Phase diagram of the $J_1$-$J_2$ Heisenberg model on the kagome lattice

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We perform an extensive density matrix renormalization group (DMRG) study of the ground-state phase diagram of the spin-1/2 $J_1$-$J_2$ Heisenberg model on the kagome lattice. We focus on the region of the phase diagram around the kagome Heisenberg antiferromagnet, i.e., at $J_2 = 0$. We investigate the static spin structure factor, the magnetic correlation lengths, and the spin gaps. Our results are consistent with the absence of magnetic order in a narrow region around $J_2 \approx 0$, although strong finite-size effects do not allow us to accurately determine the phase boundaries. This result is in agreement with the presence of an extended spin-liquid region, as it has been proposed recently. Outside the disordered region, we find that for ferromagnetic and antiferromagnetic $J_2$ the ground state displays signatures of the magnetic order of the $\sqrt{3} \times \sqrt{3}$ and the $q = 0$ type, respectively. Finally, we focus on the structure of the entanglement spectrum (ES) in the $q = 0$ ordered phase. We discuss the importance of the choice of the bipartition on the finite-size structure of the ES.

I. INTRODUCTION

The nature of the ground state of the antiferromagnetic spin-1/2 Heisenberg model on the kagome lattice (KHA) has been debated for a long time. Despite substantial analytical and numerical effort no agreement has been reached yet in the community. The proposed ground states include several valence bond crystals (VBC) 1-7, and both gapped and gapless spin liquids. 8-23

However, recent DMRG simulations 24-26 provided convincing evidence that the ground state of the KHA is a gapped spin liquid with topological entanglement entropy $\gamma = \log(2) 27-29$. This is compatible with both a spin liquid of the toric-code or the double-semion 30,31 type. Although the former appears naturally in mean field theories of the KHA 32, and for quantum dimer models on the kagome lattice 33,34 and was therefore favored, recent numerical studies suggest that the ground state of the KHA is rather in a double-semion phase 35-38. At the moment no conclusion for this issue has been reached.

Recently it has been suggested that the gapped spin liquid survives upon introducing a small antiferromagnetic next-nearest-neighbor interaction 39, i.e., in the $J_1$-$J_2$ Heisenberg model ($J_1$-$J_2$ KHA), although numerical approaches based on Gutzwiller projected fermionic states support a $U(1)$ Dirac spin liquid in a narrow region around $J_2 = 0$ 40. This is in contrast with the $T = 0$ phase diagram of the classical version of the model. At $J_2 = 0$ the ground state of the classical $J_1$-$J_2$ KHA exhibits an extensive degeneracy 41-45. This is lifted upon introducing an infinitesimal $J_2$, and the system develops magnetic order 46. Precisely, for ferromagnetic $J_2$ the so-called $\sqrt{3} \times \sqrt{3}$ order emerges, whereas in the antiferromagnetic case one has the $q = 0$ order. The two ordering patterns are shown schematically in Figure 1.

The magnetic order survives in the quantum model, at least for large enough $J_2$, as it has been established by exact diagonalization studies 47. However, the precise phase boundary between the magnetically ordered phases and the disordered spin-liquid region at $J_2 \approx 0$ has not been determined yet (see Ref. 48 for some interesting results obtained using the functional renormalization group approach). In this work by performing $SU(2)$-symmetric DMRG calculations we investigate the ground-state phase diagram of the $J_1$-$J_2$ KHA as a function of $J_1$ and $J_2$. Here we set $J_1 = 1$, considering both positive and negative $J_2$. We study the finite-size behavior of the static spin structure factor, the spin-spin correlation length, and the spin gap. For ferromagnetic $J_2$ we provide numerical evidence that magnetic order of the $\sqrt{3} \times \sqrt{3}$ type survives up to $J_2 \lesssim -0.1$. On the other hand, for antiferromagnetic $J_2$ signatures of the $q = 0$ state appear already at $J_2 \gtrsim 0.2$. In the narrow region at $-0.1 \lesssim J_2 \lesssim 0.2$, although strong finite-size effects are present, our data are compatible with an extended disordered region, suggestive of a spin liquid behavior 29. Finally, we analyze the structure of the entanglement spectrum (ES) in the $q = 0$ ordered phase at large $J_2 \gg 0.2$. Recently, it has been suggested that in presence of continuous symmetry breaking the low-lying levels in the ES are reminiscent of the so-called tower-of-states, which appear in finite-size energy spectra. 50-52. This correspondence has been checked numerically in Ref. 52 for the $J_1$-$J_2$ KHA.
at large ferromagnetic $J_2$, i.e., in presence of the $\sqrt{3} \times \sqrt{3}$ order, and for the 2D Bose Hubbard model in the superfluid phase. Here we investigate how the identification of the correct tower-of-states structure in the ES depends on the choice of the bipartition, in finite-size systems.

