Loop-Weighted Walk

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Abstract

Loop-weighted walk with parameter $\lambda \geq 0$ is a non-Markovian model of random walks that is related to the loop $O(N)$ model of statistical mechanics. A walk receives weight $\lambda^k$ if it contains $k$ loops; whether this is a reward or punishment for containing loops depends on the value of $\lambda$. A challenging feature of loop-weighted walk is that it is not purely repulsive, meaning the weight of the future of a walk may either increase or decrease if the past is forgotten. Repulsion is typically an essential property for lace expansion arguments. This article circumvents the lack of repulsion and proves, via the lace expansion, that for any $\lambda \geq 0$ loop-weighted walk is diffusive in high dimensions.

1 Introduction and Main Results

Loop-weighted walk with parameter $\lambda$, abbreviated $\lambda$-LWW, is a model of self-interacting walks that can be informally defined as follows. Formal definitions will be given in Section 1.2. Let $\omega$ be a walk on a graph. A walk is called a loop if $\omega$ begins and ends at the same vertex. The loop erasure $\text{LE}(\omega)$ is formed by chronologically removing loops from $\omega$. If $n_L(\omega)$ denotes the number of loops removed, the $\lambda$-LWW weight of a walk $\omega$ is

$$w_\lambda(\omega) = \lambda^{n_L(\omega)}. \quad (1.1)$$

For $\lambda \geq 0$ the weight $w_\lambda$ defines a probability measure on $n$-step walks by defining the probability of $\omega$ to be proportional to $w_\lambda(\omega)$. Heuristically if $0 \leq \lambda < 1$ the effect of the weight is to discourage walks from containing loops, while if $\lambda > 1$ loops are encouraged. For $\lambda \neq 1$ this weight is not Markovian. Special cases include some well-studied models: 0-LWW is self-avoiding walk and 1-LWW is simple random walk.

This paper consists of a lace expansion analysis of $\lambda$-LWW. The lace expansion, originally introduced by Brydges and Spencer [3], is now a well-developed and powerful tool for proving mean-field behaviour in high dimensions [14]. With the notable exception of [16], walk models that have been successfully studied with the lace expansion have been purely repulsive. A walk model being purely repulsive means that the weight $w$ on walks that defines the model satisfies the inequality

$$w(\omega \circ \eta) \leq w(\omega)w(\eta), \quad (1.2)$$

where $\omega \circ \eta$ is the concatenation of the walks $\omega$ and $\eta$. For example, self-avoiding walk is purely repulsive. In general, see Figure 1, $\lambda$-LWW is not purely repulsive.

The most significant new step required to analyze $\lambda$-LWW by the lace expansion is therefore a technique to overcome the lack of repulsion. This is done by resumming $\lambda$-LWW to obtain a model of self-interacting and self-avoiding walks. The particular form of the $\lambda$-LWW weight leads to a very explicit description of the self-interaction in terms of a generalization of the loop measure of [10], and this explicit description makes it clear that the self-interaction is repulsive. This enables a lace expansion to be performed. Further details about the proof follow after the statement of Theorem 1.1.

Some notation will be needed to state the results. Let $\langle \cdot \rangle^\lambda_n$ denote expectation with respect to the measure on $n$-step walks associated to $w_\lambda$. Let $c_n^\lambda$ by the normalizing factor for the expectation, i.e.,
For each diagram consider (i) the concatenation of the black (dashed) and red walks and (ii) the walks as being separate. On the left (i) results in four loops being erased while (ii) results in no loops being erased. On the right (i) results in one loop being erased while (ii) results in three loops being erased. It follows that the $\lambda$-LWW weight is not purely repulsive for $\lambda \neq 0$.

the sum over all $n$ step walks weighted by $\lambda^n L(\omega)$ as in (1.1). Let $\chi_\lambda(z) = \sum_n c_\lambda^n z^n$, and let $z_c(\lambda)$ be the radius of convergence of $\chi_\lambda(z)$. The main result of this article can be summarized as saying that, in high dimensions, $\lambda$-LWW has mean-field behaviour at criticality.

Theorem 1.1. Fix $\lambda \geq 0$ and consider loop-weighted walk with parameter $\lambda$ on $\mathbb{Z}^d$. There is a $d_0 = d_0(\lambda)$ and constants $A, D$, such that for $d \geq d_0$

1. The susceptibility diverges linearly: $\chi_\lambda(z) \sim Az_c(z_c - z)^{-1}$ as $z \nearrow z_c$,

2. $c_\lambda^n = A(z_c(\lambda))^{-n} (1 + O(n^{-\delta}))$ for any $\delta < 1$, and

3. $\lambda$-LWW is diffusive: $\langle |\omega|^2 \rangle_\lambda^n = Dn(1 + O(n^{-\delta}))$ for any $\delta < 1$.

Remark 1.2. In what follows no attempt has been made to track the value of $d_0$. The current method of proof requires $d_0 \gg 9$.

For $\lambda = 0$ Theorem 1.1 has been proven with $d_0 = 5$ by Hara and Slade [8]. It is worth stressing that Theorem 1.1 holds for $\lambda > 1$, in which case $\lambda$-LWW is attractive in the sense that the formation of loops is encouraged. Theorem 1.1 actually holds in greater generality. Let $\lambda_\ell \geq 0$ be the weight of the loop $\ell$. Suppose the set of weights $\lambda_\ell$ satisfy a mild symmetry hypothesis (see Section 1.2.1) and are uniformly bounded above. Then the weight of $\lambda$ per loop in Equation (1.1) can be replaced by a product of the $\lambda_\ell$ over the set of loops $\ell$ that are erased when performing loop erasure on $\omega$.

Let us say a few more words about the proof of Theorem 1.1. As described earlier, the key step is a resummation of $\lambda$-LWW into a self-interacting and self-avoiding walk. The self-interaction is a many-body interaction, and this leads to a lace expansion based on hypergraphs instead of the usual graph based expansion that is used for self-avoiding walk. It should be stressed that hypergraphs are merely an organizational tool, and no prior knowledge of hypergraphs is needed to understand the expansion. Once the lace expansion has been performed the various self-interacting self-avoiding walk quantities can be re-expressed in terms of $\lambda$-LWW. The diagrams that occur in analyzing the expansion generalize the diagrams for self-avoiding walk, and when $\lambda = 0$ they reduce to the diagrams for self-avoiding walk. With some effort it is possible to analyze the diagrams for $\lambda > 0$ by existing methods. Once the analysis of the diagrams is completed it is possible to apply well-established techniques to analyze $\lambda$-LWW, namely the trigonometric approach to the convergence of the lace expansion [14] and complex analytic methods for studying asymptotics.

The remainder of the introduction is as follows. Section 1.1 describes an important connection between $\lambda$-LWW and the loop $O(N)$ model of statistical physics. Section 1.2 gives a formal definition of $\lambda$-LWW, relates $\lambda$-LWW to a self-interacting and self-avoiding walk, and outlines how this enables a lace expansion analysis. Lastly, Section 1.3 establishes a few conventions used in the remainder of the article.

1.1 Motivation from Statistical Mechanics

For $N \in \mathbb{N}$ the $O(N)$ model on a finite graph $G = (V, E)$ is a generalization of the Ising model. To each vertex $x \in V$ is associated a spin $\mathbf{s}_x$ taking values in the unit sphere in $\mathbb{R}^N$. The probability of a spin
configuration is defined to be
\[
P((s_x)_{x \in V}) \propto \exp(\beta \sum_{x \sim y} s_x \cdot s_y),
\]
where \(\beta\) is a real parameter and the summation is over all edges \(\{x, y\} \in E\). A simplification of the \(O(N)\) model known as the loop \(O(N)\) model was introduced in [6]. The loop \(O(N)\) model is a graphical model defined in terms of subgraph configurations on \(G\). In the special case of a graph with degree bounded by 3, the loop \(O(N)\) model is supported on subgraphs that are disjoint unions of cycles of length at least 3, and the probability of a subgraph \(H\) is given by
\[
P(H) \propto z^{\#(E(H)) N} \#(H),
\]
where \(\#H\) denotes the number of connected components of \(H\). Note that the probability in Equation (1.4) may be negative if \(N < 0\); Equation (1.4) defines a signed measure in general.

The definition of the loop \(O(N)\) model on an arbitrary graph \(G\) involves noncyclic subgraphs, see for example [5]. The noncyclic subgraphs are predicted by non-rigorous renormalization group arguments to be irrelevant, at least for \(|N| \leq 2\) on planar graphs [12]. Call the model whose configurations are disjoint unions of cyclic subgraphs the \(O(N)\) cycle gas. For \(N \in \mathbb{N}\) this model has previously appeared in the physics literature as a model for melting transitions [13].

As will be described in Appendix A.4, \(\lambda\)-LWW is a random walk representation of the \(O(N)\) cycle gas. The two-point function of \(\lambda\)-LWW corresponds to a two-point correlation in the \(O(N)\) cycle gas for \(N = -2\lambda\). In other words, \(\lambda\)-LWW yields a probabilistic interpretation of the \(O(N)\) cycle gas for \(N < 0\). This is an example of a negative activity isomorphism theorem: an equivalence between a statistical mechanics model at negative activity (\(N < 0\)) and a probability model. An important previous example of such a theorem is the Brydges–Inhbrü isomorphism between branched polymers and the hard-core gas [4]. In the present work the isomorphism allows results about \(\lambda\)-LWW to be transferred to the \(O(N)\) cycle gas for \(N < 0\). For example, the isomorphism theorem combined with Theorem 1.1 immediately implies the following corollary.

**Corollary 1.3.** For \(d\) sufficiently large the susceptibility of the \(O(N)\) cycle gas on \(\mathbb{Z}^d\) for \(N < 0\) diverges linearly at the critical point.

To summarize, \(\lambda\)-LWW can be viewed as a random walk representation of an approximation of the \(O(N)\) model. Thus \(\lambda\)-LWW fits into a long history of random walk representations of spin models [1, 2, 7] inspired by the pioneering work of Symanzik [15].

### 1.2 Introduction to the Loop-Weighted Walk Model

The rest of the paper will be concerned with \(\mathbb{Z}^d\), the simple cubic lattice in \(d\) dimensions. Edges \(\{x, y\}\) will be abbreviated \(xy\). Two vertices \(x\) and \(y\) will be called adjacent, written \(x \sim y\), if \(xy\) is an edge in \(\mathbb{Z}^d\). Let \(\Omega = \{y \in \mathbb{Z}^d \mid o \sim y\}\), so \(|\Omega| = 2d\) is the number of vertices adjacent to the origin \(o \in \mathbb{Z}^d\).

#### 1.2.1 Model Definition

The next paragraphs establish some conventions about walks. An \(n\)-step walk \(\omega = (\omega_0, \omega_1, \ldots, \omega_n)\) is a sequence of \(n + 1\) adjacent vertices in \(\mathbb{Z}^d\). Given a walk \(\omega\), \(|\omega|\) will denote the number of steps in \(\omega\). A walk is a loop if \(|\omega| = 0\), self-avoiding if \(\omega_i = \omega_j\) implies \(i = j\), and a self-avoiding polygon if \(\omega_i = \omega_j\) and \(i \neq j\) implies \(\{i, j\} = \{0, |\omega|\}\).

A walk \(\omega\) begins at \(\omega_0\) and ends at \(\omega_{|\omega|}\). Let \(\Omega_{\text{SAW}}(x, y) = \{\omega : x \to y \mid \omega \text{ self-avoiding}\}\); if \(x = y\) this is taken to be the set of self-avoiding polygons. Let \(\Omega_{\text{SAP}} = \bigcup_x \Omega_{\text{SAW}}(x, x)\) and \(\Omega_{\text{SAW}} = \bigcup_x \bigcup_y \Omega_{\text{SAW}}(x, y)\). If \(\omega^{(i)} = (\omega^{(i)}_0, \ldots, \omega^{(i)}_{k_i})\) for \(i = 1, 2\) and \(\omega^{(1)}_{k_1} = \omega^{(2)}_0\) the concatenation \(\omega^{(1)} \circ \omega^{(2)}\) of \(\omega^{(1)}\) with \(\omega^{(2)}\) is defined by
\[
\omega^{(1)} \circ \omega^{(2)} = (\omega^{(1)}_0, \ldots, \omega^{(1)}_{k_1}, \omega^{(2)}_1, \ldots, \omega^{(2)}_{k_2}).
\]
To define $\lambda$-LWW precisely we now give an explicit description of the loop erasure of a walk $\omega$. Define

\[
\tau_\omega = \min \{ i \mid \exists j < i \text{ such that } \omega_i = \omega_j \}, \quad \tau_\omega^* = \min \{ j \mid \omega_j = \omega_{\tau_\omega} \}.
\] (1.6)

If $\omega$ is a self-avoiding walk, define $\tau_\omega = \tau_\omega^* = \infty$. The time $\tau_\omega$ is the first time a walk visits a vertex twice.

**Definition 1.4.** Let $\omega$ be a walk of length $n$. The single loop erasure $\text{LE}^1(\omega)$ of $\omega$ is given by

\[
\text{LE}^1(\omega) = (\omega_0, \ldots, \omega_{\tau_\omega^*} \wedge n, \omega_{\tau_\omega^*+1}, \ldots, \omega_n),
\] (1.8)

where $a \wedge b$ denotes the minimum of $a$ and $b$. The walk $(\omega_{\tau_\omega^*}, \omega_{\tau_\omega^*+1}, \ldots, \omega_{\tau_\omega})$ is the loop removed by loop erasure. The loop erasure $\text{LE}(\omega)$ of $\omega$ is the result of iteratively applying $\text{LE}^1$ until $\tau_\omega = \infty$.

By construction, each loop removed from a walk by loop erasure is a self-avoiding polygon.

**Definition 1.5.** The loop vector $n_\lambda(\omega)$ of $\omega$ is the vector with components

\[
n_\lambda^n(\omega) = \# \text{ of times } \eta \text{ is removed by loop erasure applied to } \omega, \quad \eta \in \Omega_{\text{SAP}}.
\] (1.9)

In what follows $\lambda$ will denote a vector of activities $\lambda_\eta \geq 0$ for $\eta \in \Omega_{\text{SAP}}$. Inequalities with respect to $\lambda$ are to be interpreted pointwise in $\eta \in \Omega_{\text{SAP}}$. Define

\[
\lambda^{n_\lambda}(\omega) = \prod_\eta \lambda_\eta^{n_\lambda}(\omega).
\] (1.10)

**Definition 1.6.** Let $\lambda \geq 0$, $z \geq 0$. The weight $w_{\lambda,z}$ of $\lambda$-LWW at activity $z$ is given by

\[
w_{\lambda,z}(\omega) = z^{\omega} \chi^\Lambda_{\omega},
\] (1.11)

**Definition 1.7.** The susceptibility $\chi_{\lambda}(z)$ of $\lambda$-LWW is

\[
\chi_{\lambda}(z) = \sum_{x} \sum_{\omega: x \rightarrow x} w_{\lambda,z}(\omega).
\] (1.12)

The critical point $z_c(\lambda)$ of $\lambda$-LWW is defined to be the radius of convergence of $\chi_{\lambda}(z)$.

If $0 \leq \lambda \leq 1$ then $\chi_{\lambda}(z) \leq \chi_1(z)$, and hence $\chi_{\lambda}(z)$ converges for $z < |\Omega|^{-1}$. The next proposition guarantees the critical point is always nontrivial.

**Proposition 1.8.** Let $\bar{\lambda} = \sup_\eta \lambda_\eta > 1$. If $z < (|\Omega| \sqrt{\bar{\lambda}})^{-1}$ then $\chi_{\lambda}(z)$ is finite.

**Proof.** An $n$-step walk contains at most $\lfloor n/2 \rfloor$ loops, and weighting each loop by $\bar{\lambda}$ yields an upper bound for $\chi_{\lambda}(z)$. The resulting sum, after cancelling common factors of $\sqrt{\bar{\lambda}}$, is $\chi_1(\bar{z})$ for some $\bar{z} < |\Omega|^{-1}$. □

If $\mathcal{R}$ is an isometry of $\mathbb{Z}^d$, and $A \subset \mathbb{Z}^d$, let $\mathcal{R}A = \{ \mathcal{R}a \mid a \in A \}$.

**Assumption 1.** Assume that $\lambda_\eta = \lambda_{R\eta}$ for any isometry $\mathcal{R}$ and any $\eta \in \Omega_{\text{SAP}}$. Further assume that $\lambda_\eta = \lambda_{\tilde{\eta}}$ if $\eta$ and $\tilde{\eta}$ are self-avoiding polygons that differ only in terms of initial vertex and orientation.

**Assumption 2.** Assume $\sup_{\eta \in \Omega_{\text{SAP}}} \lambda_\eta < \infty$.

**Theorem 1.9.** Fix $\lambda \geq 0$. If Assumption 1 and Assumption 2 hold, then there is a $d_0 = d_0(\lambda)$ and constants $A$ and $D$ such that for $d \geq d_0$ the conclusions of Theorem 1.1 hold.

The constants $A$ and $D$ have explicit expressions, see Section 6.2. In what follows it will always be assumed that Assumptions 1 and 2 hold.
1.2.2 Aspects of the Proof

This section describes the basic facts about $\lambda$-LWW that allow for a lace expansion analysis and gives an outline of the proof of Theorem 1.9.

**Definition 1.10.** The loop-erased $\lambda$-LWW weight $\bar{\omega}_{\lambda,z}$ on self-avoiding walks is

$$\bar{\omega}_{\lambda,z}(\eta) = \sum_{\omega: LE(\omega) = \eta} w_{\lambda,z}(\omega).$$  \hfill (1.13)

Note that the definition of $\bar{\omega}_{\lambda,z}$ assigns non-zero weight only to self-avoiding walks. The definition of $\bar{\omega}_{\lambda,z}$ implies that for any $x \in \mathbb{Z}^d$

$$\sum_{\omega: \omega \to x} \bar{\omega}_{\lambda,z}(\omega) = \sum_{\omega: \omega \to x} w_{\lambda,z}(\omega),$$  \hfill (1.14)

as the left-hand side is just a reorganization of the right-hand side. This identity will be important in what follows.

**Definition 1.11.** The range, range(\omega), of a walk $\omega$ is the set of vertices visited by $\omega$.

**Definition 1.12.** Let $A, B \subset \mathbb{Z}^d$. The $\lambda$-LWW loop measure $\mu_{\lambda,z}(A; B)$ is

$$\mu_{\lambda,z}(A; B) = \sum_{x} \sum_{\omega: \omega \to x \cap A \neq \emptyset} \mathbb{1}_{\{\text{range(\omega) \cap A} \neq \emptyset\}} \mathbb{1}_{\{\text{range(\omega) \cap B} = \emptyset\}} \frac{w_{\lambda,z}(\omega)}{|\omega|}. \hfill (1.15)$$

Define $\mu_{\lambda,z}(A) = \mu_{\lambda,z}(A; \emptyset)$. For singleton sets \{x\}, \{y\}, let $\mu_{\lambda,z}(x; y) = \mu_{\lambda,z}(\{x\}; \{y\})$. For the special case of $\lambda = 1$ the next theorem is [10, Proposition 9.5.1].

**Theorem 1.13.** The loop erased $\lambda$-LWW weight on self-avoiding walks can be written in terms of the $\lambda$-LWW loop measure:

$$\bar{\omega}_{\lambda,z}(\eta) = \sum_{\omega: LE(\omega) = \eta} w_{\lambda,z}(\omega) = z^{|\eta|} \exp(\mu_{\lambda,z}(\text{range}(\eta))).$$  \hfill (1.16)

**Proof.** Deferred to Appendix A. \hfill \Box

A function $f: 2^{\mathbb{Z}^d} \to \mathbb{R}$ of subsets of $\mathbb{Z}^d$ is said to be weakly increasing if $A \subset B$ implies $f(A) \leq f(B)$, and weakly decreasing if $f(A) \geq f(B)$.

**Proposition 1.14.** Assume $z \geq 0$, $\lambda \geq 0$.

1. Let $A, B \subset \mathbb{Z}^d$. Then for any isometry $\mathcal{R}$

$$\mu_{\lambda,z}(\mathcal{R}A; \mathcal{R}B) = \mu_{\lambda,z}(A; B),$$  \hfill (1.17)

2. $\mu_{\lambda,z}(A; B)$ is weakly increasing in $A$ and weakly decreasing in $B$.

**Proof.** The first item follows from the isometry invariance of $w_{\lambda,z}$, which follows from Assumption 1. The second follows as increasing $A$ (decreasing $B$) reduces (increases) the constraints on the set of walks that contribute to the defining sum, and $w_{\lambda,z}(\omega) \geq 0$. \hfill \Box

If $\eta = \eta_1 \circ \eta_2$ is self-avoiding then Theorem 1.13 and the definition of the loop measure imply

$$\bar{\omega}_{\lambda,z}(\eta) = z^{|\eta_1|} z^{|\eta_2|} \exp(\mu_{\lambda,z}(\text{range}(\eta_1))) \exp(\mu_{\lambda,z}(\text{range}(\eta_2) \cap \text{range}(\eta_1))).$$  \hfill (1.18)

Dropping the constraint in the second loop measure increases the weight, and hence loop-erased $\lambda$-LWW is purely repulsive. This enables a lace expansion analysis of $\lambda$-LWW as the two-point functions of $\lambda$-LWW and loop-erased $\lambda$-LWW coincide by Equation (1.14). This is done as follows:
(i) Section 2 derives a lace expansion for self-avoiding walk in an $\mathcal{X}$ gas. This means the weight of a self-avoiding walk is given by

$$w(\omega) = \prod_{X \in \mathcal{X}} (1 + \alpha_X)^\mathbb{1}_{\ell(X) \cap \text{range}(\omega) \neq \emptyset}, \quad (1.19)$$

where for each term $X$ in the product $\ell(X)$ is a subset of $\mathbb{Z}^d$. An important assumption is that the weight given by Equation (1.19) is invariant under isometries. The precise definition of an $\mathcal{X}$ gas is given in Section 2.

