Inverse Ingham type inequalities for the Burgers model

Paola Loreti ∗  Daniela Sforza †
February 17, 2022

Abstract

Viscoelastic materials have the properties both of elasticity and viscosity. In a previous work we investigate glass relaxation in the framework of viscoelasticity. Here we consider the Burgers model, a first but meaningful step in the general analysis, showing a reachability theorem thanks to the analysis of the gap between eigenvalues and the representation of the solution in Fourier series.

Keywords: Viscoelasticity, the Burgers model, Ingham inequalities.
Mathematics Subject Classification: 45K05.

1 Introduction

The aim of this paper is to show exact reachability in time $T_0$ of Burgers equation. Burgers equation concerns viscoelastic materials having the properties both of elasticity and viscosity. Also it can be seen as first step to study relaxation of glasses through approximation of the stretched exponential function with Prony series. As references on the topic see [9, 10, 11]. The equation of viscoelasticity of Burgers models is solved using the Fourier analysis in [8], indeed in [8] we are able to give a detailed spectral analysis and, by using Fourier expansions, the solution of the Burgers equation. Thanks to the previous results, in this paper we state and solve for sufficiently large time the reachability problem. Theorem 3.7, the inverse observability inequality is shown by using Ingham type approach, see [2] and also [3]. This together with Theorem 3.8 allow us to obtain the result. It remains an open question to find the optimal time.

Theorem 1.1 Let $b_1, b_2, r_1, r_2 > 0$ with $\frac{3}{2}(b_1 + b_2) < r_1 + r_2$ and $\frac{b_1}{r_1} + \frac{b_2}{r_2} = 1$.

Then, there exists $T_0 > 0$ such that for $T > T_0$ and $(u_0, u_1) \in L^2(0, \pi) \times H^{-1}(0, \pi)$, there exists $f \in L^2(0, T)$ such that the weak solution $u$ of the equation

$$u_{tt}(t, x) = u_{xx}(t, x) - b_1 \int_0^t e^{-r_1(t-s)} u_{xx}(s, x) ds - b_2 \int_0^t e^{-r_2(t-s)} u_{xx}(s, x) ds, \quad t \in (0, T), \quad x \in (0, \pi),$$

with boundary conditions

$$u(t, 0) = 0, \quad u(t, \pi) = f(t), \quad t \in (0, T),$$

and null initial values

$$u(0, x) = u_t(0, x) = 0, \quad x \in (0, \pi),$$

verifies the final conditions

$$u(T, x) = u_0(x), \quad u_t(T, x) = u_1(x), \quad x \in (0, \pi).$$

∗Dipartimento di Scienze di Base e Applicate per l’Ingegneria, Sapienza Università di Roma, Via Antonio Scarpa 16, 00161 Roma (Italy); e-mail: <paola.loreti@uniroma1.it>

†Dipartimento di Scienze di Base e Applicate per l’Ingegneria, Sapienza Università di Roma, Via Antonio Scarpa 16, 00161 Roma (Italy); e-mail: <daniela.sforza@uniroma1.it>
2 Preliminaries

Throughout the paper, we adopt the convention to write \( F \times G \) if there exist two positive constants \( c_1 \) and \( c_2 \) such that \( c_1 F \leq G \leq c_2 F \).

Let \( \Omega \subset \mathbb{R}^N \), \( N \geq 1 \), be a bounded open set with sufficiently smooth boundary \( \partial \Omega \). We study the integro-differential equation written in the form

\[
    u_{tt} = \gamma^2 \Delta u - b_1 \int_0^t e^{-r_1(t-s)} \gamma^2 \Delta u(s) ds - b_2 \int_0^t e^{-r_2(t-s)} \gamma^2 \Delta u(s) ds, \\
    u(t, x) = 0 \quad t \geq 0, \ x \in \partial \Omega, 
\]

where \( \gamma > 0 \) and \( b_1, r_1, b_2, r_2 \) are positive constants satisfying the conditions

\[
    r_1 + r_2 - b_1 - b_2 > 0, \quad \frac{b_1}{r_1} + \frac{b_2}{r_2} = 1. 
\]

The assumptions on the integral kernels follow from the Burgers model.

