Overcategories and free monoids for overcategories

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Abstract

An overcategory with base category \( \mathcal{C} \) is merely any functor into \( \mathcal{C} \). In this paper we extend the work of Dominique Bourn and Jacques Penon [4] on overcategories. In particular we show that Freyd’s adjoint theorem, a theorem of Barr and Wells in [6] are still valid in the overcategorical context. We also show that a free monoid construction remains valid in the context of overcategories. The motivation for this study is the development of higher categories as found in [4] and in [10].

Contents

1 Theory of Overcategories 4
  1.1 Definition of Over(co)limits 6
  1.2 Some results of over(co)completeness of overalgebras 8
  1.3 Freyd’s Adjoint Theorem in the overcategorical context 9
  1.4 A Theorem of Barr and Wells in the Overcategorical Context 12
  1.5 Beck’s theorem in the overcategorical Context 14
## Free Overmonoids

### 2.1 Liberal monoidal categories

Overcategories are just objects of the comma category $(\text{CAT} \downarrow \mathcal{C})$ where $\mathcal{C} \in \text{CAT}$ without any other requirements and in [8] the authors call it parametrized categories. Thus, as categorical structure, they are poorer than indexed categories or fibrations. It is well known that we can extend most categorical concepts to indexed categories and fibrations, and most of the usual theorems for categories (for instance, Freyd’s adjoint theorem or Beck’s monadicity theorem) have their equivalent for indexed categories and fibrations (for indexed categories, see [9]). Surprisingly, they are few studies for overcategories despite the fact that we find applications of it for example in [4] and in [10], where both of these works deal with the perspective of higher category theory.

Let us explain briefly the differences between our study of overcategories and others important approaches as we can find for example in [9] for the indexed categories or in [7, 8] for the fibrations. In [9] the authors have developed intensively the theory of indexed categories, which are fibrations with a choice of a cleavage (or if we use the "Grothendieck construction", it is a pseudofunctor with a choice of isomorphisms in Cat). In these very precise context they proved an "Adjoint Functor Theorem" á la Peter Freyd. They used an "Initial object theorem" to prove it, but all the time in the context of their "Indexed categories". In our approach we prove also an "Adjoint Functor Theorem" a la Peter Freyd and an "Initial object theorem" to prove it, but in the poorer context of overcategories. In [7] J.Bénabou use the theory of fibrations with a concept of "definability" for fibrations, and he tried to make more much clearer the problem of finding the good "logical environment" to build category theory. For him, category theory can be build with
the notion of fibration and definability. An other aspect of the paper [7] is the fact that the author said that "Indexed categories" are not the good environment to build category theory, because the choice of cleavage (using the choice axiom), which is a part of the definition of an "Indexed categories", makes things much more harder and less natural, where many confusions can be made. Thus the spirit of the paper of Benabou is completely different from our approach where we just show that we can develop some concepts of category theory for overcategories rather than to be embarrassed by the foundation of category theory itself. In the beginning of the paper [8] the authors start to speak about overcategories that they called "parametrized categories", however then they "enriched" quickly this notion with the notion of "cartesian cones", "cartesian maps" (which is here a specific case of the cartesian cones), and fibrations. In all their paper they studied properties like "wellpoweredness", "small idempotency", "generators", etc. which are the major new notions of this paper, but with results involving at less cartesian maps or more, fibrations. In particular they gave a new characterisation of an elementary toposes with their concept of wellpowerdness for parametrized categories. But despite of all these very interesting results, their paper has different perspectives from our approach where we investigate overcategories without any others structures on it.

Dominique Bourn and Jacques Penon have used overcategories in [4] (called "surcatégories" in their article) as a major tool for their studies of categorification, which is one of the most important question in higher category theory.

However we believe that the level of generality of our study of overcategories in this article and in the article [4], allows the possibility of applications to many other contexts, especially those where the categories involved have structure poorer than fibrations or indexed categories. Let us be more precise on this point. For example if for a fixed category \( \mathcal{C} \) it is difficult to see that it is complete or cocomplete, but this category is equipped with
a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ such that the fibers $F^{-1}(d)(d \in \mathcal{D})$ are complet or cocomplet and this (co)completeness has some good properties among the whole category $\mathcal{C}$, then we can ask if this underlying structure of $\mathcal{C}$ is good enough to resolve some mathematical questions which are involved. This is exactly the first motivation of the author in [10] to study overcategories because at that time he found that the cocompleteness of the category $T\text{-CAT}$ of $T$-categories was not evident.

The first section of this article deals with the overcategorical version of some classical theorems of the category theory. The "overcategorical theorems" that we establish in this paper, especially the overadjoint Freyd theorem, and a theorem overcategoric of Barr-Wells theorem, could be useful for classical category theory itself.

The second section of this article deals with the free monoid construction in the context of overcategories. We build it within framework of the overmonoidal overcategories [4]. As a matter of fact in [4] the authors establish an adjunction result (to obtain free "overmonoids") in an ideal context they label "numeral" [4]. We demonstrate a similar theorem (theorem [4]) which also results from an ideal context that I label liberal and which allows us to establish a result of free overmonoids result.

I am grateful to Jacques Penon who permitted me to access the details of his conjoint work [4] with Dominique Bourn. The research for this present paper was completed in 2009.

1 Theory of Overcategories

Let $\mathcal{G}$ be a fixed category. An overcategory is an object of the 2-category $\mathcal{CAT}/\mathcal{G}$. Thus it is given by a couple $(\mathcal{C}, A)$, where $\mathcal{C}$ is a category and $\mathcal{C} \xrightarrow{A} \mathcal{G}$ is a functor (often called "arity functor", in reference to its use in
this paper). In what follows the arity functor is often noted with the letter $A$ because there is no risk of confusion. The evident morphisms of $\mathcal{CAT}/\mathcal{G}$ are called overfunctors, but we also need in this paper morphism between overcategories with different base categories $\mathcal{G}$ and $\mathcal{G}'$. Therefore such a morphism $(\mathcal{C}, A) \xrightarrow{(F, F_0)} (\mathcal{C}', A')$ is given by two functors: $\mathcal{C} \xrightarrow{F} \mathcal{C}'$ and $\mathcal{G} \xrightarrow{F_0} \mathcal{G}'$ such as $A'F = F_0A$ (see for example section 1.3). For a fixed overcategory $(\mathcal{C}, A)$, its objects and its morphisms are respectively objects and morphisms of the domain category $\mathcal{C}$.

The pairs of adjoints morphisms and monads in the 2-category $\mathcal{CAT}/\mathcal{G}$ are called respectively pairs of adjoint overfunctors and overmonads. In fact every overcategorical concept will be expressed by using "over" before the categorical concept that it generalizes. But we sometimes forget the word "over", when the context implies that no confusion is possible. It is easy to see that the category of algebras for a given overmonad is an overcategory. The objects of this overcategory will be called overalgebras.

