Six New Mechanics corresponding to further Shape Theories

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Abstract

A suite of relational notions of shape are presented at the level of configuration space geometry, with corresponding new theories of shape mechanics and shape statistics. These further generalize two quite well known examples: –1) Kendall’s (metric) shape space with his shape statistics and Barbour’s mechanics thereupon. 0) Leibnizian relational space alias metric scale-and-shape space to which corresponds Barbour–Bertotti mechanics. This paper’s new theories include, using the invariant and group namings, 1) Angle alias conformal shape mechanics. 2) Area ratio alias affine shape mechanics. 3) Area alias affine scale-and-shape mechanics. 1) to 3) rest respectively on angle space, area-ratio space, and area space configuration spaces. Probability and statistics applications are also pointed to in outline.

4) Various supersymmetric counterparts of –1) to 3) are considered. Since supergravity differs considerably from GR-based conceptions of Background Independence, some of the new supersymmetric shape mechanics are compared with both. These reveal compatibility between supersymmetry and GR-based conceptions of Background Independence, at least within these simpler model arenas.
1 Introduction

Newtonian Mechanics, and the Newtonian paradigm of Physics more generally, are based on absolute space and time. The immovable external character of absolute space and time led to these being opposed by relationalists, most notably by Leibniz [1] and Mach [2]. On the other hand, the Newtonian paradigm of Physics sufficed to explain humankind’s observations of nature until the end of the 19th century.

Indeed, a satisfactory relational alternative to the foundations of Mechanics was not found until 1982 by Barbour and Bertotti [3]. This is a theory in which Euclidean transformations are held to be physically irrelevant. The next RPM was not formulated until 2003: it is Barbour’s ‘mechanics of pure shape’ [4]; in this case, it is similarity transformations which are held to be physically irrelevant. It is useful to term this a Shape Mechanics, and the preceding a Scale-and-Shape Mechanics. I then considered what the configuration spaces are for these two theories. It turns out that this Shape Mechanics’ configuration spaces (‘shape spaces’) are simpler, with the corresponding Shape-and-Scale Mechanics’ configuration spaces then being the cones over these [5]. E.g. the shape spaces for $N$ particles in 1-d are ${{N-2} \choose 2}$-spheres $S^{N-2}$, and those in 2-d are complex projective spaces $CP^{N-2}$; furthermore for 3 particles in 2-d, $CP^1 = S^2$: the shape sphere of all possible triangular shapes. Furthermore, these shape spaces turned out to have already arisen in Kendall’s studies of the Geometry, Probability and Statistics of shapes (6, 7, 8)). This is a very useful interdisciplinary connection which I pointed out in the Theoretical Physics literature in [9]. I further studied these two RPMs in [10], eventually summarizing the key properties of their configuration spaces within my review [11] on configuration spaces in Theoretical Physics more generally.

RPMs have a number of foundationally valuable applications in Theoretical Physics [10], including the following.

1) RPMs have a number of features in common with GR as viewed as a dynamical system [12, 13, 14, 10]. In particular, they have an energy constraint analogous to the GR Hamiltonian constraint; both are quadratic in the momenta. They also have constraints linear in the momenta (and first-class) which are analogues of the GR momentum constraint.

2) RPMs are useful in analyzing which aspects of Background Independence GR possesses. Difficulties with these then become facets of the notorious Problem of Time in Quantum Gravity [18, 19, 20, 21, 10, 22, 23, 24, 25]. Each constraint that is quadratic in the momenta can be taken to arise from the configuration space $q$ of the theory. In the case of GR, $q = Diff(\Sigma)$: the spatial diffeomorphisms on the 3-space of fixed topology $\Sigma$, whereas $\mathfrak{g} = \text{Riem}(\Sigma)$: the spatial 3-metrics on $\Sigma$. The corresponding quotient space of these is Wheeler’s Superspace($\Sigma$) = $\text{Riem}(\Sigma)/\text{Diff}(\Sigma)$ [15, 16]; RPMs then offer model arenas of other such quotient spaces as well: shape spaces or relational spaces alias scale-and-shape spaces. Some of these are then more closely analogous to further GR configuration spaces such as conformal superspace [17]; see Appendix C for more detailed comparison between RPM and GR configuration spaces.

3) I also applied [26] Kendall’s own case of Shape Statistics to Timeless Records Theory [19]. This is one of the various approaches to the Problem of Time; in this case, one sees how far one can get by addressing timeless propositions. In particular, my application makes it concrete that Shape Statistics provides the machinery requisite for rendering the classical version of Timeless Records Theory mathematically sharp.

4) RPMs are useful for modelling closed universe and quantum cosmological effects [15, 10].

5) RPMs are also useful models [38, 10] in the study of geometrical quantization [28], and affine quantum geometrodynamics [29, 30].

The current paper represents a major extension of the scope of RPMs: from one theory three decades ago [3] and a second theory one decade ago [4] to now presenting a large number of RPM theories, and moreover derived in a systematic manner. This is based on knowing the standard set of geometries on flat space, standard group theory of the corresponding transformation groups and the interplay between the two [31]. The expansion in scope then rests upon this. This is simultaneously the smallest relationally nontrivial unit. The archetype of minimal relationally nontrivial unit is the relational triangle; e.g. Barbour’s seminars have often involved demonstrations involving shuffling wooden triangles. Furthermore, the corresponding crucial technical tools are based on knowledge of the topology and geometry of the minimal relationally nontrivial unit.
corresponding configuration space of relational triangles. This shape space is the \textit{shape sphere}, or a portion thereof, depending on the exact modelling assumptions [11]. E.g. Kendall’s \textit{spherical blackboard} (Fig 6.a), which is well known in the Shape Geometry and Shape Statistics literature [6, 7]. This consists of 1/3 of a hemisphere corresponding to indistinguishable particles with mirror image triangles identified. On the other hand, a whole hemisphere is required if the former assumption is dropped, or a whole sphere if both are dropped; this furthermore becomes Montgomery’s ‘\textit{pair of pants}’ [32] (in the Celestial Mechanics literature) if both assumptions are dropped and double collisions are excised. In Sec 3, I then introduce, name and consider the configuration spaces of minimal relationally nontrivial units for a large range of further Shape and Scale-and-Shape Theories. These are also very much expected to be a technically important nucleus for the corresponding theories of RPM and of (Scale-and-)Shape Statistics, for which I also provide matching names. Due to this, the summary tables in Figs 2 and 3 are of substantial importance in all of Shape Theory, Shape Statistics and RPMs alongside its applications to understanding the foundations of GR-like theories, of Background Independence and of modelling whole(universe quantum cosmological features. This justifies presenting a number of frontiers for subsequent research directions.

Sec 4 outlines known examples of topological and geometrical structure for configuration spaces. This illustrates some of the detail that one can eventually expect in the study of the new shape spaces and scale-and-shape spaces introduced in the current paper. Sec 5 considers the issue of configuration comparers for (Scale-and-)Shape Theories. This includes three way comparison between Barbour’s approach, Kendall’s and DeWitt’s – the last of these having been foundational in the study of GR itself as a dynamical system.

This paper’s main worked-out application involves new theories of RPM corresponding to further notions of shape, or of scale-and-shape. The new such theories I present in Sec 6 are as follows.

1) \textit{Conformal Shape Mechanics}: a theory of angles alone which most readily generalizes [4] from a structural perspective due to the continued availability of the Euclidean norm.

2) \textit{Area Mechanics} in \textit{2-d} – tied to \textit{equiareal} geometry [33]. I show that Area Mechanics requires its kinetic term be built without evoking the Euclidean norm, which ceases to be a licit structure in this context.

3) \textit{Area Ratio alias Affine Shape Mechanics} (then Area Mechanics’ alias as \textit{Affine Scale-and-Shape Mechanics} becomes clear).

For each of 2) and 3), I also provide a \textit{d-dimensional} generalization of the underlying shape theory. I also then show in Sec 7 that Barbour’s Best Matching comparer used in building RPMs continues to thrive in the complex plane C. I first use this to reformulate [4] (now renamed as \textit{metric} Shape Mechanics, given all the other theories of Shape Mechanics provided in the current paper!) I then indicate how the indirect formulation of Mechanics runs into difficulty for the Shape Theory in which the M"{o}bius group is taken to be physically irrelevant.

Sec 8 then outlines new frontiers of research in Shape Mechanics, with Sec 9 following this up by outlining corresponding new frontiers of research in Shape Statistics. This is due to the current paper’s range of notions of shape exceeding that considered by Kendall, while remaining amenable to parallel shape space geometry based investigations. I.e. I generalize Kendall’s own illustrative ‘standing stones’ problem for metric Shape Statistics to affine, conformal and M"{o}bius Shape Statistics and other projective Shape Statistics besides.

Kendall and collaborators’ topological, geometrical, probabilistic and statistical work [6, 47, 8, 7], indicates the considerable value of the interdisciplinarity connection [9, 10, 26] between this and Metric Shape Mechanics. It is furthermore to be expected that such interdisciplinary connections will extend to the range of other shape and scale-and-shape theories considered in this paper. Moreover, this Shape Geometry, Probability and Statistics took Kendall’s group 20 years to develop for one notion of shape; thus it should in no way be expected for the current publication to work all of this out. The current paper already multiplies by a sizeable factor the number of known Relational Mechanics theories. The list of Shape Statistics frontiers and other interdisciplinary applications provided is then to be regarded as a source of future research papers, Such interdisciplinarities involving metric (scale-and-)shapes have indeed already been established. The current paper then points out that these have conceptual-level analogues for other theories of (scale-and-)shape as outlined in the Frontier sections. As well as the above-mentioned interdisciplinarities, further such which are established for shape theories include with

i) Robotics [34] ((for the reasons given in Secs 3 and 7.1).

ii) Image Analysis [35] (e.g. shapes photographed from whichever direction).

iii) Biology [36] (e.g. animal morphology).

iv) Astronomy (e.g. galaxies, CMB patterning, microlensing).

Additionally, I do not stop at affine and conformal type theories of Classical Mechanics. In Sec 10, I give furthermore the first ever treatment of supersymmetric RPM, alongside frontier questions concerning supershapes. This is in the context of enlarging \textit{Eucl}(d) and \textit{Sim}(d) firstly to \textit{superEucl}(d) and \textit{superSim}(d) and then onward to two competing super-apex groups: the superconformal and superaffine groups, alongside many other subgroups of these. Some context and motivation for this development is as follows.
1) Whether Relationalism and Background Independence more generally have a similar characterization in Supergravity. The answer is no ([22, 82] and Appendix D). Due to this, it was a substantial oversight to consider only GR in [18, 37] on the assumption that other theories of gravity would be similar in this regard.

2) Whether Relationalism is compatible with Supersymmetry [22, 82]. The current paper shows that the answer to this is yes.

I end in Sec 11 by providing the Dirac quantizations for a representative four among the current paper’s new RPMs. This parallels how e.g. Rovelli [38] and Smolin [39] quantized [3] around a decade after its inception (whereas I subsequently considered both Dirac and reduced quantizations of both [3] and [4]).

Appendix A supports the text by outlining the group-based approach to the foundations of flat theories of geometry. Appendix B outlines those parts of the theory of Lie algebras and Lie groups that are used in this paper, including supersymmetric counterparts. Appendix C compares the foundational variety of flat space notions of geometry with that of notions of differential geometry, Appendix C also gives an outline of GR’s Dirac algebra of constraints. This is provided firstly because RPMs’ constraint algebras share a number of features with it. Secondly, it is provided for for comparison with Appendix D’s significantly distinct supergravity algebra of constraints. This contrast is of foundational interest as regards canonical quantization, Quantum Gravity and differences in the form in which Background Independence can be manifested. Finally, the latter is also contrasted with supersymmetric RPMs’ own constraint algebras.

2 Relational configurations and the corresponding configuration spaces

First consider configuration space \(q\) [40]: the space of generalized configurations \(Q^A\) for a physical system. The rest of this Sec considers finite flat-space Mechanics examples of configurations and configuration spaces. N.B. that these in fact promptly come to involve mass-weighted configurations.

This paper considers the case of point configurations. Here \(q = q(N, d) = \times_{i=1}^{N} a(d),\) for \(a(d)\) the absolute space. For these, the relevant principles of Configurational Relationalism is that Physics involves not only a

\[ g \] acts on absolute space \(a(d)\) (usually \(\mathbb{R}^d\)).

\[ g \] acts on configuration space \(q(N, d)\), i.e. acting rather on a material entity of at least some physical content.

However, in the case of \(q = q(N, d) = \times_{i=1}^{N} a(d),\) the group action takes, particle by particle, the form of a group action on individual particles in \(a(d).\) Due to this, each kind of geometry that can be considered for \(a(d)\) corresponds to a relationalism imposed on the whole of \(q(N, d).\)

Thus b) can be resolved by resolving a), which amounts to addressing the groups acting on \(\mathbb{R}^d\) has been well-addressed, e.g. in Klein’s Erlangen approach to geometry. In this way, Appendix A’s well-known mathematics can be straightforwardly commandeered to settle b) also.

Some limitations on the choice of \(g, q\) pairs are as follows.

A) Nontriviality. \(g\) cannot be too large, by a degrees of freedom counting criterion. Then using \(q := \dim(q)\) and \(l := \dim(g),\) a theory on \(q/g\) is inconsistent if \(l > q,\) trivial if \(l = q\) and relationally trivial if \(l = q - 1.\) The last of these is because relational nontriviality requires for one degree of freedom to be expressed in terms of another. This is as opposed to it being meaningfully expressible in terms of some external or elsewise unphysical ‘time parameter’.

B) Further structural compatibility is required. A simple example of this is that in considering \(d\)-dimensional particle configurations, \(g\) is to involve the same \(d\) (or less, but certainly not more).

C) A more general structural compatibility criterion is that \(g\) is to have a group action.² A group action’s credibility may further be enhanced though its being ‘natural’, and some further mathematical advantages are conferred from it

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1 This is trivial in the sense that there are no independent degrees of freedom in the principal stratum of the orbit space. Moreover, the group action can be such that non-principal strata retain nontrivial dynamical content. Such non-principal strata occur e.g. in the standard action of \(SO(3)\) on \(\mathbb{R}^3\) and on the configuration space of \(X \times \mathbb{R}^d\) of \(N\)-particle configurations. This is one way in which the counting argument is ‘local’ rather than ‘global’. Another is how 2 \(\div\) GR manages to be trivial as regards local degrees of freedom but none the less is capable of possessing global degrees of freedom.

2 A group action \(a \) on a set \(X\) is a map \(a: g \times X \rightarrow X\) such that i) \(g_1 \circ g_2 x = g_1 \circ (g_2 x)\) (compatibility) and ii) \(e x = x \forall x \in X\) (identity). This terminology continues to apply if \(X\) carries further structures. Examples include left action \(g x,\) right action \(x g,\) and conjugate action \(g x g^{-1}\). By a natural group action, I just mean one which does not require a choice as to how to relate \(X\) and \(g\) due to there being one ‘obvious way’ in which it acts. E.g. \(\text{Perm}(X)\) acts naturally on \(X,\) or an \(n \times n\) matrix group acts naturally on the corresponding \(n\)-vectors. An action is faithful if \(g_1 \neq g_2 \Rightarrow g_1 x \neq g_2 x\) for some \(x \in X,\) whereas it is free if this is so for all \(x \in X.\) A map is proper if the inverse map of each compact set is itself compact; a particular subcase of this used below is proper group action.
being whichever of faithful or free, with the combination of free and proper conferring yet further advantages. Sec 5 contains a further compatibility condition relevant in the case of Mechanics.

One might additionally wish to choose \( g \) for a given \( q \) so as to eliminate all trace of any extraneous background entities.\(^3\) The automorphism group\(^4\) \( \text{Aut}(\mathfrak{a}) \) of absolute space \( \mathfrak{a} \) is then an obvious possibility for \( g \). However some subgroup \([41]\) of \( \text{Aut}(\mathfrak{a}) \) might also be desirable, not least because the inclusion of some such automorphisms depends on which level of mathematical structure \( \sigma \) is to be taken to be physical. I.e. \( g \subseteq \text{Aut}(\langle \mathfrak{a}, \sigma \rangle) \) is a more general possibility. Such subgroups also comply with A) and stand a good chance of suitably satisfying criteria B) and C). See Sec 5 for examples.

