An exponential estimate for the cubic partial sums of multiple Fourier series

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Abstract. We prove an exponential integral estimate for the cubic partial sums of multiple Fourier series on sets of large measure. This estimate yields some new properties of Fourier series.

Keywords: multiple Fourier series, exponential integral estimates, cubic partial sums.

§ 1. Introduction

Put $\mathbb{T} = \mathbb{R}/2\pi$ and let $\mathbb{T}^d$ denote the $d$-dimensional torus. For every function $f \in L^1(\mathbb{T}^d)$ we consider the multiple Fourier series and its conjugate:

$$
\sum_{\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d} a_{\mathbf{n}} e^{i \mathbf{n} \cdot \mathbf{x}},
$$

$$
\sum_{\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d \setminus \{0\}} \left( \prod_{k=1}^{d} (-i \cdot \text{sign } n_k) \right) a_{\mathbf{n}} e^{i \mathbf{n} \cdot \mathbf{x}},
$$

where

$$
\mathbf{n} = (n_1, \ldots, n_d), \quad \mathbf{x} = (x_1, \ldots, x_d), \quad \mathbf{n} \cdot \mathbf{x} = n_1 x_1 + \cdots + n_d x_d,
$$

$$
a_{\mathbf{n}} = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(\mathbf{x}) e^{-i \mathbf{n} \cdot \mathbf{x}} \, d\mathbf{x}.
$$

Denote the rectangular and cubic partial sums of the series (1) by

$$
S_n f(\mathbf{x}) = \sum_{-n_i \leq k_i \leq n_i} a_{\mathbf{k}} e^{i \mathbf{k} \cdot \mathbf{x}}, \quad \mathbf{n} \in \mathbb{Z}^d,
$$

$$
\tilde{S}_n f(\mathbf{x}) = \sum_{-n \leq k_i \leq n} a_{\mathbf{k}} e^{i \mathbf{k} \cdot \mathbf{x}}, \quad n \in \mathbb{N},
$$

and let $\tilde{S}_n$ and $\tilde{S}_n$ be their conjugates.

We shall consider the Orlicz class of functions corresponding to the logarithmic function

$$
\text{Log}_k(u) = |u| \max\{0, \log^k |u|\}, \quad k = 1, 2, \ldots.
$$

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This is the Banach space of functions

\[ \text{Log}_k(L)(\mathbb{T}^d) = \left\{ f \in L^1(\mathbb{T}^d) : \int_{\mathbb{T}^d} \text{Log}_k(f) < \infty \right\} \]

with the Luxemburg norm

\[ \|f\|_{\text{Log}_k(L)} = \inf \left\{ \lambda : \lambda > 0, \int_{\mathbb{T}^d} \text{Log}_k \left( \frac{f}{\lambda} \right) \leq 1 \right\} < \infty. \]

It is well known that the rectangular partial sums of the \(d\)-dimensional Fourier series of any function \(f \in \text{Log}_{d-1}(L)(\mathbb{T}^d)\) converge in measure (see [1], [2]), that is,

\[ \lim_{\min(n) \to \infty} \min_{1 \leq i \leq d} n_i \}

\[ \|f\|_{\text{Log}_k(L)} = \inf \left\{ \lambda : \lambda > 0, \int_{\mathbb{T}^d} \text{Log}_k \left( \frac{f}{\lambda} \right) \leq 1 \right\} < \infty. \]

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\[ \|f\|_{\text{Log}_k(L)} = \inf \left\{ \lambda : \lambda > 0, \int_{\mathbb{T}^d} \text{Log}_k \left( \frac{f}{\lambda} \right) \leq 1 \right\} < \infty. \]

for every \(\varepsilon > 0\), where \(\min(n) = \min_{1 \leq i \leq d} n_i\).

On the other hand, Konyagin [3] and Getsadze [4] established that \(\text{Log}_{d-1}(L)\) is the largest Orlicz space whose elements satisfy (4).

The following problem was considered in [5], [6]. Find an exact estimate for the growth of a function \(\Phi : \mathbb{R}^+ \to \mathbb{R}^+\) with \(\lim_{t \to 0} \Phi(t) = 0\) such that for every function \(f \in \text{Log}_{d-1}(L)(\mathbb{T}^d)\) and every number \(\varepsilon > 0\) one can find a set \(E_{f,\varepsilon} \subset \mathbb{T}^d\), \(|E_{f,\varepsilon}| > (2\pi)^d - \varepsilon\), satisfying the condition

\[ \lim_{\min(n) \to \infty} \int_{E_{f,\varepsilon}} \Phi(|S_n f(x) - f(x)|) \, dx = 0. \] (5)

The expected sharp bound for the growth of such functions is

\[ \limsup_{t \to \infty} \frac{\log \Phi(t)}{t^{1/d}} < \infty. \] (6)

One can observe that (5) implies convergence in measure and, moreover, it gives a quantitative characterization of the convergence rate.

This problem was considered in [6] in the one-dimensional case. The following estimate for the conjugate function \(\tilde{f}\) was proved there:

\[ \int_{\mathbb{T}} \exp \left( c_1 \frac{\tilde{f}(x)}{M f(x)} \right) \, dx < c_2, \] (7)

where \(M f(x)\) is the Hardy–Littlewood maximal function. It was then used to derive the following exponential estimate for the one-dimensional partial sums of Fourier series, which in its turn yields (5) in the one-dimensional case.

