GAUSS DIAGRAM FORMULAS OF VASSILIEV INVARIANTS OF SPATIAL 2-BOUQUET GRAPHS

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Abstract. We introduce new formulas that are Vassiliev invariants of flat vertex isotopy classes of spatial 2-bouquet graphs, which are equivalent to 2-string links. Although any Gauss diagram formula of Vassiliev invariants of spatial 2-bouquet graphs in a 3-space has been unknown, this paper gives the first and simple example.

1. Introduction

A spatial graph is a graph embedded in $\mathbb{R}^3$. It often becomes a model of molecule as an embedding of a molecular graph, or a coordination polymer (e.g. [1, Section 1]). In general, the interaction between topological graph theory and the investigation of chemical structures is a rich area. In particular, we would like to emphasize the following two points:

- Multicyclic polymers having shapes corresponding to rigid 4-valent graphs (e.g. flat vertex isotopy classes of spatial 2-bouquet graphs) are synthesized [5, 13].
- The difference between spatial graphs affects a condensed matter, e.g. it is shown that the dominance of the trefoil knot in the case of large excluded volumes [14].

On the other hand, Deguchi applied the Vassiliev invariant of order two to computational science to study random knotting or linking (1994, e.g. [2, 3]). One of his motivation, in practical application of the Jones polynomial, is to solve two problems: (1) Divergence occurs when we evaluate polynomials; (2) Computational time is growing exponentially with respect to the number of crossings of link diagrams.

Deguchi [2] gave a solution of two problems showing an algorithm by using the expansion at $q = 1$ for the Jones polynomial $V_K(q)$:

$$V_K(q) = 1 + v_2(K)q^2 + v_3(K)q^3 + \ldots,$$

where $v_i(K)$ is called Vassiliev invariant of degree $i$. We stand on the viewpoint. It is also meaningful because in general, if all Vassiliev invariants for two knots coincide, then their (Alexander, Conway, Jones, Kauffman, HOMFLY-PT, etc.) polynomial invariants coincide. Nowadays, it is known that every Vassiliev invariant is expressed by a Gauss diagram formula (2000, [4]). It seems likely that this type of formulas is the simplest for computation purposes.

In this paper, we devote ourselves to 2-component link invariants since it is known that there exists one to one correspondence between flat vertex isotopy classes of bouquet graphs to 2-string links (e.g., Figure 1 for the details of the definitions...
of bouquet graphs and flat vertex isotopy, see [9, Section 2]). In order to give the

![Figure 1](image1)

**Figure 1.** (a) An oriented bouquet graph (b) A 2-string link with base points (= a 2-string tangle) (c) A neighborhood of the flat vertex (d) A link obtained by ignoring the base points of (b).

statement of main results (Theorems 1 and 2), we use definitions of Gauss diagrams and arrow diagrams for links. For these definitions, please see [10]. Here, we give an example by Figure 2. In the following statements, an estimation “degree $\leq n$”

![Figure 2](image2)

**Figure 2.** (a) A 2-component link having components $A$ and $B$, (b) A Gauss diagram of (a), (c) Another Gauss diagram of (b).

induced by $n$ arrows is called order $n$.

**Theorem 1.** Each of $\langle \begin{array}{c} \times \\ \times \end{array}, \cdot \rangle$, $\langle \begin{array}{c} \times \\ \times \end{array}, \cdot \rangle$, $\langle \begin{array}{c} \times \\ \times \end{array}, \cdot \rangle$, and $\langle \begin{array}{c} \times \\ \times \end{array}, \cdot \rangle$ is an integer-valued nonzero function that is an invariant of order three of two-component links.

As a corollary, each of them is also an invariant of order three of spatial graphs in $\mathbb{R}^3$ up to flat vertex isotopy.

**Theorem 2.** Each of $\langle \begin{array}{c} \times \\ \times \end{array}, \cdot \rangle$, $\langle \begin{array}{c} \times \\ \times \end{array}, \cdot \rangle$, and $\langle \begin{array}{c} \times \\ \times \end{array}, \cdot \rangle - \frac{1}{3} \langle \begin{array}{c} \times \\ \times \end{array}, \cdot \rangle$ is an integer-valued nonzero function that is an invariant of order three of two-component links.