### 2.1 Direct implementation of Configurational Relationalism

Suppose we are in possession of a \( q, g \) candidate pair. Then seek to represent the generators of \( g \) in terms of \( Q^h, \frac{\partial}{\partial q^h} \), which manifestly acts on \( q \). We can then check whether some candidate objects \( O(Q^h) \) are \( g \)-invariants by explicitly checking that these are indeed preserved by \( g \)'s generators. For particles configurations, invariants are plentiful and intuitively clear for a wide range of \( g \), as a direct follow-on of the forms taken by the corresponding \( g \)-invariants in \( \mathbb{R}^d \). E.g. for \( g = \text{Euc}(d) \), separations between particles and angles between particles are of this nature; Fig 4 tabulates such for further \( g \).

### 2.2 Indirect implementation of Configurational Relationalism: ‘\( g \)-act \( g \)-all’ method

A given set of objects can also be interesting in a wider range of ways than being \( g \)-invariants. Moreover, in some cases, invariants are unknown or nonexistent. There are more general concepts of ‘good \( g \) objects, such as \( g \)-tensors (of which \( g \)-invariants are but one example: \( g \)-scalars) or objects which are one of the preceding modulo a linear function of the generators. Here one is representing auxiliaries in terms of \( q^g \) and tangent bundle auxiliary quantities \( d q^g \).

Configurational Relationalism’s broad strategy is the ‘\( g \)-act \( g \)-all method’\([42]\). Consider here consider some object \( O \) belonging to some space of objects \( S \). The \( O \) may be composites of some kind of more basic variables \( b \) (\( Q^h \) alone in this Sec’s \( q \)-only setting, though further Secs such as Sec 5 extend this). Such composites indeed cover far more than just \( S \): also e.g. notions of distance, information, correlation, and also quantum operators and quantum versions of all of the preceding. In whichever case, start by applying \( g \)-act; this can initially be conceived of as \( S \overset{g}{\rightarrow} g \times S, O \mapsto O \).

End by applying \( g \)-all: some operation \( S_{g \in g} \) is applied, which makes use of all of the \( g^2 \in g \). This has the effect of cancelling out \( g \)-act’s use of \( g^2 \), so overall a \( g \)-invariant version of each \( O \) is produced, which I denote by

\[
O_{g^{-\text{inv}}} := S_{g \in g} \circ \overset{g}{\rightarrow} \quad (1)
\]

Examples of \( S_{g \in g} \) include summing, integrating, averaging (group averaging is an important basic technique in Group and Representation Theory), taking infs or sups, and extremizing, in each case indeed meaning over \( g \). For the first two examples,

\[
S_{g \in g} \text{ include } \sum_{g \in g}, \quad \int_{g \in g} Dg. \quad (2)
\]

Barbour’s Best Matching’s own \( S_{g \in g} \) is extremization over \( g \) (see Secs 4-5).

Finally, ‘Maps’ can additionally be inserted between \( g \)-act and \( g \)-all to produce an even more general

\[
O_{g^{-\text{inv}}} := S_{g \in g} \circ \text{Maps} \circ \overset{g}{\rightarrow} \quad (3)
\]

‘Maps’ covers a very general assortment of maps, though these are to all be \( g \)-invariant; if not, \( g \) would act on a new type of object \( O' = \text{Maps} \circ O \).

### 2.3 On the variety of notions of point particle configuration

For the most commonly considered case of \( \mathfrak{a} = \mathbb{R}^d \), Appendices A and B provide many suitable \( g \)’s; see also Fig 2. What physical considerations enter these choices? A case of note is whether scale is to be physically meaningless. Moreover, in directly modelling nature, disregarding scale jeopardizes standard cosmological theory without providing a viable replacement\([44]\). Additionally, retaining scale may enable time provision\([45]\). On the other hand, the metric Shape Theory is both mathematically simpler and recurs as a subproblem within the metric Scale-and-Shape theory. Moreover, it is quite commonplace to consider physical theories with scale that possess a distinct scale-invariant phase in an ‘unbroken’ higher-energy regime, by which not matching everyday experience is not necessarily the end to a theory’s relevance.
One possibility which has hitherto not been mentioned in setting up RPMs is that preservation of inner products • is not the only possibility: an alternative to this is preservation of × products (or in dimension-independent language of forms, of exterior products ∧ ). This corresponds to whether infinitesimal Procrustean stretches (d-volume top form preserving in dimension d) and shears are to be physically meaningless. These distort relative angles and of ratios of relative separations respectively. They combine with translations and rotations to form the ‘equi-top-form group’ Equi(d), or furthermore with the dilations to form the affine group Aff(d). This represents one way of extending Sim(d) through its being a subgroup within a larger group Each of these groups corresponds to a further known type of geometry as per Appendix B.1.

Further issues involve whether the configurations are to be mirror image identified, and whether the particles are to be distinguishable. Both of these issues translate to the form taken by the configuration space topology [10]. This involves using not necessarily q but q = ∑ N j=1 q/ q′ more generally, in particular for discrete group q′ = Z2, ZN, Z2 × ZN though partial indistinguishability is also possible.

A distinct further possibility which has not yet been mentioned in setting up RPMs stems from whether to allow the ‘inversion in the sphere’ transformation (73), which also preserves angles. If so, special conformal transformations exist, providing a distinct extension of Sim(d). Note furthermore that the affine and conformal extensions are incompatible with each other as per (91), so these two extensions cannot furthermore be composed. They correspond to two different ‘apex groups’ within each of which the Sim(d) hitherto used in RPMs and Shape Statistics sits as a subgroup.

This is a good point at which to note that 1-d is too simple to support distinctions between a number of types of geometry. As well as having no nontrivial volume forms and so is bereft of an affine extension. Thus rotations and the further possibility of extension to affine transformations require dimension ≥ 2. On the other hand, 2-d has an infinite-d conformal group. In fact 1-d does too, though that one is less interesting through coinciding with the reparametrizations. See Appendix B for an outline of both of these workings). These invalidate 1 and 2-d configurations of a finite number of particles from having a nontrivial relational theory by the counting argument A) of Sec 2. Thus dimension ≥ 3 is required to investigate this possibility, though e.g. dimension 2 also provides finite subgroups of the conformal group within which Sim(2) sits as a subgroup. E.g. the Möbius group considered in Sec 5.6; this group in turn corresponds to a type of projective geometry.

This is also a good point at which to discuss the Relationalism of the above two extensions. The conformal extension’s involvement of inversion amounts to replacing A(d) = R^d by R^d ∪ ∞; this is a new consideration in the context of RPMs and Background Independence more generally. On the one hand, this is adding an extra structure which might be interpreted as absolute: Riemann’s notion of the ‘point at infinity’. On the other hand, appending this one point allows for a rather more general class of angle-preserving transformations to be well-defined. The affine extension, on the other hand, remains within the usual A(d) = R^d. It amounts to modelling situations in which configurations have no

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3This does not mean that relationalists necessarily discard structures, but rather that they are prepared to consider the outcome of entertaining more minimalist ontologies.

4A homomorphism is a map µ : S1 → S2 that is structure-preserving. In particular, if a such is invertible (equivalently bijective) it is an isomorphism, if S1 = S2 it is an endomorphism, and, if both apply, it is an automorphism.

5I use a Gothic font for spaces so as to not confuse configurations with the configuration spaces they belong to. Also Zₙ is here the cyclic group of order n.

6Further real projective variants additionally exhibit point-to-line duality, which further unusual property heralds departure from Relational Particle Mechanics. Moreover, the C case of projective geometry additionally does not discern between lines and circles.
| Invariants | Group $\mathcal{G}$ | $\dim(\mathcal{G})$ | 1-d | 2-d | 3-d |
|------------|----------------------|------------------|-----|-----|-----|
| None       | $id$                 | 0                | $q^1$ | $q^2$ | $q^3$ |
| Ratios     | $Dil$                | 1                | $q^1$ | 2 components | 3 components |
| Differences| $Tr$                 | $d$              | $p^1$, $p^2$ | support 3 ratios | 2 components |
| Dot Products| $Rot$              | $\frac{d(d-1)}{2}$ | row 1 again | $\frac{\|q^1\|\|q^2\|}{\sqrt{2}}$ | 2 components |
| - $*$ -    | $Tr \times Dil$     | $d + 1$          | $p^1$, $p^2$, $p^3$ | support 3 ratios | 2 components |
| - $\circ$ -| $Rot \times Dil$   | $\frac{d(d-1)}{2} + 1$ | row 2 again | 4 without scale supports 2 ratios |
| - $-$ -    | $Tr \times Rot$     | $\frac{d(d+1)}{2}$ | row 3 again | $\frac{\|q^1\|\|q^2\|}{\sqrt{2}}$ | 2 components |
| - $-$ -    | $Tr \times [Rot \times Dil]$ | $\frac{d(d+1)}{2} + 1$ | row 5 again | 7 without scale supports 2 ratios |

Top Forms

| $\wedge$ | $SL(d, \mathbb{R})$ | $d^2 - 1$ | row 1 again | 3 subordinated areas $(q_1 \times q_2)_3$ | 4 subordinated volumes $q_1 \times q_2 \times q_3$ |
| $\wedge$ | $GL(d, \mathbb{R})$ | $d^2$ | row 2 again | 9 supports 2 ratios of subordinated areas | 9 supports 3 ratios of subordinated volumes |
| $\wedge$ | $Tr \times SL(d, \mathbb{R})$ | $d(d+1)$ - 1 | row 3 again | 3 areas $(\mathbb{R}_A \times \mathbb{R}_B)_3$ | 4 volumes $\mathbb{E}_A \times \mathbb{E}_B \times \mathbb{E}_C$ |
| $\wedge$ | $Tr \times GL(d, \mathbb{R})$ | $d(d+1)$ | row 5 again | 11 supports 2 area ratios | 11 supports 3 volume ratios |

Cross Ratios

| $\wedge$ | $PGL(d, \mathbb{R})$ | $2(d^2 - 1)$ | row 1 again | 1 complex cross ratio: $\frac{\|q_1\|\|q_2\|}{\sqrt{2}}$ | 2 degrees of freedom |
| $\wedge$ | $M\ddot{o}b\ddot{i}us$ | | | supports 2 independent local angle freedoms $\mathbb{E}_A / \mathbb{E}_B$ | 2 degrees of freedom |

Figure 2: Invariant corresponding to each group, dimension of the group and the minimal relationally nontrivial unit in spatial dimensions 1, 2 and 3. O is an absolute origin; the axis and ruler logos denote absolute orientation and absolute scale respectively. In particular, note that the bottom six rows are interesting and new. In this way, the current paper covers enough new material, with a broad enough range of new applications for further researchers to work on, to justify introducing this useful shorthand notation for the invariants/observables, and the namings of all the corresponding configurations and of theories of Mechanics, and of Statistics in this and the next Figure.

Overall meaning of either relative angle (by equivalence under global shears), or of relative ratio (by equivalence under global Procrustean stretches). Considering these in the context of RPMs is also new to the current paper. See Sec 5 for discussion of issues of direct realization, and of applications for which that is not a consideration.

In the by now well studied relational space and (metric) shape space cases, in both 2- and 3-d the minimal relationally nontrivial unit is the triangle. Barbour’s well-known demonstrations of Best Matching with wooden triangles, and Kendall’s method of sampling in threes leading to his spherical blackboard methodology – Fig 6.a – follow. The minimal relationally nontrivial unit is all of that type of Relationalism’s smallest whole universe relational model, smallest relationally nontrivial subsystem, and smallest relationally nontrivial sampling unit for Shape Statistics. These are ways in which minimal relationally nontrivial units are important. Thus in the last three columns of Fig 2, I depict and explain the form these take, for each of this paper’s suite of subgroups of the affine group, the full conformal group in 3-d and the Möbius group in 2-d. In perusing this Figure, it may be useful to bear in mind how the simplest affine and projective geometry theorems also require use of more points than the simplest Euclidean geometry ones [33].

More generally, yet further models of absolute space $\mathfrak{a}$ might be considered, such as $S^d$ or $T^d$. E.g. for GR, both $S^3$ and $T^d$ have been substantially studied as compact models for space; $T^d$ is of course also a compactification of $\mathbb{R}^d$; Cosmology also makes use of hyperbolic space $\mathbb{H}^d$, which admits compactifications of its own. RPMs based on such are then closer to GR than RPMs based on flat space. RPMs based on $S^d$ in place of $\mathbb{R}^d$ in the role of absolute space have started to be considered elsewhere [10, 46]. Not that $S^2$ admits the additional interpretation and realization as an observed space model: the sky, due to which Shape Theory and Shape Statistics for it had already previously appeared in the literature [47, 7]. The current paper does not further explore this paragraph’s additional possibilities, nor have RPM on $T^d$, $\mathbb{H}^d$ or compactifications thereof started to be investigated to date. None the less, it is clear
that the current paper’s systematics concerning invariants, groups, families of subgroups, minimal relationally nontrivial units, configuration spaces, configuration comparers, and constructions of Mechanics and Statistics upon (generalized) shape spaces, furthermore carries over to all these other cases as well. In particular, the current paper considers the different levels of geometry on open infinite flat space, but if one passes to whichever curved space instead, one can again contemplate a comparable range of geometrical structures thereupon.]

3 Configuration space geometry

In setting up a large number of new theories of Shape Mechanics, with underlying Shape Theories and configuration spaces, many of which are new too, if is rather necessary to create names and notations for the configuration spaces in question (Figure 3). These are conceptually important entities, whose precise mathematical nature shall one day be known in detail, much as [7, 10] lay this out in the case of redundant similarity group.

| invariants | group $G$ | Configuration space name | Mechanics name | Statistics name |
|------------|-----------|--------------------------|---------------|-----------------|
| None       | $id$      | particle position space $q(N,d) = \mathbb{R}^{Nd}$ | Newtonian absolute mechanics | Standard ($\mathbb{R}^d$ multivariate) statistics |
| ratios /   | $Dil$     | ratio space $r_{Dil}(N,d) = q(N+1,d) = \mathbb{S}^{Nd-1}$ | ratio = clumping mechanics | ratio = clumping statistics |
| differences | $Tr$      | relative space $r_{Tr}(N,d) = q(n,d) = \mathbb{R}^{nd}$ | relative = barycentric mechanics | relative statistics = barycentric statistics |
| dot products | $Rot$     | dot space $D(N,d) = \mathbb{R}^{n+1,d} = \bigcup (C^d_{n-1})_{2-d}$ | dot = rotationally invariant mechanics | dot = rotationally invariant statistics |
| - / -      | $Tr \times Dil$ | preshape space $[Kendall]$ $p(N,d) = \mathbb{S}^{nd-1}$ | preshape = non-rotational mechanics | preshape = non-rotational statistics |
| - / -      | $Rot \times Dil$ | dot-ratio space $D_{Rot}(N,d) = \mathbb{S}^{n+1,d} = \bigcup (C^d_{n-1})_{2-d}$ | dot-ratio mechanics | dot-ratio statistics |
| - - / -    | $Tr \times Rot$ | relational space $[Leibniz]$ $R(N,d)$ | relational = metric shape and scale $= Leibniz$ mechanics $[BBR2]$ | relational = metric shape-and-scale $= Leibniz$ statistics |
| - - / -    | $Tr \times DiR$ | (metric) shape space $[Kendall]$ $s(N,d)$ | (metric) shape mechanics $[Barbour]$ | (metric) shape mechanics $[Kendall]$ |
| top forms  | $SL(d,\mathbb{R})$ | subtended-top-form space $\mathbb{S}^N(N,d)$ | subtended-top-form mechanics | subtended-top-form statistics |
| - / -      | $GL(d,\mathbb{R})$ | subtended-top-form-ratio space $\mathbb{S}^A(N,d)$ | subtended-top-form-ratio mechanics | subtended-top-form-ratio statistics |
| - / -      | $Tr \times SL(d,\mathbb{R})$ | top-form space $\mathbb{W}(N,d)$ | top-form mechanics | top-form statistics |
| - / -      | $Tr \times GL(d,\mathbb{R})$ | top-form-ratio = affine shape space $\mathbb{A}(N,d)$ | top-form-ratio = affine shape mechanics | top-form-ratio = affine shape statistics |
| cross ratios | $PGL(d,\mathbb{R})$ | (local) angle space $\mathbb{A}(N,d)$ | local angle = conformal shape mechanics | local angle = conformal shape statistics |

Figure 3: For each group–invariant pair, we give name and notation for the corresponding configuration space, and the names for the corresponding Mechanics and Statistics. Note the two alternative namings: by group and by the theory’s invariant objects. Note many configuration space geometry notions reduce to others, and that ones of known geometry are indicated; see [7, 10, 11] for more about these.

