UNITARITY CONSTRAINTS ON THE B AND B* FORM FACTORS
FROM QCD ANALYTICITY
AND HEAVY QUARK SPIN SYMMETRY

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Abstract
A method of deriving bounds on the weak meson form factors, based on
perturbative QCD, analyticity and unitarity, is generalized in order to fully
exploit heavy quark spin symmetry in the ground state (L = 0) doublet of
pseudoscalar (B) and vector (B*) mesons. All the relevant form factors of
these mesons are taken into account in the unitarity sum. They are treated as
independent functions along the timelike axis, being related by spin symmetry
only near the zero recoil point. Heavy quark vacuum polarisation up to three
loops in perturbative QCD and the experimental cross sections \(\sigma(e^+e^- \rightarrow \Upsilon)\)
are used as input. We obtain bounds on the charge radius of the elastic form
factor of the B meson, which considerably improve previous results derived
in the same framework.
1 Introduction

Bounds on the charge radius of the elastic form factor of the \( B \)-meson were recently derived in a number of papers [1]-[4]. The interest in this form factor comes from the fact that it coincides, in the large quark mass limit, with the renormalized Isgur-Wise function of the heavy quark effective theory [5],[6]. The short distance and finite mass corrections are in this case much smaller than for the flavor changing currents involved in the semileptonic decays of the \( B \) meson into \( D \) or \( D^* \). Therefore, rigorous bounds on this form factor are of interest for testing various nonperturbative techniques applied for the calculation of the Isgur-Wise function.

The method applied in Refs. [1]-[4], based on previous works [7], starts by exploiting the same input as the standard QCD sum rules, i.e. the QCD euclidian expansion of a polarization function, related by analyticity and unitarity to the physical states of interest. However, while in the usual formulation of the QCD sum rules one tries, by suitable methods, to enhance the contribution of the low energies in the dispersion integral and saturates the unitarity sum by the lowest lying resonances, in the approach proposed in [7] the dispersion relation is written as a rigorous integral inequality for the modulus squared of the form factors of the physical states along the time like region. By using in addition the analyticity properties of the form factors, this inequality is shown to constrain the behaviour of these functions or their derivatives near the zero recoil or other points of physical interest.

In refs. [1]-[4] the method was applied to the elastic form factor of the pseudoscalar \( B \) meson. An attempt to exploit heavy quark symmetry in the ground \( B \) meson state doublet was made in [1], where the \( B\bar{B} \) and \( B\bar{B}^* + B^*\bar{B} \) intermediate states were included in the unitarity sum, with the additional assumption that the relevant form factors of the \( B \) and \( B^* \) mesons are identical along the whole unitarity cut. However, this is an unjustified extension of the heavy quark spin symmetry, which is valid only near the zero recoil point. As illustrated in [8] by specific models, the \( B \) and \( B^* \) form factors can be indeed quite different along the time like axis, especially near thresholds. The problem was correctly solved in [9], where by means of special techniques allowing the simultaneous treatment of several analytic functions [10],[11], the inclusion the form factor of the \( B \rightarrow B^* \) transition was possible within the strict heavy quark spin symmetry hypotheses. More precisely, the elastic form factor and the \( BB^* \) form factor were treated as distinct functions along the unitarity cut, being assumed to coincide only near the zero recoil point. This led to a considerable improvement of the bounds on the charge radius of the \( B \) elastic form factor: the range \(-4.5 \leq \rho^2 \leq 6.1\), obtained in [3] without imposing spin symmetry, was narrowed in [3] to \(-0.90 \leq \rho^2 \leq 2.60\).

However, in [3] spin symmetry was not fully exploited, as the contribution of the \( B^*\bar{B}^* \) intermediate states in the unitarity sum was not included. This problem is addressed in the present paper, where we treat simultaneously all the weak form factors of the \( B \) and \( B^* \) mesons. The quadratic expression yielded by unitarity is written in a suitable "diagonal" form, which allows us to apply the optimization
theory for vector-valued analytic functions. The different thresholds in the
unitarity sum and the subthresholds singularities of the various form factors are
taken into account correctly. In this way the consequences of spin symmetry in
the ground state doublet of the $B$ mesons are exploited in an optimal way. The
present paper contains in addition two important improvements of the work done
before: we use as input the heavy quark vacuum polarization function computed
in perturbative QCD up to three loops and we include in the dispersion
relation for the polarization function the three $\Upsilon$ resonances with masses below the
threshold for $B\bar{B}$ production (these terms were neglected in previous works).

