About lifespan and the continuous dependence for the Navier-Stokes equation in $\dot{B}_{p,r}^{d-1}$

Weikui Ye$^1$*, Wei Luo$^1$† and Zhaoyang Yin$^{1,2}$‡

1Department of Mathematics, Sun Yat-sen University, Guangzhou, 510275, China
2Faculty of Information Technology, Macau University of Science and Technology, Macau, China

Abstract

In this paper, we mainly investigate the Cauchy problem for the Navier-Stokes (NS) equation. We first establish the local existence in the Besov space $\dot{B}_{p,r}^{d-1}$ with $1 \leq r, p < \infty$. We give a lower bound of the lifespan $T$ which depends on the norm of the Littlewood-Paley decomposition of the initial data $u_0$. Then we prove that if the initial data $u_0^n \to u_0$ in $\dot{B}_{p,r}^{d-1}$, then the corresponding lifespan satisfies $T_n \to T$, which implies that the common lower bound of the lifespan. Finally, we prove that the data-to-solutions map is continuous in $\dot{B}_{p,r}^{d-1}$. So the solutions of Navier-Stokes equation are well-posedness (existence, uniqueness and continuous dependence) in the Hadamard sense. Combining [2, 7, 14], we deduce that $\dot{B}_{p,\infty}^{d-1}$ with $1 \leq p < \infty$ is the critical space which solutions are ill-posedness, while $u \in \dot{B}_{p,r}^{d-1}$ with $1 \leq r, p < \infty$ are well-posedness. Moreover, if we choose the initial data in a subset $\dot{B}_{p,\infty}^{d-1}$ of $\dot{B}_{p,\infty}^{d-1}$, we can obtain the well-posedness of the solutions.

Keywords: The Navier-Stokes equation, Lifespan, Continuous dependence, Well-posedness.
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*e-mail: 904817751@qq.com
†email: luowei23@mail2.sysu.edu.cn
‡email: mcsyzy@mail.sysu.edu.cn
1 Introduction

In this paper, we mainly investigate the initial value problem of the Navier-Stokes (NS) equation.

\[
\begin{align*}
\begin{cases}
u t - \Delta u + u \nabla u + \nabla P &= 0, \\
\text{div} u &= 0 \\
|u|_{t=0} &= u_0.
\end{cases}
\tag{1.1}
\end{align*}
\]

where the unknowns are the vector fields \( u = (u_1, u_2, \ldots, u_d) \) and the scalar function \( P \). Here, \( u \) are the velocity, respectively, while \( P \) denotes the pressure.

In the seminal paper [17], Leray proved the global existence of finite energy weak solutions to the (NS) equation. Yet the uniqueness and regularity of such weak solutions are big open questions in the field of mathematical fluid mechanics except the case when the initial data have special structure. For instance, with axi-symmetric initial velocity and without swirl component, Ladyzhenskaya [16] deduced that all the global solutions in \( \dot{B}^{\frac{d}{p}, 1}_{\infty} \) are stable, which implies the stability of the system. Ladyzhenskaya [16] and independently Ukhovskii and Yudovich [18] proved the existence of weak solution along with the uniqueness and regularity of such solution to the (NS) equation. When the initial data \( u_0 \) has a slow space variable, Chemin and Gallagher [5, 7] can also prove the global well-posedness of such a system.

In the nonhomogeneous Besov space, Z Guo, J Li and Z Yin in [13] proved the uniform continuous dependence and the inviscid limit of the (NS) equation with \( \dot{B}^{s+1}_{p,r} \), \( s > \frac{d}{p} + 1 \) or \( s = \frac{d}{p} + 1, r = 1 \). In the homogeneous Sobolev space, Fujita and Kato [11] proved the global well-posedness of (NS) when the initial data \( u_0 \) is sufficiently small in \( H^\frac{d}{2} \). This result was generalized by Cannone, Meyer and Planchon [2] for initial data being sufficiently small in the homogeneous Besov space \( \dot{B}^{s}_{p,\infty} \) with \( p \in (3, \infty) \). The end-point result in this direction is due to Koch and Tataru [12], where they proved the global well-posedness of the (NS) equation with initial data being sufficiently small in BMO\(^{-1}\). Moreover, Bourgain and Pavlović [2] proved that the (NS) equation is actually ill-posed with initial data in \( \dot{B}^{1}_{\infty,r} \). Germain in [7] proves an instability result for (1.1) in \( \dot{B}_{r,\infty}^{1} \) with \( r > 2 \) by showing that the map from the initial data to the solution is not in the class \( C^2 \). Recently, Wang [14] proved that ill-posedness for (1.1) in \( \dot{B}^{1}_{\infty,r} \) with \( 1 \leq r \leq 2 \). Up to now, the ill-posedness for the (NS) equation in \( \dot{B}^{1}_{\infty,r} \) (\( 1 \leq r \leq \infty \)) has been studied. In [8], Cheskidov and Shvydkoy proved the non-continuity for (1.1) in \( \dot{B}^{1}_{\infty,\infty} \) for a special initial data in \( \dot{B}^{d-1}_{p,\infty} \). Since \( \dot{B}^{d-1}_{p,\infty} (p < d) \hookrightarrow \dot{B}^{d-1}_{p,\infty} \hookrightarrow \dot{B}^{1}_{\infty,\infty} \hookrightarrow \dot{B}^{1}_{\infty,\infty} \), this implies some ill-posedness in \( \dot{B}^{d-1}_{p,\infty} \). So \( \dot{B}^{d-1}_{p,\infty} \) seems to be the critical homogeneous Besov space in which (1.1) is ill-posed when \( 1 \leq p < \infty \).

