Idempotent systems and character algebras

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Abstract

We recently introduced the notion of an idempotent system. This linear algebraic object is motivated by the structure of an association scheme. There is a type of idempotent system, said to be symmetric. In the present paper we classify up to isomorphism the idempotent systems and the symmetric idempotent systems. We also describe how symmetric idempotent systems are related to character algebras.

1 Introduction

A symmetric association scheme (or SAS) is a combinatorial object that generalizes a distance-regular graph and a generously transitive permutation group. The concept of an SAS first arose in design theory [6,8,16] and group theory [19]. A systematic study began with [10,12]. A comprehensive treatment is given in [3,9]. The combinatorial regularity of an SAS gives it a rich algebraic structure. A (symmetric) character algebra [3,13] and an idempotent system [14] can be used to study the algebraic structure of an SAS, without having to assume the combinatorial structure. A character algebra is an abstraction of the Bose-Mesner algebra [7] of an SAS. For a vertex $x$ of an SAS the corresponding subconstituent algebra is generated by the Bose-Mesner algebra and the dual Bose-Mesner algebra with respect to $x$ [18]. A symmetric idempotent system is an abstraction of the primary module of a subconstituent algebra of an SAS [14, Section 1]. Our purpose in the present paper is two fold: (i) we classify up to isomorphism the idempotent systems and symmetric idempotent systems; (ii) we describe how symmetric idempotent systems are related to character algebras.

We recall the definition of an idempotent system [14]. Fix a field $\mathbb{F}$ and an integer $d \geq 0$. Let $V$ denote a vector space over $\mathbb{F}$ with dimension $d + 1$. Let $\text{End}(V)$ denote the $\mathbb{F}$-algebra consisting of the $\mathbb{F}$-linear maps $V \to V$. Let $\mathcal{A}$ denote an $\mathbb{F}$-algebra isomorphic to $\text{End}(V)$. An idempotent system in $\mathcal{A}$ is a sequence $\Phi = (\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ such that

(i) $\{E_i\}_{i=0}^d$ is a system of mutually orthogonal rank 1 idempotents in $\mathcal{A}$;

(ii) $\{E_i^*\}_{i=0}^d$ is a system of mutually orthogonal rank 1 idempotents in $\mathcal{A}$;

(iii) $E_0 E_i^* E_0 \neq 0 \quad (0 \leq i \leq d)$;
(iv) $E_0^*E_iE_0^* \neq 0$ \hspace{1mm} (0 \leq i \leq d).

We say that $\Phi$ is over $F$ and has diameter $d$. The idempotent system $\Phi$ is said to be symmetric whenever there exists an antiautomorphism of $A$ that fixes each of $E_i, E_i^*$ for $0 \leq i \leq d$.

We recall the concept of a character algebra \cite{3, 13}. Throughout this paper a character algebra is understood to be symmetric. A character algebra over $F$ is a sequence $(C; \{x_i\}_{i=0}^d)$ such that $C$ is a commutative $F$-algebra and $\{x_i\}_{i=0}^d$ is a distinguished basis for $C$ that satisfies the conditions in Definition 8.1 below. We are mainly interested in the semisimple case, in which $C$ has a basis $\{e_i\}_{i=0}^d$ of primitive idempotents. Among $\{e_i\}_{i=0}^d$ there exists a distinguished one, said to be trivial. A character system over $F$ is a sequence $\Psi = (C; \{x_i\}_{i=0}^d; \{e_i\}_{i=0}^d)$, where $(C; \{x_i\}_{i=0}^d)$ is a semisimple character algebra over $F$, and $\{e_i\}_{i=0}^d$ is an ordering of the primitive idempotents of $C$ with $e_0$ trivial. We say that $\Psi$ has diameter $d$.

The following notation is convenient. Let $\text{Mat}_{d+1}(F)$ denote the algebra consisting of the $d+1$ by $d+1$ matrices that have all entries in $F$. We index rows and columns by $0, 1, \ldots, d$. For $0 \leq i, j \leq d$ let $\Delta_{i,j}$ denote the matrix in $\text{Mat}_{d+1}(F)$ that has $(i, j)$-entry 1 and all other entries 0.

Our classification of idempotent systems is summarized as follows. An invertible matrix $R \in \text{Mat}_{d+1}(F)$ is said to be solid whenever for both $R$ and $R^{-1}$ all entries are nonzero in both column 0 and row 0. Two matrices $R, S$ in $\text{Mat}_{d+1}(F)$ are said to be diagonally equivalent whenever there exist invertible diagonal matrices $H, K$ in $\text{Mat}_{d+1}(F)$ such that $S = HRK$. For a solid invertible matrix $R \in \text{Mat}_{d+1}(F)$, we show that the sequence

$$\Phi_R = (\{\Delta_{i,i}\}_{i=0}^d; \{R\Delta_{i,i}R^{-1}\}_{i=0}^d)$$

is an idempotent system in $\text{Mat}_{d+1}(F)$. We show that the map $R \mapsto \Phi_R$ induces a bijection between the following two sets:

(i) the diagonal equivalence classes of solid invertible matrices in $\text{Mat}_{d+1}(F)$;

(ii) the isomorphism classes of idempotent systems over $F$ with diameter $d$.

Our classification of symmetric idempotent systems is summarized as follows. An invertible $R \in \text{Mat}_{d+1}(F)$ is said to be almost orthogonal (AO) whenever $R^t$ is diagonally equivalent to $R^{-1}$. By restricting the above bijection to AO solid invertible matrices, we get a bijection between the following two sets:

(i) the diagonal equivalence classes of AO solid invertible matrices in $\text{Mat}_{d+1}(F)$;

(ii) the isomorphism classes of symmetric idempotent systems over $F$ with diameter $d$.

The above classifications have the following alternate version. A solid invertible matrix $R \in \text{Mat}_{d+1}(F)$ is said to be normalized whenever in column 0 of $R$ all entries are equal to 1 and in column 0 of $R^{-1}$ all entries are the same. As we will show, each diagonal equivalence class of solid invertible matrices contains a unique normalized element. Thus our first bijection above induces a bijection between the following two sets:

(i) the diagonal equivalence classes of normalized solid invertible matrices in $\text{Mat}_{d+1}(F)$;

(ii) the isomorphism classes of symmetric idempotent systems over $F$ with diameter $d$. 

The above classifications have the following alternate version. A solid invertible matrix $R \in \text{Mat}_{d+1}(F)$ is said to be normalized whenever in column 0 of $R$ all entries are equal to 1 and in column 0 of $R^{-1}$ all entries are the same. As we will show, each diagonal equivalence class of solid invertible matrices contains a unique normalized element. Thus our first bijection above induces a bijection between the following two sets:
(i) the normalized solid invertible matrices in Mat_{d+1}(F);
(ii) the isomorphism classes of idempotent systems over F with diameter d.

Similarly we get a bijection between the following two sets:

(i) the AO normalized solid invertible matrices in Mat_{d+1}(F);
(ii) the isomorphism classes of symmetric idempotent systems over F with diameter d.

Shortly we will describe this bijection in more detail. Next we describe how AO normalized solid invertible matrices and symmetric idempotent systems are related to character systems. Consider the following sets:

\begin{align*}
\text{AON}_d(F) & : \text{the AO normalized solid invertible matrices in Mat}_{d+1}(F); \\
\text{SIS}_d(F) & : \text{the isomorphism classes of symmetric idempotent systems over } F \text{ with diameter } d; \\
\text{CS}_d(F) & : \text{the isomorphism classes of character systems over } F \text{ with diameter } d.
\end{align*}

We will show that these three sets are mutually in bijection, and we will describe the bijections involved.

We already gave a bijection from AON_d(F) to SIS_d(F). We now describe the inverse bijection. Let Φ = (\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d) denote a symmetric idempotent system in A. The elements \{E_i\}_{i=0}^d form a basis for a commutative subalgebra M of A. For 0 ≤ i ≤ d there exists a unique A_i ∈ M such that \(A_i E_i^* E_0 = E_i^* E_0\) for 0 ≤ i ≤ d. The elements \{A_i\}_{i=0}^d form a basis of M. Let P_Φ denote the transition matrix from the basis \{E_i\}_{i=0}^d of M to the basis \{A_i\}_{i=0}^d of M. The matrix P_Φ is AO normalized solid invertible, and called the first eigenmatrix of Φ.

For an idempotent system Φ, let [Φ] denote the isomorphism class of idempotent systems over F that contains Φ. We show that the following maps are inverses:

\[\text{AON}_d(F) \xrightarrow{Φ \mapsto [Φ]} \text{SIS}_d(F), \quad \text{SIS}_d(F) \xrightarrow{[Φ] \mapsto P_Φ} \text{AON}_d(F).\]

Next we describe the bijective correspondence between AON_d(F) and CS_d(F). Let Ψ = (C; \{x_i\}_{i=0}^d; \{e_i\}_{i=0}^d) denote a character system over F. Let P_Ψ denote the transition matrix from the basis \{e_i\}_{i=0}^d of C to the basis \{x_i\}_{i=0}^d of C. The matrix P_Ψ is AO normalized solid invertible, and called the eigenmatrix of Ψ. For an AO normalized solid invertible P ∈ Mat_{d+1}(F), we construct a character system Ψ_P as follows. Let \{e_i\}_{i=0}^d denote indeterminates, and let C denote the vector space over F with basis \{e_i\}_{i=0}^d. Turn C into an algebra such that \(e_i e_j = \delta_{i,j} e_i\) for 0 ≤ i ≤ d. View P as the transition matrix from \{e_i\}_{i=0}^d to a basis \{x_i\}_{i=0}^d. We show that the sequence

\[Ψ_P = (C; \{x_i\}_{i=0}^d; \{e_i\}_{i=0}^d)\]

is a character system over F. We show that the following maps are inverses:

\[\text{AON}_d(F) \xrightarrow{P \mapsto Ψ_P} \text{CS}_d(F), \quad \text{CS}_d(F) \xrightarrow{[Ψ] \mapsto P_Ψ} \text{AON}_d(F).\]
Next we describe the bijective correspondence between $\text{SIS}_d(\mathbb{F})$ and $\text{CS}_d(\mathbb{F})$. For a symmetric idempotent system $\Phi = (\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ over $\mathbb{F}$, we show that the sequence

$$\Psi_\Phi = (\{M; \{A_i\}_{i=0}^d; \{E_i\}_{i=0}^d\})$$

is a character system over $\mathbb{F}$. For a character system $\Psi = (\mathcal{C}; \{x_i\}_{i=0}^d; \{e_i\}_{i=0}^d)$ over $\mathbb{F}$, we construct a symmetric idempotent system $\Phi_\Psi$ as follows. For $0 \leq i \leq d$ define $E_i$, $E_i^* \in \text{End}(\mathcal{C})$ such that $E_i e_j = \delta_{i,j} e_j$ and $E_i^* x_j = \delta_{i,j} x_j$ for $0 \leq j \leq d$. We show that the sequence

$$\Phi_\Psi = (\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$$

is a symmetric idempotent system in $\text{End}(\mathcal{C})$. We show that the following maps are inverses:

$$\text{SIS}_d(\mathbb{F}) \xrightarrow{[\Phi] \mapsto [\Psi_\Phi]} \text{CS}_d(\mathbb{F}), \quad \text{CS}_d(\mathbb{F}) \xrightarrow{[\Psi] \mapsto [\Phi_\Psi]} \text{SIS}_d(\mathbb{F}).$$

For a symmetric idempotent system $\Phi = (\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$, the sequence

$$\Phi^* = (\{E_i^*\}_{i=0}^d; \{E_i\}_{i=0}^d)$$

is an idempotent system, called the dual of $\Phi$. We have a bijective involution $\text{SIS}_d(\mathbb{F}) \rightarrow \text{SIS}_d(\mathbb{F})$ that sends $[\Phi] \mapsto [\Phi^*]$, called the duality map. From the above bijections $\text{SIS}_d(\mathbb{F}) \rightarrow \text{AON}_d(\mathbb{F})$ and $\text{SIS}_d(\mathbb{F}) \rightarrow \text{CS}_d(\mathbb{F})$, the sets $\text{AON}_d(\mathbb{F})$ and $\text{CS}_d(\mathbb{F})$ inherit a duality map. We describe these duality maps in detail. We find that the duality map on $\text{CS}_d(\mathbb{F})$ is essentially the same thing as the duality map for character algebras defined by Kawada [13].

