Unified QCD evolution equations for quarks and gluons

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Abstract

Considering the BFKL and DGLAP QCD evolution equations for structure functions, we discuss the possibility of unifying them in the whole $x$ and $Q^2$ range. We emphasize that the main problem is related to the constraint of angular ordering of the radiation, and the cancellation of the related collinear singularities for inclusive processes. At the leading log $1/x$ and log $Q^2$ level, we write down a unified system of equations satisfying this cancellation constraint. At low $x$, it leads to a less singular behaviour of the structure functions than the BFKL prediction.

1. Introduction

There exist two different tools in perturbative QCD when one wants to get predictions for quark and gluon structure functions measured in Deep-Inelastic Scattering: The first one, valid when $Q^2$ is large, is a resummation to all order of the leading logarithms of $Q^2$ ($LLQ^2$), namely terms of the type $\alpha_s^n \ln(Q^2/\Lambda_{QCD}^2)^n$, due to the \textit{collinear} singularity of the radiative corrections. It leads to the so called "Altarelli-Parisi" (or DGLAP) evolution equations at leading order. The second one, valid when the Bjorken variable $x$ is small, is a resummation to all order of the leading logarithm of $x$ ($LL1/x$), namely terms like $\alpha_s^n (\ln1/x)^n$, due to the \textit{infra-red} singularities of the radiative soft part. This tedious calculation was performed twenty years ago by L.N.Lipatov and collaborators (BFKL equation) and predicts a singular behaviour of the proton structure function at small $x$. HERA revived this result, since the data on the quark structure function inside proton at very small $x$ are in qualitative agreement with the BFKL prediction.

Since the HERA experiment covers a very large range in $x$ and $Q^2$, it would be of great interest to have a unified system of these two equations. In the past, this possibility has already been discussed. It was first noticed that a system of equation taking into account both $LL1/x$ and $LLQ^2$ terms can be written down, by a precise combination of the corresponding integral kernels. More recently, it has been proved rigorously that a unified description of the gluon radiation in the whole $x$-range is possible, due to the property of \textit{angular ordering}. In this picture, one can show that...
the dominant contribution due to collinear singularities (present in any gluon production amplitude) comes from the regions satisfying the following kinematical property:

$$Q/x \gg \ldots \gg \theta_{i} \gg \theta_{i-1} \gg \ldots \gg \theta_{1} \quad (1)$$

where $\theta_{i} \approx (q_{i})/x_{i}$ are the angles of the emitted gluon with respect to the direction of the first emitted gluon momentum. The two previous regimes are recovered in two different limits: i) at $x$ of the order of unity, corresponding to non strongly ordered $x_{i}$, one recovers the $q_{i}$ ordering and the $LLQ^{2}$ resummation \([3]\). ii) at small $x$, the gluon momentum fractions $x_{i}$ are strongly ordered, and the relation \([\ref{1}]\) doesn’t imply $q_{i}$-ordering. Thus collinear singularities as well as infrared ones can contribute to $LL1/x$ singularities.

The key point is that for structure functions (this is not the case for non-inclusive quantities) the collinear singularities cancel in such a way that one recovers the BFKL evolution \([4, 5]\). We propose here a scheme of such unification for both glue and quarks, which fullfill this constraint in an effective way \([\ref{2}]\).

### 2. Unified QCD evolution equations

#### 2.1. Equations in Mellin space

We will restrict ourselves, for the sake of simplicity, to the case of a fixed coupling constant $\alpha_{S}$, as for the original BFKL derivation. We consider the double inverse Mellin transform of the singlet ($F_{S}$) and gluon ($F_{G}$) structure function with respect to $Q^{2}$ and $x$:

$$F_{S,G}(x, Q^{2}) = \int \frac{d \gamma}{2\pi i} e^{\gamma \ln Q^{2}/\Lambda^{2}}$$

$$\times \int \frac{d j}{2\pi i} e^{(j-1) \ln 1/x} \varphi_{S,G}(j, \gamma). \quad (2)$$