In the next section we present the derivation of the bounds. Section 3 contains
the numerical results and our conclusions.

2 The derivation of the bounds

We use the standard definitions of the form factors of the $B$ and $B^*$ mesons:

$$< B(p')|V^\mu|B(p) >= (p + p')^\mu F(q^2)$$

$$< B^*(p', \epsilon)|V^\mu|B(p) >= \frac{2i\epsilon^{\nu\alpha\beta}}{m_B + m_{B^*}} \epsilon_\nu p'_\alpha p_\beta V(q^2)$$

$$< B^*(p', \epsilon')|V^\mu|B^*(p, \epsilon) >= F_1(q^2)(\epsilon \cdot \epsilon')P_\mu + F_2(q^2)[\epsilon_\mu (\epsilon' \cdot P) + \epsilon'_\mu (\epsilon \cdot P)]$$

$$+ F_3(q^2)(\epsilon \cdot P)(\epsilon' \cdot P)P_\mu + F_4(q^2)[\epsilon_\alpha (\epsilon' \cdot P) - \epsilon'_\alpha (\epsilon \cdot P)] \frac{g_{\alpha\mu}q^2 - q_\mu q_\alpha}{m_{B^*}^2},$$

where $V^\mu = \bar{b}\gamma^\mu b$, $\epsilon(\epsilon')$ denote the polarization vectors of the $B^*$ mesons, $P = p + p'$
and $q = p - p'$.

The form factors defined above have cuts in the complex plane $t = q^2$, from
the threshold $t_0$ for $B\bar{B}$ production to infinity. The effect of the lower branch cuts due
to light intermediate states ($\pi\pi$, $KK$, etc) is negligible. The three resonances
$\Upsilon(1S)$, $\Upsilon(2S)$, $\Upsilon(3S)$ with masses lower than $2m_B$ produce additional singularities,
which can be approximated by poles on the real axis below $t_0$. On the other
hand, heavy quark symmetry predicts definite relations among the form factors $F_1 - F_4$
near the zero recoil point $w = 1$ ($w = v \cdot v'$, $v$ and $v'$ being the velocities of
the initial and final meson, respectively). In this region some of the form factors in
(2) and (3) are approximately equal to the elastic form factor and other vanish.
Specifically, for $w \approx 1$ one has

$$V(w) = -F_1(w) = F_2(w) = F(w), \quad F_3(w) = F_4(w) = 0,$$

and we recall that $F(w)$ satisfies the normalization condition

$$F(1) = 1.$$
We are interested in finding restrictions on the slope of this function at zero recoil,
or the so called charge radius, defined as

\[ \rho^2 = -F'(1), \]  

which differs by \( \frac{4\pi}{f_0} \log\alpha_s(m_b) \) from the charge radius \( \rho_{IW}^2 \) of the universal Isgur-Wise function \([3]\).

As in the derivation of the usual QCD sum rules, for studying the form factors (1-3) we start by considering the vacuum polarization tensor due to the current \( V^\mu \):

\[ \Pi^\mu\nu(q) = (q^\mu q^\nu - g^\mu\nu q^2)\Pi(q^2) = i \int dx e^{iqx} <0|T(V^\mu(x)V^\nu(0))|0> . \]  

The first derivative of the invariant amplitude \( \Pi(q^2) \) satisfies the dispersion relation

\[ \Pi'(q^2) = \frac{1}{\pi} \int_0^\infty \frac{\text{Im}\Pi(t)}{(t - q^2)^2} dt , \]  

the spectral function being defined by the unitarity relation

\[ (q^\mu q^\nu - g^\mu\nu q^2)\text{Im}\Pi(t + i\epsilon) = \frac{1}{2} \sum_{\Gamma} \int d\rho_{\Gamma}(2\pi)^4 \delta(t)(q - p_{\Gamma}) \times <0|V^\mu(0)|\Gamma> <\Gamma|V^\nu(0)^+|0> . \]  

