Few-Body Effects in Cold Atoms and Limit Cycles

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Physical systems with a large scattering length have universal properties independent of the details of the interaction at short distances. Such systems can be realized in experiments with cold atoms close to a Feshbach resonance. They also occur in many other areas of physics such as nuclear and particle physics. The universal properties include a geometric spectrum of three-body bound states (so-called Efimov states) and log-periodic dependence of low-energy observables on the physical parameters of the system. This behavior is characteristic of a renormalization group limit cycle. We discuss universality in the three- and four-body sectors and give an overview of applications in cold atoms.

1. INTRODUCTION

The Effective Field Theory (EFT) approach provides a powerful framework that exploits the separation of scales in physical systems. Only low-energy (or long-range) degrees of freedom are included explicitly, with the rest parametrized in terms of the most general local (contact) interactions. This procedure exploits the fact that a low-energy probe of momentum $k$ cannot resolve structures on scales smaller than $R \sim 1/k$. Using renormalization, the influence of short-distance physics on low-energy observables is captured in a small number of low-energy constants. Thus, the EFT describes universal low-energy physics independent of detailed assumptions about the short-distance dynamics. All physical observables can be described in a controlled expansion in powers of $kl$, where $l$ is the characteristic low-energy length scale of the system. The size of $l$ depends on the system under consideration: for a finite-range potential, e.g., it is given by the range of the potential. For the systems discussed here, $l$ is of the order of the effective range $r_e$ or the van der Waals scale $l_{vdW}$.

Effective Field Theories can be obtained by applying a renormalization group (RG) transformation to a (more) fundamental theory. The RG transformation integrates out high momentum modes from the fundamental theory and leads to a description of low-energy physics in terms of low-energy degrees of freedom only. It can be understood as a change of the resolution scale of the EFT. The transformation generates a flow in the space of coupling constants that determine the effective Lagrangian $\mathcal{L}$. If the RG transformation is continuous, this flow can be expressed by a differential equation for the

\footnote{Note that $\hbar = 1$ in this talk.}
coupling constants $g$:

$$\Lambda \frac{d}{d\Lambda} g = \beta(g),$$  \hspace{1cm} (1)

where $\Lambda$ is the ultraviolet cutoff which corresponds to the inverse resolution scale. The function $\beta(g)$ determines the behavior of $g$ under the RG transformation. The RG equation (1) can have many types of solutions. The two simplest ones are (a) renormalization group fixed points and (b) renormalization group limit cycles. Fixed points are ubiquitous in condensed matter and particle physics. They play an important role, e.g., in critical phenomena and the scale dependence of coupling constants in high energy physics. Renormalization group limit cycles have been suggested by Ken Wilson already in 1971 [1], but their phenomenological importance has only been realized very recently.

An important signature of a limit cycle is the appearance of a discrete scaling symmetry, the invariance of physical observables under changes of the resolution scale by a discrete scale transformation with a preferred scaling factor $\lambda$. The prime example of a RG limit cycle is the Efimov effect [2] for 3-body systems with large S-wave scattering length $a$. Vitaly Efimov discovered that in the limit $a = \pm \infty$ there can be infinitely many 3-body bound states (trimers) with an accumulation point at the scattering threshold [2]. These trimers are called Efimov states. One remarkable feature of Efimov states is that they have a geometric spectrum with preferred scaling factor $\lambda^2 \approx 515$:

$$E_T^{(n)} = \lambda^{2(n_s-n)} \hbar^2 \kappa_s^2 / m,$$  \hspace{1cm} (2)

where $\kappa_s$ is the binding wavenumber of the branch of Efimov states labeled by $n_s$. The Efimov effect is just one example of the universal phenomena characterized by the discrete scaling symmetry in the 3-body system [2]. These universal properties persist also for finite values of the scattering length as long as $a \gg l$. For a review of these phenomena, which we refer to as Efimov physics, see Ref. [3].

For a generic system, the scattering length is of the same order of magnitude as the low-energy length scale $l$. Only a very specific choice of the parameters in the underlying theory (a so-called fine tuning) will generate a large scattering length. Nevertheless, systems with large scattering length can be found in many areas of physics. Examples are the S-wave scattering of nucleons and of $^4$He atoms. For alkali atoms close to a Feshbach resonance, $a$ can be tuned experimentally by adjusting an external magnetic field. This is particularly interesting, since it allows to experimentally test the dependence of physical observables on the scattering length.