However, On the other hand, in [14] Wang proposed that the largest homogeneous Besov space on the general initial data for which (1.1) is well-posed (existence, uniqueness and continuous dependence) is still unknown, especially for the continuous dependence. As we know, we have

\[ \dot{B}^{d-1}_{p,r} (1 \leq p, r < \infty) \hookrightarrow \dot{B}^{1}_{p,\infty} (1 \leq p < \infty) \hookrightarrow \text{BMO}^{-1} \hookrightarrow \dot{B}^{1}_{\infty,\infty}. \]

In this paper, our aim is to solve this problem by establishing the local well-posedness of the solution for the Cauchy problem (1.1) in \( \dot{B}^{d-1}_{p,r} (1 \leq p, r < \infty) \). We deduce that \( \dot{B}^{d-1}_{p,\infty} \) with \( 1 \leq p < \infty \) is the critical space which solutions are ill-posedness, while \( u \in \dot{B}^{d-1}_{p,r} \) with \( 1 \leq r, p < \infty \) are well-posedness.

If we choose the initial data in a subset \( \dot{B}^{d-1}_{p,\infty} \) of \( \dot{B}^{d-1}_{p,r} \), we can also obtain the well-posedness of the solutions. This improves considerably the recent results in [4]. Moreover, by this result, it’s easy to deduce that all the global solutions in \( \dot{B}^{d-1}_{p,r} \) (\( 1 \leq p, r < \infty \)) are stable, which implies the stability results in [6][10] for the general initial data.

For convenience, we transform the system (1.1) into an equivalent form of incompressible type. By using \( \text{div} u = 0 \), we have

\[ u \nabla u = \text{div}(u \otimes u). \]
Therefore, the system (1.1) is formally equivalent to the following equations

\[
\begin{align*}
\begin{cases}
  u_t - \Delta u = - \mathbb{P}(u \nabla u), \\
  \text{div} u = 0, \\
  u|t = 0 = u_0.
\end{cases}
\end{align*}
\]

where \( \mathbb{P} = I + \nabla(-\Delta)^{-1} \text{div} \), \( \text{div} u_0 = 0 \).

In fact, the main difficulty is that the system is parabolic, which implies it's hard to get the lower bound of the lifespan \( T \). In this paper, we would like to present a general functional framework to deal with the local existence of the solutions of (1.1) in the homogeneous Besov spaces. By obtaining the expression of the lifespan, we get the uniformly lower bound of the lifespan by constructing a \( T^* \rightarrow T \) (see the key Lemma 4.1 below). Finally, we prove the continuous dependence in the Besov homogeneous space.

Our paper are as follow.

In section 2, we give some useful preliminaries. In section 3, we prove the local existence and the uniqueness of (1.2) with the expression of local time is given. In section 4, we firstly prove that if the initial data \( u_0 \) tends to \( u_0 \) in \( \dot{B}^{d-1}_{p,1}(\mathbb{R}^d) \), then their local existence times satisfy \( T_n \rightarrow T \), which implies that they have public lower bound of the lifespan \( T - \delta \) with \( n \) is sufficient large. Finally, we obtain the continuous dependence in the critical Besov space.

There are the main theorems:

**Theorem 1.1.** Let \( u_0 \in \dot{B}^{d-1}_{p,1}(\mathbb{R}^d) \) with \( r, p \in [1, \infty) \). Then there exists a positive time \( T \) such that the solutions of (1.2) are local well-posedness in \( E^p_T \) with

\[
E^p_T \colon= C([0, T]; \dot{B}^{d-1}_{p,r}(\mathbb{R}^d)) \cap L^1([0, T]; \dot{B}^{d+1}_{p,r}(\mathbb{R}^d))
\]

in the Hadamard sense. If the initial data is small enough (\( E_0 \leq \frac{1}{16c_1} \) in (3.10)), then the solutions of (1.2) are global well-posedness.

Since the solutions of \( \dot{B}^{d-1}_{p,\infty} \) with \( 1 \leq p < \infty \) may be ill-posedness, if we consider the initial data in a subset of \( \dot{B}^{d-1}_{p,\infty} \), we can obtain the well-posedness . Setting

\[
\dot{E} \colon= C_w([0, T]; \dot{B}^{d-1}_{p,r}(\mathbb{R}^d)) \cap L^1([0, T]; \dot{B}^{d+1}_{p,r}(\mathbb{R}^d))
\]

and

\[
\dot{B}^{d-1}_{p,\infty}(\mathbb{R}^d) \colon= \{ f \in \dot{B}^{d-1}_{p,\infty}(\mathbb{R}^d) \mid \exists j_0 \text{ s.t. } \sup_{|j| \leq j_0} 2^{|j|-1} \| \hat{f} \|_{L^p} \leq \frac{a}{4} \},
\]

where \( a \colon= \min\{ \frac{1}{16c_1} \frac{1}{\| \cdot \|_{\dot{B}^{d-1}_{p,\infty}}} , \frac{1}{4c_1} \} \), \( C_1 \) is the constant in Lemma 2.5 we can choose it bigger if necessary.

**Theorem 1.2.** Let \( u_0 \in \dot{B}^{d-1}_{p,\infty}(\mathbb{R}^d) \) with \( p \in [1, \infty) \). Then there exists a positive time \( T \) such that

(1) If \( \| u_0 \|_{\dot{B}^{d-1}_{p,\infty}} \leq \frac{1}{16c_1} \), then the solutions of (1.2) are global well-posedness in \( \dot{E}^p_T \) in the Hadamard sense.

(2) If \( u_0 \in \dot{B}^{d-1}_{p,\infty}(\mathbb{R}^d) \), then the solutions of (1.2) are local well-posedness in \( \dot{E}^p_T \) in the Hadamard sense.

