NON-SVEP, Right-Inversion Point Spectrum and Chaos

Marcin Moszyński

Abstract. We discuss some relations between the local existence of analytic selections of eigenvectors (LSP = “NON-SVEP”) for an operator in Banach space and some chaoticity properties of linear dynamical system (with discrete or continuous time) generated by this operator. Our main goal is to prove the existence of a strong connection of some results known for many years in the local spectral theory to some important problems (which seem to be not solved so far) in the linear chaos theory. We also find a simple particular solution of the problem formulated in “Eigenvectors Selection Conjecture” (Banasiak and Moszyński in Discrete Contin Dyn Syst A 20(3):577–587, 2008, Conjecture 4.3, p. 585) and we formulate a new convenient spectral criterion for linear chaos. To make the assumptions more clear we introduce some special parts of the point spectrum of a closed operator, including the right-inversion point spectrum. Using this new criterion we prove chaoticity of a large class of super-upper-triangular operators in \(l^p\) and \(c_0\) spaces and also of some strongly continuous semi-groups generated by such operators.

Mathematics Subject Classification. 47A11, 47A16, 47D06.

Keywords. Analytic selections of eigenvectors, SVEP, Local selection property (LSP), Linear chaos (continuous and discrete case), Sub-chaos, Chaotic, mixing, hypercyclic and frequently hypercyclic operators or \(C_0\) semi-groups of operators, Super-upper-triangular operators, Right-inversion point spectrum.

Contents

0. Introduction: Linear Chaos and Selections of Eigenvectors 2
  0.1. Notation 4
1. Local Selection Property (LSP), SVEP and NON-SVEP 5
2. LSP and Sub-chaos 6

The paper is supported by: MNiSW (Polish Ministry of Science and Higher Education) grant Nieskończoność wymiarowe układy dynamiczne asymptotyka, stabilność i chaos No. NN 201 605640 and by NCN (National Science Centre—Poland) grant Analiza spectralna i metody asymptotyczne dla skalarnych i macierzowych operatorów różnicowych No. 2013/09/B/ST1/04319.
0. Introduction: Linear Chaos and Selections of Eigenvectors

Linear chaos has recently become a really popular concept. The publication of the large monograph “Linear Chaos” [17] by Grosse-Erdmann and Peris is one of the best examples of this popularity. The number of new “chaotic notions” increases, including a large portion of some new kinds of “chaotical properties” of dynamics. However, some of the problems in this theory posed relatively long time ago remain still unsolved. The first of our aims here is to shed a new light on one of such problems—a spectral problem related to chaoticity.

Let $X$ be a Banach space. We consider here two kinds of linear dynamical systems in $X$: discrete and continuous linear dynamical systems $\mathcal{T} = \{T_t\}_{t \in \mathbb{T}}$, related to the choice of the time set $\mathbb{T}$: $\mathbb{T} := \mathbb{N}$ ($0 \in \mathbb{N}$ here) or $\mathbb{T} := [0; +\infty)$, respectively. A discrete time dynamical system is just the successive powers family $\mathcal{T}$ given by $T^n := A^n$, $n \in \mathbb{N}$, where $A$ is a fixed linear bounded operator on $X$, and $A$ is called the generator of $\mathcal{T}$ in this case. A continuous time dynamical system is any $C_0$ semigroup of linear bounded operators $\mathcal{T} = \{T_t\}_{t \geq 0}$ on $X$, and the name “generator” for such dynamical system means exactly the infinitesimal generator of the $C_0$ semigroup (see e.g., [13,19]).

Recall now that $\mathcal{T}$ is called chaotic iff it is transitive and the set of periodic points for $\mathcal{T}$ is dense in $X$ (see [10,17]). According to the common terminology for the discrete time case (only) we shall also say that “$A$ is chaotic” instead of “$\mathcal{T}$ is chaotic”, where $A$ is the generator of the discrete time dynamical system $\mathcal{T}$. For many systems the chaotic behaviour is observed only in an invariant subspace (see e.g., [4,5,7]): $\mathcal{T}$ is called sub-chaotic iff there exists a closed subspace $\tilde{X}$ invariant under $\mathcal{T}$, $\{0\} \neq \tilde{X} \subset X$, such that $\tilde{\mathcal{T}} := \{T_t |_{\tilde{X}}\}_{t \in \mathbb{T}}$ is chaotic as a linear dynamical system in $\tilde{X}$. In such a case we shall also say that $\mathcal{T}$ is chaotic in $\tilde{X}$. Each $\tilde{X}$ satisfying the above conditions is called a space of chaoticity for $\mathcal{T}$.

Many particular examples of chaotic and sub-chaotic linear systems have been discovered with an essential use of a certain spectral criterion for chaos and sub-chaos. The main assumptions of the mentioned criterion are related to eigenvectors of the generator of the system with the eigenvalues belonging to the unit circle $\mathbb{U} := \{z \in \mathbb{C} : |z| = 1\}$ (in the discrete case) or to the imaginary axis $i\mathbb{R}$ (in the continuous case), see e.g., [4–7,10], and also e.g., [8,12] for some analogous examples concerning hypercyclicity property, being tightly connected to chaoticity. We want to stress, that some similar relations to $\mathbb{U}$ or $i\mathbb{R}$ - eigenvectors exist also for other kinds of “chaoticity notions”, see...
e.g., [3] and important new results in [9], which include some measure theory “ergodic version of chaoticity”.

Hence denote:

\[ \mathcal{B} := \begin{cases} \bigcup \mathbb{R} & \text{for discrete time case} \\ \mathbb{R} & \text{for continuous time case,} \end{cases} \] (0.1)

and to recall this criterion, let us remind first the notion of selection of eigenvectors (“a potential single element” of spanning eigenvector field, see e.g., [17]). Let \( A \) be a linear operator in \( X \), and let \( f : \Omega \rightarrow X \), where \( \emptyset \neq \Omega \subset \mathbb{C} \).

