MULTI-LINEAR PRODUCTS WITH ODD FACTORS IN
PSEUDO-DIFFERENTIAL CALCULUS WITH
SYMBOLS IN MODULATION SPACES

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Abstract. We give sufficient conditions on the Lebesgue exponents for compositions of odd numbers of pseudo-differential operators with symbols in modulation spaces. As a byproduct, we obtain sufficient conditions for twisted convolutions of odd numbers of factors to be bounded on Wiener amalgam spaces.

0. Introduction

In the paper we deduce multi-linear Weyl products and other products in pseudo-differential calculus of odd factors with symbols belonging to suitable modulation spaces. Especially we improve some of the odd multi-linear products in [4]. Here we notice that for corresponding bilinear products, the results in [4] are sharp.

Suppose that \( N \geq 1 \) is an integer. Then it follows from [4, Proposition 2.5] that

\[
M^{p_1, q_1} \# M^{p_2, q_2} \# M^{p_3, q_3} \# \ldots \# M^{p_N, q_N} \subseteq M^{p_0, q_0}
\]

holds true when

\[
\max \left( R_N \left( \frac{1}{q_j} \right), 0 \right) \leq \min \left( \frac{1}{p_j}, \frac{1}{p_j'}, \frac{1}{q_j}, \frac{1}{q_j'}, R_N \left( \frac{1}{p} \right) \right).
\]

Here

\[
R_N(x) = \frac{1}{N-1} \left( \sum_{j=0}^{N} x_j - 1 \right), \quad x = (x_0, x_1, \ldots, x_N) \in [0, 1]^{N+1},
\]

and we observe that [4 Proposition 2.5] is a weighted version of (0.1) and (0.2). (See [18] and Section 1 for notations.) We notice that the conditions on \( p_j \) and \( q_j \) in (0.2) can be outlined because they are always fulfilled when the other conditions hold (see e.g. Theorem 0.1' in [4] and its proof).

The multi-linear multiplication property (0.1) with (0.2) is obtained by interpolating the case that (0.1) holds true when (0.2) is replaced by

\[
R_N \left( \frac{1}{q_j} \right) \leq 0 \leq R_N \left( \frac{1}{p} \right),
\]

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(see Proposition 2.1 in [4]) with the case
\[ M^{2,2} \# M^{2,2} \# \cdots \# M^{2,2} \subseteq M^{2,2}. \quad (0.1)' \]
Note that the last multiplication property is equivalent with the fact that compositions of Hilbert-Schmidt operators result into new Hilbert-Schmidt operators.

In the paper we improve (0.1) and (0.2) for multi-linear products with odd factors, by replacing (0.1)' with the more general
\[ M^{p,p} \# M^{p',p'} \# \cdots \# M^{p,p} \subseteq M^{p,p} \quad (0.1)'' \]
in the interpolation with (0.1) and (0.2)'. This leads to that (0.1) remains true for odd \( N \), when (0.2) is replaced by
\[ \max \left( R_N \left( \frac{1}{p'} \right), 0 \right) \leq \min \left( Q_N \left( \frac{1}{p'} \right), Q_N \left( \frac{1}{p} \right), Q_N \left( \frac{1}{p'}, \frac{1}{p} \right), R_N \left( \frac{1}{p} \right) \right). \quad (0.2)'' \]
Here
\[ Q_N(x, y) = \min_{j+k \text{ odd}} \left( \frac{1}{2} (x_j + y_k), 1 - \frac{1}{2} (x_j + y_k) \right) \text{ and } Q_N(x) = Q_N(x, x), \]
when \( x = (x_0, x_1, \ldots, x_N) \in [0, 1]^{N+1} \) and \( y = (y_0, y_1, \ldots, y_N) \in [0, 1]^{N+1} \).
In fact, in Section 2 we deduce weighted versions of (0.1) and (0.2)' (see Proposition 2.5 and Theorem 2.6 in Section 2).

We observe that (0.1) is close to certain regularization techniques for linear operators in distribution theory. In fact, a convenient way to regularizing operators might be to enclose them with regularizing operators. For example, suppose that \( T \) is a linear and continuous operator from the Schwartz space \( \mathcal{S}(\mathbb{R}^d) \) to the set of tempered distributions \( \mathcal{S}'(\mathbb{R}^d) \). The operator \( T \) possess weak continuity properties in the sense that it is only guaranteed that \( T \) maps elements from \( \mathcal{S}(\mathbb{R}^d) \) into the significantly larger space \( \mathcal{S}'(\mathbb{R}^d) \). On the other hand, by enclosing \( T \) with operators \( S_1, S_2 \) which are regularizing in the sense that they are mapping \( \mathcal{S}(\mathbb{R}^d) \) into the smaller space \( \mathcal{S}(\mathbb{R}^d) \), the resulting operator
\[ T_0 = S_1 \circ T \circ S_2 \]
becomes again regularizing. Equivalently, by using the kernel theorem of Schwartz and identifying kernels with operators one has that
\[ (K_1, K_2, K_3) \mapsto K_1 \circ K_2 \circ K_3, \]
\[ \mathcal{S}(\mathbb{R}^{2d}) \times \mathcal{S}'(\mathbb{R}^{2d}) \times \mathcal{S}(\mathbb{R}^{2d}) \to \mathcal{S}(\mathbb{R}^{2d}), \quad (0.3) \]
is sequently continuous. Here we observe that by omitting one of the compositions one may only guarantee that
\[ (K_1, K_2) \mapsto K_1 \circ K_2 \]
is sequently continuous from \( \mathcal{S}(\mathbb{R}^{2d}) \times \mathcal{S}'(\mathbb{R}^{2d}) \) or \( \mathcal{S}'(\mathbb{R}^{2d}) \times \mathcal{S}(\mathbb{R}^{2d}) \) to \( \mathcal{S}'(\mathbb{R}^{2d}) \), which is a significantly weaker continuity property.
The continuity of the map \((0.3)\) can be obtained by rewriting \(K_1 \circ K_2 \circ K_3\) as
\[
K_1 \circ K_2 \circ K_3(x, y) = R_{K_2, F}(x, y) \equiv \langle K_2, F(x, y, \cdot) \rangle,
\]
where
\[
F(x, y, x_1, x_2) = (K_1 \otimes K_3)(x, x_1, x_2, y) = K_1(x, x_1)K_3(x_2, y).
\]
The asserted continuity then follows from the facts that \((K, F) \mapsto R_{K, F}\) is continuous from \(\mathcal{S}'(\mathbb{R}^{2d}) \times \mathcal{S}(\mathbb{R}^{2d})\) to \(\mathcal{S}(\mathbb{R}^{2d})\), and that \((K_1, K_3) \mapsto K_1 \otimes K_3\) is continuous from \(\mathcal{S}(\mathbb{R}^{2d}) \times \mathcal{S}(\mathbb{R}^{2d})\) to \(\mathcal{S}(\mathbb{R}^{2d})\).

The mapping properties can also be conveniently formulated within the theory of pseudo-differential calculus, e.g. in the Weyl calculus. Recall that the Weyl product \(#\) of the elements \(a_1, a_2 \in \mathcal{S}(\mathbb{R}^{2d})\) are defined by the formula
\[
\text{Op}^w(a_1 # a_2) = \text{Op}^w(a_1) \circ \text{Op}^w(a_2),
\]
where \(\text{Op}^w(a)\) is the Weyl operator of \(a \in \mathcal{S}(\mathbb{R}^{2d})\), defined by
\[
\text{Op}^w(a)f(x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} a(\frac{1}{2}(x + y), \xi)f(y)e^{i(x-y, \xi)} \, dy \, d\xi,
\]
when \(f \in \mathcal{S}(\mathbb{R}^d)\). The definition of \(\text{Op}^w(a)\) extends to a continuous operator from \(\mathcal{S}'(\mathbb{R}^d)\) to \(\mathcal{S}^*(\mathbb{R}^{2d})\). We may also extend the definition of \(\text{Op}^w(a)\) to allow \(a \in \mathcal{S}'(\mathbb{R}^{2d})\), and then \(\text{Op}^w(a)\) is continuous from \(\mathcal{S}(\mathbb{R}^d)\) to \(\mathcal{S}'(\mathbb{R}^d)\).

By straight-forward Fourier techniques it follows that \((0.3)\) is equivalent to
\[
\mathcal{S}(\mathbb{R}^{2d}) \# \mathcal{S}(\mathbb{R}^{2d}) \# \mathcal{S}(\mathbb{R}^{2d}) \subseteq \mathcal{S}(\mathbb{R}^{2d}), \tag{0.4}
\]
which by duality gives
\[
\mathcal{S}'(\mathbb{R}^{2d}) \# \mathcal{S}'(\mathbb{R}^{2d}) \# \mathcal{S}'(\mathbb{R}^{2d}) \subseteq \mathcal{S}'(\mathbb{R}^{2d}), \tag{0.5}
\]
Again we observe that for the bilinear case we only have
\[
\mathcal{S}(\mathbb{R}^{2d}) \# \mathcal{S}(\mathbb{R}^{2d}) \subseteq \mathcal{S}'(\mathbb{R}^{2d}),
\]
and that \(\mathcal{S}'(\mathbb{R}^{2d}) \# \mathcal{S}'(\mathbb{R}^{2d})\) does not make any sense.

By similar arguments, \((0.4)\) and \((0.5)\) can be extended to any odd-linear Weyl products. That is, one has
\[
\mathcal{S}(\mathbb{R}^{2d}) \# \mathcal{S}(\mathbb{R}^{2d}) \# \mathcal{S}(\mathbb{R}^{2d}) \# \cdots \# \mathcal{S}(\mathbb{R}^{2d}) \subseteq \mathcal{S}(\mathbb{R}^{2d}), \tag{0.4}'
\]
and
\[
\mathcal{S}'(\mathbb{R}^{2d}) \# \mathcal{S}'(\mathbb{R}^{2d}) \# \mathcal{S}'(\mathbb{R}^{2d}) \# \cdots \# \mathcal{S}'(\mathbb{R}^{2d}) \subseteq \mathcal{S}'(\mathbb{R}^{2d}), \tag{0.5}'
\]
which are analogous to \((0.1)''\), taking into account that \(M^{p', p'}\) is the dual of \(M^{p, p}\) when \(p < \infty\).
1. Preliminaries

In this section we introduce notation and discuss the background on Gelfand–Shilov spaces, pseudo-differential operators, the Weyl product, twisted convolution and modulation spaces. Most proofs can be found in the literature and are therefore omitted.

