Shock formation in magnetised electron–positron plasmas: mechanism and timing

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Abstract

The shock formation process in electron–positron pair plasmas is investigated in the presence of an ambient perpendicular magnetic field. In initially unmagnetised plasmas, which are dominated by the Weibel or filamentation instability, the shock formation time is a multiple of the saturation time of the linear instability. While in weakly magnetised plasmas the mechanism is still the same, higher magnetisations induce synchrotron maser modes such that the shock formation is dominated by magnetic reflection. As a consequence the formation times are reduced. The focus is on the detailed picture of the particle kinetics, in which the transition between Weibel and magneto-hydrodynamic shocks can be clearly identified.

1. Introduction

Collisionless shocks can provide efficient particle acceleration and are thus important in the context of cosmic rays. The level of non-thermal acceleration is a critical parameter, thus non-relativistic shocks are better particle accelerators than relativistic shocks (Sironi et al 2013). Moreover, the level of non-thermal acceleration depends on the strength of the ambient, initial magnetisation, as well as on the angle of the field with respect to the particle flow. The main difference regarding particle acceleration between non-relativistic and relativistic shocks is that the latter become poor accelerators in the presence of too strong and too oblique magnetic fields. The reason for this is that in such a situation, particles flowing along the field lines cannot go fast enough to escape upstream, be scattered back to the shock front, and close the Fermi acceleration cycles (see Sironi et al 2015 and references therein). In contrast, non-relativistic shock can easily have particles outrun their front (Guo et al 2014, Park et al 2015).

Shocks mediated by the Weibel or filamentation instability (Fried 1959, Weibel 1959) are common in astrophysics and are strongest in initially unmagnetised plasmas with a large anisotropy in the momentum distribution, given by a directed bulk flow or temperature anisotropy. Electromagnetic modes are seeded from noise level and amplified to a large-scale turbulent magnetic field structure. The charged plasma particles are scattered in the magnetic turbulence, which finally leads to an isotropisation of the particle flow.

The shock formation process in initially unmagnetised plasmas has been studied in detail (Bret et al 2013, 2014) and it was found that the steady-state formation is associated with the saturation time of the magnetic instability in the plasma, $\tau_s$. The time to form a steady-state shock in an electron–positron pair plasma is given by

$$\tau_d = d \tau_s,$$

where $d$ is the dimension parameter, which is $d = 2$ in 2D and $d = 3$ in a real 3D setup. In electron-ion plasmas, the shock formation time is enhanced due to an extra merging time of the filaments (Stockem Novo et al 2015). When we speak of a steady-state shock, we refer to the compression ratio, which has saturated to a constant value...
on spatial lengths much larger than the electron skin depth $c/\omega_{pe}$ with electron plasma frequency
\[ \omega_{pe} = \sqrt{4\pi e^2 n/m_e}, \]
where $e$ is the electron charge, $m_e$ the mass of an electron and $n$ the electron density.

Long-term studies of shock formation in magnetised plasmas show that the dominant mediating process changes to magnetic reflection at high magnetisation (Sironi et al 2013). How the ambient magnetic field $B_0 \neq 0$ actually influences the particle kinetics and the shock formation time is not clear yet.

In a series of 1D simulations, highly magnetised shocks have been investigated by Gallant et al (1992), which we will call magnetised shocks in the following. Synchrotron maser modes lead to a coherent reflection of the particles in the shock front. The thermalisation in the downstream was found to be faster than the gyration time scale.

The manuscript is organised as follows. The theory of filamentation modes in a plasma with perpendicular magnetic field and the jump conditions in magnetised shocks are summarised in section 2. Particle-in-cell (PIC) simulations are presented and discussed in section 3. The findings are summarised in section 4.

2. Theory of shock formation in magnetised plasmas

2.1. Shock formation in Weibel shocks

2.1.1. The filamentation instability in magnetised plasmas

Consider two counter-streaming pair beams of identical density and Lorentz factor $\gamma_0 = \frac{1}{\sqrt[2]{1 - \beta^2}}$, with $\beta = v_0/c$. Both beams are cold, and embedded in a magnetic field $B_0 \perp v_0$. In order to investigate the Weibel instability, we look at the growth of perturbations with $k \perp v_0$. As evidenced on figure 1, in 3D, the perpendicular magnetic field breaks the axial symmetry, and one has to investigate every possible orientation of the wave-vector in the $x_1, x_2$ plane.