**Theorem A** (see [6]). For every \(f \in L^1(\mathbb{T})\) we have

\[ \int_{\mathbb{T}} \exp \left( c_1 \frac{|S_n f(x)| + |\tilde{S}_n f(x)|}{M f(x)} \right) \, dx \leq c_2, \quad n = 1, 2, \ldots, \] (8)

where \(c_1\) and \(c_2\) are absolute constants.
The sharpness of the exponent in (8) (and hence in (5)) was proved by Oskolkov [7].

The relation (5) in the two-dimensional case with a function $\Phi$ satisfying (6) was established in [5]. The case $d \geq 3$ of this problem remains open, and so is the problem of the sharpness of (6) in the two-dimensional case.

Analogous estimates for the one-dimensional Walsh system and rearranged Haar systems were established in [8]. A similar problem was considered in [9] for general orthogonal $L^2$-series.

In this paper we consider a similar problem for cubic partial sums. Our main result is the following theorem.

**Theorem 1.** For every $f \in \text{Log}_{d-1}(L)(\mathbb{T}^d)$ there is a measurable function $F(x) > 0$ on $\mathbb{T}^d$ such that

$$|\{x \in \mathbb{T}^d : F(x) > \lambda\}| \lesssim \frac{\|f\|_{\text{Log}_{d-1}(\mathbb{T}^d)}}{\lambda},$$

(9)

$$\int_{\mathbb{T}^d} \exp\left(\frac{|S_n f(x)| + |\tilde{S}_n f(x)|}{F(x)}\right) dx \lesssim 1, \quad n = 1, 2, \ldots.$$  

(10)

Here and in what follows, the relation $a \lesssim b$ stands for the inequality $a \leq c \cdot b$, where $c$ is a constant depending only on the dimension $d$.

**Corollary 1.** For every $f \in \text{Log}_{d-1}(L)(\mathbb{T}^d)$ and every $\varepsilon > 0$ there is a set $E = E_{f, \varepsilon} \subset \mathbb{T}^d$ such that

$$|E_{f, \varepsilon}| > (2\pi)^d - \varepsilon,$$

(11)

$$\int_{E_{f, \varepsilon}} \exp\left(\gamma \varepsilon \frac{|S_n f(x)| + |\tilde{S}_n f(x)|}{\|f\|_{\text{Log}_{d-1}(\mathbb{T}^d)}}\right) dx \lesssim 1, \quad n = 1, 2, \ldots,$$

(12)

where $\gamma > 0$ is a constant depending only on $d$.

**Corollary 2.** For every $f \in \text{Log}_{d-1}(L)(\mathbb{T}^d)$ and every $\varepsilon > 0$ there is a set $E_{f, \varepsilon} \subset \mathbb{T}^d$ satisfying (11) and such that the relations

$$\lim_{n \to \infty} \int_{E_{f, \varepsilon}} (\exp(A|S_n f(x) - f(x)|) - 1) \, dx = 0,$$

(13)

$$\lim_{n \to \infty} \int_{E_{f, \varepsilon}} (\exp(A|\tilde{S}_n f(x) - \tilde{f}(x)|) - 1) \, dx = 0$$

(14)

hold for any $A > 0$, where $\tilde{f}$ is the $d$-dimensional conjugate function of $f$.

**Remark 1.** The method used in our proof of Theorem 1 is also applicable to the mixed partial sums of multiple Fourier series defined by the formula

$$S_n^B f(x) = \sum_{-n_i \leq k_i \leq n_i} \left( \prod_{s \in B} (-i \cdot \text{sign } n_s) \right) a_k e^{i k \cdot x}, \quad n \in \mathbb{Z}^d,$$

where $B \subset \{1, 2, \ldots , d\}$ (see [10], Ch. 8). Namely, given any $f \in \text{Log}_{d-1}(L)(\mathbb{T}^d)$, one can find a function $F(x) > 0$ satisfying (9) and

$$\int_{\mathbb{T}^d} \exp\left(\frac{|S_n^B f(x)|}{F(x)}\right) dx \lesssim 1, \quad n = 1, 2, \ldots.$$
To avoid technical difficulties in the proofs, we consider only the typical cases when $B = \emptyset$ or \{1, 2, $\ldots$, d\} (Theorem 1).

Remark 2. The counterexamples of Konyagin [3] and Getsadze [4] show that $\text{Log}_{d-1}(L)(\mathbb{T}^d)$ is the largest Orlicz class where such properties hold.

Remark 3. We prove Theorem 1 by reducing it to the one-dimensional case. This well-known approach was first used by Sjölin [11] to prove a multidimensional version of Carleson’s theorem.

