Approximating Maximin Shares with Mixed Manna*

Rucha Kulkarni† ruchark2@illinois.edu  
Ruta Mehta‡ rutameht@illinois.edu  
Setareh Taki§ staki2@illinois.edu

Abstract

In this paper we initiate the study of fair allocations of a mixed manna under the popular fairness notion of maximin share (MMS): allocate $m$ indivisible items, goods/chores, among $n$ agents with additive valuations. A mixed manna allows an item to be a good for some agents and chore for others, and hence strictly generalizes the well-studied goods (chores) only manna. For the good manna, a remarkable result by Procaccia and Wang [PW14] showed non-existence of MMS allocation. This prompted works on finding an $\alpha$-MMS allocation, where every agent gets at least $\alpha$ times her MMS value for $\alpha \in [0,1]$. A series of works obtained efficient algorithms, improving the $\alpha$ factor to $(\sim 3/4)$ for $n \geq 5$ agents. Although computing an $\alpha$-MMS allocation for the best possible $\alpha$, i.e., maximum $\alpha$ for which it exists, is known to be NP-hard, the question of finding $\alpha$-MMS for the near best $\alpha$ remains unresolved.

We make significant progress towards this question for mixed manna by showing a striking dichotomy: We derive two conditions and show that the problem is tractable under these two conditions, while dropping either renders the problem intractable. The two conditions are: (i) number of agents is a constant, and (ii) for every agent, her total value for goods differs significantly from her total value for chores.

First, for instances satisfying (i) and (ii) we design a PTAS – an efficient algorithm to find $(\alpha - \epsilon)$-MMS allocation given $\epsilon > 0$ for the best possible $\alpha$. Second, we show that if either condition is not satisfied then finding $\alpha$-MMS for any $\alpha \in (0,1]$ is NP-hard, even when solution exists for $\alpha = 1$. Our PTAS runs in time $2^{\mathcal{O}(1/\epsilon^2)}\text{poly}(m)$ for a given $\epsilon$, and therefore gives polynomial run-time for $\epsilon$ as small as $O(1/\sqrt{\log m})$.

As a corollary, our algorithm resolves the open question of designing a PTAS for the goods only setting with constantly many agents (best known was $\alpha \sim 3/4$), and similarly also for chores only setting. In terms of techniques, for the first time, we use market equilibrium as a tool to solve an MMS problem, which may be of independent interest.

---

*Supported by NSF Award CCF-1750436 (CAREER).
†University of Illinois at Urbana-Champaign.
‡University of Illinois at Urbana-Champaign.
§University of Illinois at Urbana-Champaign. Additional support from NSF Grant CCF-1942321 (CAREER)
1 Introduction

Fair allocations of indivisible items is a fundamental problem that arises naturally in various multi-agent systems [Ste48, BT96, VB02, Mou04, EPT07, Bud11, GHS+18], for example, seats in schools/courses, spectrum allocation, air traffic management, computing resources on networks, splitting assets and liabilities in partnership dissolution, and office tasks. Many of these involve both goods that are freely disposable and chores that have to be assigned. In this paper we study the fair division of a mixed manna: fairly divide a set $\mathcal{M}$ of discrete items, goods/chores, among a set $\mathcal{N}$ of agents where each agent $i \in \mathcal{N}$ has an additive valuation function $v_i : 2^\mathcal{M} \to \mathbb{R}$. We note that a mixed manna allows an item to be a good (positively valued) for some agents, and a chore (negatively valued) for others, and thereby strictly generalizes the extensively studied good (chore) manna (see Section 6 for a discussion on related works).

Among various well-studied notions of fairness (see Section 6), maximin-share (MMS) is one of the most popular notion (e.g., see [Bud11, KPW18, AMNS17, GHS+18, FGH+19, GT20]). Inspired from the classical cut-and-choose mechanism\textsuperscript{1}, the MMS value of agent $i$ is the value she can guarantee herself if she is to partition (cut) $\mathcal{M}$ into $n = |\mathcal{N}|$ bundles, given that she can choose the last. Formally, if $\Pi_n(\mathcal{M})$ represents all possible partitions $(A_1, \ldots, A_n)$ of $\mathcal{M}$ into $n$ bundles,

$$\text{MMS}_i = \max_{(A_1, \ldots, A_n) \in \Pi_n(\mathcal{M})} \min_{k \in [n]} v_i(A_k).$$

The goal is to find an MMS allocation, i.e., an allocation where every agent gets at least her MMS value. This problem has seen extensive work in case of a good (chore) only manna, while no results are known for mixed manna. In this paper we initiate the study of finding an MMS allocation of a mixed manna.

For the good manna, a notable result of Kurokawa et al. [KPV18] showed that an MMS allocation may not always exist but $\alpha$-MMS exists for $\alpha = 2/3$, where every agent gets at least $\alpha$ times her MMS value. This prompted works on efficient computation of an $\alpha$-MMS allocation for progressively better $\alpha \in [0,1]$, namely [AMNS17, BKM17, GMT18] for $\alpha = 2/3$, improved to $\alpha = 3/4$ by Ghodsi et al. [GHS+18]. And most recently, Garg and Taki improved this to $\alpha = (3/4 + 1/(12n))$ [GT20], and this is the best known for $n \geq 5$ many agents. Similarly, for the bad (chores) manna, starting from the work of Aziz et al. [ARSW17] for $\alpha = 1/2$, a series of works improved it to 9/11 [BKM17, HL19]. Note that with negative MMS values in this case, an $\alpha$-MMS allocation gives each $i \in \mathcal{N}$ at least $\frac{1}{\alpha} \cdot \text{MMS}_i$. On the other hand, the non-existence of $\alpha$-MMS allocation for good manna is known for $\alpha$ very close to 1 [KPV18]. This large gap naturally raises the following problem:

Design an efficient algorithm to find an $\alpha$-MMS allocation for the best possible $\alpha$, i.e., the maximum $\alpha \in [0,1]$ for which it exists.

This exact problem in intractable: In case of identical agents, an $\alpha = 1$-MMS allocation exists by definition, however, finding one is known to be $\text{NP}$-hard for good manna [BL16].\textsuperscript{2} On the positive side, a polynomial-time approximation scheme (PTAS) is known for this case due to [Woe97]. Given a constant $\epsilon \in (0,1]$, the algorithm finds a $(1 - \epsilon)$-MMS allocation in polynomial time. No such result is known when the agents are not identical. In light of these results, we ask,

\textsuperscript{1}In case of divisible items and two agents, one agent cuts so that she is okay with both the bundles and the other person chooses (mentioned in the Bible).

\textsuperscript{2}Checking if a given instance admits an MMS allocation is known to be in $\text{NP}^\text{#P}$, but not known to be in $\text{NP}$ [BL16].
Question. Can we design a PTAS, namely an efficient algorithm to find \((\alpha - \epsilon)\)-MMS allocation, given \(\epsilon > 0\), for the best possible \(\alpha\)?

In this paper we make a significant progress towards this question for mixed manna by showing the following dichotomy result: We derive two conditions and show that the problem is tractable under these two conditions, while dropping either renders the problem intractable. The two conditions are: (i) number of agents \(n\) is a constant, and (ii) for every agent, her total value for goods differs significantly from her total (absolute) value for chores, i.e., if these two values are \(v_i^+\) and \(v_i^-\) then for a constant \(\tau > 0\), \(v_i^+ \geq (1 + \tau)v_i^-\) or vice-versa.

First, for instances satisfying (i) and (ii), we design a PTAS (as defined above). Second, we show that if either condition is not satisfied, then finding \(\alpha\)-MMS for any \(\alpha \in (0, 1]\) is NP-hard, even when solution exists for \(\alpha = 1\). This hardness is striking because it shows inapproximability for any non-trivial factor when either (i) or (ii) is not satisfied. This also indicates the necessity of the two conditions for efficient computation of \(\alpha\)-MMS allocation in mixed manna instances.

Our algorithm, in principle, gives a little more than a PTAS. It runs in time \(2O(1/\epsilon^2) poly(m)\) for a given \(\epsilon\), and therefore gives polynomial run-time for \(\epsilon\) as small as \(O(1/\sqrt{\log m})\), where \(m = |\mathcal{M}|\).

As a corollary, we settle the question of obtaining a PTAS for the good manna when the number of agents is a constant. For this case, efficient algorithms are known for the following \(\alpha\) values: \((9/4 + 1/12n)\) for \(n \geq 5\) [GT20], \(4/5\) for \(n = 4\) [GHS+18], \(8/9\) for \(n = 3\) [GM19], and \(1\) for \(n = 2\) [BL16]. Similarly, for the chores only case as well, where the best known factor is \(9/11\) [HL19]. For the reader’s convenience, in Table 1 we summarise the best known results including ours.

| Manna                  | n Identical agents | n Non-identical agents |
|------------------------|--------------------|------------------------|
|                        | constant \(n\)     | non-constant \(n\)     | constant \(n\) | non-constant \(n\) |
| Goods only             | PTAS [Woe97]       | PTAS (Corr. 3.2)      | 3/4 + 1/(12n) [GT20] |
| Chores only            | PTAS [AAWY98]      | PTAS (Corr. 3.3)      | 11/9 [HL19]      |
| Special Mixed          | PTAS (Algo. 4.1)   | NP-Hard (Thm. 4.2)    | PTAS (Algo. 2)  | NP-Hard (Thm. 4.2) |
| General Mixed          | NP-Hard (Thm. 4.1) |                         |                      |

Table 1: Summary of results. Every entry is the best known result for the \(\alpha\)-MMS problem in the setting specified by the column and row headers. The Special mixed manna has for every agent \(i \in \mathcal{N}\), \(\max\{v_i^+ - v_i^-\} \geq (1+\tau)\min\{v_i^+, v_i^-\}\) for a constant \(\tau\).

Next we give an overview of our approach to prove the results, and the techniques used.

### 1.1 Our Approach, and the Techniques Used

Let us first state the \(\alpha\)-MMS problem for mixed manna. A mixed manna instance is represented by \((\mathcal{N}, \mathcal{M}, (v_i)_{i \in \mathcal{N}})\), where the value of agent \(i \in \mathcal{N}\) for set \(S\) of items is \(v_i(S) = \sum_{j \in \mathcal{M}} v_{ij}\). Since \(v_{ij}\)s can be positive (goods) or negative (chores), the MMS value of an agent can be positive or negative. In this case an \(\alpha\)-MMS allocation for \(\alpha \in [0, 1]\) is defined as:

\[
(A_1, \ldots, A_n) \in \Pi_n(M) \text{ s.t. } \forall i \in \mathcal{N}, \ v_i(A_i) \geq \alpha \text{MMS}_i \text{ if MMS}_i \geq 0, \text{ otherwise } v_i(A_i) \geq \frac{1}{\alpha} \text{MMS}_i.
\]

The problem is to find \(\alpha\)-MMS allocation for the maximum possible value of \(\alpha\) that renders existence. We show both algorithmic and hardness results for this problem.

**Hardness.** We show a strong inapproximability result. Instances with identical agents, i.e., \(v_i = v, \ \forall i \in \mathcal{N}\), have an exact MMS allocation by definition. We show that, even for these instances, finding \(\alpha\)-MMS allocation for any \(\alpha \in [0, 1]\) is NP-hard even if the instance in addition satisfies one of the following two conditions (see Theorem 2.1).
\( C_1. \) \( n = |\mathcal{N}| \) is a constant.

\( C_2. \) For each \( i \in \mathcal{N}, \max\{v^+_i, v^-_i\} \geq (1 + \tau) \min\{v^+_i, v^-_i\} \) where \( v^+_i \) and \( v^-_i \) are agent \( i \)'s total (absolute) value of goods and chores respectively, and \( \tau > 0 \) is a constant.

To show the above result we reduce the NP-complete PARTITION problem [GJ90] (for definition see Section 4) to an instance \((\mathcal{N}, \mathcal{M}, v)\) with non-negative MMS value, such that MMS > 0 if and only if the PARTITION problem has a solution. This proves inapproximability for any \( \alpha \in (0, 1] \) because MMS > 0 if and only if \((\alpha \ast \text{MMS}) > 0\). This result is in sharp contrast with good/bad manna where simple greedy approaches give \( 1/2 \)-MMS [AMNS17, ARSW17].

In light of the above hardness, we consider instances that satisfy both \( C_1 \) and \( C_2 \) for the algorithmic result. For good manna with two identical agents, [BL16] showed that finding the exact MMS-allocation is NP-hard (recall that it exists in case of identical agents). This implies that even if both \( C_1 \) and \( C_2 \) hold, finding \( \alpha \)-MMS for the best \( \alpha \in [0, 1] \) is NP-hard. Therefore, we aim for a PTAS (formal details are in Section 3).

1.1.1 PTAS assuming both \( C_1 \) and \( C_2 \)

Our aim is to design a polynomial-time algorithm for the following problem.

**OPT-\( \alpha \)-MMS.** Given an instance \((\mathcal{N}, \mathcal{M}, (v_i)_{i \in \mathcal{N}})\) (possibly with non-identical agents) that satisfies \( C_1 \) and \( C_2 \), and an \( \epsilon > 0 \), find an \((\alpha - \epsilon)\)-MMS allocation for the best possible \( \alpha \).

Next we describe the high-level approach of our algorithm, the challenges it poses, and the techniques used to resolve them. The first and foremost challenge posed by having chores is that they are not free disposal. Hence they cannot be arbitrarily allocated once we have found a good enough partial allocation.

**Challenge 1: the best possible \( \alpha \) is not known.** All known algorithms for good/chore manna aim for a particular \( \alpha \) value. They are able to iteratively reduce to smaller instances by doing partial allocations while maintaining existence of \( \alpha \)-MMS for the reduced instance. Such an approach is not applicable here. We resolve this by designing an algorithm to solve the following problem instead, and then do binary search on the \( \alpha \) value using it as a subroutine.

**Sub-problem.** Given \( \alpha \), either find an \((\alpha - \epsilon)\)-MMS allocation, or correctly report that no \( \alpha \)-MMS allocation exists.

Our algorithm for the **Sub-problem** (Algorithm 2) performs the following high-level steps:

**Step 1.** Find \( \mu_i = \text{MMS}_i \) for each \( i \in \mathcal{N} \).

**Step 2.** Allocate the set of high (absolute) value items computed using \( \mu_i \)s.

**Step 3.** Allocate the remaining low (absolute) value items.

**Challenge 2: unknown MMS values of the agents.** To handle both goods and chores together, it turns out that we must target the smallest possible value that we need to guarantee each agent. This requires almost exact computation of their MMS values. The PTAS to find MMS values for the goods only [Woe97] and chores only [AAWY98] cases do not translate to the mixed manna. As the above hardness result shows, mixed manna is significantly different from these cases.

We resolve this by designing a PTAS for identical agents in Sections 5 and A for non-negative and negative MMS values respectively. These are used in **Step 1** of the algorithm to approximately compute the MMS values of the agents.
**Challenge 3:** allocating arbitrarily high (absolute) valued items in **Step 2.** When there are only goods, each high valued good can be given to any distinct agent, and both the agent and the good can be removed to get a smaller instance – \(\alpha\)-MMS allocation is not lost by this reduction [KPW18]. In case of only chore, the absolute value of every chore has to be smaller than the absolute MMS value. But under the mixed manna, there could be very high-valued goods which when combined only with the right set of high absolute valued chores (and vice-versa) ensure an \(\alpha\)-MMS allocation. Therefore, we cannot greedily allocate high-valued goods and reduce the instance.

We resolve this in **Step 2** as follows: Let \(\mu_i\) be the MMS value of agent \(i\); \(\mu_i\) can be positive, zero or negative. Using the \(\mu_i\)s for all the agents, next we partition \(\mathcal{M}\) into a set of BIG and SMALL items. For this, we first modify the instance so that every item is either a good or a chore for all the agents. This partitions \(\mathcal{M}\) into goods \(\mathcal{M}^+\) and chores \(\mathcal{M}^-\). Next we define \(\text{BIG}^+\) as the set of \(j \in \mathcal{M}^+\) such that some agent \(i\) has \(v_{ij} > \frac{|\mu_i|}{m}\) if \(\mu_i \geq 0\), and \(\text{BIG}^-\) as the set of \(j \in \mathcal{M}^-\) such that some agent \(i\) has \(v_{ij} < -\epsilon\). Set \(\text{BIG} = \text{BIG}^+ \cup \text{BIG}^-\) and items not in \(\text{BIG}\) are SMALL.

We show that \(|\text{BIG}|\) is a constant. This is not at all obvious. Since the MMS problem is scale invariant (Lemma 2.3), wlog we assume that \(|v_i(\mathcal{M})| = n\). Then using conditions \(C_1\) and \(C_2\) we show that \(v_i^+ = O(1) = v_i^-\). Then it follows that \(|\text{BIG}^-|\) is a constant. Now if \(\mu_i\) is big enough, then it is easy to see that the number of BIG goods of agent \(i\) is a constant. But \(\mu_i\) can be arbitrarily small, i.e., \(O(1/m)\) for \(m = |\mathcal{M}|\). Yet, in Lemma 3.3 we show that no matter the value of \(\mu_i\), the number of BIG goods of agent \(i\) is a constant, and therefore their union \(\text{BIG}^+\) has constantly many goods (Corollary 3.1).

