All Stable Characteristic Classes of Homological Vector Fields

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Abstract. An odd vector field $Q$ on a supermanifold $M$ is called homological, if $Q^2 = 0$. The operator of Lie derivative $L_Q$ makes the algebra of smooth tensor fields on $M$ into a differential tensor algebra. In this paper, we give a complete classification of certain invariants of homological vector fields called characteristic classes. These take values in the cohomology of the operator $L_Q$ and are represented by $Q$-invariant tensors made up of the homological vector field and a symmetric connection on $M$ by means of the algebraic tensor operations and covariant differentiation.

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1. Introduction

The unique existence theorem for solutions of ordinary differential equations ensures integrability of smooth one-dimensional distributions on differentiable manifolds. A new phenomenon arises in the category of smooth supermanifolds: Due to an extra sign factor in the definition of the supercommutator of vector fields, the classical Frobenius criterion of integrability

\[ [Q, Q] = 0 \]

is not fulfilled automatically for odd vector fields \( Q \). Rather it becomes a nontrivial condition to satisfy, \( [Q, Q] = 2Q^2 = 0 \). An odd vector field \( Q \) that squares to zero is called a **homological vector field**. The homological vector fields were first introduced by Shander [1] in his study of differential equations on supermanifolds. The local normal forms of homological vector fields were then considered by Schwarz [2] and Vaintrob [3]. In the former paper it was proposed to refer to supermanifolds with homological vector fields as \( Q \)-**manifolds**. The \( Q \)-manifolds play a prominent role both in physics and mathematics.

In theoretical physics, the homological vector fields appear usually as classical BRST differentials on the ghost-extended configuration/phase spaces of gauge theories [2], [4]. On the other hand, various mathematical concepts can be reformulated and studied in terms of \( Q \)-manifolds. An incomplete list of examples includes de Rham and Koszul complexes, \( L_\infty \) algebras [5], [6], rational homotopy types [7], Lie algebroids [8], and \( n \)-algebroids [9], [10]. The advantage of the “homological” point of view over traditional ones is its geometric clarity and flexibility. Notice that the \( Q \)-manifolds form a category, whose morphisms are just diffeomorphisms of supermanifolds that relate homological vector fields.

Having translated some class of mathematical objects in the language of \( Q \)-manifolds we get a natural definition of morphisms for the objects of interest, which may be hard to see or formulate in a classical (i.e., non-homological) approach. A typical example is the category of Lie algebroids where the homological approach offers a concise and elegant formulations for such important notions as a Lie algebroid homomorphism and an adjoint module [8]. From this perspective it is desirable to have a structure theory of \( Q \)-manifolds, which would capture both the local and global properties of homological vector fields.

In this paper, we study the global invariants of \( Q \)-manifold called **characteristic classes** [11], [12], [13]. The idea behind the construction of such invariants is as follows. Given a \( Q \)-manifold \((M, Q)\), we denote by \( T(M) \) the algebra of smooth tensor fields on \( M \) of arbitrary types \((n, m)\). The operator of Lie derivative \( \delta = L_Q \) makes the algebra \( T(M) = \bigoplus T^{n,m}(M) \) into a differential tensor algebra. Let \( H(M, Q) = \ker\delta/\operatorname{Im}\delta \) denote the group of \( \delta \)-cohomology. Since \( \delta \) respects the tensor operations – tensor product, contraction and permutation of tensor indices – the space \( H(M, Q) \) inherits the structure of tensor algebra. The algebra \( H(M, Q) \) is thus a natural invariant of the \( Q \)-manifold \( M \).

Unfortunately, this invariant is hard to compute even in a topologically trivial situation.

\(^1\)Of course, if we want to treat \( H \) as a functor from the category of \( Q \)-manifolds to the category of tensor algebras, then we should restrict ourselves to the subalgebra of covariant tensor fields on \( M \).
This is due to possible local singularities of the homological vector field. The way out is to consider a special differential subalgebra $A \subset T(M)$ called the algebra of concomitants. Given a symmetric affine connection $\nabla$ on $M$ with curvature $R$, by a concomitant associated to the triple $(M, Q, \nabla)$ we understand a tensor field on $M$ which is made up of the homological vector field $Q$, the curvature tensor $R$, and their covariant derivatives by means of the tensor operations. According to the classical reduction theorem \[14\] the concomitants exhaust all the natural tensor fields associated to $Q$ and $\nabla$. Also the set of all concomitants is invariant under the action of $\delta$. We say that a $\delta$-closed concomitant $C \in A$ is a universal cocycle if the closedness condition $\delta C = 0$ follows from the integrability condition \[\text{(1)}\] regardless of any specificity of $Q$, $\nabla$ and $M$. In other words, the universal cocycles are universal $Q$-invariant tensor polynomials in $\nabla^n Q$ and $\nabla^m R$ that can be attributed to any $Q$-manifold with connection. (This, a little bit vague, definition of “universality” can be made precise using the notion of a graph complex associated to the differential algebra of concomitants $A$, see Section \[3\]) The stable characteristic classes of $Q$-manifolds are now defined to be the elements of $H(M, Q)$ that are represented by the universal cocycles. A remarkable fact \[13\] is that the $\delta$-cohomology classes of universal cocycles do not depend on the choice of the symmetric connection and hence they are invariants of a $Q$-manifold as such. Furthermore, the nontrivial universal cocycles admit a fairly explicit description in contrast to the group $H(M, Q)$.

