Random matrix theory and spectral sum rules for the Dirac operator in QCD

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Abstract

We construct a random matrix model that, in the large $N$ limit, reduces to the low energy limit of the QCD partition function put forward by Leutwyler and Smilga. This equivalence holds for an arbitrary number of flavors and any value of the QCD vacuum angle. In this model, moments of the inverse squares of the eigenvalues of the Dirac operator obey sum rules, which we conjecture to be universal. In other words, the validity of the sum rules depends only on the symmetries of the theory but not on its details. To illustrate this point we show that the sum rules hold for an interacting liquid of instantons. The physical interpretation is that the way the thermodynamic limit of the spectral density near zero is approached is universal. However, its value, i.e. the chiral condensate, is not.

SUNY-NTG-92/45
December 1992

\textsuperscript{1}Dedicated to Hans Weidenmüller’s 60th birthday.
1. Introduction

Since our main understanding of nonperturbative phenomena in QCD comes from numerical simulations analytical results in this direction are most welcome [1, 2]. In particular the issue of quark confinement has not been resolved. What is better known [3] is the mechanism of chiral symmetry breaking. In the chiral limit the QCD partition function is invariant under $SU(N_f) \times SU(N_f) \times U(1)$, whereas the axial $U(1)$ symmetry is broken explicitly by quantum fluctuations. For more than one flavor ($N_f > 1$) the chiral symmetry is spontaneously broken down to $SU(N_f) \times U(1)$ by the formation of quark condensates. According to Goldstone’s theorem this leads to $N_f^2 - 1$ massless excitations.

At low energy, these are the relevant degrees of freedom of the QCD partition function, and it is possible to write down an effective Lagrangian for an arbitrary value of the vacuum angle $\theta$ in terms of these degrees of freedom only [1, 3, 2]. The partition function can be simplified further by suppressing the space-time dependence of the Goldstone modes altogether.

Recently, Leutwyler and Smilga [2] evaluated the QCD partition function in this limit. This enabled them to write down sum rules for the spectrum of the Dirac operator. What enters in these sum rules is what we call the microscopic spectral density as opposed to the continuum spectral density that enters in the calculation of the condensate [1]. The latter one is related to the thermodynamic limit of the spectral density, whereas the former one provides information on the way the thermodynamic limit is approached. The question we would like to address is in how far the sum rules are specific for the QCD partition function, and in what sense they are universal. Since the partition function of [2] only involves constant fields one would suspect that the detailed structure of the QCD vacuum is not important, and that the sum rules only depend on the symmetries of the theory.

This adagio is taken to the extreme by constructing a model with the correct chiral structure, but that apart from this does not contain any other information. As is well known from random matrix theory, the minimization of information leads to gaussian random matrix ensembles [7]. The appropriate ensemble will be constructed in section 2, and with the help of mathematical techniques developed in the framework of Anderson localization and compound nucleus scattering [8, 9] we are able to derive the partition function that was obtained by Leutwyler and Smilga (see section 3). Sum rules for one
and two flavors are derived in section 4. The structure of the random matrix model was inspired by the semiclassical approximation to the QCD partition function where instantons are the main degrees of freedom \[10, 11, 12\]. Therefore, it is natural to check the validity of these sum rules for an instanton liquid model \[13\] of the QCD vacuum, a task which is carried out in section 5. Concluding remarks are made in section 6. Finally, in appendix A sum rules for a finite number of degrees of freedom are derived.

2. Random matrix model

The Euclidean QCD partition function for \(N_f\) flavors and nonzero vacuum angle \(\theta\) can be written as

\[
Z_{\text{QCD}} = \sum_{\nu} e^{i\theta\nu} < \prod_{f=1}^{N_f} \prod_{n} (\lambda_n^2 + m_f^2) m_f^\nu >_{S_\nu(A)} ,
\]

