Abstract

Let $(M, I)$ be an almost complex 6-manifold. The obstruction to integrability of the almost complex structure (the so-called Nijenhuis tensor) $N : \Lambda^{0,1}(M) \to \Lambda^{2,0}(M)$ maps a 3-dimensional bundle to a 3-dimensional one. We say that Nijenhuis tensor is non-degenerate if it is an isomorphism. An almost complex manifold $(M, I)$ is called nearly Kähler if it admits a Hermitian form $\omega$ such that $\nabla(\omega)$ is totally antisymmetric, $\nabla$ being the Levi-Civita connection. We show that a nearly Kähler metric on a given almost complex 6-manifold with non-degenerate Nijenhuis tensor is unique (up to a constant). We interpret the nearly Kähler property in terms of $G_2$-geometry and in terms of connections with totally antisymmetric torsion, obtaining a number of equivalent definitions.

We construct a natural diffeomorphism-invariant functional $I \to \int_M \text{Vol}_I$ on the space of almost complex structures on $M$, similar to the Hitchin functional, and compute its extrema in the following important case. Consider an almost complex structure $I$ with non-degenerate Nijenhuis tensor, admitting a Hermitian connection with totally antisymmetric torsion. We show that the Hitchin-like functional $I \to \int_M \text{Vol}_I$ has an extremum in $I$ if and only if $(M, I)$ is nearly Kähler.
0 Introduction

0.1 Almost complex manifolds with non-degenerate Nijenhuis tensor

In geometry, two kinds of plane distributions often arise. There are integrable ones: complex structures, foliations, CR-structures. On the other hand, there are “maximally non-integrable” distributions, such as the contact structures, where the obstruction to integrability is nowhere degenerate. Looking at almost complex structures in dimension 3, one finds that the obstruction to the integrability, so-called Nijenhuis tensor

\[ N : \Lambda^2 T^{1,0}(M) \to T^{0,1}(M) \]

maps a 3-dimensional bundle to a 3-dimensional bundle. It is only natural to study the class of complex 3-manifolds such that \( N \) is nowhere degenerate.

Given such a manifold \( M \), it is possible to construct a nowhere degenerate, positive volume form \( \det N^* \otimes \det N^* \) on \( M \) (for details, see \[12\]).
We study extrema of this volume form, showing that these extrema correspond to an interesting geometric structure (Theorem 2.2).

In Hermitian geometry, one often encounters a special kind of almost complex Hermitian manifolds, called strictly nearly Kähler (NK-) manifolds, or Gray manifolds, after Alfred Gray (Definition 4.1). These manifolds can be characterized in terms of the \( G_2 \)-structure on their Riemannian cone, or in terms of a special set of equations reminiscent of Calabi-Yau equations (Subsection 4.4).

We prove that a strictly nearly Kähler 3-manifold is uniquely determined by its almost complex structure (Corollary 3.3). Moreover, such manifolds are extrema of the volume functional associated with the Nijenhuis tensor (Theorem 2.2). This reminds of a construction of Hitchin’s functional on the space of all \( SL(3, \mathbb{C}) \)-structures on a manifold, having extrema on Calabi-Yau manifolds ([Hi2]).

0.2 Contents

This paper has the following structure.

- In Section 1, we introduce the class of 3-manifolds with nowhere degenerate Nijenhuis tensor, and describe the basic structures associated with these manifolds. We give a sketch of a proof of existence of a Hermitian connection with totally antisymmetric torsion, due to Friedrich and Ivanov, and show that such a Hermitian metric is uniquely determined by the almost complex structure, if the Nijenhuis tensor is nowhere degenerate.
- In Section 2, we introduce the nearly Kähler manifolds, giving several versions of their definition and listing examples.
- In Section 3, we apply the results about connections with totally antisymmetric torsion to nearly Kähler geometry, showing that an almost complex structure determines the Hermitian structure on such a manifold uniquely, up to a constant multiplier.
- In Section 4, we give several additional versions of a definition of a nearly Kähler manifold, obtaining an explicit description of a Nijenhuis tensor in terms of an orthonormal frame. We also interpret the nearly Kähler structure on a manifold in terms of \( G_2 \)-geometry of its Riemannian cone. This is used to show that an NK-structure on a manifold \( M \) is uniquely determined by its metric, unless \( M \) is locally isometric to a 6-sphere (Proposition 4.7).
- In Section 5, we study infinitesimal variations of an almost complex structure. We prove that NK-manifolds are extrema of an intrinsic volume
functional described earlier. A partial converse result is also obtained. Given an almost complex manifold \( M \) with nowhere degenerate Nijenhuis tensor, admitting a Hermitian connection with totally antisymmetric torsion, \( M \) is an extremum of the intrinsic volume functional if and only if \( M \) is nearly Kähler.

1 Almost complex manifolds with non-degenerate Nijenhuis tensor

1.1 Nijenhuis tensor on 6-manifolds

Let \( (M, I) \) be an almost complex manifold. The Nijenhuis tensor maps two \((1,0)\)-vector fields to the \((0,1)\)-part of their commutator. This map is \( C^\infty \)-linear, and vanishes, as the Newlander-Nirenberg theorem implies, precisely when \( I \) is integrable. We write the Nijenhuis tensor as

\[
N : \Lambda^2 T^{1,0}(M) \longrightarrow T^{0,1}(M).
\]

The dual map

\[
N^* : \Lambda^{0,1}(M) \longrightarrow \Lambda^{2,0}(M)
\]

is also called the Nijenhuis tensor. Cartan’s formula implies that \( N^* \) acts on \( \Lambda^1(M) \) as the \((2,-1)\)-part of the de Rham differential.

When one studies the distributions, one is usually interested in integrable ones (such as \( T^{1,0}(M) \subset TM \otimes \mathbb{C} \) for complex or CR-manifolds) or ones where the obstruction to integrability is nowhere degenerate (such as a contact distribution).

For the Nijenhuis tensor in complex dimension \( > 3 \), non-degeneracy does not make much sense, because the space \( \text{Hom}(\Lambda^{0,1}(M), \Lambda^{2,0}(M)) \) becomes quite complicated. However, for \( n = 3 \), both sides of (1.1) are 3-dimensional, and we can define the non-degeneracy as follows.

**Definition 1.1:** Let \( (M, I) \) be an almost complex manifold of real dimension 6, and \( N : \Lambda^2 T^{1,0}(M) \longrightarrow T^{0,1}(M) \) the Nijenhuis tensor. We say that \( N \) is **non-degenerate** if \( N \) is an isomorphism everywhere. Then \( (M, I) \) is called an almost complex 6-manifold with nowhere degenerate Nijenhuis tensor.

**Remark 1.2:** Such manifolds were investigated by R. Bryant. His results were presented at a conference [Br1], but never published. The present author did not attend (unfortunately) and was not aware of his work.
The first thing one notices is that the determinant $\det N^*$ gives a section

$$\det N^* \in \Lambda^{3,0}(M)^{\otimes 2} \otimes \Lambda^{3,0}(M)^*.$$  

Taking

$$\det N^* \otimes \det N^* \in \Lambda^{3,0}(M) \otimes \Lambda^{0,3}(M) = \Lambda^6(M) \quad (1.2)$$

we obtain a nowhere degenerate real volume form $\text{Vol}_I$ on $M$. This form is called the canonical volume form associated with the Nijenhuis tensor. This gives a functional $I : \Psi \rightarrow \int_M \text{Vol}_I$ on the space of almost complex structures. One of the purposes of this paper is to investigate the critical points of the functional $\Psi$, in the spirit of Hitchin’s work ([Hi2], [Hi3]).

