Introduction. — Electromagnetic waves in a material medium propagate according to Maxwell’s equations complemented by certain relations linking strengths and induced fields – the constitutive relations. Depending on the dielectric properties of the medium and also on the presence of applied external fields, a variety of optical effects can be found. One of wide interest is the birefringence [1, 2], occurring when electromagnetic waves propagate in media exhibiting two distinct refractive indexes [1] in a same wave vector direction. The phenomenon of triple refraction has been much less investigated. By trirefringence in a given wave vector direction, we mean the existence of three distinct refractive indexes in that direction. In the realm of linear electrodynamics, trirefringence does occur [3]. Nevertheless, nothing was studied when the dielectric tensors are field dependent (nonlinear electrodynamics).

The production of novel optical materials with multirefringent properties has been measured in tailored photonic crystals [4]. However, in these media, structural details (lattice constants, defects, etc) are imperative and hence they can not be described in terms of effective dielectric tensors [5]. Developments in photonic band gap materials and the so-called metamaterials have enabled the discovery of several new phenomena [6]. For instance, it was experimentally shown [7] that nearly transparent isotropic metamaterials allow light propagation when both effective dielectric coefficients (permittivity and permeability) are negative. In fact, this phenomenon and other unusual properties displayed by isotropic media with negative dielectric coefficients were long ago proposed theoretically [8]. It is worth emphasizing these media must be dispersive and the negative coefficients are obtained for convenient frequency ranges. Some other unusual properties exhibited by specific metamaterials are negative refractive index [9], trapping of light [10, 11], perfect lens devices [12], the electromagnetic cloaking effect [13, 14] and the occurrence of asymmetry for propagation of light in opposite wave vector directions [15]. Wave propagation in indefinite metamaterials (where not all the principal components of the dielectric tensors have the same sign) has been also considered [16], showing that effects already proposed in the context of isotropic metamaterials can be obtained and possibly improved. Indefinite metamaterials can also be used for investigating certain aspects of General Relativity [16].

In this Letter, we show that nonlinear metamaterials described in terms of effective dielectric tensors, may display trirefringence. Analytical expressions describing this effect are formally obtained from Maxwell’s electromagnetism and a simple theoretical model is numerically examined. A possible experimental realization of the media expected to display this effect is also addressed. The vectorial three dimensional formalism [17] is used. The units are set such that $c = 1$.

Wave propagation. — The electrodynamics of a continuum medium at rest in the absence of sources is governed by the Maxwell field equations

\[ \nabla \cdot \vec{D} = 0, \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{1} \]
\[ \nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}. \tag{2} \]

taken together with the constitutive relations between the fundamental fields $\vec{E}$ and $\vec{B}$, and the induced ones $\vec{D}$ and $\vec{H}$, written here as

\[ D_i = \sum_{j=1}^{3} \varepsilon_{ij} E_j, \quad H_i = \sum_{j=1}^{3} \mu_{ij} B_j. \tag{3} \]

The dielectric coefficients $\varepsilon_{ij}$ and $\mu_{ij}$ are the components of the permittivity and the inverse permeability tensors, respectively, and they encompass all information about the electromagnetic properties of the medium. Further, for any vector $\vec{a}$ we denote its $i$th component by $a_i$ ($i = 1, 2, 3$).

The propagation of monochromatic electromagnetic waves is here examined within the limit of geometrical optics [1] using the method of field disturbances [18]. This method can be summarized as follows [18]. Let $\Sigma$, defined by $\phi(t, \vec{x}) = 0$, be a smooth (differentiable of class $C^n$, $n > 2$) hypersurface. The function $\phi$ is understood to
be a real-valued smooth function of the coordinates \((t, \vec{x})\) and regular in a neighborhood \(U\) of \(\Sigma\). The spacetime is divided by \(\Sigma\) into two disjoint regions \(U^+\), for which \(\phi(t, \vec{x}) < 0\), and \(U^-\), corresponding to \(\phi(t, \vec{x}) > 0\). The discontinuity of an arbitrary function \(f(t, \vec{x})\) (supposed to be a smooth function in the interior of \(U^\pm\)) on \(\Sigma\) is a smooth function in \(U\), and is given by [18]

\[
[f(t, \vec{x})]_\Sigma \equiv \lim_{(p_\Sigma) \to P} [f(P^+) - f(P^-)],
\]

