Aspects of the screw function corresponding to the Riemann zeta-function

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Abstract
We introduce a screw function corresponding to the Riemann zeta-function and study its properties from various aspects. Typical results are several equivalent conditions for the Riemann hypothesis in terms of the screw function. One of them can be considered an analog of so-called Weil's positivity or Li's criterion. In addition, we prove a few partial but unconditional results for such equivalents.

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1 INTRODUCTION

Let \( \xi(s) \) be the Riemann zeta-function and let \( \xi(s) \) be the Riemann xi-function. The latter is an entire function defined by

\[
\xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)
\]

and satisfies two functional equations \( \xi(s) = \xi(1-s) \) and \( \xi(s) = \overline{\xi(\overline{s})} \), where \( \Gamma(s) \) is the gamma-function and the bar denotes the complex conjugate.

Typical results of the present paper are several equivalent conditions for the Riemann hypothesis (RH, for short) that claims that all zeros of \( \xi(s) \) lie on the critical line \( \Re(s) = 1/2 \). The core of the interrelation among all such equivalents is the function \( \Psi(t) \) on \([0, \infty)\).
defined by
\[ \Psi(t) := 4(e^{t/2} + e^{-t/2} - 2) - \sum_{n \leq e^t} \frac{\Lambda(n)}{\sqrt{n}} (t - \log n) 
+ \frac{t}{2} \left[ \frac{\Gamma'(1/4)}{\Gamma(1/4)} - \log \pi \right] + \frac{1}{4} \left( C - e^{-t/2} \Phi(e^{-2t}, 2, 1/4) \right), \]

where \( \Lambda(n) \) is the von Mangoldt function defined by \( \Lambda(n) = \log p \) if \( n = p^k \) with \( k \in \mathbb{Z}_{>0} \) and \( \Lambda(n) = 0 \); otherwise, \( C = \pi^2 + 8G \) with the Catalan constant \( G = \sum_{n=0}^{\infty} (-1)^n (2n + 1)^{-2} \), and \( \Phi(z, s, a) = \sum_{n=0}^{\infty} (n + a)^{-s} z^n \) is the Hurwitz–Lerch zeta function. Formula (1.1) shows that \( \Psi(t) \) is real-valued and continuous on \([0, \infty)\) and that \( \Psi(0) = 0 \) by \( \Phi(1, 2, 1/4) = \zeta(2, 1/4) = C \). First, we state a few fundamental properties of \( \Psi(t) \) that are unconditionally proven in Section 2 below.

**Theorem 1.1.** The following holds for \( \Psi(t) \) defined in (1.1).

1. The one-sided Fourier transform formula
\[ \int_0^\infty \Psi(t) e^{izt} \, dt = -\frac{1}{z^2} \frac{\xi'(1/2 - iz)}{\xi(1/2 - iz)} \]
holds if \( \mathfrak{Re}(z) > 1/2 \), where \( i = \sqrt{-1} \).

2. The series representation
\[ \Psi(t) = \sum_{\gamma} \frac{1 - \cos(\gamma t)}{\gamma^2} = \sum_{\gamma} \frac{1 - e^{i\gamma t}}{\gamma^2} \]
holds for every \( t \geq 0 \), where the sums in the middle and the right-hand side range over all zeros \( \gamma \) of \( \xi(1/2 - iz) \) counting with multiplicity.

3. The estimate \( \Psi(t) \ll \exp(t/2 - c \sqrt{t}) \) holds for some constant \( c > 0 \).

In Theorem 1.1 and in what follows, we use the Vinogradov symbol \( f(x) \ll g(x) \) and the Landau symbol \( f(x) = O(g(x)) \) as symbols to mean that there exists a positive constant \( M \) such that \( |f(x)| \leq M g(x) \) holds for a prescribed range of \( x \).

By the first equality in (1.3), the function \( \Psi(t) \) is naturally extended to an even function on the real line. Therefore, we henceforth identify \( \Psi(t) \) with that extended even function, that is, we understand by replacing \( t \) with \( -t \) in the right-hand side of (1.1) when \( t \) is negative. Also, by Theorem 1.1, any of (1.1), (1.3), or the Fourier inversion of (1.2) can be chosen as the definition of \( \Psi(t) \), but in this paper, we chose definition (1.1) including prime numbers.

Formulas (1.2) and (1.3) suggest that the function \( \Psi(t) \) is related to the class of screw functions introduced by M. G. Kreîn. For \( 0 < a \leq \infty \), he introduced the class \( \mathcal{G}_a \) consisting of all continuous functions \( g(t) \) on \((-2a, 2a)\) such that \( g(-t) = \overline{g(t)} \) (hermitian) and the kernel
\[ G_g(t, u) := g(t - u) - g(t) - g(-u) + g(0) \]

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is nonnegative definite on \((-a, a)\), that is,

\[
\sum_{i,j=1}^{n} G_g(t_i, t_j) \overline{\xi_i} \overline{\xi_j} \geq 0
\]  
(1.5)

for all \(n \in \mathbb{N}\), \(\xi_i \in \mathbb{C}\), and \(|t_i| < a\), \((i = 1, 2, \ldots, n)\). (In the literature, a kernel satisfying (1.5) is often referred to as a positive definite kernel or semipositive definite kernel, but in this paper, we use the term above.) The members of \(G_a\) are called screw functions because of their relationship with screw arcs in Hilbert spaces ([16, §12]).

Let \(\mathcal{N}\) be the Nevanlinna class that consists of analytic functions in the upper half-plane \(\mathbb{C}_+ = \{z \mid \Im(z) > 0\}\) mapping \(\mathbb{C}_+\) into \(\mathbb{C}_+ \cup \mathbb{R}\). (Note that \(\mathcal{N}\) is also called the class of Pick functions, or R functions, or Herglotz functions depending on the literature.) Krein–Langer [14, Satz 5.9] ([15, Prop. 5.1]) showed that the equality

\[
\int_{0}^{\infty} g(t)e^{izt} \, dt = -\frac{i}{z^2}Q(z), \quad \Im(z) > h
\]

for some \(h \geq 0\) establishes a bijective correspondence between all functions \(g \in G_\infty\) with \(g(0) = 0\) and all functions \(Q \in \mathcal{N}\) with the property

\[
\lim_{y \to +\infty} \frac{Q(iy)}{y} = 0.  
\]  
(1.6)

On the other hand, J. C. Lagarias [18, (1.5)] proved that the RH is true if and only if

\[
\Im\left[i \frac{\xi'}{\xi} \left(\frac{1}{2} - iz\right)\right] > 0 \quad \text{when} \quad \Im(z) > 0.
\]

The latter means that the function

\[
Q_\xi(z) := i \frac{\xi'}{\xi} \left(\frac{1}{2} - iz\right)
\]  
(1.7)

belongs to \(\mathcal{N}\). It is easy to confirm that \(Q_\xi(z)\) satisfies (1.6) unconditionally by Dirichlet series expansion of \((\zeta'/\zeta)(s)\) and an asymptotic expansion of \((\Gamma'/\Gamma)(s)\) as \(|s| \to \infty\) (see (4.10)). Therefore, if we assume that the RH is true, \(Q_\xi(z)\) belongs to \(\mathcal{N}\) and satisfies (1.6), and thus, there exists a corresponding screw function \(g \in G_\infty\). Such \(g\) must be equal to the function defined by

\[
g(t) := -\Psi(t)
\]  
(1.8)

from the Fourier integral formula (1.2) and the uniqueness of the Fourier transform. Conversely, assuming that this \(g(t)\) is a screw function, the RH holds from (1.2) and the above result of Krein–Langer.

Hence, we obtain the following equivalent condition for the RH that is the starting point of other equivalent conditions for the RH in the present paper.

**Theorem 1.2.** The RH is true if and only if \(g(t)\) defined by (1.8) is a screw function on \(\mathbb{R} = (-\infty, \infty)\).

Henceforth throughout this paper, \(g(t)\) is the function defined by (1.1) and (1.8), and \(G_g(t, u)\) is the kernel defined by (1.4) for this \(g(t)\), unless stated otherwise. We call \(g(t)\) the screw function.
of the Riemann zeta-function after Theorem 1.2, although it is an abuse of words in a strict sense. The series representation of \( g(t) \) obtained from (1.3) is nothing but an integral representation of a screw function ([16, Theorem 5.1 and (7.11)]) and implies

\[
G_g(t, u) = \sum_\gamma \frac{(e^{i\gamma t} - 1)(e^{-i\gamma u} - 1)}{\gamma^2}
\]

(cf. the second line of the proof of [16, Theorem 5.1]). The kernel \( G_g(t, u) \) is nonnegative on \( \mathbb{R}^2 \) under the RH, but it can be shown unconditionally that \( G_g(t, u) \) is nonnegative if \( |t| < a \) and \( |u| < a \) for small \( a > 0 \) (see Section 4).

As a main application of Theorem 1.2, we obtain a variant of Weil’s famous criterion for the RH by the positivity of the Weil distribution (see Section 3.2) in terms of screw functions as follows. For each \( 0 < a \leq \infty \), we define the hermitian form \( \langle \cdot, \cdot \rangle_{G_g,a} \) for functions supported in \([-a, a]\) by

\[
\langle \phi_1, \phi_2 \rangle_{G_g,a} := \int_{-a}^{a} \int_{-a}^{a} G_g(t, u)\phi_1(u)\overline{\phi_2(t)} \, du \, dt
\]

when the right-hand side is absolutely convergent. We also define the space

\[
\mathcal{C}_0(a) := \left\{ \phi \in C_c^\infty(\mathbb{R}) \mid \text{supp} \phi \subset [-a, a], \int_{-a}^{a} \phi(t) \, dt = 0 \right\},
\]

where \( C_c^\infty(\mathbb{R}) \) is the space of all smooth functions on the real line having compact support. Note that \( \mathcal{C}_0(a) \) is not the class of continuous functions on \((-a, a)\).

**Theorem 1.3.** The RH is true if and only if the hermitian form \( \langle \cdot, \cdot \rangle_{G_g,a} \) is nonnegative definite on \( \mathcal{C}_0(a) \), that is, \( \langle \phi, \phi \rangle_{G_g,a} \geq 0 \) for all \( \phi \in \mathcal{C}_0(a) \) for every \( 0 < a < \infty \). Moreover, assuming that the RH is true, \( \langle \cdot, \cdot \rangle_{G_g,a} \) is positive definite on \( L^2(-a, a) \), that is, \( \langle \phi, \phi \rangle_{G_g,a} > 0 \) for all nonzero \( \phi \in L^2(-a, a) \) for every \( 0 < a < \infty \).

See Section 3 for details and more on the Weil distribution and Weil’s positivity criterion. In addition to Theorem 1.3, we obtain the following analog of the equivalent of the RH by H. Yoshida [33] (see also Section 4.2) described by the nondegeneracy of hermitian forms.

**Theorem 1.4.** The RH is true if and only if the hermitian form \( \langle \cdot, \cdot \rangle_{G_g,a} \) is nondegenerate on \( L^2(-a, a) \) for every \( 0 < a < \infty \). The latter condition is equivalent that the integral operator \( G_g[a] : L^2(-a, a) \to L^2(-a, a) \) defined by

\[
G_g[a] : \phi(t) \mapsto 1_{[-a,a]}(t) \int_{-a}^{a} G_g(t, u)\phi(u) \, du
\]

does not have zero as an eigenvalue for every \( 0 < a < \infty \), where \( 1_A(t) \) is the characteristic function of a subset \( A \subset \mathbb{R} \).

The advantage of Theorems 1.3 and 1.4 is that hermitian forms \( \langle \cdot, \cdot \rangle_{G_g,a} \) are represented by an integral operator with a continuous kernel acting on usual \( L^2 \)-spaces. This makes the strategy of Yoshida [33] and E. Bombieri [1] for the RH via the Weil distribution analytically more straightforward and simple.
It is well known that the continuity of the kernel is not sufficient to conclude that the corresponding integral operator is of trace class. If we assume that the RH is true, $G_g[a]$ is a positive definite Hilbert–Schmidt operator for every $0 < a < \infty$. Therefore, it is a trace class operator by Mercer’s theorem. Furthermore, the traceability of $G_g[a]$ does not depend on the definiteness of $G_g[a]$ as follows.

**Theorem 1.5.** For each $0 < a < \infty$, $G_g[a]$ is a trace class operator unconditionally.

Since $G_g[a]$ is a trace class operator,

$$\text{Tr} G_g[a] = \sum_{n=1}^{\infty} \lambda_{a,n} = \int_{-a}^{a} G_g(t, t) \, dt$$

holds by [9, Chapter IV, Theorem 8.1], where $\lambda_{a,1}, \lambda_{a,2}, \ldots$ are nonzero eigenvalues of $G_g[a]$ counting with multiplicity.

We state further applications of Theorem 1.2 which are somewhat secondary to Theorems 1.3 and 1.4, but they are interesting in their own right, or the relations between different subjects suggested by those results are interesting.

The function $\Psi(t)$ is bounded on $[0, \infty)$ by (1.3) under the RH, and vice versa.

**Theorem 1.6.** The RH is true if and only if $\Psi(t) = O(1)$ on $[0, \infty)$.

If $\Psi(t)$ belongs to $L^1(0, \infty)$ or $L^2(0, \infty)$, formula (1.2) holds for $\mathfrak{S}(z) > 0$ and defines an analytic function in $\mathbb{C}_+$ whose extension to the real line $\mathfrak{S}(z) = 0$ is bounded or a function of $L^2(\mathbb{R})$, respectively. Therefore, in both cases, it contradicts $z = 0$ being a simple pole of the right-hand side. Hence, $\Psi(t)$ belongs to neither $L^1(0, \infty)$ nor $L^2(0, \infty)$, regardless of whether the RH is true or false.

If the RH is true, $G_g(t, t) = 2(g(0) - g(t)) = 2\Psi(t)$ is nonnegative by (1.5) for $n = 1$ and $\xi = 1$, and vice versa.

**Theorem 1.7.** The RH is true if and only if $\Psi(t)$ is pointwise nonnegative, that is, $\Psi(t) \geq 0$ for every $t \in \mathbb{R}$. Further, if RH is true, $\Psi(t) > 0$ when $t \neq 0$.

We then discretize the pointwise positivity condition in Theorem 1.7 using the $n$th moment

$$\mu_n := \int_{0}^{\infty} 4^{-1} e^{-t/2} \Psi(t) \cdot t^n \, dt$$

for $n \in \mathbb{Z}_{\geq 0}$, where the integral is absolutely convergent by Theorem 1.1 (3).

**Theorem 1.8.** Let

$$\Delta_n := \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{pmatrix} \quad \text{and} \quad \Delta_n^{(1)} := \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \mu_2 & \mu_3 & \cdots & \mu_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n+1} & \mu_{n+2} & \cdots & \mu_{2n+1} \end{pmatrix}$$
be Hankel matrices consisting of moments $\mu_k$ defined by (1.13). Then the RH is true if and only if 
$$\det \Delta_n \geq 0$$
and 
$$\det \Delta^{(1)}_n \geq 0$$
for all $n \in \mathbb{Z}_{>0}$.

In [19], X.-J. Li proved the so-called Li’s criterion that claims that the RH is true if and only if all Li coefficients defined by

$$\lambda_{n+1} = \frac{1}{n!} \frac{d^n}{dw^n} \left[ \frac{1}{(1 - w)^2} \frac{\xi'}{\xi} \left( \frac{1}{1 - w} \right) \right]_{w=0}$$ (1.14)

are positive. Since the sequence $\{n^{-1}\lambda_n\}_{n=1}^{\infty}$ was considered by J. B. Keiper [11] before Li, the Li coefficients are also referred to as the Keiper–Li coefficients in some literature. Bombieri–Lagarias [2] found that the positivity of all Li coefficients is a discretization of the positivity of the Weil distribution. The relation of Theorems 1.3 and 1.8 can be considered as an analog of the relation of the Weil distribution and Li coefficients (see also Section 3.4). We are then interested in a direct relation between moments $\mu_n$ and Li coefficients $\lambda_n$. Changing of variables as $z = (i/2)(1 - 2X)$ in (1.2) and then expanding $\exp(itz)$ to the power series of $X$, we obtain

$$\mu_n = \frac{d^n}{dX^n} \left[ \frac{1}{(1 - 2X)^2} \frac{\xi'}{\xi} (1 - X) \right]_{X=0}.$$ (1.15)

The similarity between (1.14) and (1.15) is obvious, and it is shown in Section 8 that there is an explicit relation between them.