where the product is over the positive eigenvalues of the Dirac operator, and the average \(< \cdots >_{S_\nu(A)}\) is over gauge field configurations with topological quantum number \(\nu\) weighted by the gauge field action \(S_\nu(A)\). The topological part of the action, \(\exp i\theta\nu\), has been displayed explicitly. Here and below, the mass matrix is taken to be diagonal. According the Atiyah-Singer theorem, the Dirac equation in a gauge field with topological quantum number \(\nu\) has exactly \(\nu\) zero eigenvalues. However, configurations with zero total topological charge may actually be composed of spatially well-separated components with opposite topological charge. Such configurations will give rise to almost zero modes and thus will play an important role in the chiral dynamics of the QCD partition function. Our starting point is that the chiral properties of the QCD partition function are determined by zero modes and almost zero modes only. Although it is not necessary for the sake of the argument, it is instructive to think of the field configurations as a superposition of \(N_+\) instantons and \(N_-\) anti-instantons. Each isolated instanton or anti-instanton has exactly one fermionic zero mode with a definite chirality \[14\]. In total we have \(\nu = N_+ - N_-\) exact zero modes. At finite separation of instantons and anti-instantons the remaining modes are no longer exact zero modes of the Dirac equation and give rise to nonzero overlap matrix elements \(T\) of the Dirac operator \[15\].

A model describing the zero mode part of the QCD partition function with \(N\) zero
modes or almost zero modes in the Euclidean volume $V$ is defined by
\[
Z(\theta) = \sum_N \mu(N) \sum_{N_+} \left( \frac{N}{N_+} \right) e^{i \theta (N_+ - N_-)} \int \mathcal{D}T \mathcal{D}{T^\dagger} \prod_f \det \left( \frac{m_f}{i} T^\dagger m_f \right). \tag{2.2}
\]
Zero modes of each chirality are treated as independently distributed identical particles, hence the binomial factor. The distribution function $\mu(N)$ of the total number of zero modes $N = N_+ + N_-$ is peaked at some average value $\overline{N}$ with a width that is much smaller than $\overline{N}$. As long as a large number of different values of $N$ contribute to the partition function with roughly equal probability\footnote{If $\mu(N) \sim \delta(N - \overline{N})$ the parity of $N_+ - N_-$ is the same as the parity of $\overline{N} = N_+ + N_-$, and the partition function has the additional symmetry $Z(\theta = 0) = Z(\theta = \pi)$.}, our results do not depend on the detailed shape of $\mu(N)$. The average over all gauge field configurations in eq. (2.1) is replaced by an average over gaussian distributed overlap matrix elements with distribution function given by
\[
P(T) = \exp \left( -\frac{N}{2\lambda^2} T T^\dagger \right). \tag{2.3}
\]
The integration measure $\mathcal{D}T$ is the Haar measure. The structure of the overlap matrix, with off-diagonal blocks $T$ and $T^\dagger$ and diagonal blocks equal to the quark masses $m_f$ times the identity, is dictated by the chirality of the zero modes. The matrix $T$ is a $N_+ \times N_-$ matrix, and for zero quark masses, the total overlap matrix has $|N_+ - N_-|$ exact zero eigenvalues. The density $\overline{N}/V$ of the total number of zero modes is kept fixed. As in the instanton liquid approximation to the QCD partition function [16], it is considered to be an external parameter. The thermodynamic limit can thus be taken by letting $N \to \infty$, where here and below we will omit the bar. A similar model with $N_+ = N_-$ was considered in [17, 18].

The order parameter in the study of chiral symmetry breaking is the quark condensate $\langle \bar{q}_f q_f \rangle$ defined by
\[
\langle \bar{q}_f q_f \rangle = \lim_{m_f \to 0} \lim_{N \to \infty} -\frac{1}{N} \frac{d}{dm_f} \log Z(\theta), \tag{2.4}
\]
where the order of the limits should be taken as indicated in the formula. In general, $\langle \bar{q}_f q_f \rangle$ depends on $\theta$. Its value at $\theta = 0$ in the limit where for all flavors $m_f \to 0$ is approached from above is denoted by $-\Sigma$. By writing the determinant as the product
\[ \Pi' (\lambda_n^2 + m_f^2) \] one obtains the Banks-Casher formula for \( \Sigma \)

\[ \Sigma = \pi < \rho_C(0) >_{Z(\theta=0)}, \] (2.5)

where the average \( < \cdots >_{Z(\theta)} \) is with respect to the partition function (2.2) (or (2.1) in the case of QCD). The continuum spectral density is defined by

\[ \rho_C(\lambda) = \lim_{m_f \to 0} \lim_{N \to \infty} \frac{1}{N} \rho(\lambda), \] (2.6)

and the spectral density \( \rho(\lambda) \) is

\[ \rho(\lambda) = \sum \delta(\lambda - \lambda_n), \] (2.7)

where the eigenvalues \( \pm \lambda_n \) are the nonzero eigenvalues of the overlap matrix in the chiral limit.

