Nonuniqueness of gravity–induced fermion interaction in the Einstein–Cartan theory

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The problem of nonuniqueness of minimal coupling procedure for Einstein–Cartan (EC) gravity with matter is investigated. It is shown that the predictions of the theory of gravity with fermionic matter can radically change if the freedom of the addition of a divergence to the flat space matter Lagrangean density is exploited. The well–known gravity–induced four–fermion interaction is shown to reveal unexpected features. The solution to the problem of nonuniqueness of minimal coupling of EC gravity is argued to be necessary in order for the theory to produce definite predictions. In particular, the EC theory with fermions is shown to be indistinguishable from usual General Relativity on the effective level, if the flat space fermionic Lagrangean is appropriately chosen. Hence, the solution to the problem of nonuniqueness of minimal coupling procedure is argued to be necessary if EC theory is to be experimentally verifiable. It could also enable experimental tests of theories based on EC, such as loop approach to quantisation of gravitational field. Some ideas of how the arbitrariness incorporated in EC theory could be restricted or even eliminated are presented.

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I. INTRODUCTION

Einstein–Cartan theory (EC) is acknowledged as a viable alternative for General Relativity (GR), confirmed by all available experimental data. For an exhaustive review of the theory, see [1]. For a mathematically rigorous formulation in terms of tensor valued differential forms, see [2]. If coupled to fermions, the EC theory is claimed to differ from General Relativity by the presence of a gravity–induced four–fermion interaction. The effective action obtained by integrating out the connection contains an additional term, when compared to GR, which is proportional to the square of an axial fermion current. The equation for Dirac bispinor field is nonlinear, even in the limit of the space–time metric being Minkowski’s flat one. This nonlinearity can be interpreted as describing interaction between fermions. Some interesting properties of this interaction have been studied [3][4][5]. It is generally believed that, although in principle it is measurable, its effects cannot be measured in practice due to the smallness of coupling constant appearing in front of a new term in the effective action.

The interest in gravity–induced four–fermion interaction increased in the last few years because of the development of canonical approach to the quantisation of gravity. Since the introduction by Ashtekar of a new formalism for complex General Relativity (GR) [6][7], reducing constraints of the theory to the polynomial form, many steps forward in the program of quantisation have been made. A new formalized treatment of nonperturbative canonical gravity has emerged, known as Loop Quantum Gravity [8]. In order to avoid difficulties concerning reality conditions, necessary in Ashtekar complex approach, Barbero [9] proposed a real alternative. The relation between these two approaches was then clarified by Immirzi [10] and Holst [11]. It appears that adding a new term to the standard Palatini action of General Relativity allows a unified treatment of them. If the multiplicative parameter $\beta$ is introduced in front of the new term, the theory reduces to that of Ashtekar and Barbero for $\beta = \pm i$ and $\beta = \pm 1$ respectively. The new constant $\beta$ is called the Immirzi parameter. In the case of absence of torsion generating matter, the additional term, called the Holst term, does not influence field equations, as it vanishes on account of Bianchi identity. Hence, the Immirzi parameter drops out from the classical theory but appears to play an important role in quantum theory as it enters the spectra of area and volume operators [12]. This allowed for the establishing of theoretical bounds on possible values of the Immirzi parameter by black hole entropy calculations and comparison with the Bekenstein–Hawking formula [13]. A precise value of the parameter was given shortly after [14]. Then it was noted that even in the classical theory of gravity, the new Holst term does influence field equations when fermions are minimally coupled [15]. This initiated discussion on the role played by the Immirzi parameter in classical gravity with fermions [16][17][18][19][20]. As originally observed in [15], the Immirzi parameter enters the coupling constant in front of the four–fermion interaction term of EC gravity. It has been concluded that measuring the strength of this interaction can provide a tool to estimate the value of the Immirzi parameter independently from the quantum theory of gravity. Although the subsequent investigations [18][20] showed that the Immirzi parameter can be “hidden” in the parameters...
of more general, non–minimal coupling procedures, it should be stressed that the minimal coupling scheme has been historically successful in constructing models, which could withstand the rigors of experimental testing whenever such tests were feasible. The experimental successes of the standard model of particle physics and general theory of relativity seem to support the minimal approach. Indeed, the Yang–Mills theories, which constitute the formal basis for the standard model, employ minimal coupling scheme on the fundamental level. The necessity of using non–minimal couplings when describing effectively composed objects does not hold much relevance as long as we aim to incorporate elementary point–like fermions (quarks and leptons) into the theory of gravity. As is well known, the EC gravity can be formulated as a gauge theory of Yang–Mills type for the Poincaré group. Hence, one could hope that the application of minimal coupling would lead to the physically relevant model in this case as well. According to this viewpoint, special importance should be attached to the origina analysis of [15], rather than to the later analyses employing non–minimal couplings.

However, there is an important issue which seems to have been overseen in most considerations concerning predictions of EC gravity with fermions. The standard minimal coupling procedure (MCP) that simply means converting all partial derivatives in flat space matter Lagrangean into covariant ones and applying metric volume element to construct Lagrangean four–form is not unique in the case of torsion connections. Equivalent flat Lagrangeans (generating the same flat space field equations) give rise to curved theories which are not in general equivalent. That issue has been discussed since the very beginning of gauge formulation of gravity [21]. The nature of the problem is recalled in Section II. One approach to solve it, reconsidered here in Section III is to set up the procedure for choosing the ‘appropriate’ flat space Lagrangean from the whole class of equivalent ones. The possibility of using Noether theorem to achieve that purpose is investigated. More radical solution would be to modify MCP itself to make it give equivalent results for equivalent Lagrangeans. Such modification was proposed by Saa in [22][23] leading to interesting effects, such as propagating torsion or coupling gauge fields to torsion without breaking gauge symmetry. Although Saa’s idea provides a very interesting solution to the problem, it results in significant departures from standard GR, which are not certain to withstand the confrontation with observable data [24][25] without some assumptions of rather artificial nature, such as demanding a priori that part of the torsion tensor vanish [26].

In this paper we wish to argue that solving the MCP nonuniqueness problem is crucial if EC gravity is to produce any nontrivial definite predictions. In Section IV we briefly recall the formalism of EC theory and comment on compatibility of different possible definitions of energy–momentum and spin density tensors. In Section III we impose some reasonable restrictions on flat space fermionic Lagrangeans which leaves us with two–parameter family. We also give plausible arguments in favour of one particular choice. Then in Section V we rederive the effective action and modified Dirac equation for this family. We show that the above mentioned freedom can lead to the change of coupling constant of axial–axial four–fermion interaction, as well as appearance of a new vector–vector and even parity breaking axial–vector interaction 1. We show that even disregarding the technical limitations one cannot distinguish experimentally between EC gravity with fermions and standard torsionless GR treatment before the nonuniqueness problem of EC gravity with matter is solved, unless torsion is directly measurable. Finally, we try to understand the physical nature of the new interaction by employing the background field approximation. Our results show that the earlier statements [3][4] of universality (independence of matter type) of interaction remain valid, whereas the question of whether the interaction is repulsive or attractive for aligned and antiparallel spins remains open, the answer being dependent on the values of parameters of our generalized model.