Since both beams are cold, we can analyse the system through a four-fluids cold model. Electrons and positrons moving to the left constitute for two components and those moving to the right constitute for the other two. Initially, the system is charge and current neutral since both beams are individually so.

Similar calculations have been performed for counter-streaming electron beams with a flow-aligned field (Godfrey et al 1975), or even an oblique one (Bret et al 2006, Bret and Dieckmann 2008, Bret 2014). In the field-free case, the present system is equivalent to counter-streaming electron beams, because the linear response varies with $q^2$. An external magnetic field introduces cyclotron frequencies depending on the sign of the charges. Therefore, we expect to recover previous results in the limit $B_0 = 0$.

The derivation of the dispersion equation is standard. We start writing the matter and momentum conservation equations

\[ \frac{\partial n_i}{\partial t} + \nabla \cdot (n_i v_i) = 0, \quad (2) \]

\[ \frac{\partial p_i}{\partial t} + (v_i \cdot \nabla)p_i = q_i \left( E_i + \frac{v_i \times (B + B_0)}{c} \right), \quad (3) \]

where $n_i, v_i, i = 1 \ldots 4$, stand for the densities and velocities of the 4 species involved. All equilibrium quantities are then perturbed by a small amount, proportional to \( \exp(i k \cdot r - i \omega t) \). The self-generated magnetic field $B_i$ is expressed in terms of the self-generated electric $E_i$ field through $B_i = (c/\omega)k \times E_i$. 

Figure 1. Sketch of the system considered. In 3D, the perpendicular magnetic field breaks the axial symmetry, and one has to investigate every possible orientations of the wave-vector.
Once linearised, equation (2) allow to express the first order density perturbation \( n_{i1} \) in terms of \( \psi_{i1} \). Inserting the latter into the linearised equation (3) gives an expression for \( \psi_{i1} \), hence of the first order current \( J_i \), in terms of \( E_i \). The dispersion equation is finally obtained from a combination of Maxwell–Ampère's and Faraday's equations

\[
T(E_i) \equiv \mathbf{k} \times (\mathbf{k} \times E_i) + \frac{\omega^2}{c^2} \left( E_i + \frac{4i\pi}{\omega} J_i(E_i) \right) = 0. \tag{4}
\]

In order to admit non-trivial solutions \( E_i \neq 0 \), we need \( \det T = 0 \), which is the final dispersion equation. This tensor has been computed analytically using the Mathematica Notebook described in Bret (2007). The tensor elements are reported in appendix, in terms of the dimensionless variables

\[
x = \frac{\omega}{\omega_p}, \quad Z = \frac{k\nu_0}{\omega_p}, \quad \beta = \frac{\nu_0}{c}, \quad \Omega_b = \frac{\omega_b}{\omega_p}, \quad \text{with} \quad \omega_b = \frac{|q| B_0}{mc}. \tag{5}
\]

The dispersion equation for the Weibel modes stems from \( T_{13} = 0 \). Changing \( (Z_2, Z_3) \) to \( (Z \sin \theta_1, Z \cos \theta_1) \) with \( \theta_1 \) the orientation of the wave vector in the \( (x_1, x_3) \) plane, the numerical study displayed in figure 2 shows that as is the case for a flow-aligned field (Godfrey et al 1975, Bret 2014), the growth-rate saturates at large \( Z = |Z| \). The asymptotic value of the growth-rate can be derived in the following way.

This dispersion equation is a polynomial in \( Z \) that we shall write \( P(Z, x) = 0 \). As we let \( Z \to \infty \), the asymptotic dispersion equation \( P_\infty(x) = 0 \) is simply the coefficient of higher degree of \( Z \) in the polynomial \( P \).

The result is

\[
P_\infty(x) = 2\beta^2\Omega_b^2 \cos 2\theta_k + 2\beta^2\Omega_b^2 + \gamma_0 x^2 (\Omega_b^2 - 4\beta^2\gamma_0) - \gamma_0^3 x^4 = 0, \tag{6}
\]

which can be solved exactly, giving the growth-rate

\[
\delta_\infty^2 = \frac{\Delta^2 + 16\beta^2\gamma_0^3 \Omega_b^2 \cos^2 \theta_k - \Delta}{2\gamma_0^3}, \quad \text{with} \quad \Delta = \Omega_b^2 - 4\beta^2\gamma_0^3. \tag{7}
\]

In order to find out if the instability can be suppressed completely, we have to address the growth-rate. Since it is always smaller than \( \delta_\infty \), cancelling the Weibel instability means cancelling \( \delta_\infty \). This in turn, implies having \( \beta^2\gamma_0^3 \Omega_b^2 \cos^2 \theta_k = 0 \). When such a condition is met, \( \delta_\infty \) vanishes only if \( \Delta > 0 \), that is, \( \Omega_b > 2\beta\gamma_0^{3/2} \).