The article is organized as follows. Section II introduces the $J_1$-$J_2$ Heisenberg model on the kagome lattice, and the DMRG method. In particular, we describe in detail the geometry used in the DMRG simulations. In section III and IV we discuss the numerical results for the static spin structure factor, and the spin-spin correlation length. The energy gaps are presented in section V. Finally, in section VII we investigate the structure of the entanglement spectrum in the $q = 0$ ordered phase.

II. MODEL AND METHOD

The spin-1/2 $J_1$-$J_2$ Heisenberg model on the kagome lattice is defined by the SU(2)-invariant Hamiltonian

$$\mathcal{H} = J_1 \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + J_2 \sum_{\langle\langle i,k \rangle\rangle} \mathbf{S}_i \cdot \mathbf{S}_k.$$  

(1)

Here, $\mathbf{S}_i$ is the spin operator acting on the lattice site $i$, while $\langle i,j \rangle$ and $\langle\langle i,k \rangle\rangle$ denote nearest and next-nearest-neighbor sites, respectively. We restrict ourselves to $J_1 = 1$ in (1).

We obtain the ground state of the $J_1$-$J_2$ KHA using SU(2)-symmetric DMRG calculations. The geometry used in the simulations is depicted in Figure 2. The two basis vectors of the kagome lattice are denoted as $\mathbf{e}_1$ and $\mathbf{e}_2$. The unit cell (thicker purple lines) contains three sites. Since DMRG prefers open boundary conditions, we consider kagome cylinders, using periodic (open) boundary conditions in the $\mathbf{e}_2$ ($\mathbf{e}_1$)-direction. Here we focus on cylinders with width $W$ and length $L$, where $W$ and $L$ are the numbers of unit cells along the $\mathbf{e}_2$ and $\mathbf{e}_1$ directions, respectively. In order to alleviate spurious effects due to sharp edges, the unit cells at the right boundary of the cylinder contain only two sites. The total number of spins on the lattice is given as $W \times (3L + 2)$. Here we consider only cylinders with $W = 3$ and $W = 4$, which, following Ref. 27, are referred to as YC6 and YC8 cylinders. The computational time scales approximately linearly with $L$ and exponentially with $W$. In our DMRG calculations we keep up to $\sim 5000$ SU(2) states, which correspond to approximately 20000 U(1) states. This allows us to obtain accurate ground-state wavefunctions for cylinders with lengths $L = 4, 6, 8, 10, 12$ for both the YC6 and YC8 geometries. The largest cylinder considered in this work (with $W = 4$ and $L = 12$) contains 152 spins.

III. STATIC SPIN STRUCTURE FACTOR

Here we discuss the static spin structure factor $S(q)$ obtained from the ground state of the $J_1$-$J_2$ Heisenberg model as a function of $-0.2 \leq J_2 \leq 0.4$. The structure factor is defined as

$$S(q) = \frac{1}{N} \sum_{i,j=1}^{N} \langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle e^{i \mathbf{q} \cdot (r_j - r_i)}.$$  

(2)

Here $N$ is the total number of lattice sites, $\langle \cdot \rangle$ denotes the ground-state expectation value, $r_i$ is the position of site $i$, and $\mathbf{q}$ is a generic vector in the reciprocal lattice.