We can rewrite the integro-differential equation \( (1) \) in an abstract version. Let \( H = L^2(\Omega) \) be endowed with the usual scalar product and norm

\[
    \langle u, v \rangle := \int_{\Omega} u(x)v(x) \, dx, \quad \|u\|_H := \left( \int_{\Omega} |u(x)|^2 \, dx \right)^{1/2}, \quad u, v \in H.
\]

We define the operator \( L : D(L) \subset H \to H \) by

\[
    D(L) = H^2(\Omega) \cap H^1_0(\Omega) \\
    Lu = -\gamma^2 \Delta u \quad u \in D(L). 
\]

It is well known that \( L \) is a self-adjoint positive operator on \( H \) with dense domain \( D(L) \). The spectrum of \( L \) is composed of an increasing sequence \( \{\lambda_n\}_{n \geq 1} \) of positive eigenvalues with \( \lambda_n \to \infty \) and there exists an orthonormal basis \( \{e_n\}_{n \geq 1} \) of \( L^2(\Omega) \) consisting of the corresponding eigenvectors. Moreover, we assume that the eigenvalues \( \lambda_n \) are all distinct numbers.

Recalling that \( b_i, r_i > 0, \ i = 1, 2 \), satisfy the condition \( \frac{b_1}{r_1} + \frac{b_2}{r_2} = 1 \), we consider the following Cauchy problem:

\[
\begin{align*}
    &u''(t) + Lu(t) - b_1 \int_0^t e^{-r_1(t-s)} Lu(s) ds - b_2 \int_0^t e^{-r_2(t-s)} Lu(s) ds = 0 \quad t \geq 0, \\
    &u(0) = u_0, \quad u'(0) = u_1.
\end{align*}
\]

For any couple \( u_0 \in D(\sqrt{L}) \) and \( u_1 \in H \) of initial data we can give an expansion by means of the eigenvectors \( e_n \), that is

\[
\begin{align*}
    u_0 &= \sum_{n=1}^{\infty} u_0^n e_n, \quad u_0^n = \langle u_0, e_n \rangle, \quad \|u_0\|^2_{D(\sqrt{L})} = \sum_{n=1}^{\infty} u_0^n^2 \lambda_n, \\
    u_1 &= \sum_{n=1}^{\infty} u_1^n e_n, \quad u_1^n = \langle u_1, e_n \rangle, \quad \|u_1\|^2_H = \sum_{n=1}^{\infty} u_1^n^2.
\end{align*}
\]

**Theorem 2.1** Given \( u_0 = \sum_{n=1}^{\infty} u_0^n e_n \in D(\sqrt{L}) \) and \( u_1 = \sum_{n=1}^{\infty} u_1^n e_n \in H \), for any \( n \in \mathbb{N} \) we define the numbers

\[
    \omega_n = \sqrt{\lambda_n} + \frac{b_1 + b_2}{2} + O\left(\frac{1}{\sqrt{\lambda_n}}\right),
\]
\[\rho_n = b_1 + b_2 - r_1 - r_2 + O\left(\frac{1}{\lambda_n}\right),\]  
(8)

\[C_n = \frac{u_0 n}{2} - i\left(\frac{1}{4}\right)((b_1 + b_2)u_0 + 2u_1 n)\frac{1}{\sqrt{\lambda_n}} + (u_0 n + u_1 n)O\left(\frac{1}{\lambda_n}\right),\]  
(9)

\[R_{1,n} = \frac{r_1 r_2 u_{11n}}{(r_1 + r_2 - b_1 - b_2)\lambda_n} + (u_0 n + u_1 n)O\left(\frac{1}{\lambda_n^2}\right),\]  
(10)

\[R_{2,n} = \frac{(b_1 + b_2 - r_1)(b_1 + b_2 - r_2)(u_0 n (b_1 + b_2 - r_1 - r_2) + u_{11n})}{(b_1 + b_2 - r_1 - r_2)\lambda_n} + (u_0 n + u_1 n)O\left(\frac{1}{\lambda_n^2}\right).\]  
(11)

Then, the series
\[u(t) = \sum_{n=1}^{\infty} \left(C_n e^{i\omega_n t} + \overline{C_n} e^{-i\omega_n t} + R_{1,n} + R_{2,n} e^{\rho_n t}\right) e_n\]  
(12)

is the solution of problem [5].

