Nonsimilar solution of a boundary layer flow of a Reiner–Philippoff fluid with nonlinear thermal convection

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Abstract

The thermodynamics modeling of a Reiner–Philippoff-type fluid is essential because it is a complex fluid with three distinct probable modifications. This fluid model can be modified to describe a shear-thinning, Newtonian, or shear-thickening fluid under varied viscoelastic conditions. This study constructs a mathematical model that describes a boundary layer flow of a Reiner–Philippoff fluid with nonlinear radiative heat flux and temperature- and concentration-induced buoyancy force. The dynamical model follows the usual conservation laws and is reduced through a nonsimilar group of transformations. The resulting equations are solved using a spectral-based local linearization method, and the accuracy of the numerical results is validated through the grid dependence and convergence tests. Detailed analyses of the effects of specific thermophysical parameters are presented through tables and graphs. The study reveals, among other results, that the buoyancy force, solute and thermal expansion coefficients, and thermal radiation increase the overall wall drag, heat, and mass fluxes. Furthermore, the study shows that amplifying the...
space and temperature-dependent heat source parameters allows fluid particles to lose their cohesive force and, consequently, maximize flow and heat transfer.

**Keywords**
boundary layer flow, nonlinear convection, nonlinear radiation, Reiner–Philippoff fluid

# Introduction

Nonlinearity is an art. The rational nonlinear relationship between the shear stress and strain of a non-Newtonian fluid is the rheological distinction between Newtonian and non-Newtonian fluids. In the last six decades, the study of Newtonian fluid has been well covered and addressed in the literature.\(^1,2\) The nonlinear interaction of shear stress and strain in a non-Newtonian fluid may be represented mathematically as

\[
\mathcal{F} \left( \frac{\partial u}{\partial y}, \tau \right) = 0. \tag{1}
\]

Equation (1) gives further classifications among the non-Newtonian fluids depending on the complexity that exists between \(\tau\) and \(\frac{\partial u}{\partial y}\). These subdivisions encompass, among others, second-grade fluids, third-grade fluids, and fourth-grade fluids. The complexity of the nonlinear structure of Equation (1) makes it difficult to obtain both analytical and approximate solutions for the non-Newtonian fluid. Reiner–Philippoff, Eyring–Powell, Prandtl–Powell, Johnson–Segalman, Ellis, Casson, and Carreau–Yasuda fluids are examples of models with complex nonlinear structures. The viscosity of time-independent non-Newtonian fluids is only dependent on shear rate and temperature.\(^3\) In this study, the Reiner–Philippoff fluids with probable relaxation to dilatant, Newtonian, and pseudoplastic in varied viscoelastic conditions are of particular interest. Animasaun et al.\(^4\) further confirmed that this fluid could be characterized as shear-thinning or pseudoplastic in which viscosity decreases with an increased shear rate, shear thickening, or dilatant in which viscosity increases with a decreased shear rate, and plastic in which a certain magnitude of shear stress must be applied before flow occurs. Due to the wide variety of physical and molecular structures seen in non-Newtonian fluids, it is not easy to offer a single constitutive relation that can be utilized to characterize these fluids. At the beginning of the 20th century, an important concept that revolutionized the study of fluid flow by unifying the theoretical study with experimental observation was pioneered by Prandtl.\(^5\)

Prandtl divided the flow field into two areas: one within the boundary layer, where viscosity dominates and causes the bulk of the drag experienced by the boundary body, and one outside the boundary layer, where viscosity may be ignored without substantial consequences on the behavior of the fluid flow. There have been debates on solution techniques and behavior of fluid flow in boundary layer theory; see Pantokratoras.\(^6\) A similar solution (similar transformation) is the result of the appropriate scaling of a partial differential equation to its ordinary differential equation equivalent. This transformation means that the equation has a similar solution and is dimension and position invariant. This class of solutions, known as
similar solutions, has traditionally played an essential role because it is the only class of precise solutions for the boundary layer equations. However, this class of solutions is limited due to the level of nonlinearity specified in Equation (1). According to Hansen and Na, the similar transformation approach for a boundary layer flow of a Reiner–Philippoff fluid is only conceivable when the flow is operating at an angle of 90°. Hansen and Na reported that for a plausible similar solution to exist, the free stream velocity should take the form \( u_w = ax^{\frac{1}{3}} \). Na presented a nonsimilar solution for a Reiner–Philippoff boundary layer flow over a general body/medium. The investigation is restricted to the continuity and momentum equations with the pressure gradient as the only body force. Patil et al. extended the study in Na to a three-dimensional flow. Patil et al. argued that for the flow past a 90° wedge, similar solutions exist. Consequently, a nonsimilar transformation is suitable for the flow inclined at any arbitrary angle. Yam et al. investigated the flow of a Reiner–Philippoff fluid past a continuously expanding or contracting wedge. The authors performed stability analysis on the self-similar solution and discovered that it is unique and stable.