The paper is organized as follows. In Section 2 we fix some notation and recall some basic concepts. In Section 3 we recall the notion of an idempotent system. In Section 4 we classify the idempotent systems in terms of solid invertible matrices. In Section 5 we classify the symmetric idempotent systems in terms of AO solid invertible matrices. In Section 6 we consider the normalization of a solid invertible matrix, and we establish the bijection $\text{AON}_d(\mathbb{F}) \rightarrow \text{SIS}_d(\mathbb{F})$. In Section 7 we describe the inverse of this bijection. In Sections 8–10 we discuss character algebras and character systems. In Section 11 we show that $\text{AON}_d(\mathbb{F})$, $\text{SIS}_d(\mathbb{F})$, $\text{CS}_d(\mathbb{F})$ are mutually in bijection, and we describe the bijections involved. Section 12 is about the duality maps for $\text{AON}_d(\mathbb{F})$, $\text{SIS}_d(\mathbb{F})$, $\text{CS}_d(\mathbb{F})$.

2 Preliminaries

We now begin our formal argument. In this section we fix some notation and recall some basic concepts. Throughout this paper $\mathbb{F}$ denotes a field. All vector spaces discussed in this paper are over $\mathbb{F}$. All algebras discussed in this paper are associative, over $\mathbb{F}$, and have a multiplicative identity. For an algebra $\mathcal{A}$, by an automorphism of $\mathcal{A}$ we mean an algebra isomorphism $\mathcal{A} \rightarrow \mathcal{A}$, and by an antiautomorphism of $\mathcal{A}$ we mean an $\mathbb{F}$-linear bijection $\tau : \mathcal{A} \rightarrow \mathcal{A}$ such that $(YZ)^\tau = Z^\tau Y^\tau$ for $Y, Z \in \mathcal{A}$. For the rest of this paper, fix an integer $d \geq 0$. Let $\text{Mat}_{d+1}(\mathbb{F})$ denote the $\mathbb{F}$-algebra consisting of the $d + 1$ by $d + 1$ matrices that have all entries in $\mathbb{F}$. We index rows and columns by $0, 1, \ldots, d$. By the Skolem-Noether theorem [17] Corollary 7.125, a map $\sigma : \text{Mat}_{d+1}(\mathbb{F}) \rightarrow \text{Mat}_{d+1}(\mathbb{F})$ is an automorphism of
Mat_{d+1}(F) if and only if there exists an invertible $S \in \text{Mat}_{d+1}(F)$ such that $A^\sigma = SAS^{-1}$ for $A \in \text{Mat}_{d+1}(F)$. A map $\tau : \text{Mat}_{d+1}(F) \to \text{Mat}_{d+1}(F)$ is an antimorphism of $\text{Mat}_{d+1}(F)$ if and only if there exists an invertible $T \in \text{Mat}_{d+1}(F)$ such that $A^\tau = T^\tau A^\tau T^{-1}$ for $A \in \text{Mat}_{d+1}(F)$, where $A^\top$ denotes the transpose of $A$.

For $0 \leq i, j \leq d$ let $\Delta_{i,j}$ denote the matrix in $\text{Mat}_{d+1}(F)$ that has $(i,j)$-entry 1 and all other entries is 0. The matrices $\Delta_{i,j}$ $(0 \leq i, j \leq d)$ form a basis for the vector space $\text{Mat}_{d+1}(F)$. Let $\mathcal{D}$ denote the subalgebra of $\text{Mat}_{d+1}(F)$ consisting of diagonal matrices in $\text{Mat}_{d+1}(F)$. The algebra $\mathcal{D}$ is commutative, and the matrices $\Delta_{i,i}$ $(0 \leq i \leq d)$ form a basis for the vector space $\mathcal{D}$. Note that the identity matrix $I = \sum_{i=0}^{d} \Delta_{i,i}$.

**Lemma 2.1** A matrix in $\text{Mat}_{d+1}(F)$ is diagonal if and only if it commutes with every diagonal matrix in $\text{Mat}_{d+1}(F)$.

Let $V$ denote a vector space with dimension $d + 1$. Let $\text{End}(V)$ denote the algebra consisting of the $F$-linear maps $V \to V$. We recall how each basis $\{v_i\}_{i=0}^{d}$ of $V$ gives an algebra isomorphism $\text{End}(V) \to \text{Mat}_{d+1}(F)$. For $X \in \text{End}(V)$ and $M \in \text{Mat}_{d+1}(F)$, we say that $M$ represents $X$ with respect to $\{v_i\}_{i=0}^{d}$ whenever $Xv_j = \sum_{i=0}^{d} M_{ij}v_i$ for $0 \leq j \leq d$. The isomorphism sends $X$ to the unique matrix in $\text{Mat}_{d+1}(F)$ that represents $X$ with respect to $\{v_i\}_{i=0}^{d}$. For two bases $\{u_i\}_{i=0}^{d}$, $\{v_i\}_{i=0}^{d}$ of $V$, by the transition matrix from $\{u_i\}_{i=0}^{d}$ to $\{v_i\}_{i=0}^{d}$ we mean the matrix $P \in \text{Mat}_{d+1}(F)$ such that $v_j = \sum_{i=0}^{d} P_{ij}u_i$ for $0 \leq j \leq d$. In this case, $P$ is invertible, and $P^{-1}$ is the transition matrix from $\{v_i\}_{i=0}^{d}$ to $\{u_i\}_{i=0}^{d}$. We recall some basic facts concerning bilinear forms. By a bilinear form on $V$ we mean a map $\langle , \rangle : V \times V \to F$ that satisfies the following four conditions for $u, v, w \in V$ and $\alpha \in F$: (i) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$; (ii) $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$; (iii) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$; (iv) $\langle u, \alpha v \rangle = \alpha \langle u, v \rangle$. A bilinear form $\langle , \rangle$ on $V$ is said to be symmetric whenever $\langle u, v \rangle = \langle v, u \rangle$ for $u, v \in V$. Let $\langle , \rangle$ denote a bilinear form on $V$. The following are equivalent: (i) there exists a nonzero $u \in V$ such that $\langle u, v \rangle = 0$ for all $v \in V$; (ii) there exists a nonzero $v \in V$ such that $\langle u, v \rangle = 0$ for all $u \in V$. The form $\langle , \rangle$ is said to be degenerate whenever (i), (ii) hold and nondegenerate otherwise. Assume that $\langle , \rangle$ is nondegenerate. A basis $\{u_i\}_{i=0}^{d}$ of $V$ is said to be orthogonal with respect to $\langle , \rangle$ whenever $\langle u_i, u_j \rangle = 0$ if $i \neq j$ $(0 \leq i, j \leq d)$.

For the rest of this paper, let $A$ denote an algebra that is isomorphic to $\text{Mat}_{d+1}(F)$. The identity element of $A$ is denoted by $I$.

**Definition 2.2** By a system of mutually orthogonal rank 1 idempotents in $A$ we mean a sequence $\{E_i\}_{i=0}^{d}$ of elements in $A$ such that

$$E_iE_j = \delta_{i,j}E_i \quad (0 \leq i, j \leq d),$$

$$\text{rank}(E_i) = 1 \quad (0 \leq i \leq d).$$

**Example 2.3** The matrices $\{\Delta_{i,i}\}_{i=0}^{d}$ form a system of mutually orthogonal rank 1 idempotents in $\text{Mat}_{d+1}(F)$.

We now make a more general statement.
Lemma 2.4 For a sequence \( \{E_i\}_{i=0}^d \) of elements in \( \text{Mat}_{d+1}(F) \) the following are equivalent:

(i) \( \{E_i\}_{i=0}^d \) is a system of mutually orthogonal rank 1 idempotents;

(ii) there exists an invertible \( R \in \text{Mat}_{d+1}(F) \) such that \( E_i = R \Delta_{i,i} R^{-1} \) for \( 0 \leq i \leq d \).

The above result can be stated more abstractly as follows.

Lemma 2.5 For a sequence \( \{E_i\}_{i=0}^d \) of elements in \( A \) the following are equivalent:

(i) \( \{E_i\}_{i=0}^d \) is a system of mutually orthogonal rank 1 idempotents;

(ii) there exists an algebra isomorphism \( A \rightarrow \text{Mat}_{d+1}(F) \) that sends \( E_i \mapsto \Delta_{i,i} \) for \( 0 \leq i \leq d \).

Lemma 2.6 Let \( \{E_i\}_{i=0}^d \) denote a system of mutually orthogonal rank 1 idempotents in \( A \). Then \( \{E_i\}_{i=0}^d \) form a basis for a commutative subalgebra of \( A \). Moreover \( I = \sum_{i=0}^d E_i \).

In this paper we will occasionally speak of three sets being mutually in bijection. This means that for any ordering \( A, B, C \) of the three sets and \( a \in A, b \in B, c \in C \), if \( a, b \) correspond and \( b, c \) correspond then \( a, c \) correspond.

3 Idempotent systems

In this section we recall from [14] the notion of an idempotent system, and make some general remarks about it. Recall the algebra \( A \) that is isomorphic to \( \text{Mat}_{d+1}(F) \).

Definition 3.1 (See [14, Definition 3.1].) By an idempotent system in \( A \) we mean a sequence

\[ \Phi = (\{E_i\}_{i=0}^d; \{E^*_i\}_{i=0}^d) \]

such that

(i) \( \{E_i\}_{i=0}^d \) is a system of mutually orthogonal rank 1 idempotents in \( A \);

(ii) \( \{E^*_i\}_{i=0}^d \) is a system of mutually orthogonal rank 1 idempotents in \( A \);

(iii) \( E_0 E^*_i E_0 \neq 0 \) \quad (0 \leq i \leq d);

(iv) \( E^*_0 E_i E_0^* \neq 0 \) \quad (0 \leq i \leq d).

The idempotent system \( \Phi \) is said to be over \( F \). We call \( d \) the diameter of \( \Phi \).

