Homogenization of the Allen–Cahn equation with periodic mobility

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Abstract
We analyze the sharp interface limit for the Allen–Cahn equation with an anisotropic, spatially periodic mobility coefficient and prove that the large-scale behavior of interfaces is determined by mean curvature flow with an effective mobility. Formally, the result follows from the asymptotics developed by Barles and Souganidis for bistable reaction–diffusion equations with periodic coefficients. However, we show that the corresponding cell problem is actually ill-posed when the normal direction is rational. To circumvent this issue, a number of new ideas are needed, both in the construction of mesoscopic sub- and supersolutions controlling the large-scale behavior of interfaces and in the proof that the interfaces obtained in the limit are actually described by the effective equation.

Mathematics Subject Classification 35B27 · 35D40

1 Introduction

1.1 Overview
In this paper, we study the sharp interface limit of the Allen–Cahn equation with periodic mobility. Given \( u_0 \in UC(\mathbb{R}^d; [-1, 1]) \), we are interested in the behavior as \( \epsilon \to 0^+ \) of the solutions \( (u^\epsilon)_{\epsilon>0} \) of the following equations:

\[
\begin{aligned}
& m(\epsilon^{-1} x, \epsilon Du^\epsilon) u^\epsilon_t - \Delta u^\epsilon + \epsilon^{-2} W'(u^\epsilon) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty), \\
& u^\epsilon = u_0 \quad \text{on } \mathbb{R}^d \times \{0\}.
\end{aligned}
\]

(1)

Here \( W : [-1, 1] \to [0, \infty) \) is a double well-potential with wells of equal depth (for concreteness, \( W^{-1}(\{0\}) = \{-1, 1\} \)) and \( m : \mathbb{T}^d \times \mathbb{R}^d \to (0, \infty) \) is a mobility coefficient.

Equation (1) describes the large-scale behavior of phase transitions when the free energy is determined by the Allen–Cahn functional and energy dissipation is determined by \( m \). The
scaling results from blowing up space by $\epsilon^{-1}$ and time, by $\epsilon^{-2}$. Without going into the details yet, we only assume that $W$ and $m$ are bounded and sufficiently smooth.

Classically, in the spatially homogeneous, isotropic setting (i.e., when $m \equiv 1$), the large-scale behavior of $(u^\epsilon)_{\epsilon > 0}$ is described by the so-called sharp interface limit. Informally, this means that, as $\epsilon \to 0^+$, $u^\epsilon(\cdot, t) \to 1$ in $E_t$ and $u^\epsilon(\cdot, t) \to -1$ in $\mathbb{R}^d \setminus E_t$, where the family of open sets $(E_t)_{\epsilon > 0}$ is a mean curvature motion with $E_0 = \{u_0 > 0\}$. In the periodic setting considered here, we prove below that a similar result holds, except that the effective interface velocity $V$ has the following form:

$$\bar{m}(n_{\partial E_t})V = \kappa_{\partial E_t}.$$  \hspace{1cm} (2)

Here $n_{\partial E_t}$ and $\kappa_{\partial E_t}$ are the normal vector and mean curvature of the surface $\partial E_t$ and $\bar{m}$ is the effective mobility determined by $m$ and $W$ through averaging effects.

### 1.2 Motivation

Since the ’90s, there has been significant interest in whether or not the main features of bistable reaction–diffusion equations in spatially homogeneous media persist in the periodic setting. Beginning with Barles and Souganidis [9], some attention has been devoted to the sharp interface limit in periodic media, but the question is largely open.

In [9], the authors treated equations of the form

$$u_t^\epsilon - \text{div}(a(\epsilon^{-1}x)Du^\epsilon) + \epsilon^{-2}W'(u^\epsilon) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty),$$  \hspace{1cm} (3)

where the coefficient $a$ is a smooth, integer periodic, uniformly elliptic matrix field. They proved that, under some strong assumptions on $a$ and $W$, the asymptotic behavior of the interface between the sets $\{u^\epsilon \approx 1\}$ and $\{u^\epsilon \approx -1\}$ is described by an anisotropic curvature flow (see [9, Section 6]).

The starting point for this work is the following slight generalization of the previous equation, namely,

$$m(\epsilon^{-1}x, \epsilon Du^\epsilon)u_t^\epsilon - \text{div}(a(\epsilon^{-1}x)Du^\epsilon) + \epsilon^{-2}W'(u^\epsilon) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty).$$  \hspace{1cm} (4)

Here we have added to the previous equations the coefficient $m$, which is a smooth, positive function, integer periodic in the first (spatial) variable. To understand the roles played by $a$ and $m$, it is important to note that, at least formally, (4) is the (rescaled) gradient flow of the spatially heterogeneous free energy

$$\mathcal{F}^a(u; \Omega) = \int_{\Omega} \left( \frac{1}{2} \langle a(y)Du(y), Du(y) \rangle + W(u(y)) \right) dy$$  \hspace{1cm} (5)

with respect to the $L^2$-Riemannian metric given by

$$\langle v, w \rangle_u = \int_{\mathbb{R}^d} v(y)w(y)m(y, Du(y)) dy.$$  \hspace{1cm} (6)

Hence $a$ determines the energy while $m$ controls the rate of energy dissipation.

As is already apparent from a formal asymptotic analysis, (4) is amenable to the same techniques that were used in [9], again provided certain assumptions are made on $a$ and $W$. More precisely, the analysis in [9] is based on an asymptotic expansion of $u^\epsilon$ of the form

$$u^\epsilon(x, t) = U(\epsilon^{-1}d(x, t), \epsilon^{-1}x, Dd(x, t))$$
$$+ \epsilon P(\epsilon^{-1}d(x, t), \epsilon^{-1}x, Dd(x, t), D^2d(x, t), d_t(x, t)) + \cdots,$$  \hspace{1cm} (7)
where $d$ is the signed distance to the developing interface. Substituting the ansatz into (4) and expanding in powers of $\epsilon$, one is led to certain PDE determining the functions $U$ and $P$ in the expansion (see Appendix D for the details). The assumptions in [9] are used precisely to ensure that these PDE have smooth solutions.

Recently, the author showed that the PDE determining $U$ may not have smooth solutions in general (e.g., even if $a$ and $W$ are smooth) and there are obstructions of a variational nature that make identifying the right class of coefficients a difficult problem [32]. Nonetheless, there is ample reason to believe that [9] is a step in the right direction and it is instructive to revisit those results.

The key assumptions made in [9], stated somewhat informally, are as follows:

(i) There is a family $\{U_e\}_{e \in S^{d-1}}$, smoothly parametrized by $e \in S^{d-1}$, of smooth solutions of the pulsating standing wave equation

$$D_e^*(a(y)D_e U_e) + W'(U_e) = 0 \quad \text{in } \mathbb{R} \times \mathbb{T}^d,$$

$$\lim_{s \to \pm\infty} U_e(s, y) = \pm 1, \quad \partial_s U_e \geq 0. \quad (8)$$

Here $D_e = e\partial_s + D_y$ and $D_e^*$ is the $L^2$ adjoint of $D_e$ (see (23)). Note that since $D_e$ maps functions on $\mathbb{R} \times \mathbb{T}^d$ to vector fields taking values in $[0] \times \mathbb{R}^d$, this is a degenerate elliptic PDE.

(ii) The linearized equation associated with (8) is solvable. That is, given a bounded function $F \in C^\infty(\mathbb{R} \times \mathbb{T}^d)$ and an $e \in S^{d-1}$, there is a unique constant $\overline{F}(e) \in \mathbb{R}$ for which the following degenerate elliptic cell problem has a smooth solution $P_{e,F}$:

$$D_e^*(a(y)D_e P_{e,F}) + W''(U_e)P_{e,F} = F(s, y) - \overline{F}(e)\partial_s U_e \quad \text{in } \mathbb{R} \times \mathbb{T}^d. \quad (9)$$

Such a function $P_{e,F}$ will be referred to henceforth as a corrector.

In addition to the existence of $P_{e,F}$ and $\overline{F}(e)$, [9] assumes that $e \mapsto \overline{F}(e)$ and $e \mapsto P_{e,F}$ are smooth. This presents problems even in the present work and seems to be intractable in general (see Remark 2 below), but we will see that the loss of smoothness can be overcome in some situations.

Proposition 1 in [32] shows that assumption (i) amounts to a constraint on the energy $\mathcal{F}^a$, and the counter-examples in that reference show that this constraint may not be satisfied in general. In fact, the arguments in recent work of the author and Feldman [24] show that there are coefficients $a$ for which smooth solutions of (8) fail to exist in any direction $e$ (see [24, Remarks 13 and 50]).

This difficulty notwithstanding, the motivation for the present work stems from the additional degree of freedom in the mobility $m$. When $m$ is non-constant, there are two, independent contributors to the effective behavior in the sharp interface limit in (4): the energy landscape of $\mathcal{F}^a$ and the oscillation of $m$. Even if the role of the energy is not completely understood, that still leaves the question of the asymptotics of (1), where $a$ is constant but $m$ is oscillatory.

We answer that question here, proving that homogenization occurs and the sharp interface limit is described by (2). At a formal level, this is immediate given the approach of [9]. However, even in this setting, a number of significant problems arise when turning the formal asymptotics into a proof, and these require new ideas. We hope this paper will shed light on the difficulties that need to be overcome to treat (4) in general, although there are still substantial problems that would need to be addressed first.
1.3 Main result

Before stating the result, here are the assumptions on $m$:

$$ m \in C(\mathbb{T}^d \times \mathbb{R}^d), $$

$$ 0 < \theta := \min \left\{ m(y, v) \mid (y, v) \in \mathbb{T}^d \times \mathbb{R}^d \right\}, $$

$$ \Theta := \max \left\{ m(y, v) \mid (y, v) \in \mathbb{T}^d \times \mathbb{R}^d \right\} < \infty, $$

$$ H_1 = \sup \left\{ \frac{|m(y, v) - m(y', v)|}{\|y - y'\|} \mid (y, v), (y', v') \in \mathbb{T}^d \times \mathbb{R}^d, \ y \neq y' \right\} < \infty. $$

Concerning the potential, we assume that $W \in C^3([-1, 1])$ satisfies

$$ W = 0 = \{-1, 1\}, \quad W' = 0 = \{-1, 0, 1\}, $$

$$ (-1, 0) \subseteq [W' > 0], \quad (0, 1) \subseteq [W' < 0], $$

$$ W''(1) \land W''(-1) > 0, \quad W''(0) < 0. $$

The main result of the paper, stated next, says that the macroscopic interfaces obtained in the sharp interface limit of (1) are described by (2), with the effective mobility $\overline{m}$ defined by

$$ \overline{m}(e) = c_w^{-1} \int_{\mathbb{R} \times \mathbb{T}^d} m(y, \hat{q}(s)e)\hat{q}(s)^2 \, dy \, ds, \quad c_w = \int_{-\infty}^{\infty} \hat{q}(s)^2 \, ds. $$

Due to the exponential decay of $\hat{q}$ as $|s| \to \infty$ (see Appendix C), the assumptions on $m$ imply that

$$ \overline{m} \in C(S^{d-1}), \quad \theta \leq \overline{m} \leq \Theta. $$

**Theorem 1** Assume that $m$ satisfies (10), (11), (12), and (13) and $W$ satisfies (14), (15), and (16). Let $\overline{m} : S^{d-1} \to [\theta, \Theta]$ denote the effective mobility defined by (17). If $u_0 \in UC(\mathbb{R}^d; [-1, 1]), (u^e)_{e>0} \subseteq C(\mathbb{R}^d \times [0, \infty); [-1, 1])$ are the viscosity solutions of (1), and $\bar{u} : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ is the unique viscosity solution of the effective level set PDE

$$ \left\{ \begin{array}{l}
\overline{m}(\overline{D}u)\overline{u}_t - \text{tr} \left((Id - \overline{D}u \otimes \overline{D}u) \, D^2\overline{u}\right) = 0 \quad \text{in} \ \mathbb{R}^d \times (0, \infty), \\
\overline{u} = u_0 \quad \text{on} \ \mathbb{R}^d \times \{0\},
\end{array} \right. $$

then $u^e \to \pm 1$ locally uniformly in $\{\pm \overline{u} > 0\}$.

To prove Theorem 1, we start with the ansatz (7). As shown in Appendix D below, since $a \equiv Id$, the expansion can be rewritten in the following simplified form

$$ u^e(x, t) = q \left( \frac{d(x, t)}{\epsilon} \right) + \epsilon d_t(x, t) P_{Dd(x, t)} \left( \frac{d(x, t)}{\epsilon}, \frac{x}{\epsilon} \right) + \cdots $$

Here $d$ is the signed distance to the developing interface; $q$ is the one-dimensional standing wave solution of (1), that is, the centered heteroclinic solution of the ODE

$$ -\hat{q} + W'(q) = 0 \quad \text{in} \ \mathbb{R}, \quad \lim_{s \to \pm \infty} q(s) = \pm 1, \quad q(0) = 0; $$

and we are led to the following cell problem for the correctors $(P_e)_{e \in S^{d-1}}$ and effective mobility $\overline{m}$:

$$ m(y, \hat{q}(s)e)\hat{q}(s) + D_s^e D_e P_e + W''(q(s))P_e = \overline{m}(e)\hat{q}(s) \quad \text{in} \ \mathbb{R} \times \mathbb{T}^d. $$

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To see the connection to our previous discussion of [9], note that (20) and (21) are special cases of (8) and (9) obtained by setting \( a \equiv \text{Id} \) and \( U_e(s, y) = q(s) \).

As we briefly discuss in Remark 2 below, it turns out that the cell problem (21) is ill-posed in general when \( e \) is a rational direction, that is, \( e \in \mathbb{RZ}^d \). This is precisely the obstruction to assumption (ii) above. In some cases, it is possible to find solutions when \( e \) is an irrational direction, that is, \( e \notin \mathbb{RZ}^d \), but it is not obvious this is true in general.

Nonetheless, we show below that, as long as \( e \notin \mathbb{RZ}^d \), approximate correctors do exist. More precisely, if \( m \) and \( W \) satisfy the assumptions of Theorem 1 and \( \nu > 0 \), then there is an \( \overline{m}(e) > 0 \) and a \( \tilde{P}^\nu_e \in C^{2,\mu}(\mathbb{R} \times \mathbb{T}^d) \) such that

\[
\left| [m(y, \dot{q}(s)e) - \overline{m}(e)]\dot{q}(s) + D^\nu_e D\tilde{P}^\nu_e + W''(q(s))\tilde{P}^\nu_e \right| \leq \nu \dot{q}(s) \quad \text{in } \mathbb{R} \times \mathbb{T}^d. \tag{22}
\]

We will see that this is a good enough replacement of assumption (ii), but it leads to two problems that still need to be addressed. First, the sub- and supersolutions used in [9] to control solutions of (3) as \( \epsilon \rightarrow 0^+ \) were global constructions that required correctors in every direction \( e \). That presents a problem here since there are obstructions to approximate correctors in rational directions. To circumvent this, we build on the graphical approach of [32], showing how to construct local sub- and supersolutions controlling the developing interface in a neighborhood of any point where the normal vector is irrational.

Next, we need to show that knowledge of the behavior of the macroscopic interface at points where the normal is irrational is enough to determine its behavior globally. This is too much to ask in general (e.g., if the interface is a plane with a rational normal), but we are saved by the fact that the structure of (1) already gives us a good amount of information at locations where the interface is more-or-less flat. Using viscosity theoretic arguments, we prove that an interface that is only known to satisfy (2) in irrational directions and that is well-behaved at points where it is flat actually moves by (2) in the viscosity sense. Put another way, a solution \( \bar{u} \) of (18) “in irrational directions” is actually a viscosity solution in the usual sense.

### 1.4 Related literature

Allen and Cahn [1] made the connection between mean curvature flow and the equation bearing their name. This attracted quite a lot of attention in the mathematical community. The first rigorous proofs appeared in the work of De Mottoni and Schatzman [20], Bronsard and Kohn [10], and Chen [13] in the setting of smooth flows. The development of a viscosity theory for level set PDEs led to proofs that mean curvature motion describes the asymptotics globally in time. This was the work of Evans et al. [21], Barles et al. [8], and Barles and Souganidis [9]. Shortly thereafter, Ilmanen [26] showed how to describe the global asymptotics using Brakke flow and geometric measure theory.

In addition to the Allen–Cahn equation, considerable energy was invested in the motion of interfaces in the stochastic Ising model with Kac interactions. See the work of De Masi, Orlandi, Presutti, and Triolo [19], which described the macroscopic evolution up to the development of singularities, and Katsoulakis and Souganidis [28, 29] for the global result (also see [30]). In that context, the mesoscopic scale is described by a non-local reaction diffusion equation with a mobility term in the same spirit as (1). This partly inspired the present work.

Another source of inspiration is the discussion of phase separation phenomena by Taylor and Cahn [36]. There the authors propose equations like (1), except the mobility depends only on the direction of the gradient. As acknowledged already in [36], it is not clear that such
equations are well-posed. At any rate, from a modeling perspective, given that the Allen–Cahn functional is very sensitive to the norm of the gradient, it seems natural to consider mobility coefficients that depend on the amplitude of the gradient in addition to the direction as in (1) or (4).

Beyond [9], homogenization results for geometric motions in the parabolic scaling regime have been hard to come by. Homogenization has been proved when the initial interface is a graph evolving in a laminar medium, starting with Barles et al. [4] for a class of forced mean curvature flows and, more recently, [32] for phase field equations like (4). For plane-like initial data, Cesaroni et al. [12] prove homogenization of solutions of mean curvature flow with periodic forcing, obtaining effective front speeds that are discontinuous with respect to the normal direction (see also the paper by Chen and Lou [14]). In the companion paper [33], the author proves homogenization for a broad class of geometric motions, roughly corresponding to those for which planes are stationary solutions. In dimensions greater than two, the effective coefficients are generically discontinuous at every rational direction.

The fundamental difficulty of the present work and [33] stems from the fact that the notion of corrector is not tractable in rational directions. This is morally equivalent to the hurdles faced in the homogenization of oscillating boundary value problems. (A non-exhaustive list of references is: the papers by Barles et al. [5], Barles and Mironescu [7], Gérard-Varet and Masmoudi [25], Choi and Kim [15], Feldman [22], and Feldman and Kim [23].) In that context as well as the present one, the fact that there is no well-defined ergodic constant in the cell problem in a given rational direction is intimately linked to the non-unique ergodicity of the associated group of translations on the torus. In addition to complicating the analysis considerably, this can ultimately lead to discontinuities in the effective coefficients.

The homogenization of the general gradient flow (4) or sharp interface versions is expected to be even more difficult than the problems considered here or in [33]. To start with, we know by now that the effective interface velocity will not be as well-behaved as (2). In [24], the author and Feldman provide examples of coefficients \((a, W)\) for which homogenization can be proved with effective velocity \(V = 0\). Hence, at least heuristically, the effective mobility is infinite. They also show that pathological “bubbling” occurs generically, suggesting that either \(V = 0\) or a more complicated description of the limit is necessary. The same difficulties arise for variational curvature flows (see also the earlier paper by Novaga and Valdinoci [34]).

Finally, the use of approximate correctors in this work was very much inspired by what is by now a standard tool in the homogenization arsenal, see the papers by Ishii [27], Lios and Souganidis [31], and Caffarelli et al. [11], for example.

### 1.5 Outline of the paper

Section 2 details the strategy of the proof and states Theorem 2, which is needed to overcome the difficulty encountered in rational directions. Section 3 treats the proof of Theorem 2 and Sect. 4, the existence of approximate correctors. Section 5 describes the large-scale behavior of interfaces when the normal direction is irrational. Section 6 treats the case when the normal direction is rational and the curvature is small. In Sect. 7, the case when “the normal vanishes” is analyzed.

There are four appendices. Appendix A contains a lemma used in the proof of homogenization; Appendix B proves that (1) is well posed; and Appendix C includes regularity results needed in the construction of correctors as well as a few standard facts about the one-dimensional Allen–Cahn equation. Finally, Appendix D reviews the approach of [9] in
the context of the general problem (4), deriving the pulsating wave equation (8) and the cell problem (9) in the process.

1.6 Notation

In $\mathbb{R}^d$, $\langle \cdot, \cdot \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is the Euclidean inner product and $\| \cdot \| : \mathbb{R}^d \to [0, \infty)$, the associated norm. Given $p \in \mathbb{R}^d \setminus \{0\}$, we write $\hat{p} = \|p\|^{-1}p$.

$\mathcal{L}_d$ is the Lebesgue measure in $\mathbb{R}^d$. $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure, normalized to coincide with surface area.

Given $a, b \in \mathbb{R}$, we write $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.

For a set $A$, the characteristic function $\chi_A$ is defined by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$, otherwise. Given a finite measure $\mu$, $\int_A(\cdot) \mu(dx) = \mu(A)^{-1} \int \chi_A(\cdot) \mu(dx)$.

Finally, given $e \in S^{d-1}$, we define the differential operator $\mathcal{D}_e$ in $\mathbb{R} \times \mathbb{T}^d$ by

$$\mathcal{D}_e \varphi(s, y) = e \partial_s \varphi(s, y) + D_y \varphi(s, y) \quad \text{for } \varphi \in C^\infty_c(\mathbb{R} \times \mathbb{T}^d).$$

$\mathcal{D}^*_e$ denotes its $L^2$ adjoint, which acts on smooth $\mathbb{R}^d$-valued vector fields according to the formula

$$\mathcal{D}^*_e \Psi(s, y) = -\langle e, \partial_s \Psi(s, y) \rangle - \text{div}_y \Psi(s, y) \quad \text{for } \Psi \in C^\infty_c(\mathbb{R} \times \mathbb{T}^d; \mathbb{R}^d).$$

2 Strategy of proof

2.1 Solutions of (18) in irrational directions

As explained in the introduction, the approximate correctors used in the asymptotic analysis are only available in irrational directions. This leads to the question of whether or not an interface known to move by (2) at points where its normal vector is irrational actually has to move only in irrational directions. This leads to the question of whether or not an interface known to move by (2) at points where its normal vector is irrational actually has to move in a way globally. The case of a rational plane translating at an arbitrary velocity shows this is untenable. However, the form of (1) already rules this out. By building appropriate sub- and supersolutions, it is possible to prove that the interface obtained in the limit is a solution of (2) in a sense that is a priori weaker than the standard notion of viscosity solution, but which nonetheless turns out to be equivalent.

