We propose a modified XY model in which cascade transfer emerges from spatially local interactions, where the spin corresponds to the “velocity” of a turbulent field. For this model, we theoretically calculate the scale-to-scale energy flux and the equal time correlation function in $d$ dimensions. The result indicates an inverse energy cascade with the non-Kolmogorov energy spectrum proportional to $k^{-3}$. We also numerically confirm the result for the cases $d = 2$ and $d = 3$. We thus conclude that the cascade transfer in our model represents a different universality class from standard fluid turbulence.

Insights into the cascade transfer.—Let us consider the minimum elements required for cascade transfer to occur. Obviously, nonlinearity is indispensable because the essence of cascade transfer is strong inevitable interference between widely separated length scales. Furthermore, this nonlinearity must conserve “energy” if there is neither injection nor dissipation. To ensure the existence of the “inertial range,” the injection and dissipation must act at large (small) and small (large) scales, respectively. Thus, the minimum elements required for the “energy” cascade to occur are (i) nonlinearity that conserves “energy”; (ii) injection at large (small) scales; and (iii) dissipation at small (large) scales.

We now construct a simple model for cascade transfer by specifying these three elements. Respecting the ease of the intuitive interpretation of the nonlinear interaction, we consider the two-component “velocity” vector $v_i$ at each site $i$ on a two-dimensional square lattice. In the case shown in Figs. 1(a) and 1(b), the “energy” $\langle |v_i|^2 \rangle / 2$, where $\langle \cdot \rangle$ denotes the ensemble average, may be localized at small and large scales, respectively. For the model to evolve from the state shown in Fig. 1(a) to that shown in Fig. 1(b) while conserving energy, “ferromagnetic interactions” may be suitable nonlinearity. Because this nonlinear interaction may induce an inverse energy cascade, where the energy is transferred from small to large scales, we must incorporate into the model injection and dissipation terms that act at small and large scales, respectively. To this end, it may be suitable for the ease of analysis to choose a random force that is white in space and time and a friction dissipation.

Model.—Let $v_i(t) := (v_i^1(t), v_i^2(t)) \in \mathbb{R}^2$ be the “velocity” at site $i$ of a $d$-dimensional hypercubic lattice. For simplicity, we consider a hypercubic lattice with $N^d$ vertices and lattice constant $a$ and impose periodic boundary conditions. The collection of the nearest neighbor-
ing sites of \(i\) is denoted \(B_i\). The time evolution of \(v_i^a\), \(a \in \{1, 2\}\), is given by the following Langevin equation:

\[
\partial_t v_i^a = \lambda \sum_{j \in B_i} R^{ab}(v_i) v_j^b - \gamma v_i^a + \sqrt{\xi_i^a},
\]

where \(R^{ab}(v_i)\) represents the projection in the direction perpendicular to \(v_i\):

\[
R^{ab}(v_i) := \delta^{ab} - \frac{v_i^a v_i^b}{|v_i|^2}.
\]

Here, \(\lambda > 0\) is a coupling constant, \(\gamma \geq 0\) is a friction coefficient, and \(\epsilon > 0\) represents the strength of the random force, which is the zero-mean white Gaussian noise that satisfies

\[
\langle \xi_i^a(t) \xi_j^b(t') \rangle = \delta^{ab} \delta_{ij} \delta(t - t'),
\]

and \(|v_i|^2 := v_i^a v_i^a\). Here and hereafter, we employ the summation convention for \(a, b, c\) that repeated indices in one term are summed over \(\{1, 2\}\). A snapshot of the steady-state velocity profile of the model for the case \(d = 2\) is shown in Fig. 1(c). Below, we mainly consider the case of \(d = 2\), but the extension to any \(d\) is straightforward [30].