Relative space $v(N,d) = q(N,d)/Tr(d) = \mathbb{R}^{nd}$ for $n := N - 1$. Bases of relative inter-particle separation vectors – Lagrange coordinates – and of cluster separation vectors – relative Jacobi coordinates – are then natural thereupon (Fig 1.c–f). In an absolute worldview, these correspond to passing to centre of mass frame, whereas in a relational worldview they correspond to absolute absolute origin being meaningless.

Useful Lemma (Jacobi pairs). Within the subgroups of the affine group, the number of relational configuration spaces requiring independent study is halved, since each version with translations is the same as the version without with one particle more.

Proof. For these groups taking out the centre of mass is always equally trivial. Moreover, the diagonal form in Jacobi’s relative $\rho^A$ is identical in every respect with that of the mass-weighted point particles bar there being one $\rho^A$ less. □
Take the form of the invariants of a geometry [Appendix A] and apply to whichever of \( q \) and \( \rho \) are rendered appropriate by the assumption of material point particles [displayed in column 1 of Figs 2–3 and further laid out in Fig 5]. These are what then firstly serve as potential functional dependence, and subsequently end up being wavefunction dependencies. Figs 3 and 4 then name the corresponding configuration spaces.

If absolute axes are also to have no meaning, the remaining configuration space is

\[
\text{relational space } \mathfrak{R}(N, d) := q(N, d)/\text{Eucl}(d) ,
\]

of dimension \( nd – d(d-1)/2 = d\{2n + 1 – d\}/2 \); in particular, \( N – 1 \) in 2-d, \( 2N – 3 \) in 2-d and \( 3N – 6 \) in 3-d. If, instead, absolute scale is also to have no meaning, the configuration space is Kendall’s preshape space [7] \( \mathfrak{P}(N, d) := \mathfrak{U}(N, d)/\text{Dil} \), of dimension \( nd – 1 \). If both absolute axes and absolute scale are to have no meaning, then the configuration space is Kendall’s [7]

\[
\text{shape space } \mathfrak{S}(N, d) := q(N, d)/\text{Sim}(d) .
\]

This is of dimension \( N – 1 \) in 1-d, \( 2N – 4 \) in 2-d and \( 3N – 7 \) in 3-d; as well as featuring in accounts of RPMs [9, 10], it is well-known from the Shape Geometry and the Shape Statistics literatures [6, 47, 7]. Also note that \( \mathfrak{P}(N, 1) = \mathfrak{S}(N, 1) \), since there are no rotations in 1-d. The above quotient spaces are taken to be not just sets but also normed spaces, metric spaces, topological spaces, and, where possible, Riemannian geometries. Their analogy with GR’s configuration spaces is laid out in Fig 16.

Note that the Jacobi pairs simplification does not apply within those further groups that include the special conformal transformations \( K_i \). This is because of the commutation relation (88), by which translations cease to be so trivially removable. Also contrast the conformal case’s pure angle information with the similarity case’s mixture of angle and ratio information.

See Fig 4 for an outline of further subgroups of \( \text{Conf}(d) \), alongside indication of other combinations of generators which also fail to close as groups for the reasons stated.

The configuration space level can have a metric geometry (or in reduced cases generally a stratified such), even if the original configurations sit in a geometry with less structure than that. This is entirely possible because the map from space to the space of spaces need not be category-preserving.

Finally, while two area ratios have the range of a quadrant or a whole plane if signed, it is not a priori likely for these to carry a flat metric.

Figure 4: Layout of \( g \)-invariant contents, in the pattern following on from Fig 14’s layout of which combinations of generators are group-theoretically allowed. Note that four of the subgroups of \( \text{Conf}(d) \) have the same invariants; this is due to incorporating the special conformal transformation being rather restrictive.

Considering larger units than the minimal one is valuable not only since furtherly relational theories need some such, but also because some applications need more system complexity. It is insightful here to point out that Montgomery’s falling cat\(^8\) and the relational triangle are fully minimal robotic models. E.g. envisage the space of triangles not as a bunch of rigid wooden shapes but as the shapes that can be formed by a flexible, extendible entity. This perspective involves paths in configuration space. Models along the above lines rather quickly acquire complexity. Indeed [49, 50] considered the K-shaped clustering of three relative Jacobi coordinate vectors presentation of quadrilaterals as axe configurations.

\(^7\)Bold font here denotes configuration space objects, with indices \( iI \) and \( iA \) respectively. \n\(^8\)This involves 2-rod configurations, corresponding to configuration space \( \mathbb{RP}^2 \).
The underlying shape space in this case is $\mathbb{CP}^2$: Fig 6.b), and the relational space is $C(\mathbb{CP}^2)$. I now point out that axes are already complex enough tools to have very different applications according to angles and proportions. Thus a ‘robotic axe’ reinterpretation of the model shape space of axes is already a model of a significantly adaptable robotic tool. Finally robotic models may also eventually expected to enter foundational Theoretical Physics, along the lines of Hartle’s IGUS [53]. I.e the Information Gathering and Utilizing System model concept could well receive a classical and then quantum robotic implementation.

4 Configuration comparers

Various such can be built from $q$’s kinetic metric’s $^9 M$ inner product and norm [6, 7, 3, 54, 10]

\[
\begin{align*}
\text{(Kendall Dist)} & = (Q, Q)_M, \\
\text{(Barbour Dist)} & = ||dQ||_M^2, \\
\text{(DeWitt Dist)} & = (dQ, dQ')_M.
\end{align*}
\]

Next, if there is additionally a physically irrelevant $g$ acting upon $q$,

\[
\begin{align*}
\text{(Kendall $g$-Dist)} & = (Q, \overrightarrow{g} Q')_M, \\
\text{(Barbour $g$-Dist)} & = ||d_g Q||_M^2 \quad \text{and} \\
\text{(DeWitt $g$-Dist)} & = (\overrightarrow{g}_{d_g} Q, \overrightarrow{g}_{d_g} Q')_M.
\end{align*}
\]

Then $g$-all moves – such as integral, sum, average, inf, sup or extremum – can be applied, after insertion of Maps if necessary. This is the first publication to consider this three-way comparison, DeWitt’s own approach being well-known from the foundations of GR as a dynamical system [54].

Then e.g. (10) subjected to the $\times \sqrt{2W}$ and integration maps before a $g$-all extremum move gives Best Matching; this can furthermore now be recognized as a subcase of a weighted path metric. ‘Comparers’ then have a further issue: if $M = M(Q)$, does one use $Q_1$ or $Q_2$ in evaluating $M$ itself? This situation does not arise in the $\mathbb{R}^n$ shapes context of Kendall, but it does in DeWitt’s GR context; he resolved it in the symmetric manner, i.e. using $Q_1$ and $Q_2$ to equal extents.

\[9\text{This exists independently of whether it is contracted into velocities or changes; e.g. moment of inertia is this metric contracted into mechanical configurations themselves. It only provides a norm if it is positive-definite.}\]
Figure 6: a) Unlabelled triangles’ spherical blackboard (or half of it; triangleland is either $S^2$ or some regular portion thereof, depending on exactly how the configurations are being modelled [11]). Here ‘regular’ means that the base and median partial moments of inertia are equal, D is a double collision and M is a merger (third particle at the centre of mass of the other two). If the triangles are labelled and with mirror images distinct, the whole sphere is realized. b) Complex projective chopping board of axe configuration. This figure concentrates on the two ratio coordinates $\beta$ and $\chi$, suppressing the further variety in relative angle coordinates $\phi$ and $\psi$ ($\beta, \chi, \phi, \psi$ are Gibbons–Pope type coordinates, see e.g. [49]).

Figure 7: 5 notions of matching shapes keep the dotted one fixed and perform transformations. a) Barbour’s wooden triangles. b) The same, now for overhead projector slides. Affine matching adds c) to this and conformal matching adds d) e) is Möbius matching.

Furthermore, how good the ‘best fit’ is can be assessed in substantially geometrically general cases by making set of relational objects out of primed and unprimed vertices (Fig 8.a), to which the corresponding notion of Shape Statistics is to be applied. In the present case, these are triangles that one can test against the $\epsilon$-bluntness criterion.

As a first new example, consider affine space. Its $d$-volume top form construct is not amenable to a notion of distance out of being antisymmetric and thus not obeying symmetry, separation, or, if signed, positivity. Affine space can however be equipped as a metric space, giving a $\text{Dist}(q, Gq)$ style finite comparer [35] for $G \in GL(d, \mathbb{R})$. For instance this is a useful tool in image recognition, which is affine to leading order [35]. This corresponds to how images can be stretched and sheared as well as enlarged, translated and rotated, depending on the exact modelling assumptions made. One application of this is to nearby objects, for which many viewpoints can be attained ‘by walking to the other side’ of the object. One interesting proposition then is whether two pictures feature the same object viewed from different angles. A second application is that the former proposition also arises within hypothetical smaller closed-topology universes [55], and within the setting of multiple images from gravitational lensing [56]. However, the first of these has additional evolution effects (the multiple images would generally correspond to the object’s configuration in different aeons). The second has additional distortion effects (the geometry around a gravitational lens is not flat, taking one outside the scope of the current paper). A third application concerns studying populations of anisotropic astrophysical objects (most obviously galaxies). Then does one’s collection of images feature a single such viewed from multiple angles, or is it a mixture of objects from distinct populations? Furthermore, are we seeing these objects at entirely random orientations, or do they exhibit a statistically significant pattern of orientations?

As further new examples, see the next two Secs for many new instances of Best Matching.

5 Theories of Shape Mechanics

This application is to whole-universe models, for which Temporal Relationalism applies. I.e. there is no time at the primary level for the universe as a whole. This is best implemented by a geometrical action, which is mathematically
dual to an action making no use of parametrization, which upon introducing a parametrization is then invariant under reparametrization. At least one primary constraint must then follow from any reparametrization-invariant action due to a well-known argument of Dirac’s [59]. In the case of Mechanics, this gives an equation which is usually interpreted as an energy conservation equation. However in the present context this is to be reinterpreted as an *equation of time* [57]. Indeed rearranging it gives an expression for emergent Machian time: a concrete realization resolution of primary-level timelessness by Mach’s ‘time is to be abstracted from change’. In the case of metric Scale-and-Shape Mechanics, this emergent time amounts to a relational recovery of a quantity which is more usually regarded as Newtonian time. Note that more generally handling Mechanics models in a temporally relational manner requires a modified version of the Principles of Dynamics as laid out in [58]. The particular cases of geometrical actions in the next two sections are all *Jacobi-type* action [40], corresponding to a Riemannian notion of geometry.

The case of Relationalism most usually considered as an RPM is metric Scale-and-Shape Mechanics [3], as motivated by the direct approximate physical regime in which Mechanics is used. Note however that Sec 4’s example of the affine transformations arising in the study of images shows how the ‘direct physical regime’s geometry’ may not be the only one of relevance. Finally, if fundamental theory is under consideration, the defects of Newtonian theory itself may well not be the only useful guideline. E.g. scale-invariant and conformal models are often considered in High-Energy Physics (c.f. Sec 2.3). Also RPM has been argued to be of substantial value [18, 19, 21, 10, 11] as a model arena for GR’s own GR model arena perspective also leaves a number of other modelling assumptions for RPMs open. See Appendix C in this regard, both for metric shape RPM and for the new RPMs presented below.

### 5.1 Metric Shape-and-Scale Mechanics (alias Euclidean RPM)

Let us begin with the most familiar case of RPM: the Barbour–Bertotti 1982 theory, albeit reformulated in terms of relative Jacobi coordinates. We will then show how the other RPMs arise from various sources of geometrical variety in this setting. The action for this RPM is [10]

$$S[p^A, b] = \sqrt{2} \int ds \sqrt{W}, \; ds = ||d_b \rho||, \; d_b \rho^A := d_b \rho^A - d_b \times \rho^A,$$

(12)

for $W := E - V$ for $E$ the total energy and $V$ the potential, which in this case is of the form $V(-\cdot-)$. Also the index $A$ runs from 1 to $n := N - 1$: the number of independent inter-particle cluster vectors, where $N$ is the number of particles itself.

The quadratic constraint\(^{11}\)

$$\mathcal{E} := ||\pi||^2/2 + V(-\cdot-) = E$$

(13)

then follows as a primary constraint. This is often interpreted as an energy constraint in the nonrelational and subsystem contexts, but is to be interpreted as an *equation of time* in the relational whole-universe context. Also, the

$$\text{(zero total angular momentum of the universe)}, \; \mathcal{E} := \sum_{A=1}^n L_A^2 \times L_A = 0$$

(14)

---

\(^{10}\)A previous such theory existed [48] but was built in another manner. This previous theory fails to fit mass anisotropy bounds, and ceases to look natural once cast in reduced variables. [3] also remains simple all the way down its reduction process [10], whereas [48] does not. Instead there is a distinct [48] type theory [10] which is simple in each stage of the reduction process’ variables. These not matching up gives a further theoretical reason to favour [3], and theories built similarly to it for other \(\mathbf{g}\), rather than [48] type theories.

\(^{11}\)In this paper, I use the calligraphic font to denote constraints.
follows as a secondary constraint from varying with respect to \( \rho \); thus it results from implementing Configurational Relationism.

[The original version [3] has the \( \rho^A \) to \( q^I \) of the above, with an extra translational best matching correction \( -d_2 \) resulting in an extra]

\[
(\text{zero total momentum constraint}) \quad \mathcal{P} := \sum_{I=1}^{N} p_I = 0 ,
\]

for \( p_I \) the momentum conjugate to \( q^I \).

### 5.2 Metric Shape Mechanics (alias similarity RPM)

In this case – the theory originally due to Barbour 2003 [4] and somewhat reformulated as per [9, 10] –

\[
S[\rho^A, b, c] = \sqrt{2} \int \, ds \sqrt{W} , \quad ds = \|d_{a^c} \rho^A \| / \rho , \quad d_{a^c} \rho^A := d_2 \rho^A - d_2 - \rho^A - d_2 - \rho^A
\]

for \( V = V(-, -, -) \) and \( \rho := \sqrt{I} \) for \( I \) the total moment of inertia.

This gives

\[
\mathcal{E} := I|||\pi|||^2 / 2 + V(-, -, -) = E
\]

as a primary constraint, (14) again from variation with respect to \( b \), and, from variation with respect to \( c \) [4],

(zero total dilational momentum of the universe) ,

\[
\mathcal{D} := \sum_{I=1}^{N} \rho^I \cdot I_{A} - \rho_{A} = 0 .
\]

### 5.3 3-d conformal Shape Mechanics, alias local angle Mechanics

In fact, the further special conformal Best Matching can be appended into the \( q^I \) version of (16) to give (I runs from 1 to \( N \))

\[
S[q^I, \rho, b, c, k] = \sqrt{2} \int \, ds \sqrt{W} , \quad ds = \|d_{a^c} q^I \| / \sqrt{I} , \quad d_{a^c} q^I := dq^I - da^I - d_2 \rho^I - dq^I - \{q^I 2 \rho^I - 2q^I a q^I b \} dk
\]

for \( V \) now of the form \( V(\zeta) \). This is a first RPM theory that is new to this paper. Then variations with respect to \( a, b \) and \( c \) yield (15), and the \( q^I \) counterparts of (14) and (18) respectively. But now also variation with respect to \( k \) produces the

(zero total special conformal momentum constraint)

\[
\mathcal{K}_k := \sum_{I=1}^{N} \{q^I 2 \rho^I - 2q^I a q^I b \} p_I b = 0 .
\]

The quadratic constraint arising as a primary constraint is now

\[
\mathcal{E} := I|||\pi|||^2 / 2 + V(\zeta) = E .
\]

Then the linear constraints close as per the conformal algebra, and \( \mathcal{K}_k \) manages to commute with \( \mathcal{E} \) also. This set-up works similarly for \( d > 3 \); this just requires a different presentation for the larger amounts of \( Rot(d) \).

One further motivation for this model is that, while Similarity Mechanics is already to some extent a useful model of GR’s conformal superspace (Appendix C), Conformal Shape Mechanics is surely a better model of this.