**Notations:** Throughout, we donate \( \dot{B}^s_{p,r}(\mathbb{R}^d) = \dot{B}^s_{p,r}(\mathbb{R}^d) \), \( \| u \|_{\dot{B}^s_{p,r}(\mathbb{R}^d)} + \| v \|_{\dot{B}^s_{p,r}(\mathbb{R}^d)} = \| u, v \|_{\dot{B}^s_{p,r}} \) and \( C([0, T]; \dot{B}^s_{p,r}(\mathbb{R}^d)) = \dot{C}(\dot{B}^s_{p,r}), L_p([0, T]; \dot{B}^s_{p,r}(\mathbb{R}^d)) = L^p_T(\dot{B}^s_{p,r}). \)
2 Preliminaries

In this section, we will recall some propositions and lemmas on the Littlewood-Paley decomposition and Besov spaces.

Proposition 2.1. Let $C$ be the annulus $\{ \xi \in \mathbb{R}^d : \frac{1}{2} \leq |\xi| \leq 2 \}$. There exist radial functions $\chi$ and $\varphi$, valued in the interval $[0,1]$, belonging respectively to $\mathcal{D}(B(0,\frac{3}{2}))$ and $\mathcal{D}(C)$, and such that

$$\forall \xi \in \mathbb{R}^d, \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1,$$

$$\forall \xi \in \mathbb{R}^d \setminus \{0\}, \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1,$$

$$|j - j'| \geq 2 \Rightarrow \text{Supp } \varphi(2^{-j}.) \cap \text{Supp } \varphi(2^{-j'}.) = \emptyset,$$

$$j \geq 1 \Rightarrow \text{Supp } \chi(.) \cap \text{Supp } \varphi(2^{-j}.) = \emptyset.$$

The set $\tilde{C} = B(0,\frac{3}{2}) + C$ is an annulus, and we have

$$|j - j'| \geq 5 \Rightarrow 2^j C \cap 2^{j'} \tilde{C} = \emptyset.$$

Further, we have

$$\forall \xi \in \mathbb{R}^d, \frac{1}{2} \leq \chi^2(\xi) + \sum_{j \geq 0} \varphi^2(2^{-j}\xi) \leq 1,$$

$$\forall \xi \in \mathbb{R}^d \setminus \{0\}, \frac{1}{2} \leq \sum_{j \in \mathbb{Z}} \varphi^2(2^{-j}\xi) \leq 1.$$

Definition 2.2. Denote $\mathcal{F}$ by the Fourier transform and $\mathcal{F}^{-1}$ by its inverse. Let $u$ be a tempered distribution in $S'(\mathbb{R}^d)$. For all $j \in \mathbb{Z}$, define

$$\Delta_j u = 0 \text{ if } j \leq -2, \quad \Delta_{-1} u = \mathcal{F}^{-1}(\chi \mathcal{F} u), \quad \Delta_j u = \mathcal{F}^{-1}(\varphi(2^{-j}.) \mathcal{F} u) \text{ if } j \geq 0, \quad S_j u = \sum_{j' < j} \Delta_{j'} u.$$

Then the Littlewood-Paley decomposition is given as follows:

$$u = \sum_{j \in \mathbb{Z}} \Delta_j u \text{ in } S'(\mathbb{R}^d).$$

Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$. The nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^d)$ is defined by

$$B^s_{p,r} = B^s_{p,r}(\mathbb{R}^d) = \{ u \in S'(\mathbb{R}^d) : \| u \|_{B^s_{p,r}(\mathbb{R}^d)} = \| (2^j \| \Delta_j u \|_{L^r}) \|_{l^p(\mathbb{Z})} < \infty \}.$$ 

Similarly, we can define the homogeneous Besov space.

$$\dot{B}^s_{p,r} = \dot{B}^s_{p,r}(\mathbb{R}^d) := \{ u \in S'_h(\mathbb{R}^d) : \| u \|_{\dot{B}^s_{p,r}(\mathbb{R}^d)} := \| 2^j \| \Delta_j u \|_{L^r(\mathbb{R}^d)} \|_{l^p} \leq \infty \},$$

where the Littlewood-Paley operator $\hat{\Delta}_j$ is defined by

$$\hat{\Delta}_j u = \mathcal{F}^{-1}(\varphi(2^{-j}.) \mathcal{F} u) \text{ if } j \in \mathbb{Z}.$$

Lemma 2.3. Let $-\frac{d}{p} < s \leq \frac{d}{p} (s = \frac{d}{p}, r = 1)$. Assume $f^n$ is uniformly bounded in $\dot{B}^s_{p,\infty} \cap \dot{B}^s_{\infty,\infty} (\forall n > 0)$ or $\dot{B}^s_{p,\infty} \cap L^\infty$. Then $\varphi f^n$ is bound in $\dot{B}^s_{p,r} \cap B^s_{p,r} (\forall 0 < \epsilon < s + \frac{d}{p})$, and the map $f^n \mapsto \varphi f^n$ is compact for $\dot{B}^s_{p,r} \cap \dot{B}^{s-\epsilon}_{p,r}$ to $\dot{B}^{s-\epsilon}_{p,r}$, where $\varphi \in C_0^\infty(\mathbb{R}^d).$
Proof. The proof is based on Theorem 2.93 and Theorem 2.94 in [1], we omit it here. \( \square \)

**Definition 2.4.** [1] Let \( s \in \mathbb{R}, 1 \leq p, q, r \leq \infty \) and \( T \in (0, \infty] \). The functional space \( \tilde{L}^q_T(\dot{B}^s_{p,r}) \) is defined as the set of all the distributions \( f(t) \) satisfying \( \| f \|_{\tilde{L}^q_T(\dot{B}^s_{p,r})} := \|(2^k s)^{-1} \Delta f(t)\|_{L^q_t L^p_x} \|_{L^r_t} < \infty \).