The algebra of concomitants is naturally graded, $A = \bigoplus A^k_{n,m}$; here the subscripts $(n, m)$ refer to the tensor type of concomitants, while the superscript $k$ is the degree of homogeneity in $Q$ and its derivatives:

$$A^k \ni C[Q] \iff t^k C[Q] = C[tQ] \quad \forall t \in \mathbb{R}.$$ 

Since $\delta : A^k_{n,m} \to A^{k+1}_{n,m}$, we have a direct sum of finite-dimensional complexes. The stable characteristic classes form a subgroup $H_{st}(A)$ in the $\delta$-cohomology group $H(A) = \bigoplus H(A^k_{n,m})$. The computation of the groups $H(A)$ and $H_{st}(A)$ is somewhat facilitated by the fact that the algebra of concomitants contains a differential ideal $R \subset A$ generated by the concomitants of the symmetric connection $\{\nabla^m R\}$. The corresponding short exact sequence of complexes

$$0 \longrightarrow R \overset{i}{\longrightarrow} A \overset{p}{\longrightarrow} A/R \longrightarrow 0$$

gives rise to the exact triangle in cohomology

$$\begin{array}{ccc}
H(R) & \overset{i_*}{\longrightarrow} & H(A) \\
\downarrow \delta & & \downarrow p_* \\
H(A/R) & \overset{p_*}{\longrightarrow} & H(A) \\
\end{array}$$

Geometrically, one can view $H(A/R)$ as the space of characteristic classes of flat $Q$-manifolds, i.e., $Q$-manifolds admitting a flat symmetric connection. The universal cocycles

\[\text{We use the adjective “stable” to emphasize that there may exist other characteristic classes which are specific to $Q$-manifolds of any particular dimension, see Remark \[3.1\] below.}\]
of flat $Q$-manifolds are constructed from the $(1, n)$-tensors $\nabla^n Q$, which are symmetric in lower indices. If the connecting homomorphism $\partial$ is nonzero, not any characteristic class can be extended from the flat to arbitrary $Q$-manifolds and the obstruction to extendability is controlled by the elements of $\text{Im} \partial$. In [13], the extendable characteristic classes, or more precisely the elements of $H(A)/\text{Im}\epsilon_*$, were called intrinsic. The intrinsic characteristic classes survive on flat $Q$-manifolds, therefore they are more closely related to the structure of the homological vector field rather than the topology of $M$. In the stable situation, both the intrinsic characteristic classes and the characteristic classes of flat $Q$-manifolds were explicitly computed in [13]. What has remained an open question is whether there are nontrivial universal cocycles lying in $\mathcal{R}$. In this paper, we show that the answer is negative so that all the stable characteristic classes are in fact intrinsic. The proof of this fact is given in Section 4 and does not exploit the short exact sequence above; instead, we use a special classification of stable characteristic classes of $Q$-manifolds.

**Conventions and notation.** Throughout the paper we work in the category of smooth supermanifolds. This allows us to omit the boring prefix “super” whenever possible. So the terms like manifolds, functions, algebras and so on will actually mean the corresponding notions of supergeometry.

Given a manifold $M$, we denote by $\mathfrak{X}(M)$ the space of smooth vector fields on $M$. The space $\mathfrak{X}(M)$ carries both the structure of a real Lie algebra with respect to the commutator of vector fields and the structure of a $C^\infty(M)$-module. The endomorphisms of the module $\mathfrak{X}(M)$ form an associative algebra $\mathfrak{A}(M)$ over $C^\infty(M)$. The elements of $\mathfrak{A}(M)$ are smooth tensor fields of type $(1, 1)$. The operation of contraction of tensor indices endows $\mathfrak{A}(M)$ with the natural trace $\text{Str} : \mathfrak{A}(M) \rightarrow C^\infty(M)$. We let $\Omega(M) = \bigoplus \Omega^n(M)$ denote the algebra of exterior differential forms on $M$.

In our study of the differential tensor algebra of concomitants $A$ we will mostly deal with the smooth tensor fields of type $(1, n)$. Any such tensor field is naturally identified with a $C^\infty(M)$-linear map from $\mathfrak{X}(M)^{\otimes n}$ to $\mathfrak{X}(M)$ and, in the sequel, we will freely use this identification. We let $O_S$ denote the fully symmetric part of a $(1, n)$-tensor $O$ so that $(O_S)_S = O_S$. Then $O_S$ defines a map from the symmetrized tensor power $\mathfrak{X}(M)^{\otimes n}$ to $\mathfrak{X}(M)$. Given a $(1, n)$-tensor field $O(X_1, \ldots, X_n)$, we denote by $(\nabla^k O)(X_1, \ldots, X_k)(X_{k+1}, \ldots, X_{k+n})$ the value of the $k$th covariant derivative of $O$ evaluated on $k + n$ vector fields $X_1, \ldots, X_{k+n}$. To avoid any ambiguity concerning sign factors let us write an explicit expression in terms of local coordinates,

\[
(\nabla^k O)(X_1, \ldots, X_k)(X_{k+1}, \ldots, X_{k+n}) = (-1)^\varepsilon X_{i_k+n}^i \cdots X_{i_1+n}^i \nabla_{i_1} \cdots \nabla_{i_k} O_{i_{k+1} \cdots i_{k+n}}^j \frac{\partial}{\partial x^j},
\]

\[
\varepsilon = \sum_a \epsilon(X_a)(\epsilon_{i_{a+1}} + \cdots + \epsilon_{i_{k+n}}).
\]

All covariant or partial derivatives are assumed to act from the left.
2. The Losik-Janyska-Markl Basis of Concomitants

The elementary concomitants \( \{\nabla^n Q, \nabla^m R\} \) generating the algebra \( \mathcal{A} \) are not free; rather they satisfy an infinite number of tensor relations coming from the integrability condition for \( Q \) and the Bianchi-Ricci identities for \( \nabla \). To take into account these relations we pass on to another generating set of concomitants, which, similar to the set of elementary concomitants, consists of two parts. One part was introduced, in fact, by Losik in the context of Gelfand-Fuks cohomology [15], while the other appeared in the recent paper by Janyska and Markl [16].