As a corollary, each of them is also an invariant of order three of spatial graphs in $\mathbb{R}^3$ up to flat vertex isotopy.

**Corollary 1** (["Ostlund-Polyak-Viro formula]). "Ostlund-Polyak-Viro formula $\langle T, \cdot \rangle$, which is $\langle \begin{array}{c} \times \\ \times \end{array}, \cdot \rangle + \langle \begin{array}{c} \times \\ \times \end{array}, \cdot \rangle + \langle \begin{array}{c} \times \\ \times \end{array}, \cdot \rangle - \frac{1}{3} \langle \begin{array}{c} \times \\ \times \end{array}, \cdot \rangle$, becomes a link invariant of order three of two-component links.

Theorem 3 implies that invariants in Theorems 1 and 2 are strictly stronger than "Ostlund-Polyak-Viro formula (Corollary 1), which is known as a Vassiliev link invariant of degree three [12, 10] (the formula in [12] is misprinted and [10] gives the correct formula).
Theorem 3. There exists an infinitely many pairs \((i,j)\) of 2-component links \(L_i, L_j\) \((i \neq j)\) such that
\[
\langle 1++1, L_i \rangle \neq \langle 2++, L_j \rangle, \quad \langle 2+, L_i \rangle \neq \langle 1++, L_j \rangle, \quad \langle 2+, L_i \rangle \neq \langle 1++, L_j \rangle
\]
and \(\langle 1++, L_i \rangle \neq \langle 2++, L_j \rangle\) for our invariants as in Theorem 7 whereas for any pair \(i,j\),
\[
\langle T, L_i \rangle = \langle T, L_j \rangle
\]
on Östlund-Polyak-Viro formula \(\langle T, \cdot \rangle\) as in Corollary 7.

2. Preliminaries

If a reader is familiar with the brackets \(\langle \cdot, \cdot \rangle\) and \(\langle \cdot, \cdot \rangle\) introduced in [4] or treated in [10], the reader can skip this section except for Notation 2.

Definition 1. Let \(L\) be a two-component link. For \(L\), let \(D\) be a link diagram and \(G\) a (signed oriented) Gauss diagram, where each sign is a local writhe. In what follows, every Gauss diagram is signed and oriented. A sub-Gauss diagram of \(G\) is a Gauss diagram obtained by ignoring some arrows. Then, let \(\text{Sub}(G)\) be the set of sub-Gauss diagrams of \(G\). For Gauss diagrams \(A\) and \(z\), \(\langle \cdot, \cdot \rangle\) is an orthonormal scalar product, i.e. \(\langle A, z \rangle = 1\) if \(A = z\) and 0 otherwise. Then, \(\langle \cdot, \cdot \rangle\) is defined by
\[
\langle A, G \rangle = \sum_{z \in \text{Sub}(G)} \text{sign}(z)(A, z),
\]
here \(\text{sign}(z)\) is the product of the signs in \(z\). In general, let \(\langle S \rangle\) be a \(\mathbb{Q}\)-vector space generated by the set \(S\) of finitely many Gauss diagrams. Let \(\langle S \rangle\) be a vector space generated by the Gauss diagram having at most \(d\) arrows where \(d\) is sufficiently large. We extend \(\langle \cdot, \cdot \rangle\) to \(\langle S \rangle \times \langle S \rangle\) bilinearly.

Notation 1. In this paper, every circle of Gauss diagrams is oriented counterclockwise. When no confusion is likely arise, we omit an orientation on each circle, e.g. \(\xrightarrow{\circ}\).

Fact 1 (The linking number relation [12] Page 451, Theorem 5], [10] Section 4.1]). The linking number \(lk(L)\) of a two-component link \(L\) is given by
\[
lk(L) = \langle 1++, \cdot \rangle = \langle 0++, \cdot \rangle.
\]

In particular, for a Gauss diagram \(D\) of a two-component link \(L\), the above formula is also represented by
\[
\sum_{z \in \text{Sub}(D)} \sum_{\epsilon} \text{sign}(z)(1++, z) = \sum_{z \in \text{Sub}(D)} \sum_{\epsilon} \text{sign}(z)(0++, z).
\]

Notation 2. Let \(\epsilon\) be + or − and we fix the sign. Let \(1++1 = \xrightarrow{1++1} + \xrightarrow{1++1} + \xrightarrow{1++1} + \xrightarrow{1++1} \).

