SMALL PROBABILITY EVENTS FOR TWO-LAYER GEOPHYSICAL FLOWS UNDER UNCERTAINTY

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Abstract. The stochastics two-layer quasi-geostrophic flow model is an intermediate system between the single-layer two dimensional barotropic flow model and the continuously stratified three dimensional baroclinic flow model. This model is widely used to investigate basic mechanisms in geophysical flows, such as baroclinic effects, the Gulf Stream and subtropical gyres.

A large deviation principle for the two-layer quasi-geostrophic flow model under uncertainty is proved. The proof is based on the Laplace principle and a variational approach. This approach does not require the exponential tightness estimates which are needed in other methods for establishing large deviation principles.

1. Introduction

The continuously stratified, three dimensional (3D) baroclinic quasi-geostrophic flow model describes large scale geophysical fluid motions in the atmosphere and oceans. This model is much simpler than the primitive flow model or the rotating Navier-Stokes flow model. When the fluid density is approximately constant, this model reduces to the barotropic, single-layer, two dimensional (2D) quasi-geostrophic model. The two-layer quasi-geostrophic flow model, in which the fluid consists of two homogeneous fluid layers of uniform but distinct densities $\rho_1$ and $\rho_2$, is an intermediate system between the single-layer 2D barotropic flow model and the continuously stratified, 3D baroclinic flow model.

The two-layer quasi-geostrophic flow model has been used as a theoretical and numerical model to understand basic mechanisms in large scale geophysical flows, such as baroclinic effects [36], wind-driven circulation [6, 5], the Gulf Stream [29], fluid stability [4] and subtropical gyres [34]. Recently Salmon [45] introduced a generalized two-layer ocean flow model.

We prove a large deviation principle for this stochastic infinite dimensional system, by a recent weak convergence approach, based on a variational representation for functionals of infinite dimensional Brownian motion [2, 3]. In this approach, the large deviations for SPDEs are derived by showing some qualitative properties (well-posedness,
compactness and weak convergence) of certain perturbations of the original SPDEs. This method has been recently applied in several papers on SPDEs [44, 49, 21] or SDEs in infinite dimensions [39].

More information about this weak convergence approach for large deviations in the finite dimensional setting can be found in the book [22]. It is different from other existing approaches, which usually require extra exponential tightness estimates, for establishing large deviation principles for SPDEs [12, 13, 14, 15, 27, 30, 38, 43, 50]. An alternative approach [25] for large deviations is based on nonlinear semi-group theory and infinite dimensional Hamilton-Jacobi equations; it also requires to establish exponential tightness.

This paper is organized as follows. The mathematical formulation for the stochastic two-layer geophysical flow model is in the next section. Then the well-posedness for the model is discussed in §3. Finally, a large deviation principle is shown in §4.

2. Mathematical setup

We consider the two-layer quasi-geostrophic flow model ([36], p. 423; [45], p.87):

\[
\begin{align*}
\frac{\partial q_1}{\partial t} + J(\psi_1, q_1 + \beta y) &= \nu \Delta^2 \psi_1 + f + \sqrt{\epsilon} \sigma_1(q_1, q_2) \dot{W}^1, \\
\frac{\partial q_2}{\partial t} + J(\psi_2, q_2 + \beta y) &= \nu \Delta^2 \psi_2 - r \Delta \psi_2 + \sqrt{\epsilon} \sigma_2(q_1, q_2) \dot{W}^2,
\end{align*}
\]

(2.1)

with boundary condition

\[\psi_1 = \psi_2 = 0, \quad q_1 = q_2 = 0,\]

where potential vorticities \(q_1(x, y, t), q_2(x, y, t)\) for the top layer and the bottom layer are defined via stream functions \(\psi_1(x, y, t), \psi_2(x, y, t)\), respectively,

\[
\begin{align*}
q_1 &= \Delta \psi_1 - F_1 \cdot (\psi_1 - \psi_2), \\
q_2 &= \Delta \psi_2 - F_2 \cdot (\psi_2 - \psi_1).
\end{align*}
\]

(2.2)

Remark 2.1. The boundary conditions \(\psi_1 = \psi_2 = 0, q_1 = q_2 = 0\) give: \(\Delta \psi_1 = \Delta \psi_2 = 0\) on the boundary.

Here \(x, y\) are Cartesian coordinates in zonal (east), meridional (north) directions, respectively; \((x, y) \in D := (0, L) \times (0, L)\), where \(L\) is a positive number; \(F_1, F_2\) are positive constants defined by

\[
\begin{align*}
F_1 &= \frac{f_0^2}{g h_1 \rho_2 - \rho_1}, \\
F_2 &= \frac{f_0^2}{g h_2 \rho_2 - \rho_1},
\end{align*}
\]

with \(g\) the gravitational acceleration; \(h_1, h_2\) the depth of top and bottom layers, \(\rho_1, \rho_2\) the densities \((\rho_2 > \rho_1)\) of top and bottom layers, respectively; and \(L, \rho_0\) the
characteristic scales for horizontal length and density of the flows, respectively; \( f_0 + \beta y \) (with \( f_0, \beta \) constants) is the Coriolis parameter and \( \beta \) is the meridional gradient of the Coriolis parameter; \( \nu > 0 \) is the viscosity. Note that \( r = f_0 \frac{\delta E}{2(h_1 + h_2)} \) is the Ekman constant which measures the intensity of friction at the bottom boundary layer (the so-called Ekman layer) or the rate for vorticity decay due to the friction in the Ekman layer. Here \( \delta E = \sqrt{2\nu/f_0} \) is the Ekman layer thickness ([36], p.188).

Moreover, \( J(h,g) = h_x g_y - h_y g_x \) is the Jacobi operator and \( \Delta = \partial_{xx} + \partial_{yy} \) is the Laplace operator.

Now, we set \( W = \begin{pmatrix} W^1 \\ W^2 \end{pmatrix} \) and \( \dot{W} = \begin{pmatrix} \dot{W}^1 \\ \dot{W}^2 \end{pmatrix} \). An important part of the above flow model (2.1) is the white noise term \( \dot{W} \), which is the generalized derivative of a Wiener process \( W(t) \) with respect to time \( t \), in an appropriate function space to be specified below. This white noise term \( \left( \sqrt{\varepsilon \sigma_1} \dot{W}^1 \sqrt{\varepsilon \sigma_2} \dot{W}^2 \right) \), with \( \varepsilon > 0 \) a small parameter and \( \sigma_1 \) noise intensity, describes the fluctuating part of the external wind forcing in both of the fluid layers; see Arnold [1]. The fluctuating part is usually of a shorter time scale than the response time scale of the large scale quasi-geostrophic flows. So we neglect the autocorrelation time of this fluctuating process. We thus assume the noise is white in time but it is allowed to be colored in space, i.e., it may be correlated in space variables \( x \) and \( y \). The Wiener process (also a Gaussian process) \( W(t) \) has zero mean and is characterized by its covariance operator \( Q \). There has been some analysis on wind stress curl data from the National Aeronautics and Space Administration Scatterometer (NSCAT) and from the National Center for Environmental Prediction (NCEP); see, for example, [32, 10]. Such data analysis also involves estimating the covariance and its trace, and the trace is usually taken to be finite. In this paper, we consider the case when the covariance operator \( Q \) of the Wiener process has a finite trace.

In the following, \( L^2(D), V = H^1_0(D) \) denote the standard scalar Lebesgue and Sobolev spaces. Let \( \mathcal{D}(A) = H^2(D) \cap H^1_0(D) \), where \( H = L^2(D) \times L^2(D) \) and \( H^1_0(D) = H^1_0(D) \times H^1_0(D) \) are product vector spaces. The scalar product and the induced norm in \( L^2(D) \) or \( H \) are denoted as \( (\cdot, \cdot) \) and \( \| \cdot \| \), respectively.

\( W(t) \) is a Wiener processes defined on a filtered probability space \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \), taking values in \( H \). We denote \( Q \) as its associated covariance operator, it is a linear symmetric positive covariant operator in the Hilbert space \( H \). We assume that \( Q \) is trace class, i.e., \( \text{tr}(Q) < \infty \).

As in [17] and [44], let \( \mathbb{H}_0 = Q^{\frac{1}{2}} \mathbb{H} \). Then \( \mathbb{H}_0 \) is a Hilbert space with the scalar product

\[
(q, \psi)_0 = (Q^{-\frac{1}{2}} q, Q^{-\frac{1}{2}} \psi), \quad \forall q, \psi \in \mathbb{H}_0,
\]

together with the induced norm \( | \cdot |_0 = \sqrt{(\cdot, \cdot)_0} \). The embedding \( i : \mathbb{H}_0 \to \mathbb{H} \) is Hilbert-Schmidt and hence compact, and moreover, \( i i^* = Q \).
Let $L_Q$ be the space of linear operators $S$ such that $SQ^{1/2}$ is a Hilbert-Schmidt operator (and thus a compact operator) from $H$ to $H$. The norm in the space $L_Q$ is $\|S\|_{L_Q} = tr(SQ^*)$, where $S^*$ is the adjoint operator of $S$.

With these notations, the above two-layer system can be rewritten as:

$$\frac{\partial q}{\partial t} = [Aq + F(q)] + \sqrt{\epsilon}\sigma(q)\dot{W},$$

$$q(0) = \xi,$$

or

$$dq = [Aq + F(q)]dt + \sqrt{\epsilon}\sigma(q)dW,$$

$$q(0) = \xi,$$

where

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad \sigma(q)\dot{W} = \begin{pmatrix} \sigma_1(q_1, q_2)\dot{W}_1 \\ \sigma_2(q_1, q_2)\dot{W}_2 \end{pmatrix},$$

$$Aq = \begin{pmatrix} -\nu\Delta^2\psi_1 \\ -\nu\Delta^2\psi_2 \end{pmatrix},$$

$$F(q) = \begin{pmatrix} -J(\psi_1, q_1 + \beta y) + f \\ -J(\psi_2, q_2 + \beta y) - r\Delta\psi_2 \end{pmatrix}.$$ 

Here $q_1, q_2$ and $\psi_1, \psi_2$ are defined in [2.2].