Some notable studies on the dynamics of Reiner–Philippoff fluid include the study of Khan et al. which modeled a non-Darcian medium with an optimized framework of second law analysis and activation energy. Khan et al. employed the nonuniform heat source/sink and the generalized heat flux phenomenon. The study predicted thermal energy reduction as the modified thermal relaxation number increases and positive radiation impacts both irreversibility and entropy generation rate. By assuming a stretchable sheet with variable thickness, Ahmad et al. characterized the flow of Reiner–Philippoff fluid by employing a similar approach. Recently, Sajid et al. investigated the influence of temperature-dependent heat sink and source on the Reiner–Philippoff fluid with changeable concentration diffusivity and concluded that enhancement of heat generation increases the thermal boundary layer thickness. The second law irreversibility analysis subjected to the Darcy–Forchheimer model for Reiner–Philippoff fluid was addressed by Xiong et al. Mallikarjuna et al. employed spectral quasilinearization and statistical techniques to analyze Reiner–Philippoff fluid flow over an inclined stretchable sheet. The study concluded that the dilatant and pseudoplastic fluids exhibit different behaviors as the Bingham number increases.

In another related study of a non-Newtonian fluid, Bilal et al. investigated the nonisothermal flow of a Williamson fluid subjected to a modified heat flux (Cattaneo–Christov) hypothesis. The magnetic field suppresses the velocity field and accompanying boundary layer region. Furthermore, a drop in temperature profile and heat transfer coefficient is seen when the thermal relaxation parameter increases in magnitude. The influence of a magnetic field and different slip circumstances on a reactive Jeffery fluid trapped in an exponentially stretching leaky medium was investigated by Bilal et al.

The study of heat and mass transfer for Reiner–Philippoff fluid is a burgeoning research area in scientific literature. However, to the authors’ knowledge, no investigation has generalized the choice of utilizing an arbitrary stream function velocity on thermal and species equations. Furthermore, most studies have been limited to considering the free stream velocity admissible to similar solutions. The analysis here reveals the behavior of Reiner–Philippoff fluid flow by utilizing a nonadmissible similar free stream velocity under the effect of some thermophysical parameters.

The study is set up as follows: the mathematical analyses of the flow and generic transformation technique via the nonsimilar methods are presented in Section 2. The numerical method employed in approximating the solution of the resulting dimensionless boundary value problem is discussed in Section 3. Section 4 contains the results and discussions
of the pertinent flow properties, which include the use of graphical and tabular representation of results. The study is finalized in Section 5.

2 | MODEL FORMULATION

The wedge angle parameter \( F = \frac{2\Upsilon}{1+\Upsilon} \), which equates \( E_\theta = F\pi \) for the overall angle of the wedge, where \( \Upsilon \in [0,1] \) is the power of the stretching wedge. Nonlinear shear-stress deformation behavior is exemplified by the Reiner–Philippoff fluid. This shear-stress deformation relation is given by

\[
\frac{\partial u}{\partial y} = \frac{\tau}{\mu_\infty + \frac{\mu_\infty - \mu_\infty}{1 + (\pi)^2}}.
\]

The above fluid model is a three-parameter model that illustrates shear rate-limiting viscosities at high and low shear stress. With a significant variation in viscosity, the Reiner–Philippoff fluid behaves as a dilatant, Newtonian, and pseudoplastic fluid. In dimensionless form, the flow function takes the form

\[
f(\xi) = \frac{\xi}{1 + \frac{\lambda - 1}{1 + \xi^2}}
\]

with \( \xi = \frac{\tau_s}{\nu} \) and \( \lambda = \frac{\mu_o}{\mu_\infty} \) is the Reiner–Philippoff fluid parameter, where

\[
\lambda = \begin{cases} <1, & \text{Dilatant,} \\ =1, & \text{Newtonian,} \\ >1, & \text{Pseudoplastic.} \end{cases}
\]

Consider a two-dimensional boundary flow of a steady incompressible chemically reactive Reiner–Philippoff fluid as seen in Figure 1. The free stream velocity is assumed to vary
nonlinearly (can be ascribed to pressure gradient) in the direction of flow. The thermal conductivity and mass diffusivity are dependent on temperature and concentration, respectively. The magnetic field acting in the flow direction is nonuniformly transverse to the flow direction with strength $B(x) = B_0 x^{-1/3}$ with nonlinear thermal convection. Nonlinear thermal radiative heat flux in its full form is employed without the usual truncation of the higher-order terms in the expansion. The flow equations regarding mass, momentum, energy, and concentration conservation are similar to those in Na, Ahmed et al., and Sajid et al.

\[
\frac{\partial v}{\partial y} = -\frac{u}{\partial x},
\]

\[
u_w \frac{du_w}{dx} + \frac{1}{\rho} \frac{\partial \tau}{\partial y} - \frac{\sigma B^2 u}{\rho} + g_a [\alpha_1 (T - T_\infty) + \alpha_2 (T - T_\infty)^2]
+ g_a [\alpha_3 (C - C_\infty) + \alpha_4 (C - C_\infty)^2] = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y},
\]

\[
\frac{\sigma B^2 u^2}{\rho c_p} + \frac{1}{\rho c_p} \frac{\partial}{\partial y} \left( \chi(T) \frac{\partial T}{\partial y} \right) + \frac{1}{\rho c_p} \frac{r_w u_w}{x_v} [B_1 e^{-\eta} (T_w - T_\infty) + B_2 (T - T_\infty)] - \frac{1}{\rho c_p} \frac{\partial q_r}{\partial y} = u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y},
\]

\[
\frac{\partial}{\partial y} \left( D_k(C) \frac{\partial C}{\partial y} \right) - k_r(x)(C - C_\infty) = u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y},
\]

where $\tau$ is the shear-stress deformation relation defined in Equation (2). The appropriate boundary conditions are

\[
\begin{align*}
u(x, y) &= 0, \quad v(x, y) = 0, \quad T(x, y) = T_w, \quad C(x, y) = C_w \quad \text{as} \quad y = 0, \\
u(x, y) &\to u_w, \quad T(x, y) \to T_\infty, \quad C(x, y) \to C_\infty, \quad \text{as} \quad y \to \infty.
\end{align*}
\]

