Stability of an upwind Petrov Galerkin discretization of convection diffusion equations

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Abstract

We study a numerical method for convection diffusion equations, in the regime of small viscosity. We identify a norm for which we have both continuity and an inf-sup condition, which are uniform in mesh-width and viscosity, up to logarithmic terms, as long as the viscosity is smaller that the mesh-width. The analysis allows for the formation of a boundary layer.

1 Introduction

For many fluid flow problems of relevance to engineering, the convective term, hyperbolic in nature, is moderated by a viscous term, elliptic in nature. For large viscosity, Galerkin finite element methods yield good results. As the viscosity tends to zero, sharp gradients in the fluid velocity, as well as boundary layers will appear. When the characteristic length of boundary layers is smaller than the mesh-width, standard Galerkin methods become unstable.

This has motivated a large body of work on stabilised methods. For an overview and introduction we refer to [8].

In this paper we consider a general method, introduced in [5], that applies to differential forms on arbitrary meshes. So far we have not provided any analysis of this method in the convection dominated regime. One advantage of the method is that it treats uniformly differential forms of all degrees. Thus it fits into the framework of finite element exterior calculus (FEEC) [2]. More precisely it fits into the framework of finite element systems (FES) [4].

For scalar equations, on product grids, the method relates to exponential fitting. Variants of exponential fitting can be traced all the way back to [1], see also [10]. This method can be completely analysed in dimension one, for instance because the discrete solution turns out to interpolate the exact one. However, already in dimension two, in spite of its naturality, the method is hard to analyse, compared with Galerkin methods for large viscosity.

To be more precise, one defines upwinded finite elements that solve local problems related to the adjoint equation. Downwinded elements locally solve the original equation. In cases where these problems can be solved explicitly one often obtains exponential functions with viscosity dependent parameters, hence the name. In [7] a Galerkin method with downwinded elements is analysed. We,
on the other hand, are interested in a Petrov-Galerkin method with standard trial space, and upwinded test space. Compared with for instance [6] we point out that our methods produce conforming spaces.

One of our main sources of inspiration is [3]. This paper analyses a parabolic problem with an $H^{1/2}$ norm in time. For our purposes the time-variable corresponds to a space-variable which increases in the direction of the flow. We also point out that [9] advocates the use of a $H^{1/2}$ norm for convection diffusion problems. However that paper pertains to the one-dimensional problem.

For a certain choice of viscosity dependent norms we obtain both continuity and an inf-sup condition, up to logarithmic factors in the viscosity and mesh width. Admittedly the hypothesis required for our proof, essentially that the flow is aligned with the mesh, is very restrictive. On the other hand we do allow for the formation of a boundary layer at the outflow boundary.

Thus the novel features of our analysis concern estimates for a $H^{1/2}$ norm in multi-dimensions, and the fact that we prove stability under hypotheses that allow the formation of a boundary layer.

The paper is organized as follows. In §2 we set up the model problem we consider and discuss some numerical results. In §3 we provide a study of some parabolic problems that motivates our techniques. In §4 we provide continuity estimates for some operators acting on functions of one real variable. In §5 we put our results together to prove an inf-sup condition for convection diffusion problems.

2 Problem setup

We consider a domain $U$ in $\mathbb{R}^n$. On this domain we consider the equation:

$$-\alpha \Delta u + \beta \cdot \nabla u + \gamma u = f. \quad (1)$$

The scalar $\alpha > 0$ is constant in the domain. The vector field $\beta$ is also constant, and directed along the first axis. With a slight abuse of notations we take it of the form $\beta(1,0)$ for a scalar $\beta > 0$. We take $\gamma = 1$ for the theory we develop (but the numerics work well also in the sharper case $\gamma = 0$). The right hand side $f$ is given in $L^2(U)$. We impose homogeneous Dirichlet conditions on $u$.

This equation has a unique solution in $H^1_0(U)$, as can be deduced from the Lax-Milgram lemma.

We are interested in letting the parameter $\alpha$ tend to 0 (all other things remaining equal). If we let $u_\alpha$ denote the corresponding solution, we know that $u_\alpha$ converges in $L^2$ to some function $u_0$ as $\alpha$ tends to 0. It then follows that:

$$\alpha \int |\nabla u_\alpha|^2 + \gamma \int |u_\alpha|^2 \to \int f u_0. \quad (2)$$

In general therefore the $H^1$ norm of $u_\alpha$ blows up. One observes the formation of a boundary layer at the outflow boundary, which is the part of $\partial U$ where $\beta \cdot \nu > 0$, where $\nu$ denotes the outward pointing normal vector on $\partial U$. A boundary layer of a rather different nature appears close to the part of the boundary where $\beta \cdot \nu = 0$. Away from the boundary layers, $u_\alpha$ converges to $u_0$ in strong norms. The limit $u_0$ satisfies the homogeneous boundary condition on the inflow boundary (where $\beta \cdot \nu > 0$), but in general not elsewhere.
Figure 1: Numerical solution computed with (left) and without (right) our proposed upwinding.

In Figure 1 numerical results are shown for \( f = 1, \beta = 1 \) and \( \gamma = 0 \). The width of the domain in the horizontal direction is 1. We chose \( \alpha = 3 \times 10^{-4} \).

The standard numerical method we use, is a Galerkin finite element method with continuous \( Q_{11} \) finite elements on a square grid of width \( h \). For \( h = 1/80 \) we observe that the numerical solution is very oscillatory (right hand figure). This well-known instability appears whenever the Péclet number \( \beta h / \alpha \) exceeds 1 (for the displayed figure it is above 40). The upwinded method we propose is a Petrov-Galerkin method with \( Q_{11} \) trial space and a test space we now proceed to describe.

Generally speaking, consider a mesh consisting of cells of various dimensions: vertices (dimension 0), edges (dimension 1), faces (dimension 2), etc. arranged in a cellular complex. For definiteness one can think of simplicial complexes or product grids. Actually only the latter are considered in our numerical experiments and for the stability proof we present in the following sections.

