Quasi-exactly solvable problems and the dual (q-)Hahn polynomials

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Abstract. A second-order differential (q-difference) eigenvalue equation is constructed whose solutions are generating functions of the dual (q-)Hahn polynomials. The fact is noticed that these generating functions are reduced to the (little q-)Jacobi polynomials, and implications of this for quasi-exactly solvable problems are studied. A connection with the Azebel-Hofstadter problem is indicated.
1 Introduction.

In the present paper we shall consider some second-order differential and $q$-difference eigenvalue equations of the form:

$$a(z)f''(z) + b(z)f'(z) + c(z)f(z) = \lambda f(z)$$  \hspace{1cm} (1)

and

$$\alpha(z)f(q^n z) + \beta(z)f(q^{n+1} z) + \gamma(z)f(q^{n+2} z) = \lambda f(z)$$  \hspace{1cm} (2)

respectively, where the functions $a(z)$, $b(z)$, $c(z)$ are polynomials in $z$, while $\alpha(z)$, $\beta(z)$, $\gamma(z)$, in $z$ and $z^{-1}$. We shall be looking for polynomial solutions $f(z)$. Note that using a transformation of the type $\psi(y) = g(y)f(z(y))$ we can always reduce (1) to the Schrödinger form $-\frac{d^2}{dy^2} \psi(y) + V(y) \psi(y) = \lambda \psi(y)$.

After subjecting the coefficients of (1) and (2) to certain general conditions, we shall see that the polynomial solutions to equations (1) and (2) are given by generating functions of the dual Hahn and dual $q$-Hahn polynomials, respectively. These solutions are explicit in the sense that all the eigenvalues $\lambda$ and corresponding eigenfunctions $f(z)$ in the space of polynomials of degree at most $N$ are known explicitly.

Let us first consider equation (1). It is known \[1\] (see also \[2\] for a recent review of the subject) that the spectral problem for the operator $D = a(z)\frac{d^2}{dz^2} + b(z)\frac{d}{dz} + c(z)$ in the space $\mathcal{H}_N$ spanned by the vectors $1, z, z^2, \ldots, z^N$ is closely related to the representation theory of the algebra $sl_2$. $\mathcal{H}_N$ is a representation space of this algebra, and the generators of $sl_2$ have in this representation the following form:

$$J^+ = z^2 \frac{d}{dz} - Nz; \quad J^0 = z \frac{d}{dz} - N/2; \quad J^- = \frac{d}{dz}. \hspace{1cm} (3)$$

The necessary and sufficient condition \[3\] for the operator $D$ to leave $\mathcal{H}_N$ invariant is that $D$ be expressed in the form

$$D = c_{++}J^+J^+ + c_{+0}J^+J^0 + c_{+}J^+J^- + c_{0-}J^0J^- + \frac{c_{-}}{N}J^-J^- + c_+J^+ + c_0J^0 + c_-J^- + d, \hspace{1cm} (4)$$

where $c_{i,j}, c_i, d$ are constant parameters. Henceforth, we shall assume that (4) holds. According to the classification of Turbiner, equation (1) is called in this case quasi-exactly-solvable.\[4\]

\[1\] For the expressions of all possible potentials in the corresponding Schrödinger equations see \[4\].
If the parameters $c_{++} = c_{+0} = c_+ = 0$ then $D$ is, obviously, upper diagonal in the basis of monomials $\{z^k\}_{k=0}^N$, and hence, it preserves the flag $\mathcal{H}_0 \subset \mathcal{H}_1 \subset \cdots \subset \mathcal{H}_N$. The coefficient $a(z)$ in this case is a polynomial of no more than the second degree, $b(z)$, the first degree, and $c$ is independent of $z$. Hence, the operator $D$ preserves $\mathcal{H}_N$ for any $N$. Corresponding equation (1) is then called exactly-solvable\footnote{An important algebraic approach to the exactly solvable problems is proposed in \cite{5}.}. It is easy to verify that in this case, changing the six remaining parameters $c_{+-}, c_{0-}, c_0, c_-, d$, we can obtain for any $N$ an arbitrary operator $D$ with the just mentioned restriction on the degrees of $a(z), b(z),$ and $c(z)$. The full classification of the polynomial solutions to (1) for such an operator $D$ is available in the literature (e.g., \cite{6}). All these solutions can be written in an explicit form. In particular, the classical orthogonal polynomials (Jacobi, Laguerre, and Hermite) satisfy exactly solvable equations of the type (1). In the present paper, however, we shall be interested in a different type of solutions.

