PERIODIC SOLUTIONS OF DIFFERENTIAL-ALGEBRAIC EQUATIONS

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Abstract. In this paper, we study the existence of periodic solutions for a class of differential-algebraic equation

\[ h'(t, x) = f(t, x), \quad \frac{d}{dt}, \]

where \( h(t, x) = A(t)x(t), \ h(t, x) \) and \( f(t, x) \) are \( T \)-periodic in first variable.

Via the topological degree theory, and the method of guiding functions, some existence theorems are presented. To our knowledge, this is the first approach to periodic solutions of differential-algebraic equations. Some examples and numerical simulations are given to illustrate our results.

1. Introduction. Differential-algebraic equations (DAEs) arise naturally in many mathematical models of engineering and science, such as electric circuit [1], chemical processes [18, 22], mechanical multibody systems [20], incompressible fluids [21], molecular dynamics [19] and optimal control [6, 7]. Generally speaking, it is more difficult to find solutions of DAEs because of its typical structure and property different from ODEs. At present there are various numerical methods on the solutions of DAEs [4, 10]. The index of DAEs is an important standard for the computational complexity and the difficulty of analysis involved in the study of DAEs. Once the index is greater than two, it is extremely difficult to handle and even results in so-called ill-posed problem [3, 5], hence the study of DAEs with a high index has drawn great attentions [2, 11, 23]. At the same time, the existence of periodic solutions of systems has always been a major problem in the field of engineering and scientific research. However, to our knowledge few researchers have addressed the study of periodic solutions of DAEs.

This paper aims at researching the existence of periodic solutions for a class of nonautonomous DAEs. In particular, our existence theorems about periodic

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solutions are independent of the index of DAEs and it is also a valid method to find periodic solutions of DAEs in practical application. Theoretically, the homotopy method and topological degree theory are very powerful tools in the study of the existence of periodic solutions of differential equations [14, 15, 16]. In this paper, we apply homotopy method and topological degree theory to provide existence theorems about periodic solutions for some nonautonomous DAEs.

More precisely, we consider the following differential-algebraic system:

\[ h'(t, x) = f(t, x), \quad \frac{d}{dt}, \quad (1) \]

where \( f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) is continuous and \( T \)-periodic in its first variable, \( h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) is continuously differentiable, \( T \)-periodic in its first variable, and linear in its second variable, and assume that system (1) admits existence and uniqueness of solutions of (1) with respect to initial values. By the linearity of \( h \) in \( x \), let’s denote \( h(t, x) \) by \( A(t) x(t) \), where \( A(t) \) is a singular \((n \times n)\)-matrix. The aim is to show that system (1) admits \( T \)-periodic solutions of the form \( x(t + T) = x(t), \forall t \in \mathbb{R} \), which is equivalent to the solutions of system (1) with the boundary value condition:

\[ x(T) = x(0). \quad (2) \]

In [9], Hale and Mawhin applied the coincidence degree theory to the problem of existence of periodic solutions for some nonautonomous neutral functional differential equations. They presented the existence theorems for some neutral functional differential equations when the operator \( D \) defined by \( D(t) \varphi = \varphi(0) - A(t) \varphi \) is stable. As motivated in their works, we are concerned with the existence of periodic solutions of system (1) when \( A(t) \) is not invertible, i.e., the corresponding operator \( D \) is unstable. In our approach, we use the Lyapunov-Schmidt reduction and topological degree theory to provide some existence results to periodic solutions of system (1).

At the same time, the prior estimate of periodic solutions for system (1) is difficult to deal with. In virtue of the method of guiding function introduced and developed by Krasnosel’skii, Mawhin and others [8, 12, 13, 17], we give a theorem to estimate the prior bound of periodic solutions of system (1).

This paper is organized as follows. In Section 2, we provide some existence theorems and the priori estimate about periodic solutions for system (1). In Section 3, some examples and numerical simulations are given. Finally, we conclude the paper with a summary in Section 4.