(ii) The loop measure is of the form in Equation (1.19). More precisely, it is an $\mathcal{X}$ gas with

$$\mathcal{X} = \bigcup_{x \in \mathbb{Z}^d} \{ \omega: x \to |\omega| \geq 1 \}, \quad (1.20)$$

$$\ell(\omega) = \text{range}(\omega), \quad (1.21)$$

$$\alpha_\omega = \exp \left( \frac{w_{\lambda,z}(\omega)}{|\omega|} \right) - 1. \quad (1.22)$$

The set $\mathcal{X}$ allows distinct walks with identical ranges to be distinguished. Section 3 applies the results of Section 2 to loop-erased $\lambda$-LWW and hence $\lambda$-LWW itself.

(iii) Section 4 and Section 5 establish diagrammatic estimates and use these to prove the convergence of the lace expansion at the critical point. The analysis is based on [14].

(iv) Lastly, Section 6 proves the main theorem after establishing some further estimates on the lace expansion coefficients. The analysis is based on [11].

As mentioned in the introduction, the only previous lace expansion analysis of a walk model that is not purely repulsive is [16], which analyzes a self-avoiding walk with nearest neighbour attractions. The analysis in [16] is based on exploiting the self-avoiding nature of the walk and the argument requires that the self-avoiding walk has a step distribution with infinite range. For a more extensive discussion of the lace expansion and the models to which it has so far been applied see [14].

1.3 Notation and Conventions

For $A$ a set, let $\mathbb{1}_{\{A\}}$ denote the indicator function of $A$. For notational ease we will occasionally make use of the Kronecker delta $\delta_{x,y} = \mathbb{1}_{\{x=y\}}$. The single step distribution $D(x)$ is defined by $D(x) = |\Omega|^{-1} \mathbb{1}_{\{x \sim o\}}$, where we recall that $|\Omega| = 2d$ and $x \sim o$ indicates that $x$ is a nearest neighbour of the origin $o$.

The Fourier transform $\hat{f}: [-\pi, \pi]^d \to \mathbb{C}$ of a function $f$ on $\mathbb{Z}^d$ is defined by

$$\hat{f}(k) = \sum_{x \in \mathbb{Z}^d} e^{ik \cdot x} f(x) \quad (1.23)$$

Subwalks of a walk $\omega$ can be identified by specifying the subinterval that defines them. That is, for $0 \leq a < b \leq |\omega|$ define $\omega[a, b] = (\omega_a, \ldots, \omega_b)$, $\omega[a, b) = \omega[a, b - 1]$, $\omega(a, b] = \omega[a + 1, b]$, and $\omega(a, b) = \omega[a + 1, b - 1]$. By convention $[a, a] = \{a\}$ and $\omega[a, a] = \omega_a$. To avoid some ungainly notation let $\omega[a:] = \omega[a, |\omega|]$.

By convention $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$. The set $\{0, 1, \ldots, n\}$ will be denoted $[n]$, and $|\omega|$ will denote $|[\omega]|$ when $\omega$ is a walk. Further, $c$ will denote a positive constant independent of the dimension $d$ and activity $z$; the precise value of $c$ may change from line to line.

2 A Lace Expansion

This section derives a lace expansion that can be applied to $\lambda$-LWW. The expansion is developed in greater generality as this requires no extra effort.

Remark 2.1. The lace expansion presented here can be derived by other means. In particular, the expansion can be derived using the technique developed for self-interacting walks in [17].
2.1 Self Avoiding Walk in an $\mathcal{X}$ Gas

Let $\mathcal{X}$ be a set, and let $\ell: \mathcal{X} \to 2^{\mathbb{Z}^d}$ be a map such that $\ell(X)$ is a connected subset of $\mathbb{Z}^d$ for each $X \in \mathcal{X}$. $\mathcal{X}$ is the set of particles and $\ell(X)$ is the particle type. Associate an activity $\alpha_X \in \mathbb{R}$ to each $X \in \mathcal{X}$.

**Definition 2.2.** Self-avoiding walk in an $\mathcal{X}$ gas is the model of walks weighted by

$$w(\omega) \equiv \mathbf{1}_{\{\omega \in \Omega_{\text{SAW}}\}} \prod_{X \in \mathcal{X}} (1 + \alpha_X)^{\mathbf{1}_{\{\text{range}(\omega) \cap \ell(X) \neq \emptyset\}}}.$$  \hspace{1cm} (2.1)

Equation (2.1) weights walks depending on the particle types that intersect the range of the walk. Heuristically, the choices of $\mathcal{X}$ and $\alpha_X$ determine a way of measuring the volume of a self-avoiding walk.

**Definition 2.3.** Let $A$ be a set. For $s, t \in A$, $s \neq t$, the pair $\{s, t\} \equiv st$ is called an edge. A graph $\Gamma$ on $A$ is a set of edges.

The condition $\omega \in \Omega_{\text{SAW}}$ that a walk $\omega$ is self-avoiding can be expressed using graphs.

$$\mathbf{1}_{\{\omega \in \Omega_{\text{SAW}}\}} = \prod_{0 \leq s < t \leq |\omega|} \mathbf{1}_{\{\omega_s \neq \omega_t\}} = \prod_{0 \leq s < t \leq |\omega|} (1 - \mathbf{1}_{\{\omega_s = \omega_t\}}) = \sum_{\Gamma} \prod_{st \in \Gamma} (-\mathbf{1}_{\{\omega_s = \omega_t\}}),$$ \hspace{1cm} (2.3)

where the sum is over all graphs $\Gamma$ on $[\omega]$.

2.2 Graphical Representations

This section provides a representation of the weight of a self-avoiding walk in an $\mathcal{X}$ gas in terms of hypergraphs. Let us emphasize that no prior knowledge of hypergraphs is needed to understand the expansion. The utility of the representation is that it allows recursive identities for self-avoiding walks in an $\mathcal{X}$ gas to be derived.

2.2.1 Graph Representation of Self-Avoidance

**Definition 2.3.** Let $A$ be a set. For $s, t \in A$, $s \neq t$, the pair $\{s, t\} \equiv st$ is called an edge. A graph $\Gamma$ on $A$ is a set of edges.

The condition $\omega \in \Omega_{\text{SAW}}$ that a walk $\omega$ is self-avoiding can be expressed using graphs.

$$\mathbf{1}_{\{\omega \in \Omega_{\text{SAW}}\}} = \prod_{0 \leq s < t \leq |\omega|} \mathbf{1}_{\{\omega_s \neq \omega_t\}} = \prod_{0 \leq s < t \leq |\omega|} (1 - \mathbf{1}_{\{\omega_s = \omega_t\}}) = \sum_{\Gamma} \prod_{st \in \Gamma} (-\mathbf{1}_{\{\omega_s = \omega_t\}}),$$ \hspace{1cm} (2.3)

where the sum is over all graphs $\Gamma$ on $[\omega]$.

2.2.2 Hypergraph Decomposition of an $\mathcal{X}$ Gas Weight

A representation of the weight on self-avoiding walks due to the particles in $\mathcal{X}$ is less straightforward than the graph representation of self-avoidance. This is because the condition of self-avoidance only involves two distinct times, while the condition that a $\text{range}(\omega) \cap \ell(X) \neq \emptyset$ involves many distinct times. This can be handled by using inclusion-exclusion.

**Lemma 2.4.** Let $\omega$ be a walk, and let $X \in \mathcal{X}$. Then

$$(1 + \alpha_X)^{\mathbf{1}_{\{\ell(X) \cap \text{range}(\omega) \neq \emptyset\}}} = \prod_{J \subseteq [\omega]} (1 + F_{J,X}(\omega)),$$ \hspace{1cm} (2.4)
where

\[ F_{J,X}(\omega) = \begin{cases} \alpha X \prod_{j \in J} \mathbb{1}_{\{\omega_j \in \ell(X)\}}, & |J| \in 2\mathbb{N} + 1 \\ -\frac{\alpha}{1+\alpha} \prod_{j \in J} \mathbb{1}_{\{\omega_j \in \ell(X)\}}, & |J| \in 2\mathbb{N}. \end{cases} \]  

(2.5)

In Equation (2.5) 0 is included in 2\mathbb{N}.

Proof. Apply inclusion-exclusion to the condition \( \ell(X) \cap \text{range}(\omega) \neq \emptyset \): 

\[ \mathbb{1}_{\{\ell(X) \cap \text{range}(\omega) \neq \emptyset\}} = 1 - \prod_{j=0}^{\lfloor |\omega| / 2 \rfloor} (1 - \mathbb{1}_{\{\omega_j \in \ell(X)\}}) = \sum_{J \subseteq [\omega]} (-1)^{|J|+1} \prod_{j \in J} \mathbb{1}_{\{\omega_j \in \ell(X)\}}. \]  

(2.6)

The weights \( F_{J,X} \) arise by using the identities \((1 + \alpha)^{-1} = 1 - \frac{\alpha}{1+\alpha} \) and \((1 + \alpha)^{1A} = 1 + \alpha \mathbb{1}_A. \) \( \square \)

**Definition 2.5.** A hypergraph \( G \) on a set \( A \) is a collection of non-empty subsets of \( A \). Each element of \( G \) is called a hyperedge. In particular, 

\[ \prod_{a \in A} (1 + F(a)) = \sum_{G} \prod_{a \in G} F(a) \]  

(2.7)

where \( F(a) \) is a weight associated to the hyperedge \( a \), and the sum on the right-hand side of (2.7) is over all hypergraphs on \( A \).

To represent the product over \( \mathcal{X} \) in Equation (2.1) in terms of hypergraphs take \( A \) in Definition 2.5 to be \( 2^{[n]} \setminus \emptyset \times \mathcal{X} \). If \( a \in A \) then \( a = (J,X) \) for \( J \) a non-empty subset of \([n]\) and \( X \in \mathcal{X} \). Defining \( F(a) = F_{J,X} \) gives the formula 

\[ \prod_{X \in \mathcal{X}} (1 + \alpha_X)^{\mathbb{1}_{\{\ell(X) \cap \text{range}(\omega) \neq \emptyset\}}} \prod_{X \in \mathcal{X}} \prod_{|J| \in [\omega]} \prod_{|J| \geq 1} (1 + F_{J,X}(\omega)) \]  

(2.8)

\[ = \sum_{G} \prod_{(J,X) \in G} F_{J,X}(\omega) \]  

(2.9)

where the sum in (2.9) is over all hypergraphs.

The next corollary is a useful hypergraph representation of the weight carried by a subwalk.

**Corollary 2.6.** Let \( \omega \) be an \( n \)-step walk. For \( k \leq n \), \( X \in \mathcal{X} \), 

\[ (1 + \alpha_X)^{\mathbb{1}_{\{\text{range(}(\omega[0,k]) \cap \ell(X) = \emptyset\}}} \mathbb{1}_{\{\text{range(}(\omega[k,n]) \cap \ell(X) \neq \emptyset\}}} = \prod_{J \subseteq [n], |J| \geq 1 \atop J \cap [k,n] \neq \emptyset} (1 + F_{J,X}(\omega)). \]  

(2.10)

Proof. Note that \( \mathbb{1}_{\{\text{range}(\omega[0,k]) \cap \ell(X) = \emptyset\}} \mathbb{1}_{\{\text{range}(\omega[k,n]) \cap \ell(X) \neq \emptyset\}} \) can be rewritten as \( \mathbb{1}_{\{\text{range}(\omega) \cap \ell(X) \neq \emptyset\}} - \mathbb{1}_{\{\text{range}(\omega[0,k]) \cap \ell(X) = \emptyset\}} \). The corollary follows by applying Lemma 2.4 to both \( \omega \) and \( \omega[0,k] \) and dividing. \( \square \)

**2.2.3 Graphical Representation of Self-Avoiding Walk in an \( \mathcal{X} \) Gas**

**Definition 2.7.** Let \( J \subseteq [n] \) be non-empty and let \( X \) denote an element of \( \mathcal{X} \cup \{\emptyset\} \). A pair \((J,X)\) is timelike if \( |J|=2 \), \( X = \emptyset \). A pair \((J,X)\) is spacelike if \( X \neq \emptyset \). The use of spacelike and timelike as labels has no relation to the use of these terms in physics.

Extend the definition of \( F_{J,X} \) by defining \( F_{J,X} \) via (2.5) if \((J,X)\) is spacelike, and defining \( F_{st,\emptyset} = -\mathbb{1}_{\{\omega_s=\omega_t\}} \) for timelike hyperedges \((st,\emptyset)\). Let \( \mathcal{G}(n) \equiv \mathcal{G}[0,n] \) denote the set of hypergraphs whose
hyperedges are pairs \((J, X)\) such that (i) \(X \in \mathcal{X} \cup \{\emptyset\}\), (ii) \(J \subset [n]\), \(|J| \geq 1\), and (iii) \(X = \emptyset\) implies \(|J| = 2\). The decompositions of Sections 2.2.2 and 2.2.1 imply that

\[
c_n(a, x) = \sum_{\omega : a \rightarrow x} \mathbb{1}_{\omega \in \Omega_{\text{SAW}}} \prod_{X \in \mathcal{X}} (1 + \alpha_X)^{1_{\ell(X) \cap \text{range}(\omega) \neq \emptyset}} \\
= \sum_{\omega : a \rightarrow x} \sum_{G \in G(n)} F_{L,X}(\omega). \tag{2.12}
\]

### 2.3 Lace Graphs

**Definition 2.8.** A graph \(\Gamma\) on \([a, b]\) is (lace) connected if (i) \(b > a + 1\), (ii) for all \(a < j < b\) there is an edge \(st \in \Gamma\) such that \(s < j < t\) and (iii) there are \(j_1, j_2\) such that \(aj_1, j_2b \in \Gamma\). Let \(G[a, b]\) and \(G^c[a, b]\) denote the set of graphs and connected graphs, respectively, on \([a, b]\).

We caution the reader that the definition of lace connectedness is not the same as the graph theoretic definition of connectedness. The adjective lace will be dropped in what follows, as the graph-theoretic notion of connectedness is not relevant in this section.

For \(a \leq b\) and \(w\) a function on graphs on \([a, b]\) define

\[
K[a, b] = \sum_{G \in G[a, b]} w(G), \quad J[a, b] = \sum_{G \in G^c[a, b]} w(G), \tag{2.13}
\]

and let \(K[a, b] = J[a, b] = 0\) if \(a > b\). For \(a < b\) the observation that a graph on \([a, b]\) either contains \(a\) in a connected subgraph or does not and the definition of connectedness imply

\[
K[a, b] = K[a, a + 1] K[a + 1, b] + \sum_{j \geq 2} J[a, a + j] K[a + j, b]. \tag{2.14}
\]

**Definition 2.9.** A graph is a lace graph if the removal of any edge results in a graph which is not connected.

A **labelled graph** is a graph where each edge is given a label of either spacelike or timelike; a labelled graph may contain both the edge \((st, \text{spacelike})\) and \((st, \text{timelike})\). The definition of a lace graph applies to labelled graphs as the notion of connectedness does not depend on the labelling. The following procedure associates a unique lace \(L_{\Gamma}\) to each labelled connected graph \(\Gamma\) on \([a, b]\). The labelled lace \(L_{\Gamma}\) consists of the set of edges \(s_t t_i\) along with their labellings, where \(s_t t_i\) are determined by

\[
s_1 = a, \quad t_1 = \max\{v : s_1 v \in \Gamma\}, \quad t_{i+1} = \max\{v : \exists s < t_i \exists sv \in \Gamma\}, \quad s_{i+1} = \min\{s : st_{i+1} \in \Gamma\}. \tag{2.15}
\]

In the case this does not specify \(s_t t_i\) uniquely at some step \(s_t t_i\) is chosen to have the label spacelike. The procedure terminates when \(t_{i+1} = b\). See Figure 2.

---

**Figure 2:** The left-hand side depicts a connected labelled graph, while the right-hand side depicts the corresponding labelled lace graph. The black solid edges are labelled spacelike, while the red zigzag edges are labelled timelike.

A labelled edge \(st\) is said to be *compatible* with a lace \(L\) if \(L_{\Gamma \cup \{st\}} = L\), i.e., if the addition of the labelled edge \(st\) does not alter the outcome of the above algorithm. Let \(L[a, b]\) denote the set of labelled lace graphs on \([a, b]\) and \(C(L)\) the set of compatible labelled edges for a lace \(L \in L\).
Lemma 2.10. Let \( w \) be a weight on labelled edges \( s t \). Then
\[
\sum_{\Gamma \in \mathcal{G}[a,b]} \prod_{s \in \Gamma} w(st) = \sum_{L \in \mathcal{L}[a,b]} \prod_{s \in L} w(st) \prod_{s t \in \mathcal{L}} (1 + w(st)),
\]
where the sums are over labelled connected graphs and labelled laces, respectively.

Proof. The proof is the same as the proof for unlabelled graphs, see [3], [14], or [19]. \( \square \)

Remark 2.11. Definition 2.8 is not the definition of connectedness typically used for the lace expansion for self-avoiding walk due to \( \{a,a+1\} \) not being considered connected. This change in definition is entirely cosmetic for self-avoiding walk as graphs of the form \( \{a,a+1\} \) do not contribute.

2.4 Laces and Hypergraphs

This section obtains an analogue of Lemma 2.10 for hypergraphs that can be applied to obtain a recursion for the \( n \)-step connectivity.

2.4.1 Recursion Relation for Hypergraphs

Definition 2.12. For a hyperedge \( (J,X) \) define \( \text{span}(J,X) = \{\min J, \max J\} \). If \( (J,X) \) is spacelike label \( \text{span}(J,X) \) spacelike, and if \( (J,X) \) is timelike label \( \text{span}(J,X) \) timelike. If \( G \) is a hypergraph the labelled graph \( G_L \) with labelled edges \( \{\text{span}(J,X) \mid (J,X) \in G\} \) will be called the graph of spans of \( G \).

Definition 2.13. A hypergraph \( G \) on \( [a,b] \) is connected if the graph of spans of \( G \) is connected on \( [a,b] \). The set of connected hypergraphs on \( [a,b] \) is denoted \( \mathcal{G}^+ \).

The objects \( \alpha \) and \( \alpha_0 \) in the next definition have interpretations in the context of \( \lambda \)-LWW, but for now should be thought of as convenient shorthand.

Definition 2.14. Let \( \mathcal{X}_0 = \{X \in \mathcal{X} \mid 0 \in \ell(X)\} \) and let \( y \in \Omega \) be a vertex adjacent to 0. Define
\[
\alpha_0 = \alpha_0(\mathcal{X}) = \prod_{X \in \mathcal{X}_0} (1 + \alpha_X), \quad \alpha = \alpha(\mathcal{X}) = \prod_{X \in \mathcal{X}_0} (1 + \alpha_X)^{\frac{1}{w(\ell(X))}},
\]
Assumption 3 implies that \( \alpha \) is well defined, i.e., independent of the vertex \( y \in \Omega \) chosen.

The definition of \( \alpha_0 \) implies that
\[
\alpha_0 = \sum_{G \in \mathcal{G}[1,1]} w(G),
\]
where \( w(G) = \prod_{(J,X) \in G} F_{J,X} \). Using Equation (2.18) and the definition of connectedness for hypergraphs implies that for \( n \geq 1 \)
\[
\sum_{G \in \mathcal{G}[0,n]} w(G) = \alpha_0^{-1} \left( \sum_{G_1 \in \mathcal{G}[0,1]} \sum_{G_2 \in \mathcal{G}[1,n]} w(G_1) w(G_2) + \sum_{j \geq 2} \sum_{G_1 \in \mathcal{G}[0,j]} \sum_{G_2 \in \mathcal{G}[j,n]} w(G_1) w(G_2) \right),
\]
The factor of \( \alpha_0^{-1} \) multiplying the first term arises since the hypergraphs \( G \in \mathcal{G}[1,1] \) are double counted due to being present in both \( \mathcal{G}[0,1] \) and \( \mathcal{G}[1,n] \). The factor of \( \alpha_0^{-1} \) multiplying the second factor arises similarly, due to double counting of the sum over \( \mathcal{G}[j,j] \); translation invariance of the \( \mathcal{X} \) gas implies this is the same as the sum over \( \mathcal{G}[1,1] \). The next lemma simplifies Equation (2.19) by computing the sum over \( \mathcal{G}[0,1] \).

