbb{Z} \text{ with } k > l\} \text{ is a successor of the lacunary set } E_1 := \{\pm 2^j\}_{j \in \mathbb{Z}} \text{ and } S_{\mathcal{I}_{E_1}} \text{ satisfies } (1.1), \text{ one deduces that for each } p \in (1, \infty) \text{ there exist positive constants } A_{E_1} \text{ and } B_{E_1} \text{ such that}

\begin{equation}
A_{E_1} \leq \|S_{\mathcal{I}_{E_1}}(f)\|_{L^p(\mathbb{R})} \leq B_{E_1} \|f\|_{L^p(\mathbb{R})} \quad (1 < p < \infty).
\end{equation}

Here we adopt the following convention: if } K = \{a_n\}_{n \in \mathbb{Z}} \text{ in } \mathbb{R} \text{ can be written as } K = \{a_n\}_{n \in \mathbb{Z}} \text{ and } a_n < a_{n+1} \text{ for all } n \in \mathbb{Z} \text{ and moreover, } a_n \rightarrow -\infty \text{ as } n \rightarrow -\infty \text{ and } a_n \rightarrow +\infty \text{ as } n \rightarrow +\infty, \text{ then } \mathcal{I}_K \text{ denotes the collection }

\{\{a_n, a_{n+1}\} : n \in \mathbb{Z}\}.

The main goal of this note is to determine the behaviour of the best constant } B_{E_1} \text{ in } (1.2) \text{ ‘near’ } p = 1 \text{ or, in other words, to establish sharp asymptotic estimates for the } L^p - L^p \text{ operator norm of } S_{\mathcal{I}_{E_1}} \text{ as } p \rightarrow 1^+. \text{ Before stating our results, let us mention that it follows from the work of Bourgain } [4] \text{ that in the lacunary case } E_1 = \{\pm 2^j\}_{j \in \mathbb{Z}} \text{ the best constant } B_{E_1} \text{ in the classical Littlewood-Paley inequality } (1.1) \text{ behaves like } (p-1)^{-3/2} \text{ as } p \rightarrow 1^+, \text{ namely there exist absolute constants } c_1, c_2 > 0 \text{ such that}

\begin{equation}
\frac{c_1}{(p-1)^{3/2}} \leq \|S_{\mathcal{I}_{E_1}}\|_{L^p(\mathbb{R})} \leq \frac{c_2}{(p-1)^{3/2}} \quad (1 < p \leq 2).
\end{equation}

To be more precise, in [4], Bourgain established a periodic version of (1.3) and his proof was obtained in [4] by using a classical inequality due to S.-Y. A. Chang, J. M. Wilson, and T. H. Wolff on dyadic martingales [6] combined with explicit formulas for translations of dyadic systems [3] and appropriate vector-valued inequalities. An alternative proof of the upper estimate in (1.3) in the periodic setting was given by the author in [1] by using the work of T. Tao and J. Wright on Marcinkiewicz multiplier operators [26] and Tao’s converse extrapolation theorem [25]. Recently, in [18], A. K. Lerner showed that

\begin{equation}
\|S_{\mathcal{I}_{E_1}}\|_{L^2(w) \rightarrow L^2(w)} \lesssim \|w\|_{A_2}^{3/2}
\end{equation}

for any } A_2 \text{ weight } w \text{ on the real line; see } [18, \text{ Theorem 1.1}]. \text{ Moreover, as explained in } [18], \text{ by combining (1.4) with an extrapolation result due to J. Duoandikoetxea } [9] \text{ one obtains yet another proof of (1.3); see } [18, \text{ Remark 4.2}]. \text{ The main ingredients in Lerner’s proof of (1.3) in } [18] \text{ were an appropriate variant of the Chang-Wilson-Wolff inequality } (2.7) \text{ as well as sharp weighted estimates for multiplier operators of the form } P_T T_m \text{ with } T_m \text{ being a multiplier operator whose symbol } m \text{ satisfies a Marcinkiewicz-type condition on } I; \text{ see } [18, \text{ Lemma 3.2}]. \text{ For sparse bounds for Rubio de Francia-type operators } S_{\mathcal{I}} \text{ and for Marcinkiewicz-type multiplier operators in the Walsh-Fourier setting, we refer the reader to the recent papers } [10] \text{ and } [1], \text{ respectively.}
As mentioned above, this note focuses on the behaviour of the $L^p - L^p$ operator norm of $S_{IE_2}$ ‘near’ $p = 1^+$. More specifically, our main result is the following sharp asymptotic estimate as $p \to 1^+$.

