The Helmholtz equation in random media: well-posedness and a priori bounds

O. R. Pembery∗, E. A. Spence∗

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Abstract

We prove well-posedness results and a priori bounds on the solution of the Helmholtz equation
\[ \nabla \cdot (A\nabla u) + k^2 nu = -f, \]
posed either in \( \mathbb{R}^d \) or in the exterior of a star-shaped Lipschitz obstacle, for a class of random \( A \) and \( n \), random data \( f \), and for all \( k > 0 \). The particular class of \( A \) and \( n \) and the conditions on the obstacle ensure that the problem is nontrapping almost surely. These are the first well-posedness results and a priori bounds for the stochastic Helmholtz equation for arbitrarily large \( k \) and for \( A \) and \( n \) varying independently of \( k \).

These results are obtained by combining recent bounds on the Helmholtz equation for deterministic \( A \) and \( n \) and general arguments (i.e. not specific to the Helmholtz equation) presented in this paper for proving a priori bounds and well-posedness of variational formulations of linear elliptic stochastic PDEs. We emphasise that these general results do not rely on either the Lax-Milgram theorem or Fredholm theory, since neither are applicable to the stochastic variational formulation of the Helmholtz equation.

Keywords: Helmholtz equation, random media, well-posedness, a priori bounds, high frequency, resolvent, nontrapping

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1 Introduction

The goals of this paper are to prove results on the well-posedness of variational formulations of the stochastic Helmholtz equation
\[ \nabla \cdot (A(\omega)\nabla u(\omega)) + k^2 n(\omega)u(\omega) = -f(\omega), \] (1.1)
as well as a priori bounds on its solution that are explicit in the wavenumber \( k \) and the material coefficients \( A \) and \( n \).

We consider (1.1) with physical domain either \( \mathbb{R}^d \), \( d = 2, 3 \), or \( \mathbb{R}^d \setminus \overline{D} \), where \( D \) (referred to as the obstacle) is a bounded, Lipschitz, open set such that \( \mathbb{R}^d \setminus \overline{D} \) is connected, and

- \( \omega \) is an element of the underlying probability space,
- \( A \) is a symmetric-positive-definite matrix-valued random field such that ess supp \((I - A)\) is compact,
- \( n \) is a positive real-valued random field such that ess supp \((1 - n)\) is compact,
- \( f \) is a real-valued random field such that ess supp \( f \) is compact, and
- \( k > 0 \) is the wavenumber,

and we are particularly interested in the case where the wavenumber \( k \) is large.

The motivation for establishing well-posedness and proving a priori bounds on the solution of (1.1) is the growing interest in Uncertainty Quantification (UQ) for the Helmholtz equation; see e.g. [59, 54, 9, 24, 21, 22, 33, 4]. (In this PDE context, by ‘UQ’ we mean theory and algorithms for computing statistics of quantities of interest involving PDEs either posed on a random domain or having random coefficients.) There is a large literature on UQ for the stationary diffusion equation
\[ - \nabla \cdot (\kappa(\omega)\nabla u(\omega)) = f(\omega), \] (1.2)
due in part to its large number of applications (e.g. in modelling groundwater flow), and a priori bounds on the solution are vital for the rigorous analysis of UQ algorithms; see e.g. [3, 2, 26, 45, 17]. In contrast, whilst (1.1) has many applications (e.g. in geophysics and electromagnetics), there is much less rigorous theory of UQ for the Helmholtz equation. The main reason for this is that the (deterministic) PDE theory of (1.1) when \( k \) is large is much more complicated that the analogous theory for (1.2).

∗Department of Mathematical Sciences, University of Bath, Claverton Down, Bath, BA2 7AY, UK, O.R.Pembery@bath.ac.uk, E.A.Spence@bath.ac.uk
To our knowledge, the only work that considers (1.1) with large \( k \) and attempts to establish either (i) well-posedness of variational formulations or (ii) a priori bounds is [21], which considers both (i) and (ii) for (1.1) posed in a bounded domain with an impedance boundary condition. We discuss the results of [21] further in [2], but we highlight here that (a) [21] considers \( A = I \) and \( n = 1 + \eta \), with \( \eta \) random and the magnitude of \( \eta \) decreasing with \( k \), whereas we consider classes of \( A \) and \( n \) that allow \( k \)-independent random perturbations, and (b) in its well-posedness result, [21] invokes Fredholm theory to conclude existence of a solution, but this relies on an incorrect assumption about compact containment of Bochner spaces—see Appendix A below. In \( \S 2 \) we also discuss the papers [9, 35, 36, 33] on the theory of UQ for either (1.1) or the related time-harmonic Maxwell’s equations; in these papers either the \( k \)-explicit well-posedness is not a primary concern or \( k \) is assumed to be small. Our hope is that the results in the present paper can be used in the rigorous theory of UQ for Helmholtz problems with large \( k \).

The main results in this paper, namely well-posedness and a priori bounds explicit in \( k, A, \) and \( n \) (Theorems 1.4 and 1.8 below), are proved by combining:

1. bounds for the Helmholtz equation in [27] with \( A \) and \( n \) deterministic but spatially-varying, with
2. general arguments (i.e. not specific to the Helmholtz equation) presented here for proving a priori bounds and well-posedness of variational formulations of linear elliptic SPDEs.

Regarding 1: the \( k \)-dependence of the bounds on \( u \) in terms of \( f \) depends crucially on whether or not \( A, n, \) and \( D_+ \) are such that there exist trapped rays. In the trapping case, the solution operator can grow exponentially in \( k \) (see [49, 10, 48, 12, 6] and [7, \( \S 2.5 \)], and the reviews in [44, \S 6], [15, \S 1.1], and [27, \S 1]). In the nontrapping case, the solution operator is bounded uniformly in \( k \) (see [55, 43, 11]). The bounds in [27] are under conditions on \( A, n, \) and \( D_+ \) that ensure nontrapping of rays; the significance of these bounds is that they are the first (deterministic) bounds for the Helmholtz scattering problem in which both \( A \) and \( n \) vary and the bound is explicit in \( A \) and \( n \) (as well as in \( k \)); this feature allows us to prove results when \( A \) and \( n \) are random fields.

Regarding 2: the main reason these general arguments are needed is the fact that the standard variational formulations of the (deterministic) Helmholtz equation are not coercive. In the deterministic case, the remedy for the lack of coercivity is to use Fredholm theory, but this is not applicable to the stochastic variational formulation of the stochastic Helmholtz equation because the necessary compactness results do not hold in Bochner spaces (see Appendix A below). A nontechnical summary of the ideas behind our general well-posedness results is given in Remark 3.12 below. Some of these results are similar in spirit to the results in [26, 45] about the PDE (1.2) (in the case when the coefficient \( \kappa \) is not uniformly bounded above and below); moreover, we use some of the ideas and technical tools from these two papers. One reason we state our well-posedness results in general (i.e. not only in the specific case of the Helmholtz equation) is that we expect that they can be used in the future to prove well-posedness results for the time-harmonic Maxwell’s equations in random media.

### 1.1 Statement of main results

#### Notation and basic definitions

Let either (i) \( D_+ \subset \mathbb{R}^d, d = 2, 3 \), be a bounded Lipschitz open set such that \( \emptyset \subset \subset D_- \) and the open complement \( D_+ := \mathbb{R}^d \setminus D_- \) is connected or (ii) \( D_+ = \emptyset \). Let \( \Gamma_D = \partial D_+ \). Fix \( R > 0 \) and let \( B_R \) be the ball of radius \( R \) centred at the origin. Define \( \Gamma_R := \partial B_R \) and \( D_R = D_+ \cap \Gamma_R \) (see Figure 1). Let \( \gamma \) denote the trace operator from \( D_R \) to \( \partial D_R = \Gamma_D \cup \Gamma_R \) and define \( H^0_{0,D}(D_R) := \{ v \in H^1(\Gamma_D) : \gamma v = 0 \text{ on } \Gamma_R \} \).

Let \( \mathcal{T}_R : H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R) \) be the Dirichlet-to-Neumann map for the deterministic equation \( \Delta u + k^2 u = 0 \) posed in the exterior of \( B_R \) with the Sommerfeld radiation condition

\[
\frac{\partial u}{\partial r}(x) - iku(x) = a \left( \frac{1}{r^{(d-1)/2}} \right) \text{ as } r := |x| \to \infty, \text{ uniformly in } \frac{x}{|x|} \quad (1.3)
\]

see [46, \S 2.6.3] and [14, Equations 3.5 and 3.6] for an explicit expression for \( \mathcal{T}_R \) in terms of Hankel functions and Fourier series (\( d = 2 \))/spherical harmonics (\( d = 3 \)). Let \( (\cdot, \cdot)_{\Gamma_R} \) be the duality pairing on \( \Gamma_R \) between \( H^{-1/2}(\Gamma_R) \) and \( H^{1/2}(\Gamma_R) \) and write \( d\lambda \) for Lebesgue measure.

Let \( L^\infty(D_+; \mathbb{R}^{d \times d}) \) be the set of matrix-valued functions \( A : D_+ \to \mathbb{R}^{d \times d} \) such that \( A_{ij} \in L^\infty(D_+; \mathbb{R}) \) for all \( i, j = 1, \ldots, d \). We define other matrix-valued spaces on \( D_+ \) and \( D_R \), e.g. \( W^{1,\infty}(D_+; \mathbb{R}^{d \times d}) \), analogously.

The range of functions is \( \mathbb{C} \) we suppress the second argument in a function space, e.g. we write \( L^\infty(D_+) \) for \( L^\infty(D_+; \mathbb{C}) \). We write \( D_1 \subset \subset D_2 \) if \( D_1 \) is a compact subset of the open set \( D_2 \). Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete \( \sigma \)-finite probability space. Let

- \( f : \Omega \to L^2(D_+) \) be such that \( \text{ess sup} f \subset \subset B_R \) almost surely,
- \( n : \Omega \to L^\infty(D_+; \mathbb{R}) \) be such that \( \text{ess sup}(1 - n) \subset \subset B_R \) almost surely and there exist \( n_{\min}, n_{\max} : \Omega \to \mathbb{R} \) such that
  \[
  0 < n_{\min}(\omega) \leq n(x)(\omega) \leq n_{\max}(\omega) \text{ for almost every } x \in D_+ \text{ almost surely},
  \]
- \( A : \Omega \to L^\infty(D_+; \mathbb{R}^{d \times d}) \) be such that \( \text{ess sup}(1 - 0) \subset \subset B_R \), \( A_{ij} \) are \( \text{ess sup}(1 - A) \subset \subset B_R \), \( A_{ij} = A_{ji} \) almost surely, and there exist \( A_{\min}, A_{\max} : \Omega \to \mathbb{R} \) such that
  \[
  0 < A_{\min}(\omega) < A_{\max}(\omega) \text{ almost surely and } A_{\min}(\omega)|\xi|^2 \leq (A(\omega, x)\xi) \cdot \xi \leq A_{\max}(\omega)|\xi|^2 \text{ for almost every } x \in D_+ \text{ and for all } \xi \in \mathbb{C}^d \text{ almost surely},
  \]
Figure 1: The domains $D_-$ and $D_R$, the set $\Gamma_R$, and essential supports of $I-A$, $1-n$ and $f$ in the definition of the Helmholtz stochastic EDP.

If $v : \Omega \to Z$ for some function space $Z$ of functions on $\mathbb{R}^d$, we abuse notation slightly and write $v(\omega, x)$ instead of $v(\omega)$. 

**Variational Formulations** We consider three different formulations of the *Helmholtz stochastic exterior Dirichlet problem* (stochastic EDP); Problems 1–3 below.

Define the sesquilinear form $a(\omega)$ on $H^1_{0,D}(D_R) \times H^1_{0,D}(D_R)$ by

$$[a(\omega)](v_1, v_2) := \int_{D_R} \left((A(\omega)\nabla v_1) \cdot \nabla \overline{v_2} - k^2 n(\omega) v_1 \overline{v_2}\right) d\lambda - \langle T_R \gamma v_1, \gamma v_2 \rangle_{\Gamma_R}, \quad (1.4)$$

and the antilinear functional $L(\omega)$ on $H^1_{0,D}(D_R)$ by

$$[L(\omega)](v_2) := \int_{D_R} f(\omega) \overline{v_2} d\lambda. \quad (1.5)$$

Define the sesquilinear form $a$ on $L^2(\Omega; H^1_{0,D}(D_R)) \times L^2(\Omega; H^1_{0,D}(D_R))$ by

$$a(v_1, v_2) := \int_{\Omega} [a(\omega)](v_1(\omega), v_2(\omega)) d\mathbb{P}(\omega) \quad (1.6)$$

and the antilinear functional $\mathcal{L}$ on $L^2(\Omega; H^1_{0,D}(D_R))$ by

$$\mathcal{L}(v_2) := \int_{\Omega} [L(\omega)](v_2(\omega)) d\mathbb{P}(\omega). \quad (1.7)$$

We consider the following three problems:

**Problem 1 (Measurable EDP almost surely)** Find a measurable function $u : \Omega \to H^1_{0,D}(D_R)$ such that

$$[a(\omega)](u(\omega), v) = [L(\omega)](v) \text{ for all } v \in H^1_{0,D}(D_R) \text{ almost surely.}$$

**Problem 2 (Second-order EDP almost surely)** Find $u \in L^2(\Omega; H^1_{0,D}(D_R))$ such that

$$[a(\omega)](u(\omega), v) = [L(\omega)](v) \text{ for all } v \in H^1_{0,D}(D_R) \text{ almost surely.}$$

**Problem 3 (Stochastic variational EDP)** Find $u \in L^2(\Omega; H^1_{0,D}(D_R))$ such that

$$a(u, v) = \mathcal{L}(v) \text{ for all } v \in L^2(\Omega; H^1_{0,D}(D_R)).$$
Remark 1.1 (Why consider all of Problems 1–3?) The analogues of Problems 1 and 3 for the stationary diffusion equation (1.2) are well-studied in the UQ literature. For example, [58, 2, 47, 16, 17, 53, 39, 32] consider Problem 1, and [3, 38, 5, 30] consider Problem 3. Furthermore, Problem 3 is the foundation of the Stochastic Galerkin method (a finite element method in $\Omega \times D$, where $D$ is the spatial domain), see, e.g., [3]. In the context of the Helmholtz equation, Problem 3 is studied for the Interior Impedance Problem in [21].

In the case of the stationary diffusion equation (1.2), the well-posedness and relationships between the solutions of Problem 1-3 are well-understood. Indeed, when the diffusion coefficient $\kappa$ in (1.2) is uniformly bounded below (over $\omega$), the bilinear forms for both Problems 1 and 3 are coercive, and existence and uniqueness of a solution can be concluded using the Lax–Milgram theorem. In the case of log-normal coefficients $\kappa$, however, one cannot use the Lax–Milgram theorem to conclude existence and uniqueness of a solution to Problem 3, and one must use the techniques in [26, 45].

