Dynamics of nonlocal diffusion problems with a free boundary and a fixed boundary\textsuperscript{1}

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Abstract

In this paper, we first consider two scalar nonlocal diffusion problems with a free boundary and a fixed boundary. We obtain the global existence, uniqueness and longtime behaviour of solution of these two problems. The spreading-vanishing dichotomy and sharp criteria for spreading and vanishing are established. We also prove that accelerated spreading could happen if and only if a threshold condition is violated by kernel function. Then we discuss a classical Lotka-Volterra predator-prey model with nonlocal diffusions and a free boundary which can be seen as nonlocal diffusion counterpart of the model in the work of Wang (2014 J. Differential Equations 256, 3365-3394).

Keywords: Nonlocal diffusion; Free boundary; Spreading-vanishing; Spreading speed; Accelerated spreading.

AMS Subject Classification (2000): 35K57, 35R09, 35R20, 35R35, 92D25

1 Introduction

Recently the authors of [1] proposed the following nonlocal diffusion model with free boundaries to study the population dispersal

\[
\begin{aligned}
    u_t &= d \int_{g(t)}^{h(t)} J(x-y)u(t,y)dy - du + f(t, x, u), \quad t > 0, \quad g(t) < x < h(t), \\
    u(t, x) &= 0, \quad t > 0, \quad x \notin (g(t), h(t)), \\
    h'(t) &= \mu \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x-y)u(t, x)dydx, \quad t > 0, \\
    g'(t) &= -\mu \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(x-y)u(t, x)dydx, \quad t > 0, \\
    h(0) &= -g(0) = h_0 > 0, \quad u(0, x) = u_0(x), \quad |x| \leq h_0, \\
\end{aligned}
\]

where kernel function $J$ satisfies

(J) $J \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}), \ J \geq 0, \ J(0) > 0, \ \int_{\mathbb{R}} J(x)dx = 1, \ \sup_{x \in \mathbb{R}} J(x) < \infty,$

and the growth term $f$ satisfies

(F1) (i) $f \in C(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+), \ f(t, x, 0) \equiv 0, \ \text{and} \ f(t, x, u) \ \text{is locally Lipschitz continuous in} \ u \in \mathbb{R}^+, \ \text{i.e., for any given} \ M, T > 0, \ \text{there exists a constant} \ L(M, T) > 0 \ \text{such that}

\[
|f(t, x, u_1) - f(t, x, u_2)| \leq L(M, T)|u_1 - u_2|
\]

for all $u_1, u_2 \in [0, M], \ t \in [0, T] \ \text{and all} \ x \in \mathbb{R};$

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(ii) there exists $K > 0$ such that for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, $f(t, x, u) < 0$ when $u > K$.

This model is mainly derived from two aspects: dispersal term and free boundary condition. As in [2, 3], we assume that $u(t, x)$ is the population density at point $x$ and time $t$, and $J(x - y)$ is the probability distribution function of jumping from location $y$ to $x$. Thus $J(x - y)u(t, y)$ is the rate at which individuals reach $x$ from position $y$, and by integrating over the entire survival area of the species we see that $\int_{y(t)}^{h(t)} J(x - y)u(t, y)dy$ is the rate at which individuals arrive at $x$ from all other locations. Similarly, $J(x - y)u(t, x)$ is the rate at which individuals leave location $x$ to jump to site $y$, and $u(t, x) = \int_{-\infty}^{\infty} J(x - y)u(t, x)dy$ is the rate at which the individuals depart from location $x$ to go to all other positions. Taking $d$ as dispersal coefficient, we get the dispersal term

$$d \left( \int_{y(t)}^{h(t)} J(x - y)u(t, x)dy - u(t, x) \right).$$

For the meaning of free boundary condition, similar to the corresponding local diffusion model [4, 5], we assume that the expanding rate of free boundary is proportional to the outward flux at the boundary. Please refer to [4] for more related details.

Based on the above analysis, let us now introduce our first model. Suppose that the initial habitat of the species is the spatial domain $[0, h_0]$. It is well-known to us that the species will instinctively expand their own habitat for the sake of their survival. We assume that the species can move to the area $(-\infty, 0)$, but once they enter this area, they will die immediately, which means that this region is highly lethal to the species. Thus the species can only enlarge their habitat through the right boundary, and our model can be formulated by the following problem

$$\begin{cases}
u_t = d \int_0^{h(t)} J(x - y)u(t, y)dy - d u + f(t, x, u), & t > 0, \ 0 \leq x < h(t), \\
u(t, h(t)) = 0, & t > 0, \\
h'(t) = \mu \int_0^{h(t)} \int_{h(t)}^{\infty} J(x - y)u(t, x)dydx, & t > 0, \\
h(0) = h_0, & x \in [0, h_0],
\end{cases} \tag{1.2}$$

where $d, \mu$ and $h_0$ are positive constants, kernel function $J$ satisfies (J), and $u_0(x)$ satisfies

$$(\text{H}) \quad u_0 \in C([0, h_0]), \quad u_0 > 0 \text{ in } [0, h_0], \quad u_0(h_0) = 0.$$

Unlike problem [1.2], in our second model we assume that the species know that the area $(-\infty, 0)$ is seriously fatal to them, so they will not jump to this area. This indicates that the condition imposed at $x = 0$ is analogous to the usual homogeneous Neumann boundary condition. Hence we can derive the following model

$$\begin{cases}
u_t = d \int_0^{h(t)} J(x - y)u(t, y)dy - d \left( \int_0^{\infty} J(x - y)dy \right) u + f(t, x, u), & t > 0, \ 0 \leq x < h(t), \\
u(t, h(t)) = 0, & t > 0, \\
h'(t) = \mu \int_0^{h(t)} \int_{h(t)}^{\infty} J(x - y)u(t, x)dydx, & t > 0, \\
h(0) = h_0, & x \in [0, h_0],
\end{cases} \tag{1.3}$$
where $J$ and $u_0$ satisfy (J) and (H), respectively.

At last, we take the classical Lotka-Volterra predator-prey model as an example to study the interaction between two species, namely, we consider the following problem

$$
\begin{align*}
    u_{1t} &= d_1 \int_0^{h(t)} J_1(x-y) u_1(t, y) dy - d_1 u_1 + u_1 (a_1 - b_1 u_1 - c_1 u_2), & t > 0, \ 0 \leq x < h(t), \\
    u_{2t} &= d_2 \int_0^{h(t)} J_2(x-y) u_2(t, y) dy - d_2 u_2 + u_2 (a_2 - b_2 u_2 + c_2 u_1), & t > 0, \ 0 \leq x < h(t), \\
    u_i(t, h(t)) &= 0, & t > 0, \\
    h'(t) &= \sum_{i=1}^{2} \mu_i \int_0^{h(t)} \int_{h(t)}^{\infty} J_i(x-y) u_i(t, x) dy dx, & t > 0, \\
    h(0) &= h_0, \ u_i(0, x) = u_{i0}(x), & 0 \leq x \leq h_0, \ i = 1, 2,
\end{align*}
$$

where all parameters are positive, $J_i$ satisfy the condition (J), and $u_{i0}$ meet the condition (H).

For these nonlocal diffusion problems, there are two main differences from the corresponding local diffusion free boundary problems [4, 6]. Firstly, there is usually no regularity effect. The lack of regularity makes it difficult to derive some important uniform estimates which are crucial to study the dynamics for these models. Secondly, it is easily seen from the above equations that the change rate of species density at $(t, x)$ is affected not only by the density of species at $(t, x)$, but also by the value near site $x$, which leads to some difficulties and differences from local diffusion model when considering the longtime behaviour of solution. It is worthy mentioning that to overcome difficulties caused by the above differences, some new techniques are recently introduced in [7].

Before starting our research, we now give a brief introduction for the recent works on nonlocal diffusion free boundary problems. By considering a semi-wave problem, Du et.al [8] proved that when spreading happens, problem (1.1) has a finite spreading speed if and only if $J$ satisfies

$$(J1) \ \int_0^{\infty} x J(x) dx < \infty.$$  

Then Du and Ni [9] gave comprehensive and delicate results on spreading speed by discussing the conditions satisfied by $J$. In [10], the authors showed that the local diffusion free boundary problem in one dimension space can be approximated by a slightly modified version of nonlocal diffusion free boundary problem in [1]. In addition, there are many other related works for nonlocal diffusion free boundary problems, such as [7, 11] for two species models, [12] for high dimension and radial symmetry version, [13, 14, 15] for systems with nonlocal and local diffusions, [16] for two species models with different free boundaries, [17] for logistic model in time periodic environment and [18, 19] for the epidemic model with partial degenerate diffusion.

This paper is arranged as follows. In Section 2, some preparatory results are given, such as eigenvalue problem, steady state problems on half space and a technical lemma; Section 3 is devoted to dynamics of the model (1.2), and we prove the well-posedness, spreading-vanishing dichotomy and spreading or vanishing criteria, which also are obtained for the model (1.3) in Section 4. Moreover, the spreading speed for model (1.2) is discussed in Section 3. More precisely, for the model (1.2), we prove that its spreading speed has lower and upper bounds if (J1) holds, and accelerated spreading happens if (J1) is violated. As for the model (1.3), we show that it has a finite spreading speed if and only if (J1) holds in Section 4; Section 5 is concerned with two species.
model (1.4). The well-posedness, longtime behavior and the estimates of spreading speed are given. At last, a brief discussion is stated.

In this paper, we assume that the function $J$ satisfies the condition (J), and $0 < T < \infty$.

2 Some preparatory results

In order to save space, we let $\mathbb{R}^+ = (0, \infty)$ and $\mathbb{R}^+ = [0, \infty)$, and define

$$\mathbb{H}^T_{h_0} = \{ h \in C([0, T]) : h(0) = h_0, \inf_{0 \leq t_1 < t_2 \leq T} \frac{h(t_2) - h(t_1)}{t_2 - t_1} > 0 \},$$

and for $h \in \mathbb{H}^T_{h_0}$, we define

$$D^T_h = \{(t, x) : 0 < t \leq T, 0 \leq x < h(t)\}, \quad D^\infty_h = \{(t, x) : 0 \leq t < \infty, 0 \leq x \leq h(t)\}$$

and $\overline{D}^T_h$ is the closure of $D^T_h$. In this section we first study the maximum principle and comparison principle, and then investigate the corresponding eigenvalue problem to (1.3). Finally, we discuss the steady state problems corresponding to problems (1.2), (1.3) and (1.4) respectively.

2.1 The maximum principle and comparison principle

Lemma 2.1 (Maximum principle). Let $h \in \mathbb{H}^T_{h_0}$, $c \in L^\infty(D^T_h)$ and $\psi$, $\psi_t \in C(\overline{D}^T_h)$ satisfy

$$\left\{ \begin{array}{l}
\psi_t \geq \int_0^{h(t)} J(x-y) \psi(t, y)dy + cv, \quad 0 < t \leq T, 0 \leq x < h(t), \\
\psi(t, h(t)) \geq 0, \quad 0 < t \leq T; \quad \psi(0, x) \geq 0, \quad 0 \leq x \leq h_0.
\end{array} \right.$$

Then $\psi \geq 0$ in $\overline{D}^T_h$. Moreover, if $\psi(0, x) \neq 0$ in $[0, h_0]$, then $\psi > 0$ in $D^T_h$.

Proof. Let $\Psi = \psi e^{-kt}$ with constant $k > \|c(t, x)\|_{L^\infty(D^T_h)} + d$. Assume on the contrary that there exists $(t^*, x^*) \in (0, T] \times [0, h(t))$ such that $\Psi(t^*, x^*) = \inf_{D^T_h} \Psi < 0$. Then $\Psi_t(t^*, x^*) \leq 0$, and

$$d \int_0^{h(t^*)} J(x^*-y) \Psi(t^*, y)dy - d \int_0^{h(t^*)} J(x^*-y)dy \Psi(t^*, x^*) \geq 0.$$

Therefore,

$$0 \geq \Psi_t(t^*, x^*) - d \int_0^{h(t^*)} J(x^*-y) \Psi(t^*, y)dy + d \int_0^{h(t^*)} J(x^*-y)dy \Psi(t^*, x^*)$$

$$\geq d \int_0^{h(t^*)} J(x^*-y)dy \Psi(t^*, x^*) + c(t^*, x^*) \Psi(t^*, x^*) - k \Psi(t^*, x^*) > 0.$$

This is a contradiction, and so $\psi \geq 0$ in $\overline{D}^T_h$. If $\psi(0, x) \neq 0$ in $[0, h_0]$, we can prove that $\psi > 0$ in $D^T_h$ by similar arguments in the proof of Lemma 2.2. The details are omitted here. \qed

Lemma 2.2 (Maximum principle). Let $v, c \in L^\infty([0, T] \times \mathbb{R}^+)$, $v, v_t \in C([0, T] \times \overline{\mathbb{R}}^+)$ and satisfy

$$\left\{ \begin{array}{l}
v_t \geq \int_0^\infty J(x-y) v(t, y)dy + cv, \quad 0 < t \leq T, x \geq 0, \\
v(0, x) \geq 0,
\end{array} \right.$$

Then $v \geq 0$ in $[0, T] \times \mathbb{R}^+$. Moreover, if $v(0, x) \neq 0$ in $\mathbb{R}^+$, then $v > 0$ in $(0, T] \times \overline{\mathbb{R}}^+$. 

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Define \( T_0 = \sup \{ s \in (0, T] : w > 0 \text{ in } [0, s] \times \mathbb{R}^+ \} \). Since we have, for \((t, x) \in (0, T] \times \mathbb{R}^+\),

\[
w_t - d \int_0^\infty J(x - y)w(t, y)dy - cw \geq (A - c - d) \int_0^\infty J(x - y)dy \varepsilon e^{At} > \varepsilon e^{At},
\]

\( w_t \) has a finite lower bound independent of \( x \in \mathbb{R}^+ \). This together with the fact that \( w(0, x) \geq \varepsilon \) for \( x \in \mathbb{R}^+ \) indicates that \( T_0 > 0 \) is well defined. If \( T_0 = T \), then (2.1) holds. Otherwise we have \( T_0 < T \) and \( \inf_{x \in \mathbb{R}^+} w(T_0, x) = 0 \) since if \( \inf_{x \in \mathbb{R}^+} w(T_0, x) > 0 \), because \( w_t(T_0, x) \) has a finite lower bound independent of \( x \in \mathbb{R}^+ \), we can find some \( 0 < \delta \ll 1 \) such that \( w(t, x) > 0 \) for \((t, x) \in [T_0, T_0 + \delta] \times \mathbb{R}^+ \) which is a contradiction to the definition of \( T_0 \). Thus for any \( 0 < \varepsilon_n \ll 1 \) with \( \varepsilon_n \to 0 \) as \( n \to \infty \), there exist \( x_n \in \mathbb{R}^+ \) such that \( w(T_0, x_n) < \varepsilon_n \).