For each cell \( T \), let \( \beta_T \) be the tangential component of \( \beta \) on \( T \). We construct an upwinded basis function \( v \) attached to a given vertex by first assigning the value 1 to this vertex, and the value 0 to all others. Next we extend recursively, from vertices to edges, from edges to faces, etc, each time solving the equation, on say the cell \( T \) (with prescribed boundary values):

\[-\alpha \Delta_T v - \beta_T \cdot \nabla_T v = 0.\]  

We may remark that if \( \beta = 0 \) the method of recursive harmonic extension produces the piecewise linears on simplicial meshes, and the tensor product \( Q_{1...1} \) functions on orthogonal product meshes. On such meshes the obtained functions are simple also in the case where \( \beta \) is non-zero (but constant on the domain): On an edge the solutions to (3) are linear combinations of the constant function and a certain exponential:

\[ v(x) = c_1 + c_2 \exp\left(-\frac{\beta_T}{\alpha} x\right). \]  

If \( \beta_T = 0 \) one replaces of course the exponential by a linear function. Globally one obtains, from the recursive extension procedure, tensor products of such functions.
Figure 2: Relative distance from the solution obtained with exact and approximate upwinding

We develop our theory for the case when the domain is of the form $U = [0, T] \times V$ and the mesh is aligned with the first axis, which is also the direction of the vectorfield $\beta$. In this case we obtain, with the above method, the tensor products of functions which in the direction of the first axis are piecewise of the form $4$, and which in the direction of $V$ are standard $Q_{1,1}$ functions. This defines our upwinded test space. Recall that the trial space is just $Q_{1,1}$.

For applications it is important that the numerical method be able to treat variable $\beta$, not necessarily aligned with the mesh. This is done by replacing (3) by:

$$\text{div}_T \exp(\frac{\beta_T \cdot x}{\alpha}) \text{grad}_T v(x) = 0,$$

(5)

and solving this equation approximately on a sub-grid. The particular sub-grid we advocate consists in adding one point to each cell of the mesh, and taking the corresponding simplicial refinement. The added points are placed taking into account the expected singular behaviour of the upwinded basis functions. In other words we do a barycentric refinement, where barycenters are computed with weights involving $\alpha/\beta h$. The main reason for preferring (5) to (3) when doing the discrete upwinding, is that in this form the method extends nicely to differential forms, as explained in [5].

In Figure 2 we have plotted the relative distance between the numerical solutions $u_{ex}$ and $u_{ap}$ obtained with exact and approximate unwinding respectively,
with respect to the norm $V^\alpha$ defined by:

$$\|u\|_{V^\alpha}^2 = \int |u|^2 + \alpha \int |\nabla u|^2.$$  \hfill (6)

We first notice that the relative error stays below 0.03 in this experiment, indicating that the use of approximate unwinding does not change too much the computed solution. In the following sections we just analyse the case of exact upwinding.

Interestingly we also notice that the relative error in the computed solution, between exact and discrete upwinding seems to be maximal along a certain line, here given by $\alpha = 0.15h$. We also noticed that the relative error between the exact and upwinded basis functions is maximal along such a line, but with a different proportionality constant.

### 3 A study of parabolic problems

In this section we give our reading of [3]. It serves mainly to motivate the techniques of the next sections. Some improvements occur, because in our setting we have Lemma 3.3. We shall restrict our attention to the Crank-Nicolson scheme, whereas [3] treats some other discretizations as well.

**Inf-Sup condition.** We state some ways in which inf-sup conditions may be obtained, for bilinear forms on Hilbert spaces.

**Proposition 3.1.** Suppose $X$ and $Y$ are Hilbert spaces and that $a : X \times Y \to \mathbb{R}$ is a continuous bilinear form. Suppose we have a continuous linear map $A : X \to Y$ such that:

$$\|Au\| \leq C_1\|u\|,$$  \hfill (7)

$$|a(u, Au)| \geq \frac{1}{C_2}\|u\|^2.$$  \hfill (8)

Then $a$ satisfies the inf-sup condition:

$$\inf_{u \in X} \sup_{v \in Y} \frac{|a(u, v)|}{\|u\| \|v\|} \geq \frac{1}{C_1 C_2}.$$  \hfill (9)

**Proof.** For non-zero $u \in X$ we have $Au \neq 0$ and we may write:

$$\sup_{v \in Y} \frac{|a(u, v)|}{\|v\|} \geq \frac{|a(u, Au)|}{\|Au\|} \geq \frac{1}{C_1 C_2}\|u\|.$$  \hfill (10)

The inf-sup estimate follows. \hfill $\square$

**Proposition 3.2.** Suppose we have two Hilbert spaces $X$ and $Y$, and two continuous bilinear forms $a$ and $b$ on $X \times Y$. Suppose we have two continuous operators $A$ and $B$ from $X$ to $Y$ such that for some $C_1 > 0$:

$$b(u, Bu) + a(u, Au) \geq \frac{1}{C_1}\|u\|^2.$$  \hfill (11)
Suppose moreover that we have the compatibility conditions:

\[ b(u, Au) \geq 0, \quad (12) \]
\[ |a(u, Bu)| \leq C_2 a(u, Au). \quad (13) \]

Then \( b + a \) satisfies an inf-sup condition on \( X \times Y \).

Proof. We introduce a parameter \( \lambda > 0 \). We remark that \( B + \lambda A : X \to Y \) is continuous and that:

\[ (b + a)(u, Bu + \lambda Au) = b(u, Bu) + \lambda b(u, Au) + a(u, Bu) + \lambda a(u, Au), \quad (14) \]
\[ \geq b(u, Bu) + \lambda a(u, Au) - |a(u, Bu)|, \quad (15) \]
\[ \geq b(u, Bu) + a(u, Au) + (\lambda - C_2 - 1)a(u, Au). \quad (16) \]

We choose \( \lambda \geq C_2 + 1 \). Then we apply Proposition \[3.1\] of Crank-Nicolson.

Crank-Nicolson. We let \( O \) be a Hilbert space, with scalar product \( \langle \cdot, \cdot \rangle \). Let \( X \) be a Hilbert space contained in \( O \). Let \( a : X \times X \to \mathbb{R} \) be a continuous linear form, which is also coercive.

