1. INTRODUCTION

This paper rests on three, a priori quite distinct domains, namely, noncommutative geometry, stochastic calculus (StC) and symplectic mechanics (SyM). Hence we give below a very brief outline of some basic facts concerning StC and SyM.

StC is motivated by the desire to put in a firm mathematical basis, physically relevant but ill-defined differential equations (e.g. Langevin’s equation in Brownian motion), that finally have been interpreted as stochastic differential equations. Accordingly, the appropriate tool for their study has been developed, namely, stochastic integration of Itô, Stratonovich and others (e.g. [1], [2], [3]). The essential point to be emphasized in this context, is that stochastic differentiation involves 2nd order derivatives, e.g. in Itô’s formula (in its simplest form) for the differential of a function $f$ of a Wiener stochastic process $X_t$ (e.g. [1] §5.3, [3] §4.3)

$$df(X_t,t) = dt(\partial_t f + \frac{\gamma}{2}\partial_x^2 f) + dX_t \partial_x f.$$  (1.1)

It is a general feature of StC that stochastic differentiation obeys a generalized Leibniz rule

$$d(X_t Y_t) = dX_t Y_t + dY_t X_t + d[X_t, Y_t]$$  (1.2)

(e.g. [4] p.134), where the 3rd term is the so-called bracket of the processes (semimartingales), related to the quadratic variation of a process $X_t$

$$[X_t, X_t] = \lim_{k \to \infty} \sum_k (X_{t_k} - X_{t_{k-1}})^2$$  (1.3)

the limit being taken with respect to the width of the partition of the (time) interval, where $t$ varies.

There have been efforts to extend these considerations to a semimartingale theory on differentiable manifolds, by appropriately introducing basic concepts like vector field, connection etc. ([4], [5]). We will come back to this later on.

In SyM we consider an even-dimensional manifold $M$ extended by $\mathbb{R}$, $(M \times \mathbb{R}, \omega)$, where $\omega$ is a closed 2-form of maximal rank

$$\omega = \frac{1}{2} dx_i \wedge dx_j \omega_{ij} + dt \wedge dH ,$$  (1.4)

that is $\omega_{ij}$ is nondegenerate and $H$ is a function. Then $\omega$ has a 1-dimensional kernel, which by definition gives the Hamiltonian vector fields $X_H$ using the insert operator

$$X_H : \quad X_H \cdot \omega = 0 .$$  (1.5)

*In physics $M \times \mathbb{R}$ is the extended phase-space of a physical system and $H$ its Hamiltonian.
Then Hamilton’s equations are given by the action of $X_H$ on a smooth function $A$ defined on $M \times \mathbb{R}$

$$X_H(A) = 0 \iff \partial_t A = -\{H, A\} ,$$

\[ (1.6) \]

$\{,\}$ being the Poisson bracket of two functions. This is the starting point of both Hamiltonian dynamics (e.g. [6] ch.9) and statistical mechanics. In the latter, making approximations based on iteration schemes applied to (1.6), kinetic equations are derived, giving the (probabilistic) time evolution of physical quantities in a system with many degrees of freedom (e.g. [7] ch.II). These are often generalized diffusion (or Fokker-Planck) equations (e.g. [8], [9] §2)

$$\partial_t A = -\{H_0, A\} + \partial_i (a^{ij} \partial_j A) - \beta a^{ij} (\partial_j H_0) \partial_i A ,$$

\[ (1.7) \]

summation convention always assumed\[†\]. Let us notice here that it is possible to derive kinetic equations via StC, however at the expense of modifying Hamiltonian dynamics ([9] Section 3, [3] §4.3). In fact a common characteristic of both statistical mechanics and StC is the appearance of 2nd order differential operators that cannot be given easily a geometrical meaning in a natural way. Therefore, our objective is to formulate an appropriate geometric framework for this to be possible and develop SyM in it by retaining the Hamiltonian scheme and making only a minimal modification of the differential structure of the manifold. Specifically we shall retain the manifold structure as this is encoded in the ordinary algebra of smooth functions on it and modify the differential calculus (DC). This leads us to noncommutative geometry and we shall see that tensor calculus and SyM can be developed in this framework as a direct extension of what is known in ordinary differential geometry.

