CHARACTERIZATION OF LINEAR GROUPS WHOSE REDUCED $C^*$-ALGEBRAS ARE SIMPLE

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Abstract. The reduced $C^*$-algebra of a countable linear group $\Gamma$ is shown to be simple if and only if $\Gamma$ has no nontrivial normal amenable subgroups. Moreover, these conditions are shown to be equivalent to the uniqueness of tracial state on the aforementioned $C^*$-algebra.

CONTENTS

1. Introduction 2
2. Preliminaries and recollections 3
   2.1. Notations 3
   2.2. On the criteria of Bekka, Cowling, and de la Harpe 4
   2.3. The ping-pong lemma 5
   2.4. Some fundamental lemmas of Tits 6
   2.5. Attractor families 7
   2.6. Cellular decomposition 8
   2.7. Transversality 11
   2.8. Unipotent one-parameter subgroups 13
3. Simultaneously proximal elements 15
   3.1. Density theorems 16
   3.2. Existence 19
4. Strong transversality for simply laced groups 20
   4.1. Combinatorial data 20
   4.2. Weight spaces for simply laced groups 21
   4.3. Exterior powers 25
   4.4. Reduction modulo $k$ 29
5. Producing representations with contractive dynamics 30
   5.1. Producing representations with proximal behavior 30
   5.2. Quasi-projective transformations and quasi-proximal unipotents 32
6. Proof of the main theorem 35
References 41

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1. Introduction

It is of some interest to understand the structure of the reduced $C^*$-algebra of a countable group $\Gamma$. Quite a natural question is: Under which conditions is the reduced $C^*$-algebra of $\Gamma$ simple? This question was posed, for example, in [11]. (For a more complete history of this question, please see [12] and the references contained therein.) Along the same lines, one might wonder under what circumstances the reduced $C^*$-algebra of $\Gamma$ has several nonproportional traces. The purpose of this note is to resolve these questions in the case where $\Gamma$ is linear.

This note has as its purpose the demonstration of the following result.

**Theorem 1.1.** Let $\Gamma$ be a countable linear group. The following are equivalent:

(i) The reduced $C^*$-algebra of $\Gamma$ is simple.
(ii) The reduced $C^*$-algebra of $\Gamma$ has a unique trace, up to normalization.
(iii) $\Gamma$ has no nontrivial normal amenable subgroups.

Here is an outline of the structure of the paper: In §2.2, we review the criteria of Bekka, Cowling, and de la Harpe for the simplicity of the reduced $C^*$-algebra. We then present a modification of their criteria, which we will eventually verify. In §2.3, we present a modified ping-pong lemma, which is better adapted to our purposes: Namely, it will produce ping-pong partners for given elements, rather than for some powers thereof. In §§2.4 and 2.8, we collect some lemmas of Tits, and some facts about exterior powers of representations of rank-one groups, respectively.

In §2.5, we introduce a tool from dynamics, which will eventually enable us to make sense of ‘perturbations’ in the context of a Zariski-dense subgroup. Namely, this tool exploits compactness, and gives a priori, though not effective, bounds on powers of conjugating elements in certain algebraic expressions.

In §2.6, we use the Bruhat decomposition to study the orbit structure on products of Grassmann varieties. In §2.7, we apply the results of §2.6 to a special case arising from the study of dynamics of actions on vector spaces over local fields. Roughly speaking, we explore the subject of transversality for the characteristic subspaces of semisimple elements of a Zariski-dense subgroup of a simple group defined over a local field.

Eventually, we will encode certain algebraic properties in the dynamical language of proximality. For example, the ‘very proximality’ of a semisimple element will allow us show that it lies in a free subgroup (cf. the proof of Theorem 6.5). Similarly, we can encode higher-order algebraic information about subgroups of linear groups using elements proximal with respect to several representations simultaneously. We tell this story in §3.

In §4, we revisit the subject of transversality for characteristic subspaces of non-torsion, semisimple elements. Using special facts about simply laced groups, we obtain very precise control over the characteristic subspaces of conjugate semisimple elements in a Zariski-dense subgroup in any sufficiently large, irreducible representation on a vector space over a local field. The proof proceeds by reduction modulo a local field of a group scheme over $\mathbb{Z}$. Hence §§4.2 and 4.3 take place in the setting of complex reductive groups. In §4.4, we reap the rewards for the case of a general local field. (For arguments in a similar spirit, cf. [4] and [21].)

In §5, we consider the problem of finding, for a fixed non-torsion element of a linear group, a projective representation over a local field with the property that the fixed element has very contractive dynamics. In §5.1, we consider the case of a
semisimple element. In particular, we can write down a very satisfying criterion for
the existence of an irreducible representation of $G(k)$ satisfying the constraint that
a prescribed semisimple element act very proximally. (See Proposition 5.1, below.)
In §5.2, the case of a nontorsion unipotent.
In the remaining section, §6, we prove Theorem 1.1, and offer several related
formulations, which might prove of some utility.

2. Preliminaries and recollections

2.1. Notations. In this paper, all fields mentioned will be made to lie in a universal
field $\Omega$—that is, an algebraically closed field of infinite transcendence degree over
its prime field. All fields under discussion will be infinite. By the phrase “a vector
space” we will always mean a vector space over $\Omega$ unless we specify otherwise,
writing for instance “a vector space over $k$,” where $k$ is another field. Unless
specified otherwise, by the phrase “an algebraic group,” we shall intend an affine
algebraic group defined over $\Omega$, which we implicitly identify with the set of its
points over $\Omega$. Otherwise, we will use the terminology, “an algebraic group defined
over $k$,” or “a $k$-group.” We will call a $k$-group $G$ absolutely almost simple (resp.
almost $k$-simple) if it has no proper normal, connected, algebraic subgroups (resp.
k-subgroups).

Let $k$ be a field. A vector space $E$ is said to have a $k$-structure if we have
implicitly associated to $E$ a $k$-submodule $E_k$ of $E$ with the property that $E \cong E_k \otimes_k \Omega$. If $V$ is an algebraic variety defined over $k$, we shall denote by $V(k)$ the
set of its $k$-points. We will habitually use the following observations: If $E$ is a
vector space with a $k$-structure $E_k$, then we can identify $\text{GL}(E)(k)$ with $\text{GL}(E_k)$.
If we denote by $\text{Gr}_m(E)$ (resp. $P(E)$) the Grassmann variety of $m$-dimensional
subspaces of $E$ (resp. the projective space of $E$), then we can identify $\text{Gr}_m(E)(k)$
with $\text{Gr}_m(E_k)$ (resp. $P(E)(k)$ with $P(E_k)$).

Let $G$ be a $k$-group, $E$ a vector space endowed with a $k$-structure, $\rho : G \to \text{GL}(E)$
a $k$-rational representation (i.e., a homomorphism which is a $k$-morphism). Then
$\rho$ is said to be absolutely irreducible if $\rho(G)$ leaves invariant no proper, nontrivial
subspace of $E$. It is said to be irreducible over $k$ if $\rho(G)$ leaves invariant no proper,
nontrivial subspace of $E$ which is defined over $k$. In other words, the representation
$\rho_k : G(k) \to \text{GL}(E_k)$ is irreducible in the usual sense.

Occasionally, we shall work with local fields $k$. In this case, there might be some
ambiguity between the Zariski topology on a $k$-variety and the topology coming
from the locally compact field. We shall endeavor to be explicit, but we usually
mean the Zariski topology. Therefore, terms like $k$-dense and $k$-open always refer to
the Zariski topology, even when $k$ is local. ‘Connected’ will always mean ‘connected
in the Zariski topology.’ On the other hand, the terms ‘interior’ and ‘compact’ will
always refer to those notions relative to the topology induced from the absolute
value on a normed field. We denote by $\text{Int } X$ the interior of a set $X$.

If we have a field extension, $k \subset \ell$ and a $k$-variety $V$, then confusion might arise
about whether a subset of $X$ is open (or closed) in the Zariski topology coming from
$k$, or from $\ell$ (or from $\Omega$, for that matter). Fortunately, this is not the case, since
the $k$-topology on $V(k)$ coincides with the restriction to $V(k)$ of the $\ell$-topology on
$V(\ell)$.

If $g$ and $h$ are elements of any group whatsoever, we introduce the notation
$^gh = ghg^{-1}$ and $^h = g^{-1}hg$. If $g$ is an element of an algebraic group, we denote by
$g_s$ and $g_u$ its semisimple and unipotent Jordan components. An algebraic $k$-group is said to be unipotent if all of its elements are unipotent. We recall that a connected algebraic $k$-group is called semisimple if it contains no nontrivial, connected, solvable algebraic subgroup (equivalently, $k$-subgroup). It is called reductive if it contains no nontrivial, connected, unipotent algebraic subgroup (equivalently, $k$-subgroup). We refer to [21] for the representation theory of semisimple algebraic groups, and to [9] for generalities about root systems.

If $d$ is a distance on the $k$-points of an affine $k$-variety which is comparable to the affine distance coming from the absolute value on $k$, then we call $d$ admissible. If $X$ and $Y$ are subsets of affine $k$-space, and $f : X \to Y$ a function, then we denote by $\|f\|$ the supremum

$$\sup_{p,q \in X} \frac{d(f(p), f(q))}{d(p, q)}.$$

Let now $E$ denote a finite-dimensional vector space over a local field $k$, $P$ its projective space. If $g$ is a linear automorphism of $E$ (and $\dim E > 1$), we denote by $A(g) \subset P$ (resp. $A'(g)$) the projectivization of the sum of the eigenspaces of $g$ corresponding to eigenvalues of maximal norm (resp., not of maximal norm). Write $\text{Cr}(g) = A'(g) \cup A'(g^{-1})$. We note that if $g, h \in \text{GL}(E)$, then $A(\lambda h) = g \cdot A(h)$, $A'(\lambda h) = g \cdot A'(h)$; and, for any $z \in \mathbb{N}$, $A(g^z) = A(g)$, $A'(g^z) = A'(g)$. Also, if $g_s \in \text{GL}(E)$, then $A(g) = A(g_s)$, $A'(g) = A'(g_s)$.

**Definition 2.1.** We say that $g \in \text{GL}(E)$ is **proximal** if $A(g)$ is a singleton set. We say that $g$ is very proximal if both $g$ and $g^{-1}$ are proximal.

If $\rho$ is a representation of a group $\Gamma$ on a finite-dimensional vector space over a local field $k$, and $S$ is a subset of $\Gamma$, then we write

$$\Omega_+(\rho, S) = \{s \in S \mid \rho(s) \text{ is proximal}\},$$

$$\Omega_0(\rho, S) = \{s \in S \mid \rho(s) \text{ is very proximal}\}.$$

We note that if $\rho$ is a $k$-rational representation of a $k$-group $G$ on a finite-dimensional vector space (over $\Omega$, that is), then we can regard the restriction of $\rho$ to $G(k)$ as a representation on a finite-dimensional $k$-space. Thus, if $S \subset G(k)$, then we will use the notation $\Omega_+(\rho, S), \Omega_0(\rho, S)$ as described above.

Finally, suppose $E$ is a vector space endowed with a $k$-structure, $P$ its projectivization. If $h \in \text{GL}(E)(k)$, it makes sense to talk about $A(h) \subset P(k)$. It also makes sense, if $g \in \text{GL}(E)$, to talk about $g \cdot A(h) \subset P$. We caution that this is not the same thing as $A(\lambda h)$, which need not be defined.

### 2.2. On the criteria of Bekka, Cowling, and de la Harpe.

In the paper [2], Bekka, Cowling, and de la Harpe provide several sufficient conditions on a countable group $\Gamma$ for the reduced $C^*$-algebra $C_r^*(\Gamma)$ of $\Gamma$ to be simple, and to have a unique normalized trace.

For a countable group $\Gamma$, denote by $\lambda_{\Gamma} : \Gamma \to U(\ell^2(\Gamma))$ the left regular representation of $\Gamma$ on the Hilbert space $\ell^2(\Gamma)$. For a square-summable sequence $a = \langle a_j \rangle$ of complex numbers, denote by $\|a\|_2$ the $\ell^2$-norm ($\sum_{j=1}^\infty |a_j|^2)^{1/2}$. The following appears as Lemma 2.1 in [2].
Lemma 2.2. If for every finite set $F$ of nonidentity elements of $\Gamma$, there exist an element $g \in \Gamma$ and a number $C$ such that

$$(1) \quad \left\| \sum_{j=1}^{\infty} a_j \lambda_{F}(g^{-j} h g^j) \right\| \leq C \|a\|_2$$

for each $h \in F$ and every square-summable sequence $a$, then the reduced $C^*$-algebra $C^r(\Gamma)$ is simple, and has a unique trace up to normalization.

In turn the other sufficient conditions of the paper [2] entail the one above. The following condition—and the proof of its sufficiency—is a slight modification of some of those other sufficient conditions.

Given a representation $\rho$ of $\Gamma$ on a vector space over a local field, and a positive number $c$, denote by $A_c(\rho(g))$ the projectivization of the sum of those generalized eigenspaces of $\rho(g)$ corresponding to eigenvalues of norm $c$. When $\rho$ is clear from the context, we shall omit it from our notation.

Lemma 2.3. Suppose that for every finite subset $F \subset \Gamma$, there exists a decomposition $F = F_p \cup F_r$; a representation $\rho_h$ of $\Gamma$ on a finite-dimensional vector space over a local field for each element $h \in F_r$; and an element $g \in \Gamma$ of infinite order, such that

(i) for each $h \in F_p$, the subgroup $\langle g,h \rangle$ generated by $g$ and $h$ is canonically isomorphic to the free product $\langle g \rangle \ast \langle h \rangle$; and

(ii) for each $h \in F_r$ and every pair of positive numbers $c$ and $d$, we have $h \cdot A_c(\rho_h(g)) \cap A_d(\rho_h(g)) = \emptyset$.

Then $C^r(\Gamma)$ is simple, and has a unique trace up to normalization.

Proof. First, fix $h \in F_p$. In the proof of Lemma 2.2 of [2], Bekka, Cowling, and de la Harpe show that there exists $C$ such that equation (1) holds for $g$, our fixed $h$, and every square-summable sequence $a$.

Next, applying the proofs of Lemmas 2.3 and 2.4 of [2] to the set $F_r$, we arrive at the same conclusion for each element $h \in F_r$; and an element $g \in \Gamma$ of infinite order, such that

(i) for each $h \in F_r$, the subgroup $\langle g,h \rangle$ generated by $g$ and $h$ is canonically isomorphic to the free product $\langle g \rangle \ast \langle h \rangle$; and

(ii) for each $h \in F_r$ and every pair of positive numbers $c$ and $d$, we have $h \cdot A_c(\rho_h(g)) \cap A_d(\rho_h(g)) = \emptyset$.

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Proof. First, fix $h \in F_p$. In the proof of Lemma 2.2 of [2], Bekka, Cowling, and de la Harpe show that there exists $C$ such that equation (1) holds for $g$, our fixed $h$, and every square-summable sequence $a$.

Next, applying the proofs of Lemmas 2.3 and 2.4 of [2] to the set $F_r$, we arrive at the same conclusion for each element $h \in F_r$. Since $F$ is a finite set, we can take $C$ large enough to work for all $h \in F$ simultaneously. The result now follows from Lemma 2.2.

2.3. The ping-pong lemma. The following formulation of the ping-pong lemma is better adapted for our purposes than the usual formulation. The usual formulation in practice requires one to raise given group elements to some powers. But the statement of our problem necessitates finding a ping-pong partner for a fixed element.

Lemma 2.4 (Ping-pong). Let $L$ be a group generated by a subgroup $K$ and an element $h$, with $K$ of cardinality exceeding 2. Assume given a subset $U$ of an $L$-space $X$, satisfying the following conditions:

(i) $h \cdot U \neq U$; and

(ii) For all integers $j$ such that $h^j \neq 1$, and all $g \neq 1$ in $K$, we have $gh^j \cdot U \subseteq U$.

Then $L$ is isomorphic to the free product $K \ast \langle h \rangle$.

Proof. If $\bar{K} \cong K$ and $\bar{\langle h \rangle} \cong \langle h \rangle$, then there is a canonical epimorphism $\eta$ from $\bar{K} \ast \bar{\langle h \rangle}$ onto $L$. Explicitly, $\eta$ sends $\bar{g}$ to $g$ for $\bar{g} \in \bar{K}$, and sends $\bar{h}$ to $h$. To show that the kernel of this epimorphism is trivial, it is necessary and sufficient to show that
any nontrivial reduced word in the free product $\tilde{K} * \langle \tilde{h} \rangle$ remains nontrivial when evaluated in $L$.

Let $w$ be such a word. Our hypothesis on the order of $K$ guarantees that we can conjugate $w$ to a reduced word $w'$ which begins and ends with nontrivial elements from $\tilde{K}$. Since $\eta(w) = 1$ if and only if $\eta(w') = 1$, we may assume $w$ has the form

$$w = \tilde{g}_\ell \tilde{h}^{j_\ell} \cdots \tilde{h}^{j_1} \tilde{g}_1 \tilde{h}^{j_1} \tilde{g}_0$$

where $h^{j_i} \neq 1$, all $i = 1, \ldots, \ell$, and $g_i \in K \setminus \{1\}$, all $i = 0, \ldots, \ell$.

Consider the union $(h \cdot U) \cup (h^{-1} \cdot U)$. We prove by induction on $m$ that the subword $w_m = \tilde{g}_m \tilde{h}^{j_m} \cdots \tilde{h}^{j_1} \tilde{g}_1 \tilde{h}^{j_1} \tilde{g}_0$ acts (via $\eta$) on this union by sending it into $U$.

First of all, $g_0 \cdot ((h \cdot U) \cup (h^{-1} \cdot U))$ is contained in $U$ by (ii). Next

$$\eta(w_m) \cdot ((h \cdot U) \cup (h^{-1} \cdot U)) = (g_m h^{j_m}) \eta(w_{m-1}) \cdot ((h \cdot U) \cup (h^{-1} \cdot U))$$

$$\subset (g_m h^{j_m}) \cdot U$$

$$\subset U$$

by the induction hypothesis and another application of (ii).

We conclude that $\eta(w) \cdot ((h \cdot U) \cup (h^{-1} \cdot U)) \subset U$. But the hypothesis that $h \cdot U \neq U$ is equivalent to $(h \cdot U) \cup (h^{-1} \cdot U) \not\subset U$. It follows that $w \notin \ker \eta$. \hspace{1cm} \Box

In our applications, the subgroup $K$ will always be infinite cyclic.

2.4. Some fundamental lemmas of Tits. In this section, we simply collect for reference in the sequel several powerful observations of Tits regarding the relationship between the norm of a projective transformation and proximality. Let $E$ be a finite-dimensional vector space over a local field $k$, and $P$ its projectivization. All topological terminology in this section (e.g., interior, compact) will refer to the topology induced from the locally compact topology of $k$. Given a projective transformation $b$ on $P$ and a subset $K \subset P$, we understand $\|p|_K\|$ to be the norm relative to an admissible distance on $P$.

**Lemma 2.5** (Lemma 3.5 of [22]). A projective transformation has finite norm.

**Lemma 2.6** (Lemma 3.8 of [22]). Let $g \in GL(E)$, let $K \subset P$ be a compact set, and let $q > 0$.

(i) Suppose that $A(g)$ is a singleton and that $K \cap A'(g) = \emptyset$. Then there exists an integer $N$ such that $\|\tilde{g}^z|_K\| < q$ for all $z > N$; and for every neighborhood $U$ of $A(g)$, there exists an integer $N'$ such that $\tilde{g}^z K \subset U$ for all $z > N'$.

(ii) Assume that, for some $m \in \mathbb{N}$, we have $\tilde{g}^m K \subset \text{Int} K$ and $\|\tilde{g}^m|_K\| < 1$. Then $g$ is proximal and $A(g) \subset \text{Int} K$.

For linear subspaces $V$ and $W$ of $P$, let us denote by $V \vee W$ the join of $V$ and $W$. If $V \vee W = E$ and $V \cap W = 0$, then we can define a map $\text{proj}(V, W) : P \setminus V \to W$ by sending $p \in P$ to $(\{p\} \vee V) \cap W$.

**Lemma 2.7** (Lemma 3.9 of [22]). Let $g \in GL(E)$ be semisimple. Let $K$ be a compact subset of $P \setminus A'(g)$. Set $\pi = \text{proj}(A'(g), A(g))$. Let $U$ be a neighborhood of $\pi(K)$ in $P$.

(i) The set $\{\|\tilde{g}^z|_K\| : z \in \mathbb{N}\}$ is bounded.
Suppose \( N \subset \mathbb{N} \) is any infinite set such that, for any pair \( \lambda, \mu \) of eigenvalues of \( g \) whose absolute value is maximum, we have \((\lambda^{-1}\mu)^2 \to 1\) as \( z \) tends to infinity along \( N \). Then \( \hat{g}^zK \subset U \) for almost all \( z \in N \).

2.5. Attractor families. When working with a given projective transformation, it sometimes helps to conjugate by a projective transformation with more favorable dynamical properties. If we conjugate the former element by a large enough power of the latter, the dynamics of the resulting transformation resemble the latter more than the former. The content of this section is a result which allows us bound this “large enough” at the expense of introducing non-constructiveness.

