Non-realizability of the pure braid group as area-preserving homeomorphisms

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Abstract. Let Homeo_+(D^n_2) be the group of orientation-preserving homeomorphisms of D^2 fixing the boundary pointwise and n marked points as a set. The Nielsen realization problem for the braid group asks whether the natural projection p_n : Homeo_+(D^n_2) → B_n := π_0(Homeo_+(D^n_2)) has a section over subgroups of B_n. All of the previous methods use either torsion or Thurston stability, which do not apply to the pure braid group P B_n, the subgroup of B_n that fixes n marked points pointwise. In this paper, we show that the pure braid group has no realization inside the area-preserving homeomorphisms using rotation numbers.

Key words: group actions, low-dimensional dynamics, topological dynamics
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1. Introduction
Denote by D^2 the two-dimensional disk. Let Homeo_+(D^n_2) be the group of orientation-preserving homeomorphisms of D^2 fixing the boundary pointwise and n marked points as a set. Denote B_n := π_0(Homeo_+(D^n_2)). The Nielsen realization problem for B_n asks whether the natural projection

p_n : Homeo_+(D^n_2) → B_n

has a section over subgroups of B_n. For the whole group B_n, this question has several previous results. Salter and Tshishiku [12] used Thurston stability to show that B_n has no realization in Diff_+(D^n_2) and the author [1] used ‘hidden torsion’ and Markovic’s machinery [9] to show that B_n has no realization in Homeo_+(D^n_2). Let P B_n < B_n be the subgroup that preserves n marked points pointwise. The Nielsen realization problem for P B_n is widely open since the two methods in [12] and [1] fail to work and have no hope to repair. The following question is asked by [8, Question 3.12] and [12, Remark 1.4].
PROBLEM 1.1. (Realization of pure braid group) Does $PB_n$ have realization as diffeomorphisms or homeomorphisms? In other words, does $p_n$ have sections over $PB_n$?

Denote by $Homeo_+^a(D_n^2)$ the group of orientation-preserving, area-preserving homeomorphisms of $D^2$ fixing the boundary pointwise and $n$ marked points as a set. In this paper, we make progress by proving the following result.

THEOREM 1.2. The pure braid group cannot be realized as area-preserving homeomorphisms on $D_n^2$ for $n \geq 9$. In other words, the natural projection $p_n^a : Homeo_+^a(D_n^2) \to B_n$ has no sections over $PB_n$.

We remark that the Nielsen realization problem is closely related to the existence of flat structures on a surface bundle. We refer the reader to [8] for more history and background.

Comparing with the method in [2], the novelty of this paper is to provide a different proof towards the final contradiction of the result in [2]. The original contradiction is to use the fact that a certain Dehn twist is a product of commutators in its centralizer. However, such structure does not hold in $PB_n$. Instead, we prove a stronger dynamical property about Dehn twists about non-separating curves. In the beginning of §4, we present an outline of the proof. Since this paper has a lot of overlap with [2], we omit or sketch many proofs to reduce redundancy.

This paper is organized as follows.
- In §2, we discuss rotation numbers.
- In §3, we discuss the pure braid group and the minimal decomposition theory.
- In §4, we give an outline of the proof and finish the argument.

2. Rotation numbers of annulus homeomorphisms

In this section, we discuss the properties of rotation numbers on annuli.

2.1. Rotation number of an area-preserving homeomorphism of an annulus. Firstly, we define the rotation number for geometric annuli. Let

$N = N(r) = \left\{ w \in \mathbb{C} : \frac{1}{r} < |w| < r \right\}$

be the geometric annulus in the complex plane $\mathbb{C}$. Denote the geometric strip in $\mathbb{C}$ by

$P = P(r) = \left\{ x + iy = z \in \mathbb{C} : |y| < \frac{\log r}{2\pi} \right\}$. 

The map $\pi(z) = e^{2\pi iz}$ is a holomorphic covering map $\pi : P \to N$. The deck transformation on $P$ is $T(x, y) = (x + 1, y)$.