We will also consider a different limit of the derivative with respect of \( m \) of the partition function, namely,

\[ \lim_{N \to \infty} \frac{1}{N} \frac{d^p}{d m_f^p} \log Z(\theta). \] (2.8)

Let us consider the case \( p = 1 \) in more detail. Again writing the determinant as a product over eigenvalues, one finds

\[ \lim_{N \to \infty} \lim_{N m_f \to \infty} \frac{1}{N} \rho(\lambda) = \lim_{N \to \infty} \int_0^\infty dx < \frac{1}{N} \rho(\frac{x}{N}) >_{Z(\theta)} \frac{N m_f}{x^2 + m_f^2 N^2}. \] (2.9)

What enters in this expression is what we call the microscopic spectral density defined by

\[ \rho_S(x) = \lim_{N \to \infty} \frac{1}{N} \rho(\frac{x}{N}), \] (2.10)

as opposed to the continuum spectral density defined by in eq. (2.6). In this limit, the spectral density function near zero is enlarged proportional to the size (given by \( N \)) of the system. Note that \( \rho_S(x) \) depends on \( N m_f \).
3. Calculation of the partition function

In order to evaluate the partition function (2.2) the determinant is written as an integral over Grassmann variables

$$\prod_f \det \begin{pmatrix} m_f & i T^f_m \end{pmatrix} = \int \prod_f \mathcal{D}\psi^f \mathcal{D}\phi^f \exp \sum_f \left( \psi^{f*}_i \phi^{f*}_i \right) \left( \begin{pmatrix} m_f & i T^f_m \end{pmatrix} \begin{pmatrix} \psi^f_i \\ \phi^f_i \end{pmatrix} \right),$$

(3.1)

where the measure of the Grassmann integration is as usual

$$\mathcal{D}\psi^f = \prod_i d\psi^f_i d\psi^{f*}_i,$$

(3.2)

and the conjugation $^*$ is the conjugation of the second kind (i.e., $\psi^{**} = -\psi$). The integral over $T$ is gaussian and can be performed easily. In the partition function this results in the factor

$$\exp \frac{2\lambda^2}{N} \psi_i^{f*} \phi_i^g \phi_j^g \phi_j^f,$$

(3.3)

which represents a 4-fermion interaction.

The quartic term can be written as a sum of two squares

$$\psi_i^{f*} \psi_i^g \phi_j^g \phi_j^f = \frac{1}{4} (\psi_i^{f*} \psi_i^g + \phi_i^{f*} \phi_i^g)(\psi_j^{g*} \psi_j^f + \phi_j^{g*} \phi_j^f) - \frac{1}{4} (\psi_i^{f*} \psi_i^g - \phi_i^{f*} \phi_i^g)(\psi_j^{g*} \psi_j^f - \phi_j^{g*} \phi_j^f).$$

(3.4)

Each of the two squares can be linearized with the help of a Hubbard-Stratonovich transformation. This allows us to perform the Grassmann integrations at the expense of the introduction of the new real valued integration variables $\sigma^f$ and $\bar{\sigma}^f$, respectively. Apart from an irrelevant overall constant, the partition function reduces to

$$Z = \sum_N \mu(N) \sum_{N_+} \binom{N}{N_+} \int \mathcal{D}\sigma \mathcal{D}\bar{\sigma} \det^{N_+}(\sigma + i\bar{\sigma} + m) \det^{N_-}(\sigma - i\bar{\sigma} + m) \times \exp -\frac{N}{2\lambda^2} \text{Tr}(\sigma + i\bar{\sigma})(\sigma - i\bar{\sigma}).$$

(3.5)

As always, the measure of the integral over the matrices $\sigma$ and $\bar{\sigma}$ is the Haar measure. The diagonal mass matrix is denoted by $m$.