### 5.4 Affine Scale-and-Shape Mechanics in 2-d, alias area mechanics

As a second theory of RPM new to this paper, suppose that one proceeds by extending Sec 5.2’s construct to include affine Best Matching, so

\[
S[\rho^A, b, e, f] = \sqrt{2} \int \, ds \sqrt{W} , \quad ds = \|d_{S_L} \rho^A \|
\]

for \( d_{S_L} \rho^A := d_2 \rho^A - d_2 \rho^A \) the \( SL(2, \mathbb{R}) \) best-matched derivative, for \( d_2 \) the 3-vector \([df, de, db]\) of auxiliaries and

\[
S := \left[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right] ,
\]

alongside \( V = V(-, -) \) in this case. Then

\[
\mathcal{E} := ||\pi||^2 / 2 + V(-, -) = E
\]
arises as a primary constraint, and the linear zero total \( SL(2, \mathbb{R}) \) momentum constraint

\[
\dot{S} := \sum_{A=1}^{n} \rho_A \pi_A \tag{25}
\]
arises from variation with respect to \( s \). This includes \( \mathcal{L} \) again from \( b \)-variation alongside new zero total Procrustean momentum \( \mathcal{P}_r \) and shear momentum \( \mathcal{S}_h \) constraints. Explicitly,

\[
\mathcal{P}_r := \rho_x \pi_x - \rho_y \pi_y \quad , \quad \mathcal{S}_h := \rho_x \pi_y + \rho_y \pi_x .
\]

Then these last two linear constraints fail to Poisson-brackets close with the candidate theory’s \( \mathcal{E} \). Thus this constitutes an example of Best Matching failing as an implementation due to the outcome of the Dirac Algorithm [59]. In such cases, one is to decide which part(s) of the triple \( q, \pi, S \) are to be modified. The present case has a clear solution: the \( || || \) structure ceases to have any business in an affine theory! In this way, the id to Conf family of further Best Matching appendings within the normed form of kinetic line element is not a general procedure. What is more general is my ‘good \( g \) objects’ approach (Sec 2.2), by which \( || || \) is recognized to be illicit at the outset for a \( g \) as redundant as \( Equi(2) \).

According to that, take instead

\[
ds^2 = \sum_{A,B=1}^{n} (d_{SL} \rho^A \times d_{SL} \rho^B)_3 .
\]

Then e.g. focusing on the smallest relationally nontrivial case, \( \sum_{cycles \ A,B=1}^{3} (d_{SL} \rho^A \times d_{SL} \rho^B)_3 \) inverts nicely in the change to momentum sense, giving the primary constraint

\[
\mathcal{E} := \sum_{cycles \ A,B=1}^{3} \pi^A \times \pi^B / 2 + V(-x-) = E ,
\]

alongside the same linear \( \dot{S} \) as before. This mechanical model is indeed consistent.

### 5.5 Affine Shape Mechanics in 2-d, alias Area Ratio Mechanics

A third theory of RPM new to this paper has

\[
S[q^A, b, c, e, f] = \sqrt{2} \int ds \sqrt{V} \quad , \quad ds^2 = \sum_{A,B=1}^{n} d_{GL} \rho^A \times d_{GL} \rho^B / \sum_{C,D=1}^{n} \rho^C \times \rho^D .
\]

Here, \( d_{GL} \rho^A := d \rho^A - d_{G} \rho^A \) the \( GL(2, \mathbb{R}) \) best-matched derivative, for \( d_{G} := [df, dc, db, dc] \) auxiliaries and

\[
G := \left[ \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right]^T .
\]

Also \( V = V(-x-) / (-x-) \).

This results in a the same linear constraints as in the previous Subsec plus \( D \), the four of which can be packaged as the zero total \( GL(2, \mathbb{R}) \) momentum constraint

\[
\dot{G} := \sum_{A=1}^{n} \rho^A \pi_A .
\]

In the smallest relationally nontrivial case \( n = 3 \), using once again the sum of cycles combination, the primary constraint is

\[
\mathcal{E} := \sum_{cycles \ A,B=1}^{3} (\rho^A \times \rho^B)_3 \sum_{cycles \ C,D=1}^{3} (\pi^C \times \pi^D)_3 / 2 + V(-x-) = E .
\]

### 5.6 Some notions of 2-d relational configuration also admit a \( \mathbb{C} \) formulation

As well as the two components of \( Tr(2) \) being an obligatory pairing in this setting (keep both or none), \( Rot(2) \) and \( Dil \) are also an obligatory pairing: as the modulus and phase parts of a single complex number.

The simplest notions of relational configuration that admit a \( \mathbb{C} \) formulation have configuration spaces forming the diamond array \( \mathbb{C}^N \), \( \mathbb{C}^n \), \( \mathbb{C}^p \), \( \mathbb{C}^p^{-1} \) corresponding to quotienting out by none, one or both of \( Tr(2) \) and \( Rot(2) \times Dil \).

In one sense, complex representability extends to the 2-d affine case: its Procrustean stretch and shear can be co-represented in complex form On the other hand, this is a \( \gamma \bar{z} \) representation, i.e. antiholomorphic, and furthermore area is not particularly natural in the complex plane, since it is an \( \text{Im} \) part rather than a whole complex entity: \( \text{Im}(z^A \bar{z}^B) \).

For sure, Möbius Configurational Relationalism continues to lie firmly within the complex domain.

Complex examples of Best Matching include the following. The translation correction is \( da = da_x + i da_y \) and the rotation-and-dilation correction is

\[
dB z^I \quad \text{for} \quad dB = dc + i db .
\]

Affine best matching – new to this paper – completes this with the Procrustean-and-shear correction

\[
dC \bar{z}^I \quad \text{for} \quad dC = df + i de .
\]
On the other hand, Möbius best matching – also new to this paper – completes the above with a holomorphic quadratic correction:

\[ dM \overline{z}^2 \quad \text{for} \quad M \in \mathbb{C} \]  

(35)

The new geometrical entity of particular interest are the cross-ratio spaces \( c(N, 2) \), In particular the minimal relationally nontrivial unit is \( c(4, 2) \). Whereas each cross ratio is itself a complex number-valued quantity, it is not yet clear which geometrical structure is natural to cross-ratio space.

### 5.7 Examples of complex Mechanics formulations

The zero total momentum constraint, if present, is just \( P^x = \sum_{I=1}^{N} p_I \), for each \( p_I \) of form \( p_x + ip_y \). On the other hand, the complex zero total dilational-and-angular momentum constraint is (new to this paper)

\[ Q^x := \sum_{I=1}^{N} \overline{z}^I p_I = 0 \]  

(36)

A complex action for \( \text{Sim}(2) \) is then (using complex vector norm, and new to this paper,

\[ S = \sqrt{2} \int \left| \frac{dA,BZ}{z} \right| \sqrt{E-V} , \]

(37)

for \( V = V(-/-) \). The corresponding form of the quadratic primary constraint is

\[ \mathcal{E} := |z|^2 ||p||^2 / 2 + V(-/-) = E \]  

(38)

Finally, with \( \text{Conf}(d) \) of course becoming infinite-dimensional in 2-d, I also briefly consider a finite Lie subgroup in that case, i.e. the Möbius group as a finite substitute. [An infinite group of physically irrelevant transformations would trivialize any finite particle configuration.] The zero total Möbius momentum constraint – new to this paper – is

\[ \mathcal{M}^x := \sum_{I=1}^{N} \overline{z}^I z^I p_I = 0 \]  

(39)

Moreover, provision of an indirectly formulated action for this is blocked. I.e. in this case, for now we reach a new impasse rather than a new theory of RPM. This occurs due to the following reason. Whereas one can readily enter changes into a ratio so as to produce a function that is homogeneous linear in change, in a cross ratio everything which enters in the numerator also features in the denominator. This forces the change to feature homogeneously with degree zero.

### 6 Some research frontiers in Shape Mechanics

**Frontier 1.** An important next step is to work out reduced versions of the RPM actions. For metric shape and scale-and-shape theories, this was covered in [9, 5, 10]. The GR counterpart of this procedure is also well known, in general leading to the impasse known as the Thin Sandwich Problem [14, 60]. Moreover, reduction is a means of getting at least some candidates for the (generalized) shape space geometries upon which to base the both the corresponding Shape Statistics and the geometrical reduced quantization scheme.

**Frontier 2.** Distinct direct considerations can on some occasions permit finding the relational configuration space and then building a Mechanics thereupon. E.g. this occurs for 1- and 2-d metric shape RPM [9, 10]; moreover, in this case
direct consideration and indirect consideration followed by reduction coincide. To what extent does this coincidence extend among the new theories laid out in this paper?

For instance, the direct implementation of Möbius RPM remains open in this case, due to the inherent problem with making a homogeneous linear function out of cross-ratios. I.e. that converting any number of $z_i$ in $[z_1, z_2; z_3, z_4]$ into $dw_i$ quantities does not succeed in giving a quantity homogeneous linear in the $d$’s.

Frontier 3. Some projective geometries have point-to-line duality. An interesting question for the foundations of Mechanics then is if one sets up a ‘point particle Mechanics’ here, can it be reinterpreted as a dual ‘line Mechanics’ thereupon? How does this impact upon our preconceptions of classical dynamics?

7 Shape Statistics

7.1 Metric Shape Statistics

Clumping Statistics investigates hypotheses concerning ratios of relative separations (detailed information which can be attributed locally and to subsystems). These already exist in 1-d and in settings simpler than metric shape spaces, so this topic is well-known. Astrophysical situations modelled by this include tight binary stars, globular clusters, galaxies and voids: absence of clumping. E.g. Roach [61] provided a discrete statistical study of clumping; this can in turn be interpreted in terms of coarse-grainings of RPM configurations. Also note that Geometrical Probability on the shape space $S(N, 1) = S^{n-1}$ provides an alternative method to this.

Next, consider completing the above at the metric shape space level to probing relative angle information, the existence of which requires $d \geq 2$. E.g. Kendall [6, 47] investigated the relative angle question of whether the locations of the standing stones of Land’s End in Cornwall contained more alignments than could be put down to to random chance.12 This involves the following procedures.

1) Sample in threes.
2) Consider whether there were a statistically significant number of almost collinear triangles quantified by a bluntness angle $< \epsilon$ some small value (Fig 10.a).
3) Use probability distributions based on the corresponding shape space geometry (i.e. on Kendall’s spherical black-board).

Another application of Relative Angle Statistics is disproving claims of quasar alignment.13 Shape Statistics has also been applied to biological modelling of specific 3-d objects (e.g. skulls) viewed as ratios and relative angles based upon some approximating collection of ‘marker points’ [8].

Significant results for different values of $\epsilon$ carry different implications [62]. Were the standing stones laid out skillfully by the epoch’s standards for e.g. astronomical or religious reasons ($\epsilon \leq 10$ minutes of arc), or were they just the markers of routes or plots of land ($\epsilon \leq 1$ degree)?

I also point out here that metric Shape Statistics (whether or not scaled) is likely to be a useful tool for Robotics. For instance, this can be used to analyze the extent to which robots adopt approximately the same configuration in response to similar external conditions. Or as regards the extent to which there is success in robots mimicking (sequences of) animal configurations (fall like a cat, run like a horse...). I.e. (toy model) robotic configuration spaces not only involve determining the path properties outlined in Sec 3, but are also the basis for theories of firstly Geometrical Probability and secondly of Shape Statistics.

Frontier 4. As regards specific Mechanics questions to be settled by Shape Statistics, consider whether a given Celestial Mechanics configurations exhibit a shape statistically significant number of eclipses: clumping-based questions include whether it exhibits a shape statistically significant number of tight binaries. Whether it contains a shape statistically significant globular cluster. Finally, whether two images of globular clusters have sufficiently similar clustering detail to be of the same system or to be members of similarly formed populations.

N.B. that the metric case of Shape Statistics is a well-established approach [6, 47, 8, 7]. Whereas the current paper concentrates on laying out many new theories of background-independent Mechanics, these are equally tied to being

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12Kendall’s work is a solution to Broadbent’s [62] previous posing of the standing stones problem as one to be addressed by some kind of Geometrical Probability.

13Moreover for ‘observed sky’ applications, the space of spherical triangles [47] is even better. See [7] for a review of the corresponding shape geometry and Shape Statistics, and [46] for the related topic of Shape Mechanics built from stripping down $S^d$ rather than $\mathbb{R}^d$ absolute spaces.
Figure 10: a) $\epsilon$-bluntness in probing metric shapes. b) and c) are assessments of almost-colinearity in fours in the conformal and affine settings. In the Möbius setting, cross ratios are to be used in this assessment and the method has to consider almost-cocircularity in fours alongside almost collinearity in fours.

geometrical theories of shape with statistical applications. If the reader wishes to see what these look like for metric shapes – i.e. similarity-redundant geometry – they should cast a look through [7]. In the current article, I point out there is an analogue of this where it is each of conformal, affine, equivoluminal groups that are redundant, or where these, the Euclidean and the similarity group are supersymmetrized. It is on this basis that I point to a number of frontiers. These are likely to be of interest to applied topologists, to geometers, to people working in Probability and Statistics, as well as to people working in Theoretical Mechanics and on the Foundations of Physics, in particular on models of Background Independence for use in Quantum Gravity.

7.2 More general Shape Statistics

I next entertain the idea of the above Shape Statistics being but the first of a larger family, each corresponding to a distinct notion of shape. A more generalized methodology is as follows.

1) Find the probe unit, as tabulated in the last three columns of Fig 2.
2) Find the geometry of the space of the probe unit (the next paper in this series); by the discussion in Sec 4 this extends to comparison of two ‘best fit’ configurations.
3) Build geometrical probability theory thereupon. Note that this being geometrical is not always necessary since some shape spaces turn out to be flat (see [11] for examples).
4) Build Shape Statistics – using a restricted region of 2) – corresponding to the probe unit taking some particular distinctive form.

For each Shape Statistics, one application is to the classical Records Theory of the corresponding Shape Mechanics.

7.3 Affine and equi-top-form Shape Statistics

Frontier 5. One likely application of the 2-d case – Area and Area Ratio Statistics – is image recognition. In looking at images of point configurations, or approximating images using sampling points, the flat space affine Shape Statistics of images only makes sense if probing at least in fours. The minimal equivalent of the spherical blackboard is the affine shape space $\mathfrak{A}(4, 2)$, of dimension 2.

Our illustrative question then is whether Thompson’s fish are affine shape statistically significantly coincident under affine transformations, or whether they ‘just look that way’ much like the standing stones of Land’s End ‘look to the eye’ to have a lot of collinearities in threes. For now I present a 2-d analogue of the fish problem (as per Fig 8.b), which points to use of equiareal Shape Statistics in this case.

7.4 Conformal Shape Statistics, alias Local Angle Statistics

This concerns quantitative analysis of propositions concerning local angles exhibited by a point distribution. In particular in 3-d, the minimum sampling unit here is probing in sixes, corresponding to a 3-d angle space $\mathfrak{A}(4, 3)$.

7.5 Möbius Shape Statistics as a Cross-Ratio Statistics, and other such

This is but one Cross-Ratio Statistics, since cross ratios are invariants in a wider range of projective geometries. In the present case in the complex plane, the minimum probing size is 4 points, corresponding to the 2-d cross-ratio space $\mathfrak{C}(4, 2)$.

As per Fig 10.b-c), this case has a very natural analogue of the Euclidean case’s ‘collinearity in threes’ hypothesis. Firstly note that in this geometrical setting lines and circles can be mapped to each other, so collinear is to be upgraded to that, and four points are needed. Secondly, with these changes made, the question becomes how many almost-real cross ratios there are. This makes use of the well-known fact that real cross ratios correspond to a line-or-circle passing through the four points, with ‘almost real’ in the role of bluntness parameter.
7. Using multiple Shape Statistics, or one as general as needed

Frontier 7. Investigate whether some instances of evolving objects – e.g. the shape of a particular animal’s skull as that animal ages, of skulls over the course of the evolution of species, or of galaxies – preserve relative angle information to a greater extent than ratio information. Investigating whether this is significant to the extent of each such sequence involving conformal maps would require setting up a conformal Shape Statistics. Other hypotheses would include that assessing angle information within metric Shape Statistics suffices, or that ageing and evolution are better modelled affinely. For instance, what type of geometrical transformation is general enough to model each of the three examples given above?

In conclusion, having a Shape Statistics for each type of geometry multiplies opportunities of spotting significant patterns in nature.

8 Supersymmetric RPM

GR has a clear-cut Temporal and Configurational Relationalism split as laid out in Appendix C. On the other hand, Appendix D explains how Supergravity does not. This difference stems from Supergravity’s constraint algebraic structure being more complicated than GR’s. The above distinction between GR and Supergravity then has further knock-on effects as regards other aspects of Background Independence, such as 1) as regards which notions of observables remain meaningful. 2) Supergravity does not have a direct analogue of Superspace. Moreover, Supersymmetry is itself a source of constraints, and the example of Supergravity shows that capable of overriding the importance of other sources of constraints such as Temporal Relationalism and Configurational Relationalism. Due to these observations, whether it is possible for Supersymmetry to be compatible with Relationalism and with Background Independence more generally is an interesting question. Below, we settle this matter in the affirmative by constructing supersymmetric RPMs for the first time.