1.2 Connections with totally antisymmetric torsion

Let $(M, g)$ be a Riemannian manifold, $\nabla : TM \rightarrow TM \otimes \Lambda^1 M$ a connection, and $T \subset \Lambda^2 M \otimes TM$ its torsion. Identifying $TM$ and $\Lambda^1 M$ via $g$, we may consider $T$ as an element in $\Lambda^2 M \otimes \Lambda^1 M$, that is, a 3-form on $TM$. If $T$ is totally skew-symmetric as a 3-form on $TM$, we say that $\nabla$ is a connection with totally skew-symmetric (or totally antisymmetric) torsion. If, in addition, $M$ is Hermitian, and $\nabla$ preserves the Hermitian structure, we say that $\nabla$ is a Hermitian connection with totally antisymmetric torsion.

Connections with totally skew-symmetric torsion are extremely useful in physics and differential geometry. An important example of such a connection is provided by a theorem of Bismut ([Bi]).

**Theorem 1.3:** Let $(M, I)$ be a complex manifold, and $g$ a Hermitian metric. Then $M$ admits a unique connection with totally skew-symmetric torsion preserving $I$ and $g$.

Connections with totally skew-symmetric torsion were studied at great length by Friedrich, Ivanov and others (see e.g. [FI], [F], [AF]). Bismut’s theorem requires the base manifold to be complex. Motivated by string theory, Friedrich and Ivanov generalized Bismut’s theorem to non-integrable almost complex manifolds ([FI]). For completeness, we sketch a proof of their theorem below.
**Theorem 1.4:** Let \((M, I, \omega)\) be an almost complex Hermitian manifold, and
\[
N : \Lambda^2 T^{1,0}(M) \rightarrow T^{0,1}(M).
\]
the Nijenhuis tensor. Consider the 3-linear form \(\rho : T^{1,0}(M) \times T^{1,0}(M) \times T^{1,0}(M) \rightarrow \mathbb{C},\)
\[
\rho(x, y, z) := \omega(N(x, y), z) \quad (1.3)
\]
Then \(M\) admits a connection \(\nabla\) with totally skew-symmetric torsion preserving \((\omega, I)\) if and only if \(\rho\) is skew-symmetric. Moreover, such a connection is unique.

**Sketch of a proof:** [Theorem 1.4] is proven essentially in the same way as one proves Bismut’s theorem and existence and uniqueness of a Levi-Civita connection. Let \((M, I, g)\) be a Hermitian manifold, and \(\nabla_0\) a Hermitian connection. Then all Hermitian connections can be obtained by taking \(\nabla(A) := \nabla_0 + A\), where \(A\) is a 1-form with coefficients in the algebra \(u(TM)\) of all skew-Hermitian endomorphisms. The torsion \(T_A\) of \(\nabla(A)\) is written as
\[
T_A = T_0 + \text{Alt}_{12}(A),
\]
where \(T_0\) is a torsion of \(\nabla_0\), and \(\text{Alt}_{12}\) denotes the antisymmetrization of
\[
\Lambda^1(M) \otimes u(TM) \subset \Lambda^1(M) \otimes \Lambda^1(M) \otimes TM
\]
over the first two indices. We identify \(u(TM)\) with \(\Lambda^{1,1}(M)\) in a standard way. Then [Theorem 1.4] can be reinterpreted as a statement about linear-algebraic properties of the operator
\[
\text{Alt}_{12} : \Lambda^1(M) \otimes \Lambda^{1,1}(M) \rightarrow \left(\Lambda^2(M) \otimes \Lambda^1(M)\right)^{(2,1)+(1,2)}. \quad (1.4)
\]
(the superscript \((\ldots)^{(2,1)+(1,2)}\) means taking \((2, 1) + (1, 2)\)-part with respect to the Hodge decomposition), as follows.

By definition, the Nijenhuis tensor \(N\) is a section of \(\Lambda^{2,0} \otimes T^{0,1}\). Identifying \(T^{0,1}\) with \(\Lambda^{1,0}\) via \(g\), we can consider \(N\) as an element of \(\Lambda^{2,0} \otimes \Lambda^{1,0}\). By Cartan’s formula, \(N\) is equal to the \((3,0)\)-part of the torsion. Therefore, existence of a connection with totally skew-symmetric torsion implies that \((1.3)\) is skew-symmetric.

Conversely, assume that \((1.3)\) is skew-symmetric. Since \((1.4)\) maps \(\Lambda^1(M) \otimes \Lambda^{1,1}(M)\) to \((2, 1) + (1, 2)\)-tensors, the \((3, 0)\) and \((0, 3)\)-parts of torsion
stay skew-symmetric if we modify the connection by adding \( A \in \Lambda^1 \otimes \mu(TM) \). Denote by \( T_1 \) the \((2, 1) \oplus (1, 2)\)-part of the torsion \( T_0 \). To prove Theorem 1.4 we need to find \( A \in \Lambda^1(M) \otimes \Lambda^{1,1}(M) \) such that \( T_1 - \text{Alt}_{12}(A) \) is totally skew-symmetric.

The map \( \text{Alt}_{12} : \Lambda^1(M) \otimes \Lambda^2(M) \rightarrow \Lambda^2(M) \otimes \Lambda^1(M) \) is an isomorphism, as a dimension count implies (this map has no kernel, which is easy to see). Therefore, (1.4) is injective. Using dimension count again, we find that cokernel of (1.4) projects isomorphically into
\[
\Lambda^{2,1}(M) \oplus \Lambda^{1,2}(M) \subset \Lambda^2(M) \otimes \Lambda^1(M).
\]
Therefore, for any \( T_1 \) in \((2, 1) \oplus (1, 2)\)-part of \( \Lambda^2(M) \otimes \Lambda^1(M) \) there exists \( A \in \Lambda^1(M) \otimes \Lambda^{1,1}(M) \) and \( B \in \Lambda^{2,1}(M) \oplus \Lambda^{1,2}(M) \) such that \( T_1 = \text{Alt}_{12}(A) + B \).

1.3 Connections with antisymmetric torsion on almost complex 6-manifolds

Let \((M, I)\) be an almost complex manifold, \( N \) its Nijenhuis tensor. To obtain all Hermitian connections with totally skew-symmetric torsion on \((M, I)\), one needs to find all metrics \( g \) for which the tensor \( \omega(N(x, y), z) \) is skew-symmetric. As Theorem 1.4 implies, these metrics are precisely those for which such a connection exists.

We also prove the following proposition

**Proposition 1.5:** Let \((M, I)\) be an almost complex 6-manifold with Nijenhuis tensor which is non-degenerate in a dense subset of \( M \), and \( g \) a Hermitian metric admitting a connection with totally antisymmetric torsion. Then \( g \) is uniquely determined by \( I \), up to conformal equivalence. Moreover, the Riemannian metric \( g \) determines \( I \) uniquely, unless \((M, g)\) is locally isometric to a 6-sphere.