To do so, we therefore need

- Either \( \Omega_b = 0 \), that is, no magnetic field. In such a case, \( \Delta < 0 \) and \( \delta_\infty = 2\beta/\sqrt{|\gamma_0|} \) which is the field-free result (Bret et al 2006).
- Or, \( \beta = 0 \), that is, no relative motion. Here, \( \Delta > 0 \) gives the expected result \( \delta_\infty = 0 \).
- Or, \( \cos \theta_k = 0 \), that is, \( \theta_k = \pm \pi/2 \). Here, we have

\[
\delta_\infty = \begin{cases} 
0, & \text{when } \Delta > 0, \\
\frac{2\beta}{\sqrt{|\gamma_0|}} \sqrt{1 - \frac{16}{4\beta^2\gamma_0}}, & \text{when } \Delta < 0.
\end{cases} \tag{8}
\]

Figure 2. Numerical resolution in terms of \( Z = |Z| \) of the dispersion equation \( T_{13} = 0 \) for \( \gamma_0 = 2 \), \( \Omega_b = 5 \) and various angles \( \theta_1 \).
Let us now turn to the largest growth-rate in terms of $\theta_0$. We see from equation (7) that it is reached for $\cos \theta_k = \pm 1$, that is $\theta_k = 0[\pi]$. Setting $\cos \theta_k = 1$ in equation (7) gives

$$\delta_{\infty, \text{max}}^2 = \frac{2 \beta}{\gamma_0}$$

which is the growth-rate in the absence of magnetic field. Among all the Weibel modes which grow, those with $\theta_k = 0[\pi]$ grow as if there were no magnetic field. This stands in stark contrast with the case of a flow-aligned field, where the Weibel instability can be completely suppressed (Godfrey et al 1975, Bret et al. 2006). A similar effect has been found for the case of two counter-streaming electron beams (Bret 2014). It arises from the fact that the Weibel instability has the particles move essentially sideways. In the presence of a perpendicular magnetic field, such motions may simply be found parallel to the field, cancelling the Lorentz force, and the field effect with it.

### 2.1.2. The saturation phase in magnetised plasmas

The filamentation instability grows exponentially only in the early stage and enters a saturation phase. At that time, the magnetic field has been amplified from its fluctuation value, which is discussed in (Bret et al. 2013), up to equipartition with the gyro-kinetic motion of the electrons and positrons. Thus, the saturation time of the filamentation instability is given by

$$\tau_s = 6^{-1} \Pi \omega_{pe}^{-1}$$

where $\Pi$ is the number of e-foldings of the instability and $\delta \omega_{pe}^{-1}$ its growth rate. By this time, the field has grown to nearly equipartition with the saturation magnetic field (Medvedev and Loeb 1999, Silva et al 2003)

$$B_s^2 = 8 \pi n_0 m_e e^2.$$  

From the cyclotron frequency $\omega_c = qB/\gamma_m c$, we derive the size of the filaments at saturation, which is also the electronic Larmor radius in $B_s$ (we set $v_0 \sim c$)

$$L_s = \frac{c}{\omega_c} = \sqrt{\frac{\gamma}{2}} \frac{c}{\omega_{pe}}.$$  

### 2.1.3. Shock formation time

Now, the same argument holds as in the case of initially unmagnetised pair shocks (Bret et al. 2014). In order to know if the plasma flow is stopped in the overlapping region within the time $\tau_s$, we need to compare $L = 2c\tau_s$, the size of this region at this time, with the electron or positron Larmor radius at saturation $L_s$. Therefore, we get the condition

$$\frac{L}{L_s} = \frac{\sqrt{8 \Pi \gamma}}{\delta} \gg 1$$

for shock formation. The downstream density is still only twice the upstream density at this time, thus, the density compression of 3 according to the Rankine-Hugoniot jump conditions for the 2D case needs another $\tau_s$ since the overlapping region no longer expands (Bret et al. 2014). In the 3D case, the expected density jump is $\approx 4$ so the time $2\tau_s$ is required to bring enough material in the central region. The shock formation time in Weibel shocks $\tau_{f,W}$ finally reads

$$\tau_{f,W} = \frac{d}{\delta} \delta^{-1} \Pi \omega_{pe}^{-1},$$

where $d$ is the dimensionality of the system ($d = 2$ (3) for a 2D (3D) setup). Formally, equation (14) is the same as for unmagnetised plasmas with a modified growth rate $\delta$ in the case of magnetised plasma.