**Definition 0.1.** \( f \) is a selection of eigenvectors for \( A \) (on \( \Omega \)) iff \( f(\lambda) \in \ker(A - \lambda I) \) for any \( \lambda \in \Omega \). We usually abbreviate this name to: *e.v. selection* (for \( A \), on \( \Omega \)). An e.v. selection \( f \) is:

- non-trivial iff \( f \neq 0 \) (where 0 denotes the constant-zero function on \( \Omega \));
- analytic iff \( \Omega \) is an open set and \( f \) is an \( X \)-vector-valued analytic function.

Using this terminology we can shortly formulate the spectral chaoticity criterion mentioned previously as follows.

**Criterion 0.2.** Assume that \( X \) is a Banach space, \( A \) is the generator of a linear dynamical system \( T \) in \( X \) and \( \Omega \subset \mathbb{C} \) is open and connected. If there exists a non-trivial analytic selection \( f \) of eigenvectors of \( A \) on \( \Omega \) such that \( \Omega \cap \mathcal{B} \neq \emptyset \), then \( T \) is sub-chaotic. Moreover, for \( f \) as above \( \overline{\text{lin}} f(\Omega) \) is a space of chaoticity for \( T \).

This formulation is a joint reformulation of several particular results. In the continuous time case, see [4, Criterion 3.3] (being a “sub-chaotic extension” of a “full-chaoticity case” from [10]). The discrete time case was not explicitly formulated in similar form, but one can easily obtain it, e.g., using the Godefroy and Shapiro result [17, Theorem 3.1] and repeating almost the same arguments as in the continuous case.

In fact, with such assumptions also stronger assertions can be easily obtained. For continuous time case one can prove the mixing property of the system restricted to this subspace, using [17, Theorem 7.32]. For discrete time case one can prove the mixing and the frequent hypercyclicity property of the restricted system using [17, Theorem 9.22], see the proof of Remark 4.4.

Note that there exist also some similar criteria in which some weaker than analyticity regularity conditions on the selection are assumed (see [5,8,12]). A question related to such kinds of criteria was posed in—let’s call it—“Eigenvectors Selection Conjecture” [5, Conjecture 4.3, p. 585]. It concerned some sufficient “general” conditions for the closed operator \( A \), which could guarantee that the “richness” of a set for which we know that it is contained in the point spectrum, gives automatically the existence of a non-trivial selection of eigenvectors being regular in a sense (eg. measurable, continuous etc.). In the present paper we partially solve this problem in Theorem 1.6 (“Eigenvectors Selection Theorem Ω”). Our solution sounds surprisingly simple: it suffices to know only that one point \( \lambda_0 \) is in the point spectrum of the
operator $A$ and that $A - \lambda_0 I$ is surjective, and we obtain the existence of a non-trivial selection of eigenvectors with very strong regularity—being analytic! More precisely, we explain that in some sense the problem was solved many years before it was actually posed. . . . However, this solution was hidden in a different part of operator theory—the local spectral theory—usually not being associated with linear chaos theory.

The paper consists of two parts. Our main goal in the first part (Sects. 1 and 2) is to show an essential intersection of the two seemingly distant areas of operator theory. Starting from a classical Finch result of local spectral theory and making several simple steps—remarks, we get interesting new results for linear chaos theory! These simple remarks, however, show some of the concepts known for a long time in quite a new light. In Sect. 1 we recall some necessary notions and results of local spectral theory, which are related to our selection existence problems, including a reformulation of the Finch result of 1975 mentioned above [15]. In Sect. 2 we simply formulate the sub-chaos results directly following from Sect. 1. To make our considerations less abstract we illustrate them several times by the simplest classical example of linear chaotic operator—Rolewicz operator ([20] and, e.g., [17]).

In the second part of the paper (Sects. 3 and 4) we formulate a weaker version of the solution of the “Eigenvectors Selection Conjecture” mentioned above—Corollary 3.3 (“Eigenvectors Selection Theorem II”). Yet, this time it is obtained in a different manner—without the use of local spectral theory. We study here some special parts of the point spectra of closed operators which are related to the problems investigated above. The most important here is the part of $\sigma_p(A)$ which we call the right-inversion point spectrum (RIPS), denoted here by $\sigma_{p^*}(A)$ [see (3.3)]. The single point of the RIPS of the generator $A$ of a linear dynamical system $T$ can guarantee sub-chaos or even chaos for $T$. The appropriate result—“RIPS Chaoticity Criterion” (Theorem 3.4) seems to be a new and convenient tool in linear chaos theory. Its assumptions and the assertion turn out to be a good compromise for some interesting applications (compare, e.g., with Theorem 2.2 and see the comments in Example 2.4). Our main application of this tool is presented in Sect. 4, where we prove the chaoticity of a large class of super-upper-triangular operators in $l^p$ and $c_0$ spaces (Theorem 4.1). Note that those operators can be treated as generalizations (and also as perturbations) of “classical” weighted left-shifts which in turn generalize the mentioned Rolewicz type operators. We also study here discrete and continuous dynamical systems generated by some related operators and we make remarks on the more detailed “chaotic properties”—mixing and frequent hypercyclicity.