Let \( k \) be an infinite field, with local extensions \( k_1, \ldots, k_r \). Let \( G \) be a connected \( k \)-group. Let \( \rho_1, \ldots, \rho_r \) be absolutely irreducible representations of \( G \) on vector spaces \( E_1, \ldots, E_r \), respectively, and suppose \( \rho_i \) is \( k_i \)-rational, \( i = 1, \ldots, r \). Denote by \( P_i \) the projective space of \( E_i \).

In this section, we shall be concerned with the compact Hausdorff topological space \( X = P_1(k_1) \times \cdots \times P_r(k_r) \). For brevity, we shall write \( \phi_i = \rho_i(\phi) \) when \( \phi \in G(k) \). We wish to associate to an element \( \phi \in \cap_{i=1}^r \Omega_+(\rho_i, G(k)) \) an open subset of \( X \). For such an element \( \phi \), set

\[
\mathcal{O}_\phi = (P_1(k_1) \setminus A'(\phi_1)) \times \cdots \times (P_r(k_r) \setminus A'(\phi_r)).
\]

**Definition 2.8.** We shall say a subset \( \Phi \) of \( \cap_{i=1}^r \Omega_+(\rho_i, G(k)) \) is an attractor family if \( \{ \mathcal{O}_\phi \mid \phi \in \Phi \} \) covers \( X \).

We remark that the notion of an attractor family is defined relative to a given family of representations. This will play an important role later, but we suppress mention of the family of representations when no ambiguity can arise.

We also remark that a conjugate of an attractor family is again an attractor family.

Suppose \( S \subset G(k) \) is a Zariski-dense subgroup of \( G \).

**Lemma 2.9.**

(i) If \( C \) is the conjugacy class in \( S \) of some \( \phi \in \cap_{i=1}^r \Omega_+(\rho_i, S) \), then \( C \) contains a finite attractor family.

(ii) Given an attractor family \( \Phi \), and a neighborhood \( U \subset X \) of

\[
\bigcup_{\phi \in \Phi} A(\phi_1) \times \cdots \times A(\phi_r),
\]

then there exists a number \( N' \) such that for any \( p \in X \) there exists \( \phi \in \Phi \) such that \((\hat{\rho}_1 \times \cdots \times \hat{\rho}_r)(\phi^z)p \subset U \) for all \( z > N' \).

**Proof.** Let \( \phi \) be as in part (i) of the lemma. Suppose, contrary to fact that the collection \( \{ \mathcal{O}_\phi \mid x \in S \} \) does not cover \( X \); namely, that there exists \( p \in X \setminus \bigcup_{x \in S} \mathcal{O}_\phi \). By definition, \( p = (p_1, \ldots, p_r) \in X \setminus \bigcup_{x \in S} \mathcal{O}_\phi \) if and only if for every \( x \in S \) there exists an index \( i \) such that \( x \cdot p_i \subset A'(\phi_i) \). Let \( N(p_i, A'(\phi_i)) \) denote the subvariety \( \{ x \in G \mid x \cdot p_i \subset A'(\phi_i) \} \). Then \( S \subset \bigcup_{i=1}^r N(p_i, A'(\phi_i)) \).

Since \( S \) is Zariski-dense, we have also \( G \subset \bigcup_{i=1}^r N(p_i, A'(\phi_i)) \). But \( G \) is irreducible as an algebraic variety, whence there exists an index \( i \) such that \( G = N(p_i, A'(\phi_i)) \). In particular, we have also \( S \subset N(p_i, A'(\phi_i)) \). In other words, \( p_i \) lies in the subspace

\[
\bigcap_{x \in S} A'(\rho_i(\phi^z)).
\]
But this intersection is a proper and $S$-invariant subspace and hence also $G$-invariant. It follows from irreducibility of $\rho_i$ that this intersection is empty, providing our contradiction. Therefore, $\{O_{\varphi,x} \mid x \in S\}$ covers $X$. Part (i) of the lemma now follows from compactness of $X$.

Suppose now that $\Phi$ and $U$ are as in part (ii) of the lemma. By compactness, it is necessary and sufficient to prove the statement under the additional hypothesis that $\Phi$ is finite. We may assume that $U = \bigcup_{\varphi \in \Phi} U_{\varphi}$ is a union of “cubes”

$$U_{\varphi} = U(\varphi,1) \times \cdots \times U(\varphi,r)$$

with $U(\varphi,i) \subset P_i(k_i)$ is a neighborhood of $A(\phi_i)$, $i = 1, \ldots, r$. Because $X$ is compact Hausdorff, it follows that we can take neighborhoods $V_{\varphi}$ of $X \setminus \mathcal{O}_{\varphi}$ with the property that $\bigcap_{i \in \Phi} V_{\varphi}$ is empty. By compactness, we can cover $X \setminus V_{\varphi}$ with a finite number of cubes, each of which lies in the complement of $\mathcal{O}_{\varphi}$. By construction of $\mathcal{O}_{\varphi}$, the closure of the $i$th “side” of each of these cubes lies outside $A'(\phi_i)$. Part (ii) now follows from Lemma 2.9.(i). \qed

If $E$ is a vector space, then we denote by $E^\vee$ the dual space of $E$. We begin with the useful observation that if $n = \dim E$ is finite, then the projective space of $E^\vee$ is naturally isomorphic to the Grassmann variety $Gr_{n-1}(E)$ of hyperspaces in $E$. Explicitly, this isomorphism is given by sending $t \in P(E^\vee)$ to $\ker t$, where $t \in E^\vee \setminus \{0\}$ is a representative of the line $t$. Moreover, if $E$ has a $k$-structure, then this isomorphism is defined over $k$; and if $k$ is local, then the restriction to the $k$-points is a homeomorphism.

If $\rho : G \to GL(E)$ is a representation of a group $G$ on a space $E$, let us denote by $\rho^\vee$ the dual representation $G \to GL(E^\vee)$. We return to the notation introduced at the beginning of this section. Let $\rho_1, \ldots, \rho_r, P_1, \ldots, P_r$ be as above.

**Corollary 2.10.** Let $\Phi \subset \bigcap_{i=1}^r \Omega_0(\rho_i, G(k))$ be an attractor family for the the family of representations $\{\rho_1, \ldots, \rho_r, \rho_1^\vee, \ldots, \rho_r^\vee\}$. Assume given subsets $U_i, K_i \subset P_i(k_i)$ for $i = 1, \ldots, r$, where $U_i$ is a neighborhood of $\bigcup_{\varphi \in \Phi} A(\rho_i(\varphi))$ and $K_i$ is compact with $K_i \cap \bigcup_{\varphi \in \Phi} A'(\rho_i(\varphi^{-1})) = \emptyset$, all $i = 1, \ldots, r$.

Then there exists a number $N'$ such that for any $\gamma_i \in \Omega_0(\rho_i, G(k))$ there exists a $\phi \in \Phi$ such that

$$\phi^z \cdot A(\gamma_i) \subset U_i$$

$$\phi^z \cdot A'(\gamma_i^{-1}) \cap K_i = \emptyset$$

all $i = 1, \ldots, r$ and $z > N'$.

**Proof.** The corollary follows from two observations: First, for any representation $\rho$ on a vector space $E$ over a local field, $\rho(\gamma^{-1})$ is proximal implies that $\rho^\vee(\gamma)$ is proximal. Moreover, under the correspondence $P(E^\vee) \cong Gr_{\dim E-1}(E)$ described in the previous paragraph, the image in $P(E)$ of the “hyperspace” $A(\rho^\vee(\gamma))$ is exactly $A'(\rho(\gamma^{-1}))$.

The second observation is that if $K \subset P(E)$ and $w \cap K = \emptyset$, then the condition $v \cap K = \emptyset$ defines a neighborhood of $w$ in the locally compact topology on $Gr_{\dim E-1}(E)$. The result now follows from Lemma 2.9.(ii). \qed

2.6. **Cellular decomposition.** In this section, we take $G$ to be a connected, $k$-split, almost $k$-simple $k$-group. We begin by recalling some standard facts about parabolic subgroups and the Weyl group. Fix a maximal $k$-split torus $T$ and a Borel
subgroup $B$ defined over $k$ and containing $T$. We denote by $H$ the set of simple roots relative to the notion of positivity induced on $\Delta(T, G)$ from $B$. For each simple root $\alpha \in H$ we have an associated simple reflection $w_\alpha \in W = W(T, G)$. The set $\{w_\alpha \mid \alpha \in H\}$ is a minimal generating set for $W$. Relative to this set of generators, there is a unique element $w_0 \in W$ whose length in the word metric $\ell_H$ is maximal. In particular, $\ell_H(w_0) = |\Delta^+|$.

We call a $k$-subgroup $P$ of $G$ parabolic if it contains a Borel subgroup defined over $k$, and standard parabolic if it contains our fixed Borel $B$. The notion of “standard” therefore is not invariant, but it will be clear from the context which Borel is the standard one. In any case, every standard parabolic $P$ is of the form

$$P_I = \bigcup_{w \in W_I} BwB$$

where $I$ is a subset of $H$ and $W_I$ the subgroup of $W$ generated by the simple reflections associated to roots in $I$. From the characterization of $\{w_\alpha\}$ as a minimal generating set, it follows that $I \subseteq H$ implies $W_I < W$ and $P_I < G$. Also, it follows that every $k$-subgroup containing $B$ is $k$-closed.

We recall the fact that the unipotent radical $B_u$ of $B$ is isomorphic as a $k$-variety to a ‘pointed’ affine space of dimension $|\Delta^+|$. Explicitly, the map

$$t = (t_\beta)_{\beta \in \Delta^+} \mapsto \prod_{\beta \in \Delta^+} u_{\beta}(t_\beta)$$

is an isomorphism of algebraic $k$-varieties, where the product is taken in any fixed order on $\Delta^+$.

Let $U^-$ be the unipotent radical of an ‘opposite’ Borel subgroup, $B^{\overline{w_0}}$. Having fixed a total ordering on $\Delta^+$, we denote by $\pi$ the projection of $U^-B \cong U^- \times B$ onto the pointed affine space $A|\Delta^+|$ of dimension $|\Delta^+|$ given by

$$\prod_{\beta < 0} u_{\beta}(t_\beta)b \mapsto t = (t_\beta)_{\beta < 0}.$$ 

Here the product on the left-hand side is taken with respect to the total ordering on $\Delta$. In particular, the leftmost term comes from $U^-\lambda$, where $\lambda$ is the highest root.

**Proposition 2.11.** If $h \in G(k)$ is not central in an absolutely almost simple $k$-subgroup of full rank in $G$, then $h$ is conjugate to an element of $Bw_0B$ by an element of $G(k)$.

**Proof.** For $w \in W$, let us denote by $X(w)$ the closure of the Bruhat cell $BwB$. We can identify the tangent space to $X(w)$ at the identity with a certain subspace of the Lie algebra of $G$. For a root $\alpha \in \Delta$, let us denote by $L_\alpha$ the corresponding root subspace of Lie $G$, and by $X_\alpha \subseteq L_\alpha$ the normalized derivation corresponding to conjugation by the root subgroup $U_\alpha$. Since $X(w)$ is invariant under the conjugation action of $T$, it follows that this subspace is spanned by the root subspaces it contains. Notably, if $w \neq w_0$, then $T_1X(w) \subseteq \text{Lie } B \oplus \bigoplus_{-\lambda < \nu < 0} L_\nu$, where $\lambda$ is the highest root of $\Delta$ with respect to the partial ordering induced from $B$. Let $\{e_\nu(a) \mid \nu < 0\}$ be the ‘standard’ basis of the tangent space to $A|\Delta^+|$ at $a$.

If $s \in T$, then the pair $(X(w), s)$ is invariant under the conjugation action of $T$. It follows that if $w \neq w_0$, then $d_\pi \pi$ maps the tangent space $T_sX(w)$ into the span of those vectors $e_\nu(o)$ with $-\lambda < \nu < 0$. 

For an element $g \in G$, let us denote by $\phi_g$ the morphism $G \to A^{|\Delta^+|}$ given by $x \mapsto \pi(xg)$. We shall be interested in the differential $d_1\phi_1$ of $\phi$ at the identity. If $s \in T \setminus \ker \lambda$, then $d_1\phi_s(X_{-\lambda})$ is a nonzero multiple of $e_{-\lambda}(0)$. But this implies that $d_s\pi(T_s[s]) = \im d_1\phi_s \not\subset d_s\pi(\bigcup_{w \neq w_0} T_sX(w))$, whence $T_s[s] \not\subset \bigcup_{w \neq w_0} T_sX(w)$. Thus in this case we have that $[s] \not\subset \bigcup_{w \neq w_0} X(w)$, namely that $[s]$ meets $Bw_0B$.

But the orbit of $-\lambda$ under the Weyl group is the set of all (long) roots in $\Delta$. This implies that $[s]$ meets $Bw_0B$ unless $s$ lies in the intersection of the kernels of all (long) roots. Namely, $[s]$ meets $Bw_0B$ unless $Z_G(s)$ contains the connected reductive subgroup $M$ of $G$ generated by those subgroups $U_\alpha$ with $\alpha \in \Delta$ (long).

(Put otherwise, $M$ is the smallest absolutely almost simple, closed $k$-subgroup of $G$ containing $T$.) Incidentally, this completes the proof of the proposition in the special case where $h$ is semisimple.

We recall the fact that if $s$ is the semisimple Jordan component of an element $h \in G$, then $[s]$ lies in the closure of $[h]$. If $w \in W \setminus \{w_0\}$, then $\bigcap_{x \in G} \Ad x(T_xX(w)) = 0$. Equivalently, the connected component of $\bigcap_{x \in \mathbb{G}} x X(w)$ containing the identity is trivial. Combining the preceding remarks leads us to conclude that $X(w)$ contains no nontrivial unipotent classes. By irreducibility, the same holds for $\bigcup_{w \neq w_0} X(w) = G \setminus Bw_0B$. In particular, the proposition holds in the special case where $h$ is unipotent.

Now let $s$ (resp. $u$) be the semisimple (resp. unipotent) Jordan component of a fixed but arbitrary element $h \in G$. If the centralizer $Z_G(s)$ of $s$ does not contain a conjugate of $M$, then $B[s]B = G$; since $[s]$ lies in the closure of $[h]$, it follows that $B[h]B$ is dense.

On the other hand, suppose that $Z_G(s)$ contains a conjugate of $M$. Denote by $H$ the connected reductive (in fact simple) subgroup $Z_G(s)$. Upon conjugation, we may assume that $B_H = B \cap H$ is a Borel subgroup of $H$. If $h$ is not central in $H$, then $u \neq 1$. By the preceding remarks, $B_H(h^H)B_H = B_H(u^H)B_H$ is dense in $H$. But then $B[h]B$ is dense in $BHB$. The latter is a closed, irreducible algebraic variety, hence coincides with a Bruhat cell closure $X(w)$. Since $L_{-\lambda} \subset \Lie M$, it follows that $w = w_0$.

\begin{remark}
Suppose $h = su$. As a consequence of the proof of Proposition 2.11, we see that $[h]$ meets $Bw_0B$ if either $u \neq 1$, or $s$ is not itself central in a simple subgroup of full rank in $G$.

Suppose $h = s \in T$ lies in the intersection of the kernels of all (long) roots of $\Delta(G,T)$. Thus, if $\Delta$ is simply laced, then $s$ is central. On the other hand, if $\Delta$ is not simply laced, then either $s$ is central; or $s$ lies in the kernel of no short root of $\Delta$, and $Z_G(s)^o$ is that maximal proper connected simple subgroup of $G$ containing $T$ whose root system coincides with the set of long roots of $\Delta$.

If $\alpha$ is a short root contained in a $G_2$-subsystem (resp. $B_2$-subsystem) of $\Delta$, then there exist a pair of long roots of $\Delta$ whose difference is $3\alpha$ (resp. $2\alpha$). It follows that $\alpha(s)$ is a cube root (resp. square root) of 1. Note that any root in a root system of type $C_3$ can be written as the sum of two short roots. If $s$ lies outside the kernels of all short roots of $\Delta$, it follows that $\Delta$ contains no subsystem of type $C_3$, which excludes from consideration the possibility that $\Delta$ is of type $C_n$, $n \geq 3$, or of type $F_4$. In other words, $B[h]B$ is dense unless either:

(i) $\Delta$ is of type $B_n$, $n \geq 2$; $k$ is not of characteristic two; and $\alpha(s) = -1$ for all short roots $\alpha$; or
(ii) $\Delta$ is of type $G_2$; $k$ is not of characteristic three; $\alpha(s) = \epsilon$ for an equilateral triangle of short roots $\alpha \in \Delta$, where $\epsilon$ is a root of the polynomial $x^2 + x + 1$; and $\alpha(s) = \epsilon^2 = \epsilon^{-1}$ for the other three short roots $\alpha \in \Delta$.

This proposition is useful in analyzing the diagonal action of $G$ on the variety $G/P \times G/Q$, where $P$ and $Q$ are standard parabolic subgroups. To be precise, this is the action $g \cdot (x, y) = (gx, gy)$; or, analogously, $g \cdot (x, y) = (\sigma x, \sigma y)$, if we prefer to think of points of $G/P$ as subgroups conjugate to $P$. We write $[(xP, yQ)]$ for the $G$-orbit of a point of $G/P \times G/Q$.

The fundamental fact about this action is that the $G$-orbits are in one-to-one correspondence with double cosets $PwQ$, or equivalently, with the set $W_P \backslash W/W_Q$ (cf. §3 of [20] and §5 of [6]). Here $W_P$ is the subgroup of the Weyl group which indexes the union in (2), and $W_Q$ is similarly defined. The following observation will be helpful. For this next lemma, we can relax our standing assumption that $G$ is almost $k$-simple. It will be sufficient to assume that $G$ is reductive.

**Lemma 2.13.** The orbit $[(P, \omega_0 Q)]$ is open and dense in $G/P \times G/Q$.

**Proof.** The orbit $[(P, \omega_0 Q)]$ coincides with the image of $G \times P \omega_0 Q$ under the morphism $G \times G \to G/P \times G/Q$ sending $(x, y)$ to $(xP, xyQ)$. This morphism is the composition of the $k$-variety isomorphism $G \times G \to G \times G$ given by $(x, y) \mapsto (x, xy)$ with the usual quotient morphism $G \times G \to G/P \times G/Q$. The result follows from the remark that $G \times P \omega_0 Q$ is open.

**Corollary 2.14.** If $k$ is a local field, then $G(k) \cdot (P, \omega_0 Q)$ is open and closed in $[(P, \omega_0 Q)](k)$.

**Proof.** Lemma 2.13 implies that the orbit morphism $g \mapsto g \cdot (P, \omega_0 Q)$ is separable. The result follows from paragraph 3.18 of [7].

2.7. **Transversality.** The main aim of this section is to prove Proposition 2.17, which will later enable us to choose elements with favorable properties generically.

Let $G$ be a $k$-group, and let $\rho : G \to GL(E)$ be a finite-dimensional representation on a vector space $E$. Suppose moreover that $E$ is endowed with a $k$-structure with respect to which $\rho$ is rational. In this section we also assume that $\rho$ is irreducible over $k$—namely, $\rho(G)$ leaves invariant no subspace of $E$ which is defined over $k$.

We shall suppress mention of $\rho$ whenever possible, writing $\rho(g)e = g \cdot e$ when $g \in G$ and $e \in E$.

Let $V$ and $W$ be subspaces of $E$ defined over $k$ with $V > 0$ and $W < E$. Let us recall the following lemma of Tits.

**Lemma 2.15** (Lemma 3.10 of [22]). The set $\{g \in G \mid g \cdot V \not\subset W\}$ is $k$-open and nonempty.

The representation $\rho$ induces a $k$-morphism $G \times G \to \text{Gr}_{\dim V}(E) \times \text{Gr}_{\dim W}(E)$, where $\text{Gr}_m(E)$ denotes the Grassmann variety of $m$-dimensional subspaces of $E$. Suppose now that $P$ is the stabilizer in $G$ of $V$ (i.e., $P = \{p \in G \mid p \cdot V = V\}$), and $Q$ is the stabilizer in $G$ of $W$. If $P$ and $Q$ are parabolic subgroups, then this morphism descends to a $G$-equivariant $k$-morphism $G/P \times G/Q \to \text{Gr}_{\dim V}(E) \times \text{Gr}_{\dim W}(E)$,
given explicitly by \((g_1 P, g_2 Q) \mapsto (g_1 \cdot V, g_2 \cdot W)\). The morphism is equivariant under the diagonal \(G\)-action.

Denote by \(\text{Inc}\) the preimage under this morphism of \(\{(v, w) \mid v \subset w\}\). (We intentionally suppress mention of the dimensions \(\dim V\) and \(\dim W\).) Note that \(\text{Inc}\) is a \(k\)-closed and \(G\)-invariant subvariety of \(G/P \times G/Q\).

For the remainder of this section we assume \(G\) is \(k\)-split, \(k\)-connected, and almost \(k\)-simple. Fix a maximal \(k\)-split torus \(T\). Suppose \(B\) is a Borel subgroup of \(G\) defined over \(k\) and containing \(T\). As we discuss above, \(B\) gives rise to a word length on the Weyl group \(W = W(T, G)\). For clarity, we shall refer to the longest word according to \(B\) as \(w_B\) (rather than the more conventional \(w_0\) used above).

We say that \(B\) stabilizes a subset \(X \subset E\) if \(\rho(b)X = X\) for all \(b \in B\).

**Corollary 2.16.** Take \(V\) and \(W\) as in the previous lemma, with \(V > 0\) and \(W < E\), and take an element \(h \in Bw_B B\). Suppose that \(B\) stabilizes both \(V\) and \(W\). Then \(h \cdot V \not\subset W\).