The complex matrix $\sigma + i\bar{\sigma}$ can be decomposed in 'polar coordinates' as

$$\sigma + i\bar{\sigma} = U\Lambda V^{-1},$$

(3.6)
where $U$ and $V$ are unitary matrices and $\Lambda$ is a diagonal real positive definite matrix. Since the r.h.s has $N_f$ more degrees of freedom than the l.h.s., one has to impose constraints on the new integration variables. This can be achieved \cite{19} by restricting $U$ to the coset $U(N_f)/U(1)^{N_f}$, where $U(1)^{N_f}$ is the diagonal subgroup of $U(N_f)$. In terms of the new variables the partition function reads

$$Z = \sum_N \mu(N) \sum_{N_+} \left( \begin{array}{c} N \\ N_+ \end{array} \right) \int J(\Lambda) \mathcal{D}\Lambda \mathcal{D}U \mathcal{D}V \times \det^{N_+}(U\Lambda V^{-1} + me^{i\theta/N_f}) \det^{N_+}(V\Lambda U^{-1} + me^{-i\theta/N_f}) \exp(-\frac{N}{2\lambda^2} \text{Tr}\Lambda^2),$$

(3.7)

where the integral over $U$ is over $U(N_f)/U(1)^{N_f}$ and the integral over $V$ is over $U(N_f)$. A phase factor $\exp(-i\theta N_f)$ has been absorbed in the integration over $V$.

For $N_f$ flavors we have $N_f$ condensates which break down the symmetry of the action to $U(N_f)/U(1)^{N_f}$ leaving us with $N_f^2 - 1$ Goldstone modes for $N_+ \neq N_-$. When the total topological charge is zero the phase of the determinant also cancels which provides us with an additional zero mode.

The term proportional to $m$ plays the role of a small symmetry breaking term. The integrals over the nonzero modes will be performed by a saddle point integration at $m = 0$, whereas the integrals over the soft modes will be accounted for exactly at a fixed value of $mN$. There are two types of nonzero modes, the phase $\exp(i\alpha)$ of the determinant $\det VU^{-1}$ for $N_+ \neq N_-$, and the eigenvalues $\Lambda$. The partition function at $m = 0$ factorizes accordingly,

$$Z(m = 0) = \sum_N \mu(N) \int d\alpha (\exp(i\alpha) + \exp(-i\alpha))^N \int J(\Lambda) d\Lambda \det^{N_+}\Lambda \exp(-\frac{N}{2\lambda^2} \text{Tr}\Lambda^2).$$

(3.8)

The leading order contribution in $1/N$ of the integral over $\Lambda$ can be obtained by a saddle point approximation. The saddle point equations for the $\Lambda$ integrals read

$$\Lambda_i = \pm \lambda.$$

(3.9)

The negative solution is not inside the integration manifold and can be omitted.

The integral over $\alpha$, which ranges from 0 until $2\pi$, can be executed either before or after the summation over $N_+$. In the first case we find zero for odd values of $N$ while for
even values of $N$ only the term $N_+ = N_- = N/2$ contributes in which case the $\alpha$ integral becomes soft. However, in the second case the interference between contributions of all different topologies results in the integrand $(2 \cos \alpha)^N$ which allows us to perform the integral by a saddle point approximation in order to obtain the leading order contribution in $1/N$. The saddle points are located at $\alpha = 0$ and $\alpha = \pi$. It is at this point that the distribution function $\mu(N)$ plays a role: the contribution of the latter saddle point, $(-1)^N$, can be ignored after the summation over $N$, whereas the contribution at $\alpha = 0$ yields a overall factor $\mu(N)$ that does not contribute to the $m$ dependent part of the partition function to be discussed below.

At the saddle points in $\Lambda$ and $\alpha$, the $U-$dependence can be absorbed into $V$. The the $U-$integration yields a finite irrelevant constant. Since the phase $\exp i\alpha$ has already been extracted the remaining integral over $V$ is over $SU(N_f)$ instead of $U(N_f)$. We treat $m$ as a small parameter and expand the determinants up to first order in $m$. At the saddle point $\alpha = 0$ the result for $m$ dependent part of the partition function is

$$Z(m, \theta) = \int_{\det V=1} \mathcal{D}V \exp \left( N^2 \lambda \frac{1}{2} \text{Tr}(mV^{-1} \exp(-i\theta/N_f) + mV \exp(i\theta/N_f)) \right), \quad (3.10)$$

which coincides with the result derived by Smilga and Leutwyler [2] for the QCD partition function using chiral perturbation theory.

