Age of Changed Information: Content-Aware Status Updating in the Internet of Things

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Abstract—In Internet of Things (IoT), the freshness of status updates is crucial for mission-critical applications. In this regard, it is suggested to quantify the freshness of updates by using Age of Information (AoI) from the receiver’s perspective. Specifically, the AoI measures the freshness over time. However, the freshness in the content is neglected. In this paper, we introduce an age-based utility, named as Age of Changed Information (AoCI), which captures both the passage of time and the change of information content. By modeling the underlying physical process as a discrete time Markov chain, we investigate the AoCI in a time-slotted status update system, where a sensor samples the physical process and transmits the update packets to the destination. With the aim of minimizing the weighted sum of the AoCI and the update cost, we formulate an infinite horizon average cost Markov Decision Process. We show that the optimal updating policy has a special structure with respect to the AoCI and identify the condition under which the special structure exists. By exploiting the special structure, we provide a low complexity relative policy iteration algorithm that finds the optimal updating policy. We further investigate the optimal policy for two special cases. In the first case where the state of the physical process transits with equiprobability, we show that optimal policy is of threshold type and derive the closed-form of the optimal threshold. We then study a more generalized periodic Markov model of the physical process in the second case. Lastly, simulation results are laid out to exhibit the performance of the optimal updating policy and its superiority over the zero-wait baseline policy.

Index Terms—Internet of things, information freshness, Markov decision processes, structural analysis.

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I. INTRODUCTION

With the sharp proliferation of the Internet of Thing (IoT) devices and the rising need of mission-critical services, timely delivery of information has become increasingly important in real-time status update systems, such as environmental monitoring in smart city, vehicle tracking in autonomous driving, and video surveillance in smart home, whose performance strongly depends on the freshness of the status updates received by the destination [2]–[4]. The Age of Information (AoI) has been recently introduced to measure information freshness from the receiver’s perspective [5]. Particularly, it is defined as the time elapsed since the generation of the most recent status update packet received by the destination. In general, the smaller the AoI at the destination, the fresher the received status update. The AoI jointly characterizes the packet delay and the packet intergeneration time, which distinguishes AoI from conventional metrics, such as delay and throughput. Such a superiority of AoI for evaluating the information freshness in various wireless networks has been demonstrated in recent studies [5]–[8] by resorting to queueing theory.

An IoT network mainly consists of three components, i.e. the IoT device, the communication network, and the destination node. As such, to optimize the information freshness in terms of the AoI for the IoT, it is of great importance to control the status update process, which has attracted significant research attention in recent years [9]–[18]. Particularly, authors in [9] studied the AoI optimal packet transmission policy with random information arrivals under a transmission capacity constraint. In the case where the IoT device generates status updates at will, the authors in [10] studied the AoI minimization problem with a single device, where a zero-wait policy was shown to be non-optimal. Minimizing the average AoI in a status update system with multiple devices was further studied in [11]. Two age-optimal data collection problems were investigated to minimize the average AoI and peak AoI for UAV enabled wireless sensor networks in [12]. An optimal status updating scheme with hybrid Automatic Repeat request (ARQ) was proposed in [13] to minimize the average AoI under a constraint on the average number of transmissions. By considering the restriction from bandwidth and power consumption constraints, a dynamic scheduling algorithms was developed to minimize the average AoI of industrial IoT networks in [14]. The authors in [15] designed a joint status sampling and updating to minimize the average AoI under an average energy cost constraint. Further, by empowering the sensor nodes with energy harvesting techniques,
recent work [16] developed the age-aware primary spectrum sensing and update strategy for an energy harvesting cognitive radio. Meanwhile, the wireless energy transfer procedure and scheduling of update packet transmissions were jointly optimized by authors in [17], who further extended their research by considering the sampling cost in [18].

As seen above, the AoI has been widely used as a performance metric to characterize the information freshness over time. It does, however, disregard the content carried by the updates and the current knowledge at the receiver. A natural question that then emerges is whether measuring the freshness updates and the current knowledge at the receiver. A natural question that then emerges is whether measuring the freshness of updates through the AoI alone is sufficient. Several recent attempts have been made to answer this question [19]–[23]. The pros and cons of these metrics are elaborated in the following.

- Authors in [19] utilized mutual information between the state of the physical process and the received updates at the destination to evaluate the information freshness. The mutual information quantifies the amount of information that the received updates carry about the current value of the physical process. Although the destination has no knowledge of the current value of the physical process, it is proved that the mutual information is a non-negative and non-increasing function of AoI if the physical process is a stationary Markov chain and the sampling times are independent of the value of the physical process. Therefore, the mutual information can be computed by the destination via the AoI. However, for more general cases where the sampling policy needs to be devised based on the causal knowledge of the value of the physical process, the mutual information is not necessarily a function of the age. In this light, how to compute the mutual information at the destination is unknown.

- In [20], the authors proposed a metric, called sampling age, which is the time difference between the last ideal sampling time and the first actual sampling time. The ideal sampling time is the most recent time at which the state of the physical process changed relative to the last received update. However, the ideal sampling time is not available to the destination and hence the sampling age cannot be obtained by the destination.

- The Age of Synchronization (AoS) was proposed in [21] to measure how long the information at the receiver has become desynchronized compared with the physical process. It is defined as the time difference between the current time and the earliest update generation time after the previous synchronization time [22]. Similar to the AoI, the AoS drops when the destination receives a status update packet. However, unlike the AoI which begins to increase immediately after the reception of a status update packet at the destination, the AoS remains to be zero and does not increase until the sensor generates a new status update packet. However, the destination does not know the generation time of the earliest update after the previous synchronization until it receives this update. Hence, the AoS cannot be calculated at the destination.

- The Age of Incorrect Information (AoII) was proposed in [23] to address the real-time remote estimation problem. The AoII combines a time penalty function and an estimation error penalty function that reflects the difference between the current estimate at the destination and the actual state of the physical process. As such, the AoII will increase with time when the receiver stays in an erroneous state. Computing the AoII at the destination requires that the state of the physical process is available to the destination in any time slot. Otherwise, the estimation at the destination and the state of the physical process cannot be compared to compute the estimation error penalty function in the AoII. However, the destination cannot observe the state of the physical process until it receives the status update packet. Therefore, the AoII cannot be computed by the destination.

In summary, mutual information cannot be computed if the sampling times are determined by using causal knowledge of the value of the physical process, while the other three metrics are only available to the transmitter rather than the destination. Since the last three metrics cannot be computed by the destination, they cannot be applied in the scenario where the sensor has no computing capability and the destination is in charge of decision-making. This is also the scenario we focus on in this work. Moreover, even if the sensor has the computing capability, the above three metrics require the continuous sensing of the physical process, which would induce noticeable energy consumption.