In order to obtain the weak solution, the noise intensity $\sigma : H \to L_Q(H_0; H)$ is assumed to satisfy the following conditions.

**Assumption A:**

A.1. The noise intensity $\sigma(\cdot) : H \to L_Q(H_0; H)$ is continuous.

A.2. $\|\sigma(q)\|_{L_Q(H_0; H)}^2 \leq K(1 + \|q\|_H^2)$, for some positive constant $K$.

A.3. There exists a constant $L$ such that for all $q, \psi \in H$, we have $\|\sigma(q) - \sigma(\hat{q})\|_{L_Q(H_0; H)}^2 \leq L\|q - \hat{q}\|_H^2$.

For strong solution, the noise intensity $\sigma$ is assumed to satisfy the following additional conditions.

**Assumption A’:**

A’.1. The noise intensity $\sigma(\cdot) : H \to L_Q(H_0; H^1)$ is continuous.

A’.2. $\|\sigma(q)\|_{L_Q(H_0; H^1)}^2 \leq K'(1 + \|q\|_{H_0^1}^2)$, for some positive constant $K'$. 

3. Well-posedness

3.1. Well-posedness of two-layer system. To treat the nonlinearity in the two-layer fluid model we need the following lemmas:

Lemma 3.1. The Jacobian operator has the following properties:

\[ J(u, v) = -J(v, u), \quad (J(u, v), v) = 0, \]
\[ (J(u, v), w) = (J(v, w), u), \]

for \( u, v, w \) in \( H^1 \). Moreover the following estimates hold:

\[ |(J(u, v), \Delta u)| \leq c_0 \| \Delta v \| \cdot \| \nabla u \| \cdot \| \Delta u \|, \quad u, v \in H^2; \]
\[ |(J(u, v), w)| \leq c_1 \| \Delta u \| \cdot \| \Delta v \| \cdot \| w \|, \quad u, v \in H^2, w \in L^2; \]
\[ |(J(u, v), w)| \leq c_1 \| \nabla u \| \cdot \| \Delta v \| \cdot \| \nabla w \|, \quad u, w \in H^1, v \in H^2. \]

The detailed proof of the above lemma can be found in [19].

Lemma 3.2. Let \( F_1 \) and \( F_2 \) be positive constants, and assume that \( q_i, \psi_i, (i = 1, 2) \) satisfy

\[ q_1 = \Delta \psi_1 - F_1 \cdot (\psi_1 - \psi_2), \]
\[ q_2 = \Delta \psi_2 - F_2 \cdot (\psi_2 - \psi_1). \]

Then we have

\[ \| \psi_1 \|^2_{H^2} + \| \psi_2 \|^2_{H^2} \leq C (\| q_1 \|^2 + \| q_2 \|^2). \]

Proof. On any finite interval \( t \in [0, T] \), we take inner product on both sides of the equations with \( \Delta \psi_1 \) and \( \Delta \psi_2 \) respectively:

\[ \int_D q_1 \Delta \psi_1 dx = \int_D |\Delta \psi_1|^2 dx + F_1 \int_D |\nabla \psi_1|^2 dx - F_1 \int_D \nabla \psi_2 \nabla \psi_1 dx, \]
\[ \int_D q_2 \Delta \psi_2 dx = \int_D |\Delta \psi_2|^2 dx + F_2 \int_D |\nabla \psi_2|^2 dx - F_2 \int_D \nabla \psi_2 \nabla \psi_1 dx. \]

Multiply both sides of first equation by \( F_2 \), second equation by \( F_1 \), then add them together:

\[ F_2 \int_D |\Delta \psi_1|^2 dx + F_1 \int_D |\Delta \psi_2|^2 dx \]
\[ + \quad F_1 F_2 \int_D (|\nabla \psi_2|^2 + |\nabla \psi_1|^2 - 2 \nabla \psi_2 \nabla \psi_1) dx \]
\[ = \quad F_2 \int_D q_1 \Delta \psi_1 dx + F_1 \int_D \Delta \psi_2 q_2 dx. \]
Noticing that \( F_1 F_2 \int_{D} (|\nabla \psi_2|^2 + |\nabla \psi_1|^2 - 2 \nabla \psi_2 \nabla \psi_1) \, dx \) is always non-negative, we have:

\[
F_2 \int_{D} |\Delta \psi_1|^2 \, dx + F_1 \int_{D} |\Delta \psi_2|^2 \, dx \\
\leq F_2 \int_{D} q_1 |\Delta \psi_1| \, dx + F_1 \int_{D} |\Delta \psi_2| q_2 \, dx \\
\leq F_2 \int_{D} |q_1| |\Delta \psi_1| \, dx + F_1 \int_{D} |\Delta \psi_2| |q_2| \, dx
\]

By the Young’s Inequality:

\[
F_2 \int_{D} |\Delta \psi_1|^2 \, dx + F_1 \int_{D} |\Delta \psi_2|^2 \, dx \\
\leq \frac{1}{2} F_2 \int_{D} q_1^2 \, dx + \frac{1}{2} F_2 \int_{D} |\Delta \psi_1|^2 \, dx + \frac{1}{2} F_1 \int_{D} |q_2|^2 \, dx + \frac{1}{2} F_1 \int_{D} |\Delta \psi_2|^2 \, dx.
\]

Let \( C = \max\{F_1, F_2\} \), we get:

\[
\int_{D} |\Delta \psi_1|^2 \, dx + \int_{D} |\Delta \psi_2|^2 \, dx \leq C(\int_{D} |q_1|^2 \, dx + \int_{D} |q_2|^2 \, dx),
\]

which is simply

\[
(3.5) \quad \|\Delta \psi_1\|^2_{L^2} + \|\Delta \psi_2\|^2_{L^2} \leq C(\|q_1\|^2_{L^2} + \|q_2\|^2_{L^2}).
\]

Since \( D = (0, L) \times (0, L) \), by the Poincaré’s inequality and the inequality (6.4) in [51], we know that \( \|\psi_i\|_{H^2} \) can be upper bounded by \( \|\Delta \psi_i\|_{L^2} \) multiplying by a positive constant. Thus there exists a positive constant \( C \), such that

\[
(3.6) \quad \|\psi_1\|^2_{H^2} + \|\psi_2\|^2_{H^2} \leq C(\|q_1\|^2_{L^2} + \|q_2\|^2_{L^2}).
\]

In the proof of well-posedness of the system, we need the following random version of the Gronwall’s inequality, which is Lemma 3.9 in [21] with minor modification.

**Lemma 3.3.** ([21]) Let \( X, Y \) and \( I \) be non-decreasing, non-negative processes, \( \varphi \) be a non-negative processes and \( Z \) be a non-negative integrable random variable. Assume that \( \int_{0}^{T} \varphi(r) \, dr \leq M \) almost surely and there exist positive constants \( \alpha \) and \( \beta \leq \frac{1}{2(1 + M^{N})} \), \( \tilde{C} > 0 \) and \( \bar{C} > 0 \) (depending on \( M \)) such that

\[
X(t) + \alpha Y(t) \leq Z + \int_{0}^{t} \varphi(r) X(r) \, dr + I(t), \quad \text{a.s.}
\]

\[
\mathbb{E}(I(t)) \leq \beta \mathbb{E}(X(t)) + \tilde{C} \int_{0}^{t} \mathbb{E}(X(r)) \, dr + \bar{C}.
\]

Then if \( X \in L^\infty([0, T] \times \Omega) \), we have for \( t \in [0, T] \),

\[
\mathbb{E}X(t) + \alpha \mathbb{E}Y(t) \leq \bar{C}(1 + \mathbb{E}(Z)).
\]
3.2. Well-posedness of perturbed two-layer system. The solution for the stochastic two-layer geophysical flow problem under random influences is denoted as \( q^\varepsilon \), although often we omit the \( \varepsilon \) here. The goal for this paper is to show the large deviation principle (or equivalently, the Laplace principle) for the family \( q^\varepsilon \).

Let \( \mathcal{A} \) be the class of \( \mathbb{H}_0 \)-valued \((\mathcal{F}_t)\)-predictable stochastic processes \( q \) with the property \( \int_0^T \|q(s)\|^2 \, ds < \infty, \) a.s. Let

\[
S_M = \left\{ h \in \mathcal{A} \text{ and } h \in L^2(0, T; H_0) : \int_0^T \|h(s)\|^2 \, ds \leq M \right\}.
\]

The set \( S_M \) endowed with the following weak topology is a Polish space (complete separable metric space) \([3]\):

\[
d_{\mathbb{H}}: d_1(h, k) = \sum_{i=1}^\infty \frac{1}{2^i} \left( \int_0^T |h(s) - \tilde{e}_i(s)\rangle s - k(s), \tilde{e}_i(s)\rangle_0 ds \right),
\]

where \( \{\tilde{e}_i(s)\}_{i=1}^\infty \) is a complete orthonormal basis for \( L^2(0, T; \mathbb{H}_0) \). Define

\[
(3.7) \quad \mathcal{A}_M = \{ q \in \mathcal{A} : q(\omega) \in S_M, \text{a.s.} \}.
\]

As in \([44]\), we prove existence and uniqueness of the solution to the stochastic two-layer geophysical flow equation. However, in the sequel, we will need some precise bounds on the norm of the solution to a more general equation, which contains an extra forcing term driven by an element of \( \mathcal{A}_M \). More precisely, let \( h \in \mathcal{A} \) and consider the following generalized two-layer system with initial condition \( q_h(0) = \xi \),

\[
(3.8) \quad dq_h(t) + [Aq_h(t) + F(q_h(t))] \, dt = \sigma(q_h(t)) \, dW(t) + \tilde{\sigma}(q_h(t)) h(t) \, dt.
\]

**Remark 3.4.** The noise intensity \( \tilde{\sigma} : \mathbb{H} \to L_Q(\mathbb{H}_0, \mathbb{H}) \) is assumed to satisfy the same conditions as \( \sigma \) (Assumption A).