$k_r(x)$ represent variable chemical reaction rate define as $k_r(x) = k_r x^{-2}$ (see Vaidya et al.,), $k(T)$ and $D_k(C)$ are temperature-dependent thermal conductivity and concentration-dependent mass-diffusivity defined as

\[
k(T) = r_w \left( 1 + \Lambda_1 \frac{T - T_\infty}{T_w - T_\infty} \right) \quad \text{and} \quad D_k(C) = D_\infty \left( 1 + \Lambda_2 \frac{C - C_\infty}{C_w - C_\infty} \right).
\]

The choice of concentration-dependent mass-diffusivity is justified following the experimental study of Bardow et al. The exact form of the variation is supported by Sajid et al. and some of the references therein. The radiative heat flux resulting from the Rosseland approximation is defined as

\[
q_r = -\frac{4 \sigma^*}{3k_3} \frac{\partial T^4}{\partial y} = -\frac{16 \sigma^*}{3k_1} \left[ T^3 \frac{\partial T}{\partial y} \right].
\]

The search for a similar solution for a Reiner–Philippoff fluid has generated some research attention. Na and Hansen and Hansen and Na showed that flow over a stretching wedge for
reduced equations (5)–(7) has a similar solution only if the referencing velocity \( u_w = x \). However, due to other flow conditions, the need to investigate a nonsimilar solution becomes important.

### 2.1 Nonsimilar solution

In the broader sense, if the boundary layer flow over any body has to be examined, the nonsimilar transformation is the best fit. By applying the following nonsimilar transformation on the model equations

\[
\begin{align*}
\zeta &= x, \quad \eta = \sqrt{\frac{u_w}{X}} y, \quad f(\eta, \zeta) = -\psi \sqrt{\frac{u_w}{X}} g(\eta, \zeta), \quad \tau = \rho \sqrt{\frac{u_w}{X}} y g(\eta, \zeta), \quad \theta(\eta, \zeta) = \frac{T - T_\infty}{T_w - T_\infty}, \quad C(\eta, \zeta) = \frac{C - C_\infty}{C_w - C_\infty}.
\end{align*}
\]

Equations (2) and (5)–(8) yield

\[
\begin{align*}
g' + \frac{1 + P(\zeta)}{2} f'' &= \frac{u_w^3}{R} + \gamma \frac{g'}{g'}, \\
(1 + \Lambda_1 \theta) \theta'' + \Lambda_1 \theta^2 &= \frac{4}{3} R \left[ (1 + (\theta_w - 1) \theta)^3 \theta'' + 3(1 + (\theta_w - 1) \theta)^2 (\theta_w - 1) \theta'' \right] \\
&= -B_1 e^{-\eta} - B_2 \theta - \frac{1 + P(\zeta)}{2} Prf \theta' - PrME \epsilon^2 + Pr^2 \left( f' \frac{\theta'}{\zeta} - \theta' \frac{f'}{\zeta} \right), \\
(1 + \Lambda_2 \phi) \phi'' + \Lambda_2 \phi^2 &= \frac{1 + P(\zeta)}{2} Scf \phi' + k_s Sc \phi + Sc^2 \left( f' \frac{\phi'}{\zeta} - \phi' \frac{f'}{\zeta} \right),
\end{align*}
\]

where \( P(\zeta) = \frac{\zeta}{u_w} \frac{du_w}{d\zeta}, \lambda = \frac{\mu_w}{\mu}, \) and \( \gamma = \left( \frac{\tau_1}{\rho_1 \sqrt{\gamma}} \right) \). The equations are subject to the following boundary conditions:

\[
\begin{align*}
f'(\eta, \zeta) &= 0, \quad f(\eta, \zeta) = 0, \quad \theta(\eta, \zeta) = 1, \quad \phi(\eta, \zeta) = 1, \quad \text{at} \quad \eta = 0, \\
f'(\eta, \zeta) &\rightarrow 1, \quad \theta(\eta, \zeta) &\rightarrow 0, \quad \phi(\eta, \zeta) &\rightarrow 0, \quad \text{as} \quad \eta \rightarrow \infty.
\end{align*}
\]

The engineering parameters that are of interest in their nonsimilar form become

\[
\begin{align*}
\frac{1}{2} Re_x^2 C_i &= g(0, \zeta), \quad Re_x^{-\frac{1}{2}} Nu_x = -\left( 1 + \Lambda_1 \theta(0, \zeta) + \frac{4}{3} R \left[ (1 + (\theta_w - 1) \theta(0, \zeta))^3 \right] \right) \\
\theta'(0, \zeta), \\
Re_x^{-\frac{1}{2}} Sh_x &= -(1 + \Lambda_2 \phi(0, \zeta)) \phi'(0, \zeta),
\end{align*}
\]
where \( Re = \frac{u_w x}{\nu} \) is the Reynolds number and the dimensionless parameters are defined as follows:

