New Fermions at $e^+e^-$ Colliders:

I. Production and Decay.

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ABSTRACT

We analyze the production in $e^+e^-$ collisions of new heavy fermions stemming from extensions of the Standard Model. We write down the most general expression for the production of two heavy fermions and their subsequent decays, allowing for the polarization of the $e^+e^-$ initial state and taking into account the final polarization of the fermions. We then discuss the various decay modes including cascade and three body decays, and the production mechanisms, both pair production and single production in association with ordinary fermions.
1. Introduction

Despite its tremendous success in describing all experimental data available today, the Standard Model of the electromagnetic, weak and strong interactions based on the gauge symmetry $SU(3)_C \times SU(2)_L \times U(1)_Y$, is widely believed not to be the ultimate truth. Besides the fact that it has too many parameters which are merely incorporated by hand, the Standard Model does not unify the electroweak and strong forces in a satisfactory way since the coupling constants of these interactions are different and seem to be independent. Therefore, one expects the existence of a more fundamental theory which describes the three forces within the context of a single gauge group and hence, with only one coupling constant. This grand unified theory will be based on a gauge group containing $SU(3)_C \times SU(2)_L \times U(1)_Y$ as a subgroup and will reduce to this symmetry at present energies.

The grand unified groups [1–4] provide fermion representations in which a complete generation of Standard Model quarks and leptons can be naturally embedded. In most of the cases these representations are large enough to accommodate additional new fermions which, in fact, are needed to have anomaly–free theories. It is conceivable that these new fermions, if for instance they are protected by some symmetry, acquire masses not much larger than the Fermi scale. This is very likely, and even necessary if the new gauge bosons which are generic predictions of the unified theories, are relatively light [5].

Besides the SU(5) group [1] [the simplest Lie group containing $SU(3)_C \times SU(2)_L \times U(1)_Y$ as a subgroup and with two representations to accommodate the 15 Standard Model fermions] which has no room for “light” new fermions or gauge bosons, the SO(10) group [3] has received much attention. It is the simplest group in which the 15 Weyl spinors of each Standard Model generation of fermions can be embedded into a single multiplet. This representation has dimension 16 and, to have an anomaly–free theory, contains a right–handed Majorana neutrino. In fact, heavy isosinglet neutrinos have been discussed in various models, such as left–right symmetric models [6], in attempts to explain the small masses of the three observed neutrinos.

Another popular unifying group is $E_6$ [4] which contains SU(5) and SO(10) as subgroups and is the next anomaly–free choice after SO(10). The interest in $E_6$ is mainly due the fact that superstring theories, which attempt to unify all fundamental forces including gravity, suggest that this symmetry is an acceptable four dimensional field theoretical limit [7]. In $E_6$, each quark–lepton generation lies in the representation of dimension 27; to complete this representation, twelve new fields are needed in addition the Standard Model fermion fields. For each family one has two additional isodoublets of leptons, two isosinglets neutrinos [which can be either of the Dirac or Majorana type] and an isosinglet quark with charge $-1/3$.

Several other gauge groups have been considered with various theoretical motivations and most of them predict the existence of new fermions. For instance, schemes of grand unification based on large orthogonal groups have been proposed to explain the origin of parity violation in weak interactions [8]: in these models, weak interactions are parity symmetric but fermions with left–handed and right–handed couplings acquire different masses. They
predict a rich spectrum of fermions, called mirror fermions [9], which have the opposite chiral properties of the ordinary ones. In the simplest version of these models [10], the gauge symmetry and the symmetry breaking pattern are the same as in the Standard Model: one simply adds to the spectrum of the latter three families of heavy fermions with opposite chiralities. Theoretical arguments based on the unitarity of scattering amplitudes [11] suggest that the masses of these mirror fermions should not exceed a few hundred GeV.

The direct search for these new fermions and, in case of discovery, the study of their basic properties, will be a major goal of the next generation of accelerators. In this paper, we analyze in detail the production of these new fermions at $e^+e^-$ colliders.

The new leptons and quarks will mix with the ordinary fermions of the Standard Model [12, 13]. This mixing will give rise to new currents which determine to a large extent their decay properties and allows for new production mechanisms. If the new particles have non–zero electromagnetic and weak charges, they can be pair produced if their masses are smaller than the beam energy. In general the reactions are built–up by a superposition of photon and $Z$ boson exchange [additional contributions could come from extra gauge bosons if their masses are not much larger than the c.m. energy of the collider]. The cross sections are large [14] and, up to phase space suppression factors, of the order of the point-like QED cross section for muon pair production.

Fermion mixing allows an additional production mechanism for the new fermions: single production in association with their light partners. In the case of quarks [and for second and third generation new leptons if inter–generational mixing is neglected] the production process is mediated by $s$–channel gauge boson exchange; since the mixing angles are restricted to values smaller than $O(10^{-1})$ by present experimental data [13], the cross sections are rather small. But in the case of [the first generation] heavy leptons, additional $t$–channel exchanges, $W$ exchange for neutral leptons and $Z$ exchange for charged leptons, are present increasing the cross sections by several orders of magnitude. This results in large production rates [14] which permit to probe lepton masses close to the total c.m. energy.

The new fermions will decay through mixing into light fermions and gauge bosons. Depending on the particle masses, the gauge bosons can be real or virtual and will decay into light quarks and leptons. Therefore, in the production of the new fermions the final states are rather complicated: six particles in the case of pair production and four particles for single production. However, it is very important to have at hand the information on the correlation between all the particles involved in the process. Indeed, these correlations will be very helpful to optimize the experimental cuts which permit to suppress the various backgrounds without affecting drastically the signal cross sections. Furthermore, they permit to discriminate between particles of different nature [e.g. Majorana or Dirac neutrinos] or with different couplings [vector, “mirror” or standard couplings in pair production, or with left and right–handed mixing in single production] and therefore shed some light on the origin of the new fermion.
Several analyses of the production of new fermions in $e^+e^-$ collisions have been conducted in the recent years [14-20] in various special cases. In this paper we will extend on these analyses in the following ways:

(i) We present the most general expressions for the production of two heavy fermions, with different masses to allow for the production of two different fermions and to treat the cases of single and pair production in the same footing, of any flavor and for arbitrary couplings of the new fermions, including all possible channels and the polarization of the initial $e^+e^-$ state; complete formulae are given for angular distributions and total cross sections.

(ii) We give the most general expression of the decay of the heavy fermions into three body final states, allowing for cascade decays, including all channels and the possibility of off-shell intermediate vector bosons; complete formulae for the angular and energy distributions of the final decay products as well as the total decay widths of the heavy fermions are given.

(iii) We systematically take into account the polarization of the heavy fermions in both the production and the decay processes; due to the factorization of the two sequences, the information on the final polarization permits an easy reconstruction of the full correlations between the initial state and the final particles from the decays of the heavy fermions.

Note that since our analytical results are general, they can also be used to discuss, at $e^+e^-$ colliders, the details of top quark pair production, or the production of heavy fermions of a fourth generation with a heavy neutrino.

The paper is organized as follows. In the next section, we summarize the interactions of the new fermions. In section 3, we analyze the general case of the production of two heavy fermions and discuss the cases of pair production and the single production in association with ordinary fermions. In section 4, we discuss in detail the decay modes of the new fermions including the cascade decays and the three body decays with off–shell gauge bosons. Section 5 contains our conclusions. For completeness, we summarize in the Appendix the formalism for combining the spin–dependent cross sections and decay distributions which permits to obtain the full correlations between all the particles involved in the process.

2. Interactions

The new fermions couple to the photon and the electroweak gauge bosons $W/Z$ with full strength, except for singlet neutrinos which have zero electromagnetic and weak charges and therefore couple to the latter only through mixing as will be discussed later. They similarly couple to extra gauge bosons when the latter are present. Allowing for these extra currents, the interaction is given by the Lagrangian

$$\mathcal{L} = \sum_{V=\gamma,Z,W,\ldots} g_V J^\mu_V V_\mu$$

(2.1)
where the complex conjugation of the charged currents is understood. The currents $J^V_{\mu}$ can be expressed in terms of left–handed and right–handed charges as

$$J^V_{\mu} = \sum_{f} \bar{\psi}_f \gamma_\mu \left[ Q^f_{L} (1 - \gamma_5) + Q^f_{R} (1 + \gamma_5) \right] \psi_f$$ \hspace{1cm} (2.2)

For the minimal Standard Model gauge bosons, the coupling constants $g_V$ are simply

$$g_\gamma = e = \sqrt{4\pi\alpha}, \quad g_Z = e/s_W c_W, \quad g_W = e\sqrt{2}/s_W$$ \hspace{1cm} (2.3)

with $e$ the proton charge and $s^2_W = 1 - c^2_W \equiv \sin^2 \theta_W$; the couplings $Q^f_{f'V}^{L,R}$ read

$$Q^f_{L,R} = e^f, \quad Q^f_{L,R} = I^f_{3L,3R} - e^f s^2_W, \quad Q^f_{L,R} = |I^f_{3L,3R}|$$ \hspace{1cm} (2.4)

with $e^f$ the electric charge in units of $e$ and $I_{3L}, I_{3R}$ the third components of weak isospin.