Recall (1.7) and define $q_{\zeta}(z) := Q_{\zeta}(\sqrt{z})/\sqrt{z}$. Then, $q_{\zeta}(z)$ belongs to the subclass of $\mathcal{N}$ corresponding to Krein’s strings under the RH (see Section 9). The corresponding string is the one named Zeta string by S. Kotani [12].

On the other hand, the series representation (1.3) suggests that $g(t) = -\Psi(t)$ is a mean-periodic function by appropriate choice of function space. In fact, it is unconditionally shown that $g(t)$ is mean periodic using the function spaces used in Fesenko–Ricotta–Suzuki [8] (see Section 10).

The functions in class $\mathcal{C}_a$ have various fruitful connections with many different mathematical objects, for example, due to a relation with positive-definite functions described in [16]. Therefore, more interesting discoveries and connections are expected for the screw function of the Riemann zeta-function from the origin mentioned above. Furthermore, although this paper focused only on the Riemann zeta-function for simplicity, it would be natural to extend the results of this paper to other general $L$-functions like automorphic $L$-functions or $L$-functions in the Selberg class.

However, we leave such studies for other papers [22, 25–28] and future research.

This paper is organized as follows. In Section 2, we prove Theorem 1.1 and a lemma needed in later sections. In Section 3, we prove Theorem 1.3 after reviewing the Weil distribution and preparing for the notation. In Section 4, we establish the pointwise nonnegativity of $\Psi(t)$, and the nonnegativity of $G_{\alpha}(t, u)$ for small $t, u$, as a preliminary step toward proving Theorem 1.4. Moreover, we state and prove a lower bound for $\langle \phi, \phi \rangle_{G_{\alpha}}$ under restrictions to $\phi$. In Section 5, we prove Theorem 1.4 using results in Section 4 and make a comparison with Yoshida’s results. In Section 6, we prove Theorem 1.5 and study eigenvalues of the kernel $G_{\alpha}(t, u)$. In Section 7, we prove Theorems 1.6–1.8. In Section 8, we describe explicit relations between the moments $\mu_n$ and the Li coefficients. In Section 9, we describe a Krein string corresponding to $Q_{\zeta}(z)$ in (1.7) under the RH. In Section 10, we discuss the mean periodicity of the screw function $g(t)$. In Section 11, we introduce variants $\Psi_{\omega}(t)$ of $\Psi(t)$ and state an analog of Theorem 1.7. Further, we study relations among different $\Psi_{\omega}(t)$'s.
2 | PROOF OF THEOREM 1.1 AND A LEMMA

2.1 | Proof of Theorem 1.1 (1)

We use the change of variables $s = 1/2 - iz$ for convenience of writing. We first note that the equality

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s - 1} + \frac{1}{s} - \frac{1}{2} \log \pi + \frac{1}{2} \frac{\Gamma'(s/2)}{\Gamma(s/2)} - \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s}$$

holds for $\Re(s) > 1$, which is equivalent to $\Im(z) > 1/2$ in terms of $z$. We have

$$\int_{0}^{\infty} 4(e^{t/2} + e^{-t/2} - 2) e^{izt} \, dt = -\frac{1}{z^2}\left(\frac{1}{s - 1} + \frac{1}{s}\right) \text{ if } \Im(z) > 1/2,$$

(2.1)

$$\int_{0}^{\infty} t e^{izt} \, dt = -\frac{1}{z^2} \text{ if } \Im(z) > 0,$$

(2.2)

$$\int_{0}^{\infty} \frac{(t - \log n)}{\sqrt{n}} 1_{[\log n, \infty)}(t) e^{izt} \, dt = -\frac{1}{z^2} n^{-s} \text{ if } \Im(z) > 0$$

by direct and simple calculation. The third equality leads to

$$\int_{0}^{\infty} \left( - \sum_{n \leq e^t} \frac{\Lambda(n)}{\sqrt{n}} (t - \log n) \right) e^{izt} \, dt$$

$$= -\int_{0}^{\infty} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} (t - \log n) 1_{[\log n, \infty)}(t) e^{izt} \, dt$$

(2.3)

$$= -\frac{1}{z^2}\left(- \sum_{n=2}^{\infty} \frac{\Lambda(n)n^{-s}}{n}\right) \text{ if } \Im(z) > 1/2.$$

Hence, the proof is completed if it is proved that

$$\int_{0}^{\infty} \frac{1}{4} \left(C - e^{-t/2}\Phi(e^{-2t}, 2, 1/4)\right) e^{izt} \, dt = -\frac{1}{2z^2}\left(\frac{\Gamma'(s)}{\Gamma(s/2)} - \frac{\Gamma'(1/4)}{\Gamma(1/4)}\right)$$

(2.4)

holds for $\Im(z) > 0$. To prove this, we recall the well-known series expansion

$$\frac{\Gamma'(w)}{\Gamma(w)} = -\gamma_0 - \sum_{n=0}^{\infty} \left(\frac{1}{w + n} - \frac{1}{n + 1}\right),$$

(2.5)

where $\gamma_0$ is the Euler–Mascheroni constant. On the other hand, we have

$$\int_{0}^{\infty} \frac{1 - e^{-2(n + \frac{1}{4})t}}{2(n + \frac{1}{4})^2} e^{izt} \, dt = \frac{1}{z^2}\left(\frac{1}{s/2 + n} - \frac{1}{n + 1/4}\right)$$

for nonnegative integers $n$ and $\Im(z) > 0$ by direct and simple calculation. In addition, noting that $C = \pi^2 + 8G$ is a special value of the Hurwitz zeta function, precisely $C = \zeta(2, 1/4) = \pi^2 + 8G$. 

\[ \Phi(1, 2, 1/4) = \sum_{n=0}^{\infty} (n + 1/4)^{-2}, \]

we get

\[
\int_0^\infty \frac{1}{4} \left[ C - e^{-t/2} \Phi(e^{-2t}, 2, 1/4) \right] e^{izt} dt
\]

\[
= \frac{1}{2} \int_0^\infty \left( \sum_{n=0}^{\infty} \frac{1 - e^{-2(n+1/4)^2}}{2(n + 1/4)^2} \right) e^{izt} dt
\]

\[
= \frac{1}{2z^2} \sum_{n=0}^{\infty} \left( \frac{1}{s/2 + n} - \frac{1}{n + 1} \right) + \frac{1}{2z^2} \sum_{n=0}^{\infty} \left( \frac{1}{n + 1} - \frac{1}{n + 1/4} \right)
\]

\[
= -\frac{1}{2z^2} \left( \frac{\Gamma'(s/2)}{\Gamma(s/2)} + \gamma_0 \right) + \frac{1}{2z^2} \left( \frac{\Gamma'(1/4)}{\Gamma(1/4)} + \gamma_0 \right).
\]

We used (2.5) in the last equation. Hence, we obtain equality (2.4).

\section{2.2 \ Proof of Theorem 1.1 (2)}

Since \( \xi(s) \) is an order one entire function, Hadamard’s factorization theorem gives

\[
\xi \left( \frac{1}{2} - iz \right) = e^{a + bz} \prod_{\gamma} \left[ \left( 1 - \frac{z}{\gamma} \right) e^{\frac{z}{\gamma}} \right].
\]

Taking the logarithmic derivative of both sides,

\[
\frac{\xi'(s)}{\xi(s)} \left( \frac{1}{2} - iz \right) = ib + i \sum_{\gamma} \left( \frac{1}{z - \gamma} + \frac{1}{\gamma} \right), \quad (2.6)
\]

where the sum on the right-hand side converges absolutely and uniformly on every compact subset of \( \mathbb{C} \) outside the zeros \( \gamma \). Substituting \( z = 0 \) into (2.6), we have \( ib = (\xi'/\xi)(1/2) \). On the other hand, taking the logarithmic derivative of \( \xi(s) = \xi(1-s) \), \( (\xi'/\xi)(s) = -(\xi'/\xi)(1-s) \). Substituting \( s = 1/2 \) in this equality, we get \( (\xi'/\xi)(1/2) = 0 \); thus, \( ib = (\xi'/\xi)(1/2) = 0 \). For each term on the right-hand side of (2.6), we have

\[
-\frac{i}{z^2} \left( \frac{1}{z - \gamma} + \frac{1}{\gamma} \right) = \int_0^\infty \frac{1 - e^{-ity}}{\gamma^2} e^{izt} dt, \quad \Im(z) > \Im(\gamma)
\]

by direct calculation of the right-hand side, where \( |\Im(\gamma)| \leq 1/2 \) ([29, Theorem 2.12]).

Therefore, if \( \Psi(t) \) is the function defined by the right-hand side of (1.3), then (1.2) holds when \( \Im(z) > 1/2 \) by interchanging summation and integration, which is justified by the absolute convergence of the sum on the right-hand side of (1.3), and noting the symmetry \( \gamma \rightarrow -\gamma \) coming from the functional equation \( \xi(s) = \xi(1-s) \). The two functions \( \Psi(t) \) and \( \tilde{\Psi}(t) \) have the same Fourier transform, so they are identical. The second equality in (1.3) also follows from the symmetry \( \gamma \rightarrow -\gamma \). \qed
The aimed assertion follows from the following proposition and (1.1). The result is weaker than the one that follows from the best known zero-free region of \( \zeta(s) \), but it is sufficient to guarantee the convergence of the integral in (1.13).

**Proposition 2.1.** There exists \( c > 0 \) such that

\[
\sum_{n \leq e^t} \frac{\Lambda(n)}{\sqrt{n}} (t - \log n) = 4e^{t/2} + O\left( e^{t/2} e^{-c\sqrt{t}} \right)
\]  

holds for \( t > 0 \).

**Proof.** We write the left-hand side of (2.7) as \( \varphi(t) \). For \( X > 0 \) and \( c > 0 \), we have

\[
-\frac{1}{2\pi} \int_{-X+ic}^{X+ic} \frac{e^{-izt}}{z^2} \, dz = \begin{cases} 
    t + O\left( e^{ct} \min\left\{ \frac{1}{|t|X^2}, \frac{1}{\sqrt{X^2 + c^2}} \right\} \right) & (t > 0), \\
    0 + O\left( \frac{2X}{X^2 + c^2} \right) & (t = 0), \\
    0 + O\left( e^{ct} \min\left\{ \frac{1}{|t|X^2}, \frac{1}{\sqrt{X^2 + c^2}} \right\} \right) & (t < 0)
\end{cases}
\]

by a standard way of analytic number theory as in [29, §3.12]. Applying this to each term of the Dirichlet series expansion \(-\left( \zeta'/\zeta \right)(s) = \sum_{n=1}^{\infty} \Lambda(n)n^{-s}\), we have

\[
-\frac{1}{2\pi} \int_{-X+ic}^{X+ic} \frac{1}{z^2} \left[ -\frac{\zeta'}{\zeta} \left( \frac{1}{2} - iz \right) \right] e^{-izt} \, dz = \varphi(t) + O\left( \frac{\Lambda(e^t)}{e^{t/2} X^2 + c^2} \right)
\]  

for \( c > 1/2 \). In the remaining part of the proof, we take \( c \) as \( 1/2 + 1/t \).

In the sum on the right-hand side of (2.8), the contribution from \( n \) in the range \( n \leq e^{t/2} \) or \( n \geq 3e^{t/2} \) is

\[
\ll \frac{1}{X^2} \sum_n \frac{\Lambda(n)}{\sqrt{n}} e^{c(t-log n)} = \frac{e^{ct}}{X^2} \left[ -\frac{\zeta'}{\zeta} \left( \frac{1}{2} + c \right) \right] \ll \frac{t \cdot e^{t/2}}{X^2},
\]

since \( |t - \log n| \gg 1, e^{ct} = e \cdot e^{t/2}, \) and \((\zeta'/\zeta)(1/2 + c) = O(t)\) (by \( \zeta(s) = (s-1)^{-1} + O(1) \) near \( s = 1 \)). For \( n \) in the range \( e^{t/2} < n < e^t - 2 \), if we put \( [e^{t}] - n = r \),

\[
t - \log n = \log \frac{e^t}{n} = \log \frac{e^t}{[e^t] - r} \geq -\log \left( 1 - \frac{r}{[e^t]} \right) \geq \frac{r}{[e^t]}.
\]
Therefore, the contribution from \( n \) in this range is

\[
\ll \frac{e^t}{X^2} \sum_{2 \leq r < e^t/2} \frac{\Lambda([e^t] - r)}{\sqrt{[e^t] - r}} \cdot \frac{1}{r} \ll \frac{t \cdot e^t}{X^2} \sum_{2 \leq r < e^t/2} \frac{1}{\sqrt{e^t - r}} \cdot \frac{1}{r} \ll \frac{t \cdot e^{t/2}}{X^2} \sum_{2 \leq r < e^t/2} \frac{1}{r} \ll \frac{t^2 \cdot e^{t/2}}{X^2}.
\]

(2.10)

The same applies to the contribution from \( n \) in the range \( e^t + 2 < n < 3e^t/2 \). Finally, the contribution from \( n \) in the range \( e^t - 2 < n < e^t + 2 \) is

\[
\ll \frac{\Lambda(e^t)}{\sqrt{e^t}} e^{\frac{1}{c\log(1+O(1))}} \frac{1}{X} \ll \frac{t \cdot e^{-t/2}}{X},
\]

(2.11)

because

\[
\min \left\{ \left| \frac{1}{t - \log n} \right| X^2, \frac{1}{\sqrt{X^2 + c^2}} \right\} \leq \frac{1}{\sqrt{X^2 + c^2}} \leq \frac{1}{X}.
\]

From the estimates (2.9), (2.10), and (2.11) for the sum in (2.8), we obtain

\[
\varphi(t) = -\frac{1}{2\pi} \int_{-X+ic}^{X+ic} \frac{1}{z^2} \left[ -\frac{\zeta'}{\zeta} \left( \frac{1}{2} - iz \right) \right] e^{-izt} dz + O \left( \frac{t^2 \cdot e^{t/2}}{X^2} \right) + O \left( \frac{t \cdot e^{-t/2}}{X} \right)
\]

for \( c = 1/2 + 1/t \). Referring the zero-free region \( \sigma > 1 - c(\log |t| + 3)^{-1} \) \((c > 0)\) of \( \zeta(\sigma + it) \) ([29, Theorem 3.8]), we put \( \psi(x) = c'(\log |t| + 3)^{-1} \) \((0 < c' < c)\) and let \( C_X \) be the curve from \(-X + i(1/2 - \psi(-X))\) to \(X + i(1/2 - \psi(X))\) along \( y = 1/2 - \psi(x)\), where \( z = x + iy \). Then, moving the path of integration downward,

\[
\varphi(t) = 4e^{t/2} + O \left( \frac{t^2 \cdot e^{t/2}}{X^2} \right) + O \left( \frac{t \cdot e^{-t/2}}{X} \right)
\]

\[-\frac{1}{2\pi} \left( \int_{-X+ic}^{X+ic} + \int_{C_X} \right) \frac{1}{z^2} \left[ -\frac{\zeta'}{\zeta} \left( \frac{1}{2} - iz \right) \right] e^{-izt} dz.
\]

The first and third integrals on the right-hand side are estimated as \( \ll X^{2} e^{it} \log X \ll X^{-2} e^{t/2} \log X \) by the estimate \((\zeta'/\zeta)(\sigma + it) \ll \log(|t| + 3)\) ([29, (3.11.7)]). The integral on \( C_X \) is estimated as

\[
\ll \int_{-X}^{X} \frac{\log(|x| + 3)}{(|x| + 3)^2} e^{t(1/2 - \psi(x))} dx \ll e^{t/2} \exp \left( -\frac{c't}{\log(X + 3)} \right) \int_{-X}^{X} \frac{\log(|x| + 3)}{(|x| + 3)^2} dx \ll e^{t/2} \exp \left( -\frac{c't}{\log(X + 3)} \right) \log(X + 3).
\]
Therefore, we obtain
\[
\varphi(t) = 4e^{t/2} + O\left(\frac{e^{t/2}\log X}{X^2}\right) + O\left(\frac{e^{t/2} \exp \left(-\frac{c't}{\log(X+3)}\right) \log(X+3)}{X^2}\right) + O\left(\frac{t^2 \cdot e^{t/2}}{X^2}\right) + O\left(\frac{t \cdot e^{-t/2}}{X}\right).
\]

Finally, choosing \(X\) as \(\log X = t^{1/2}\), we obtain (2.7) for a properly taken \(c > 0\).

\[\square\]

### 2.4 A necessary lemma

The following lemma is often used in subsequent sections.