Employing the level set formulation of (2), we are interested in functions $u$ defined in a space-time domain $U \subseteq \mathbb{R}^d \times (0, \infty)$ solving the level-set PDE

$$\bar{m}(\bar{D}u)u_t - \text{tr} \left( (\text{Id} - \bar{D}u \otimes \bar{D}u) \, D^2u \right) = 0 \quad \text{in } U. \quad (24)$$

The question then becomes identifying a notion of solution of (24) that is well-adapted to the study of our homogenization problem (1). The next definition serves that purpose.

**Definition 1** Given an open set $U \subseteq \mathbb{R}^d \times (0, \infty)$, we say that a locally bounded, upper semi-continuous function $v : U \to \mathbb{R}$ is a *subsolution* of (24) in irrational directions in $U$ if there is a constant $K(v) > 0$ such that, given any smooth function $\varphi : \mathbb{R}^d \times (0, \infty) \to \mathbb{R}$ and any point $(x_0, t_0) \in U$ at which the difference $v - \varphi$ has a strict local maximum, the following conditions are met:

(a) If $D\varphi(x_0, t_0) \in \mathbb{R}^d \setminus \mathbb{R}^d$, then

$$\bar{m}(\bar{D}\varphi(x_0, t_0)) \varphi_t(x_0, t_0) - \text{tr} \left( (\text{Id} - \bar{D}\varphi(x_0, t_0) \otimes \bar{D}\varphi(x_0, t_0)) \, D^2\varphi(x_0, t_0) \right) \leq 0.$$
(b) If $D\varphi(x_0, t_0) \in \mathbb{R}^d \setminus \{0\}$, then
$$\varphi_t(x_0, t_0) \leq K(v) \left\| (\text{Id} - \overline{D\varphi}(x_0, t_0) \otimes \overline{D\varphi}(x_0, t_0)) D^2\varphi(x_0, t_0) \right\|.$$ 
(c) If $\|D\varphi(x_0, t_0)\| = \|D^2\varphi(x_0, t_0)\| = 0$, then
$$\varphi_t(x_0, t_0) \leq 0.$$ 

Similarly, a locally bounded, lower semi-continuous function $w : U \rightarrow \mathbb{R}$ is a supersolution of (24) in irrational directions in $U$ if $-w$ is a subsolution in irrational directions. A locally bounded, continuous function is a solution of (24) in irrational directions in $U$ if it is both a subsolution and a supersolution.

Using approximate correctors, we show that the interface obtained from (1) in the limit $\epsilon \rightarrow 0^+$ satisfies (a), (b) and (c) then follow directly from the structure of (1) without correctors. It only remains to determine whether or not Definition 1 is enough to identify the viscosity solution of (24). That is the content of the next theorem.

**Theorem 2** Given an open set $U \subseteq \mathbb{R}^d \times (0, \infty)$, if $u : U \rightarrow \mathbb{R}$ is a subsolution (resp. supersolution) of (24) in irrational directions in $U$, then it is a viscosity subsolution (resp. supersolution) of (24) in $U$ in the usual sense.

In view of the theorem, to prove Theorem 1 it suffices to show that the interface obtained from (1) is a solution of (18) in irrational directions. The rest of this section lays out the main steps of that argument, while the proof of Theorem 2 is deferred to Sect. 3.

### 2.2 Proof of Theorem 1

Henceforth $u_0 \in UC(\mathbb{R}^d; [-1, 1])$ is fixed and $(u^\epsilon)_{\epsilon>0}$ are the solutions of (1) in $\mathbb{R}^d \times (0, \infty)$. The existence and uniqueness of these solutions is sketched in Appendix B.

The macroscopic phases that develop in the sharp interface limit are described by the following open sets, parametrized by $t > 0$:

$$\Omega_t^{(1)} = \left\{ x \in \mathbb{R}^d \mid \liminf_{\epsilon} u^\epsilon(x, t) = 1 \right\},$$

$$\Omega_t^{(2)} = \left\{ x \in \mathbb{R}^d \mid \limsup_{\epsilon} u^\epsilon(x, t) = -1 \right\}.$$ 

Recall that the half-relaxed limits in the definition are given by

$$\liminf_{\epsilon} u^\epsilon(x, t) = \lim_{\delta \rightarrow 0^+} \inf \left\{ u^\epsilon(y, s) \mid \epsilon + \|x - y\| + |t - s| \leq \delta \right\},$$

$$\limsup_{\epsilon} u^\epsilon(x, t) = \lim_{\delta \rightarrow 0^+} \sup \left\{ u^\epsilon(y, s) \mid \epsilon + \|x - y\| + |t - s| \leq \delta \right\}.$$ 

We proceed by proving that $(\Omega_t^{(1)})_{t>0}$ and $(\Omega_t^{(2)})_{t>0}$ define super- and subflows of (2), respectively, in the sense of [9]. It will not be necessary to know what that means in this paper. Instead, we associate phase indicator functions $\chi_*, \chi^* : \mathbb{R}^d \times (0, \infty) \rightarrow \mathbb{R}$ to the sets $(\Omega_t^{(1)})_{t>0}$ and $(\Omega_t^{(2)})_{t>0}$ and prove these are respectively discontinuous super- and subsolutions of (18). $\chi_*$ and $\chi^*$ are defined for $(x, t) \in \mathbb{R}^d \times (0, \infty)$ via the formulae

$$\chi_*(x, t) = \begin{cases} 1, & \text{if } x \in \Omega_t^{(1)}, \\ -1, & \text{otherwise}, \end{cases} \quad \chi^*(x, t) = \begin{cases} 1, & \text{if } x \in \mathbb{R}^d \setminus \Omega_t^{(2)}, \\ -1, & \text{otherwise}, \end{cases}$$

and at $t = 0$ by...
\[
\chi_*(x, 0) = \lim_{\delta \to 0^+} \inf \{ \chi_*(y, s) \mid \|x - y\| + s < \delta, \ s > 0 \}, \\
\chi^*(x, 0) = \lim_{\delta \to 0^+} \sup \{ \chi^*(y, s) \mid \|x - y\| + s < \delta, \ s > 0 \}.
\]

The main thrust of the paper is the proof of the following result:

**Proposition 1** \( \chi^* \) (resp. \( \chi_* \)) is a subsolution (resp. supersolution) of (18) in irrational directions in \( \mathbb{R}^d \times (0, \infty) \). Furthermore, \( \chi^*(\cdot, 0) = -1 \) in \( \{u_0 < 0\} \) and \( \chi_*(\cdot, 0) = 1 \) in \( \{u_0 > 0\} \).

**Proof** Where \( \chi_* \) is concerned, the first statement follows from Propositions 4, 9, and 11, and the second, from Proposition 12 below. The statements concerning \( \chi^* \) then follow by replacing \( u^\epsilon \) by \(-u^\epsilon \), \( W \) by \( u \mapsto W(-u) \), \( \chi^* \) by \(-\chi^* \), etc. since this has the effect of transforming supersolutions into subsolutions. \( \square \)

In view of Theorem 2, we can remove the “irrational directions” qualifier and instead treat \( \chi^* \) and \( \chi_* \) as viscosity sub- and supersolutions. It only remains to prove that, even though these functions are discontinuous, \( \chi^* \) and \( \chi_* \) can still be compared to \( \bar{u} \). This part is classical.

**Theorem 3** If \( \bar{u} \) is the solution of (18), then \( \chi^* \leq -\chi_{[\bar{u} < 0]} \) and \( \chi_* \geq \chi_{[\bar{u} > 0]} \) in \( \mathbb{R}^d \times [0, \infty) \).

**Proof** Due to the positivity and continuity of \( \bar{m} \), the existence and uniqueness of \( \bar{u} \) is standard (see [8]). Since \( u_0 \) is uniformly continuous and the equation is translationally invariant, a well-known (e.g., approximation) argument shows that \( \bar{u} \) is uniformly continuous in both variables. Thus, [9, Proposition 2.1] applies, giving the desired conclusion. \( \square \)

Finally, notice that Theorem 3 implies Theorem 1 by definition of \( \chi^* \) and \( \chi_* \). Therefore, it only remains to prove Proposition 1 and Theorem 2.

### 3 Proof of Theorem 2

#### 3.1 Sketch of proof

In this section, we prove Theorem 2. Here it is important to recall a very convenient (equivalent) notion of viscosity solution of (24) that goes back to Barles and Georgelin [6].

**Definition 2** Given an open set \( U \subseteq \mathbb{R}^d \times (0, \infty) \), a locally bounded, upper semi-continuous function \( v : U \to \mathbb{R} \) is a viscosity subsolution of (24) in \( U \) if, given any smooth function \( \varphi : \mathbb{R}^d \times (0, \infty) \to \mathbb{R} \) and any point \((x_0, t_0) \in U\) at which the difference \( v - \varphi \) has a strict local maximum, the following conditions are met:

(a) If \( D\varphi(x_0, t_0) \neq 0 \), then
\[
\bar{m}(D\varphi(x_0, t_0))\varphi_t(x_0, t_0) - \text{tr}(\langle (1 - D\varphi(x_0, t_0) \otimes D\varphi(x_0, t_0)) D^2\varphi(x_0, t_0) \rangle) \leq 0.
\]

(b) If \( \|D\varphi(x_0, t_0)\| = \|D^2\varphi(x_0, t_0)\| = 0 \), then
\[
\varphi_t(x_0, t_0) \leq 0.
\]

Similarly, a locally bounded, lower semi-continuous function \( w : U \to \mathbb{R} \) is a viscosity supersolution of (24) in \( U \) if \(-w\) is a viscosity subsolution. A locally bounded, continuous function is a viscosity solution of (24) in \( U \) if it is both a subsolution and a supersolution.
With Definition 2 in mind, formally, the reason Theorem 2 is true is if the solution in question were smooth, then it would be clear. Indeed, we only need to check points where $Du(x_0, t_0) \neq 0$ since otherwise Definition 2 (b) and Definition 1, (iii) are in agreement. If $Du(x_0, t_0) \in \mathbb{R}^d$, then either $Du(x, t) \not\in \mathbb{R}^d$ for some $(x, t)$ arbitrarily close to $(x_0, t_0)$, in which case the necessary differential inequality follows by continuity, or $Du(x, t) = Du(x_0, t_0)$ in a neighborhood of $(x_0, t_0)$. In the latter case, differentiation shows

$$(\text{Id} - \widehat{Du}(x_0, t_0) \otimes \widehat{Du}(x_0, t_0)) D^2u(x_0, t_0) = 0$$

and now we are in the purview of Definition 1, (ii).

The sub- and supersolutions we work with in the proof are discontinuous, being indicator functions of open sets, and, thus, far from smooth. To circumvent this, we show that the sketch above is correct when $u$ is semi-convex or semi-concave and then use sup- and inf-convolutions to pass to the general case.

### 3.2 Proof of Theorem 2

In order to make the previous sketch rigorous in the case of a semi-convex/semi-concave function, we will invoke properties of the derivatives of such functions.

First, we recall that a semi-convex/semi-concave function has a derivative in $BV_{\text{loc}}$.

**Lemma 1** If $\Omega \subseteq \mathbb{R}^d$ is a bounded open set and $u : \Omega \rightarrow \mathbb{R}$ is semi-convex or semi-concave, then $Du \in BV_{\text{loc}}(\Omega; \mathbb{R}^d)$ and the absolutely continuous part of the derivative of $Du$ coincides with $D^2 u \mathcal{L}^d$-a.e. in $\Omega$.

Next, we show that the differentiation step in the sketch can be made rigorous even in the semi-convex/semi-concave case. In the lemma below, we have in mind that $V$ is the derivative of a semi-convex/semi-concave function.

**Proposition 2** Suppose $\Omega \subseteq \mathbb{R}^d$ is a bounded open set and $V \in BV_{\text{loc}}(\Omega; \mathbb{R}^m)$ for some $m \in \mathbb{N}$. Let $D^{ac} V \in L^1_{\text{loc}}(\Omega; \mathbb{R}^{d \times m})$ denote the Radon-Nikodym derivative of the Radon measure $DV$ with respect to $\mathcal{L}^d$. Given any $v \in \mathbb{R}^d$ and $e \in S^{m-1}$,

$$D^{ac} V = 0 \quad \mathcal{L}^d\text{-a.e. in } \{V = v\}$$

and

$$(\text{Id} - \widehat{V} \otimes \widehat{V}) D^{ac} V = 0 \quad \mathcal{L}^d\text{-a.e. in } \{\widehat{V} = e\}.$$  

**Proof** Let $\mathcal{D}_V$ denote the set of approximate differentiability points of $V$, that is, $x \in \mathcal{D}_V$ if and only if there is a linear map $A_x : \mathbb{R}^d \rightarrow \mathbb{R}^m$ such that

$$\lim_{r \rightarrow 0^+} r^{-d} \int_{B(x, r)} \frac{\|V(y) - V(x) - A_x(y - x)\|}{r} \, dy = 0.$$  

Since $V \in BV_{\text{loc}}(\Omega; \mathbb{R}^m)$, it follows that $\mathcal{L}^d(\Omega \setminus \mathcal{D}_V) = 0$ and $A_x = D^{ac} V(x)$ for a.e. $x \in \Omega$ (see [3, Theorem 3.83]).

A straightforward computation shows $D^{ac} V = 0$ a.e. in $\{V = v\}$ (see also [3, Proposition 3.73]).

Define $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ by $f(p) = \widehat{p}$ if $p \neq 0$ and $f(0) = 0$. It is not hard to see that each $x \in \mathcal{D}_V$ with $A_x \neq 0$ is an approximate differentiability point of $f(V)$. Furthermore, a straightforward computation shows its approximate derivative is given by
\[ \|V\|^{-1} (\text{Id} - \tilde{V} \otimes \tilde{V}) D^{\text{ac}} V \text{ a.e.} \] (Both statements can be found in [3, Proposition 3.71]). As in the case of \( \{V = v\} \), it is not hard to see that the approximate derivative of \( f(V) \) has to vanish a.e. in \( \{f(V) = e\} = \{\tilde{V} = e\} \). \( \square \)

Now we are prepared for the

**Proof of Theorem 2** The theorem has two halves, one for subsolutions and the other, for supersolutions. We will only prove the half concerning supersolutions. Since the transformation \( u \mapsto -u \) takes a subsolution into a supersolution, the subsolution half follows directly.

Let \( V \) be a bounded, open subset of \( U \) such that \( \tilde{V} \) is compactly contained in \( U \). For convenience, pick a \( T > 0 \) such that \( V \subseteq \mathbb{R}^d \times (0, T) \).

For each \( \eta > 0 \), define the inf-convolution \( u_\eta \) of \( u \) by

\[
 u_\eta(x,t) = \inf \left\{ u(y,s) + \frac{\|y-x\|^2}{2\eta} + \frac{(t-s)^2}{2\eta} \mid (y,s) \in U \cap (\mathbb{R}^d \times (0,T)) \right\}.
\]

A classical argument shows that \( u_\eta \) is a semi-concave function in \( U \cap (\mathbb{R}^d \times (0,T)) \).

Another well-known argument shows that there is an \( \eta_0 > 0 \) such that if \( 0 < \eta < \eta_0 \), then, for each \( (x,t) \in V \), there is a \( (y,s) \in U \cap (\mathbb{R}^d \times (0,T)) \) such that

\[
 u_\eta(x,t) = u(y,s) + \frac{\|y-x\|^2}{2\eta} + \frac{(t-s)^2}{2\eta}.
\]

From this, we can argue as in [16] to show that \( u_\eta \) is a subsolution of (24) in irrational directions in \( V \). In fact, we claim that, for each \( \eta \in (0, \eta_0) \),

\[
 u_\eta \text{ is a viscosity subsolution of (24) in } V. \quad (25)
\]

Here we remind the reader that we will work with the (equivalent) definition of viscosity subsolution provided by Definition 2.

Henceforth, fix \( \eta \in (0, \eta_0) \) and let us proceed to the proof of (25). Assume that \( \varphi \) is smooth and \( u_\eta - \varphi \) has a strict local minimum at \( (x_0, t_0) \in V \). If \( \|D\varphi(x_0, t_0)\| = \|D^2\varphi(x_0, t_0)\| = 0 \) or \( \overline{D\varphi(x_0)} \notin \mathbb{R}^d \), then there is nothing to show since \( u_\eta \) is a supersolution in irrational directions in \( V \). Thus, it remains to consider the case when \( D\varphi(x_0) \neq 0 \) and \( \overline{D\varphi(x_0)} \in \mathbb{R}^d \).

In what follows (subtracting a constant from \( \varphi \) if necessary), let \( s > 0 \) be such that \( B((x_0, t_0), s) \subseteq V, u_\eta(x,t) > \varphi(x,t) \) for \( x \in B((x_0, t_0), s) \setminus \{(x_0, t_0)\} \), and \( u_\eta(x_0, t_0) = \varphi(x_0, t_0) \).

Since \( u_\eta \) is semi-concave and \( \varphi \) is smooth, it follows that \( u_\eta - \varphi \) is semi-concave in \( B((x_0, t_0), s) \). Thus, for each \( \delta > 0 \) small enough, we can apply Jensen’s Lemma [16, Lemma A.3], thereby obtaining a set \( K_\delta \subseteq B((x_0, t_0), s) \) such that \( \mathcal{L}^{d+1}(K_\delta) > 0 \) and, for each \( (x,t) \in K_\delta \),

(i) \( u_\eta \) is twice punctually differentiable at \( (x,t) \).

(ii) There is an \( (a_{(x,t)}, p_{(x,t)}) \in B(0, \delta) \) such that the function \( (y,r) \mapsto u_\eta(y,r) - \varphi(y,r) - \langle p_{(x,t)}, y \rangle - a_{(x,t)}r \) has a local minimum in \( B((x_0, t_0), s) \).

We claim that we can find an \( (x_1, t_1) \in K_\delta \) such that

\[
- \delta \Theta \leq \overline{m} \left( \frac{D\varphi(x_1, t_1) + p(x_1, t_1)}{\|D\varphi(x_1, t_1) + p(x_1, t_1)\|} \psi_1(x_1, t_1) \right)
- \text{tr} \left( \left( \text{Id} - \frac{[D\varphi(x_1, t_1) + p(x_1, t_1)] \otimes [D\varphi(x_1, t_1) + p(x_1, t_1)]}{\|D\varphi(x_1, t_1) + p(x_1, t_1)\|^2} \right) D^2\varphi(x_1, t_1) \right). \quad (26)
\]
Making $\delta$ and $s$ smaller if necessary, we can assume that
\[
Du_\eta(x_1, t_1) = D\varphi(x_1, t_1) + p_{(x_1, t_1)} \neq 0 \quad \text{for all } (x_1, t_1) \in K_\delta.
\]

We will prove (26) by studying the structure of the spatial derivatives of $u_\eta$ in $K_\delta$. We only need to consider the following two cases:

(a) For a.e. $(x, t) \in K_\delta$, $Du_\eta(x, t) \in \mathbb{R}^d \setminus \{0\}$.
(b) There is a measurable $A_\delta \subseteq K_\delta$ such that $Du_\eta(x, t) \in \mathbb{R}^d \setminus \mathbb{R}^d$ for a.e. $(x, t) \in A_\delta$ and $\mathcal{L}^{d+1}(A_\delta) > 0$.

The easier case is (b). If (b) holds, then we can fix an $(x_1, t_1) \in A_\delta$ and invoke the supersolution property of $u_\eta$ at $(x_1, t_1)$ to find
\[
0 \leq \overline{m}(Du_\eta(x_1, t_1))u_{\eta,t}(x_1, t_1) - \text{tr}\left(\left(\text{Id} - \widehat{Du_\eta}(x_1, t_1) \otimes \widehat{Du_\eta}(x_1, t_1)\right)D^2u_\eta(x_1, t_1)\right).
\]

Now we recall that, by (ii), the following relations hold:
\[
Du_\eta(x_1, t_1) = D\varphi(x_1, t_1) + p_{(x_1, t_1)}, \quad D^2u_\eta(x_1, t_1) \geq D^2\varphi(x_1, t_1),
\]
\[
u_{\eta,t}(x_1, t_1) = \varphi_t(x_1, t_1) + a_{(x_1, t_1)}.
\]

By ellipticity and (11), this gives (26).

Next, we turn to case (a). Given $t \in (0, T)$, let $U_t = \{x \in \mathbb{R}^d \mid (x, t) \in U\}$. Recall from Proposition 2 that the map $Du_\eta(\cdot, t) \in BV_{\text{loc}}(U_t; \mathbb{R}^d)$ for each fixed $t$ and $D^{ac}(Du_\eta(\cdot, t)) = D^2u_\eta(\cdot, t)$ a.e. Let us define $\{B_e\}_{e \in S^{d-1} \cap \mathbb{R}^d}$ by
\[
B_e = \left\{ (x, t) \in K_\delta \mid \frac{Du_\eta(x, t)}{\|Du_\eta(x, t)\|} = e \right\}.
\]

Since we assumed (a) holds, it follows that $\sum_e \mathcal{L}^{d+1}(B_e) > 0$.