We define \( Y \) as follows:

\[ Y = H^{1/2}_{00}((\mathbb{R}+, O) \cap L^2(\mathbb{R}+, X). \quad (17) \]

Given \( f \in Y' \), we are interested in finding \( u \in Y \) such that for all \( v \):

\[ \langle \dot{u}, v \rangle + a(u,v) = \langle f, v \rangle. \quad (18) \]

This is an abstract parabolic equation. We consider the initial condition \( u(0) = 0 \).

Given a Galerkin space \( X_\sigma \) and a time-step \( \tau \), the Crank-Nicolson scheme is defined as follows. We let \( u : \mathbb{R}+ \to X_\sigma \) be continuous and \( \tau \)-piecewise linear. We denote \( u_i = u(i\tau) \). We impose \( u(0) = 0 \) and, for all \( v \in X_\sigma \):

\[ \langle \frac{u_{i+1} - u_i}{\tau}, v \rangle + a(u_{i+1/2}, v) = \langle f_{i+1/2}, v \rangle. \quad (19) \]

Here we have put:

\[ u_{i+1/2} = \frac{1}{2}(u_i + u_{i+1}), \quad (20) \]

and:

\[ f_{i+1/2} = \frac{1}{\tau} \int_{i\tau}^{(i+1)\tau} f. \quad (21) \]

More generally, for any function \( u \) on \( \mathbb{R} \), we let \( \bar{u} \) be the function which is \( \tau \)-piecewise constant, with the same piecewise averages as \( u \). The Crank-Nicolson scheme then yields, for all \( v : \mathbb{R}+ \to X_\sigma \):

\[ \int \langle \dot{\bar{u}}, v \rangle + \int a(\bar{u}, v) = \int \langle \bar{f}, v \rangle. \quad (22) \]

We are particularly interested in problems with a parameter \( \alpha \). That is, the space \( X \) is replaced by \( X^\alpha \), and \( a \) by \( a^\alpha \). As a set \( X = X^\alpha \), but we suppose
that the norm of $X^\alpha$ is equivalent to the one defined by $a^\alpha$, uniformly in $\alpha$. Explicitly for all $v \in X$:

$$\frac{1}{C} \|u\|_\alpha^2 \leq a^\alpha(u, u) \leq C \|u\|_\alpha^2.$$  \hfill (23)

As $\alpha$ converges to 0, that norm converges to the norm on $O$, which is denoted $\|\cdot\|_0$.

**Lemma 3.3.** Suppose we have $\tau \ll \sigma$ and $\alpha \ll \sigma$. Then we have an estimate:

$$|u|_{H^{1/2}(O)}^2 + \|u\|_{L^2(X^\alpha)}^2 \ll |u|_{H^{1/2}(O)}^2 + \|\bar{u}\|_{L^2(X^\alpha)}^2.$$  \hfill (24)

**Proof.** We have (approximation and inverse inequality):

$$\|u - \bar{u}\|_{L^2(X^\alpha)} \ll \tau^{1/2} |u|_{H^{1/2}(X^\alpha)},$$  \hfill (25)

$$\ll \tau^{1/2} \alpha^{1/2} \sigma^{-1} |u|_{H^{1/2}(O)},$$  \hfill (26)

$$\ll |u|_{H^{1/2}(O)}.$$  \hfill (27)

This gives the estimate.

We now derive a stability estimate for (22). It uses the Hilbert transform, which we denote by $H$. The essential property of the Hilbert transform, in addition to its various continuity properties, is that:

$$\int \dot{u} H u = |u|_{H^{1/2}(O)}^2.$$  \hfill (28)

This can be most easily seen using the Fourier transform, which we denote by $F$.

Compared with notations of Proposition 3.2, we use $B = H$ and $A = \text{id}$. The bilinear form $b$ in that proposition corresponds to the first term on the left hand side of (18). We chose notations so that $a$ keeps its meaning.

With $v = H\bar{u} + \lambda u$ we get:

$$\int \langle \dot{u}, v \rangle + \int a^\alpha(\bar{u}, v) \geq |u|_{H^{1/2}(O)}^2 + \lambda \int a^\alpha(\bar{u}, \bar{u}) - \int |a^\alpha(\bar{u}, H\bar{u})|.$$  \hfill (29)

We have:

$$\int |a^\alpha(\bar{u}, H\bar{u})| \leq C \|\bar{u}\|_{L^2(X^\alpha)} \|u\|_{L^2(X^\alpha)},$$  \hfill (30)

$$\leq \frac{C}{2\epsilon} \|\bar{u}\|_{L^2(X^\alpha)}^2 + \frac{C\epsilon}{2} \|u\|_{L^2(X^\alpha)}^2.$$  \hfill (31)

So we get, using Lemma 3.3

$$\int \langle \dot{u}, v \rangle + \int a^\alpha(\bar{u}, v) \geq \frac{1}{C'} (|u|_{H^{1/2}(O)}^2 + \|u\|_{L^2(X^\alpha)}^2) +$$

$$\lambda \int a^\alpha(\bar{u}, \bar{u}) - \frac{C}{2\epsilon} \|\bar{u}\|_{L^2(X^\alpha)}^2 - \frac{C\epsilon}{2} \|u\|_{L^2(X^\alpha)}^2.$$  \hfill (32)

Choose $\epsilon$ so small that the last term is dominated by the first. Then choose $\lambda$ so big that the second term dominates the third.

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We get:

\[
\int \langle \dot{u}, v \rangle + \int a^\alpha(\bar{u}, v) \geq \frac{1}{C}(|u|^2_{H^{1/2}(O)} + \|u\|^2_{L^2(X^\alpha)}).
\] (34)

Together with (22) this gives a stability estimate for the Crank-Nicolson scheme:

\[
|u|^2_{H^{1/2}(O)} + \|u\|^2_{L^2(X^\alpha)} \lesssim \|\bar{f}\|^2_{V^\alpha}.
\] (35)

The operation \( f \to \bar{f} \) is well defined on the (non-closed) subspace \( L^2(\mathbb{R}_+, X^{\alpha'}) \) of \( Y^{\alpha'} \).