**Basic properties.**—Let \(|v_i|^2/2\) be the “energy” at site \(i\). A crucial property of the nonlinear term of the model (1) is that the term does not contribute to the energy exchange:

\[
v_i^a \left( \lambda \sum_{j \in B_i} R^{ab}(v_i) v_j^b \right) = 0.
\]

Therefore, the time evolution of \(|v_i|^2/2\) is governed only by the dissipation rate \(\gamma |v_i|^2\) and injection rate \(\sqrt{\epsilon \xi_i^c} \circ \xi_i^c\):

\[
\partial_t \frac{1}{2} |v_i|^2 = -\gamma |v_i|^2 + \sqrt{\epsilon \xi_i^c} \circ \xi_i^c,
\]

where the symbol \(\circ\) denotes multiplication in the sense of Stratonovich [31]. Thus, if there is neither injection nor dissipation (i.e., \(\epsilon = \gamma = 0\)), the energy at site \(i\), \(|v_i|^2/2\), is conserved without any averaging. If \(\epsilon > 0\) and \(\gamma > 0\), it follows that \((|v_i|^2)^2 = 2T\) in the steady-state, where we introduced the “temperature” as \(T := \epsilon/2\gamma\).

It becomes easier to understand the behavior of the model by introducing the amplitude \(A_i\) and the phase \(\theta_i\) as \(v_i = A_i (\cos \theta_i, \sin \theta_i)\). In terms of \(A_i\) and \(\theta_i\), (1) can be expressed as

\[
\partial_t A_i = -\gamma A_i + \frac{\epsilon}{2A_i} + \sqrt{\epsilon} \xi_i^a,
\]

\[
A_i \partial_t \theta_i = -\lambda \sum_{j \in B_i} A_j (\sin \theta_i - \sin \theta_j) + \sqrt{\epsilon} \xi_i^b.
\]

Here, \(\xi_i^A := \xi_i^1 \cos \theta_i + \xi_i^2 \sin \theta_i\) and \(\xi_i^B := -\xi_i^1 \sin \theta_i + \xi_i^2 \cos \theta_i\), where the multiplication is interpreted in the Itô sense [31]. Note that (7) has the form of the random-bond XY model with asymmetric coupling. If \(A_i\) is frozen uniformly in space, the system exhibits the Kosterlitz-Thouless transition [32–34]. Therefore, we can say that this model is a modified XY model with amplitude (energy) fluctuations. We emphasize that, in contrast with the standard XY model, the detailed balance is necessary for the cascade transfer to occur in the steady-state.

In the following, we use the property that the energy dissipation and injection act at large and small scales, respectively. Let \(K_i \equiv \ell_i^{-1}\) be the energy injection scale. Since the injection due to the noise \(\xi_i^a\) acts with uniform strength on each Fourier mode, \(K_i\) can be defined, for instance, as

\[
K_i := \frac{2 \pi}{L} \frac{1}{N^d} \sum_{n=-N/2+1}^{N/2} \sum_{n=-N/2+1}^{N/2} \sqrt{n_1^2 + \ldots + n_d^2},
\]

where \(L := Na\). The energy injection due to the “thermal noise” mainly acts at scales \(\ll \ell_i\). Similarly, let \(K_{\gamma} \equiv \ell_{\gamma}^{-1}\) be the dissipation scale. This scale may depend on the friction coefficient \(\gamma\) and dissipation rate \(\gamma \langle |v_i|^2 \rangle = \epsilon\). Therefore, \(K_{\gamma}\) is defined as \(K_{\gamma} := \gamma^{3/2} \epsilon^{-1/2}\) [35–37]. We thus expect that the dissipation is dominant at scales \(\gg \ell_{\gamma}\). Note that \(K_{\gamma} \to 0\) as \(\gamma \to 0\).