**Theorem 1.** There exist absolute constants $c_1, c_2 > 0$ such that
\[
\frac{c_1}{(p - 1)^2} \leq \|S_{IE_2}\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \leq \frac{c_2}{(p - 1)^2}
\]
for all $1 < p \leq 2$.

The lower estimate in Theorem 1 is obtained by adapting the corresponding argument of Bourgain that establishes the lower estimate in [1, Theorem 1]. The upper estimate in Theorem 1 is a consequence of the following result.

**Theorem 2.** There exists an absolute constant $c_0 > 0$ such that
\[
\|S_{IE_2}\|_{L^{2} \to L^2(w)} \leq c_0[w]_{A_2}^2
\]
for all $A_2$ weights $w$ on the real line.

Notice that the lower estimate in Theorem 1 shows that the exponent $r = 2$ in $[w]_{A_2}$ in Theorem 2 is best possible. At this point, we remark that by carefully examining the proof of [22, Theorem 1.2] for the case of $S_{IE_2}$ (and by invoking results of Tao and Wright on Marcinkiewicz multiplier operators [26]) one gets $\|S_{IE_2}\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \lesssim (p - 1)^{-5/2}$ for $p$ ‘close’ to $1^+$.

Theorem 2 is established by suitably modifying Lerner’s proof of [13] and the ‘scheme’ of its proof is roughly as follows: after reformulating the problem by using duality, one employs Lerner’s variant of the Chang-Wilson-Wolff inequality [18, Theorem 2.7]. Then, the idea is to perform suitable ‘frequency translations’ of the Littlewood-Paley projections appearing in the definition of $S_{IE_2}$ so that one can effectively use [18, Theorem 2.7] again. After doing so, one completes the proof by using an appropriate modification of [18, Lemma 3.2]; see Lemma 3 below. The details of the proof of Theorem 2 are presented in Section 2. In Section 3 we give the proof of Theorem 1 and in Section 4 we make some further remarks related to the present work and more specifically, we present periodic versions of Theorems 1 and 2 and extensions for certain lacunary sets of order $N \in \mathbb{N}$, $N \geq 2$.

**Notation.** If $\alpha \in \mathbb{C}$, then $|\alpha|$ stands for the modulus of the complex number $\alpha$, whereas if $A \subset \mathbb{R}$ is measurable, $|A|$ denotes the Lebesgue measure of $A$.

Given two positive quantities $A$ and $B$, if there exists an absolute constant $c_0 > 0$ such that $A \leq c_0 B$, we shall write $A \preceq B$ or $B \succeq A$. If $A \preceq B$ and $B \preceq A$, we write $A \sim B$.

As usual, the class of Schwartz functions on the real line is denoted by $\mathcal{S}(\mathbb{R})$ and $C_c^{\infty}(\mathbb{R})$ stands for the class of $C_c^{\infty}(\mathbb{R})$-functions with compact support in $\mathbb{R}$.

If $g$ is an integrable function on the torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, then its Fourier coefficient at $n \in \mathbb{Z}$ is given by $\hat{g}(n) := \int_\mathbb{T} g(x) e^{-i2\pi nx} \, dx$. If $f \in \mathcal{S}(\mathbb{R})$, then its Fourier transform is given by $\hat{f}(\xi) := \int_\mathbb{R} f(x) e^{-i2\pi \xi x} \, dx$, $\xi \in \mathbb{R}$. If $m \in L^{\infty}(\mathbb{R})$, then $T_m$ denotes the multiplier operator with symbol $m$, namely $T_m(f)(\xi) = m(\xi) \hat{f}(\xi)$, $\xi \in \mathbb{R}$ for $f \in \mathcal{S}(\mathbb{R})$. If $I \subseteq \mathbb{R}$, we write $P_I = T_{\chi_I}$.