**Proof.** By hypothesis, \(P = \text{Stab}_G(V)\) and \(Q = \text{Stab}_G(W)\) are (standard) parabolic subgroups whose intersection contains \(B\). If \(h \cdot V \subset W\), then \((hP, Q) \in \text{Inc}\). But \(\text{Inc}\) is \(G\)-invariant, and hence contains the orbit \(\{(hP, Q)\} = [(P, w_B Q)\}\). Because \(\text{Inc}\) is closed and \(\{(P, w_B Q)\}\) is dense in \(G/P \times G/Q\), it follows that \(G/P \times G/Q \subset \text{Inc}\).

In particular, we have that \((gP, Q)\in \text{Inc}\) for all \(g \in G\). Hence \(g \cdot V \subset W\) for all \(g\). The desired conclusion now follows from Lemma 2.15. \(\square\)

To simplify notation, we write \(A(g) = A(\rho(g))\) for \(g \in G(k)\), and similarly for \(A'(g)\) and \(\text{Cr}(g)\).

**Proposition 2.17.** Fix \(h \in G(k)\). If \(g \in G(k)\) is semisimple and \(A(g) \neq E\), then the set
\[U_{h,g} = \{u \in G \mid ^u h \cdot A(g) \not\subset \text{Cr}(g)\}\]
is \(k\)-open; and is \(k\)-dense provided \(h\) is not central in an absolutely almost simple \(k\)-subgroup of full rank in \(G\).

Given an element \(g \in T(k)\) we can choose a positive system \(\Delta^+\) for \(\Delta = \Delta(T, G)\) with the following property: If \(\alpha \in \Delta\) is such that \(|\alpha(g)| > 1\), then \(\alpha \in \Delta^+\). If \(B\) is the Borel subgroup associated to this choice of positive system, then \(B\) stabilizes \(A(g)\) as well as \(A'(g^{-1})\). In fact, if \(\lambda\) is the highest weight of \(\rho\) (with respect to the notion of positivity \(\Delta^+\)), then we can write \(A(g)\) explicitly as the direct sum of those weight spaces of the form \(E^{\lambda - \mu}\) where \(\mu\) is a nonnegative integral sum of simple roots \(\alpha\) with \(|\alpha(g)| = 1\). Likewise, we can realize \(A'(g^{-1})\) as the direct sum of the weight spaces \(not\) of the form \(E^{w_\mu \lambda + \mu}\), where \(\mu\) ranges over the same set of sums.

**Proof of Proposition 2.17.** The condition defining \(U_{h,g}\) is an open condition. To complete the proof, it suffices to show that the open sets \(U_1 = \{u \in G \mid ^u h \cdot A(g) \not\subset A'(g)\}\) and \(U_2 = \{u \in G \mid ^u h \cdot A(g) \not\subset A'(g^{-1})\}\) are nonempty.

Let \(T\) be a maximal \(k\)-split torus containing \(g\). By the remarks in the paragraph following the statement of Proposition 2.17, we see that \(A(g)\) and \(A'(g^{-1})\) are stabilized by a common \(k\)-defined Borel subgroup \(B\) containing \(T\). If \(u\) is such that \(^u h \in Bw_B B\), then it follows from Corollary 2.16 that \(^u h \cdot A(g) \not\subset A'(g^{-1})\). It follows from Proposition 2.11 that the set \(U_2\) of such \(u\) is nonempty.

On the other hand it is even easier to see that the set of \(u \in G\) such that \(^u h \cdot A(g) \not\subset A'(g)\) is large. In fact, \(U_1\) is nonempty simply by virtue of the conjugacy
(over $k$) of all Borel subgroups over $k$. Namely, $h$ can be conjugated into the stabilizer of $A(g)$, and we certainly have that $A(g) \not\subset A'(g)$. \hfill $\square$

In particular, if $h$ is a fixed element of $G(k)$ not central in an absolutely almost simple $k$-subgroup of full rank in $G$, then there is a nonempty open set of elements $g$ satisfying the condition: If $g \in G(k)$ is proximal, then $h \cdot A(g) \notin Cr(g)$. This follows from the dominance of the morphism $T \times G \to G$ given by $(t,v) \mapsto t^v$.

We also note that it is not necessary in Proposition 2.17 to assume that the element $g$ is semisimple. Jordan decomposition tells us that the Proposition holds without this assumption either if the field $k$ is perfect; or if the representation $\rho$ is absolutely irreducible and $G$ is absolutely almost simple.

We conclude this section with a useful result in a similar spirit. Denote by $P$ the projective space of our $G$-module $E$. The following results immediately from Lemma 2.15.

**Corollary 2.18** (Lemma 11 of [16]). If $g_1, g_2 \in G(k)$ are such that $A(g_1)$ and $A(g_2)$ are both proper subspaces of $P(k)$, then the set

$$\{ x \in G \mid x \cdot A(g_1), x \cdot A(g_1^{-1}) \notin Cr(g_2) \text{ and } A(g_2), A(g_2^{-1}) \notin x \cdot Cr(g_1) \}$$

is $k$-open and $k$-dense in $G$.

### 2.8. Unipotent one-parameter subgroups

Let $G$ be a connected, reductive complex algebraic group. We fix a maximal torus $T$ and a system $\Pi$ of simple roots for $\Delta = \Delta(T,G)$.

For a root $\alpha \in \Delta$, we denote by $U_\alpha$ the corresponding root subgroup; and by $S_\alpha$, the simple rank one subgroup generated by $U_\alpha$ together with $U_{-\alpha}$. Also, if $S \subset \Pi$, let us denote by $G_S$ the subgroup generated by all $S_\alpha$ for all $\alpha \in S$.

Let $E$ be a finite-dimensional module for $G$. Denote by $E^\mu$ the weight space of $E$ corresponding to weight $\mu$.

Suppose that $\mu_0$ is a weight of $E$ with the property that $\mu_0 - \alpha$ is not a weight of $E$. Consider the maximal $\alpha$-string through $\mu_0$: Namely, the longest possible string

$$\mu_0, \mu_0 + \alpha, \ldots, \mu_0 + m\alpha$$

of weights of $E$. Since $S_\alpha$ stabilizes $\bigoplus_{i=0}^{m} E^{\mu_0+i\alpha}$, we can infer that if $M$ is any $S_\alpha$-irreducible submodule of $E$, then $M$ is spanned by weight vectors for the action of $G$ on $E$, or equivalently, with its intersections with the weight spaces $E^\mu$.

It follows that for fixed $\alpha \in \Delta$, we can choose a basis $\{ e_i^\mu \}$ of $E$ with the properties that

(i) For any particular weight $\mu$, the vectors $e_i^\mu$ for $i = 1, \ldots, \dim E^\mu$ form a basis for $E^\mu$;

(ii) If $M$ is any $S_\alpha$-submodule of $E$, then $\{ e_i^\mu \}$ contains a basis for $M$; and

(iii) For all $i = 1, \ldots, \dim E^\mu$, either $U_\alpha$ fixes $e_i^\mu$ or there exists a $j$, $1 \leq j \leq \dim E^{\mu+\alpha}$, such that for every $t \in \mathbb{C}^\times$, we have

$$u_\alpha(t) \cdot e_i^\mu = e_j^\mu + t e_j^{\mu+\alpha} + \text{correction},$$

where the correction vector lies in $\bigoplus_{i=2}^{\infty} E^{\mu+i\alpha}$ and is of quadratic magnitude in $t$.

Consider the family of nilpotent endomorphisms $u_\alpha(t) - I$, where $I$ is the identity matrix. Given $e \in E^\mu$, there is a number $z$ such that the vector-valued function $(u_\alpha(t) - I)^z \cdot e$ is not zero identically in $t$, but this vector lies in the kernel of $u_\alpha(t) - I$.
for all $t \in \mathbb{C}$. It follows from property (iii) that for this $z$, the vector $(u_\alpha(t) - I)z \cdot e$ is independent of $t$, up to scalar.

For $M$ as in item (ii), we observe that $M$ is spanned by its intersections with the weight spaces $E^\mu$. Also, if $\dot{w}_\alpha \in G$ represents the simple reflection $w_\alpha \in W$ corresponding to $\alpha$, then $\dot{w}_\alpha$ normalizes $S\alpha$. It follows that $\dot{w}_\alpha \cdot (M \cap E^\mu) = M \cap E^{\mu - \nu}$.

The set of pairs $(\mu, i)$ indexing the aforementioned basis \{e^\mu_i \mid 1 \leq i \leq \dim E^\mu\} of $E$ possesses additional structure. By property (iii) above, we have that either $U_\alpha$ fixes $e^\mu_i$, or for some $j$ and every $t \in \mathbb{C}^\times$, we have $u_\alpha(t) \cdot e^\mu_i = e^\mu_i + t u_\alpha(t) \cdot e^\mu_{i+\alpha}$. In the latter case, let us say that $(\mu, i)$ is $\alpha$-linked to $(\mu + \alpha, j)$. It is true—though no mere consequence of notation—that $(\mu_1, i_1)$ is $\alpha$-linked to $(\mu_2, i_2)$ if and only if $(\mu_2, i_2)$ is $(-\alpha)$-linked to $(\mu_1, i_1)$. By an $\alpha$-chain of $E$ we intend a sequence of pairs $\langle(\mu_j, i_j)\rangle_{j=1,\ldots,t}$ such that each $(\mu_j, i_j)$ is $\alpha$-linked to its successor $(\mu_{j+1}, i_{j+1})$.

Let us introduce a bit of notation. Let $A$ be the weight lattice of $G$. Let $\mathcal{L} : A \times \mathbb{N} \to A$ be the projection onto the first coordinate. We observe that if $I$ is an $\alpha$-chain of $E$, then $\mathcal{L} I$ is an $\alpha$-string of weights of $E$.

If $X \subset E$, we define the set $\mathcal{L}_E X$ to be the smallest subset $K$ of $A$ satisfying

$$X \subset \bigoplus_{\mu \in K} E^\mu.$$ 

For example, for a basis vector $e^\mu_i$, we have the identity $\mathcal{L}_E(S\alpha \cdot e^\mu_i) = \mathcal{L} I$, where $I$ is the maximal $\alpha$-chain of $E$ containing $(\mu, i)$. As we observed above, $w_\alpha$ acts as an involution on the set $\mathcal{L}_E(S\alpha \cdot e^\mu_i)$.

Suppose $M$ is a subspace of a $E$ which is spanned by its intersections with the weight spaces $E^\mu$. Write $M^\mu = M \cap E^\mu$. By hypothesis, $M = \bigoplus_{\mu \in \mathcal{L}_E M} M^\mu$.

**Lemma 2.19.** Let $M$ be an $S\alpha$-submodule of $E$. Suppose $\mu$ and $\nu$ are weights of $\mathcal{L}_E M$ and satisfying

$$|\langle \mu, \alpha \rangle| \leq \langle \nu, \alpha \rangle.$$

Upon restriction to $E^\mu$ and projection to $E^\nu$, for $t \in \mathbb{C}^\times$, $u_\alpha(t)$ induces a linear map $M^\mu \cap \text{Span} U_{-\alpha} \cdot M^\nu \to M^\nu$. This map is injective for all but finitely many values of $t$.

Likewise, the map $M^\nu \to M^\mu$ induced from $u_{-\alpha}(t)$ is injective for all but finitely many $t \in \mathbb{C}$.

**Proof.** Without loss of generality, we assume that $G$ has rank one, $G = S\alpha$, and that $E$ is irreducible. In this case, the weight spaces are one-dimensional.

We consider the assertion about the map $E^\mu \to E^\nu$. Since $w_\alpha$ induces an involution on the set of weights of $E$ the hypothesis entails that $E^\nu$ is nonzero. The map in question is linear, and hence completely determined by the image of $e^\mu$. Having chosen a basis correctly, this map is given by $e^\mu \mapsto t^{\langle \nu - \mu, \alpha \rangle} e^\nu$, hence has full rank for almost every $t$.

The second assertion follows by symmetry. \hfill $\square$

We conclude with a remark about exterior powers of representations of rank-one groups. For simplicity we assume that $G$ itself has rank one, $G = S\alpha$. In this case, the weight lattice is just $\mathbb{Z} \omega_\alpha$, so we shall identify a weight $\mu = z \omega_\alpha$ with the number $\langle \mu, \alpha \rangle \| \omega_\alpha \| = z/\sqrt{2}$. If $E$ is an irreducible, finite-dimensional representation of $G$, let us denote by $e^i$ a nonzero weight vector of weight $i$. 
Suppose now \( m \leq n = \dim E \) and let \( I \) be an \( m \)-element subset of \( \{ \sqrt{\frac{n}{m}}, \sqrt{\frac{n}{m}} + \sqrt{\frac{2}{m}}, \ldots, \sqrt{\frac{n-1}{m}} \} \). Then we define
\[
e_i = \bigwedge_{i \in I} e_i.
\]
Denote by \([I]\) the weight \( \sum_{i \in I} i \). We observe that the vectors \( e_I \) form a basis of the \( m \)-fold exterior power \( \bigwedge^m E \) of \( E \). Moreover, the vector \( e_I \) is a weight vector of weight \([I]\) for the induced representation of \( G \).

**Lemma 2.20.** Let \( E \) be an irreducible, finite-dimensional module for a rank-one group \( G \). Let \( k \) be an arbitrary nonzero vector in \( E \), and \( m < n \). If \( E_0 \) is an irreducible component of \( \bigwedge^m E \), then the map \( x \mapsto x \wedge k \) does not annihilate \( E_0 \).

**Proof.** If \( m = n - 1 \), then \( \bigwedge^m E \) is irreducible as a \( G \)-module, and the result follows from Lemma 2.15, for example. Therefore, we assume \( m < n - 1 \).

Let \( h = \sum c_i e_i \) be a nonzero vector in \( E_0 \). If \( i \) is a weight of \( E \), and \( I \) is a set of such satisfying \( i \notin I \) and \( c_i \neq 0 \), then \( e_i \wedge h \neq 0 \).

In turn, we can write \( e_i \) as a sum \( \sum c_i u_\alpha(t) \cdot k \) or \( \sum c_i u_{-\alpha}(t) \cdot k \) over a finite set of times \( t \). (Up to exchanging \( \alpha \) with \(-\alpha \), we may assume the former.) It follows that \( u_\alpha(t) \cdot k \wedge h \) is nonzero for some/almost every \( t \in \mathbb{C} \). The same holds for \( k \wedge u_{-\alpha}(-t) \cdot h \).

Combining with Lemma 2.15, we obtain the following corollary:

**Corollary 2.21.** Let \( G \), \( E \) and \( E_0 \) be as in the Lemma 2.20. If \( k \) and \( h \) are arbitrary nonzero vectors in \( E \) and \( E_0 \), respectively, then the set of \( g \in G \) satisfying the equation
\[
k \wedge g \cdot h \neq 0
\]
is nonempty and open.

**Proof.** Lemma 2.20 implies that the set of \( x \) satisfying \( k \wedge x = 0 \) is a proper subspace.

### 3. Simultaneously proximal elements

We begin this chapter with a lemma due to Margulis and Soifer. Let \( k_1, \ldots, k_r \) be local fields. We denote by \( E_1, \ldots, E_r \) vector spaces over the fields \( k_1, \ldots, k_r \), respectively; and by \( P_1, \ldots, P_r \) their respective projective spaces.

**Lemma 3.1** (Compare Lemma 3 of [16]). Assume given for \( i = 1, \ldots, r \) a sequence of semisimple elements \( h_i \in \text{GL}(E_i) \). There exists an infinite subset \( N \subset \mathbb{N} \) such that for any \( i \), if \( \lambda \) and \( \mu \) are eigenvalues of \( h_i \) whose absolute value is maximum, we have \( (\lambda^{-1}\mu)^z \to 1 \) as \( z \) tends to infinity along \( N \).

**Proof.** We prove by induction on \( q \) that there exists a set \( N_q \subset \mathbb{N} \) with the properties described in the statement of the lemma satisfied for \( h_i, i = 1, \ldots, q \). The base case is given in Lemma 3.9.(i) of [22].

Assume given \( N_{q-1} \) and suppose \( \lambda \) and \( \mu \) are eigenvalues of \( h_q \) of absolute value maximal among eigenvalues of \( h_q \). By local compactness, the unit ball in \( k_q \) is compact. Since \( |\lambda\mu^{-1}| = 1 \), there exists a sequence \( \{ z_m \}_{m \in \mathbb{N}} \) of elements of \( N_{q-1} \) such that \( z_m \to \infty \) and \( (\lambda^{-1}\mu)^{z_m} \) converges as \( m \to \infty \). Extracting a subsequence \( \{ z_{m_n} \} \) such that successive differences \( z_{m_n} - z_{m_{n-1}} \) tend to infinity in \( n \), we see that the set \( N_q = \{ z_m - z_{m-1} \mid n \in \mathbb{N} \} \) satisfies all the conditions set forth in the statement of the lemma.
3.1. Density theorems. Eventually, we will want to choose some elements of $G(k)$ which act very proximally with respect to several given representations. Speaking informally, we shall call such elements \textit{simultaneously very proximal} elements, and likewise for simultaneously proximal elements. In this section, we will show that if we can find simultaneously proximal elements, then not only are such elements abundant, but so are simultaneously very proximal elements. In the next section, we will show that there exist simultaneously proximal elements.

Let $k$ be an infinite field (not necessarily local); $k_1, \ldots, k_r$, locally compact, valued extensions of $k$; $E_1, \ldots, E_r$ vector spaces, with each $E_i$ endowed with a fixed $k_i$-structure, $i = 1, \ldots, r$. Denote by $P_i$ the projective space of $E_i$. Let $G$ be a connected, reductive $k$-group, $S$ a Zariski-dense subgroup of $G(k)$, and assume given a collection $\{\rho_i : G \to \text{GL}(E_i)\}_{i=1,\ldots,r}$ of finite-dimensional representations of $G$. Suppose moreover that each $\rho_i$ is $k_i$-rational and irreducible over $k_i$.

**Proposition 3.2** (Compare to Lemma 6 of [16]). Suppose that the intersection $\bigcap_{i=1}^r \Omega_+(\rho_i, S)$ is nonempty. Then $\bigcap_{i=1}^r \Omega_+(\rho_i, S)$ is $k$-dense in $G(k)$.

**Proof.** Let $g$ be an element of the intersection $\bigcap_{i=1}^r \Omega_+(\rho_i, S)$, the existence of which is guaranteed by our hypothesis. For brevity of notation, we write $g_i = \rho_i(g) \in \text{GL}(E_i)(k_i)$. Consider the set

$$U_i = \{x \in G \mid x \cdot A(g_i) \not\subset A'(g_i)\}.$$ 

Each $U_i$ is a Zariski-open and nonempty subset of $G$. By connectedness, the intersection $U = \bigcap_{i=1}^r U_i$ is Zariski-dense. Since $G$ is reductive, it follows that $G(k)$—and hence also $S$—is Zariski-dense in $G$ (cf. Lemma 13.3.9 of [18]). Therefore $S \cap U$ is $k$-dense in $G(k)$.

Let $h$ be taken from the intersection $S \cap U$. Then for all $i = 1, \ldots, r$ we have $h \cdot A(g_i) \subset P_i \setminus A'(g_i)$. Let $K_i$ be a compact subset of $P_i(k_i)$ such that the singleton $A(g_i)$ is contained in the interior of $K_i$. (Here we mean ‘compact’ and ‘interior’ in the topology induced from the topology of the local field $k_i$.)

By Lemma 2.5, there exists a number $q > 0$ such that $\|g_i^z\|_{K_i} < q$, all $i = 1, \ldots, r$. Let us write $L_i = h \cdot K_i$. Since $L_i \subset P_i \setminus A'(g_i)$, by Lemma 2.6, it follows that there exists an integer $N_h$ such that $\|g_i^z\|_{L_i} < 1/q$ and $g_i^z \cdot L_i \subset \text{Int} K_i$, all $i = 1, \ldots, r$ and $z > N_h$. Let $N \subset \mathbb{N}$ be given so that the set $\{g^z \mid z \in N\}$ is $k$-connected, and let $N'(h) = \{m \in N \mid m > N_h\}$. By definition of $q$, we have $\|\hat{\rho}_i(g^z h)\|_{K_i} < 1$ and $g_i^z h \cdot K_i \subset \text{Int} K_i$, all $i = 1, \ldots, r$ and $z$ exceeding $N_h$.

Therefore Lemma 2.6.(ii) implies that $g^z h$ lies in the intersection $\bigcap_{i=1}^r \Omega_+(\rho_i, S)$ for all $z > N_h$. Denote by $\text{Cl}_k(\cdot)$ the $k$-closure operator. Since $N \setminus N'(h)$ is finite and the set $\{g^z h \mid z \in N\}$ is $k$-connected, it follows that

$$g^z h \in \text{Cl}_k\left(\bigcap_{i=1}^r \Omega_+(\rho_i, S)\right)$$

for all $z \in N$. That this is true for every $h$ in the $k$-dense set $S \cap U$ implies that $G \subset \text{Cl}_k(\bigcap_{i=1}^r \Omega_+(\rho_i, S))$. This completes the proof. \hfill \Box

**Proposition 3.3** (Compare Proposition 3.11 of [22]). If $\bigcap_{i=1}^r \Omega_+(\rho_i, S)$ is $k$-dense in $G(k)$, then so is $\bigcap_{i=1}^r \Omega_0(\rho_i, S)$.