**Definition 3.5. (Weak solution)**

Recall that a stochastic process \( q_h(t, \omega) \) is called the weak solution for the generalized stochastic two-layer quasi-geostrophic flow problem \([3.8]\) on \([0, T]\) with initial condition \( \xi \) if \( q_h \) is in \( C([0, T]; \mathbb{H}) \cap L^2((0, T); \mathbb{H}_0^1) \), a.s., and satisfies

\[
(3.9) \quad (q_h(t), \psi) - (\xi, \psi) + \int_0^t [(q_h(s), A\psi) + (F(q_h(s)), \psi)] \, ds
\]

for all \( \psi \in \mathcal{D}(A) \) and all \( t \in [0, T] \).

In the following, we work in the Banach space \( X := C([0, T]; \mathbb{H}) \cap L^2((0, T); \mathbb{H}_0^1) \) with the norm

\[
(3.10) \quad \|q\|_X = \left\{ \sup_{0 \leq s \leq T} \|q(s)\|^2 + \int_0^T \sup_{0 \leq \tau \leq s} \|\nabla q(\tau)\|^2 \, ds \right\}^{\frac{1}{2}}.
\]

**Definition 3.6. (Mild solution and strong solution)**

Given \( \xi \in \mathbb{H} \), mild solution to the stochastic evolutionary equation \([3.8]\) on the stochastic space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) is a \( \mathcal{F}_t \)-adapted process \( q_h(t) \in C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathbb{H}_0^1) \),
satisfying the following integral form:

\[ q_h(t) = S(t)\xi + \int_0^t S(t-s)\hat{\sigma}(q_h(s)) h(s) - F(q_h(s)) \, ds + \int_0^t S(t-s)\sigma(q_h(s)) \, dW(s), \]

where \( S(t) \) is the semigroup generated by the linear (unbounded) operator \( A \). Moreover, \( q_h \) is called a strong solution if \( u \in C([0,T]; H^1_0) \cap L^2(0,T; D(A)) \) for \( q_h(0) = \xi \in H^1_0 \).

**Theorem 3.7.** (Well-posedness)
If Assumptions \((A1, A.2, A.3)\) hold and the initial datum satisfies that \( E\|\xi\|^2 < \infty \), then the mild solution to equation (3.8) is unique. Moreover, for any \( h \in \mathcal{A}_M \), there exists a pathwise unique weak solution \( q_h \) of the generalized stochastic two-layer quasi-geostrophic flow problem (3.8) with initial condition \( q_h(0) = \xi \in H^1 \) such that \( q_h \in X \) a.s. There also exists a constant \( C_1 = C_1(T, M, K, F_1, F_2, \nu, r) \) such that

\[ E\left( \sup_{0 \leq t \leq T} \|q_h(t)\|^2 + \int_0^T \sup_{0 \leq \tau \leq s} \|\nabla q_h(\tau)\|^2 \, d\tau \right) \leq C_1 \left( 1 + E\|\xi\|^2 \right). \]

Furthermore, if the initial condition \( q_h(0) = \xi \in H^1 \) and the additional Assumptions \((A'1, A'.2)\) hold, then the solution is strong and unique.

**Remark 3.8.** Note that when \( h \equiv 0 \) and \( \sigma \) is multiplied by \( \sqrt{\varepsilon} \) with any positive constant \( \varepsilon \), we deduce that the stochastic two-layer geophysical flow equation has a unique weak solution. Note that here \( \varepsilon \) does not have to be small. On the other hand, if the covariance operator \( Q \equiv 0 \), i.e., when the Gaussian process \( W \) vanishes, we also deduce the existence and uniqueness of the solution to the deterministic control equation defined in terms of an element \( h \in L^2((0,T); H) \) and an initial condition \( \xi \in H^1 \)

\[ dq(t) + [Aq(t) + F(q(t))] dt = \sigma(q(t))h(t) dt, \quad q(0) = \xi. \]

If \( h \in S_M \), the solution \( q \) to (3.12) satisfies

\[ \sup_{0 \leq t \leq T} \|q(s)\|^2 + \int_0^T \sup_{0 \leq \tau \leq s} \|\nabla q(\tau)\|^2 \, d\tau \leq C_2(F_1, F_2, \nu, r, K, T, M, E\|\xi\|). \]

**Proof.** We refer to [17] and [21] for the existence of the mild solution. Here we only give a priori estimates for the solutions of (3.8) to guarantee the existence of the strong solutions. For simplicity, we suppress the \( h \)-dependence in \( q \)'s and we also omit the subscript \( L^2 \) in various norms below.

We define \( \|q\|_{L^2} = \|q_1\|^2_{L^2} + \|q_2\|^2_{L^2} \). The Ito formula for \( \|q_1\|^2_{L^2} \) and using Assumption
A gives:

\[ \|q_1\|^2 = \|q_1(0)\|^2 + 2\nu \int_0^t (\Delta^2 \psi_1, q_1) ds \]
\[ -2 \int_0^t (J(\psi_1, q_1 + \beta y), q_1) ds + \int_0^t (f, q_1) ds + \int_0^t (\tilde{\sigma}(q_1, q_2) h_1, q_1) ds \]
\[ + 2 \int_0^t \|\sigma_1(q_1, q_2)\|^2_{L^2(H_0; H)} ds + 2 \int_0^t (\sigma_1(q_1, q_2) dW(s), q_1) ds \]
\[ \leq \|q_1(0)\|^2 - \nu \int_0^t \|\nabla q_1\|^2 ds + \int_0^t \|f\|^2 ds \]
\[ + C \int_0^t (1 + \|h_1\|)(\|q_1\|^2 + \|q_2\|^2) ds + |M_1(\tau)|, \]

where

\[ M_1(\tau) = 2 \int_0^\tau (\sigma_1(q_1, q_2) dW(s), q_1) ds. \]

Therefore

\[ \|q_1\|^2 + \nu \int_0^t \|\nabla q_1(\tau)\|^2 d\tau \]
\[ \leq \|q_1(0)\|^2 + \int_0^t \|f\|^2 d\tau \]
\[ + C \int_0^t (1 + \|h_1\|)(\|q_1(\tau)\|^2 + \|q_2(\tau)\|^2) d\tau + |M_1(\tau)|. \]

Similarly,

\[ \|q_2\|^2 + \nu \int_0^t \|\nabla q_2(\tau)\|^2 d\tau \]
\[ \leq \|q_2(0)\|^2 \]
\[ + C \int_0^t (1 + \|h_2\|)(\|q_1(\tau)\|^2 + \|q_2(\tau)\|^2) d\tau + |M_2(\tau)|. \]

where

\[ M_2(\tau) = 2 \int_0^\tau (\sigma_2(q_1, q_2) dW(s), q_2) ds. \]

Adding (3.14) and (3.15) and taking sup,

\[ \sup_{0 \leq \tau \leq t} \|q(\tau)\|^2 + \nu \int_0^t \sup_{0 \leq \tau \leq s} \|\nabla q(\tau)\|^2 ds \]
\[ \leq \|q(0)\|^2 + \int_0^t \|f\|^2 ds \]
\[ + C \int_0^t (1 + \|h\|) \sup_{0 \leq \tau \leq s} \|q(\tau)\|^2 ds + \sup_{0 \leq \tau \leq t} |\tilde{M}(\tau)|, \]

(3.16)
where \( \hat{M} = M_1 + M_2 \). By the Burkholder-Davis-Gaudy inequality,

\[
\mathbb{E} \sup_{0 \leq \tau \leq t} |\hat{M}(\tau)| \leq \mathbb{E} \left( \int_0^t \|q(t)\|^2 \|\sigma(q)\|_{L^2(H_0;H)}^2 ds \right)^{1/2} \\
\leq \mu \mathbb{E} \sup_{0 \leq \tau \leq t} \|q(t)\|^2 + C \int_0^t \left( 1 + \mathbb{E} \sup_{0 \leq \tau \leq s} \|q(\tau)\|^2 \right) ds.
\]

(3.17)

Noticing that \( \|h\|_{L^2(0,T;\mathbb{H})}^2 \leq M \), taking \( \mu \) depending on \( M \) small enough such that the condition of Lemma 3.3 is satisfied and using Lemma 3.3, there exists a constant \( C_1 = C_1(T, M, K, F_1, F_2, \nu, r) \) such that

\[
\mathbb{E} \left( \sup_{0 \leq \tau \leq t} \|q(\tau)\|^2 + \nu \int_0^t \sup_{0 \leq \tau \leq s} \|\nabla q(\tau)\|^2 ds \right) \leq C_1 \left( 1 + \mathbb{E} \|\xi\|^2 \right), \text{ for all } 0 \leq t \leq T.
\]

Next, we derive the estimates of \( \mathbb{E} \sup_{0 \leq \tau \leq t} \|\nabla q_1(\tau)\|^2 \) and \( \mathbb{E} \sup_{0 \leq \tau \leq t} \|\nabla q_1(\tau)\|^2 \). Ito formula for \( \|\nabla q_2\|^2 \) gives:

\[
\|\nabla q_1(t)\|^2 = \|\nabla q_1(0)\|^2 + 2\nu \int_0^t (\nabla^2 \psi_1, \nabla q_1) ds + 2 \int_0^t (f, \nabla q_1) ds \\
- 2 \int_0^t ((J(\psi_1, q_1 + \beta y), \nabla q_1) ds + 2 \int_0^t (\sigma_1(q_1, q_2) h_1, \nabla q_1) ds \\
+ 2 \int_0^t \|\nabla \sigma_1(q_1, q_2)\|^2 \|L^2(H_0;H) ds + 2 \int_0^t (\sigma_1(q_1, q_2) dW(s), \nabla q_1) ds.
\]