Define the curvature tensor of \( \nabla \) by
\[
R(X_1, X_2, X_3) = ([\nabla_{X_3}, \nabla_{X_2}]X_1 - \nabla_{[X_3, X_2]}X_1).
\]

Losik’s part of the generating set concerns the concomitants that involve the covariant derivatives of \( Q \). Following [15], we introduce a sequence of symmetric \((1,n)\)-tensors \( Q_n \) of the form
\[
Q_n(X_1, \ldots, X_n) = (\nabla^n Q)_S(X_1, \ldots, X_n) - (\nabla^{n-2} R_Q)_S(X_1, \ldots, X_{n-2})(X_{n-1}, X_n),
\]
where the \((1,2)\)-tensor \( R_Q \) is defined by
\[
R_Q(X_1, X_2) = (-1)^{\epsilon(X_1) + \epsilon(X_2)} R(X_1, X_2, Q)
\]
and the second term in the r.h.s. of (3) is absent for \( n = 0, 1 \). Besides the total symmetry in lower indices, the concomitants (3) satisfy an infinite sequence of algebraic relations of the form
\[
Q_n(Q, X_1, \ldots, X_{n-1}) + (\text{terms involving } Q_m \text{ with } m < n) = 0.
\]

All these relations are quadratic in \( Q \)'s and obtained by repeated differentiation of integrability condition (1).

The special convenience of the concomitants (3) is that they generate a tensor algebra, which is “almost” closed under the action of the differential. We have
\[
\delta Q_0 = 0, \quad (\delta Q_1)(X) = Q_1(Q_1(X)) - \frac{1}{2} R(X, Q, Q), \quad (\delta Q_2)(X_1, X_2) = 0,
\]
\[
(\delta Q_n)(X_1, \ldots, X_n) = O_S(X_1, \ldots, X_n), \quad n > 2,
\]
where
\[
O(X_1, \ldots, X_n) \equiv - \sum_{k=1}^{n-2} \binom{n}{k} (-1)^{\sum_i \epsilon(x_i)} Q_{k+1}(X_1, \ldots, X_k, Q_{n-k}(X_{k+1}, \ldots, X_n)).
\]

The only “bad” concomitant is \( Q_1 \). Excluding \( Q_1 \), we get a differential tensor subalgebra \( \mathcal{Q} \subset \mathcal{A} \) generated by all \( Q_n \)’s with \( n \neq 1 \). The algebra \((\mathcal{Q}, \delta)\) enjoys an increasing filtration \( 0 \subset \mathcal{Q}_2 \subset \mathcal{Q}_3 \subset \cdots \subset \mathcal{Q}_\infty = \mathcal{Q} \), where the \( n \)th differential subalgebra \( \mathcal{Q}_n \) is generated by the concomitants \( Q_0, Q_2, \ldots, Q_n \). Each \( \mathcal{Q}_n \) contains a differential subalgebra \( \mathcal{Q}'_n \) constituted by all the concomitants of \( \mathcal{Q}_n \) that do not involve tensor contractions of
the homological vector field \( Q = Q_0 \) with the other generators \( Q_2, \ldots, Q_n \). With these definitions the main result of the paper can be expressed by the relation

\[
H_{st}(\mathcal{A}) = Q'_2/(Q'_2 \cap \delta Q'_3).
\]

In particular, the concomitants of the symmetric connection \( \{\nabla^n R\} \), being considered as a set of generators completing \( (3) \) to a multiplicative basis in \( \mathcal{A} \), do not contribute to the stable cohomology at all. The proof will be given in Section 4.

Let us now turn to Janyska-Markl’s part of the multiplicative basis in \( \mathcal{A} \). It was shown in \([16]\) that for any symmetric connection with curvature \( (2) \) one can associate a sequence of \((1, n)\)-tensor fields \( R_n \) of the form

\[
R_n(X_1, \ldots, X_n) = (\nabla^{n-3} R)(X_1, \ldots, X_{n-3})(X_{n-2}, X_{n-1}, X_n) + K_n,
\]

such that \( K_n \) is made up of \( R_k \) with \( k < n \) and the tensors \( (6) \) enjoy the following symmetries:

(S1): the antisymmetry in \( X_{n-1} \) and \( X_n \),
(S2): the cyclic symmetry in \( X_{n-2}, X_{n-1} \) and \( X_n \):

\[
(-1)^{\varepsilon(X_{n-1})+\varepsilon(X_{n-2})+\varepsilon(X_n)} R_n(X_1, \ldots, X_{n-3}, X_{n-2}, X_{n-1}, X_n)
\]

\[
+(-1)^{\varepsilon(X_{n-2})+\varepsilon(X_n)+\varepsilon(X_{n-1})} R_n(X_1, \ldots, X_{n-3}, X_n, X_{n-2}, X_{n-1})
\]

\[
+(-1)^{\varepsilon(X_n)+\varepsilon(X_{n-1})+\varepsilon(X_{n-2})} R_n(X_1, \ldots, X_{n-3}, X_{n-1}, X_n, X_{n-2}) = 0,
\]
(S3): for \( n \geq 4 \), the cyclic symmetry in \( X_{n-3}, X_{n-1} \) and \( X_n \):

\[
(-1)^{\varepsilon(X_{n-3})+\varepsilon(X_{n-2})+\varepsilon(X_n)} R_n(X_1, \ldots, X_{n-3}, X_{n-2}, X_{n-1}, X_n)
\]

\[
+(-1)^{\varepsilon(X_{n-1})+\varepsilon(X_{n-2})+\varepsilon(X_{n-3})} R_n(X_1, \ldots, X_{n-3}, X_{n-1}, X_{n-2}, X_n)
\]

\[
+(-1)^{\varepsilon(X_n)+\varepsilon(X_{n-1})+\varepsilon(X_{n-2})} R_n(X_1, \ldots, X_n, X_{n-2}, X_{n-3}, X_{n-1}) = 0,
\]
(S4): for \( n \geq 5 \), the total symmetry in \( X_1, \ldots, X_n \):

\[
R_3 = R, \quad R_4 = \nabla R.
\]

In this case, property (S1) follows from the standard antisymmetry of the curvature tensor \( (2) \), and the properties (S2), (S3) reduce to the first and second Bianchi identities. Therefore one can regards the properties (S2), (S3) as the higher-order generalization of the Bianchi identities for the curvature tensor. The explicit calculation of the tensors \( K_n \) appears to be quite a difficult task even for \( n = 5 \) and the complexity grows rapidly with \( n \). Fortunately, the concrete form of \( K_n \)'s is absolutely inessential for our subsequent considerations. What we will actually use is two facts: (i) the tensors \( R_n \) generate the whole algebra of concomitants associated to the symmetric connection and (ii) the generators \( R_n \) obey no universal algebraic relations except for linear relations (S1)-(S4).
Applying the Lie derivative $\delta$ to the concomitants (6) yields

$$\begin{align*}
(\delta R_n)(X_1, \ldots, X_n) &= R_{n+1}(Q, X_1, \ldots, X_n) \\
&+ \sum_{k=1}^{n} (-1)^{\sum_{i<k} \epsilon(X_i)} R_n(\ldots, Q_1(X_k), \ldots) - Q_1(R_n(X_1, \ldots, X_n)) + \cdots.
\end{align*}$$

Here the dots stand for terms that are at least bilinear in $R_k$’s with $k < n$. Again, the explicit form of the omitted terms is inessential for our subsequent calculations.