In this paper, we concentrate on the scenario where the receiver aims to conduct timely detection of status changes in the underlying physical process only based on its received update packets. In practice, a status change won’t be detected until an update generated after the change point is successfully delivered to the destination for the first time. However, it is impossible to know the exact time instant of a status change unless the physical process is monitored continuously. Therefore, it is challenging to design the optimal updating policy to balance the information freshness and the energy consumption. On the one hand, sampling and transmitting at a higher frequency incurs a higher energy consumption of the sensor. On the other hand, sampling at a lower frequency results in staleness in detecting a status change or even a miss detection. The error-prone wireless channel further worsens the situation, since the update packet may be dropped due to channel outage. As a result, the receiver could be fooled into believing that no change in state has taken place. Motivated by all this, we introduce a utility function from the receiver’s perspective that depicts both the passage of time and the change of information content. We further investigate this utility in a status update system consisting of a sensor and a destination. In particular, the sensor monitors the real-time status of a physical process, which is modeled by a discrete time Markov chain with uniform stationary distribution, and transmits status update packets to the destination. In our earlier work [1], we investigated the effects of content change on the information freshness and designed the optimal status updating policy in an IoT system. However, the model of the physical process is limited to the two-state Markov chain with the equal
transition probabilities. The key contributions of this paper are summarized as follows:

- Motivated by the fact that a status change will not be perceived by the destination until an update generated after the change instant is successfully delivered, we introduce a new age-based utility, referred to as Age of Changed Information (AoCI), that characterizes the information freshness via the updates received by the destination. The word “changed” refers to the newly received update that brings new content different from the previous one at the destination. The AoCI takes into account the information content of the updates and the current knowledge at the destination. It will increase when the update with the same status information is received.

- We formulate the status updating problem as an infinite horizon average cost Markov Decision Process (MDP) with the goal of minimizing the weighted sum of the AoCI and the update cost. By incorporating the AoCI into the cost function of the MDP, the sensor is made to sample and transmit at a higher frequency when the same status information is continuously received, thereby potentially reducing the miss detection. We analyze the properties of the value function without specifying the state transition model of the physical process. Armed with these properties, we show that the optimal updating policy has a special structure with respect to the AoCI and identify the condition on the return probability of the physical process under which the special structure exists. A structure-aware relative policy iteration algorithm is then proposed to obtain the optimal updating policy with low complexity.

- We study two special cases, where the return probability satisfies the condition. In the first case, by giving an example that the state of the underlying physical process transits with equiprobability, we simplify the MDP and prove that the optimal policy is of threshold type. We also derive the optimal threshold in closed-form, which sheds insight on how the system parameters affect the threshold policy. Particularly, we prove that the optimal threshold is non-increasing with transmission success probability and the number of states of the physical process, but is non-decreasing with the update cost. We generalize the example in the first case by studying the periodic Markov model of the physical process in the second case. Simulation results highlight interesting insights on the effects of the system parameters and show the superiority of the optimal updating policy over the zero-wait policy.

The rest of the paper is organized as follows: Section II provides a description of the system model and a definition for the proposed performance metric. In Section III, we present the MDP formulation and analyze the structure of the optimal policy. Two special cases are then analyzed in Section IV. Simulation results are presented in Section V, followed by the conclusion in Section VI.

![Figure 1. An illustration of a status update system monitoring a physical process.](image-url)
next slot, a new status update will be generated.\footnote{The reason why we sample and transmit a new status update rather than retransmit a failed update lies in two aspects. On the one hand, according to the studies in \cite{13}, in the context of AoI, it is better not to retransmit an uncoded packet with the classical ARQ protocol, where failed transmissions are discarded at the destination and the receiver tries to decode each retransmission as a new message. This is because the probability of a successful transmission is the same for a retransmission and for the transmission of a new update. On the other hand, in this work we focus on the scenario that the sensor performs simple monitoring tasks, such as reading temperature, and hence, the cost for generating status packets is assumed to be negligible compared to that of transmission. Then, the energy costs for transmitting a new update and retransmitting a failed update are almost the same. Altogether, we choose to transmit a new update when the transmission failure occurs.}

**B. Freshness Metric**

We assume that at the beginning of a slot a status update is generated and transmitted, and the destination will receive it at the end of the slot if the transmission succeeds. The AoI, commonly used to quantify the freshness of the information, is specified as the time elapsed since the generation of the latest status update received by the destination. Suppose that the latest status update successfully received by the destination was generated at the time instants $U(t)$, i.e., $U(t) = \max\{g_{i} \mid d_{i} \leq t\}$, where $g_{i}$ and $d_{i}$ represent the time instants when the update $i$ is generated and delivered, respectively. Then, the AoI at the beginning of slot $t$ is given by

$$\delta_{t} = t - U(t).$$  \hspace{1cm} (1)

The proposed metric, AoCI, is different from the AoI in that the AoCI not only captures the time lag of the update received at the destination, but also includes variations in the information content of these updates. In particular, the AoCI only declines when the newly received update content differs from the previous one, and boosts otherwise. Let us denote by $n(t) = \max\{i \mid d_{i} \leq t\}$ the index of the latest update the destination receives at the end of slot $t$. We let $Y_{j}$ denote the information content of update $j$. It is worth noting that $Y_{j}$ is equal to the state of the physical process in the slot when update $j$ was generated, e.g., $Y_{n(t)} = X_{U(t)}$. Let us denote by $m(t) = \max\{j \mid Y_{j} \neq Y_{n(t)} \wedge d_{j} \leq d_{n(t)}\}$ the index of the most recently updated packet which has different content from the latest update $n(t)$ got. Then, we can define the AoCI at the beginning of slot $t$ as

$$\Delta_{t} = t - U'(t),$$  \hspace{1cm} (2)

where $U'(t) = \min\{g_{k} \mid d_{n(t)} < d_{k} \leq d_{m(t)}\}$ represents the generation time of the next successfully received update after $m(t)$. Noting that all the update packets that have been successfully received after $m(t)$ have the same content as the latest one received. We set the upper limits to the AoCI and the AoI, which are denoted by $\hat{\Delta}$ and $\hat{\delta}$, respectively.

In slot $t$ where a status update is received successfully, we denote by $D_{t} \in \{0, 1\}$ an indicator for whether the content of the newly received update varies from that of the previously received update. If $D_{t} = 1$, then the newly received update has different content. Otherwise, it has the same content. Particularly, the content change probability is defined as

$$\Pr(D_{t} = 1) \overset{(a)}{=} \Pr(Y_{n(t)} \neq Y_{n(t) - 1}) \overset{(b)}{=} 1 - p_{r}(\delta_{t}),$$  \hspace{1cm} (3)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure2.png}
\caption{An illustration of the AoCI and the AoI in a time-slotted status update system.}
\end{figure}

where $p_{r}(\delta_{t}) = \Pr(X_{U(t)} = X_{U(t) - \delta_{t}})$ is the return probability that the state of the physical process remains the same after $\delta_{t}$ steps. In the above equation, (a) holds due to the definition of $D_{t}$, and (b) holds because of the fact that $Y_{n(t)} = X_{U(t)}$ and $Y_{n(t) - 1} = X_{U(t) - \delta_{t}}$.\footnote{Note that the time difference between two consecutive samples at the sensor is $\delta_{t}$. Therefore, the update $n(t) - 1$ was generated at $U(t) - \delta_{t}$, and its content $Y_{n(t) - 1}$ is the same as the state of the physical process at $U(t) - \delta_{t}$, i.e., $X_{U(t) - \delta_{t}}$.} According to (2), if a new status update generated by the sensor is received successfully by the destination (i.e., $a_{t} = 1, h_{t} = 1$) and it contains different content from the update previously received (i.e., $D_{t} = 1$), then the AoCI decreases to one; otherwise, the AoCI increases by one. Then, the dynamics of the AoCI is given by

$$\Delta_{t+1} = \begin{cases} 1, & a_{t} = 1, h_{t} = 1, D_{t} = 1; \\ \min\{\Delta_{t} + 1, \hat{\Delta}\}, & \text{otherwise}. \end{cases}$$  \hspace{1cm} (4)

For ease of exposition, we demonstrate how the AoCI and the AoI evolve over time in Fig. 2, where the dotted line represents the AoI and the solid line represents the AoCI.