Since

\[
J(\psi_1, q_1 + \beta y) = J(\psi_1, q_1) + \beta \frac{\partial \psi_1}{\partial x},
\]

the Cauchy-Schwartz inequality, the Young’s inequality and \( \|\nabla \psi_1\|_{L^\infty} \leq C \|\nabla \psi_1\| \) imply:

\[
2|(J(\psi_1, q_1 + \beta y), \nabla q_1)| \leq 2|(J(\psi_1, q_1), \nabla q_1)| + 2\beta \left| \frac{\partial \psi_1}{\partial x}, \nabla q_1 \right| \\
\leq C_1 \|\nabla \psi_1\| \|\nabla q_1\| \|\Delta q_1\| + 2\beta \left| \frac{\partial \psi_1}{\partial x}, \Delta q_1 \right| \\
\leq \frac{\nu}{4} \|\Delta q_1\|^2 + C_2 \|\nabla \psi_1\|^2 \|\nabla q_1\|^2 + C_3 \left| \frac{\partial \psi_1}{\partial x} \right|^2 \\
= C_2 \|\nabla (q_1 + F_1(\nabla \psi_1 - \nabla \psi_2))\|^2 \|\nabla q_1\|^2 \\
+ C_3 \left| \frac{\partial \psi_1}{\partial x} \right|^2 + \frac{\nu}{4} \|\Delta q_1\|^2 \\
\leq C_4 (\|\nabla q_1\|^4 + \|\nabla q_2\|^4) + \frac{\nu}{4} \|\Delta q_1\|^2.
\]
Moreover, using Assumption we have:

\[ 2\|\nabla \sigma_1(q_1, q_2)\|^2_{L^2(H_0; H)} = 2\|\sigma_1(q_1, q_2)\|^2_{L^2(H_0; V)} \leq 2K'(1 + \|\nabla q_1\|^2 + \|\nabla q_2\|^2), \]

\[ (f, \triangle q_1) \leq \frac{\nu}{4}\|\triangle q_1\|^2 + \frac{1}{\nu}\|f\|^2, \]

\[ |(\tilde{\sigma}(q_1, q_2) h_1, \triangle q_1)| \leq \frac{\nu}{4}\|\triangle q_1\|^2 + \frac{1}{\nu}\|\tilde{\sigma}(q_1, q_2) h_1\|^2 \]

\[ \leq \frac{\nu}{4}\|\triangle q_1\|^2 + C(1 + \|q_1\|^2 + \|q_2\|^2)\|h_1\|^2. \]

Using the above estimates, Poincare inequality and Lemma 3.2, we get:

\[ \|\nabla q_1(t)\|^2 + 2\nu \int_0^t \|\triangle q_1(s)\|^2 ds \]

\[ \leq C + \|\nabla q_1(0)\|^2 + c \int_0^t \|\nabla q\|^4 ds \]

\[ + C \int_0^t \|\nabla q(\tau)\|^2 \|h_1\|^2 ds + 2 \int_0^t (\triangle q_1, \sigma_1(q_1, q_2))dW(s) \]

\[ \leq C + \|\nabla q_1(0)\|^2 + c \int_0^t \|\nabla q\|^4 ds \]

\[ (3.19) + C \int_0^t \|\nabla q(\tau)\|^2 \|h_1\|^2 ds + 2 \sup_{0 \leq \tau \leq t} |M_3(\tau)|, \]

where

\[ M_3(\tau) = \int_0^\tau (\triangle q_1, \sigma_1(q_1, q_2)dW(s)). \]

Similarly, we have:

\[ \|\nabla q_2(\tau)\|^2 + 2\nu \|\triangle q_2(\tau)\|^2 ds \]

\[ \leq C + \|\nabla q_2(0)\|^2 + c \int_0^t \|\nabla q\|^4 ds \]

\[ (3.20) + C \int_0^t \|\nabla q(\tau)\|^2 \|h_1\|^2 ds + 2 \sup_{0 \leq \tau \leq t} |M_4(\tau)|, \]

where

\[ M_4(\tau) = \int_0^\tau (\triangle q_2, \sigma_1(q_1, q_2)dW(s)). \]

Adding (3.19) and (3.21), we have

\[ \|\nabla q(\tau)\|^2 + 2\nu \|\triangle q(\tau)\|^2 ds \leq C + \|\nabla \xi\|^2 \]

\[ (3.21) + c \int_0^t (\|h_1\|^2 + \|\nabla q(\tau)\|^2)\|\nabla q(\tau)\|^2 ds + 2 \sup_{0 \leq \tau \leq t} |\tilde{M}(\tau)|, \]
where
\[ \tilde{M}(\tau) = M_3(\tau) + M_4(\tau). \]

By the Burkholder-Davis-Gaudy inequality and (3.18):
\[
E \sup_{0 \leq \tau \leq t} |\tilde{M}(\tau)| \leq K' E\left( \int_0^t \|\nabla q\|^2 (1 + \|\nabla q\|^2) ds \right)^{1/2}
\leq \mu_1 E \sup_{0 \leq \tau \leq t} \|\nabla q_1\|^2 + C \int_0^t (1 + E \sup_{0 \leq \tau \leq s} \|\nabla q\|^2) ds.
\]
Taking \( \mu \) (depending on \( M \) and the bound in (3.18)) small enough such that the condition of Lemma 3.3 is satisfied and using Lemma 3.3, there exists a constant \( C_2 = C_2(T, M, K, K', F_1, F_2, \nu, r) \) such that
\[
E \sup_{0 \leq \tau \leq t} \|\nabla q(\tau)\|^2 + 2\nu \int_0^t E \sup_{0 \leq \tau \leq s} \|\Delta q(\tau)\|^2 ds \leq C_2 (E\|\nabla \xi\|^2 + 1).
\]
Thus the mild solution, is also strong solution, when \( q(0) = \xi \in \mathbb{H}_0^1 \).

The proof of uniqueness is standard, we omit the proof here.

4. Large deviations

4.1. Definition. We consider large deviations via a weak convergence approach [2, 3], based on variational representations of infinite dimensional Wiener processes. In this approach, the large deviations for SPDEs are derived by showing some qualitative properties (well-posedness, compactness and weak convergence) of certain perturbations of the original SPDEs [44, 49, 21]. More information about this weak convergence approach for large deviations in the finite dimensional setting can be found in the book [22]. It is different from other existing approaches, which usually require extra exponential tightness estimates, for establishing large deviation principles for SPDEs [12, 13, 15, 14, 16, 27, 30, 38, 43, 50]. An alternative approach [25] for large deviations is based on nonlinear semi-group theory and infinite dimensional Hamilton-Jacobi equations; it also requires to establish exponential tightness.

We rewrite the stochastic two-layer model to indicate its dependence on the small parameter \( \varepsilon \):
\[
dq^\varepsilon(t) + [Aq^\varepsilon(t) + F(q^\varepsilon(t))] dt = \sqrt{\varepsilon} \sigma(q^\varepsilon(t)) dW(t), \quad q^\varepsilon(0) = \xi.
\]
The solution is denoted as \( q^\varepsilon = G^\varepsilon(\sqrt{\varepsilon} W) \) for a Borel measurable function \( G^\varepsilon : C([0, T]; \mathbb{H}) \to X \). We show a large deviation principle for \( q^\varepsilon \).

The space \( X = C([0, T]; \mathbb{H}) \cap L^2((0, T); \mathbb{H}_0^1) \) endowed with the metric associated with the norm defined in (3.10) is Polish. Let \( \mathcal{B}(X) \) denote its Borel \( \sigma \)-field. The theory
of large deviations \[3, 26\] is about the exponential decay of \(P(q^\varepsilon \in A)\) for events \(A \in \mathcal{B}(X)\) as \(\varepsilon \to 0\); this decay is described in terms of a rate function. We recall some definitions \[3\].

**Definition 4.1.** (Good Rate function)
A function \(I : X \to [0, \infty]\) is called a good rate function on \(X\) if for each \(M < \infty\) the level set \(\{q \in X : I(q) \leq M\}\) is a compact subset of \(X\). For \(A \in \mathcal{B}(X)\), we define \(I(A) = \inf_{q \in A} I(q)\).

**Definition 4.2.** (Large deviation principle)
Let \(I\) be a rate function on \(X\). The random sequence \(\{q^\varepsilon\}\) is said to satisfy a large deviation principle on \(X\) with rate function \(I\) if the following two conditions hold.

1. **Large deviation upper bound.** For each closed subset \(F\) of \(X\):
   
   \[
   \limsup_{\varepsilon \to 0} \varepsilon \log P(q^\varepsilon \in F) \leq -I(F).
   \]

2. **Large deviation lower bound.** For each open subset \(G\) of \(X\):
   
   \[
   \liminf_{\varepsilon \to 0} \varepsilon \log P(q^\varepsilon \in G) \geq -I(G).
   \]

The hypothesis on the growth condition and the Lipschitz property of \(\sigma\) are still the same as (A.1) (A.2) (A.3).

The proof of the large deviation principle will use the following technical lemma which studies time increments of the solution to the stochastic control equation. For any integer \(k = 0, \cdots, 2^n - 1\), and \(s \in [kT2^{-n}, (k + 1)T2^{-n}]\), set \(s_n = kT2^{-n}\) and \(\bar{s}_n = (k + 1)T2^n\). Given \(N > 0\), \(h \in \mathcal{A}_M\), \(\varepsilon \geq 0\) small enough, let \(q^\varepsilon_h\) denote the solution to (3.8) given by Theorem 3.7, and for \(t \in [0, T]\), let

\[
G_N(t) = \left\{ \omega : \left( \|q_h(t)\|^2 \leq N \right) \text{ and } \left( \int_0^T \|\nabla q_h(t)\|^2 \, dt \leq N \right) \right\}.
\]

**Lemma 4.3.** Let \(M, N > 0\), \(\sigma\) and \(\tilde{\sigma}\) satisfy the Assumptions (A.1), (A.2) and (A.3), \(\xi \in \mathbb{H}\). Then there exists a positive constant \(C := C(\nu, \kappa, K, L, T, M, N, \varepsilon_0)\) such that for any \(h \in \mathcal{A}_M\), \(\varepsilon \in [0, \varepsilon_0]\),

\[
I_n(h, \varepsilon) := \mathbb{E} \left[ 1_{G_N(T)} \int_0^T \|q^\varepsilon_h(s) - q_h(\bar{s}_n)\|^2 \, ds \right] \leq C 2^{-n}. \tag{4.2}
\]

This lemma may be similarly proved as in \[21\].

