INTERIOR GRADIENT BOUND FOR MINIMAL GRAPHS IN A PRODUCT MANIFOLD $M \times R$

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ABSTRACT. Let $(M, g)$ be an $n$-dimensional complete Riemannian manifold with $\text{Ric}(M) \geq -(n-1)Q$, where $Q \geq 0$ is a constant. We obtain an interior gradient bound for minimal graphs in $M \times R$ under some technical assumptions. For details, see Theorem 2.

1. Introduction

Let $(M, g)$ be an $n$-dimensional complete Riemannian manifold. For each $u \in C^3(M)$, we can define a graph in the product manifold $(M \times R, g + dt^2)$ as follows:

$$\Sigma(u) := \{(x, u(x))|x \in M\}.$$ 

Naturally, we can equip the graph $\Sigma(u)$ with the metric induced from $M \times R$. Then, it is well known that $\Sigma(u)$ is a minimal hyper-surface in the product manifold $M \times R$ iff $u$ satisfies

$$\text{div}_M \frac{\nabla_M u}{\sqrt{1 + |\nabla_M u|^2}} = 0,$$

where $\nabla_M$ and $\text{div}_M$ are the gradient and divergence of $(M, g)$. Minimal surface theory in $R^{n+1}$ has been studied by many famous mathematicians. Minimal surfaces in a general compact Riemannian manifold $(N, \bar{g})$ are also interesting subjects with rich applications. There is relatively few results of minimal surfaces in a general complete Riemannian manifold except the cases when $N = H^{n+1}(-1)$ or $M \times R$, see [7] and [9].

In the case of $M$ being a domain of the Euclidean space $R^n$, many powerful techniques have already been developed to derive a priori gradient bounds for solutions to (1). The key tool is the use of the Maximum Principle for suitable quantities. In [5], N. Korevaar gave a beautiful application of the maximum principle proof of the following

**Theorem 1.** Let $M$ be the ball $B_2(0)$ of radius 2 with center at the origin 0 of $R^n$. Then there exist two constants $K_i$, $i = 1, 2$, depending only on $n$, ...
such that
\[ v(0) \leq K_2 \exp(K_1 u(0)^2), \]
where \( u \in C^3(B_1(0)) \) is a negative solution of (1), \( v = \sqrt{1 + |\nabla u|^2} \).

Later, this kind of argument was used by Ecker and Huisken \[4\] and Colding and Minicozzi \[3\] to study the mean curvature flow.

In this note, we will use the maximum principle argument to prove the following

**Theorem 2.** Let \( M \) be an \( n \)-dimensional complete Riemannian manifold with \( \text{Ric}(M) \geq -(n-1)Q \), where \( Q \geq 0 \) is a constant. \( B_r(p) \) is a geodesic ball with center \( p \) and radius \( r \) such that for any geodesic \( \gamma : [0, s] \to B_r(p), 0 \leq s < r \), parametrized by arc length with \( \gamma(0) = p \), we have at \( \gamma(t), 0 \leq t \leq s \),
\[ K(X, \frac{\partial}{\partial \gamma}) \leq C_+, \]
where \( X \) is any vector of length less than one in \( T_{\gamma(t)}M \) linearly independent of \( \frac{\partial}{\partial \gamma} \), \( K(X, \frac{\partial}{\partial \gamma}) \) is the sectional curvature of the 2-plane spanned by \( X \) and \( \frac{\partial}{\partial \gamma} \). Then there exist \( K_i = K_i(n, Q, r, C_+) \), \( i = 1, 2 \), such that if \( u \in C^3(B_r(p)) \) is a negative solution of (1), then
\[ v(p) \leq K_2 \exp(K_1 u(p)^2). \]

Our argument is to generalize the proof of Korevaar to our case. However, many details are much more involved. We believe the estimate in the theorem above can be improved with the use of the idea in \[3\]. Another interesting subject is to study the Mean Curvature Flow in \( M \times R \), see \[8\].

### 2. PROOF OF THEOREM 2

Our proof is divided into six steps.