Take once again the general case (4) and let now $c_{++} = c_{--} = 0$. Then it is seen from (3) and (4) that the operator $D$ is represented in the basis of monomials $\{z^k\}_{k=0}^N$ by a tridiagonal matrix. The equation (1) takes on the following form (\(f(z) = \sum_{k=0}^N z^k p_k\))\footnote{The quantity $\sum_{k=0}^N a_k z^k p_k$, where $a_k$ are some parameters, is called a generating function of the sequence $\{p_k\}_{k=0}^N$.}:

\[
\begin{pmatrix}
    a_0 - \lambda & b_0 & 0 \\
    c_1 & a_1 - \lambda & b_1 & 0 \\
    & c_2 & a_2 - \lambda & b_2 & 0 \\
    & & \ddots & \ddots & \ddots \\
    & & & c_N & a_N - \lambda \\
\end{pmatrix}
\begin{pmatrix}
p_0 \\
p_1 \\
\vdots \\
p_N
\end{pmatrix} = 0
\]

(5)

We see that the quantities $p_k$ satisfy the three-term recurrence relation

\[
    c_k p_{k-1} + (a_k - \lambda) p_k + b_k p_{k+1} = 0, \quad p_{-1} = 0.
\]

(6)

Thus [e.g., \cite{3}], $p_k(\lambda)$ form a finite system of orthogonal polynomials. (See \cite{7,8,9,10} for studies related to this aspect of quasi-exact solvability.)

The matrix elements $a_k, b_k, c_k$ are polynomials of degree 2 in the index $k$. The coefficients of these polynomials are expressed in terms of the 7 free parameters $c_{++}, c_{+0}, c_{0-}, c_+, c_0, c_-, d$. It is easy to verify that $a_k, b_k, c_k$ can
be obtained in the following way. Assume \( A(k), B(k), C(k) \) to be arbitrary polynomials in \( k \) of degree 2 and impose the boundary conditions \( C(N+1) = B(-1) = 0 \). Then \( a_k = A(k), b_k = B(k), c_k = C(k), k = 0, \ldots, N \).

Now consider a particular case of (5). Namely, impose the Askey-Wilson condition for the transposed matrix \( a_k + b_{k-1} + c_{k+1} = 0 \) and the restriction \( c_{+0} = c_{0-} \). Then the polynomials \( \{\hat{p}_k(\lambda/c_{+0})\}_{k=0}^N \) associated with the transposed matrix, are the dual Hahn hypergeometric orthogonal polynomials.

One of their generating functions provides an explicit solution to (1). In Section 2 we show that the corresponding equation (1) is reduced to the exactly solvable (see above) equation for the Jacobi polynomials by means of the transformation \( \psi(y) = (y+1)^N f(y) \). (Note that this is a particular case of the transformation that connects various forms of a quasi-exactly solvable equation [10]. Such transformations comprise the irreducible representation of the group \( GL(2) \) in the space of polynomials.) Equation (1) in this case has an infinite number of formal explicit solutions \( f(z) \), but only \( N + 1 \) of them are guaranteed to be polynomials.

The above considerations can be generalized for equation (2). Equation (2) is related to quantum deformations of the \( sl_2 \) algebra in a similar way as equation (1) is related to \( sl_2 \). For connections between (quantum) groups and orthogonal polynomials see [12,13].