\[
\begin{align*}
\omega_1 &= \frac{g\alpha_1 (T_w - T_\infty) \zeta}{u_w^2}, \quad A_1 = \frac{\omega_1}{Re}, \quad \omega_2 = \frac{g\alpha_3 (C_w - C_\infty) \zeta}{u_w^2}, \quad A_2 = \frac{\omega_2}{Re}, \quad \alpha_t \\
\alpha_c &= \frac{\alpha_4 (C_w - C_\infty)}{\alpha_3}, \quad M = \frac{\sigma B_0^2 u_w \zeta}{\rho}, \quad \gamma = \frac{4\sigma \eta^2}{k_n r_\infty}, \quad Pr = \frac{c_p \mu}{r_\infty}, \quad \theta_w = \frac{T_w}{T_\infty}, \quad Sc = \frac{\nu}{D_\infty}, \\
E_c &= \frac{u_w^2}{c_p (T_w - T_\infty)}, \quad k' = \frac{k_1 \zeta^{\frac{3}{2}}}{u_w}.
\end{align*}
\]

(19)

3 | NUMERICAL METHOD

The systems of equations from the nonsimilar transformation were solved numerically using a spectral-based local linearization method. The equations were linearized locally using the linearization technique of Bellman and Kalaba.\(^{27}\) The velocity, temperature, and concentration are approximated as linear combinations of the Lagrange interpolation polynomials on the transformed Chebyshev–Gauss–Lobatto nodes:

\[
\{\eta_j, \zeta_f\} = \left\{\eta_\infty, \zeta_f\right\} = \frac{1}{2} \left(-\cos \frac{\pi \{i, j\}}{\{M_\eta, M_\zeta\}} + 1\right), \quad i = 0, 1, ..., M_\eta; \quad j = 0, 1, ..., M_\zeta,
\]

(20)

where \( \eta_\infty \) is the truncation of the infinite \( \eta \)-domain, \( \zeta_f \) is the finite value of the standardized axial coordinate, and \( M_\eta \) and \( M_\zeta \) are the numbers of grid points in the \( \eta \) and \( \zeta \) domains, respectively. The ensuing detached linearized algebraic systems are solved iteratively for a set number of iterations. See Motsa\(^{28}\) and Motsa and Animasaun\(^{29}\) for an exhaustive implementation of this numerical approach to bivariate systems of differential equations. The spectral-based local linearization scheme for the nonsimilar equations is obtained as

\[
\begin{align*}
G_{s+1,j} &= \frac{u_w^3 (\eta_j)}{u_w^3 (\zeta_f)} G_{s,j}^2 + \frac{\lambda y^2 \zeta_f}{\gamma^2 \zeta_f} F_{s,j}^r \\
&= K_{2n,s,j} \\
[\text{diag}[a_{0n,s,j}] D^2 + \text{diag}[a_{1n,s,j}] D + \text{diag}[a_{2n,s,j}] + \text{diag}[a_{3n,s,j}]] F_{s+1,j} \\
+ \text{diag}[a_{4n,s,j}] \sum_{m=0}^{M_\zeta} d_{j,m} F_{s+1,j} \\
&= K_{1n,s,j} \\
[\text{diag}[b_{0n,s,j}] D^2 + \text{diag}[b_{1n,s,j}] D + \text{diag}[b_{2n,s,j}] \Theta_{s+1,j} + \text{diag}[b_{3n,s,j}] \sum_{m=0}^{M_\zeta} d_{j,m} \Theta_{s+1,j}] \\
&= K_{3n,s,j} \\
[\text{diag}[c_{0n,s,j}] D^2 + \text{diag}[c_{1n,s,j}] D + \text{diag}[c_{2n,s,j}] \Phi_{s+1,j} + \text{diag}[c_{3n,s,j}] \sum_{m=0}^{M_\zeta} d_{j,m} \Phi_{s+1,j}] \\
&= K_{3n,s,j},
\end{align*}
\]

(21)
where

\[
\begin{align*}
    a_{0ns,j} &= -\frac{u_w^3(\zeta)}{u_w^3(\zeta)}G_{s+1,j} + \lambda \gamma^2 \zeta_j, \\
    a_{1ns,j} &= \frac{2G_{s+1,j} u_w^3(\zeta)\gamma^2 \zeta_j (1 - \lambda)}{\left(u_w^3(\zeta)G_{s+1,j} + \gamma^2 \zeta_j\right)^2} + \frac{1 + P(\zeta)}{2} F_{s,j} + \zeta_j \sum_{m=0}^{M_s} d_{j,m} F_{s,m}, \\
    a_{2ns,j} &= -2P(\zeta) F_{s,j} M - \zeta_j \sum_{m=0}^{M_s} d_{j,m} F_{s,m}, \\
    a_{3ns,j} &= \frac{1 + P(\zeta)}{2} F_{s,j}, \\
    a_{4ns,j} &= -\zeta_j F_{s,j},
\end{align*}
\]

\[
\begin{align*}
    b_{0ns,j} &= (1 + A_1 \Theta_{s,j}) + \frac{4}{3} R (1 + (\theta_w - 1) \Theta_{s,j})^3, \\
    b_{1ns,j} &= 2A_1 \Theta_{s,j}^2 + 8R (\theta_w - 1) (1 + (\theta_w - 1) \Theta_{s,j})^2 \Theta_{s,j} + \frac{1 + P(\zeta)}{2} P_{s+1,j} + P_{s+1,j} \sum_{m=0}^{M_s} d_{j,m} F_{s+1,m}, \\
    b_{2ns,j} &= A_1 \Theta_{s,j}^2 + 4R (\theta_w - 1) (1 + (\theta_w - 1) \Theta_{s,j})^2 \Theta_{s,j}^2 + 8R (\theta_w - 1)^2 (1 + (\theta_w - 1) \Theta_{s,j}) \Theta_{s,j}^2 + B_2, \\
    b_{3ns,j} &= -P_{s+1,j} \Theta_{s,j}^2
\end{align*}
\]