II. NONUNIQUENESS OF MINIMAL COUPLING PROCEDURE

A classical field theory in flat Minkowski space is defined by the action functional

$$S = \int \mathcal{L},$$

where $\mathcal{L}$ is a Lagrangean density and $\mathcal{L} = \mathcal{L} d^4x = \mathcal{L} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ a Lagrangean four–form. It is well known that the addition of a divergence of a vector field $V$ to $\mathcal{L}$ changes $\mathcal{L}$ by a differential

$$\partial \mu V^\mu d^4x = L_V d^4x = d(V^\mu d^4x), \quad (II.1)$$

1 Although it was mentioned in [27] that an axial–vector interaction might appear if flat space fermion Lagrangean were appropriately chosen, no explanation of the nature of this alternative flat Lagrangean can be found, nor were the reasons for not treating it seriously given. No mention of possibility of vector–vector interaction, or the change of coupling constant appeared, either.
where $L$ denotes Lie derivative and $\partial$ the internal product. Thus, such a transformation does not change field equations generated by $S$. In order to proceed from Minkowski space to the general Riemann–Cartan (RC) manifold with metric $g_{\mu\nu}$ and the metric compatible connection $\nabla$ (not necessarily torsion–free)\(^2\), we can apply MCP

$$\int L(\phi, \partial_{\mu} \phi, \ldots) d^4x \longrightarrow \int L(\phi, \nabla_{\mu} \phi, \ldots) \epsilon,$$

(II.2)

where $\epsilon = \sqrt{g} d^4x$ is the metric volume form, $g$ being the determinant of a matrix of components $g_{\mu\nu}$ of the metric tensor in the basis $\partial_{\mu}$, and $\phi$ represents fields of the theory. Dots in (II.2) correspond to the possibility of $\nabla$ to depend on higher derivatives of fields. Had we used the modified flat space Lagrangean $\mathcal{L} + \partial_{\mu} V^\mu$, we would have obtained different Lagrangean four–form on RC manifold, the difference being

$$\nabla_{\mu} V^\mu \epsilon = \nabla_{\mu} V^\mu \epsilon - T_{\mu} V^\mu \epsilon,$$

(II.3)

where $\nabla$ is the torsion–free Levi–Civita connection and $T_{\mu} = T^\nu_{\mu\nu}$ the torsion trace vector. The first term in (II.3) is a differential, $\nabla_{\mu} V^\mu \epsilon = d(V_{\mu} \epsilon)$, whereas the second is not. Hence, the equivalent flat Lagrangeans yield nonequivalent theories on RC space.

One could hope differential forms formalism would fix the problem and argue that the last expression of (II.1), rather than the first, should be adopted to curved space. Then, $d(V_{\mu} d^4x)$ would transform into $d(V_{\mu} \epsilon)$, which is again a differential. However, this is not a good solution, since decomposition of a given Lagrangean four–form $L_1 d^4x$ to the sum of another Lagrangean four–form $L_2 d^4x$ and the term $d(V_{\mu} d^4x)$ is by no means unique. We should rather use the identity $d(V_{\mu} d^4x) = -\ast d\mu_{\mu} \wedge dV^\mu$, where $\ast$ is a hodge star (see Section VII), and minimally couple gravity by the passage $dV^\mu \longrightarrow DV^\mu = dV^\mu + \omega_{\mu\nu} V^\nu$ (where $\omega_{\mu\nu}$ are connection one–forms) and by the change of a hodge star of flat Minkowski metric to the one of curved metric on the finall manifold, but this would give the result identical to (II.3).

III. ENERGY–MOMENTUM AND SPIN TENSORS FOR EQUIVALENT LAGRANGEANS

In this section we will investigate transformation properties of Noether currents of physical interest under the addition of a divergence to the Lagrangean. The leading idea is that demanding these currents to have the most 'reasonable' form may help us restrict the class of equivalent Minkowski space Lagrangeans and thus limit the number of nonequivalent theories on RC space–time which are worth further considerations. Let $\mathcal{L}(\phi^A, \partial_{\mu} \phi^A)$ be a Lagrangean density of a field theory in Minkowski space. The invariance of $\mathcal{L}$ under the global action of a Lie group of transformations $x^\mu \rightarrow x^\mu + \delta x^\mu$, $\phi^A \rightarrow \phi^A + \delta \phi^A$ which do not change the volume–form $d^4x$ of Minkowski metric implies that the Noether current

$$j^\mu = -t_{\nu}^{\mu} \delta x^\nu + \frac{\partial \mathcal{L}}{\partial(\partial_{\mu} \phi^A)} \delta \phi^A$$

is conserved, i.e. $\partial_{\mu} j^\mu = 0$, if field equations are satisfied. Here

$$t_{\mu\nu} := \frac{\partial \mathcal{L}}{\partial(\partial_{\mu} \phi^A)} \partial_{\nu} \phi^A - \eta_{\mu\nu} \mathcal{L}.$$

(III.1)

is the canonical energy–momentum tensor, constituting the set of currents which are conserved due to the symmetry of $\mathcal{L}$ under space–time translations $x^\mu \rightarrow x^\mu + a^\mu$. In the case of $\mathcal{L}$ being invariant under proper Lorentz transformations (all Minkowski space Lagrangean densities considered in this paper posses that property), the corresponding conserved currents comprise an angular momentum tensor $\mathcal{M}^\nu_{\mu\nu}$

$$\mathcal{M}^\nu_{\mu\nu} \epsilon_{\nu\alpha} = 2 j^\mu = (x^\nu t^\mu_{\nu} - x^\mu t^\nu_{\mu} + S^\nu_{\mu\nu}) \epsilon_{\nu\alpha},$$

where $\epsilon_{\mu\nu}$ are parameters of a Lorentz transformation $(\Lambda(\epsilon))^\mu_{\nu} \equiv \delta^\mu_{\nu} + \epsilon^\mu_{\nu}$ for small $\epsilon$. Here the spin density tensor $S^\nu_{\mu\nu}$ is a part of $\mathcal{M}^\nu_{\mu\nu}$ which depends on transformation properties of fields $\phi^A$. If $\delta \phi^A$ is known, it can be computed from

$$S^\nu_{\mu\nu} \epsilon_{\nu\alpha} = 2 \frac{\partial \mathcal{L}}{\partial(\partial_{\mu} \phi^A)} \delta \phi^A.$$

(III.2)

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\(^2\) For more general considerations concerning not necessarily metric connections see [26].
Apart from obvious scaling freedom, we can construct new conserved currents from a given one via the transformation

$$j^\mu = j^\mu + \partial_\nu f^{\mu\nu},$$  \hspace{1cm} (III.3)

where $f^{\mu\nu} = -f^{\nu\mu}$. In addition to giving another conserved current, transformation (III.3) does not change integrated charges $Q = \int_{t=\text{const.}} J^0 d^3 \vec{x}$, if $f^{\mu\nu}$ vanishes sufficiently fast at spatial infinity.