Let \( \alpha(z), \beta(z), \gamma(z) \) in (2) be the first order polynomials in \( z \) and \( z^{-1} \). Then, obviously, in the basis of monomials \( \{z^k\}_{k=0}^N \), equation (2) has the form (5) where \( a_k, b_k, c_k \) are expressed in terms of the coefficients of the polynomials \( \alpha(z), \beta(z), \gamma(z) \). For the space spanned by \( \{z^k\}_{k=0}^N \) to be an eigenspace, these coefficients should be such that \( b_{-1} = c_{N+1} = 0 \). This condition implies that only 7 out of total 9 coefficients are independent. We shall fix 3 more by requiring the Askey-Wilson condition for the transposed matrix to hold: \( a_k + b_{k-1} + c_{k+1} = 0 \). Fixing then one of the remaining 4 free parameters in an appropriate way and putting \( s = 0 \), we obtain the dual q-Hahn basic hypergeometric polynomials \( \{\hat{p}_k(\lambda/\epsilon)\}_{k=0}^N \) as a system generated by the recurrence relation \( b_{k-1}\hat{p}_{k-1} + (a_k - \lambda/\epsilon)\hat{p}_k + c_{k+1}\hat{p}_{k+1} = 0 \).

In Section 3 we shall consider equations of the type (2) whose solutions will be given by the generating functions of the dual q-Hahn polynomials.

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\(^4\) In addition to the results for differential and \( q \)-difference equations reviewed in [3], it is also possible [11] to obtain similar results for difference equations of the form \( \sum_i A_i(x)f(x + \delta_i) = \lambda f(x) \).
These equations are related to the \(q\)-difference equation for the little \(q\)-Jacobi polynomials.

It is interesting to note that the zeros of polynomial solutions of equations (1) and (2) are connected with the eigenvalues \(\lambda\) by a set of Bethe-ansatz type algebraic equations \([15,16]\).

Thus, the message of the present paper can be summarized as follows. The dual Hahn (dual \(q\)-Hahn) polynomials are the most general system in the Askey-scheme of the known hypergeometric (basic hypergeometric) orthogonal polynomials \([14]\) whose generating function provides an explicit polynomial solution to the eigenvalue equation (1) (equation (2)). This generating function of the dual Hahn (dual \(q\)-Hahn) polynomials is reduced to the Jacobi (little \(q\)-Jacobi) polynomials. (The corresponding problems are, thus, exactly solvable.)

2 Dual Hahn polynomials and a differential equation

Unlike in the introduction, we shall now begin with the dual Hahn polynomials rather than with the differential equation.

The dual Hahn polynomials are defined by the formula (e.g.,\([14]\))

\[
p_n(\lambda(x)) = \sum_{k=0}^{n} \frac{(-n)_k(-x)_k(x + \gamma + \delta + 1)_k}{(\gamma + 1)_k(-N)_k k!}, \quad n = 0, 1, \ldots, N, \quad (7)
\]

where \(\gamma\) and \(\delta\) are fixed parameters and the “shifted” factorial is defined as \((a)_0 = 1, \ (a)_k = a(a + 1) \cdots (a + k - 1), \ k = 1, 2, \ldots\). The polynomials \((7)\) satisfy the recurrence relation (which we will formally consider for an arbitrary integer \(n\))

\[
\lambda(x)p_n = A_n p_{n+1} - (A_n + C_n)p_n + C_n p_{n-1},
\]

\[
A_n = (n - N)(n + \gamma + 1), \quad C_n = n(n - \delta - N - 1), \quad (8)
\]

\(
\lambda(x) = x(x + \gamma + \delta + 1).
\)