2. Main results. To investigate the existence of periodic solutions of system (1), we consider the following auxiliary system,

\[ h'(t, x) = \lambda f(t, x), \quad (3) \]

where \( \lambda \in [0, 1] \). Our main results are as follows.

**Theorem 2.1.** Let \( \Omega \subseteq \mathbb{R}^n \) be a bounded open set. Assume the followings hold for system (3):

(H1) \( \partial \Omega \) contains no \( T \)-periodic solution of system (3) for each \( \lambda \in [0, 1] \), i.e., any possible \( T \)-periodic solution \( x \) of system (3) satisfies \( x(t) \notin \partial \Omega \);

(H2) \( k^{-1}_p(t) : \mathbb{R}^n \to \mathbb{R}^n \) is continuous, where \( k^{-1}_p(t) \) is the inverse of \( A(t)|_{\text{Im}A(t)} \);

(H3) \( \partial \Omega \) contains no zero of the map \( g : \mathbb{R}^n \to \mathbb{R}^n \) defined by

\[ g(a) = \frac{1}{T} \int_0^T f(t, a_k + k^{-1}_p(t)(I - P(t))A(0)a)ds, \]

where \( \lambda \in [0, 1] \). Our main results are as follows.
where \( a_k = P(0)a \in \ker A(0) \), \( P: \mathbb{R}^n \to \ker A(t) \) is continuous and \( \mathbb{R}^n = \ker A(t) \oplus \text{Im} A(t) \) for all \( t \);

(H4) The Brouwer degree \( \deg(g(a), \Omega, 0) \neq 0 \).

Then system (1) has at least one \( T \)-periodic solution \( x^*(t) \in \Omega \) for all \( t \).

Proof. Consider the auxiliary system:

\[
\begin{align*}
h'(t, x(t)) &= \lambda f(t, x(t)) \quad (4) \\
h(T, x(T)) &= h(0, x(0)) + \lambda \int_0^T f(s, x(s))ds. \quad (5)
\end{align*}
\]

Thus,

\[
A(T)x(T) = A(0)x(0) + \lambda \int_0^T f(s, x(s))ds, \quad (6)
\]

and hence

\[
\int_0^T f(s, x(s))ds = 0. \quad (7)
\]

Rewriting (3) in the form of an equivalent integral equation, we have

\[
A(t)x(t) = A(0)x(0) + \lambda \int_0^t f(s, x(s))ds. \quad (8)
\]

For fixed \( t \) in \([0, T]\),

\[
A(t)x(t) = A(t)(x_k(t) + x_I(t))
\]

\[
= P(t)(A(0)x(0)) + \lambda \int_0^t f(s, x(s))ds \quad (9)
\]

\[
+ (I - P(t))(A(0)x(0) + \lambda \int_0^t f(s, x(s))ds),
\]

where \( x_k(t) \in \ker A(t), x_I(t) \in \text{Im} A(t) \), and \( x(t) \in \mathbb{R}^n = \ker A(t) \oplus \text{Im} A(t) \).

Hence, we have

\[
A(t)x_k(t) = P(t)A(0)x(0) + \lambda P(t) \int_0^t f(s, x(s))ds = 0, \quad (10)
\]

and

\[
A(t)x_I(t) = (I - P(t))A(0)x(0) + \lambda(I - P(t)) \int_0^t f(s, x(s))ds, \quad (11)
\]

or equivalently,

\[
x_I(t) = k_p^{-1}(t)(I - P(t))A(0)x(0) + \lambda k_p^{-1}(t)(I - P(t)) \int_0^t f(s, x(s))ds. \quad (12)
\]