Since \( w \) is bounded below and continuous in \([0, T_0] \times \mathbb{R}^+\), by Ekeland’s variational principle, for these \( \varepsilon_n \), \((T_0, x_n)\) and \( \sigma = \min\{T_0/2, 1\} \), there exist \((t_n, y_n) \in [0, T_0] \times \mathbb{R}^+ \) such that

\[
\begin{cases}
  w(t_n, y_n) \leq w(T_0, x_n) < \varepsilon_n, \\
  |T_0 - t_n| + |x_n - y_n| < \sigma,
\end{cases}
\]

and

\[
w(t_n, y_n) - w(t, y_n) \leq \frac{|t_n - t| \varepsilon_n}{\sigma} \text{ for } t \in [0, T_0].
\]

Due to the choice of \( \sigma \), we have \( t_n > 0 \), and thus

\[
w(t_n, y_n) \leq \varepsilon_n/\sigma \to 0 \text{ as } n \to \infty.
\]

Moreover, by \( w \geq 0 \) in \([0, T_0] \times \mathbb{R}^+\), we have

\[
w(t_n, y_n) \geq c(t_n, y_n)w(t_n, y_n) + \varepsilon e^{At_n} \geq -\|c\|_{\infty} \varepsilon_n + \varepsilon e^{At_n} \geq \frac{1}{2} \varepsilon \text{ for all large } n,
\]

which contradicts to (2.2). So (2.1) holds for any \( 0 < \varepsilon \ll 1 \). By letting \( \varepsilon \to 0 \) we derive \( v \geq 0 \) in \([0, T] \times \mathbb{R}^+\). If \( v(0, x) \neq 0 \), we can easily get the desired result by [11 Lemma 3.3].

By the maximum principles obtained above we can get the standard comparison principles.

### 2.2 Eigenvalue problem corresponding to (1.3)

In the following, we discuss the principal eigenvalue of the following eigenvalue problem

\[
(\mathcal{L}_{(0, l)}^N + a_0)\phi := d \int_0^l J(x - y)\phi(y)dy - dj(x)\phi + a_0 \phi = \lambda \phi \text{ in } [0, l],
\]

where \( j(x) = \int_0^\infty J(x - y)dy \) and \( a_0, l \) are positive constants. It follows from (J) that \( j(x) \) is nondecreasing and continuously differentiable in \( x \in \mathbb{R}^+ \), \( j(0) = 1/2 \) and \( j(x) > 0 \) for \( x > 0 \). As

\[
\lim_{x \to 0^+} \frac{x}{j(x) - 1/2} = \frac{1}{J(0)} > 0,
\]

it yields that

\[
\int_0^l \frac{1}{j(x) - 1/2}dx = \infty \text{ for all } l > 0.
\]

In view of [20] Theorem 2.1, problem (2.3) has a principal eigenvalue \( \lambda_p(\mathcal{L}_{(0, l)}^N + a_0) \) defined by

\[
\lambda_p(\mathcal{L}_{(0, l)}^N + a_0) := \inf \{ \lambda \in \mathbb{R} : (\mathcal{L}_{(0, l)}^N + a_0)\phi \leq \lambda \phi \text{ in } [0, l] \text{ for some } 0 < \phi \in C([0, l]) \}.
\]
Lemma 2.3. Assume that \( l > 0 \). Then the followings hold true:

1. \( \lambda_p(\mathcal{L}^N_{(0, t)} + a_0) \) is strictly increasing and continuous in \( l \);
2. \( \lim_{l \to \infty} \lambda_p(\mathcal{L}^N_{(0, t)} + a_0) = a_0 \);
3. \( \lim_{l \to 0} \lambda_p(\mathcal{L}^N_{(0, t)} + a_0) = a_0 - d/2. \)

Proof. (1) Since the continuity of \( \lambda_p(\mathcal{L}^N_{(0, t)} + a_0) \) in \( l \) can be proved by a similar method in [1, Proposition 3.4], we only prove the monotonicity.

Let \( l_2 > l_1 > 0 \), and \( \phi_1 \) be the corresponding positive eigenfunction to \( \lambda_p(\mathcal{L}^N_{(0, t)} + a_0) \). Then

\[
\left[ dj(x) - a_0 + \lambda_p(\mathcal{L}^N_{(0, t_1)} + a_0) \right] \phi_1(x) = d \int_0^{l_1} J(x - y) \phi_1(y) dy > 0 \text{ in } [0, l_1],
\]

which implies \( dj(x) - a_0 + \lambda_p(\mathcal{L}^N_{(0, t_1)} + a_0) > 0 \) in \([0, l_1]\). Together with the monotonicity of \( j(x) \), we can define the following nonnegative continuous function

\[
\tilde{\phi}(x) = \begin{cases} 
\phi_1(x), & x \in [0, l_1], \\
\frac{d \int_0^{l_1} J(x - y) \phi_1(y) dy}{dj(x) - a_0 + \lambda_p(\mathcal{L}^N_{(0, t_1)} + a_0)}, & x \in [l_1, l_2].
\end{cases}
\]

It then follows that

\[
d \int_0^{l_2} J(x - y) \tilde{\phi}(y) dy - dj(x) \tilde{\phi} + a_0 \tilde{\phi} \geq \lambda_p(\mathcal{L}^N_{(0, t_1)} + a_0) \tilde{\phi} \text{ in } [0, l_2],
\]

and this inequality holds strictly in \( x = l_1 \). Combing with \( \tilde{\phi}(l_1) > 0 \), by the variational characterization of \( \lambda_p(\mathcal{L}^N_{(0, t)} + a_0) \), we have

\[
\lambda_p(\mathcal{L}^N_{(0, t_2)} + a_0) = \sup_{0 \neq \psi \in L^2([0, t_2])} \frac{d \int_0^{l_2} \int_0^{l_2} J(x - y) \psi(y) \psi(x) dy dx - d \int_0^{l_2} j(x) \psi^2(x) dx}{\int_0^{l_2} \psi^2(x) dx} + a_0
\]

\[
\geq \frac{d \int_0^{l_2} \int_0^{l_2} J(x - y) \tilde{\phi}(y) \tilde{\phi}(x) dy dx - d \int_0^{l_2} j(x) \tilde{\phi}^2(x) dx}{\int_0^{l_2} \tilde{\phi}^2(x) dx} + a_0
\]

\[
> \lambda_p(\mathcal{L}^N_{(0, t_1)} + a_0).
\]

(2) Since \( (\mathcal{L}^N_{(0, t)} + a_0)1 \leq a_0 \), it follows that \( \lambda_p(\mathcal{L}^N_{(0, t)} + a_0) \leq a_0 \) for all \( l > 0 \) by taking \( \lambda = a_0 \) and the test function \( \phi \equiv 1 \) in [2.4]. On the other hand, due to the condition (\text{J}), for any given \( 0 < \varepsilon \ll 1 \), there exists \( L > 0 \) such that \( \int_{-L}^{L} J(x) dx > 1 - \varepsilon \). For any \( l > L \), taking \( \phi \equiv 1 \) as the test function in the variational characterization of \( \lambda_p(\mathcal{L}^N_{(0, t)} + a_0) \) we have

\[
\lambda_p(\mathcal{L}^N_{(0, t)} + a_0) \geq \frac{d \int_0^{l} \int_0^{l} J(x - y) dy dx - d \int_0^{l} j(x) dx}{l} + a_0
\]

\[
> \frac{d \int_{-L}^{l} \int_0^{l} J(x - y) dy dx - d \int_{-L}^{l} j(x) dx}{l} + a_0
\]

\[
= \lambda_p(\mathcal{L}^N_{(0, t)} + a_0) + a_0.
\]
\[
\geq \frac{d(l - 2L)(1 - \varepsilon)}{l} - \frac{d}{l} \int_0^l j(x)dx + a_0 \rightarrow -\varepsilon d + a_0 \text{ as } l \rightarrow \infty.
\]

By the arbitrariness of \(\varepsilon\), \(\lim_{l \to \infty} \lambda_p(\mathcal{L}^N_{(0, l)} + a_0) \geq a_0\). The conclusion (2) holds.

3. Let \(\phi_1\) be the corresponding positive eigenfunction to \(\lambda_p(\mathcal{L}^N_{(0, l)} + a_0)\). Then
\[
|\lambda_p(\mathcal{L}^N_{(0, l)} + a_0) - a_0 + \frac{d}{2} - \frac{d}{2} \int_0^l \phi_1^2(x)dx + \frac{d}{2} \int_0^l \phi_1^2(x)dx \geq \frac{d}{2} \int_0^l \phi_1^2(x)dx.
\]

The proof is complete. \(\square\)

2.3 The steady state problems

In the sequel, we further assume that nonlinear term \(f\) satisfies

(F2) \(f \in C^1\) is independent of \((t, x)\), and \(f(u)/u\) is strictly decreasing for \(u > 0\);

(F3) \(f'(0) > 0\).

From conditions (F1)-(F3), we know that \(f(u) = 0\) has a unique positive root \(u^*\). To study the longtime behavior of solution of problem \((1.2)\), we next give two lemmas about a steady state problem and a fixed boundary problem on half space respectively.

Lemma 2.4. Suppose that \(f\) satisfies conditions (F1)-(F3). Then the problem
\[
d \int_0^\infty J(x - y)U(y)dy - dU + f(U) = 0 \quad \text{in } \mathbb{R}^+    \tag{2.5}
\]

has a unique bounded positive solution \(U \in C(\mathbb{R}^+)\), and \(\lim_{x \to \infty} U(x) = u^*\). Moreover, \(0 < U < u^*\) and \(U\) is non-decreasing in \(\mathbb{R}^+\).

Proof. Step 1: Existence of bounded positive solution. It is well known (e.g. see \cite{1} Proposition 3.5) that for large \(l > 0\), the following problem
\[
d \int_0^l J(x - y)u(y)dy - du + f(u) = 0 \quad \text{in } [0, l]    \tag{2.6}
\]

has a unique positive solution \(u_l \in C([0, l])\). If \(u_l(x_0) = \max_{x \in [0, l]} u_l(x) > u^*\) for \(x_0 \in [0, l]\), then
\[
du_l(x_0) < du_l(x_0) - f(u_l(x_0)) = d \int_0^l J(x_0 - y)u_l(y)dy \leq du_l(x_0),
\]

which implies \(0 < u_l \leq u^*\) for large \(l\).
We claim that \( u_l \) is non-decreasing in \( l \) for large \( l \). Since \( u_l \) is positive and continuous in \([0, l]\), for any \( l_2 > l_1 > 0 \), we can define

\[
\rho_\ast = \inf\{\rho > 1 : \rho u_{l_2} \geq u_{l_1} \text{ in } [0, l_1]\}.
\]

If \( \rho_\ast > 1 \), it follows from the definition of \( \rho_\ast \) that there exists \( \bar{x} \in [0, l_1] \) such that \( \rho_\ast u_{l_2}(\bar{x}) = u_{l_1}(\bar{x}) \).

Thanks to \( \rho_\ast u_{l_2}(x) \geq u_{l_1}(x) \) in \([0, l_1]\), we have

\[
d \int_0^{l_1} J(\bar{x} - y) \rho_\ast u_{l_2}(y) dy - d\rho_\ast u_{l_2}(\bar{x}) + f(\rho_\ast u_{l_2}(\bar{x})) \geq 0.
\]

Meanwhile, we easily see that

\[
d \int_0^{l_1} J(\bar{x} - y) \rho_\ast u_{l_2}(y) dy - d\rho_\ast u_{l_2}(\bar{x}) + \rho_\ast f(u_{l_2}(\bar{x})) \leq 0.
\]

Hence \( f(\rho_\ast u_{l_2}(\bar{x})) \geq \rho_\ast f(u_{l_2}(\bar{x})) \). Since \( \rho_\ast > 1 \) and \( u_{l_2}(\bar{x}) > 0 \), one can derive a contradiction from the condition \((F2)\). So our claim is true. Hence we can define \( U(x) = \lim_{n \to \infty} u_l(x) \) for \( x \in \mathbb{R}^+ \).

Then \( 0 < U \leq u^* \). By the dominated convergence theorem, \( U \) satisfies \( (2.5) \).

**Step 2:** \( \lim_{x \to \infty} U(x) = u^* \). Arguing indirectly, there exist \( \varepsilon_1 > 0 \) and \( x_n \not
\to \infty \) such that \( U(x_n) \leq u^* - \varepsilon_1 \). Taking \( w_n(x) = u_{2x_n}(x + x_n) \), then we have

\[
d \int_{-x_n}^{x_n} J(x - y)w_n(y) dy - dw_n(x) + f(w_n(x)) = 0 \quad \text{for } x \in [-x_n, x_n].
\]

By [1] Proposition 3.6, we have \( \lim_{n \to \infty} w_n(x) \to u^* \) locally uniformly in \( \mathbb{R} \). Thus there exists \( N > 0 \) such that \( u_{2x_n}(x_n) \geq u^* - \varepsilon_1/2 \) for any \( n \geq N \), and we further have that \( u^* - \varepsilon_1 \geq U(x_N) \geq u_{2x_N}(x_N) \geq u^* - \varepsilon_1/2 \). This is a contradiction, and so \( \lim_{x \to \infty} U(x) = u^* \).

**Step 3:** The continuity of \( U \). Firstly, it is deduced from \( J \in L^1(\mathbb{R}) \) and \( U \in L^\infty(\mathbb{R}^+) \) that \( \int_0^\infty J(x - y)U(y)dy \) is continuous in \( \mathbb{R}^+ \). By virtue of \( u_l \not
\to U \) and \( \lim_{x \to \infty} U(x) = u^* \), we see that \( U \) has a finite positive lower bound. Hence there exists a positive constant \( C \) such that

\[
d - \frac{f(U(x))}{U(x)} \geq C > 0, \quad \forall x \in \mathbb{R}^+.
\]

(2.7)

We shall show that for any \( x_n \to x \in \mathbb{R}^+ \), there holds \( U(x_n) \to U(x) \). Without loss of generality, we assume that \( U(x_n) \geq U(x) \) for all \( n \). Then, by (2.7), we obtain

\[
d \int_0^\infty [J(x_n - y) - J(x - y)]U(y)dy = [d - c(x)][U(x_n) - U(x)] \geq C[U(x_n) - U(x)] \geq 0,
\]

(2.8)

where

\[
c(x) = \begin{cases} 
\frac{f(U(x_n)) - f(U(x))}{U(x_n) - U(x)}, & U(x_n) \neq U(x), \\
0, & U(x_n) = U(x),
\end{cases}
\]

and we have used the fact that \( U(x_n) \geq U(x) \) implies \( \frac{f(U(x_n)) - f(U(x))}{U(x_n) - U(x)} \leq \frac{f(U(x)) - f(U(x))}{U(x) - U(x)} \). Since the left side of (2.8) goes to 0 as \( n \to \infty \), we have \( U(x_n) \to U(x) \), and so \( U \in C(\mathbb{R}^+) \).
Step 4: Uniqueness. Suppose that \( V \) is another bounded positive solution of (2.5). We claim \( V \leq u^* \). If \( \sup_{x \in \mathbb{R}^+} V(x) := \bar{V} > u^* \), then there exist \( z_n \in \mathbb{R}^+ \) such that \( V(z_n) \to \bar{V} \). Moreover,

\[
d\bar{V} < d\bar{V} - f(\bar{V}) = \lim_{n \to \infty} d \int_{0}^{\infty} J(z_n - y)V(y)dy \leq d\bar{V}.
\]

Hence \( V \leq u^* \). We now prove that for any \( l > 0 \), \( V \) has a positive lower bound in \([0, l]\). If there exist \( l_0 > 0 \) and \( \bar{x}_n \to \bar{x}_0 \in [0, l_0] \) such that \( V(\bar{x}_n) \to 0 \). Since \( V \) satisfies (2.5), we have

\[
d \int_{0}^{l_0} J(\bar{x}_n - y)V(y)dy \leq dV(\bar{x}_n) + f(V(\bar{x}_n)).
\]

Letting \( n \to \infty \), we derive that

\[
\int_{0}^{l_0} J(\bar{x}_0 - y)V(y)dy \leq 0,
\]

which implies

\[
\int_{0}^{l_0} J(\bar{x}_0 - y)V(y)dy = 0.
\]

Hence \( J(\bar{x}_0 - x)V(x) = 0 \) almost everywhere on \([0, l_0]\). By (J) and positivity of \( V \), we immediately obtain a contradiction. Thus \( \inf_{x \in [0, l]} V(x) > 0 \) for all \( l > 0 \). Then for large \( l > 0 \), we can define

\[
\eta_* = \inf \{ \eta > 1 : \eta V(x) \geq u_l(x) \text{ in } [0, l] \}.
\]

We claim \( \eta_* = 1 \). Once our claim is true, we can show \( u_l \leq V \) for large \( l > 0 \), and thus \( U \leq V \). Assume that \( \eta_* \geq 1 \). Then \( \eta_* V(x) > u_l(x) \) in \([0, l] \). If there exists \( \bar{x}_0 \in [0, l] \) such that \( \eta_* V(\bar{x}_0) = u_l(\bar{x}_0) \), then by arguing as in Step 1 we can get a contradiction. So \( \eta_* V(x) > u_l(x) \) in \([0, l] \). If \( \inf_{x \in [0, l]} \eta_* V(x) - u_l(x) = 0 \), then there exist \( \bar{x}_n \) and \( \bar{x}_0 \in [0, l] \) such that \( \bar{x}_n \to \bar{x}_0 \) and \( \eta_* V(\bar{x}_n) - u_l(\bar{x}_n) \to 0 \) as \( n \to \infty \). Clearly, we have

\[
d \int_{0}^{l} J(\bar{x}_n - y)\eta_* V(y)dy - d\eta_* V(\bar{x}_n) + f(\eta_* V(\bar{x}_n)) < 0.
\]

Therefore, by (2.6), we deduce that

\[
\lim_{n \to \infty} d \int_{0}^{l} J(\bar{x}_n - y)u_l(y)dy = \lim_{n \to \infty} du_l(\bar{x}_n) - f(u_l(\bar{x}_n)) = \lim_{n \to \infty} d\eta_* V(\bar{x}_n) - f(\eta_* V(\bar{x}_n)) > \lim_{n \to \infty} d \int_{0}^{l} J(\bar{x}_n - y)\eta_* V(y)dy,
\]

which yields

\[
\int_{0}^{l} J(\bar{x}_0 - y) [u_l(y) - \eta_* V(y)] dy \geq 0.
\]

This contradicts to \( \eta_* V(x) > u_l(x) \) in \([0, l] \). Hence \( \inf_{x \in [0, l]} \eta_* V(x) - u_l(x) > 0 \), and by the definition of \( \eta_* \) and \( \eta_* > 1 \) we easily show a contradiction. So we prove our claim.