The inclusion of additional gauge bosons in the previous equations is straightforward once their couplings to fermions are specified.

The new fermions will mix with the ordinary fermions which have the same $U(1)_Q$ and $SU(3)_C$ quantum numbers [12, 13]. The mixing will determine the decay systematics of the new leptons and quarks and allows for their single production in association with light fermions. In principle, one has to treat the three generations of fermions on the same footing; this, leads to rather complicated mixing patterns. For instance in $E_6$, the mixing in the general case would be described by $6 \times 6$ non–diagonal matrices in the quark and charged lepton sectors and by $15 \times 15$ matrices in the neutrino sector [4]. The matrix elements will depend on the vacuum expectation values of the Higgs fields and on arbitrary Yukawa couplings, making a general analysis rather complicated.

In order not to commit ourselves to any particular model, we will allow for the mixing between different generations and treat the mixing angles as phenomenological parameters. For instance, in the case of the interaction of the electron and its associated neutrino with exotic charged and neutral heavy leptons, the general Lagrangian describing the transitions between $e, \nu_e$ and the heavy leptons $N^k, E^k$ where $k$ is a generation index reads

$$\mathcal{L} = \frac{1}{2} \sum_{k=1}^{3} \sum_{i=L,R} \left[ g_W \zeta_i^{E^kW} \bar{\nu}_i \gamma_\mu E^k_i W^\mu + g_Z \zeta_i^{eE^kZ} \bar{\nu}_i \gamma_\mu E^k_i Z^\mu \right] + \text{h.c.}$$

$$+ \frac{1}{2} \sum_{k=1}^{3} \sum_{i=L,R} \left[ g_W \zeta_i^{N^kW} \bar{\nu}_i \gamma_\mu N^k_i W^\mu + g_Z \zeta_i^{eN^kZ} \bar{\nu}_i \gamma_\mu N^k_i Z^\mu \right] + \text{h.c.}$$ \hspace{1cm} (2.5)

where we have allowed for both left–handed and right–handed mixing and assumed small angles so that one can write $\sin \zeta_{L,R} \simeq \zeta_{L,R}$. From this Lagrangian, where $g_{W,Z}$ are given in eq. (2.3), the charges $Q^f_{f'V}^{L,R}$ as in eq. (2.2) can be easily derived. The generalization to the other light leptons and to quarks as well as to extra gauge bosons is straightforward.

\footnote{However, the inter–generational mixing will induce at the tree level, flavor changing neutral currents which are severely constrained by existing data [4]; neglecting the latter allows for an enormous simplification: the mixing can be parametrized by a few angles which can be treated as phenomenological parameters.}
3. Production in $e^+e^-$ Collisions

3.1 General Case

In this subsection, we give the most general expression of the differential cross section for the production of two fermions with different masses, to treat pair and single production on the same footing, including the longitudinal polarization of the initial $e^+/e^-$ beams and the polarization of the final fermions.

Consider the process where a pair of heavy fermions is produced in $e^+e^-$ annihilation through gauge boson exchange

$$ e^+(l, \xi) e^-(l, \xi) \rightarrow F(p, \overline{p}) F(p, n) $$

(3.1)

$\xi, \overline{\xi}$ denote the degrees of longitudinal polarization of the initial electron and positron and $l, \overline{l}$ their momenta; $p(\overline{p})$, $n(\overline{n})$ and $m(\overline{m})$ are the four–momentum, spin vector and mass of the final fermion $F(\overline{F})$. The most general form of the differential cross section, $d\sigma$ can be written in terms of the scalar products of the spin four–vectors and momenta of the particles, as

$$ d\sigma = \frac{1}{2} N_c e^4 (2\pi)^4 \delta^4(l + \overline{l} - p - \overline{p}) \frac{d^3 p}{(2\pi)^3 2p^0} \frac{d^3 \overline{p}}{(2\pi)^3 2\overline{p}^0} \left[ (1 - \xi \overline{\xi}) A + (\xi - \overline{\xi}) A' \right] $$

(3.2)

where $e$ is the proton charge, $N_c$ the color factor of the final fermions and the squared amplitudes $A$ and $A'$ can be expressed in terms of generalized charges \cite{21} $Q_{1,2,3}$ and $Q'_{1,2,3}$

$$ A = (p.l)(\overline{p}.l)(Q_1 + Q_3) + (p.l)(\overline{p}.l)(Q_1 - Q_3) + m\overline{m}(l.l)Q_2 $$

$$ - (n.l)[\overline{m}(p.l)Q'_2 + m(\overline{p}.l)(Q'_1 - Q'_3)] + (n.\overline{l})[m(p.l)Q'_2 + m(\overline{p}.l)(Q'_1 + Q'_3)] $$

$$ - (\overline{n}.l)[m(p.l)Q'_2 + m(\overline{p}.l)(Q'_1 + Q'_3)] + (\overline{n}.\overline{l})[m(p.l)Q'_2 + m(\overline{p}.l)(Q'_1 - Q'_3)] $$

$$ + (n.\overline{n})Q_2[(l.l)(p.\overline{p}) - (l.l)(\overline{p}.l)] - 2Q_2(n.l)(\overline{p}.l) + (n.\overline{l})(\overline{n}.\overline{l})(l.\overline{l}) $$

$$ - (n.\overline{n})[Q_2(l.l) + (p.\overline{p}) + (\overline{p}.l) + (p.l)] + m\overline{m}(Q_1 + Q_3) $$

$$ - (n.l)[\overline{m}(p.\overline{p}) + (p.l) + (\overline{p}.l) + (p.l)] + m\overline{m}(Q_1 - Q_3) $$

$$ A' = A ( Q_1 \leftrightarrow Q'_1 , Q_2 \leftrightarrow Q'_2 , Q_3 \leftrightarrow Q'_3 ) $$

(3.3)

In terms of the helicity amplitudes $Q_{ij}$ with $i, j = L, R$, the charges $Q$ and $Q'$ are

$$ Q_1 = \frac{1}{4} \left[ |Q_{LL}|^2 + |Q_{RR}|^2 + |Q_{RL}|^2 + |Q_{LR}|^2 \right] $$

$$ Q_2 = \frac{1}{2} \text{Re} \left[ Q_{LL}Q_{RL}^* + Q_{RR}Q_{LR}^* \right] $$

$$ Q_3 = \frac{1}{4} \left[ |Q_{LL}|^2 + |Q_{RR}|^2 - |Q_{RL}|^2 - |Q_{LR}|^2 \right] $$
\[
Q'_1 = \frac{1}{4} \left[ |Q_{LL}|^2 + |Q_{RL}|^2 - |Q_{RR}|^2 - |Q_{LR}|^2 \right]
\]
\[
Q'_2 = \frac{1}{2} \text{Re} \left[ Q_{LL}Q'_{RL} - Q_{RR}Q'_{LR} \right]
\]
\[
Q'_3 = \frac{1}{4} \left[ |Q_{LL}|^2 + |Q_{LR}|^2 - |Q_{RR}|^2 - |Q_{RL}|^2 \right]
\] (3.4)

The helicity amplitudes \( Q_{ij} \) depend on the process under consideration. For \( s = (l + \bar{t})^2, t = (p - l)^2 \) and \( u = (p - \bar{t})^2 \) channel vector bosons \( V_S, V_T \) and \( V_U \) exchange, respectively, the general form is

\[
Q_{ij} = \sum_{V_S} \frac{g^2_{V_S}}{e^2} \frac{Q^{FFV_S}Q^{eV_S}}{s - M^2_{V_S} + i\Gamma_{V_S}M_{V_S}} + \sum_{V_T} \frac{g^2_{V_T}}{e^2} \frac{Q^{eFV_T}Q^{eFV_T}}{t - M^2_{V_T}} + \sum_{V_U} \frac{g^2_{V_U}}{e^2} \frac{Q^{eFV_U}Q^{eFV_U}}{u - M^2_{V_U}}
\] (3.5)

The normalization factors \( g_V \) and the reduced couplings \( Q^{ffV}_{LR} \) can be derived from the Lagrangian describing the \( ffV \) interaction; see section 2.

