SHEAVES ON $\mathbb{P}^1 \times \mathbb{P}^1$, BIGRADED RESOLUTIONS, AND COADJOINT ORBITS OF LOOP GROUPS

ROGER BIELAWSKI & LORENZ SCHWACHHÖFER

Abstract. We construct a canonical linear resolution of acyclic 1-dimensional sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ and discuss the resulting natural Poisson structure.

1. Introduction

The goal of this paper is to present a (yet another) variation on a theme developed by several authors, notably Moser, Adams, Harnad, Hurtubise, Previato [13], [1]–[5], and relating integrable systems, rank $r$ perturbations, spectral curves and their Jacobians, and coadjoint orbits of loop groups.

Let us briefly recall that, given matrices $A, Y, F, G$ of size, respectively, $N \times N$, $r \times r$, $N \times r$, and $r \times N$, one defines a $\mathfrak{gl}_r(\mathbb{C})$-valued rational map

$$Y + G(A - \lambda)^{-1}F,$$

i.e. an element of the loop algebra $\tilde{\mathfrak{gl}}(r)^{-}$, consisting of loops extending holomorphically to the outside of some circle $S^1 \subset \mathbb{C}$. This determines a (shifted) reduced coadjoint orbit in $\tilde{\mathfrak{gl}}(r)^{-}$ (see Remark 4.5 for a definition). On the other hand, the polynomial (1.1) also determines (generically) a curve $S$ and a line bundle $L$ of degree $\deg + r - 1$: the curve is defined as the spectrum of (1.1), and $L$ is the dual of the eigenbundle of (1.1). This describes $S$ as an affine curve in $\mathbb{C}^2$, and the isospectral flows, corresponding to Hamiltonians on the space of rank $r$ perturbations, linearise on the Jacobian of the projective model of $S$.

In fact, as shown by Adams, Harnad, and Hurtubise [1, 2], it is more convenient to compactify $S$ inside a Hirzebruch surface $F_d$, $d \geq 1$. This results in singularities, which may be partially resolved, but it gives a particularly nice description of $\text{Jac}^0(S)$, i.e. of the flow directions.

In this paper, we consider a different compactification of $S$, namely inside $\mathbb{P}^1 \times \mathbb{P}^1$ and defined as

$$S = \left\{ (z, \lambda) \in \mathbb{P}^1 \times \mathbb{P}^1; \det \begin{pmatrix} Y - z & G \\ F & A - \lambda \end{pmatrix} = 0 \right\}.$$

This is a very natural thing to do, but we know of only one occurrence in the literature: the paper of Sanguinetti and Woodhouse [17] (we are grateful to Philip Boalch for this reference). In that paper, in addition to other results, the authors use the above compactification to give a nice picture of the duality phenomenon discussed in [3]. Our application is to another subtlety of the rank $r$ perturbation isospectral flow: the fact that the flow may leave the set where $\text{rank } F = \text{rank } G = r$, without becoming singular. More precisely, we have:

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Theorem 1.1. Let $S$ be a smooth curve in $\mathbb{P}^1 \times \mathbb{P}^1$, defined by \([12]\) and corresponding to a (shifted) rank $r$ perturbation of the matrix $A$ ($r \leq N$). A line bundle $L \in \text{Jac}^{0-r+1}(S)$ corresponds to $(A,Y,F,G)$ with $\text{rank} F = \text{rank} G = r$ if and only if $L$ satisfies:

$$H^0(S, L(0,-1)) = H^1(S, L(0,-1)) = 0, \quad H^0(S, L(-1,0)) = 0, \quad H^1(S, L(1,-2)) = 0.$$ 

We are interested in more than line bundles on smooth curves in $\mathbb{P}^1 \times \mathbb{P}^1$. The above approach generalises to acyclic (i.e. semistable) 1-dimensional sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$, with a fixed bigraded Hilbert polynomial. In Sections 2 and 3, we construct a natural linear resolution of such a sheaf, very much in the spirit of Beauville \([6]\). This gives us a linear polynomial matrix $M(z,\lambda)$ (up to certain group action). If the support of the sheaf is a smooth curve of bidegree $(r,N)$, then the matrix has size $r \times N$. As long as the point $(\infty, \infty)$ does not belong to the support of the sheaf, then matrices $M(z,\lambda)$ can be identified with the quadruples $(A,Y,F,G)$. The space $\mathcal{M}(k,l)$ of the $(A,Y,F,G)$ has a natural Poisson structure, obtained by identifying it with $\mathfrak{gl}_N(\mathbb{C})^* \oplus \mathfrak{gl}_r(\mathbb{C})^* \oplus T^* M_{N \times r}(\mathbb{C})$. Thus we obtain a Poisson structure on the quotient of an open subset of $\mathcal{M}(N,r)$ by $GL_N(\mathbb{C}) \times GL_r(\mathbb{C})$. The (generic) symplectic leaves are known, from \([3,1]\), to be reduced coadjoint orbits of loop groups. Our aim is to describe these symplectic leaves directly in terms of sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$. We show that they correspond to symplectic leaves of a particular Mukai-Tyurin-Bottacin Poisson structure \([14,13,8,9,10,11]\) on the moduli space $M_Q(r,N)$ of simple sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ with (bigraded) Hilbert polynomial $N x + ry$. The surface $Q = \mathbb{P}^1 \times \mathbb{P}^1$ is an example of a Poisson surface \([8]\), and consequently, for every choice of a Poisson structure on $Q$, i.e. a section $s$ of the anticanonical bundle $K_Q^* \simeq \mathcal{O}(2,2)$, one obtains a Poisson structure on $M_Q(r,N)$ as a map

$$T_M Q(r,N) \simeq \text{Ext}_Q^1(\mathcal{F}, \mathcal{F} \otimes K_Q) \rightarrow \text{Ext}_Q^1(\mathcal{F}, \mathcal{F}) \simeq T_M Q(r,N).$$

We show that the (generic) symplectic leaves $\mathfrak{gl}_N(\mathbb{C})^* \oplus \mathfrak{gl}_r(\mathbb{C})^* \oplus T^* M_{N \times r}(\mathbb{C})$, i.e. reduced coadjoint orbits in $\mathfrak{gl}(r)^{-}$, are the symplectic leaves of the Mukai-Tyurin-Bottacin structure corresponding to $s(z,\lambda) = 1$, i.e. to the anticanonical divisor $2(\infty) \times \mathbb{P}^1 + \mathbb{P}^1 \times \{\infty\}$.