Thanks to the above analysis, it is easily shown that \( \lim_{x \to \infty} V(x) = u^* \) and \( V \) is continuous in \( \mathbb{R}^+ \). Similarly, we define

\[
\gamma_* = \inf \{ \gamma > 1 : \gamma U \geq V \text{ in } \mathbb{R}^+ \}.
\]

Obviously, \( \gamma_* \) is well defined and \( \gamma_* \geq 1 \). Next we show that \( \gamma_* = 1 \). Once it is done, we have \( U \geq V \), and so \( U \equiv V \). Assume on the contrary that \( \gamma_* > 1 \). By the definition of \( \gamma_* \) and the
fact that \( \lim_{x \to \infty} U(x) = \lim_{x \to \infty} V(x) = u^* \), there must exist \( \hat{x} \in \mathbb{R}^+ \) so that \( \gamma_* U(\hat{x}) = V(\hat{x}) \).

Similar to the above, it can be shown that \( f(\gamma_* U(\hat{x})) \geq \gamma_* f(U(\hat{x})) \), which is a contradiction to the assumptions of \( f \). Hence \( \gamma_* = 1 \), and further \( U \equiv V \).

**Step 5:** The proof of \( U < u^* \) and the monotonicity of \( U \). If there exists \( z_0 \in \mathbb{R}^+ \) with \( U(z_0) = u^* \), since \( u^* \) is not a solution of (2.5), then we can find some \( \bar{z} \in \mathbb{R}^+ \) such that \( U(\bar{z}) = u^* \) and \( U(x) < u^* \) in some left (right) neighborhood of \( \bar{z} \). Furthermore, by (2.5),

\[
u^* = \int_0^\infty J(\bar{z} - y)U(y)dy < u^*\]

which implies that \( U < u^* \).

Define \( \bar{U}(x) := U(x + \delta) \) with any \( \delta > 0 \). Then \( \bar{U} \) satisfies

\[
d \int_0^\infty J(x - y)\bar{U}(y)dy - d\bar{U} + f(\bar{U}) \leq 0 \quad \text{in} \quad \mathbb{R}^+.
\]

Similarly to the above, we can show that \( U(x + \delta) = \bar{U}(x) > U(x) \) in \( \mathbb{R}^+ \), and hence \( U \) is nondecreasing in \( \mathbb{R}^+ \). The proof is ended. \( \square \)

**Lemma 2.5.** Let \( f \) satisfy conditions (F1)–(F3) and \( U \) be the unique bounded positive solution of (2.5). Then, for any nonnegative function \( w_0 \in C(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+) \), the following problem

\[
\begin{aligned}
w_t &= d \int_0^\infty J(x - y)w(t, y)dy - dw + f(w), \quad t > 0, \quad x \geq 0, \\
w(0, x) &= w_0(x), \quad \neq 0, \quad x \geq 0 
\end{aligned} \tag{2.9}
\]

has a unique solution \( w \), and \( \lim_{t \to \infty} w(t, x) = U(x) \) locally uniformly in \( \mathbb{R}^+ \).

**Proof.** We first show the existence and uniqueness of \( w \). Clearly, for large \( l > 0 \), problem

\[
\begin{aligned}
w^l_t &= d \int_0^l J(x - y)w^l(t, y)dy - dw^l + f(w^l), \quad t > 0, \quad 0 \leq x \leq l, \\
w^l(0, x) &= w_0(x), \quad 0 \leq x \leq l 
\end{aligned}
\]

has a unique positive solution \( w^l \). By the maximum principle, \( 0 \leq w^l \leq \max\{\|w_0\|_\infty, K\} \), and \( w^l \) is nondecreasing in \( l \). Define \( w = \lim_{l \to \infty} w^l \). Then \( w(0, x) = w_0(x) \). For any \( (t, x) \in \mathbb{R}^+ \times \mathbb{R}^+ \), we have that, for large \( l \),

\[
\begin{aligned}
w^l(t, x) - w_0(x) &= \int_0^t \int_0^l J(x - y)w^l(\tau, y)dyd\tau - d \int_0^t w^l(\tau, x)d\tau + \int_0^t f(w^l(\tau, x))d\tau. 
\end{aligned}
\]

Letting \( l \to \infty \), it follows from the dominated convergence theorem that

\[
\begin{aligned}
w(t, x) - w_0(x) &= \int_0^t \int_0^\infty J(x - y)w(\tau, y)dyd\tau - d \int_0^t w(\tau, x)d\tau + \int_0^t f(w(\tau, x))d\tau. 
\end{aligned}
\]

This implies that \( w \) satisfies (2.9) and \( w \in C(\mathbb{R}^+ \times \mathbb{R}^+) \). The uniqueness can be deduced by the maximum principle.

Now we prove that longtime behaviour of \( w \). Clearly, \( \lim_{t \to \infty} w^l(t, x) = u_l(x) \) uniformly in \([0, l]\), which is given in the proof of Lemma 2.4 and \( \lim_{t \to \infty} u_l(x) \to U(x) \) locally uniformly in \( \mathbb{R}^+ \) by Dini’s theorem. Since \( w \geq w^l \), we have \( \lim \inf_{t \to \infty} w(t, x) \geq U(x) \) locally uniformly in \( \mathbb{R}^+ \).
Let $W$ be the unique solution of (2.10) with initial data $W_0(x) = \max\{\|w_0\|_{\infty}, K\}$. The maximum principle asserts that $W \geq w$ and $W$ is non-increasing in $t$. It follows from Lemma 2.4 and Dini’s theorem that $\lim_{t \to \infty} W(t, x) = U(x)$ locally uniformly in $\mathbb{R}^+$. So $\limsup_{t \to \infty} w(t, x) \leq U(x)$ locally uniformly in $\mathbb{R}^+$. The proof is ended.

To carry out the discussion on longtime behavior of solution of problem (1.3), we consider the fixed boundary problem

$$
\begin{align*}
&\begin{cases}
  u_t = d \int_{0}^{l} J(x-y)u(t,y)dy - dj(x)u + f(u), & t > 0, \ 0 \leq x \leq l, \\
  u(0, x) = u_0(x) \geq 0, & 0 \leq x \leq l,
\end{cases}
\end{align*}
$$

(2.10)

where $j(x) = \int_{0}^{\infty} J(x-y)dy$.

Lemma 2.6. Let conditions (F1)-(F3) hold, $u_0 \in C([0, l])$ and $u$ be the unique solution of (2.10). Then we have

(1) problem (2.10) has a unique positive steady state $\bar{u}_t \in C([0, l])$ if and only if $\lambda_p(L^N_{(0, l)} + f'(0)) > 0$, and when $u_0 \neq \bar{u}_t$ in $C([0, l])$;

(2) if $\lambda_p(L^N_{(0, l)} + f'(0)) \leq 0$, then $0$ is the only nonnegative bounded steady state of problem (2.10), and $\lim_{t \to \infty} u = 0$ in $C([0, l])$;

(3) $\lim_{t \to \infty} \bar{u}_t(x) = u^* \quad \text{locally uniformly in } \mathbb{R}^+,$ where $u^*$ is the unique positive root of $f(u) = 0$.

Proof. (1) Let $\bar{u}_t$ be a positive steady state of (2.10). Similarly to Lemma 2.4, one can show that $\bar{u}_t \in C([0, l])$. By the condition (F2),

$$
d \int_{0}^{l} J(x-y)\bar{u}_t(y)dy - dj(x)\bar{u}_t(x) + f'(0)\bar{u}_t(x) > 0 \quad \text{in } [0, l].
$$

Thanks to the variational characterization of $\lambda_p(L^N_{(0, l)} + f'(0))$, we have $\lambda_p(L^N_{(0, l)} + f'(0)) > 0$.

Suppose that $\lambda_p(L^N_{(0, l)} + f'(0)) > 0$ with the corresponding positive eigenfunction $\phi$. Let $\beta = \max_{u \in [0, K]} |f'(u)| + d + 1$, and define the operator $\Gamma$ by

$$
\Gamma(u) = \frac{1}{\beta} \left( d \int_{0}^{l} J(x-y)u(y)dy - dj(x)u(x) + f(u) + \beta u \right) \quad \text{for } u \in C([0, l]).
$$

Obviously, $\Gamma$ is nondecreasing in $\{u \in C([0, l]) : 0 \leq u \leq K\}$ and $\Gamma(K) \leq K$. As $\lambda_p(L^N_{(0, l)} + f'(0)) > 0$, we can find $0 < \varepsilon \ll 1$ such that $\Gamma(\varepsilon \phi) \geq \varepsilon \phi$ and $\varepsilon \phi \leq K$. Then by a simple iteration argument and the dominated convergence theorem, we can find a steady state of (2.10) denoted by $\bar{u}_t$ with $\varepsilon \phi \leq \bar{u}_t \leq K$. Proof of the uniqueness and continuity is similar to that of Lemma 2.4.

Now we study the longtime behaviour of solution $u$ of (2.10). Since $u_0 \geq 0$, the maximum principle gives $u(t, x) > 0$ for $t > 0$ and $0 \leq x \leq l$. So, we may assume $u_0 > 0$ in $[0, l]$. There is $0 < \varepsilon \ll 1$ such that $\varepsilon \phi \leq u_0$ in $[0, l]$. Let $\underline{u}$ and $\bar{u}$ be the unique solution of (2.10) with initial data $\varepsilon \phi$ and $\max\{K, \|u_0\|_{\infty}\}$, respectively. By the comparison principle, $\underline{u} \leq u \leq \bar{u}$, and $\underline{u}$ and $\bar{u}$ are, respectively, nondecreasing and nonincreasing in $t$. It then follows from our early analysis that $\lim_{t \to \infty} \underline{u} = \lim_{t \to \infty} \bar{u} = \bar{u}_t$ in $C([0, l])$. Hence $\lim_{t \to \infty} u = \bar{u}_t$ in $C([0, l])$.

(2) Arguing indirectly, let $u_t \neq 0$ be a nonnegative steady state of (2.10). Clearly, $u_t \in C([0, l])$, and $u_t > 0$ in $[0, l]$ by the condition (J). Similarly to the above,

$$
d \int_{0}^{l} J(x-y)u_t(y)dy - dj(x)u_t(x) + f'(0)u_t(x) > 0 \quad \text{in } [0, l].$$
This implies $\lambda_p(L^N_{(0,l)} + f'(0)) > 0$. So, problem (2.10) has no nontrivial and nonnegative bounded steady state if $\lambda_p(L^N_{(0,l)} + f'(0)) \leq 0$. Similar to the argument in (1), $\lim_{t \to \infty} u = 0$ in $C([0,l])$.

(3) Let $\tilde{u}_t$ be the unique positive steady state of (2.10). By arguing as in the proof of Lemma 2.4, we conclude that $\tilde{u}_t \leq u^*$ and $\tilde{u}_t$ is nondecreasing in $l$. Then $\bar{U} := \lim_{t \to \infty} \tilde{u}_t$ exists and satisfies

$$\lambda^* \int_0^\infty J(x-y)\bar{U}(y)dy - dj(x)\bar{U} + f(\bar{U}) = 0.$$ 

Similarly to the proof of Lemma 2.4, $\bar{U}$ is continuous in $\mathbb{R}^+$ and $\lim_{x \to \infty} \bar{U}(x) = u^*$, and then we further have $\bar{U} \equiv u^*$. Then by Dini’s theorem, conclusion (3) holds.

The rest of this section is prepared for the study of longtime behavior of solution of (1.4).

**Lemma 2.7.** Suppose that $\lambda$ is a positive constant and the function $k$ satisfies

$$k \in C(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+), \ k := \inf_{x \in \mathbb{R}^+} k(x) > 0 \text{ and } k_\infty := \lim_{x \to \infty} k(x) < \infty.$$ 

Consider problem

$$d \int_0^\infty J(x-y)U(y)dy - du + U(k(x) - \lambda U) = 0 \quad \text{in } \mathbb{R}^+.$$ 

(2.11)

Then the following results hold:

1. problem (2.11) has a unique bounded positive solution $U_k \in C(\mathbb{R}^+)$, and $\lim_{x \to \infty} U_k(x) = k_\infty/\lambda$;

2. let functions $k_i$ satisfy the condition (K) and $k_1 \leq k_2$ in $\mathbb{R}^+$, then $U_{k_1} \leq U_{k_2}$ in $\mathbb{R}^+$.

**Proof.** (1) We show the existence by a monotone iteration argument. By Lemma 2.4 problem

$$d \int_0^\infty J(x-y)u(y)dy - du + u(k - \lambda u) = 0 \quad \text{in } \mathbb{R}^+$$

has a unique positive solution $u$ and $0 < u < k/\lambda$. Let $\bar{u} = \|k\|_\infty/\lambda$ and $\bar{\beta} = d + 2\|k\|_\infty + 1$.