\[
\begin{align*}
    c_{0ns,j} &= 1 + A_2 \Phi_{s,j}, \\
    c_{1ns,j} &= 2A_2 \Phi_{s,j} + \frac{1 + P(\zeta)}{2} Sc_{s+1,j} + Sc_{s+1,j} \sum_{m=0}^{M_s} d_{j,m} F_{s+1,m}, \\
    c_{2ns,j} &= A_2 \Phi_{s,j}^2 - k_r Sc, \\
    c_{3ns,j} &= -Sc_{s+1,j} \Phi_{s,j}
\end{align*}
\]

and

\[
\begin{align*}
    K_{1ns,j} &= -P(\zeta_j) \left(1 + F_{s,j}^2 + \frac{1 + P(\zeta)}{2} F_{s,j} \sum_{m=0}^{M_s} d_{j,m} F_{s,m} + \zeta_j \sum_{m=0}^{M_s} d_{j,m} F_{s,m}\right) - A_1 (1 + \alpha_i \Theta_{s,j}) \Theta_{s,j} - A_2 (1 + \alpha_r \Phi_{s,j}) \Phi_{s,j}, \\
    K_{2ns,j} &= A_1 \Theta_{s,j}^2 + A_1 \Theta_{s,j} \Theta_{s,j} - B_1 e^{-\eta} - PrMec F_{s+1,j}^2 + 4R (\theta_w - 1) (1 + (\theta_w - 1) \Theta_{s,j}) \Theta_{s,j}^2 \\
    &+ 4R (\theta_w - 1) (1 + (\theta_w - 1) \Theta_{s,j}) \Theta_{s,j}^2 + 8R (\theta_w - 1)^2 (1 + (\theta_w - 1) \Theta_{s,j}) \Theta_{s,j}^2 \Theta_{s,j}, \\
    K_{3ns,j} &= A_2 \Phi_{s,j}^2 + A_2 \Phi_{s,j} \Phi_{s,j}.
\end{align*}
\]

In the above set of linearized algebraic equations, \(D\) and \(d\) are the scaled Chebyshev differentiation matrices in \(\eta\) and \(\zeta\), respectively, based on the mapping \([-1, 1]^1 \rightarrow [0, \eta_{\infty}] \times [0, \zeta_{\infty}]\). See Trefethen.\(^{30}\)

### 3.1 | Grid independency

To establish that the results in the current study are independent of the number of grid points, numerical simulations were carried out with \(50 \times 10, 50 \times 20, 50 \times 30, 50 \times 40, 50 \times 50\) grids. The skin friction, Nusselt number, and Sherwood number at \(\zeta = 1\) and 2 are calculated using
the following parameter values: \( \lambda = 1, \gamma = 0.5, A_1 = A_2 = 0.01, \alpha_1 = \alpha_c = 0.3, \Lambda_1 = \Lambda_2 = 0.01, R = \theta_w = Sc = k_r = 0.1, B_1 = B_2 = 0.01, Pr = 0.7, M = Ec = 0. \\

These results are presented in Table 1, and based on the values in the table, it is preferable to use a \( 50 \times 30 \) grid to solve the nonsimilar system of equations. These equations are solved successively for a defined number of iterations using suitable initial guesses.

### 3.2 Numerical validation

The accuracy of the computational results obtained from the present numerical simulation is validated by comparing the results from Na\(^8\) with the present study. These comparisons are presented in Table 2 and are seen to agree. It is noted that the Newtonian flow for the nonsimilar case is similar when the reference velocity, \( u_w \), is uniform, and this alludes to why the skin friction for Newtonian flow is the same for all values of \( \zeta \) in Table 2 as purported by Na\(^8\).

### 3.3 Numerical convergence

To establish the convergence of the numerical solution, the convergence of the skin friction, Nusselt, and Sherwood numbers after each iteration of the numerical simulations is presented. The simulation was carried out for 25 iterations. The convergence of the aforementioned physical properties is depicted in Figure 2. As these physical properties converge quickly, the

| Grids \( M_y \times M_x \) | \( \frac{1}{2} \text{Re}^2 \text{C}_{\text{f}} \) \( \zeta = 1 \) | \( \zeta = 2 \) | \( \text{Re}^{-1} \text{Nu}_{\text{x}} \) \( \zeta = 1 \) | \( \zeta = 2 \) | \( \text{Re}^{-2} \text{Sh}_{\text{x}} \) \( \zeta = 1 \) | \( \zeta = 2 \) |
|-------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( 50 \times 10 \)     | 0.6092          | 0.6383          | 0.3358          | 0.3428          | 0.1888          | 0.1906          |
| \( 50 \times 20 \)     | 0.5957          | 0.6432          | 0.3336          | 0.3436          | 0.1880          | 0.1909          |
| \( 50 \times 30 \)     | 0.6019          | 0.6442          | 0.3347          | 0.3438          | 0.1883          | 0.1909          |
| \( 50 \times 40 \)     | 0.5981          | 0.6447          | 0.3340          | 0.3439          | 0.1881          | 0.1910          |
| \( 50 \times 50 \)     | 0.6008          | 0.6449          | 0.3345          | 0.3439          | 0.1883          | 0.1910          |

| \( \zeta \) | \( \text{g} \) for all \( \zeta \)s |
|------------|-------------------------------|
| Na\(^8\)   | 0.33206                        |
| Present    | 0.3320574                      |
numeral results used in the analysis presented in Section 4 are the numerical results that are computed after 10 iterations.