We are going to see that most of the notions in the 2-category $\mathcal{CAT}/\mathcal{G}$ can be done again in the 2-category $\mathcal{CAT}/\mathcal{G}$, and it is very likely that most of the concepts and theorems in $\mathcal{CAT}$ extend to $\mathcal{CAT}/\mathcal{G}$. We will particularly demonstrate three theorems in $\mathcal{CAT}/\mathcal{G}$ coming from three important theorems in $\mathcal{CAT}$: Freyd’s Overadjoint Theorem (which is the overcategorical version of the classical Freyd Adjoint Theorem. See theorem 1), Barr-Wells’s Overcategorical Theorem (which is the overcategorical version of the result that we can find in [6]. See theorem 2), and Beck’s Overcategorical Theorem (which is the overcategorical version of Beck’s classical theorem. See theorem 3). These theorems are the obvious generalisations of the classical ones.
1.1 Definition of Over(co)limits

In [4] the notions of limits and colimits in $\text{CAT}/\mathbb{G}$ are defined and these notions will be used afterwards. To facilitate the reader we will recall the definitions.

If $\mathcal{C}$ is a small category and if $\mathcal{E}$ is a category, then we have the classical diagonal functor $\mathcal{E} \xrightarrow{\Delta} \mathcal{E}^{\mathcal{C}}$, which sends an object to a constant functor and which sends a morphism to a constant natural transformation.

Moreover if $(\mathcal{E}, A)$ is a overcategory, let $\mathcal{E}(\mathcal{C})$ be the subcategory of $\mathcal{E}^{\mathcal{C}} \times \mathbb{G}$ given by:

$\mathcal{E}(\mathcal{C})(0) = \{(F, B) \in \mathcal{E}^{\mathcal{C}} \times \mathbb{G}/AF = \Delta(B)\}$,

$\mathcal{E}(\mathcal{C})(1) = \{(F, B) \xrightarrow{(\tau, b)} (F', B')/b \in \mathbb{G}(1) \text{ and } \tau \text{ is a natural transformation such as } A\tau = \Delta(b)\}$.

$\mathcal{E}(\mathcal{C})$ has a natural overcategory structure given by the second projection: $\mathcal{E}(\mathcal{C}) \xrightarrow{\Delta} \mathbb{G}$, $(F, B) \mapsto B$. In fact $(\mathcal{E}(\mathcal{C}), A)$ is a cotensor of the $\text{CAT}$-enriched category $\text{CAT}/\mathbb{G}$ (in $\text{CAT}/\mathbb{G}$ is a $\text{CAT}$-enriched category because it is a 2-category), because we have the following isomorphism in $\text{CAT}$

$$\text{CAT}/\mathbb{G}((\mathcal{E}', A); (\mathcal{E}(\mathcal{C}), A)) \simeq \text{Funct}(\mathcal{C}; \text{CAT}/\mathbb{G}((\mathcal{E}', A); (\mathcal{E}, A))).$$

We also have the diagonal overfunctor (also noted $\Delta$): $(\mathcal{E}, A) \xrightarrow{\Delta} (\mathcal{E}(\mathcal{C}), A)$ defined by $x \mapsto (\Delta(x), A(x))$. If $(F, B) \in (\mathcal{E}(\mathcal{C}), A)$, an overcone of $(F, B)$ is a morphism $\Delta(x) \xrightarrow{(\tau, b)} (F, B)$ ($x \in \mathcal{E}$) of $(\mathcal{E}(\mathcal{C}), A)$, where $A(x) \xrightarrow{b} B$ is a morphism of $\mathbb{G}$. In the same way we define overcocones.

It is easy to see that if $\mathcal{C}$ is connected, then every overcone is a cone in the classical sense (respectively, every overcocone is a cocone in the classical sense).

The overcategory $(\mathcal{E}, A)$ has $\mathbb{C}$-overlimits (that [4] calls $(\mathbb{C})$-limits) if every $(F, B) \in (\mathcal{E}(\mathcal{C}), A)$ has a universal overcone $\Delta(x) \xrightarrow{(\tau, b_1)} (F, B)$ such as
\( A(x) = B \), i.e. if we give ourselves another overcone \( \Delta(y) \xrightarrow{(\sigma, b)} (F, B) \) (where \( A(y) \xrightarrow{b} B \) is a morphism of \( C \)) then there is a unique morphism \( y \xrightarrow{f} x \) in \( \mathcal{E} \) such that \( (\tau, 1_B)\Delta(f) = (\sigma, b) \). The definition of \( C \)-overcolimit is dual. The definition of overlimits and overcolimits enable us to include the case where \( C \) is the empty category, which gives an alternative definition of overinitial objects (see section 1.3) and overfinal objects.

If \( C \) is connected and nonempty then it is easy to see that the following definitions are equivalent

- \((\mathcal{E}, A)\) has \( C \)-overlimits.
- \( \forall (F, B) \in (\mathcal{E}(C), A) \), \((F, B)\) has a universal overcone \( \Delta(x) \xrightarrow{(\tau, 1_B)} (F, B) \) such as \( A(x) = B \).
- \( \forall (F, B) \in (\mathcal{E}(C), A) \), the functor \( C \xrightarrow{F} \mathcal{E}_B \) has a limit which is preserved by the canonical inclusion \( \mathcal{E}_B \hookrightarrow \mathcal{E} \).
- The diagonal overfunctor \( (\mathcal{E}, A) \xrightarrow{\Delta} (\mathcal{E}(C), A) \) has a right overadjoint.

In the same way, if \( C \) is connected and nonempty we have dual definitions for \( C \)-overcolimits.

**Remark 1** Let \( \mathbb{N} \) be the category of non-negative integers with the natural order. In the terminology adopted in [4], \( \mathbb{N} \)-limits are colimits. We prefer to adopt the word \( \mathbb{N} \)-colimit for this specific kind of filtered colimit. And in the overcategorical context we prefer the word \( \mathbb{N} \)-overcolimits instead of \((\mathbb{N})\)-colimits (as adopted by [4]).

We are now going to define \( K \)-equalizers and \( K \)-coequalizers which are important notions because with them we get a overadjonction result similar to Freyd’s Adjoint theorem (theorem 1), but more general.
A overcategory \((\mathcal{E}, A)\) has \(K\)-equalizers if every pair \(a \xrightarrow{f} b\) in \(\mathcal{E}\), which has the property \(A(f) = A(g)\), has an equalizer \(e\) in \(\mathcal{E}\).

\[
\begin{array}{ccc}
c & \xleftarrow{e} & a \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
a & \xrightarrow{f} & b \\
\end{array}
\]
such as \(A(e) = A(1_a)\). The definition of \(K\)-coequalizers is dual.

If \(T\) is a overmonad on \((\mathcal{E}, A)\), the Eilenberg-Moore algebra category \(\mathcal{E}^T\) is trivially an overcategory \((\mathcal{E}^T, A)\) where its objects are called overalgebras, not only to emphasise the overcategorical context, but also to focus on the fact that an overalgebra is an algebra which lives in a fiber.

