PRIME GROUP GRADED RINGS WITH APPLICATIONS TO
PARTIAL CROSSED PRODUCTS AND LEAVITT PATH ALGEBRAS

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Abstract. In this article we generalize a classical result by Passman on primeness of unital
strongly group graded rings to the class of nearly epsilon-strongly group graded rings which
are not necessarily unital. Using this result, we obtain (i) a characterization of prime $s$-unital
strongly group graded rings, and, in particular, of infinite matrix rings and of group rings
over $s$-unital rings, thereby generalizing a well-known result by Connell; (ii) characterizations
of prime $s$-unital partial skew group rings and of prime unital partial crossed products; (iii)
a generalization of the well-known characterizations of prime Leavitt path algebras, by Larki
and by Abrams-Bell-Rangaswamy.

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1. Introduction

Let $S$ be a ring. By this we mean that $S$ is associative but not necessarily unital. Unless otherwise stated, ideals of $S$ are assumed to be two-sided. Recall that a proper ideal $P$ of $S$ is called prime if for all ideals $A$ and $B$ of $S$, $A \subseteq P$ or $B \subseteq P$ holds whenever $AB \subseteq P$. The ring $S$ is called prime if $\{0\}$ is a prime ideal of $S$. The class of prime rings contains many well-known constructions, for instance left or right primitive rings, simple rings and matrix rings over integral domains. Prime rings also generalize integral domains to a non-commutative setting. Indeed, a commutative ring is prime if and only if it is an integral domain.

Throughout this article, $G$ denotes a multiplicatively written group with neutral element $e$. Recall that $S$ is called $G$-graded, if for each $x \in G$ there is an additive subgroup $S_x$ of $S$ such that $S = \bigoplus_{x \in G} S_x$, as additive groups, and for all $x, y \in G$, the inclusion $S_x S_y \subseteq S_{xy}$ holds. If in addition, $S_x S_y = S_{xy}$ holds for all $x, y \in G$, then $S$ is said to be strongly $G$-graded. An interesting problem, studied for the past 50 years, concerns finding necessary and sufficient conditions for different classes of group graded rings to be prime, see [4] [9] [10] [28] [29] [35] [36] [37] [38] [39] [40]. In the case when $S$ is unital and strongly $G$-graded, Passman has completely solved this problem by proving the following rather involved result:
**Theorem 1.1** (Passman [40, Thm. 1.3]). Suppose that $S$ is a unital and strongly $G$-graded ring. Then $S$ is not prime if and only if there exist:

(i) subgroups $N < H \subseteq G$ with $N$ finite,
(ii) an $H$-invariant ideal $I$ of $S_e$ such that $I^x I = \{0\}$ for every $x \in G \setminus H$, and
(iii) nonzero $H$-invariant ideals $A, B$ of $S_N$ such that $A, B \subseteq I S_N$ and $AB = \{0\}$.

Let us briefly explain the notation used in the formulation of this result as well as some technical aspects of Passman’s proof of it. Suppose that $x \in G$ is cancellative. If $x \in G$, then $I^x$ denotes the $S_e$-ideal $S_{x-1} I S_x$. Let $H, N$ be subgroups of $G$. The ideal $I$ is called $H$-invariant if $I^x \subseteq I$ holds for every $x \in H$; $S_N$ denotes $\bigoplus_{x \in N} S_x$, which is clearly a subring of $S$. In [40] Passman provided a “combinatorial” proof of Theorem 1.1 by combining two main ideas. First, a coset counting method, also known as the “$\Delta$-method”, developed by Passman [35] and Connell [10], secondly, the “bookkeeping procedure” introduced by Passman in [35] which involves a careful study of the action of the group $G$ on the lattice of ideals of $S_e$. In [40] Passman also showed that analogous criteria exist for semiprimeness of strongly graded rings. In this article, however, only the concept of primeness will be studied.

In a subsequent article [39] Passman obtained an analogue of Theorem 1.1 for the slightly larger class of unital $G$-graded rings which are cancellative, that is, rings $S$ having the property that for all $x, y \in G$ and all homogeneous subsets $U, V \subseteq S$, the implication $US_x S_y V = \{0\} \Rightarrow US_{xy} V = \{0\}$ holds. It is clear that strongly $G$-graded rings are cancellative. However, not all cancellative $G$-graded rings are strongly graded. For instance, the first Weyl algebra $A_1(\mathbb{F})$, over a field $\mathbb{F}$ of characteristic zero, is a $\mathbb{Z}$-graded ring which is not strongly graded but still cancellative, since it is a domain (see e.g. [10, Chap. 2]).

The motivation for the present article is the observation that many important examples of group graded rings are not cancellative, but may still be prime. Indeed, suppose that $R$ is a unital ring and let $S := M_n(R)$ denote the ring of $n \times n$-matrices with entries in $R$. Then it is easy to see that $S$ is prime if and only if $R$ is prime. On the other hand, one can construct group gradings on $S$ that are not cancellative. Consider the case $n = 2$. Let $e_{ij}$ denote the matrix with 1 in position $ij$ and zeros elsewhere. If $G = \mathbb{Z}$ and we put $S_0 := Re_{11} + Re_{22}$, $S_1 := Re_{12}, S_{-1} := Re_{21}$, and $S_x := \{0\}$, for $x \in \mathbb{Z} \setminus \{0, 1, -1\}$, then this defines a $G$-grading on $S$ satisfying $S_1 \cdot S_1 S_{-1} \cdot S_0 = \{0\}$ but $S_1 \cdot S_0 \cdot S_0 = Re_{12} \neq \{0\}$. This shows that the grading is not cancellative. In a similar fashion, one may define non-cancellative $\mathbb{Z}$-gradings on $M_n(R)$, for every $n \geq 2$.

This phenomenon is not confined to rings of matrices. In fact, these structures can be considered as special cases of so-called Leavitt path algebras $L_R(E)$, over a unital ring $R$, defined by directed graphs $E$ (for the details, see Section 14). All Leavitt path algebras carry a canonical $\mathbb{Z}$-grading. One can show that with this grading, a Leavitt path algebra defined by a finite graph $E$, is cancellative if and only if it is strongly graded (see Proposition 2.24 and Proposition 14.4). However, it is easy to give examples of Leavitt path algebras which are not strongly $\mathbb{Z}$-graded. Nevertheless, the question of primeness of such structures has been completely resolved in the case when $R$ is commutative, for any directed graph $E$, by Larki [25] building upon previous work by Abrams, Bell and Rangaswamy [3, Thm. 1.4]. Their results involve a certain “connectedness” property on the set $E^0$ of vertices of $E$. Namely, a directed graph $E$ is said to satisfy condition (MT-3) if for all $u, v \in E^0$, there exist $w \in E^0$ and paths from $u$ to $w$ and from $v$ to $w$. 
There exist:
are unreachable by the results of [39, 40]. Thirdly, we would like to motivate why assertions
are not necessarily unital
crossed products [11, 12, 13]. Thus, Theorem 1.3 allows us to consider classes of rings which
\{ \text{If} \exists s \in G \}\text{such that the equalities } \epsilon_x(s)s = s = se_x'(s) \text{ hold. Note that every nearly epsilon-strongly G-graded ring } S \text{ is necessarily non-degenerately G-graded, i.e. for every } x \in G \text{ and every nonzero } s \in S_x, \text{ we have } ss_{x-1} \neq \{0\} \text{ and } S_{x-1}s \neq \{0\}. \text{ In loc. cit. it is shown that every Leavitt path algebra, equipped with its canonical Z-grading, is nearly epsilon-strongly graded. In addition, s-unital partial skew group rings and unital partial crossed products are nearly epsilon-strongly graded (see Section I.3).}
Here is the main result of this article:

**Theorem 1.3.** Suppose that \(G\) is a group and that \(S\) is a \(G\)-graded ring. Consider the following five assertions:

(a) \(S\) is not prime.
(b) There exist:
   (i) subgroups \(N \triangleleft H \subseteq G\),
   (ii) an \(H\)-invariant ideal \(I\) of \(S_e\) such that \(I^xI = \{0\}\) for every \(x \in G \setminus H\), and
   (iii) nonzero ideals \(A, B\) of \(S_N\) such that \(A, B \subseteq IS_N\) and \(AIS_HB = \{0\}\).
(c) There exist:
   (i) subgroups \(N \triangleleft H \subseteq G\) with \(N\) finite,
   (ii) an \(H\)-invariant ideal \(I\) of \(S_e\) such that \(I^xI = \{0\}\) for every \(x \in G \setminus H\), and
   (iii) nonzero ideals \(A, B\) of \(S_N\) such that \(A, B \subseteq IS_N\) and \(AIS_HB = \{0\}\).
(d) There exist:
   (i) subgroups \(N \triangleleft H \subseteq G\) with \(N\) finite,
   (ii) an \(H\)-invariant ideal \(I\) of \(S_e\) such that \(I^xI = \{0\}\) for every \(x \in G \setminus H\), and
   (iii) nonzero \(H\)-invariant ideals \(A, B\) of \(S_N\) such that \(A, B \subseteq IS_N\) and \(AIS_HB = \{0\}\).
(e) There exist:
   (i) subgroups \(N \triangleleft H \subseteq G\) with \(N\) finite,
   (ii) an \(H\)-invariant ideal \(I\) of \(S_e\) such that \(I^xI = \{0\}\) for every \(x \in G \setminus H\), and
   (iii) nonzero \(H/N\)-invariant ideals \(\tilde{A}, \tilde{B}\) of \(S_N\) such that \(\tilde{A}, \tilde{B} \subseteq IS_N\) and \(\tilde{A}\tilde{I}S_H\tilde{B} = \{0\}\).

The following assertions hold:

1. If \(S\) is non-degenerately G-graded, then (e) \implies (d) \implies (c) \implies (b) \implies (a).
2. If \(S\) is nearly epsilon-strongly G-graded, then (a) \iff (b) \iff (c) \iff (d) \iff (e).

Let us make four remarks on Theorem 1.3. First of all, this result is applicable to rings which are not necessarily unital. Secondly, unital strongly G-graded rings (see Lemma 2.15) and cancellatively G-graded rings (see [39, Lem. 1.2]) satisfy \(r \cdot \text{Ann}_x(S_x) = \{0\}\), and \(l \cdot \text{Ann}_x(S_x) = \{0\}\), for every \(x \in G\). However, many important classes of group graded rings rarely satisfy such annihilator conditions, for instance Leavitt path algebras [11, 12, 13, 25, 13] and partial crossed products [11, 12, 13]. Thus, Theorem 1.3 allows us to consider classes of rings which are unreachable by the results of [39, 40]. Thirdly, we would like to motivate why assertions
(b), (c) and (d) appear in Theorem 1.3. By allowing $N$ to be infinite, assertion (b) creates more flexibility when attempting to prove that $S$ is non-prime. Assertion (c) is identical to the assertion in [39, Thm. 2.3], and assertion (d) is essentially identical to the assertion in [40, Thm. 1.3]. Finally, it might be possible to generalize assertion (2) of Theorem 1.3 beyond the class of nearly epsilon-strongly graded rings (see Remark 12.8).

Here is a detailed outline of this article.

In Section 2, we state our conventions on groups, rings and modules. We also provide preliminary results on different types of graded rings such as epsilon-strongly graded rings, nearly epsilon-strongly graded rings and cancellatively graded rings. In Section 3, we consider $H$-invariant ideals and record some of their basic properties. In Section 4, we obtain a one-to-one correspondence between graded ideals of $S$ and $G$-invariant ideals of the principal component $S_e$. We also give a characterization of prime nearly epsilon-strongly $G$-graded rings in the case when $G$ is an ordered group. In Section 5, we prove the implication (b)$\Rightarrow$(a) of Theorem 1.3 for non-degenerately $G$-graded rings. In Section 6, we obtain some technical results that will be necessary in Section 7, where we provide the bulk of results needed to establish Theorem 1.3. Our approach is very much influenced by Passman [40]. In particular, we utilize a version of the $\Delta$-method. In Section 8, we prove the implication (a)$\Rightarrow$(e) of Theorem 1.3 for nearly epsilon-strongly graded rings. In Section 9, the proof of Theorem 1.3 is finalized. We also show that Theorem 1.1 can be recovered from Theorem 1.3. In Section 10, we use Theorem 1.3 to obtain the following generalization of a result by Passman (see [40, Cor. 4.6]):

**Theorem 1.4.** Suppose that $G$ is torsion-free and that $S$ is nearly epsilon-strongly $G$-graded. Then $S$ is prime if and only if $S_e$ is $G$-prime.

The remaining sections are devoted to applications of our findings. In Section 11, we obtain an $s$-unital analogue of Passman’s Theorem 1.1 (see Corollary 11.1) and consider $\mathbb{Z}$-graded Morita context algebras and $\mathbb{Z}$-graded infinite matrix rings. In Section 12, we apply Theorem 1.3 to group rings. Notably, we obtain the following non-unital generalization of Connell’s [10] classical characterization:

**Theorem 1.5.** Suppose that $R$ is an $s$-unital ring and that $G$ is a group. Then the group ring $R[G]$ is prime if and only if $R$ is prime and $G$ has no non-trivial finite normal subgroup.

In Section 13, we apply our results to $s$-unital partial skew group rings (see Theorem 13.5 and Theorem 13.7) and to unital partial crossed products (see Theorem 13.9 and Theorem 13.10). In Section 14, we use Theorem 1.3 to obtain a characterization of prime Leavitt path algebras, thereby generalizing Theorem 1.2 by allowing the coefficient ring $R$ to be non-commutative:

**Theorem 1.6.** Suppose that $E$ is a directed graph and that $R$ is a unital ring. Then the Leavitt path algebra $L_R(E)$ is prime if and only if $R$ is prime and $E$ satisfies condition (MT-3).

2. Preliminaries

In this section, we recall some useful notions and conventions on groups, rings and modules. We also provide some preliminary results on different types of graded rings such as epsilon-strongly graded rings, nearly epsilon-strongly graded rings and cancellatively graded rings. These results will be utilized in subsequent sections.
2.1. Groups. For the entirety of this article, $G$ denotes a multiplicatively written group with neutral element $e$. Let $H$ be a subgroup of $G$. The index of $H$ in $G$ is denoted by $[G : H]$. Take $g \in G$. The order of $g$ is denoted by $\text{ord}(g)$. The centralizer of $g$ in $G$ is defined to be the subgroup $C_G(g) := \{x \in G \mid xg = gx\}$ of $G$. Recall that the finite conjugate center of $G$ is the subgroup $\Delta(G) := \{g \in G \mid [G : C_G(g)] < \infty\}$ of $G$. The almost centralizer of $H$ in $G$ is the subgroup $D_G(H) := \{x \in G \mid [H : C_H(x)] < \infty\}$ of $G$. Note that $D_G(H) \cap H = \Delta(H)$.

By the orbit-stabilizer theorem, $\Delta(G)$ can equivalently be described as the set of elements of $G$ with only finitely many conjugates in $G$. If $G$ is equipped with a total order relation $\leq$ such that for all $a, b, x, y \in G$ the inequality $a \leq b$ implies the inequality $xay \leq xby$, then $G$ is called an ordered group.

2.2. Rings and modules. Throughout this article, all rings are assumed to be associative but not necessarily unital. Let $R$ be a ring. If $U$ and $V$ are subsets of $R$, then $UV$ denotes the set of finite sums of elements of the form $uv$ where $u \in U$ and $v \in V$. We say that $R$ is unital if it has a nonzero multiplicative identity element. In this article, we will also consider the following weaker notion of unitality. The ring $R$ is called $s$-unital if for every $r \in R$ the inclusion $r \in rR \cap rR$ holds. For future reference, we recall the following:

**Proposition 2.1** (Tominaga [30, Prop. 12], [44]). A ring $R$ is $s$-unital if and only if for any finite subset $V$ of $R$ there is $u \in R$ such that for every $v \in V$ the equalities $uw = vu = v$ hold.

If $M$ is a left $R$-module and $U$ is a subset of $M$, then the left annihilator of $U$ is defined to be the set $\text{Ann}_R(U) := \{r \in R \mid r \cdot u = 0, \forall u \in U\}$. If $N$ is a right $R$-module and $V$ is a subset of $N$, then the right annihilator $r.\text{Ann}_R(V)$ is defined analogously.

2.3. Group graded rings. For the rest of this article $S$ denotes a nonzero $G$-graded ring. Note that the principal component $S_e$ is a subring of $S$ and every $x \in G$, the set $S_xS_{x^{-1}}$ is an ideal of $S_e$. The support of $S$, denoted by $\text{Supp}(S)$, is the set of $x \in G$ with $S_x \neq \{0\}$. In general, $\text{Supp}(S)$ need not be a subgroup of $G$ (see [32, Rmk. 46]). Take $s \in S$. Then $s = \sum_{x \in G} s_x$, for unique $s_x \in S_x$, such that $s_x = 0$ for all but finitely many $x \in G$. The support of $s$, denoted by $\text{Supp}(s)$, is the set of $x \in G$ with $s_x \neq 0$.

**Proposition 2.2.** The ring $S$ is strongly $G$-graded if and only if for every $x \in G$ the equalities $S_xS_e = S_eS_x = S_x$ and $S_xS_{x^{-1}} = S_e$ hold.

**Proof.** Suppose that for every $x \in G$ the equalities $S_xS_e = S_eS_x = S_x$ and $S_xS_{x^{-1}} = S_e$ hold. Take $x, y \in G$. Then $S_{xy} = S_{xy}S_e = S_{xy}S_{y^{-1}}S_y \subseteq S_{xy}S_{y^{-1}}S_y = S_{xy}S_y \subseteq S_{xy}$. Thus $S_xS_y = S_{xy}$.

The converse statement is trivial. \(\square\)

**Remark 2.3.** Suppose that $S$ is unital strongly $G$-graded. Then, for every $x \in G$, the relations $0 \neq 1_S \in S_e = S_xS_{x^{-1}}^{-1}$ hold (see e.g. [20, Prop. 1.1.1]). Therefore, $\text{Supp}(S) = G$.

The following notion was first introduced by Clark, Exel and Pardo in the context of Steinberg algebras [8, Def. 4.5]:

**Definition 2.4.** The ring $S$ is said to be symmetrically $G$-graded if for every $x \in G$, the equality $S_xS_{x^{-1}}S_x = S_x$ holds.

**Remark 2.5.** If $S$ is symmetrically $G$-graded, then $\text{Supp}(S)^{-1} = \text{Supp}(S)$.

Note that strongly $G$-graded rings are symmetrically $G$-graded. As the following example shows, a grading which is not strong may fail to be symmetrical:
Example 2.6. Let $R$ be a unital ring and consider the standard $\mathbb{Z}$-grading on the polynomial ring $R[x] = \bigoplus_{i \in \mathbb{Z}} S_i$ where $S_i := R x^i$ for $i \geq 0$, and $S_i := \{0\}$ for $i < 0$. Clearly, $\text{Supp}(S)^{-1} \neq \text{Supp}(S)$ and thus, by Remark 2.5, it follows that the grading is not symmetrical.

Next, we will consider another special type of grading. Passman appears to have been the first to give the following definition (see also [9, 34]):

Definition 2.7 ([41] p. 32]). The ring $S$ is said to be non-degenerately $G$-graded if for every $x \in G$ and every nonzero $s \in S_x$, we have $s S_x \neq \{0\}$ and $S_{x^{-1}} s \neq \{0\}$.

Clearly, every unital strongly $G$-graded ring is non-degenerately $G$-graded.

2.4. Epsilon-strongly graded rings. Now, we consider a generalization of unital strongly graded rings, introduced by Nystedt, Öinert and Pinedo [32, Def. 4, Prop. 7].

Definition 2.8. The ring $S$ is called epsilon-strongly $G$-graded if for every $x \in G$ there exists $\epsilon_x \in S_x S_{x^{-1}}$ such that for all $s \in S_x$ the equalities $\epsilon_x s = s = s \epsilon_{x^{-1}}$ hold.

Example 2.9. Let $R$ be a unital ring and consider the following $\mathbb{Z}$-grading on the ring $M_2(R)$ of $2 \times 2$-matrices with entries in $R$:

\[ (M_2(R))_0 := \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}, \quad (M_2(R))_1 := \begin{pmatrix} 0 & 0 \\ R & 0 \end{pmatrix}, \quad (M_2(R))_{-1} := \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}, \]

and $(M_2(R))_i$ zero if $|i| > 1$. Clearly, this grading is not strong, but epsilon-strong with

\[ \epsilon_1 = \begin{pmatrix} 1_R & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \epsilon_{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1_R \end{pmatrix}. \]

Suppose that $R$ is prime. Then $M_2(R)$ is also prime, but $(M_2(R))_0$ is not prime. This is an example of a prime epsilon-strongly $\mathbb{Z}$-graded ring whose principal component is not prime.

Moreover, unital partial crossed products (see [32]), Leavitt path algebras of finite graphs (see [31]), and certain Cuntz-Pimsner rings (see [24]) are classes of graded rings that are epsilon-strongly graded. A further generalization was introduced by Nystedt and Öinert:

Definition 2.10 ([31] Def. 10]). The ring $S$ is called nearly epsilon-strongly $G$-graded if for every $x \in G$ and every $s \in S_x$ there exist $\epsilon_x(s) \in S_x S_{x^{-1}}$ and $\epsilon_x(s)^\prime \in S_{x^{-1}} S_x$ such that the equalities $\epsilon_x(s) s = s = s \epsilon_x(s)^\prime$ hold.

Notably, every Leavitt path algebra with its natural $\mathbb{Z}$-grading is nearly epsilon-strongly $\mathbb{Z}$-graded whereas only Leavitt path algebras of finite graphs are epsilon-strongly $\mathbb{Z}$-graded (cf. [31] Thm. 28, Thm. 30])

Proposition 2.11 ([31] Prop. 11]). The ring $S$ is nearly epsilon-strongly $G$-graded if and only if $S$ is symmetrically $G$-graded and for every $x \in G$ the ring $S_x S_{x^{-1}}$ is $s$-unital.

Remark 2.12. The following implications hold for all $G$-graded rings:

unital strong $\implies$ epsilon-strong $\implies$ nearly epsilon-strong $\implies$ symmetrical

Proposition 2.13. Suppose that $S$ is nearly epsilon-strongly $G$-graded. Then, $s \in s S_e \cap S_e s$ for every $s \in S$. In particular, $S$ is $s$-unital and $S_e$ is an $s$-unital subring of $S$. 
Proof. Proposition 2.11 yields, in particular, that (i) \( S_e = S_eS_eS_e \) and (ii) \( S_eS_e = S_e^2 \) is an \( s \)-unital ring. But (i) gives that \( S_e = S_e^3 \subseteq S_e^2 \subseteq S_e \). Thus, \( S_e = S_e^2 \) is \( s \)-unital.

Let \( s = \sum_{y \in G} s_y \in S \) with \( s_y \in S_{y^{-1}} \). Fix \( x \in \text{Supp}(s) \). By Proposition 2.11, there are finitely many elements \( a_i \in S_xS_{x^{-1}} \subseteq S_e, \ b_j \in S_{x^{-1}}S_x \subseteq S_e \) and \( s_i, s_j \in S \) such that \( s_x = \sum_i a_is_i = \sum_j s_j b_j \). Now, since \( S_e \) is \( s \)-unital, there is some \( e_x \in S_e \) such that \( e_xa_i = a_i \) and \( b_je_x = b_j \) for all \( i, j \) (see Proposition 2.1). Then, \( e_xSx = s_x = s_xe_x \). Hence, we can find such an \( e_x \in S_e \) for every \( x \in \text{Supp}(s) \). Since \( \text{Supp}(s) \) is a finite set, it follows from Proposition 2.1 that there is some \( e_x \in S_e \) such that \( e_xe_x = e_x = e_xe_x \) for every \( x \in \text{Supp}(s) \). Then \( e_xSx = \sum e_x s_x = \sum e_x(e_x s_x) = \sum (e_x e_x) s_x = \sum e_x s_x = \sum s_x = s \), where the sum runs over \( \text{Supp}(s) \). Similarly, \( se_x = s \) \( \square \).

Not every symmetrically \( G \)-graded ring is nearly epsilon-strongly \( G \)-graded:

Example 2.14. Let \( R \) be an idempotent ring that is not \( s \)-unital (see e. g. [30, Expl. 2.5]). Consider the \( G \)-graded ring \( S \) defined by \( S_e := R \) and \( S_x := \{0\} \) if \( x \in G \setminus \{e\} \). Clearly, \( S \) is symmetrically \( G \)-graded, but by Proposition 2.13 \( S \) is not nearly epsilon-strongly \( G \)-graded.

Proposition 2.15 ([31 Prop. 3.4]). If \( S \) is nearly epsilon-strongly \( G \)-graded, then \( S \) is non-degenerately \( G \)-graded.

Lemma 2.16. If \( S \) is \( s \)-unital strongly \( G \)-graded, then the following assertions hold:

(a) \( S \) is nearly epsilon-strongly \( G \)-graded.
(b) \( s \in sS_e \cap S_es \) for every \( s \in S \).
(c) \( r \cdot \text{Ann}_S(S_x) = \{0\} \) for every \( x \in G \).

Proof. (a): Clearly, \( S \) is symmetrically \( G \)-graded. By [23, Lem. 6.8], \( S_xS_{x^{-1}} = S_e \) is \( s \)-unital for every \( x \in G \). The desired conclusion now follows from Proposition 2.11.

(b): This follows from (a) and Proposition 2.13.

(c): Take \( x \in G \) and \( s \in r \cdot \text{Ann}_S(S_x) \). Then \( \{0\} = S_{x^{-1}}S_x = S_es \). Thus, \( s = 0 \) by (b). \( \square \)

Remark 2.17. If \( S \) is strongly \( G \)-graded, then \( S \) is \( s \)-unital if and only if \( S_e \) is \( s \)-unital.