Integrating over the variables of one of the final fermions as well as on the azimuthal angle of the remaining one, the differential cross section \( d\sigma/d\cos \theta \), where \( \theta \) specifies the direction of the latter particle with respect to the incoming electron, reads

\[
\frac{d\sigma}{d\cos \theta} = \frac{3}{8} \sigma_0 N_c \lambda^\frac{3}{2} \frac{1}{4} \left[ (1 - \xi\bar{\xi})A + (\xi - \bar{\xi})A' \right]
\] (3.6)

where \( \sigma_0 = 4\pi\alpha^2/3s \) is the point-like QED cross section for muon pair production and \( \lambda \) the usual two body phase space function

\[
\lambda = (1 - \mu^2 - \bar{\mu}^2)^2 - 4\mu^2\bar{\mu}^2, \quad \text{with} \quad \mu = m/\sqrt{s}, \quad \bar{\mu} = \bar{m}/\sqrt{s}
\] (3.7)

In terms of the charges \( Q_i \) and \( Q'_i \), the reduced amplitudes squared \( \mathcal{A} \) and \( \mathcal{A}' \) read

\[
\mathcal{A} = \left[ 1 - (\mu^2 - \bar{\mu}^2)^2 + \lambda \cos^2 \theta \right] Q_1 + 4\mu\bar{\mu}Q_2 + 2\lambda^\frac{3}{2}\cos \theta Q_3
- \frac{2m}{s} n \cdot (l - \bar{t}) \left[ (1 - \mu^2 + \bar{\mu}^2)Q'_1 + (1 + \mu^2 - \bar{\mu}^2)\frac{m}{m}Q'_2 + \lambda^\frac{3}{2}\cos \theta Q'_3 \right]
+ \frac{2m}{s} n \cdot (l + \bar{t}) \left[ (1 - \mu^2 - \bar{\mu}^2)Q'_3 + \lambda^\frac{3}{2}\cos \theta (Q'_1 - \frac{m}{m}Q'_2) \right]
+ \frac{2\bar{m}}{s} \bar{\mu} \cdot (\bar{l} - l) \left[ (1 - \bar{\mu}^2 + \mu^2)Q'_1 + (1 + \bar{\mu}^2 - \mu^2)\frac{m}{m}Q'_2 + \lambda^\frac{3}{2}\cos \theta Q'_3 \right]
- \frac{2\bar{m}}{s} \bar{\mu} \cdot (\bar{l} + l) \left[ (1 - \bar{\mu}^2 - \mu^2)Q'_3 + \lambda^\frac{3}{2}\cos \theta (Q'_1 - \frac{m}{m}Q'_2) \right]
+ n \cdot \bar{\mu} (1 - \cos^2 \theta) \lambda Q_2 - \frac{8}{s} Q_2 [n \cdot l \bar{\mu} \cdot (l + \bar{t}) + \bar{\mu} \cdot \bar{\bar{l}} n \cdot (l + \bar{t})]
+ \frac{4}{s} n \cdot l \bar{\mu} \cdot \bar{\bar{l}} \left[ (1 + \mu^2 + \bar{\mu}^2 + \lambda^\frac{3}{2}\cos \theta)Q_2 + 2\mu\bar{\mu}(Q_3 - Q_1) \right]
- \frac{4}{s} \bar{\mu} \cdot l n \cdot \bar{\bar{l}} \left[ (3 - \mu^2 - \bar{\mu}^2 + \lambda^\frac{3}{2}\cos \theta)Q_2 + 2\mu\bar{\mu}(Q_3 + Q_1) \right]
\]
\[ A' = A \left( Q_1 \leftrightarrow Q'_1, \ Q_2 \leftrightarrow Q'_2, \ Q_3 \leftrightarrow Q'_3 \right) \] (3.8)

The polarization four–vector \( P_\mu \) of the final state fermion \( F \) is defined by \( d\sigma^{pol}/d\cos\theta \sim d\sigma^{unpol}/d\cos\theta \times [1 + P_\mu n^\mu] \), with \( n_\mu \) the spin vector which satisfies the relations \( n \cdot n = -1 \) and \( n \cdot p = 0 \); see Appendix. In the \( F \) rest frame, assuming CP–conservation, the components are \((0, P_1, P_2)\) with \( P_1 \) and \( P_2 \) the transverse and longitudinal polarizations with respect to the flight direction. Summing over the polarizations of one of the final fermions, e.g. \( \overline{F} \), the longitudinal and transverse components of the polarization vector of the other fermion, in its own rest frame, are given by

\[
\begin{align*}
\mathcal{P}_|| &= \frac{[1 - \mu^2 + \overline{\mu}^2 + (1 + \mu^2 - \overline{\mu}^2)\cos^2\theta] \lambda Q_1 Q'_2 + \cos\theta[(1 + \mu^2 - \overline{\mu}^2 - \lambda)\overline{\mu} Q'_2 + 2(1 - \mu^2 - \overline{\mu}^2)Q'_1]}{[1 - (\mu^2 - \overline{\mu}^2)^2 + \lambda \cos^2\theta] Q_1 - 4\mu\overline{\mu} Q_2 - 2\lambda^2 \cos\theta Q_3} \\
\mathcal{P}_\perp &= 2\mu \sin\theta \frac{[1 - \mu^2 + \overline{\mu}^2]Q'_1 + (1 + \mu^2 - \overline{\mu}^2)\overline{\mu} Q'_2 + \lambda \cos\theta Q'_3}{[1 - (\mu^2 - \overline{\mu}^2)^2 + \lambda \cos^2\theta] Q_1 + 4\mu\overline{\mu} Q_2 + 2\lambda^2 \cos\theta Q_3} \quad (3.9)
\end{align*}
\]

In the next subsections, the special cases of heavy fermion pair production and single production in association with massless fermions will be discussed.

### 3.2. Pair Production

In \( e^+e^- \) collisions, the pair production of new fermions proceeds through \( s \)–channel gauge boson exchange; there are also contributions from \( t \)–channel exchange in the case of heavy lepton production, but they are quadratically suppressed by mixing angle factors and therefore, rather small. The unpolarized differential cross section \( d\sigma/d\cos\theta \) for the process \( e^+e^- \to F\overline{F} \) is (see also Ref. [20])

\[
\begin{align*}
\frac{d\sigma}{d\cos\theta} = \frac{3}{8} \sigma_0 N_c \beta_F \left[ (1 + \beta_F^2 \cos^2\theta)Q_1 + (1 - \beta_F^2)Q_2 + 2\beta_F \cos\theta Q_3 \right] \quad (3.10)
\end{align*}
\]

with \( \beta_F = (1 - 4m_F^2/s)^{1/2} \) the velocity of the fermion in the final state; the charges \( Q_1, Q_2 \) and \( Q_3 \) are given by in eq. (3.4) with the helicity amplitudes \( Q_{ij} \) with \( i, j = L, R \) in the general case in eq. (3.5). If only \( s \)–channel photon and Z boson exchange is present, these helicity amplitudes are simply given by

\[
Q_{ij} = e^F e^e + \frac{Q^{FFZ}_{ij} Q^{eeZ}_{ij}}{s_W^2 c_W^2} \frac{s}{s - M_Z^2 + i\Gamma_Z M_Z} \quad (3.11)
\]

where \( Q^{FFZ}_{ij} \) are the reduced couplings of the left and right–handed fermions to the Z boson.