**Proof:** This is Proposition 3.1 and Proposition 4.7.

1.4 Correspondence with the results of R. Bryant

Since the first version of this paper was written, the previously unpublished results of R. Bryant appeared in a fundamental and important preprint [Br2]. There is a significant overlap with our research, although the presentation and terminology is different. The property (1.3) (which is equivalent to existence of Hermitian connection with totally antisymmetric curvature) is
called “Nijenhuis tensor of real type” in [Br2]. The main focus of [Br2] is
the so-called “quasi-integrable almost complex manifold”: manifolds with
Nijenhuis tensor of real type, which is at every point of $M$ either non-
degenerate (of constant signature) or zero. Examples of such structures
are found. In particular, all twistor spaces of Kaehler surfaces with sign-
definite holomorphic bisectional curvature are shown to be quasi-integrable.
A variant of [Theorem 2.2] is also proven. It is shown ([Br2], Proposition 8)
that nearly Kaehler manifolds are critical points of the functional $\text{Vol}_I$.

2 Nearly Kähler manifolds: an introduction

Nearly Kähler manifolds (also known as $K$-spaces or almost Tachibana
spaces) were defined and studied by Alfred Gray ([Gr1], [Gr2], [Gr3], [Gr4])
in a general context of intrinsic torsion of $U(n)$-structures and weak ho-
lonomies. An almost complex Hermitian manifold $(M, I)$ is called nearly
Kähler if $\nabla_X(I)X = 0$, for any vector fields $X$ ($\nabla$ denotes the Levi-Civita
connection). In other words, the tensor $\nabla \omega$ must be totally skew-symmetric,
for $\omega$ the Hermitian form on $M$. If $\nabla_X(\omega) \neq 0$ for any non-zero vector field
$X$, $M$ is called strictly nearly Kähler.

In this section, we give an overview of known results and “folk theo-
rems” of nearly Kähler geometry. Most of this theory was known (in a
different context) since 1980-ies, when the study of Killing spinors was ini-
tiated ([BFGK]).

2.1 Splitting theorems for nearly Kähler manifolds

As V. F. Kirichenko proved, nearly Kähler manifolds admit a connection
with totally antisymmetric, parallel torsion ([K]). This observation was
used to prove a splitting theorem for nearly Kähler manifolds: any nearly
Kähler manifold is locally a Riemannian product of a Kähler manifold and
a strictly nearly Kähler one ([Gr4], [N1]).

A powerful classification theorem for Riemannian manifolds admitting
an orthogonal connection with irreducible connection and parallel torsion
was obtained by R. Cleyton and A. Swann in [CS]. Cleyton and Swann
proved that any such manifold is either locally homogeneous, has vanishing
torsion, or has weak holonomy $G_2$ (in dimension 7) or $SU(3)$ (in dimension
6).

Using Kirichenko theorem, this result can be used to obtain a classifi-
cation of nearly Kähler manifolds. P.-A. Nagy ([N2]) has shown that that
any strictly nearly Kähler manifold is locally a product of locally homogeneous manifolds, strictly nearly Kähler 6-manifolds, and twistor spaces of quaternionic Kähler manifolds of positive Ricci curvature, equipped with the Eells-Salamon metric.

These days the term “nearly Kähler” usually denotes strictly nearly Kähler 6-manifolds. In sequel we shall follow this usage, often omitting “strictly” and “6-dimensional”.

In dimension 6, a manifold is (strictly) nearly Kähler if and only if it admits a Killing spinor ([Gru]). Therefore, such a manifold is Einstein, with positive Einstein constant.

As one can easily show (see Theorem 4.2), strictly nearly Kähler 6-manifolds can be defined as 6-manifolds with structure group SU(3) and fundamental forms \( \omega \in \Lambda^{1,1}(M) \), \( \Omega \in \Lambda^{3,0}(M) \), satisfying \( d\omega = 3\lambda \text{Re} \Omega \), \( d\text{Im} \Omega = -2\lambda \omega^2 \). An excellent introduction to nearly Kähler geometry is found in [MNS].

The most puzzling aspect of nearly Kähler geometry is a complete lack of non-homogeneous examples. With the exception of 4 homogeneous cases described below (Subsection 2.3), no other compact examples of strictly nearly Kähler 6-manifolds are known to exist.

2.2 Nearly Kähler manifolds in \( G_2 \)-geometry and physics

Nearly Kähler manifolds have many uses in geometry and physics. Along with Calabi-Yau manifolds, nearly Kähler manifolds appear as target spaces for supersymmetric sigma-models, solving equations of type II string theory. These manifolds are the only 6-manifolds admitting a Killing spinor. This implies that a Riemannian cone \( C(M) \) of a nearly Kähler manifold has a parallel spinor.

Let \((M, g)\) be a Riemannian manifold. Recall that the Riemannian cone of \((M, g)\) is a product \( M \times \mathbb{R}^>0 \), with a metric \( g t^2 \oplus \lambda \cdot dt^2 \), where \( t \) is a unit parameter on \( \mathbb{R}^>0 \), and \( \lambda \) a constant. It is well known that \( M \) admits a real Killing spinor if and only if \( C(M) \) admits a parallel spinor (for appropriate choice of \( \lambda \)). Then, \( C(M) \) has restricted holonomy, for any nearly Kähler 6-manifold. It is easy to check that in fact \( C(M) \) has holonomy \( G_2 \). This explains a tremendous importance that nearly Kähler manifolds play in \( G_2 \)-geometry.

We give a brief introduction of \( G_2 \)-geometry, following [Hi2] and [J2]. Let \( V^7 \) be a 7-dimensional real vector space. The group \( GL(7, \mathbb{R}) \) acts on \( \Lambda^3(V^7) \) with two open orbits. For \( \nu \) in one of these orbits, its stabilizer \( St(\nu) \subset GL(7, \mathbb{R}) \) is 14-dimensional, as a dimension count insures. It is easy
to check that $St(\nu)$ is a real form of a Lie group $G_2$. For one of these orbits, $St(\nu)$ is a compact form of $G_2$, for another one it is non-compact. A 3-form $\nu \in \Lambda^3(V^7)$ is called **stable** if its stabilizer is a compact form of $G_2$.

A 7-manifold $X$ equipped with a 3-form $\rho$ is called a **$G_2$-manifold** if $\rho$ is stable everywhere in $X$. In this case, the structure group of $X$ is reduced to $G_2$. Also, $X$ is equipped with a natural Riemannian structure:

$$x, y \mapsto \int_X (\rho \lrcorner x) \wedge (\rho \lrcorner x) \wedge \rho \quad (x, y \in TM).$$

(2.1)

A $G_2$-manifold is called **parallel** if $\nabla \rho = 0$, where $\nabla$ is the Levi-Civita connection associated with this Riemannian structure.

Isolated singularities of $G_2$-manifolds are of paramount importance in physics ([AG], [AW]). A simplest example of an isolated singular point is a conical singularity.

A metric space $X$ with marked points $x_1, \ldots, x_n$ is called a **space with isolated singularities**, if $X \setminus \{x_1, \ldots, x_n\}$ is a Riemannian manifold. Consider a space $(X, x)$ with a single singular point. The singularity $x \in X$ is called **conical** if $X$ is equipped with a flow acting on $X$ by homotheties and contracting $X$ to $x$. In this case, $X \setminus x$ is isomorphic to a Riemannian cone of a Riemannian manifold $M$.

It is easy to check that the cone $C(M)$ of a nearly Kähler manifold is equipped with a parallel $G_2$-structure, and, conversely, every conical singularity of a parallel $G_2$-manifold is obtained as $C(M)$, for some nearly Kähler manifold $M$ ([Hi3], [IPP]). For completeness’ sake, we give a sketch of a proof of this result in Proposition 4.5.