In the rest of this article, we will freely use the fact that nearly epsilon-strongly graded rings are symmetrically graded, non-degenerately graded, and \( s \)-unital without further comment. For additional characterizations of (nearly) epsilon-strongly graded rings, we refer to [27].

2.5. Induced gradings. Now, we recall two important functorial constructions. For more details, we refer the reader to [22]. The first construction assigns a subring of \( S \) with an inherited grading. Let \( H \) be a subgroup of \( G \) and put \( S_H := \bigoplus_{x \in H} S_x \). Note that \( S_H \) is an \( H \)-graded ring that is also a subring of \( S \). Consider the map \( \pi_H : S \to S_H \) defined by

\[
\pi_H \left( \sum_{x \in G} s_x \right) = \sum_{x \in H} s_x.
\]

The following result is well-known (see e. g. [33, Lem. 2.4]):

Lemma 2.18. The map \( \pi_H : S \to S_H \) is an \( S_H \)-bimodule homomorphism.

We can “map down” nonzero ideals when the ring is non-degenerately \( G \)-graded:

Lemma 2.19. Suppose that \( H \) is a subgroup of \( G \). If \( A \) is a left (resp. right) ideal of \( S \), then \( \pi_H(A) \) is a left (resp. right) ideal of \( S_H \). If, in addition, \( S \) is non-degenerately \( G \)-graded and \( A \) is nonzero, then \( \pi_H(A) \) is nonzero.
Proof. The first statement immediately follows from Lemma 2.18. For the second statement suppose that $S$ is non-degenerately $G$-graded and that $A$ is a nonzero left $S$-ideal. Pick a nonzero $a \in A$ and $x \in \text{Supp}(a)$. Then, since $S$ is non-degenerately $G$-graded, $\{0\} \neq S_{x^{-1}}a = \pi_{\{e\}}(S_{x^{-1}}a) \subseteq \pi_{\{e\}}(A) \subseteq \pi_H(A)$. The case when $A$ is a right ideal is proved similarly.

We now describe the second construction: Given a normal subgroup $N$ of $G$, we define the induced $G/N$-grading on $S$ in the following way. For every $C \in G/N$, put $S_C := \text{⨁}_{x \in C} S_x$. This yields a $G/N$-grading on $S$. The following non-trivial result, proved by Lännström, will be essential later on in this article:

**Proposition 2.20** ([22, Prop. 5.8]). Suppose that $S$ is nearly epsilon-strongly $G$-graded and that $N$ is a normal subgroup of $G$. Then the induced $G/N$-grading on $S$ is nearly epsilon-strong.

We will also need the following result:

**Proposition 2.21.** Suppose that $S$ is non-degenerately $G$-graded and that $N$ is a normal subgroup of $G$. Then the induced $G/N$-grading on $S$ is non-degenerate.

**Proof.** Take $x \in G$ and a nonzero $a \in S_{xN}$. Write $a = a_{x_1} + a_{x_2} + \ldots + a_{x_k}$ where $n_1, \ldots, n_k \in N$ are all distinct and $a_{xn_i} \neq 0$ for every $i$. By non-degeneracy of the $G$-grading there is some $c_{n_1^{-1}x^{-1}} \in S_{N^{x^{-1}}} = S_{x^{-1}N}$ such that $c_{n_1^{-1}x^{-1}}a_{xn_1} \neq 0$. Note that

$$\pi_{\{e\}}(c_{n_1^{-1}x^{-1}}a) = c_{n_1^{-1}x^{-1}}a_{xn_1} \neq 0.$$ 

Hence, $c_{n_1^{-1}x^{-1}}a \neq 0$. This shows that $S_{x^{-1}N}a \neq \{0\}$. Similarly, $aS_{x^{-1}N} \neq \{0\}$. Thus, the induced $G/N$-grading on $S$ is non-degenerate.

2.6. **Cancellatively graded rings.** We now briefly discuss Passman’s notion of cancellatively graded rings. We will, however, not work with this class of rings outside of this section.

In [30] Passman extended his results from [20] to the class of cancellatively group graded rings which generalizes the class of unital strongly group graded rings. To avoid any confusion, we wish to point out that Passman’s notion of $H$-stability is used interchangeably with our notion of $H$-invariance. Recall from the introduction that a unital $G$-graded ring $S$ is called **cancellative** if for all $x, y \in G$ and all homogeneous subsets $U, V \subseteq S$, the implication $US_xV = \{0\} \Rightarrow US_yV = \{0\}$ holds. Clearly, all strongly graded rings are cancellative. However, e.g. canonical $Z$-gradings on Leavitt path algebras (see Section 14) need not be cancellative.

**Remark 2.22.** Lännström has observed that if $S$ is epsilon-strongly $G$-graded, then $S$ must be unital (see [22, Prop. 3.8]). Moreover, $\epsilon_x$ is central in $S_e$ for every $x \in G$ (see [32]).

**Lemma 2.23.** The following assertions hold for each $x \in G$:

(a) If $S$ is symmetrically $G$-graded, then $r.\text{Ann}_S(S_x) = r.\text{Ann}_S(S_{x^{-1}}S_x)$.

(b) If $S$ is epsilon-strongly $G$-graded, then $r.\text{Ann}_S(S_x) = r.\text{Ann}_S(S_{x^{-1}}S_x) = r.\text{Ann}_S(\epsilon_x^{-1})$.

**Proof.** (a): Suppose that $S$ is symmetrically $G$-graded. If $s \in r.\text{Ann}_S(S_x)$, then $S_{x^{-1}}S_xs = \{0\}$, which implies that $r.\text{Ann}_S(S_x) \subseteq r.\text{Ann}_S(S_{x^{-1}}S_x)$. If, conversely, $s \in r.\text{Ann}_S(S_{x^{-1}}S_x)$, then $S_{x}s = S_{x}S_{x^{-1}}S_{x}s = \{0\}$. Thus, $r.\text{Ann}_S(S_{x^{-1}}S_x) \subseteq r.\text{Ann}_S(S_x)$.

(b): Suppose that $S$ is epsilon-strongly $G$-graded. Then $S_{x^{-1}}S_{\epsilon} = \epsilon_{x^{-1}}S_{\epsilon} = S_{\epsilon_{x^{-1}}}$, which entails that $r.\text{Ann}_S(S_{x^{-1}}S_x) = r.\text{Ann}_S(S_{\epsilon_{x^{-1}}}) = r.\text{Ann}_S(\epsilon_{x^{-1}})$, where the last equality follows from the fact that $1_S = 1_{S_e}$.
Proposition 2.24. Suppose that $S$ is epsilon-strongly $G$-graded. Then the following assertions are equivalent:

(a) the grading on $S$ is strong;
(b) for every $x \in G$, the equality $r.\text{Ann}_S(S_x) = \{0\}$ holds;
(c) the grading on $S$ is cancellative.

Proof. (a)$\Rightarrow$(b): Take $x \in G$. For $s \in S$, we note that

$$S_x s = \{0\} \implies S_x^{-1} S_x s = \{0\} \implies S_c s = \{0\} \implies 1_S \cdot s = 0.$$ 

Hence, $r.\text{Ann}_S(S_x) = \{0\}$.

(b)$\Rightarrow$(c): By [39, Lem. 1.2], $S$ is cancellative if and only if, for every $x \in G$, (i) $S_x S_{x^{-1}}$ is a so-called middle cancellable ideal of $S_e$ and (ii) $r.\text{Ann}_S(S_x) = \{0\}$. In the special case of epsilon-strongly graded rings, (ii) actually implies (i). Let $x \in G$ and recall that $S_x S_{x^{-1}}$ being middle cancellable means that $US_x S_{x^{-1}} V = \{0\}$ implies that $UV = \{0\}$ for all subsets $U, V \subseteq S_c$. Moreover, note that $S_x S_{x^{-1}} = \epsilon_x S_e$ for some central element $\epsilon_x \in S_e$ and

$$US_x S_{x^{-1}} V = \{0\} \iff U \epsilon_x S_c V = \{0\} \iff \epsilon_x U S_c V = \{0\} \iff \epsilon_x U V = \{0\}.$$ 

Now, note that, using Lemma 2.23, we get

$$\{0\} = r.\text{Ann}_S(S_{x^{-1}}) = r.\text{Ann}_S(S_x S_{x^{-1}}) = r.\text{Ann}_S(\epsilon_x S_e) \supseteq r.\text{Ann}_S(\epsilon_x).$$ 

Hence, $UV = \{0\}$ whenever $US_x S_{x^{-1}} V = \{0\}$. Thus, $S_x S_{x^{-1}}$ is middle cancellable for every $x \in G$. In other words, (ii) implies (i).

(c)$\Rightarrow$(a): Suppose that the grading on $S$ is not strong. There is some $x \in G$ such that $\epsilon_x \neq 1_S$. Put $U = V := \{1_S - \epsilon_x\}$ and note that $1 - \epsilon_x$ is an idempotent. Clearly, $UV = \{1_S - \epsilon_x\} \neq \{0\}$, since $\epsilon_x \neq 1_S$. However, we also have that $US_x S_{x^{-1}} V = U \epsilon_x S_c V = \{0\}$ which shows that $S_x S_{x^{-1}}$ is not a middle cancellable ideal of $S_c$. By [39, Lem. 1.2] (see also the above proof of (b)$\Rightarrow$(c)), the grading is not cancellative. \qed

Proposition 2.25. If $S$ is unital and cancellatively $G$-graded, then $\text{Supp}(S) = G$.

Proof. Take $x \in G$. Since $S$ is unital, we get that $S_c S_{x^{-1}} S_c = S_c S_c S_c = S_c \neq \{0\}$. Thus, by cancellativity, we get $S_e S_x S_{x^{-1}} S_c = \neq \{0\}$. Hence, $S_x \neq \{0\}$. \qed

Recall that a unital strongly $G$-graded ring $S$ also satisfies $\text{Supp}(S) = G$ (see Remark 2.3). However, $\text{Supp}(S) = G$ need not hold, in general, for nearly epsilon-strongly graded rings.

Remark 2.26. Proposition 2.24 demonstrates that epsilon-strongly graded rings which can be reached by Passman’s “cancellative results” [39] are, in fact, unital strongly graded. Thus, that case has already been treated by Passman in [40].

3. Invariant ideals

Recall that $S$ is a $G$-graded ring. If $S$ is strongly $G$-graded, then there is an action of $G$ on the lattice of ideals of $S_N$ for any normal subgroup $N$ of $G$ (see [40 Sec. 5.2]). The purpose of this section is to investigate this construction for more general classes of $G$-graded rings.

Definition 3.1. If $I$ is a subset of $S$ and $x \in G$, then we define $I^x := S_{x^{-1}} IS_x$.

Lemma 3.2. If $x \in G$ and $I$ is an ideal of $S_c$, then $I^x$ is an ideal of $S_e$.

Proof. Clearly, $I^x$ is an additive subgroup of $S_e$. Since $S_{x^{-1}}$ and $S_x$ are $S_c$-bimodules, it follows that $S_c I^x = S_c S_{x^{-1}} IS_x \subseteq S_{x^{-1}} IS_x = I^x$. Similarly, $I^x S_c \subseteq I^x$. \qed
Recall that if \( H, K \) are subsets of \( G \), then \( K \) is said to be normalized by \( H \) if \( Kx = xK \) for every \( x \in H \).

**Definition 3.3** (cf. [39, p. 406]). Suppose that \( H \) is a subgroup of \( G \) and that \( I \) is a subset of \( S \). Then \( I \) is called \( H \)-invariant if \( I^x \subseteq I \) for every \( x \in H \). Furthermore, if \( K \) is a subset of \( G \) which is normalized by \( H \), then we say that \( I \) is \( H/K \)-invariant if \( S_{x^{-1}}KIS_x \subseteq I \) for every \( x \in H \).

In the special case of \( s \)-unital (and in particular unital) strongly \( G \)-graded rings, our definition coincides with Passman’s notion of invariance used in [40].

**Lemma 3.4.** Suppose that \( H \) is a subgroup of \( G \) and that \( S \) is \( s \)-unital strongly \( G \)-graded. Then a subset \( I \) of \( S \) (cf. [39, p. 406])

**Definition 3.3** Suppose that \( H \) is a subgroup of \( G \) and that \( I \) is a subset of \( S \). Then \( I \) is called \( H \)-invariant if \( I^x \subseteq I \) for every \( x \in H \). Furthermore, if \( K \) is a subset of \( G \) which is normalized by \( H \), then we say that \( I \) is \( H/K \)-invariant if \( S_{x^{-1}}KIS_x \subseteq I \) for every \( x \in H \).

In the special case of \( s \)-unital (and in particular unital) strongly \( G \)-graded rings, our definition coincides with Passman’s notion of invariance used in [40].

**Lemma 3.4.** Suppose that \( H \) is a subgroup of \( G \) and that \( S \) is \( s \)-unital strongly \( G \)-graded. Then a subset \( I \) of \( S \) is \( H \)-invariant if and only if \( I^x = I \) for every \( x \in H \).

**Proof.** Suppose that \( I \) is \( H \)-invariant. Take \( x \in H \). By Lemma 2.16(b) we have

\[
I \subseteq S_eJS_e = (S_{x^{-1}}S_x)I(S_{x^{-1}}S_x) = S_{x^{-1}}(S_xIS_x)S_x \subseteq S_{x^{-1}}IS_x = I^x \subseteq I.
\]

This shows that \( I^x = I \). The converse statement is trivial. \( \square \)

**Example 3.5.** Let us again look at Example 2.9. Let \( J, J' \) be nonzero \( R \)-ideals and consider the following ideals of \((M_2(R))_e\):

\[
I = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \quad \text{and} \quad I' = \begin{pmatrix} J & 0 \\ 0 & J' \end{pmatrix}.
\]

It is easily checked that \( I \) is \( \mathbb{Z} \)-invariant but \( I^x = I \) does not hold for every \( x \in \mathbb{Z} \). Moreover, if \( J \nsubseteq J' \), then a quick verification shows that \( I' \) is not \( \mathbb{Z} \)-invariant. However, \( I' \) is invariant with respect to any proper non-trivial subgroup of \( \mathbb{Z} \).

More examples of invariant ideals may be found in Example 13.11 and Example 14.3. The following result is essential and will often be used implicitly in the rest of this article:

**Lemma 3.6** (cf. [40, Lem. 5.7]). Suppose that \( I \) and \( J \) are subsets of \( S \). Then the following assertions hold for all \( x, y \in G \):

(a) \( (I^y)^x \subseteq I^{xy} \)

(b) \( I^xJ^y \subseteq (IJ)^x \) if \( I \) or \( J \) is an ideal of \( S_e \).

(c) If \( I \subseteq J \), then \( I^x \subseteq J^x \).

**Proof.**

(a): \( (I^y)^x = S_{y^{-1}}S_xS_y = (S_{y^{-1}}S_x)I(S_xS_y) \subseteq S_{xy^{-1}}IS_{xy} = I^{xy} \).

(b): \( I^xJ^y = S_{x^{-1}}S_eJS_x \subseteq S_{x^{-1}}IS_eJS_x \subseteq S_{x^{-1}}IJS_x = (IJ)^x \).

(c): \( I^x = S_{x^{-1}}IS_x \subseteq S_{x^{-1}}JS_x = J^x \). \( \square \)

For unital strongly \( G \)-graded rings, the inclusions in (a) and (b) of Lemma 3.6 are, in fact, equalities (see [40, Lem. 5.7]). However, Example 3.15 below shows that the inclusion in Lemma 3.6(a) can be strict for some nearly epsilon-strongly graded rings. We now prove that the inclusion in Lemma 3.6(b) is actually an equality for nearly epsilon-strongly graded rings.

**Definition 3.7.** If \( I \) is a subset of \( S \), then we say that \( I \) is \( \epsilon \)-invariant if for every \( x \in G \), the equality \( S_xS_{x^{-1}}I = IS_xS_{x^{-1}} \) holds.

**Remark 3.8.** If \( S \) is epsilon-strongly \( G \)-graded, \( H \) is a subgroup of \( G \) and \( I \) is an ideal of \( S_H \), then the statement

\[
(1) \quad S_xS_{x^{-1}}I = IS_xS_{x^{-1}}, \quad \forall x \in G
\]
is equivalent to the statement

\[(2) \quad \epsilon_x I = I \epsilon_x, \quad \forall x \in G.\]

Note that if \(H = \{e\}\), then \(G\) (and hence also \(H\)) is true since the elements \(\epsilon_x\), for \(x \in G\), are central idempotents in \(S_e\) (see Remark 2.22). This justifies our usage of the term "\(e\)-invariant".

**Lemma 3.9.** If \(S\) is nearly epsilon-strongly \(G\)-graded, then every ideal of \(S_e\) is \(e\)-invariant.

**Proof.** Take \(x \in G\) and let \(I\) be an ideal of \(S_e\). We prove that \(S_x S_{x^{-1}} I \subseteq I S_x S_{x^{-1}}\). The reversed inclusion can be shown in an analogous fashion and is therefore left to the reader. Take \(s_x \in S_x\), \(s_{x^{-1}} \in S_{x^{-1}}\) and \(a \in I\). Since \(s_{x^{-1}} a \in S_{x^{-1}}\), there is \(\epsilon_{x^{-1}} (s_{x^{-1}} a) \in S_x S_{x^{-1}}\) such that \(s_{x^{-1}} a = s_{x^{-1}} a \cdot \epsilon_{x^{-1}} (s_{x^{-1}} a)\). Using that \(s_{x} s_{x^{-1}} a \subseteq I\), it follows that \(s_{x} s_{x^{-1}} a = (s_{x} s_{x^{-1}} a) \cdot \epsilon_{x^{-1}} (s_{x^{-1}} a) \in I S_x S_{x^{-1}}\).

**Proposition 3.10.** Suppose that \(S\) is symmetrically \(G\)-graded, \(N\) is a normal subgroup of \(G\), and that \(I, J\) are ideals of \(S_N\). If \(I\) or \(J\) is \(e\)-invariant, then \((IJ)^x = I^x J^x\) for every \(x \in G\).

**Proof.** Suppose that \(I\) is \(e\)-invariant. Then, since \(S\) is symmetrically \(G\)-graded, we get \(I^x J^x = S_{x^{-1}} I S_x S_{x^{-1}} J S_x = S_{x^{-1}} S_x S_{x^{-1}} I J S_x = S_{x^{-1}} I J S_x = (IJ)^x\)

for every \(x \in G\). The case when \(J\) is \(e\)-invariant can be treated analogously.

Combining Lemma 3.9 and Proposition 3.10 we obtain the following result:

**Corollary 3.11.** If \(S\) is nearly epsilon-strongly \(G\)-graded and \(I, J\) are ideals of \(S_e\), then \((IJ)^x = I^x J^x\) for every \(x \in G\).

**Proposition 3.12.** If \(S\) is nearly epsilon-strongly \(G\)-graded and \(I\) is an ideal of \(S_e\), then the following assertions hold:

(a) \(IS_y = S_y I^y\) and \(S_{y^{-1}} I = I^y S_{y^{-1}}\) for every \(y \in G\).

(b) If \(H\) is a subgroup of \(G\) and \(I\) is \(H\)-invariant, then \(IS_y = S_y I\) for every \(y \in H\).

**Proof.** (a): Take \(y \in G\). Since \(S\) is symmetrically \(G\)-graded and \(I\) is \(e\)-invariant by Lemma 3.9, we have \(IS_y = I (S_y S_{y^{-1}} S_y) = I (S_y S_{y^{-1}} S_y) = S_y (S_{y^{-1}} I S_y) = S_y I^y\). Similarly, \(S_{y^{-1}} I = (S_{y^{-1}} S_y S_y) = S_{y^{-1}} I (S_{y^{-1}} S_y) = S_{y^{-1}} I (S_{y^{-1}} S_y) = S_{y^{-1}} I^y S_{y^{-1}}\).

(b): Take \(y \in H\). By (a), we get \(IS_y = S_y I^y \subseteq S_y I = I^y S_y \subseteq IS_y\). Thus, \(S_y I = IS_y\).

In the following lemma we use the induced quotient grading described in Section 2.5.

**Lemma 3.13.** Suppose that \(N\) is a normal subgroup of \(G\). If \(I\) is a \(G/N\)-invariant subset of \(S_N\), then \(I\) is \(G\)-invariant.

**Proof.** Suppose that \(I\) is \(G/N\)-invariant. Take \(x \in G\). Then \(S_{x^{-1}} IS_x \subseteq S_{x^{-1}} N IS_{xN} \subseteq I\).

**Remark 3.14.** In Passman’s original setting of unital strongly \(G\)-graded rings an important property that is repeatedly used is that, for \(y \in G\), \(S_y I = IS_y\) if and only if \(I^y = I\) for any ideal \(I\) of \(S_H\), where \(H\) is a subgroup of \(G\). In our generalized setting, we will have to make do with the result in Proposition 3.12 which only holds for ideals of the principal component.

The identity \((I^y)^y = I^{y^2}\), for all \(x, y \in G\), does not hold in general when working with nearly epsilon-strongly \(G\)-graded rings. Before giving an example for which this identity fails, note that if \(x \not\in \text{Supp}(S)\), then \(I^x = \{0\}\) for every ideal \(I\) of \(S_e\).
Example 3.15. Let $R$ be an s-unital ring and let $G$ be a non-trivial group. Consider the nearly epsilon-strong $G$-graded ring $S$ defined by $S_e := R$ and $S_x := \{0\}$ for $x \in G \setminus \{e\}$. Now, consider the nonzero ideal $R$ of $R$ and let $x \in G \setminus \{e\}$. Then $\{0\} \neq R = R^{xx^{-1}} \neq (R^2)^{x^{-1}} = \{0\}$, because $x \notin \text{Supp}(S)$.

Lemma 3.16. Suppose that $S$ is nearly epsilon-strongly $G$-graded, $K$ is a subgroup of $G$ and that $I$ and $J$ are ideals of $S_e$. Then the following assertions hold:

(a) If $I, J$ are $K$-invariant, then $IJ$ is $K$-invariant.
(b) If $I$ is $K$-invariant, then $r \cdot \text{Ann}_{S_e}(I)$ is $K$-invariant.

Proof. (a): This follows from Corollary 3.11. (b): Take $x \in G$. From Proposition 3.12 it follows that

\[ I \cdot S_{x^{-1}}(r \cdot \text{Ann}_{S_e}(I))S_x \subseteq S_{x^{-1}}I(r \cdot \text{Ann}_{S_e}(I))S_x = S_{x^{-1}}(I \cdot r \cdot \text{Ann}_{S_e}(I))S_x = \{0\}. \]

Lemma 3.17. If $x \in G$ and $F$ is a family of subsets of $S$, then $\left( \sum_{I \in F} I \right)^x = \sum_{I \in F} I^x$.

Proof. $\left( \sum_{I \in F} I \right)^x = S_{x^{-1}}(\sum_{I \in F} I)S_x = \sum_{I \in F} S_{x^{-1}}IS_x = \sum_{I \in F} I^x$.

Definition 3.18. For $H \subseteq G$ and $M \subseteq S$ we define $M^H := \sum_{h \in H} S_h^{-1}MS_h$.

Lemma 3.19. With the above notation the following assertions hold:

(a) If $H$ is a subgroup of $G$ and $M \subseteq S$, then $M^H$ is an $H$-invariant subset of $S$.
(b) If $S_e$ is $s$-unital and $I$ is an ideal of $S_e$, then $I^G$ is the smallest $G$-invariant ideal of $S_e$ containing $I$.

Proof. (a): Take $x \in H$. Combining Lemma 3.6(a) and Lemma 3.17 we deduce that $(M^H)^x = \left( \sum_{y \in H} M^y \right)^x = \sum_{y \in H} (M^y)^x \subseteq \sum_{y \in H} M^{yx} = M^H$.

(b): From (a), it follows that $I^G$ is $G$-invariant. Clearly, $I^G$ is an ideal of $S_e$ and $I = I^e \subseteq I^G$ by $s$-unitality of $S_e$. Suppose now that $J$ is a $G$-invariant $S_e$-ideal such that $I \subseteq J$. Then, by Lemma 3.6(c), $I^x \subseteq J^x \subseteq J$ for every $x \in G$ and hence we get $I^G = \sum_{x \in G} I^x \subseteq J$.

Lemma 3.20. The following assertions hold:

(a) Suppose that $S$ is non-degenerately $G$-graded. Let $I$ be a subset of $S_e$ and let $x \in \text{Supp}(S)$ be such that $I(S_xS_{x^{-1}}) = I$ or $(S_xS_{x^{-1}})I = I$. If $I \neq \{0\}$, then $I^x \neq \{0\}$.
(b) Suppose that $S$ is symmetrically $G$-graded. Then for every $S_e$-ideal $I$ and every $x \in G$, we have $I^x(S_{x^{-1}}S_x) = I^x$.

Proof. (a): Suppose that $I^x = \{0\}$. Since $S$ is non-degenerately $G$-graded, we have $IS_x = \{0\}$ or $S_{x^{-1}}I = \{0\}$. Hence, $\{0\} = IS_xS_{x^{-1}} = I$ or $\{0\} = S_xS_{x^{-1}}I = I$.

(b): For every $x \in G$, we have $I^x(S_{x^{-1}}S_x) = S_{x^{-1}}IS_x(S_{x^{-1}}S_x) = S_{x^{-1}}IS_x = I^x$.

Later on, we need to consider ideals $I$ satisfying $I^xI = \{0\}$ for every $x \in G \setminus H$ for some subgroup $H$ of $G$. The following result will allow us to replace $I$ with $I^H$.