Let \( \Phi = (\{E_i\}_{i=0}^d; \{E^*_i\}_{i=0}^d) \) denote an idempotent system in \( A \). Then the sequence

\[ \Phi^* = (\{E^*_i\}_{i=0}^d; \{E_i\}_{i=0}^d) \]

is an idempotent system in \( A \), called the dual of \( \Phi \). For an object \( \phi \) attached to \( \Phi \), the corresponding object attached to \( \Phi^* \) is denoted by \( \phi^* \).
For an algebra \( A' \) and an algebra isomorphism \( \sigma : A \rightarrow A' \), define the sequence

\[
\Phi^\sigma = (\{ E_i^\sigma \}_{i=0}^d; \{ (E_i^\sigma)^* \}_{i=0}^d).
\]

Then \( \Phi^\sigma \) is an idempotent system in \( A' \). Let \( \Phi' = (\{ E_i' \}_{i=0}^d; \{ E_i'^* \}_{i=0}^d) \) denote an idempotent system in \( A' \). By an isomorphism of idempotent systems from \( \Phi \) to \( \Phi' \) we mean an algebra isomorphism \( \sigma : A \rightarrow A' \) such that \( \Phi^\sigma = \Phi' \). The idempotent systems \( \Phi \) and \( \Phi' \) are said to be isomorphic whenever there exists an isomorphism of idempotent systems from \( \Phi \) to \( \Phi' \).

**Definition 3.2** Let \( \Phi = (\{ E_i \}_{i=0}^d; \{ E_i^* \}_{i=0}^d) \) denote an idempotent system in \( A \). Recall from Lemma 2.6 that \( \{ E_i \}_{i=0}^d \) form a basis for a commutative subalgebra of \( A \); we denote this subalgebra by \( M \).

4 A classification of the idempotent systems

In this section we classify the idempotent systems up to isomorphism, in terms of a type of invertible matrix said to be solid. To motivate the classification, we first construct an example of an idempotent system. Given an invertible \( R \in \text{Mat}_{d+1}(F) \), consider the following matrices in \( \text{Mat}_{d+1}(F) \):

\[
E_i = \Delta_{i,i}, \quad E_i^* = R\Delta_{i,i}R^{-1} \quad (0 \leq i \leq d).
\]

(1)

Note that each of \( \{ E_i \}_{i=0}^d \) and \( \{ E_i^* \}_{i=0}^d \) is a system of mutually orthogonal rank 1 idempotents in \( \text{Mat}_{d+1}(F) \). Our next goal is to find a necessary and sufficient condition on \( R \) such that the conditions Definition 3.1(iii), (iv) are satisfied.

**Lemma 4.1** Referring to (1), the entries of \( E_0^*E_i^*E_0 \) and \( E_i^*E_0E_0^* \) are described as follows. For \( 0 \leq r, s \leq d \) their \((r,s)\)-entry is

\[
(E_0^*E_i^*E_0)_{r,s} = \delta_{r,0}\delta_{s,0}R_{0,i}(R^{-1})_{i,0},
\]

(2)

\[
(E_i^*E_0E_0^*)_{r,s} = R_{r,0}(R^{-1})_{0,i}R_{i,0}(R^{-1})_{0,s}.
\]

(3)

**Proof.** By matrix multiplication.

\( \square \)

**Lemma 4.2** Referring to (1), for \( 0 \leq i \leq d \) the following are equivalent:

(i) \( E_0^*E_i^*E_0 \neq 0 \);

(ii) \( R_{0,i} \neq 0 \) and \( (R^{-1})_{i,0} \neq 0 \).

**Proof.** Use (2).

\( \square \)
Lemma 4.3 Referring to (1), for $0 \leq i \leq d$ the following are equivalent:

(i) $E_0^* E_i E_0^* \neq 0$;
(ii) $R_{i,0} \neq 0$ and $(R^{-1})_{0,i} \neq 0$.

Proof. Use (3). □

Definition 4.4 An invertible matrix $R \in \text{Mat}_{d+1}(\mathbb{F})$ is said to be solid whenever the following hold:

(i) in column 0 and row 0 of $R$ all entries are nonzero;
(ii) in column 0 and row 0 of $R^{-1}$ all entries are nonzero.

Proposition 4.5 Referring to (1), the following are equivalent:

(i) the sequence $(\{E_i\}_{i=0}^{d}; \{E_i^*\}_{i=0}^{d})$ is an idempotent system in $\text{Mat}_{d+1}(\mathbb{F})$;
(ii) the matrix $R$ is solid.

Proof. By Definitions 4.1, 4.4 and Lemmas 4.2, 4.3. □

Definition 4.6 For a solid invertible matrix $R \in \text{Mat}_{d+1}(\mathbb{F})$ define the sequence

$$
\Phi_R = (\{\Delta_{i,i}\}_{i=0}^{d}; \{R\Delta_{i,i}R^{-1}\}_{i=0}^{d}).
$$

Note by Proposition 4.5 that $\Phi_R$ is an idempotent system in $\text{Mat}_{d+1}(\mathbb{F})$.

Proposition 4.7 Every idempotent system in $A$ is isomorphic to $\Phi_R$ for some solid invertible $R \in \text{Mat}_{d+1}(\mathbb{F})$.

Proof. Let $\Phi = (\{E_i\}_{i=0}^{d}; \{E_i^*\}_{i=0}^{d})$ denote an idempotent system in $A$. By Lemma 2.5 there exists an algebra isomorphism $\sigma : A \to \text{Mat}_{d+1}(\mathbb{F})$ that sends $E_i \mapsto \Delta_{i,i}$ for $0 \leq i \leq d$. Then $\Phi^\sigma$ is an idempotent system in $\text{Mat}_{d+1}(\mathbb{F})$, and $\sigma$ is an isomorphism of idempotent systems from $\Phi$ to $\Phi^\sigma$. By Lemma 2.3 there exists an invertible $R \in \text{Mat}_{d+1}(\mathbb{F})$ such that $(E_i^\sigma) = R\Delta_{i,i}R^{-1}$ for $0 \leq i \leq d$. The idempotent system $\Phi^\sigma$ has the form

$$
\Phi^\sigma = (\{\Delta_{i,i}\}_{i=0}^{d}; \{R\Delta_{i,i}R^{-1}\}_{i=0}^{d}).
$$

The matrix $R$ is solid by Proposition 4.5 and $\Phi^\sigma = \Phi_R$ by Definition 4.6. The result follows. □
Definition 4.8 Matrices $S, T$ in $\text{Mat}_{d+1}(\mathbb{F})$ are said to be \textit{diagonally equivalent} whenever there exist invertible diagonal matrices $H, K$ in $\text{Mat}_{d+1}(\mathbb{F})$ such that $T = HSK$.

Note that diagonal equivalence is an equivalence relation on $\text{Mat}_{d+1}(\mathbb{F})$.

Let $R$ denote a solid invertible matrix in $\text{Mat}_{d+1}(\mathbb{F})$ and let $H, K$ denote invertible diagonal matrices in $\text{Mat}_{d+1}(\mathbb{F})$. For $0 \leq r, s \leq d$ the $(r, s)$-entries of $HRK$ and $(HRK)^{-1}$ are

\[
(HRK)_{r,s} = R_{r,s}H_{r,r}K_{s,s}, \quad ((HRK)^{-1})_{r,s} = \frac{(R^{-1})_{r,s}}{H_{s,s}K_{r,r}}.
\]

Lemma 4.9 Let $R$ denote a solid invertible matrix in $\text{Mat}_{d+1}(\mathbb{F})$. Then every matrix that is diagonally equivalent to $R$ is solid invertible.

\begin{proof}
Use (4).
\end{proof}

Proposition 4.10 For solid invertible matrices $R, S$ in $\text{Mat}_{d+1}(\mathbb{F})$ the following are equivalent:

(i) $R$ and $S$ are diagonally equivalent;

(ii) the idempotent systems $\Phi_R$ and $\Phi_S$ are isomorphic.

Suppose (i), (ii) hold. Let $H, K$ denote invertible diagonal matrices in $\text{Mat}_{d+1}(\mathbb{F})$ such that $S = HRK$. Then the automorphism of $\text{Mat}_{d+1}(\mathbb{F})$ that sends $A \mapsto HAH^{-1}$ is an isomorphism of idempotent systems from $\Phi_R$ to $\Phi_S$.

\begin{proof}
(i) $\Rightarrow$ (ii) There exist invertible diagonal matrices $H, K$ in $\text{Mat}_{d+1}(\mathbb{F})$ such that $S = HRK$. Consider the automorphism $\sigma$ of $\text{Mat}_{d+1}(\mathbb{F})$ that sends $A \mapsto HAH^{-1}$ for $A \in \text{Mat}_{d+1}(\mathbb{F})$. We show that $\sigma$ is an isomorphism of idempotent systems from $\Phi_R$ to $\Phi_S$. For $0 \leq i \leq d$ we have $H\Delta_{i,i}H^{-1} = \Delta_{i,i}$ by Lemma 2.1 and so $\Delta_{i,i} = \Delta_{i,i}$. Using $HR = SK^{-1}$ and Lemma 2.1 we find that for $0 \leq i \leq d$,

\[
HR\Delta_{i,i}R^{-1}H^{-1} = SK^{-1}\Delta_{i,i}KS^{-1} = S\Delta_{i,i}S^{-1}.
\]

So $\sigma$ sends $R\Delta_{i,i}R^{-1} \mapsto S\Delta_{i,i}S^{-1}$. By these comments $\sigma$ is an isomorphism of idempotent systems from $\Phi_R$ to $\Phi_S$.

(ii) $\Rightarrow$ (i) Let $\sigma$ denote an isomorphism of idempotent systems from $\Phi_R$ to $\Phi_S$. By the Skolem-Noether theorem, there exists an invertible $H \in \text{Mat}_{d+1}(\mathbb{F})$ such that $A\sigma = HAH^{-1}$ for $A \in \text{Mat}_{d+1}(\mathbb{F})$. For $0 \leq i \leq d$, $\sigma$ fixes $\Delta_{i,i}$ and so $H\Delta_{i,i}H^{-1} = \Delta_{i,i}$. By this and Lemma 2.1 $H$ is diagonal. For $0 \leq i \leq d$, $\sigma$ sends $R\Delta_{i,i}R^{-1} \mapsto S\Delta_{i,i}S^{-1}$, so

\[
HR\Delta_{i,i}R^{-1}H^{-1} = S\Delta_{i,i}S^{-1}.
\]

Thus $R^{-1}H^{-1}S$ commutes with $\Delta_{i,i}$ for $0 \leq i \leq d$. By this and Lemma 2.1 $R^{-1}H^{-1}S$ is diagonal; denote this diagonal matrix by $K$. Then $S = HRK$. The matrices $H, K$ are diagonal, so $R$ and $S$ are diagonally equivalent.

Suppose (i), (ii) hold. In the proof of (i) $\Rightarrow$ (ii), we have shown the last assertion of the proposition statement.
\end{proof}
**Definition 4.11** Let \( \rho \) denote a diagonal equivalence class of solid invertible matrices in \( \text{Mat}_{d+1}(F) \). By Proposition 4.10 the set \( \{ \Phi_R \mid R \in \rho \} \) is contained in an isomorphism class of idempotent systems; denote this isomorphism class by \( \Phi_\rho \).

In the next result we classify the idempotent systems up to isomorphism.

**Theorem 4.12** Consider the following sets:

(i) the diagonal equivalence classes of solid invertible matrices in \( \text{Mat}_{d+1}(F) \);

(ii) the isomorphism classes of idempotent systems over \( F \) with diameter \( d \).

The map \( \rho \mapsto \Phi_\rho \) is a bijection from (i) to (ii).

**Proof.** The given map is surjective by Proposition 4.7 and injective by Proposition 4.10.

We have some comments.