Let us now consider two Lagrangean densities differing by a divergence of a vector field $V^\mu (\phi^A)$ (we wish $V^\mu$ not to depend on derivatives of $\phi^A$ in order for both Lagrangeans to depend on first derivatives only)

$$\mathcal{L} - \mathcal{L}' = \partial_\mu V^\mu = \frac{\partial V^\mu}{\partial \phi^A} \partial_\mu \phi^A .$$  \hspace{1cm} (III.4)

Here, $V$ is required to transform as a vector under proper Lorentz transformations: if $\phi^A \to \phi'^A$ represents the action of a relevant representation of a proper Lorentz group in the space of fields, we have $V^\mu (\phi^A) \to V^\mu (\phi'^A) = \Lambda^\mu_\nu V^\nu (\phi^A)$. Hence, $\partial_\mu V^\mu$ is a Lorentz scalar and $\mathcal{L}'$ is a Lorentz scalar (if $\mathcal{L}$ is a scalar). All Lagrangean densities considered by us are also required to be real, which implies reality of $V$. The difference in energy–momentum and spin tensors corresponding to (III.4) will be

$$t'_{\mu \nu} - t_{\mu \nu} = \partial_\rho f^{\mu \nu \rho}, \hspace{1cm} (S'^{\alpha \beta \mu} - S^{\alpha \beta \mu}) \varepsilon_{\alpha \beta} = 2 \frac{\partial V^\mu}{\partial \phi^A} \delta \phi^A ,$$  \hspace{1cm} (III.5)

where $f^{\mu \nu \rho} = \delta^{\rho}_{\nu} V^{\mu} - \delta^{\rho}_{\mu} V^{\nu}$. Since $f^{\mu \nu \rho} = -f^{\mu \rho \nu}$, the change of $t_{\mu \nu}$ corresponds to the usual freedom (III.3) left by Noether procedure and integrated energy and momenta do not transform. Hence, the energy–momentum tensor cannot help us choose among Lagrangeans differing by a divergence. Let us then focus our attention on spin tensor. For the purposes of this paper, we will confine ourselves to the case of a Dirac bispinor field $\psi$ and the vector field of the form

$$V^\mu = \bar{\psi} B^\mu \psi ,$$  \hspace{1cm} (III.6)

where $B^\mu = a^{\gamma \mu} + b^{\gamma \mu \gamma \delta}$ for some real numbers $a$ and $b$ (recall that $V$ is required to transform as a vector under proper Lorentz transformations). Here $\gamma^\mu$ are the Dirac matrixes obeying $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 \eta^{\mu \nu}$, $\gamma^5 := -i \gamma^0 \gamma^1 \gamma^2 \gamma^3$ and $\bar{\psi} := \psi^\dagger \gamma^0$, where $\psi^\dagger$ is hermitian conjugation of a column matrix. Hence,

$$V^\mu = a J^\mu_{(V)} + b J^\mu_{(A)} ,$$  \hspace{1cm} (III.7)

where $J^\mu_{(V)} = \bar{\psi} \gamma^\mu \psi$, $J^\mu_{(A)} = \bar{\psi} \gamma^\mu \gamma^5 \psi$ denote Dirac vector and axial current. For the relevant representation of the Lorentz group

$$S (\Lambda (\varepsilon)) = \exp \left( -\frac{i}{4} \varepsilon_{\alpha \beta} \Sigma^\alpha \Sigma^\beta \right) , \hspace{1cm} \Sigma^\alpha \Sigma^\beta = \frac{i}{2} [\gamma^\alpha, \gamma^\beta] , \hspace{1cm} \psi \to S (\Lambda) \psi ,$$  \hspace{1cm} (III.8)

we have

$$\delta \psi = -\frac{i}{4} \varepsilon_{\alpha \beta} \Sigma^\alpha \Sigma^\beta \psi , \hspace{1cm} \bar{\delta} \psi = \frac{i}{4} \varepsilon_{\alpha \beta} \bar{\psi} \Sigma^\alpha \Sigma^\beta ,$$  \hspace{1cm} (III.9)

which yields

$$S'^{\alpha \beta \mu} - S^{\alpha \beta \mu} = \frac{i}{2} \bar{\psi} [\Sigma^\alpha \Sigma^\beta, B^\mu] \psi ,$$  \hspace{1cm} (III.10)

on account of (III.5). One can observe that the expression vanishes for densities of spatial components of spin: $S'^{ij0} = S^{ij0}$. Hence, for all Lagrangeans differing by divergence from the standard one

$$\mathcal{L}_{F0} = \frac{i}{2} \left( \bar{\psi} \gamma^\mu \partial_\mu \psi - \partial_\mu \bar{\psi} \gamma^\mu \psi \right) - m \bar{\psi} \psi$$  \hspace{1cm} (III.11)

(throughout the paper we use the $c = \hbar = 1$ units) we have

$$S^{ij0} = \frac{1}{4} \bar{\psi} \{ \Sigma^{ij}, \gamma^0 \} \psi = \frac{1}{2} \bar{\psi} \Sigma^{ij} \psi .$$
If integrated over the space, this yields an expected value at a state represented by a wave function \( \psi(x) \) of the correct spin operator (first quantisation interpretation of \( \psi \) is applied). Hence, densities of space–space spin components cannot be used to choose appropriate Lagrangean. As far as time–space components are concerned, the situation is more interesting as their densities do transform according to

\[
S_{\mu_0 j_0}^{\alpha} - S_{\nu_0 j_0}^{\alpha} = \mathbb{T}B^j \psi .
\]

We can see that corresponding integrated charges are also different. If we were able to measure them, we could choose between \( \mathcal{L} \) and \( \mathcal{L}' \). Unfortunately, as pointed out by Kibble \[21\], it is not clear whether any physical significance should be attached to the separation of time–space components of angular momentum into orbital and spin parts. On the other hand, one could claim that the \((0j)\) components of spin do not have any physical meaning and should not appear at all as nonzero quantities. We could then postulate them to vanish. In the case of a Dirac field, such a rule would make the choice of a Lagrangean perfectly unique, leaving us with \( \mathcal{L}_{F0} \) \[III.1\]. Although the rule may seem to be rather artificial, it is an example of how one can try to lower the degree of arbitrariness incorporated in \( EC \) theory. It could be interesting to test it for other types of matter fields.