Let

\[
X = \{ x : [0, T] \to \mathbb{R}^n | x(t) \text{ is continuous on } [0, T], x_k(0) = x_k(T) \}
\]

with norm \( ||x|| = \sup_{t \in [0, T]} |x(t)| \). It is easy to see that \( X \) is a Banach space with the norm \( || \cdot || \). For \( x \in X \) with \( x(t) \in \overline{\Omega} \) for all \( t \in [0, T] \), we consider the following
operator $T(x(0), x(t), \lambda)(t)$:

$$
T(x(0), x(t), \lambda)(t) = \begin{pmatrix}
    x(0) + \frac{1}{T} \int_0^T f(s, x(s))ds \\
x_k(t) + k_p^{-1}(t)(I - P(t))A(0)x(0) \\
    + \lambda k_p^{-1}(t)(I - P(t)) \int_0^t f(s, x(s))ds
\end{pmatrix},
$$

(13)

where $\lambda \in [0, 1]$.

We claim that each fixed point $x$ of $T$ in $X$ is a solution of (3) with $x(T) = x(0)$. In fact, if $x$ is a fixed point of $T$, we have

$$
\begin{pmatrix}
    x(0) \\
x(t)
\end{pmatrix} = \begin{pmatrix}
x(0) + \frac{1}{T} \int_0^T f(s, x(s))ds \\
x_k(t) + k_p^{-1}(t)(I - P(t))A(0)x(0) \\
    + \lambda k_p^{-1}(t)(I - P(t)) \int_0^t f(s, x(s))ds
\end{pmatrix}.
$$

(14)

Hence

$$
\frac{1}{T} \int_0^T f(s, x(s))ds = 0,
$$

(15)

and

$$
x(t) = x_k(t) + k_p^{-1}(t)(I - P(t))A(0)x(0) + \lambda k_p^{-1}(t)(I - P(t)) \int_0^t f(s, x(s))ds.
$$

(16)

By equation (16),

$$
x(T) = x_k(T) + k_p^{-1}(T)(I - P(T))A(T)x(0) + \lambda k_p^{-1}(T)(I - P(T)) \int_0^T f(s, x(s))ds.
$$

(17)

Notice that $A(T) = A(0)$. We have $\ker A(T) = \ker A(0)$. Thus

$$
x(T) = x_k(T) + x_I(0) + \lambda k_p^{-1}(T)(I - P(T)) \int_0^T f(s, x(s))ds = x(0).
$$

(18)

In consequence, the fixed point $x$ of operator $T$ satisfies the boundary value condition of (3). As we will see, it also satisfies (3). In fact, by equation (12) and (16), we have

$$
x(t) = x_k(t) + k_p^{-1}(t)(I - P(t))A(0)x(0) + \lambda k_p^{-1}(t)(I - P(t)) \int_0^t f(s, x(s))ds
$$

(19)

$$
= x_k(t) + x_I(t).
$$

In conclusion, the fixed point $x$ of operator $T$ is a solution of (3) with $x(T) = x(0)$. Next, it is sufficient to prove the existence of the fixed point of $T$.

Let

$$
X_\lambda = \left\{ x \in X \left| \sup_{t \neq s} \left| \frac{x_k(t) - x_k(s)}{t - s} \right| \leq \lambda \right\}
$$

Then $X_\lambda$ is a closed and convex subspace of $X$, hence there exists a retraction $\alpha_\lambda : X \to X_\lambda$. 


Thus, that (20) as

Similarly we can prove that

By the definition of \( \lambda k \in (I) \).

We claim that

On the contrary, suppose that there exists \((x(t), \lambda) \in \Omega \times \tilde{\Omega} \times [0, 1] \).

We have

Consider the following homotopy:

Hence equation (25) can be rewritten as:

By the definition of \( X \), we obtain \( x_k(t) \in X \), which means that \( \alpha x_k(t) = x_k(t) \).

Hence equation (25) can be rewritten as:

Similarly we can prove that \( x(t) \) is a solution of system (3) with \( x(T) = x(0) \). It is in contradiction to hypothesis (H1).