The following propositions are immediate.

**Proposition 1** Let us call split overfork, a split fork in the overcategorical context, i.e. it is a diagram \(a \xrightarrow{f} b \xrightarrow{g} c\) which is a fork in a fiber \(\mathcal{E}_B\). Then such split overforks are absolute overcoequalizers. ☐

**Proposition 2** Every overalgebra (for a fixed overmonad) is a overcoequalizer. ☐

**Proposition 3** \((\mathcal{E}, A)\) is overcomplete iff \((\mathcal{E}, A)\) has overequalizers and overproducts. ☐

**Proposition 4** \((\mathcal{E}, A)\) is overcocomplete iff \((\mathcal{E}, A)\) has overcoequalizers and oversums. ☐

### 1.2 Some results of over(co)completeness of overalgebras

The following propositions are very similar to the classical ones and so do not require detailed proof.
Proposition 5 Let $T$ be an overmonad on $(\mathcal{E}, \mathcal{A})$. In this case:

$(\mathcal{E}, \mathcal{A})$ is overcomplete $\implies (\mathcal{E}^T, \mathcal{A})$ is overcomplete

Proposition 6 Let $T$ be an overmonad on $(\mathcal{E}, \mathcal{A})$. In this case:

$(\mathcal{E}, \mathcal{A})$ has $K$-equalizers $\implies (\mathcal{E}^T, \mathcal{A})$ has $K$-equalizers

Proposition 7 Let $T$ be an overmonad on $(\mathcal{E}, \mathcal{A})$. Suppose that $(\mathcal{E}, \mathcal{A})$ is overcocomplete. In this case:

$(\mathcal{E}^T, \mathcal{A})$ has overcoequalizers $\iff (\mathcal{E}^T, \mathcal{A})$ is overcocomplete

1.3 Freyd’s Adjoint Theorem in the overcategorical context

As we are going to see, Freyd’s Adjoint Theorem remains true in the context of $\mathbb{C}AT/\mathbb{G}$. We call it "Freyd’s Overadjoint Theorem" to refer to its overcategorical nature. Freyd’s Overadjoint Theorem can be used for example for the proof of the theorem 2 which allows us to prove some overcocompleteness results. But as we will demonstrate, unlike "Beck’s Theorem in the overcategorical context" (see section 1.5), Freyd’s overadjoint theorem requires in addition $K$-equalizers (see theorem 1).

Let $(\mathcal{A}, \mathcal{A}) \xrightarrow{F} (\mathcal{B}, \mathcal{A})$ be an overfunctor and $B \in (\mathcal{B}, \mathcal{A})$. An object of the comma category $(B \downarrow F)$ is given by a couple $(A, a)$ consisting of an object $A$ of $\mathcal{A}$ and to a morphism $B \xrightarrow{a} F(A)$ in $\mathcal{B}$, and a morphism of $(B \downarrow F)$ is given by an arrow $(A, a) \xrightarrow{f} (A', a')$ such that $F(f)a = a'$.

The comma category $(B \downarrow F)$ is an overcategory. Indeed we have the arity functor $(B \downarrow F) \xrightarrow{\Lambda} \Lambda(B)/\mathbb{G}$ defined on the objects as: $(A, a) \mapsto \Lambda(a)$ and defined on the morphism as: $f \mapsto \Lambda(f)$ ($\Lambda$ is here the arity functor of the overcategory $(\mathcal{B}, \mathcal{A})$).
Furthermore we have the following canonical morphism of overcategories, given by the first projection

\[
\begin{array}{ccc}
(B \downarrow F) & \xrightarrow{Q} & \mathfrak{A} \\
\downarrow \mathfrak{A} & & \downarrow \mathfrak{A} \\
\mathfrak{A}(B)/\mathbb{G} & \xrightarrow{Q_0} & \mathbb{G}
\end{array}
\]

**Proposition 8** Let \((\mathfrak{A}, \mathfrak{A}) \xrightarrow{G} (\mathcal{D}, \mathfrak{A})\) be a overfunctor such that \((\mathfrak{A}, \mathfrak{A})\) is overcomplete and has K-equalizers. We suppose that \(G\) preserves overlimits and K-equalizers. Then \(\forall B \in \mathcal{D}, \) the comma overcategory \(((B \downarrow G), \mathfrak{A})\) is overcomplete and has K-equalizers. ✷

**Proof** It is enough to prove that the functor \(((B \downarrow G), \mathfrak{A}) \xrightarrow{Q} (\mathfrak{A}, \mathfrak{A})\) creates small overproducts, overequalizers, and K-equalizers. First we consider all functors \(J \xrightarrow{F} (B \downarrow G)\) such that \(F \in (B \downarrow G)^{(J)}\). Thus \(QF \in A^{(J)}\) and if \(J\) is a small discret category, then \(\lim QF\) exists because \((\mathfrak{A}, \mathfrak{A})\) is overcomplete. It is easy to prove (as in [3]) that \(\lim F\) exists and that it is unique such that \(Q(\lim F) = \lim QF\). If \(J = \downarrow\), we use a similar argument to prove that \(Q\) creates overequalizers.

To prove that \(Q\) creates K-equalizers we use a similar argument, but we must take \(J = \downarrow\) and \(F\) such that the image of the functor \(\mathfrak{A}F\) is a fixed arrow in \(\mathfrak{A}(B)/\mathbb{G}\). ■

Let \((\mathcal{D}, \mathfrak{A})\) be an overcategory and let \(G \in \mathcal{G}\). The object \(0_G \in \mathcal{D}_G\) is overinitial if for all objects \(d \in \mathcal{D}\), and for all \(G \xrightarrow{b} \mathfrak{A}(d)\) in \(\mathcal{G}(1)\), there is a unique morphism \(0_G \xrightarrow{x} d\) of \(\mathcal{D}\) over \(b\).

**Proposition 9** Let \((\mathfrak{A}, \mathfrak{A}) \xrightarrow{F} (\mathcal{D}, \mathfrak{A})\) be an overfunctor, \(B \in (\mathcal{D}, \mathfrak{A})\), and \((R_B, v)\) be an object of \(((B \downarrow F), \mathfrak{A})\) such that \(\mathfrak{A}(v) = 1_{\mathfrak{A}(B)}\). In this case:

\((R_B, v)\) is overinitial in \(((B \downarrow F), \mathfrak{A})\) \iff \(v\) is initial in \((B \downarrow F)\)
Lemma 1 (Lemma of the overinitial object) Let \((\mathcal{D}, A)\) a overcategory over-complete with \(K\)-equalizers, and let \(G \in \mathcal{G}\).