Similarly, the operator $\hat{\Gamma}$ defined by

$$\hat{\Gamma}(u) = \frac{1}{\bar{\beta}} \left( d \int_0^\infty J(x-y)u(y)dy - du + u(k(x) - \lambda u) + \hat{\beta}u \right), \ u \in C(\mathbb{R}^+)$$

is nondecreasing in $\{u \in C(\mathbb{R}^+) : 0 \leq u \leq \bar{u}\}$, and $\hat{\Gamma}(u) \geq u$ and $\hat{\Gamma}(\bar{u}) \leq \bar{u}$. Hence, by a iteration process, problem (2.11) has a positive solution $U_k$ with $u \leq U_k \leq \bar{u}$. Since $d \int_0^\infty J(x-y)U_k(y)dy$ and $k$ are continuous, it follows from a quadratic formula that $U_k$ is also continuous in $\mathbb{R}^+$. Now we prove $\lim_{x \to \infty} U_k(x) = k_\infty/\lambda$. Let $u_* := \inf_{x \to \infty} U_k(x)$, then $u_* > 0$ due to $U_k \geq u$. If $u_* < k_\infty/\lambda$, then there exists $x_n \not\to \infty$ such that $U_k(x_n) \to u_*$. Hence, as $n \to \infty$,

$$d \int_0^\infty J(x-y)U_k(y)dy = dU_k(x_n) - U_k(x_n)(k(x_n) - \lambda U_k(x_n)) \to du_* - u_*(k_\infty - \lambda u_*) < du_*.$$ 

Let $\chi_{[-x_n,\infty)}(y)$ be the characteristic function of the set $[-x_n,\infty)$. Then, by Fatou’s lemma,

$$\liminf_{n \to \infty} \int_0^\infty J(x-y)U_k(y)dy = \liminf_{n \to \infty} \int_{-x_n}^\infty J(y)U_k(x_n + y)dy$$
\[
= \lim_{n \to \infty} \inf \int_{-\infty}^{\infty} J(y)U_k(x_n + y)\chi_{[-x_n, \infty)}(y)\,dy
\]
\[
\geq \int_{-\infty}^{\infty} \lim_{n \to \infty} J(y)U_k(x_n + y)\chi_{[-x_n, \infty)}(y)\,dy
\]
\[
\geq \int_{-\infty}^{\infty} J(y)u_*\,dy = u_*. \]

We get a contradiction, and so \( u_* \geq k_\infty/\lambda \).

Assume \( u^0 := \limsup_{x \to -\infty} U_k(x) > k_\infty/\lambda \), and there is \( x_0 \to \infty \) such that \( U_k(x_0) \to u^0 \). So

\[
d \int_{0}^{\infty} J(\tilde{x}_n - y)U_k(y)\,dy = du_k(\tilde{x}_n) - U_k(\tilde{x}_n)(k(\tilde{x}_n) - \lambda U_k(\tilde{x}_n)) \to du^0 - u^0(k_\infty - \lambda u^0) > du^0
\]
as \( n \to \infty \). On the other hand, by the dominated convergence theorem,

\[
\lim_{n \to \infty} \int_{0}^{\infty} J(\tilde{x}_n - y)U_k(y)\,dy = \lim_{n \to \infty} \int_{-\tilde{x}_n}^{\infty} J(y)U_k(\tilde{x}_n + y)\,dy
\]
\[
= \lim_{n \to \infty} \int_{-\tilde{x}_n}^{\infty} J(y)U_k(\tilde{x}_n + y)\chi_{[-\tilde{x}_n, \infty)}(y)\,dy
\]
\[
\leq \lim_{n \to \infty} \int_{-\tilde{x}_n}^{\infty} J(y) \sup_{x \geq \tilde{x}_n + y} U_k(x)\chi_{[-\tilde{x}_n, \infty)}(y)\,dy
\]
\[
= \int_{-\infty}^{\infty} J(y)u^0\,dy = u^0. \]

We get a contradiction, and so \( u^0 \leq k_\infty/\lambda \). Therefore, \( \lim_{x \to -\infty} U_k(x) = k_\infty/\lambda \).

We now prove the uniqueness. Let \( v \) be another bounded positive solution of \( 2.11 \). Similarly, \( v \) is continuous and \( v \leq k_\infty/\lambda \). Similar to the proof of Lemma \( 2.4 \), \( v \geq u \). Thanks to the above iteration process, it yields \( v \geq U_k \), and thus \( \lim_{x \to -\infty} v(x) = k_\infty/\lambda \). So similarly to the methods in Lemma \( 2.4 \), we have \( U_k \equiv v \). The uniqueness is obtained.

(2) By the conclusion (1), we have that \( U_k \) is positive and continuous, and \( \lim_{x \to -\infty} U_k(x) = k_1/\lambda \leq k_2/\lambda = \lim_{x \to -\infty} U_k(x) \). Similarly to the proof of Lemma \( 2.4 \), one can show that \( U_k \leq U_k \) in \( \mathbb{R}^+ \), and the details are omitted here. \( \square \)

**Remark 2.8.** In Lemma \( 2.7 \), if we further suppose that \( k \) is strictly increasing in \( \mathbb{R}^+ \), so is \( U_k \). In fact, taking advantage of the analogous arguments in the proof of Lemma \( 2.4 \), we can show that \( U_k \) is nondecreasing in \( \mathbb{R}^+ \). If \( U_k(x_1) = U_k(x_2) \) for some \( 0 < x_1 < x_2 \), it then follows that

\[
\int_{0}^{\infty} J(x_1 - y)U_k(y)\,dy > \int_{0}^{\infty} J(x_2 - y)U_k(y)\,dy.
\]
Thus

\[
\int_{-x_1}^{\infty} J(y)U_k(y + x_1)\,dy > \int_{-x_2}^{\infty} J(y)U_k(y + x_2)\,dy,
\]
which is impossible because \( U_k \) is nondecreasing and positive in \( \mathbb{R}^+ \).

**Remark 2.9.** For any \( 0 < \varepsilon \ll 1 \), let \( U^\varepsilon_k \) be the unique bounded positive solution of \( 2.11 \) with \( k(x) \) replaced by \( k(x) \pm \varepsilon \). By the boundness and monotone convergence of \( U^\varepsilon_k \), and uniqueness of bounded positive solution of \( 2.11 \), we have that \( U^\varepsilon_k \to U_k \) as \( \varepsilon \to 0 \).
Lemma 2.10. Let \( k \) satisfy the condition (K), and \( U_k \) be the unique bounded positive solution of (2.11). Assume \( w_0 \in C(R^+ \cap L^\infty(R^+)) \) and \( w_0 \geq, \neq 0 \). Then the problem
\[
\begin{cases}
    w_t = d \int_0^\infty J(x-y)w(t,y)dy - dw + w(k(x) - \lambda w), & t > 0, x \geq 0, \\
    w(0,x) = w_0(x) \geq, \neq 0, & x \geq 0
\end{cases}
\]
has a unique solution \( w \), and \( \lim_{t \to \infty} w(t,x) = U_k(x) \) locally uniformly in \( R^+ \).

The proof of Lemma 2.10 is similar to that of Lemma 2.5, and we omit it here.

Lemma 2.11. Assume that \( h(t) \) is continuous in \( R^+ \) and increasing to \( \infty \), and \( k \) satisfies the condition (K) and \( U_k \) is the unique bounded positive solution of (2.11). Let \( v \) be a nonnegative and bounded continuous function in \( R^+ \times R^+ \) and satisfy
\[
\limsup_{t \to \infty} v(t,x) \leq k(x) \text{ locally uniformly in } R^+.
\]

Suppose that \( u \) satisfies
\[
\begin{cases}
    u_t = d \int_0^{h(t)} J(x-y)u(t,y)dy - du + u(v(t,x) - \lambda u), & t > 0, 0 \leq x < h(t), \\
    u(t, h(t)) = 0, & t > 0, \\
    h(0) = h_0, \ u(0, x) = u_0(x), & 0 \leq x \leq h_0.
\end{cases}
\]

Then \( \limsup_{t \to \infty} u(t,x) \leq U_k(x) \) locally uniformly in \( R^+ \).

Proof. The idea of this proof comes from [7, Lemma 3.14]. For any integer \( n \geq 1 \), by (2.12), there exist \( T_n \nearrow \infty \) such that, for \( t \geq T_n \) and \( x \in [0, n+1] \), we have \( v(t,x) \leq k(x) + 1/n \). Define
\[
k_n(x) = \begin{cases}
    k(x) + 1/n, & 0 \leq x \leq n, \\
    k(x) + 1/n + 2(K_0 + 1 - k(x) - 1/n)(x-n), & n < x \leq n + 1/2, \\
    K_0 + 1, & x > n + 1/2,
\end{cases}
\]
where \( K_0 > \max\{\|k\|_\infty, \|v\|_\infty\} \). It is clear that \( k_n \in C(R^+ \cap L^\infty(R^+)) \), \( k \leq k_n \leq K_0 + 1 \), \( k_n \) is nonincreasing in \( n \), \( \lim_{n \to \infty} k_n = k \) and \( \lim_{x \to \infty} k_n(x) = K_0 + 1 \). Fix \( K_1 > \max\{\|u_0\|_\infty + K_0 + 1, (K_0 + 1)/\lambda\} \). By Lemma 2.7 there is \( 0 < \varepsilon \ll 1 \) such that \( \varepsilon U_k(x) < K_1 \). Let \( z_n \) be a solution of
\[
\begin{cases}
    z_t = d \int_0^\infty J(x-y)z(t,y)dy - dz + z(k_n(x) - \lambda z), & t > T_n, x \geq 0, \\
    z(T_n, x) = K_1, & x \geq 0.
\end{cases}
\]

It follows from a comparison argument that
\[
\varepsilon U_k(x) \leq z_n(t,x) \leq K_1, \ u(t,x) \leq z_n(t,x) \quad \text{for } t > T_n, x \in R^+.
\]

By virtue of Lemma 2.10 we have
\[
\lim_{t \to \infty} z_n(t,x) = Z_n(x) \quad \text{locally uniformly in } R^+,
\]
where \( Z_n \) is the unique bounded positive solution of (2.11) with \( k \) replaced by \( k_n \). Since \( k_n \) is nonincreasing in \( n \), from Lemma 2.10 and \( z_n(t,x) \geq \varepsilon U_k(x) \) we see that \( Z_n \) is also nonincreasing in \( n \) and \( Z_n \geq \varepsilon U_k \). Define \( Z = \lim_{n \to \infty} Z_n \). Then \( Z > 0 \) and satisfies (2.11). The uniqueness implies \( Z = U_k \). Thus combining (2.13) with (2.14), we prove our conclusion. \[\square\]
3 The dynamics of the problem \([1,2]\)

Inspired by [1, Theorem 2.1], we can utilize the ODE theory, the contraction mapping principle and maximum principle to prove the global existence and uniqueness of solution of problem \([1,2]\), and the details are omitted here.

**Theorem 3.1** (Existence and uniqueness). Problem \([1,2]\) has a unique global solution \((u, h)\). Moreover, \(u \in C(\mathcal{D}^T), h \in C^1([0, T])\) and \(0 < u \leq \max\{\|u_0\|_\infty, K\}\) in \(D^T_h\) for any \(T > 0\).

Obviously, \(h'(t) > 0\) for \(t > 0\) by the third equation in \([1,2]\). Thus \(h_\infty := \lim_{t \to \infty} h(t) \leq \infty\). In the following, we further suppose that \(f\) satisfies conditions \((F2)\) and \((F3)\).

For positive constants \(d, l\) and \(a_0\), we define an operator \(\mathcal{L}^d_{(0, l)} + a_0\) by

\[
(\mathcal{L}^d_{(0, l)} + a_0) \varphi := d \int_0^l J(x - y)\varphi(y)dy - \varphi + a_0\varphi \quad \text{for} \quad \varphi \in C([0, l]).
\]

The generalized principal eigenvalue of \(\mathcal{L}^d_{(0, l)} + a_0\) is given by

\[
\lambda_p(\mathcal{L}^d_{(0, l)} + a_0) := \inf \{\lambda \in \mathbb{R} : (\mathcal{L}^d_{(0, l)} + a_0)\phi \leq \lambda \phi \text{ in } [0, l], \phi > 0\}.
\]

Please see [1] Proposition 3.4 for some useful properties of \(\lambda_p(\mathcal{L}^d_{(0, l)} + a_0)\).

**Lemma 3.2.** If \(h_\infty < \infty\), then \(\lambda_p(\mathcal{L}^d_{(0, h_\infty)} + f'(0)) \leq 0\) and \(\lim_{t \to \infty} \|u(t, \cdot)\|_{C([0, (h(t))]}) = 0\).

**Proof.** We first prove \(\lambda_p(\mathcal{L}^d_{(0, h_\infty)} + f'(0)) \leq 0\). Arguing indirectly, suppose \(\lambda_p(\mathcal{L}^d_{(0, h_\infty)} + f'(0)) > 0\). By the condition \((J)\), there are \(\varepsilon_0, \sigma_0 > 0\) such that \(J(x) > \sigma_0\) for \(|x| \leq \varepsilon_0\). Due to the continuity of \(\lambda_p(\mathcal{L}^d_{(0, l)} + f'(0))\) about \(l\), there exists \(0 < \varepsilon < \varepsilon_0/2\) such that \(\lambda_p(\mathcal{L}^d_{(0, h_\infty - \varepsilon)} + f'(0)) > 0\). For such \(\varepsilon\), there is \(T > 0\) such that \(h(t) > h_\infty - \varepsilon\) for all \(t \geq T\). Let \(w\) be the unique solution of

\[
\begin{align*}
\begin{cases}
t = d \int_0^{h_\infty - \varepsilon} J(x - y)w(t, y)dy - dw + f(w), \quad t > T, \quad 0 \leq x \leq h_\infty - \varepsilon, \\
w(T, x) = u(T, x) \quad 0 \leq x \leq h_\infty - \varepsilon.
\end{cases}
\end{align*}
\]

In view of \(\lambda_p(\mathcal{L}^d_{(0, h_\infty - \varepsilon)} + f'(0)) > 0\) and [1, Proposition 3.5], we see that \(w(t, x)\) converges to a positive steady state \(W(x)\) uniformly in \([0, h_\infty - \varepsilon]\) as \(t \to \infty\). Moreover, it follows from a simple comparison argument that \(u(t, x) \geq w(t, x)\) for \(t \geq T\) and \(x \in [0, h_\infty - \varepsilon]\). So there exists \(T_1 > T\) such that \(u(t, x) \geq \frac{1}{2}W(x)\) for \(t > T_1\) and \(x \in [0, h_\infty - \varepsilon]\). Therefore, for \(t > T_1\),

\[
h'(t) = \mu \int_0^{h(t)} \int_{h(t)}^{h(t)} J(x - y)u(t, x)dydx \geq \mu \int_{h_\infty - \frac{\delta_0}{2}}^{h_\infty - \frac{\delta_0}{2}} \int_{h_\infty - \frac{\delta_0}{2}}^{h_\infty + \frac{\delta_0}{2}} J(x - y)u(t, x)dydx
\]

\[
\geq \mu \int_{h_\infty - \frac{\delta_0}{2}}^{h_\infty + \frac{\delta_0}{2}} \int_{h_\infty - \frac{\delta_0}{2}}^{h_\infty} \sigma_0 \frac{1}{2}W(x)dydx > 0,
\]

which contradicts to \(h_\infty < \infty\). And so \(\lambda_p(\mathcal{L}^d_{(0, h_\infty)} + f'(0)) \leq 0\).

Next we prove \(\lim_{t \to \infty} \|u(t, \cdot)\|_{C([0, h(t)])} = 0\). Let \(\bar{w}\) be the unique solution of

\[
\begin{align*}
\begin{cases}
\bar{w}_t = d \int_0^{h_\infty} J(x - y)\bar{w}(t, y)dy - d\bar{w} + f(\bar{w}), \quad t > 0, \quad 0 \leq x \leq h_\infty, \\
\bar{w}(0, x) = u(0, x), \quad 0 \leq x \leq h_0; \quad \bar{w}(0, x) = 0, \quad h_0 \leq x \leq h_\infty.
\end{cases}
\end{align*}
\]

Since \(\lambda_p(\mathcal{L}^d_{(0, h_\infty)} + f'(0)) \leq 0\), we see from [1, Proposition 3.5] that \(\lim_{t \to \infty} \bar{w} = 0\) in \(C([0, h_\infty])\).