If $g$ is a locally integrable function on the real line and $I \subset \mathbb{R}$ is an interval, we use the standard notation
\[
\langle g \rangle_I := |I|^{-1} \int_I g(x) \, dx.
\]
A non-negative, locally integrable function \( w \) on \( \mathbb{R} \) is said to be an \( A_2 \) weight if, and only if,
\[
[w]_{A_2} := \sup_{I \subset \mathbb{R}: \text{interval}} \langle w \rangle_I (w^{-1})_I < \infty.
\]

If \( f \) is a locally integrable function on \( \mathbb{R} \) then its non-centred Hardy-Littlewood maximal function \( M(f) \) is given by
\[
M(f)(x) := \sup_{I \text{ interval}: x \in I} |I|^{-1} \int_I |f(y)|dy \quad (x \in \mathbb{R}).
\]
The maximal Hilbert transform \( H^*(g) \) of \( g \in \mathcal{S}(\mathbb{R}) \) is given by
\[
H^*(g)(x) := \sup_{\varepsilon > 0} \left| \int_{|x-y| > \varepsilon} \frac{g(y)}{x-y} dy \right| \quad (x \in \mathbb{R}).
\]

2. Proof of Theorem 2

Arguing as in [18], it follows from duality that, to prove Theorem 2, it suffices to show that there exists an absolute constant \( A_0 > 0 \) such that
\[
\| \sum_{k \in \mathbb{Z}} \sum_{l < k} P_{I_{k,l}^+}(\psi_{k,l}) \|_{L^2(\sigma)} \leq A_0[\sigma]_{A_2}^{1/2} \left( \sum_{k \in \mathbb{Z}} \sum_{l < k} |\psi_{k,l}|^2 \right)^{1/2}
\]
and
\[
\| \sum_{k \in \mathbb{Z}} \sum_{l < k} P_{I_{k,l}^-}(\psi_{k,l}) \|_{L^2(\sigma)} \leq A_0[\sigma]_{A_2}^{1/2} \left( \sum_{k \in \mathbb{Z}} \sum_{l < k} |\psi_{k,l}|^2 \right)^{1/2}
\]
for all \( A_2 \) weights \( \sigma \) on \( \mathbb{R} \) and for any collection of Schwartz functions \( \{\psi_{k,l}\}_{k > l} \), where only finitely many of the functions \( \psi_{k,l} \) are non-zero. Here for \( k, l \in \mathbb{Z} \) with \( k > l \), we use the notation \( I_{k,l}^+ := [2^{-k} - 2^{-l}, 2^{-k} - 2^{-l-1}) \) and \( I_{k,l}^- := [-2^{-k} + 2^{-l-1}, -2^{-k} + 2^{-l}) \).

By using [18, Theorem 2.7], one deduces that there exist dyadic lattices \( I_j, j \in \{1, 2, 3\} \), and a \( \phi \in C_{c}^\infty(\mathbb{R}) \) with \( \text{supp}(\phi) \subseteq [-1/2, 1/2] \cup [1/2, 2] \) so that
\[
\| \sum_{k \in \mathbb{Z}} \sum_{l < k} P_{I_{k,l}^+}(\psi_{k,l}) \|_{L^2(\sigma)} \leq [\sigma]_{A_2}^2 \left( \sum_{k \in \mathbb{Z}} \sum_{l < k} |\psi_{k,l}|^2 \right)^{1/2} S_{\phi,I_j} \left( \sum_{k \in \mathbb{Z}} \sum_{l < k} P_{I_{k,l}^-}(\psi_{k,l}) \right)_{L^2(\sigma)}
\]
where
\[
S_{\phi,I_j}(g)(x) := \left( \sum_{\nu \in \mathbb{Z}} \sum_{l \in I_j} \left( \frac{1}{|I|} \int_I |\hat{P}_\nu(g)(x')|^2 dx' \right) \chi_l(x) \right)^{1/2}
\]
and \( \hat{P}_\nu \) denotes the multiplier operator with symbol \( \phi(2^{-\nu}\xi) \), \( \xi \in \mathbb{R} \). For the definition of dyadic lattices and their basic properties, we refer the reader to [19]. Notice that
\[
S_{\phi,I_j} \left( \sum_{k \in \mathbb{Z}} \sum_{l < k} P_{I_{k,l}^+}(\psi_{k,l}) \right)(x) \leq \Gamma_{I_j,0}(x) + \Gamma_{I_j,1}(x) \quad \text{for } j \in \{1, 2, 3\},
\]
where
\[
\Gamma_{I_{j},r}(x) := \left( \sum_{\nu \in \mathbb{Z}} \sum_{I \in \mathcal{L}_j: |I| = 2^{-\nu}} \left[ \frac{1}{|I|} \int_I \tilde{P}_\nu \left( \sum_{l < \nu + r} P_{I_{\nu-l}}(\psi_{\nu+l})(y) \right) \right]^{2} dy \right)^{1/2} \chi_J(x),
\]