**Lemma 4.13** For a solid invertible \( R \in \text{Mat}_{d+1}(F) \) the matrix \( R^{-1} \) is solid invertible. Moreover the map \( \text{Mat}_{d+1}(F) \rightarrow \text{Mat}_{d+1}(F) \), \( A \mapsto RAR^{-1} \) is an isomorphism of idempotent systems from \( \Phi_{R^{-1}} \) to \( (\Phi_R)^* \).

**Proof.** The matrix \( R^{-1} \) is solid invertible by Definition 4.4. We have

\[
\Phi_{R^{-1}} = (\{ \Delta_{i,i} \}_{i=0}^d; \{ R^{-1} \Delta_{i,i} R \}_{i=0}^d), \quad (\Phi_R)^* = (\{ R \Delta_{i,i} R^{-1} \}_{i=0}^d; \{ \Delta_{i,i} \}_{i=0}^d).
\]

Thus the map \( A \mapsto RAR^{-1} \) is an isomorphism of idempotent systems from \( \Phi_{R^{-1}} \) to \( (\Phi_R)^* \).

For a solid invertible \( R \in \text{Mat}_{d+1}(F) \), note by Definition 4.3 that \( R^t_i \) is solid invertible. In Section 5 we consider the case in which \( R^t_i \) is diagonally equivalent to \( R^{-1} \).

## 5 Symmetric idempotent systems

In [14] we introduced a type of idempotent system, said to be symmetric. In this section we classify the symmetric idempotent systems in terms of solid invertible matrices.

**Definition 5.1** (See [14] Definition 5.1.) Let \( \Phi = (\{ E_i \}_{i=0}^d; \{ E_i^* \}_{i=0}^d) \) denote an idempotent system in \( \mathcal{A} \). We say that \( \Phi \) is symmetric whenever there exists an antiautomorphism \( \dagger \) of \( \mathcal{A} \) that fixes each of \( E_i, E_i^* \) for \( 0 \leq i \leq d \).

**Lemma 5.2** (See [14] Lemma 5.2.) Referring to Definition 5.1, the antiautomorphism \( \dagger \) is unique and \( (A^\dagger)^\dagger = A \) for \( A \in \mathcal{A} \).
Proposition 5.3 For a solid invertible \( R \in \text{Mat}_{d+1}(\mathbb{F}) \) the following are equivalent:

(i) \( R^t \) is diagonally equivalent to \( R^{-1} \);

(ii) the idempotent system \( \Phi_R \) is symmetric.

Suppose (i), (ii) hold. Let \( H, K \) denote invertible diagonal matrices in \( \text{Mat}_{d+1}(\mathbb{F}) \) such that \( R^t = HR^{-1}K \). Let \( \dagger \) denote the antiautomorphism of \( \text{Mat}_{d+1}(\mathbb{F}) \) corresponds to \( \Phi_R \). Then \( \dagger \) sends \( A \mapsto KA^tK^{-1} \) for \( A \in \text{Mat}_{d+1}(\mathbb{F}) \).

Proof. For \( 0 \leq i \leq d \) define

\[ E_i = \Delta_{i,i}, \quad \quad E_i^* = R\Delta_{i,i}R^{-1}. \]

Note that \( \Phi_R = \{ (E_i^d)_{i=0}^d; (E_i^* d)_{i=0}^d \} \).

(i) \( \Rightarrow \) (ii) Let \( H, K \) denote invertible diagonal matrices in \( \text{Mat}_{d+1}(\mathbb{F}) \) such that \( R^t = HR^{-1}K \). Let \( \dagger \) denote the antiautomorphism of \( \text{Mat}_{d+1}(\mathbb{F}) \) that sends \( A \mapsto KA^tK^{-1} \). We show that \( \dagger \) fixes each of \( E_i \) and \( E_i^* \) for \( 0 \leq i \leq d \). Using Lemma 2.1 we find that \( \dagger \) fixes \( E_i \) for \( 0 \leq i \leq d \). Observe that \( \dagger \) sends \( E_i^* \) to \( K(R^{-1})^t\Delta_{i,i}R^tK^{-1} \) for \( 0 \leq i \leq d \). Using \( R^t K^{-1} = HR^{-1} \) and Lemma 2.1

\[ K(R^{-1})^t\Delta_{i,i}R^tK^{-1} = R\Delta_{i,i}R^{-1} \]

(0 \leq i \leq d).

By these comments \( \dagger \) fixes \( E_i^* \) for \( 0 \leq i \leq d \). We have shown that \( \dagger \) fixes each of \( E_i \) and \( E_i^* \) for \( 0 \leq i \leq d \). Therefore \( \Phi_R \) is symmetric.

(ii) \( \Rightarrow \) (i) By Definition 5.1 there exists an antiautomorphism \( \dagger \) of \( \text{Mat}_{d+1}(\mathbb{F}) \) that fixes each of \( E_i \) and \( E_i^* \) for \( 0 \leq i \leq d \). By our comments in Section 2 there exists an invertible \( K \in \text{Mat}_{d+1}(\mathbb{F}) \) such that \( A^\dagger = KA^tK^{-1} \) for \( A \in \text{Mat}_{d+1}(\mathbb{F}) \). The matrix \( K \) is diagonal by Lemma 2.1 and since \( \dagger \) fixes \( E_i \) for \( 0 \leq i \leq d \). Observe that \( \dagger \) sends \( E_i^* \) to \( K(R^{-1})^t\Delta_{i,i}R^tK^{-1} \) for \( 0 \leq i \leq d \). By this and since \( \dagger \) fixes \( E_i^* \),

\[ R\Delta_{i,i}R^{-1} = K(R^{-1})^t\Delta_{i,i}R^tK^{-1} \quad (0 \leq i \leq d). \]

Thus \( R^t K^{-1} R \) commutes with \( \Delta_{i,i} \) for \( 0 \leq i \leq d \). By this and Lemma 2.1 \( R^t K^{-1} R \) is diagonal; denote this diagonal matrix by \( H \). Then \( R^t = HR^{-1}K \). So \( R^t \) is diagonally equivalent to \( R^{-1} \).

Suppose (i), (ii) hold. In the proof of (i) \( \Rightarrow \) (ii), we have shown the last assertion of the proposition statement.

In view of Proposition 5.3 we make a definition.

Definition 5.4 A matrix \( R \in \text{Mat}_{d+1}(\mathbb{F}) \) is said to be almost orthogonal (AO) whenever \( R \) is invertible and \( R^t \) is diagonally equivalent to \( R^{-1} \).

Lemma 5.5 Let \( R \) denote an AO matrix in \( \text{Mat}_{d+1}(\mathbb{F}) \). Then every matrix in \( \text{Mat}_{d+1}(\mathbb{F}) \) that is diagonally equivalent to \( R \) is AO.

Proof. By Definition 5.3 there exist invertible diagonal matrices \( H_1, K_1 \) in \( \text{Mat}_{d+1}(\mathbb{F}) \) such that \( R^t = H_1 R^{-1} K_1 \). Let \( S \) denote a matrix in \( \text{Mat}_{d+1}(\mathbb{F}) \) that is diagonally equivalent to \( R \). Then there exist invertible diagonal matrices \( H_2, K_2 \) in \( \text{Mat}_{d+1}(\mathbb{F}) \) such that
\[ S = H_2 R K_2. \] One routinely finds that \( S^t = K_2^2 H_1 S^{-1} K_1 H_2^2. \) Therefore \( S^t \) is diagonally equivalent to \( S^{-1}. \)

In the next result we classify up to isomorphism the symmetric idempotent systems.

**Theorem 5.6** Consider the following sets:

(i) the diagonal equivalence classes of AO solid invertible matrices in \( \text{Mat}_{d+1}(\mathbb{F}) \);

(ii) the isomorphism classes of symmetric idempotent systems over \( \mathbb{F} \) with diameter \( d. \)

The map \( \rho \mapsto \Phi_\rho \) is a bijection from (i) to (ii).

**Proof.** By Theorem 4.12, Proposition 5.3 and Lemma 5.5

6 Normalized solid invertible matrices

In Section 4 we classified the idempotent systems up to isomorphism. We showed that the isomorphism classes are in bijection with the diagonal equivalence classes of solid invertible matrices. In this section we introduce a type of solid invertible matrix, said to be normalized. We show that each diagonal equivalence class of solid invertible matrices contains a unique normalized element.

**Definition 6.1** A solid invertible matrix \( R \in \text{Mat}_{d+1}(\mathbb{F}) \) is said to be normalized whenever the following (i), (ii) hold:

(i) in column 0 of \( R \) all entries are equal to 1;

(ii) in column 0 of \( R^{-1} \) all entries are the same.

**Lemma 6.2** Let \( R \) denote a solid invertible matrix in \( \text{Mat}_{d+1}(\mathbb{F}) \) and let \( H, K \) denote invertible diagonal matrices in \( \text{Mat}_{d+1}(\mathbb{F}) \). Then \( HRK \) is normalized if and only if

\[ H_{r,r} = \frac{1}{R_{r,0}K_{0,0}}, \quad K_{r,r} = \frac{(R^{-1})_{r,0}K_{0,0}}{(R^{-1})_{0,0}} \quad (0 \leq r \leq d). \]  

**Proof.** Use (4) with \( s = 0 \), along with Definition 6.1

**Proposition 6.3** Each diagonal equivalence class of solid invertible matrices in \( \text{Mat}_{d+1}(\mathbb{F}) \) contains a unique normalized element.

**Proof.** Let \( R \) denote a solid invertible matrix in \( \text{Mat}_{d+1}(\mathbb{F}) \). We show that there exists a unique normalized solid invertible matrix in \( \text{Mat}_{d+1}(\mathbb{F}) \) that is diagonally equivalent to \( R \). Observe that there exist invertible diagonal matrices \( H, K \) in \( \text{Mat}_{d+1}(\mathbb{F}) \) that satisfy (5). Then \( HRK \) is a normalized solid invertible matrix in \( \text{Mat}_{d+1}(\mathbb{F}) \) that is diagonally equivalent to \( R \). Concerning uniqueness, let \( H', K' \) denote invertible diagonal matrices in \( \text{Mat}_{d+1}(\mathbb{F}) \) such that \( H'RK' \) is normalized. By Lemma 6.2 there exists a nonzero \( \alpha \in \mathbb{F} \) such that \( H' = \alpha H \) and \( K' = \alpha^{-1} K \). Therefore \( HRK = H'RK' \).
**Definition 6.4** For an idempotent system $\Phi$ over $\mathbb{F}$ with diameter $d$, let $[\Phi]$ denote the isomorphism class that contains $\Phi$.

**Corollary 6.5** Consider the following sets:

(i) the normalized solid invertible matrices in $\text{Mat}_{d+1}(\mathbb{F})$;

(ii) the isomorphism classes of idempotent systems over $\mathbb{F}$ with diameter $d$.

The map $R \mapsto [\Phi_R]$ is a bijection from (i) to (ii).

**Proof.** By Theorem 4.12 and Proposition 6.3.

**Definition 6.6** Let $\text{AON}_d(\mathbb{F})$ denote the set consisting of the AO normalized solid invertible matrices in $\text{Mat}_{d+1}(\mathbb{F})$. Let $\text{SIS}_d(\mathbb{F})$ denote the set consisting of the isomorphism classes of symmetric idempotent systems over $\mathbb{F}$ with diameter $d$.