The above three-term recurrence relation can be viewed as the eigenvalue equation for an infinite tridiagonal matrix \(M\), \(p_n\’s\) being components of an eigenvector. For what follows, we would need to demand that the finite dimensional space \(L_N\) corresponding to the indices \(n = 0, 1, \ldots, N\) be
invariant under the action of the matrix $M$. This would be the case if the matrix elements $M_{-10} = M_{N+1N} = 0$. Since for our matrix $M_{0-1} = C_0 = 0$ and $M_{N+N} = A_N = 0$, the transposed matrix $M^T$ will have the desired property of preserving $L_N$. The polynomials associated with $M^T$ satisfy the recurrence

$$\lambda(x)\tilde{p}_n = C_{n+1}\tilde{p}_{n+1} - (A_n + C_n)\tilde{p}_n + A_{n-1}\tilde{p}_{n-1}; \quad (9)$$

and it is easy to show by induction that

$$\tilde{p}_n = \frac{A_0 A_1 \cdots A_{n-1}}{C_1 C_2 \cdots C_n} p_n = \frac{(-N)_n (\gamma + 1)_n}{(-\delta - N)_n n!} p_n.$$ 

Now multiply both sides of (9) by $z^n$ and perform summation over $n$ from $n = 0$ to $N$. We obtain

$$\lambda f(z) = z(z-1)^2 f''(z) + \{(\gamma - N + 2)z^2 - (\gamma - \delta - 2N + 2)z - (\delta + N)\} f'(z) - N(\gamma + 1)(z-1)f(z),$$

where $f(z) = \sum_{n=0}^{N} z^n \tilde{p}_n$. To get the homogenous equation (11), it was necessary to put $\tilde{p}_{-1} = 0$ and $\tilde{p}_{N+1} = 0$ (we can do this because we are looking for solutions in $L_N$).

We can represent (11) in the form $\lambda f(z) = Df(z)$ as the eigenvalue equation for a second-order differential operator $D$ in the space $H_N$ spanned by monomials $\{z^k\}_{k=0}^{N}$. One can already notice that (11) can be reduced to a hypergeometric equation. However, we shall follow another approach which can be easier generalized to $q$-difference equations.

Since $M^T$ in $L_N$ is just the matrix representation of the operator $D$ in the basis $\{z^k\}_{k=0}^{N}$, the eigenvalues of $D$ in $H_N$ and $M^T$ in $L_N$ are the same. To find them, first replace the parameter $N$ in (7) and (8) by $N + \epsilon, \epsilon \neq 0$. Then (7) will be valid not only for $n = 0, 1, \ldots, N$, but also for $n = N + 1$. We find the eigenvalues from the equation:

$$0 = \det(M^T - \lambda I) = \det(M - \lambda I) = \lim_{\epsilon \to 0} A_0 A_1 \cdots A_N p_{N+1}(\lambda). \quad (11)$$

Here (only) one of the factors $A_i$ goes to zero as $\epsilon \to 0$: $A_N = -\epsilon(N + \gamma + 1)$. Furthermore, only the addend with the index $k = N + 1$ in the expression

$$p_{N+1}(\lambda(x)) = \sum_{k=0}^{N+1} \frac{(-N - 1)_k (-x)_k (x + \gamma + \delta + 1)_k}{(\gamma + 1)_k (-N - \epsilon)_k k!}$$
is not bounded as $\epsilon \to 0$ (growing as $1/\epsilon$). Hence (11) is equivalent to 
$(-x)_{N+1}(x + \gamma + \delta + 1)_{N+1} = 0$. From here, using the definition of $\lambda$ in (8), we obtain the eigenvalues:

$$\lambda(m) = m(m + \gamma + \delta + 1), \quad m = 0, 1, \ldots, N. \tag{12}$$

The corresponding eigenvectors of $D$ are

$$f_m(z) = \sum_{n=0}^{N} z^n \frac{(-N)_n(\gamma + 1)_n}{(-\delta - N)_n n!} p_n(\lambda(m))$$

Notice that this generating function is one of those admitting representation in terms of the hypergeometric series [14]:

$$f_m(z) = (1 - z)^m \sum_{k=0}^{N-m} \frac{(m - N)_k(m + \gamma + 1)_k}{(-\delta - N)_k k!} z^k. \tag{13}$$