**Step 1.** Let \( \eta(x, z) \) be a nonnegative continuous function in \( B_r(p) \times R^- \), which vanishes on \( \{ |x| = r, z < 0 \} \) and is smooth where it is positive. Then \( \eta v \) has a positive maximum in the interior of \( B_r \), say at \( P \). If \( \{x^1\} \) is a normal coordinate system at \( P \) with \( \frac{\partial}{\partial x^1} = \frac{\partial}{\partial \gamma} \), then we have at \( P \)
\[ (\eta v)_i = 0, i = 1, \cdots, n, \]
\[ [\eta v]_{ij} \leq 0, \]
where the subscript \( i \) on the outside of parenthesis means the covariant derivative w.r.t. \( \frac{\partial}{\partial x^i} \). By the chain rule,
\[ (\eta)_i = \eta_t + \eta_z u_i, \]
\[ (\eta)_{ij} = \eta_{ij} + \eta_{iz} u_j + \eta_{ez} u_i + \eta_{zz} u_i u_j + \eta_{zz} u_i u_j. \]
Define
\[ g^{ij} = \delta^{ij} - \nu^i \nu^j, \]
where \( \delta^{ij} \) is the standard Kronecker’s symbol and \( \nu^i = \frac{\nu_i}{\nu} \). Note that \( |\nu| < 1 \).

So \([g^{ij}]\) is positive definite. Then the trace of the product of \([(\eta \nu)_{ij}]\) with \([g^{ij}]\) is non-positive, i.e.,

\[
g^{ij}(\eta \nu)_{ij} \leq 0.
\]

But

\[
(\eta \nu)_{ij} = (\eta)_{ij}v + (\eta)_{ij}v_i + \eta v_{ij}.
\]

Hence

\[
0 \geq g^{ij}(\eta)_{ij}v + g^{ij}(\eta)_{ij}v_j + g^{ij}(\eta)_{ij}v_i + g^{ij}\eta v_{ij} = g^{ij}(\eta)_{ij}v + 2g^{ij}\left(-\frac{\eta}{v}\right)v_i v_j + \eta g^{ij} v_{ij}
\]

\[
= v[g^{ij}(\eta)_{ij} + \eta g^{ij}\left(-\frac{2}{v}v_i v_j + v_{ij}\right)].
\]

Here we have substituted \(-\eta v v_{ij} \) for \((\eta)_{ij}\). Therefore

\[
(2) \quad g^{ij}(\eta)_{ij} + \eta g^{ij}\left(-\frac{2}{v}v_i v_j + v_{ij}\right) \leq 0.
\]

We compute

\[
v_i = \nu^j u_{ji},
\]

\[
v_{ij} = \nu^k u_{kij} + \frac{1}{v}(u_{kij} u_{ki} - \nu^k u_{ki} \nu^l u_{lj}).
\]

So

\[
(3) \quad g^{ij} v_{ij} = g^{ij} u_{kij} \nu^k + \frac{1}{v} g^{ij}(u_{kij} u_{ki} - \nu^k u_{ki} \nu^l u_{lj}).
\]

Note that the second term is nonnegative for \(|\nu| < 1\). From

\[
u_{ki} = u_{ik}
\]

and the Ricci formula

\[
u_{ikj} = u_{ijk} + R_{kj} u_i
\]

where \( R_{kj} \) is the Ricci curvature of \((M, g)\), it follows that

\[
 g^{ij} u_{kij} \nu^k = g^{ij}(u_{ijk} + R_{kj} u_i) \nu^k
\]

\[
(4) \quad g^{ij} u_{ijk} \nu^k + \frac{g^{ij}}{v} R_{kj} u_i u_k.
\]

For \( u \) satisfies \((1)\), we have

\[
 g^{ij} u_{ij} = 0.
\]

Integrating by parts,

\[
(5) \quad g^{ij} u_{ijk} = (g^{ij} u_{ij})_k - (g^{ij})_k u_{ij} = -(g^{ij})_k u_{ij}.
\]

But

\[
-(g^{ij})_k = (\nu^j \nu^i)_k = \frac{u_{ik} u_j}{v^2} + \frac{u_i u_{jk}}{v^2} - \frac{2}{v^4} u_{ij} u_k u_{ij} u_{ij}.
\]

So

\[
(6) \quad -(g^{ij})_k u_{ij} \nu^k = 2\left(\frac{u_{ik} u_{ij} u_j u_k}{v^3} - \frac{u_i u_{jk} u_{ij} u_{ij} u_{ij}}{v^3}\right) = \frac{2}{v} g^{ij} v_i v_j.
\]
Note that
g^{ij} R_{kj} u_i u_k = R_{kj} \nu^j \nu^k.