In the case of unpolarized initial beams, the cross section eq. (3.10) allows for three independent measurements: the total production cross section \( \sigma_F \), the forward–backward asymmetry \( A_{FB}^F \) and the parameter \( \alpha_F \) defined as \( d\sigma/d\cos\theta \sim 1 + \alpha_F \cos^2\theta \). These three parameters, and their asymptotic values, are given by
\[ \sigma_F = \frac{3}{2} \sigma_0 N_{\beta_F} \left[ (1 + \frac{1}{3} \beta_F^2) Q_1 + (1 - \beta_F^2) Q_2 \right] \sqrt{\frac{m_F}{s}} \rightarrow N_{\sigma_0 Q_1} \]  

(3.12)

\[ A_F^{FB} = \beta_F Q_3 \left[ (1 + \frac{1}{3} \beta_F^2) Q_1 + (1 - \beta_F^2) Q_2 \right]^{-1} \sqrt{\frac{\pi m_F}{4 N_{Q_1}}} \rightarrow \frac{3 Q_3}{4 Q_1} \]  

(3.13)

\[ \alpha_F = \beta_F^2 Q_1 \left[ Q_1 + (1 - \beta_F^2) Q_2 \right]^{-1} \sqrt{\frac{m_F}{s}} \rightarrow \frac{1}{Q_1} \]  

(3.14)

For Majorana neutrinos one has to symmetrize eq. (3.10) because of the two identical particles in the final state. This symmetrization makes that the Majorana neutrino has only axial–vector couplings so that the charges are \( Q_2 = -Q_1 \) and \( Q_3 = 0 \), leading to the simple expression for the cross section

\[ \sigma_{\text{N}_{\text{maj}}} = \frac{1}{2} \sigma_0 |Q_{LL}|^2 + |Q_{RL}|^2 \]  

(3.15)

The total cross section is proportional to \( \beta^3 \) and thus, strongly suppressed near threshold; the angular distribution behaves like \( d\sigma/d\cos \theta \sim 1 + \cos^2 \theta \) and therefore, there is no forward–backward asymmetry and the \( \alpha \) parameter is equal to one. Note that since the isosinglet Majorana neutrinos do not couple to the photon and Z boson, they can be pair produced only through the exchange of an extra gauge boson and thus, the helicity amplitudes \( Q_{ij} \) have to be altered.

Let us now discuss the polarization of the heavy fermions. Summing over the polarizations of \( \tilde{F} \), the two components of the polarization vector of \( F \) in its own rest frame, \( P_\parallel \) and \( P_\perp \) can be written as (see also refs. [20, 22])

\[ P_\parallel = -\frac{(1 + \cos^2 \theta) \beta_F Q_3' + \cos \theta [(1 - \beta_F^2) Q_2' + (1 + \beta_F^2) Q_1']}{(1 + \beta_F^2 \cos^2 \theta) Q_1 + (1 - \beta_F^2) Q_2 + 2\beta_F \cos \theta Q_3} \]

\[ P_\perp = \sqrt{1 - \beta_F^2} \sin \theta Q_1' + Q_2' + \beta_F \cos \theta Q_3' \]

(3.16)

where the helicity amplitudes \( Q_{ij} \) are given in eq. (3.5) and (3.10) and the charges \( Q_{1,2,3}' \) in eq. (3.4). Averaged over the polar angle, the two components become

\[ < P_\parallel > = -\frac{4}{3} \frac{\beta_F Q_3'}{(1 + \beta_F^2/3) Q_1 + (1 - \beta_F^2) Q_2} \]

\[ < P_\perp > = \frac{3\pi m_F}{4} \frac{Q_1' + Q_2'}{\sqrt{s} (1 + \beta_F^2/3) Q_1 + (1 - \beta_F^2) Q_2} \]  

(3.17)

Again, for Majorana neutrinos one has to symmetrize the previous expressions. This makes that the currents become purely axial-vector, and therefore there is no polarization effect. The polarization of the final particles, together with their angular distributions, permits to discriminate between Majorana and Dirac neutrinos, or between particles with different couplings [standard, mirror or vector couplings].

9
4. 2. Single Production

In $e^+e^-$ collisions one can also have access to the new fermions via single production in association with their light partners if fermion mixing is not too small. Assuming that extra gauge bosons are too heavy to affect the production, the process proceeds through $s$–channel $Z$ exchange for all fermions, but for heavy leptons [only the first family if inter–generational mixing is neglected] one has additional $t$–channel gauge boson exchanges: $W$ exchange for neutral leptons and $Z$ exchange for charged leptons. Neglecting the mass of the light fermion partner, the differential cross section for the process $e^+e^- \rightarrow F\bar{f}$ is

$$\frac{d\sigma}{d\cos\theta} = \frac{3}{8} \sigma_0 N_c (1 - \mu^2)^2 \left[ (1 + \mu^2 + (1 - \mu^2) \cos^2 \theta) Q_1 + 2 \cos \theta Q_3 \right]$$

(3.18)

where $\mu^2 = m_F^2/s$. The charges $Q_{1,3}$ as given in eq. (3.5) are built–up by the helicity amplitudes

$$Q_{ij} = \frac{1}{2} \frac{\zeta_{ij}^{E_kZ} Q_{eeZ}}{s_W^2 c_W^2} \left[ \frac{1}{1 - z} + \frac{1}{t/s - z} \right]$$

for $E^k$

$$Q_{ij} = \frac{\zeta_{ij}^{N_kZ} Q_{eeZ}}{2 s_W^2 c_W^2} \frac{1}{1 - z} + \frac{\zeta_{ij}^{N_kW} Q_{eW}}{s_W^2} \frac{1}{t/s - w}$$

for $N^k$

$$Q_{ij} = \frac{1}{2} \frac{\zeta_{ij}^{FZ} Q_{eeZ}}{s_W^2 c_W^2} \frac{1}{1 - z}$$

for quarks

(3.19)

with the reduced masses $z = M_Z^2/s$, $w = M_W^2/s$ and $t/s = -(1 - \cos \theta)(1 - \mu^2)/2$. Note again that in the case of Majorana neutrinos, the $Q_{ij}$ have to be symmetrized.

For quarks [and for second and third family leptons if inter–generational mixing is neglected] the total cross section $\sigma(e^+e^- \rightarrow F\bar{f})$ takes the simple form

$$\sigma(e^+e^- \rightarrow F\bar{f}) = \sigma_0 N_c (1 - \mu^2)^2 \left( 1 + \frac{1}{2} \mu^2 \right) Q_1$$

(3.20)

The cross sections for the production of the conjugate states $\sigma(e^+e^- \rightarrow \bar{F}\bar{f})$ is the same. For first generation heavy leptons, the analytical expressions are much more involved because of the $t$–channel contributions. Denoting by $V_S$ the gauge boson exchanged in the $s$–channel [$Z$ for both types of leptons] and by $V_T$ the one exchanged in the $t$–channel [$W$ for $N$ and $Z$ for $E$], one can write a common expression for heavy lepton $L = N^k, E^k$ single production

$$\sigma \left( e^+e^- \rightarrow L\bar{L} \right) = 3\sigma_0 \left\{ \frac{1}{3} (1 - \mu^2)^2 \left[ 1 + \frac{\mu^2}{2} \right] \frac{q^{+V_S} q^{-V_S}}{(1 - v_S)^2} \right\} \frac{q^{+V_T} q^{-V_T}}{1 - v_T}$$

$$+ \left\{ (1 - \mu^2)(3 + 2v_T - \mu^2) - 2(1 - \mu^2 + v_T)(1 + v_T) \log \frac{1 - \mu^2 + v_T}{v_T} \right\} \frac{q^{+V_T} q^{-V_T}}{1 - v_S}$$

$$+ \left\{ -(1 - \mu^2)(1 + \mu^2 - 2v_T - 2v_T(\mu^2 - v_T) \log \frac{1 - \mu^2 + v_T}{v_T} \right\} \frac{q^{+V_S} q^{-V_S}}{(1 - v_S)^2}$$
\[ + \left[ (1 - \mu^2) \frac{1 + 2v_T}{v_T} - (2 - \mu^2 + 2v_T) \log \frac{1 - \mu^2 + v_T}{v_T} \right] q^{V^* V} \]

\[ + \left[ (1 - \mu^2) \left( 2 - \frac{1}{1 + v_T - \mu^2} \right) - (2v_T - \mu^2) \log \frac{1 - \mu^2 + v_T}{v_T} \right] q^{V V}_+ \}

(3.21)

where \( v_S = M^2_{V_S}/s \), \( v_T = M^2_{V_T}/s \) and the charges \( q^{VV'}_\pm = q^{V^* V}_\pm \ldots \), are defined by

\[ q^\pm = \frac{1}{4} \left[ |q_{LL}^{V^* V'}|^2 + |q_{RR}^{V^* V'}|^2 \right] \]

\[ q^V = \frac{1}{4} \left[ |q_{LR}^{V^* V'}|^2 + |q_{RL}^{V^* V'}|^2 \right] \]

with \( q_{ij}^{V_S} = (g^2_{V_S}/e^2)Q^L_{ij}Q^S_{j} \) etc.

