ON A SEMILINEAR TIMOSHENKO-COLEMAN-GURTIN SYSTEM: QUASI-STABILITY AND ATTRACTORS

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Abstract. A semilinear Timoshenko-Coleman-Gurtin system is studied. The system describes a Timoshenko beam coupled with a temperature with Coleman-Gurtin law. Under some assumptions on nonlinear damping terms and nonlinear source terms, we establish the global well-posedness of the system. The main result is the long-time dynamics of the system. By using the methods developed by Chueshov and Lasiecka, we get the quasi-stability property of the system and obtain the existence of a global attractor which has finite fractal dimension. Result on exponential attractors of the system is also proved.

1. Introduction. In this paper, we consider the following semilinear Timoshenko-Coleman-Gurtin system:

\begin{align*}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x + g_1(\varphi_t) + f_1(\varphi, \psi) &= h_1, \\
\rho_2 \psi_{tt} - b \psi_{xx} + k(\varphi_x + \psi) + \delta \theta_x + g_2(\psi_t) + f_2(\varphi, \psi) &= h_2, \\
\rho_3 \theta_t - \frac{1 - \alpha}{\beta} \theta_{xx} - \frac{\alpha}{\beta} \int_0^\infty g(s) \theta_{xx}(t - s) ds + \delta \psi_t x &= 0,
\end{align*}

where \((x, t) \in (0, L) \times \mathbb{R}^+, \varphi(x, t)\) is the transverse displacement, \(\psi(x, t)\) denotes the rotation angle of the center of mass of a beam element and \(\theta(x, t)\) is the relative temperature. The physical coefficients \(\rho_1, \rho_2, k, b, \rho_3, \delta, \beta\) and \(\alpha \in (0, 1)\) are positive fixed constants. The functions \(g_1(\varphi_t)\) and \(g_2(\psi_t)\) are nonlinear damping terms, the functions \(f_1(\varphi, \psi)\) and \(f_2(\varphi, \psi)\) denote nonlinear source terms, and \(h_1, h_2\) are external force terms.

We consider the following initial data

\begin{align*}
\begin{cases}
(\varphi(x, 0), \psi(x, 0), \theta(x, 0)) = (\varphi_0(x), \psi_0(x), \theta_0(x)), \\
(\varphi_1(x, 0), \psi_1(x, 0)) = (\varphi_1(x), \psi_1(x)), \theta(-s)|_{s > 0} = \vartheta_0(s), x \in (0, L),
\end{cases}
\end{align*}

where the past history function \(\vartheta_0\) on \(\mathbb{R}^+ = (0, \infty)\) is a given datum, and the following boundary conditions

\begin{align*}
\varphi(0, t) = \varphi(L, t) = \psi(0, t) = \psi(L, t) = \theta(0, t) = \theta(L, t) = 0, \quad \forall \ t > 0,
\end{align*}

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The original derivation of Timoshenko model was presented in Timoshenko [39], which is given by

\[ \rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x = 0, \tag{6} \]
\[ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) = 0, \tag{7} \]

where \( \varphi(x,t) \) is the transverse displacement and \( \psi(x,t) \) denotes the rotation angle of the center of mass of a beam element. The coefficients \( \rho_1, \rho_2, k \) and \( b \) are positive constants which denote density, mass moment of inertia, shear coefficient and flexural rigidity, respectively. In fact the Timoshenko model (6)-(7) is the asymptotic limit of Bresse system, see, for example, Ma and Monteiro [24]. There are so many results concerning Timoshenko system by now. Those results are mainly concerned with global existence and stability to Timoshenko system, which are linear by adding suitable damping effects, such as internal damping, (boundary) frictional damping, viscoelastic damping and heat damping, and so on. It is well-known that if the damping term is added in one of the two equations, the system (6)-(7) is exponentially stable under the “equal wave speeds” assumption

\[ \frac{\rho_1}{\rho_2} = \frac{k}{b}. \]

But if the damping terms are added in the two equations (6)-(7), then the system decays exponentially without the “equal wave speeds” assumption. We refer the reader to [1, 2, 3, 4, 7, 23, 25, 27, 26, 30, 32, 36, 37, 40, 41], among others. For dynamics on Timoshenko system, we mention the work of Fatori et al. [13]. They considered the Timoshenko of the form

\[ \rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x + f_1(\varphi, \psi) + \varphi_t = h_1, \]
\[ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + f_2(\varphi, \psi) + \psi_t = h_2, \]

By adopting the following assumptions on \( f_i(\varphi, \psi), i = 1, 2, \)

(I) \( f_i : \mathbb{R}^2 \rightarrow \mathbb{R} \) is locally Lipschitz continuous on each of its arguments;

(II) \[ F(s, r) \geq -\theta_2 - \mu_1 |r|^2 - \theta_1 |s|^2, \]
\[ F(s, r) \leq f_1(s, r)s + f_2(s, r)r + \theta_1 |s|^2 + \mu_1 |r|^2 + \theta_2, \]

where \( \frac{\partial F}{\partial s}(s, \cdot) = f_1(s, \cdot) \) and \( \frac{\partial F}{\partial r}(\cdot, r) = f_2(\cdot, r) \), and \( \theta_1, \theta_2, \mu_1, \mu_2 > 0 \) satisfy some conditions, the authors proved the existence of the global attractors and exponential attractors for the system. Grasselli et al. [21] studied a non-autonomous viscoelastic Timoshenko system of which is added two memory terms on both equations, they obtained the uniform attractors of the system. Feng and Yang [16] established the existence of global attractors and exponential attractors for a nonlinear Timoshenko beam with a time-delay term.

The general model of the thermoelastic beam model of Timoshenko reads

\[ \begin{align*}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x &= 0, \\
\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \delta \theta_x &= 0, \\
\rho_3 \theta_t + q_x + \delta \psi_{tx} &= 0,
\end{align*} \tag{8} \]

where \( \theta(x,t) \) is the relative temperature and \( q(x,t) \) represents the heat flux vector.

If we assume the Coleman-Gurtin law for the heat flux, i.e.,

\[ \beta q(t) + (1 - \alpha)\theta(t) + \alpha \int_0^\infty g(s)\theta_x(t - s)ds = 0, \quad \alpha \in (0, 1), \]
then we can get the desired Timoshenko-Coleman-Gurtin system \([1, 2]\). The limit cases \(\alpha = 0\) and \(\alpha = 1\) correspond to the Fourier case and Gurtin-Pipkin case \([12]\). For results on thermoelastic Timoshenko system, one can refer to \([14, 15, 17, 28, 29, 34, 35]\), and the references therein. It is worthy mentioning that all above cited works on thermoelastic Timoshenko systems were concerned with global well-posedness and stability. There no results concerning long-time dynamics of thermoelastic Timoshenko system by now.

Our goal of the present work is to study the long-time dynamics of Timoshenko-Coleman-Gurtin system \([1, 2]\), and obtain the global attractors and their properties to the system. We also establish the existence of exponential attractors. Since our problem has damping terms in all of the equations \([11, 2]\), we shall not assume the equal speeds assumption. The main features of this work are as follows:

(i) We consider a nonlinear damped system. By using the gradient system and asymptotic smoothness of the system, we prove the existence of a global attractor, which are characterized as unstable manifold of the set of stationary solutions.

(ii) We use multiplier functional to establish a stabilizability inequality to get the quasi-stability of the system and prove the global attractors have finite fractal dimension. We also show the existence of exponential attractors.

By virtue of the the memory with past history, system do not correspond to autonomous system. To deal with the memory, motivated by Giorgi et al. \([18, 19]\), we define a new variable \(\eta_t\) by

\[
\eta_t(x, s) = \int_0^s \theta(t - \tau) d\tau, \quad (t, s) \in [0, \infty) \times \mathbb{R}^+.
\]  

Therefore the past history of \(\theta\) satisfies

\[
\eta_t + \eta_s = \theta, \quad (x, t, s) \in \mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{R}^+,
\]  

where

\[
\eta_t(0) = 0 \quad \text{in} \quad \mathbb{R}^n, \quad t \geq 0,
\]

and initial condition

\[
\eta^0(s) = \eta_0(s) \quad \text{in} \quad \mathbb{R}^n, \quad s \in \mathbb{R}^+,
\]

with

\[
\eta_0(s) = \int_0^s \theta_0(\tau) d\tau, \quad s \in \mathbb{R}^+.
\]

By using \([11]\), we have

\[
\frac{\alpha}{\beta} \int_0^\infty g(s) \theta_{xx}(t - s) d\sigma = -\frac{\alpha}{\beta} \int_0^\infty g'(s) \eta_{xx}^t ds.
\]

If we denote

\[
\xi(s) = -\frac{\alpha}{\beta} g'(s) \quad \text{and} \quad l = \frac{1 - \alpha}{\beta} > 0,
\]

then we get the following system, which is equivalent to problem \([1, 2]\)

\[
\rho_1 \varphi_{tt} - k(\varphi_x + \psi_x) + g_1(\varphi_t) + f_1(\varphi, \psi) = h_1,
\]

\[
\rho_2 \psi_{tt} - b(\varphi_x + \psi) + k(\varphi_x + \psi_x) + \delta \theta_x + g_2(\psi_t) + f_2(\varphi, \psi) = h_2,
\]

\[
\rho_3 \theta_t - l \theta_{xx} - \int_0^\infty \xi(s) \eta_{xx}^t(s) ds + \delta \psi_{tx} = 0,
\]

\[
\eta_t + \eta_s = \theta,
\]
with the following initial data
\[
\begin{aligned}
(\varphi(x,0), \psi(x,0), \theta(x,0)) &= (\varphi_0(x), \psi_0(x), \theta_0(x)), \\
(\varphi_1(x,0), \psi_1(x,0)) &= (\varphi_1(x), \psi_1(x)), \quad \eta^0(x,s) = \eta_0(s),
\end{aligned}
\] (15)
and the following boundary conditions
\[
\varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = \theta(0,t) = \theta(L,t) = \eta'(0,0) = \eta'(L,0) = 0.
\] (16)

The plan of this paper is as follows: In Section 2, we give some assumptions used in this work and space setting. In Section 3, we shall study the global well-posedness of the system by using classical semigroup methods. In Section 4, we study the existence of a global attractor and its property, which contains finite fractal dimension. Section 5 is devoted to the existence of exponential attractors.