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2. ACYCLIC SHEAVES ON $\mathbb{P}^1 \times \mathbb{P}^1$ AND THEIR RESOLUTIONS

Definition 2.1. Let $X$ be a complex manifold and let $\mathcal{F}$ be a coherent sheaf on $X$.

(i) The support of $\mathcal{F}$ is the complex subspace $\text{supp} \mathcal{F}$ of $X$ defined as the zero-locus of the annihilator (in $\mathcal{O}_X$) of $\mathcal{F}$. The dimension $\dim \mathcal{F}$ of $\mathcal{F}$ is the dimension of its support.

(ii) $\mathcal{F}$ is pure, if $\dim \mathcal{E} = \dim \mathcal{F}$ for all non-trivial coherent subsheaves $\mathcal{E} \subset \mathcal{F}$.

(iii) $\mathcal{F}$ is acyclic if $H^*(\mathcal{F}) = 0$.

Remark 2.2. In the case of 1-dimensional sheaves on a smooth surface $X$, purity of $\mathcal{F}$ means that, at at every point $x \in \text{supp} \mathcal{F}$, the skyscraper sheaf $\mathbb{C}_x$ does not embed into $\mathcal{F}_x$. In addition, a 1-dimensional sheaf $\mathcal{F}$ on a smooth surface $X$ is pure if and only if it is reflexive, i.e. after performing the duality $\mathcal{F} \mapsto \mathcal{E}xt^1_X(\mathcal{F}, K_X)$ twice, we obtain back $\mathcal{F}$ (up to isomorphism) (see \([9\ §1.1]\)).
In the remainder of the paper, all sheaves are coherent.

We shall now consider sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$. For any $p, q \in \mathbb{Z}$ we denote by $\mathcal{O}(p, q)$ the line bundle $\mathcal{O}(p) \otimes \mathcal{O}(q)$, where $\pi_1, \pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ are the two projections. We shall also denote by $\zeta$ and $\eta$ the two affine coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$.

Let $\mathcal{F}$ be a sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$. Associated to $\mathcal{F}$ is its bigraded Hilbert polynomial

$$P_\mathcal{F}(x, y) = \sum_{x, y \in \mathbb{Z}} \chi(\mathcal{F}(x, y)).$$

The sheaf $\mathcal{F}$ is 1-dimensional if and only if $P_\mathcal{F}$ is linear.

We begin by describing a canonical resolution of acyclic 1-dimensional sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$.

**Theorem 2.3.** Let $\mathcal{F}$ be a 1-dimensional acyclic sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$. Then $\mathcal{F}$ has a linear resolution by locally free sheaves of the form

$$0 \to \mathcal{O}(-2, -1)^{\oplus k} \oplus \mathcal{O}(-1, -2)^{\oplus l} \xrightarrow{M(\zeta, \eta)} \mathcal{O}(-1, -1)^{\oplus (k+l)} \to \mathcal{F} \to 0,$$

for some $k, l \geq 0$.

Conversely, any $\mathcal{F}$ defined as cokernel of a map $M(\zeta, \eta)$ as above with det $M(\zeta, \eta) \neq 0$ is acyclic and 1-dimensional.

**Remark 2.4.** Let $\mathcal{F}$ be a 1-dimensional acyclic sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$ with $P_\mathcal{F}(x, y) = lx + ky$. Then $\mathcal{F}$ is semistable with respect to $\mathcal{O}(1, 1)$.

**Remark 2.5.** This resolution is canonical, but not necessarily minimal, in the sense of being obtained from the minimal resolution of the bigraded module $\bigoplus_{i,j \in \mathbb{Z}} H^0(\mathcal{F}(i, j))$.

**Proof.** Let $h^0(\mathcal{F}(0, 1)) = k$ and $h^0(\mathcal{F}(1, 0)) = l$, so that $P_\mathcal{F} = lx + ky$. Let $\mathcal{E} = \mathcal{F}(1, 1)$, and let $\Gamma_*(\mathcal{E}) = \bigoplus_{i,j \in \mathbb{Z}} H^0(\mathcal{E}(i, j))$ be the associated bigraded module over the bigraded ring $S = \bigoplus_{i,j \in \mathbb{Z}} H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(i, j))$. Furthermore, let $\Gamma_*(\mathcal{E})|_{\geq 0} = \bigoplus_{i,j \geq 0} H^0(\mathcal{E}(i, j))$ be its truncation. Owing to [12 Lemma 6.8], the sheaf associated to $\Gamma_*(\mathcal{E})|_{\geq 0}$ is again $\mathcal{E}$. Moreover, [12 Theorem 6.9] implies, as $\mathcal{E}(-1, -1)$ is acyclic, that the natural map

$$H^0(\mathcal{E}) \otimes H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(p, q)) \to H^0(\mathcal{E}(p, q))$$

is surjective for any $p, q \geq 0$. Therefore, we have a surjective homomorphism

$$S^{\oplus (k+l)} \to \Gamma_*(\mathcal{E})|_{\geq 0} \to 0$$

of bigraded $S$-modules. Since $\mathcal{E}$ is of pure dimension 1, its projective dimension is 1, and, hence, the above homomorphism extends to a linear free resolution

$$0 \to \bigoplus_{i=1}^{k+l} S(-p_i, -q_i) \to \bigoplus_{i=1}^{k+l} S \to \Gamma_*(\mathcal{E})|_{\geq 0} \to 0,$$

where $p_i, q_i \geq 0$ and $p_i + q_i > 0$ for each $i$. The corresponding sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ give us a locally free resolution of $\mathcal{E}$:

$$0 \to \bigoplus_{i=1}^{k+l} \mathcal{O}(-p_i, -q_i) \to \bigoplus_{i=1}^{k+l} \mathcal{O} \to \mathcal{E} \to 0.$$

Since $H^*\mathcal{E}(-1, -1) = 0$, either $p_i = 0$ or $q_i = 0$ for every $i$. Since $h^0(\mathcal{E}(-1, 0)) = k$, we deduce, after tensoring (2.3) with $\mathcal{O}(-1, 0)$, that $\sum p_i = k$. Similarly $\sum q_i = k$. 


There exists a canonical biholomorphism free sheaves) on $S$.

Corollary 2.8.

Proposition 2.7.

Similarly, let $\text{det}_S(r, d)$ be the moduli space of semistable vector bundles (locally free sheaves) on $S$. For $d = r(q-1)$ define the generalised theta divisor $\Theta$ as the set of bundles with nonzero section. Then we have:

Corollary 2.9.