\( r \in \{0, 1\} \). In view of (2.3), to prove (2.1) it suffices to show that
\[
\|\Gamma_{I_{j},r}\|_{L^2(\sigma)} \leq \left[ \sigma \right]^{3/2} \left( \sum_{k \in \mathbb{Z}} \sum_{l < k} |\psi_{k,l}|^2 \right)^{1/2} \left( \sum_{k \in \mathbb{Z}} \sum_{l < k} |\psi_{k,l}|^2 \right)^{1/2} L^2(\sigma)
\]

for \( j \in \{1, 2, 3\} \) and \( r \in \{0, 1\} \).

Fix a \( j \in \{1, 2, 3\} \). We shall only focus on the proof of (2.5) for \( r = 0 \), as the other case is treated similarly. Observe that one can write
\[
\left| \tilde{P}_\nu \left( \sum_{l < \nu} P_{I_{\nu-l}}(\psi_{\nu,l}) \right) \right| = \left| \sum_{l < \nu} P_{I_{\nu-l}}(\rho_{\nu,l}) \right|,
\]

where \( I_{\nu-l} = [-2^{\nu-l}, -2^{\nu-l-1}) \), \( \rho_{\nu,l} : = \tilde{\theta}_\nu \ast \tilde{\psi}_{\nu,l} \) with \( \tilde{\theta}_\nu(x) = \hat{\phi}(2^{-\nu}(\xi + 2^{\nu})) \), \( \xi \in \mathbb{R} \), and \( \tilde{\psi}_{\nu,l}(x) = e^{-i2\pi 2^{\nu}x} \psi_{\nu,l}(x) \), \( x \in \mathbb{R} \). Hence, to prove (2.5) for \( r = 0 \), it suffices to show that
\[
\sum_{\nu \in \mathbb{Z}} \left\| \sum_{l < \nu} P_{I_{\nu-l}}(\rho_{\nu,l}) \right\|_{L^2(\sigma, I_j)}^2 \leq \left[ \sigma \right]^{3/2} \left( \sum_{k \in \mathbb{Z}} \sum_{l < k} |\psi_{k,l}|^2 \right)^{1/2} \left( \sum_{k \in \mathbb{Z}} \sum_{l < k} |\psi_{k,l}|^2 \right)^{1/2} L^2(\sigma)
\]

for \( j \in \{1, 2, 3\} \), where
\[
\sigma_{\nu,I_{j}}(x) := \sum_{I \in \mathcal{L}_j: |I| = 2^{-\nu}} \langle \sigma \rangle_I \chi_I(x).
\]

To prove (2.7), fix a \( \nu \in \mathbb{Z} \) and note that by employing [18, Lemma 3.1] and [18, Theorem 2.7] one deduces that
\[
\left\| \sum_{l < \nu} P_{I_{\nu-l}}(\rho_{\nu,l}) \right\|_{L^2(\sigma, I_{j})} \leq \left[ \sigma \right]^{1/2} \sum_{j' = 1}^{3} \left\| \sum_{l < \nu} P_{I_{\nu-l}}(\rho_{\nu,l}) \right\|_{L^2(\sigma, I_{j})}
\]

for \( j' \in \{1, 2, 3\} \). Observe that since \( \text{supp}(\phi) \subseteq [-2, -1/2] \cup [1/2, 2] \), one has
\[
S_{\phi,I_{j'}} \left( \sum_{l < \nu} P_{I_{\nu-l}}(\rho_{\nu,l}) \right) \leq \sum_{r' \in (-1,0,1)} K_{I_{j'}, \nu, r'},
\]