(4)

with \(P^+\), \(P^-\) and \(P\) belonging to \(U^+, U^-\) and \(\Sigma\), respectively. The electromagnetic fields are supposed to be smooth functions in the interior of \(U^+\) and \(U^-\) and continuous across \(\Sigma\) (\(\phi\) is now taken as the eikonal \([\text{1}]\) of the wave). However they have a discontinuity in their first derivatives, such that [18]

\[
[\partial_t E_i]_\Sigma = \omega e_i, \quad [\partial_t B_i]_\Sigma = \omega b_i, \quad [\partial_t E_j]_\Sigma = -q_i e_j, \quad [\partial_t B_j]_\Sigma = -q_j b_j,
\]

(5) (6)

where \(e_i\) and \(b_i\) are related to the derivatives of the electric and magnetic fields on \(\Sigma\), and correspond to the components of the polarization of the propagating waves \([\text{1}9]\). The quantities \(\omega\) and \(q_i\) are the angular frequency and the \(i\)th component of the wave vector. (Incidentally, we note that the negative signs appearing in Eq. (6) are missing in the corresponding equations in Ref. \([\text{17}].\))

For the cases of interest in this work, the permittivity of the media under study will be described by real diagonal tensors (losses have been neglected) whose components \(\varepsilon_{ij}\) are dependent only upon the constant frequency of the wave. We set the magnetic permeability of these media to be functions of the modulus of the electric field, such that \(\mu_{ij}(|\vec{E}|) = \delta_{ij}/\mu(|\vec{E}|)\), where \(\delta_{ij} = \text{diag}(1, 1, 1)\). Thus, applying the boundary conditions stated by Eqs. \([\text{5}7, \text{6}]\) to the field equations \([\text{12}]\) we obtain the eigenvalue equation \([\text{20, 21}]

\[
\sum_{j=1}^{3} Z_{ij} e_j = 0,
\]

(7)

where we defined the traces

\[
Z_1 \doteq \sum_{i=1}^{3} Z_{ii}, \quad Z_2 \doteq \sum_{i,j=1}^{3} Z_{ij} Z_{ji}, \quad Z_3 \doteq \sum_{i,j,l=1}^{3} Z_{ij} Z_{jl} Z_{li}.
\]

(10)

As a requirement of the geometrical optics limit, the wave fields are considered to be negligible when compared with the external fields. Thus, we assume from now on that the fields are approximated by their external counterparts \(\vec{E}_{ext}\) and \(\vec{B}_{ext}\). We set \(\vec{E} \approx \vec{E}_{ext} = \vec{E} \vec{x}\) and \(\vec{B} \approx \vec{B}_{ext} = \vec{B} \vec{y}\), which could be arbitrary functions of space and time coordinates. Let us examine the particular case of uniaxial media \([\text{1, 2}]\) where \(\varepsilon_{ij} = \text{diag}(\varepsilon_{||}, \varepsilon_{\perp}, \varepsilon_{\perp})\). Using Eqs. \([8\) and \(10],\) straightforward calculations show that Eq. \((9)\) results in the following algebraic fourth degree equation for the phase velocity \(v = \omega/q\) of the propagating waves,

\[
a v^4 + b v^3 + c v^2 + d v + e = 0,
\]

(11)

with

\[
a = 6 \varepsilon_{\perp}^2 \varepsilon_{||},
\]

(12)

\[
b = 6 \mu' \varepsilon_{\perp} \varepsilon_{||} E B \hat{q}_z,
\]

(13)

\[
c = -6 \frac{\mu'}{\mu} \varepsilon_{\perp} [2 \varepsilon_{\perp} \hat{q}_x^2 + (\varepsilon_{\perp} + \varepsilon_{||})(\hat{q}_y^2 + \hat{q}_z^2)],
\]