Let $0 < h, s \in \mathbb{R}$ be fixed. The space $\mathcal{S}_{s,h}(\mathbb{R}^d)$ consists of all $f \in C^\infty(\mathbb{R}^d)$ such that

$$
\|f\|_{s,h} \equiv \sup_{x} \frac{|x^\beta \partial^\alpha f(x)|}{h^{\alpha_1 + |\beta|} \alpha! \beta! s} \leq 1
$$

is finite, with supremum taken over all $\alpha, \beta \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$.

The space $\mathcal{S}_{s,h} \subseteq \mathcal{S}$ ($\mathcal{S}$ denotes the Schwartz space) is a Banach space which increases with $h$ and $s$. Inclusions between topological spaces are understood to be continuous. If $s > 1/2$, or $s = 1/2$ and $h$ is sufficiently large, then $\mathcal{S}_{s,h}$ contains all finite linear combinations of Hermite functions. Since the space of such linear combinations is dense in $\mathcal{S}$, it follows that the topological dual $(\mathcal{S}_{s,h})'(\mathbb{R}^d)$ of $\mathcal{S}_{s,h}(\mathbb{R}^d)$ is a Banach space which contains $\mathcal{S}'(\mathbb{R}^d)$.

1.1. Gelfand-Shilov spaces of functions and distributions. The Gelfand–Shilov spaces $\mathcal{S}_s(\mathbb{R}^d)$ and $\Sigma_s(\mathbb{R}^d)$ (cf. [13]) are the inductive and projective limits, respectively, of $\mathcal{S}_{s,h}(\mathbb{R}^d)$, with respect to the parameter $h$. Thus

$$
\mathcal{S}_s(\mathbb{R}^d) = \bigcup_{h > 0} \mathcal{S}_{s,h}(\mathbb{R}^d) \quad \text{and} \quad \Sigma_s(\mathbb{R}^d) = \bigcap_{h > 0} \mathcal{S}_{s,h}(\mathbb{R}^d),
$$

where $\mathcal{S}_s(\mathbb{R}^d)$ is equipped with the the strongest topology such that the inclusion map from $\mathcal{S}_{s,h}(\mathbb{R}^d)$ into $\mathcal{S}_s(\mathbb{R}^d)$ is continuous, for every choice of $h > 0$. The space $\Sigma_s(\mathbb{R}^d)$ is a Fréchet space with seminorms $\| \cdot \|_{s,h}$, $h > 0$. We have $\Sigma_s(\mathbb{R}^d) \neq \{0\}$ if and only if $s > 1/2$, and $\Sigma_s(\mathbb{R}^d) \neq \{0\}$ if and only if $s \geq 1/2$. From now on we assume that $s > 1/2$ when we consider $\Sigma_s(\mathbb{R}^d)$, and $s \geq 1/2$ when we consider $\mathcal{S}_s(\mathbb{R}^d)$.

The Gelfand–Shilov distribution spaces $\mathcal{S}'_s(\mathbb{R}^d)$ and $\Sigma'_s(\mathbb{R}^d)$ are the projective and inductive limit respectively of $\mathcal{S}'_{s,h}(\mathbb{R}^d)$.

$$
\mathcal{S}'_s(\mathbb{R}^d) = \bigcap_{h > 0} \mathcal{S}'_{s,h}(\mathbb{R}^d) \quad \text{and} \quad \Sigma'_s(\mathbb{R}^d) = \bigcup_{h > 0} \mathcal{S}'_{s,h}(\mathbb{R}^d).
$$

In [13][19][21] it is proved that $\mathcal{S}'_s(\mathbb{R}^d)$ is the topological dual of $\mathcal{S}_s(\mathbb{R}^d)$, and $\Sigma'_s(\mathbb{R}^d)$ is the topological dual of $\Sigma_s(\mathbb{R}^d)$.

For each $\varepsilon > 0$ and $s > 1/2$ we have

$$
\mathcal{S}_{1/2}(\mathbb{R}^d) \subseteq \Sigma_s(\mathbb{R}^d) \subseteq \mathcal{S}_s(\mathbb{R}^d) \subseteq \Sigma_{s+\varepsilon}(\mathbb{R}^d)
$$

and

$$
\mathcal{S}'_{s+\varepsilon}(\mathbb{R}^d) \subseteq \Sigma'_s(\mathbb{R}^d) \subseteq \mathcal{S}'_s(\mathbb{R}^d) \subseteq \mathcal{S}'_{1/2}(\mathbb{R}^d).
$$
The Gelfand–Shilov spaces are invariant under several basic operations, e.g. translations, dilations, tensor products and (partial) Fourier transformation.

We normalize the Fourier transform of \( f \in L^1(\mathbb{R}^d) \) as \[
(\mathcal{F}f)(\xi) = \hat{f}(\xi) \equiv (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x)e^{-i\langle x,\xi \rangle} \, dx,
\]
where \( \langle \cdot, \cdot \rangle \) denotes the scalar product on \( \mathbb{R}^d \). The map \( \mathcal{F} \) extends uniquely to homeomorphisms on \( \mathcal{S}'(\mathbb{R}^d) \), \( \mathcal{S}'(\mathbb{R}^d) \), \( \Sigma(\mathbb{R}^d) \), and restricts to homeomorphisms on \( \mathcal{S}(\mathbb{R}^d) \), \( \mathcal{S}_s(\mathbb{R}^d) \), and \( \Sigma_s(\mathbb{R}^d) \), and to a unitary operator on \( L^2(\mathbb{R}^d) \).

1.2. Modulation spaces. Next we turn to the basic properties of modulation spaces, and start by recalling the conditions for the involved weight functions. Let \( 0 < \omega \in L^\infty_{\text{loc}}(\mathbb{R}^d) \). Then \( \omega \) is called moderate if there is a function \( 0 < v \in L^\infty_{\text{loc}}(\mathbb{R}^d) \) such that
\[
\omega(x+y) \lesssim \omega(x)v(y), \quad x, y \in \mathbb{R}^d.
\]
(1.3)

Then \( \omega \) is also called \( v \)-moderate. Here the notation \( f(x) \lesssim g(x) \) means that there exists \( C > 0 \) such that \( f(x) \leq Cg(x) \) for all arguments \( x \) in the domain of \( f \) and \( g \). If \( f \lesssim g \) and \( g \lesssim f \) we write \( f \asymp g \). The function \( v \) is called submultiplicative if it is even and (1.3) holds when \( \omega = v \). We note that if (1.3) holds then
\[
v^{-1} \lesssim \omega \lesssim v.
\]

For such \( \omega \) it follows that (1.3) is true when
\[
v(x) = Ce^{c|x|},
\]
for some positive constants \( c \) and \( C \) (cf. [15]). In particular, if \( \omega \) is moderate on \( \mathbb{R}^d \), then
\[
e^{-c|x|} \lesssim \omega(x) \lesssim e^{c|x|},
\]
for some \( c > 0 \).

The set of all moderate functions on \( \mathbb{R}^d \) is denoted by \( \mathcal{P}(\mathbb{R}^d) \). If \( v \) in (1.3) can be chosen as \( v(x) = \langle x \rangle^s = (1 + |x|^2)^{s/2} \) for some \( s \geq 0 \), then \( \omega \) is said to be of polynomial type or polynomially moderate. We let \( \mathcal{P}(\mathbb{R}^d) \) be the set of all polynomially moderate functions on \( \mathbb{R}^d \).

Let \( \phi \in \mathcal{S}_s(\mathbb{R}^d) \setminus 0 \) be fixed. The short-time Fourier transform (STFT) \( V_\phi f \) of \( f \in \mathcal{S}'(\mathbb{R}^d) \) with respect to the window function \( \phi \) is the Gelfand–Shilov distribution on \( \mathbb{R}^{2d} \) defined by
\[
V_\phi f(x,\xi) \equiv \mathcal{F}(\phi \cdot f)(\xi).
\]

For \( a \in \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \) and \( \Phi \in \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \setminus 0 \) the symplectic short-time Fourier transform \( V_{\phi a} \) of \( a \) with respect to \( \Phi \) is the defined similarly as
\[
V_{\phi a}(X,Y) = \mathcal{F}_{\sigma}(a \Phi(\cdot - X))(Y), \quad X, Y \in \mathbb{R}^{2d}.
\]
We have
\[ V_\Phi a(X, Y) = 2^d V_\Phi a(x, \xi, -2\eta, 2y), \]
which shows the close connection between \( V_\Phi a \) and \( V_\Phi a \). The Wigner distribution \( W_f,\phi \) and \( V_\phi f \) are also closely related.

If \( f, \phi \in \mathcal{S}_s(\mathbb{R}^d) \) and \( a, \Phi \in \mathcal{S}_s(\mathbb{R}^{2d}) \) then
\[ V_\phi f(x, \xi) = (2\pi)^{-d} \int f(y) \phi(y - x) e^{-i\langle y, \xi \rangle} \, dy \]
and
\[ V_\Phi a(X, Y) = \pi^{-d} \int a(Z) \Phi(Z - X) e^{2i\sigma(Y, Z)} \, dZ. \]

Let \( \omega \in \mathcal{P}_E(\mathbb{R}^{2d}) \), \( p, q \in [1, \infty] \) and \( \phi \in \mathcal{S}_{1/2}(\mathbb{R}^d) \setminus \{0\} \) be fixed. The modulation space \( M_{\omega}^{p,q}(\mathbb{R}^d) \) consists of all \( f \in \mathcal{S}'_{1/2}(\mathbb{R}^d) \) such that
\[ \| f \|_{M_{\omega}^{p,q}} \equiv \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_\phi f(x, \xi) \omega(x, \xi)|^p \, dx \right)^{q/p} \, d\xi \right)^{1/q} \]  
(1.5)
is finite, and the Wiener amalgam space \( W_{\omega}^{p,q}(\mathbb{R}^d) \) consists of all \( f \in \mathcal{S}'_{1/2}(\mathbb{R}^d) \) such that
\[ \| f \|_{W_{\omega}^{p,q}} \equiv \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_\phi f(x, \xi) \omega(x, \xi)|^q \, dx \right)^{p/q} \, d\xi \right)^{1/p} \]  
(1.6)
is finite (with obvious modifications in (1.5) and (1.6) when \( p = \infty \) or \( q = \infty \)).

**Remark 1.1.** As follows from Proposition 1.3 (2) below we have that in fact \( M_{\omega}^{p,q}(\mathbb{R}^d) \) contains the superspace \( \Sigma_1(\mathbb{R}^d) \) of \( \mathcal{S}'_{1/2}(\mathbb{R}^d) \), and is contained in the subspace \( \Sigma'_1(\mathbb{R}^d) \) of \( \mathcal{S}'_1(\mathbb{R}^d) \), when \( \omega \in \mathcal{P}_E(\mathbb{R}^{2d}) \). Hence we could from the beginning have assumed that \( f \in \Sigma'_1(\mathbb{R}^d) \) in (1.5) and (1.6).