**Remark 3.1.** In [3], which treats the case of fixed \( \alpha \), the obtained stability estimate concerns the discrete norm with square:

\[
|u|^2_{H^{1/2}(O)} + \|\bar{u}\|^2_{L^2(X)} \lesssim \|\bar{f}\|^2_{V^\alpha}.
\] (36)

The appearance of \( \bar{u} \) makes this norm slightly weaker. The interpretation is that some oscillations in time are not so well controlled in \( X \) norm.

### 4 Continuity estimates of some operators

In this section we prove uniform continuity estimates for the operators we use in our stability proofs. These operators act on functions of a real variable.

We will use \( H^{1/2} \) spaces on \( \mathbb{R} \) with values in Hilbert spaces. We refer to [11] (especially chapter 35) for definitions pertaining to scalar valued functions. Until now we have used the characterisation with the Fourier transform, but now we will also use the Slobodetski seminorm, as it appears in particular in Lemma 35.2 in [11].

We will use also the space \( H^{1/2}_w \) consisting of functions \( u \in L^2 \) such that for some \( C \geq 0 \), we have, for all \( y \in \mathbb{R} \):

\[
\|u - \tau_y u\|_{L^2} \leq C|y|^{1/2}.
\] (37)

Here, \( \tau_y \) denotes translation by the vector \( y \). The best constant \( C \) in this estimate is denoted:

\[
|u|_{H^{1/2}_w} = C.
\] (38)

The \( w \) stands for "weak", reflecting that the Banach space \( H^{1/2}_w \) is slightly bigger than the Hilbert space \( H^{1/2} \). The space \( H^{1/2}_w \) is nothing but the Besov space \( B^{1/2,2}_w \), see for instance Lemma 35.1 in [11]. We will use that it is big enough to contain piecewise constant functions. On the other hand it is small enough to be included in all \( H^{1/2-\epsilon} \) spaces, for \( \epsilon > 0 \).

**Proposition 4.1.** Consider the map \( u \mapsto \bar{u} \), which projects onto \( \tau \)-piecewise constant functions. It is bounded from \( H^{1/2} \) to \( H^{1/2}_w \), uniformly in \( \tau \).

**Proof.** In this proof we use the Slobodetski seminorm on \( H^{1/2} \).

Let \( P_\tau : u \mapsto \bar{u} \) denote the \( L^2 \) projection onto \( \tau \)-piecewise constant functions.

On the reference interval \([-1,1]\] we have an estimate, for the jump at 0:

\[
|P_\tau u(0+) - P_\tau u(0-)| \lesssim |u|_{H^{1/2}(-1,1)}.
\] (39)
Scaling to the interval \([-\tau, \tau]\) we get:
\[
|P_{\tau}u(0+) - P_{\tau}u(0-)| \leq |u|_{H^{1/2}(-\tau, \tau)}.
\] (40)

For \(|y| < \tau\) we have:
\[
\|P_{\tau}u - \tau y P_{\tau}u\|_{L^2} \leq \sum_{k \in \mathbb{Z}} |y| |P_{\tau}u(k\tau^+) - P_{\tau}u(k\tau^-)|^2,
\] (41)
\[
\leq |y| \sum_{k \in \mathbb{Z}} |u|_{H^{1/2}((k-1)\tau, (k+1)\tau)}^2,
\] (42)
\[
\leq |y| |u|_{H^{1/2}}^2.
\] (43)

For \(|y| \geq \tau\) we have:
\[
\|P_{\tau}u - \tau y P_{\tau}u\|_{L^2} \leq \|P_{\tau}u - u\|_{L^2} + \|u - \tau y u\|_{L^2} + \|\tau y u - \tau y P_{\tau}u\|_{L^2},
\] (44)
\[
\leq \left(\tau^{1/2} + |y|^{1/2}\right)|u|_{H^{1/2}},
\] (45)
\[
\leq |y|^{1/2}|u|_{H^{1/2}}.
\] (46)

Together these two estimates conclude the proof. \(\square\)

**Proposition 4.2.** Consider the canonical injection of \(H^{1/2}_w\) into \(H^{1/2-\epsilon}\). Its norm is of order \(1/\epsilon^{1/2}\) for small \(\epsilon\).

**Proof.** We write:
\[
|u|_{H^{1/2-\epsilon}}^2 = \iint \frac{|u(x+y) - u(x)|^2}{|y|^{2-2\epsilon}} \, dx \, dy,
\] (47)
\[
\leq \left(\int \min\{|y|, 1\} \frac{1}{|y|^{2-2\epsilon}} \, dy\right) \|u\|_{H^{1/2}}^2.
\] (48)

The integral over \(y\) is bounded by (twice):
\[
\int_0^1 \frac{1}{|y|^{2-2\epsilon}} \, dy + \int_1^{\infty} \frac{1}{|y|^{2-2\epsilon}} \, dy = \frac{1}{2\epsilon} + \frac{1}{1 - 2\epsilon}.
\] (49)

This yields the claimed result. \(\square\)

**Proposition 4.3.** Given a function \(u\) which is 0 on \(\mathbb{R}_-\), we define a function \(v\) which is also 0 on \(\mathbb{R}_-\), and solves:
\[
\alpha \dot{v} + \beta v = \beta u.
\] (50)

The map \(u \mapsto v\), from \(H^{1/2-\epsilon}\) to \(H^{1/2}\), has a norm of order \(1/\epsilon^{1/2}\) for small \(\epsilon\) and \(\alpha\).

**Proof.** We use the formula:
\[
v(t) = \int_{-\infty}^{t} \beta \exp\left(\frac{\beta(s-t)}{\alpha}\right) u(s) \, ds.
\] (51)

We introduce the function \(G_\alpha\) defined by:
\[
G_\alpha(s) = \begin{cases} 
\frac{\beta}{\alpha} \exp\left(-\frac{\beta s}{\alpha}\right) & \text{for } s \geq 0, \\
0 & \text{for } s \leq 0.
\end{cases}
\] (52)
With this formula we have:

\[ v = G_\alpha * u. \]  

(53)

Notice that:

\[ \|G_\alpha\|_{L^1} = 1. \]  

(54)

This gives uniform boundedness, from \( L^2 \) to \( L^2 \), for convolution by \( G_\alpha \).