Combining (3)-(7) yields

\begin{equation}
\frac{1}{v} g^{ij} (-\frac{2}{v} v_i v_j + v_{ij}) \geq \frac{R_{kj} \nu^j \nu^k}{v^2}.
\end{equation}

**Step 2.** Let

\[ u_0 = -u(p) > 0. \]

Define \( \phi(x, z) = \left( \frac{1}{2u_0} z + (r^2 - \rho(x)^2) \right)^+ . \)

Here "+" means positive part and

\( \rho(x) = \text{dist}(p, x) \)

for all \( x \in B_r(p) \), where \( \text{dist}(\cdot, \cdot) \) denotes the geodesic distance. Then

\[ 0 \leq \phi \leq r^2, \quad \phi_z = \frac{1}{2u_0}, \quad \phi_{zz} = \phi_{iz} = 0, \]

\[ |\nabla \phi|^2 = 4\rho^2 |\nabla \rho|^2 = 4\rho^2 < 4r^2. \]

Let

\[ f(\phi) = \exp(C_1 \phi) - 1, \]

where \( C_1 > 0 \) is to be determined. It is easy to see that

\[ f(0) = 0, f' > 0, f'' > 0. \]

Let

\[ \eta(x, u(x)) = f \circ \phi(x, u(x)). \]

Then \( \eta \) has the properties required at the beginning of **Step 1.** In this setting,

\[ (\eta)_i = f' \cdot (\phi_i + \phi_z u_i), \]

\[ (\eta)_{ij} = f'' \cdot (\phi_j + \phi_z u_j)(\phi_i + \phi_z u_i) \]

\[ + f' \cdot (\phi_{ij} + \phi_{iz} u_j + \phi_{zj} u_i + \phi_{zz} u_j u_i + \phi_{zz} u_{ij}). \]

Hence

\[ g^{ij}(\eta)_{ij} = g^{ij} f'' \cdot (\phi_i \phi_j + \phi_z \phi_i u_j + \phi_z \phi_j u_i + \phi_z^2 u_i u_j) \]

\[ + g^{ij} f' \cdot (\phi_{ij} + \phi_{zij}). \]

\[ = f'' \cdot (g^{ij} \phi_i \phi_j + 2\phi_z g^{ij} \phi_i u_j + \phi_z^2 g^{ij} u_i u_j) + f' \cdot g^{ij} \phi_{ij}. \]

But

\[ g^{ij} \phi_i \phi_j = \phi_i^2 - \phi_i \nu^j \phi_j \nu^j \geq 0, \]

\[ g^{ij} \phi_i u_j = \frac{\phi_i \nu^j}{\nu}, \]

\[ g^{ij} u_i u_j = \frac{|\nabla u|^2}{v^2}. \]
So
\[ g^{ij}(\eta)_{ij} = f'' \cdot (\phi_{\eta}^2 \frac{|\nabla u|^2}{1 + |\nabla u|^2} + 2\phi_{\eta} \phi_{ij}^{\nu} + \phi_{\nu}^2 - \phi_{\nu}^{\nu} \phi_{\nu}^{\nu}) + f' \cdot g^{ij} \phi_{ij} \]
(9)
\[ \geq f'' \cdot \frac{|\nabla u|^2 + 4u_0 u_i \phi_i}{4u_0 (1 + |\nabla u|^2)} + f' \cdot g^{ij} \phi_{ij}. \]
Combining (2), (8) and (9) yields
\[ f'' \cdot \frac{|\nabla u|^2 + 4u_0 u_i \phi_i}{4u_0 (1 + |\nabla u|^2)} + f' \cdot g^{ij} \phi_{ij} + \frac{\eta R_{k\nu} \nu^k \nu^j}{v^2} \leq 0. \]

**Step 3.** Note that
\[ \phi_{ij} = (-\rho^2)_{ij} = -2\rho_i \rho_j - 2\rho_{ij}. \]
So
\[ g^{ij} \phi_{ij} = -2(\rho_i^2 - \rho_i \rho_j \nu^i) - 2\rho(\rho_{ij} - \rho_{ij} \nu^i \nu^j) \]
\[ = -2(1 - \rho_i \nu^i \rho_j \nu^j) - 2\rho(\Delta \rho - \rho_{ij} \nu^i \nu^j). \]
For
\[ |\rho_i \nu^j| \leq 1, |\rho_i \nu^i \rho_j \nu^j| \leq 1, \]
we get
\[ -2(1 - \rho_i \nu^i \rho_j \nu^j) \geq -2(1 - (-1)) = -4. \]