**Lemma 2.1.** Let \(F(z)\) be an entire function of the exponential type and let \(A\) be a complex number. Suppose that \(F(\gamma) = A\) for all zeros \(\gamma\) of \(\xi(1/2 - iz)\). Then \(F(z) = A\) as a function.

**Proof.** It suffices to deal only with the case of \(A = 0\). The number of zeros of \(F(z)\) in the disc \(|z| \leq r\) counted with multiplicity is \(O(r)\) by the assumption. On the other hand, the number of distinct zeros of \(\xi(1/2 - z)\) in the disc \(|z| \leq r\) is not \(O(r)\) by [29, Section 9.12], which was first proved by Littlewood. This is a contradiction if \(F \neq 0\).

The known result that the number of simple zeros of \(\zeta(s)\) on the critical line up to height \(T\) is bounded below by \(T \log T\) is more convenient for the proof of Lemma 2.1 (see Conrey [6] and his comments on the result of Levinson (1974) in the introduction).

### 3 PROOF OF THEOREM 1.3

We denote the Fourier transform of \(\phi(t)\) as
\[
\hat{\phi}(z) = \int_{-\infty}^{\infty} \phi(t) e^{itz} \, dt
\]
in this and the latter sections even when \(z\) is not a real number.

#### 3.1 Proof of necessity

Assuming that the RH is true, \(G_g(t, u)\) is nonnegative definite on the real line by Theorem 1.2. Therefore, \(\langle \phi, \phi \rangle_{G_g,a} = \langle G_g[a] \phi, \phi \rangle_{L^2} \geq 0\) for every \(\phi \in L^2(-a, a)\), since all eigenvalues of \(G_g[a]\) are nonnegative ([23, §8]). Moreover, the positive definiteness is directly proved as follows.

For integrable functions \(\phi_1(t)\) and \(\phi_2(t)\) with \(\text{supp} \phi_i \subset [-a, a] \ (i = 1, 2)\),
\[
\langle \phi_1, \phi_2 \rangle_{G_g,a} = \sum_{\gamma} \frac{\hat{\phi}_1(-\gamma) - \hat{\phi}_1(0)}{\gamma} \cdot \frac{\hat{\phi}_2(\gamma) - \hat{\phi}_2(0)}{\gamma} \tag{3.1}
\]
by (1.9). If we take $\phi = \phi_1 = \phi_2$ and suppose that the RH is true. Then all $\gamma$ are real, and thus $\hat{\gamma}\phi(\gamma) = \hat{\phi}(-\gamma)$. Therefore the right-hand side of (3.1) is equal to

$$\sum_{\gamma} \left| \frac{\hat{\phi}(\gamma) - \hat{\phi}(0)}{\gamma} \right|^2 \geq 0.$$  \hspace{1cm} (3.2)

The sum is positive for every nonzero $\phi(t)$ in $L^2(-a, a)$ by Lemma 2.1, since $\hat{\phi}(z)$ is a nonconstant entire function of the exponential type. Hence, the hermitian form $\langle \cdot, \cdot \rangle_{G_0, a}$ is positive definite for every $0 < a < \infty$ under the RH.

To prove the sufficiency of the nonnegative definiteness of the hermitian forms $\langle \cdot, \cdot \rangle_{G_0, a}$, we use Weil’s positivity criterion.

### 3.2  Weil’s positivity

The linear functional $W : C_c^\infty(\mathbb{R}) \to \mathbb{C}$ defined by

$$\psi \mapsto W(\psi) := \sum_{\gamma} \hat{\psi}(\gamma)$$  \hspace{1cm} (3.3)

is called the Weil distribution. A. Weil [30] (see also [33]) showed that the RH is true if and only if the distribution $W$ is nonnegative definite, that is,

$$W(\psi * \bar{\psi}) \geq 0 \quad \text{for every } \psi \in C_c^\infty(\mathbb{R}),$$  \hspace{1cm} (3.4)

where

$$(\psi_1 * \psi_2)(x) := \int_{-\infty}^{\infty} \psi_1(y)\psi_2(x - y)dy, \quad \bar{\psi}(x) := \overline{\psi(-x)}.$$  \hspace{1cm} (3.5)

Note that Weil does not mention that it is sufficient to take compactly supported functions as test functions at least in [30, 31] and his collected papers, but in Yoshida [33], the criterion has been formulated in the form above, although it is not sure whether it is the first literature.

### 3.3  Reduction of sufficiency to Weil’s positivity

For $0 < a < \infty$, we define

$$C(a) := \{ \psi \in C_c^\infty(\mathbb{R}) \mid \text{supp } \psi \subset [-a, a] \}.$$  \hspace{1cm} (3.6)

Let we set $D = d/dt$ and

$$\left( I_{b}^{(a)} \phi \right)(t) := \int_{-a}^{t} \phi(u)du + b \quad (b \in \mathbb{C}).$$

Then the maps

$$C(a) \xrightarrow{D} C_0(a), \quad C_0(a) \xrightarrow{I_{b}^{(a)}} C(a)$$  \hspace{1cm} (3.7)
are bijective and are inverse to each other, where $\mathcal{C}_0(a)$ is the space defined in (1.11). Therefore, from the following proposition, we find that the nonnegativity of $\langle \cdot, \cdot \rangle_{G,a}$ on $\mathcal{C}_0(a)$ for every $0 < a < \infty$ implies that the RH is true.

**Proposition 3.1.** Let $0 < a < \infty$ and let $C(a)$ be the space defined in (3.6). Then,

$$\langle D\psi_1, D\psi_2 \rangle_{G,a} = W(\psi_1 \ast \tilde{\psi}_2) \quad (3.8)$$

for every $\psi_1, \psi_2 \in C(a)$.

**Proof.** If $\phi = D\psi$ and $\hat{\phi}(0) = 0$, we have $z^{-1}(\hat{\phi}(z) - \hat{\phi}(0)) = -i\tilde{\psi}(z)$ by integration by parts. Therefore, $\langle D\psi_1, D\psi_2 \rangle_{G,a} = \sum_{\gamma} \hat{\psi}_1(-\gamma)\hat{\psi}_2(\gamma)$ by (3.1). The right-hand side is equal to $W(\psi_1 \ast \tilde{\psi}_2)$ by (3.3) and (3.5). □

### 3.4 Relation between $W$ and $\Psi$

The pointwise positivity of Theorem 1.7 turns out to be a special case of the Weil positivity. For $t > 0$, we consider the triangular function

$$\Delta_t(x) = \begin{cases} 
\frac{1}{2}(t - |x|), & |x| \leq t; \\
0, & |x| > t.
\end{cases} \quad (3.9)$$

Then $W(\Delta_t) = \Psi(t)$, because

$$\int_{-\infty}^{\infty} \Delta_t(x) e^{i\omega x} dx = \frac{1 - \cos(zt)}{\omega^2} \quad (3.10)$$

for any $z \in \mathbb{C}$. The triangular function $\Delta_t$ is expressed as $\Delta_t = R_t \ast \tilde{R}_t$ by the rectangular function

$$\begin{align*}
R_t(x) &= \begin{cases} 
1, & |x| < t/2; \\
\sqrt{2}, & |x| = t/2; \\
0, & |x| > t/2.
\end{cases}
\end{align*} \quad (3.11)$$

Therefore,

$$\Psi(t) = W(\Delta_t) = W(R_t \ast \tilde{R}_t). \quad (3.11)$$

This one is similar to the formula of Li coefficients by the Weil distribution $W$ in [2].
3.5 | Relation between $W$ and $g$

For a test function $\phi \in C^\infty_c(\mathbb{R})$,

$$\int_{-\infty}^{\infty} (-g(t))\phi''(t) \, dt = -\sum_{\gamma} \frac{1}{\gamma^2} (\hat{\phi}''(\gamma) - \hat{\phi}''(0)) = \sum_{\gamma} \hat{\phi}(\gamma) = W(\phi),$$

where the prime means differentiation. Therefore, we find that

$$-g''(t) = \Psi''(t) = \sum_{\gamma} e^{i\gamma t} = W(t)$$

as a distribution, where we understand the distribution as $W(\phi) = \int W(t)\phi(t) \, dt$. From this relation, the Weil distribution $W(t)$ can be regarded as an “accelerant” of the screw function $g(t)$ of $\xi(s)$ in the sense of [15, Introduction].

4 | PREPARATIONS FOR THE PROOF OF Theorem 1.4

4.1 | Positivity of $\Psi(t)$

It is observed that $\Psi(t)$ is positive in a small range $[0, t_0)$ of $t$ by numerical calculation by a computer (Figure 1). This observation is unconditionally proven and used to prove Theorem 1.4.

**Theorem 4.1.** There exists $t_0 > \log 2$ such that $\Psi(t) > 0$ for $0 < t < t_0$.

**Proof.** This follows from Theorem 4.2 below for the weaker result $t_0 > 0$, which is sufficient for Theorem 1.4, but here we prove it in another direct way with the help of numerical calculation by computer. Differentiating (1.1) and noting $(\Gamma'/\Gamma)(1/4) = -\gamma_0 - (\pi/2) - 3 \log 2$, we have $\Psi'(t) = 2(e^{t/2} - e^{-t/2}) + c - \arctan(e^{t/2}) + \text{arctanh}(e^{-t/2})$ with $c = (\pi/4) - (\gamma_0 + 3 \log 2)/2$, since there is no contribution from the primes for $\Psi(t)$ on $[0, \log 2)$. Then we find that all zeros of $\Psi'(t)$ on $[0, \log 2)$ are $t_1 = 0.152631 \ldots$ and $t_2 = 0.464002 \ldots$ by numerical calculation. Since $\text{arctanh}(e^{-t/2}) \rightarrow +\infty$ as $t \rightarrow 0^+$, the value of $\Psi''(t)$ is positive on $[0, t_1)$ and $(t_2, \log 2)$ and is negative on $(t_1, t_2)$. Further, $\Psi(0) = 0$ by definition and $\Psi(t_2) = 0.0396618 \ldots > 0$. Therefore, $\Psi(t)$ is positive on $[0, \log 2]$ by the continuity. $\square$
4.2 Yoshida’s results

Yoshida [33] studied a hermitian form on $C^\infty_c(\mathbb{R})$ defined by

$$\langle \phi, \psi \rangle_W := W(\phi \ast \bar{\psi}).$$

The space $C(a)$ in (3.6) was first introduced by him to localize the positivity (3.4). For $0 < a < \infty$ and $N \in \mathbb{Z}_{\geq 0}$, he also introduced the spaces

$$K(a) := \left\{ \psi(t) = \begin{cases} h(t), & (|t| \leq a) \\ 0, & (|t| > a) \end{cases} \right. \text{for some } h \in C^\infty(\mathbb{R}) \text{ with the period } 2a,$$

and

$$K_N(a) := \left\{ \psi \in K(a) \left| \int_{-a}^{a} \psi(x) \exp\left(\frac{\pi i nx}{a}\right) \, dx = 0 \text{ for all } n \in \mathbb{Z}, |n| \leq N \right. \right\}. \quad (4.1)$$

Yoshida proved without any assumption that the hermitian form $\langle \cdot, \cdot \rangle_W$ is positive definite on $K(a)$ if $a > 0$ is sufficiently small ([33, Lemma 2]). Connes–Consani [4] provides an operator theoretic conceptual reason for this result. Yoshida also proved that, for given $a_0 > 0$ and $\mu > 0$, there exists $N \geq 0$ such that $\langle \phi, \phi \rangle_W \geq \mu \| \phi \|^2_{L^2}$ for every $\phi \in K_N(a)$ and $0 < a \leq a_0$ ([33, Lemma 3]).

4.3 Positivity of $G_g(t, u)$

If the RH is true, $g(t)$ of (1.1) and (1.8) belongs to the class $G_\infty$ by Theorem 1.2. Therefore, the kernel $G_g(t, u)$ is nonnegative definite on $\mathbb{R}$, but we can directly confirm that it is nonnegative definite on $\mathbb{R}$ as

$$\sum_{i,j} G_g(t_i, t_j) \overline{\xi_i} \xi_j = \sum_{y} \frac{1}{y^2} \sum_i (1 - e^{iy_i}) \xi_i^2 \geq 0$$

by using (1.9), since all zeros $y$ of $\xi(1/2 - iz)$ are real. Hence, the restriction $g|_{[-2a, 2a]}$ belongs to the class $C_a$ for every $a > 0$.

As an analog of Yoshida’s result above, we unconditionally prove that the restriction $g|_{[-2a, 2a]}$ belongs to $C_a$ when $a > 0$ is sufficiently small. If $g|_{[-2a, 2a]}$ belongs to $C_a$, the kernel $G_g(t, u)$ is nonnegative definite on $(-a, a)$. It implies the nonnegativity of $\Psi(t)$ on $(-a, a)$, because

$$0 \leq G_g(t, t) = g(0) - g(t) - g(-t) = 2g(t) = 2\Psi(t)$$

for $t \in (-a, a)$. In particular, Theorem 4.1 in a weak form $t_0 = a > 0$ follows from this.

**Theorem 4.2.** There exists $a_0 > 0$ such that $G_g(t, u)$ is positive definite on $L^2(-a, a)$ if $0 < a < a_0$. In particular, the restriction $g|_{[-2a, 2a]}$ belongs to the class $C_a$ for every $0 < a < a_0$.

**Proof.** We first introduce a few auxiliary functions. For $\phi \in L^2(-a, a)$, we define the transformation

$$\Phi_t(\phi, z) := \frac{\hat{\phi}(z) - \hat{\phi}(0)}{z} = \int_{-a}^{a} \phi(t) e^{itz} - 1 \, dt.$$  \hspace{1cm} (4.2)
For reals \( t, u \), and a nonzero real \( y \), we define

\[
K(t, u) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{izt} - 1}{z} \frac{1}{z} \, dz,
\]

\[
K(t, u; y) := \frac{1}{2\pi} \int_{-\infty + iy}^{\infty + iy} \frac{e^{izt} - 1}{z} \frac{1}{z} \, dz.
\]

Then, we find that

\[
K(t, u) = \frac{1}{2} (|t| + |u| - |t - u|) = \begin{cases} 
  u, & \text{if } t > 0, u > 0, t - u > 0, \\
  t, & \text{if } t > 0, u > 0, t - u < 0, \\
  0, & \text{if } t < 0, u > 0, t - u < 0, \\
  -u, & \text{if } t < 0, u < 0, t - u < 0, \\
  -t, & \text{if } t < 0, u < 0, t - u > 0, \\
  0, & \text{if } t > 0, u < 0, t - u > 0,
\end{cases} \tag{4.3}
\]

and

\[
K(t, u; y) = \frac{1}{2y} \left( -e^{-y(|t| + |u|)} - e^{-y(u + |u|)} + e^{-y(t + u + |t - u|)} + 1 \right) \]

\[
= \begin{cases} 
  \frac{1 - e^{-2uy}}{2y}, & \text{if } t > 0, u > 0, t - u > 0, \\
  \frac{1 - e^{-2iy}}{2y}, & \text{if } t > 0, u > 0, t - u < 0, \\
  0, & \text{if } t < 0, u > 0, t - u < 0, \\
  e^{-2uy} - 1, & \text{if } t < 0, u < 0, t - u < 0, \\
  e^{-2ty} - 1, & \text{if } t < 0, u < 0, t - u > 0, \\
  0, & \text{if } t > 0, u < 0, t - u > 0,
\end{cases} \tag{4.4}
\]

by an elementary calculus.