2. Assumptions and space setting. In this paper we use standard Lebesgue and Sobolev spaces
\[
L^q(0,L), \quad 1 \leq q \leq \infty, \quad \text{and} \quad H^1_0(0,L).
\]
Moreover, we denote the norm in the space $B$ by $\| \cdot \|_B$. In the case $q = 2$ we write $\|u\|$ instead of $\|u\|_2$. By using Poincaré’s inequality, we have
\[
\|u\| \leq L\|u_x\| \quad \text{and} \quad \|u\|_{H^1_0} = \|u_x\|, \quad \forall \ u \in H^1_0(0,L).
\]

Next we shall give some assumptions used in this paper. For nonlinear source terms $f_1$ and $f_2$, we use the assumptions in [24].

(F1) There exist a $C^2$ function $F : \mathbb{R}^2 \to \mathbb{R}$ satisfying
\[
\nabla F = (f_1, f_2),
\] (17)
and there exist two positive constants $\beta$ and $m_F$ such that for any $\varphi, \psi \in \mathbb{R}$
\[
F(\varphi, \psi) \geq -\beta(\|\varphi\|^2 + |\psi|^2) - m_F
\] (18)
where $0 \leq \beta < \min\left\{ \frac{b}{4L^2 + 2L^2}, \frac{k}{8L^2} \right\}$.

(F2) There exist positive constants $p \geq 1$ and $C_f$ such that for any $u, \psi, s \in \mathbb{R}$,
\[
|\nabla f_i(\varphi, \psi)| \leq C_f(1 + |\varphi|^{p-1} + |\psi|^{p-1}), \quad i = 1, 2,
\] (19)
which gives us that there exists a positive constant $C_F$ such that
\[
F(\varphi, \psi) \leq C_F(1 + |\varphi|^{p+1} + |\psi|^{p+1}),
\]

Concerning to the damping functions $g_i$ $(i = 1, 2, 3)$, we assume that

(G1)\[
g_i \in C^1(\mathbb{R}), \quad g_i(0) = 0 \quad \text{and} \quad \text{g}_i \text{ is increasing}.
\] (20)

(G2) There exist constants $m_i > 0$ and $M_i > 0$ such that for any $s \in \mathbb{R}$,
\[
m_i \leq g'_i(s) \leq M_i, \quad \forall \ s \in \mathbb{R}, \quad i = 1, 2,
\] (21)
which gives us the monotonicity property
\[
(g_i(u) - g_i(v))(u - v) \geq m_i|u - v|^2, \quad \forall \ u, v \in \mathbb{R}, \quad i = 1, 2.
\]

For the relaxation function $\xi(s)$, we assume

(R1) $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function such that
\[
\xi(0) > 0, \quad \int_0^\infty \xi(s)ds = l_0 < \infty.
\] (22)
(R2) There exists a positive constant \( \mu \) such that
\[
\xi'(t) + \mu \xi(t) \leq 0, \quad \text{for } t \geq 0. \tag{23}
\]

To this end we define the following weighted spaces with respect to the variable \( \eta \),
\[
\mathcal{M} = L^2_\xi(\mathbb{R}^+, H^1_0) = \left\{ \eta : \mathbb{R}^+ \to H^1_0 \left| \int_0^\infty \xi(s) \| \eta_x(s) \|^2 ds < \infty \right. \right\},
\]
which is a Hilbert space with the norm
\[
\| \eta \|^2_{\mathcal{M}} = \int_0^\infty \xi(s) \| \eta_x(s) \|^2 ds,
\]
and the inner-product
\[
(\eta, \chi)_{\mathcal{M}} = \int_0^\infty \xi(s) \int_0^L \eta_x(s) \chi_x(s) dx ds.
\]

Finally we introduce the phase space
\[
\mathcal{H} = H^1_0(0, L) \times L^2(0, L) \times H^1_0(0, L) \times L^2(0, L) \times L^2(0, L) \times \mathcal{M},
\]
equipped with the norm
\[
\| (\varphi, \psi, \theta, \eta) \|^2_{\mathcal{H}} = \rho_1 \| \dot{\varphi} \|^2 + \rho_2 \| \dot{\psi} \|^2 + \rho_3 \| \theta \|^2 + b \| \psi_x \|^2 + k \| \varphi_x + \psi \|^2 + \| \eta \|^2_{\mathcal{M}}.
\]

3. Well-posedness. In this section, we shall prove the global well-posedness of system \([11]-[16]\). First we give the following energy identities.

3.1. Energy identities. Now we define the energy functional of solutions of system \([11]-[16]\) as
\[
E(t) = \frac{1}{2} \| (\varphi, \varphi_t, \psi, \psi_t, \theta, \eta) \|^2_{\mathcal{H}}
= \frac{1}{2} \left[ \rho_1 \| \varphi_t \|^2 + \rho_2 \| \psi_t \|^2 + \rho_3 \| \theta \|^2 + b \| \psi_x \|^2 + k \| \varphi_x + \psi \|^2 + \| \eta \|^2_{\mathcal{M}} \right], \tag{24}
\]
and define the modified energy functional by
\[
\mathcal{E}(t) = E(t) + \int_0^L F(\varphi, \psi) dx - \int_0^L h_1 \varphi dx - \int_0^L h_2 \psi dx. \tag{25}
\]
Then we can get the following lemma.

**Lemma 3.1.** The modified energy functional defined in \([25]\) satisfies for all \( t \geq 0 \),
\[
\frac{d}{dt} \mathcal{E}(t) = -t \int_0^\infty \theta_x^2 dx + \frac{1}{2} \int_0^\infty \xi'(s) \| \eta_x \|^2 ds. \tag{26}
\]
Moreover, if the functions \( h_1, h_2 \in L^2(0, L) \), then there exists a positive constant \( K = K(\| h_1 \|, \| h_2 \|) \) such that for any \( t \geq 0 \),
\[
\mathcal{E}(t) \geq \frac{1}{4} \| (\varphi, \varphi_t, \psi, \psi_t, \theta, \eta) \|^2_{\mathcal{H}} - K. \tag{27}
\]

**Proof.** Multiplying \([11]\) by \( \varphi_t \), \([12]\) by \( \psi_t \) and \([13]\) by \( \theta \), respectively, integrating the results over \((0, L)\) and using integration by parts, we have
\[
\frac{d}{dt} \mathcal{E}(t) = -t \int_0^L \theta_x^2 dx + \int_0^L \theta(t) \left( \int_0^\infty \xi(s) \eta_{xx}(s) ds \right) dx + \frac{1}{2} \frac{d}{dt} \| \eta \|^2_{\mathcal{M}}. \tag{28}
\]
It follows that
\[
\frac{1}{2} \frac{d}{dt} \|\eta\|_M^2 = - \int_0^\infty \xi(s) \int_0^L \theta(t) \eta_{xx} dx ds - (\eta_s, \eta)_M,
\]
which, together with (28), gives us (26).

Using (18) and Poincaré’s inequality, we derive
\[
\int_0^L F(\varphi, \psi) dx \geq - \beta L^2 \| \varphi_x \|^2 - \beta L^2 \| \psi_x \|^2 - Lm_F
\]
where we used the Cauchy-Schwarz inequality
\[
\| \varphi_x \|^2 = \| \varphi_x + \psi - \psi \|^2 \leq 2 \| \varphi_x + \psi \|^2 + 2 \| \psi \|^2.
\]

It follows from Young’s inequality and Poincaré’s inequality that
\[
\int_0^L h_1 \varphi dx \leq \| h_1 \| \cdot L \| \varphi_x \|
\]
\[
\leq L \| h_1 \| \| \varphi_x + \psi \| + L^2 \| h_1 \|^2 \| \psi_x \|^2
\]
\[
\leq \frac{2L^2}{k} \| h_1 \|^2 + \frac{k}{8} \| \varphi_x + \psi \|^2 + \frac{4L^4}{b} \| h_1 \|^2 + \frac{b}{16} \| \psi_x \|^2.
\]

Analogously, we have
\[
\int_0^L h_2 \psi dx \leq \frac{4L^2}{b} \| h_2 \|^2 + \frac{b}{16} \| \psi_x \|^2.
\]

Combining (29) with (25), we can obtain (27) with \( K = Lm_F + \left( \frac{2L^2}{k} + \frac{4L^4}{b} \right) \| h_1 \|^2 + \frac{4L^4}{b} \| h_2 \|^2 \). The proof is now complete.

3.2. Well-posedness. To prove the global well-posedness of (11)-(16) by using semigroup method, we write the derivative \( \eta_s \) as an operator form, see, e.g., [20].