Taken together the concomitants $\{Q_n\}$ and $\{R_n\}$ constitute a multiplicative basis in the differential tensor algebra $A$.

### 3. THE GRAPH COMPLEX

The differential tensor algebra of concomitants $(A, \delta)$ admits a very helpful visualization in terms of finite graphs with legs. The relevant graphs are composed of black and white vertices assigned to the basis concomitants:

$$Q_n(X_1, \ldots, X_n) \leftrightarrow \quad R_n(X_1, \ldots, X_n) \leftrightarrow$$

As is seen the edges incident to the vertices are directed and each vertex is allowed to have the only outgoing and several incoming edges. The edges represent, respectively, the contravariant and covariant tensor indices. The planar embedding of the vertex graphs induces the natural left-to-right ordering of the incoming edges. This ordering, however, is crucial only for the white vertices, since the basis concomitants corresponding to black vertices are fully symmetric in covariant indices. Gluing together incoming and outgoing edges of the vertices above one can produce more general graphs, which will describe contraction schemes for the tensor indices of composite concomitants. The graphs we define in such a way need not be connected and loops are allowed. The remaining uncontracted tensor indices correspond to legs, i.e., edges bounded by a vertex from one side and having “free end” on the other. To indicate the order of the indices, the incoming and outgoing legs are numbered, separately, by consecutive integers and this defines a decoration of a graph. Permutations of tensor indices result then in permutations of labels on the legs. Since the algebra of functions on a supermanifold is supercommutative rather than commutative in the usual sense, we should take into account the sign factors arising upon permutation of different tensor components in composite concomitants. This leads us to the concept of orientation. By definition, an orientation on a graph $\Gamma$ is determined by ordering the vertices of $\Gamma$. Two orientations are the same if they obtained from one another by a permutation of numbers of black and white vertices with even number of black-vertex swaps. Finally, to allow for the integrability condition $Q^2 = 0$ and its differential consequences (4) we exclude from consideration the graphs that have at least one edge joining a univalent black vertex with another black vertex. The remaining graphs will be referred to as $A$-graphs.
(S1) Antisymmetry in the last two incoming edges

(S2) The first Bianchi identity

(S3) The second Bianchi identity

(S4) Total symmetry in the first $n-3$ incoming edges

Figure 1. Symmetries of white vertices

Now we are ready to define a graph complex $(G, \partial)$ associated to the differential tensor algebra of concomitants $\mathcal{A}$. The group of $k$-cochains $G^k$ is, by definition, a quotient of the real vector space spanned by $\mathcal{A}$-graphs with $k$ black vertices and arbitrary number of white vertices:

$$G^k = \mathbb{R}[\text{\mathcal{A}-graphs with } k \text{ black vertices}] / \text{relations},$$

where the relations are of two sorts:

1. (Orientation) $(\Gamma, -\text{or}) = - (\Gamma, \text{or})$.
2. (Vertex symmetries) The order of edges coming to the black vertices is considered to be inessential (total symmetry), while the symmetry of white vertices is described by the equivalence relations shown in Fig. 1.

Thus, $G^k$ is spanned by $\mathcal{A}$-graphs with $k$ black vertices and arbitrary number of white vertices. We set

$$G = \bigoplus_{k \geq 0} G^k.$$

The coboundary operator $\partial : G^k \to G^{k+1}$ is defined to be the graphical representation of the differential $\delta$ given by Rel. (5), (7). Namely, let $\Gamma$ be an $\mathcal{A}$-graph and let $v$ be a
All Stable Characteristic Classes

\[ \partial \begin{pmatrix} \lambda \end{pmatrix} = \sum_{I' \cup I'' = I \atop |I'| > 0, |I''| > 1} \begin{pmatrix} \lambda \end{pmatrix}, \quad \partial \begin{pmatrix} \lambda \end{pmatrix} = 0, \]

\[ \partial \begin{pmatrix} \lambda \end{pmatrix} = \begin{pmatrix} 1 & 2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 3 \end{pmatrix}, \quad \partial \begin{pmatrix} 1 \end{pmatrix} = 0, \]

\[ \partial \begin{pmatrix} \lambda \end{pmatrix} = \begin{pmatrix} 1 & 2 & \cdots & n \end{pmatrix} - \begin{pmatrix} 1 & 2 & \cdots & n \end{pmatrix} + \sum_{k} \begin{pmatrix} 1 & 2 & \cdots & n \end{pmatrix} + \cdots \]

**Figure 2.** Action of the coboundary operator on the black and white vertices. In the first equality \(|I| \geq 3\). The omitted terms in the r.h.s. of the last equality are given by graphs with two or more white vertices.

vertex of \( \Gamma \). Define an element \( \Gamma_v \in \mathcal{G} \) as follows. If the valency of \( v \) is 1 or 3, then \( \Gamma_v \) is zero. In the opposite case, \( \Gamma_v \) is obtained from \( \Gamma \) by replacing the vertex \( v \) with a linear combination of graphs \( \partial v \) as shown in Fig. 2. The orientation on \( \Gamma_v \) is determined by the following rule: choose a representative of the orientation on \( \Gamma \) so that \( v \) is the first vertex, then the new vertices of \( \partial v \) are numbered as in Fig. 2 and the numbers of all other vertices increase by 1. With this orientation convention the map

\[ \partial \Gamma = \sum_{v \in \Gamma} \Gamma_v \]

is a coboundary operator. Denote the corresponding cohomology groups by \( H(\mathcal{G}) = \bigoplus H^k(\mathcal{G}) \).