The idea of this correspondence is quite clear. Let $X = C(M)$ be a parallel $G_2$-manifold, and $\omega_C$ its 3-form. Unless $X$ is flat, we may assume that $X$ has holonomy which is equal to $G_2$ and not its proper subgroup. Indeed, if holonomy of $X$ is less than $G_2$, by Berger’s classification of irreducible holonomies $X$ is represented (as a Riemannian manifold) as a product of manifolds of smaller dimension. However, the singular point of the metric completion $\overline{X}$ is isolated, and this precludes such a decomposition, unless $\overline{X}$ is smooth. In the latter case, $X$ is flat.

Since holonomy of $X$ is (strictly) $G_2$, the 3-form can be reconstructed from the Riemannian structure uniquely. After rescaling, we may assume that the Riemannian structure structure on $X = C(M)$ is homogeneous of weight 2, with respect to the action of $\mathbb{R}^{>0}$ on $C(M)$. Then $\omega_C$ is homogeneous of weight 3. Homogeneous $G_2$-structures on $C(M)$ correspond naturally to $SU(3)$-structures on $M$. We write $\omega_C$ as $t^2 \pi^* \omega \wedge dt + t^3 \pi^* \rho$,
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where $\rho$, $\omega$ are forms on $M$, and $\pi : C(M) \to M$ is the standard projection. From a local coordinate expression of a $G_2$-form, we find that $\omega$ is a Hermitian form corresponding to an almost complex structure $I$, and $\rho = \text{Re} \Omega$ for a nowhere degenerate $(3,0)$-form $\Omega$ on $(M,I)$.

The converse is proven by the same computation: given an $SU(3)$-manifold $(M,I,\omega,\Omega)$, we write a 3-form

$$\omega_C := t^2 \pi^* \omega \wedge dt + t^3 \pi^* \rho,$$

(2.2)
on $C(M)$, and show that it is a $G_2$-structure, using a coordinate expression for a $G_2$-form.

As Fernandez and Gray proved in [FG], a $G_2$-manifold $(X,\omega_C)$ is parallel if and only if $\omega_C$ is harmonic. For the form (2.2), $d\omega_C = 0$ is translated into

$$d\omega = 3\rho.$$  (2.3)

Since $*\rho = I\rho$ and $*\omega = \omega^2$, the condition $d^*\omega_C = 0$ becomes $dI\rho = -2\omega^2$. After an appropriate rescaling, we find that this is precisely the condition defining the nearly Kähler structure (Theorem 4.2). Therefore, $C(M)$ is a $G_2$-manifold if and only if $M$ is nearly Kähler (see Subsection 4.3 for a more detailed argument).

The correspondence between conical singularities of $G_2$-manifolds and nearly Kähler geometry can be used further to study the locally conformally parallel $G_2$-manifolds (see also [IPP]). **Locally conformally parallel $G_2$-manifold** is a 7-manifold $M$ with a covering $\tilde{M}$ equipped with a parallel $G_2$-structure, with the deck transform acting on $\tilde{M}$ by homotheties. Since homotheties preserve the Levi-Civita connection $\tilde{\nabla}$ on $\tilde{M}$, $\tilde{\nabla}$ descends to a torsion-free connection on $M$, which is no longer orthogonal, but preserves the conformal class of a metric. Such a connection is called a **Weyl connection**, and a conformal manifold of dimension $> 2$ equipped with a torsion-free connection preserving the conformal class is a **Weyl manifold**. The Weyl manifolds are a subject of much study in conformal geometry (see e.g. [DO] and the reference therein).

The key theorem of Weyl geometry is proven by P. Gauduchon ([Ga1]). He has shown that any compact Weyl manifold is equipped with a privileged metric in its conformal class. This metric (called a **Gauduchon metric** now) is defined as follows.

Let $(M,[g],\nabla)$ be a compact Weyl manifold, where $[g]$ is a conformal class, and $g \in [g]$ any metric within this conformal class. Since $\nabla[g] = 0$, we have $\nabla(g) = g \otimes \theta$, where $\theta$ is a 1-form, called a **Lee form**. A metric $g$
is called Gauduchon if $\theta$ satisfies $d^*\theta = 0$. A Gauduchon metric is unique (up to a complex multiplier).

Let now $(M, \nabla, [g])$ be a Weyl manifold with a Ricci-flat connection $\nabla$. In [Ga2], Gauduchon has shown that the Lee form $\theta$ of the Gauduchon metric on $M$ is parallel with respect to the Levi-Civita connection associated with this metric.

Applying this argument to a compact locally conformally parallel $G_2$-manifold $M$, we obtain that the Lee form is parallel. From this one infers that the parallel $G_2$-covering $\tilde{M}$ of $M$ is a cone over some Riemannian manifold $S$ (see e.g. [V], Proposition 11.1, also see [KO] and [GOP]). Using the argument stated above, we find that this manifold is in fact nearly Kähler. Therefore, $S$ is Einstein, with positive Ricci curvature. Since $M$ is compact, $S$ is complete, and by Myers theorem, $S$ is actually compact (see [V], Remark 10.7). Now, the argument which proves Theorem 12.1 of [V] can be used to show that $\dim H^1(M, \mathbb{Q}) = 1$, and $M = C(S)/\mathbb{Z}$. This gives the following structure theorem, which is proven independently in [IPP].

**Theorem 2.1:** Let $M$ be a compact locally conformally parallel $G_2$-manifold. Then $M = C(S)/\mathbb{Z}$, where $S$ is a nearly Kähler manifold, and the $\mathbb{Z}$-action on $C(S) \cong S \times \mathbb{R}^+ \mathbb{R}$ is generated by a map $(x, t) \mapsto (\varphi(x), qt)$, where $|q| > 1$ is a real number, and $\varphi: S \rightarrow S$ an automorphism of nearly Kähler structure.

2.3 Examples of nearly Kähler manifolds

Just as the conical singularities of parallel $G_2$-manifolds correspond to nearly Kähler manifolds, the conical singularities of $Spin(7)$-manifolds correspond to the so-called “nearly parallel” $G_2$-manifolds (see [I]). A $G_2$-manifold $(M, \omega)$ is called nearly parallel if $d\omega = c^*\omega$, where $c$ is some constant. The analogy between nearly Kähler 6-manifolds and nearly parallel $G_2$-manifolds is almost perfect. These manifolds admit a connection with totally antisymmetric torsion and have weak holonomy $SU(3)$ and $G_2$ respectively. N. Hitchin realized nearly Kähler 6-manifolds and nearly parallel $G_2$-manifolds as extrema of a certain functional, called Hitchin functional by physicists (see [Hi3]).

However, examples of nearly parallel $G_2$-manifolds are found in profusion (every 3-Sasakian manifold is nearly parallel $G_2$), and compact nearly Kähler manifolds are rare.

Only 4 compact examples are known (see the list below); all of them
homogeneous. In \[Bu\] it was shown that any homogeneous nearly Kähler 6-manifold belongs to this list.

1. The 6-dimensional sphere $S^6$. Since the cone $C(S^6)$ is flat, $S^6$ is a nearly Kähler manifold, as shown in Subsection 2.2. The almost complex structure on $S^6$ is reconstructed from the octonion action, and the metric is standard.

2. $S^3 \times S^3$, with the complex structure mapping $\xi_i$ to $\xi'_i$, $\xi'_i$ to $-\xi_i$, where $\xi_i, \xi'_i, i = 1, 2, 3$ is a basis of left invariant 1-forms on the first and the second component.