Notation 3 (Terminological remark for Reidemeister moves). We use the minimal generating set \(\{\Omega_{1a}, \Omega_{1b}, \Omega_{2a}, \Omega_{2a}\}\) of Reidemeister moves for oriented link diagrams by Polyak [11] Theorem 1.1]. For them, it is convenient to use Östlund’s notations because [10] Table 1 includes a version involving two component links, \(\Omega_{2a}\) and \(\Omega_{3a}\) [11] Theorem 1.1], represented by Gauss diagrams. Concretely, \(\Omega_{2a}\) corresponds to \(\Omega_{1+--}\), and \(\Omega_{3a}\) corresponds to the three presentations \(\Omega_{1+--}\), \(\Omega_{1+-++}\), and \(\Omega_{1++--}\), depending on connectedness of components, for two component links.

For Reidemeister moves involving one link component, \(\Omega_{1a}\) (\(\Omega_{1b}\), resp.) is denoted by \(\Omega_{1++}\) (\(\Omega_{1+-}\), resp.)
3. PROOF OF MAIN RESULTS

We will show Theorem 1 after proving Theorem 2. In this section, we freely use Östlund’s notation \[10\] of \(\text{arrow diagrams}\) and their moves (in particular, cf. \[10\, \text{Section 1.6 and Table 1}\]) since \[10\] is the paper giving the proof of all statements of \[12\]. If a reader is familiar with word-theoretic approach to Gauss diagrams, an advantage for computation is given as in \[6\]. Denoted by \(D_{\Omega}^{\gamma}\) (\(D_{r}^{\Omega}\), resp.) the left (right, resp.) Gauss diagram of in each Reidemeister move \(\Omega\) as in \[10, \text{Table 1}\].

3.1. Proof of Theorem 2. Since the invariance under each of \(\Omega_{1+\pm}\), \(\Omega_{3+\pm}\), and \(\Omega_{3+\pm}\) is obvious, we show the invariance under \(\Omega_{II+\pm}\), \(\Omega_{III+\pm}\), \(\Omega_{III+\pm}\), and \(\Omega_{III+\pm}\).

- \(\Omega_{II+\pm}\). First we consider \(\varnothing\).

\[
\langle \varnothing, D_{l}^{\Omega_{II+\pm}} \rangle - \langle \varnothing, D_{r}^{\Omega_{II+\pm}} \rangle = \sum_{z^{(l)} \in \operatorname{Sub}(D_{l}^{\Omega_{II+\pm}})} \operatorname{sign}(z^{(l)})(\varnothing, z^{(l)}) - \sum_{z^{(r)} \in \operatorname{Sub}(D_{r}^{\Omega_{II+\pm}})} \operatorname{sign}(z^{(r)})(\varnothing, z^{(r)}).
\]

Since \(D_{l}^{\Omega_{II+\pm}}\) has two more arrows than \(D_{r}^{\Omega_{II+\pm}}\), \(z^{(l)}\) is denoted by \(z_{2}^{(l)}\) if \(z^{(l)}\) contains these two arrows. Similarly, if \(z^{(l)}\) contains \(i\) \((i = 0, 1)\) arrow of these two arrows, \(z^{(l)}\) is denoted by \(z_{i}^{(l)}\). Then,

\[
\sum_{z^{(l)} \in \operatorname{Sub}(D_{l}^{\Omega_{II+\pm}})} \operatorname{sign}(z^{(l)})(\varnothing, z^{(l)}) - \sum_{z^{(r)} \in \operatorname{Sub}(D_{r}^{\Omega_{II+\pm}})} \operatorname{sign}(z^{(r)})(\varnothing, z^{(r)})
\]