Next we want to estimate \( \rho_{ij} \). Denote the Hessian of \( \rho \) by \( H(\rho) \). Also denote \( \frac{\partial}{\partial x^i} \) by \( X_i \). Let \( \tilde{X}_i, i = 1, \ldots, n, \) be the Jacobi fields along \( \gamma \) satisfying
\[ \tilde{X}_i(\gamma(\rho)) = X_i(\gamma(\rho), \tilde{X}_i(\gamma(0)) = X_i(\gamma(0)), [\tilde{X}_i, \frac{\partial}{\partial \gamma}] = 0. \]
Since \( \tilde{X}_i \) is a Jacobi field, it satisfies the Jacobi equation
\[ \nabla \frac{\partial}{\partial \gamma} \tilde{X}_i + R(\tilde{X}_i, \frac{\partial}{\partial \gamma}) \frac{\partial}{\partial \gamma} = 0. \]
We compute
\[ \rho_{ij} = H(\rho)(X_i, X_j) \]
\[ = X_i X_j \rho - \langle \nabla X_i, X_j \rangle \rho \]
\[ = X_i \langle X_j, \frac{\partial}{\partial \rho} \rangle - \langle \nabla X_i, X_j, \frac{\partial}{\partial \rho} \rangle \]
\[ = \langle X_j, \nabla X_i, \frac{\partial}{\partial \rho} \rangle \]
\[ = \langle X_j, \nabla \frac{\partial}{\partial \rho} X_i \rangle \]
\[ = \int_0^\rho \frac{d}{dt} \langle \tilde{X}_j, \nabla \frac{\partial}{\partial \gamma} \tilde{X}_i \rangle dt \]
\[ = \int_0^\rho (\langle \nabla \frac{\partial}{\partial \gamma} \tilde{X}_j, \nabla \frac{\partial}{\partial \gamma} \tilde{X}_i \rangle + \langle \tilde{X}_j, \nabla \frac{\partial}{\partial \gamma} \nabla \frac{\partial}{\partial \gamma} \tilde{X}_i \rangle) dt \]
\[
\int_0^\rho \left( \langle \nabla \frac{\partial}{\partial t} \tilde{X}_j, \nabla \frac{\partial}{\partial t} \tilde{X}_i \rangle - \langle \tilde{X}_j, R(\tilde{X}_i, \frac{\partial}{\partial t}) \frac{\partial}{\partial t} \rangle \right) dt
\]

Note that \( X_n = \frac{\partial}{\partial \rho} \). Then the fourth equality shows that

\[ \rho_{nj} = 0 \]

for \( j = 1, \ldots, n \).

Define the "index form" for a vector field \( \tilde{X}_i \) along \( \gamma \) as follows

\[ I_{\rho}^0(\tilde{X}_i) := \int_0^\rho \left( \| \nabla \frac{\partial}{\partial t} \tilde{X}_i \|^2 - \langle \tilde{X}_i, R(\tilde{X}_i, \frac{\partial}{\partial t}) \frac{\partial}{\partial t} \rangle \right) dt. \]

By this definition,

\[ \rho_{ii} = I_{\rho}^0(\tilde{X}_i). \]

Let \( E_i(\gamma(t)), 0 \leq t \leq \rho, i = 1, \ldots, n \), be the parallel transport of \( X_i(\gamma(\rho)) \) along \( \gamma \). Since Jacobi field minimizes the index form among all vector fields along the same geodesic with the same boundary values (see [2]), we have

\[ \rho_{ii} \leq I_{\rho}^0(\tilde{X}_i). \]

It follows that

\[ \Delta \rho = \sum_{i=1}^{n-1} \rho_{ii} \]

\[ = \frac{n-1}{\rho} - \frac{1}{\rho^2} \int_0^\rho t^2 \text{Ric}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) dt \]

\[ \leq \frac{n-1}{\rho} + \frac{1}{\rho^2} \int_0^\rho t^2(n-1)Q dt \]

\[ = \frac{n-1}{\rho} + \frac{(n-1)Q}{3} \rho. \]