Absolute space is here a $\mathbb{Z}_2$-graded version of $\mathbb{R}^d$: $\{ \mathbf{a} = \mathbb{R}^{(d|2p)} \}^N$ for $N = p$ Supersymmetry, meaning with $p$ supercharges each accompanied by conjugates. The $\mathbf{a} \to x^a_0 \mathbf{a}$ construct then continues to apply: $\mathbf{q} = (d|2p)$ $\mathbb{R}^{(N|d|2p)}$. Then see Fig 15 for eleven supersymmetric $\mathfrak{g}$ which are subgroups of one or both of the superconformal and superaffine ‘superapex groups’. Each provides a corresponding notion of Relationalism and a reduced configuration space geometry. It is possible also at least in principle to consider a supersymmetric $\mathfrak{g}$ subject to a ‘merely bosonic’ $\mathfrak{g}$ (such as those tabulated in Fig 5).

Finite models including fermions attain Temporal Relationalism [10] through being homogeneous linear geometries

$$\text{d}s = \sqrt{m_{AB}dQ^A dQ^B} \sqrt{W} + \text{I}_c dQ^C. \quad (40)$$

Note that the action (40) is no longer of Jacobi type but rather of Randers type [65] (a subcase of Finsler geometry if it is additionally nondegenerate, and of Jacobi–Synge type action). Here $\mathbf{m}$ is a quadratic ‘bosonic’ contribution to the overall notion of metric involved, and $\text{I}$ is a linear ‘fermionic’ contribution. This is in the further context of the species indexed by $A, B$ on the one hand and by $C$ on the other are not to be overlapping in this setting (i.e. a partition into distinct bosonic and fermionic species respectively). (40) is also a model arena for the relational form of Einstein–Dirac Theory [66]. The above presentation is to prior to applying Best Matching with respect to the $\mathfrak{g}$ in question.

Let us next apply that specifically in the case of $N = 1$, $d = 1$ and $\mathfrak{g} = \text{super}Tr(1)$. Here the particles indexed by $I$ each have coordinates $q^I, \theta^I, \bar{\theta}^I$, due to the usual 1-d $x$ being accompanied by two Grassmann coordinates $\theta$ and $\bar{\theta}$.

Then

$$S_{\text{susy}}[q^I, \theta^I, \bar{\theta}^I, a, \alpha] = \sqrt{2} \int \{ \|d_{a,\alpha}q\| \sqrt{W} + i \sum_{I=1}^{N} (\bar{\theta}^I d_{a,\alpha} \theta^I - d_{a,\alpha} \bar{\theta}^I \theta^I) \} \quad (41)$$

14For sure, this is meant here in Wheeler’s sense, rather than in the entirely technically different sense used in the Supersymmetry literature.

15In 1-d, the version with only one Grassmann coordinate is also possible. The non-relational version of Supersymmetric Mechanics with two Grassmann coordinates was first studied by Nicolai [63] and Witten [64].
for fermionic best-matched derivatives
\[ d_{a,\alpha} \theta^I := d\theta^I - da + id\alpha , \quad d_{a,\alpha} \bar{\theta}^I := d\bar{\theta}^I - da - id\alpha \]  
(42)

and bosonic best-matched derivatives
\[ d_{a,\alpha} q^I := dq^I - da - \bar{\theta}^I d\alpha - d\alpha \bar{\theta}^I . \]  
(43)

The \(d\alpha\) and \(d\bar{\alpha}\) corrections to the fermionic species are ‘Grassmann translations’. Furthermore, upon imposing Supersymmetry these also feature as corrections to the bosonic changes in the Grassmann-linear manner indicated.

Then variation with respect to \(a\) gives a new form of 1-d zero total momentum of the universe constraint
\[ S_{\text{susy}} := \sum_{I=1}^N \{ p_I + p_\theta I - p_\bar{\theta} I \} = 0 . \]  
(44)

The new form just reflects that fermions also carry momentum. On the other hand, variation with respect to \(\alpha\) and \(\bar{\alpha}\) give the zero total supersymmetric exchange momentum of the universe constraints
\[ S := - \sum_{I=1}^N \{ p_I + i\theta^I p_I \} = 0 , \quad S^I := \sum_{I=1}^N \{ p_\bar{\theta} I + i\bar{\theta}^I p_I \} = 0 . \]  
(45)

Note that these gain one piece from the fermionic sector and one piece from the bosonic sector. These constraints are accompanied by the standard quasi-bosonic \(E\), except that now \(V\) contains fermionic species also:
\[ E := ||p||^2 / 2 + V(q^I, \theta^I, \bar{\theta}^I) = E . \]  
(46)

Taking for now the stance of not knowing the supersymmetric analogues of shape, the incipient form of \(V\) is
\[ V(q^I, \theta^I, \bar{\theta}^I) = V_B(q^K) + \sum_{I=1}^N \{ \theta^I u_I(q^K) - \bar{\theta}^I v_I(q^K) + \sum_{J=1}^N \theta^I \bar{\theta}^J w_{IJ}(q^K) \} , \]  
(47)

by virtue of the automatic truncation in Grassmann polynomials afforded by the underlying anticommutativity. Then demanding algebraic closure gives the conditions on \(V\) for \(V\) to be a function of the \(\text{superTr}(1)\) notion of shape as
\[ \sum_{I=1}^N \bar{\theta}^I \left\{ \frac{\partial V_B(q^K)}{\partial q^I} + \sum_{J=1}^N \theta^J \frac{\partial u_{IJ}(q^K)}{\partial q^I} \right\} = 0 , \quad \sum_{I=1}^N \theta^I \left\{ - \frac{\partial V_B(q^K)}{\partial q^I} + \sum_{J=1}^N \bar{\theta}^J \frac{\partial v_{IJ}(q^K)}{\partial q^I} \right\} = 0 . \]  
(48)

Frontier 8. Gain an understanding of the notion of ‘supershapes’.

Furthermore, the above implementation of best matched Configurational Relationalism readily extends to all dimensions and, concurrently, to apply to all the other supersymmetric \(\mathfrak{g}\) listed whose Supersymmetry is tied to the momentum generators \(P_I\).

Frontier 9. Work in excess to that presented here is required for those theories in which Supersymmetry is tied to special conformal transformations \(K_i\), instead, with the principal remaining question of interest being the production of an explicit superconformal RPM.

The above implementation proceeds firstly by correcting terms with spatial vector indices \(dq^I\) by subtracting products of\(^{16}\) \(\theta^{I\alpha}\) and \(d\alpha^\alpha\), and of \(\theta^{I\alpha}\) and \(d\bar{\alpha}^\alpha\), necessitating ‘interconversion arrays’ \(A^{I\alpha\bar{\alpha}}\). Then familiarity with standard spinorial formulations points to these arrays being e.g. the vector of Pauli matrices in dimension 3, and corresponding generalizations in further dimensions based on that dimension’s corresponding Dirac and Clifford mathematics. This case also requires use of the well-known distinction between \(\dagger\) and ‘. Additionally, it is clear from super-brackets relations at the level of the algebra that fermionic species are rotational spinors and also carriers of nontrivial homothetic weight. These give corresponding matching corrections to the fermionic changes. Upon variation with respect to the translational and rotational auxiliaries, the preceding two best matching terms contribute, respectively, fermionic angular momentum to the zero total angular momentum constraint and fermionic dilational momentum to the zero total dilational momentum constraint. In the \(\text{SL}(d, \mathbb{R})\) case, fermionic species are indeed \(\text{SL}(d, \mathbb{R})\) spinors, giving corresponding matching conditions, and contributing fermionic angular momentum, shear and Procrustean stretch to the corresponding zero total angular momentum, shear and Procrustean stretch constraints.

The given \(\text{superTr}(1)\) theory suffices to establish that Supersymmetry is another setting in which a simple removal of the centre of mass by passing to relative coordinates is not possible. Nor does preemptively taking out the centre of mass allow for one’s philosophical worldview and subsequent physical paradigm to avoid the possibility of Supersymmetry. This is since relative translations still exist within that setting, and these are sufficiently similar mathematically to absolute translations to enable Supersymmetry in the usual manner.

\(^{16}\)Dotted and undotted Greek indices here are a standard spinorial index notation.
Frontier 10. Understand the extent to which superRPMs can be cast in reduced form.

Frontier 11. Whether directly or by the preceding reduction, what is the topology and geometry of each \( g \)’s super-shape space?

[The theory of supershapes is not well-known enough for questions about ‘supersymmetric Shape Statistics’ to be posed for now.]

Frontier 12. The range of supersymmetric RPMs sketched out in the current paper is of anticipated future value in assessing the extent to which Relationalism and other aspects of Background Independence can be reconciled with Supersymmetry. This is in anticipation of more detailed investigation of how the former are substantially altered in passing form GR to Supergravity [22].

Finally N.B. that in the above theories

\[ \{ S, \bar{S} \} \sim \mathcal{P} \quad \text{and not} \quad \mathcal{E}, \quad (49) \]

signifying that these supersymmetric RPMs are not models of Supergravity’s principal alteration (Appendix D) as compared to GR (Appendix C) as regards aspects of Background Independence and subsequent Problem of Time facets. This is relevant to the discussion of differences between GR and Supergravity in Appendices C and D. In particular, at least the super-RPM arena in which the Supersymmetry is tied to \( P_i \) is one in which Configurational Relationalism can indeed include Supersymmetry, and do so without interfering with the ‘usual’ separate provision of Temporal Relationalism.

9 Quantum RPMs

All of this paper’s relational theories of Mechanics make for interesting quantum schemes. In each case, if a model is relational to this extent, how is the corresponding QM affected? It is not that different [10] for metric RPM with and without scale! This is due to relative angular momentum (and relative dilational momentum [70], and mixtures [71, 49]) having the same mathematics as angular momentum. Moreover, the quantum metric shape quadrilateral [50] did produce a more unusual and distinctive combination of features of the Periodic Table and of Gell-Mann’s eightfold way. Whereas the conformal case is well known for one absolutist particle and in QFT setting, it is not known as an \( N \)-body relational problem as posed here.

I make use of Dirac quantization, much as Smolin [39] did for the original RPM [3]. Within Dirac quantization, one works with the standard position coordinates or Jacobi coordinates based kinematical quantization [28] (first done by Rovelli [38] for the original RPM). [In contrast, kinematical quantization becomes a nontrivial geometrical issue in reduced quantization, though understanding the geometry and topology of the reduced configuration space in question leads to this being resolved also.] I also make use of the Laplacian operator ordering, which in the current flat redundant configuration space case is equivalent also to the conformal operator ordering and the \( \xi \) operator orderings more generally. [These differ from the Laplacian by \(-\xi R\) for \( R \) the Ricci scalar of the configuration space. A particular configuration space dimension dependent value of \( \xi \) renders the overall operator conformally invariant, hence constituting the conformal operator ordering.] These operator orderings were originally proposed by DeWitt [67] for use in what became GR Quantum Cosmology through the pioneering works of Misner [68] (see also [69] for uses of such operator orderings).

Example 1) The metric scale-and-shape RPM has

\[
\hat{\mathcal{E}} \Psi = \frac{\hbar}{i} \sum_{A=1}^{n} \mathcal{L}_{A}^{A} \times \frac{\partial \Psi}{\partial \mathcal{L}_{A}^{A}} = 0 ,
\]

meaning that \( \Psi = \Psi(-\cdot-) \). The ‘main wave equation’ is

\[
\hat{\mathcal{E}} \Psi = -\frac{\hbar^2}{2} \triangle_{\mathbb{R}^{nd}} \Psi + V(-\cdot-) \Psi = E \Psi ,
\]

(in this case, conformal operator ordering collapses to Laplacian ordering since \( \mathbb{R}^{nd} \) is flat). This example is but a slight upgrade of Smolin’s [39] (which involved particle coordinates rather than Jacobi coordinates).

Example 2) Metric shape RPM has (50) and [10]

\[
\hat{D} \Psi = \frac{\hbar}{i} \sum_{A=1}^{n} \mathcal{L}_{A}^{A} \cdot \frac{\partial \Psi}{\partial \mathcal{L}_{A}^{A}} = 0 ;
\]

together, these mean that \( \Psi = \Psi(-\cdot-/\cdot-) \). The ‘main’ wave equation can then be expressed as

\[
\hat{\mathcal{E}} \Psi = -\frac{\hbar^2}{2} \triangle_{\mathbb{R}^{nd}} \Psi + V(-\cdot-/\cdot-) \Psi = E \Psi
\]
by use of conformal flatness of the configuration space metric.

Example 3) New to the current paper, 3-d Conformal shape RPM has

$$\hat{K}_i \Psi = \frac{\hbar}{i} \sum_{I=1}^{N} (q^I \delta_i^j - 2q_i q^j) \frac{\partial \Psi}{\partial q^j} = 0 \ ,$$  

(54)

which in conjunction with

$$\hat{P}_\Psi = \frac{\hbar}{i} \sum_{I=1}^{N} \frac{\partial \Psi}{\partial q^I}$$  

(55)

and the q-versions of (50) and (52) signify that \(\Psi = \Psi(\angle)\). The main wave equation is then

$$\hat{E} \Psi = -\frac{\hbar^2}{2} I \triangle_{R^N} \Psi + V(\angle) \Psi = E \Psi \ .$$  

(56)

Example 4) Also new to the current paper, 2-d Area RPM has

$$\hat{S} \Psi = \frac{\hbar}{i} \sum_{A=1}^{n} \rho^A S \frac{\partial \Psi}{\partial q^A} = 0 \ .$$  

(57)

This means that \(\Psi = \Psi(-x-)\). Noting that this and the next two examples lie outside the levels of structure upon which Laplacians and conformal Laplacians are defined, so it is even less clear for now in these cases how to operator order, the ‘main wave equation’ for 4-particle area RPM is

$$\hat{E} \Psi = -\frac{\hbar^2}{2} \sum_{A,B=1}^{3 \text{ cycles}} (\frac{\partial}{\partial q^A} \times \frac{\partial}{\partial q^B})_3 \Psi + V(-x-) \Psi = E \Psi \ .$$  

(58)

Example 5) New again to the current paper, 2-d affine shape RPM has linear constraints (57) and (52), or, formulating them together,

$$\hat{Q} \Psi = \frac{\hbar}{i} \sum_{A=1}^{n} (\rho^A G \frac{\partial \Psi}{\partial q^A} = 0 \ .$$  

(59)

This means that \(\Psi = \Psi(-x- - x-\ldots)\). The main wave equation for 4-particle affine shape RPM is

$$\hat{E} \Psi = -\frac{\hbar^2}{2} \sum_{A,B=1}^{3 \text{ cycles}} (\rho^C \times \rho^D)_3 \sum_{A,B=1}^{3 \text{ cycles}} (\rho^C \times \rho^D)_3 \Psi + V(-x- - x-\ldots) \Psi = E \Psi \ .$$  

(60)

Example 6) As a final new Dirac quantization in this paper, in the case of superTr(1) RPM,

$$\hat{S} = -\frac{\hbar}{i} \sum_{I=1}^{N} \{ \frac{\partial}{\partial q^I} + i\theta^I \frac{\partial}{\partial \theta^I} \} \Psi = 0 \ ,$$  

(61)

$$\hat{S}^\dagger = \frac{\hbar}{i} \sum_{I=1}^{N} \{ \frac{\partial}{\partial q^I} + i\theta^I \frac{\partial}{\partial \theta^I} \} \Psi = 0 \ ,$$  

(62)

$$\hat{P}_{\text{susy}} \Psi = \frac{\hbar}{i} \sum_{I=1}^{N} \{ \frac{\partial}{\partial q^I} + \frac{\partial}{\partial \theta^I} - \frac{\partial}{\partial \theta^I} \} \Psi = 0 \ ,$$  

(63)

$$\hat{E} \Psi - \frac{\hbar^2}{2} \triangle_{R^N} \Psi + V(q^I, \theta^I, \bar{\theta}^I) \Psi = E \Psi \ .$$  

(64)

with \(V\) within the form allowed by (48).

Frontier 13. Study this paper’s new theories’ Dirac quantization schemes.

Frontier 14. Obtain the reduced quantization schemes, from first obtaining the underlying corresponding geometry and then proceeding along the lines of i) Isham’s geometrical kinematical quantization and ii) formulation and solution of the wave equations.