where for \( r' \in \{0, 1\} \) we have
\[
K_{I_{j}, \nu, r'}(x) := \left( \sum_{\mu < \nu - 1} \sum_{J \in \mathcal{L}_{j}: |J| = 2^{-\mu}} \left[ \frac{1}{|J|} \int_J \tilde{P}_\mu \left( P_{I_{\mu+r}}(\rho_{\nu,\mu+r}) \right)(y) \right]^{2} dy \right)^{1/2} \chi_J(x)
\]

and for \( r' = -1 \),
\[
K_{I_{j}, \nu, -1}(x) := \left( \sum_{J \in \mathcal{L}_{j}: |J| = 2^{-(\nu-1)}} \left[ \frac{1}{|J|} \int_J \tilde{P}_{-1} \left( P_{I_{\mu-1}}(\rho_{\nu,\mu-1}) \right)(y) \right]^{2} dy \right)^{1/2} \chi_J(x).
\]
We shall prove that there exists an absolute constant $C_0 > 0$ such that for each $\nu \in \mathbb{Z}$ one has
\begin{equation}
(2.9) \quad \|K_{I_j, \nu, r'}\|_{L^2(\sigma, x_j)}^2 \leq C_0 [\sigma]_{A_2} \left( \sum_{I_k < I_j - 1} |\psi_{r,l}|^2 \right)^{1/2} \|L^2(\sigma)\|_2 \quad \text{for } r' \in \{0, 1\}
\end{equation}
and
\begin{equation}
(2.10) \quad \|K_{I_j, \nu, r'}\|_{L^2(\sigma, x_j)}^2 \leq C_0 [\sigma]_{A_2} \|\psi_{r,\nu-1}\|_{L^2(\sigma)}^2
\end{equation}
for all $j' \in \{1, 2, 3\}$. We shall only provide the details of the proof of (2.9) for $r' = 0$, as the other cases, i.e., (2.9) for $r' = 1$ and (2.10) are treated similarly. To prove (2.10) for $r' = 0$, fix a $j' \in \{1, 2, 3\}$ and write
\begin{equation}
(2.11) \quad \|K_{I_j, \nu, r'}\|_{L^2(\sigma, x_j')}^2 = \sum_{\mu \leq \nu - 1} \|T_{m_{\nu, \mu}}(P_{I_j}(\tilde{\psi}_{\nu, \mu}))\|_{L^2(\sigma, x_{j'})}^2
\end{equation}
where $m_{\nu, \mu}(\xi) := \phi(2^{-\nu} \xi) \phi(2^{-\nu}(\xi + 2^r))$, $\xi \in \mathbb{R}$ and
\begin{equation}
(\sigma, x_j)_{\mu, x_j'}(x) := \sum_{j \in \mathbb{Z}: |j| = 2^{-\mu}} \langle \sigma_{x_j}, j \rangle_{j' x_j'}(x).
\end{equation}
Observe that since $\mu < \nu - 1$ one has
\begin{equation}
(2.12) \quad |(m_{\nu, \mu})(\xi)| \leq 2^{-\mu} |\phi'(2^{-\mu} \xi)| \|\phi\|_{L^\infty(\mathbb{R})} + 2^{-\mu} |\phi(2^{-\mu} \xi)| \|\phi'\|_{L^\infty(\mathbb{R})}
\end{equation}
for all $\xi \in \mathbb{R}$. Hence, by using (2.12) and an appropriate variant of [18, Lemma 3.2]; see Section 2.1, below, one gets
\begin{equation}
(2.13) \quad \|T_{m_{\nu, \mu}}(P_{I_j}(\tilde{\psi}_{\nu, \mu}))\|_{L^2((\sigma, x_j)_{\nu, x_j'})} \leq [\sigma]_{A_2} \|\psi_{\nu, \mu}\|_{L^2(\sigma)}^2,
\end{equation}
where we also used the fact that $|\tilde{\psi}_{\nu, \mu}| = |\psi_{\nu, \mu}|$.