\[
= \frac{2}{i=0} \sum_{z_{i}^{(l)} \in \operatorname{Sub}(D_{l}^{\Omega_{II+\pm}})} \operatorname{sign}(z_{i}^{(l)})(\varnothing, z_{i}^{(l)}) - \sum_{z^{(r)} \in \operatorname{Sub}(D_{r}^{\Omega_{II+\pm}})} \operatorname{sign}(z^{(r)})(\varnothing, z^{(r)}).
\]
Here, by definition, the set of elements labeled by $z_0^{(l)}$ corresponds bijectively to that of $z^{(r)}$. Then,

$$
\sum_{z_0^{(l)} \in \text{Sub}(D_1^\Omega_{II+-})} \text{sign}(z_0^{(l)})(\bigotimes_0, z_0^{(l)}) = \sum_{z^{(r)} \in \text{Sub}(D_0^\Omega_{II+-})} \text{sign}(z^{(r)})(\bigotimes_0, z^{(r)}).
$$

In general, for any linear sum $A$ of Gauss diagrams,

$$(3) \quad \sum_{z_0^{(l)} \in \text{Sub}(D_1^\Omega_{II+-})} \text{sign}(z_0^{(l)})(A, z_0^{(l)}) = \sum_{z^{(r)} \in \text{Sub}(D_0^\Omega_{II+-})} \text{sign}(z^{(r)})(A, z^{(r)}).$$

Note also that two arrows in the difference between $D_1^\Omega_{II+-}$ and $D_0^\Omega_{II+-}$ have + and − signs, respectively. Then if $z_0^{(l)}$ includes + sign in the difference between $D_1^\Omega_{II+-}$ and $D_0^\Omega_{II+-}$, we denote it by $z_1^+$. Then, there exists another $z_1^-$ with − sign in the difference, and we denote it by $z_1^-$. Then,

$$
\sum_{z_1^{(l)} \in \text{Sub}(D_1^\Omega_{II+-})} \text{sign}(z_1^{(l)})(\bigotimes_0, z_1^{(l)})
= \sum_{z_1^+ \in \text{Sub}(D_1^\Omega_{II+-})} \text{sign}(z_1^+)(\bigotimes_0, z_1^+) + \sum_{z_1^- \in \text{Sub}(D_1^\Omega_{II+-})} \text{sign}(z_1^-)(\bigotimes_0, z_1^-)
= \sum_{z_1^+ \in \text{Sub}(D_1^\Omega_{II+-})} \text{sign}(z_1^+)(\bigotimes_0, z_1^+) - \sum_{z_1^+ \in \text{Sub}(D_1^\Omega_{II+-})} \text{sign}(z_1^+)(\bigotimes_0, z_1^+)
= 0.
$$

In general, for any linear sum $A$ of Gauss diagrams,

$$(4) \quad \sum_{z_1^{(l)} \in \text{Sub}(D_1^\Omega_{II+-})} \text{sign}(z_1^{(l)})(A, z_1^{(l)}) = 0.$$

Next, by (3),

$$
(\bigotimes_0 - \frac{1}{3} \bigotimes_0, D_1^\Omega_{II+-}) - (\bigotimes_0 - \frac{1}{3} \bigotimes_0, D_0^\Omega_{II+-})
= (\bigotimes_0, D_1^\Omega_{II+-}) - \frac{1}{3} (\bigotimes_0, D_0^\Omega_{II+-}).
$$

Further, by (3),

$$
(\bigotimes_0, D_1^\Omega_{II+-}) - \frac{1}{3} (\bigotimes_0, D_1^\Omega_{II+-})
= \sum_{z_2^{(l)} \in \text{Sub}(D_1^\Omega_{II+-})} \sum_{\epsilon = +, -} \text{sign}(z_2^{(l)})(\bigotimes_2, z_2^{(l)})
- \sum_{z_2^{(l)} \in \text{Sub}(D_1^\Omega_{II+-})} \sum_{\epsilon = +, -} \text{sign}(z_2^{(l)})(\bigotimes_2, z_2^{(l)})
= 0 \quad (: \text{ The linking number relation (2)}).$$
We note also that by (3) and (4), it is easy to see that the following two formulas hold
\[ \langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, D_{l}^{\Omega_{Ii}^{+++}} - D_{r}^{\Omega_{Ii}^{+++}} \rangle = 0, \]
and
\[ \langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, D_{l}^{\Omega_{Ii}^{+++}} - D_{r}^{\Omega_{Ii}^{+++}} \rangle = 0. \]