By a comparison argument again, we get \(u \leq \bar{w}\) in \(D^\infty_h\). Hence \(\lim_{t \to \infty} \|u(t, \cdot)\|_{C([0, h(t)])} = 0\). □
Lemma 3.3. Let $U \in C(\mathbb{R}^+)$ be the unique bounded positive solution of (2.5). If $h_\infty = \infty$, then \( \lim_{t \to \infty} u(t, x) = U(x) \) locally uniformly in \([0, \infty)\).

Proof. Firstly, let $w$ be the unique solution of (2.9) with initial value $w(0, x) = \|u_0\|_\infty$. It follows from Lemma 2.5 that $\lim_{t \to \infty} w(t, x) = U(x)$ locally uniformly in $\mathbb{R}^+$. By a comparison consideration, $u \leq w$ in $D_h^\infty$. Thus $\limsup_{t \to \infty} u(t, x) \leq U(x)$ locally uniformly in $\mathbb{R}^+$.

By [1 Proposition 3.4], we know that for large $l$, $\lambda_p(L^d_{(0, t)} + f'(0)) > 0$. For such large $l > 0$, there exists $T > 0$ such that $h(t) > l$ for $t \geq T$. Consider the following problem

\[
\begin{align*}
\frac{u}{\partial t} &= d \int_0^t J(x-y)u(y, y)dy - du + f(u), \quad t > T, \quad 0 \leq x \leq l,
\quad 0 \leq x \leq l.
\end{align*}
\]

Again, by the comparison principle, $u(t, x) \geq w(t, x)$ for $t \geq T$ and $0 \leq x \leq l$. Let $u_t$ be the unique positive solution of (2.6). Then $\lim_{t \to \infty} u(t, x) = u_t(x)$ uniformly in $[0, l]$, and $\lim_{t \to \infty} u_t = U$ locally uniformly in $\mathbb{R}^+$. Hence, $\liminf_{t \to \infty} u(t, x) \geq U(x)$ locally uniformly in $\mathbb{R}^+$. □

Due to the above two lemmas, we immediately obtain the following conclusion.

Theorem 3.4 (Spreading-vanishing dichotomy). One of the following alternatives happens for (1.2):

1. **Spreading**: $h_\infty = \infty$ and $\lim_{t \to \infty} u(t, x) = U(x)$ locally uniformly in $\mathbb{R}^+$;
2. **Vanishing**: $h_\infty < \infty$ and $\lim_{t \to \infty} \|u(t, \cdot)\|_{C([0, h(t)])} = 0$.

Next we discuss when spreading or vanishing happens, and some criteria will be given. Thanks to Lemma 3.2 and [1 Proposition 3.4], we easily derive the following lemma.

Lemma 3.5. If $f'(0) \geq d$, then spreading happens for (1.2).

If $f'(0) < d$, by [1 Proposition 3.4] there is a unique $\ell_D^* > 0$ such that $\lambda_p(L^d_{(0, \ell_D^*)} + f'(0)) = 0$ and $\lambda_p(L^d_{(0, t)} + f'(0))(\ell_D - \ell_D^*) > 0$ for all $l > 0$.

Lemma 3.6. Suppose $f'(0) < d$. If $h_0 \geq \ell_D^*$, then spreading happens for (1.2). Moreover, if $h_\infty < \infty$, then $h_\infty \leq \ell_D^*$.

For later discussion, we now give two comparison principles which can be proved by analogous considerations with [1 Theorem 3.1] and [12 Lemma 3.4].

Theorem 3.7 (Comparison principle). Let conditions (J) and (F1) hold, and $\bar{h} \in C^1([0, T])$, $\bar{u}, \bar{u}_t \in C(D_h^\infty)$ satisfy

\[
\begin{align*}
\bar{u}_t &\geq d \int_0^{h(t)} J(x-y)\bar{u}(y, y)dy - d\bar{u} + f(t, x, \bar{u}), \quad 0 < t \leq T, \quad 0 \leq x \leq \bar{h}(t), \\
\bar{u}(t, \bar{h}(t)) &\geq 0, \quad 0 < t \leq T, \\
\bar{h}'(t) &\geq \mu \int_0^{h(t)} \int_{h(t)}^\infty J(x-y)\bar{u}(y, x)dydx, \quad 0 < t \leq T, \\
\bar{h}(0) &\geq h_0, \quad \bar{u}(0, x) \geq u_0(x), \quad x \in [0, h_0].
\end{align*}
\]

Then the solution $(u, h)$ of (1.2) satisfies

\[
\begin{align*}
u(t, x) &\leq \bar{u}(t, x), \quad h(t) \leq \bar{h}(t) \text{ for } 0 < t \leq T, \quad 0 \leq x \leq h(t).
\end{align*}
\]
Theorem 3.8 (Comparison principle). Let conditions (J) and (F1) hold, and \( \bar{h} \in C^1([0,T]) \), \( r \in C([0,T]) \) be nondecreasing and \( 0 \leq r(t) < \bar{h}(t) \). If \( \bar{u}, \bar{h} \in C(D^r_{\bar{h}}) \) satisfy

\[
\begin{array}{l}
\bar{u}_t \geq d \int_0^{\bar{h}(t)} J(x - y)\bar{u}(t,y)dy - d\bar{u} + f(t,x,\bar{u}), \quad 0 < t \leq T, \quad r(t) < x < \bar{h}(t), \\
\bar{u}(t,\bar{h}(t)) \geq 0, \quad 0 < t \leq T, \\
\bar{h}'(t) \geq \mu \int_0^{\bar{h}(t)} \int_{\bar{h}(t)}^\infty J(x - y)\bar{u}(t,x)dydx, \quad 0 < t \leq T, \\
\bar{u}(t,x) \geq u(t,x), \quad 0 \leq t \leq T, \quad 0 \leq x \leq r(t), \\
\bar{h}(0) \geq h_0, \quad \bar{u}(0,x) \geq u_0(x), \quad x \in [0,h_0],
\end{array}
\]

(3.2)

then the solution \((u,h)\) of (1.2) satisfies

\[ u(t,x) \leq \bar{u}(t,x), \quad h(t) \leq \bar{h}(t) \quad \text{for} \quad 0 < t \leq T, \quad 0 \leq x \leq \bar{h}(t). \]

It can be easily seen that the above comparison principles are still valid if \( \int_\mathbb{R} J(x)dx \neq 1 \) and condition (ii) in (F1) is not satisfied. The similar results hold for the solution \((u,h)\) of (1.3). The pair \((\bar{u},\bar{h})\) is usually called an upper solution of (1.2) or (1.3). We can also define a lower solution and derive analogous conclusions by reversing all the inequalities of (3.1) or (3.2). Moreover, it follows from Theorem 3.7 that the solution \((u,h)\) of (1.2) or (1.3) is strictly increasing in \( \mu > 0 \).

Lemma 3.9. Let \( f'(0) < d \). If \( h_0 < \ell^*_D \), then there exists \( \mu_D > 0 \) such that vanishing happens for (1.2) when \( 0 < \mu \leq \mu_D \).

Proof. For any given \( h_1 \in (h_0, \ell^*_D) \), from Proposition 3.4 we know that \( \lambda_1 := \lambda_p(J_{[0,h_1]} + f'(0)) < 0 \). Let \( \phi_1 \) be the corresponding positive eigenfunction to \( \lambda_1 \) with \( \|\phi_1\|_\infty = 1 \). For constant \( C > 0 \), we define \( u_1 = Ce^{\lambda_1 t/2} \phi_1 \), and easily see that, for \( t > 0 \) and \( x \in [0,h_1] \),

\[ u_{1t} - d \int_0^{h_1} J(x - y)u_1(t,y)dy + du_1 - f'(0)u_1 = -\frac{\lambda_1}{2} Ce^{\lambda_1 t/2} \phi_1 > 0. \]

Let \( \bar{u} \) be the unique solution of

\[
\begin{array}{l}
\bar{u}_t = d \int_0^{h_1} J(x - y)\bar{u}(t,y)dy - d\bar{u} + f(\bar{u}), \quad t > 0, \quad 0 \leq x \leq h_1, \\
\bar{u}(0,x) = u(0,x) \quad \text{in} \quad [0,h_0], \quad \bar{u}(0,x) = 0 \quad \text{in} \quad [h_0,h_1].
\end{array}
\]

Thanks to the assumptions on \( f \), we have \( f(\bar{u}) \leq f'(0)\bar{u} \). Choose \( C > 0 \) such that \( C\phi_1 > u_0 \) in \([0,h_0] \). Then, by the comparison principle,

\[ \bar{u} \leq u_1 = Ce^{\lambda_1 t/2} \phi_1 \leq Ce^{\lambda_1 t/2} \quad \text{in} \quad \mathbb{R}^+ \times [0,h_1]. \]

Define

\[ \bar{h}(t) = h_0 + \mu h_1 C \int_0^t e^{\lambda_1 s/2} ds \quad \text{for} \quad t \geq 0. \]

We now prove that \((\bar{u},\bar{h})\) is an upper solution of (1.2). Clearly, for \( t > 0 \),

\[ \bar{h}(t) = h_0 - \frac{2\mu h_1 C}{\lambda_1} (1 - e^{\lambda_1 t/2}) < h_0 - \frac{2\mu h_1 C}{\lambda_1} \leq h_1 \]

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provided that

\[ 0 < \mu \leq \mu_d := \frac{-\lambda_1 (h_1 - h_0)}{2h_1 C}. \]

Thus by the equation of \( \bar{u} \), we obtain that, for \( t > 0 \) and \( x \in [0, \bar{h}(t)) \),

\[ \bar{u}_t \geq d \int_0^{\bar{h}(t)} J(x - y) \bar{u}(t, y) dy - d \bar{u} + f(\bar{u}). \]

Moreover, it is easy to check that

\[ \mu \int_0^{\bar{h}(t)} \int_{\bar{h}(t)}^{\infty} J(x - y) \bar{u}(t, x) dy dx \leq \mu C h_1 e^{\lambda_1 t/2} = \bar{h}'(t). \]

Therefore, \((\bar{u}, \bar{h})\) is an upper solution of \((1.2)\). By Theorem 3.7

\[ u(t, x) \leq \bar{u}(t, x), \quad h(t) \leq \bar{h}(t) \quad \text{for} \quad t > 0, \quad x \in [0, \bar{h}(t)]. \]

Consequently, \( h_\infty \leq \lim_{t \to \infty} \bar{h} \leq h_1. \]

From the above proof we easily see that, when \( f'(0) < d \) and \( h_0 < \ell_d^* \), vanishing happens for \((1.2)\) if \( u_0(x) \) is small sufficiently.

**Lemma 3.10.** Let \( c \in C(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+) \), \( g_0, H > 0 \), \( w_0 \in C([0, g_0]) \) with \( w_0(g_0) = 0 \) and \( w_0 > 0 \) in \([0, g_0)\). Then there exists \( \mu^0 > 0 \) depending on \( J, d, c, w_0, g_0 \) and \( H \) such that when \( \mu \geq \mu^0 \) and \((w, g)\) satisfies

\[
\begin{align*}
    w_t &\geq d \int_0^{g(t)} J(x - y)w(t, y) dy - dw + c(x)w, \quad t > 0, \quad 0 \leq x < g(t), \\
    w(t, 0) &\geq 0, \quad t > 0, \\
    g'(t) &\geq \mu \int_0^{g(t)} \int_{g(t)}^{\infty} J(x - y)w(t, x) dy dx, \quad t > 0, \\
    w(0, x) &\leq w_0(x), \quad g(0) = g_0, \quad x \in [0, g_0],
\end{align*}
\]

we must have \( \lim_{t \to \infty} g(t) \geq H. \)

**Proof.** By the maximum principle, \( w > 0 \) and \( g' > 0 \) for \( t > 0 \) and \( 0 \leq x < g(t) \). Choose a function \( s \in C^1([0, 1]) \) with \( s(0) = g_0, \ s(1) = H \) and \( s' > 0 \), and consider the problem

\[
\begin{align*}
    z_t &\geq d \int_0^{s(t)} J(x - y)z(t, y) dy - dz + c(x)z, \quad t > 0, \quad 0 \leq x < s(t), \\
    z(t, 0) &\geq 0, \quad t > 0, \\
    z(0, x) &\leq w_0(x), \quad s(0) = g_0, \quad x \in [0, g_0].
\end{align*}
\]

Similar to the arguments in [11, Lemma 2.3], this problem has a unique solution \( z \in C([0, 1] \times [0, s(t)]) \), and \( z > 0 \) in \([0, 1] \times [0, s(t)]\). Thus \( \int_0^{s(t)} \int_0^{\infty} J(x - y)z(t, x) dy dx \) is continuous and has a positive lower bound in \([0, 1]\). Since \( s \in C^1([0, 1]) \), there exists \( \mu^0 > 0 \) such that, for \( \mu \geq \mu^0 \),

\[ s'(t) \leq \mu \int_0^{s(t)} \int_{s(t)}^{\infty} J(x - y)z(t, x) dy dx \quad \text{for} \quad t \in [0, 1]. \]

Comparing \((w, g)\) with \((z, s)\) yields \( g \geq s \) in \([0, 1]\). So \( \lim_{t \to \infty} g(t) \geq s(1) = H \) since \( g'(t) > 0. \) \( \square \)
By Lemmas 3.6 and 3.10 we easily have the following result which implies that when initial habitat $h_0$ and growth rate $f'(0)$ are small, spreading can happen if expanding rate $\mu$ is large enough. The details of proof are omitted here.

**Lemma 3.11.** Suppose $f'(0) < d$. If $h_0 < \ell_D^*$, then there exists $\bar{\mu}_D > 0$ such that spreading happens for (1.2) when $\mu \geq \bar{\mu}_D$.

Based on the above lemmas, when initial habitat $h_0$ and growth rate $f'(0)$ are small, we may find a critical value of expanding rate $\mu$ by monotonicity and continuous dependence of solution on $\mu$, which governs spreading and vanishing for problem (1.2).

**Lemma 3.12.** Suppose $f'(0) < d$. If $h_0 < \ell_D^*$, then there exists $\mu^*_D > 0$ such that spreading happens for (1.2) when $\mu > \mu^*_D$, and vanishing happens when $0 < \mu \leq \mu^*_D$.

In conclusion, we have the spreading-vanishing criteria as below.

**Theorem 3.13** (Spreading-vanishing criteria). Let $(u, h)$ be the unique solution of (1.2).

1. If $f'(0) \geq d$, then spreading happens;
2. If $f'(0) < d$, then there exists a unique $\ell_D^* > 0$ such that spreading happens when $h_0 \geq \ell_D^*$;
3. If $f'(0) < d$ and $h_0 < \ell_D^*$, then there is $\mu_D^* > 0$ such that spreading happens for (1.2) if and only if $\mu > \mu_D^*$.

Next we discuss the spreading speed of (1.2), and the following semi-wave problem is vital to our arguments.

**Proposition 3.14** ([8, Theorem 1.2]). Assume that $J$ satisfies the condition (J) and $f$ satisfies conditions (F1)-(F3). Then the problem

\[
\begin{cases}
  d \int_{-\infty}^{0} J(x-y)\phi(y)dy - d\phi + c\phi' + f(\phi) = 0, & -\infty < x < 0, \\
  \phi(-\infty) = u^*, \quad \phi(0) = 0, \quad c = \mu \int_{-\infty}^{0} \int_{0}^{\infty} J(x-y)\phi(x)dydx 
\end{cases}
\]

has a unique solution pair $(c_0, \phi^{c_0})$ with $c_0 > 0$ and $\phi^{c_0}(x)$ nonincreasing in $(-\infty, 0]$ if and only if the condition $(J1)$ is satisfied. We usually call $\phi^{c_0}$ the semi-wave solution of (3.3) with speed $c_0$.