Let $P(\zeta, \eta)$ be an irreducible polynomial of bidegree $(k, l)$, and $S = \{ (\zeta, \eta); P(\zeta, \eta) = 0 \}$ the corresponding integral curve of genus $g = (k-1)(l-1)$. There exists a canonical biholomorphism

$$\text{Jac}^{-1}(S) - \Theta \simeq \{ M \in \mathcal{A}(k, l); \text{det } M = P \}/GL_n(\mathbb{C}) \times GL_k(\mathbb{C}) \times GL_l(\mathbb{C}).$$

Let us write $n = k + l$. The polynomial matrix $M(\zeta, \eta)$ in (2.3) has size $n \times n$ and is of the form

$$M(\zeta, \eta) = \begin{pmatrix} A_0 + A_1 \zeta & B_0 + B_1 \eta \end{pmatrix},$$

with $A_0, A_1 \in \text{Mat}_{n, k}(\mathbb{C})$, $B_0, B_1 \in \text{Mat}_{n, l}(\mathbb{C})$. Let us denote by $\mathcal{A}(k, l)$ the space of such matrices with nonzero determinant. The group $GL_n(\mathbb{C}) \times GL_k(\mathbb{C}) \times GL_l(\mathbb{C})$ acts on $\mathcal{M}(k, l)$ via:

$$F \mapsto (g, h_1, h_2). (A(\zeta) \ B(\eta)) = g \{ A(\zeta) \ B(\eta) \left( \begin{array}{cc} h_1^{-1} & 0 \\ 0 & h_2^{-1} \end{array} \right) \},$$

and we can restate Theorem 2.3 as follows:

**Corollary 2.6.** There exists a natural bijection between

(a) isomorphism classes of 1-dimensional acyclic sheaves $\mathcal{F}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ such that $h^0(\mathcal{F}(0, 1)) = k$, $h^0(\mathcal{F}(1, 0)) = l$,

and

(b) orbits of $GL_{k+1}(\mathbb{C}) \times GL_k(\mathbb{C}) \times GL_l(\mathbb{C})$ on $\mathcal{A}(k, l)$.

For a sheaf define by (2.2), we can describe its support as follows. As a set, the support of $\mathcal{F}$ is

$$S = \{ (\zeta, \eta) \in \mathbb{P}^1 \times \mathbb{P}^1; \text{det } M(\zeta, \eta) = 0 \}. $$

Let us write $\text{det}_S M(\zeta, \eta) = \prod_{i=1}^r q_i(\zeta, \eta)^{r_i}$, where $q_i$ are irreducible polynomials. We define the minimal polynomial $p_M(\zeta, \eta)$ of $M$ as $\prod_{i=1}^r q_i(\zeta, \eta)^{r_i}$, where

$$r_i = \max \{ a_i b_i; \text{at a generic point, } M(\zeta, \eta) \text{ has a Jordan block of size } a_i, \text{ with eigenvalue } q_i(\zeta, \eta)^{b_i} \}. $$

Then:

**Proposition 2.7.** The support of $\mathcal{F}$ is the curve $(S, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}/(p_M))$. 

Let us now fix the support $S$. For simplicity, we shall assume that it is an integral curve in the linear system $|O(k, l)|$ on $\mathbb{P}^1 \times \mathbb{P}^1$, i.e. $S$ is given by an irreducible polynomial $P(\zeta, \eta)$ of bidegree $(k, l)$, $k, l \geq 1$. This immediately implies that the rank of $\mathcal{F}$ is constant, i.e. $\mathcal{F}$ is locally free. Theorem 2.3 and Corollary 2.3 imply

**Corollary 2.8.** Let $P(\zeta, \eta)$ be an irreducible polynomial of bidegree $(k, l)$, and $S = \{ (\zeta, \eta); P(\zeta, \eta) = 0 \}$ the corresponding integral curve of genus $g = (k-1)(l-1)$. There exists a canonical biholomorphism

$$\text{Jac}^{-1}(S) - \Theta \simeq \{ M \in \mathcal{A}(k, l); \text{det } M = P \}/GL_n(\mathbb{C}) \times GL_k(\mathbb{C}) \times GL_l(\mathbb{C}).$$

Similarly, let $U_S(r, d)$ be the moduli space of semistable vector bundles (locally free sheaves) on $S$. For $d = r(q-1)$ define the generalised theta divisor $\Theta$ as the set of bundles with nonzero section. Then we have:

**Corollary 2.9.** Let $P(\zeta, \eta)$ be an irreducible polynomial of bidegree $(k, l)$, and $S = \{ (\zeta, \eta); P(\zeta, \eta) = 0 \}$ the corresponding integral curve of genus $g = (k-1)(l-1)$. There exists a canonical biholomorphism

$$U_S(r, r(g-1)) - \Theta \simeq \{ M \in \mathcal{A}(kr, lr); \text{det } M = P' \}/GL_{nr}(\mathbb{C}) \times GL_{kr}(\mathbb{C}) \times GL_{lr}(\mathbb{C}).$$
3. A geometric resolution

There is a much more geometric way of constructing resolution (2.2), which works under mild assumptions on the sheaf $F$ (cf. [7] for the case of $\sigma$-sheaves).

**Definition 3.1.** Let $F$ be a 1-dimensional sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$ and let $\pi_1, \pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the two projections. We say that $F$ is bipure, if $F$ has no nontrivial coherent subsheaves supported on $\{z\} \times \mathbb{P}^1$ or on $\mathbb{P}^1 \times \{z\}$ for any $z \in \mathbb{P}^1$.

Let now $F$ be an acyclic and bipure sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$ with Hilbert polynomial $lx + ky$. As in the proof of Theorem 2.3, we consider the sheaf $E = F(1, 1)$. Let $D_\zeta$ and $D_\eta$ denote the divisors $\{\zeta\} \times \mathbb{P}^1$, $\mathbb{P}^1 \times \{\eta\}$. We set

(3.1) $V_\zeta = \{s \in H^0(E); s|_{D_\zeta} = 0\}$,  $W_\eta = \{s \in H^0(E); s|_{D_\eta} = 0\}$.