Here the summation is over all possible hadron states \( \Gamma \) with appropriate flavor quantum numbers, with an integral over the phase space allowed to each intermediate state. We shall include in this sum the three \( \Upsilon \) resonances with masses lower than the threshold of the \( B\bar{B} \) production and the contribution of the two-particle states \( |B\bar{B}>, |B\bar{B}^* + B^*\bar{B}> \) and \( |B^*\bar{B}^*> \) above this threshold (the \( \Upsilon(4S) \) resonance is not included, in order to avoid double counting \([3]\)). This contribution can be evaluated in a straightforward way, by using the definitions (1-3) of the form factors, performing the phase-space integration and the summation over the polarizations of the \( B^* \) intermediate states. Taking into account the positivity of the spectral function of \( \Pi \), which follows from (9) we obtain the following inequality:

\[ \frac{1}{\pi}\text{Im}\Pi(t + i\epsilon) \geq \frac{27}{4\pi\alpha^2} \sum_i M_{\Upsilon_i}\Gamma_{\Upsilon_i}\delta(t - M_{\Upsilon_i}^2) \]

\[ + \frac{n_f}{48\pi} \left( 1 - \frac{t_0}{t} \right)^{3/2} |F(t)|^2 \theta(t - t_0) + (1 - \frac{t_0}{t})^{3/2} (1 - \frac{t_1^*}{t})^{3/2} \frac{2t}{t_0} |V(t)|^2 \theta(t - t_0) \]

\[ + (1 - \frac{t_0^*}{t})^{3/2} \left[ 2|F_1(t)|^2 + \frac{4t}{t_0} |F_2(t)|^2 + |\hat{F}_3(t)|^2 + \left( \frac{4t}{t_0} \right)^2 |F_4(t)|^2 \right] \theta(t - t_0^*) \right) , \]

where \( \hat{F}_3(t) = \left( \frac{2t}{t_0^*} - 1 \right) F_1(t) + \frac{2t}{t_0^*} F_2(t) + \frac{2t}{t_0^*} \left( \frac{2t}{t_0^*} - 1 \right) F_3(t) \).
In (10) the widths $\Gamma_{\Upsilon_i}$ are defined through the parametrization

$$\sigma(e^+e^- \to \Upsilon_i) = 12\pi^2 \delta(t - M_{\Upsilon_i}^2) \frac{\Gamma_{\Upsilon_i}}{M_{\Upsilon_i}},$$

(12)

deriving from the cross section for $\Upsilon$ production, $t_0 = 4m_B^2$, $t^*_1 = (m_B + m_B^*)^2$ and $t^*_0 = 4m_B^2$ are the thresholds for $B\bar{B}$, $B\bar{B}^*$ and $B^*\bar{B}^*$ production, respectively. We used the notation $t_1 = (m_B^* - m_B)^2$ and $n_f = 3$ is the number of light quark flavors which give identical contribution in the unitarity sum [2]. It was convenient to write the expressions [12] since

$$\int_0^\infty \frac{1}{(t_0 - q^2)^2(1 - t_0^*/t)^{3/2}} |F(t)|^2 dt$$

+ \int_0^\infty \frac{1}{(t_0 - q^2)^2} |V(t)|^2 dt + \int_0^\infty \frac{1}{(t_0 - q^2)^2} |\tilde{V}(t)|^2 dt$$

\begin{align*}
\Pi' (q^2) &> \frac{n_f}{48\pi^2} \left\{ \int_0^\infty \frac{1}{(t_0 - q^2)^2} \left[ (1 - t_0^*/t)^{3/2} \right] |F(t)|^2 dt \right. \\
&+ \int_0^\infty \frac{1}{(t_0 - q^2)^2} |V(t)|^2 dt + \int_0^\infty \frac{1}{(t_0 - q^2)^2} |\tilde{V}(t)|^2 dt \\
&\times \left[ 2 |F_1(t)|^2 + \frac{4t_0}{t_0^*} |F_2(t)|^2 + 4t_0^* |F_3(t)|^2 + (4t_0^*)^2 |F_4(t)|^2 \right] \right\},
\end{align*}

(13)

where

$$\Pi' (q^2) = \Pi' (q^2) - \frac{27}{4\pi^2} \sum_i \frac{M_{\Upsilon_i}\Gamma_{\Upsilon_i}}{(q^2 - M_{\Upsilon_i}^2)^2}.$$