**Main result.**—Let \(\Pi(k)\) be the scale-to-scale energy flux, which represents the energy transfer from scales \(k^{-1}\) to scales \(k^{-1}\) (The precise definition is given below.) In the steady-state, \(\Pi(k)\) becomes scale-independent in the “inertial range” \(K_{\gamma} \ll k \ll K_i\):

\[
\Pi(k) \approx -\epsilon < 0.
\]

Since \(\Pi(k)\) is negative, (9) states that the model exhibits an inverse energy cascade; i.e., the energy is transferred conservatively and continuously from small to large scales. Correspondingly, the equal-time correlation function \(C(\ell) := \langle v_i^a v_{i+\ell}^a \rangle\), where \(\ell := r_i - r_l\) and \(r_i\) denotes
the position of site \( i \), follows a power-law:
\[
C(\ell) \sim \frac{1}{16} (\lambda a^2)^{-1} \ell^2 \quad \text{for} \quad \ell_i \ll \ell \ll \ell_\gamma.
\]  

From (10), the one-dimensional energy spectrum \( E^{(1D)}(k) \) reads
\[
E^{(1D)}(k) \sim C(\lambda a^2)^{-1} k^{-3} \quad \text{for} \quad K_\gamma \ll k \ll K_i,
\]
where \( C \) is a positive dimensionless constant.

**Numerical simulation.**—We here present the results of numerical simulation for the case \( d = 2 \) [30]. Time integration is performed using the simplest discretization method with \( \Delta t = 0.01 \). The initial value of \( v_i^0 \) is set as \( v_i^0(0) = \sqrt{\epsilon} \Delta w_i^0 \), where \( \{ \Delta w_i^0 \} \) denote the independent Wiener processes with variance \( \Delta \). The parameter values are chosen as \( \lambda = 1, \epsilon = 0.002 \), and \( \gamma = 0.001 \), so that \( T = 1 \). The system size is fixed as \( N = 1024 \) with \( a = 1 \). In this case, the injection and dissipation scales are estimated as \( K_\gamma a \simeq 2.41 \) and \( K_i a \simeq 1 \times 10^{-3} \), respectively. Note that \( K_i \) does not increase but approaches a constant value as \( N \) increases.

Figure 2(a) shows the scale dependence of the scale-to-scale energy flux \( \Pi(k) \) at different times. As expected from the result (9), \( \Pi(k) \) is negative and scale independent in the inertial range \( K_\gamma \ll k \ll K_i \). The magnitude of \( \Pi(k) \) in the inertial range is on the order of \( \epsilon \), i.e., \( \Pi(k)/\epsilon \simeq -1 \), which is consistent with (9). Furthermore, the scale range over which \( \Pi(k) \) is nearly constant extends to larger scales as time increases. This result also supports that the energy is continuously transferred from small to large scales. In Fig. 2(b), we plot the one-dimensional energy spectrum \( E^{(1D)}(k) \) for the same times as in Fig. 2(a). In the inertial range, \( E^{(1D)}(k) \) follows the power-law \( \propto k^{-3} \), which is consistent with the theoretical prediction (11). At scales smaller than the injection scale \( K_i \), \( E^{(1D)}(k) \) is proportional to \( k \). This result implies that the “equipartition of energy” is realized for small scales \( \gtrsim K_i \). We can also confirm the existence of the inverse energy cascade by noting that the spectrum extends to larger scales as time passes. Note that the range over which \( \Pi(k) \) is flat does not exactly correspond to the range over which \( E^{(1D)}(k) \propto k^{-3} \). This discrepancy is similar to that observed in ordinary fluid turbulence [38].