Define the operator \( T \) by
\[
T \eta = - \eta_s, \quad \eta \in D(T),
\]
with
\[
D(T) = \{ \eta \in M | \eta_s \in M, \eta(0) = 0 \},
\]
is the infinitesimal generator of a translation semigroup. In particular,
\[
(T \eta, \eta)_M = \int_0^\infty \xi'(s) \| \eta_x(s) \|^2 ds, \quad \eta \in D(T),
\]
and the solution of
\[
\eta_t = T \eta + \theta, \quad \eta(0) = 0,
\]
has an explicit representation formula.

Now we introduce two new dependent variables \( \tilde{\varphi} = \varphi_t \) and \( \tilde{\psi} = \psi_t \), then system (11)-(16) is equivalent to the following problem for an abstract Cauchy problem
\[
\begin{align*}
\frac{d}{dt} U(t) &= (A + B) U(t) + F(U(t)), \quad t > 0, \\
U(0) &= U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, \eta_0)^T,
\end{align*}
\]
where
\[
U(t) = (\varphi(t), \tilde{\varphi}(t), \psi(t), \tilde{\psi}(t), \theta(t), \eta)^T \in \mathcal{H},
\]
and
\[
AU(t) = \begin{pmatrix}
\frac{k}{\rho_1}(\phi_x + \psi)_x \\
\frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\phi_x + \psi) - \frac{\delta}{\rho_2}\theta_x - \frac{\lambda}{\rho_3}\psi_{xt} \\
\frac{l}{\rho_3}\theta_{xx} + \frac{1}{\rho_3}\int_0^x \xi(s)\eta_{xx}(s)ds - \frac{\delta}{\rho_3}\psi_{xt}
\end{pmatrix},
\]
\[
BU(t) = \begin{pmatrix}
0 \\
-\frac{g_1(\tilde{\phi})}{\rho_1} \\
-\frac{g_2(\tilde{\psi})}{\rho_2} \\
0
\end{pmatrix},
\]
with the domain
\[
D(A) = \left\{ U(t) \in \mathcal{H} | \phi, \psi \in (H^2(0, L) \cap H^1_0(0, L)), \phi_t, \psi_t \in H^1_0(0, L), \theta \in H^1_0(0, L), \eta \in D(T), \ l\theta_{xx} + \int_0^\infty \xi(s)\eta_{xx}(s)ds \in L^2(0, L) \right\},
\]
and
\[
D(B) = \mathcal{H}.
\]
The source terms are represented by a nonlinear function \( F : \mathcal{H} \to \mathcal{H} \) defined by
\[
F(U(t)) = \begin{pmatrix}
0 \\
-\frac{f_1(\phi, \psi)}{\rho_1} \\
-\frac{f_2(\phi, \psi)}{\rho_2} \\
0
\end{pmatrix}.
\]
Next we will show that the operator is maximal monotone. Let \( U(t) = (\phi(t), \tilde{\phi}(t), \psi(t), \tilde{\psi}(t), \theta(t), \eta) \) be in \( D(A) \), it is easy to get that for any \( t > 0 \),
\[
(AU, U)_\mathcal{H} = -l \int_0^L \theta_x^2 dx + \frac{1}{2} \int_0^\infty \xi(s)\|\eta_x(s)\|^2 ds \leq 0,
\]
which implies that the operator \( A \) is a dissipative operator. Furthermore the operator \( A \) has the property \( R(T - A) = \mathcal{H} \). Indeed, let \( U^* = (\phi^*, \tilde{\phi}^*, \psi^*, \tilde{\psi}^*, \theta^*, \eta^*) \in \mathcal{H} \), we must solve the problem \( U - AU = U^* \) for some \( U \in D(A) \). The equation becomes the system
\[
\phi - \tilde{\phi} = \phi^*,
\]
\[
\tilde{\phi} - \frac{k}{\rho_1}(\phi_x + \psi)_x = \tilde{\psi}^*,
\]
\[
\psi - \tilde{\psi} = \psi^*,
\]
\[
\tilde{\psi} - \frac{b}{\rho_2}\psi_{xx} + \frac{k}{\rho_2}(\phi_x + \psi) + \frac{\delta}{\rho_2}\theta_x = \tilde{\psi}^*,
\]
\[
\theta - \frac{l}{\rho_3}\theta_{xx} - \frac{1}{\rho_3}\int_0^\infty \xi(s)\eta_{xx}(s)ds + \frac{\delta}{\rho_3}\tilde{\psi}_x = \theta^*,
\]
\[
\eta - \theta - T\eta = \eta^*.
\]
From \( 39 \) and \( \eta(0) = 0 \), we infer that
\[
\eta(s) = (1 - e^{-s})\theta + \int_0^s e^{s-x}\eta^*(s)ds.
\]
By using (40), (34) and (36), we can get the following problem
\[ \rho_1 \varphi - k(\varphi_x + \psi)_x = \rho_1(\varphi^* + \hat{\varphi}^*), \]
\[ \rho_2 \psi - b \psi_{xx} + k(\varphi_x + \psi) + \delta \theta_x = \rho_2(\psi^* + \hat{\psi}^*), \]
\[ \rho_3 \theta - \left(1 + \int_0^\infty g(s)(1 - e^{-s})ds\right) \theta_{xx} + \delta \psi_x = \vartheta, \]
where
\[ \vartheta = \int_0^\infty g(s) \int_0^\infty e^{\tau - s} \eta^*_x(s) d\tau ds + \delta \psi_x^* + \theta^*. \]

The same arguments as in [18], we know that \( \vartheta \in H^{-1}(0, L) \). In the sequel we define a bilinear from
\[ \mathcal{B} : (H^1_0(0, L))^2 \times L^2(0, L) \rightarrow \mathbb{R} \]
given by
\[ \mathcal{B}((\varphi, \psi, \theta), (\varphi^*, \hat{\varphi}^*, \hat{\theta}^*)) = \int_0^L \left[ \rho_1 \varphi \varphi^* + \rho_2 \psi \psi^* + \rho_3 \theta \theta^* + k(\varphi_x + \psi)(\varphi^*_x + \hat{\varphi}^*_x) + b \psi_x \hat{\psi}_x \right] dx. \]

It is easy to verify that \( \mathcal{B} \) is continuous and coercive. By using the Lax-Milgram theorem, the elliptic problem (41) has a unique weak solution \((\varphi, \psi, \theta) \in (H^1_0(0, L))^2 \times L^2(0, L)\). It follows from (40) that
\[ \int_0^\infty \xi(s) \|\eta_x(s)\|^2 ds \leq 2 \lambda_0 \|\theta_x\|^2 + 2 \int_0^\infty \xi(s) \left( \int_0^s e^{\tau - s} \eta^*_x(\tau) d\tau \right)^2 ds \]
\[ \leq 2 \lambda_0 \|\theta_x\|^2 + 2 \int_0^\infty \left( \int_0^s e^{\tau - s} \sqrt{\xi(\tau)} \|\eta^*_x(\tau)\|^2 d\tau \right)^2 ds \]
\[ \leq 2 \lambda_0 \|\theta_x\|^2 + 2 \|\eta^*\|^2_{\mathcal{M}}, \]
which gives us \( \eta \in \mathcal{M} \). Then we can obtain
\[ T \eta = \eta - \theta - \eta^* \in \mathcal{M} \]

Therefore \( U(t) \in D(A) \) solves the problem \( U - AU = U^* \).

By using the same method as Theorem 2.2 in Charles et al. [18] and also in Barbu [31], we can prove the operator \( B \) is monotone, hemicontinuous and a bounded operator.

From above we know that the operator \( A + B \) is maximal monotone in \( \mathcal{H} \), and then we conclude that the operator \( A + B \) generates a semigroup of contractions in \( \mathcal{H} \) by Lumer-Phillips theorem, see, for example, Pazy [31] and Barbu [31].

Now we claim that \( \mathcal{F} \) is locally Lipschitz on \( \mathcal{H} \). Indeed, we denote \( U^i = (\varphi^i, \hat{\varphi}^i, \psi^i, \hat{\psi}^i, \theta^i, \eta^i), \ i = 1, 2 \). Let \( B \) be a bounded set of \( \mathcal{H} \), and \( U^1, U^2 \in B \).

Under the assumption (19), we infer that for \( j = 1, 2 \),
\[ |f_j(\varphi^1, \psi^1) - f_j(\varphi^2, \psi^2)|^2 \]
\[ = |\nabla f_j(\nu(\varphi^1, \psi^1) + (1 - \nu)(\varphi^2, \psi^2))|^2 \cdot |(\varphi^1, \psi^1) - (\varphi^2, \psi^2)|^2 \]
\[ \leq C_B^2 \left( 1 + |\varphi^1|^{p-1} + |\varphi^2|^{p-1} + |\psi^1|^{p-1} + |\psi^2|^{p-1} \right) \left( |\varphi^1 - \varphi^2|^2 + |\psi^1 - \psi^2|^2 \right). \]

Then we know that there exists a constant \( C_B > 0 \) such that
\[ \| \mathcal{F}(U^1) - \mathcal{F}(U^2) \|^2_{\mathcal{H}} \leq C_B \|U^1 - U^2\|^2_{\mathcal{H}}, \]
which yields \( \mathcal{F} \) is locally Lipschitz on \( \mathcal{H} \).
In light of the above results, we infer that the abstract Cauchy problem (33) admits a unique local mild solution which is given by

\[ U(t) = e^{(A+B)t}U_0 + \int_0^t e^{(t-s)(A+B)}F(U(s))ds, \]  
(42)

defined in a maximal interval \([0, t_{\text{max}}])\). If \(t_{\text{max}} < \infty\), then

\[ \lim_{t \to t_{\text{max}}} \|U(t)\|_H = \infty. \]  
(43)

We use contradiction method to see that the solution is global. Let \(U(t)\) be a mild solution with \(U_0 \in D(A+B)\). By using Theorem 6.1.5 in Pazy [31], we know that the solution is a strong solution. It follows from (27) that for all \(t \geq 0\),

\[ \|U(t)\|^2_H \leq 4\mathcal{E}(0) + K, \]

which, by using density arguments, holds for mild solutions. Then it is a contradiction with (43) and therefore \(t_{\text{max}} = \infty\), i.e., the solution \(U(t)\) is a global solution.