Proposition 3.21 (cf. [40], Lem. 5.5]). Suppose that $S$ is nearly epsilon-strongly $G$-graded and that $H$ is a subgroup of $G$. Let $I$ be an ideal of $S_e$ such that $I^2I = \{0\}$ for every $x \in G \setminus H$. Then $(I^H)^x(I^H) = \{0\}$ for every $x \in G \setminus H$.

Proof. Take $x \in G$ such that $(I^H)^xI^H \neq \{0\}$. There exist $h_1, h_2 \in H$ such that $\{0\} \neq (I^{h_1x})^2I^{h_2} \subseteq I^{h_1x}I^{h_2}$, by Lemma 3.6(a). By Lemma 3.20(b), we have $I^{h_1x}((I^{h_2}(S_{h_2^{-1}}S_{h_2})) = I^{h_1x}I^{h_2}$. Hence, Lemma 3.20(a) applies to the $S_e$-ideal $I^{h_1x}I^{h_2}$. Thus, $\{0\} \neq (I^{h_1x}I^{h_2})h_2^{-1} \subseteq I^{h_1x}h_2^{-1}I$. By assumption, $h_1xh_2^{-1} \in H$ and hence $x \in H$. □
Lemma 3.22. Suppose that $S$ is nearly epsilon-strongly $G$-graded and that $S_e$ is $G$-semiprime. Furthermore, let $H$ be a subgroup of $G$ and let $I$ be an $H$-invariant ideal of $S_e$ such that $I^x I = \{0\}$ for every $x \in G \setminus H$. Then the following assertions hold:

(a) The ideal $I$ does not contain any nonzero nilpotent $H$-invariant ideal.
(b) Let $W$ be a subgroup of $H$ of finite index. Then $I$ does not contain any nonzero nilpotent $W$-invariant ideal.

Proof. (a): Seeking a contradiction, suppose that $J$ is a nonzero $H$-invariant ideal of $S_e$ such that $J^2 = \{0\}$ and $J \subseteq I$. First we show that $J^x J = \{0\}$ for every $x \in G$. Indeed, for $x \in H$ we have $J^x J \subseteq J^2 = \{0\}$ while for $x \in G \setminus H$ we have $J^x J \subseteq I^x I = \{0\}$ by Lemma 3.6(c). Next, note that $J^G = \sum_{x \in G} J^x$ is a nonzero $G$-invariant ideal of $S_e$ by Lemma 3.19. We claim that $J^G J^G = \{0\}$. If we assume that the claim holds, then we get the desired contradiction, since $S_e$ is assumed to be $G$-semiprime. Now, we prove the claim. Seeking a contradiction, suppose that $J^G J^G \neq \{0\}$. Then $J^G J^G = (\sum_{x \in G} J^x) (\sum_{y \in G} J^y) = \sum_{x,y \in G} J^x J^y = \{0\}$. Hence there are $x,y \in G$ such that $J^x J^y \neq \{0\}$. By Lemma 3.20(b), we have $J^x J^y (S_{y^{-1} y}) = J^x J^y$, and therefore Lemma 3.20(a) implies that $\{0\} \neq (J^x J^y)^{y^{-1}}$. Moreover, by Corollary 3.11, we have $\{0\} \neq (J^x J^y)^{y^{-1}} = (J^x)^{y^{-1}} (J^y)^{y^{-1}} \subseteq J^2 = \{0\}$, which is a contradiction.

(b): Seeking a contradiction, suppose that $J \subseteq I$ is a nonzero $W$-invariant ideal of $S_e$ such that $J^2 = \{0\}$. Let $Wx_1, Wx_2, \ldots, Wx_n$ be a set of representatives of the right cosets of $W$ in $H$ and, for every $i \in \{1, \ldots, n\}$, let $J^W x_i := \sum_{y \in W} J^y x_i$. We wish to prove that $J' := J^W x_1 + J^W x_2 + \ldots + J^W x_n$ is a nonzero $H$-invariant nilpotent ideal contained in $I$.

To begin with, note that for all $y_1, y_2 \in W$ and $i \in \{1, \ldots, n\}$ we have

$$J^{y_1 x_i} J^{y_2 x_i} = S_{(y_1 x_i)^{-1}} J S_{y_1 x_i} S_{(y_2 x_i)^{-1}} J S_{y_2 x_i} \subseteq S_{(y_1 x_i)^{-1}} J S_{y_1 y_2^{-1}} J S_{y_2 x_i}.$$ 

Using that $y_1 y_2^{-1} \in W$ and that $J$ is $W$-invariant, Proposition 3.12(b) yields $JS_{y_1 y_2^{-1}} J = S_{y_1 y_2^{-1}} JJ = \{0\}$. Hence, $J^{y_1 x_i} J^{y_2 x_i} = \{0\}$, and therefore it follows that

$$(J^W x_i)^2 = \left( \sum_{y_1 \in W} J^{y_1 x_i} \right) \left( \sum_{y_2 \in W} J^{y_2 x_i} \right) = \sum_{y_1, y_2 \in W} J^{y_1 x_i} J^{y_2 x_i} = \{0\}. $$

In other words, $J^W x_i$ is a nilpotent ideal for every $i \in \{1, \ldots, n\}$. Since $J'$ is a finite sum of nilpotent ideals, we conclude that $J'$ is also a nilpotent ideal.

Next, we prove that $J'$ is $H$-invariant. For this we repeatedly use Lemma 3.6. Note that for all $i \in \{1, \ldots, n\}$ and $y \in H$, we have $(J^W x_i)^y \subseteq J^W x_i y = J^W x_j$ for some $j \in \{1, \ldots, n\}$ with $W x_i y = W x_j$. Now, by Lemma 3.17 $(J')^y = (J^W x_1)^y + \ldots + (J^W x_n)^y \subseteq J'$ and hence $J'$ is $H$-invariant. Finally, we show that $J' \subseteq I$. Note that $J \subseteq I$ implies $J^W x_i \subseteq I^W x_i$ for every $y \in W$. In addition, we have $J^W x_i \subseteq I$, since $I$ is $H$-invariant. It follows that $J^W x_i \subseteq I$ for every $i \in \{1, \ldots, n\}$, which gives the inclusion $J' \subseteq I$.

Summarizing, we have established that $J'$ is indeed an $H$-invariant nilpotent ideal contained in $I$, but by virtue of (a) we must have $J' = \{0\}$. However, writing $W x_j$ for the right coset containing $e$, we get $\{0\} \neq J = J^e \subseteq J^W x_j \subseteq J'$. This contradiction proves the assertion. □
4. Graded prime ideals

Recall that $S$ is a $G$-graded ring. In this section, we obtain a correspondence between graded prime ideals of $S$ and $G$-prime ideals of $S_e$, in the case when $S$ is nearly epsilon-strongly $G$-graded. Using that correspondence, we establish a primeness result in the case when $G$ is ordered (see Corollary 4.14). That result will be generalized in Section 10 using more elaborate methods. We wish to emphasize that the rest of this article does not depend on the results of this section.

**Definition 4.1.** An ideal $I$ of $S$ is called graded if $I = \bigoplus_{x \in G} (I \cap S_x)$.

**Example 4.2.** This example illustrates that a graded ring may have infinitely many ideals but only trivial graded ideals. Indeed, consider the complex Laurent polynomial ring equipped with the standard $\mathbb{Z}$-grading, that is, $\mathbb{C}[t, t^{-1}] = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}t^i$. This is clearly a strong $\mathbb{Z}$-grading and hence also nearly epsilon-strong. Every point of the circle gives rise to a maximal ideal of $\mathbb{C}[t, t^{-1}]$. On the other hand, the only graded ideals are $\{0\}$ and $\mathbb{C}[t, t^{-1}]$.

Let $I$ be an ideal of $S$. Then $I_e := I \cap S_e$ is an $S_e$-ideal. Conversely, if $J$ is an $S_e$-ideal, then $SJS$ is a graded ideal of $S$. For strongly graded rings we have the following bijection:

**Proposition 4.3** (cf. [29, Prop. 2.11.7]). If $S$ is unital strongly $G$-graded, then the map $I \mapsto I_e$ is a bijection between the set of graded ideals of $S$ and the set of $G$-invariant ideals of $S_e$.

We now generalize Proposition 4.3 to nearly epsilon-strongly graded rings (see Theorem 4.7). To this end, we need three lemmas.

**Lemma 4.4** (cf. [10, Expl. 2.7.3]). If $S_e$ is $s$-unital and $I$ is an ideal of $S_e$, then $I$ is $G$-invariant if and only if $(SIS)_e = I$.

**Proof.** Suppose that $I^x = S_{x-1}IS_x \subseteq I$ for every $x \in G$. Then $(SIS)_e = SIS \cap S_e \subseteq I$. The reversed inclusion follows since $S_e$ is $s$-unital. Conversely, suppose that $(SIS)_e = I$. Then $S_{x-1}IS_x \subseteq SIS \cap S_e = (SIS)_e = I$ for every $x \in G$. Thus, $I$ is $G$-invariant. \qed

**Lemma 4.5.** If $I$ is a graded ideal of $S$, then $I_e$ is a $G$-invariant ideal of $S_e$.

**Proof.** Take $x \in G$. Then $S_{x-1}I_eS_x \subseteq (S_{x-1}IS_x) \cap (S_{x-1}S_eS_x) \subseteq I \cap S_e = I_e$. \qed

**Lemma 4.6** (cf. [10, Prop. 1.1.34]). Suppose that $S$ is nearly epsilon-strongly $G$-graded. If $I$ is a graded ideal of $S$, then $SI_eS = SI_e = I_eS = I$.

**Proof.** Using that $S$ is $s$-unital, we get $I_e \subseteq I_eS$ and $I \subseteq SI_e$. Hence, $SI_e \subseteq SI_eS \subseteq I$ and similarly $I_eS \subseteq SI_eS \subseteq I$. Next, we prove that $I \subseteq SI_e$. Since $I$ is graded, it is enough to show that $a_x \in SI_e$ for every homogeneous $a_x \in I \cap S_x$. Indeed, since $S$ is nearly epsilon-strongly $G$-graded, we have $a_x = e_x(a_x) \cdot a_x$ for some $e_x(a_x) \in S_xS_{x-1}$. Write $e_x(a_x) = \sum_i c_i b_i$ for finitely many $c_i \in S_x$ and $b_i \in S_{x-1}$. Then $a_x = \sum_i c_i b_i a_x$. Note that for any $i$ we have $b_i a_x \in S_{x-1}S_x \subseteq S_x$ and $b_i a_x \in I$ thus yielding $a_x \in I \cap S_e = I_e$. Hence, $a_x = \sum_i c_i b_i a_x \in S_eI_e \subseteq SI_e$. By an analogous argument the inclusion $I \subseteq I_eS$ follows. We conclude that $I = SI_e = I_eS$. Consequently, $I = SI_e \subseteq SI_eS \subseteq I$. \qed

**Theorem 4.7.** Suppose that $S$ is nearly epsilon-strongly $G$-graded. The map $I \mapsto I_e$ is a bijection between the sets $\{\text{graded ideals of } S\}$ and $\{\text{$G$-invariant ideals of } S_e\}$. The inverse map is given by $J \mapsto SJS$. 
We proceed by induction to show that Theorem 4.11.

Proof. Let $I$ be a graded ideal. By Lemma 4.5, $I_e$ is a $G$-invariant ideal of $S_e$. In other words, the map $I \mapsto I_e$ is well-defined. Furthermore, by Lemma 4.6 we have $SI_e S = I$ establishing that $I \mapsto I_e$ is injective. Next, suppose that $J$ is a $G$-invariant ideal of $S_e$. By Lemma 4.4 $(SJS)_e = J$ proving that $I \mapsto I_e$ is surjective.

Later on we will apply Theorem 4.7 to Leavitt path algebras (see Section 14).

Definition 4.8. A proper graded ideal $P$ of $S$ is called graded prime if for all graded ideals $A, B$ of $S$, we have $A \subseteq P$ or $B \subseteq P$ whenever $AB \subseteq P$. A proper $G$-invariant ideal $Q$ of $S_e$ is called $G$-prime if for all $G$-invariant ideals $A, B$ of $S_e$, we have $A \subseteq Q$ or $B \subseteq Q$ whenever $AB \subseteq Q$. The ring $S_e$ is called $G$-prime if $\{0\}$ is a $G$-prime ideal of $S_e$.

For unital strongly $G$-graded rings, the bijection $I \mapsto I_e$ from Theorem 4.7 restricts to a bijection between graded prime ideals of $S$ and $G$-prime ideals of $S_e$ (see [29, Prop. 2.11.7]). We proceed to show that the same holds for nearly epsilon-strongly $G$-graded rings.

Lemma 4.9. Suppose that $S$ is nearly epsilon-strongly $G$-graded. If $I$ is a graded ideal of $S$ such that $I_e$ is a $G$-prime ideal of $S_e$, then $I$ is graded prime.

Proof. Suppose that $A, B$ are graded ideals of $S$ such that $AB \subseteq I$. Then $A_e B_e \subseteq AB \subseteq I \cap S_e = I_e$. By Theorem 4.7 the $S_e$-ideals $A_e, B_e$ are $G$-invariant. Since $I_e$ is $G$-prime, we have $A_e \subseteq I_e$ or $B_e \subseteq I_e$. Assume w.l.o.g. that $A_e \subseteq I_e$. Then $A = SA_e S \subseteq SI_e S = I$ by Theorem 4.7. Thus, $I$ is a graded prime ideal of $S$.

Lemma 4.10. Suppose that $S$ is nearly epsilon-strongly $G$-graded. If $I$ is a graded prime ideal of $S$, then $I_e$ is a $G$-prime ideal of $S_e$.

Proof. Clearly, $I_e$ is an ideal of $S_e$. Suppose that $A, B$ are $G$-invariant ideals of $S_e$ such that $AB \subseteq I_e$. We need to show that $A \subseteq I_e$ or $B \subseteq I_e$. By Theorem 4.7, $A = (SAS)_e$, $B = (SBS)_e$ and $S(AB)S \subseteq SI_e S = I$. Clearly, $SAS$ and $SBS$ are graded ideals of $S$. By Lemma 4.6, $SASSBS = S(AB)S \subseteq I$. Since $I$ is graded prime, we have $SAS \subseteq I$ or $SBS \subseteq I$. Assume w.l.o.g. that $SAS \subseteq I$. Then $A \subseteq I_e$ and thus $I_e$ is $G$-prime.

By combining Lemma 4.9 and Lemma 4.10 we get the desired bijection:

Theorem 4.11. Suppose that $S$ is nearly epsilon-strongly $G$-graded. The map $I \mapsto I_e$ restricts to a bijection between the sets \{graded prime ideals of $S$\} and \{$G$-prime ideals of $S_e$\}.

We now generalize a well-known result by Năstăsescu and Van Oystaeyen to the setting of $s$-unital group graded rings:

Proposition 4.12 (cf. [28, Prop. II.1.4]). Suppose that $G$ is an ordered group and that $S$ is $s$-unital. If $I$ is a graded ideal of $S$, then $I$ is graded prime if and only if $I$ is prime.

Proof. Suppose that $I$ is graded prime. For every $k \geq 0$, let $P(k)$ be the following statement:

\[ a, b \in S \text{ satisfy } aSb \subseteq I \text{ and } |\text{Supp}(a)| + |\text{Supp}(b)| \leq k \implies a \in I \text{ or } b \in I. \]

We proceed by induction to show that $P(k)$ holds for every $k \geq 0$.

Base case: $k = 0$. If $|\text{Supp}(a)| + |\text{Supp}(b)| = 0$, then $a = b = 0 \in I$.

Inductive step: Take $k \geq 0$ such that $P(k)$ holds. Suppose that $aSb \subseteq I$ and $|\text{Supp}(a)| + |\text{Supp}(b)| = k + 1$. Put $m := |\text{Supp}(a)|$ and $n := |\text{Supp}(b)|$. Then we can write $a = \sum_{i=1}^{m} a_{x_i}$ and $b = \sum_{j=1}^{n} b_{y_j}$ where $x_1, \ldots, x_m \in G$ and $y_1, \ldots, y_n \in G$ satisfy $x_1 < \cdots < x_m$ and $y_1 < \cdots < y_n$. Take $z \in G$. For any $s_z \in S_z$ we have $as_z b \in I$. Using that $G$ is an ordered
group and that $I$ is a graded ideal, we get $a_{x_m}s_2b_{y_n} \in I$. This shows that $a_{x_m}Sb_{y_n} \subseteq I$. By graded primeness of $I$, and $s$-unitality of $S$, we get $a_{x_m} \in Sa_{x_m}S \subseteq I$ or $b_{y_n} \in Sb_{y_n}S \subseteq I$.

Case 1: $a_{x_m} \in I$. Put $a' = a - a_{x_m}$. Then $a'Sb = aSb - a_{x_m}Sb \subseteq I - I = I$. Since $|\text{Supp}(a')| + |\text{Supp}(b)| < k + 1$, the induction hypothesis yields that $a' \in I$ or $b \in I$, and hence that $a = a' + a_{x_m} \in I$ or $b \in I$.

Case 2: $b_{y_n} \in I$. Put $b' = b - b_{y_n}$. Then $aSb' = aSb - a_{x_m}Sb \subseteq I - I = I$. Since $|\text{Supp}(a')| + |\text{Supp}(b')| < k + 1$, the induction hypothesis yields that $a \in I$ or $b' \in I$, and hence that $a \in I$ or $b = b' + b_{y_n} \in I$.

Therefore, $P(k + 1)$ holds.

Now, let $A, B$ be nonzero ideals of $S$ with $AB \subseteq I$. Seeking a contradiction, suppose that there are $a \in A \setminus I$ and $b \in B \setminus I$. Since $A$ and $B$ are ideals, it follows that $aSb \subseteq AB \subseteq I$. Since $P(k)$ holds for every $k \geq 0$, we get that $a \in I$ or $b \in I$, which is a contradiction.

The converse statement is trivial. \hfill \square

By Proposition [4.12] we immediately obtain the following partial generalization of a result by Abrams and Haefner [4. Thm. 3.2]:

**Corollary 4.13.** Suppose that $G$ is an ordered group and that $S$ is $s$-unital. Then $S$ is graded prime if and only if $S$ is prime.

Combining the above result with Theorem [4.11] we immediately get the following:

**Corollary 4.14.** Suppose that $G$ is an ordered group and that $S$ is nearly epsilon-strongly $G$-graded. Then $S$ is prime if and only if $S_e$ is $G$-prime.

**Example 4.15.** Let $R$ be a unital ring.

(a) Consider the Laurent polynomial ring $R[t, t^{-1}] = \bigoplus_{i \in \mathbb{Z}} R t^i$ equipped with its canonical strong $\mathbb{Z}$-grading. Since $t$ is central in $R[t, t^{-1}]$, any ideal $I$ of $R$ satisfies $t^{-n}It^n = I$. Thus, every ideal of $R$ is $\mathbb{Z}$-invariant. Hence, $R$ being $\mathbb{Z}$-prime is equivalent to $R$ being prime. Therefore, Corollary [4.14] implies that $R[t, t^{-1}]$ is prime if and only if $R$ is prime.

(b) More generally, let $G$ be an ordered group and consider the group ring $R[G]$. Note that for any ideal $I$ of $R = (R[G])_e$ we have $\delta_{x-1}I\delta_x = \delta_{x-1}\delta_xI = I$ for every $x \in G$. Thus, every ideal of $R$ is $G$-invariant. By Corollary [4.14] it follows that $R[G]$ is prime if and only if $R$ is prime (see e.g. [21 Thm. 6.29]).

5. The “easy” direction

Recall that $S$ is a $G$-graded ring. In this section, we prove the implication (b)$\Rightarrow$(a) of Theorem [5.3] for non-degenerately $G$-graded rings (see Proposition [5.3]).

**Lemma 5.1** (cf. [40 Lem. 1.4]). Suppose that $S$ is non-degenerately $G$-graded, that $H$ is a subgroup of $G$, and that $I$ is an ideal of $S_e$ which satisfies $I^xI = \{0\}$ for every $x \in G \setminus H$. Then the following two assertions hold:

(a) $IS_xI = \{0\}$ for every $x \in G \setminus H$.

(b) $ISI \subseteq IS_H \subseteq S_H$.

**Proof.**

(a): Take $x \in G \setminus H$ and $s \in IS_xI$. By assumption, $S_{x-1}IS_xI = I_xI = \{0\}$ and hence $S_{x-1}s = \{0\}$. Using that $S$ is non-degenerately $G$-graded, we get that $s = 0$.

(b): Employing part (a), we get $ISI = \bigoplus_{x \in G} IS_xI = \bigoplus_{x \in H} IS_xI \subseteq S_H$. \hfill \square

**Lemma 5.2.** Suppose that $S$ is non-degenerately $G$-graded and that $N$ is a subgroup of $G$. If $\bar{A}$ is a nonzero subset of $S_N$, then $S\bar{A}S$ is a nonzero ideal of $S$. 

Proof. Clearly, $S\tilde{A}S$ is an ideal of $S$. Choose a nonzero $a \in \tilde{A}$. Let $n \in \text{Supp}(a) \subseteq N$. By non-degeneracy of the $G$-grading, there is some $s_{n-1} \in S_{n-1}$ and some $t_e \in S_e$ such that $t_eas_{n-1} \neq 0$. Therefore, $t_eas_{n-1} \in S\tilde{A}S \setminus \{0\}$. This shows that $S\tilde{A}S$ is nonzero. \qed

Proposition 5.3 (cf. [10] Thm. 1.3)]. Suppose that $S$ is non-degenerately $G$-graded and that there exist

(i) subgroups $N \triangleleft H \subseteq G$,

(ii) an $H$-invariant ideal $I$ of $S_e$ such that $I^2 I = \{0\}$ for every $x \in G \setminus H$, and

(iii) nonzero ideals $\tilde{A}, \tilde{B}$ of $S_N$ such that $\tilde{A}, \tilde{B} \subseteq IS_N$, and $\tilde{A}S_H \tilde{B} = \{0\}$.

Then $S$ is not prime.

Proof. If $x \in H$, then the second condition in (iii) implies that $\tilde{A}S_x \tilde{B} = \{0\}$.

If $x \in G \setminus H$, then the first condition in (iii) implies that $\tilde{A}S_x \tilde{B} \subseteq (IS_N)S_x(IS_N)$. Since $S_NS_x = \bigoplus_{n \in N} S_nS_x \subseteq \bigoplus_{n \in N} S_{nx}$ and $nx \in G \setminus H$, it follows from Lemma 5.1 that

$$IS_NS_xI \subseteq \bigoplus_{n \in N} IS_{nx}I = \{0\}.$$ 

Hence, $\tilde{A}S_x \tilde{B} = \{0\}$ for every $x \in G$, and thus $\tilde{A}S \tilde{B} = \{0\}$. Now, by (iii) and Lemma 5.2 it follows that $A := S\tilde{A}S$ and $B := S\tilde{B}S$ are nonzero ideals of $S$ satisfying $AB = (S\tilde{A}S)(S\tilde{B}S) \subseteq S(\tilde{A}\tilde{S}B)S = \{0\}$. This shows that $S$ is not prime. \qed

Remark 5.4. Note that $N$ is not required to be finite in Proposition 5.3.

In an attempt to ease the technical notation, we now introduce the following notion.

Definition 5.5 (NP-datum). Let $S$ be a $G$-graded ring. An NP-datum for $S$ is a quintuple $(H, N, I, \tilde{A}, \tilde{B})$ with the following three properties:

(NP1) $H$ is a subgroup of $G$, and $N$ is a finite normal subgroup of $H$,

(NP2) $I$ is a nonzero $H$-invariant ideal of $S_e$ such that $I^2 I = \{0\}$ for every $x \in G \setminus H$, and

(NP3) $\tilde{A}, \tilde{B}$ are nonzero ideals of $S_N$ such that $\tilde{A}, \tilde{B} \subseteq IS_N$, and $\tilde{A}\tilde{B} = \{0\}$.

An NP-datum $(H, N, I, \tilde{A}, \tilde{B})$ is said to be balanced if it satisfies the following property:

(NP4) $\tilde{A}, \tilde{B}$ are nonzero ideals of $S_N$ such that $\tilde{A}, \tilde{B} \subseteq IS_N$, and $\tilde{A}S_H \tilde{B} = \{0\}$.

Remark 5.6. (a) If $S$ is nearly epsilon-strongly $G$-graded, then (NP4) implies (NP3).

(b) Suppose that $S$ is s-unital strongly $G$-graded. An NP-datum $(H, N, I, \tilde{A}, \tilde{B})$ for $S$ is necessarily balanced whenever $\tilde{A}$ or $\tilde{B}$ is $H$-invariant. Indeed, suppose that $\tilde{A}$ is $H$-invariant. For any $h \in H$, we get that $\tilde{A}S_h \tilde{B} = S_e \tilde{A}S_h \tilde{B} = S_h S_{h^{-1}} \tilde{A}S_h \tilde{B} \subseteq S_h \tilde{A}\tilde{B} = \{0\}$ by Lemma 2.16. The proof of the case when $\tilde{B}$ is $H$-invariant is analogous.

Corollary 5.7. Suppose that $S$ is non-degenerately $G$-graded. If $S_e$ is not $G$-prime, then $S$ has a balanced NP-datum $(H, N, I, \tilde{A}, \tilde{B})$ for which $\tilde{A}, \tilde{B}$ are $H/N$-invariant.