See Appendices C and D for this paper’s final Frontiers.

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A Flat \(\mathbb{R}^d\) geometries

A.1 Real geometries

I develop this here from a simple Kleinian position – based on invariants corresponding to transformation groups – by considering \(\mathbf{G} \leq \text{Aut}(\mathbb{R}^d, \sigma)\) for various layers of mathematical structures.\(^\text{17}\) \(\sigma\) could be · (scalar products, i.e. the

\(^\text{17}\) See [33] for an in-depth account of the foundations of geometry, albeit not based on group theory. See also [31] for comparison between four approaches to the foundations of geometry, including the group-theoretic approach. Finally, I use \(\langle \cdot , \cdot \rangle\) to demarcate a space (prior to the comma) that is equipped with further structures (after the comma).
Euclidean metric $\delta_{ij}$, but also $/ \not\subseteq$ denoting ratios, $- \not\subseteq$ denoting differences, as feature e.g. in the Euclidean notion of distance, or $\angle \not\subseteq$ denoting angles. $\sigma$ could also be $\land$: the top form wedge product supported in dimension $d$, e.g. area built out of cross products $\times$ in 2-$d$ or volumes built out of scalar triple products $[\times \cdot \cdot]$ in 3-$d$. Some geometries additionally allow for a number of combinations of these structures; see in particular column 1 of Fig 2.

To be clear about the above shorthands’ definitions, let $u, v, w, y \in \mathbb{R}^d$. Then the scalar product is a 2-slot operation $u \cdot v$. The Euclidean norm alias magnitude is then a special case of the square root of this: $||v|| := \sqrt{v \cdot v}$. Also

$$( \text{Euclidean distance between } u \text{ and } w ) := ||u - w||,$$  \tag{65}

i.e. the Euclidean norm of the difference between the two vectors $u - w$. Ratio is then a 2-slot operation acting on scalars, e.g. a ratio of two components of vectors

$$( \text{ratio of magnitudes of } u \text{ and } w ) := ||u|| / ||w||,$$  \tag{66}

$$( \text{ratio of distances } ) := ||u - v|| / ||w - y||,$$  \tag{67}

$$( \text{ratio of scalar products } ) := (u \cdot v) / (w \cdot y).$$  \tag{68}

The angle between $u$ and $w$ is then the arccos of the particular combination

$$( \text{scalar product of unit vectors } \hat{u} \text{ and } \hat{w} ) = (\hat{u} \cdot \hat{w}) = \frac{(u \cdot v)}{||u|| \cdot ||v||},$$  \tag{69}

which is a product of square roots of 2 subcases of (68). Finally, the $d$-volume top form is

$$( \text{areas of parallelograms formed by vectors } u, v ) = (u \wedge v)_3 \text{ in } 2-$d,$$  \tag{70}

and

$$( \text{volumes of parallelepipeds formed by vectors } u, v, w ) = |u \times v \cdot w| \text{ in } 3-$d.$$  \tag{71}

Then possible $g$ include the following; cases whose corresponding geometry is well-known are indicated. See Figs A and 12 for the meanings of the types of transformations involved, and columns 1 and 2 of Fig 2 for a summary. $g = id$: a trivial limiting case corresponding to no transformations being available. $g = Aut(\mathbb{R}^d, -)$: translations $x \rightarrow x + q$, which form a $d$-dimensional Abelian group $\langle \mathbb{R}^d, + \rangle$. $g = Aut(\mathbb{R}^d, /)$: dilations, $g = Aut(\mathbb{R}^d, -)$: rotations $x \rightarrow Rx$ forming the special orthogonal group $SO(d) := \{ B \in GL(d, \mathbb{R}) : B^T B = I, \det B = 1 \}$, which is of dimension $d(d - 1)/2$. $g = Aut(\mathbb{R}^d, -)$: isometries, corresponding to Euclidean geometry itself. $g = Aut(\mathbb{R}^d, /)$: rotations $x \rightarrow Rx$ forming the special linear group $SL(d, \mathbb{R})$:

$g = Aut(\mathbb{R}^d, \langle \rangle) := SL(d, \mathbb{R}) ;$ the $d^2 - 1$ dimensional special linear group, consisting of the $d(d - 1)/2$ rotations, $d(d - 1)/2$ shears and $d - 1$ ‘Procrustean stretches’. $g = Aut(\mathbb{R}^d, \langle \rangle)$: $\text{Tr}(d) \times SL(d, \mathbb{R})$; the $d(d + 1)/2$-dimensional ‘equi-top-form group’ corresponding to ‘equi-top-form geometry’ (for $d = 2, \land = \times$ and this is the quite well known equiareal geometry). $g = Aut(\mathbb{R}^d, \langle \rangle / \langle \rangle) := GL(d, \mathbb{R})$; the $d^2$-dimensional general linear group, consisting of rotations, shears and Procrustean stretches now alongside dilations. $g = Aut(\mathbb{R}^d, \langle \rangle / \langle \rangle)$: $\text{Tr}(d) \times SL(d, \mathbb{R})$; affine group of linear transformations, corresponding to affine geometry.

So far, the above transformations can all be summarized within the form of the eq at the top of Fig 11 The most general case included here is affine geometry, within which all the other $g$ above are realized as subgroups.

Reflections could also be involved in each case. These are a third elementary type of isometry about an invariant mirror hyperplane (line in 2-$d$, plane in 3-$d$). Unlike translations and rotations, they are a discrete operation. For a mirror through the origin, characterized by a normal $n$, the explicit form for the corresponding reflection is the linear transformation

$$Ref : v \rightarrow v - 2(v \cdot n)n \quad \tag{72}$$

Moreover, a further direction in $d$-dimensional geometry can be taken by introducing inversions in $\mathbb{S}^{d - 1}$

$$Inv : v \rightarrow \frac{v}{||v||^2}.$$  \tag{73}

These also preserve angles – but not other ratios of scalar products (Fig 12.b) – paving the way to the yet larger group of transformations that specifically preserves just angles.

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18The semidirect product group $g = \mathcal{N} \rtimes \mathcal{H}$ is given by $(n_1 h_1) \circ (n_2, h_2) = (n_1 \varphi_{h_1}(n_2), h_1 \circ h_2)$ for $\mathcal{N} \subseteq g$. $\mathcal{N}$ is a normal subgroup of $g$, $\mathcal{H}$ a subgroup of $g$ and $\varphi : \mathcal{N} \rightarrow Aut(\mathcal{N})$ a group homomorphism. Compare the direct product’s $(g_1, k_1) \circ (g_2, k_2) = (g_1 \circ g_2, k_1 \circ k_2)$: this has no normal group specification, and trivial automorphism.
Figure 11: Elementary transformations. 2-d illustration of translation, rotation, dilation, shear, and Procrustean stretch (i.e. \(d\)-volume top form preserving stretches, in particular area-preserving in 2-d and volume-preserving in 3-d). I also indicate the relation of the last four of these to the irreducible pieces of the general linear matrix \(G\), and which geometrically illustrious groups these transformations form part of. The \(T\) superscript denotes ‘tracefree part’. Note that Procrustean stretches do not respect ratios and shears do not respect angles.

Figure 12: 2-d renditions of a) reflection, which in this case is about a mirror line. b) Inversion in the circle. This transformation requires a grid of squares to envisage – rather than a single square – since it has a local character which differs from square to square. N.B. also that this can map between circles and lines, with the sides of the squares depicted often mapping to circular arcs.

Another perspective on geometry involves weakening the five axioms of Euclidean geometry [33, 31]. The best-known such weakening is absolute geometry, which involves dropping just Euclid’s parallel postulate. This leads firstly to hyperbolic geometry arising as an alternative to Euclid’s, and then more generally to such as Riemannian differential geometry. In contrast with this, affine geometry is that this retains Euclid’s parallel postulate, and indeed places central importance upon developing its consequences (‘parallelism’). This approach drops instead Euclid’s right-angle and circle postulates. These two initially contrasting themes continues to run strong in the eventual generalization to affine differential geometry.

Two furtherly primary types of geometry are, firstly, ordering geometry, which involves just a ‘intermediary point’ variant of Euclid’s line postulates. By involving neither the parallel postulate nor the circle and right-angle pair of postulates, this can be seen as serving as a common foundation for both absolute and affine geometry [33]. On the other hand, projective geometry involves ceasing to be able to distinguish between lines and circles in addition to angles being meaningless and no parallel postulate. From a group-theoretic perspective, this is evoking the projective linear group \(PGL(d, \mathbb{R}) = GL(d, \mathbb{R})/Z(GL(d, \mathbb{R}))\).\(^{19}\)

Probably the best-known example of projective group is the Möbius group \(PGL(2, \mathbb{C})\) acting upon \(\mathbb{C} \cup \infty\) as the fractional linear transformations \(z \mapsto \frac{az + b}{cz + d}\) for \(a, b, c, d \in \mathbb{C}\) such that \(ad - bc \neq 0\). This is 6-d, because there is one complex

\(^{19}\)For a matrix group \(\mathcal{G} = GL(\mathcal{V})\) (for \(\mathcal{V}\) a vector space) or subgroups thereof, the centre \(Z(\mathcal{G})\) of \(\mathcal{G}\) consists of whichever \(k I\) are allowed by the definition of that subgroup and the field \(\mathcal{F}\) that \(\mathcal{V}\) is based upon.
For practical use within Euclidean theories of space, note in particular that 'spatial' measurements in our experience lie within the forms (67) and (69), i.e. measuring tangible objects against a ruler and measuring angles between tangible entities. On the other hand, more advanced, if indirect, physical applications make use of (extensions of) the other notions of geometry above.

B Lie groups and Lie algebras

A Lie group [75] is simultaneously a group and a differentiable manifold; its composition and inverse operations are differentiable. Working with the corresponding infinitesimal ('tangent space') around \(g\)'s identity element – the Lie algebra \(g\) – is more straightforward due to vector spaces' tractability, whilst very little information is lost in doing so. [E.g. the representations of \(g\) determine those of \(\mathfrak{g}\).] More formally, a Lie algebra is a vector space equipped with a product (bilinear map) \([\cdot,\cdot]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\) that is antisymmetric and obeys the Leibniz (product) rule and the Jacobi identity

\[
[g_1,[g_2,g_3]] + \text{cycles} = 0
\]

(75)

\(\forall g_1, g_2, g_3 \in \mathfrak{g}\). This is an example of algebraic structure: equipping a set with one or more product operations. In the present case, Lie brackets are exemplified by Poisson brackets and commutators. Particular subcases of Lie brackets then include the familiar Poisson brackets and quantum commutators.

Moreover, a Lie algebra’s generators (Lie group generating infinitesimal elements) \(\tau_p\) obey

\[
[[\tau_p,\tau_q]] = C^r_{pq}\tau_r ,
\]

(76)

where \(C^r_{pq}\) are the structure constants of that Lie algebra. It readily follows that the structure constants with all indices lowered are totally antisymmetric, and also obey

\[
C^o_{[pq]C^r_{si]} = 0 .
\]

(77)

Next suppose that it is hypothesized that some subset of the generators, \(K_k\), is significant. Denote the rest of the generators by \(H_h\). On now needs to check to what extent the algebraic structure in question actually complies with this assignation of significance. Such checks place limitations on how generalizable some intuitions and concepts which hold for simple examples of algebraic structures are. In general, the split algebraic structure is of the form

\[
[[K_k,K'_k]] = C^o_{kk'}K_kK'_k + C^e_{kk'}H_h ,
\]

(78)

\[
[[K_k,H_h]] = C^o_{kh}K_kK'_h + C^e_{kh}H'_h ,
\]

(79)

\[
[[H_h,H'_h]] = C^o_{hh'}K_k + C^e_{hh'}H_hH'_h .
\]

(80)

Denote the second to fifth right hand side terms by 1) to 4). 1) and 4) being zero are clearly subgroup closure conditions. 2) and 3) are ‘interactions between’ \(\mathfrak{h}\) and \(\mathfrak{g}\). The following cases of this are then realized in this paper.

I) Direct product. If 1) to 4) are zero, then \(g = \mathfrak{g} \times \mathfrak{h}\).

II) Semi-direct product. If 2) alone is nonzero, then \(g = \mathfrak{g} \rtimes \mathfrak{h}\).

III) Thomas integrability. If 1) is nonzero, then \(\mathfrak{g}\) is not a subalgebra: attempting to close it leads to some \(K_k\) are discovered to be integrabilities. I denote this by \(s\subset \mathfrak{h}\); the arrow points to the part of the split which arises as an integrability of the other part. A simple example of this occurs in splitting the Lorentz group’s generators up into rotations and boosts; this is indeed the group-theoretic underpinning [76] of Thomas precession (see Appendix B.1).

IV) Two-way integrability If 1) and 4) are nonzero, neither \(\mathfrak{g}\) nor \(\mathfrak{h}\) are subalgebras, due to their imposing integrabilities on each other. I denote this by \(s\Theta\mathfrak{h}\), with the double arrow indicating that the two parts of the split are integrabilities of each other. In this case, any wishes for \(\mathfrak{h}\) to play a significant role by itself are almost certainly dashed by the mathematical reality of the algebraic structure in question.

Note that this classification is important as regards understanding how GR’s constraints are more subtle than Gauge Theory’s, and Supergravity’s than GR’s; this is further developed in Appendices C and D.

B.1 Examples of Lie groups and Lie algebras

For Abelian Lie groups, the structure constants are all zero. Examples of this include \(Tr(d)\) and \(Dil(d)\) – which are both noncompact – and the compact \(Rot(2) = SO(2)\). The corresponding Lie algebras’ generators are \(P_i = -\partial x^i\), \(D = -x^i \partial x^i\) and \(L = y \partial y - x \partial x\).

On the other hand, for \(d > 2\) the \(Rot(d) = SO(d)\) are non-Abelian Lie groups. Compare these with the \(O(d)\) Lie groups: \(SO(d)\) is a Lie subgroup of \(O(d)\). The former has 2 connected components related by a discrete reflection. The
corresponding Lie algebra sees only the connected component that contains the identity, so is the same for each of $O(d)$ and $SO(d)$. $GL(\mathbf{U})$ and $SL(\mathbf{U})$ are also Lie groups [non-Abelian for $\dim(\mathbf{U}) > 1$].

Some Lie algebras used in this paper are as follows. The general linear algebras $gl(\mathbf{U})$ of $d \times d$ matrices over $\mathbb{F}$, the real cases of which have dimension $d^2$. The special linear algebras $sl(\mathbf{U})$ are the zero-trace such, the real cases of which have dimension $d^2 - 1$. The generators for $gl(d, \mathbb{R})$ are, very straightforwardly, $G_{ij} = x^i \frac{\partial}{\partial x^j}$. The special orthogonal algebras $so(d) := \{ A \in gl(d, \mathbb{R}) \mid A + A^T = 0 \}$ of dimension $d(d - 1)/2$. These are generated by $M_{ij} = x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}$ subject to the schematic noncommutation relation
\[ [[M, M]] \sim M . \tag{81} \]

Among these, $so(2)$ is the above-mentioned Abelian algebra and $so(3)$ has the alternating symbol $\epsilon^i_{jk}$ for its structure constants. The $3$-d case also simplifies by the duality between $M_{ij}$ and $L_i$. $SO(4)$ [and, more famously, $SO(3,1)$: the Lorentz group] satisfy accidental relations linking them to a direct product of two copies of $SO(3)$. Moreover, in the $SO(3,1)$ case – whereas linear combinations can be taken so as to obtain this split, the original presentation’s generators differ in physical significance (3 rotations and 3 boosts) – which physical meanings are not preserved by taking said linear combinations. Adhering then to the physically meaningful split into $J_i$ and $K_i$ is the setting of the Thomas precession mentioned in the previous sub-appendix. Schematically, this decomposes (81) into
\[ [[J, J]] \sim J , \quad [[J, K]] \sim K , \quad [[K, K]] \sim K + J , \tag{82} \]

the key bracket being the last one by which the boosts are not a subalgebra, the precession in question referring to the rotation arising thus from a combination of boosts.