An immediate application of Lemma 1, Proposition 2, and Fubini’s Theorem shows that
\[
\left(\text{Id} - \widehat{Du_\eta} \otimes \widehat{Du_\eta}\right)D^2u_\eta = 0 \quad \text{a.e. in } \bigcup_{e \in S^{d-1} \cap \mathbb{R}^d} B_e.
\]

Thus, we can fix a point $(x_1, t_1) \in \bigcup_e B_e$ such that
\[
\left(\text{Id} - \widehat{Du_\eta}(x_1, t_1) \otimes \widehat{Du_\eta}(x_1, t_1)\right)D^2u_\eta(x_1, t_1) = 0.
\]

Since $\bigcup_{e \in S^{d-1} \cap \mathbb{R}^d} B_e \subseteq K_\delta$ and $u_\eta$ is a supersolution of (18) in irrational directions in $V$, we have
\[
0 \leq \overline{m}\left(Du_\eta(x_1, t_1)\right)u_{\eta,t}(x_1, t_1)
\]
\[
\leq \overline{m}\left(\frac{D\varphi(x_1, t_1) + p_{(x_1, t_1)}}{\|D\varphi(x_1, t_1) + p_{(x_1, t_1)}\|}\right)\varphi_t(x_1, t_1) + \delta \Theta
\]
\[
- \text{tr}\left(\left(\text{Id} - \frac{[D\varphi(x_1, t_1) + p_{(x_1, t_1)}] \otimes [D\varphi(x_1, t_1) + p_{(x_1, t_1)}]}{\|D\varphi(x_1, t_1) + p_{(x_1, t_1)}\|^2}\right)D^2\varphi(x_1, t_1)\right).
\]

This is exactly (26).

We conclude that, no matter which of cases (a) or (b) occur, there is an $(x_1, t_1) \in K_\delta$ such that (26) holds. Next, by recalling that $(x_0, t_0)$ is a strict local minimum of $u_\eta - \varphi$ in $B((x_0, t_0), s)$ and $K_\delta \subseteq B((x_0, t_0), s)$, a straightforward argument shows that
(x_1, t_1) \to (x_0, t_0)$ as $\delta \to 0^+$. Thus, sending $\delta \to 0^+$ in (26) and recalling that $\overline{m}$ is continuous, we obtain the desired inequality:

$$0 \leq \overline{m}(D\varphi(x_0, t_0))\varphi_t(x_0, t_0) - \text{tr} \left( (\text{Id} - \overline{D\varphi}(x_0, t_0) \otimes \overline{D\varphi}(x_0, t_0)) D^2\varphi(x_0, t_0) \right).$$

Since $\varphi$ was arbitrary, we proved that (25) holds as long as $\eta \in (0, \eta_0)$. At the same time, we know that, for each $(x, t) \in V$,

$$u(x, t) = \lim_{\delta \to 0^+} \inf_{\varphi} \{ u_\eta(y, s) \mid \|x - y\| + |t - s| + \eta \leq \delta \}.$$

Thus, the stability properties of viscosity solutions (see [16, Section 6]) imply that $u$ is also a supersolution of (24) in $V$. Since $V$ was arbitrary, we conclude that $u$ is a viscosity supersolution of (24) in $U$. \hfill \Box

### 4 Approximate correctors

The purpose of this section is to prove the existence of approximate correctors, that is, solutions of the differential inequality (22). When $e \in S^{d-1} \setminus \mathbb{R}Z^d$, this is possible because the diffusion in the $\langle e \rangle^\perp$ directions explores the entire torus. We will show below that the same strategy does not work in rational directions precisely because this is no longer the case. In Remark 3, we show that when $m$ is sufficiently regular, (21) has solutions in certain Diophantine directions; Remark 2 shows that there is an obstruction when $e \in \mathbb{R}Z^d$.

Recall that the effective mobility $\overline{m}(e)$ is given by

$$\overline{m}(e) = c_W^{-1} \int_{\mathbb{R} \times \mathbb{T}^d} m(y, \dot{q}(s)e)\dot{q}(s)^2 \, ds, \quad c_W = \int_{-\infty}^{\infty} \dot{q}(s)^2 \, ds. \quad (27)$$

The main result of this section concerning existence of approximate correctors is stated next.

**Theorem 4** Fix $e \in S^{d-1} \setminus \mathbb{R}Z^d$. If $m$ and $W$ satisfy the assumptions of Theorem 1 and $\nu > 0$, then there is a $\overline{P}_e^\nu \in C^{2,\mu}(\mathbb{R} \times \mathbb{T}^d)$ such that (22) holds.

To prove the theorem, we start by regularizing $m$: given $\mu \in (0, 1)$, fix $\tilde{m} \in C^\mu(\mathbb{R} \times \mathbb{T}^d)$ such that

$$\sup \left\{ |m(y, \dot{q}(s)e) - \tilde{m}(s, y)| \mid (s, y) \in \mathbb{R} \times \mathbb{T}^d \right\} \leq \frac{1}{3} \nu, \quad (28)$$

$$\|\tilde{m}\|_{C^\mu(\mathbb{R} \times \mathbb{T}^d)} + \|D\tilde{m}\|_{C^\mu(\mathbb{R} \times \mathbb{T}^d)} + \|D^2\tilde{m}\|_{C^\mu(\mathbb{R} \times \mathbb{T}^d)} < \infty. \quad (29)$$

We decompose $\tilde{m}$ in the following way:

$$\tilde{m}(s, y) = \int_{\mathbb{T}^d} \tilde{m}(s, y') \, dy' + \left( \tilde{m}(s, y) - \int_{\mathbb{T}^d} \tilde{m}(s, y') \, dy' \right) =: \tilde{m}_1(s) + \tilde{m}_2(s, y). \quad (30)$$

Correspondingly, we define a corrector $\overline{P}_e$ and penalized correctors $(\overline{P}_2^\delta)_{\delta > 0}$ solving the following PDE:

$$\tilde{m}_1(s)\dot{q}(s) - \overline{P}_e + W''(q(s))\overline{P}_e = \overline{m}(e)\dot{q}(s) \quad \text{in } \mathbb{R}, \quad (31)$$

$$\tilde{m}_2(s, y)\dot{q}(s) + \delta \overline{P}_2^\delta + D^\mu\overline{P}_2^\delta + W''(q(s))\overline{P}_2^\delta = 0 \quad \text{in } \mathbb{R} \times \mathbb{T}^d, \quad (32)$$

$$\overline{m}(e) = c_W^{-1} \int_{\mathbb{R} \times \mathbb{T}^d} \tilde{m}(s, y)\dot{q}(s)^2 \, dy \, ds. \quad (33)$$
Here, as above, \( D_e = e \partial_s + D_y \) and \( D^*_e \) is its \( L^2 \) adjoint (see (23)).

The existence and regularity of \( \mathcal{P}_e \) and \( (P^\delta_e)_{\delta > 0} \) is discussed in Appendix C. Theorem 4 is proved as soon as we establish the following:

**Proposition 3** \( \|q^{-1}(\delta P^\delta_2)\|_{L^\infty(\mathbb{R} \times \mathbb{T}^d)} \to 0 \) as \( \delta \to 0^+ \). In particular, given \( v > 0 \), if \( \delta > 0 \) is small enough, then \( \tilde{P}^\delta_e = \mathcal{P}_e + P^\delta_e \) satisfies (22).

### 4.1 Convergence of \( \delta P^\delta_2 \)

Since \( (P^\delta_2)_{\delta > 0} \subseteq C^{2,\mu}(\mathbb{R} \times \mathbb{T}^d) \), straightforward manipulations show that the functions \( (V^\delta_\zeta)_{\zeta \in \mathbb{R}} \) obtained by the rule

\[
V^\delta_\zeta(x) = V^\delta_2((x, e) - \zeta, x), \quad V^\delta_2(s, y) = q(s)^{-1}P^\delta_2(s, y)
\]

are solutions of the following family of PDE:

\[
\tilde{m}_2((x, e) - \zeta, x) + \delta V^\delta_\zeta - \Delta V^\delta_\zeta - \frac{2q((x, e) - \zeta)}{q((x, e) - \zeta)}(e, D\nu^\delta_\zeta) = 0 \quad \text{in } \mathbb{R}^d.
\]

Thus, the asymptotic behavior of \( (\delta P^\delta_2)_{\delta > 0} \) is captured by that of \( (\delta V^\delta_\zeta)_{\delta > 0} \).

Notice that if we define \( (\tilde{V}^\delta_\zeta)_{\zeta \in \mathbb{R}} \) by

\[
\tilde{V}^\delta_\zeta(x) = V^\delta_\zeta(x + \zeta e), \quad (34)
\]

then the functions \( (\tilde{V}^\delta_\zeta)_{\zeta \in \mathbb{R}} \) satisfy the “centered” family of PDE:

\[
\tilde{m}_2((x, e), x + \zeta e) + \delta \tilde{V}^\delta_\zeta - \Delta \tilde{V}^\delta_\zeta - \frac{2q((x, e))}{q((x, e))}(e, D\tilde{V}^\delta_\zeta) = 0 \quad \text{in } \mathbb{R}^d. \quad (35)
\]

Therefore, letting \( \tilde{p} : \mathbb{R} \times \mathbb{R} \times (0, \infty) \to (0, \infty) \) and \( g : \langle e \rangle^\perp \times \langle e \rangle^\perp \times (0, \infty) \to (0, \infty) \) denote the fundamental solutions of the operators \( \partial_t^\perp \partial_s^\perp + W''(q(s)) \) and \( \partial_t^\perp - \Delta_{\langle e \rangle^\perp} \), respectively, we find

\[
\tilde{V}^\delta_\zeta(x) = -\int_0^\infty e^{-\delta t}U_\zeta(x, t) \, dt,
\]

\[
U_\zeta(x, t) := \int_{-\infty}^\infty Q^\delta_\zeta(x + (\tilde{s} - (x, e))e, t)\tilde{p}((x, e), \tilde{s}, t)d\tilde{s},
\]

\[
Q^\delta_\zeta(x, t) := \int_{\langle e \rangle^\perp} \tilde{m}_2(\tilde{s}, x + y + \zeta e)g(0, y, t)\mathcal{H}^{d-1}(dy). \quad (36)
\]

(Above \( \Delta^\perp_{\langle e \rangle} \) is the Laplacian in the \( \langle e \rangle^\perp \) directions, that is, \( \Delta_{\langle e \rangle^\perp} = \text{tr}((\text{Id} - e \otimes e)D^2) \)).

Thus, \( g(0, y, t) = (4\pi t)^{-\frac{d+1}{2}} \exp\left(-\frac{1}{4t}\|y\|^2\right) \).

We will prove that \( \delta V^\delta_\zeta \to 0 \) uniformly using the averaging induced by the diffusion in \( \langle e \rangle^\perp \). Toward that end, the following observation will play a decisive role.

**Lemma 2** If \( \mathcal{F} \) is a compact subset of \( C(\mathbb{T}^d) \) in the uniform norm topology, then there is a modulus \( \eta : [0, \infty) \to [0, \infty) \) with \( \lim_{\delta \to 0^+} \eta(\delta) = 0 \) such that

\[
\sup\left\{ \left| \int_{\langle e \rangle^\perp} u(x + y)g(0, y, t) \, dt - \int_{\mathbb{T}^d} u(y) \, dy \right| \mid x \in \mathbb{T}^d, u \in \mathcal{F} \right\} \leq \eta(t^{-1}).
\]
then an elementary computation shows that
\[ q \frac{\partial q}{\partial t} + \mathcal{H}^d - \sum_{k \in \mathbb{Z}^d} |\hat{u}(k)| < \infty. \]
If we define \( Q : \mathbb{T}^d \times (0, \infty) \to \mathbb{R} \) by
\[ Q(x, t) = \int_{(e)^\perp} u(x + y) g(0, y, t) \mathcal{H}^{d-1}(dy), \]
then an elementary computation shows that
\[ \hat{Q}(k, t) = \hat{u}(k) e^{-2\pi^2 (k-e)e^2 t}. \]
Thus, the fact that \( k \neq (k, e)e \) for all \( k \in \mathbb{Z}^d \) readily implies
\[ \lim_{t \to \infty} \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |\hat{Q}(k, t)| = 0. \]
In particular,
\[ \limsup_{t \to \infty} \left\{ \left| \int_{(e)^\perp} u(x + y) g(0, y, t) \mathcal{H}^{d-1}(dy) - \int_{\mathbb{T}^d} u(y) dy \right| \mid x \in \mathbb{T}^d \right\} = 0. \]
Finally, the general case follows by approximation.

With Lemma 2 in hand, Proposition 3 follows readily:

**Proof of Proposition 3** Recall that \( V^\delta_2 = \hat{q}^{-1} P^\delta_2 \). Hence our previous computations yield
\[ \|q^{-1}(\delta P^\delta_2)\|_{L^\infty(\mathbb{R} \times \mathbb{T}^d)} = \sup \left\{ |\delta \hat{v}^\delta(x)| \mid x \in \mathbb{R}^d, \ z \in \mathbb{R} \right\} \leq \int_0^\infty e^{-\delta t} \|Q^\delta_z(z, t)\|_{L^\infty(\mathbb{T}^d)} dt. \]
Observe that \( \{m_2(\tilde{z}, \cdot + \zeta e) \mid (\tilde{z}, \zeta) \in \mathbb{R}^2 \} \) is relatively compact in \( C(\mathbb{T}^d) \) by the choice of \( m_2 \). Thus, by Lemma 2, there is a modulus \( \eta : [0, \infty) \to [0, \infty) \) with \( \lim_{\tilde{z} \to 0^+} \eta(\delta) = 0 \) such that
\[ \sup \left\{ \|Q^\delta_z(z, t)\|_{L^\infty(\mathbb{T}^d)} \mid (\tilde{z}, \zeta) \in \mathbb{R}^2 \right\} \leq \eta(t^{-1}). \]
Putting it all together, we conclude by observing that
\[ \limsup_{\delta \to 0^+} \|q^{-1}(\delta P^\delta_2)\|_{L^\infty(\mathbb{R} \times \mathbb{T}^d)} \leq \lim_{\delta \to 0^+} \int_0^\infty e^{-\delta t} \eta(t^{-1}) dt = 0. \]

Now that Theorem 4 is proved, a few remarks are in order.

**Remark 1** The diffusion in \( (e)^\perp \) directions is needed in Proposition 3. More precisely, the same is no longer true if the forcing only depended on \( s \).

Here is an example. Set \( W(u) = (1 - u^2)^2 \) and consider the penalized correctors solving the following:
\[ q(s)\hat{q}(s) + \delta P^\delta - \tilde{P}^\delta + W''(q(s)) P^\delta = 0 \text{ in } \mathbb{R}. \]
In this case, \( q \) is an even function so \( q \hat{q} \) is orthogonal to \( \hat{q} \). Thus, one can show that \( \delta P^\delta \to 0 \)
uniformly as \( \delta \to 0^+ \). However, this is no longer true when we renormalize by \( \hat{q} \).
In this case, \( v^\delta = \hat{q}^{-1} P^\delta \) solves the PDE:
\[
q(s) + \delta v^\delta - \ddot{v}^\delta + 2 \tanh(s) \dot{v}^\delta = 0 \quad \text{in } \mathbb{R}.
\]
For a given \( \delta > 0 \), as \( \bar{s} \to \infty \), we find that \( v^\delta(\cdot + \bar{s}) \) converges to the bounded solution \( v_+^\delta \) of
\[
1 + \delta v_+^\delta - \ddot{v}_+^\delta + 2 \dot{v}_+^\delta = 0 \quad \text{in } \mathbb{R}.
\]
Since the coefficients are constant, this gives \(-\delta v_+^\delta = 1\). In particular,
\[
\|\hat{q}^{-1}(\delta P^\delta)\|_{L^\infty(\mathbb{R})} = \sup \{ \delta |v^\delta(s)| \mid s \in \mathbb{R} \} \geq 1 \quad \text{for all } \delta > 0.
\]

It is worth noting that, in the previous example, any bounded solution \( P \) of \( q(s) \hat{q}(s) - \dot{P} + W''(q(s)) P = 0 \) in \( \mathbb{R} \) necessarily grows much faster than \( \hat{q} \) as \( s \to \pm \infty \). That is, in this case, the function \( \hat{q}^{-1} P \) is an unbounded solution of the associated PDE.

Not only was the diffusion in orthogonal directions necessary in the proof of Theorem 4, but irrationality of \( e \) was also.

**Remark 2** If \( e \in \mathbb{R} \mathbb{Z}^d \), then \( (\hat{q}^{-1} \delta P_2)_{\delta>0} \) converges uniformly to a non-constant function in general. To see this, notice that, if \( e \in \mathbb{R} \mathbb{Z}^d \), then, in Lemma 2, the conclusion changes to the following one:
\[
\lim_{t \to \infty} \int_{(e)^\perp} u(x + y, t) g(0, y, t) \mathcal{H}^{d-1}(dy) = \int_{T^d(e)^\perp} u(\xi) \mathcal{H}^{d-1}(d\xi).
\]
Here the sub-tori \( (T^d(e)^\perp)_{r \in (0, r)} \) are defined by
\[
T^d(e)^\perp(r) = \{ y \in T^d \mid \langle y, e \rangle = r + \langle k, e \rangle \text{ for some } k \in \mathbb{Z}^d \},
\]
\[
r_e = \min \{ \langle k, e \rangle \mid k \in \mathbb{Z}^d \} \cap (0, \infty).
\]
Geometrically, this becomes transparent, for example, when \( e \) is a coordinate vector.

As a consequence of the previous observation, we see that if \( (P^\delta)_{\delta>0} \) are the bounded solutions of \( \hat{m}(s, y) + \delta P^\delta + D_\xi D_e P^\delta + W''(q(s)) P^\delta = 0 \) in \( \mathbb{R} \times T^d \) and \( e \in \mathbb{R} \mathbb{Z}^d \), then
\[
\lim_{\delta \to 0^\downarrow} \hat{q}^{-1}(\delta P^\delta(s, y)) = \tilde{m}_e(\langle y, e \rangle - s) \quad \text{uniformly in } \mathbb{R} \times T^d,
\]
where \( \tilde{m}_e : [0, r_e) \to \mathbb{R} \) is given by
\[
\tilde{m}_e(\xi) = c_w^{-1} \int_{-\infty}^{\infty} \int_{T^d(e)^\perp(\xi)} \hat{m}(s, \xi) \hat{q}(s)^2 \mathcal{H}^{d-1}(d\xi) \, ds.
\]
While \( \tilde{m}_e \) certainly extends to a periodic function in \( \mathbb{R} \), it need not be constant.

Similarly, (21) cannot have a (e.g., weak) solution unless \( \tilde{m}_e \) is constant, and one can show that (22) cannot hold unless \( 2v \) is larger than the oscillation of \( \tilde{m}_e \).

Finally, we describe a situation where (21) does have solutions in certain directions.

**Remark 3** If we impose enough regularity assumptions on \( m \) and arithmetic conditions on \( e \), and if \( \{ Q^\xi \} \) are defined as above, then it is possible to show the following estimate
\[
\sup \left\{ \int_0^\infty \| Q^\xi(\cdot, t) \|_{L^\infty(T^d)} \, dt \mid (\bar{s}, \xi) \in \mathbb{R} \right\} < \infty.
\]
This can be made precise by following [33, Proposition 22]. With this estimate, we use (36) to see that \( \|v^d_\cdot\|_{L^\infty(\mathbb{R}^d)} \) is bounded independently of \((\delta, \zeta)\). Therefore, we can send \( \delta \to 0^+ \) to obtain a solution of (21).

### 5 Irrational contact points

The remainder of the paper is devoted to the proof of Proposition 1. This section establishes that the phase indicator function \( \chi_* \) satisfies condition (a) in the definition of a supersolution in irrational directions (see Definition 1).

Put another way, the goal of this section is to prove the following:

**Proposition 4** If \( \varphi \) is a smooth function in \( \mathbb{R}^d \times (0, \infty) \); \( (x_0, t_0) \in \mathbb{R}^d \times (0, \infty) \) is a point where \( \chi_* - \varphi \) has a strict local minimum; and \( D\varphi(x_0, t_0) \in \mathbb{R}^d \setminus \mathbb{R}^d \), then

\[
\overline{m}(D\varphi(x_0, t_0))\varphi_t(x_0, t_0) - tr \left( \left( Id - D\varphi(x_0, t_0) \otimes D\varphi(x_0, t_0) \right) D^2\varphi(x_0, t_0) \right) \geq 0. \tag{37}
\]

The proof of Proposition 4 proceeds by contradiction and is divided into three steps. The first step involves the construction of a suitable local subsolution of (18). The second step, the so-called initialization step, shows that the solutions \( (u^\varepsilon)_{\varepsilon > 0} \) develop a relatively sharp interface around the level surface \( \{\varphi = \varphi(x_0, t_0)\} \) after a short macroscopic time. As in [9], this initial step allows us to convert the macroscopic subsolution of the first step into a subsolution of (1). This conversion is precisely the third step. If \( \varphi \) does not satisfy (37), these subsolutions slip underneath the solutions \( (u^\varepsilon)_{\varepsilon > 0} \) and force \( (x_0, t_0) \) to be an interior point of the evolution \( t \mapsto \Omega_i^{(1)} \), a contradiction.

#### 5.1 Macroscopic subsolution

Here we recall some useful observations that follow from the assumption that \( D\varphi(x_0, t_0) \neq 0 \). It will be useful to introduce some notation.

Throughout the section, we let \( e = \overline{D}\varphi(x_0, t_0) \). Let \( \{e_1, \ldots, e_{d-1}\} \) be an orthonormal basis for \( \mathbb{R}^{d-1} \) and \( O_e : \mathbb{R}^{d-1} \to \mathbb{R}^d \) be a linear isometry with \( O_e(\mathbb{R}^{d-1}) = \langle e \rangle^\perp \). Given \( R > 0 \), we define the open cube \( Q(0, R) \subseteq \mathbb{R}^{d-1} \) by

\[
Q(0, R) = \{x' \in \mathbb{R}^{d-1} \mid \max\{|\langle x', e_1 \rangle|, \ldots, |\langle x', e_{d-1} \rangle|\} < R/2\}.
\]

For an \( \tilde{x}' \in \mathbb{R}^{d-1} \), we set \( Q(\tilde{x}', R) = \tilde{x}' + Q(0, R) \).