0.1. Notation

The symbols $\mathcal{L}(X)$, $\mathcal{C}(X)$, $\mathcal{B}(X)$ denote the sets of linear, closed, and bounded operators on the normed space $X$, respectively (here the domain $D(A)$ of $A$ need not be dense for $A \in \mathcal{L}(X)$ or $\mathcal{C}(X)$, but $D(A) = X$ for $A \in \mathcal{B}(X)$). If $A \in \mathcal{L}(X)$ and $X \neq \{0\}$, then $\sigma(A)$ is its spectrum and $\sigma_p(A)$ is the set of its all eigenvalues (the point spectrum).
If $X$ is a Banach space and $\Omega$ is an open subset of $\mathbb{C}$, then $\mathcal{A}(\Omega, X) := \{ f : f : \Omega \rightarrow X \ and \ is \ an \ analytic \ X\text{-vector-valued \ function} \}$; for $f \in \mathcal{A}(\Omega, X)$, $\lambda_0 \in \Omega$, $n \in \mathbb{N}$ by $f_{\lambda_0, n}$ we denote the $n$-th coefficient in $X$ in the power series expansion of $f$ with the center at $\lambda_0$, i.e.

$$f(\lambda) = \sum_{n=0}^{+\infty} (\lambda - \lambda_0)^n f_{\lambda_0, n} \tag{0.2}$$

for $\lambda$ in a neighborhood of $\lambda_0$.

1. Local Selection Property (LSP), SVEP and NON-SVEP

We are interested here in such linear operators acting in a Banach space, that possess a non-trivial analytic e.v. selection on a neighborhood of a given point $\lambda_0 \in \mathbb{C}$.

Let $X$ be a Banach space, $A \in \mathcal{L}(X)$ and $\lambda_0 \in \mathbb{C}$. For any open subset $\Omega$ of $\mathbb{C}$ denote

$$S(\Omega, A) := \{ f : f \ is \ an \ analytic \ e. \ v. \ selection \ for \ A \ on \ \Omega \}.$$

For $r > 0$ let $D(\lambda_0, r)$ be the open disc in $\mathbb{C}$ of radius $r$ centered at $\lambda_0$.

**Definition 1.1.** A has the local selection property at $\lambda_0$ iff there exists $\epsilon > 0$ such that $S(D(\lambda_0, \epsilon), A) \neq \{ 0 \}$. The above will be abbreviated to: A has the LSP at $\lambda_0$.

**Remark 1.2.** If $A$ has the LSP at $\lambda_0$, then $\lambda_0$ is an accumulation point of $\sigma_p(A)$, since in the opposite case, any $f \in S(D(\lambda_0, \epsilon), A)$ with $\epsilon > 0$ would be zero on $D(\lambda_0, \delta) \setminus \{ \lambda_0 \}$ for some $\delta \in (0; \epsilon)$, and then, by analyticity, $f = 0$.

The above assertion can be much improved if $A$ is closed.

**Remark 1.3.** Suppose that $A \in \mathcal{C}(X)$ has the LSP at $\lambda_0$. Then there exists $\epsilon > 0$ and such $f \in S(D(\lambda_0, \epsilon), A)$, that $f(\lambda_0) \neq 0$. In particular $\lambda_0 \in \text{Int } \sigma_p(A)$.

**Proof.** Using the definition of the LSP choose $\epsilon > 0$ and $g \in S(D(\lambda_0, \epsilon), A)$, $g \neq 0$. So, we have

$$g(\lambda) = \sum_{n=0}^{+\infty} (\lambda - \lambda_0)^n g_{\lambda_0, n}, \quad \lambda \in D(\lambda_0, \epsilon)$$

and $N(g) := \{ n \in \mathbb{N} : g_{\lambda_0, n} \neq 0 \} \neq \emptyset$. Defining $n_0 := \min N(g)$, for any $\lambda \in D(\lambda_0, \epsilon)$ we have $g(\lambda) = (\lambda - \lambda_0)^{n_0} \sum_{n=0}^{+\infty} (\lambda - \lambda_0)^n g_{\lambda_0, n+n_0}$, and obviously the expansion is also convergent. Now let us define $f : D(\lambda_0, \epsilon) \rightarrow X$ by the formula

$$f(\lambda) := \sum_{n=0}^{+\infty} (\lambda - \lambda_0)^n g_{\lambda_0, n+n_0}, \quad \lambda \in D(\lambda_0, \epsilon).$$

Thus $f \in \mathcal{A}(D(\lambda_0, \epsilon), X)$, moreover $f(\lambda) = (\lambda - \lambda_0)^{-n_0} g(\lambda) \in \text{Ker } (A - \lambda I)$ for $\lambda \in D(\lambda_0, \epsilon) \setminus \{ \lambda_0 \}$. But we also have $0 \neq g_{\lambda_0, n_0} = f(\lambda_0) \in \text{Ker } (A - \lambda_0 I)$ by the closedness of $A$ and by the continuity of $f$ at $\lambda_0$. Hence $f \in S(D(\lambda_0, \epsilon), A)$ and $f(\lambda_0) \neq 0$. So, by the continuity of $f$ we get $\lambda_0 \in \text{Int } \sigma_p(A)$. \qed
As one can easily see, the LSP is closely related to the well-known SVEP (single-valued extension property). The SVEP is mostly defined only for bounded operators (see e.g., [1, 18]), and we need it for more general case (in fact—for some closed operators mainly). But the classical definition itself can be extended to any $A \in \mathcal{L}(X)$, with no difficulty (for closed operator case see e.g., [14, 15] in general Banach spaces, and [2] in Hilbert space).

**Definition 1.4.** $A$ has the SVEP at $\lambda_0$ iff $S(\Omega, A) = \{0\}$ for any open connected neighborhood $\Omega$ of $\lambda_0$.