On the other hand, in [30], certain weight classes containing \( \mathcal{P}_E(\mathbb{R}^{2d}) \) and superexponential weights are introduced. For any \( s > 1/2 \), the corresponding families of modulation spaces are large enough to contain superspaces of \( \mathcal{S}'_s(\mathbb{R}^d) \) and subspaces of \( \mathcal{S}_s(\mathbb{R}^d) \).

However, we are not dealing with these large families of modulation spaces because we need (1) and (2) in Proposition 1.3 which are not known to be true for weights of this generality.

**Remark 1.2.** The literature contains slightly different conventions concerning modulation and Wiener amalgam spaces. Sometimes our definition of a Wiener amalgam space is considered as a particular case of a general class of modulation spaces (cf. [5–7]). Our definition is adapted to give the relation (1.9) that suits our purpose to transfer continuity.
for the Weyl product on modulation spaces to continuity for twisted convolution on Wiener amalgam spaces.

On the even-dimensional phase space $\mathbb{R}^{2d}$ we may define modulation spaces based on the symplectic STFT. Thus if $\omega \in \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \setminus 0$ are fixed, then the symplectic modulation spaces $M^{p,q}_{(\omega)}(\mathbb{R}^{2d})$ and Wiener amalgam spaces $W^{p,q}_{(\omega)}(\mathbb{R}^{2d})$ are obtained by replacing the STFT $a \mapsto \mathcal{V}\Phi a$ by the corresponding symplectic version $a \mapsto \mathcal{V}_{\Phi}a$ in (1.5) and (1.6). (Sometimes the word symplectic before modulation space is omitted for brevity.) By (1.4) we have

$$ M^{p,q}_{(\omega)}(\mathbb{R}^{2d}) = M^{p,p}_{(\omega)}(\mathbb{R}^{2d}), \quad \omega(x, \xi, y, \eta) = \omega_0(x, \xi, -2\eta, 2y). $$

It follows that all properties which are valid for $M^{p,q}_{(\omega)}$ carry over to $M^{p,q}_{(\omega)}$.

From

$$ V_{\omega} f(x, -x) = e^{i(x, \xi)} V_\Phi f(x, \xi) $$

it follows that

$$ f \in W^{p,q}_{(\omega)}(\mathbb{R}^{2d}) \iff \hat{f} \in M^{p,q}_{(\omega)}(\mathbb{R}^{2d}), \quad \omega_0(\xi, -x) = \omega(x, \xi). $$

In the symplectic situation these formulas read

$$ \mathcal{V}_{\mathcal{F},\Phi}(\mathcal{F}_{\sigma}a)(X, Y) = e^{2i\sigma(Y,X)} \mathcal{V}_\Phi a(Y, X) $$

and

$$ \mathcal{F}_{\sigma} M^{p,q}_{(\omega)}(\mathbb{R}^{2d}) = W^{p,p}_{(\omega)}(\mathbb{R}^{2d}), \quad \omega_0(X, Y) = \omega(Y, X). $$

For brevity we denote $M^{p}_{(\omega)} = M^{p,p}_{(\omega)}$, $W^{p}_{(\omega)} = W^{p,p}_{(\omega)}$, and when $\omega \equiv 1$ we write $M^{p,q} = M^{p,q}_{(1)}$ and $W^{p,q} = W^{p,q}_{(1)}$. We also let $M^{p,q}_{(\omega)}(\mathbb{R}^{2d})$ be the completion of $\mathcal{S}_{s}(\mathbb{R}^{2d})$ with respect to the norm $\| \cdot \|_{M^{p,q}_{(\omega)}}$.

In the following proposition we list some basic facts on invariance, growth and duality for modulation spaces. For any $p \in (0, \infty]$, its conjugate exponent $p' \in [1, \infty]$ is defined by

$$ p' = \begin{cases} 
\infty, & p \in (0, 1], \\
\frac{p}{p-1}, & p \in (1, \infty), \\
1, & p = \infty.
\end{cases} $$

Since our main results are formulated in terms of symplectic modulation spaces, we state the result for them instead of the modulation spaces $M^{p,q}_{(\omega)}(\mathbb{R}^{2d})$.

**Proposition 1.3.** Let $p, q, p_j, q_j \in [1, \infty]$ for $j = 1, 2$, and $\omega, \omega_1, \omega_2, v \in \mathcal{P}_E(\mathbb{R}^{4d})$ be such that $v = \hat{v}$, $\omega$ is $v$-moderate and $\omega_2 \lesssim \omega_1$. Then the following is true:
1. $a \in M_{(\omega)}^{p,q}(R^{2d})$ if and only if (1.5) holds for any $\phi \in M_{(\omega)}^{1,1}(R^{2d}) \setminus 0$. Moreover, $M_{(\omega)}^{p,q}(R^{2d})$ is a Banach space under the norm in (1.5) and different choices of $\phi$ give rise to equivalent norms;

2. if $p_1 \leq p_2$ and $q_1 \leq q_2$ then

\[ \Sigma_1(R^{2d}) \subseteq M_{(\omega)}^{p,q}(R^{2d}) \subseteq \Sigma_1(R^{2d}) \quad \text{and} \quad M_{(\omega_1)}^{p_1,q_1}(R^{2d}) \subseteq M_{(\omega_2)}^{p_2,q_2}(R^{2d}). \]

If in addition $v \in \mathcal{P}(R^{2d})$, then

\[ \mathcal{P}(R^{2d}) \subseteq M_{(\omega)}^{p,q}(R^{2d}) \subseteq \mathcal{P}(R^{2d}); \]

3. the $L^2$ inner product $(\cdot, \cdot)_{L^2}$ on $S_{1/2}$ extends uniquely to a continuous sesquilinear form $(\cdot, \cdot)$ on $M_{(\omega)}^{p,q}(R^{2d}) \times M_{(1/\omega)}^{p,q}(R^{2d})$. On the other hand, if $\|a\| = \sup \{|a,b|\}$, where the supremum is taken over all $b \in S_{1/2}(R^{2d})$ such that $\|b\|_{M_{(1/\omega)}^{p,q}} \leq 1$, then $\|\cdot\|$ and $\|\cdot\|_{M_{(\omega)}^{p,q}}$ are equivalent norms;

4. if $p, q < \infty$, then $S_{1/2}(R^{2d})$ is dense in $M_{(\omega)}^{p,q}(R^{2d})$ and the dual space of $M_{(\omega)}^{p,q}(R^{2d})$ can be identified with $M_{(1/\omega)}^{p',q'}(R^{2d})$, through the form $(\cdot, \cdot)$. Moreover, $S_{1/2}(R^{2d})$ is weakly dense in $M_{(\omega)}^{p,q}(R^{2d})$ with respect to the form $(\cdot, \cdot)$ provided $(p, q) \neq (\infty, 1)$ and $(p, q) \neq (1, \infty)$;

5. if $p, q, r, s, u, v \in [1, \infty]$, $0 \leq \theta \leq 1$,

\[ \frac{1}{p} = \frac{1}{r} + \frac{\theta}{u} \quad \text{and} \quad \frac{1}{q} = \frac{1}{s} + \frac{\theta}{v}, \]

then complex interpolation gives

\[ (M_{(\omega)}^{\theta,s}, M_{(\omega)}^{u,v})[\theta] = M_{(\omega)}^{p,q}. \]

Similar facts hold if the $M_{(\omega)}^{p,q}$ spaces are replaced by the $W_{(\omega)}^{p,q}$ spaces.

The proof of Proposition 1.3 can be found in [2, 5, 6, 8, 10, 11, 27, 30].

In fact, (1) follows from Gröchenig’s argument verbatim in [14 Proposition 11.3.2 (c)]. Note that the window class $M_{(\omega)}^{1,1}(R^{2d})$ in (1) contains $\Sigma_1(R^{2d})$, which in turn contains $S_{1/2}(R^{2d})$. Furthermore, if in addition $v \in \mathcal{P}(R^{4d})$, then $M_{(\omega)}^{1,1}(R^{2d})$ contains $\mathcal{P}(R^{2d})$.

The proof of (2) in [14 Chapter 12] is based on Gabor frames and formulated for polynomial type weights $\mathcal{P}(R^{4d})$. These arguments also hold for the broader weight class $\mathcal{P}_E(R^{4d})$. Another way to prove this is by means of [14 Lemma 11.3.3] and Young’s inequality.

The assertions (3)–(5) in Proposition 1.3 can be found for more general weights in Theorem 4.17, and a combination of Theorem 3.4 and Proposition 5.2 in [30].

Remark 1.4. For $p, q \in (0, \infty]$ (instead of $p, q \in [1, \infty]$ as in Proposition 1.3), $a \in M_{(\omega)}^{p,q}(R^{2d})$ if and only if (1.5) holds for any $\phi \in \Sigma_1(R^{2d}) \setminus 0$.\
Moreover, \( M^{p,q}(\mathbb{R}^{2d}) \) is a quasi-Banach space under the quasi-norm in \((1.5)\) and different choices of \( \phi \) give rise to equivalent quasi-norms. (See e.g. \([12, 32]\).)

**Remark 1.5.** Let \( \mathcal{P} \) be the set of all \( \omega \in \mathcal{P}_E(\mathbb{R}^{2d}) \) such that

\[
\omega(X, Y) = e^{c(|X|^{1/s} + |Y|^{1/s})},
\]

for some \( c > 0 \). (Note that this implies that \( s \geq 1 \).) Then

\[
\bigcap_{\omega \in \mathcal{P}} M_{(1/\omega)}^{p,q}(\mathbb{R}^{2d}) = \Sigma_s(\mathbb{R}^{2d}),
\]

\[
\bigcup_{\omega \in \mathcal{P}} M_{(1/\omega)}^{p,q}(\mathbb{R}^{2d}) = \Sigma'_s(\mathbb{R}^{2d}),
\]

\[
\bigcup_{\omega \in \mathcal{P}} M_{(1/\omega)}^{p,q}(\mathbb{R}^{2d}) = \mathcal{S}_s(\mathbb{R}^{2d}),
\]

\[
\bigcup_{\omega \in \mathcal{P}} M_{(1/\omega)}^{p,q}(\mathbb{R}^{2d}) = \mathcal{S}'_s(\mathbb{R}^{2d}),
\]

and for \( \omega \in \mathcal{P} \)

\[
\Sigma_s(\mathbb{R}^{2d}) \subseteq M_{(1/\omega)}^{p,q}(\mathbb{R}^{2d}) \subseteq \Sigma'_s(\mathbb{R}^{2d}) \quad \text{and} \quad \mathcal{S}_s(\mathbb{R}^{2d}) \subseteq M_{(1/\omega)}^{p,q}(\mathbb{R}^{2d}) \subseteq \Sigma'_s(\mathbb{R}^{2d}).
\]

( Cf. \([3, \text{Prop. 4.5}], [16, \text{Prop. 4}], [22, \text{Cor. 5.2}] \) and \([25, \text{Thm. 4.1}]\). See also \([30, \text{Thm. 3.9}]\) for an extension of these inclusions to broader classes of Gelfand–Shilov and modulation spaces.)