3. Given a self-dual Einstein Riemannian 4-manifold $M$ with positive Einstein constant, one defines its \textit{twistor space} $\text{Tw}(M)$ as a total space of a bundle of unit spheres in $\Lambda^2(M)$ of anti-self-dual 2-forms. Then $\text{Tw}(M)$ has a natural Kähler-Einstein structure $(I_+, g)$, obtained by interpreting unit vectors in $\Lambda^2(M)$ as complex structure operators on $TM$. Changing the sign of $I_+$ on $TM$, we obtain an almost complex structure $I_-$ which is also compatible with the metric $g$ [ES]. A straightforward computation insures that $(\text{Tw}(M), I_-, g)$ is nearly Kähler [M].

As N. Hitchin proved [Hi1], there are only two compact self-dual Einstein 4-manifolds: $S^4$ and $\mathbb{C}P^2$. The corresponding twistor spaces are $\mathbb{C}P^3$ and the flag space $F(1, 2)$. The almost complex structure operator $I_-$ induces a nearly Kähler structure on these two symmetric spaces.

2.4 Nearly Kähler manifolds are extrema of volume on almost complex manifolds with nowhere degenerate Nijenhuis tensor

Let $(M, I, \omega)$ be a nearly Kähler manifold, and $N^* : \Lambda^{0,1}(M) \to \Lambda^{2,0}(M)$ the Nijenhuis tensor. By Cartan’s formula, $N^*$ is the $(2, -1)$-part of the de Rham differential (with respect to the Hodge decomposition). In Theorem 4.2 it is shown that $d\omega$ is a real part of a nowhere degenerate $(3, 0)$-form $\Omega$. Therefore, the 3-form

$$\omega(N(x, y), z) = d\omega(x, y, z) = \text{Re} \Omega(x, y, z)$$

is nowhere degenerate on $T^{1,0}(M)$. We obtain that the Nijenhuis tensor $N$ is nowhere degenerate.
The main result of this paper is the following theorem, which is analogous to \[\text{Hi3}\].

**Theorem 2.2:** Let \((M, I)\) be a compact almost complex 6-manifold with nowhere degenerate Nijenhuis tensor admitting a Hermitian connection with totally antisymmetric torsion. Consider the functional

\[
I \rightarrow \int_M \text{Vol}_I
\]

on the space of such manifolds constructed in Subsection 1.1. Then (2.4) has a critical point at \(I\) if and only if \((M, I)\) admits a nearly Kähler metric.

**Proof:** Follows from Proposition 5.6 and Theorem 4.2. \(\blacksquare\)

**Remark 2.3:** As follows from Corollary 3.3, the nearly Kähler metric on \((M, I)\) is uniquely determined by the almost complex structure.

### 3 Almost complex structures and connections with totally antisymmetric torsion

Let \((M, I)\) be a 6-dimensional almost complex manifold, and

\[
N^* : \Lambda^{0,1}(M) \rightarrow \Lambda^{2,0}(M)
\]

its Nijenhuis tensor. Given a point \(x \in M\), the operator \(N^*|_{\Lambda^{0,1}(M)}\) can a priori take any value within \(\text{Hom}(\Lambda^{0,1}(M), \Lambda^{2,0}(M))\). For \(N^*|_{\Lambda^{0,1}(M)}\) generic, the stabilizer \(St(N^*_x)\) of \(N^*_x\) within \(GL(T_xM)\) is 2-dimensional. If we fix a complex parameter, the eigenspaces of \(N^*_x\) (taken in appropriate sense) define a frame in \(TM\). Thus, a geometry of a “very generic” 6-dimensional almost complex manifold is rather trivial.

However, for a \(N^*_x\) inside a 10-dimensional subspace

\[
W_0 \subset \text{Hom}(\Lambda^{0,1}(M), \Lambda^{2,0}(M)),
\]

(\text{Remark 3.2}), the stabilizer \(St(N^*_x)\) contains \(SU(3)\), and the geometry of \((M, I)\) becomes more interesting.

**Proposition 3.1:** Let \((M, I)\) be an almost complex 6-manifold with Nijenhuis tensor which is non-degenerate in a dense set. Assume that \((M, I)\)
admits a Hermitian structure $\omega$ and a Hermitian connection with totally antisymmetric torsion. Then $\omega$ is uniquely determined by $I$, up to conformal equivalence.

**Proof:** Consider the map

$$C := \text{Id} \otimes N^* : \Lambda^{1,1}(M) \to \Lambda^{1,0}(M) \otimes \Lambda^{2,0}(M)$$

(3.1)

obtained by acting with the Nijenhuis tensor $N^* : \Lambda^{0,1}(M) \to \Lambda^{2,0}(M)$ on the second tensor multiplier of $\Lambda^{1,1}(M) \cong \Lambda^{1,0}(M) \otimes \Lambda^{0,1}(M)$. Then $C$ maps $\omega$ to a 3-form

$$x, y, z \mapsto \omega(N(x, y), z).$$

As Theorem 1.4 implies, $(M, I, \omega)$ admits a Hermitian connection with totally antisymmetric torsion if and only if $C(\omega)$ lies inside a 1-dimensional space

$$\Lambda^{3,0}(M) \subset \Lambda^{1,0}(M) \otimes \Lambda^{2,0}(M).$$

However, $C$ is an isomorphism in a dense subset of $M$, hence, all $\omega$ which satisfy the conditions of Theorem 1.4 are proportional. ■

**Remark 3.2:** The same argument proves that an almost complex manifold admits a Hermitian connection with totally antisymmetric torsion if and only if $C^{-1}(\Lambda^{3,0}(M))$ contains a Hermitian form. This is the space $W_0$ alluded to in the beginning of this section.

**Proposition 3.1** leads to the following corollary.

**Corollary 3.3:** Let $(M, I)$ be an almost complex 6-manifold. Then $(M, I)$ admits at most one strictly nearly Kähler metric, up to a constant multiplier.

**Proof:** Let $\omega_1$ and $\omega_2$ be nearly Kähler metrics on $(M, I)$. Since $(M, I, \omega_i)$ is strictly nearly Kähler, the 3-form $C(\omega_i) \in \Lambda^{3,0}(M)$ is nowhere degenerate (see (3.1)). Therefore, $(M, I)$ has nowhere degenerate Nijenhuis tensor. Then, by Proposition 3.1, $\omega_i$ are proportional: $\omega_1 = f \omega_2$. However, $d\omega_i^2 = 0$ on any nearly Kähler 3-manifold (see e.g. Theorem 4.2 (ii)). Then $2f df \wedge \omega_2^2 = 0$. This implies $df = 0$, because the map $\eta \mapsto \eta \wedge \omega_2^2$ is an isomorphism on $\Lambda^1(M)$. ■

**Remark 3.4:** The converse is also true: unless $(M, g)$ is locally isometric to a 6-sphere, the Riemannian metric $g$ determines the nearly Kähler almost complex structure $I$ uniquely (Proposition 4.7).
4 Nearly Kähler geometry and Hermitian connections with totally antisymmetric torsion

4.1 Hermitian structure on $\Lambda^{3,0}(M)$ and nearly Kähler manifolds

Let $(M, I)$ be an almost complex 6-manifold, and $\Omega \in \Lambda^{3,0}(M)$ a non-degenerate $(3,0)$-form. Then $\Omega \wedge \overline{\Omega}$ is a positive volume form on $M$. This gives a $\text{Vol}(M)$-valued Hermitian structure on $\Lambda^{3,0}(M)$. If $M$ is in addition Hermitian, then $M$ is equipped with a natural volume form $\text{Vol}_h$ associated with the metric, and the map

$$\Omega \mapsto \frac{\Omega \wedge \overline{\Omega}}{\text{Vol}_h}$$

can be considered as a Hermitian metric on $\Lambda^{3,0}(M)$. This metric agrees with the usual Riemann-Hodge pairing known from algebraic geometry, when $I$ is integrable. The following definition is a restatement of the classical one, see Subsection 2.

