Almost complex circle actions with few fixed points

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Let $M$ be a smooth manifold with a smooth action of a Lie group $G$. If the action has exactly two fixed points, then the conjecture of Smith in [1] states that the corresponding representations of $G$ at these two points are equivalent. The case when $G$ is cyclic was proved by Milnor. Since the paper [2] of Conner and Floyd the theory of smooth actions of Lie groups has extensively involved the use of extraordinary cohomology theories, formal group laws, and localization theorems for both ordinary and equivariant cohomology (see [3]–[5]). The present paper is concerned with circle actions preserving some almost complex structure on the given manifold and having only isolated fixed points.

As shown in [6], if $M^n$ is a manifold with an almost complex circle action and an odd number of isolated fixed points, then $n = 4k$. The case $M^4 = \mathbb{C}P^2$ provides us with an example of a smooth almost complex action with exactly three fixed points.

**Theorem 1.** There does not exist an 8-dimensional manifold $M$ with a smooth almost complex circle action having exactly three isolated fixed points.

We recall that the quaternionic projective space $\mathbb{H}P^2$ admits a circle action with exactly three fixed points but does not carry any almost complex structure.

If a smooth action of $S^1$ on an $n$-dimensional manifold is almost complex and has exactly two fixed points, then $n = 2$ or 6; furthermore, the weights at the fixed points of the action are $\{a, b, -a - b\}$ and $\{-a, -b, a + b\}$ for some positive integers $a$ and $b$ if $n = 6$. The proof can be found in [6] under the assumption that the action is symplectic, but it is easily extended to the almost complex case. The main tool is the localization theorem for Chern numbers or for the universal toric genus [7].

**Theorem 2.** For every almost complex manifold $M^4$ the manifold $(S^6) \#_{S^1} (S^1 \times S^1 \times M^4)$ admits an almost complex structure and a circle action preserving that structure and having exactly two fixed points.

Here by $(S^6) \#_{S^1} (S^1 \times S^1 \times M^4)$ we mean a fiberwise connected sum of the manifolds along a tubular neighborhood of two free orbits of the $S^1$-action on each manifold.

**Proof of Theorem 1.** Let $M = M^8$ be a smooth manifold with a smooth almost complex circle action such that its fixed points are all isolated and there are exactly three of them, let $a_1 = m$ be the maximum in absolute value among all the weights of the action, and let $x \in M$ be the fixed point corresponding to $a_1$. We can assume that $m > 0$, otherwise consider the conjugate action. Any $2k$-manifold with a smooth almost complex circle action and weights equal to ±1 has exactly $2^k$ fixed points if they are all isolated [8], [9]. Hence $m \neq \pm 1$. Furthermore, the weights of $S^1 / \mathbb{Z}/m$ acting on the manifold $M^{2/m}$ are also ±1, so the number of fixed points of the $S^1 / \mathbb{Z}/m$-action is also a power of two. The number of fixed points cannot be equal to one: this would contradict the localization theorem for Chern numbers. Thus, the component of $M^{2/m}$ containing the fixed points of the action is two-dimensional. Let $y \neq x, y \in M^{2/m}$ be a second fixed point such that $b_1 = -a_1 = -m$ (the $b_i$’s are the weights at $y$). Then the sets of weights $(a_2, a_3, a_4)$ and $(b_2, b_3, b_4)$ are equal modulo $m$, since the residuals are the weights of $\mathbb{Z}/m \subset S^1$ acting on the space of the normal bundle in $M$ of the component of $M^{2/m}$. We denote the weights at the third fixed point $z \in M$ by $c_i$. □
Lemma 1 [10]. The weights $a_1, \ldots, c_4$ can be split into pairs of weights opposite in sign and equal in absolute value, and moreover, the weights in any pair can be assumed to belong to different fixed points.

Lemma 2 [6]. If the number of fixed points of an almost complex circle action on $M^{2n}$ is less than $n + 1$, then for each fixed point $x \in M^{2n}$ there exists another fixed point having the same sum of weights as $x$.

Lemma 2 implies that the sums of the weights at all the fixed points are equal, and Lemma 1 implies that the sum of all the weights is zero, and therefore the sum at each fixed point is zero. This means that the weights $b_2, b_3, b_4$ are of the form $a_2 + m, a_3 + m, a_4$. The weight $a_4$ cannot be opposite to either of the weights $a_2 + m, a_3 + m$ because in that case we would have $m + a_2 + a_3 + a_4 \neq 0$. Moreover, $a_2 + a_4 \neq 0$ and $a_3 + a_4 \neq 0$. So the only remaining possibility is $c_1 = c_2 = -a_4$. Using similar arguments, we conclude that $a_2 + m = -a_2, c_3 = -a_3, c_4 = -a_3 - m$. But then the manifold $M^{2/(m/2)}$ (more precisely, its component carrying the fixed points of the action) is four-dimensional and has exactly two fixed points. The weights of the $S^1/Z/(m/2)$-action on this manifold are $(2, -1)$ and $(-2, 1)$, and this contradicts the localization theorem.

Remark 3. The three sets of ‘weights’

$$(-7, -1, 10, 12), \quad (-8, 3, 5, 14), \quad (-8, 3, 5, 14)$$

provide an example satisfying every condition of the localization theorem, but they still cannot be the weights of some almost complex circle action on $M^8$. Note that sum of the weights is equal to 14 in each set, so the implication of Lemma 2 still holds, but the statement of Lemma 1 is obviously false.

Proof of Theorem 2. It is well known that $S^6 = G_2/SU(3)$ admits an almost complex structure that is preserved under the action of the maximal torus $T^2 \subset SU(3)$ [1]. There are exactly two fixed points, and one can find a one-dimensional Abelian subgroup acting on $S^6$ with the weights $\{a + b, -a, -b\}$ and $\{-a - b, a, b\}$. □

Let $M^6$ and $N^6$ be manifolds with almost complex circle actions, and let $U_M$ and $U_N$ be tubular neighbourhoods of some orbits of free actions on $M^6$ and $N^6$. We fix two equivariant diffeomorphisms $\phi_M, \phi_N : S^1 \times S^4 \to \partial U_M, \partial U_N$ and denote by $X$ the smooth manifold $(M^6 - U_M) \cup_{\phi_M} (S^1 \times S^4 \times I) \cup_{\phi_N} (N^6 - U_N)$. Theorem 2 is therefore reduced to the following statement.

Lemma 3. The manifold $X$ admits an almost complex structure that is itself an extension of the structures on $M^6 - U_M$ and $N^6 - U_N$. This structure is invariant under the natural extension of the $S^1$-action.

Proof of Lemma 3. The theorem on the existence of an equivariant tubular neighborhood enables us to extend the smooth structures from $M^6 - U_M$ and $N^6 - U_N$ to $X$. The obstruction to equivariant extension of the almost complex structures lies in the group $\pi_4(SO(6)/U(3)) = \pi_4(SO/U)$, but this group is trivial [12]. □

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