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§ 2. Notation and lemmas

By Theorem 9.5 in Ch. 2 of [12], the Luxemburg norm satisfies the relations

$$\|f\|_{\text{Log}_k(L)} \leq 1 \implies \int_{\mathbb{T}^d} \text{Log}_k(f) \leq \|f\|_{\text{Log}_k(L)},$$

(15)

$$\|f\|_{\text{Log}_k(L)} \geq 1 \implies \int_{\mathbb{T}^d} \text{Log}_k(f) \geq \|f\|_{\text{Log}_k(L)}.$$  

(16)

In fact, these inequalities hold not only for logarithmic but also for general Luxemburg norms. Using (15) and (16), one can easily check that

$$\|f\|_{\text{Log}_k(L)} \lesssim 1 + \int_{\mathbb{T}^d} \text{Log}_k(f)$$

(17)

for every $f \in \text{Log}_k(\mathbb{T}^d)$. Clearly, if $\|f\|_{\text{Log}_k(L)} = 1$, then we have both upper and lower bounds

$$1 + \int_{\mathbb{T}^d} \text{Log}_k(f) \lesssim \|f\|_{\text{Log}_k(L)} = 1 \lesssim 1 + \int_{\mathbb{T}^d} \text{Log}_k(f).$$

(18)

The one-dimensional conjugate function of $f \in L^1(\mathbb{T})$ is defined as

$$\tilde{f}(x) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(x + t)}{2\tan(t/2)} \, dt = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|t| < \varepsilon} \frac{f(x + t)}{2\tan(t/2)} \, dt.$$  

(19)

It is well known that $\tilde{f}(x)$ is defined a.e. for every Lebesgue integrable function and satisfies the inequality

$$\int_{\mathbb{T}} \text{Log}_{k-1}(\tilde{f}) \lesssim 1 + \int_{\mathbb{T}} \text{Log}_k(f), \quad k = 1, 2, \ldots$$

(20)

(see [13], Ch. 7). We shall need this inequality in the following form.

Lemma 1. If $f \in \text{Log}_k(L)(\mathbb{T}^d)$, $k = 0, 1, \ldots$, then the function

$$g(x_1, x_2, \ldots, x_d) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(x_1 + t, x_2 + t, x_3, \ldots, x_d)}{\tan(t/2)} \, dt$$

is defined a.e. on $\mathbb{T}^d$ and satisfies the bound

$$\int_{\mathbb{T}^d} \text{Log}_{k-1}(g) \lesssim 1 + \int_{\mathbb{T}^d} \text{Log}_k(f).$$
We define the $d$-dimensional conjugate of a function $f \in \text{Log}_{d-1}(\mathbb{T}^d)$ as an iterated integral:

\[
\tilde{f}(x) = \text{p.v.} \frac{1}{\pi^d} \int_{\mathbb{T}^d} f(x + t) \prod_{k=1}^{d} \frac{1}{2\tan(t_k/2)} dt_1 \ldots dt_d
\]
\[
= \text{p.v.} \frac{1}{\pi} \int_T \left( \ldots \left( \text{p.v.} \frac{1}{\pi} \int_T f(x + t) \prod_{k=1}^{d} \frac{1}{2\tan(t_k/2)} dt \right) \ldots \right) dt_1,
\]

where the variables of integration are taken in the reverse order $t_d, t_{d-1}, \ldots, t_1$.

Note that the $d$-dimensional conjugate $\tilde{f}$ is defined a.e. for $f \in \text{Log}_{d-1}(\mathbb{T}^d)$. In what follows we understand all integrals in the sense of the principal value and omit the symbol p.v. before them. The two-dimensional case of the following lemma was proved in [14]. This lemma enables us to use the modified partial sums

\[
S_n^* f(x) = \frac{1}{\pi^d} \int_{\mathbb{T}^d} \prod_{k=1}^{d} \frac{\sin nt_k}{2\tan(t_k/2)} f(x + t) dt,
\]
\[
\tilde{S}_n^* f(x) = \frac{1}{\pi^d} \int_{\mathbb{T}^d} \prod_{k=1}^{d} \frac{\cos nt_k - 1}{2\tan(t_k/2)} f(x + t) dt
\]
in the proof of the theorem.

**Lemma 2.** If $f \in \text{Log}_{d-1}(L(\mathbb{T}^d))$, then

\[
\int_{\mathbb{T}^d} \sup_n |S_n f(x) - S_n^* f(x)| \, dx \lesssim \|f\|_{\text{Log}_{d-1}(L(\mathbb{T}^d))}, \tag{21}
\]
\[
\int_{\mathbb{T}^d} \sup_n |\tilde{S}_n f(x) - \tilde{S}_n^* f(x)| \, dx \lesssim \|f\|_{\text{Log}_{d-1}(L(\mathbb{T}^d))}. \tag{22}
\]

**Proof.** One can clearly assume that

\[
\|f\|_{\text{Log}_{d-1}(L(\mathbb{T}^d))} = 1. \tag{23}
\]

We shall only prove (21). (22) can be proved similarly. We have

\[
S_n f(x) = \frac{1}{\pi^d} \int_{\mathbb{T}^d} \prod_{k=1}^{d} D_n(t_k) f(x + t) dt, \tag{24}
\]

where

\[
D_n(x) = \frac{\sin(n + 1/2)x}{2 \sin(x/2)} = \frac{\sin n x}{2 \tan(x/2)} + \frac{1}{2} \cos nx \tag{25}
\]
is the Dirichlet kernel. Substituting (25) into (24), we see that the difference $S_n f(x) - S_n^* f(x)$ is the sum of several integrals of the form