Some composite Lie groups of particular relevance to this paper are then $Eucl(d)$ and $Sim(d)$. Here e.g. $Eucl(d)$’s semidirect product structure is due to the bracket $[[J, P]] \sim P$, signifying that $P$ is a $Rot(d)$-vector. Also, $Tr(d) \rtimes Dil$’s semidirect product structure rests upon $[[P, D]] \sim P$. On the other hand, $Rot(d)$–$Dil$ independence as found e.g. within the family of subgroups of $Aff(d)$ [and thus in particular $Eucl(d)$ and $Sim(d)$] is based upon rotation and dilation generators commuting: a direct product split
\[ [[M, D]] = 0 \quad ( [[L, D]] = 0 \text{ in } 3-d \text{ and } [[L, D]] = 0 \text{ in } 2-d ) . \tag{83} \]

### B.2 Killing vectors and isometries: plain, homothetic and conformal

![Figure 13: Decomposition of special conformal transformation into an inversion, translation and another inversion.](image)

One way of getting at $Eucl(d)$ which usefully extends to further groups is by solving Killing’s equation in flat space. In this case, given Killing’s Lemma [77], the form
\[ \xi^i = a^i + B_{j}^i x^j \tag{84} \]
for the Killing vectors readily follows. Repeating for the homothetic Killing equation in flat space,
\[ \xi^i = a^i + B_{j}^i x^j + cx^j ^{\prime} \tag{85} \]
enjours. Finally, repeating for the conformal Killing equation in flat space
\[ \xi^i = a^i + B_{j}^i x^j + cx^j + \{ 2k^j x^i - k^i x^j \} \delta_{jk} x^k . \tag{86} \]
enjours for $d \geq 3$ [75]. The $k^i$ correspond to special conformal transformations
\[ x^i \rightarrow \frac{x^i - k^i x^j}{1 - 2k \cdot x + k^2 x^2} \tag{87} \]
formed from an inversion, a translation and then a second inversion. The infinitesimal generator is $K_i := x^2 \frac{\partial}{\partial x^i} - 2x_i x^j \frac{\partial}{\partial x^j}$. Thus conformal group $Conf(d)$ of dimension $d(d + 2)/2$ (in particular 10 for $d = 3$) arises.
For $d = 2$, the conformal Killing equation famously collapses to the Cauchy–Riemann equations, causing an infinity of solutions: any holomorphic function $f(z)$ will do. In 1-$d$, the conformal Killing equation collapses to $d v / d x = \phi(x)$, amounting to reparametrization by a 1-$d$ coordinate transformation $v = \Phi(x) + a$ for $\Phi := \int \phi(x) d x$. This case is not subsumed within (86): $Conf(1)$ is also infinite-dimensional, albeit rather less interesting than the 2-$d$ version. (1-$d$ has no angles to preserve, though conformal factors can be defined for it none the less; note also that the metric drops out of the 1-$d$ conformal Killing equation.)

Due to an integrability of the schematic form

$$[[K, P]] \sim M + D ,$$

the conformal algebra is $(P, K)\ominus(M, D)$ Thomas. I.e. a translation and an inverted translation compose to give both a rotation (‘conformal precession’) and an overall expansion. Elsewise $K_i$ behaves much like $P_i$ does.

### B.3 Some further subgroups acting upon $\mathbb{R}^d$

The above three Sections can be viewed as introducing $P_i, M_{ij}, D$ and $K_i$ generators.

One can furthermore consider e.g. shears and $d$-dimensional volume preserving stretches ($G_{(ij)}^T$ generators); each of these are only nontrivial for $d \geq 2$. Alongside the rotations, these form $SL(d, \mathbb{R})$; then the equi-top-form group $E(d) := Tr(d) \times SL(d, \mathbb{R})$, corresponding to the eponymous geometry (equiareal in 2-$d$ [33]). Also, $GL(d, \mathbb{R}) = Dil \times SL(d, \mathbb{R})$; then the affine group $E(d) := Tr(d) \times GL(d, \mathbb{R})$, corresponding to affine geometry [33]. $dim(E(d)) = d\{d + 1\} - 1$ and $dim(A(d)) = d\{d + 1\}$. The unsplit nonzero affine brackets are, schematically

$$[G, G] \sim G , \quad [G, P] \sim P ,$$

signifying closure of the $GL(d, \mathbb{R})$ subgroup and that $P_i$ is a $GL(d, \mathbb{R})$ vector. As regards general $sl(d, \mathbb{R})$ generators, perform the antisymmetric–symmetric and trace–tracefree splits on $gl(d, \mathbb{R})$’s generators (Fig A). Then the antisymmetric part is just the rotations, the tracefree symmetric part is $E_{ij} = x^i \frac{\partial}{\partial x^j} + x^j \frac{\partial}{\partial x^i} - \frac{2}{n} \delta_{ij} x^k \frac{\partial}{\partial x^k}$ and the trace part (the usual dilation) is discarded. E.g. Corresponding infinitesimal matrices for $sl(2, \mathbb{R})$ these are $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which form the triple (23) with the infinitesimal rotation matrix. The corresponding generators are $x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$ for Procrustean stretches and $x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ for shears. It is also important to note that

$$[\text{Shear}, \text{Shear}'] \sim \text{Rotation}$$

by which the non-rotational parts of $SL(d, \mathbb{R})$ cannot be included in the absence of the rotations.

Figure 14: Summary sketch, of groups including further groups acting upon $\mathbb{R}^d$. These are arrived at by adding generators as per the labelled arrows. Moreover, the group relations involved do not permit all combinations of generators to be included. In particular, absences marked $X$ are due to integrability (88). Absences marked * are due to integrability (90). Finally, absences marked $\dagger$ are due to obstruction (91). Figures 4 and 5 then use a matching layout.

Moreover, the $K_i$ and $G_{(ij)}^T$ generators are not compatible with each other, as is clear from

$$\text{conformal transformations only preserving angles whereas shears do not preserve angles} .$$

(91)
Thus there are two distinct ‘apex groups’: \(Conf(d)\) from including \(K_i\) and \(Aff(d)\) from including \(S_{ij}\). ‘Apex’ is used here in the sense that the other possibilities are contained within as Lie subgroups. These include a number of subgroups not yet considered (Fig 13).

Finally, some of the above groups also support nontrivially distinct projective versions, obtained by quotienting out by the centre of the group in question. E.g. if this is performed upon the affine group, projective geometry [33, 31] ensues.

### B.4 Superconformal and superaffine algebras and some of their subalgebras

Some algebraic structures involve anticommutators \([A, B]_+ := AB + BA\). These enter models of fermionic species. See e.g. [78, 79] for an extensive ‘mathematical methods for physicists’ treatments of these and of the ensuing notion of spinors.

As compared to the conformal group \(Conf(d)\), the superconformal group \(superConf(d)\) [80] has additional fermionic generators \(S\) and \(Q\) (and conjugates), in which sense it is doubly supersymmetric (denoted \(N = 2\)). Here,

\[
[[S, \bar{S}]]_+ \sim P
\]

and

\[
[[Q, Q]]_+ \sim K ,
\]

so both the momentum and the special conformal transformation arise as integrabilities.

On the other hand, the superaffine algebra \(superAff(d)\) has just the one additional fermionic generator \(S\) (and conjugate). Then (92) holds, by which \(superAff(d)\) is \(S\Theta(P, G)\) Thomas (composition of supersymmetries results in a translation). Examining the table of subgroups in Fig 15, the exhibited supersymmetric subgroups of \(superAff(d)\) share this property. In contrast, the superconformal group has a more complicated sequence of integrabilities, with \((S, Q)\Theta(P, K)\Theta(M, D)\); this is a tripartition to (78-80)’s bipartition. Finally note that all the groups in the table bar \(superConf(d)\) have just the one supersymmetric generator (and conjugate): \(N = 1\). 8 are supersymmetric subgroups of \(superConf(d)\) and 5 of \(superAff(d)\) [4 of which are also among the previous 8].

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**Figure 15:** Some subgroups, and overruled combinations, of supersymmetric groups, within the superconformal and superaffine apex groups. These are again laid out according to 5, except now with at least one supersymmetric generator as well. The cases marked with an F fail to support Supersymmetry due to Supersymmetry requiring at least one of \(P_i\) and \(K_i\) as an integrability: (92-93).

### C Flat to differential geometry modelling of space, and GR-level counterpart of this paper

A) Firstly, metric geometry is one of the outcomes of weakening the axioms of Euclidean geometry. In particular, a first route to such is through assuming only absolute geometry and then finding a large multiplicity of such to exist. Moreover, some aspects of ‘metric geometry’ remain upon ceasing to assume a metric. For instance, some such aspects form topological manifold geometry and differentiable manifold geometry. Furthermore, many of the outcomes
of stripping down the structure of Euclidean geometry have close counterparts in differential geometry. E.g., similarity, conformal, affine and projective differential geometries exist [81, 41]. Additionally, it is possible to reconcile e.g. affine and metric (or conformal metric) notions in differential geometry. E.g. the standard modelling assumptions in GR include that the metric connection serve as a (and the only) affine connection.

In this way, stripping away the layers of structure assumed in GR (whether to model spacetime, or to model space within a geometrodynamical perspective) leads to closely analogous first ports of call to those in stripping away the layers of structure assumed in RPM’s based upon $\mathbb{R}^d$. In the specific case of GR, these first ports of call [82] are conformal geometry and affine geometry. Upon Supersymmetry’s extra structure, Supergravity is another first port of call. In the case of studying GR configurations, physical irrelevance of $\text{Diff}(\Sigma)$ is also under consideration; the most common corresponding configuration spaces are depicted in Fig 16.b). Here $\Sigma$ some spatial topology taken in the current paper to have some fixed compact without boundary form. The Figure juxtaposes these with analogous RPM configuration spaces as laid out in the main part of the current paper. One can also consider an analogous diamond of affine spaces, with variants on GR allowing for the additional possibility of both retaining a metric and introducing an affine connection other than the metric connection. Indeed, one means of such an alternative theory having torsion is through the difference of two distinct affine connections constituting a torsion tensor. Superconformal Supergravity is an example of a further combination of the elements of these first ports of call. All of these are open to relational analyses and yet wider consideration of Background Independence leading to whether they exhibit significant differences from GR as regards the Problem of Time’s many facets.

A second port of call is considering Topological Relationalism (whether in the context of variants on GR allowing for topology change, or in the context of Topological Field Theories being a further generalization of Conformal Field Theories). See [73] for discussion of the second and subsequent ports of call from a relational perspective. What about modelling this second port of call with RPMs? In fact, this is relatively straightforward. E.g. in one sense change in particle number in RPMs is analogous to topology change in GR. (I.e. theories which retain the upper layers of structure whilst letting subsequent layers of structure also be dynamical rather than absolute backgrounds). In another sense (stripping away the upper layers), topological RPMs involve distinguishing only those configurations which are topologically distinct. Then e.g. scaled triangleland collapses to a small finite number of points: the total collision, the double collision (or three such if labelled) and the general configuration that is not any of the previous. This is rather simpler than the space of all topological manifolds in some given dimension! Thus it is very plausible for RPMs to be able to model multiple layers of structure (at the cost of resembling GR rather less at the lower levels).

B) Let us next return to the upper layers of structure, so as to justify some of the many GR–RPM analogies that occur there. GR can be cast in Temporal and Configurational Relationalism form [20], the latter being tied to the group of spatial diffeomorphisms $\text{Diff}(\Sigma)$. The underlying configuration space is $\text{Riem}(\Sigma)$, i.e. the space of spatial (positive-definite) 3-metrics on $\Sigma$. Here the constraint provided by Temporal Relationalism [20] gives – via Dirac’s argument that reparametrization invariance necessarily implies a primary constraint [59] – a relational recovery of the quadratic GR Hamiltonian constraint $\mathcal{H}$ [12]. On the other hand, Configurational Relationalism [20, 10] provides the linear GR momentum constraint $\mathcal{M}_i$ [12]. In this sense, Temporal and Configurational Relationalism remain distinct Background Independence aspects in GR, much as they also are in RPMs. Each provides a constraint of its own.

These constraints then close in accord with the Dirac algebroid of GR, which is schematically of the form

\[ \{ \mathcal{M}, \mathcal{M} \} \sim \mathcal{M} \quad , \quad \{ \mathcal{M}, \mathcal{H} \} \sim \mathcal{H} \quad , \quad \{ \mathcal{H}, \mathcal{H} \} \sim \mathcal{M} \ . \]

These are the classical theory’s Poisson brackets; this closure is a classical realization of the further Constraint Closure aspect of Background Independence.

The first bracket means that the $\mathcal{M}_i$ – which correspond to the spatial diffeomorphisms $\text{Diff}(\Sigma)$ – themselves close to form a true infinite-$d$ Lie algebra. The second bracket signifies that $\mathcal{H}$ is a $\text{Diff}(\Sigma)$ scalar density. The third
bracket is not only the one expressing an integrability [83] but also the one containing both a structure function and the derivative of its RHS constraint. Its integrability means that in GR, Temporal Relationalism obliges the existence of nontrivial Configurational Relationalism. This feature does not occur in RPMs, in which Temporal and Configurational Relationalism can each be modelled in the absense of the other. The third bracket can furthermore be viewed as an algebroid counterpart [89] of Thomas precession: $H \{ M_i \}$. In more detail, compare how composing two boosts results in a rotation: Thomas precession, whereas composing two time evolutions – or pure hypersurface deformations in [84]'s interpretation of $H$ – results in a spatial diffeomorphism: Moncrief–Teitelboim on-slice Lie dragging. [Lie dragging is the motion corresponding to $Diff(\Sigma)$ in the same manner as precession is a name for a motion corresponding to $Rot(d)$.] This analogy is (as far as the Author is aware) new to this paper. As well as rendering the Dirac Algebroid more pedagogically accessible to students, this analogy is useful in the below revelations about Supergravity being substantially different from GR as regards the form taken by its Relationalism and Background Independence more generally.

Further consequences of the above algebraic form as regards background independent features are as follows.

1) that Wheeler’s Superspace ($\Sigma$) := $\text{Riem}(\Sigma)/Diff(\Sigma)$ – central to geometrodynamics – is a meaningful intermediary configuration space. This follows from $M_i$ forming a subalgebra of the Dirac algebroid by the first bracket, so it is meaningful to reduce out $M_i$ by itself.  

2) Expression in terms of Beables or Observables is a fourth aspect of Background Independence; the general failure of which in classical and quantum gravitational theories then constitutes the well-known ‘Problem of Observables’ facet of the Problem of Time. Observables or beables are objects which commute with constraints. In the case of commuting with all constraints – $H$ and $M_i$ in GR – one is dealing with Dirac observables. In the case of commuting with linear constraints only – $M_i$ in GR – one is dealing with Kuchař observables [72]. Moreover, concepts of observables or beables as objects which commute with subsets of constraints only make sense if the subset in question algebraically closes [22]. Thus $M_i$ forming a subalgebra of the Dirac algebroid by the first bracket guarantees the meaningfulness of Kuchař observables in the case of GR. See [87] for the form taken by the Kuchař observables for the current paper’s new non-supersymmetric theories.

Further conformal Configurational Relationalism has also been considered in the case of GR and of alternative theories of conformogeometrodynamics [44, 17, 86]. The original conception of this used metric shape RPM as a model arena; however, the current paper’s conformal shape RPM is surely a comparable or enhanced source of insights.

Frontier 15. Finally, returning to topic A), perform the differentiable manifold level counterpart of the current paper’s comparative relational study of the diverse levels of structure to be found in flat geometries.

The current paper and Frontier 15 can furthermore be seen as part of a wider program in which an increasing number of levels of mathematical structure assumed in Physics are taken to be dynamical rather than fixed background structures. This program was initiated by Isham [74], who considered replacing geometrical quantization based upon the usual configuration space with that based upon generalized configuration spaces. I then provided a classical precursor for this program in [73], in particular sketching out an even wider range of classical Shape Statistics theories than the current paper’s (at the levels of topological manifolds and of topological spaces). That work also posed Shape Statistics on GR’s Superspace ($\Sigma$) and conformal superspace, $CS(\Sigma)$. Those remaining a distant dream in the general case, I also posed the more tractable analogues in the settings of anisotropic minisuperspace and of inhomogeneous perturbations about minisuperspace [11, 25].

D Contrast with canonical formulation of Supergravity

Supersymmetry and split space-time GR can each separately be envisaged as Thomas-type effects at the level of algebraic structures. Furthermore, upon considering both at once, the integrabilities involved go in opposite directions, forming the more complicated ‘two-way’ integrability case). [Contrast also with the superconformal group’s composition of integrabilities being tripartite with two aligned one-way integrabilities instead.] The schematic form of the key new relation for Supergravity is (see e.g. [88])

$$\{ S, S \}_C \sim H + M_i \ .$$  \hspace{1cm} (97)

This forms the second integrability of the ‘two-way’ pair, to the first integrability being of form (96).\footnote{An algebroid allows for ‘structure functions’ – including derivative operators in the present case – of constraints to appear on the right-hand side.}

The implications of the two-way integrability case include that the linear constraints do not close by themselves; thus

1) they cannot be quotiented out as a unit (in this sense no Supergravity counterpart of Wheeler’s Superspace).