Proof. If $S_e$ is not $G$-prime, then there are nonzero $G$-invariant ideals $\tilde{A}, \tilde{B}$ of $S_e$ such that $\tilde{A}\tilde{B} = \{0\}$. We claim that $(G, \{e\}, S_e, \tilde{A}, \tilde{B})$ is a balanced NP-datum. Conditions (NP1), (NP2) and (NP3) are trivially satisfied. We now check condition (NP4). Take $x \in G$. Seeking a contradiction, suppose that $\tilde{A}S_x \tilde{B} \neq \{0\}$. Note that $\tilde{A}S_x \tilde{B} \subseteq S_x$. By non-degeneracy of the $G$-grading, $S_{x^{-1}} \cdot \tilde{A}S_x \tilde{B} \neq \{0\}$. Since $\tilde{A}$ is $G$-invariant, we get that $S_{x^{-1}} \tilde{A}S_x \tilde{B} \subseteq \tilde{A}\tilde{B} = \{0\}$, which is a contradiction. Note that, trivially, $\tilde{A}, \tilde{B}$ are both $G/\{e\}$-invariant. \qed

By combining the above results we get the following.
Corollary 5.8. Suppose that $S$ is non-degenerately $G$-graded. If $S$ is prime, then $S_e$ is $G$-prime.

6. Passman pairs and the Passman replacement argument

In this section, we generalize a technical result by Passman [40]. Recall that $S$ is a $G$-graded ring. We are interested in pairs $(J, M)$ where $J$ is a nonzero ideal of $S_e$ and $M \subseteq G$ is a subset such that $J^x J = \{0\}$ for every $x \in G \setminus M$. Given such a pair $(J, M)$, where $M$ is of a certain type, we will find another pair $(K, L)$ where $K \subseteq J$ is a nonzero ideal of $S_e$ and $L$ is a subgroup of $G$. Crucially, the new pair $(K, L)$ satisfies $K^x K = \{0\}$ for $x \in G \setminus L$. Passman’s original proof relies on $S$ being unital and strongly $G$-graded, and provides a construction of the ideal $K$. As we will see, his main argument generalizes to our extended setting, although we do not get an explicit description of the ideals.

Definition 6.1. If $I$ is a nonzero ideal of $S_e$ and $M \subseteq G$ is such that $I^x I = \{0\}$ for every $x \in G \setminus M$, then we call $(I, M)$ a Passman pair.

Proposition 6.2 (cf. [40, Lem. 2.1]). Suppose that $S$ is nearly epsilon-strongly $G$-graded and that $(J, M)$ is a Passman pair where $M = \bigcup_{k=1}^n g_k G_k$ for some subgroups $G_1, \ldots, G_n$ of $G$ and $g_1, \ldots, g_n \in G$. Then there exist a nonzero ideal $K \subseteq J$ of $S_e$ and a subgroup $L$ of $G$ such that $(K, L)$ is a Passman pair. In addition, $[L : L \cap G_k] < \infty$ for some $k \in \{1, \ldots, n\}$.

We now fix a group $G$ and a finite family $\{G_1, \ldots, G_n\}$ of subgroups of $G$. To establish Proposition 6.2 we need the following:

Lemma 6.3. Suppose that $S$ is nearly epsilon-strongly $G$-graded. Let $B = \{A_1, A_2, \ldots, A_l\}$ be a family of subgroups of $G$ such that for all $i, j \in \{1, \ldots, l\}$ there is some $k \in \{1, \ldots, n\}$ such that $A_j \subseteq G_k$, and $A_i \cap A_j \in B$. Let $(J, M)$ be a Passman pair. Suppose that $M = \bigcup_{k=1}^n g_k A_{n_k}^t$ where $n_1, \ldots, n_l \in \{1, \ldots, l\}$ and $g_1, \ldots, g_l \in G$. Then there exist a nonzero ideal $K \subseteq J$ of $S_e$ and a subgroup $L$ of $G$ such that $(K, L)$ is a Passman pair. In addition, if $B$ is non-empty then $[L : L \cap A_j] < \infty$ for some $j \in \{1, \ldots, l\}$.

Proof. The proof proceeds by induction on $|B|$. If $|B| = 0$, then the assumption that $(J, \emptyset)$ is a Passman pair implies that $(J, \{e\})$ is a Passman pair. Next, suppose that $|B| \geq 1$. Let $A$ be a maximal element of $B$ ordered by inclusion and note that $B' = B \setminus \{A\}$ is closed under intersections. We consider the following set of Passman pairs:

$$P := \{(K, N) \mid \{0\} \neq K \subseteq J, \quad N = \bigcup_{j=1}^8 g_j A_{k_j} \text{ for some } k_1, \ldots, k_8 \in \{1, \ldots, l\}, \quad g_1, \ldots, g_8 \in G\}$$

Note that $P$ is non-empty since $(J, M) \in P$. For $(K, N) \in P$ with $N = \bigcup_{j=1}^8 g_j A_{k_j}$, we let $\text{Supp}(K, N)$ be the subset $\{A_{k_1}, \ldots, A_{k_8}\} \subseteq B$. Let $\deg(K, N)$ be the number of times that $A = A_{k_j}$ in the expression of $N$.

Now, choose $(K, N) \in P$ of minimal degree. We consider two mutually exclusive cases:

Case 1: $\deg(K, N) = 0$. In this case $\text{Supp}(K, N) \subseteq B'$. Hence, the induction hypothesis applies and we conclude that there exists some Passman pair $(I, L)$ such that $\{0\} \neq I \subseteq K \subseteq J$ and $L$ is a subgroup of $G$. 

Case 2: \( \deg(K,N) = m > 0 \). Let \( N = z_1A \cup z_2A \cup \ldots \cup z_mA \cup T \) where \( T \) is a finite union of cosets of groups in \( B' \). Put

\[
L := \left\{ g \in G \mid g \left( \bigcup_{i=1}^{m} z_iA \right) = \bigcup_{i=1}^{m} z_iA \right\}.
\]

Our goal is to prove that \((K,L)\) is a Passman pair. Note that \( L \) is the stabiliser of \( \bigcup_{i=1}^{m} z_iA \). Thus, \( L \) is in fact a subgroup of \( G \). Take \( x \in G \) such that \( K^xK \neq \{0\} \). We will show that \( x \in L \). Indeed, if \( h = x^{-1}h' \) for some \( h' \in G \setminus N \), then

\[
(K^xK)^h(K^xK) = ((K^x)^hK^h)(K^x) \subseteq K^{xh}K^hK \subseteq K^{xh}K = K^{xh-1}h' = K^{h'} = \{0\}
\]

where we have used Lemma \[3.2\] Lemma \[3.6\] and Corollary \[3.11\]. Similarly, if \( h \in G \setminus N \), then \((K^xK)^h(K^xK) \subseteq (K^{xh}K^h)(K^xK) \subseteq K^hK = \{0\}\) for every \( h \in (G \setminus N) \cup x^{-1}(G \setminus N) = G \setminus (N \cap x^{-1}N) \). Thus, \((K^xK,N \cap x^{-1}N)\) is a Passman pair. Since \( K^xK \subseteq K \subseteq J \) and \( N \cap x^{-1}N \) is a finite union of cosets in \( B \), it follows that \((K^xK,N \cap x^{-1}N) \subseteq P \). Let \( m' := \deg(K^xK,N \cap x^{-1}N) \). By minimality of \( m \), we have \( m' \geq m > 0 \). Note that \( x^{-1}N = x^{-1}z_1A \cup x^{-1}z_2A \cup \ldots \cup x^{-1}z_mA \cup x^{-1}T \). Since \( A \) is maximal, the \( m' \) cosets of \( N \cap x^{-1}N \) must come from \( (\cup_{i=1}^{m} z_iA) \cap (\cup_{i=1}^{m} x^{-1}z_iA) \). Moreover, cosets are either equal or disjoint, and hence \( m' \leq m \). This shows that \( m' = m \) and \( \cup_{i=1}^{m} z_iA = x^{-1}(\cup_{i=1}^{m} z_iA) \) which in turn shows that \( x \in L \). Summarizing, we have established that \( K^xK = \{0\} \) for every \( x \in G \setminus L \), i.e. \((K,L)\) is a Passman pair.

Now, suppose that \( B \) is non-empty. It remains to show that \( |L : L \cap A_j| < \infty \) for some \( j \in \{1, \ldots, l\} \). Consider \( G \) acting on the left on the cosets of \( A_i \), i.e. \( G \acts \{ gA \mid g \in G \} \) by \( g \cdot g_1A = g_1g_2A \) for all \( g_1, g_2 \in G \). Note that \( L \) acts on the finite set of cosets \( D = \{ z_1A, z_2A, \ldots, z_mA \} \). Let \( i \in \{1, \ldots, m\} \) be arbitrary. A short computation shows that \( \text{Stab}_G(z_iA) = z_iAz_i^{-1} \). Thus, \( \text{Stab}_L(z_iA) = z_iAz_i^{-1} \cap L \). Hence, by the orbit-stabilizer theorem we have \( |L : z_iA| = |L : L \cap z_iA| \). Using that \( D \) is a finite set, we conclude that the orbit of \( z_iA \) is finite, i.e. \( |L : z_iA| < \infty \). Thus, we have \( |L : L \cap z_iA| < \infty \) for every \( i \in \{1, \ldots, m\} \).

We consider two mutually exclusive cases.

Case A: \( L \cap z_iA = \emptyset \) for every \( i \). Note that \( K^xK = \{0\} \) for every \( x \in G \setminus N \cup G \setminus L = G \setminus (N \cup L) \). By the case assumption, we have \( N \cap L = T \cap L \). We see that \( T \cap L \) is a finite union of cosets from the set \( B'' = \{ A' \cap T \mid A' \in B' \} \).

Note that \( |B''| \leq |B'| < |B| \). By the induction hypothesis, it follows that there is a Passman pair \((I,L')\) satisfying the required properties.

Case B: \( L \cap z_iA \neq \emptyset \) for some \( i \). Let \( a \in A \) be such that \( z_iA \cap L \neq \emptyset \). Since \( (z_iA)A = z_iA \), we may assume that \( z_i \in L \) by choosing another representative of the coset. It follows that \( L \cap A \equiv L \cap z_iAz_i^{-1} \) via the map defined by \( a \mapsto z_iaz_i^{-1} \) for every \( a \in A \cap L \). As noted above we have \( |L : L \cap z_iAz_i^{-1}| < \infty \) and hence \( |L : L \cap A| < \infty \). Consequently, \( |L : L \cap A_j| < \infty \) with \( A_j := A \) as required.

We are now ready to give a proof of Proposition \[6.2\].

**Proof of Proposition \[6.2\]** Let \((J,M)\) be a Passman pair where \( M = \bigcup_{k=1}^{n} g_kG_k \) for some subgroups \( G_1,G_2,\ldots,G_n \) of \( G \). Furthermore, let \( B \) denote the closure of \( \{G_1,G_2,\ldots,G_n\} \) with respect to intersections. Then \( M \) is a finite union of left cosets of subgroups of \( B \), and we may apply Lemma \[6.3\]. Hence, there is a Passman pair \((K,L)\) where \( L \) is a subgroup of \( G \) and \( K \subseteq J \) is a nonzero ideal of \( S_e \). In addition, using that \( B \) is non-empty, we have
By Proposition 6.2, there is a nonzero ideal $H_0$. Hence, by assumption we have $i \in \{\}$.

Consequently, $[L : L \cap G_k] < \infty$ for some $k$. By Proposition 2.13, there is a subgroup $L$ of $G$ and a nonzero ideal $I \subseteq J$ of $S_e$ such that $(I, L)$ is a Passman pair of $S$. In other words, $I^2 I = \{0\}$ for every $x \in G \setminus L$. In addition, $[L : L \cap H_k] < \infty$ for some $k \in \{1, \ldots, n\}$.

Proof. For each positive integer $m$ let $A_m$ be the set consisting of all $(h_1, h_2, \ldots, h_m) \in G^m$ such that

- $J^{h_1} J^{h_2} \cdots J^{h_m} \neq \{0\}$,
- $e = h_i$ for some $i \in \{1, \ldots, m\}$, and
- $Wh_j = Wh_i$ if and only if $i = j$.

By Proposition 2.13, $S_e$ is $s$-unital and hence $J = J^e \neq \{0\}$. This shows that $e \in A_1$. Now, by assumption $[G : W] < \infty$, and hence there is a greatest integer $s$ such that $A_s$ is non-empty. Pick $s \in \{h_1, h_2, \ldots, h_s\} \in A_s$ and put $K := J^{h_1} J^{h_2} \cdots J^{h_s}$. Using that $s \in A_s$ and that $J^e$ is an ideal of $S_e$, we get that $K \subseteq J^e = J$. We will construct a set $M \subseteq G$ such that $(K, M)$ is a Passman pair of $S$ where $M$ has the required form for Proposition 6.2.

Take $x \in G$ such that $K^x K \neq \{0\}$. We begin by showing that $\{h_1 x, h_2 x, \ldots, h_s x\}$ represents the same set of right cosets of $W$ as $\{h_1, h_2, \ldots, h_s\}$. Seeking a contradiction, suppose that there is some $i \in \{1, \ldots, s\}$ such that $Wh_i x \neq Wh_j$ for each $j \in \{1, \ldots, s\}$. By Corollary 3.11 and Lemma 3.6(4), we get that

$$\{0\} \neq K^x K \subseteq (J^{h_1 x} J^{h_2 x} \cdots J^{h_s x})(J^{h_1} J^{h_2} \cdots J^{h_s}) \subseteq J^{h_1 x} J^{h_1} \cdots J^{h_s}.$$

Hence, $(h_i x, h_1, h_2, \ldots, h_s) \in A_{s+1}$ which contradicts the assumption on $s$. Thus, $\{Wh_1 x, Wh_2 x, \ldots, Wh_s x\} = \{Wh_1, Wh_2, \ldots, Wh_s x\}$. In particular, $h_i x \in W$ for some $i \in \{1, \ldots, s\}$. By a computation similar to that in (3), we get that $\{0\} \neq K^x K \subseteq J^{h_i x} J$.

Hence, by assumption we have $h_i x \in \bigcup_{k=1}^n w_k H_k$. We have thus proved that

$$K^x K = \{0\}, \quad \forall x \in G \setminus \left( \bigcup_{i=1}^n \bigcup_{k=1}^n h_i^{-1} w_k H_k \right).$$

By Proposition 6.2 there is a nonzero ideal $I \subseteq K \subseteq J$ of $S_e$ and a subgroup $L$ of $G$ such that $(I, L)$ is a Passman pair. Moreover, $[L : L \cap H_k] < \infty$ for some $k \in \{1, \ldots, n\}$.

Remark 6.5. Let $S$ be nearly epsilon-strongly $G$-graded and let $W$ be a subgroup of $G$. Then $S_W$ is a nearly epsilon-strongly $W$-graded ring and $J \cup k=1 w_k H_k$ is a Passman pair of $S_W$. By Proposition 6.2 there is a subgroup $L$ of $W$ and a nonzero ideal $K \subseteq J$ of $S_e$ such that $(K, L)$ is a Passman pair of $S_W$. In other words, $K^x K = \{0\}$ for every $x \in W \setminus L$. In contrast, note that Proposition 6.4 gives a Passman pair $(K, L)$ of the larger ring $S$, i.e. we have $K^x K = \{0\}$ for every $x \in G \setminus L$. 

7. Passman forms and the $\Delta$-method

Let $S$ be a graded ring. For nonzero graded ideals $A, B$ of $S$, we have that $AB = \{0\}$ implies $\pi_N(A)\pi_N(B) \subseteq AB = \{0\}$ for every normal subgroup $N$ of $G$. Moreover, if $S$ is non-degenerately $G$-graded, then $\pi_N(A) \neq \{0\}$ and $\pi_N(B) \neq \{0\}$ by Lemma 2.19. In this section, we consider nonzero ideals $A, B$ of $S$ such that $AB = \{0\}$ and show that there exist a normal subgroup $N$ of $G$ and nonzero ideals $\tilde{A}, \tilde{B}$ of $S_N$ such that $\tilde{A}\tilde{B} = \{0\}$.

Recall that a ring is called \textit{semiprime} if it contains no nonzero nilpotent ideal. Analogously, we make the following definition:

**Definition 7.1.** If for every $G$-invariant ideal $I$ of $S_e$, $I^2 = \{0\}$ implies $I = \{0\}$, then the ring $S_e$ is called \textit{G-semiprime}.

**Remark 7.2.** (a) $S_e$ is $G$-semiprime if and only if $S_e$ contains no nonzero nilpotent $G$-invariant ideal.

(b) If $S_e$ is $G$-prime, then $S_e$ is $G$-semiprime.

We record the following result which follows directly from Remark 7.2(b) and Corollary 5.7:

**Corollary 7.3.** Suppose that $S$ is nearly epsilon-strongly $G$-graded. If $S_e$ is not $G$-semiprime, then $S$ has a balanced NP-datum.

Our main task for the remainder of this section is to establish Proposition 7.4 below. Recall that, for a given group $H$, $\Delta(H) := \{h \in H \mid [H : C_H(h)] < \infty\}$ denotes its finite conjugate center (cf. Section 2).

**Proposition 7.4 (cf. [40] Prop. 3.1)).** Suppose that $S$ is nearly epsilon-strongly $G$-graded and that $S_e$ is $G$-semiprime. Let $A, B$ be nonzero ideals of $S$ such that $AB = \{0\}$. Then there exist a subgroup $H$ of $G$, a nonzero $H$-invariant ideal $I$ of $S_e$ and an element $\beta \in B$ such that the following assertions hold:

(a) $I^2I = \{0\}$ for every $x \in G \setminus H$;

(b) $I\pi_{\Delta(H)}(A) \neq \{0\}$, $I\pi_{\Delta(H)}(\beta) \neq \{0\}$;

(c) $I\pi_{\Delta(H)}(A) \cdot I\beta = \{0\}$.

Using Connell’s result (cf. [41] Lem. 5.2), we show that Proposition 7.4 holds for the special case of group rings in the following example.

**Example 7.5.** Let $R$ be a unital semiprime ring, and consider the group ring $R[G] = \bigoplus_{x \in G} R\delta_x$ with its natural strong $G$-grading. Let $\Delta := \Delta(G)$ and let $a, b \in R[G]$. The $\Delta$-argument was used by Connell to prove that if $a\delta_x b = 0$ for every $x \in G$, then $\pi_\Delta(a)b = 0$.

We show that Proposition 7.4 holds in this special case:

Let $A, B$ be nonzero ideals of $R[G]$ such that $AB = \{0\}$. Put $H := G$ and $I := R$. Since $R[G]$ is non-degenerately $G$-graded, we can choose $\beta \in B$ such that $\beta_e \neq 0$. Now, note that (a) is trivially satisfied. Moreover, (b) follows from Lemma 2.19 and the fact that $\beta_e \neq 0$. Next, note that (c) asserts that $R\pi_\Delta(A) \cdot R\beta = \{0\}$. Also note that $R\pi_\Delta(A)R\beta = R\pi_\Delta(AR)\beta = R\pi_\Delta(A)\beta$. Now, let $\alpha \in A$ and let $x \in G$. Then $\alpha\delta_x\beta \subseteq AB = AB = \{0\}$. Applying Connell’s $\Delta$-result, we have $\pi_\Delta(\alpha)\beta = 0$, and since $\alpha$ is arbitrary it follows that $\pi_\Delta(A)\beta = \{0\}$. Thus, $R\pi_\Delta(A)\beta = \{0\}$ which shows that (c) is satisfied.

Before proving Proposition 7.4 we show that it also holds in the following special case:

**Example 7.6.** Suppose that $G$ is an FC-group and that $S$ is nearly epsilon-strongly $G$-graded. Let $A, B$ be nonzero ideals of $S$ such that $AB = \{0\}$. Put $H := G$, $I := S_e$ and choose a
nonzero $\beta \in B$. Since $G$ is an FC-group, it follows that $\Delta := \Delta(G) = G$. Note that (a) is trivially satisfied. Moreover, $I_{\pi_\Delta}(A) = S_eA = A \neq \{0\}$ and $I_{\pi_\Delta}(\beta) = S_e\beta \not\supset \beta \neq 0$. Thus, (b) holds. Finally, $I_{\pi_\Delta}(A) \cdot I_\beta = S_eA \cdot S_e\beta \subseteq AS_eB \subseteq AB = \{0\}$. Hence, (c) is satisfied.

The key bookkeeping device used by Passman [40] is the notion of a form. We extend his definition to our generalized setting:

**Definition 7.7.** Let $S$ be a $G$-graded ring. Suppose that $A, B$ are nonzero ideals of $S$ such that $AB = \{0\}$. We say that the quadruple $(H, D, I, \beta)$ is a Passman form for $(A, B)$ if the following conditions are satisfied:

(a) $H$ is a subgroup of $G$ and $D = D_G(H) = \{x \in G \mid [H : C_H(x)] < \infty\}$;
(b) $I$ is an $H$-invariant ideal of $S_e$ such that $I^2 I = \{0\}$ for every $x \in G \setminus H$;
(c) $0 \neq \beta \in B$, $IA \neq \{0\}$, and $IA \neq \{0\}$.

The size of a Passman form $(H, D, I, \beta)$ is defined to be the number of right $D$-cosets in $G$ meeting $\text{Supp}(\beta)$.

**Remark 7.8.** Passman (see [40, Prop. 7.1]) only considers forms coming from unital strongly $G$-graded rings. For that class of rings our definition coincides with his original definition.

**Example 7.9.** Here are two examples of Passman forms:

(a) In Example 7.5, $(G, \Delta(G), R, \beta)$ is a Passman form. Let $g_1, g_2, \ldots, g_n \in G$ be such that $\text{Supp}(\beta) \subseteq \bigcup_{i=1}^n g_i \Delta(G)$ is a minimal cover (meaning that it is not possible to choose elements $h_1, \ldots, h_m \in G$ such that $\text{Supp}(\beta) \subseteq \bigcup_{i=1}^m h_i \Delta(G)$, for any $m < n$). The size of the Passman form $(G, \Delta(G), R, \beta)$ is $n$.

(b) In Example 7.6, $(G, G, S_e, \beta)$ is a Passman form of size 1.

Later in this section we will consider Passman forms of minimal size, whose existence is guaranteed by the following:

**Proposition 7.10** (cf. [40, Lem. 7.2]). Suppose that $S$ is nearly epsilon-strongly $G$-graded. If $A, B$ are nonzero ideals of $S$ such that $AB = \{0\}$, then $(A, B)$ has a Passman form.

**Proof.** Put $H := G, D := \Delta(G)$, and $I := S_e$. Note that $I$ is $G$-invariant. Furthermore, $IA = S_eA = A \neq \{0\}$. Now, let $\beta \in B \setminus \{0\}$. It remains to show that $I\beta \neq \{0\}$. To this end, write $\beta = \sum_{x \in G} \beta_x$. Since $S$ is nearly epsilon-strongly $G$-graded, for every $x \in \text{Supp}(\beta)$, there exists some $\epsilon_x(\beta_x) \in S_xS_{x-1} \subseteq S_e = I$ such that $\epsilon_x(\beta_x)\beta_x = \beta_x$. Moreover, there is some $s \in S_e = I$ such that $s\epsilon_x(\beta_x) = \epsilon_x(\beta_x)$ for every $x \in \text{Supp}(\beta)$ (see Proposition 2.13 and Proposition 2.1). Thus,

$$I\beta \ni s\beta = s \sum \beta_x = \sum s(\epsilon_x(\beta_x)\beta_x) = \sum (s\epsilon_x(\beta_x))\beta_x = \sum \epsilon_x(\beta_x)\beta_x = \beta \neq 0,$$

where all sums run over $\text{Supp}(\beta)$. This shows that $(G, \Delta(G), S_e, \beta)$ is a Passman form. \qed

**Proposition 7.11** (cf. [40, Lem. 3.3(ii)]). Suppose that $S$ is non-degenerately $G$-graded and that $A, B$ are nonzero ideals of $S$ such that $AB = \{0\}$. Let $(H, D, I, \beta)$ be a Passman form for $(A, B)$. Then the following assertions hold:

(a) $I_{\pi_\Delta(H)}(A) \neq \{0\}$

(b) There exists a Passman form $(H, D, I, \beta')$ for $(A, B)$ such that $I_{\pi_\Delta(H)}(\beta') \neq \{0\}$, and hence $I_{\pi_D}(\beta') \neq \{0\}$. Moreover, the size of $(H, D, I, \beta')$ is not greater than the size of $(H, D, I, \beta)$.
Proof. (a): Note that \( IA \neq \{0\} \) is a right \( S \)-ideal. By Lemma 2.18 and Lemma 2.19, we have \( \{0\} \neq \pi_{\Delta(H)}(IA) = I\pi_{\Delta(H)}(A) \).

(b): We construct a Passman form with the required properties. Write \( \beta = \sum_{x \in G} \beta_x \). By assumption, \( I \beta \neq \{0\} \). Hence there is some \( r \in I \subseteq S \) and \( x \in G \) such that \( r \beta_x \neq 0 \). By non-degeneracy of the \( G \)-grading, we have \( (r \beta_x) S_{x-1} \neq \{0\} \), i.e. there is some \( \sigma_{x-1} \in S_{x-1} \) such that \( r \beta_x \sigma_{x-1} \neq 0 \). Thus, \( I \beta_x \sigma_{x-1} \neq \{0\} \). Hence, \( (H, D, I, \beta) \) with \( \beta' := \beta \sigma_{x-1} \) is a Passman form for \( (A, B) \) such that \( I\pi_{\Delta(H)}(\beta') \neq \{0\} \). We now show that the size of \( (H, D, I, \beta') \) is less than or equal to the size of \( (H, D, I, \beta) \). Suppose that \( (H, D, I, \beta) \) has size \( m \) and that \( Dg_1, \ldots, Dg_m \) form a minimal set of right \( D \)-cosets covering \( \text{Supp}(\beta) \). Then

\[
\text{Supp}(\beta \sigma_{x-1}) \subseteq \text{Supp}(\beta)_{x-1} \subseteq Dg_1 x^{-1} \cup Dg_2 x^{-1} \cup \ldots \cup Dg_m x^{-1}
\]

and hence the \( m \) right \( D \)-cosets \( \{Dg_i x^{-1}\}_{i=1}^m \) cover \( \text{Supp}(\beta \sigma_{x-1}) \). Thus, the size of \( (H, D, I, \beta') \) is less than or equal to \( m \). Finally, since \( \Delta(H) \subseteq D \), we get \( \{0\} \neq I\pi_{\Delta(H)}(\beta') \subseteq I\pi_D(\beta') \). \( \square \)

Lemma 7.12. Suppose that \( S \) is nearly epsilon-strongly \( G \)-graded and that \( S_e \) is \( G \)-semiprime.

