The multiple effects of gradient coupling on network synchronization

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(Dated: November 5, 2018)

Recent studies have shown that synchronizability of complex networks can be significantly improved by asymmetric couplings, and increase of coupling gradient is always in favor of network synchronization. Here we argue and demonstrate that, for typical complex networks, there usually exists an optimal coupling gradient under which the maximum network synchronizability is achieved. After this optimal value, increase of coupling gradient could deteriorate synchronization. We attribute the suppression of network synchronization at large gradient to the phenomenon of network breaking, and find that, in comparing with sparsely connected homogeneous networks, densely connected heterogeneous networks have the superiority of adopting large gradient. The findings are supported by indirect simulations of eigenvalue analysis and direct simulations of coupled nonidentical oscillator networks.

PACS numbers: 05.45. Xt, 89.75.-k

Complex networks have attracted a great deal of interest since the discoveries of the small-world [1] and scale-free [2] properties. Roughly, small-world networks are characterized by a locally highly regular connecting structure and a globally small network distance, while the defining characteristic of scale-free networks is a power-law distribution \( p(k) \sim k^{-\gamma} \) in the node degree. Signatures of small-world and scale-free networks have been discovered in many natural and man-made systems [3, 4, 5], and they constitute the cornerstones of modern network science.

At a systems level, synchronization is one of the most common dynamical processes. For instance, in biology, synchronization of oscillator networks is fundamental [6]. In a computer network designed for large scale, parallel computation, to achieve synchronous timing is essential. Recent studies of the synchronizability of complex networks have revealed that small-world and scale-free networks, due to their small network distances, are generally more synchronizable than regular networks [7, 8]. A somewhat surprising finding is that a scale-free network, while having smaller network distances than a small-world network of the same size, is actually more difficult to synchronize [9]. Considering the ubiquity of scale-free networks and the importance of synchronization in network functions, the finding seems to have generated a paradox. However, it is recently found that, with weighted and asymmetric couplings, the synchronizability of scale-free networks can be significantly improved and, in general, can be much higher than the small-world networks [10, 11, 12, 13, 14, 15, 16].

For a pair of connected nodes on the network, the mutual couplings between them are usually unbalanced. One direction will weight over the other direction and generate a coupling gradient on the link. To enhance network synchronization, both the direction and the weight of coupling gradient should be properly set according to the network properties such as node degree [10, 11] and betweenness [12]. In Refs. [10, 11] it has been shown that, by setting the coupling gradient flow from the higher-degree node to the smaller-degree node, the synchronizability of scale-free networks can be significantly improved and higher than that of homogeneous networks. With the same scheme of gradient direction, in Refs. [11, 16] it has been shown that the synchronizability of scale-free networks can be further improved by increasing the gradient weight, and larger gradient in general assumes higher synchronizability. This enhancing role of coupling gradient is further highlighted in Ref. [13], where nodes are proposed to be connected by only gradient couplings, i.e. the one-way-coupling configuration.

While most of the previous studies are focusing on the enhancing role of coupling gradient on synchronization, there are accumulating evidences showing the opposite: large gradient may also deteriorates synchronization. For example, in Ref. [10, 13] it is observed that as gradient increases, network synchronizability is firstly enhanced, gradually reaching to its maximum at an optimal gradient; then, after this optimal value, increase of gradient will suppress synchronization. The similar phenomenon is also briefly reported in Ref. [11], there it is found that the suppressing effect of large gradient has a close relation to the network parameters. Despite of these observations, a detail study on the multiple effects of coupling gradient on network synchronization is still absent.

In this paper, we will study in detail the multiple effects of coupling gradient on network synchronization, and investigate the problem of synchronization optimization in asymmetrically coupled scale-free networks. Our main findings are the following. (1) The destructive role of large gradient comes from the phenomenon of network breaking, increasing gradient will also increase the breaking probability. (2) In general, small and densely-connected heterogeneous networks have a lower breaking probability than large and sparsely-connected










































































































































































































































































































































































































































































































































































































































































































































































































































































































































































































































homogeneous networks. (3) While large gradient deteriorates the propensity of global synchronization, partial synchronization of node clusters is enhanced.