Lemma 2.15. Fix \( n \geq 1 \). The following equation holds:
\[
\sum_{G \in \mathcal{G}[0,n]} \prod_{(J,X) \in G} F_{J,X} = \alpha \sum_{G \in \mathcal{G}[1,1]} \prod_{(J,X) \in G} F_{J,X} + \alpha_0^{-1} \sum_{j \geq 2} \sum_{G_1 \in \mathcal{G}[0,j]} \sum_{G_2 \in \mathcal{G}[j,n]} \prod_{(J,X) \in G_1} F_{J,X} \prod_{(J,X) \in G_2} F_{J,X}.
\]
Proof. Let \( \omega \) be a walk. Lemma 2.4 and (2.3) imply that

\[
\sum_{G \in \mathcal{G}[a,b]} \prod_{(J,X) \in G} F_{J,X}(\omega) = \mathbb{1}_{\{\omega_i=\omega_t\}} \prod_{X \in X} \left(1 + \alpha_X \right)^1_{\{\omega_i \in \ell(X) \cap \ell(X) \neq \emptyset\}}.
\]  

The constraint that \( \omega_0 \neq \omega_1 \) is irrelevant as \( \omega_{j+1} \neq \omega_j \) for any walk. Using the representation of \( \alpha_0 \) in Equation (2.18) gives

\[
\sum_{G_1 \in \mathcal{G}[0,1]} \prod_{(J,X) \in G} F_{J,X}(\omega) = (1 + \alpha_X)^{1}_{\{\omega_1 \in \ell(X) \cap \ell(X) \neq \emptyset\}}.
\]

and this last quantity is \( \alpha \) by Assumption 3. Using (2.18) and (2.22) to rewrite (2.19) gives the claim. \( \square \)

2.4.2 Laces for Hypergraphs and Weights on Lace Edges

The weight \( w(G) = \prod F_{J,X} \) on hypergraphs can be pushed forward to a weight \( w^\omega_\ast(st) \) on labelled graphs:

\[
w^\omega_\ast(st, \text{timelike}) = -\mathbb{1}_{\{\omega_s=\omega_t\}} \tag{2.23}
w^\omega_\ast(st, \text{spacelike}) = (1 - \mathbb{1}_{\{\omega_s=\omega_t\}}) \sum_{\{\{J_i, X_i\}\}} \prod_{i} F_{J_i, X_i}(\omega). \tag{2.24}
\]

The sum in (2.24) is over all non-empty collections of hyperedges, each of whose span is the labelled edge \((st, \text{spacelike})\). The factor \((1 - \mathbb{1}_{\{\omega_s=\omega_t\}})\) accounts for the possibility that a timelike hyperedge exists when the edge \(st\) is given the label spacelike. Note that this weight neglects hyperedges \((J,X)\) with \(|J|=1\). For notational ease let \( F_{J,X} = F_{\{J\},X} \).

Lemma 2.16. The following identity holds for \( a < b \):

\[
\sum_{G \in \mathcal{G}[a,b]} \prod_{(J,X) \in G} F_{J,X}(\omega) = \sum_{L \in \mathcal{L}[a,b]} \prod_{m \in L} w^\omega_\ast(st) \prod_{\text{span}(J,X) \in \mathcal{C}(L)} (1 + F_{J,X}(\omega)) \prod_{X \in X} (1 + F_{J,X}(\omega)). \tag{2.25}
\]

The left-hand sum is over all connected hypergraphs on \([a,b]\), while the right-hand sum is over labelled laces.

Proof. Apply Lemma 2.10 with the weight \( w_\ast \), and take the product of this equation with the last term on the right-hand side of (2.25). Expanding the product over connected labelled graphs with weight \( w_\ast \), gives the left-hand side of (2.25) as hyperedges of the form \(\{J\}, X\) play no role in connectivity, and for each \(st\) the weight \( w_\ast \) is a sum of the possible collections of hyperedges whose span is \(st\). Similarly, \(1 + w_\ast(ij)\) for \(ij \in \mathcal{C}(L)\) can be written in the product form used above, giving the right-hand side. \( \square \)

The next definition and lemma simplifies the sum over laces in (2.25) by resumming the contributions to the product over \(st \in L\).

Definition 2.17. For \( 0 \leq s < t \) define

\[
I^X_\ast(s,t) = \begin{cases} 
1 & \omega_s = \omega_t \\
1 - \prod_{X \in X} \left(1 - \frac{\alpha_X}{1+\alpha_X} \mathbb{1}_{\{\omega_s \in \ell(X)\}} \mathbb{1}_{\{\omega_t \in \ell(X)\}} \mathbb{1}_{\{\ell(X) \cap \text{range}(\omega(s,t)) = \emptyset\}} \right) & \omega_s \neq \omega_t.
\end{cases}
\]

(2.26)

Lemma 2.18. Let \( st \) be an edge. Then

\[
w^\omega_\ast(st, \text{spacelike}) + w^\omega_\ast(st, \text{timelike}) = -I^X_\ast(s,t) \tag{2.27}
\]
Proof. The case \( \omega_s = \omega_t \) corresponds to the timelike edge. Consider the spacelike term. Any non-empty collection of spacelike hyperedges \( \{(J_i, X_i)\} \) may be chosen, so

\[
w_s^*(st, \text{spacelike}) = (1 - \mathbb{1}_{\{\omega_s = \omega_t\}}) \left[ \prod_{(J, X) \in \mathcal{J}_s(X)} (1 + F_{J, X}(\omega)) - 1 \right].
\] (2.28)

A hyperedge with span \( st \) is equivalent to a possibly empty subset \( J \) of \( (s, t) \), so using \( F_{J \cup \{ab\}, X} = \mathbb{1}_{\{\omega_s \in \ell(X)\}} \mathbb{1}_{\{\omega_t \in \ell(X)\}} F_{J, X} \) gives

\[
w_s^*(st) = \mathbb{1}_{\{\omega_s \neq \omega_t\}} \left[ \prod_{X \in \mathcal{X}} \prod_{J \subset (s, t)} \left( 1 + \mathbb{1}_{\{\omega_s \in \ell(X)\}} \mathbb{1}_{\{\omega_t \in \ell(X)\}} F_{J, X}(\omega) \right) - 1 \right],
\] (2.29)

where we recall that \( F_{\emptyset, X}(\omega) = -\alpha X (1 + \alpha X)^{-1} \). Putting the condition that \( \omega_s \) and \( \omega_t \) are in \( \ell(X) \) into the product, separating the case \( J = \emptyset \), and then applying Lemma 2.4 yields

\[
w_s^*(st, \text{spacelike}) = \mathbb{1}_{\{\omega_s \neq \omega_t\}} \left[ \prod_{X \in \mathcal{X}} \prod_{s, t \in \ell(X)} \left( 1 - \frac{\alpha X}{1 + \alpha X} \right) \prod_{J \subset (s, t)} \left( 1 + F_{J, X}(\omega) \right) \right] - 1
\] (2.30)

\[
= \mathbb{1}_{\{\omega_s \neq \omega_t\}} \left[ \prod_{X \in \mathcal{X}} \prod_{s, t \in \ell(X)} (1 + \alpha X)^{-1} (\text{range}(\omega_s, \omega_t) \cap \ell(X) = \emptyset) - 1 \right],
\] (2.31)

which is the second half of (2.26). \( \square \)

### 2.5 The Lace Expansion Equation for Self-Avoiding Walk in an \( \mathcal{X} \) Gas

This section uses Lemmas 2.15 and 2.16 to obtain a recursion for the \( n \)-step connectivity. By summing the resulting recursion over \( n \) the lace expansion for a self-avoiding walk in an \( \mathcal{X} \) gas is obtained.

#### 2.5.1 Lace Expansion Equation

For \( m \geq 2 \) define \( \pi^{(N)}_m(x) \) to be

\[
(-1)^N z^m \alpha_0^{-1} \sum_{\omega: 0 \to x} \sum_{L \in \mathcal{C}^{(N)}[0,m]} \left( \prod_{s \in L} -F_s^\mathcal{X}(s, t) \right) \prod_{(J, X) \in \mathcal{C}(L)} (1 + F_{J, X}(\omega)) \prod_{a \leq j \leq b} (1 + F_{J, X}(\omega)),
\] (2.32)

where \( \mathcal{C}^{(N)}[0, m] \) is the set of laces with \( N \) edges on the interval \([0, m]\). Let \( \pi_m \) denote \( \sum_{N \geq 1} (-1)^N \pi^{(N)}_m \).

Define the \( n \)-step connectivity \( c_n \) to be zero for \( n < 0 \). Equation (2.12) combined with Lemmas 2.15 and 2.16 imply

\[
z^n c_n(a, x) = \sum_{0 \to x} \sum_{y \sim a} z^{n-1} c_{n-1}(y, x) + \sum_{j \geq 2} \sum_{y} \pi_j(y) z^{n-j} c_{n-j}(y, x) \quad n \geq 1, \quad n = 0.
\] (2.33)

Let \( G_z(x) = \sum_n c_n(a, x) \). Summing (2.33) over \( n \), using the translation invariance of \( G_z(x) \), and taking the Fourier transform yields

\[
\hat{G}_z(k) = \alpha_0 + \alpha z \hat{D}(k) \hat{G}_z(k) + \hat{\Pi}_z(k) \hat{G}_z(k),
\] (2.34)

where \( \Pi_z(x) = \sum_{m \geq 2} \pi_m(x) \).

The next two sections give expressions for \( \pi^{(N)}_m(x) \) in terms of self-avoiding walks in an \( \mathcal{X} \) gas.
2.5.2 Walk Representation of $\pi^{(X)}_m(x)$ for $N = 1$

If $N = 1$ the lace consists of a single edge $0m$. If $x = 0$ then $\omega_0 = \omega_m$, $I^0_2(0, m) = 1$, and

$$
\pi^{(1)}_m(0) = z^m \alpha_0^{-1} \sum_{\omega: \omega = 0 \to x \atop |\omega| = m} \mathbb{1}_{\{\omega \in \Omega_{SAW}\}} \prod_X (1 + \alpha_X)^{1_{\{\text{range}(\omega) \cap \ell(X) \neq \emptyset\}}}.
$$
(2.35)

If $x \neq 0$ the set of incompatible hyperedges are those that contain both $0$ and $m$. Let $m_1 = m - 1$ and fix $X \in \mathcal{X}$. Corollary 2.6 implies that for $\omega \in \Omega_{SAW}$

$$
\prod_{(J,X) \in \mathcal{C}(0m)} (1 + F_{J,X}(\omega)) = \prod_{X \in \mathcal{X}} (1 + \alpha_X)^{1_{\{\text{range}(\omega) \cap \ell(X) \neq \emptyset\}}} + 1_{\{\text{range}(\omega) \cap \ell(X) \neq \emptyset\}} \prod_{\omega \in \ell(X)} (1 + \alpha_X)^{1_{\{\text{range}(\omega[1, m_1]) \cap \ell(X) = \emptyset\}}} - 1
$$
(2.36)

while the right-hand side is zero if $\omega$ is not self-avoiding. Precisely, Equation (2.36) follows by using Corollary 2.6 to compute the products over hyperedges $(J, X)$ with (i) $J \subset [1, m_1]$, (ii) $J \subset [1, m]$ with $m \in J$, and (iii) $J \subset [0, m_1]$ with $0 \in J$. The product over compatible hyperedges is the product of these terms. The definition of $I^0_2(0, m)$ when $\omega_m = x \neq 0$ then gives

$$
\pi^{(1)}_m(x) = z^m \alpha_0^{-1} \sum_{\omega: \omega = 0 \to x \atop |\omega| = m} \mathbb{1}_{\{\omega \in \Omega_{SAW}\}} \prod_X (1 + \alpha_X)^{1_{\{\text{range}(\omega) \cap \ell(X) \neq \emptyset\}}} (\prod_{X \in \mathcal{X}} (1 + \alpha_X)^{1_{\{\text{range}(\omega) \cap \ell(X) \neq \emptyset\}}} - 1).
$$
(2.37)

2.5.3 Walk Representation of $\pi^{(N)}_m(x)$ for $N \geq 2$

For $N \geq 2$ the central observation is that the edges of a lace on $[a, b]$ divides the interval $[a, b]$ into $2N - 1$ subintervals, see Figure 2.

**Definition 2.19.** Let $m \in \mathbb{N}$. A vector $\vec{m}$ with components $m_1, \ldots, m_{2N-1}$ is called valid if (i) $m_1 \geq 1$, $m_{2N-1} \geq 1$, and $m_{2j} \geq 1$ for $1 \leq j \leq N - 1$, (ii) $m_{2j+1} \geq 0$ for $1 \leq j \leq N - 1$, and (iii) $\sum m_i = m$.

The lengths of the subintervals determined by a lace form valid vector $\vec{m}$. The restrictions on which $m_i$ are strictly positive arise from the definition of connectedness, see [14, Section 3.3] for more details. The subintervals are given by

$$
\vec{I}_1 = [0, m_1], \quad \vec{I}_2 = [m_1, m_1 + m_2], \ldots \quad \vec{I}_{2N-1} = [m_1 + \ldots m_{2N-2}, m_1 + \ldots m_{2N-1}].
$$
(2.39)

To each interval $\vec{I}_k$ associate a walk $\omega^{(k)}$, e.g. $\omega^{(2)} = (\omega_{m_1}, \omega_{m_1+1}, \ldots, \omega_{m_1+m_2})$. The walks $\omega^{(k)}$ interact with one another through the compatible edges.

To the $k^{th}$ interval associate (i) all hyperedges whose span is contained in $\vec{I}_k$ and (ii) all compatible hyperedges $(J, X)$ such that span$(J, X)$ is not contained in $\vec{I}_k$ with max $J \in \vec{I}_k$ and max $J \neq \max \vec{I}_k$.

For the subinterval $2N - 1$ omit the last condition: if a hyperedge has max $J$ being the maximal index possible, i.e., $m$, associate this edge to $\vec{I}_{2N-1}$. Subintervals $\vec{I}_k$ for $k < 2N - 1$ are missing hyperedges of the form $(\max \vec{I}_k, X)$. Including them and dividing by $\alpha_0$ to correct for this shows the weight associated to the interval $\vec{I}_k$ is

$$
\alpha_0^{-1} \prod_{(J, X) \subset \vec{I}_k} (1 + F_{J,X}) \prod_{(J, X) \in \mathcal{C}_k} (1 + F_{J,X}),
$$
(2.40)

where the factor of $\alpha_0^{-1}$ for $k = 2N - 1$ comes from the prefactor $\alpha_0^{-1}$ in the definition of $\pi^{(N)}_m$.

The last two factors together are easy to evaluate. A compatible hyperedge must have its minimum index be at least the second index of either $\omega^{(k-2)}$ or $\omega^{(k-3)}$. Suppose the first case; the second is
similar. Corollary 2.6 implies the product in (2.40) forces $\omega^{(k)}$ to be self-avoiding, $\omega^{(k)}$ to avoid $\omega^{(k-1)}$ and $\omega^{(k-2)}$ [1:], and assigns $\omega^{(k)}$ the weight
\[
\alpha_{0}^{-1} \prod_{X \in \mathcal{X}} (1 + \alpha_{X})^{i_{\text{range}(\omega^{(k)} \cap \tau(X), \mathcal{X})}} \left(1^{i_{\text{range}(\omega^{(k-2)}, \omega^{(k-1)}, [1:]) \cap \tau(X), \mathcal{X})}} \right),
\]
(2.41)

**Figure 3:** The diagrammatic representation of $\pi_{m}^{(N)}(x)$ for $m = \sum m_{i}$. The vertices $x_{1}, \ldots, x_{4}$ and $x'_{0}, \ldots, x'_{3}$ are summed over.

As an explicit formula for $\pi_{m}^{(N)}$ detailing the constraints is unwieldy, let us explain the formula with a brief discussion of the diagrammatic representation of $\pi_{m}^{(N)}$ in Figure 3. The solid lines represent a subdivision of a walk $\omega$ into subwalks; these subwalks are subject to self-avoidance constraints detailed below. Pairs of zigzag lines represent $I_{X}^{i}(t, t_{i+1})$, where $t_{i}$ is the time $x_{i}$ occurs in the walk $\omega = \omega^{[1]} \circ \ldots \circ \omega^{[2N-1]}$. Each walk $\omega^{(i)}$ has length $m_{i}$ and is self-avoiding. Further, each walk $\omega^{(i)}$ avoids some of the previous walks $\omega^{(j)}$ for $j < i$, excluding the endpoint of $\omega^{(i-1)}$. The walk $\omega^{(j)}$ is weighted by those particles in $\mathcal{X}$ that do not intersect the subwalks which $\omega^{(i)}$ is forbidden to intersect; for example, in Equation (2.41) the walk $\omega^{(k)}$ is being weighted by all particles that do not intersect $\omega^{(k-1)}$ or $\omega^{(k-2)}$ [1:].

**Remark 2.20.** If the $\mathcal{X}$ gas is repulsive in the sense that $\alpha_{X} \geq 0$ for each $X$, then ignoring the fact that some particles do not weight a subwalk gives an upper bound for the weight on the subwalks $\omega^{(i)}$. Ignoring the constraint of avoiding $\omega^{(j)}$ for some $j < i$ gives a further upper bound on $\pi_{m}^{(N)}(x)$.

## 3 Application of the $\mathcal{X}$ gas Lace Expansion to $\lambda$-LWW

As was remarked earlier, Theorem 1.13 shows that loop-erased $\lambda$-LWW is a self-avoiding walk in an $\mathcal{X}$ gas, and hence the lace expansion of Section 2 can be applied. The relevant $\mathcal{X}$ gas was defined in Equations (1.20)–(1.22). Quantities such as $\alpha_{0}(\mathcal{X})$ and $I_{X}^{i}$ will be written as $\alpha_{0}(\lambda, z)$, $I_{\lambda, z}^{i}$, and so forth in what follows. The arguments $\lambda$ and $z$ will be omitted on occasion to lighten the notation. Note that $\lambda \geq 0$ and $z \geq 0$ implies $w_{\lambda, z}(\omega) \geq 0$, and hence $\alpha_{\omega} \geq 0$. In particular, $\lambda$-LWW is repulsive in the sense of Remark 2.20.

**Definition 3.1.** The two point function $G_{\lambda, z}(x, y)$ for $\lambda$-LWW is defined by
\[
G_{\lambda, z}(x, y) = \sum_{\omega: x \rightarrow y} w_{\lambda, z}(\omega).
\]
(3.1)

Theorem 1.13 and Equation (1.14) imply that the two-point function of the self-avoiding walk in the $\mathcal{X}$ gas defined by Equations (1.20)–(1.22) is identical to the two-point function of $\lambda$-LWW. For $\lambda$-LWW Equation (2.34) specializes to the following proposition.

**Proposition 3.2.**
\[
\hat{G}_{\lambda, z}(k) = \frac{\alpha_{0}(\lambda, z)}{1 - \alpha(\lambda, z) [\Omega] D(k) - \Pi_{\lambda, z}(k)},
\]
(3.2)
To analyze the recursion (3.2) it will be convenient to re-express the quantities in (3.2) in terms of $w_{\lambda,z}$ and the loop measure $\mu_{\lambda,z}$.

$$\alpha_0(\lambda, z) = \exp(\mu_{\lambda,z}(0)),$$
$$\alpha(\lambda, z) = \exp(\mu_{\lambda,z}(0; y)).$$

Note that $\alpha_0 \geq \alpha \geq 1$. Let $I_{\lambda,z}^A = I_{\lambda,z}^B$. $I_{\lambda,z}^A(a, b)$ can be written in a loop measure like way:

$$I_{\lambda,z}^A(a, b) = \mathbb{1}_{\{\omega_u = \omega_b\}} + \mathbb{1}_{\{\omega_u \neq \omega_b\}} \left(1 - e^{-\mu_{\lambda,z}(\omega_u; \omega_b; \omega(\omega(a, b)))}\right),$$

where

$$\mu_{\lambda,z}(A; B; C) = \sum_{x} \sum_{\omega: x \rightarrow x} \frac{w_{\lambda,z}(\omega)}{|\omega|} \mathbb{1}_{\{\omega(\omega) \cap A = \emptyset\}} \mathbb{1}_{\{\omega(\omega) \cap B = \emptyset\}} \mathbb{1}_{\{\omega(\omega) \cap C = \emptyset\}}.$$  

As with the loop measure, define $\mu_{\lambda,z}(A, B) = \mu_{\lambda,z}(A, B; \emptyset)$. The effect of this more complicated loop measure is to require that both an element from $A$ and $B$ are in the range of the walk.

4 Convergence of the Lace Expansion I. Preliminaries

This section establishes the basic facts used to prove the convergence of the lace expansion. The strategy is that of [14], adapted for $\lambda$-LWW. An important role is played by the function $H_{\lambda,z}$ in the next definition.

**Definition 4.1.** The reduced two point function $H_{\lambda,z}(x, y)$ is defined by

$$H_{\lambda,z}(x, y) = (1 - \delta_{x,y})G_{\lambda,z}(x, y).$$

A useful fact that will be used repeatedly is that

$$G_{\lambda,z}(x, y) = \delta_{x,y} \alpha_0(\lambda, z) + H_{\lambda,z}(x, y).$$

The two-point functions $G_{\lambda,z}$ and $H_{\lambda,z}$ inherit the isometry invariance of the weight $w_{\lambda,z}$. By translation invariance $G_{\lambda,z}(x, y) = G_{\lambda,z}(o, y - x)$, and $G_{\lambda,z}(x)$ will denote $G_{\lambda,z}(o, x)$.