**Proof.** $G$ contains a Zariski-open set of semisimple elements [19]. Because $G$ is reductive, $G(k)$—and hence also $\bigcap_{i=1}^r \Omega_+(\rho_i, S)$—is Zariski-dense in $G$. Let $g^{-1}$
be a semisimple element of \( \bigcap_{i=1}^r \Omega_+(\rho_i, S) \). For convenience we use the notation \( g_i = \rho_i(g) \).

By Lemma 2.15, the set
\[
\bigcap_{i=1}^r \left( \{ x \in G \mid x \cdot A(g_i) \not\subset A'(g_i^{-1}) \} \cap \{ x \in G \mid x^{-1} \cdot A(g_i) \not\subset A'(g_i^{-1}) \} \right)
\]
is nonempty and open. Let \( h \in S \) be an element of this intersection, and set
\[
B_i = (h \cdot A'(g_i)) \cup (h \cdot A(g_i) \cap A'(g_i^{-1})),
\]
\[
B'_i = A'(g_i) \cup (A(g_i) \cap h \cdot A'(g_i^{-1})),
\]
and \( U_i = \{ x \in G \mid x \cdot A(g_i^{-1}) \not\subset B_i \) and \( h \cdot A(g_i^{-1}) \not\subset x \cdot B_i' \}. Because of the conditions set on \( h \), we have \( B_i \neq P_i \) and \( B'_i \neq P_i \), and it follows from Lemma 2.15 that \( U = \bigcap_{i=1}^r U_i \) is open and dense in \( G \). Therefore \( S \cap U \) is \( k \)-dense in \( G(k) \). Let \( u \in S \cap U \).

Set \( \pi_i = \text{proj}(A'(g_i), A(g_i)) \) and \( \pi'_i = \text{proj}(h \cdot A'(g_i), h \cdot A(g_i)) \). We have \( u \cdot A(g_i^{-1}) \not\subset h \cdot A'(g_i) \) and \( \pi'_i(u \cdot A(g_i^{-1})) \not\subset A'(g_i^{-1}) \). Similarly, \( u^{-1}h \cdot A(g_i^{-1}) \not\subset A'(g_i) \) and \( \pi_i(u^{-1}h \cdot A(g_i^{-1})) \not\subset h \cdot A'(g_i) \). Let \( \gamma_i \) (resp. \( \gamma'_i \)) denote the set of all \( \bar{A}(g_i^{-1}) \) (resp. \( u^{-1}h \cdot A(g_i^{-1}) \)) such that
\[
u \cdot Y_i \cap h \cdot A'(g_i^{-1}) = \emptyset, \quad \pi'_i(u \cdot Y_i) \cap A'(g_i^{-1}) = \emptyset,
\]
\[
Y'_i \cap A'(g_i) = \emptyset, \quad \pi_i(Y'_i) \cap h \cdot A'(g_i^{-1}) = \emptyset.
\]
Let \( Z_i \) (resp. \( Z'_i \)) be a compact neighborhood of \( \pi'_i(u \cdot Y_i) \) (resp. \( \pi_i(Y'_i) \)) in \( P_i \) whose intersection with \( A'(g_i^{-1}) \) (resp. \( h \cdot A'(g_i^{-1}) \)) is empty. By Lemma 2.7.(i) and Lemma 2.5, there exists a number \( q > 0 \) such that
\[
\|\hat{\rho}_i(hg^zh^{-1}u)|\gamma_i\| < q \quad \text{and} \quad \|\hat{\rho}_i(g^z)|\gamma'_i\| < q
\]
for all \( i = 1, \ldots, r \) and \( z \in \mathbb{N} \).

Lemma 3.1 implies that there exists an infinite subset \( N \subseteq \mathbb{N} \) such that for every pair \( \lambda, \mu \in \Omega(g_i) \), and every \( i = 1, \ldots, r \) we have \( (\lambda \mu^{-1})^z \to 1 \) as \( z \to \infty \) along \( N \). The linear transformation \( \rho_i(hgh^{-1}) \) has the same eigenvalues as \( g_i \). Therefore, Lemma 2.7.(ii) implies that, for almost all \( z \in N \) we have
\[
hg^zh^{-1}u \cdot Y_i \subseteq Z_i \quad \text{and} \quad g^z \cdot Y'_i \subseteq Z'_i.
\]
Upon replacing \( N \) by a subset, we may assume in addition that the set \( \{ g^z \mid z \in N \} \) is \( k \)-connected.

By Lemma 2.6.(i), we have for almost all \( z \in N \) that
\[
\|\hat{\rho}_i(hg^zh^{-1})|Z_i\| < 1/q, \quad g^{-z} \cdot Z_i \subseteq u \cdot \text{Int} \gamma'_i,
\]
\[
\|\hat{\rho}_i(hg^zh^{-1})|Z'_i\| < 1/q\|u^{-1}\|, \quad h^{-z}h^{-1} \cdot Z'_i \subseteq u \cdot \text{Int} \gamma'_i.
\]
Let \( N'(u) \) denote the set of all \( z \in N \) such that (3), (4), and (5) hold simultaneously, all \( i = 1, \ldots, r \).

For all \( z \in N'(u) \), we have
\[
g^{-z}hg^zh^{-1}u \cdot Y_i \subseteq \text{Int} \gamma_i, \quad \|\hat{\rho}_i(g^{-z}hg^zh^{-1}u)|\gamma_i\| < 1,
\]
\[
u^{-1}hg^zh^{-1}g^z \cdot Y'_i \subseteq \text{Int} \gamma'_i, \quad \|\hat{\rho}_i(u^{-1}hg^zh^{-1}g^z)|\gamma'_i\| < 1;
hence, by Lemma 2.6.(ii),

\[ g^{-z}h_g h^{-1} u \in \bigcap_{i=1}^{r} \Omega_0(\rho_i, S). \]

Since \( N \setminus N'(u) \) is finite and the set \( \{g^z \mid z \in N\} \) is \( k \)-connected, it follows that

\[ g^{-z}h_g h^{-1} u \in \text{Cl}_k \left( \bigcap_{i=1}^{r} \Omega_0(\rho_i, S) \right) \]

for all \( z \in N \). That this is true for every \( u \) in the \( k \)-dense set \( S \cap U \) implies that \( G \subset \text{Cl}_k(\bigcap_{i=1}^{r} \Omega_0(\rho_i, S)) \). This completes the proof. \( \square \)

**Proposition 3.4** (Compare to Lemma 9 of [16]). Let \( H \) be a finite-index subgroup of \( S \) and \( Hg \) a coset of \( H \) in \( S \). If \( \bigcap_{i=1}^{r} \Omega_+ (\rho_i, S) \) is nonempty, then so is \( \bigcap_{i=1}^{r} \Omega_0 (\rho_i, Hg) \).

**Proof.** Since \( G \) is \( k \)-connected, \( H \) is Zariski-dense in \( G \). The previous two propositions tell us that we can find an element \( h \) of \( \bigcap_{i=1}^{r} \Omega_0(\rho_i, H) \). Set \( h_i = \rho_i(h) \).

Consider the set

\[ U_i = \{ x \in G \mid xg \cdot A(h_i) \not\subset A'(h_i) \} \cap \{ x \in G \mid A(h_i^{-1}) \not\subset xg \cdot A'(h_i^{-1}) \}. \]

It follows from Lemma 2.15 that \( U = \bigcap_{i=1}^{r} U_i \) is open and dense in \( G \). Therefore \( H \cap U \) is \( k \)-dense in \( G(k) \). Let \( x \in H \cap U \).

Set \( h_0 = xg \in Hg \). By construction, we have for all \( i \) that \( h_0 \cdot A(h_i) \not\subset A'(h_i) \) and \( h_0^{-1} \cdot A(h_i^{-1}) \not\subset A'(h_i^{-1}) \). Since the sets \( A(h_i^\pm 1) \) are singletons, there exist compact sets \( K_i \) and \( K_i^{-} \) in \( P_i \) such that

\[ A(h_i) \subset \text{Int} K_i, \quad h \cdot K_i \subset P_i \setminus A'(h_i), \]

\[ A(h_i^{-1}) \subset \text{Int} K_i^{-}, \quad h^{-1} \cdot K_i^{-} \subset P_i \setminus A'(h_i^{-1}), \]

all \( i = 1, \ldots, r \). Write \( M_i = h_0 \cdot K_i \) and \( M_i^{-} = h_0^{-1} \cdot K_i^{-} \). By Lemma 2.5, there exists a number \( q > 0 \) such that

\[ \max_{1 \leq i \leq r} \{ \| \hat{\rho}_i(h_0) \|_{K_i}, \| \hat{\rho}_i(h_0^{-1}) \|_{M_i} \} < q. \]

Since \( M_i \subset P_i \setminus A'(h_i) \) and \( M_i^{-} \subset P_i \setminus A'(h_i^{-1}) \), it follows from Lemma 2.6.(i) that there exists a number \( N_1 \) such that \( \| \hat{\rho}_i(h^z) \|_{M_i} < 1/q \) and \( \| \hat{\rho}_i(h^{-z}) \|_{M_i^{-}} < 1/q \) for each \( z > N_1 \) and \( i = 1, \ldots, r \). Moreover, there exists a number \( N_2 > 0 \) such that \( h^z \cdot M_i \subset \text{Int} K_i \) and \( h^{-z} \cdot M_i^{-} \subset \text{Int} K_i^{-} \) for each \( z > N_2 \) and \( i = 1, \ldots, r \).

By choice of \( q \), we have that

\[ h^z h_0 \cdot K_i \subset \text{Int} K_i, \quad \| \hat{\rho}_i(h^z h_0) \|_{K_i} < 1, \]

\[ h^{-z} h_0^{-1} \cdot K_i^{-} \subset \text{Int} K_i^{-}, \quad \| \hat{\rho}_i(h^{-z} h_0^{-1}) \|_{K_i^{-}} < 1, \]

for each \( z > \max(N_1, N_2) \) and \( i = 1, \ldots, r \). So from Lemma 2.6.(ii) it follows that \( h^z h_0, h^{-z} h_0^{-1} \in \bigcap_{i=1}^{r} \Omega_+ (\rho_i, S) \) for each \( z > \max(N_1, N_2) \). But \( h^{-z} h_0^{-1} \) is conjugate to \( h_0^{-1} h^{-z} = (h^z h_0)^{-1} \). It follows that for such \( z \) we have \( h^z h_0 \in \bigcap_{i=1}^{r} \Omega_0(\rho_i, Hg) \). \( \square \)

We conclude this section with a result on density, though not about proximality. It says that for a subset of a finitely generated linear group, profinite density is better than Zariski density, and almost as good as Zariski openness.
Lemma 3.5. Let \( \Gamma \) be a finitely generated, Zariski-dense subgroup of a reductive algebraic group \( G \); \( Hg \) a coset in \( \Gamma \) of a finite-index subgroup \( H < \Gamma \); and \( Y \) a Zariski-open subset of \( G \). Then \( Y \cap Hg \) is Zariski dense in \( G \).

Proof. Upon replacing \( Y \) by \( Y \cdot g^{-1} \), we may reduce to the case \( g = 1 \). Since \( H \) is of finite index in \( \Gamma \), it is Zariski dense, and we reduce to the case \( \Gamma = H = Hg \). The result now follows from the proof of Proposition 3 of [16], for example. \( \square \)

3.2. Existence. As promised, in this section we will produce elements which act proximally with respect to each of several given representations.

Fix an infinite field \( k \) and assume that \( k_1, \ldots, k_r \) is a family of local fields extending \( k \). As before, \( G \) is a connected, reductive \( k \)-group, and \( S \) a Zariski-dense subgroup of \( G(k) \). For \( i = 1, \ldots, r \), let \( E_i \) be a finite-dimensional vector space endowed with a \( k_i \)-structure. Assume given a family \( \rho_i : G \to \text{GL}(E_i) \) of \( k_i \)-rational, \( k_i \)-irreducible representations.

Lemma 3.6. If \( \Omega_+(\rho_i, S) \neq \emptyset \) for all \( i = 1, \ldots, r \), then also

\[
\bigcap_{i=1}^r \Omega_+(\rho_i, S) \neq \emptyset.
\]

Proof. By induction on \( r \). For \( r = 1 \), there is nothing to prove. Now suppose that both \( \bigcap_{i=1}^{r-1} \Omega_+(\rho_i, S) \) and \( \Omega_+(\rho_r, S) \) are nonempty. By Proposition 3.2, we can find semisimple elements \( g \) and \( h \) such that \( g \in \bigcap_{i=1}^{r-1} \Omega_+(\rho_i, S) \) and \( h \in \Omega_+(\rho_r, S) \). For convenience, we shall write \( h_i = \rho_i(h) \) and \( g_i = \rho_i(g) \) for \( i = 1, \ldots, r \). Also, set \( \pi_i = \text{proj}(A'(h_i), A(h_i)) \) for \( i = 1, \ldots, r-1 \), and \( \pi_r = \text{proj}(A'(g_r), A(g_r)) \).

Since the \( \rho_i \) are irreducible and \( S \) is dense, we can find \( v \in S \) be such that \( v \cdot A(g_i) \subset P_i \setminus A'(g_i) \) for \( i = 1, \ldots, r-1 \). Consider the set of elements \( u \in G \) simultaneously satisfying the following \( r + 1 \) conditions:

\[
\begin{align*}
&u \cdot \pi_i (v \cdot A(g_i)) \subset P_i \setminus A'(g_i), \quad i = 1, \ldots, r-1, \\
&u \cdot A(h_r) \subset P_r \setminus A'(g_r).
\end{align*}
\]

(6) \( u \cdot A(h_r) \not\subset (v^{-1} \cdot A'(h_r) \cap A(g_r)) \cup A'(g_r). \)

By Lemma 2.15, the set of such \( u \) is Zariski-open. Since \( S \) is dense, we can choose such an element \( u \) in \( S \). By (6), we have \( v \cdot \pi_r (u \cdot A(h_r)) \subset P_r \setminus A'(h_r) \).

Choose compact sets \( L_i \) such that

\[
\begin{align*}
L_i &\subset P_i \setminus A'(g_i) \\
u \cdot \pi_i(v \cdot A(g_i)) &\subset \text{Int} L_i,
\end{align*}
\]

for \( i = 1, \ldots, r-1 \); and

\[
\begin{align*}
L_r &\subset P_r \setminus A'(h_r) \\
v \cdot \pi_r(u \cdot A(h_r)) &\subset \text{Int} L_r.
\end{align*}
\]

Let \( N \subset \mathbb{N} \) be the infinite set whose existence is guaranteed by Lemma 3.1, where the set of given semisimple elements is \( h_1, h_2, \ldots, h_{r-1}, g_r \). We choose compact sets \( K_i \) such that

\[
\begin{align*}
A(g_i) &\subset \text{Int} K_i \\
v \cdot K_i &\subset P_i \setminus A'(h_i) \\
u \cdot \pi_i(v \cdot K_i) &\subset \text{Int} L_i
\end{align*}
\]
for $i = 1, \ldots, r - 1$; and

$$A(h_r) \subset \text{Int } K_r$$

$$u \cdot K_r \subset P_r \setminus A(g_r)$$

$$v \cdot \pi_r(u \cdot K_r) \subset \text{Int } L_r.$$  

By Lemma 2.7, we have that there exist numbers $N_0$ and $q$ such that, if we set $N' = \{ z \in N \mid z > N_0 \}$, then

$$uh^z v \cdot K_i \subset \text{Int } L_i, \quad \text{all } z \in N'; \quad \text{and} \quad$$

$$\| \hat{\rho}_i(h^z) \|_{\pi_i K_i} < q \quad \text{for all } z \in \mathbb{N}$$

for $i = 1, \ldots, r - 1$; and

$$vg^z u \cdot K_r \subset \text{Int } L_r, \quad \text{all } z \in N'; \quad \text{and} \quad$$

$$\| \hat{\rho}_r(g^z) \|_{\pi_r K_r} < q \quad \text{for all } z \in \mathbb{N}.$$  

By Lemma 2.6.(i), there exists a number $N_1$ such that

$$\| \hat{\rho}_i(g^z) \|_{L_i} < (q \| \hat{\rho}_i(u) \|_{u^{-1} L_i} \| \hat{\rho}_r(v) \|_{K_r})^{-1}$$

$$g^z \cdot L_i \subset K_i$$

for all $z > N_1$ and $i = 1, \ldots, r - 1$; and

$$\| \hat{\rho}_r(h^z) \|_{L_r} < (q \| \hat{\rho}_r(v) \|_{u^{-1} L_r} \| \hat{\rho}_r(u) \|_{K_r})^{-1}$$

$$h^z \cdot L_r \subset K_r$$

for all $z > N_1$.

It follows from Lemma 2.6.(ii) that if $z > N_1$ and $s \in N'$, that $\rho_i(g^z u h^z v)$ is proximal for $i = 1, \ldots, r - 1$, as is $\rho_r(h^z v g^z u)$. But $h^z v g^z u$ is conjugate to $g^z u h^z v$. It follows that if $s, z > \max(N_0, N_1)$ are both taken in $N$, then we have $g^z u h^z v \in \bigcap_{i=1}^r \Omega_+(\rho_i, S).$ \hfill \qed

Momentarily breaking with our notational convention, we denote by $\hat{S}$ the profinite completion of $S$.

**Corollary 3.7.** If $\Omega_+(\rho_i, S) \neq \emptyset$ for all $i = 1, \ldots, r$, then the set

$$\bigcap_{i=1}^r \Omega_0(\rho_i, S)$$

of simultaneously very proximal elements is Zariski-dense in $G$, as well as dense in the profinite topology on $\hat{S}$.

4. Strong transversality for simply laced groups

4.1. **Combinatorial data.** Let $\Delta$ be a reduced root system, and $\Lambda$ the associated weight lattice. Suppose given an affine hyperspace $K$ of $\Lambda \otimes \mathbb{R}$, and denote by $\mathcal{K}$ the parallel linear hyperspace. The goal of this section is to define some combinatorial data, depending on (the linearization of) our affine hyperspace $K$, which develop consistently as we restrict our attention to smaller and smaller root subsystems.

Fix for reference a system $\Pi$ of simple roots. As usual, we identify $\Pi$ with the vertex set of the Dynkin diagram of $\Delta$. Let us call a simple root $\omega$ *extremal* if the subset $\Pi \setminus \{ \omega \}$ corresponds to the vertex set of a connected subgraph of the Dynkin diagram of $\Delta$. Given an extremal vertex $\omega$ and an element $w \in W(\Delta)$ such that
Let $\Delta$ be an irreducible, reduced root system of simply laced type and of higher rank; $K$, $\mathcal{D}K$, $\Pi$, and $\omega$ as above. Then there exists an element $w \in W$ such that

1. $w \cdot \omega$ is not parallel to $K$;
2. $\Pi \setminus \{\omega\} \not\subseteq w^{-1} \cdot \mathcal{D}K$; and
3. $w \cdot \alpha_w$ is not orthogonal to $K$.

Proof. It follows from irreducibility of $\Delta$ that the Weyl group cannot stabilize the union of a proper subspace and its orthogonal complement. In particular, there exists $w \in W$ such that $w \cdot \omega$ lies outside $\mathcal{D}K \cup \mathcal{D}K^\perp$. For a nonzero root $\alpha \in \Delta$, let $w_\alpha \in W$ be the corresponding simple reflection. Suppose first that $w$ does not satisfy condition (ii). Then $w_{\omega} w_{\alpha_w + \beta_{q-1}}$ verifies the conclusion of the lemma.

Therefore, we suppose that $w$ already verifies condition (ii). If $\alpha_w$ is orthogonal to $\omega$, then we have that $w \cdot \alpha_w$ also lies outside of $\mathcal{D}K \cup \mathcal{D}K^\perp$. We are left with the case where $\alpha_w \sim \omega$, i.e. $\langle \alpha_w, \omega \rangle = -1$ (using the normalization that $\langle \alpha, \alpha \rangle = 2$ for all roots $\alpha \neq 0$). In the case where $w$ fails to verify condition (iii), then $\alpha_w w = \alpha_w$, and $w_{\omega} w$ verifies the conclusion of the lemma.

Let $w$ be as in Lemma 4.1. Upon replacing $\Pi$ by $w \cdot \Pi$, we may assume $w = 1$. That is,

1. $\omega$ is not parallel to $K$;
2. $\Pi \setminus \{\omega\} \not\subseteq \mathcal{D}K$;
3. $\alpha = \alpha_w$ is not orthogonal to $K$; and
4. the combinatorial path $\alpha = \beta_1 \sim \cdots \sim \beta_q = \omega$ connecting $\alpha$ to $\omega$ in the Dynkin diagram of $\Delta$ passes entirely through vertices $\beta_2, \ldots, \beta_{q-1} \in \Pi$ lying parallel to $K$.

Define $\Delta' = \Delta \cap \text{Span}(\Pi \setminus \{\omega\})$. By construction, $\Delta'$ is again irreducible and of simply laced type. By definition, $\alpha$ is not parallel to $K$. It follows that $K' = K \cap \text{Span} \Delta'$ has codimension one in $\text{Span} \Delta'$. Provided $\Delta'$ has higher rank, we can choose a set $\Pi'$ of simple roots for $\Delta'$ and a simple root $\omega' \in \Pi'$ such that properties (CD1)-(CD4) hold with $\omega$, $K$, $\Pi$, and $\alpha$ replaced by $\omega'$, $K'$, etc. If rank $\Delta' = 1$, then we simply set $\alpha' = \omega'$ to be a nonzero root, and $\Pi' = \{\alpha'\}$.