By combining (2.11) with (2.13), one establishes (2.9) for $r' = 0$. One shows (2.9) for $r' = 1$ and (2.10) similarly. Notice that it follows from (2.8), (2.9), (2.10), and (2.7) that (2.6) holds. We thus obtain (2.5) for $r = 0$. The proof of (2.5) for $r = 1$ is similar. Therefore, the proof of (2.1) is now complete, in view of (2.3), (2.4), and (2.5). One shows (2.2) in an analogous way. We have thus established Theorem 2.

2.1. A variant of [18, Lemma 3.2]. In the previous section, inequality (2.13) was obtained by using the following variant of [18, Lemma 3.2].

**Lemma 3.** Let $\mathcal{I}, \mathcal{J}$ be two given dyadic lattices in $\mathbb{R}$ and let $\mu, \nu \in \mathbb{Z}$ be such that $\mu < \nu$.

If $m$ is bounded on an interval $K$ and differentiable in the interior of $K$ with
\begin{equation}
C_{m, K} := \|m\|_{L^\infty(K)} + \int_K |m'(\xi)| \, d\xi < \infty,
\end{equation}
then there exists an absolute constant $c_0 > 0$ such that
\begin{equation}
\|P_{K}(T_m(f))\|_{L^2((\sigma, x)_{\mu, \gamma})} \leq c_0 C_{m, K} \|f\|_{L^2(\sigma)}
\end{equation}
for every $A_2$ weight $\sigma$ on $\mathbb{R}$. 

[18]
Proof. Let $\mu, \nu$ be two given integers such that $\mu < \nu$ and let $\sigma$ be a given $A_2$ weight. We have
\[
\langle \sigma, \chi_J \rangle_{\mu, J}(x) = \sum_{j \in J, |j| = 2^{-\mu}} \langle \sigma, \chi_J \rangle_{J}(x) = \sum_{j \in J, |j| = 2^{-\mu}} \sum_{I \in J, |I| = 2^{-\nu}} \langle \sigma \rangle_{I \cap J} \frac{|I \cap J|}{|J|} \chi_J(x).
\]
Fix a $J \in \mathcal{J}$ with $|J| = 2^{-\nu}$. Notice that, by arguing as in the proof of [18, Lemma 3.2], one has
\[
|P_K(T_m(f))(y)| \leq A_0 C_{m,K} T_K(f)(x) + \int_K H^*(M_{-\sigma} f)(x) |m'(t)| dt
\]
for all $x, y \in 2J$, where $A_0 > 0$ is an absolute constant, $C_{m,K}$ is as in the statement of the lemma, and
\[
T_K(f)(x) := H^*(M_{-\sigma} f)(x) + H^*(M_{-\sigma} f)(x) + (2^{-\mu} |K| + 1) M(f)(x).
\]
Here $a$ and $b$ are the left and right endpoints of $K$, respectively. We write
\[
u(x) := A_0 C_{m,K} T_K(f)(x) + \int_K H^*(M_{-\sigma} f)(x) |m'(t)| dt \quad \text{for } x \in 2J.
\]
As in [18], one has
\[
\frac{1}{|J|} \int_J |P_K(T_m(f))(y)|^2 dy \leq \inf_{x \in 2J} [\nu(x)]^2
\]
and hence, for every $I \in \mathcal{I}$ with $|I| = 2^{-\nu}$ and $I \cap J \neq \emptyset$ one has
\[
\langle \sigma \rangle_{I \cap J} \frac{|I \cap J|}{|J|} \int_J |P_K(T_m(f))(y)|^2 dy \leq \int_I [\nu(x)]^2 \sigma(x) dx,
\]
where we used the fact that if $I \cap J \neq \emptyset$ and $2|I| \leq |J|$, then $I \subseteq 2J$. Hence, by using (2.15) and the fact that the intervals $I \in \mathcal{I}$ with $I \subseteq 2J$ and $|I| = 2^{-\nu}$ have mutually disjoint interiors and their union is contained in $2J$, we deduce that
\[
\sum_{I \in \mathcal{I}, |I| = 2^{-\nu}} \langle \sigma \rangle_{I \cap J} \frac{|I \cap J|}{|J|} \int_J |P_K(T_m(f))(y)|^2 dy \leq \int_{2J} [\nu(x)]^2 \sigma(x) dx
\]
for every $J \in \mathcal{J}$ with $|J| = 2^{-\nu}$.