(14)

In the euclidean region $q^2 < 0$ the function $\Pi' (q^2)$ can be calculated by applying QCD perturbation theory, with nonperturbative corrections included by means of OPE. Due to the large value of $m_b$, the QCD expression of $\Pi' (q^2)$ can be used also at $q^2 = 0$ or even at positive $q^2$ much less than $4m_b^2$. Moreover, in this case the nonperturbative corrections are shown to be entirely negligible [13]. In the previous works [4]-[6] only the lowest order (one-loop) perturbative polarization function was used as input in eq. (14) (the terms containing the $\Upsilon$ poles being also omitted). In the present analysis we introduce explicitly in (14) the contribution of the $\Upsilon$ resonances, using the experimental information on $\Gamma_{\Upsilon_i}$. In the same time we use as input the expression of the polarization function up to three loops [12]-[14]:

$$\Pi' (q^2) = \Pi'^{(0)} (q^2) + \frac{\alpha_s(\mu^2)}{\pi} \Pi'^{(1)} (q^2) + \left( \frac{\alpha_s(\mu^2)}{\pi} \right)^2 \Pi'^{(2)} (q^2, \mu^2)$$

(15)

with the $\overline{MS}$ coupling $\alpha_s(\mu^2)$ defined in the conventional way. We use the standard expressions [12]

$$\Pi'^{(0)} (q^2) = \frac{1}{32\pi^2 m_b^2} \int_0^1 \frac{v(3 - v^2)}{(1 - q^2x/4m_b^2)^2} dx,$$

(16)
$$\Pi'(q^2) = \frac{1}{24\pi m_b^2} \int_0^1 \frac{v(3-v^2)}{(1-q^2x/4m_b^2)^2} \left[ \frac{\pi}{2v} - \frac{v + 3}{4} \right] dx,$$  
(17)

with $v = \sqrt{1-x}$. As concerns the last term in (15), we shall use the Taylor series around $q^2 = 0$:

$$\Pi'(2)(q^2, \mu^2) = \frac{3}{64\pi^2 m_b^2} \sum_n nC_n \left( \frac{q^2}{4m_b^2} \right)^{n-1},$$  
(18)

the coefficients $C_n$ being given in eq.(11) of ref. [14] (we recall that they depend explicitly on the normalization scale $\mu$).

With the above expressions, the input entering (13) is completely specified and this inequality can be viewed as an integral quadratic condition for the form factors of interest along the unitarity cut. By applying standard techniques of analytic functions [10], extended to "vector-valued functions" (see [11] and the references therein) we shall obtain from this condition a quadratic inequality relating the values of the form factors and their derivatives at the zero recoil point. Using then the relations (4-6) we shall finally express the derived inequality as a constraint on the charge radius (6).

We first conformally map the cut $t = q^2$ plane onto the unit disk in the complex plane $z$, such that the unitarity cut becomes the boundary $|z| = 1$. Actually, since the integrals appearing in (13) have different thresholds, we shall use for them different conformal mappings. More precisely, we take

$$z(t) = \frac{\sqrt{t_0 - t} - \sqrt{t_0}}{\sqrt{t_0 - t} + \sqrt{t_0}}$$  
(19)

for the first integral in the right hand side of (13) and similar expressions, with $t_0$ replaced by $t_0^*$ and $t_0^{**}$, respectively, for the second and the third integral. By the mapping (19) the threshold $t_0$ becomes $z = -1$ and the zero recoil point $w = 1$ (equivalent to $t = 0$, since $w = 1 - \frac{t}{2m_B^2}$) is applied onto the origin $z = 0$. Similarly, using the mappings suitable for the other integrals in (13) as explained above, the thresholds $t_0^*$ and $t_0^{**}$ become also $z = -1$ and the corresponding zero recoil point is applied on the origin. It is easy to see that the conformal mappings used for the second and the third integrals transform the threshold $t_0$ into a point situated inside the unit circle, close to $-1$. By performing the above changes of variable, all the integrals in (13) become integrals along the same contour, i.e. the boundary $z = e^{i\theta}$ of the unit disk.

