The Quantum Liquid of Alpha Clusters
-a variational approach

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Abstract. Within the variational approach of Bose liquids we analyze the g.s. energy of charge neutral alpha matter at $T=0$. As a prerequisite for such calculation we take from the literature or propose new $\alpha - \alpha$ potentials that are particularly suitable for this task, i.e. possess a repulsive core and/or reproduce the low energy scattering data and the resonance properties of the $\alpha - \alpha$ system. The alpha matter EOS is then obtained with the HNC method using Pandharipande-Bethe correlation derived variationally in the lowest order expansion of the energy functional or a simple gaussian function with a healing range determined by the normalization of the radial distribution function in the lowest order. We show that saturation is achieved only via repulsive and shallow potentials that are not consistent with the scattering and resonance constraints.

Keywords: Effective $n - n$ interactions; Alpha Matter, Hypernetted Chain Approximation, Resonant Reactions, SUSY Potential

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INTRODUCTION

The pioneering alpha-matter calculations reported in [1] made use of $\alpha - \alpha$ potentials characterized by a strong or even infinite repulsive component. These interactions were constructed to fit the elastic-scattering phase shifts deduced from experiment. The Equation of State (EOS) depends strongly on the shape and strength of these potentials and it is usually derived within the frame of the paired-phonon analysis (PPA) or the HNC/n method. The EOS calculated with hard-core potentials, trivially saturates at densities and energies close to the nuclear matter saturation point ($\rho_\alpha \approx 0.04$ fm$^{-3}$, $E/N_\alpha \approx -11$-16 MeV). On the other hand, for a typical soft-core potential such as the one proposed by Ali and Bodmer (AB) [2] the alpha matter almost fails to saturate [1]. In fact a deep minimum in a very soft EOS at a high density is predicted with AB potential. However at such high densities the alpha-condensate is almost completely depleted due to Pauli blocking. Somehow this disappointing result is conflicting with what one would expect based on the manifestation of alpha clustering in real nuclei. The clusterization of alpha particles on the surface of nuclei at densities around half the central nuclear density, as revealed by $\alpha$-decay, $\alpha$-transfer reactions or the putative dilute three-alphas condensate in the Hoyle state of $^{12}$C, are pointing to a high stability of alpha matter at lower densities.

We recently addressed the problem of alpha matter saturation and concluded that in order to avoid the collapse of the EOS very shallow potentials with a strong repulsive core are necessary [3]. It would be then of interest to extent this study to other potentials, especially those in agreement to elastic phase-shift data or the first resonant states in the
short lived $^8$Be. It is therefore timely to gain a better understanding of the role played by
the $\alpha - \alpha$ potentials in a many-body approach to $\alpha$-matter.

**ALPHA-ALPHA POTENTIALS**

The popular $S$-state Ali-Bodmer $\alpha - \alpha$ potential consists of a short-ranged (1.43 fm)
repulsive part and a long-ranged attractive part (2.50 fm) [2],

$$V_{\alpha\alpha}(r) = 475 \exp \left[-(0.7r)^2\right] - 130 \exp \left[-(0.475r)^2\right].$$

This potential reproduce the $\alpha - \alpha$ elastic scattering phase-shifts at low energies. The
same phase-shift properties are shared by the deep Buck et al. (BFW) potential [4],

$$V_{\alpha\alpha}(r) = V_0 \exp \left[-(\mu r)^2\right],$$

where $V_0 = -122.6225$ MeV, $\mu = 0.469$ fm$^{-1}$. In addition the BFW potential reproduces
the energy and the width of the first $0^+$ resonant state in $^8$Be.