**C. Problem Formulation**

The aim of this paper is to find an update policy $\pi = (\alpha_{0}, \alpha_{1}, \ldots)$ that minimizes the total average cost, which is defined as the weighted sum of the AoCI and the update cost. By defining $\Pi$ as a set of stationary and deterministic policies,\footnote{A policy is said to be stationary and deterministic if it is time invariant and chooses an action with probability one.} our problem is formulated as follows:

$$\min_{\pi \in \Pi} \lim_{T \to \infty} \sup_{T} \frac{1}{T} \sum_{t=0}^{T} \mathbb{E}[\Delta_{t} + \omega a_{t} C_{u}],$$  \hspace{1cm} (5)

where $\omega$ is the weighting factor and the expectation is taken with respect to the distribution over trajectories induced by $\pi$ together with the transition probabilities.

**III. OPTIMAL UPDATE POLICY DESIGN**

**A. MDP Formulation**

The optimization problem in (5) can be cast into an infinite horizon average cost Markov decision process
(S, A, Pr(·|·), C(·, ·)), where each element is described as follows:

- States: The state of the MDP at time slot t is defined to be the tuple of the AoCI and the AoI, i.e., $s_t \triangleq (\Delta_t, \delta_t)$. Since both the AoCI and the AoI are bounded by their upper limits, the state space S is finite.

- Actions: The action at time slot t is $a_t$ and the action set is $A = \{0, 1\}$.

- Transition Probability: Let $Pr(s_{t+1}|s_t, a_t)$ denote the transition probability that state transits from $s_t$ to $s_{t+1}$ by taking action $a_t$ at slot t. Because the event of packet transmission and that of content change are independent, according to the AoCI evolution dynamics (4), the transition probability is represented as in (6) and $Pr(s_{t+1}|s_t, a_t) = 0$ otherwise.

- Cost: We let $C(s_t, a_t) = \Delta_t + \omega a_t C_u$ denote the instantaneous cost at state $s_t$ given action $a_t$.

The above MDP is a finite-state finite-action average-cost MDP. According to [24, Theorem 8.4.5], there exists a deterministic stationary average optimal policy for the finite-state finite-action average-cost MDP if the cost function is bounded and the MDP is unichain, i.e., the Markov chain corresponding to every deterministic stationary policy consists of a single recurrent class plus a possibly empty set of transient states. Below, we examine these two conditions. First, the cost of the above MDP is bounded since the instantaneous cost is defined as the weighted sum of the AoCI and the energy consumption. Second, since the state $(\Delta, \delta)$ is reachable from all other states, the induced Markov chain has a single recurrent class. Hence, the MDP is unichain. Altogether, there exists a stationary and deterministic optimal policy. The optimal policy $\pi^*$ to minimize the total average cost can be obtained by solving the following Bellman equation [25]:

$$
\theta + V(s) = \min_{a \in \{0, 1\}} \left\{ C(s, a) + \sum_{s' \in S} Pr(s'|s, a)V(s') \right\}, \forall s \in S,
$$

(7)

where $\theta$ is the optimal value to (5) for all initial state and $V(s)$ is the value function which is a mapping from s to real values. Moreover, for any $s \in S$, the optimal policy can be given by

$$
\pi^*(s) = \arg \min_{a \in \{0, 1\}} \left\{ C(s, a) + \sum_{s' \in S} Pr(s'|s, a)V(s') \right\}.
$$

(8)

We can see from (8) that the optimal policy $\pi^*$ depends on the value function $V(\cdot)$. Unfortunately, there is usually no closed-form solution for $V(\cdot)$ [25]. Therefore, numerous numerical algorithms have been proposed in the literature, such as policy iteration and value iteration. Nonetheless, owing to the curse of dimensionality these approaches are typically computationally exhausting, and few insights can be leveraged for optimal policy. Hence, in the sequel, we study the structural properties of the optimal updating policy.

To analyze the structure of $\pi^*$, we introduce the state-action value function $Q(s, a)$, which is defined as

$$
Q(s, a) = C(s, a) + \sum_{s' \in S} Pr(s'|s, a)V(s'),
$$

(9)

for all $s \in S$ and $a \in A$. Note that $Q(s, a)$ is related to the RHS of the Bellman equation in (7). The optimal policy can also be expressed in terms of $Q(s, a)$, i.e.,

$$
\pi^*(s) = \arg \min_{a \in \{0, 1\}} Q(s, a), \ \forall s \in S.
$$

(10)

B. Structural Analysis and Optimal Policy

Before we present the main theorem, we first show the key properties of the value function $V(\Delta, \delta)$ in the following lemmas.

**Lemma 1.** The value function $V(\Delta, \delta)$ is non-decreasing with $\Delta$ for any $\delta$.

*Proof:* See Appendix A.

**Lemma 2.** Given $\delta$ we have $V(\Delta_2, \delta) - V(\Delta_1, \delta) \geq \Delta_2 - \Delta_1$ for any $\Delta_2 \geq \Delta_1$.

*Proof:* See Appendix B.

**Lemma 3.** For any $\Delta$ and $\delta_1 \leq \delta_2$, if $p_r(\delta_1) - p_r(\delta_2) \leq \frac{1}{v_4(\Delta+1)} - \frac{1}{v_4(\Delta)}$, for any $k \in \mathbb{Z}_{\geq 0}$, we have $V(\Delta, \delta_1) - V(\Delta, \delta_2) \leq \delta_2 - \delta_1$, where $v_4(\cdot)$ is the value function obtained in the value iteration algorithm.

*Proof:* See Appendix C.

Then, we present the structure of the optimal updating policy in the following theorem.

**Theorem 1.** Given $\delta = (\Delta, \delta)$, if $p_r(1) - p_r(\delta + 1) \leq \frac{1}{v_4(\Delta+1)} - \frac{1}{v_4(\Delta)}$, for any $\Delta$ and $k \in \mathbb{Z}_{\geq 0}$, the optimal updating policy has a special structure for any $\delta$, that is, if $\Delta \geq \Delta(\delta)$, then $\pi^*(s) = 1$, where $\Delta(\delta)$ is the minimum integer value satisfying $p_r(1) - p_r(\delta)\Delta - p_r\delta - \omega C_u \geq 0$.

*Proof:* See Appendix D.

It is noteworthy that the special structure in Theorem 1 is different from the threshold structure defined in available studies [15], [23], where the optimal policy is to update when $\Delta$ is no less than the threshold and the optimal policy is to keep idle otherwise.