Calling the NON-SVEP the opposite to the SVEP, more precisely, defining: $A$ has the NON-SVEP at $\lambda_0$ iff $A$ does not have the SVEP at $\lambda_0$, and using the fact that two analytic functions on a open connected set $\Omega$ coincide, if they take the same values on a subset possessing accumulation point in $\Omega$, we obviously get:

**Remark 1.5.** $A$ has the LSP at $\lambda_0$ iff $A$ has the NON-SVEP at $\lambda_0$.

The result below is in fact a reformulation of a theorem by J.K. Finch [15, Th. 2, p. 61]. It can be also treated as one of possible and natural solutions of Eigenvectors Selection Conjecture of [5].

**Theorem 1.6.** (Eigenvectors Selection Theorem I) Suppose that $A \in \mathcal{C}(X)$. If $\lambda_0 \in \sigma_p(A)$ and $A - \lambda_0 I$ is surjective, then $A$ has the LSP at $\lambda_0$.

**Proof.** It suffices to use the above remark, and the above mentioned original Finch result to the operator $A - \lambda_0 I$. □

**Example 1.7.** (The Rolewicz type operators) Let $X = l^p = l^p(\mathbb{N})$—the standard power $p$-summable sequences space with $1 \leq p < +\infty$ and consider $A = \mu T_{-1}$, where $\mu \in \mathbb{C} \setminus \{0\}$ and $T_{-1}$ denotes the backward shift operator, given by $(T_{-1}f)_n = f_{n+1}$ for $f \in l^p$ and $n \in \mathbb{N}$. Then $A$ satisfies the assumptions of Theorem 1.6 with $\lambda_0 = 0$ and so $A$ has the LSP at $0$. Surely, in this simple particular case one can easily obtain the LSP result without the above theorem, just by constructing the appropriate selection of eigenvalues. Note that in the case $p = 2$ and $|\mu| > 1$ this operator is known in the literature as Rolewicz operator (see [20] and, e.g., [17]).

2. LSP and Sub-chaos

Assume that $X$ is a Banach space and $A$ is the generator of a linear dynamical system $T$ in $X$ (discrete or continuous system, with the sense of the generator described in Sect. 0 and with $\mathcal{B}$ defined by (0.1)).

Using the terminology of Sect. 1 we can again shortly formulate the first part of Criterion 0.2 as follows.

**Criterion 2.1.** (Sub-chaos criterion) If $A$ has the LSP at $\lambda_0$ for some $\lambda_0 \in \mathcal{B}$, then $T$ is sub-chaotic.

**Proof.** It suffices to “mix” Criterion 0.2 with Definition 1.1. □
As an immediate corollary following this criterion we get a convenient result allowing to prove sub-chaoticity with a relatively simple way for many examples of dynamical systems.

**Theorem 2.2.** If \( \lambda_0 \in \sigma_p(A) \) and \( A - \lambda_0 I \) is surjective for some \( \lambda_0 \in \mathbb{B} \), then \( T \) is sub-chaotic.

*Proof.* Observe first that in both cases of the time set the operator \( A \) is closed, because it is a generator of \( T \) (see e.g., [19] for the cont. time case, being the \( C_0 \) semi-group case). Now we get the assertion by Criterion 2.1 and Theorem 1.6. \( \square \)

We illustrate the above theorem by two examples—of the discrete and of the continuous dynamical system generated by the same operator. In both cases sub-chaoticity, and in fact even chaoticity, is a well-known result (see e.g., [17]), but our goal here is only to illustrate a new method. Note also an important technical difference related to the use of the above theorem as a practical tool in these two cases.

**Example 2.3.** (Rolewicz type continuous system) Consider the continuous dynamical systems \( T \) generated by the Rolewicz type operators \( A = \mu T_{-1} \) from Example 1.7. If \( \mu \neq 0 \) then Theorem 2.2 gives sub-chaos of \( T \) immediately, if we consider \( \lambda_0 = 0 \), which is in \( \mathbb{B} = i\mathbb{R} \) in this case.

**Example 2.4.** (Rolewicz type discrete system) Consider now the analogous discrete dynamical systems \( T \) generated by the Rolewicz type operators as above. For this system we must assume that \( |\mu| > 1 \). Then \( \frac{1}{\mu} \in D(0,1) = \sigma_p(T_{-1}) \), and we can try to use Theorem 2.2, taking e.g., \( \lambda_0 = 1 \in \mathbb{U} \cap \sigma_p(A) \) in this case. So, we would obtain sub-chaos of \( T \), if we only proved that \( \mu T_{-1} - I \) is surjective. Although this is true for any \( p \in [1;+\infty) \), checking it is not just trivial and it needs some “technical work”. However, as we shall show in the next section, this work can be replaced by the fast use of a certain convenient “tool”, which is formulated in Theorem 3.4. 

3. **Right-Inversion Point Spectrum and the RIPS Chaoticity Criterion**

We introduce here three special parts of the point spectrum of a closed operator in Banach space. The first two are related directly to problems considered in the previous sections, and the third corresponds to a new idea, playing a crucial role here. The results of this section, excluding Proposition 3.1 (i), are obtained independently of the local spectral theory results of Sect. 1, i.e. the appropriate analytic selections of eigenvectors are constructed here in some simple, elementary ways, in particular, without using Eigenvectors Selection Theorem 1.6.

For a Banach space \( X \) and \( A \in \mathcal{C}(X) \) we define:

\[
\sigma_{\text{LSP}}(A) := \{ \lambda \in \mathbb{C} : A \text{ has the LSP at } \lambda \}, \tag{3.1}
\]

\[
\sigma_{\text{ps}}(A) := \{ \lambda \in \sigma_p(A) : \text{Ran} (A - \lambda I) = X \}, \tag{3.2}
\]

\[
\sigma_{p^*}(A) := \{ \lambda \in \sigma_p(A) : \exists C \in \mathcal{B}(X) \ (A - \lambda I)C = I \}. \tag{3.3}
\]
We call the above subsets the LSP point spectrum, the surjective point spectrum and the right-inversion point spectrum (or RIPS) of \( A \), respectively. Note, that the inclusion

\[
\sigma_{\text{LSP}}(A) \subseteq \sigma_p(A)
\]

follows immediately from Remark 1.3.