By Minkowski’s inequality, it is easy to find that
\[
\| f \|_{\tilde{L}^q_T(\dot{B}^s_{p,r})} \leq \| f \|_{L^q_T(\dot{B}^s_{p,r})} \quad q \leq r; \quad \| f \|_{\tilde{L}^q_T(\dot{B}^s_{p,r})} \geq \| f \|_{L^q_T(\dot{B}^s_{p,r})} \quad q \geq r.
\]

Finally, we state some useful results in the heat equation
\[
\begin{aligned}
&\left\{ \begin{array}{l}
  u_t + \Delta u = G, \ x \in \mathbb{R}^d, \ t > 0, \\
  u(0, x) = u_0(x).
\end{array} \right.
\end{aligned}
\] (2.1)

**Lemma 2.5.** [2] Let \( s \in \mathbb{R}, 1 \leq q, q_1, p, r \leq \infty \) with \( q_1 \leq q \). Assume \( u_0 \) in \( \dot{B}^s_{p,r} \), and \( G \) in \( \tilde{L}^q_T(\dot{B}^s_{p,r}) \).

Then \( (2.2) \) has a unique solution \( u \) in \( \tilde{L}^q_T(\dot{B}^{s+\frac{d}{p}-1}_{p,r}) \) satisfying
\[
\| u \|_{\tilde{L}^q_T(\dot{B}^{s+\frac{d}{p}-1}_{p,r})} \leq C\left( \| u_0 \|_{\dot{B}^s_{p,r}} + \| G \|_{\tilde{L}^q_T(\dot{B}^{s+\frac{d}{p}-1}_{p,r})} \right).
\]

In particular, if \( s = \frac{d}{p} - 1 \), we have
\[
\| u \|_{\tilde{L}^q_T(\dot{B}^{\frac{d}{p}-1}_{p,r}) \cap \tilde{L}^q_T(\dot{B}^{\frac{d}{p}+\frac{1}{2}}_{p,r}) \cap \tilde{L}^q_T(\dot{B}^{\frac{d}{p}+\frac{3}{2}}_{p,r}) \cap \tilde{L}^q_T(\dot{B}^{\frac{d}{p}+1}_{p,r})} \leq C\left( \| u_0 \|_{\dot{B}^{\frac{d}{p}-1}_{p,r}} + \| G \|_{\tilde{L}^q_T(\dot{B}^{\frac{d}{p}+1}_{p,r})} \right).
\]

3 Local existence and uniqueness

We divide the proof of Theorem 3.1 into 4 steps:

**Step 1: Local existence and uniqueness.**

Different to the proof in [10], we will prove the local existence and uniqueness for \( (1.2) \) in a more precise way, which is necessary to prove the continuous dependence later. Now set \( u_0 \in \dot{B}^{\frac{d}{p}-1}_{p,r} \) \((1 \leq p < \infty)\) and define the first term \( u^0 := e^{t \Delta} u_0 \). Then we introduce a sequence \( \{ u^{n+1} \} \) with the initial data \( u^n \) by solving the following heat equation:
\[
\begin{aligned}
&\left\{ \begin{array}{l}
  u^{n+1} - \Delta u^{n+1} = \mathcal{P}(-u^n \nabla u^n), \\
  \text{div} u^{n+1} = 0, \\
  u^{n+1} |_{t=0} = \hat{S}_n u_0, \end{array} \right.
\end{aligned}
\] (3.1)

where \( \hat{S}_n g := \sum_{k<n} \hat{\Delta}_k g \), it makes sense if \( s < \frac{d}{p} \) or \( s = \frac{d}{p}, r = 1 \).

Taking advantage of Lemmas 2.3, we shall bound the approximating sequences in \( E^p_T \). Now we claim that there exists some \( T \) independent of \( n \) such that the solutions \( u^n \) satisfies the following inequalities:
\[
\begin{aligned}
(H_1) : & \quad \| u^n \|_{\tilde{L}^p_T(\dot{B}^{\frac{d}{p}-1}_{p,r})} \leq 2E_0, \\
(H_2) : & \quad \| u^n \|_A \leq 2a, \qquad A := \tilde{L}^2_T(\dot{B}^{\frac{d}{p}+1}_{p,r}) \cap \tilde{L}^1_T(\dot{B}^{\frac{d}{p}+1}_{p,r}),
\end{aligned}
\] (3.2)

where \( E_0 := \| u_0 \|_{\dot{B}^{\frac{d}{p}-1}_{p,r}} \).

Now we suppose that \( T \) and \( a \) satisfies the following inequality:
\[
a \leq \min\{ \frac{1}{16C_1^2 E_0}, \left( \frac{\sqrt{E_0}}{4C_1} \right)^{\frac{d}{p}}, \frac{1}{4C_1} \}.
\] (3.3)
where $C_1 \geq C$, $C$ is the constant of Lemma 2.5.

It’s easy to check that $(H_1) - (H_2)$ hold true for $n = 0$. Now we will show that if $(H_1) - (H_2)$ hold true for $n$, then they hold true for $n + 1$. In fact, combine (3.3) and Lemma 2.5 we have

$$
\|e^{t\Delta}u_0\|_A \leq a.
$$

(3.4)

and

$$
\|u^{n+1}\|_A \\
\leq \|e^{t\Delta}u_0\|_A + \|\mathbb{P}\text{div}(-u^n \otimes u^n)\|_{L_t^1(B^{\mathbb{R}^d-1}_p)B^{\mathbb{R}^d-1}_p)} \\
\leq a + C_1\|u^n\|_{L_t^1(B^{\mathbb{R}^d-1}_p)} \|u^n\|_{L_t^2(B^{\mathbb{R}^d-1}_p)} \\
\leq a + C_1\|u^n\|_{L_t^1(B^{\mathbb{R}^d-1}_p)} \|u^n\|_{L_t^2(B^{\mathbb{R}^d-1}_p)} \\
\leq a + C_1(2E_0)^{\frac{1}{2}}(2a)^{\frac{1}{2}} \\
\leq 2a
$$

(3.5)

This implies $(H_1) - (H_2)$ hold true for $n + 1$.