**Derivation of the main result.**—Let \( \hat{v}_k \) be the discrete Fourier transform of \( v_i^0 \) with \( k := 2\pi n/L \), where \( n^1, n^2 \in \{-N/2 + 1, \ldots, 0, 1, \ldots, N/2\} \). We define the low-pass filtering operator by
\[
\mathcal{P}^{<K} : v_i \mapsto v_i^{<K} := \sum_{|k|<K} \hat{v}_k e^{ik \cdot r_i},
\]
where \( \sum_{|k|<K} \) denotes the sum over all possible \( k \) that satisfy \( |k| < K \). This operator sets to zero all Fourier components with a wavenumber greater than \( K \). By applying this operator to both sides of (1) and taking the average, we obtain the low-pass filtered energy balance equation:
\[
\partial_t \frac{1}{2} \langle |v_i^{<K}|^2 \rangle = -\Pi(K) - \gamma \langle |v_i^{<K}|^2 \rangle + \sqrt{\epsilon} \langle v_i^{<K} \circ \xi_i^{<K} \rangle,
\]
where
\[
\Pi(K) := -\lambda \left\langle v_i^{<K} \cdot \mathcal{P}^{<K} \left( \sum_{j \in B_i} R(v_i) \cdot v_j \right) \right\rangle.
\]
steady-state, we obtain
\[ \Pi(K) = -\gamma \langle |v_i^K|^2 \rangle + \sqrt{\epsilon} \langle v_i^K \circ \xi_i^K \rangle \]
\[ \approx -\gamma \langle |v_i|^2 \rangle = -\epsilon<0 \quad \text{for } K_\gamma \ll K \ll K_\iota. \quad (15) \]
The model thus exhibits the inverse energy cascade; i.e.,
the energy is transferred conservatively from small to large
scales in the “inertial range” \( K_\gamma \ll K \ll K_\iota \). Note
that the above argument is essentially the same as that
for the two-dimensional fluid turbulence [35–37, 39].

We now determine the functional form of the energy
spectrum. To this end, we express the energy flux in
the terms of the velocity correlation function as in the derivation
of the Kolmogorov 4/5-law [8]. We first note that
\[ \Pi(K) \text{ can be rewritten as} \]
\[ \Pi(K) = -\partial_t \frac{1}{2} \langle |v_i^K|^2 \rangle_{NL} \]
\[ = -\sum_{|k|<K} \frac{1}{N^2} \sum_{r_j-r_l} e^{-ik(r_j-r_l)} \partial_t \frac{1}{2} \langle v_j v_l \rangle_{NL}, \]
(16)
where \( \partial_t \big{|}_{NL} \) denotes the time evolution due to the non-
linear term. By taking the continuum limit, (16) can be
expressed as
\[ \Pi(K) = -\int_{|k|<K} \frac{d^2k}{(2\pi)^2} \int d^2k e^{-ik \cdot \ell} \epsilon(\ell) \]
\[ = -\int_0^\infty K d\ell J_1(K \ell) \epsilon(\ell). \quad (17) \]
Here, \( J_1 \) is the Bessel function of the first
kind and we have assumed the homogeneity
\( \epsilon(\ell) := \partial_t \langle v(\ell) v(0) \rangle_{NL} = \partial_t \langle v(r_j) v(r_l) \rangle_{NL} \)
and isotropy \( \epsilon(\ell) = \epsilon(\ell) \) with \( \ell := r_j - r_l \). We now substitute
(17) into the relation (15) to find
\[ \int_0^\infty dx J_1(x) \epsilon \left( \frac{x}{K} \right) \simeq \epsilon \quad \text{for } K_\gamma \ll K \ll K_\iota. \quad (18) \]
By taking first the limit \( \gamma \to 0 \) (\( K_{\gamma} \to 0 \)) and then
the limit \( K \to 0 \), we obtain, for large \( \ell_1 \), [8]
\[ \epsilon(\ell) \simeq \epsilon, \quad (19) \]
where we have used the identity \( \int_0^\infty dx J_1(x) = 1 \). A
simple expression for \( \epsilon(\ell) \) can be obtained by noting that
\( v_i \) tends to align with \( \langle \langle u_i \rangle \rangle := \sum_{j \in B_i} v_j / 4 \) because of
the nonlinearity of the model. In other words, for
the angle \( \alpha_i \) between \( \hat{v_i} \) and \( \langle \langle \hat{u}_i \rangle \rangle \), we
conjecture that \( \alpha_i \ll 1 \) in the steady-state. Therefore,
by assuming that each angle between \( \hat{v_i} \) and its nearest
neighbor \( \hat{v}_j \) is on the order of \( \alpha_i \ll 1 \), we find that
\[ R^{ab}(u_i) \langle \langle u_j \rangle \rangle = \langle \langle u_i^a \rangle \rangle \cos \alpha_i \]
\[ \approx \langle \langle u_i^a \rangle \rangle - \cos \alpha_i \langle \langle u_i \rangle \rangle \]
\[ \approx \langle \langle u_i^a \rangle \rangle - u_i^a \cos \alpha_i (A_i - \langle \langle A_i \rangle \rangle). \quad (20) \]
Since \( \{ A_i \} \) are independent and identically distributed
random variables, we obtain from (20) that
\[ \partial_t \left( \frac{1}{2} \langle \langle v_i^a v_i^a \rangle \rangle_{NL} \right) \]
\[ = 2\lambda \left[ \langle \langle v_i^a R^{ac}(v_j) \langle \langle v_j^c \rangle \rangle \rangle \rangle + \langle \langle v_i^a R^{ac}(v_l) \langle \langle v_l^c \rangle \rangle \rangle \rangle \right] \]
\[ \approx 2\lambda \left[ \langle \langle v_i^a \rangle \rangle \langle \langle v_j^c \rangle \rangle - \langle \langle v_i^a \rangle \rangle \langle \langle v_l^c \rangle \rangle \right], \quad (21) \]
for \( |r_j - r_l| > a \). Note that \( \langle \langle \cdot \rangle \rangle \cdot \) is the discrete Lapla-
cian. Therefore, \( \epsilon(\ell) \) in (19) can be expressed in terms of
\( C(\ell) := \langle \langle v_i^c(r_j) v_i^c(r_l) \rangle \rangle - \langle \langle v_i^c \rangle \rangle \langle \langle v_i^c \rangle \rangle \]
\[ 4\lambda a^2 \left( \frac{\partial^2}{\partial \ell^2} + \frac{1}{\ell} \frac{\partial}{\partial \ell} \right) C(\ell) \simeq \epsilon. \quad (22) \]
It follows from this equation that
\[ C(\ell) \simeq \frac{1}{16} (\lambda a^2)^{-1} \ell^2 \quad \text{for } \ell_i \ll \ell \ll \ell_\gamma. \quad (23) \]
Correspondingly, the asymptotic behavior of the one-
dimensional energy spectrum \( E^{(1D)}(k) \) in the inertial
range reads
\[ E^{(1D)}(k) \sim C(\lambda a^2)^{-1} k^{-3} \quad \text{for } K_\gamma \ll k \ll K_\iota, \quad (24) \]
where \( C \) is a dimensionless positive constant.