Moreover, for some constant \(M > 0\) we have
\[|R_{1,n}| + |R_{2,n}| \leq \frac{M}{\sqrt{\lambda_n}} |C_n| \quad \forall n \in \mathbb{N},\]  

\[\sum_{n=1}^{\infty} \lambda_n |C_n|^2 \leq \|u_0\|^2_{D(\sqrt{T})} + \|u_1\|^2_{\mathcal{H}}.\]  
(13)

3 An Ingham type inequality

Our goal is to prove an inverse inequality for the function
\[u(t) = \sum_{n=1}^{\infty} \left(C_n e^{i\omega_n t} + \overline{C_n} e^{-i\omega_n t} + R_{1,n} + R_{2,n} e^{\rho_n t}\right),\]  
(14)

with \(C_n, \omega_n \in \mathbb{C} \) and \(R_{1,n}, R_{2,n}, \rho_n \in \mathbb{R}\). Throughout this section we assume that there exist \(\gamma, \alpha_\omega > 0\) and \(\alpha_\rho < 0\) such that
\[
\liminf_{n \to \infty} (\Re \omega_{n+1} - \Re \omega_n) = \gamma, \\
\lim_{n \to \infty} \Im \omega_n = \alpha_\omega, \quad \lim_{n \to \infty} \rho_n = \alpha_\rho.
\]  
(15)

(16)

Moreover, we suppose that there exist \(\nu > 1/2\) and \(M > 0\) such that
\[|R_{1,n}| + |R_{2,n}| \leq \frac{M}{n^{\nu}} |C_n| \quad \forall n \in \mathbb{N}.\]  
(17)

3.1 Some auxiliary results

Here we list some results to be used later.

For any \(T > 0\) we consider the function defined by
\[g(t) := \begin{cases} 
\sin\frac{\pi t}{T} & \text{if } t \in [0, T], \\
0 & \text{otherwise}.
\end{cases}\]  
(18)

Some properties of \(g\) are now in order.
Lemma 3.1 Set
\[ G(w) := -\frac{T\pi}{T^2 w^2 - \pi^2}, \quad w \in \mathbb{C}, \] (19)
the following properties hold.

(i) For any \( w \in \mathbb{C} \)
\[ \int_0^\infty g(t)e^{iwt}dt = (1 + e^{iwT})G(w). \] (20)
\[ |G(w)| = |G(\overline{w})|, \] (21)

(ii) Let \( \sigma > 0 \) and \( n \in \mathbb{N} \). Then, for \( T > \frac{2\pi}{\sigma} \) and \( w \in \mathbb{C}, |w| \geq \sigma n \), we have
\[ |G(w)| \leq \frac{4\pi}{T\sigma^2(4n^2 - 1)}. \] (22)

Lemma 3.2 For any \( \varepsilon \in (0, 1) \) there exists \( n_0 \in \mathbb{N} \) such that
\[ |\Re \omega_n - \Re \omega_m| \geq \gamma \sqrt{1 - \varepsilon}|n - m|, \quad \forall n, m \geq n_0, \] (23)
\[ \Re \omega_n \geq \gamma \sqrt{1 - \varepsilon}n, \quad \forall n \geq n_0. \] (24)

Lemma 3.3 For any \( \varepsilon \in (0, 1) \), \( T > \frac{2\pi}{\gamma \sqrt{1 - \varepsilon}} \) and \( a > 0 \) there exists \( n_0 \in \mathbb{N} \) such that
\[ a|G(\omega_n)| \leq \frac{\pi \varepsilon}{T\gamma^2(1 - \varepsilon)} \quad \forall n \geq n_0, \] (25)
\[ a \sum_{n=n_0}^\infty |G(\omega_n)| \leq \frac{\pi \varepsilon}{T\gamma^2(1 - \varepsilon)}. \] (26)