Figure 3 (a) and (b) show the expected structure factors (the circles denote the positions of the peaks in momentum space) for the classical $\sqrt{3} \times \sqrt{3}$ state and the $q = 0$ state, respectively. Panels (i)-(iii) plot the DMRG result for $S(q)$ for $J_2 = 0.2$, $J_2 = 0.1$, and $J_2 = 0.4$. The data are for a YC6 cylinder (with $3 \times 12$ unit cells, cf. Figure 2). Clearly, for $J_2 = 0.2$ sharp peaks with $S(q_{K}) \approx 6$ are visible at the $K$-points $q_{K}$ of the extended Brillouin zone (see Figure 3 (b) for the definition of the high-symmetry points), in agreement with what is expected for the $\sqrt{3} \times \sqrt{3}$ state. Notice that the much smaller peaks at the $M$-points of the first Brillouin zone cannot be resolved with the available system sizes. We observe that at $J_2 = 0$ $S(q)$ is featureless (shown in Figure 4), which signals the absence of magnetic order. On the other hand, already at $J_2 = 0.1$ some peaks start developing at the $M$-points $q_{M}$, as expected for the classical $q = 0$ state (cf. Figure 3 (b)). These become sharper upon increasing $J_2$ (one has $S(q_{M}) \approx 5$ for $J_2 = 0.4$).

All these features are more quantitatively discussed in Figure 4 plotting the (squared) antiferromagnetic order parameter $m_{Q}^2 = S(Q)/N$ versus $J_2$. Here $Q$ denotes the positions of the peaks of the structure factors. Data are for both YC6 and YC8 cylinders (panels (a) and (b) in the Figure) with lengths $L = 4, 6, 8, 10, 12$. Precisely, Figure 4 plots $m_{q_{M}}^2$ for the $\sqrt{3} \times \sqrt{3}$ order for $J_2 < 0$ (empty symbols) and $m_{q_{M}}^2$ (i.e., the order parameter for the $q = 0$ order) for $J_2 \geq 0$. In the region $-0.1 < J_2 < 0.2$, $m_{Q}^2$ is almost featureless and $S(Q)$ itself is nearly size independent. This is compatible with a

FIG. 2. (Color online) The $J_1$-$J_2$ Heisenberg model on kagome cylinders. The spins are located at the vertices of the lattice. The two basis vectors of the lattice are denoted as $\mathbf{e}_1$ and $\mathbf{e}_2$. Periodic (PBC) and open (OBC) boundary conditions are imposed along the $\mathbf{e}_2$ and $\mathbf{e}_1$ directions, respectively. The unit cells consist of three sites and are denoted by the thicker (purple) triangles. The spins are located at the vertices of the lattice. The spin-$\frac{1}{2}$ in order to alleviate edge effects in the DMRG simulation.

Here, $\mathbf{e}_1$ and $\mathbf{e}_2$ are the interaction strengths between nearest and next-nearest neighbors, respectively. Notice that the unit cells at the right boundary are incomplete and are denoted by the thicker (purple) triangles. The spins are located at the vertices of the lattice. The spin-$\frac{1}{2}$ in order to alleviate edge effects in the DMRG simulation.
Finally, from the structure factor $S(q)$ one can define a correlation length $\xi(Q,q_{\text{min}})$ as
\[
\xi(Q,q_{\text{min}}) = \frac{1}{q_{\text{min}}} \sqrt{\frac{S(Q)}{S(Q + q_{\text{min}})}} - 1,
\]
where $q_{\text{min}}$ is the point next to the peak (at $Q$) of the structure factor. Here we choose $q_{\text{min}} = b_1/L$, with $b_1$ being the reciprocal lattice vector corresponding to the long direction of the cylinder (see Figure 3 (b) for its definition). Other choices of $q_{\text{min}}$ are expected to be equivalent in the 2D limit $W, L \to \infty$.

Figure 5 plots $\xi(Q,q_{\text{min}})$ for the YC6 and YC8 cylinders (a) and (b) in the Figure) and various cylinder lengths $L$. In the Figure we show $\xi(q_K,q_{\text{min}})$ (empty symbols) and $\xi(q_M,q_{\text{min}})$ (full symbols) in the region with $J_2 < 0$ and $J_2 \geq 0$, respectively. The qualitative behavior is the same for both YC6 and YC8 cylinders. We obtain small correlation lengths with weak dependence on the cylinder length for $-0.1 \lesssim J_2 \lesssim 0.15$. In particular, at $J_2 = 0$ both correlation lengths are of the order of the lattice constant, as expected in a spin liquid.28 This behavior reflects that of the order parameter $m_Q^2$ (cf. Figure 4). Outside the disordered region the correlation lengths show an increasing trend as a function of the cylinder length $L$. For the extremal values $J_2 = -0.2$ and $J_2 = 0.4$ considered in this work $\xi(Q,q_{\text{min}})$ is of the order of the system size. Notice that since $\xi(Q,q_{\text{min}})/\sqrt{N}$ is expected to be size-independent at a second order phase transition, it could be useful in order to determine the phase boundary of the $q = 0$ and the $\sqrt{3} \times \sqrt{3}$ ordered phases. Notice that, although the same result can be obtained from the scaling law $m_Q^2 \sim \xi^{-1+\eta}$ of the order parameter, this would require the knowledge of the critical exponent $\eta$.