Concluding remarks.—One of the fundamental properties
of cascades that we have not discussed here is scale
locality [40–42]. An energy cascade is scale-local if modes
that make a significant contribution to energy transfer at
each scale are limited to those in the vicinity of that scale.
From the fact that the energy flux and spectrum gradu-
ally extend to larger scales as time passes (see Fig. 2),
it seems that the inverse cascade is scale-local. However,
a numerical study of scale locality implies that it is not
scale-local [43], although there remains a problem of how
to define the scale locality. A more detailed study on
the scale locality should be carried out in the future.

Interestingly, the behavior of the energy spectrum
\( E^{(1D)}(k) \sim k^{-3} \) at large scales is also observed in atmo-
spheric turbulence. In the upper troposphere and lower
stratosphere, \( E^{(1D)}(k) \sim k^{-5/3} \) at scales between 10 and
500 km while \( E^{(1D)}(k) \sim k^{-3} \) at scales between 500 and
3000 km [39, 44–49]. We also note that turbulent be-
behavior similar to that of our model is found in so-called
spin turbulence [20–25] and Fibonacci turbulence [50]. It
would thus be interesting to investigate the relationship
between these systems and our model.

In conclusion, we constructed a modified XY model in
which cascade transfer emerges. Because this inverse cas-
cade induces the non-Kolmogorov spectrum \( E^{(1D)}(k) \sim k^{-3} \),
represents a different universality class from standard
fluid turbulence. We thus hope that our model trig-
gers further investigation of cascade transfer in various
systems such as condensed matter, active matter, and
other statistical mechanical systems.