Denote by $p_1 : P \to \mathbb{R}$ the projection to the $x$-coordinate, and by $Homeo_+(N)$ the group of homeomorphisms of $N$ that preserves orientation and the two ends. Fix $f \in Homeo_+(N)$ and $x \in N$, and let $\tilde{x} \in P$ and $\tilde{f} \in Homeo_+(P)$ denote lifts of $x$ and $f$, respectively. We define the translation number of the lift $\tilde{f}$ at $\tilde{x}$ by

$$\rho(\tilde{f}, \tilde{x}, P) = \lim_{n \to \infty} \frac{p_1(\tilde{f}^n(\tilde{x})) - p_1(\tilde{x})}{n}. \quad (1)$$
The rotation number of \( f \) at \( x \) is then defined as
\[
\rho(f, x, N) = \rho(\tilde{f}, \tilde{x}, P) \quad \text{(mod 1).}
\]

The rotation number is not defined everywhere (see, e.g., [4] for more background on rotation numbers). The closed annulus \( N_c \) is
\[
N_c = \left\{ \omega \in \mathbb{C} : \frac{1}{r} \leq |\omega| \leq r \right\},
\]
For \( f \in \text{Homeo}_+(N_c) \), the rotation and translation numbers are defined analogously.

Let \( A \) be an open annulus embedded in a Riemann surface (in particular this endows \( A \) with the complex structure). By the Riemann mapping theorem [7, Ch. 3.2], there is a unique \( r \) such that there is a biholomorphic map \( u_A : A \rightarrow N(r) =: N \). For any \( f \in \text{Homeo}_+(A) \) (the group of end-preserving homeomorphisms), we define the rotation number of \( f \) on \( A \) by
\[
\rho(f, x, A) := \rho(g, u_A(x), N),
\]
where \( g = u_A \circ f \circ u_A^{-1} \).

We have the following theorems of Poincaré–Birkhoff and Handel about rotation numbers [6] (See also Franks [4].)

**Theorem 2.1.** (Properties of rotation numbers) If \( f : N_c \rightarrow N_c \) is an orientation-preserving, boundary component preserving, area-preserving homeomorphism and \( \tilde{f} : P_c \rightarrow P_c \) is any lift, then:
\begin{itemize}
  \item (Handel) the translation set
    \[
    R(\tilde{f}) = \bigcup_{\tilde{x} \in P_c} \rho(\tilde{f}, \tilde{x}, P_c)
    \]
    is a closed interval;
  \item (Poincaré–Birkhoff) if \( r \in R(\tilde{f}) \) is rational, then there exists a periodic orbit of \( f \) realizing the rotation number \( r \) mod 1.
\end{itemize}

2.2. **Separators and their properties.** We let \( A \) continue to denote an open annulus embedded in a Riemann surface. Then \( A \) has two ends and we choose one of them to be the left end and the other one to be the right end. We call a subset \( X \subset A \) separating (or essential) if every arc \( \gamma \subset A \) which connects the two ends of \( A \) must intersect \( X \).

**Definition 2.2.** (Separator) We call a subset \( M \subset A \) a separator if \( M \) is compact, connected and separating.

The complement of \( M \) in \( A \) is a disjoint union of open sets. We have the following lemma.

**Lemma 2.3.** Let \( M \) be a separator. Then there are exactly two connected components \( A_L(M) \) and \( A_R(M) \) of \( A - M \) which are open annuli homotopic to \( A \) and with the property that \( A_L(M) \) contains the left end of \( A \) and \( A_R(M) \) contains the right end of \( A \). All other components of \( A - M \) are simply connected.
Proof. We compactify the annulus $A$ by adding points $p_L$ and $p_R$ to the corresponding ends of $A$. The compactifications is a two-sphere $S^2$. Moreover, $M$ is a compact and connected subset of $S^2 - \{p_L, p_R\}$.

Now we observe that every component of $S^2 - M$ is simply connected. Denote by $\Omega_L$ and $\Omega_R$ the connected components of $S^2 - M$ containing $p_L$ and $p_R$, respectively. Since $M$ is separating, we conclude that these are two different components. We define $A_L(M) = \Omega_L - p_L$ and $A_R(M) = \Omega_R - p_R$. It is easy to verify that these are required annuli.

We now prove another property of a separator. Let $\pi : \tilde{A} \to A$ be the universal cover.

**Proposition 2.4.** Let $M \subset A$ be a compact domain with smooth boundary. Then $\pi^{-1}(M)$ is connected; i.e., $M$ is a separator.