\( (II) \). \( \tilde{x} = 0 \). In this case, since \( 0 = (id - H) x(0) \), we can rewrite equation (20) as

\[ \begin{pmatrix} x(0) + \frac{1}{T} \int_0^T f(s, x(s))ds \\ \alpha x_k(t) + k_p^{-1}(t)(I - P(t))A(0)x(0) + \lambda k^{-1}(t)(I - P(t)) \int_0^t f(s, x(s))ds \end{pmatrix} = (20) \]
Thus,
\[
\frac{1}{T} \int_0^T f(s, \varpi(s))ds = 0, \tag{28}
\]
and
\[
\varpi(t) = \varpi_k(0) + k_p^{-1}(t)(I - P(t))A(0)\varpi(0). \tag{29}
\]
Substituting (29) into (28), we can obtain:
\[
\frac{1}{T} \int_0^T f(s, \varpi_k(0) + k_p^{-1}(s)(I - P(s))A(0)\varpi(0))ds = 0. \tag{30}
\]
By the definition of the map \(g : \mathbb{R}^n \to \mathbb{R}^n\), we have
\[
g(\varpi(0)) = \frac{1}{T} \int_0^T f(s, \varpi_k(0) + k_p^{-1}(s)(I - P(s))A(0)\varpi(0))ds
= 0, \tag{31}
\]
where \(\varpi(0) \in \partial\Omega\). It is a contradiction to hypothesis (H3).

From (I) and (II), we prove that:
\[
0 \neq (id - H)(\partial(\Omega \times \tilde{\Omega}) \times [0, 1]). \tag{32}
\]
Therefore, by the homotopy invariance of Brouwer degree, we have
\[
\text{deg}(id - H(\cdot, \cdot, 1), \Omega \times \tilde{\Omega}, 0)
= \text{deg}(id - H(\cdot, \cdot, 0), \Omega \times \tilde{\Omega}, 0)
= \text{deg}(g, \Omega, 0)
\neq 0. \tag{33}
\]
By the regularity of topological degree, there exists \(x^* \in \tilde{\Omega}\), such that
\[
\begin{pmatrix}
x^*(0) \\
x^*(t)
\end{pmatrix} = \mathcal{T}(x^*(0), x^*(t), 1). \tag{34}
\]
Thus \(x^*(t)\) is a fixed point of \(T\) in \(X\), and hence \(x^*(t)\) is a solution of system (1) with the boundary value condition \(x(T) = x(0)\). The proof is completed. \(\Box\)

In order to find a \(T\)-periodic solution of system (1) via Theorem 2.1, we now face on two difficult problems: finding \(\Omega\) (in fact, it is a priori bound problem) and estimating the Brouwer degree \(\text{deg}(g(a), \Omega, 0) \neq 0\). For the first one, the method of guiding functions introduced and developed by Krasnosel’skii, Mawhin and others is a very useful tool to design the priori bound. Next, we will use the method of guiding function to find a \(T\)-periodic solution for system (1). For the second one, there are a few special cases in which it is easy to estimate \(\text{deg}(g(a), \Omega, 0)\).

**Definition 2.2.** A continuously differentiable function \(V : \mathbb{R}^n \to \mathbb{R}\) is called a guiding function for the periodic problem of system (1), if there exists a constant \(M > 0\), such that
\[
|\langle \nabla V(h(t,x)), f(t,x) \rangle| > 0, \ |x| \geq M. \tag{35}
\]

**Lemma 2.3.** Assume there exists a guiding function \(V : \mathbb{R}^n \to \mathbb{R}\), such that the followings hold.

(H5) For \(M\) large enough,
\[
|\langle \nabla V(h(t,x)), f(t,x) \rangle| > 0 \text{ for all } |x| \geq M, t \in \mathbb{R}; \tag{36}
\]
(H6) \(V(h(t,x)) \to \infty\), as \(|x| \to \infty\).
Then there exists a constant $\rho^* > 0$, such that any $T$-periodic solution of system (3) satisfies $x(t) \in B(0, \rho^*)$ for $t \in \mathbb{R}$, where $B(0, \rho^*)$ denotes a ball with the radius of $\rho^*$.