For the rest of the proof we suppose, without loss of generality, that \( \beta = 1 \).

We have:

\[ FG_\alpha(\xi) = \frac{1}{1 + \alpha^2 \xi^2}. \]  

(55)

It follows that:

\[ |FG_\alpha(\xi)|^2 = \frac{1}{1 + \alpha^2 |\xi|^2}. \]  

(56)

We can therefore write:

\[ |u|_{H^{1/2}}^2 = \int \frac{|\xi|^2}{1 + \alpha^2 |\xi|^2} |\mathcal{F}u(\xi)|^2 d\xi, \]  

(57)

\[ \leq C_\alpha |u|_{H^{1/2}}^2, \]  

(58)

with:

\[ C(\alpha, \epsilon) = \max\{ \frac{|\xi|^{2\epsilon}}{1 + \alpha^2 |\xi|^2} : \xi \in \mathbb{R} \}. \]  

(59)

Calculus gives that the maximum is achieved when:

\[ |\xi|^2 = \frac{\epsilon}{(1 - \epsilon)\alpha^2}. \]  

(60)

Then:

\[ C(\alpha, \epsilon) = \frac{(1 - \epsilon)^{1/2}}{(1 - \epsilon)\alpha^{2\epsilon}}. \]  

(61)

This concludes the proof.

Corollary 4.4. When we compose the three operators defined in Propositions 4.1, 4.2 and 4.3, we get an operator from \( H^{1/2} \) to itself, with norm of order \( |\log(\alpha)|^{1/2} \).

Proof. We get a norm of order:

\[ \frac{1}{\epsilon^{1/2} \alpha^\epsilon}, \]  

(62)

Then we chose \( \epsilon = -1/|\log(\alpha)| \).

Proposition 4.5. There exists a \( C > 0 \) such that for all \( \tau > 0 \), all \( u \) which are continuous and \( \tau \)-piecewise linear:

\[ \|u\|_{L^\infty} \leq C |\log(\tau)|^{1/2} |u|_{H^{1/2}}. \]  

(63)
Proof. We have, for small \( s > 0 \):

\[
\|u\|_{L^\infty} \lesssim \|\mathcal{F}u\|_{L^1},
\]

\[
\lesssim \int (1 + |\xi|^2)^{-s/2} (1 + |\xi|^2)^{1/2} |\mathcal{F}u|,
\]

\[
\lesssim \left( \int (1 + |\xi|^2)^{-s/2} \right)^{1/2} \left( \int (1 + |\xi|^2)^{1/2} |\mathcal{F}u|^2 \right)^{1/2},
\]

\[
\lesssim \frac{1}{s^{1/2}} \|u\|_{H^{(1+s)/2}}.
\]

Then we use an inverse inequality:

\[
\|u\|_{L^\infty} \lesssim \tau^{-s/2} \|u\|_{H^{1/2}}.
\]

Finally we let:

\[
s = -\log(\tau)^{-1}.
\]

This gives the estimate.

Proposition 4.6. There exists a \( C > 0 \) such that for all \( \tau > 0 \), all \( u \) which are continuous and \( \tau \)-piecewise linear:

\[
\|\mathcal{H}u\|_{L^\infty} \leq C |\log(\tau)| \|u\|_{L^\infty}.
\]

Proof. We use that for large \( p \), the Hilbert transform is continuous from \( L^p \) to \( L^p \), with norm of order \( p \). We write, for large \( p < \infty \):

\[
\|\mathcal{H}u\|_{L^\infty} \lesssim \|\mathcal{H}u\|_{L^p}^{1-1/p} \|\mathcal{H}\dot{u}\|_{L^p}^{1/p},
\]

\[
\lesssim p \|u\|_{L^p}^{1-1/p} \|\dot{u}\|_{L^p}^{1/p},
\]

\[
\lesssim p \tau^{-1/p} \|u\|_{L^\infty}.
\]

Then we choose:

\[
p = -\log(\tau).
\]

This concludes the proof.

Proposition 4.7. For each \( \epsilon > 0 \) there exists \( C > 0 \) such that for all \( \alpha \leq \tau/C \), and all \( u \) which is \( \tau \)-piecewise constant, the solution of the equation:

\[
\alpha \dot{v} + \beta v = \beta u,
\]

satisfies:

\[
\|u - v\|_{L^2} \leq \epsilon \|u\|_{L^2}.
\]

Proof. We use the notations of the proof of Proposition 4.3.

We first fix \( \tau = 1 \).

For \( \delta > 0 \) we decompose \( G_\alpha \) as follows:

\[
G_\alpha = G_\delta^\alpha + (G_\alpha - G_\delta^\alpha),
\]

with:

\[
G_\delta^\alpha(s) = \begin{cases} \frac{\delta}{\pi} \exp\left(\frac{-\delta s}{\alpha}\right) & \text{for } 0 \leq s \leq \delta, \\ 0 & \text{for } s < 0 \text{ and } s > \delta. \end{cases}
\]
We have:
\[ \int_{0}^{\delta} G_{\alpha}^{\delta} ds = 1 - \exp\left(\frac{-\beta \delta}{\alpha}\right). \]  
(79)

Fix \( \delta \in ]0, 1[ \). As \( \alpha \to 0 \) the above number tends to 1.

Choose \( \epsilon > 0 \). For small enough \( \alpha \) we have, for any \( u \) that is constant on \( ]-1, 0[ \) and \( ]0, 1[ \):
\[ \|u - G_{\alpha}^{\delta} \|_{L^2(-1, 1)} \leq \frac{\epsilon}{4} \|u\|_{L^2(0, 1)}. \]  
(80)

Therefore, for \( u \in L^2(\mathbb{R}) \) which is constant on each interval \( ]k, k+1[ \) for \( k \in \mathbb{Z} \):
\[ \|u - G_{\alpha}^{\delta} \|_{L^2} \leq \frac{\epsilon^2}{16} \sum_{k \in \mathbb{Z}} \|u\|_{L^2(k-1, k+1)}, \]  
(81)
\[ \leq \frac{\epsilon^2}{8} \|u\|_{L^2}^2. \]  
(82)

For small enough \( \alpha \) we also have:
\[ \|G_{\alpha} - G_{\alpha}^{\delta}\|_{L^1} = \exp\left(\frac{-\beta \delta}{\alpha}\right) \leq \frac{\epsilon}{4}. \]  
(83)

Therefore:
\[ \|u - G_{\alpha} \|_{L^2} \leq \frac{\epsilon (2^{1/2} + 1)}{4} \|u\|_{L^2}. \]  
(84)

This gives the result for \( \tau = 1 \). One concludes by scaling.