### 2.2. Magnetised shocks

The analysis of strongly magnetised shocks showed that they are parametrised purely by the upstream magnetisation

$$\sigma_e = \frac{B_0 e}{m_e c \omega_{pe}} \left( \frac{1}{\gamma_0} \right)$$

which is the ratio of the upstream Poynting flux to the upstream kinetic energy flux. Here, $B_0$ is the initial magnetic field strength in the far upstream. The total magnetisation $\sigma$ is then given by the contribution of the electron ($\sigma_e$) and positron species ($\sigma_p$),
\[ \frac{1}{\sigma} = \frac{1}{\sigma_c} + \frac{1}{\sigma_e} \]  

(16)

which in case of symmetric beams is just \( \sigma = \sigma_c/2 \). In the downstream region the plasma quickly thermalises and the jump conditions for magnetised plasma apply. The magnetohydrodynamic jump conditions can be derived from (see Stockem et al 2012)

\[
\gamma_0 \left( 1 + \beta_{h0} \beta_0 (1 + \sigma) - \frac{\beta_0}{\beta_{h0}} \right) \left( 1 + \frac{\Gamma_{ad}}{\Gamma_{ad} - 1} \frac{\beta_{h0}^2}{1 - \beta_{h0}^2} \right) - 1 = 0
\]

(17)

\[
= \frac{\Gamma_{ad}}{\Gamma_{ad} - 1} \gamma_0 \left( \frac{\beta_{h0}}{1 - \beta_{h0}^2} (\beta_0 + \beta_{h0}) - \frac{\sigma}{2} \frac{\beta_0}{\beta_0 + \beta_{h0}} \frac{\beta_0}{\beta_{h0}} (1 + \beta_{h0}^2) + 2 \right) = 0,
\]

where a perpendicular magnetic field and a cold upstream were assumed. In the limit of very high relativistic beams \( \gamma_0 \gg 1 \) the above equation (17) agrees with the approximation given in Gallant et al (1992)

\[
\left( 1 + \frac{1}{\sigma} \right) \beta_{h0}^2 - \left[ \frac{\Gamma_{ad}}{2} + \left( \frac{1}{\Gamma_{ad} - 1} \right) \right] \beta_{h0} - \left( 1 - \frac{\Gamma_{ad}}{2} \right) = 0.
\]

(18)

In our 2D geometry the ideal adiabatic constant is \( \Gamma_{ad} = 3/2 \). Since we are interested in weakly magnetised flows \( (\sigma \ll 1) \), the shock velocity \( \beta_{sh} \) and compression ratio \( n_2/n_1 \) can be approximated to

\[
\beta_{sh} \approx (\Gamma_{ad} - 1) + \frac{(2 - \Gamma_{ad}) \Gamma_{ad} \sigma}{2(\Gamma_{ad} - 1)}
\]

(19)

\[
\frac{n_2}{n_1} = B_{2L} B_1 \approx \frac{\Gamma_{ad}}{\Gamma_{ad} - 1} - \frac{(2 - \Gamma_{ad}) \Gamma_{ad}}{2(\Gamma_{ad} - 1)^3} \sigma
\]

(20)

with the downstream to upstream magnetic field ratio \( B_2/B_1 \). In 1D simulations, a magnetic overshoot was found at the leading edge of the shock, which in the downstream frame can be approximated as (Gallant et al 1992)

\[
B_{max} = B_0 \sqrt{\frac{1 + \frac{\gamma_0}{\Gamma_{ad} - 1} - \beta_{sh}}{1 - \beta_{sh}}},
\]

(21)

In the case of weak magnetisations \( \sigma \ll 1 \) we can simplify this expression with \( \beta_{sh} \approx 1/2 \) to

\[
B_{max} \approx \frac{\sqrt{2} B_0}{\sqrt{\sigma(1 - \beta_{sh})}} \approx \frac{8}{\sqrt{\sigma}} B_0 \approx 4 \sqrt{\gamma_0} m_e c \omega_{pe} \frac{\mu_s c}{e}
\]