It is convenient to introduce a compact notation by defining the following functions of the variable $z$:

$$f_1(z) = F(t), \quad f_2(z) = V(t),$$  
$$f_3(z) = F_1(t), \quad f_4(z) = F_2(t), \quad f_5(z) = \tilde{F}_3(t), \quad f_6(z) = F_4(t),$$  
(20)

where $\tilde{F}_3$ is defined in (11). Using the conformal mappings defined above, the normalization condition (3) and the definition (6) of the charge radius, one can
show easily that the functions $f_i(z)$ satisfy the relations
\begin{equation}
  f_i(0) = 1, \quad f_i'(0) = -8\rho^2, \quad i = 1, \ldots, 5; \quad f_6(0) = 0,
\end{equation}
the derivative being with respect to $z$. Moreover, following the standard technique presented in [1]-[3] we shall define a set of functions $\phi_i(z)$ analytic and without zeros in the unit disk, whose moduli squared on the boundary are proportional to the positive weights appearing in the integrals (13), multiplied by the Jacobian $|\frac{dt}{dz}|$ of the conformal mapping (19). These functions can be constructed easily and unambiguously, by using the relations
\begin{equation}
  t = \frac{4t_0}{(1-z)^2}, \quad \left(1 - \frac{t_0}{t}\right)^{3/2} = \frac{(1+z)^3}{8},
\end{equation}
which follow from (19), with
\begin{equation}
  d = \frac{\sqrt{t_0 - q^2} - \sqrt{t_0}}{\sqrt{t_0 - q^2} + \sqrt{t_0}}
\end{equation}
With these definitons, we can write the inequality (13) in the equivalent form
\begin{equation}
  \frac{1}{2\pi} \int_0^{2\pi} \sum_{i=1}^6 |\phi_i(\theta)f_i(\theta)|^2 d\theta \leq 1
\end{equation}
where the functions $\phi_i(z)$, obtained using (22) can be written in a compact form as
\begin{equation}
  \phi_i(z) = \phi_i(0) \frac{(1+z)^{a_i}(1-z)^{b_i}}{(1-zd_i)^{c_i}}.
\end{equation}
The parameters entering this expression are as follows:
\begin{align*}
  \phi_1(0) &= \frac{(1-d)^2}{32m_B} \sqrt{\frac{n_f}{6\pi\Pi'(q^2)}}, \quad a_1 = 2, \ b_1 = 1/2, \ c_1 = 2, \ d_1 = d \\
  \phi_2(0) &= 2\sqrt{2}\phi_1(0), \quad a_2 = 2, \ b_2 = -3/2, \ c_2 = 2, \ d_2 = d \\
  \phi_3(0) &= \frac{(1-d^*)^2}{32m_{B^*}} \sqrt{\frac{n_f}{3\pi\Pi'(q^2)}}, \quad a_3 = 2, \ b_3 = 1/2, \ c_3 = 2, \ d_3 = d^* \\
  \phi_4(0) &= 2\sqrt{2}\phi_3(0), \quad a_4 = 2, \ b_4 = -3/2, \ c_4 = 2, \ d_4 = d^* \\
  \phi_5(0) &= \phi_3(0) \sqrt{2}, \quad a_5 = 2, \ b_5 = 1/2, \ c_5 = 2, \ d_5 = d^* \\
  \phi_6(0) &= 8\sqrt{2}\phi_3(0), \quad a_6 = 2, \ b_6 = -3/2, \ c_6 = 2, \ d_6 = d^*,
\end{align*}
with $\Pi'$ defined in (14) and $d$ in (23) ($d^*$ is obtained from $d$ by replacing $m_B$ by $m_{B^*}$).
As discussed above, the form factors appearing in (13) have three poles on the real axis below the threshold \( t_0 = 4m_B^2 \), due to the three \( bb \) bound states \( \Upsilon(1S), \Upsilon(2S) \) and \( \Upsilon(3S) \) with masses smaller than the threshold for \( B\bar{B} \) production. The positions of these poles are known from the experimental masses of the \( \Upsilon \) resonances, but the residues are unknown, containing the unphysical \( \Upsilon B\bar{B} \) or \( \Upsilon B\bar{B}^* \) couplings [2].