Using one of the representations of the Jacobi polynomials (see, e.g., [17]) we can rewrite (13) in the form:

$$f_m(z) = \frac{(N - m)!}{(-N - \delta)_{N-m}} (1 - z)^N P_{N-m}^{(-\delta-N-1,-\gamma-N-1)} \left(1 + \frac{z}{1 - z}\right), \tag{14}$$

where $P_k^{(\alpha,\beta)}(x)$ is the usual notation for the Jacobi polynomial. Thus, this formula expresses the generating function of the dual Hahn polynomials in terms of the Jacobi polynomials. It is easy to transform (10) into the hypergeometric equation for the Jacobi polynomials. Since the equation for the Jacobi polynomials is valid for an arbitrary large index $k$, formula (14) provides an infinite number of nonpolynomial solutions to (10) for $m = -1, -2, \ldots$.

The operator $D$ is expressed in the following form in terms of the generators (4):

$$D = J^+J^0 - 2J^+J^- + J^0J^- + (\gamma + 1 + N/2)J^+ + (\delta - \gamma - 2)J^0 - (N/2 + \delta)J^- + N(\delta + \gamma)/2 \tag{15}$$

After the transformation

$$\psi(y) = f(\coth^{2}y/2) \left\{ \sqrt{\sinh y \sinh^2 \gamma \frac{y}{2} \cosh^4 \frac{y}{2} \coth^N \frac{y}{2}} \right\}^{-1} \tag{16}$$
equation (10) is reduced to the Schrödinger-type equation

\[-d^2\frac{d^2}{dy^2}\psi(y) + V(y)\psi(y) = \varepsilon\psi(y),\]

\[V(y) = \frac{1}{2\sinh^2 y} \{(\gamma - \delta)(2N + \gamma + \delta + 2) \cosh y + (N + \gamma)^2 + (N + \delta)^2 + 2(2N + \gamma + \delta) + \frac{3}{2}\} + \frac{1}{4}(1 + \gamma + \delta)^2,\]  

with formal solutions:

\[\varepsilon_m = -m(m + \gamma + \delta + 1), \quad m = \ldots, -1, 0, 1, \ldots, N,\]

\[\psi_m(y) = \frac{c_m P_{N-m}^{(-\delta-N-1,-\gamma-N-1)}(-\cosh y)}{\sqrt{\sinh y \sinh \frac{\gamma}{2} \cosh \frac{\delta}{2} \coth N \frac{\gamma}{2} (1 - \cosh y)^N}},\]  

where \(c_m\) is a constant factor. This is the exactly solvable Schrödinger equation related to the Jacobi polynomials. It is easy to verify that if \(\gamma + N < 0\) and \(2m + \gamma + \delta + 1 > 0\), then the function \(\psi_m(y)\) belongs to the space \(L^2(-\infty, \infty)\) (that is \(\int_{-\infty}^{\infty} |\psi_m|^2 dy < \infty\)). In this case, since the operator \(-d^2/dy^2 + V(y)\) is symmetric with respect to the inner product \((f, g) = \int_{-\infty}^{\infty} f(y) g(y) dy\), such eigenfunctions \(\psi_m(y)\) corresponding to different \(\varepsilon_m\) are orthogonal.

Note that, generally, \(\psi_m(y)\) and \(\psi'_m(y)\) are discontinuous at \(y = 0\). Consider the physically more reasonable Schrödinger equation (17) in the space \(L^2(0, \infty)\) with the boundary condition \(\psi(0) = 0\). The functions \(\psi_m(y)\) for \(m = 0, 1, \ldots, N\) belong to this space if \(\gamma + N < -1/2, \gamma + \delta + 1 > 0\). The corresponding \(\varepsilon_m\) are the levels of the discrete spectrum because they are less then the asymptotics \((1 + \gamma + \delta)^2/4\) of the potential as \(y \to \infty\); and \(V(y)\) goes to \(-\infty\) as \(y \to 0\). Note that \(\varepsilon_m, m = N, \ldots, 1, 0\) are the first \(N + 1\) lowest eigenvalues of the Schrödinger operator.