In the Helmholtz case, the sesquilinear form $a$ in Problem 3 is not coercive on $\Omega \times D$, and Fredholm theory (which can be used in the deterministic case to conclude existence and uniqueness) is not applicable, because the Bochner space $L^2(\Omega; H^1(D))$ is not compactly contained in $L^2(\Omega; L^2(D))$—see Appendix A.

Given these difficulties in proving well-posedness of a solution to Problem 3, we consider Problem 2 as an intermediary between Problems 1 and 3; studying Problem 2 will allow us to move from well-posedness and a priori bounds for Problem 1 to well-posedness and a priori bounds for Problem 3 (under additional assumptions on $A$ and $n$).

We highlight that it is not automatic that a solution of Problem 1 is also a solution of Problem 2. In the case of the stationary diffusion equation, showing that a solution to Problem 1 is also a solution of Problem 2 is possible in the case of uniformly coercive and bounded coefficients $\kappa$ using the Lax–Milgram Theorem (see e.g. [3, 32.2]), and is possible in the case of log-normal coefficients by [17, Theorem 2.2 and Proposition 2.4] using well-known bounds on the solution of the deterministic problem. In contrast, obtaining conditions under which a solution of Problem 1 is a solution of Problem 2 in the Helmholtz case requires the newly-proved deterministic bounds in [27], since none of the previously-existing bounds on the solution were explicit in $A$ or $n$.

Remark 1.2 (Measurability of $u$ in Problem 1) It is natural to construct the solution of Problem 1 pathwise; that is, one defines $u(\omega)$ to be the solution of the deterministic problem with coefficients $A(\omega)$ and $n(\omega)$. However, is it then not obvious that $u$ is measurable. In the proof of Theorem 1.4 below, we show that the measurability of $u$ follows from

(i) a natural condition on the measurability of the coefficients and data (Condition C1 below), and

(ii) the continuity of the map taking the coefficients of the deterministic PDE to the solution of the deterministic PDE (see Lemma 5.12 below).

In Theorems 1.4 and 1.8 we prove results on the well-posedness of Problems 1–3 under conditions on $A$, $n$, $f$, and $D_\omega$. Although $A$, $n$, and $f$ are defined on $D_\omega$, since $\text{ess supp}(I - A)$, $\text{ess supp}(1 - n)$, and $\text{ess supp} f$ are compactly contained in $D_R$, we can consider $A,n,$ and $f$ as functions on $D_R$.

Condition 1.3 (Regularity and stochastic regularity of $f$, $A$, and $n$) The map $f \in L^2(\Omega; L^2(D))$, the map $A : \Omega \rightarrow W^{1,\infty}(D_R; \mathbb{R}^{d \times d})$ with $A \in L^\infty(\Omega; L^\infty(D_R; \mathbb{R}^{d \times d}))$, and the map $n \in L^\infty(\Omega; L^\infty(D_R; \mathbb{R}))$.

Theorem 1.4 (Equivalence of variational problems) Under Condition 1.3:

- The maps $a$ and $\Sigma$ (defined by (1.6) and (1.7)) are well-defined.
- $u \in L^2(\Omega; H^1(D))$ solves Problem 2 if and only if $u$ solves Problem 3.
- If $u \in L^2(\Omega; H^1(D))$ solves Problem 2, then any member of the equivalence class of $u$ solves Problem 1.
- The solution of Problem 1 exists and is unique up to modification on a set of measure zero in $\Omega$.

The solution of Problems 2 and 3 is unique in $L^2(\Omega; H^1(D))$.

Observe that the only relationship between formulations not proved in Theorem 1.4 is: if $u : \Omega \rightarrow H^1(D)$ solves Problem 1 then $u \in L^2(\Omega; H^1(D))$ and $u$ solves Problem 2. Theorem 1.8 below includes this relationship, but we need additional assumptions on $A,n,$ and $D_\omega$.

Definition 1.5 (A particular class of (deterministic) nontrapping coefficients) Let $\mu_1, \mu_2 > 0$, $A_0 \in W^{1,\infty}(D_R; \mathbb{R}^{d \times d})$ with $\text{ess supp}(I - A_0) \subset \subset B_R$, and $n_0 \in W^{1,\infty}(D_R; \mathbb{R})$ with $\text{ess supp}(1 - n_0) \subset \subset B_R$. We write $A_0 \in \text{NT}_A(\mu_1)$ if

$$A_0(x) - (x \cdot \nabla)A_0(x) \geq \mu_1$$

in the sense of quadratic forms for almost every $x \in D_R$. We write $n_0 \in \text{NT}_n(\mu_2)$ if

$$n_0(x) + x \cdot \nabla n_0(x) \geq \mu_2$$

for almost every $x \in D_R$.

The significance of the class of coefficients in Definition 1.5 is that [27, Theorem 2.10(ii)] proves bounds on the solution of (1.1) for such $A$ and $n$, where the constant in the bound only depends on $\mu_1, \mu_2, k, R,$ and $d$. 


Condition 1.6 ($k$-independent nontrapping conditions on (random) $A$ and $n$) The maps $A: \Omega \to W^{1,\infty}(D;\mathbb{R}^{d\times d})$, $n: \Omega \to W^{1,\infty}(D;\mathbb{R})$, and there exist $\mu_1, \mu_2: \Omega \to \mathbb{R}$, independent of $f$, with $\mu_1(\omega), \mu_2(\omega) > 0$ almost surely and $1/\mu_1, 1/\mu_2 \in L^2(\Omega; \mathbb{R})$ such that $A(\omega) \in NT_A(\mu_1(\omega))$ almost surely and $n(\omega) \in NT_n(\mu_2(\omega))$ almost surely.

Definition 1.7 (Star-shaped) The set $D \subseteq \mathbb{R}^d$ is star-shaped with respect to the point $x_0$ if for any $x \in \overline{D}$ the line segment $[x_0, x] \subseteq D$.

Theorem 1.8 (Equivalence of variational problems in a nontrapping case) Let $D_-$ be star-shaped with respect to the origin. Under Conditions 1.3 and 1.6:

- The maps $a$ and $\mathcal{L}$ (defined by (1.6) and (1.7)) are well-defined.
- Problems 1–3 are all equivalent.
- The solution $u \in L^2(\Omega; H^1_0(D))$ of these problems exists, is unique, and, given $k_0 > 0$, satisfies the bound
  \[ \|\nabla u\|_{L^2(\Omega; \mathbb{R}^d)}^2 + k^2 \|u\|_{L^2(\Omega; \mathbb{R}^d)}^2 \leq \|C_1\|_{L^2(\Omega; \mathbb{R}^d)}^2 \|f\|_{L^2(\Omega; \mathbb{R}^d)}^2 \]  
  for all $k \geq k_0$, where $C_1: \Omega \to \mathbb{R}$ is given by
  \[ C_1 = \max \left\{ \frac{1}{\mu_1}, \frac{1}{\mu_2} \right\} \left( \frac{R^2}{\mu_1} + \frac{2}{\mu_2} \left( R + \frac{d - 1}{2k_0} \right)^2 \right). \]

As highlighted above, Theorem 1.8 is obtained from combining deterministic a priori bounds from [27] with the general arguments in §3 about well-posedness of variational formulations of stochastic PDEs. Theorem 1.8 uses the most basic a priori bound proved in [27] (from [27, Theorem 2.10(i)]), but [27] contains several extensions of this bound. Remarks 1.9, 1.10, 1.12, 1.13, and 1.14 outline the implications that these (deterministic) extensions have for the stochastic Helmholtz equation.

Remark 1.9 (Dirichlet boundary conditions on $\Gamma_D$ and plane-wave incidence) The formulations of the stochastic EDP above assume that $u = 0$ on the boundary $\Gamma_D$. An important scattering problem for which $u \neq 0$ on $\Gamma_D$ is when $u$ is the field scattered by an incident plane wave; in this case $\gamma u = -\gamma u_1$, where $u_1$ is the incident plane wave [13, p. 107].

The results in this paper can be easily extended to the case when $u \neq 0$ on $\Gamma_D$ using [27, Theorem 2.10(iv)], which proves a priori (deterministic) bounds in this case. One subtlety, however, is that $f$ is then not necessarily independent of $\mu_1$ and $\mu_2$. Indeed in this case
  \[ f = -\nabla \cdot (A\nabla u_1) - k^2 n u_1; \]  
if $\mu_1$ depends on $A$ and $\mu_2$ depends on $n$ then (1.12) shows that $f$ may be not be independent of $\mu_1$ and $\mu_2$.

One can produce an analogue of Theorem 1.8 in the case where $f, \mu_1, \mu_2$ are dependent, but one requires $1/\mu_1, 1/\mu_2 \in L^1(\Omega)$ and $f \in L^1(\Omega; L^1(\Omega))$; see Remark 5.17 below.

Remark 1.10 (The case when either $n = 1$ or $A = I$) When either $n = 1$ or $A = I$, [27, Theorem 2.10(ii) and (iii)] give deterministic bounds under weaker conditions on $A$ and $n$ respectively; the corresponding results for the stochastic case are that:

- When $n = 1$ almost surely, the condition $A(\omega) \in NT_A(\mu_1(\omega))$ in Condition 1.6 can be improved to
  \[ 2A(\omega) - (x \cdot \nabla)A(\omega) \geq \mu_1(\omega) \]  
  for almost every $x \in D_+$, almost surely.

- When $A = I$ almost surely, the condition $n(\omega) \in NT_n(\mu_2(\omega))$ in Condition 1.6 can be improved to
  \[ 2n(\omega) + x \cdot \nabla n(\omega) \geq \mu_2(\omega) \]  
  for almost every $x \in D_+$, almost surely.

Remark 1.11 (Geometric interpretation of the conditions on $A$ and $n$ in Definition 1.5) Recall that the $k \to \infty$ asymptotics of solutions of the Helmholtz equation are governed by the behaviour of rays (see, e.g., [11]). Given (deterministic) $A_0$ and $n_0$, the Helmholtz EDP is nontrapping if all rays starting in $D_R$ and evolving according to the Hamiltonian flow defined by the symbol of $\nabla \cdot (A_0\nabla u) + k^2 n_0 u = -f_0$ escape from $D_R$ after some uniform time (see, e.g., [11, Definition 1.1]); the EDP is trapping otherwise. The $k$-dependence of the solution operator depends strongly on whether the problem is trapping, and the type of trapping present; see, e.g., the overview discussions in [27, §1], [15, §1.1].

The conditions on $A$ and $n$ in Condition 1.6 and the star-shapedness restriction on $D_-$ are sufficient for the Helmholtz stochastic EDP to be nontrapping almost surely. As noted in Remark 1.10, when $A = I$ almost surely the condition on $n$ can be improved from (1.9) to (1.13) using [27, Theorem, Part (iii)]. The condition (1.13) is equivalent to nontrapping when $n$ is radial, i.e. $n(\omega, x) = n(\omega, |x|)$. Indeed, if $n$ is radial and $2n(\omega, x) + x \cdot \nabla n(\omega, x) < 0$ at a point $x \in \mathbb{R}^3$, then the deterministic Helmholtz EDP given by $n(\omega, x)$ is trapping; see [49] and [27, Theorem 6.7].
Remark 1.12 (The Helmholtz stochastic truncated exterior Dirichlet problem) When applying the Galerkin method to Problems 1–3, the Dirichlet-to-Neumann map $T_\nu$ is expensive to compute. Therefore, it is common to approximate the DtN map on $\Gamma_R$ by an ‘absorbing boundary condition’ (see, e.g., [34, §3.3] and the references therein), the simplest of which is the impedance boundary condition $\partial u/\partial n - i ku = 0$. We call the Helmholtz stochastic EDP posed in $D_R$ with an impedance boundary condition on $\Gamma_R$ the stochastic truncated exterior Dirichlet problem (stochastic TEDP). In fact, since we no longer need to know the DtN map explicitly on the truncation boundary, the truncation boundary can be arbitrary (i.e. it does not have to be just a circle/sphere). Note that in the case where the obstacle is the empty set, the TEDP is just the Interior Impedance Problem.

The results in this paper also hold for the stochastic TEDP (with arbitrary Lipschitz truncation boundary) under an analogue of Condition 1.6 based on the deterministic bounds in [27, Theorem 2.21] instead of [27, Theorem 2.10].

Remark 1.13 (Discontinuous $A$ and $n$) The requirements on $A$ and $n$ in Definition 1.5 require them to be continuous (since $W^{1,\infty}(D_R) = C^{0,1}(D_R)$ as $D_R$ is Lipschitz; see, e.g., [20, §4.2.3, Theorem 5]). In addition to proving deterministic a priori bounds for the class of $A$ and $n$ in Definition 1.5, the paper [27] proves deterministic bounds for discontinuous $A$ and $n$ satisfying (1.8) and (1.9) in a distributional sense; see [27, Theorem 2.27]. In this case, when moving outward from the obstacle to infinity, $A$ can jump downwards and $n$ can jump upwards on interfaces that are star-shaped. (When the jumps are in the opposite direction, the problem is trapping; see [48] and [44, §6].) The well-posedness results and a priori bounds in this paper can therefore be adapted to prove results about the stochastic Helmholtz equation for a class of random $A$ and $n$ that allows nontrapping jumps on randomly-placed-star-shaped interfaces.

Remark 1.14 ($k$-dependent $A$ and $n$) In this paper we focus on random fields $A$ and $n$ varying independently of $k$; this corresponds to a fixed physical medium, characterised by $A$ and $n$, with waves of frequency $k$ passing through. In §1.2 below we construct a $A$ and $n$ as ($k$-independent) $W^{1,\infty}$ perturbations of random fields $A_0$ and $n_0$ satisfying Condition 1.6.

We note, however, that results for $A$ and $n$ being $k$-dependent $L^\infty$ perturbations of $A_0$ and $n_0$ satisfying Condition 1.6 can easily be obtained. The basis for this is observing that deterministic a priori bounds hold when (a) $A \in \text{NT}_A(\mu_1)$, $n = n_0 + \eta$, where $n_0 \in \text{NT}_n(\mu_2)$ and $\|\eta\|_{L^\infty(D_R;\mathbb{R})}$ is sufficiently small, and (b) $A = A_0 + B$, $n = n_0 + \eta$, where $A_0 \in \text{NT}_A(\mu_1)$, $n_0 \in \text{NT}_n(\mu_2)$, $\|\eta\|_{L^\infty(D_R;\mathbb{R})}$ and $\|B\|_{W^{1,\infty}(D_R;\mathbb{R}^d)}$ are both sufficiently small, and $A$, $n$, and $D_-$ are such that $u \in H^2(D_R)$ (see, e.g., [42, Theorem 4.18(a)] or [29, Theorems 2.3.3.2 and 2.4.2.5] for these latter requirements). Given these deterministic bounds, the general arguments in this paper can then be used to prove well-posedness of the analogous stochastic problems.