**Corollary 6.7** The map $\text{AON}_d(\mathbb{F}) \rightarrow \text{SIS}_d(\mathbb{F})$, $R \mapsto [\Phi_R]$ is a bijection.

**Proof.** By Theorem 5.6 and Proposition 6.3.

In the next section we will consider the inverse of the bijection in Corollary 6.7.

### 7 The inverse of the bijection in Corollary 6.7

In this section we describe the inverse of the bijection in Corollary 6.7. Let $\Phi = (\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ denote a symmetric idempotent system in $\mathcal{A}$, and let the antiautomorphism $\dagger$ of $\mathcal{A}$ be from Definition 5.1. Let the algebra $\mathcal{M}$ be from Definition 3.2.

**Definition 7.1** (See [14, Definition 4.1].) For $0 \leq i \leq d$ define

$$m_i = \text{tr}(E_i^*E_0),$$

(6)

where $\text{tr}$ means trace.

**Lemma 7.2** (See [14, Lemma 4.4].) The following hold:

(i) $m_i \neq 0$ ($0 \leq i \leq d$);

(ii) $\sum_{i=0}^d m_i = 1$.

**Definition 7.3** (See [14, Definition 4.5].) Setting $i = 0$ in (6) we find that $m_0 = m_0^*$; let $\nu$ denote the multiplicative inverse of this common value. We call $\nu$ the size of $\Phi$. We emphasize that $\nu = \nu^*$.

**Lemma 7.4** (See [14, Lemmas 6.3, 7.4].) For $0 \leq i \leq d$ there exists a unique $A_i \in \mathcal{M}$ such that

$$A_iE_0^*E_0 = E_i^*E_0.$$
Lemma 7.5 (See [14 Lemma 7.5].) We have $A_0 = I$.

Lemma 7.6 (See [14 Lemma 7.7].) The elements $\{A_i\}_{i=0}^d$ form a basis for the vector space $M$.

By Lemma 7.6 there exist scalars $p_{ij}^h$ ($0 \leq h, i, j \leq d$) in $F$ such that

$$A_i A_j = \sum_{h=0}^d p_{ij}^h A_h \quad (0 \leq i, j \leq d). \quad (7)$$

Definition 7.7 For $0 \leq i \leq d$ define $k_i = \nu m_i^*$. 

Lemma 7.8 (See [14 Lemma 8.4].) The following hold:

(i) $k_i \neq 0$ ($0 \leq i \leq d$);

(ii) $\nu = \sum_{i=0}^d k_i$;

(iii) $k_0 = 1$.

Lemma 7.9 (See [14 Lemma 10.9].) For $0 \leq i, j \leq d$, $p_{ij}^0 = \delta_{i,j} k_i$.

Lemma 7.10 (See [14 Lemma 10.10].) For $0 \leq i, j \leq d$, $k_i k_j = \sum_{h=0}^d p_{ij}^h k_h$.

Definition 7.11 Each of $\{E_i\}_{i=0}^d$ and $\{A_i\}_{i=0}^d$ is a basis for the vector space $M$. Let $P = P_\Phi$ denote the transition matrix from $\{E_i\}_{i=0}^d$ to $\{A_i\}_{i=0}^d$. We call $P$ the first eigenmatrix of $\Phi$. Let $Q = Q_\Phi$ denote the first eigenmatrix of $\Phi^*$. We call $Q$ the second eigenmatrix of $\Phi$.

Lemma 7.12 (See [14 Lemmas 12.7, 12.8].) For $0 \leq i, j \leq d$ the following hold:

(i) $P_{i,0} = 1$;

(ii) $P_{0,j} = k_j$;

(iii) $(P^{-1})_{i,0} = \nu^{-1}$;

(iv) $(P^{-1})_{0,j} = \nu^{-1} k_j^*$. 

Definition 7.13 (See [14 Definition 14.1].) Let $K$ (resp. $K^*$) denote the diagonal matrix in $\text{Mat}_{d+1}(F)$ that has $(i, i)$-entry $k_i$ (resp. $k_i^*$) for $0 \leq i \leq d$.

Lemma 7.14 (See [14 Lemma 14.2].) The following hold:

(i) $PQ = \nu I$;

(ii) $P^* K^* = KQ$. 

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Lemma 7.15 (See [14, Lemma 14.4, Proposition 18.1].) There exists an algebra isomorphism \( A \rightarrow \text{Mat}_{d+1}(F) \) that sends \( E_i \mapsto \Delta_{i,i} \) and \( E_i^* \mapsto P\Delta_{i,i}P^{-1} \) for \( 0 \leq i \leq d \).

Corollary 7.16 The first eigenmatrix \( P \) is AO normalized solid invertible. Moreover the idempotent system \( \Phi_P \) is isomorphic to \( \Phi \).

**Proof.** By Lemma 7.12 \( P \) is normalized solid invertible. By Lemma 7.14 \( P \) is AO. By Lemma 7.15 \( \Phi_P \) is isomorphic to \( \Phi \). \( \square \)

Proposition 7.17 Referring to the bijection in Corollary 6.7, the inverse bijection sends \([\Phi] \mapsto P_{\Phi} \).

**Proof.** By Corollary 7.16 \( \square \)

Corollary 7.18 Two symmetric idempotent systems over \( F \) are isomorphic if and only if they have the same first eigenmatrix.

**Proof.** By Proposition 7.17 \( \square \)

8 Character algebras

Our next goal is to explain how AO normalized solid invertible matrices and symmetric idempotent systems are related to character algebras. Traditionally a character algebra is defined over the complex number field \([3, 13]\). In the present paper we define a character algebra over an arbitrary field.

Definition 8.1 (See [3, Section II.2.5].) By a character algebra over \( F \) with diameter \( d \) we mean a sequence

\[(C; \{x_i\}_{i=0}^d),\]

where \( C \) is a commutative \( F \)-algebra, and \( \{x_i\}_{i=0}^d \) are elements in \( C \) that satisfy the following (i)--(iv).

(i) \( x_0 = 1 \).

(ii) \( \{x_i\}_{i=0}^d \) is a basis of the vector space \( C \).

(iii) Define scalars \( p_{ij}^h \) \( (0 \leq h, i, j \leq d) \) such that

\[x_i x_j = \sum_{h=0}^d p_{ij}^h x_h \quad (0 \leq i, j \leq d). \] (8)

Then there exist nonzero scalars \( \{k_i\}_{i=0}^d \) such that

\[p_{ij}^0 = \delta_{i,j} k_i \quad (0 \leq i, j \leq d). \] (9)

(iv) There exists an algebra homomorphism \( \varphi : C \rightarrow F \) such that \( \varphi(x_i) = k_i \) for \( 0 \leq i \leq d \). For historical reasons, we call the scalars \( p_{ij}^h \) the intersection numbers.

We refer the reader to \([2, 5, 11, 13, 15]\) for background information on character algebras.
Next we discuss the notion of isomorphism for character algebras. Consider two character algebras \((C; \{x_i\}_{i=0}^{d})\) and \((C'; \{x'_i\}_{i=0}^{d})\) over \(F\). By an isomorphism of character algebras from \((C; \{x_i\}_{i=0}^{d})\) to \((C'; \{x'_i\}_{i=0}^{d})\) we mean an algebra isomorphism \(C \rightarrow C'\) that sends \(x_i \mapsto x'_i\) for \(0 \leq i \leq d\). The character algebras \((C; \{x_i\}_{i=0}^{d})\) and \((C'; \{x'_i\}_{i=0}^{d})\) are said to be isomorphic whenever there exists an isomorphism of character algebras from \((C; \{x_i\}_{i=0}^{d})\) to \((C'; \{x'_i\}_{i=0}^{d})\).

**Lemma 8.2** Two character algebras over \(F\) are isomorphic if and only if they have the same intersection numbers.

**Proof.** Use (S). \(\square\)

**Lemma 8.3** Referring to the character algebra in Definition (5) the intersection numbers satisfy the following (i)--(iii):

(i) \(k_0 = 1\);

(ii) \(p^h_{0i} = \delta_{h,i}\) for \(0 \leq h, i \leq d\);

(iii) \(p^h_{ij} = p^h_{ji}\) for \(0 \leq h, i, j \leq d\).

**Proof.** (i), (ii) Since \(x_0 = 1\).

(iii) Since \(C\) is commutative. \(\square\)

As an illustration, we describe the character algebras of diameter \(d = 1\).

**Lemma 8.4** For a character algebra \((C; \{x_i\}_{i=0}^{1})\) over \(F\), the intersection numbers satisfy the following:

(i) \(p^1_{11} = k_1 - 1\);

(ii) \(x_{1}^2 = k_1 x_0 + (k_1 - 1)x_1\);

(iii) \(e^2 = (k_1 + 1)e\), where \(e = x_0 + x_1\).

**Proof.** By (9), \(p^0_{11} = k_1\). By this and (8),

\[
x_{1}^2 = k_1 x_0 + p^{1}_{11} x_1.
\]  

In this equation, apply \(\varphi\) to each side to get \(k^2_1 = k_1 k_0 + p^{1}_{11} k_1\). By this and since \(k_0 = 1\), \(k_1 \neq 0\) we get (i). By (i) and (10) we get (ii). Using (ii) we get (iii). \(\square\)

In the next result, we classify up to isomorphism the character algebras with diameter \(d = 1\).
Proposition 8.5 Let $0 \neq k \in \mathbb{F}$. Then up to isomorphism there exists a unique character algebra over $\mathbb{F}$ that has diameter $d = 1$ and $k_1 = k$.

Proof. First we show the uniqueness. By (9) and Lemma 8.3, we find that all the intersection numbers are uniquely determined by $k_1$ and $p_{11}^1$. By Lemma 8.4(i), $p_{11}^1$ is determined by $k_1$. By these comments, all the intersection numbers are determined by $k_1$.

Now the uniqueness follows by Lemma 8.2. Next we show the existence. Consider the quotient algebra $\mathcal{C} = \mathbb{F}[x]/I$, where $\mathbb{F}[x]$ is the $\mathbb{F}$-algebra of polynomials in a variable $x$, and $I$ is the ideal of $\mathbb{F}[x]$ generated by $(x + 1)(x - k)$. Define $x_0 = 1 + I$ and $x_1 = x + I$. We show that $(\mathcal{C}; \{x_i\}_{i=0}^d)$ is a character algebra. To do this, we verify conditions (i) – (iv) in Definition 8.1. Condition (i) holds since by construction $x_0$ is the multiplicative identity in $\mathcal{C}$. Condition (ii) holds since by construction $x_0, x_1$ form a basis for $\mathcal{C}$. Now consider condition (iii). We just mentioned that $x_0$ is the identity in $\mathcal{C}$. By construction $x_2 = kx_0 + (k - 1)x_1$. By these comments, $p_{00}^0 = 1$, $p_{01}^0 = p_{10}^0 = 0$, $p_{11}^0 = k$.

Thus (9) holds with $k_0 = 1$, $k_1 = k$. We have verified condition (iii). Concerning condition (iv), note that $k$ is a root of the polynomial $(x + 1)(x - k)$, so there exists an algebra homomorphism $\varphi : \mathcal{C} \to \mathbb{F}$ that sends $x_1$ to $k$. The map $\varphi$ satisfies the requirements of condition (iv). We have shown that $(\mathcal{C}; \{x_i\}_{i=0}^d)$ is a character algebra. By construction this character algebra has $k_1 = k$. $\blacksquare$

9 Semisimple character algebras and character systems

In this section we discuss a type of character algebra, said to be semisimple. Motivated by this type of character algebra, we introduce the notion of a character system.