We consider oscillator networks of the following form

\[ \dot{x}_i = F(x_i) - \varepsilon \sum_{j=1}^{N} G_{i,j} H(x_j), \quad i = 1, \ldots, N, \]  

where \( F(x_i) \) governs the local dynamics of uncoupled node \( i \), \( H(x) \) is the coupling function, \( \varepsilon \) is the coupling strength, and \( G_{i,j} \) is an element of the coupling matrix \( G \) which takes form

\[ G_{i,j} = -\frac{A_{i,j} k_i^\beta}{\sum_{j=1}^{N} A_{i,j} k_j^\beta}, \quad \text{for } i \neq j. \]  

with \( k_i \) the degree of node \( i \) and \( A = \{a_{i,j}\} \) the adjacency matrix of the network, \( a_{i,j} = 1 \) if nodes \( i \) and \( j \) are connected, \( a_{i,j} = 0 \) otherwise, and \( a_{i,i} = 0 \). To keep the synchronization state a solution of the system, we choose \( G_{i,i} = 1 \).

It is worthy to note that the parameter \( \beta \) in Eq. \( 2 \) modulates both the direction and weight of the coupling gradient on each link \( 16 \). Statistically, if \( \beta > 0 \), gradient is flowing from the higher-degree node to the smaller-degree node; while if \( \beta > 0 \), gradient is flowing in the opposite direction. Meanwhile, by tuning the absolute value \( |\beta| \), we are also able to control the gradient weight: larger \( |\beta| \) generates larger gradient \( 16 \). Please also note that changing \( \beta \) does not change the total coupling cost of the network, it only redistribute the weight of the couplings. Besides the flexibility of gradient control, the coupling scheme of Eq. \( 2 \) is also representative to a variety of network models proposed in previous studies. For instance, the symmetric network model in Refs. \( 7,9 \) can be realized by replacing \( G \) with \( A \) in Eq. \( 1 \); the weighted asymmetric network model (constructed based on the information of node degree) in Ref. \( 10 \) can be realized by setting \( \beta = 0 \) in Eq. \( 2 \); and the directed tree-structure network model in Ref. \( 13 \) in principle can be realized by setting \( \beta \to \infty \) in Eq. \( 2 \). A schematic plot on the realization of these different network models by changing \( \beta \) is illustrated in Fig. 1.

The limiting case of \( \beta \to \infty \) in Eq. \( 2 \) is of special interest [Fig. 1(c)], since it represents the extreme situation of node connection: the one-way coupling format, i.e., \( G_{i,j} = -1 \) and \( G_{i,i} = 0 \) when \( k_j > k_i \), and \( G_{i,j} = 0 \) and \( G_{i,i} = -1 \) otherwise. In this case, each node only receives coupling from one of its neighbors who has the largest degree. (Strictly speaking, each node is receiving coupling from one of its neighbors who has the largest node-scalar \( \beta \) associated with the increased gradient effect. However, if there are two or more large-degree nodes on the network which are not directly connected, like the case of Fig. 1(c)], the network will break into several sub-networks of tree-structure, and at the root of each sub-network locates a local largest-degree node. Once is broken, the network can never be globally synchronized whatever the coupling strength, manifesting the destructive role of large gradient.

To have a global picture on the gradient effects, we have investigated the variation of network synchronizability as a function of the gradient parameter \( \beta \) for different network topologies. The synchronizability of coupled network can be evaluated by the method of master stability function (MSF) \( 8 \), which states that a network is generally more synchronizable when the spread of the eigenvalue spectrum of its coupling matrix is narrow. In particular, let \( 0 = \lambda_1 \leq \lambda_2 \ldots \leq \lambda_N \) be the eigenvalue spectrum of the coupling matrix \( G \). Then the smaller the ratio \( R = \lambda_N / \lambda_2 \), the more likely synchronous dynamics is to occur on the network. In general, the matrix constructed by Eq. \( 2 \) is asymmetric and its eigenvalues are complex. Noticing that the coupling matrix \( G \) can be written as \( G = QLD^\beta \), with \( L = D - A \), \( D = \text{diag}(k_1, k_2, \ldots, k_N) \) the diagonal matrix of degrees, and \( Q = \text{diag}(1 / \sum_j L_{i,j} k_j^\beta, \ldots, 1 / \sum_k L_{N,j} k_j^\beta) \) the normalization factors on rows of \( G \). From the following identity

\[ \det(QLD^\beta - \lambda I) = \det(Q^{1/2}D^{\beta/2}LD^{\beta/2}Q^{1/2} - \lambda I) \]  

it is found that the eigenvalues of the asymmetric matrix \( G \) are equal to that of the symmetric matrix \( H = Q^{1/2}D^{\beta/2}LD^{\beta/2}Q^{1/2} \), which are real and nonnegative.