## 2. NONCOMMUTATIVE DIFFERENTIAL CALCULUS ON MANIFOLDS\[‡\]

We consider an $(N + 1)$-dimensional manifold $M \times \mathbb{R}$ and the ordinary (commutative) algebra of (smooth) functions on it, $\mathcal{A}$. The universal differential enveloppe of $\mathcal{A}$ is an $\mathbb{N}_0$-graded algebra $\Omega_u = \bigoplus_{k \in \mathbb{N}_0} \Omega_u^k$ with $\Omega_u^0 = \mathcal{A}$ equipped with an exterior derivative operator $d_u$, satisfying well-known axioms (e.g. [10])

- $d_u 1 = 0$, $d_u^2 = 0$.
- $d_u (\psi \psi') = (d_u \psi) \psi' + (-1)^k \psi (d_u \psi')$, $\psi \in \Omega_u^k$.
- $\mathcal{A}$ and $d_u \mathcal{A}$ generate $\Omega_u$.

A simple representation of forms $\phi \in \Omega_u^n$, as functions on $(M \times \mathbb{R})^{r+1}$ is given by ([11])

$$\phi(x_0, \ldots, x_{r+s}) = \phi(x_0, \ldots, x_r) \psi(x_{r+1}, \ldots, x_{r+s}) ,$$

$$\phi(x_0, \ldots, x_{r+1}) = \sum_{k=0}^{r+1} (-1)^k \phi(x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{r+1}) ,$$

\[†\]The terminology stems from the fact that in the case of Brownian motion, time evolution is given by the superposition of a diffusion process and a systematic drift (friction) term — 2nd and 3rd term in (1.7) respectively (cf. [3] ch.I).

\[‡\]Lack of space does not allow for detailed proofs, which will be presented elsewhere.
ψ ∈ Ω^s_u. Then we readily get \((d_u g)f \neq gd_u f\) in general, hence the name noncommutative DC. Universality of \((Ω_u, d_u)\) means that for any other DC \((Ω, d)\) over \(A\), it can be proved that there exists a unique graded algebra homomorphism

\[
\pi : Ω_u \rightarrow Ω, \quad \pi|_A = \text{id}_A, \quad d \circ \pi = \pi \circ d_u
\]

(2.1)

conserving the grade of forms. It is possible to make 1-forms \(Ω^1_u\) a commutative, associative algebra, by defining

\[
(\alpha \bullet_u \beta)(x, y) := -\alpha(x, y) \beta(x, y).
\]

(2.2)

This implies

\[
(d_u f) \bullet_u \alpha = [f, \alpha] := f\alpha - \alpha f, \quad f \in A.
\]

(2.3)

Thus \(\bullet_u\) measures deviations from ordinary DC due to noncommutativity. In what follows we consider a minimal deformation of it, namely a DC \((Ω, d)\) on \(M \times \mathbb{R}\) in which

\[
df \bullet dg \bullet dh = 0, \quad dt \bullet df = 0, \quad f, g, h \in A.
\]

(2.4)

Such a DC can be obtained by taking the quotient of \(Ω_u\) by the differential ideal \(J\), generated by

\[
d_u f \bullet_u d_u g \bullet_u d_u h, \quad d_u t \bullet_u d_u f, \quad f, g, h \in A,
\]

that is

\[
\pi : Ω_u \rightarrow Ω = Ω_u/J, \quad \pi(\alpha) \bullet \pi(\beta) = \pi(\alpha \bullet_u \beta),
\]

(2.5)

\(\pi\) being the canonical projection of \(Ω_u\) on \(Ω\). By (2.4)

\[
d(fg) = (df)g + (dg)f + df \bullet dg
\]

(2.6)

formally identical to (1.2) with \(df \bullet dg\) corresponding to \(d[f, g]\) (see also section 5 below).