The value of $\Sigma$ at $\theta = 0$ can be obtained from eq. (2.4)

$$\Sigma(\theta = 0) = \lim_{m \to 0} \lim_{N \to \infty} \frac{1}{2\lambda} \left< \text{Tr}(V + V^{-1}) \right>_{Z(\theta=0)}. \quad (3.11)$$

For $N_f = 1$ the integral over $V$ is absent, and the sign of the quark condensate is independent of the sign of $m$. For more than one flavor the order of the limits allows us to perform the $V$ integral by a saddle point approximation. In the case of equal positive masses the saddle point for $\theta = 0$ is at $V = 1$ which allows us to identify the parameter $\lambda$ as

$$\lambda = \frac{1}{\Sigma(\theta = 0)}, \quad (3.12)$$

which completes the calculation of the partition function.

As observed in [2], for more than one flavor the value of the condensate depends on the sign of the quark mass. For example, in the case of two flavors with equal negative masses and $\theta = 0$, the saddle point is at $V = -1$. 

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It is also possible to introduce a complex mass in eq. (2.2) with the mass in the lower block equal to the complex conjugate of the mass in the upper block of the overlap matrix. In this case the final result for the partition function depends only on the combination $me^{i\theta}$ and its complex conjugate which makes it possible [2] to derive relations between the $m$ and the $\theta$ derivatives of the partition function.

4. Sum rules for one and two flavors

Sum rules for moments of the inverse squares of the eigenvalues of the Dirac operator can be derived for an arbitrary number of flavors and for any value of the total topological charge [2]. In this section we only present a derivation for the simplest two cases of one flavor and of two flavors with equal masses. Proofs of the general results can be found in [2].

For $N_f = 1$ there is no integration and the result for the ratio of the massive and massless partition function is particularly simple

$$Z_{N_f=1} = \exp(Nm\Sigma \cos \theta),$$

which was first obtained in [2].

In the case of two flavors the integral over $SU(2)$ can be performed for an arbitrary mass matrix. Here, we restrict ourselves to the simpler case of equal quark masses. In this case the ratio of the massive and the massless partition function reduces to

$$Z_{N_f=2} = \int_0^{2\pi} \frac{d\phi}{\pi} \sin^2 \frac{\phi}{2} \exp(2Nm\Sigma \cos \frac{\phi}{2} \cos \theta),$$

where the factor $\sin^2 \frac{\phi}{2}$ results from the invariant measure of $SU(2)$. The integral is elementary and results in

$$Z_{N_f=2} = \frac{I_1(2mN\Sigma \cos \frac{\theta}{2})}{mN\Sigma \cos \frac{\theta}{2}}. \tag{4.3}$$

The partition function can also be evaluated by writing the fermion determinant as a product over eigenvalues. For $N_f = 1$, we obtain sum rule

$$\exp(\Sigma mN \cos \theta) = \langle m^\nu \prod_n' (1 + \frac{m^2}{\lambda^2_n}) \rangle_{Z(m=0)}, \tag{4.4}$$
where the average is with respect to the partition function $Z$ with $m = 0$, which also includes the weight factor $\prod' \lambda_n^2$ involving the product of the nonzero eigenvalues of the Dirac operator. The exclusion of zero eigenvalues in the product is denoted by a prime. For two flavors a similar sum rule can be derived

$$
\prod_{x_{1n} \geq 0} \left(1 + \frac{4m^2 N^2 \cos^2 \frac{\theta}{2}}{x_{1n}^2} \right) = < m^{2\nu} \prod_n' \left(1 + \frac{m^2}{\lambda_n^2} \right)^2 >_{Z(m=0)} \ . \tag{4.5}
$$

Here, the $x_{1n}$ is the $n$’th zero of the Bessel function $J_1$. By expanding both the r.h.s. and the l.h.s. of eqs. (4.4) and (4.5) in powers of $m$ we obtain sum rules for the inverse moments of the eigenvalues $\lambda_n$. Additional sum rules for $N_f = 2$ can be derived from the expansion in powers of both $m_u$ and $m_d$ of the general expression for the partition function.