Using the fact that \(|\text{BIG}|\) is a constant we enumerate all possible allocations \(B^\pi = (B_1, \ldots, B_n)\) of BIG among \(n\) agents – again constantly many such \(B^\pi\) since \(n\) is a constant. The next step is to allocate SMALL items, given \(B^\pi\).

**Challenge 4:** allocating low valued items. In case of good manna, when items are low-valued, previous works use bag-filling like approaches [GHS+18, GMT18, GT20], where iteratively, a bag is greedily filled with items and given to the first agent who values it at least \(\alpha\) times her MMS value. Due to the greedy nature, enough value is left for the remaining agent after this allocation. Such algorithms do not work for a mixed manna because there are also chores remaining to be allocated, and it is not clear how to fill the bag so that \(\alpha\)-MMS allocation is retained for the remaining agents. This makes the \(\alpha\)-MMS problem non-trivial even when all items have small absolute value.

We resolve this in **Step 3** by designing a two-step process (details in Section 3.2):

**Step 3a** Assign SMALL goods on top of \(B^\pi\) using market equilibrium: Given that agent \(i\) already got \(B_i\) as per \(B^\pi\), we still need to give her \(c_i = \alpha \mu_i - v_i(B_i)\). Using this we design a market with budget-additive utilities [BGHM17] with SMALL goods where every agent has an additive (linear) valuation function \(v_i\) and a cap \(c_i\) on their utility. We note that goods are considered divisible in the market. We show that, if \(B^\pi\) is optimal (corresponds to an \(\alpha\)-MMS allocation) then any market equilibrium allocation gives value \(c_i\) to each agent \(i\). This together with Bundle \(B_i\) gives her \(\alpha \mu_i\). Therefore, if this step fails to give \(\alpha\)-MMS value to every agent then we can safely discard \(B^\pi\).

The equilibrium allocation is fractional and needs be rounded. By proving that the equilibrium allocation can be made acyclic (Claim 3.1) we ensure that there are at most \(O(n)\) shared goods. Using this we round the allocation so that no one losses by more than \(\epsilon \mu_i\) and thereby guarantee \((\alpha - \epsilon)\)-MMS value to all at the end of this step (Lemma 3.5).

**Step 3b** Next we assign SMALL chores using envy graph: Repeatedly compute the envy graph on agents with their current allocations, and make it cycle free. Then it allocates a small chore
to any sink agent in the graph. This process works, because the sink agents have the following property. As they do not envy any other agent and the total value for unallocated items is negative, every sink agent values their own allocation at least \( v_i(M)/n \). Since \( v_i(M)/n \) is an upper bound on MMS, giving her a small chore still keeps her allocation \( \alpha \)-MMS valued (Lemma 3.6).

If Step 3a discards all \( B^\alpha \) then we know that \( \alpha \)-MMS allocation does not exist for the given \( \alpha \). Putting all the above steps together, we show in Theorem 3.1 that Algorithm 2 solves the Subproblem in time \( 2^{O(1/\epsilon^2)}O(m^2) \) and therefore allows \( \epsilon \) to be as small as \( O(1/\sqrt{\log m}) \). Finally, by doing binary search on \( \alpha \) using Algorithm 2 as a subroutine, we can solve the \( \text{OPT-} \alpha \)-MMS problem in polynomial time (see Theorem 3.2). And we also get PTAS for good (bad) only manna as corollaries (Corollaries 3.2 and 3.3). We note that, for the case of good (bad) manna the algorithm becomes simpler since it does not need Step 3b (Step 3a).

**Organization.** In Section 2 we formally define the problem and some of its properties. In Section 3 we discuss our first main result, the PTAS (for non-identical agents). Section 4 shows the second main result on the hardness of approximation. Sections 5 and A, discuss the PTAS for the identical agents \( \alpha \)-MMS problems for agents respectively with non-negative and negative MMS values. All missing proofs, unless otherwise stated, are in Appendix B.

## 2 Preliminaries

In this section, we formally define mixed instances and other relevant notions of maximin share. We also show how agents can be divided into two classes based on the sign of their maximin share values, and this demarcation is crucial in the algorithms.

**Notations.** We use \([k]\) to denote the set \( \{1, 2, \ldots, k\} \). For \( c \in \mathbb{R} \), \( c^+ \) denotes \( \max\{c, 0\} \).

**Definition 2.1.** An MMS instance is a tuple \((N, M, (v_i)_{i \in N})\), where \( N \) is a set of \( n \) agents, \( M \) is a set of \( m \) indivisible items, and for each agent \( i \in N \), \( v_i : 2^M \rightarrow \mathbb{R} \) is her additive valuation function, represented by \( v_i(S) = \sum_{j \in S} v_{ij} \) for \( S \subseteq M \). If the agents have identical valuation functions, i.e., all the \( v_i \)'s are same, then we denote the MMS instance by \((N, M, v)\).

The valuation functions allow every item to be negatively valued by some agents and positively valued by some. We partition the items into three sets based on how agents value them as follows. 

- **Goods** are the items valued positively by all the agents. The set of goods is denoted by \( M^+ = \{ j \in M \mid v_{ij} \geq 0, \forall i \in N \} \).
- **Chores** are items valued negatively by all the agents, and the set of chores is termed \( M^- = \{ j \in M \mid v_{ij} \leq 0, \forall i \in N \} \).
- **Mixed** items are both positively and negatively valued by the agents, and their set is denoted by \( M^{gc} = \{ j \in M \mid \exists i, k \in N, v_{ij} > 0, \text{ and } v_{kj} < 0 \} \). Note that, \( M = M^+ \cup M^- \cup M^{gc} \). We call an instance pure if \( M^{gc} = \emptyset \), and mixed otherwise.

A partition of all items among all agents is referred as an allocation, and denoted by \( A^\alpha = \{A_1, A_2, \ldots, A_n\} \). Thus, \( A_i \cap A_{i'} = \emptyset \) for all distinct \( i, i' \in N \), and \( \cup_i A_i = M \).

**Definition 2.2 (MMS value).** Given an MMS instance, let \( \Pi_n(M) \) be the set of all possible allocations of \( M \) into \( n \) sets. Let \( A^\alpha = \{A_1, A_2, \ldots, A_n\} \) be any allocation in \( \Pi_n(M) \). The maximin share (MMS) value of an agent \( i \), denoted by \( M^{\alpha}_n(M) \), is defined as,

\[
M^{\alpha}_n(M) = \max_{A^\alpha \in \Pi_n(M)} \min_{A_k \in A^\alpha} v_i(A_k).
\]

We refer to \( \Pi_n(M) \) by MMS when the qualifiers \( n \) and \( M \) are clear, and by MMS when agents have identical valuation functions. Note that \( M^{\alpha}_n \) can be negative too. In all discussion
hence forth, we divide agents into two types based on whether their MMS values are negative or not. This is important even to define the approximate notion of MMS. We prove that an agent’s MMS value is negative only if the sum of her values for all items is negative. Hence, the sign of any agent’s MMS value can be determined easily.

Lemma 2.1. \( v_i(M) \geq 0 \) iff \( \text{MMS}_i \geq 0 \).

Proof. If the sum of valuations of all items \( v_i(M) \) is negative, there can be no allocation where every bundle has non-negative valuation. Hence, \( \text{MMS}_i \) is negative. If the sum of valuations is positive, then adding all items to one bundle and no item in other bundles makes the least-valued bundle have zero value. Thus, in this case, \( \text{MMS}_i \geq 0 \). \( \square \)

An allocation which gives every agent \( i \) a set of items worth at least \( \text{MMS}_i \) is called an MMS allocation. We define approximate MMS allocations as follows.

Definition 2.3 \((\alpha\text{-MMS)} \text{ allocation}. \) \( A^\alpha \) is called an \( \alpha\text{-MMS allocation} \) for an \( \alpha \in (0,1] \), if for each agent \( i \in N \) we have \( v_i(A_i) \geq \alpha\text{MMS}_i \) if \( \text{MMS}_i \geq 0 \), \( v_i(A_i) \geq (1/\alpha)\text{MMS}_i \), if \( \text{MMS}_i < 0 \). Equivalently,

\[
v_i(A_i) \geq \min\{\alpha\text{MMS}_i, (1/\alpha)\text{MMS}_i\}.
\]

Definition 2.4 \((\text{GENERAL } \alpha\text{-MMS)} \text{ problem.} \) Given an MMS instance \((N,M,(v_i)_{i\in N})\), and a number \( \alpha \in (0,1] \), the GENERAL \( \alpha\text{-MMS problem} \) is to either find an \( \alpha\text{-MMS allocation} \) of the items \( M \) among the agents \( N \), or correctly report that an \( \alpha\text{-MMS allocation} \) does not exist for the given instance.

Even for the goods only case \((M^- = M^{gc} = \emptyset)\), we know that an MMS allocation may not exist \([KPW18]\), and that checking existence is at least \( \text{NP-hard} \) even with two agents and is in the 2\textsuperscript{nd} level of the polynomial-hierarchy (not known to be in \( \text{NP} \)). Therefore, the GENERAL \( \alpha\text{-MMS} \) problem is essentially intractable. To make the problem amenable to polynomial-time algorithms, we make two assumptions. We discuss the (absolute) necessity of both of these assumptions in Section 4 by proving hardness of finding \( \alpha\text{-MMS allocation} \) for any \( \alpha \in (0,1] \) if either of these assumptions are dropped. Formally, we prove the following result.

Theorem 2.1. Given an instance \((N,M,v)\) with identical agents and \( v(M) > 0 \) such that one of the following two holds: either \( |N| = 2 \) or \( v(M^+) \geq (1+\tau)|v(M^-)| \) for a constant \( \tau \). Finding \( \alpha\text{-MMS allocation} \) of \((N,M,v)\) for any \( \alpha \in (0,1] \) is \( \text{NP-hard} \).

To prove the theorem, we show two reductions of the known \( \text{NP-Hard} \) problem \( \text{PARTITION} \) to the GENERAL \( \alpha\text{-MMS} \) problems where either \( |N| = 2 \) or \( v(M^+) \geq (1+\tau)|v(M^-)| \). Section 4 discusses the formal proof of this theorem. Theorem 2.1 motivate the following problem. Note that, when agents are identical, \( M^{gc} = \emptyset \), and therefore \( v(M^+) \) and \( v(M^-) \) are respectively every agent’s total positive values and total negative values. To handle non-identical agents,

\[
\text{for each agent } i \in N, \text{ define } v^+_i = \sum_{j \in M, v_{ij} \geq 0} v_{ij} \text{ and } v^-_i = \sum_{j \in M, v_{ij} < 0} |v_{ij}|. \tag{1}
\]

Definition 2.5 \((\alpha\text{-MMS problem).} \) The \( \alpha\text{-MMS problem} \) is the special case of the GENERAL \( \alpha\text{-MMS problem} \) where,

1. the number of agents \( n \) is constant, and
2. for some constant \( \tau > 0 \), for every agent \( i \in N \), either \( v^+_i \geq (1+\tau)v^-_i \) or \( v^-_i \geq (1+\tau)v^+_i \).

Equivalently, \( \max\{v^+_i, v^-_i\} \geq (1+\tau)\min\{v^+_i, v^-_i\} \).
As discussed in Section 1.1 α-MMS problem is NP-hard to solve exactly (due to [BL16]). We will design a PTAS for the α-MMS problem in Sections 3 and 5. By PTAS we mean, in polynomial-time it either returns an \((\alpha - \epsilon)^+\)-MMS allocation for a given constant \(\epsilon > 0\), or correctly reports that no α-MMS allocation exists. Recall that by \((\alpha - \epsilon)^+\) we mean \(\max\{\alpha - \epsilon, 0\}\). Our final goal is to solve the following α-MMS problem where α is not specified, and has to be maximized.

**Definition 2.6 (OPT-α-MMS Problem).** Given an instance of the α-MMS problem, find an α-MMS allocation for an \(\alpha > 0\) such that there is no \(\alpha'\)-MMS allocation for any \(\alpha' > \alpha\) for the given instance.

It is easy to see that we can solve OPT-α-MMS (up to a small error) efficiently using the polynomial-time algorithm for α-MMS by doing a binary search on the value of \(\alpha\).

The following simple observations will be useful in what follows.

**Lemma 2.2.** Given an MMS instance \((N, M, (v_i)_{i \in N})\), \(MMS_i \leq v_i(M)/|N|\) for all \(i \in N\).

**Proof.** If \(MMS_i > v_i(M)/n\), it implies that there exists a partition of items in \(\Pi_n(M)\) where all bundles have value greater than \(v_i(M)/n\). Therefore, \(v_i(M) \geq n \cdot MMS_i > n \cdot \frac{v_i(M)}{n} = v_i(M)\), which is a contradiction.

**Lemma 2.3.** α-MMS problem is scale-free. That is, given an instance \((N, M, (v_i)_{i \in N})\), we scale each \(v_i\) by some factor \(c_i > 0\) to get a new instance \((N, M, (v_i^{new})_{i \in N})\) where \(v_i^{new} = c \cdot v_i\). Then we will have

1. \(MMS_i^{new} = c_i \cdot MMS_i\) and
2. If \(A^\pi\) is an α-MMS allocation for \((N, M, (v_i)_{i \in N})\), then for \((N, M, (v_i^{new})_{i \in N})\) too, \(A^\pi\) is an α-MMS allocation.

**Proof.** For any bundle \(S \subseteq M\), we have \(v_i^{new}(S) = c \cdot v_i(S)\). Therefore, \(MMS_i^{new} \geq c \cdot MMS_i\). If \(MMS_i^{new} > c \cdot MMS_i\), then we take a partition of items in \(\Pi_n(M)\) that obtains \(MMS_i^{new}\) and re-scale the valuations back by \(1/c_i\) and we obtain a better \(MMS_i\) for \((N, M, (v_i)_{i \in N})\) which is a contradiction. Therefore, \(MMS_i^{new} = c_i \cdot MMS_i\).

Further, \(v_i^{new}(A_k) = c_i \cdot v_i(A_k) \geq c_i \cdot \alpha \cdot MMS_i = \alpha \cdot MMS_i^{new}, \forall A_k \in A^\pi\).

### 3 PTAS for α-MMS Problem with Non-identical Agents

In this section we obtain a PTAS for the OPT-α-MMS problem (Definition 2.6). For this we will design a PTAS for the α-MMS problem and then do binary search to find the best α. We use the results from Section 5 and Appendix A, that we have algorithms to find \((1 - \frac{\alpha}{2})\) approximate MMS values of all agents (See lines 3 in Algorithm 1). We use these values to find an allocation that gives every agent a bundle of value \((\alpha - \frac{\alpha}{2})\) times this approximate MMS value. Finally we will combine the approximation factors to get an \((\alpha - \epsilon)^+\)-MMS allocation for the problem.

First we formally state the result we are aiming for. We then show how the problem instance can be reduced to one that satisfies a number of nice properties – a pre-processing step. Then,
by leveraging these nice properties we finally design the algorithm and discuss its correctness and running-time. Recall the Problem Definition 2.5. Since this exact problem in intractable [BL16] we aim for a PTAS. We will use $n$ to denote $|\mathcal{N}|$ and $m$ to denote $|\mathcal{M}|$.

**Theorem 3.1.** Given an $\epsilon > 0$, and an $\alpha$-MMS problem instance $(\mathcal{N}, \mathcal{M}, (v_i)_{i \in \mathcal{N}})$ with $\alpha \in [0, 1]$, there is an algorithm that runs in time $f(n, \frac{1}{\epsilon^2})O(m^2)$ and outputs either

1. an $(\alpha - \epsilon)^+$-MMS allocation, or
2. correctly outputs that an $\alpha$-MMS allocation does not exist.

The proof of Theorem 3.1 is constructive, and we discuss the algorithm shortly.

**High-level Idea of the Algorithm.** We will need the following definition.

**Definition 3.1 (Pure Instance).** We call an instance $(\mathcal{N}, \mathcal{M}, (v_i)_{i \in \mathcal{N}})$ pure if $\mathcal{M}^{gc} = \emptyset$. That is, all agents agree on if an item is a good or a chore.

We first discuss why assuming that $\max\{v_i^+, v_i^-\} \geq (1+\tau) \cdot \min\{v_i^+, v_i^-\}$ helps.

**Lemma 3.1.** Given an instance $(\mathcal{N}, \mathcal{M}, (v_i)_{i \in \mathcal{N}})$ of the $\alpha$-MMS problem with $|v_i(\mathcal{M})| = n$ for all $i \in \mathcal{N}$, the following holds.

**Property**: $v_i^+ = O(n) = v_i^-$, $\forall i \in \mathcal{N}$.

**Proof.** If $v_i^+ \geq (1+\tau)v_i^-$, we have $v_i^+ - v_i^- = n$.

$$n = v_i^+ - v_i^- \geq (1+\tau)v_i^- - v_i^- \Rightarrow v_i^- \leq \frac{n}{\tau}, \text{ and, } n = v_i^+ - v_i^- \geq v_i^+ - \frac{v_i^+}{1+\tau} \Rightarrow v_i^+ \leq \frac{n(1+\tau)}{\tau}.$$ 

As $n$ and $\tau$ are constant, $v_i^+ = O(n) = v_i^-$. 