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Supplemental Material: A Simple XY Model for Cascade Transfer
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1. Three-dimensional case

In this section, we present the numerical result for the three-dimensional case with two-component velocity $\mathbf{v}_i$. Even for this case, we can derive the inverse energy cascade with $k^{-3}$ spectrum: for $K_\gamma \ll k \ll K_1$,

$$\Pi(k) \simeq -\epsilon$$  \hspace{1cm} (S1)

and

$$E^{(1D)}(k) \sim C(\lambda a^2)^{-1} \epsilon k^{-3},$$  \hspace{1cm} (S2)

where $C$ is a positive dimensionless constant. In Fig. S1, we plot the scale dependence of the energy flux $\Pi(k)$ and spectrum $E^{(1D)}(k)$. The parameter values and the system size are chosen as $\lambda = 1$, $\epsilon = 0.002$, $\gamma = 0.001$, and $N = 128$ with $a = 1$. In this case, the injection and dissipation scales are estimated as $K_1 a \simeq 3.02$ and $K_\gamma a \simeq 1 \times 10^{-3}$. This result is consistent with the prediction (S1) and (S2). Note that, in this case, $E^{(1D)}(k) \propto k^2$ at small scales, whereas it is proportional to $E^{(1D)}(k) \propto k$ in the two-dimensional case. This result also implies that the “equipartition of energy” is realized for small scales.

![FIG. S1. Left: the scale-to-scale energy flux $\Pi(k)/\epsilon$. Right: the energy spectrum $E^{(1D)}(k)$. The dash-dotted and dotted lines represent the power-laws $\propto k^{-3}$ and $\propto k^2$, respectively.](image)

2. Scale locality

In this section, we numerically investigate the scale locality of the inverse energy cascade. An energy cascade is scale-local if modes that make a significant contribution to energy transfer at each scale are limited to those in the vicinity of that scale. More precisely, we define the scale locality of the inverse energy cascade for our model as follows [1–3]. We first note that the scale-to-scale energy flux $\Pi(K) := -\lambda \langle v_i^{<K} \cdot \mathcal{P}^{<K}[\sum_{j \in B_i} R(\mathbf{v}_i) \cdot \mathbf{v}_j] \rangle$ has a form of “velocity” $v_i^{<K}$ times “force” $\lambda \mathcal{P}^{<K}[\sum_{j \in B_i} R(\mathbf{v}_i) \cdot \mathbf{v}_j]$. We describe the energy flux as infrared local if $|\lambda \langle v_i^{<Q} \cdot \mathcal{P}^{<K}[\sum_{j \in B_i} R(\mathbf{v}_i) \cdot \mathbf{v}_j] \rangle|$ and $|\lambda \langle v_i^{<K} \cdot \mathcal{P}^{<K}[\sum_{j \in B_i} R(\mathbf{v}_i^{<Q}) \cdot \mathbf{v}_j^{<Q}] \rangle|$ gives an asymptotically negligible contribution for $Q \ll K$. That is, the energy flux satisfies the infrared locality if $|\lambda \langle v_i^{<Q} \cdot \mathcal{P}^{<K}[\sum_{j \in B_i} R(\mathbf{v}_i) \cdot \mathbf{v}_j] \rangle|$ and $|\lambda \langle v_i^{<K} \cdot \mathcal{P}^{<K}[\sum_{j \in B_i} R(\mathbf{v}_i^{<Q}) \cdot \mathbf{v}_j^{<Q}] \rangle|$ decay as fast as $(Q/K)^\alpha$ with $\alpha > 0$ for $Q \ll K$. Here, we have used the fact that $v_i^{<Q} \propto v_i^{<Q}$ for $Q \leq K$. Similarly, we describe the energy flux as ultraviolet local if $|\lambda \langle v_i^{<K} \cdot \mathcal{P}^{<K}[\sum_{j \in B_i} R(\mathbf{v}_i^{>Q}) \cdot \mathbf{v}_j^{>Q}] \rangle|$ decays as fast as
(Q/K)^{-\alpha} \text{ with } \alpha > 0 \text{ for } Q \gg K. \text{ Here, we do not need to consider } |\lambda\langle v_i^{>Q} \cdot \mathcal{P}^{<K} \sum_{j \in B_i} R(v_i) \cdot v_j \rangle|/\Pi(K)| \text{ because } \langle v_i^{>Q} \rangle^{<K} = 0 \text{ for } Q \gtrsim K.