In this talk, we discuss some applications of an EFT for few-body systems with large scattering length $a \gg l$ [4]. In this theory, the limit cycle is manifest in the RG behavior of the 3-body interaction which makes it ideally suited to study the Efimov effect and its consequences for other 3-body observables.

2. THREE-BODY SYSTEM WITH LARGE SCATTERING LENGTH

We first give a very brief review of the EFT for few-body systems with large scattering length $a$, focusing on S-waves. A more detailed treatment is given in Ref. [3].
For typical momenta $k \sim 1/a$, the EFT expansion is in powers of $l/a$ so that higher order corrections are suppressed by powers of $l/a$. We consider a 2-body system of non-relativistic bosons (referred to as atoms) with large scattering length $a$ and mass $m$. At sufficiently low energies, the most general Lagrangian may be written as:

$$
\mathcal{L} = \psi^\dagger \left( i \partial_t + \frac{\vec{\nabla}^2}{2m} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2 - \frac{D_0}{6} (\psi^\dagger \psi)^3 + \ldots ,
$$

where the $C_0$ and $D_0$ are nonderivative 2- and 3-body interaction terms, respectively. The strength of the $C_0$ term is determined by the scattering length $a$, while $D_0$ depends on a 3-body parameter to be introduced below. The dots represent higher-order terms which are suppressed at low-energies. For momenta $k$ of the order of the inverse scattering length $1/a$, the problem is nonperturbative in $ka$. The exact 2-body scattering amplitude can be obtained analytically by summing the so-called bubble diagrams with the $C_0$ interaction term. The $D_0$ term does not contribute to 2-body observables. After renormalization, the resulting amplitude reproduces the leading order of the well-known effective range expansion for the atom-atom scattering amplitude: $f_{AA}(k) = (-1/a - ik)^{-1}$, where the total energy is $E = k^2/m$. If $a > 0$, $f_{AA}$ has a pole at $k = i/a$ corresponding to a shallow dimer with the universal binding energy $B_2 = 1/(ma^2)$. Higher-order interactions are perturbative and give the momentum-dependent terms in the effective range expansion.

We now turn to the 3-body system. At leading order, the atom-dimer scattering amplitude is given by the integral equation shown in Fig. 1. A solid line indicates a single atom and a double line indicates an interacting two-atom state (including rescattering corrections). The integral equation contains contributions from both the 2-body and the 3-body interaction terms. The inhomogeneous term is given by the first two diagrams on the right-hand side: the one-atom exchange diagram and the 3-body term. The integral equation simply sums these diagrams to all orders. After projecting onto S-waves, we obtain the equation

$$
\mathcal{T}(k; p; E) = \frac{16}{3a} M(k, p; E) + \frac{4}{\pi} \int_0^\Lambda dq \frac{q^2 M(q, p; E)}{-1/a + \sqrt{3q^2/4 - mE - i\epsilon}} \mathcal{T}(k, q; E) ,
$$

for the off-shell atom-dimer scattering amplitude with the inhomogeneous term

$$
M(k; p; E) = \frac{1}{2pk} \ln \left( \frac{p^2 + pk + k^2 - mE}{p^2 - pk + k^2 - mE} \right) + \frac{H(\Lambda)}{\Lambda^2} .
$$

Figure 1. Integral equation for the atom-dimer scattering amplitude. Single (double) line indicates single atom (two-atom state).
The first term is the S-wave projected one-atom exchange, while the second term comes from the 3-body interaction. The physical atom-dimer scattering amplitude $f_{AD}$ is given by the solution $\mathcal{T}$ evaluated at the on-shell point: $f_{AD}(k) = \mathcal{T}(k, k; E)$ where $E = 3k^2/(4m) - 1/(ma^2)$. The 3-body binding energies $B_3$ are given by those values of $E$ for which the homogeneous version of Eq. (4) has a nontrivial solution.