To understand why bounds hold in the case (a), observe that one can write the PDE as
\[
\nabla \cdot (A\nabla u) + k^2 n_0 u = -f - k^2 \eta u;
\]
if $\|\eta\|_{L^\infty(D_R;\mathbb{R})}$ is sufficiently small then the contribution from the $k^2 \eta u$ term on the right-hand side of (1.14) can be absorbed into the $k^2 \|u\|_{H^2(D_R)}^2$ term appearing on the left-hand side of the bound (the deterministic analogue of (1.10)). In the case $n_0 = 1$, this is essentially the argument used to prove the a priori bound in [21, Theorem 2.4] (see [27, Remark 2.15]). The reason bounds hold in the case (b) is similar, except now we need the $H^2$ norm of $u$ on the left-hand side of the bound (as well as the $H^1$ norm) to absorb the contribution from the $\nabla \cdot (B\nabla u)$ term on the right-hand side.

1.2 Random fields satisfying Condition 1.6

The main focus of this paper is proving well-posedness of the variational formulations of the stochastic Helmholtz equation, and a priori bounds on the solution, for the most-general class of $A$ and $n$ allowed by the deterministic bounds in [27]. However, in this section, motivated by the Karhunen-Loève Expansion (see e.g. [41, p. 201ff.]) and similar expansions of material coefficients for the stationary diffusion equation [39, §2.1], we consider $n$ and $A$ as finite-series expansions around known random fields $n_0$ and $A_0$ satisfying Condition 1.6. Let $m \in \mathbb{N}$ and define
\[
n(\omega, \mathbf{x}) = n_0(\mathbf{x}) + \sum_{j=1}^{m} Y_j(\omega)\psi_j(\mathbf{x}) \quad \text{and} \quad A(\omega, \mathbf{x}) = A_0(\mathbf{x}) + \sum_{j=1}^{m} Z_j(\omega)\Psi_j(\mathbf{x}),
\]
where:
- $\text{ess supp}(1 - n_0)$, $\text{ess supp}(I - A_0) \subset \subset B_R$,
- $Y_j, Z_j \sim \text{Unif}(-1/2, 1/2)$ i.i.d.,
- $\psi_j \in W^{1,\infty}(D_R;\mathbb{R})$ with $\text{ess supp} \psi_j \subset \subset B_R$ for all $j = 1, \ldots, m$, and
- $\Psi_j \in W^{1,\infty}(D_R;\mathbb{R}^d)$ with $\text{ess supp} \Psi_j \subset \subset B_R$ for all $j = 1, \ldots, m$.

Regarding the measurability of $n$ and $A$ defined by (1.15); the proof that the sum of measurable functions is measurable is standard, but we have not been able to find this result for this particular setting of mappings into a separable subspace of a general normed vector space, and so we briefly give it in Lemma C.7.

The following lemmas give sufficient conditions for the series in (1.15) to satisfy Condition 1.6.
Lemma 1.15 (Series expansion of $A$ satisfies Condition 1.6) Let $\mu > 0$, $\delta \in (0, 1)$. If $A_0 \in \text{NT}_A(\mu)$, and
\[
\sum_{j=1}^{\infty} \text{ess sup}_{x \in D_R} \| \Psi_j(x) - (x \cdot \nabla)\Psi_j(x) \|_{\text{op}, C^d} \leq 2\delta \mu,
\] (1.16)
where $\| \cdot \|_{\text{op}, C^d}$ is the operator norm induced by the Euclidean vector norm on $\mathbb{C}^d$, then $A \in \text{NT}_A((1 - \delta)\mu)$ almost surely.

Proof of Lemma 1.15 Since $A_0 \in \text{NT}_A(\mu)$, we have
\[
\left( (A(\omega, x) - (x \cdot \nabla)A(\omega, x)) \xi \right) \cdot \xi \geq \mu |\xi|^2 + \sum_{j=1}^{\infty} \left( Z_j(\omega)(\Psi_j(x) - (x \cdot \nabla)\Psi_j(x)) \xi \right) \cdot \xi
\] (1.17)
for all $\xi \in \mathbb{C}^d$, for almost every $x \in D_R$, almost surely. As $Z_j \sim \text{Unif}(-1/2, 1/2)$ for all $j$ and the bound (1.16) holds, the right-hand side of (1.17) is bounded below by
\[
\mu |\xi|^2 - \frac{1}{2} 2\delta \mu |\xi|^2 = (1 - \delta)\mu |\xi|^2
\] almost surely.

Since $\xi \in \mathbb{C}^d$ was arbitrary, it follows that $A(\omega) \in \text{NT}_A((1 - \delta)\mu)$ almost surely, as required. \qed

Lemma 1.16 (Series expansion of $n$ satisfies Condition 1.6) Let $\mu > 0$ and $\delta \in (0, 1)$. If $n_0 \in \text{NT}_n(\mu)$ and
\[
\sum_{j=1}^{\infty} \| \psi_j(x) + x \cdot \nabla \psi_j(x) \|_{L^\infty(D_R, \mathbb{R})} \leq 2\delta \mu,
\] (1.18)
then $n \in \text{NT}_n((1 - \delta)\mu)$.

The proof of Lemma 1.16 is omitted, since it is similar to the proof of Lemma 1.15; in fact it is simpler, because it involves scalars rather than matrices.

Outline of the paper In §2 we discuss our results in the context of related literature. In §3 we state general results on a priori bounds and well-posedness for stochastic variational formulations. In §4 we prove the results in §3. In §5 we prove Theorems 1.4 and 1.8. In Appendix A we discuss the failure of Fredholm theory for the stochastic variational formulation of Helmholtz problems. In Appendix B we recap results from measure theory and the theory of Bochner spaces. In Appendix C we collect together results on measurability in separable subspaces of arbitrary normed vector spaces.

2 Discussion of the main results in the context of other work on UQ for time-harmonic wave equations

In this section we discuss existing results on well-posedness of (1.1), as well as analogous results for the elastic wave equation and the time-harmonic Maxwell’s equations. The most closely-related work to the current paper is [21] (and its analogue for elastic waves [23]), in that a large component of [21] consists of attempting to prove well-posedness and a priori bounds for the stochastic variational formulation (i.e. Problem 3) of the Helmholtz Interior Impedance Problem; i.e., (1.1) with $A = I$ and stochastic $n$ posed in a bounded domain with an impedance boundary condition $\partial u / \partial v - ik u = g$ (recall that this boundary condition is a simple approximation to the Dirichlet-to-Neumann map $T_R$ defined above (1.3)). Under the assumption of existence, [21] shows that for any $k > 0$ the solution is unique and satisfies an a priori bound of the form (1.10) (with different constant $C_1$), provided $n = 1 + \eta$ where the random field $\eta$ satisfies (almost surely) $\| \eta \|_{L^\infty} \leq C / k$ for some $C > 0$ independent of $k$. [21] then invokes Fredholm theory to conclude existence, but this relies on an incorrect assumption about compact containment of Bochner spaces—see Appendix A below. However, combining Theorem 1.4 and Remarks 1.12 and 1.14 with $A = I$ and $n_0 = 1 + \eta$ (with $\eta$ as above) produces an analogous result to Theorem 1.8, and gives a correct proof of [21, Theorem 2.5]. Therefore the analysis of the Monte Carlo interior penalty discontinuous Galerkin method in [21] can proceed under the assumptions of Theorem 1.4 and Remarks 1.12 and 1.14.

The paper [33] considers the Helmholtz transmission problem with a stochastic interface, i.e. (1.1) posed in $\mathbb{R}^d$ with both $A$ and $n$ piecewise constant and jumping on a common, randomly-located interface. A component of this work is establishing well-posedness of Problem 1 for this setup. To do this, the authors make the assumption that $k$ is small (to avoid problems with trapping mentioned above—see the comments after [33, Theorem 4.3]); the sesquilinear form $a$ is then coercive and an a priori bound (in principle explicit in $A$ and $n$) follows [33, Lemma 4.5]. By Remark 1.13, the results of this paper can be used to obtain the analogous well-posedness result for large $k$ in the case of nontrapping jumps.

The paper [9] studies the Bayesian inverse problem associated to (1.1) with $A = I$ and $n = 1$ posed in the exterior of a Dirichlet obstacle. That is, [9] analyses computing the posterior distribution of the shape of the obstacle given noisy observations of the acoustic field in the exterior of the obstacle. A component of the analysis in [9] is the well-posedness of the forward problem for an obstacle with a variable boundary [9, Proposition 3.5]. Instead of mapping the problem to one with a fixed domain and variable $A$ and $n$, [9] instead
works with the variability of the obstacle directly, using boundary-integral equations. The \( k \)-dependence of the solution operator is not considered, but would enter in [9, Lemma 3.1].

The papers [36] and [35] consider the time-harmonic Maxwell’s equations with (i) the material coefficients \( \varepsilon, \mu \) constant in the exterior of a perfectly-conducting random obstacle and (ii) \( \varepsilon, \mu \) piecewise constant and jumping on a common randomly located interface; in both cases these problems are mapped to problems where the domain/interface is fixed and \( \varepsilon \) and \( \mu \) are random and heterogeneous. The papers [36] and [35] essentially consider the analogue of Problem 1 for the time-harmonic Maxwell’s equations, obtaining well-posedness from the corresponding results for the related deterministic problems.

3  General results on proving a priori bounds and well-posedness of stochastic variational formulations

In this section we state general results for proving a priori bounds and well-posedness results for variational formulations of linear elliptic SPDEs.

3.1  Notation and definitions of the variational formulations

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete \( \sigma \)-finite probability space. Let \( X \) and \( Y \) be separable Banach spaces over a field \( \mathbb{F} \) (where \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{C} \)). Let \( B(X, Y^*) \) denote the space of bounded linear maps \( X \rightarrow Y^* \). Let \( \mathcal{C} \) be a topological space with topology \( \mathcal{T} \). Given maps
\[
c : \Omega \rightarrow \mathcal{C}, \quad A : \mathcal{C} \rightarrow B(X, Y^*), \quad \text{and} \quad L : \mathcal{C} \rightarrow Y^*,
\]
let \( \mathfrak{A} : L^2(\Omega; X) \rightarrow L^2(\Omega; Y)^* \) be defined by
\[
[\mathfrak{A}(u)](v) := \int_{\Omega} [A(c(\omega))u(\omega)](v(\omega)) \, d\mathbb{P}(\omega) \tag{3.1}
\]
for \( v \in L^2(\Omega; Y) \), and let \( \mathcal{L} \in L^2(\Omega; Y)^* \) be defined by
\[
\mathcal{L}(v) := \int_{\Omega} L(c(\omega))(v(\omega)) \, d\mathbb{P}(\omega) \tag{3.2}
\]
for \( v \in L^2(\Omega; Y) \). Recall that a bounded linear map \( X \rightarrow Y^* \) is equivalent to a sesquilinear (or bilinear) form on \( X \times Y \); see e.g. [51, Lemma 2.1.38]. To keep notation compact, we write \( A(c) = (A \circ c)(\omega) \) and \( L(c) = (L \circ c)(\omega) \).

Remark 3.1 (Interpretation of the space \( \mathcal{C} \)) The space \( \mathcal{C} \) is the ‘space of inputs’. For the stochastic Helmholtz EDP in §1.1 the space \( \mathcal{C} \) is defined in Definition 5.5 below, but the upshot of this definition is that for any \( \omega \in \Omega \) the triple \((A(\omega), n(\omega), f(\omega))\) is an element of \( \mathcal{C} \). The maps \( c, A, \) and \( L \) are given by \( c = (A, n, f), A = a, \) and \( L = L, \) where \( a \) and \( L \) are given by (1.4) and (1.5) respectively and the equality \( A = a \) is meant in the sense of the one-to-one correspondence between \( B(X, Y^*) \) and sesquilinear forms on \( X \times Y \).

The following three problems are the analogues in this general setting of Problems 1–3 in §1.

Problem MAS (Measurable variational formulation almost surely) Find a measurable function \( u : \Omega \rightarrow X \) such that
\[
A(c(\omega))u(\omega) = L(c(\omega) \text{ in } Y^* \tag{3.3}
\]
amost surely.

Problem SOAS (Second-order moment variational formulation almost surely) Find \( u \in L^2(\Omega; X) \) such that (3.3) holds almost surely.

Problem SV (Stochastic variational formulation) Find \( u \in L^2(\Omega; X) \) such that
\[
\mathfrak{A}u = \mathcal{L} \text{ in } L^2(\Omega; Y)^*. \tag{3.4}
\]

Remark 3.2 (Immediate relationships between formulations) Since \( L^2(\Omega; X) \subset B(\Omega, X) \) (the space of all measurable functions \( \Omega \rightarrow X) \) it is immediate that if \( u \) solves Problem SOAS then every member of the equivalence class of \( u \) solves Problem MAS.

3.2  Conditions on \( A, L, \) and \( c \)

We now state the conditions under which we prove results about the equivalence of Problems MAS–SV.

Condition A1 (\( A \) is continuous) The function \( A : \mathcal{C} \rightarrow B(X, Y^*) \) is continuous, where we place the norm topology on \( X \), the dual norm topology on \( Y^* \), and the operator norm topology on \( B(X, Y^*) \).

Condition A2 (Regularity of \( A \circ c \)) The map \( A \circ c \in L^\infty(\Omega; B(X, Y^*)) \).
We note that Condition A2 is violated in the well-studied case of a log-normal coefficient $\kappa$ for the stationary diffusion equation (1.2); in order to ensure the stochastic variational formulation is well-defined in this case, one must change the space of test functions as in [26, 45].

**Condition L1 (L is continuous)** The function $L : \mathcal{C} \to Y^*$ is continuous, where we place the dual norm topology on $Y^*$.

**Condition L2 (Regularity of $L \circ c$)** The map $L \circ c \in L^2(\Omega; Y^*)$.

**Condition C1 (c is measurable)** The function $c : \Omega \to \mathcal{C}$ is measurable.

To state the next condition, we need to recall the following definition.

**Definition 3.3 (P-essentially separably valued [50, p26])** Let $(S, T_0)$ be a topological space. A function $h : \Omega \to S$ is P-essentially separably valued if there exists $E \in \mathcal{F}$ such that $\mathbb{P}(E) = 1$ and $h(E)$ is contained in a separable subset of $S$.

**Condition C2 (c is P-essentially separably valued)** The map $c : \Omega \to \mathcal{C}$ is P-essentially separably valued.

**Remark 3.4 (Why do we need Condition C2?)** The theory of Bochner spaces requires strong measurability of functions (see Definitions B.9 and B.14 below). However, the proof techniques used in this paper rely heavily on the measurability of functions (see Definition B.1 below). In separable spaces these two notions are equivalent (see Corollary B.19). However, some of the spaces we encounter (such as $L^\infty(D; \mathbb{R})$) are not separable. Therefore, in our arguments we use Condition C2 along with the Pettis Measurability Theorem (Theorem B.18 below) to conclude that measurable functions are strongly measurable.