On the other hand, given \(T > 0\) and any \(t \in (0, T)\), we consider two mild solutions \(U^1\) and \(U^2\) with initial data \(U^1(0)\) and \(U^2(0)\), respectively. By (42), we get

\[ \|U^1(t) - U^2(t)\|_H \leq \|e^{(t-s)(A+B)}(U^1(0) - U^2(0))\|_H \]

\[ + \int_0^t \|e^{(t-s)(A+B)}(F(U^1(s)) - F(U^2(s)))\|_Hds, \]

which, using the local Lipschitz property of \(F\) and (27), gives us

\[ \|U^1(t) - U^2(t)\|_H \leq \|U^1(0) - U^2(0)\|_H + \int_0^t \|U^1(s) - U^2(s)\|_Hds. \]

Then applying the Gronwall inequality, we can obtain for any \(t \in [0, T]\),

\[ \|U^1(t) - U^2(t)\|_H \leq e^{C_0T}\|U^1(0) - U^2(0)\|_H, \]

where \(C_0 = C(U^1(0), U^2(0))\).

By using Theorem 6.1.5 in Pazy [31], we can get that any mild solutions with initial data in \(D(A+B)\) are strong.

Therefore we proved the global well-posedness of problem (11)-(16), which is given by the following theorem.

**Theorem 3.2.** Assume that (17)-(23) hold, then we have the following results.

(i) If initial data \(U_0 \in H\), then problem (27) has a unique mild solution \(U(t) \in C([0, \infty), H)\) with \(U(0) = U_0\) given by

\[ U(t) = e^{(A+B)t}U_0 + \int_0^t e^{(t-\tau)(A+B)}F(U(\tau))d\tau. \]

(ii) If \(U^1(t)\) and \(U^2(t)\) are two mild solutions of problem (33) then there exists a positive constant \(C_0 = C(U^1(0), U^2(0))\), such that

\[ \|U^1(t) - U^2(t)\|_H \leq e^{C_0T}\|U^1(0) - U^2(0)\|_H, \]

for any \(0 \leq t \leq T\).

(iii) If \(U_0 \in D(A + B)\), then the above mild solution is a strong solution.

**Remark 1.** We know from the well-posedness of problem (33) given by Theorem 3.2 that the one-parameter family of operators \(S(t) : H \to H\) defined by

\[ S(t)U_0 = U(t), \]  
for all \(t \geq 0\),

where \(U(t)\) is the unique solution of problem (33), satisfies the semigroup properties

\[ S(0) = I \]  
and \(S(t+s) = S(t) \circ S(s)\), for all \(t, s \geq 0\),
and defines a nonlinear \( C_0 \)-semigroup, which is locally Lipschitz continuous on the space \( \mathcal{H} \). Thus we can study the long-time dynamics of problem (33) through the dynamical system \((\mathcal{H}, \mathcal{S}(t))\).

4. **Global attractor.** In this section, we will establish the existence of a global attractor and its fractal dimension to problem (33)-(36).

4.1. **Abstract results.** First of all, we recall some basic theories concerning global attractor. For most of results, we would like to refer the reader to Chueshov [9], Chueshov & Lasiecka [11], [11], Hale [22], Robinson [33], Temam [38], among others.

Let \((\mathcal{H}, \mathcal{S}(t))\) be a dynamical system given by a strongly continuous semigroup \( \mathcal{S}(t) \) on a Banach space \( \mathcal{H} \). A compact set \( \mathcal{A} \subset \mathcal{H} \) is called a global attractor of the semigroup \( \mathcal{S}(t) \) if \( \mathcal{A} \) is strictly invariant with respect to \( \mathcal{S}(t) \), i.e., \( \mathcal{S}(t)\mathcal{A} = \mathcal{A} \) for all \( t \geq 0 \), and \( \mathcal{A} \) attracts any bounded set \( \mathcal{B} \subset \mathcal{H} \), that is,

\[
\lim_{t \to \infty} \text{dist}_H(\mathcal{S}(t)\mathcal{B}, \mathcal{A}) = 0,
\]

where \( \text{dist}_H \) is the Hausdorff semidistance in \( \mathcal{H} \).

Let \( \mathcal{N} \) be the set of stationary points of \( \mathcal{S}(t) \). Then the unstable manifold \( \mathcal{M}_+ (\mathcal{N}) \) is the family of \( \mathcal{y} \in \mathcal{H} \) such that there exists a full trajectory \( u(t) \) satisfying

\[
u(0) = \mathcal{y}, \quad \lim_{t \to -\infty} \text{dist}(u(t), \mathcal{N}) = 0.
\]

The following theorem is well-known, which can be found in Chueshov & Lasiecka [11] (Corollary 7.5.7), also in Hale [22] (Theorem 2.4.6).

**Theorem 4.1.** Assume that a dynamical system \((\mathcal{H}, \mathcal{S}(t))\) is an asymptotically smooth gradient system, with the corresponding Lyapunov functional denoted by \( \Phi \). Suppose that

\[
\Phi(S(t)\mathcal{z}) \to \infty \quad \text{if and only if} \quad \| \mathcal{z} \|_{\mathcal{H}} \to \infty,
\]

and that the set of stationary points \( \mathcal{N} \) is bounded. Then the system \((\mathcal{H}, \mathcal{S}(t))\) has a compact global attractor which characterized by \( \mathcal{A} = \mathcal{M}_+ (\mathcal{N}) \).

Given a compact set \( \mathcal{M} \) in a metric space \( \mathcal{H} \), the fractal dimension of \( \mathcal{M} \) is defined by

\[
\dim^f_{\mathcal{H}} \mathcal{M} = \lim_{\varepsilon \to 0} \sup \frac{\ln n(\mathcal{M}, \varepsilon)}{\ln(1/\varepsilon)},
\]

where \( n(\mathcal{M}, \varepsilon) \) is the minimal number of closed balls of radius \( \varepsilon \) which covers \( \mathcal{M} \).

Let \( \mathcal{X}, \mathcal{Y} \) be two reflexive Banach spaces with \( \mathcal{X} \) compactly embedded in \( \mathcal{Y} \) and we denote \( \mathcal{H} = \mathcal{X} \times \mathcal{Y} \). Consider the dynamics system \((\mathcal{H}, \mathcal{S}(t))\) given by an evolution operator

\[\mathcal{S}(t)U_0 = (\mathcal{u}, \mathcal{u}_t), \quad U_0 = (\mathcal{u}(0), \mathcal{u}_t(0)) \in \mathcal{H},\]

where the function \( \mathcal{u} \) have regularity

\[\mathcal{u} \in C(\mathbb{R}^+; \mathcal{X}) \cap C^1(\mathbb{R}^+; \mathcal{Y}).\]

The dynamical system \((\mathcal{H}, \mathcal{S}(t))\) is quasi-stable on a set \( \mathcal{B} \subset \mathcal{H} \) if there exists a compact seminorm \( n_{\mathcal{X}} \) on \( \mathcal{X} \) and nonnegative scalar functions \( a(t) \) and \( c(t) \), locally bounded in \([0, \infty)\), and \( b(t) \in L^1(\mathbb{R}^+) \) with \( \lim_{t \to \infty} b(t) = 0 \), such that,

\[\| \mathcal{S}(t)\mathcal{U}^1 - \mathcal{S}(t)\mathcal{U}^2 \|_{\mathcal{H}} \leq a(t)\| \mathcal{U}^1 - \mathcal{U}^2 \|_{\mathcal{H}},\]

and

\[\| \mathcal{S}(t)\mathcal{U}^1 - \mathcal{S}(t)\mathcal{U}^2 \|_{\mathcal{H}} \leq b(t)\| \mathcal{U}^1 - \mathcal{U}^2 \|_{\mathcal{H}} + c(t) \sup_{0 < s < t} [n_{\mathcal{X}}(\mathcal{U}^1(s) - \mathcal{U}^2(s))]^2,\]
for any \( U^1, U^2 \in B \). The inequality (47) is often called stabilizability inequality.

We usually use the following property of quasi-stable system to prove the asymptotic smoothness of semigroup. One can find the theorem in Chueshov and Lasiecka [11] (Theorem 7.9.4).

**Theorem 4.2.** Let the dynamical system \((H, S(t))\) be given by (44) and satisfying (45). If the system \((H, S(t))\) is quasi-stable on every bounded positively invariant set \( B \subset H \), then the dynamical system \((H, S(t))\) is asymptotically smooth.

One can find the following theorem in Hale [22] (Theorem 2.4.6) and Chueshov and Lasiecka [11] (Corollary 7.5.7).

**Theorem 4.3.** Assume the dynamical system \((H, S(t))\) is asymptotically smooth and gradient with Lyapunov functional \( \Phi \). Assume further
(i) \( \Phi \) is bounded from above on any bounded subset of \( H \),
(ii) the set \( \Phi_R = \{ U | \Phi(U) \leq T \} \) is bounded for every \( R \),
(iii) the set of stationary points \( \mathcal{N} \) is bounded.
Then the dynamical system \((H, S(t))\) has a compact global attractor characterized by \( \mathfrak{A} = \mathcal{M}^+ \mathcal{N} \).