V. THE SPIN TRIPLET GAPS

Using $SU(2)$-invariant DMRG simulations we obtain the lowest-energy eigenstate in both the $S = 0$ and $S = 1$ sectors. We extrapolate their energies in the single-site DMRG
correlation length \( \xi \)

\[ \xi \propto \frac{1}{L \cdot W} \]

behavior in the limit of an ordered phase. It is interesting to investigate the behavior of the system in the thermodynamic limit. A less pronounced feature is also visible at \( J_2 \approx -0.1 \). For both \( J_2 \approx -0.1 \) and \( J_2 \approx 0.2 \), the correlation length shows a strong dependence on the system size with a decreasing trend as a function of \( L, W \), suggesting a vanishing behavior in the limit of \( L, W \to \infty \), as expected in a magnetically ordered phase. It is interesting to investigate the behavior of \( \Delta_t \) in the limit \( L \to \infty \), i.e., for infinitely long cylinders. This is illustrated in Figure 7, plotting \( \Delta_t \) as a function of \( 1/L \) for \( J_2 = 0, 0.1, 0.4 \) and both YC6 and YC8 cylinders. The dotted lines are the linear extrapolations to the infinite cylinder limit. The extrapolated gaps are shown in Figure 8. The triplet gap shows a peak at \( J_2 \approx 0.1 \) with a value of approximately \( \Delta_t \approx 0.14 \) for the YC6 and \( \Delta_t \approx 0.18 \) for the YC8 cylinder. It is interesting to observe that the maximum of the gap is not at \( J_2 = 0 \), where the structure factor is featureless (cf. Figure 4). For larger \( |J_2| \) the extrapolated gap exhibits decreasing behavior as a function of \( J_2 \). We should remark that, although the extrapolated gaps seem to vanish outside the disordered region, this should not be associated with the presence of Goldstone modes, as infinite long cylinders are expected to exhibit 1D behavior and no symmetry breaking.

VI. DISCUSSION

Here we discuss the physical implications of the numerical results presented in section III, section IV, and section V.
We divide the discussion in three parts for different parameter ranges. First we consider the case $J_2 \lesssim -0.1$, then $-0.1 \lesssim J_2 \lesssim 0.2$, and finally $J_2 \gtrsim 0.2$.

a. $J_2 \lesssim -0.1$. The static spin structure factor at $J_2 \approx -0.2$ (see Figure 3 (i)) exhibits sharp peaks at the $K$-points of the extended Brillouin zone. The peak positions are in agreement with what is expected for the classical $\sqrt{3} \times \sqrt{3}$ order. Moreover, the DMRG data suggest a sudden increase of the antiferromagnetic order parameter $m_Q^2$. The corresponding spin-spin correlation length is of the order of the system size, and it increases upon increasing $|J_2|$. This could suggest magnetic order of the $\sqrt{3} \times \sqrt{3}$ type in the thermodynamic limit. This is also weakly confirmed by the behavior of the triplet gap $\Delta_t$. We numerically observe that $\Delta_t$ decreases upon increasing $L$ and $W$ for $J_2 \lesssim -0.1$, which is consistent with a vanishing behavior in the 2D limit (cf. Figure 6 and Figure 8), as expected in a magnetically ordered phase, due to the presence of the Goldstone modes.

b. $-0.1 \lesssim J_2 \lesssim 0.2$. In this region we observe a dome-shaped triplet gap. For both the YC6 and YC8 geometries

c. $0.2 \lesssim J_2$. We find sharp peaks in the static spin structure factor (cf. Figure 3 (c)) at the $M$-points of the extended Brillouin zone. This is in agreement with what is expected for the $q = 0$ magnetic order. The triplet gap exhibits a decreasing behavior upon increasing $W$ and $L$. Correspondingly, the spin-spin correlation length rapidly increases with $J_2$ (cf. Figure 5).