In this case we have the following equivalence

\[(\mathcal{D}, A)\text{ has an overinitial object in one fiber } \mathcal{D}_G \iff \exists \text{ a set } I \text{ and a family of objects } k_i \in \mathcal{D}_G (i \in I) \text{ such that } \forall d \in (\mathcal{D}, A), \forall G \xrightarrow{h} A(d) \text{ in } \mathcal{G}, 	ext{ there is an } i \in I, \text{ and there is a morphism } k_i \rightarrow d \text{ in } \mathcal{D} \text{ over } h (\text{via the arity functor}).\]

The proof of this lemma is very similar to the classical one (see [4, proposition 1.8 page 25]) and thus it is not necessary to give the details of the demonstration. It is useful to note that this demonstration requires \(K\)-equalizers.

Let \((\mathcal{A}, A) \xrightarrow{F} (\mathcal{B}, A)\) be an overfunctor. An object \(B \in (\mathcal{B}, A)\) has the solution set condition for \(F\) if there is a set \(I\) and a set of objects \(\{(A_i, b_i) / i \in I \text{ and } A(b_i) = 1_{A(B)}\} \subset (B \downarrow F)\), such that \(\forall (A, b) \in (B \downarrow F), \exists i \in I, \exists A_i \xrightarrow{a_i} A\) in \((\mathcal{A}, A)\), such that \(F(a_i)b_i = b\).

Theorem 1 (Freyd’s Overadjoint theorem) Let \((\mathcal{A}, A)\) be an overcomplete overcategory with \(K\)-equalizers, and let \((\mathcal{A}, A) \xrightarrow{F} (\mathcal{B}, A)\) be an overfunctor. In that case the following properties are equivalent

\[F \text{ has a left overadjunction} \iff F \text{ preserve overlimits and } K-\text{equalizers and every object } B \in (\mathcal{B}, A) \text{ has a solution set condition for } F\]

PROOF First we suppose that \(F\) preserves overlimits and \(K\)-equalizers and every object \(B \in (\mathcal{B}, A)\) has a solution set condition for \(F\). Let \(B \in \text{Ob}(\mathcal{B})\), the overcategory \((\mathcal{A}, A)\) is overcomplete and have \(K\)-equalizers which are preserved by \(F\), thus thanks to the proposition 8 we know that \(((B \downarrow F), A)\) is
overcompletness and have $K$-equalizers. Therefore $((B \downarrow F), \mathcal{A})$ verifies in addition the hypothesis "solution set condition" of the lemma of the overinitial object in the fiber $(B \downarrow F)_{1, \kappa(B)}$. Thus $((B \downarrow F), \mathcal{A})$ has a overinitial object in the fiber $(B \downarrow F)_{1, \kappa(B)}$. If we write down $B \xrightarrow{\eta_B} F(R_B)$ this overinitial object, then thanks to the proposition 6, it is initial in $(B \downarrow F)$. Then $F$ has a left adjoint: $G \dashv F$, and it is clearly an overadjoint. The converse is trivial.

1.4 A Theorem of Barr and Wells in the Overcategorical Context

As we are going to see, we have an overcategorical version of the result that we can find in [2]. This theorem is an overcategorical adaptation of some results that we can find in [6], [??].

**Theorem 2 (Barr-Wells’s Overcategorical Theorem)** Let $(\mathcal{C}, \mathcal{A})$ be an overcomplete and overcocomplete overcategory with $K$-equalizers. Let $T$ be an overmonad on $(\mathcal{C}, \mathcal{A})$, which preserves $\kappa$-filtered overcolimits for some regular cardinal $\kappa$. In this case the overcategory $(\mathcal{C}_T, \mathcal{A})$ of overalgebras is overcomplete, overcocomplete, and has $K$-equalizers.

**Proof** The overcompletness of $(\mathcal{C}_T, \mathcal{A})$ and the fact that $(\mathcal{C}_T, \mathcal{A})$ has $K$-equalizers is a direct consequence of proposition 5 and proposition 6.

Thanks to proposition 5, we also know that it is sufficient to prove that $(\mathcal{C}_T, \mathcal{A})$ has overcoequalizers to demonstrate that it is overcocomplete. To prove the existence of overcoequalizers in $(\mathcal{C}_T, \mathcal{A})$, it suffices to show that the diagonal overfunctor

$$
\Delta : (\mathcal{C}_T, \mathcal{A}) \xrightarrow{\text{colim}} ((\mathcal{C}_T)_{(1)}, \mathcal{A})
$$

has a left overadjoint $\text{colim} \dashv \Delta$. We are in a position to apply Freyd’s Overadjoint Theorem (see theorem 1), because $(\mathcal{C}_T, \mathcal{A})$ is overcomplete and has
$K$-equalizers and $\Delta$ preserves overlimits and $K$-equalizers. This last point is easy because limits in $(\mathcal{C}^T)^{(\Pi)}$ are computed pointwise. We need to show that every object of $((\mathcal{C}^T)^{(\Pi)}, A)$ has a solution set condition $S_F$ for $\Delta$. In particular if $(F, G_0) \in (\mathcal{C}^T)^{(\Pi)}$, then this solution set condition $S_F$ must be in $\mathcal{C}^T_{\lambda((F,G_0)=G_0}$. More precisely $F$ is the following data: It is a pair of morphism of $\mathcal{C}^T$: $(A, \alpha) \xrightarrow{f} (B, \beta)$, which is in the fiber $\mathcal{C}^T_{G_0}$. A solution set condition $S_F$ for $\Delta$ is given by

$$S_F = \{ (B, \beta) \xrightarrow{b_i} (D_i, \delta_i) \in \mathcal{C}^T_{G_0}/i \in I \}$$

such that we give ourselves the natural transformation $F \xrightarrow{\sigma} \Delta(C, \gamma)$ (where $(C, \gamma) \in (\mathcal{C}^T, A)$), then there are a $(D_i, \delta_i) \in S_F$, a morphism of overalgebras $(D_i, \delta_i) \xrightarrow{a_i} (C, \gamma)$, and a natural transformation $F \xrightarrow{\tau} \Delta(D_i, \delta_i)$, such that $\Delta(a) \tau = \sigma$. Therefore it means that when we consider the following diagram (where $j$ is a morphism of $\mathcal{G}$ and $h$ is not necessary in the same fiber as $f$ and $g$; here we have $\lambda(f) = \lambda(g) = 1_{G_0}$ and $\lambda(h) = j$)

$$\begin{align*}
(A, \alpha) & \xrightarrow{f} (B, \beta) \\
\downarrow & \downarrow h \\
(C, \gamma) & \xrightarrow{j} (C, \gamma)
\end{align*}$$

such that $hf = hg$, then $\exists i \in I, \exists (D_i, \delta_i) \xrightarrow{a_i} (C, \gamma)$, such that the following
If we build such a solution set then Freyd’s Overadjoint Theorem (see theorem 1) shows that \((\mathcal{C}^T, A)\) has overcoequalizers.

This solution set condition is built as in the classical case (see [author?] [2, proposition 4.3.6 page 206]), i.e. by transfinite induction, and there is no difficulty in transcribing it from the classical case to the overcategorical context.