**Theorem 3.15** (Spreading speed). Assume that (F1)-(F3) hold and spreading happens for (1.2). Then we have

\[
\frac{U(0)}{u^*}c_0 \leq \liminf_{t \to \infty} \frac{h(t)}{t} \leq \limsup_{t \to \infty} \frac{h(t)}{t} \leq c_0 \quad \text{if } (J1) \text{ holds},
\]

\[
\lim_{t \to \infty} \frac{h(t)}{t} = \infty \quad \text{if } (J1) \text{ does not hold},
\]

where $U$ is given by Lemma 2.4.

**Proof. Step 1:** The proof of the last inequality of (3.4). For any small $\varepsilon > 0$ and $L > 0$ to be determined later, define

\[
\bar{h}(t) = (1 + \varepsilon)c_0t + L, \quad \bar{u}(t, x) = (1 + \varepsilon)\phi^{c_0}(x - \bar{h}(t)).
\]
We are going to check that there exists $T > 0$ such that

$$
\begin{cases}
\bar{u}_t \geq d \int_0^{	ilde{h}(t)} J(x - y)\bar{u}(t, y)dy - d\bar{u} + f(\bar{u}), & t > 0, \ 0 \leq x < \tilde{h}(t), \\
\bar{u}(t, \tilde{h}(t)) \geq 0, & t > 0, \\
\dot{\tilde{h}}(t) \geq \mu \int_0^{	ilde{h}(t)} \int_{\tilde{h}(t)}^\infty J(x - y)\bar{u}(t, x)dydx, & t > 0, \\
\tilde{h}(0) \geq h(T), \ \bar{u}(0, x) \geq u(T, x), & x \in [0, h(T)].
\end{cases}
$$

(3.6)

Clearly, from a simple comparison argument, we have $\limsup_{t \to \infty} u(t, x) \leq u^*$ uniformly in $\mathbb{R}^+$, which yields that there is $T > 0$ such that $u(t, x) \leq (1 + \varepsilon/2)u^*$ for $t \geq T$ and $x \in \mathbb{R}^+$. Due to $\phi^{c_0}(-\infty) = u^*$, we may let $L$ large sufficiently such that $h(0) = L > h(T)$ and $\bar{u}(0, x) = (1 + \varepsilon)\phi^{c_0}(x - L) \geq (1 + \varepsilon/2)u^* \geq u(T, x)$ for $x \in [0, h(T)]$.

Now we show that the first and third inequations of (3.6) hold. Direct calculations yield

$$
\bar{u}_t = -(1 + \varepsilon)^2 c_0 \phi^{c_0'}(x - \tilde{h}(t)) \geq -(1 + \varepsilon)c_0 \phi^{c_0'}(x - \tilde{h}(t))
$$

$$
= (1 + \varepsilon) \left( d \int_{-\infty}^{\tilde{h}(t)} J(x - y)\phi^{c_0}(y - \tilde{h}(t))dy - d\phi^{c_0}(x - \tilde{h}(t)) + f(\phi^{c_0}(x - \tilde{h}(t))) \right)
$$

$$
\geq d \int_0^{\tilde{h}(t)} J(x - y)\bar{u}(t, y)dy - d\bar{u} + f(\bar{u}).
$$

Moreover,

$$
(1 + \varepsilon)\mu \int_0^{\tilde{h}(t)} \int_{\tilde{h}(t)}^\infty J(x - y)\phi^{c_0}(x - \tilde{h}(t))dydx = (1 + \varepsilon)\mu \int_0^{\tilde{h}(t)} \int_{\tilde{h}(t)}^\infty J(x - y)\phi^{c_0}(x)dydx
$$

$$
\leq (1 + \varepsilon)\mu \int_{-\infty}^\infty \int_0^{\tilde{h}(t)} J(x - y)\phi^{c_0}(x)dydx
$$

$$
= (1 + \varepsilon)c_0 = \tilde{h}'(t).
$$

By virtue of Theorem 3.7, we have

$$
u(t + T, x) \leq \bar{u}(t, x), \ \ h(t + T) \leq \tilde{h}(t) \quad \text{for} \ t \geq 0, \ x \in [0, h(t)].
$$

Thus

$$
\limsup_{t \to \infty} \frac{\tilde{h}(t)}{t} \leq \lim_{t \to \infty} \frac{\tilde{h}(t) - T}{t} = (1 + \varepsilon)c_0.
$$

Letting $\varepsilon \to 0$, we obtain the desired conclusion.

**Step 2: The proof of first inequality of (3.4).** Define

$$
\xi(x) = \begin{cases}
1, & |x| \leq 1, \\
2 - |x|, & 1 \leq |x| \leq 2, \\
0, & |x| \geq 2,
\end{cases}
$$

and $J_n(x) = \xi(\frac{x}{n})J(x)$. Clearly, $J_n$ are supported compactly and nondecreasing in $n$, and $J_n \leq J$.

Also we have $\lim_{n \to \infty} J_n(x) = J(x)$ in $L^1(\mathbb{R})$ and locally uniformly in $\mathbb{R}$. Thus we can choose $n$ large enough, say $n \geq N > 0$, such that $d(\|J_n\|_1 - 1)u + f(u)$ still meets conditions (F1)-(F3).
For any given \( n \geq N \), let \((u_n, h_n)\) be the unique solution of

\[
\begin{cases}
  u_{nt} = d \int_0^{h_n(t)} J_n(x - y)u_n(t, y)dy - du_n + f(u_n), & t > 0, \ 0 \leq x < h_n(t), \\
  u_n(t, h_n(t)) = 0, & t > 0, \\
  h_n'(t) = \mu \int_0^{h_n(t)} \int_{h_n(t)}^{\infty} J_n(x - y)u_n(t, x)dydx, & t > 0, \\
  h_n(0) = h(T), \ u_n(0, x) = u(T, x), & x \in [0, h(T)].
\end{cases}
\]

(3.7)

Since spreading happens for (1.2), we can conclude from Theorem 3.13 that spreading happens for \((u_n, h_n)\) by choosing \( T \) large enough.

1. For any \( n \geq N \), we claim that the semi-wave problem

\[
\begin{cases}
  d \int_{-\infty}^{0} J_n(x - y)\phi_n(y)dy - d\phi_n + c\phi'_n + f(\phi_n) = 0, \ -\infty < x < 0, \\
  \phi_n(-\infty) = u^*_n, \ \phi_n(0) = 0, \ c_n = \mu \int_{-\infty}^{0} \int_{0}^{\infty} J_n(x - y)\phi_n(x)dydx
\end{cases}
\]

(3.8)

has a unique solution pair \((c_n, \phi_n)\) with \( c_n > 0 \) and \( \phi_n \) nonincreasing, where \( u^*_n \) is the unique positive root of equation \( d(\|J_n\|_1 - 1)u + f(u) = 0 \). Clearly, \( u^*_n \leq u^*_{n+1} \leq u^* \) and \( \lim_{n \to \infty} u^*_n = u^* \).

We may rewrite the above semi-wave problem as the equivalent form

\[
\begin{cases}
  d_n \int_{-\infty}^{0} \hat{J}_n(x - y)\phi_n(y)dy - d_n\phi_n + c\phi'_n + d(\|J_n\|_1 - 1)\phi_n + f(\phi_n) = 0, \ -\infty < x < 0, \\
  \phi_n(-\infty) = u^*_n, \ \phi_n(0) = 0, \ c_n = \mu_n \int_{-\infty}^{0} \int_{0}^{\infty} \hat{J}_n(x - y)\phi_n(x)dydx,
\end{cases}
\]

where \( d_n = d(\|J_n\|_1) \), \( \hat{J}_n = J_n/\|J_n\|_1 \) and \( \mu_n = \mu(\|J_n\|_1) \). Since \( J_n \) has a compact support, the condition \((J1)\) holds for \( J_n \). Then our conclusion directly follows from [8, Theorem 1.2].

2. We prove \( \lim_{n \to \infty} c_n = c_0 \) and \( \lim_{n \to \infty} \phi_n = \phi^{c_0} \) locally uniformly in \((-\infty, 0] \). Since \( J_n \) is nondecreasing in \( n \) and \( J_n \leq J \), we can see from [11, Lemma 2.8] that \( c_n \leq c_{n+1} \leq c_0 \) and \( \phi_n \leq \phi_{n+1} \leq \phi^{c_0} \). By the monotonicity and boundedness of \( \phi_n \), there exists \( \phi_\infty \geq \phi_N \) such that \( \phi_n \) converges to \( \phi_\infty \) in \((-\infty, 0] \). Besides, due to \( 0 \leq \phi_n \leq u^* \) and

\[
\sup_{n \geq N} |\phi'_n(x)| \leq \frac{2du^* + \max_{u \in [0, u^*]} f(u)}{c_N} < \infty \text{ in } (-\infty, 0],
\]

in view of a consideration of compactness we have \( \phi_\infty \in C(\mathbb{R}) \) and \( \lim_{n \to \infty} \phi_n = \phi_\infty \) locally uniformly in \((-\infty, 0] \). Let \( c_\infty = \lim_{n \to \infty} c_n \leq c_0 \). Next we show \( c_\infty = c_0 \) and \( \phi_\infty = \phi^{c_0} \).

For any given \( x \in (-\infty, 0) \), it can be seen from (3.8) that

\[
c_n\phi_n(x) - c_n\phi_n(0) = -\int_{0}^{x} \left( d \int_{-\infty}^{0} J_n(z - y)\phi_n(y)dy - d\phi_n(z) + f(\phi_n(z)) \right) dz.
\]

Thanks to the dominated convergence theorem, we have

\[
c_\infty\phi_\infty(x) - c_\infty\phi_\infty(0) = -\int_{0}^{x} \left( d \int_{-\infty}^{0} J(z - y)\phi_\infty(y)dy - d\phi_\infty(z) + f(\phi_\infty(z)) \right) dz.
\]

Therefore,

\[
d \int_{-\infty}^{0} J(x - y)\phi_\infty(y)dy - d\phi_\infty + c_\infty\phi_\infty' + f(\phi_\infty) = 0 \text{ in } (-\infty, 0].
\]
Combining this with $\phi_\infty(-\infty) \geq \phi_N(-\infty) > 0$, we easily derive $\phi_\infty(-\infty) = u^*$. Moreover, by the monotone convergence theorem,

$$c_\infty = \lim_{n \to \infty} c_n = \lim_{n \to \infty} \mu \int_{-\infty}^{0} \int_{0}^{\infty} J_n(x-y)\phi_n(x)dydx = \mu \int_{-\infty}^{0} \int_{0}^{\infty} J(x-y)\phi_\infty(x)dydx.$$

Using [8, Theorem 1.2], we get $c_\infty = c_0$ and $\phi_\infty = \phi^{c_0}$.

(3) Let $U \in C(\mathbb{R}^+)$ be the unique bounded positive solution of (2.3). For $n \geq N$, by Lemma 2, we easily know that the steady state problem

$$d \int_{0}^{\infty} J_n(x-y)U(y)dy - dU + f(U) = 0 \quad \text{in } \mathbb{R}^+$$

has a unique bounded positive solution $U_n \in C(\mathbb{R}^+)$, which is non-decreasing in $\mathbb{R}^+$. Moreover, $0 < U_n < u^*_n$ and $\lim_{x \to \infty} U_n(x) = u^*_n$. Since $J_n \leq J_{n+1}$, by similar considerations with Lemma 2, we have $U_n \leq U_{n+1} \leq U$. Define $U_\infty = \lim_{n \to \infty} U_n$. Then $U_\infty > 0$ and is nondecreasing. By the dominated convergence theorem,

$$d \int_{0}^{\infty} J(x-y)U_\infty(y)dy - dU_\infty + f(U_\infty) = 0 \quad \text{in } \mathbb{R}^+.$$

Then we easily deduce $U_\infty(\infty) = u^*$. By Lemma 2 and Dini’s theorem, $U_\infty \equiv U$ and $\lim_{n \to \infty} U_n = U$ locally uniformly in $\mathbb{R}^+$.

(4) Now we prove the first inequality of (3.4). For $0 < \varepsilon \ll 1$ and $L \gg 1$, we define

$$\mathbf{h}(t) = (1 - \varepsilon)\frac{U_n(0)}{u^*_n}c_nt + 2L, \quad \mathbf{u}(t, x) = (1 - \varepsilon)\frac{U_n(0)}{u^*_n}\phi_n(x - \mathbf{h}(t)).$$

We will check that, for some $T_1 > 0$, $(\mathbf{u}, \mathbf{h})$ satisfies

$$\begin{cases}
\mathbf{u}_t \leq d \int_{0}^{\mathbf{h}(t)} J_n(x-y)\mathbf{u}(t, y)dy - d\mathbf{u} + f(\mathbf{u}), & t > 0, \quad L < x < \mathbf{h}(t), \\
\mathbf{u}(t, \mathbf{h}(t)) \leq 0, & t > 0, \\
\mathbf{h}'(t) \leq \mu \int_{0}^{\mathbf{h}(t)} \int_{\mathbf{h}(t)}^{\infty} J_n(x-y)\mathbf{u}(t, x)dydx, & t > 0, \\
\mathbf{u}(t, x) \leq u_n(t + T_1, x), & t > 0, \quad 0 \leq x \leq L, \\
\mathbf{h}(0) \leq h_n(T_1), \quad \mathbf{u}(0, x) \leq u_n(T_1, x), & x \in [0, \mathbf{h}(0)].
\end{cases} \quad (3.9)$$

As spreading happens for $(u_n, h_n)$, by Theorem [8], there exists a $T_1 > 0$ such that $h_n(T_1) > 2L = \mathbf{h}(0)$, and $\mathbf{u}(t, x) \leq (1 - \varepsilon)U_n(0) \leq (1 - \varepsilon)U_n(x) \leq u_n(t + T_1, x)$ for $t \geq 0$ and $x \in [0, 2L]$ which implies $\mathbf{u}(0, x) \leq u_n(T_1, x)$ in $[0, \mathbf{h}(0)]$. Since $J_n(x) = 0$ for $|x| \geq L$, we have

$$\mu \int_{0}^{\mathbf{h}(t)} \int_{\mathbf{h}(t)}^{\infty} J_n(x-y)\mathbf{u}(t, x)dydx = \mu (1 - \varepsilon)\frac{U_n(0)}{u^*_n} \int_{0}^{\mathbf{h}(t)} \int_{\mathbf{h}(t)}^{\infty} J_n(x-y)\phi_n(x - \mathbf{h}(t))dydx$$

$$= \mu (1 - \varepsilon)\frac{U_n(0)}{u^*_n} \int_{-\mathbf{h}(t)}^{0} \int_{0}^{\infty} J_n(x-y)\phi_n(x)dydx$$

$$= \mu (1 - \varepsilon)\frac{U_n(0)}{u^*_n} \int_{-\mathbf{h}(t)}^{0} \int_{0}^{\infty} J_n(x-y)\phi_n(x)dydx$$
Moreover, by denoting \( z := z(x,t) = x - h(t) \), we have

\[
\dot{u}_n = - (1 - \varepsilon)^2 \left[ \frac{U_n(0)}{u_n^*} \right]^2 c_n \phi_n'(z) \leq - (1 - \varepsilon) \frac{U_n(0)}{u_n^*} c_n \phi_n'(z)
\]

\[
= (1 - \varepsilon) \frac{U_n(0)}{u_n^*} \left( d_n \int_{-\infty}^{h(t)} \hat{J}_n(x-y) \phi_n(z(y,t))dy - d_n \phi_n(z) + d(\|J_n\|_1 - 1) \phi_n(z) \right) \\
+ (1 - \varepsilon) f(\phi_n(z)) \\
\leq d \int_{0}^{h(t)} J_n(x-y) u(t,y)dy - du + d(\|J_n\|_1 - 1)u + f(u).
\]

Hence, (3.9) holds. Applying Theorem 3.8 to (3.7) and (3.9), we derive (Spreading-vanishing criteria).