4.2. Weight spaces for simply laced groups. Let $G$ be a connected, reductive complex algebraic group. We fix a maximal torus $T$ and a system $\Pi$ of simple roots for $\Delta(T, G)$. Denote by $B$ the corresponding Borel subgroup of $G$.

In this section, we will also assume that $G$ is a simply laced group, i.e. a group of type $A$, $D$, or $E$ (modulo its unipotent radical). We begin essentially by formulating statements analogous to those in §2.8, but in the higher rank situation. We recall the fact that the unipotent radical $B_u$ of $B$ is isomorphic as an algebraic variety to
affine space of dimension $|\Delta^+|$. Explicitly, the map

$$t = (t_\beta)_{\beta \in \Delta^+} \mapsto \prod_{\beta \in \Delta^+} u_\beta(t_\beta)$$

is an isomorphism of complex algebraic varieties, where the product is taken in any fixed order on $\Delta^+$.

Fix an numbering $\langle \beta_1, \beta_2, \ldots \rangle$ of $\Delta^+$ compatible with the poset structure: that is to say, with the property that $\beta_i < \beta_j$ implies $i > j$. Fix an irreducible representation of $G$ on a finite-dimensional complex vector space $E$. Write $u_i = u_{\beta_i}$ and $t_i = t_{\beta_i}$. To an arbitrary but fixed nonzero vector $e \in E$, we can associate an endomorphism $\prod_{i=1}^{\mid \Delta^+ \mid} (u_i(t_i) - I)^{z_i} \cdot e$ is not zero identically in $(t_1, \ldots, t_j)$, but this vector itself lies in the kernel of $u_j(t_j) - I$ for all $t_j \in \mathbb{C}$. If $e$ is a weight vector, then so is $\prod_{i=1}^{\mid \Delta^+ \mid} (u_i(t_i) - I)^{z_i} \cdot e$.

**Lemma 4.2.** The vector $\prod_{i=1}^{\mid \Delta^+ \mid} (u_i(t_i) - I)^{z_i} \cdot e$ is independent of $t = (t_i)_{i=1, \ldots, \mid \Delta^+ \mid}$, up to scalar. If $E$ is irreducible as a complex representation of $G$, then the line in this direction is the unique $B$-invariant line in $E$.

**Proof.** The first assertion follows from the analogous phenomenon in rank one. For the second, we need only show invariance under $B$.

Taking $e \in E^\mu$, we see that for almost all $t$, we have that $\mu + \sum_{i=1}^{\mid \Delta^+ \mid} z_i \beta_i$ is a lowest weight in $\ell_E \left( \prod_{i=1}^{\mid \Delta^+ \mid} (u_i(t_i) - I)^{z_i} \cdot e \right)$, and likewise when $e$ is replaced by any element of $B \cdot e$. Therefore

$$\ell_E \left( b \cdot \prod_{i=1}^{\mid \Delta^+ \mid} (u_i(t_i) - I)^{z_i} \cdot e \right) = \ell_E \left( \prod_{i=1}^{\mid \Delta^+ \mid} (b(u_i(t_i)) - I)^{z_i} \cdot e \right)$$

for any $b \in B$. The result now follows from the compatibility of the numbering with the poset structure on $\Delta^+$, together with the fact that $\prod_{i=1}^{\mid \Delta^+ \mid} (u_i(t_i) - I)^{z_i} \cdot e$ is independent of $t$, up to scalar. \[\square\]

Given a set of simple roots $S \subset \Pi$, we can identify $S$ with a subgraph of the Dynkin diagram of $G$. We shall be interested in those $S$ for which $S$ and $S' = \Pi \setminus S$ correspond to connected (nonempty) subgraphs.

Denote by $G_S$ the group generated by the root subgroups $U_\alpha$, $U_{-\alpha}$ for $\alpha \in S$. Set $\Delta_S = \Delta \cap \text{Span} \, S$ and $\Delta_S^+ = \Delta^+ \cap \Delta_S$. By symmetry, we can adopt similar notations for the situation where $S$ and $S'$ exchange rôles.

Before turning to the next lemma, it is useful to recall the commutation relations for unipotent subgroups $U_\alpha$, $\alpha' \in \Delta^+$. If $\alpha, \beta \in \Delta^+$ are such that $\alpha + \beta$ is not a root, then $U_\alpha$ and $U_\beta$ commute. Otherwise, $u_\alpha(t)$ and $u_\beta(s)$ commute to $u_{\alpha + \beta}(cst)$ for some nonzero constant $c \in \mathbb{C}$ depending only on $\alpha$ and $\beta$. Though not strictly necessary for our purposes, we recall that, upon normalizing the unipotent subgroups $U_\alpha$, $\alpha' \in \Delta$, we can take $c = c(\alpha, \beta) = \pm 1$ for all positive roots $\alpha$ and $\beta$.

Suppose that $E' < E$ is a $G_S$-submodule of $E$. If we assume moreover that $E'$ is irreducible as a $G_S$-submodule, then there is a unique line in $E'$ which is
stable under all \( U_\beta, \beta \in \Delta^+_S \). Let \( e' \in E' \) denote a nonzero vector in the invariant direction. Since \( S' \) is the vertex set of a connected subgraph, we can apply the preceding considerations to produce numbers \( y_\alpha, \alpha \in \Delta^+_S \), such that the vector \( \prod_{\alpha \in \Delta^+_S} (u_\alpha(t_\alpha) - I)^{y_\alpha} \cdot e' \) (in a fixed ordering on \( \Delta^+_S \)) is independent of \( t = (t_\alpha)_{\alpha \in \Delta^+_S} \), up to scalar. Let \( e'' \) be a nonzero vector lying in this direction. Applying Lemma 4.2 to the group \( G_{S'} \), we obtain that \( e'' \) is invariant under those subgroups \( U_\alpha \) with \( \alpha \in \Delta^+_S \).

\textbf{Lemma 4.3.} Let \( S \) be the complement in \( \Pi \) of an extremal \( \omega \). Suppose \( E', e', e'' \) are as above. Then \( e'' \) is invariant under \( U_\beta \) for all \( \beta \in \Delta^+ \setminus \Delta_S \).

\textit{Proof.} We know that \( e'' \) is a weight vector, say \( e'' \in E'' \). Then \( \mu \) is the highest weight in the set \( \mathcal{L}_E(S_\omega \cdot e') = \mathcal{L}_E(U_\omega \cdot e') \cup \mathcal{L}_E(U_{-\omega} \cdot e') \). Fix a simple root \( \alpha \in S \). We compute

\[ u_\omega(t_\omega) \cdot e' = u_\omega(t_\omega) u_\alpha(t) \cdot e' \]

\[ = u_{\alpha + \omega}(t_\omega) u_\alpha(t) \cdot e' \]

(7)

Now consider a positive root \( \beta \notin \Delta_S \) such that \( U_\beta \) stabilizes \( e'' \). If \( \alpha + \beta \) is a root, then \( u_\alpha(-t) u_\beta(-s/t) u_\alpha(t) = u_{\alpha + \beta}(s) u_\beta(-s/t) \). By hypothesis on \( \beta \), it follows that \( u_{\alpha + \beta}(s) \cdot e'' = u_\alpha(t) u_\beta(-s/t) u_\alpha(t) \cdot e'' \). Combining with equation (7), we have

\[
\mathcal{L}_E(u_{\alpha + \beta}(s) \cdot e'') = \mathcal{L}_E(u_\alpha(-t) u_\beta(-s/t) u_\alpha(t) \cdot e'') \\
\subseteq \mathcal{L}_E(U_\alpha U_\beta u_\alpha(t_\omega) u_\alpha(t) \cdot e'') \\
= \mathcal{L}_E(U_\alpha U_\beta u_\alpha(t_\omega) u_\alpha(t) u_\omega(t_\omega) \cdot e') \\
\subseteq \mathcal{L}_E(U_\alpha U_\beta u_\alpha(t_\omega) u_\alpha(t) u_\omega(t_\omega) \cdot e')
\]

for almost every \( t_\omega \in \mathbb{C} \).

Assume contrary to fact that \( \mu + \alpha + \beta \) lies in \( \mathcal{L}_E(U_\alpha U_\beta U_\omega \cdot e') \). Then either \( \mu + \alpha + \beta \) lies in \( \mathcal{L}_E(U_\beta U_\omega \cdot e') \) or \( \mu + \alpha + \beta \) lies in \( \mathcal{L}_E(U_\omega \cdot e') \) for some \( \ell > 0 \). The first option is impossible because \( U_\beta \) fixes \( e'' \). We proceed to show the second is impossible by computing the scalar products of \( \alpha - \ell \beta \) against the fundamental weights.

Since \( \mu \) is the highest weight of \( \mathcal{L}_E(U_\omega \cdot e') \), we see that \( \ell \beta - \alpha \) must be a positive integer multiple of \( \omega \). In particular, \( \ell \beta - \alpha \) must be orthogonal to \( \varpi_\alpha' \) for \( \alpha' \in S \). Taking \( \alpha = \alpha' \), we obtain

\( \ell \langle \beta, \varpi_\alpha \rangle = 1 \),

whence \( \ell = 1 \). For the \( \alpha' \in S \) different from \( \alpha \), the orthogonality condition simply implies that \( \langle \beta, \varpi_\alpha' \rangle = 0 \). It follows that \( \beta = \alpha + \ell' \omega \) for some positive integer \( \ell' \). But since \( \beta \in \Delta^+ \) and \( \omega \) is extremal, we must have \( \ell' = 1 \). But then it cannot happen that \( \alpha + \beta \) is a root.

On the other hand, suppose that \( \beta \) is as above, but \( \alpha \in \Pi \setminus S = \{ \omega \} \). Then we have that \( U_\alpha U_\beta U_\omega \) fixes \( e'' \). But \( u_\alpha(t_\omega) u_\beta(s) u_\alpha(t_\omega) = u_{\alpha + \beta}(s) u_\beta(s) \). Combining these observations yields

\[ \mathcal{L}_E(U_{\alpha + \beta} \cdot e'') = \mathcal{L}_E(U_\alpha U_\beta U_\omega \cdot e'') = \mathcal{L}_E(e'') = \{ \mu \} \]

So far, we have shown that if a positive root \( \beta \) is such that \( U_\beta \) leaves \( e'' \) invariant, then the same is true for all roots for the form \( \alpha + \beta, \alpha \in \Pi \). By construction, \( e'' \) is invariant under \( U_\omega \). The result now follows. \( \square \)
Definition 4.4. Let \( k \) be a field. Given a positive integer \( \ell \), we shall say that a \( k \)-rational, absolutely irreducible representation of a \( k \)-split, reductive algebraic \( k \)-group \( G \) on a finite-dimensional vector space \( E \) is \( \ell \)-large if the set of its weights contains a root string of length \( \ell \); otherwise, we shall say that \( E \) is \( \ell \)-small.

We remark that \( E \) is \( \ell \)-large if and only if there exist a weight \( \mu \) of \( E \) and a root \( \alpha \) of \( G \) with

\[
\frac{2\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \geq \ell - 1.
\]

Using the normalization that \( \langle \alpha, \alpha \rangle = 2 \) for all roots \( \alpha \neq 0 \), the previous inequality reads simply \( \langle \mu, \alpha \rangle \geq \ell - 1 \).

On the other hand, if \( \mu \) is a weight of an \( \ell \)-small representation of \( G \), then we can write \( \mu = \sum_{\alpha \in \Pi} n_{\alpha} \omega_{\alpha} \), with \( |n_{\alpha}| < \ell - 1 \) for all \( \alpha \in \Pi \). It follows that there exists a constant \( \kappa \), depending only on the type of the root system \( \Delta \), such that if \( \mu \) is a weight of an arbitrary \( \ell \)-small representation of a reductive group with absolute root system \( \Delta \), then \( ||\mu|| < \kappa(\ell - 1) \). Let \( \kappa_{r} \) be a positive number no greater than the minimum value of \( \kappa(\Delta) \) as \( \Delta \) ranges over all root systems of type ADE and rank \( r \).

Given a representation of \( G \) on a finite-dimensional complex vector space \( E \) and a set \( S \) of simple roots, we can decompose \( E \) as an internal direct sum of irreducible \( G_{S} \)-submodules. If \( S \) is the vertex set of a connected subgraph, then it makes sense to speak of \( \ell \)-small or \( \ell \)-large \( G_{S} \)-submodules of \( E \).

Given an extremal vertex \( \omega \), consider the set \( D_{\omega} = \{ \beta \in \Delta^{+} \mid \langle \beta, \omega \rangle = 1 \} \). We are interested in a certain convex cone in \( \Lambda \otimes \mathbb{R} \), where \( \Lambda \) denotes the root lattice associated to \( \Delta \). Namely, define \( C_{\omega} \) to be the cone whose base is the convex hull of \( D_{\omega} \setminus \{ \omega \} \), and whose vertex is 0. Choose positive constants \( \theta_{r} \) such that for any root system of type ADE and rank \( i \), we have that the angle between \( \omega \) and \( C_{\omega} \) is no less than \( \theta_{r} \) for any extremal root \( \omega \).

Let \( r \) denote the rank of \( G \).

Lemma 4.5. Let \( S \) be the complement in \( \Pi \) of an extremal vertex \( \omega \), as above. Let \( E \) be an irreducible complex representation of \( G \) whose set of weights contains a root string of length \( \ell \). Suppose that \( e \) is a nonzero element of an \( \ell' \)-small submodule \( E' \subseteq E \) for the action of \( G_S \). Then \( \mathcal{L}_E(S_{\omega} \cdot e) \) contains a root string of length exceeding \( \ell - 4\kappa_{r-1}(\ell' - 1) \tan \theta_{r} \).

Proof. We may assume without loss of generality that \( e \) is a weight vector, say \( e \in E'' \). Define corresponding vectors \( e' \) and \( e'' \) as in Lemma 4.3, and set \( \nu' \) and \( \mu \) to be their respective weights. In this case, \( \mu - \nu' \) is a nonnegative multiple of \( \omega \).

Denote by \( \lambda \) the highest weight of the representation \( E \). By irreducibility we have \( \lambda \in \mathcal{L}_E(B \cdot e'') \). It follows from Lemma 4.3 that \( \lambda - \mu \) lies in the span of \( S \). Therefore, we have the identity \( \langle \mu, \omega \rangle = \langle \lambda, \omega \rangle \).

Let \( \mu^{+} \) be the highest weight in the string \( \mathcal{L}_E(U_{\omega} \cdot e) \). Consider the trapezoid with corners \( \mu, \nu, \nu', \) and \( \mu^{+} \). Denote by \( U_{S} \) the subgroup generated by the root subgroups associated to roots in \( S \). In other words, \( U_{S} \) is the product of the root subgroups associated to roots in \( \Delta_{S}^{+} \). By definition, \( \mu \) lies in \( \mathcal{L}_E(U_{\omega} U_{S} \cdot e) \). The commutation relations tell us in particular that \( \mathcal{L}_E(U_{\omega} U_{S} \cdot e) \subset \mathcal{L}_E(\prod_{\beta \in \Delta^{+}} U_{\beta}) U_{S} U_{\omega} \cdot e \), where \( D = D_{\omega} \setminus \{ \omega \} \). By definition, we have that \( \mu - \mu^{+} \) forms with \( \omega \) an angle no smaller than \( \theta_{r} \).
Now since \( E' \) is \( \ell' \)-small, it follows that \( \|\nu - \nu'\| < 2k_{r-1}(\ell' - 1) \). Since \( \langle \nu, \varpi_\omega \rangle = \langle \nu', \varpi_\omega \rangle \), we conclude that

\[
\langle \mu^+, \varpi_\omega \rangle \geq \langle \mu, \varpi_\omega \rangle - 2k_{r-1}(\ell' - 1) \tan \theta_r = \langle \lambda, \varpi_\omega \rangle - 2k_{r-1}(\ell' - 1) \tan \theta_r.
\]

The same argument used for the opposite system \( -\Delta^+ \) tells us that the lowest weight \( \mu^- \) of \( \mathcal{L}_E(U_{-\omega} \cdot e) \) satisfies \( \langle \mu^-, \varpi_\omega \rangle \leq \langle w_0 \cdot \lambda, \varpi_\omega \rangle + 2k_{r-1}(\ell' - 1) \tan \theta_r \). The result follows. \( \square \)

For a subset \( S \subset \Pi \) which is the vertex set of a connected subgraph of the Dynkin diagram, we denote by \( E_S, \ell\text{-small} \) the sum of all \( \ell \)-small \( G_S \)-submodules of \( E \). In this case, we can canonically identify \( E_S, \ell\text{-small} \) as a \( G_S \)-submodule of \( E \). Set \( \tau_\ell = 2k_{r-1} \tan \theta_r \).

**Corollary 4.6.** Let \( S \) and \( \omega \) be as in Lemma 4.5. If \( E \) is an \((\ell' + 2\ell \tau_r)\)-large representation of \( G \), then

\[
E_S, \ell\text{-small} \cap E_{(\omega)} = 0.
\]

Let us conclude with the observation that \( \mu \in \mathcal{L}_E(E_{\ell\text{-small}}) \) implies that \( \langle \mu, \beta \rangle < \ell \) for all \( \beta \in \Delta_S \).

### 4.3. Exterior powers.

We begin with a very general lemma. Let \( G \) be a connected algebraic group and \( E \) a \( G \)-module with basis \( \mathcal{B}_E \). Given a finite subset \( I \subset \mathcal{B}_E \), we define \( e_I = \bigwedge_{v \in I} v \). (We acknowledge yet ignore the subtlety that \( e_I \) is only defined modulo sign.) Then we have that the set \( \{e_I \mid I \subset \mathcal{B}_E, |I| = m \} \) forms a basis for the \( m \)-fold exterior power \( \bigwedge^m E \) of \( E \).

**Lemma 4.7.** Let \( E \) and \( F \) be modules for an connected algebraic \( k \)-group \( G \), with bases \( \mathcal{B}_E \) and \( \mathcal{B}_F \), respectively. Take finite subsets \( I \subset \mathcal{B}_E, J \subset \mathcal{B}_F, e \in \bigwedge^{|I|} E, \) and \( f \in \bigwedge^{|J|} F \). Write \( m = |I| + |J| \). Let \( H \) be an irreducible subvariety of \( G \) defined over \( k \).

Suppose that, for some \( h \in H \), we have that \( h \cdot e \) (resp. \( h \cdot f \)) has nonzero component in the \( e_I \)-direction (resp. the \( e_J \)-direction). Then for an open, dense set of \( h \in H \), we have the following:

(i) The vector \( h \cdot (e \wedge f) \in \bigwedge^m (E \oplus F) \) has nonzero component in the \( e_{I \cup J} = e_I \wedge e_J \)

direction with respect to the basis \( \bigwedge^m (\mathcal{B}_E \cup \mathcal{B}_F) \); and

(ii) likewise for \( h \cdot (e \wedge e_J) \).

**Proof.** The set of \( g \) such that \( g \cdot e_I \) has nonzero component in the \( e_I \)-direction is open and contains the identity. Thus, the second assertion is a special case of the first.

We now treat assertion (i). Consider the set \( H_e \) (resp. \( H_f \)) of \( g \in H \) such that \( g \cdot e \) (resp. \( g \cdot f \)) has nonzero component in the \( e_I \)-direction (resp. the \( e_J \)-direction). By hypothesis, \( H_e \) and \( H_f \) are nonempty. But they are also open subsets of \( H \). By irreducibility, their intersection is nonempty, again open, and therefore dense. If \( h \) lies in the intersection \( H_e \cap H_f \), then \( h \) satisfies the conclusion of assertion (i). \( \square \)

Suppose now that \( G \) is a reductive complex algebraic group, and that \( E \) is a finite-dimensional complex vector space on which \( G \) acts. Suppose furthermore
that our basis $\mathcal{B}$ of $E$ consists of weight vectors for the action of a fixed maximal torus $T$, say $e_i^\mu$, $1 \leq i \leq \dim E^\mu$. If $I$ is a subset of $\{(\mu, i) \mid 1 \leq i \leq \dim E^\mu\}$, then define $$e_I = \bigwedge_{(\mu, i) \in I} e_i^\mu.$$ If $K$ is a set of weights of $E$ (or more generally, a subset of $\Lambda \otimes \mathbb{R}$) and $m = \sum_{\mu \in K} \dim E^\mu$, then let us denote by $e_K$ the distinguished basis element of the $m$-fold exterior power $\bigwedge^m E$. Also, let us write $[K]$ for the weight $\sum_{\nu \in K} L^\nu$. In this notation, $e_K$ is a weight vector of weight $[K]$ in the representation $\bigwedge^m E$.

Given a nonzero root $\beta \in \Delta(T, G)$, we form the basis $\{e_i^\mu\}$ as usual. Likewise, when $I$ is a subset of $\{(\mu, i) \mid 1 \leq i \leq \dim E^\mu\}$ we write $e_I$ for the corresponding monomial basis element for the exterior power. As we have seen, given a $G$-module $E$, the index set $\{(\mu, i) \mid 1 \leq i \leq \dim E^\mu\}$ carries some additional structures, depending on the choice of $\beta$: Namely, the relation of $\beta$-linkedness and the notion of $\beta$-chains. If $C$ is a $\beta$-chain of $E$, then let us denote by $\ell(C)$ the length (i.e. cardinality) of such a chain.