The sum rules (4.4) and (4.5) are for a fixed value of $\theta$. However, we will investigate the validity of the Leutwyler-Smilga sum rules in an interacting instanton vacuum where, for technical reasons, the total topological is taken to be zero. The corresponding partition function $Z_0$ is given by the Fourier transform of $Z(\theta)$,

$$
Z_0 = \int_0^{2\pi} \frac{d\theta}{2\pi} Z(\theta) \ . \tag{4.6}
$$

which can be derived from the the decomposition

$$
Z = \sum_\nu e^{i\nu\theta} Z_\nu . \tag{4.7}
$$

For one and two flavors, the integrals over $\theta$ are elementary. The results

$$
Z_0 = I_0(Nm\Sigma) \quad \text{for} \quad N_f = 1, \tag{4.8}
$$

$$
Z_0 = I_0^2(Nm\Sigma) - I_1^2(Nm\Sigma) \quad \text{for} \quad N_f = 2, \tag{4.9}
$$

were first derived by Leutwyler and Smilga \cite{3]. Sum rules are obtained by equating (4.8) and (4.9) to the r.h.s. of (4.4) and (4.5) restricted to zero topological charge, respectively. For $N_f = 1$ one finds

$$
I_0(Nm\Sigma) = < \prod_n (1 + \frac{m^2}{\lambda_n^2}) >_{\nu=0} , \tag{4.10}
$$

and for $N_f = 2$ we have

$$
I_0^2(Nm\Sigma) - I_1^2(Nm\Sigma) = < \prod_n (1 + \frac{m^2}{\lambda_n^2})^2 >_{\nu=0} . \tag{4.11}
$$
The expansion of both sides of this equation in powers of $mN$ yields sum rules for moments of the inverse eigenvalues of the Dirac operator. The first two sum rules are

\[ \langle \sum_{n>0} \frac{1}{N^2 \lambda_n^2} \rangle_{\nu=0} = \frac{\Sigma^2}{4N_f}, \]

\[ \langle (\sum_{n>0} \frac{1}{N^2 \lambda_n^2})^2 \rangle_{\nu=0} - \langle \sum_{n>0} \frac{1}{N^2 \lambda_n^2} \rangle_{\nu=0}^2 = \langle \sum_{n>0} \frac{1}{N^4 \lambda_n^4} \rangle_{\nu=0} - \frac{\Sigma^4}{16N_f(N_f+1)}. \]

(4.12)  

(4.13)

As was shown by Leutwyler and Smilga, both sum rules hold for arbitrary $N_f$ and also be worked out for arbitrary topological charge. The only modification is to replace $N_f$ by $N_f + |\nu|$. In the case of two or more flavors additional sum rules can be derived by varying the quarks masses independently \[2\]. We only quote the result

\[ \langle \frac{1}{N^4 \lambda_n^4} \rangle_{\nu=0} = \frac{\Sigma^4}{16N_f(N_f^2 - 1)}. \]

(4.14)

Again, the result for arbitrary topological charge is obtained by replacing $N_f$ by $N_f + |\nu|$. Formally, this sum rule diverges for $N_f = 1$. That this is indeed the case can be understood from the behavior of the spectral density at small $\lambda$. For $N_+ = N_-$ we expect $\rho(\lambda) \sim \lambda^{N_f+2}$, where the factor $\lambda^{N_f}$ originates from the fermion determinant and the factor $\lambda^2$ from the Jacobian of the unitary transformation that diagonalizes the overlap matrix.

The above sum rules diverge for $N_f = 0$. Consequently, we expect that in this case, and therefore in the quenched approximation, that the thermodynamic limit of the spectral density near zero is approached in a completely different way.

The sum rules (4.12-14) can be expressed in the microscopic spectral density by writing the sum over the eigenvalues as the integral

\[ \sum_{n} \frac{1}{N^p \lambda_n^p} = \int \frac{dx}{x^p} \rho_S(x)_{mN=0}. \]

(4.15)

The second sum rule expresses an integral over the correlation function $\langle \rho_S(x) \rho_S(x') \rangle_{\nu=0}$ in terms of integrals over the average microscopic spectral density $\langle \rho_S(x) \rangle_{\nu=0}$.