(22)

The Larmor radius of electrons in this field is then simply given by \( r_L = \sqrt{\gamma_0} c/(4\omega_{pe}) \). In order to form a shock, the particles need to be gyrating several times in this peak field. We estimate a number of \( \Psi = 20 \), which will give an estimate of the shock formation time due to magnetic reflection

\[
\tau_{f,M} = 5 \sqrt{\gamma_0} \omega_{pe}^{-1}.
\]

(23)

3. PIC simulations of shock formation in magnetised plasmas

In order to compare with the theory, we performed PIC simulations of shock formation in initially magnetised plasmas. The simulations are 2D in space and 3D in momentum space. In the beginning, the 2D simulation box is homogeneously filled with a quasi-neutral plasma of electrons and positrons. We use the reflecting wall setup in order to save computation time, see figure 3. The momentum distributions of the particles are Maxwellians with very low thermal velocity \( v_{th}/c = 10^{-3} \) and a plasma bulk flow with Lorentz factor \( \gamma_0 = 15 \). We use the definition \( \gamma_0 = (1 - v_0^2/c^2)^{-1/2} \) with \( v_0 \) the flow velocity along the \( x_3 \) direction and \( c \) the speed of light. The initial magnetic field is oriented along \( x_3 \) and thus perpendicular to the plasma flow. The magnetisation is varied in the simulations with values \( \sigma_e = 10^{-3} \) to \( 10^{-1} \), which is the critical transition range from Weibel to magnetised shocks. The simulation box size is \( L_1 = L_2 = 400 \times 400 c^2/\omega_{pe}^2 \), with a resolution \( \Delta x = 0.026 \gamma_0^{1/2} c/\omega_{pe} \). We use 18 particles per cell and periodic boundary conditions along \( x_2 \).

Listed in table 1 are the exact calculations for the shock velocity taken from equation (17) which are necessary to examine the small changes. Note that the simulations were performed in a 2D spatial setup which results in an adiabatic gas constant \( \Gamma_{ad} = 3/2 \) and a shock velocity of approximately \( v_{sh} = 0.5c \). The shock formation time from the simulations are plotted in figure 4.

In order to estimate the shock formation time from the simulations, we average the 2D density over the \( x_2 \) direction. We plot the averaged density versus the shock propagation direction \( x_1 \) and time \( t \) for the different magnetisations, see figure 5. The Weibel shock for magnetisation \( \sigma_e = 10^{-3} \) develops on the longest time scale and shows a wide shock front region (yellow and orange region). The downstream region is almost
homogeneous (dark red region). The shock propagation is constant after $t = 90 \omega_{pe}^{-1}$. The line over plotted in figure 5 follows the shock front and its linear extrapolation for small times defines the formation time of the collisionless shock, which is in this case $t_{sf} = 90 \omega_{pe}^{-1}$.

If the magnetisation is increased, the shock formation time decreases as listed in Table 1. It is lowest for $\sigma_e = 10^{-3}$, $t_{sf} = 53 \omega_{pe}^{-1}$, and increases again for the magnetised shock $\sigma_e = 10^{-1}$, $t_{sf} = 30 \omega_{pe}^{-1}$. The downstream region of the magnetised shocks shows an oscillatory structure, which is already present for $\sigma_e = 10^{-2}$. For $\sigma_e = 10^{-3}$ a strong precursor develops at $t = t_{sf} = 30 \omega_{pe}^{-1}$ which fades out at $t = 130 \omega_{pe}^{-1}$. The strength of this precursor is on the order of the estimate of the overshoot, equation (22), but slightly reduced because of the 2D geometry. A 2D perpendicular magnetic field has been self-consistently generated during the shock formation (see figure 6). In agreement with previous findings (Sironi and Spitkovsky 2009), the shock front width decreases when the magnetisation is increased from $\sigma_e = 10^{-5}$ to $\sigma_e = 10^{-3}$ but still shows the typical filamentary structure. For higher magnetisations this filamentary structure in the shock front disappears.

In order to identify the filamentation instability as the mediator for Weibel shocks and to investigate the influence of the ambient magnetic field, we analyse the magnetic energy density during the early stage of interaction in a narrow slab close to the reflecting wall in the interval $x \in [395, 400]$. The growth rate does not vary with the ambient field strength for magnetisations $\sigma_e \lesssim 10^{-3}$, in agreement with the theoretical model (see Figure 6).