We remark that, while the index set $\{(\mu, i) \mid 1 \leq i \leq \dim E^\mu\}$ is fixed, the very basis which it indexes depends on the choice of a nonzero root. We will soon have occasion to compare these constructions for different $\beta$. We will resolve eventual ambiguities by explicitly appending the root to the notation. For example, we shall speak of the “component in the $e_I(\beta)$-direction” to specify that component with respect to the basis of monomials in the vectors $e_i^\mu = e_i^\mu(\beta)$ associated to $\beta$.

The substance of the next lemma is that a certain property is invariant under changing the basis corresponding to one nonzero root to the basis corresponding to another.

**Definition 4.8.** Let us say that a subset $I \subset \{(\mu, i) \mid 1 \leq i \leq \dim E^\mu\}$ is $(\beta, \ell)$-sparse if $|I \cap C| < \min(\ell, \ell(C))$ for every maximal $\beta$-chain $C$ of $E$.

**Lemma 4.9.** Let $G$ be a simply laced, connected, reductive complex algebraic group whose Dynkin diagram is a connected graph. Let $\beta$, $\beta'$ be nonzero roots. Let $I$ be a $(\beta, \ell)$-sparse subset of $\{(\mu, i) \mid 1 \leq i \leq \dim E^\mu\}$. Then $g \cdot e_K$ has a nonzero component in the $e_I(\beta)$-direction for some/almost every $g$ if and only if there exists a $(\beta', \ell')$-sparse subset $J \subset \{(\mu, i) \mid 1 \leq i \leq \dim E^\mu\}$ such that $g \cdot e_K$ has a nonzero component in the $e_J(\beta')$-direction for some/almost every $g$.

**Proof.** Let $w$ be a representative in $G$ of an element $w \in W$ sending $\beta$ to $\beta'$. Since $w$ conjugates $U_\beta$ to $U_{\beta'}$, it follows that $w \cdot e_i^\mu(\beta) = e_i^{w, \mu}(\beta')$, up to scalar multiple, for some $i'$, $1 \leq i' \leq \dim E^{w, \mu} = \dim E^\mu$. Thus $w$ sends $\beta$-chains to $\beta'$-chains of the same length. The result follows. \qed

We begin to specialize the situation. We take for $K$ an affine hyperspace of $\Lambda \otimes \mathbb{R}$, and denote by $\mathcal{P}K$ the parallel linear hyperspace. For a fixed nonzero root $\alpha \in \Delta$, we denote by $\{e_i^\mu\}$ the basis constructed in §2.8. When a construction involving basis vectors depends on the choice of $\alpha$, we shall use the notational convention of Lemma 4.9 to make the choice explicit.

Set $\ell_s = 2^s \prod_{i=1}^s [1 + \tau_i]$. Set $S = \Pi \setminus \{\omega\}$. We denote by $I_{s, \text{small}}$ the set of indices $(\mu, i)$ with $e_i^\mu \in E_S, \text{small}$. Since $E_{S, \text{small}}$ is an $S_{\alpha}$-submodule of $E$, it
Proposition 4.10. Suppose $G$ is a simply laced group of rank $r$; $K$ an affine hyperspace of $\Lambda \otimes \mathbb{R}$; and $\Pi$ and $\alpha = \beta_1 \sim \cdots \sim \beta_4 = \omega$ chosen as in properties (CD) of §4.1.

If $E$ is an $\ell_r$-large $G$-module, then there exists an $(\alpha, 2\ell_{r-1})$-sparse set $I$ such that the $e_I$-component of $g \cdot e_K$ is nonzero for some (equiv., almost every) $g \in G$.

Proof. Induction on $r$. The rank one case follows from Corollary 2.21, for instance. Thus, assume $r > 1$.

For convenience, we shall denote by $w_j$ the simple reflection through the simple root $\beta_j$ of the combinatorial path of property (CD4) of §4.1. We observe that the subsystem $\Delta'$ of §4.1 coincides with $\Delta_S$. Set $w = w_2 \cdots w_q - 1 \in W_S$. Denote by $K''$ the affine hyperspace $(w^{-1} \cdot \alpha^+ \cap K) \oplus \mathbb{R}\omega$. Let $\alpha'$ and $\omega'$ be as in §4.1.

Let $E'$ be an irreducible, $\ell_{r-1}$-large $G_S$-submodule of $E$. By the induction hypothesis, there exists an $(\alpha', 2\ell_{r-2})$-sparse set $I'$ (to use the notation of Lemma 4.9) such that $g \cdot e_K \cap L 
subseteq E'$ has nonzero component in the $e_I(\alpha')$-direction for almost every $g \in G_S$. Let $w' \in W_S$ transport $w^{-1} \cdot \alpha$ to $\alpha'$. Since $w^{-1} \cdot \alpha$ is not parallel to $K''$, it follows from Corollary 2.21 that we can take $I'$ to satisfy $I' \cap w' \cdot K'' = \emptyset$.

Now Lemma 4.9 tells us that there exists a $(w^{-1} \cdot \alpha, 2\ell_{r-2})$-sparse set $J'$ satisfying $\ell J' \cap K'' = \emptyset$, such that $g \cdot e_K \cap L 
subseteq E'$ has nonzero component in the $e_I(\alpha^{-1} \cdot \alpha)$-direction for almost every $g \in G_S$.

Collecting the preceding facts using Lemma 4.7, we see that there exists a set $I$ satisfying

1. $I \cap I_{\ell_{r-1}} - \text{small}$ coincides with $I_{w^{-1} \cdot K} \cap I_{\ell_{r-1}} - \text{small} = I_K \cap I_{\ell_{r-1}} - \text{small}$
2. $I \cap I_{\ell_{r-1}} - \text{small}$ is $(w^{-1} \cdot \alpha, 2\ell_{r-2})$-sparse
3. $I \setminus I_{\ell_{r-1}} - \text{small}$ is $(w^{-1} \cdot \alpha, 2\ell_{r-2})$-sparse

such that $g \cdot e_{w^{-1} \cdot K} = g \cdot e_K$ has nonzero component in the $e_I(\alpha)$-direction for almost every $g \in G_S$.

Corollary 4.6 tells us that $L_E(S \omega \cdot e_I^I) = L_E(U \omega \cdot e_I^I) \cup L_E(U \omega \cdot e_I^I)$ is an $\omega$-string of length at least $2\ell_{r-1}$ for each index $(\mu, i) \in I_{\ell_{r-1}} - \text{small}$. In fact, combining Corollary 4.6 with Lemma 2.19 tells us even more: For $\mu \in L I_{\ell_{r-1}} - \text{small}$ fixed and any integer $m$ satisfying $|\langle \omega, \mu \rangle | 2m \cdot 2\ell_{r-1} - 1$, the map

$$\text{Span}(e_I^I)_{(\mu, i) \in I_{\ell_{r-1}} - \text{small}} \rightarrow E^{\mu + m \omega}$$

is injective. This map (defined up to scalar multiple) is simply the action of $u_{\pm \omega}(t)$ followed by projection to $E^{\mu + m \omega}$. Since any $\omega$-string in $K''$ meets $L I$ at most once, it follows from Lemma 2.15 and Lemma 4.7.(ii) that there exists a set $J$ satisfying

1. $J \setminus I_{K''} \text{ coincides with } I \setminus I_{K''}$
2. If $\mu \in K'' \cap L J$, then $|\langle \omega, \mu \rangle | \geq \ell_{r-1} - 1$

such that $sg \cdot e_K$ has nonzero component in the $e_J(\alpha)$-direction for almost every $s \in S \omega$, $g \in G_S$. In fact, we can say more:

2. If $\mu \in K'' \cap L J$, then $|\langle \omega, \mu \rangle | \text{ equals } \ell_{r-1} - 1 \text{ or } \ell_{r-1}$.

In particular, every $w^{-1} \cdot \alpha$-chain $C$ meeting $I_{K''} \cap J$ has length at least $\ell_{r-1}$, and satisfies $|J \cap C | < 2\ell_{r-2} + 2$. On the other hand, if $C$ does not meet $I_{K''} \cap J$, then $J \cap C = I \cap C \setminus I_{\ell_{r-1}} - \text{small}$. The result now follows from $(w^{-1} \cdot \alpha)$-sparsity of $I \setminus I_{\ell_{r-1}} - \text{small}$ and a final application of Lemma 4.9. \qed
Proposition 4.10 provides a roundabout method to derive some information about the direct sum decomposition of certain exterior power representations of simple groups. Let $G$, $K$ and $E$ be as in that statement. Denote by $V_K$ the projectivization of the linear subspace $\bigoplus_{\mu \in K} E^\mu$. A consequence of Proposition 4.10 is that $V_K \cap g \cdot V_K = \emptyset$ for almost all $g \in G$. In fact, it follows from that proposition and the fact that $E$ is $\ell_r$-large that
\[
\dim E - 2 \sum_{\mu \in K} \dim E^\mu \geq \ell_r - 2 \ell_{r-1}
\]
for any affine hyperspace $K$ of $A \otimes \mathbb{R}$.

**Proposition 4.11.** Take $G$, $K$, and $E$ as in Proposition 4.10, and take $h \in T$. Let $h = h_s h_u = h_u h_s$ be the Jordan decomposition of $h$. If the semisimple component $h_s$ is not torsion, then the set of $u \in G$ satisfying
\[ h^u \cdot e_K \land e_K \neq 0 \]
is nonempty and open.

**Proof.** Fix for the moment a basis $\mathcal{B}$ of $E$ consisting of weight vectors for the action of $T$. Given a finite subset $I \subset \mathcal{B}$, we define $e_I = \bigwedge_{\upsilon \in I} \upsilon$. Although only defined up to sign, the vector $e_I$ is again a weight vector for the action of $T$. We have already seen that the set $\mathcal{B}(m) = \{ e_I \mid I \subset \mathcal{B}, |I| = m \}$ forms a basis for the $m$-fold exterior power $\Lambda^m E$ of $E$.

Take $m = 2 \sum_{\mu \in K} \dim E^\mu = 2 \dim V_K + 2$. Given elements $u \in G$ and $X \in \text{Lie} T$, it makes sense to consider the quantity $\chi_{\mathcal{B},I,u}(X)$, defined by
\[ X \cdot u \cdot e_K \land u \cdot e_K = \sum_{I \subset \mathcal{B}, |I| = m} \chi_{\mathcal{B},I,u}(X) e_I. \]
We note that if we regard elements of $A$ as functionals on Lie $T$, then $\chi_{\mathcal{B},I,u}$ lies in $A \otimes \mathbb{C}$.

By Proposition 4.10, $g \cdot e_K \land e_K$ is nonzero for almost all $g \in G$. By dominance of the morphism $G \times T \rightarrow G$ given by $(u,t) \mapsto t^u$, there exist $I$ and $u$ such that $\chi_{\mathcal{B},I,u}$ is not identically zero on Lie $T$.

If $w \in G$ represents $w \in W(T,G)$, then we have the relation
\[ w \cdot \chi_{\mathcal{B},I,u} = \chi_{\mathcal{B}',I,w \cdot I,wu}, \]
up to sign. We are now ready to vary our basis $\mathcal{B}$. Since $\chi_{\mathcal{B},I,u} \neq 0$ as a member of $A \otimes \mathbb{C}$, it follows that $W \cdot \chi_{\mathcal{B},I,u}$ spans $A \otimes \mathbb{C}$ over $\mathbb{C}$.

Upon conjugation, we may assume that $h_s \in T$. We first prove the result in the special case of a semisimple element $h = h_s$. By hypothesis, $h$ does not lie in the intersection of the kernels of all characters of $T$. It follows from the preceding discussion that there exists a $w \in W$ such that $hiw \cdot e_K \land iw \cdot e_K$ has nonzero component in the $e_{\mathcal{B}',I}$-direction with respect to the basis $(\mathcal{B}'(m))$ of $\Lambda^m E$. The result in the case $h = h_s$ now follows from connectedness of $G$.

Finally, we turn to the general case, $h_u \neq 1$. By the preceding argument, we see that there exists a $u \in G$ satisfying the equation $h_u^w \cdot e_K \land e_K \neq 0$. The computation
\[
\begin{pmatrix}
\lambda & 0 \\
0 & \lambda^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & \mu \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\lambda^{-1} & 0 \\
0 & \lambda
\end{pmatrix}
= \begin{pmatrix}
1 & \lambda^2 \mu \\
0 & 1
\end{pmatrix}
\]
for \( \lambda, \mu \in \mathbb{C}^\times \) implies that there exists a sequence of matrices \( t_i \in T \) such that \( h_s^{t_i} \to 1 \) as \( i \to \infty \). Thus
\[
h_s^{t_i} \cdot e_K \land e_K = (h_s^{t_i}) h_s^u \cdot e_K \land e_K \to h_s^u \cdot e_K \land e_K
\]
since \( h_s \) commutes with \( T \). \( \square \)

4.4. **Reduction modulo** \( k \). We wish to generalize the results of the previous section to the context of a simple group \( G \) over a local field \( k \). To apply those results, we must first construct universal objects corresponding to \( G \), its representations, etc. Roughly speaking, we can realize \( G \) as the ‘extension of scalars’ by \( k \) of a Chevalley group defined over \( \mathbb{Z} \). Then we can extend scalars of the large representation of Proposition 4.10 to obtain a representation of \( G \) with some favorable properties. We will follow a somewhat different procedure.

To begin, let \( k \) be an arbitrary field. Every \( k \)-split, absolutely almost simple algebraic \( k \)-group \( G \) comes from a group scheme \( H \) defined over \( \mathbb{Z} \) by change of base \( \mathbb{Z} \to k \) [10]. As usual, we identify \( H \) with the group of its points over \( \mathbb{C} \). We can endow the universal enveloping algebra of \( H \) with a \( \mathbb{Z} \)-structure such that any finite-dimensional, \( k \)-rational representation \( \rho \) of \( G \) is obtained by extending scalars of the corresponding representation of \( H \) from \( \mathbb{Q} \) to \( k \), and then factoring out the unique maximal invariant \( k \)-subspace [15].

The following lemma is a special case of the corollary to Proposition 14, Chapter II, §7.7 of [8].

**Lemma 4.12.** Let \( V \) be a rational vector space. If \( W \) and \( W' \) are linear subspaces of \( V \), then
\[
(W \cap W') \otimes k = (W \otimes k) \cap (W' \otimes k).
\]

Let \( T \) be a maximal \( k \)-split torus, and write \( \Delta \) for the root system \( \Delta(T, G) \), which coincides with the root system of the Chevalley scheme \( H \). Denote by \( \Lambda \) the corresponding weight lattice. Given a \( G \)-module \( E \) and a subset \( K \subset \Lambda \otimes \mathbb{R} \), denote by \( V_K \) the projectivization of the linear subspace \( \bigoplus_{\mu \in K} E^\mu \).

Let \( \ell_r \) be as in the statement of Proposition 4.10.

**Corollary 4.13.** Let \( G \) be a connected, simply laced, \( k \)-split, absolutely almost simple algebraic \( k \)-group of rank \( r \); \( T \) a maximal \( k \)-split torus of \( G \); and \( \rho \) an \( \ell_r \)-large, \( k \)-rational representation of \( G \) on a vector space \( E \). Suppose \( h \in G(k) \) has semisimple Jordan component of infinite order.

Denote by \( X \) the set of elements \( u \in G \) such that
\[
V_K \cap h^u \cdot V_{K'} = \emptyset
\]
for every pair of parallel affine hyperspaces \( K, K' \) of \( \Lambda \otimes \mathbb{R} \). Then \( X \) is nonempty and open.

**Proof.** Since \( G \) is \( k \)-split, the subspaces \( V_K \) and \( V_{K'} \) are defined over \( k \). It follows that the condition (8) is an open condition on \( u \in G \). Therefore, we need only show that this condition is nonvacuous for each pair \( K, K' \) of parallel affine hyperspaces.

Suppose first that \( K \neq K' \). Then there exists a choice of a system \( \Delta^+ \) of positive roots of \( G \) with respect to \( T \) such that \( \mu' - \mu > 0 \) for any weights \( \mu \in K, \mu' \in K' \). If \( B \) is the Borel subgroup corresponding to this choice of \( \Delta^+ \), then any \( u \) which conjugates \( h \) into \( B \) will satisfy condition (8). We are left with the case where \( K = K' \).
Let us denote by $\rho_C, E_C$ the corresponding complex representation of the associated Chevalley scheme $H$, which we identify with its points over $C$. Denote by $E_Q$ the rational structure coming from the $\mathbb{Z}$-structure on the universal enveloping algebra of $H$. Write
\[ Y = \{ v \in H \mid W_K \cap h^v \cdot W_K = \emptyset \}, \]
where $W_K = \bigoplus_{\mu \in K} E^\mu_Q$.

It follows from Lemma 4.12 that
\[ \mathbb{C}(Y) = \mathbb{Q}(Y) \otimes \mathbb{C} \]
\[ k(X) = \mathbb{Q}(Y) \otimes k. \]
But $Y$ is nothing but $\{ v \in H \mid e_K \wedge h^v \cdot e_K \neq 0 \}$. Thus Proposition 4.11 implies that $\mathbb{C}(Y) = \mathbb{C}(H)$. The result now follows from dimension considerations. \qed

5. Producing Representations with Contractive Dynamics

5.1. Producing representations with proximal behavior. We begin this section by outlining some of the elementary representation theory of semisimple groups, which can be found in more detail in §24.B of [5]. We restore $k$ to being a local field. Let $G$ be a connected, semisimple, $k$-split $k$-group, $T$ a maximal $k$-split torus. We write $\Delta = \Delta(T, G)$ and $W = W(T, G)$ for the root system and Weyl group associated to $T$.

We fix for the moment a system $\Delta^+$ of positive roots for $\Delta$. Let $B$ denote the corresponding Borel $k$-subgroup containing $T$, and $U$ its unipotent radical. If $\lambda_0$ is a character of $T$, we can regard it as a function on $B = TU$ by extending it trivially on $U$.

Fix a $k$-rational character $\lambda$ of $T$, dominant with respect to $\Delta^+$. Let $E(\lambda)$ denote the Weyl module associated to $\lambda$. Namely,
\[ E(\lambda) = \{ f \in \Omega[G] \mid f(gb) = (-w_0 \cdot \lambda)(b) \cdot f(g), g \in G, b \in B \}. \]

First we suppose that $k$ has characteristic zero. Then $E(\lambda)$ is an absolutely irreducible $G$-module with highest weight $\lambda$. Because $G$ is $k$-split, we can endow $E(\lambda)$ with a $k$-structure with respect to which the representation $\rho : G \to \text{GL}(E(\lambda))$ is rational. That $E(\lambda)$ is finite-dimensional (and nonzero) follows from dominance of $\lambda$.

On the other hand, if $k$ has positive characteristic, then the Weyl module $E(\lambda)$ need not be irreducible. Nevertheless, $E(\lambda)$ contains a unique absolutely irreducible subrepresentation $F(\lambda)$. As before, we have that $F(\lambda)$ is finite-dimensional and nonzero; and that $\lambda$ is still the highest weight of $F(\lambda)$. In this case it is also true (cf. Theorem 2.5 of [21]) that $F(\lambda)$ can be endowed with a $k$-structure with respect to which $\rho : G \to \text{GL}(F(\lambda))$ is rational.

In the remainder of this section, we consider the problem of producing a representation wherein a fixed semisimple element of $G(k)$ acts very proximally. Take $h$ in $T(k)$.

We wish to associate some root data with $h$. Choose a system $\Delta^+$ of positive roots containing all roots $\phi \in \Delta$ such that $|\phi(h)| > 1$, where $|\cdot|$ denotes the absolute value on $k$. Let $\Pi$ be the associated set of simple roots. Define the set $\Delta_{\text{im}} = \{ \phi \in \Delta \mid |\phi(h)| = 1 \}$. This is easily seen to be a root subsystem of $\Delta$, and $\Delta^+_{\text{im}} = \Delta_{\text{im}} \cap \Delta^+$ is a valid choice of positive system. It is also true that the simple roots associated to this choice of positivity is exactly $\Pi_{\text{im}} = \Delta_{\text{im}} \cap \Pi$. 

30 TAL POZNANSKY
The main tool for constructing projective actions where a given nontorsion semi-simple element admits a ping-pong partner is the following observation.

**Proposition 5.1.** Suppose there exist an element \( w \in W \) and a dominant weight \( \lambda \) satisfying the following conditions.

(i) \( w \cdot \lambda = -\lambda \)

(ii) \( w \cdot \Delta_{im} = \Delta_{im} \)

(iii) \( \lambda \perp \Pi_{im} \)

Then there exists an absolutely irreducible, finite-dimensional \( G \)-module \( E \) and a \( k \)-structure on \( E \) with respect to which the map \( \rho : G \to GL(E) \) is \( k \)-rational and \( \rho(h) \in GL(E_k) \) is very proximal.

**Proof.** Let \( E = E(\lambda) \) if \( \text{char} \, k = 0 \); otherwise, let \( E = F(\lambda) \). By the discussion above, the representation \( \rho : G \to GL(E) \) is \( k \)-rational, absolutely irreducible, and finite-dimensional. We need only show that \( \rho(h) \) is very proximal.

Any weight \( \mu \) of \( \rho \) must be of the form

\[
\mu = \lambda - \sum_{\alpha \in \Pi} c_{\alpha} \alpha
\]

with \( c_{\alpha} \geq 0 \). The definition of \( \Delta^+ \) immediately implies that \( \lambda(h) \) is an eigenvalue of maximal norm, and that if \( |\mu(h)| = |\lambda(h)| \), then all coefficients \( c_{\alpha} \) in (9) vanish for \( \alpha \in \Pi \setminus \Pi_{im} \).