We now prove weak convergence and compactness in the following two subsections.

**4.2. Weak convergence.** Let \(\varepsilon_0\) be defined as in Theorem 3.7 and \(h_\varepsilon\) be a family of random elements taking values in \(\mathcal{A}_M\). Let \(q_{h_\varepsilon}\), or more strictly speaking, \(q^\varepsilon_h\), be the solution of the corresponding stochastic control equation with initial condition \(q_{h_\varepsilon}(0) = \xi \in \mathbb{H}\):

\[
dq_{h_\varepsilon} + [Aq_{h_\varepsilon} + F(q_{h_\varepsilon})] \, dt = \sigma(q_{h_\varepsilon})h_\varepsilon \, dt + \sqrt{\varepsilon} \sigma(q_{h_\varepsilon}) \, dW(t). \tag{4.3}
\]
In component form:

\begin{align*}
\frac{\partial q_{h,1}}{\partial t} + J(\psi_{h,1}, q_{h,1} + \beta y) &= \nu \Delta^2 \psi_{h,1} \\
&+ f + \sigma_1(q_{h,1}, q_{h,2}) h_{\varepsilon,1} + \sqrt{\varepsilon} \sigma_1(q_{h,1}, q_{h,2}) \dot{W}_1, \\
\frac{\partial q_{h,2}}{\partial t} + J(\psi_{h,2}, q_{h,2} + \beta y) &= \nu \Delta^2 \psi_{h,2} \\
&- r \Delta \psi_2 + \sigma_2(q_{h,1}, q_{h,2}) h_{\varepsilon,2} + \sqrt{\varepsilon} \sigma_2(q_{h,1}, q_{h,2}) \dot{W}_2.
\end{align*}

Note that \( q_{h,\varepsilon} \equiv \mathcal{G}^\varepsilon(\sqrt{\varepsilon} W + \int_0^{T} h_{\varepsilon}(s) ds) \) due to the uniqueness of the solution.

For all \( h \in L^2(0, T; \mathbb{H}_0) \), let \( q_{h} \) be the solution of the corresponding control equation (4.10) with initial condition \( q_{h}(0) = \xi \):

\begin{equation}
(4.4) \quad dq_{h} + [A q_{h} + F(q_{h})] dt = \sigma(q_{h}) h dt.
\end{equation}

In component form:

\begin{align*}
\frac{\partial q_{h,1}}{\partial t} + J(\psi_{h,1}, q_{h,1} + \beta y) &= \nu \Delta^2 \psi_{h,1} \\
&+ f + \sigma_1(q_{h,1}, q_{h,2}) h_{1}, \\
\frac{\partial q_{h,2}}{\partial t} + J(\psi_{h,2}, q_{h,2} + \beta y) &= \nu \Delta^2 \psi_{h,2} \\
&- r \Delta \psi_2 + \sigma_2(q_{h,1}, q_{h,2}) h_{2}.
\end{align*}

Noting that \( \int_0^{T} h(s) ds \in C([0, T]; \mathbb{H}_0) \), we define \( \mathcal{G}^0 : C([0, T]; \mathbb{H}_0) \to C([0, T]; \mathbb{H}) \cap L^2((0, T); \mathbb{H}_0^1) \) by

\[ \mathcal{G}^0(g) = q_{h} \quad \text{if} \quad g = \int_0^{T} h(s) ds \quad \text{for some} \quad h \in L^2(0, T; \mathbb{H}_0). \]

If \( g \) cannot be represented as above, we define \( \mathcal{G}^0(g) = 0 \).

**Lemma 4.4. (Weak convergence)**

Suppose that \( \sigma \) satisfies the Assumptions (A.1), (A.2) and (A.3). Let \( \xi \) be \( \mathcal{F}_0 \)-measurable such that \( \mathbb{E}\|\xi\|_{\mathcal{H}}^2 < +\infty \), and let \( h_{\varepsilon} \) converge to \( h \) in distribution as random elements taking values in \( \mathcal{A}_M \) (Note that here \( \mathcal{A}_M \) is endowed with the weak topology induced by the norm \( \|\cdot\|_{\mathcal{H}} \)). Then as \( \varepsilon \to 0 \), \( q_{h,\varepsilon} \) converges in distribution to \( q_{h} \) in \( X = C([0, T]; \mathbb{H}) \cap L^2((0, T); \mathbb{H}_0^1) \) endowed with the norm \( \|\cdot\|_{\mathcal{S}_M} \). That is, \( \mathcal{G}^\varepsilon(\sqrt{\varepsilon} W + \int_0^{T} h_{\varepsilon}(s) ds) \) converges in distribution to \( \mathcal{G}^0(\int_0^{T} h(s) ds) \) in \( X \), as \( \varepsilon \to 0 \).

**Proof.** Since \( \mathcal{A}_M \) is a Polish space (complete separable metric space), by the Skorokhod representation theorem, we can construct processes \((\tilde{h}_{\varepsilon}, \tilde{h}, \tilde{W})\) such that the joint distribution of \((\tilde{h}_{\varepsilon}, \tilde{W})\) is the same as that of \((h_{\varepsilon}, W)\), the distribution of \( \tilde{h} \) coincides with that of \( h \), and \( \tilde{h}_{\varepsilon} \to \tilde{h} \), a.s., in the (weak) topology of \( \mathcal{S}_M \).
Let \( \tilde{q}^\varepsilon = qh - q_h \), or in component form \( \tilde{q}^\varepsilon = (\tilde{q}_1^\varepsilon, \tilde{q}_2^\varepsilon) = (q_{1h}^\varepsilon - q_{1h}^0, q_{2h}^\varepsilon - q_{2h}^0) \). We first derive

\[
\begin{align*}
\frac{d\tilde{q}^\varepsilon}{dt} + [A\tilde{q}^\varepsilon + F(q_{1h}^\varepsilon) - F(q_h)] dt \\
= [\sigma(q_{1h}) \tilde{h}^\varepsilon - \sigma(q_h) \tilde{h}] dt + \sqrt{\varepsilon} \sigma(q_{1h}) dW(t), \\
\tilde{q}^\varepsilon(0) = 0.
\end{align*}
\]

In component form, \( \tilde{q}_1^\varepsilon(0) = 0, \tilde{q}_2^\varepsilon(0) = 0 \) and

\[
\begin{align*}
\frac{\partial \tilde{q}_1^\varepsilon}{\partial t} + J(\psi_{1h}^\varepsilon, q_{1h}^\varepsilon + \beta y) - J(\psi_{1h}^0, q_{1h}^0 + \beta y) &= \nu \Delta^2 \tilde{\psi}_1^\varepsilon \\
&+ \left[ (\sigma_1(q_{1h}) \tilde{h}_1^\varepsilon - \sigma_1(q_h) \tilde{h}_1) \right], \\
\frac{\partial \tilde{q}_2^\varepsilon}{\partial t} + J(\psi_{2h}^\varepsilon, q_{2h}^\varepsilon + \beta y) - J(\psi_{2h}^0, q_{2h}^0 + \beta y) &= \nu \Delta^2 \tilde{\psi}_2^\varepsilon \\
&+ \left[ (\sigma_2(q_{2h}) \tilde{h}_2^\varepsilon - \sigma_2(q_h) \tilde{h}_2) \right].
\end{align*}
\]

Similar to the definition of \( \|q\|_{L_2} \) in the proof of theorem 3.7, we define that \( \|\tilde{q}\|^2 = \|\tilde{q}_1\|^2 + \|\tilde{q}_2\|^2 \). On any finite time interval \([0, t]\) with \( t \leq T \), the Itô’s formula, Lemmas
Similarly to 3.1 and 3.2, Assumption (A.2) and (A.3) yield

\[\|\tilde{q}_1(t)\|^2 + 2\nu \int_0^t \|\nabla \tilde{q}_1(s)\|^2 ds\]

\[\begin{align*}
&= \left( -2 \int_0^t (J(\psi^\varepsilon_1, q_1^h), \tilde{q}_1^e) ds - 2\beta \int_0^t \left( \frac{\partial \psi^\varepsilon_1}{\partial x}, \tilde{q}_1^e \right) ds \\
&\quad + 2\nu F_1 \int_0^t (\Delta \tilde{\psi}_1^e - \tilde{\psi}_2^e, \tilde{q}_1^e) ds + 2 \int_0^t (\sigma_1(q_{h_e})h_1^e - \sigma_1(q_{h_1})Q_1^e) ds \\
&\quad + 2\sqrt{\varepsilon} \int_0^t (\tilde{q}_1^e, \sigma_1(q_{h_e})dW^1(s)) + \varepsilon \int_0^t \|\sigma_1(q_{h_e})Q_1^e\|^2_{H, S} ds \right) \\
&\leq C_1 \int_0^t \|\tilde{q}_1^e\|^2 ds + C_2 \int_0^t \|\Delta \tilde{\psi}_1^e\|^2 + \|\nabla q_1^h\|^2 + \|\tilde{q}_1^e\|^2 ds \\
&\quad + \nu F_1 \int_0^t \left( \|\Delta \tilde{\psi}_1^e\|^2 + \|\Delta \tilde{\psi}_2^e\|^2 + 2\|\tilde{q}_1^e\|^2 \right) ds \\
&\quad + 2 \int_0^t \sqrt{L} \|\tilde{q}_1^e\| \|h_1^e\| \|\tilde{q}_1^e\| ds + \int_0^t (\sigma_1(q)(h_1^e - h_1), \tilde{q}_1^e) ds \\
&\quad + 2\sqrt{\varepsilon} \int_0^t (\tilde{q}_1^e, \sigma_1(q_{h_e})dW^1(s)) + \varepsilon K \int_0^t (1 + |q_{h_e}|^2) ds \\
&\leq \frac{1}{2} C_1 \left( \int_0^t \|\tilde{q}_1^e\|^2 ds + \int_0^t \left( \int_0^t \left( \frac{\partial \tilde{\psi}_1^e}{\partial x} \right)^2 ds \right) + \frac{1}{2} C_2 \left( \int_0^t \|\Delta \tilde{\psi}_1^e\|^2 ds + \int_0^t \|\nabla q_1^h\|^2 \|\tilde{q}_1^e\|^2 ds \right) \right) \\
&\quad + \nu F_1 \int_0^t \left( \|\Delta \tilde{\psi}_1^e\|^2 + \|\Delta \tilde{\psi}_2^e\|^2 + 2\|\tilde{q}_1^e\|^2 \right) ds \\
&\quad + \frac{1}{2} \left( \int_0^t \|\tilde{q}_1^e\|^2 ds + L \int_0^t \|h_1^e\|^2 \|\tilde{q}_1^e\|^2 ds + \int_0^t (\sigma_1(q)(h_1^e - h_1), \tilde{q}_1^e) ds \right) \\
&\quad + 2\sqrt{\varepsilon} \int_0^t (\tilde{q}_1^e, \sigma_1(q_{h_e})dW^1(s)) + \varepsilon K \int_0^t (1 + |q_{h_e}|^2) ds. \\
\end{align*}\]