Since the intervals $J$ in $\mathcal{J}$ with $|J| = 2^{-\mu}$ ‘tile’ $\mathbb{R}$, one deduces from (2.10) that
\[
\|P_K(T_m(f))\|_{L^2(\sigma, \chi_J), \mu, J} \leq 2 \|\nu\|_{L^2(\sigma)}.
\]
Arguing as in [18], one completes the proof of the Lemma 3 by using (2.17), Minkowski’s inequality, as well as the well-known bounds $\|M\|_{L^2(\sigma), \mu, J} \lesssim [\sigma]_{A_2}$ and $\|H^*\|_{L^2(\sigma), \mu, J} \lesssim [\sigma]_{A_2}$; see [14] and [15] for more general results involving two-weighted inequalities. \qed

3. Proof of Theorem 1

Arguing as in [18, Remark 4.1], the upper estimate in Theorem 1 follows from Theorem 2 combined with [3, Theorem 3.1].

The lower estimate in Theorem 1 is obtained by adapting ideas from Bourgain’s paper [4] to our case. To be more specific, fix a Schwartz function $\eta$ satisfying the properties $\supp(\eta) \subseteq [1/2, 4]$ and $\eta_{[1,2]} \equiv 1$. For $N \in \mathbb{N}$, define $\eta_N$ by
\[
\eta_N(\xi) := \hat{\eta}(N^{-1} \xi), \quad \xi \in \mathbb{R}.
\]
Notice that, since $\|\eta_N\|_{L^1(\mathbb{R})} = \|\eta\|_{L^1(\mathbb{R})}$ and $\|\eta_N\|_{L^2(\mathbb{R})} = N^{1/2} \|\eta\|_{L^2(\mathbb{R})}$, one deduces
\begin{equation}
(3.1) \quad \|\eta_N\|_{L^p(\mathbb{R})} \lesssim N^{(p-1)/p} \quad (1 < p < 2).
\end{equation}
Fix a $p \in (1, 2)$ ‘close’ to $1^+$ and choose $N \in \mathbb{N}$ such that
\begin{equation}
(3.2) \quad \log N \sim (p - 1)^{-1}.
\end{equation}
By using (3.1) and (3.2), Minkowski’s inequality, and Hölder’s inequality, we have
\begin{align*}
\|S_{\mathcal{F}_2} \|_{L^p(\mathbb{R})} \lesssim \|S_{\mathcal{F}_2}(\eta_N)\|_{L^p(\mathbb{R})} \geq \|S_{\mathcal{F}_2}(\eta_N)\|_{L^p([0,1])}

& \geq \left( \sum_{k \in \mathbb{Z}, l < k} \left\| P_{I_{k,l}}(\eta_N) \right\|^2_{L^p([0,1])} \right)^{1/2}

& \geq \left( \sum_{k = 2}^{\log N} \sum_{l = 1}^{k-1} \left\| P_{I_{k,l}}(\eta_N) \right\|^2_{L^p([0,1])} \right)^{1/2},
\end{align*}
where $[x]$ denotes the integer part of $x \in \mathbb{R}$. Since $\eta_N(\xi) = 1$ for $\xi \in [N, 2N]$, one can easily check that for all integers $k, l$ with $2 \leq k \leq [\log N]$ and $1 \leq l < k$ one has
\[ |P_{I_{k,l}}(\eta_N)(x)| \sim \left| \frac{\sin(\pi lx)}{x} \right| \text{ for all } x \neq 0.\]
Hence, a standard computation yields that
\[ \|P_{I_{k,l}}(\eta_N)\|_{L^1([0,1])} \sim \log \left( 4|I_{k,l}| \right) \sim l \]
for all $k, l \in \mathbb{N}$ with $2 \leq k \leq [\log N]$ and $1 \leq l < k$. We thus have
\[ \|S_{\mathcal{F}_2} \|_{L^p(\mathbb{R})} \lesssim \left( \sum_{k = 2}^{\log N} \sum_{l = 1}^{k-1} 1 \right)^{1/2} \sim (\log N)^2,\]
which, together with (3.2), shows that the lower estimate in Theorem 1 holds true.