On the other hand, quasistability also implies that global attractors have finite fractal-dimension. One can find the following theorem in Chueshov and Lasiecka [11] (Theorem 7.9.6).

**Theorem 4.4.** Let \((H, S(t))\) be given by (44) and satisfying (45). If \((H, S(t))\) possesses a compact global attractor \( \mathfrak{A} \) and is quasi-stable on \( \mathfrak{A} \), then the attractor \( \mathfrak{A} \) has finite fractal dimension.

4.2. **Global attractor and its fractal dimension.** In this subsection, we shall prove the existence of a global attractor. The main result is as follows.

**Theorem 4.5.** Assume the hypotheses (17) - (23) hold. Then the dynamical system \((H, S(t))\) generated by the problem (11) - (16) has a compact global attractor \( \mathfrak{A} \), with finite fractal dimension, which is characterized by
\[
\mathfrak{A} = \mathcal{M}^+ \mathcal{N},
\]
where \( \mathcal{N} \) is the set of stationary points of \( S(t) \) and \( \mathcal{M}^+ \mathcal{N} \) is the unstable manifold emanating from \( \mathcal{N} \).

Next we use Theorem 4.3 to prove Theorem 4.5 which will be divided into two parts.

4.2.1. **Gradient system and stationary solutions.** A dynamical system \((H, S(t))\) is called a gradient system if it possesses a strict Lyapunov functional. More precisely, a functional \( \Phi : H \to \mathbb{R} \) is a strict Lyapunov function for a system \((H, S(t))\) if,
(i) the map \( t \to \Phi(S(t)z) \) is non-increasing for each \( z \in H \),
(ii) if \( \Phi(S(t)z) = \Phi(z) \) for some \( z \in H \) and for all \( t \), then \( z \) is a stationary point of \( S(t) \), that is, \( S(t)z = z \).

Then we have the following lemma.

**Lemma 4.6.** The dynamical system \((H, S(t))\) corresponding to problem (11) - (16) is gradient.
Proof. Let us take the modified energy \( \mathcal{E}(t) \) defined in (25) as a Lyapunov function \( \Phi \). Then for \( U = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, \eta_0) \in \mathcal{H} \), we can obtain from (26) that for all \( t > 0 \),

\[
\frac{d}{dt} \Phi(S(t)U) = -l \int_0^L \theta_x^2 dx + \frac{1}{2} \int_0^\infty \xi'(s) \| \eta_x(s) \|^2 dx
\]

\[
- \int_0^L (g_1(\varphi_t) \varphi_t + g_2(\psi_t) \psi_t) dx \leq 0,
\]

which gives us \( \Phi(S(t)U) \) is non-increasing. Now let us suppose \( \Phi(S(t)U) = \Phi(U) \) for all \( t \geq 0 \), then we have

\[
0 = \Phi(S(t)U) - \Phi(U)
\]

\[
= -l \int_0^t \int_0^L \theta_x^2 dx ds + \frac{1}{2} \int_0^t \int_0^\infty \xi'(s) \| \eta_x(s) \|^2 dx ds
\]

\[
- \int_0^t \int_0^L (g_1(\varphi_t) \varphi_t + g_2(\psi_t) \psi_t) dx ds \leq 0,
\]

which implies for any \( t \geq 0 \),

\[
\int_0^L (g_1(\varphi_t) \varphi_t + g_2(\psi_t) \psi_t) dx = 0,
\]

and

\[
l \int_0^L \theta_x^2 dx + \frac{1}{2} \int_0^\infty (-\xi'(s)) \| \eta_x(s) \|^2 ds = 0.
\]

By using the hypothesis (21), we can conclude from (49) that for any \( t \geq 0 \),

\[
\varphi_t(t) = 0, \quad \psi_t(t) = 0, \quad \text{a.e. in } (0, L),
\]

which gives us that for all \( t \geq 0 \),

\[
\varphi(t) = \varphi_0, \quad \psi(t) = \psi_0.
\]

On the other hand, noting that the two terms in the left-hand side of (50) are nonnegative, we can get for all \( t \geq 0 \),

\[
\int_0^\infty \xi'(s) \| \eta_x(s) \|^2 ds = 0.
\]

By using the condition \( \xi'(s) + \mu \xi(s) \leq 0 \), we know that for all \( t \geq 0 \),

\[
\eta(x, s) = 0.
\]

Therefore \( \theta(t) = 0 \) follows from the equation (14).

This gives us that \( S(t)U_0 = U(t) = (\varphi_0, 0, \psi_0, 0, 0, 0) \) is a stationary solution, i.e., \( S(t)U_0 = U_0 \), for all \( t \geq 0 \). The proof of this lemma is therefore done.

Lemma 4.7. Under the hypotheses of Theorem 4.5, the set of equilibrium points \( \mathcal{N} \) is bounded in \( \mathcal{H} \).

Proof. Let \( U \in \mathcal{N} \) be a stationary solution of problem (11)-(16), we know that \( U = (\varphi, 0, \psi, 0, 0, 0) \) and satisfies the following equations:

\[
\begin{aligned}
- k(\varphi_x + \psi)_x + f_1(\varphi, \psi) &= h_1, \\
- b\psi_{xx} + k(\varphi_x + \psi) + f_2(\varphi, \psi) &= h_2.
\end{aligned}
\]

(51)
Now multiplying the first equation in (51) by \( \varphi \), the second equation by \( \psi \), respectively, and integrating the resultant over \((0,L)\), we have
\[
\int_0^L \left[ k(\varphi_x + \psi)^2 + b\psi_x^2 \right] dx
= -\int_0^L \left[ f_1(\varphi,\psi)\varphi + f_2(\varphi,\psi)\psi \right] + \int_0^L (h_1\varphi + h_2\psi) dx.
\]
By using (17)-(18) and (29), we can obtain
\[
-\int_0^L \left[ f_1(\varphi,\psi)\varphi + f_2(\varphi,\psi)\psi \right] \leq Lm_F + (\beta L^2 + 2\beta L^4)\|\psi_x\|^2 + 2\beta L^2\|\varphi_x + \psi\|^2.
\]
By using Young’s inequality and Poincaré’s inequality, we have
\[
\int_0^L (h_1\varphi + h_2\psi) dx \leq \frac{k}{2}\|\varphi_x + \psi\|^2 + \frac{b}{2}\|\psi_x\|^2 + \left( \frac{L^2}{2k} + \frac{L^4}{b} \right)\|h_1\|^2 + \frac{L}{b}\|h_2\|^2.
\]
Combining the above results, we can infer that
\[
\frac{1}{4}\|U\|^2_{\mathcal{H}} = \frac{k}{4}\|\varphi_x + \psi\|^2 + \frac{b}{4}\|\psi_x\|^2 \leq Lm_F + \left( \frac{L^2}{2k} + \frac{L^4}{b} \right)\|h_1\|^2 + \frac{L}{b}\|h_2\|^2,
\]
which shows that the set \( \mathcal{N} \) is bounded in \( \mathcal{H} \). The proof of this lemma is complete.

4.2.2. Quasi-stability. In this subsection, we will establish the quasi-stability of semigroup generated by global solution of problem (11)-(16).

Lemma 4.8. Assume that (17)-(23) hold, given a bounded set \( B \) of \( \mathcal{H} \), then there exist constants \( \gamma, b_0 > 0 \) and \( C_B > 0 \) such that
\[
\|S(t)U^1 - S(t)U^2\|^2_{\mathcal{H}} \leq b_0e^{-\gamma t}\|U^1 - U^2\|^2_{\mathcal{H}} + C_B \int_0^t e^{-\gamma(t-s)}\left( \|\varphi\|^2_{2p} + \|\psi\|^2_{2p} \right) ds,
\]
where \( S(t)U^i = (\varphi^i, \varphi^i_0, \psi^i_0, \theta^i_0, \eta^i_0) \) is the weak solution of problem (11)-(16) with respective to initial conditions \( U^i \) in \( B \), \( i = 1, 2 \).

Proof. For any \( (\varphi_0^i, \psi_0^i, \varphi_0^i_0, \psi_0^i_0, \theta_0^i, \eta_0^i) \in B \), \( i = 1, 2 \), where the set \( B \) is a bounded subset of \( \mathcal{H} \). Let \( U^i(t) = (\varphi^i, \varphi^i_0, \psi^i_0, \theta^i, \eta^i) \) be, with \( i = 1, 2 \), the two corresponding solutions with respect to initial data \( (\varphi_0^i, \varphi_0^i_0, \psi_0^i_0, \theta_0^i, \eta_0^i) \). We denote
\[
\varphi = \varphi^1 - \varphi^2, \quad \psi = \psi^1 - \psi^2, \quad \theta = \theta^1 - \theta^2, \quad \eta = \eta^1 - \eta^2,
\]
and
\[
G_1(\varphi) = g_1(\varphi^1) - g_1(\varphi^2), \quad G_2(\psi) = g_2(\psi^1) - g_2(\psi^2),
F_i(\varphi, \psi) = f_i(\varphi^1, \psi^1) - f_i(\varphi^2, \psi^2),
\]
then \( (\varphi, \varphi^1, \psi, \psi^1, \theta, \eta) \) satisfies
\[
\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = -F_1(\varphi, \psi) - G_1(\varphi^1), \quad \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \delta\theta_x = -F_2(\varphi, \psi) - G_2(\psi),
\]
\[
\rho_3\theta_t - \theta_{xx} - \int_0^\infty \xi(s)\eta_{xx}(s) ds + \delta\psi_{xt} = 0, \quad \eta_t + \eta_x = \theta,
\]
with Dirichlet boundary conditions and initial conditions
\[
U^1(0) - U^2(0) = (\varphi_0, \varphi^1_0, \psi^1_0, \theta_0, \eta_0).
\]
The corresponding energy $E(t)$ for (53)-(56) is the same as (24).