4.1 Random Walk Quantities and Bounds

**Definition 4.2.** The random walk $2$-point function $C_z(x)$ and its Fourier transform $\hat{C}_z(k)$ are given by

$$C_z(x) = \sum_{\omega: o \rightarrow x} \mathbb{1}_{\{\omega\}} z^{\omega},$$
$$\hat{C}_z(k) = \frac{1}{1 - z |\Omega|} D(k).$$

The following facts about the random walk two-point function will be useful. For notational clarity, let $\beta$ be a quantity that is $O(|\Omega|^{-1})$. $\beta$ should be thought of as a small parameter.

**Lemma 4.3** (Lemma 5.5 of [11]). Assume $d > 4$. Then for $0 \leq z \leq |\Omega|^{-1}$

$$\sup_x D(x) \leq \beta,$$
$$\|C_z\|_2^2 \leq 1 + c\beta,$$
$$\|(1 - \cos(k \cdot x))C_z(x)\|_{\infty} \leq 5(1 + c\beta)(1 - D(k)).$$

**Proposition 4.4.** Let $r \in \mathbb{N}$. There is a constant $K$ independent of $d$ such that for $d > 2r$.

$$\int_{[-\pi, \pi]^d} \left(\frac{1}{1 - D(k)}\right)^r \frac{d^dk}{(2\pi)^d} \leq 1 + c\beta.$$  

**Proof.** This is a small modification of the proof of [11, Lemma 4.3].
4.2 Convergence Strategy and Basic Bounds

The proof of convergence is based on comparing the behaviour of simple random walk and $\lambda$-LWW. Define $p(z)$ by

$$
\frac{\hat{G}_{\lambda,z}(0)}{\alpha_0(\lambda,z)} = \frac{1}{1 - p(z)} \frac{1}{|\Omega|} = \hat{C}_{p(z)}(0).
$$

Roughly speaking, the intuition is that $\lambda$-LWW should behave like simple random walk. The definition of $p(z)$ serves to determine the activity of the simple random walk that matches $\lambda$-LWW with activity $z$. The following bootstrap lemma is what enables conclusions to be drawn for $z < \zeta_0(\lambda)$.

**Lemma 4.5** ([14, Lemma 5.9]). Let $a < b$, let $f$ be a continuous function on the interval $[z_1, z_2]$, and assume that $f(z_1) \leq a$. Suppose for each $z \in (z_1, z_2)$ that if $f(z) \leq b$ then in fact $f(z) \leq a$. Then $f(z) \leq a$ for all $z \in [z_1, z_2]$.

To describe the function $f$ used in applying Lemma 4.5 some definitions are needed.

**Definition 4.6.** Define $\Delta_k \hat{A}(\ell)$ by

$$
-\frac{1}{2} \Delta_k \hat{A}(\ell) = \hat{A}(\ell) - \frac{1}{2} \left( \hat{A}(\ell + k) + \hat{A}(\ell - k) \right),
$$

and define

$$
U_{p(z)}(k, \ell) = 16\hat{\epsilon}_{p(z)}^{-1}(k) \left( \hat{C}_{p(z)}(\ell - k)\hat{C}_{p(z)}(\ell) + \hat{C}_{p(z)}(\ell + k)\hat{C}_{p(z)}(\ell) + \hat{C}_{p(z)}(\ell - k)\hat{C}_{p(z)}(\ell + k) \right).
$$

The quantity $U_{p(z)}$ is in fact a convenient upper bound for $\frac{1}{2} \left| \Delta_k \hat{C}_{p(z)}(\ell) \right|$, this can be seen by using [14, Lemma 5.7]. Define $f(z) = \max\{f_1(z), f_2(z), f_3(z)\}$, where

$$
f_1(z) = z\alpha(\lambda, z) |\Omega|, \quad f_2(z) = \sup_{k,\ell \in [-\pi,\pi]^d} \frac{\hat{G}_{\lambda,z}(k)}{C_{p(z)}(k)}, \quad f_3(z) = \sup_{k,\ell \in [-\pi,\pi]^d} \frac{\Delta_k \hat{G}_{\lambda,z}(\ell)}{U_{p(z)}(k, \ell)}.
$$

The next lemma will be useful for estimating $G_{\lambda,z}$.

**Lemma 4.7.** Assume $y \neq x$. The following inequality holds:

$$
G_{\lambda,z}(x, y) \leq z\alpha(\lambda, z) |\Omega| \sum_u D(u)G_{\lambda,z}(u, y).
$$

**Proof.** This can be proven using the loop measure representation. For $a$ a walk beginning at $u \sim o$, let $o\eta = (o, u) \eta$.

$$
G_{\lambda,z}(o, y) = \sum_{\eta: o \sim y} \mathbb{1}_{\{\eta \in \Omega_{SAW}\}} z^{\eta} \exp(\mu_{\lambda,z}(\text{range}(\eta)))
$$

(4.13)

$$
= \sum_{u \sim o} \sum_{\eta: u \sim y} \mathbb{1}_{\{o\eta \in \Omega_{SAW}\}} z \exp(\mu_{\lambda,z}(o; \text{range}(\eta))) z^{\eta} \exp(\mu_{\lambda,z}(\text{range}(\eta)))
$$

(4.14)

$$
\leq z\alpha(\lambda, z) |\Omega| \sum_u D(u) \sum_{\eta: u \sim y} \mathbb{1}_{\{\eta \in \Omega_{SAW}\}} z^{\eta} \exp(\mu_{\lambda,z}(\text{range}(\eta)))
$$

(4.15)

$$
= z\alpha(\lambda, z) |\Omega| \sum_u D(u)G_{\lambda,z}(u, y).
$$

(4.16)

The inequality follows as (a) Proposition 1.14 implies $\mu_{\lambda,z}(o; \text{range}(\eta))$ is bounded above by $\mu_{\lambda,z}(o; u) = \alpha_0$ and (b) $\mathbb{1}_{\{\eta \in \Omega_{SAW}\}}$ is bounded above by $\mathbb{1}_{\{\eta \in \Omega_{SAW}\}}$. □
Proposition 4.8. Assume \( d > 4 \). Fix \( z \in (0, z_c) \) and assume \( f \leq K \). Then there is a constant \( c_K \) independent of \( z \) and \( d \) such that

\[
\| ( 1 - \cos(k \cdot x) ) H_{\lambda,z} \|_\infty \leq c_K ( 1 + \beta ) \hat{C}_p(z)^{-1},
\]

\[
\| H_{\lambda,z} \|_2^2 \leq c_K \beta \tag{4.17}
\]

\[
\| H_{\lambda,z} \|_\infty \leq c_K \beta. \tag{4.19}
\]

Proof. The general fact that \( \| g \|_\infty \leq \| g \|_1 \) and the identity

\[
\sum_x \cos(k \cdot x) f(x) e^{i\ell \cdot x} = \frac{1}{2} \left( \hat{f}(\ell + k) + \hat{f}(\ell - k) \right)
\]

imply that

\[
\| ( 1 - \cos(k \cdot x) ) H_{\lambda,z}(x) \|_\infty = \| ( 1 - \cos(k \cdot x) ) G_{\lambda,z}(x) \|_\infty \leq \frac{1}{2} \| \Delta_k \hat{G}_{\lambda,z}(\ell) \|_1. \tag{4.20}
\]

The definition of \( U \), the fact that \( f_3 \leq K \), and the Cauchy-Schwarz inequality then imply

\[
\| ( 1 - \cos(k \cdot x) ) H_{\lambda,z}(x) \|_\infty \leq 16K \hat{C}_p(z)^{-1} \| \hat{C}_p(z) \|_2^2, \tag{4.22}
\]

which yields (4.17) after using (4.5).

To estimate the \( \| H_{\lambda,z} \|_2^2 \) note that Lemma 4.7 implies

\[
H_{\lambda,z}(x) \leq z_\alpha(\lambda, z) |\Omega| D * G_{\lambda,z}(x) \tag{4.23}
\]

The factor \( z_\alpha |\Omega| \) is estimated using \( f_1(z) \leq K \). To estimate \( D * G_{\lambda,z} \) use Parseval’s identity, \( f_2(z) \leq K \), and Lemma 4.3.

\[
\| H_{\lambda,z} \|_2^2 \leq K^2 \| D * G_{\lambda,z} \|_2^2 \leq K^4 \| \hat{D} \hat{C}_p(z)^{-1} \|_2^2 = K^4 \| \hat{C}_p(z)^{-1} \|_2^2 - 1 \leq K^4 \beta. \tag{4.24}
\]

For the last inequality use the fact that \( \sup_x H_{\lambda,z}(x) = \sup_{x \neq 0} G_{\lambda,z}(x) \), Lemma 4.7, Equation (4.2) and then Lemma 4.7 again. Using \( f_1 \leq K \) gives

\[
H_{\lambda,z}(x) \leq K \alpha_0(\lambda, z) D(x) + K^2 D * D * G_{\lambda,z}(x). \tag{4.25}
\]

A little manipulation shows that \( \| D * D * G_{\lambda,z} \|_\infty \leq \| \hat{D} \hat{C}_p(z)^2 \|_1 \), so Lemma 4.3 implies

\[
\| D * D * G_{\lambda,z} \|_\infty \leq K \beta. \tag{4.26}
\]

Equation (4.4) implies \( D(x) \leq \beta \) so it suffices to show \( \alpha_0(\lambda, z) \) is bounded above. This follows from \( f_2 \leq K \):

\[
\alpha_0 = \int_{[-\pi, \pi]^d} \hat{G}_{\lambda,z}(k) \frac{d^d k}{(2\pi)^d} \leq K \int_{[-\pi, \pi]^d} \hat{C}_p(z)(k) \frac{d^d k}{(2\pi)^d} \leq K \| \hat{C}_p(z) \|_1, \tag{4.27}
\]

and this last integral is finite for \( d \geq 3 \), and decreases as the dimension \( d \) increases. \( \square \)

5 Convergence of the Lace Expansion II. Diagrammatic Bounds and Convergence

To control the lace expansion it is necessary to show that \( \hat{\Pi}_{\lambda,z} \) is small. This is done by obtaining bounds on norms of \( \Pi^{(N)}_{\lambda,z} \) in terms of \( H_{\lambda,z}, G_{\lambda,z} \), and \( I_{\lambda,z} \); these bounds are known as diagrammatic bounds. Coupled with Proposition 4.8 diagrammatic bounds are what make the hypothesis \( f(z) \leq K \) powerful.

Obtaining diagrammatic bounds requires bounding the weight of walks constrained to have \( \omega_j = x \) in terms of unconstrained walks. This is best illustrated by an example. Consider obtaining a bound
for \( \frac{d}{dz} G_{\lambda,z}(o,x) \). For self-avoiding walk (\( \lambda = 0 \)) this is straightforward: the Leibniz rule implies the derivative is a sum over all self-avoiding walks from \( o \) to \( x \) together with a marked edge. Splitting the walk at the marked edge and using the fact that self-avoiding walk is purely repulsive yields

\[
\frac{d}{dz} G_{0,z}(o,x) \leq z^{-1} G_{0,z} \ast H_{0,z}(o,x).
\] (5.1)

For \( \lambda > 0 \) a similar argument is possible, but the weight on the second half of the walk is not \( w_{\lambda,z} \): knowledge of the first half of the walk is needed to know when loops are erased. Section 5.1 derives identities for walks that play the role of Equation (5.1) for \( \lambda > 0 \) in terms of the weight \( w_{\lambda,z} \) alone. Section 5.2 uses these identities to derive the diagrammatic bounds necessary to apply Lemma 4.5.

### 5.1 Decompositions for \( \lambda \)-LWW

The formulas presented in this section are simply the result of tracking what happens when loop erasure is performed.

#### 5.1.1 Decompositions from Loop Erasure

The loop erasure of a walk can be thought of as a last exit decomposition: if \( \omega : x \to y \) then the second vertex in the loop erasure is the first vertex visited after the last visit to \( x \). Iterating this implies the next proposition.

**Proposition 5.1.** Let \( \omega \) be a walk. Define \( \ell_0 = 0 \), and \( \ell_k = \sup \{ j \mid \omega_j = \omega_{\ell_{k-1}} \} + 1 \). Suppose there are \( n + 1 \) finite values of \( \ell_k \) such that \( \ell_k \leq |\omega| \). Then

\[
\operatorname{LE}(\omega) = (\omega_{\ell_0}, \omega_{\ell_1}, \ldots, \omega_{\ell_n}).
\] (5.2)

In Proposition 5.1 the restriction to finite values at most \( |\omega| \) is due to the fact that there will be an \( \ell_k = |\omega| + 1 \), and then \( \ell_{k+1} = -\infty \). The loop erasure of a walk \( \omega \) induces a decomposition of \( \omega \) itself. Let \( \eta = \operatorname{LE}(\omega) = (\omega_{\ell_0}, \ldots, \omega_{\ell_n}) \). Define, for \( 0 \leq r < s \leq k \),

\[
\eta^{-1}[r,s] = \omega[\ell_r, \ell_s - 1],
\] (5.3)

where, recalling Proposition 5.1, \( \ell_{k+1} = |\omega| + 1 \). See Figure 4.

**Remark 5.2.** It would be notationally more accurate to write \( \operatorname{LE}(\omega)^{-1}[r,s] \) as the definition requires knowledge of the walk \( \omega \) whose loop erasure is \( \eta \). As the walk \( \omega \) will be clear from context this will not cause any confusion.

![Figure 4: An illustration of the definition of \( \eta^{-1}[r,s] \) and of the subdivision of a walk given by Equation (5.4). The blue walk, \( \eta^{-1}[0,t_1] \), is the preimage of the blue portion of the loop erasure of the walk. The red and green curves are \( \eta^{-1}[t_1,t_2] \) and \( \eta^{-1}[t_2,t_3] \) respectively. The small gaps in curves indicate the flow of time.](image-url)

The following extension of the notion of the concatenation of two walks will be notationally convenient. If \( \omega^1 : x_1 \to y_1 \) and \( y_1 \sim x_2 \) write \( \omega^1 \circ \omega^2 \) for the walk that consists of \( \omega^1 \) followed by a step from \( y_1 \) to \( x_2 \) followed by the walk \( \omega^2 \).
For a walk $\omega$ whose loop erasure has $k$ steps and a sequence of times $0 = t_0 < t_1 < t_2 < \cdots < t_n = k$
Equation (5.3) yields a decomposition of $\omega$.

$$\omega = \eta^{-1} [t_0, t_1] \cdots \eta^{-1} [t_{n-1}, t_n].$$
(5.4)

The decomposition of $\omega$ given by (5.4) has two notable features. First, the loop erasure of the segments of the decomposition yield $\eta [t_i, t_{i+1} - 1]$. Second, each segment, barring perhaps the first segment, never returns to its starting vertex. See Figure 4.

The next definitions serve to formalize the idea that given the loop erasure $\eta = \text{LE}(\omega [0, j])$ of a walk $\omega$ up to time $j$, the remainder of $\omega$ has the effect of erasing some of $\eta$, and then extending the remainder of $\eta$ to complete the formation of $\text{LE}(\omega)$.

**Definition 5.3.** Let $A \subset \mathbb{Z}^d$. The hitting time $\tau_\omega(A)$ of $A$ by $\omega$ is $\tau_\omega(A) = \inf\{j \geq 0 \mid \omega_j \in A\}$.

**Definition 5.4.** Let $\eta; x \to y$ be a self-avoiding walk, and let $\omega$ be a walk beginning at $y$. Let $\eta^0 = \eta[0, |\eta|]$. For $k \geq 1$ inductively define

$$s_k^0(\omega) = \tau_\omega(\eta^{k-1}), \quad t_k^0(\eta) = \eta^{-1}(\omega_{s_k^0(\omega)}), \quad \eta^k = \eta[0, t_k^0(\eta)].$$
(5.5)

The times $s_k^0(\omega)$ are the shrinking times of $\eta$ by $\omega$.

See Figure 5 for an illustration of shrinking times. The walks $\eta^k$ in the definition are decreasing in length, and it follows that the times $t_k^0(\eta)$ are decreasing in $k$.

**Figure 5:** An illustration of the shrinking times of the self-avoiding walk $\eta$ (dashed black) by $\omega$ (red). The gaps in $\omega$ are to indicate the progress of time. Note that the second hitting time of $\eta$ is not a shrinking time as it occurs on a portion of $\eta$ that is erased at the first hitting time.

### 5.1.2 Expected Visits of $\lambda$-LWW

The next proposition gives a formula for the expected number of visits of a closed $\lambda$-LWW to a given vertex $y$. As the notation in the proposition is somewhat involved a description of the formula in advance may be beneficial. The number of visits by a walk $\omega$ to a vertex $y$ can be expressed as

$$|\{j \geq 1 \mid \omega_j = y\}| = \sum_{j \geq 1} I_{\{\omega_j = y\}}.$$  
(5.6)

Consider a walk with $\omega_j = y$. This naturally splits into two pieces: the walk $\omega^{(a)}$ up to time $j$, and the walk $\omega^{(b)}$ after time $j$. The splitting times introduced in Section 5.1 then splits each of $\omega^{(a)}$ and $\omega^{(b)}$ into $k$ segments if there are $k$ splitting times. In Proposition 5.5 the segments of $\omega^{(a)}$ are called $\omega^{(a)}_i$ for $i = 1, \ldots, k$, and the segments of $\omega^{(b)}$ are called $\omega^{(b)}_{i+k}$ for $i = 1, \ldots, k$. The conditions $A_i$ and $B_i$ are formalizations of the fact that these subwalks arise from splitting times.

**Proposition 5.5.** Fix $x, y \in \mathbb{Z}^d$, $y \neq x$. Then

$$\sum_{\omega: x \to x} |\{j \geq 1 \mid \omega_j = y\}| w_{\lambda, z}(\omega) = \alpha_0 \sum_{k \geq 1} \sum_{x_0, \ldots, x_k \text{ distinct}} \sum_{i=1}^k I_{\{x_0 = x\}} I_{\{x_k = y\}} \lambda^k$$
(5.7)

$$\sum_{\omega^{(a)}: x_{i-1} \to x_i} \prod_{i=1}^k w_{\lambda, z}(\omega^{(i)}) I_{\{\omega^{(i)} \in A_i\}} [\prod_{i=1}^k w_{\lambda, z}(\omega^{(k+i)}) I_{\{\omega^{(k+i)} \in B_i\}}]$$
(5.8)
where $A_i$ and $B_i$ are defined as follows. A walk $\omega$ is in $A_i$ if $\omega[1:]$ does not hit $\text{LE}(\omega^{(j)})$ for any $j < i$. A walk $\omega$ is in $B_i$ if

1. $\omega$ does not hit $\omega_{k-j}$ for $j > i + 1$,
2. $\omega[1:]$ hits $\omega_{k-i}$ at $\omega_{0}^{k-i}$,
3. $\omega$ hits $\omega_{k-i-1}$ at $\omega_{0}^{k-i}$, and $\omega$ does not hit $\text{LE}(\omega^{(k-1)}) \setminus \{\omega_{0}^{k-i}\}$.

Proof. Rewrite $|\{j \geq 1 \mid \omega_j = y\}|$ as $\sum_{j \geq 1} 1_{\{\omega_j = y\}}$. To prove the claim it suffices to show that walks with $\omega_j = y$ are in bijection with the summands such that $|\omega^{(1)} \cdots \omega^{(k)}| = j$.

Suppose $\omega_j = y$, and let $\eta = \text{LE}(\omega[0, j])$. Let $t^i, s^k$ be $t^i_\omega(\eta)$ and $s^k_\omega(\omega)$, respectively. Assume there are $k$ shrinking times for the walk $\omega$. Observing that $\omega$ closed implies $t^k = 0$, $s^k = |\omega|$ implies

$$\begin{align*}
\omega[0, j] &= \eta^{-1} [t^k, t^k-1] \cdots \eta^{-1} [t^2, t^1] \eta^{-1} [t^1, |\eta|] \\
\omega[j:] &= \omega[j, s^1] \cdots \omega [s^{k-1}, s^k].
\end{align*}$$

(5.9) (5.10)

Call the subwalks on the right-hand sides of (5.9) and (5.10) the constituents of $\omega[0, j]$ and $\omega[j:]$, respectively. Call a walk $\omega$: $x \rightarrow x$ an excursion if the only occurrences of $x$ in $\omega$ are $\omega_0$ and $\omega_{|\omega|}$.

Separating any initial excursions from $x$ to $x$ from the first subwalk comprising $\omega[0, j]$ gives the factor $\alpha_0$. To complete the claim, notice that any excursions immediately after a shrinking time that occur prior to the next hitting time of $\eta^i$ can be transferred to the previous subwalk comprising $\omega[j:]$. In the case of the first constituent of $\omega[j:]$ the excursions can be transferred to the last constituent of $\omega[0, j]$.

The next proposition handles the case of visits to the initial vertex of a walk.

Proposition 5.6.

$$\sum_{\omega: x \rightarrow x \atop |\omega| \geq 1} |\{j \geq 1 \mid \omega_j = x\}| \omega_{\lambda, z}(\omega) = \alpha_0(\alpha_0 - 1).$$

(5.11)

Proof. Write $|\{j \geq 1 \mid \omega_j = x\}|$ as $\sum_{j \geq 1} 1_{\{\omega_j = x\}}$. Inserting this into the left-hand side of (5.11) and split each walk $\omega$ at time $j$. Summing the remainder after time $j$ gives a factor $\alpha_0$. Summing over $j$ gives $\alpha_0 - 1$ as $j \geq 1$ implies the empty walk is excluded.