\(Ω^1\) is an \(A\)-bimodule. Then it is natural to define the space of vector fields \(X\) as its dual \(A\)-bimodule\(^3\)

\[
\langle fXh, \alpha \rangle = f\langle X, h\alpha \rangle, \quad f, h \in A, \quad X \in X, \quad \alpha \in Ω^1.
\]

(2.7)

\(\langle , \rangle\) denoting duality and the action of \(X\) on \(A\) is naturally given by

\[
X(f) := \langle X, df \rangle
\]

(2.8)

implying

\[
(fX)(g) = fX(g), \quad (Xf)(g) = X(fg) - X(f)g.
\]

(2.9)

It can be shown that \(X\) is characterized by \(X \in X \iff X(fgh) = fX(gh) + gX(fh) + hX(fg) - fgX(h) - fhX(g) - ghX(f)\)

(2.10)

\(^3\)We define \(X\) as the left \(A\)-module, dual to \(Ω^1\) taken as a right \(A\)-module only. The full \(A\)-bimodule structure of \(X\) is then defined by (2.7).
a condition which is equivalent to
\[ X(f^3) = 3fX(f^2) - 3f^2X(f) \, . \] (2.11)
Moreover \[ \text{[2.4]} \] gives
\[ X(tf) = tX(f) + fX(t) \, . \] (2.12)
Duality and \[ \text{[2.10], [2.11]} \] give in local coordinates \((x^\mu, t)\)
\[ X = X^t \partial_t + X^\mu \partial_\mu + \frac{1}{2} X^\mu_\nu \partial_\mu \partial_\nu \, , \] (2.13)
\[ df = dt \partial_t f + dx^\mu \partial_\mu f + \frac{1}{2} \xi^\mu_\nu \partial_\mu \partial_\nu f \, , \quad \xi^\mu_\nu := dx^\mu \bullet dx^\nu \] (2.14)
in an obvious notation. Because of \[ \text{(2.13), (2.14)} \] we may call \((\Omega, d)\) a 2nd order DC. Moreover \[ \text{[2.14]} \] reduces formally to \[(1.1)\] in 1-dimension, if \(\xi^\mu_\nu\) is proportional to \(dt\) (see section 4 below). Thus on \(\mathcal{X}, \Omega^1\) we have respectively noncovariant bases \(\{\partial_t, \partial_\mu, \partial_\mu \partial_\nu\}\), \(\{dt, dx^\mu, \xi^\mu_\nu\}\). To get covariant expressions, the concept of a connection has to be introduced.

3. CONNECTIONS

By considering the middle \(A\)-linear tensor product \(\otimes\) of \(\Omega^1\) with any \(A\)-bimodule \(M\) (see e.g. \[ \text{[1]} \] §3.7) it is possible to define a (right) connection and on its dual \(M^*\), a (left) connection
\[ \nabla : M \rightarrow M \otimes \Omega^1 : \quad \nabla(\mu f) = (\nabla \mu)f + \mu \otimes df \, , \] (3.1)
\[ \nabla : M^* \rightarrow \Omega^1 \otimes M^* : \quad \nabla(fm) = f(\nabla m) + df \otimes m \, , \] (3.2)
so that
\[ d\langle m, \mu \rangle = \langle \nabla m, \mu \rangle + \langle m, \nabla \mu \rangle \, . \] (3.3)

\(\nabla\) can be extended to tensor fields, though the details will not be given here. Moreover defining
\[ \nabla(\mu \otimes \psi) := \mu \otimes d\psi + (\nabla \mu)\psi \, , \] (3.4)
\[ \nabla(\psi \otimes m) := d\psi \otimes m + (-1)^r \psi(\nabla m) \, , \] (3.5)
\(\nabla\) is extended to \(M \otimes \Omega, \Omega \otimes M^*\). For \(M = \Omega^1, M^* = \mathcal{X}\) curvature and torsion are defined by
\[ \nabla^2 : \Omega^1 \rightarrow \Omega^1 \otimes \Omega^2 \, , \quad \Theta : \Omega^1 \rightarrow \Omega^2 : \quad \Theta(\alpha) := d\alpha + \pi(\nabla \alpha) \, , \] (3.6)
\(\pi\) being given by \[ \text{[2.5]} \].