**Condition B (A priori bound almost surely)** There exist $C_j, f_j : \Omega \to \mathbb{R}$, $j = 1, \ldots, m$ such that $C_j f_j \in L^1(\Omega)$ for all $j = 1, \ldots, m$ and the bound

$$\|u(\omega)\|_X^2 \leq \sum_{j=1}^m C_j(\omega)f_j(\omega) \quad (3.5)$$

holds almost surely.

**Remark 3.5 (Notation in the a priori bound)** We use the notation $f_j$ in the right-hand side of (3.5) to emphasise the fact that typically these terms relate to the right-hand sides of the PDE in question. For the stochastic Helmholtz EDP, $m = 1$, $f_1 = \|f\|_L^2(\Omega; D^*)$, and $C_1$ is given by (1.11).

**Condition U (Uniqueness almost surely)** $\ker(A_{c(\omega)}) = \{0\}$ $\mathbb{P}$-almost surely.

The condition $\ker(A_{c(\omega)}) = \{0\}$ $\mathbb{P}$-almost surely can be stated as: given $\mathcal{G} \in L^2(\Omega; Y)^*$, for $\mathbb{P}$-almost every $\omega \in \Omega$ the deterministic problem $A_{c(\omega)} u_0 = \mathcal{G}$ has a unique solution.

### 3.3 Results on the equivalence of Problems MAS–SV

**Theorem 3.6 (Measurable solution implies second-order solution)** Under Condition B, if $u$ solves Problem MAS then $u$ solves Problem SOAS and satisfies the stochastic a priori bound

$$\|u\|_{L^2(\Omega; X)}^2 \leq \sum_{j=1}^m \|C_j f_j\|_{L^1(\Omega)} \quad (3.6)$$

Note that the stochastic a priori bound (3.6) is the expectation of the right-hand side of the bound (3.5).

**Lemma 3.7 (Stochastic variational formulation well-defined)** Under Conditions A1, A2, L1, L2, C1, and C2, the maps $\mathfrak{A}$ and $\mathfrak{L}$ defined by (3.1) and (3.2) are well-defined in the sense that

$$[\mathfrak{A}(v_1)](v_2), \mathfrak{L}(v_2) < \infty \quad \text{for all } v_1 \in L^2(\Omega; X), \text{ for all } v_2 \in L^2(\Omega; Y). \quad (3.7)$$

**Theorem 3.8 (Second-order solution implies stochastic variational solution)** Under Conditions L1, L2, C1, and C2, if $u$ solves Problem SOAS then $u$ solves Problem SV.

**Theorem 3.9 (Stochastic variational solution implies second-order solution)** If Problem SV is well-defined and $u$ solves Problem SV, then $u$ solves Problem SOAS.

Theorems 3.6, 3.8, and 3.9 and Lemma 3.7 are summarised in Figure 2.

**Remark 3.10 (Condition L2 in Theorem 3.8)** In Theorem 3.8 we could replace Condition L2 with Condition A2, and the result would still hold—see the proof for further details. However, Condition L2 is less restrictive than Condition A2, as it only requires $L^2$ integrability of $L \circ c$ as opposed to essential boundedness of $A \circ c$.

**Lemma 3.11 (Showing uniqueness of the solution to Problems MAS–SV)** If Condition U holds, then
Figure 2. The relationship between the variational formulations. An arrow from Problem P to Problem Q with Conditions R indicates ‘under Conditions R, the solution of Problem P is a solution of Problem Q’

1. the solution to Problem MAS (if it exists) is unique up to modification on a set of $\mathbb{P}$-measure 0 in $\Omega$,
2. the solution to Problem SOAS (if it exists) is unique in $L^2(\Omega; X)$, and
3. if Problem SV is well-defined, the solution to Problem SV (if it exists) is unique in $L^2(\Omega; X)$.

**Remark 3.12 (Informal discussion on the ideas behind the equivalence results)** The diagram in Figure 2 summarises the relationships between the variational formulations, and the conditions under which they hold. Moving ‘up’ the left-hand side of the diagram, we prove a solution of Problem SV is a solution of Problem SOAS in Theorem 3.9; the key idea in this theorem is to use a particular set of test functions and the general measure-theory result of Lemma B.22 below; this approach was used for the stationary diffusion equation (1.2) with log-normal coefficients in [26], and for a wider class of coefficients in [45].

Moving ‘down’ the right-hand side, we prove a solution of Problem MAS is a solution of Problem SOAS in Theorem 3.6; the key part of this proof is that the bound in Condition B gives information on the integrability of the solution $u$. (In the case of (1.2) with uniformly coercive and bounded coefficient $\kappa$, the analogous integrability result follows from the Lax–Milgram theorem; [16, Proposition 2.4] proves an equivalent result for (1.2) with lognormal coefficient $\kappa$ with an isotropic Lipschitz covariance function.) Proving a solution of Problem SOAS is a solution of Problem SV in Theorem 3.8 essentially amounts to posing conditions such that the quantities $[A(c(\omega))(u(\omega))](v(\omega))$ and $L(c(\omega))(v(\omega))$ are Bochner integrable for any $v \in L^2(\Omega; Y)$, so that (3.4) makes sense. Lemma 3.7 shows that the stronger property (3.7) holds, and requires stronger assumptions than Theorem 3.8, since the proof of Theorem 3.8 uses the additional information that $u$ solves Problem SOAS.

**Remark 3.13 (Changing the condition $u \in L^2(\Omega; X)$)** Here we seek the solution $u \in L^2(\Omega; X)$ but we could instead require $u \in L^p(\Omega; X)$, for some $p > 0$ and require $A u = \Sigma$ in $L^q(\Omega; Y)^*$, for some $q > 0$ (i.e. use test functions in $L^q(\Omega; Y)$). In this case, the proof of Theorem 3.9 would be nearly identical, as the space $D$ of test functions used there (see (4.7) below) is a subset of $L^q(\Omega; Y)$ for all $q > 0$. One could also develop analogues of Theorems 3.6 and 3.8 and Lemma 3.7 in this setting—see e.g. [26, Theorem 3.20] for an example of this approach for the stationary diffusion equation with lognormal diffusion coefficient.

**Remark 3.14 (Non-reliance on the Lax-Milgram theorem)** The above results hold for an arbitrary sesquilinear form and hence are applicable to a wide variety of PDEs; their main advantage is that they apply to PDEs whose stochastic variational formulations are not coercive. For example, as noted in §1, for the stationary diffusion equation (1.2) with coefficient $\kappa$ bounded uniformly below in $\omega$, the bilinear form of Problem SV is coercive; existence and uniqueness follow from the Lax-Milgram theorem, and hence the chain of results above leading to the well-posedness of Problem SV is not necessary.

**Remark 3.15 (Overview of how these results are applied to the Helmholtz equation in §5)** We obtain the results for the Helmholtz equation via the following steps (which could also be applied to other SPDEs fitting into this framework):

1. Define the map $c$ (via $A, n$, and $f$) such that for almost every $\omega \in \Omega$ there exists a solution of the deterministic Helmholtz EDP corresponding to $c(\omega)$.
2. Define $u : \Omega \to X$ to map $\omega$ to the solution of the deterministic problem corresponding to $c(\omega)$.
3. Prove that Conditions A1, A2, L1, L2, C1, C2, B, and $U$ hold, so that one can apply Theorems 3.6, 3.8, and 3.9 along with Lemmas 3.7 and 3.11 to show Problem 3 is well-defined and $u$ is unique and satisfies Problems 1–3.

Steps 1 and 2 can be thought of as constructing a solution pathwise.
4.1 Preliminary lemmas

To simplify notation, we introduce the following definition.

Definition 4.1 (Pairing map) For fixed $c: \Omega \rightarrow \mathcal{C}$, $A: \Omega \rightarrow B(X,Y^*)$, given $v: \Omega \rightarrow X$ we define the map $\pi_v: \Omega \rightarrow Y^*$ by

$$\pi_v(\omega) := \langle (A \circ c)(\omega), v(\omega) \rangle.$$

A key ingredient in proving that the stochastic variational formulation is well-defined (Lemma 3.7) is showing that the maps $\pi_u$ and $\mathcal{A} \circ c$ are measurable. Showing that $\mathcal{A} \circ c$ is measurable is straightforward (see Lemma 4.2 below), but showing that $\pi_u$ is measurable is not. This is because $\mathcal{A} \circ c$ depends on $\omega$ only through its dependence on $c$, but $\pi_u$ depends on $\omega$ through both the dependence of $\mathcal{A} \circ c$ on $\omega$ and the dependence of $u$ on $\omega$; it is this dual dependence that causes the extra complication.

Lemma 4.2 ($\mathcal{A} \circ c$ is measurable) Under Conditions L1 and C1 the function $\mathcal{A} \circ c$ is measurable.

Proof of Lemma 4.2 The map $c$ is measurable (by Condition C1) and $\mathcal{L}$ is continuous (by Condition L1), therefore Lemma B.4 implies that $\mathcal{A} \circ c$ is measurable. □

We now move on to the more-involved process of showing $\pi_v$ is measurable.

Definition 4.3 (Product map) For $v: \Omega \rightarrow X$, let $P_v: \Omega \rightarrow B(X,Y^*) \times X$ be defined by $P_v(\omega) = ((A \circ c)(\omega), v(\omega))$.

Lemma 4.4 (Product map is strongly measurable) When $B(X,Y^*) \times X$ is equipped with the product topology, if Conditions A1 and C1 hold, and if $v: \Omega \rightarrow X$ is measurable, then $P_v: \Omega \rightarrow B(X,Y^*) \times X$ is measurable.

Proof of Lemma 4.4 By the result on the measurability of the Cartesian product of measurable functions (Lemma B.6), $P_v$ is measurable with respect to $(\mathcal{F}, B(B(X,Y^*)) \otimes B(X))$ (where $\mathcal{F}$ denotes the Borel $\sigma$-algebra—see Definition B.2), as both of the coordinate functions $A \circ c$ and $v$ are measurable. Since $B(X,Y^*)$ and $X$ are both metric spaces, they are both Hausdorff. As $X$ is separable, Lemma B.7 on the product of Borel $\sigma$-algebras implies $B(B(X,Y^*)) \otimes B(X) = B(B(X,Y^*) \times X)$. Hence $P_v$ is measurable with respect to $(\mathcal{F}, B(B(X,Y^*) \times X))$. □

Definition 4.5 (Evaluation map) Let $Z$ be a separable Banach space. The function $\eta_{Z^*}: B(X,Z^*) \times X \rightarrow Z^*$ is defined by

$$\eta_{Z^*}(\langle H, v \rangle) := H(v).$$

Observe that the pairing, product, and evaluation maps ($\pi_v, P_v$, and $\eta_{Z^*}$ respectively) are related by $\pi_v = \eta_{Z^*} \circ P_v$.

Lemma 4.6 (Evaluation map is continuous) Let $Z$ be a separable Banach space. The map $\eta_{Z^*}$ is continuous with respect to the product topology on $B(X,Z^*) \times X$ and the dual norm topology on $Z^*$.

The proof of Lemma 4.6 is straightforward and omitted.

Lemma 4.7 ($\pi_v$ is measurable) If Conditions A1 and C1 hold and $v$ is measurable, then the function $\pi_v$ as defined by (4.1) is measurable.

Proof of Lemma 4.7 By Lemma 4.4 $P_v$ is measurable and by Lemma 4.6 $\eta_{Z^*}$ is continuous. Therefore Lemma B.4 implies that $\pi_v = \eta_{Z^*} \circ P_v$ is measurable. □

4.2 Proofs of Theorems 3.6, 3.8, and 3.9 and Lemmas 3.7 and 3.11

Proof of Theorem 3.6 We need to show $u: \Omega \rightarrow X$ is Bochner integrable, satisfies the bound (3.6), and has $\|u\|_{L^2(\Omega; X)} < \infty$. Our plan is to use Corollary B.12 to show $u$ is Bochner integrable, and establish (3.6) as a by-product. Since $u$ solves Problem MAS, $u$ is measurable. As $X$ is separable, it follows from Corollary B.19 that $u$ is strongly measurable. Define $N : X \rightarrow \mathbb{R}$ by

$$N(u) := \|u\|_X^2;$$

since $N$ is continuous, Lemma B.4 implies $N \circ u : \Omega \rightarrow \mathbb{R}$ is measurable. Therefore, since both the left- and right-hand sides of (3.5) are measurable and (3.5) holds for almost every $\omega \in \Omega$ we can integrate (3.5) over $\Omega$ with respect to $\mathbb{P}$ and obtain

$$\int_{\Omega} \|u(\omega)\|_X^2 \, d\mathbb{P}(\omega) \leq \sum_{j=1}^m \|C_j f_j\|_{L^1(\Omega)},$$

(4.3)

the right-hand side of which is finite since Condition B includes that $C_j f_j \in L^1(\Omega)$ for all $j = 1, \ldots, m$. Since $u$ is strongly measurable, the bound (4.3) and Corollary B.12 with $p = 2$ imply that $u$ is Bochner integrable. The norm $\|u\|_{L^2(\Omega; X)}$ is thus well-defined by Definition B.13 and (4.3) shows that (3.6) holds, and so in particular $\|u\|_{L^2(\Omega; X)} < \infty$. □

Proof of Lemma 3.7 We must show that for any $v_1 \in L^2(\Omega; X)$ and any $v_2 \in L^2(\Omega; Y)$:
The quantities \( \{A_{\omega}(\omega)\} \{v_2(\omega)\} \) and \( L_{\omega}(v_2(\omega)) \) are Bochner integrable, so that the definitions of \( \mathfrak{A} \) and \( \mathfrak{B} \) as integrals over \( \Omega \) make sense.

The maps \( \mathfrak{A}(v_1) \) and \( \mathfrak{B} \) are linear and bounded on \( L^2(\Omega; Y) \), that is, \( \mathfrak{A} : L^2(\Omega; X) \to L^2(\Omega; Y)^* \) and \( \mathfrak{B} \in L^2(\Omega; Y)^* \).

It follows from these two points that \( \mathfrak{A} \) and \( \mathfrak{B} \) are well-defined.