Proof. Let $x(t)$ be a $T$-periodic function of auxiliary system (3) for $\lambda \in [0, 1]$. Then $|V(h(t, x))|$ is also a $T$-periodic function, and thus there exists $\tau > 0$ such that $|V(h(\tau, x(\tau)))| = \max_{t \in [0, T]} |V(h(t, x(t)))|$. Naturally we have

$$|\langle \nabla V(h(\tau, x(\tau))), f(\tau, x(\tau)) \rangle| = 0.$$ 

By the definition of a guiding function, we have $|x(t)| < M$ for all $t \in [0, T]$. Thus we can take a large enough constant $\rho^*$ such that $x(t) \in B(0, \rho^*)$ for all $t \in [0, T]$. \hfill \Box

**Theorem 2.4.** Assume that there exists $(m + 1)$ guiding functions $V_i(x)$ for system (1), such that the followings hold:

(H7) For $M_i$ large enough,

$$|\langle \nabla V_i(h(t, x)), f(t, x) \rangle| > 0, \quad |x| \geq M_i, \quad i = 0, 1, \ldots, m;$$

(H8) $\sum_{i=0}^{m} |\nabla V_i(h(t, x))| \to \infty$, as $|x| \to \infty$;

(H9) $\deg(\nabla V_0, B_{M_0}, 0) \neq 0$, where $B_{M_0} = \{ p \in \mathbb{R}^n : |p| < M_0 \}$.

Then system (1) has at least one $T$-periodic solution.

Proof. Consider the auxiliary system (3), where $\lambda \in [0, 1]$. Set

$$L_i = \sup\{|V_i(h(t, x))| : |x| \leq M_i\}, \quad L = \sum_{i=0}^{m} L_i;$$

$$\Omega = \{ p \in \mathbb{R}^n : \sum_{i=0}^{m} |V_i(h(t, p))| < L + 1 \}, \quad V(h(t, x)) = \sum_{i=0}^{m} |V_i(h(t, x))|.$$ 

By (H8), $\Omega$ is bounded for $\lambda \in (0, 1]$. We claim that every possible $T$-periodic solution $x(t)$ of (3) satisfies:

$$x(t) \in \Omega.$$ 

In fact, since $x(t)$ is a $T$-periodic solution of (3), there exists a sequence $\{t_j\} \subset \mathbb{R}$ such that

$$|V(h(t_j, x(t_j)))| \to \sup_{t \in \mathbb{R}} |V(h(t, x(t)))|, \quad \text{as } j \to \infty.$$ 

Hence, for some $i$,

$$|V_i(h(t_j, x(t_j)))| \to \sup_{t \in \mathbb{R}} |V_i(h(t, x(t)))|, \quad \text{as } j \to \infty.$$ 

It follows that

$$\frac{d}{dt} V_i(h(t_j, x(t_j))) = \langle \nabla V_i(h(t_j, x(t_j))), \lambda f(t_j, x(t_j)) \rangle \to 0, \quad \text{as } j \to \infty.$$ 

By (H7), we have

$$|x(t_j)| < M_i, \quad \text{as } j \to \infty.$$ 

Consequently, by the definition of $\Omega$, we have

$$x(t) \in \Omega.$$ 

Thus, (H1) holds. Next, it is sufficient to show that the topological degree

$$\deg(g(a), \Omega, 0) \neq 0.$$ 

In fact, consider the following homotopy,
\[
H(a, \lambda) = \lambda \text{sgn}((\nabla V_0(h(t, \cdot)), f(t, \cdot))|_{\partial B_{M_0}}) \nabla V_0(h(t, a)) + (1 - \lambda)g(a),
\] (37)
where \((a, \lambda) \in \overline{B_{M_0}} \times [0, 1].\) It follows that
\[
\langle \nabla V_0(h(t, a)), H(a, \lambda) \rangle = \lambda \text{sgn}((\nabla V_0(h(t, \cdot)), f(t, \cdot))|_{\partial B_{M_0}}) |\nabla V_0(h(t, a))|^2
+ (1 - \lambda)\langle \nabla V_0(h(t, a)), g(a) \rangle.
\] (38)