\footnote{The $C$ subscript stands for Casalbuoni Poisson bracket [89]. The time component $R_0$ arising within (92) in the indefinite case’s super-Poincaré subgroup can be seen as a precursor of (97).}
2) Temporal and Configurational Relationalism become a fused notion as opposed to separate notions.

Moreover, two more possibilities for splits manifest themselves. The main idea here is that the Temporal to Configurational Relationalism split, the quadratic and linear constraints split and the notion of Kuchař observables correspond to treating the linear constraints \( \mathcal{LIN}' \) differently in GR. However, in a wider range of theories including Supergravity, this consideration is to be supplanted more generally by splits which respect the subalgebraic structures contained within the constraint algebraic structure.

Then as a second possibility for a split [82], one can also consider the \( \mathcal{S}, \mathcal{H} \) to \( \mathcal{LIN}' \) split (for \( \mathcal{LIN}' \) the linear non-supersymmetric constraints); this is Thomas with a \( \mathcal{LIN}' \) subalgebra.

As a third possibility, the \( \mathcal{LIN}' \) to \( \mathcal{S} \) split is Thomas, exhibiting a ‘non-supersymmetric’ subalgebraic structure. In particular, these other splits permit a meaningful notion of quotienting out \( \mathcal{LIN}' \), giving a well-defined quotient space and a well defined notion of observables in this modified sense. I.e.

1.A) extend \( \text{Riem}(\Sigma) \) to include the space of gravitino fields, and then quotient out solely by the non-supersymmetric linear constraints.

2.A) A notion of observables or beables can be defined as commuting with solely the non-supersymmetric linear constraints.

The yet further versions 1.B) and 2.B) removing the word ‘linear’ from the preceding definitions are likely to be harder to handle. I term 2) and 2.A) non-supersymmetric Kuchař and Dirac observables or beables respectively. I term 1) and 1.A) non-supersymmetric Superspace and True-space respectively; True-space is a formal reference to the space of true dynamical degrees of freedom itself. Finally, whether any of the entities termed ‘non-supersymmetric’ are in violation of the spirit of Supersymmetry may be a matter further relevance from viewpoints which take Supersymmetry sufficiently seriously.

Furthermore, due to \( \mathcal{H} \)'s ties to Temporal Relationalism in GR, and of (some) linear constraints’ ties to Configurational Relationalism, Supergravity’s change of status as regards which of these constraints can be entertained independently of which others also concerns how Relationalism is to be viewed. Additionally, Temporal Relationalism, Configurational Relationalism, Constraint Closure and Expression in terms of Beables are four of the aspects of Background Independence [24] underlying four of the Problem of Time facets [18, 37, 85]. Thus Supergravity exhibits a very different realization of these from GR [22], which could herald one or both of foundational problems for Supersymmetry or a hitherto insufficiently general conceptualization of Background Independence and the Problem of Time. One possibility here is that Supersymmetry breaks down the divide between Temporal and Configurational Relationalism as separate providers of constraints. Another possibility is that Supersymmetry renders constraint provision tripartite, by itself constituting a third provider of constraints. The current paper then shows that none of the above happen in superRPMs whose supersymmetry is tied to the translations. In this case, Supersymmetry is compatible with Relationalism and is implemented as a subcase of Configurational Relationalism. Of course, the current paper also points out that superRPMs lack an analogue of (96). In this manner they realize the opposite single plank to that realized by non-supersymmetric GR. It is then rather interesting that the existing conception of Background Independence can be combined with Supersymmetry without requiring modification of either, at least in these nontrivial and complementary model arenas.

Frontier 16. Resolve the above matter in full Supergravity. Or at least do so in some model arena exhibiting specifically the ‘two-way’ pair of intergabilities, by which \( \mathcal{S} \) implies \( \mathcal{H} \), and \( \mathcal{H} \) imply \( \mathcal{LIN}' \), with the Temporal to Configurational Relationalism divide hitherto having been between \( \mathcal{H} \) and the full set of linear constraints. Does this have any further consequences for the ‘space and configuration primality’ versus ‘spacetime primality’ debate? [13, 20, 24].

References

[1] See The Leibnitz–Clark Correspondence, ed. H.G. Alexander (Manchester 1956).

[2] E. Mach, Die Mechanik in ihrer Entwicklung, Historisch-kritisch dargestellt (J.A. Barth, Leipzig 1883). An English translation is The Science of Mechanics: A Critical and Historical Account of its Development Open Court, La Salle, Ill. 1960).

[3] J.B. Barbour and B. Bertotti, Proc. Roy. Soc. Lond. A382 295 (1982).

[4] J.B. Barbour, Class. Quant. Grav. 20 1543 (2003), gr-qc/0211021.

[5] E. Anderson, arXiv:1001.1112.

[6] D.G. Kendall, Bull. Lond. Math. Soc. 16 81 (1984).

[7] D.G. Kendall, D. Barden, T.K. Carne and H. Le, Shape and Shape Theory (Wiley, Chichester 1999).
[8] C.G.S. Small, *The Statistical Theory of Shape* (Springer, New York, 1996).

[9] E. Anderson, Class. Quant. Grav. **25** 025003 (2008), arXiv:0706.3934.

[10] E. Anderson, arXiv:1111.1472.

[11] E. Anderson, arXiv:1503.01507.

[12] R. Arnowitt, S. Deser and C. Misner, in *Gravitation: An Introduction to Current Research* ed. L. Witten (Wiley, New York 1962), arXiv:gr-qc/0405109.

[13] J.A. Wheeler, in *Battelle Rencontres: 1967 Lectures in Mathematics and Physics* ed. C. DeWitt and J.A. Wheeler (Benjamin, New York 1968).

[14] R.F. Baierlein, D.H. Sharp and J.A. Wheeler, Phys. Rev. **126** 1864 (1962); J.A. Wheeler, in *Groups, Relativity and Topology* ed. B.S. DeWitt and C.M. DeWitt (Gordon and Breach, New York 1964).

[15] B.S. DeWitt, Phys. Rev. **160** 1113 (1967).

[16] A.E. Fischer, in *Relativity* (Proceedings of the Relativity Conference in the Midwest, held at Cincinnati, Ohio June 2-6, 1969), ed. M. Carmeli, S.I. Fickler and L. Witten (Plenum, New York 1970); A.E. Fischer, J. Math. Phys **27** 718 (1986); D. Giulini, Gen. Rel. Grav. **41** 785 (2009) 785, arXiv:0902.3923.

[17] J.W. York Jr., Ann. Inst. Henri Poincaré **21** 319 (1974); A.E. Fischer and V. Moncrief, Gen. Rel. Grav. **28**, 207 (1996).

[18] J.B. Barbour, *The End of Time* (Oxford University Press, Oxford 1999).

[19] J.B. Barbour, B.Z. Foster and N. Ó Murchadha, Class. Quant. Grav. **19** 3217 (2002), gr-qc/0012089; E. Anderson and F. Mercati, arXiv:1311.6541.

[20] C. Kiefer, *Quantum Gravity* (Clarendon, Oxford 2004).

[21] E. Anderson, SIGMA **10** 092 (2014), arXiv:1312.6073.

[22] E. Anderson, arXiv:1310.1524.

[23] E. Anderson, arXiv:1409.4117.

[24] E. Anderson, *The Problem of Time between General Relativity and Quantum Mechanics*, forthcoming book.

[25] E. Anderson, Stud. Hist. Phil. Mod. Phys. **51** 1 (2015), arXiv:1307.1923.

[26] D.N. Page and W.K. Wootters, Phys. Rev. **D27**, 2885 (1983); M. Gell-Mann and J.B. Hartle, Phys. Rev. **D47** 3345 (1993); J.J. Halliwell, Phys. Rev. **D60** 105031 (1999), quant-ph/9902008; in *The Future of Theoretical Physics and Cosmology* (Stephen Hawking 60th Birthday Festschrift volume) ed. G. Kunstatter, D. Vincent and J. Williams (World Scientific, Singapore, 1992), reprinted as Int. J. Mod. Phys. Proc. Suppl. **D20** 3 (2011).

[27] A.W. Fitzgibbon and A. Zisserman, in *Proceedings of Computer Vision and Pattern Recognition, 2003*.

[28] C.J. Isham, in *Relativity, Groups and Topology II* ed. B. DeWitt and R. Stora (North-Holland, Amsterdam 1984).

[29] C.J. Isham and A. Kakas, Class. Quant. Grav. **1** 621 (1984); Class. Quant. Grav. **1** 633 (1984).

[30] J.R. Klauder, Int. J. Geom. Meth. Mod. Phys. **3** 81 (2006), gr-qc/0507113.

[31] J. Stillwell, *The Four Pillars of Geometry* (Springer, New York 2005).

[32] H.S.M. Coxeter, *Introduction to Geometry* (Wiley, New York 1989).

[33] A. Abrams and R. Ghrist, arXiv:math/0009118.

[34] A.W. Fitzgibbon and A. Zisserman, in *Proceedings of Computer Vision and Pattern Recognition, 2003*.

[35] D.W. Thompson, *On Growth and Form* (Cambridge University Press, New York 1984).

[36] C.J. Isham, in *Integrable Systems, Quantum Groups and Quantum Field Theories* ed. L.A. Ibort and M.A. Rodriguez (Kluwer, Dordrecht 1993), gr-qc/9210011.

[37] C. Rovelli, p. 292 in *Conceptual Problems of Quantum Gravity* ed. A. Ashtekar and J. Stachel (Birkhäuser, Boston 1991).

[38] L. Smolin, in *Conceptual Problems of Quantum Gravity* ed. A. Ashtekar and J. Stachel (Birkhäuser, Boston 1991).

[39] C. Lanczos, *The Variational Principles of Mechanics* (University of Toronto Press, Toronto 1949).

[40] S. Kobayashi, *Transformation Groups in Differential Geometry* (Springer–Verlag, Berlin 1972).

[41] E. Anderson, arXiv:1205.1256.

[42] E. Anderson, Class. Quant. Grav. **23** (2006) 2469, gr-qc/0511068.

[43] E. Anderson, J.B. Barbour, B.Z. Foster and N. Ó Murchadha, Class. Quant. Grav. **20** 157 (2003), gr-qc/0211022.
[45] E. Anderson, Class. Quant. Grav. 31 (2014) 025006, arXiv:1305.4685.
[46] E. Anderson, arXiv:1505.02448.
[47] D.G. Kendall, Statistical Science 4 87 (1989).
[48] J.B. Barbour and B. Bertotti, Nuovo Cim. B38 1 (1977); For an account of prior discoveries of this theory, see Mach’s principle: From Newton’s Bucket to Quantum Gravity ed. J.B. Barbour and H. Pfister (Birkhäuser, Boston 1995).
[49] E. Anderson, Int. J. Mod. Phys. D23 1450001 (2014), arXiv:1202.4186.
[50] E. Anderson and S.A.R. Kneller, Int. J. Mod. Phys. D23 1450052 (2014), arXiv:1303.5645.
[51] See e.g. C. Marchal, Celestial Mechanics (Elsevier, Tokyo 1990).
[52] R.G. Littlejohn and M. Reinsch, Rev. Mod. Phys. 69 213 (1997).
[53] J.B. Hartle, The Physics of ’Now’, Am. J. Phys. 73 101 (2005), gr-qc/0403001.
[54] E. Anderson, Int. J. Mod. Phys. D23 1450014 (2014), arXiv:1202.4186.
[55] E. Anderson and S.A.R. Kneller, Int. J. Mod. Phys. D23 1450052 (2014), arXiv:1303.5645.
[56] See e.g. C. Marchal, Celestial Mechanics (Elsevier, Tokyo 1990).
[57] R.G. Littlejohn and M. Reinsch, Rev. Mod. Phys. 69 213 (1997).
[58] J.B. Hartle, The Physics of ’Now’, Am. J. Phys. 73 101 (2005), gr-qc/0403001.
[59] E. Anderson, arXiv:1505.02448.
[60] J.B. Barbour and B. Bertotti, Nuovo Cim. B38 1 (1977); For an account of prior discoveries of this theory, see Mach’s principle: From Newton’s Bucket to Quantum Gravity ed. J.B. Barbour and H. Pfister (Birkhäuser, Boston 1995).
[61] M. Lachi`eze-Rey and J.P. Luminet, Phys. Rept. 254 (1995) 135, arXiv:gr-qc/9605010.
[62] E. Anderson, arXiv:1505.02448.
[63] P.A.M. Dirac, Lectures on Quantum Mechanics (Yeshiva University, New York 1964).
[64] R. Bartnik and G. Fodor, Phys. Rev. D48 3596 (1993).
[65] S.A. Roach, The Theory of Random Clumping (Methuen, London 1968).
[66] S. Broadbent, J.R. Statist. Soc. A 143 109 (1980).
[67] H. Nicolai, J. Phys. A. Math. Gen. 9 1497 (1976).
[68] E. Witten, Nucl. Phys. B 188 513 (1981); B202 253 (1982).
[69] B.S. DeWitt, Rev. Mod. Phys. 29 377 (1957).
[70] C.W. Misner, in Magic Without Magic: John Archibald Wheeler ed. J. Klauder (Freeman, San Francisco 1972);
[71] S.W. Hawking and D.N. Page, Nucl. Phys. B264 185 (1986); J.J. Halliwell, Phys. Rev. D38 2468 (1988); D.L. Wiltshire, in Cosmology: the Physics of the Universe ed. B. Robson, N. Visvanathan and W.S. Woolcock (World Scientific, Singapore 1996), gr-qc/0101003. E. Anderson, Class. Quant. Grav. 27 045002 (2010), arXiv:0905.3357.
[72] E. Anderson, Gen. Rel. Grav. 43 1529 (2011), arXiv:0909.2439.
[73] K.V. Kuchař, in General Relativity and Gravitation 1992, ed. R.J. Gleiser, C.N. Kozamah and O.M. Moreschi M (Institute of Physics Publishing, Bristol 1993), gr-qc/9304012.
[74] E. Anderson, arXiv:1412.0239.
[75] See e.g. Y. Choquet-Bruhat, C. DeWitt-Morette and M. Dillard-Bleick, Analysis, Manifolds and Physics Vol. 1 (Elsevier, Amsterdam 1982).
[76] R. Gilmore, Lie Groups, Lie Algebras, and Some of Their Applications (Dover, New York 2006).
[77] See e.g. R.M. Wald General Relativity (University of Chicago Press, Chicago 1984).
[78] M.E. Peskin and D.V. Schroeder, An Introduction to Quantum Field Theory (Perseus Books, Reading, Massachusetts 1995).
[79] T. Frenkel, The Geometry of Physics: An Introduction (Cambridge University Press, Cambridge 2011).
[80] S. Weinberg, The Quantum Theory of Fields. Vol III. Supersymmetry. (Cambridge University Press, Cambridge 2000).
[81] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry Volumes 1 and 2 (Wiley, New York 1963).
[82] E. Anderson, arXiv:1411.4316.
[83] V. Moncrief and C. Teitelboim, Phys. Rev. D6 966 (1972).

[84] S.A. Hojman, K.V. Kuchař and C. Teitelboim, Ann. Phys. N.Y. 96 88 (1976).

[85] E. Anderson, in Classical and Quantum Gravity: Theory, Analysis and Applications ed. V.R. Frignanni (Nova, New York 2012), arXiv:1009.2157; Annalen der Physik, 524 757 (2012), arXiv:1206.2403.

[86] J.W. York Jr., Phys. Rev. Lett. 28 1082 (1972). Phys. Rev. Lett. 26 1656 (1971); J. Math. Phys. 14 456 (1973). E. Anderson, J.B. Barbour, B.Z. Foster, B. Kelleher and N. ó Murchadha, Class. Quant. Grav 22 1795 (2005), gr-qc/0407104. J.B. Barbour, for proceedings of the conference Quantum Field Theory and Gravity (Regensburg, 2010) arXiv:1105.0183. F. Mercati, arXiv:1409.0105.

[87] E. Anderson, arXiv:1505.03551.

[88] See e.g. P. Vargas Moniz, Quantum Cosmology - The Supersymmetric Perspective – Vols. 1 and 2 (Springer, Berlin 2010).

[89] R. Casalbuoni, Nuovo Cim. 33A 115 (1976).