On the other hand,

\[ -\rho_{ij} \nu^i \nu^j = - \int_0^\rho \langle \nabla \frac{\partial}{\partial t} \tilde{X}_j, \nabla \frac{\partial}{\partial t} \tilde{X}_i \rangle \nu^i \nu^j dt + \int_0^\rho \langle \nu^j \tilde{X}_j, R(\nu^i \tilde{X}_i, \frac{\partial}{\partial t}) \frac{\partial}{\partial t} \rangle dt. \]

The first term on the r.h.s. can be bounded as follows

\[ - \int_0^\rho \langle \nabla \frac{\partial}{\partial t} \tilde{X}_j, \nabla \frac{\partial}{\partial t} \tilde{X}_i \rangle \nu^i \nu^j dt \]

\[ \leq \int_0^\rho |\nabla \frac{\partial}{\partial t} \tilde{X}_j, \nabla \frac{\partial}{\partial t} \tilde{X}_i | dt \]

\[ \leq \int_0^\rho |\nabla \frac{\partial}{\partial t} \tilde{X}_j| \cdot |\nabla \frac{\partial}{\partial t} \tilde{X}_i| dt \]

\[ \leq \left( \int_0^\rho |\nabla \frac{\partial}{\partial t} \tilde{X}_j|^2 dt \right)^{\frac{1}{2}} \cdot \left( \int_0^\rho |\nabla \frac{\partial}{\partial t} \tilde{X}_i|^2 dt \right)^{\frac{1}{2}} \]
\[
\leq \frac{1}{2} \int_0^\rho |\nabla_\partial \tilde{X}_j|^2 dt + \frac{1}{2} \int_0^\rho |\nabla_\partial \tilde{X}_i|^2 dt
= (n-1) \int_0^\rho |\nabla_\partial \tilde{X}_j|^2 dt
= (n-1)(I_0^\rho (\tilde{X}_j) + \int_0^\rho \langle \tilde{X}_j, R(\tilde{X}_j, \frac{\partial}{\partial t}) \frac{\partial}{\partial t} \rangle dt)
\leq (n-1)(\frac{\rho}{\rho} E_j + \int_0^\rho \langle \tilde{X}_j, R(\tilde{X}_j, \frac{\partial}{\partial t}) \frac{\partial}{\partial t} \rangle dt)
\leq \frac{(n-1)^2}{\rho} + \frac{(n-1)^2 Q}{3} \rho + (n-1) \int_0^\rho \langle \tilde{X}_j, R(\tilde{X}_j, \frac{\partial}{\partial t}) \frac{\partial}{\partial t} \rangle dt.
\]

To go further, we need to estimate $|\tilde{X}_j|$. This is done in the next step.

**Step 4.** Generally, let $J = J(t)$ be a Jacobi field along $\gamma$ satisfying

$|J(0)| = 0, |J(\rho)| = 1$.

We compute

\[
\frac{d}{dt} |J(t)|^2 = 2\langle J, \nabla_\partial J \rangle,
\]

\[
\frac{d^2}{dt^2} |J(t)|^2 = 2|\nabla_\partial J|^2 + 2\langle J, \nabla_\partial \nabla_\partial J \rangle
\geq -2\langle J, R(J, \frac{\partial}{\partial t}) \frac{\partial}{\partial t} \rangle
\geq -2|J|^2 C_+,
\]
i.e.,

\[
\frac{d^2}{dt^2} |J(t)|^2 + 2C_+ |J(t)|^2 \geq 0.
\]

Define

\[
L := \frac{d^2}{dt^2} + 2C_+
\]

be an ordinary differential operator. Let

\[
w(t) = \cos \sqrt{2C_+} t.
\]

Then $w$ satisfies

\[
Lw = 0.
\]

Moreover, note that $0 \leq C_+ < \frac{\pi^2}{8\rho^2}$, so $w$ also satisfies

\[
w(t) > 0, 0 \leq t \leq \rho.
\]

By Theorem 2.11 in [6], we have

\[
\max_{0 \leq t \leq \rho} \{ |J(t)|^2 \over w(t) \} = \max \{ |J(0)|^2 \over w(0), |J(\rho)|^2 \over w(\rho) \}
= \max \{ 0, 1 \over w(\rho) \}
= 1 \over w(\rho).
\]
Hence
\[ |J(t)|^2 \leq \frac{w(t)}{w(\rho)} = \frac{\cos \sqrt{2C_+t}}{\cos \sqrt{2C_+\rho}} < \frac{1}{\cos \sqrt{2C_+r}}. \]