FIG. 6. (Color online) The spin triplet gap $\Delta_t$ of the $J_1$-$J_2$ Heisenberg model on the kagome lattice as a function of next-nearest neighbor interaction $J_2$. Spin gaps for the YC6 (see (a)) and the YC8 cylinder (see (b)), and various cylinder lengths are shown. The gaps are obtained by subtracting the energies of the lowest-energy states in the $S = 0$ and $S = 1$ symmetry sectors, which can be directly accessed by $SU(2)$-symmetric DMRG simulations.

FIG. 7. (Color online) The spin triplet gap in the $J_1$-$J_2$ Heisenberg model on the kagome lattice plotted versus $1/L$ for both YC6 and YC8 cylinders. Data for $L = 4, 6, 8, 10, 12$ and $J_2 = -0.2, 0.1, 0.4$ are shown in the figure. The lines denote the linear extrapolations to the infinite cylinder limit.
VII. ENTANGLEMENT SPECTROSCOPY IN THE $q = 0$ PHASE

Given a spatial bipartition of the cylinder in parts $A$ and $B$, the so-called entanglement spectrum (ES) levels \cite{Qian2016} \{$\xi_i$\} are constructed from the Schmidt decomposition of the ground-state wavefunction $|\psi\rangle$ as

$$|\psi\rangle = \sum_i e^{\xi_i/2} |\psi_i^A\rangle \otimes |\psi_i^B\rangle,$$

where $|\psi_i^A(B)\rangle$ form an orthonormal basis set for subsystem $A(B)$. Alternatively, the ES can be thought of as the spectrum of an effective entanglement Hamiltonian $\mathcal{H}_E$ that is defined as

$$\mathcal{H}_E \equiv \exp(-\rho_A),$$

where $\rho_A$ is the reduced density matrix of subsystem $A$. Since the DMRG algorithm works directly in the Schmidt basis the ES is available essentially for free during a ground state simulation and provides another useful tool to characterize the properties of the ground state.

It has been proposed recently \cite{Qian2016} that in a model that breaks a continuous symmetry in the thermodynamic limit the low-lying part of the ground-state entanglement spectrum (ES) exhibits the tower-of-states structure, which describes the finite-size energy spectrum of the model. In particular, for a spin model that fully breaks the $SU(2)$ symmetry, many features of the low-lying ES levels can be understood in terms of the entanglement Hamiltonian

$$\mathcal{H}_E \propto \frac{S_A^2}{W} + \cdots,$$

where $S_A$ is the total spin in subsystem $A$ and $W \sim \sqrt{N}$ the cylinder width (cf. Figure 2). The low-lying spectrum of (6) is shown schematically in Figure 9, plotting ES levels versus $S_A(S_A + 1)$. In each sector with fixed $S_A$ there are $(2S_A + 1)^2$ levels (rhombi in the Figure) forming the tower-of-states, which are divided from higher-lying levels by an entanglement gap. The tower-of-states levels exhibit linear behavior with respect to $S_A(S_A + 1)$. Notice that, although (6) gives $(2S_A + 1)^2$ degenerate levels in each spin sector, this degeneracy is in general lifted, as shown in Figure 9. The correspondence between ES and tower-of-states has been numerically verified in the $J_1$-$J_2$ KHA in Ref. 52 for $J_2 = -1.0$, i.e. deep in the $\sqrt{3} \times \sqrt{3}$ ordered phase.

Notice that both the $\sqrt{3} \times \sqrt{3}$ and the $q = 0$ ordering patterns correspond to full breaking of the $SU(2)$ symmetry (see Figure 1), as they contain three ferromagnetic sublattices. As a consequence, deep in the $q = 0$ phase one should expect the same tower-of-states structure shown in Figure 9 in the ES. However, here we provide numerical evidence that the identification of the correct tower-of-states depends on the choice of the bipartition, at least for small system sizes.