We then prove the equality

\[
\langle \phi, \phi \rangle_{G_g, a} = I_0 + I_1 + I_\infty
\]

with

\[
I_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty + ic} \left( \frac{1}{s} - \frac{1}{s+1} \right) \left( \Phi_1(\phi; z)\Phi_1(\phi; \bar{z}) + \Phi_1(\phi; -z)\Phi_1(\phi; -\bar{z}) \right) \, dz, \tag{4.5}
\]

\[
I_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty + ic} \left( \frac{1}{s} - \frac{1}{s+1} \right) \left( \Phi_1(\phi; z)\Phi_1(\phi; \bar{z}) + \Phi_1(\phi; -z)\Phi_1(\phi; -\bar{z}) \right) \, dz, \tag{4.6}
\]

\[
I_\infty = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Re \left[ \frac{1}{2} \Gamma' \left( \frac{s}{2} \right) - \frac{1}{2} \log \pi \right] \left( |\Phi_1(\phi; z)|^2 + |\Phi_1(\phi; -z)|^2 \right) \, dz, \tag{4.7}
\]
where \( s = 1/2 - iz \) as elsewhere. We put

\[
g_0(t) := -4\left(e^{t/2} + e^{-t/2} - 2\right), \quad g_1(t) := \sum_{n \in \mathbb{N}} \frac{\Lambda(n)}{\sqrt{n}}(|t| - \log n),
\]

\[
g_\infty(t) := \frac{|t|}{2} \left[ \Gamma\left(\frac{1}{4}\right) - \log \pi \right] - \frac{1}{4} \left( C - e^{-|t|/2} \Phi(e^{-2|t|}, 2, 1/4) \right)
\]

so that \( I_* = \langle \phi, \phi \rangle_{G_{g_*a}} \) for \( * \in \{0, 1, \infty\} \) with the same notation as (1.10) and \( \langle \phi, \phi \rangle_{G_{g_*a}} = I_0 + I_1 + I_\infty \). We will prove (4.5), (4.6), and (4.7) for \( I_0, I_1, \) and \( I_\infty \), respectively.

For \( t > 0 \) and \( c > 1 \), taking the inversion of (2.1) and noting \( g_0(0) = 0 \),

\[
g_0(t) = \frac{1}{2\pi} \int_{-\infty+ic}^{\infty+ic} \left( \frac{1}{s-1} + \frac{1}{s} \right) \frac{e^{-ist} - 1}{z^2} \, dz.
\]

Because we find that the integral on the right-hand side is zero for \( t < 0 \) by moving \( c \to +\infty \), we can write

\[
g_0(t) = \frac{1}{2\pi} \int_{-\infty+ic}^{\infty+ic} \left( \frac{1}{s-1} + \frac{1}{s} \right) \left( \frac{e^{ist} - 1}{z^2} + \frac{e^{-ist} - 1}{z^2} \right) \, dz
\]

for any \( t \in \mathbb{R} \). Therefore,

\[
G_{g_0}(t, u) = g_0(t - u) - g_0(t) - g_0(-u) + g_0(0)
\]

\[
= \frac{1}{2\pi} \int_{-\infty+ic}^{\infty+ic} \left( \frac{1}{s-1} + \frac{1}{s} \right) \left( \frac{e^{istu} - 1}{z} - \frac{e^{-istu} - 1}{z} + \frac{e^{-istu} - 1}{z} - \frac{e^{istu} - 1}{z} \right) \, dz.
\]

From this formula and definitions (1.10) and (4.2), we obtain (4.5). In the same way, (4.6) and (4.7) are obtained by using (2.2), (2.3), and (2.4), but for (4.7), we move the horizontal line of the integration to the real line in the final step by noting the absence of poles \( (\Gamma'/\Gamma)(s/2) \) in \( \Re(s) > 0 \).

Now we prove the positivity of \( \langle \phi, \phi \rangle_{G_{g_*a}} \) for small \( a > 0 \). We first note the equality

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} |\Phi_1(\phi; z)|^2 \, dz
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \phi(u) \frac{e^{izu} - 1}{z} \, du \right) \left( \int_{-\infty}^{\infty} \phi(t) \frac{e^{-izt} - 1}{z} \, dt \right) \, dz
\]

\[
= \int_{-\infty}^{a} \int_{-\infty}^{a} \phi(u)\overline{\phi(t)} K(t, u) \, dudt.
\]

We fix \( c > 1 \) in (4.5) and (4.6) and set

\[
C_1 = \max_{\Re(s)=1+c} \left\{ \left| \frac{1}{s-1} + \frac{1}{s} \right|, \left| \frac{\zeta'}{\zeta}(s) \right| \right\}.
\]
ASPECTS OF THE SCREW FUNCTION OF $\zeta$

Then, for $F(s) = ((s - 1)^{-1} + s^{-1})$ or $(\zeta'/\zeta)(s)$,

$$\left| \frac{1}{2\pi} \int_{-\infty+ic}^{\infty+ic} F(s) \Phi_1(\phi; \pm z) \Phi_1(\phi; \pm \bar{z}) \, dz \right| \leq C_1 \cdot \left[ \frac{1}{2\pi} \int_{-\infty+ic}^{\infty+ic} |\Phi_1(\phi; \pm z)|^2 \, dz \right]^{1/2} \left[ \frac{1}{2\pi} \int_{-\infty+ic}^{\infty+ic} |\Phi_1(\phi; \pm \bar{z})|^2 \, dz \right]^{1/2}$$

by the Schwartz inequality. For the quantities on the right-hand side, we have

$$\frac{1}{2\pi} \int_{-\infty+ic}^{\infty+ic} |\Phi_1(\phi; \pm z)|^2 \, dz = \int_{-a}^{a} \int_{-a}^{a} \phi(u)\phi(t) K(t, u; \pm c) \, du \, dt$$

as in (4.9). If we fix a real number $a_1 > 0$, there exists $C_2 = C_2(a_1) > 0$ such that

$$K(t, u; \pm c) \leq C_2 K(t, u)$$

for any $|t| \leq a$ and $|u| \leq a$ if $0 < a \leq a_1$ by (4.3) and (4.4). Therefore,

$$\frac{1}{2\pi} \int_{-\infty+ic}^{\infty+ic} |\Phi_1(\phi; \pm z)|^2 \, dz \leq C_2 \int_{-a}^{a} \int_{-a}^{a} \phi(u)\phi(t) K(t, u) \, du \, dt$$

$$= C_2 \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} |\Phi_1(\phi; \pm z)|^2 \, dz.$$ 

This leads to

$$I_0 + I_1 \leq 2 C_1 C_2 \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} |\Phi_1(\phi; z)|^2 \, dz + 2 C_1 C_2 \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} |\Phi_1(\phi; -z)|^2 \, dz.$$ 

Following the above, we take $C > 0$ such that $C > 3C_1 C_2$. Since

$$\Re \left[ \frac{\Gamma'}{\Gamma} (\sigma + it) \right] = \log |\sigma + it| + O(|\sigma + it|^{-1}) \quad (4.10)$$

for a fixed $\sigma$, we can take $t_0 > 0$ so that

$$\Re \left[ \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} - \frac{i}{2} \right) - \frac{1}{2} \log \pi \right] \geq C$$

if $z \in \mathbb{R}$ and $|z| \geq t_0$. We put

$$C_0 = \max_{|z| \leq t_0} \Re \left[ \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} - \frac{i}{2} \right) - \frac{1}{2} \log \pi \right].$$
Then we have
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \Re \left[ \left( \frac{1}{2} \gamma^2 - \frac{1}{2} \log \pi \right) |\Phi_1(\phi; z)|^2 \right] dz \\
\geq C \cdot \frac{1}{2\pi} \int_{|z| > t_0} |\Phi_1(\phi; z)|^2 \, dz - C_0 \cdot \frac{1}{2\pi} \int_{|z| \leq t_0} |\Phi_1(\phi; z)|^2 \, dz
\]
\[
\geq C \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} |\Phi_1(\phi; z)|^2 \, dz - (C + C_0) \cdot \frac{1}{2\pi} \int_{|z| \leq t_0} |\Phi_1(\phi; z)|^2 \, dz.
\]

From the above calculations for \(I_0, I_1, I_\infty\), we obtain
\[
\langle \phi, \phi \rangle_{G_{g, a}} \geq ((C - 2C_1C_2) - a(C + C_0)C_3) \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} |\Phi_1(\phi; z)|^2 \, dz
\]
\[
+ ((C - 2C_1C_2) - a(C + C_0)C_3) \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} |\Phi_1(\phi; -z)|^2 \, dz.
\]

For the second and fourth integrals on the right-hand side,
\[
\frac{1}{2\pi} \int_{|z| \leq t_0} |\Phi_1(\phi; z)|^2 \, dz = \int_{-a}^{a} \int_{-a}^{a} \phi(u) \overline{\phi(t)} \left( \frac{1}{2\pi} \int_{|z| \leq t_0} \frac{e^{iuz} - 1 - e^{-it} - 1}{z} \, dz \right) dudt.
\]

If \(0 < a < a_1\) is sufficiently small, we have
\[
\frac{1}{2} \left( \frac{e^{iuz} - 1 - e^{-it} - 1}{z} + \frac{e^{-iuz} - 1 - e^{it} - 1}{z} \right) = ut (1 + O(|z|)) \ll aK(t, u)
\]
for \(z \in \mathbb{R}\) with \(|z| \leq t_0\), since both \(|uz|\) and \(|tz|\) are small in the range \(|z| \leq t_0\). Therefore, there exists \(0 < a_2 < a_1\) and \(C_3 > 0\) such that
\[
\frac{1}{2\pi} \int_{|z| \leq t_0} \frac{e^{iuz} - 1 - e^{-it} - 1}{z} \, dz \leq a \cdot C_3 \cdot K(t, u)
\]
for every \(0 < a < a_2\). Hence,
\[
\frac{1}{2\pi} \int_{|z| \leq t_0} |\Phi_1(\phi; z)|^2 \, dz \leq a \cdot C_3 \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} |\Phi_1(\phi; z)|^2 \, dz.
\]

Applying this to (4.11), we obtain
\[
\langle \phi, \phi \rangle_{G_{g, a}} \geq ((C - 2C_1C_2) - a(C + C_0)C_3) \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} |\Phi_1(\phi; z)|^2 \, dz
\]
\[
+ ((C - 2C_1C_2) - a(C + C_0)C_3) \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} |\Phi_1(\phi; -z)|^2 \, dz
\]
for \(0 < a < a_2\). The coefficients of the integrals on the right-hand side are positive if \(0 < a < a_2\) is sufficiently small. □
To state another analog of Yoshida’s result, we define

\[ \mathfrak{K}_{N,0}(a) := \left\{ \phi \in L^2(-a, a) \right\| \hat{\phi}(0) = 0, I_0^{(a)}(\phi) \in K_N(a) \right\}, \]

where \( K_N(a) \) is the space in (4.1). The following is also used to prove Theorem 1.4.

**Theorem 4.3.** Let \( a_0 > 0 \) and \( \mu > 0 \) be given numbers. Then there exists \( N \geq 0 \) such that

\[ \langle \phi, \phi \rangle_{G_{g,a}} \geq \mu \int_{-\infty}^{\infty} |\Phi_1(\phi, z)|^2 \, dz \quad (4.12) \]

for every \( \phi \in \mathfrak{K}_{N,0}(a) \) and \( 0 < a \leq a_0 \), where \( \Phi_1(\phi, z) \) is the transform of \( \phi \) in (4.2).

**Proof.** By taking \( a_1 \) larger than \( a_0 \) and taking \( C \) as \( C > 3C_1C_2 + \mu \) in the proof of Theorem 4.2, we obtain the lower bound (4.11) for \( 0 < a \leq a_0 \) with \( C > C_1C_2 + \mu \). Therefore, the proof is completed if the estimate

\[ \int_{|z| \leq t_0} |\Phi_1(\phi; z)|^2 \, dz \ll \frac{1}{N} \int_{-\infty}^{\infty} |\Phi_1(\phi; z)|^2 \, dz \quad (4.13) \]

is proven for \( \phi \in \mathfrak{K}_{N,0}(a) \) and \( 0 < a \leq a_0 \), where the implied constant depends only on \( a_0 \). This is shown in the same way as [33, Lemma 3] as follows.

Let \( \phi \in \mathfrak{K}_{N,0}(a) \) and put \( \psi = I_0^{(a)}(\phi) (\in K_N(a)) \). Then \( \psi(-a) = 0, \psi(a) = \hat{\phi}(0) = 0, \Phi_1(\phi, z) = -i\hat{\psi}(z) \), and

\[ \int_{-\infty}^{\infty} |\Phi_1(\phi, z)|^2 \, dz = \int_{-\infty}^{\infty} |\hat{\psi}(z)|^2 \, dz = \|\psi\|_{L^2}^2 \]

by the Parseval identity. Let \( \psi(t) = (2a)^{-1/2} \sum_{|n| > N} c_n e^{i\pi t/a} \) be the Fourier expansion of \( \psi \) in \( L^2(-a, a) \). Then \( \|\psi\|_{L^2}^2 = \sum_{|n| > N} |c_n|^2, c_n \ll n^{-k} \) for any \( k \in \mathbb{Z}_{\geq 0} \), and

\[ \hat{\psi}(z) = \frac{1}{\sqrt{2a}} \left[ \sum_{|n| > N} c_n \frac{a}{\pi\operatorname{in}} e^{i\pi t/a} e^{izt} \right]^{a} - \frac{1}{\sqrt{2a}} \int_{-a}^{a} \sum_{|n| > N} c_n \frac{a}{\pi\operatorname{in}} e^{i\pi t/a} iz e^{izt} \, dt \]

by applying integration by parts to the Fourier integral. Hence, we obtain

\[ \hat{\psi}(t) \leq \sqrt{2a(1 + a|z|)} \sum_{|n| > N} \left| \frac{c_n}{\pi n} \right| \leq \sqrt{2a(1 + a|z|)} \left( \sum_{|n| > N} \frac{1}{|n|^2} \right)^{1/2} \|\psi\|_{L^2} \]

for \( z \in \mathbb{R} \). For the second inequality, we used the Schwartz inequality. This inequality implies (4.13), since \( \Phi_1(\phi, z) = -i\hat{\psi}(z) \).

Theorem 4.3 can also be proved using (3.8) and [33, Lemma 3], but here we performed a direct proof as above.
5 | PROOF OF THEOREM 1.4

First, we show the second half of Theorem 1.4. Since \( g(t) \) is continuous and hermitian on the real line, the integral operator \( G_g[a] \) on \( L^2(-a, a) \) defined in (1.12) is a self-adjoint Hilbert–Schmidt operator for every \( 0 < a < \infty \). Therefore, the spectrum of \( G_g[a] \) consists of eigenvalues and zero. Whether or not zero is an eigenvalue is not determined by the general theory. The hermitian form \( \langle \cdot, \cdot \rangle_{G_g[a]} \) is nonnegative definite if and only if all eigenvalues of \( G_g[a] \) are nonnegative, and the nondegeneracy for \( \langle \cdot, \cdot \rangle_{G_g[a]} \) is equivalent to that \( G_g[a] \) does not have zero as an eigenvalue, since \( \langle \phi_1, \phi_2 \rangle_{G_g[a]} = \langle G_g[a] \phi_1, \phi_2 \rangle_{L^2} \) and there exists an orthonormal basis for \( L^2(-a, a) \) consisting of eigenfunctions of \( G_g[a] \).

5.1 | Proof of necessity

If the RH is true, \( G_g(t, u) \) is nonnegative definite on \((-a, a)\) for every \( 0 < a < \infty \) by Theorem 1.2. Therefore, all eigenvalues \( G_g[a] \) are nonnegative ([23, §8]). If there exists an eigenfunction for eigenvalue zero, it must be zero as a function by (3.1), (3.2), and Lemma 2.1. This is a contradiction. Hence, \( \langle \cdot, \cdot \rangle_{G_g[a]} \) is nondegenerate.