**Step 5.** Applying the results in **Step 4** to \( \tilde{X}_i \), we get
\[
-\rho_{ij} \nu^i \nu^j \leq \frac{(n-1)^2}{\rho} + \frac{(n-1)^2Q}{3} \rho + (n-1) \int_0^\rho |\tilde{X}_j|^2K(\tilde{X}_j, \partial_{\tilde{t}})dt
\]
\[
+ \int_0^\rho |\nu^i \tilde{X}_i|^2K(\nu^j \tilde{X}_j, \partial_{\tilde{t}})dt
\]
\[
\leq \frac{(n-1)^2}{\rho} + \frac{(n-1)^2Q}{3} \rho + n(n-1)C_+ \cos \sqrt{2C_+r} \rho.
\]

Therefore
\[
\triangle \rho - \rho_{ij} \nu^i \nu^j
\]
\[
\leq \frac{n-1}{\rho} + \frac{(n-1)Q}{3} \rho + \frac{(n-1)^2Q}{3} \rho + \frac{n(n-1)C_+}{\cos \sqrt{2C_+r}} \rho
\]
\[
(12) = \frac{n(n-1)}{\rho} + \frac{n(n-1)Q}{3} \rho + \frac{n(n-1)C_+}{\cos \sqrt{2C_+r}} \rho.
\]

Combining (11) and (12) yields
\[
g^{ij} \phi_{ij} \geq -4 - 2n(n-1) - \frac{2n(n-1)Q}{3} \rho^2 - \frac{2n(n-1)C_+}{\cos \sqrt{2C_+r}} \rho^2
\]
\[
\geq -4 - 2n(n-1) - \frac{2n(n-1)Q}{3} r^2 - \frac{2n(n-1)C_+}{\cos \sqrt{2C_+r}} r^2
\]
\[
(13) := -C_2.
\]

**Step 6.** When \( |\nabla u|(P) \geq 16u_0 \), we have
\[
|\nabla u|^2 + 4u_0 u_i \phi_i \geq |\nabla u|^2 - 8|\nabla u|u_0 \geq \frac{1}{2} |\nabla u|^2.
\]
Moreover, when \( |\nabla u|(P) \geq \max\{3,16u_0\} \), we have
\[
(14) \frac{|\nabla u|^2 + 4u_0 u_i \phi_i}{4u_0^2(1 + |\nabla u|^2)} \geq \frac{1}{8u_0} \cdot \frac{|\nabla u|^2}{1 + |\nabla u|^2} > \frac{1}{10u_0^2}.
\]

Note that
\[
(15) \frac{R_{k^i l^k \nu^j}}{v^2} \geq -\frac{(n-1)Q |\nu|^2}{v^2} \geq -(n-1)Q,
\]
and
\[
f' = C_1 \exp(C_1 \phi), f'' = C_1^2 \exp(C_1 \phi).
\]
Combining (10) and (13)-(15), we get
\[
(16) \frac{1}{10u_0^2} C_1^2 - C_2 C_1 - (n-1)Q \leq 0.
\]
It follows that for large $C_1$ (depending only on $n$, $Q$, $r$, $C_+$, $u_0$), (15) is contradicted if $|\nabla u|(P) \geq \max\{3,16u_0\} := C_3$. Therefore

$$|\nabla u|(P) \leq C_3.$$ 

$$v(P) \leq 1 + C_3 := C_4.$$ 

$$\eta(x,u(x))v(x) \leq \eta(P,u(P))v(P) \leq C_4 \exp(C_1r^2).$$

At point $p$,

$$\left(\exp(r^2 - \frac{1}{2}) - 1\right)v(p) \leq C_4 \exp(C_1r^2).$$

For large $u_0$ it is easy to see that $C_1$ may be taken to be a multiple of $u_0^2$, so that the interior gradient bound has the form

$$v(p) \leq K_2 \exp(K_1u_0^2),$$

where $K_i, i = 1, 2,$ depend only on $n$, $Q$, $C_+$, $r$.

This completes the proof of Theorem 2.

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