For any $\zeta$ and $\eta$, consider the maps

$$E(-1, 0) \to E, \quad E(0, -1) \to E,$$

given by multiplication by global non-zero sections of $O(1, 0)$ and $O(0, 1)$, vanishing at $\zeta$ and $\eta$, respectively. Since $E$ is bipure, these maps are injective, and therefore $V_\zeta \simeq H^0(E(-1, 0))$, $W_\eta \simeq H^0(E(0, -1))$ for any $\zeta, \eta$. In particular, $\dim V_\zeta = k$, $\dim W_\eta = l$, for any $\zeta$ and $\eta$. Therefore, $\zeta \mapsto V_\zeta$ and $\eta \mapsto W_\eta$ are subbundles of $H^0(E) \otimes O$ on $\mathbb{P}^1$. They are isomorphic to $H^0(E(-1, 0)) \otimes O(-1)$, and to $H^0(E(0, -1)) \otimes O(-1)$. The isomorphism is realised explicitly via the map:

$$H^0(E(-1, 0)) \otimes O(-1) \ni (s, (a, b)) \mapsto (b\zeta - a)s \in H^0(E)$$

(here $(a, b) \in \mathbb{I}$, where $\mathbb{I}$ is the fibre of $O(-1)$ over $[t]$), and similarly for the subbundle $W$. We now define a vector bundle $U$ on $\mathbb{P}^1 \times \mathbb{P}^1$, the fibre of which at $\zeta, \eta$ is $V_\zeta \oplus W_\eta$, i.e.:

$$U \simeq (H^0(E(-1, 0)) \otimes O(-1, 0)) \oplus (H^0(E(0, -1)) \otimes O(0, -1)).$$

We obtain an injective map of sheaves $U \to H^0(E) \otimes O$. Let $G$ be the cokernel, i.e.:

(3.2) $0 \to U \longrightarrow H^0(E) \otimes O \longrightarrow G \longrightarrow 0$.

We claim that $G \simeq E$, and so (3.2) is a natural resolution of $E$. To prove this, tensor the resolution (2.2) by $O(1, 1)$ to obtain:

(3.3) $0 \to O(-1, 0)^{\oplus k} \oplus O(0, -1)^{\oplus l} \overset{M(\zeta, \eta)}{\longrightarrow} O^{\oplus (k + l)} \to E \to 0$.

Clearly, the middle term is identified with $H^0(E) \otimes O$. For any $\zeta_0$, consider the image of $M(\zeta_0, \eta)$ restricted to $O(-1, 0)^{\oplus k}|_{\zeta_0} \oplus 0$. This image does not depend on $\eta$, and since $F$ is bipure, it is exactly $V_{\zeta_0}$, defined in (3.1), i.e. sections vanishing on $\zeta_0 \times \mathbb{P}^1$. Similarly, for any $\eta_0$, the image of $M(\zeta, \eta_0)$ restricted to $0 \oplus O(0, -1)^{\oplus l}|_{\eta_0}$ is precisely $W_{\eta_0}$. Hence, there are canonical isomorphisms between both first and second terms in resolutions (3.2) and (3.3), which commute with the horizontal maps. Therefore $G \simeq E$.

4. Poisson structure and orbits of loop groups

According to Corollary 2.6, acyclic sheaves with Hilbert polynomial $lx + ky$ correspond to orbits of $GL_{k+l}(\mathbb{C}) \times GL_k(\mathbb{C}) \times GL_l(\mathbb{C})$ on $A(k, l)$, where $A(k, l)$ is the set of polynomial matrices defined in (2.4) and the action is given in (2.5).
We now make the following assumption about the sheaf $\mathcal{F}$:

$$(\infty, \infty) \notin \text{supp} \mathcal{F}.$$  

This can be, of course, always achieved via an automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$. In terms of the matrix $M(\zeta, \eta)$ corresponding to $\mathcal{F}$, (4.1) means that $\det(A_1, B_1) \neq 0$. We can, therefore, use the action of $GL_{k+l}(\mathbb{C})$ to make $(A_1, B_1)$ equal to minus the identity matrix, so that $M(\zeta, \eta)$ becomes

$$X \in \text{Mat}_{k,k}(\mathbb{C}), \ Y \in \text{Mat}_{l,l}(\mathbb{C}), \ G, F^T \in \text{Mat}_{l,k}(\mathbb{C}).$$

The residual group action is that of conjugation by the block-diagonal $GL_k(\mathbb{C}) \times GL_l(\mathbb{C})$. We denote this group by $K$.

**Remark 4.1.** We are, essentially, in the situation of [5]. The only difference is that we do not fix $X$ or $Y$.

We denote by $\mathcal{M}(k,l)$ the space of all matrices of the form (4.2), which we identify with quadruples $(X, Y, F, G)$ as above. The action of $K = GL_k(\mathbb{C}) \times GL_l(\mathbb{C})$ on $\mathcal{M}(k,l)$ is given by

$$(g,h).(X, Y, F, G) = (gXg^{-1}, hYh^{-1}, gFh^{-1}, hGg^{-1}).$$

Let us also write $\mathcal{S}(k,l)$ for the set of isomorphism classes of acyclic sheaves with Hilbert polynomial $lx + ky$ on $\mathbb{P}^1 \times \mathbb{P}^1$, which satisfy (4.1). The content of Corollary 2.6 is that there exists a natural bijection

$$(4.4) \quad \mathcal{M}(k,l)/K \simeq \mathcal{S}(k,l).$$

**4.1. Poisson structure.** The vector space $\text{Mat}_{k,l} \times \text{Mat}_{l,k}$ has a natural $K$-invariant symplectic structure: $\omega = \text{tr}(dF \wedge dG)$. On the other hand, $\text{Mat}_{k,k} \simeq \mathfrak{gl}_k(\mathbb{C})^*$ and $\text{Mat}_{l,l} \simeq \mathfrak{gl}_l(\mathbb{C})^*$ have canonical Poisson structures, and therefore, $\mathcal{M}(k,l)$ has a natural $K$-invariant Poisson structure. If $\mathcal{M}(k,l)^0$ is the subset of $\mathcal{M}(k,l)$, on which the action of $K$ is free and proper, then $\mathcal{M}(k,l)^0/K$ is a Poisson manifold, and, consequently, we obtain a Poisson structure on the corresponding subset of acyclic sheaves with Hilbert polynomial $lx + ky$ and satisfying (4.1). We shall now want to describe symplectic leaves of $\mathcal{M}(k,l)^0/K$ in terms of sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$.