Definition 9.1 A character algebra $(\mathcal{C}; \{x_i\}_{i=0}^d)$ is said to be semisimple whenever there exists a basis $\{e_i\}_{i=0}^d$ of the vector space $\mathcal{C}$ such that $e_ie_j = \delta_{i,j}e_i$ ($0 \leq i, j \leq d$) and $1 = \sum_{i=0}^d e_i$. In this case, $\{e_i\}_{i=0}^d$ are unique up to permutation, and called the primitive idempotents of $\mathcal{C}$.

We are mainly interested in the semisimple character algebras.

Note 9.2 Consider a character algebra from Definition 8.1. In [3 Section II.2.5], the intersection numbers are assumed to be real with $k_i > 0$ ($0 \leq i \leq d$), and under this assumption, it is shown that the character algebra is semisimple [3 Proposition II.5.4].

Lemma 9.3 Let $(\mathcal{C}; \{x_i\}_{i=0}^d)$ denote a semisimple character algebra over $\mathbb{F}$. Then there exists a unique primitive idempotent of $\mathcal{C}$ that is not sent to zero by $\varphi$. This primitive idempotent is sent to $1$ by $\varphi$.

Proof. Concerning the existence, observe that the sum of the primitive idempotents of $\mathcal{C}$ is equal to the multiplicative identity of $\mathcal{C}$, and that $\varphi$ sends this identity to $1$. Concerning
the uniqueness, let $e_0$ denote a primitive idempotent of $C$ such that $\varphi(e_0) \neq 0$. Pick any other primitive idempotent $f$ in $C$. We have $e_0f = 0$, so $0 = \varphi(e_0f) = \varphi(e_0)\varphi(f)$. By this and $\varphi(e_0) \neq 0$ we get $\varphi(f) = 0$. We have shown the uniqueness. The last assertion of the lemma statement follows from our above remarks.

**Definition 9.4** Referring to Lemma 9.3 let $e_0$ denote the unique primitive idempotent of $C$ that is not sent to 0 by $\varphi$. The primitive idempotent $e_0$ is said to be trivial. Note that $\varphi(e_0) = 1$.

In Proposition 8.5 we described the character algebras with diameter $d = 1$. In the next result we determine which of these character algebras is semisimple.

**Lemma 9.5** Let $(C, \{x_i\}_{i=0}^1)$ denote the character algebra from Proposition 8.5. Then $(C, \{x_i\}_{i=0}^1)$ is semisimple if and only if $k \neq -1$. In this case the primitive idempotents satisfy

\[
\begin{align*}
x_0 &= e_0 + e_1, & x_1 &= ke_0 - e_1, \\
e_0 &= \frac{x_0 + x_1}{k+1}, & e_1 &= \frac{ke_0 - x_1}{k+1}.
\end{align*}
\]

**Proof.** To prove the lemma in one direction, we assume that $k = -1$ and show that $(C, \{x_i\}_{i=0}^1)$ is not semisimple. To show this, we assume that $(C, \{x_i\}_{i=0}^1)$ is semisimple and get a contradiction. By Lemma 8.4(iii) we have $e^2 = 0$, where $e = x_0 + x_1$. Write $e = \alpha_0e_0 + \alpha_1e_1$, where $\alpha_0, \alpha_1 \in F$ and $e_0, e_1$ are the primitive idempotents of $C$. Using $e^2 = 0$ we find $\alpha_0^2 = 0$ and $\alpha_1^2 = 0$, forcing $\alpha_0 = \alpha_1 = 0$ so $e = 0$, contradicting the fact that $x_0, x_1$ are linearly independent. We have proved the lemma in one direction. To prove the lemma in the other direction, we assume that $k \neq -1$ and show that $(C, \{x_i\}_{i=0}^1)$ is semisimple. Define elements $\{e_i\}_{i=0}^1$ by (12). Using Lemma 8.4 with $k_1 = k$ we find that $e_0, e_1$ form a basis for $C$ such that

\[
e_0 + e_1 = 1, \quad e_0^2 = e_0, \quad e_0e_1 = 0, \quad e_1^2 = e_1.
\]

By these comments $(C, \{x_i\}_{i=0}^1)$ is semisimple with primitive idempotents $e_0, e_1$. Line (11) is obtained from (12). \hfill \square

**Definition 9.6** By a character system over $F$ with diameter $d$, we mean a sequence

$\Psi = (C; \{x_i\}_{i=0}^d; \{e_i\}_{i=0}^d)$,

where $(C, \{x_i\}_{i=0}^d)$ is a semisimple character algebra over $F$, and $\{e_i\}_{i=0}^d$ are the primitive idempotents of $C$ with $e_0$ trivial. We say that $(C; \{x_i\}_{i=0}^d)$ and $\Psi$ are associated.

Next we discuss the notion of isomorphism for character systems. Suppose we are given two character systems over $F$, denoted $\Psi = (C; \{x_i\}_{i=0}^d; \{e_i\}_{i=0}^d)$ and $\Psi' = (C'; \{x'_i\}_{i=0}^d; \{e'_i\}_{i=0}^d)$. By an **isomorphism of character systems** from $\Psi$ to $\Psi'$ we mean an algebra isomorphism $C \to C'$ that sends $x_i \mapsto x'_i$ and $e_i \mapsto e'_i$ for $0 \leq i \leq d$. We say $\Psi$ and $\Psi'$ are isomorphic whenever there exists an isomorphism of character systems from $\Psi$ to $\Psi'$. 

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Definition 9.7 Referring to the character system $\Psi$ in Definition 9.6 let $[\Psi]$ denote the isomorphism class that contains $\Psi$.

Definition 9.8 Let $\text{CS}_d(F)$ denote the set consisting of the isomorphism classes of character systems over $F$ with diameter $d$.

Definition 9.9 (See [3, p. 90].) Let $\Psi = (\mathfrak{C}; \{x_i\}_{i=0}^d; \{e_i\}_{i=0}^d)$ denote a character system over $F$. Let $P = P_\Psi$ denote the transition matrix from the basis $\{e_i\}_{i=0}^d$ of $\mathfrak{C}$ to the basis $\{x_i\}_{i=0}^d$ of $\mathfrak{C}$. We call $P$ the eigenmatrix of $\Psi$.

Lemma 9.10 For the character system associated with the semisimple character algebra in Lemma 9.5, the eigenmatrix $P$ satisfies

$$P = \begin{pmatrix} 1 & k \\ 1 & -1 \end{pmatrix}, \quad P^{-1} = \frac{1}{k+1} \begin{pmatrix} 1 & k \\ 1 & -1 \end{pmatrix}. $$

Proof. By (11) and (12). □

Proposition 9.11 Two character systems over $F$ are isomorphic if and only if they have the same eigenmatrix.

Proof. Let $\Psi, \Psi'$ denote the character systems in question. First assume that $\Psi$ and $\Psi'$ are isomorphic. Then $\Psi$ and $\Psi'$ have the same eigenmatrix by Definition 9.9. Next assume that $\Psi$ and $\Psi'$ have the same eigenmatrix. Then $\Psi$ and $\Psi'$ have the same diameter. Write $\Psi = (\mathfrak{C}; \{x_i\}_{i=0}^d; \{e_i\}_{i=0}^d)$ and $\Psi' = (\mathfrak{C}'; \{x'_i\}_{i=0}^d; \{e'_i\}_{i=0}^d)$. Consider the $F$-linear map $\gamma: \mathfrak{C} \rightarrow \mathfrak{C}'$ that sends $e_i \mapsto e'_i$ for $0 \leq i \leq d$. Then $\gamma$ is an algebra isomorphism. By Definition 9.9 $\gamma$ sends $x_i \mapsto x'_i$ for $0 \leq i \leq d$. Thus $\gamma$ is an isomorphism of character systems from $\Psi$ to $\Psi'$. Therefore the character systems $\Psi$ and $\Psi'$ are isomorphic. □

Lemma 9.12 Let $\Psi = (\mathfrak{C}; \{x_i\}_{i=0}^d; \{e_i\}_{i=0}^d)$ denote a character system over $F$. Then the eigenmatrix $P = P_\Psi$ satisfies the following (i), (ii) for $0 \leq i, j \leq d$:

(i) $P_{i,0} = 1$;

(ii) $P_{0,j} = k_j$.

Proof. (i) Since $x_0 = 1 = \sum_{i=0}^d e_i$.

(ii) We have $x_j = \sum_{i=0}^d P_{i,j} e_i$. In this equation, apply $\varphi$ to each side. By Definition 8.1(iv) we have $\varphi(x_j) = k_j$, and by Definition 9.4 we have $\varphi(e_i) = \delta_{i,0}$ for $0 \leq i \leq d$. By these comments we get the result. □
10 Some properties of the eigenmatrix of a character system

In this section we show that the eigenmatrix of a character system is AO normalized solid invertible. Throughout this section let \((\mathfrak{C}; \{x_i\}_{i=0}^d)\) denote a character algebra over \(\mathbb{F}\). The following definition is a variation on [11, p. 145].

**Definition 10.1** Define a bilinear form \(\langle \ , \rangle : \mathfrak{C} \times \mathfrak{C} \to \mathbb{F}\) such that

\[
\langle x_i, x_j \rangle = \delta_{i,j} k_i \quad (0 \leq i, j \leq d).
\]  

(13)

**Lemma 10.2** The bilinear form \(\langle \ , \rangle\) is symmetric and nondegenerate.

**Proof.** By [13] and since \(k_i \neq 0\) for \(0 \leq i \leq d\).

The following result is a variation on [4, Proposition 2.5].

**Lemma 10.3** For \(u, v \in \mathfrak{C}\), \(\langle u, v \rangle = \langle uv, x_0 \rangle\).

**Proof.** Without loss of generality, we assume that \(u = x_i\) and \(v = x_j\) for some integers \(i, j \leq d\). Using (8), (13), (9), (13) in order,

\[
\langle x_i x_j, x_0 \rangle = \sum_{h=0}^d p_{ij}^h \langle x_h, x_0 \rangle = \sum_{h=0}^d p_{ij}^h \delta_{h,0} k_0 = p_{ij}^0 = \delta_{i,j} k_i = \langle x_i, x_j \rangle.
\]

The result follows. \(\square\)

For the rest of this section, assume that \((\mathfrak{C}; \{x_i\}_{i=0}^d)\) is semisimple, and let \(\Psi = (\mathfrak{C}; \{x_i\}_{i=0}^d; \{e_i\}_{i=0}^d)\) denote an associated character system. Recall the eigenmatrix \(P = P_\Psi\) from Definition 9.9

**Lemma 10.4** The basis \(\{e_i\}_{i=0}^d\) of \(\mathfrak{C}\) is orthogonal with respect to \(\langle \ , \rangle\).

**Proof.** For \(0 \leq i, j \leq d\) with \(i \neq j\) we have \(e_i e_j = 0\). By this and Lemma 10.3 we have \(\langle e_i, e_j \rangle = \langle e_i e_j, x_0 \rangle = 0\). The result follows. \(\square\)

**Definition 10.5** For \(0 \leq i \leq d\) define \(m_i = \langle e_i, e_i \rangle\) and note that \(m_i\) is nonzero. Define \(\nu = m_0^{-1}\). We call \(\nu\) the size of \(\Psi\).