Similarly, if $v_i^+ \geq (1+\tau)v_i^-$, we have $v_i^- - v_i^+ = n$. By swapping $v_i^+$ and $v_i^-$ in the above expressions, we have $v_i^+ = O(n) = v_i^-$ for this case too. $\square$

Lemma 2.3 shows we can assume the problem is scale-free, hence $|v_i(\mathcal{M})| = n$ is without loss of generality. Using Property we show that there can only be a constant number of goods valued at least $O(1/n)$ by any agent. Next, we note that it is without loss of generality to assume that every mixed item, namely $j \in \mathcal{M}^{gc}$, will be assigned to an agent who values it positively at an $\alpha$-MMS allocation. We therefore reduce the given instance to a pure instance by modifying the valuations so that agents with negative values for the mixed items (items in $\mathcal{M}^{gc}$) now have zero value for them. By modifying the valuations, we may lose the relation $|v_i^+ - v_i^-| \geq \tau \cdot \min\{v_i^+, v_i^-\}$ for agents who have higher value for their set of chores. However, we show that Property still holds. We then leverage Property and the fact that the instance is pure crucially to get an efficient algorithm. The algorithm uses tools such as market equilibrium, envy-cycle elimination and bag-filling, and combines them carefully to obtain an allocation that gives every agent at least her $(\alpha - \epsilon)^+$-MMS share according to the old valuation functions.

In the next section we discuss the pre-processing of converting to a pure instance and computation of MMS$_i$ for every agent $i \in \mathcal{N}$. In Section 3.2, we discuss the algorithm, and analyze it’s correctness and running time.
3.1 Pre-processing

The pre-processing algorithm, specified in Algorithm 1, takes as input an instance \((\mathcal{N}, \mathcal{M}, (v_i)_{i \in \mathcal{N}})\) of the \(\alpha\)-MMS problem, and a constant \(\gamma \in [0, 1]\). It has three steps. First, we normalize all valuations so that the absolute sum of values for all items is \(n\) for every agent. That is, if \(v_i^+ > v_i^-\), scale the valuation function of agent \(i\) so that \(v_i(\mathcal{M}) = n\), and scale to \(-n\) otherwise. Then, we find the MMS values of all agents within accuracy factor \(\gamma\). To do this, for each agent \(i \in \mathcal{N}\), we create an identical agents instance with MMS find the \((1 - \gamma)\)-approximate MMS allocation using Algorithms 4 and 5. In the last step, we reduce the instance to a pure instance by modifying the values for mixed items as \(u_{ij} = \max\{0, v_{ij}\}\) for all \(j \in \mathcal{M}^g\) and \(i \in \mathcal{N}\). The algorithm returns the (pure) instance with the modified valuations, and the set of \((1 - \gamma)\)-approximate MMS values of the agents.

| Algorithm 1: Pre-processing Algorithm |
|---------------------------------------|
| **Input**: \((\mathcal{N}, \mathcal{M}, (v_i)_{i \in \mathcal{N}})\) of \(\alpha\)-MMS problem, \(\gamma \in (0, 1]\) |
| **Output**: Pure instance \((\mathcal{N}', \mathcal{M}', (v_i)_{i \in \mathcal{N}'})\), approximate MMS values within \(\gamma\) error |
| 1 For all agents \(i \in \mathcal{N}\), normalize the valuations so that \(|v_i(\mathcal{M})| = n\) |
| 2 \(u_i \leftarrow v_i\) for all agents \(i \in \mathcal{N}\) // valuation functions that will be returned |
| 3 for \(i \in \mathcal{N}\) // find \((1 - \gamma)\)-approximate MMS values using Algorithm 4 or 5 |
| 4 do |
| 5 if MMS\(_i\) \(\geq 0\) then |
| 6 \(\bar{\mu}_i \leftarrow \text{Algorithm} \ 4((\mathcal{N}, \mathcal{M}, v_i), \gamma)\) |
| 7 else |
| 8 \(\mu_i \leftarrow \text{Algorithm} \ 5((\mathcal{N}, \mathcal{M}, v_i), \gamma)\) |
| 9 for all mixed items \(j \in \mathcal{M}^g\) and agents \(i \in \mathcal{N}\) do |
| 10 if \(v_{ij} < 0\) then |
| 11 \(u_{ij} \leftarrow 0\) |
| 12 end |
| 13 end |
| 14 end |
| 15 return \((\mathcal{N}, \mathcal{M}, (u_i)_{i \in \mathcal{N}}), \{\bar{\mu}_i | i \in \mathcal{N}\})\) |

Next we show the guarantee from the pre-processing step that is crucial for the final algorithm.

**Lemma 3.2.** Given an instance \((\mathcal{N}, \mathcal{M}, (v_i)_{i \in \mathcal{N}})\) of \(\alpha\)-MMS problem and \(\gamma \in (0, 1]\), Algorithm 1 returns \((\mathcal{N}, \mathcal{M}, (u_i)_{i \in \mathcal{N}})\) and \(\{\bar{\mu}_i | i \in \mathcal{N}\}\) that satisfy, \(\forall i \in \mathcal{N}, \ u_i^+ = O(n) = u_i^-\) and

\[
\begin{align*}
    &\text{if } \text{MMS}_i > 0, \bar{\mu}_i \in [(1 - \gamma)\text{MMS}_i, \text{MMS}_i], \\
    &\text{if } \text{MMS}_i = 0, \bar{\mu}_i = 0, \\
    &\text{if } \text{MMS}_i < 0, \bar{\mu}_i \in [(1/(1 - \gamma))\text{MMS}_i, \text{MMS}_i].
\end{align*}
\]  

**Proof.** By construction of \(u_i\) we know that \(u_i^+ \leq v_i^+\) and \(u_i^- \leq v_i^-\). Therefore, using Lemma 3.1 it follows that for each agent \(i\), \(u_i^+ = O(n) = u_i^-\).

Next, note that if \(\text{MMS}_i = 0\), the PTAS of Section 5 will give the exact MMS allocation. Hence, using Theorems 5.1 and A.1 for the \(\bar{\mu}_i\) values returned, we have (2) for each agent \(i \in \mathcal{N}\). \(\square\)

3.2 The PTAS

We first define all notation and concepts, and provide intuition on how they are used.
The Markets. The algorithm will solve the following market equilibrium problem as an intermediate step, to allocate small goods. The problem and a polynomial time algorithm for it have been described in [BGHM17]. Briefly, the problem takes a set of buyers with budgets and utility cap values, a set of divisible items available in one unit each, and valuation functions of the buyers for the items. A market equilibrium is a set of prices for all items, and an allocation of all the items among the buyers. The allocation is such that for every agent, the value of the bundle she has received, if less than her utility cap, is the highest among all bundles that cost at most her budget.

Formally, the input to the problem, popularly known as a market with budget-additive utilities, is a tuple \((B^m, G^m, (vb_i)_{i \in B^m}, C)\), where \(B^m\) is a set of buyers, \(G^m\) is a set of divisible goods, \((vb_i)_{i \in B^m}\) are linear valuation functions \(vb_i : S \subseteq \mathcal{M} \to \mathbb{R}_+\) of all buyers, and \(C = \{c_i \in \mathbb{R}_+ \mid i \in B^m\}\) are utility caps of all agents. Every buyer has a budget of 1 dollar. As proved in [BGHM17], the market equilibrium allocation is a solution of the following convex program. The variables are \(x_{ij}\) for all \(i \in B^m, j \in G^m\), where \(x_{ij}\) denotes the fraction of good \(j\) assigned to buyer \(i\).

\[
\begin{align*}
\max \; & \sum_i \log \sum_j vb_{ij} x_{ij} \\
\text{s.t.} \; & \sum_j vb_{ij} x_{ij} \leq c_i, \; \forall i \in B^m \\
& \sum_i x_{ij} \leq 1, \; \forall j \in G^m \\
& x_{ij} \geq 0, \; \forall i, j
\end{align*}
\]

The following two key properties of the market equilibrium allocation enable us to use it to allocate the small goods. Consider the following undirected bipartite graph corresponding to every market equilibrium allocation. The buyers and goods are the nodes in the two parts, and an edge is drawn between buyer \(i\) and good \(j\) only if \(j\) has been allocated in non-zero amount to \(i\), i.e., \(x_{ij} > 0\). Call it the market equilibrium allocation graph. Then we can claim the following for every such graph.

Claim 3.1. The market equilibrium allocation graph can be assumed to be a tree graph.

Proof-Idea of Claim 3.1. We know from [BGHM17] that all goods assigned to a buyer in a market equilibrium allocation have the highest ratio of valuation to price, i.e., \(v_{ij}/p_j\). We use this to prove that for every cycle in the graph, we can re-allocate the goods along the cycle and eliminate one edge, without changing the utilities and money spent of any buyer and the money spent on and amount sold of any good. That is, every allocation after removing one cycle, along with the same set of prices, is also a market equilibrium. □

A full proof of Claim 3.1 is provided in Appendix B for completeness.

Claim 3.2. There are at most \(n - 1\) shared goods in the market equilibrium allocation graph.

Proof. Suppose there are \(k\) shared goods. Consider the subgraph of the market allocation graph with the \(n + k\) nodes corresponding to all the buyers and only the shared goods. From Claim 3.1, we know this subgraph is acyclic. Hence, there are at most \(n + k - 1\) edges. Further, each item is shared, meaning there are at least two edges incident to each node representing a good. Thus, there are at least \(2k\) edges. The inequality \(n + k - 1 \geq 2k\) is satisfied only when \(k \leq n - 1\), hence there are at most \(n - 1\) shared goods. □
Intuitively, the market equilibrium allocation helps as follows. We divide the items into BIG and SMALL (details below), and design a market problem with the SMALL goods, considered as divisible here. The fractional market equilibrium allocation, over an optimal partition of the BIG items, will prove to be an optimal $\alpha$-MMS allocation, if an $\alpha$-MMS allocation exists. As there are at most $n - 1$ shared goods, after arbitrarily rounding off the allocation, every agent still has an $(\alpha - \epsilon)^+$-MMS allocation.

**Envy Graph.** This is a directed graph that captures the envy of agents for other agents’ allocations. Each node in this graph represents an agent of the $\alpha$-MMS problem. There is a directed edge $(i \rightarrow k)$ corresponding to agents $i$ and $k$ if agent $i$ values agent $k$’s allocation more than her own. It is shown in [LMMS] that all cycles in an envy graph can be eliminated without reducing the valuation of any agent. Intuitively, this is done by exchanging the bundles of agents along a cycle, by giving each agent in the cycle the bundle of her successor, thereby making her happier. This process is repeated until all cycles are eliminated.

The envy graph will be used to allocate the SMALL chores, after allocating all the SMALL goods and rounding their allocation. The key idea is that until the value of the unallocated items is negative, the sink agents of the graph will have large value for their current bundle, hence can be given an additional SMALL chore.

**Big/Small Items.** Given an $\alpha$-MMS instance $(\mathcal{N}, \mathcal{M}, (v_i)_{i \in \mathcal{N}})$, the algorithm first applies the pre-processing subroutine with a particular $\gamma$. Let the pure instance returned by the algorithm be $(\mathcal{N}, \mathcal{M}, (\tilde{v}_i)_{i \in \mathcal{N}})$ and the approximate MMS values of the original instance be $\{\tilde{\mu}_i \mid i \in \mathcal{N}\}$. In order to get an error of $\epsilon$ in total, the algorithm will aim for an error of $\beta \in (0, 1]$. We fix

$$\gamma = \beta = \frac{\epsilon}{2}.$$  

The next definition classifies items as big or small using this parameter $\beta$.

**Definition 3.2** (Big and Small items.) Any good that has value more than $\beta \tilde{\mu}_i/n$ for some agent $i$ with $\tilde{\mu}_i \geq 0$, or more than $\beta/n$ for some agent with $\tilde{\mu}_i < 0$, is called a big good. The set of all big goods, called $\text{BIG}^+$, is,

$$\text{BIG}^+ = \{j \in \mathcal{M}^+ \mid \forall i \in \mathcal{N} : (\tilde{\mu}_i \geq 0 \text{ and } u_{ij} > \beta \tilde{\mu}_i/n) \text{ or } (\tilde{\mu}_i < 0 \text{ and } u_{ij} > \beta/n)\}.$$  

(4)

Any chore that has absolute value more than $\beta$ for some agent $i$ is called a big chore. The set of all big chores, called $\text{BIG}^-$, is,

$$\text{BIG}^- = \{j \in \mathcal{M}^- \mid \exists i \in \mathcal{N} : |u_{ij}| > \beta\}.$$  

(5)

The set of all big items, denoted by BIG, is $\text{BIG}^+ \cup \text{BIG}^-$. Any item that is not a big item, is called a small item. The sets of small goods, small chores, and small items, are respectively termed and defined as $\text{SMALL}^+ = \mathcal{M}^+ \setminus \text{BIG}^+$, $\text{SMALL}^- = \mathcal{M}^- \setminus \text{BIG}^-$, $\text{SMALL} = \text{SMALL}^+ \cup \text{SMALL}^-$.  

Analogously, for each $i \in \mathcal{N}$, $\text{BIG}^+_i$ is the set of big goods for agent $i$, i.e., if $\tilde{\mu}_i \geq 0$, $\text{BIG}^+_i := \{j \in \mathcal{M}^+ \mid u_{ij} > \beta \tilde{\mu}_i/n\}$, and $\text{BIG}^-_i = \{j \in \mathcal{M}^+ \mid u_{ij} > \beta/n\}$, otherwise. We similarly define $\text{BIG}^-_i$.

We abuse notation slightly by calling an item in BIG (or SMALL) a BIG (resp. SMALL) item. The idea is to enumerate all partitions of BIG items, and design an approach that will allocate the SMALL items optimally when given an optimal partition of BIG. If the value of $\tilde{\mu}_i$ is big enough, then it is easy to show that $|\text{BIG}^+|$ is constant. We next show that no matter how small $\tilde{\mu}_i$ is, $|\text{BIG}^+_i|$ and $|\text{BIG}^-_i|$ are constant for all $i$, after applying the pre-processing algorithm.
After the pre-processing algorithm is applied, from Lemma 3.2, we have $u_i^+ + O(n) = u_i^-$. We use this to prove the next two lemmas. Hence forth, we sometimes refer to MMS$_i$ by $\mu_i$ for brevity. The next Lemma will use a bag-filling algorithm 3 defined in Section 5. Briefly, Algorithm 3 takes as input an MMS instance $(N', M, (v_i)_{i \in N'})$, and a partition of the BIG items, what are termed as Big there, from $M$ into $|N'|$ bundles, and an $\epsilon > 0$. If the input satisfies certain conditions (one of (11) or (12)), then the algorithm returns an allocation where every agent gets a bundle of value $(1 - \epsilon)$.

**Lemma 3.3.** For the instance returned by pre-processing algorithm $(N', M, (u_i)_{i \in N'})$, the number of big goods of every agent, i.e., $|\text{BIG}_i^+| = O(n^2/\beta)$.

**Proof.** First we show this when $\tilde{\mu}_i \geq 0$.

If $\tilde{\mu}_i \geq 1/2$, $|\text{BIG}_i^+| \leq |\{ j \in M : u_{ij} > \beta/(2n) \}| \leq \frac{u_i^+}{\beta/(2n)} = \frac{2n}{\beta}O(n) = O(n^2/\beta).$ \hspace{1cm} (6)

Otherwise, if $\tilde{\mu}_i \in [0, 1/2)$, then from (2), $\mu_i < (1 - (1 - \gamma))$. Divide $\text{BIG}_i^+$ into two sets as follows.

$$\text{BIG}_i^+ = \{ j \in M^+ : u_{ij} \geq \beta/(2n) \} \cup \{ j \in M^+ : \beta\tilde{\mu}_i/n < u_{ij} \leq \beta/(2n) \}.$$ \hspace{1cm} (7)

Let us call the first set in Equation (7) LARGE and the second set MEDIUM. Similarly as for the case of $\tilde{\mu}_i \geq 1/2$, we can prove the size of LARGE is $O(n^2/\beta)$, a constant. We now prove that the number of items in MEDIUM is at most $\frac{n(n-1)}{(1-\gamma)^2} - (n - 2)$. We prove that if this is not true then there is a partition of all the items where all parts have value more than MMS$_i$ for agent $i$; in other words, the MMS value of agent $i$ is strictly larger than MMS$_i$. The partition is as follows. Add all items except the goods from MEDIUM to the first bundle. If $i$'s value for this bundle is more than 1, divide MEDIUM to make $n - 1$ bundles with at least $n/((1 - \gamma)\beta) + 1$ items in each. Then all the remaining bundles too have value at least

$$\left(\frac{n}{(1-\gamma)\beta} + 1\right) \left(\frac{\beta}{n}\right) \tilde{\mu}_i > \frac{\tilde{\mu}_i}{(1-\gamma)} \geq \mu_i.$$

If the value of the first bundle is less than 1 for $i$, then first add enough goods from MEDIUM to this bundle so that its value is just above $\mu_i$. As every item in MEDIUM has value at most $\beta/(2n)$, and $\mu_i < 1/(2(1 - \gamma))$, the value of the first bundle is at most,

$$\frac{1}{2(1-\gamma)} + \frac{\beta}{2n} = \frac{n + \beta}{2(1-\gamma)n} < \frac{1}{2(1-\gamma)} = \frac{1}{2(1 - \epsilon/2)} \leq \frac{1}{2 - \alpha} \leq 1,$$ \hspace{1cm} (8)

where we use $\gamma = \epsilon/2$, and $\epsilon \leq \alpha \leq 1$ to get the last three inequalities.