\[
\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \prod_{k \in A} \frac{\sin nt_k}{\tan(t_k/2)} \prod_{k \in A^c} \cos(nt_k) \cdot f(x + t) dt, \tag{26}
\]
where \( A \subseteq \{1, 2, \ldots, d\} \) is a subset of integers. Applying the product formulae for
trigonometric functions, we split each integral (26) into a sum of integrals of the form
\[
\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \phi(n(\pm t_1 \pm t_2 \pm \cdots \pm t_d)) \frac{2^{d-1}}{\prod_{k \in A} \tan(t_k/2)} \, f(x + t) \, dt,
\]
where the function \( \phi \) is either the sine or cosine. This reduces the proof of
the lemma to an estimation of the integrals (27). When \( A = \emptyset \), the desired estimate is
\[
\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} |f(x + t)| \, dt = \frac{\|f\|_{L^1}}{(2\pi)^d} \lesssim \|f\|_{L\log^{d-1} L}.
\]
When \( A \neq \emptyset \), the integrals (27) are estimated in a similar way. Therefore we
estimate only the integral
\[
I_n f(x) = \int_{\mathbb{T}^d} \frac{\sin n(t_1 + t_2 + \cdots + t_d)}{\prod_{k=l+1}^d \tan(t_k/2)} \, f(x + t) \, dt,
\]
which corresponds to \( A = \{1, \ldots, l\}, \, l \geq 1 \). After the change of variables
\[
u_1 = t_1 + t_2 + \cdots + t_d, \quad u_2 = t_2, \ldots, \quad u_d = t_d
\]
we obtain from (28) that
\[
|I_n f(x)| = \left| \int_{\mathbb{T}^l} \frac{\sin n u_1}{\prod_{k=l+1}^d \tan(u_k/2)} \, G(x, u) \, du \right|
\leq \left| \int_{\mathbb{T}^l} \frac{\sin n u_1 \left( \int_{\mathbb{T}^{d-l}} \frac{G(x, u)}{\prod_{k=l+1}^d \tan(u_k/2)} \, du_{l+1} \cdots du_d \right) \, du_1 \cdots du_l} {\prod_{k=l+1}^d \tan(u_k/2)} \, du_{l+1} \cdots du_d \, |du_1 \cdots du_l|,
\]
where
\[
G(x, u) = f(x_1 + u_1 - u_2 - \cdots - u_d, x_2 + u_2, \ldots, x_d + u_d).
\]
The inner integral may be regarded as a function of the variables \( x_k, k = 1, 2, \ldots, d, \) and
\( u_j, j = 1, 2, \ldots, l \). Moreover, applying Lemma 1 \((d - l)\) times, we have
\[
\int_{\mathbb{T}^d} \sup_n |I_n(x)| \, dx
\leq 1 + \int_{\mathbb{T}^{d+1}} \int_{\mathbb{T}^{d-l}} \cdots \int_{\mathbb{T}^{d-l}} \left| \frac{G(x, u)}{\prod_{k=l+1}^d \tan(u_k/2)} \, du_{l+1} \cdots du_d \right| \, du_1 \cdots du_l \, dx_1 \cdots dx_d
\lesssim 1 + \int_{\mathbb{T}^{d+1}} \cdots \int_{\mathbb{T}^{d-l}} \log_{d-l} \left( |G(x, u_1, \ldots, u_l, 0, \ldots, 0)| \right) \, du_1 \cdots du_l \, dx_1 \cdots dx_d
\approx 1 + (2\pi)^{l} \int_{\mathbb{T}^d} \log_{d-l}(f) \lesssim \|f\|_{\log_{d-l}(L)(\mathbb{T}^d)} = 1,
\]
which yields (21). Here we have used the inequality (18), which holds under the condition (23). \( \square \)
§ 3. Proofs of the main results

Proof of Theorem 1. We first prove the estimate (10) for the operators

\[ U_n f(x) = \frac{1}{\pi^d} \int_{T^d} \prod_{k=1}^{d} \phi_k(t_k) f(x + t) \, dt, \]  

where each \( \phi_k \) is one of the four functions

\[ \frac{\sin nt}{2 \tan(t/2)}, \quad \frac{\cos nt}{2 \tan(t/2)}, \quad \sin nt, \quad \cos nt. \]  

We call them operators of type \( U \). When all the \( \phi_k \) are of the form (33), the estimate (10) for \( U_n \) holds trivially. One can take \( F(x) \equiv c \cdot \| f \|_1 \) with an appropriate absolute constant \( c > 0 \). It is also easy to prove (10) in the case when only one function of the form (32) occurs in (31). Indeed, we can assume without loss of generality that

\[ U_n f(x) = \frac{1}{\pi^d} \int_{T^d} \sin nt_d \prod_{k=1}^{d-1} \sin nt_k \cdot f(x + t) \, dt. \]  

Observe that

\[ U_n f(x) = \frac{1}{\pi} \int_T \sin n(t_d - x_d) \frac{\sin n((t_d - x_d)/2)}{2 \tan((t_d - x_d)/2)} g(x_1, \ldots, x_{d-1}, t_d) \, dt_d, \]  

where

\[ g(x_1, \ldots, x_{d-1}, t_d) \]
\[ = \int_{T^{d-1}} \prod_{k=1}^{d-1} \sin nt_k \cdot f(x_1 + t_1, \ldots, x_{d-1} + t_{d-1}, t_d) \, dt_1 \ldots dt_{d-1}. \]