Another possible restriction of the freedom of choice of Lagrangean density could be to require the spin tensor to have as few independent components as possible. In general, \( S^{\mu
u} = -S^{\nu\mu} \) represents \( 4 \times 6 = 24 \) independent components. If \( \[III.11\] \) is chosen as a Lagrangean density, the spin tensor appears to be totally antisymmetric, thus having only 4 independent components. Hence, this criterion would again distinguish \( \mathcal{L}_{F0} \) as an appropriate Lagrangean density, in a unique manner.

### IV. EINSTEIN–CARTAN GRAVITY

#### A. Field equations

The Lagrangean four–form of the theory is \( \mathcal{L} = \mathcal{L}_G + \mathcal{L}_m \), where \( \mathcal{L}_G = -\frac{1}{8\pi G} \epsilon_{abcd} e^a \wedge e^b \wedge \Omega^{cd} \) represents gravitational part and \( \mathcal{L}_m \) the matter part. Here \( k = 8\pi G \), where \( G \) is gravitational constant, \( e^a = e^a dx^\mu \) is an orthonormal coframe, \( \omega^a_b = \Gamma^a_{bc} e^c \) are connection one–forms (spin connection) obeying the antisymmetry condition \( \omega_{ab} = -\omega_{ba} \) and \( \Omega^{ab} := \omega_\alpha^a \wedge \omega_\beta^b \wedge \omega_\gamma^c = \frac{1}{6} R^{abcd} e^a \wedge e^b \wedge e^c \) the curvatures two–forms. The connection coefficients \( \Gamma^a_{bc} \) are related to the metric connection \( \nabla \) on RC manifold by \( \nabla_c e_b = \Gamma^a_{bc} e_a \), where \( e_a = e^a \partial_x^\mu \) is an orthonormal tetrad (a basis of vector fields which is dual to one–form field basis \( e^a \)). Variation is given by

\[
\delta \mathcal{L} = \delta e^a \wedge \left( \frac{\delta \mathcal{L}_G}{\delta e^a} + \frac{\delta \mathcal{L}_m}{\delta e^a} \right) + \delta \omega^{ab} \wedge \left( \frac{\delta \mathcal{L}_G}{\delta \omega^{ab}} + \frac{\delta \mathcal{L}_m}{\delta \omega^{ab}} \right) + \delta \phi^A \wedge \frac{\delta \mathcal{L}_m}{\delta \phi^A} ,
\]

\( \phi^A \) representing matter fields (we used the independence of \( \mathcal{L}_G \) on \( \phi^A \)). Explicitly,

\[
\frac{\delta \mathcal{L}_G}{\delta e^a} = -\frac{1}{2k} \epsilon_{abcd} e^b \wedge \Omega^{cd} , \quad \frac{\delta \mathcal{L}_G}{\delta \omega^{ab}} = \frac{1}{2k} \epsilon^{ab} e^c Q^c \wedge e^d ,
\]

where \( Q^a := D e^a = \frac{1}{k} T^a \epsilon e^a \wedge e^c \) is a torsion two–form, whose components in a tetrad basis we have denoted by \( T^a \). The field equations are

\[
\frac{\delta \mathcal{L}_G}{\delta e^a} + \frac{\delta \mathcal{L}_m}{\delta e^a} = 0 \quad \Leftrightarrow \quad G^a_b := R^a_b - \frac{1}{2} R \delta^a_b = k \tilde{t}^a_b ,
\]

\[
\frac{\delta \mathcal{L}_G}{\delta \omega^{ab}} + \frac{\delta \mathcal{L}_m}{\delta \omega^{ab}} = 0 \quad \Leftrightarrow \quad T^{cab} - T^a \eta^{bc} - T^b \eta^{ac} = k S^{abc} ,
\]

where \( R^a_b := \eta^{ac} R^{cd} e^d_b \), \( R := R^a_a \), \( T^a := T^a \) and the dynamical definitions of energy–momentum and spin density tensors on Riemann–Cartan space (for calculational convenience each of them given below in two equivalent forms) are

\[
\tilde{t}^{a \mu}_b := - \frac{\delta \mathcal{L}_m}{\delta e^a} \equiv \tilde{t}_a^b \epsilon := \frac{\delta \mathcal{L}_m}{\delta e^a} \wedge e^b , \quad S^{abc} e_c := 2 \star \frac{\delta \mathcal{L}_m}{\delta \omega^{ab}} = \frac{1}{2} \tilde{S}^{abc} \star e_c := \frac{\delta \mathcal{L}_m}{\delta \omega^{ab}} .
\]

#### B. Compatibility between Noether and dynamical definitions of spin density and energy–momentum tensors

It seems also natural to promote to energy–momentum and spin density tensors the expressions obtained from Noether currents \[III.1\], \[III.2\] via MCP. We will denote the resulting tensors by \( t^{a \mu}_b \) and \( S^{abc} \) (the corresponding
objects in (IV.2) were denoted with ˜ to distinguish from what we consider now). There is no reason in general for \( t^a_b \) and \( S^{abc} \) to be equal to \( \tilde{t}^a_b \) and \( \tilde{S}^{abc} \). Perhaps one could demand such equality to hold and thus restrict the freedom of choice of the flat space Lagrangean and the resulting EC theory?

When the transformation (III.4) is applied to the flat space Lagrangean, the Lagrangean four–form on RC manifold obtained by standard MCP transforms as

\[
\mathcal{L}' = \mathcal{L} - e_a \wedge dV^a - e_a \wedge \omega^a_b V^b .
\]

This induces transformation rules for energy–momentum and spin density tensor components

\[
\tilde{t}^a_b = \tilde{\tilde{t}}^a_b + \nabla_a V_b - \delta^a_b \nabla_c V^c ,
\]

\[
\tilde{S}^{abc} = \tilde{\tilde{S}}^{abc} + \eta^{ac} V^b - \eta^{bc} V^a .
\]

(IV.3)

Comparison with (III.5) allows to conclude that \( t^a_b = \tilde{t}^a_b \iff \tilde{t}^a_b = \tilde{\tilde{t}}^a_b \). Hence, the two definitions of energy–momentum tensor give the same result either for all Lagrangeans related by the equivalence relation (III.4) or for none of them. In the case of a Dirac field, the equivalence class defined by (III.11) appears to work well. In particular, if gravity is minimally coupled to (III.11) itself, the resulting Lagrangean four–form is

\[
\hat{\Sigma}_{F0} = -\frac{i}{2} e_a \wedge (\bar{\psi} \gamma^a D\psi - D\bar{\psi} \gamma^a \psi) - m\bar{\psi}\psi \epsilon ,
\]

(IV.4)

and the energy–momentum tensor is

\[
t^a_b = \tilde{\tilde{t}}^a_b = \frac{i}{2} (\bar{\psi} \gamma^b \nabla_a \psi - \nabla_a \bar{\psi} \gamma^b \psi) - \delta^a_b \left[ \frac{i}{2} \left( \bar{\psi} \gamma^c \nabla_c \psi - \nabla_c \bar{\psi} \gamma^c \psi \right) - m\bar{\psi}\psi \right]
\]

(here \( \nabla_a \psi \) are components of a one–form \( D\psi \) in the cotetrad basis: \( D\psi = (\nabla_a \psi) e^a \)). Note that the cannonical energy–momentum tensor is the one which appears in the ‘Einstein equation’ (IV.1), not the symmetric one obtained by Belinfante–Rosenfeld method.