Below, we discuss \( \Omega_{III}^{+++} \) \((\neq b, m, t)\). The difference between \( D_{l}^{\Omega_{III}^{+++}} \) and \( D_{r}^{\Omega_{III}^{+++}} \) is the addition of three arrows. Let \( z^{(l)} \) \((z^{(r)}, \text{resp.})\) be an element of \( \text{Sub}(D_{l}^{\Omega_{III}^{+++}}) \) \((\text{Sub}(D_{r}^{\Omega_{III}^{+++}}), \text{resp.})\). Then, if \( z^{(l)} \) \((z^{(r)}, \text{resp.})\) contains \( i \) \((i = 0, 1, 2, 3)\) arrow(s) of these three arrows, \( z^{(l)} \) \((z^{(r)}, \text{resp.})\) is denoted by \( z_{i}^{(l)} \) \((z_{i}^{(r)}, \text{resp.})\).

By the same argument as the case \( \Omega_{II}^{+} \), for \( i = 0, 1 \), we have
\[
\sum_{z_{i}^{(l)} \in \text{Sub}(D_{l}^{\Omega_{III}^{+++}})} \text{sign}(z_{i}^{(l)})(A, z_{i}^{(l)}) = \sum_{z_{i}^{(r)} \in \text{Sub}(D_{r}^{\Omega_{III}^{+++}})} \text{sign}(z_{i}^{(r)})(A, z_{i}^{(r)}).
\]

Then, we note that by (3) and (4), the following two formulas hold. For any \( \neq b, m, t \),
\[
\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, D_{l}^{\Omega_{III}^{+++}} - D_{r}^{\Omega_{III}^{+++}} \rangle = 0,
\]
and
\[
\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, D_{l}^{\Omega_{III}^{+++}} - D_{r}^{\Omega_{III}^{+++}} \rangle = 0.
\]

Hence, we discuss \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{3} \langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, D_{l}^{\Omega_{III}^{+++}} - D_{r}^{\Omega_{III}^{+++}} \rangle \) in the following.

- \( \Omega_{II}^{+++} \).

By (3),
\[
\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, D_{l}^{\Omega_{III}^{+++}} \rangle - \frac{1}{3} \langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, D_{l}^{\Omega_{III}^{+++}} \rangle = \sum_{z_{2}^{(l)} \in \text{Sub}(D_{l}^{\Omega_{III}^{+++}})} \text{sign}(z_{2}^{(l)})(\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, z_{2}^{(l)} \rangle) \delta_{++,}^{+}
- \sum_{z_{2}^{(l)} \in \text{Sub}(D_{l}^{\Omega_{III}^{+++}})} \text{sign}(z_{2}^{(l)})(\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, z_{2}^{(l)} \rangle)
\]

(for the second term, \( \frac{1}{3} \cdot 3 \) appears as in [10] Sec. 4.8.3] by symmetry of \( \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \))

\[ = 0 \tag{\text{3 appears as in [10] Sec. 4.8.3] by symmetry of \( \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \)} \]

- \( \Omega_{II}^{++} \).

Since there is no element corresponding to \( z_{2}^{(l)} \) or \( z_{2}^{(r)} \),
\[
\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, D_{l}^{\Omega_{III}^{+++}} - D_{r}^{\Omega_{III}^{+++}} \rangle = 0,
\]
and
\[
\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, D_{l}^{\Omega_{III}^{+++}} - D_{r}^{\Omega_{III}^{+++}} \rangle = 0.
\]

- \( \Omega_{II}^{+} \).