Note that an ultraviolet cutoff $\Lambda$ has been introduced in (4). This cutoff is required to insure that Eq. (4) has a unique solution. All physical observables, however, must be invariant under changes of the cutoff, which determines the behavior of $H$ as a function of $\Lambda$ [4]:

$$H(\Lambda) = \frac{\cos[s_0 \ln(\Lambda/\Lambda_*) + \arctan s_0]}{\cos[s_0 \ln(\Lambda/\Lambda_*) - \arctan s_0]},$$

(6)

where $s_0 = 1.00624$ is a transcendental number and $\Lambda_*$ is a 3-body parameter introduced by dimensional transmutation. This parameter cannot be predicted by the EFT and must be taken from experiment. It is evident that $H(\Lambda)$ is periodic and runs on a limit cycle. When $\Lambda$ is changed by the preferred scaling factor $\lambda = \exp(\pi/s_0) \approx 22.7$, $H$ returns to its original value. Note that the definition of the 3-body parameter is not unique. The parameter $\Lambda_*$ from Eq. (4) arises naturally in the EFT description, while the parameter $\kappa_*$ from Eq. (2) is defined via the 3-body Efimov spectrum in the limit $a \to \pm \infty$. Both definitions are related by a constant factor: $\Lambda_* = 2.62\kappa_* [3]$.

In summary, two parameters are required in the 3-body system at leading order in $l/a$: the scattering length $a$ (or the dimer binding energy $B_2$) and the 3-body parameter $\Lambda_*$ or $\kappa_* [4]$. The EFT reproduces the universal aspects of the 3-body system first derived by Efimov [2] and is a very efficient calculational tool to calculate those properties.

3. APPLICATIONS TO COLD ATOMS

We now turn to some applications of this EFT to systems of cold atoms. First we discuss universal scaling functions. Since only two parameters enter at leading order, different 3-body observables show correlations. These correlations must appear in all 3-body systems with short-range interactions and large scattering length. In the left panel of Fig. 2 we display the scaling function relating the trimer excited and ground state energies $B_3^{(1)}$ and $B_3^{(0)}$, respectively [5]. The data points give various calculations using realistic $^4\text{He}$ potentials while the solid line gives the universal prediction from EFT. Different points on this line correspond to different values of $\kappa_*$. The small deviations of the potential calculations from the universal curve are mainly due to effective range corrections and can be calculated at next-to-leading order in EFT [6]. The calculation corresponding to the data point far off the universal curve can easily be identified as problematic. In the right panel of Fig. 2 we display a similar correlation between the triton binding $B_t$ and the spin-doublet $S$-wave neutron-deuteron scattering length $a_{nd}^{(1/2)}$ from nuclear physics taken from [7], which is known as the Phillips line. The dash-dotted and dashed lines give the results excluding and including the leading corrections in $l/a$, respectively. It is evident that nuclear and atomic systems show the same universal scaling behavior. While these systems have very different structure at short distances, all that matters here is that $|a| \gg l$ but not the details of the mechanism leading to the large scattering length.
Figure 2. Left panel: The scaling function $B_3^{(1)}/B_2$ versus $B_3^{(0)}/B_2$ for $^4$He atoms [5]. Right panel: The scaling function $a_{nd}^{(1/2)}$ versus $B_t$ from nuclear physics [7]. The dash-dotted and dashed lines exclude and include the leading correction in $l/a$, respectively. The data points are calculations using realistic potentials.

There are also observable features directly related to the occurrence of a limit cycle in the 3-body system. The discrete scale invariance is manifest in the log-periodic dependence of 3-body observables on the scattering length $a$. This dependence can be tested in experiments with cold atoms close to a Feshbach resonance. For this purpose we consider 3-body recombination, which is the process when three atoms scatter to form a dimer and the third atom balances energy and momentum. This is one of the main loss processes for trapped atoms and condensates of atoms near a Feshbach resonance. The event rate $\nu$ per unit time and unit volume can be parametrized as $\nu = \alpha \rho^3$, where $\rho$ is the density of the atoms and $\alpha$ is the recombination constant.

Unfortunately, heavy alkali atoms, such as Rb and Cs, form many deeply-bound diatomic molecules. Therefore, Efimov states are resonances rather than sharp states, because they can decay into a deep molecule and an energetic atom. The presence of deep molecules also affects other 3-body observables, but using unitarity their influence can be accounted for by one real parameter $\eta_* [9]$. This parameter describes how much probability is lost by scattering into the deep bound states. For $\eta_* = 0$ the observables are not modified, while for $\eta_* = \infty$ all probability is lost into the deep states. The consequences of the limit cycle become totally washed out for $\eta_* \gtrsim 1$. The 3-body recombination coefficient $\alpha$ was first calculated in Refs. [10, 11, 12]. Analytical expressions for $\alpha$ as a function of $a$, $\kappa_*$, and $\eta_*$ have been obtained in Refs. [9, 3].