(14)

\[
d = -6 \frac{\mu'}{\mu^3} \varepsilon_{\perp} E B \hat{q}_z,
\]

(15)

\[
e = 6 \frac{\mu'}{\mu^2} \varepsilon_{||} (\hat{q}_x^2 + \hat{q}_y^2 + \hat{q}_z^2).
\]

(16)

We defined \(\hat{q}_x \doteq (\hat{q} \cdot \hat{\alpha})\), where \(\hat{q} \doteq \hat{q}/q\), for any unit vector \(\hat{\alpha}\). Then, \(\hat{q} = \hat{q}_x \hat{x} + \hat{q}_y \hat{y} + \hat{q}_z \hat{z}\) and \(q^2 \doteq \hat{q} \cdot \hat{q} = 1\). Notice that when the propagation occurs in the \(xy\)-plane (spanned by the external electric and magnetic fields), the coefficients \(b\) and \(d\) are null and the generalized Fresnel equation reduces to a quadratic equation in \(v^2\), therefore allowing only birefringence. The same behavior occurs if \(\mu' = 0\). In fact in this situation we recover linear electrodynamics.

Solving Eq. \((11)\) we obtain

\[
v_o = \pm \frac{1}{\sqrt{\mu \varepsilon_{||}}},
\]

(17)

\[
v_{e \pm} = -\sigma \hat{q}_z \pm \sqrt{(\sigma \hat{q}_z)^2 + \frac{1}{\mu \varepsilon_{||}} \left( \varepsilon_{||} \hat{q}_x^2 + \hat{q}_y^2 + \hat{q}_z^2 \right)},
\]

(18)

where

\[
\sigma = \frac{\mu' E B}{2 \mu^2 \varepsilon_{||}}.
\]

(19)

The solution \(v_o\) does not depend on direction of the wave propagation and will be called the ordinary wave,
whereas \( v^e_z \) depend on direction of wave propagation and will be called extraordinary waves \([1]\). By definition, the velocities of the waves are given by \( \vec{v} = v \hat{q} \), where \( v \) is given by Eqs. (17) and (18). In order to achieve more simplicity in the following analysis we assume the external fields to be constant and set the wave vector in a given direction of the \( xz \)-plane, i.e., \( \hat{q}_y = 0 \), \( \hat{q}_x = \sin \theta \) and \( \hat{q}_z = \cos \theta \). In this notation \( \theta \) indicates the angle between \( \hat{q} \) and the \( z \) direction.

**Trirefringence.** — Two distinct solutions for \( \vec{v}^e \) in a same given direction \( \hat{q} \) can be obtained from Eq. (18) if

\[
-1 < \frac{1}{\mu \varepsilon_\parallel \sigma^2} \left( \frac{\varepsilon_\parallel}{\varepsilon_\perp} \tan^2 \theta + 1 \right) < 0.
\]  

(20)

Thus, taking into account the ordinary wave, Eq. (20) defines a region inside which trirefringence occurs in any chosen direction \( \hat{q} \). Let us examine this effect closer. In order to guarantee the existence of an ordinary wave we must set \( \mu \varepsilon_\perp > 0 \), otherwise \( v_o \) is not real. This is true when both coefficients \( \mu \) and \( \varepsilon_\perp \) present the same sign, which can be positive for usual media or negative for left handed materials (in this case a negative refractive index occurs \([8, 9, 23]\)). Let us set these coefficients to be positive. Now, in order to satisfy Eq. (20) we set \( \varepsilon_\parallel = -\varepsilon_\perp < 0 \). Hence, trirefringence occurs in directions determined by

\[
\frac{\varepsilon_\perp}{\varepsilon_\parallel} \frac{\tan^2 \theta}{\varepsilon_\parallel} > 1 - \varepsilon_\parallel \mu \sigma^2.
\]  

(21)

The above discussed phenomenon is displayed in Fig. 1 where the normal surfaces \([1, 2]\) associated with the ordinary and extraordinary waves are depicted for some specific values of the quantities appearing in Eq. (21). For any given direction encompassed by the angles between the two dashed straight lines, defined by Eq. (21), there are three distinct solutions: the dashed and dot-dashed curves representing the extraordinary waves, and the circular solid curve representing the ordinary wave. For angles between the dashed and the solid straight lines only birefringence occurs, and finally only one refraction occurs for directions encompassed by the angles between the two solid straight lines. It is also worth noticing from Fig. 1 that in the sectors where more then one refractive index occur, the medium under consideration behaves as a positive or negative medium \([1, 2]\), depending on sub-sectors and extraordinary waves.