Using the coordinates determined by \( O_e \), we define open rectangular prisms \( Q^e(0, R, \rho) \) of base length \( R > 0 \) and height ratio \( \rho \) by

\[
Q^e(0, R, \rho) = \left\{ O_e(x') + se \mid x' \in Q(0, R), \ s \in (-\rho R/2, \rho R/2) \right\}.
\]

Given \( x \in \mathbb{R}^d \), we define \( Q^e(x, R, \rho) = x + Q^e(0, R, \rho) \).

Finally, an arbitrary point \( x \in \mathbb{R}^d \) will frequently be written as \( x = (x_e, x') \) with the understanding that \( x_e \in \mathbb{R} \) and \( x' \in \mathbb{R}^{d-1} \) are such that \( x = O_e(x') + x_e e \). In particular, we will write \( x_0 = (x_0e, x_0') \).

We will use the fact that \( \{\varphi = 0\} \) is locally a one-parameter family of graphs near \( (x_0, t_0) \). Specifically, we have

**Proposition 5** There are constants \( \rho, \nu, S, V > 0 \) and a smooth function \( g : Q(x_0', S) \times (t_0 - \nu, t_0 + \nu) \to \mathbb{R} \) such that
(i) \( \varphi(x, t) > 0 \) (resp. \( \varphi(x, t) \leq 0 \)) for some \( (x, t) \in Q^e(x_0, S, \rho) \times (t_0 - v, t_0 + v) \) if and only if \( x_e > g(x', t) \) (resp. \( x_e \leq g(x', t) \)).

(ii) \( |g(x'_1, t) - g(x'_2, t)| \leq \frac{1}{2} \rho(d - 1) \frac{1}{2} \| x'_1 - x'_2 \| \) no matter the choice of \( x'_1, x'_2 \in Q(x'_0, S) \) or \( t \in (t_0 - v, t_0 + v) \).

(iii) \( |g(x', t) - g(x', s)| \leq V|t - s| \) for all \( x' \in Q(x'_0, S) \) and \( t, s \in (t_0 - v, t_0 + v) \).

Further, we can assume that \( \rho < 1 \) and \( (x_0, t_0) \) is a strict local minimum of \( \chi_* - \varphi \) in \( Q^e(x_0, S, \rho) \times (t_0 - v, t_0 + v) \).

**Proof** The construction of \( g \) is a classical application of the implicit function theorem. The fact that \( D\varphi(x_0, t_0) = \|D\varphi(x_0, t_0)\| \epsilon \) implies that \( Dg(x_0, t_0) = 0 \), and hence the existence of \( \rho < 1 \). \( \square \)

By assumption, there is an \( \alpha > 0 \) such that
\[
\overline{m}(D\varphi(x_0, t_0)) \varphi_t(x_0, t_0) - \text{tr}\left( (\text{Id} - D\varphi(x_0, t_0) \otimes D\varphi(x_0, t_0)) D^2 \varphi(x_0, t_0) \right) 
\leq -10\alpha \| D\varphi(x_0, t_0) \|.
\]

In particular, since \( g \) is smooth, this implies there is an \( 0 < S_1 < S \) and a \( 0 < v_1 < v \) such that
\[
\overline{m}\left( \frac{e - Dg}{\sqrt{1 + \| Dg \|^2}} \right) g_t - \text{tr}\left( \left( \text{Id} - \frac{Dg \otimes Dg}{1 + \| Dg \|^2} \right) D^2 g \right)
\geq 9\alpha \quad \text{in} \quad Q(x'_0, S_1) \times (t_0 - v_1, t_0 + v_1). \tag{38}
\]

Next, given a free variable \( c > 0 \) to be determined, define \( \tilde{g} : Q(x'_0, S) \times (t_0 - v, t_0 + v) \to \mathbb{R} \) by
\[
\tilde{g}(x', t) = g(x', t) + \frac{c\| x' - x'_0 \|^2}{2}.
\]

Notice that there is a \( c_1 > 0 \) such that if \( c \in (0, c_1) \), then
\[
\overline{m}\left( \frac{e - D\tilde{g}}{\sqrt{1 + \| D\tilde{g} \|^2}} \right) \tilde{g}_t - \text{tr}\left( \left( \text{Id} - \frac{D\tilde{g} \otimes D\tilde{g}}{1 + \| D\tilde{g} \|^2} \right) D^2 \tilde{g} \right)
\geq 8\alpha \quad \text{in} \quad Q(x'_0, S_1) \times (t_0 - v_1, t_0 + v_1). \tag{39}
\]

Finally, let \( d : Q^e(x_0, S_1, \rho) \times (t_0 - v_1, t_0 + v_1) \to \mathbb{R} \) and \( d_e : Q^e(x_0, S_1, \rho) \times (t_0 - v_1, t_0 + v_1) \to \mathbb{R} \) be the signed distance functions to \( \{ x_e = g(x', t) \} \) and \( \{ x_e = \tilde{g}(x', t) \} \), respectively. Specifically, we define \( d \) by
\[
d(x, t) = \begin{cases} 
\text{dist}(x, \{ \tilde{x} \in Q^e(x_0, S_1, \rho) \mid \tilde{x}_e = g(\tilde{x}', t) \}), & \text{if } x_e > g(x', t), \\
-\text{dist}(x, \{ \tilde{x} \in Q^e(x_0, S_1, \rho) \mid \tilde{x}_e = g(\tilde{x}', t) \}), & \text{otherwise},
\end{cases}
\]

and we define \( d_e \), similarly, but with \( g \) replaced by \( \tilde{g} \).

Let \( a_0 > 0 \) be another free variable. Arguing as in [32, Appendix C], we deduce the following facts about \( d_e \):

**Lemma 3** Making \( v_1 \) smaller if necessary, there is a constant \( \gamma > 0 \) depending on \( c_1, \varphi, \) and \( S_1 \) but not on \( c \) such that
\[
M := \sup \left\{ \| \partial_t^i D^j d_e(x, t) \| + \| \partial_t^i D^j d(x, t) \| \mid x \in Q^e(x_0, S_1/2, \rho), \right. \\
\left. t \in (t_0 - v_1, t_0 + v_1), \ |d_e(x, t)| < \gamma, \ i + j \leq 4 \right\} < \infty.
\]

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Furthermore, we can choose $0 < S_1' < S_1$ and $0 < v_1' < v_1$ in such a way that (making $\gamma$ smaller, if necessary)

$$\bar{m}(Dd_c) \Delta d_c - \Delta d_c \leq -7\bar{\alpha}, \quad |\bar{m}(Dd_c) - \bar{m}(e)| \leq \alpha_0 \quad \text{in} \quad Q^e(x_0, S_1', \rho) \times (t_0 - v_1', t_0 - v_1') \cap |d| < \gamma.$$ 

Finally, we let $\eta, \beta > 0$ be free variables to be determined and pick $0 < S_2 < S_1'$ and $0 < v_2 < v_1'$ and define $\tilde{d}_c : Q^e(x_0, S_2, \rho) \times (t_0 - v_2, t_0 + v_2) \to \mathbb{R}$ by

$$\tilde{d}_c(x, t) = d_c(x + \eta(t - (t_0 - v_2))e, t).$$

Notice that $\tilde{d}_c(\cdot, t)$ is the signed distance to the surface $\{x = \tilde{g}(x', t) - \eta(t - (t_0 - v_2))\}$, and it is well-defined by making $S_2$ and $v_2$ smaller if necessary.

$\tilde{d}_c$ is the subsolution of (24) that will be used in the sequel. Its key properties are summarized next. In the statement, we use the notation $\partial_p$ for the parabolic boundary. Specifically, for a space-time domain $Q \times (a, b)$ that means

$$\partial_p[Q \times (a, b)] = \partial Q \times (a, b] \cup \bar{Q} \times \{a\}.$$

**Proposition 6** If $\eta \in (0, \Theta^{-1}\bar{\alpha})$, $\beta \in (0, \frac{1}{2}\eta v_2 \land 1 \land \frac{1}{2}S_2)$, $c \in (0, c_1)$, $v_2 \in (0, v_1')$, and $S_2 \in (0, S_1'/2)$ satisfy the inequalities

$$(3\eta + 2V)v_2 < \frac{\rho S_2}{4} + \frac{cS_2^2}{8},$$

then

$$\bar{m}(D\tilde{d}_c) \tilde{d}_c, - \Delta \tilde{d}_c \leq -6\bar{\alpha}, \quad |\bar{m}(D\tilde{d}_c) - \bar{m}(e)| \leq \alpha_0 \quad \text{in} \quad \{|\tilde{d}_c| < \gamma\},$$

$$\chi[\tilde{d}_c \geq \beta] \leq \chi[d \geq \beta] \quad \text{on} \quad \partial_p[Q^e(x_0, S_2, \rho) \times (t_0 - v_2, t_0 + v_2)].$$

**Proof** The inequality $2\eta v_2 < \frac{\rho S_2}{2} < \frac{\rho(S_1 - S_0)}{4}$ implies that $x + \eta(t - (t_0 - v_2))e \in Q^e(x_0, S_1', \rho)$ whenever $x \in Q^e(x_0, S_2, \rho)$. Thus, the first statement follows from Lemma 3 and the inequality $\bar{m} \leq \Theta$.

Concerning the second statement, we know that $\tilde{d}_c(\cdot, t_0 - v_2) = d_c(\cdot, t_0 - v_2)$. Therefore, the ordering between $g$ and $\tilde{g}$ implies $\tilde{d}_c \leq d$ on the surface $\{t = t_0 - v_2\}$.

Next, we check the remaining inequalities, namely,

$$\chi[\tilde{d}_c \geq 2\beta] \leq \chi[d \geq 2\beta] \quad \text{on} \quad \partial Q^e(x_0, S_2, \rho) \times [t_0 - v_2, t_0 + v_2].$$

We will start by examining points $x = (x_e, x')$ with $x' \in \partial Q(x_0, S_2)$.

Assume that $\tilde{d}_c(x, t) \geq \beta$ and $x' \in \partial Q(x_0, S_2)$. To show that $d(x, t) \geq \beta$, we proceed point by point. We claim that if $d(y, t) = 0$, then $\|y - x\| \geq \beta$. If $\|y' - x'\| \geq \beta$, we are done. Therefore, assume that $\|y' - x'\| \leq \beta$.

By the definition of $\tilde{d}_c$ and $d_c$, we have

$$g(x', t) + \frac{cS_2^2}{8} - 2\eta v_2 \leq g(x', t) + \frac{c\|x' - x_0\|^2}{2} - \eta(t - (t_0 - v_2)) < x_e. \quad (41)$$

At the same time, $|g(y', t) - g(x', t)| \leq \frac{1}{2}\rho \beta$. Therefore, since $y_e = g(y', t)$, we find

$$y_e \leq g(x', t) + \frac{\rho \beta}{2} < x_e - \frac{cS_2^2}{8} + 2\eta v_2 + \frac{\beta}{2}. \quad$$

(41)
Recalling that $2\beta < \eta \nu_2$ and appealing to (40), we conclude
\[
\|y - x\| \geq |y_e - x_e| = x_e - y_e \geq \frac{c S_e^2}{2} - \left(2\eta \nu_2 + \frac{\beta}{2}\right) \geq \beta.
\]
Hence $|d(x, t)| \geq \beta$. Similarly, (41) shows that $g(x', t) < x_e$ so $d(x, t) > 0$. Thus, $d(x, t) \geq \beta$ as claimed.

Finally, we consider points $(x, t)$ with $x$ on the top or bottom of the box $Q^c(x_0, S_2, \rho)$, that is, points for which $|x_e - x_e| = \rho S_2/2$. To start with, observe that if $x_e = x_0, e - \rho S_2/2$, then $\tilde{d}_e(x, t) < 0 < \beta$. Indeed, it suffices to show that $x_e + \eta (t - (t_0 - \nu_2)) < \tilde{g}(x', t)$, which is true since, by (40),
\[
x_e + \eta (t - (t_0 - \nu_2)) \leq g(x_0, t_0) - \frac{\rho S_2}{2} + 2\eta \nu_2
\]
\[
\leq \tilde{g}(x', t) + 2(V + \eta)\nu_2 - \frac{\rho S_2}{4} < \tilde{g}(x', t).
\]

It remains to consider the case when $x_e = x_0, e + \rho S_2/2$. We claim that $d(x, t) \geq \beta$ in this case. To see this, we first show that if $d(y, t) = 0$ and $y \in Q^c(x_0, S_2, \rho)$, then $\|y - x\| \geq \beta$. Indeed, in case $y \in Q^c(x_0, S_2 + 2\beta, \rho)$, we apply (40) to find
\[
\|y - x\| \geq |y_e - x_e| = |g(x_0, t_0) - g(y', t)|
\]
\[
\geq \frac{\rho S_2}{2} - |g(x_0, t_0) - g(y', t)|
\]
\[
\geq \frac{\rho S_2}{2} - \frac{\rho (S_2 + 2\beta)}{4} - 2V\nu_2 \geq \beta.
\]

On the other hand, if $y \in Q^c(x_0, S_1, \rho) \setminus Q^c(x_0, S_2 + 2\beta, \rho)$, then we can consider two cases: (i) $y' \in Q(x_0, S_1) \setminus Q(x_0, S_2 + 2\beta)$ or (ii) $\rho (S_2 + 2\beta)/2 \leq |y_e| < \rho S_1/2$. In case (i), $\|x - y\| \geq \|x' - y'\| \geq \beta$ follows immediately. On the other hand, in case (ii), we can assume that both $|y_e - x_0| \geq \rho (S_2 + 2\beta)/2$ and $y' \in Q(x_0, S_2 + 2\beta)$, but then this contradicts Proposition 5 and $2V\nu_2 < \rho S_2/2$. So only case (i) is possible and then $|d(x, t)| \geq \beta$ follows. Similarly, we find that $d(x, t) > 0$ so $d(x, t) \geq \beta$. \hfill \Box

In the remainder of this section, we will adjust the constants if necessary so that the hypotheses of Proposition 6 hold. In addition, we impose the following constraint on $\eta$:
\[
\eta \nu_2 < \gamma.
\]

The justification for this restriction comes in the remark that follows. Henceforth, $\eta$, $c$, $\nu_2$, and $S_2$ are fixed. We reserve the right to make $\beta > 0$ smaller later. Also note that $\alpha_0$ remains undetermined at this stage and so far no restrictions have been imposed on it.

**Remark 4** Notice that the boundary inequality in Proposition 6 has the following (trivial) consequence: for each $\epsilon > 0$,
\[
(1 - \beta \epsilon) \chi_{[d \leq \beta]} \leq (1 - \beta \epsilon) \chi_{[d \leq \beta]} - \chi_{[d < \beta]} \leq \chi_{[d \geq \beta]} - \chi_{[d < \beta]} \quad \text{on} \quad \partial P [Q(x_0, S_2, \rho) \times (t_0 - \nu_2, t_0 + \nu_2)].
\]

Further, notice that since $Dd_e(x_0, t_0) = e$ and $\eta \nu_2 < \gamma$, it follows that
\[
\tilde{d}_e(x_0, t_0) = d_e(x_0 + \eta \nu_2 e, t_0) = \eta \nu_2 > 2\beta.
\]

In particular, $\{\tilde{d}_e > 2\beta\}$ contains a neighborhood of $(x_0, t_0)$. \hfill \square
5.2 Initialization

In this section, we prove an initialization result that shows that the solutions \((u^\epsilon)_{\epsilon>0}\) develop a sharp transition along the interface \(\{\varphi = 0\}\) when \(\epsilon > 0\) is sufficiently small. Here we follow [9] using the result of Appendix A.

**Proposition 7** Given \(\delta \in (0, \frac{1}{2})\), there is a \(\tau = \tau(\delta, \beta, \varphi) > 0\) and an \(\epsilon_0 = \epsilon_0(\beta, \varphi, u_0) > 0\) such that if \(\epsilon \in (0, \epsilon_0)\) and \(t \in [t_0 - v_2, t_0 + v_2]\), then

\[
u^\epsilon(\cdot, t + \epsilon^2 \log(e^{-1})) \geq (1 - \beta \epsilon) \gamma(\cdot, t) \geq \gamma(\cdot, t)\quad \text{in } Q^\epsilon(x_0, S_2, \rho).
\]

**Proof** First of all, since \([d \geq \beta] \subseteq \{x \in \mathbb{R}^d|\varphi(x) = 1\}\), there is an \(\epsilon_0 > 0\) such that

\[
u^\epsilon(x) \leq \gamma(\cdot, t) \quad \text{for } \epsilon \in (0, \epsilon_0).
\] (42)

For the rest of the proof, fix \(t \in [t_0 - v_2, t_0 + v_2]\).

Let \(\psi: \mathbb{R}^d \to \mathbb{R}\) be the function given by

\[
\psi(x) = \begin{cases} 
1 - \delta, & \text{if } x \in Q^\epsilon(x_0, (S_1 + S_2)/2, \rho) \cap \{d(\cdot, t) \geq \beta/2\}, \\
-1, & \text{otherwise}.
\end{cases}
\]

Fix a smooth, symmetric non-negative function \(\rho\) such that \(\rho = 0\) in \(\mathbb{R}^d \setminus B(0, 1)\) and \(\int_{\mathbb{R}^d} \rho(x)\ dx = 1\). Let \(\rho_{\nu}(x)\) be the mollifying family in \(\mathbb{R}^d\) given by \(\rho_{\nu}(x) = v^{-d} \rho(v^{-1}x)\).

In view of Lemma 3, there is a \(\nu > 0\) small and independent of \(t\) with the following property:

\[
x \in [d \geq \beta] \cap Q^\epsilon(x_0, S_2, \rho) \implies B(x, \nu) \subseteq [d(\cdot, t) \geq \beta/2] \cap Q^\epsilon(x_0, (S_1 + S_2)/2, \rho),
\]

\[
x \in [d \leq \beta] \cap Q^\epsilon(x_0, S_2, \rho) \implies B(x, \nu) \subseteq [d(\cdot, t) \leq \beta/2] \cap Q^\epsilon(x_0, (S_1 + S_2)/2, \rho).
\]

Define \(\psi = \rho_{\nu} \ast \psi\). Recall that

\[
\|D\psi\|_{L^\infty(\mathbb{R}^d)} \leq C_{\rho} \nu^{-1},
\]

\[
\|D^2\psi\|_{L^\infty(\mathbb{R}^d)} \leq C_{\rho} \nu^{-2}.
\]

Notice that, by the choice of \(\nu\), if \(x \in Q^\epsilon(x_0, S_2, \rho)\) and \(d(x, t) \geq \beta\), then

\[
\psi(x) = \int_{B(x, \nu)} \psi(y) \rho_{\nu}(x - y)\ dy = 1 - \delta.
\]

Similarly, if \(x \in Q^\epsilon(x_0, S_2, \rho)\) and \(d(x, t) \leq 0\), then \(\psi(x) = -1\). In summary,

\[
[d \geq \beta] \cap Q^\epsilon(x_0, S_2, \rho) \subseteq \{\psi = 1 - \delta\}, \quad [d \leq \beta] \cap Q^\epsilon(x_0, S_2, \rho) \subseteq \{\psi = -1\}.
\]

Let \(\tilde{\chi}^\epsilon\) be the function from Appendix A, Lemma 12; let \(K > 0\) be a free variable; and define a family \((u^\epsilon)_{\epsilon>0}\) in \(\mathbb{R}^d \times [t, \infty)\) by

\[
u^\epsilon(x, t') = \tilde{\chi}^\epsilon(\psi(x) - \epsilon^{-1}K(t' - t), \epsilon^{-1}(t' - t)).
\]

The construction of Appendix A gives that \(\tilde{\chi}^\epsilon \leq -\tilde{h}(\chi^\epsilon)\), where \(\tilde{h}\) is defined in (51). Thus, by arguing as in [9, Lemma 4.1] and invoking (11) and (12), we find that \(u^\epsilon\) is a subsolution of (1) if \(K\) is large enough. (Notice that \(K\) depends only on \(\nu\) through (43) and (44) and, thus, is independent of \(t\)). Further, \(u^\epsilon(\cdot, 0) \leq \psi \leq u^\epsilon(\cdot, t)\) by (42). Therefore,

\[
u^\epsilon(x, t') \leq u^\epsilon(x, t' + t)\quad \text{if } (x, t') \in \mathbb{R}^d \times [0, \infty).
\]

Now the conclusion follows from the properties of \(\chi\) arguing exactly as in [9]. Note that \(\tau\) depends only on \(K\) and so is independent of \(t\).
5.3 Mesoscopic subsolutions

Finally, we use the macroscopic subsolution of Sect. 5.1, namely $\tilde{d}_c$, to build mesoscopic subsolutions of (1) that converge to 1 in the sets $\{\tilde{d}_c > 2\beta\}$. Appealing to Remark 4, we will then conclude that $u^\epsilon \to 1$ uniformly in a neighborhood of $(x_0, t_0)$, a patent contradiction.

Recall that $e = D\bar{\varphi}(x_0, t_0)$. Let $\alpha_1$ be one last free variable, for convenience. Invoking Theorem 4, we fix an approximate corrector $P_\epsilon = \bar{P}_\epsilon + P_2^\beta \in C^{2,\mu}(\mathbb{R} \times \mathbb{T}^d)$ such that (22) holds with $v = \alpha_1$.

Define a family $(v^\epsilon)_{\epsilon > 0}$ in $\{(x, t) \in Q^\epsilon(x_0, S_2, \rho) \times [t_0 - v_2, t_0 + v_2] \mid |\tilde{d}_c(x, t)| < \gamma\}$ by

$$v^\epsilon(x, t) = q\left(\frac{\tilde{d}_c(x, t) - 2\beta}{\epsilon}\right) + \epsilon \left(\frac{\tilde{d}_c(x, t) - 2\beta}{\epsilon}, \frac{x}{\epsilon}\right) - 2\beta).$$

We show below that, provided $\alpha_0, \alpha_1,$ and $\beta$ are chosen appropriately, $v^\epsilon$ is a subsolution of (1) as soon as $\epsilon > 0$ is small enough.