Denote by \( E^\mu \) the weight space of \( E \) corresponding to weight \( \mu \). In this language, the projectivization \( \hat{E}^\mu \) of \( E^\mu \) lies in \( A(\rho(h)) \). We work to establish the claim that if

\[
\mu = \lambda - \sum_{\alpha \in \Pi_{im}} c_{\alpha} \alpha
\]

is a weight for some integers \( c_{\alpha} \), then all of the coefficients \( c_{\alpha} \) are zero. It will follow that \( A(\rho(h)) \) coincides with the projectivized weight space \( \hat{E}^\lambda \). Since \( \lambda \) has multiplicity one as a weight of \( \rho \), it will follow that the \( \rho(h) \) is proximal.

We have already observed that \( \Delta_{im} \) is itself an abstract root system; that \( \Delta^+_{im} = \Delta_{im} \cap \Delta^+ \) is a valid notion of positivity in \( \Delta_{im} \); and that \( \Pi_{im} = \Pi \cap \Delta_{im} \) is a system of simple roots. Denote by \( W_{im} \) the Weyl group of this system. In particular, for any element \( z = \sum_{\alpha \in \Pi_{im}} z_{\alpha} \alpha \) of the root lattice of \( \Delta_{im} \), there exists an element \( w_z \in W_{im} \) such that \( w_z \cdot z \) lies in the closed fundamental Weyl chamber corresponding to \( \Pi_{im} \).

We note that the Weyl group \( W_{im} \) of the subsystem \( \Delta_{im} \) is generated by the reflections \( \sigma_\alpha \) through hyperspaces perpendicular to simple roots \( \alpha \in \Pi_{im} \). Although \( W_{im} \) was defined abstractly in terms of a root system, we may identify \( W_{im} \) with the corresponding subgroup of \( W \); namely, the subgroup generated by the root reflections \( \sigma_\alpha \) (a minor abuse of notation) associated to simple roots in \( \Pi_{im} \). Now for all \( \alpha \in \Pi_{im} \), condition (iii) implies that \( \sigma_\alpha \) fixes \( \lambda \). So if we take

\[
z = -\sum_{\alpha \in \Pi_{im}} c_{\alpha} \alpha,
\]

then

\[
w_z \cdot \mu = \lambda + w_z \cdot z \gtrless \lambda,
\]

with respect to the notion of positivity coming from \( \Delta^+ \). If \( \mu \) is a weight of \( \rho \), then of course so is \( w_z \cdot \mu \). However, \( w_z \cdot \mu \) cannot be a higher weight than \( \lambda \). Therefore, \( z = 0 \) and \( \mu = \lambda \). This proves our claim, and establishes the proximality of \( \rho(h) \).
Condition (i) implies in particular that $-\lambda$ is a weight of $\rho$. If there were a lower weight $\mu$, then $w_0 \cdot \mu$ would be higher than $\lambda$. By construction of $\Delta^+$, we see that $|\mu(h)|$ is minimized over all weights $\mu$ of $\rho$ when $\mu = -\lambda$, that is, when $|\mu(h)| = |\lambda(h)|^{-1}$. Moreover, if $\mu$ is a weight realizing $|\mu(h)| = |\lambda(h)|^{-1}$, then $\mu$ satisfies $\mu + \lambda = \sum_{\alpha \in \Pi_{im}} c_\alpha \alpha$. Applying $w$ to both sides yields

$$w \cdot \mu = \lambda + w \sum_{\alpha \in \Pi_{im}} c_\alpha \alpha.$$ 

But condition (ii) implies that $w$ preserves the root lattice corresponding to the subsystem $\Delta_{im}$. Therefore, $w \cdot \sum_{\alpha \in \Pi_{im}} c_\alpha \alpha = \sum_{\alpha \in \Pi_{im}} c_\alpha' \alpha$. But we have already observed that the only weight of the form (10) is $\lambda$ itself. Thus $w \cdot \mu = \lambda$, or equivalently, $\mu = -\lambda$. Thus $A(\rho(h^{-1}))$ coincides with the projectivization of $E_{-\lambda}$. Since $\lambda$ has multiplicity one, so does $w \cdot \lambda = -\lambda$, and the result follows. 

\[\Box\]

**Remark 5.2.** It often happens that a Weyl group contains the linear transformation $-1$ sending each root to its negative. In this case, we note that the hypotheses of this theorem are much easier to verify. Conditions (i)–(iii) can be replaced with the much simpler condition that $\Pi_{im} \subseteq \Pi$.

Of course, this Proposition is not very useful if $\Pi_{im}$ coincides with $\Pi$. We conclude this section by quoting a lemma which will eventually serve as an “escape hatch” from this situation.

**Lemma 5.3** (Lemma 4.1 of [22]). Let $\ell$ be a finitely generated field, and $t \in \ell^\times$ an element of infinite multiplicative order. Then there exists a locally compact field $k$ endowed with an absolute value $\omega$, and a homomorphism $\sigma : \ell \to k$ such that $\omega(\sigma(t)) \neq 1$.

5.2. **Quasi-projective transformations and quasi-proximal unipotents.** If $G$ is a connected, semisimple algebraic group over an archimedean local field $k$, $S \subset G(k)$ a $k$-dense subgroup, and $u$ a unipotent (and nontorsion) element of $S$, then our eventual goal is to produce an irreducible, finite-dimensional, $k$-rational linear representation $\rho$ of $G$ such that $\rho(u)$ behaves “as if it were proximal,” and such that there exists genuinely proximal elements in $\rho(S)$. By the phrase “as if it were proximal,” we mean roughly that $\rho(u)$ acts on the projective space with an isolated fixed point; that the basin of attraction for this fixed point is large; and that convergence to this fixed point occurs uniformly on compact sets in the appropriate sense.

We begin with a brief review of the notions of quasi-projective transformations and contractions. The notion was introduced by Furstenberg [13], in the same paper where he introduced proximality. The basic references for this section are the papers of Gol’dsheĭd and Margulis [14], and Abels, Margulis, and Soifer [1].

In this section, we let $E$ denote a finite-dimensional vector space over an archimedean local field $k$, $P$ its projectivization. Endow $P$ with the angle metric

$$d(p, q) = \arccos \frac{|\langle p, q \rangle|}{|p||q|},$$

where $p$ and $q$ are representatives in $E \setminus \{0\}$ of $p$ and $q$ respectively; $|\cdot, \cdot|$ is a fixed Euclidean (resp. Hermitian) scalar product on $E$ if $k = \mathbb{R}$ (resp., if $k = \mathbb{C}$); and $||\cdot||$ is the norm on $E$ associated to the scalar product $|\langle \cdot, \cdot \rangle|$. 
Definition 5.4. A map \( b : P \to P \) is called a quasi-projective transformation if it is the pointwise limit of a sequence of projective transformations of \( P \).

If \( b \) is a quasi-projective transformation, we denote by \( M_0(b) \) the image under \( b \) of the set of points of continuity for \( b \). If \( s = (s_n) \) is a a sequence in \( \text{GL}(E) \) with \( b = \lim \hat{s}_n \) then we set \( M_0(s) = M_0(b) \). Let \( L_1(s) \) denote the topmost space in the Lyapunov filtration associated to \( s \): Namely, if \( \beta \) is the linear operator on \( E \) given by

\[
\beta(x) = \lim_{n \to \infty} \|s_n\|^{-1}s_n
\]

(which exists thanks to compactness of the unit ball in \( E \)), then \( L_1(s) \) is by definition the projectivization in \( P \) of \( \ker \beta \).

Definition 5.5. We call \( b \) a contraction if \( M_0(b) \) is a point.

Definition 5.6. We call a sequence \( s = (s_n)_{n>0} \) in \( \text{GL}(E) \) is contractive if the following conditions are satisfied:

(i) The associated sequence \( (\hat{s}_n) \) converges pointwise on \( P \); and

(ii) there is a sequence of scalars \( (B_n) \) in \( k^\times \) such that \( (B_n s_n) \in \text{GL}(E) \) converges pointwise to a linear map \( \beta \) of rank 1.

Remark 5.7. If \( s \) is a contractive sequence, then \( b \) is a contraction, and moreover agrees with \( \hat{\beta} \) on the complement of \( L_1(s) \). Conversely, it is true that any contraction is the limit of a contractive sequence.

The following lemma is evident.

Lemma 5.8. Let \( b \) be a quasi-projective transformation with \( b = \lim \hat{s}_n \) for some sequence \( s = (s_n) \) in \( \Sigma \); and take \( h \in \Sigma \).

(i) \( M_0(hs) = h \cdot M_0(s) \)

If in addition \( s \) is a contractive sequence, then

(ii) \( L_1(hs) = L_1(s) \)

We shall exploit the interplay between the notions of proximality and contraction. For a subsemigroup \( \Sigma \) of \( \text{GL}(E) \), let us denote by \( \Sigma^\circ \) the set of quasi-projective transformations arising as limits of sequences of projective transformations induced by elements of \( \Sigma \). We quote the following result.

Lemma 5.9 (Lemma 3.13 of [1]). If \( \Sigma \) is an irreducible subsemigroup of \( \text{GL}(E) \) and \( s = (s_n) \) is a contractive sequence in \( \Sigma \), then \( \Sigma \) contains a proximal element.

Proof. Upon replacing \( s \) by \( hs \) for some \( h \in \Sigma \), we may assume by Lemma 5.8 and Lemma 2.15 that \( M_0(s) \notin L_1(s) \). It now follows from Lemma 2.6.(ii) that \( s_n \) is proximal for \( n \) sufficiently large.

Suppose that \( u \in \text{GL}(E) \) is a unipotent linear transformation. If \( x \in P \) then \( \hat{u}^nx \) converges pointwise to an eigendirection for \( u \); i.e., to a point in the projectivization in \( P \) of \( \ker u - I \).

Definition 5.10. We shall say a unipotent element \( u \) is quasi-proximal if for some integer \( d \), the endomorphism \( u_d = (u - I)^d : E \to E \) has rank one.
We shall assume for the remainder of this section that \( u \) is quasi-proximal. Then if \( d \) is as in the definition, \( (u_d)^{-1} u^n \) converges pointwise to a nonzero scalar multiple of \( u_d \). Thus, the sequence \( s = (u^n) \) is contractive. Denote by \( B(u) \) and \( B'(u) \) the projectivizations of \( u_d \) and \( \ker u_d \), respectively. We have \( M_0(s) = B(u) \) and \( L_1(s) = B'(u) \). We note that we have the following analogue for quasi-proximal unipotents of Tits’ Lemma 2.6.(i):

**Lemma 5.11.** Suppose that \( u \in \text{GL}(E) \) is a quasi-proximal unipotent. Let \( K \subset P \setminus B'(u) \) be a compact set. For any \( q > 0 \) there exists an integer \( N \) such that \( \|u^z|_K\| < q \) for all \( z > N \); and for every neighborhood \( U \) of \( B(u) \) there exists an integer \( N' \) such that \( u^z \cdot K \subset U \) for all \( z > N' \).

**Proof.** The second assertion follows from the observation that the quasi-projective transformation \( \lim_{z \to \infty} \hat{u}^z \) agrees with \( \hat{u}_d \) on the complement in \( P \) of \( L_1(\{u^n\}) \).

The first assertion requires more care. Let \( p,q \in P \), and pick representatives \( p,q \in E \setminus \{0\} \). We first observe that if \( p,q \in E \setminus \{0\} \), then we can regard \( \langle u^p, u^q \rangle \) as a polynomial in \( z \) with coefficients in \( k \). Moreover, if \( p,q \in P \setminus B'(u) \), then the degree of this polynomial is precisely \( 2d \). It follows that the expression \( \cos^2 d(\hat{u}^z p, \hat{u}^z q) \) is a ratio of polynomials of degree \( 4d \) in \( z \), with identical leading coefficients. (In a suitable basis the coefficients are homogeneous polynomials of degree four in \( p \) and \( q \).)

Let \( K \) be as in the statement of the theorem. From the above description of \( d(\hat{u}^z p, \hat{u}^z q) \), we see that if \( z \) is large relative to the distance \( d(K,B'(u)) \), then

\[
\lim_{p \to q} \frac{d(\hat{u}^z p, \hat{u}^z q)}{d(p,q)} = 0
\]

for \( q \in K \).

Consider the functions \( f_z : K \to \mathbb{R}_+ \) given by

\[
f_z(p) = \sup_{q \in K \setminus \{p\}} \frac{d(\hat{u}^z p, \hat{u}^z q)}{d(p,q)}.
\]

The function \( f_z \) is continuous because if \( p \) is near \( q \), then either

\[
\frac{d(\hat{u}^z p, \hat{u}^z p')}{d(p,p')} \text{ is close to } \frac{d(\hat{u}^z q, \hat{u}^z p')}{d(q,p')}
\]

or \( q = p' \).

We observe that rational functions in \( z \) have the property that they are eventually monotone. From the remarks above, it follows that the sequence \( f_z \) eventually decreases in \( z \) monotonically to zero. It follows from Dini’s theorem that this convergence is uniform. The result follows.

Let us summarize the results of this section.

**Corollary 5.12.** If \( \Sigma \) is an absolutely irreducible subsemigroup of \( \text{GL}(E) \) which contains quasi-proximal unipotent elements, then it contains proximal elements.

**Proof.** Apply Lemma 5.9.

It may be illuminating to supply a more direct proof of Corollary 5.12. Suppose \( u \) is a quasi-proximal unipotent. By irreducibility, there exists an element \( h \in \Sigma \) such that \( h \cdot B(u) \not\subset B'(u) \). We will see that \( hu^z \) is proximal for all sufficiently large \( z \).
Let $K \subset P$ be a compact set such that $B(u) \subset \text{Int} K$ and $h \cdot K \subset P \setminus B'(u)$. By Lemma 5.11 there exists a number $N$ such that $u^zh \cdot K \subset \text{Int} K$ and $\|\hat{h}^z|_{h \cdot K}\| < 1/\|\hat{h}|_K\|$ for all $z > N$. Then $hu^zh \cdot K \subset h \cdot \text{Int} K$ and $\|\hat{hu}^z|_{h \cdot K}\| < 1$. We conclude by invoking Lemma 2.6.(ii).

In closing, we note that if a unipotent element is quasi-proximal, then so is its inverse. Moreover, $B(u) = B(u^{-1})$ and $B'(u) = B'(u^{-1})$. We also have for any $x \in \text{GL}(E)$ and $z \in \mathbb{N}$ that $B(zu) = x \cdot B(u)$, $B(u^z) = B(u)$, and likewise for $B'$.

6. Proof of the main theorem

We say a group is $\text{icc}$ if all of its nonidentity conjugacy classes are infinite. Bekka and de la Harpe [3] showed the following:

**Lemma 6.1.** Suppose that $\Gamma_0$ is a finite-index subgroup of an icc group $\Gamma$. If $C^*_r(\Gamma_0)$ is simple (resp. has a unique trace up to normalization), then the same is true of $C^*_r(\Gamma)$.

**Lemma 6.2.** Let $H$ denote the union of the finite conjugacy classes of a group $\Gamma$. Then $H$ is a characteristic, amenable subgroup of $\Gamma$.

**Proof.** The conjugacy class of a product lies in the product of conjugacy classes. Thus, it remains only to prove that $H$ is amenable.

If $h \in \Gamma$ lies in a finite conjugacy class, then its centralizer $Z_\Gamma(h)$ is of finite index in $\Gamma$. If $S$ is a finite subset of $H$, then $\bigcap_{h \in S} Z_\Gamma(h)$ is likewise of finite index in $\Gamma$. But this intersection contains the center of the subgroup $\langle S \rangle$. It follows that every finitely generated subgroup of $H$ is virtually abelian, and hence amenable. The result follows. \hfill $\square$

We begin by deducing Theorem 1.1 from the following result:

**Theorem 6.3.** Let $G$ be a connected, semisimple algebraic group with trivial center, and let $\Gamma < G$ be a finitely generated, Zariski-dense subgroup. Then $C^*_r(\Gamma)$ is simple, and has a unique trace up to normalization.

**Proof of Theorem 1.1.** Paschke and Salinas [17] showed that if $\Gamma$ has a nontrivial normal amenable subgroup, then $C^*_r(\Gamma)$ is not simple, nor is its trace unique up to normalization.

Suppose $\Gamma$ has no nontrivial normal amenable subgroup. Let $G$ be the Zariski closure of $\Gamma$. If we denote by $R$ the radical of $G$, then $\Gamma \cap R$ is a solvable normal subgroup of $\Gamma$, hence trivial. Therefore, upon projecting modulo $R$, we may suppose that $G$ is semisimple. Likewise, up to projecting to the adjoint group, we may assume that $G$ is center-free.

For convenience, for a subgroup $H$ of $G$, let us denote by $\overline{\text{Cl}(H)}$ the Zariski closure of $H$ and by $H^\circ$ and the connected component of $H$ containing the identity. For any infinite subgroup $\Gamma_0$ of $\Gamma$, we have that the dimension of $\text{Cl}(\Gamma_0)$ is positive. Since $\Gamma$ is nonamenable, it must contain an infinite, finitely generated subgroup. Write

$$d = \max\{\dim \text{Cl}(\Gamma_0) \mid \Gamma_0 \text{ is finitely generated}\}.$$ 

We can write $\Gamma$ as the union of an increasing chain $\Gamma_0 \subset \Gamma_1 \subset \cdots$ of finitely generated subgroups. Write $G_0 = \text{Cl}(\Gamma_0)^\circ$. Without loss of generality, we may suppose that $\dim G_0 = d$. \hfill $\square$
A theorem of Schur has it that if a linear torsion group is finitely generated, then it is finite. In particular, \( \Gamma \) that is connected, we must have \( \Gamma \) is closed, while this last is closed, while \( \Gamma \) is dense. Hence \( \Gamma \) is a normal subgroup of \( G \).

For an element \( a \in G \), we have \( a^{|G_0|} = a^{|\Gamma_0|} \). But \( a^{|G_0|} \) is Zariski dense in \( a^G \). In particular, if \( a \) lies in a finite conjugacy class of \( \Gamma_0 \), then \( a \in Z_G(G_0) = 1 \). Therefore, \( \Gamma_0 \) is icc.

The remainder of the proof consists in exhibiting an arbitrary Zariski-dense subgroup \( \Gamma \) of an arbitrary semisimple adjoint group \( G \) as the increasing union of subgroups whose reduced \( C^* \)-algebras are simple and have a unique tracial state. The result will then follow by taking inductive limits.

We can write \( G^0 \) as an internal direct product of simple groups, say \( G^0 = G_1 \times \cdots \times G_r \). Induction on \( r \): If \( r = 1 \), then \( G_0 = G^0 \). We can take \( \Gamma_0 \) Zariski dense for \( i \geq |G/G^0| \). The result in this case follows from Theorem 6.3.

Therefore, suppose that \( r > 1 \), and indeed that \( G_0 \) is a simple, proper subgroup of full rank in \( G^0 \). Then \( \Gamma_0 \) is Zariski dense in \( G_0 \) and of finite index in \( \Gamma_0 \). By the induction hypothesis, \( C^*_\gamma(\Gamma_0 \cap G_0) \) is simple and has a unique trace, up to normalization. But then the algebra \( C^*_\gamma(\Gamma_0) \) enjoys the same properties by Lemma 6.1.

We shall prove that groups as in the statement of Theorem 6.3 ‘virtually’ satisfy the hypotheses of Lemma 2.3. Namely, if \( \Gamma \) is such a group, then there exists a finite index subgroup \( \Gamma_0 \) such that for any finite subset \( F \subset \Gamma_0 \), there exists an element \( g \in \Gamma \) of infinite order such that etc. But, one may wish to know more: For a fixed finite subset \( F \), how large is the set of \( g \in \Gamma \) verifying the conditions of that lemma? We shall prove the following generalization of Theorem 6.3; it includes the statement that the set of such \( g \) is very large indeed.

**Theorem 6.4.** Let \( G \) and \( \Gamma \) be as in Theorem 6.3, and let \( F \) be a finite subset of \( G \). Suppose that for every simple direct factor \( H \) of \( G \), and every torsion element \( h \in F \), the centralizer \( Z_H(h)^0 \) is not a simple, proper subgroup of full rank in \( H \). Then the set of \( g \in \Gamma \) which verify conditions (i) and (ii) of Lemma 2.3 with respect to some decomposition \( F = F_p \cup F_r \) is dense in the profinite topology on \( \Gamma \).

Before turning to the proofs of Theorems 6.3 and 6.4, we prove an important special case.

**Theorem 6.5.** Let \( G \), \( \Gamma \), and \( F \) be as in Theorem 6.4. Write \( G \) as the almost direct product \( G_\gamma \), where \( G_\gamma \) is the largest normal subgroup of \( G \) whose Weyl group contains the linear transformation \( -1 \). Suppose that \( G_\gamma \) contains no nontrivial power of any element of \( F \) whose semisimple part is nontorsion.

Denote by \( Y_F \) the set of elements \( \gamma \in \Gamma \) of infinite order such that the subgroup \( \langle \gamma, h \rangle \) generated by \( \gamma \) and \( h \) is canonically isomorphic to the free product \( \langle \gamma \rangle * \langle h \rangle \), all \( h \in F \). Then \( Y_F \) is dense in the profinite topology on \( \Gamma \).