This is illustrated in Figure 10 plotting the half-system ES for a kagome cylinder with $4 \times 12$ unit cells (YC8 geometry) at $J_2 = 1.0$ and for two different bipartitions. The bipartitions are shown in (a) and (b): The three ferromagnetic sublattices forming the $q = 0$ state (cf. Figure 2) are denoted as $A$, $B$, $C$, bonds connecting spins on different sublattices are shown with different colors. While (a) corresponds to a straight cut, (b) has a zigzag structure. One should observe that the straight cut crosses only $B$-$C$ and $A$-$B$ bonds, whereas all the three types of bonds ($A$-$B$, $B$-$C$, and $A$-$C$) are crossed by the zigzag cut in (b). This suggests that the straight cut might not capture the
quantum correlations between sublattices \(A\) and \(C\). Notice that for the \(\sqrt{3} \times \sqrt{3}\) state this is not the case as the straight cut would cross all the three different types of bonds. The difference between the two cuts is reflected in the corresponding entanglement spectra.

The ES obtained using the straight cut (a) is reported in Figure 10 (c). The ES levels are plotted versus \(S_A(S_A + 1)\). Full symbols denote the lowest \((2S_A + 1)^2\) levels in each spin sector. Strong deviations from the expected picture in Figure 9 are visible. In particular, no gap between the tower-of-states levels and the rest of the spectrum is visible. Better agreement with Figure 9 is found using the zigzag cut, as it is clear from Figure 10 (d). For instance, the low-lying levels now show a clear linear behavior with respect to \(S_A(S_A + 1)\). Moreover, in the \(S_A = 0\) and \(S_A = 1\) sectors the tower-of-states levels are well separated from higher-lying levels by an entanglement gap, although this becomes smaller for \(S_A = 2\), when the low-lying levels start mixing with the rest of the spectrum. Finally, we should mention that, despite the numerical evidence in Figure 10, within the available system sizes we cannot exclude that the difference between the ES in (c) (d) disappears considering larger cylinders.

**VIII. CONCLUSION**

We performed an extensive DMRG study of the ground-state phase diagram of the \(J_1-J_2\) Heisenberg model on kagome cylinders. We restricted ourselves to \(J_2 = 1\), considering both antiferromagnetic and ferromagnetic \(J_2\). In particular, we investigated the behavior of the model around the pure kagome point at \(J_2 = 0\). To this purpose, we monitored the behavior of the spin triplet gap, the static structure factor, and the magnetic correlation length, as a function of \(J_2\). We should remark that our results are based on finite-size cylinders. Strong finite-size effects do not allow us to provide conclusive results about the phase diagram of the model in the thermodynamic limit.

By comparing the finite-size behaviors of the spin gap, the structure factor, and the correlation lengths, we found numerical evidence suggesting that the ground state of the model displays magnetic order for \(J_2 \lesssim -0.1\) and \(J_2 \gtrsim 0.2\). Precisely, for \(J_2 \lesssim -0.1\) the structure factor exhibits sharp peaks at the \(K\)-points of the extended Brillouin zone, in agreement with what is expected for the classical \(\sqrt{3} \times \sqrt{3}\) state, whereas at \(J_2 \gtrsim 0.2\) one observes peaks at the \(M\)-points, which signal the \(q = 0\) magnetic pattern. In both cases the correlation lengths associated with the two structures show a rapid increase upon increasing \(|J_2|\) and the system size. Correspondingly, the triplet gap decreases, suggesting a vanishing gap in the thermodynamic limit. Within the system sizes accessible to the simulations our results are consistent with the presence of a magnetically disordered phase for \(-0.1 \lesssim J_2 \lesssim 0.2\), which is compatible with spin-liquid behavior\(^{29}\). In this region the spin gap shows a weaker dependence on the cylinder size. Moreover, the DMRG data support a finite gap for infinitely long cylinders. The static structure factor is featureless at the \(J_2 = 0\) point, and it exhibits not very pronounced structures in the whole region \(-0.1 \lesssim J_2 \lesssim 0.2\). The magnetic correlation lengths associated with the \(\sqrt{3} \times \sqrt{3}\) and the \(q = 0\) order are of the order of the lattice unit.

As a final point, we investigated the structure of the ground state entanglement spectrum (ES) in the \(q = 0\) ordered phase. We found that the identification of the tower-of-states structure, which is associated with the \(SU(2)\) symmetry breaking in the thermodynamic limit, depends dramatically on the...
choice of the spatial bipartition of the state, at least for small system sizes.

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