In the case of spin, we will confine our considerations to the Dirac field and to the vector field of the form (III.6). For (III.11), we find by straightforward calculations based on (IV.4), (IV.2) and (III.2) that

\[
\tilde{S}^{abc} = \tilde{\tilde{S}}^{abc} = \frac{1}{4} \bar{\psi} \left\{ \Sigma^{ab} , \gamma^c \right\} \psi .
\]

Then, for the Lagrangean density \( \mathcal{L}_{F0} + \partial_{\mu} V^\mu \), we get from (III.10) and (IV.3)

\[
\tilde{S}'^{abc} - S'^{abc} = \bar{\psi} \left( \frac{i}{2} \left[ \Sigma^{ab} , B^c \right] - \eta^{ac} B^b + \eta^{bc} B^a \right) \psi ,
\]

which vanishes identically on account of \( [\gamma^a , \Sigma^{bc}] = 4i\eta^{[a}[\gamma^\epsilon] . \) Hence, both definitions are perfectly compatible, independently of the choice of flat Lagrangean from the equivalence class of (III.11) (the equivalence relation being given by (III.4)). It is worth noting that in the case of gauge fields such compatibility would not occur, unless we use MCP in a naive manner, \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + [A_{\mu} , A_{\nu}] \rightarrow \nabla_{\mu} A_{\nu} - \nabla_{\nu} A_{\mu} + [A_{\mu} , A_{\nu}] \), which breaks gauge symmetry.

**V. EFFECTIVE ACTION, MODIFIED DIRAC EQUATION AND THEIR PHYSICAL MEANING**

**A. Derivation of the effective action**

Let us define the *contorsion one–forms*

\[
K^a_b = K^a_{bc} e^c := \omega^a_b - \omega^a_b
\]

(objects with \( \circ \) above will always denote torsion–free objects, related to LC connection). The curvature two–form decomposition

\[
\Omega^a_b = \Omega^a_b + D K^a_b + K^a_c \wedge K^c_b
\]
results in
\[ \mathcal{L}_G := -\frac{1}{4k} \epsilon_{abcd} e^a \wedge e^b \wedge \Omega^{cd} = \mathcal{L}_G - \frac{1}{4k} \epsilon_{abcd} e^a \wedge e^b \wedge K^c_e \wedge K^{ed} - \frac{1}{4k} \overset{\circ}{D} \left( \epsilon_{abcd} e^a \wedge e^b \wedge K^{cd} \right), \]
where \( \overset{\circ}{D} \epsilon_{abcd} = 0 \) was used. Here \( k = 8\pi G \), where \( G \) is a gravitational constant. Since all Lorentz indexes in the last term are contracted, \( \overset{\circ}{D} \) acts like a usual differential. Using the relation between components of contorsion and torsion tensors
\[ K_{abc} = \frac{1}{2}(T_{cab} + T_{bac} - T_{abc}) \]
and decomposing torsion into its irreducible parts
\[ T_{abc} = \frac{1}{3}(\eta_{ac} T_b - \eta_{ab} T_c) + \frac{1}{6} \epsilon_{abcd} S^d + q_{abc}, \quad T_a := T^b_{ab}, \quad S_a := \epsilon_{abcd} T^{bcd}, \]
we can finally obtain
\[ \mathcal{L}_G = \overset{\circ}{\mathcal{L}}_G + \frac{1}{2k} \left( \frac{2}{3} T_a T^a - \frac{1}{24} S_a S^a - \frac{1}{2} q_{abc} q^{abc} \right) \epsilon + d(\ldots) \]
(the last term is a differential whose particular form will not be needed). Similarly, the Dirac Lagrangean \([V.4]\) decomposes as
\[ \overset{\circ}{\mathcal{L}}_{F0} = \overset{\circ}{\mathcal{L}}_{F0} - \frac{1}{8} S_a J^a_{(A)} \epsilon. \]
The addition of a divergence of a vector field \( V \) to the flat space Lagrangean results in one more term \([IV.3]\). Ultimately, we have the following four–form on RC space representing EC gravity with fermions
\[ \mathcal{L} = \overset{\circ}{\mathcal{L}}_G + \overset{\circ}{\mathcal{L}}_{F0} + d(\ldots) + \frac{1}{2k} \left( \frac{2}{3} T_a T^a - \frac{1}{24} S_a S^a - \frac{1}{2} q_{abc} q^{abc} \right) - \frac{k}{4} S_a J^a_{(A)} - 2k T_a V^a \right) \epsilon. \]
Variation of the resulting action with respect to \( T_a, S_a \) and \( q_{abc} \) yields the equations
\[ T^a = \frac{3k}{2} V^a, \quad S^a = -3k J^a_{(A)}, \quad q_{abc} = 0. \]
Inserting these results into \([V.1]\), we finally get the effective Lagrangean four–form
\[ \mathcal{L}_{eff} = \overset{\circ}{\mathcal{L}}_G + \overset{\circ}{\mathcal{L}}_{F0} + \frac{3k}{16} \left( J^a_{(A)} J^a_{(A)} - 4V_a V^a \right) \epsilon. \]
The total differential has been omitted in the final formula. Note that beyond the well–known axial–axial interaction term \([28]\) we have an additional one due to the ambiguity \([III.4]\). In the case of \( V \) being a linear combination of axial and vector currents \([III.7]\) we have
\[ \mathcal{L}_{eff} = \overset{\circ}{\mathcal{L}}_G + \overset{\circ}{\mathcal{L}}_{F0} + \left( C_{AA} J^a_{(A)} J^a_{(A)} + C_{AV} J^a_{(A)} J^a_{(V)} + C_{VV} J^a_{(V)} J^a_{(V)} \right) \epsilon, \]
\[ C_{AA} = \frac{3k}{16} (1 - 4b^2), \quad C_{AV} = \frac{3k}{2} a b, \quad C_{VV} = -\frac{3k}{4} a^2, \]
where \( a, b \) are completely arbitrary real numbers! Hence, in the most generic case we have three types of possible contact interactions and we cannot claim that none of them is small. As far as we cannot eliminate the ambiguity \([III.4]\), there is nothing on the basis of which their smallness could be conjectured. We can only establish experimentally some bounds on the values of \( a \) and \( b \). To see how this could possibly be done, observe that the nonlinear equation for \( \psi \)
\[ \left( i \gamma^a \nabla_a - m \right) \psi + \left[ -2C_{AA} J^a_{(A)} \gamma^5 + C_{AV} \left( J^a_{(A)} - J^a_{(V)} \gamma^5 \right) + 2C_{VV} J^a_{(V)} \right] \gamma^a \psi = 0 \]
obtained from \( \mathbf{V.4} \) via variational procedure does not reduce to the usual Dirac equation in the limit of vanishing Riemannian curvature. For the space–time metric being flat, the first term of \( \mathbf{V.6} \) reduces to the usual Dirac one, but the remaining two preserve their forms. One could try to interpret physically the resulting equation \([3][27][4][3]\), aiming ultimately at measuring physical effects produced by \( a \) and \( b \) by flat space experiments.