Clearly, the above relation between \( \mathcal{A} \)-graphs and concomitants gives rise to an epimorphism \( \phi : \mathcal{G} \to \mathcal{A} \) of vector spaces. Moreover, the way we have defined the operator \( \partial \) shows that \( \phi \) is a cochain map. The following definition is central for our consideration.

**Definition 3.1.** The stable characteristic classes of \( Q \)-manifolds are the \( \delta \)-cohomology classes belonging to the image of the homomorphism

\[ H(\phi) : H(\mathcal{G}) \to H(\mathcal{A}). \]

**Remark 3.1.** One can view each element \( g \in \mathcal{G} \) as defining a (nonlinear) differential operator \( \widehat{g} \) on homological vector fields and symmetric connections with values in tensor fields. The assignment \( g \mapsto \widehat{g} \) defines a homomorphism \( \widehat{\phi} : \mathcal{G} \to \mathfrak{Mat}_{Q, \nabla} \) from the space of graphs to the space of natural differential operators [17], [18] that act on a homological vector field \( Q \) and a symmetric connection \( \nabla \). The homomorphism \( \phi \) decomposes as
\[ \phi = \text{ev}_{Q,\nabla} \circ \hat{\phi}, \]
where \( \text{ev}_{Q,\nabla} \) is the evaluation map. The map \( \hat{\phi} \) is known to be surjective and so is the map \( \phi \). The kernel of \( \text{ev}_{Q,\nabla} \) depends on a particular form of \( Q \) and \( \nabla \), while the kernel of \( \hat{\phi} \) is completely determined by the dimension of the underlying supermanifold \( M \).

Namely, adapting the Main Theorem of Invariant Theory [17], [19] to our situation, one can argue that the map \( \hat{\phi} \) is an isomorphism in stable range of dimensions. More precisely, if \( G^{(m,n)} \subset G \) is the subspace of graphs with \( m \) vertices and \( n \) legs, then the restriction of \( \hat{\phi} \) to \( G^{(m,n)} \) is a bijection provided that \( \min(s, t) \gg \min(m, n) \), where \( (s, t) = \dim M \).

Some exact evolutions of the low bound of stable dimensions can be found in [18], [20], [21]. Beyond the stable range the map \( H(\phi) \) is neither surjective nor injective. In the Introduction, the space of stable characteristic classes \( \text{Im} H(\phi) \) was denoted by \( H_{st}(A) \).

Remark 3.2. The graph complex \( G \) underlying the definition of stable characteristic classes is not the only possible or most natural choice. In Sec. 4.2 we will also consider a quotient complex \( G/P \) associated to a certain subcomplex \( P \subset G \). The homomorphism \( \phi \) passes through the quotient for an appropriate choice of symmetric connection, giving rise to an additional series of characteristic classes (A-series). It is the A-series of invariants of \( Q \)-manifolds that was originally discovered in [11] and called the principal series of characteristic classes.

The rest of the paper is devoted to computation of the graph cohomology.

Since \( \partial \) does not affect the legs of \( A \)-graphs, the graph complex is decomposed into the direct sum of subcomplexes

\[ G = \bigoplus_{n,m} G_{n,m}, \]

where the subscripts \( n \) and \( m \) refer to the number of incoming and outgoing legs of graphs. Another remarkable property of the coboundary operator \( \partial \) is that it neither permutes the connected components of an \( A \)-graph nor changes their number. This leads to the further decomposition of \( G_{n,m} \) into the direct sum of subcomplexes \( G_{n,m}^{I_1,...,I_k;J_1,...,J_k} \), where \( \{I_1, ..., I_k\} \) and \( \{J_1, ..., J_k\} \) are partitions of the sets \( \{1, ..., n\} \) and \( \{1, ..., m\} \). The complex \( G_{n,m}^{I_1,...,I_k;J_1,...,J_k} \) is generated by graphs with \( k \) connected components such that the incoming and outgoing legs of the \( l \)th component are labelled by the element of \( I_l \) and \( J_l \), respectively. Notice that the sets \( \{I_1, ..., I_k\} \) and \( \{J_1, ..., J_k\} \) are defined up to simultaneous permutations of \( I_q \) with \( I_p \) and \( J_q \) with \( J_p \), and some of the sets \( I_1, ..., I_k, J_1, ..., J_k \) may be empty. Let \( \bar{G} = \bigoplus \bar{G}_{n,m} \) denote the subcomplex of connected graphs. It is clear that

\[ G_{n,m}^{I_1,...,I_k;J_1,...,J_k} \cong \bigotimes_{l=1}^{k} G_{|I_l|,|J_l|}, \]

and by the Künneth formula the computation of the graph cohomology boils down to the computation of the groups \( H^k(\bar{G}_{n,m}) \).

The characteristic classes that belong to the image of the connected graph cohomology under the map (9) will be called primitive. A linear basis in the space of all characteristic classes is made up of the primitive characteristic classes by means of tensor products and permutations of tensor indices.
4. The cohomology of the connected graph complex

We start with the observation that the complex of connected graphs \( \tilde{G} \) splits into a direct sum of four subcomplexes,

\[
\tilde{G} = \tilde{G}^{(1)} \oplus \tilde{G}^{(2)} \oplus \tilde{G}^{(3)} \oplus \tilde{G}^{(4)}.
\]

Here \( \tilde{G}^{(1)} \) is the one dimensional complex spanned by the graph \( \bullet \to \). To describe the complex \( \tilde{G}^{(2)} \) it is convenient to introduce a special notation \(^3\) for a graph entering the third relation in Fig. 2, namely,

\[
\begin{array}{c}
\frac{1}{2} \\
2 \\
1
\end{array} \equiv \begin{array}{c}
\frac{1}{2} \\
1
\end{array}.
\]

(10)

The action of the differential on the bivalent white vertex reads

\[
\partial \left( \begin{array}{c}
\frac{1}{2} \\
1
\end{array} \right) = \begin{array}{c}
\frac{1}{2} \\
\frac{1}{2}
\end{array} - \begin{array}{c}
1 \\
1
\end{array}.
\]

As is seen the vector space of graphs composed of black and white bivalent vertices is invariant under the action of \( \partial \) and we identify this space with \( \tilde{G}^{(2)} \). The graphs generating \( \tilde{G}^{(3)} \) contain only black vertices of valency \( \geq 3 \). Finally, the linear span of all other \( \mathcal{A} \)-graphs defines the complex \( \tilde{G}^{(4)} \).