We then propose a low-complexity relative policy iteration algorithm to compute the optimal policy based on the special structure of the optimal updating policy presented in Theorem 1. Although the exact value of the threshold is
Algorithm 1 Relative Policy Iteration based on the Threshold Structure

1: Initialization: Set \( k = 0 \) and \( \pi_0(s) = 0 \) for all state \( s = (\Delta, \delta) \in S \), select a reference state \( s^\dagger \) and set \( V_0(s^\dagger) = 0 \).
2: repeat
3: \( \pi_{k+1}(s) \leftarrow 0 \).
4: Policy Evaluation:
5: Given policy \( \pi_k(s) \), compute the value of \( \theta_k \) and \( V_k(s) \) by solving the following \(|S|\) linear equations:
6: \[
\begin{align*}
\theta_k + V_k(s) &= C(s, \pi_k(s)) + \\
\sum_{s' \in S} \Pr(s'|s, \pi_k(s))V_k(s'),
\end{align*}
\]
7: \( V_k(s^\dagger) = 0 \).
8: Policy Improvement:
9: for \( s = (\Delta, \delta) \in S \) do
10: if \( p_r(1) - p_r(\delta + 1) \leq \frac{\delta}{V_k(\Delta, 1) - V_k(1, 1)} \) and \( p_s(1 - p_r(\delta))\Delta - p_{s, \delta} - \omega C_u \geq 0 \) then
11: \( \pi_{k+1}(s) \leftarrow 1 \).
12: else
13: \( \pi_{k+1}(s) \leftarrow \arg\min_{\alpha \in A} \{ C(s, \alpha) + \\
\sum_{s' \in S} \Pr(s'|s, \alpha)V_k(s') \} \).
14: end if
15: end for
16: \( k \leftarrow k + 1 \).
17: until \( \pi_{k+1}(s) = \pi_k(s) \) for all \( s \in S \).
18: \( \pi^* \leftarrow \pi_{k+1} \).
19: return the optimal policy \( \pi^* \).

In this section, we study two special cases, where the return probability of the physical process satisfies certain conditions.

A. Case 1

We first consider a special case where the return probability \( p_r(\delta) \) is irrespective of \( \delta \), i.e., \( p_r(\delta_1) = p_r(\delta_2) \neq 0 \) for any \( \delta_1 \neq \delta_2 \). It is easy to see that the condition in Theorem 1 is satisfied in this special case and the optimal updating policy has a threshold structure with respect to the AoCI.

One example for this special case is that the state of the underlying physical process transits with equiprobability. The one-step state transition probability matrix for \( M \)-state discrete time Markov chain is given by

\[
\Pr(X_{t+1}|X_t) = \begin{bmatrix}
p_c & p_c & \cdots & p_c \\
p_c & p_c & \cdots & p_c \\
\vdots & \vdots & \ddots & \vdots \\
p_c & p_c & \cdots & p_c
\end{bmatrix},
\]

where \( p_c = \frac{1}{M} \). Since the \( \delta \)-step state transition probability matrix \( \Pr(X_{t+\delta}|X_t) \) is the same with \( \Pr(X_{t+1}|X_t) \) for all \( \delta \), we have \( p_r(\delta) = 1/M \).

In this case, we can simplify the MDP formulated in Section III.A. In particular, the state at slot \( t \) is only the AoCI, i.e., \( s_t = \Delta_t \), and the state transition probability in (6) can be simplified as

\[
\begin{align*}
\Pr(s_{t+1} = \min\{\Delta + 1, \hat{\Delta}\}|s_t = \Delta, a_t = 0) &= 1, \\
\Pr(s_{t+1} = \min\{\Delta + 1, \hat{\Delta}\}|s_t = \Delta, a_t = 1) &= p_f + p_sp_r, \\
\Pr(s_{t+1} = 1|s_t = \Delta, a_t = 1) &= p_s(1 - p_r),
\end{align*}
\]

and \( \Pr(s_{t+1} | s_t, a_t) = 0 \) otherwise.

Based on the simplified state and transition probability, we present the monotonicity property of \( V(s) \) in the following lemma.

Lemma 4. The value function \( V(s) \) is non-decreasing with \( s \).

Proof: See Appendix E.

Next, in the following theorem, we give results on the structure of the optimal updating policy.

Theorem 2. The optimal policy has a threshold structure, that is, if \( \pi^*(s_1) = 1 \), then \( \pi^*(s_2) = 1 \) for all \( s_2 \geq s_1 \).

Proof: See Appendix F.

According to Theorem 2, the optimal policy can be represented as a threshold policy, which is given by

\[
\pi^*(s) = \begin{cases}
1, & \text{if } s \geq \Omega, \\
0, & \text{otherwise},
\end{cases}
\]

where \( \Omega \) is the threshold at which the switching occurs. Under the threshold policy, we proceed with analyzing the total average cost of any threshold \( \Omega \) in the asymptotic regime.
Lemma 5. Let \( p_z \triangleq p_f + p_s p_r \). When \( \bar{\Delta} \) goes to infinity, for any given threshold \( \Omega \), the total average cost \( J(\Omega) \) of the threshold policy approaches to \( J(\Omega) = J_1(\Omega) + J_2(\Omega) \), where

\[
J_1(\Omega) = \frac{1 - p_z}{\Omega(1 - p_z)} p_z \left( \frac{\Omega(\Omega - 1)}{2} + \frac{\Omega}{1 - p_z} + \frac{p_z}{(1 - p_z)^2} \right),
\]

and

\[
J_2(\Omega) = \frac{\omega C_u}{\Omega(1 - p_z) + p_z}.
\]

Proof: See Appendix G.

By leveraging the above results, we can proceed to find the optimal threshold value \( \Omega^* \).

Theorem 3. The asymptotically optimal threshold \( \Omega^* \) of the optimal updating policy is given by

\[
\Omega^* = \arg \min (J([\Omega'])), \quad \Omega' = \frac{\sqrt{p_r + 2\omega C_u (1 - p_s)} - p_s}{1 - p_z},
\]

where \( \Omega' \) is a one-dimensional random walk with 4-state and 6-state, respectively. The period of both Markov chain is 2. In both cases, the Markov model is periodic with period 2. Hence, the return probability \( p_r \) is large. We can also observe that the asymptotically optimal threshold is non-decreasing with \( C_u \). This indicates that the sensor will remain idle until the AoCI is large. Hence, the optimal policy is able to achieve a balance between the AoCI and the update cost.

Fig. 3 illustrates the asymptotically optimal threshold \( \Omega^* \) of the optimal updating policy with respect to \( p_s \) under different \( C_u \). The asymptotically optimal threshold is shown to be non-increasing with \( p_s \). This is because, before the destination successfully receives an update packet, the sensor has to sample and transmit more times when \( p_s \) is small. Hence, updating the status is productive only when the AoCI is large. We can also observe that the asymptotically optimal threshold is non-decreasing with \( C_u \). This indicates that the sensor will remain idle until the AoCI is large, if the update cost is high. Hence, the optimal policy is able to achieve a balance between the AoCI and the update cost.

Fig. 4 shows the asymptotically optimal threshold \( \Omega^* \) of the optimal updating policy with respect to \( M \) under different \( C_u \). We can observe that the asymptotically optimal threshold is non-increasing with \( M \). This is due to the fact that the return probability \( p_r \) is large, when \( M \) is small. In other words, the received status update is more likely to contain the same content with the previous one. Hence, it is more cost-efficient to have a larger threshold at a smaller \( M \). We note that the reason why \( \Omega^* \) becomes constant when \( M \) becomes large is due to the fact that \( \Omega' \) converges as \( M \) grows and so do \([\Omega']\) and \([\Omega']\).