3 Continuous dual \(q\)-Hahn (dual \(q\)-Hahn) polynomials and a \(q\)-difference equation
3.1 \( q \) a root of unity

The continuous dual \( q \)-Hahn polynomials (which depend on parameters \( a, b, \) and \( c \)) are defined by the expression (e.g.,\([14]\))

\[
p_n(x) = \sum_{k=0}^{n} \frac{(q^{-n}; q)_k (at; q)_k (at^{-1}; q)_k q^k}{(ab; q)_k (ac; q)_k (q; q)_k k!}, \quad 2x = t + t^{-1}, \ n = 0, 1, \ldots,
\]

where \((d; q)_0 = 1\) and \((d; q)_k = (1 - d)(1 - dq) \cdots (1 - dq^{k-1})\), \(k = 1, 2, \ldots\)  
(In fact the \( n \)'th continuous dual \( q \)-Hahn polynomial differs from \( p_n(x) \) by a constant.) They satisfy the recurrence relation

\[
2xp_n = A_np_{n+1} + (a + a^{-1} - A_n - C_n)p_n + C_n p_{n-1},
\]

\[
A_n = a^{-1}(1 - abq^n)(1 - acq^n), \quad C_n = a(1 - q^n)(1 - bcq^{n-1}).
\]

(20)  

As in the previous section, introduce the matrix \( M_q \) associated with the eigenvalue problem (20) and the space \( L_k \) corresponding to the indices \( n = 0, 1, \ldots, k \). Let us first consider the case when \( q \) is an \( N \)'th primitive root of unity, that is \( q = e^{2\pi i S/N} \), where \( S \) and \( N \) are positive integers which do not have a common divisor other than 1. Let us set furthermore \( ac = q \). Then, obviously, \( M_q \) preserves \( L_{N-1} \). (Moreover, the orthogonal complement of \( L_{N-1} \) to the whole space where \( M_q \) acts is also invariant with respect to \( M_q \).)

Multiplying both sides of the recurrence relation (20) by \( z^n \) and performing summation from \( n = 0 \) to \( N - 1 \), we obtain

\[
2xf(z) = \{(az)^{-1} + az\}f(z) + \{-(a^{-1} + bq^{-1})z^{-1} + a + 2b + qa^{-1} - (a + b)qz\}f(qz) + b\{((qz)^{-1} - q - 1 + q^2z)f(q^2z),
\]

(21)  

where \( f(z) = \sum_{n=0}^{N-1} z^n p_n \).

Proceeding in a similar way as in Section 2, we obtain the following set of solutions to (21) in the space spanned by \( \{z^k\}_{k=0}^{N-1} \):

\[
2x_m = aq^m + a^{-1}q^{-m},
\]

(22)  

\(^{5}\)Note that the basic hypergeometric polynomials for \( q \) a root of unity have a number of interesting properties and applications \([15]\).
where in the last formula we used the equivalence of the continuous dual $q$-Hahn polynomials at $ac = q^{-N+1}$ and the dual $q$-Hahn polynomials (to be verified below) and the expression for a generating function of the dual $q$-Hahn polynomials [4].

The solution is especially simple for $m = 0$: $2x_0 = a + a^{-1}$, $f_0(z) = 1 + z + z^2 + \ldots + z^{N-1}$. In this case we also know explicitly the zeros of $f_0(z)$: $z_i = q^i$, $i = 1, 2, \ldots, N - 1$. Note that the zeros of all $N$ solutions $f_m(z)$ can be found in the case when $b = 0$. Then it is a simple exercise to obtain, using (21), the set of zeros $z(m) = \{z_1, z_2, \ldots, z_{N-1}\}$ of $f_m(z)$:

\[ z(m) = \{q^{m+1}, q^{m+2}, \ldots, q^{N-1}, a^{-2}q^{-m+1}, a^{-2}q^{-m+2}, \ldots, a^{-2}\}, \]

\[ z(0) = \{q, q^2, \ldots, q^{N-1}\}, \quad z(N-1) = \{a^{-2}q^2, a^{-2}q^3, \ldots, a^{-2}q^{N}\} \]  