Similarly...
\[ \| \tilde{q}_2(t) \|^2 + 2\nu \int_0^t \| \nabla \tilde{q}_2(s) \|^2 ds \\
= -2 \int_0^t (J(\tilde{\psi}_2(q_2^h), q_2^h)) ds - r \int_0^t (\nabla \tilde{\psi}_2, \tilde{q}_2) ds - 2\beta \int_0^t (\nabla \tilde{\psi}_2, \tilde{q}_2) ds \\
+ 2\nu F_2 \int_0^t (\Delta(\tilde{\psi}_2 - \tilde{\psi}_1), \tilde{q}_2) ds + 2 \int_0^t (\sigma_2(q_{h_c}) h^c_2 - \sigma_2(q_{h_c}) h_2, \tilde{q}_2) ds \\
+ 2\sqrt{\varepsilon} \int_0^t (\tilde{q}_2, \sigma_2(q_{h_c}) dW^2(s)) + \varepsilon \int_0^t \| \sigma_2(q_{h_c}) Q_2^{\frac{1}{2}} \|^2_{H.S.} ds \\
\leq C_1 \int_0^t \| \tilde{q}_2 \| \| \nabla \tilde{q}_2 \| ds + C_2 \int_0^t \| \Delta \tilde{q}_2 \| \| \nabla q_{h_c} \| \| \tilde{q}_2 \| ds \\
+ C_3 \int_0^t \| \nabla \tilde{q}_2 \| \| \tilde{q}_2 \| ds + 2\nu F_2 \int_0^t (\| \nabla \tilde{q}_2 \| + \| \nabla \tilde{q}_2 \|) \| \tilde{q}_2 \| ds \\
+ 2 \int_0^t \left\{ \| (\sigma_2(q_{h_c}) - \sigma_2(q_{h_c})) h^c_2 \| \right\} \| \tilde{q}_2 \| ds + \int_0^t (\sigma_2(q)(h^c_2 - h_2), \tilde{q}_2) ds \\
+ 2\sqrt{\varepsilon} \int_0^t (\tilde{q}_2, \sigma_2(q_{h_c}) dW^2(s)) + \varepsilon \int_0^t \| \sigma_2(q_{h_c}) Q_2^{\frac{1}{2}} \|^2_{H.S.} ds \\
\leq \frac{1}{2} C_1 \left( \int_0^t \| \tilde{q}_2 \|^2 ds + \int_0^t \| \nabla \tilde{q}_2 \|^2 ds \right) \\
+ \frac{1}{2} C_2 \left( \int_0^t \| \nabla \tilde{q}_2 \|^2 ds + \int_0^t \| \nabla q_{h_c} \|^2 \| \tilde{q}_2 \|^2 ds \right) \\
+ \frac{1}{2} C_3 \int_0^t \left( \| \nabla \tilde{q}_2 \|^2 + \| \tilde{q}_2 \|^2 \right) ds + \nu F_2 \int_0^t \left( \| \nabla \tilde{q}_2 \|^2 + \| \nabla \tilde{q}_1 \|^2 + 2 \| \tilde{q}_2 \|^2 \right) ds \\
+ 2 \int_0^t \int (\tilde{q}_2, \sigma_2(q_{h_c}) ds + \int_0^t \left( (\sigma_2(q)(h^c_2 - h_2), \tilde{q}_2) ds \\
+ 2\sqrt{\varepsilon} \int_0^t (\tilde{q}_2, \sigma_2(q_{h_c}) dW^2(s)) + \varepsilon K \int_0^t \left( 1 + \| q_{h_c} \|^2 \right) ds \\
\leq \frac{1}{2} C_1 \left( \int_0^t \| \tilde{q}_2 \|^2 ds + \int_0^t \| \nabla \tilde{q}_2 \|^2 ds \right) \\
+ \frac{1}{2} C_2 \left( \int_0^t \| \nabla \tilde{q}_2 \|^2 ds + \int_0^t \| \nabla q_{h_c} \|^2 \| \tilde{q}_2 \|^2 ds \right) \\
+ \frac{1}{2} C_3 \int_0^t \left( \| \nabla \tilde{q}_2 \|^2 + \nu F_2 \int_0^t \left( \| \nabla \tilde{q}_2 \|^2 + \| \nabla \tilde{q}_1 \|^2 + 2 \| \tilde{q}_2 \|^2 \right) ds \\
+ \int_0^t \int \| \tilde{q}_2 \|^2 ds + L \int_0^t \| h^n \|^2 \| \tilde{q}_2 \|^2 ds + \int_0^t \left( (\sigma_2(q)(h^c_2 - h_2), \tilde{q}_2) ds \\
+ 2\sqrt{\varepsilon} \int_0^t (\tilde{q}_2, \sigma_2(q_{h_c}) dW^2(s)) + \varepsilon K \int_0^t \left( 1 + \| q_{h_c} \|^2 \right) ds . \right) \]
Adding (4.6) and (4.7), we obtain an integral inequality for \( \|\tilde{q}(t)\| \) which involves \( q_h = (q_1^h, q_2^h) \):

\[
\|\tilde{q}(t)\|^2 + 2\nu \int_0^t \|\nabla \tilde{q}(s)\|^2 ds \leq \int_0^t \left\{ C_4 + C_5 \|q_h\|^2 + L \|h_\varepsilon\|_0^2 \right\} \|\tilde{q}\|^2 ds
\]

(4.8) \hspace{1cm} + T_1(t, \varepsilon) + T_2(t, \varepsilon) + T_3(t, \varepsilon),

where

\[
T_1(t, \varepsilon) = 2\sqrt{\varepsilon} \int_0^t \left( \tilde{q}_\varepsilon(s), \sigma(q_{h_\varepsilon}(s)) \right) dW(s),
\]

\[
T_2(t, \varepsilon) = \varepsilon K \int_0^t (1 + |q_{h_\varepsilon}(s)|^2)^2 ds,
\]

\[
T_3(t, \varepsilon) = 2 \int_0^t \left( \sigma(q_h(s)) \left( h_\varepsilon(s) - h(s) \right), \tilde{q}_\varepsilon(s) \right) ds.
\]

Our goal is to show that as \( \varepsilon \to 0 \), \( \|\tilde{q}(t)\|^2 + \int_0^t \|\nabla \tilde{q}(s)\|^2 ds \to 0 \) in probability, which implies that \( q_{h_\varepsilon} \to q_h \) in distribution in \( C([0, T]; H^1) \cap L^2((0, T); H_0^1) \), as \( \varepsilon \to 0 \).

Fix \( N > 0 \) and for \( t \in [0, T] \) let

\[
G_N(t) = \left\{ \omega : \left( \|q_h(t)\|^2 \leq N \right) \text{ and } \left( \int_0^T \|\nabla q_h(t)\|^2 dt \leq N \right) \right\},
\]

\[
G_{N, \varepsilon}(t) = G_N(t) \cap \left\{ \|q_{h_\varepsilon}(t)\|^2 \leq N \right\} \cap \left\{ \int_0^T \|\nabla q_{h_\varepsilon}(t)\|^2 dt \leq N \right\}.
\]

**Claim 1.** For any \( \varepsilon > 0 \), \( \sup_{0 < \varepsilon \leq \varepsilon_0} \sup_{h, h_\varepsilon \in A_M} \mathbb{P}(G_{N, \varepsilon}(T)^c) \to 0 \) as \( N \to \infty \).