1.5 Beck’s theorem in the overcategorical Context

It is easy to see that Beck theorem remains true in \(\text{CAT}/\mathbb{G}\). We call this theorem "over-Beck’s theorem" to refer to its overcategorical nature. Like in the classical case, we use two lemmas which facilitate the demonstration of the over-Beck’s theorem (see [3]). But the proof of these two lemmas and of the over-Beck’s theorem are very similar to the classical one (see [3]), and thus it is not necessary to give the details of the demonstrations. Contrary to the Freyd’s overadjoint theorem and the Barr-Wells’s overcategorical Theorem, we notice that we do not need the presence of \(K\)-equalizers.

Lemma 2 Let \((\mathcal{A}, A) \xrightarrow{G} (\mathcal{X}, A)\), \((\mathcal{A}', A) \xrightarrow{G'} (\mathcal{X}', A)\), two overadjunc-

\[\text{Diagram commutes}\]

\[
\begin{array}{c}
\text{(A, \alpha)} \\
\downarrow f \\
\text{(B, B)} \\
\downarrow g \\
\text{(D, \delta)} \\
\downarrow h \\
\text{(C, \gamma)}
\end{array}
\]

\[
\begin{array}{c}
G_0 \\
\downarrow j \\
G_1
\end{array}
\]
tions which generate the same overmonad $T$. If we suppose that $G$ satisfies hypothesis 3 of theorem 3 then there is a unique overfunctor $(\mathcal{A}', A) \xrightarrow{M} (\mathcal{A}, A)$ such that the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{F'} & \mathcal{A}' \\
\downarrow^1 & & \downarrow^M \\
\mathcal{X} & \xrightarrow{F} & \mathcal{A} \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{G'} & \mathcal{A} \\
\downarrow^1 & & \downarrow^G \\
\mathcal{X} & \xrightarrow{G} & \mathcal{A} \\
\end{array}
\]

\[\square\]

**Lemma 3** In the situation $(\mathcal{A}, A) \xrightarrow{G} (\mathcal{X}, A)$, $G^T$ creates overcoequalizers of $(\mathcal{X}^T, A)$ for absolute overcoequalizers, that is given the diagram

\[
(x, h) \xrightarrow{d_0} \xrightarrow{d_1} (y, k)
\]

in one fiber of $(\mathcal{X}^T, A)$ such that the pair

\[
G^T((x, h)) \xrightarrow{G^T(d_0)} \xrightarrow{G^T(d_1)} G^T((y, k)), \text{ i.e } x \xrightarrow{d_0} \xrightarrow{d_1} y
\]

has an absolute overcoequalizer $y \xrightarrow{e} z$, so there is a unique $T$-algebra $(z, m)$ and a unique morphism $(y, k) \xrightarrow{f} (z, m)$ of $(\mathcal{X}^T, A)$ such that $G^T(f) = e$ and furthermore $(y, k) \xrightarrow{f} (z, m)$ is a overcoequalizer of the pair $(x, h) \xrightarrow{d_0} \xrightarrow{d_1} (y, k)$ \square

**Theorem 3 (Beck’s Overmonadicity Theorem)** Let us consider the overadjunction $(\mathcal{A}, A) \xrightarrow{G} (\mathcal{X}, A)$ with overmonad $T$, the canonical final overadjunction $(\mathcal{X}^T, A) \xrightarrow{G^T} (\mathcal{X}, A)$, and the comparison overfunctor $(\mathcal{A}, A) \xrightarrow{K} (\mathcal{X}^T, A)$.
\((\mathcal{X}^T, \lambda)\) which is the unique overfunctor such that the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{F} & \mathcal{X}^T \\
1_{\mathcal{X}} \downarrow & & \downarrow K \\
\mathcal{X}^T & \xrightarrow{G} & \mathcal{X}
\end{array}
\]

In this case the following conditions are equivalent

1. \(K\) is an isomorphism in \(\text{CAT}/\mathcal{G}\) (i.e there is a overfunctor \((\mathcal{X}^T, \lambda) \xrightarrow{L} (\mathcal{A}, \kappa)\) such as \(KL = 1_{\mathcal{X}^T}\) and \(LK = 1_{\mathcal{A}}\)).

2. \((\mathcal{A}, \kappa) \xrightarrow{G} (\mathcal{X}, \lambda)\) creates overcoequalizers of \(a \xrightarrow{f} g \xrightarrow{b}\) for which the pair \(G(a) \xrightarrow{G(f)} G(b)\) has an absolute overcoequalizer.

3. \((\mathcal{A}, \kappa) \xrightarrow{G} (\mathcal{X}, \lambda)\) creates overcoequalizers of \(a \xrightarrow{f} g \xrightarrow{b}\) for which the pair \(G(a) \xrightarrow{G(f)} G(b)\) has split overcoequalizers.

\[\blacksquare\]

2 Free Overmonoids

In [4] the authors suggest two constructions of the free monoid associated with an object of a monoidal category. This first construction \((\text{author})\ [4, \text{proposition 1.2 page 14}]\) requires further properties on the underlying monoidal category that the authors call "numérale" (for the overcategorical context; see \(\text{author}\) [4, proposition 1.3.3 page 24]). The second construction of the free monoid such as it is found in \(\text{author}\) [4, proposition 1.3 page 16] fits well with the pointed case (see result 2) and we are especially
interested in this case (but in the overcategorical context). As the first construction, this second construction requires further properties on the underlying monoidal category. Therefore we call "liberal" those useful properties by which the free monoid can be obtained from this second construction. We shall make a small reminder of the main results but the reader is deeply encouraged to see the details of these constructions in [4] because we greatly use them at the end of the proof of the theorem 4. After we will show that all of these constructions apply in the overmonoidal context (which is the overcategorical version of the monoidal context), where overmonoidals overcategories are for monoidal categories what overcategories are for categories. Although techniques used here are close to those we find in [4], some concepts like Liberal Overmonoidal Overcategories and Pointed Overmonoidal Overcategories are new. In particular the proof of the theorem 4 is similar to the proposition 10 below which is in [4].

2.1 Liberal monoidal categories

Let \( \mathcal{V} = (\mathcal{V}, \otimes, I, u_l, u_r, \text{ass}) \) be a monoidal category. We sometimes denote it by its underlying category \( \mathcal{V} \). \( \mathcal{V} \) is liberal if the following properties hold:

- \( \mathcal{V} \) has \( \bigwedge \) -colimits and coequalizer;
- \( \forall X \in \mathcal{V}(0), (\_ \otimes X) \) and \( X \otimes (\_ \) preserves \( \bigwedge \) -colimits;
- \( \forall X \in \mathcal{V}(0), (\_ \otimes X) \) preserves coequalizers.