For any \( G \)-invariant ideal \( I \) of \( S_e \) the following assertions hold:

(a) \( r. \text{Ann}_{S_e}(I) = r. \text{Ann}_{S_e}(I^2) \).

(b) \( r. \text{Ann}_{S}(I) = r. \text{Ann}_{S}(I^2) \).

Proof. (a): Put \( J := r. \text{Ann}_{S}(I^2) \). Clearly, \( r. \text{Ann}_{S}(I) \subseteq J \). By Corollary 3.11, \( (IJ)^x = I^x J^x \) for every \( x \in G \), and hence \( IJ \) is a \( G \)-invariant ideal of \( S_e \) by Lemma 3.16. Moreover, by definition \( I^2 J = \{0\} \), and hence \( (IJ)^2 = (IJ)(IJ) \subseteq I(IJ) = I^2 J = \{0\} \). Since \( S_e \) is \( G \)-semiprime, it follows that \( IJ = \{0\} \). Thus, \( J \) annihilates \( I \), i.e. \( J \subseteq r. \text{Ann}_{S}(I) \).

(b): Similarly, the inclusion \( r. \text{Ann}_{S}(I) \subseteq r. \text{Ann}_{S}(I^2) \) is immediate. We now show the reversed inclusion. Take \( \gamma = \sum_{x \in G} \gamma_x \in r. \text{Ann}_{S}(I^2) \). Since \( I^2 \subseteq S_e \), we have \( I^2 \gamma_x = \{0\} \) for every \( x \in G \). Next, let \( x \in G \). Using (a), we obtain that \( \gamma_x S_{x-1} \subseteq r. \text{Ann}_{S}(I) \). In other words, \( I \gamma_x \subseteq \{0\} \) which, by non-degeneracy of the \( G \)-grading, yields \( I \gamma_x = \{0\} \), and hence \( \gamma_x \in r. \text{Ann}_{S}(I) \). Since \( x \in G \) is arbitrary, it follows that \( \gamma \in r. \text{Ann}_{S}(I) \). \( \square \)

Lemma 7.13 (cf. [40] Lem. 3.3(iii))). Suppose that \( S \) is nearly epsilon-strongly \( G \)-graded and that \( S_e \) is \( G \)-semiprime.

Furthermore, let \( A, B \) be nonzero ideals of \( S \) such that \( AB = \{0\} \). If \( (H, D, I, \beta) \) is a Passman form for \( (A, B) \) of minimal size with \( I\pi_D(\beta) \neq \{0\} \), then for every \( \gamma \in S_D \) we have \( I\gamma \beta = \{0\} \) if and only if \( I\gamma \pi_D(\beta) = \{0\} \).

Proof. Suppose that \( I\gamma \beta = \{0\} \). Since \( \pi_D \) is an \( S_D \)-bimodule homomorphism by Lemma 2.18, it follows that \( \{0\} = \pi_D(I\gamma \beta) = I\pi_D(\beta) \).

Conversely, suppose that \( I\gamma \pi_D(\beta) = \{0\} \). Take \( s \in I \) and note that \( s \gamma \beta \in IS_DB \subseteq B \). Seeking a contradiction, suppose that \( (H, D, I, s \gamma \beta) \) is a Passman form for the pair \( (A, B) \). We show that \( (H, D, I, s \gamma \beta) \) has less than minimal size. Indeed, suppose that \( n \in \mathbb{N} \) is the size of \( (H, D, I, \beta) \), i.e. the minimal number such that \( \text{Supp}(\beta) \subseteq \bigcup_{i=1}^n Dg_i \) for some \( g_1, \ldots, g_n \in G \). Since \( I\pi_D(\beta) \neq \{0\} \), we have \( \pi_D(\beta) \neq 0 \). Hence, we may w.l.o.g. assume that \( g_1 = e \). Moreover, it is immediate that

\[
\text{Supp}(s \gamma \beta) \subseteq \text{Supp}(s \gamma) \text{Supp}(\beta) \subseteq D \left( \bigcup_{i=1}^n Dg_i \right) \subseteq \left( \bigcup_{i=1}^n Dg_i \right).
\]

By assumption, however, \( 0 = s \gamma \pi_D(\beta) = \pi_D(s \gamma \beta) \) which entails that \( \text{Supp}(s \gamma \beta) \subseteq \bigcup_{i=1}^n Dg_i \). This is a contradiction since \( (H, D, I, \beta) \) is assumed to be minimal. Thus, \( (H, D, I, s \gamma \beta) \) is not a Passman form, and hence \( I\gamma \beta = \{0\} \) (cf. Definition 7.7). As this holds for every \( s \in I \), we have \( I^2 \gamma \beta = \{0\} \). By Lemma 7.12(b), this yields \( I\gamma \beta = \{0\} \). \( \square \)
7.1. Properties of a Passman form of minimal size. In what follows, we fix a nearly epsilon-strongly $G$-graded ring $S$ such that $S_e$ is $G$-semi-prime, nonzero ideals $A, B$ of $S$ with $AB = \{0\}$, and a Passman form $(H, D, I, \beta)$ for $(A, B)$ of minimal size. Throughout this section we assume that $I\pi_{\Delta(H)}(A) \cdot I\beta \neq \{0\}$.

**Lemma 7.14** (cf. [10] Lem. 3.4)). The following assertions hold:
(a) There exists $\alpha \in A \cap S_H$ such that $I\pi_{\Delta(H)}(\alpha) \beta \neq \{0\}$.
(b) For every $\alpha \in A$ there is a subgroup $W$ of $H$ of finite index that centralizes $\text{Supp}(\pi_D(\alpha))$ and $\text{Supp}(\pi_D(\beta))$.

**Proof.** (a): By assumption, we have $I\pi_{\Delta(H)}(A) \cdot I\beta \neq \{0\}$. In other words, $\pi_{\Delta(H)}(A) \cdot I\beta$ is not contained in $r.\text{Ann}_S(I)$. Furthermore, by Lemma 7.12(b), we have $r.\text{Ann}_S(I) = r.\text{Ann}_S(I^2)$. Applying Lemma 2.18 we get $I\pi_{\Delta(H)}(r.I\beta) = I^2\pi_{\Delta(H)}(A) \cdot I\beta \neq \{0\}$. Hence, there exists some $\alpha \in IAI \subseteq A$ such that $I\pi_{\Delta(H)}(\alpha) \beta \neq \{0\}$. Additionally, we have $\alpha \in ISI \subseteq S_H$ by Lemma 5.1(b). Since $D \cap H = \Delta(H)$, we get $\pi_D(\alpha) = \pi_{\Delta(H)}(\alpha)$ and thus $I\pi_{\Delta(H)}(\alpha) \beta \neq \{0\}$.

(b): Note that $P := \text{Supp}(\pi_D(\alpha)) \cup \text{Supp}(\pi_D(\beta))$ is a finite subset of $D = D_G(H)$ and consider $W := \bigcap_{x \in P} C_H(x)$. Since $P \subseteq D$ and $[H : C_H(x)] < \infty$ for every $x \in D$, we get that $[H : W] < \infty$. □

**Lemma 7.15** (cf. [10] Lem. 3.4)). Suppose that $\alpha \in A \cap S_H$ is such that $I\pi_D(\beta) \neq \{0\}$. Let $W$ be given by Lemma 7.14. Then there are $d_0 \in \text{Supp}(\pi_D(\alpha))$ and $u \in W$ such that $I(S_e \alpha_d S_{d-1})^u \pi_D(\alpha \beta) \neq \{0\}$.

**Proof.** First put $\gamma := \pi_D(\alpha) \beta$ and write $\alpha = \sum_{x \in G} \alpha_x, \beta = \sum_{x \in G} \beta_x$, and $\gamma = \sum_{x \in G} \gamma_x$. Furthermore, let $J := \sum_{d \in D}(S_e \alpha_d S_{d-1})W \subseteq S_e$. Note that $J$ is a $W$-invariant ideal of $S_e$. Using that $S$ is nearly-epsilon strongly $G$-graded, note that for all $d \in D, y \in G$, we have

\[(4) \alpha_d S_{y-1} S_{d-1} \subseteq S_e \alpha_d (S_{d-1} S_e) S_{y-1} S_{d-1} \subseteq S_e \alpha_d S_{d-1} \cdot S_d \beta_y S_{y-1} S_{d-1} \subseteq J \cdot S_e \subseteq J.\]

Take $x \in G$. Then by (4), $\gamma_x S_{x-1} \subseteq J$. Seeking a contradiction, suppose that $IJ\gamma = \{0\}$. Then $IJ \gamma_x S_{x-1} = \{0\}$. Hence $\gamma_x S_{x-1} \subseteq r.\text{Ann}_S(IJ)$. Using that $IJ \subseteq IJ$, we get

\[(5) \gamma_x S_{x-1} \subseteq IJ \cap r.\text{Ann}_S(IJ).\]

By Lemma 3.16 we know that $IJ \cap r.\text{Ann}_S(IJ)$ is a $W$-invariant nilpotent ideal of $S_e$ contained in $I$, and hence $IJ \cap r.\text{Ann}_S(IJ) = \{0\}$ by Lemma 3.22(b). By non-degeneracy of the $G$-grading, (5) implies that $I \gamma_x = \{0\}$. Since $x \in G$ is arbitrary, this yields $I \gamma = \{0\}$, i.e. $I\pi_D(\alpha) \beta = \{0\}$. This contradicts the properties of $\alpha$. Consequently, $I\pi_D(\alpha) \beta \neq \{0\}$, i.e. there exist some $d_0 \in D$ and some $u \in W$ such that $I(S_e \alpha_d S_{d-1})^u \pi_D(\alpha) \beta \neq \{0\}$. □

For the remainder of this section, we fix $\alpha \in A \cap S_H$ such that $|\text{Supp}(\pi_D(\alpha))|$ is minimal subject to $I\pi_D(\alpha) \beta \neq \{0\}$. We also fix $W$ given by Lemma 7.14.

**Lemma 7.16** (cf. [10] Lem. 3.4)). For every $y \in W$ and every $d \in D$, we have

\[IS_{y-1} \alpha_d S_{d-1} y \pi_D(\alpha) \pi_D(\beta) = IS_{y-1} \pi_D(\alpha) S_{d-1} y \alpha_d \pi_D(\beta).\]

**Proof.** Take $y \in W, d \in D, a_{y-1} \in S_{y-1}$, and $b_{d-1} \in S_{d-1}$. Note that if $d \notin \text{Supp}(\alpha)$, then the claim trivially holds. Therefore, we now suppose that $d \in D \cap \text{Supp}(\alpha)$. Define $\gamma := a_{y-1} \alpha_d b_{d-1} - a_{y-1} \alpha_d b_{d-1} \alpha_d$. 

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A short computation, using Lemma 7.14(b), shows that \( \gamma \in A \cap S_H \). Moreover, since \( \pi_D \) is an \( S_e \)-bimodule homomorphism by Lemma 2.18, we get

\[ \pi_D(\gamma) = a_{y^{-1}}a_db_{d-1}y\pi_D(\alpha) - a_{y^{-1}}\pi_D(\alpha)b_{d-1}y\alpha_d. \]

From this we get that \( \text{Supp}(\pi_D(\gamma)) \subseteq \text{Supp}(\pi_D(\alpha)) \). We claim that the minimality assumption on \( \alpha \) implies that \( I\pi_D(\gamma)\pi_D(\beta) = \{0\} \). If the claim holds, then we get that

\[ Ia_{y^{-1}}(\alpha_db_{d-1}y\pi_D(\alpha) - \pi_D(\alpha)b_{d-1}y\alpha_d)\pi_D(\beta) = \{0\} \]

and hence that

\[ IS_{y^{-1}}\alpha_dS_{d-1}y\pi_D(\alpha)\pi_D(\beta) = IS_{y^{-1}}\pi_D(\alpha)S_{d-1}y\alpha_d\pi_D(\beta). \]

Now we show the claim. Write \( \gamma = \sum_{x \in G} \gamma_x \). By considering the cases when \( x \in \text{Supp}(\alpha) \) and \( x \notin \text{Supp}(\alpha) \) separately, for each \( x \in G \) we get that

\[ \gamma_x = a_{y^{-1}}a_db_{d-1}y\alpha_x - a_{y^{-1}}\alpha_xb_{d-1}y\alpha_d. \]

Now, recall that \( \pi_D(\gamma) = \sum_{x \in D} \gamma_x \). However, due to (6), \( \gamma_d = 0 \), and thus, \( |\text{Supp}(\pi_D(\gamma))| < |\text{Supp}(\pi_D(\alpha))| \), since \( \alpha_d \neq 0 \). The minimality assumption on \( \alpha \) therefore implies that \( I\pi_D(\gamma)\beta = \{0\} \). Applying the map \( \pi_D \) to the former equation yields \( \pi_D(\gamma)\pi_D(\beta) = \{0\} \).

**Lemma 7.17.** There are elements \( x_1, \ldots, x_n \in W \) and \( g_1, \ldots, g_n \in \text{Supp}(\beta) \setminus D \) such that if \( S_{y^{-1}}I\pi_D(\alpha)S_{y}\pi_D(\beta) \neq \{0\} \), then \( y \in \bigcup_{k=1}^n x_kH_k \) whenever \( y \in W \). Here, \( H_k := C_W(g_k) \).

**Proof.** Let \( \tilde{\alpha} := \alpha - \pi_D(\alpha) \) and let \( \tilde{\beta} := \beta - \pi_D(\beta) \). Then

\[ S_{y^{-1}}I(\pi_D(\alpha) + \tilde{\alpha})S_{y}(\pi_D(\beta) + \tilde{\beta}) = S_{y^{-1}}I\alpha S_{y}\beta \subseteq S_{y^{-1}}IAS_{y}B \subseteq AB = \{0\}. \]

Note that \( S_{y^{-1}}I\pi_D(\alpha)S_{y}\beta \) and \( S_{y^{-1}}I\tilde{\alpha}S_{y}\pi_D(\beta) \) have support disjoint from \( D \). On the other hand, \( \{0\} \neq S_{y^{-1}}I\pi_D(\alpha)S_{y}\pi_D(\beta) \subseteq S_{D} \). Hence, \( S_{y^{-1}}I\pi_D(\alpha)S_{y}\pi_D(\beta) \) must be additively cancelled out by \( S_{y^{-1}}I\tilde{\alpha}S_{y}\beta \). In particular, these two expressions must have a support element in common, i.e., there exist \( a \in \text{Supp}(\tilde{\alpha}) \), \( b \in \text{Supp}(\beta) \), \( g \in \text{Supp}(\pi_D(\alpha)) \), and \( f \in \text{Supp}(\pi_D(\beta)) \) such that \( y^{-1}ab = y^{-1}gf \). Multiplying with \( y \) from the left and with \( y^{-1} \) from the right gives \( ayby^{-1} = gffy^{-1} = gf \), where we have used the fact that \( y \in W \) commutes with both \( \text{Supp}(\pi_D(\alpha)) \) and \( \text{Supp}(\pi_D(\beta)) \). Consequently, \( yby^{-1} = a^{-1}gf \), and hence \( y \in xC_W(b) \) for some fixed \( x \) depending on \( a, b, g, f \). Since there are only a finite number of choices for the parameters \( a, b, g, f \) and \( b \in \text{Supp}(\beta) \setminus D \), the desired conclusion follows.

Next, we will construct an ideal \( J \) of \( S_e \) that allows us to apply the Passman replacement argument (see Section 6). In the following two lemmas we make use of the notation introduced in Lemma 7.17.

**Lemma 7.18 (cf. [40] Lem. 3.5).** For every \( d \in D \),

\[ I(S_e\alpha_dS_{d-1})^{y} \cdot \pi_D(\alpha)\beta = \{0\}, \quad \forall y \in W \setminus \bigcup_{k=1}^n x_kH_k. \]

**Proof.** Take \( y \in W \) such that \( I(S_e\alpha_dS_{d-1})^{y} \cdot \pi_D(\alpha)\beta \neq \{0\} \). Expanding this expression, we get \( IS_{y^{-1}}\alpha_dS_{d-1}S_{y}\pi_D(\alpha)\beta \neq \{0\} \). Since \( S_{y^{-1}}\alpha_dS_{d-1}S_{y}\pi_D(\alpha) \subseteq S_{D} \), Lemma 7.13 implies that \( IS_{y^{-1}}\alpha_dS_{d-1}S_{y}\pi_D(\alpha)\pi_D(\beta) \neq \{0\} \). As a consequence, \( IS_{y^{-1}}\alpha_dS_{d-1}y\pi_D(\alpha)\pi_D(\beta) \neq \{0\} \), since \( S_{d-1}S_{y} \subseteq S_{d-1}y \). By Lemma 7.16, we get that \( IS_{y^{-1}}\pi_D(\alpha)S_{d-1}y\alpha_d\pi_D(\beta) \neq \{0\} \) and, due to
There exists an ideal $J$ of $S_e$ such that $J^y J = \{0\}$ for every $y \in W \setminus \bigcup_{k=1}^{n} u^{-1} x_k H_k$.

Proof. Set $\gamma := \pi_D(\alpha) \beta$ and write $\gamma = \sum_{x \in G} \gamma_x$. By Lemma 7.15, there exist $d_0 \in D$ and $u \in W$ such that $I(S_e \alpha d_0 S_{d_0}^{-1})^u \gamma \neq \{0\}$. Hence, there exists $x \in G$ such that $I(S_e \alpha d_0 S_{d_0}^{-1})^u \gamma_x \neq \{0\}$.

By non-degeneracy of the $G$-grading, we have that $J := I(S_e \alpha d_0 S_{d_0}^{-1})^u \gamma_x S_{x^{-1}} = \{0\}$ is an ideal of $S_e$ contained in $I$. Recall that $I$ is $W$-invariant, since $W$ is a subgroup of $H$. Now, combining the fact that $J \subseteq I(S_e \alpha d_0 S_{d_0}^{-1})^u S_e = (S_e \alpha d_0 S_{d_0}^{-1})^u$ with Lemma 3.6, for every $y \in W$ we get

$$J^{u^{-1}} y \subseteq I^{u^{-1}} y ((S_e \alpha d_0 S_{d_0}^{-1})^u)^{u^{-1}} y \subseteq I^{u^{-1}} y (S_e \alpha d_0 S_{d_0}^{-1})^u \subseteq I(S_e \alpha d_0 S_{d_0}^{-1})^u.$$

By Lemma 7.18, it follows that $J^{u^{-1}} y \pi_D(\alpha) \beta = \{0\}$ for every $y \in W \setminus \bigcup_{k=1}^{n} x_k H_k$ or, equivalently, that $J^{y} \gamma = \{0\}$ for every $y \in W \setminus \bigcup_{k=1}^{n} u^{-1} x_k H_k$. In particular, we have $J^{y} \gamma_x = \{0\}$, and hence, $J^{y} (S_e \gamma_x S_{x^{-1}}) = \{0\}$. This shows that $J^{y} J = \{0\}$ for every $y \in W \setminus \bigcup_{k=1}^{n} u^{-1} x_k H_k$.

7.2. Establishing Proposition 7.4 We still assume that $I_{\pi_{\Delta(H)}(A)} \cdot I \beta \neq \{0\}$. Combining that assumption with the following lemma, we will establish Proposition 7.4.

Lemma 7.20. There is a Passman form for $(A, B)$ of size smaller than the size of $(H, D, I, \beta)$.

Proof. By Lemma 7.19, we have $[H : W] < \infty$. Let $J$ be the ideal of $S_e$ from Lemma 7.19. By Proposition 6.4, there exists a subgroup $L$ of $H$ and a nonzero ideal $K \subseteq J$ of $S_e$ such that $K^y K = \{0\}$ for every $y \in H \setminus L$. Furthermore, we have $[L : L \cap H_k] < \infty$ for some subgroup $H_k$ of $W$. We claim that $(L, D_G(L), K^L, \pi_D(\alpha) \beta)$ is a Passman form of size smaller than the size of $(H, D, I, \beta)$. We first check that it satisfies the conditions in Definition 7.7.

Note that condition (a) is trivially satisfied. Moreover, it follows from Lemma 3.19 that $K^L$ is an $L$-invariant ideal of $S_e$. Since $K \subseteq I$, we have $K^x K = \{0\}$ for every $x \in G \setminus L$. This shows that $K^x K = \{0\}$ for every $x \in G \setminus L$. Thus, by Proposition 3.21, $(K^L)^x (K^L) = \{0\}$ for every $x \in G \setminus L$. Hence, condition (b) is satisfied. Next, note that $\gamma := \pi_D(\alpha) \beta \in B$. It remains to show that $K^L \gamma \neq \{0\}$ and $K^L A \neq \{0\}$. Seeking a contradiction, suppose that $K^L \gamma = \{0\}$. Then $K^L \gamma_x = \{0\}$, and hence $K^L (S_e \gamma_x S_{x^{-1}}) = \{0\}$. We get that $K^L J = \{0\}$. This implies that $J \subseteq r \cdot \text{Ann}_{S_e}(K^L)$. But since $r \cdot \text{Ann}_{S_e}(K^L)$ is an $L$-invariant ideal by Lemma 3.16, we deduce from Lemma 3.19 that $J \subseteq J^L \subseteq r \cdot \text{Ann}_{S_e}(K^L)$ and hence that $K^L J^L = \{0\}$. As $K \subseteq J$, this yields $(K^L)^2 = \{0\}$, which is a contradiction by Lemma 3.22(a). Therefore, $K^L \pi_D(\alpha) \beta \neq \{0\}$. It follows that $K^L \pi_A(\alpha) \neq \{0\}$, and hence $K^L A \neq \{0\}$, by Lemma 2.18. Summarizing, we have shown that $(L, D_G(L), K^L, \pi_D(\alpha) \beta)$ is a Passman form.

To proceed, let $n$ be the size of the Passman form $(H, D, I, \beta)$, i.e. the number of cosets of $D$ in $H$ meeting $\text{Supp}(\beta)$. Furthermore, let $m$ denote the size of $(L, D_G(L), K^L, \pi_D(\alpha) \beta)$. We claim that $m < n$. To show this, first note that $D = D_G(H) \subseteq D_G(L)$ and that $\text{Supp}(\pi_D(\alpha) \beta) \subseteq D \cdot \text{Supp}(\beta)$. Hence, $m \leq n$. Combining the facts that $[L : L \cap H_k] < \infty$ for some $H_k = C_W(g)$ with $g \in \text{Supp}(\beta) \setminus D$ and $L \cap H_k = L \cap C_W(g) = C_L(g)$, we infer that
\[ L : C_L(g) \] < \infty and hence that \( g \in D_G(L) \). This means that the two distinct \( D \)-cosets \( Dg \) and \( D \) are contained in \( D_G(L) \). Consequently, \( m < n \), as claimed.

We are now fully prepared to prove the following:

**Proof of Proposition 7.4.** Let \( S \) be nearly epsilon-strongly \( G \)-graded such that \( S_e \) is \( G \)-semiprime. Furthermore, let \( A, B \) be nonzero ideals of \( S \) such that \( AB = \{0\} \). We now show that conditions (a)-(c) in Proposition 7.4 are satisfied. By Proposition 7.10, \( S \) admits a minimal Passman form for \( (A, B) \), say \( (H, D, I, \beta) \). Moreover, by Proposition 7.11, we may assume that \( I_{\pi \Delta(H)}(A) \neq \{0\} \) and that \( I_{\pi \Delta(H)}(\beta) \neq \{0\} \). Hence, conditions (a) and (b) hold. Seeking a contradiction, suppose that \( I_{\pi \Delta(H)}(A) \cdot I_{\beta} \neq \{0\} \). Then the previous results, in particular Lemma 7.20, yields a Passman form of size smaller than that of \( (H, D, I, \beta) \), which is the desired contradiction. Hence, \( I_{\pi \Delta(H)}(A) \cdot I_{\beta} = \{0\} \) which shows that condition (c) holds.

8. The “hard” direction

Recall that \( S \) is a \( G \)-graded ring. In this section, we prove the implication \( (a) \Rightarrow (e) \) of Theorem 1.3 for nearly epsilon-strongly \( G \)-graded rings (see Proposition 8.9). We remind the reader that if \( H, K \) are subgroups of \( G \), then \( \pi \) normalizes \( K \) if \( Kx = xK \) for every \( x \in H \). In that case it follows that \( H \subseteq N_G(K) \), where \( N_G(K) := \{ x \in G \mid xK = Kx \} \) denotes the normalizer of \( K \) in \( G \), and we allow ourselves to speak of \( H/K \)-invariance in the sense of Definition 3.3.

**Lemma 8.1.** Suppose that \( H, K \) are subgroups of \( G \) such that \( H \) normalizes \( K \). If \( x \in H \), \( k_1, k_2 \in K \), \( r \in S_{xk_1} \), \( \alpha \in S \) and \( s \in S_{k_2x^{-1}} \), then \( \pi_K(\alpha s) = r\pi_K(\alpha)s \).

**Proof.** Write \( \alpha = \sum_{y \in G} \alpha_y \), where \( \alpha_y \in S_y \) for \( y \in G \). Take \( y \in G \). Note that \( xk_1 \cdot y \cdot k_2x^{-1} \in K \) if and only if \( y \in k_1^{-1}x^{-1}Kk_2^{-1} = k_1^{-1}Kk_2^{-1} = K \). Thus, \( \pi_K(\alpha s) = \sum_{y \in G} \pi_K(\alpha y s) = \sum_{y \in K} \pi_K(\alpha y s) = \sum_{y \in K} r\alpha_y s = r\pi_K(\alpha)s \).