Thanks to the groundwork laid in §4.1, the measurability of \( \{A_{\omega}(\omega)\} \{v_2(\omega)\} \) and \( L_{\omega}(v_2(\omega)) \) follows from Lemmas 4.7 and 4.2 (which need Conditions A1, L1, and C2). Their \( \mathbb{P} \)-essential separability follows from Conditions A1, L1, and C2 and Lemma B.20 and thus their strong measurability follows from Corollary B.19 on the equivalence of measurability and strong measurability when the image is separable. Their Bochner integrability then follows from the Bochner integrability condition in Theorem B.11 (with \( V = \mathbb{P} \)) and the Cauchy–Schwartz inequality since

\[
\int_{\Omega} |L_{\omega}(v_2(\omega))| \mathbb{d}\mathbb{P}(\omega) \leq \int_{\Omega} \|\mathfrak{B} \circ c\|_{Y,Y} \|v_2(\omega)\|_Y \, \mathbb{d}\mathbb{P}(\omega) \leq \|\mathfrak{B} \circ c\|_{L^2(\Omega; Y)^*} \|v_2\|_{L^2(\Omega; Y)},
\]

which is finite by Condition L2, and

\[
\int_{\Omega} \left| \{A_{\omega}(\omega)\} \{v_2(\omega)\} \right| \mathbb{d}\mathbb{P}(\omega) \leq \int_{\Omega} \|A_{\omega}(\omega)\|_{Y,\mathbb{P}} \|v_2(\omega)\|_Y \, \mathbb{d}\mathbb{P}(\omega)
\]

\[
\leq \operatorname{ess sup}_{\omega \in \Omega} \|A_{\omega}(\omega)\|_{L^2(X,Y)} \int_{\Omega} \|v_1(\omega)\|_X \|v_2(\omega)\|_Y \, \mathbb{d}\mathbb{P}(\omega)
\]

\[
\leq \|A \circ c\|_{L^\infty(\Omega; (X,Y))} \|v_1\|_{L^2(\Omega; X)} \|v_2\|_{L^2(\Omega; Y)},
\]

which is finite by Condition A2.

We now show \( \mathfrak{B} \in L^2(\Omega; Y)^* \) and \( \mathfrak{A} : L^2(\Omega; X) \to L^2(\Omega; Y)^* \). Observe that \( |\mathfrak{B}(v_2)| \leq \int_{\Omega} |L_{\omega}(v_2(\omega))| \mathbb{d}\mathbb{P}(\omega) \) and \( \|\mathfrak{B}(v_1)\| \leq \int_{\Omega} \|A_{\omega}(\omega)\|_{L^2(\Omega; Y)} \|v_2(\omega)\|_Y \mathbb{d}\mathbb{P}(\omega) \) and thus by (4.4) and (4.5) \( \mathfrak{B} \) and \( \mathfrak{A}(v_1) \) are bounded. They are clearly linear, and so follows \( \mathfrak{B} \in L^2(\Omega; Y)^* \) and \( \mathfrak{A}(v_1) \in L^2(\Omega; Y)^* \), i.e., \( \mathfrak{A} : L^2(\Omega; X) \to L^2(\Omega; Y)^* \).

**Proof of Theorem 3.8** In order to show that \( u \) solves Problem SV, we must show:

1. either the functional \( \mathfrak{B} \in L^2(\Omega; Y)^* \) or the functional \( \mathfrak{A}(u) \in L^2(\Omega; Y)^* \), and
2. the equality (3.4) holds.

For Point 1 we show that \( \mathfrak{B} \in L^2(\Omega; Y)^* \), (since this is easier than showing \( \mathfrak{A}(u) \in L^2(\Omega; Y)^* \)); in fact the proof is contained in the proof of Lemma 3.7.

For Point 2, since \( u \) solves Problem SOAS, for \( \mathbb{P} \)-almost every \( \omega \in \Omega \) we have

\[ A_{\omega}(u)(\omega) = L_{\omega}(u) \]

in \( Y^* \). Hence, for any \( v \in L^2(\Omega; Y) \) we have

\[
\{A_{\omega}(u)(\omega)\}(v(\omega)) = L_{\omega}(u)(v(\omega))
\]

for \( \mathbb{P} \)-almost every \( \omega \in \Omega \). Since \( \mathfrak{B} \in L^2(\Omega; Y)^* \), the right-hand side of (4.6) is a strongly measurable function with finite integral. Hence the left-hand side of (4.6) is as well, and we can integrate over \( \Omega \) to conclude

\[
[\mathfrak{A}u](v) = \mathfrak{B}(v)
\]

for all \( v \in L^2(\Omega; Y) \), that is, \( \mathfrak{A}u = \mathfrak{B} \in L^2(\Omega; Y)^* \).

The following lemma is needed for the proof of Theorem 3.9.

**Lemma 4.8** Let \( \delta : \Omega \times Y \to \mathbb{P} \). For \( y \in Y \), define \( \Omega_y := \{\omega \in \Omega : \delta(\omega, y) = 0\} \) and define \( \Omega := \{\omega \in \Omega : \delta(\omega, y) = 0 \text{ for all } y \in Y\} \). If

- for all \( \omega \in \Omega \), \( \delta(\omega, \cdot) \) is a continuous function on \( Y \) and
- for all \( y \in Y \), the map \( \delta(\cdot, y) : \Omega \to \mathbb{P} \) is measurable and \( \mathbb{P}(\emptyset) = 1 \),

then \( \mathbb{P}(\Omega) = 1 \).

**Proof of Lemma 4.8** We must show that the set \( \Omega \in \mathcal{F} \), and \( \mathbb{P}(\Omega) = 1 \). Observe that, for any \( y \in Y \), the set \( \Omega_y \in \mathcal{F} \), since \( \Omega_y = \mathbb{P}_y(\emptyset) = 0 \) which is the preimage under a measurable map of a measurable set.

Since \( Y \) is a Hilbert space, it is separable, and therefore it has a countable dense subset \( \{y_n\}_{n \in \mathbb{N}} \). We will show that \( \mathbb{P}(\cap_{n \in \mathbb{N}} \Omega_{y_n}) = 1 \) and \( \Omega = \cap_{n \in \mathbb{N}} \Omega_{y_n} \). The set \( \cap_{n \in \mathbb{N}} \Omega_{y_n} \in \mathcal{F} \), as \( \mathcal{F} \) is a \( \sigma \)-algebra and \( \mathbb{P}(\cap_{n \in \mathbb{N}} \Omega_{y_n}) = \sup_{n \in \mathbb{N}} \mathbb{P}(\Omega_{y_n}) = 0 \), and hence \( \mathbb{P}(\cap_{n \in \mathbb{N}} \Omega_{y_n}) = 1 \). To first show \( \Omega = \cap_{n \in \mathbb{N}} \Omega_{y_n} \) we observe that \( \Omega = \cap_{y \in Y} \Omega_y \) and \( \cap_{y \in Y} \Omega_y \subseteq \cap_{n \in \mathbb{N}} \Omega_{y_n} \). It therefore suffices to show \( \cap_{n \in \mathbb{N}} \Omega_{y_n} \subseteq \cap_{y \in Y} \Omega_y \) to conclude \( \Omega = \cap_{n \in \mathbb{N}} \Omega_{y_n} \).
Fix $y \in Y$. By density of $(y_m)_{m \in \mathbb{N}}$, there exists a subsequence $(y_{n_m})_{m \in \mathbb{N}}$ such that $y_{n_m} \to y$ as $m \to \infty$. Fix $\omega \in \cap_{n \in \mathbb{N}} \Omega_{n_m}$; that is, for all $m \in \mathbb{N}$, $\omega \in \cap_{n \in \mathbb{N}} \Omega_{n_m}$. As $\delta(\cdot, y_{n_m})$ is a continuous function on $Y$, $\delta(\omega, y_{n_m}) \to \delta(\omega, y)$ as $m \to \infty$. But as previously noted, $\delta(\omega, y_{n_m}) = 0$ for all $m \in \mathbb{N}$. Hence we must have $\delta(\omega, y) = 0$, and thus $\omega \in \Omega_y$. Since $\omega \in \cap_{n \in \mathbb{N}} \Omega_{n_m}$ was arbitrary, it follows that $\cap_{n \in \mathbb{N}} \Omega_{n_m} \subseteq \Omega_y$, and since $y \in Y$ was arbitrary, it follows that $\cap_{n \in \mathbb{N}} \Omega_{n_m} \subseteq \cap_{y \in Y} \Omega_y$ as required.

**Proof of Theorem 3.9.** Let $u \in L^2(\Omega; X)$ solve Problem SV. We need to show that $u$ solves Problem SOAS. Observe that $u$ solving Problem SOAS means $A_{\delta(\cdot)}(u(\omega)) = (L_{\delta(\cdot)}(\omega))(\cdot)$ in $Y^*$ for almost every $\omega \in \Omega$. We now use an idea from [26, Theorem 3.3]. Our plan is to use test functions of the form $y \mathbb{1}_E$, where $y \in Y$ and $E \in \mathcal{F}$ to reduce Problem SV to the statement

$$
\int_E \left[A_{\delta(\cdot)}(u(\omega))\right](y(\omega)) \, d\mathbb{P}(\omega) = \int_E \left[L_{\delta(\cdot)}(\omega)\right](y(\omega)) \, d\mathbb{P}(\omega)
$$

for all $E \in \mathcal{F}$ and then show this implies $u$ satisfies Problem SOAS via Lemma B.22.

First define the space

$$
\mathcal{D} := \{y \mathbb{1}_E : y \in Y, E \in \mathcal{F}\};
$$

it is straightforward to see that the elements of $\mathcal{D}$ are maps from $\Omega$ to $Y$. The fact that $\mathcal{D} \subseteq L^2(\Omega; Y)$ follows via the following three steps:

1. The elements of $\mathcal{D}$ are measurable, indeed the indicator function of a measurable set is a measurable function $\Omega \to \mathbb{R}$, and multiplication by $y \in Y$ is a continuous function $\mathbb{R} \to Y$. Hence elements of $\mathcal{D}$ are measurable by Lemma B.4.

2. As $Y$ is a separable Hilbert space, it follows from Corollary B.19 that the elements of $\mathcal{D}$ are strongly measurable.

3. $\|y \mathbb{1}_E\|_{L^2(\Omega; Y)} = \mathbb{P}(E)^{1/2} \|y\|_Y < \infty$ for all $y \in Y, E \in \mathcal{F}$.

Since Problem SV is well-defined, and $u$ solves Problem SV, and $\mathcal{D} \subseteq L^2(\Omega; Y)$, we have that $|\mathbb{A}u|(\omega) = \mathcal{L}(\cdot)$ for all $v \in \mathcal{D}$. Therefore, we have

$$
\int_{\mathcal{D}} \left[A_{\delta(\cdot)}(u(\omega))\right](y \mathbb{1}_E(\omega)) \, d\mathbb{P}(\omega) = \int_{\mathcal{D}} \left[L_{\delta(\cdot)}(\omega)\right](y \mathbb{1}_E(\omega)) \, d\mathbb{P}(\omega)
$$

for all $y \in Y$ and $E \in \mathcal{F}$. If we define $\delta : \Omega \times Y \to \mathcal{F}$ by $\delta(\omega, y) := A_{\delta(\cdot)}(u(\omega)) - L_{\delta(\cdot)}(\omega)(y)$ then, by the definition of $\mathbb{1}_E$, (4.8) becomes

$$
\int \delta(\omega, y) \, d\mathbb{P}(\omega) = 0 \quad \text{for all } E \in \mathcal{F}.
$$

To conclude $u$ solves Problem SOAS we must show $\delta(\omega, y) = 0$ for all $y \in Y$, almost surely. We will use Lemma B.22, so the first step is to show that for all $y \in Y$ $\delta(\cdot, y)$ is Bochner integrable. This follows from the fact that Problem SV is well-defined, and thus the quantities $\left[A_{\delta(\cdot)}(v_1(\omega))\right](v_2(\omega))$ and $L_{\delta(\cdot)}(\omega)(v_2(\omega))$ are Bochner integrable for any $v_1 \in L^2(\Omega; X), v_2 \in L^2(\Omega; Y)$. In particular, they are Bochner integrable when $v_1 = u$, and $v_2 = y \mathbb{1}_E$ and thus their difference $\delta$ is Bochner integrable. Secondly, $\delta(\cdot, y)$ is a continuous function on $Y$ since $A_{\delta(\cdot)}(\cdot)$ is $L_{\delta(\cdot)}(\omega)$ in $Y^*$, for all $\omega \in \Omega$.

We now show $\delta(\omega, y) = 0$ for all $y \in Y$, almost surely. For $y \in Y$ define the set $\Omega_y := \{\omega \in \Omega : \delta(\omega, y) = 0\}$; by (4.9) and Lemma B.22 we have that $\mathbb{P}(\Omega_y) = 1$ for all $y \in Y$. By Lemma 4.8, $\delta(\cdot, y)$ is $0$ for all $y \in Y$, almost surely, that is, $A_{\delta(\cdot)}(u(\omega)) = L_{\delta(\cdot)}(\omega)$ almost surely; it follows that $u$ solves Problem SOAS.

**Remark 4.9.** (Connection with the argument in [45, Remark 2.2]) The argument in Lemma 4.8 and the final part of Theorem 3.9 closely mirrors the result in [45, Remark 2.2]. That is, we prove in general

$$
\mathbb{P}(\delta(\omega, y) = 0) = 1 \quad \text{for all } y \in Y \quad \text{implies} \quad \mathbb{P}(\delta(\omega, y) = 1 \quad \text{for all } y \in Y),
$$

and [45, Remark 2.2] shows an analogous result for the stationary diffusion equation (1.2) with non-uniformly coercive and unbounded coefficient $\kappa$.

**Proof of Lemma 3.11.** Proof of Part 1. Suppose $u_1, u_2 : \Omega \to X$ solve Problem MAS. Let $E = \{\omega \in \Omega : u_1(\omega) \neq u_2(\omega)\}$. By the definition of Problem MAS there exist $E_1, E_2 \subseteq \mathcal{F}$ such that $\mathbb{P}(E_1) = \mathbb{P}(E_2) = 0$ and

$$
A_{\delta(\omega)}(u_1(\omega)) \neq L_{\delta(\omega)} \quad \text{iff } \omega \in E_1, \quad \text{and } \quad A_{\delta(\omega)}(u_2(\omega)) \neq L_{\delta(\omega)} \quad \text{iff } \omega \in E_2.
$$

As $\ker(A_{\delta(\omega)}) = \{0\}$ $\text{P}$-almost surely, there exists $E_3 \in \mathcal{F}$ such that $\mathbb{P}(E_3) = 0$ and

$$
\ker(A_{\delta(\omega)}) = \{0\} \quad \text{iff } \omega \in E_3.
$$

We claim $E \subseteq E_1 \cup E_2 \cup E_3$. Indeed, if $u_1(\omega) \neq u_2(\omega)$ then either: (i) at least one of $u_1$ and $u_2$ does not solve Problem MAS at $\omega$ or (ii) $u_1$ and $u_2$ both solve Problem MAS at $\omega$, but $\ker(A_{\delta(\omega)}) \neq \{0\}$.
Since $P(E_j) = 0, j = 1, 2, 3$, we have $P(E_1 \cup E_2 \cup E_3) = 0$. Therefore $E \in \mathcal{F}$ and $P(E) = 0$ since $(\Omega, \mathcal{F}, P)$ is a complete probability space; hence $u_1 = u_2$ almost surely, as required.