It is worth noting that in ordinary DC, there is a basis of 1-forms, \(\{dx^\mu\}\) so that \(d(dx^\mu) = 0\). This implies that torsion is completely reducible to the connection. However in 2nd order DC this is not true, since in general there is no coordinate system where \(d\xi^\mu_\nu\) vanish. Below we shall see that a covariant basis of \(\Omega^2\) includes components of \(\Theta\) irreducible to \(\nabla\).

*In StC on manifolds, \[ \text{[2.11]} \] has been used as a characterization of vector fields (\[ \text{[4]} \] Lemma 6.1).
Further progress follows by extending the •-product to $\Omega^2$ and $\Omega^1 \otimes \Omega^1$

\[
(\alpha \otimes \beta) \cdot \gamma := \alpha \otimes (\beta \cdot \gamma) \quad , \quad (\alpha \otimes \beta) \cdot (\gamma \otimes \delta) := (\alpha \cdot \gamma) \otimes (\beta \cdot \delta) \quad ,
\]

\[
\omega \cdot (gdf \ h) := (gdf \ h) \cdot \omega := g[f, \omega]h \quad , \quad (\alpha \beta) \cdot (\gamma \delta) := (\alpha \cdot \gamma)(\beta \cdot \delta) \quad ,
\]

\[
(\alpha \otimes \beta) \cdot (\gamma \otimes \delta) := (\alpha \cdot \gamma) \otimes (\beta \cdot \delta) \quad ,
\]

$\alpha, \beta, \gamma, \delta \in \Omega^1, \omega \in \Omega^2, f, g, h \in \mathcal{A}$. Then it is possible to obtain a right $\mathcal{A}$-linear product of 1-forms through

\[
\alpha \circ \beta := \alpha \beta - \pi(\nabla \alpha \cdot \beta)
\]

since by noncommutativity the product in $\Omega$ in general is not right $\mathcal{A}$-linear in both factors $(\alpha f)\beta \neq \alpha \beta f$. Then a lengthy calculation gives the identity

\[
\Theta(\alpha \cdot \beta) = \alpha \circ \beta + \beta \circ \alpha - \alpha \cdot \Theta(\beta) - \Theta(\alpha) \cdot \beta - \Theta(\alpha) \cdot \Theta(\beta) + \pi B(\alpha, \beta) \quad ,
\]

where

\[
B(\alpha, \beta) := \nabla(\alpha \cdot \beta) - \alpha \cdot \nabla \beta - \nabla \alpha \cdot \beta + (\nabla \alpha) \cdot (\nabla \beta)
\]

is a tensor field.

Motivated by (3.10) an antisymmetric and right $\mathcal{A}$-linear wedge product $\wedge$ is defined by

\[
\alpha \wedge \beta := \alpha \circ \beta + \frac{1}{2}[\Theta(\alpha) \cdot \beta + \alpha \cdot \Theta(\beta)] + \pi B(\alpha, \beta) \quad ,
\]

an indispensable tool for SyM and for obtaining a covariant basis of $\Omega^2$ as well.