B. Case 2

In this case, we consider that \( p_r(1) = 0 \) and \( p_r(\delta) \geq 0 \) for \( \delta > 1 \). It is also easy to see that the condition in Theorem 1 is satisfied in this special case and the optimal updating policy has a special structure with respect to the AoCI as given in Theorem 1. The optimal updating policy can also be obtained via Algorithm 1.

The example for this case is the physical process modeled by periodic Markov chain. For instance, the Markov model could be a \( M \)-state one-dimensional random walk with the state transition probability matrix given by

\[
\Pr(X_{t+1}|X_t) = \begin{bmatrix}
p_r & 0 & \ldots & 0 & 1 - p_r \\
1 - p_r & p_r & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
p_r & 0 & \ldots & 1 - p_r & 0
\end{bmatrix},
\]

where \( p_r \in (0, 1) \) and \( M \) is even. Every state of this Markov chain is periodic with period 2. Hence, the return probability \( p_r(\delta) \) is 0 when \( \delta \) is odd, and is positive when \( \delta \) is even.

In Figs. 5 and 6, we illustrate the analytical results of Theorem 1 when Markov model of the physical process is a one-dimensional random walk with 4-state and 6-state, respectively. The period of both Markov chain is 2. In both figures, the optimal updating policy is shown to have a special
structure with respect to the AoCI for any $\delta$. In fact, the structure of the optimal policy unveils a tradeoff between the AoCI and the update cost. Particularly, if the AoCI is small, it is not efficient for the sensor to send the status update to the destination due to the update cost. It can also be seen that the optimal action $\pi^* = 0$ does not appear in the whole state space of the AoCI if the AoI is high. This is due to the fact that the AoCI is always no less than the AoI and it is more efficient to generate a new status update due to the outdated information at the destination with the high AoI.

V. Simulation Results

In this section, the simulation results of the optimal updating policy are presented to examine the effects of system parameters. The performance of the optimal updating policy is also compared with that of two baseline policies, i.e., zero-wait policy and sample-at-change policy. In zero-wait policy, the sensor samples and transmits the status update at each time slot. While in sample-at-change policy, a new update is generated only when the state changes relative to the previous received update at the destination. Note that the sample-at-change policy is genie-aided and can achieve the minimum AoCI.

A. Performance Evaluation in the Special Case 1

In Fig. 7, we compare the total average cost of the optimal policy and two baseline policies with respect to $p_s$. It can be observed in Fig. 7(a) that the optimal policy outperforms the zero-wait policy. Moreover, as $p_s$ increases, there is a larger reduction in the total average cost. The reason can be explained with the aid of Fig. 7(b). Although the zero-wait policy gains a smaller AoCI, it bears a constant update cost. In contrast, the optimal policy can trade off the AoCI for the update
cost. Particular, in the optimal policy, the sensor remains idle until the AoCI is larger than a threshold, thereby inducing a large AoCI. However, the optimal policy has a smaller update cost than the zero-wait policy. We also observe that the sample-at-change policy outperforms the optimal policy at first. As $p_s$ increases, the optimal policy beats the sample-at-change policy, because the update cost of the optimal policy declines as $p_s$ increases. Hence, the optimal policy is more cost-efficient.

In Fig. 8, we compare the total average cost of the optimal policy and two baseline policies with respect to $M$, where $p_c$ is set to be $1/M$. We can see in Fig. 8(a) that the optimal policy outperforms both baseline policies. Moreover, for both the optimal policy and the zero-wait policy, when $M$ grows, the total average cost decreases. As shown in Fig. 8(b), the average AoCI of the optimal policy is larger than that of the zero-wait policy, whereas the average update cost of the optimal policy is smaller than that of the zero-wait policy. The average AoCI decreases with the increasing of $M$ because the received status update is more likely to have a new content with a large $M$. The average update cost is affected by $M$ and the optimal threshold simultaneously. The increase of the average update cost when $M = 4$ is due to the decrease of the optimal threshold as shown in Fig. 4. When the optimal threshold is fixed, the average update cost decreases as $M$ increases. We also observe that the total average cost of the sample-at-change policy increases with $M$. This is because the return probability decreases with $M$ and the sample-at-change policy approaches the zero-wait policy as $M$ increases.

B. Performance Evaluation in the Special Case 2

In Fig. 9, we compare the total average cost of the optimal policy and two baseline policies with respect to $p_s$. It can be observed that the optimal policy outperforms both two
policies are coincident when $\omega$ increases, both the AoCI and the update cost of the optimal policy decline. This is because the AoCI can be reset with fewer transmission when $p_s$ is large. The AoCI of both baseline policy decreases with the increase of $p_s$. However, the update cost of the zero-wait policy remains constant and the update cost of the sample-at-change policy increases with $p_s$. In particular, the sample-at-change policy degenerates to zero-wait policy when $p_s = 1$. As a result, the performance gain of the optimal policy is larger when $p_s$ is larger.

In Fig. 10, we compare the total average cost of the optimal policy and two baseline policies with respect to $\omega$. From Fig. 10(a), we can observe that the total average cost of three policies increase with the weighting factor $\omega$. The three policies are coincident when $\omega = 0$, which implies that the sensor under the optimal policy would sample and transmit in each time slot. As $\omega$ increases, the total average cost of the zero-wait policy and the sample-at-change policy grow linearly. This is because these two baseline policies cannot adapt to the weighting factor and its average AoCI and average update cost are constant, as shown in Fig. 10(b). On the contrary, the optimal policy is able to adjust according to $\omega$. When $\omega$ grows larger, the AoCI is traded off to obtain a smaller update cost. Therefore, the optimal policy can strike a balance between the AoCI and the update cost.

## VI. Conclusion

In this paper, by identifying the ignorance of information content variation in the conventional AoI, we have proposed the AoCI as an age-based utility to quantify information freshness. The AoCI not only measures the freshness by the passage of time but also captures the information content of the updates at the destination. We have further investigated the updating policy in the status update system by taking into account both the AoCI and the update cost, and formulated the status updating problem as an infinite horizon average cost MDP. We have analyze the properties of the value function without specifying the state transition model of the physical process. Based on these properties, we have proved that the optimal updating policy has a special structure with respect to the AoCI and identify the condition on the return probability of the physical process under which the special structure exists. Equipped with this, we have provided a structure-aware relative policy iteration algorithm to obtain the optimal updating policy with low complexity. We have also studied two special cases where the condition holds. In the special case where the state of the underlying physical process transits with equiprobability, we have proved that the optimal policy is of threshold type and derived the closed-form of the optimal threshold. We have also proved that the optimal threshold is non-increasing with respect to transmission success probability and the number of states of the physical process, respectively, but is non-decreasing with respect to the update cost. Results from the simulation have shown the impacts of the unreliable channel and the physical process on the total average cost. By comparing the optimal updating policy with the zero-wait policy and the sample-at-change policy, it is shown that the optimal updating policy achieves a balance between the AoCI and the update cost and yields a substantial performance boost in terms of the total average cost compared to zero-wait policy.