To simulate, we generate scale-free network of \( N = 2^{10} \) nodes and average degree \( \langle k \rangle = 6 \) by the generalized model introduced in Ref. \( 17 \). In this model, the degree exponent \( \gamma \) can be adjusted via a parameter \( B \). Defining the new preferential attachment function as \( p \sim (k_i + B) / \sum_j (k_j + B) \), it can be proven that \( \gamma = 3 + B / m \), with \( m = 3 \) the number of new links that associated to each new added node in the model. Using \( B = 0, 12 \) and \( 51 \), we generate scale-free networks of degree exponents \( \gamma = 3, 7 \) and \( 20 \), respectively.
and average degree (the black curve). The extreme values of realizations under parameters \( \gamma = 3 \) (the lower curve) and \( \gamma = 7 \) (the middle curve) and \( \gamma = 20 \) (the upper curve). The optimal gradient \( \beta_o \) is about 0.9 in the later two cases. Each data is an average result of 100 network realizations. (b) For the case of \( \gamma = 7 \) in (a), the value of \( R_i \) for different network realizations under parameters \( \beta = 0.5 \) (the red curve) and \( \beta = 5 \) (the black curve). The extreme values of \( R_i \) indicate the broken of the network topology in the corresponding realizations.

The variations of \( R \) as a function of \( \beta \) for these three networks are plotted in Fig. 2(a). It is found that, in the case of \( \gamma = 3 \), increasing \( \beta \) will enhance synchronizability monotonically; however, in the case of \( \gamma = 7 \) or 8, as \( \beta \) increases from zero, the network synchronizability is firstly enhanced, and, after reaching its maximum at value about \( \beta_o \approx 0.9 \) [19], it begin to be suppressed. Another interesting finding in Fig. 2(a) is that, for the fixed gradient parameter \( \beta \), increasing the degree exponent \( \gamma \) will always decrease the network synchronizability, indicating the superior synchronizability of scale-free networks under gradient couplings.

To gain insight on the transition of gradient effect from enhancing to suppressing synchronization, we go on to investigate the changes happening in the neighboring region of the optimal gradient \( \beta_o \). With \( \gamma = 7 \) (the middle curve in Fig. 2(a)), we plot in Fig. 2(b) the individual value of \( R_i \) for a large number of network realizations under gradient parameters \( \beta_1 = 0.5 < \beta_o \) and \( \beta_5 = 5 > \beta_o \). It is found that, for \( \beta_1 = 0.5 \), the eigenratio \( R_i \) is oscillating around its mean value \( R \approx 6 \) with very small fluctuations; while for \( \beta_5 = 5 \), the eigenratio \( R_i \) occasionally bursts into some extreme values of order \( 10^3 \). Since of \( R \equiv \lambda_N/\lambda_2 \), a divergent \( R_i \) thus indicates \( \lambda_2 \to 0 \), which, from the eigenvalue analysis, implies the breaking of network topology in the corresponding realization. Therefore the suppression effect of large gradient can be attributed to the phenomenon of network breaking, and the optimal gradient can be understood as a balance between the enhancing and the suppressing effect of gradient coupling.

By knowing that synchronization suppression at large gradient is induced by network breaking, we next to investigate the relationship between the breaking probability and the network parameters. To facilitate the analysis, we consider again the limiting case of \( \beta \to \infty \) in Eq. (2). As shown in Fig. 1(c), network breaking happens when there are more than one local-maximum-degree nodes coexist on the network. To break the network, the “breaking nodes” do not have to be possessing very large degree, they are only required to have the largest degree among its neighbors. Once is broken, the network will be divided into several tree-structured subnetworks, with each subnetwork is led by one of such “breaking nodes”. Due to the complicated configurations of the “breaking nodes”, we are unable to give an analytical prediction on the relationship between the breaking probability and the network parameters. The numerical results on the variation of \( p_b \) as functions of \( \gamma \) and \( \langle k \rangle \) are plotted in Fig. 3, which show that, in comparing sparsely connected homogeneous networks, densely connected heterogeneous networks are more difficult to break down [13]. It can be also found from Fig. 3(b) that the increase of system size will always increase the breaking probability. These findings (calculated for the limiting case) are coincident with the findings in Fig. 2 (calculated for the general case), both tell us that heterogeneous networks are more sustainable to large gradient.