The difference operator $D_q$ (defined by the equation (21) written in the form $2xf(z) = D_qf(z)$) can be expressed in terms of the generators of the $U_{q^{1/2}}(sl_2)$ algebra represented in $\mathcal{H}_{N-1}$. In a certain representation in this space the generators have the form (we use the notation from [16]):

\[ A = q^{-\frac{N+1}{4}}T_+, \quad D = q^{\frac{N+1}{4}}T_-, \]

\[ B = z(q^{1/2} - q^{-1/2})^{-1}(q^{\frac{N+1}{2}}T_- - q^{-\frac{N+1}{2}}T_+), \]

\[ C = -z^{-1}(q^{1/2} - q^{-1/2})^{-1}(T_- - T_+), \]  

(recall that $q = e^{2\pi i S/N}$) where the operators $T_+$ and $T_-$ act on a vector $g(z) \in \mathcal{H}_{N-1}$ as follows: $T_\pm g(z) = g(q^{\pm 1/2}z)$.

As is easy to verify,

\[ D_q = A^2\{-b(1 + q^{-1})A^2 + (q^{1/2} - q^{-1/2})(bq^{-\frac{N+1}{4}}CA + aq^{-\frac{N+1}{4}}BD - bq^{\frac{N+1}{4}}BA - a^{-1}q^\frac{N+1}{4}CD) + (a + 2b + a^{-1}q)q^{\frac{N+1}{4}}\}. \]  

(26)
### 3.1.1 Azbel-Hofstadter problem

It was recently shown \[19\] that part of the spectrum of the Hamiltonian in the Azbel-Hofstadter problem (of an electron on a square lattice subject to a perpendicular uniform magnetic field) can be obtained as \((N)\) solutions \(\lambda\) of the following equation in \(H_{N-1}\):

\[
i(z^{-1} + qz)f(qz) - i(z^{-1} + q^{-1}z)f(q^{-1}z) = \lambda f(z),
\]

where \(q = e^{i\Phi}/2\). \(\Phi = 4\pi S/N\) is the flux of the magnetic field per plaquette of the lattice. (Henceforth, we assume that \(N\) is odd.) The spectrum has particularly interesting properties when \(S, N \to \infty\) so that \(S/N \to \alpha\), where \(\alpha\) is an irrational number (see, e.g., \[20,21\]). Representation of (27) in the basis of monomials gives:

\[
i(q^{n+1} - q^{-(n+1)})\hat{p}_{n+1} + i(q^n - q^{-n})\hat{p}_{n-1} = \lambda \hat{p}_n, \quad n = 0, 1, \ldots, N-1,
\]

where the polynomials \(\hat{p}_n(\lambda)\) are defined by the formula \(f(z) = \sum_{n=0}^{N-1} z^n \hat{p}_n(\lambda)\).

On the other hand, setting in (20) \(a = iq^{1/2}\) (hence, \(c = -iq^{1/2}\)), \(b = 0\), we reduce (20) to

\[
(1 - q^{n+1})\hat{p}_{n+1} + (1 - q^n)\hat{p}_{n-1} = 2x\hat{p}_n,
\]

\[
\hat{p}_n = a^{-n}p_n, \quad n = 0, 1, \ldots, N-1,
\]

If we denote the \(N \times N\) matrices corresponding to eigenvalue equations (28) and (29) by \(H\) and \(M\), respectively, then the following expression holds:

\[
H = (M - M^*)/i.
\]

In other words, \(H\) is the imaginary part of \(2M\). (Note that \(M\) and its adjoint \(M^*\) do not commute.) The spectrum of \(M\) is given by (22) with \(a = iq^{1/2}: 2x_k = 2\sin\frac{2\pi k}{N}, k = 0, 1, \ldots, N-1\).

Expression (30) provides a connection between the results of Section 3 and the Azbel-Hofstadter problem.