By Proposition 1.3 (4) we have norm density of \( \mathcal{S}_{1/2} \) in \( M_{(1/\omega)}^{p,q} \) when \( p, q < \infty \). We may relax the assumptions on \( p \), provided we replace the norm convergence with narrow convergence. This concept, that allows us to approximate elements in \( M_{(1/\omega)}^{p,q}(\mathbb{R}^{2d}) \) for \( 1 \leq q < \infty \), is treated in \([23, 27, 28]\), and, for the current setup of possibly exponential weights, in \([30]\). (Sjöstrand’s original definition in \([23]\) is somewhat different.)

Narrow convergence is defined by means of the function

\[
H_{a, \omega, p}(Y) \equiv \| \nu a(\cdot, Y) \omega(\cdot, Y) \|_{L^p(\mathbb{R}^{2d})}, \quad Y \in \mathbb{R}^{2d},
\]

for \( a \in \mathcal{S}_{1/2}'(\mathbb{R}^{2d}), \omega \in \mathcal{P}_E(\mathbb{R}^{2d}), \Phi \in \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \) and \( p \in [1, \infty] \).

**Definition 1.6.** Let \( p, q \in [1, \infty] \), and \( a, a_j \in M_{(\omega)}^{p,q}(\mathbb{R}^{2d}) \), \( j = 1, 2, \ldots \), then \( a_j \) is said to converge narrowly to \( a \) with respect to \( p, q, \Phi \) if \( a \in \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \) and \( \omega \in \mathcal{P}_E(\mathbb{R}^{2d}) \), if there exist \( g_j, g \in L^q(\mathbb{R}^{2d}) \) such that:

1. \( a_j \to a \) in \( \mathcal{S}_{1/2}'(\mathbb{R}^{2d}) \) as \( j \to \infty \);
2. \( H_{a_j, \omega, p} \leq g_j \) and \( g_j \to g \) in \( L^q(\mathbb{R}^{2d}) \) and a.e. as \( j \to \infty \).

**Proposition 1.7.** If \( \omega \in \mathcal{P}_E(\mathbb{R}^{2d}) \) and \( 1 \leq q < \infty \) then the following is true:

1. \( \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \) is dense in \( M_{(\omega)}^{\infty,q}(\mathbb{R}^{2d}) \) with respect to narrow convergence;
2. \( M_{(\omega)}^{\infty,q}(\mathbb{R}^{2d}) \) is sequentially complete with respect to the topology defined by narrow convergence.

We refer to \([4]\) for a proof of Proposition 1.7.
1.3. Pseudo-differential operators. Next we recall some basic facts from pseudo-differential calculus (cf. [18]). Let \( M(d, \Omega) \) be the set of all \( d \times d \) matrices with entries in \( \Omega \), \( s \geq 1/2 \), \( a \in \mathcal{S}_s(\mathbb{R}^{2d}) \), and \( A \in M(d, \mathbb{R}) \) be fixed. Then the pseudo-differential operator \( \text{Op}_A(a) \) defined by

\[
\text{Op}_A(a) f(x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} a(x-A(x-y), \xi) f(y)e^{i(x-y,\xi)} \, dyd\xi
\]  

(1.10)
is a linear and continuous operator on \( \mathcal{S}_s(\mathbb{R}^d) \). For \( a \in \mathcal{S}'_s(\mathbb{R}^{2d}) \) the pseudo-differential operator \( \text{Op}_A(a) \) is defined as the continuous operator from \( \mathcal{S}_s(\mathbb{R}^d) \) to \( \mathcal{S}'_s(\mathbb{R}^d) \) with distribution kernel given by

\[
K_{a,A}(x,y) = (2\pi)^{-d} (\mathcal{F}_2^{-1}a)(x-A(x-y), x-y).
\]  

(1.11)

Here \( \mathcal{F}_2 F \) is the partial Fourier transform of \( F(x, y) \in \mathcal{S}'_s(\mathbb{R}^{2d}) \) with respect to the variable \( y \in \mathbb{R}^d \). This definition generalizes (1.10) and is well defined, since the mappings

\[
\mathcal{F}_2 \quad \text{and} \quad F(x,y) \mapsto F(x-A(x-y), y-x)
\]  

(1.12)

are homeomorphisms on \( \mathcal{S}'_s(\mathbb{R}^{2d}) \). The map \( a \mapsto K_{a,A} \) is hence a homeomorphism on \( \mathcal{S}'_s(\mathbb{R}^{2d}) \).

If \( A = 0 \), then \( \text{Op}_A(a) \) is the standard or Kohn-Nirenberg representation \( a(x, D) \). If instead \( A = \frac{1}{2} I \), then \( \text{Op}_A(a) \) agrees with the Weyl operator or Weyl quantization \( \text{Op}^w(a) \).

For any \( K \in \mathcal{S}'_s(\mathbb{R}^{d_1+2d_2}) \), let \( T_K \) be the linear and continuous mapping from \( \mathcal{S}_s(\mathbb{R}^{d_1}) \) to \( \mathcal{S}'_s(\mathbb{R}^{d_2}) \) defined by

\[
(T_K f, g)_{L^2(\mathbb{R}^{d_2})} = (K, g \otimes \overline{f})_{L^2(\mathbb{R}^{d_1+2d_2})}, \quad f \in \mathcal{S}_s(\mathbb{R}^{d_1}), \quad g \in \mathcal{S}_s(\mathbb{R}^{d_2}).
\]  

(1.13)

It is a well known consequence of the Schwartz kernel theorem that if \( t \in \mathbb{R} \), then \( K \mapsto T_K \) and \( a \mapsto \text{Op}_A(a) \) are bijective mappings from \( \mathcal{S}'(\mathbb{R}^{2d}) \) to the space of linear and continuous mappings from \( \mathcal{S}_s(\mathbb{R}^d) \) to \( \mathcal{S}'(\mathbb{R}^d) \) (cf. e.g. [18]).

Likewise the maps \( K \mapsto T_K \) and \( a \mapsto \text{Op}_A(a) \) are uniquely extendable to bijective mappings from \( \mathcal{S}'_s(\mathbb{R}^{2d}) \) to the set of linear and continuous mappings from \( \mathcal{S}_s(\mathbb{R}^d) \) to \( \mathcal{S}'_s(\mathbb{R}^d) \). In fact, the asserted bijectivity for the map \( K \mapsto T_K \) follows from the kernel theorem [20, Theorem 2.3] (cf. [13, vol. IV]). This kernel theorem corresponds to the Schwartz kernel theorem in the usual distribution theory. The other assertion follows from the fact that \( a \mapsto K_{a,A} \) is a homeomorphism on \( \mathcal{S}'_s(\mathbb{R}^{2d}) \).

In particular, for each \( a_1 \in \mathcal{S}'_s(\mathbb{R}^{2d}) \) and \( A_1, A_2 \in M(d, \mathbb{R}) \), there is a unique \( a_2 \in \mathcal{S}'_s(\mathbb{R}^{2d}) \) such that \( \text{Op}_{A_1}(a_1) = \text{Op}_{A_2}(a_2) \). The relation between \( a_1 \) and \( a_2 \) is given by

\[
\text{Op}_{A_1}(a_1) = \text{Op}_{A_2}(a_2) \iff a_2(x, \xi) = e^{i((A_1-A_2)D_\xi) / 2} a_1(x, \xi).
\]  

(1.14)

(Cf. [18].) Note that the right-hand side makes sense, since it means \( \tilde{a}_2(\xi, x) = e^{i((A_1-A_2)x, \xi)} \tilde{a}_1(\xi, x) \), and since the map \( a(\xi, x) \mapsto e^{i(x, \xi)} a(\xi, x) \) is continuous on \( \mathcal{S}'_s(\mathbb{R}^{2d}) \).
For future references we observe the relationship
\[
|(V_\Phi K_{a,A})(x, y, \xi, -\eta)| = |(V_\Phi a)(x - A(x - y), A^*\xi + (I - A^*)\eta, \xi - \eta, y - x)|,
\]
\[
\Phi(x, y) = (\mathcal{F}_2 \Psi)(x - A(x - y), x - y)
\]
which follows by straightforward applications of Fourier inversion formula (see also the proof of Proposition 2.5 in [33]). We observe that for the Weyl case, (1.15) takes the convenient form
\[
|(V_\Phi K_w^a)(x, y, \xi, -\eta)| = |(V_\Phi a)((\frac{1}{2})(Y + X), \frac{1}{2}(Y - X))|,
\]
\[
\Phi(x, y) = (\mathcal{F}_2 \Psi)((\frac{1}{2})x + y), x - y),
\]
\[
X = (x, \xi) \in \mathbb{R}^{2d}, \quad Y = (y, \eta) \in \mathbb{R}^{2d},
\]
when using symplectic short-time Fourier transforms. Here \(K_w^a\) is the kernel of \(\text{Op}_w^a\).

Next we discuss symbol products in pseudo-differential calculi, twisted convolution and related operations (see [11, 18]). Let \(A \in M(d, \mathbb{R}), s \geq 1/2\) and let \(a, b \in \mathcal{S}'_s(\mathbb{R}^{2d})\). The pseudo product with respect to \(A\) or the \(A\)-pseudo product \(a \#_A b\) between \(a\) and \(b\) is the function or distribution which satisfies
\[
\text{Op}_A(a \#_A b) = \text{Op}_A(a) \circ \text{Op}_A(b),
\]
provided the right-hand side makes sense as a continuous operator from \(\mathcal{S}_s(\mathbb{R}^d)\) to \(\mathcal{S}'_s(\mathbb{R}^d)\). Since the Weyl case is especially important, we put \(a \# b = a \#_A b\) when \(A = \frac{1}{2}I_d\), where \(I_d\) is the unit matrix of order \(d\). That is, we have
\[
\text{Op}_w^a(a \# b) = \text{Op}_w^a(a) \circ \text{Op}_w^w(b),
\]
provided the right-hand side makes sense.