**Proof.** Since $M$ is a compact domain with boundary which separates the two ends of $A$, we can find a circle $\gamma \subset M$ which is essential in $A$ (i.e., $\gamma$ is a separator itself) (note that $M$ has only finitely many boundary components). Denote by $T$ the deck transformation of $\tilde{A}$. Thus, the lift $\pi^{-1}(\gamma)$ is a $T$-invariant, connected subset of $\tilde{A}$. Let $C$ be the component of $\pi^{-1}(M)$ which contains $\pi^{-1}(\gamma)$. Then $C$ is $T$ invariant. We show that $\pi^{-1}(M) = C$.

Let $p \in M$. Since $M$ is a compact domain with smooth boundary, we can find an embedded closed arc $\alpha \subset M$ which connects $p$ and $\gamma$. Let $\tilde{p}$ be a lift of $p$ and let $\tilde{\alpha}$ be the corresponding lift of $\alpha$ such that $\tilde{p}$ is one of its end points. Then the other end point of $\tilde{\alpha}$ is in $\pi^{-1}(\gamma)$ and this shows that $\tilde{p} \in C$. This concludes the proof. \hfill \Box

Now we discuss an ordering on the set of separators.

**Proposition 2.5.** Suppose that $M_1, M_2 \subset A$ are two disjoint separators. Then either $M_1 \subset A_L(M_2)$ or $M_1 \subset A_R(M_2)$. Moreover, $M_1 \subset A_L(M_2)$ implies that $M_2 \subset A_R(M_1)$.

**Proof.** Since $M_1$ is connected, it follows that $M_1$ is a subset of a connected component $C$ of $A - M_2$. Since $C$ is open, we know that there is a neighborhood $N_1$ of $M_1$ with smooth boundary such that $N_1 \subset C$ (It is elementary to construct such $N_1$.) If $C$ is simply connected, the cover $\pi^{-1}(C) \to C$ is a trivial cover. Let $\tilde{C}$ be a connected component of $\pi^{-1}(C)$. By Proposition 2.4, the set $\pi^{-1}(N_1)$ is connected, so it is contained in a single connected component of $\pi^{-1}(C)$. However, this contradicts the fact that $\pi^{-1}(N_1)$ is also translation invariant. Thus, either $M_1 \subset A_L(M_2)$ or $M_1 \subset A_R(M_2)$.

Suppose that $M_1 \subset A_L(M_2)$. Then $A_L(M_1) \subset A_L(M_2)$ as well. On the other hand, by the first part of the proposition, we already know that either $M_2 \subset A_L(M_1)$ or $M_2 \subset A_R(M_1)$. If $M_2 \subset A_L(M_1)$, then $A_L(M_2) \subset A_L(M_1)$. This shows that $A_L(M_1) \subset A_L(M_2)$, which implies that $M_2 \subset A_L(M_2)$. This is absurd, so we must have $M_2 \subset A_R(M_1)$. \hfill \Box

**Definition 2.6.** The inclusion $M_1 \subset A_L(M_2)$ is denoted as $M_1 < M_2$.

2.3. The rotation interval of an annular continuum and prime ends. Let $K \subset A$ be a separator (in the literature, also known as an essential continuum). We call $K$ an essential annular continuum if $A - K$ has exactly two components. Observe that an
essential annular continuum can be expressed as a decreasing intersection of essential closed topological annuli in $A$.

It is possible to turn any separator $M \subset A$ into an essential annular continuum. Let $M$ be a separating connected set. By Lemma 2.3, we know that $A - M$ has exactly two connected annular components $A_L(M)$ and $A_R(M)$, and all other components of $A - M$ are simply connected. We call a simply connected component of $A - M$ a bubble component. Then the annular completion $K(M)$ of $M$ is defined as the union of $M$ and the corresponding bubble components of $A - M$.