First of all, let us describe sheaves corresponding to symplectic leaves in $\mathcal{M}(k,l)$. Such a leaf is determined by fixing conjugacy classes of $X$ and $Y$. On the other hand, conjugacy classes of $k \times k$ matrices correspond to isomorphism classes of torsion sheaves on $\mathbb{P}^1$, of length $k$. This correspondence is given by associating to a matrix $X \in \text{Mat}_{k,k}(\mathbb{C})$ the sheaf $\mathcal{G}$ via

$$0 \to \mathcal{O}(-1)^{\oplus k} \xrightarrow{X-\zeta} \mathcal{O}^{\oplus k} \to \mathcal{G} \to 0.$$  

If, for example, $X$ is diagonalisable with distinct eigenvalues $\zeta_1, \ldots, \zeta_r$ of multiplicities $k_1, \ldots, k_r$, then $\mathcal{G} \simeq \bigoplus_{i=1}^r \mathcal{O}^{k_i}|_{\zeta_i}$, i.e. $\mathcal{G}|_{\zeta_i}$ is the skyscraper sheaf of rank $k_i$.

**Proposition 4.2.** Let $P$ be a conjugacy class of $k \times k$ matrices. The bijection (4.4) induces a bijection between

(i) orbits of $GL_k(\mathbb{C}) \times GL_l(\mathbb{C})$ on \{(X, Y, F, G) \in \mathcal{M}(k,l); X \in P\}, and
(ii) isomorphism classes of sheaves $\mathcal{F}$ in $\mathcal{S}(k,l)$ such that $\mathcal{F}|_{\eta=\infty}$ is isomorphic to $\mathcal{G}$ defined by (4.5).


Proof. At \( \eta = \infty \), the matrix (4.2) becomes
\[
\begin{pmatrix}
X - \zeta & 0 \\
G & -1
\end{pmatrix}.
\]
The statement follows from (4.5) and (2.2).

Therefore symplectic leaves on \( \mathcal{M}(k, l) \) correspond to fixing isomorphism classes of \( F|_{\eta=\infty} \) and of \( F|_{\xi=\infty} \). Symplectic leaves on \( \mathcal{M}(k, l)^0/K \) are of course smaller than \( K \)-orbits of symplectic leaves on \( \mathcal{M}(k, l)^0 \). They are obtained by fixing \( X \) and \( Y \) and taking the symplectic quotient of \( \text{Mat}_{k,l} \times \text{Mat}_{l,k} \) by \( \text{Stab}(X) \times \text{Stab}(Y) \). We shall describe sheaves corresponding to a particular symplectic leaf in the case when \( X \) and \( Y \) are diagonalisable.

4.2. Orbits of \( GL_k(\mathbb{C}) \) and matrix-valued rational maps. We consider now only the action of \( GL_k(\mathbb{C}) \simeq GL_k(\mathbb{C}) \times \{1\} \subset K \) on \( \mathcal{M}(k, l) \). We fix a semisimple conjugacy class of \( X \), i.e. we suppose that \( X \) is diagonalisable, with distinct eigenvalues \( \zeta_1, \ldots, \zeta_r \) of multiplicities \( k_1, \ldots, k_r \). The stabiliser of \( X \) is then isomorphic to \( \prod_{i=1}^r GL_{k_i}(\mathbb{C}) \). If the action of \( GL_k(\mathbb{C}) \) is to be free, we must have \( k_i \leq l \), \( i = 1, \ldots, r \). Let us diagonalise \( X \), so that \( X \) has the block-diagonal form \((\zeta_1 \cdot 1_{k_1 \times k_1}, \ldots, \zeta_r \cdot 1_{k_r \times k_r})\), and let \( F_i, G_i \) denote the \( k_i \times l \) and \( l \times k_i \) submatrices of \( F, G \) such that rows of \( F \) and the columns of \( G \) have the same coordinates as the block \( \zeta_i \cdot 1_{k_i \times k_i} \). The action of \( GL_k(\mathbb{C}) \) is free and proper at \((X, Y, F, G)\) if and only if \( \text{rank} F_i = \text{rank} G_i = k_i \) for \( i = 1, \ldots, r \).

As in [5 1], we can associate to each element of \( \mathcal{M}(k, l) \) a Mat\(_{l,l}(\mathbb{C})\)-valued rational map:

\[
R(\zeta) = Y + G(\zeta - X)^{-1}F.
\]
The mapping \((X, Y, F, G) \mapsto R(\zeta)\) is clearly \( GL_k(\mathbb{C})\)-invariant. If \( X \) is diagonalisable, as above, i.e. \( X = (\zeta_1 \cdot 1_{k_1 \times k_1}, \ldots, \zeta_r \cdot 1_{k_r \times k_r}) \), then

\[
R(\zeta) = Y + \sum_{i=1}^r G_i F_i. \tag{4.7}
\]

We clearly have:

Lemma 4.3. Let \( P \) be a semisimple conjugacy class of \( k \times k \) matrices with eigenvalues \( \zeta_1, \ldots, \zeta_r \) of multiplicities \( k_1, \ldots, k_r \). The map \((X, Y, F, G) \mapsto R(\zeta)\) induces a bijection between

(i) \( GL_k(\mathbb{C})\)-orbits on \( \{(X, Y, F, G) \in \mathcal{M}(k, l)^0; X \in P \} \), and

(ii) the set \( \mathcal{R}_d(P) \) of all rational maps of the form

\[
R(\zeta) = Y + \sum_{i=1}^r \frac{R_i}{\zeta - \zeta_i},
\]

where \( \text{rank} R_i = k_i \).

\[\square\]

4.3. Orbits of loop groups. A rational map of the form (4.6) may be viewed as an element of a loop Lie algebra \( \mathfrak{g}(l)^- \), consisting of maps from a circle \( S^1 \) in \( \mathbb{C} \), containing the points \( \zeta_i \) in its interior, which extend holomorphically outside \( S^1 \) (including \( \infty \)). The group \( GL(l)^+ \), consisting of smooth maps \( g : S^1 \to GL_l(\mathbb{C}) \), extending holomorphically to the interior of \( S^1 \), acts on \( \mathfrak{g}(l)^- \) by pointwise conjugation, followed by projection to \( \widetilde{\mathfrak{g}(l)}^- \). In particular, if all eigenvalues of \( X \) are
distinct, then the action is
\[ g(\zeta). \left( Y + \sum_{i=1}^{r} \frac{R_i}{\zeta - \zeta_i} \right) = Y + \sum_{i=1}^{r} \frac{g(\zeta)R_i g(\zeta)^{-1}}{\zeta - \zeta_i}. \]

Therefore, if we fix conjugacy classes of the \( R_i \), we obtain an orbit of \( GL(l)^+ \) in \( \mathfrak{gl}(l)^- \). We shall now consider quotients of such orbits by \( \text{Stab}(Y) \) and describe which sheaves correspond to elements of such an orbit. Let us give a name to such quotients:

**Definition 4.4.** The quotient of an orbit of \( GL(l)^+ \) in \( \mathfrak{gl}(l)^- \) by \( GL_l(\mathbb{C}) \) is called a semi-reduced orbit.