We first find such a basis of $\Omega^1$, by noticing that $\Omega^1 \cdot \Omega^1 =: \Omega^1_2$ is an ideal of $(\Omega^1, \cdot)$. Its annihilator in $\mathcal{X}$ is a submodule $\mathcal{X}_1$ of $\mathcal{X}$, that can be shown to consist of all derivations of $\mathcal{A}$. Thus, by introducing complementary projections $p_1, p_2$ on $\Omega^1$ and their duals $p^*_1, p^*_2$ on $\mathcal{X}$ we write

\[
\Omega^1 = \Omega^1_1 \oplus \Omega^1_2 \quad , \quad \mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2 \quad , \quad \Omega^1_i = p_i(\Omega^1) \quad , \quad \mathcal{X}_i = p^*_i(\mathcal{X})
\]

Writing

\[
p_1(dx^\mu) = dx^\mu + \frac{1}{2} \xi^{\rho \sigma} P^\mu_{\rho \sigma} =: \tilde{dx}^\mu
\]

we get

\[
p^*_2(\partial_\mu \partial_\nu) = \partial_\mu \partial_\nu - P^\rho_{\mu \nu} \partial_\rho =: \tilde{\partial}_{\mu \nu}
\]

and from this, that $P^\rho_{\mu \nu}$ are the components of an ordinary, symmetric (torsionless) connection, hence \{dt, $\tilde{dx}^\mu$, $\xi^{\mu \nu}$\}, \{dt, $\partial_\mu$, $\tilde{\partial}_{\mu \nu}$\} are covariant bases of $\Omega^1, \mathcal{X}$ respectively. Then

\[
df = dt \partial_t f + \tilde{dx}^\mu \partial_\mu f + \frac{1}{2} \xi^{\mu \nu} \tilde{\partial}_{\mu \nu} f
\]

Projections $p^*_i$ correspond to the introduction of a connection in StC on manifolds, where it is defined as a mapping ([4] p.32) $\Gamma : \mathcal{A} \rightarrow$ symmetric bilinear forms on the manifold

\[
\Gamma(f^2) = 2f \Gamma(f) + 2df \otimes df
\]
In fact $\Gamma(f) = p_2(df)$, where we identify $\xi^{\mu\nu}$ with the symmetrized $dx^\mu \otimes dx^\nu$. Equivalently, $p_1^*$ is a mapping of 2nd order differential operators to derivations, its kernel $p_2^*(\mathcal{X})$ giving the distribution of horizontal subspaces on the manifold (\ref{section3}).

A deeper look at the concept of a connection, presupposes the following remarks: In analogy with\ref{section3}, we require\ref{section3} (\ref{section3}), \ref{section3}\ref{section3}, \ref{section3}\ref{section3} and the 1st term in (\ref{section3}), \ref{section3}\ref{section3} the remaining parts can be expressed via $K, L, A, B$ and the 1st term in (\ref{section3}) which represents a 1st order connection. More explicitly, we write

\begin{align}
\nabla(d\bar{x}^\mu) &= -\bar{d}x^\nu \otimes \bar{d}x^\nu \Gamma^\mu_{\rho\nu} - \frac{1}{2} \bar{d}x^\nu \otimes \xi^{\rho\sigma} \Gamma^\mu_{\rho\sigma\nu} - \frac{1}{2} \xi^{\rho\sigma} \otimes \bar{d}x^\nu \Gamma^\mu_{\rho\sigma\nu} - \frac{1}{4} \xi^{\rho\sigma} \otimes \xi^{\kappa\lambda} \Gamma^\mu_{\kappa\lambda\rho\sigma}, \\
\nabla(\xi^{\mu\nu}) &= -\bar{d}x^\rho \otimes \bar{d}x^\sigma \Gamma^\mu_{\sigma\rho} - \frac{1}{2} \bar{d}x^\rho \otimes \xi^{\kappa\lambda} \Gamma^\mu_{\kappa\lambda\rho} - \frac{1}{2} \xi^{\rho\sigma} \otimes \bar{d}x^\kappa \Gamma^\mu_{\kappa\rho\sigma} - \frac{1}{4} \xi^{\rho\sigma} \otimes \xi^{\kappa\lambda} \Gamma^\mu_{\kappa\rho\sigma},
\end{align}
where \( t \) is considered as the \((N+1)\)-coordinate and an index of the form \( \tilde{\sigma} \) is symmetric and corresponds to a \( \xi^{\mu\rho} \)-component. Then lengthy calculations show that all \( \Gamma \)'s are expressed via \( \Gamma_{\alpha\nu}^\mu \) and the components of \( A, B, K, L \). The expressions are complicated and will be given elsewhere.