Let \( \text{Mon}(\mathcal{V}) \) be the category of monoids in \( \mathcal{V} \). We have a forgetful functor \( \text{Mon}(\mathcal{V}) \stackrel{U}{\longrightarrow} \mathcal{V}, (M, e, m) \longrightarrow M \) and we have in [4, proposition 1.3 page 16]:

**Proposition 10** If \( \mathcal{V} \) is liberal and if \( I \) is an initial object then the preceding forgetful functor has a left adjoint

\[
\begin{array}{c}
\text{Mon}(\mathcal{V}) \xrightarrow{U} \mathcal{V} \\
\downarrow Mo \\
\end{array}
\]

□

17
In order to construct this free monoid functor \( \text{Mo}(\cdot) \), we use the notion of graded monoid (defined in \textbf{(author?) [4, page 12]}). A graded monoid in a monoidal category \( \mathcal{V} \) is given by a triple \((X_n)_{n \in \mathbb{N}}, (t_n)_{n \in \mathbb{N}}, (k_{n,m})_{n,m \in \mathbb{N}}\) where \((X_n)_{n \in \mathbb{N}}\) is a family of objects of \( \mathcal{V} \), \((X_n \xrightarrow{t_n} X_{n+1})_{n \in \mathbb{N}}\) is a family of morphisms of \( \mathcal{V} \), and \((X_n \otimes X_n \xrightarrow{k_{n,m}} X_{n+m})_{n,m \in \mathbb{N}}\) is a family of morphisms of \( \mathcal{V} \), verifying some axioms that we can find in \textbf{(author?) [4, page 12]}. In \[4\] it is proved that every monoid has an underlying graded monoid and every graded monoid \((X_n)_{n \in \mathbb{N}}, (t_n)_{n \in \mathbb{N}}, (k_{n,m})_{n,m \in \mathbb{N}}\) is linked with a free monoid. Then the strategy to built the free monoid \( \text{Mo}(X) \) for every \( X \in \mathcal{V} \) is first to built a graded monoid \( \Psi_X \) where this construction also requires the construction by induction of a secondary family of morphisms \((X \otimes X_n \xrightarrow{q_n} X_{n+1})_{n \in \mathbb{N}}\), then \( \text{Mo}(X) \) is also the free monoid associated with this graded monoid \( \Psi_X = ((X_n)_{n \in \mathbb{N}}, (t_n)_{n \in \mathbb{N}}, (k_{n,m})_{n,m \in \mathbb{N}}) \).

We must remember that \( \text{Mo}(X) = \text{colim}_n X_n \) and \((X_n)_{n \in \mathbb{N}}\) is built by induction with morphisms \( X_n \xrightarrow{t_n} X_{n+1} \), \( X \otimes X_n \xrightarrow{q_n} X_{n+1} \), by considering the coequalizer \( q_{n+1} := \text{coker}(\gamma_n^0, \gamma_n^1) \), where

\[
\begin{align*}
\gamma_n^0 &= \text{Id} \otimes t_n, \\
\gamma_n^1 &= (X \otimes X_n \xrightarrow{q_n} X_{n+1} \xrightarrow{u_{n+1}} I \otimes X_{n+1} \xrightarrow{\text{Id} \otimes \text{Id}} X \otimes X_{n+1}), \\
t_{n+1} &= (X_{n+1} \xrightarrow{u_{n+1}} I \otimes X_{n+1} \xrightarrow{\text{Id} \otimes \text{Id}} X \otimes X_{n+1} \xrightarrow{q_{n+1}} X_{n+2}),
\end{align*}
\]

and where the initialization is given by \( X_0 = I, X_1 = X, I \xrightarrow{t_0 = \text{Id}_X} X, \)

\( X \otimes I \xrightarrow{q_0 = u_0} X \).

Morphisms \( k_{n,m} \) are built by induction (see \textbf{(author?) [4, page 16 and page 17]}), but we do not describe it here because we do not explicitly need them anymore. Let \((X_n \xrightarrow{l_n} \text{Mo}(X))_{n \in \mathbb{N}}\), the universal cocone defining \( \text{Mo}(X) \). The associated universal arrow is \( X \xrightarrow{l(X) = l_1} \text{Mo}(X) \). Let us remind that the multiplication \( \text{Mo}(X) \otimes \text{Mo}(X) \xrightarrow{m} \text{Mo}(X) \) is the unique arrow such as \( \forall n, m \in \mathbb{N}: m(l_n \otimes l_m) = l_{n+m}k_{n,m} \). When \( n = 1 \) we have \( k_{1,m} = q_m \) which
gives the equality $m(l_1 \otimes l_m) = l_{m+1}q_m$ and which will be useful for the construction of the free overmonoid (see result 2).

### 2.2 Liberal monoidal overcategories

Let $G$ be some fixed category.

We shall expand further on the "overmonoidal" context, what we have made for the monoidal context.

**Remark 2** As application we will see in [? ] that the results of $\mathcal{C}$AT, which has enabled to build the free contractible operad of weak $\omega$-categories of Batanin (see [1]) are true in $\mathcal{C}$AT/$\infty$-$\text{Gr}$, which gives us many kind of free colored operads and especially the free contractible colored operads for weak higher transformations.

Let us now briefly recall the definition of monoidal overcategory.

Let $G$ be a fixed category. An monoidal overcategory (over $G$) is a monoidal object of the 2-category $\mathcal{C}$AT$\setminus G$. A monoidal overcategory is thus given by a 7-uple: $\mathcal{E} = (E, A, \otimes, I, u_l, u_r, ass)$ where:

- $A$ is a functor: $E \xrightarrow{A} G$;
- $(E \times_G E, A) \xrightarrow{\otimes} (E, A)$ is a morphism of $\mathcal{C}$AT$\setminus G$, where $E \times_G E$ is the kernel pair of $A$;
- $G \xrightarrow{I} E$ is a functor and a section (i.e we have $AI = 1_G$);
- $u_r$ and $u_l$ are natural isomorphisms: $\otimes(1_E, IA) \xrightarrow{u_r} 1_E$, $\otimes(IA, 1_E) \xrightarrow{u_l} 1_E$;
- $ass$ is a natural isomorphism: $\otimes(\otimes \times 1_E) \xrightarrow{ass} \otimes(1_E \times \otimes)$.

And these data satisfy the usual conditions of coherence i.e those given by the axioms of monoidal categories. A simple consequence of this definition
is that for every object $B$ of $G$ each fiber $E_B$ is a monoidal category. We write with the same notation in each fiber the tensor product because the context will prevent any confusion.

**Remark 3** Obviously, strict monoidal overcategories means that $u_l, u_r$ and $ass$ are natural identities. \[\square\]

Let $\mathcal{E} = (E, \Lambda, \otimes, I, u_l, u_r, ass)$ and $\mathcal{E}' = (E', \Lambda', \otimes', I', u'_l, u'_r, ass')$ be two monoidal overcategories with respective base categories $G$ and $G'$. A strict morphism $\mathcal{E} \xrightarrow{(F,F_0)} \mathcal{E}'$, is given by two functors $E \xrightarrow{F} E'$ and $G \xrightarrow{F_0} G'$ such that $F_0 \Lambda = \Lambda' F$, $FI = I' F_0$ and $F \otimes = \otimes' (F \times F_0)$. Let $\mathcal{E}$ be a monoidal overcategory. An overmonoid in $\mathcal{E}$ is given by a pair $(C; C_0)$ where $C_0 \in G(0)$ and $\mathcal{E} = (C, m, e)$ is a monoid in $E_{C_0}$ ($m$ is the multiplication and $e$ is the unity). Thus $(\mathcal{E}; C_0)$ is more properly written as $(C, m, e; C_0)$. It is a monoid in a fibre.