Remark 4.1. For the above propositions, minimal changes occur when we replace functions from \( \mathbb{R} \) to \( \mathbb{R} \), by functions from \( \mathbb{R} \) to some Hilbert space.

5 Convection diffusion

For a function \( u \) on \( U = [0, T] \times V \), derivation along the first axis will be denoted \( \dot{u} \), and derivation along the remaining axes (in \( V \)) will be denoted \( \partial_v u \).

The variational form of equation (1) can be written:
\[ \int \langle \dot{u}, \alpha \dot{v} \rangle + \int a^\alpha(u, v) = \int \langle f, v \rangle. \]  
(85)

Here, integration is on \( [0, T] \), and for functions on \( V \) we denote:
\[ \langle u, v \rangle = \int_V uv. \]  
(86)

Moreover \( a^\alpha \) denotes the bilinear map defined on functions on \( V \) by:
\[ a^\alpha(u, v) = \int_V uv + \alpha \int_V \partial_v u \cdot \partial_v v. \]  
(87)

We let \( X_\sigma \) denote some standard finite element space of functions on \( V \), such as \( Q_{1,1} \) with respect to a mesh of width \( \sigma \). Let \( Z^0_\tau \) denote the space of \( \tau \)-piecewise constant functions on \( [0, T] \). Also let \( Z^1_\tau \) denote the space of continuous \( \tau \)-piecewise linear ones, which are 0 at the extremities of the interval. Finally let \( Z^1_{\tau}(\alpha) \) denote the space of continuous functions which are \( \tau \)-piecewise of the upwinded form (4), which are also 0 at the extremities.

As already indicated in (2) we solve (85) by a Petrov Galerkin method, where the trial space is \( Z^1_\tau \otimes X_\sigma \) and the test space is \( Z^1_{\tau}(\alpha) \otimes X_\sigma \).
Given $u \in Z^1 \otimes X_\sigma$, we construct a quasi-optimal test function for (85) in $Z^1(\alpha) \otimes X_\sigma$ in several steps:

- $v_0 = \mathcal{H}u + \lambda u$, with $\lambda \geq 1$ to be determined,
- $v_1 = \bar{v}_0 \in Z^0_\tau \otimes X_\sigma$ (projection onto $\tau$-piecewise constants in the first direction),
- $\alpha \hat{v}_2 + \beta v_2 = \beta v_1$, with $v_2(0) = 0$,
- $v_3$ is defined by putting:

$$v_3 = v_2 \text{ on } [0, T - \tau],$$  
$$\alpha \hat{v}_3 + \beta \hat{v}_3 = 0 \text{ on } [T - \tau, T], \text{ with } v_3(T) = 0.$$ 

This last $v_3$ is in $Z^1(\alpha) \otimes X_\sigma$ and will be our candidate for an optimal test function. Our first task is to show how relevant norms of $v_3$ can be controlled.

We then write:

$$\int \langle \dot{u}, \alpha \hat{v}_3 + \beta v_3 \rangle + \int a^\alpha(u, v_3) = I_1 + I_2 + I_3 + I_4,$$  

with:

$$I_1 = \int_{[0, T - \tau]} \langle \dot{u}, \beta v_1 \rangle,$$  
$$I_2 = \int_{[T - \tau, T]} \langle \dot{u}, \alpha \hat{v}_3 + \beta v_3 \rangle,$$  
$$I_3 = \int_{[0, T - \tau]} a^\alpha(u, v_2),$$  
$$I_4 = \int_{[T - \tau, T]} a^\alpha(u, v_3).$$

We estimate the four terms successively. The overall plan is to show that $I_1$ and $I_3$ are big and together dominate the norm squared of $u$, whereas the terms $I_2$ and $I_4$ will be shown not to deteriorate this estimate.

We have four parameters: $\alpha, \tau, \sigma$ and $\lambda$. All our constants are independent of these parameters. Notice also that we will let $\lambda$ vary, contrary to the theory provided for parabolic problems, where it was just chosen big enough.

**Controlling $v_3$.** We first remark that $v_3$ is not too big. More precisely we have the following estimates. By Corollary 4.4 we have:

$$\|v_2\|_{H^{1/2}} \leq |\log(\tau)|^{1/2} \lambda \|u\|_{H^{1/2}}.$$  

We also have:

$$\|v_2\|_{L^2(X^\sigma)} \leq \lambda \|u\|_{L^2(X^\sigma)}.$$  

To estimate $v_3$ we use the explicit formula, for $t \in [T - \tau, T]$:

$$v_3(t) = \frac{\exp(\frac{\beta(T - t)}{\alpha}) - 1}{\exp(\frac{\beta}{\alpha}) - 1} v_2(T - \tau).$$
We can deduce the following formula on $[T-\tau,T]$:  
\[
\alpha \dot{v}_3 + \beta v_3 = -\frac{\beta}{\exp(\frac{2\tau}{\alpha}) - 1} v_2(T-\tau). \tag{98}
\]

We remark that for the characteristic function of $[T-\tau,T]$ we have:  
\[
\|\chi_{[T-\tau,T]}\|_{\dot{H}^{1/2}_0}^2 = \|\chi_{[T-\tau,T]}\|_{L^2}^2 + |\chi_{[T-\tau,T]}|_{H^{1/2}_0}^2, \tag{99}
\]
\[
\leq \tau + 1. \tag{100}
\]