The next definition is to avoid explicitly writing the cumbersome right-hand side of (5.7) repeatedly.

Definition 5.7. The bubble chain $\text{BC}_{\lambda, z}(x, y)$ from $x$ to $y$ is defined to be $\alpha_0(\alpha_0 - 1)$ if $x = y$ and the right-hand side of (5.7) if $x \neq y$.

The last decomposition formula is the analogue of Proposition 5.5 for walks $\omega$ that are not closed. Some notation will be needed: for $\eta$ a self-avoiding walk ending at $x$ define $\text{BC}_{\lambda, z}(x, y)$ to be the bubble chain in $\mathbb{Z}^d \setminus \{\eta_0, \ldots, \eta_{|\eta|-1}\}$. See Figure 6.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.png}
\caption{An illustration of a contribution to the sum in Equation (5.12). For clarity, only the loop erasure of each walk is shown. The dashed black path is the bubble chain portion of the walk. The vertex $v$ indicates the division between the path prior to the bubble chain and after the bubble chain.}
\end{figure}
Proposition 5.8. Fix \( x, y, b \in \mathbb{Z}^d, x \neq y, b \neq x \). Then
\[
\sum_{\omega: x \to y} \mathbb{1}_{\{x \notin \omega[1:1]\}} \{j \geq 1 \mid \omega_j = b\} w_{\lambda, z}(\omega) = \\
\sum_{a \in \mathbb{Z}^d} \sum_{x \to a} \sum_{\omega(1): a \to y} \sum_{\omega(2): \omega(1) \mid LE(\omega(1))} \left( \delta_{a,b} + BC_{\lambda, z}(\omega(1)) (a, b) \right) w_{\lambda, z}(\omega(1)) w_{\lambda, z}(\omega(2)).
\] (5.12)

Proof. This follows by writing \(|\{j \geq 1 \mid \omega_j = b\}| = \sum_{j \geq 1} \mathbb{1}_{\{\omega_j = b\}}\) and noting that this splits, by applying Equation (5.4) with \( \eta = LE(\omega[0, j]) \), a walk \( \omega \) into (i) an initial segment \( \omega^{(1)} \) whose loop erasure is the subset of \( LE(\omega[0, j]) \) that is contained in \( LE(\omega) \), (ii) a bubble chain from the endpoint of \( \omega^{(1)} \) to \( b \) whose walks do not hit \( \text{LE}(\omega^{(1)}) \); if the endpoint of \( \omega^{(1)} \) is \( b \) then it is also possible this walk is null, and (iii) a walk \( \omega^{(2)} \) from the endpoint of \( \omega^{(1)} \) to \( y \) that does not, after the first vertex, hit \( \text{LE}(\omega^{(1)}) \).

The restriction in Proposition 5.8 to walks \( \omega \) that do not return to their initial vertex is simply because this is the type of sum that will occur most frequently in what follows.

5.1.3 Two-Point Functions and Their Derivatives

The quantity \( I_{\lambda, z}^\lambda(a, b) \) defined in Equation (3.4) is inconvenient due to its dependence on the details of \( \omega \); the next definition introduces a simple upper bound.

Definition 5.9. The interaction two-point function \( I_{\lambda, z}(x, y) \) is the function
\[
I_{\lambda, z}(x, y) = \mathbb{1}_{\{x = y\}} + \mathbb{1}_{\{x \neq y\}} \left( 1 - e^{-\mu_{\lambda, z}(x, y) \delta}\right).
\] (5.14)

Lemma 5.10. Let \( \omega \) be a walk of length \( n \), and let \( 0 \leq a < b \leq n \).

\[
I_{\lambda, z}^\lambda(a, b) \leq I_{\lambda, z}(\omega_a, \omega_b).
\] (5.15)

Proof. If \( \omega_a = \omega_b \) then (5.15) is an equality. If \( \omega_a \neq \omega_b \) the inequality follows because the loop measure is decreasing in its final argument.

The important point of the next bound is that it is independent of \( \eta \).

Proposition 5.11. Let \( \eta: x \to y \) be a self-avoiding walk. Then
\[
\frac{d}{dz} I_{\lambda, z}^\lambda(x, y) \leq \mathbb{1}_{\{x \neq y\}} z^{-1} \sum_{a \in \mathbb{Z}^d} \sum_{\omega: a \to a} \mathbb{1}_{\{x \in \omega\}} \mathbb{1}_{\{y \in \omega\}} w_{\lambda, z}(\omega).
\] (5.16)

Further, the right-hand side of (5.16) is an upper bound for \( \frac{d}{dz} I_{\lambda, z}(x, y) \) as well.

Proof. Differentiate, and then use \( e^{-x} \leq 1 \) for \( x \geq 0 \).

Definition 5.12. The scaled two-point functions \( G_{\lambda, z}(x, y) \) and \( H_{\lambda, z}(x, y) \) are defined by
\[
G_{\lambda, z}(x, y) = \alpha_0(\lambda, z)^{-1} G_{\lambda, z}(x, y), \quad H_{\lambda, z}(x, y) = \alpha_0(\lambda, z)^{-1} H_{\lambda, z}(x, y).
\] (5.17)

Let \( \bar{B}_{\lambda, z}(x) = \bar{H}_{\lambda, z}(x)^2 \). An upper bound on \( BC_{\lambda, z} \) is obtained by dropping the constraints \( A_i \) and \( B_i \).

Definition 5.13. Define \( B^*_{\lambda, z}(x, y) \) by
\[
B^*_{\lambda, z}(x, y) = \begin{cases} 
\sum_{k \geq 1} \lambda^k \frac{\bar{B}_{\lambda, z} \circ \cdots \circ \bar{B}_{\lambda, z}(x, y)}{k \text{ terms}} & x \neq y \\
\alpha_0 - 1 & x = y
\end{cases}
\] (5.18)
Proposition 5.14. Let \( \eta \) be any self-avoiding walk ending at \( x \). Then
\[
\text{BC}^\eta_{\lambda,z}(x,y) \leq \text{BC}_{\lambda,z}(x,y) \leq B^*_{\lambda,z}(x,y). \tag{5.19}
\]

Proof. The first inequality follows as the set of summands is increasing from left to right and all summands are non-negative. For the second inequality note that relaxing the conditions \( A_i \) and \( B_i \) increases the set of summands. Using \( H_{\lambda,z}(x,y) = H_{\lambda,z}(y,x) \), which follows from Theorem 1.13, to reverse the direction of the walks \( \omega^{(k+1)} \) gives the upper bound \( B^*_{\lambda,z}(x,y) \).

The next lemma is trivial but will be used frequently. The lemma shows that if a sum over walks satisfying some constraints is upper bounded by relaxing the constraints, an upper bound on the derivative is obtained by differentiating the upper bound.

Lemma 5.15. Suppose \( A, B \) are two sets of walks, and \( A \subset B \). Then
\[
\frac{d}{dz} \sum_{\omega \in A} w_{\lambda,z}(\omega) \leq \frac{d}{dz} \sum_{\omega \in B} w_{\lambda,z}(\omega). \tag{5.20}
\]

Proof. Each summand is non-negative and the set of summands on the right-hand side is larger.

The formulas of Section 5.1.2 yield diagrammatic bounds on derivatives of two-point functions by applying the identity
\[
|\omega| = \sum_{a \in \mathbb{Z}^d} |\{ j \geq 1 \mid \omega_j = a \}|, \tag{5.21}
\]
where \( j = 0 \) is not included because there \(|\omega| + 1 \) vertices in a walk.

Proposition 5.16. For \( x, y \in \mathbb{Z}^d, x \neq y \),
\[
\frac{d}{dz} \bar{G}_{\lambda,z}(x) = \frac{d}{dz} \bar{H}_{\lambda,z}(x) \leq z^{-1}(1 + \|B^*_{\lambda,z}\|_1) \bar{G}_{\lambda,z} \ast \bar{H}_{\lambda,z}(x). \tag{5.22}
\]

Proof. The first equality is straightforward as \( \bar{G}_{\lambda,z}(x) = \delta_{0,x} + \bar{H}_{\lambda,z}(x) \) by Equation (4.2). For the inequality observe that
\[
\frac{d}{dz} \bar{H}_{\lambda,z}(x) = z^{-1} \sum_{\omega: x \to y \atop x \notin \omega[1:]} |\omega| w_{\lambda,z}(\omega).
\]

Applying (5.21) and Proposition 5.8 yields
\[
z^{-1} \sum_{b \in \mathbb{Z}^d} \sum_{a \in \mathbb{Z}^d} \sum_{\omega^{(1)}: x \to a \atop x \notin \omega[1:]} \sum_{\omega^{(2)}: a \to y \atop \omega^{(2)}[1:] \cap \text{LE}(\omega^{(1)}) = \emptyset} (\delta_{a,b} + \text{BC}_{\lambda,z}^{\text{LE}(\omega^{(1)})} (a,b)) w_{\lambda,z}(\omega^{(1)}) w_{\lambda,z}(\omega^{(2))} \tag{5.23}.
\]

By Proposition 5.14 removing the restriction on the bubble chain gives an upper bound. The sum over \( b \) then gives the factor \( 1 + \|B^*_{\lambda,z}\|_1 \). Dropping the constraint that \( \omega^{(2)} \) does not intersect \( \text{LE}(\omega^{(1)}) \) gives the claim.

Proposition 5.17.
\[
\frac{d}{dz} \alpha_0(\lambda, z) = z^{-1} \alpha_0(\alpha_0 - 1). \tag{5.24}
\]

Proof. As a zero step walk does not survive being differentiated,
\[
\frac{d}{dz} \alpha_0(\lambda, z) = z^{-1} \sum_{\omega: a \to \phi \atop |\omega| \geq 1} |\omega| w_{\lambda,z}(\omega). \tag{5.25}
\]

The proposition follows by applying (5.21) and then Proposition 5.6.
Proposition 5.18.

\[
\frac{d}{dz}||BC_{\lambda,z}||_1 \leq z^{-1} \alpha_0 (\alpha_0 (\alpha_0 - 1) + (\alpha_0 - 1)^2) + 2\lambda z^{-1} \alpha_0 (1 + ||B^*_{\lambda,z}||_1) \left( \|H_{\lambda,z} \cdot (\bar{G}_{\lambda,z} \star \bar{H}_{\lambda,z})\|_1 + \| (\bar{H}_{\lambda,z} \cdot (\bar{G}_{\lambda,z} \star \bar{H}_{\lambda,z})) \star B^*_{\lambda,z}\|_1 \right)
\]

(5.26)

Proof. By Lemma 5.15 it suffices to obtain bounds on the derivative of \(||B^*_{\lambda,z}||_1\). For the summand with \(x = y\) the result is \(z^{-1} \alpha_0^2 (\alpha_0 - 1) + z^{-1} \alpha_0 (\alpha_0 - 1)^2\) by Proposition 5.17.

For \(x \neq y\) differentiating Equation (5.18) and using Proposition 5.16 gives \(z^{-1} (\alpha_0 - 1) (||B^*_{\lambda,z}||_1 - \alpha_0 (\alpha_0 - 1))\) if the derivative is applied to \(\alpha_0\). Otherwise

\[
\frac{d}{dz}||B^*_{\lambda,z}||_1 = \alpha_0 \frac{d}{dz} \sum_{k \geq 1} \sum_{y} \lambda^k \bar{H}^2_{\lambda,z} \cdots \bar{H}^2_{\lambda,z} (y)
\]

(5.28)

\[
\leq 2\alpha_0 \sum_{k \geq 1} \sum_{y} \lambda^k \left( \bar{H}_{\lambda,z} \frac{d}{dz} \bar{H}_{\lambda,z} \right) \star \bar{H}^2_{\lambda,z} \cdots \bar{H}^2_{\lambda,z} (y)
\]

(5.29)

\[
= 2\alpha_0 z^{-1} (1 + ||B^*_{\lambda,z}||_1) \lambda^k \left( \| \bar{H}_{\lambda,z} \cdot (\bar{G}_{\lambda,z} \star \bar{H}_{\lambda,z})\|_1 + \| (\bar{H}_{\lambda,z} \cdot (\bar{G}_{\lambda,z} \star \bar{H}_{\lambda,z})) \star B^*_{\lambda,z}\|_1 \right).
\]

(5.30)

5.2 Diagrammatic Bounds

The bounds derived in this section will be obtained under the assumption that \(f(z) \leq K\) for \(z < z_c(\lambda)\). In particular the results of Proposition 4.8 hold. It will also be assumed that the dimension \(d\) is sufficiently large, i.e., \(\beta\) is sufficiently small.

5.2.1 Initial Diagrammatic Bounds

Proposition 5.19. If \(z < z_c\) then \(\alpha_0 (\lambda, z) \leq 1 + c\beta\).

Proof. By definition and Theorem 1.13

\[
\alpha_0 (\lambda, z) = \exp (\mu_{\lambda,z}(\omega)) = 1 + \sum_{\omega: \text{a} \to \text{a}, |\omega| \geq 1} w_{\lambda,z}(\omega).
\]

(5.31)

The walks contributing to the sum have their last vertex a neighbour of \(o\), so an argument like the one showing (5.37) for \(x = o\) implies

\[
\sum_{\omega: \text{a} \to \text{a}, |\omega| \geq 1} w_{\lambda,z}(\omega) = z\lambda |\Omega| D \star H_{\lambda,z}(o),
\]

(5.32)

which is bounded by \(z\lambda |\Omega| \|H_{\lambda,z}\|_{\infty}\). The claim follows from \(z |\Omega| \leq f_1(z) \leq K\) and (4.19).

\[
\square
\]

Proposition 5.20. If \(z < z_c\) then \(||B^*_{\lambda,z}||_1 \leq c\beta\).

Proof. Repeatedly using \(||f * g||_1 \leq ||f||_1 ||g||_1\) implies

\[
||B^*_{\lambda,z}||_1 \leq \alpha_0 \left( \alpha_0 - 1 + \sum_{k \geq 1} \lambda^k ||\bar{H}_{\lambda,z}||^2_2 \right),
\]

(5.33)

The interchange of summations is valid as each term is non-negative. By Proposition 5.19 \(\alpha_0 \leq 1 + c\beta\) so \(\alpha_0 - 1 \leq c\beta\). Since \(\alpha_0 \geq 1\), \(\|\bar{H}_{\lambda,z}\|_2^2 \leq \|H_{\lambda,z}\|_2^2\), so Equation (4.18) implies that for \(\beta\) sufficiently small

\[
\sum_{k \geq 1} \lambda^k ||\bar{H}_{\lambda,z}||^2_2 \leq c\beta.
\]

(5.34)

\[
\square
\]
Proposition 5.21. Let \( I_{\lambda,z}(x) = I_{\lambda,z}(o,x) \). If \( z < z_c \) then \( \| I_{\lambda,z} \|_1 \leq 1 + c\beta \).

Proof. The inequality \( 1 - e^{-x} \leq x \) implies that \( 1 + \| \mathbb{1}_{\{x \neq 0\}} \mu_{\lambda,z}(o,x) \|_1 \) is an upper bound for \( \| I_{\lambda,z} \|_1 \).

The factor of 1 is from the term \( \mathbb{1}_{\{x = 0\}} \) in \( I_{\lambda,z} \). Observe that \( \| \mathbb{1}_{\{x \neq 0\}} \mu_{\lambda,z}(o,x) \|_1 \) is bounded by

\[
\sum_{x \neq o} \sum_{y} \sum_{\omega: y \rightarrow y} \mathbb{1}_{\{o \in \omega\}} \mathbb{1}_{\{x \in \omega\}} \frac{w_{\lambda,z}(\omega)}{d},
\]

as \( \sum_{x \neq o} \mathbb{1}_{\{x \in \omega\}} \leq |\text{range(\omega)}| \leq |\omega|. \) By translation invariance this is

\[
\sum_{y} \sum_{\omega: o \rightarrow o} \mathbb{1}_{\{-y \in \omega\}} w_{\lambda,z}(\omega) = \| \sum_{\omega: o \rightarrow o} \mathbb{1}_{\{y \in \omega\}} w_{\lambda,z}(\omega) \|_1,
\]

where the norm is with respect to \( y \). To establish the proposition (i) bound \( \mathbb{1}_{\{y \in \omega\}} \) by \( |\{j \geq 1 | \omega_j = y\}| \), (ii) apply Proposition 5.6 for the summands with \( y = 0 \), (iii) apply Proposition 5.5 and Proposition 5.14 for the summands with \( y \neq o \), and (iv) observe that the sum of these two bounds is \( \| B^*_{\lambda,z} \|_1 \) and apply Proposition 5.20.

5.2.2 Bounds for \( \pi^{(1)} \)

Proposition 5.22.

\[
\pi^{(1)}(x) = \sum_{m \in \mathbb{Z}} \pi^{(1)}_m \left\{ \begin{array}{ll}
z\lambda|\Omega| D \ast \bar{H}_{\lambda,z}(0) & x = 0 \\
\leq \bar{H}_{\lambda,z}(x) I_{\lambda,z}(0,x)e^{\mu_{\lambda,z}(o,x)} & x \neq 0,
\end{array} \right.
\]

Proof. For \( x = o \) the claim follows from Equation (2.35) and Equation (5.32). For \( x \neq o \) Equation (2.37), forgetting the constraint in the loop measure, and the rearrangement \( e^{\mu_{\lambda,z}(o,x)} - 1 = e^{\mu_{\lambda,z}(o,x)} I_{\lambda,z}(o,x) \) imply the claim.

Proposition 5.23. The following bounds hold for \( u = 0, 1 \) and \( k \in [-\pi, \pi]^d \).

\[
\| x \|^{2u} \pi^{(1)} \|_1 \leq c\beta \left( \mathbb{1}_{\{u = 0\}} + \| x \|^{2u} H_{\lambda,z} \|_\infty \right),
\]

\[
\| (1 - \cos k \cdot x) \pi^{(1)} \|_1 \leq c\beta \| (1 - \cos k \cdot x) \bar{H}_{\lambda,z} \|_\infty.
\]

Proof. The triangle inequality, \( \| f \|_1 \leq \| f \|_\infty \| g \|_1 \) with \( g = I_{\lambda,z} \), and \( 1 - \cos 0 = 0 \) imply

\[
\| x \|^{2u} \pi^{(1)} \|_1 \leq \mathbb{1}_{\{u = 0\}} \| \Omega \| \| G_{\lambda,z} \|_\infty + \| I_{\lambda,z}(o,x)e^{\mu_{\lambda,z}(o,x)} \mathbb{1}_{\{x \neq o\}} \|_1 \| x \|^{2u} \bar{H}_{\lambda,z} \|_\infty
\]

\[
\| (1 - \cos k \cdot x) \pi^{(1)} \|_1 \leq \| I_{\lambda,z}(o,x)e^{\mu_{\lambda,z}(o,x)} \mathbb{1}_{\{x \neq o\}} \|_1 \| (1 - \cos k \cdot x) \bar{H}_{\lambda,z} \|_\infty.
\]

Using \( z \| \Omega \| \leq f_1(z) \leq K \) by \( f_3 \leq K \) and sup_{x} e^{\mu_{\lambda,z}(o,x)} \leq \alpha_0(\lambda, z) \) implies

\[
\| I_{\lambda,z}(o,x)e^{\mu_{\lambda,z}(o,x)} \mathbb{1}_{\{x \neq o\}} \|_1 \leq \alpha_0 \| I_{\lambda,z}(o,x) \mathbb{1}_{\{x \neq o\}} \|_1 \leq \alpha_0 \| I_{\lambda,z}(o,x) \|_1 - 1.
\]

The conclusion now follows from Propositions 5.19 and 5.21.

5.2.3 Bounds for \( \pi^{(N)} \), \( N \geq 2 \)

Proposition 5.24. Let \( m \geq 2 \), \( x \in \mathbb{Z}^d \), and \( N \geq 2 \). Let \( x_0 = o, x_{N-1} = x \). Then

\[
\left| \pi^{(N)}_m(x) \right| \leq \sum_{m \in \mathbb{Z}} \sum_{\omega^{(1)}} \sum_{\omega^{(2)}} \cdots \sum_{\omega^{(N-2)}} \sum_{\omega^{(2N-2)}} \sum_{\omega^{(2N-1)}} \sum_{\omega^{(2N)}} \sum_{\omega^{(2N+1)}} 
\]

\[
\prod_{j=0}^{N-1} |I_{\lambda,z}(x_j, x_j^*)| \prod_{k=1}^{2N-1} \omega^{(k)} \exp \left( \mu_{\lambda,z}(\text{range}(\omega^{(k)})) \right)
\]

\[
(5.43)
\]
where the summation is over valid vectors $\vec{m}$ (recall Definition 2.19) of subinterval lengths such that $\sum m_i = n$.

Proof. This follows from Section 2.5.3. By Lemma 5.10 the factors of $I^{\omega}_{\lambda,z}$ can be replaced by $I_{\lambda,z}$. In the language of an $X$ gas, as $\alpha_\omega \geq 0$ for any walk $\omega$, the $X$ gas for $\lambda$-LWW is repulsive in the sense of Remark 2.20, and this proves the claim. \hfill $\square$

Upper bounds on $\|\pi^{(N)}(x)\|_1$ can be efficiently found by formulating Proposition 5.24 in terms of multiplication and convolution operators. Let $\mathcal{M}_g$ and $\mathcal{C}_g$ denote multiplication and convolution by $g$, respectively.