4.1. The cohomology of \( \tilde{G}^{(1)} \). It is clear that \( H(\tilde{G}^{(1)}) \cong \tilde{G}^{(1)} \cong \mathbb{R} \).

4.2. The cohomology of \( \tilde{G}^{(2)} \). It was shown in [13] that the cohomology of \( \tilde{G}^{(2)} \) is trivial. This a little bit disappointing fact means that we have no nontrivial characteristic classes associated to the graph complex \( \tilde{G}^{(2)} \). The situation, however, is not so hopeless as might appear. Observe that the complex \( \tilde{G}^{(2)} \) contains the subcomplex \( \mathcal{P} \) spanned by the cocycles

\[
\Pi_{2n-1} = -(\frac{n-3}{2n-1}) \begin{array}{c}
1 \\
1
\end{array}, \quad \partial \Pi_{2n-1} = 0, \quad \forall n \in \mathbb{N}.
\]

Let \( \pi: \tilde{G}^{(2)} \to \tilde{G}^{(2)}/\mathcal{P} \) denote the canonical projection. The acyclicity of \( \tilde{G}^{(2)} \) implies two facts: (i) there exists a cochain \( \Psi_{2n-1} \in \tilde{G}^{(2)} \) such that \( \Pi_{2n-1} = \partial \Psi_{2n-1} \) and (ii) \( \Psi_{2n-1}' = \pi(\Psi_{2n-1}) \) is a nontrivial cocycle of \( \tilde{G}^{(2)}/\mathcal{P} \). The relative cocycles \( \Psi_{2n-1} \) admit an explicit description [11]. Namely, each \( \Psi_{2n-1} \) is represented by a linear combination of cyclic graphs

\[
\Psi_{2n-1} = \psi_1 + \psi_2 + \ldots + \psi_{2n-1}
\]

composed of the black and white bivalent vertices, where the graph

---

\(^3\)This cannot cause a confusion since the basis white vertices \(^8\) have valency \( \geq 4 \).
ψ_1 = \[ \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array} \]

consists of \(4n - 3\) black vertices numbered sequentially and the other \(2n - 2\) terms in (11) are obtained from each other by the successive action of three linear operators:

\[ \psi_m = CBA(\psi_{m-1}) . \]

To describe the action of the operators \(A, B,\) and \(C\) we need the following terminology. The white vertices divide a cyclic graph \(\Gamma\) into several arcs endowed with black vertices. The length of an arc is, by definition, the number of its black vertices.

A: The operator \(A\) acts successively on the black vertices of \(\Gamma\) by inverting their color and multiplying the result by \((-1)^l\), where \(l\) is the number of black vertices preceding the inverted one. The labels on the vertices remain the same. Adding all such graphs up one gets \(A(\Gamma)\).

B: The operator \(B\) multiplies the graph \(\Gamma\) by \(1/k\), where \(k\) is the number of arcs of nonzero length.

C: The operator \(C\) acts successively on the nonzero-length arcs of \(\Gamma\) by decreasing their length by 1. More precisely, it removes the first black vertex of an arc and multiplies the result by \((-1)^l\), where \(l\) is the number of black vertices preceding the removed one. If the removed vertex was labelled by \(k\), then the labels less than \(k\) remain intact while the labels greater than \(k\) are shifted down by 1. By summing over all nonzero-length arcs of \(\Gamma\) one gets \(C(\Gamma)\).

Let us now explain the relevance of the cocycles \(\Psi'_{2n-1} \in \bar{G}^{(2)}/\mathcal{P}\) to the characteristic classes of \(Q\)-manifolds. Recall that to any symmetric connection \(\nabla\) one can associate a sequence of closed \(2m\)-forms

\[ P_m = \text{Str}(R^m) \in \Omega^{2m}(M) \]

called the \(m\)th Pontryagin character of \(M\). On any supermanifold, there exists a special symmetric connection \(\nabla\) with the property that \(P_{2n-1} = 0\) for all \(n \in \mathbb{N}\) (see e.g. [13]). This results in triviality of all Pontryagin’s characters with odd \(m\)’s, while the other characters may well be nontrivial. Denote by \(\mathcal{A}'\) the algebra of concomitants constructed by this special connection. It is clear that

\[ \phi(\Pi_{2n-1}) = \left(4^{n-3}_{2n-1}\right)P_{2n-1}(Q^{\otimes(4n-2)}) . \]

Since the last expression vanishes in \(\mathcal{A}'\), we have a well-defined cochain map \(\phi : \bar{G}/\mathcal{P} \to \mathcal{A}'\), which takes \(\Psi'_{2n-1}\) to the \(\delta\)-cocycle \(A_{2n-1} = \phi(\Psi'_{2n-1}) \in \mathcal{A}'\). Following the prescriptions above, one can easily write down explicit expressions for the scalar concomitants \(A_{2n-1}\) with small \(n\)’s. To do this in a compact way, it is convenient to identify the covariant derivative \(Q_1 = \nabla Q \in \mathcal{A}'\) with the right endomorphism \(\Lambda \in \mathfrak{A}(M)\) defined by
the rule: $\Lambda(X) = \nabla_X Q$ for all $X \in \mathfrak{X}(M)$. Similarly, define the right endomorphism $R \in \mathfrak{A}(M)$ by setting $R(X) = \frac{1}{2} R(X, Q, Q)$. With these definitions we have

$A_1 = \text{Str}(\Lambda),$ 

$A_3 = \text{Str}(\Lambda^5 + 5RA^3 + 10R^2\Lambda),$ 

$A_5 = \text{Str}(\Lambda^9 + 9RA^7 + 18R^2\Lambda^5 + 9R^2RA^4 + 9R^2RA^3 + 45R^3\Lambda^3 + 21R^2ARA^2 + 15R^2A^2RA + 3RARARA + 126R^4\Lambda).$

The infinite sequence of $\delta$-cohomology classes $[A_{2n-1}]$ is called the $A$-series of characteristic classes [13]. In the particular case of homological vector fields on $NQ$-manifolds of degree 1 (see e.g. [8], [9], [10]) the corresponding characteristic classes of $A$-series add up to the secondary characteristic classes of Lie algebroids [22], [23].