Using the notation of the proof of Proposition 4.3 we introduce $v = v_2 - v_3$ and write:
\[
v = G_\alpha \ast \frac{1}{\beta} (\alpha \dot{v} + \beta v), \tag{101}
\]
\[
v = G_\alpha \ast (v_1 + \frac{1}{\exp(\frac{2\tau}{\alpha}) - 1} v_2(T-\tau)) \chi_{[X-\sigma,X]}. \tag{102}
\]

We deduce, combining Propositions 4.2 and 4.3 that:
\[
\|v\|_{H^{1/2}_0} \lesssim \exp(\frac{1}{\epsilon^{1/2} \alpha}) (\|v_1\|_{L^\infty} + \|v_2\|_{L^\infty}). \tag{103}
\]

We let $\epsilon = -1/|\log(\alpha)|$ and combine with Proposition 4.6 to deduce:
\[
\|v\|_{H^{1/2}_0} \lesssim |\log(\tau)|^{1/2} (|\log(\tau)|^{1/2} + \lambda) \|u\|_{H^{1/2}_0}. \tag{104}
\]

We conclude:
\[
\|v_3\|_{H^{1/2}_0} \lesssim |\log(\tau)|^{1/2} (|\log(\tau)|^{1/2} + \lambda) \|u\|_{H^{1/2}_0}. \tag{105}
\]

We also remark:
\[
\|v_3\|_{L^2(X^\alpha)} \lesssim \lambda \|u\|_{L^2(X^\alpha)}. \tag{106}
\]

**Estimating $I_1$.**

\[
I_1 = \int_{[0,T-\tau]} \beta \langle \dot{u}, \mathcal{H}u + \lambda u \rangle, \tag{107}
\]
\[
= \beta |u|_{H^{1/2}(O)}^2 - \beta \int_{[T-\tau,T]} \langle \dot{u}, \mathcal{H}u \rangle + \frac{\lambda \beta}{2} \|u(T-\tau)\|_0^2. \tag{108}
\]

In this equation we remark that:
\[
\int_{[T-\tau,T]} |\langle \dot{u}, \mathcal{H}u \rangle| = \int_{[T-\tau,T]} |\langle \frac{u(T-\tau)}{\tau}, \mathcal{H}u \rangle|, \tag{109}
\]
\[
\leq \frac{1}{2\epsilon} \|u(T-\tau)\|_0^2 + \frac{\epsilon}{2\tau} \int_{[T-\tau,T]} \|\mathcal{H}u\|_0^2. \tag{110}
\]

Moreover, Propositions 4.5 and 4.6 give a constant $C_1$ so that:
\[
\frac{1}{\tau} \int_{[T-\tau,T]} \|\mathcal{H}u\|_0^2 \leq \|\mathcal{H}u\|_{L^\infty(O)}^2, \tag{111}
\]
\[
\leq C_1 |\log(\tau)|^{3/2} \|u\|_{H^{1/2}(O)}^2. \tag{112}
\]
Choosing: 
\[ \epsilon = \frac{1}{C_1 C_2 |\log(\tau)|^2}, \tag{113} \]
we get:
\[ \left| \int_{[T-\tau,T]} \langle \dot{u}, \mathcal{H}u \rangle \right| \leq \frac{C_1 C_2 |\log(\tau)|^3}{2} \|u(T-\tau)\|_0^2 + \frac{1}{2C_2} \|u\|_{H^{1/2}(O)}^3. \tag{114} \]
All in all, we get:
\[ I_1/\beta \geq \|u\|_{H^{1/2}(O)}^2 - \frac{1}{2C_2} \|u\|_{H^{1/2}(O)}^2 + \frac{\lambda - C_1 C_2 |\log(\tau)|^3}{2} \|u(T-\tau)\|_0^3. \tag{115} \]
In the following we suppose that \( \lambda \) satisfies:
\[ \lambda \geq 2C_1 C_2 |\log(\tau)|^3. \tag{116} \]

**Estimating \( I_2 \).** Integration by parts, using (98) gives:
\[ I_2 = \frac{\beta}{\exp(\frac{\beta \tau}{\alpha}) - 1} \langle u(T-\tau), v_3(T-\tau) \rangle. \tag{117} \]
Using also Propositions 4.5, 4.6, we deduce:
\[ |I_2| \lesssim \frac{1}{\exp(\frac{\beta \tau}{\alpha}) - 1} \|u(T-\tau)\|_0 \|v_2(T-\tau)\|_0, \tag{118} \]
\[ \lesssim \frac{1}{\exp(\frac{\beta \tau}{\alpha}) - 1} \|u(T-\tau)\|_0 |\log(\tau)|^{3/2} \|u\|_{H^{1/2}(O)}, \tag{119} \]
\[ \lesssim \frac{\lambda}{\exp(\frac{\beta \tau}{\alpha}) - 1} (|\log(\tau)|^3 \|u(T-\tau)\|_0^2 + \|u\|_{H^{1/2}(O)}^2). \tag{120} \]
Suppose we have an inequality:
\[ \alpha \leq \frac{\beta \tau}{|\log(\tau)|}, \tag{121} \]
then we have:
\[ \frac{1}{\exp(\frac{\beta \tau}{\alpha}) - 1} \leq \frac{\tau}{1 - \tau}. \tag{122} \]
In the following we suppose also that we have an inequality:
\[ \lambda \leq \frac{1}{C\tau}, \tag{123} \]
for some large enough \( C \).

Then we deduce that \( I_2 \) does not deteriorate the estimate for \( I_1 \). That is, assuming (116), (123) and (121), and arbitrarily large \( C \), we get an estimate:
\[ I_1 + I_2 \gtrsim \|u\|_{H^{1/2}(O)}^2 - \frac{1}{C} \|u\|_{L^2(O)}^2 + \lambda \|u(T-\tau)\|_0^2. \tag{124} \]
Estimating $I_3$. We write:

$$I_3 = \int_{[0,T-\tau]} a^\alpha(u,v_1) + \int_{[0,T-\tau]} a^\alpha(u,v_2 - v_1).$$  \hspace{1cm} (125)

For the first term on the right hand side:

$$\int_{[0,T-\tau]} a^\alpha(u,v_1) = \lambda \int_{[0,T-\tau]} a^\alpha(\bar{u},\bar{u}) + \int_{[0,T-\tau]} a^\alpha(\bar{u},\mathcal{H}u).$$  \hspace{1cm} (126)