If $(\mathcal{E}; C_0)$ and $(\mathcal{E}'; C'_0)$ are overmonoids, a morphism $(\mathcal{E}; C_0) \xrightarrow{(f,f_0)} (\mathcal{E}'; C'_0)$, is given by a pair $(f,f_0)$ where $C_0 \xrightarrow{f_0} C'_0$ is an arrow in $G$ and $\mathcal{E} \xrightarrow{f} \mathcal{E}'$ is given by an arrow $C \xrightarrow{f} C'$ in $E$ such as $\Lambda(f) = f_0$ and $fm = m'(f \otimes f_0)$. $fe = e' I(f_0)$. We note $/\mathbb{Mon}(E, \Lambda)$ the category of overmonoids of $\mathcal{E}$.

Let $\mathcal{E}$ be a monoidal overcategory. It is liberal if the following two conditions are satisfied

- $\forall B \in G(0)$, the fiber $E_B$ is a liberal monoidal category.
- $\forall B \in G(0)$, the canonical inclusion functor $E_B \hookrightarrow E$ preserves coequalizer and $\mathbb{N}$-colimits.

Let $(\mathcal{E}; C_0)$ be a overmonoid of $\mathcal{E}$, then
Proposition 11 The pair \((\mathcal{E} / C, \hat{A})\), such as \(\mathcal{E} / C \xrightarrow{\hat{A}} \mathcal{G} / \Lambda(C)\), \(x \mapsto \Lambda(x)\), produces a overmonoidal overcategory

\[
\mathcal{E} / C = (\mathcal{E} / C, \hat{A}, \hat{\otimes}, \hat{I}, \hat{u}_l, \hat{u}_r, \hat{a}_{\text{so}})
\]

The proof is in (author?) [4, page 22] but let us recall that if \((X, x), (Y, y) \in \mathcal{E} / C\) then \((X, x) \hat{\otimes} (Y, y) := (X \otimes Y, m(x \otimes y))\). If \(b \in \mathcal{G} / \Lambda(C)\) then \(\hat{I}(b) := eI(b)\). The 2-cells \(\hat{u}_l, \hat{u}_r, \hat{a}_{\text{so}}\) are also provided with the corresponding data of \(\mathcal{E}\).

When overcoequalizers exist in \(\mathcal{E}\), it is not difficult to see that overcoequalizers in \(\mathcal{E} / C\) are computed by it, and we have the same phenomenon for \(\mathring{\mathbb{N}}\)-overcolimits. So we have the following easy proposition that is left for the reader.

Proposition 12 If \(\mathcal{E} = (\mathcal{E}, \Lambda, \otimes, I, u_l, u_r, a_{\text{so}})\) is a liberal monoidal overcategory then \(\mathcal{E} / C = (\mathcal{E} / C, \hat{A}, \hat{\otimes}, \hat{I}, \hat{u}_l, \hat{u}_r, \hat{a}_{\text{so}})\) is a liberal monoidal overcategory, and the morphism \(\mathcal{E} / C \xrightarrow{(S, S_0)} \mathcal{E}\) given by the functor \(\mathcal{E} / C \xrightarrow{S} \mathcal{E}\), \((X, x) \mapsto X\), is a strict morphism of overmonoidal overcategories which preserves the liberal structure.

We have the following proposition too

Proposition 13 If \(\mathcal{E}\) is a liberal monoidal overcategory and if \((\mathcal{E}', C_0) \xrightarrow{(h, h_0)} (\mathcal{E}', C_0')\) is a morphism of overmonoids, then the morphism

\[
\mathcal{E} / C \xrightarrow{(h^*, h_0^*)} \mathcal{E}' / C'
\]

is a strict morphism of monoidal overcategories which preserves the liberal structure.

PROOF The fact that \((h^*, h_0^*)\) is a strict morphism of overmonoidal overcategories has already been shown ((author?) [4, page 25]) and the fact that \(h^*\) preserves \(\mathring{\mathbb{N}}\)-overcolimits has already been proved for the numeral context [see 4, page 25]. We only have to show that \(h^*\) preserves overcoequalizers and it is evident by construction.
Now we have enough material to give the main theorem of this paragraph.

**Theorem 4** Let $\mathcal{E} = (E, A, \otimes, I, u_l, u_r, aso)$ be a liberal monoidal overcategory such that $\forall B \in G(0)$ the object $I(B)$ is initial in the fiber $\mathbb{E}_B$ and such that $\forall b \in G(1)$ the object $I(b)$ is initial in the fiber $\mathbb{E}_b$, then the forgetful overfunctor
\[
(\mathcal{M}on(E, A), A) \xrightarrow{U} (E, A)
\]
has a left overadjoint $M \dashv U$ and it is overmonadic. \qed

**Proof** It is similar to the proof of proposition \[10\] and we just need to adapt it to the overcategorical context. In particular we use proposition \[10\], the reminders in section \[2.1\], proposition \[13\] plus the following two results (The first result below is a refinement of proposition \[13\]. We prove these two results by induction):

**Result 1** Let $(\mathcal{E}; C_0) \xrightarrow{(h, h_0)} (\mathcal{E}'; C_0')$ be a morphism of overmonoids, then if $(X, x) \in \mathbb{E}/C$ then $\forall n \in \mathbb{N}$, $h^*((X, x)_n) = (h^*((X, x)))_n$, where $(X, x)_n$ is the $n^{th}$ object of the graded monoid associated with $(X, x)$ (see \[4, 1.2.3 page 12\] for definition and results about graded monoids).

**Result 2** $\forall n \in \mathbb{N}$, $(X, l(X))_n = (X_n, l_n)$, where $X_n \xrightarrow{l_n} Mo(X)$ is an arrow of the colimit cocone defining $Mo(X)$, and where $(X, l(X))_n$ is the $n^{th}$ object of the graded monoid associated with $(X, l(X))$.

The overmonadicity of $U$ is a simple consequence of theorem \[3\]. In particular this overmonadicity has already been proved in the numeral context \[4, see proposition 1.3.1, page 20\]. \[\blacksquare\]

Now we can study the important case of pointed overmonoidal overcategories. In \[4\] it is proved that to any monoidal category $\mathcal{V} = (V, \otimes, I, u_l, u_r, aso)$ we associate its pointed monoidal category
\[
Pt(\mathcal{V}) = (Pt(V), \otimes, \tilde{I}, u_\tilde{I}, u_r, aso)
\]
and if \( V \) was liberal then \( Pt(V) \) remained liberal.