In geometrical optics the directions of light rays are given by the directions of the group velocities \( \vec{u} = \partial \omega / \partial \vec{q} = v \hat{q} + q(\partial v / \partial \hat{q}) \), which are considered as the physical velocities of propagation of the rays \([1]\). As we see from Eq. (18), the extraordinary waves depend on the wave vector. Thus, the directions of the extraordinary light rays do not in general coincide with the directions of the extraordinary phase velocities. Taking

\[
v = \{ v_o, v_o^e, v_e^e \}
\]

in the definition of the group velocity

\[
\vec{u} = \partial \omega / \partial \vec{q} = v \hat{q} + q(\partial v / \partial \hat{q})
\]

and denoted by \( \vec{v}^o \), the ordinary and extraordinary waves depend on the wave vector. Thus, the directions of the extraordinary light rays do not in general coincide with the directions of the extraordinary phase velocities. Taking

\[
v = \{ v_o, v_o^e, v_e^e \}
\]
we obtain that the ordinary group velocity \( \vec{u}_o = v_0 \hat{q} \). Nevertheless, the extraordinary group velocities are given by

\[
u'_e = u_x \hat{x} + u_z \hat{z}, (22)\]

with

\[
u_x = \frac{v^3_e \sin \theta + \sigma v^2_e \cos 2\theta + \frac{\nu}{\varepsilon} (\varepsilon - 1) \sin \theta \cos^2 \theta}{v^2_e + (1 - \eta) \sigma v_e \cos \theta + \frac{\eta}{2 \varepsilon} \cos^2 \theta}, (23)\]

\[
u_z = \frac{v^3_e \cos \theta + \sigma v^2_e \cos 2\theta - \frac{\nu}{\varepsilon} (\varepsilon - 1) \cos \theta \sin^2 \theta}{v^2_e + (1 - \eta) \sigma v_e \cos \theta + \frac{\eta}{2 \varepsilon} \cos^2 \theta}, (24)\]

and where we define \( \eta = \omega/\varepsilon \theta \). In these equations \( v_e \equiv v^\pm \) represents the extraordinary wave velocities in the direction \( \hat{q} \), which can be obtained from Fig. 1 by taking the distances from the associated points on its wave vector surface to the origin of the coordinate system. For the particular model set in Fig. 1 the corresponding ray surfaces \[1, 2\] are depicted in Fig. 2 as a straight lines in Fig. 1, there are two different extraordinary rays associated with it in Fig 2. For instance, \( \sigma v_e \cos \theta \), and dispersive \( (\varepsilon_1 = \varepsilon_1(\omega)) \) medium. The other one is composed of a liquid medium whose permeability and permittivity are dependent upon the modulus of the resultant electric field as \( \varepsilon_2 = 1 - g(|E|) \) and \( \varepsilon_2 = 1 - f(|E|) \), where \( f(|E|) \) and \( g(|E|) \) are usually much smaller than unity \[27\]. Additionally, if convenient external fields are present, then it is always possible to guarantee that our constraints are fulfilled.

Refraction analyzers were disregarded in our work once they are just a straightforward extension of the analyzers valid for birefringent crystals \[2\]. The only difference now is the existence of two extraordinary waves.

Closing, trirefringence is a phenomenon allowed to occur in some specific nonlinear metamaterials in terms of effective dielectric tensors and in principle could be formulated both in terms of wave and ray propagation. With the present technology in manipulating the dielectric coefficients in metamaterials \[16, 20\], we hope that the effect here derived can be experimentally tested and if verified could lead to applications.

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