### 3.2 Arbitrary \(q\)

Equations (13) and (24) are valid for an arbitrary complex \(q\) (except for certain fixed values which one can treat on the basis of continuity considerations). In this general case, in order to obtain a \(q\)-difference equation with
the largest number of free parameters, we shall use the approach of Section 2. Namely, consider the polynomials associated with the transposed matrix $M_q^T$. Put $ac = q^{N-1}$, then the space $L_{N-1}$ will be invariant with respect to $M_q^T$. (Note that unlike for $q^N = 1$, in the general case the orthogonal complement of $L_{N-1}$ to the whole infinite-dimensional space is not invariant with respect to $M_q^T$.) The polynomials associated with $M_q^T$ are connected with the dual q-Hahn polynomials as follows (c.f. Section 2):

$$
\tilde{p}_n = \frac{(ab;q)_n(q^{-N+1};q)_n}{a^{2n}(q;q)_n(ba^{-1}q^{-N+1};q)_n} p_n.
$$

Proceeding as in Section 2, we obtain the following equation for the generating function $f(z) = \sum_{n=0}^{N-1} z^n \tilde{p}_n$:

$$
2xf(z) = \{az^{-1} + a^{-1}z\}f(z) + \{-a + bq^{-N}\}z^{-1} + a + b + bq^{-N} + a^{-1}q^{-N+1} - (a^{-1}q^{-N+1} + b)z \}f(qz) + bq^{-N}\{z^{-1} - q - 1 + qz\}f(q^2z),
$$

Its solutions in the space spanned by $\{z^k\}_{k=0}^{N-1}$ are

$$
2xm = aq^m + a^{-1}q^{-m}, \quad m = 0, 1, \ldots, N - 1,
$$

$$
f_m(z) = \sum_{n=0}^{N-1} z^n \frac{(ab;q)_n(q^{-N+1};q)_n}{a^{2n}(q;q)_n(ba^{-1}q^{-N+1};q)_n} \sum_{k=0}^{n} \frac{(q^{-n};q)_k(a^2q^m;q)_k(q^{-m};q)_kq^k}{(ab;q)_k(q^{-N+1};q)_k(q;q)_k}
$$

$$
(z;q)_m \sum_{k=0}^{N-1-m} \frac{(q^{m-N+1};q)_k(abq^m;q)_kq^{-mk}z^k}{(ba^{-1}q^{-N+1};q)_k(q;q)_ka^2k} =
$$

$$
(z;q)_m P_{N-1-m}(za^{-2}q^{-m-1}, ba^{-1}q^{-N}, a^2q^{-2m}|q),
$$

(33)

where $P_k(x, \alpha, \beta|q)$ are the little $q$-Jacobi polynomials. Thus the equation (31) is related to the $q$-difference equation for these polynomials [14] by the transformation $P_{N-1-m}(x) = f_m(xa^2q^{m+1})/(xa^2q^{m+1};q)_m$.

Finally, consider the dual q-Hahn polynomials. (Other known basic hypergeometric polynomials leading by the procedure of this section to equations of the type (2) can be considered as particular cases of the continuous dual q-Hahn or dual q-Hahn polynomials.) These polynomials are defined by the recurrence relation (we use $N - 1$ instead of $N$ in the usual definition
\[ (14) \]
\[
\mu(y)p_n = A_n p_{n+1} + (1 + \gamma \delta q - A_n - C_n)p_n + C_n p_{n-1},
\]
\[
A_n = (1 - q^{n-N+1})(1 - \gamma q^{n+1}), \quad C_n = \gamma q(1 - q^n)(\delta - q^{n-N}),
\]
\[
\mu(y) = q^{-y} + \gamma \delta q^{y+1}, \quad p_{-1} = 0, \quad n = 0, 1, \ldots, N - 1.
\]

Setting \( \gamma = abq^{-1}, \delta = ab^{-1}, q^{-N+1} = ac \), and multiplying the recurrence relation (14) by \( a^{-1} \), we obtain (20) where \( 2x = t + t^{-1}, t = aq^y \). Thus, the first \( N \) continuous dual q-Hahn polynomials at \( ac = q^{-N+1} \) and the dual q-Hahn polynomials are the same (up to renaming the parameters).

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