Modular Momentum of the Aharonov—Bohm Effect on Noncommutative Lattices

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Abstract

Based on the technique of noncommutative geometry, it is shown that, by means of the concept of \( \theta \)-quantization, there is an equivalence between the notion of the modular momentum of the Aharonov-Bohm effect and the notion of a noncommutative lattice over a circle poset.

I. Introduction

The Aharonov-Bohm effect’s modular momentum on noncommutative lattices shall be constructed. We have studied the construction of fields over the Penrose Tiling(The Penrose Lattices) for a plane[1]. As a very simple example of a quantum mechanical system ,the \( \theta \)-quantization of the wave function for a particle on a lattice is shown in [3]. We noticed that by means of \( \theta \)-quantization, the theoretical structure of the Aharonov-Bohm effect’s modular momentum is equivalent to the theory of the above example[1].

In this article, the equivalence of the both above theories shall be described concretely, such as the theoretical structure of the Aharonov-Bohm effect can be studied with the techniques of non-commutative geometry on non-commutative lattices.

For the self-consistency of the explanation in the article, we make reference to the summary of [2] for the modular momentum and also to the summary of [3] for the \( \theta \)-quantization of wave functions on a lattice for the circle.

II. The Aharonov-Bohm effect and Quantum nonlocality[2]

II.1. Consider electrons of wavelength \( \lambda \) diffracting through a heavy grating, the grating consists of narrow slits.

1. The electrons scatter into discrete directions defined by angles \( \theta_n \) (n : an integer) \( \sin \theta_n = n \lambda / \ell \), in these directions the partial electron waves interfere constructively. These diffraction satisfy the following conditions where the n slits are spaced a distance \( \ell \) apart, and because of the grating is heavy, the energy of the electrons is practically the same before and after diffraction : hence their momentum \( p = h/\lambda \) and wavelength remain the same, according to the de Broglie relation.

2. The transverse momentum \( p_x \) of an electron scattered through an angle \( \theta_n \) is \( p_x = p \sin \theta_n = n h/\ell \). These satisfy the following conditions where the incident electrons wave move parallel to the y-axis and diffract in the xy-plane, and the grating is free to
move in the x-direction, the x-component of momentum is conserved during diffraction, and the grating acquires momentum $n\hbar/\ell$ in the x-direction from the electron. The electron and grating exchange transverse momentum only in multiples of $\hbar/\ell$.

II-2. Next, we place a solenoid between neighboring slits, the solenoids must move independently of the grating.

1. If all the solenoids carry the same flux $\Phi_B$, then electrons, after passing the grating, will scatter into a new set of angles $\theta'_n$ defined by $\sin \theta'_n = (n + (e \Phi_B / 2\pi \hbar c))(\lambda / \ell)$. These satisfy the following condition where a solenoid carrying a flux $\Phi_B$ contributes $e\Phi_B / \hbar c$ to the relative phase of partial waves passing on either side of it.

2. The constructive interference now corresponds to a change in the electron's transverse momentum of $p_x = p \sin \theta'_n = (n + (1/2))(\hbar / \ell)$. This satisfies the following condition where if the extra phase due to the solenoids is $e\Phi_B / \hbar c = \pi$, then the pattern of lines of constructive interference will be shifted by half the separation between neighboring lines.

II-3. We can arrange for the electric and magnetic fields of each solenoid to vanish wherever the electrons go. Thus the electrons and solenoids must exchange momentum non-locally. Nonlocal exchange of momentum is apparently an unobservable effect, for details, see[2].

II-4. Modular momentum[2]

The operator $\exp(ip\ell/\hbar)$ reveals the relative phase $\alpha$, a nonlocal aspect of $\varphi_a$ ($\varphi_a = \varphi_1 + e^{i\alpha} \varphi_2$, this equation is interference wave equation), because it translates the wave function. If we replace $p$ by $p - n\hbar/\ell$ in $\exp(ip\ell/\hbar)$, the operator remains invariant, since $\exp(i2\pi n) = 1$. So $\exp(ip\ell/\hbar)$ does not depend on all of $p$; it depends only on $p$ mod $n\hbar/\ell$.

Thus, this quantity is called the modular momentum

$$p \text{ mod } p_0 \quad p = p_1 + np_0, \quad 0 \leq p_1 \leq p_0, \quad p_0 = \hbar/\ell.$$ 

then, $2\pi np_0 \leq 2\pi p \leq 2\pi (n+1)p_0$. This modular momentum is a circle length $2\pi p_0$.

III. Line bundles on a circle poset and $\theta$-quantization[3]

The modular momentum of the Aharonov-Bohm effect can be studied with techniques of noncommutative geometry on noncommutative lattices. We shall construct the $\theta$-quantization of a particle on a lattice for a circle. We shall do so by constructing an appropriate 'line bundle' with a connection.

The real line $\mathbb{R}^1$ is the universal covering space of the circle $S^1$, and the fundamental group $\pi_1(S^1) = \mathbb{Z}$ acts on $\mathbb{R}^1$ by translation.
\[ R^1 \ni x \rightarrow x + N, \quad Z \ni N, \]

The quotient space of this action is \( S^1 \) and the projection \( x \rightarrow \exp(i2\pi x) \). Now, the domain of a typical Hamiltonian for a particle on \( S^1 \) need not consist of functions on \( S^1 \). Rather it can be obtained from functions \( \varphi_\theta \) on \( R^1 \) transforming under an irreducible representation of \( \pi_1(S^1) = \mathbb{Z} \). According to \( \varphi_\theta(x + N) = \exp(iN\theta)\varphi_\theta(x) \), projection is \( \rho_\theta : N \rightarrow \exp(iN\theta) \).