Then we can write

\[ U_n f(x) = \frac{\cos nx_d}{\pi} \int_T \sin nt_d \cdot g(x_1, \ldots, x_{d-1}, t_d) \frac{\sin n((t_d - x_d)/2)}{2 \tan((t_d - x_d)/2)} \, dt_d \]
\[ - \frac{\sin nx_d}{\pi} \int_T \cos nt_d \cdot g(x_1, \ldots, x_{d-1}, t_d) \frac{\cos n((t_d - x_d)/2)}{2 \tan((t_d - x_d)/2)} \, dt_d. \]

Let \( M_d g(x) \) be the maximal function of \( g(x) \) with respect to the variable \( x_d \). It follows easily from (7) that

\[ \int_{T^d} \exp\left( c_1 \frac{|U_n f(x)|}{M_d g(x)} \right) \, dx < c_2. \]

Since the maximal functions satisfies the weak \( L^1 \) inequality, the operators (34) and the function \( F(x) = M_d g(x) \) satisfy (10) and (9), as required.
To prove this for the general operators (31), we use induction on the dimension \( d \). According to the approach above, the required assertion holds when \( d = 1 \). To make the induction step, we assume that the exponential estimate holds for all operators (31) in dimension \( d - 1 \geq 1 \). Take a function \( f \in \text{Log}_{d-1}(T^d) \) such that
\[
\|f\|_{\text{Log}_{d-1}(L)(T^d)} = 1.
\] (35)
According to the approach above, we can assume that at least two functions \( \phi_k \) of type (32) occur in (31). Hence there is no loss of generality in assuming that
\[
U_n f(x) = \frac{1}{\pi^d} \int_{T^d} \prod_{k=1}^{d-2} \phi_k(t_k) \sin(nt_{d-1}) \sin(n_k) \frac{f(x + t)}{2 \tan(t_{d-1}/2)} \, dt.
\]
Thus we obtain
\[
U_n f(x) = \frac{1}{2\pi^d} \int_{T^d} \prod_{k=1}^{d-2} \phi_k(t_k) \frac{\cos n(t_{d-1} - t_d)}{4 \tan(t_{d-1}/2) \tan(t_d/2)} f(x + t) \, dt.
\]
\[
- \frac{1}{2\pi^d} \int_{T^d} \prod_{k=1}^{d-2} \phi_k(t_k) \frac{\cos n(t_{d-1} + t_d)}{4 \tan(t_{d-1}/2) \tan(t_d/2)} f(x + t) \, dt
\]
\[
= U_n^{(1)} f(x) - U_n^{(2)} f(x).
\]
We estimate only the first integral \( U_n^{(1)} f(x) \). The second can be estimated in a similar way. By making the change of variables
\[
u_1 = t_1, \quad u_2 = t_2, \quad \ldots, \quad u_{d-2} = t_{d-2}, \quad u_{d-1} = t_{d-1} - t_d, \quad u_d = t_d
\]
in the expression for \( U_n^{(1)} f(x) \), we obtain
\[
U_n^{(1)} f(x) = \frac{1}{2\pi^d} \int_{T^d} \prod_{k=1}^{d-2} \phi_k(u_k) \frac{\cos nu_{d-1}}{4 \tan((u_{d-1} + u_d/2)) \tan(u_d/2)} G(x, u) \, du,
\]
where
\[
G(x, u) = f(x_1 + u_1, \ldots, x_{d-2} + u_{d-2}, x_{d-1} + u_{d-1} + u_d, x_d + u_d). \tag{36}
\]
Using the identity
\[
\frac{1}{\tan(u + v) \tan v} = \frac{1}{\tan u \tan v} - \frac{1}{\tan u \tan(u + v)} - 1,
\]
we obtain that
\[
U_n^{(1)} f(x) = \frac{1}{2\pi^d} \int_{T^d} \prod_{k=1}^{d-2} \phi_k(u_k) \frac{\cos nu_{d-1}}{2 \tan(u_{d-1}/2)} \frac{1}{2 \tan(u_d/2)} G(x, u) \, du
\]
\[
- \frac{1}{2\pi^d} \int_{T^d} \prod_{k=1}^{d-2} \phi_k(u_k) \frac{\cos nu_{d-1}}{2 \tan(u_{d-1}/2)} \frac{1}{2 \tan((u_{d-1} + u_d)/2)} G(x, u) \, du
\]
\[
- \frac{1}{2\pi^d} \int_{T^d} \prod_{k=1}^{d-2} \phi_k(u_k) \cos nu_{d-1} G(x, u) \, du
\]
\[
= U_n^{(1,1)} f(x) - U_n^{(1,2)} f(x) - U_n^{(1,3)} f(x).
\]
For each \( i = 1, 2, 3 \) we shall find a function \( F^{(i)}(x) \geq 0 \) such that

\[
\{|x \in \mathbb{T}^d : F^{(i)}(x) > \lambda| \} \lesssim \frac{\|f\|_{\log_d-1}(\mathbb{T}^d)}{\lambda},
\]  
(37)

\[
\int_{\mathbb{T}^d} \exp\left( \frac{|U^{(1,i)}_n f(x)|}{F^{(i)}(x)} \right) dx \lesssim 1.
\]  
(38)