We can expand to the overmonoidal context this construction and this result. Let \( \mathcal{E} = (E, \Lambda, \otimes, I, u_l, u_r, aso) \) be a monoidal overcategory over a fixed category \( G \).

Let \( Pt(E) \) the category with objects the pairs \((X, x)\) where \( X \in E(0) \) and \( I(\Lambda(X)) \xrightarrow{\Delta} X \in E(1) \), and which has for arrows \((X, x) \xrightarrow{f} (Y, y)\), given by morphism \( X \xrightarrow{f} Y \) of \( E \) such as \( fx = yI(\Lambda(f)) \). In this case we have

**Proposition 14** The pair \((Pt(E), \tilde{\Lambda})\) such that:

\[
\tilde{\Lambda}(X, x) \xrightarrow{\tilde{u}_l} I(\Lambda(X))
\]

produces a structure of monoidal overcategory

\[
Pt(\mathcal{E}) = (Pt(E), \tilde{\Lambda}, \otimes, \tilde{I}, \tilde{u}_l, \tilde{u}_r, \tilde{aso})
\]

**Proof**

- Its tensor is the bifunctor \( Pt(E) \times_G Pt(E) \xrightarrow{\otimes} Pt(E) \),
  \[
  ((X, x), (Y, y)) \mapsto (X, x) \otimes (Y, y) := (X \otimes Y, (x \otimes y)u_l^{-1}).
  \]

- Its "unity" functor is \( G \xrightarrow{I} Pt(E), G \mapsto (I(G), 1_{I(G)}) \).

- Left and right isomorphisms of unity: For all \((X, x)\) of \( Pt(E)(0) \) the tensor \( \tilde{I}(\tilde{\Lambda}(X, x)) \otimes (X, x) \) is given by the morphism \((1_{I(\Lambda(X))} \otimes x)u_l^{-1}\) of \( E \), and we have \( u_l((1_{I(\Lambda(X))} \otimes x)u_l^{-1}) = x \) thanks to the equality

\[
u_l(X)(1_{I(\Lambda(X))} \otimes x) = xu_l(I(\Lambda(X))).\]

Thus we get

\[
\tilde{I}(\tilde{\Lambda}(X, x)) \otimes (X, x) \xrightarrow{\tilde{u}_l(X, x)} (X, x)
\]

and \( \tilde{u}_l(X, x) \) given by \( u_l(X) \) is a good candidate to define \( \tilde{u}_l \). Thus we obtain the natural transformation \( \tilde{\otimes}(\tilde{I}, Id) \xrightarrow{\tilde{u}_l} Id \) which is in fact, an underlying datum of its 2-cell \( \tilde{u}_l \). In the same way we obtain the 2-cell

\[
\tilde{\otimes}(Id, \tilde{I}) \xrightarrow{\tilde{u}_l} Id.
\]
The tensor products

\[((X,x)\otimes(Y,y))\otimes(Z,z)\text{ and } (X,x)\otimes((Y,y)\otimes(Z,z))\]

are respectively given by

\[[((x\otimes y)u^{-1}_i)\otimes z]u^{-1}_i \text{ and } [x\otimes ((y\otimes z)u^{-1}_i)]u^{-1}_i,\]

and we have the equality

\[aso[((x\otimes y)u^{-1}_i)\otimes z]u^{-1}_i = [x\otimes ((y\otimes z)u^{-1}_i)]u^{-1}_i\]

due to the naturality of \(aso\) and the underlying overmonoid structure of \(I(A(X))\). We consequently obtain

\[((X,x)\otimes(Y,y))\otimes(Z,z) \overset{aso}{\longrightarrow} (X,x)\otimes((Y,y)\otimes(Z,z))\]

where in particular \(aso\) is given by \(aso\), and is the good candidate to be the 2-cells of associativity. Thus we obtain the natural transformation \(\otimes(\otimes \times Id) \overset{aso}{\longrightarrow} \otimes(Id \times \otimes)\) which in reality is an underlying datum of the 2-cell \(aso\), and with this description of \(Pt(\mathcal{E})\) it is now easy to see that it is a monoidal overcategory.

As for \(\mathcal{E}/\mathcal{C}\), when overcoequalizers exist in \(\mathcal{E}\), then we can see that overcoequalizers in \(Pt(\mathcal{E})\) are computed by it, and we have the same phenomenon for \(\overset{\overline{N}}\mathcal{N}\)-overcolimits. So we have the following easy proposition.

**Proposition 15** If \(\mathcal{E} = (E,A,\otimes,I,u_l,u_r,aso)\) is a liberal monoidal overcategory then \(Pt(\mathcal{E}) = (Pt(E),\overline{A},\overline{\otimes},\overline{I},\overline{u_l},\overline{u_r},\overline{aso})\) stays a liberal overmonoidal category, and trivially the functor \(\overline{I}\) send objects and arrows of \(G\) to initial objects in the corresponding fibers.
The following proposition is easy. It is the overmonoidal version of the result in (author?) [4, 1.2.1 page 10].

**Proposition 16** If \( \mathcal{E} = (\mathcal{E}, A, \otimes, I, u_l, u_r, aso) \) is a monoidal overcategory, then we have the commutative triangle

\[
\begin{array}{ccc}
/\text{Mon}(\mathcal{E}, A) & \xrightarrow{\varphi} & /\text{Mon}(\text{Pt}(\mathcal{E}), \tilde{A}) \\
U & \downarrow & U' \\
\text{Pt}(\mathcal{E}) & & \\
\end{array}
\]

such that \( \varphi \) is an isomorphism given by \( \varphi((C, e, m; C_0)) = ((C, e), e, m; C_0) \), and with \( U((C, e, m; C_0)) = (C, e) \) and \( U'(((C, x), e, m; C_0)) = (C, e) \).

With the theorem and the previous propositions we have at once:

**Theorem 5** If \( \mathcal{E} = (\mathcal{E}, A, \otimes, I, u_l, u_r, aso) \) is a liberal monoidal overcategory then the forgetful overfunctor

\[
(\text{Pt}(\mathcal{E}), \tilde{A}) \xrightarrow{U} (\text{Pt}(\mathcal{E}), \tilde{A}), \quad (C, e, m; C_0) \xrightarrow{(C, e)}
\]

has a left overadjoint and is overmonadic.

**Remark 4** Let us denote by \( M \) the left overadjoint of \( U \), then if we applied \( (X, x) \in \text{Pt}(\mathcal{E}) \) to the unity \( 1_{\text{Pt}(\mathcal{E})} \xrightarrow{l} UM \) of this overadjunction, we obtain the morphism \( (X, x) \xrightarrow{l((X, x))} U(M(X, x)) \) of \( \text{Pt}(\mathcal{E}) \) i.e \( (X, x) \xrightarrow{l((X, x))} U(X, e, m; X_0) = (X, e) \). And in particular this morphism gives us the equality \( l((X, x))x = e \). This equality is important because it shows, in the particular context of colored operads of [5] and [?], that the operads of weak higher transformations are well-provided with a system of operations.
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