1.4. The Weyl product and the twisted convolution. The symplectic Fourier transform \(\mathcal{F}_\sigma\) is continuous on \(\mathcal{S}_s(\mathbb{R}^{2d})\) and extends uniquely to a homeomorphism on \(\mathcal{S}'_s(\mathbb{R}^{2d})\), and to a unitary map on \(L^2(\mathbb{R}^{2d})\), since similar facts hold for \(\mathcal{F}\). Furthermore \(\mathcal{F}_\sigma^2\) is the identity operator.

Let \(s \geq 1/2\) and \(a, b \in \mathcal{S}_s(\mathbb{R}^{2d})\). The twisted convolution of \(a\) and \(b\) is defined by
\[
(a \ast_\sigma b)(X) = (2/\pi)^d \int_{\mathbb{R}^{2d}} a(X - Y)b(Y)e^{2i\sigma(X,Y)} dY.
\]
continuous map from $S'_s(R^{2d}) \times S'_s(R^{2d})$ to $S'_s(R^{2d})$. If $a, b \in S'_s(R^{2d})$, then $a \# b$ makes sense if and only if $a \ast \hat{b}$ makes sense, and
\[ a \# b = (2\pi)^{-\frac{d}{2}} a \ast \sigma \left( \mathcal{F}_\sigma b \right). \tag{1.18} \]

For the twisted convolution we have
\[ \mathcal{F}_\sigma (a \ast \sigma b) = (\mathcal{F}_\sigma a) \ast \sigma b = \tilde{a} \ast \sigma \left( \mathcal{F}_\sigma b \right), \tag{1.19} \]
where $\tilde{a}(X) = a(-X)$ (cf. [26]). A combination of (1.18) and (1.19) gives
\[ \mathcal{F}_\sigma (a \# b) = (2\pi)^{-\frac{d}{2}} (\mathcal{F}_\sigma a) \ast \sigma \left( \mathcal{F}_\sigma b \right). \tag{1.20} \]

We have the following result for the map $e^{it \langle AD_\xi, D_x \rangle}$ in (1.14) when the domains are modulation spaces. We refer to [29, Proposition 1.7] for the proof (see also [30, Proposition 6.14]).

**Proposition 1.8.** Let $\omega_0 \in \mathcal{S}_E(R^{2d})$, $p, q \in [1, \infty]$, $A_1, A_2 \in \mathcal{M}(d, R)$, and set
\[ \omega_{A}(x, \xi, \eta, y) = \omega_0(x - Ay, \xi - A^*\eta, \eta, y). \]
The map $e^{it \langle AD_\xi, D_x \rangle}$ on $S'_{1/2}(R^{2d})$ restricts to a homeomorphism from $M_{p,q}(\omega_{A_1})(R^{2d})$ to $M_{p,q}(\omega_{A_2})(R^{2d})$.

In particular, if $a_1, a_2 \in S'_{1/2}(R^{2d})$ satisfy (1.14), then $a_1 \in M_{p,q}(\omega_{A_1})(R^{2d})$, if and only if $a_2 \in M_{p,q}(\omega_{A_2})(R^{2d})$.

(Note that in the equality of (2) in [30, Proposition 6.14], $y$ and $\eta$ should be interchanged in the last two arguments in $\omega_0$.)

2. CONTINUITY FOR THE WEYL PRODUCT ON MODULATION SPACES

In this section we deduce results on sufficient conditions for continuity of the Weyl product on modulation spaces, and the twisted convolution on Wiener amalgam spaces. The main results are Theorem 2.6 and 2.10 concerning the Weyl product and more general products in pseudo-differential calculus, and Theorem 2.12 concerning the twisted convolution.

When proving Theorem 2.6 we first need norm estimates. Then we prove the uniqueness of the extension, where generally norm approximation not suffices, since the test function space may fail to be dense in several of the domain spaces. The situation is saved by a comprehensive argument based on narrow convergence. First we prove the important special cases Propositions 2.2 and 2.3 and then we deduce Theorem 2.6.
For $N \geq 2$ we let $R_N$ be the function on $[0,1]^{N+1}$, given by

$$R_N(x) = (N-1)^{-1} \left( \sum_{j=0}^{N} x_j - 1 \right),$$  \hspace{1cm} (2.1)

and we consider mappings of the form

$$(a_1, \ldots, a_N) \mapsto a_1 \# \cdots \# a_N,$$ \hspace{1cm} (2.2)

or, more generally, mappings of the form

$$(a_1, \ldots, a_N) \mapsto a_1 \#_A \cdots \#_A a_N,$$ \hspace{1cm} (2.2)'

We observe that

$$R_N \left( \frac{1}{p} \right) + R_N \left( \frac{1}{p} \right) = 1. \hspace{1cm} (2.3)$$

We first show a formula for the STFT of $a_1 \# \cdots \# a_N$ expressed with

$$F_j(X,Y) = V_{\Phi_j} a_j(X+Y, X-Y). \hspace{1cm} (2.4)$$

The following lemma is a restatement of [4, Lemma 2.3]. The proof is therefore omitted.

**Lemma 2.1.** Let $\Phi_j \in S_{1/2}(\mathbb{R}^{2d})$, $j = 1, \ldots, N$, $a_k \in S_{1/2}(\mathbb{R}^{2d})$ for some $1 \leq k \leq N$, and $a_j \in S_{1/2}(\mathbb{R}^{2d})$ for $j \in \{1, \ldots, N\} \setminus k$. Suppose

$$\Phi_0 = \pi^{(N-1)d} \Phi_1 \# \cdots \# \Phi_N \quad \text{and} \quad a_0 = a_1 \# \cdots \# a_N.$$  

If $F_j$ are given by (2.3) then

$$F_0(X_N, X_0)$$

$$= \int \cdots \int_{\mathbb{R}^{2(N-1)d}} e^{2iQ(X_0, \ldots, X_N)} \prod_{j=1}^{N} F_j(X_j, X_{j-1}) dX_1 \cdots dX_{N-1} \hspace{1cm} (2.5)$$

with

$$Q(X_0, \ldots, X_N) = \sum_{j=1}^{N-1} \sigma(X_j - X_0, X_{j+1} - X_0).$$

Next we use the previous lemma to find sufficient conditions for the extension of (2.2) to modulation spaces. The integral representation of $V_{\Phi_0} a_0$ in the previous lemma leads to the weight condition

$$1 \lesssim \omega_0(X_N + X_0, X_N - X_0) \prod_{j=1}^{N} \omega_j(X_j + X_{j-1}, X_j - X_{j-1}),$$

$$X_0, X_1, \ldots, X_N \in \mathbb{R}^{2d}. \hspace{1cm} (2.6)$$

The following result is a restatement of [4, Proposition 2.2]. The proof is therefore omitted.
Proposition 2.2. Let \( p_j, q_j \in [1, \infty], j = 0, 1, \ldots, N \), and suppose
\[
R_N(\frac{1}{q'}) \leq 0 \leq R_N(\frac{1}{p}).
\]
Let \( \omega_j, j = 0, 1, \ldots, N \), and suppose (2.6) holds. Then the map (2.2) from \( \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \times \cdots \times \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \) to \( \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \) extends uniquely to a continuous and associative map from \( \mathcal{M}_{(\omega_1)}^{p_1,q_1}(\mathbb{R}^{2d}) \times \cdots \times \mathcal{M}_{(\omega_N)}^{p_N,q_N}(\mathbb{R}^{2d}) \) to \( \mathcal{M}_{(1/\omega)}^{p_0,q_0}(\mathbb{R}^{2d}) \).

The associativity means that for any product (2.2), where the factors \( a_j \) satisfy the hypotheses, the subproduct
\[
a_{k_1} \# a_{k_1+1} \# \cdots \# a_{k_2}
\]
is well defined as a distribution for any \( 1 \leq k_1 \leq k_2 \leq N \), and
\[
a_1 \# \cdots \# a_N = (a_1 \# \cdots \# a_k) \# (a_{k+1} \# \cdots \# a_N),
\]
for any \( 1 \leq k \leq N - 1 \).

For appropriate weights \( \omega \) the space \( \mathcal{M}_{(\omega)}^d(\mathbb{R}^{2d}) \) consists of symbols of Hilbert–Schmidt operators acting between certain modulation spaces (cf. [29][31]).

The next result is an extension of this fact.

Proposition 2.3. Let \( N \geq 3 \) be odd, \( p, p_j \in (0, \infty], j = 1, \ldots, N \), and let \( \omega_j \in \mathcal{P}_E(\mathbb{R}^{4d}), j = 0, 1, \ldots, N \), and suppose (2.6) holds. Then the following is true:

1. if \( p_0 = p_N = p, p_j = \max(1, p) \) when \( j \in [3, N-2] \) is odd and \( p_j = p' \) when \( j \) is even, then the map (2.2) from \( \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \times \cdots \times \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \) to \( \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \) extends uniquely to a continuous and associative map from \( \mathcal{M}_{(\omega_1)}^{p_1}(\mathbb{R}^{2d}) \times \cdots \times \mathcal{M}_{(\omega_N)}^{p_N}(\mathbb{R}^{2d}) \) to \( \mathcal{M}_{(1/\omega)}^{p}(\mathbb{R}^{2d}) \);

2. if \( p_j = p \) when \( j \) is even and \( p_j = p' \) when \( j \) is odd, then the map (2.2) from \( \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \times \cdots \times \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \) to \( \mathcal{S}_{1/2}(\mathbb{R}^{2d}) \) extends uniquely to a continuous and associative map from \( \mathcal{M}_{(\omega_1)}^{p_1}(\mathbb{R}^{2d}) \times \cdots \times \mathcal{M}_{(\omega_N)}^{p_N}(\mathbb{R}^{2d}) \) to \( \mathcal{M}_{(1/\omega)}^{p'}(\mathbb{R}^{2d}) \).

Proposition 2.3 follows by combining (1.16) with the following result for kernel operators. The details are left for the reader.