Then the remaining total value of all goods is at least $(n - 1)$. Consider an instance $(N', M, v)$ with $|N'| = n - 1$ identical agents, $M$ is the set of unallocated items from MEDIUM, and $v = u_i$. There are no Big items. This instance satisfies condition set (11) of the bag-filling algorithm 3 as follows. The total value of all items is $n - 1$, at least the number of agents. There are no Big items, hence by default, the value from Big items in every bag from any partition of Big is $0 \leq 1$. The value of every Small item is at most $\epsilon = \beta/(2n)$. Calling Bag-filling algorithm with the input $((N', M, v), B^x = [B_i \mid i \in N', B_i = \emptyset])$ forms $n - 1$ bundles with value at least $1 - \beta/(2n)$. From Equation (8), for $\gamma \leq 1/2$, this value is strictly greater than $1/(2(1 - \gamma)) > \mu_i$.

Therefore, $|\text{MEDIUM}| \leq \frac{n(n-1)}{(1-\gamma)^2} + (n - 2) = O(n^2/\beta)$. Hence $|\text{BIG}_i^+| = |\text{LARGE}| + |\text{MEDIUM}| = O(n^2/\beta)$, for any $i$ with $\tilde{\mu}_i \geq 0$.

If $\tilde{\mu}_i < 0$, $u_i(M) < 0$. Then by the definition of a BIG good for this case, $|\text{BIG}_i^+| \leq u_i^+/\beta/n = \frac{n}{\beta}O(n) = O(n^2/\beta)$. \hfill \square
Lemma 3.4. For the instance returned by pre-processing algorithm \((N, M, (u_i)_{i \in N})\), the number of big chores of every agent \(i \in N\), i.e., \(|BIG^-| = O(n/\beta)|.

Proof. By definition of a BIG chore, \(|BIG^-| \leq u_i^-/\beta = O(n/\beta)|.

Corollary 3.1. For the instance returned by pre-processing algorithm \((N, M, (u_i)_{i \in N})\), the number of big items, i.e., \(|BIG| = O(n^3/\beta)\).

Proof. For every agent \(i \in N\), \(|BIG_i| = |BIG^+_i| + |BIG^-_i| = O(n^2/\beta)|, from Lemmas 3.3 and 3.4. Then \(|BIG| \leq \sum_i |BIG_i| = O(n \cdot n^2/\beta) = O(n^3/\beta)|.

Let \(\pi^{\ast}\) be an \(\alpha\)-MMS allocation, and \(\pi^{\ast}\) be the partition of BIG items corresponding to \(\pi^{\ast}\). Then as \(n\) is constant, Corollary 3.1 implies that we can find \(\pi^{\ast}\) in constant time.

We are now ready to discuss the main algorithm that uses all the notions defined above.

3.2.1 Algorithm

Algorithm 2 applies the pre-processing algorithm 1, and gets a pure instance \((N, M, (u_i)_{i \in N})\) and a set of approximate MMS values \(\{\tilde{\mu}_i\}_i\). Every agent’s \(\alpha\)-MMS value is defined as \(\alpha\) (or \(1/\alpha\)) times \(\tilde{\mu}_i\) for agents with positive (resp. negative) \(\tilde{\mu}_i\). This gives us our target values. The algorithm now enumerates all possible partitions \(B^\pi\) of the BIG items (Line 11).

For each \(B^\pi\), it defines a market problem \((B^m, G^m, (vb_i)_{i \in B^m}, C)\) (Line 12). Here the buyers are the agents \(N\), and the goods \(G^m\) are all SMALL goods, now considered divisible. The valuation functions are the functions returned by the pre-processing algorithm, extended to fractional allocations of the goods by interpolation. That is, for all \(i \in B^m, x_i \in \mathbb{R}^k\) where \(k = |G^m|, vb_i(x_i) = \sum_j u_{ij}x_{ij}\). The main parameters in the market definition are the utility caps of agents. The algorithm finds the \(\alpha\)-MMS value minus the value obtained from the BIG items for every agent. For all \(i\), it sets the utility cap \(c_i\) as the maximum of this value and zero. Effectively, we include an agent as a buyer in the market only if she has not received an \(\alpha\)-MMS bundle from the BIG items. An equilibrium of the above market is computed (Line 13).

Intuitively, if \(B^\pi\) corresponds to an \(\alpha\)-MMS allocation then there is an (integral) equilibrium allocation in the market where every agent \(i\) gets a bundle of value \(c_i\) (see proof of Lemma 3.5). Therefore, if the equilibrium allocation does not give every agent a bundle of value \(c_i\), the algorithm discards \(B^\pi\). Otherwise, the allocation is rounded off by assigning shared goods arbitrarily to any buyer (Line 17). After rounding, any unallocated SMALL goods are arbitrarily allocated.

The next step assigns the remaining items, which are the SMALL chores. For this, the algorithm constructs the envy graph (Line 26). It then repeatedly assigns any unallocated chore to any sink agent and modifies the envy graph.

If any partition of BIG items \(B^\pi\), followed by allocating SMALL goods and chores as described, results in an \((\alpha - \epsilon)^\ast\)-MMS allocation, the following last step is performed. Consider the original valuation functions \((v_i)_{i \in N}\) before the pre-processing step was applied. For every item \(j \in M^g\) according to \((v_i)_{i \in N}\), if it is allocated to some agent who values it negatively, re-allocate it to any agent who has positive value for it. Return the allocation after all such re-allocations.

If an \((\alpha - \epsilon)^\ast\)-MMS allocation was not obtained for any partition \(B^\pi\), the algorithm reports that no \(\alpha\)-MMS allocation exists for the given instance.
3.2.2 Analysis of Algorithm 2

We use $A^{\pi*}$ to denote an $\alpha$-MMS allocation, and $B^{\pi*}$ for the corresponding partition of BIG items.

We first analyze the market phase of the algorithm. The following claim will be used to prove a lower bound on the valuations of agents after the market phase.

Claim 3.3. If there is an $\alpha$-MMS allocation for an instance $(N, M, (v_i)_{i \in N})$, then for any instance $(N, M', (v_i)_{i \in N})$ obtained by removing a subset of chores, i.e., $M' = M^{-} \subseteq M^{-}$, and $M'^+ = M'^{gc} = M^{gc}$, there is an allocation which gives every agent a bundle of value $\alpha\text{MMS}_i$ when $\text{MMS}_i \geq 0$ (or $(1/\alpha)\text{MMS}_i$ when $\text{MMS}_i < 0$). Here $\text{MMS}_i$ is the MMS value of agent $i$ for $(N, M, (v_i)_{i \in N})$.

Proof. Let $A^{\pi*} = [A_i^*]_{i \in N}$ be the $\alpha$-MMS allocation for $(N, M, (v_i)_{i \in N})$. Then consider the allocation $A^{\pi}$ for the modified instance obtained by giving every agent $i$ all the items in $A_i^* \cap M'$. Every agent values their bundle in $A^{\pi}$ at least as much as in $A^{\pi*}$, hence at least $\min\{\alpha\text{MMS}_i, (1/\alpha)\text{MMS}_i\}$. \qed

Let us call a bundle of items $A_i$ given to agent $i$ an $(\alpha - \beta)\overline{\text{MMS}}$ bundle, if $u_i(A_i) \geq \min\{(\alpha - \beta)\bar{\mu}_i, (1/\alpha - \beta)\bar{\mu}_i\}$. An allocation where every agent gets an $(\alpha - \beta)\overline{\text{MMS}}$ bundle is called an $(\alpha - \beta)\overline{\text{MMS}}$ allocation.

Lemma 3.5. In Algorithm 2, if the partition of BIG items being considered is $B^{\pi*}$, then the allocation $A^{\pi}$ obtained after solving the convex program of equation (3) and rounding the equilibrium tree allocation arbitrarily is an $(\alpha - \beta)\overline{\text{MMS}}$ allocation.

Proof. The market equilibrium allocates all SMALL goods. All BIG items have already been allocated, and only the (SMALL) chores are left unallocated after the market step. Therefore, from Claim 3.3, if $B^{\pi*}$ is the partition of BIG items being examined, there must exist an integral allocation of SMALL goods, and hence a fractional one too, that gives every agent an $\alpha$-MMS bundle. As $\bar{\mu}_i \leq \mu_i$ for all $i$, there is an allocation $A^\pi = [A_1, \ldots, A_n]$ that gives agent $i$ a bundle of value at least $\min\{\alpha\bar{\mu}_i, (1/\alpha)\bar{\mu}_i\} - u_i(A_i)$. This value is the utility cap $c_i$.

The highest value of the objective function of the market program (3) is reached when all agents get a bundle of value equal to their utility caps $c_i$. As such an allocation does exist, solving the program will give a fractional assignment of SMALL goods that gives every agent a $\min\{\alpha\bar{\mu}_i, (1/\alpha)\bar{\mu}_i\}$ valued bundle.

The market equilibrium allocation is rounded off arbitrarily. As each good is SMALL, every agent values it at most $\beta\bar{\mu}_i/n$ if $\bar{\mu}_i \geq 0$, and at most $\beta/n$ if $\bar{\mu}_i < 0$. In the worst case, assume an agent loses all her shared items in the rounding step. From Claim 3.2, she loses at most $n - 1$ items.

If $\bar{\mu}_i \geq 0$, her loss is at most $(n - 1)(\beta\bar{\mu}_i/n) < \beta\bar{\mu}_i$. After rounding the bundle received from the market equilibrium, she still values her bundle at least $\alpha\bar{\mu}_i - \beta\bar{\mu}_i = (\alpha - \beta)\bar{\mu}_i$.

If $\bar{\mu}_i < 0$, then after scaling in repressing we have $u(M) = -n$. From Lemma 2.2, $\bar{\mu}_i \leq \mu_i \leq -1$. By losing all shared items, an agent’s loss is at most $(n - 1)\beta/n \leq \beta$. She still receives a bundle of value $(1/\alpha)\bar{\mu}_i - \beta \geq (1/\alpha + \beta)\bar{\mu}_i$. As $1/\alpha + \beta \leq 1/(\alpha - \beta)$, $((1/\alpha + \beta)\bar{\mu}_i) \geq (1/(\alpha - \beta))\bar{\mu}_i$. \qed

Next, we argue correctness of the envy cycle elimination step.
Lemma 3.6. If $B^\pi^*$ is the partition of BIG items being examined, after the rounding step of Algorithm 2, every unallocated item can be allocated so that the resulting allocation is an $(\alpha - \beta)$-MMS allocation.

Proof. From Lemma 3.5, after rounding, every agent has an $(\alpha - \beta)$-MMS bundle. Assigning any goods that are unallocated after the market step arbitrarily still retains this property.

Consider any sink agent $i$ of the (acyclic) envy graph. As $i$ does not envy any other agent’s bundle, her value for her own bundle is the highest. This is at least the average of her total valuation for all assigned items. Until there is at least 1 unassigned chore, if $\tilde{\mu}_i \geq 0$, each agent has value strictly greater than $n$ for all the assigned items. Hence $i$ has value greater than 1 for her own bundle. Otherwise if $\tilde{\mu}_i < 0$, $i$’s total value of all assigned items is greater than $-n$. Hence, her value for her own bundle is greater than $-1$. Now every unassigned chore has value at most $\beta$ for any agent. After assigning any chore $b \in $ SMALL$^{-}$ to $i$, her value for her resulting bundle is,

$$\begin{align*}
\text{if } \tilde{\mu}_i &\geq 0, \ u_i(A_i \cup \{b\}) \geq 1 - \beta \geq (1 - \beta)\tilde{\mu}_i \geq (\alpha - \beta)\tilde{\mu}_i, \\
\text{if } \tilde{\mu}_i &< 0, \ u_i(A_i \cup \{b\}) \geq -1 - \beta \geq (1 + \beta)\tilde{\mu}_i \geq (1/\alpha + \beta)\tilde{\mu}_i \geq (1/(\alpha - \beta))\tilde{\mu}_i.
\end{align*}$$

This argument is true until there is at least one unallocated chore. Hence, the process of updating the envy graph and allocating a chore to any sink agent can be repeated until all chores are allocated. In the end, we still have an $(\alpha - \beta)$-MMS allocation.

Proof of Theorem 3.1. When $(\alpha - \epsilon) < 0$, by definition 2.5, when agents with $\text{MMS}_i \geq 0$ receive bundles of non-negative value, and those with $\text{MMS}_i < 0$ receive bundles of value higher than $-\infty$, the allocation is an $(\alpha - \epsilon)^+$-MMS bundle. Hence, by giving all items to one agent, the Algorithm returns an $(\alpha - \epsilon)^+$-MMS allocation.

For the case when $(\alpha - \epsilon) \geq 0$, assume an $\alpha$-MMS allocation $A^\pi^*$ exists. Algorithm 2 enumerates all partitions of BIG items, hence it considers $B^\pi^*$, the partition corresponding to $A^\pi^*$. In the iteration when it considers $B^\pi^*$, from Lemmas 3.5 and 3.6, we get an $(\alpha - \beta)$-MMS allocation $A^\pi^*$ after assigning all items. After the last step of re-allocating items in $M^{gc}$, if given to agents who value them negatively according to $(v_i)_{i \in X}$, we have that $v_i(A_i) \geq u_i(A_i)$.

Hence, if an $\alpha$-MMS allocation exists, the algorithm returns an allocation with $v_i(A_i) \geq (\alpha - \beta)\tilde{\mu}_i$ if $\tilde{\mu}_i \geq 0$, and $v_i(A_i) \geq (1/(\alpha - \beta))\tilde{\mu}_i$ if $\tilde{\mu}_i < 0$.

To ensure this is an $(\alpha - \epsilon)^+$-MMS allocation, we want

$$\begin{align*}
(\alpha - \beta)\tilde{\mu}_i &\geq (\alpha - \epsilon)\mu_i, \text{ when } \tilde{\mu}_i \geq 0, \\
(1/(\alpha - \beta))\tilde{\mu}_i &\geq (1/(\alpha - \epsilon))\mu_i, \text{ when } \tilde{\mu}_i < 0.
\end{align*}$$

Simplifying these equations using Equation (2), we get,

$$\begin{align*}
\tilde{\mu}_i \geq 0 : \ (\alpha - \beta)\tilde{\mu}_i &\geq (\alpha - \beta)(1 - \gamma)\mu_i \\
\Rightarrow \text{ we want: } (\alpha - \beta)(1 - \gamma) &\geq (\alpha - \epsilon). \\
\tilde{\mu}_i < 0 : \ 1 \geq (\alpha - \beta)(1 - \gamma)\tilde{\mu}_i \\
\Rightarrow \text{ we want: } (\alpha - \epsilon) &\leq (\alpha - \beta)(1 - \gamma).
\end{align*}$$

Both in (9) and (10) we want $(\alpha - \beta)(1 - \gamma) \geq (\alpha - \epsilon)$. Replacing $\beta = \gamma = \frac{\epsilon}{2}$, we get

$$\begin{align*}
(\alpha - \beta)(1 - \gamma) = (\alpha - \frac{\epsilon}{2})(1 - \frac{\epsilon}{2}) = \alpha - \alpha\frac{\epsilon}{2} - \frac{\epsilon}{2} + (\frac{\epsilon}{2})^2 &\geq \alpha - \frac{\epsilon}{2} - \frac{\epsilon}{2} = (\alpha - \epsilon)
\end{align*}$$

where the inequality follows since $\alpha \leq 1$.  

15
If an $\alpha$-MMS allocation exists, an $(\alpha - \epsilon)^{+}$-MMS allocation always does, and is given as output. Hence, if an $(\alpha - \epsilon)^{+}$-MMS allocation is not given, then an $\alpha$-MMS allocation does not exist, as is correctly reported. Thus, Algorithm 2 gives a $(\alpha - \epsilon)^{+}$-MMS allocation, else correctly reports that an $\alpha$-MMS allocation does not exist.

Let us finally look at the running time of Algorithm 2. First, the pre-processing step calls the PTAS for identical agents (from Section 5 and Appendix A) $n$ times, hence from the proofs of Theorem 5.1 and A.1, requires $O(2^{1/\epsilon}nm + 2^{1/\epsilon^2}n^2 \log m)$ time. Next, the main for loop enumerating all partitions of BIG runs for $|\text{BIG}|$ iterations. From Corollary 3.1, this is $n^{O(n^3/\epsilon)}$. In each iteration, the market equilibrium, from [BGHM17], requires $O(mn^6 \log (m+n))$ time. From [LMMS], the time to compute an acyclic envy graph is $O(mn^3)$. For each partition of BIG items, there are $O(|\text{SMALL}|)$ envy graph computations. Thus, in the worst case, the envy graph phase takes $O(m^2n^3)$ time. Finally, looking at all items in $M^{gc}$ and re-assigning if necessary takes a further $O(m)$ time.