**Lemma 5.25.** Fix $N \geq 2$ and let $\bar{H} = \bar{H}_{\lambda,z}$, $\bar{G} = \bar{G}_{\lambda,z}$, and $I = I_{\lambda,z}$. Then

$$
\sum_x |\pi^{(N)}(x)| \leq \| (\mathcal{C}_I \mathcal{M}_I) (\mathcal{C}_I \mathcal{M}_I)^{N-2} \bar{H} * I \|_{\infty}.
$$

(5.44)

Proof. The definition of a valid vector of lengths implies that summing (5.42) over all valid vectors of lengths results in the sums over walks with indices $1, 2j$, and $2N - 1$ being replaced by $\bar{H}_{\lambda,z}$, and the remaining sums of walks are replaced by $\bar{G}_{\lambda,z}$. Consulting Figure 3, this means that all horizontal solid lines except the leftmost and rightmost are weighted by $\bar{G}_{\lambda,z}$, while the rest are weighted by $\bar{H}_{\lambda,z}$. Formally,

$$
|\pi^{(N)}(x)| \leq \sum_{x_1, \ldots, x_{N-1}} \left( \prod_{j=0}^{N-1} I_{\lambda,z}(x_j, x_j') \right) \bar{H}_{\lambda,z}(x_0, x_1)
$$

(5.45)

$$
\left( \prod_{j=0}^{N-2} \bar{H}_{\lambda,z}(x_j', x_{j+1}) \right) \left( \prod_{j=0}^{N-3} \bar{G}_{\lambda,z}(x_j', x_{j+2}) \right) \bar{H}_{\lambda,z}(x_{N-2}, x'_{N-1}).
$$

(5.46)

Replace the factor $I_{\lambda,z}(x_0, x_0')$ by $I_{\lambda,z}(y, x_0')$ in (5.45) and call the resulting function $F(x, y)$. As $\sum_x |F(x, y)| = \sum_x |\pi^{(N)}(x)|$ the quantity $\sup_y \sum_x |F(x, y)|$ is an upper bound for the left-hand side of (5.44). The associativity of convolution implies

$$
\sum_x |F(x, y)| = \left( (\mathcal{C}_I \mathcal{M}_I) (\mathcal{C}_I \mathcal{M}_I)^{N-2} \bar{H} * I \right)(y).
$$

(5.47)

Equation (5.44) follows as $\mathcal{C}_G \mathcal{C}_I = \mathcal{G}_I$ and $\mathcal{C}_I \mathcal{C}_I = \mathcal{H}_I$. \hfill $\square$

The right-hand side of Lemma 5.25 can be easily estimated with the help of the next lemma.

**Lemma 5.26 (Lemma 4.6 of [14]).** Given non-negative even functions $f_0, f_1, \ldots, f_{2M}$ on $\mathbb{Z}^d$, define $\mathcal{C}_j$ and $\mathcal{M}_j$ to be the operations of convolution with $f_{2j}$ and multiplication by $f_{2j-1}$ for $j = 1, \ldots, M$. Then for any $k \in \{0, \ldots, 2M\}$,

$$
\|\mathcal{C}_M \mathcal{M}_M \ldots \mathcal{C}_1 \mathcal{M}_1 f_0\|_{\infty} \leq \|f_k\|_{\infty} \prod \|f_j * f_{j'}\|_{\infty},
$$

(5.48)

where the product is over disjoint consecutive pairs $j, j'$ taken from $\{0, \ldots, 2M\} \setminus \{k\}$.

The strange formatting of the bounds in the next proposition are strictly for typographic convenience; in applications we multiply through by the denominators of the left-hand sides.

**Proposition 5.27.** Let $N \geq 2$. Then for $z < z_c$ and $u \in \{0, 1\}$

$$
\|x^{2u} \pi^{(N)}(x)\|_1 \leq \|x^{2u} H_{\lambda,z}\|_{\infty} (c\beta)^{N-2+1(z=2)} (1 + c\beta)^{N+4(z\geq 3)}
$$

(5.49)

$$
\|(1 - \cos(k \cdot x)) \pi^{(N)}(x)\|_1 \leq \|(1 - \cos(k \cdot x)) H_{\lambda,z}(x)\|_{\infty} (c\beta)^{N-2+1(z=2)} (1 + c\beta)^{N+4(z\geq 3)}
$$

(5.50)
Lemma 5.28. Fix $c$.

Proof. Suppose that both

$$\frac{1}{(2N-1)^a} \leq \frac{1}{(2N-1)^a} \left(\sum_{j=1}^{n} |x_j|^2 \pi^{(N)}(x_j)\right) \leq \sum_{j=1}^{n} \frac{1}{(2N-1)^a} |x_j|^2 \pi^{(N)}(x_j)$$

and

$$\sum_{j=1}^{n} \frac{1}{(2N-1)^a} \left(\pi^{(N)}(x_j)\right) \leq \sum_{j=1}^{n} \frac{1}{(2N-1)^a} \left(\pi^{(N)}(x_j)\right)$$

Suppose further that if $N = 2$ the same bounds hold with each term $i$.

Lemma 5.26 and putting the sup norm on the term involving the factor of $\lambda, z$.

Then (see [14, Section 4.2.3])

For $u = 1$, note that $x = x_1 + \ldots x_{2N-1}$, where $x_j$ is the displacement along the $j$th subwalk in a summand contributing to $\pi^{(N)}$. As $|x|^2 \leq \sum |x_j|^2$ it follows that an upper bound is given by

$$\sum_{j=1}^{n} \frac{1}{(2N-1)^a} \left(\pi^{(N)}(x_j)\right) \leq \sum_{j=1}^{n} \frac{1}{(2N-1)^a} \left(\pi^{(N)}(x_j)\right)$$

where the $j$th two-point function $\lambda, z$ is replaced with $|x|^2 \eta, z$. The claim follows by (i) applying Lemma 5.26 and putting the sup norm on the term involving the factor of $\lambda, z$.

The rest of the proof establishes Equations (5.51) and (5.52).

First observe that the difference between $N = 2$ and $N \geq 3$ is only that all two-point functions in Lemma 5.25 are $\lambda, z$ for $N = 2$, while for $N \geq 3$ factors of $G^{a,b}$ arise.

If $u = 0$ Equation (5.51) follows by applying Lemma 5.26 to the right-hand side of Lemma 5.25, putting the sup norm on the final $I_{\lambda, z} \ast \eta, z$, and using the inequality

$$\|\lambda, z \ast \eta, z \ast I_{\lambda, z}\| \leq \|\lambda, z \ast \eta, z\| \leq \|\lambda, z\|.$$
Proof. This follows by combining the bounds of Propositions 5.23 and 5.27 for \( u = 0 \) with the bound 
\[ \| (1 - \cos k \cdot x) H_{\lambda,z}(x) \|_\infty \leq c_K (1 + \beta)^{-1}(k) \] of Equation (4.17).

The remainder of this section is devoted to verifying the hypothesis of Lemma 4.5 for \( z_1 = 0, \ z_2 = z_c(\lambda), \ a = 4 \) and \( b = 1 + O(\beta) \).

**Lemma 5.29.** The function \( f \) obeys \( f(0) = 1 \).

**Proof.** Clearly \( f_1(0) = 0 \). The definition of \( p(z) \) implies \( p(0) = 0 \) as \( \alpha_0(\lambda, 0) = 1 \), so \( f_2(0) = 1 \). Lastly, \( f_3(0) = 0 \). \( U_0 = 16 \) while \( \Delta_k \hat{G}_{\lambda,0} = 0 \).

**Lemma 5.30.** The function \( f \) is continuous on \( (0, z_c) \).

**Proof.** It suffices to show \( f_1, f_2, f_3 \) are continuous on \([0, r]\) for any \( r < z_c \). For \( f_1 \) this follows as \( \alpha(\lambda, z) \leq \alpha_0(\lambda, z) \leq \chi(\lambda) \).

Recall (see [14, Lemma 5.13]) that the supremum of an equicontinuous family of functions over a compact interval is a continuous function, provided this supremum is finite. It follows that it is enough to prove a bound uniform in \( k \) on the derivative of \( f_2 \) (resp. \( f_3 \)) with respect to \( z \). Since equicontinuity of a family \( \{g_\alpha\} \) is equivalent to equicontinuity of \( \{g_\alpha\} \), the absolute value on \( \hat{G}_{\lambda,z} \) (resp. \( \Delta_k \hat{G}_{\lambda,z} \)) can be ignored. For \( f_2 \) the derivative is

\[
\frac{d}{dz} \hat{G}_{\lambda,z}(k) = \frac{1}{C_p(z)(k)^2} \left[ C_p(z)(k) \frac{d\hat{G}_{\lambda,z}(k)}{dz} - \hat{G}_{\lambda,z}(k) \frac{dC_p(z)(k)}{dp} \right] \quad (5.59)
\]

Now note: \( |\hat{G}_{\lambda,z}(k)| \leq \chi(\lambda), \ |\frac{d}{dz} \hat{G}_{\lambda,z}(k)| \leq \left| \frac{d}{dz} \chi(\lambda) \right|, \ \left| \frac{d}{dz} \hat{C}_p(k) \right| \leq |\Omega| \chi(\lambda)^2 \). Further,

\[
\left| \frac{d}{dz} p(z) \right| = \left| \frac{d}{dz} |\Omega|^{-1} \left( 1 - \frac{\alpha_0(\lambda, z)}{\chi(\lambda)} \right) \right| 
\leq |\Omega|^{-1} \alpha_0(\lambda, r) \frac{d}{dz} \chi(\lambda) + |\Omega|^{-1} \alpha_0(\lambda, r)
\]

and \( \frac{d}{dz} \alpha_0(\lambda, r) \) is bounded above by \( \frac{d}{dz} \chi(\lambda) \) by Lemma 5.15. A uniform bound on the derivative then follows from

\[
\frac{1}{2} \leq \hat{C}_p(z)(k) \leq \hat{C}_p(z)(0) = \frac{\chi(z)}{\alpha_0(\lambda, z)} \leq \chi(\lambda)
\]

where the second inequality follows from the definition of \( p(z) \), and the last inequality from \( \alpha_0(\lambda, z) \geq 1 \).

For \( f_3 \) the calculation is essentially the same. Calculating the derivative shows that what is needed is upper bounds on \( |\hat{G}_{\lambda,z}(k)|, \left| \frac{d}{dz} \hat{G}_{\lambda,z}(k) \right|, \left| \frac{d}{dz} \hat{C}_p(k) \right| , \) and \( \left| \frac{d}{dz} p(z) \right| \), along with upper and lower bounds on \( \hat{C}_p(z) \). These bounds have already been obtained.

The next lemma completes the bootstrap argument.

**Lemma 5.31.** Suppose \( d \) is sufficiently large. Fix \( z \in (0, z_c) \), and suppose that \( f(z) \leq 4 \). Then there is a constant \( c \) independent of \( z \) and \( d \) such that \( f(z) \leq 1 + c\beta \).

**Proof.** We prove \( f_j(z) \leq 1 + c\beta \) for \( j = 1, 2, 3 \) in sequence. Since \( \alpha_0(\lambda, z) \) and \( \chi(\lambda) \) are both positive and finite it follows that

\[
\frac{\alpha_0(\lambda, z)}{\chi(\lambda)} = 1 - z \alpha(\lambda, z) |\Omega| - \hat{N}_{\lambda,z}(0) > 0.
\]

Equation (5.63) and Lemma 5.28 together imply

\[
f_1(z) = z \alpha(\lambda, z) |\Omega| \leq 1 + \hat{N}_{\lambda,z}(0) \leq 1 + \epsilon_4 \beta.
\]
Proposition 5.19 implies \( \alpha_0 \leq 1 + \tilde{c}\beta \), so \( f_2 \leq 1 + O(\beta) \) follows if
\[
\frac{\hat{G}_{\lambda,z}(k)}{\alpha_0(\lambda, z)C_{p(z)}(k)} = 1 + \frac{1 - p(z) |\Omega| \hat{D}(k) - \hat{F}_{\lambda,z}(k)}{\hat{F}_{\lambda,z}(k)} \tag{5.65}
\]
is \( 1 + O(\beta) \), where
\[
\hat{F}_{\lambda,z}(k) \equiv \hat{G}_{\lambda,z}(k)^{-1} = 1 - z\alpha(\lambda, z) |\Omega| \hat{D}(k) - \hat{\Pi}_{\lambda,z}(k). \tag{5.66}
\]
By definition, \( p(z) |\Omega| = z\alpha(\lambda, z) |\Omega| + \hat{\Pi}_{\lambda,z}(0) \). Hence the numerator of the right-hand side of (5.65) is
\[
1 - p(z) |\Omega| \hat{D}(k) - \hat{F}_{\lambda,z}(k) = \hat{\Pi}_{\lambda,z}(0) \left( 1 - \hat{D}(k) \right) - \left( \hat{\Pi}_{\lambda,z}(0) - \hat{\Pi}_{\lambda,z}(k) \right), \tag{5.67}
\]
which is bounded above by \( 4\tilde{c}_4\beta \). An alternative upper bound of the right hand side of (5.67) follows from Equations (5.57) and (5.58):
\[
\hat{\Pi}_{\lambda,z}(0) \left( 1 - \hat{D}(k) \right) - \left( \hat{\Pi}_{\lambda,z}(0) - \hat{\Pi}_{\lambda,z}(k) \right) \leq \tilde{c}_4\beta \left( 1 - \hat{D}(k) \right) + \tilde{c}_4\beta \left( 1 - p(z) |\Omega| \hat{D}(k) \right). \tag{5.68}
\]
Since
\[
\left( 1 - \hat{D}(k) \right) \hat{C}_{p(z)}(k) = 1 + \hat{D}(k) \left( \frac{p(z) |\Omega| - 1}{1 - p(z) |\Omega| |\Omega| \hat{D}(k)} \right) \leq 2, \tag{5.69}
\]
the numerator of (5.65) is bounded by
\[
3\tilde{c}_4\beta \left( 1 - p(z) |\Omega| \hat{D}(k) \right) \leq 3\tilde{c}_4\beta \left( \hat{F}_{\lambda,z}(0) + \left( 1 - \hat{D}(k) \right) \right). \tag{5.70}
\]
The denominator of (5.65) is
\[
\hat{F}_{\lambda,z}(k) = \hat{F}_{\lambda,z}(0) + \left( \hat{F}_{\lambda,z}(k) - \hat{F}_{\lambda,z}(0) \right) \tag{5.71}
\]
\[
= \hat{F}_{\lambda,z}(0) + z\alpha(\lambda, z) |\Omega| \left( 1 - \hat{D}(k) \right) + \left( \hat{\Pi}_{\lambda,z}(0) - \hat{\Pi}_{\lambda,z}(k) \right). \tag{5.72}
\]
Let \( \bar{\lambda} = \sup_{\eta \in \Omega_{0\eta}} \lambda_{\eta} \), and \( \lambda^* = \max(1, \bar{\lambda}) \). For \( z \leq (2 |\Omega| \sqrt{\lambda^*})^{-1} \) Proposition 1.8 (if \( \lambda^* > 1 \) or neglecting loops (if \( \lambda^* \leq 1 \)) implies \( \hat{F}_{\lambda,z}(0) \geq \hat{C}_z / \sqrt{\lambda^*} \geq \frac{1}{2} \). Then \( 1 - \hat{D}(k) \geq 0 \) and (5.58) imply
\[
\hat{F}_{\lambda,z}(k) \geq \hat{F}_{\lambda,z}(0) - 2\tilde{c}_4\beta \geq \frac{1}{2} - 2\tilde{c}_4\beta. \tag{5.73}
\]
For \( (2 |\Omega| \lambda^*)^{-1} \leq z < z_c(\lambda) \) Equation (5.58), \( \hat{F}_z(0) > 0 \), and \( \alpha(\lambda, z) \geq 1 \) imply
\[
1 - p(z) |\Omega| \hat{D}(k) = 1 - \left( 1 - \hat{F}_{\lambda,z}(0) \right) \hat{D}(k) \leq 1 - \hat{D}(k) + \hat{F}_{\lambda,z}(0) \tag{5.74}
\]
and hence
\[
\hat{F}_{\lambda,z}(k) \geq \hat{F}_{\lambda,z}(0) + \frac{1}{2\sqrt{\lambda^*}} \left( 1 - \hat{D}(k) \right) - \tilde{c}_4\beta \left( 1 - p(z) |\Omega| \hat{D}(k) \right) \tag{5.75}
\]
\[
\geq \left( \frac{1}{2\sqrt{\lambda^*}} - \tilde{c}_4\beta \right) \left( \hat{F}_{\lambda,z}(0) + \left( 1 - \hat{D}(k) \right) \right). \tag{5.76}
\]
For \( z \leq (2 |\Omega| \lambda^*)^{-1} \) or \( (2 |\Omega| \lambda^*)^{-1} \leq z < z_c \) these lower and upper bounds combine to imply the right-hand side of (5.65) is \( 1 + O(\beta) \), and hence \( f_2(z) = 1 + O(\beta) \).

Lastly consider \( f_3(z) \). As for \( f_2 \), it suffices to prove the claim for \( f_3 / \alpha_0 \). Let \( \tilde{g}_{\lambda,z}(k) \) be \( z\alpha(\lambda, z) |\Omega| \hat{D}(k) + \hat{\Pi}_{\lambda,z}(k) \), so
\[
\frac{\hat{G}_{\lambda,z}(k)}{\alpha_0(\lambda, z)} = \frac{1}{1 - \tilde{g}_{\lambda,z}(k)}. \tag{5.77}
\]
The symmetry of $D(x)$ and $\Pi_{\lambda,z}(x)$ implies that $g_{\lambda,z}(x) = g_{\lambda,z}(-x)$, so applying Lemma 5.7 of [14] gives:

$$\frac{1}{2} |\Delta_k \hat{G}_{\lambda,z}(\ell)| \leq \frac{1}{2} \left( \hat{G}_{\lambda,z}(\ell - k) + \hat{G}_{\lambda,z}(\ell + k) \right) \hat{G}_{\lambda,z}(\ell) \left( |\hat{g}_{\lambda,z}(0)| - |\hat{g}_{\lambda,z}(k)| \right)$$

$$+ 4 \hat{G}_{\lambda,z}(\ell - k) \hat{G}_{\lambda,z}(\ell) \hat{G}_{\lambda,z}(\ell + k) \left( |\hat{g}_{\lambda,z}(0)| - |\hat{g}_{\lambda,z}(k)| \right) \left( |\hat{g}_{\lambda,z}(0)| - |\hat{g}_{\lambda,z}(\ell)| \right).$$

(5.78)

(5.79)

Using $f_2(z) \leq 1 + O(\beta)$ bounds each factor of $\hat{G}_{\lambda,z}$ by $(1 + O(\beta)) \hat{C}_{\ell}(z)$. Further,

$$|\hat{g}_{\lambda,z}(0)| - |\hat{g}_{\lambda,z}(k)| \leq \sum_x (1 - \cos(k \cdot x)) (\alpha(\lambda, z)|\Omega| + |\Pi_x(x)|)$$

$$\leq z \alpha(\lambda, z)|\Omega| \left( 1 - \hat{D}(k) \right) + \bar{c}_4 \beta \hat{C}_{\ell}(z) k^{-1}$$

$$\leq (2 + O(\beta)) \hat{C}_{\ell}(z) k^{-1},$$

(5.80)

(5.81)

(5.82)

where the second inequality is by (5.58) and the third is by $f_1(z) \leq 1 + O(\beta)$ and (5.69). Combining the bounds and using the definition of $U_{\ell}(z)$ gives $f_3(z) \leq 1 + O(\beta)$.

**Corollary 5.32.** For $d$ sufficiently large, $\lambda$-LWW satisfies a $k$-space infrared bound: there is a constant $K$ such that for $0 \leq z \leq z_c(\lambda)$

$$\hat{G}_{\lambda,z}(k) \leq K \hat{C}_{\ell}(z).$$

(5.83)

**Proof.** The proof of Lemma 5.31 showed that $f_2(z) \leq 1 + O(\beta)$ without absolute values on $\hat{G}_{\lambda,z}$, uniformly for $z < z_c$. Taking a limit gives the claim.

The fact that the quantities $T_{\lambda,z}$ and $S_{\lambda,z}$ defined below are small will be important in what follows.

**Definition 5.33.** The triangle diagram $T_{\lambda,z}$ and square diagram $S_{\lambda,z}$ are the quantities

$$T_{\lambda,z} = \|\hat{H}_{\lambda,z}^{3}\|_1, \quad S_{\lambda,z} = \|\hat{H}_{\lambda,z}^{4}\|_1.$$

(5.84)

**Corollary 5.34.** For $d$ sufficiently large and $z \leq z_c$ the triangle and square diagrams are bounded above by $c\beta$.

**Proof.** For notational convenience write $\hat{H}_{\lambda,z} = \alpha_0^{-1} H_{\lambda,z}$, and similarly for $\hat{G}_{\lambda,z}$. By Equation (4.2) $\alpha_0^{-1} H_{\lambda,z} = \alpha_0^{-1} \hat{G}_{\lambda,z} - 1$. Corollary 5.32 implies $\alpha_0^{-1} \hat{G}_{\lambda,z} \leq (1 + O(\beta)) \hat{C}_{\ell}(z)$ since $\alpha_0 \leq 1 + O(\beta)$. The claim follows from Proposition 4.4.