**Proposition 2.7.** Let $M \subset A$ be a separator. Then the annular completion $K(M)$ is an annular continuum.

**Proof.** We can again compactify $A$ by adding the points $p_L$ and $p_R$, one at each end. The compactification is the two-sphere $S^2$. Then $A_L(M)$ and $A_R(M)$ are two disjoint open disks in $S^2$, and $K(M) = S^2 - (A_L(M) \cup A_R(M))$. But the complement of two disjoint open disks in $S^2$ is connected. This proves the proposition. □

Now let $f$ be a homeomorphism of $A$ that leaves an annular continuum $K$ invariant. If $\mu$ is an invariant Borel probability measure supported on $K$, we define the $\mu$-rotation number

$$\sigma(f, \mu) = \int_A \phi \, d\mu,$$

where $\phi : A \to \mathbb{R}$ is the function which lifts to the function $p_1 \circ f - p_1$ on $\tilde{A}$ (recall that $p_1 : \tilde{A} \to \mathbb{R}$ is the projection onto the first coordinate).

The set of $f$ invariant Borel probability measures on $K$ is a non-empty, convex and compact set (with respect to the weak topology on the space of measures), which is denoted by $M(K)$. We define the rotation interval of $K$

$$\sigma(f, K) = \{\sigma(f, \mu) | \mu \in M(K)\},$$

which is a non-empty segment $[\alpha, \beta]$ of $\mathbb{R}$. The interval is non-empty because there exists at least one $f$ invariant measure, and it is an interval because the set of $f$ invariant measures is convex.

The following is a classical result of Franks and Le Calvez [5, Corollary 3.1].

**Proposition 2.8.** If $\sigma(f, K) = \{\alpha\}$, the sequence

$$\frac{p_1 \circ f^n(x) - p_1(x)}{n}$$

converges uniformly for $x \in \pi^{-1}(K)$ to the constant function $\alpha$.

We remark that this implies that points in $K$ all have the rotation number $\alpha$.

The following theorem of Franks and Le Calvez [5, Proposition 5.4] is a generalization of the Poincaré–Birkhoff theorem.

**Theorem 2.9.** If $f$ is area-preserving and $K$ is an annular continuum, then every rational number in $\sigma(f, K)$ is realized by a periodic point in $K$. 
The theory of prime ends is an important tool in the study of two-dimensional dynamics which can be used to transform a two-dimensional problem into a one-dimensional problem. Recall that we assume that $A$ is an open annulus embedded in a Riemann surface $S$. Suppose that $f$ is a homeomorphism of $S$ which leaves $A$ invariant. Furthermore, let $K \subset A$ be an annular continuum and suppose that $f$ leaves $K$ invariant. Then both $A_L(K)$ and $A_R(K)$ are $f$ invariant.

Since $A$ is embedded in $S$, we can define the frontiers of $A$, $A_L(K)$ and $A_R(K)$. By Carathéodory’s theory of prime ends (see, e.g., [11, Ch. 15]), the homeomorphism $f$ yields an action on the frontiers of $A_L(K)$ and $A_R(K)$. Consider the right-hand frontier of $A_L(K)$ (the one which is contained in $A$). Then the set of prime ends on this frontier is homeomorphic to the circle, and we denote by $f_L$ the induced homeomorphism of this circle. Likewise, the set of prime ends on the left-hand frontier of $A_R(K)$ is homeomorphic to the circle, and we denote by $f_R$ the induced homeomorphism of this circle.

The rotation number of a circle homeomorphism (defined by equation (2)) is well defined everywhere and is the same number for any point on the circle. The rotation numbers of $f_L$ and $f_R$ are called $r_L$ and $r_R$. We refer to them as the left and right prime end rotation numbers of $f$. We have the following theorem of Matsumoto [10].

**Theorem 2.10.** (Matsumoto’s theorem) If $K$ is an annular continuum, then its left and right prime end rotation numbers $r_L, r_R$ belong to the rotation interval $\sigma(f, K)$.

3. Minimal decompositions and characteristic annuli

3.1. Minimal decompositions. We recall the theory of minimal decompositions of surface homeomorphisms. This is established in [9]. Firstly we recall the definition of the upper semi-continuous decomposition of a surface and the minimal decomposition theory; see also Markovic [9, Definition 2.1]. Let $M$ be a surface.

**Definition 3.1.** (Upper semi-continuous decomposition) Let $S$ be a collection of closed, compact, connected subsets of $M$. We say that $S$ is an upper semi-continuous decomposition of $M$ if the following holds.