**Remark 4.5.** In the literature (see, e.g. [1]–[5]) a reduced orbit is the symplectic quotient of an orbit by \( H_Y = \text{Stab}(Y) \). The \( GL_l(\mathbb{C}) \)-moment map on \( \mathfrak{gl}(l)^- \) is identified with \( Y + \sum_{i=1}^{r} R_i \), so that a reduced orbit is obtained by fixing the value of \( a = \pi(\sum_{i=1}^{r} R_i) \), where \( \pi \) is the projection \( \mathfrak{gl}_l(\mathbb{C}) \to \mathfrak{gl}_l(\mathbb{C})/h_{\mathfrak{p}}^l \) (with \( \perp \) is taken with respect to \( tr \), and dividing by \( \text{Stab}(a) \subset \text{Stab}(Y) \)). Therefore, if \( \text{Stab}(Y) \) fixes \( a \), then a reduced orbit can be identified with a subset of a semi-reduced orbit.

Let us, therefore, fix a semi-reduced orbit of \( GL(l)^+ \). We choose \( r \) distinct points \( \zeta_1, \ldots, \zeta_r \) in \( \mathbb{C} \). Furthermore, we choose \( r + 1 \) conjugacy classes \( Q_0, Q_1, \ldots, Q_r \) of \( l \times l \) matrices. This data determines a semi-reduced orbit \( \Upsilon = \Upsilon(Q_0, \ldots, Q_r) \) of \( GL(l)^+ \) defined as

\[ \Upsilon = \left\{ R(\zeta) = Y + \sum_{i=1}^{r} \frac{R_i}{\zeta - \zeta_i}; \ Y \in Q_0, \forall i \geq 1 R_i \in Q_i \right\} / GL_l(\mathbb{C}). \]

Let
\[ k_i = \text{rank} Q_i, \ i = 1, \ldots, r, \ \ k = \sum_{i=1}^{r} k_i. \]

In the notation of Lemma 4.3, \( \Upsilon \subset R_l(P) \), where \( P \) is the semisimple conjugacy class of \( k \times k \) matrices with eigenvalues \( \zeta_i \) of multiplicities \( k_i \).

Thanks to Proposition 4.2 the conjugacy class \( P \) determines \( \mathcal{F}|_{\eta=\infty} \), which, in the case at hand, is \( \bigoplus_{i=1}^{r} \mathbb{C}^{k_i}|_{\zeta_i=\infty} \). Similarly, \( Q_0 \) determines the isomorphism class of \( \mathcal{F}|_{\zeta=\infty} \). We now discuss the significance of the other conjugacy classes \( Q_1, \ldots, Q_r \).

We claim that they determine the isomorphism class of \( \mathcal{F}|_{\eta^2=\infty} \), i.e. of \( \mathcal{F} \) restricted to the first order neighbourhood of \( \eta = \infty \). Indeed, consider again the canonical resolution (2.2) of \( \mathcal{F} \) with \( M(\zeta, \eta) \) given by (4.2). Let \( \tilde{\eta} = 1/\eta \) be a local coordinate near \( \eta = \infty \), so that
\[ M(\zeta, \tilde{\eta}) = \begin{pmatrix} X - \zeta & \tilde{\eta}F \\ G & \tilde{\eta}Y - 1 \end{pmatrix}. \]

Using action (2.2), we can multiply \( M(\zeta, \tilde{\eta}) \) on the right by \( \begin{pmatrix} 1 & 0 \\ 0 & (1 - \tilde{\eta}Y)^{-1} \end{pmatrix} \). On the scheme \( \tilde{\eta}^2 = 0 \), we have \( (1 - \tilde{\eta}Y)^{-1} = 1 + \tilde{\eta}Y \), and so \( M(\zeta, \tilde{\eta}) \) becomes (on \( \tilde{\eta}^2 = 0 \)):
\[ \begin{pmatrix} X - \zeta & \tilde{\eta}F \\ G & -1 \end{pmatrix}. \]
To describe $\mathcal{F}|_{\bar{y}^2=0}$, it is enough to describe it near each $\zeta_i$, i.e. to describe $\mathcal{G}_i = \mathcal{F}|_{U_i \times \{\bar{y}^2=0\}}$, where $U_i$ is an open neighbourhood of $\zeta_i$ (not containing the other $\zeta_j$). The resolution (2.12) of $\mathcal{F}$ restricted to $U_i \times \{\bar{y}^2=0\}$ becomes

$$0 \to \mathcal{O}(-2, -1)^{\oplus k_i} \oplus \mathcal{O}(-1, -2)^{\oplus l} \xrightarrow{M_i(\zeta, \eta)} \mathcal{O}(-1, -1)^{\oplus (k_i + l)} \to \mathcal{G}_i \to 0,$$

where

$$M_i(\zeta, \eta) = \begin{pmatrix} \zeta_i - \zeta & \bar{\eta} F_i \\ G_i & -1 \end{pmatrix}.$$  

This implies that we have an exact sequence

$$0 \to \mathcal{O}(-2, -1)^{\oplus k_i} \to \mathcal{O}(-1, 0)^{\oplus k_i} \to \mathcal{G}_i \to 0,$$

on $U_i \times \{\bar{y}^2=0\}$. Therefore $\mathcal{G}_i$ is determined by the $GL_{k_i}(\mathbb{C})$-conjugacy class of $F_i G_i$, which is the same as the $GL_{l_i}(\mathbb{C})$-conjugacy class of $G_i F_i$. Lemma 4.3 and formula (4.7) imply that the conjugacy class of $G_i F_i$ is $Q_i$. Thus, the conjugacy classes $Q_1, \ldots, Q_r$, which determine the orbit (4.5), correspond to the isomorphism class of $\mathcal{F}|_{\bar{y}^2=0}$. Observe that the support of $\mathcal{G}_i$ is given by $\text{det}((\zeta_i - \zeta) + \bar{\eta} F_i G_i) = 0$. In other words, the eigenvalues of $F_i G_i$ give $\frac{\zeta_i}{\eta}$ at $(\zeta, \eta) = (\zeta_i, 0)$, i.e. the first order neighbourhood of $\text{supp} \mathcal{F}$ at $(\zeta_i, \infty)$.

Summing up, we have:

**Theorem 4.6.** There exists a natural bijection between elements of the semi-reduced rational orbit (4.5) of $\overline{GL(l)}^+$ in $\mathfrak{gl}(l)^-$ and isomorphism classes of 1-dimensional acyclic sheaves $\mathcal{F}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ such that

(i) the Hilbert polynomial of $\mathcal{F}$ is $P_\mathcal{F}(x, y) = lx + ky$.