To obtain the cross section for the production of \( E \) and \( N \) in the case of left–handed or right–handed mixing is then straightforward: one has simply to specify \( V_S \) and \( V_T \) and choose the proper combination of charges. For instance, in the case of the production of a heavy neutrino with a left–handed mixing denoted\(^3\) \( N_L \), one recovers up to a factor of two, the formula given in Ref. [18] for the production of Majorana neutrinos, by setting

\[ v_S = \frac{M^2_Z}{s}, \quad v_T = \frac{M^2_W}{s}, \quad q^{WW}_- = q^{ZZ}_- = 0 \]

(3.22)

This is due to the fact that the production cross section of a Majorana neutrino is just the sum of the production cross sections of a Dirac neutrino and its anti–neutrino; this also holds true for the angular distributions.

For \( E \) production, we have a \( Z \) exchange in both \( s \) and \( t \)–channels. This simplifies considerably the expression eq. (3.21) since in addition to the fact that \( v_S = v_T \), the charges factorize. Note that for \( s^2_W = 1/4 \), the left–handed and right–handed couplings of the electron to the \( Z \) are equal in magnitude but with opposite signs. This leads to

\[ q^{ZZ}_- \simeq 0, \quad q^{ZZ}_+ \simeq \frac{1}{2} |q^{LL}_Z|^2 \simeq \frac{1}{2} |q^{RR}_Z|^2 \]

(3.23)

which translates into the fact the cross sections for \( E_L \) and \( E_R \) are approximately equal.

Finally, the longitudinal and transverse components of the polarization vector of the heavy fermion produced in association with a massless partner reads

\[ P_{\parallel} = -\frac{[1 - \mu^2 + (1 + \mu^2) \cos^2 \theta]Q'_3 + 2 \cos \theta Q'_1}{[1 + \mu^2 + (1 - \mu^2) \cos^2 \theta] Q_1 + 2 \cos \theta Q_3} \]

\[ P_{\perp} = 2\mu \sin \theta \frac{Q'_1 + \cos \theta Q'_3}{[1 + \mu^2 + (1 - \mu^2) \cos^2 \theta] Q_1 + 2 \cos \theta Q_3} \]

(3.24)

where the charges \( Q'_{1,2,3} \) are given in eq. (3.4) with the helicity amplitudes eq. (3.19).

\(^2\)In principle, the indices \( L, R \) refer to the handedness of the heavy lepton mixing with its light partner. However, due to the fact that the latter is massless, they are also the chirality of the heavy lepton.
4. Decays and Correlations

4.1 Two–body Decays

The heavy heavy fermion $F$, with a mass $m$ and spin four–vector $n$, will decay into a lighter fermion $f_0$ and virtual or real gauge bosons which subsequently decay into two massless fermions $f_1$ and $\bar{f}_2$

$$F(p, n) \rightarrow f_0(l_0) \ V(l_V) \rightarrow f_0(l_0) \ f_1(l_1) \ \bar{f}_2(l_2) \quad (4.1)$$

In most of the cases the fermion $f_0$ is just the ordinary partner of $F$ [which can be considered as massless] and the decay occurs through mixing. But it is possible that $f_0$ is also a heavy fermion and the process eq. (4.1) is a “cascade” decay. For instance, for fermions belonging to the same isodoublets, the heavier fermion can decay through the exchange of a $W$ boson into its lighter isospin partner and the latter subsequently decays through mixing into three massless particles. Moreover, it is also possible that the mixing between heavy fermions is much larger than the mixing between the new and ordinary fermions in which case the decay of the heavy fermion first into a lighter one, is more important although kinematically disfavored.

If the mass difference between $F$ and $f_0$ is larger than the mass of the exchanged gauge boson, the latter will be on mass–shell and the decay is a two–body decay. Assuming that $f_0$ is also polarized [the mass and spin four–vector will be denoted by $m_0$ and $n_0$ respectively], the differential decay width in terms of the momenta and spin–vectors of $F$ and $f_0$ is given by

$$d\Gamma = (2\pi)^4 \delta^4(p - l_0 - l_V) \ \frac{d^3l_0}{(2\pi)^3} \ \frac{d^3l_V}{(2\pi)^3} \ \frac{e^2}{2m} \ d\Gamma^0$$

(4.2)

with

$$d\Gamma^0 = \left[ m^2 + m_0^2 - 2M_V^2 + \frac{(m^2 - m_0^2)^2}{M_V^2} \right] q_1 - 6m_0mq_2$$

$$+ (n \cdot n_0) \left[ \frac{m^2 + m_0^2 - (m^2 - m_0^2)^2}{M_V^2} \right] q_2 + 2mm_0q_1$$

$$+ 2(n \cdot l_0) \frac{m}{M_V^2} (m^2 - m_0^2 - 2M_V^2)q_3 + 2(n_0 \cdot p)(n_0 \cdot l_0) \frac{(m - m_0)^2}{M_V^2} q_2$$

(4.3)

In terms of the left and right–handed $Ff_0V$ couplings, the charges $q_{1,2,3}$ are

$$q_{1,3} = \frac{1}{4} \frac{g^2}{e^2} \left[ (Q_L^{Ff_0V})^2 \pm (Q_R^{Ff_0V})^2 \right] \quad , \quad q_2 = \frac{1}{2} \frac{g^2}{e^2} Q_L^{Ff_0V} Q_R^{Ff_0V}$$

(4.4)

Summing over the polarization of $f_0$, the angular distribution $d\Gamma/d\cos \theta_0$ where $\theta_0$ is the angle between the spin vector $n$ and the flight direction of $f_0$, can be written as

$$\frac{d\Gamma}{d\cos \theta_0} = \frac{1}{2} (\Gamma_{tot} + \cos \theta_0 \Gamma_{ang})$$

(4.5)
\( \Gamma_{\text{tot}} \) is obtained by integrating the previous expression over the angle \( \theta_0 \)

\[
\Gamma_{\text{tot}} = \int_{-1}^{+1} d\cos\theta_0 \frac{d\Gamma}{d\cos\theta_0} \tag{4.6}
\]

and corresponds to the partial decay width for on–shell gauge bosons. The \( \cos\theta_0 \) term can be isolated by integrating \( d\Gamma/d\cos\theta_0 \) asymmetrically

\[
\Gamma_{\text{ang}} = \int_{0}^{+1} d\cos\theta_0 \frac{d\Gamma}{d\cos\theta_0} - \int_{-1}^{0} d\cos\theta_0 \frac{d\Gamma}{d\cos\theta_0} \tag{4.7}
\]

The expressions of \( \Gamma_{\text{tot}} \) and \( \Gamma_{\text{ang}} \) are

\[
\Gamma_{\text{tot}} = \frac{\alpha}{2} \frac{m^3}{M_V^2} \lambda^2 \left\{ \left( 1 - \mu_0^2 \right)^2 + \mu_v^2 \left( 1 + \mu_0^2 - 2\mu_v^2 \right) \right\} q_1 - 6\mu_0\mu_v^2 q_2 \tag{4.8}
\]

\[
\Gamma_{\text{ang}} = \frac{\alpha}{2} \frac{m^3}{M_V^2} \lambda (1 - \mu_0^2 - 2\mu_v^2) q_3 \tag{4.9}
\]

where

\[
\lambda = (1 - \mu_0^2 - \mu_v^2)^2 - 4\mu_0^2\mu_v^2 \quad \text{, with } \mu_0 = m_0/m, \mu_v = M_V/m \tag{4.10}
\]

The partial decay width and the angular distribution in the case where the heavy fermion directly decays into its light partner and a gauge boson \( V \) can be obtained from the last expression by simply setting \( m_0 = 0 \) in the previous expressions; one has

\[
\Gamma_{\text{tot}} = \frac{\alpha g^2}{8} \frac{m^3}{e^2 M_V^2} \left( (1 - \mu_v^2)^2(1 + 2\mu_v^2) \right) \left[ (Q^{F_0V}_L)^2 + (Q^{F_0V}_R)^2 \right] \tag{4.11}
\]

\[
\Gamma_{\text{ang}} = \frac{\alpha g^2}{8} \frac{m^3}{e^2 M_V^2} \left( (1 - \mu_v^2)^2(1 - 2\mu_v^2) \right) \left[ (Q^{F_0V}_L)^2 - (Q^{F_0V}_R)^2 \right] \tag{4.12}
\]

in agreement with Ref. [20], once the charges are specified.

In the next subsection, we will discuss the case where the exchanged gauge bosons are off–shell, leading to three–body decays of the heavy fermions.