Below we collect some more properties of those spectra.

**Proposition 3.1.** If \( A \in \mathcal{C}(X) \), then

(i) \( \sigma_{p^*}(A) \subseteq \sigma_{ps}(A) \subseteq \sigma_{\text{LSP}}(A) \);

(ii) \( \sigma_{\text{LSP}}(A) \) and \( \sigma_{p^*}(A) \) are open subsets of \( \mathbb{C} \);

(iii) If \( \lambda_0 \in \sigma_{p^*}(A) \) and \( B_0 \in \mathcal{B}(X) \) is such that \( (A - \lambda_0 I)B_0 = I \), then

\[
U := D(\lambda_0, \frac{1}{\|B_0\|}) \subseteq \sigma_{p^*}(A).
\]

Moreover, for any \( f_0 \in \ker (A - \lambda_0 I) \setminus \{0\} \)

there exists \( f \in \mathcal{S}(U, A) \) such that \( f(\lambda_0) = f_0 \) and \( f(\lambda) \neq 0 \) for any \( \lambda \in U \). Such \( f \) can be given by

\[
\lambda \in U.
\]

**Proof.** Definitions (3.1), (3.2), (3.3) and Theorem 1.6 make both inclusions of (i) obvious. Also (ii) is obvious for \( \sigma_{\text{LSP}} \) by its definition, and (ii) for \( \sigma_{p^*} \) follows from (iii). So it remains to prove (iii). Suppose that \( \lambda_0 \in \sigma_{p^*}(A) \). In particular we have \( X \neq \{0\} \). Let \( 0 \neq f_0 \in \ker (A - \lambda_0 I) \) and let \( B_0 \in \mathcal{B}(X) \) be such that \( (A - \lambda_0 I)B_0 = I \). The formula (3.4) defines \( f \in \mathcal{A}(U, X) \), because

\[
\|B_0^n f_0\| \leq \|B_0\|^n \|f_0\| \quad \text{for} \quad n \in \mathbb{N}.
\]

Moreover \( f(\lambda_0) = f_0 \). Suppose that \( \lambda \in U \). Using (3.5) and the closedness of \( A \) one can easily verify that \( f(\lambda) \in D(A) \) and

\[
(A - \lambda_0 I)f(\lambda) = (A - \lambda_0 I)f_0 + \sum_{n=0}^{+\infty} (\lambda - \lambda_0)^n B_0^{n-1} f_0 = (\lambda - \lambda_0)f(\lambda),
\]

which gives \( Af(\lambda) = \lambda f(\lambda) \). Thus we proved that \( f \in \mathcal{S}(U, A) \). Observe that (3.4) means exactly that for any \( \lambda \in U \)

\[
f(\lambda) = (I - (\lambda - \lambda_0)B_0)^{-1} f_0,
\]

but \( (I - (\lambda - \lambda_0)B_0)^{-1} \) is an isomorphism of \( X \), which gives \( f(\lambda) \neq 0 \). So, the second part of (iii) is proved and in particular \( U \subseteq \sigma_p(A) \). The proof of the first part is completed by using the following simple lemma.

**Lemma 3.2.** If \( X \neq \{0\} \), \( A \in \mathcal{L}(X) \), \( \lambda_0 \in \mathbb{C} \) and \( B_0 \in \mathcal{B}(X) \) is such that \( (A - \lambda_0 I)B_0 = I \), then there exists \( B \in \mathcal{A} \left( D(\lambda_0, \frac{1}{\|B_0\|}), \mathcal{B}(X) \right) \) satisfying \( B(\lambda_0) = B_0 \) and \( (A - \lambda I)B(\lambda) = I \) for any \( \lambda \in D(\lambda_0, \frac{1}{\|B_0\|}) \).

**Proof of the lemma.** Note first that \( \|B_0\| \neq 0 \), because \( X \neq \{0\} \). One would easily check that the required analytic operator-valued function \( B \) can be defined by
Moreover, by Proposition 3.1 (iii) we have: if
\[ B(\lambda) = B_0 (I - (\lambda - \lambda_0)B_0)^{-1} = \sum_{n=0}^{+\infty} (\lambda - \lambda_0)^n B_0^{n+1}, \quad (3.6) \]
for \( \lambda \in D \left( \lambda_0, \frac{1}{\|B_0\|} \right) \).

Obviously, the proposition above allows to formulate a more particular solution of Eigenvectors Selection Conjecture of [5].

**Corollary 3.3.** (Eigenvectors Selection Theorem II) Suppose that \( A \in \mathcal{C}(X) \). If \( \lambda_0 \in \sigma_{p^*}(A) \), then \( A \) has the LSP at \( \lambda_0 \).

Note that (3.4) is the explicit power expansion for the selection \( f \) and the coefficients of this expansion are expressed in terms of \( B_0 \) and \( f_0 \). We can use this fact to get one more result of linear chaos theory. Recall that a vector \( x \in X \) is cyclic for \( C \in \mathcal{B}(X) \) iff \( \overline{\text{lin}} \{ C^n x : n \in \mathbb{N} \} = X \).

Assume, as in the previous section, that \( A \) is the generator of a linear dynamical system \( T \) in \( X \), and \( \mathcal{B} \) is given by (0.1). We can now formulate a convenient tool—a new formulation of an abstract chaoticity criterion for \( T \).