4. SOME FURTHER REMARKS

4.1. Periodic versions of Theorems 1 and 2. If $f$ is a trigonometric polynomial on $\mathbb{T}$, define $S_2(f)$ by
\[ S_2(f)(\theta) := \left( |\hat{f}(0)|^2 + \sum_{k,l \in \mathbb{N}_0: k > l} \left[ |\Delta_{I_{k,l}^+}(f)(\theta)|^2 + |\Delta_{I_{k,l}^-}(f)(\theta)|^2 \right] \right)^{1/2} \quad (\theta \in \mathbb{T}),\]
where
\[ \Delta_{I_{k,l}^+}(f)(\theta) := \sum_{n \in I_{k,l}^+} \hat{f}(n)e^{2\pi n\theta} \quad (\theta \in \mathbb{T}) \]
and
\[ \Delta_{I_{k,l}^-}(f)(\theta) := \sum_{n \in I_{k,l}^-} \hat{f}(n)e^{2\pi n\theta} \quad (\theta \in \mathbb{T}) \]
with $I_{k,l}^+$ and $I_{k,l}^-$ being as in the Euclidean case; $I_{k,l}^+ := [2^k - 2^l, 2^k - 2^l - 1)$ and $I_{k,l}^- := [2^k - 2^l - 1, 2^k - 2^l)$. A straightforward adaptation of the argument of Section 2 to the periodic setting yields
\begin{equation}
(4.1) \quad \|S_2(f)\|_{L^2(w)} \lesssim [w]_{A_2(\mathbb{T})}^2 \|f\|_{L^2(\mathbb{T})},
\end{equation}
for all periodic $A_2$ weights $w$ and for every trigonometric polynomial $f$ on $\mathbb{T}$, where the implied constant in (4.1) is independent of $w$ and $f$. Recall that a non-negative integrable function $\sigma$ on $\mathbb{T}$ is said to be a periodic $A_2$ weight if, and only if,

$$\sup_{I \subseteq \mathbb{T}^*} \text{arc} x_\sigma^y I_x^\sigma < \infty.$$  

Moreover, by using (4.1) and a periodic version of [9, Theorem 3.1] as well as an adaptation of the argument of the previous section to the periodic setting, one deduces that there exist absolute constants $c_1, c_2 > 0$ such that

$$\frac{c_1}{(p-1)^2} \leq |S_2|_{L^p(\mathbb{T}) \rightarrow L^p(\mathbb{T})} \leq \frac{c_2}{(p-1)^2} \quad (1 < p \leq 2).$$  

4.2. Littlewood-Paley operators formed with respect to certain lacunary sets of finite order. By arguing as in the proof of Theorem 2, one can show that

$$\|S_{E_2}\|_{L^2(w) \rightarrow L^2(w)} \lesssim [w]_{A_2}^2,$$  

where $E_2 := \{ \pm (2^k + 2^l) : k, l \in \mathbb{Z} \text{ with } k > l \}$. More generally, if one considers the lacunary set of order $N \in \mathbb{N}$ (with $N \geq 2$) given by

$$E_N := \{ \pm (2^{k_1} + \cdots + 2^{k_N}) : k_1, \ldots, k_N \in \mathbb{Z} \text{ with } k_1 > \cdots > k_N \}$$

and $S_{E_N}$ denotes the corresponding Littlewood-Paley operator, then by suitably modifying and iterating the first part of the proof of Theorem 2 and next, by using an appropriate extension of Lemma 3 one can show that

$$\|S_{E_N}\|_{L^2(w) \rightarrow L^2(w)} \lesssim [w]_{A_2}^{1+N/2}$$  

for any $A_2$ weight $w$ on $\mathbb{R}$, where the implied constant in 4.3 depends only on $N$; we omit the details. By using 4.3 and an adaptation of the argument of Section 3 one gets

$$\|S_{E_N}\|_{L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})} \sim (p-1)^{-(1+N/2)} \quad (1 < p \leq 2),$$  

where the implied constants in 4.4 depend only on $N$. Analogous results hold in the periodic setting.

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