- If $S_1, S_2 \in S$, then $S_1 \cap S_2 = \emptyset$.
- If $S \in S$, then $S$ does not separate $M$; i.e., $M - S$ is connected.
- We have $M = \bigcup_{S \in S} S$.
- If $S_n \in S$, $n \in \mathbb{N}$ is a sequence that has the Hausdorff limit equal to $S_0$, then there exists $S \in S$ such that $S_0 \subset S$.

Now we define acyclic sets on a surface.

**Definition 3.2.** (Acyclic sets) Let $S \subset M$ be a closed, connected subset of $M$ which does not separate $M$. We say that $S$ is acyclic if there is a simply connected open set $U \subset M$ such that $S \subset U$ and $U - S$ is homeomorphic to an annulus.

The simplest examples of acyclic sets are points, embedded closed arcs and embedded closed disks in $M$. Let $S \subset M$ be a closed, connected set that does not separate $M$. Then $S$ is acyclic if and only if there is a lift of $S$ to the universal cover $\tilde{M}$ of $M$, which is a compact subset of $\tilde{M}$. The following theorem is a classical result called Moore’s theorem; see, e.g., [9, Theorem 2.1].
THEOREM 3.3. (Moore’s theorem) Let $M$ be a surface and $S$ be an upper semi-continuous decomposition of $M$ so that every element of $S$ is acyclic. Then there is a continuous map $\phi : M \to M$ that is homotopic to the identity map on $M$ and such that for every $p \in M$, we have $\phi^{-1}(p) \in S$. Moreover, $S = \{\phi^{-1}(p) | p \in M\}$.

We call the map $M \to M/\sim$ the Moore map, where $x \sim y$ if and only if $x, y \in S$ for some $S \in S$. The following definition appears in [9, Definition 3.1]

Definition 3.4. (Admissible decomposition) Let $S$ be an upper semi-continuous decomposition of $M$. Let $G$ be a subgroup of $\text{Homeo}(M)$. We say that $S$ is admissible for the group $G$ if the following holds.

- Each $f \in G$ preserves setwise every element of $S$.
- Let $S \in S$. Then every point in every frontier component of the surface $M - S$ is a limit of points from $M - S$ which belong to acyclic elements of $S$.

If $G$ is a cyclic group generated by a homeomorphism $f : M \to M$, we say that $S$ is an admissible decomposition of $f$.

An admissible decomposition for $G < \text{Homeo}(M)$ is called minimal if it is contained in every admissible decomposition for $G$. We have the following theorem [9, Theorem 3.1].

THEOREM 3.5. (Existence of minimal decompositions) Every group $G < \text{Homeo}(M)$ has a unique minimal decomposition.

Denote by $A(G)$ the subcollection of acyclic sets from $S(G)$. By a mild abuse of notation, we occasionally refer to $A(G)$ as a subset of $M$ (the union of all sets from $A(G)$). To distinguish the two notions, we do as follows. When we refer to $A(G)$ as a collection, then we consider it as the collection of acyclic sets. When we refer to as a set (or a subsurface of $M$), we have in mind the other meaning.

We have the following result [9, Proposition 2.1].

Proposition 3.6. Every connected component of $A(G)$ (as a subset of $S_g$) is a subsurface of $M$ with finitely many ends.

Lemma 3.7. For $H < G < \text{Homeo}(M)$, we have that $A(G) \subset A(H)$.

Proof. The inclusion $A(G) \subset A(H)$ is because the minimal decomposition of $G$ is also an admissible decomposition of $H$ and the minimal decomposition of $H$ is finer than that of $G$. \qed

3.2. Lifting through hyperelliptic branched cover. Denote by $S_{g;n,b}$ the surface of genus $g$ with $b$ boundary components and $n$ marked points. To make the analysis easier, we take the following hyperelliptic $\mathbb{Z}/2$ branched cover:

$$\pi_n : S = S_{(n-1)/2;n,1} \to S_{0;n,1}$$

for $n$ odd or

$$\pi_n : S = S_{(n/2)-1;n,2} \to S_{0;n}$$

for $n$ even.

The cover is shown by Figures 1 and 2. The hyperelliptic involution on $S$ is denoted by $\tau$. 