Finally, let us write down a sum rule for nonzero values of $mN$. In the case of one flavor one obtains from the infinite product expansion for the Bessel function $I_0$,

\[ \sum_n \frac{mN \Sigma^2}{x_{4n}^2 + m^2 N^2 \Sigma^2} = \int_0^\infty \frac{dx}{x^2} \langle \rho_S(x) \rangle_{mN \text{ fixed}}_{\nu=0} \frac{mN}{x^2 + m^2 N^2}. \]

(4.16)
where $x_{0n}$ is the $n$’th zero of the Bessel function $J_0$. One is tempted to invert this equality in order to obtain an analytical answer for the microscopic spectral density. Since $\rho_S$ depends on $mN$ there is no unique solution, but a trivial solution can be written down readily,

$$\rho_S(x) = \sum_n \delta(x - \frac{x_{0n}}{\Sigma}). \quad (4.17)$$

Because there is no reason to believe that the dispersion (4.13) is zero, we do not expect that this is the exact microscopic level density.

In appendix A it is shown that for one flavor sum rules can also be derived at finite $N$. These results provide an explicit proof that the thermodynamic limit has been taken correctly.

5. Sum rules in the instanton liquid

The partition function (2.1) can be approximated semiclassically by a liquid of instantons. Instead of averaging over all gauge field configurations, we average over the collective coordinates of the instantons only, whereas 1-loop quantum fluctuations about the instantons are included in the measure. The action in (2.1) is the instanton action, which also includes the interaction between instantons. We use the so called streamline [20, 21] interaction supplemented by a core in order to stabilize the instanton liquid. The fermion determinant is calculated in the space spanned by the fermionic zero modes with overlap matrix elements that can be derived from the streamline configuration [22]. More details on the above instanton liquid model can be found in [13].

The numerical simulations were carried out for a liquid of 64 instantons in a Euclidean space time volume of $(2.378)^3 \times 4.756$ in units of $\Lambda^{-4}$. Averages were obtained from 125 statistically independent configurations. Our main results are presented in Table 1. Calculations were done for one, two and three massless flavors (see heading). The first row shows the smallest eigenvalue and its average level motion (between brackets). The condensate $\Sigma$ (row 2) is obtained from an extrapolation of the spectral density to $\lambda = 0$. The results for the sum rule (4.12) (row 3) are compared to the analytical result (row 4). We find complete agreement inside the error bars. The remaining rows involve numerical
and theoretical values for the quantities $S_1$ and $S_2$ defined by

$$S_1 = \frac{< (\sum \frac{1}{\lambda n} )^2 >_{\nu=0} - < \sum \frac{1}{\lambda n} >_{\nu=0}}{< \sum \frac{1}{\lambda n} >_{\nu=0}^2}, \quad S_2 = \frac{< (\sum \frac{1}{\lambda n} )^2 >_{\nu=0}}{< \sum \frac{1}{\lambda n} >_{\nu=0}^2}. \quad (5.1)$$

Also in this case we find excellent agreement with the theoretical values

$$S_{1\text{th}} = \frac{N_f}{N_f + 1}, \quad S_{2\text{th}} = \frac{N_f^2}{N_f^2 - 1}. \quad (5.2)$$

where the derivation for $S_2$ does not hold for $N_f = 1$. We consider this a strong argument in favor of the universality of the sum rules.

6. Discussion and conclusions

The main conclusion of this work is that the low energy chiral limit of the QCD partition function as derived by Leutwyler and Smilga coincides with a large $N$ limit of a random matrix theory. Therefore all conclusions of [2] also pertain to the chiral random matrix theory proposed in section 2. The random matrix theory resembles the partition function of a gas of instantons. The main difference is that instead of inheriting its disorder from the distribution of the collective coordinates, the overlap matrix elements are taken to be independently distributed according to a gaussian random matrix ensemble.

From random matrix theory it is know that [23] microscopic level correlations such as for example the nearest neighbor level spacing distribution or the variance of the number of levels as a function of the average number of levels in a given interval are universal. They only depend on the symmetries of the Hamiltonian but not on the details of the matrix elements. The spectral sum rules put forward by Smilga and Leutwyler involve the microscopic level density. Since they can be obtained from a chiral random matrix theory we expect that the sum rules are also universal, and that they do not depend on the details of the low-energy structure of the theory but only on its symmetries. In particular, they depend on the number of flavors and, for example, diverge in the quenched approximation. This conclusion has been confirmed by simulations of an instanton liquid with overlap matrix elements that differ strongly from those of a gaussian random matrix ensemble. Inside the statistical uncertainty of the calculations all sum rules were reproduced. The
physical interpretation is that the way the thermodynamic limit of the spectral density near zero is approached is universal. However, its value, \textit{i.e.} the chiral condensate, depends on the details of the theory.