Proof. Thanks to (24) and (22) we observe that for \( \varepsilon \in (0, 1) \) and \( T > \frac{2\pi}{\gamma \sqrt{1 - \varepsilon}} \), fixed \( a > 0 \), there exists \( n_0 \in \mathbb{N} \) such that
\[ a|G(\omega_n)| \leq \frac{\pi}{T\gamma^2(1 - \varepsilon)} \frac{4a}{4n^2 - 1} \leq \frac{\pi}{T\gamma^2(1 - \varepsilon)} \varepsilon \quad \forall n \geq n_0, \]
\[ a \sum_{n=n_0}^\infty |G(\omega_n)| \leq \frac{\pi}{T\gamma^2(1 - \varepsilon)} \sum_{n=n_0}^\infty \frac{4a}{4n^2 - 1} \leq \frac{\pi}{T\gamma^2(1 - \varepsilon)} \varepsilon, \]
and hence we get our statement. \( \square \)

We can state the following result, for the proof see e.g. [5].

Proposition 3.4 Let \( g \) be the weight function defined by (18). Suppose that
\[ \liminf_{n \to \infty} (\Re \omega_{n+1} - \Re \omega_n) = \gamma > 0 \]
and \( \{C_n\} \) is a complex number sequence with \( \sum_{n=1}^\infty |C_n|^2 < +\infty \).

Then for any \( \varepsilon \in (0, 1) \) and \( T > \frac{2\pi}{\gamma \sqrt{1 - \varepsilon}} \) there exists \( n_0 = n_0(\varepsilon) \in \mathbb{N} \) independent of \( T \) and \( C_n \) such that we have
\[ \int_0^\infty g(t) \left( \sum_{n=n_0}^\infty C_n e^{i\omega_n t} + \overline{C_n} e^{-i\omega_n t} \right)^2 dt \]
\[ \geq 2\pi T \sum_{n=n_0}^\infty \left( \frac{1}{\pi^2 + 4T^2(3\omega_n)^2} - \frac{4}{T^2\gamma^2(1 - \varepsilon)} \right) (1 + e^{-23\omega_n T}) |C_n|^2. \] (27)
Proposition 3.5 Assume that
\[
\lim_{n \to \infty} \left( \Re \omega_{n+1} - \Re \omega_n \right) = \gamma > 0, \quad (28)
\]
\[
\lim_{n \to \infty} \rho_n = \alpha \rho < 0, \quad (29)
\]
and there exist \( \nu > 1/2 \) and \( M > 0 \) such that
\[
|R_{1,n}| + |R_{2,n}| \leq \frac{M}{n^\nu} |C_n| \quad \forall \ n \in \mathbb{N}. \quad (30)
\]
Then for any \( \varepsilon \in (0,1) \) and \( T > \frac{2\pi}{\gamma \sqrt{1-\varepsilon}} \) there exists \( n_0 = n_0(\varepsilon) \in \mathbb{N} \) independent of \( T \) and \( C_n \) such that we have
\[
\int_0^\infty \left( \sum_{n=n_0}^\infty C_n e^{i\omega_n t} + \overline{C_n} e^{-i\omega_n t} + R_{1,n} + R_{2,n} e^{\rho_n t} \right)^2 dt 
\geq 2\pi T \sum_{n=n_0}^\infty \left( \frac{1}{\pi^2 + 4T^2(3\omega_n)^2} - \frac{4(1+\varepsilon)}{T^2\gamma^2(1-\varepsilon)} \right) (1 + e^{-2\omega_n T}) |C_n|^2. \quad (31)
\]

Proof. We consider the weight function \( g \) defined by [18].
\[
\int_0^\infty g(t) \left( \sum_{n=n_0}^\infty C_n e^{i\omega_n t} + \overline{C_n} e^{-i\omega_n t} + R_{1,n} + R_{2,n} e^{\rho_n t} \right)^2 dt 
= \int_0^\infty g(t) \left( \sum_{n=n_0}^\infty C_n e^{i\omega_n t} + \overline{C_n} e^{-i\omega_n t} \right)^2 dt + 2 \sum_{n,m=n_0}^\infty \int_0^\infty g(t) \left( C_n e^{i\omega_n t} + \overline{C_n} e^{-i\omega_n t} \right) \left( R_{1,m} + R_{2,m} e^{\rho_m t} \right) dt 
+ \int_0^\infty g(t) \left( \sum_{n=n_0}^\infty R_{1,n} + R_{2,n} e^{\rho_n t} \right)^2 dt.
\]