4.3. The cohomology of $\bar{G}^{(3)}$. It was found in [13] that the computation of $H(\bar{G}^{(3)})$ is essentially equivalent to the computation of the stable cohomologies of a Lie algebra of formal vector fields with tensor coefficients. The latter was done by Fuks in [24]. Before stating the result let us note that the graphs spanning $\bar{G}^{(3)}$ may have at most one outgoing leg. Since the coboundary operator does not mix graphs with different numbers of incoming and outgoing legs, we have the following decomposition:

\begin{equation}
H(\bar{G}^{(3)}) = \left( \bigoplus_{q,n} H^q(\bar{G}^{(3)}_{1,n}) \right) \oplus \left( \bigoplus_{p,m} H^p(\bar{G}^{(3)}_{0,m}) \right).
\end{equation}

The space $\bar{G}^{(3)}_{1,n}$ is spanned by tree connected graphs with $n$ incoming and 1 outgoing legs, while the graphs from $\bar{G}^{(3)}_{0,m}$ contain exactly one cycle, $m$ incoming and no outgoing legs.

**Theorem 4.1.** All nontrivial groups in the sum (12) have the following dimensions:

$$\dim H^{n-1}(\bar{G}^{(3)}_{1,n}) = (n-1)!, \quad \dim H^n(\bar{G}^{(3)}_{0,m}) = (n-1)!. $$

As a basis of the nontrivial cocycles one can take the graphs depicted in Fig. [3]. By definition, the $B$-series of characteristic classes is spanned by the tree graphs with $n$ incoming legs and $n-1$ trivalent vertices. The left-most leg of these graphs is labelled by 1 and the labels on the other $n-1$ incoming legs may be chosen arbitrary. The total number of different decorated graphs of type $B_n$ is thus $(n-1)!$. The graphs of $C$-series have the form of a cycle composed of $n$ trivalent vertices with $n$ incoming legs. Since the cyclic permutations of labels on the incoming legs preserve the isomorphism class of a decorated graph $C_n$, there are exactly $(n-1)!$ different decorations.

The closedness of the graphs $B_n$ and $C_n$ readily follows from the fact that the coboundary operator $\partial$ annihilates the trivalent black vertex as is seen from Fig. [2]. So any trivalent graph is a cocycle. Interestingly enough the vertices of valency higher than 3 do not contribute to the cohomology. In analytical terms this means that the corresponding tensor cocycles of $A$ can be chosen to involve no more than second covariant derivatives of the
homological vector field. Using the definition of the basis concomitants \( \mathcal{B} \), one can easily assign the tensor expressions for the graphs in Fig. 3. Let us interpret the basis generator \( Q_2 \), which corresponds to the trivalent black vertex, as a \( C^\infty(M) \)-module homomorphism \( \mathfrak{X}(M) \to \mathfrak{A}(M) \) that takes a vector field \( X \) to the right endomorphism \( Q_2(X) \):

\[
Q_2(X)(Y) = Q_2(X,Y) \quad \forall Y \in \mathfrak{X}(M).
\]

Now, identifying the \( (1, n+1) \)-tensors with the homomorphisms \( \mathfrak{X}(M)^\otimes n \to \mathfrak{A}(M) \), we can write

\[
B_n(X_1, X_2, \ldots, X_n) = (-1)^{\sum_k \epsilon(X_{2k})} Q_2(X_n)Q_2(X_{n-1}) \cdots Q_2(X_1).
\]

The concomitants \( C_n \) are then defined by

\[
C_n(X_1, X_2, \ldots, X_n) = \text{Str}B_n(X_1, X_2, \ldots, X_n).
\]

Permuting the arguments \( X_1, X_2, \ldots, X_n \) one gets the basis of nontrivial \( \delta \)-cocycles.

4.4. The cohomology of \( \bar{G}^{(4)} \). Below we prove that the complex \( \bar{G}^{(4)} \) is acyclic and this gives the main result of the paper.

Let us introduce the following terminology. By multivalent vertices we will mean the black and white vertices of valency \( \geq 3 \). Besides multivalent vertices, each graph \( \Gamma \in \bar{G}^{(4)} \) is allowed to have some number of univalent and bivalent black vertices. A branch is, by definition, a connected subgraph of \( \Gamma \) given by a maximal string of bivalent vertices bounded by one or two multivalent vertices. Graphically, a typical branch looks like

\[
\gamma_{\alpha\beta}^{\alpha\beta} = \quad \gamma_{\alpha\beta}^{\alpha\beta}
\]

The marks \( \alpha, \beta \in \{\bullet, \circ, \varnothing\} \) denote three possible types of boundary vertices (the symbol \( \varnothing \) is used to indicate that the branch ends with a leg).

**Lemma 4.2.** For any graph of \( \bar{G}^{(4)} \) at least one of the following statements is true:

1. there is a branch of nonzero length;
2. there is an edge joining black and white multivalent vertices;
3. there is a leg adjacent to a white vertex.