---

6 Proofs of the Main Results

To go beyond the $k$-space infrared bound of Corollary 5.32 requires control of the derivatives of $G_{\lambda,z}$ and $\Pi_{\lambda,z}$ with respect to $z$. This control is established in Section 6.1. The remainder of the section establishes Theorem 1.9 using arguments based on [11, Chapter 6]. Throughout let $z_c = z_c(\lambda)$.

6.1 Further Diagrammatic Bounds

Having verified that the bounds of Section 5.2 holds for $z < z_c$, the monotone convergence theorem implies they continue to hold at $z_c$.

**Proposition 6.1.** For $d$ sufficiently large and $0 < z \leq z_c$

$$\frac{d}{dz} \|\hat{B}_{\lambda,z}\|_1 \leq z_c^{-1} c\beta.$$

(6.1)

---

[1] Lemma 5.7 is a general fact about even functions.
Proof. The left-hand side is a polynomial with positive coefficients, so it suffices to obtain an upper bound at \( z = z_c \). By Proposition 5.18, \( \alpha_0(\lambda, z_c) \leq 1 + c\beta \), \( \|B^*_{\lambda, z_c}\|_1 \leq c\beta \), and Corollary 5.34, the claim follows.

**Proposition 6.2.** Let \( d \) be sufficiently large, \( 0 < z \leq z_c \), and \( v = 1, 2 \). Then

\[
\|\partial_z^v G_{\lambda, z}\|_\infty = \|\partial_z^v \tilde{H}_{\lambda, z}\|_\infty \leq c\beta z_c^{-v} \tag{6.2}
\]

Proof. As for Proposition 6.1 it suffices to consider \( z = z_c \). The equality of the first two terms follows from Equation (4.2). Proposition 5.16 implies

\[
\frac{d}{dz} \tilde{H}_{\lambda, z} \leq z^{-1}(1 + \|B^*_{\lambda, z}\|_1)\tilde{H}_{\lambda, z} \ast \hat{G}_{\lambda, z}.
\tag{6.3}
\]

The claim follows for \( v = 1 \) as \( \|\tilde{H}_{\lambda, z} \ast \hat{G}_{\lambda, z}\|_\infty \leq c\beta \) by Equation (5.53) and \( \|B^*_{\lambda, z}\|_1 \leq c\beta \) by Proposition 5.20.

For \( v = 2 \) apply Lemma 5.15. After computing the derivative and using the triangle inequality (i) argue as for \( v = 1 \) for the term from differentiating \( z^{-1} \) (ii) use Proposition 6.1 when differentiating \( \|B^*_{\lambda, z}\|_1 \) and (iii) when differentiating either of the two-point functions use Proposition 5.16 and \( \|\tilde{H}_{\lambda, z} \ast \hat{H}_{\lambda, z} \ast \hat{G}_{\lambda, z}\|_\infty \leq \|\tilde{H}_{\lambda, z} \ast \hat{H}_{\lambda, z}\|_\infty + \|\hat{H}_{\lambda, z} \ast \hat{H}_{\lambda, z} \ast \hat{G}_{\lambda, z}\|_\infty \), and Corollary 5.34 to see that this is bounded by \( c\beta \). Each term is therefore bounded by \( c\beta z_c^{-2} \).

**Proposition 6.3.** Let \( d \) be sufficiently large, \( 0 < z \leq z_c \), and \( v = 1, 2 \). Then

\[
\|\partial_z^v I_{\lambda, z}\|_1 \leq c\beta z_c^{-v} \tag{6.4}
\]

Proof. For \( v = 1 \) note

\[
\frac{d}{dz} I_{\lambda, z} = \frac{d}{dz} (1 - e^{-\mu_{\lambda, z}(0, x)}) \leq \frac{d}{dz} \mu_{\lambda, z}(a, x). \tag{6.5}
\]

This bound is increasing in \( z \), so considering \( z_c \) is enough. Translation invariance, as in the proof of Proposition 5.21, implies this is equal to the derivative in \( z \) of \( \|B^*_{\lambda, z}\|_1 \). The claim follows for \( v = 1 \) from Proposition 6.1.

For \( v = 2 \) it is enough to bound the derivative of the bound of Proposition 5.18. This is similar to the arguments already given: the only new terms that arise occur when differentiating \( \|\tilde{H}_{\lambda, z} \ast \hat{G}_{\lambda, z} \ast \hat{H}_{\lambda, z}\|_1 \), which is \( \tilde{H}_{\lambda, z} \ast \hat{G}_{\lambda, z} \ast \hat{H}_{\lambda, z}(o) \). By Proposition 5.16 after taking a derivative the result is, up to a factor of \( (1 + O(\beta)) \), a square diagram \( \tilde{H}_{\lambda, z} \ast \hat{G}_{\lambda, z} \ast \hat{H}_{\lambda, z} \ast \hat{G}_{\lambda, z}(0) \). Repeatedly using Equation (4.2) and Corollary 5.34 shows this is at most \( c\beta \).

**Proposition 6.4.** For \( d \) sufficiently large, \( 0 < z < z_c \), and \( v = 1, 2 \)

\[
\|\partial_z^v \Pi_{\lambda, z}\|_1 \leq c\beta z_c^{-v} \tag{6.6}
\]

Proof. The Leibniz rule and Lemma 5.15 imply that the result of differentiating \( \Pi \) is a sum of terms of the form of the bounds of Proposition 5.11, but where each term has one of the factors of \( G_{\lambda, z} \), \( \tilde{H}_{\lambda, z} \) or \( I_{\lambda, z} \) differentiated. Given this, the argument is as in the proofs of Proposition 5.23 and Proposition 5.27. Let us describe the proof for \( N = 2 \). For \( N = 1 \) the proof is similar as \( e^{\mu_{\lambda, z}(o, x)} \leq \alpha_0 \).

Consider \( v = 1 \). There are \( 3N - 1 \) terms arising when differentiating \( \pi_{\lambda, z}^{(N)} \). If \( \tilde{G}_{\lambda, z} \) or \( \tilde{H}_{\lambda, z} \) is differentiated apply Proposition 5.16 and place the sup norm on this term when applying Lemma 5.26, and then use Proposition 6.2 to bound this norm. If \( I_{\lambda, z} \) is differentiated use Lemma 5.26 placing the sup norm on a term \( \tilde{H}_{\lambda, z} \) and use Proposition 6.3 to bound the one norm of the derivative of \( I_{\lambda, z} \). This yields the claim as the factor of \( 3N - 1 \) is irrelevant for the convergence of the series.

If \( v = 2 \) there are \( (3N - 1)^2 \) terms. If both derivatives fall on a single factor proceed as in the previous paragraph and use Proposition 6.2 or Proposition 6.3. If the derivatives fall on distinct factors, one factor being \( I_{\lambda, z} \), proceed as before. For the remaining case, where two distinct factors of \( \tilde{H}_{\lambda, z} \) (or \( \tilde{G}_{\lambda, z} \)) are differentiated, place a sup norm on one term. The new term to bound when applying Lemma 5.26 is of the form \( \|\tilde{H}_{\lambda, z} \ast \hat{G}_{\lambda, z} \ast \hat{G}_{\lambda, z} \ast I_{\lambda, z}\|_\infty \). It suffices to bound \( \|\tilde{H} \ast \hat{G} \ast \hat{G}\|_\infty \), and this is bounded above by \( \|\tilde{H} \ast \hat{G}\|_\infty + \|\hat{H} \ast \hat{G}\|_\infty + \|\tilde{H} \ast \hat{H} \ast \hat{H}\|_\infty \), all of which are bounded by \( c\beta \) by Corollary 5.34.

\( \square \)
Corollary 6.5. Let $d$ be sufficiently large and $0 < z \leq z_c$. Then $-\frac{d}{dz} \hat{F}_z(0) \geq c > 0$.

Proof. The derivative is

$$-\frac{d}{dz} \hat{F}_z(0) = |\Omega| \alpha(\lambda, z) + z |\Omega| \frac{d}{dz} \alpha(\lambda, z) + \frac{d}{dz} \hat{\Pi}_{\lambda, z}(0).$$

(6.7)

By Proposition 6.4 $\left| \frac{d}{dz} \hat{\Pi}_{\lambda, z}(k) \right|$ is bounded above by a constant since $z_c$ is bounded below by a term of order $\beta$ by Proposition 1.8. An argument as in the proof of Proposition 5.17 shows the magnitude of the second term is bounded by a constant. As $\alpha(\lambda, z) \geq 1$ the first term dominates for $d$ sufficiently large. □

6.1.1 Derivatives of Moments

The next proposition (for $\lambda = 0$) is [14, Exercise 5.17].

Lemma 6.6. For $d$ sufficiently large and $0 \leq z < z_c$

$$\| |x|^2 \Pi_{\lambda, z}(x) \|_1 \leq c \beta$$

(6.8)

Proof. This follows from $\hat{G}^{-1}_\mu(z) \leq 1 - \hat{D}(k)$ and (5.58). □

Proposition 6.7. For $0 \leq z \leq z_c$ the following bounds hold:

$$\| |x|^2 H_{\lambda, z}(x) \|_\infty \leq c \beta,$$

(6.9)

$$\| |x|^2 H_{\lambda, z}(x) \|_2 \leq c.$$

(6.10)

Proof. The proof relies on the identity

$$|x|_\mu^2 H_{\lambda, z}(x) = - \int_{[-\pi, \pi]^d} \partial^2_{k\mu} \hat{H}_{\lambda, z}(k) e^{-i k \cdot x} \frac{d^d k}{(2\pi)^d},$$

(6.11)

where $\mu$ is a unit basis vector of $\mathbb{Z}^d$. Omitting the subscripts $\lambda$ and $z$ and letting a subscript $\mu$ denote partial differentiation with respect to $k_\mu$ the derivative can be calculated:

$$\hat{G}_{\mu, \mu}(k) = z \alpha |\Omega| \frac{\hat{D}_{\mu, \mu}(k)}{F^2(k)} + 2(z \alpha |\Omega|)^2 \frac{\hat{D}_{\mu, \mu}(k)}{F^2(k)} + \hat{\Pi}_{\mu, \mu}(k) + 4z \alpha |\Omega| \frac{\hat{D}_\mu(k) \hat{\Pi}_\mu(k)}{F^3(k)} + 2 \hat{\Pi}_\mu^2(k) F^3(k).$$

(6.12)

To obtain an estimate of $\| |x|^2 H_{\lambda, z} \|_\infty$ take the absolute value of (6.11) inside of the integral and estimate the resulting one norms. Using $z \alpha |\Omega| \leq 1 + O(\beta)$ an upper bound for the first two terms is

$$(1 + O(\beta)) \left( \| \frac{\hat{D}_{\mu, \mu}(k)}{(1 - \hat{D}(k))^2} \|_1 + 2 \| \frac{\hat{D}_{\mu}(k)}{(1 - \hat{D}(k))^3} \|_1 \right) \leq c \beta,$$

(6.13)

where the second inequality follows by estimating the integrals, see [11, Appendix A].

For the remaining terms, $\| |x|^2 \Pi_{\lambda, z} \|_1 \leq c \beta$ implies $\| \Pi_{\mu, \mu} \|_\infty \leq c \beta$. Since $\Pi_{\mu}(k) = 0$ when $k_\mu = 0$ Taylor’s theorem and the above bound on $\| \Pi_{\mu, \mu} \|_\infty$ imply $\| \hat{\Pi}_{\mu} \|_\infty \leq c \beta |k_\mu|$. Lastly, $\| \frac{\hat{D}_\mu(k)}{(1 - \hat{D}(k))^3} \|_\infty \leq c |k_\mu|$. These bounds combined with the $k$-space infrared bound Corollary 5.32 imply each of the remaining terms are bounded by $c \beta$. This proves Equation (6.9).

For $\| |x|^2 H_{\lambda, z} \|_2$ use Parseval’s identity: $\| |x|^2 H_{\lambda, z} \|_2 = \| \partial^2_{\mu} \hat{H}_{\lambda, z} \|_2$. The previously described bounds for the numerators along with Corollary 5.32 and Proposition 4.4 imply that $\hat{G}_{\mu, \mu}(k)$ is square integrable in sufficiently high dimensions. This implies Equation (6.10).

□

Proposition 6.8. For $d$ sufficiently large and $0 < z \leq z_c$

$$\| \partial^\nu |x|^2 \Pi_{\lambda, z} \|_1 \leq c \beta z^v_z$$

(6.14)
Proof. Distribute the factor $|x|^2$ along the factors of $\bar{H}_{\lambda,z}$ and $\bar{G}_{\lambda,z}$ as in the proof of Proposition 5.27. The proof is now essentially the same as for Proposition 6.4. For each term place the sup norm on the factor with the term $|x|^2$.

If a factor $|x|^2 G_{\lambda,z}$ has been differentiated once or twice the resulting term whose norm must be estimated has the form of either $\bar{H}_{\lambda,z} \ast \bar{G}_{\lambda,z}$ or $\bar{H}_{\lambda,z} \ast \bar{G}_{\lambda,z} \ast \bar{G}_{\lambda,z}$. In either case the factor $|x|^2$ can again be split along the factors in the convolution. In the first case use Equation (4.2), the triangle inequality, and Young’s inequality to obtain

$$
\|(|x|^2 \bar{H}_{\lambda,z}) \ast \bar{G}_{\lambda,z}\|_{\infty} \leq \| |x|^2 \bar{H}_{\lambda,z}\|_{\infty} + \| |x|^2 \bar{H}_{\lambda,z}\|_2 \| \bar{G}_{\lambda,z}\|_2, \quad (6.15)
$$

and then use Proposition 6.7 to see that this is bounded by $c\beta$. For the second case arguing similarly gives

$$
\|(|x|^2 \bar{H}_{\lambda,z}) \ast \bar{G}_{\lambda,z} \ast \bar{G}_{\lambda,z}\|_{\infty} \leq \| |x|^2 \bar{H}_{\lambda,z}\|_{\infty} + \| |x|^2 \bar{H}_{\lambda,z} \ast \bar{H}_{\lambda,z}\|_{\infty} + \| |x|^2 \bar{H}_{\lambda,z}\|_2 \| \bar{H}_{\lambda,z}\|_2. \quad (6.16)
$$

The first case analysis implies the first two terms are bounded above by $c\beta$. The rest of the analysis of these terms is in the proof of Proposition 6.4.

The cases in which all derivatives fall on factors without the term $|x|^2$ can be handled in the same manner as in the proof of Proposition 6.4 by using Young’s inequality, the triangle inequality, and Corollary 5.34.

6.2 Linear Divergence of $\chi_\lambda(z)$ as $z \nearrow z_c$

Before proving the linear divergence of the susceptibility it will be helpful to verify that it is only infinite at the critical point $z = z_c$ itself.

Lemma 6.9. For $d$ sufficiently large and $|z| \leq z_c$ the inverse susceptibility $\hat{F}_{\lambda,z}(0)$ satisfies

$$
\left| \hat{F}_{\lambda,z}(0) \right| \geq \frac{\Omega}{2} |z_c - z|, \quad (6.17)
$$

Proof. As $\hat{F}_{\lambda,z_c}(0) = 0$ the fundamental theorem of calculus implies

$$
\left| \hat{F}_{\lambda,z}(0) \right| = \left| \int_{z_c}^z - \frac{d}{dz} \hat{F}_z(0) \, dz \right|. \quad (6.18)
$$

Using $\hat{F}_{\lambda,z_c}(0) = 0$, Equation (6.7), and integrating from $z_c$ to $z$ along a straight line implies

$$
\left| \hat{F}_{\lambda,z}(0) \right| = |\Omega| |z - z_c| \left| \int_0^1 \alpha(\lambda, z_t) + z_t \frac{d}{dz} \alpha(\lambda, z_t) + |\Omega|^{-1} \frac{d}{dz} \hat{\Pi}_{\lambda,z}(0) \, dt \right|, \quad z_t = (1 - t)z_c + tz. \quad (6.19)
$$

The last two terms are bounded by $c\beta$, see the proof of Corollary 6.5. The claim follows by taking the dimension sufficiently large as $\int \alpha = 1 + O(\beta)$.

Define constants $A = A(\lambda)$ and $D = D(\lambda)$ by

$$
A(\lambda) = z_c^{-1} \left( \alpha(\lambda, z_c) |\Omega| + z_c |\Omega| \frac{d}{dz} \alpha(\lambda, z_c) + \frac{d}{dz} \hat{\Pi}_{\lambda,z_c}(0) \right)^{-1}, \quad (6.20)
$$

$$
D(\lambda) = A(\lambda) \left( -z_c |\Omega| \alpha(\lambda, z_c) \nabla_k^2 \hat{D}(0) - \nabla_k^2 \hat{\Pi}_{\lambda,z_c}(0) \right). \quad (6.21)
$$

Theorem 6.10. For $d$ large enough, the susceptibility of $\lambda$-LWW diverges linearly as $z \nearrow z_c$:

$$
\chi_\lambda(z) \sim \frac{Az_c}{z_c - z}. \quad (6.22)
$$

The constant $A$ in (6.22) is as in Equation (6.20).
Proof. Recall $\hat{F}_{\lambda,z}(0) = \hat{G}_{\lambda,z}(0)^{-1}$ is zero at $z_c$ since $\chi_{\lambda}(z) \nrightarrow \infty$ as $z \nearrow z_c$.

\[
\chi_{\lambda}(z) = \frac{1}{\hat{F}_{\lambda,z}(0) - \hat{F}_{\lambda,z}(0)} = \frac{1}{z_c - z} \left[ \alpha(\lambda, z_c) |\Omega| + z |\Omega| \frac{\alpha(\lambda, z_c) - \alpha(\lambda, z)}{z_c - z} + \hat{\Pi}_{\lambda,z}(0) - \hat{\Pi}_{\lambda,z}(0) \right]^{-1}.
\]

The claim follows from Proposition 6.8 and Proposition 5.17 combined with $\alpha_0 \leq 1 + c\beta$ for $z \leq z_c$, which implies differentiability of $\alpha_0$ at $z_c$.

6.3 Growth Rate and Diffusive Scaling

To establish the growth rate of $\lambda$-LWW, as well as the diffusive scaling, a Tauberian type theorem is needed. The statement and proof of the next lemma in [11] involve fractional derivatives of order $1 + \epsilon$ for $0 < \epsilon < 1$, but the arguments apply without modification for two ordinary derivatives.

**Lemma 6.11** (Lemma 6.3.4 of [11]). Let

\[
f(z) = \frac{1}{\phi(z)} = \sum_{n=0}^{\infty} b_n z^n,
\]

where $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$. Suppose that

\[
\sum_{n=0}^{\infty} n^2 |a_n| R^n < \infty,
\]

so in particular, $\phi(z)$, $\phi'(z)$, and $\phi''(z)$ are finite when $|z| = R$. Assume in addition that $\phi'(R) \neq 0$. Suppose that $\phi(R) = 0$ and $\phi(z) \neq 0$ for $|z| \leq R$, $z \neq R$. Then

\[
f(z) = \frac{1}{\phi'(R)} \frac{1}{R - z} + O(1)
\]

uniformly in $|z| \leq R$, and

\[
b_n = R^{-n-1} \left[ \frac{1}{-\phi'(R)} + O(n^{-\alpha}) \right] \text{ as } n \to \infty,
\]

for every $\alpha < 1$.

Recall that $c_{\lambda}^{n}$ is the total mass of $n$-step $\lambda$-LWW, i.e.,

\[
c_{\lambda}^{n} = \sum_{x} \sum_{\omega : \sigma \rightarrow x} \lambda^{n\mathcal{L}(\omega)}.
\]

**Theorem 6.12.** For $d$ sufficiently large and any $\delta < 1$

\[
c_{\lambda}^{n} = A(\lambda) z_c(\lambda)^{-n} (1 + O(n^{-\delta})).
\]

Proof. Apply Lemma 6.11 to $\hat{F}_{\lambda,z}(0)$. The verification of the hypotheses of the theorem are the conclusions of Proposition 6.4, Corollary 6.5, and Lemma 6.9.

The proof of the next theorem is essentially the proof for self-avoiding walk in [11] verbatim; it is reproduced here for the sake of completeness. The next lemma, which will be used several times, is stated here for the convenience of the reader.
Lemma 6.13 (Lemma 6.3.2 of [11]). Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \). Let \( R > 0 \), and suppose \( f'(R) = \sum_{n=0}^{\infty} n |a_n| R^{n-1} < \infty \), so in particular \( f(z) \) converges for \( |z| \leq R \). Then for \( |z| \leq R \)

\[
|f(z) - f(R)| \leq f'(R)|R - z|.
\] (6.29)

If \( f''(z)(R) < \infty \), then for \( |z| \leq R \)

\[
|f(z) - f(R) - f'(R)(z - R)| \leq \frac{1}{2} f''(R)|R - z|^2.
\] (6.30)

Theorem 6.14. For \( d \) sufficiently large \( \lambda \)-LWW is diffusive:

\[
(\langle |\omega(n)|^2 \rangle_n^{\lambda} = Dn(1 + O(n^{-\delta}))
\] (6.31)
as \( n \to \infty \) for any \( \delta < 1 \). The constant \( D \) is that of (6.21).