4 | RESULTS AND DISCUSSIONS

In this section, analyses of the effects of parametric values of some parameters on the flow profiles are presented. Unless varied in the figures and tables or stated otherwise, the following base values for the parameters $\lambda = 2$, $\gamma = 0.5$, $M = 0.02$, $A_1 = A_2 = \alpha_1 = \alpha_c = 0.6$, $\Lambda_1 = \Lambda_2 = 0.01$, $R = 0.1$, $\partial_w = 0.3$, $B_1 = B_2 = 0.2$, $Pr = 0.7$, $Ec = 0.1$, $Sc = 0.1$, $k_r = 0.5$ were chosen.

This study aimed to identify the impacts of some flow parameters and their importance on the free stream velocity. The solution is highly stable and ascertains the spectral method’s computational ability.

Figure 3 profiled the Reiner–Philippoff rheological characterization in Figure 3A the flow momentum, Figure 3B energy, and Figure 3C concentration profiles. The characterization of pseudoplastic, that is, shear-thinning ($\lambda > 1$) fluid, maintains low velocity near the surface, which results in higher temperature and concentration of the fluid (see Figure 3A,B). The velocity improves into the free stream due to the injected heat source and radiation impact that decompose the frictional force within the particle, hence neutralizing in the region where viscous forces are no longer significant. Dilatant ($\lambda < 1$), that is, a shear-thickening fluid, on the other hand, displayed a higher flow field near the surface compared with the Newtonian and the pseudoplastic fluids with a minimal temperature, concentration, and velocity into the

**FIGURE 2** Convergence of the skin friction, Nusselt, and Sherwood numbers for the nonsimilar solution.
free stream region. However, Newtonian fluid ($\lambda = 1$) finds its behavior between the shear-thickening and shear-thinning fluids.

With the enormous usage of pseudoplastic fluids, such as paint, honey, toothpaste, blood rheology, hydraulic brakes, engine oil in industries, medicine, and engineering, this investigation modeled its findings for $\lambda = 2$. Figures 4 and 5, respectively, depict the graphical representation of the impacts of linear and nonlinear convection parameters on the velocity field. For $\alpha_t = \alpha_c = 0$, the buoyancy forces in the model equation (14) reduce to linear forms, the instance in which the Boussinesq approximation is valid. However, due to the viscosity of the moving fluid, and the interaction between the surface and the fluid fluxes, the current model proved the linear model void; hence the quadratic model. Evidently, an increase in the values of $A_t, A_c, \alpha_t, \alpha_c$ predicts a higher flow rate as depicted in Figures 4 and 5 because the linear convection parameters are ratios of the buoyancy force to the viscous force. However, the nonlinear convection terms are expected to portray the same behavior as the linear convection because they are just additional terms to the latter term. Meanwhile, the introduction of $\alpha_t, \alpha_c$ indicates how what extent the convection terms affect the flow field since it is well known that higher-order terms increase the contribution of convection.
The effects of space-dependent and temperature-dependent heat source parameters on the energy fields are, respectively, presented in Figure 6A,B. An increase in $B_1$ and $B_2$ elevates the temperature distribution. Physically, $B_1^{+ve}$ and $B_2^{+ve}$ signify the generation of heat within the flow regime while $B_1^{-ve}$ and $B_2^{-ve}$ connote heat absorption. The result obtained here is not far-fetched; as more heat is being generated in the system, the fluid particles are broken down, thus, losing their cohesive force and resulting in an increased flow and energy rate. The effect of the injected heat source on the flow rate and solute profiles is shown as the chemical reaction parameter, $k'_r$, is improved in Figure 7A,B, respectively. As the chemical reaction parameter increases, the flow regime experience a reduction in both flow rate and solute concentration profiles. This behavior is ascribed to the injected heat, which reduced cohesive molecule force, making them less compact, thus lessening the reaction rate.
The effects of $\Lambda_1$ and $\Lambda_2$ on the temperature and concentration profiles are illustrated in Figure 8A,B, respectively. Here, the heightening of $\Lambda_1$ and $\Lambda_2$ elevates both the thermal and solute distributions. Physically, the thermal boundary layer widens as $\Lambda_1$ increases, attributed to the intensification of shearing stress of the pseudoplastic fluid due to viscous diffusion. Moreover, this result predicts that; for further mass transfer enhancement, more injection of mass-diffusivity number, $\Lambda_2$, is required.

In Figure 9A,B, the impact of $R$ is seen on $f'$ and $\theta$ profiles of the model solution. $R$ immensely influenced the velocity and energy field a few distances away from the surface. Mathematically, $R$ is the ratio of Stefan–Boltzmann to mean absorption and thermal conductivity. Thus, its introduction is expected to inject more heat into the flow system. Physically, a thickened thermal boundary layer and rise in flow hydrodynamic result in a rise in
emitting more heat into the flow region. Hence, a rise in fluid velocity and temperature as thermal radiation is enhanced.