Denote by $\widetilde{PB}_n$ the lifts of mapping classes under $\pi_n$, where they satisfy the following exact sequence:

$$1 \to \mathbb{Z}/2 \to \widetilde{PB}_n \xrightarrow{L} PB_n \to 1.$$  

Let $c$ be a simple closed curve on $S_{0; n, 1}$ and denote by $T_c$ the Dehn twist about $c$. For every simple closed curve $c$ on $S_{0; n, 1}$, we have the following easy fact about its preimage under $\pi_n$.

**Fact 3.8.**

1. If $c$ bounds an odd number of points, then the lift is a single curve $c'$. The preimages of $T_c^2$ under $L$ are $T_{c'}$ and $T_{c'} \tau$.
2. If $c$ bounds an even number of points, then the lift is two curves $c_1, c_2$. The preimages of $T_c$ under $L$ are $T_{c_1} T_{c_2}$ and $T_{c_1} T_{c_2} \tau$. In particular, if $c$ bounds 2 points, then $c_1 = c_2$.

From the above fact, we know that if $c$ bounds two points and $c_1 = c_2$ are the lifts, we have that $T_{c_1}^2 \in \widetilde{PB}_n$. We have the following fact.

**Fact 3.9.** If $\alpha$ is a non-separating simple closed curve that is invariant under $\tau$, then a square of the Dehn twist about $c$ is in $\widetilde{PB}_n$. We call such element an invariant Dehn twist square.

Let $b$ be the curve in $D_n^2$ bounding five points $P_1, \ldots, P_5$. The lift of $b$ under the cover $\pi_n$ is a curve $c$ bounding a genus-two subsurface as Figure 3.

If a curve $\alpha$ is on the genus-two subsurface of $S$ that is cut out by $c$, then we call the invariant Dehn twist square about $\alpha$ a left invariant Dehn twist square. We have the following important relation in $\widetilde{PB}_n$.

**Proposition 3.10.** The element $T_c \in \widetilde{PB}_n$ is a product of left invariant Dehn twist squares in $\widetilde{PB}_n$. 
Proof. We have the basic fact that $P B_n$ is generated by Dehn twists about curves in the interior of $b$ bounding two points; see, e.g., [3, Ch. 9]. Take a lift of all of the elements; we obtain a product of squares of Dehn twists about non-separating curves that are disjoint from $c$ and on the left of $c$ in $\tilde{P}B_n$. After taking a square of the equation, we obtain the proposition.

□

3.3. Characteristic annuli. From now on, we work with the assumption that there exists a realization of the pure braid group

$$E' : P B_n \to \text{Homeo}_{+}^\tau(D_n^2).$$

Lifting by the hyperelliptic involution, we obtain a new realization

$$E : \tilde{P}B_n \to \text{Homeo}_{+}^\tau(S_g)\,^\tau,$$

where the image lies in the centralizer of the hyperelliptic involution $\tau$. We now only work with the new realization $\tilde{E}$.

For an element $f \in \tilde{P}B_n$ or a subgroup $F < \tilde{P}B_n$, we shorten $A(E(f))$ to $A(f)$, and $A(E(F))$ to $A(F)$, denoting the corresponding collections of acyclic components. Denote by $S$ the hyperelliptic cover we defined in §3.2. Recall that $c \subset S$ is a separating curve that is invariant under $\tau$ and divides $S$ into subsurfaces $S_L$ of genus two and $S_R = S - S_L$ (see more about $c$ in the previous section). We know that $T_c \in \tilde{P}B_n$. We have the following theorem about the minimal decomposition of $E(T_c)$.

**Theorem 3.11.** The set $A(T_c)$ has a component $L(c)$ which is homotopic to $S_L$ and a component $R(c)$ homotopic to $S_R$.

**Proof sketch.** The proof is the same as the proof of [2, Theorem 4.1]. We use the fact that there are pseudo-Anosov elements on the left and on the right of $c$ in $\tilde{P}B_n$. In this theorem, we need $n \geq 9$.

For the rest of paper, we write

$$B := S - L(c) - R(c).$$

Let $p_L : L(c) \to L(c)/\sim$ and $p_R : R(c) \to R(c)/\sim$ be the Moore maps of $L(c)$ and $R(c)$ corresponding to the decomposition $S(c)$. Let $L \subset L(c)/\sim$ be an open annulus bounded by the end of $L(c)'$ on one side, and by a simple closed curve on the other. The open annulus $R \subset R(c)/\sim$ is defined similarly. We have the following definition (see [9, Ch. 5]).
Definition 3.12. An annulus of the form \( A = p^{-1}_L(L) \cup B \cup p^{-1}_R(R) \) is called a characteristic annulus.