Indeed, for \( \varepsilon > 0 \), \( h, h_\varepsilon \in A_M \), the Markov inequality and the estimate (3.11) imply

\[
\mathbb{P}(G_{N, \varepsilon}(T)^c) \leq \mathbb{P}\left( \|q_h(t)\|^2 > N \right) + \mathbb{P}\left( \|q_{h_\varepsilon}(t)\|^2 > N \right) + \mathbb{P}\left( \int_0^T \|\nabla q_h(t)\|^2 dt > N \right) + \mathbb{P}\left( \int_0^T \|\nabla q_{h_\varepsilon}(t)\|^2 dt > N \right)
\]

\[
\leq \frac{1}{N} \sup_{h, h_\varepsilon \in A_M} \mathbb{E}\left( \|q_h(t)\|^2 + \|q_{h_\varepsilon}(t)\|^2 + \int_0^T \|\nabla q_h(t)\|^2 dt + \int_0^T \|\nabla q_{h_\varepsilon}(t)\|^2 dt \right)
\]

\[
\leq C_1(\nu, \kappa, K, L, T, M) \left( 1 + \mathbb{E}[\xi^2] \right) N^{-1}.
\]

**Claim 2.** For fixed \( N > 0 \), \( h, h_\varepsilon \in A_M \) such that as \( \varepsilon \to 0 \), \( h_\varepsilon \to h \) a.s. in the weak topology of \( L^2([0, T], H_0) \), one has as \( \varepsilon \to 0 \)

\[
\mathbb{E}\left[ 1_{G_{N, \varepsilon}(T)} \left( \|\tilde{q}_h(t)\|^2 + \int_0^T \|\nabla \tilde{q}_h(t)\|^2 dt \right) \right] \to 0.
\]

The Claim 2 can be similarly proved as in [21].
To conclude the proof of the Lemma 4.4 let $\delta > 0$ and $\alpha > 0$ and set
\[
\Lambda_{\varepsilon} := |\tilde{q}_{\varepsilon}|^2_X = \|\tilde{q}_{h}(t)\|^2 + \int_0^T \|\nabla \tilde{q}_{h}(t)\|^2 dt.
\]
Then the Markov inequality implies that
\[
\mathbb{P}(\Lambda_{\varepsilon} > \delta) = \mathbb{P}(G_{N,\varepsilon}(T)^c) + \frac{1}{\delta} \mathbb{E}\left(1_{G_{N,\varepsilon}(T)}|\tilde{q}_{\varepsilon}|^2_X\right).
\]
By Claim 1, we can choose $N$ large enough so that $\mathbb{P}(G_{N,\varepsilon}(T)^c) < \alpha$ for every $\varepsilon$. Fix $N$, Claim 2 then implies that for $\varepsilon$ small enough, $\mathbb{E}\left(1_{G_{N,\varepsilon}(T)}|\tilde{q}_{\varepsilon}|^2_X\right) < \delta \alpha$. This concludes the proof of the Lemma 4.4. □

4.3. Compactness. The following compactness result will show that the rate function of the LDP satisfied by the solution to (4.3) is a good rate function.

**Lemma 4.5.** (Compactness)
Let $M$ be any fixed finite positive number. Define
\[
K_M = \{q_h \in C([0,T];\mathbb{H}) \cap L^2((0,T);\mathbb{H}^1_0) : h \in S_M\},
\]
where $q_h$ is the unique solution of the control equation:
\[
dq_h(t) + [Aq_h(t) + F(q_h(t))] dt = \sigma(q_h(t)) h(t) dt, \quad q_h(0) = \xi.
\]
Then $K_M$ is a compact subset in $X$.

**Proof.** Let $q^n$ be a sequence in $K_M$, corresponding to solutions of (4.10) with controls $h^n$ in $S_M$:
\[
dq^n(t) + [Aq^n(t) + F(q^n(t))] dt = \sigma(q^n(t)) h^n(t) dt, \quad q^n(0) = \xi.
\]
Since $S_M$ is a bounded closed subset in the Hilbert space $L^2((0,T);\mathbb{H}^1_0)$, it is weakly compact. So there exists a subsequence of $h^n$, still denoted as $h^n$, which converges weakly to a limit $h$ in $L^2((0,T);\mathbb{H}^1_0)$. Note that in fact $h \in S_M$ as $S_M$ is closed. We now show that the corresponding subsequences of solutions, still denoted as $q^n$, converges in $X$ to $q$ which is the solution of the following “limit” equation
\[
dq(t) + [Aq + F(q)] dt = \sigma(q) h dt, \quad q(0) = \xi.
\]
This will complete the proof of the compactness of $K_M$.
Let $\tilde{q} = q^n - q$, or in component form $\tilde{q} = (\tilde{q}_1, \tilde{q}_2) = (q_1^n - q_1, q_2^n - q_2)$.
\[
d\tilde{q} + [A\tilde{q} + F(q^n) - F(q)] dt = [\sigma(q^n) h^n - \sigma(q) h] dt, \quad \tilde{q}(0) = 0.
\]
In component form, $\tilde{q}_1(0) = 0$, $\tilde{q}_2(0) = 0$ and
\[
\frac{\partial \tilde{q}_1}{\partial t} + J(\psi_1^n, q_1^n + \beta y) - J(\psi_1, q_1 + \beta y) = \nu \Delta^2 \tilde{\psi}_1 + \left[(\sigma_1(q^n) - \sigma_1(q)) h_1^n + \sigma_1(q)(h_1^n - h_1)\right],
\]
\[
\frac{\partial \tilde{q}_2}{\partial t} + J(\psi_2^n, q_2^n + \beta y) - J(\psi_2, q_2 + \beta y) = \nu \Delta^2 \tilde{\psi}_2 - \tau \Delta \tilde{\psi}_2 + \left[(\sigma_2(q^n) - \sigma_2(q)) h_2^n + \sigma_2(q)(h_2^n - h_2)\right].
\]
After the following transformations:

\[
\frac{\partial \tilde{q}_1}{\partial t} + J(\psi_1^n, q_1^n + \beta y) - J(\psi_1, q_1 + \beta y) = \nu \Delta^2 \tilde{\psi}_1 + \left[ (\sigma_1(q^n) - \sigma_1(q)) h_1^n + \sigma_1(q) (h_1^n - h_1) \right].
\]

Since the Jacobian operator is bilinear:

\[
\frac{\partial \tilde{q}_1}{\partial t} + J(\psi_1^n, q_1^n) - J(\psi_1^n, q_1) + J(\psi_1, q_1) - J(\psi_1, q_1 + \beta \partial_x \tilde{\psi}_1) = \nu \Delta^2 \tilde{\psi}_1 + \left[ (\sigma_1(q^n) - \sigma_1(q)) h_1^n + \sigma_1(q) (h_1^n - h_1) \right].
\]

By adding and subtracting the same item \(J(\psi_1^n, q_1)\), we get:

\[
\frac{\partial \tilde{q}_1}{\partial t} + J(\psi_1^n, q_1) - J(\psi_1^n, q_1 + \beta \partial_x \tilde{\psi}_1) + J(\psi_1, q_1) - J(\psi_1, q_1 + \beta \partial_x \tilde{\psi}_1) = \nu \Delta^2 \tilde{\psi}_1 + \left[ (\sigma_1(q^n) - \sigma_1(q)) h_1^n + \sigma_1(q) (h_1^n - h_1) \right].
\]

Finally, we have:

\[
\frac{\partial \tilde{q}_1}{\partial t} + J(\psi_1^n, \tilde{q}_1) + J(\tilde{\psi}_1, q_1) + \beta \partial_x \tilde{\psi}_1 = \nu \Delta (\tilde{q}_1 + F_1(\tilde{\psi}_1 - \tilde{\psi}_2)) + \left[ (\sigma_1(q^n) - \sigma_1(q)) h_1^n + \sigma_1(q) (h_1^n - h_1) \right].
\]

Thus, on any finite time interval \([0, T]\), the Itô’s formula, (A.3) and the Young’s inequality imply

\[
\|\tilde{q}_1(t)\|^2 + 0 + 2 \int_0^t (J(\tilde{\psi}_1, q_1), \tilde{q}_1) ds + 2\beta \int_0^t (\partial_x \tilde{\psi}_1, \tilde{q}_1) ds = 2\nu \int_0^t (\Delta \tilde{q}_1(s), \tilde{q}_1(s)) ds + 2\nu F_1 \int_0^t (\Delta (\tilde{\psi}_1 - \tilde{\psi}_2), \tilde{q}_1) ds + 2 \int_0^t (\sigma_1(q^n) h_1^n - \sigma_1(q) h_1, \tilde{q}_1) ds.
\]
\[ \| \tilde{q}_1(t) \|^2 + 2\nu \int_0^t \| \nabla \tilde{q}_1(s) \|^2 ds \]
\[ = -2 \int_0^t (J(\tilde{\psi}_1, q_1), \tilde{q}_1) ds - 2\beta \int_0^t (\frac{\partial \tilde{\psi}_1}{\partial x}, \tilde{q}_1) ds \]
\[ + 2\nu F_1 \int_0^t (\Delta(\tilde{\psi}_1 - \tilde{\psi}_2), \tilde{q}_1) ds + 2 \int_0^t (\sigma_1(q^n) h_1^n - \sigma_1(q) h_1, \tilde{q}_1) ds \]
\[ \leq C_1 \int_0^t \| \tilde{q}_1 \| \| \frac{\partial \tilde{\psi}_1}{\partial x} \| ds + C_2 \int_0^t \| \Delta \tilde{\psi}_1 \| \| \nabla q_1 \| \| \tilde{q}_1 \| ds \]
\[ + 2\nu F_1 \int_0^t (\| \Delta \tilde{\psi}_1 \| + \| \Delta \tilde{\psi}_2 \|) \| \tilde{q}_1 \| ds \]
\[ + 2 \int_0^t \{ \| (\sigma_1(q^n) - \sigma_1(q)) h_1^n \| \} \| \tilde{q}_1 \| ds + 2 \int_0^t (\sigma_1(q)(h_1^n - h_1), \tilde{q}_1) ds \]
\[ \leq \frac{1}{2} C_1 (\int_0^t \| \tilde{q}_1 \|^2 ds + \int_0^t \| \frac{\partial \tilde{\psi}_1}{\partial x} \|^2 ds) + \frac{1}{2} C_2 (\int_0^t \| \Delta \tilde{\psi}_1 \|^2 ds + \int_0^t \| \nabla q_1 \|^2 \| \tilde{q}_1 \|^2 ds) \]
\[ + \nu F_1 (\int_0^t (\| \Delta \tilde{\psi}_1 \|^2 + \| \Delta \tilde{\psi}_2 \|^2) + 2\| \tilde{q}_1 \|^2 ds) \]
\[ + 2 \int_0^t \sqrt{L} \| \tilde{q} \| \| h^n \| \| \tilde{q}_1 \| ds 2 \int_0^t (\sigma_1(q)(h_1^n - h_1), \tilde{q}_1) ds \]
\[ \leq \frac{1}{2} C_1 (\int_0^t \| \tilde{q}_1 \|^2 ds + \int_0^t \| \frac{\partial \tilde{\psi}_1}{\partial x} \|^2 ds) + \frac{1}{2} C_2 (\int_0^t \| \Delta \tilde{\psi}_1 \|^2 ds + \int_0^t \| \nabla q_1 \|^2 \| \tilde{q}_1 \|^2 ds) \]
\[ + \nu F_1 (\int_0^t (\| \Delta \tilde{\psi}_1 \|^2 + \| \Delta \tilde{\psi}_2 \|^2) + 2\| \tilde{q}_1 \|^2 ds) \]
\[ + \int_0^t \| \tilde{q} \|^2 ds + L \int_0^t \| h^n \|^2 \| \tilde{q}_1 \|^2 ds + 2 \int_0^t (\sigma_1(q)(h_1^n - h_1), \tilde{q}_1) ds. \]