A microscopic approach to the $\alpha - \alpha$ potential is provided by the double-folding
method [5] where the interaction between two alpha ions is calculated as an overlap
of the single particle densities of the interacting objects smeared by the local two-body
potential $v_{nn}$.

$$V_{\alpha\alpha}(r) = \int dr_1 \int dr_2 \rho_\alpha(r_1) \rho_\alpha(r_2) v_{nn}(\rho, r - r_1 + r_2)$$

The effective $n - n$ interaction $v_{nn}$ is taken to be dependent on the density $\rho$ of the
nuclear matter where the two nucleons are embedded. It should also consist of a density
independent finite-range part with preferably two ranges such that a potential similar to
the Ali-Bodmer is obtained at least for the direct component. A choice satisfying these
requirement is provided by the Gogny [6] interaction. Very recently we proposed a bare
$\alpha - \alpha$ interaction based on the double-folding method at energies around the barrier,
using realistic densities of the $\alpha$-particle and the density dependent Gogny nucleon-
nucleon effective interaction [3]. The bare $\alpha - \alpha$ potential consisted only in the direct
term. In the present work we add the knock-on nucleon exchange term including recoil
corrections and the non-local kernels are localized in the lowest order of the Perey-Saxon
approximation at energies around the barrier [7]. Since all three parametrizations of the
Gogny force (D1 [8], D1S [9], D1N [10]) are dominated by exchange the resulted folded
potentials are very deep. It is a matter of evidence that deep potentials such as BFW and
Gogny are never saturating alpha matter because of the large amount of attraction that
will contribute to the total energy.

For our alpha matter investigations let us first consider the direct part in the double-
folding potential (3). The motivation behind such a simplification was recently discussed
in connection with the necessity of accounting for the incompressibility of nuclear matter
in cold clustering processes [11] and extreme sub-barrier fusion [12]. In this framework
a double-folding repulsive potential with a zero-range interaction is added to the direct
and exchange potential such that the energy cost for overlapping two pieces of nuclear
matter are payed-off. The strength of this repulsive $\delta$-like potential is in a simplified picture proportional to the nuclear incompressibility at the corresponding density of total overlap.

We assume a Gaussian nuclear matter distribution of the $\alpha$-particle,

$$\rho_\alpha(r) = 4 \left( \frac{1}{\pi b^2} \right)^{3/2} \ e^{-r^2/b^2}, \quad (4)$$

where the length parameter $b$ is determined from the root mean square radius (rms) $1.58 \pm 0.002$ fm extracted via a Glauber analysis of experimental interaction cross sections [13].

Using the analytical expression of the direct component $n-n$ force in the Gogny parametrization, as listed in [3], and inserting the Gaussian density distribution (4) in the double folding integral (3) the direct part of the $\alpha-\alpha$ interactions reads,

$$V_{\alpha\alpha}^d(r) = 4 \sum_{i=1}^{2} (4W_i + 2B_i - 2H_i - M_i) \left( \frac{\mu_i^2}{\mu_i^2 + 2b^2} \right)^{3/2} e^{-\mu_i^2 + 2b^2 r^2}$$

$$+ \frac{3}{2} \frac{4\gamma + 2}{t^3 (\gamma + 2)^{3/2} (\sqrt{\pi} b)^{3(\gamma+1)}} e^{-\frac{2\gamma + 2}{4b^2 r^2}} \quad (5)$$

Above, $W_i, B_i, H_i, M_i$ are strength of the $i = 1, 2$-th finite-range term of the Gogny parametrization whereas $\mu_i$ the corresponding ranges, while the parameter $\gamma$ characterizes the density dependence of the force. In the left panel of Fig.1 we represent the AB, D1 and D1N potentials on a magnified scale around the minimum. The two Gogny potentials display pockets that are shallower and shifted to larger radii compared to the Ali-Bodmer potential. The minima of these potentials are close to the classically touching configuration of the $\alpha-\alpha$ system.