Proposition 2.4. Let \( N \geq 3 \) be odd, \( p, p_j \in (0, \infty], j = 1, \ldots, N \), and let \( \omega_j \in \mathcal{P}_E(\mathbb{R}^{4d}), j = 0, 1, \ldots, N \), be such that
\[
\inf_{x_j, \xi_j \in \mathbb{R}^d} \left( \omega_0(x_0, x_N, \xi_0, -\xi_N) \prod_{j=1}^{N} \omega_j(x_{j-1}, x_j, \xi_{j-1}, -\xi_j) \right) > 0. \tag{2.7}
\]
Then the following is true:
Proof. We observe that (2.7) is the same as

\[(K_1, K_2, \ldots, K_N) \mapsto K_1 \circ K_2 \circ \cdots \circ K_N \quad (2.8)\]

from \(S_{1/2}(\mathbb{R}^{2d}) \times \cdots \times S_{1/2}(\mathbb{R}^{2d})\) to \(S_{1/2}(\mathbb{R}^{2d})\) extends uniquely to a continuous and associative map from \(M_{p_1}^{\omega_1}(\mathbb{R}^{2d}) \times \cdots \times M_{p_N}^{\omega_N}(\mathbb{R}^{2d})\) to \(M_{(1/o)}^{(1/o)}(\mathbb{R}^{2d})\), and

\[\|K_1 \circ K_2 \circ \cdots \circ K_N\|_{M_{(1/o)}^{p_{(1/o)}}} \lesssim \prod_{j=1}^N \|K_j\|_{M_{\omega_j}^{p_j}}, \quad (2.9)\]

\[K_j \in M_{\omega_j}^{p_j}(\mathbb{R}^{2d}), \ j = 1, \ldots, N;\]

(2) if \(p_j = p\) when \(j\) is even and \(p_j = p'\) when is odd, then the map (2.8) extends uniquely to a continuous and associative map from \(M_{p_1}^{\omega_1}(\mathbb{R}^{2d}) \times \cdots \times M_{p_N}^{\omega_N}(\mathbb{R}^{2d})\) to \(M_{(1/o)}^{p'}(\mathbb{R}^{2d})\).

First suppose that \(K_j \in S_{1/2}(\mathbb{R}^{2d})\) for every \(j\). Let \(p_0 = \max(p, 1)\),

\[\tilde{K}_j = \begin{cases} K_j, & j \text{ odd} \\ \bar{K}_j, & j \text{ even} \end{cases}\]

\[\tilde{\omega}_j(x, y, \xi, \eta) = \begin{cases} \omega_j(x, y, \xi, \eta), & j \text{ odd} \\ \omega_j(x, y, -\xi, -\eta), & j \text{ even} \end{cases}\]

\[y = (x_2, x_3, \ldots, x_{N-2}), \quad \eta = (\xi_2, \xi_3, \ldots, \xi_{N-2}),\]

\[G(x_0, x_N, x_1, x_{N-1}) = \tilde{K}_1(x_0, x_1)\tilde{K}_N(x_{N-1}, x_N),\]

\[H_1(x_1, x_{N-1}, y) = \prod_{j=1}^{(N-1)/2} \tilde{K}_2j(x_{2j-1}, x_{2j}),\]

\[H_2(y) = \prod_{j=1}^{(N-3)/2} \tilde{K}_{2j+1}(x_{2j}, x_{2j+1}),\]

\[H(x_0, x_N) = (H_2, H_1(x_1, x_{N-1}, \cdot))_{L^2},\]
\[ \vartheta_0(x_0, x_N, x_1, x_{N-1}, x_0, \xi_0, \xi_1, \xi_N) = \omega_1(x_0, x_1, \xi_0, \xi_1)\omega_N(x_{N-1}, x_N, \xi_{N-1}, \xi_N), \]

\[ \vartheta_1(x_1, x_{N-1}, y, \xi_1, \xi_{N-1}, \eta) = \prod_{j=1}^{(N-1)/2} \varpi_{2j}(x_{2j-1}, x_{2j}, \xi_{2j-1}, \xi_{2j}), \]

and

\[ \vartheta_2(y, \eta) = \prod_{j=1}^{(N-3)/2} \varpi_{2j+1}(x_{2j}, x_{2j+1}, \xi_{2j}, \xi_{2j+1}). \]

Then it follows by straight-forward computations that

\[ (K_1 \circ \cdots \circ K_N)(x_0, x_N) = (G(x_0, x_N, \cdot), H)_{L^2}, \tag{2.10} \]

and that the right-hand side makes sense as an element in \( S_{1/2}(\mathbb{R}^{2d}) \) when \( K_j \in S'_{1/2}(\mathbb{R}^{2d}) \) for even \( j \). In the same way, the right-hand side of (2.10) makes sense as an element in \( S'_{1/2}(\mathbb{R}^{2d}) \) when \( K_j \in S'_{1/2}(\mathbb{R}^{2d}) \) for odd \( j \). Hence the map (2.8) extends to continuous mappings from

\[ S_{1/2}(\mathbb{R}^{2d}) \times S'_{1/2}(\mathbb{R}^{2d}) \times S_{1/2}(\mathbb{R}^{2d}) \times \cdots \times S_{1/2}(\mathbb{R}^{2d}) \]

to \( S'_{1/2}(\mathbb{R}^{2d}) \), and from

\[ S'_{1/2}(\mathbb{R}^{2d}) \times S_{1/2}(\mathbb{R}^{2d}) \times S'_{1/2}(\mathbb{R}^{2d}) \times \cdots \times S'_{1/2}(\mathbb{R}^{2d}) \]
to \( S'_{1/2}(\mathbb{R}^{2d}) \). The uniquenesses of these extensions follows by approximating those \( K_j \) which belong to \( S'_{1/2}(\mathbb{R}^{2d}) \), by taking sequences of elements \( S_{1/2}(\mathbb{R}^{2d}) \) which converge to those \( K_j \) in \( S'_{1/2}(\mathbb{R}^{2d}) \).

We have that \( S_{1/2}(\mathbb{R}^{2d}) \) is dense in \( M^p_{(\omega_j)}(\mathbb{R}^{2d}) \subseteq S'_{1/2}(\mathbb{R}^{2d}) \) when \( p < \infty \), and that \( p' = 1 < \infty \) when \( p = \infty \) and \( p \leq 1 < \infty \) when \( p' = \infty \). Hence it follows from the recent uniqueness properties, that the result follows if we prove that (2.10) holds when \( K_j \in S_{1/2}(\mathbb{R}^{2d}) \) for every \( j \).

We have

\[ \| \tilde{K}_j \|_{M^p_{(\omega_j)}} = \| K_j \|_{M^p_{(\omega_j)}}, \tag{2.11} \]

\[ \| G \|_{M^p_{(\omega_0)}} \lesssim \| K_1 \|_{M^p_{(\omega_1)}} \| K_N \|_{M^p_{(\omega_N)}}, \tag{2.12} \]

\[ \| H_1 \|_{M^p_{(\omega_1)}(\mathbb{R}^{(N-1)d})} = \prod_{j=1}^{(N-1)/2} \| \tilde{K}_j \|_{M^p_{(\omega_2j)}(\mathbb{R}^{2d})}, \]

\[ \| H_2 \|_{M^p_{(\omega_2)}(\mathbb{R}^{(N-3)d})} = \prod_{j=1}^{(N-3)/2} \| \tilde{K}_{2j+1} \|_{M^p_{(\omega_{2j+1})}(\mathbb{R}^{2d})}, \]

and

\[ \| \tilde{K}_j \|_{M^p_{(\omega_j)}(\mathbb{R}^{2d})} \lesssim \| K_j \|_{M^p_{(\omega_j)}(\mathbb{R}^{2d})}, \tag{2.13} \]
which implies

\[
\|\mathcal{H}\|_{M_{(\omega)}^{p}}} \lesssim \prod_{j=2}^{N-1} \|\mathcal{K}_{j}\|_{M_{(\omega_{j})}^{p_{j}}} = \prod_{j=2}^{N-1} \|\mathcal{K}_{j}\|_{M_{(\omega_{j})}^{p_{j}}}
\]  

(2.13)

A combination of this estimate with (2.10), (2.12) and (2.13) gives (2.9), and (1) follows.

The assertion (2) follows by similar arguments and is left for the reader. \(\square\)

The following result now follows by interpolation between Propositions 2.2 and 2.3. Here and in what follows we let

\[I_{N} = [0, N] \cap \mathbb{Z}, \quad \text{and} \quad \Omega_{N} = \{(j, k) \in I_{N}^2; j + k \in 2\mathbb{Z} + 1\},\]

and

\[
Q_{0,N}(x, y) = \min_{j+k \in 2\mathbb{Z}+1} \left(\frac{x_j + y_k}{2}\right),
\]

\[
Q_{N}(x, y) = \min_{j+k \in 2\mathbb{Z}+1} \left(\frac{x_j + y_k}{2}, 1 - \frac{x_j + y_k}{2}\right),
\]

\[
Q_{0,N}(x) = Q_{0,N}(x, x), \quad Q_{N}(x) = Q_{N}(x, x),
\]

\[x = (x_0, x_1, \ldots, x_N) \in [0, 1]^{N+1},\]

\[y = (y_0, y_1, \ldots, y_N) \in [0, 1]^{N+1}.\]

Proposition 2.5. Let \(N \geq 3\) be odd, \(R_{N}\) be as in (2.1), \(Q_{N}\) be as in (2.14), and let \(p_{j}, q_{j} \in [1, \infty], j = 0, 1, \ldots, N,\) be such that

\[
\max \left(\frac{R_{N}}{q_{j}}, 0\right) \leq \min \left(\frac{Q_{N}}{p_{j}}, Q_{N}(\frac{1}{p_{j}}), R_{N}(\frac{1}{p_{j}})\right). \tag{2.15}\]

Also let \(\omega_{j} \in \mathcal{P}_{E}(R^{4d}), j = 0, 1, \ldots, N,\) and suppose (2.6) holds. Then the map (2.1) from \(S_{1/2}(R^{2d}) \times \cdots \times S_{1/2}(R^{2d})\) to \(S_{1/2}(R^{2d})\) extends uniquely to a continuous and associative map from \(\mathcal{M}^{p_{1}, q_{1}}(R^{2d}) \times \cdots \times \mathcal{M}^{p_{N}, q_{N}}(R^{2d})\) to \(\mathcal{M}^{p_{0}, q_{0}}(R^{2d}).\)

We observe that \(Q_{N}(\frac{1}{q_{j}}) = Q_{N}(\frac{1}{q})\) when \(q\) is the same as in the previous proposition.