Table 3 displayed the variation in the physical engineering parameters of interest, that is, the skin friction, heat, and mass transfer rates. The computational results of \( \text{Re} \), \( \text{Nu} \), and \( \text{Sh} \) for the solution model are presented in Table 3. The impacts of several parameters were tabled, among which it is observed that a rise in \( \gamma \), \( \alpha_1 \), \( \alpha_2 \), \( \Lambda_1 \), \( \Lambda_2 \), \( R \), \( \theta_w \) corresponds to skin friction improvement and reduction as \( k_r \) grows. \( A_1, A_2, \alpha_1, \alpha_2, \Lambda_1, \Lambda_2, R, \theta_w, k_r, B_1, B_2 \) produced a corresponding increment in heat transfer rate and a reduction for more injection of \( k_r \), \( \gamma \), \( B_1 \), and \( B_2 \). Mass transfer rate is enhanced as parameters \( A_1, A_2, \alpha_1, \alpha_2, \Lambda_1, \Lambda_2, R, \theta_w, k_r, B_1, B_2 \) increase and depreciated for an increase in \( \gamma \).

**Figure 8** Effects of the variable thermal conductivity \( \Lambda_1 \) and the variable mass-diffusivity parameter \( \Lambda_2 \) on the temperature and concentration profiles, respectively.

**Figure 9** Effect of radiation parameter \( R \) on the velocity and energy profile.
Table 3  Variation in the skin friction, heat, and mass transfer rates when \( u_w = x^{1/2} + 1 \) at \( \zeta = 2 \)

| Parameters | \( \lambda \) | \( \gamma \) | \( A_1 \) | \( A_2 \) | \( \alpha_t \) | \( \alpha_c \) | \( \Lambda_1 \) | \( \Lambda_2 \) | \( R \) | \( \theta_w \) | \( k_r' \) | \( B_1 \) | \( B_2 \) | \( \frac{1}{2} \text{Re}^{3/2} C_f \) | \( \text{Re}^{1/3} N_u \) | \( \text{Re}^{1/3} S_h \) |
|------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|----------------|----------------|----------------|
| 0.50       | 0.50      | 0.60      | 0.60      | 0.60      | 0.01      | 0.01      | 0.10      | 0.30      | 0.50      | 0.20      | 0.20      | 1.710334   | 0.160110   | 0.296223   |
| 1.00       |           |           |           |           |           |           |           |           |           |           |           | 2.276684   | 0.158161   | 0.295956   |
| 1.50       |           |           |           |           |           |           |           |           |           |           |           | 2.302944   | 0.156569   | 0.295730   |
| 2.00       | 0.50      |           |           |           |           |           |           |           |           |           |           | 2.290765   | 0.155246   | 0.295513   |
| 1.50       |           |           |           |           |           |           |           |           |           |           |           | 2.389418   | 0.147013   | 0.294458   |
| 2.50       |           |           |           |           |           |           |           |           |           |           |           | 2.494707   | 0.137806   | 0.293204   |
| 0.50       | 0.01      |           |           |           |           |           |           |           |           |           |           | 1.604471   | 0.084973   | 0.286958   |
| 0.20       |           |           |           |           |           |           |           |           |           |           |           | 1.851742   | 0.111543   | 0.290086   |
| 0.60       |           |           |           |           |           |           |           |           |           |           |           | 2.290405   | 0.155244   | 0.295513   |
| 0.01       |           |           |           |           |           |           |           |           |           |           |           | 1.602397   | 0.065972   | 0.284144   |
| 0.20       |           |           |           |           |           |           |           |           |           |           |           | 1.878886   | 0.098191   | 0.288009   |
| 0.60       |           |           |           |           |           |           |           |           |           |           |           | 2.290407   | 0.155243   | 0.295513   |
| 0.01       |           |           |           |           |           |           |           |           |           |           |           | 2.271903   | 0.138816   | 0.293505   |
| 0.20       |           |           |           |           |           |           |           |           |           |           |           | 2.334098   | 0.144372   | 0.294210   |
| 0.60       |           |           |           |           |           |           |           |           |           |           |           | 2.290400   | 0.155244   | 0.295513   |
| 0.01       |           |           |           |           |           |           |           |           |           |           |           | 2.222224   | 0.134162   | 0.292792   |
| 0.20       |           |           |           |           |           |           |           |           |           |           |           | 2.286336   | 0.141192   | 0.293713   |
| 0.60       |           |           |           |           |           |           |           |           |           |           |           | 2.290390   | 0.155244   | 0.295513   |
| 0.01       |           |           |           |           |           |           |           |           |           |           |           | 2.290765   | 0.155246   | 0.295513   |
| 0.20       |           |           |           |           |           |           |           |           |           |           |           | 2.281279   | 0.179549   | 0.295873   |