Since \( g(t) \) is positive we have
\[
\int_0^\infty g(t) \left( \sum_{n=n_0}^\infty C_n e^{i\omega_n t} + \overline{C_n} e^{-i\omega_n t} + R_{1,n} + R_{2,n} e^{\rho_n t} \right)^2 dt \geq \int_0^\infty g(t) \left( \sum_{n=n_0}^\infty C_n e^{i\omega_n t} + \overline{C_n} e^{-i\omega_n t} \right)^2 dt 
+ 2 \sum_{n,m=n_0}^\infty \int_0^\infty g(t) \left( C_n e^{i\omega_n t} + \overline{C_n} e^{-i\omega_n t} \right) \left( R_{1,m} + R_{2,m} e^{\rho_m t} \right) dt.
\]

We estimate the first term on the right-hand side of the previous inequality by means of Proposition 3.4 and hence, thanks to (27) we deduce
\[
\int_0^\infty g(t) \left( \sum_{n=n_0}^\infty C_n e^{i\omega_n t} + \overline{C_n} e^{-i\omega_n t} + R_{1,n} + R_{2,n} e^{\rho_n t} \right)^2 dt 
\geq 2\pi T \sum_{n=n_0}^\infty \left( \frac{1}{\pi^2 + 4T^2(3\omega_n)^2} - \frac{4(1+\varepsilon)}{T^2\gamma^2(1-\varepsilon)} \right) (1 + e^{-2\omega_n T}) |C_n|^2 
+ 2 \sum_{n,m=n_0}^\infty \int_0^\infty g(t) \left( C_n e^{i\omega_n t} + \overline{C_n} e^{-i\omega_n t} \right) \left( R_{1,m} + R_{2,m} e^{\rho_m t} \right) dt. \quad (32)
\]
As regards the second term, we note that
\[
\sum_{n,m=0}^{\infty} \int_0^{\infty} g(t) \left( e^{i\omega_n t} + e^{-i\omega_n t} \right) \left( R_{1,m} + R_{2,m} e^{\rho_n t} \right) \, dt
\]
\[
= \sum_{n,m=0}^{\infty} R_{1,m} \Re \left[ e^{i\omega_n T} G(\omega_n) \right] + \sum_{n,m=0}^{\infty} R_{2,m} \Re \left[ e^{i(\omega_n + \rho_n) T} G(\omega_n - i\rho_n) \right]
\]
\[
\geq - \sum_{n,m=0}^{\infty} |R_{1,m}| |C_n| (1 + e^{-3\omega_n T}) |G(\omega_n)| - \sum_{n,m=0}^{\infty} |R_{2,m}| |C_n| (1 + e^{(\rho_n - 3\omega_n) T}) |G(\omega_n - i\rho_n)|. \quad (33)
\]

Thanks to (17) we have
\[
\sum_{n=0}^{\infty} |R_{1,m}| |C_n| (1 + e^{-3\omega_n T}) |G(\omega_n)| \leq M \sum_{n=0}^{\infty} \frac{|C_n|^2}{m^2} |G(\omega_n)| + M \sum_{n=0}^{\infty} |C_n|^2 (1 + e^{-2\omega_n T}) |G(\omega_n)| \sum_{m=0}^{\infty} \frac{1}{m^{2\nu}}.
\]