**Definition 4.1:** Let $(M, I, \omega)$ be an almost complex Hermitian manifold, and $\nabla$ the Levi-Civita connection. Then $(M, I, \omega)$ is called nearly Kähler if the tensor $\nabla \omega$ is totally antisymmetric:

$$\nabla \omega \subset \Lambda^{3}(M).$$

The following theorem is a main result of this section.

**Theorem 4.2:** Let $(M, I, \omega)$ be an almost complex Hermitian 6-manifold equipped with a $(3,0)$-form $\Omega$. Assume that $\Omega$ satisfies $3\lambda \text{Re } \Omega = d\omega$, and $|\Omega|_\omega = 1$, where $\lambda$ is a constant, and $|\cdot|_\omega$ is the Hermitian metric on $\Lambda^{3,0}(M)$ constructed above. Then the following conditions are equivalent.

(i) $M$ admits a Hermitian connection with totally antisymmetric torsion.

(ii) $d\Omega = -2\sqrt{-1}\lambda \omega^2$

(iii) $(M, I, \omega)$ is nearly Kähler, and $d\omega = \nabla \omega$.

The equivalence of (ii) and (iii) is known (see e.g. [H3], the second part of the proof of Theorem 6).
The existence of Hermitian connections with totally antisymmetric torsion on nearly Kähler manifolds is also well known (see Section 2). This connection is written as $\nabla_{NK} = \nabla + T$, where $\nabla$ is the Levi-Civita connection on $M$, and $T$ the operator obtained from the 3-form $3\lambda \text{Im} \Omega$ by raising one of the indices. The torsion of $\nabla_{NK}$ is totally antisymmetric by construction (it is equal $T$). Also by construction, we find that $T(\omega) = -3\lambda \text{Re} \Omega$, hence $\nabla_{NK}(\omega) = 0$. Therefore, $\nabla_{NK}$ is a Hermitian connection with totally antisymmetric torsion. This takes care of the implication (iii) $\Rightarrow$ (i).

To prove Theorem 4.2, it remains to prove that (i) implies (ii); we do that in Subsection 4.2. For completeness’ sake, we sketch the proof of the implication (ii) $\Rightarrow$ (iii) in Subsection 4.4.

**Remark 4.3:** As Corollary 3.3 shows, a non-Kähler nearly Kähler metric on $M$ is uniquely determined by the almost complex structure $I$.

### 4.2 Connections with totally antisymmetric torsion and Nijenhuis tensor

**Lemma 4.4:** In assumptions of Theorem 4.2 (i) implies (ii).

**Proof.**

**Step 1:** We show that $d\Omega \in \Lambda^{2,2}(M)$.

Were $(M, I)$ integrable, the differential $d$ would have only $(0,1)$- and $(1,0)$-part with respect to the Hodge decomposition: $d = d^{1,0} + d^{0,1}$. For a general almost complex manifold, $d$ splits onto 4 parts:

$$d = d^{2,-1} + d^{1,0} + d^{0,1} + d^{-1,2}.$$  

This follows immediately from the Leibniz rule. However,

$$0 = d^2 \omega = d(\Omega + \overline{\Omega}) = d\Omega + d\overline{\Omega}. \tag{4.1}$$

Since $\Lambda^{p,q}(M)$ vanishes for $p$ or $q > 3$, we also have

$$d\Omega + d\overline{\Omega} = d^{0,1} \Omega + d^{-1,2} \Omega + d^{2,-1} \overline{\Omega} + d^{1,0} \overline{\Omega} \tag{4.2}$$

The four terms on the right hand side of (4.2) have Hodge types $(3, 1)$, $(2, 2)$, $(2, 2)$ and $(1, 3)$. Since their sum vanishes by (4.1), we obtain

$$d^{0,1} \Omega = 0, \quad d^{1,0} \overline{\Omega} = 0, \quad d^{2,-1} \overline{\Omega} = -d^{-1,2} \Omega.$$
Then (1.2) gives
\[ d\Omega = -d^2 \Omega = d^{-1,2}\Omega. \] (4.3)

**Step 2:**
\[ d^{2,1} \bigg|_{\Lambda^{1,1}(M)} = \bigwedge \circ \text{Id} \otimes N^*, \] (4.4)
where \( N^* : \Lambda^{0,1}(M) \to \Lambda^{2,0}(M) \) is the Nijenhuis tensor,
\[ \text{Id} \otimes N^* : \Lambda^{1,1}(M) \to \Lambda^{2,0}(M) \otimes \Lambda^{1,0}(M) \]
acts as \( N^* \) on the second multiplier of \( \Lambda^{1,1}(M) \cong \Lambda^{1,0}(M) \otimes \Lambda^{0,1}(M) \), and \( \bigwedge \) denotes the exterior product. (4.4) is immediately implied by the Cartan’s formula for the de Rham differential.

**Step 3:** From the existence of Hermitian connection with totally anti-symmetric torsion we obtain that the form
\[ \omega(N(x, y), z) : T^{1,0}M \times T^{1,0}M \times T^{1,0}M \to \mathbb{C} \]
is totally antisymmetric (see Theorem 1.4). From (4.4) it follows that
\[ \omega(N(x, y), z) = d\omega = 3\lambda \text{Re} \Omega. \] (4.5)
Consider an orthonormal frame \( dz_1, dz_2, dz_3 \) in \( \Lambda^{1,0}(M) \), satisfying \( \Omega = dz_1 \wedge dz_2 \wedge dz_3 \) (such a frame exists because \( |\Omega|_{\omega} = 1 \)). Then (4.5) gives
\[ N^*(d\bar{z}_i) = \lambda d\bar{z}_i, \] (4.6)
where \( d\bar{z}_1 = dz_2 \wedge dz_3, d\bar{z}_2 = -dz_1 \wedge dz_3, d\bar{z}_3 = dz_1 \wedge dz_2 \).

**Step 4:** Using Cartan’s formula as in Step 2, we express \( d^{-1,2}\Omega \) through the Nijenhuis tensor. Then (4.3) can be used to write \( d\Omega = d^{-1,2}\Omega \) in terms of \( N^* \). Finally, (4.4) allows to write \( d^{-1,2}\Omega \) in coordinates, obtaining
\[ d\Omega = -2 \sqrt{-1} \lambda \omega^2. \]

### 4.3 \( G_2 \)-structures on cones of Hermitian 6-manifolds

**Proposition 4.5:** Let \((M, I, \omega)\) be an almost complex Hermitian manifold, \( \Omega \in \Lambda^{3,0}(M) \) a \((3,0)\)-form which satisfies \( d\omega = 3\lambda \text{Re} \Omega \), for some real constant, and \( |\Omega|_{\omega} = 1 \). Assume, in addition, that \( d\Omega = -2 \sqrt{-1} \lambda \omega^2 \). Consider the cone \( C(M) = M \times \mathbb{R}^{>0} \), equipped with a 3-form \( \rho = 3t^2 \omega \wedge dt + t^3 d\omega \), where \( t \) is the unit parameter on the \( \mathbb{R}^{>0} \)-component. Then \((C(M), \rho)\) is
a parallel, $G_2$-manifold (see Subsection 2.2). Moreover, any parallel $G_2$-structure $\rho'$ on $C(M)$ is obtained this way, assuming that $\rho'$ is homogeneous of weight 3 with respect to the the natural action of $\mathbb{R}^>0$ on $C(M)$.