The form factors \( V \) and \( F_i \) have in addition branch points at the threshold \( t_0 \) of the \( B\bar{B} \) production, below the beginning of the corresponding unitarity cut. If an estimate of the discontinuity across these cuts were available, the treatment of these subthreshold singularities in the present formalism could be done exactly [16] (the method was applied recently in [17] to the \( B \rightarrow D \) form factors). In what follows we shall resort to a pole approximation, keeping only the contribution of the \( \Upsilon \) resonances situated below the thresholds \( t^*_{0} \) and \( t^{**}_{0} \), respectively. Using \( m_B = 5.379 \) GeV, \( m_{B^*} = 5.324 \) GeV and the masses of the \( \Upsilon \) resonances (\( M_{\Upsilon^1} = 9.460 \) GeV, \( M_{\Upsilon^2} = 10.023 \) GeV, \( M_{\Upsilon^3} = 10.355 \) GeV, \( M_{\Upsilon^4} = 10.580 \) GeV) one can easily see that the form factor \( V(t) \) has only three poles below its unitarity threshold, much like \( F(t) \), while \( F_i(t) \) have four poles. Passing to the functions \( f_i(z) \) according to (20) and using the conformal transformation (19), we find that the functions \( f_1(z) \) and \( f_2(z) \) have inside \(|z| < 1\) three poles situated at the points

\[
\begin{align*}
z_1 &= -0.38, \quad z_2 = -0.52, \quad z_3 = -0.67. \quad (27)
\end{align*}
\]

We neglected here the difference between \( m_B \) and \( m_{B^*} \), which is entirely justified as long as the singularities remain the same. As concerns the remaining functions \( f_i, i \geq 3, \) they have four poles, with positions

\[
\begin{align*}
z_1^* &= -0.37, \quad z_2^* = -0.49, \quad z_3^* = -0.62, \quad z_4^* = -0.79, \quad (28)
\end{align*}
\]

obtained by using the conformal mapping (19) with \( t_0 \) replaced by \( t_0^* \) and \( t \) by \( M_{\Upsilon_i} \).

The inequality (24) has the form of an \( L^2 \) norm condition [10] involving several functions. We derive from it constraints on the functions \( f_i \) and their derivatives at the origin \( z = 0 \), which corresponds through the conformal mapping to the point of zero recoil \( w = 1 \). If the functions \( f_i \) were analytic, this would be very easily done, by applying standard techniques in the Hilbert space \( H^2 \) [10]. However, as shown above, the functions have a finite number of poles, with known positions but unknown residua. The simplest treatment of this situation is based on the technique of Blaschke functions [10] (the method was applied previously in [3]-[4]). We define the following functions

\[
\begin{align*}
B(z) &= \prod_{k=1}^{3} \frac{(z - z_k)}{(1 - zz_k)}, \quad B^*(z) = \prod_{k=1}^{4} \frac{(z - z_k^*)}{(1 - zz_k^*)} \quad (29)
\end{align*}
\]

where we took into account that \( z_k \) and \( z_k^* \) are real.