## Appendix

A. Proof of Lemma 1

We prove Lemma 1 through mathematical induction and the value iteration algorithm (VIA) [25]. We first briefly introduce VIA. For each state $s$, let $V_k(s)$ be the value function at iteration $k$. In VIA, the value function can be updated as follows:

$$V_{k+1}(s) = \min_{a} \{Q_k(s, a)\}, \forall s \in S.$$  \hspace{1cm} (18)

Under any initialization of the initial value $V_0(s)$, the sequence $\{V_k(s)\}$ converges to the value function in the Bellman equation (7) [25], i.e.,

$$\lim_{k \to \infty} V_k(s) = V(s), \forall s \in S.$$  \hspace{1cm} (19)
Therefore, the monotonicity of \( V(s) \) in \( S \) can be guaranteed by proving that for any \( s_1, s_2 \in S \), such that \( s_1 \leq s_2 \),
\[
V_k(s_1) \leq V_k(s_2), \quad k = 0, 1, \ldots
\]  
(20)

According to (19), the monotonicity of \( V(s) \) with respect to the \( \text{AoCI} \) can be guaranteed by proving that for any \( s_1 = (\Delta_1, \delta), s_2 = (\Delta_2, \delta) \in S \), such that \( \Delta_1 \leq \Delta_2 \),
\[
V_k(s_1) \leq V_k(s_2), \quad k = 0, 1, \ldots
\]  
(21)

Then, we prove (21) via mathematical induction. Without loss of generality, we initialize \( V_0(s) = 0 \) for all \( s \in S \). Thus, (21) holds for \( k = 0 \). Next, we assume that (21) holds up till \( k > 0 \) and we examine whether it holds for \( k + 1 \).

When \( a = 0 \), we have \( Q_k(s_1, 0) = \Delta_1 + V_k(s'_1) \) and \( Q_k(s_2, 0) = \Delta_2 + V_k(s'_2) \), where \( s'_1 = (\Delta_1 + 1, \delta + 1) \) and \( s'_2 = (\Delta_2 + 1, \delta + 1) \). Since \( \Delta_1 + 1 \leq \Delta_2 + 1 \) and \( V_k(s'_1) \leq V_k(s'_2) \), we can easily see that \( Q_k(s_1, 0) \leq Q_k(s_2, 0) \).

When \( a = 1 \), we have
\[
Q_k(s_1, 1) = \Delta_1 + \omega C_u + p_1 (1 - p_r(\delta)) V_k(1, 1) + p_f V_k(\Delta_1 + 1, \delta + 1) + p_a p_r(\delta) V_k(\Delta_1 + 1, 1)
\]  
(22)

and
\[
Q_k(s_2, 1) = \Delta_2 + \omega C_u + p_1 (1 - p_r(\delta)) V_k(1, 1) + p_f V_k(\Delta_2 + 1, \delta + 1) + p_a p_r(\delta) V_k(\Delta_2 + 1, 1).
\]  
(23)

Since \( V_k(\Delta_1, \delta) \leq V_k(\Delta_2, \delta) \) for any \( \delta \), we can also verify that \( Q_k(s_1, 1) \leq Q_k(s_2, 1) \).

Altogether, we can assert that \( V_{k+1}(s_1) \leq V_{k+1}(s_2) \) for any \( k \). By taking limits on both sides of (21) and (19), we complete the proof of Lemma 1.

**B. Proof of Lemma 2**

Let \( s_1 = (\Delta_1, \delta) \) and \( s_2 = (\Delta_2, \delta) \). Based on Lemma 1, we have
\[
Q(s_2, 0) - (\Delta_2 - \Delta_1) = \Delta_1 + V(\Delta_2 + 1, \delta + 1) \geq \Delta_1 + V(\Delta_1 + 1, \delta + 1) = Q(s_1, 0),
\]  
(24)

and
\[
Q(s_2, 1) - (\Delta_2 - \Delta_1) = \Delta_1 + \omega C_u + p_1 (1 - p_r(\delta)) V(1, 1) + p_f V(\Delta_2 + 1, \delta + 1) + p_a p_r(\delta) V(\Delta_1 + 1, 1) \geq Q(s_1, 1).
\]  
(25)

Since \( V(s) = \min Q(s, a) \), we can prove \( V(s_2) - V(s_1) \geq \Delta_2 - \Delta_1 \) in four cases as follows:

- **Case 1:** If \( V(s_1) = Q(s_1, 0) \) and \( V(s_2) = Q(s_2, 0) \), then \( V(s_2) - V(s_1) = Q(s_2, 0) - Q(s_1, 0) \geq \Delta_2 - \Delta_1 \).
- **Case 2:** If \( V(s_1) = Q(s_1, 1) \) and \( V(s_2) = Q(s_2, 1) \), then \( V(s_2) - V(s_1) = Q(s_2, 1) - Q(s_1, 1) \geq \Delta_2 - \Delta_1 \).
- **Case 3:** If \( V(s_1) = Q(s_1, 0) \) and \( V(s_2) = Q(s_2, 1) \), then \( V(s_2) - V(s_1) = Q(s_2, 1) - Q(s_1, 0) \leq \Delta_2 - \Delta_1 \).
- **Case 4:** If \( V(s_1) = Q(s_1, 1) \) and \( V(s_2) = Q(s_2, 0) \), then \( V(s_2) - V(s_1) = Q(s_2, 0) - Q(s_1, 1) \geq \Delta_2 - \Delta_1 \).

This completes the proof of Lemma 2.

**C. Proof of Lemma 3**

Let \( s_1 = (\Delta_1, \delta_1) \) and \( s_2 = (\Delta_2, \delta_2) \). We use mathematical induction and the VIA to prove Lemma 3. Specifically, we need to prove that for any \( \delta_1 \leq \delta_2 \),
\[
V_k(s_1) - V_k(s_2) \leq \delta_2 - \delta_1, \quad k = 0, 1, \ldots
\]  
(26)

if \( p_r(\delta_1) - p_r(\delta_2) \leq \frac{\delta_2 - \delta_1}{V_k(\Delta + 1, 1) - V_k(1, 1)} \) for any \( k \).

Without loss of generality, we initialize \( V_0(s) = 0 \) for all \( s \in S \). Thus, (26) holds for \( k = 0 \). Next, we assume that (26) holds up till \( k > 0 \) and we examine whether it holds for \( k + 1 \).

When \( a = 0 \), we have
\[
Q_k(s_2, 0) + (\delta_2 - \delta_1) = \Delta_1 + V(\Delta + 1, \delta_2 + 1) + \delta_2 - \delta_1 \geq \Delta_1 + V(\Delta + 1, \delta_1 + 1) = Q_k(s_1, 0).
\]  
(27)

When \( a = 1 \), we have
\[
Q_k(s_2, 1) + (\delta_2 - \delta_1) = \Delta_1 + V(\Delta + 1, \delta_2 + 1) + \delta_2 - \delta_1 \geq \Delta_1 + V(\Delta + 1, \delta_1 + 1) = Q_k(s_1, 0).
\]  
(28)

where (a) holds if \( p_r(\delta_1) - p_r(\delta_2) \leq \frac{\delta_2 - \delta_1}{V_k(\Delta + 1, 1) - V_k(1, 1)} \).