**Definition 10.6** For \(0 \leq i \leq d\) define \(k_i^* = \nu m_i\). By construction \(k_0^* = 1\) and

\[
\langle e_i, e_i \rangle = \nu^{-1} k_i^* \quad (0 \leq i \leq d).
\]  

(14)

**Lemma 10.7** For \(0 \leq i, j \leq d\),

\[
\langle x_i, e_j \rangle = \nu^{-1} P_{ji} k_j^* = k_i (P^{-1})_{i,j}.
\]  

(15)

**Proof.** We have \(\langle x_i, e_j \rangle = P_{ji} \langle e_j, e_j \rangle\) by Lemma 10.4 and since \(P\) is the transition matrix from \(\{e_i\}_{i=0}^d\) to \(\{x_i\}_{i=0}^d\). We have \(\langle x_i, e_j \rangle = (P^{-1})_{i,j} \langle x_i, x_i \rangle\) by (13) and since \(P^{-1}\) is the transition matrix from \(\{x_i\}_{i=0}^d\) to \(\{e_i\}_{i=0}^d\). By these comments and (13), (14) we get the result. \(\square\)
Lemma 10.8 For $x \in \mathbb{C}$ we have $\varphi(x) = \nu \langle x, e_0 \rangle$.

Proof. Write $x = \sum_{i=0}^{d} \alpha_i e_i$ with $\alpha_i \in \mathbb{F}$ for $0 \leq i \leq d$. By Definition 9.4 we get $\varphi(x) = \alpha_0$. Using Lemma 10.4 and Definition 10.5, $\langle x, e_0 \rangle = \alpha_0 \langle e_0, e_0 \rangle = \alpha_0 \nu^{-1}$. By these comments we get the result. $\square$

Lemma 10.9 We have $\sum_{i=0}^{d} x_i = \nu e_0$.

Proof. Write $e_0 = \sum_{\ell=0}^{d} \alpha_\ell x_\ell$ with $\alpha_\ell \in \mathbb{F}$ for $0 \leq \ell \leq d$. Pick an integer $i$ ($0 \leq i \leq d$). In the previous equation, take the inner product with $x_i$, and simplify the result using (13) to get $\langle x_i, e_0 \rangle = \alpha_i k_i$. By Definition 8.1(iv) and Lemma 10.8, $k_i = \varphi(x_i) = \nu \langle x_i, e_0 \rangle$. By these comments, $\alpha_i = \nu^{-1}$. The result follows. $\square$

Lemma 10.10 For $0 \leq i, j \leq d$ the following hold:

(i) $(P^{-1})_{i,0} = \nu^{-1}$;

(ii) $(P^{-1})_{0,j} = \nu^{-1} k_0^*$.

Proof. (i) Set $j = 0$ in (15) and evaluate the result using $k_0^* = 1$ along with Lemma 9.12(ii).

(ii) Set $i = 0$ in (15) and evaluate the result using $k_0 = 1$ along with Lemma 9.12(i). $\square$

The following two propositions are variations on [3, Theorem II.5.5].

Proposition 10.11 The eigenmatrix $P$ of $\Psi$ is normalized solid invertible.

Proof. By Lemmas 9.12 and 10.10 $\square$

Proposition 10.12 The eigenmatrix $P$ of $\Psi$ is AO.

Proof. By Definition 5.4 it suffices to show that $P^{-1}$ and $P^t$ are diagonally equivalent. Let $K$ (resp. $K^*$) denote the diagonal matrix in $\text{Mat}_{d+1}(\mathbb{F})$ that has $(i, i)$-entry $k_i$ (resp. $k_i^*$) for $0 \leq i \leq d$. Note that $K$ and $K^*$ are invertible. By (15) we have $\nu^{-1} P^t K^* = K P^{-1}$, and so $P^{-1} = \nu^{-1} K^{-1} P^t K^*$. Thus $P^{-1}$ and $P^t$ are diagonally equivalent. The result follows. $\square$

By Propositions 9.11, 10.11, 10.12 there exists a map $CS_d(\mathbb{F}) \to AON_d(\mathbb{F})$ that sends $[\Psi] \mapsto P_\Psi$. In the next section we will show that this map is a bijection.
11 Symmetric idempotent systems, character systems, and AO normalized solid invertible matrices

Recall the sets $\text{AON}_d(\mathbb{F})$, $\text{SIS}_d(\mathbb{F})$ from Definition 6.6 and the set $\text{CS}_d(\mathbb{F})$ from Definition 9.8. As we mentioned in Section 1, our goal is to show that these three sets are mutually in bijection, and to describe the bijections involved. In Sections 6 and 7, we obtained a bijection $\text{AON}_d(\mathbb{F}) \rightarrow \text{SIS}_d(\mathbb{F})$, and described its inverse. At the end of Section 10 we obtained a map $\text{CS}_d(\mathbb{F}) \rightarrow \text{AON}_d(\mathbb{F})$. In the present section we show that this map is a bijection, and describe its inverse. We also describe the bijective correspondence between $\text{SIS}_d(\mathbb{F})$ and $\text{CS}_d(\mathbb{F})$.

Proposition 11.1 Let $\Phi = (\{E_i\}_{i=0}^d; \{E^*_i\}_{i=0}^d)$ denote a symmetric idempotent system over $\mathbb{F}$. Define a sequence

$$\Psi = (\{M_i\}_{i=0}^d; \{E_i\}_{i=0}^d),$$

where the algebra $M$ is from Definition 7.2 and the elements $\{A_i\}_{i=0}^d$ are from Lemma 7.4. Then $\Psi$ is a character system over $\mathbb{F}$ whose eigenmatrix is the first eigenmatrix of $\Phi$.

Proof. We first show that $(\{M_i\}_{i=0}^d; \{A_i\}_{i=0}^d)$ is a character algebra over $\mathbb{F}$. We verify the conditions (i)–(iv) in Definition 8.1. Condition (i) holds by Lemma 7.5 and condition (ii) holds by Lemma 7.6. Recall the scalars $\{k_i\}_{i=0}^d$ from Definition 7.7 and the scalars $p_{ij}^h$ from (7). Condition (iii) holds by Lemmas 7.8(i) and 7.9. The elements $\{E_i\}_{i=0}^d$ form a basis of $M$, so there exists an $\mathbb{F}$-linear map $\varphi : M \rightarrow \mathbb{F}$ such that $\varphi(E_i) = \delta_{i,0}$ for $0 \leq i \leq d$. One routinely checks that $\varphi$ is an algebra homomorphism. Recall the first eigenmatrix $P = P_{\Phi}$ from Definition 7.11. By Definition 7.11, $A_j = \sum_{i=0}^d P_{i,j}E_i$ for $0 \leq j \leq d$. In this equation, apply $\varphi$ to each side to find that $\varphi(A_j) = P_{i,j}^0$ for $0 \leq j \leq d$. By this and Lemma 7.12(ii), $\varphi(A_j) = k_j$ for $0 \leq j \leq d$. Thus condition (iv) holds. We have shown that $(\{M_i\}_{i=0}^d; \{A_i\}_{i=0}^d)$ is a character algebra over $\mathbb{F}$. By construction, $\{E_i\}_{i=0}^d$ are the primitive idempotents of $M$. So $(\{M_i\}_{i=0}^d; \{A_i\}_{i=0}^d)$ is semisimple. By construction, $\varphi(E_i) = \delta_{i,0}$ for $0 \leq i \leq d$, so the primitive idempotent $E_0$ is trivial. Now $\Psi$ is a character system over $\mathbb{F}$, in view of Definition 9.6. By Definitions 7.11, 9.9, $P$ is the eigenmatrix of $\Psi$. The result follows. \hfill \square

By Proposition 11.1 together with Corollary 7.18 and Proposition 9.11 we have a map $\text{SIS}_d(\mathbb{F}) \rightarrow \text{CS}_d(\mathbb{F})$, $[\Phi] \mapsto [\Psi]$. Proposition 11.2 The following (i)–(iii) hold.

(i) The map $\text{SIS}_d(\mathbb{F}) \rightarrow \text{AON}_d(\mathbb{F})$, $[\Phi] \mapsto P_{\Phi}$ is equal to the composition

$$\text{SIS}_d(\mathbb{F}) \xrightarrow{[\Phi] \mapsto [\Psi]} \text{CS}_d(\mathbb{F}) \xrightarrow{[\Psi] \mapsto P_{\Psi}} \text{AON}_d(\mathbb{F}).$$

(ii) The map $\text{CS}_d(\mathbb{F}) \rightarrow \text{AON}_d(\mathbb{F})$, $[\Psi] \mapsto P_{\Psi}$ is a bijection.

(iii) The map $\text{SIS}_d(\mathbb{F}) \rightarrow \text{CS}_d(\mathbb{F})$, $[\Phi] \mapsto [\Psi]$ is a bijection.

Proof. (i) By the last assertion of Proposition 11.1.
(ii) By Proposition 7.17, the map $SIS_d(F) \to AON_d(F)$, $[\Phi] \mapsto P_\Phi$ is bijective. By this and (i) above, the map $CS_d(F) \to AON_d(F)$, $[\Psi] \mapsto P_\Psi$ is surjective. By Proposition 9.11 the map $CS_d(F) \to AON_d(F)$, $[\Psi] \mapsto P_\Psi$ is injective. The result follows.

(iii) By (i), (ii) above and since the map $SIS_d(F) \to AON_d(F)$, $[\Phi] \mapsto P_\Phi$ is a bijection.

Our next goal is to describe the inverse of the bijection in Proposition 11.2(ii).

Lemma 11.3 Let $P$ denote an AO normalized solid invertible matrix in $\text{Mat}_{d+1}(F)$. Let $\{e_i\}_{i=0}^d$ denote indeterminates, and let $\mathcal{C}$ denote the vector space over $F$ with basis $\{e_i\}_{i=0}^d$. Turn $\mathcal{C}$ into an algebra such that $e_ie_j = \delta_{i,j}e_i$ for $0 \leq i, j \leq d$. View $P$ as the transition matrix from the basis $\{e_i\}_{i=0}^d$ of $\mathcal{C}$ to a basis $\{x_i\}_{i=0}^d$ of $\mathcal{C}$. Then $\Psi_P = (\mathcal{C}; \{x_i\}_{i=0}^d; \{e_i\}_{i=0}^d)$ is a character system over $F$ that has eigenmatrix $P$.

Proof. Recall the idempotent system $\Phi = \Phi_P$ from Definition 4.6 and the character system $\Psi_\Phi$ from Proposition 11.1. Note by Corollary 7.16 that $\Phi$ has first eigenmatrix $P$. By this and Proposition 11.1, $\Psi_\Phi$ has eigenmatrix $P$. By construction

$$\Psi_\Phi = (\mathcal{D}; \{A_i\}_{i=0}^d; \{E_i\}_{i=0}^d),$$

where $E_i = \Delta_{i,i}$ for $0 \leq i \leq d$, and $\{A_i\}_{i=0}^d$ are from Lemma 7.4. Consider the $F$-linear map $\pi : \mathcal{C} \to \mathcal{D}$ that sends $e_i \mapsto E_i$ for $0 \leq i \leq d$. By construction, $\pi$ is an algebra isomorphism. We have $\pi(x_i) = A_i$ for $0 \leq i \leq d$, since $P$ is the transition matrix from $\{e_i\}_{i=0}^d$ to $\{x_i\}_{i=0}^d$ and also the transition matrix from $\{E_i\}_{i=0}^d$ to $\{A_i\}_{i=0}^d$. Therefore $\Psi_P$ is a character system over $F$, and $\pi$ is an isomorphism of character systems from $\Psi_P$ to $\Psi_\Phi$. The result follows.