Similarly,
\[ \| \tilde{q}_2(t) \|^2 + 0 + 2 \int_0^t (J(\tilde{\psi}_2, q_2), \tilde{q}_2) ds + 2\beta \int_0^t (\frac{\partial \tilde{\psi}_2}{\partial x}, \tilde{q}_2) ds \]
\[ = -r \int_0^t (\Delta \tilde{\psi}_2, \tilde{q}_2) ds + 2\nu \int_0^t (\Delta \tilde{q}_2(s), \tilde{q}_2(s)) ds \]
\[ + 2\nu F_2 \int_0^t (\Delta(\tilde{\psi}_2 - \tilde{\psi}_1), \tilde{q}_2) ds + 2 \int_0^t (\sigma_2(q^n) h_2^n - \sigma_2(q) h_2, \tilde{q}_2) ds. \]
\[ \|\tilde{q}(t)\|^2 + 2\nu \int_0^t \|\nabla \tilde{q}(s)\|^2 ds \]

\[ = -2 \int_0^t (J(\tilde{\psi}, q_2), \tilde{q}_2)ds - r \int_0^t (\Delta \psi_2, \tilde{q}_2)ds - 2\beta \int_0^t (\tilde{\psi}_2, \tilde{q}_2)ds + 2\nu F_2 \int_0^t (\tilde{\psi}_2 - \tilde{\psi}_1, \tilde{q}_2)ds + 2 \int_0^t (\sigma_2(q^n) h_n^2 - \sigma_2(q) h_n, \tilde{q}_2)ds \]

\[ \leq C_1 \int_0^t \|\tilde{q}\| \|\tilde{\psi}_2\| ds + C_2 \int_0^t \|\tilde{\psi}_2\| \|\nabla q_2\| \|\tilde{q}_2\| ds \]

\[ + C_3 \int_0^t \|\Delta \psi_2\| \|\tilde{q}_2\| ds + 2\nu F_2 \int_0^t (\|\Delta \tilde{\psi}_2\| + \|\Delta \tilde{\psi}_2\|) \|\tilde{q}_2\| ds \]

\[ \leq \frac{1}{2} C_1 \left( \int_0^t \|\tilde{q}\|^2 ds + \int_0^t \|\tilde{\psi}_2\|^2 ds \right) + \frac{1}{2} C_2 \left( \int_0^t \|\Delta \tilde{\psi}_2\|^2 ds + \int_0^t \|\nabla q_2\|^2 \|\tilde{q}_2\|^2 ds \right) \]

Adding \([4.14]\) and \([4.15]\), and by the Theorem \([3.7]\) we obtain an integral inequality for \(\|\tilde{q}(t)\|^2 = \|\tilde{q}_1(t)\|^2 + \|\tilde{q}_2(t)\|^2\) which involves \(q = \{q_1, q_2\}\):

\[ \|\tilde{q}(t)\|^2 + 2\nu \int_0^t \|\nabla \tilde{q}(s)\|^2 ds \leq 2 \int_0^t (\sigma(q)(h_n - h), \tilde{q})ds \]

\[ + \int_0^t \left\{ C_4 + C_5 \|q\|^2 + L \|h_n\|^2 \right\} \|\tilde{q}\|^2 ds. \]

As in \([21]\), for \(N \geq 1\) and \(k = 0, \ldots, 2^N\), set \(t_k = k2^{-N}\). For \(s \in [t_{k-1}, t_k], 1 \leq k \leq 2^N\), let \(s_N = t_k\). The inequality \([3.13]\) implies that there exists a constant \(\bar{C} > 0\) such that

\[ \sup_n \left[ \|q(t)\|^2 + \int_0^T \|\nabla \tilde{q}(t)\|^2 dt + \|\tilde{q}_n(t)\|^2 + \int_0^T \|\nabla \tilde{q}_n(t)\|^2 dt \right] = \bar{C} < +\infty. \]
Thus the Gronwall’s inequality implies

\[
(4.17) \quad \sup_{t \leq T} |\bar{q}_n(t)|_X^2 \leq \exp \left( \int_0^T \left\{ C_4 + C_5 \|q\|_1^2 + L \|h_n\|_1^2 \right\} \, ds \right) \sum_{i=1}^4 I_{n,N}^i,
\]

where

\[
I_{n,N}^1 = \int_0^T \left| (\sigma(q(s)) [h_n(s) - h(s)] , \bar{q}_n(s) - \tilde{q}_n(\tilde{s}_N)) \right| \, ds,
\]

\[
I_{n,N}^2 = \int_0^T \left| (\sigma(q(s)) - \sigma(q(\tilde{s}_N))) [h_n(s) - h(s)] , \bar{q}_n(\tilde{s}_N) \right| \, ds,
\]

\[
I_{n,N}^3 = \sup_{1 \leq k \leq 2N} \sup_{t_k-1 \leq t \leq t_k} \left| (\sigma(q(t_k)) \int_{t_{k-1}}^t (h_\varepsilon(s) - h(s)) \, ds , \bar{q}_\varepsilon(t_k)) \right|,
\]

\[
I_{n,N}^4 = \sum_{k=1}^{2N} \left( \sigma(q(t_k)) \int_{t_{k-1}}^{t_k} [h_n(s) - h(s)] \, ds , \bar{q}_n(t_k) \right).
\]

The Cauchy-Schwarz inequality, (A.2), (A.3) and Lemma 4.3 imply that for some constant $C$ which does not depend on $n$,

\[
I_{n,N}^1 \leq \left( \int_0^T \left( 1 + C \right) \|h_n(s) - h(s)\|_0^2 \, ds \right)^{\frac{1}{2}} \left( 2 \int_0^T \left( \|q_n(s) - q_n(\tilde{s}_N)\|^2 + \|q(s) - q(\tilde{s}_N)\|^2 \right) \, ds \right)^{\frac{1}{2}} \leq C 2^{-\frac{N}{4}},
\]

(4.18)

\[
I_{n,N}^2 \leq \left( L \int_0^T \|q(s) - q(\tilde{s}_N)\|^2 \, ds \right)^{\frac{1}{2}} \left( \tilde{C} \int_0^T \|h_n(s) - h(s)\|_0^2 \, ds \right)^{\frac{1}{2}} \leq C 2^{-\frac{N}{4}},
\]

(4.19)

\[
I_{n,N}^3 \leq K (1 + \|q(t)\|) \left( \|q(t)\| + \|q_n(t)\| \right) 2^{-\frac{N}{2}} 2M \leq C \tilde{C} 2^{-\frac{N}{2}}.
\]

(4.20)

We now use a time discretization argument from Proposition 4.4 of \cite{21}. For given $\alpha > 0$, one may choose $N$ large enough to have $\sum_{i=1}^4 I_{n,N}^i \leq \alpha$. Then, for fixed $N$ and $k = 1, \ldots, 2^N$, as $n \to \infty$, the weak convergence of $h_n$ to $h$ implies that of $\int_{t_{k-1}}^{t_k} (h_n(s) - h(s)) \, ds$ to 0 weakly in $H_0$. Since $\sigma(q(t_k))$ is a compact operator, we deduce that for fixed $k$ the sequence $\sigma(q(t_k)) \int_{t_{k-1}}^{t_k} (h_n(s) - h(s)) \, ds$ converges to 0 strongly in $H$ as $n \to \infty$. Since $\sup_n \sup_k \|\bar{q}_n(t_k)\| \leq 2\tilde{C}$, we have $\lim_n I_{n,N}^3 = 0$. Thus as $n \to \infty$, $\|\bar{q}_n(t)\|^2 \to 0$. Using this convergence and (4.16), we deduce that $|\bar{q}|_X \to 0$ as $n \to \infty$. This shows that every sequence in $K_M$ has a convergent subsequence. Hence $K_M$ is a compact subset of $X$.

With the above results, we have the following theorem.
Theorem 4.6. (Large deviation principle)
Let $q^\varepsilon$ be the solution of the stochastic two-layer problem

$$
\frac{dq^\varepsilon}{dt} + [Aq^\varepsilon + F(q^\varepsilon)] = \sqrt{\varepsilon} \sigma(q^\varepsilon) dW(t), \quad q^\varepsilon(0) = \xi \in \mathbb{H}.
$$

(4.21)

Then $\{q^\varepsilon\}$ satisfies the large deviation principle, in $C([0, T]; \mathbb{H}) \cap L^2((0, T); \mathbb{H}^1_0)$ with the good rate function

$$
I_\xi(\psi) = \inf_{\{h \in L^2([0, T], \mathbb{H}^1_0); \psi = G^0(\int_0^t h(s) \, ds)\}} \left\{ \frac{1}{2} \int_0^T \|h(s)\|_{\mathbb{H}^1_0}^2 \, ds \right\}.
$$

(4.22)

Here the infimum of an empty set is taken as infinity.

Proof. Lemma [13] and Lemma [14] imply that $\{q^\varepsilon\}$ satisfies the Laplace principle which is equivalent to the large deviation principle in $X = C([0, T]; \mathbb{H}) \cap L^2((0, T); \mathbb{H}^1_0))$ with the above-mentioned rate function; see Theorem 4.4 in [2] or Theorem 4.4 in [3].

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