The derivation of the exchange part of the potential will be given elsewhere. Here we only reproduce the closed expression obtained after the localization via the Perey-Saxon
method of the non-local kernel,

\[ V_{\alpha\alpha}^{ex}(r) = -32 \sum_{i=1}^{2} (W_i + 2B_i - 2H_i - 4M_i) \left( \frac{\beta_i}{b} \right)^3 e^{-\frac{1}{2} \left[ 1 - \frac{1}{4} \left( \frac{\beta_i}{b} \right)^2 \right] r^2} e^{\pm \frac{i}{2} |K|^2 \beta_i^2} \]

\[ \times \begin{cases} 
\exp \left[ -\frac{1}{2} \left( \frac{\beta_i}{b} \right)^2 K r \right] & \text{for } K^2 < 0 \\
\cos \left[ \frac{1}{2} \left( \frac{\beta_i}{b} \right)^2 |K| r \right] & \text{for } K^2 \geq 0
\end{cases} \tag{6} \]

where,

\[ \frac{1}{\beta_i^2} = \frac{8}{\mu_i^2} + \frac{9 + \frac{1}{4}}{b^2}. \tag{7} \]

and,

\[ K^2(r) = \frac{2\mu}{\hbar^2} (E_{c.m.} - V_{\alpha\alpha}(r)) \tag{8} \]

Above we distinguish between a sub-barrier branch of the potential \((K^2 < 0)\) and an over-barrier one \((K^2 > 0)\). Note that in the exchange kernel we used an alpha single-particle density matrix constructed from the 0s harmonic-oscillator orbitals. Only a few iterations are needed to obtain convergence in the localization procedure. In the left panel of Fig.2 the full (direct+exchange) \( \alpha - \alpha \) potential for three parametrizations of the Gogny interaction is displayed at zero relative momentum \((K_{rel} = 0)\) and compared to the BFW potential.

### Supersymmetric Potentials

In a previous work we concluded that a shallow potential (in any case shallower than the traditional AB) is needed in order to obtain the saturation alpha matter at physically reasonable densities [3]. On the other hand, a shallow potential results naturally from the requirement that \(^8\text{Be}\) has no bound states. An alternative way to the Ali-Bodmer phenomenological ansatz to construct a shallow potential was proposed by Baye [14]. A sequence of supersymmetric transformations, which removes the Pauli-forbidden states in the sense of orthogonality condition model (OCM), is applied to the original Schrödinger equation in such a way that the phase-shifts corresponding to the transformed Hamiltonian coincide to the phase-shifts of the original Hamiltonian. In this manner one obtains a new shallow potential that is exactly phase-equivalent (modulo \(2\pi\)) to the original deep potential. In the particular case of \( \alpha - \alpha \) system the phase-equivalent potential is obtained from the suppression of the two lowest \((l=0)\) bound forbidden states. A compact formula provides the needed shallow potential

\[ V_{\alpha\alpha}^{(2)}(r) = V_{\alpha\alpha}^{(0)}(r) - \frac{\hbar^2}{m} d^2 \log \det \Psi^{(2)}(r) \tag{9} \]
where the elements of the $2 \times 2$ matrix $\Psi^{(2)}$ read

$$
(\Psi^{(2)})_{ij}(r) = \int_0^r u_i(r')u_j(r')dr'
$$

and $V_{aa}^{(0)}$ is any one of the deep potentials discussed earlier. Above we denote by $u_{0s,1s}$ the lowest two bound state wave functions of the deep nuclear+Coulomb ($l=0$) potential. We obtain these states using the Runge-Kutta-Nystrom (RKN) numerical integration [15].

The supersymmetric shallow partners of the BFW potential were determined already by Baye [14] and more recently in a semi-classical approach in [16] for $l=0, 2$ and 4. The first two bound states in this potential are located according to this reference at -72.8 MeV and -25.9 MeV. With the RKN method we obtain the values -72.62 MeV and -25.61 MeV. The minimum of the shallow BFW-SUSY potential is located at 2.85 fm and has a depth of -7.56 MeV and is similar to the Baye estimation and about 1 MeV shallower than the semiclassical estimation of Horiuchi et al. [16].

The transformed potential displays a $20/r^2$ singularity at the origin. In practical calculations we use an interpolation with gaussians that is non-singular at the origin.

