Matrix Models, Quantum Penner Action and Two-Dimensional String Theory

Camillo Imbimbo

INFN, Sezione di Genova
I-16146 Genova, Italy

Sunil Mukhi

Tata Institute of Fundamental Research
Bombay 400 005, India

ABSTRACT

A very elementary model of a single positive hermitian random matrix coupled to an external matrix is defined and studied. Expanding the exact effective action around its classical solution leads to the “quantum Penner action”, from which a rich structure of correlation functions is obtained. These are shown to be equal to the all-orders perturbative expansion of tachyon amplitudes in the two-dimensional string at self-dual radius.

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1 Introduction

Random-matrix models have been intensively studied in the last few years[1]. Although the bulk of the attention has been paid to models describing dynamically triangulated random surfaces[2], and their double-scaling limits[3], it has turned out that many of the results obtained can be equivalently deduced from a far simpler class of matrix models, which moreover require no double-scaling limit. These are sometimes called “topological matrix models” because the random-matrix integral can be thought of as a generating function for certain topological invariants.

The first such model was the one constructed by Penner[4][5] to study the virtual Euler characteristic of the moduli space of Riemann surfaces. Subsequently, Kontsevich[6] obtained a model which generates intersection numbers on this moduli space (see also Ref.[7]), and generalizations of the Kontsevich model[8] were soon found which describe more complicated topological problems associated to vector bundles over moduli space.

The simplest of this class of models will be the subject of this article. We will show that a simple (almost trivial) model of hermitian positive random matrices coupled to an external matrix gives rise to a fascinating theory[9] which can eventually be seen to generalize both the Penner and Kontsevich models. Moreover, our theory will be shown to satisfy $W_{\infty}$ identities which uniquely fix its perturbative expansion to be that of noncritical $c = 1$ string theory at self-dual radius[10][11].

The two-dimensional string at this radius has been recently argued[12] to determine universal properties of type II superstrings compactified on Calabi-Yau manifolds with conifold singularities. This suggests the exciting possibility that the matrix model described here could have a direct bearing on physical properties of type II superstrings. We will make some speculations in this and other directions below.

2 A Topological Matrix Model

Consider a single $N \times N$ hermitian matrix $M$ whose eigenvalues are constrained to be positive-semidefinite. We will choose a linear action, describing the coupling of $M$ to a fixed external matrix $A$. Remarkably, this almost trivial choice leads us to a matrix model which describes a well-known string theory, as we will see below.

The random-matrix integral is

$$Z(A) = \int dM \ e^{-\nu \text{tr} MA}$$

(1)

where $\nu$ is a coupling constant. The eigenvalue part of the integral is taken over the range $[0, \infty)$. Thus this integral is convergent.
The redefinition $M \rightarrow MA^{-1}$ transfers the $A$-dependence to a prefactor:

$$Z(A) = (\det A)^{-N} \int dM \ e^{-\nu \text{tr} M}$$

(2)

We would like to think of this as a toy version of a quantum field theory. Treating $A$ as a fixed source (a background), we add an extra source term and compute the Legendre transform of the free energy to obtain the effective action. Thus we start now with

$$Z_A(J) = \int dM \ e^{-\nu \text{tr} MA - \text{tr} JM} = (\det (A + \frac{J}{\nu}))^{-N} \int dM \ e^{-\nu \text{tr} M}$$

(3)

The free energy as a function of the background $A$ and the source $J$ is minus the log of this integral, which is (dropping additive constants):

$$F_A(J) = N \text{ tr log } (A + \frac{J}{\nu})$$

(4)

To find the effective action, define the “classical field” $\hat{M}$ by

$$\hat{M} \equiv \frac{\partial F_A(J)}{\partial J} = \frac{N}{\nu} (A + \frac{J}{\nu})^{-1}$$

(5)

Then make the Legendre transformation

$$\Gamma(\hat{M}) = F_A(J) - \text{tr} \hat{M} J$$

(6)

and eliminate $J$ from Eq.(5) to get (again dropping an additive constant)

$$\Gamma(\hat{M}) = \nu \text{ tr } \hat{M} A - N \text{ tr log } \hat{M}$$

(7)

It is perhaps surprising that in such a trivial theory, the effective action is not identically equal to the classical one! The difference is an additive logarithmic term, which is generated dynamically — essentially by the boundary of the integration region at 0. The coefficient $N$ in front of the log term is undesirable since we ultimately intend to take the limit of large $N$ with the coupling $\mu$ fixed. It can be easily removed by including in the “bare” action a term of the same logarithmic type, with coefficient $(\nu - N)$.