We shall divide into the following five steps to prove this lemma.

**Step 1.** We first claim that there exists a positive constant $C_B$ such that for any $t > 0$,

$$E'(t) \leq -l \int_0^L \theta_s^2 dx - \frac{m_1}{2} \int_0^L \psi_t^2 dx - \frac{m_2}{2} \int_0^L \psi_t^2 dx + \frac{1}{2} \int_0^\infty \xi'(s) \| \eta_x(s) \|^2 ds + C_B \left( \| \varphi \|_{2p}^2 + \| \psi \|_{2p}^2 \right). \quad (57)$$

To prove this, we multiply equations (53)-(55) by $\varphi_t, \psi_t$ and $\theta$, respectively, integrate the result over $(0, L)$ and use (56) to conclude that

$$E'(t) = -l \int_0^L \theta_s^2 dx + \frac{1}{2} \int_0^\infty \xi'(s) \| \eta_x \|^2 ds - \int_0^L \left( F_1(\varphi, \psi) \varphi_t + F_2(\varphi, \psi) \psi_t \right) dx$$

$$- \int_0^L \left( G_1(\varphi_t) \varphi_t + G_2(\psi_t) \psi_t \right) dx. \quad (58)$$

By using (19), Hölder’s inequality and Young’s inequality, we can get

$$\int_0^L F_1(\varphi, \psi) \varphi_t dx = \int_0^L \left[ f_1(\varphi^1, \psi^1) - f_1(\varphi^2, \psi^2) \right] \varphi_t dx$$

$$\leq C_1 \int_0^L \left(1 + |\varphi^1|^{p-1} + |\varphi^2|^{p-1} + |\psi^1|^{p-1} + |\psi^2|^{p-1})(|\varphi| + |\psi|) \| \varphi_t \| dx$$

$$\leq C_1 \left( \| \varphi^1 \|_{2p}^{p-1} + \| \varphi^2 \|_{2p}^{p-1} + \| \psi^1 \|_{2p}^{p-1} + \| \psi^2 \|_{2p}^{p-1} \right)(\| \varphi \|_{2p} + \| \psi \|_{2p}) \| \varphi_t \|$$

$$+ C_1 \left( \| \varphi \|_{2p} + \| \psi \|_{2p} \right) \| \varphi_t \|$$

$$\leq C_B \left( \| \varphi \|_{2p} + \| \psi \|_{2p} \right) \| \varphi_t \|$$

$$\leq \frac{m_1}{2} \| \varphi_t \|^2 + C_B \left( \| \varphi \|_{2p}^2 + \| \psi \|_{2p}^2 \right). \quad (59)$$

Similarly, we have

$$\int_0^L F_2(\varphi, \psi) \psi_t dx \leq \frac{m_2}{2} \| \psi_t \|^2 + C_B \left( \| \varphi \|_{2p}^2 + \| \psi \|_{2p}^2 \right). \quad (60)$$

It follows from (21) that

$$\int_0^L G_1(\varphi_t) \varphi_t dx \geq m_1 \| \varphi_t \|^2, \quad (61)$$

and

$$\int_0^L G_2(\psi_t) \psi_t dx \geq m_2 \| \psi_t \|^2. \quad (62)$$

Inserting (59) and (60) into (58), we can obtain the desired estimate (57).

**Step 2.** Let us define the functional

$$\phi(t) = \rho_1 \int_0^L \varphi \varphi_t dx + \rho_2 \int_0^L \psi \psi_t dx. \quad (63)$$
Taking the derivative of $\phi$ and using (53)-(54), we can obtain

$$
\phi'(t) = \int_0^L \rho_1 \varphi_t \varphi dx + \rho_1 \int_0^L \varphi_t^2 dx + \int_0^L \rho_2 \psi_t \psi dx + \rho_2 \int_0^L \psi_t^2 dx
$$

$$
= \rho_1 \int_0^L \varphi_t^2 dx + \int_0^L [(\varphi_x + \psi)_x - F_1(\varphi, \psi) - G_1(\varphi_t)] \varphi dx
$$

$$
+ \rho_2 \int_0^L \psi_t^2 dx + \int_0^L [b \psi_{xx} - k(\varphi_x + \psi) - \delta \theta_x - F_2(\varphi, \psi) - G_2(\psi_t)] \psi dx
$$

$$
= -k \int_0^L (\varphi_x + \psi)^2 dx - b \int_0^L \psi_t^2 dx + \delta \int_0^L \theta \psi_x dx + \rho_1 \int_0^L \varphi_t^2 dx
$$

$$
+ \rho_2 \int_0^L \psi_t^2 dx - \int_0^L (F_1(\varphi, \psi) \varphi + F_2(\varphi, \psi) \psi) dx
$$

$$
- \int_0^L (G_1(\varphi_t) \varphi + G_2(\psi_t) \psi) dx.
$$

(64)

By using Hölder’s and Young’s inequalities, we can get

$$
\int_0^L F_1(\varphi, \psi) \varphi dx
$$

$$
\leq C_1 (\|\varphi\|^2_{2p} + \|\varphi^2\|^2_{2p} + \|\psi\|^2_{2p} + \|\varphi\|^2_{2p})(\|\varphi\|_{2p} + \|\psi\|_{2p})\|\varphi\|_{2p}
$$

$$
+ C_1 L \frac{p-1}{p} (\|\varphi\|_{2p} + \|\psi\|_{2p})\|\varphi\|_{2p}
$$

$$
\leq C_B (\|\varphi\|^2_{2p} + \|\psi\|^2_{2p}),
$$

(65)

and

$$
\int_0^L F_2(\varphi, \psi) \psi dx \leq C_B (\|\varphi\|^2_{2p} + \|\psi\|^2_{2p}).
$$

(66)

Applying (21), we can obtain

$$
\int_0^L G_1(\varphi_t) \varphi dx = \int_0^L [g_1(\varphi_t^1) - g_1(\varphi_t^2)] \varphi dx
$$

$$
\leq L M_1 \rho_1 \|\varphi_t\|_{2p}^{p-1} \|\varphi\|_{2p}
$$

$$
\leq \rho_1 \|\varphi_t\|^2 + C_1 \|\varphi\|^2_{2p},
$$

(67)

and

$$
\int_0^L G_2(\psi_t) \psi dx \leq \rho_2 \|\psi_t\|^2 + C_1 \|\psi\|^2_{2p},
$$

(68)

It follows from Young’s inequality that

$$
\int_0^L \theta \psi_x dx \leq \frac{b}{2} \int_0^L \psi_x^2 dx + C_1 \int_0^L \theta^2 dx.
$$

(69)

Combining (65)-(69) with (64), we can conclude that

$$
\phi'(t) \leq -k \int_0^L (\varphi_x + \psi)^2 dx - \frac{b}{2} \int_0^L \psi_x^2 dx + 2 \rho_1 \int_0^L \varphi_t^2 dx + 2 \rho_2 \int_0^L \psi_t^2 dx
$$

$$
+ C_1 \int_0^L \theta^2 dx + C_B (\|\varphi\|^2_{2p} + \|\psi\|^2_{2p}).
$$

(70)
Step 3. Now we define the functional $\Psi(t)$ by
\[
\Psi(t) = \rho_2 \int_0^L \psi_t u dx,
\] (71)
where $-u_x = \theta$ with $u(0) = u(L) = 0$. Then the functional $\Psi(t)$ satisfies for any $t > 0$,
\[
\Psi'(t) \leq \frac{\rho_2 \delta}{4\rho_2} \int_0^L \varphi_t^2 dx + \left(C_1 + \frac{\delta}{2}\right) \int_0^L \theta_x^2 dx + \frac{b}{2} \int_0^L \psi_x^2 dx + C_B \int_0^L \theta^2 dx
+ \frac{k}{2} \int_0^L (\varphi_x + \psi)^2 dx + C_1 \|\eta\|^2_M + C_B (\|\varphi\|_{L^p} + \|\psi\|_{L^p}).
\] (72)
Indeed,
\[
\Psi'(t) = \rho_2 \int_0^L \psi_t u dx + \rho_2 \int_0^L \psi_t u dx := \Psi_1(t) + \Psi_2(t).
\] (73)
By using (55) and Young’s inequality, we shall see that
\[
\Psi_1(t) = \rho_2 \int_0^L \psi_t u dx
= \rho_2 \int_0^L \psi_t \delta^{-1} \left(-\frac{l}{\rho_3} \theta_x^2 - \frac{1}{\rho_3} \int_0^L \xi(s) \eta_x(s) ds - \frac{\delta}{\rho_3} \psi_t^2\right) dx
= \rho_2 \int_0^L \psi_t \left(-\frac{l}{\rho_3} \theta_x^2 - \frac{1}{\rho_3} \int_0^L \xi(s) \eta_x(s) ds - \frac{\delta}{\rho_3} \psi_t^2\right) dx
= -\frac{\rho_2 \delta}{\rho_3} \int_0^L \psi_t^2 dx - \frac{\rho_2 l}{\rho_3} \int_0^L \psi_t \theta_x dx - \frac{\rho_2}{\rho_3} \int_0^L \psi_t \int_0^L \xi(s) \eta_x(s) ds dx
\leq -\frac{\rho_2 \delta}{2\rho_3} \int_0^L \psi_t^2 dx + C_1 \int_0^L \theta_x^2 dx + C_1 \|\eta\|^2_M.
\] (74)
It follows from (54) that
\[
\Psi_2(t) = \rho_2 \int_0^L \psi_t u dx
= \int_0^L \left[b \psi_{xx} - k(\varphi_x + \psi) - \delta \theta_x - F_2(\varphi, \psi) - G_2(\psi_t)\right] u dx
= -b \int_0^L \psi_x u_x dx - k \int_0^L (\varphi_x + \psi) u dx - \delta \int_0^L \theta_x u dx
- \int_0^L F_2(\varphi, \psi) u dx - \int_0^L G_2(\psi_t) u dx.
\] (75)
By using Young’s and Poincaré’s inequalities, we arrive at
\[
-b \int_0^L \psi_x u_x dx \leq \frac{b}{2} \int_0^L \psi_x^2 dx + \frac{b}{2} \int_0^L \theta_x^2 dx,
\] (76)
\[
-k \int_0^L (\varphi_x + \psi) u dx \leq \frac{k}{2} \int_0^L (\varphi_x + \psi)^2 dx + \frac{kL^2}{2} \int_0^L \theta_x^2 dx,
\] (77)
\[
-\delta \int_0^L \theta_x u dx \leq \frac{\delta}{2} \int_0^L \theta_x^2 dx + \frac{\delta L^2}{2} \int_0^L \theta_x^2 dx.
\] (78)
From \[19\] we get
\[- \int_0^L F_2(\varphi, \psi)udx \leq C_f \int_0^L (1 + |\varphi|^p + |\psi|^p + |\varphi^2|^p + |\psi|^p)(|\varphi| + |\psi|)u dx \]
\[\leq C_1(1 + \|\varphi\|_2 + \|\varphi^2\|_2 + \|\psi\|_2)(|\varphi| + |\psi|)u \]
\[\leq C_B(\|\varphi\|_2^2 + \|\psi\|_2^2) + C_B L^2 \int_0^L \theta^2 dx. \tag{79}\]