By using respectively (26) with \( a = M \) and (25) with \( a = M \sum_{m=1}^{\infty} \frac{1}{m^{2\nu}} \) we obtain
\[
M \sum_{m=0}^{\infty} |C_m|^2 \sum_{n=0}^{\infty} |G(\omega_n)| \leq \frac{\pi \varepsilon}{T \gamma^2 (1 - \varepsilon)} \sum_{n=0}^{\infty} |C_n|^2 (1 + e^{-2\omega_n T}),
\]
\[
M \sum_{n=0}^{\infty} |C_n|^2 (1 + e^{-2\omega_n T}) |G(\omega_n)| \sum_{m=0}^{\infty} \frac{1}{m^{2\nu}} \leq \frac{\pi \varepsilon}{T \gamma^2 (1 - \varepsilon)} \sum_{n=0}^{\infty} |C_n|^2 (1 + e^{-2\omega_n T}),
\]
whence
\[
\sum_{n,m=0}^{\infty} |R_{1,m}| |C_n| (1 + e^{-3\omega_n T}) |G(\omega_n)| \leq \frac{2\pi \varepsilon}{T \gamma^2 (1 - \varepsilon)} \sum_{n=0}^{\infty} |C_n|^2 (1 + e^{-2\omega_n T}). \quad (34)
\]

In a similar way, taking also into account (10) and \( \alpha \rho < 0 \), one can get
\[
\sum_{n,m=0}^{\infty} |R_{2,m}| |C_n| (1 + e^{(\rho_n - 3\omega_n) T}) |G(\omega_n - i\rho_n)| \leq \frac{2\pi \varepsilon}{T \gamma^2 (1 - \varepsilon)} \sum_{n=0}^{\infty} |C_n|^2 (1 + e^{-2\omega_n T}). \quad (35)
\]

Plugging (34) and (35) into (33) we obtain
\[
\sum_{n,m=0}^{\infty} \int_0^{\infty} g(t) \left( e^{i\omega_n t} + e^{-i\omega_n t} \right) \left( R_{1,m} + R_{2,m} e^{\rho_n t} \right) \, dt \geq - \frac{4\pi \varepsilon}{T \gamma^2 (1 - \varepsilon)} \sum_{n=0}^{\infty} |C_n|^2 (1 + e^{-2\omega_n T}).
\]

By (32) we get
\[
\int_0^{\infty} g(t) \left( \sum_{n=0}^{\infty} C_n e^{i\omega_n t} + C_n e^{-i\omega_n t} + R_{1,n} + R_{2,n} e^{\rho_n t} \right)^2 \, dt \geq 2\pi T \sum_{n=0}^{\infty} \left( \frac{1}{\pi^2 + 4T^2 (3\omega_n)^2} - \frac{4(1 + \varepsilon)}{T^2 \gamma^2 (1 - \varepsilon)} \right) (1 + e^{-2\omega_n T}) |C_n|^2.
\]

In conclusion, taking into account the definition of \( g \) (31) follows from the above inequality. \( \square \)
To make meaningful the inequality (31), we need to discuss the right-hand side.

**Theorem 3.6** Under the assumptions of Proposition 3.5, if \( \gamma > 4\alpha_\omega \) and \( T > \frac{2\pi}{\sqrt{\gamma^2(1-\varepsilon) - 16\alpha_\omega^2(1+\varepsilon)}} \), then there exists \( n_0 = n_0(\varepsilon) \in \mathbb{N} \) independent of \( T \) and \( C_n \) such that we have

\[
\int_0^T \left( \sum_{n=n_0}^{\infty} C_n e^{i\omega_n t} + \overline{C_n} e^{-i\omega_n t} + R_{1,n} + R_{2,n} e^{\rho_n t} \right)^2 dt 
\geq 2\pi T \left( \frac{1}{\pi^2 + 4T^2 \alpha_\omega^2(1+\varepsilon)} - \frac{4}{T^2 \gamma^2(1-\varepsilon)} \right) \sum_{n=n_0}^{\infty} (1 + e^{-2\varepsilon\omega_n T})|C_n|^2. \tag{36}
\]

**Proof.** We start by using (16) and replacing in (31) \( \varepsilon \) with \( \varepsilon' \in (0, \frac{\varepsilon}{2}) \); taking into account that \( \frac{1+\varepsilon'}{1-\varepsilon'} < \frac{1}{1-\varepsilon} \), we have for \( n \geq n_0 \)

\[
\frac{1}{\pi^2 + 4T^2 (3\omega_n)^2} - \frac{4(1+\varepsilon')}{T^2 \gamma^2(1-\varepsilon')} \geq \frac{1}{\pi^2 + 4T^2 \alpha_\omega^2(1+\varepsilon)} - \frac{4}{T^2 \gamma^2(1-\varepsilon)}
\]