By Lemma 8.1 with \( k_1 = k_2 = e \), we get the following result (cf. [40, p. 721]).

**Corollary 8.2.** Suppose that \( H, K \) are subgroups of \( G \) such that \( H \) normalizes \( K \). For every \( \alpha \in S \) and \( x \in H \), we have \( S_{x^{-1}} \pi_K(\alpha)S_x = \pi_K(S_{x^{-1}} \alpha S_x) \).

Given ideals \( A, B \) of \( S \) such that \( AB = \{0\} \), we will find new ideals \( A_1, A_2, B_1, B_2 \) of subrings of \( S \) satisfying \( A_1B_1 = \{0\} \) and \( A_2B_2 = \{0\} \).

**Lemma 8.3.** Suppose that \( S \) is nearly epsilon-strongly \( G \)-graded and that \( S_e \) is \( G \)-semiprime. If \( S \) is not prime, then there exists a subgroup \( H \) of \( G \) such that \( S_{\Delta(H)} \) is not prime. In fact, there exist nonzero \( H/\Delta(H) \)-invariant ideals \( A_1, B_1 \) of \( S_{\Delta(H)} \) such that \( A_1, B_1 \subseteq IS_{\Delta(H)} \) and \( A_1B_1 = \{0\} \).

**Proof.** Let \( A, B \) be nonzero ideals of \( S \) such that \( AB = \{0\} \). By Proposition 7.4, there are a subgroup \( H \) of \( G \), a nonzero \( H \)-invariant ideal \( I \) of \( S_e \), and \( \beta \in B \) such that:

(a) \( I_{\pi \Delta(H)}(A) \neq \{0\} \);
(b) \( I_{\pi \Delta(H)}(\beta) \neq \{0\} \);
(c) \( I_{\pi \Delta(H)}(A) \cdot I_{\beta} = \{0\} \).

By Lemma 8.1 with \( k_1 = k_2 = e \), we get the following result (cf. [40, p. 721]).

**Corollary 8.2.** Suppose that \( H, K \) are subgroups of \( G \) such that \( H \) normalizes \( K \). For every \( \alpha \in S \) and \( x \in H \), we have \( S_{x^{-1}} \pi_K(\alpha)S_x = \pi_K(S_{x^{-1}} \alpha S_x) \).

By Lemma 8.1 with \( k_1 = k_2 = e \), we get the following result (cf. [40, p. 721]).

**Corollary 8.2.** Suppose that \( H, K \) are subgroups of \( G \) such that \( H \) normalizes \( K \). For every \( \alpha \in S \) and \( x \in H \), we have \( S_{x^{-1}} \pi_K(\alpha)S_x = \pi_K(S_{x^{-1}} \alpha S_x) \).
Consider the set \( A_1 := I_{\Delta(H)}(A) \subseteq IS_{\Delta(H)} \). Clearly, \( A_1 \) is nonzero by (a). Take \( h \in H \). Then Proposition 3.12 states the fact that \( \Delta(H) \subseteq H \), and Lemma 8.1 yields
\[
S_{h^{-1}\Delta(H)}A_1S_{h\Delta(H)} = S_{h^{-1}\Delta(H)}I_{\Delta(H)}(A)S_{h\Delta(H)} = IS_{h^{-1}\Delta(H)}\pi_{\Delta(H)}(A)S_{h\Delta(H)}
= I_{\Delta(H)}(S_{h^{-1}\Delta(H)}AS_{h\Delta(H)}) \subseteq I_{\Delta(H)}(A) = A_1.
\]
By taking \( h = e \), the above computation yields \( S_{\Delta(H)}A_1S_{\Delta(H)} \subseteq A_1 \). Thus, \( A_1 \) is an \( H/\Delta(H) \)-invariant ideal of \( S_{\Delta(H)} \). Next, we define \( B_1 := \sum_{h \in H} IS_{h^{-1}\Delta(H)}\pi_{\Delta(H)}(\beta)S_{h\Delta(H)} \). Clearly, \( B_1 \subseteq IS_{\Delta(H)} \) and \( B_1 \) is nonzero. Take \( h_1 \in H \). Using that \( \Delta(H) \) is a normal subgroup of \( H \) and that \( I \) is \( H \)-invariant, we get
\[
S_{h_1^{-1}\Delta(H)}B_1S_{h_1\Delta(H)} = \sum_{h \in H} S_{h_1^{-1}\Delta(H)} \cdot IS_{h^{-1}\Delta(H)}\pi_{\Delta(H)}(\beta)S_{h\Delta(H)} \cdot S_{h_1\Delta(H)}
= \sum_{h \in H} IS_{h_1^{-1}\Delta(H)}S_{h^{-1}\Delta(H)} \cdot \pi_{\Delta(H)}(\beta) \cdot S_{h\Delta(H)}S_{h_1\Delta(H)}
\subseteq \sum_{h \in H} IS_{h_1^{-1}\Delta(H)} \cdot \pi_{\Delta(H)}(\beta) \cdot S_{h_1\Delta(H)} = B_1.
\]
By taking \( h_1 = e \), the above computation yields \( S_{\Delta(H)}B_1S_{\Delta(H)} \subseteq B_1 \). Thus, \( B_1 \) is an \( H/\Delta(H) \)-invariant ideal of \( S_{\Delta(H)} \). By Proposition 2.20, the induced \( H/\Delta(H) \)-grading on \( S_H \) is nearly epsilon-strong and hence \( S_{h^{-1}\Delta(H)}S_{h\Delta(H)} \cdot \pi_{\Delta(H)}(A) = \pi_{\Delta(H)}(A) \). Using that \( I \) is \( H \)-invariant, it follows from Lemma 2.18 Lemma 8.1 and (c), that
\[
A_1B_1 = I_{\Delta(H)}(A) \cdot \sum_{h \in H} IS_{h^{-1}\Delta(H)}\pi_{\Delta(H)}(\beta)S_{h\Delta(H)}
= \sum_{h \in H} I_{\Delta(H)}(A)S_{h^{-1}\Delta(H)}\pi_{\Delta(H)}(\beta)S_{h\Delta(H)}
= \sum_{h \in H} I \cdot S_{h^{-1}\Delta(H)}S_{h\Delta(H)}\pi_{\Delta(H)}(A) \cdot S_{h^{-1}\Delta(H)}\pi_{\Delta(H)}(\beta)S_{h\Delta(H)}
= \sum_{h \in H} IS_{h^{-1}\Delta(H)} \cdot \pi_{\Delta(H)}(S_{h\Delta(H)}AS_{h^{-1}\Delta(H)}) \cdot \pi_{\Delta(H)}(\beta)S_{h\Delta(H)}
\subseteq \sum_{h \in H} IS_{h^{-1}\Delta(H)} \cdot \pi_{\Delta(H)}(A) \cdot \pi_{\Delta(H)}(\beta)S_{h\Delta(H)}
= \sum_{h \in H} S_{h^{-1}\Delta(H)} \cdot I_{\Delta(H)}(A) \cdot \pi_{\Delta(H)}(\beta)S_{h\Delta(H)}
= \sum_{h \in H} S_{h^{-1}\Delta(H)} \cdot \pi_{\Delta(H)}(I_{\Delta(H)}(A)\beta) \cdot S_{h\Delta(H)} = \{0\}.
\]
As a result, \( S_{\Delta(H)} \) is not prime. \( \square \)

**Lemma 8.4.** Suppose that we are in the setting of Lemma 8.3. Then there exists a finitely generated normal subgroup \( W \) of \( H \) such that \( W \subseteq \Delta(H) \). Moreover, there exist nonzero \( H/W \)-invariant ideals \( A_2, B_2 \) of \( S_W \) such that \( A_2, B_2 \subseteq IS_W \) and \( A_2B_2 = \{0\} \).

**Proof.** Let \( A_1, B_1 \subseteq IS_{\Delta(H)} \) be as in Lemma 8.3. Then there exist nonzero elements \( a_1 \in A_1 \) and \( b_1 \in B_1 \). Putting \( P := \text{Supp}(a_1) \cup \text{Supp}(b_1) \) and using that \( A_1, B_1 \) are ideals of \( S_{\Delta(H)} \), we see that \( P \subseteq \Delta(H) \). Moreover, let \( W \) be the normal closure of \( P \) in \( H \). Then \( W \) is clearly a finitely generated normal subgroup of \( H \) with \( W \subseteq \Delta(H) \). Now, consider \( A_2 := A_1 \cap S_W \) and
By restricting to torsion elements, we see that \( W/N \) is torsion-free abelian. By the fundamental theorem of finitely generated abelian groups, \( W/N \) is a finitely generated abelian group. By the fundamental theorem of finitely generated abelian groups, \( W/W' \) has a finite maximal torsion subgroup \( K \), i.e. \( W/W' \cong \mathbb{Z}^n \oplus K \) for some \( n \geq 0 \). Thus, \( N/W' \cong K \). By restricting to torsion elements, we see that \( N/W' \cong K \). Thus, \( N \) is a finite subgroup of \( W \). Since every automorphism of \( W \) preserves element order, it follows that \( N \) is a characteristic subgroup of \( W \). We also get that \( W/N \) is torsion-free abelian, because \( W' \subseteq N \). 

Definition 8.6 (cf. [40, p. 14]). Suppose that \( A \) is a nonzero ideal of a nearly epsilon-strongly \( W \)-graded ring \( S_W \) and that \( N \triangleleft W \). For any nonzero \( a \in A \) we define \( \text{meet}_N(a) \) to be the number of cosets of \( N \) in \( W \) that meet \( \text{Supp}(a) \). Define \( m := \min \{ \text{meet}_N(b) \mid b \in A \setminus \{0\} \} \). Let \( \text{min}_N(A) \) denote the additive span of all nonzero elements \( a \in A \) such that \( \text{meet}_N(a) = m \).

Lemma 8.7 (cf. [40, Lem. 4.1]). Suppose that \( S \) is nearly epsilon-strongly \( G \)-graded and that \( H \) is a subgroup of \( G \). Furthermore, suppose that \( N \triangleleft W \) are subgroups of \( G \) that are normalized by \( H \) and that \( A \) is a nonzero \( H/W \)-invariant ideal of \( S_W \). Then the following assertions hold:

(a) \( \text{min}_N(A) \) is a nonzero \( H/W \)-invariant ideal of \( S_W \).

(b) \( \pi_N(A) \) is a nonzero \( H/W \)-invariant ideal of \( S_N \).

Proof. (a): Note that \( \text{min}_N(A) \) is nonzero by definition. We show that \( \text{min}_N(A) \) is an ideal of \( S_W \). Let \( \alpha \neq 0 \) be a generator of \( \text{min}_N(A) \) and take \( w \in W \). It is enough to show that \( S_w \alpha \) and \( \alpha S_w \) are contained in \( \text{min}_N(A) \). To this end, note that \( \text{Supp}(\alpha S_w) \subseteq (\text{Supp}(\alpha))w \) and that \( \text{Supp}(S_w \alpha) \subseteq w(\text{Supp}(\alpha)) \). Since \( N \triangleleft W \), right and left cosets of \( N \) in \( W \) coincide. Let \( \{w_1 N, w_2 N, \ldots, w_m N\} \) be a minimal set of cosets of \( N \) that covers \( \text{Supp}(\alpha) \). That is, \( \text{Supp}(\alpha) \subseteq w_1 N \cup \ldots \cup w_m N = N w_1 \cup \ldots \cup N w_m \) with \( m \) minimal among such covers. Hence, \( \text{Supp}(\alpha S_w) \subseteq N w_1 w \cup \ldots \cup N w_m w \) and \( \text{Supp}(S_w \alpha) \subseteq w_1 N \cup \ldots \cup w_m N \). Consequently, \( \alpha S_w \) and \( S_w \alpha \) meet less than or exactly \( m \) cosets of \( N \). It follows that \( \alpha S_w, S_w \alpha \in \text{min}_N(A) \) and therefore \( \text{min}_N(A) \) is an ideal of \( S_W \).

Next, let \( \alpha \in A \) be a generator of \( \text{min}_N(A) \) and take \( h \in H \). To show that \( \text{min}_N(A) \) is \( H/W \)-invariant, it is enough to show that \( S_{h^{-1} 1} \alpha S_{h W} \subseteq \text{min}_N(A) \). Take \( k_1, k_2 \in W \). We will show that \( S_{h^{-1} k_1} \alpha S_{h k_2} \subseteq \text{min}_N(A) \).

Using that \( A \) is assumed to be \( H/W \)-invariant, we have \( S_{h^{-1} k_1} \alpha S_{h k_2} \subseteq A \). Hence, it only remains to show that \( S_{h^{-1} k_1} \alpha S_{h k_2} \) meets a minimal number of cosets of \( N \). As before, let \( w_1 N \cup \ldots \cup w_m N \) be a minimal cover of \( \text{Supp}(\alpha) \). Then

\[
\text{Supp}(S_{h^{-1} k_1} \alpha S_{h k_2}) \subseteq h^{-1} k_1(\text{Supp}(\alpha))hk_2 \subseteq h^{-1} k_1(w_1 N)hk_2 \cup h^{-1} k_1(w_2 N)hk_2 \cup \ldots \cup h^{-1} k_1(w_m N)hk_2.
\]
Since both $H$ and $W$ normalize $N$, we get that $h^{-1}k_1(w_iN)hk_2 = (h^{-1}k_1w_ihk_2)N$. Moreover, since $H$ normalizes $W$, and $k_1w_i \in W$, we have $h^{-1}(k_1w_i)h \in W$. Thus, $h^{-1}k_1w_ih \cdot k_2 \in W$. Hence, $\text{Supp}(S_{h^{-1}k_1} \alpha S_{hk_2})$ meets less than or exactly $m$ cosets of $N$ in $W$. Thus, $\min_N(A)$ is $H/W$-invariant.

(b): By Lemma 2.19 and Proposition 2.15 it follows that $\pi_N(A)$ is a nonzero ideal of $S_N$. Take $\alpha \in A$ and $h \in H$. Since $H$ normalizes $N$, Lemma 8.1 yields $S_{h^{-1}W} \pi_N(\alpha)S_{hW} = \pi_N(S_{h^{-1}W} \alpha S_{hW}) \subseteq \pi_N(S_{h^{-1}W}AS_{hW}) \subseteq \pi_N(A)$, where the last inclusion follows by the $H/W$-invariance of $A$. This shows that $\pi_N(A)$ is $H/W$-invariant.

Lemma 8.8 (cf. [10] Lem. 4.2]). Suppose that $S$ is nearly epsilon-strongly $G$-graded and that $H$ is a subgroup of $G$. Let $N \triangleleft W$ be subgroups of $G$ such that $N,W$ are normalized by $H$ and $W/N$ is a unique product group. Furthermore, let $A,B$ be nonzero ideals of $S_W$ such that $AB = \{0\}$. Then there exist nonzero ideals $A',B'$ of $S_N$ such that $A'B' = \{0\}$. Moreover, the following assertions hold:

(a) If $A$ (resp. $B$) is $H/W$-invariant, then $A'$ (resp. $B'$) is $H/W$-invariant.
(b) If $A,B \subseteq IS_W$ for some ideal $I \subseteq S_c$, then $A',B' \subseteq IS_N$.

Proof. Put $A' := \pi_N(\min_N(A))$ and $B' := \pi_N(\min_N(B))$, and note that they are both ideals of $S_N$ by Lemma 8.7. Let $\alpha = \sum_{x \in G} \alpha_x \in A$ and $\beta = \sum_{x \in G} \beta_x \in B$ be generators of $\min_N(A)$ and $\min_N(B)$, respectively.

Consider the induced $W/N$-grading on $S_W$ (see Section 2.5). With this grading, $S_W$ has principal component $S_N$. Moreover, it follows from Proposition 2.20 that $S_W$ is a nearly epsilon-strongly $W/N$-graded ring. Thus, we may w.l.o.g. assume that $N = \{e\}$.

Now, using the fact that $W$ is a unique product group, we write $x_{0}y_{0}$ for the unique product of $(\text{Supp}(\alpha))(\text{Supp}(\beta))$ and deduce from $\alpha \beta \subseteq AB = \{0\}$ that $\alpha_{x_{0}}\beta_{y_{0}} = 0$, since no cancelling can occur. But then $\alpha\beta_{y_{0}} = \sum_{x \in G} \alpha_x \beta_{y_{0}}$ has smaller support size than that of $\alpha$. Since $\alpha$ meets a minimal number of cosets of $N$, it follows that $\alpha\beta_{y_{0}} = 0$. Hence, $\alpha_x \beta_{y_{0}} = 0$ for every $x \in W$, which in turn implies that $\alpha_x \beta$ has smaller support size than that of $\beta$. As a result, we must have $\alpha_x \beta = 0$. In consequence, we have $\alpha_x \beta_y = 0$ for all $x, y \in W$, and hence $\pi_N(\alpha)\pi_N(\beta) = \alpha_e \beta_e = 0$. Thus, $A'B' = \{0\}$.

Finally, we prove (a) and (b). If $A$ is $H/W$-invariant, then it follows from Lemma 8.7 that $A'$ is $H/W$-invariant. Next, suppose that $A \subseteq IS_W$. Then, $\min_N(A) \subseteq A \subseteq IS_W$. Hence, by Lemma 2.18 $A' = \pi_N(\min_N(A)) \subseteq \pi_N(IS_W) \subseteq IS_N$. The proof of the corresponding statements for $B$ and $B'$ is completely analogous.

Proposition 8.9. Suppose that $S$ is nearly epsilon-strongly $G$-graded. If $S$ is not prime, then it has an NP-datum $(H,N,I,A,B)$ for which $A,B$ are $H/N$-invariant.

Proof. If $S_e$ is not $G$-semiprime, then the desired conclusion follows from Corollary 7.3. Now, suppose that $S_e$ is $G$-semiprime. Then Proposition 7.4 provides us with a subgroup $H$ of $G$ and an $H$-invariant ideal $I$ of $S_e$ such that $I^2I = \{0\}$ for every $x \in G \setminus H$. In particular, condition (NP2) holds.

To proceed, we apply Lemma 8.3 which yields nonzero $H/\Delta(H)$-invariant ideals $A_1,B_1$ of $S_{\Delta(H)}$ such that $A_1B_1 = \{0\}$. Moreover, by Lemma 8.4 there exists a finitely generated normal subgroup $W$ of $H$ with $W \subseteq \Delta(H)$ and nonzero $H/W$-invariant ideals $A_2,B_2$ of $S_W$ such that $A_2B_2 = \{0\}$.

Next, by Proposition 8.5 there is a finite characteristic subgroup $N \triangleleft W$ such that $W/N$ is torsion-free abelian. Since $N$ is a characteristic subgroup, we get that $N \triangleleft W \triangleleft H$. This establishes condition (NP1). Moreover, by a well-known result by Levi [26], $W/N$ is an
ordered group, and hence a unique product group. Note that $H$ normalizes $N$ and $W$. This means that Lemma 8.8 is at our disposal, i.e. there are nonzero $H/W$-invariant, and in particular $H/N$-invariant, ideals $\tilde{A}, \tilde{B}$ of $S_N$ such that $\tilde{A}, \tilde{B} \subseteq IS_N$ and $\tilde{A}\tilde{B} = \{0\}$. Hence, condition (NP3) holds. This shows that $(H, N, I, \tilde{A}, \tilde{B})$ is an NP-datum for $S$.

9. Proof of the main theorem

In this section, we finish the proof of Theorem 1.3 and show that Passman’s result (see Theorem 1.1) can be recovered from it.

Proof of Theorem 1.3. (1) Suppose that $S$ is non-degenerately $G$-graded.

(e)$\Rightarrow$(d): Suppose that (e) holds. By Lemma 3.13, $\tilde{A}, \tilde{B}$ are $H$-invariant. It only remains to show that $\tilde{A}S_H\tilde{B} = \{0\}$. Take $x \in H$. Seeking a contradiction, suppose that $\tilde{A}S_xN\tilde{B} \neq \{0\}$. Note that $\tilde{A}S_xN\tilde{B} \subseteq S_xN$. By non-degeneracy of the $G$-gradation on $S$, it follows that $S_H$ is non-degenerately $H$-graded. Hence, by Proposition 2.21 the $H/N$-grading on $S_H$ is also non-degenerate. Consequently, $S_{x^{-1}}^{-1}\tilde{A}S_xN\tilde{B} \neq \{0\}$. By the $H/N$-invariance of $\tilde{A}$ we get that $\{0\} \neq S_x^{-1}\tilde{A}S_xN\tilde{B} \subseteq \tilde{A}\tilde{B} = \{0\}$ which is a contradiction. We conclude that $\tilde{A}S_x\tilde{B} \subseteq \tilde{A}\tilde{S}_xN\tilde{B} = \{0\}$. Thus, $\tilde{A}S_H\tilde{B} = \{0\}$.

(d)$\Rightarrow$(c)$\Rightarrow$(b): This is trivial.

(b)$\Rightarrow$(a): This follows from Proposition 5.3.

(2) Suppose that $S$ is nearly epsilon-strongly $G$-graded. By Proposition 2.15, $S$ is non-degenerately $G$-graded. Hence, by (1) we get that (e)$\Rightarrow$(d)$\Rightarrow$(c)$\Rightarrow$(b)$\Rightarrow$(a). The remaining implication, (a)$\Rightarrow$(e), follows from Proposition 8.9.

Proof of Theorem 1.4. Let $S$ be a unital strongly $G$-graded ring. The claim of Theorem 1.1 follows immediately from Remark 5.6 and the equivalence (a)$\Leftrightarrow$(d) in Theorem 1.3.

10. Applications for torsion-free grading groups

Recall that $S$ is a $G$-graded ring. In this section, we pay special attention to the case when $G$ is torsion-free. The following result generalizes Corollary 4.14 and establishes Theorem 1.4.

Theorem 10.1 (cf. [10, Cor. 4.6]). Suppose that $G$ is torsion-free and that $S$ is nearly epsilon-strongly $G$-graded. Then $S$ is prime if and only if $S_e$ is $G$-prime.

Proof. Suppose that $S$ is not prime. By Theorem 1.3 there is a balanced NP-datum $(H, N, I, \tilde{A}, \tilde{B})$ for $S$. Using that $G$ is torsion-free, we conclude that $N = \{e\}$. In consequence, $S_N = S_e$ and $I, \tilde{A}, \tilde{B}$ are all ideals of $S_e$. Consider the sets $\tilde{A}^G$ and $\tilde{B}^G$. By Proposition 3.19 they are nonzero $G$-invariant ideals of $S_e$. Note that $\tilde{A}S_x\tilde{B} = \{0\}$ for every $x \in G$ by the same argument as in the proof of Proposition 5.3. Using this, we get that $\tilde{A}^G\tilde{B}^G = \{0\}$ and hence $S_e$ is not $G$-prime.

Now suppose that $S$ is prime. By Corollary 5.8 it follows that $S_e$ is $G$-prime.

Remark 10.2. Note that a strongly $G$-graded ring with local units is necessarily nearly epsilon-strongly $G$-graded (see Lemma 2.16). Hence, [4] Thm. 3.1 by Abrams and Haefner follows from Theorem 10.1.

The following corollary is similar to a result by Öinert [33, Thm. 4.4]:

Corollary 10.3. Suppose that $G$ is torsion-free and that $S$ is nearly epsilon-strongly $G$-graded. If $S_e$ is prime, then $S$ is prime.
Example 10.4. Let $R$ be a unital ring, let $u$ be an idempotent of $R$, and let $\alpha : R \to uRu$ be a corner ring isomorphism. In this example we consider the corner skew Laurent polynomial ring $R[t_+, t_-, \alpha]$ which was introduced by Ara, Gonzalez-Barroso, Goodearl and Pardo in [5]. For the convenience of the reader we now briefly recall its definition: $R[t_+, t_-, \alpha]$ is the universal unital ring satisfying the following two conditions:

(a) there is a unital ring homomorphism $i : R \to R[t_+, t_-, \alpha]$;
(b) $R[t_+, t_-, \alpha]$ is the $R$-algebra satisfying the following equations for every $r \in R$:

$$t_- t_+ = 1, \quad t_+ t_- = i(u), \quad rt_- = t_- \alpha(r), \quad t_+ r = \alpha(r)t_+.$$ 

Assigning degrees $-1$ to $t_-$ and $1$ to $t_+$ turns $R[t_+, t_-, \alpha]$ into a $\mathbb{Z}$-graded ring with principal component $R$. By [24, Prop. 8.1], $R[t_+, t_-, \alpha]$ is nearly epsilon-strongly $\mathbb{Z}$-graded. Hence, if $R$ is prime, then it follows from Corollary [10.3] that $R[t_+, t_-, \alpha]$ is also prime. Of course, when $u = 1$ and $\alpha$ is the identity map, then $R[t_+, t_-, \alpha]$ is the familiar ring $R[t, t^{-1}]$.

11. Applications to $s$-unital strongly graded rings

In this section, we apply our results to $s$-unital strongly $G$-graded rings. Recall that, by Lemma [2.16], every $s$-unital strongly $G$-graded ring is nearly epsilon-strongly $G$-graded. Thus, by Theorem [1.3] we obtain the following $s$-unital generalization of Passman’s Theorem [1.1].

Corollary 11.1. Suppose that $S$ is an $s$-unital strongly $G$-graded ring. Then $S$ is not prime if and only if it has an NP-datum $(H, N, I, A, B)$ for which $A, B$ are both $H$-invariant.