As seen from (29) the functions \( B(z) \) and \( B^*(z) \) have modulus equal to 1 on the boundary of the unit disk (i.e. for \( z = e^{i\theta} \)). Therefore, we can insert the modulus
squared of the function $B(\theta)$ (or $B^*(\theta)$) in the integral appearing in (24), without spoiling the inequality or losing information. The relation (24) is thus equivalent to
\[
\frac{1}{2\pi} \int_0^{2\pi} \sum_{i=1}^{6} |\phi_i(\theta)B_i(\theta)f_i(\theta)|^2 d\theta \leq 1,
\] (30)
where we denoted
\[
B_i(z) = B(z) \quad (i = 1, 2) ; \quad B_i(z) = B^*_i(z) \quad (i = 3, 6).
\] (31)
But the products $B_i(z)f_i(z)$ are functions analytic in $|z| < 1$, the poles of the form factors $f_i$ being compensated by the zeros of the functions $B_i(z)$. We can apply therefore the well-known results of interpolation theory for vector-valued analytic functions (see [11] and references therein) to obtain from (30) constraints on the form factors at points inside the analyticity domain. In particular, being interested in finding bounds on the charge radius (6) which appear in (21), we shall apply an inequality of the Schur-Caratheodory type [10] at $z = 0$:
\[
\sum_{i=1}^{6} \left[ \phi_i B_i f_i \right]^2(0) + \left( \phi_i B_i f_i \right)'^2(0) \leq 1.
\] (32)
It is important to note that up to now the form factors $f_i$ were treated as independent functions, without assuming that they coincide along the unitarity integrals. We use now heavy quark spin symmetry, which imply the relations (21). Then (32) can be written as an inequality for the charge radius
\[
\sum_{i=1}^{5} \phi_i^2(0) B_i^2(0) + \sum_{i=1}^{5} \left[ \phi_i(0) B_i'(0) + \phi_i'(0) B_i(0) - 8\rho^2 \phi_i(0) B_i(0) \right]^2 \leq 1.
\] (33)
The function $f_6$ does not contribute, due to the last condition in (21). The inequality (33) can be written as
\[
a(\rho^2)^2 - 2b\rho^2 + c \leq 0,
\] (34)
where
\[
a = 64 \sum_{i=1}^{5} B_i^2(0) \phi_i^2(0)
\]
\[
b = 8 \sum_{i=1}^{5} B_i(0) \phi_i(0) [\phi_i'(0) B_i(0) + \phi_i(0) B_i'(0)]
\]
\[
c = 5 \sum_{i=1}^{5} [\phi_i'(0) B_i(0) + \phi_i(0) B_i'(0)]^2 + 5 \sum_{i=1}^{5} B_i^2(0) \phi_i^2(0) - 1.
\] (35)
The quantities $\phi_i(0)$, $\phi_i'(0)$, $B_i(0)$ and $B_i'(0)$, entering the above coefficients, are calculable from the relations (23), (24) and (25) and contain all the dynamical information in the problem.
We discuss now the lower and upper bounds on the charge radius $\rho^2$ calculated from the above equation (34). First we recall that the results previously reported in [2] and the second reference [3] can be obtained by restricting the sums in the expressions (35) to a single term, $i = 1$. In the above works only the lowest order term $\Pi^0$ in the expansion (35) of $\Pi^\prime$ was retained and the contribution of the $\Upsilon$ poles in the relation (14) was dropped out. Also, for simplicity the choice $m_b = m_B$ for the mass of the $b$ quark was made, and the value of $q^2$ which enters as a parameter in eq. (13) was taken $q^2 = 0$. With these restrictions, eq. (34) gives the interval $-4.5 \leq \rho^2 \leq 6.1$ already reported in [2]. Keeping two terms ($i = 1, 2$) in the sums appearing in (35), with the same numerical input as just described, we recover the interval $-0.9 \leq \rho^2 \leq 2.60$ obtained in [9]. Finally, with all the five terms in the sums, i.e. by including all the form factors of the ground states $B$ and $B^*$, we obtain with the same input the range $-0.35 \leq \rho^2 \leq 2.15$. This result shows the improvement which can be obtained by fully exploiting spin symmetry in the ground state $B$ doublet.

As we mentioned, the above results were obtained with some simplifying assumptions concerning the input. It is therefore of interest to perform the analysis with a more realistic input, according to the complete formulas given above. The main improvement is the QCD expression (15) of the polarization function up to three loops corrections. This expression depends on the scale $\mu$ which appears in the $\overline{MS}$ coupling $\alpha_s(\mu)$ and in the coefficients $C_n$ of the Taylor expansion (18). We shall use in our analysis two scales, namely $\mu = m_b$, for which the coefficients $C_n$ are (14)

$$C_1 = 32.73, \quad C_2 = 33.24, \quad C_3 = 29.61, \quad C_4 = 26.94,$$

and $\mu = 2m_b$, which gives

$$C_1 = 49.57, \quad C_2 = 43.31, \quad C_3 = 37.91, \quad C_4 = 33.92. \quad (37)$$

We note that for the above choices of $\mu$ the coefficients $C_n$ do not depend on the specific value of $m_b$. Although the coefficients in (36) and (37) are quite different, the final results, i.e. the bounds on $\rho^2$, turn out to be practically the same.