We here remark that the energy flux \Pi(K) in our model cannot be scale-local without averaging. That is, because \Pi(K) is not Galilean invariant, a \( k = 0 \) mode can directly contribute to the unaveraged energy flux if we boost the flow with a uniform velocity \( \mathbf{U} \), i.e., \( v_i \rightarrow v_i + \mathbf{U} \) for all \( i \). Note that this property is the same as the \textit{unsubtracted flux} for fluid turbulence \cite{Tanaka1996}. Even though the unaveraged flux is not scale-local, the averaged flux \Pi(K) may become scale-local because of the cancellation of the large-scale contribution.

We now present the results of numerical simulation. The parameter values and system size are the same as in the main text: \( \lambda = 1, \epsilon = 0.002, \gamma = 0.001, \) and \( N = 1024 \) with \( a = 1 \), so that the injection and dissipation scales are estimated as \( K_i a \simeq 2.41 \) and \( K_\gamma a \simeq 1 \times 10^{-3} \). We first consider the infrared locality. For the infrared locality, we investigate the \( Q \)-dependence of the following quantities:

\[ \left| \lambda \left\langle v_i^{<Q} \cdot \mathcal{P}^{<K} \left[ \sum_{j \in B_i} R(v_i) \cdot v_j \right] \right\rangle \right|, \tag{S3} \]

\[ \left| \lambda \left\langle v_i^{<K} \cdot \mathcal{P}^{<K} \left[ \sum_{j \in B_i} R(v_i^{<Q}) \cdot v_j^{<Q} \right] \right\rangle \right|. \tag{S4} \]

We calculated these two quantities for \( Ka \simeq 0.31 \) and \( Ka \simeq 0.18 \) (\( K = 2\pi n/Na \) with \( n = 50 \) and 30, respectively), which is within the inertial range: \( K_\gamma \ll K \ll K_i \). Figures S2 and S3 show the \( Q \)-dependencies of these quantities at \( t = 50000 \) normalized by \( |\Pi(K)| \) with \( Q < K \) for \( Ka \simeq 0.31 \) and \( Ka \simeq 0.18 \), respectively. Although both quantities decay as \( Q \rightarrow 0 \), they are almost flat in the inertial range \( K_\gamma \ll Q < K \ll K_i \). This result implies that the inverse
cascade is not strictly infrared local. In other words, the contributions to the energy flux from large-scale modes may not be ignored.

For the ultraviolet locality, we investigate the $Q$-dependence of the following quantity:

$$
|\lambda \left< \mathcal{P}^{<K} \cdot \mathcal{P}^{<K} \left[ \sum_{j \in B_i} R(\nu_i^{>Q}) \cdot \nu_j^{>Q} \right] \right> |. \tag{S5}
$$

The result is shown in Fig. S4. As in the case of the infrared locality, $|\lambda \left< \mathcal{P}^{<K} \cdot \mathcal{P}^{<K} \left[ \sum_{j \in B_i} R(\nu_i^{>Q}) \cdot \nu_j^{>Q} \right] \right>/\Pi(K)$ does not decay rapidly in the inertial range $K_\gamma \ll K < Q \ll K_i$. Therefore, small-scale modes may contribute significantly to the energy flux.

FIG. S4. $Q$-dependence of $|\lambda \left< \mathcal{P}^{<K} \cdot \mathcal{P}^{<K} \left[ \sum_{j \in B_i} R(\nu_i^{>Q}) \cdot \nu_j^{>Q} \right] \right>/\Pi(K)$ for $K a \simeq 0.31$ (left) and $K a \simeq 0.18$ (right).

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