Together, every iteration of the main for loop takes $O(mn^6 \log (m+n) + m^2n^3)$ time. Hence, the total time for the algorithm is $O(2^{1/\epsilon}nm + 2^{1/\epsilon^2}n^2 \log m) + O(2^{O(n^3 \log n/\epsilon)}(mn^6 \log (m+n) + m^2n^3))$, which is,

$$2^{O(\log n/\epsilon^2)}O(\log m) + 2^{O(n^3 \log n/\epsilon)}O(m^2) = f(n, \frac{1}{\epsilon^2})O(m^2).$$

As discussed in Section 2, a binary search for the highest $\alpha$ can be done using the algorithm, which leads to a PTAS for OPT-$\alpha$-MMS.

**Theorem 3.2.** There is a PTAS for the OPT-$\alpha$-MMS problem.

**Proof.** For any $\alpha^* \in (0, 1]$, the PTAS for the $\alpha$-MMS problem can check if an $\alpha^*$-MMS allocation exists, and if it does not then we know that the maximum possible $\alpha$ is less than $\alpha^*$. Hence, a binary search on the value of $\alpha$, where for each value we run the PTAS, suffices to solve the OPT-$\alpha$-MMS problem.

As corollaries of Theorem 3.2, we get the following results for the only goods and only chores settings.

**Corollary 3.2.** There is a PTAS for the problem of finding the best approximate MMS allocation when distributing goods among a constant number of agents.

**Corollary 3.3.** There is a PTAS for the problem of finding the best approximate MMS allocation when distributing chores among a constant number of agents.
Algorithm 2: \((\alpha - \epsilon)^{+}\)-MMS allocation of mixed items to non-identical agents

**Input**: Instance \((\mathcal{N}, \mathcal{M}, (v_i)_{i \in \mathcal{N}})\) of \(\alpha\)-MMS problem, \(\alpha \in (0, 1]\), and \(\epsilon > 0\)

**Output**: \((\alpha - \epsilon)^{+}\)-MMS allocation or report \(\alpha\)-MMS allocation does not exist.

1. if \(\epsilon \leq \alpha\) // all items to one agent is \((\alpha - \epsilon)^{+}\)-MMS allocation
   2. then
      3. if there is some agent \(i\) with \(v_i(M) \geq 0\) then
         4. \(A_i \leftarrow M; A_k \leftarrow \emptyset \forall k \neq i\)
      5. else
         6. \(A_i \leftarrow M; A_k \leftarrow \emptyset \forall k \neq 1\)
      7. return \(A^\pi = [A_1, A_2, \ldots, A_n]\)
   8. Fix \(\gamma = \beta = \frac{\epsilon}{2}\) // to get \(\leq \epsilon\) error overall. See Eq. (9) and (10)
   9. \((\mathcal{N}, \mathcal{M}, (u_i)_{i \in \mathcal{N}}), \{\tilde{\mu}_i | i \in \mathcal{N}\} \leftarrow \text{Preprocessing}((\mathcal{N}, \mathcal{M}, (v_i)_{i \in \mathcal{N}}), \gamma)\) // Algorithm 1
10. Define \(\text{BIG}^+, \text{BIG}^-, \text{SMALL}^+\) and \(\text{SMALL}^-\) as in Definition 3.2
11. for all allocations \(B^\pi = [B_1, B_2, \ldots, B_n]\) of \(\text{BIG}\)
12. Define a market \((B^m, G^m, (v^m_b)_{b^m \in B^m}, C)\) as follows:

   \[ B^m := \mathcal{N}, \quad G^m := \text{SMALL}^+, \quad (v^m_b(x_i)) = \sum_j u_{ij} x_{ij}, \]

   \[ C := \{c_i | c_i = \max\{0, a_i - u_i(B_i)\}\} \text{ where } a = \min\{\alpha \tilde{\mu}, (1/\alpha) \tilde{\mu}\}. \]
13. Find \(A^* = [A_1^* \cdot A_n^*]\) := market equilibrium tree allocation // Solve Equation 3
14. if some agent received less than \(c_i\) in \(A^*\) then
15.   continue // discard this allocation of BIG
16. else
17.   for all \(\{j | \exists i, 0 < x_{ij} < 1\}\) // round off allocation of all shared goods \(j\)
18.      do
19.         for \(k \in \mathcal{N}\setminus\{i\}\) do
20.             \[ A_k^* \leftarrow A_k^* \setminus [x_{kj}] \] // remove all allocations of \(j\)
21.         For any \(i \in \mathcal{N}: A_i^* \leftarrow A_i^* \cup \{j\} \) // assign \(j\) to any agent \(i\)
22.      \[ A^\pi = [A_1, A_2, \ldots, A_n] \leftarrow [A_1^*, A_2^*, \ldots, A_n^*] \]
23.      if \((\mathcal{M}^+\setminus A^\pi) \neq \emptyset\) // there is an unallocated good
24.         then
25.             \[ A_1 = A_1 \cup (\mathcal{M}^+\setminus A^\pi) \] // can also assign remaining goods arbitrarily
26.         Construct envy graph and eliminate all cycles
27.      while \((\mathcal{M}^-\setminus A^\pi) \neq \emptyset\) // there is an unallocated chore
28.         do
29.             Add any unallocated chore to a sink agent
30.             Update envy graph and eliminate cycles
31.         if \(v_i(A_i) \geq \min\{(\alpha - \epsilon)\tilde{\mu}_i, (1/(\alpha - \epsilon))\tilde{\mu}_i\}\) for all \(i\) then
32.             for \(j \in \mathcal{M}^g\) according to \((v_i)_{i \in \mathcal{N}}\) and \(i: j \in A_i\) // reallocate mixed items if given to agents who consider them chores
33.             do
34.                 if \(v_{ij} < 0\) then
35.                     \[ A_i \leftarrow A_i \setminus \{j\}; \] For any \(i': v_{i'j} > 0: A_i' \leftarrow A_i' \cup \{j\} \]
36.             return \(A^\pi\)
37. return "\(\alpha\)-MMS allocation does not exist"
4 Hardness of Approximation

Theorem 3.1 makes two assumptions on the GENERAL $\alpha$-MMS problem with a mixed manna. First, the number of agents is assumed to be a constant. Second, the total absolute valuation of every agent for the goods is assumed to be at least $(1 + \tau)$ times the total valuation for chores (or vice versa), for some constant $\tau$. In this section we show that relaxing either of these two assumptions makes the $\alpha$-MMS problem NP-hard for any $\alpha \in (0, 1]$ even when agents are identical.

When agents are identical, the allocation that decides the MMS value of the agents is also an MMS allocation for the instance. Thus, for $\alpha = 1$, the GENERAL $\alpha$-MMS problem should return an MMS allocation. Furthermore, given $v(M) > 0$ we are guaranteed to have $\text{MMS} \geq 0$ due to Lemma 2.1. However next we show that when either assumption of problem $\alpha$-MMS is dropped, deciding if the inequality is indeed strict is NP-hard.

We show two NP-hardness results in Theorems 4.1 and 4.2, stated in Section ?? and recalled here, and both reduce from the known NP-hard PARTITION problem.

**Theorem 4.1.** Given an instance $(N, M, v)$ with constantly many (two) identical agents and $v(M) > 0$, checking if MMS > 0 is NP-hard.

**Proof.** We reduce an instance of PARTITION to an MMS instance $(N, M, v)$ with two identical agents. Let $N = \{1, 2\}$. $M = [m + 2]$, where the first $m$ items are goods and the last two are chores. The valuation function $v$ is defined as follows, where $\beta = 1/4$.

$$v_j = \begin{cases} e_j, & \forall j \in [m] \\ -(\sum e_i/2) + \beta, & j \in \{m + 1, m + 2\} \end{cases}.$$

That is, the goods correspond to PARTITION elements, and have the same value as the weight of the element, and the chores are $\beta$ more than the negated weight of each set in an equal distribution of the elements. Note that, (a) the trivial partition where all items are in the same bundle has the smaller bundle valued zero, and (b) the average of values of all items is $\beta$, and MMS cannot be higher than the average (Lemma 2.2). Hence, $0 \leq \text{MMS} \leq \beta$.

We prove the correctness of the reduction in the following two claims.

**Claim 4.1.** PARTITION has a solution $\Rightarrow$ MMS $\geq \beta$.

**Proof.** Divide the goods into two bundles as per the PARTITION solution, and add one chore to each set. This gives us two bundles of equal value $\beta$, implying that $\text{MMS} \geq \beta$. \qed

**Claim 4.2.** MMS > 0 $\Rightarrow$ PARTITION has a solution.

**Proof.** We prove the contrapositive by contradiction. Suppose PARTITION does not have a solution, but MMS > 0 for the instance $(N, M, v)$. Let $A' = (A_1, A_2)$ be the allocation achieving the MMS value, and let $u_1 = v(A_1)$ and $u_2 = v(A_2)$. Then we have $u_1, u_2 > 0$.

First we prove that both the chores cannot be in the same bundle. If they are, and if all goods are not in this bundle, then the value of the bundle with chores is at most the sum of all except the smallest good. This is $(-\sum e_i + 2\beta) + (\sum e_i - \min e_i) \leq 1/4 - 1 < 0$. If every good and chore is
in the same bundle, the value of the other bundle is 0. But \( v_1 > 0 \), hence the chores are in separate bundles.

But then the value of the goods in each bundle is at least the total value minus the chore’s value, i.e., for \( i = 1, 2 \), \( v(A_i \cap \mathcal{M}^+) = u_i - \left(-\frac{1}{2} \sum_{e_i \in [m]} e_i + \beta \right) \geq \text{MMS} - \beta + \frac{1}{2} \sum_{i} e_i > \frac{1}{2} \sum_i e_i - \beta. \)

Since \( \beta = 1/4 \) while \( v(A_i \cap \mathcal{M}^+) \) and \( \frac{1}{2} \sum_i e_i \) are integers, it follows that \( v(A_i \cap \mathcal{M}^+) \geq \frac{1}{2} \sum_i e_i. \) Then partition \( (A_1 \cap \mathcal{M}^+, A_2 \cap \mathcal{M}^+) \) of \( E = (e_1, \ldots, e_m) \) is a solution of the PARTITION problem, a contradiction. \( \square \)

Claims 4.1 and 4.2 show that \( \text{MMS} > 0 \iff \) there is a solution to PARTITION.

**Remark 4.1.** When \( v(\mathcal{M}) < 0 \), a similar reduction shows the \( \alpha \)-MMS problem is NP-Hard for \( \alpha \in (1/2, 1] \). We swap the signs of all the partition elements, and add goods with value \( \sum_j e_j/2 - \beta \). Then we show that \( \text{MMS} = -\beta \) if PARTITION has a solution, \( \text{MMS} = -2\beta \) otherwise. Hence, a better than 2-approximate MMS allocation would be able to distinguish between the two cases.

When agents are identical, they agree on every item if it is a good or a chore, and therefore \( \mathcal{M}^{gc} = \emptyset \). Therefore, \( v_i^+ \) and \( v_i^- \) as defined in Definition 2.5 are same as \( v(\mathcal{M}^+) \) and \( |v(\mathcal{M}^-)| \) respectively. \( \square \)

**Theorem 4.2.** Given a fixed constant \( \tau > 0 \), even if instance \( (\mathcal{N}, \mathcal{M}, v) \) with identical agents satisfies \( v(\mathcal{M}^+) \geq (1 + \tau)|v(\mathcal{M}^-)| \), checking if \( \text{MMS} > 0 \) is NP-hard.

**Proof.** Again, we give a reduction from PARTITION. Let \( E = \{e_1, e_2, \ldots, e_m\} \) be the set of elements given as input for PARTITION. Create an instance \( (\mathcal{N}, \mathcal{M}, v) \) as follows: \( \mathcal{N} \) has \( n \) agents, where \( n \) will be fixed later based on the value of \( \tau \). \( \mathcal{M} = \{1, 2, \ldots, m+n\} \) where the first \( m+(n-2) \) items are goods, and the last 2 are chores. The valuation function \( v \) is defined as follows, where \( \beta = 1/4. \)

\[
v_j = \begin{cases} 
    e_j & \forall j \in [m] \\
    \beta & \text{for } j \in \{m+1, \ldots, m+(n-2)\} \\
    - \sum_{i \in [m]} e_i/2 + \beta & \text{for } j \in \{m+n-1, m+n\}.
\end{cases}
\]

That is, the first \( m \) goods have values equal to the weights of the corresponding elements of PARTITION. The remaining \( (n-2) \) goods have value \( \beta \) each, and both the chores have value \( -(\sum_{i} e_i/2) + \beta \). Fix \( n \) to satisfy \( v(\mathcal{M}^+) \geq (1 + \tau)|v(\mathcal{M}^-)| \) that is, \( ((n-2)\beta + \sum_{i} e_i) \geq (1 + \tau)(\sum_{i} e_i + 2\beta). \)

We again have \( 0 \leq \text{MMS} \leq \beta \). The lower bound because \( v(\mathcal{M}) > 0 \) and Lemma 2.1, and the upper bound because the average \( v(\mathcal{M})/n \) is \( \beta \) and Lemma 2.2. The correctness is argued in the next two claims.

**Claim 4.3.** PARTITION has a solution \( \Rightarrow \text{MMS} \geq \beta. \)

**Proof.** Divide the first \( m \) goods as per the division of the elements of PARTITION into equal valued sets, and add one chore to each bundle. From the remaining goods \( \{m+1, \ldots, m+(n-2)\} \) give one each to the remaining \( (n-2) \) bundles. The value of every bundle created is \( \beta \). Hence, \( \text{MMS} \geq \beta. \) \( \square \)

**Claim 4.4.** \( \text{MMS} > 0 \Rightarrow \) PARTITION has a solution.
Proof. We prove the contrapositive of the statement, by contradiction. Suppose PARTITION instance $E = e_1, \ldots, e_m$ does not have a solution, but $\text{MMS} < 0$ for $(N, M, v)$.

Given that there are exactly two chores, at least $(n - 2)$ bundles have only goods and has to have at least one good. Furthermore, since $e_i$s are positive integers and $\beta = 1/4$, each of these $(n - 2)$ bundles have value at least $\beta$. Now, $\beta$ being the upper bound on the MMS value, wlog we can assume that these $(n - 2)$ bundles have exactly one good of the minimum value, namely $\beta$. This exhaust the goods $\{m + 1, \ldots, m + (n - 2)\}$ with value $\beta$. Therefore, the two chores and all goods corresponding to the PARTITION problem elements, and no other good, are in the remaining two bundles. Let these be the first two bundles $A_1$ and $A_2$.

Now by the same argument as in the proof of Claim 4.2, we can show that both $A_1$ and $A_2$ have positive value only if each contains exactly one chore and the total value of goods in each, namely $v(A_i \cap E)$ for $i = 1, 2$, is at least $\frac{1}{2} \sum_{i \in [m]} e_i$. Thus, $(A_1 \cap E, A_2 \cap E)$ is a solution to the PARTITION instance $E$, a contradiction.

Claims 4.3 and 4.4 show that $\text{MMS} > 0$ for $(N, M, v) \iff$ there is a solution to PARTITION. $\Box$

Remark 4.2. If we change the relation of valuation of goods and chores to $|v(M^-)| \geq (1 + \tau)v(M^+)$, the $\alpha$-MMS problem is NP-hard for $\alpha \in (1/2, 1]$. A similar reduction as for Theorem 4.2 shows this. The modifications are we swap the signs of the PARTITION elements and corresponding items, add two goods of value $\sum j e_j/2 - \beta$, and $(n - 2)$ chores of value $-\beta$. We show $\text{MMS} = -\beta$ if PARTITION has a solution, and $-2\beta$ otherwise.

Theorems 4.1 and 4.2 show that even if we know that $\text{MMS} \geq 0$ checking if it is strictly positive is NP-hard. Since for $\alpha \in (0, 1]$, $\text{MMS} > 0 \iff \alpha \text{MMS} > 0$, this essentially means, we can not find an $\alpha$-MMS allocation for any value of $\alpha \in (0, 1]$ if either of the two conditions in $\alpha$-MMS problem is dropped. The next theorem formalizes this.

Theorem 2.1. Given an instance $(N, M, v)$ with identical agents and $v(M) > 0$ such that one of the following two holds: either $|N| = 2$ or $v(M^+) \geq (1 + \tau)v(M^-)$ for a constant $\tau$. Finding $\alpha$-MMS allocation of $(N, M, v)$ for any $\alpha \in (0, 1]$ is NP-hard.

Even though an instance with identical agents is guaranteed to have an allocation where every agent gets at least the MMS value, i.e., 1-MMS allocation exists, Theorem 2.1 ruling out efficient algorithm for finding $\alpha$-MMS allocation any $\alpha \in (0, 1]$ is very striking. In light of this result, it is evident that even getting a PTAS, in other words finding $(1 - \epsilon)$-MMS allocation, in case of identical agents is non-trivial and important. In the next section, we resolve this for the $\alpha$-MMS problem.

5 PTAS for $\alpha$-MMS with Identical Agents

We discuss the $\alpha$-MMS problem for identical agents in this section. Note that an exact MMS allocation always exists for this case. Also, as agents are identical, they have the same MMS values. From Lemma 2.1, finding the sign of these values is easy. Hence, we describe two algorithms, one for each case, when MMS $\geq 0$ and otherwise. In this section, we describe a PTAS for the case with non-negative MMS values. The PTAS for the other case is similar, and is discussed in Appendix B.