5.2 | Proof of sufficiency

We prove the sufficiency with the same strategy as in the proof of [33, Theorem 2]. That is, we prove that \( \langle \cdot, \cdot \rangle_{G_g[a]} \) degenerates on \( L^2(-a, a) \) at \( a = a_0 \) for some \( a_0 > 0 \) if the RH is false.

Let \( A \) be the set of all positive real numbers \( a \) such that \( \langle \cdot, \cdot \rangle_{G_g[a]} \) is nonnegative definite on \( L^2(-a, a) \) and set \( B = (0, \infty) \setminus A \). If \( b \in B \), there exists \( \phi \in L^2(-b, b) \) such that \( \langle \phi, \phi \rangle_{G_g[b]} < 0 \). Then \( \langle \phi, \phi \rangle_{G_g,b'} < 0 \) if \( b' \) is sufficiently close to \( b \) by the continuity of the kernel \( G_g(t, u) \). Hence, \( B \) is open and \( A \) is a closed subset of \((0, \infty)\). Assume that the RH is false. Then \( A \) is bounded by Theorem 1.3. Therefore, we prove that \( \langle \cdot, \cdot \rangle_{G_g[a]} \) degenerates on \( L^2(-a, a) \) at \( a = a_0 \) for the maximum element \( a_0 \) of \( A \).

Let \( a_1 \) be a real number greater than \( a_0 \) and let \( \mu > 0 \). For \( n \in \mathbb{Z} \), we put

\[
\chi_n[a](t) := \frac{1}{\sqrt{2a}} \exp \left( \frac{\pi i t}{a} \right).
\]

Then \( \{\chi_n[a]\}_{n \in \mathbb{Z}} \) forms an orthonormal basis of \( L^2(-a, a) \). We set

\[
\chi_{n,m}[a](t) := \frac{\pi in}{a} \chi_n[a](t) - (-1)^{n+m} \frac{\pi im}{a} \chi_m[a](t)
\]

for \( m, n \neq 0 \). Then \( \chi_{n,m}[a](0) = 0 \) and integrals \( I^{(a)}_0(\chi_{n,n+1}[a]) \) and \( I^{(a)}_0(\chi_{n,n-(n+1)}[a]) \) belong to \( K_N(a) \) for every \( n > N \in \mathbb{N}_0 \), because

\[
I^{(a)}_0(\chi_{n,m}[a]) = \chi_n[a] - (-1)^{m+n} \chi_m[a].
\]
We denote by $\mathcal{N}_N(a)$ the closed subspace of $L^2(-a, a)$ spanned by $\chi_{n,n+1}[a]$ and $\chi_{-n,-(n+1)}[a]$ for all $n > N$. Based on Theorem 4.3, we can take $N$ such that

$$\langle v, v \rangle_{G,g,a} \geq \mu \|f_0^{(a)}(v)\|_{L^2(-a,a)}$$

for every $v \in \mathcal{N}_N(a)$ and $0 < a \leq a_1$, because inequality (4.12) extends to $v$ in the $L^2$-closure $\mathcal{N}_N(a)$ of a subspace of $\mathcal{K}_{N,0}(a)$ and $\Phi_1(v, \cdot) = f_0^{(a)}(v)$. In particular, $\langle v, v \rangle_{G,g,a} > 0$ for every nonzero $v \in \mathcal{N}_N(a)$.

Let $\mathcal{N}_N(a)^\perp$ be the orthogonal complement of $\mathcal{N}_N(a)$ in $L^2(-a, a)$ with respect to $\langle \cdot, \cdot \rangle_{L^2(-a,a)}$. We put

$$u_n[a] := \chi_n[a] \quad \text{if } |n| \leq N,$$

$$u_{\pm(N+1)}[a] := \sum_{k=1}^{\infty} \frac{(-1)^k}{N+k} \cdot \chi_{\pm(N+k)},$$

$$u_{\pm n}[a] := \chi_{\pm n, \pm(n+1)}[a] \quad \text{if } n \geq N+1.$$ 

(5.2)

Then, $u_{\pm(N+1)}[a]$ are orthogonal to $u_n[a]$ ($|n| \neq N+1$), $u_{N+1}[a]$ and $u_{-(N+1)}[a]$ are orthogonal, and $u_{\pm(N+1)}[a](0) = 0$. Further, the space $\mathcal{N}_N(a)^\perp$ is a $(2N + 3)$-dimensional space spanned by $\{u_n[a]\}_{|n| \leq N+1}$. As we will show later, for each $u_n[a]$ ($|n| \leq N+1$), there exists $v_n[a] \in \mathcal{N}_N(a)$ such that

$$\langle v, u_n[a] \rangle_{G,g,a} = \langle v, v_n[a] \rangle_{G,g,a} \quad \text{for every } v \in \mathcal{N}_N(a).$$

(5.3)

Each $u_n[a] - v_n[a]$ is orthogonal to $\mathcal{N}_N(a)$ for $\langle \cdot, \cdot \rangle_{G,g,a}$. Let

$$A_N(a) = \{ \phi \in L^2(-a, a) \mid \langle \phi, v \rangle_{G,g,a} = 0 \text{ for all } v \in \mathcal{N}_N(a) \}$$

be the annihilator of $\mathcal{N}_N(a)$ for $\langle \cdot, \cdot \rangle_{G,g,a}$. Then $\langle \cdot, \cdot \rangle_{G,g,a}$ is nonnegative definite on $L^2(-a, a)$ if and only if it is nonnegative definite on $A_N(a)$, since any element of $L^2(-a, a)$ can be written as a linear combination of $u_n[a] - v_n[a] \in A_N(a)$ ($|n| \leq N+1$) and $v \in \mathcal{N}_N(a)$.

By definition of $a_0$, $\langle \cdot, \cdot \rangle_{G,g,a}$ is nonnegative definite on $A_N(a)$ for $0 < a \leq a_0$ and is not nonnegative definite for $a_0 < a \leq a_1$. The hermitian form $\langle \cdot, \cdot \rangle_{G,g,a}$ on $A_N(a)$ is represented by matrix coefficients

$$\langle u_m[a] - v_m[a], u_n[a] - v_n[a] \rangle_{G,g,a}, \quad |m|, |n| \leq N+1.$$ 

(5.4)

By (5.3), we have

$$\langle u_m[a] - v_m[a], u_n[a] - v_n[a] \rangle_{G,g,a} = \langle u_m[a], u_n[a] \rangle_{G,g,a} - \langle v_m[a], v_n[a] \rangle_{G,g,a}.$$ 

The continuity of $\langle u_m[a], u_n[a] \rangle_{G,g,a}$ is trivial. Therefore, the proof of Theorem 1.4 is reduced to the continuity of $\langle v_m[a], v_n[a] \rangle_{G,g,a}$ in a neighborhood $U_{a_0}$ of $a = a_0$ as in the proof of [33, Theorem 2], since such fact implies that $\langle \cdot, \cdot \rangle_{G,g,a}$ degenerates on $L^2(-a, a)$ at $a = a_0$. 
We show the existence of an orthogonal projection $v \in V_N(a)$ of $u \in V_N(a)'^\perp$ with respect to $\langle \cdot, \cdot \rangle_{G,a}$. Put $d = \dim V_N(a)^\perp = 2N + 3$. Take a basis $\{\phi_k\}_{k=1}^d$ of $V_N(a)^\perp$ and denote $\psi_k$ the orthogonal projection of $\phi_k$ in the closed subspace $G_g[a](V_N(a))$ of $L^2(-a, a)$ with respect to $\langle \cdot, \cdot \rangle_{L^2(-a,a)}$. Then each $\phi_k - \psi_k$ belongs to $A_N(a)$, since $\langle \phi_k - \psi_k, v \rangle_{G,a} = \langle \phi_k - \psi_k, G_g[a]v \rangle_{L^2(-a,a)} = 0$. We decompose it as $\phi_k - \psi_k = u_k - v_k$ by $u_k \in V_N(a)^\perp$ and $v_k \in V_N(a)$ according to the direct sum $L^2(-a,a) = V_N(a)^\perp \oplus V_N(a)$.

Then $\{u_k\}_{k=1}^d$ is also a basis of $V_N(a)^\perp$, because, if $\{u_k\}_{k=1}^d$ is linearly dependent, a nonzero linear combination of $v_k$'s must belongs to $N(a) \cap V_N(a)$, which is $\{0\}$ by the positivity of $\langle \cdot, \cdot \rangle_{G,a}$ on $V_N(a)$. As a result, we obtain the direct sum $L^2(-a,a) = N(a) \oplus V_N(a)$ that is also orthogonal for $\langle \cdot, \cdot \rangle_{G,a}$, that is, for each $u \in V_N(a)^\perp$, there exists $v \in V_N(a)$ such that $u - v \in N(a)$.

We back to the proof of the continuity of $\langle v_m[a], v_n[a] \rangle_{G,a}$. Hereafter in this proof, we abbreviate $\chi_n[a]$ as $\chi_n$, $u_n[a]$ as $u_n$, $v_n[a]$ as $v_n$, $\langle \cdot, \cdot \rangle_{G,a}$ as $\langle \cdot, \cdot \rangle$, $I_0(a)$ as $I(\phi)$, and $\langle \cdot, \cdot \rangle_{L^2(-a,a)}$ as $\langle \cdot, \cdot \rangle_{L^2}$.

Let $\{\tau_+^k\}$ and $\{\tau_-^k\}$ be orthonormal systems obtained from $\{I(u_n)\}_{n \geq N+1} = \chi_n + \chi_{n+1}$ and $\{I(u_n)\}_{n \geq N+1} = \chi_{-n} + \chi_{-(n+1)}$ by the Gram–Schmidt orthogonalization process with respect to $\langle \cdot, \cdot \rangle_{L^2}$, respectively. We have

$$\tau_\pm^j = \frac{(-1)^{j-1}}{\sqrt{j(j+1)}} \left( \sum_{k=1}^j (-1)^k \chi_{\pm(N+k)} - j(-1)^{(j+1)-1} \chi_{\pm(N+j+1)} \right)$$

By the determinant formula

$$\tau_\pm^j = \frac{1}{\sqrt{G_{j-1}G_j}} \begin{vmatrix} \langle \omega_1, \omega_1 \rangle_{L^2} & \langle \omega_1, \omega_2 \rangle_{L^2} & \cdots & \langle \omega_1, \omega_{j-1} \rangle_{L^2} & \omega_1 \\ \langle \omega_2, \omega_1 \rangle_{L^2} & \langle \omega_2, \omega_2 \rangle_{L^2} & \cdots & \langle \omega_2, \omega_{j-1} \rangle_{L^2} & \omega_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle \omega_j, \omega_1 \rangle_{L^2} & \langle \omega_j, \omega_2 \rangle_{L^2} & \cdots & \langle \omega_j, \omega_{j-1} \rangle_{L^2} & \omega_j \end{vmatrix},$$

where $\omega_k = \chi_{\pm(N+k)} + \chi_{\pm(N+k+1)}$, $G_0 = 1$, and

$$G_j = \begin{vmatrix} \langle \omega_1, \omega_1 \rangle_{L^2} & \langle \omega_1, \omega_2 \rangle_{L^2} & \cdots & \langle \omega_1, \omega_j \rangle_{L^2} \\ \langle \omega_2, \omega_1 \rangle_{L^2} & \langle \omega_2, \omega_2 \rangle_{L^2} & \cdots & \langle \omega_2, \omega_j \rangle_{L^2} \\ \vdots & \vdots & \ddots & \vdots \\ \langle \omega_j, \omega_1 \rangle_{L^2} & \langle \omega_j, \omega_2 \rangle_{L^2} & \cdots & \langle \omega_j, \omega_j \rangle_{L^2} \end{vmatrix} = (j + 1).$$

We write $\tau_{2n-1} = \tau_+^n$ and $\tau_{2n} = \tau_-^n$ and define the basis $\{\eta_j\}$ of $V_N(a)$ by

$$\eta_j = D(\tau_j), \quad j = 1, 2, \ldots.$$  

Then, there exists a sequence of positive numbers $\mu(M)$, $M \in \mathbb{N}$ independent of $a \in U_{a_0}$ such that $\lim_{M \to \infty} \mu(M) = \infty$ and that

$$\langle v, v \rangle \geq \mu(M) ||I(v)||_{L^2}$$

if $v \in V_N(a)$ satisfies $\langle I(v), I(\eta_i) \rangle_{L^2} = \langle I(v), \tau_i \rangle_{L^2} = 0$ for every $i \leq M$ when $a \in U_{a_0}$ by the orthogonality of $\{\tau_k\}$ for $\langle \cdot, \cdot \rangle_{L^2}$ and Theorem 4.3.
We have \( \langle \eta_j, \eta_l \rangle = \langle D(\tau_j), D(\tau_l) \rangle = \langle \tau_j, \tau_l \rangle_W \),
\[
\langle u_n, \eta_j \rangle = \langle X_n, \eta_j \rangle = \frac{a}{\pi i n} \langle X_n - (-1)^n X_0, \tau_j \rangle_W \quad (0 < |n| \leq N),
\]
\[
\langle u_{\pm(N+1)}, \eta_j \rangle = \frac{a}{\pi i} \sum_{k=1}^{\infty} \frac{(-1)^k}{(N+k)^2} \langle X_{\pm(N+k)} - (-1)^{N+k} X_0, \tau_j \rangle_W
\]
by \( X_n = D(a(\pi i n)^{-1}(X_n - (-1)^n X_0)) \),
\[
\langle u_{\pm(N+1)}, \eta_j \rangle = D \left[ \frac{a}{\pi i} \sum_{k=1}^{\infty} \frac{(-1)^k}{(N+k)^2} (X_{N+k} - (-1)^{N+k} X_0) \right],
\]
\( X_k - (-1)^k X_0 \in C(\mathbb{Z}) \), the extension of (3.8) to \( \mathcal{V}_N(a) \), and the definition \( \langle u, v \rangle_W = W(u * \bar{v}) \). Therefore, by definition (5.2), estimates of \( \langle \eta_j, \eta_l \rangle \) and \( \langle u_n, \eta_j \rangle \) (\( 0 < |n| \leq N + 1 \)) are reduced to the estimates of \( \langle X_m, X_n \rangle_W \) \((m, n \in \mathbb{Z})\), but they are known as \( |\langle X_m, X_n \rangle_W| \ll |m - n|^{-1} \) if \( m \neq n \) and \( |\langle X_n, X_n \rangle_W| \sim \log |n| \) as \( |n| \to \infty \) in [33, (5.17) and (5.18)], where the implied constants can be taken independent of \( a \in U_{a_0} \). From these estimates and the explicit formula (5.5), we can easily deduce the estimates
\[
|\langle \eta_j, \eta_l \rangle| \ll \frac{1}{|j - l|} + \frac{1}{\max\{j, l\}} \log \max\{j, l\} \quad (5.9)
\]
for \( j, l \in \mathbb{N} \) with \( j \neq l \) and
\[
|\langle u_n, \eta_j \rangle| = |\langle X_n, \eta_j \rangle| \ll \frac{\log j}{j} \quad (0 < |n| \leq N), \quad |\langle u_{\pm(N+1)}, \eta_j \rangle| \ll \frac{\log j}{j} \quad (5.10)
\]
for \( j \in \mathbb{N} \), where the implied constants independent of \( a \in U_{a_0} \). Further, we obtain
\[
|\langle u_0, \eta_j \rangle| = |\langle X_0, \eta_j \rangle| \ll \frac{(\log j)^2}{j} \quad (5.11)
\]
for \( j \in \mathbb{N} \). This estimate cannot be reduced to the estimate of \( \langle X_m, X_n \rangle_W \) and must be treated separately. But we leave that for later (Section 5.3 below) and continue with the proof of Theorem 1.4.