**Proof of Part 2.** By Remark 3.2, if $u_1, u_2 \in L^2(\Omega; X)$ solve Problem SOAS, then all the representatives of the equivalence classes of $u_1$ and $u_2$ solve Problem MAS. Hence, by Part 1, any representative of $u_1$ and any representative of $u_2$ differ only on some set (depending on the representatives) of $\mathbb{P}$-measure zero in $\Omega$. Therefore $u_1 = u_2$ in $L^2(\Omega; X)$, by definition of $L^2(\Omega; X)$.

**Proof of Part 3.** As Problem SV is well-defined, by Remark 3.2 and Theorem 3.9, if $u_1$ and $u_2$ solve Problem SV, then $u_1$ and $u_2$ also solve Problem MAS. We then repeat the reasoning in the proof of Part 2 to show $u_1 = u_2$ in $L^2(\Omega; X)$. \hfill \square

# 5 Proofs of Theorems 1.4 and 1.8

In §5.1 we place the Helmholtz stochastic EDP into the framework developed in §3. In §5.2 we give sufficient conditions for the Helmholtz stochastic EDP to satisfy Conditions A1, L1, C1, etc. In §5.3 we apply the general theory developed in §3 to prove Theorems 1.4 and 1.8.

## 5.1 Placing the Helmholtz stochastic EDP into the framework of §3

Recall $R > 0$ is fixed. We let $X = Y = H^1_{0,D}(D_R)$ and define the norm $\|v\|_{L^2(D_R)} := \|\nabla v\|_{L^2(D_R)}^2 + k^2 \|v\|_{L^2(D_R)}^2$ on $H^1_{0,D}(D_R)$. Throughout this section, $A_n, n_0, f_0$ will be deterministic functions. Recall that since the supports of $1-n$, $I-A$, and $f$ are compactly contained in $B_R$, we can consider $A_n$ and $f$ as functions on $D_R$ rather than on $D$. In order to define the space $\mathcal{C}$ and the maps $c, A, L$ we define the following function spaces on $D_R$.

**Definition 5.1 (Compact-support spaces)** Let

$$L^2_R(D_R) := \left\{ f_0 \in L^2(D_R) : \text{ess supp}(f_0) \subset B_R \right\}.$$  

$$L^\infty_{R,\min}(D_R; \mathbb{R}) := \left\{ n_0 \in L^\infty(D_R; \mathbb{R}) : \text{ess supp}(1-n_0) \subset B_R, \text{there exists } \alpha_{n_0} > 0 \text{ such that } n_0(x) \geq \alpha_{n_0} \text{ almost everywhere } \right\},$$

$$L^\infty_{R,\min}(D_R; \mathbb{R}^{d \times d}) := \left\{ A_0 \in L^\infty(D_R; \mathbb{R}^{d \times d}) : A_0(x) \text{ is symmetric almost everywhere,} \right.$$  

$$\text{ess supp}(I-A_0) \subset B_R, \text{there exists } \alpha_{A_0} > 0 \text{ such that } \alpha_{A_0} \leq A_0(x) \text{ almost everywhere,} \right.$$  

$$\text{in the sense of quadratic forms} \right\},$$

and

$$W^{1,\infty}_{R,\min}(D_R; \mathbb{R}^{d \times d}) := \left\{ A_0 \in L^\infty_{R,\min}(D_R; \mathbb{R}^{d \times d}) : A_0 \in W^{1,\infty}(D_R; \mathbb{R}^{d \times d}) \right\}.$$  

Observe that the norm on $L^\infty(D_R; \mathbb{R})$ induces a metric on $L^\infty_{R,\min}(D_R; \mathbb{R})$, and similarly for $L^\infty_{R,\min}(D_R; \mathbb{R}^{d \times d})$, $W^{1,\infty}_{R,\min}(D_R; \mathbb{R}^{d \times d})$, and $L^2_R(D_R)$. These spaces are not vector spaces, and are not complete, but completeness and being a vector space is not required in what follows—we only need them to be metric spaces.

**Definition 5.2 (Deterministic form and functional)** For $(A_0, n_0, f_0) \in L^\infty_{R,\min}(D_R; \mathbb{R}) \times L^\infty_{R,\min}(D_R; \mathbb{R}) \times L^2_R(D_R)$ let the sesquilinear form $a_{A_0, n_0}$ on $H^1_{0,D}(D_R) \times H^1_{0,D}(D_R)$ and the bilinear functional $L_{f_0}$ on $H^1_{0,D}(D_R)$ be given by

$$a_{A_0, n_0}(v_1, v_2) := \int_{D_R} \left( A_0 \nabla v_1 \cdot \nabla \overline{v_2} - k^2 n_0 v_1 \overline{v_2} \right) d\lambda - \langle T_R \gamma v_1, \gamma v_2 \rangle_{T_R} \quad \text{and} \quad L_{f_0}(v_2) := \int_{D_R} f_0 \overline{v_2} d\lambda,$$

for $v_1, v_2 \in H^1_{0,D}(D_R)$.

**Problem 5.3 (Helmholtz EDP)** For $(A_0, n_0, f_0) \in L^\infty_{R,\min}(D_R; \mathbb{R}) \times L^\infty_{R,\min}(D_R; \mathbb{R}) \times L^2_R(D_R)$ find $u_0 \in H^1_{0,D}(D_R)$ such that $a_{A_0, n_0}(u_0, v) = L_{f_0}(v)$ for all $v \in H^1_{0,D}(D_R)$.

**Definition 5.4 (d_\infty metric)** Let $(X_1, d_1), \ldots, (X_m, d_m)$ be metric spaces. The $d_\infty$ metric on the Cartesian product $X_1 \times \cdots \times X_m$ is defined by

$$d_\infty((x_1, \ldots, x_m), (y_1, \ldots, y_m)) := \max_{j=1, \ldots, m} d_j(x_j, y_j).$$

**Definition 5.5 (The input space $\mathcal{C}$)** We let

$$\mathcal{C} := W^{1,\infty}_{R,\min}(D_R; \mathbb{R}^{d \times d}) \times L^\infty_{R,\min}(D_R; \mathbb{R}) \times L^2_R(D_R)$$

with topology given by the $d_\infty$ metric.
Definition 5.6 (The input map $c$) Define $c : \Omega \to C$ by
\[
c(\omega) = (A(\omega), n(\omega), f(\omega)).
\] (5.1)

Definition 5.7 (The maps $A$ and $\mathcal{L}$ for the Helmholtz stochastic EDP) Let
\[
A((A_0, n_0, f_0)) := a_{A_0, n_0} \quad \text{and} \quad \mathcal{L}((A_0, n_0, f_0)) := L_{f_0},
\] (5.2)
where the definition of $A$ is understood in terms of the equivalence between $B(X, Y^*)$ and sesquilinear forms on $X \times Y$.

5.2 Verifying the Helmholtz stochastic EDP satisfies the general conditions in §3

Lemma 5.8 (Conditions C1 and C2 for Helmholtz stochastic EDP) If $A, n,$ and $f$ are strongly measurable, then $c$ defined by (5.1) satisfies Conditions C1 and C2.

Proof Since $A, n,$ and $f$ are strongly measurable, by Theorem B.18 they are measurable and $\mathbb{P}$-essentially separately valued. By Lemma B.23, it follows that $c$ is $\mathbb{P}$-essentially separately valued, so $c$ satisfies Condition C1. By Lemma B.23, it follows that $c$ is $\mathbb{P}$-essentially separately valued, so $c$ satisfies Condition C2. □

Lemma 5.9 (Conditions A1 and L1 for Helmholtz stochastic EDP) The maps $A$ and $\mathcal{L}$ given by (5.2) satisfy Conditions A1 and L1.

Proof of Lemma 5.9 We need to show that if $(A_m, n_m, f_m) \to (A_0, n_0, f_0)$ in $C$ then $A((A_m, n_m, f_m)) \to A((A_0, n_0, f_0))$ in $B(X, Y^*)$, and similarly for $\mathcal{L}$. We have, for $v_1, v_2 \in X$, $v_1, v_2 \in Y$,\[
\left| \int_{D_R} \left( (A_m - A_0) \nabla v_1 \cdot \nabla \overline{v_2} - k^2 (n_m - n_0) v_1 \overline{v_2} \right) \, d\lambda \right| \leq \left\| A_m - A_0 \right\|_{L^\infty(D_R \times \mathbb{R}^{d+1})} \left\| \nabla v_1 \right\|_{L^2(D_R)} \left\| \nabla v_2 \right\|_{L^2(D_R)} + k^2 \left\| n_m - n_0 \right\|_{L^\infty(D_R)} \left\| v_1 \right\|_{L^2(D_R)} \left\| v_2 \right\|_{L^2(D_R)} \leq 2d\omega \left( (A_m, n_m, f_m), (A_0, n_0, f_0) \right) \left\| v_1 \right\|_{1,k} \left\| v_2 \right\|_{1,k}.
\]
Hence if $(A_m, n_m, f_m) \to (A_0, n_0, f_0)$ in $C$, then $A((A_m, n_m, f_m)) \to A((A_0, n_0, f_0))$ in $B(X, Y^*)$. We also have\[
\left| \int_{D_R} (f_m - f_0) \overline{v_2} \, d\lambda \right| \leq \left\| f_m - f_0 \right\|_{L^2(D_R)} \left\| v_2 \right\|_{L^2(D_R)} \leq \frac{1}{Z} \left\| f_m - f_0 \right\|_{L^2(D_R)} \left\| v_2 \right\|_{1,k}.
\]
Hence if $(A_m, n_m, f_m) \to (A_0, n_0, f_0)$ in $C$, then $\mathcal{L}((A_m, n_m, f_m)) \to \mathcal{L}((A_0, n_0, f_0))$ in $Y^*$.

Definition 5.10 (The solution operator $S$) Define $S : C \to H^1_{0,D}(D_R)$ by letting $S(A_0, n_0, f_0) \in H^1_{0,D}(D_R)$ be the solution of the Helmholtz EDP (Problem 5.3).

Theorem 5.11 ($S$ is well defined) For $(A_0, n_0, f_0) \in C$ the solution $S((A_0, n_0, f_0))$ of the Helmholtz EDP (Problem 5.3) exists, is unique, and depends continuously on $f_0$.

Proof of Theorem 5.11 Since $\mathcal{R}(-(\mathcal{H}^2, \mathcal{H}^2), \gamma, \gamma) \geq 0$ for all $v \in H^1_{0,D}(D_R)$ (see, e.g., [46, Theorem 2.6.4]), $a_{A_0, n_0}$ satisfies a Garding inequality. Since the inclusion $H^1_{0,D}(D_R) \subset L^2(D_R)$ is compact, Fredholm theory shows that uniqueness implies well-posedness (see, e.g., [42, Theorem 2.34]). Since $A$ is Lipschitz and $n$ is $L^\infty$, uniqueness follows from the unique continuation results in [37, 25, 57]; see [28, §2] for these results specifically applied to Helmholtz problems. □

Lemma 5.12 (Continuity of solution operator for Helmholtz stochastic EDP) For the Helmholtz stochastic EDP, the solution operator $S : C \to H^1_{0,D}(D_R)$ is continuous.

Sketch Proof of Lemma 5.12 Let $(A_0, n_0, f_0), (A_1, n_1, f_1) \in C$, with $S((A_0, n_0, f_0)) = u_0$ and $S((A_1, n_1, f_1)) = u_1$. Then for any $v \in H^1_{0,D}(D_R)$ we have\[
\left[ [A((A_0, n_0, f_0))]_{(u_0)} \right](v) = [L((A_0, n_0, f_0))]_2(v) \quad \text{and} \quad [A((A_1, n_1, f_1))]_{(u_1)}(v) = [L((A_1, n_1, f_1))]_2(v).
\]
Continuity of $S$ then follows from:
1. Deriving the Helmholtz equation with coefficients $A_0$ and $n_0$ satisfied by $u_d := u_0 - u_1$.
2. Recalling that the well-posedness result of Theorem 5.11 holds when $f_0 \in L^2(D_R)$ is replaced by a right-hand side in $(H^1_{0,D}(D_R))^*$; see, e.g., [42, Theorem 2.34].
3. Applying the result in Point 2 to obtain a bound
\[
\|u_d\|_{1,k} \leq C(A_0, n_0) \|F\|_{(H^1_{0,D}(D_R))^*}.
\]
4. Showing $\|F\|((H^1_0(D))')$ depends on $\|\nabla u_1\|_{L^2(D^*)}$, $\|u_1\|_{L^2(D^*)}$, $\|A_1 - A_0\|_\infty(D^*_R)$, $\|n_1 - n_0\|_{L^\infty(D^*_R)}$, and $\|f_0 - f_1\|_{L^2(D^*)}$.

5. Eliminating the dependence on $u_1$ by writing $u_1 = u_0 - u_d$ and moving terms in $u_d$ to the left-hand side, to obtain a bound on $u_d$ of the form

$$\|\nabla u_d\|_{L^2(D^*)} + k\|u_d\|_{L^2(D^*)} \leq \tilde{C}(u_0, A_0, n_0, A_1 - A_0, \|n_1 - n_0\|_{L^\infty(D^*_R)}, \|f_0 - f_1\|_{L^2(D^*)}).$$

6. Concluding that $u_d \to 0$ in $H^1_0(D^*_R)$ as $(A_1, n_1, f_1) \to (A_0, n_0, f_0)$ in $C$.

Lemma 5.13 (Condition U for the Helmholtz stochastic EDP) The Helmholtz stochastic EDP satisfies Condition U.

Proof of Lemma 5.13 The fact that this uniqueness condition holds is immediate from Theorem 5.11.

Condition 5.14 (Nontrapping condition for Helmholtz EDP [27, Condition 2.6]) $d = 3, D^*$ is star-shaped with respect to the origin, $A_0 \in W^{1,\infty}(D^*_R; \mathbb{R}^{d \times d})$, $n_0 \in W^{1,\infty}(D^*_R; \mathbb{R})$, and there exist $\tau_1, \tau_2 > 0$ such that

$$A_0(x) - (x \cdot \nabla)A_0(x) \geq \tau_1, \text{ in the sense of quadratic forms, for almost every } x \in D^+_R, \tag{5.3}$$

and

$$n_0(x) + x \cdot \nabla n_0(x) \geq \tau_2 \text{ for almost every } x \in D^+_R. \tag{5.4}$$

Theorem 5.15 (Well-posedness of the Helmholtz EDP under Condition 5.14 [27, Theorem 2.10(i)]) Let $(A_0, n_0, f_0) \in C$ and suppose $A_0$ and $n_0$ satisfy Condition 5.14. Then the solution of the Helmholtz EDP (Problem 5.3) exists and is unique. Furthermore, given $k_0 > 0$ for all $k \geq k_0$, the solution $u_0$ of the Helmholtz EDP satisfies the bound

$$\tau_1\|\nabla u_0\|^2_{L^2(D^*_R)} + \tau_2 k^2\|u_0\|^2_{L^2(D^*_R)} \leq C_1\|f_0\|^2_{L^2(D^*_R)}, \text{ where } C_1 := 4 \left( \frac{R^2}{\tau_1} + \frac{1}{\tau_2} + \frac{R + d - 1}{2k_0} \right)^2. \tag{5.5}$$

We can now prove Condition B holds for the Helmholtz stochastic EDP.