This finally leads us to consider the model with matrix integral

$$Z(A) = \int dM \ e^{-\nu \text{tr} MA + (\nu - N) \text{ tr log } M}$$

(8)

whose effective action is

$$\Gamma(\hat{M}) = \nu \text{ tr } \hat{M} A - \text{ tr log } \hat{M}$$

(9)
Continuing to treat this as a toy field theory, we will show right away that this action gives rise to a rich set of “amplitudes”. Later we will see that these turn out to be precisely the tachyon scattering amplitudes of \(c = 1\) string theory at the self-dual radius, with \(\frac{1}{\nu^2}\) as the genus-expansion parameter!

To evaluate amplitudes, we need to first solve the equations of motion coming from the effective action \(\Gamma(\hat{M})\):

\[
0 = \frac{\partial \Gamma}{\partial \hat{M}} = A - \hat{M}^{-1}
\]

and then shift the “field” \(\hat{M}\) about the classical solution \(\hat{M} = A^{-1}\) by \(\hat{M} = A^{-1} + \hat{m}\). Then \(\hat{m}\) is the “quantum field”, and its amplitudes are determined by the expansion of the effective action \(\Gamma(\hat{M})\) about the classical solution, which leads to the “quantum Penner action”:

\[
\Gamma(\hat{m}) = \nu \text{tr log } A + \nu \sum_{k=2}^{\infty} (-1)^k \left( \frac{1}{k} \text{tr } (A\hat{m})^k \right)
\]

For the special case \(A = 1\) this coincides with the classical action of the Penner model, whose partition function computes the virtual Euler characteristic of the moduli space of Riemann surfaces. Here we see that a general background \(A\) gives rise to a full-fledged theory of amplitudes. To compute these, we read off the 1PI vertices, for example the two- and three-point vertices are

\[
\Gamma_{i_1,j_1;i_2,j_2}^{(2)} = \nu A_{i_2,j_1} A_{i_1,j_2}
\]

\[
\Gamma_{i_1,j_1;i_2,j_2;i_3,j_3}^{(3)} = -\nu \left[ A_{i_3,j_1} A_{i_1,j_2} A_{i_2,j_3} + A_{i_2,j_1} A_{i_3,j_2} A_{i_1,j_3} \right]
\]

while the propagator is the inverse of the two-point vertex:

\[
G_{i_1,j_1;i_2,j_2}^{(2)} = \langle \hat{m}_{i_1,j_1} \hat{m}_{i_2,j_2} \rangle = \frac{1}{\nu} A_{i_1,j_2}^{-1} A_{i_2,j_1}^{-1}
\]

Now we can compute \(n\)-point functions in terms of the background \(A\). At this point we take the limit of large \(N\), and parametrize the background in terms of an infinite number of independent parameters \(t_n\) via the Kontsevich-Miwa transform

\[
t_n = \frac{1}{\nu} \text{tr } A^{-n}
\]

Then the two- and three-point functions are easily computed to be

\[
\langle \text{tr } \hat{M}^2 \rangle = \nu (2t_2 + (t_1)^2)
\]

\[
\langle \text{tr } \hat{M}^3 \rangle = \nu (3t_3 + 6t_1 t_2 + (t_1)^3 + \frac{1}{\nu^2} 3t_3)
\]

Higher point functions follow using the higher vertices calculated from the action, by computing all connected and disconnected tree diagrams.
The structure of the amplitudes already begins to look interesting. Because of obvious homogeneity properties of the quantum Penner action, the \( n \)-point function will be a quasi-homogeneous function of the \( t_i \) where each term has \( \sum i = n \). Moreover, apart from an overall factor of \( \nu \), the amplitudes have the form of a power series expansion in \( \nu \) which \textit{terminates} at order \( \left( \frac{1}{\nu^2} \right)^{[\frac{n-1}{2}]} \) where \([ \cdot ]\) denotes the integer part. In the next section we show that the amplitudes that we obtain in this way represent the complete perturbative solution of \( c = 1 \) string theory at the self-dual radius, with the operators \( tr \hat{M}^n \) representing the tachyons of momentum \( -n \) in that theory, and the parameters \( t_n \) representing the couplings to tachyons of positive momentum.

3 Relation to \( c = 1 \) string

The \( c = 1 \) string is a background of bosonic string theory where the spacetime is two-dimensional. One of these dimensions may be compact, and we focus on the case where this compact direction has radius unity in suitable units, the self-dual value under \( T \)-duality. The perturbative solution of this theory takes the form of a generating function for “tachyon” scattering amplitudes, satisfying a set of recursive \( W_\infty \) ward identities which completely determine it (at the self-dual radius, these have been obtained from matrix models in Ref.\[10\] and from the topological Landau-Ginzburg model in Ref.\[11\]). Denote the partition function of this theory \( Z_{W_\infty}(t, \bar{t}) \) where \( t_n, \bar{t}_n \) \((n = 1, 2, \ldots)\) are respectively the couplings to tachyons of positive and negative momentum \( |n| \). The \( W_\infty \) identities can be written

\[
\frac{1}{\mu^2} \frac{\partial Z_{W_\infty}}{\partial t_n}(t, \bar{t}) = \frac{1}{(n + 1)(\mu n + 1)} \oint dz W_{n+1}(z, t, \frac{\partial}{\partial t}) Z_{W_\infty} \tag{18}
\]