Specifically, we describe an algorithm that finds a $(1 - \epsilon)$-approximate MMS allocation for agents with non-negative MMS values, for any constant $\epsilon > 0$. This algorithm will be used as a
subroutine in the $\alpha$-MMS problem with non-identical agents for finding the MMS value of every agent approximately within any constant factor.

Note that the $\alpha$-MMS problem is scale invariant. Therefore, without loss of generality, here on we assume $v(M) = n = |\mathcal{N}|$. Due to Lemma 2.2, this also implies $\text{MMS} \leq v(M)/n = 1$. Also, since valuations are identical $\mathcal{M}^{gc} = \emptyset$.

**Definition 5.1.** For an instance $(\mathcal{N}, \mathcal{M}, v)$ with identical agents and constant $\epsilon > 0$, let $\text{Big}$ be the set of items in $\mathcal{M}$ which have absolute value bigger than $\frac{\epsilon}{2}$, i.e.,

$$\text{Big} = \{ j \in \mathcal{M} : |v_j| \geq \frac{\epsilon}{2} \}.$$  

Let $\text{Big}^+$ and $\text{Big}^-$ be respectively the sets of big goods and big chores in $\mathcal{M}$, i.e., $\text{Big}^+ = \text{Big} \cap \mathcal{M}^+$, and $\text{Big}^- = \text{Big} \cap \mathcal{M}^−$. Let $\text{Small}$ be the set of small items, $\mathcal{M} \setminus \text{Big}$, and similarly define small goods ($\text{Small}^+$) and small chores ($\text{Small}^-$).

We abuse notation slightly and call items in the set $\text{Big}$ (or $\text{Small}$) as $\text{Big}$ (resp. $\text{Small}$) items. We use a bag filling algorithm discussed next in our final PTAS (Algorithm 4).

### 5.1 A Bag-Filling Algorithm

In this section we design a bag-filling algorithm that generalizes algorithms in [GHS+18, GMT18, GT20] to the mixed setting.

Our algorithm (Algorithm 3) takes as input an MMS instance $(\mathcal{N}, \mathcal{M}, v)$, and a partition of the $\text{Big}$ items of $\mathcal{M}$ into $n$ bags, denoted by $B^\pi = [B_1, B_2, \ldots, B_n]$ such that they satisfy one of the following two condition sets. And it outputs an allocation of items $A^\pi = [A_1, \ldots, A_n]$ where $v(A_i) \geq 1-\epsilon$, for all $i \in [n]$.

\begin{align*}
&v(\text{Small}) + \sum_{k=1}^{n} v(B_k) \geq n, \quad \text{and} \quad v(\text{Small}) + \sum_{k=1}^{n} v(B_k) \geq n(1 - \frac{\epsilon}{2}). \tag{11} \\
v(B_k) \leq 1 \quad \forall k \in [n]. \\
|v_j| < \epsilon \quad \forall j \in \text{Small}. &\quad \text{and} \quad v(B_k) \leq 1 - \frac{\epsilon}{2} \quad \forall k \in [n]. \tag{12} \\
|v_j| < \frac{\epsilon}{2} \quad \forall j \in \text{Small}. 
\end{align*}

Next we explain the algorithm: Algorithm 3 has $n - 1$ rounds. Each round starts with a bag/bundle from $B$. If the bag is valued at least $(1-\epsilon)$, then it is assigned to some agent. If not, first we add all unallocated $\text{Small}$ chores to this bag. Then one by one we add unallocated goods from $\text{Small}$ until it is valued at least $(1-\epsilon)$. This bag is then assigned to some agent, and the current round ends. After all rounds are done, in the last step, all remaining items from $\text{Small}$ are added to the bag $B_n$.

Next we prove the correctness of the algorithm.

**Lemma 5.1.** If an MMS instance with identical agents satisfies condition set (11) or (12), then Algorithm 3 gives a $(1-\epsilon)$-Allocation.

**Proof.** First, we prove the claim for condition set (11). By induction on $k \in [n - 1]$, we prove that the value of each assigned bundle after $k$ rounds is in $[1-\epsilon, 1]$. Assume the value of all bundles assigned to the first $k - 1$ agents are in this range. Now $v(B_j) \leq 1$ for all $j \in \{k, \ldots, n\}$. If $v(B_k) \geq 1-\epsilon$, we are done. If not, then while the value of $B_k$ is less than $1-\epsilon$, the value of
Algorithm 3: Bag-filling to find \((1 - \epsilon)\)-MMS Allocation for identical agents

**Input:** \((N', M, v)\), Partition of \(\text{Big} \subseteq M\): \(B^\pi = [B_1, B_2, \ldots, B_n]\). Input satisfies Condition set (11) or (12).

**Output:** \(A^\pi = \{A_1, \ldots, A_n\}\) such that \(v(A_i) \geq 1 - \epsilon, \forall i \in [n]\).

1. \(A^\pi \leftarrow \emptyset, \text{Small} \leftarrow M \setminus \text{Big}\)
2. for \(k \in \{1, \ldots, n - 1\}\) do
   3. while \(v(B_k) < 1 - \epsilon\) do
      4. \(j \leftarrow \text{argmin}_{j \in \text{Small}} v_j\)
      5. \(B_k \leftarrow B_k \cup \{j\}, \text{Small} \leftarrow \text{Small} \setminus \{j\}\)
      6. \(A^\pi \leftarrow A^\pi \cup \{B_k\}\)
3. \(B_n \leftarrow B_n \cup \text{Small}, A^\pi \leftarrow A^\pi \cup \{B_n\}\)
4. return \(A\)

the unallocated items from \(\text{Small}\) is at least the value of all items minus that of all the allocated bundles and unallocated bags of \(\text{Big}\) items. This can be bounded as,

\[
v(M) - \sum_{i<k} v(A_i) - v(B_k) - \sum_{i>k} v(B_i) > n - (k - 1) - (1 - \epsilon) - (n - k) > \epsilon.
\]

Hence, there is at least one unallocated \(\text{Small}\) good. Before adding the last \(\text{Small}\) good to \(A_k\), its value was strictly less than \(1 - \epsilon\). Adding the last item increases the value by at most \(\epsilon\). Hence, the value of \(A_k\) is at most 1. Thus, \(v(A_k) \in [1 - \epsilon, 1]\), for all \(k \in [n - 1]\). As the total value of all items is at least \(n\), and the total value of the \(n - 1\) assigned bundles is at most \(n - 1\), the last agent also gets a bundle of value at least 1.

Similarly, if the instance satisfies condition set (12), we can prove by induction that after \(k \in [n - 1]\) rounds, all assigned bundles have value in \([1 - \epsilon, 1 - \epsilon/2]\). In every round, while this is not true, the value of unallocated goods is at least \(\epsilon/2\). Thus, there is at least one \(\text{Small}\) good. Finally, after assigning \(n - 1\) bundles, the total value remaining is at least \(n(1-\epsilon/2) - (n-1)(1-\epsilon/2)\), hence the last bundle also has value at least \((1 - \epsilon)\). \(\square\)

5.2 The PTAS

In this section we design an algorithm, namely Algorithm 4, to find \((1 - \epsilon)\)-MMS allocation for instance \((N', M, v)\) when it satisfies conditions of Definition 2.5. We first discuss two crucial properties of MMS allocations that provide the main intuition behind the algorithm.

Suppose we find an allocation for a subset of agents \(N'\) where everyone in \(N'\) gets at least \((1 - \epsilon)\). Since the remaining agents are identical, an MMS allocation exists for the smaller instance with the remaining agents and the remaining items too. Now suppose the MMS value of the smaller instance was equal or higher than that of the original instance. Then a \((1 - \epsilon)\)-MMS allocation of the smaller instance, combined with the allocation of the agents in \(N'\), would be a \((1 - \epsilon)\)-MMS allocation overall.

We describe a sub-case of the problem where we can use this intuition. Denote partitions of \(\text{Big}\) items into \(n\) bundles by \(B^\pi = \{B_1, B_2 \cdots, B_n\}\). Classify the bundles based on their value, into
sets \( \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \) and \( \mathcal{B}_4 \) as follows.

\[
\begin{align*}
\mathcal{B}_1 & := \{ B_k \in B^\pi : v(B_k) > 1 \} \\
\mathcal{B}_2 & := \{ B_k \in B^\pi : 1 - \epsilon \leq v(B_k) \leq 1 \} \\
\mathcal{B}_3 & := \{ B_k \in B^\pi : 0 \leq v(B_k) < 1 - \epsilon \} \\
\mathcal{B}_4 & := \{ B_k \in B^\pi : v(B_k) < 0 \}
\end{align*}
\] (13)

Let \( A^\pi \) be some MMS allocation, and \( B^\pi = \{ B_1^*, B_2^*, \ldots, B_n^* \} \) be the allocation of Big items according to \( A^\pi \). We prove that if \( \mathcal{B}_4 \) is empty for \( B^\pi \), then by assigning the bundles \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) to the corresponding agents, the MMS values of the remaining agents for the remaining items does not decrease.

**Lemma 5.2.** If \( B^\pi \) has \(|\mathcal{B}_4| = 0 \) and all SMALL chores can be added to bundles in \( \mathcal{B}_1 \) without reducing their value to less than \( 1 - \epsilon \), then,

\[
\text{MMS}^n(M) \leq \text{MMS}^n - |\mathcal{B}_1| - |\mathcal{B}_2| \left( \bigcup_{B \in \mathcal{B}_3} B \cup \text{Small}^+ \right).
\]

**Proof.** Consider the allocation of Small items in \( A^\pi \). Allocate items in \( M' = \bigcup_{B \in \mathcal{B}_3} B \cup \text{Small}^+ \) among the set of agents who have received bundles in \( \mathcal{B}_3 \), say \( \mathcal{N}' \), as they are allocated in \( A^\pi \). Call this allocation \( A^{\pi'} \). Now the allocation \( A^\pi \) may also have some Small chores assigned to agents in \( \mathcal{N}' \), but no other goods. The lowest valued bundle in \( A^{\pi'} \) thus has value at least that of the lowest valued bundle in \( A^\pi \) (since no SMALL chore is added to these bundles in \( A^{\pi'} \)). The MMS value of agents in \( \mathcal{N}' \), when partitioning \( M' \) among them, is at least that of the lowest valued bundle of \( A^{\pi'} \), hence is at least \( \text{MMS}^n(M) \).

\( \square \)

Lemma 5.2 implies that if we are given \( B^\pi \) where \( \mathcal{B}_4 \) is empty and all SMALL chores can be exhausted by adding them to bundles in \( \mathcal{B}_1 \) while their value remains at least \( 1 - \epsilon \), we can allocate \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) to the corresponding agents, and reduce the problem to the smaller instance with the remaining agents and items. The lemma assumes we know \( B^\pi \). One way to get the right \( B^\pi \) is to enumerate all possible partitions of Big, and somehow optimally allocate Small items over these. But there could be wrong partitions that cannot apply the algorithm that allocates the Small items. The next property of MMS allocations helps to identify one such type of wrong partitions of Big.

**Lemma 5.3.** For a partition \( B^\pi \) of Big items, suppose sets \( \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4 \) defined in (13) satisfy the following three conditions:

(a) \( \mathcal{B}_1 \) is non-empty and \( \text{Small}^- = \emptyset \)

(b) \(|\mathcal{B}_4| > 0 \)

(c) \( v(\mathcal{B}_3 \cup \mathcal{B}_4 \cup \text{Small}^+) < (1 - \frac{\epsilon}{2})(|\mathcal{B}_3| + |\mathcal{B}_4|) \).

Then either \( B^\pi \) is not a partition of the Big items in any MMS allocation, or there exists another MMS where the allocation of big items has strictly smaller value for \(|\mathcal{B}_1| + |\mathcal{B}_4| \).

**Proof.** Let \( B^\pi \) be the partition of big items in an MMS allocation that has lowest \(|\mathcal{B}_1| + |\mathcal{B}_4| \). And to the contrary suppose \( B^\pi \) also satisfy all the three conditions of (a), (b) and (c). Then \( \mu^* = \text{MMS} < (1 - \frac{\epsilon}{2}) \), because the total valuation is \( v(\mathcal{B}_3 \cup \mathcal{B}_4 \cup \text{Small}^+) < (1 - \frac{\epsilon}{2})(|\mathcal{B}_3| + |\mathcal{B}_4|) \).
Take any bundle $B$ from $B_1$ that has $v(B) > 1$. Let $v(B) = (1 + x)$ for some $x > 0$. Take $B'$ from $B_4$ with $v(B') = -y$ for some $y > 0$. In an MMS allocation, $B'$ must be bundled with a set of Small goods of value at least $y + \mu^*$.

We make two new bundles using $B$, $B'$ and the Small goods added to $B'$ in the MMS allocation. First merge $B$ and $B'$ into one bundle (with value $1 + x - y$) and have no items in the other bundle.

If $1 + x - y < \mu^*$, start adding the Small goods one by one to a bundle that has a lower value at each time. Let $A_1$ and $A_2$ be the resulting bundles after adding the Small goods. Without loss of generality, let $v(A_1) \geq v(A_2)$. We have

$$v(A_1) + v(A_2) \geq 1 + x - y + \mu^* = 1 + x + \mu^* \text{ and } v(A_1) - v(A_2) \leq \frac{\epsilon}{2} \Rightarrow v(A_2) \geq \frac{1}{2}(1 + x + \mu^* - \frac{\epsilon}{2}) > \frac{1}{2}(1 + \mu^* - \frac{\epsilon}{2}) > \mu^* \Rightarrow v(A_1) > \mu^*.$$

The second inequality holds because each Small good has value at most $\frac{\epsilon}{2}$ and Small goods are always added to the lower valued bundle.

If $1 + x - y > \mu^*$, add all remaining Small goods to the empty bundle and now each bundle values at least $\mu^*$.

Depending on the value of $1 + x - y$, we removed one bundle from either $B_1$ and/or one bundle from $B_4$, without reducing the value of the lowest valued bundle in the allocation. Therefore, either $B^\pi$ is not the partition of Big items in an MMS allocation, or there exists another MMS allocation where the partition of big items is such that corresponding ($|B_1| + |B_4|$) is strictly smaller. A contradiction.

**Remark 5.1.** The statement and the proof of the Lemma 5.3 hold even if Small$^-$ was non-empty initially, but all the Small chores can be allocated so that the conditions hold after their allocation. Particularly, we add chores to bundles in $B_1$ while maintaining the value of bundles to be at least $1 - \epsilon$. If, all chores are exhausted but there are still bundles with value more than 1, we update the definitions in (13) and then we use the lemma.

Lemma 5.3 gives us the condition when a partition of Big can be safely discarded. Using this, the high-level idea for Algorithm 4 is as follows: Let the set of all partitions of Big into $n$ bundles be denoted by $\Pi_n(Big)$. For each allocation $B^\pi \in \Pi_n(Big)$, algorithm either ends up with an allocation $A^\pi$ of all the items, or discards $B^\pi$ because the conditions of Lemma 5.3 are satisfied. If we are sure that allocation $A^\pi$ will give every agent a bundle of value at least $(1 - \epsilon)$, then $A^\pi$ is returned as an output, otherwise it is stored in a list $A$ of allocations. In the end, we return the allocation with the highest maximin value from $A$. The former case automatically ensures a $(1 - \epsilon)$-MMS allocation, since $MMS \leq 1$. On the other hand, when $MMS << 1$, we show that the later case still manages to guarantee a $(1 - \epsilon)$-MMS valued bundle to everyone.

We are now ready to discuss the algorithm and its analysis formally.

**Algorithm description.** The main for loop in Algorithm 4 considers every partition $B^\pi$ of the Big items. For each partition $B^\pi$, it first classifies the bundles into the sets $B_1$, $B_2$, $B_3$, and $B_4$ as defined in (13).

If $B_1$ is not empty, then add the Small chores to each bag in $B_1$ until the bag has value at most 1. This process ends either when all bundles in $B_1$ now have value at most 1, or all Small chores are exhausted. In the first case, condition set (11) is satisfied and we run the bag-filling algorithm 3 (Line 10).