Proposition 11.4 The following maps are inverses:

(i) $AON_d(F) \to CS_d(F)$, $P \mapsto [\Psi_P]$;

(ii) $CS_d(F) \to AON_d(F)$, $[\Psi] \mapsto P_\Psi$.

Proof. By Proposition 11.2 the map (ii) is a bijection. Pick any AO normalized solid invertible $P \in \text{Mat}_{d+1}(F)$. By Lemma 11.3 $\Psi_P$ has eigenmatrix $P$. So the composition of (i) and (ii) is the identity. By these comments we get the result.

Our next goal is to describe the inverse of the bijection in Proposition 11.2(iii).
Lemma 11.5 Let $\Psi = (\mathcal{C}; \{x_i\}_{i=0}^d; \{e_i\}_{i=0}^d)$ denote a character system over $\mathbb{F}$. For $0 \leq i \leq d$ define $E_i, E_i^* \in \text{End}(\mathcal{C})$ such that

$$E_i e_j = \delta_{i,j} e_j, \quad E_i^* x_j = \delta_{i,j} x_j$$

for $0 \leq j \leq d$. Then the sequence

$$\Phi\Psi = (\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$$

is a symmetric idempotent system in $\text{End}(\mathcal{C})$ whose first eigenmatrix is the eigenmatrix of $\Psi$.

Proof. Recall that $P = P_\Psi$ is the transition matrix from the basis $\{e_i\}_{i=0}^d$ of $\mathcal{C}$ to the basis $\{x_i\}_{i=0}^d$ of $\mathcal{C}$. By Propositions 10.11, 10.12, $P$ is AO normalized solid invertible. By this and Theorem 5.6, $\Phi P$ is an idempotent system in $\text{Mat}_{d+1}(\mathbb{F})$. We show that $\Phi P$ is isomorphic to $\Phi P$. Let $\pi : \text{End}(\mathcal{C}) \to \text{Mat}_{d+1}(\mathbb{F})$ denote the algebra isomorphism that sends each $A \in \text{End}(\mathcal{C})$ to the matrix in $\text{Mat}_{d+1}(\mathbb{F})$ that represents $A$ with respect to the basis $\{e_i\}_{i=0}^d$. By the equation on the left in (16), $\pi(E_i) = \Delta_i,i$ for $0 \leq i \leq d$. By the equation on the right in (16) along with linear algebra we obtain $\pi(E_i^*) = P\Delta_i,iP^{-1}$ for $0 \leq i \leq d$. By these comments, $\Phi P$ is an idempotent system in $\text{End}(\mathcal{C})$, and $\pi$ is an isomorphism of idempotent systems from $\Phi P$ to $\Phi P$. The idempotent system $\Phi P$ is symmetric since $\Phi P$ is symmetric. By Corollary 7.16, $\Phi P$ has first eigenmatrix $P$. Thus $\Phi P$ has first eigenmatrix $P$. 

By Corollary 7.18, Proposition 9.11, and Lemma 11.5, we have a map $\text{CS}_d(\mathbb{F}) \to \text{SIS}_d(\mathbb{F})$, $[\Psi] \mapsto [\Phi P]$.

Proposition 11.6 The following maps are inverses:

(i) $\text{SIS}_d(\mathbb{F}) \to \text{CS}_d(\mathbb{F})$, $[\Phi] \mapsto [\Psi]$;

(ii) $\text{CS}_d(\mathbb{F}) \to \text{SIS}_d(\mathbb{F})$, $[\Psi] \mapsto [\Phi P]$.

Proof. By Corollary 7.18 and Proposition 9.11.

In summary, we have shown that the three sets $\text{AON}_d(\mathbb{F})$, $\text{SIS}_d(\mathbb{F})$, $\text{CS}_d(\mathbb{F})$ are mutually in bijection, and we described the bijections involved. They are displayed in the following diagram.

```
\begin{tikzpicture}
\node (A) {$\text{AON}_d(\mathbb{F})$};
\node (B) [above of=A] {$\text{P} \rightarrow [\Phi_P]$};
\node (C) [right of=A] {$\text{SIS}_d(\mathbb{F})$};
\node (D) [above of=C] {$\text{P}_\Psi \leftarrow [\Psi]$};
\node (E) [below of=A] {$\text{AON}_d(\mathbb{F})$};
\node (F) [below of=B] {$\text{P}_\Psi \leftarrow [\Psi]$};
\node (G) [left of=C] {$\text{id}$};
\node (H) [above of=G] {$\text{[\Phi_P]}$};
\node (I) [below of=G] {$\text{[\Psi]}$};
\node (J) [right of=G] {$\text{[\Psi]}$};
\node (K) [above of=J] {$\text{[\Phi]}$};
\node (L) [below of=J] {$\text{[\Psi_P]}$};
\draw[->] (A) -- (B) node[midway,above] {$P \rightarrow [\Phi_P]$};
\draw[->] (B) -- (C) node[midway,above] {$\text{P}_\Psi \leftarrow [\Psi]$};
\draw[->] (C) -- (D) node[midway,above] {$\text{[\Phi_P]}$};
\draw[->] (D) -- (E) node[midway,above] {$\text{P}_\Psi \leftarrow [\Psi]$};
\draw[->] (E) -- (F) node[midway,above] {$\text{P} \rightarrow [\Phi_P]$};
\draw[->] (F) -- (G) node[midway,above] {$\text{id}$};
\draw[->] (G) -- (H) node[midway,above] {$\text{[\Phi_P]}$};
\draw[->] (H) -- (I) node[midway,above] {$\text{id}$};
\draw[->] (I) -- (J) node[midway,above] {$\text{[\Phi]}$};
\draw[->] (J) -- (L) node[midway,above] {$\text{[\Psi]}$};
\end{tikzpicture}
```
12 Duality

Recall the sets $\text{AON}_d(F)$, $\text{SIS}_d(F)$ from Definition 9.6 and the set $\text{CS}_d(F)$ from Definition 9.8 So far, we have shown that these three sets are mutually in bijection, and we described the bijections involved. For an idempotent system $\Phi$ we have its dual $\Phi^*$ from below Definition 3.1. By construction we have the bijection $\text{SIS}_d(F) \to \text{SIS}_d(F)$, $[\Phi] \mapsto [\Phi^*]$, which we call the duality map on $\text{SIS}_d(F)$. The sets $\text{AON}_d(F)$, $\text{CS}_d(F)$ each inherit a duality map via the above bijections. In this section we describe these duality maps in detail.

**Definition 12.1** By the duality map on $\text{AON}_d(F)$ we mean the composition

$$\text{AON}_d(F) \xrightarrow{P \mapsto [\Phi_P]} \text{SIS}_d(F) \xrightarrow{[\Phi] \mapsto [\Phi^*]} \text{SIS}_d(F) \xrightarrow{[\Phi] \mapsto P_{\Phi}} \text{AON}_d(F).$$

**Theorem 12.2** The duality map $\text{AON}_d(F) \to \text{AON}_d(F)$ sends $P \mapsto \nu P^{-1}$, where $\nu^{-1}$ is the $(0,0)$-entry of $P^{-1}$.

**Proof.** Pick an AO normalized solid invertible matrix $P \in \text{Mat}_{d+1}(F)$, and set $\Phi = \Phi_P$. By Definition 12.1 the duality map $\text{AON}_d(F) \to \text{AON}_d(F)$ sends $P \mapsto \Phi_P^*$. So it suffices to show that $\Phi^*$ has first eigenmatrix $\nu P^{-1}$. By Proposition 7.17, $\Phi$ has first eigenmatrix $P$. Note by Lemma 7.12(iii) that the scalar $\nu$ is equal to the size of $\Phi$ from Definition 7.3. By Definition 7.11 and Lemma 7.14(i), the first eigenmatrix of $\Phi^*$ is $\nu P^{-1}$. The result follows. $\Box$

**Definition 12.3** Let $P$ denote an AO normalized solid invertible matrix in $\text{Mat}_{d+1}(F)$. By the dual of $P$ we mean the matrix $\nu P^{-1}$ from Theorem 12.2. Let $P^*$ denote the dual of $P$.

**Definition 12.4** By the duality map on $\text{CS}_d(F)$ we mean the composition

$$\text{CS}_d(F) \xrightarrow{[\Psi] \mapsto [\Phi_\Psi]} \text{SIS}_d(F) \xrightarrow{[\Phi] \mapsto [\Phi^*]} \text{SIS}_d(F) \xrightarrow{[\Phi] \mapsto [\Psi_{\Phi}]} \text{CS}_d(F).$$

**Definition 12.5** [11, Definition 3.2] Let $\Psi$ (resp. $\Psi'$) denote a character system over $F$ with eigenmatrix $P$ (resp. $P'$). We say that $\Psi$ and $\Psi'$ are dual whenever $PP' \in FI$.

**Lemma 12.6** Referring to Definition 12.5, assume that $\Psi$, $\Psi'$ are dual. Then $P' = P^*$.

**Proof.** Use Lemmas 9.12(i) and 10.10(i). $\Box$

The following lemma is a variation on a result by Kawada, see [3, Theorem II.5.9].

**Lemma 12.7** For each character system over $F$, its dual exists and is unique up to isomorphism of character systems.

**Proof.** Concerning existence, let $\Psi$ denote a character system over $F$ with eigenmatrix $P$. By Theorem 12.2 $P^*$ is AO normalized solid invertible, so by Lemma 11.3 $\Psi_{P^*}$ is a character system over $F$ that has eigenmatrix $P^*$. By construction $PP^* \in FI$ so $\Psi$, $\Psi_{P^*}$ are dual. We have shown existence. The uniqueness assertion follows from Proposition 9.11 and Lemma 12.6. $\Box$
Definition 12.8 A pair of isomorphism classes in $\text{CS}_d(\mathbb{F})$ are said to be dual whenever each character system in the first isomorphism class is dual to each character system in the second isomorphism class.

Lemma 12.9 For each isomorphism class in $\text{CS}_d(\mathbb{F})$ there exists a unique dual isomorphism class in $\text{CS}_d(\mathbb{F})$.

Proof. By Lemma 12.7

Theorem 12.10 The duality map $\text{CS}_d(\mathbb{F}) \rightarrow \text{CS}_d(\mathbb{F})$ sends each isomorphism class in $\text{CS}_d(\mathbb{F})$ to the dual isomorphism class in $\text{CS}_d(\mathbb{F})$.

Proof. By construction the composition

$$
\text{CS}_d(\mathbb{F}) \xrightarrow{[\Psi] \mapsto P_{\Psi}} \text{AON}_d(\mathbb{F}) \xrightarrow{P \mapsto P^*} \text{AON}_d(\mathbb{F}) \xrightarrow{P \mapsto [\Psi_P]} \text{CS}_d(\mathbb{F})
$$

sends each isomorphism class in $\text{CS}_d(\mathbb{F})$ to the dual isomorphism class in $\text{CS}_d(\mathbb{F})$. By Proposition 11.2(i) and Definition 12.3 the map (17) is equal to the duality map on $\text{CS}_d(\mathbb{F})$. The result follows.

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