Here, \( \mu \) is the cosmological constant of the theory, and the \( W \)-generators are differential operators defined in terms of certain free-fermion operators by

\[
W_{n+1} = : \Psi \partial_{\bar{z}}^{n+1} \Psi : \tag{19}
\]

The free-fermion operators in turn are defined through bosonization,

\[
\Psi(z, t, \frac{\partial}{\partial t}) = e^{i\mu \phi(z, t, \frac{\partial}{\partial t})} \tag{20}
\]

\[
\bar{\Psi}(z, t, \frac{\partial}{\partial t}) = e^{-i\mu \phi(z, t, \frac{\partial}{\partial t})} \tag{21}
\]

\[
\partial \phi(z, t, \frac{\partial}{\partial t}) = \frac{1}{z} + \sum_{n>0} nt_n z^n - \frac{1}{\mu^2} \sum_{n>0} \frac{\partial}{\partial t_n} z^{-n-1} \tag{22}
\]

To make contact with the matrix model of the previous section, we replace the couplings \( t_n \) by an external hermitian matrix \( A \) defined through Eq.(13) above. The
limit $N \to \infty$ on this matrix is implicit, since this is required for the parameters $t_n$ to be all independent. A lengthy but straightforward computation shows that the $W_\infty$ constraint can be rewritten

$$
\frac{1}{i\mu} \frac{\partial Z_{W_\infty}(t, \bar{t})}{\partial t_n} = \frac{1}{(i\mu)^n} (\det A)^{-i\mu} \text{tr} \left( \frac{\partial}{\partial A} \right)^n (\det A)^{i\mu} Z_{W_\infty}(t, \bar{t})
$$

(23)

It is easy to check explicitly that this constraint is solved by the random-matrix model with partition function

$$
Z(t, \bar{t}) = (\det A)^\nu \int dM \ e^{-\nu \text{tr} MA + (\nu - N) \text{tr} \log M - \sum_{k>0} \bar{t}_k \text{tr} M^k}
$$

(24)

where $\nu = -i\mu$. Differentiating in $\bar{t}_n$ and setting $\bar{t} = 0$, and dividing by the partition function, we find the normalized correlator

$$
\nu \langle \text{tr} M^n \rangle \equiv \frac{1}{Z(A)} \int dM \nu \text{tr} M^n e^{-\nu \text{tr} MA + (\nu - N) \text{tr} \log M}
$$

(25)

where now $Z$ is the partition function of the matrix model defined in Eq.(8) above. This shows that the expectation values of $\text{tr} M^n$ computed via the quantum Penner action are (after multiplying by $\nu$) just the expectation values of tachyons of negative momentum, $\langle T_{-n} \rangle$, in the $c = 1$ string at self-dual radius. The same can be done for higher derivatives in $\bar{t}$, so we have proved that the simple matrix model defined in the previous section is (at least perturbatively) equivalent to this string theory.

4 Discussion and Conclusions

A more detailed discussion of the model above, and in particular its relation to previous attempts at finding a topological matrix model for the $c = 1$ string, can be found in Ref.[9]. Here we point out some of the interesting open questions that follow from this work.

Although the equivalence of our matrix model to the $c = 1$ string has been demonstrated only perturbatively, the matrix model seems perfectly well-defined outside perturbation theory (it is really no more than the matrix analogue of the Gamma-function). Thus one may ask whether this provides new insight into the tricky question of giving a nonperturbative definition to the $c = 1$ string.

Recently Ghoshal and Vafa have argued that the $c = 1$ string at self-dual radius describes universal properties of type II superstrings compactified on Calabi-Yau manifolds which are developing conifold singularities. Their identification makes use only of the $c = 1$ partition function. It would be interesting to understand whether the
amplitudes of the $c = 1$ string likewise tell us something about conifold singularities and related issues concerning Ramond-Ramond states. It might, for example, be possible to address some recent speculations due to Shenker\cite{13} using our topological matrix model.

Finally, the simple way in which the background matrix $A$ enters into our model suggests that one might understand something more about background-independent string field theory from this point of view. In the absence of a background of tachyons, the potential for our matrix model is 0, rather analogous to the situation in topological field theories, which are believed to contain background-independent information about quantum gravity. It has been argued by us in Ref.\cite{9} that this model contains other “vacua” corresponding to string theory in the background of $c < 1$ minimal models, so that one could at least hope to find a background-independent picture for all $c \leq 1$ string backgrounds.

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