In a recent experiment with cold $^{133}$Cs atoms, the Innsbruck group has presented the first experimental evidence for Efimov physics [13]. They used a Feshbach resonance to control the scattering length of $^{133}$Cs atoms. Since inelastic 2-body losses were energetically forbidden, the dominant loss mechanisms were inelastic 3-body losses. By varying the external magnetic field, they were able to change the scattering length from $-2500 a_0$.

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See Ref. [8] for the conjecture of a limit cycle in a deformed version of Quantum Chromodynamics.
through 0 to +1600 $a_0$, where $a_0$ is the Bohr radius. They observed a giant loss feature at a magnetic field that corresponds to the scattering length $a = -850(20) \, a_0$. They also measured the 3-body recombination rate for positive values of $a$ reached by increasing the magnetic field through a zero of the scattering length. They observed a local minimum in the inelastic loss rate at a magnetic field that corresponds to a scattering length of 210(10) $a_0$. Their data could well described by the universal expressions from Refs. [9, 3], therefore giving the first indirect evidence for Efimov states in cold atoms [13].

In the left panel of Fig. 3, we compare the universal prediction for negative scattering length at $T = 0$ with the Innsbruck data for the loss coefficient $K_3 = 3\alpha$ at $T = 10 \, nK$ [13]. For $\kappa_* = 0.945/\alpha_0$ and $\eta_* = 0.06$, one obtains a good fit. Both parameters are well determined by the resonance: $\kappa_*$ by the position of the Efimov resonance and $\eta_*$ by the height of the peak. The positive scattering length data from Ref. [13] are shown in the right panel of Fig. 3. One complication here is that the lowest temperature that was reached for positive $a$ was 250 nK. To take account the thermal effects properly would require knowing the 3-body recombination rate as a function of the collision energy. We compare the universal expression for $T = 0$ with the data for $T \approx 250...400 \, nK$ [13]. For $\kappa_* = 0.707/\alpha_0$ and $\eta_* = 0.06$, one obtains a good fit to the data above $a \approx 400 \, a_0$. However, the value of $\eta_*$ is not well determined by the data and only the upper bound $\eta_* < 0.2$ can be given [13].

We note that the values of $\kappa_*$ and $\eta_*$ for positive and negative scattering length need not be the same in the present case since both regions are divided by a nonuniversal region with small scattering length. Moreover the universal expressions are rigorously only valid for $|a| \gg l$ with $l = l_{vdW} \approx 200 \, a_0$ for Cs atoms. Therefore the minimum

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**Figure 3.** Left panel: The 3-body loss coefficient $K_3 = 3\alpha$ in $^{133}$Cs for $a < 0$. The curves are for $\kappa_* = 0.945/\alpha_0$ and three different values of $\eta_*$. The data points are for $T = 10 \, nK$ [13]. Right panel: $K_3$ in $^{133}$Cs for $a > 0$. The curves are for $\kappa_* = 0.707/\alpha_0$ and three different values of $\eta_*$. The data points are for $T \approx 250...400 \, nK$ [13].
around \( a = 210(10) a_0 \) is outside the region of validity of the universal theory.

4. SUMMARY & OUTLOOK

We have discussed the universal properties of few-body systems with large scattering length \( a \gg l \). In the 3-body system these properties include the Efimov effect and a discrete scaling symmetry leading to log-periodic dependence of 3-body observables on the scattering length. These features can be understood as manifestations of a RG limit cycle in the 3-body system. Moreover, we have presented an EFT that is designed to exploit the separation of scales in few-body systems with large scattering length. In this EFT, the limit cycle is manifest through the RG behavior of the 3-body interaction required for proper renormalization. The EFT is very general and has applications in few-body systems from atomic to nuclear and particle physics. As an example, we have shown universal scaling functions for \(^4\)He atoms and the Phillips line. Furthermore, we have discussed the recent 3-body recombination data for cold \(^{133}\)Cs atoms by the Innsbruck group that provided the first evidence of Efimov states in cold atoms [13]. Future challenges include the extension of the EFT to the four-body system [14, 15] and the proper calculation of finite temperature effects [16, 17, 18].

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