Since we have the same formulas as in the above \( \Omega_{II}^{++} \) case except for replacing \( b \) with \( t \), we omit its proof. \( \square \)
3.2. Proof of Theorem 1. Since the invariance under each of $\Omega_{1+\pm}$, $\Omega_{2+\pm}$, and $\Omega_{3+--\pm}$ is obvious, we show the invariance under $\Omega_{t+\pm}$, $\Omega_{t+\pm+b}$, $\Omega_{t+\pm+m}$, and $\Omega_{t+\pm+t}$. Below, the same notations $z^{(1)}_i$, $z^{(r)}$, and $z^\pm_1$ as those of the proof of Theorem 2 apply.

- $\Omega_{t+\pm}$.

\[
\langle \begin{array}{c}
\sigma_0
\end{array}, D^{\Omega_{t+\pm}}\rangle - \langle \begin{array}{c}
\sigma_0
\end{array}, D^{\Omega_{t+\pm}}\rangle = \sum_{z^{(1)} \in \text{Sub}(D^{\Omega_{t+\pm}})} \text{sign}(z^{(1)})(\begin{array}{c}
\sigma_0
\end{array}, z^{(1)}) - \sum_{z^{(r)} \in \text{Sub}(D^{\Omega_{t+\pm}})} \text{sign}(z^{(r)})(\begin{array}{c}
\sigma_0
\end{array}, z^{(r)})
\]

\[
= \sum_{i=0}^{2} \sum_{z^{(1)}_i \in \text{Sub}(D^{\Omega_{t+\pm}})} \text{sign}(z^{(1)}_i)(\begin{array}{c}
\sigma_0
\end{array}, z^{(1)}_i) - \sum_{z^{(r)} \in \text{Sub}(D^{\Omega_{t+\pm}})} \text{sign}(z^{(r)})(\begin{array}{c}
\sigma_0
\end{array}, z^{(r)})
\]

\[
= \sum_{z^{(1)}_2 \in \text{Sub}(D^{\Omega_{t+\pm}})} \text{sign}(z^{(1)}_2)(\begin{array}{c}
\sigma_0
\end{array}, z^{(1)}_2) \quad (\because [\text{4}, \text{4}])
\]

\[
= 0 \quad (\because \text{there is no } z_2 \text{ that takes non-zero value}).
\]

Next, we consider $\Omega_{t+\pm+\pm} (* = b, m, t)$. We note that $z^{(1)}_2$ corresponds to $z^{(r)}_2$ under the Reidemeister move $\Omega_{t+\pm+\pm} (* = b, m, t)$. Then, each $\Omega_{t+\pm+\pm}$ corresponds to a sum of the three canonical subtractions “$z^{(1)}_2 - z^{(r)}_2$” of pairings $(z^{(1)}_2, z^{(r)}_2)$ as in Figure 4. Further, by the definition of $(\begin{array}{c}
\sigma_0
\end{array}, \cdot)$, each term, which survives in subtractions “$z^{(1)}_2 - z^{(r)}_2$”, consists of two arrows relevant to $\Omega_{t+\pm+\pm}$ and the other one with a sign $\epsilon$ (for * = b, t) or + (for * = m) as in the case “ccw” (i.e., “counterclockwise as in $\begin{array}{c}
\sigma_0
\end{array}$” embedded in $\begin{array}{c}
\sigma_0
\end{array}$) of Figure 5.

**Figure 4.** Three pairings corresponding to three subtractions for each Reidemeister move $\Omega_{t+\pm+\pm}$.

- $\Omega_{t+\pm+b}$.

\[
\langle \begin{array}{c}
\sigma_0
\end{array}, D^{\Omega_{t+\pm+b}}\rangle - \langle \begin{array}{c}
\sigma_0
\end{array}, D^{\Omega_{t+\pm+b}}\rangle = \sum_{z^{(1)} \in \text{Sub}(D^{\Omega_{t+\pm+b}})} \text{sign}(z^{(1)})(\begin{array}{c}
\sigma_0
\end{array}, z^{(1)}) - \sum_{z^{(r)} \in \text{Sub}(D^{\Omega_{t+\pm+b}})} \text{sign}(z^{(r)})(\begin{array}{c}
\sigma_0
\end{array}, z^{(r)})
\]