At the end of this step, we want to find the time $T$ and $a$ satisfying (3.3): We discuss $\|u_0\|_{\dot{B}^{\mathbb{R}^d-1}_p}$ in categories to meets the conditions $I$ and $II$ of (3.3):

(1) For $E_0 \leq \frac{1}{4C_1}$, we let $a = 2E_0$, which implies (3.3).

Then we have

$$
\|e^{t\Delta}u_0\|_A \leq 2\|u_0\|_{\dot{B}^{\mathbb{R}^d-1}_p} = 2E_0 = a
$$

That is (3.4) and we have

$$
T = \infty.
$$

(2) For $E_0 \geq \frac{1}{4C_1}$, we let $a = \tilde{c} := \min\left\{\frac{1}{16C_1E_0}, \frac{1}{4C_1}\right\}$, which implies (3.3).

Then we let $T$ is small enough so that

$$
\|e^{t\Delta}u_0\|_A \leq a.
$$

In fact, since $u_0 \in \dot{B}^{\mathbb{R}^d-1}_p$, let $j_0$ be an integer such that

$$
\left[\sum_{|j| \geq j_0} (\|\Delta_j u_0\|_{L^p(2^{|j|+1})})^\frac{1}{2} \leq \frac{a}{4}.
$$

(3.7)

Then we have

$$
\|e^{t\Delta}u_0\|_{L_t^1\dot{B}^{\mathbb{R}^d-1}_p} \\
\leq \left[\sum_{j \in \mathbb{Z}} (2^{|j|+1})^\frac{1}{2}\|\Delta_j u_0\|_{L^p} \int_0^T e^{-2^{|j|}t} dt\right]^{\frac{1}{2}}
$$
Remark 3.1. By (3.10), we know that if $u \in \dot{B}^{\frac{d}{2}+1}_{p,r}(\dot{L}^\infty_T(\mathbb{R}^n))$, \(u_{B^{\frac{d}{2}+1}_{p,r}(\dot{L}^\infty_T(\mathbb{R}^n))}\) implies (3.3). Step 3: Existence of a solution.

In this step, we use the compactness argument in Besov spaces for the approximate sequence $\{u^n\}$ to get some solution $u$ of (1.2). Since $u^n$ is uniformly bounded in $E^n_T$, the interpolation inequality yields that $u^n$ is also uniformly bounded in $L^q_T(\dot{B}^{\frac{d}{2}+1}_{p,r})$ for $1 \leq q \leq \infty$. Then, by Bony decomposition, we can easily get that for $\epsilon > 0$ small enough:

\[
\partial_t u^n \text{is uniformly bounded in } L^\infty_T(\dot{B}^{\frac{d}{2}+1-\epsilon}_{p,r}).
\]
Let \( \{\chi_j\}_{j \in \mathbb{N}} \) be a sequence of smooth functions with value in \([0, 1]\) supported in the ball \( B(0, j + 1) \) and equal to 1 on \( B(0, j) \). The above argument ensures that \( u^{n + 1} \) is uniformly bounded in \( C_T^{\sigma(\epsilon)}(\dot{B}^{\frac{d}{p}-1}_{p,r}) \cap \mathcal{C}_T(\dot{B}^{\frac{d}{p}-1}_{p,r}) \) for any \( \epsilon > 0 \). Then by Lemma 2.3, we deduce that embedding relation \( \dot{B}^{\frac{d}{p}-1}_{p,r} \cap \dot{B}^{\frac{d}{p}-1-2\epsilon}_{p,r} \hookrightarrow \dot{B}^{\frac{d}{p}-1}_{p,r} \) is local compact. So we get that for any \( j \in \mathbb{N} \), \( \{\chi_j u^{n+1}\}_{j \in \mathbb{N}} \) is uniformly bounded in \( C_T^{\sigma(\epsilon)}(\dot{B}^{\frac{d}{p}-1}_{p,r}) \cap \mathcal{C}_T(\dot{B}^{\frac{d}{p}-1}_{p,r} \cap \dot{B}^{\frac{d}{p}-1-2\epsilon}_{p,r}) \). Thus, by applying Ascoli’s theorem and Cantor’s diagonal process, there exists some functions \( u_j \) such that for any \( j \in \mathbb{N} \), \( \chi_j u^n \) tends to \( u_j \). As \( \chi_j \chi_{j+1} = \chi_j \), we have \( u_j = \chi_j u_{j+1} \). From that, we can easily deduce that there exists \( u \) such that for all \( \chi \in D(\mathbb{R}^d) \),

\[
\chi u^n \rightharpoonup \chi u \quad \text{in} \quad C_T(\dot{B}^{\frac{d}{p}-1}_{p,r}),
\]  

as \( n \) tends to \( \infty \)(up to a subsequence). By the interpolation, we have

\[
\chi u^n \rightharpoonup \chi u \quad \text{in} \quad L^1_T(\dot{B}^{1+\delta}_{p,r}), \quad 0 < \delta \leq 2 + \epsilon.
\]  

Combining the Fatou property for Besov spaces with \( u^n \) is uniformly bounded in \( E^{p}_T = C_T(\dot{B}^{\frac{d}{p}-1}_{p,r}) \cap L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,r}) \), we readily get

\[
u \in (L^{\infty}_T \dot{B}^{\frac{d}{p}-1}_{p,r} \cap L^1_T \dot{B}^{\frac{d}{p}+1}_{p,r}).
\]

Finally, following the argument of Theorem 2.94 and Theorem 3.19 in [1], it is a routine process to verify that \( u \) satisfies the system (1.2). Thus, we show that \( u \in E^{p}_T \).

Step 4: Uniqueness.