**Case \( i = 1 \).** Consider the operator

\[
U'_n g(x_1, \ldots, x_d) = \frac{1}{2\pi^{d-1}} \int_{\mathbb{T}^d} \prod_{k=1}^{d-2} \phi_k(u_k) \frac{\cos nu_{d-1}}{2\tan(u_{d-1}/2)} \times g(x_1 + u_1, \ldots, x_{d-1} + u_{d-1}, x_d) \, du_1 \ldots du_{d-1}
\]

acting on the function

\[
g(x_1, \ldots, x_d) = \frac{1}{\pi} \int_{\mathbb{T}} f(x_1, \ldots, x_{d-2}, x_{d-1} + t, x_d + t) \frac{dt}{2\tan(t/2)}
\]  
(39)

In view of (36), we get

\[
U^{(1,1)}_n f(x) = \frac{1}{2\pi^{d-1}} \int_{\mathbb{T}^d} \prod_{k=1}^{d-2} \phi_k(u_k) \frac{\cos nu_{d-1}}{2\tan(u_{d-1}/2)} \times \left( \frac{1}{\pi} \int_{\mathbb{T}} \frac{1}{2\tan(u_{d-1}/2)} G(x, u) \, du \right) \, du_1 \ldots du_{d-1}
\]

\[
= \frac{1}{2\pi^{d-1}} \int_{\mathbb{T}^d} \prod_{k=1}^{d-2} \phi_k(u_k) \frac{\cos nu_{d-1}}{2\tan(u_{d-1}/2)} \times g(x_1 + u_1, \ldots, x_{d-1} + u_{d-1}, x_d) \, du_1 \ldots du_{d-1}
\]

\[
= U'_n g(x_1, \ldots, x_{d-1}, x_d).
\]  
(40)

For every fixed \( x_d \) we may regard \( U'_n \) as a \((d-1)\)-dimensional operator (31) of type \( U \). Thus, by the induction hypothesis, for every \( x_d \in \mathbb{T} \) there is a function \( F_{x_d}(x_1, \ldots, x_{d-1}) = F^{(1)}(x_1, \ldots, x_d) \) such that

\[
\{|x_1, \ldots, x_{d-1} \in \mathbb{T}^{d-1} : F_{x_d}(x_1, \ldots, x_{d-1}) > \lambda| \} \lesssim \frac{\|g_{x_d}\|_{\log_d-2}(\mathbb{T}^{d-1})}{\lambda},
\]  
(41)

\[
\int_{\mathbb{T}^{d-1}} \exp\left( \frac{|U'_n g_{x_d}(x_1, \ldots, x_{d-1})|}{F_{x_d}(x_1, \ldots, x_{d-1})} \right) dx_1 \ldots dx_{d-1} \lesssim 1, \quad n = 1, 2, \ldots.
\]  
(42)

Here \( g_{x_d} \) is the function \( g(x_1, \ldots, x_d) \) regarded as a function of the variables \( x_1, \ldots, x_{d-1} \). On the other hand, it follows from Lemma 1 that

\[
\int_{\mathbb{T}^d} \log_{d-2}(g) \lesssim 1 + \int_{\mathbb{T}^d} \log_{d-1}(f) \lesssim \|f\|_{\log_{d-1}(\mathbb{T}^d)} = 1.
\]  
(43)
Applying (17), (18), (43) and (41), we obtain

\[
\left| \{ x \in T^d : F^{(1)}(x) > \lambda \} \right| \lesssim \frac{1}{\lambda} \int_T \|g_{x_d}\|_{\text{Log}_{d-2}(T^d-1)} \, dx_d
\]

\[
\lesssim \frac{1}{\lambda} \int_T \left( 1 + \int_{T^{d-1}} \text{Log}_{d-2}(g) \, dx_1 \ldots dx_{d-1} \right) \, dx_d
\]

\[
\lesssim \frac{1}{\lambda} \left( 1 + \int_{T^d} \text{Log}_{d-2}(g) \, du \right)
\]

\[
\lesssim \frac{1}{\lambda} \left( 1 + \int_{T^d} \text{Log}_{d-1}(f) \, du \right)
\]

\[
\lesssim \|f\|_{\text{Log}_{d-1}(T^d)} / \lambda.
\]

Using (40) and integrating the inequality (42) with respect to \(x_d\), we get

\[
\int_{T^d} \exp \left( \frac{c|U_n^{(1,1)} f(x)|}{F^{(1)}(x)} \right) \, dx \lesssim 1.
\]

This yields (37) and (38) when \(i = 1\).

**Case \(i = 2\).** The estimate for \(U_n^{(1,2)} f(x)\) can be proved in a similar way. We have

\[
U_n^{(1,2)} f(x) = \frac{1}{2\pi^{d-1}} \int_{T^{d-1}} \prod_{k=1}^{d-2} \phi_k(u_k) \frac{\cos n u_{d-1}}{2 \tan(u_{d-1}/2)}
\]

\[
\times \left( \frac{1}{\pi} \int_T \frac{G(x,u)}{2 \tan((u_{d-1} + u_d)/2)} \, du \right) \, du_1 \ldots du_{d-1}.
\]