11.1. Morita context algebras. Let $S$ be an $s$-unital strongly $G$-graded ring. For every $x \in G$ the canonical multiplication map $m_x : S_x \otimes_{S_e} S_{x^{-1}} \to S_e$, $a \otimes b \mapsto ab$ is an isomorphism of $S_e$-bimodules. Indeed, $m_x$ is well-defined and surjective, using that $S$ is strongly $G$-graded. Moreover, the injectivity is a consequence of the $s$-unitality. Noteworthily, by associativity of the multiplication, for every $x \in G$ we also have

$$m_x \otimes \text{id} = \text{id} \otimes m_{x^{-1}} : S_x \otimes_{S_e} S_{x^{-1}} \otimes_{S_e} S_x \to S_x$$

$$m_{x^{-1}} \otimes \text{id} = \text{id} \otimes m_x : S_{x^{-1}} \otimes_{S_e} S_x \otimes_{S_e} S_{x^{-1}} \to S_{x^{-1}}.$$ 

Thus, for every $x \in G$ we get a quintupel $(S_e, S_x, S_{x^{-1}}, m_x, m_{x^{-1}})$ which is usually referred to as a strict Morita context.

Next, let us consider an $s$-unital ring $R$ and a strict Morita context $(R, M, N, \mu_1, \mu_-)$, i.e. we have $R$-bimodules $M, N$ and $R$-bimodule isomorphisms

$$\mu_1 : M \otimes_R N \to R, \quad \mu_- : N \otimes_R M \to R$$

satisfying the mixed associativity conditions $\mu_1 \otimes \text{id} = \text{id} \otimes \mu_-$ and $\mu_- \otimes \text{id} = \text{id} \otimes \mu_1$. Furthermore, we assume that $RM = MR = M$ and $RN = NR = N$. We form a $\mathbb{Z}$-graded module $S$ by putting

$$S_n := \left\{ \begin{array}{ll} R & n = 0 \\ M \otimes_R^n & n > 0 \\ N \otimes_R^{-n} & n < 0. \end{array} \right.$$ 

We wish to turn $S$ into a $\mathbb{Z}$-graded ring. The product of two positively graded elements is just the usual tensor product $\otimes_R$ of tensor products of $M$’s, and similarly the product of two negatively graded elements is just the usual tensor product of $N$’s. To deal with products of mixed elements, we repeatedly make use of the maps $\mu_1$ and $\mu_-$. By the mixed associativity conditions, this multiplication becomes associative, and hence $S$ is a $\mathbb{Z}$-graded ring as desired.
In addition, as the maps $\mu_1$ and $\mu_{-1}$ are surjective, we may infer that $S$ is strongly $\mathbb{Z}$-graded. Clearly, $S$ is $s$-unital. Finally, Theorem 10.1 implies that if $R$ is $\mathbb{Z}$-prime, then $S$ is prime.

11.2. $s$-unital strongly graded matrix rings. In what follows, let $R$ be an $s$-unital ring. Let $M_{\mathbb{Z}}(R)$ denote the ring of infinite $\mathbb{Z} \times \mathbb{Z}$-matrices with only finitely many nonzero entries in $R$. For $r \in R$ and $i, j \in \mathbb{Z}$ we write $re_{i,j}$ for the matrix in $M_{\mathbb{Z}}(R)$ with $r$ in the $ij$th position and zeros elsewhere. We regard $M_{\mathbb{Z}}(R)$ as a $\mathbb{Z}$-graded ring with respect to

$$(7) \quad \deg(re_{i,j}) := i - j \quad \text{for all } i, j \in \mathbb{Z} \text{ and all nonzero } r \in R.$$

The corresponding homogeneous components of the $\mathbb{Z}$-grading are given by

$$(M_{\mathbb{Z}}(R))_k = \bigoplus_{i \in \mathbb{Z}} Re_{i+k,i}, \quad k \in \mathbb{Z}.$$

In particular, $(M_{\mathbb{Z}}(R))_0 = \bigoplus_{i \in \mathbb{Z}} Re_{i,i}$ is the main diagonal.

Lemma 11.2. The ring $M_{\mathbb{Z}}(R)$ is $s$-unital and strongly $\mathbb{Z}$-graded with respect to the grading defined by (7).

Proof. Put $S := M_{\mathbb{Z}}(R)$. By Proposition 2.1 and $s$-unitality of $R$, it follows that $S$ is $s$-unital and that $S_0S_n = S_nS_0 = S_n$, for every $n \in \mathbb{Z}$. Take $k \in \mathbb{Z}$. Since $R$ is $s$-unital, and hence idempotent, we get that $S_0S_{-k} = \langle \sum_{i \in \mathbb{Z}} Re_{i+k,i} \rangle = \sum_{i \in \mathbb{Z}} R^{2e_{i+k,i}}e_{i,i+k} = \sum_{i \in \mathbb{Z}} Re_{i+k,i+k} = S_0$. The claim now follows from Proposition 2.2. \qed

Corollary 11.3. The ring $M_{\mathbb{Z}}(R)$ is prime if and only if $R$ is prime.

Proof. Suppose that $R$ is not prime, i.e. there are nonzero ideals $A, B$ of $R$ such that $AB = \{0\}$. Then $M_{\mathbb{Z}}(A) \cdot M_{\mathbb{Z}}(B) = \{0\}$ which shows that $M_{\mathbb{Z}}(R)$ is not prime. Conversely suppose that $R$ is prime. Note that any ideal $I$ of $(M_{\mathbb{Z}}(R))_0$ of the form $I = \bigoplus_{i \in \mathbb{Z}} I_i e_{i,i}$ for some family of $R$-ideals $I_i$, $i \in \mathbb{Z}$, and it is $\mathbb{Z}$-invariant if and only if $I_i = I_0$ for every $i \in \mathbb{Z}$. Next, let $A, B$ be $\mathbb{Z}$-invariant ideals of $(M_{\mathbb{Z}}(R))_0$ such that $AB = \{0\}$. There are $R$-ideals $A_0, B_0$ such that $A = \bigoplus_{i \in \mathbb{Z}} A_0 e_{i,i}$ and $B = \bigoplus_{i \in \mathbb{Z}} B_0 e_{i,i}$. Since $AB = \{0\}$, we see that $A_0B_0 = \{0\}$ and thus $A_0 = \{0\}$ or $B_0 = \{0\}$ due to the primeness of $R$. Hence, $A = \{0\}$ or $B = \{0\}$. Consequently, $(M_{\mathbb{Z}}(R))_0$ is $\mathbb{Z}$-prime and hence $M_{\mathbb{Z}}(R)$ is prime by Theorem 10.1. \qed

Remark 11.4. The above result shows that primeness of the principal component is not a necessary condition for primeness of a strongly graded ring. Nevertheless, by Corollary 5.8, $G$-primeness of $S_{\mathbb{C}}$ is a necessary condition.

Now we fix $n \in \mathbb{N}$ and consider $M_n(R)$, the ring of $n \times n$-matrices with entries in $R$. The ring $M_n(R)$ comes equipped with a natural $\mathbb{Z}$-grading defined by

$$(8) \quad \deg(re_{i,j}) := i - j \quad \text{for all } i, j \in \{1, \ldots, n\} \text{ and all nonzero } r \in R.$$

Lemma 11.5. The ring $M_n(R)$ is nearly epsilon-strongly $\mathbb{Z}$-graded with respect to the grading defined by (8).

Proof. Put $S := M_n(R)$. Take $k \in \mathbb{Z}$ and $r \in R$. Note that for $i, j$ such that $i - j = k$, and $a, b, c \in R$ such that $abc = r$, we have $ae_{i,j}, ce_{i,j} \in S_k$, $be_{j,i} \in S_{-k}$ and $ae_{i,j}be_{j,i} = re_{i,j}$. Take $s \in S_k$. Then $s = \sum_{i - j = k} r_{i,j} e_{i,j} \in S_k$ for some $r_{i,j} \in R$. By Proposition 2.1 and $s$-unitality of $R$, there is $u \in R$ such that $ur_{i,j} = r_{i,j}u = r_{i,j}$ for all $i, j$. Put $v := \sum_{i - j = k} u e_{i,j} u e_{j,i} \in S_k S_{-k}$ and $w := \sum_{i - j = k} u e_{j,i} u e_{i,j} \in S_{-k} S_k$. Then $uv = s$ and $sw = s$. This shows that $S$ is nearly epsilon-strongly $\mathbb{Z}$-graded. \qed
Note that if \( R \) is prime, then \((M_n(R))_0\) is \(\mathbb{Z}\)-prime. Hence, by Corollary 11.6 and Lemma 11.5 we obtain the following \( s \)-unital generalization of a well-known result:

**Corollary 11.6** (cf. [21] Prop. 10.20). The ring \( M_n(R) \) is prime if and only if \( R \) is prime.

The \(\mathbb{Z}\)-grading on \( M_n(R) \) defined above induces a \(\mathbb{Z}/n\mathbb{Z}\)-grading on \( M_n(R) \) (see Section 2.5). By Lemma 11.5 and Proposition 2.20 this turns \( M_n(R) \) into a nearly epsilon-strongly \(\mathbb{Z}/n\mathbb{Z}\)-graded ring. By using an argument similar to the one in the proof of Lemma 11.2 it is not difficult to see that this grading is, in fact, strong. Hence, Corollary 11.6 is applicable but presently it is not clear to the authors how to use it to prove Corollary 11.6.

### 12. Applications to \( s \)-unital skew group rings

Connell [10] famously gave a characterization of when a unital group ring \( R[G] \) is prime. In this section, we generalize and recover his result from our main theorem. More precisely, we describe when an \( s \)-unital group ring \( R[G] \) is prime.

Let \( R \) be a (possibly non-unital) ring and let \( \alpha : G \to \text{Aut}(R) \) be a group homomorphism. We define the **skew group ring** \( R \star_\alpha G \) as the set of all formal sums \( \sum_{x \in G} r_x \delta_x \) where \( \delta_x \) is a symbol for each \( x \in G \) and \( r_x \in R \) is zero for all but finitely many \( x \in G \). Addition on \( R \star_\alpha G \) is defined in the natural way and multiplication is defined by linearly extending the rules \( r_x r_y \delta_y = r_x (r_y \delta_y) \), for all \( r, r' \in R \) and \( x, y \in G \). This yields an associative ring structure on \( R \star_\alpha G \). Moreover, \( S = R \star_\alpha G \) is canonically \( G \)-graded by putting \( S_x := R \delta_x \) for every \( x \in G \). If \( \alpha_x = \text{id}_R \) for every \( x \in G \), then we simply write \( R[G] \) for \( R \star_\alpha G \) and call it a **group ring**. Note that \( R \star_\alpha G \) is a so-called **partial skew group ring** (see Section 13).

**Proposition 12.1.** Suppose that \( R \) is a ring and that \( \alpha : G \to \text{Aut}(R) \) is a group homomorphism. The following assertions are equivalent:

(a) \( R \) is idempotent;
(b) \( R \star_\alpha G \) is strongly \( G \)-graded;
(c) \( R \star_\alpha G \) is symmetrically \( G \)-graded.

**Proof.** (a)\( \Rightarrow \) (b): Suppose that \( R \) is idempotent, i.e. \( R^2 = R \). Then for all \( x, y \in G \) we have \( (R \delta_x)(R \delta_y) = R \alpha_x(R \delta_y) \delta_{xy} = R \delta_{xy} \). In other words, \( R \star_\alpha G \) is strongly \( G \)-graded.

(b)\( \Rightarrow \) (c): This holds in general for strongly \( G \)-graded rings (see [23] Prop. 4.45).

(c)\( \Rightarrow \) (a): This holds in general for symmetrically \( G \)-graded rings (see [23] Prop. 4.47). \( \square \)

**Proposition 12.2.** Suppose that \( R \) is a ring and that \( \alpha : G \to \text{Aut}(R) \) is a group homomorphism. The following assertions are equivalent:

(a) \( R \) is \( s \)-unital;
(b) \( R \star_\alpha G \) is \( s \)-unital strongly \( G \)-graded;
(c) \( R \star_\alpha G \) is nearly epsilon-strongly \( G \)-graded.

**Proof.** (a)\( \Rightarrow \) (b): Suppose that \( R \) is \( s \)-unital. In particular, \( R \) is idempotent. Hence, \( R \star_\alpha G \) is strongly \( G \)-graded by Proposition 12.1. It is easy to see that \( R \star_\alpha G \) is \( s \)-unital.

(b)\( \Rightarrow \) (c): This follows from Lemma 2.16

(c)\( \Rightarrow \) (a): This holds for any nearly epsilon-strongly graded ring (see Proposition 2.13). \( \square \)

**Example 12.3.** In this example we consider the \( s \)-unital ring \( M_n(\mathbb{R}) \) of \( \mathbb{R} \times \mathbb{R} \)-matrices with only finitely many nonzero entries in \( \mathbb{R} \). Recall that the group \( \text{SO}_3(\mathbb{R}) \) of rotations in \( \mathbb{R}^3 \) contains a subgroup \( F \) isomorphic to free group of rank 2 (see e.g. [17] [42]). For every \( x \in F \subseteq \text{SO}_3(\mathbb{R}) \) we may define a diagonal matrix \( \text{diag}(x, x, x, \ldots) \) which is row-finite.
and column-finite but does not belong to $M_N(\mathbb{R})$. We thus obtain a group homomorphism $\alpha : F \to \text{Aut}(M_N(\mathbb{R}))$ by putting

$$\alpha_x(a) := \text{diag}(x, x, x, \ldots) a \text{ diag}(x^{-1}, x^{-1}, x^{-1}, \ldots)$$

for $x \in F$ and $a \in M_N(\mathbb{R})$. Since $M_N(\mathbb{R})$ is simple and $F$ is torsion-free, it follows from Corollary 10.3 that the $s$-unital skew group ring $M_N(\mathbb{R}) \star_\alpha F$ is prime.

We proceed to prove Theorem 1.3 by using our main theorem:

**Theorem 12.4.** Suppose that $R$ is an $s$-unital ring. Then the group ring $R[G]$ is prime if and only if $R$ is prime and $G$ has no non-trivial finite normal subgroup.

**Proof.** We prove the converse statement: $R[G]$ is not prime if and only if $R$ is not prime or $G$ has a non-trivial finite normal subgroup. By Proposition 12.2, (a)$\Leftrightarrow$(c) in Theorem 1.3 holds for $S = R[G]$. In other words, the group ring $R[G]$ is not prime if and only if it has a balanced NP-datum. We prove that the $G$-graded ring $R[G]$ has a balanced NP-datum if and only if $R$ is not prime or $G$ has a non-trivial finite normal subgroup. First note that for any ideal $I$ of $R$ we have $I^2 = R\delta_{x-1}IR\delta_x = RIR\delta_e = I\delta_e$ for every $x \in G$. In particular, every ideal of $R$ is $G$-invariant.

Suppose that $(H, N, I, \tilde{A}, \tilde{B})$ is a balanced NP-datum for $R[G]$.

Case I: $H = G$. Note that $N < H = G$ is a finite normal subgroup of $G$. Condition (NP4) proves that $R[N]$ is not prime. Then either $N = \{e\}$ and $R$ is not prime or there exists a non-trivial finite normal subgroup $N$ of $G$.

Case 2: $H \subsetneq G$. Note that condition (NP2) implies that there is a nonzero ideal $I$ of $R$ such that $I^2 = \{0\}$. Thus $R$ is not prime.

Now we prove the converse statement.

Case I: $R$ is not prime. There are nonzero ideals $\tilde{A}, \tilde{B}$ of $R$ such that $\tilde{A}\tilde{B} = \{0\}$. This implies that $A R\delta_x \tilde{B} = R\delta_x \tilde{A}\tilde{B} = \{0\}$ for every $x \in G$. Therefore, $\tilde{A} \cdot R[G] \cdot \tilde{B} = \{0\}$. We note that $(G, \{e\}, R, \tilde{A}, \tilde{B})$ is a balanced NP-datum.

Case II: there exists a non-trivial finite normal subgroup $N$ of $G$.

Consider $H := G$ and $I := R$. Pick a nonzero $a \in R$. Let $A$ be the ideal of $S_N$ generated by the element $\sum_{n \in N} a\delta_n$ and let $\tilde{B}$ be the ideal of $S_N$ generated by the set $\{r\delta_n - r\delta_e \mid n \in N, r \in R\}$. Since $N$ is non-trivial, it follows that $\tilde{A}$ and $\tilde{B}$ are nonzero ideals of $S_N$. Next, let $t \in R$, $x \in G$ and $n_1 \in N$. Then, since $N$ is finite and normal in $G$,

$$\left(\sum_{n \in N} a\delta_n\right) t\delta_x (r\delta_{n_1} - r\delta_e) = \left(\sum_{n \in N} at\delta_{nx}\right) (r\delta_{n_1} - r\delta_e) = \left(\sum_{n \in N} at\delta_{x n}\right) (r\delta_{n_1} - r\delta_e)$$

$$= a\delta_x \left(\sum_{n \in N} t\delta_n\right) (r\delta_{n_1} - r\delta_e) = a\delta_x \left(\sum_{n \in N} tr\delta_{n_1} - \sum_{n \in N} tr\delta_{ne}\right)$$

$$= a\delta_x \left(\sum_{n \in N} tr\delta_n - \sum_{n \in N} tr\delta_n\right) = 0.$$  

This shows that $\tilde{A} \cdot R[G] \cdot \tilde{B} = \{0\}$. Hence, $(H, N, I, \tilde{A}, \tilde{B})$ is a balanced NP-datum. □

**Remark 12.5.** Note that Theorem 12.4 applies to $s$-unital group rings $R[G]$ which are not necessarily unital. Hence, this application shows that our results indeed reach farther than Passman’s results [38, 39, 40] which are only concerned with unital rings.
Remark 12.6. The above result can not be generalized to s-unital (unital) skew group rings. Indeed, neither primeness of $R$ nor the non-existence of non-trivial finite normal subroups of $G$ are necessary conditions for primeness of an s-unital skew group ring $R \star_{\alpha} G$. To see this, consider the matrix algebra $M_4(\mathbb{R}) \cong \mathbb{R}^4 \star_{\alpha} \mathbb{Z}/4\mathbb{Z}$ as a unital skew group ring. It is well-known that $M_4(\mathbb{R})$ is prime, but $\mathbb{R}^4$ is not prime and $\mathbb{Z}/4\mathbb{Z}$ contains a non-trivial finite normal subgroup. Note, however, that in this case $\mathbb{R}^4$ is actually $\mathbb{Z}/4\mathbb{Z}$-prime.

Example 12.7. Suppose that $G$ is torsion-free and let $F(G, \mathbb{C})$ be the algebra of all complex-valued functions on $G$ with finite support, under pointwise addition and multiplication. Note that $F(G, \mathbb{C})$ is s-unital. We define a map $\alpha : G \rightarrow \text{Aut}(F(G, \mathbb{C}))$ by putting $\alpha_x(f)(y) := f(x^{-1}y)$ for all $x, y \in G$ and $f \in F(G, \mathbb{C})$. Clearly, $F(G, \mathbb{C})$ is $G$-prime. Using Theorem 10.1 we get that $F(G, \mathbb{C}) \star_{\alpha} G$ is prime.

Remark 12.8. Consider the non-unital group ring $R[G]$ where $R := 2\mathbb{Z}$ and $G := \mathbb{Z}$. Note that $R$ is not s-unital and hence $R[G]$ is not nearly epsilon-strongly $G$-graded (see Proposition 2.13). It is, however, non-degenerately $G$-graded. In fact, it is not difficult to see that $R[G]$ is a domain and hence prime. From this we easily see that the equivalences (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c)$\Leftrightarrow$(d)$\Leftrightarrow$(e) in Theorem 1.3 hold for $R[G]$. This example suggests that it might be possible to generalize Theorem 1.3.

13. Applications to crossed products defined by partial actions

A significant development in the study of $C^*$-algebras was the introduction of the notion of a partial action by Exel [15]. Various algebraic analogues of this notion were developed and studied during the last two decades (see e.g. [7, 12, 13]).

In this section, we apply our main theorem to obtain results on primeness of s-unital partial skew group rings (see Section 13.1) and of unital partial crossed products (see Section 13.2). We also apply our results to some particular examples of partial skew group rings associated with partial dynamical systems (see Section 13.3).

13.1. Partial skew group rings. Recall that a partial action of $G$ on an s-unital ring $R$ (see [12, p. 1932]) is a pair $\{\{\alpha_g\}_{g \in G}, \{D_g\}_{g \in G}\}$, where for all $g, h \in G$, $D_g$ is a (possibly zero) s-unital ideal of $R$, $\alpha_g : D_{g^{-1}} \rightarrow D_g$ is a ring isomorphism. We require that the following conditions hold for all $g, h \in G$:

(P1) $\alpha_e = \text{id}_R$;
(P2) $\alpha_g(D_{g^{-1}}D_h) = D_gD_{gh}$;
(P3) if $r \in D_{g^{-1}}D_{(gh)^{-1}}$, then $\alpha_g(\alpha_h(r)) = \alpha_{gh}(r)$.

Given a partial action of $G$ on $R$, we can form the s-unital partial skew group ring $R \star_{\alpha} G := \bigoplus_{g \in G} D_g \delta_g$ where the $\delta_g$’s are formal symbols. For $g, h \in G, r \in D_g$ and $r' \in D_h$ the multiplication is defined by the rule:

$$(r\delta_g)(r'\delta_h) = \alpha_g(\alpha_{g^{-1}}(r)r')\delta_{gh}$$

It can be shown that $R \star_{\alpha} G$ is an associative ring (see e.g. [12, Cor. 3.2]). Moreover, $S := R \star_{\alpha} G$ is canonically $G$-graded by putting $S_g := D_g \delta_g$ for every $g \in G$.

Proposition 13.1. The canonical $G$-grading on $R \star_{\alpha} G$ is nearly epsilon-strong.

Proof. Take $g \in G$. Note that

$$S_gS_{g^{-1}} = D_g\delta_gD_{g^{-1}}\delta_{g^{-1}} = \alpha_g(\alpha_{g^{-1}}(D_g)D_{g^{-1}})\delta_e = \alpha_g(D_{g^{-1}}D_{g^{-1}})\delta_e = \alpha_g(D_{g^{-1}})\delta_e = D_g\delta_e$$
and hence

\[ S_gS_{g^{-1}}S_g = (S_gS_{g^{-1}})S_g = D_g \delta_e D_g \delta_g = D_g^2 \delta_g = D_g \delta_g = S_g. \]

This shows that the \( G \)-grading is symmetrical and that \( S_gS_{g^{-1}} \) is \( s \)-unital for every \( g \in G \).

By Proposition 2.11 the desired conclusion follows. \( \square \)

**Remark 13.2.** We will identify \( R \) with \( R \delta_e \) via the canonical isomorphism.

**Definition 13.3.** Let \( H \) be a subgroup of \( G \). An ideal \( I \) of \( R \) is called \( H \)-invariant if \( \alpha_h(ID_{h^{-1}}) \subseteq I \) for every \( h \in H \). The ring \( R \) is called \( G \)-prime if for all \( G \)-invariant ideals \( I, J \) of \( R \), we have \( I \{ 0 \} \) or \( J \{ 0 \} \), whenever \( IJ \{ 0 \} \).

**Remark 13.4.** Consider \( S := R \star G \) with its canonical \( G \)-grading.

(a) Let \( H \) be a subgroup of \( G \). Note that, for \( h \in H \), we have

\[ I^h \subseteq I \iff D_h^{-1} \delta_h^{-1} \cdot I \cdot D_h \delta_h \subseteq I \delta_e \iff \alpha_{h^{-1}}(\alpha_h(ID_h)\delta_e) \subseteq I \delta_e \]

\[ \iff \alpha_{h^{-1}}(D_hID_h) \delta_e \subseteq I \delta_e \iff \alpha_{h^{-1}}(D_hID_h) \subseteq I \iff \alpha_{h^{-1}}(ID_h) \subseteq I. \]

This shows that \( G \)-invariance in the sense of Definition 13.3 is equivalent to \( G \)-invariance defined by the \( G \)-grading (see Definition 13.3).

(b) By (a) we note that \( R \) is \( G \)-prime if and only if \( S_e \) is \( G \)-prime.

**Theorem 13.5.** Suppose that \( G \) is torsion-free and that \( R \star G \) is an \( s \)-unital partial skew group ring. Then \( R \star G \) is prime if and only if \( R \) is \( G \)-prime.

**Proof.** This follows from Proposition 13.1, Theorem 10.1 and Remark 13.4(b). \( \square \)

We proceed to characterize prime \( s \)-unital partial skew group rings for general groups.

**Lemma 13.6.** Suppose that \((\{a_g\}_{g \in G}, \{D_g\}_{g \in G})\) is a partial action of \( G \) on \( R \), and that \( I \) is an ideal of \( R \). For any subgroup \( H \) of \( G \), the following holds:

\[ \alpha_h(ID_{h^{-1}}) \subseteq I, \quad \forall h \in H \iff \alpha_h(ID_{h^{-1}}) = ID_h, \quad \forall h \in H \]

**Proof.** Take \( h \in G \).

\( \iff \): Clear, since \( ID_h \subseteq I \).

\( \Rightarrow \): Note that \( I \cap D_h = I \cdot D_h \), by \( s \)-unitality of \( D_h \). Thus, \( \alpha_h(ID_{h^{-1}}) \subseteq I \) implies \( \alpha_h(ID_{h^{-1}}) \subseteq I \cap D_h = ID_h \). By applying \( \alpha_{h^{-1}} \) to both sides, and using that \( h \) is arbitrary, we get \( ID_{h^{-1}} \subseteq \alpha_{h^{-1}}(ID_h) \subseteq ID_{h^{-1}} \). Hence, \( \alpha_{h^{-1}}(ID_h) = ID_{h^{-1}} \). \( \square \)

**Theorem 13.7.** The \( s \)-unital partial skew group ring \( R \star G \) is not prime if and only if there are:

(i) subgroups \( N < H \subseteq G \) with \( N \) finite,

(ii) an ideal \( I \) of \( R \) such that

- \( \alpha_h(ID_{h^{-1}}) = ID_h \) for every \( h \in H \),
- \( ID_g \cdot \alpha_g(ID_{g^{-1}}) = \{ 0 \} \) for every \( g \in G \setminus H \), and

(iii) nonzero ideals \( \tilde{A}, \tilde{B} \) of \( R \star G \) such that \( \tilde{A}, \tilde{B} \subseteq I \delta_e(R \star G) \) and \( \tilde{A} \cdot D_h \delta_h \cdot \tilde{B} = \{ 0 \} \) for every \( h \in H \).