**Theorem 3.4.** (RIPS Chaoticity Criterion) Suppose that \( \lambda_0 \in \sigma_{p^*}(A) \) and \( B_0 \in \mathcal{B}(X) \) is such that \( (A - \lambda_0 I)B_0 = I \) and \( \text{dist}(\mathcal{B}, \lambda_0) < \frac{1}{\|B_0\|} \). Then:

(i) \( T \) is sub-chaotic and \( \overline{\text{lin}} \{ B_0^n f_0 : n \in \mathbb{N} \} \) is a space of chaoticity for \( T \) for any \( f_0 \in \text{Ker} (A - \lambda_0 I) \setminus \{0\} \).

(ii) if some \( f_0 \in \text{Ker} (A - \lambda_0 I) \) is cyclic for \( B_0 \), then \( T \) is chaotic.

**Proof.** Consider \( U := D(\lambda_0, \frac{1}{\|B_0\|}) \) and a vector-valued function \( f \) defined on \( U \) by (3.4) with some \( f_0 \in \text{Ker} (A - \lambda_0 I) \setminus \{0\} \). By Proposition 3.1 (iii) \( f \in \mathcal{S}(U, A) \) and \( f \) is non-trivial, hence by Criterion 0.2 \( \overline{\text{lin}} f(U) \) is a space of chaoticity for \( T \). From the other hand [4, Lemma 3.4] gives \( \overline{\text{lin}} \{ B_0^n f_0 : n \in \mathbb{N} \} = \overline{\text{lin}} f(U) \), which finishes the proof of (i) and (ii) of the theorem. \( \square \)

Observe that the choice of the above operator \( B_0 \)—the right inverse to \( (A - \lambda_0 I) \)—is not unique, and thus also the value \( \|B_0\| \) can be not uniquely determined by \( A \in \mathcal{C}(X) \) and \( \lambda \in \mathbb{C} \). Therefore, it is convenient to define:

\( \text{Rinv}(A, \lambda) := \{ B \in \mathcal{B}(X) : (A - \lambda I)B = I \} \)

and for \( \lambda \in \sigma_{p^*}(A) \):

\[ r_*(A, \lambda) := \inf \{ \|B\| : B \in \text{Rinv}(A, \lambda) \}. \quad (3.7) \]

We then have: \( \sigma_{p^*}(A) = \{ \lambda \in \sigma_p(A) : \text{Rinv}(A, \lambda) \neq \emptyset \} \). Obviously, if \( B \in \text{Rinv}(A, \lambda) \) and \( B' \in \mathcal{B}(X) \), then

\[ B' \in \text{Rinv}(A, \lambda) \iff \text{Ran} (B' - B) \subset \text{Ker} (A - \lambda I). \]

Moreover, by Proposition 3.1 (iii) we have: if \( \lambda_0 \in \sigma_{p^*}(A) \), then

\[ D(\lambda_0, r_*(A, \lambda_0)^{-1}) \subset \sigma_{p^*}(A). \]

Using the above notation we can slightly reformulate the above criterion for chaos.
Corollary 3.5. Suppose that $\lambda_0 \in \sigma_{p^*}(A)$ and $\text{dist} (\mathbb{B}, \lambda_0) < (r_*(A, \lambda_0))^{-1}$. Then:

(i) $T$ is sub-chaotic and $\text{lin}\{B^nf_0 : n \in \mathbb{N}\}$ is a space of chaoticity for $T$ for any $f_0 \in \text{Ker}(A - \lambda_0 I) \setminus \{0\}$ and $B \in \text{Rinv}(A, \lambda_0)$.

(ii) if there exists $f_0 \in \text{Ker}(A - \lambda_0 I)$ which is cyclic for some $B \in \text{Rinv}(A, \lambda_0)$, then $T$ is chaotic.

As an easy illustration of our new tool we can consider one of the best known classical examples of linear chaos.

Example 3.6. (chaos for Rolewicz type operators) The discrete dynamical system $T$ from Example 2.4 generated by the Rolewicz type operator $\mu T_{-1}$ with $|\mu| > 1$ is chaotic, and not only sub-chaotic. It suffices to use Theorem 3.4 with $\lambda_0 = 0$, $B_0 = \mu^{-1}T_{+1}$ and $f_0 = e_0$, where $(e_0)_n = 0$ for $n > 0$, $(e_0)_0 = 1$, and $T_{+1}$ is the forward shift operator, given by $(T_{+1}f)_n = f_{n-1}$ for $n > 0$ and $(T_{+1}f)_0 = 0$. Observe also that $r_*(\mu T_{-1}, 0) = |\mu^{-1}|$ (one can easily check, that the ,,inf" in the definition (3.7) is ,,reached" just by $B_0 = \mu^{-1}T_{+1}$).

4. Chaotic Super-Upper-Triangular Operators

In this section $X = l^p = l^p(\mathbb{N})$ with $1 \leq p < +\infty$ or $X = c_0$—the standard space of complex sequences on $\mathbb{N}$ converging to 0. We study here a large class of super-upper-triangular operators in $X$ and we prove a generalization of the chaoticity result for Rolewicz type operators from Example 3.6.

By $T_{-1}, T_{+1}$ we denote the backward and the forward shift operator in $X$, respectively. For $A \in \mathcal{B}(X)$ we consider its matrix terms $A(k, l) \in \mathbb{C}$ given by $A(k, l) := (Ae_l)_k$ for $k, l \in \mathbb{N}$, where $e_l \in X$ is the $l$-th standard “base” vector, i.e. $(e_l)_n = 0$ for $n \neq l$, $(e_l)_l = 1$. Recall that $A \in \mathcal{B}(X)$ is upper-triangular iff $A(k, l) = 0$ for any $k > l, k, l \in \mathbb{N}$ and $A \in \mathcal{B}(X)$ is super-upper-triangular iff $A(k, l) = 0$ for any $k \geq l, k, l \in \mathbb{N}$ (i.e. $A$ possesses only zero matrix terms below the first super diagonal). By $A_{\text{sup-dia}}$ we denote the superdiagonal part of $A$ (i.e. “the first super diagonal” of $A$) and by $A_{\text{off}}$—its “off-superdiagonal part”:

$$(A_{\text{sup-dia}}x)_j := A(j, j+1)x_{j+1} \quad \text{for} \quad x \in X, j \in \mathbb{N}, \quad A_{\text{off}} := A - A_{\text{sup-dia}}. \quad (4.1)$$

We obviously have $A_{\text{sup-dia}}, A_{\text{off}} \in \mathcal{B}(X)$, if $A \in \mathcal{B}(X)$.