By (H7), for any \(a \in \partial B_{M_0},\) the sign of \(\langle \nabla V_0(t, \cdot), f(t, \cdot) \rangle\) does not change. By the definition of \(g(a),\) we have
\[
\langle \nabla V_0(h(t, a), g(a)) \rangle
= \langle \nabla V_0(h(t, a)), \frac{1}{T} \int_0^T f(t, a_k + k_p^{-1}(t)(I - P(t))A(0)a)ds \rangle
= \frac{1}{T} \int_0^T \langle \nabla V_0(h(t, a)), f(t, a_k + k_p^{-1}(t)(I - P(t))A(0)a) \rangle ds,
\] (39)
and we can easily obtain that the sign of \(\langle \nabla V_0(h(t, \cdot)), f(t, \cdot) \rangle\) and \(\langle \nabla V_0(h(t, a), g(a) \rangle)\) are same. Thus, equation (38) is nonzero for all \((a, \lambda) \in \partial(B_{M_0} \times [0, 1]),\) which means that \(0 \neq H(\partial(B_{M_0} \times [0, 1])).\)

By the homotopy invariance of topology degree, we have
\[
\deg(g(a), B_{M_0}, 0)
= \deg(\text{sgn}((\nabla V_0(h(t, \cdot), f(t, \cdot))|_{\partial B_{M_0}}) \nabla V_0(h(t, a)), B_{M_0}, 0)
\] (40)
\[
\neq 0.
\]

All the conditions of Theorem 2.1 hold, then system (1) has at least one \(T\)-periodic solution \(x^*(t) \in \Omega.\) \(\square\)

**Corollary 1.** Assume that there exists a guiding function \(V\) for system (1), such that (H5) and (H6) hold, and the following holds:
(H10) \(\deg(\nabla V, B_{\rho^*}, 0) \neq 0\) and \(\nabla V(p) \neq 0\) for \(|p| \geq \rho^*\).
Then there exists at least one \(T\)-periodic solution of (1).

**Proof.** By Lemma 2.3, for each \(\lambda \in (0, 1),\) any \(T\)-periodic solution of system (3) can not reach the boundary of \(B(0, \rho^*)\) for \(\rho^* \gg 1.\) Thus (H1) holds. By Theorem 2.4, likewise it can be proved. \(\square\)

**Corollary 2.** Let \(\Omega \subset \mathbb{R}^n\) be an open and bounded neighborhood of 0 which is symmetric with respect to 0, such that (H1) holds. The definition of \(k_p^{-1}(t)\) and \(g(a)\) is the same as in Theorem 2.1. Assume that the following holds:
(H11) \(g(-a) = -g(a), g(a) \neq 0, \forall a \in \partial \Omega.\)
Then there exists at least one \(T\)-periodic solution \(x^*(t) \in \Omega\) for system (1).