In order to invoke the comparison principle, we extend $(v^\epsilon)_{\epsilon > 0}$ to $Q^\epsilon(x_0, S_2, \rho) \times (t_0 - v_2, t_0 + v_2).$ The construction again follows [9]. First, we define $(\overline{v}^\epsilon)_{\epsilon > 0}$ by

$$\overline{v}^\epsilon(x, t) = \begin{cases} \max\{v^\epsilon(x, t), -1\}, & \text{if } \tilde{d}_c(x, t) \geq - (\gamma + 2\beta)/2, \\ -1, & \text{if } \tilde{d}_c(x, t) < - (\gamma + 2\beta)/2. \end{cases}$$

Finally, we fix a smooth, non-decreasing function $f : \mathbb{R} \to [0, 1]$ such that

$$f(\xi) = 1 \quad \text{if} \quad \xi \geq \frac{7\gamma}{8} + \frac{\beta}{4}, \quad f(\xi) = 0 \quad \text{if} \quad \xi \leq \frac{3\gamma}{4} + \frac{\beta}{2}$$

and define $(w^\epsilon)_{\epsilon > 0}$ by

$$w^\epsilon(x, t) = (1 - f(\tilde{d}_c(x, t)))\overline{v}^\epsilon(x, t) + f(\tilde{d}_c(x, t))(1 - \beta \epsilon).$$

Here is the main result we will need to proceed:

**Proposition 8** There is an $\epsilon_1 > 0$ and a choice of the parameters $\alpha_0$, $\alpha_1$, and $\beta$ such that $w^\epsilon$ satisfies

$$\begin{cases} m(\epsilon^{-1}x, D\bar{\varphi}\epsilon)w^\epsilon_t - \Delta w^\epsilon + \epsilon^{-2}W'(w^\epsilon) \leq 0 & \text{in } Q^\epsilon(x_0, S_2, \rho) \times (t_0 - v_2, t_0), \\ w^\epsilon \leq (1 - \beta \epsilon)\chi_{[\tilde{d}_c \geq \beta]} - \chi_{[\tilde{d}_c < \beta]} & \text{on } \partial_p[Q^\epsilon(x_0, S_2, \rho) \times (t_0 - v_2, t_0)]. \end{cases}$$

Furthermore, if $(x, t) \in Q^\epsilon(x_0, S_2, \rho) \times (t_0 - v_2, t_0)$ satisfies $\tilde{d}_c(x, t) > 2\beta$, then

$$\liminf_{\epsilon} w^\epsilon(x, t) = 1.$$
Hence $\chi_\ast = 1$ in a neighborhood of $(x_0, t_0)$. This contradicts the assumption that $D\varphi(x_0, t_0) \neq 0$. \hfill $\square$

Now we proceed with the proof of Proposition 8. The proof will be presented through a series of lemmas. The first deals with $v^\epsilon$ near the interface.

**Lemma 4** There is a choice of $\beta$, $\alpha_1$, and $\alpha_0$ such that: (i) $\beta$ is small enough to satisfy the constraints of the previous section and (ii) there is a constant $v(\beta, \tilde{\alpha})$ such that, for all $\epsilon > 0$ small enough, we have

$$m(\epsilon^{-1}x, Dv^\epsilon)v^\epsilon_t - \Delta v^\epsilon + \epsilon^{-2}W'(v^\epsilon) \leq -\frac{v(\beta, \tilde{\alpha})}{3\epsilon} \text{ in } \{|d| < \gamma\}.$$  

The selection of $\beta$, $\alpha_1$, and $\alpha_0$ below is a little delicate. The reason is, at some stage, the fact that $D\tilde{d}_c$ is not constant introduces errors. Let $\omega_\epsilon$ be a modulus of continuity for $m$ and $\overline{m}$ at $e$, that is,

$$\omega_\epsilon(\chi) = \sup \left\{ |m(y, v) - m(y, e)| + |\overline{m}(v) - \overline{m}(e)| \mid \|v - e\| \leq \chi, \ y \in \mathbb{T}^d \right\}.$$  

The errors $D\tilde{d}_c$ are proportional to $\omega_\epsilon(\alpha_0)$ with proportionality constants depending on the choice of $\alpha_1$ through the magnitudes of the derivatives of $P_\epsilon$. Therefore, to control these errors, we need to choose $\alpha_1$ before $\alpha_0$. Given that $\beta$ depends on $\alpha_0$ through Proposition 6, it has to be chosen last.

**Proof** In what follows, to declutter the notation, it will be convenient to define $s = s(x, t) = \tilde{d}_c(x, t) - 2\beta$ and $p_\epsilon$ by

$$p_\epsilon(x, t) = \epsilon D\tilde{d}_c(x, t) P_\epsilon(\epsilon^{-1}s(x, t), \epsilon^{-1}x) + \epsilon^2 D\tilde{d}_c(x, t) P_\epsilon(\epsilon^{-1}s(x, t), \epsilon^{-1}x).$$  

By definition of $v^\epsilon$, the regularity of $P_\epsilon^{\delta}$, and the definition of $M$ in Lemma 3, we have

$$m(\epsilon^{-1}x, Dv^\epsilon)v^\epsilon_t = \epsilon^{-1}\hat{q}(\epsilon^{-1}s)m(\epsilon^{-1}x, \hat{q}(\epsilon^{-1}s))D\tilde{d}_c(x, t) + p_\epsilon)\tilde{d}_c(x, t) + O(1) \leq \epsilon^{-1}\hat{q}(\epsilon^{-1}s) \left( m(\epsilon^{-1}x, \hat{q}(\epsilon^{-1}s))e + M\omega_\epsilon(\alpha_0 + O(\epsilon)) \right) + O(1).$$  

Note, in addition, that, no matter the choice of $e' \in S^{d-1}$, the function $P_2^{\delta} = V_2^{\delta}\hat{q}$ defined in (32) satisfies

$$D_{e'}D_{e'}P_2^{\delta} + \frac{2\hat{q}(s)}{\hat{q}'(s)}(e', D_{e'}V_2^{\delta}) \in \mathbb{R} \times \mathbb{T}^d.$$  

Combining this with the estimate $\|D\tilde{d}_c - e\| \leq \alpha_0$ from Proposition 6, we find

$$-\Delta v^\epsilon = -\epsilon^{-2}\hat{q}(\epsilon^{-1}s) - \epsilon^{-1}\hat{q}(\epsilon^{-1}s) \Delta \tilde{d}_c(x, t) + \epsilon^{-1}D_{\tilde{d}_c(x, t)}D_{\tilde{d}_c(x, t)} P_\epsilon(\epsilon^{-1}s, \epsilon^{-1}x) \tilde{d}_c(x, t) + O(1) \leq -\epsilon^{-2}\hat{q}(\epsilon^{-1}s) + \epsilon^{-1}\hat{q}(\epsilon^{-1}s) \left( |\overline{m}(e) - m(\epsilon^{-1}x, \hat{q}(\epsilon^{-1}s))e| \tilde{d}_c(x, t) - \Delta \tilde{d}_c(x, t) + M\alpha_1 + O(\alpha_0) \right) - \epsilon^{-1}W'(\hat{q}(\epsilon^{-1}s)) P_\epsilon + O(1).$$
Therefore, writing $\epsilon^{-2} W'(v^\epsilon) = \epsilon^{-2} W'(q) + \epsilon^{-1} W''(q) P^{\beta} - 2\beta \epsilon^{-1} W''(q) + O(1)$ yields

$$m(\epsilon^{-1} x, eDv^\epsilon)v^\epsilon_t - \Delta v^\epsilon + \epsilon^{-2} W'(v^\epsilon) = \epsilon^{-1} \left\{ \hat{q}(\epsilon^{-1} s, \epsilon^{-1} x) \left( \overline{D}\tilde{d}_c(x, t) \tilde{d}_c(x, t) - \Delta \tilde{d}_c(x, t) + 2M\omega_c(\alpha_0 + O(\epsilon)) + O(\alpha_0) + M\alpha_1 \right) - 2\beta W''(q(\epsilon^{-1} s)) \right\} + O(1).$$

Thus, appealing once more to Proposition 6,

$$m(\epsilon^{-1} x, eDv^\epsilon)v^\epsilon_t - \Delta v^\epsilon + \epsilon^{-2} W'(v^\epsilon) \leq \epsilon^{-1} \left\{ -(6\tilde{\alpha} + 2M\omega_c(\alpha_0 + O(\epsilon)) + O(\alpha_0) + M\alpha_1)\hat{q}(\epsilon^{-1} s) - 2\beta W''(q(\epsilon^{-1} s)) \right\} + O(1).$$

Finally, we choose $\alpha_1, \alpha_0,$ and $\beta,$ in that order. To start with, choose $\alpha_1$ so that $M\alpha_1 < \tilde{\alpha}$. Next, choose $\alpha_0$ so that, in the expression above, as soon as $\epsilon$ is small enough, we have

$$2M\omega_c(\alpha_0 + O(\epsilon)) + O(\alpha_0) \leq 2M\omega_c(2\alpha_0) + O(\alpha_0) \leq \tilde{\alpha}. $$

Note that this choice depends on $\alpha_1$ through the magnitude of the derivatives of $V_2^{\beta}$, which contribute to the $O(\alpha_0)$ term. However, it does not depend on any of the parameters introduced in Sect. 5.1 (and, in particular, introduces no new restrictions on $\beta$) so there is no risk of circular reasoning.

By [9, Lemma 4.3], there is a $\beta(\tilde{\alpha}) > 0$ such that if $\beta \in (0, \beta(\tilde{\alpha}))$, then

$$v(\beta, \tilde{\alpha}) := \sup \left\{ 3\tilde{\alpha}\hat{q}(s) + 2\beta W''(q(s)) \mid s \in \mathbb{R} \right\} > 0. \tag{46}$$

At last, fix such a $\beta$ consistently with the restrictions of Sect. 5.1. Note that, with this choice of $(\alpha_0, \alpha_1, \beta)$, for small enough $\epsilon > 0$, we find

$$m(\epsilon^{-1} x, eDv^\epsilon)v^\epsilon_t - \Delta v^\epsilon + \epsilon^{-2} W'(v^\epsilon) \leq -\frac{\nu(\beta, \tilde{\alpha})}{2\epsilon} + O(1) \quad \text{in } \{|\tilde{d}_c| < \gamma\}. $$

Henceforth, we assume that $\beta, \alpha_0,$ and $\alpha_1$ have been chosen so that Lemma 4 holds. These three parameters will remain fixed throughout the rest of this section.

Next, we show that the functions $(\overline{v}^\epsilon)_{\epsilon > 0}$ are subsolutions away from $\{\tilde{d}_c > 0\}$.

**Lemma 5** If $\epsilon > 0$ is small enough, then $\overline{v}^\epsilon$ is a subsolution of (1) in $\{\tilde{d}_c < \gamma\}$.

**Proof** It is clear that $\overline{v}^\epsilon$ is a subsolution in $\{-\frac{(\gamma + 2\beta)}{2} < \tilde{d}_c < \gamma\}$, being the maximum of two subsolutions there. At the same time, we claim that $\overline{v}^\epsilon$ is a subsolution in $\{\tilde{d}_c < -(\frac{3\beta}{2} + \frac{\gamma}{4})\}$ as soon as $\epsilon > 0$ is small enough. Indeed, if $\tilde{d}_c(x, t) < -(\frac{3\beta}{2} + \frac{\gamma}{4})$, then the exponential estimates of Propositions 13 and 16 and Corollary 2 imply that

$$v^\epsilon(x, t) \leq q \left( -\frac{\gamma}{4\epsilon} \right) + \epsilon \left( M \left| P_c \left( \tilde{d}_c(x, t) - \frac{2\epsilon \beta}{\epsilon - 2\beta} \right) \right| - 2\beta \right) \leq -1 + C \exp \left( -\frac{\gamma}{4C\epsilon} \right) - 2\beta \epsilon. $$

Hence $\overline{v}^\epsilon = -1$ in $\{\tilde{d}_c < -(\frac{3\beta}{2} + \frac{\gamma}{4})\}$ as soon as $\epsilon$ is sufficiently small. In particular, that makes $\overline{v}^\epsilon$ is a subsolution in $\{d_c < \gamma\}$. \hfill \Box

Finally, we verify that $(w^\epsilon)_{\epsilon > 0}$ remains a subsolution inside $\{d_c > 0\}$ and has the right boundary behavior.
Lemma 6 If $\epsilon > 0$ is small enough, then $w^\epsilon$ is a subsolution of (I) in $Q^\epsilon(x_0, S_2, \rho) \times (t_0 - v_2, t_0 + v_2)$ and

$$w^\epsilon \leq (1 - \beta \epsilon) \chi_{\{\tilde{d}_c \geq \beta\}} - \chi_{\{\tilde{d}_c < \beta\}} \text{ on } \partial P [Q^\epsilon(x_0, S_2, \rho) \times (t_0 - v_2, t_0)].$$

**Proof** Arguing as in Lemma 5, we see that $\tilde{\sigma}^\epsilon = v^\epsilon$ in $\{\tilde{d}_c > \frac{\tilde{v} + 2\beta}{2}\}$ as soon as $\epsilon > 0$ is sufficiently small. In fact, we can assume that $1 - v^\epsilon \leq 2\beta \epsilon$ in $\{\tilde{d}_c > \frac{\tilde{v} + 2\beta}{2}\}$.

Plugging $w^\epsilon$ into the equation in $\{\tilde{v} + 2\beta \epsilon < \tilde{d}_c < \gamma\}$, we find

$$m \left( \epsilon^{-1} x, \tilde{D} w^\epsilon \right) w^\epsilon_i - \Delta w^\epsilon + \epsilon^{-2} W'(w^\epsilon) = m \left( \epsilon^{-1} x, \tilde{D} w^\epsilon \right) w^\epsilon_i - (1 - f(\tilde{d}(x, t))) \Delta v^\epsilon + 2f'(\tilde{d}(x, t))(D\tilde{d}, Dv^\epsilon) + \epsilon^{-2} W'(w^\epsilon) + (f'(\tilde{d}(x, t)) \Delta \tilde{d} + 2f''(\tilde{d}(x, t)))$$

$$\times (v^\epsilon(x, t) - (1 - \beta \epsilon)) = (I) + (II) + (III) + (IV) + (V) + (VI),$$

where, in view of the exponential estimates in Propositions 13 and 16 and Corollary 2, the error terms can be estimated as follows:

$$\begin{align*}
(I) &= (1 - f(\tilde{d}(x, t)))[m(\epsilon^{-1} x, \tilde{D} v^\epsilon)v^\epsilon_i - \Delta v^\epsilon + \epsilon^{-2} W'(v^\epsilon)] \\
&\leq -\frac{1}{3} (1 - f(\tilde{d}(x, t))) v(\beta, \tilde{\alpha}) \epsilon^{-1},
\end{align*}$$

$$\begin{align*}
(II) &= f(\tilde{d}(x, t)) \epsilon^{-2} W'(1 - \beta \epsilon) \leq -f(\tilde{d}(x, t)) \beta W''(1) \epsilon^{-1} + O(1),
\end{align*}$$

$$\begin{align*}
(III) &= \epsilon^{-2} W'(1 - f) v^\epsilon + f(1 - \beta \epsilon) - (1 - f) \epsilon^{-2} W'(v^\epsilon) - f \epsilon^{-2} W'(1 - \beta \epsilon) \\
&= \epsilon^{-2} O(|v^\epsilon - 1|^2 + \epsilon^2) = O(1),
\end{align*}$$

$$\begin{align*}
(IV) &= (1 - f(\tilde{d}(x, t)))(m(\epsilon^{-1} x, \tilde{D} w^\epsilon) - m(\epsilon^{-1} x, \tilde{D} v^\epsilon)) v^\epsilon_i \\
&\leq C \left[ \epsilon^{-1} \hat{q} \left( \frac{\tilde{d}(x, t) - 2\beta \epsilon}{\epsilon} \right) + \partial_x Pe \left( \frac{\tilde{d}(x, t) - 2\beta \epsilon}{\epsilon} \right) \right] + \epsilon \\
&\leq C \exp \left( -(C \epsilon)^{-1} \left( \frac{\gamma - 2\beta \epsilon}{2} \right) \right),
\end{align*}$$

$$\begin{align*}
(V) &= (1 - f(\tilde{d}(x, t))) m(\epsilon^{-1} x, \tilde{D} w^\epsilon)(w^\epsilon_i - v^\epsilon_i) \\
&\leq C \exp \left( -(C \epsilon)^{-1} \left( \frac{\gamma - 2\beta \epsilon}{2} \right) \right) + C f'(\tilde{d}(x, t)) \partial_t|v^\epsilon(x, t) - (1 - \beta \epsilon)| \\
&\leq C \left[ \exp \left( -(C \epsilon)^{-1} \left( \frac{\gamma - 2\beta \epsilon}{2} \right) \right) + \epsilon \right],
\end{align*}$$

$$\begin{align*}
(VI) &= 2f'(\tilde{d}(x, t))(D\tilde{d}, Dv^\epsilon) + (f'(\tilde{d}(x, t)) \Delta \tilde{d} + 2f''(\tilde{d}(x, t))) (v^\epsilon(x, t) - (1 - \beta \epsilon)) \\
&\leq C \left( \epsilon^{-1} + 1 \right) \exp \left( -(C \epsilon)^{-1} \left( \frac{\gamma - 2\beta \epsilon}{2} \right) \right) + C \epsilon.
\end{align*}$$

In particular, we find, in the limit $\epsilon \to 0^+$,

$$m \left( \epsilon^{-1} x, \tilde{D} w^\epsilon \right) w^\epsilon_i - \Delta w^\epsilon + \epsilon^{-2} W'(w^\epsilon) \leq -\min \left\{ \frac{1}{3} v(\beta, \tilde{\alpha}), \beta W''(1) \right\} \epsilon^{-1} + O(1).$$
Thus, \( w^\epsilon \) is a subsolution in the domain \( \{ \frac{\gamma + 2\beta}{2} < d_c < \gamma \} \) as soon as \( \epsilon \) is small enough. At the same time, \( w^\epsilon = \overline{v}_\epsilon \) in \( \{ d_c < \frac{3\gamma + \beta}{4} \} \) and \( w^\epsilon = 1 - \beta \epsilon \) in \( \{ \frac{7\gamma}{8} + \frac{\beta}{4} < d_c \} \) so \( w^\epsilon \) is actually a subsolution in \( Q^\epsilon(x_0, S_2, \rho) \times (t_0 - v_2, t_0 + v_2) \).

Finally, we check the boundary condition. We claim that, for all \( \epsilon > 0 \) small enough,

\[
v^\epsilon \leq (1 - \beta \epsilon)\chi_{[d_c \geq \beta]} - \chi_{[d_c < \beta]} \quad \text{in} \quad Q^\epsilon(x_0, S_2, \rho) \times (t_0 - v_2, t_0 + v_2). \tag{47}
\]

To see this, first, choose \( \kappa > 0 \) such that

\[
\max \{ \dot{q}(s) \mid s \geq \kappa \} < \frac{\beta}{\Theta_1} - 1.
\]

Now notice that if \( \tilde{d}_c(x, t) \leq 2\beta + \kappa \epsilon \), then

\[
v^\epsilon(x, t) \leq q(\kappa) + \epsilon \Theta \| \dot{q} \|_{L^\infty(\mathbb{R})} - 2\beta \epsilon,
\]

while \( \tilde{d}_c(x, t) > 2\beta + \kappa \epsilon \) implies, by the choice of \( \kappa \),

\[
v^\epsilon(x, t) \leq 1 - \beta \epsilon.
\]

Thus, there is an \( \bar{\epsilon} > 0 \) such that, for each \( \epsilon \in (0, \bar{\epsilon}) \),

\[
v^\epsilon \leq 1 - \beta \epsilon \quad \text{in} \quad Q^\epsilon(x_0, S_2, \rho) \times (t_0 - v_2, t_0 + v_2).
\]

Finally, if \( \tilde{d}_c(x, t) < \beta \), then, making \( \bar{\epsilon} > 0 \) smaller if necessary, we find, for each \( \epsilon \in (0, \bar{\epsilon}) \),

\[
v^\epsilon(x, t) \leq -1 + C \exp \left( -\frac{\beta}{C \epsilon} \right) - 2\beta \epsilon \leq -1.
\]

This completes the proof of (47). Since \( f(\xi) = 0 \) if \( \xi \leq 2\beta \), the claimed boundary behavior of \( w^\epsilon \) follows.

\[\boxdot\]

### 6 Rational contact points

In this section, we prove the analogue of Proposition 4 for rational directions assuming in addition that the level set of \( \phi \) is nearly flat at the contact point. That is, we tackle condition (b) in Definition 1. The main result is stated below:

**Proposition 9** Fix \( \delta \in (0, 1) \). If \( \phi \) is a smooth function in \( \mathbb{R}^d \times (0, \infty) \); \( (x_0, t_0) \in \mathbb{R}^d \times (0, \infty) \) is a point where \( \chi_* - \phi \) has a strict local minimum; \( D\phi(x_0, t_0) \in \mathbb{R}^d \setminus \{0\} \); and the level set of \( \phi \) has is \( \delta \)-flat at \( (x_0, t_0) \) in the following sense

\[
\left\| \left( I_d - \widehat{D\phi(x_0, t_0)} \otimes \widehat{D\phi(x_0, t_0)} \right) D^2 \phi(x_0, t_0) \right\| \leq \delta \| D\phi(x_0, t_0) \|,
\]

then

\[
\phi_t(x_0, t_0) \geq -10\theta^{-1}d\delta \| D\phi(x_0, t_0) \|. \tag{48}
\]

The proof of Proposition 9 is a minor modification of the proof of Proposition 4. Let us summarize the details.
Again, proceed by contradiction. If (48) fails, then we can construct $\tilde{d}_c$ once more in such a way that
$$\tilde{d}_{c,t} \leq -9\theta^{-1}d\delta \text{ in } |\tilde{d}_c| < \gamma.$$ Further, by continuity, we can make $S_2, v_2, c,$ and $\gamma$ so small that
$$|\Delta\tilde{d}_c| \leq 2d\delta \text{ in } |\tilde{d}_c| < \gamma.$$ Hence, with these changes, the conclusions of Proposition 6 still hold except $-6\tilde{\alpha}$ should be replaced by $-6d\delta$ and the mobility $m$ by the constant $\theta$.