Denote \( f = \mathcal{E}(T_c) \). Every characteristic annulus is invariant under \( f \). We observe that \( B \) is a separator in \( A \), that is, \( B \) is an essential, compact and connected subset of \( A \). Note that a characteristic annulus \( A \) is invariant under \( f \), but it may not be invariant under homeomorphisms which are lifts (with respect to \( \mathcal{E} \)) of other elements from \( \tilde{P}B_n \). However, \( B \) is invariant under these lifts of elements from the image under \( \mathcal{E} \) of the centralizer of \( T_c \in \tilde{P}B_n \). As we see from the next lemma, the dynamical information about \( f \) is contained in \( B \).

Lemma 3.13. Fix a characteristic annulus \( A \). Then:

1. every number \( 0 < r < 1 \) appears as the rotation number \( \rho(f, x, A) \) for some \( x \in A \); 
2. if \( 0 < \rho(f, x, A) < 1 \), then \( x \in B \).

The proof of the above lemma can be seen in [2, Lemma 4.5], which is a result of two facts. One is that \( f \) is homotopic to a Dehn twist and the other is that the realization is area-preserving.

4. The proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2. We first discuss the main strategy.

4.1. Outline of the proof. Recall that \( c \) is a separating simple closed curve that divides the surface \( S \) (the hyperelliptic cover of \( S_{0;1,n} \)) into a genus-two subsurface and its complement. Fix a characteristic annulus \( A \) of \( T_c \). Let \( E_r \) be the set of points in \( A \) that have rotation numbers equal to \( r \) under \( \mathcal{E}(T_c) \). Lemma 3.13 states that the set \( E_r \) is not empty when \( 0 < r < 1 \).

The key observation of the proof lies in the analysis of connected components of \( E_r \). Let \( E \) be a component of \( E_r \). We show the following results:

1. \( E \) is \( \mathcal{E}(h) \)-invariant for \( h \) a left invariant Dehn twist square; 
2. \( \overline{E} \) is a separator in \( A \); 
3. if \( E \) contains a periodic orbit, then \( E \) contains a separator.

Denote by \( K(\overline{E}) \) the annular completion of \( \overline{E} \), and let \( \rho(\mathcal{E}(T_c), K(\overline{E})) \) be the rotation interval of \( K(\overline{E}) \). We claim that \( \rho(\mathcal{E}(T_c), K(\overline{E})) = \{ r \} \). First of all, we know that \( r \in \rho(\mathcal{E}(T_c), K(\overline{E})) \). If \( \rho(\mathcal{E}(T_c), K(\overline{E})) \neq \{ r \} \), then \( \rho(\mathcal{E}(T_c), K(\overline{E})) \) contains infinitely many rational numbers. By Theorem 2.9, there exist three periodic points \( x_1, x_2, x_3 \in K(\overline{E}) \) with different rational rotation numbers \( r_1, r_2, r_3 \). Let \( F_i \) denote the connected component of \( E_{r_i} \), containing \( r_i \), and let \( M_i \subset F_i \) be a separator.

By Proposition 2.5, there is an ordering on disjoint separators. Without loss of generality, we assume that \( M_1 < M_2 < M_3 \). Based on a discussion about the position of \( E \) with respect to the \( M_i \), we obtain a contradiction. Thus, \( \rho(\mathcal{E}(T_c), K(E)) \) is the singleton \( \{ r \} \). We know from Theorem 2.10 that the left and right prime end rotation numbers of \( K(\overline{E}) \) are both \( r \). In the group of circle homeomorphisms, the centralizer of an irrational rotation is essentially \( \text{SO}(2) \).
We then show a new ingredient of the proof: the rotation numbers of the realization of a left invariant Dehn twist square on the set of prime ends of \( K(\overline{E}) \) are all zero. This contradicts the fact that \( T_c \) is a product left invariant Dehn twist square as in Proposition 3.10.