The constant

\[
\frac{1}{\pi^2 + 4T^2 \alpha_\omega^2(1+\varepsilon)} - \frac{4}{T^2 \gamma^2(1-\varepsilon)}
\]

is positive if

\[
T^2 \left[ \gamma^2(1-\varepsilon) - 16\alpha_\omega^2(1+\varepsilon) \right] > 4\pi^2. \tag{37}
\]

Since \( \gamma > 4\alpha_\omega \) we have \( \gamma^2(1-\varepsilon) - 16\alpha_\omega^2(1+\varepsilon) > 0 \) if \( \varepsilon < \frac{\gamma^2-16\alpha_\omega^2}{\gamma^2+16\alpha_\omega^2} \). If we assume the more restrictive condition \( T > \frac{2\pi}{\sqrt{\gamma^2(1-\varepsilon) - 16\alpha_\omega^2(1+\varepsilon)}} \) with respect to that \( T > \frac{2\pi}{\gamma \sqrt{1-\varepsilon}} \), then (37) holds true. \( \square \)

If we take the sequences \( \{\omega_n\}_{n \in \mathbb{Z}} \) and \( \{\rho_n\}_{n \in \mathbb{Z}} \) composed by pairwise distinct non null numbers such that \( \rho_n \neq i\omega_m \) for any \( n, m \in \mathbb{Z} \) and argue in a similar way as in [5, 6], where the Haraux method [1] is successfully applied, we get the following inverse inequality.

**Theorem 3.7** For any \( T > \frac{2\pi}{\sqrt{\gamma^2(1-\varepsilon) - 16\alpha_\omega^2(1+\varepsilon)}} \) there exists a positive constant \( C = C(T) \) such that

\[
\int_0^T \left( \sum_{n=1}^{\infty} C_n e^{i\omega_n t} + \overline{C_n} e^{-i\omega_n t} + R_{1,n} + R_{2,n} e^{\rho_n t} \right)^2 dt \geq C \sum_{n=1}^{\infty} (1 + e^{-2\varepsilon\omega_n T})|C_n|^2. \tag{38}
\]

If we assume that \( k : [0, \infty) \to [0, \infty) \) is a locally absolutely continuous function such that \( k(0) > 0 \), \( k'(t) \leq 0 \) for a.e. \( t \geq 0 \) and \( \int_0^t k(s) \, ds < 1 \), \( t \geq 0 \), then for the Cauchy problem

\[
\begin{cases}
u(t, x) = \Delta u(t, x) + \int_0^t k(t-s) \Delta u(s, x) ds = 0, \quad t \geq 0, \ x \in \Omega, \\
u(t, x) = 0 \quad t \geq 0, \ x \in \partial \Omega, \\
u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \Omega.
\end{cases} \tag{39}
\]

we can adapt the argumentations done in [7, Theorem 1.3] to obtain the following result.

**Theorem 3.8** For \( T > 0 \), there exists a constant \( C_0 > 0 \) depending on \( T \) and \( k \) such that for every \( u_0 \in H_0^1(\Omega) \) and \( u_1 \in L^2(\Omega) \), denoted by \( u \) the weak solution of (39), one can define \( \partial_\nu u \) such that the following inequality holds

\[
\int_0^T \int_{\partial \Omega} |\partial_\nu u|^2 \, d\sigma dt \leq C_0 (\|u_0\|_{H_0^1(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2).
\tag{40}
\]
4 Controllability

Theorem 4.1 Let $b_1, b_2, r_1, r_2 > 0$ with $\frac{3}{2}(b_1 + b_2) < r_1 + r_2$ and $\frac{b_1}{r_1} + \frac{b_2}{r_2} = 1$.