### 4.2 Three Body Decays

The amplitude for the decay, eq. (4.1), in the general case where the fermion \( f_0 \) is also massive and polarized can be obtained from the amplitude eq.(3.3) of the production of two heavy fermions by crossing symmetry: one simply has to change the labels of the four–momenta and to reverse the sign of the mass \( m \). One obtains for the differential decay width

\[
d\Gamma = (2\pi)^4 \delta^4(p - l_0 - l_1 - l_2) \frac{d^3 l_0}{(2\pi)^3 2 l_0^0} \frac{d^3 l_1}{(2\pi)^3 2 l_1^0} \frac{d^3 l_2}{(2\pi)^3 2 l_2^0} \frac{8N_c e^4}{m^5} d\Gamma^0 \tag{4.13}
\]
where, after some simplifications, \( d\Gamma^0 \) reads

\[
2d\Gamma^0 = \left( p \cdot l_2 \right) \left( l_0 \cdot l_1 \right) (Q_1 + Q_3) + \left( p \cdot l_1 \right) \left( l_0 \cdot l_2 \right) (Q_1 - Q_3) - mm_0 (l_1 . l_2) Q_2 \\
+ \left( n . l_1 \right) [m_0 (p . l_3) Q'_2 - m (l_0 . l_2) (Q'_1 - Q'_3)] - \left( n . l_2 \right) [m_0 (p . l_1) Q'_2 - m (l_0 . l_1) (Q'_1 + Q'_3)] \\
- \left( n . l_1 \right) [m (l_0 . l_2) Q'_2 - m_0 (p . l_2) (Q'_1 + Q'_3)] + \left( n . l_2 \right) [m (l_0 . l_1) Q'_2 - m_0 (p . l_1) (Q'_1 - Q'_3)] \\
+ \left( n . l_2 \right) \left[ m^2 Q_2 + mm_0 (Q_1 + Q_3) \right] - \left( n . l_1 \right) \left[ m^2 Q_2 + mm_0 (Q_1 - Q_3) \right] \\
+ \left( n . n_0 \right) \left[ (l_1 . l_2) (p . l_0) - (p . l_1) (l_0 . l_2) - (p . l_2) (l_0 . l_1) \right] Q_2 \\
- 2 \left[ (n . l_1) (n_0 . l_2) + (n . l_2) (n_0 . l_1) \right] (l_1 . l_2) Q_2
\]

(4.14)

The generalized charges \( Q_i \) and \( Q'_i \) are the same as those given for the production amplitudes, eq. (3.4), but they are built–up with the helicity amplitude

\[
Q_{ij} = \frac{g_{\nu}^2}{e^2} Q_{ij}^{f_0 V} Q_{f_0 f_2 V} \left( \frac{m^2}{(l_1 + l_2)^2 - M_V^2 + i \Gamma_V M_V} \right)
\]

(4.15)

assuming that the decay is mediated by only one gauge boson exchange; other channels can be easily included.

In the case where the fermion \( f_0 \) is also massless, as it happens in most of the cases: the heavy fermion directly decays through mixing into its light partners and real or virtual gauge bosons which subsequently decay into two massless fermions, one can sum over its polarization and the expression \( d\Gamma^0 \) simplifies considerably

\[
d\Gamma^0 = \left( p \cdot l_2 \right) \left( l_0 \cdot l_1 \right) (Q_1 + Q_3) + \left( p \cdot l_1 \right) \left( l_0 \cdot l_2 \right) (Q_1 - Q_3) \\
- m (n \cdot l_1) (l_0 \cdot l_2) (Q'_1 - Q'_3) + m (n \cdot l_2) (l_0 \cdot l_1) (Q'_1 + Q'_3)
\]

(4.16)

However, since in this case several decay channels are possible, the helicity amplitudes are involved and one has in the general case where \( s = (p - l_0)^2 \), \( t = (p - l_1)^2 \) and \( u = (p - l_2)^2 \) channels \( V_S, V_T \) and \( V_U \) exchanges are present

\[
Q_{ij} = \sum_{V_S} \frac{g_{\nu}^2}{e^2} Q_{ij}^{f_1 f_2 V_S} Q_{j}^{f_2 F V_S} m^2 s - M_{V_S}^2 \\
+ \sum_{V_T} \frac{g_{\nu}^2}{e^2} Q_{ij}^{f_0 f_2 V_T} Q_{j}^{f_2 F V_T} m^2 t - M_{V_T}^2 \\
+ \sum_{V_U} \frac{g_{\nu}^2}{e^2} Q_{ij}^{f_0 f_1 V_U} Q_{j}^{f_2 F V_U} m^2 u - M_{V_U}^2
\]

(4.17)

where the widths of the gauge bosons have been omitted for simplicity. These expressions may need to be supplemented by statistical factors.

However, in most physical situations there are at most two decay “channels” only, although in the same channel, several gauge bosons can be exchanged. These two decay channels

\footnote{An example of a situation where all the three channels occur is the decay of a heavy Majorana neutrino into an \( e^+ e^- \) pair and a neutrino or antineutrino. There is an “s–channel” \( Z \)–boson exchange \( N \rightarrow \nu_e Z \rightarrow \nu_e e^+ e^- \), a “t–channel” \( W \)–boson exchange \( N \rightarrow e^- W^+ \rightarrow e^+ e^- \nu_e \) and a “u–channel” exchange \( N \rightarrow e^+ W^- \rightarrow e^+ e^- \bar{\nu}_e \); if the light neutrino is also a Majorana particle, the three amplitudes and the one due to \( N \rightarrow \bar{\nu}_e Z \rightarrow \bar{\nu}_e e^+ e^- \) add coherently.}
occur for instance, when two identical particles are present in the final state [e.g. $E \rightarrow e^- Z \rightarrow e^- e^+ e^-$] hence requiring the symmetrization of the amplitudes or for decays involving both the isospin up and down light partners of the heavy fermions, leading to both neutral and charged gauge boson exchanges [e.g. $E \rightarrow e^- Z + \nu_e W^- \rightarrow e^- \nu_e \bar{\nu}_e$]. In the rest of the discussion, we will therefore assume that only two channels are present and for simplicity, that there is only one gauge boson exchanged in each channel.

Integrating over the momentum of one of the final fermions, e.g. $f_2$, as well as on the azimuthal angle dependence, and using the usual scaled variables

$$x_1 = \frac{2 \, (p \cdot l_1)}{m^2}, \quad x_2 = \frac{2 \, (p \cdot l_2)}{m^2}, \quad x_0 = \frac{2 \, (p \cdot l_0)}{m^2} = 2 - x_1 - x_2$$

the differential decay width writes

$$\frac{d\Gamma}{d\cos \theta_1 dx_1 dx_2} = \frac{N_c \alpha^2}{8\pi} m \, d\Gamma_0$$

with $\theta_1$ the angle between the fermion $f_1$ and the spin vector $n$ of the heavy fermion [the projections of $n$ onto the flight directions of $f_1$ and $f_2$ are simply $2n \cdot l_1/m = x_1 \cos \theta_1$ and $2n \cdot l_2/m = - x_2 \cos \theta_1$]; neglecting the widths of the exchanged gauge bosons and using the scaled masses $v_S = M_{\nu_S}^2/m^2$ and $v_T = M_{\nu_T}^2/m^2$, $d\Gamma_0$ is given by

$$d\Gamma_0 = \frac{x_1 (1 - x_1) q_-^{SS} + x_2 (1 - x_2) q_+^{SS} + x_0 (1 - x_0) q_0^{TT} + x_2 (1 - x_2) q_+^{TT}}{(1 - x_0 - v_S)^2} \frac{x_0 (1 - x_0) q_0^{TT} + x_2 (1 - x_2) q_+^{TT}}{(1 - x_1 - v_T)^2} \cos \theta_1$$

$$- \left[ \frac{x_1 (1 - x_1) q_-^{SS} + x_2 (1 - x_2) q_+^{SS}}{(1 - x_0 - v_S)^2} + \frac{x_0 (1 - x_0) q_0^{TT} + x_2 (1 - x_2) q_+^{TT}}{(1 - x_1 - v_T)^2} \right] \cos \theta_1 \right.$$  

$$- 2 \eta \, q_+^{ST} \frac{x_2 (1 - x_2)}{(1 - x_0 - v_S)(1 - x_1 - v_T)}$$

(4.20)