The DGLAP equations (for fixed $\alpha_{S}$) can be written in matrix form for $\varphi_{S}$ and $\varphi_{G}$ as follows:

$$\left( \begin{array}{c} \varphi_{S} \\ \varphi_{G} \end{array} \right) = \left( \begin{array}{c} \varphi_{S}^{(0)} \\ \varphi_{G}^{(0)} \end{array} \right) + \frac{\alpha_{S}}{4\pi \gamma} \left( \begin{array}{cc} \nu_{F} & 2n_{F} \phi_{F}^{G} \\ \frac{2n_{F}}{\nu_{G}} & \nu_{G} \end{array} \right) \left( \begin{array}{c} \varphi_{S} \\ \varphi_{G} \end{array} \right), \quad (3)$$

where $\{\nu_{G}, \nu_{F}, \phi_{F}^{G}, \phi_{G}^{(0)}\}$ are the usual ($j$-dependent) DGLAP weights \([\ref{3}]\), and $\varphi_{S,G}^{(0)}$ are the initial conditions.

The equation \([\ref{3}]\) must be modified so as to take into account the BFKL contribution in the gluon sector, due to soft singularities. This dominant contribution can be expressed as a singularity in the $j$–plane at the value

$$j_{L} = 1 + \frac{\alpha_{S} N_{C}}{\pi} \chi(\gamma) \quad (4)$$

where

$$\chi(\gamma) \equiv 2\psi(1) - \psi(1 - \gamma) ; \quad \psi(\gamma) = \frac{d \ln \Gamma(\gamma)}{d \gamma} \quad (5)$$

is the eigenvalue-function of the BFKL kernel. Closing the path of integration in $j$ around the rightest pole given by \([\ref{4}]\), one gets:

$$F_{S,G}(x, q^{2}) = \int \frac{d \gamma}{2\pi i} e^{\gamma \ln Q^{2}/\Lambda^{2}} e^{\alpha_{S} \gamma \ln 1/x}$$

$$\simeq \left( \frac{Q^{2}}{\Lambda^{2}} \right)^{1/2} x^{-\frac{\alpha_{S}}{4\pi} + 4 \ln 2}, \quad (6)$$

where a saddle point method is used and gives the dominant contribution at $\gamma_{c} = 1/2$, corresponding to $\chi(\gamma_{c}) = 4 \ln 2$.

In order to implement this singular behaviour \([\ref{4}]\), we replace the gluonic contribution to the anomalous dimension by the following \([\ref{5}]\):

$$\nu_{G(j)} \rightarrow \nu_{G(j)}^{*} = \gamma \chi(\gamma) \{ \nu_{G} + \Psi \} - \Psi, \quad (7)$$

where $\Psi$ is an arbitrary function holomorphic in the $j$-plane near $j = 1$ and below. Inserted in equation \([\ref{3}]\), this modification provides a system of unified equations mixing the DGLAP and BFKL kernels. Indeed, inverting the relation (3), after the replacement $\nu_{G} \rightarrow \nu_{G}^{*}$, one gets

$$\left( \begin{array}{c} \varphi_{G} \\ \varphi_{S} \end{array} \right) = \frac{1}{D(j, \gamma)}$$

$$\times \left( \begin{array}{cc} \nu_{F} & \frac{\alpha_{S}}{4\pi \gamma} \phi_{F}^{G} \\ \frac{1}{\nu_{G}} - \frac{\alpha_{S}}{4\pi \gamma} \nu_{G} & 1 - \frac{\alpha_{S}}{4\pi \gamma} \phi_{G}^{(0)} \end{array} \right) \left( \begin{array}{c} \varphi_{G}^{(0)} \\ \varphi_{S}^{(0)} \end{array} \right), \quad (8)$$

with

$$D(j, \gamma) = 1 - \frac{\alpha_{S}}{4\pi \gamma} (\nu_{F}^{*} + \nu_{F})$$

$$+ \left( \frac{\alpha_{S}}{4\pi \gamma} \right)^{2} (\nu_{F} \nu_{G}^{*} - 2n_{F} \phi_{F}^{G} \phi_{G}^{(0)}). \quad (9)$$

Before studying this denominator \([\ref{6}]\), let us note that in principle $\phi_{G}^{G}$ also gets a contribution from infrared singularities, such that it should be changed into $\gamma \chi(\gamma) \{ \nu_{G} + \Psi_{1} \} - \Psi_{1}$ where $\Psi_{1}$ has the same properties as $\Psi$. This will be discussed elsewhere \([\ref{6}]\).