Proof. Evidently, the result holds true when \(R_{N}(q) \leq 0\) in view of Proposition 2.2. We need to prove the result when \(R_{N}(\frac{1}{q}) \geq 0.\)

We use the same notations as in Lemma 2.7 and its proof. By Propositions 2.2 and 2.3 we have

\[
\mathcal{M}^{\omega_{1}, s_{1}}(\omega_{1}) \times \cdots \times \mathcal{M}^{\omega_{N}, s_{N}}(\omega_{N}) \rightarrow \mathcal{M}^{\omega_{0}, s_{0}}(1/\omega_{0}) \tag{2.16}
\]

and

\[
\mathcal{M}^{u_{1}, a_{1}}(\omega_{1}) \times \cdots \times \mathcal{M}^{u_{N}, a_{N}}(\omega_{N}) \rightarrow \mathcal{M}^{u_{0}, a_{0}}(1/\omega_{0}) \tag{2.17}
\]
when \( r_j, s_j, u_j \in [1, \infty] \), \( j \in I_N \), satisfy
\[
\sum_{j=0}^{N} \frac{1}{s'_j} \leq 1 \leq \sum_{j=0}^{N} \frac{1}{r_j},
\]
and
\[
u_j = \begin{cases} 
    v', & j \in 2\mathbb{Z}, \\
    v, & j \in 2\mathbb{Z} + 1,
\end{cases}
\]
for some \( v \in [1, \infty] \). Hence, by combining Proposition 1.3 (5) with multi-linear interpolation in [1, Chapter 4], we get
\[
\mathcal{M}_{(\omega_1)}^{p_1,q_1} \times \cdots \times \mathcal{M}_{(\omega_N)}^{p_N,q_N} \hookrightarrow \mathcal{M}_{(1/\omega_0)}^{p_0,q_0}
\]
when
\[
\frac{1}{p_j} = \frac{1 - \theta}{r_j} + \frac{\theta}{v'}, \quad \frac{1}{q_j} = \frac{1 - \theta}{s_j} + \frac{\theta}{v'}, \quad j \in 2\mathbb{Z}
\]
and
\[
\frac{1}{p_k} = \frac{1 - \theta}{r_k} + \frac{\theta}{v}, \quad \frac{1}{q_k} = \frac{1 - \theta}{s_k} + \frac{\theta}{v}, \quad k \in 2\mathbb{Z} + 1.
\]
This gives
\[
\sum_{j=0}^{N} \frac{1}{p_j} = (1 - \theta) \sum_{j=0}^{N} \frac{1}{r_j} + \theta \cdot \frac{N + 1}{2} \left( \frac{1}{v} + \frac{1}{v'} \right)
\]
\[
= (1 - \theta) \sum_{j=0}^{N} \frac{1}{r_j} + \theta \cdot \frac{N + 1}{2} \geq 1 + \theta \cdot \frac{N - 1}{2}
\]
and
\[
\sum_{j=0}^{N} \frac{1}{q'_j} = (1 - \theta) \sum_{j=0}^{N} \frac{1}{s'_j} + \theta \cdot \frac{N + 1}{2} \left( \frac{1}{v} + \frac{1}{v'} \right)
\]
\[
= (1 - \theta) \sum_{j=0}^{N} \frac{1}{s'_j} + \theta \cdot \frac{N + 1}{2} \leq 1 + \theta \cdot \frac{N - 1}{2},
\]
which implies
\[
R_N(\frac{1}{q'}) \leq \frac{\theta}{2} \leq R_N(\frac{1}{p}).
\]
In particular we have \( R_N(\frac{1}{q'}) \leq \frac{\theta}{2} \).
By (2.20) and (2.21) we also get
\[
\frac{1}{p_j} + \frac{1}{p_k} = \frac{1 - \theta}{r_j} + \frac{1 - \theta}{r_k} + \frac{\theta}{v} + \frac{\theta}{v'} \geq \frac{\theta}{v} + \frac{\theta}{v'} = \theta,
\]
when \( j + k \) is odd. That is,
\[
\frac{1}{2} \left( \frac{1}{p_j} + \frac{1}{q_k} \right) \geq \frac{\theta}{2} \geq R_N(\frac{1}{q}) \tag{2.23}
\]

In the same way we get
\[
\frac{1}{2} \left( \frac{1}{q_j} + \frac{1}{q_k} \right) \geq \frac{\theta}{2} \geq R_N(\frac{1}{p}) \text{ and } \frac{1}{2} \left( \frac{1}{p_j} + \frac{1}{q_k} \right) \geq \frac{\theta}{2} \geq R_N(\frac{1}{q}),
\]

when \( j + k \) is odd. We also have
\[
\frac{1}{p_j'} + \frac{1}{p_k'} = \frac{1 - \theta}{r_j'} + \frac{1 - \theta}{r_k'} + \frac{\theta}{v'} + \frac{\theta}{v} \geq \frac{\theta}{p} + \frac{\theta}{p'} = \theta,
\]
when \( j + k \) is odd, and it follows that (2.23) and its two following inequalities hold true with \( p_j', p_k', q_j' \) and \( q_k' \) in place of \( p_j, p_k, q_j \) and \( q_k \), respectively, at each occurrence. By combining these inequality we get (2.15).

In order for verify the interpolation completely, we need to prove that if \( p, q \in [1, \infty)^{N+1} \) satisfy (2.15), then there are \( r, s, u \in [1, \infty)^{N+1}, v \in [1, \infty] \) and \( \theta \in [0, 1] \) such that (2.16)–(2.21) hold. As remarked above, the result holds true if \( R_N(\frac{1}{p'}) \leq 0 \). Therefore assume that \( R_N(\frac{1}{q'}) > 0 \). By (2.15) it follows that \( R_N(\frac{1}{q}) \leq \frac{1}{2} \). Choose \( \theta \in (0, 1] \) such that \( R_N(\frac{1}{q}) = \frac{\theta}{2} \). By reasons of symmetry we may assume that
\[
p_0 = \min(j \in I_N)(p_j, q_j),
\]
and we shall consider the two cases when \( \frac{1}{p_0} \geq \frac{\theta}{2} \) and when \( \frac{1}{p_0} < \frac{\theta}{2} \), separately.

First suppose that \( \frac{1}{p_0} \geq \frac{\theta}{2} \). Then
\[
\min \left( \frac{1}{p}, \frac{1}{p'}, \frac{1}{q}, \frac{1}{q'}, R_N(\frac{1}{p}) \right) \geq R_N(\frac{1}{q}),
\]
and the result follows from [4, Proposition 2.5].

Therefore assume that \( \frac{1}{p_0} < \frac{\theta}{2} \) and let \( v > 2 \) be chosen such that
\[
\frac{1}{p_0} = \frac{\theta}{v}.
\]
Then
\[
\frac{1}{p_j}, \frac{1}{q_j} \geq \frac{1}{p_0} = \frac{\theta}{v}, \quad j \in 2\mathbb{Z} \tag{2.24}
\]
and since
\[
Q_N(\frac{1}{p}), Q_N(\frac{1}{q}), Q_N(\frac{1}{p}, \frac{1}{q}) \geq R_N(\frac{1}{p}) = \frac{\theta}{2},
\]
we get
\[
\frac{\theta}{v} + \frac{1}{p_k} = \frac{1}{p_0} + \frac{1}{p_k} \geq \theta
\]
and
\[
\frac{\theta}{v} + \frac{1}{q_k} = \frac{1}{p_0} + \frac{1}{q_k} \geq \theta.
\]
when \( k \) is odd. This implies that
\[
\frac{1}{p_k} \cdot \frac{1}{q_k} \geq \frac{\theta}{\nu'}, \quad k \in 2\mathbb{Z} + 1.
\] (2.25)

By (2.24) and (2.25), there are \( r, s \in [1, \infty]^{N+1} \) such that (2.20) and (2.21) hold. We have
\[
(1 - \theta)R_N\left(\frac{1}{p}\right) = R_N\left(\frac{1}{p}\right) - \theta R_N\left(\frac{1}{p}, \frac{1}{q}, \ldots, \frac{1}{q}, \frac{1}{v'}\right)
\geq R_N\left(\frac{1}{p}\right) - \theta \left(\frac{1}{N - 1} \left(\frac{N + 1}{2}, \frac{1}{v'} + \frac{1}{v}\right) - 1\right)
= \frac{\theta}{2} - \theta \left(\frac{1}{N - 1} \left(\frac{N + 1}{2}, \frac{1}{v'} + \frac{1}{v}\right) - 1\right) = 0
\]
and
\[
(1 - \theta)R_N\left(\frac{1}{q}\right) = R_N\left(\frac{1}{q}\right) - \theta R_N\left(\frac{1}{p}, \frac{1}{v}, \ldots, \frac{1}{v'}, \frac{1}{v}\right)
= \frac{\theta}{2} - \theta \left(\frac{1}{N - 1} \left(\frac{N + 1}{2}, \frac{1}{v'} + \frac{1}{v}\right) - 1\right) = 0.
\]

Consequently, if \( p, q \in [1, \infty]^{N+1} \) satisfy (2.15), we have found \( r, s, u \in [1, \infty]^{N+1} \) and \( \theta \in [0, 1] \) such that (2.16)–(2.21) hold. Hence the interpolation works out properly and the result follows. \( \square \)

Next we polish up Proposition 2.5 by purging away some superfluous conditions. More precisely we have the following.

**Theorem 2.6.** Let \( N \geq 3 \) be odd, \( R_N \) be as in (2.1), \( Q_{0,N} \) and \( Q_N \) be as in (2.14), and let \( p_j, q_j \in [1, \infty], j = 0, 1, \ldots, N \), be such that
\[
\max \left( R_N\left(\frac{1}{p}\right), 0 \right) \leq \min \left( Q_N\left(\frac{1}{p}\right), Q_{0,N}\left(\frac{1}{q}\right), Q_N\left(\frac{1}{p}, \frac{1}{q}\right), R_N\left(\frac{1}{p}\right) \right). \tag{2.26}
\]
Also let \( \omega_j \in \mathcal{P}_E(\mathbb{R}^d), j = 0, 1, \ldots, N \), and suppose (2.6) holds. Then the map (2.2) from \( S_{1/2}(\mathbb{R}^d) \times \cdots \times S_{1/2}(\mathbb{R}^d) \) to \( S_{1/2}(\mathbb{R}^d) \) extends uniquely to a continuous and associative map from \( M_{\mathbb{R}^d}^{p_0,q_0} \) to \( M_{\mathbb{R}^d}^{p_0,q_0} \).

We need some preparations for the proof of Theorem 2.6. First we have the following analogy of [4 Lemma 2.7].