Since \( V_{k+1}(s) = \min Q_k(s, a) \), we can show that \( V_{k+1}(s_1) - V_{k+1}(s_2) \leq \frac{\delta_2 - \delta_1}{V_k(\Delta + 1, 1) - V_k(1, 1)} \) for any \( k \) in \( \mathbb{Z}_{\geq 0} \) in four cases as follows:

- **Case 1:** If \( V_{k+1}(s_1) = Q_k(s_1, 0) \) and \( V_{k+1}(s_2) = Q_k(s_2, 0) \), then \( V_{k+1}(s_1) - V_{k+1}(s_2) = Q_k(s_1, 0) - Q_k(s_2, 0) \leq \delta_2 - \delta_1 \).
- **Case 2:** If \( V_{k+1}(s_1) = Q_k(s_1, 1) \) and \( V_{k+1}(s_2) = Q_k(s_2, 1) \), then \( V_{k+1}(s_1) - V_{k+1}(s_2) = Q_k(s_1, 1) - Q_k(s_2, 1) \leq \delta_2 - \delta_1 \).
- **Case 3:** If \( V_{k+1}(s_1) = Q_k(s_1, 0) \) and \( V_{k+1}(s_2) = Q_k(s_2, 1) \), then \( V_{k+1}(s_1) - V_{k+1}(s_2) = Q_k(s_1, 0) - Q_k(s_2, 1) \leq Q_k(s_1, 1) - Q_k(s_1, 1) \leq \delta_2 - \delta_1 \).
- **Case 4:** If \( V_{k+1}(s_1) = Q_k(s_1, 1) \) and \( V_{k+1}(s_2) = Q_k(s_2, 0) \), then \( V_{k+1}(s_1) - V_{k+1}(s_2) = Q_k(s_1, 1) - Q_k(s_2, 0) \leq \delta_2 - \delta_1 \).
• Case 4: If \( V_{k+1}(s_1) = Q_k(s_1, 1) \leq Q_k(s_1, 0) \) and \( V_{k+1}(s_2) = Q_k(s_2, 0) \), then \( V_{k+1}(s_1) - V_{k+1}(s_2) = Q_k(s_1, 1) - Q_k(s_2, 0) \leq Q_k(s_1, 0) - Q_k(s_2, 0) \leq \delta_2 - \delta_1 \).

By taking limits on both sides of (26) and by (19), we complete the proof of Lemma 3.

D. Proof of Theorem 1

The optimal updating policy can be obtained by leveraging the VIA. In particular, we investigate the difference of the state-action value function. Let \( s = (\Delta, \delta) \). According to Lemmas 1-3, if \( p_r(1) - p_r(\delta + 1) \leq \frac{\delta}{\nu_k(\Delta+1,1) - \nu_k(1,1)} \) for any \( k \in \mathbb{Z}_{\geq 0} \), we have

\[
Q_k(s, 0) - Q_k(s, 1) = p_s \left( V_k(\Delta + 1, \delta + 1) - p_r(\delta) V_k(\Delta + 1, 1) \right) \\
- p_s \left( V_k(\Delta + 1, \delta + 1) - V_k(\Delta + 1, 1) \right) \\
+ p_s \left( V_k(\Delta, \delta) - V_k(\Delta, 1) \right) - \omega C_u \\
\geq p_s \left( 1 - p_r(\delta) \right) \Delta - p_r \delta - \omega C_u. \tag{29}
\]

We can see that \( Q_k(s, 0) - Q_k(s, 1) \) is lower bounded by \( p_s \left( 1 - p_r(\delta) \right) \Delta - p_r \delta - \omega C_u \), which is the sum of an increasing positive function with respect to \( \Delta \) and two negative constants. It is evident that there exists a positive integer \( \Delta(\delta) \) such that \( \Delta(\delta) \) is the minimum value satisfying \( p_s \left( 1 - p_r(\delta) \right) \Delta - p_r \delta - \omega C_u \geq 0 \). Therefore, if \( \Delta \geq \Delta(\delta) \), then we have \( Q_k(s, 0) - Q_k(s, 1) \geq 0 \). According to the definition of the optimal policy in (10), the optimal policy for a given \( \delta \) is to update when \( \Delta \geq \Delta(\delta) \).

E. Proof of Lemma 4

The proof is similar to that of Lemma 1. By initializing \( V_0(s) = 0 \) for all \( s \in S \), it is easy to see that \( V_k(s_1) \leq V_k(s_2) \) holds for \( k = 0 \). Next, we assume that \( V_k(s_1) \leq V_k(s_2) \) holds up till \( k > 0 \) and we examine whether it holds for \( k + 1 \).

When \( \alpha = 0 \), we have \( Q_k(s_1, 0) = s_1 + V_k(s_1 + 1) \) and \( Q_k(s_2, 0) = s_2 + V_k(s_2 + 1) \). Since \( s_1 \leq s_2 \) and \( V_k(s_1) \leq V_k(s_2) \), we can easily see that \( Q_k(s_1, 0) \leq Q_k(s_2, 0) \).

When \( \alpha = 1 \), the state-action value functions at iteration \( k \) are given by

\[
Q_k(s_1, 1) = s_1 + \omega C_u + p_s \left( 1 - p_r \right) V_k(1) \\
+ \left( p_f + p_s p_r \right) V_k(s_1 + 1), \tag{30}
\]

and

\[
Q_k(s_2, 1) = s_2 + \omega C_u + p_s \left( 1 - p_r \right) V_k(1) \\
+ \left( p_f + p_s p_r \right) V_k(s_2 + 1). \tag{31}
\]

Bearing in mind that \( V_k(s_1) \leq V_k(s_2) \), we can also verify that \( Q_k(s_1, 1) \leq Q_k(s_2, 1) \).

Altogether, we can assert that \( V_{k+1}(s_1) \leq V_{k+1}(s_2) \) for any \( k \). By taking limits on both sides of \( V_k(s_1) \leq V_k(s_2) \) and by (19), we complete the proof of Lemma 4.

F. Proof of Theorem 2

Suppose \( \pi^*(s_1) = 1 \), we have \( Q(s_1, 0) - Q(s_1, 1) \geq 0 \). Therefore, the optimal updating policy has a threshold structure if \( Q(s, a) \) has a sub-modular structure, that is,

\[
Q(s_1, 0) - Q(s_1, 1) \leq Q(s_2, 0) - Q(s_2, 1), \tag{32}
\]

for any \( s_1, s_2 \in S \) and \( s_1 \leq s_2 \).

According to the definition of \( Q(s, a) \), we have

\[
Q(s_1, 0) - Q(s_1, 1) = p_s(1 - p_r) [V(s_1 + 1) - V(1)] - \omega C_u, \tag{33}
\]

and

\[
Q(s_2, 0) - Q(s_2, 1) = p_s(1 - p_r) [V(s_2 + 1) - V(1)] - \omega C_u. \tag{34}
\]

Since \( V(s_1 + 1) \leq V(s_2 + 1) \), it is easy to see that (32) holds. Along with \( Q(s_1, 0) - Q(s_1, 1) \geq 0 \), we complete the proof of Theorem 2.