Let \( \{\psi_k\} \) be the basis of \( \mathcal{V}_N(a) \) obtained from \( \{\eta_k\} \) by the Gram–Schmidt orthogonalization process with respect to \( \langle \cdot, \cdot \rangle \). The positivity of \( \langle \cdot, \cdot \rangle \) on \( \mathcal{V}_N(a) \) ensures that this orthogonalization process works. By (5.3), we have
\[
u_n = \sum_{i=1}^{\infty} f_{ni} \psi_i, \quad f_{ni} = \langle v_n, \psi_i \rangle = \langle u_n, \psi_i \rangle \quad (|n| \leq N + 1).
\]
Then \( f_{ni} \) is a continuous function of \( a \) by (5.2), (5.6), (5.7), and the obvious continuity of \( \langle X_m, X_n \rangle \) for \( a \). We have
\[
\langle v_m, v_n \rangle = \sum_{i=1}^{\infty} f_{mi} f_{ni}, \quad \left( \sum_{i>M} |f_{mi} f_{ni}| \right)^2 \leq \sum_{i>M} |f_{mi}|^2 \sum_{i>M} |f_{ni}|^2.
\]
Therefore, the continuity of \( \langle v_m, v_n \rangle \) for \( a \) reduces to the uniformity of convergence of \( \sum_{i=1}^{\infty} |f_{ni}|^2 = \sum_{i=1}^{\infty} |\langle u_n, \psi_i \rangle|^2 \) in \( U_{a_0} \) for every \( |n| \leq N + 1 \).
To prove such convergence, we write
\[
\psi_i = \sum_{j=1}^i d_{ij} \eta_j, \quad \eta_i = \sum_{j=1}^i c_{ij} \psi_j
\]
by infinite-dimensional lower triangular matrices \(C = (c_{ij})\) and \(D = (d_{ij})\). For a positive integer \(M\), we set
\[
C = \begin{bmatrix} X_1 & 0 \\ X_3 & X_4 \end{bmatrix}, \quad D = \begin{bmatrix} Y_1 & 0 \\ Y_3 & Y_4 \end{bmatrix},
\]
where \(X_1\) and \(Y_1\) denote the first \(M \times M\)-blocks. Let \(\|A\|\) be the operator norm of the operator \(A\) on a Hilbert space \(\ell^2\) of row vectors defined by \(\|A\| = \sup_{\|x\|_{\ell^2} = 1} \|xA\|_{\ell^2}\). Using the estimates (5.1), (5.8), and (5.9)–(5.11), we obtain
\[
\|X_1^{-1}\| \ll 1, \quad \|X_3\| \ll 1, \quad \|X_4^{-1}\| \ll (\mu(M) + O(1))^{-1}
\]
with implied constants independent of \(a \in U_{a_0}\) in the same way as the proof of [33, Lemma 9], because the difference between (5.9) and (II) of [33, Lemma 9] and the difference between (5.10)–(5.11) and (III) of [33, Lemma 9] do not affect the calculations to prove (5.12).

Finally, we estimate \(\sum_{i>M} |\langle u_n, \psi_i \rangle|^2\) for \(|n| \leq N + 1\). Since
\[
\sum_{i>M} |\langle u_n, \psi_i \rangle|^2 = \sum_{i>M} \left| \sum_{j=1}^i d_{ij} \langle u_n, \eta_j \rangle \right|^2,
\]
we get
\[
\sum_{i>M} |\langle u_n, \psi_i \rangle|^2 = \|\xi \begin{bmatrix} tY_3 \\ tY_4 \end{bmatrix} \|_{\ell^2}^2,
\]
where \(\xi = (\langle u_n, \eta_1 \rangle, \langle u_n, \eta_2 \rangle, \ldots, \langle u_n, \eta_j \rangle, \ldots)\). By (5.10) and (5.11),
\[
\|\xi\|_{\ell^2} \ll 1.
\]
From \(CD = 1\), we get \(Y_4 = X_4^{-1}, Y_3 = -X_4^{-1}X_3X_1^{-1}\). Therefore,
\[
\sum_{i>M} |\langle u_n, \psi_i \rangle|^2 = \|\xi \begin{bmatrix} tX_1^{-1}X_3 \\ tX_4^{-1} \end{bmatrix} \|_{\ell^2}^2.
\]
By (5.8), we obtain
\[
\|\xi \begin{bmatrix} tX_1^{-1}X_3 \\ tX_4^{-1} \end{bmatrix} \|_{\ell^2}^2 \ll (\mu(M) + O(1))^{-1}\|\xi\|_{\ell^2}^2
\]
with implied constants independent of \(a \in U_{a_0}\). From (5.13) and (5.14), \(\sum_{i=1}^\infty |\langle u_n, \psi_i \rangle|^2\) converges uniformly on \(U_{a_0}\), so the proof is complete. \(\square\)
To complete the proof of Theorem 1.4, we prove (5.11) using the same notation as in the second half of Section 5.2. We use the Weil explicit formula in the following form:

$$\lim_{X \to \infty} \sum_{|\gamma| \leq X} \hat{\phi}(\gamma) = \hat{\phi}(i/2) + \hat{\phi}(-i/2) - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \phi(\log n) - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \phi(-\log n)$$

$$-(\log \pi)\phi(0) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \Re \left[ \frac{\Gamma(z)}{\Gamma(1/4 + iz/2)} \right] \hat{\phi}(z) \, dz,$$

where the sum on the left-hand side ranges over all zeros $\gamma$ of $\xi(1/2 - iz)$ counting with multiplicity, that is, $\rho = 1/2 - i\gamma$ for complex zeros $\rho$ of the Riemann zeta-function. Formula (5.15) is obtained from the explicit formula in [1, p. 186] by taking $\phi(t) = e^{t/2}f(e^t)$ for $f(x)$ in that formula and noting the symmetry of zeros $\gamma \mapsto -\gamma$. For the conditions for test functions, we use the conditions in [2, Section 3].

We prove (5.11) by calculating $\langle \chi_0, \eta_j \rangle$ directly using (5.15) as in [33, (5.15)–(5.18)]. However, we only give the outlines, because the argument of the proof is similar to [33] except for the choice of test functions $\phi(t)$.

By integrating by parts using (1.9) and (1.10),

$$\langle \chi_0, \chi_k \rangle = \sum_{\gamma} \frac{(\cos(\alpha \gamma) - 1)}{\gamma^2} \cdot \frac{(-1)^k 2a \sin(\gamma)}{k\pi + \alpha \gamma}$$

$$- \sum_{\gamma} \frac{\alpha \gamma \cos(\alpha \gamma) - \sin(\alpha \gamma)}{\alpha \gamma^3} \cdot \frac{(-1)^k 2a \sin(\gamma)}{k\pi + \alpha \gamma}.$$ 

The functions on the right-hand side are expressed as Fourier transforms as follows:

$$\int_{-a}^{a} \frac{1}{2} (|t| - a) e^{i\alpha t} \, dt = \frac{\cos(az) - 1}{z^2},$$

$$\int_{-a}^{a} \frac{1}{4a} (t^2 - a^2) e^{i\alpha t} \, dt = \frac{az \cos(az) - \sin(az)}{az^3},$$

$$\int_{-a}^{a} e^{\pi i k/\alpha} e^{i\alpha t} \, dt = \frac{(-1)^k 2a \sin(az)}{\pi k + az}.$$ 

Therefore, if we set $\phi_{1,k}(t)$ and $\phi_{2,k}(t)$ as

$$\phi_{1,k}(t) = \int_{-a}^{a} \exp \left( \frac{\pi i (t - u)}{\alpha} \right) \mathbf{1}_{(-a,a)}(t-u) \frac{1}{2} (|u| - a) \, du,$$

$$\phi_{2,k}(t) = \int_{-a}^{a} \exp \left( \frac{\pi i (t - u)}{\alpha} \right) \mathbf{1}_{(-a,a)}(t-u) \frac{1}{4a} (u^2 - a^2) \, du,$$
we have
\[ \hat{\phi}_{1,k}(z) = (-1)^k 2a \frac{(\cos(az) - 1) \sin(az)}{z^2(k\pi + az)}, \]
\[ \hat{\phi}_{2,k}(z) = (-1)^k 2 \frac{az \cos(az) - \sin(az)) \sin(az)}{z^3(k\pi + az)}. \]

These functions are compactly supported and continuously differentiable functions as follows:

\[ \phi_{1,k}(t) = \begin{cases} 0, & |t| \geq 2a; \\ \frac{a(-1)^k}{2\pi^2k^2} \left[ a \exp \left( \frac{\pi ikt}{a} \right) - a - \pi ik(t - 2a) \right], & a \leq t \leq 2a; \\ \frac{a}{2\pi^2k^2} \left[ a((-1)^k - 2) \exp \left( \frac{\pi ikt}{a} \right) + (-1)^k(a + \pi ikt) \right], & -a < t < a; \\ \frac{a(-1)^k}{2\pi^2k^2} \left[ a \exp \left( \frac{\pi ikt}{a} \right) - a - \pi ik(t + 2a) \right], & -2a \leq t \leq -a, \end{cases} \]

\[ \phi_{2,k}(t) = \begin{cases} 0, & |t| \geq 2a; \\ \frac{i(-1)^{k+1}}{4\pi^3k^3} \left[ 2a^2(1 + \pi ik) \exp \left( \frac{\pi ikt}{a} \right) - 2a^2 - 2iak(t - a) + \pi^2k^2t(t - 2a) \right], & 0 \leq t \leq 2a; \\ \frac{i(-1)^k}{4\pi^3k^3} \left[ 2a^2(1 - \pi ik) \exp \left( \frac{\pi ikt}{a} \right) - 2a^2 - 2iak(t + a) + \pi^2k^2t(t + 2a) \right], & -2a \leq t \leq 0. \end{cases} \]

Applying the Weil explicit formula (5.15) to \( \phi_{1,k}(t) \) and \( \phi_{2,k}(t) \), we obtain

\[ \sum_{\gamma} \frac{(\cos(ay) - 1)}{y^2} \cdot \frac{(-1)^k 2a \sin(ay)}{k\pi + ay} = (-1)^k 32a^2i(\cos(ia/2) - 1) \frac{\sin(ia/2)}{4\pi^2k^2 + a^2} \]

\[ - \sum_{1 \leq n \leq e^a} \frac{\Lambda(n)}{\sqrt{n}} \frac{a^2}{\pi^2k^2} \left[ ((-1)^k - 2) \cos \left( \frac{\pi k \log n}{a} \right) + (-1)^k \right] \]

\[ - \sum_{e^a < n \leq e^{2a}} \frac{\Lambda(n)}{\sqrt{n}} \frac{a^2(-1)^k}{\pi^2k^2} \left[ \cos \left( \frac{\pi k \log n}{a} \right) - 1 \right] \]

\[ - \frac{a^2}{\pi^2k^2}((-1)^k - 1) \log \pi \]

\[ + 8a^2 \sum_{n=0}^{\infty} \frac{(e^{-a(4n+1)} - 2e^{-a(4n+1)/2})(-1)^{k+1}}{(4n + 1)(a^2(4n + 1)^2 + 4\pi^2k^2)} \]
\[
+ \frac{a^2(-1)^k(1 - 2(-1)^k)}{4\pi^2 k^2} \left[ \Gamma' \left( \frac{1}{4} + \frac{\pi ik}{2a} \right) + \Gamma' \left( \frac{1}{4} - \frac{\pi ik}{2a} \right) \right]
+ \frac{a^2(-1)^k \Gamma'}{2\pi^2 k^2} \frac{1}{\Gamma} \left( \frac{1}{4} \right)
\]

(5.17)

and

\[
\sum_{\gamma} \frac{a \gamma \cos(a \gamma) - \sin(a \gamma)}{a \gamma^3} \cdot \frac{(-1)^k 2a \sin(a \gamma)}{k \pi + a \gamma}
= 64a((ai/2) \cos(ai/2) - \sin(ai/2))\frac{(-1)^k \sin(ia/2)}{4\pi^2 k^2 + a^2}
\]

\[
- a^2 \sum_{1 \leq n \leq e^{2a}} \frac{\Lambda(n)(-1)^k}{\sqrt{n} \pi^3 k^3} \sin \left( \frac{\pi k \log n}{a} \right)
+ a \sum_{1 \leq n \leq e^{2a}} \frac{\Lambda(n)(-1)^k}{\sqrt{n} \pi^2 k^2} \left( (\log n - a) - a \cos \left( \frac{\pi k \log n}{a} \right) \right)
- \frac{a^2(-1)^k}{\pi^2 k^2} \log \pi
+ 8a \sum_{n=0}^{\infty} \frac{(a(4n + 1) + 2)e^{-\alpha(4n+1)}(-1)^{k+1}}{(4n + 1)^2(a^2(4n + 1)^2 + 4\pi^2 k^2)}
+ \frac{a^2i(-1)^k}{4\pi^3 k^3} \left[ \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} + \frac{\pi ik}{2a} \right) (1 - \pi ik) - \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} - \frac{\pi ik}{2a} \right) (1 + \pi ik) \right]
+ \frac{a(-1)^k}{4\pi^2 k^2} \left[ 2a \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} \right) + \psi^{(1)} \left( \frac{1}{4} \right) \right],
\]

respectively, where \( \psi^{(1)}(z) := \frac{d^2}{dz^2} \log \Gamma(z) \). The calculation for the integral on the right-hand side of (5.15) is performed in the same way as in [33, §5]. By substituting (5.17) and (5.18) into (5.16),

\[
\langle \chi_0, \chi_k \rangle \ll \frac{\log k}{k^2},
\]

(5.19)

where the implied constant depends only on \( a \). On the other hand, we have

\[
\langle \chi_0, D(\tau_{\pm j}) \rangle = \pm \frac{(-1)^{j-1} \pi i}{a \sqrt{j(j+1)}} \sum_{k=1}^{j} (-1)^{k-1} (N + k) \langle \chi_0, \chi_{\pm(N+k)} \rangle
- j(-1)^{(j+1)-1} (N + j + 1) \langle \chi_0, \chi_{\pm(N+j+1)} \rangle
\]

by (5.5). Therefore,

\[
\langle \chi_0, D(\tau_{\pm j}) \rangle \ll \frac{(\log j)^2}{j}
\]

by (5.19), where the implied constant depends only on \( a \). This implies (5.11) by definition of \( \eta_j \). \( \Box \)
5.4 Comparison with Yoshida’s results

For every $0 < a < \infty$, there exists $N_0 = N_0(a) > 0$ such that $\langle \cdot, \cdot \rangle_W$ is positive definite on $K_N(a)$ if $N > N_0$ ([33, Lemma 3]). Therefore, we can take the completion $\hat{K}(a)$ of $K_N(a)$ with respect to $\langle \cdot, \cdot \rangle_W$. Define $W(a)$ by $K(a) = W(a) \oplus K_N(a)$ and set $\hat{K}(a) = W(a) \oplus \hat{K}_N(a)$. Then the hermitian form can be extended to $\hat{K}(a)$. Yoshida proved that the RH is true if and only if $\langle \cdot, \cdot \rangle_W$ is nondegenerate on $\hat{K}_N(a)$ for every $a > 0$. Further, he proved that $\langle \cdot, \cdot \rangle_W$ is nondegenerate on $C(a)$ and $K(a)$ for every $a > 0$ ([33, Propositions 2 and 7]).

Theorem 1.4 is an analog of Yoshida’s result, but it has the advantage that the space is simpler than $\hat{K}(a)$ and the hermitian forms are described by an integral operator with continuous kernel. Further, Proposition 3.1 and Yoshida’s result [33, Proposition 2] imply that $\langle \cdot, \cdot \rangle_G$ is nondegenerate on $\mathbb{C}_0(a)$.

6 PROPERTIES OF THE KERNEL $G_g(t, u)$

In this part, we prove Theorem 1.5 and study a little about the eigenvalues of $G_g[a]$.