**Proof:** As Fernandez and Gray has shown ([FG]), to show that a $G_2$-structure $\rho$ is parallel it suffices to prove that $d\rho = d^*\rho = 0$. Clearly, $d\rho = 0$, because

$$d\rho = 3t^2d\omega \wedge dt + 3t^2dt \wedge d\omega = 0.$$  

On the other hand, $*(\omega \wedge dt) = \frac{1}{2}t^2\omega^2$, and $*d\omega = -3dt \wedge I(d\omega)$, where * is taken with respect to the cone metric on $C(M)$. This is clear, because $(\omega, \Omega)$ defines an $SU(3)$-structure on $M$, and $d\omega = 3\lambda \text{Re} \Omega$. Then

$$* \rho = \frac{3}{2} t^4 \omega^2 - 3t^3 dt \wedge I(d\omega). \quad (4.7)$$

Since $d\Omega = -2\sqrt{1-\lambda} \omega^2$ and $3\lambda d \text{Re} \Omega = d^2 \omega = 0$, we obtain $d\text{Im} \Omega = -2\lambda \omega^2$. This gives $dI(d\omega) = -2\omega^2$, because $\lambda I(d\omega) = \text{Im} \Omega$. Then (4.7) implies

$$d(*\rho) = 6t^3 dt \wedge \omega^2 + 3t^3 dt \wedge dI(d\omega) = 6t^3 dt \wedge \omega^2 - 6t^3 dt \wedge \omega^2 = 0.$$  

We proved that $C(M)$ is a parallel $G_2$-manifold. The converse statement is straightforward. □

In Subsection 2.2 it is shown that the holonomy of $C(M)$ is strictly $G_2$, unless it is flat (in the latter case, $M$ is locally isometric to a sphere). Therefore, Proposition 4.5 implies the following corollary.

**Corollary 4.6:** In assumptions of Proposition 4.5, the almost complex structure is uniquely determined by the metric, unless $M$ is locally isometric to a 6-sphere. □

### 4.4 Near Kählerness obtained from $G_2$-geometry

Now we can conclude the proof of Theorem 4.2, implying Theorem 4.2 (iii) from Theorem 4.2 (ii). Let $M$ be a 6-manifold satisfying assumptions of Theorem 4.2 (ii). Consider the cone $C(M)$ equipped with a parallel $G_2$-structure $\rho$ as in Proposition 4.5. Let $g_0$ be a cone metric on $C(M)$. From the argument used to prove Proposition 4.5 it is clear that $g_0$ is a metric induced by the 3-form $\rho$ as in (2.1).
Consider the map \( C(M) \xrightarrow{\tau} M \times \mathbb{R} \) induced by \((m, t) \mapsto (m, \log t)\), and let \( g_1 = \tau^* g_\pi \) be induced by the product metric \( g_\pi \) on \( M \times \mathbb{R} \). Denote by \( \nabla_0, \nabla_1 \) the corresponding Levi-Civita connections. We know that \( \nabla_0(\rho) = 0 \), and we need to show that
\[
\nabla_1(\omega) = d\omega. \tag{4.8}
\]

The metrics \( g_0, g_1 \) are proportional: \( g_1 = g_0 e^{-t} \). This allows one to relate the Levi-Civita connections \( \nabla_1 \) and \( \nabla_0 \) (see e.g. [Or]):
\[
\nabla_1 = \nabla_0 + \frac{1}{2} A,
\]
where \( A : TM \to \text{End}(\Lambda^1(M)) \) is an \( \text{End}(\Lambda^1(M)) \)-valued 1-form mapping \( X \in TM \) to
\[
(\theta, X) \text{Id} - X \otimes \theta + X^2 \otimes \theta^2 \tag{4.9}
\]
and \( \theta \) the 1-form defined by \( \nabla_0(g_1) = g_1 \otimes \theta \), \( X \otimes \theta \) the tensor product of \( X \) and \( \theta \) considered as an endomorphism of \( \Lambda^1(M) \), and \( X^2 \otimes \theta^2 \) the dual endomorphism.

From \eqref{eq:4.9} and \( \nabla_0(\rho) = 0 \) we obtain
\[
(\nabla_1)_X(\rho) = (X, \theta)\rho - (\rho \lrcorner X) \wedge \theta + (\rho \lrcorner \theta^2) \wedge X^2. \tag{4.10}
\]
Since \( \theta = \frac{dt}{t} \), we have \( \nabla_1(\theta) = 0 \), and \( \nabla_1 \) preserves the decomposition \( \Lambda^*(C(M)) \cong \Lambda^*(M) \oplus dt \wedge \Lambda^*(M) \). Restricting ourselves to the \( dt \wedge \Lambda^*(M) \)-summand of this decomposition and applying \eqref{eq:4.10}, we find
\[
(\nabla_1)_X(t^3 \omega \wedge \theta) = t^3 (d\omega \lrcorner X) \wedge \theta.
\]
for any \( X \) orthogonal to \( dt \). Since \( g_1 \) is a product metric on \( C(M) \cong M \times \mathbb{R} \), this leads to \( \nabla \omega = d\omega \), where \( \nabla \) is the Levi-Civita connection on \( M \). This implies \eqref{eq:4.8}. We deduced Theorem 4.2 (iii) from Theorem 4.2 (ii). The proof of Theorem 4.2 is finished.

Using Corollary 4.6, we also obtain the following useful proposition.

**Proposition 4.7**: Let \((M, I, g)\) be a nearly Kähler manifold. Then the almost complex structure is uniquely determined by the Riemannian structure, unless \( M \) is locally isometric to a 6-sphere.
5 Almost complex structures on 6-manifolds and their infinitesimal variations

5.1 Hitchin functional and the volume functional

Let \((M, I)\) be an almost complex 6-manifold with nowhere degenerate Nijenhuis tensor \(N\), and \(\text{Vol}_I = \det N^* \otimes \det N^*\) the corresponding volume form (see 1.2). In this section we study the extrema of the functional \(I \xrightarrow{\Psi} \int_M \text{Vol}_I\).

A similar functional was studied by N. Hitchin for 6- and 7-manifolds equipped with a stable 3-form (see [Hi3]). Since then, this functional acquired a pivotal role in string theory and M-theory, under the name “Hitchin functional”.

Our first step is to describe the variation of \(\Psi\). We denote by \(\mathfrak{M}\) the space of all almost complex structures with nowhere degenerate Nijenhuis tensor on \(M\).

Let \((M, I, \omega)\) be an almost complex manifold with nowhere degenerate Nijenhuis tensor
\[
N \in \text{Hom}(\Lambda^2 T^{1,0}(M), T^{0,1}(M)),
\]
\(\delta \in T_I \mathfrak{M}\) an infinitesimal variation of \(I\), and
\[
N_\delta \in \text{Hom}(\Lambda^2 T^{1,0}(M), T^{0,1}(M))
\]
the corresponding variation of the Nijenhuis tensor. Consider the form \(\rho := \omega(N(x, y), z)\) associated with the Hermitian structure on \(M\) as in Theorem 1.4. After rescaling \(\omega\), we assume that
\[
|\rho|_\omega = 1.
\]

Since the Nijenhuis tensor is nowhere degenerate, \(\rho\) is also nowhere degenerate. Therefore, \(\rho\) can be used to identify \(T^{0,1}(M)\) and \(\Lambda^2 T^{1,0}(M)\), and we may consider \(N_\delta\) as an endomorphism of \(\Lambda^{0,1}(M)\). Notice that this identification maps \(N\) to the identity automorphism of \(\Lambda^{0,1}(M)\).