**Proof.** By Proposition 13.1 we may apply Theorem 1.3 to \( S := R \star G \). For \( g \in G \), we get

\[ I^g \cdot I = \{ 0 \} \iff D_g^{-1} \delta_g^{-1} \cdot I \cdot D_g \delta_g \cdot I \delta_e = \{ 0 \} \iff \alpha_{g^{-1}}(\alpha_g(D_{g^{-1}}) \cdot ID_g) \delta_e \cdot I \delta_e = \{ 0 \} \]

\[ \iff (\alpha_{g^{-1}}(D_{g}ID_g) \cdot I) \delta_e = \{ 0 \} \iff \alpha_{g^{-1}}(D_{g}ID_g) \cdot I = \{ 0 \} \]

\[ \iff \alpha_{g^{-1}}(D_{g}ID_g) \cdot ID_{g^{-1}} = \{ 0 \} \iff D_gID_g \cdot \alpha_g(ID_{g^{-1}}) = \{ 0 \}. \]
Using that $I, D_g$ are ideals of $R$ and that $D_g$ is $s$-unital, we get that $D_gID_g \subseteq ID_g \subseteq D_g(ID_g)$. Hence, $D_gID_g = ID_g$. We conclude that $I^g \cdot I = \{0\}$ if and only if $ID_g \cdot \alpha_g(ID_g^{-1}) = \{0\}$. The desired conclusion now follows by Remark 13.4(a) and Lemma 13.6. □

13.2. **Unital partial crossed products.** Recall that a unital twisted partial action of $G$ on a unital ring $R$ (see [32, p. 2]) is a triple $(\{\alpha_g\}_{g \in G}, \{D_g\}_{g \in G}, \{w_{g,h}\}_{(g,h) \in G \times G})$, where for all $g, h \in G$, $D_g$ is a unital ideal of $R$, $\alpha_g : D_g^{-1} \to D_g$ is a ring isomorphism and $w_{g,h}$ is an invertible element in $D_gD_h$. Let $1_g \in \mathbb{Z}(R)$ denote the (not necessarily nonzero) multiplicative identity element of the ideal $D_g$. We require that the following conditions hold for all $g, h \in G$:

1. **(UP1)** $\alpha_e = \text{id}_R$;
2. **(UP2)** $\alpha_g(D_g^{-1}D_h) = D_gD_h$;
3. **(UP3)** if $r \in D_{g^{-1}}D_{(gh)^{-1}}$, then $\alpha_g(\alpha_h(r)) = w_{g,h}\alpha_{gh}(r)w_{g,h}^{-1}$;
4. **(UP4)** $w_{e,g} = w_{g,e} = 1_g$;
5. **(UP5)** if $r \in D_g^{-1}D_{h}D_{hl}$, then $\alpha_g(rw_{hl})w_{g,hl} = \alpha_g(r)w_{g,h}w_{gh,l}$.

Given a unital twisted partial action of $G$ on $R$, we can form the unital partial crossed product $R \ast^w G := \bigoplus_{g \in G} D_g\delta_g$ where the $\delta_g$’s are formal symbols. For $g, h \in G, r \in D_g$ and $r' \in D_h$ the multiplication is defined by the rule:

$$(r\delta_g)(r'\delta_h) = r\alpha_g(r'1_{g^{-1}})w_{g,h}\delta_{gh}$$

It can be shown that $R \ast^w G$ is an associative ring (see e.g. [13, Thm. 2.4]). Moreover, Nystedt, Öinert and Pinedo established in [32, Thm. 35] that its natural $G$-grading is epsilon-strong, and in particular nearly epsilon-strong. Thus, Theorem 1.3 is applicable.

**Remark 13.8.** (a) Let $H$ be a subgroup of $G$. Note that an ideal $I$ of $R$ is $H$-invariant (in the sense of Definition 13.3) if and only if $\alpha_h(I1_h^{-1}) \subseteq I$ for every $h \in H$. 

(b) We also define $G$-primeness of $R$ according to Definition 13.3. By a computation, similar to the one in Remark 13.4, we note that $G$-primeness of $R$ is equivalent to $G$-primeness of $S_e$.

The next result partially generalizes Theorem 13.5.

**Theorem 13.9.** Suppose that $G$ is torsion-free and that $R \ast^w G$ is a unital partial crossed product. Then $R \ast^w G$ is prime if and only if $R$ is $G$-prime.

**Proof.** Using the fact that unital partial crossed products are epsilon-strongly graded (see [32, Thm. 35]), the desired conclusion follows from Theorem 10.1 and Remark 13.8(b). □

The proof of the following result is similar to the proof of Theorem 13.5 and is therefore omitted.

**Theorem 13.10.** The unital partial crossed product $R \ast^w G$ is not prime if and only if there are:

(i) subgroups $N < H \subseteq G$ with $N$ finite,
(ii) an ideal $I$ of $R$ such that
   - $\alpha_h(I1_h^{-1}) = I1_h$ for every $h \in H$,
   - $I \cdot \alpha_g(Ig^{-1}) = \{0\}$ for every $g \in G \setminus H$, and
(iii) nonzero ideals $\mathcal{A}, \mathcal{B}$ of $R \ast^w N$ such that $\mathcal{A}, \mathcal{B} \subseteq I \cdot (R \ast^w N)$ and $\mathcal{A} \cdot 1_h\delta_h \cdot \mathcal{B} = \{0\}$ for every $h \in H$. 
13.3. Partial dynamical systems. In this section we consider several examples of partial skew group rings coming from a particular type of partial dynamical system (cf. [14]).

Let $X$ be a topological space and let $A_1, A_2, B_1, B_2$ be subspaces of $X$. Furthermore, let $h_1: A_1 \to B_1$ and $h_2: A_2 \to B_2$ be homeomorphisms. For the remainder of this section, $G$ denotes the free group $\mathbb{F}_2 = \langle g_1, g_2 \rangle$. For $g \in G$ we define,

$$
\theta_g = \left\{ \begin{array}{ll}
  h_j & \text{if } g = g_j \\
  h_j^{-1} & \text{if } g = g_j^{-1} \\
  \theta_{g_{k_1}} \circ \cdots \circ \theta_{g_{k_m}} & \text{if } g = g_{k_1} \cdots g_{k_m} \text{ is in reduced form,}
\end{array} \right.
$$

where $\circ$ denotes partial function composition. Moreover, we let $X_g$ denote the domain of the function $\theta_{g^{-1}}$. We thus obtain a partial action of $G$ on the space $X$ which we denote by $(\{\theta_g\}_{g \in G}, \{X_g\}_{g \in G})$. This induces a partial action of $G$ on the $s$-unital ring $R := C_c(X)$, of continuous compactly supported complex-valued functions on $X$, by putting $D_g := C_c(X_g)$ and defining $\alpha_g: D_{g^{-1}} \to D_g$ by $\alpha_g(f) := f \circ \theta_{g^{-1}}$ for every $g \in G$. Therefore, we may define the $s$-unital partial skew group ring $S := R \star \alpha G$.

Example 13.11. (a) First, we consider $X = \mathbb{R}$ with

- $h_1: [0, \infty) \to (-\infty, 0], \ t \mapsto -t$, and
- $h_2: \mathbb{R} \to \mathbb{R}, \ t \mapsto t + 1$.

It is not difficult to see that $R = C_c(\mathbb{R})$ is not $G$-prime. Hence, by Theorem 13.10 $C_c(\mathbb{R}) \star \alpha G$ is not prime.

(b) Now we consider $X = \mathbb{R}$ with

- $h_1: [0, \infty) \to [0, \infty), \ t \mapsto 2t$, and
- $h_2: \mathbb{R} \to \mathbb{R}, \ t \mapsto t + 1$.

It is not difficult to see that $R = C_c(\mathbb{R})$ is $G$-prime. Hence, by Theorem 13.10 $C_c(\mathbb{R}) \star \alpha G$ is prime.

Example 13.12. Now we consider $X$ with its discrete topology.

(a) Consider $X = \{x_1, x_2, x_3, x_4\}$ with

- $h_1: \{x_1, x_2\} \to \{x_3, x_4\}$ given by $h_1(x_1) = x_3$ and $h_1(x_2) = x_4$, and
- $h_2: \{x_1, x_3\} \to \{x_2, x_4\}$ given by $h_2(x_1) = x_2$ and $h_2(x_3) = x_4$.

Note that the ideals of $C_c(X) \cong \mathbb{C}^4$ correspond bijectively to the $2^4$ subsets of $X$. For arbitrary elements $x, y, z \in X$ there is $g \in G$ such that $\theta_g(x) = y$. From this we conclude that $R = C_c(X)$ is $G$-prime. Hence, by Theorem 13.10 $C_c(X) \star \alpha G$ is prime.

(b) Consider $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ with

- $h_1: \{x_1, x_2\} \to \{x_3, x_4\}$ given by $h_1(x_1) = x_3$ and $h_1(x_2) = x_4$, and
- $h_2: \{x_1, x_3\} \to \{x_2, x_4\}$ given by $h_2(x_1) = x_2$ and $h_2(x_3) = x_4$.

The nonzero ideals $J_1 := C_c(\{x_5\})$ and $J_2 := C_c(\{x_6\})$ of $C_c(X)$ are $G$-invariant. Clearly, $J_1J_2 = \{0\}$ and hence $R = C_c(X)$ is not $G$-prime. By Theorem 13.10 $C_c(X) \star \alpha G$ is not prime.

14. Applications to Leavitt path algebras

In this section, we use our main theorem to obtain a characterization of prime Leavitt path algebras with coefficients in an arbitrary, possibly non-commutative, unital ring (see Theorem 14.12). Our result generalizes previous results by Abrams, Bell and Rangaswamy [3 Thm. 1.4], and Larki [25 Prop. 4.5].
The Leavitt path algebra $L_K(E)$ over a field $K$ associated with a directed graph $E$ was introduced by Ara, Moreno and Pardo in [6] and independently by Abrams and Aranda Pino in [2]. These algebras are algebraic analogues of graph $C\star$-algebras. For a thorough account of the history and theory of Leavitt path algebras, we refer the reader to the excellent monograph [1]. Recall that a directed graph $E$ is a tuple $(E^0, E^1, s, r)$ where $E^0$ is the set of vertices, $E^1$ is the set of edges and $s: E^1 \to E^0$ and $r: E^1 \to E^0$ are maps specifying the source respectively range of each edge. For an arbitrary $v \in E^0$, the set $s^{-1}(v) = \{ e \in E^1 \mid s(e) = v \}$ is the set of edges emitted from $v$. If $s^{-1}(v) = \emptyset$, then $v$ is called a sink. If $s^{-1}(v)$ is an infinite set, then $v$ is called an infinite emitter. A vertex that is neither a sink nor an infinite emitter is called regular. A path in $E$ is a series of edges $\alpha := f_1 f_2 \ldots f_n$ such that $r(f_i) = s(f_{i+1})$ for $i \in \{1, \ldots, n-1\}$, and such a path has length $n$ which we denote by $|\alpha|$. By convention, we consider a vertex to be a path of length zero. The set of all paths in $E$ is denoted by $E^\ast$.

Leavitt path algebras with coefficients in a commutative unital ring was introduced by Tomforde [43] and further studied in [20]. A further generalization was studied by Hazrat [18], and Nystedt and Öinert [31]. Following their lead, we consider Leavitt path algebras with coefficients in a general (possibly non-commutative) unital ring:

**Definition 14.1.** Let $E$ be a directed graph and let $R$ be a unital ring. The Leavitt path algebra of the graph $E$ with coefficients in $R$, denoted by $L_R(E)$, is the free associative $R$-algebra generated by the symbols $\{v \mid v \in E^0\} \cup \{f \mid f \in E^1\} \cup \{f^\ast \mid f \in E^1\}$ subject to the following relations:

(a) $vw = \delta_{v,w}v$ for all $v, w \in E^0$;

(b) $s(f)f = fr(f) = f$ for every $f \in E^1$;

(c) $r(f)f^\ast = f^\ast s(f) = f^\ast$ for every $f \in E^1$;

(d) $f^\ast f' = \delta_{f,f'}r(f)$ for all $f, f' \in E^1$;

(e) $\sum_{f \in E^1, s(f) = v} ff^\ast = v$ for every $v \in E^0$ for which $0 < |s^{-1}(v)| < \infty$.

We let every element of $R$ commute with the generators.

**Remark 14.2.** By (a), $\{v \mid v \in E^0\}$ is a set of pairwise orthogonal idempotents in $L_R(E)$.

In the following example, we use Theorem 4.7 to describe the $\mathbb{Z}$-invariant ideals of $L_K(E)$.

**Example 14.3.** Let $K$ be a field and let $E$ be a row-finite directed graph. Recall that there exists a bijection between graded ideals of the Leavitt path algebra $L_K(E)$ and hereditary subsets of $E^0$ (see [1, Thm. 2.5.9]). Furthermore, since $L_K(E)$ is naturally nearly epsilon-strongly $\mathbb{Z}$-graded (see [31, Thm. 30]), we can apply Theorem 4.7 to infer that there is a bijection between hereditary subsets of $E^0$ and $\mathbb{Z}$-invariant ideals of $(L_K(E))_0$. More precisely, we obtain the following explicit description of the $\mathbb{Z}$-invariant ideals of $(L_K(E))_0$:

$$\{I(H) \mid H \subseteq E^0 \text{hereditary vertex set}\},$$

where $I(H)$ denotes the ideal of $(L_K(E))_0$ generated by the elements $v \in H$.

Recall that $L_R(E)$ comes equipped with a canonical $\mathbb{Z}$-grading defined by $\deg(f) := 1$ and $\deg(f^\ast) := -1$ for every $f \in E^1$, and $\deg(v) := 0$ for every $v \in E^0$ (cf. [1, Cor. 2.1.5]). In the sequel, the following result will become useful:

**Proposition 14.4** (Nystedt and Öinert [31]). Suppose that $E$ is a directed graph and that $R$ is a unital ring. Consider $L_R(E)$ with its canonical $\mathbb{Z}$-grading.

(a) $L_R(E)$ is nearly epsilon-strongly $\mathbb{Z}$-graded.
If $E$ is finite, then $L_{R}(E)$ is epsilon-strongly $\mathbb{Z}$-graded.

In order to begin understanding when a Leavitt path algebra is prime, we consider a few examples.

**Example 14.5.** Let $R$ be a unital ring and let $E_1$ be the directed graph below:

$$E_1 : \bullet_e$$

In this case, $L_{R}(E_1) = vR \cong R$ is prime if and only if $R$ is prime.

**Example 14.6.** Let $R$ be a unital ring and let $E_2$ be the directed graph below:

$$E_2 : \bullet_{v_1} \rightarrow \bullet_{v_2}$$

We have $L_{R}(E_2) = v_1 R + v_2 R \cong R \oplus R$. Note that $v_1 R, v_2 R$ are nonzero ideals of $L_{R}(E_2)$ such that $(v_1 R)(v_2 R) = \{0\}$. Thus, $L_{R}(E_2)$ is never prime, for any ring $R$.

From the above examples, it is clear that a criterion for primeness of $L_{R}(E)$ must depend on properties of both the coefficient ring $R$ and the graph $E$. To describe such a criterion, we need to introduce a preorder $\geq$ on the set of vertices $E^0$ of the directed graph $E$ in the following way: We write $u \geq v$ if there is a path (possibly of length zero) from $u$ to $v$. Note that $v \geq v$ for every $v \in E^0$, i.e. the preorder is reflexive. Transitivity of the preorder follows by concatenating the paths.

**Definition 14.7.** A directed graph $E$ is said to satisfy condition (MT-3) if the above defined preorder $\geq$ is downward directed, i.e. if for every pair of vertices $u, v \in E^0$, there is some $w \in E^0$ such that $u \geq w$ and $v \geq w$.

The graph $E_2$ in Example 14.6 does not satisfy condition (MT-3) since there is no vertex $u$ such that $v_1 \geq u$ and $v_2 \geq u$. The next example shows a graph satisfying condition (MT-3):

**Example 14.8.** Let $R$ be a unital ring and let $E_3$ be the directed graph below:

$$E_3 : \bullet_{v_1} \rightarrow \bullet_{v_2}$$

$E_3$ satisfies condition (MT-3). Indeed for $v_1, v_2$ we have $v_1 \geq v_2$ and $v_2 \geq v_2$. A computation yields $L_{R}(E_3) \cong M_2(R)$ (see [24, Exp. 2.6]). By Corollary 11.6, it follows that $L_{R}(E_3)$ is prime if and only if $R$ is prime.

In the case when $K$ is a field, Abrams, Bell and Rangaswamy have shown that $L_{K}(E)$ is prime if and only if $E$ satisfies condition (MT-3) (see [3, Thm. 1.4]). For Leavitt path algebras with coefficients in a commutative unital ring, the following generalization was proved by Larki [25, Prop. 4.5]:

**Proposition 14.9.** Suppose that $E$ is a directed graph and that $R$ is a unital commutative ring. Then $L_{R}(E)$ is prime if and only if $R$ is an integral domain and $E$ satisfies condition (MT-3).

We aim to generalize Proposition 14.9 to Leavitt path algebras with coefficients in a general (possibly non-commutative) unital ring. Since Leavitt path algebras are nearly epsilon-strongly $\mathbb{Z}$-graded (see Proposition 14.4), we will be able to obtain this generalization as a corollary to Theorem 1.3. We begin with the following result:

**Proposition 14.10.** Suppose that $E$ is a directed graph and that $R$ is a unital ring. Consider the Leavitt path algebra $S = L_{R}(E)$. The following assertions hold:
(a) There exist $v, w \in E^0$ such that $SvSwS = \{0\}$ if and only if $E$ does not satisfy condition (MT-3).

(b) If $R$ is prime and there exist nonzero $r, s \in R$ and $v, w \in E^0$ such that $SrvSwS = \{0\}$, then $E$ does not satisfy condition (MT-3).

Proof. (a): Suppose that $E$ does not satisfy condition (MT-3). There exist $v, w \in E^0$ such that for every $y \in E^0$ we have $v \not\geq y$ or $w \not\geq y$. Take a monomial $r\alpha\beta^*$ in $L_R(E)$. From the properties of $v$ and $w$, it follows that $vbr\alpha\beta^*w = 0$. Therefore $SvSwS = \{0\}.$

Now suppose that $E$ satisfies condition (MT-3). Take $v, w \in E^0$. There exist $y \in E^0$ and paths $\alpha, \beta$ from $v$ to $y$ and from $w$ to $y$, respectively. Then $SvSwS \supseteq v\cdot v\cdot \alpha\beta^*w\cdot w = \alpha\beta^* \neq 0$.

(b): Suppose that $R$ is prime and that there exist nonzero $r, s \in R$ and $v, w \in E^0$ such that $SrvSwS = \{0\}$. Let $P = RrR$ and $Q = RsR$. Then $P$ and $Q$ are nonzero ideals of $R$. Hence, from primeness of $R$ it follows that $PQ$ is a nonzero ideal of $R$. Take $p_i \in P$ and $q_i \in Q$, for $i \in \{1, \ldots, n\}$, such that $\sum_{i=1}^{n} p_i q_i \neq 0$. Seeking a contradiction, suppose that $E$ satisfies condition (MT-3). There exist $y \in E^0$ and paths $\alpha, \beta$ from $v$ to $y$ and from $w$ to $y$, respectively. We get $\{0\} = SrvSwS \supseteq S(PrR)v \cdot S(RsR)w \cdot S \supseteq \sum_{i=1}^{n} p_i v \cdot \alpha\beta^*q_i w \cdot w = \sum_{i=1}^{n} p_i q_i \alpha \beta^* \neq 0$, which is a contradiction. 

The following result is a special case of [11] Thm. 2.2.11. Tomforde [13] established this result for Leavitt path algebras with coefficients in a commutative unital ring. His proof generalizes verbatim to Leavitt path algebras with coefficients in a general unital ring. For the convenience of the reader, we include a full proof:

**Proposition 14.11** (cf. [13] Lem. 5.2). Suppose that $E$ is a directed graph and that $R$ is a unital ring. If $a \in (L_R(E))_0$ is nonzero, then there exist $\alpha, \beta \in E^*$, $v \in E^0$ and a nonzero $t \in R$ such that $a\alpha\beta = tv$.

Proof. If we for every $N \in \mathbb{N}$ put $G_N := \text{Span}_R\{\alpha\beta^* | \alpha, \beta \in E^*, |\alpha| = |\beta| \leq N\}$, then $(L_R(E))_0 = \bigcup_{N=0}^{\infty} G_N$. The proof proceeds by induction over $N$. Base case: $N = 0$. Take a nonzero $a \in G_0$. Then $0 \neq a = \sum_{i=1}^{n} r_i v_i$ for some nonzero $r_i \in R$ and distinct vertices $v_i \in E^0$. If we put $a = \beta := v_1$, then $a\alpha\beta = r_1 v_1$.

Inductive step: suppose that $N > 0$ and that the statement of the proposition holds for every nonzero element in $G_{N-1}$. Take a nonzero $a \in G_N$. Then $a = \sum_{i=1}^{M} r_i \alpha_i \beta_i + \sum_{j=1}^{M'} s_j v_j$, where $\alpha_i, \beta_i \in E^*$ with $|\alpha_i| = |\beta_i| \geq 1$ and $v_j \neq v_j'$ for all $j \neq j'$. We consider two mutually exclusive cases.

Case 1: some $v_j$ is not regular. If $v_j$ is an infinite emitter, then there is some edge $f \in E^1$ with $s(f) = v_j$ such that $f$ is not included in any path $\alpha_i, \beta_i$. Put $\alpha = \beta := f$. Then $a\alpha\beta = 0 + f^* s_j v_j f = s_j v_j$. If $v_j$ is a sink, then put $\alpha = \beta := v_j$ and note that $a\alpha\beta = s_j v_j$.

Case 2: every $v_j$ is regular. Then $v_j = \sum_{s(f) = v_j} f f^*$ for every $j$. Hence, we may write $a = \sum_{i=1}^{M'} r_i \alpha_i \beta_i$ where $\gamma_i, \delta_i \in E^*$ with $|\gamma_i| = |\delta_i| \geq 1$. By regrouping the elements of the sum, we may rewrite it as $a = \sum_{i=1}^{P} e_i x_{i,j} f_{j}^*$, where

- $e_i, f_i \in E^1$ with $e_i \neq e_{i'}$ for $i \neq i'$ and $f_j \neq f_{j'}$ for $j \neq j'$, and
- $x_{i,j} \in G_{N-1}$ with $e_i x_{i,j} f_{j}^* \neq 0$ for all $i, j$.

Note that $e_1 x_{1,1} f_{1}^* \neq 0$ implies $r(e_1) x_{1,1} r(f_1) \neq 0$. By the induction hypothesis, there are $\alpha', \beta' \in E^*$ such that $(\alpha')^* r(e_1) x_{1,1} r(f_1) \beta' = tv$ for some $v \in E^0$ and $t \in R$. Put $\alpha := e_1 \alpha'$ and $\beta := f_1 \beta'$. Then $a\alpha\beta = (\alpha')^* e_1 x_{1,1} f_1^* f_1 \beta' = (\alpha')^* r(e_1) x_{1,1} r(f_1) \beta' = tv$. 

We can now establish Theorem [1.6].
Theorem 14.12. Suppose that $E$ is a directed graph and that $R$ is a unital ring. The Leavitt path algebra $L_R(E)$ is prime if and only if $R$ is prime and $E$ satisfies condition (MT-3).

Proof. Put $S := L_R(E)$. Suppose that $R$ is not prime. There exist nonzero ideals $I, J$ of $R$ such that $IJ = \{0\}$. Let $A$ and $B$ be the nonzero ideals in $S$ consisting of sums of monomials with coefficients in $I$ and $J$, respectively. Then $AB = \{0\}$ which implies that $S$ is not prime. Suppose now that $E^0$ does not satisfy condition (MT-3). By Proposition 14.10(a), there exist $v, w \in E^0$ such that $SvSwS = \{0\}$. Consider the nonzero ideals $C := SvS$ and $D := SwS$ of $S$. Then $CD = SvSSwS \subseteq SvSwS = \{0\}$ which shows that $S$ is not prime.

Suppose that $S$ is not prime and that $R$ is prime. By Proposition 14.4, $S$ is nearly epsilon-strongly $\mathbb{Z}$-graded. Theorem 1.3 implies that there exist nonzero ideals $\tilde{A}, \tilde{B}$ of $S_0$ such that $\tilde{A}\tilde{B} = \{0\}$. Take $a \in \tilde{A} \setminus \{0\}$ and $b \in \tilde{B} \setminus \{0\}$. By Proposition 14.11, there exist $v, w \in E^0$, $r, s \in R \setminus \{0\}$ and $\alpha, \beta, \gamma, \delta \in E^*$ such that $\alpha^*a\beta = rv$ and $\gamma^*b\delta = sw$. Now, $SrvSwS \subseteq S\tilde{A}\tilde{B}S = \{0\}$ and hence $SrvSwS = \{0\}$. Employing Proposition 14.10(b), we conclude that $E$ does not satisfy condition (MT-3). □

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