Lemma 5.16 (Condition B for Helmholtz stochastic EDP) If Conditions 1.3 and 1.6 hold, then Condition B holds for the Helmholtz stochastic EDP.

Proof of Lemma 5.16 As Condition 1.6 holds, Condition 5.14 holds for $\omega$-almost every $\omega \in \Omega$ (with $A_0 = A(\omega), n_0 = n(\omega)$, $\tau_1 = \mu_1(\omega)$, and $\tau_2 = \mu_2(\omega)$). Hence, by Theorem 5.15 the bound (3.5) holds for all $k \geq k_0$, with $X = H^1_0(D^*_R), m = 1$, $C_1(\omega) = \max\{\mu_1(\omega), \mu_2(\omega)\} \left[ \frac{R^2}{\mu_1(\omega)} + \frac{1}{\mu_2(\omega)} + \frac{R + d - 1}{2k_0} \right]^2$, and $f_1 = \|f(\omega)\|_{L^2(D^*_R)}$. It now remains to show that $C_1 \|f\|^2_{L^2(D^*_R)} \in L^1(\Omega)$. We first show $C_1 \|f\|^2_{L^2(D^*_R)}$ is measurable and then show that it lies in $L^1(\Omega)$. To show measurability, we rewrite $C_1(\omega)$ as

$$C_1(\omega) = \max\left\{ \frac{2R^2}{\mu_1^2(\omega)} + \frac{2}{\mu_1(\omega)\mu_2(\omega)} \left( R + \frac{d - 1}{2k_0} \right)^2, \frac{2R^2}{\mu_1(\omega)\mu_2(\omega)} + \frac{2}{\mu_2^2(\omega)} \left( R + \frac{d - 1}{2k_0} \right)^2 \right\}.$$

The functions $\mu_1^{-1}$ and $\mu_2^{-1}$ are measurable by assumption; to conclude $C_1$ is measurable we use the facts (see e.g., [31, Theorems 19.2C, 20.4A]): (i) the square of a measurable function is measurable, and (ii) the product, sum, and maximum of two measurable functions are measurable. Under Condition 1.3, the function $f$ lies in the Bochner space $L^2(\Omega; L^2(D^*_R))$. Therefore, $f$ is strongly measurable and hence $f$ is measurable by Theorem B.16. The map $f \mapsto \|f\|^2_{L^2(D^*_R)}$ is clearly continuous, and therefore $f_1$ is measurable by Lemma B.4. As the product of two measurable functions is measurable, it follows that $C_1 \|f\|^2_{L^2(D^*_R)}$ is measurable.

We now show that $C_1 \|f\|^2_{L^2(D^*_R)} \in L^1(\Omega)$. The assumptions $1/\mu_1, 1/\mu_2 \in L^2(\Omega)$ and the Cauchy–Schwarz inequality imply $1/(\mu_1\mu_2) \in L^1(\Omega)$. Therefore the maps,

$$\omega \mapsto \frac{2R^2}{\mu_1^2(\omega)} + \frac{2}{\mu_1(\omega)\mu_2(\omega)} \left( R + \frac{d - 1}{2k_0} \right)^2 \text{ and } \omega \mapsto \frac{2R^2}{\mu_1(\omega)\mu_2(\omega)} + \frac{2}{\mu_2^2(\omega)} \left( R + \frac{d - 1}{2k_0} \right)^2$$

are in $L^1(\Omega)$. Since the maximum of two functions in $L^1(\Omega)$ is also in $L^1(\Omega)$, it follows that $C_1 \in L^1(\Omega)$. Condition 1.3 implies that $\|f\|^2_{L^2(D^*_R)} \in L^1(\Omega)$.
To conclude $C_1\|f\|_{L^2(D_R)}^2 \in L^1(\Omega)$, observe that the only dependence of $C_1$ on $\omega$ is through $\mu_1$ and $\mu_2$. As $\mu_1$ and $\mu_2$ are assumed independent of $f$, and measurable functions of independent random variables are independent [40, p.230] it follows that $C_1$ and $\|f\|_{L^2(D_R)}^2$ are independent, and therefore
\[
\left\|C_1\|f\|_{L^2(D_R)}^2\right\|_{L^1(\Omega)} = \int_\Omega C_1(\omega)\|f(\omega)\|^2_{L^2(D_R)}d\mu(\omega) = \left(\int_\Omega C_1(\omega)d\mu(\omega)\right)\left(\int_\Omega \|f(\omega)\|^2_{L^2(D_R)}d\mu(\omega)\right) = \|C_1\|_{L^1(\Omega)}\|\|f\|_{L^2(D_R)}^2\|_{L^1(\Omega)} < \infty. \tag{5.6}
\]
Therefore $C_1\|f\|_{L^2(D)}^2 \in L^1(\Omega)$ as required. We take the expectation (equivalently, the $L^1$ norm) of (5.5) (with $A_0 = A(\omega)$ etc.) and use (5.6) to obtain (1.10). □

**Remark 5.17 (The case when $f$, $\mu_1$, and $\mu_2$ are not independent)** Remark 1.9 shows that for the physically relevant example of scattering by a plane wave, $f$, $\mu_1$, and $\mu_2$ may not be independent. In this case, if we replace the requirements in Condition 1.6 that $f \in L^2(\Omega; L^2(D))$ and $1/\mu_1, 1/\mu_2 \in L^2(\Omega)$ with the stronger requirements $f \in L^4(\Omega; L^4(D))$ and $1/\mu_1, 1/\mu_2 \in L^2(\Omega)$, then one can obtain the bound
\[
\|\nabla u\|^2_{L^2(D)} + k^2\|\nabla u\|^2_{L^2(D)} \leq \|C_1\|_{L^2(\Omega)}\|f\|^2_{L^2(\Omega)}. \tag{5.6}
\]
Indeed, instead of independence, we use the Cauchy–Schwartz inequality in (5.6) to conclude
\[
\left\|C_1\|f\|_{L^2(D_R)}^2\right\|_{L^1(\Omega)} \leq \|C_1\|_{L^2(\Omega)}\|f\|^2_{L^2(\Omega)} = \|C_1\|_{L^2(\Omega)}\|f\|^2_{L^1(\Omega)}.
\]

**Lemma 5.18 (Condition L2 for Helmholtz stochastic EDP)** If $f \in L^2(\Omega; L^2(D_R))$ and $A$ and $n$ are strongly measurable, then Condition L2 holds for the Helmholtz stochastic EDP.

**Proof of Lemma 5.18** Since $A$, $n$, and $f$ are strongly measurable, Conditions C1 and C2 hold by Lemma 5.8; i.e., $c$ is both measurable and $\mathbb{P}$-essentially separably valued. Furthermore, by Theorem B.18 $c$ is strongly measurable. By Lemma 5.9, Condition L1 holds, so the map $L$ is continuous. Hence, by Lemma B.21, $L \circ c$ is strongly measurable. We also have that
\[
\|(L \circ c)(\omega)\|_{Y^*} = \frac{1}{K}\|f(\omega)\|_{L^2(D_R)},
\]
and thus $L \circ c \in L^2(\Omega; Y^*)$ since $f \in L^2(\Omega; L^2(D_R))$, i.e. Condition L2 holds. □

**Lemma 5.19 (Condition A2 for the Helmholtz stochastic EDP)** If $A \in L^\infty(\Omega; L^\infty(D_R; \mathbb{R}^{d \times d}))$, $n \in L^\infty(\Omega; L^\infty(D_R; \mathbb{R}))$, and $f$ is strongly measurable, then Condition A2 holds for the Helmholtz stochastic EDP.

**Proof of Lemma 5.19** A near-identical argument to that at the beginning of the proof of Lemma 5.18 shows $A \circ c$ is strongly measurable. Recall that the Dirichlet-to-Neumann operator $T_R$ is continuous from $H^{1/2}(\Gamma_R)$ to $H^{-1/2}(\Gamma_R)$, see e.g. [46, Theorem 2.6.4]. Let $v_1 \in X, v_2 \in Y$, and observe that the Cauchy–Schwarz inequality and these properties of $T_R$ imply that there exists $C(k) > 0$ such that
\[
\left\|\left[A_{\omega}(\omega)\right](v_1)v_2\right\|_{Y^*} = \int_{D_R} \left(\langle A(\omega)\nabla v_1, \nabla v_2 \rangle - k^2 n(\omega)v_1v_2\right)d\lambda - \langle T_Rv_1, v_2\rangle_{\Gamma_R} \leq \|A(\omega)\|_{L^\infty(D_R; \mathbb{R}^{d \times d})} + \|\nabla v_1\|_{L^2(D_R)}\|\nabla v_2\|_{L^2(D_R)} + k^2\|n(\omega)\|_{L^\infty(D_R; \mathbb{R})}\|v_1\|_{L^2(D_R)}\|v_2\|_{L^2(D_R)} + C(k)\|\nabla v_1\|_{H^{1/2}(\Gamma_R)}\|\nabla v_2\|_{H^{-1/2}(\Gamma_R)}.
\]
Since the trace operator $\gamma$ is continuous from $H^1(D_R)$ to $H^{1/2}(\Gamma_R)$ (see, e.g. [42, Theorem 3.38]), there exists $\tilde{C} > 0$ such that
\[
\|(A \circ c)(\omega)\|_{B(X, Y^*)} \leq \tilde{C}\max\left\{\|A(\omega)\|_{L^\infty(D_R; \mathbb{R}^{d \times d})}, \|n(\omega)\|_{L^\infty(D_R; \mathbb{R})}, C(k)\right\}\|v_1\|_{1,k}\|v_2\|_{1,k}.
\]
and hence $A \circ c \in L^\infty(\Omega; B(X, Y^*))$. □

### 5.3 Proofs of Theorems 1.4 and 1.8

**Proof of Theorem 1.4** We construct a solution of Problem 1 by letting $u = S \circ c$ (which is well-defined by Theorem 5.11), and observe that, by construction, $[a(\omega)](u(\omega), v) = [L(\omega)](v)$ for all $v \in H^1_0(D_R)$ almost surely. It follows that $u$ is measurable by Condition 1.3, Lemma 5.12, Lemma 5.12, and Lemma B.4, and so $u$ solves Problem 1. We therefore proceed to apply the general theory.

Conditions A1 and L1 hold by Lemma 5.9; Condition A2 holds by Lemma 5.19; Condition L2 holds by Lemma 5.18; Conditions C1 and C2 hold by Lemma 5.8 and Condition 1.3; and Condition U holds by Lemma 5.13. Therefore we can apply Theorems 3.8 and 3.9 and Lemmas 3.7 and 3.11 to conclude the results. □

**Proof of Theorem 1.8** All the conclusions of Theorem 1.4 hold, and we only need to show that if $u$ solves Problem 1 then it also solves Problem 2. Condition B holds by Conditions 1.3, 1.6, and Lemma 5.16. The result then follows from Theorem 3.6. □
A Failure of Fredholm theory for the stochastic variational formulation of Helmholtz problems

The standard approach to proving existence and uniqueness of a (deterministic) Helmholtz BVP is to show that the associated sesquilinear form satisfies a Gårding inequality, and then apply Fredholm theory to deduce that existence and uniqueness are equivalent; see, e.g., [42, Theorem 4.10]. This procedure relies on the fact that the inclusion $H^1_0(D_R) \hookrightarrow L^2(D_R)$ is compact; see, e.g., [42, Theorem 3.27].

As noted in [22], the analysis in [21] of Problem 3 for the Helmholtz Interior Impedance Problem mimics this approach and assumes that $L^2(\Omega; H^1(D))$ is compactly contained in $L^2(\Omega; L^2(D))$, where $D$ is the spatial domain. Here we briefly show that $L^2(\Omega; H^1(D))$ is not compactly contained in $L^2(\Omega; L^2(D))$ by giving an explicit example of a bounded sequence in $L^2(\Omega; H^1(D))$ that has no convergent subsequence in $L^2(\Omega; L^2(D))$.

Example A.1 Let $(\Omega, \mathcal{F}, \mathbb{P}) = ([0, 1], B([0, 1]), \lambda)$. Let $D$ be a compact subset of $\mathbb{R}^d$. Since $L^2(\Omega)$ is separable, it has an orthonormal basis, which we denote by $(f_m)_{m \in \mathbb{N}}$. Let $u_m \in L^2(\Omega; H^1(D))$ be defined by

$$u_m(\omega)(x) := f_m(\omega), \quad \text{for all } x \in D,$$

i.e., for each value of $\omega$, $u_m(\omega)$ is a constant function on $D$ and so $\|u_m(\omega)\|_{H^1(D)} = \|f_m(\omega)\|_{L^2(D)}$. Then

$$\|u_m\|^2_{L^2(\Omega; H^1(D))} = \int_{\Omega} \|u_m(\omega)\|^2_{H^1(D)} \, d\mathbb{P}(\omega) = \lambda(D)^2 \int_{\Omega} |f_m(\omega)|^2 \, d\mathbb{P}(\omega) = \|f_m\|^2_{L^2(D)} \lambda(D)^2,$$

and so $u_m$ is a bounded sequence in $L^2(\Omega; H^1(D))$. However, for $n \neq m$, we have

$$\|u_m - u_n\|^2_{L^2(\Omega; L^2(D))} = \int_{\Omega} \|u_m(\omega) - u_n(\omega)\|^2_{L^2(D)} \, d\mathbb{P}(\omega)$$

$$= \lambda(D)^2 \int_{\Omega} |u_m(\omega) - u_n(\omega)|^2 \, d\mathbb{P}(\omega) = \lambda(D)^2 \|f_m - f_n\|^2_{L^2(D)} = 2\lambda(D)^2, \quad \text{if } n \neq m,$$

since the $f_m$ form an orthonormal basis for $L^2(D)$. Therefore $(u_m)_{m \in \mathbb{N}}$ is bounded in $L^2(\Omega; H^1(D))$ but does not have a convergent subsequence in $L^2(\Omega; L^2(D))$, and thus the inclusion of $L^2(\Omega; H^1(D))$ into $L^2(\Omega; L^2(D))$ cannot be compact.

B Recap of basic material on measure theory and Bochner spaces

We include this section, not only for completeness, but also to aid readers of this paper who are more familiar with deterministic, as opposed to stochastic, Helmholtz problems.

B.1 Recap of measure theory results

We first recall some results from measure theory, our main reference for which is [8]. Even though [8] mainly considers mappings with image $\mathbb{R}$, the results we quote for more general images are straightforward generalisations of the results in [8].

Definition B.1 (Measurable map) If $(M, \mathcal{M})$ and $(N, \mathcal{N})$ are measurable spaces, we say that $f : M \to N$ is measurable (with respect to $(\mathcal{M}, \mathcal{N})$) if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$.