Here we remark that:

$$\int_{[0,T-\tau]} a^\alpha(\bar{u},\bar{u}) \geq \frac{1}{C_3} \|\bar{u}\|^2_{L^2(X^\alpha)};$$  \hspace{1cm} (127)

and that:

$$\left| \int_{[0,T-\tau]} a^\alpha(\bar{u},\mathcal{H}u) \right| \leq \|\bar{u}\|_{L^2(X^\alpha)}\|u\|_{L^2(X^\alpha)}.$$  \hspace{1cm} (128)

For the second term we have:

$$\left| \int_{[0,T-\tau]} a^\alpha(u,v_2 - v_1) \right| \leq \|u\|_{L^2(X^\alpha)}\|v_2 - v_1\|_{L^2(X^\alpha)}.$$  \hspace{1cm} (129)

Now, according to Proposition 4.7 under assumption (121), we get estimates, for arbitrarily large $C$:

$$\|v_2 - v_1\|_{L^2(X^\alpha)} \leq \frac{1}{C} \|v_1\|_{L^2(X^\alpha)}.$$  \hspace{1cm} (130)

We combine this with the estimate:

$$\|v_1\|_{L^2(X^\alpha)} \leq \|u\|_{L^2(X^\alpha)} + \lambda \|\bar{u}\|_{L^2(X^\alpha)}.$$  \hspace{1cm} (131)

All in all we deduce that for arbitrarily large $C$ we may get an estimate:

$$I_3 \geq \lambda \|\bar{u}\|^2_{L^2(X^\alpha)} - \frac{1}{C} \|u\|^2_{L^2(X^\alpha)}.$$  \hspace{1cm} (132)

Estimating $I_4$. We use the explicit formula (97). We compute:

$$\int_{[T-\tau,T]} \frac{\exp\left(\frac{\alpha(T-t)}{\beta}\right) - 1}{\exp\left(\frac{\alpha}{\beta}\right) - 1} dt = \frac{\alpha}{\beta} - \frac{\tau}{\exp\left(\frac{\tau \alpha}{\beta}\right) - 1}.$$  \hspace{1cm} (133)

We therefore have:

$$|I_4| \leq \alpha \|u(T-\tau)\|_{X^\alpha}\|v_3(T-\tau)\|_{X^\alpha}.$$  \hspace{1cm} (134)

Here we substitute:

$$\|v_3(T-\tau)\|_{X^\alpha} \leq \|v_1\|_{L^\infty(X^\alpha)},$$  \hspace{1cm} (135)

$$\leq \|\mathcal{H}u\|_{L^\infty(X^\alpha)} + \lambda \|\bar{u}\|_{L^\infty(X^\alpha)},$$  \hspace{1cm} (136)

$$\leq |\log(\tau)|^{1/2}\|u\|_{L^\infty(X^\alpha)} + \lambda \|\bar{u}\|_{L^\infty(X^\alpha)},$$  \hspace{1cm} (137)

$$\leq \tau^{-1/2}\|u\|_{L^2(X^\alpha)} + \lambda \|\bar{u}\|_{L^2(X^\alpha)}.$$  \hspace{1cm} (138)
Therefore:

\[ |I_4| \leq \frac{\alpha}{\tau} (|\log(\tau)|^{1/2} \|u\|_{L^2(X^\alpha)}^2 + \frac{\alpha^{3/2}}{\sigma_t^{1/2}} \lambda \|u(T - \tau)\|_0 \|\bar{u}\|_{L^2(X^\alpha)}). \tag{139} \]

Under hypothesis (116) we deduce:

\[ \leq \frac{1}{|\log(\tau)|^{1/2}} \|u\|_{L^2(X^\alpha)}^2 + \frac{\lambda}{|\log(\tau)|^{3/2}} (\|u(T - \tau)\|_0^2 + \|\bar{u}\|_{L^2(X^\alpha)}^2). \tag{140} \]

Therefore \(|I_4|\) is eventually dominated by \(I_1 + I_3\).

Remark 5.1. Our stability estimate depends on getting \(|I_4|\) smaller than say \((I_1 + I_3)/2\). Now \(1/|\log(\tau)|\) converges very slowly to 0. However the corresponding terms in the above estimate can be made small simply by choosing a better constant in (116). That is, we replace that condition by:

\[ \alpha \leq \frac{\beta \tau}{C|\log(\tau)|}, \tag{141} \]

for a large \(C\). Then \(|I_4|\) is dominated by \(I_1 + I_3\) without having to wait for \(|\log(\tau)|\) to become very large in (140).

Combination of estimates. All in all we get:

\[ I_1 + I_2 + I_3 + I_4 \geq \|u\|_{H^{1/2}(\Omega)}^2 + \lambda \|u(T - \tau)\|_0^2 + \lambda \|\bar{u}\|_{L^2(X^\alpha)}^2 \tag{142} \]

\[ - \frac{1}{C} \|u\|_{L^2(\Omega)}^2 - \frac{1}{C} \|\bar{u}\|_{L^2(X^\alpha)}^2. \tag{143} \]

Combined with Lemma 3.3 we deduce in particular:

\[ I_1 + I_2 + I_3 + I_4 \geq \|u\|_{H^{1/2}(\Omega)}^2 + \|\bar{u}\|_{L^2(X^\alpha)}^2. \tag{144} \]

We also have:

\[ |v_3|_{H^{1/2}(\Omega)}^2 + \|v_3\|_{L^2(X^\alpha)}^2 \leq C(\alpha, \tau)(\|u\|_{H^{1/2}(\Omega)}^2 + \|\bar{u}\|_{L^2(X^\alpha)}^2), \tag{145} \]

\[ C(\alpha, \tau) = (|\log(\alpha)| + |\log(\tau)|^{6})|\log(\alpha)|. \tag{146} \]

Summing up we get:

**Theorem 5.1.** The convection diffusion equation (1), discretized by the above Petrov-Galerkin method on quasi uniform grids, assuming a condition (141), satisfies a uniform discrete inf-sup condition up to logarithmic terms in \(\alpha\) and \(\tau\).

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