To prove the uniqueness, we let \( u, v \in E^{p}_T \) be two solutions of (1.2) with the same initial data. Let \( w = u - v \), then we have

\[
\begin{cases}
  w_t - \Delta w = -\mathcal{P} \text{div}[w \otimes u + v \otimes w], \\
  \text{div} w = 0, \\
  w|_{t=0} = 0.
\end{cases}
\]  

By lemma 2.3, we have

\[
\|w\|_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,r})} + \|w\|_{L^1_T(\dot{B}^{\frac{d}{p}+\frac{1}{2}}_{p,r})} \leq C_1(\|u\|_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,r})} + \|v\|_{L^1_T(\dot{B}^{\frac{d}{p}+\frac{1}{2}}_{p,r})}) \leq C_1(2E_0)^{\frac{1}{2}}(2a)^{\frac{d}{4}} \|w\|_{L^1_T(\dot{B}^{\frac{d}{p}+\frac{1}{2}}_{p,r})} + 2aC_1\|w\|_{L^1_T(\dot{B}^{\frac{d}{p}+\frac{1}{2}}_{p,r})} \leq \frac{1}{2}\|w\|_{L^1_T(\dot{B}^{\frac{d}{p}+\frac{1}{2}}_{p,r})} + \frac{1}{2}\|w\|_{L^1_T(\dot{B}^{\frac{d}{p}+\frac{1}{2}}_{p,r})},
\]  

where \( C_1(2E_0)^{\frac{1}{2}}(2a)^{\frac{d}{4}} \leq \frac{1}{2} \) and \( 2aC_1 \leq \frac{1}{2} \) are based on (3.3). This implies that \( u = v \), which completes the proof of uniqueness.

4 Continuous dependence

Before we prove the continuous dependence of (1.2), firstly, we need to prove that let \( T \) be a lifespan corresponding to the initial data \( u_0 \) by (3.14), if \( u^n \) tends to \( u_0 \) in \( \dot{B}^{\frac{d}{p}-1}_{p,r} \), then there exists a lifespan \( T^n \) corresponding to \( u^n_0 \) such that \( T^n \rightharpoonup T \). This implies a common lifespan both for \( u^n \) and \( u \) when \( n \) is sufficient large. Here is the key lemma:
**Lemma 4.1.** Let \( u_0 \in \dot{B}^\frac{d}{p} _{r,p} \) be the initial data of \( u \) with \( 1 \leq r, p < \infty \), if there exists another initial data \( u_0' \in \dot{B}^\frac{d}{p} _{r,p} \) such that \( \| u_0' - u_0 \|_{\dot{B}^\frac{d}{p} _{r,p}} \to 0 \) \( (n \to \infty) \), then we can construct a life-span \( T^n \) corresponding to \( u_0' \) such that

\[
T^n \to T, \quad n \to \infty,
\]

where the life-span time \( T \) is corresponding to \( u_0 \).

**Proof.** By virtue of Remark 3.1 since \( T = \infty \) when \( E_0 \leq \frac{1}{16C_4} \), we only consider the large initial data. Thus, we need to prove \( T^n \to T \) when \( E_0 > \frac{1}{16C_4} \). For convenience, since \( T = \min\{T_0, T_1\} \), we write down the definitions of \( T_0 \) and \( T_1 \):

\[
T_0 = \frac{a}{4} \frac{1}{2^{2j_0} E_0}, \quad T_1 = \frac{a^2}{4^2} \frac{1}{2^{2j_0} E_0^2},
\]

where \( j_0 \) is a fixed integer such that

\[
\left| \sum_{|j| \geq j_0} (\| \Delta_j u_0 \|_{L^p} 2^{(\frac{d}{2} - 1)j} ) \right|^\frac{1}{p} < \frac{a}{4}.
\]

As \( u_0 \in \dot{B}^\frac{d}{p} _{r,p} \), we can suppose that \( j_0 \) is the smallest integer such that the above inequality holds true. Since \( E_0^n \to E_0 \), in order to prove \( T_0^n \to T_0 \) and \( T_1^n \to T_1 \), it sufficient to show that there exists a corresponding sequence \( j_0^n \) satisfying

\[
\left| \sum_{|j| \geq j_0^n} (\| \Delta_j u_0 \|_{L^p} 2^{(\frac{d}{2} - 1)j} ) \right|^\frac{1}{p} < \frac{a}{4},
\]

and \( j_0^n \to j_0 \).

Now we begin to construct a subsequence \( \{j_0^n\}_{n \in \mathbb{N}} \geq j_0 \). Fix a positive constant \( \epsilon = \frac{a}{8} < \frac{a}{4} \).

For this \( \epsilon \), there exists \( N_\epsilon \) such that when \( n \geq N_\epsilon \), we have

\[
\| u_0^n - u_0 \|_{\dot{B}^\frac{d}{p} _{r,p}} \leq \epsilon.
\]

Then we define that \( j_0^\epsilon \) be the smallest integer that

\[
\left| \sum_{|j| \geq j_0^\epsilon} (\| \Delta_j u_0 \|_{L^p} 2^{(\frac{d}{2} - 1)j} ) \right|^\frac{1}{p} < \frac{a}{4} - \epsilon.
\]

By the definition of \( j_0 \), we have \( j_0 \leq j_0^\epsilon \).