The construction of mesoscopic subsolutions proceeds as before, except this time $v^\epsilon$ is simply given by
$$v^\epsilon (x, t) = q\left(\frac{\tilde{d}_c(x, t) - 2\beta}{\epsilon}\right) - 2\beta\epsilon.$$ When it comes time to check that $v^\epsilon$ is a subsolution, we use
$$m(\epsilon^{-1} x, D\tilde{d}_c)\tilde{d}_{c,t} - \Delta\tilde{d}_c \leq \theta\tilde{d}_{c,t} - \Delta\tilde{d}_c \leq -6d\delta \text{ in } |\tilde{d}_c| < \gamma.$$ The remainder of the construction goes through exactly as before.

## 7 Shrinking subsolutions

In this section, we construct mesoscopic subsolutions of (1) that approximate characteristic functions of shrinking balls. Using these, we prove that the limiting evolution satisfies the remaining differential inequality in Definition 1, namely condition (c). Employing similar ideas, we also prove that the phase indicator functions $\chi_*$ and $\chi^*$ are compatible with the initial datum.

### 7.1 Finite speed of shrinking

As shown in [9], to prove $\chi_*$ satisfies the right differential inequality when the gradient vanishes, it suffices to check that a ball contained in $\{\chi_* = 1\}$ cannot shrink too fast. Toward that end, we begin by proving the next result:

**Proposition 10** Fix $R > 0$ and $t_0 \geq 0$ and assume that $B(x_0, R) \subseteq \Omega_{t_0}^{(1)}$. Given $\theta \in (0, \theta)$, there is an $h > 0$ depending continuously on $R$ (and independent of $(x_0, t_0)$) such that
$$B\left(x_0, \sqrt{R^2 - 2\theta^{-1}(d - 1)s}\right) \subseteq \Omega_{t_0+s}^{(1)} \text{ if } s \in [0, h].$$

By invoking the proposition, we can prove

**Theorem 5** Fix $R > 0$ and $t_0 > 0$ and assume that $B(x_0, R) \subseteq \Omega_{t}^{(1)}$. If $\theta < \theta$, then
$$B\left(x_0, \sqrt{R^2 - 2\theta^{-1}(d - 1)s}\right) \subseteq \Omega_{t_0+s}^{(1)} \text{ for each } s \in \left[0, \frac{\theta R^2}{2(d - 1)}\right].$$
Proof For each \( s \in [0, \frac{\theta R^2}{2(d-1)}] \), define \( R : (0, \infty) \to [0, \infty) \) by
\[
R(s) = \sup \left\{ r \geq 0 \mid B(x_0, r) \subseteq \Omega_{t_0+s}^{(1)} \right\}.
\]
Note that the definition of \( R \) implies that \( B(x_0, R(s)) \subseteq \Omega_{t_0+s}^{(1)} \). Moreover, by assumption, \( R(0) \geq R \). Let \( T = \inf \{ s > 0 \mid R(s) = 0 \} \).

We claim that \( s \mapsto R(s) \) is a lower semi-continuous viscosity supersolution of the ODE
\[
\frac{\partial}{\partial t} R + \frac{(d-1)}{R} \geq 0 \quad \text{in} \quad (0, T).
\]
The lower semi-continuity follows from the fact that \( \chi_s \) is lower semi-continuous.

Notice that, given an \( s \in (0, T) \), Proposition 10 yields an \( h > 0 \) such that if \( s' \in (s-h, s) \), then
\[
R(s) \geq \sqrt{R(s')^2 - \frac{2(d-1)(s-s')}{\theta}}.
\]
From this, it follows easily that if \( \varphi \) is a smooth function and \( s' \mapsto R(s') - \varphi(s') \) has a local minimum at \( s \), then
\[
\frac{\partial}{\partial t} \varphi(s) + \frac{(d-1)}{R(s)} \geq 0.
\]
By the comparison principle for viscosity solutions, we deduce that \( s \mapsto R(s) \) is at least as large as the solution of the ODE with initial condition \( R(0) \). In particular,
\[
R(s) \geq \sqrt{R(0) - \frac{2(d-1)s}{\theta}} \geq \sqrt{R - \frac{2(d-1)s}{\theta}} \quad \text{in} \quad (0, T).
\]
Note, in addition, that this inequality yields \( T \geq \frac{\theta R^2}{2(d-1)}. \)

Now we prove Proposition 10. First, observe that the function \( d : \mathbb{R}^d \times [0, \frac{\theta R^2}{2(d-1)}] \to \mathbb{R} \) given by
\[
d(x, t) = \sqrt{R^2 - 2\theta^{-1}(d-1)t} - \|x\|
\]
satisfies
\[
\theta d_t - \text{tr}((\text{Id} - \widetilde{D}d \otimes \widetilde{D}d) D^2 d) = -\frac{\theta(d-1)}{\theta \sqrt{R^2 - 2\theta^{-1}(d-1)(t-t_0)}} + \frac{(d-1)}{\|x\|}.
\]
Note, in addition, that \( d_t \leq 0 \). A direction computation yields the following lemma:

Lemma 7 Fix \( R > 0 \) and \( t_0 > 0 \). For each \( \rho \in (0, 1) \) and \( \nu \in (0, \frac{\theta}{R} - 1) \), the function \( d \) above satisfies
\[
\frac{\partial}{\partial t} d_t - \text{tr}((\text{Id} - \widetilde{D}d \otimes \widetilde{D}d) D^2 d) \leq -\frac{1}{R} \left[ \frac{\theta}{\theta - 1} - 1 - \nu \right] \quad \text{in} \quad A_{\rho, \nu} \times \left( 0, \frac{R^2 \theta}{2(d-1)} \right),
\]
where \( A_{\rho, \nu} = B(0, (1-\rho)^{-1} R) \setminus B(0, (1+\nu)^{-1} R) \).
We use $d$ to construct global mesoscopic subsolutions arguing as in Sect. 5. To start with, define $u^\epsilon : A_{\rho, v} \times \left(0, \frac{R^2\theta}{2(d-1)} \right) \to \mathbb{R}$ by

$$v^\epsilon(x, t) = q \left( \frac{d(x, t) - 2\beta}{\epsilon} \right) - 2\beta \epsilon.$$ 

Observe that, using the sign of $d_t$, we can compute

$$m(\epsilon^{-1}x, Dv^\epsilon)v^\epsilon_t - \Delta v^\epsilon + \epsilon^{-2}W'(v^\epsilon) = \epsilon^{-1}m(\epsilon^{-1}x, Dd(x, t))d_t - \epsilon^{-2}\dot{q} - \epsilon^{-1}\dot{q}\Delta d + \epsilon^{-2}W'(q) - 2\beta\epsilon^{-1}W''(q) \leq -\epsilon^{-1}\left(C_R\dot{q} + 2\beta W''(q)\right),$$

where $C_R = \frac{1}{R} \left[ \frac{\theta}{\sigma^2} - 1 - v \right] > 0$. As in [9], we can choose $\beta = \beta(v) > 0$ so that, for each $\beta \in (0, \bar{\beta})$,

$$\mu_{\beta} := \min \left\{ C_R\dot{q}(s) + 2\beta W''(q(s)) \mid s \in \mathbb{R} \right\} > 0.$$

This gives

$$m(\epsilon^{-1}x, Dv^\epsilon)v^\epsilon_t - \Delta v^\epsilon + \epsilon^{-2}W'(v^\epsilon) \leq -\mu_{\beta} \epsilon^{-1} \text{ in } A_{\rho, v} \times \left(0, \frac{R^2\theta}{2(d-1)} \right).$$

We will not be able to proceed in the entire time interval $\left(0, \frac{R^2\theta}{2(d-1)} \right)$ since the interface $\{d = 0\}$ does not remain in $A_{\rho, v}$. Therefore, we restrict attention to $\mathbb{R}^d \times (0, T)$ for some $T > 0$ and choose $\gamma > 0$ so that

$$\{(x, t) \in \mathbb{R}^d \times (0, T) \mid |d(x, t)| < \gamma \} \subseteq A_{\rho, v} \times [0, T].$$

Clearly, it is possible to do this by continuity. A concrete choice of $T$ and $\gamma$ is

$$T = \frac{\theta}{4(d-1)} \cdot \frac{R^2v(v + 2)}{(v + 1)^2}, \quad \gamma = \left[ \frac{Rv}{2(v + 1)} \right] \wedge \left[ \frac{\rho R}{2(1 - \rho)} \right].$$

Notice that, for a fixed $(\theta, \rho, v)$, $T$ and $\gamma$ depend continuously on $R$.

Next, we define $(\tilde{\nu}^\epsilon)_{\epsilon > 0}$ and $(w^\epsilon)_{\epsilon > 0}$ in $\mathbb{R}^d \times [0, T]$ as before with the choice of $\gamma$ just selected. To get things started, we need the following variant of Lemma 6:

**Lemma 8** Under the assumptions of Proposition 10, given $\beta > 0$, there are constants $\tau, \epsilon_0 > 0$ depending only on $\beta$ such that, for each $\epsilon \in (0, \epsilon_0)$,

$$u^\epsilon(\cdot, t_0 + \tau \epsilon^2 \log(\epsilon^{-1})) \geq (1 - \beta \epsilon)\chi_{\{||x-x_0|| \leq R-\beta\}} - \chi_{\{||x-x_0|| > R-\beta\}} \text{ in } \mathbb{R}^d.$$ 

The proof follows by arguing exactly in [9, Lemma 4.1], replacing the function $\chi$ used there by the same function $\tilde{\chi}^\epsilon$ used in the proof of Proposition 7.

Comparing $u^\epsilon$ and $w^\epsilon$ as in Sect. 5, we find

$$w^\epsilon(x, t) \leq u^\epsilon(x, t + \tau \epsilon^2 \log(\epsilon^{-1})) \text{ if } (x, t) \in \mathbb{R}^d \times [0, T].$$

Combined with the fact that $\liminf_{\epsilon} w^\epsilon(x, t) = 1$ if $d(x, t) \geq 2\beta$, this gives

$$\{d(\cdot, t) \geq 2\beta\} \subseteq \Omega_t^{(1)} \text{ if } t \in [0, T].$$

Sending $\beta \to 0^+$, we obtain the conclusion of Proposition 10 with $h = T$. 

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7.2 Supersolution property at zero

The finite shrinking speed of the previous section is intimately related to the final differential inequality in Definitions 1 and 2. In fact, it implies it, as shown in the next result.

Proposition 11 If \( \varphi : \mathbb{R}^d \times (0, \infty) \to \mathbb{R} \) is smooth, \( \chi_* - \varphi \) has a strict local minimum at \( (x_0, t_0) \in \mathbb{R}^d \times (0, \infty), \) and \( \|D\varphi(x_0, t_0)\| = \|D^2\varphi(x_0, t_0)\| = 0, \) then

\[ \varphi_t(x_0, t_0) \geq 0. \]

The proof below is based on an insight from [9].

Proof First, notice that if \( \chi_*(x_0, t_0) = 1 \), then \( \chi_* = 1 \) in a neighborhood of \( (x_0, t_0) \), and this implies \( \varphi_t(x_0, t_0) = 0 \) directly.

Assume now that \( \chi_*(x_0, t_0) = -1 \) and, without loss of generality, that \( \varphi(x_0, t_0) = 0 \). It follows that there is an open ball \( B \subseteq \mathbb{R}^d \times (0, \infty) \) containing \( (x_0, t_0) \) such that

\[ \chi_*(x, t) - \varphi(x, t) \geq -1 \quad \text{if} \quad (x, t) \in B. \]

In particular, since \( \varphi(x_0, t_0) = 0 \), this gives

\[ \chi_*(x, t) - \varphi_t(x_0, t_0)(t - t_0) + o(\|x - x_0\|^2 + |t - t_0|) \geq -1 \quad \text{if} \quad (x, t) \in B. \quad (49) \]

Let \( C = \frac{4(d-1)}{2} \). We claim that there is a sequence \((x_n, t_n)_{n \in \mathbb{N}}\) such that

\[ \chi_*(x_0, t_0) = \lim_{n \to \infty} (x_n, t_n), \quad t_n < t_0, \]

\[ \|x_n - x_0\|^2 \leq C|t_n - t_0|, \quad \chi_*(x_n, t_n) = -1. \]

Assuming the claim is true for now, we set \( (x, t) = (x_n, t_n) \) in \((49)\) to find

\[ \varphi_t(x_0, t_0)(t_0 - t_n) + o(|t_n - t_0|) \geq 0. \]

Dividing by \( t_0 - t_n \) and sending \( n \to \infty \), this yields

\[ \varphi_t(x_0, t_0) \geq 0. \]

It remains to prove the claim. We argue by contradiction, assuming that it is false. We can then fix an \( s \in (0, t_0) \) such that \( B(x_0, \sqrt{C(t_0 - s)}) \subseteq \Omega_*^{(1)} \). Now Theorem 5 implies that

\[ B \left( x_0, \sqrt{C(t_0 - s) - \frac{2(d-1)(t-s)}{\theta}} \right) \subseteq \Omega_*^{(1)} \quad \text{if} \quad t \in \left[ 0, \frac{C\theta(t_0 - s)}{2(d-1)} \right] + s. \]

At the same time, notice that, by the choice of \( C \),

\[ s + \frac{C\theta(t_0 - s)}{2(d-1)} = s + 2(t_0 - s) > t_0. \]

Thus, we deduce that

\[ x_0 \in B \left( x_0, \sqrt{C(t_0 - s) - \frac{2(d-1)(t-s)}{\theta}} \right) \subseteq \Omega_*^{(1)} \]

but this contradicts the assumption that \( \chi_*(x_0, t_0) = -1. \) \( \square \)
### 7.3 Initial datum

The proof of Proposition 10 can be modified slightly to prove that \( \{ \chi_\ast (\cdot, 0) = 1 \} \supseteq \{ u_0 > 0 \} \), as claimed in Proposition 1.

**Proposition 12** \( \chi_\ast (\cdot, 0) = 1 \) in \( \{ u_0 > 0 \} \).

To prove this, we will use the following variant of Lemma 8.

**Lemma 9** Given \( \beta, r \in (0, 1) \) and \( x_0 \in \mathbb{R}^d \), if \( B(x_0, r) \subseteq \{ u_0 > 0 \} \), then there is a \( \tau > 0 \) depending only on \( \beta \) and \( r \) and \( \epsilon_0 \in (0, 1) \) such that, for each \( \epsilon \in (0, \epsilon_0) \),

\[
\{ u^\epsilon (\cdot, \tau \epsilon^2 \log(\epsilon^{-1})) \geq (1 - \beta \epsilon) \chi_{\|x - x_0\| \leq r - \beta} - \chi_{\|x - x_0\| > r - \beta} \}.
\]

Now we prove the proposition.

**Proof of Proposition 12** Fix \( x_0 \in \{ u_0 > 0 \} \). We need to prove that \( \chi_\ast (x_0, 0) = 1 \). Toward this end, first, fix an \( r > 0 \) such that \( B(x_0, r) \subseteq \{ u_0 > 0 \} \).

Arguing exactly as in the proof of Proposition 10 with Lemma 9 in place of Lemma 8 and \( \theta = 2^{-1} \theta \), we find an \( h > 0 \) such that

\[
B \left( x_0, \sqrt{r^2 - 4\theta^{-1}(d - 1)r} \right) \subseteq \Omega_i^{(1)} \quad \text{for each } t \in (0, h).
\]

It follows that there is an \( h' > 0 \) such that \( B \left( x_0, 2^{-1}r \right) \subseteq \Omega_i^{(1)} \) for each \( t \in (0, h') \). By the definition of \( \Omega_i^{(1)} \) and \( \chi_\ast \), this implies that

\[
\chi_\ast (x_0, 0) = \lim_{\delta \to 0^+} \inf \{ \chi_\ast (y, s) \mid \|x - y\| + s \leq \delta, \ s > 0 \}
\]

\[
\geq \inf \{ \chi_\ast (y, s) \mid y \in B(x_0, 2^{-1}r), \ s \in (0, h') \} = 1.
\]

Since \( \chi_\ast \leq 1 \), the proof is complete. \( \square \)

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### Appendix A: Initialization

We are interested in proving an “initialization” type result for the phase field equation

\[
m(x, Du)u_t - \Delta u + W'(u) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty).
\]

Defining the modified forcing \( \tilde{f} : [-1, 1] \to \mathbb{R} \) by

\[
\tilde{f}(u) = \begin{cases} \Theta^{-1} W'(u), & \text{if } u \in [0, 1], \\ \theta^{-1} W'(u), & \text{if } u \in [-1, 0]. \end{cases}
\]

we will construct a “universal subsolution” \( \tilde{x}^\epsilon \) of the ODE

\[
\tilde{x}_t^\epsilon \leq -\tilde{f}(\tilde{x}^\epsilon) \quad \text{in } [-1, 1]
\]

with the same properties as the specific subsolution used in [9, Lemma 4.1]. As shown in Proposition 7 above, the choice of \( \tilde{f} \) enables us to build certain subsolutions of (1) that are used to analyze the development of sharp interfaces as \( \epsilon \to 0^+ \).

In this section, we build \( \tilde{x}^\epsilon \) by proceeding by analogy with Chen’s paper [13, Section 3]. The main result is Lemma 12 below.
A.1 Preliminaries

The assumptions on $W$ imply that we can fix a $\mu \in (0, \frac{1}{4})$ such that

\[
\frac{W''(-1)}{2} \leq W''(u) \leq \frac{3W''(-1)}{2} \quad \text{if } u \in [-1 - 2\mu, -1 + 2\mu],
\]

\[
\frac{W''(1)}{2} \leq W''(u) \leq \frac{3W''(1)}{2} \quad \text{if } u \in [1 - 2\mu, 1],
\]

\[
\frac{3W''(0)}{2} \leq W''(u) \leq \frac{W''(0)}{2} \quad \text{if } u \in [-\mu, \mu].
\]

From this, we deduce that

\[
\theta^{-1} W'(u) \leq W'(u) \leq \Theta^{-1} W'(u) \quad \text{if } u \in [0, 1] \cup [-1 - 2\mu, -1],
\]

\[
\Theta^{-1} W'(u) \leq W'(u) \leq \theta^{-1} W'(u) \quad \text{if } u \in [-1, 0] \cup [1, 1 + 2\mu].
\]

A.2 Regularization

Fix $\epsilon \in (0, 1)$. To start with, let $\tilde{f} : [-3, 3] \to \mathbb{R}$ be the function defined by

\[
\tilde{f}(u) = \begin{cases} 
\Theta^{-1} W'(u), & \text{if } u \in [0, 3] \cup [-3, -1], \\
\theta^{-1} W'(u), & \text{if } u \in [-1, 0]. 
\end{cases}
\]

(51)

Notice that $\tilde{f}$ is Lipschitz continuous.

Next, we make some adjustments. To start with, let $\rho : [-1, 1] \to (0, \infty)$ be a smooth function with $\rho(s) = \rho(-s)$, $\rho(1) = \rho(-1) = 0$, and $\int_{-1}^{1} \rho(s) ds = 1$, and, given $\epsilon \in (0, 1)$, define $\rho^\epsilon(s) = \epsilon^{-1} \rho(\epsilon^{-1} s)$. Define $\tilde{f}_\epsilon$ by $\tilde{f}_\epsilon = \rho^\epsilon * \tilde{f} + 2\text{Lip}(\tilde{f}; [-3, 3]) \epsilon$. Notice that, by construction, for each $u \in [-2, 2]$, we have

\[
\tilde{f}_\epsilon(u) - \tilde{f}(u) \geq \text{Lip}(\tilde{f}; [-3, 3]) \epsilon.
\]

In particular, $\tilde{f}_\epsilon \geq \tilde{f}$ in $[-2, 2]$.

Finally, fix a cut-off function $\eta \in C^\infty(\mathbb{R}; [0, 1])$ such that

\[
\eta(u) = 1 \text{ if } u \in [-\frac{1}{4}, \infty), \quad \eta(u) = 0 \text{ if } u \in (-\infty, -\frac{3}{4}],
\]

\[
|\eta'(u)| \leq 4,
\]

and define $f_\epsilon : \mathbb{R} \to \mathbb{R}$ by

\[
f_\epsilon(u) = \eta(u) \tilde{f}(u) + (1 - \eta(u)) \tilde{f}_\epsilon(u).
\]

Some properties of $f_\epsilon$ are summarized next:

**Lemma 10** There is a constant $M_1 > 0$ and an $\epsilon_0 > 0$ with $\epsilon_0 < \frac{1}{2}$ such that if $\epsilon \in (0, \epsilon_0)$, then the following statements hold:

(i) $\frac{W''(-1)}{2\Theta} \leq f'_\epsilon(u) \leq \frac{3W''(-1)}{2\Theta}$ if $u \in [-1 - \mu, -1 + \mu]$.

(ii) There is a $z_\epsilon \in [-1 - \mu, -1]$ such that

\[
[u \in [-1 - \mu, 1] \mid f_\epsilon(u) = 0] = \{z_\epsilon, 0, 1\}.
\]

(52)

(iii) $\|f'_\epsilon\|_{L^\infty([-1-\mu, 1])} \leq M_1$.