\footnote{By abuse of notation, we use the same symbol for a graph cocycle and its image under the map \( \mathfrak{g} \).}
Proof. Suppose a connected graph \( \Gamma \in \bar{G}^{(4)} \) fails to meet the first and second conditions. Then, there are two options: either \( \Gamma \) involves only black vertices or it consists of white multivalent and black univalent vertices. In the former case \( \Gamma \) must belong to \( \bar{G}^{(3)} \) and not to \( \bar{G}^{(4)} \). (In the absence of white vertices, black univalent vertices cannot coexist with black multivalent ones in a connected \( A \)-graph.) Turning to the second possibility, we note that the symmetries of white vertices (see Fig. 1) lead to the identities

\[
\sum_{\alpha} = 0,
\]

where the dots stand for other possible incoming edges/legs. As a consequence no more than two univalent black vertices may join with a white vertex. Having no legs, the graph \( \Gamma \) must take the form of a cyclic graph composed of the bivalent white vertices (10). It remains to note that all such graphs belong to \( \bar{G}^{(2)} \) rather than \( \bar{G}^{(4)} \). □

Theorem 4.3. The complex \( \bar{G}^{(4)} \) is acyclic.

Proof. The complex \( \bar{G}^{(4)} \) admits a decreasing filtration

\[
\bar{G}^{(4)} = F_1 \bar{G}^{(4)} \supset F_2 \bar{G}^{(4)} \supset \cdots \supset F_\infty \bar{G}^{(4)} = 0,
\]

where \( F_k \bar{G}^{(4)} \) spans the graphs with \( k \) and more multivalent and univalent vertices. Define the corresponding spectral sequence \( \{ E_r, d_r \} \). The zero differential \( d_0 \) increases the number of bivalent vertices, leaving the other vertices intact. More precisely, if we prescribe the boundary vertices of the branch (13) the degrees \( | \bullet | = | \varnothing | = 0 \) and \( | \circ | = 1 \), then

\[
d_0 \gamma_\alpha^{\alpha \beta} = \frac{1}{2} (-1)^{|\alpha|} (1 + (-1)^{k+|\alpha|+|\beta|}) \gamma_\alpha^{\alpha \beta}_{k-1},
\]

where we assume that \( h \gamma_0^{\alpha \beta} = 0 \). In other words, the operator \( h \) either annihilates the branch or shortens it by one bivalent vertex. The \( k - 1 \) bivalent vertices of the resulted (nonzero) branch appear to be numbered in order and the labels on the other vertices of the graph \( \Gamma \) are reduced by one. In this way the new graph gets an orientation.

It follows from relations (14) and (15) that the operator \( \Delta = h d_0 + d_0 h \) is diagonal in the natural basis of \( \bar{G}^{(4)} \). More precisely, \( \Delta \Gamma = (n_1 + n_2 + n_3) \Gamma \), where the eigenvalue depends on the structure of the graph \( \Gamma \in \bar{G}^{(4)} \). Namely, \( n_1 \) is the number of branches of nonzero length, \( n_2 \) is the number of edges joining black and white multivalent vertices, and \( n_3 \) is the number of legs adjacent to white vertices. By Lemma 4.2, the sum \( n_1 + n_2 + n_3 \) is strictly positive and the operator \( \Delta \) is invertible. Taking the composition \( h \Delta^{-1} \) as a contracting
homotopy for $d_0$, we see that the complex $(E_0, d_0)$ is acyclic and so is the graded complex associated to the filtered complex $\mathcal{G}^{(4)}$. On the other hand,

$$\mathcal{G}^{(4)} = \lim_{k} \mathcal{G}^{(4)} / F_k \mathcal{G}^{(4)},$$

where $\lim$ denotes the projective limit. (The last equality follows from the fact that we consider graphs with finite number of edges and vertices.) It remains to note that acyclicity of the associated graded complex of $\mathcal{G}^{(4)}$ implies acyclicity of the r.h.s. of (16).

We close this section with two theorems. The first theorem was proved in [13], while the second one summarizes the results of the present paper.

**Theorem 4.4.** The characteristic classes of homological vector fields are independent of the choice of symmetric connection.

**Theorem 4.5.** All the primitive characteristic classes of a homological vector field $Q$ are grouped into the two infinite series $B$ and $C$ plus the $\delta$-cohomology class $[Q]$. For a special choice of symmetric connection one can also define the $A$-series of characteristic classes.

4.5. **Example.** To show nontriviality of the constructed characteristic classes we consider the homological vector field associated to a Lie algebra $\mathcal{L}$. Let $\{t_a\}$ be a basis in $\mathcal{L}$ with commutation relations

$$[t_a, t_b] = f_{ab}^c t_c.$$

Then the homological vector field on $\Pi \mathcal{L}$ reads

$$Q = \frac{1}{2} c^{a_1} \cdots c^{a_{2n-1}} \partial \frac{\partial}{\partial c^b}.$$

By the definition of the parity reversing functor, $\epsilon(c^a) = \epsilon(t_a) + 1$. The linear space of functions on $\Pi \mathcal{L}$ endowed with the differential $Q$ gives us a model for the Chevalley-Eilenberg complex of the Lie algebra $\mathcal{L}$. Upon choosing a flat affine connection on $\Pi \mathcal{L}$, we see that the characteristic classes of $A$-series are nothing but primitive elements of the Lie algebra cohomology:

$$A_{2n-1} = \text{tr}(\text{ad}_{a_1} \cdots \text{ad}_{a_{2n-1}}) c^{a_1} \cdots c^{a_{2n-1}} \quad \forall n \in \mathbb{N}.$$

Here $\text{ad} = \{f_{ab}^c\}$ are the matrices of the adjoint representation of $\mathcal{L}$.

The universal cocycles of $B$- and $C$-series are given by the following ad-invariant tensors on $\Pi \mathcal{L}$:

$$B_n = (\text{ad}_{a_1} \cdots \text{ad}_{a_n})_{a_{n+1}}^b d c^{a_1} \otimes \cdots \otimes d c^{a_n} \otimes \frac{\partial}{\partial c^b},$$

$$C_n = \text{tr}(\text{ad}_{a_1} \cdots \text{ad}_{a_n}) d c^{a_1} \otimes \cdots \otimes d c^{a_n} \quad \forall n \in \mathbb{N}.$$

Since any coboundary of (17) is necessarily proportional to $c^a$, the tensor cocycles above are either zero or nontrivial. For instance, if $\mathcal{L}$ is semi-simple, then the one-form $C_1$ is zero, while the two-form $C_2$ is non-degenerate (the Killing metric).


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