(ii) $(\infty, \infty) \not\in \text{supp} \mathcal{S}$, and $\mathcal{F}|_{y=\infty} \simeq \bigoplus_{i=1}^r \mathbb{C}^{k_i}|_{(\zeta_i, \infty)}$.

(iii) The isomorphism class of $\mathcal{F}|_{\zeta=\infty}$ corresponds to $Q_0$, as in Proposition 4.3.

(iv) The isomorphism class of $\mathcal{F}|_{y^2=\infty}$ corresponds to conjugacy classes $Q_1, \ldots, Q_r$, as described above.

□

**Remark 4.7.** A variation of this result is probably well known to the integrable systems community (at least when $\mathcal{F}$ is a line bundle supported on a smooth curve $S$). We think it useful, however, to state it in this language and in full generality.

### 4.4. Symplectic leaves of $\mathcal{M}(k, l)^0/K$.

We can finally describe symplectic leaves of $\mathcal{M}(k, l)^0$, i.e. sheaves corresponding to a particular symplectic leaf $L$ in $\mathcal{M}(k, l)/K$, at least in the case when $L \subset \mathcal{M}(k, l)^0/K$, and $X$ and $Y$ are semisimple. As we already mentioned in (4.1) a symplectic leaf in $\mathcal{M}(k, l)^0/K$ is obtained by fixing $X$ and $Y$, as well as a coadjoint orbit $\Lambda \subset \mathfrak{h}^*$ of $H = \text{Stab}(X) \times \text{Stab}(Y)$. If $\mu : \text{Mat}_{k,l} \times \text{Mat}_{l,k} \to \mathfrak{h}^*$ is the moment map for $H$, then the symplectic leaf determined by these data is:

$$L = \{(X, Y, F, G) \in \mathcal{M}(k, l)^0; X \text{ and } Y \text{ are given, } \mu(F, G) \in \Lambda\}/H.$$

Let $X$ be diagonal, written as in (4.2), i.e. $X = (\zeta_1 \cdot 1_{k_1 \times k_2}, \ldots, \zeta_r \cdot 1_{k_r \times k_r})$ and let $F_i, G_i$, $i = 1, \ldots, r$, be the corresponding submatrices of $F$ and $G$. Then $\text{Stab}(X) \simeq \prod_{i=1}^r GL_{k_i}(\mathbb{C})$, and the moment map is the projection of the $GL_{k_i}(\mathbb{C})$-moment map, i.e. $(F, G) \mapsto FG$, onto the Lie algebra of $\text{Stab}(X)$. In other words, the $(\text{Stab}(X))$-moment map can be identified with (5):

$$\mu_X(F, G) = (F_1 G_1, \ldots, F_r G_r).$$
Similarly, if $Y$ is diagonal with $s$ distinct eigenvalues of multiplicities $l_1, \ldots, l_s$, then we obtain $l_i \times k$ and $k \times l_i$ submatrices $G^i, F^i$. The stabiliser of $Y$ is isomorphic to $\prod_{i=1}^{s} GL_{l_i}(\mathbb{C})$ and the moment map is
\begin{equation}
\mu_Y(F, G) = (G^1 F^1, \ldots, G^s F^s).
\end{equation}
Therefore, an orbit $\Lambda$ corresponds to $r + s$ conjugacy classes $\pi_1, \ldots, \pi_r, \rho_1, \ldots, \rho_s$ of $k_i \times k_i$ matrices for the $\pi_i$, and $l_j \times l_j$ matrices for the $\rho_j$. The leaf $L$ will be contained in $\mathcal{M}(k, l)^0/K$ if and only if each conjugacy class consists of matrices of maximal rank ($k_i$ or $l_j$). From the discussion in the previous subsection, we immediately obtain:

**Proposition 4.8.** Let $L$ be a symplectic leaf of the Poisson manifold $\mathcal{M}(k, l)^0/K$, defined as in (4.11) with semisimple $X$ and $Y$. Then the image of $L$ under the bijection (4.4) consists of isomorphism classes of sheaves $F$ in $\mathcal{S}(k, l)$ such that the isomorphism class of $F|_{\zeta^2 = \infty}$ and of $F|_{\eta^2 = \infty}$ is fixed (and determined by $L$).

Spelling things out, $X$ determines $F|_{\eta^2 = \infty} \simeq \bigoplus_{i=1}^{r} \mathbb{C}^{k_i}|_{\zeta_i, \infty}$, and each $\pi_i, i = 1, \ldots, r$, determines $F$ restricted to a neighbourhood of $(\zeta_i, \infty)$ in $\{\eta^2 = \infty\}$ via (4.10). Similarly, $Y$ and the $\rho_j$ determine $F|_{\zeta^2 = \infty}$.

**Remark 4.9.** Symplectic leaves of $\mathcal{M}(k, l)^0/K$ can be also identified with reduced orbits (cf. Definition [15]) of $GL(l)^+ \oplus \mathfrak{gl}(l)^-$. Therefore, the last proposition describes sheaves corresponding to a reduced orbit with $Y$ semisimple. Furthermore, if we view $\mathcal{M}(k, l)^0/K$ as an open subset of the moduli space of semistable sheaves with Hilbert polynomial $lx + ky$, then this map is a symplectomorphism between the Mukai-Tyurin-Bottacin symplectic structure, described in the introduction, and the Kostant-Kirillov form on a reduced orbit of a Lie group. For an open dense set, where $F$ is a line bundle on a smooth curve, this follows from results in [2, 4]. Since both symplectic structures extend everywhere, they are must be isomorphic everywhere.

**Example 4.10.** If we want $F$ to be a line bundle over its support, then we must require that all $k_i$ and all $l_j$ are equal to 1. A symplectic leaf in $\mathcal{M}(k, l)^0/K$ is now given by fixing diagonal matrices $X = \text{diag}(\zeta_1, \ldots, \zeta_k)$ and $Y = \text{diag}(\eta_1, \ldots, \eta_l)$ with all $\zeta_i$ and all $\eta_j$ distinct, as well as the diagonal entries of $FG$ and $GF$, and quotienting by the group of $(k + l) \times (k + l)$ diagonal matrices (acting as in (4.3)). If the diagonal entries of $FG$ are fixed to be $\alpha_1, \ldots, \alpha_k$, and the diagonal entries of $GF$ are $\beta_1, \ldots, \beta_l$, then the corresponding subset of $\mathcal{S}(k, l)$ consists of sheaves $F$ supported on a 1-dimensional scheme $S$ such that
\begin{align*}
S \cap \{\eta^2 = \infty\} &= \bigcup_{i=1}^{k} \left\{ \zeta - \zeta_i = \frac{\alpha_i}{\eta} \right\}, \\
S \cap \{\zeta^2 = \infty\} &= \bigcup_{j=1}^{l} \left\{ \eta - \eta_j = \frac{\beta_j}{\zeta} \right\}
\end{align*}
and the rank of $F$ restricted to $S \cap \{\eta^2 = \infty\}$ and $S \cap \{\eta^2 = \infty\}$ is everywhere 1.