The change of the variable \(t = u_d + u_{d-1}\) in the inner integral yields that

\[
\frac{1}{\pi} \int_T \frac{G(x,u)}{2 \tan((u_{d-1} + u_d)/2)} \, du
\]

\[
= \frac{1}{\pi} \int_T f(x_1 + u_1, \ldots, x_{d-2} + u_{d-2}, x_{d-1} + t, x_d - u_{d-1} + t) \, dt
\]

\[
= g(x_1 + u_1, \ldots, x_{d-2} + u_{d-2}, x_{d-1}, x_d - u_{d-1}),
\]

where \(g\) is again the function (39). Thus we obtain

\[
U_n^{(1,2)} f(x) = U_n'' g(x_1, \ldots, x_{d-1}, x_d) = \frac{1}{2\pi^{d-1}} \int_{T^{d-1}} \prod_{k=1}^{d-2} \phi_k(u_k) \frac{\cos n u_{d-1}}{2 \tan(u_{d-1}/2)}
\]

\[
\times g(x_1 + u_1, \ldots, x_{d-2} + u_{d-2}, x_{d-1}, x_d - u_{d-1}) \, du_1 \ldots du_{d-1}.
\]

For every fixed \(x_{d-1}\), we may regard this as a \((d-1)\)-dimensional operator of type \(U\) acting on the function \(g\) of the remaining variables \(x_1, \ldots, x_{d-2}, x_d\). By the
induction hypothesis, as in the case when \( i = 1 \), we obtain a function \( F^{(2)}(x) \) satisfying (37) and (38) when \( i = 2 \).

Case \( i = 3 \). Observe that \( U_n^{(1,3)} \) is also a \((d - 1)\)-dimensional operator of type \( U \) acting on the function (36). As in the previous cases, we can then easily obtain (37) and (38) when \( i = 3 \).

Thus we have established the desired estimate for \( U_n \).

Since \( S_n^{*} \) is an operator of type \( U \), we can find a function \( F_1(x) \) such that

\[
\{|x \in \mathbb{T}^d: F_1(x) > \lambda\}| \lesssim \frac{\|f\|_{\text{Log}_{d-1}(\mathbb{T}^d)}}{\lambda} \quad (44)
\]

\[
\int_{\mathbb{T}^d} \exp\left(\frac{|S_n^{*} f(x)|}{F_1(x)}\right) dx \lesssim 1, \quad n = 1, 2, \ldots .
\] (45)

As to \( \tilde{S}_n^{*} \), we have the bound

\[
|\tilde{S}_n^{*} f(x)| \leq |U_n f(x)| + G(x),
\]

where

\[
U_n f(x) = \frac{1}{\pi^d} \int_{\mathbb{T}^d} \prod_{k=1}^{d} \frac{\cos nt_k}{2 \tan(t_k/2)} f(x + t) dt,
\]

\[
G(x) = \frac{1}{\pi^d} \left| \text{p.v.} \int_{\mathbb{T}^d} \prod_{k=1}^{d} \frac{f(x + t)}{2 \tan(t_k/2)} dt \right|.
\]

It is well known that \( G(x) \) satisfies

\[
|\{G(x) > \lambda\}| \lesssim \frac{\|f\|_{\text{Log}_{d-1}(\mathbb{T}^d)}}{\lambda}.
\] (46)

Since \( U_n \) is an operator of type \( U \), there is a function \( F_2(x) \) such that

\[
\{|x \in \mathbb{T}^d: F_2(x) > \lambda\}| \lesssim \frac{\|f\|_{\text{Log}_{d-1}(\mathbb{T}^d)}}{\lambda},
\] (47)

\[
\int_{\mathbb{T}^d} \exp\left(\frac{|U_n f(x)|}{F_2(x)}\right) dx \lesssim 1, \quad n = 1, 2, \ldots .
\] (48)

Finally, by Lemma 2 we have

\[
|S_n f(x)| + |\tilde{S}_n f(x)| \leq |S_n^{*} f(x)| + |\tilde{S}_n^{*} f(x)| + F_3(x)
\]

\[
\leq |S_n^{*} f(x)| + |U_n f(x)| + G(x) + F_3(x),
\]

where the function \( F_3(x) \geq 0 \) satisfies

\[
\|F_3\|_{L^1(\mathbb{T}^d)} \lesssim \|f\|_{\text{Log}_{d-1}(L)(\mathbb{T}^d)}.
\] (49)
We claim that all the conclusions of Theorem 1 hold for $F = 4(F_1 + F_2 + F_3 + G)$. Indeed, (9) follows immediately from (44), (46), (47) and (49) (using Chebyshev’s inequality for $F_3$). To prove (10), observe that

$$\exp\left(\left|S_n f(x)\right| + \left|\tilde{S}_n f(x)\right|\right) \leq \exp\left(\left|S^*_n f(x)\right| + \left|U_n f(x)\right| + G(x) + F_3(x)\right) F(x)$$

$$\leq \exp\left(\frac{4\left|S^*_n f(x)\right|}{F(x)}\right) + \exp\left(\frac{4\left|U_n f(x)\right|}{F(x)}\right) + \exp\left(\frac{4G(x)}{F(x)}\right) + \exp\left(\frac{4F_3(x)}{F(x)}\right)

\leq \exp\left(\frac{\left|S^*_n f(x)\right|}{F_2(x)}\right) + \exp\left(\frac{\left|U_n f(x)\right|}{F_2(x)}\right) + \exp\left(\frac{\left|G(x)\right|}{F_2(x)}\right) + \exp\left(\frac{\left|F_3(x)\right|}{F_2(x)}\right) + 2\varepsilon.$$