The lemma follows directly from the properties of $\tilde{f}$ and the definition of $\tilde{f}_\epsilon$. Therefore, the proof is omitted.
A.3 Modification

In what follows, let $M = \theta^{-1} ||W''||_{L^{\infty}([-3,3])}$. Following [13, Section 3], we now fix a family of cut-off functions $(\tilde{\zeta}_\epsilon)_{\epsilon \in (0,1)} \subseteq C_c^1(\mathbb{R}; [0,1])$ such that, for each $\epsilon \in (0,1)$,

(a) $\tilde{\zeta}_\epsilon(u) = 1$ if $u \in [0, 2\epsilon |\log(\epsilon)|]$,
(b) $\tilde{\zeta}_\epsilon(u) = 0$ if $u \in (-\infty, -\frac{\epsilon}{M}] \cup [3\epsilon |\log(\epsilon)|, \infty)$,
(c) $\tilde{\zeta}_\epsilon$ satisfies the bounds

$$0 \leq \tilde{\zeta}_\epsilon'(u) \leq \frac{2M}{\epsilon} \quad \text{if } u \in [-\frac{\epsilon}{M}, 0], \quad -\frac{2}{\epsilon |\log(\epsilon)|} \leq \tilde{\zeta}_\epsilon'(s) \leq 0 \quad \text{if } u \in [0, 3\epsilon |\log(\epsilon)|].$$

Now we define $\tilde{f}_\epsilon : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{f}_\epsilon(u) = (1 - \tilde{\zeta}_\epsilon(u)) f_\epsilon(u) + \tilde{\zeta}_\epsilon(u) \left( \frac{\epsilon |\log(\epsilon)| - u}{|\log(\epsilon)|} \right).$$

To start with, we record some properties of the family $(\tilde{f}_\epsilon)_{\epsilon \in (0,1)}$:

**Lemma 11** There are positive constants $c, M_2, \epsilon_1 > 0$ such that if $\epsilon \in (0, \epsilon_0 \wedge \epsilon_1)$, then the following statements hold:

(a) $\tilde{f}_\epsilon \geq f_\epsilon \geq \tilde{f}$ in $[-1 - \mu, 1]$.

(b) The following inequalities hold away from 0:

$$\tilde{f}_\epsilon(u) \leq -c \epsilon \quad \text{if } u \in [2\epsilon |\log(\epsilon)|, 3\epsilon |\log(\epsilon)|], \quad \tilde{f}_\epsilon(u) \geq c \epsilon \quad \text{if } u \in \left[-\frac{\epsilon}{M}, 0\right].$$

(c) $\|\tilde{f}_\epsilon\|_{L^{\infty}([-1-\mu, 1])} \leq M_2$.

**Proof** To see that (a) holds, observe that the identity $f_\epsilon(0) = \tilde{f}(0) = 0$ implies we can write

$$\tilde{f}_\epsilon(u) = f_\epsilon(u) + \tilde{\zeta}_\epsilon(u) \left( \epsilon - \left( \frac{1}{|\log(\epsilon)|} + \frac{f_\epsilon(u) - f_\epsilon(0)}{u} \right) u \right). \quad (53)$$

Recall that the $\zeta_\epsilon$ term only has to be dealt with when $u \in [-\epsilon/M, 0] \cup [0, 3\epsilon |\log(\epsilon)|]$.

Fix $\epsilon_1' > 0$ such that $\epsilon/M \leq 1/4$ if $\epsilon \in (0, \epsilon_1')$. If $\epsilon \in (0, \epsilon_1')$ and $u \in [-\epsilon/M, 0]$, then the definition of $M$ gives

$$f_\epsilon(u) = |f_\epsilon(u)| = |\tilde{f}(u)| \leq \theta^{-1} ||W''||_{L^{\infty}([-3,3])} |u| = M |u|.$$ 

Hence $f_\epsilon(u) \leq M |u| \leq \epsilon$, which gives

$$\tilde{f}_\epsilon(u) = f_\epsilon(u) + \tilde{\zeta}_\epsilon(u) \left( \epsilon - f_\epsilon(u) - \frac{u}{|\log(\epsilon)|} \right) \geq f_\epsilon(u).$$

Making $\epsilon_1'$ smaller if necessary, we can assume that $3\epsilon |\log(\epsilon)| \leq \mu$ if $\epsilon \in (0, \epsilon_1')$. Now note that if $u \in [2\epsilon |\log(\epsilon)|, 3\epsilon |\log(\epsilon)|]$, then we can write

$$- \left( \frac{1}{|\log(\epsilon)|} + \frac{f_\epsilon(u) - f_\epsilon(0)}{u} \right) u \geq -3\epsilon - f_\epsilon(u) \geq -3\epsilon - \frac{W''(0)}{2\Theta} u \geq -3\epsilon + \frac{|W''(0)| \epsilon |\log(\epsilon)|}{\Theta}.$$
Finally, we let $\epsilon_1'' = \exp(-\frac{3\theta}{|W'(0)|})$ and $\epsilon_1 = \epsilon'_1 \wedge \epsilon_1'' \wedge \frac{1}{2}$. The previous string of inequalities implies that if $\epsilon \in (0, \epsilon_1 \wedge \epsilon_0)$, then

$$-\left(\frac{1}{|\log(\epsilon)|} + \frac{f_\epsilon(u) - f_\epsilon(0)}{u}\right)u \geq 0 \text{ if } u \in [2\epsilon|\log(\epsilon)|, 3\epsilon|\log(\epsilon)|].$$

From this and (53), it follows that $\tilde{f}_\epsilon(u) \geq f_\epsilon(u)$ for all $u \in [2\epsilon|\log(\epsilon)|, 3\epsilon|\log(\epsilon)|]$. Next, we note that if $u \in [0, 2\epsilon|\log(\epsilon)|]$ and $\epsilon \in (0, \epsilon_1 \wedge \epsilon_0)$, then a direct computation shows that

$$f_\epsilon(u) \leq -\frac{|W''(0)||u|}{2\Theta} \leq \epsilon - \frac{u}{|\log(\epsilon)|} = \tilde{f}_\epsilon(u).$$

This completes the proof that $\tilde{f}_\epsilon \geq f_\epsilon$ in $[-2, 2]$ and then the inequality $f_\epsilon \geq \tilde{f}$ in the same interval follows from the construction of $f_\epsilon$.

Next, we prove (b). Recall that if $u \in [0, \mu]$, then

$$f_\epsilon(u) \leq -\frac{|W''(0)||u|}{2\Theta}$$

and, thus, for all $u \in [2\epsilon|\log(\epsilon)|, 3\epsilon|\log(\epsilon)|]$ and $\epsilon \in (0, \epsilon_1 \wedge \epsilon_0)$,

$$\tilde{f}_\epsilon(u) \leq -(1 - \epsilon_\epsilon(u))\Theta^{-1}|W''(0)||\epsilon|\log(\epsilon)| - \epsilon \leq -\epsilon.$$

Let $\epsilon'' = M\mu$. If $u \in [-\frac{\epsilon}{M}, 0]$ and $\epsilon \in (0, \epsilon''')$, then

$$\tilde{f}_\epsilon(u) \geq (1 - \epsilon_\epsilon(u))\frac{|W''(0)|}{2\Theta}|u| + \epsilon \epsilon(u)\epsilon.$$

When $u \in [-\frac{\epsilon}{M}, 0]$, this gives (by property (c) of $\epsilon_\epsilon$ above),

$$\tilde{f}_\epsilon(u) \geq \left(1 - \frac{2M}{\epsilon}, \frac{\epsilon}{4M}\right)\epsilon = \frac{\epsilon}{2}$$

while the case $u \in [-\frac{\epsilon}{M}, -\frac{\epsilon}{4M}]$ yields

$$\tilde{f}_\epsilon(u) \geq (1 - \epsilon_\epsilon(u))\frac{|W''(0)|\epsilon}{8M\Theta} + \epsilon \epsilon(u)\epsilon \geq \frac{|W''(0)|\epsilon}{8M\Theta}.$$

Therefore, if we replace $\epsilon_1$ above by $\epsilon_1 \wedge \epsilon''$, we conclude that there is a $c > 0$ such that (b) holds.

(c) follows directly from the choice of $\epsilon_\epsilon$, conclusion (b) of Lemma 10, and (50). 

Henceforth, we let $\chi^\epsilon : [-1 - \mu, 1] \times [0, \infty) \to [-1 - \mu, 1]$ denote the solution map of the ODE associated with $-\tilde{f}_\epsilon$, that is,

$$\left\{ \begin{array}{ll}
\chi^\epsilon(\xi, s) + \tilde{f}_\epsilon(\chi^\epsilon(\xi, s)) = 0 & \text{if } (\xi, s) \in [-1 - \mu, 1] \times (0, \infty), \\
\chi^\epsilon(\xi, 0) = \xi & \text{if } \xi \in [-1 - \mu, 1].
\end{array} \right.$$

**Lemma 12**

(i) For each $\beta > 0$, there is an $\epsilon(\beta), \tau(\beta) > 0$ such that if $\epsilon \in (0, \epsilon(\beta))$, then

$$\chi^\epsilon(\xi, s) \geq 1 - \beta \epsilon \text{ if } \xi \geq 3\epsilon|\log(\epsilon)|, s \geq \tau(\beta)|\log(\epsilon)|. \quad (54)$$

(ii) $\chi^\epsilon > 0$ in $[-1 - \mu, 1] \times [0, \infty)$, independently of $\epsilon > 0$.

(iii) For each $a > 0$, there is an $\epsilon(a) > 0$ and $B(a) > 0$ such that if $\epsilon \in (0, \epsilon(a))$, then

$$\left|\frac{\chi^\epsilon(\xi, s)}{\chi^\epsilon(\xi, s)}\right| \leq \frac{B(a)}{\epsilon} \text{ if } (\xi, s) \in [-1 - \mu, 1] \times [0, a|\log(\epsilon)|]. \quad (55)$$
**Proof** The proof proceeds exactly as in [13, Lemma 3.1]. The main difference is $\tilde{f}_\epsilon''$ can grow like $C\epsilon^{-1}$ near $-1$, which, upon inspection of the proof in [13], only has the effect of increasing the constant $B(a)$ in (55). □

**Appendix B: Comparison principle**

After a multiplication by $m^{-1}$, (1) is a special case of the following class of equations:

$$
\begin{cases}
  u_t - G(x, Du)\text{tr}(D^2 u) + B(x, u, Du) = 0 & \text{in } \mathbb{R}^d \times (0, T), \\
  u = u_0 & \text{on } \mathbb{R}^d \times \{0\}.
\end{cases}
$$

(56)

In what follows, we assume that $G : \mathbb{R}^d \times \mathbb{R}^d \to (0, \infty)$ and $B : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ are bounded, continuous functions for which there are constants $C_1, K, m, M > 0$ such that, for each $(y, u, v) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$, $y' \in \mathbb{R}^d$, and $u' \in \mathbb{R}$, we have

$$
|G(y, v) - G(y', v)| + |B(y, u, u) - B(y', v, u)| \leq C_1|y - y'|, \quad (57)
$$

$$
M \leq G(y, v) \leq M, \quad (58)
$$

$$
|B(y, u, v) - B(y, u', v)| \leq K|u - u'|. \quad (59)
$$

As is standard in the theory of viscosity solutions, to obtain well-posedness of (56), we start with a comparison principle:

**Theorem 6** Fix $T > 0$. If $u$ is a bounded, upper semi-continuous function satisfying $u_t - G(x, Du)\text{tr}(D^2 u) + B(x, u, Du) \leq 0$ in $\mathbb{R}^d \times (0, T)$ and $v$ is a bounded, lower semi-continuous function satisfying $v_t - G(x, Dv)\text{tr}(D^2 v) + B(x, v, Dv) \geq 0$ in $\mathbb{R}^d \times (0, T)$ and if $M$ is defined by

$$
M = \lim_{\delta \to 0^+} \sup \left\{ u^\delta(x, 0) - v^\delta(y, 0) \mid x, y \in \mathbb{R}^d, \|x - y\| \leq \delta \right\},
$$

then

$$
u(x, t) - v(x, t) \leq Me^{Kt} \lor 0 \text{ for all } (x, t) \in \mathbb{R}^d \times (0, T).
$$

**Sketch of the proof** The Lipschitz assumption on $G$ implies that a comparison argument can be carried out in the spirit of [16]. To get the exponential bound, we note that, given $\delta > 0$ and $K$ as in (59), if we define the functions $\tilde{u}^\delta$ and $\tilde{v}^\delta$ by $\tilde{u}^\delta(x, t) = e^{-(K+\delta)t}u(x, t)$ and $\tilde{v}^\delta(x, t) = e^{-(K+\delta)t}v(x, t)$ and write $\tilde{w}^\delta = \tilde{u}^\delta - \tilde{v}^\delta$, then a standard argument shows that $\tilde{w}^\delta$ satisfies $\tilde{w}^\delta \leq M \lor 0$ in $\mathbb{R}^d \times (0, T)$. The result is recovered after sending $\delta \to 0^+$. □

Existence now follows using Perron’s Method and regularization:

**Corollary 1** Given $u_0 \in BUC(\mathbb{R}^d)$, there is a unique, bounded viscosity solution of (56).

**Proof** First, assume that $u_0$ is a bounded function in $\mathbb{R}^2$ with bounded second derivative. It follows that $\tilde{u}(x, t) = u_0(x) + Ct$ and $\underline{u}(x, t) = u_0(x) - Ct$ define super- and subsolutions of (56) provided $C > 0$ is large enough. Thus, an application of Perron’s Method gives a bounded, continuous solution $u$ with $u(\cdot, 0) = u_0$ in $\mathbb{R}^d$.

If $u_0$ is not so regular, then nonetheless we can find a sequence $(u_{0,n})_{n \in \mathbb{N}}$ of functions as above such that $\|u_{0,n} - u_0\|_{L^\infty(\mathbb{R}^d)} \to 0$ as $n \to \infty$. The bound in Theorem 6 implies that the associated solutions $(u_n)_{n \in \mathbb{N}}$ are uniformly Cauchy in $\mathbb{R}^d \times [0, T]$. Therefore, their limit $u = \lim_{n \to \infty} u_n$ exists and, by stability, is a solution of (56). □
Appendix C: Construction of correctors

In this section, we discuss the standing waves and correctors used in the ansatz (19). Throughout we assume that $W$ satisfies satisfies (14), (15), and (16) as in Theorem 1.

C.1 Standing waves

We begin by recalling some standard facts concerning the standing waves of the Allen–Cahn equation. Up to translations, this is the function $q : \mathbb{R} \to (-1, 1)$ such that

$$-\ddot{q} + W'(q) = 0 \quad \text{in} \ \mathbb{R}, \quad \lim_{s \to \pm \infty} q(s) = \pm 1, \quad q(0) = 0. \quad (60)$$

**Proposition 13** There is a unique, strictly increasing $q : \mathbb{R} \to (-1, 1)$ satisfying (60). Further, there is a constant $C > 0$ such that

$$|q(s) - 1| \leq C \exp \left(-\frac{s}{C}\right), \quad |q(s) + 1| \leq C \exp \left(\frac{s}{C}\right).$$

**Proof** (60) has a Hamiltonian structure. In particular, the expression $\frac{1}{2}\dot{q}(s)^2 - W(q(s))$ is independent of $s \in \mathbb{R}$. Since this should clearly be zero at infinity, we deduce that $\frac{1}{2}\dot{q}(0)^2 = W(0)$. Thus, $|\dot{q}(0)|$ is uniquely determined a priori. We solve the ODE $\dot{q} = W'(q)$ in $\mathbb{R}$ with $q(0) = 0$ and $\dot{q}(0) = \sqrt{2W(0)}$, and then an exercise shows that the solution satisfies $\lim_{s \to \pm \infty} q(s) = \pm 1$. Further, the identity $\dot{q}(s) = \sqrt{2W(q(s))}$ implies $q$ is strictly increasing.

The exponential convergence to $\pm 1$ can be proved using a stability analysis or the maximum principle. Here we are using (15). \qed

The standing wave $q$ generates the solutions of (8). More precisely, for each $e \in S^{d-1}$, the function $U_e(s, y) = q(s)$ is a solution of (8) with $a \equiv \text{Id}$. The penalized correctors constructed in Sect. C.3 will be approximate solutions of (9). It will therefore be helpful to know some properties of the principal eigenfunction $\partial_e U_e$, which in this case equals $\dot{q}$.

**Proposition 14** $\dot{q} \in C^{2,\alpha}(\mathbb{R})$ solves the linearized Allen–Cahn equation:

$$-\ddot{q} + W''(q(s))\dot{q} = 0 \quad \text{in} \ \mathbb{R}.$$  

Furthermore, there is a constant $C > 0$ such that $|\dot{q}(s)| \leq C \exp(-C^{-1}|s|)$.

**Proof** The PDE is obtained directly by differentiating $q$. The exponential convergence can be proved using the maximum principle and (15) (cf. [32, Proposition 31]). \qed

Finally, we will need the following fact concerning the drift appearing in the renormalized Allen–Cahn operator:

**Proposition 15** The function $s \mapsto \frac{\dot{q}(s)}{q(s)}$ is bounded and uniformly Lipschitz continuous in $\mathbb{R}$.

**Proof** Applying Schauder estimates to $\dot{q}$, we find a constant $C > 0$ such that

$$|\dot{q}(s)| + |\ddot{q}(s)| \leq C \left(\sup \left\{|\dot{q}(s')| \mid s' \in (s - 1, s + 1)\right\}\right).$$

Further, by the Harnack inequality, there is no loss of generality in assuming that

$$\sup \left\{|\dot{q}(s')| \mid s' \in (s - 1, s + 1)\right\} \leq C\dot{q}(s).$$

Thus, $|\dot{q}(s)| + |\ddot{q}(s)| \leq C^2$. This gives the boundedness of $\frac{\ddot{q}}{q}$ directly and the uniform Lipschitz continuity after differentiation. \qed
C.2 Linearized Allen–Cahn equation

In the construction of approximate correctors, we used the linearized Allen–Cahn operator \(-\partial_s^2 + W''(q(s))\). In particular, the next solvability result was used:

**Proposition 16** If \( f \in C^{\alpha}(\mathbb{R}) \) for some \( \alpha \in (0, 1) \), then there is a unique \( P^f \in C^{2,\alpha}(\mathbb{R}) \) and a unique \( \overline{f} \in \mathbb{R} \) solving the PDE:

\[
-\overline{f} + W''(q(s))P^f = (f(s) - \overline{f})\dot{q}(s) \quad \text{in} \quad \mathbb{R}, \quad P^f(0) = 0.
\]

Furthermore, there is a \( C > 0 \) such that \( |P^f(s)| \leq |\overline{f}| \| L_{\infty}(\mathbb{R}) \exp(-C^{-1}|s|) \).

**Proof** The operator \( \mathcal{L} = -\partial_s^2 + W''(q(s)) \) with domain \( H^2(\mathbb{R}) \) is self-adjoint in \( L^2(\mathbb{R}) \) with closed range. Therefore, \( \text{Ran}(\mathcal{L}) = \text{Ker}(\mathcal{L})^\perp \). Since \( \dot{q} \) is a positive eigenfunction, it follows that, for each \( g \in H^2(\mathbb{R}) \),

\[
\langle \mathcal{L}g, g \rangle_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} \dot{h}(s)^2 \dot{q}(s)^2 \, ds,
\]

where \( h(s) = \dot{q}(s)^{-1}g(s) \). From this, it follows that \( \text{Ker}(\mathcal{L}) = \text{span}[\dot{q}] \).

Thus, if \( f \in C^{\alpha}(\mathbb{R}) \), then \( f\dot{q} \in L^2(\mathbb{R}) \) and there is a \( P^f \in H^2(\mathbb{R}) \) with \( \mathcal{L}P^f = (f - \overline{f})\dot{q} \) provided \( \overline{f} \in \mathbb{R} \) is given by

\[
\overline{f} = c_{\dot{W}}^{-1} \int_{-\infty}^{\infty} f(s)\dot{q}(s)^2 \, ds.
\]

The local maximum principle implies \( P^f \in L^\infty(\mathbb{R}) \) and then Schauder estimates give \( P^f \in C^{2,\alpha}(\mathbb{R}) \). Since \( \dot{q}(s) \to 0 \) exponentially as \( s \to \pm\infty \), a maximum principle argument shows that \( P^f \) does also. \( \square \)

C.3 Penalized correctors

Finally, we prove the existence and regularity of the penalized correctors used in Sect. 4. Recall that these are the solutions \((P^\delta_2)_{\delta>0}\) of the penalized cell problem

\[
\tilde{m}_2(s, y)\dot{q}(s) + \delta P^\delta_2 + \partial_x^\delta D_x P^\delta_2 + W''(q)P^\delta_2 = 0 \quad \text{in} \quad \mathbb{R} \times \mathbb{T}^d. \tag{61}
\]

We search for a solution of the form \( P^\delta_2(s, y) = V^\delta_2(s, y)\dot{q}(s) \), employing the so-called Doob–Legendre transform. Plugging this ansatz into the equation, we find that \( P^\delta_2 \) solves (61) if and only if \( V^\delta_2 \) solves

\[
\tilde{m}_2(s, y) + \delta V^\delta_2 + \partial_x^\delta D_x V^\delta_2 - \frac{2\dot{q}(s)}{\dot{q}(s)}(e, D_x V^\delta_2) = 0 \quad \text{in} \quad \mathbb{R} \times \mathbb{T}^d. \tag{62}
\]

It is clear that (62) has a solution since it is degenerate elliptic and \( \tilde{m}_2 \) is bounded. However, we will not proceed in this way.