Analogously,
\[- \int_0^L G_2(\psi_t)udx \leq \rho \delta^4 \rho_3^2 \int_0^L \psi_t^2 dx + C_1 \int_0^L \theta^2 dx. \tag{80}\]

Then the estimate \[72\] follows from \[73\] \- \[80\].

**Step 4.** We define the functional \( J(t) \) by
\[ J(t) = - \int_0^\infty \xi(s) \left( \int_0^L \rho_3 \theta(t) \eta(s) dx \right) ds. \tag{81}\]

Taking the derivative of \( J(t) \), we have
\[ J'(t) = - \int_0^\infty \xi(s) \left( \int_0^L \rho_3 \theta(t) \eta(s) dx \right) ds - \rho_3 \int_0^L \theta^2 dx + \int_0^\infty \left( \int_0^L \rho_3 \theta(t) \eta(s) dx \right) ds. \tag{82}\]

From \[13\] we get
\[ \int_0^\infty \xi(s) \left( \int_0^L \rho_3 \theta(t) \eta(s) dx \right) ds = -l \int_0^\infty \xi(s) \int_0^L \theta_{xx}(t) \eta(s) dx ds + \delta \int_0^\infty \psi_{xt}(t) \eta(s) dx ds \]
\[ - \int_0^L \left( \int_0^\infty \xi(s) \eta_{xx} ds \right) \left( \int_0^\infty \xi(s) \eta(s) ds \right) dx \]
\[ := J_1 + J_2 + J_3. \tag{83}\]

We observe that for any \( 0 < \varepsilon < 1 \),
\[ J_1 \leq l_0 \int_0^L \theta_{xx}^2 dx + \frac{1}{4} \|\eta\|_{L^4}^2, \quad J_2 \leq l_0 \|\eta\|_{L^4}^2, \tag{84}\]
and
\[ J_3 \leq \varepsilon \int_0^L \psi_t^2 dx + C \varepsilon \|\eta\|_{L^4}^2, \]
which, together with \[83\] \- \[84\], implies for any \( 0 < \varepsilon < 1 \),
\[ \int_0^\infty \xi(s) \left( \int_0^L \rho_3 \theta(t) \eta(s) dx \right) ds \leq l_0 \int_0^L \theta_{xx}^2 dx + \varepsilon \int_0^L \psi_t^2 dx + C_1 \|\eta\|_{L^4}^2. \tag{85}\]
It follows from Young’s inequality that
\[
\int_0^\infty \left( \int_0^L \rho_3 \theta(t) \eta_x(s) ds \right) ds
= \rho_3 \int_0^\infty (-\xi'(s)) \left( \int_0^L \theta(t) \eta(s) dx \right) ds
\leq \frac{\rho_3 l_0}{2} \int_0^L \theta^2 dx - \frac{\rho_3 l_1 L}{2 l_0} \int_0^\infty \xi'(s) \|\eta_x(s)\|^2 ds,
\] (86)
where
\[
l_1 = - \int_0^\infty g'(s) ds.
\]
It follows from (82)-(86) that for any \(0 < \varepsilon < 1\),
\[
J'(t) \leq - \frac{\rho_3}{2} \int_0^L \theta^2 dx + \varepsilon \int_0^L \psi_1^2 dx + \varepsilon \int_0^L \theta^2 dx - C_1 \int_0^\infty \xi'(s) \|\eta_x(s)\|^2 ds. \tag{87}
\]

**Step 5.** In the sequel we define the Lyapunov functional \(\mathcal{L}(t)\) by
\[
\mathcal{L}(t) = NE(t) + \varepsilon_1 \phi(t) + \varepsilon_2 \Psi(t) + J(t), \tag{88}
\]
where \(\varepsilon_1, \varepsilon_2 \in (0, 1)\) and \(N > 0\) are constants to be determined later. Then we can obtain that there exist two positive constants \(\beta_1\) and \(\beta_2\) such that
\[
\beta_1 E(t) \leq \mathcal{L}(t) \leq \beta_2 E(t). \tag{89}
\]
Indeed, Young’s inequality and Poincaré’s inequality give us
\[
|\phi(t)| \leq \frac{\rho_3}{2} \int_0^L \varphi_1^2 dx + \frac{\rho_3}{2} \int_0^L \psi_1^2 dx + \frac{\rho_1 L^2}{2} \int_0^L \varphi_x^2 dx + \frac{\rho_2 L^2}{2} \int_0^L \psi_x^2 dx,
\]
\[
|\Psi(t)| \leq \frac{\rho_2}{2} \int_0^L \psi_1^2 dx + \frac{\rho_2 L^2}{2} \int_0^L \theta^2 dx
\]
\[
|J(t)| \leq \frac{l_0}{2} \int_0^L \theta^2 dx + \frac{1}{2} \|\eta\|_M^2.
\]
Thus there exists a constant \(\beta_0 > 0\) such that
\[
|\mathcal{L}(t) - NE(t)| = |\varepsilon_1 \phi(t) + \varepsilon_2 \Psi(t) + J(t)| \leq \beta_0 E(t),
\]
Then we can get (89) with \(\beta_1 = N - \beta_0\) and \(\beta_2 = N + \beta_0\) by choosing \(\beta_0 > 0\) such that \(N - \beta_0 > 0\).

It follows from (51), (70), (72) and (87) that for any \(t > 0\),
\[
\mathcal{L}'(t) \leq - \left( lN - \varepsilon_2 \left( C_1 + \frac{\delta}{2} \right) - l\eta_0 \right) \int_0^L \theta_x^2 dx - \left( \frac{k}{2} \varepsilon_1 - \frac{k}{2} \varepsilon_2 \right) \int_0^L (\varphi_x + \psi_x)^2 dx
- \frac{m_1}{2} N + \varepsilon_2 \frac{\rho_2 \delta}{4 \rho_3} - 2 \rho_1 \varepsilon_1 \int_0^L \varphi_x^2 dx - \left( \frac{m_2}{2} N - 2 \rho_2 \varepsilon_1 - \varepsilon \right) \int_0^L \psi_x^2 dx
- \left( \frac{b}{2} \varepsilon_1 - \frac{b}{2} \varepsilon_2 \right) \int_0^L \psi_x^2 dx - \left( \frac{\rho_1 l_0}{2} - C_B \varepsilon_2 - C_1 \varepsilon_1 \right) \int_0^L \theta^2 dx
+ \left( \frac{N}{2} - C_1 \right) \int_0^\infty \xi'(s) \|\eta_x(s)\|^2 ds + C_1 \varepsilon_2 \|\eta\|_M^2
+ C_B \left( \|\varphi\|_p^2 + \|\psi\|_p^2 \right). \tag{90}
\]
For any fixed \( \varepsilon \in (0, 1) \), we first choose \( \varepsilon_1 > 0 \) satisfying
\[
\varepsilon_1 < \frac{\rho_3 l_0}{8C_1}.
\]
Then for any fixed \( \varepsilon, \varepsilon_1 > 0 \), we take \( \varepsilon_2 > 0 \) small enough such that
\[
\varepsilon_2 < \min \left\{ \frac{1}{2} \varepsilon_1, \frac{\rho_3 l_0}{8C_B} \right\},
\]
which implies
\[
\frac{b}{2} \varepsilon_1 - \frac{b}{2} \varepsilon_2 > 0, \quad \frac{k}{2} \varepsilon_1 - \frac{k}{2} \varepsilon_2 > 0,
\]
and
\[
\frac{\rho_3 l_0}{2} - C_B \varepsilon_2 - C_1 \varepsilon_1 > 0.
\]
For any \( \varepsilon, \varepsilon_1, \varepsilon_2 > 0 \) fixed, we at last pick the constant \( N > 0 \) large enough so that
\[
lN - \varepsilon_2 \left( C_1 + \frac{\delta}{2} \right) - ll_0 > 0, \quad \frac{m_1}{2} N + \varepsilon_2 \frac{\rho_2 \delta}{4 \rho_3} - 2 \rho_1 \varepsilon_1 > 0, \quad \frac{m_2}{2} N - 2 \rho_2 \varepsilon_1 - \varepsilon > 0,
\]
and
\[
\frac{N}{2} - C_1 > 0.
\]
Then we conclude that there exist a constant \( \gamma_0 > 0 \) and a constant \( C_B > 0 \), depending on \( B \), such that
\[
L'(t) \leq -\gamma_0 E(t) + C_B \left( \| \varphi \|_{2p}^2 + \| \psi \|_{2p}^2 \right).
\]
By using \( (\ref{89}) \), we have
\[
L(t) \leq L(0) e^{-\frac{\gamma_0}{2} t} + C_B \int_0^t e^{-\frac{\gamma_0}{2} (t-s)} \left( \| \varphi \|_{2p}^2 + \| \psi \|_{2p}^2 \right) ds.
\]
Using \( (\ref{89}) \) again, we get
\[
E(t) \leq \frac{\beta_2}{\beta_1} E(0) e^{-\frac{\gamma_0}{2} t} + C_B \int_0^t e^{-\frac{\gamma_0}{2} (t-s)} \left( \| \varphi \|_{2p}^2 + \| \psi \|_{2p}^2 \right) ds.
\]
Therefore by renaming the constants, we can get \( (\ref{52}) \). The proof is done.