The results exemplified in Figs. 2 and 3 are based on the eigenvalue and topology analysis. It is useful to examine the gradient effects in real oscillator networks. For this purpose we have check the synchronization of scale-free networks of coupled nonidentical chaotic Rössler oscillators, a typical model employed in studying network synchronization [8, 10, 11, 12]. The dynamics of a single oscillator is described by \( \dot{F}_i(x_i) = -\omega_i y_i - z_i, \omega_i x_i + 0.15 y_i, z_i(x_i - 8.5) + 4 \), where \( \omega_i \) is the natural frequency of the \( i \)th oscillator. In simulations we choose \( \omega_i \) randomly from the range \([0.9, 1.1]\), so as to make the oscillators nonidentical. The coupling function is chosen to be \( H(x) = x \). The degree of synchronization can be characterized by monitoring the amplitude \( A \) of the mean field \( X(t) = \sum_{i=1}^{N} x_i(t)/N \) [10]. For small coupling strength \( \epsilon \), \( X(t) \) oscillates irregularly and \( A \) is approximately zero, indicating lack or a lower degree of synchronization. As the coupling parameter is increased, synchronization sets in and \( A \) is increased gradually from zero (nonsynchronous state) to its maximum (synchronous state). By \( \epsilon = 0.15 \), we plot in Fig. 4(a) the variation of \( A \) as a function of \( \beta \). It is found that as \( \beta \) increases \( A \) is firstly increased, and reaching its maximum at about \( \beta_o \approx 1 \), manifesting the
constructive role coupling gradient. Then, as $\beta$ increases from $\beta_o$, $A$ begins to decrease, manifesting the destructive role of coupling gradient. An interesting finding is that increasing $\beta$ further does not decreases $A$ continuously, the system always keeps on high coherence at about $A \approx 8$. Please note that for the adopted coupling function $H$, the stable region of the MSF function [8] has only a lower boundary, which confirms that the decrease of $A$ at $\beta \geq \beta_o$ in Fig. 4(b) is induced exclusively by the breaking effect (not induced by the instability of the shortest wave mode [8]).

The finding that $A$ keeps on large values at very large $\beta$ indicates that, despite of the increased probability of network breaking, nodes are still strongly correlated, with a manner that is different to the situation of small $\beta$. To gain insight, we choose a network realization in Fig. 4(a), and plot the frequency distribution $\omega_i$ of the oscillators under different gradient parameters: $\beta = 0.1$ in Fig. 4(b), $\beta = 1$ in Fig. 4(c) and $\beta = 10$ in Fig. 4(d). It is found that, for small gradient [Fig. 4(b)], $\omega_i$ is distributed randomly around the mean value $\omega = 1$, indicating the low system coherence; around the optimal gradient [Fig. 4(c)], most of the oscillators are synchronized to be having the same frequency, with few exceptions which are synchronized to another frequency, indicating a higher coherence of the system dynamics under this gradient; for large gradient [Fig. 4(d)], the oscillators are separated into two clusters of the similar size, the frequency of one cluster is different to that of the other one. Fig. 4(d) indicates that, under the large gradient, although the probability of network breaking is high, network nodes are still strongly correlated due to the existence of synchronous clusters. The clusters are a direct result of the network breaking. Led by the “breaking nodes”, each subnetwork develops into a synchronous cluster.

In summary, we have studied the multiple effects of coupling gradient on network synchronization, and investigated the dependence of these effects to the network parameters. Our findings suggest that, in comparing with sparsely connected homogeneous networks, densely connected heterogeneous networks take more advantages from the gradient couplings.

YCL and CTZ thank the great hospitality of National University of Singapore, where part of the work was done during their visits. YCL was supported by AFOSR under Grants No. FA9550-06-1-0024 and No. FA9550-07-1-0045. CTZ was supported by the National Natural Science Foundation of China under Grant No. 10575013.

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[18] Breaking phenomenon is also found in other types of complex networks such as the modular and assortative networks, there breaking probabilities are also dependent on the network modularity and assortativity.
[19] Numerically we find that $\beta_o$ is insensitive to $\gamma$. This can be heristically understood as a competing result of the breaking probability and the total gradient. Under the same parameter $\beta$, heterogeneous networks have small breaking probability but possess large gradient, the former enhances synchronization while the latter in general suppresses synchronization.