Instead, notice that \( V^\delta_2 \) is a classical solution of (62) if and only if the one-parameter family of functions \((v^\delta_\zeta)_{\zeta \in \mathbb{R}}\) determined by

\[
v^\delta_\zeta(x) = V^\delta_2((x, e) - \zeta, x)
\]

gives rise to solutions of the following family of equations:

\[
\tilde{m}_2((x, e) - \zeta, x) + \delta v^\delta_\zeta - \Delta v^\delta_\zeta - \frac{2\dot{q}((x, e) - \zeta)}{\dot{q}((x, e) - \zeta)}(e, D v^\delta_\zeta) = 0 \quad \text{in} \quad \mathbb{R}^d. \tag{63}
\]
Proof of Theorem 7

If \( \tilde{m} \) satisfies (29), then, for each \( \delta > 0 \) and \( \zeta \in \mathbb{R} \), there is a \( v^\delta \in C^{2,\alpha}(\mathbb{R}^d) \) solving (63). Furthermore, the map \( (\zeta, x) \mapsto v^\delta(x) \) is twice continuously differentiable with respect to \( \zeta \) and there is a constant \( C > 0 \) independent of \( \delta \) such that

\[
\| v^\delta \|_{L^\infty(\mathbb{R}^d)} + \| \frac{\partial v^\delta}{\partial \zeta} \|_{C^{1,\alpha}(\mathbb{R}^d)} + \| \frac{\partial^2 v^\delta}{\partial \zeta^2} \|_{C^\alpha(\mathbb{R}^d)} \leq C(1 + \delta^{-1}).
\]

This leads immediately to a regularity result for \( V^\delta \):

Corollary 2

If \( \tilde{m} \) satisfies (29), then the unique viscosity solution \( V^\delta_2 \) of (62) is in \( C^{2,\mu}(\mathbb{R} \times \mathbb{T}^d) \) and there is a \( \delta \)-independent constant \( C > 0 \) depending only on \( f \) such that

\[
\| V^\delta_2 \|_{C^{2,\alpha}(\mathbb{R} \times \mathbb{T}^d)} \leq C(1 + \delta^{-1}).
\]

Furthermore, making \( C \) larger if necessary, we also have:

\[
\| P^\delta_2 \|_{L^\infty(\mathbb{T}^d)} + \| \partial_s P^\delta_2 \|_{L^\infty(\mathbb{T}^d)} \leq C(1 + \delta^{-1}) \exp(-C^{-1}|s|).
\]

Proof of Theorem 7

Since the drift \( \tilde{q} \) is bounded and uniformly Lipschitz continuous by Proposition 15, (63) has a unique solution \( v^\delta \in C^{2,\mu}(\mathbb{R}^d) \) and Schauder estimates give

\[
\| v^\delta \|_{L^\infty(\mathbb{R}^d)} \leq \| f \|_{L^\infty(\mathbb{R} \times \mathbb{T}^d)} \delta^{-1},
\]

\[
\| v^\delta \|_{C^{2,\mu}(\mathbb{R}^d)} \leq C' \left( \| v^\delta \|_{L^\infty(\mathbb{R}^d)} + \| \tilde{m} \|_{C^\mu(\mathbb{R} \times \mathbb{T}^d)} \right) \leq C' \| \tilde{m} \|_{C^\alpha(\mathbb{R} \times \mathbb{T}^d)} (1 + \delta^{-1}).
\]

By uniqueness, it is easy to see that \( \zeta \mapsto v^\delta \) is continuous with respect to the topology of local uniform convergence.

Recall the functions \( (\tilde{v}_\zeta)_{\zeta \in \mathbb{R}} \) defined in (34) by translation. As pointed out above, these functions satisfy (35). Thus, employing the method of difference quotients, we deduce that the functions \( (w_\zeta)_{\zeta \in \mathbb{R}} \) defined by \( w_\zeta = \frac{\partial^2 q}{\partial \zeta^2} \) satisfy

\[
(D_y m)((x, e), x + \zeta e), e) + \delta w_\zeta - \Delta w_\zeta + \frac{2\tilde{q}((x, e))}{\tilde{q}'((x, e))}(e, D w_\zeta) = 0 \quad \text{in} \ \mathbb{R}^d.
\]

Similarly, the functions \( (p_\zeta)_{\zeta \in \mathbb{R}} \) given by \( p_\zeta = \frac{\partial^2 q}{\partial \zeta^2} \) satisfy

\[
(D_y^2 m)((x, e), x + \zeta e), e) + \delta p_\zeta - \Delta p_\zeta + \frac{2\tilde{q}((x, e))}{\tilde{q}'((x, e))}(e, D p_\zeta) = 0 \quad \text{in} \ \mathbb{R}^d.
\]

Thus, since \( D_y m \) and \( D_y^2 m \) are just as regular as \( \tilde{m} \), there is a \( C > 0 \) such that

\[
\left\| \frac{\partial \tilde{v}_\zeta}{\partial \zeta} \right\|_{C^{2,\alpha}(\mathbb{R}^d)} + \left\| \frac{\partial^2 \tilde{v}_\zeta}{\partial \zeta^2} \right\|_{C^\alpha(\mathbb{R}^d)} \leq C(1 + \delta^{-1}).
\]

Furthermore, if \( \zeta, \zeta' \in \mathbb{R} \), then the Hölder regularity of \( D_y^2 m \) yields

\[
\left\| \frac{\partial^2 \tilde{v}_\zeta}{\partial \zeta^2} - \frac{\partial^2 \tilde{v}_{\zeta'}}{\partial \zeta'^2} \right\|_{L^\infty(\mathbb{R}^d)} \leq \delta^{-1} \| D_y^2 m \|_{C^\alpha(\mathbb{R} \times \mathbb{T}^d)} |\zeta - \zeta'|^\nu.
\]

These bounds readily carry over to \( (v_\zeta)_{\zeta \in \mathbb{R}} \), giving the desired estimates. \( \square \)
We proceed with the proof of Corollary 2:

**Proof of Corollary 2** Define \( V_2^\delta : \mathbb{R} \times \mathbb{T}^d \to \mathbb{R} \) by
\[
V_2^\delta (s, y) = v_{(y,e)}^{(y,e)} s(y).
\]

An exercise shows this is well-defined. Differentiating, we eventually find \( V_2^\delta \in C^{2,\alpha} (\mathbb{R} \times \mathbb{T}^d) \). A calculus exercise shows that \( V_2^\delta \) is a solution of (62).

The exponential convergence of \( P_2^\delta \) follows from a maximum principle argument as in [32, Proof of Proposition 30]. Using the fact that \( \partial_s P_2^\delta \) satisfies a structurally similar linear PDE, we obtain a similar exponential estimate on \( \partial_s P_2^\delta \).

\( \Box \)

**Appendix D: Formal asymptotics**

In this appendix, we review the approach of [9] as it pertains to (1). The aim is to show, in particular, how the assumptions made in that work arise naturally from a formal asymptotic expansion. While [9] only treats the case when \( m \equiv 1 \), it should be stressed the approach adapts readily to the case of a general \( m \).

The key assumption in the analysis of (1) in [9] is the existence of smooth families of pulsating waves and correctors. Since there are currently no known examples of non-constant coefficients \((a, m)\) for which this assumption holds, we will only present the formal asymptotics that underlie the arguments rather than a rigorous proof.

**D.1 Preliminaries**

By analogy with the well-studied case where \((a, m) \equiv (\text{Id}, 1)\), we expect that there are open sets \( \{E_t\}_{t \geq 0} \) such that
\[
\{u^\epsilon(\cdot, t) \approx 1\} \to E_t \quad \text{and} \quad \{u^\epsilon(\cdot, t) \approx -1\} \to \mathbb{R}^d \setminus E_t \quad \text{as} \quad \epsilon \to 0^+.
\]

The question is the identification of \( \{E_t\}_{t \geq 0} \). Let us hypothesize that it is governed by a geometric flow of the following form:
\[
\bar{M}^{(a,m)}(n_{\partial E_t}) V_{\partial E_t} = \text{tr} (\bar{S}^a (n_{\partial E_t}) A_{\partial E_t}) \quad (64).
\]

Here \( \bar{S}^a \) and \( \bar{M}^{(a,m)} \) are effective coefficients, which we expect to appear in the sharp interface limit due to averaging. The formal arguments below suggest that the influence of \( m \) only appears in the effective mobility \( \bar{M}^{(a,m)} \), hence the notation.

Since we are arguing formally, we will assume the sets \( \{E_t\}_{t \geq 0} \) have smooth boundaries, which vary smoothly as functions of \( t \).

To relate the solution \( u^\epsilon \) of (1) to the macroscopic interface \( \partial E_t \), we will use the signed distance function \( d : \mathbb{R}^d \times (0, \infty) \to \mathbb{R} \) to \( \{E_t\}_{t \geq 0} \), defined by
\[
d(x, t) = \begin{cases} 
\text{dist}(x, \partial E_t), & \text{if} \ x \in E_t, \\
-\text{dist}(x, \partial E_t), & \text{otherwise}.
\end{cases}
\]

Note that the smoothness of the evolution \( \{E_t\}_{t \geq 0} \) implies the smoothness of \( d \) locally near points on the interface, and (64) holds if and only if \( d \) satisfies
\[
\bar{M}^{(a,m)}(Dd)_{\partial E_t} - \text{tr} (\bar{S}^a (Dd) D^2 d) = 0 \quad \text{on} \quad \bigcup_{t \geq 0} \partial E_t \times \{t\}. \quad (65)
\]
D.2 Asymptotic expansion

Let us rewrite the ansatz (7) already introduced in the introduction in a more suggestive form:

$$
u (x, t) = u_D (x, t, \epsilon) + \epsilon Q (x, t) \frac{d}{dt} (x, t) + \epsilon^2 P (x, t) \frac{d}{dt} (x, t) + \cdots$$

Here \( \{ Q \} \) and \( \{ P \} \) are functions to be determined.

Next, we set \( d = \) the dependence of these functions on the parameters \( (e, X, q) \).

For the ansatz (66) to produce a solution of (1), we require that

$$0 = m \epsilon^{-1} (x, e) \partial_t u - \text{div} (a (x, e) \partial_t u) + \epsilon^{-2} W (u) = \epsilon^{-2} A_1 + \epsilon^{-1} A_2 + \cdots$$

where the neglected terms are of lower order in \( \epsilon \).

D.3 Vanishing to order \( \epsilon^{-2} \)

Setting \( A_1 = 0 \) and substituting \( y = \epsilon^{-1} x \) and \( e = Dd (x, t) \) leads to the following equations for \( \{ U \} \):

$$D^* (a (y) D U) + W (U) = 0 \quad \text{in} \quad \mathbb{R} \times \mathbb{T}^d.$$  

At the same time, away from the interface \( \partial E_t \), we know that \( u \approx 1 \) in \( E_t \) and \( u \approx -1 \) outside. That suggests the limiting condition

$$\lim_{s \to \pm \infty} U s, y = \pm 1.$$  

Finally, it is convenient to add the monotonicity assumption \( \partial_t U \neq 0 \).

Next, we set \( A_2 = 0 \) and proceed similarly. At this stage, derivatives of the map \( e \mapsto U_e \) appear. In order to avoid calculus on manifolds, it is convenient to extend \( \{ U \} \) to \( \{ U \} \) according to the rule (cf. \[ 32, \text{Remark 2} \])

$$U_v (s, y) = U_e (\| v \|, y) \quad \text{for} \quad e = \frac{v}{\| v \|}.$$  

Next, to ease the notation, we define the vector-valued functions \( \{ R \} \) to be the derivative of \( v \mapsto U_v \), hence

$$\langle R_v (s, y), \xi \rangle = \lim_{h \to 0} \frac{U_{v+h \xi} (s, y) - U_v (s, y)}{h} \quad \text{for} \quad (s, y) \in \mathbb{R} \times \mathbb{T}^d, \quad \xi \in \mathbb{R}^d.$$  

D.4 Vanishing to order \( \epsilon^{-1} \)

We now proceed to investigate the consequences of the identity \( A_2 = 0 \). Making the substitutions

$$y = \epsilon^{-1} x, \quad e = Dd (x, t), \quad X = D^2 d (x, t), \quad q = d_t (x, t).$$
we derive the equation
\[
D_e^*(a(y)D_e(Q_e^X + P_e^q)) + W'(U_e)(Q_e^X + P_e^q) = G(s, y, e, X, q) \quad \text{in } \mathbb{R} \times \mathbb{T}^d,
\]
where \(G\) is given by
\[
G(s, y, e, X, q) = G_1(s, y, e, X) - G_2(s, y, e, q),
\]
\[
G_1(s, y, e, X) = \text{tr} (a(y)X) \partial_s U_e + 2(a(y)e, e \partial_s R_e(s, y)) + 2\text{tr}(a(y)D_3R_e(s, y)X) + ((\text{div} a)(y), XR_e(s, y)),
\]
\[
G_2(s, y, e, q) = qm(y, D_eU_e)\partial_s U_e.
\]

The question now is the solvability of the linear equation (70).

Here is where \(\tilde{M}^{(a,m)}\) and \(\tilde{S}^a\) come into the picture. There is a natural solvability condition associated with equations of the form
\[
D_e^*(a(y)D_eP) = F(s, y) \quad \text{in } \mathbb{R} \times \mathbb{T}^d.
\]

To see this, first, notice that differentiating (68) with respect to \(s\) shows that the function \(V_e := \partial_s U_e\) solves the linear PDE
\[
D_e^*(a(y)D_eV_e) + W''(U_e)V_e = 0 \quad \text{in } \mathbb{R} \times \mathbb{T}^d.
\]

Hence, multiplying the previous equation by \(V_e\) and integrating by parts, we obtain
\[
0 = \int_{\mathbb{R} \times \mathbb{T}^d} (D_e^*(a(y)D_eV_e) + W''(U_e)V_e) P \, dy \, ds = \int_{\mathbb{R} \times \mathbb{T}^d} F(s, y)V_e \, dy \, ds.
\]

Due to this solvability condition, we are led to the following equations for \(Q_e^X\) and \(P_e^q\):
\[
D_e^*(a(y)D_eQ_e^X) + W''(U_e)Q_e^X = G_1(s, y, e, X) - \bar{G}_1(e, X)\partial_s U_e \quad \text{in } \mathbb{R} \times \mathbb{T}^d,
\]
\[
D_e^*(a(y)D_eP_e^q) + W''(U_e)P_e^q = -\left[ G_2(s, y, e, q) - \bar{G}_2(e, q)\partial_s U_e \right] \quad \text{in } \mathbb{R} \times \mathbb{T}^d.
\]

Notice these equations are of the form (9). Invoking the solvability condition, we compute the necessary equations for \(\bar{G}_1\) and \(\bar{G}_2\):
\[
\bar{G}_1(e, X) = \|V_e\|_{L^2(\mathbb{R} \times \mathbb{T}^d)}^2 \int_{\mathbb{R} \times \mathbb{T}^d} V_e \left( \text{tr}(a(y)X) + 2(a(y)e, X \partial_s R_e) + 2\text{tr}(a(y)D_3R_eX) + ((\text{div} a)(y), XR_e) \right) \, dy \, ds,
\]
\[
\bar{G}_2(e, q) = q\|V_e\|_{L^2(\mathbb{R} \times \mathbb{T}^d)}^{-2} \int_{\mathbb{R} \times \mathbb{T}^d} m(y, D_eU_e)|V_e|^2 \, dy \, ds.
\]

By linearity in \(X\) and \(q\), we can fix a symmetric matrix-valued function \(\tilde{S}^a\) and a positive function \(\tilde{M}^{(a,m)}\) such that
\[
\text{tr}(\tilde{S}^a(e)X) = \bar{G}_1(e, X)\|V_e\|_{L^2(\mathbb{R} \times \mathbb{T}^d)}^2, \quad \tilde{M}^{(a,m)}(e)q = \bar{G}_2(e, q)\|V_e\|_{L^2(\mathbb{R} \times \mathbb{T}^d)}^2.
\]

Finally, in order for the sum \(P_e^X + P_e^q\) to solve (70), we require that \(\bar{G}_1(e, X) - \bar{G}_2(e, q) = 0\), which, rewritten in terms of \(d\), yields the equation
\[
\tilde{M}^{(a,m)}(Dd(x, t))d_t(x, t) - \text{tr}(\tilde{S}^a(Dd(x, t)))D^2d(x, t) = 0,
\]
which is precisely the PDE we sought to derive.
**Remark 5** As shown in [9], it is possible to build a proof from the previous arguments provided that solutions of (68), (71), and (72) do, in fact, exist and are regular enough as functions of \( (s, y, e, X, q) \). (See also Theorem 1 above and [32, Theorem 8] for rigorous proofs in special cases.) Of course, as mentioned already in the introduction, in general, existence and smoothness need not hold.

**D.5 Interpretation of \( \tilde{S}^a \)**

It is worth noting that the coefficient \( \tilde{S}^a \) computed above has a variational interpretation.

Recall that the equation (1) is the gradient flow of the energy \( \mathcal{F}^a \) given by (5) with respect to the metric determined by \( m \). The energy \( \mathcal{F}^a \) has been studied in its own right. More precisely, in [2], Ansini, Braides, and Chiadò Piat prove that the large scale behavior of \( \mathcal{F}^a \) is an anisotropic perimeter functional \( \tilde{F}^a \) of the form

\[
\tilde{F}^a(F; \Omega) = \int_{\partial F \cap \Omega} \tilde{\sigma}(n_{\partial E}(\xi)) \mathcal{H}^{d-1}(d\xi).
\]

(See also [17, 18].) We refer to \( \tilde{\sigma} \) as the surface tension.

It has been shown in [32, Proposition 1] that if \( U_e \) is a smooth solution of (68) with \( \lim_{s \to \pm \infty} U_e(s, y) = \pm 1 \) and \( \partial_s U_e \geq 0 \), then

\[
\tilde{\sigma}(e) = \int_{\mathbb{R}^d} \left( \frac{1}{2} \langle a(y) D_e U_e, D_e U_e \rangle + W(U_e) \right) dy ds.
\]

Furthermore, \( U_e \) generates a continuous one-parameter family of minimizers of \( \mathcal{F}^a \), the graphs of which foliate \( \mathbb{R}^d \times (-1, 1) \). This fact is precisely the obstruction to the existence of smooth solutions of (68) alluded to in the introduction: [32] provides examples where such foliations do not exist (cf. [24]).

It turns out that the surface tension reappears in the macroscopic velocity law \( \tilde{M}^{(a, m)}(n)V = \text{tr}(\tilde{S}^a(n)A) \). Specifically, in the (smooth) context of the preceding discussion, it is possible to prove \( \tilde{\sigma} \) is smooth away from the origin and \( \tilde{S}^a \) is given by

\[
\tilde{S}^a(e) = D^2\tilde{\sigma}(e).
\]

Indeed, as long as everything is smooth, this follows from manipulations of the integral representation of \( \tilde{S}^a \) computed above, exactly as in [32, Proposition 39].

The appearance of the surface tension in (73) shows that, at least according to this formal analysis, the gradient flow and homogenization “commute.” For more on this, see [32] and the references therein.

**D.6 Special case \( a \equiv \text{Id} \)**

Let us show, for the sake of completeness, that the formal derivation above coincides with the approach taken in the paper when \( a \) is constant.

When \( a \) is constant, say, \( a \equiv \text{Id} \), one can prove that the only possible choices of \( \{U_e\}_{e \in S^{d-1}} \) that depend continuously on \( e \) are necessarily given by

\[
U_e(s, y) = q(s + s_0(e)),
\]

where, as above, \( q \) is the standing wave solution of the spatially homogeneous Allen–Cahn equation with \( q(0) = 0 \) and \( s_0 : S^{d-1} \to \mathbb{R} \) is an arbitrary continuous function.
This claim follows from [32, Sections 6.3–6.5], the key point being that solutions of (68) inherit symmetry from $a$ when $e \notin \mathbb{R}Z^d$.

The choice of $s_0$ is irrelevant in what follows. Hence let us set $s_0 \equiv 0$. Notice this implies $U_e$ does not depend on $e$ and, thus, a direct computation shows that the functions $\{R_e\}$ of (69) are given by

$$R_e(s, y) = s \dot{q}(s)e.$$ 

Using the shorthand $c_W = \int_{-\infty}^{\infty} \dot{q}(s)^2 \, ds$, let us compute $\tilde{S}^{\text{ld}}$:

$$\langle \tilde{S}^{\text{ld}}(e)v, v \rangle = \text{tr}(\tilde{S}^{\text{ld}}(e)(v \otimes v))$$

$$= \text{tr}(v \otimes v) \int_{-\infty}^{\infty} \dot{q}(s)^2 \, ds + 2|\langle e, v \rangle|^2 \int_{-\infty}^{\infty} \frac{d}{ds}[\dot{s}(s)] \, ds$$

$$= c_W \|v\|^2.$$ 

This proves $\tilde{S}^{\text{ld}}(e) = c_W \text{Id}$. We similarly compute $\tilde{M}^{(\text{ld}, m)}$:

$$\tilde{M}^{(\text{ld}, m)}(e) = \int_{\mathbb{R} \times \mathbb{T}^d} m(y, \dot{q}(s)e) \dot{q}(s)^2 \, dy \, ds.$$ 

In particular, $\tilde{M}^{(\text{ld}, m)} = c_W \bar{m}$, where $\bar{m}$ is defined in (17). Thus, after factoring out $c_W$, the limiting motion is precisely the one obtained in Theorem 1.

Finally, let us show how one arrives at the simplified ansatz (19) starting from (66). To start with, notice that the function $G_2$ in the cell problem (71) depends linearly on $q$. Thus, if $P^q_e$ is a solution when $q = 1$, then $P^q_e \equiv q P^1_e$ defines a solution for all $q \in \mathbb{R}$. This gives rise to the the linear dependence on the time derivative $d_t$ in (19).

Where $G_1$ is concerned, recall that since $d$ is a (signed) distance function, the following identity holds at any point $(x, t)$ where $d$ is smooth:

$$D^2 d(x, t) D d(x, t) = 0.$$ 

Hence $G_1(s, y, Dd(x, t), D^2d(x, t)) = \text{tr}(a(y)X_e(s, y))$ at points $(x, t)$ near the interface. Our computation above shows that $G_1(e, X) = \text{tr}(a(y)X)$, and, therefore, for the values of $(e, X) = (Dd(x, t), D^2d(x, t))$ appearing in (66), the cell problem (71) for $Q^X_e$ is the trivial equation $D^2 e(a(y)D e Q^X_e) + W''(U_e)Q^X_e = 0$. Of course, the zero function $Q^X_e \equiv 0$ gives a solution of this PDE, and, thus, the term $Q^X_e$ can be dropped from the ansatz (66).

Combining all these observations, the expansion in (66) simplifies to the following

$$u^e(x, t) = q \left( \frac{d(x, t)}{\epsilon} \right) + \epsilon d_t(x, t) P^1_e(\epsilon^{-1} d(x, t), \epsilon^{-1} x) + \cdots,$$

which is precisely the form proposed in (19).

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