The expressions given in (15-18) were obtained using on shell renormalization, which means that $m_b$ is the pole mass. In the present work we shall treat this mass as a parameter in the reasonable range $4.7 \text{GeV} - 5.0 \text{GeV}$. For these values of $m_b$ and the choices of $\mu$ made above, the two-loops correction in the expansion (15) for $q^2 = 0$ represents about 30% of the lowest order term, while the contribution of the three-loops diagrams is of about 10% (we used $\alpha_s(5.0 \text{GeV}) = 0.21$ and $\alpha_s(10.0 \text{GeV}) = 0.18$). As we already pointed out, for heavy quarks one can extend the validity of the QCD perturbative expansion of the polarization function even at positive values of $q^2$, below the threshold $t_0$. As an example, for $q^2 = 50 \text{ GeV}^2$ the two-loops term represents a correction of about 45% of the one loop term, while the three loops
contribute in addition with approximately 20%. In the present formalism better results, i.e. stronger bounds on $\rho^2$, are obtained for larger $q^2$. On the other hand, the increased contribution of the higher order QCD corrections for the polarization function prevents us taking $q^2$ too close to the hadronic singularities. We shall take in what follows $q^2$ in the range $0 - 50\text{GeV}^2$, noticing that the relative magnitude of the perturbative corrections does not dramatically change in this domain.

We recall that much smaller values for the QCD perturbative corrections to the polarization function of heavy quarks were reported in [12] (see also [13]). The idea applied in these works was to express the pole mass $m_b$ in (16) and (17) in terms of an euclidian mass defined to first order in $\alpha_s$. This had the effect of reducing the procentual contribution of the two-loop correction, especially in the high order derivatives of the function $\Pi(q^2)$, of interest in the QCD sum rules for heavy quarks. The recent calculation of the polarization function up to three loops [14] allowed us to use a more exact expression of $\Pi'$, without resorting to the rather arbitrary procedure adopted in [12].

The contribution of the $\Upsilon$ poles in the expression (14) was evaluated using the numerical values $\Gamma_{\Upsilon_1} = 1.34\text{ keV}$, $\Gamma_{\Upsilon_2} = 0.56\text{ keV}$ and $\Gamma_{\Upsilon_1} = 0.44\text{ keV}$ [18]. The poles bring a positive contribution to the spectral function according to (10) and their inclusion improves the bounds in a significant way.

In Fig.1 we present the upper and lower bounds on the charge radius $\rho^2$ of the $B$ meson elastic form factor, computed from (34), with the input described above, for $m_b$ in the range $4.7\text{GeV} - 5.\text{GeV}$. As we mentioned, the two choices of the scale $\mu$ adopted above give almost identical results. The solid curve corresponds to the choice $q^2 = 0$, the dashed one to $q^2 = 50\text{ GeV}^2$. Taking larger values of $q^2$ we obtain much stronger bounds, but inconsistencies appear around $60\text{ GeV}^2$ (the pole contribution exceeds the QCD expression of $\Pi'(q^2)$, signaling that a better estimation of the input is necessary). As seen from Fig.1, the predictions for the charge radius are rather sensitive to the value of the pole mass $m_b$, larger values of the mass leading to stronger bounds.

The upper and lower bounds given in Fig.1 represent the best results that can be derived, using a realistic input and fully exploiting heavy quark spin symmetry for the ground state $B$ and $B^*$ mesons. We recall that the present derivation was possible by resorting to a a more powerful technique of analytic functions, which allowed the simultaneous treatment of several form factors as independent functions. The specific unitarity thresholds of the different form factors and their subthreshold singularities were correctly taken into account. Heavy quark spin symmetry was invoked finally by assuming that various form factors coincide near the zero recoil point, which is entirely legitimate.

The technique applied in this paper can be easily generalized (see [3], [17]) to include higher derivatives of the form factors at the zero recoil point. In this way, for instance, quite strong correlations among the slope and the convexity of the elastic form factor $F(t)$ can be derived. A second, more interesting generalization is to include in the unitarity sum the contribution of the excited states ($B^{**}$) with orbital
momentum $L = 1$. By applying the techniques used in this work, it is possible to derive an inequality connecting the form factors of the ground states $B$ and $B^*$ to the transition form factors between $B^{**}$ and the ground states. A new sum rule for these form factors, similar to the well-known inequalities of Bjorken \[13\] and Voloshin \[20\], will be reported in a future paper \[21\].

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Figure caption

FIG.1: Upper and lower bounds on the charge radius of the elastic form factor of the $B$ meson for various values of the pole mass $m_b$. The solid line corresponds to $q^2 = 0$, the dashed one to $q^2 = 50 \text{ GeV}^2$.

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