\[
= \sum_{z^{(1)}_2 \in \text{Sub}(D^{\Omega_{t+\pm+b}})} \text{sign}(z^{(1)}_2)(\begin{array}{c}
\sigma_0
\end{array}, z^{(1)}_2) \quad (\because [\text{4}, \text{4}])
\]

\[
= 0 \quad (\because \text{there is no } z_2 \text{ that takes non-zero value}).
\]
Figure 5. This is a list of the possible terms that take non-trivial values in the form as in Figure 4, where \( \epsilon = + \text{ or } - \).
In the same way as the above, using Figure 4 and the “case ccw” as in Figure 5, 
\[ \langle ✐ ✐ ✲ ✲ ✲ ✲ +, D^Ω III + + + t \rangle - \langle ✐ ✐ ✲ ✲ ✲ ✲ +, D_r^Ω III + + + t \rangle = 0 \]
and 
\[ \langle ✐ ✐ ✲ ✲ ✲ ✲ −, D^Ω III + + + m \rangle - \langle ✐ ✐ ✲ ✲ ✲ ✲ −, D_r^Ω III + + + m \rangle = 0. \]
Hence, ✐ ✐ ✲ ✲ ✲ ✲ ± is a link invariant. This fact together with Theorem 2 implies 
that ✐ ✐ ✲ ✲ ✲ ✲ − (= ✐ ✐ ✲ ✲ ✲ ✲ + ✐ ✐ ✲ ✲ ✲ ✲ −) is also a link invariant.

By Figure 4 and “case cw” (i.e. clockwise) as in Figure 5, the same argument 
of the proof of the invariance of ✐ ✐ ✲ ✲ ✲ ✲ ± applies, we have a proof of case ✐ ✐ ✲ ✲ ✲ ✲ ±. 
The invariance of ✐ ✐ ✲ ✲ ✲ ✲ ± implies that ✐ ✐ ✲ ✲ ✲ ✲ − (= ✐ ✐ ✲ ✲ ✲ ✲ + ✐ ✐ ✲ ✲ ✲ ✲ −) is also a link 
invariant. □

4. PROOF OF THEOREM 3

Let \( m \) and \( n \) be odd positive integers. Let \( L(m, n) \) be a 2-component link with 
\( m + n + 8 \) as in Figure 6 (e.g. for \( L(1, 1) \), see Figure 4).

![Figure 6](image)

**Figure 6.** For \( L(m, n) \) with a fixed pair \((m, n)\), the left figure is a 
link diagram and the right figure is its Gauss diagram; \( m \) (\( n \), resp.) 
denotes \( m \) (\( n \), resp.) crossings.

Let \( L_n = L(n, n) \). By Table 1 for \( L_n \), the Östlund-Polyak-Viro formula (Corol-
lar 1) takes the same value (= 0) as that of \( L_m \) even if \( m \neq n \) whereas our 
invariants take values \(-n \) or \( n \), which implies the statement of Theorem 3. □

| Invariants of Theorem 1 | Values of \( L(m, n) \) |
|-------------------------|----------------------|
| ✐ ✐ ✲ ✲ ✲ ✲ ±           | \(-n\)              |
| ✐ ✐ ✲ ✲ ✲ ✲ +           | \(-n\)              |
| ✐ ✐ ✲ ✲ ✲ ✲ −           | \(m\)               |
| ✐ ✐ ✲ ✲ ✲ ✲ +           | \(m\)               |

| Invariants of Theorem 2 | Values of \( L(m, n) \) |
|-------------------------|----------------------|
| ✐ ✐ ✲ ✲ ✲ ✲ ±           | \(m - n\)            |
| ✐ ✐ ✲ ✲ ✲ ✲ +           | \(m - n\)            |
| ✐ ✐ ✲ ✲ ✲ ✲ −           | 0                    |

| Ostlund-Polyak-Viro formula of Corollary 1 |
|--------------------------------------------|
| ✐ ✐ ✲ ✲ ✲ ✲ ± + ✐ ✐ ✲ ✲ ✲ ✲ + ✐ ✐ ✲ ✲ ✲ ✲ − ✐ ✐ ✲ ✲ ✲ ✲ ± ± |

**Table 1.**

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Figure 7. $L(1,1)$. 

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