Theorem 4.1. Let $A$ be a bounded super-upper-triangular operator in $X$. If for some $r > 0$

$$|A(j, j+1)| \geq 1 + r, \quad \text{for any} \quad j \in \mathbb{N} \quad (4.2)$$

and

$$\|A_{\text{off}}\| < r, \quad (4.3)$$

then $A$ is chaotic.
Proof. Denote $\mu_j := A(j, j + 1)$ for $j \in \mathbb{N}$, $\Lambda := A_{\text{sup-diag}} T_{+1}$ and $R := A_{\text{off}} T_{+1}$. By (4.1) $\Lambda$ is the diagonal operator with the sequence $\{\mu_n\}_{n \geq 0}$ on the main diagonal, i.e.

$$(\Lambda x)_j := \mu_j x_j \quad \text{for} \; x \in X, \; j \in \mathbb{N}. \quad (4.4)$$

Moreover $A$ is a super-upper-triangular operator, hence in particular $A = AT_{+1}T_{-1}$, so by (4.1)

$$A = (\Lambda + R)T_{-1}. \quad (4.5)$$

By (4.2), (4.3) we see, that $\Lambda$ is invertible with $\|\Lambda^{-1}\| \leq \frac{1}{1 + r}$ and $\|R\| < r$. Hence $\|\Lambda^{-1} R\| < \frac{r}{1 + r} < 1$ and $\Lambda + R = \Lambda(I + \Lambda^{-1} R)$ is also invertible with

$$\|(\Lambda + R)^{-1}\| = \|(I + \Lambda^{-1} R)^{-1}\Lambda^{-1}\| \leq \frac{1}{1 - \|\Lambda^{-1} R\|} \frac{1}{1 + r} < \frac{1}{1 - \frac{r}{1 + r}} \frac{1}{1 + r} = 1. \quad (4.6)$$

Let us define $B_0 := T_{+1}(\Lambda + R)^{-1}$. Using (4.5) we get $AB_0 = I$ and $e_0 \in \ker(A)$, so in particular $0 \in \sigma_{p^*}(A)$. We also have $\|B_0\| < 1$ by (4.6), i.e. dist $(\mathbb{B}, 0) = 1 < \frac{1}{\|B_0\|}$. Now, choosing $\lambda_0 := 0$ and $f_0 := e_0$ we can apply Theorem 3.4, if we only prove that $e_0$ is cyclic for $B_0$. Let $C = \Lambda + R$. Observe that $C$ is upper-triangular and $C(j, j) = \Lambda(j, j) = \mu_j \neq 0$ for any $j \in \mathbb{N}$, and hence $C^{-1}$ is also upper-triangular and $(C^{-1})(j, j) = \mu_j^{-1} \neq 0$. Therefore $B_0 = T_{+1}C^{-1}$ satisfies the conditions of the lemma below, and the use of this lemma finishes the proof. \hfill \Box

Lemma 4.2. If $B \in \mathcal{B}(X)$ and

$$j = k + 1 \implies B(j, k) \neq 0, \quad j > k + 1 \implies B(j, k) = 0$$

for any $j, k \in \mathbb{N}$, then $e_0$ is cyclic for $B$.

Proof. Observe first, that for any $n \in \mathbb{N}$ the following two conditions hold:

(i) lin $\{B^0 e_0, \ldots, B^n e_0\} = \text{lin} \{e_0, \ldots, e_n\}$

(ii) $B^n e_0 = \sum_{j=0}^n c_j^{(n)} e_j$ for some $c_0^{(n)}, \ldots, c_n^{(n)} \in \mathbb{C}$, with $c_n^{(n)} \neq 0$.

To prove this it suffices to use the standard induction applied to both conditions simultaneously. Hence, by (i) (for all $n$) lin $\{B^n x : n \in \mathbb{N}\} = \text{lin} \{e_n : n \in \mathbb{N}\}$. \hfill \Box

Note that the proof of Theorem 4.1 can be slightly generalized, by admitting adding of some multiples of the identity to the generator, and by considering also the continuous system case. One just needs to consider $A + \mu I$ instead of $A$ and $\lambda_0 := \mu$ instead of $0$. We have then $(A + \mu I) - \lambda_0 I = A$, and the above proof works with the same operator $B_0$, providing dist $(\mathbb{B}, \mu) \leq 1$. Hence we get the following result.

Theorem 4.3. Assume that $A$ satisfies all the assumptions of Theorem 4.1 and $\mu \in \mathbb{C}$. The linear dynamical system $T$ in $X$ (discrete or continuous one) generated by $A + \mu I$ is chaotic, providing

$$|\text{Re} \mu| \leq 1 \quad \text{at the continuous case},$$

$$|\mu| \leq 2 \quad \text{at the discrete case}.$$
We end with a remark related to the above results, concerning some additional “chaotic properties” (mixing and frequent hypercyclicity, see e.g., [17]).

Remark 4.4. The dynamical system considered in Theorem 3.4 (ii), and thus also in Theorems 4.1 and 4.3 is mixing. Moreover, in the discrete time case such system is frequently hypercyclic.