Claim 5.1: In these assumptions,
\[
\frac{d\Psi}{dI}(\delta) = 2 \text{Re} \int_M \text{Tr} N_\delta \text{Vol}_I.
\]
Proof: It is well known that
\[ \frac{d(\det A)}{d t} = \det A \text{Tr} \left( A^{-1} \frac{dA}{dt} \right) \]
for any matrix \( A \). Applying that to the map
\[ N^* \otimes N^* : \Lambda^{1,0}(M) \otimes \Lambda^{0,1}(M) \rightarrow \Lambda^{0,2}(M) \otimes \Lambda^{2,0}(M), \]
we obtain that
\[ \frac{d(\det(N^* \otimes N^*))}{d I}(\delta) = \text{Tr} \left( \frac{(N^*_\delta \otimes N^* + N^* \otimes N^*_\delta)}{(N^* \otimes N^*)} \right) \cdot \det(N^* \otimes N^*) \]  (5.3)
However, after we identify \( \Lambda^{1,0}(M) \) and \( \Lambda^{0,2}(M) \) as above, \( N \) becomes an identity, and (5.3) gives
\[ \frac{d(\det(N^* \otimes N^*))}{d I}(\delta) = 2 \text{Re Tr} N_\delta \text{Vol}_I \]  (5.4)

Remark 5.2: We find that the extrema of the functional \( \Psi(M, I) = \int_M \text{Vol}_I \) are precisely those almost complex structures for which \( \text{Re Tr} N_\delta = 0 \) for any infinitesimal variation \( \delta \) of \( I \).

5.2 Variations of almost complex structures and the Nijenhuis tenor

It is convenient, following Kodaira and Spencer, to consider infinitesimal variations of almost complex structures as tensors \( \delta \in \Lambda^{0,1}(M) \otimes T^{1,0}(M) \). Indeed, a complex structure on a vector space \( V \), \( \dim \mathbb{R} V = 2d \), can be considered as a point of the Grassmanian of \( d \)-dimensional planes in \( V \otimes \mathbb{C} \). The tangent space to a Grassmanian at a point \( W \subset V \otimes \mathbb{C} \) is given by \( \text{Hom}(W, V \otimes \mathbb{C}/W) \).

Consider the \((0,1)\)-part \( \nabla^{0,1} \) of the Levi-Civita connection
\[ \nabla^{0,1} \delta \in \Lambda^{0,1}(M) \otimes T^{1,0}(M) \otimes \Lambda^{0,1}(M), \]
and let \( \overline{\nabla} : \Lambda^{0,1}(M) \otimes T^{1,0}(M) \otimes \Lambda^{0,1}(M) \rightarrow \Lambda^{0,2}(M) \otimes T^{1,0}(M) \) denote the composition of \( \nabla^{0,1} \) with the exterior multiplication map
\[ \Lambda^{0,1}(M) \otimes T^{1,0}(M) \otimes \Lambda^{0,1}(M) \rightarrow \Lambda^{0,2}(M) \otimes T^{1,0}(M). \]
The following claim is well known.

**Claim 5.3:** Let \((M, I)\) be an almost complex manifold, and

\[
\delta \in \Lambda^{0,1}(M) \otimes T^{1,0}(M)
\]

an infinitesimal variation of almost complex structure. Denote by \(N_\delta \subset \Lambda^{2,0}(M) \otimes T^{0,1}(M)\) the corresponding infinitesimal variation of the Nijenhuis tensor (see Subsection 5.1). Then \(N_\delta = \overline{\delta}\), where

\[
\overline{\delta} : \Lambda^{0,1}(M) \otimes T^{1,0}(M) \to \Lambda^{0,2}(M) \otimes T^{1,0}(M)
\]

is the differential operator defined above.

**Proof:** The proof of **Claim 5.3** follows from a direct computation (see e.g. [KS]). ■

**Claim 5.3** can be used to study the deformation properties of the functional \(I \xrightarrow{\Psi} \int_M \text{Vol}_I\) constructed above (see Subsection 5.1). Indeed, from [Remark 5.2] it follows that \(\Psi\) has an extremum at \(I\) if and only if \(\text{Re} \, \text{Tr} \, N_\delta = 0\) for any \(\delta \in \Lambda^{0,1}(M) \otimes T^{1,0}(M)\). Using the identification \(T^{1,0}(M) \cong \Lambda^{2,0}(M)\), provided by the non-degenerate \((3,0)\)-form as above, we can consider \(\delta\) as a \((2,1)\)-form on \(M\). Then

\[
\overline{\partial} \delta \in \Lambda^{0,2}(M) \otimes \Lambda^{2,0}(M) = \Lambda^{2,2}(M)
\]

is the \((2,2)\)-part of \(d\delta\). Under these identifications, and using \(|\rho|_\omega = 1\) from (5.1), we can express \(\text{Tr} \, N_\delta\) as

\[
\text{Tr} \, N_\delta = \frac{\overline{\partial} \delta \wedge \omega}{\text{Vol}_I},
\]  

(5.5)

where \(\overline{\partial}\) is a \((0,1)\)-part of the de Rham differential. This gives the following claim.

**Claim 5.4:** Let \((M, I, \omega)\) be an almost complex Hermitian 6-manifold with nowhere degenerate Nijenhuis tensor. Assume that the corresponding 3-form \(\rho\) satisfies \(|\rho|_\omega = 1\) (see (5.1)). Consider the functional \(\Psi(I) = \int_M \text{Vol}_I\) on the space of such almost complex structures. Then

\[
\frac{d\Psi}{dI}(\delta) = 2 \text{Re} \int_M \overline{\partial} \delta \wedge \omega,
\]

(5.6)
where \( \delta \in \Lambda^{0,1}(M) \otimes T^{1,0}(M) \) is an infinitesimal deformation of an almost complex structure \( I \), considered as a \((2,1)\)-form on \( M \).

**Proof:** Claim 5.4 is implied immediately by (5.5) and Claim 5.1

Comparing Claim 5.4 with Remark 5.2, we find the following

**Corollary 5.5:** In assumptions of Claim 5.4, \( I \) is an extremum of \( \Psi \) if and only if

\[
\text{Re} \int_M \overline{\partial} \delta \wedge \omega = 0
\]

(5.7)

for any \( \delta \in \Lambda^{2,1}(M) \).

Integrating by parts, we find that (5.7) is equivalent to

\[
\text{Re} \int_M \delta \wedge \overline{\partial} \omega = 0
\]

and to \( \overline{\partial} \omega = 0 \). This gives the following proposition

**Proposition 5.6:** Let \((M, I, \omega)\) be an almost complex Hermitian 6-manifold with nowhere degenerate Nijenhuis tensor. Consider the functional \( \Psi(I) = \int_M \text{Vol}_I \) on the space of such almost complex structures on \( M \). Then \( I \) is an extremum of \( \Psi \) if and only if \( d\omega \) lies in \( \Lambda^{3,0}(M) \oplus \Lambda^{0,3}(M) \).

Now, Proposition 5.6 together with Theorem 4.2 implies Theorem 2.2.

Notice that by Corollary 3.3, the nearly Kähler Hermitian structure on \((M, I)\) is (up to a constant multiplier) uniquely determined by \( I \).

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M. Verbitsky, July 7, 2005

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