**Proof.** It is easily proved by Borsuk theorem. \(\square\)

3. **Application and numerical simulation.** DAEs can be used to describe many models naturally in application fields. In this section, we consider the following three examples of motion with constraint conditions. In our approach, periodic solutions can be found, and some numerical simulations are given.
Example 3.1. Consider the motion of a ‘body’ on a given plane, and the corresponding differential-algebraic system is constructed as follows:
\[
\begin{align*}
\dot{x} &= f_1(t,\mathbf{x}) = -x^3 + y + \sin z + \sin t, \\
\dot{y} &= f_2(t,\mathbf{x}) = -y^5 - x + \cos t, \\
x + y - z &= 0,
\end{align*}
\]  
(41)
where \(\mathbf{x} = (x, y, z)\) is the position of the body, and \(t\) is the time. Now we will prove that there exists at least one periodic solution.
Consider the auxiliary system of (41):
\[
\begin{align*}
\dot{x} &= \lambda f_1(t,\mathbf{x}), \\
\dot{y} &= \lambda f_2(t,\mathbf{x}), \\
x + y - z &= 0.
\end{align*}
\]  
(42)
In order to design the priori bound of (42), we use the method of guiding function. Set the guiding function 
\[
V(x,y) = \frac{1}{2}(x^2 + y^2),
\]
then (H6) naturally holds. Notice that if \(|x|^2 + |y|^2 \gg 1\), we have
\[
\langle \nabla V, f \rangle = xf_1(t,\mathbf{x}) + yf_2(t,\mathbf{x}) \\
= x(-x^3 + y + \sin z + \sin t) + y(-y^5 - x + \cos t) \\
= -x^4 - y^6 + x \sin(x + y) + x \sin t + y \cos t
\]  
(43)
then (H5) holds. Therefore, all the conditions of Lemma 2.3 are satisfied. Hence there exists a constant \(\rho^* > 0\), such that any possible periodic solution of system (42) satisfies \(x(t) \in B_{\rho^*}(0)\).

For a large enough \(\rho^*\), we have
\[
g(a_1, a_2) = (-a_1^5 + a_2 + \sin(a_1 + a_2), -a_2^5 - a_1).
\]  
(44)
By Borsuk theorem, \(\text{deg}(g(a), B_{\rho^*}(0), 0) = odd \neq 0\).
In conclusion, by Theorem 2.1, there exists at least one periodic solution \( x^*(t) \in B_{\rho^*}(0) \) for all \( t \).

Numerical simulations are given in Fig. 1. The periodic solution \( x^*(t) = (x(t), y(t), z(t))^T \), where \( x(t) \) is denoted by purple solid line, \( y(t) \) is denoted by blue dashed line, and \( z(t) \) is denoted by cyan solid line, is shown in Fig. 1(a). We take \( x(0) = -0.8538, y(0) = 1.0020, z(0) = 0.1482 \) as the initial values of system (41). According to the feature of the image of periodic solutions, the minimal period \( T = 2\pi \). Further more, we exhibit intuitively the trajectory of motion on the constrained plane in Fig. 1(b). From Fig. 1(b), system (41) has a periodic solution \( x^*(t) \), with \(|x^*| < \rho^* \) (\( \rho^* \) is a given large enough constant), rotating clockwise on the constrained plane, and minimal period \( T = 2\pi \) (obtained in Fig. 1(a)).

In conclusion, the results of numerical experiments show intuitively existence of periodic solutions of system (41).

**Example 3.2.** Consider the motion of a ‘body’ on a given sphere, and the corresponding differential-algebraic system is constructed as follows:

\[
\begin{aligned}
\dot{x} &= f_1(t, x) = -3.5x - \sin z^2 + \sin t, \\
\dot{y} &= f_2(t, x) = -3y + \cos z^2 + \cos t, \\
x^2 + y^2 + z^2 &= 4,
\end{aligned}
\]  

(45)

where \( x = (x, y, z) \) denotes the position of motion, and \( t \) denotes the time.