It is worthy to make two further observations concerning EC theory with fermions. Firstly, a parity violating term in \( \mathbf{V.4} \) appeared. Secondly, for \( V = \frac{1}{2} J^2(A) \) \( a = 0, b = \frac{1}{2} \) interaction terms in \( \mathbf{V.3} \) cancel out and the effective action of EC theory appears to be the same as the usual GR one. However, we should not think prematurely of these two theories as being indistinguishable, since the EC one introduces nonvanishing torsion on space–time, \( T^a = \frac{3k}{2} J^a(A), S^a = -3k J^a(A) \), as follows from \( \mathbf{V.2} \) for \( V = \frac{1}{2} J^2(A) \). As for parity violation, one would avoid it by demanding the flat space Dirac Lagrangean density taken as starting point to be parity invariant. This would result in \( V \sim J^2(V) \) \( b = 0 \). In this case, \( \mathbf{V.3} \) is parity invariant and contains two interaction terms with axial–axial interaction having the fixed, well–known, small coupling constant and vector–vector one having unknown coupling constant (possibly high enough to be measurable in the near future). One could argue that the case \( V \sim J^2(V) \) is what we should expect, since introduction of \( V \) having axial or mixed axial–vector transformation properties seems rather unnatural. Although the effective action is parity invariant both in the case of \( V \) being vector, as well as an axial vector (but not their combination), the first equation of \( \mathbf{V.2} \) will not violate parity if and only if \( V \) is a vector (since \( T^a \) is).

B. Is it possible to distinguish between GR and EC by measuring the strength of fermion interactions?

In the previous section we have pointed out that we could choose between EC theory and GR by torsion measurements. However, it seems very difficult to measure torsion directly \([29]\). There is another, much more promising possibility. First of all, note that the Lagrangean four–form \( \mathbf{V.4} \) is relevant for both GR and EC theory, but for GR all coupling constants \( C_{AA}, C_{AV}, C_{VV} \) vanish, whereas for EC theory they are given by \( \mathbf{V.5} \). These constants are much more likely to be measurable than torsion, as they are responsible for the strength of corresponding point interactions between fermions. Let us imagine that we can separate the contributions coming from different types of these interactions in experiments that we perform. As long as we get values of all constants indistinguishable from zero, we are not able to say which of the two theories of gravitation is correct. Measuring a non–zero value of at least one of them would provide an argument against standard GR. Of course, for any values of \( C \)'s, one could produce the effective Lagrangean \( \mathbf{V.4} \) on the base of the torsionless approach of standard GR by simply adopting from the beginning

\[
\mathcal{L}_{F_{ab}} = \mathcal{L}_{F0} + C_{AA} J^2(A) J^a + C_{AV} J^2(A) J^a + C_{VV} J^2(V) J^a
\]

as flat space Lagrangean density for fermions. However, on the ground of EC theory the interaction terms arise naturally as a necessary consequence of the relation between torsion and matter, as explained in Subsection \[V.A\].

In this paper, we aim to treat both the theories in the most natural manner, adopting the simplest Dirac theory \[\mathcal{L}_{F0} \] as flat space Lagrangean for fermions. We shall assume that the test Dirac particle \( \psi \) has a negligible influence on space–time torsion, while compared to that of a background Dirac field \( \psi_{bg} \). Hence, \( \mathbf{V.1} \) is still valid, but whereas \( \psi \) appearing in \( \mathbf{L}_{F_{0}} \), \( J(A) \) and \( V \) is the test field whose dynamics we wish to describe, the torsion components are totally determined by \( \psi_{bg} \) via the formulas.

C. The background field approximation

It seems extremely difficult to extract any information about the physical nature of a gravity–induced fermion interaction without some simplifying assumptions. The background field approach suggests \([2][3]\) that in the case of \( C_{AA} = C_{AV} = C_{VV} \) being the flat space Lagrangean \( a = b = 0 \), the interaction is repulsive for particles with aligned spins, attractive for antiparallel spins and universal (independent on whether particles or antiparticles are considered). Let us investigate what will change if possibility of divergence addition to the flat Lagrangean is taken into account. We shall assume that the test Dirac particle \( \psi \) has a negligible influence on space–time torsion, while compared to that of a background Dirac field \( \psi_{bg} \). Hence, \( \mathbf{V.1} \) is still valid, but whereas \( \psi \) appearing in \( \mathbf{L}_{F_{0}} \), \( J(A) \) and \( V \) is the test field whose dynamics we wish to describe, the torsion components are totally determined by \( \psi_{bg} \) via the formulas.
For both the test and background field, the relation \( V = aJ(V) + bJ(A) \) holds for some real numbers \( a \) and \( b \) (necessarily the same in both cases, in order for our approach to be logically consistent – according to our viewpoint, the values of these constants are determined by the most reasonable choice of flat space Lagrangean for fermions). Explicitly, the Lagrangean four–form is now

\[
\mathfrak{L}_{bgApp} = \mathfrak{L}_G - \frac{3k}{16} \left( J_a^{(A)bg} J_a^{(B)bg} - 4V^{bg} V^{bg} \right) \epsilon + \frac{3k}{8} \left( J_a^{(A)bg} J_a^{(V)bg} - 4V^{bg} V^{bg} \right) \epsilon
\]  

(for total differential discarded). Only the two last terms depend on the test field and variation with respect to it yields a **linear** equation for \( \psi \). It looks the same as (V.6) with \( J^{(A)} \), \( J^{(V)} \) replaced by \( J^{(A)bg} \), \( J^{(V)bg} \). In the limit of the space–time metric being flat, it can be rewritten in the form

\[
i\partial_t \psi = \mathcal{H} \psi
\]  

for the Hamilton operator

\[
\mathcal{H} = -i\gamma^0 \gamma^j \partial_j + \mathcal{H} \, , \quad \mathcal{H} = m\gamma^0 - \left[ 2C_{AA} J_a^{(A)bg} \gamma^5 + C_{AV} \left( J_a^{(A)bg} + J_a^{(V)bg} \gamma^5 \right) + 2C_{VV} J_a^{(V)bg} \gamma^0 \right] \gamma^a \, , \quad j = 1, 2, 3 \, .
\]  