Then, there exists $T_0 > 0$ such that for $T > T_0$ and $(u_0, u_1) \in L^2(0, \pi) \times H^{-1}(0, \pi)$, there exists $f \in L^2(0, T)$ such that the weak solution $u$ of the equation

$$u_{tt}(t, x) = \gamma^2 u_{xx}(t, x) - b_1 \int_0^t e^{-r_1(t-s)} \gamma^2 u_{xx}(s, x) ds - b_2 \int_0^t e^{-r_2(t-s)} \gamma^2 u_{xx}(s, x) ds, \quad t \in (0, T), \ x \in (0, \pi),$$

with boundary conditions

$$u(t, 0) = 0, \quad u(t, \pi) = f(t), \quad t \in (0, T),$$

and null initial values

$$u(0, x) = u_t(0, x) = 0, \quad x \in (0, \pi),$$

verifies the final conditions

$$u(T, x) = u_0(x), \quad u_t(T, x) = u_1(x), \ x \in (0, \pi).$$

Proof. To prove the statement we apply the Hilbert Uniqueness Method, following a similar strategy to that used in [5].

Let $\Omega = (0, \pi)$ and $X = L^2(0, \pi)$ be endowed with the usual scalar product and norm

$$\|u\| := \left( \int_0^\pi |u(x)|^2 \, dx \right)^{1/2} \quad u \in L^2(0, \pi).$$

We consider the operator $L : D(L) \subset L^2(0, \pi) \to L^2(0, \pi)$ defined by

$$Lu = -\gamma^2 u_{xx} \quad u \in D(L) := H^2(0, \pi) \cap H^1_0(0, \pi).$$

It is well known that $L$ is a self-adjoint positive operator on $L^2(0, \pi)$ with dense domain $D(L)$ and

$$D(\sqrt{T}) = H^1_0(0, \pi).$$

Moreover, the eigenvalues of $L$ are $\gamma^2 n^2, \ n \in \mathbb{N}$, and the corresponding eigenvectors are given by $\sqrt{\frac{2}{\pi}} \sin(nx)$ that constitute a Hilbert basis for $L^2(0, \pi)$.

We can apply our spectral analysis to the adjoint problem. First, we consider the adjoint equation given by

$$z_{tt}(t, x) = \gamma^2 z_{xx}(t, x) - b_1 \int_t^T e^{-r_1(s-t)} \gamma^2 z_{xx}(s, x) ds - b_2 \int_t^T e^{-r_2(s-t)} \gamma^2 z_{xx}(s, x) ds,$$

with the Dirichlet boundary condition

$$z(t, 0) = z(t, \pi) = 0 \quad t \in (0, T),$$

and final data

$$z(T, \cdot) = z_0 \in H^1_0(\Omega), \quad z_t(T, \cdot) = z_1 \in L^2(\Omega).$$

The backward problem (42)-(44) is equivalent to a Cauchy problem with $u(t, x) = z(T-t, x)$. Therefore we can apply the conclusions of the previous sections. First, we write the solution $z(t, x)$ of the adjoint problem as a Fourier series. Indeed, the solution $z$ of the adjoint problem can be written in the form

$$z(t, x) = \sum_{n=1}^{\infty} \left( C_n e^{i\omega_n(T-t)} + \overline{C_n} e^{-i\omega_n(T-t)} + R_{1,n} + R_{2,n} e^{i\omega_n(T-t)} \right) \sin(nx), \quad (t, x) \in [0, T] \times [0, \pi],$$
whence
\[ z_x(t, \pi) = \sum_{n=1}^{\infty} (-1)^n n \left( C_n e^{i\omega_n(T-t)} + \bar{C}_n e^{-i\omega_n(T-t)} + R_{1,n} + R_{2,n} e^{\rho_n(T-t)} \right) \quad t \in [0, T]. \]

We can apply theorems 3.7 and 3.8 to function \( z_x(t, \pi) \). By estimates (38), (13) and (40) we have that
\[ \int_0^T \left| z_x(t, \pi) \right|^2 dt \approx \| z_0 \|^2_{H_0^1(\Omega)} + \| z_1 \|^2_{L^2(\Omega)}. \]

The proof is complete. □

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