4.2. The set \( E_r \). Once again we use abbreviation \( f = \mathcal{E}(T_c) \). For a characteristic annulus \( A \), we let

\[
E_r = \{ x \in A : \rho(f, x, A) = r \}.
\]

By Lemma 3.13, if \( 0 < r < 1 \), we know that \( E_r \) is non-empty and \( E_r \subset B \).

Next, we have the following key lemmas, which correspond to [2, Lemmas 5.1, 5.3 and 5.4].

**Lemma 4.1.** Fix \( 0 < r < 1 \) and let \( E \) denote a connected component of \( E_r \). Fix a left invariant Dehn twist square \( h \) in \( \tilde{P}B_n \). For \( x \in E \), let \( C(x) \in \mathcal{A}(h) \) be the corresponding acyclic set. Then \( C(x) \subset E \). In particular, \( E \) is \( \mathcal{E}(C(T_c))-\)invariant.

**Lemma 4.2.** The closed set \( \overline{E} \) is a separator (as defined in §2).

**Lemma 4.3.** Let \( x \) be a periodic orbit of \( f \) such that \( \rho(f, x, A) = p/q \) and \( 0 < p/q < 1 \). Then the connected component \( E \) of \( E_{p/q} \), which contains \( x \), also contains a separator (as a subset).

Fix an irrational number \( r \in (0, 1) \). By Lemma 3.13, we know that \( E_r \) is not empty. Let \( E \) be a connected component of \( E_r \). By Lemma 4.1, we know that \( E \) is invariant under \( \mathcal{E}(C(T_c)) \). By Lemma 4.2, we know that \( \overline{E} \) is a separator. The annular completion \( K(\overline{E}) \) of \( \overline{E} \) is also \( \mathcal{E}(C(T_c))-\)invariant since the definition is canonical. The following claim is at the heart of the entire construction.

**Claim 4.4.** Let \( r_L \) and \( r_R \) be the left and right prime end rotation numbers of \( f \) on \( K(\overline{E}) \). Then \( r_L = r_R = r \).

**Remark.** We refer the reader to [2, Claim 5.2] for the proof. The only property we use about \( \tilde{P}B_n \) is Proposition 3.10.

4.3. Finishing the proof. We need to show a new property of a left invariant Dehn twist square \( h \in \tilde{P}B_n \).

**Claim 4.5.** The action of \( \mathcal{E}(T_b^2) \) on the set of prime ends of \( K(\overline{E}) \) has rotation number zero.

**Proof.** Now we consider the rotation set of \( \mathcal{E}(T_b^2) \) on \( K(\overline{E}) \). We claim that the rotation set satisfies

\[
\sigma(\mathcal{E}(T_b), K(\overline{E})) = \{0\}.
\]

Since \( \overline{E} \subset B \), and the fact that \( S - B \) is a union of two open subsurfaces, we know that \( K(\overline{E}) \subset B \). This means that for every point \( x \in K(\overline{E}) \subset B \), there exists \( C(x) \in \mathcal{A}(T_b^2) \) such that \( C(x) \subset B \) by Lemma 4.1. However, \( C(x) \) is acyclic and fixed by \( \mathcal{E}(T_b^2) \). Therefore, we know that the rotation number of \( \mathcal{E}(T_b^2) \) on points in \( C(x) \) is zero. By
Non-realizability of the pure braid group

Theorem 2.10, we know that the rotation number of the action of $\mathcal{E}(T^2_b)$ on the set of prime ends is also zero.

We now finish the proof.

**Proof.** Since the rotation number of $\mathcal{E}(T_c)$ on the prime ends of $K(E)$ is an irrational number $r$, it is semiconjugate to an irrational rotation. Then, up to the same semiconjugacy, the image of the centralizer of $T_c$ under $E$ is $SO(2)$. The image of each element is determined by its rotation number. However, $\mathcal{E}(T_c)$ is a product of $\mathcal{E}(T^2_b)$ for $b$ non-separating and invariant under $\tau$ by Proposition 3.10. By Lemma 4.5, we know that the rotation number of $\mathcal{E}(T^2_b)$ is zero. Thus, their product should also have zero rotation number. This contradicts the fact that the rotation number of $\mathcal{E}(T_c)$ is $r$, which is non-zero.

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