In the following we will refer to particles and antiparticles as electrons and positrons. From now on we shall adopt the Dirac representation for \( \gamma \)'s

\[
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \, , \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \, , \quad \gamma^5 = -i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \, ,
\]  

where \( \sigma^j \) are Pauli matrices. Following [4] and [3], we shall consider the background Dirac field of the form

\[
\psi_{bg} = \sqrt{n} e^{-imt} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \, , \quad n \in \mathbb{R}_+ \, ,
\]  

which simulates an ‘electron distribution of number density \( n \)’ [3] with their spins directed upwards along a fixed axes of quantisation. We will consider test particles at rest, \(-i\partial_t \psi = 0\), for which only \( \mathcal{H} \) part of \( \mathcal{H} \) is important. For the postulated background it equals

\[
\mathcal{H} = \begin{pmatrix} m + 2n(C_{AA} - C_{VV}) & 0 & 0 & 0 \\ 0 & m - 2n(C_{AA} + C_{VV}) & 0 & 0 \\ 0 & 0 & m - 4nC_{AA} + 2nC_{VV} & 0 \\ 0 & 0 & 0 & m - 4n(C_{AA} + C_{VV}) \end{pmatrix}
\]  

Let us recall that the theory does not break parity if and only if \( b = 0 \) (see [V.2]). In this case \( C_{AA} = \frac{3k}{16} \) and \( C_{VV} = -\frac{3k}{4} a^2 \) and the vectors

\[
\psi_1 = e^{-iE_1 t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \, , \quad \psi_\uparrow = e^{-iE_1 t} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \, , \quad \tilde{\psi}_1 = e^{-iE_\uparrow t} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \, , \quad \tilde{\psi}_\uparrow = e^{-iE_\uparrow t} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},
\]  

provide the set of eigenvectors of \( \mathcal{H} \). They solve equation (V.9) for \( E_1 = m + \frac{3}{8} kn(1 + 4a^2) \) , \( E_\uparrow = m - \frac{3}{8} kn(1 - 4a^2) \) , \( E_\downarrow = -m + \frac{3}{8} kn(1 + 4a^2) \) , \( E_\uparrow = -m - \frac{3}{8} kn(1 - 4a^2) \) being the corresponding eigenvalues of \( \mathcal{H} \). Hence, in the case of \( |a| < \frac{1}{2} \), the energies are shifted upwards for aligned spins and downwards for antiparallel spins, which agrees with the conclusions of [3][4] concerning attractivity and repulsivity, as well as universality of gravity–induced fermion interactions. Note, however, that the magnitude of interaction is different for aligned and antiparallel spins. For \( |a| = \frac{1}{2} \) the antiparallel spins seem not to interact at all, whereas for \( |a| > \frac{1}{2} \) all interactions are repulsive. It is important to note that the independence of results from whether particles or antiparticles are considered remains preserved in all cases, which allows us to believe in a genuine gravitational nature of the investigated phenomenon. One can easily find out that our conclusions do not change under the replacement of \( \psi_{bg} \) by \( \psi_{bg1} \), \( \tilde{\psi}_{bg1} \) or \( \tilde{\psi}_{bg1} \), which is an indispensable consistency test of our reasoning.

If \( b \neq 0 \), we have necessarily \( C_{AV} \neq 0 \) and parity symmetry is broken. Although the third spatial component of spin operator \( S^z = \frac{1}{2} \Sigma^{12} \) still commutes with \( \mathcal{H} \) and one can construct the basis of \( \mathbb{C}^4 \) from common eigenvectors of...
\( \tilde{H} \) and \( S^3 \), it is difficult to say which of them describe electrons and which correspond to positrons, as they are not in general of the form \( \left( \begin{array}{c} \kappa \\ \eta \end{array} \right) \) or \( \left( \begin{array}{c} 0 \\ \chi \end{array} \right) \) in Dirac representation, nor are their energies of the form \( m + \delta E \) or \( m - \delta E \), for \( \delta E \) independent from \( m \).

It should be stressed that the background field approach is an approximate technique, not only because of neglecting the influence of test field on space–time geometry and all matter fields on the space–time metric, but also because the Dirac field was not second quantised, which could significantly change our conclusions as it does in the standard \( a = b = 0 \) case \([5]\).

**VI. CONCLUSIONS**

Minkowski space Lagrangean densities differing by divergence give rise to generically nonequivalent theories, when Einstein–Cartan gravity is minimally included. Hence, it is important to choose carefully among equivalent flat space Lagrangeans. In the case of the Dirac field, a very natural requirement for Lagrangean density to be real, invariant under proper Lorentz transformations and not dependent on higher derivatives or higher powers of fields, leads to the two–real–parameter family \([V.4]\) of EC theories for gravity with fermions. There are some plausible arguments in favour of choosing particular values of the parameters (namely \( a = b = 0 \)), motivated by the resulting form of the spin density tensor. This choice leads to the theory discussed in the earlier papers. However, these arguments are not completely convincing. Rejecting them, we are obliged to take the entire two–parameter family seriously. Then, the gravity–induced fermion interaction acquires a three–fold structure described by the axial–axial, vector–vector and axial–vector term appearing in the effective action \([V.4]\). The strength of these constituent parts of the total interaction is governed by the coupling constants, which depend on the parameters of the theory \([V.5]\). Such values of the parameters can be chosen that the interaction do not occur at all and thus EC theory is indistinguishable from the standard GR on the level of effective action. The theories still differ by the presence of nonzero torsion in EC theory. A reasonable requirement for the flat fermionic Lagrangean density to be parity invariant reduces the number of parameters to one. The resulting EC theory is then parity invariant and differs from GR on the effective level. Although the axial–axial constituent of the fermion interaction has now a well–known, small coupling constant, the strength of the vector–vector part of the interaction is not restricted and may achieve significant values, possibly leading to the effects observable in practice. The background field approximation analysis suggests that the vector–vector constituent of the interaction is repulsive, for both aligned and antiparallel spins of particles. If significantly strong, it will dominate over the axial–axial part, making all fermions to repulse each other. Hence, the possibility of modifying EC theory with fermions by the freedom of adding a divergence to the flat fermionic Lagrangean can lead to the meaningful, interesting and new physical effects that should not be disregarded until some reasonable procedure for restricting this freedom is proposed and commonly accepted.