Replacing \( \epsilon \) by \( \frac{\epsilon}{m} (m \in \mathbb{N}^+) \), there exists \( N_\epsilon \) such that when \( n \geq N_\epsilon \), we have

\[
\| u_0^n - u_0 \|_{\dot{B}^\frac{d}{p} _{r,p}} \leq \frac{\epsilon}{m}.
\]

Then we define that \( j_0^{\frac{\epsilon}{m}} \) be the smallest integer that

\[
\left| \sum_{|j| \geq j_0^{\frac{\epsilon}{m}}} (\| \Delta_j u_0 \|_{L^p} 2^{(\frac{d}{2} - 1)j} ) \right|^\frac{1}{p} < \frac{a}{4} - \frac{\epsilon}{m}.
\]

Since \( \frac{a}{4} - \frac{\epsilon}{m} > \frac{a}{4} - \frac{\epsilon}{m-1} \), it follows that

\[
j_0 \leq j_0^{\frac{\epsilon}{m}} \leq j_0^{\frac{\epsilon}{m-1}}.
\]
Now letting \( \tilde{j}_0^m := j_0^m \), we deduce that when \( n \geq N_m^\frac{4}{m} \),
\[
\left[ \sum_{|j| \geq j_0^m} (\| \tilde{\Delta}_j u_0^n \|_{L^p} 2^{(\frac{4}{p} - 1)j})^+ \right]^\frac{1}{2} < \| u_0^n - u_0 \|_{L^p}(\bar{d})_{j_0^m} - 1 + \left[ \sum_{|j| > j_0^m} (\| \tilde{\Delta}_j u_0^n \|_{L^p} 2^{(\frac{4}{p} - 1)j})^+ \right]^\frac{1}{2} < \frac{\epsilon}{m} + \frac{a}{4} - \frac{\epsilon}{m} = \frac{a}{4}.
\]
(4.1)

Since \( \{j_0^m\} \) is a monotone and bounded sequence, we deduce that \( \tilde{j}_0^m \to \tilde{j}_0 \) (\( m \to \infty \)) for some integer \( \tilde{j}_0 \geq j_0 \). For \( 0 < \epsilon < 1 \) there exists \( N \) such that when \( m \geq N \) we have
\[
|\tilde{j}_0^m - \tilde{j}_0| < \epsilon < 1.
\]
Note that \( \tilde{j}_0^m, \tilde{j}_0 \in \mathbb{N} \), we deduce that \( \tilde{j}_0^m = \tilde{j}_0 \) when \( m \geq N \) and \( \tilde{j}_0 \) is the smallest integer that
\[
\left[ \sum_{|j| \geq \tilde{j}_0^m} (\| \tilde{\Delta}_j u_0^n \|_{L^p} 2^{(\frac{4}{p} - 1)j})^+ \right]^\frac{1}{2} < \frac{a}{4} - \frac{\epsilon}{m},
\]
We claim that \( \tilde{j}_0 = j_0 \). Otherwise, if \( \tilde{j}_0 > j_0 \), we deduce from the above inequality that
\[
\left[ \sum_{|j| \geq \tilde{j}_0} (\| \tilde{\Delta}_j u_0^n \|_{L^p} 2^{(\frac{4}{p} - 1)j})^+ \right]^\frac{1}{2} \geq \frac{a}{4}, \quad \forall m \geq N.
\]
Since the left hand-side of the above inequality is independent of \( m \), we have
\[
\left[ \sum_{|j| \geq \tilde{j}_0^m} (\| \tilde{\Delta}_j u_0^n \|_{L^p} 2^{(\frac{4}{p} - 1)j})^+ \right]^\frac{1}{2} \geq \frac{a}{4}, \quad \forall m \geq N.
\]
This contradicts the definition of \( j_0 \). So we have \( \tilde{j}_0^m \to \tilde{j}_0 = j_0(m \to \infty) \).

Finally, we construct a sequence \( j_0^n \) by \( j_0^m \) when \( n \geq N_2 \):
\[
j_0^n := \begin{cases} 
\tilde{j}_0^1, & N_2 \leq n < N_2^2 \\
\tilde{j}_0^2, & N_2^2 \leq n < N_2^3 \\
\vdots & \\
\tilde{j}_0^m, & N_\frac{m}{m+1} \leq n < N_\frac{m}{m+1} \\
\vdots &
\end{cases}
\]
(4.2)

By virtue of (4.1), one can check that
\[
\left[ \sum_{|j| \geq j_0^n} (\| \tilde{\Delta}_j u_0^n \|_{L^p} 2^{(\frac{4}{p} - 1)j})^+ \right]^\frac{1}{2} < \frac{a}{4}.
\]
Using the monotone bounded theorem, one can prove that \( j_0^n \to j_0 \) (\( n \to \infty \)). Therefore, we have
\[
T_0 \to T_0, \quad T_1 \to T_1 \quad \Rightarrow T^n \to T, \quad n \to \infty.
\]
This completes the lemma.

\textbf{Remark 4.2.} By Lemma 4.1, let \( T^\infty \) be the life-span time of \( u^\infty \), then we can fine a \( T^n \) corresponding with \( u^n \) such that \( T^n \to T^\infty \) when \( n \to \infty \). That is say, fix some small \( \delta > 0 \), there exists an integer \( N_\delta \), when \( n \geq N \), we have
\[
|T^n - T^\infty| < \delta.
\]
Thus, we can choose \( T = T^\infty - \delta \), which is also the common lifespan both for \( u^\infty \) and \( u^n \), but independent of \( n \).
Now we begin to prove the continuous dependence.

**Theorem 4.3.** Let $p < \infty$. Assume that $u^n$ be the solutions to the system \( (1.2) \) with the initial data $u_0^n$. If $u_0^n$ tends to $u_0^\infty$ in $\dot{B}^{\frac{4}{3}-1}_{p,r}$, then there exists a positive $T$ independent of $n$ such that $u^n$ tends to $u^\infty$ in $L_T(\dot{B}^{\frac{4}{3}-1}_{p,r}) \cap L^1_T(\dot{B}^{\frac{4}{3}+1}_{p,r})$.

**Proof.** By Lemma 4.1 we can easily find that $T = T^\infty - \delta$ is the common lifespan both for $u^n$ and $u^\infty$ for $n$ large enough. Since $u_0^n$ tends to $u_0^\infty$ in $\dot{B}^{\frac{4}{3}-1}_{p,r}$, by the argument of step 2, we can easily get that

$$
\|u^n\|_{L^p_t(\dot{B}^{\frac{4}{3}-1}_{p,r})} \leq C E_0^n \leq C E_0, \quad \|u^n\|_{L^p_t(\dot{B}^{\frac{4}{3}-1}_{p,r})} \cap L^1_t(\dot{B}^{\frac{4}{3}+1}_{p,r}) \leq 2a \leq \frac{1}{4C_1}, \quad \forall n
$$

where $E_0^n := \|u_0^n\|_{\dot{B}^{\frac{4}{3}-1}_{p,r}}$.