Finally the torsion-like field

\[
S(\alpha) := (d \lrcorner + \nabla \lrcorner) p_1(\alpha) \tag{3.23}
\]

has components (cf. (3.13))

\[
S_{\alpha}^\mu \sim = -\frac{1}{2} \Gamma_{(\alpha\beta)}^\mu + P_{\alpha\beta}^\mu. \tag{3.24}
\]

Thus we conclude: (a) pure connection is ordinary connection \( \Gamma_{\alpha\beta\gamma} \) only, all corrections due to 2nd order DC being the tensor fields \( K, L, B, A \), (b) any connection induces a projection on \( \Omega^1 \), by taking \( S = 0 \), (c) in view of (a), (b), and making a minimal deformation of ordinary differential geometry, we make the minimal choice

\[
K = L = B = A = S = 0. \tag{3.25}
\]

To simplify the calculations we further assume that \( \Gamma_{\alpha\beta\gamma} = \Gamma_{\alpha\gamma\beta} \).

To get a covariant basis of \( \Omega^2 \), we notice that

\[
\{ \tilde{\tilde{dx}}^\mu \circ \tilde{\tilde{dx}}^\nu, \tilde{\tilde{dx}}^\mu \circ \xi^{\rho\sigma}, \xi^{\mu\nu} \circ \tilde{\tilde{dx}}^\rho, \xi^{\mu\nu} \circ \xi^{\rho\sigma} \}
\]

together with \( dt \tilde{\tilde{dx}}^\mu, dt \xi^{\mu\nu} \) form a covariant set spanning \( \Omega^2 \). However it is not a linearly independent set, because by applying \( d \) to (2.4) it can be shown that the following relations hold

\[
\begin{aligned}
d\xi^{\mu\nu} &= [dx^\mu, dx^\nu]_+, \\
dt &= 0, \\
dt = 0, \\
\end{aligned}
\tag{3.26}
\]

where \([.,.]_+\) denotes the anticommutator with respect to the graded product. Moreover no other 2-form relations exist. Putting

\[
\Theta^\mu := \Theta(\tilde{\tilde{dx}}^\mu), \quad \Theta^{\mu\nu} := \Theta(\xi^{\mu\nu})
\]

we can show that the torsion-components irreducible to the connection are

\[
\Theta^{\mu\nu}, \quad \Theta^{\mu[\nu} \bullet \Theta^{\rho]\sigma}, \quad \Theta^{\mu[\nu} \bullet \tilde{\tilde{dx}}^\rho] .
\]

These, together with \( \tilde{\tilde{dx}}^\mu \wedge \tilde{\tilde{dx}}^\nu, \tilde{\tilde{dx}}^\mu \wedge \xi^{\rho\sigma}, \xi^{\mu\nu} \wedge \xi^{\rho\sigma}, dt \wedge \tilde{\tilde{dx}}^\mu, dt \wedge \xi^{\mu\nu} \) form a covariant basis of \( \Omega^2 \) via the relations implied by (3.12)

\[
\begin{aligned}
\tilde{\tilde{dx}}^\mu \wedge \tilde{\tilde{dx}}^\nu &= \tilde{\tilde{dx}}^\mu \circ \tilde{\tilde{dx}}^\nu - \frac{1}{2} \Theta^{\mu\nu} + \frac{1}{2} \Theta^{(\mu} \bullet \tilde{\tilde{dx}}^{\nu)} , \\
\tilde{\tilde{dx}}^\mu \wedge \xi^{\rho\sigma} &= \tilde{\tilde{dx}}^\mu \circ \xi^{\rho\sigma} + \frac{1}{6} (\Theta^{[\rho\sigma} \bullet \tilde{\tilde{dx}}^{\mu]} + \Theta^{\sigma]\rho} \bullet \tilde{\tilde{dx}}^{\mu]} , \\
\xi^{\mu\nu} \wedge \xi^{\rho\sigma} &= \xi^{\mu\nu} \circ \xi^{\rho\sigma} + \frac{1}{6} \Theta^{\mu[\nu} \bullet \Theta^{\rho]\sigma} + \Theta^{\mu[\nu} \bullet \Theta^{\sigma]\rho} .
\end{aligned}
\tag{3.28}
\tag{3.29}
\tag{3.30}
\]