Similarly, taking the guiding function \( V(x, y) = \frac{1}{2}(x^2 + y^2) \), we have

\[
\langle \nabla V, f \rangle = x(-3.5x - \sin z^2 + \sin t) + y(-3y + \cos z^2 + \cos t)
= -3.5x^2 - 3y^2 + x(\sin t - \sin z^2) + y(\cos t + \cos z^2)
\leq -2.5x^2 - (|x| - 1)^2 - (|y| - 1)^2 - 2y^2 + 2 \tag{46}
\leq -2.5x^2 - 2y^2 + 2.
\]

**Figure 2.** (a) The periodic solution of system (45). (b) The trajectory of particle motion of system (45)

Notice that if \(|x|^2 + |y|^2 \geq 1 + \delta\) and \( \delta \ll 1 \), we have \( \langle \nabla V, f \rangle < 0 \). So all the assumptions of Lemma 2.3 are satisfied. Furthermore, we can easily obtain that

\[
\deg(\nabla V, B_{1+\delta}(0), 0) = 1,
\]
then (H10) holds. By Corollary 1, there exists at least one periodic solution of system (45).

Numerical simulations are given in Fig. 2. In Fig. 2(a), the periodic solutions $x^*(t) = (x(t), y(t), z(t))^T$ with the initial values: $x(0) = -0.3406, y(0) = -0.4391, z(0) = 1.3921$ are shown. $x(t)$ is denoted by green solid line, $y(t)$ is denoted by purple dashed line, and $z(t)$ is denoted by blue solid line. According to the feature of the image of periodic solutions, the minimal period $T = 2\pi$. The trajectory of motion on the given sphere is shown in Fig. 2(b). From Fig. 2(b), system (45) has a periodic solution $x^*(t)$, with $|x^*| < \rho^*$ ($\rho^*$ is a given large enough constant), rotating clockwise on the given sphere, and minimal period $T = 2\pi$ (obtained in Fig. 2(a)). Therefore, numerical simulations show the validity of our existence theorems.

Example 3.3. Assume that there is a ‘body’ moving on the constrained surface: $z = x^2 \pm y^2$. The relevant equation for motion is exhibited in the following form of DAE:

$$
\begin{cases}
\dot{x} = f_1(t, x) = -4x + y + \sin z + \sin t, \\
\dot{y} = f_2(t, x) = -3y + \cos z + \cos t, \\
\dot{z} = x^2 \pm y^2,
\end{cases}
(47)
$$

where $x(t) = (x(t), y(t), z(t))$ is the position of the body at time $t$.

In a similar manner, take the guiding function $V(x, y) = \frac{1}{2}(x^2 + y^2)$. When $|x|^2 + |y|^2 \gg 1$, we have

$$
\langle \nabla V, f \rangle = x f_1(t, x) + y f_2(t, x)
= -4x^2 - 3y^2 + xy + x \sin z + y \cos z + x \sin t + y \cos t
< 0.
$$

Furthermore, for a large enough $\rho^*$, we have

$$
\deg(\nabla V, B_{\rho^*}(0), 0) = 1.
$$
Then all the conditions of Corollary 1 are satisfied. By Corollary 1, system (47) has at least one periodic solution $x^*(t) \in B_{\rho^*}(0)$ for all $t$.

Numerical simulations for system (47) are given in Fig. 3 and Fig. 4. In Fig. 3, the corresponding constraint surface is $z = x^2 - y^2$, the periodic solutions and trajectory are shown in Fig. 3(a) and Fig. 3(b), respectively. In Fig. 4, the constraint surface is $z = x^2 + y^2$. In conclusion, numerical simulations are consistent with our existence results.

![Figure 4](image)

**Figure 4.** (a) The periodic solution of system (47) with $z = x^2 + y^2$. (b) The trajectory of particle motion of system (47) with $z = x^2 + y^2$.

4. Conclusion. The existence of periodic solutions for differential-algebraic systems is studied. We present some existence results with homotopy method, topological degree theory and guiding function method. The index of DAEs is an important standard for the computational complexity and the difficulty of analysis. Specifically, it is an ill-posed problem if the index is greater than two. However, our approach does not depend on the index of DAEs, i.e., our results remains valid for the case of higher index. At last, DAEs are widely used to describe many models in application fields, and we use some examples to illustrate our results.

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