In the latter case, when all Small chores are exhausted but there are one or more bundles with value more than 1, consider the set $B_4$. If $|B_4| = 0$, then reduce to a simpler $\alpha$-MMS problem
Algorithm 4: \((1-\epsilon)-\text{MMS} \) Allocation for Identical Agents

**Input**: \((\mathcal{N}, \mathcal{M}, v) \) such that \(v(\mathcal{M}) = n, \epsilon \in [0, 1] \)

**Output**: \((1-\epsilon)-\text{MMS} \) Allocation

1. \(\mathcal{A} \leftarrow \emptyset. \quad \Pi_n(\text{Big}) \leftarrow \) all partitions of Big into \(n\) sets.
2. for \(B^\pi \in \Pi_n(\text{Big})\) do
   3. Define \(B_1, B_2, B_3, B_4\) as in (13).
   4. while \(B_1 \neq \emptyset\) and \(\text{Small}^- \neq \emptyset\) do
      5. \(j \leftarrow \) any unallocated chore, \(\text{Small}^- \leftarrow \text{Small}^- \setminus \{j\}\)
      6. for any \(B \in B_1 : B \leftarrow B \cup \{j\}\), \(\text{add a small chore to any bundle in } B_1\)
      7. if \(v(B) \leq 1\) then
         8. \(B_1 \leftarrow B_1 \setminus B, \quad B_2 \leftarrow B_2 \cup B\)
      9. if \(B_1 = \emptyset\) then
         10. \(A^\pi \leftarrow \text{Bag-Fill}((\mathcal{N}, \mathcal{M}, v), (B_i \in B_2 \cup B_3 \cup B_4))\)
         11. return \(A^\pi\)
   12. else if \(|B_4| > 0\) then
      13. \(A^\pi \leftarrow (B_1 \cup B_2), \quad \mathcal{N}' \leftarrow \text{agents who received } B_3 \cup B_1, \quad \mathcal{M}' \leftarrow B_3 \cup B_4 \cup \text{Small}^+\),
         \(n' \leftarrow |B_3| + |B_4|\)
      14. if \(v(\mathcal{M}) \geq n'(1-\frac{\epsilon}{2})\) then
         15. \(A^\pi \leftarrow A^\pi \cup \text{Bag-Fill}((\mathcal{N}', \mathcal{M}', v_i, (B_i)_{i \in \mathcal{N}'})\)
         16. return \(A^\pi\)
      17. else
         18. continue \(\quad \text{// conditions of Lemma 5.3 are met so discard } B^\pi.\)
   19. else
      20. \(\mathcal{N}' \leftarrow \text{agents who received } B_3 \cup B_1, \quad \mathcal{M}' \leftarrow \text{Small}^+, \quad \forall j \in \mathcal{M}', \quad v'_j = v_j.\)
      21. for \(B \in B_3\) do
         22. introduce a new good \(b\) with \(v'_b = v(B)\)
         23. \(\mathcal{M}' \leftarrow \mathcal{M}' \cup \{b\}\)
      24. \(A^\pi \leftarrow B_1 \cup B_2 \cup (1-\epsilon)-\text{MMS allocation for } (\mathcal{N}', \mathcal{M}', v')\) using the algorithm in [Woe97] \(\quad \text{// goods-only setting}\)
      25. \(\mathcal{A} \leftarrow \mathcal{A} \cup \{A^\pi\}\)
   26. return \(\arg \max_{A^\pi \in \mathcal{A}} \min_{A_i \in A^\pi} v(A_i)\)

\((\mathcal{N}', \mathcal{M}', v')\) in a goods only setting, defined as follows. \(\mathcal{N}'\) is the set of agents who received bundles from \(B_3\) or \(B_1\). The item set \(\mathcal{M}'\) contains all the \text{Small} goods with same value in \(v'\) as in \(v\), and for each bundle \(B \in B_3\) it has a new item \(b\) with value \(v'_b = v(B)\). Run the PTAS from [Woe97] on \((\mathcal{N}', \mathcal{M}', v')\) to find a \((1-\epsilon)-\text{MMS}\) allocation of these goods among the remaining \(n' = n - (|B_1| + |B_2|)\) agents, and return its output.

For the final case when \(B_1 \neq \emptyset\) and \(|B_4| > 0\), first check if the remaining unallocated goods and agents in \(B_3 \cup B_4\) fulfill the condition set 12. If they do, apply the bag-filling algorithm and return the \((1-\epsilon)-\text{MMS}\) allocation. If not, then conditions of Lemma 5.3 are satisfied so discard \(B^\pi\) and proceed to the next allocation of Big items. At the end of for loop, if no output allocation was returned and \(B^\pi\) was not discarded, then it stores the final allocation \(A^\pi\) in \(\mathcal{A}\). Finally, once done with the for loop it returns the best allocation from set \(\mathcal{A}\).
Analysis

Lemma 5.4. $|\text{Big}| = O(n/\epsilon)$.

\textbf{Proof.} As the sum of values of all items is $n$, we have $n = v(\mathcal{M}^+) - |v(\mathcal{M}^-)|$. Then, as $v(\mathcal{M}^+) \geq (1 + \tau)|v(\mathcal{M}^-)|$,

$$n \geq v(\mathcal{M}^+) - \frac{v(\mathcal{M}^+)}{1 + \tau} \implies v(\mathcal{M}^+) \leq \frac{n(1 + \tau)}{\tau}. \quad (14)$$

Finally, by the definition of $\text{Big}^+$ we have

$$|\text{Big}^+| \leq \frac{v(\mathcal{M}^+)}{\epsilon} \leq \frac{2n(1 + \tau)}{\epsilon \tau}.$$

Similarly, we have

$$n \geq (1 + \tau)|v(\mathcal{M}^-)| - |v(\mathcal{M}^-)| \implies |v(\mathcal{M}^-)| \leq n/\tau. \quad (15)$$

Thus, the number of $\text{Big}$ chores is bounded as $|\text{Big}^-| \leq \frac{2n}{\epsilon \tau}$. As $\tau$ is constant, $|\text{Big}^+|$ and $|\text{Big}^-|$ are $O(n/\epsilon)$. Hence, $|\text{Big}| = |\text{Big}^+ \cup \text{Big}^-| = O(n/\epsilon)$.

As $n$ and $\epsilon$ are also constant, and partitioning a constantly many items into constantly many sets take constant time, we get the following as a corollary of Lemma 5.4.

Corollary 5.1. The set of all possible allocations of $\text{Big}$ items, namely $\Pi_n(\text{Big})$, can found in constant time, and the main for loop of Algorithm 4 runs constantly many times.

Next we state and prove the main theorem of this section.

Theorem 5.1. Given an MMS instance $(N, \mathcal{M}, v)$ with identical agents, Algorithm 4 returns a $(1 - \epsilon)$-MMS allocation in $f(n, \frac{1}{\epsilon^2})O(\log m) + f'(n, \frac{1}{\epsilon})O(m)$ time.

\textbf{Proof.} First, we prove the correctness of the algorithm. Consider an iteration of the for loop for partition $B^\pi \in \Pi_n(\text{Big})$ of Big items, and corresponding sets $B_1, B_2, B_3$ and $B_4$ as defined in (13).

If both $B_1$ and $\text{Small}^-$ are non-empty, then the algorithm adds $\text{Small}$ chores one by one to any $B_1$ bundle, until the value of the bundle falls below 1. Note that as every $\text{Small}$ chore has absolute value at most $\epsilon/2$, upon adding the last chore before the value falls below 1, the value of every modified $B_1$ bundle is still at least $1 - \epsilon/2$, hence it is now a $B_2$ bundle.

If all $B_1$ bundles get modified to a $B_2$ bundle in this process, that is if $B_1$ is empty, then condition set (11) holds for $B_2 \cup B_3 \cup B_4$ bundles. Lemma 5.1 implies all agents get a $(1 - \epsilon)$ valued bundle which is a $(1 - \epsilon)$-MMS allocation.

When $B_1 \neq \emptyset$, and $\text{Small}^- = \emptyset$, the algorithm next checks if $B_4$ is empty or not. Suppose $B_4$ is not empty. If the condition set (12) holds for agents corresponding to bundles in $B_3$ and $B_4$ for the items not in $B_1$ and $B_2$, then Lemma 5.1 implies the bag-filling algorithm 3 gives them a $(1 - \epsilon)$ valued bundle. After allocating the $B_1$ and $B_2$ bundles to the respective agents and combining with the result of the bag-filling algorithm, we get a $(1 - \epsilon)$-MMS allocation.

If $B_4$ is not empty and the condition set (12) does not hold for the agents corresponding to $B_3 \cup B_4$, then all three conditions of Lemma 5.3 holds for (modified) $B^\pi$ (after adding $\text{Small}$ chores) implying that it is safe to discard $B^\pi$.

If $B_2$ is empty, Lemma 5.2 proves the Algorithm returns a $(1 - \epsilon)$-MMS allocation if the partition of $\text{Big}$ being examined is the optimal partition $B^\pi_*$. The set $A$ stores all the resulting allocations,
and hence it also contains the allocation corresponding to $B^{π^*}$. The allocation that gives the highest maximin value must correspond to $B^{π^*}$. Hence, Algorithm 4 returns a $(1 - \epsilon)$-MMS allocation here too.

For running time, note that every iteration of the for loop first allocates all Small chores, then either runs a bag-filling algorithm, which takes $O(m)$ time, discards the iteration, or runs the PTAS of [Woe97] which takes $O(2^{1/\epsilon^2} n \log m)$ time. In the worst case, every iteration takes $O(m + O(2^{1/\epsilon^2} n \log m))$ time. The for loop runs for $n^{[BIG]}$ iterations, which from Lemma 5.4 is $O(n^{\epsilon^2}) = 2^O(n\log n/\epsilon)$. Hence, the total run time of the algorithm is $O(2^{2\log n/\epsilon}(2^{1/\epsilon^2} n \log m + m)) = f(n, \frac{1}{\epsilon})O(\log m) + f'(n, \frac{1}{\epsilon})O(m)$ time.

6 Related Work

The first study of fair division was a cake cutting problem [Ste48]. Two of most well known notions of fairness are proportionality, where each agent gets a bundle which worth at least $1/n$ (of her value for all items, and envy-freeness, where each agent values her own bundle at least as much as other agents’. However, when items in $M$ are indivisible, proportionality and envy-freeness cannot always be attained. A simple example is allocating a single good between two agents; there is no allocation that is proportional or envy-free. This motivated the search for new notions of fairness for indivisible items.

MMS for Goods. Budish defined the maximin share (MMS) allocation for allocating indivisible items in a fair manner [Bud11]. In recent years, MMS has attracted a lot of interest [AMNS17, BKM17, BL16, FGH19, GMT18, GHS+18, KPW16, PW14, KPW18, GT20]. Bouveret and Lemaître [BL16] showed that in some restricted cases MMS allocation always exists. In a notable result, Procaccia and Wang [PW14] showed that MMS allocation might not always exist but 2/3-MMS allocation, where each agent receives at least 2/3 of her MMS value always exists. A series of works studied efficient computation of 2/3-MMS allocations for any $n$ [AMNS17, BKM17, GMT18]. Ghodsi et al. [GHS+18] showed that 3/4-MMS allocation always exists. Most recently, Garg and Taki [GT20] showed that $(3/4 + 1/(12n))$-MMS allocation always exists.

Finding MMS value is hard but a PTAS exists [Woe97]. This PTAS can be used to find $(3/4 + 1/(12n) - \epsilon)$-MMS allocation for $\epsilon > 0$ in polynomial time. Also, there is a strongly polynomial time algorithm to find 3/4-MMS allocation [GT20].

Constant number of agents For three agents, [AMNS17] showed that a 7/8-MMS allocation always exists. This factor was later improved to 8/9 in [GM19]. For four agents, [GHS+18] showed that a 4/5-MMS allocation always exist.

In all these works it has been assumed that items have non-negative valuations, i.e, $M$ consists of all goods. However, there has been impressive series of works for the case when $M$ is consist of all chores.

MMS for Chores. The problem of MMS allocation for chores was studied for the first time by Aziz et al [ARSW17], where they introduced an algorithm for 2-MMS allocation. We later introduce a general notion for $\alpha$-MMS allocation that is for mixed items where $\alpha < 1$. It is consistent for agents with positive as well as negative MMS value. Here, to keep the original results of the works, we state the approximation factor as the same as the definition of the original paper.
[BKM17] improved the previous result by showing an algorithm for 4/3-MMS allocation. Later, Huang and Lu [HL19] improved this result and showed an algorithm to obtain 11/9-MMS allocation. They also show a PTAS to find (11/9 + ε)-MMS allocation and a polynomial time algorithm to find a 5/4-MMS allocation.

Other variants of MMS. The MMS problem has been studied under various other models in the goods only setting like with asymmetric agents [FGH+19], group fairness [BBKN18, CKMS20], beyond additive valuations [BKM17, GHS+18, LV18], in matroids [GM19], with additional constraints [GM19, BB18], for agents with externalities [BMR+, AEG+13], with graph constraints [BILS19, LT19], and with strategic agents [BGJ+19]. In the chores only setting too, weighted MMS [ACL19], and asymmetric agents [ACL] notions have been investigated.

Fairness for Mixed Items Finding fair allocations of mixed items has recently caught a lot of attention for both divisible [BMSY17, BMSY19] and indivisible [ACIW19, AW19, AW20, Ale20, GM20], items. However, to the best of our knowledge, ours is the first study on MMS allocations for mixed items.

Other Fairness Notions There are other notions of fairness, beside MMS, for indivisible items. Here, we introduced the most notable work on these notions for good only setting.

EF1. This notion was first introduced by [Bud11] which is a relaxation of envy-free allocation. An allocation is EF1 if for any two agents i₁ and i₂, agent i₁ prefers (or equally likes) her own bundle to agent i₂’s bundle after removing some item from the bundle of agent i₂. An EF1 allocation can be found efficiently using envy cycle removal procedure introduced by [LMMS].

EFX. Envy free up to any good (EFX) is a stronger relaxation of envy-free allocation which was first introduced by [CKM+16]. An allocation is EFX if for any two agents i₁ and i₂, agent i₁ prefers (or equally likes) her own bundle to agent i₂’s bundle after removing any item from the bundle of agent i₂. Plant and Roughgarden [PR] showed the existence of EFX allocation for two restricted cases 1) when n = 2 and 2) when agents are identical. Until recently, the existence of an EFX allocation was unknown even for 3 agents. Chaudhury et al. [CGM20a], showed that for three agents EFX allocation always exist. The partial EFX allocation and approximate EFX allocation has also been vastly studied [CGH, CKMS20].

NSW. Nash Social Welfare (NSW) is the geometric mean of the valuation of the agents. The NSW problem is to find an allocation of indivisible items that maximizes NSW. Even though this notion is not applicable for mixed-manna, it is a popular notion of fairness in the goods only setting. The problem is APX-Hard [MG20], and remarkable approximation results for the linear valuations case have been proven by a connection of the problem with markets [CG, CDG+, BKV18, CCG+18] or real stable polynomials [AGSS17]. The best known result is a 1.45 approximation factor [BKV18]. Similar results are known, again by exploiting the market connection, for popular valuation functions like budget-additive [GHM19], separable piece-wise linear concave (SPLC) [AMGV], and their combination [CCG+18]. Recent results give an O(n) approximation when agents have subadditive valuations, a far more general class than all the earlier ones [BBKS20, CGM20b]. Recent work has also been done on the general version of the problem with asymmetric agents, where the aim is to maximize the weighted geometric mean, for given weights, and submodular utilities [GKK20].
7 Discussion

For the OPT-$\alpha$-MMS problem, we obtain a PTAS for mixed manna when the number of agents is a constant, and for every agent her total value for goods differ significantly from her total absolute value for chores. As a corollary we get a PTAS for goods only manna with constant number of agents. It would be interesting to see if the tools we develop can be used to get PTAS to solve OPT-$\alpha$-MMS for the good manna with non-constant number of agents. Similarly, for bad manna.

A second immediate direction is to study what happens when the valuation functions are more general than additive. For instance, the MMS problem for a mixed manna with submodular valuation functions is interesting. Finally, connecting market equilibria to MMS is a new relation. Like for the Nash social welfare problem, this relation could be exploited to solve other questions.

Acknowledgments. We would like to thank Prof. Jugal Garg for several valuable discussions.

A PTAS for $\alpha$-MMS of identical agents with negative MMS value

In this section we introduce an Algorithm to find $(1 - \epsilon)$-MMS allocation for $\epsilon > 0$ and identical agents when $\text{MMS} < 0$ (Algorithm 5). It is shown in Lemma 2.1 that it is easy to check whether $\text{MMS} < 0$. From Definition 2.3, a $(1 - \epsilon)$-MMS allocation gives each agent a bundle with value at least $(1/(1 - \epsilon))\text{MMS}_i$. First we scale the valuations so that $v(M) = -n$. Same as in Algorithm 4, we then enumerate over all possible allocations of big items. While there are unallocated Small goods, we add them one by one to the bundle with the least value. Once all Small goods are exhausted, we iteratively add Small chore to the bundle with the highest value.

Algorithm 5: $(1 - \epsilon)$-MMS Allocation for identical agents with negative MMS value

1. $\sigma \leftarrow \frac{1}{1 - \epsilon} - 1$
2. Normalize the valuations so that $v(M) = -n$.
3. $X \leftarrow M, A \leftarrow \emptyset$
4. $\text{Big} := \{j \in X : |v_j| \geq \sigma\}, \text{Small} := M \setminus \text{Big}, \text{Small}^+ = \text{Small} \cap M^+, \text{Small}^- = \text{Small} \cap M^-$
5. for $B \in \Pi_n(\text{Big})$ do
6.     while $\text{Small}^+ \neq \emptyset$ do
7.         for any Small good $j$: $X \leftarrow X \setminus \{j\}, B \leftarrow \arg\min_{B_k \in B} v(B_k)$
8.         $B \leftarrow B \cup \{j\}$ // add a small good to the bundle with lowest value
9.     while $X \neq \emptyset$ do
10.        for any Small chore $j$, $X \leftarrow X \setminus \{j\}, B \leftarrow \arg\max_{B_k \in B} v(B_k)$
11.        $B \leftarrow B \cup \{j\}$ // add a small chore to the bundle with highest value
12.    $A \leftarrow A \cup \{B\}$
13. return $\arg\max_{A \in A} \min_{A_i \in A} v(A_i)$
Remark A.1. Since $v(\mathcal{M}) = -n$, we can conclude that $\text{MMS}_i \leq -1$.