(Continues)
| Parameters | $\lambda$ | $\gamma$ | $A_1$ | $A_2$ | $\alpha_t$ | $\alpha_c$ | $\Lambda_1$ | $\Lambda_2$ | $R$ | $\varphi_w$ | $k_r$ | $B_1$ | $B_2$ | $\frac{1}{2}Re_C$ | $Re_{x}^{-\frac{1}{3}}Nu_x$ | $Re_{x}^{-\frac{1}{3}}Sh_x$ |
|-----------|---------|---------|-------|-------|-----------|-----------|-------------|-------------|-----|-----------|-------|-------|-------|----------------|----------------|----------------|
| 0.60      | 2.27333 | 0.22756 | 0.29656 | 0.01  | 2.29076 | 0.155246  | 0.295513   | 0.20 | 2.32227 | 0.159520  | 0.314435|
| 0.60      | 2.40031 | 0.16698 | 0.35117 | 0.01  | 2.29076 | 0.155246  | 0.295513   | 0.50 | 2.34712 | 0.176835  | 0.295738|
| 0.90      | 2.38965 | 0.19713 | 0.29594 | 0.10  | 2.28726 | 0.153307  | 0.295488   | 0.30 | 2.29032 | 0.155244  | 0.295513|
| 0.50      | 2.29310 | 0.15854 | 0.29555 | 0.30  | 2.29076 | 0.155246  | 0.295513   | 1.00 | 2.22597 | 0.145928  | 0.366063|
| 1.50      | 2.18703 | 0.13887 | 0.42650 | 0.50  | 2.32593 | 0.293162  | 0.294663   | 0.20 | 2.29041 | 0.155243  | 0.295513|
| 0.50      | 2.11417 | -0.06012| 0.29690 | 0.01  | 2.31830 | 0.346060  | 0.294183   | 0.20 | 2.29045 | 0.155244  | 0.295513|

Note: Parametric values of $M$, $Pr$, $Ec$, and $Sc$ are fixed at 0.02, 0.7, 0.1, and 0.1, respectively.
5 | CONCLUSION

The Reiner–Philippoff rheology, which can characterize a pseudoplastic, Newtonian, and dilatant fluid, was considered in this study. The novel investigation of the effects of variable thermal conductivity and mass diffusivity, nonlinear convection, Joule heating, and chemical reaction was presented. The Reiner–Philippoff liquid was subjected into motion over a surface as shown in Figure 1 and modeled under the conditions mentioned above. The system was analyzed by employing the nonsimilar group of transformations and the spectral-based local linearization technique. The results established in the present study were in line with existing literature, and the convergence of the solutions was established. The impacts of parameters of interest were presented through tables and figures, and the results indicated that:

i. The solution is highly stable and ascertains the computational ability of the spectral methods.

ii. A rise in the values of both linear and nonlinear thermal and solutal convection numbers predicts a higher flow rate since linear convection parameters are ratios of the buoyancy force to the viscous force.

iii. The space and temperature-dependent heat source terms positively enhanced the flow rate and energy fields.

iv. The profiles show that more injection of thermal conductivity and mass diffusivity is expected for further heat and mass transfer enhancement.

v. Chemical reaction parameter, \( k_r' \), is identified as a mechanism for downsizing both flow rate and solute concentration fields.

vi. For further heat and mass transfer enhancement, more radiation emission, thermal conductivity injection, and mass-diffusivity numbers are required.

vii. An increase in chemical reaction parameters negates the skin friction coefficient and heat flux rate.

NOMENCLATURE

\( A_1 \) and \( A_2 \) mixed convection parameters

\( B \) intensity of magnetic field

\( B_0 \) uniform magnetic field

\( B_1 \) coefficients of space-dependent heat

\( B_2 \) coefficient of temperature-dependent heat

\( C \) fluid concentration

\( c_p \) specific heat capacity

\( D_\infty \) ambient Brownian diffusion coefficient

\( Ec \) Eckert number

\( g_a \) gravitational acceleration

\( k \) thermal conductivity

\( k_1 \) mean absorption coefficient

\( k_r \) chemical reaction parameter

\( k_r(x) \) variable chemical reaction rate

\( M \) magnetic parameter

\( Pr \) Prandtl number
\( q_r \) radiative term
\( R \) radiation parameter
\( Re \) Reynolds number
\( \rho_{\infty} \) Reiner–Philippoff fluid thermal conductivity
\( Sc \) Schmidt number
\( T \) fluid temperature
\( u \) and \( v \) velocities along \( x \) and \( y \)
\( u_w \) free stream velocity
\( w \) conditions at the wall

**GREEK SYMBOLS**

\( \alpha_1 \) linear thermal expansion coefficients
\( \alpha_2 \) nonlinear thermal expansion coefficients
\( \alpha_3 \) linear solutal expansion coefficients
\( \alpha_4 \) nonlinear solutal expansion coefficients
\( \alpha_c \) solutal nonlinear convection parameter
\( \alpha_t \) thermal nonlinear convection
\( \Lambda_1 \) variable thermal conductivity parameter
\( \Lambda_2 \) variable mass-diffusivity parameter
\( \mu, \mu_\infty \) viscosity, limiting viscosity
\( \mu_0 \) zero shear viscosity
\( \nu \) kinematic viscosity
\( \omega_1 \) Grashof number
\( \omega_2 \) modified Grashof number
\( \phi \) dimensionless concentration
\( \rho \) fluid density
\( \sigma^* \) Stefan–Boltzmann constant
\( \tau \) shear stress
\( \tau_s \) reference shear stress
\( \theta \) dimensionless temperature
\( \theta_w \) temperature ratio

**SUBSCRIPT SYMBOL**

\( \infty \) far-field limit condition

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**CONFLICTS OF INTEREST**

The authors declare no conflicts of interest.

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