Here, $\eta$ is a statistical factor: $\eta = -1$ in the case where there are two identical Dirac fermions in the final state otherwise $\eta = +1$; for quarks one has also to divide by the color factor, i.e. $\eta = \pm 1/3$. The charges $q_{MN}^+$ and $q_{MN}^-'$ where $M, N = S, T$ describe the channels with $V_S, V_T$ exchange and the interference term; they read

$$q_{MN}^+ = \frac{1}{2} \left[ |q_{LL} M | q_{LL} M | + |q_{RR} M | q_{RR} M | + |q_{LR} M | q_{LR} M | \right], \quad q_{MN}^- = \frac{1}{2} \left[ |q_{RL} M | q_{RL} M | + |q_{LR} M | q_{LR} M | \right]$$

$$q_{MN}^-' = \frac{1}{2} \left[ |q_{LL} M | q_{LL} M | - |q_{RR} M | q_{RR} M | \right], \quad q_{MN}^-' = \frac{1}{2} \left[ |q_{RL} M | q_{RL} M | - |q_{LR} M | q_{LR} M | \right]$$

(4.21)

where the helicity amplitudes $q_{ij}^M$ with $i, j = L, R$ are given by

$$q_{ij}^S = \frac{g_2^2}{e^2} Q_i^F f_i V_S Q_j^F f_j V_S, \quad q_{ij}^T = \frac{g_2^2}{e^2} Q_i^F f_i V_T Q_j^F f_j V_T$$

(4.22)
Integrating over the energy of the particle \( f_2 \) with the boundaries \( 1 - x_1 \leq x_2 \leq 1 \), one has

\[
\frac{d\Gamma}{dx_1 d\cos \theta_1} = N_e \frac{\alpha^2}{8\pi} m \left\{ \frac{x_1^2(1-x_1)}{v_S(v_S - x_1)} [q_-^{SS} - q_-^{SS} \cos \theta_1] + \frac{x_1^2(1 - 5x_1/3)}{2(1 - x_1 - v_T)^2} q_-^{TT} \cos \theta_1 \right\} \\
+ \left[ (2x_1 - 2v_S - 1) \frac{v_S - x_1}{v_S} + \frac{x_1}{v_S}(x_1 - 1 - 2v_S) \right] [q_+^{SS} - q_+^{SS} \cos \theta_1] \\
+ \frac{x_1^2(1 - 2x_1/3)}{2(1 - x_1 - v_T)^2} \left[ q_+^{TT} - q_+^{TT} \cos \theta_1 + q_-^{TT} - q_-^{TT} \cos \theta_1 \right] + \frac{2 \eta^{ST}_I}{1 - x_1 - v_T} \\
\times \left[ (1 - x_1 + v_S)(v_S - x_1) \log \frac{v_S - x_1}{v_S} - x_1 \left( \frac{3}{2} x_1 - 1 - v_S \right) \right] \right\} \quad (4.23)
\]

Finally, when the energy of \( f_1 \) is also integrated out, one has for the decay distribution

\[
\frac{d\Gamma}{d\cos \theta_1} = \frac{1}{2} \left( \Gamma_{tot} + \cos \theta_1 \Gamma_{ang} \right) \quad (4.24)
\]

where \( \Gamma_{tot} \) corresponds to the partial decay width for off-shell intermediate vector bosons and \( \Gamma_{ang} \) to the angular distribution. They are given by

\[
\Gamma_{tot} = N_e \frac{\alpha^2}{8\pi} m \left[ (q_-^{SS} + q_+^{SS}) R_T(v_S) + (q_-^{TT} + q_+^{TT}) R_T(v_T) + \eta^{ST}_I R_I(v_S, v_T) \right] \quad (4.25)
\]

\[
\Gamma_{ang} = -N_e \frac{\alpha^2}{8\pi} m \left[ (q_-^{SS} + q_+^{SS}) R_T(v_S) + (q_-^{TT} + q_+^{TT}) R_T(v_T) + q_+^{TT} R_A(v_T) \right] \quad (4.26)
\]

where \( R_T(v) \) describes the decay width in a given channel,

\[
R_T(v) = v(v - 1) \log \frac{v - 1}{v} + \frac{1}{6v}(6v^2 - 3v - 1) \quad (4.27)
\]

\( R_I(v_S, v_T) \) describes the interference between the two channels when simultaneously present. In terms of the Spence function, \( Li_2(x) = -\int_{0}^{1} dy y^{-1} \log(1 - xy) \), it is given by

\[
R_I(v, v) = (1 - v_T)(1 - 2v_S - 3v_T) \log \frac{v_T}{v_T - 1} + \log \frac{v_T}{v_T - 1} \left[ \log \frac{v_T}{v_T - 1} \log \frac{v_S + v_T - 1}{v_S} \right] \\
\times \log \frac{v_S}{v_S - 1} + 2(v_S + v_T)(1 - v_S - v_T) \left[ \log \frac{v_T}{v_T - 1} \log \frac{v_S + v_T - 1}{v_S} \right] \\
+ \left( \frac{v_T - 1}{v_S + v_T - 1} \right) - \left[ \frac{v_T - 1}{v_S + v_T - 1} \right] + 3 - 5v_S - 5v_T \quad (4.28)
\]

and \( R_A \) describes, for only \( V_T \) exchange, the deviation of the angular distribution from the familiar \( d\Gamma/d\cos \theta_1 \sim (1 - \cos \theta_1) \) form,

\[
R_A(v) = \frac{1}{2}(v - 1)(3 - 5v) \log \frac{v - 1}{v} - \frac{1}{12v}(6v^2 - 21v - 8) \quad (4.29)
\]

Once the charges are specified, the expressions of \( R_V \) and \( R_I \) agree with those obtained in Ref. [19] and the expression of \( R_A \) [and \( R_V \)] with the result of Ref. [20].
5. Summary

In this paper, we have analyzed in detail the decay modes and the production mechanisms in $e^+e^-$ collisions, of new heavy fermions predicted by extensions of the Standard Model and in particular by Grand Unified Theories.

We have given analytical expressions for the production of two heavy fermions of any flavor in a general case: we have allowed for arbitrary couplings of the new fermions, the presence of several production channels and we have taken into account the polarization of the initial $e^+e^-$ state and the final polarization of the heavy fermions. We have treated the case where the two fermions have different masses, to discuss the pair production and the single production in association with ordinary fermions on the same footing, and to account for the possibility of producing two different heavy leptons. Complete and compact formulae were given for angular distributions, total cross sections and polarization vectors.

We have then discussed in detail the decay modes of the new fermions including the cascade decays and the three–body decays with off–shell gauge bosons. Complete analytical expressions were given for total widths as well as for angular and energy distributions. We have also taken into account the polarization of the decaying fermion; combined with the spin–dependent cross sections, the spin–dependent decay distributions allow to obtain the full correlations between all the particles involved in the process. These correlations are very useful to discriminate between different types of particles and will be very important when discussing the signals for heavy fermion production since they help to optimize the experimental cuts which permit to suppress the various backgrounds without affecting drastically the signal cross sections.

More phenomenological aspects of heavy fermion production at future high–energy $e^+e^-$ linear colliders, including a detailed analysis of the various signals and backgrounds, will be presented in a subsequent paper [23].

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Appendix: Correlated Production and Decay

In this Appendix, we briefly summarize the formalism introduced by Tsai [15] to describe the production of heavy fermions in $e^+e^-$ collisions

$$e^+e^- \to \overline{F} F \quad (A.1)$$

which subsequently decay through real or virtual gauge boson exchange, into three body massless final states

$$F \to f_0 f_1 f_2$$

$$\overline{F} \to \overline{f}_0 \overline{f}_1 \overline{f}_2 \quad (A.2)$$

including the spin correlations. This formalism allows an easy reconstruction of the correlation between the initial $e^+,e^-$ states and the final decay products of the heavy fermions.