To obtain the SUSY partner of the Gogny potentials we constrain first the original deep potential to reproduce the properties of the first resonant state in $^8$Be, i.e. $E_{\text{res}} = 92.12 \pm 0.05$ keV, $\Gamma_0 = 6.8 \pm 1.7$ eV [17]. The renormalized Gogny potentials are next subjected to a sequence of two supersymmetric transformations in order to eliminate the Pauli-forbidden states. In the right panel of Fig.2 we display the supersymmetric partners of the BFW and the three Gogny interactions (including the Coulomb component). We make the remark that the deep potentials BFW, D1 and D1S(D1N) have very different depths but upon renormalization and supersymmetric transformations all four potentials have similar depths. Thus, the physical contraint imposed by the $0^+$ resonance in $^8$Be and the removal of the forbidden states, lead to an almost unique potential for the $\alpha-\alpha$ system.

**EOS OF ALPHA MATTER**

We derive the EOS of alpha matter using the variational approach, described in [3]. In Fig.3 we compare the EOS for AB potential with BFW-SUSY and D1-SUSY and we conclude that for this last two cases the EOS curves are collapsing, while the EOS for the AB interaction saturates at too large densities. Thus shallow potentials constrained to reproduce the phase-shift elastic data and/or the first resonant state in $^8$Be do not saturate alpha matter in the HNC/0 approximation. The reason for this disappointing result is that the two-body correlation functions obtained with the Pandharipande-Bethe prescription display large overshootings near the healing distance with increasing alpha matter density. The density dependence of the healing distance $d$ does not bring sufficient density dependence in order to ensure saturation.

In order to demonstrate this effect we consider a simple phenomenological two-body Gaussian correlation function [18],

$$
f(r) = 1 - e^{-\beta^2 r^2}
$$
FIGURE 2. Bare direct+exchange $\alpha - \alpha$ potentials for the BFW and the three $n-n$ Gogny interactions used in the text (left panel) and their supersymmetric partners (right panel).

Evidently this function has no overshooting. The parameter $\beta$ is determined from the normalization condition [19],

$$4\pi \rho \int_0^\infty dr \ r^2 (f^2(r) - 1) = -1$$

(12)

where $\rho$ is the density of alpha matter. We perform a selfconsistent HNC/4 calculation using these correlation functions and the bare AB potential. Both Jackson-Feenberg
FIGURE 4. Equation of state for alpha matter in the HNC/4 approximation. The bare AB potential is scaled with a factor $N$ in order to see the variation of the saturation point as a function of the potential strength.

$(E_{JF})$ and Pandharipande-Bethe $(E_{PB})$ are displayed in Fig.4 for different renormalizations $N$ of the AB potential. The intrinsic binding energy of the $\alpha$ particle $(B_\alpha \sim 7$ MeV) is not included. The PB energy is calculated using the Kirkwood approximation for the three-body radial distribution. The values obtained in this approximation are quite similar with those obtained with JF approximation since the contribution from the three-body term is negligibly small. The saturation density does not change with the renormalization of the potential since this is determined exclusively by the density dependence of the range parameter $\beta \sim \rho^{1/3}$ (Eq.12). Similar results have been obtained for the equa-
tion of state using a 4th order cluster expansion and the HNC/0 method. Regardless of the normalization factor the EOS saturation point is very close to the saturation point of normal nuclear matter, i.e. $\rho_{NM} = 4\rho_\alpha \approx 0.16 \text{ fm}^{-3}$.

In conclusion, shallow phenomenological and semimicroscopic $\alpha - \alpha$ potentials which reproduce low energy $\alpha - \alpha$ scattering phase shifts and/or the resonance in $^8\text{Be}$ lead to a collapsing alpha matter equation of state if two-body correlation functions are obtained with the Pandharipande-Bethe prescription. In this prescription, the assumption of a finite weakly density dependent healing distance leads to overshooted two-body correlation functions and enhances the importance of the long attractive tail of the potential.

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