where we can find that

\[
\Theta^\mu \bullet \tilde{\tilde{dx}}^\nu = \frac{1}{2} (\xi^{\rho\sigma} \wedge \xi^{\alpha\beta} + \frac{1}{12} \Theta^{\nu[\rho} \bullet \Theta^{\rho]\alpha} \Theta^{\beta]\sigma}) R_\alpha^\mu \beta\rho , \tag{3.31}
\]


$R^\mu_{\alpha\beta\rho}$ being the Riemann tensor of $\Gamma^\alpha_{\beta\gamma}$.

4. SYMPLECTIC MECHANICS

Motivated by the discussion in section 1 (cf. (2.14), (1.1)), we consider the special case $N = 2n$ and

$$\xi^{\mu\nu} = -dt b^{\mu\nu},$$  \hspace{1cm} (4.1)

$b^{\mu\nu}$ being a symmetric matrix function (under (4.1), (2.14) reduces formally to (1.1)). Application of the results of section 3 gives

$$\Theta^{\mu\nu} \cdot \Theta^{\rho\sigma} = \Theta^{\mu\nu} \cdot \tilde{dx}^\rho = 0,$$  \hspace{1cm} (4.2)

$$\Theta^\mu = \frac{1}{2} dx^\mu dt b^{\alpha\beta} R^\mu_{\alpha\beta\rho}, \hspace{1cm} \Theta^{\mu\nu} = dtdx^\kappa \nabla^\kappa b^{\mu\nu}$$  \hspace{1cm} (4.3)

so that

$$\tilde{dx}^\mu \wedge dt = dx^\mu dt,$$  \hspace{1cm} (4.4)

$$\tilde{dx}^\mu \wedge \tilde{dx}^\nu = \tilde{dx}^\mu \tilde{dx}^\nu + dtdx^\kappa (b^{\nu\alpha} \Gamma^\mu_{\kappa\alpha} - \frac{1}{2} \nabla^\kappa b^{\mu\nu})$$  \hspace{1cm} (4.5)

form a basis of $\Omega^2$ — here $\nabla^\kappa$ is defined by $\Gamma^\alpha_{\beta\gamma}$. Moreover, $\wedge$ can be extended to any forms in $\Omega$ (eq.(4.25)). Then writing for any $\omega \in \Omega^2$

$$\omega = \frac{1}{2} \tilde{dx}^\mu \wedge \tilde{dx}^\nu \omega_{\mu\nu} + dtdx^\mu \omega_{\mu}$$  \hspace{1cm} (4.6)

a lengthy calculation gives that $d\omega = 0$ is equivalent to $\omega_{\mu\nu}$ being closed in the ordinary sense and

$$\omega_{\mu} = -\frac{1}{2} b^{\rho\nu} \nabla^\nu \omega_{\rho\mu} + \partial_{\mu} H$$  \hspace{1cm} (4.7)

for some function $H$. Actually, (1.5), (1.7) imply that (1.6) reduces to (1.4) as in conventional SyM. With $\omega$ having maximal rank, and applying the Hamiltonian scheme (1.5), (1.6) we get Hamilton’s equations in the 2nd order DC $X_H(A) = 0 \iff$

$$\partial_t A = -(\{H, A\} + F_\mu \omega^{\mu\nu} \partial_\nu A) + \frac{1}{2} \partial_{\mu} (b^{\mu\nu} \partial_\nu A) + \frac{1}{2} b^{\mu\nu} \Gamma^\rho_{\rho\nu} \partial_{\mu} A,$$  \hspace{1cm} (4.8)

$$F_\mu := -\frac{1}{2} \nabla^\nu (b^{\nu\rho} \omega_{\rho\mu}), \hspace{1cm} \omega^{\mu\rho} \omega_{\nu\rho} = \delta^\mu_\nu.$$  \hspace{1cm} (4.9)