One obtains a distinct quantization, called \( \theta \cdot \)quantization, for each choice of \( \exp(i\theta) \).

Equivalently, wave functions can be single valued functions on \( S^1 \) while adding a ‘gauge potential’ term to the Hamiltonian. To be more precise, one constructs a line bundle over \( S^1 \) with a connection one-form given by \( i\theta \, dx \).

IV. \( \theta \cdot \)quantization(p\( _0 \)-quantization) of Modular momentum

Now, we give an equivalence between modular momentum and circle variables, this equivalence is a simple key idea of our research. We showed in II-4 that the modular momentum of the Aharonov-Bohm effect could be represented over the lattices on a circle. The identification of both representation are attributed to replacements by variables as follows :

\[ \theta \rightarrow h/\ell = p_0, \quad x = x_1 + N \rightarrow p/p_0 = p_1/p_0 + n, \]

\( \varphi_{p_0} \) on \( p \) was transformed by irreducible representation \( \rho_{p_0} \),

\[ \rho_{p_0} : p/p_0 \rightarrow \varphi_{p_0}(p/p_0), \quad \rho_{p_0} : (p_1/p_0) + n \rightarrow \exp(inp_0) \cdot \varphi_{p_0}(p_1/p_0) \]

V. A representation of the noncommutative lattices on a circle poset

V-1. Quantization of noncommutative lattices on a circle[3]

The algebra \( A \) of a noncommutative lattice on a circle, as it is AF(approximate finite dimensional), can be approximated by algebras of matrices. The simplest approximation is a commutative algebra \( C(A) \), this is \( P_{2N}(S^1) \), of the form \( C(A) = C_N = \{c=(\lambda_1, \lambda_2, \ldots, \lambda_N) : \lambda_i \in \mathbb{C}\} \).

\( P_{2N}(S^1) \) can produce a noncommutative lattice with \( 2N \) points by considering particular class of not necessarily irreducible representations. The additional condition is \( N = \)
N+1. The partial order, or equivalently the topology, is determined by the inclusion of the corresponding kernels as in [3].

We have, over $C(A)$, a K-Cycle($\mathcal{K}, \mathcal{D}$), and for $\mathcal{K}$ (the Hilbert space), we take $C^N$, on which we represent elements of $C(A)$ as diagonal matrices. $C(A) \ni c \rightarrow \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N) \in \mathcal{B}(C^N) = M_N(C)$. Elements of sections $\mathcal{E}$ will be realized in the same manner, $\mathcal{E} \ni \eta \rightarrow \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N) \in \mathcal{B}(C^N) = M_N(C)$.

We need a Laplacian $\Delta$ and a potential $W$ for a quantum theory, as a Hamiltonian can be constructed from these ingredients. In order to define a Laplacian, we need a self-adjoint operator $\mathcal{D}$ to define the ‘exterior derivative’ $d$ and a matrix of one-form $\rho$ as a connection.

By identifying $N+j$ with $j$, we take for the operator $\mathcal{D}$, the $N \times N$ self-adjoint matrix with elements

$$D_{ij} = (1/\varepsilon \sqrt{2})(m^* \delta_{i+1,j} + m \delta_{i,j+1}), \quad i,j = 1, \cdots, N.$$ 

As for the connection 1-form $\rho$ on the bundle $\mathcal{E}$, we take it to be the hermitian matrix with elements

$$\rho_{ij} = (1/\varepsilon \sqrt{2})(\sigma m^* \delta_{i+1,j} + \sigma m \delta_{i,j+1}), \quad i,j = 1, \cdots, N.$$ 

$$\omega = \exp(-i \theta) - 1, \quad i,j = 1, \cdots, N$$

The curvature of $\rho$ vanishes, $d\rho + \rho^2 = 0$. It is also possible to prove that $\rho$ is a ‘pure gauge’ for $\theta = 2\pi k$, with $k$ any integer, there exists $\rho = c^{-1} dc$. If $c = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$, then any such $c$ will be given by $\lambda_1 = \lambda, \lambda_2 = \exp(i \theta /N) \cdot \lambda, \cdots, \lambda_N = \exp(i \theta (N-1)/N) \cdot \lambda$, with $\lambda \neq 0$. $\lambda$ is the eigenvalue of Laplacian, $\Delta g, g = \lambda, \eta$, see[3].

V.2. The generation of the Weyl form of commutation relation[1],[4]

We get the Weyl form of matrix operators with AF-algebra on the above $C(A)$. $c$ is the diagonal $N \times N$ matrix with the above eigenvalues, and $D$ can be separated as $D = D_1 + D_1^*$. $c$ is a multiplicative operator and $D_1$ is a shift operator. We get the Weyl form of commutation relation,

$$c \cdot D_1 = \exp(-i \theta /N) \cdot D_1 \cdot c, \quad \theta = \hbar /\ell,$$

By the existence of this equation, the construction of the covering (as noncommutative poset), over $S^1$-space can be proved.
VI. Conclusion
We showed that the modular momentum of the Aharonov-Bohm effect could be constructed on non-commutative lattices. The non-commutative lattices imply the discreteness of space. Recently, at Fermi Laboratory (in USA), an experiment for the assurance of the discreteness of ‘space and time’ has been preparing[5].

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