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**VII. APPENDIX: NOTATION AND CONVENTIONS**

Throughout the paper \( a, b, \ldots \) are orthonormal tetrad indexes and \( \mu, \nu, \ldots \) correspond to a holonomic frame. For inertial frame of flat Minkowski space, which is both holonomic and orthonormal, we use \( \mu, \nu, \ldots \). The metric components in an orthonormal tetrad basis \( \tilde{\epsilon}_a \) are \( g (\tilde{\epsilon}_a, \tilde{\epsilon}_b) = (\eta_{ab}) = \text{diag}(1, -1, -1, -1) \). Lorentz indexes are shifted by \( \eta_{ab} \). \( \epsilon = \epsilon^0 \wedge \epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3 \) denotes the canonical volume four–form whose components in orthonormal tetrad basis obey \( \epsilon_{0123} = -\epsilon^{0123} = 1 \). The action of a covariant exterior differential \( D \) on any \( (r, s) \)-tensorial type differential \( m \)-form

\[
T^{a_1 \ldots a_r b_1 \ldots b_s} = \frac{1}{m!} T^{a_1 \ldots a_r} b_1 \ldots b_s \mu_1 \ldots \mu_m \, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_m}
\]

is given by

\[
DT^{a_1 \ldots a_r b_1 \ldots b_s} := dT^{a_1 \ldots a_r b_1 \ldots b_s} + \sum_{i=1}^{r} \omega^{a_1 \ldots a_r} b_1 \ldots b_s \wedge T^{a_1 \ldots c \ldots a_r b_1 \ldots b_s} - \sum_{i=1}^{s} \omega^{c \ldots a_r b_1 \ldots c \ldots b_s} \wedge T^{a_1 \ldots a_r b_1 \ldots c \ldots b_s}.
\]
The hodge star action on external products of orthonormal cotetrad one-forms is given by

\[ \star e_a = \frac{1}{3!} e_{abcd} e^b \wedge e^c \wedge e^d , \quad \star (e_a \wedge e_b) = \frac{1}{2!} e_{abcd} \wedge e^d , \quad \star (e_a \wedge e_b \wedge e_c) = \epsilon_{abcd} e^d , \]

which by linearity determines the action of \( \star \) on any differential form.

### VIII. APPENDIX: NOETHER THEOREM

Let \( S[\phi^A] = \int \mathcal{L}(\phi^A, \partial_\mu \phi^A) \, d^4x \) represent the action of a field theory in Minkowski space \( \mathbb{M} \). Consider a Lie group \( \mathcal{G} \) which acts on space–time, as well as a target space \( \mathbb{T} \) in which the fields \( \phi^A \) take their values, as a group of transformations. Let

\[
\begin{align*}
\mu \rightarrow \mu' = \mu + \delta \mu , \\
\phi^A \rightarrow \phi'^A = \phi^A + \delta \phi^A 
\end{align*}
\]

represent the infinitesimal form of the action of \( \mathcal{G} \) on \( \mathbb{M} \) and \( \mathbb{T} \) respectively. The transformations are symmetry transformations of the theory if they do not change the action, up to possibly surface terms (and thus leave the form of field equations invariant). This is equivalent to the condition

\[
\mathcal{L}(\phi'^A, \partial_\mu \phi'^A) \, d^4x' = \mathcal{L}(\phi^A, \partial_\mu \phi^A) \, d^4x + \partial_\mu W^\mu d^4x 
\]

for some vector field \( W \). Note that in our approach, \( \phi^A \) transform only under the action of \( \mathcal{G} \) in target space – the coordinates \( x^\mu \) ‘hidden’ in \( \phi^A \) do not undergo any transformation. For infinitesimal transformations we have

\[
\mathcal{L}(\phi'^A, \partial_\mu \phi'^A) \approx \mathcal{L}(\phi^A, \partial_\mu \phi^A) + \frac{\partial \mathcal{L}}{\partial \phi^A} \delta \phi^A + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^A)} (\partial_\mu \delta \phi^A - \partial_\nu \phi^A \partial_\mu \delta x^\nu) ,
\]

\[
d^4x' \approx d^4x + \partial_\mu \delta x^\mu d^4x
\]

and (VIII.2) appears to be equivalent to

\[
\frac{\partial \mathcal{L}}{\partial \phi^A} \delta \phi^A + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^A)} (\partial_\mu \delta \phi^A - \partial_\nu \phi^A \partial_\mu \delta x^\nu) + \mathcal{L} \partial_\mu \delta x^\mu = \partial_\mu W^\mu
\]

which can be finally expressed in the form

\[
\partial_\mu j^\mu = \left( \frac{\partial \mathcal{L}}{\partial \phi^A} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^A)} \right) \left( \partial_\nu \phi^A \delta x^\nu - \delta \phi^A \right) ,
\]

where

\[
j^\mu = -t_\nu \delta x^\nu + \partial_\nu \delta \phi^A - W^\mu , \quad t_\nu^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^A)} \partial_\nu \phi^A - \delta_\nu^\mu \mathcal{L}
\]

is a Noether current associated to the symmetry transformation (VIII.1), which is clearly conserved, i.e. \( \partial_\mu j^\mu = 0 \), if the Euler–Lagrange equations for fields are satisfied. Note that in the case of transformations which do not change the volume–form \( d^4x \), such as space–time translations and Lorentz transformations, the condition for them to be symmetries (VIII.2) is fulfilled if and only if the Lagrangean density changes by a divergence only. For simplicity, in Section III of this paper we consider only the situation when \( \mathcal{L} \) is left just invariant by Lorentz transformations (invariance under space–time translations is obvious, since Lagrangean densities under consideration do not depend explicitly on \( x \)). In particular, this is the case for (III.11) and all Lagrangean densities related to (III.11) via (III.4). The possibility of construction of Lagrangean densities violating such invariance by the addition of a divergence of a vector field having weird transformation properties is not interesting in the context of this paper. What we wish to do is to limit the multiplicity of equivalent flat space Lagrangean densities. The requirement of Lorentz invariance should be understood as the first restriction that we impose on them in order to find the most appropriate one.
Some references (e.g. [30]) adopt a different approach, in which the derivatives \( \partial_a \) and measure \( \text{d}^4 x \) remain untouched by the transformation. Instead, \( x^\mu \) that is hidden in the fields \( \phi^A \) does transform. Hence, \( \phi^A (x^\mu) \) transforms under the combined action of \( G \) on both \( M \) and \( T \):

\[
\phi^A (x^\mu) \to \phi'^A (x'^\mu) = \phi^A (x^\mu - \delta x^\mu) + \delta \phi^A (x^\mu - \delta x^\mu) \approx \phi^A - \partial_\mu \phi^A + \delta \phi^A = \phi^A + \tilde{\delta} \phi^A ,
\]

\[
\tilde{\delta} \phi^A = \delta \phi^A - \partial_\mu \phi^A \delta x^\mu .
\]

The condition for the transformation to be a symmetry is now

\[
\frac{\partial L}{\partial \phi^A} \tilde{\delta} \phi^A + \frac{\partial L}{\partial (\partial_\mu \phi^A)} \partial_\mu \tilde{\delta} \phi^A = \partial_\mu \tilde{W}^\mu .
\]

One can easily find this condition to be equivalent to (VIII.3) by putting \( \tilde{W}^\mu = W^\mu - L \delta x^\mu \). Note however that Lagrangean densities that remain invariant under Lorentz transformations according to the first viewpoint acquire an additional divergence term \( \partial_\mu (L \delta x^\mu) \) according to the second interpretation. This is why some references [30] claim the Poincaré transformations to change the simplest Lagrangean densities of field theory, which is not the case in the approach adopted here.

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