The zeroes of $D(j, \gamma)$ depend on the region in the complex $j$–plane involved in the inverse Mellin transform \([\ref{2}]\), and thus on the region in $x$ one is looking at:

i) for $x$ of the order of unity, $\alpha_{S} \ln 1/x \ll 1$, the modification \([\ref{5}]\) has no effect, since the zeroes of $D(j, \gamma)$
are obtained for small values of $\gamma$ (of order $\pi S$). In that limit, one gets from the definition of $\chi(\gamma)$:

$$\chi(\gamma) \approx 1/\gamma + O(\gamma^2); \quad \nu_G \approx \nu_G + O(\pi^3),$$  

(10)

and one recovers the ordinary DGLAP equations [3] (at fixed $\pi S$).

ii) When $\pi S \ln 1/x = O(1)$, the singular structure of the BFKL kernel drives the relevant domain of the integration over $\gamma$ in [8] near the "critical" value $\gamma_c = 1/2$. One recovers the singular behaviour compatible with the BFKL calculations. Taking the appropriate limit $j \to 1$, $\pi S/(j - 1) = O(1)$:

$$D(j, \gamma) \propto 1 - \frac{\pi N C \chi(\gamma)}{4 \pi (j - 1)} j \to 1 - \frac{\pi N C}{4 \pi} \frac{4 \ln 2}{j - 1}$$

(11)

2.2. Constraint on the $\psi$ function

The $\psi(j)$ function we have introduced seems to be arbitrary when we try to unify DGLAP and BFKL equations. In fact, because of the precise cancellation of collinear singularities that we emphasized in the introduction, the collinear singularities arising from quark-loop contribution should cancel at small-x. Indeed, angular ordering for radiated quarks [10] can in principle modify the $LL1/x$ singularity. Despite the fact that this cancellation is well established for gluons, this is not proved when "finite parts" are included [11].

We thus impose this cancellation around $j = j_i$, namely, considering $D(j, \gamma)$ at first order in $\pi S$ (a complete discussion will be presented elsewhere [11]):

$$\psi(j_L) \approx \nu_F(j_L) \quad \nu_G(j_L) + \nu_F(j_L) = \left[\frac{\pi N C \log 2}{\pi}\right]^{-1}.$$  

(12)

An equivalent constraint is to impose that the conformal properties of the BFKL kernel [11] would be preserved, namely that the critical conformal weight should stay at $\gamma = 1/2$ [8]. This is a statement at $LL1/x$ level.

The system of equations [12] leads to a decrease of the effective dominant singularity with respect to the BFKL one [8]. This is in agreement with HERA data [8].

Note that the energy-momentum conservation can be implemented in the system of equations [8] in a consistent way [8]. It imposes that $\psi(2) = \nu_G(2) = 2$. This constraint is not related to the previous one at $j_i$ and is not responsible in our model for the decrease of the $j-$plane singularity (in contrary to the discussion of Ref. [2]).

3. Conclusion

The unification of the evolution equations for structure functions is possible, combining the leading-logarithmic contributions in both $x$ and $Q^2$ variables at fixed $\pi S$. The constraint due to the mixing of $LL1/x$ and $LLQ^2$ leads to a shift down of the $LL1/x$ prediction.

It would be interesting to know the phenomenological consequences of our system of unified equations. Work is in progress in order to take also into account the running of $\alpha_S$ and the next-leading order terms.

Acknowledgements

All these results come from a fruitful collaboration with R. Peschanski. It is a pleasure to thank L.N.Lipatov for discussion during the Cambridge meeting about conformal properties of unified equations.

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