Clearly (4.8) is identical to (1.7), provided

$$F_\mu = \partial_{\mu} F, \hspace{1cm} H_0 = H + F, \hspace{1cm} \Gamma^\rho_{\rho\mu} = -\beta \partial_{\mu} H_0$$  \hspace{1cm} (4.10)

for some function $F$. It can be shown that (4.10) are equivalent to $\omega_{\mu\nu}$ being harmonic with respect to the generalized Laplace-Beltrami operator of $b^{\mu\nu}$ and to the canonical volume $e^{-\beta H} dx^1 \cdots dx^{2n}$

\footnote{If $b^{\mu\nu}$ is nondegenerate, and $\Gamma^\rho_{\rho\mu}$ its metric connection then $\omega_{\mu\nu}$ must be harmonic.}
being covariantly constant (see section 4).

5. DISCUSSION

Below we comment on a possible realization of 2nd order DC: By (2.2) for \( f_1, \ldots, f_k \in \mathcal{A} \)

\[
(d_u f_1 \cdot u \cdots \cdot u d_u f_k)(a,b) = \Delta f_1(a,b) \cdots \Delta f_k(a,b), \quad \Delta f_i(a,b) := f_i(b) - f_i(a) .
\]

(5.1)

In general (5.1) cannot be valid in a DC \((\Omega, d, \cdot)\), being incompatible with its defining relations, (equivalently with the projection \((\Omega_u, d_u, \cdot_u) \to (\Omega, d, \cdot)\), cf. (2.1), (2.5)). However if it is to be somehow retained, it leads to some interesting suggestions:

For instance in the ordinary DC the nontrivial defining relation is \(df \cdot dg = 0\), so that in one dimension

\[
(dx \cdot dx)(a,b) = 0 \implies (b - a)^2 = 0 .
\]

(5.2)

Excluding the trivial case \(b = a\), we may reinterpret the above result by saying that \(a, b\) differ by an infinitesimal of first order. When \(a, b\) are not neighbouring, for \(\alpha \in \Omega^1\), \(\alpha(a,b)\) may be interpreted as a kind of integral

\[
\int_a^b df \cdot dg = \lim \sum_{k=1}^n \Delta f(x_{k-1},x_k) \Delta g(x_{k-1},x_k)
\]

the limit being taken with respect to the width of the partition \(a = x_0 < x_1 < \cdots < x_n = b\). Thus \(df \cdot df = 0\) means, that \(f\) must be of zero quadratic variation, hence of bounded variation. Since \(f, g\) are continuous, this ensures the existence of \(\int_a^b fg\) as a Riemann-Stieltjes integral.

In the case of the 2nd order DC, (2.4) gives similarly \((b - a)^3 = 0\), or by the above reasoning, that \(a, b\) differ by an infinitesimal of 2nd order. For arbitrary \(a, b\)

\[
\int_a^b df \cdot dg \cdot dh = 0 \implies \lim \sum_{k=1}^n \Delta f(x_{k-1},x_k) \Delta g(x_{k-1},x_k) \Delta h(x_{k-1},x_k) = 0 ,
\]

hence \(f, g, h\) must have zero cubic variation, hence finite quadratic variation given by \(\int_a^b df \cdot df\) etc. This corresponds directly to stochastic integration, (see (1.2), (1.3) and the comments following (2.6)).

The above nonrigorous remarks suggest that StC may be a possible realization of 2nd DC, but it is a still unsolved problem, whether it is the only one, and much remains to be done in this direction.

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