**Lemma 2.7.** Let \( N \geq 3 \) be odd, \( x_j \in [0, 1], y_{j,k} = \frac{1}{2}(x_j + x_k) \) \( j, k = 0, \ldots, N \), and consider the inequalities:

1. \( (N - 1)^{-1} \left( \sum_{k=0}^{N} x_k - 1 \right) \leq \min_{(j,k) \in \Omega_N} y_{j,k}; \)
2. \( y_{j,k} \leq \frac{1}{2}, \) for all \( (j, k) \in \Omega_N; \)
\[(N - 1)^{-1} \left( \sum_{k=0}^{N} x_k - 1 \right) \leq \min_{(j,k) \in \Omega_N} (1 - y_{j,k}). \]

Then
\[(1) \Rightarrow (2) \Rightarrow (3). \]

**Remark 2.8.** We notice the similarities between the previous lemma and [4, Lemma 2.7]. In fact, let \( N \geq 2, \ x_j \in [0, 1], \ j = 0, \ldots, N \) and consider the inequalities:

1. \((N - 1)^{-1} \left( \sum_{k=0}^{N} x_k - 1 \right) \leq \min_{0 \leq j \leq N} x_j;\)
2. \(x_j + x_k \leq 1, \) for all \( k \neq j;\)
3. \((N - 1)^{-1} \left( \sum_{k=0}^{N} x_k - 1 \right) \leq \min_{0 \leq j \leq N} (1 - x_j).\)

Then Lemma [4, Lemma 2.7] shows that
\[(1) \Rightarrow (2) \Rightarrow (3). \]

**Proof.** We shall use similar ideas as in the proof of Lemma [4, Lemma 2.7]. Let

\[ J_{1,N} = \{0, \ldots, N\} \cap 2\mathbb{Z}, \quad \text{and} \quad J_{2,N} = \{0, \ldots, N\} \cap (2\mathbb{Z} + 1), \]

and assume that (1) holds but (2) fails. Then \( x_j + x_k > 1 \) for some \((j, k) \in \Omega_N.\) By renumbering we may assume that \( x_2 + x_3 \leq x_j + x_{j+1} \) for every \( j \in J_{1,N}, \) and that \( x_0 + x_1 > 1. \) Then (1) and the fact that there are \((N - 1)/2\) pairs of \((j, j + 1)\) with \( j \in J_{1,N} \setminus 0\) give

\[(N - 1)y_{2,3} = \frac{N - 1}{2}(2y_{2,3}) \leq \sum_{m=1}^{(N-1)/2} 2y_{2m,2m+1} = \sum_{j=2}^{N} x_j < \sum_{j=0}^{N} x_j - 1 \leq (N - 1)y_{2,3}, \]

which is a contradiction. Hence the assumption \( x_0 + x_1 > 1 \) must be wrong and it follows that (1) implies (2).

Now suppose that (2) holds, and let \( j_0 \in J_{1,N} \) and \( k_0 \in J_{2,N}. \) Then

\[ x_j \leq 1 - x_k \quad \text{and} \quad x_j \leq 1 - x_k, \quad j \in J_{1,N}, \ k \in J_{2,N}. \]

This gives

\[ \sum_{j \in J_{1,N} \setminus j_0} x_j \leq \frac{N - 1}{2}(1 - x_{k_0}) \quad \text{and} \quad \sum_{k \in J_{2,N} \setminus k_0} x_k \leq \frac{N - 1}{2}(1 - x_{j_0}), \]

giving that

\[ \sum_{j \in J \setminus \{j_0, k_0\}} x_j \leq (N - 1) \left( 1 - \frac{1}{2}(x_{j_0} + x_{k_0}) \right) = (N - 1)(1 - y_{j_0, k_0}). \]
Since
\[ x_{j_0} + x_{k_0} - 1 = 2y_{j_0,k_0} - 1 \leq 0, \]
we obtain
\[
\sum_{j \in I_N} x_j - 1 = (x_{j_0} + x_{k_0} - 1) + \sum_{j \in I_N \setminus \{j_0,k_0\}} x_j
\leq \sum_{j \in I_N \setminus \{j_0,k_0\}} x_j \leq (N - 1)(1 - y_{j_0,k_0}).
\]

Since \( j_0 \in J_{1,N} \) and \( k_0 \in J_{2,N} \) was chosen arbitrary, it follows that (3) holds. \( \square \)

**Proof of Theorem 2.6.** If \( I_N = \{0, 1, \ldots, N\} \) as before and \( j, k \in I_N \) satisfies \( j + k \in 2\mathbb{Z} + 1 \). Then the assumptions and Lemma 2.7 implies that
\[
0 \leq R_N(\frac{1}{q_j}) \leq \min \left( \frac{1}{2} \left( \frac{1}{q_j} + \frac{1}{q_k} \right) \right), \quad j + k \in 2\mathbb{Z} + 1.
\]
Hence (2.26) implies (2.15), and the result follows from Proposition 2.5. \( \square \)

**Remark 2.9.** We observe that Theorem 2.6 implies that the inclusion
\[
M^{\infty,1} \# M^{2,2} \# M^{2,2} \subseteq M^{2,2}. \tag{2.29}
\]
In this context we observe that Theorem 0.1′ in [4] does ensure the validity of this inclusion, while Theorem 2.9 in [4] does.

We may use (1.14) and Proposition 1.8 to extend Theorem 2.6 to involve more general products arising in the pseudo-differential calculi. More precisely, the let \( M(d, \Omega) \) be the set of all \( d \times d \) matrices with entries in the set \( \Omega \), and let \( A \in M(d, \mathbb{R}) \). By (1.14) we have
\[
a_1 \# \cdots \# a_N = e^{-i(A_0 D_\xi, D_x)}(e^{i(A_0 D_\xi, D_x)} a_1) \# \cdots \# (e^{i(A_0 D_\xi, D_x)} a_N),
\]
where \( A_0 = A - \frac{1}{2} I_d \). If we combine this relation with Proposition 1.8 and Theorem 2.6, we get the following result. The condition on the weight functions is
\[
1 \preceq \omega_0(T_A(X_N, X_0)) \prod_{j=1}^N \omega_j(T_A(X_j, X_{j-1})), \quad X_0, \ldots, X_N \in \mathbb{R}^{2d}, \tag{2.30}
\]
where

\[ T_A(X, Y) = (y + A(x - y), \xi + A^*(\eta - \xi), \eta - \xi, x - y), \]

\[ X = (x, \xi) \in \mathbb{R}^{2d}, \ Y = (y, \eta) \in \mathbb{R}^{2d}. \]  

(2.31)

(See (2.16) and (2.17) in [33].)

**Theorem 2.10.** Let \( A \in \mathbf{M}(d, \mathbb{R}) \), \( N \geq 3 \) be odd, \( R_N \) be as in (2.1), \( Q_{0,N} \) and \( Q_N \) be as in (2.14), and let \( p_j, q_j \in [1, \infty] \), \( j = 0, 1, \ldots, N \), be such that

\[
\max \left( R_N \left( \frac{1}{q} \right), 0 \right) \leq \min \left( Q_{0,N} \left( \frac{1}{p} \right), Q_N \left( \frac{1}{p}, \frac{1}{q} \right), Q_N \left( \frac{1}{p}, \frac{1}{q}, R_N \left( \frac{1}{p} \right) \right) \right),
\]

(2.32)

Also let \( \omega_j \in \mathcal{P}_E(\mathbb{R}^{4d}), j = 0, 1, \ldots, N \), and suppose (2.30) and (2.31) hold. Then the map (2.2)’ from \( S_{1/2}(\mathbb{R}^{2d}) \times \cdots \times S_{1/2}(\mathbb{R}^{2d}) \) to \( S_{1/2}(\mathbb{R}^{2d}) \) extends uniquely to a continuous and associative map from \( M^{p_1,q_1}(\mathbb{R}^{2d}) \times \cdots \times M^{p_N,q_N}(\mathbb{R}^{2d}) \) to \( M^{p_0,q_0}(\mathbb{R}^{2d}) \).

In the same way we get the following result by combining (1.14) and Propositions 1.8 and 2.3. The details are left for the reader.

**Proposition 2.11.** Let \( A \in \mathbf{M}(d, \mathbb{R}) \), \( N \geq 3 \) be odd, \( p, p_j \in (0, \infty] \), \( j = 1, \ldots, N \), and let \( \omega_j \in \mathcal{P}_E(\mathbb{R}^{4d}), j = 0, 1, \ldots, N \), and suppose (2.30) and (2.31) hold. Then the following is true:

1. If \( p_0 = p = p_j \) when \( j \in [3, N - 2] \) is odd and \( p_j = p' \) when \( j \) is even, then the map (2.2)’ from \( S_{1/2}(\mathbb{R}^{2d}) \times \cdots \times S_{1/2}(\mathbb{R}^{2d}) \) to \( S_{1/2}(\mathbb{R}^{2d}) \) extends uniquely to a continuous and associative map from \( M^{p_1}(\mathbb{R}^{2d}) \times \cdots \times M^{p_N}(\mathbb{R}^{2d}) \) to \( M^{p_0}(\mathbb{R}^{2d}) \);

2. If \( p_j = p \) when \( j \) is even and \( p_j = p' \) when \( j \) is odd, then the map (2.2)’ from \( S_{1/2}(\mathbb{R}^{2d}) \times \cdots \times S_{1/2}(\mathbb{R}^{2d}) \) to \( S_{1/2}(\mathbb{R}^{2d}) \) extends uniquely to a continuous and associative map from \( M^{p_1}(\mathbb{R}^{2d}) \times \cdots \times M^{p_N}(\mathbb{R}^{2d}) \) to \( M^{p'}(\mathbb{R}^{2d}) \).

Finally we prove a continuity result for the twisted convolution. The map (2.2) is then replaced by

\[
(a_1, a_2, \ldots, a_N) \mapsto a_1 \ast_\sigma a_2 \ast_\sigma \cdots \ast_\sigma a_N.
\]

(2.33)

The following result follows immediately from Theorem 2.6. Here the condition (2.5) is replaced by

\[
1 \lesssim \omega_0(X_N - X_0, X_N + X_0) \prod_{j=1}^N \omega_j(X_j - X_{j-1}, X_j + X_{j-1}),
\]

\[ X_0, X_1, \ldots, X_N \in \mathbb{R}^{2d}. \]

(2.34)
Theorem 2.12. Let $p_j, q_j \in [1, \infty]$, $j = 0, 1, \ldots, N$, and suppose that
\[
\max \left( R_N \left( \frac{1}{p_j} \right), 0 \right) \leq \min \left( Q_N \left( \frac{1}{q_j} \right), Q_{0,N} \left( \frac{1}{p_j}, \frac{1}{q_j} \right), R_N \left( \frac{1}{q_j} \right) \right).
\]
Suppose $\omega_j \in \mathcal{P}_E(\mathbb{R}^{2d})$, $j = 0, 1, \ldots, N$, satisfy (2.34). Then the map (2.33) from $S_{1/2}(\mathbb{R}^{2d}) \times \cdots \times S_{1/2}(\mathbb{R}^{2d})$ to $S_{1/2}(\mathbb{R}^{2d})$ extends uniquely to a continuous and associative map from $W_{(\omega_1)}^{p_1,q_1}(\mathbb{R}^{2d}) \times \cdots \times W_{(\omega_N)}^{p_N,q_N}(\mathbb{R}^{2d})$ to $W_{(1/\omega_0)}^{p_0,q_0}(\mathbb{R}^{2d})$.

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