Proof. Let \( \nabla_k^2 \) denote the \( k \)-space Laplacian. Then

\[
\langle |\omega(n)|^2 \rangle_{\lambda,n} = - \frac{\nabla_k^2 \hat{c}_n^{\lambda}(0)}{c_n^{\lambda}}.
\] (6.32)

Since \( \hat{c}_n^{\lambda}(k) \) is the coefficient of \( z^n \) in \( \hat{G}_{\lambda,n}(k) \) Cauchy’s formula implies

\[
- \nabla_k^2 \hat{c}_n^{\lambda}(0) = \frac{1}{2\pi i} \oint \frac{\nabla_k^2 \hat{F}_{\lambda,n}(0)}{\hat{F}_{\lambda,n}(0)^2} \frac{dz}{z^{n+1}},
\] (6.33)

where the integral is around a small circle centred at the origin. Define \( E(z) \) by

\[
\nabla_k^2 \hat{F}_{\lambda,n}(0) \hat{F}_{\lambda,n}(0)^2 = \frac{\nabla_k^2 \hat{F}_{\lambda,n}(0)}{\hat{F}_{\lambda,n}(0)^2} + E(z).
\] (6.34)

Making this substitution into Equation (6.33) and calculating the first integral implies

\[
- \nabla_k^2 \hat{c}_n^{\lambda}(0) = \frac{\nabla_k^2 \hat{F}_{\lambda,n}(0)}{\hat{F}_{\lambda,n}(0)^2} (n+1)z_c^{-n-2} + \frac{1}{2\pi i} \oint E(z) \frac{dz}{z^{n+1}}.
\] (6.35)

Assuming the integral of \( E(z) \) is \( O(n^\delta z^{-n}) \) for every \( \delta > 0 \) implies the theorem by inserting the behaviour of \( c_n^{\lambda} \) given by Theorem 6.12.

To verify the assumption it suffices by Lemma 6.13 to prove \( |E(z)| \leq \text{const.} |z_c - z|^{-1} \) for all \( |z| \leq z_c \). Split \( E(z) \) as \( E(z) = T_1(z) + T_2(z) \) with

\[
T_1(z) = \left[ \frac{d}{dz} \hat{F}_{\lambda,n}(z) \right]^{n+1} \frac{\nabla_k^2 \hat{F}_{\lambda,n}(z)}{(z_c - z)^2},
\] (6.36)

\[
T_2(z) = \left[ \left( \frac{d}{dz} \hat{F}_{\lambda,n}(z) \right)^2 \hat{F}_{\lambda,n}(z) - \left( \frac{d}{dz} \hat{F}_{\lambda,n}(z) \right)^2 (z_c - z)^2 \right] \left( \frac{d}{dz} \hat{F}_{\lambda,n}(z) \right)^2 (z_c - z)^2.
\] (6.37)

The numerator of \( T_1(z) \) is differentiable in \( z \) by Proposition 6.8, so (i) of Lemma 6.13 implies the numerator is bounded above by a constant times \( |z_c - z|^{-1} \). It follows that \( |T_1| \leq O(|z_c - z|^{-1}) \).

For \( T_2 \) note that \( \hat{F}_{\lambda,n}(z)^2 \geq \text{const.} |z_c - z|^2 \) by Lemma 6.9 so

\[
|T_2(z)| \leq \text{const.} |z_c - z|^{-4} \left[ \hat{F}_{\lambda,n}(z) + \frac{d}{dz} \hat{F}_{\lambda,n}(z)(z_c - z) \right] \left[ \hat{F}_{\lambda,n}(z) - \frac{d}{dz} \hat{F}_{\lambda,n}(z)(z_c - z) \right],
\] (6.38)
as \( \nabla_k^2 \hat{F}_{\lambda,n}(z) \) is bounded by a constant by Proposition 6.7. By (ii) of Lemma 6.13, Proposition 6.8, and \( \hat{F}_{\lambda,n}(z) = 0 \), the middle term is \( O(|z_c - z|^2) \). Using (i) of Lemma 6.13 for \( \hat{F}_{\lambda,n}(z) \) in the last term shows the last term is \( O(|z_c - z|) \). Thus \( |T_2(z)| \leq O(|z_c - z|^{-1}) \), which proves the claim. \(\square\)
A Loop Measure Representation of $\lambda$-LWW

A fundamental property of $\lambda$-LWW is that it admits a loop measure representation. The representation follows from a theorem of Viennot [18] and is proved via the theory of heaps of pieces in Appendix A.3. The purpose of this appendix is to prove Theorem 1.13, i.e., the loop measure representation.

Remark A.1. The methods of [10, Chapter 9] are sufficient to derive formulas that would suffice for the lace expansion analysis of $\lambda$-LWW. While such an approach would have the benefit of brevity, it would not reveal the connection with the loop $O(N)$ model that are contained in Section A.4.

The rest of this section will take place in the context of an arbitrary graph $G$, as specializing to $\mathbb{Z}^d$ does not provide any simplification. The theory of heaps of pieces will be freely used; see [18] or [9] for an introduction.

A.1 Viennot’s Theorem

Definition A.2. A trivial cycle is a single edge of $G$. An oriented cycle is either (i) an oriented cyclic subgraph of $G$ or (ii) a trivial cycle in $G$.

An oriented cycle corresponds to an equivalence class of self-avoiding polygons in which a self-avoiding polygon $\omega = (\omega_0, \ldots, \omega_k = \omega_0)$ is equivalent to any cyclic permutation $\tilde{\omega} = (\omega_r, \omega_{r+1}, \ldots, \omega_k, \omega_1, \ldots, \omega_r)$. For example, a trivial cycle $\{x, y\}$ corresponds to the self-avoiding polygons $(x, y, x)$ and $(y, x, y)$, while an oriented 3-cycle corresponds to walks of the form $(x, y, z, x)$ and cyclic permutations thereof for $x, y, z$ distinct.

Definition A.3. A heap of (oriented) cycles is a heap of pieces whose labels are oriented cycles. Two oriented cycles $C_1, C_2$ are concurrent if $V(C_1) \cap V(C_2) \neq \emptyset$, i.e., if the cycles share a vertex.

Definition A.4. A pair $(\eta, H)$ where $\eta$ is a self-avoiding walk from $a$ to $b$ and $H$ is a heap of cycles whose maximal elements’ labels each contain a vertex in $\eta$ is called a legal $(a, b)$ pair. Let $V(a, b)$ denote the set of legal $(a, b)$ pairs, and $\mathcal{V}$ denote the set of all legal pairs.

Theorem 1.13, the loop measure representation of $\lambda$-LWW, is a byproduct of the proof of the following theorem of Viennot.

Theorem A.5 ([18, Proposition 6.3]). There is a bijection $\phi_{ab}$ from the set $V(a, b)$ of legal $(a, b)$ pairs to the set of walks $\Omega(a, b)$ from $a$ to $b$. Further,

1. The multi-set of edges in a legal $(a, b)$ pair $(\eta, H)$ is the same as the multi-set of edges in the walk $\tilde{\phi}_{ab}((\eta, H))$.

2. The multi-set of oriented cycles $\{\ell(x) \mid x \in H\}$ for a heap $(H, \ell, \preceq)$ is the same as the multi-set of oriented cycles that are erased by applying loop erasure to $\tilde{\phi}_{ab}((\eta, H))$.

Theorem A.5 is not proven in [18]. For the sake of completeness and the convenience of the reader a proof is given in Appendix A.2. The remainder of this section consists of a heuristic description of the proof; see also Figure 7 which depicts the proof strategy.

Let $\omega$ be a walk from $a$ to $b$. Trace $\omega$ until the first time a vertex is visited twice. This identifies a first closed subwalk $C_1 = (\omega_{\tau_1}, \ldots, \omega_{\tau_2})$. Remove $C_1$ by performing a single loop erasure, and form a heap of pieces consisting of a single piece labelled $C_1$. The first time a vertex is visited twice by the walk LE$^1(\omega)$ identifies a second closed subwalk, call this $C_2$. Remove $C_2$ and form a heap of pieces by adding a second piece labelled $C_2$ to the heap consisting of $C_1$. Continuing in this manner removes all of the closed subwalks from $\omega$, resulting in a self-avoiding walk $\eta$ from $a$ to $b$. Each maximal piece in the heap is labelled by a cycle that shares a vertex with $\eta$. In other words, this procedure converts each walk from $a$ to $b$ into a legal pair $(\eta, H)$.

Conversely, consider a legal pair $(\eta, H)$. To invert the procedure what is required is a way to reduce the heap to the empty heap one piece at a time, while inserting the labels of the removed pieces into the (initially) self-avoiding walk $\eta$. This is relatively straightforward: the maximal pieces of the heap $H$
Figure 7: The figure illustrates the bijection between valid pairs $(\eta, H)$ and walks $\omega$ whose loop erasure is $\eta$. The left-hand side shows the results of successive applications of LE$, culminating in a self-avoiding walk. The right hand side shows the heaps of oriented cycles generated, with the walk displayed in dotted gray. Each heap has been given a distinguished vertex. The vertex indicates the oriented cycle that is maximal in the walk order as well as the location at which this oriented cycle is glued in to the corresponding walk when performing loop addition.
have labels that share a vertex with $\eta$, and hence the maximal pieces can be ordered by using the linear order on vertices in $\eta$. Take the maximal piece in this order, remove it from the heap to get a heap $H'$, and glue the corresponding label into $\eta$ to get a walk $\eta'$. The maximal elements of $H'$ have labels that share a vertex with $\eta'$, and hence this procedure can be iterated.

These operations are in fact inverses of one another. The next section makes the preceding discussion precise.

### A.2 Proof of Viennot’s Theorem

The theorem requires two algorithms, one which inserts oriented cycles into a given walk, and one which removes oriented cycles from a walk. Removing oriented cycles is achieved by loop erasure. The other algorithm is introduced now.

**Definition A.6.** Let $\omega$ be a walk of length $n$, and let $C$ be an oriented cycle of length $k$. Assume that $C$ and $\omega$ have a vertex in common, and let $i$ be the minimal index such that $\omega_i$ is a vertex in $C$. Let $(c_0, \ldots, c_k)$ be the unique representative of $C$ such that $c_0 = \omega_i$. The loop insertion $\omega \oplus C$ of $C$ into $\omega$ is the walk $(\omega_0, \ldots, \omega_{i-1}, c_0, \ldots, c_k, \omega_{i+1}, \ldots, \omega_n)$.

In words, to insert a loop $C$ into a walk $\omega$ we find the first vertex $\omega_i$ in $\omega$ that is contained in $C$. $C$ is then rooted at $\omega_i$. $\omega$ is traversed until just before reaching $\omega_i$, $C$ is traversed, and then the remainder of $\omega$ is traversed.

**Lemma A.7.** Let $\omega$ be a walk, and let $C$ be the oriented cycle removed to create $\text{LE}^1(\omega)$. Then $\text{LE}(\omega) \oplus C = \omega$.

**Proof.** The definition of $\tau_\omega^*$ and the definition of loop erasure implies that the first vertex in common between $\text{LE}^1(\omega)$ and $C$ is $\omega_{\tau_\omega}$, and hence the closed self-avoiding walk representing $C$ that is inserted by loop insertion is $(\omega_{\tau_\omega}, \ldots, \omega_n)$.

Given a collection of oriented cycles that intersect a walk it is necessary to determine the order in which the cycles should be inserted. The next definition gives the correct order for inverting loop erasure.

**Definition A.8.** Let $\omega$ be a walk, and $C_1, \ldots, C_k$ a collection of oriented cycles that each share a vertex with $\omega$. Let $t_j = \min\{i \mid \omega_i \in C_j\}$. The walk order on the oriented cycles is given by setting $C_m \geq C_n$ if $t_m \geq t_n$.

The following algorithm, called the loop addition algorithm, constructs a walk beginning at the vertex $a$ and ending at the vertex $b$ from a legal $(a, b)$ pair $(\eta, H)$.

1. Set $\omega^0 = \eta$.

2. Suppose $H^{i-1} \neq \emptyset$. Set $\omega^i = \omega^{i-1} \oplus C$, where $C$ is maximal in the walk order among the labels of the maximal pieces of $H^{i-1}$. Let $y$ be the piece whose label is $C$, and set $H^i = H^{i-1} \setminus \{y\}$.

3. If $H^{i-1} = \emptyset$, output $\omega = \omega^{i-1}$. Otherwise go to 2.

The algorithm is well-defined as the labels of the maximal pieces in a heap must be vertex disjoint, so the walk order is a strict total order on the maximal pieces of the heap. Note that at each step of the algorithm the walk $\omega^i$ begins at the vertex $a$ and ends at $b$, so $\omega$ is a walk from $a$ to $b$ as claimed.

**Lemma A.9.** Suppose $(\eta, H) \in \mathcal{V}$. Suppose the output of the loop addition algorithm is $\omega$. If $C$ is the last oriented cycle inserted, then the oriented cycle removed by loop erasure applied to $\omega$ is $C$.

**Proof.** The proof is by induction on the size of $H$. Suppose $C$ was the $(k + 1)^{st}$ oriented cycle added.

1. If $C$ was the label of a maximal element in $H^{k-1}$ then $C$ is vertex disjoint from the $k^{th}$ added oriented cycle $C'$. The definition of the walk order implies that the first vertex $C$ shares with $\omega^{k-1}$ occurs prior to the first vertex in $C'$ because $C$ is disjoint from $C'$. It follows that $C$ is the oriented cycle erased by loop erasure, as $C$ closes prior to $C'$, which was previously (by induction) the first oriented cycle to close.
2. If $C$ was not the label of a maximal piece in $H^{k-1}$ then $C$ is the label of a piece that was below
the $k^{th}$ inserted piece. Suppose the $k^{th}$ piece had label $C'$. As $C$ intersects $C'$, $C$ is inserted into
the subwalk $C'$ of $\omega^{k-1}$. By induction $C'$ was the first oriented cycle to close in $\omega^{k-1}$, so $C$ is the
first oriented cycle to close in $\omega^k$. □

To construct a legal pair $(\eta, H)$ from a walk is fairly straightforward. By applying loop erasure
oriented cycles are removed, and they naturally form a heap by using the heap composition operation.
More precisely, we have the (total) loop erasure algorithm:

1. Set $\omega^0 = \eta$, and $H^0 = \emptyset$, where $\emptyset$ is the empty heap of oriented cycles.
2. If $\omega^{i-1}$ is not a self-avoiding walk, set $\omega^i = \text{LE}^1(\omega^{i-1})$, and if $C$ is the closed self-avoiding walk
removed from $\omega^{i-1}$, let $H^i = H \circ \{C\}$ where $C$ is the oriented cycle corresponding to $C$.
3. If $\omega^{i-1}$ is a self-avoiding walk, output $(\omega^{i-1}, H^{i-1})$. Otherwise go to 2.

Single loop erasure removes a subwalk of length at least 2 from any non-simple walk at each step,
so iteratively applying $\text{LE}^1$ stabilizes on a self-avoiding walk in a finite number of iterations. It follows
that the total loop erasure is well defined.

Lemma A.10. The output of the loop erasure algorithm applied to a walk $\omega = (\omega_0, \ldots, \omega_n)$ is a pair
$(\eta, H) \in V(\omega_0, \omega_n)$.

Proof. At each step of the algorithm the maximal pieces of the heap $H^i$ share a vertex with the remaining
walk $\omega^i$, and the algorithm only terminates once the remaining walk is self-avoiding. Removing a cycle
cannot change the initial vertex of a walk, so $\eta_{n} = \omega_n$. If the final vertex of $\omega$ is removed it must be that
visiting the final vertex completes a cycle, and hence $\eta$ ends at $\omega_n$. □

Proof of Theorem A.5. We claim that loop erasure and loop addition are inverses of one another, and
prove the claim by induction. Suppose the claim holds between walks whose loop erasure removes $k$
oriented cycles and pairs $(\eta, H) \in V(a, b)$ whose heap $H$ has $k$ pieces.

On the one hand, inserting the final oriented cycle $C$ in the loop addition algorithm yields a walk,
and $C$ is the first oriented cycle removed by loop erasure by Lemma A.9. By induction it follows that
loop erasure applied to the loop addition of a pair $(\eta, H) \in V(a, b)$ returns $(\eta, H)$.

On the other hand when a single oriented cycle $C$ is removed from $\omega$ the cycle $C$ is minimal in the
walk order and the removed oriented cycle is the label of a maximal piece. So the reconstruction of the
heap formed by loop erasure proceeds as if the piece with label $C$ was not present, and hence (by induction) recreates $\text{LE}^1(\omega)$ correctly. Lemma A.7 then implies that $\omega$ is the output of applying loop erasure and then loop addition. □

A.3 Proof of Theorem 1.13

The proof of Theorem 1.13 follows from two calculations. The first is a straightforward consequence of
the fact that the bijection between walks and legal pairs is given by loop erasure. Let $\mathcal{T}$ denote the
set of trivial heaps of oriented cycles, and $\mathcal{H}$ the set of heaps of oriented cycles. Let $\mathcal{C}(\eta)$ denote the
set of oriented cycles that do not share a vertex with the set $\eta$, and let $\mathcal{H}_\eta$ denote the set of heaps $H$
such that $(\eta, H)$ is a legal pair. The definition of $\lambda$-LWW, Theorem A.5, and the heap theorem [18,
Proposition 5.3] imply

\[ \bar{w}_{\lambda,z}(\eta) = \sum_{\omega: \text{LE}(\omega) = \eta} w_{\lambda,z}(\omega) \]  
\[ = z^{\vert \eta \vert} \sum_{H \in \mathcal{H}_\eta} w_{\lambda,z}(H) \]  
\[ = z^{\vert \eta \vert} \frac{\sum_{T \in \mathcal{T}(\mathcal{C}(\eta))} (-1)^{|T|} w_{\lambda,z}(T)}{\sum_{T \in \mathcal{T}} (-1)^{|T|} w_{\lambda,z}(T)}, \]
where \( w_{\lambda,z}(H) = \prod_{x \in H} w_{\lambda,z}(\ell(x)) \) for a heap \((H, \ell, \preceq)\). In particular note that this definition assigns a weight \( z^2 \lambda \) to a trivial cycle.

The second calculation is an expression for sums over trivial heaps of oriented cycles. Theorem 1.13 follows by applying Proposition A.11 to the numerator and denominator of (A.3) and cancelling common factors. This calculation is a calculation involving formal power series; to see that it holds as a relation between power series, note that for \( z \) sufficiently small the final expressions are bounded by random walk quantities, which converge.

**Proposition A.11.**

\[
\sum_{T \in T(C(A))} (-1)^{|T|} w_{\lambda,z}(T) = \exp \left( - \sum_{x \in \mathbb{Z}^d} \sum_{\omega: x \rightarrow x, |\omega| \geq 1} \mathbb{1}_{\{\text{range}(\omega) \cap A = \emptyset\}} \frac{w_{\lambda,z}(\omega)}{|\omega|} \right)
\]  

(A.4)

**Proof of Theorem 1.13.** Let \( \bar{z} = sz \). Then \( w_{\lambda,z}(\omega) = w_{\lambda,\bar{z}}(\omega) \) when \( s = 1 \). Using this observe that

\[
\sum_{T \in T(C(A))} (-1)^{|T|} w_{\lambda,z}(T) = \exp \int_0^1 \frac{d}{ds} \log \sum_{T \in T(C(A))} (-1)^{|T|} w_{\lambda,\bar{z}}(T).
\]  

(A.5)

In calculating the derivative the Leibniz rule for differentiating \( s^k \) can be interpreted as selecting one of the \( k \) vertices contained in the cycles of a trivial heap. The selected vertex distinguishes a self-avoiding polygon. Theorem A.5 can be applied to transform this into a walk weighted by \( w_{\lambda,z} \). The factor of \(-1\) in the exponent arises from the application of Theorem A.5, as the distinguished cycle carried a factor of \(-1\). Lastly, the term \(|\omega|^{-1}\) arises from the integration of \( s^{|\omega|} \) from 0 to 1; the missing factor of \( s \) is due to the differentiation which distinguished a vertex.

**A.4 Relation to Correlations of the \( O(N) \) Cycle Gas**

Note that Equation (A.1) and Equation (A.3) imply that \( G_{\lambda,z}(o,x) \) is given by a ratio of partition functions. The denominator is a sum over oriented mutually disjoint cyclic subgraphs, where the weight of a subgraph \( H \) is \( z^{|E(H)|}(-\lambda)^{|H|} \), where \(|H|\) denotes the number of cyclic subgraphs contained in \( H \). The numerator is a sum over self-avoiding walks from \( o \) to \( x \) along with disjoint cyclic subgraphs; the weight is the same as for the denominator except for the fact that the walk does not receive a factor of \( \lambda \).

For each cycle of length at least 3 summing over the possible orientations of the cycles results in a model of unoriented cycle, where each unoriented cycle has weight \(-2\lambda\), except for trivial cycles, which have weight \(-\lambda\). Defining the two-point correlation in the \( O(N) \) cycle gas to be the ratio described in the previous paragraph gives the relation between the \( O(N) \) cycle gas and \( \lambda \)-LWW. Note that if cycles of length two are assigned loop activity 0 this yields a precise correspondence between \( \lambda \)-LWW and the \( O(-2\lambda) \) cycle gas.

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