By the expression of \( h(t) \) and arbitrariness of \( \varepsilon \), it follows that

\[
\liminf_{t \to \infty} \frac{h(t)}{t} \geq \frac{U_n(0)}{u_n^*} c_n.
\]

Noticing \( \lim_{n \to \infty} (U_n, u_n^*, c_n) = (U, u^*, c_0) \), by \( n \to \infty \) in (3.10) we deduce the desired result.

**Step 3:** The proof of (3.5). From the arguments of Step 2 we see that (3.10) still holds and \( \lim_{n \to \infty} (U_n, u_n^*) = (U, u^*) \) when the condition (J1) does not hold. Similar to the proof of conclusion (2) (ii) of [12 Proposition 5.1] we can show \( \lim_{n \to \infty} c_n = \infty \) (the details are omitted here). Hence, by (3.10), the conclusion (3.5) holds.

## 4 Dynamics of the problem (1.3)

Similarly to Section 3, one can directly derive analogous results on solution \((u,h)\) of problem (1.3). So we just give these main conclusions, and omit the details of their proofs. Besides, we will pay our attention to the spreading speed for (1.3).

**Theorem 4.1** (Existence and uniqueness). Problem (1.3) has a unique global solution \((u,h)\). Moreover, for any \( T > 0, u \in C(\mathbb{R}^+), h \in C^1([0,T]), \) and \( 0 \leq u \leq \max \{\|u_0\|_\infty, K\} \).

**Theorem 4.2** (Spreading-vanishing dichotomy). One of the following alternatives holds for (1.3):

1. **Spreading:** \( h_\infty = \infty \) and \( \lim_{t \to \infty} u = u^* \) locally uniformly in \( \mathbb{R}^+ \);
2. **Vanishing:** \( h_\infty < \infty \) and \( \lim_{t \to \infty} \|u(t,\cdot)\|_{C([0,h(t)])} = 0 \).

**Theorem 4.3** (Spreading-vanishing criteria). Let \((u,h)\) be the unique solution of (1.3).

1. If \( f'(0) \geq d/2 \), then spreading happens;
2. If \( f'(0) < d/2 \), then there exists a unique \( \ell^*_N > 0 \) such that spreading happens when \( h_0 \geq \ell^*_N \);
3. If \( f'(0) < d/2 \) and \( h_0 < \ell^*_N \), then there exists \( \mu^*_N > 0 \) such that spreading happens if and only if \( \mu > \mu^*_N \), where \( \ell^*_N \) is determined by similar arguments with \( \ell^*_D \) in Section 3. Indeed, it follows from Lemma 2.2 and \( f'(0) < d/2 \) that there exists a unique \( \ell^*_N > 0 \) such that \( \lambda_p(L^0_{(0,\ell^*_N)} + f'(0)) = 0 \) and \( \lambda_p(C^N_{(0,l)} + f'(0))(l - \ell^*_N) > 0 \) for all \( l > 0 \).
Now we study the spreading speed of problems (1.3). The following theorem will be proved by several lemmas.

**Theorem 4.4** (Spreading speed). Let (F1)-(F3) hold and spreading happen for (1.3). Then

\[
\lim_{t \to \infty} \frac{h(t)}{t} = \begin{cases} 
  c_0 & \text{if (J1) is satisfied}, \\
  \infty & \text{if (J1) is not satisfied},
\end{cases}
\]

where \(c_0\) is uniquely given by the semi-wave problem (3.3).

**Lemma 4.5.** Under the same assumptions with Theorem 4.4, if the condition (J1) holds then

\[
\liminf_{t \to \infty} \frac{h(t)}{t} \geq c_0.
\]  
(4.1)

**Proof.** This proof is similar to Step 2 in the proof of Theorem 3.15, but some obvious and crucial modifications are need. So we give the sketch.

**Step 1:** Define \(J_n(x)\) as in the proof of Theorem 3.15. Clearly,

\[
j_n(x) := \int_{-x}^{\infty} J_n(y)dy \to j(x) = \int_{-x}^{\infty} J(y)dy \quad \text{uniformly for } x \in \mathbb{R} \text{ as } n \to \infty.
\]

Moreover, there is \(0 < \delta_0 \ll 1\) such that \(\tilde{f}(u) := -\delta u + f(u)\) satisfies conditions (F1)-(F3) for all \(\delta \in (0, \delta_0)\), and has a unique positive zero \(u^*_\delta\). For such \(\delta\), we can choose \(n\) large enough, say \(n \geq N_1 \) > 0, such that \(d(||J_n||_1 - 1)u + \tilde{f}(u)\) still meets conditions (F1)-(F3) and

\[
d(j_n(x) - j(x)) + \delta \geq 0 \quad \text{for all } x \in \mathbb{R}.
\]  
(4.2)

For any given \(n \geq N_1\), let \((u_n, h_n)\) be the unique solution of

\[
\begin{align*}
  u_{nt} &= d \int_0^{h_n(t)} J_n(x-y)u_n(t,y)dy - d j_n(x)u_n + \tilde{f}(u_n), \quad t > 0, \ 0 \leq x < h_n(t), \\
  u_n(t,h_n(t)) &= 0, \quad t > 0, \\
  h_n'(t) &= \mu \int_0^{h_n(t)} \int_{h_n(t)}^{\infty} J_n(x-y)u_n(t,x)dy dx, \quad t > 0, \\
  h_n(0) &= h(T), \quad u_n(0, x) = u(T, x), \quad x \in [0, h(T)].
\end{align*}
\]

Analogously, we can conclude from Theorem 4.4 that spreading happens for \((u_n, h_n)\) by choosing \(T\) large enough.

**Step 2:** Similar to the proof of Theorem 3.15 ((1) of Step 2), the semi-wave problem

\[
\begin{align*}
  d \int_{-\infty}^{0} J_n(x-y)\phi_n(y)dy - d \phi_n + c \phi_n' + \tilde{f}(\phi_n) &= 0, \quad -\infty < x < 0, \\
  \phi_n(-\infty) &= u^*_{n,\delta}, \quad \phi_n(0) = 0, \quad c_n = \mu \int_{-\infty}^{0} \int_{0}^{\infty} J_n(x-y)\phi_n(x)dy dx
\end{align*}
\]

has a unique solution pair \((c_n, \phi_n)\) with \(c_n > 0\) and \(\phi_n\) nonincreasing, where \(u^*_{n,\delta}\) is the unique positive root of \(d(||J_n||_1 - 1)u + \tilde{f}(u) = 0\). Clearly, \(u^*_{n,\delta} \leq u^*_{n+1,\delta} \leq u^*_\delta\) and \(\lim_{n \to \infty} u^*_{n,\delta} = u^*_\delta\).  

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**Step 3:** By the same lines in the proof of Theorem 3.15 ((2) of Step 2), we have \( \lim_{n \to \infty} c_n = c_0^\delta \) and \( \lim_{n \to \infty} \phi_n = \phi^\delta \) locally uniformly in \( (-\infty, 0] \), where \((c_0^\delta, \phi^\delta)\) is the unique solution pair of

\[
\begin{cases}
    d \int_{-\infty}^{0} J(x-y) \phi(y) dy - d\phi + c\phi' + \bar{f}(\phi) = 0, & -\infty < x < 0, \\
    \phi(-\infty) = u^*_n, & \phi(0) = 0, \\
    c = \mu \int_{-\infty}^{0} \int_{0}^{\infty} J(x-y) \phi(x) dy dx.
\end{cases}
\]

(4.3)

**Step 4:** We now prove that \( c_0^\delta \to c_0 \) and \( \phi^\delta \to \phi^{\delta_0} \) as \( \delta \to 0 \), where \((c_0, \phi^{\delta_0})\) is given by (3.8). Let \( \delta_n \) be a sequence decreasing to 0, and \((c_0^{\delta_n}, \phi^{\delta_n})\) be the unique solution pair of (4.3) with \( \delta \) replaced by \( \delta_n \). With the similar arguments in the proof of [9] Lemma 2.8, we have \( c_0^{\delta_n} \leq c_0^{\delta_{n+1}} \leq c_0 \) and \( \phi^{\delta_n} \leq \phi^{\delta_{n+1}} \leq \phi^{\delta_0} \). Then by arguing as in (2) of Step 2 in the proof of Theorem 3.15 we can obtain the desired result.

**Step 5:** We are going to prove \( \liminf_{t \to \infty} \frac{h_n(t)}{t} \geq c_n \). For any given \( 0 < \varepsilon \ll 1 \) and \( L \) large enough such that \( J_n(x) = 0 \) for \( |x| \geq L \), we define

\[ h(t) = c_n(1-\varepsilon)t + 2L, \quad u(t, x) = (1-\varepsilon)\phi_n(x - h(t)). \]

Analogously we easily examine that there exists \( T_1 > 0 \) such that

\[
\begin{align*}
    u_t &\leq d \int_{0}^{\frac{h(t)}{t}} J_n(x-y) u(t, y) dy - d\mu_j(x) u + \tilde{f}(u), \quad t > 0, \quad L < x < h(t), \\
    u(t, h(t)) &\leq 0, \quad t > 0, \\
    h'(t) &\leq \mu \int_{0}^{\frac{h(t)}{t}} \int_{h(t)}^{\infty} J_n(x-y) u(t, x) dy dx, \quad t > 0, \\
    u(t, x) &\leq u_n(t + T_1, x), \quad t > 0, \quad 0 \leq x \leq L, \\
    h(0) &\leq h_n(T_1), \quad u(0, x) \leq u_n(T_1, x), \quad x \in [0, h(0)].
\end{align*}
\]

By Theorem 3.8 we can get the desired result.

**Step 6:** Now we prove (4.1). Making use of \( J_n \leq J \) and (4.2) we have

\[ u_t = d \int_{0}^{\frac{h(t)}{t}} J(x-y) u(t, y) dy - d\mu j(x) u + f(u) \]

\[ = d \int_{0}^{\frac{h(t)}{t}} J(x-y) u(t, y) dy - d(1-j(x)) u + f(u) \]

\[ \geq d \int_{0}^{\frac{h(t)}{t}} J_n(x-y) u(t, y) dy - d(1-j_n(x)) u + \tilde{f}(u). \]

On the other hand,

\[ u_{nt} = d \int_{0}^{\frac{h_n(t)}{t}} J_n(x-y) u_n(t, y) dy - d\mu_j(x) u_n + \tilde{f}(u_n) \]

\[ = d \int_{0}^{\frac{h_n(t)}{t}} J_n(x-y) u_n(t, y) dy - d u_n + d(1-j_n(x)) u_n + \tilde{f}(u_n). \]

By the comparison principle (Theorem 3.7),

\[ u_n(t, x) \leq u(t + T, x), \quad h_n(t) \leq h(t + T) \] for \( t \geq 0, \ x \in [0, h_n(t)] \).

Combining this with the results in Steps 3, 4 and 5 allows us to derive (4.1).
Lemma 4.6. Under assumptions of Theorem 4.4, if the condition (J1) holds then we have
\[
\limsup_{t \to \infty} \frac{h(t)}{t} \leq c_0.
\]

Proof. This lemma can be proved by similar arguments in Step 1 of Theorem 4.4 and thus we only give the difference. For any small \( \varepsilon > 0 \) and \( L > 0 \) to be determined later, define
\[
\tilde{h}(t) = (1 + \varepsilon)c_0 t + L, \quad \tilde{u}(t, x) = (1 + \varepsilon)\phi^{\varepsilon}(x - \tilde{h}(t)).
\]

We are going to check that there exists \( T > 0 \) such that
\[
\begin{align*}
\tilde{u}_t &\geq d \int_0^{\tilde{h}(t)} J(x - y) \tilde{u}(t, y) dy - dj(x) \tilde{u} + f(\tilde{u}), \quad t > 0, \quad 0 \leq x < \tilde{h}(t), \\
\tilde{u}(t, \tilde{h}(t)) &\geq 0, \quad t > 0, \\
\tilde{h}'(t) &\geq \mu \int_0^{\tilde{h}(t)} \int_{\tilde{h}(t)}^\infty J(x - y) \tilde{u}(t, x) dy dx, \quad t > 0, \\
\tilde{u}(0, x) &\geq u(T, x), \quad \tilde{h}(0) \geq h(T), \quad x \in [0, h(T)].
\end{align*}
\]

We only show the first inequality of (4.4). Since \( \phi \) is nonincreasing in \((-\infty, 0]\), one has
\[
\begin{align*}
\tilde{u}_t &= -(1 + \varepsilon)^2 c_0 \phi^{\varepsilon}(x - \tilde{h}(t)) - (1 + \varepsilon)c_0 \phi^{\varepsilon}(x - \tilde{h}(t)) \\
&= (1 + \varepsilon) \left( d \int_{-\infty}^{\tilde{h}(t)} J(x - y) \phi^{\varepsilon}(y - \tilde{h}(t)) dy - d\phi^{\varepsilon}(x - \tilde{h}(t)) + f(\phi^{\varepsilon}(x - \tilde{h}(t))) \right) \\
&\geq d \int_0^{\tilde{h}(t)} J(x - y) \tilde{u}(t, y) dy - dj(x) \tilde{u} + d \int_0^{\tilde{h}(t)} J(x - y) \tilde{u}(t, y) dy \\
&\quad + d \int_{-\infty}^{\tilde{h}(t)} J(x - y) dy \tilde{u}(t, x) - d\tilde{u}(t, x) \\
&= d \int_0^{\tilde{h}(t)} J(x - y) \tilde{u}(t, y) dy - dj(x) \tilde{u} + d \int_0^{\tilde{h}(t)} J(x - y) [\tilde{u}(t, y) - \tilde{u}(t, x)] dy \\
&\geq d \int_0^{\tilde{h}(t)} J(x - y) \tilde{u}(t, y) dy - dj(x) \tilde{u} + f(\tilde{u}).
\end{align*}
\]

By virtue of Theorem 3.7, \( h(t + T) \leq \tilde{h}(t) \) for \( t \geq 0 \). Thus
\[
\limsup_{t \to \infty} \frac{h(t)}{t} \leq \lim_{t \to \infty} \frac{\tilde{h}(t - T)}{t} = (1 + \varepsilon)c_0.
\]

Letting \( \varepsilon \to 0 \), we complete the proof. \( \square \)

Lemma 4.7. Under assumptions of Theorem 4.4, if the condition (J1) is not true then
\[
\lim_{t \to \infty} \frac{h(t)}{t} = \infty.
\]

Proof. The proof is similar to Step 3 in the proof of Theorem 3.15 and we omit the details here. \( \square \)

Theorem 4.4 can be directly deduced from Lemmas 4.5, 4.6 and 4.7.
5 The dynamics of problem (1.4)

Now we are in a position to study the dynamics of problem (1.4). Firstly, by following the analogous lines of [7, Theorem 2.1] we can obtain the following well-posedness result.

**Theorem 5.1 (Existence and uniqueness).** Problem (1.4) has a unique global solution \((u_1, u_2, h)\). Moreover, \((u_1, u_2) \in [C(T)^2], h \in C^1([0, T]), 0 < u_1 \leq A_1 := \max\{\|u_{10}\|_\infty, a_1/b_1\} \) and \(0 < u_2 \leq \max\{\|u_{20}\|_\infty, (a_2 + c_2A_1)/b_2\}\) in \(D^T_h\) for any \(T > 0\).