Definition B.2 (Borel $\sigma$-algebra) If $(S, T_S)$ is a topological space, the Borel $\sigma$-algebra $B(S)$ on $S$ is the $\sigma$-algebra generated by $T_S$.

If $V$ is any topological space (including a Hilbert, Banach, metric, or normed vector space) then we will take always the Borel $\sigma$-algebra on $V$ unless stated otherwise.

Lemma B.3 (Continuous maps are measurable [8, Theorem 2.1.2]) Any continuous function between two topological spaces is measurable.

Lemma B.4 (The composition of a measurable and a continuous map is measurable [8, Text at top of p. 146]) Let $(M, \mathcal{M})$ be a measurable space and let $(S, T_S)$ and $(T, T_T)$ be topological spaces. Let $f : M \to S$ be measurable and let $h : S \to T$ be continuous. Then $h \circ f$ is measurable.

Definition B.5 (Product $\sigma$-algebra [19, §IV.11]) Let $(M_1, \mathcal{M}_1), \ldots, (M_m, \mathcal{M}_m)$ be measurable spaces. The product $\sigma$-algebra $\mathcal{M} = \mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_m$ is defined as the $\sigma$-algebra generated by the set of measurable rectangles

$$\{R_1 \times \cdots \times R_m \subseteq M_1 \times \cdots \times M_m : R_i \in \mathcal{M}_i, \ldots, R_m \in \mathcal{M}_m\}. \quad (B.1)$$
Lemma B.6 (Measurability of the Cartesian product of measurable functions) Let $(M_1, \mathcal{M}_1), \ldots, (M_m, \mathcal{M}_m)$ be measurable spaces and $h_j : \Omega \to M_j$, $j = 1, \ldots, m$ be measurable functions. Then the product map $P : \Omega \to M_1 \times \cdots \times M_m$ given by
\[
P(\omega) := (h_1(\omega), \ldots, h_m(\omega))
\]
is measurable with respect to $(\mathcal{F}, \mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_m)$.

Sketch proof of Lemma B.6
Let $\text{Rect}(M_1, \ldots, M_m)$ denote the set of measurable rectangles, as in (B.1). Define the set $\mathcal{P}$ by
\[
\mathcal{P} := \{C \subseteq M_1 \times \cdots \times M_m : P^{-1}(C) \in \mathcal{F}\}.
\]
The proof of the lemma consists of the following straightforward steps, whose proofs are omitted:
1. Show $\text{Rect}(M_1, \ldots, M_m) \subseteq \mathcal{P}$;
2. Show $\mathcal{P}$ is a $\sigma$-algebra,
3. Deduce $M_1 \otimes \cdots \otimes M_m \subseteq \mathcal{P}$ (since $M_1 \otimes \cdots \otimes M_m$ is generated by measurable rectangles), and
4. Conclude $P$ is measurable with respect to $(\mathcal{F}, \mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_m)$.

Lemma B.7 (Product of Borel $\sigma$-algebras is Borel $\sigma$-algebra of the product [8, Lemma 6.2.1 (i)]) Let $H_1, H_2$ be Hausdorff spaces and let $H_2$ have a countable base (e.g. $H_2$ could be a separable metric space). Then $\mathcal{B}(H_1 \times H_2) = \mathcal{B}(H_1) \otimes \mathcal{B}(H_2)$, where $\mathcal{B}(H_1 \times H_2)$ is the Borel $\sigma$-algebra of the product topology on $H_1 \times H_2$.

B.2 Recap of results on Bochner spaces
We now recap the theory of Bochner spaces, using [18] as our main reference. In what follows the space $V$ is always a Banach space.

Definition B.8 (Simple function) A function $v : \Omega \to V$ is simple if there exist $v_1, \ldots, v_m \in V$ and $E_1, \ldots, E_m \in \mathcal{F}$ such that
\[
v = \sum_{i=1}^{m} v_i \chi_{E_i},
\]
where $\chi_{E_i}$ is the indicator function on $E_i$.

Definition B.9 (Strongly measurable) A function $v : \Omega \to V$ is strongly measurable\footnote{In [18] the authors use the term $\mu$-measurable instead of strongly measurable (where $\mu$ is the measure on the domain of the functions under consideration).} if there exists a sequence of simple functions $(v_n)_{n \in \mathbb{N}}$ such that
\[
\lim_{n \to \infty} \|v_n - v\|_V = 0 \quad \mathbb{P}-\text{almost everywhere}.
\]

Definition B.10 (Bochner integrable [18, p. 49]) A strongly measurable function $v : \Omega \to V$ is called Bochner integrable if there exists a sequence of simple functions $(v_n)_{n \in \mathbb{N}}$ such that
\[
\lim_{n \to \infty} \int_{\Omega} \|v_n(\omega) - v(\omega)\|_V \, d\mathbb{P}(\omega) = 0.
\]

Theorem B.11 (Condition for Bochner integrability [18, Theorem II.2.2]) A strongly measurable function $v : \Omega \to V$ is Bochner integrable if and only if $\int_{\Omega} \|v\|_V^p \, d\mathbb{P} < \infty$.

Corollary B.12 (Sufficient condition for Bochner integrability) Let $p \geq 1$. If a strongly measurable function $v : \Omega \to V$ has $\int_{\Omega} \|v\|_V^p \, d\mathbb{P} < \infty$, then $v$ is Bochner integrable.

Definition B.13 (Bochner norm) For a Bochner integrable function $v : \Omega \to V$, let
\[
\|v\|_{L^p(\Omega; V)} := \left( \int_{\Omega} \|v(\omega)\|_V^p \, d\mathbb{P}(\omega) \right)^{1/p}, \text{ for } 1 \leq p < \infty, \text{ and } \|v\|_{L^\infty(\Omega; V)} := \text{ess sup}_{\omega \in \Omega} \|v(\omega)\|_V.
\]

Definition B.14 (Bochner space) Let $1 \leq p \leq \infty$. Then
\[
L^p(\Omega; V) := \left\{ v : \Omega \to V : v \text{ is Bochner integrable}, \|v\|_{L^p(\Omega; V)} < \infty \right\}.
\]

Definition B.15 (Complete probability space) A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is complete if for every $E_1 \in \mathcal{F}$ with $\mathbb{P}(E_1) = 0$, the inclusion $E_2 \subseteq E_1$ implies that $E_2 \in \mathcal{F}$.

Definition B.16 (Separable space) A topological space is separable if it contains a countable, dense subset.

Definition B.17 ($\sigma$-finite) A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is $\sigma$-finite if there exist $E_1, E_2, \ldots \in \mathcal{F}$ with $\mathbb{P}(E_n) < \infty$ for all $m \in \mathbb{N}$ such that $\Omega = \bigcup_{n=1}^{\infty} E_n$. 

\[ \square \]
Theorem B.18 (Pettis measurability theorem [50, Proposition 2.15]) Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete \(\sigma\)-finite measure space. The following are equivalent for a function \(v : \Omega \to V\):

1. \(v\) is strongly measurable,
2. \(v\) is measurable and \(\mathbb{P}\)-essentially separably valued.

Corollary B.19 (Equivalence of measurable and strongly measurable when the image is separable) Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a \(\sigma\)-finite measure space. If \(V\) is a separable Banach space, then a function \(v : \Omega \to V\) is strongly measurable if, and only if, it is measurable.

Lemma B.20 (The composition of a continuous map and a \(\mathbb{P}\)-essentially separably valued map)
Let \((S, T_1), (T, T_2)\) be topological spaces. If \(f_1 : S \to T_1\) and \(f_2 : T_1 \to T_2\) are such that \(f_1\) is \(\mathbb{P}\)-essentially separably valued and \(f_2\) is continuous, then \(f_2 \circ f_1\) is \(\mathbb{P}\)-essentially separably valued.

Proof of Lemma B.20 As \(f_1\) is \(\mathbb{P}\)-essentially separably valued, there exists \(E \in \mathcal{F}\) such that \(\mathbb{P}(E) = 1\) and \(f_1(E) \subseteq G \subseteq S\), where \(G\) is separable. As \(f_2\) is continuous, \(f_2(G)\) is separable [56, Theorem 16.4(a)]. Therefore, since \((f_2 \circ f_1)(E) \subseteq f_2(G)\), it follows that \(f_2 \circ f_1\) is \(\mathbb{P}\)-essentially separably valued. \(\square\)

Lemma B.21 (The composition of a continuous map and a strongly measurable map)
If \(B_1\) and \(B_2\) are Banach spaces and there exist \(f_1 : \Omega \to B_1\) and \(f_2 : B_1 \to B_2\) such that \(f_1\) is strongly measurable and \(f_2\) is continuous, then \(f_2 \circ f_1\) is strongly measurable.

Proof of Lemma B.21 By Theorem B.18, \(f_1\) is both measurable and \(\mathbb{P}\)-essentially separably valued. Therefore we can apply Lemmas B.4 and B.20 to conclude \(f_2 \circ f_1\) is both measurable and \(\mathbb{P}\)-essentially separably valued. Hence by Theorem B.18 \(f_2 \circ f_1\) is strongly measurable. \(\square\)

Lemma B.22 (Zero in all integrals implies zero almost everywhere [18, Corollary II.2.5]) If \(\alpha\) is Bochner integrable and \(\int_{\omega} \alpha(\omega) \, d\mathbb{P}(\omega) = 0\) for each \(E \in \mathcal{F}\) then \(\alpha = 0\) \(\mathbb{P}\)-almost everywhere.

Lemma B.23 (Cartesian product of \(\mathbb{P}\)-essentially separably valued maps) Let \((C_1, T_{C_1}), \ldots, (C_m, T_{C_m})\) be topological spaces, and let \(s_j : \Omega \to C_j, j = 1, \ldots, m\) be \(\mathbb{P}\)-essentially separably valued. Define \(C := \prod_1^m C_j\) and equip \(C\) with the product topology. Then the map \(f : \Omega \to C\) given by \(s(\omega) := (s_1(\omega), \ldots, s_m(\omega))\) is \(\mathbb{P}\)-essentially separably valued.

The proof of Lemma B.23 is straightforward and omitted.

C Measurability of finite series expansions (used in §1.2)

Here we collect together results from measure theory that allow us to conclude in Lemma C.7 that the series expansions for \(A\) and \(n\) in §1.2 are measurable. As mentioned in §1.2, the proof that the sum of measurable functions is measurable is standard, but we have not been able to find this result stated in the literature for this particular setting of mappings into a separable subspace of a general normed vector space.

Lemma C.1 If \(U\) is a separable normed vector space, \(m \in \mathbb{N}\), and \(\phi_j : \Omega \to U, j = 1, \ldots, m\) are measurable functions, then \(\phi_1 + \cdots + \phi_m : \Omega \to U\) is measurable.

Sketch proof of Lemma C.1 By induction, it is sufficient to show the result for \(m = 2\). We let \(B_C^U(v)\) denote the ball of radius \(r > 0\) about \(v \in U\). To show \(\phi_1 + \phi_2\) is measurable, we let \(v, r > 0\) and we show \((\phi_1 + \phi_2)^{-1}(B_C^U(v)) \in \mathcal{F}\). Let \(Q_U\) denote a countable dense subset of \(U\), which exists as \(U\) is separable.

For \(s \in Q, q \in Q_U\) let
\[
S_{s, q} = \{ \omega \in \Omega : \|\phi_1(\omega) - \frac{1}{2}v - q\|_U < s \} \cap \{ \omega \in \Omega : \|\phi_2(\omega) - \frac{1}{2}v + q\|_U < r - s \}.
\]
We claim
\[
(\phi_1 + \phi_2)^{-1}(B_C^U(v)) = \bigcup_{s \in Q} \bigcup_{q \in Q_U} S_{s, q},
\]
and the result then follows as the right-hand side is an element of the \(\sigma\)-algebra \(\mathcal{F}\). To show (C.1), let \(\omega \in \bigcup_{s \in Q} \bigcup_{q \in Q_U} S_{s, q}\), and let \(s(q) \in Q_U\) be such that \(\omega \in S_{s(q)}\). Then it follows from the triangle inequality that \(\omega \in (\phi_1 + \phi_2)^{-1}(B_C^U(v))\). Now let \(\omega \in (\phi_1 + \phi_2)^{-1}(B_C^U(v))\), define \(r_{\omega} := r - \|\phi_1(\omega) + \phi_2(\omega) - v\|_U > 0\), fix \(s \in Q \cap (0, r_{\omega}/2)\), and choose \(q \in Q_U\) such that \(\|\phi_1(\omega) - v/2 - q\|_U < s\). Then again it follows from the triangle inequality that \(\omega \in S_{s, q}\), and thus (C.1) holds, as required. \(\square\)

Corollary C.2 If \(V\) is a normed vector space, \(U \subseteq V\) is a separable subspace, and \(\phi_j : \Omega \to U, j = 1, \ldots, m\) are measurable functions, then \(\phi_1 + \cdots + \phi_m : \Omega \to U\) is measurable.

Lemma C.3 Let \(V\) be a normed vector space. If \(v \in V\) and \(Y : \Omega \to F\) is a measurable function, then \(Yv : \Omega \to V\) is a measurable function.

Proof of Lemma C.3 The map \(M_v : F \to V\) given by \(M_v(x) = xv\) is continuous. As \(Yv = M_v \circ Y\), it follows from Lemma B.4 that \(Yv\) is measurable. \(\square\)

Lemma C.4 If \(V\) is a normed vector space and \(U \subseteq V\), then the inclusion map \(\iota : U \to V\) is measurable.
Proof of Lemma C.4 As ε is continuous, it immediately follows that it is measurable.

Corollary C.5 If V is a normed vector space, U ⊆ V and φ : Ω → U is measurable, then φ : Ω → V is measurable.

Proof of Corollary C.5 This is immediate from Lemma C.4 and Lemma B.4.

Lemma C.6 If V is a normed vector space, m ∈ N, and φ1, . . . , φm ∈ V for j = 1, . . . , m then span{φ1, . . . , φm} is a separable subspace of V.

Sketch Proof of Lemma C.6 As F = R or C, it has a separable subset Qτ. Since a finite product of countable sets is countable, the set

\[ \left\{ B^V_{1/n}(q_1\phi_1 + \cdots + q_m\phi_m) : n \in \mathbb{N}, q_1, \ldots, q_m \in Q^\tau \right\} \]

is a countable base for the topology on span{φ1, . . . , φm} induced by the norm ||·||V.

Lemma C.7 The functions A and n defined by (1.15) are measurable.

Proof of Lemma C.7 The proofs for A and n are identical, and so we only give the proof for n.

The subspace U = span{ψ0, ψ1, . . . , ψm} is separable by Lemma C.6, and it is clear that the image of n lies in U. By Lemma C.3 and Corollary C.2, n : Ω → U is measurable, and therefore n : Ω → V is measurable by Corollary C.5.

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