G. Proof of Lemma 5

When \( \Delta = \infty \), for any threshold policy with the threshold of \( \Omega \), the MDP can be modeled through a Discrete Time Markov Chain (DTMC) with the same states, which is illustrated in Fig. 11. Let \( \varphi_s \) denote the steady state probability of state \( s \). The balance equations of the DTMC are given as follows:

\[
\begin{align*}
\varphi_s &= \varphi_{s-1}, & 2 \leq s \leq \Omega, \\
\varphi_s &= p_z \varphi_{s-1}, & s > \Omega,
\end{align*}
\]

where \( p_z = p_f + p_s p_r \). Then, the steady-state probability of the DTMC can be expressed with \( \varphi_1 \). Specifically,

\[
\varphi_s = \begin{cases} 
\varphi_1, & \text{if } 2 \leq s \leq \Omega, \\
p_z^{s-\Omega} \varphi_1, & \text{otherwise}.
\end{cases}
\]

Since \( \sum_{s=1}^{\infty} \varphi_s = 1 \), we can derive \( \varphi_1 = \frac{1 - p_z}{\Omega (1 - p_z) + p_z} \). By substitute \( \varphi_1 \) into Eq. (36), we can obtain the closed-form of the steady-state probability, which is given by

\[
\varphi_s = \begin{cases} 
\frac{1 - p_z}{\Omega (1 - p_z) + p_z}, & \text{if } 2 \leq s \leq \Omega, \\
\frac{p_z^{s-\Omega}}{1 - p_z}, & \text{otherwise}.
\end{cases}
\]

Then, the average cost under the threshold policy can be computed as \( J(\Omega) = J_1(\Omega) + J_2(\Omega) \), where

\[
J_1(\Omega) = \sum_{s=1}^{\infty} \varphi_s s = \frac{1 - p_z}{\Omega (1 - p_z) + p_z} \left( \frac{\Omega (\Omega - 1)}{2} \right) + \frac{\Omega}{1 - p_z} + \frac{p_z^2}{(1 - p_z)^2}, \tag{37}
\]

Figure 11. The states transitions under a threshold policy with the threshold of \( \Omega \).
and \[ J_2(\Omega) = \sum_{s=\Omega}^\infty \varphi_s \omega C_u = \frac{\omega C_u}{\Omega(1-p_z) + p_z}. \] (38)

This completes the proof of Lemma 5.

H. Proof of Theorem 3

According to Lemma 5, we have
\[ J(\Omega) = J_1(\Omega) + J_2(\Omega) \]
\[ = \frac{1 - p_z}{\Omega(1-p_z) + p_z} \left( \frac{\Omega^2 - \Omega}{2} + \frac{\Omega + \omega C_u}{1 - p_z} + \frac{p_z}{(1-p_z)^2} \right). \] (39)

We derive the optimal threshold \( \Omega^* \) by relaxing \( \Omega \) to a continuous variable. We first calculate the second order derivative of \( J(\Omega) \) as follow,
\[ \frac{\partial^2 J(\Omega)}{\partial \Omega} = \frac{1 - p_z}{\Omega(1-p_z) + p_z} \left[ \frac{1 - p_z}{\Omega(1-p_z) + p_z} \right]^2 \times \left( \frac{\Omega^2 - \Omega}{2} + \frac{\Omega + \omega C_u}{1 - p_z} + \frac{p_z}{(1-p_z)^2} \right) \]
\[ + 2 \left[ \frac{1 - p_z}{\Omega(1-p_z) + p_z} \right]^3 \times \left( \frac{\Omega^2 - \Omega}{2} + \frac{\Omega + \omega C_u}{1 - p_z} + \frac{p_z}{(1-p_z)^2} \right) \]
\[ = \frac{(1-p_z)[p_z + 2\omega C_u(1-p_z)]}{\Omega(1-p_z) + p_z} \]. (40)

where \( p_z = p_f + p_s p_r \leq 1 \). Since \( \frac{\partial^2 J(\Omega)}{\partial \Omega} \geq 0 \), \( J(\Omega) \) is a convex function with respect to \( \Omega \). Then, we calculate the first order derivative of \( J(\Omega) \) as follow,
\[ \frac{\partial J(\Omega)}{\partial \Omega} = \frac{1 - p_z}{\Omega(1-p_z) + p_z} \left( \frac{\Omega^2 - \Omega}{2} + \frac{1}{1 - p_z} \right) \]
\[ - \left[ \frac{1 - p_z}{\Omega(1-p_z) + p_z} \right]^2 \times \left( \frac{\Omega^2 - \Omega}{2} + \frac{\Omega + \omega C_u}{1 - p_z} + \frac{p_z}{(1-p_z)^2} \right). \] (41)

The optimal threshold can be obtained by setting \( \frac{\partial J(\Omega)}{\partial \Omega} \) to zero. The solution to \( \frac{\partial J(\Omega)}{\partial \Omega} = 0 \) is
\[ \Omega' = \sqrt{p_z + 2\omega C_u(1-p_z) - p_z}. \] (42)

Since \( \Omega' \) may not be an integer, the optimal threshold can be expressed as
\[ \Omega^* = \arg \min(J([\Omega']), J([\Omega'])). \] (43)

I. Proof of Corollary 1

The properties of \( \Omega^* \) in terms of \( C_u, p_s, \) and \( M \) can be proved by analyzing \( \Omega' \), which is given by
\[ \Omega' = \sqrt{p_z + 2\omega C_u(1-p_z) - p_z}. \] (44)

- According to Eq. (44), it is easy to see that \( \Omega' \) is an increasing function of \( C_u \). Then, \( \Omega^* \) is a non-decreasing function of \( C_u \) due to rounding.
- To analyze the relationship between \( \Omega' \) and \( p_s \), we first investigate how \( \Omega' \) varies with \( p_z \). Particularly, we calculate the derivative of \( \Omega' \) with respect to \( p_z \) as follow,
\[ \frac{\partial \Omega'}{\partial p_z} = \frac{1}{2p_z - p_z^2 - 2p_z \omega C_u - 2\omega C_u}{1 - p_z}. \] (45)

where \( A = \frac{1}{2}(1 - p_z)(1 - 2\omega C_u) + p_z \) and \( B = [p_z + 2\omega C_u(1-p_z)]^{\frac{1}{2}} \). Since both \( A \) and \( B \) are positive, we show that \( A \leq B \) by comparing \( A^2 \) and \( B^2 \). Specifically,
\[ A^2 - B^2 = \frac{1}{4}(1 - p_z)^2(1 + 2\omega C_u)^2 + p_z^2 \]
\[ + (1 - p_z)(1 + 2\omega C_u)p_z - 2\omega C_u(1-p_z) \]
\[ = \frac{1}{4}(1 - p_z)^2(1 + 2\omega C_u)^2 - 2\omega C_u(1-p_z)^2 \]
\[ = (1 - p_z)^2(\omega C_u - \frac{1}{2})^2 \geq 0. \] (46)

Hence, \( \Omega' \) is a non-decreasing function of \( p_z \). Furthermore, \( p_z = p_f + p_s p_r = 1 - p_s + \frac{p_s}{p_z} = 1 - \frac{M-1}{M} p_z \) is a decreasing function of \( p_s \) and \( M \). Therefore, \( \Omega' \) is non-increasing of \( p_s \) and \( M \). According to Eq. (43), \( \Omega^* \) is a non-increasing function of \( p_s \) and \( M \).

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