Similar to the proof of the uniqueness, let

$$
\begin{cases}
(u^n - u^\infty)_t - \Delta(u^n - u^\infty) = -\mathbb{P} div[(u^n - u^\infty) \otimes u^n + u^\infty \otimes (u^n - u^\infty)], \\
div(u^n - u^\infty) = 0,
\end{cases}
$$

(4.3)

By lemma 2.5 we have

$$
\begin{align*}
&\|u^n - u^\infty\|_{L^p_t(\dot{B}^{\frac{4}{3}-1}_{p,r})} \cap L^1_t(\dot{B}^{\frac{4}{3}+1}_{p,r}) \\
&\leq \|u_0^n - u_0^\infty\|_{\dot{B}^{\frac{4}{3}-1}_{p,r}} + C_1(\|u^n\|_{L^4_t(\dot{B}^{\frac{4}{3}+1}_{p,r})} + \|u^n - u^\infty\|_{L^4_t(\dot{B}^{\frac{4}{3}+1}_{p,r})}) \\
&\leq \|u_0^n - u_0^\infty\|_{\dot{B}^{\frac{4}{3}-1}_{p,r}} + C_1(2E_0)^\frac{1}{2}(2a)^\frac{1}{2}\|u^n - u^\infty\|_{L^4_t(\dot{B}^{\frac{4}{3}+1}_{p,r})} + 2aC_1\|u^n - u^\infty\|_{L^4_t(\dot{B}^{\frac{4}{3}+1}_{p,r})}) \\
&\leq \|u_0^n - u_0^\infty\|_{\dot{B}^{\frac{4}{3}-1}_{p,r}} + \frac{1}{2}(\|u^n - u^\infty\|_{L^4_t(\dot{B}^{\frac{4}{3}+1}_{p,r})} + \frac{1}{2}\|u^n\|_{L^4_t(\dot{B}^{\frac{4}{3}+1}_{p,r})})
\end{align*}
$$

(4.4)

where $C_1(2E_0)^\frac{1}{2}(2a)^\frac{1}{2} \leq \frac{1}{2}$ and $2aC_1 \leq \frac{1}{2}$ are based on (3.3). Then we have

$$
\|u^n - u^\infty\|_{L^p_t(\dot{B}^{\frac{4}{3}-1}_{p,r})} \cap L^1_t(\dot{B}^{\frac{4}{3}+1}_{p,r}) \leq C\|u_0^n - u_0^\infty\|_{\dot{B}^{\frac{4}{3}-1}_{p,r}} \to 0, \quad n \to \infty.
$$

(4.5)

This complete the proof of continuous dependence.

\[ \square \]

**Proof of Theorem 1.2**

For $\|u_0\|_{\dot{B}^{\frac{4}{3}-1}_{p,\infty}} \leq \frac{1}{10C_1}$, the proof similar to Theorem 1.1. For $u_0 \in \dot{B}^{\frac{4}{3}-1}_{p,\infty}$, By the definition of $\dot{B}^{\frac{4}{3}-1}_{p,\infty}$, we have

$$
\begin{align*}
&\|e^{t\Delta}u_0\|_{L^p_t(\dot{B}^{\frac{4}{3}-1}_{p,\infty})} \\
&\leq \sup_{j \in \mathbb{Z}} 2(\frac{4}{3}+1)^j||\Delta_j u_0||_{L^p} \int_0^{T_0} e^{-2\Delta t}dt \\
&\leq \sup_{|j| \leq j_0} 2(\frac{4}{3}+1)^j||\Delta_j u_0||_{L^p} T_0 + \sup_{|j| > j_0} 2(\frac{4}{3}-1)^j(1 - e^{-2\Delta T_0})||\Delta_j u_0||_{L^p} \\
&\leq 2^{2j_0} T_0 \|u_0\|_{\dot{B}^{\frac{4}{3}-1}_{p,r}} + \sup_{|j| > j_0} 2(\frac{4}{3}-1)^j||\Delta_j u_0||_{L^p} \\
&\leq \frac{1}{2a},
\end{align*}
$$

(4.6)
and
\[
\|e^{t\Delta}u_0\|_{L^2_tB^\frac{d}{2p}_{p,\infty}} \\
\leq \sup_{j \in \mathbb{Z}} (2^j T_1 \|\hat{\Delta}_j u_0\|_{L^p} \left(\int_0^{T_1} e^{-2^{2j} t} \, dt\right)^{\frac{1}{2}} \\
\leq \sup_{|j| \leq j_0} 2^j T_1 \|\hat{\Delta}_j u_0\|_{L^p} + \sup_{|j| > j_0} 2^j (2^{j-1} T_0) \frac{1}{2} \|\hat{\Delta}_j u_0\|_{L^p} \\
\leq 2^{j_0} T_1^{\frac{1}{4}} \|u_0\|_{B^{\frac{d}{2p}}_{p,1}} \sup_{|j| > j_0} 2^j (2^{j-1} T_0) \|\hat{\Delta}_j u_0\|_{L^p} \\
\leq \frac{1}{2} a,
\]
(4.7)

where \(T_0 = \frac{a}{4} \frac{1}{2^{2j_0} E_0}\) and \(T_1 = \frac{a^2}{4^2} \frac{1}{2^{2j_0} E_0}\). Letting \(T = \min\{T_0, T_1\}\), the proof is then similar to Theorem 1.1. So we can prove that the solutions of (1.2) are local well-posedness in \(\tilde{E}^p_T\) in the Hadamard sense.

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