The advantage of a random matrix model is that it is amenable to powerful mathematical techniques \cite{24,9} that makes it possible to obtain explicit analytical results for \textit{e.g.} the microscopic spectral density, which, by virtue of the universality arguments discussed above, also hold for the QCD partition function. Work in this direction is under way.

\textbf{Acknowledgements}

The reported work was partially supported by the US DOE grant DE-FG-88ER40388. We acknowledge the NERSC at Lawrence Livermore where most of the computations presented in this paper were performed. We would like to thank A. Smilga, M.A. Nowak and I. Zahed for useful discussions.

\textbf{Appendix A}

In this appendix we derive the sum rules (4.12) and (4.13) for one flavor without relying on a saddle point approximation. For \(N_+ = N_- = N/2\) and \(N_f = 1\) the partition function simplifies to

\[
Z = \int d\sigma d\bar{\sigma} (\sigma + i\bar{\sigma} - m)^{\frac{N}{2}} (\sigma - i\bar{\sigma} - m)^{\frac{N}{2}} \exp\left(-\frac{N}{2\lambda^2} (\sigma + i\bar{\sigma}) (\sigma - i\bar{\sigma})\right),
\]

where an irrelevant overall constant has been suppressed. At finite \(N\) the pre-exponential factors can be expanded as a binomial series which provides us with an expansion in powers of \(m\). The coefficients are elementary integrals, and for the \(m\)-dependent part of the partition function we find

\[
\frac{Z(m)}{Z(0)} = 1 + \frac{m^2 N^2 \Sigma^2}{4} + \frac{m^4 N^4 \Sigma^4}{64} (1 - \frac{2}{N}) + \cdots.
\]

It should be noted that no approximations have been made. On the other hand, the fermion determinant can be written as a product over the eigenvalues which leads to the
expansion

\[
\frac{Z(m)}{Z(0)} = 1 + m^2 \left\langle \sum_n \frac{1}{\lambda_n^2} \right\rangle_{\nu=0} + m^4 \frac{1}{2} \left\langle \sum_{n \neq n'} \frac{1}{\lambda_n^2 \lambda_{n'}^2} \right\rangle_{\nu=0} + \cdots, \tag{A.3}
\]

where the average is with respect to the massless partition function. By equating the coefficients of the powers of \( m^2 \) we obtain sum rules for the inverse powers of the eigenvalues that are valid for any value of \( N \). The sum rules (4.12) and (4.13) for \( N_f = 1 \) are reproduced by keeping only the leading order terms in \( 1/N \). We observe that the first sum rule is valid for any value of \( N \). The second sum rule is modified by the factor \((1 - 2/N)\). For \( N = 2 \) we find zero which is correct because in this case there are no terms that contribute to the sum \( n' \neq n \) in (A.3).
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|                | $N_f = 1$       | $N_f = 2$       | $N_f = 3$       |
|----------------|----------------|----------------|----------------|
| $\lambda_1$   | 0.0255(98)     | 0.0531(164)    | 0.1078(335)    |
| $\Sigma$       | 1.90(5)        | 1.33(5)        | 0.90(10)       |
| $\Sigma^2/\lambda_i^2$ | 0.89(3)  | 0.21(1)        | 0.066(2)       |
| $\Sigma^2/4N_f$| 0.90(3)        | 0.22(1)        | 0.068(7)       |
| $S_{1\uparrow}$| 0.53(3)        | 0.70(5)        | 0.83(7)        |
| $S_{1\uparrow}^{th}$ | 1/2    | 2/3            | 3/4            |
| $S_{2\uparrow}$ | 2.0(4)         | 1.20(12)       | 1.24(12)       |
| $S_{2\uparrow}^{th}$ | $\infty$ | 4/3            | 9/8            |

Table 1. Numerical results for the spectral properties of the Dirac operator in the gauge field of a liquid of instantons. From the comparison of the 3rd and 4th, the 5th and 6th and the 7th and 8th rows we conclude that the Leutwyler-Smilga sum rules are observed by this model. For the definition and further discussion of the observable, we refer to section 5.