The polarization vector $\mathcal{P}(\overline{\mathcal{P}})$ of the fermion $F(\overline{F})$ is defined with the help its spin four-vector $n(\overline{n})$. The latter satisfies the relations $n \cdot n = \overline{n} \cdot \overline{n} = -1$ and $n \cdot p = \overline{n} \cdot \overline{p} = 0$, and is introduced by replacing the usual projection operators by

$$u(p) \overline{u}(p) \to (\not{p} + m) \frac{1 + \gamma_5 \not{n}}{2} \quad \text{for particles}$$

$$v(\overline{p}) \overline{v}(\overline{p}) \to (\not{\overline{p}} - m) \frac{1 + \gamma_5 \not{\overline{n}}}{2} \quad \text{for antiparticles} \quad (A.3)$$

In the production process, the polarization vector $\mathcal{P}$ is defined as

$$d\sigma^{\text{pol}} = \frac{1}{2} d\sigma^{\text{unpol}} (1 + n \cdot \mathcal{P}) \quad (A.4)$$

Choosing the $(x,z)$ plane as the scattering plane with the electron along the $+z$ direction, the covariant spin vector $n$ can be decomposed, in general, along three directions: the $F$ direction $\vec{p} (n_\parallel)$, the transverse direction with respect to $\vec{p}$ but within the scattering plane $(n_\perp)$ and the transverse direction with respect to $\vec{p}$ but along a normal to the scattering plane $(n_N)$. The projection of the spin vector along these three directions defines the corresponding degrees of polarization

$$n^\mu = P_\parallel n_\parallel^\mu + P_\perp n_\perp^\mu + P_N n_N^\mu \quad (A.5)$$

In the case where there is no CP violation, as we will assume for the process (A1), and since the very small imaginary parts from width or loop effects can be safely neglected, there is no polarization transverse to the scattering plane, $P_N = 0$. We can therefore set the azimuthal angle to zero and take $p_\mu = (E, |\vec{p}| \sin \theta, 0, |\vec{p}| \cos \theta)$ where $\theta$ is the scattering angle; in this case the components of the polarization vector are simply

$$\mathcal{P}_\parallel = E/m \left( |\vec{p}|/E, \sin \theta, 0, \cos \theta \right) \quad , \quad \mathcal{P}_\perp = (0, \cos \theta, 0, -\sin \theta) \quad (A.6)$$
Taking into account the polarization of both final fermions, the cross section for the pair production, eq. (A1), can be written as

\[ d\sigma^{pol} = \frac{1}{4} d\sigma^{unp} \left( 1 + \mathcal{P} \cdot n + \mathcal{P} \cdot \pi + C^{\mu\nu} n_\mu n_\nu \right) \]  

(A.7)

with \( C_{\mu\nu} \) the spin correlation.

In the decay processes, eq. (A2), the polarization vectors of the heavy fermions \( \mathcal{P}' \), are defined similarly to eq. (A2) but without the factor 1/2; for instance

\[ d\Gamma^{pol} = d\Gamma^{unpol} (1 + n \cdot \mathcal{P}') \]  

(A.8)

In the narrow width approximation, the full cross section for the production and subsequent decay of the heavy fermions,

\[ e^+ e^- \rightarrow f_0 f_0 f_1 f_1 f_2 f_2 \]  

(A.9)

can be written as

\[ d\sigma = \frac{1}{4} d\sigma^{unpol} d\Gamma^{unpol} \left( 1 + \eta_{\mu\nu} \mathcal{P}^\mu \mathcal{P}'^\nu + \overline{\eta}_{\mu\nu} \overline{\mathcal{P}}^\mu \overline{\mathcal{P}}'^\nu + \eta_{\mu\alpha} \overline{\eta}_{\nu\beta} C^{\mu\nu} \mathcal{P}^\alpha \mathcal{P}'^\beta \right) \]  

(A.10)

with

\[ \eta_{\mu\nu} = -g_{\mu\nu} + p_\mu p_\nu / m^2 , \quad \overline{\eta}_{\mu\nu} = -g_{\mu\nu} + \overline{p}_\mu \overline{p}_\nu / m^2 \]  

(A.11)

Summing over the polarization of one fermion, e.g. \( \mathcal{F} \), the full cross section simplifies to

\[ d\sigma = \frac{1}{2} d\sigma^{unpol} d\Gamma^{unpol} \left( 1 + \eta_{\mu\nu} \mathcal{P}^\mu \mathcal{P}'^\nu \right) \]  

(A.12)

Note that \( \eta_{\mu\nu} \mathcal{P}^\mu \mathcal{P}'^\nu = \mathcal{P}^\mu \mathcal{P}'^\mu \), where the * refers to the components in the rest frame of the heavy fermion.

The above formalism, based on the factorization of the production and decay sequences, permits an easy reconstruction of the full correlations between the initial and the final particles from the decay of the heavy fermions, which would be very difficult to obtain directly from the full process eq. (A9) with six particles in the final state. Furthermore, it is well adapted for the setting of a Monte Carlo generator.
References

[1] H. Georgi and S. L. Glashow, Phys. Rev. Lett. 32 (1974) 438.

[2] J.C. Pati and A. Salam, Phys. Rev. D10 (1974) 275.

[3] R. W. Robinett and J. L. Rosner, Phys. Rev. D25 (1982) 3036;
P. Langacker, R. W. Robinett and J. L. Rosner, Phys. Rev. D30 (1984) 1470;
V. Barger et al., Phys. Lett. 118B (1982) 68.

[4] For a review on E6, and for a complete set of references on grand unified groups see, J.
Hewett and T. G. Rizzo, Phys. Rep. 183 (1989) 193.

[5] M. Drees and X. Tata, Phys. Lett. 196B (1987) 65;
M. Drees, Nucl. Phys. B298 (1988) 333;
W. Buchmüller, C. Greub and P. Minkowski, Phys. Lett. 267B (1991) 395.

[6] See R. N. Mohapatra, Unification and Supersymmetry, Springer, New York, 1986.

[7] For a review on superstrings phenomenology see, J. Ellis, Proc. Workshop on Grand
Unification, Icoban Japan, 1986.

[8] J. C. Pati and A. Salam, Phys. Lett. 58B (1975) 333;
G. Senjanovic, F. Wilczek and A. Zee, Phys. Lett. 141B (1984) 389.

[9] For a review see, J. Maalampi and M. Roos, Phys. Rep. C186 (1990) 53.

[10] I. Montvay, Phys. Lett. 205B (1988) 315.

[11] M. Chanowitz, M. Furman and I. Hinchliffe, Phys. Lett. B78 (1978) 285;
F. Csikor and I. Montvay, Phys. Lett. 231B (1990) 503.

[12] M. Gronau, C.N. Lung and J. L. Rosner, Phys. Rev. D29 (1984) 2539;
J. L. Rosner, Com. Nucl. Part. Phys. 15 (1986) 195;
R. W. Robinett, Phys. Rev. D33 (1986) 1908;
I. Vendramin, Nuo. Cim. 100A (1988) 63.

[13] P. Langacker and D. London, Phys. Rev. D38 (1988) 244;
E. Nardi, E. Roulet and D. Tommasini, Nucl. Phys. 386 (1992) 239;
E. Nardi, E. Roulet and D. Tommasini, Phys. Rev. D46 (1992) 3040;
W. Buchmüller, C. Greub and H. Kohrs, Nucl. Phys. B370 (1992) 3;
F. Del Aguila et al., Phys. Rev. Lett 26 (1991) 23;
G. Bhattacharyya et al., Mod. Phys. Lett. A6 (1991) 2921.

[14] A. Djouadi, D. Schaile and C. Verzegnassi [conv.] et al., Proc. of the Workshop “e+e−
Collisions at 500 GeV: the Physics Potential”, Report DESY 92–123B, P. Zerwas, ed.
[15] Y. S. Tsai, Phys. Rev. D4 (1971) 2821.

[16] J. L. Rosner, Nucl. Phys. B248 (1984) 503;
    V. Barger et al., Phys. Rev. D33 (1986) 1912;
    T. G. Rizzo, Phys. Rev. D34 (1986) 1438;
    R. W. Robinett, Phys. Rev. D33 (1986) 1908;
    F. Gilman, Comm. Nucl. Part. Phys. 16 (1986) 231;
    K. Hagiwara and D. Zeppenfeld, Nucl. Phys. B274 (1986) 1;
    C. Ahn et al., SLAC–PUB–0329 (1988);
    F. M. L. Almeida et al., Phys. Rev. D44 (1991) 2836;
    W. Buchmüller and C. Greub, Nucl. Phys. B381 (1992) 109.

[17] F. del Aguila, E. Laermann and P. M. Zerwas, Nucl. Phys. B297 (1988) 1.

[18] W. Buchmüller and C. Greub, Nucl. Phys. B363 (1991) 345.

[19] M. Dittmar et al., Nucl. Phys. B332 (1990) 1.

[20] J. Maalampi, K. Mursula and R. Viopionperä, Nucl. Phys. B372 (1992) 23.

[21] L. Sehgal and P. M. Zerwas, Nucl. Phys. B183 (1981) 417.

[22] J.H. Kühn, A. Reiter and P. M. Zerwas, Nucl. Phys. B272 (1986) 360.

[23] G. Azuelos and A. Djouadi, Preprint UdeM–LPN–TH–93–158.