Combining this with (45) and (48), we complete the proof of the theorem. □

Proof of Corollary 1. Suppose that $f \in \text{Log}_{d-1}(L)(\mathbb{T}^d)$ and let $F(x)$ be the function provided by Theorem 1. We define

$$E_{f,\varepsilon} = \left\{ x \in \mathbb{T}^d : F(x) \leq \frac{\|f\|_{\text{Log}_{d-1}(\mathbb{T}^d)}}{\gamma \varepsilon} \right\},$$

where $\gamma$ is a constant. By (9), there is a constant $\gamma$ (depending only on $d$) such that $|\overline{E_{f,\varepsilon}}| \leq \varepsilon$. This yields (11). Moreover, it follows from (10) that

$$\int_{E_{f,\varepsilon}} \exp\left(\frac{\left|S_n f(x)\right| + \left|\tilde{S}_n f(x)\right|}{\|f\|_{\text{Log}_{d-1}(\mathbb{T}^d)}}\right) dx \leq \int_{\mathbb{T}^d} \exp\left(\frac{\left|S_n f(x)\right| + \left|\tilde{S}_n f(x)\right|}{F(x)}\right) dx \lesssim 1,$$

so that (12) holds. □

Proof of Corollary 2. Suppose that $f \in \text{Log}_{d-1}(L)(\mathbb{T}^d)$. It is well known that the $(C,1)$-means $\sigma_n f$ of the Fourier series (1) of $f$ and its conjugate (2) converge almost everywhere to $f$ and $\tilde{f}$ respectively. There is also the convergence in norm

$$\lim_{\min(n) \to \infty} \|\sigma_n f - f\|_{\text{Log}_{d-1}(\mathbb{T}^d)} = 0.$$

Using this, one can find a set $G \subset \mathbb{T}^d$ and a $d$-dimensional trigonometric polynomial $P_k$ such that

$$|G| > (2\pi)^d - \frac{\varepsilon}{2}, \quad (50)$$

$$\|f - P_k\|_{L^\infty(G)} < \frac{1}{2^k}, \quad (51)$$

$$\|\tilde{f} - \tilde{P}_k\|_{L^\infty(G)} < \frac{1}{2^k}, \quad (52)$$

$$\|f - P_k\|_{\text{Log}_{d-1}(\mathbb{T}^d)} < \frac{\gamma \varepsilon k}{2^k}. \quad (53)$$

Applying Corollary 1 with $\varepsilon_k = \varepsilon/2^{k+1}$, we find sets $E_k \subset \mathbb{T}^d$ with

$$|E_k| > (2\pi)^d - \varepsilon_k, \quad k = 1, 2, \ldots, \quad (54)$$

$$\int_{E_k} \exp\left(\frac{\gamma \varepsilon_k \left|S_n(f - P_k)\right| + \left|\tilde{S}_n(f - P_k)\right|}{\|f - P_k\|_{\text{Log}_{d-1}(\mathbb{T}^d)}}\right) \leq c, \quad n = 1, 2, \ldots, \quad (55)$$

$$\int_{E_k} \exp\left(\frac{\gamma \varepsilon_k \left|S_n(f - P_k)\right| + \left|\tilde{S}_n(f - P_k)\right|}{\|f - P_k\|_{\text{Log}_{d-1}(\mathbb{T}^d)}}\right) \leq c, \quad n = 1, 2, \ldots, \quad (55)$$
Define
\[ E_{f,\varepsilon} = G \cap \left( \bigcap_k E_k \right). \]

Then (11) follows from (50) and (54). Put \( \phi(t) = \exp t - 1 \). We easily see that \( \phi(ab) \leq a\phi(b) \) for \( 0 < a < 1 \) and \( b > 0 \). Thus, using (51), (53) and (55), we get
\[
\lim_{n \to \infty} \int_{E_{f,\varepsilon}} \left( \exp(A|S_nf - f|) - 1 \right)
\leq A_k \sup_n \int_{E_{f,\varepsilon}} \left( \exp \left( k(|S_nf - P_k| + |f - P_k|) \right) \right)
\leq A_k \left( \sup_n \int_{E_{f,\varepsilon}} \exp \left( \frac{\gamma \varepsilon_k |S_nf - P_k|}{\|f - P_k\|_{\text{Log}_{d-1}(T^d)}} \right) + \int_{E_{f,\varepsilon}} \exp(2k|f - P_k|) \right) \approx \frac{A_k}{k}.
\]

Since the last quantity can be arbitrarily small, we obtain (13). In a similar way, we arrive at the inequalities
\[
\lim_{n \to \infty} \int_{E_{f,\varepsilon}} \left( \exp(A|\tilde{S}_nf - \tilde{f}|) - 1 \right)
\leq A_k \left( \sup_n \int_{E_{f,\varepsilon}} \exp \left( \frac{\gamma \varepsilon_k |\tilde{S}_nf - P_k|}{\|f - P_k\|_{\text{Log}_{d-1}(T^d)}} \right) + \int_{E_{f,\varepsilon}} \exp(2k|\tilde{f} - \tilde{P}_k|) \right) \approx \frac{A_k}{k}
\]
and, therefore, at (14). □

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