**Proof.** Let \( Hg \) be a coset of a finite-index subgroup \( H \subset \Gamma \), \( g \in \Gamma \). Upon replacing \( H \) by a finite-index subgroup, we may assume \( H \) is normal in \( \Gamma \); and we note that \( H \) is itself Zariski-dense in \( G \), by virtue of connectedness of \( G \).
Let $k$ be a finitely generated field over which $G$ is defined and such that $\Gamma \subset G(k)$. Upon extending $k$, we may also assume that $k$ contains all eigenvalues of all elements of $F$. Let $T$ be a maximal $k$-split torus of $G$. We write $F$ as a disjoint union $F_s \cup F_t \cup F_u$, where

$$F_s = \{ h \in F \mid h \text{ has nontorsion semisimple part } h_s \},$$

$$F_t = \{ h \in F \mid h \text{ is torsion} \},$$

$$F_u = \{ h \in F \mid h \text{ is not torsion, } h_s \text{ is torsion} \}.$$

Suppose first that $h \in F_s$. Upon conjugating, we may assume $T$ contains $h_s$. Then writing $h = h_{-1} h_{+1}$ with $h_{-1} \in G_{-1}$, then some eigenvalue $\tau$ of $h_{-1}$ has infinite multiplicative order. By Lemma 5.3, there exists a local field $k_h$ extending $k$ such that $|\tau| \neq 1$. If $\Pi_{im} = \Pi_{im}(h_s)$ is as in §5.1, then $\rho_h(h)$ is proximal in the irreducible representation $\rho_h$ of highest weight $\lambda$ for any positive integral sum $\lambda = \sum_\alpha \alpha$, with $\alpha \in \Pi \setminus \Pi_{im}$. By choice of $k_h$, there exists a connected component of the Dynkin diagram, not contained in $\Pi_{im}$, whose Weyl group contains $-1$. In particular, by constraining the coefficients $(\lambda, \alpha)$ in the sum to be nonzero only for those $\alpha$ lying in this connected component, we obtain that $\rho_h(h)$ is very proximal, and that $\rho_h$ factors through an absolutely simple factor of $G$.

Suppose $h \in F_u$. It follows that $h_u$ is nontorsion, whence $k$ has characteristic zero. Let $k_h$ be an archimedean local field extending $k$. Let $m_h$ be the order of $h_s$. Let $\rho_h$ be any $k_h$-rational, absolutely irreducible, finite-dimensional representation of $G$. Upon replacing $\rho_h$ by an exterior power and taking the correct irreducible component, we can guarantee that $\rho_h(h_{m_h})$ is a quasi-proximal unipotent. By Corollary 5.12, $\Omega_+(\rho_h, \Gamma)$ is nonempty. As before, we may assume that $\rho_h$ factors through an absolutely simple factor of $G$.

Finally, we turn to the torsion elements $F_t$. Denote by $\bar{F}_t$ the set of all nontrivial projections of all elements of $F_t$ to all simple direct factors of $G$. We remark that, in order to find an element of $\Gamma$ which satisfies no nontrivial relation with any $h \in F_t$, it suffices to find an element of $\Gamma$ which satisfies no nontrivial relation with any $h \in \bar{F}_t$. We also remark that $k$ already contains all eigenvalues of all elements of $\bar{F}_t$. Therefore, we assume that $\bar{F}_t = F_t$.

Fix an element $h \in F_t$. By the previous paragraph, $h$ lies in an absolutely simple direct factor $G_h$ of $G$. If $k_h$ is any local field extending $k$, and $\rho$ any $k_h$-rational, absolutely irreducible, finite-dimensional, nonzero representation of $G$ factoring through $G_h$, then it follows from absolute simplicity that $G_h \cap \ker \rho \subset Z(G_h) = 1$. Since a finitely generated linear torsion group is finite, it follows that $\rho(\Gamma)$ contains nontorsion elements. Hence, the preceding paragraphs imply that there exists such a representation $\rho_h$ with $\Omega_+(\rho_h, \Gamma)$ nonempty.

To summarize the proof so far, we have constructed a family

$$\{ \rho_h : G \to GL(E_h) \mid h \in F \}$$

of absolutely irreducible, finite-dimensional, nonzero linear representations of $G$ rational over local extensions of $k$, factoring through absolutely simple direct factors of $G$, with the properties that

(i) $\rho_h(h)$ is very proximal if $h \in F_s$;
(ii) $\langle \rho_h(h) \rangle \cong (h)$ if $h \in F_t$;
(iii) $\rho_h(h)$ has a power which is a quasi-proximal unipotent if $h \in F_u$; and
\( (iv) \) \( \Omega_r(\rho_h, F) \neq \emptyset \) for all \( h \in F \).

Let \( P_h \) denote the compact Hausdorff topological space \( P(E_h)(k_h) \).

Write \( F_i = F_s \cup F_u \). For \( h \in F_i \), let us set the notation

\[
(11) \quad \text{Kr}(h) = \begin{cases} 
\text{Cr}(\rho_h(h)) & \text{if } h \in F_s, \\
\bigcup_{v=1}^{\infty} h^v \cdot B'(\rho_h(h_u)) & \text{if } h \in F_u.
\end{cases}
\]

We note that the apparently infinite union on the right-hand side of (11) is actually a finite union of projective hyperspaces by the hypothesis that \( h_s \) is torsion.

For convenience, we introduce the notation for \( \gamma \in G \) that \( \gamma_h = \rho_h(\gamma) \). In analogy with the critical set defined by equation (11), we set for \( \gamma \in G \)

\[
C(\gamma_h) = \begin{cases} 
A(\gamma_h) \cup A(\gamma_h^{-1}) & \text{if } A(\gamma_h) \neq E_h, \\
\bigcup_{v=1}^{\infty} \gamma^v \cdot B(\rho_h(\gamma_u)) & \text{if } \gamma_s \text{ is torsion and } \rho_h(\gamma_u) \text{ quasi-proximal},
\end{cases}
\]

noting again that the union on the right-hand side is a finite union of singleton sets.

When \( h \in F_i \), we will write \( C(h) = C(h_h) \) for simplicity.

Denote by \( R \) the set of representations \( \{ \rho_h, \rho_h^* \mid h \in F \} \), where we denote by \( \rho^* \) the dual of the representation \( \rho \). Since \( \rho^*(\gamma) \) is proximal if \( \rho(\gamma^{-1}) \) is, we see that \( \bigcap_{\rho \in R} \Omega_0(\rho, S) \) coincides with \( \bigcap_{h \in F} \Omega_0(\rho_h, S) \). Lemma 2.9.(i) implies the existence of a finite attractor family \( \Phi \subset \bigcap_{h \in F} \Omega_0(\rho_h, S) \) for the family \( R \). Upon conjugation, we may assume

\[
C(h) \subset P_h \setminus A'(\phi_h^{-1}) \quad \text{and} \quad A(\phi_h) \subset P_h \setminus \text{Kr}(h)
\]

all \( h \in F_i \) and \( \phi \in \Phi \).

For each \( h \in F_i \), choose compact subsets \( U_h \) and \( K_h \subset P_h \) satisfying

\[
K_h \subset P_h \setminus \bigcup_{\phi \in \Phi} A'(\phi_h^{-1}) \quad C(h) \subset \text{Int} \, K_h
\]

\[
U_h \subset P_h \setminus \text{Kr}(h) \quad \bigcup_{\phi \in \Phi} A(\phi_h) \subset \text{Int} \, U_h.
\]

By Lemma 2.6.(i) there exist numbers \( N_h \) such that \( h^z \cdot U_h \subset \text{Int} \, K_h \), all \( h \in F_i \) and \( |z| \geq N_h \). If \( h \in F_i \), denote by \( N_h \) the order of \( h \).

Upon replacing \( \Phi \) by a high (elementwise) power of itself, Corollary 2.10 implies the following: For any \( \gamma \in \bigcap_{h \in F} \Omega_0(\rho_h, H) \) there exists \( \phi \in \Phi \) such that for all \( h \in F_i \) we have

\[
(12) \quad \phi \cdot A(\gamma_h) \subset \text{Int} \, U_h
\]

\[
(13) \quad \phi \cdot A'(\gamma_h^{-1}) \cap K_h = \emptyset.
\]

Now fix \( \gamma \in \bigcap_{h \in F} \Omega_0(\rho_h, H) \). By virtue of Lemma 2.15 and Proposition 2.17, there exists \( y \in S \) satisfying

\[
y^{-1} \phi^{-1} \cdot C(h) \subset P_h \setminus A'(\gamma_h^{-1})
\]

\[
y \cdot C(\gamma_h) \subset P_h \setminus (\phi^{-1} \cdot \text{Kr}(h))
\]

for all \( h \in F_i \) and all \( \phi \in \Phi \), and

\[
h^{\phi yz} \cdot A(\gamma_h) \subset P_h \setminus A'(\gamma_h^{-1})
\]
for all \( h \in F, \phi \in \Phi, \) and \( 0 < |z| < N_h. \) Since \( H \) is normal in \( S, \) we may as well replace \( \gamma \) by \( \delta \gamma. \) Thus we obtain an element \( \gamma \in \bigcap_{h \in F} \Omega_0(\rho_h, H) \) satisfying
\[
\phi^{-1} \cdot C(h) \subset P_h \setminus A'(\gamma_h^{-1}) \\
C(\gamma_h) \subset P_h \setminus (\phi^{-1} \cdot \text{Kr}(h))
\]
for all \( h \in F, \) and all \( \phi \in \Phi, \) and
\[
h^{\phi z} \cdot A(\gamma_h) \subset P_h \setminus A'(\gamma_h^{-1})
\]
for all \( h \in F, \phi \in \Phi, \) and \( 0 < |z| < N_h. \) Let \( \phi \in \Phi \) be that element satisfying conditions (12) and (13) for this choice of \( \gamma. \)

On the other hand, if \( h \in F, \phi \in \Phi, \) and \( 0 < |z| < N_h, \) we may replace \( \gamma \) by \( \delta \gamma. \) Then it follows in this case by (13) that \( (h^{\delta \gamma} \cdot A(\gamma_h)) \cap (\phi \cdot A'(\gamma_h^{-1})) = \emptyset. \) Thus if we write \( \beta = \delta \gamma, \) then \( \beta \in \bigcap_{h \in F} \Omega_0(\rho_h, H) \) satisfies the following conditions:
\[
C(h) \subset P_h \setminus A'(\beta_h^{-1}) \\
C(\beta_h) \subset P_h \setminus \text{Kr}(h)
\]
for all \( h \in F, \) and
\[
h^{\beta z} \cdot A(\beta_h) \subset P_h \setminus A'(\beta_h^{-1})
\]
for all \( h \in F, \) and all nontrivial powers \( h^z \) of \( h. \)

By Lemma 2.6.(i) and Lemma 5.11, we have that \( h^{\beta z} \cdot U_h \subset \text{Int} K_h \) for all \( h \in F, \) and for almost all \( z \in \mathbb{N}. \) Equations (12) and (13) imply that \( A(\beta_h) \subset \text{Int} U_h \) and \( K_h \subset P_h \setminus A'(\beta_h^{-1}), h \in F. \) Upon shrinking \( U_h, \) enlarging \( K_h, \) and using (14) if necessary, we obtain compact sets \( K_h^+, \Omega_h \subset P_h, h \in F, \) with
\[
K_h^+ \subset P_h \setminus \text{Kr}(h) \\
A(\beta_h) \subset \text{Int} K_h^+ \\
M_h \subset P_h \setminus A'(\beta_h^{-1}) \\
C(h) \subset \text{Int} M_h.
\]
and \( h^{\beta z} \cdot K_h^+ \subset \text{Int} M_h \) for all \( h \in F \) and all \( z \neq 0. \) For \( h \in F, \) by virtue of (14), we can choose compact sets \( K_h^+ \subset P_h \) such that
\[
A(\beta_h) \subset \text{Int} K_h^+ \\
h^{\beta z} \cdot K_h^+ \subset P_h \setminus A'(\beta_h^{-1})
\]
for all nontrivial powers \( h^z \) of \( h. \) For the sake of consistency, when \( h \in F, \) we shall write \( M_h = \bigcup_{h^z \neq 1} h^{\beta z} \cdot K_h^+. \) With this definition, we have \( M_h \subset P_h \setminus A'(\beta_h^{-1}) \) for all \( h \in F. \)

Now by Proposition 3.4, there exists \( \delta \in \bigcap_{h \in F} \Omega_0(\rho_h, Hg). \) By Lemma 2.15 and the fact that \( H \) is Zariski-dense, there exists an element \( x \in H \) such that for all \( h \in F \) we have
\[
x \cdot A(\beta_h^{-1}) \subset P_h \setminus \text{Cr}(\delta_h) \\
x^{-1} \cdot C(\delta_h) \subset P_h \setminus A'(\beta_h)
\]
Since \( x \in H \) and \( H \) is normal, we have \( \delta^z \in Hg. \) Therefore, we may as well suppose that \( \delta = \delta^z \in \bigcap_{h \in F} \Omega_0(\rho_h, Hg) \) satisfies the following conditions
\[
A(\beta_h^{-1}) \subset P_h \setminus \text{Cr}(\delta_h) \\
C(\delta_h) \subset P_h \setminus A'(\beta_h)
\]
all \( h \in F. \)
By equations (15) and (16), for each $h \in F$ and we can choose compact sets $K_h^-$ and $L_h$ with

\[ K_h^- \subset P_h \setminus \text{Cr}(\delta_h), \quad A(\beta_h^{-1}) \subset \text{Int} K_h^- \]

\[ L_h \subset P_h \setminus A'(\beta_h) \quad C(\delta_h) \subset \text{Int} L_h. \]

By Lemma 2.6.(i) there exists a number $\nu$ such that

\begin{align*}
(17) & \quad \beta^{-\nu} \cdot M_h \subset \text{Int} K_h^- \\
(18) & \quad \beta^\nu \cdot L_h \subset \text{Int} K_h^+ \]

for all $h \in F$. Finally, by Lemma 2.6.(i), there exists a number $\mu_0$ such that $\delta^\mu \cdot K_h^- \subset \text{Int} L_h$ for all $h \in F$, whenever $|\mu| > \mu_0$.

Consider the expression $\beta^{-\nu} h z^\nu \cdot L_h$. Using equations (17) and (18), we have

\[ \beta^{-\nu} h z^\nu \cdot L_h \subset \text{Int} \beta^{-\nu} h z^\nu \cdot K_h^+ \]

\[ \subset \text{Int} \beta^{-\nu} M_h \]

\[ \subset \text{Int} K_h^- \]

In particular, we have for $\mu > \mu_0$ and $j \neq 0$ that

\[ \delta^j \beta^{-\nu} h z^\nu \cdot L_h \subset \text{Int} L_h. \]

Thus whenever $h \in F$ and $z \in Z$ is such that $h z \neq 1$, we have

\[ \langle \delta^\mu \rangle \langle (\beta^{-\nu} h z^\nu) \rangle \cdot L_h \subset \text{Int} L_h. \]

Also, taking $z = 1$ in the above sequence of inclusions yields

\[ \beta^{-\nu} h \beta^\nu \cdot L_h \subset \text{Int} K_h^- \subset P_h \setminus \text{Cr}(\delta_h), \]

whence $\beta^{-\nu} h \beta^\nu \cdot L_h \not\subset L_h$. By Lemma 2.4, we have for all $h \in F$ and $\mu > \mu_0$ that

\[ \langle \delta^\mu, \beta^{-\nu} h \beta^\nu \rangle \cong \langle \delta^\mu \rangle * \langle \beta^{-\nu} h \beta^\nu \rangle. \]

Conjugating both sides yields

\[ \langle \beta^\nu \delta^\mu \beta^{-\nu}, h \rangle \cong \langle \beta^\nu \delta^\mu \beta^{-\nu} \rangle * \langle h \rangle. \]

If we take $\mu > \mu_0$ such that $\mu \equiv 1 \mod |H \setminus \Gamma|$ then $H \beta^\nu \delta^\mu \beta^{-\nu} = H \delta^\mu = H g$, because $H$ was assumed normal, $\beta \in H$, and $\delta \in H g$.

Finally, we return to the general case.

Lemma 6.6. Let $C$ be a nonunipotent conjugacy class in a connected, reductive group $G$. If $\Gamma$ is a finitely generated subgroup of $G$, then there exists a finite index subgroup $H < \Gamma$ which does not meet $C$.

Proof. Since $\Gamma$ lies in the points of $G$ over a finitely generated field, the identity map $\Gamma \to \Gamma$ extends to a continuous map from the profinite completion of $\Gamma$ into the points of $G$ over a local field. As the (Zariski) closure of $C$ does not contain the identity, it follows that the identity does not lie in the profinite closure of $C \cap \Gamma$. □

Proof of Theorem 6.3. Let $H$ and simple direct factor of $G$. Denote by $\pi_H$ the projection $G \to H$. If $H$ is neither of type $B_n$, $n \geq 2$, nor of type $G_2$, then Remark 2.12 implies that there exists no class of elements $h \in H$ with the property that $Z_H(h)^{\circ}$ is a simple, proper subgroup of full rank in $H$. If $H$ is of of type $B_n$, $n \geq 2$, or of type $G_2$, then there exists a unique such class, say $C_H$. In this case, $C_H$ is semisimple torsion.
It follows from Lemma 6.6 that there exists a subgroup $I_{0}$ of finite index in $I$ with the property that $\pi_{H}(I_{0}) \cap C_{H} = \emptyset$ for every $H$ of type $B_{n}$, $n \geq 2$, or $G_{2}$. The result now follows from Theorem 6.4 and Lemma 6.1.

Proof of Theorem 6.4. Let $G_{\pm1}$ be as in the statement of Theorem 6.5. Note that $G_{\pm1}$ lies in the product of the simply laced, absolutely simple direct factors of $G$. Let $Hg$ be a coset of a finite-index subgroup $H \subset I$, $g \in I$.

Let $F_{r}$ be the set of all elements $h \in F$ whose semisimple parts $h_{s}$ satisfy the condition

$$h_{s}^{j} \in G_{+1} \implies j = 0.$$  

Fix for the moment an element $h \in F_{r}$. Upon conjugating, we may assume $T$ contains $u_{s}$. Then writing $h = h_{-1}h_{+1}$ with $h_{-1} \in G_{-1}$, then some eigenvalue $\tau$ of $h_{-1}$ has infinite multiplicative order. By Lemma 5.3, there exists a local field $k_{h}$ extending $k$ such that $|\tau| \neq 1$. If $\Pi_{sm} = \Pi_{sm}(h_{s})$ is as in \S 5.1, then there exists a simply laced component of the Dynkin diagram of $G$ which is not contained in $\Pi_{sm}$. If $\rho$ is any $k_{h}$-rational, absolutely irreducible, finite-dimensional, nonzero representation of $G$ which factors through the corresponding absolutely simple direct factor of $G$, then $\rho(h)$ lies outside every compact subgroup of $\rho(G_{kh})$.

Choose such a representation $\rho_{h}$ which is $\ell_{r(h)}$-large, where $\ell(h)$ is the rank of the simple factor of $G$ through which $\rho_{h}$ factors. Given an affine hyperspace $K$ of $\Lambda \otimes \mathbb{R}$, denote by $V_{h,K}$ the projectivization of $\bigoplus_{\mu \in K} E_{\mu}^{\nu}$. By Corollary 4.13, we have that the set $X$ of $u \in G$ satisfying $V_{h,K} \cap h^{u} \cdot V_{h,K'} = \emptyset$ for every pair $K, K'$ of parallel affine hyperspaces of $\Lambda \otimes \mathbb{R}$ and every $h \in F_{r}$ is nonempty and open.

Set $F_{p} = F \setminus F_{r}$. Denote by $Y$ the set of elements $\gamma \in I$ of infinite order such that the subgroup $\langle \gamma, h \rangle$ generated by $\gamma$ and $h$ is canonically isomorphic to the free product $\langle \gamma \rangle \ast \langle h \rangle$, all $h \in F_{p}$. By Theorem 6.5, $Y$ is dense in the profinite topology on $I$. Since $I$ is finitely generated it follows from Lemma 3.5 that $Y \cap Hg$ is Zariski-dense. For convenience, denote by $\phi$ the morphism $T \times G \to G$ given by $(t, u) \mapsto ^{u}t$. Then there exists an element $g \in Y \cap Hg$ in the image of $T(k) \times X$ under the dominant morphism $\phi$.

Fix $h \in F_{r}$, and positive a number $c$. Writing $g = ^{u}t$ for $u \in X$ and $t \in T(k)$ and $K(h, c)$ for the affine hyperspace $\{ \sum_{\mu \in \Lambda} \mu \otimes q_{\mu} \in \Lambda \otimes \mathbb{R} | \sum_{\mu \in \Lambda} q_{\mu} \log |\mu(h)| = \log c \}$, we observe that $A^{c}(\rho_{h}(t))$ coincides with $V_{h,K(h,c)}$. In particular, $A^{c}(\rho_{h}(t)) \cap h^{u} \cdot A^{d}(\rho_{h}(t)) = \emptyset$ for every pair of positive numbers $c$ and $d$ and every $h \in F_{r}$. Finally, we compute

$$A^{c}(\rho_{h}(g)) \cap h \cdot A^{d}(\rho_{h}(g)) = u \cdot A^{c}(\rho_{h}(t)) \cap hu \cdot A^{d}(\rho_{h}(t)) = \emptyset$$

for every pair of positive numbers $c$ and $d$ and every $h \in F_{r}$. □

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