6.1 Proof of Theorem 1.5

Using $g_n(t)$ in (4.8), we define $G_g(t, u) = g_n(t - u) - g_n(t - u) + g_n(0)$ for $s \in \{0, 1, \infty\}$ so that $G_g(t, u) = G_0(t, u) + G_1(t, u) + G_\infty(t, u)$. It is known that a continuous integral kernel $K(t, u)$ on $[-a, a] \times [-a, a]$ is of trace class if it satisfies the Lipschitz condition $|K(t, u_1) - K(t, u_2)| \leq C|u_1 - u_2|^\alpha$ for some $C > 0$ and $1/2 < \alpha \leq 1$ ([9, Chapter IV, Theorem 8.2]). Therefore, $G_0(t, u)$ and $G_1(t, u)$ are trace class kernels, but the Lipschitz condition cannot be applied to $G_\infty(t, u)$, because

$$C - e^{-t/2} \Phi(2t - 1, 1/4) = 2t \log(1/2t) + A,$$

for $t > 0$ by using the series expansion of $\Phi(z, s, a)$ for $z$ in [7, p.30, (9)], where $A = -2(\log 2 + \psi(1/4) - \psi(2)) = \pi + 4 \log 2 + 2$ and $\zeta(2 - k, 1/4) = B_{k-1}(1/4)/(k - 1)$ with Bernoulli polynomials $B_n(x)$. However, if we decompose the kernel as $G_\infty(t, u) = G_{\infty,1}(t, u) + G_{\infty,2}(t, u)$ according to $g_\infty(t) = 2|t| \log(1/|t|) + (g_\infty(t) - 2|t| \log(1/|t|)) = g_{\infty,1}(t) + g_{\infty,2}(t)$, then $G_{\infty,2}(t, u)$ is a trace class kernel. Therefore, it remains to show that $G_{\infty,1}(t, u)$ is of trace class.

If a Hilbert–Schmidt kernel $K(t, u)$ on $[-a, a] \times [-a, a]$ has the derivative in the mean $(\partial/\partial u)K(t, u)$ that is also a Hilbert–Schmidt kernel, then $K(t, u)$ defines a trace class operator ([10, p. 120, 3]). Since $g_{\infty,1}(t) = 2|t| \log(1/|t|)$ is a piecewise continuously differentiable function and its absolute derivative is integrable on $[-a, a]$, the kernel $G_{\infty,1}(t, u)$ is of trace class.

If $\hat{\phi}(0) = 0$ for $\phi \in L^2(-a, a)$, we have

$$\langle \phi, \phi \rangle_{G_g[a]} = \int_{-a}^a \left( \int_{-a}^a g(t - u)\phi(u) du \right) \hat{\phi}(t) dt.$$

From this, if $G_g(t, u)$ is nonnegative definite, one may expect $g(t - u)$ to be so. However, it is not the case, because $g(t - u)$ is a trace class kernel without any assumptions by the proof of Theorem 1.5, and $\int g(t - t) dt = \int g(0) dt = 0$. Therefore, if $g(t - u)$ is assumed to be nonnegative definite, we have $g(t) = 0$. This is a contradiction.
6.2 | Eigenvalues

For $0 < a < \infty$, we study the eigenvalues of the integral operator $G_g[a]$ defined in (1.12). Let $\phi(t)$ be the eigenfunction of $G_g[a]$ with the eigenvalue $\lambda$. Then

$$\lambda \phi(t) = 1_{[-a,a]}(t) \sum_{\gamma} \left( \frac{\hat{\phi}(\gamma) - \hat{\phi}(0)}{\gamma} \right) e^{\gamma t} - \frac{1}{\gamma}$$

by formula (1.9) of the kernel $G_g(t, u)$. If $\lambda \neq 0$, we can write

$$\phi(t) = 1_{[-a,a]}(t) \sum_{\gamma} w_{\gamma} e^{i\gamma t} - \frac{1}{\gamma}$$

with $w_{\gamma} = \frac{1}{\lambda} \frac{\hat{\phi}(\gamma) - \hat{\phi}(0)}{\gamma} \in \mathbb{C}$.

In this case, we have $\phi(0) = 0$ and

$$\frac{\hat{\phi}(\gamma) - \hat{\phi}(0)}{\gamma} = \sum_{\mu} w_{\mu} \int_{-a}^{a} \frac{e^{i\mu t} - 1}{\mu} e^{-i\gamma t} - \frac{1}{\gamma} dt$$

$$= \frac{2a}{\gamma} \sum_{\mu} \frac{w_{\mu}}{\mu} \left( \frac{\sin(a(\gamma - \mu))}{a(\gamma - \mu)} - \frac{\sin(a\gamma)}{a\gamma} - \frac{\sin(a\mu)}{a\mu} + 1 \right).$$

Therefore,

$$\lambda w_{\gamma} = \frac{2a}{\gamma} \sum_{\mu} \frac{w_{\mu}}{\mu} \left( \frac{\sin(a(\gamma - \mu))}{a(\gamma - \mu)} - \frac{\sin(a\gamma)}{a\gamma} - \frac{\sin(a\mu)}{a\mu} + 1 \right).$$

Hence, if we put

$$H(x, y; a) = \frac{2a}{xy} \left( \frac{\sin(a(x - y))}{a(x - y)} - \frac{\sin(ax)}{ax} - \frac{\sin(ay)}{ay} + 1 \right),$$

then the nonzero eigenvalues of $G_g[a]$ correspond to the eigenvalues of the linear system

$$\lambda w_{\gamma} = \sum_{\mu} H(\gamma, \mu; a) w_{\mu}$$

for $w_{\gamma}$’s, where $\gamma, \mu$ runs over the zeros of $\xi(1/2 - iz)$ counting with multiplicity. The system is an analog of the linear system introduced in Bombieri [1, Section 7] to study Weil’s hermitian form $\langle \cdot, \cdot \rangle_W$.

On the other hand, if $\lambda = 0$, we have

$$\sum_{\gamma} w_{\gamma} e^{i\gamma t} - \frac{1}{\gamma^2} = 0$$

for $t \in (-a, a)$ by writing $w_{\gamma} = \hat{\phi}(\gamma) - \hat{\phi}(0)$. Further, $w_{\gamma} \neq 0$ for some $\gamma$ by Lemma 2.1, since $\hat{\phi}(z)$ is a nonconstant entire function of the exponential type. By Theorem 1.4, $\lambda = 0$ is actually an eigenvalue when the RH is false. Therefore, we obtain the following result similar to (iii) in the introduction or the corollary of Theorem 11 in [1].
Theorem 6.1. Suppose that the RH is false. Then there exists $a_0 > 0$ and a nonidentically vanishing sequence of complex numbers $\{w_\gamma\}_\gamma$ such that

$$\sum_\gamma w_\gamma e^{i\gamma t} - \frac{1}{\gamma^2} = 0$$
on the interval $(-a_0, a_0)$ as a function of $t$.

7  PROOFS OF THEOREMS 1.6–1.8

7.1  Proof of Theorem 1.6

Suppose that the RH is true. Then all $\gamma$ in (1.3) are real. Therefore, noting that $\xi(1/2 - iz)$ is an entire function of order one, $|\Psi(t)| \leq 2 \sum_\gamma |\gamma|^{-2} < \infty$.

Conversely, suppose that $\Psi(t)$ is bounded on $[0, \infty)$. Then, the integral on the left-hand side of (1.2) converges absolutely and uniformly on any compact subset in $C_+$. Hence, $(\xi'/\xi)(1/2 - iz)$ has no poles in $C_+$, which implies that the RH is true.

From Theorem 1.6, we are interested in the supreme of values of $\Psi(t)$. It is clearly less than or equal to $2 \sum_\gamma \gamma^{-2} (< 0.094)$ by (1.3) under the RH. However, it is not easy to determine the limit superior and limit inferior of $\Psi(t)$ even if assuming the RH. Assuming that all zeros $\gamma$ of $\xi(1/2 - iz)$ are real (RH) and simple and that the set of all positive zeros is linearly independent over the rationals, we can prove

$$\limsup_{t \to \infty} \Psi(t) = 2 \sum_\gamma \frac{1}{\gamma^2}$$

and

$$\liminf_{t \to \infty} \Psi(t) = 0$$

by Kronecker’s theorem in Diophantine approximations. The conjectural value of the limit inferior of $\Psi(t)$ suggests the difficulty of proving the nonnegativity of $\Psi(t)$ by approximation.

7.2  Proof of Theorem 1.7

Suppose that the RH is true. Then all $\gamma$ in (1.3) are real. Therefore, each term of the middle sum is nonnegative. Also, for each $t > 0$, not all terms are simultaneously zero by Lemma 2.1. Hence, $\Psi(t) > 0$ for $t > 0$.

Conversely, suppose that $\Psi(t) \geq 0$ for $t > 0$. Because $\xi(s)$ has no zeros in the half-line $[1/2, \infty)$ by the series expansion $(1 - 2^{1-s})\xi(s) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s}$ for $\Re(s) > 0$, the logarithmic derivative $(\xi'/\xi)(1/2 - iz)$ has no poles in the half-line $i \cdot [0, \infty)$ of the imaginary axis. Then, by the integral formula (1.2) and the well-known result for the Laplace transform for nonnegative functions ([32, Theorem 5b in Chap. II]), $(\xi'/\xi)(1/2 - iz)$ has no poles in $C_+$, which implies that the RH is true.

7.3  Proof of Theorem 1.8

Suppose that the RH is true. Then, $\Psi(t) > 0$ for $t > 0$ by Theorem 1.7. Therefore, $\{\mu_n\}_{n \geq 0}$ is a Stieltjes moment sequence for the measure $4^{-1}e^{-t/2}\Psi(t) \, dt$ on $[0, \infty)$. Hence, $\det \Delta \geq 0$ and $\det \Delta^{(1)} \geq$
0 for all $n \in \mathbb{Z}_{>0}$ ([17, Chapter V, §1]). Conversely, suppose that $\det \Delta_n \geq 0$ and $\det \Delta_n^{(1)} \geq 0$ for all $n \in \mathbb{Z}_{>0}$. Then $\{\mu_n\}_{n \geq 0}$ is a Stieltjes moment sequence of some measure on $[0, \infty)$ ([17, Chapter V, §1]). On the other hand, the estimate $4^{-1} e^{-t/2} \Psi(t) \ll e^{-c \sqrt{t}}$ obtained from Theorem 1.1(3) implies that the Stieltjes moment problem for $\{\mu_n\}_{n \geq 0}$ is uniquely determined by [20, Theorem 2]. Hence, $\{\mu_n\}_{n \geq 0}$ must be equal to the moment sequence of the measure $4^{-1} e^{-t/2} \Psi(t) \, dt$, and thus, $\Psi(t) \geq 0$ for $t > 0$. □

8 | RELATIONS WITH LI COEFFICIENTS

Comparing formulas (1.14) and (1.15), the similarity between the moments $\{\mu_n\}$ and the Li coefficients $\{\lambda_{n+1}\}$ is obvious. In fact, they have the following explicit relation.

**Theorem 8.1.** Li coefficients (1.14) are represented by the moments (1.13) as

$$
\lambda_1 = \mu_0, \quad \lambda_2 = 6 \mu_0 - \mu_1, \quad \lambda_3 = 19 \mu_0 - 7 \mu_1 + \frac{1}{2} \mu_2
$$

(8.1)

and

$$
\frac{1}{n!} \lambda_n = \sum_{k=0}^{n-1} \frac{1}{k! (k+3)! (n-k-1)!} \left( k - \frac{4n-1}{2} \right)^2 + 2n + \frac{7}{4} \right) (-1)^k \mu_k
$$

(8.2)

for $n \geq 4$. Conversely, the moments are represented by the Li coefficients as

$$
\frac{1}{n!} \mu_n = \sum_{j=1}^{n+1} \sum_{k=1}^{n-j+2} k 2^{k-1} \binom{n-k+2}{j} \left( -1 \right)^{j+1} \lambda_j
$$

(8.3)

for all nonnegative integers $n$. The relations (8.1)–(8.2) and (8.3) are inverse to each other.

**Proof.** We first prove (8.1) and (8.2). Formula (1.14) means that Li coefficients are defined by the power series expansion

$$
\sum_{n=0}^{\infty} \lambda_{n+1} w^n = \frac{1}{(1-w)^2} \frac{\xi'}{\xi} \left( \frac{1}{1-w} \right).
$$

(8.4)

By the changing of variables $z = (i/2)(1+w)/(1-w)$, we have

$$
\frac{1}{(1-w)^2} \frac{\xi'}{\xi} \left( \frac{1}{1-w} \right) = - \frac{z^2}{(1-w)^2} \left[ \frac{1}{z^2} \xi' \left( \frac{1}{2} - iz \right) \right]
$$

$$
= \frac{1}{4} \frac{(1+w)^2}{(1-w)^4} \int_0^\infty \Psi(t) \exp \left( -\frac{t}{2} \cdot \frac{1+w}{1-w} \right) \, dt
$$

$$
= \sum_{n=0}^{\infty} \int_0^\infty 4^{-1} e^{-t/2} \Psi(t) P_n(t) \, dt \cdot w^n,
$$

where $P_n(t)$ are some polynomials. The exchanging of the order of the integral and sum is justified by the estimate in Theorem 1.1(3). Comparing (8.4) and (8.5), we have $\lambda_{n+1} = \sum_k c_{nk} \mu_k$ if $P_n(t) =$
Therefore, we then calculate $P_n(t)$ explicitly to conclude (8.1) and (8.2). Using the power series of the exponential,

$$
\sum_{k} c_{nk} t^k. \text{ exp} \left( -\frac{t}{2} \cdot \frac{1 + w}{1 - w} \right) = e^{-t/2} \exp \left( -\frac{wt}{1-w} \right)
$$

$$
= e^{-t/2} + e^{-t/2}(-t) \sum_{l=0}^{\infty} w^{l+1} + e^{-t/2} \sum_{k=2}^{\infty} \frac{(-t)^k}{k!} w^k \frac{1}{(k-1)!} \sum_{l=0}^{\infty} \prod_{i=1}^{k-1} (l + i) w^l.
$$

On the other hand,

$$
\frac{(1 + w)^2}{(1 - w)^4} = \frac{1}{3} \sum_{m=0}^{\infty} (m + 1)(2m^2 + 4m + 3)w^m.
$$

Multiplying these two expansions,

$$
\sum_{n=0}^{\infty} P_n(t) w^n = e^{t/2} \cdot \frac{(1 + w)^2}{(1 - w)^4} \exp \left( -\frac{t}{2} \cdot \frac{1 + w}{1 - w} \right)
$$

$$
= 1 + (6 - t)w + \frac{1}{3} \sum_{m=2}^{\infty} (m + 1)(2m^2 + 4m + 3)w^m
$$

$$
+ (-7t) \sum_{l=0}^{\infty} w^{l+2} + (-t) \frac{1}{3} \sum_{m=2}^{\infty} (m + 1)(2m^2 + 4m + 3) \sum_{l=0}^{\infty} w^{m+l+1}
$$

$$
+ \sum_{k=2}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} (-t)^k \frac{1}{3k!(k-1)!} \sum_{i=1}^{k-1} (m + 1)(2m^2 + 4m + 3) \cdot w^{k+l+m}.
$$

Rearranging the right-hand side concerning monomials $w^k$, we obtain

$$
\sum_{n=0}^{\infty} P_n(t) w^n = 1 + (6 - t)w + \left( 19 - 7t + \frac{t^2}{2} \right)w^2
$$

$$
+ \sum_{n=3}^{\infty} \left[ \frac{(n+1)(2(n+1)^2 + 1)}{3} - t \left( 7 + \sum_{m=2}^{n-1} \frac{(m+1)(2(m+1)^2 + 1)}{3} \right) \right.
$$

$$
+ \sum_{k=2}^{n} \frac{(-t)^k}{k!(k-1)!} \sum_{m=0}^{n-k} \frac{(m+1)(2(m+1)^2 + 1)}{3} \prod_{i=1}^{k-1} (n - k - m + i) \left. \right] w^n.
$$

On the right-hand side, we have

$$
7 + \sum_{m=2}^{n-1} \frac{(m+1)(2(m+1)^2 + 1)}{3} = \frac{n(n+1)(n^2 + n + 1)}{6},
$$

$$
\prod_{i=1}^{k-1} (n - k - m + i) = \frac{(n - m - 1)!}{(n - k - m)!},
$$
and
\[ \sum_{m=0}^{n-k} \frac{(m+1)(2(m+1)^2+1)}{3} \frac{(n-m-1)!}{(n-k-m)!} \]
\[= \frac{1}{k(k+1)(k+2)(k+3)} \frac{(n+1)!}{(n-k)!} \left( \left( k - \frac{4(n+1)-1}{2} \right)^2 + 2(n+1) + \frac{7}{4} \right) \]
by elementary calculations. Therefore,
\[ \sum_{n=0}^{\infty} P_n(t) w^n = 1 + (6-t)w + \left( 19 - 7t + \frac{t^2}{2} \right) w^2 \]
\[+ \sum_{n=3}^{\infty} \left[ \frac{(n+1)(2(n+1)^2+1)}{3} - t \frac{n(n+1)(n^2+n+1)}{6} \right. \]
\[\left. + \sum_{k=2}^{n} \frac{(-t)^k (n+1)!}{k!(k+3)! (n-k)!} \left( \left( k - \frac{4(n+1)-1}{2} \right)^2 + 2(n+1) + \frac{7}{4} \right) \right] w^n. \]
In the sum for \( n \geq 3 \), we find that the first and second terms are equal to cases of \( k = 0 \) and \( k = 1 \), respectively, for the formula of coefficients of \( t^k \) for \( k \geq 2 \). Hence, we obtain the relations (8.1) and (8.2).