Proof. It suffices to observe that Criterion 0.2, which was used to get the chaos in the proof of Theorem 3.4 (ii), gives in fact much more than the chaos only for the restriction of T to any of the chaoticity spaces $\text{lin} f(\Omega)$. For the discrete time case one can prove the mixing and the frequent hypercyclicity property. To do this one should first choose $z_0 \in U$ and such $\epsilon$, that $1/2 > \epsilon > 0$ and $D(z_0, 2\epsilon) \subset \Omega$. Then one defines a selection $g$ of eigenvectors of the generator on $B$, taking $g(z) = f(z)$ if $z \in B \cap D(z_0, 2\epsilon)$ and $g(z) = 0$ if $z \in B \setminus D(z_0, 2\epsilon)$. The selection $g$ can be easily “regularized”, using any $C^\infty$ scalar function $\varphi$ on $B$ such that $\varphi(z) = 1$ for $z \in B \cap D(z_0, \epsilon)$ and $\varphi(z) = 0$ if $z \in B \setminus D(z_0, 2\epsilon)$, and defining $\tilde{g} := \varphi g$. Now $\tilde{g}$ is a $C^\infty$ vector-valued function (it suffices for us that it is of the $C^2$ class). Denoting $V(h, \Omega') := \text{lin} h(\Omega')$ for any $h : \Omega \longrightarrow X$, $\Omega' \subset \Omega$ we can write:

$$V(f, \Omega) \supset V(f, D(z_0, 2\epsilon)) \supset V(\tilde{g}, B \cap D(z_0, \epsilon)) \supset V(f, B \cap D(z_0, \epsilon)) = V(f, \Omega),$$

where the RHS equality follows from the analyticity of $f$ (see e.g., [4, Lemma 3.4]). Hence $\text{lin} f(\Omega) = \text{lin} g(B)$, and so we get the desired properties using [17, Theorem 9.22]. For the continuous time case we immediately obtain the mixing property e.g., by [17, Theorem 7.32], using only the continuity of the above $\tilde{g}$. □

Open Access. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

[1] Aiena, P.: Fredholm and Local Spectral Theory, with Applications to Multipiers. Kluwer, Dordrecht (2004)
[2] Aiena, P., Trapani, C., Triolo, S.: SVEP and local spectral radius for unbounded operators. Filomat 2(2), 263–273 (2014)
[3] Badea, C., Grivaux, S.: Unimodular eigenvalues uniformly distributed sequences and linear dynamics. Adv. Math. 211, 766–793 (2007)
[4] Banasiak, J., Moszyński, M.: A generalization of Desch–Schappacher–Webb criteria for chaos. Discrete Contin. Dyn. Syst. A 12(5), 959–972 (2005)
[5] Banasiak, J., Moszyński, M.: Hypercyclicity and chaoticity spaces of $C_0$—semigroups. Discrete Contin. Dyn. Syst. A 20(3), 577–587 (2008)
[6] Banasiak, J., Moszyński, M.: Dynamics of birth-and-death processes with proliferation stability and chaos. Discrete Contin. Dyn. Syst. A 29(1), 67–79 (2011)

[7] Banasiak, J., Lachowicz, M., Moszyński, M.: Chaotic behavior of semigroups related to the process of gene amplification–deamplification with cells’ proliferation. Math. Biosci. 206, 200–215 (2007)

[8] Bayart, F., Grivaux, S.: Hypercyclicity and unimodular point spectrum. J. Funct. Anal. 226(2), 281–300 (2005)

[9] Bayart, F., Matheron, E.: Mixing operators and small subsets of the circle. J. Reine Angew. Math. 715, 75–123 (2016)

[10] Desch, W., Schappacher, W., Webb, G.F.: Hypercyclic and chaotic semigroups of linear operators. Ergod. Theory Dyn. Syst. 17, 793–819 (1997). 2nd edn. Addison-Wesley, New York (1989)

[11] Edwards, R.E.: Functional Analysis. Theory and Applications. Dover Publications Inc, New York (1995)

[12] El Mourchid, S.: The imaginary point spectrum and hypercyclicity. Semigroup Forum 73(2), 313–316 (2006)

[13] Engel, K.-J., Nagel, R., Engel, K.-J., Nagel, R.: One-Parameter Semigroups for Linear Evolution Equations, Graduate Texts in Mathematics. Springer, New York (2000)

[14] Erdely, I., Shengwang, W.: A Local Spectral Theory for Closed Operators. London Mathematical Society Lecture Note Series, vol. 105. Cambridge University Press, Cambridge (1985)

[15] Finch, J.K.: The single-valued extension property on a Banach space. Pac. J. Math. 58(1), 61–69 (1975)

[16] Godefroy, G., Shapiro, J.H.: Operators with dense, invariant, cyclic manifolds. J. Funct. Anal. 98, 229–269 (1991)

[17] Grosse-Erdmann, K.-G., Peris Manguillot, A.: Linear Chaos. Springer, London (2011)

[18] Laursen, K.B., Neumann, M.M.: An Introduction to Local Spectral Theory. London Mathematical Society Monographs, vol. 20. Clarendon Press, Oxford (2000)

[19] Pazy, A.: Semigroups of Linear Operators and Applications to PDE. Springer, New York (1983)

[20] Rolewicz, S.: On orbits of elements. Stud. Math. 32, 17–22 (1969)

Marcin Moszyński
Wydział Matematyki Informatyki i Mechaniki
Uniwersytet Warszawski
ul. Banacha 2
02-097 Warsaw
Poland
e-mail: mmoszyns@mimuw.edu.pl

Received: June 30, 2016.
Revised: March 3, 2017.