### Table 1. Upstream magnetisations $\sigma_e$, shock formation time $t_{sf}$, saturation time of the filamentation instability $t_s$, measured shock velocity $v_{sh}$, and predicted value from theoretical model $v_{sh,th}$, and measured growth rate $\omega_{pe}/\omega_{pe}$

| $\sigma_e$ | $t_{sf}/\omega_{pe}$ | $t_s/\omega_{pe}$ | $v_{sh}/c$ | $v_{sh,th}/c$ | $\omega_{pe}/\omega_{pe}$ |
|-----------|---------------------|-------------------|------------|---------------|-----------------|
| $10^{-1}$ | 30                  | —                 | 0.50       | 0.55          | —               |
| $10^{-2}$ | 23                  | 74                | 0.47       | 0.472         | 0.15            |
| $10^{-3}$ | 65                  | 53                | 0.47       | 0.468         | 0.27            |
| $10^{-4}$ | 88                  | 46                | 0.47       | 0.468         | 0.27            |
| $10^{-5}$ | 90                  | 45                | 0.46       | 0.468         | 0.27            |

Figure 3. Initial simulation setup for counterstreaming electron positron flow with reflecting wall setup.

Figure 4. Shock formation time from the simulations versus upstream magnetisation.
figure 7 and table 1). For $\sigma_z = 10^{-2}$ a short quasi-linear growth time of the magnetic field exists with a reduced growth rate, while for $\sigma_z = 10^{-1}$ no linear behaviour at all can be found anymore.

The phase space plots reveal interesting information about the interplay of the filamentation instability and magnetic reflection (see figure 8). Typical for the filamentation instability is a rather slow development of the instability. The positron phase space is not perturbed for a long period while the modes continue to grow and finally saturate. Only after saturation of the linear phase of the magnetic field amplification, the particles are scattered and isotropised in the magnetic turbulence (Stockem Novo et al 2015). At this stage, the phase space is widened and the two beams, the incoming and the beam reflected from the wall, starts to narrow and overlap until the shock downstream region forms. This narrowing of the phase space does not happen at the leading edge of the reflected particle beam but a few tens of plasma skin depths behind, which guarantees a sufficient time of interacting counterflows.

For a typical Weibel shock this behaviour is observed in the particle phase spaces (see figure 8 for $\sigma_z = 10^{-5}$). In an magnetised shock a strong overshoot of the magnetic field is build up at the interface of the counterstreaming beams which immediately reflects almost all particles. The particle mean free path is
drastically reduced and the particles are accumulated, isotropised and the downstream region of the shock forms quickly (see figure 8 for $\sigma_e = 10^{-1}$).

In case of intermediate magnetisations both processes are present at the same time. In figure 8 for $\sigma_e = 10^{-3}$ the slowly evolving filamentary modes cause the spreading of the beams in the region $360 \leq x_1 \leq 400$ while magnetic reflection leads to a winding up in phase space at $340 \leq x_1 \leq 360$. This accumulation occurs ahead of the shock and is responsible for the compressed precursor in figure 5 at a distance $50-100 \ c/\omega_{pe}$ from the reflecting wall at times $30-200 \ \omega_{pe}^{-1}$. The magnetic field strength is yet not strong enough to reflect a significant amount of particles. Eventually, the filamentary modes dominate and introduce magnetic turbulence, which is why the shock character for $\sigma_e = 10^{-3}$ is still of Weibel type. For $\sigma_e = 10^{-2}$ magnetic reflection already dominates.

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**Figure 6.** Magnetic field out of the plane for magnetisations $\sigma_e = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$ (from top to bottom).
Figure 7. Magnetic energy density normalised to the initial kinetic energy in the simulations for magnetisations $\sigma = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$ (from top to bottom). The dashed line indicates the growth rate during the linear phase of the filamentation instability.
4. Conclusions

We investigated collisionless shock formation of an electron–positron plasma flow with ambient magnetic field. The initial magnetisation was increased in order to analyse the impact of the magnetic field on the shock formation time. In analogy to shock formation in initially unmagnetised plasmas, we derived a theoretical model for the shock formation time. Since the growth rate of the filamentation instability depends only weakly on a perpendicular magnetic field, the shock formation time for very low magnetisations $\sigma$ is almost the same as for unmagnetised plasmas. The shock formation time for high magnetisations, which is dominated by magnetic reflection, was also estimated.