**Lemma 4.9.** (Quasi-stability) The dynamical system \( (\mathcal{H}, S(t)) \) is quasi-stable on any bounded positively invariant set \( B \subset \mathcal{H} \).

**Proof.** Since the dynamics system \( (\mathcal{H}, S(t)) \) is defined as the solution operator of problem \( (\ref{33}) \), we conclude that \( (\ref{14})-(\ref{15}) \) hold with \( X = [H^1_0(0,L)]^2, Y = [L^2(0,L)]^2 \times \mathcal{M} \). Moreover, from (ii) in Theorem 3.2, we know that condition \( (\ref{40}) \) also holds true.

Let \( B \subset \mathcal{H} \) be a bounded set positively invariant with respect to \( S(t) \). Let \( S(t)U^i = (\varphi^i, \psi^i, \psi^1, \theta^i, \eta^i) \) be the solution of problem \( (\ref{11})-(\ref{16}) \) with respective initial conditions \( U^i \) in \( B, i = 1, 2 \). We define the seminorm
\[
n_X(\varphi, \psi) = \| \varphi \|_{2p}^2 + \| \psi \|_{2p}^2,
\]
where \( (\varphi, \psi) = (\varphi^1 - \varphi^2, \psi^1 - \psi^2) \). By using the compact embedding \( H^1 \hookrightarrow L^{2p} \), we know that \( n_X(\cdot) \) is a compact in \( X \). From \( (\ref{52}) \), we infer that
\[
\| S(t)U^1 - S(t)U^2 \|_H^2 \leq b(t)\| U^1 - U^2 \|_H^2 + c(t) \sup_{0 < s < t} \left[ n_X(\varphi(s) - \psi(s)) \right]^2,
\]
For any bounded positively invariant set \( X \) of problem \( (\ref{33}) \), we conclude that \( (\ref{44})-(\ref{45}) \) hold with
where
\[ b(t) = b_0 e^{-\gamma t}, \quad c(t) = C_B \int_0^t e^{-\gamma (t-s)} ds, \quad t \geq 0. \]

It is easy to get that
\[ b(t) \in L^1(\mathbb{R}^+) \quad \text{and} \quad \lim_{t \to \infty} b(t) = 0. \]

Since \( B \subset H \) is bounded, we know that \( c(t) \) is locally bounded on \([0, \infty)\). Therefore, the stabilizability inequality (47) holds, that is, the dynamics system \((H, S(t))\) is quasi-stable on any bounded positively invariant set \( B \subset H \). The proof is hence complete.

**Proof of Theorem 4.5.** From Theorem 5.2 and Lemma 4.9, we know that the dynamical system \((H, S(t))\) is asymptotically smooth.

For any \( R > 0 \), the set \( \Phi_R = \{ U \in H | \Phi(U) \leq R \} \), it follows from (27) that for any \( U(t) \in \Phi_R \),
\[ \| U(t) \|_H^2 \leq 4(\Phi(U(t))) + 4K \leq 4(R + K), \]
which gives us the set \( \Phi_R \) is bounded.

Then by using Lemma 4.7 and Theorem 4.3 we can get the problem (11)-(16) has a compact global attractor given by \( A = M + N \).

Moreover, from Lemma 4.9, the dynamical system \((H, S(t))\) is quasi-stable on the attractor \( A \), then by using Theorem 4.4, we know that the attractor \( A \) has finite fractal dimension. The proof is done.

5. **Exponential attractor.** In this section, we shall prove the existence of exponential attractor, and the main result is stated in the following theorem.

**Theorem 5.1.** Assume that (17)-(23) hold, then the dynamical system \((H, S(t))\) possesses a generalized exponential attractor. More precisely, for any \( \delta \in (0,1] \), there exists a generalized exponential attractor \( \mathfrak{A}_{exp,\delta} \subset H \), with finite fractal dimension in extended space \( \tilde{H}_{-\delta} \), defined as interpolation of
\[ \tilde{H}_0 := H, \quad \text{and} \quad \tilde{H}_{-1} := [L^2(0, L) \times H^{-1}(0, L)]^2 \times H^{-1}(0, L) \times \mathcal{M}_0, \]
where \( \mathcal{M}_0 = L^2(L^2(0, L))^2 \).

Before the proof of **Theorem 5.1**, we first briefly introduce some basic theory to exponential attractor. An exponential attractor of a dynamical system \((X, S(t))\) is a compact set \( \mathfrak{A}_{exp} \subset X \), that satisfies three characteristic properties: (i) it has finite fractal dimension, (ii) it is positively invariant, and (iii) it attracts exponentially fast the trajectories from any bounded set of initial data. That is, for any bounded set \( B \subset X \), there exist constants \( t_B, C_B > 0 \) and \( \gamma_B > 0 \) such that for all \( t \geq t_B \),
\[ \text{dist}_H(S(t)B, \mathfrak{A}_{exp}) \leq C_B \exp(-\gamma_B (t - t_B)). \]

In the paper we consider the concept of generalized exponential attractors which stated in Chueshov and Lasiecka \[10, 11\]. Its main difference is that the set \( \mathfrak{A}_{exp} \) has finite fractal dimension in an extended phase space \( \tilde{H} \supseteq H \). Hence we can consider exponential attractors in weaker phase spaces. Moreover, we can obtain the existence of generalized exponential attractors if the dynamical system is quasi-stable.

We use the following theorem which can be found in Chueshov and Lasiecka \[11\], Theorem 7.9.9, to prove **Theorem 5.1**.
Theorem 5.2. Let $(\mathcal{H}, S(t))$ be a dynamical system satisfying \(^{(44)}\) and \(^{(45)}\) and quasi-stable on some bounded absorbing set $\mathfrak{B}$. In addition assume there exists an extended space $\tilde{\mathcal{H}} \supseteq \mathcal{H}$ such that, for any $T > 0$,

$$\|S(t_1)y - S(t_2)y\|_{\tilde{\mathcal{H}}} \leq C_{\mathcal{B}T}|t_1 - t_2|^\gamma, \quad t_1, t_2 \in [0, T], \ y \in \mathfrak{B},$$

(93)

where $C_{\mathcal{B}T} > 0$ and $\gamma \in (0, 1)$ are constants. Then the dynamical $(\mathcal{H}, S(t))$ has a generalized exponential attractor $\mathfrak{A}_{exp} \subset \mathcal{H}$ with finite fractal dimension in $\mathcal{H}$.

Proof of Theorem 5.1. Now we take

$$\mathfrak{B} = \{U | \Phi(U) \leq R\},$$

where $\Phi$ is the strict Lyapunov functional considered in Lemma \(^{(4.6)}\). Then we know that the set $\mathfrak{B}$ is a positively invariant absorbing set for $R$ large enough. Hence the system $(\mathcal{H}, S(t))$ is quasi-stable on $\mathfrak{B}$.

For solution $U(t)$ with initial data $y = U(0) \in \mathfrak{B}$, we can conclude from the positive invariance of $\mathfrak{B}$ that there exists $C_B > 0$ such that for any $0 \leq t \leq T$,

$$\|U(t)\|_{\tilde{\mathcal{H}}^-} \leq C_B,$$

which gives us for any $0 \leq t_1 < t_2 \leq T$,

$$\|S(t_1)y - S(t_2)y\|_{\tilde{\mathcal{H}}^-} \leq \int_{t_1}^{t_2} \|U(\tau)\|_{\tilde{\mathcal{H}}^-} d\tau \leq C_B|t_1 - t_2|.$$

(94)

It follows from (94) that for any $y \in \mathfrak{B}$, the map $t \mapsto S(t)y$ is Hölder continuous in the extended space $\tilde{\mathcal{H}}$ with exponent $\delta = 1$. Hence (93) holds and then we obtain the existence of a generalized exponential attractor whose fractal dimension is finite in $\tilde{\mathcal{H}}^-$. The existence of exponential attractors in $\tilde{\mathcal{H}}^-\delta$ with $\delta \in (0, 1)$, one can refer to the same arguments in \(^{[5]}\);\(^{[24]}\). The proof is therefore complete.

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