We second prove (8.3). By the changing of variables \( w = X/(1-X) \) in (8.4), we have
\[ \frac{1}{(1-2X)^2(1-X)^2} \sum_{n=0}^{\infty} \lambda_{n+1} \left( \frac{X}{X-1} \right)^n = \sum_{n=0}^{\infty} \mu_n \frac{X^n}{n!}. \] \hspace{1cm} (8.6)
Applying \( \lambda_n = \sum_{\rho}(1 - (1 - \rho^{-1})^n) \) in [19, (1.4)] to the left-hand side,
\[ \frac{1}{(1-2X)^2(1-X)^2} \sum_{n=0}^{\infty} \lambda_{n+1} \left( \frac{X}{X-1} \right)^n = \sum_{n=0}^{\infty} \left[ \sum_{k=1}^{n} k 2^{k-1} \sum_{\rho} \rho^{-(n-k+2)} \right] X^n. \] \hspace{1cm} (8.7)
On the other hand, \( \lambda_n = \sum_{\rho}(1 - (1 - \rho^{-1})^n) \) gives
\[ \sum_{\rho} \rho^{-m} = \sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} \lambda_k \] \hspace{1cm} (8.8)
in an inductive procedure. Substituting (8.8) into (8.7) and comparing its coefficients with (8.6), we obtain the relation (8.3).
Since both relations (8.1)–(8.2) and (8.3) are obtained by different power series expansions of (8.4), it is clear that they are inverse to each other.

If the RH is true, both \( \{\lambda_n\} \) and \( \{\mu_n\} \) are positive, but unfortunately, in (8.1)–(8.2) and (8.3), the positivity of one does not directly lead to the positivity of the other. However, as an application of the relation (8.1)–(8.2), the following recurrence formula of the moments is obtained.
Theorem 8.2. Let \( \{a_j\}_{j \geq 1} \) be coefficients of the power series expansion \( \xi(1/(1-w)) = 1 + \sum_{j=1}^{\infty} a_j w^j \). For \( n \in \mathbb{Z}_{>0} \) and \( k \in \mathbb{Z}_{\geq 0} \) with \( n \geq k \), we set

\[
 b_{n,k} := \frac{n!}{k!(k+3)!} \left[ \frac{(n+1)!}{(n-k)!} \left( \left( k - \frac{4n+3}{2} \right)^2 + 2n + \frac{15}{4} \right) 
+ \sum_{j=k+1}^{n} \frac{j!}{(j-k-1)!} \left( (2j-k)^2 + k + 2 \right) a_{n-j+1} \right].
\]

Then, all \( b_{n,k} \) are positive and the recurrence relation

\[
(-1)^n \mu_n = (n+1)! a_{n+1} - \sum_{k=0}^{n-1} (-1)^k b_{n,k} \mu_k \tag{8.9}
\]

holds for all nonnegative \( n \).

Proof. The positivity of \( b_{n,k} \) follows from the positivity of \( a_j \) in [19, p. 327]. We recall that the recurrence formula

\[
\lambda_{n+1} = (n+1) a_{n+1} - \sum_{j=1}^{n} a_{n-j+1} \lambda_j \tag{8.10}
\]

holds for every nonnegative integer \( n \) in [19, p. 327]. Substituting (8.1) to (8.10) and then calculating a little, (8.9) is obtained for \( n = 1 \) and 2. For \( n \geq 3 \), substituting (8.1) and (8.2) for both sides of (8.10), then moving the terms other than \((-1)^n \mu_n/n!\) on the left-hand side to the right-hand side, and finally, grouping the right-hand side concerning moments \( \mu_k \), we obtain (8.9). \( \square \)

9 | RELATION WITH STRINGS UNDER THE RH

For a Kreĭn string \( S[m,L] \) that consists of \( 0 < L \leq \infty \) and a right-continuous nondecreasing nonnegative function \( m(x) \) on \( [0,L) \) with \( m(0_-) = 0 \), we take solutions \( \phi(x,z) \) and \( \psi(x,z) \) of the string equation \( dy'(x) + zy(x)dm(x) = 0 \) on \([0,L)\) satisfying the initial condition \( \phi'(0,z) = \psi(0,z) = 0 \), \( \phi(0,z) = \psi'(0,z) = 1 \). Then the Titchmarsh–Weyl function \( q(z) = \lim_{x \to L} \psi(x,z)/\phi(x,z) \) exists and belongs to the subclass \( \mathcal{N}_S \) of the Nevanlinna class \( \mathcal{N} \) consisting of \( q(z) \) such that

\[
q(z) = b + \int_{0}^{\infty} \frac{d\sigma(\lambda)}{\lambda - z}
\]

for some \( b \geq 0 \) and a measure \( \sigma \) on \([0,\infty)\) with \( \int_{0}^{\infty} d\sigma(\lambda)/(\lambda + 1) < \infty \). Kreĭn [13] proved that the correspondence \( S[m,L] \to \mathcal{N}_S \) is bijective.

Using the meromorphic function \( Q_\xi(z) \) in (1.7), we define

\[
q_\xi(z) = \frac{Q_\xi(\sqrt{z})}{\sqrt{z}}
\]
and suppose that the RH is true. Then, both $Q_\xi(z)$ and $q_\xi(z)$ belong to $\mathcal{N}$. Moreover, we have

$$q_\xi(z) = \frac{i}{\sqrt{z}} \xi'\left(\frac{1}{2} - i\sqrt{z}\right) = \sum_{\gamma > 0} \frac{2}{\gamma^2 - z} = b + \int_0^\infty \frac{d\sigma(\lambda)}{\lambda - z} \quad (b = 0),$$

where $\sigma$ is a measure on $[0, \infty)$ supported on points $\gamma^2$. In other words, $q_\xi \in \mathcal{N}_S$. Hence, there exists a string $S[m_\xi, L_\xi]$ corresponding to $q_\xi(z)$. Such string is called Zeta string in Kotani [12, §3.2], where he proved that $m_\xi(x) \sim 4x(\log x)^{-2}$ as $x \to 0^+$. 

### 10 MEAN PERIODICITY

Here, we explain that the screw function $g(t)$ in (1.1) and (1.8) is mean periodic even without assuming the RH. Let

$$\eta(t) := 2e^{-t/2} \sum_{n=1}^\infty (2\pi^2 n^4 e^{-4t} - 3\pi n^2 e^{-2t}) \exp(-\pi n^2 e^{-2t}).$$

Then $\xi(1/2 - iz) = \hat{\eta}(z)$ for all $z \in \mathbb{C}$ ([29, §10.1]). In addition, $-iz \xi(1/2 - iz) = \hat{\eta}'(z)$ for all $z \in \mathbb{C}$, since $\eta(t)$ is smooth and decays faster than any exponential as $|t| \to +\infty$. Therefore, by using (1.3) and putting $\phi = \eta'$, we have

$$(g * \phi)(t) = \sum_{\gamma} \frac{1}{\gamma^2} (e^{\gamma t} \hat{\phi}(-\gamma) - \hat{\phi}(0)) = 0.$$ 

Hence, if we choose a suitable function space $X$ consisting of functions on $\mathbb{R}$ such that $g \in X$ and $\phi = \eta'$ belong to the dual space $X^*$, then $g$ is a $X$-mean-periodic function (see [8, §2] for a quick overview of mean-periodic functions). In fact, we can choose the spaces $C^\infty_{\exp}(\mathbb{R})$ and $S(\mathbb{R})^*$ in [8] as such a space $X$ by Theorem 1.1 (3). For the same reason that the convolution equation $g * \phi = 0$ holds, if we choose the space $X$ such that the dual space $X^*$ contains the space

$$Z := \left\{ \frac{d}{dt} \left( e^{-t/2} \sum_{n=1}^\infty f(ne^{-t}) \right) \left| f \in S(\mathbb{R}), f(x) = f(-x), f(0) = \hat{f}(0) = 0 \right. \right\}$$

like $C^\infty_{\exp}(\mathbb{R})$ and $S(\mathbb{R})^*$, then the space $T(g) \subset X$ spanned by all translations $g(t - r)$ by $r \in \mathbb{R}$ is orthogonal to $Z$ with respect to the pairing $(\cdot, \cdot) : X \times X^* \to \mathbb{C}$. The space $Z$ consists of the first derivatives of functions in the space introduced by A. Connes in [3, Appendix I] (see also [5, §6.1] and R. Meyer [21]) to study the spectral realization of the zeros of $\xi(s)$. In the above sense, the screw function $g$ “generates” a subspace of the orthogonal complement of Connes’ space possibly dependent on $X$. This is a situation similar to [24, Theorem 3.1].

Independent of the choice of function space, the convolution equation $g * \phi = 0$ yields the integral representation

$$\frac{g * \phi^+(z)}{\phi^+(z)} = -\frac{g * \phi^-(z)}{\phi^-(z)} = \frac{1}{z^2} \xi'\left(\frac{1}{2} - iz\right),$$

which holds for all $z \in \mathbb{C}$ by the Fourier–Carleman transform ([8, Definition 2.8]), where $\phi^+(t) = \phi(t)$ if $t \geq 0$ and $= 0$ otherwise and $\phi^-(t) = 0$ if $t \geq 0$ and $= \phi(t)$ otherwise.
Although the screw function $g(t)$ attached to $\xi(s)$ is outside the framework of mean-periodic functions studied in [8], we may regard the positivity in Theorem 1.7 as an example of the single sign property of mean-periodic functions in [8, §4.3], and the equality obtained from (1.1) and (1.3) as an analog of the summation formula [8, Corollary 4.6].

11  |  VARIANTS OF $\Psi(t)$

We consider “shifted” variants of $\Psi(t)$. For a real number $\omega$, we define $\Psi_\omega(t)$ by

$$
\Psi_\omega(t) := e^{-\omega t} \Psi(t) + 2\omega \int_0^t e^{-\omega u} \Psi(u) \, du + \omega^2 \int_0^t (t-u)e^{-\omega u} \Psi(u) \, du \tag{11.1}
$$

for $t > 0$ and extend it as an even function on $\mathbb{R}$. Then we have $\Psi_0(t) = \Psi(t)$ and

$$
\int_0^\infty \Psi_\omega(t) e^{izt} \, dt = -\frac{1}{z^2} \frac{\xi'}{\xi}(1/2 + \omega - iz), \quad \Im(z) > 1/2 - \omega \tag{11.2}
$$

by (1.2) and a little calculation. Therefore, a nontrivial estimate of $\Psi_\omega(t)$ leads to an expansion of the zero-free region of $\xi(s)$ as in the case of $\Psi(t)$.

On the right-hand side of (11.2), the expansion

$$
\Im \left( \frac{\xi'}{\xi}(s + \omega) \right) = \sum_\rho \frac{\Re(s + \omega) - \Re(\rho)}{|s + \omega - \rho|^2}
$$

shows that $i(\xi'/\xi)(1/2 + \omega - iz)$ belongs to the class $\mathcal{N}$ if $\xi(s) \neq 0$ for $\Re(s) > 1/2 + \omega$, where $\rho$ runs over all zeros of $\xi(s)$ counting with multiplicity. Therefore, if $\xi(s) \neq 0$ for $\Re(s) > 1/2 + \omega$, the function $\Psi_\omega(t)$ is nonnegative by (11.2) and [14, Satz 5.9] as in the case of $g(t) = -\Psi(t)$. Furthermore, from formula (11.2) and the fact that $\xi(s) \neq 0$ on the positive real line, if we use the same result of Laplace transforms as in the proof of Theorem 1.7, the following seemingly milder equivalent condition is obtained.

**Theorem 11.1.** Let $\omega \in \mathbb{R}$. Then $\xi(s) \neq 0$ for $\Re(s) > 1/2 + \omega$ if and only if there exists $t_0 > 0$ such that $\Psi_\omega(t)$ is nonnegative when $t \geq t_0$.

The above result shows that $\Psi_\omega(t) \geq 0$ unconditionally if $\omega \geq 1/2$, because it is well known that $\xi(s) \neq 0$ for $\Re(s) > 1$. In contrast, $\Psi_\omega(t)$ changes the sign infinitely many if $\omega$ is negative, because it is known that $\xi(s)$ has infinitely many zeros on the critical line $\Re(s) = 1/2$. Similar to the relation between $\Psi(t)$ and $\Psi_\omega(t)$, we have

$$
\Psi_{\omega+\eta}(t) = e^{-\eta t} \Psi_\omega(t) + 2\eta \int_0^t e^{-\eta u} \Psi_\omega(u) \, du + \eta^2 \int_0^t (t-u)e^{-\eta u} \Psi_\omega(u) \, du
$$

for two reals $\omega$ and $\eta$. Therefore, the nonnegativity of $\Psi_\omega(t)$ on some interval $[0, t_0]$ implies the nonnegativity of $\Psi_{\omega+\eta}(t)$ on the same range when $\eta > 0$. 
For $t > 0$, the analogs of (1.1) and (1.3) are as follows:

$$\Psi(\omega)(t) = \sum_{\gamma} \frac{1}{(\gamma^2 + \omega^2)^2} \left[ \omega t(\gamma^2 + \omega^2) + (\gamma^2 - \omega^2) - e^{-\omega t} \cdot (\gamma^2 - \omega^2) \cos(\gamma t) - e^{-\omega t} \cdot 2\gamma \omega \sin(\gamma t) \right]$$

$$= 4 \left[ \frac{e^{(\frac{1}{2} - \omega)t}}{(1 - 2\omega)^2} + \frac{e^{-(\frac{1}{2} + \omega)t}}{(1 + 2\omega)^2} - \frac{4 - 2(1 - \omega t)(1 - 4\omega^2)}{(1 - 4\omega^2)^2} \right]$$

$$+ \frac{t}{2} \left[ \psi\left(\frac{1}{4} + \frac{\omega}{2}\right) - \log \pi \right]$$

$$+ \frac{1}{4} \left[ \psi^{(1)}\left(\frac{1}{4} + \frac{\omega}{2}\right) - e^{-(\frac{1}{2} + \omega)t} \Phi\left(e^{-2t}, 2, \frac{1}{4} + \frac{\omega}{2}\right) \right]$$

$$- \sum_{n \leq e^t} \frac{\Lambda(n)}{n^{\frac{1}{2} + \omega}} (t - \log n),$$

where $\psi^{(k)}(s) = \frac{d^{k+1}}{ds^{k+1}} \log \Gamma(s)$ and $\psi(s) = \psi^{(0)}(s)$. This is obtained by calculating the right-hand side of (11.1) using (1.1) and (1.3). We have $\Psi(\omega)(0) = 0$ as well as $\Psi(t)$.

As a special case,

$$\Psi_{1/2}(t) = \frac{1}{2} (t + 1)^2 - \frac{3}{2} + e^{-t}$$

$$- \frac{t}{2} (\gamma_0 + 2 \log 2 + \log \pi) + \frac{1}{4} \left[ \frac{\pi^2}{2} - e^{-t} \Phi\left(e^{-2t}, 2, \frac{1}{2}\right) \right]$$

$$- \sum_{n \leq e^t} \frac{\Lambda(n)}{n} (t - \log n)$$

for $t > 0$. In this case, $g_{1/2}(t) := -\Psi_{1/2}(t)$ belongs to the class $\mathcal{C}_\infty$, because $Q_{1/2}(z) = i(\xi'/\xi)(1 - iz)$ belongs to $\mathcal{N}$ and satisfies (1.6).

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