Theorem A.1. Algorithm 5 gives $(1 - \epsilon)$-MMS allocation.

Proof. We prove this lemma by contradiction. First, Let $B^\pi*$ be a partition of big items in an MMS allocation. We have

$$v(\text{Small}^+) \geq \sum_{B \in B^* : v(B) < \text{MMS}_i} \text{MMS}_i - v(B^\pi*) .$$

Now, let $A^\pi = \{A_1, \ldots, A_n\}$ be the output of Algorithm 5. If there exist $A_k \in A$ such that $v(A_k) < (1 + \sigma)\text{MMS}_i$. Consider each $A_k \supset B_k$ with $v(B_k) < \text{MMS}_i$. Before adding the last $\text{Small}$ good, the value of this bundle is less than $(1 + \sigma)\text{MMS}_i$ (because the algorithm adds goods to the bundle with least value). The last good added to these bundles value at most $\sigma$. Therefore, all such $A_k$'s value at most $\text{MMS}_i$. From (17) and the fact that algorithm adds goods to the least valued bundle, we have

$$v(\text{Small}^+ \setminus (\bigcup_{k \in [n]} A_k)) \geq \sum_{B \in B^* : v(B) < \text{MMS}_i} \text{MMS}_i - v(B) + \sigma = \sigma$$

which is a contradiction.

Now we prove that after adding the $\text{Small}$ chores the value of all bundles remains at least $(1 + \sigma)\text{MMS}_i$. This is true because while there exists an unallocated chore, the value of the most valued bundle is greater than $-1$ (because $v(\mathcal{M}) = -n$). Adding a chore to such bundle will decrease the value by at most $\sigma$. Therefore, the value of such bundle is at least $-(1 + \sigma) \geq (1 + \sigma)\text{MMS}_i$. By definition of $\sigma, (1 + \sigma) = 1/(1 - \epsilon)$. \hfill \qed

B  Missing proofs

We will use the following in the proof of Lemma B.1. The ratio of the valuation of an agent for a good to its price is called its bang-per-buck value for the agent. From [BGHM17], in a market equilibrium, every buyer only buys goods that give her the highest bang-per-buck value among all goods, that is belong in what is called her maximum bang-per-buck (MBB) set. Denote the MBB value of buyer $i$ by $\lambda_i$.

Lemma B.1. All cycles in the market equilibrium allocation graph defined in Section 3.2 can be eliminated efficiently to get a new equilibrium whose allocation graph is acyclic.

Proof. To prove this lemma, assume the allocation graph is not acyclic. We first show how to eliminate one cycle in the graph, so that the resulting allocation is also a market equilibrium. We first show how to eliminate one cycle in the graph, so that the resulting allocation is also a market equilibrium.

Denote the cycle by $[i_1, j_1]_{i \in [k]} = \{i_1, j_1, i_2, j_2, \ldots, i_k, j_k, i_1\}$, where $i_1$ are buyers and $j_l$ are goods. That is, buyer $i_1$ has non zero amount of goods $j_1$ and $j_k$, and every other buyer $i_l, l \in [k]\{1\}$ has non zero amount of goods $j_{l-1}$ and $j_l$. Add weights to this cycle graph, where weight of edge $(i, j)$ is the amount of money spent by buyer $i$ on good $j$, denoted by $w_{ij}$. Then consider the edge $(i, j)$ that has the minimum weight, i.e., $(i, j) = \text{argmin}_{i,j \in [k]} w_{ij}$. Without loss of generality, suppose this edge is $(i_1, j_1)$, and the minimum weight is $w$.

Change the weights of the edges as follows. For buyer $i_1$, reduce the weight of edge $(i_1, j_1)$ to 0, and increase weight of $(i_1, j_k)$ by $w$. For every other buyer $i_a$, increase the weight of edge $(i_a, j_{a-1})$
by \( w \), and decrease the weight of \((i_a, j_a)\) by \( w \). That is, on alternate edges of the cycle, we increase and decrease weights by \( w \).

Let the money spent by each buyer on the goods along the cycle be changed to the edge weight of this new graph. As every good and buyer has \( w \) amount reduced on one neighbor and the same increased on the other, the money spent on every good, and that spent by every buyer, is the same. Also change the allocations of goods so that buyers receive the fraction of goods worth the (changed) money they spend on them. Note that agent \( i_1 \) now spends no money on \( j_1 \), hence the allocation graph does not have this edge, nor the cycle \([i_t, j_t]_{t \in [k]}\).

Now for each buyer, her neighbor goods in the cycle are in her MBB set. Hence, for each buyer \( i_a \), her change in total valuation is the added value from \( j_{a-1} \) and reduced value from \( j_a \). This is given by \( \lambda_{i_a} w - \lambda_{i_a} w = 0 \), as both \( j_{a-1} \) and \( j_a \) are in her MBB set. Hence, the total valuation of every buyer remains the same.

In summary, for the same set of prices and the new allocation after removing one cycle, the total valuation of every buyer, the money spent by every buyer and the money spent on every good remains the same in the new allocation. Also, every buyer still buys goods only from her MBB set. Hence, the new allocation is an equilibrium allocation for the same set of prices.

We can similarly eliminate all cycles in the allocation graph. After every cycle elimination, we remove one edge. Hence, in at most \( O(mn) \) time, we can make the graph acyclic.

\[ \square \]

References

[AAWY98] Noga Alon, Yossi Azar, Gerhard J Woeginger, and Tal Yadid. Approximation schemes for scheduling on parallel machines. *Journal of Scheduling*, 1(1):55–66, 1998.

[ACIW19] Haris Aziz, Ioannis Caragiannis, Ayumi Igarashi, and Toby Walsh. Fair allocation of indivisible goods and chores. In *Proceedings of the 28th International Joint Conference on Artificial Intelligence (IJCAI)*, pages 53–59, 2019.

[ACL] Haris Aziz, Hau Chan, and Bo Li. Maxmin share fair allocation of indivisible chores to asymmetric agents. In *Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems*, pages 1787–1789.

[ACL19] Haris Aziz, Hau Chan, and Bo Li. Weighted maxmin fair share allocation of indivisible chores. *arXiv preprint arXiv:1906.07602*, 2019.

[AEG+13] Nima Anari, Shayan Ehsani, Mohammad Ghodsi, Nima Haghpanah, Nicole Immorlica, Hamid Mahini, and Vahab S. Mirrokni. Equilibrium pricing with positive externalities. *Theor. Comput. Sci.*, 476:1–15, 2013.

[AGSS17] Nima Anari, Shayan Oveis Gharan, Amin Saberi, and Mohit Singh. Nash Social Welfare, Matrix Permanent, and Stable Polynomials. In *8th Innovations in Theoretical Computer Science Conference (ITCS)*, pages 1–12, 2017.

[Ale20] Martin Aleksandrov. Jealousy-freeness and other common properties in fair division of mixed manna. *arXiv preprint arXiv:2004.11469*, 2020.

[AMGV] Nima Anari, Tung Mai, Shayan Oveis Gharan, and Vijay V. Vazirani. Nash social welfare for indivisible items under separable, piecewise-linear concave utilities. In *Pro-
ceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2018, pages 2274–2290.

[AMNS17] Georgios Amanatidis, Evangelos Markakis, Afshin Nikzad, and Amin Saberi. Approximation algorithms for computing maximin share allocations. *ACM Transactions on Algorithms (TALG)*, 13(4):52, 2017.

[ARSW17] Haris Aziz, Gerhard Rauchecker, Guido Schryen, and Toby Walsh. Algorithms for max-min share fair allocation of indivisible chores. In *Thirty-First AAAI Conference on Artificial Intelligence*, 2017.

[AW19] Martin Aleksandrov and Toby Walsh. Greedy algorithms for fair division of mixed manna. *CoRR*, abs/1911.11005, 2019.

[AW20] Martin Aleksandrov and Toby Walsh. Two algorithms for additive and fair division of mixed manna, 2020.

[BB18] Arpita Biswas and Siddharth Barman. Fair division under cardinality constraints. In *IJCAI*, pages 91–97, 2018.

[BBKN18] Siddharth Barman, Arpita Biswas, Sanath Kumar Krishnamurthy, and Yadati Narahari. Groupwise maximin fair allocation of indivisible goods. In *Thirty-Second AAAI Conference on Artificial Intelligence*, 2018.

[BBKS20] Siddharth Barman, Umang Bhaskar, Anand Krishna, and Ranjani G Sundaram. Tight approximation algorithms for p-mean welfare under subadditive valuations. *arXiv preprint arXiv:2005.07370*, 2020.

[BGHM17] Xiaohui Bei, Jugal Garg, Martin Hoefer, and Kurt Mehlhorn. Earning limits in fisher markets with spending-constraint utilities. In *International Symposium on Algorithmic Game Theory*, pages 67–79. Springer, 2017.

[BGJ+19] Siddharth Barman, Ganesh Ghalme, Shweta Jain, Pooja Kulkarni, and Shivika Narang. Fair division of indivisible goods among strategic agents. In *Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems*, pages 1811–1813, 2019.

[BILS19] Xiaohui Bei, Ayumi Igarashi, Xinhang Lu, and Warut Suksompong. Connected fair allocation of indivisible goods. *arXiv:1908.05433*, 2019.

[BKM17] Siddharth Barman and Sanath Kumar Krishna Murthy. Approximation algorithms for maximin fair division. In *Proceedings of the 2017 ACM Conference on Economics and Computation*, pages 647–664. ACM, 2017.

[BKV18] Siddharth Barman, Sanath Kumar Krishnamurthy, and Rohit Vaish. Finding fair and efficient allocations. In *Proceedings of the 2018 ACM Conference on Economics and Computation, Ithaca, NY, USA, June 18-22, 2018*, pages 557–574. ACM, 2018.

[BL16] Sylvain Bouveret and Michel Lemaître. Characterizing conflicts in fair division of indivisible goods using a scale of criteria. *Autonomous Agents and Multi-Agent Systems*, 30(2):259–290, 2016.
[BMR+] Simina Brânzei, Tomasz P. Michalak, Talal Rahwan, Kate Larson, and Nicholas R. Jennings. Matchings with externalities and attitudes. In *International conference on Autonomous Agents and Multi-Agent Systems, AAMAS ’13*.

[BMSY17] Anna Bogomolnaia, Herve Moulin, Fedor Sandomirskiy, and Elena Yanovskaya. Competitive division of a mixed manna. *volume 85*, pages 1847–1871. Wiley Online Library, 2017.

[BMSY19] Anna Bogomolnaia, Hervé Moulin, Fedor Sandomirskiy, and Elena Yanovskaia. Dividing bads under additive utilities. *volume 52*, pages 395–417, 2019.

[BT96] Steven J Brams and Alan D Taylor. *Fair Division: From cake-cutting to dispute resolution*. Cambridge University Press, 1996.

[Bud11] Eric Budish. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy*, 119(6):1061–1103, 2011.

[CCG+] Yun Kuen Cheung, Bhaskar Chaudhuri, Jugal Garg, Naveen Garg, Martin Hoefer, and Kurt Mehlhorn. On fair division of indivisible items. In *FSTTCS*, 2018.

[CDG+] Richard Cole, Nikhil R. Devanur, Vasilis Gkatzelis, Kamal Jain, Tung Mai, Vijay V. Vazirani, and Sadra Yazdanbod. Convex program duality, fisher markets, and nash social welfare. In *Proceedings of the 2017 ACM Conference on Economics and Computation, EC ’17*.

[CG] Richard Cole and Vasilis Gkatzelis. Approximating the nash social welfare with indivisible items. In *Proceedings of the Forty-Seventh Annual ACM on Symposium on Theory of Computing, STOC 2015*.

[CGH] Ioannis Caragiannis, Nick Gravin, and Xin Huang. Envy-freeness up to any item with high nash welfare: The virtue of donating items. In *Proceedings of the 2019 ACM Conference on Economics and Computation, EC 2019*.

[CGM20a] Bhaskar Ray Chaudhury, Jugal Garg, and Kurt Mehlhorn. Efχ exists for three agents. In *Proceedings of the 21st ACM Conference on Economics and Computation, EC å¿20*, page 1â§19, New York, NY, USA, 2020. Association for Computing Machinery.

[CGM20b] Bhaskar Ray Chaudhury, Jugal Garg, and Ruta Mehta. Fair and efficient allocations under subadditive valuations. *arXiv preprint arXiv:2005.06511*, 2020.

[CKM+16] Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel D. Procaccia, Nisarg Shah, and Junxīng Wang. The unreasonable fairness of maximum nash welfare. In *Proceedings of the 2016 ACM Conference on Economics and Computation, EC ’16, Maastricht, The Netherlands, July 24-28, 2016*, pages 305–322. ACM, 2016.

[CKMS20] Bhaskar Ray Chaudhury, Telikepalli Kavitha, Kurt Mehlhorn, and Alkmini Sgouritsa. A little charity guarantees almost envy-freeness. In *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2658–2672. SIAM, 2020.

[EPT07] Raul Etkin, Abhay Parekh, and David Tse. Spectrum sharing for unlicensed bands. *IEEE Journal on selected areas in communications*, 25(3):517–528, 2007.
Alireza Farhadi, Mohammad Ghodsi, Mohammad Taghi Hajiaghayi, Sébastien Lahaie, David M. Pennock, Masoud Seddighin, Saeed Seddighin, and Hadi Yami. Fair allocation of indivisible goods to asymmetric agents. *J. Artif. Intell. Res.*, 64:1–20, 2019.

Jugal Garg, Martin Hoefer, and Kurt Mehlhorn. Approximating the Nash Social Welfare with budget-additive valuations. arxiv:1707.04428; Preliminary version appeared in the proceedings of SODA 2018, 2019.

Mohammad Ghodsi, Mohammadtaghi Hajiaghayi, Masoud Seddighin, Saeed Seddighin, and Hadi Yami. Fair allocation of indivisible goods: Improvements and generalizations. In *Proceedings of the 2018 ACM Conference on Economics and Computation*, EC âĂŹ18, 2018.

Michael R. Garey and David S. Johnson. *Computers and Intractability; A Guide to the Theory of NP-Completeness*. W. H. Freeman and Co., USA, 1990.

Jugal Garg, Pooja Kulkarni, and Rucha Kulkarni. Approximating nash social welfare under submodular valuations through (un) matchings. In *Proceedings of the fourteenth annual ACM-SIAM symposium on discrete algorithms*, pages 2673–2687. SIAM, 2020.

Laurent Gourvès and Jérôme Monnot. On maximin share allocations in matroids. *Theor. Comput. Sci.*, 754:50–64, 2019.

Jugal Garg and Peter McGlaughlin. Computing competitive equilibria with mixed mana. In *Proceedings of the 19th International Conference on Autonomous Agents and MultiAgent Systems*, pages 420–428, 2020.

Jugal Garg, Peter McGlaughlin, and Setareh Taki. Approximating maximin share allocations. In 2nd Symposium on Simplicity in Algorithms (SOSA 2019). Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018.

Jugal Garg and Setareh Taki. An improved approximation algorithm for maximin shares. In *Proceedings of the 21st ACM Conference on Economics and Computation*, EC âĂŹ20, page 379âĂŞ380, 2020.

Xin Huang and Pinyan Lu. An algorithmic framework for approximating maximin share allocation of chores. *CoRR*, abs/1907.04505, 2019.

David Kurokawa, Ariel D Procaccia, and Junxing Wang. When can the maximin share guarantee be guaranteed? In *AAAI*, volume 16, pages 523–529, 2016.

David Kurokawa, Ariel D. Procaccia, and Junxing Wang. Fair enough: Guaranteeing approximate maximin shares. *J. ACM*, 65(2):8:1–8:27, 2018.

Richard J. Lipton, Evangelos Markakis, Elchanan Mossel, and Amin Saberi. On approximately fair allocations of indivisible goods. In *Proceedings 5th ACM Conference on Electronic Commerce (EC-2004).*

Zbigniew Lonc and Mirosław Truszczynski. Maximin share allocations on cycles. arXiv:1905.03038, 2019.

Zhentao Li and Adrian Vetta. The fair division of hereditary set systems. In *International Conference on Web and Internet Economics*, pages 297–311. Springer, 2018.
[MG20] Peter McGlaughlin and Jugal Garg. Improving nash social welfare approximations. volume 68, pages 225–245, 2020.

[Mou04] Hervé Moulin. Fair division and collective welfare. MIT press, 2004.

[PR] Benjamin Plaut and Tim Roughgarden. Almost envy-freeness with general valuations. In Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2018.

[PW14] Ariel D Procaccia and Junxing Wang. Fair enough: Guaranteeing approximate maximin shares. In Proceedings of the fifteenth ACM conference on Economics and computation, pages 675–692. ACM, 2014.

[Ste48] Hugo Steinhaus. The problem of fair division. Econometrica, 16:101–104, 1948.

[VB02] Thomas Vossen and Michael Ball. Fair allocation concepts in air traffic management. PhD thesis, PhD thesis, Supervisor: MO Ball, University of Martyland, College Park, Md, 2002.

[Woe97] Gerhard J Woeginger. A polynomial-time approximation scheme for maximizing the minimum machine completion time. Operations Research Letters, 20(4):149–154, 1997.