**Remark 4.11.** We expect that Proposition 4.8 remains true if $X$ or $Y$ are not semisimple.

5. Rank $k$ perturbations

Let us now assume that $k \leq l$. In [11], the authors consider Hamiltonian flows on a subset $\mathcal{M}$ of $\mathcal{M}^0(k, l)/K$, where rank $F = \text{rank}G = k$. It is clear from the
previous section that a generic symplectic leaf of $\mathcal{M}^0(k, l)/K$ is not contained in $\mathcal{M}$. Therefore a flow may leave $\mathcal{M}$ without becoming singular. Since such Hamiltonian flows on a particular symplectic leaf can be linearised on the Jacobian of a spectral curve, it is interesting to know which points of the (affine) Jacobian are outside of $\mathcal{M}$. We are going to give a very satisfactory answer to this, in terms of cohomology of line bundles.

Let us therefore define the following set:

$$\mathcal{M}(k, l)^{1} = \{ M \in \mathcal{M}(k, l) ; \text{ rank } F = \text{ rank } G = k \}. \tag{5.1}$$

**Remark 5.1.** The manifold of $GL_k(\mathbb{C})$-orbits in $\mathcal{M}(k, l)^{1}$ with $X = 0$ and fixed $Y$, can be identified with the set $\{ Y + GF \}$, i.e. with the space of rank $k$ perturbations of the matrix $Y$, as considered first by Moser [13] ($k = 2$), and, then by many other authors, in particular Adams, Harnad, Hurtubise, Previato [21].

We now ask which acyclic sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ correspond to orbits of $K = GL_k(\mathbb{C}) \times GL_l(\mathbb{C})$ on $\mathcal{M}(k, l)^{1}$. We have:

**Proposition 5.2.** Let $k \leq l$. The bijection of Corollary [21,5.2] induces a bijection between:

1. orbits of $GL_k(\mathbb{C}) \times GL_l(\mathbb{C})$ on $\mathcal{M}(k, l)^{1}$, and
2. isomorphism classes of acyclic sheaves $\mathcal{F}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ with Hilbert polynomial $P_{\mathcal{F}}(x, y) = lx + ky$, which satisfy, in addition, (5.1) and

$$H^0(\mathcal{F}(-1, 1)) = 0 \text{ and } H^1(\mathcal{F}(1, -1)) = 0.$$  

**Proof.** Consider short exact sequences

$$0 \to \mathcal{O}(-1)^{\oplus k} \xrightarrow{(X - \zeta, G)^T} \mathcal{O}^{\oplus (k+1)} \to W_1 \to 0,$$

$$0 \to \mathcal{O}(-1)^{\oplus l} \xrightarrow{(F, Y - y)^T} \mathcal{O}^{\oplus (k+1)} \to W_2 \to 0.$$  

The condition that $G$ has rank $k$ is equivalent to $W_1$ being a vector bundle, isomorphic to $\mathcal{O}(1)^{\oplus k} \oplus \mathcal{O}(l-k)$. This is equivalent to $H^0(W_1 \otimes \mathcal{O}(-2)) = 0$. On the other hand, we claim that the condition that $F$ has rank $k$ is equivalent to $H^1(W_2 \otimes \mathcal{O}(-2)) = 0$. Indeed, any coherent sheaf on $\mathbb{P}^1$ splits into sum of line bundles $\mathcal{O}(i)$ and a torsion sheaf [10]. Since $W_2$ has a resolution as above, we know that all degrees $i$ in the splitting are nonnegative, and $F$ has rank $k$ if and only if all $i$ are strictly positive, which is equivalent to $H^1(W_2 \otimes \mathcal{O}(-2)) = 0$.

We can use the above exact sequences to obtain two further resolutions of $\mathcal{E} = \mathcal{F}(1, 1)$:

$$0 \to \mathcal{O}(-1, 0)^{\oplus k} \to \pi_2^* W_2 \to \mathcal{E} \to 0, \tag{5.2}$$

$$0 \to \mathcal{O}(0, -1)^{\oplus l} \to \pi_1^* W_1 \to \mathcal{E} \to 0, \tag{5.3}$$

where the maps between first two terms are given by the embedding in $\mathcal{O}^{\oplus (k+l)}$ followed by the projection onto the quotients $W_2, W_1$. Tensoring (5.2) with $\mathcal{O}(0, -2)$ shows that $H^1(W_2(-2)) = 0$ if and only if $H^1(\mathcal{E}(0, -2)) = 0$, i.e. $H^1(\mathcal{F}(1, -1)) = 0$. Similarly, tensoring (5.3) with $\mathcal{O}(-2, 0)$ shows that $H^0(W_1(-2)) = 0$ if and only if $H^0(\mathcal{E}(-2, 0)) = 0$, i.e. $H^0(\mathcal{F}(-1, 1)) = 0$. }
Remark 5.3. In the case \( k = l \), \( H^0(\mathcal{E}(-2,0)) = 0 \) implies that \( \mathcal{E}(-2,0) \) is acyclic (and similarly, \( H^1(\mathcal{E}(0,-2)) = 0 \) implies that \( \mathcal{E}(0,-2) \) is acyclic). In other words \( \mathcal{G} = \mathcal{E}(-1,0) \) satisfies \( H^*(\mathcal{G}(-1,0)) = H^*(\mathcal{G}(0,-1)) = 0 \). Furthermore, the resolution (5.3) becomes the following resolution of \( \mathcal{G} \):

\[
0 \to \mathcal{O}(-1,-1)^{\oplus k} \to \mathcal{O}^k \to \mathcal{G} \to 0.
\]

In the case when \( S = \text{supp} \mathcal{G} \) is smooth and \( \mathcal{G} \) is a line bundle, the corresponding part of \( \text{Jac}^{p+k-1}(S) \) and the resolution (5.4) have been considered by Murray and Singer in [15].

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