2D PIC simulations have been performed in order to investigate the transition between both types of shocks. It has been found that the estimated shock formation times in the limiting cases of Weibel and magnetised shocks fit well. For intermediate magnetisations, a mix of both mechanisms was observed. The shock formation

Figure 8. Positron phase spaces at $\omega_{pe} = 78$ for magnetisations $\sigma = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$ (from top to bottom). Transition from unstable beams for low magnetisations to magnetic reflection (curls winding up in phase space) for high magnetisations.
time in Weibel shocks is smaller for magnetised shocks, since magnetic reflection is a very quick process. In between, we observed the reflection of a small fraction of particles at the leading edge of the shock, which propagated ahead of the shock as a precursor and finally faded out. In the meanwhile, the filamentation modes had time to evolve and to form a shock of Weibel type. However, the shock formation time was reduced compared to a pure Weibel shock, while the theoretical estimate predicted a constant shock formation time since the filamentation growth rate is constant for higher magnetisations.

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Appendix. Tensor elements

The tensor $T$ defined by equation (4) reads in terms of the dimensionless variables (5)

$$T = \begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{21} & T_{22} & 0 \\ 0 & 0 & T_{33} \end{bmatrix},$$

where the first column stand for the $x_3$ direction, the second column for $x_2$, and the third for $x_1$, the flow axis. The tensor elements read,

$$T_{11} = 1 - \frac{1}{\gamma_0^2} \left( \frac{4}{\gamma_0^2} + \frac{Z_\perp^2}{\beta^2} \right),$$

$$T_{22} = 1 - \frac{\gamma_0^2}{\gamma_0^2 x^2 - \Omega_B^2} - \frac{Z_\parallel^2}{\beta^2 x^2},$$

$$T_{33} = \frac{2(x^2 + \gamma_0^2 Z_\perp^2)}{\gamma_0 x^2 (\Omega_B^2 + \gamma_0^2 x^2)} - \frac{2(x^2 + \gamma_0^2 Z_\parallel^2)}{\gamma_0 x^2 (\Omega_B^2 + \gamma_0^2 x^2)} - \frac{4Z_\parallel^2}{\gamma_0 x^2} + \frac{Z_\perp^2}{\beta^2 x^2},$$

$$T_{21} = \frac{Z_\parallel Z_\perp}{x^2 \beta^2},$$

$$T_{12} = T_{21}. \tag{A2}$$

References

Bret A 2007 Comput. Phys. Commun. 176 362
Bret A 2014 Phys. Plasmas 21 022106
Bret A and Dieckmann M E 2008 Phys. Plasmas 15 062102
Bret A, Dieckmann M E and Deutsch C 2006 Phys. Plasmas 13 082109
Bret A, Stockem A, Fiuza F, Ruyer C, Gremillet L, Narayan R and Silva L O 2013 Phys. Plasmas 20 042102
Bret A, Stockem A, Narayan R and Silva L O 2014 Phys. Plasmas 21 072301
Fried B D 1959 Phys. Fluids 2 337
Gallant Y A, Hoshino M, Langdon A B, Arons J and Max C E 1992 Astrophys. J. 391 73
Godfrey B B, Shanahan W R and Thode L E 1975 Phys. Fluids 18 346
Guo X, Sironi L and Narayan R 2014 Astrophys. J. 794 153
Medvedev M V and Loeb A 1999 Astrophys. J. 526 697
Park J, Caprioli D and Spitkovsky A 2015 Phys. Rev. Lett. 114 085003
Silva L O, Fonseca R A, Tonge J W, Dawson J M, Mori W B and Medvedev M V 2003 Astrophys. J. 596 L121
Sironi L, Keshet U and Lemoine M 2015 Space Sci. Rev. 191 519
Sironi L and Spitkovsky A 2009 Astrophys. J. 698 1523
Sironi L, Spitkovsky A and Arons J 2013 Astrophys. J. 771 54
Silva L O 2015 Astrophys. J. 837 L28
Silva L O, Fonseca R A and Silva L O 2014 Sci. Rep. 4 3934
Silva L O, Fonseca R A and Silva L O 2012 Plasma Phys. Control. Fusion 54 125004
Stockem Novo A, Bret A, Fonseca R A and Silva L O 2015 Astrophys. J. 803 L29
Weibel E S 1959 Phys. Rev. Lett. 2 83