**Lemma 5.2.** Suppose that \(J_i\) satisfy the condition (J) and \(J_i > 0\) in \(\mathbb{R}\). If \(h_\infty < \infty\), then
\[
\lim_{t \to \infty} \|u_i(t, \cdot)\|_{C([0, h(t))]} = 0, \quad \lambda_p(L^{d_i}_{(0, h_\infty)} + a_i) \leq 0.
\]

**Proof.** This lemma can be established by the approach in the proof of [7, Theorem 3.3] with some obvious modifications, and we omit the details here. \(\square\)

**Lemma 5.3.** Assume that \(a_1b_1b_2 > a_2b_1c_1 + a_1c_1c_2\). If \(h_\infty = \infty\), then there exist four positive functions \(\bar{u}_i\) and \(\underline{u}_i\), \(i = 1, 2\), such that
\[
\begin{align*}
\underline{u}_1(x) & \leq \liminf_{t \to \infty} u_1(t, x) \leq \limsup_{t \to \infty} u_1(t, x) \leq \bar{u}_1(x) \quad \text{locally uniformly in } \mathbb{R}^+, \\
\underline{u}_2(x) & \leq \liminf_{t \to \infty} u_2(t, x) \leq \limsup_{t \to \infty} u_2(t, x) \leq \bar{u}_2(x) \quad \text{locally uniformly in } \mathbb{R}^+.
\end{align*}
\]

**Proof.** Step 1: Let \(w\) be the unique solution of
\[
\begin{aligned}
w_t &= d_1 \int_0^\infty J_1(x-y)w(t, y)dy - d_1w + w(a_1 - b_1w), \quad t > 0, \quad x \geq 0, \\
w(0, x) &= u_{10}(x), \quad 0 \leq x < h_0; \quad w(0, x) = 0, \quad x \geq h_0.
\end{aligned}
\]
(5.1)
The comparison principle gives \(u_1 \leq w\) in \(D^\infty_h\). On the other hand, in view of Lemma 2.4, problem (5.1) has a unique bounded positive steady state \(\bar{u}_1\) with \(0 < \bar{u}_1 < a_1/b_1\) such that \(\lim_{t \to \infty} w(t, x) = \bar{u}_1(x)\) locally uniformly in \(\mathbb{R}^+\). Hence \(\limsup_{t \to \infty} u_1(t, x) \leq \bar{u}_1(x)\) locally uniformly in \(\mathbb{R}^+\).

Step 2: Obviously, \(a_2 + c_2\bar{u}_1\) satisfies the condition (K) and \(\limsup_{t \to \infty}[a_2 + c_2u_1(t, x)] \leq a_2 + c_2\bar{u}_1(x)\) locally uniformly in \(\mathbb{R}^+\). By Lemma 2.11 \(\limsup_{t \to \infty} u_2(t, x) \leq \bar{u}_2(x)\) locally uniformly in \(\mathbb{R}^+\), where \(\bar{u}_2(x)\) is a unique bounded positive solution of
\[
d_2 \int_0^\infty J_2(x-y)u(y)dy - d_2u + u(a_2 + c_2\bar{u}_1 - b_2u) = 0 \quad \text{in } \mathbb{R}^+.
\]

Step 3: It can be learned from Lemmas 2.4 and 2.7 that \(\bar{u}_2 \leq (a_2b_1 + c_2a_1)/b_1b_2\) in \(\mathbb{R}^+\) and \(\lim_{x \to \infty} \bar{u}_2(x) = (a_2b_1 + c_2a_1)/b_1b_2\). Together with the assumption \(a_1b_1b_2 > a_2b_1c_1 + a_1c_1c_2\), we can find \(\sigma_1 > 0\) such that \(a_1 - c_1\bar{u}_2 + \sigma_1\) has a positive lower bound \(\bar{\beta}_0\) in \(\mathbb{R}^+\). Thus there exists \(L > 0\) such that, for any \(l > L\), \(\lambda_p(L^{d_{(0, \bar{\beta}_0)}} + \bar{\beta}_0) > 0\). By similar arguments in the proof of Lemma 2.6 we easily see that, for any \(\sigma \in (0, \sigma_1)\), the problem
\[
d_1 \int_0^l J_1(x-y)u(y)dy - d_1u + u(a_1 - c_1(\bar{u}_2(x) + \sigma) - b_1u) = 0 \quad \text{in } [0, l]
\]
has a unique positive solution, denoted by \(u_l\).
Moreover, for any \( l > L \) and \( \sigma \in (0, \sigma_1) \), there exists \( T > 0 \) such that \( h(t) > l \) and \( u_2(t, x) \leq \bar{u}_2(x) + \sigma \) for \( t \geq T \) and \( 0 \leq x \leq l \). Let \( u \) be the unique solution of

\[
\begin{aligned}
    \begin{cases}
    u = d_1 \int_0^1 J_1(x - y) u(t, y) dy - d_1 u + u(a_1 - c_1(\bar{u}_2(x) + \sigma) - b_1 u), & t > T, 0 \leq x \leq l, \\
    u(T, x) = u(T, x), & 0 \leq x \leq l.
    \end{cases}
\end{aligned}
\]

It is easy to show that \( \lim_{t \to \infty} u(t, x) = u(x) \) uniformly in \([0, l]\) as \( t \to \infty \) and \( \lim_{t \to \infty} u = u^* \) locally uniformly in \( \mathbb{R}^+ \), where \( u^* \) is the unique bounded positive solution of

\[
    d_1 \int_0^\infty J_1(x - y) u(y) dy - d_1 u + u(a_1 - c_1(\bar{u}_2(x) + \sigma) - b_1 u) = 0 \quad \text{in} \quad \mathbb{R}^+.
\]

In view of Remark \ref{rmk:5.4} \( \lim_{\sigma \to 0} u^*_1 = u_1 \), where \( u_1 \) is the unique bounded positive solution of

\[
    d_1 \int_0^\infty J_1(x - y) u(y) dy - d_1 u + u(a_1 - c_1 u_2(x) - b_1 u) = 0 \quad \text{in} \quad \mathbb{R}^+.
\]

By the comparison principle, \( u_1 \geq u \) for \( t > T \) and \( x \in [0, l] \). Therefore, \( \liminf_{t \to \infty} u_1(t, x) \geq u_1(x) \) locally uniformly in \( \mathbb{R}^+ \).

Similarly, we can also show that \( \liminf_{t \to \infty} u_2(t, x) \geq u_2(x) \) locally uniformly in \( \mathbb{R}^+ \), where \( u_2 \) is a unique bounded positive solution of

\[
    d_2 \int_0^\infty J_2(x - y) u(y) dy - d_2 u + u(a_2 + c_2 u_1(x) - b_2 u) = 0 \quad \text{in} \quad \mathbb{R}^+.
\]

The proof is complete. \( \square \)

Next we always suppose that \( J_i \) satisfy the condition \((J)\) and \( J_i > 0 \) in \( \mathbb{R} \) for \( i = 1, 2 \), and then give the criteria to determine whether \( h_\infty = \infty \). The following conclusions can be proved by similar approaches in \cite{7}, and only some minor modifications are needed. Firstly, by Lemma \ref{lem:5.2} we have \( h_\infty = \infty \) if either \( a_1 \geq d_1 \) or \( a_2 \geq d_2 \).

Assume that \( a_i < d_i \) for \( i = 1, 2 \), we easily see that there exist unique \( \ell^*_i > 0 \) such that \( \lambda_p(\mathcal{L}_{(0, \ell^*_i)}^{(c_i, \eta_i)}) + a_i = 0 \) and \( \lambda_p(\mathcal{L}_{(0, \ell^*_i)}^{(c_i, \eta_i)}) (l - \ell^*_i) > 0 \) for all \( l > 0 \). Hence, if \( h_0 \geq \ell^* := \min\{\ell^*_1, \ell^*_2\} \), we must have \( h_\infty = \infty \). Also we obtain that \( h_\infty < \infty \) implies \( h_\infty \leq \ell^* \).

Suppose that \( a_i < d_i \) for \( i = 1, 2 \) and \( h_0 < \ell^* \). It can be analogously derived from Lemma \ref{lem:3.10} that there exists \( \mu^* > 0 \) such that \( h_\infty = \infty \) when \( \mu_1 + \mu_2 \geq \mu^* \); Moreover, similarly, to the proofs of Lemma \ref{lem:3.9} and \cite{7} Lemma 3.6, we can find \( \mu_\ast > 0 \) such that \( h_\infty < \infty \) when \( \mu_1 + \mu_2 < \mu_\ast \).

To sum up, we obtain the following result concerned with when \( h_\infty < \infty \).

**Theorem 5.4.** Suppose that \( J_i \) satisfy the condition \((J)\) and \( J_i > 0 \) in \( \mathbb{R} \) for \( i = 1, 2 \). Let \((u_1, u_2, h)\) be the unique solution of \((1.3)\). Then the followings hold:

1. if \( a_1 \geq d_1 \) or \( a_2 \geq d_2 \), then \( h_\infty = \infty \);
2. if \( a_i < d_i \) for \( i = 1, 2 \), then there exists a unique \( \ell^* > 0 \) such that \( h_\infty = \infty \) when \( h_0 \geq \ell^* \);
3. if \( a_i < d_i \) for \( i = 1, 2 \) and \( h_0 < \ell^* \), then there exists \( \mu^* > \mu_\ast > 0 \) such that \( h_\infty = \infty \) when \( \mu_1 + \mu_2 > \mu^* \), and \( h_\infty < \infty \) when \( \mu_1 + \mu_2 \leq \mu_\ast \).

We now discuss the spreading speed and accelerated spreading for \((1.3)\). To this end, we first give a comparison principle which can be proved by similar methods in \cite{11} Theorem 3.1 and \cite{6} Lemma 4.1. So the details are omitted here.
Theorem 5.5. Let $J_i$ satisfy the condition (J) for $i = 1, 2$. If for any $T > 0$, $\bar{h} \in C^1([0, T])$, $\bar{u}_i, \bar{u}_{it} \in C(D_{h}^{\bar{T}})$ and satisfy
\[
\begin{aligned}
\bar{u}_{1t} &\geq d_1 \int_0^{\bar{h}(t)} J_1(x-y)\bar{u}_1(t,y)dy - d_1 \bar{u}_1 + \bar{u}_i(a_1 - b_1 \bar{u}_1), & 0 < t \leq T, & 0 \leq x < \bar{h}(t), \\
\bar{u}_{2t} &\geq d_2 \int_0^{\bar{h}(t)} J_2(x-y)\bar{u}_2(t,y)dy - d_2 \bar{u}_2 + \bar{u}_2(a_2 - b_2 \bar{u}_2 + c_2 \bar{u}_1), & 0 < t \leq T, & 0 \leq x < \bar{h}(t), \\
u_i(t, \bar{h}(t)) &\geq 0, & 0 < t \leq T, \\
\bar{h}'(t) &\geq \sum_{i=1}^{2} \mu_i \int_0^{\bar{h}(t)} \int_{h(t)}^\infty J_i(x-y)\bar{u}_i(t,x)dydx, & 0 < t \leq T, \\
\bar{h}(0) &\geq h_0, & \bar{u}_i(0, x) \geq u_0(x), & x \in [0, h_0],
\end{aligned}
\]
then the solution $(u_1, u_2, h)$ of (1.4) satisfies
\[
u_1(t, x) \leq \bar{u}_1(t, x), \quad u_2(t, x) \leq \bar{u}_2(t, x), \quad h(t) \leq \bar{h}(t) \quad \text{for} \quad t \geq 0, \quad x \in [0, h(t)].
\]

Theorem 5.6. Under the same assumptions with Theorem 5.4, the accelerate spreading happens for (1.4) if one of the followings holds:

1. $J_1$ violates (J1) and $a_1 b_1 b_2 > a_2 b_1 c_1 + a_1 c_1 c_2$;
2. $J_2$ does not satisfy (J1).

Proof. We only prove the conclusion (1) since the conclusion (2) can be proved by analogous methods. By simple comparison arguments, one easily derive that $\limsup_{t \to \infty} u_1(t, x) \leq a_1/b_1$ and $\limsup_{t \to 0} u_2(t, x) \leq (a_2 b_1 + a_1 c_2)/b_1 b_2$ uniformly in $[0, \infty)$. Due to $a_1 b_1 b_2 > a_2 b_1 c_1 + a_1 c_1 c_2$, for any small $\varepsilon > 0$ with $a_1 b_1 b_2 > a_2 b_1 c_1 + a_1 c_1 c_2 + b_1 b_2 \varepsilon$, we can find some $T > 0$ such that
\[
u_2(t, x) \leq \frac{a_2 b_1 + a_1 c_2}{b_1 b_2} + \varepsilon \quad \text{for} \quad t \geq T, \quad x \in \mathbb{R}^+.
\]
Then $(u_1, h)$ satisfies
\[
\begin{aligned}
u_{1t}(t, x) &\geq d_1 \int_0^{h(t)} J_1(x-y)\nu_1(t,y)dy - d_1 \nu_1 \\
&\quad + \nu_1 \left[a_1 - c_1 \left(\frac{a_2 b_1 + a_1 c_2}{b_1 b_2} + \varepsilon\right) - b_1 \nu_1\right], & t > T, & 0 \leq x < h(t), \\
u(t, h(t)) &\geq 0, & t > T, \\
h'(t) &\geq \mu_1 \int_0^{h(t)} \int_{h(t)}^\infty J_1(x-y)\nu_1(t,x)dydx, & t > T, \\
u_1(T, x) &\geq u_1(T, x), & x \in [0, h(T)].
\end{aligned}
\]
Let $(\nu, h)$ be the unique solution of
\[
\begin{aligned}
u = d_1 \int_0^{h(t)} J_1(x-y)\nu(t,y)dy - d_1 \nu \\
&\quad + \nu \left[a_1 - c_1 \left(\frac{a_2 b_1 + a_1 c_2}{b_1 b_2} + \varepsilon\right) - b_1 \nu\right], & t > 0, & 0 \leq x < h(t), \\
u(t, h(t)) &\geq 0, & t > 0, \\
h'(t) &\geq \mu_1 \int_0^{h(t)} \int_{h(t)}^\infty J_1(x-y)\nu(t,x)dydx, & t > 0, \\
h(0) &\geq h(T_1), & \nu(0, x) = u_1(T_1, x), & x \in [0, h(T_1)].
\end{aligned}
\]
Since \( h_\infty = \infty \), from Theorem 3.13 we may choose \( T_1 > T \) large enough such that \( h(t) = \infty \). By comparison principle, we have that \( h(t) \leq h(t + T_1) \). Noting that \( J_1 \) violates \((J1)\), owing to Theorem 3.15 \( \lim_{t \to \infty} \frac{h(t)}{t} = \infty \). So \( \lim_{t \to \infty} \frac{h(t)}{t} = \infty \). \( \square \)

Next we show that accelerated spreading cannot occur if \( J_1 \) and \( J_2 \) satisfy \((J1)\).

**Theorem 5.7.** Let \( J_i \) satisfy the condition \((J1)\) for \( i = 1, 2 \). If \( h_\infty = \infty \), then

\[
\limsup_{t \to \infty} \frac{h(t)}{t} \leq c_1 + c_2, \tag{5.2}
\]

where \( c_i \) for \( i = 1, 2 \) are determined by the following two semi-wave problems respectively,

\[
\begin{align*}
\phi_1(-\infty) &= \frac{a_1}{b_1}, \quad \phi_1(0) = 0, \quad c_1 = \mu_1 \int_0^\infty \int_{-\infty}^\infty J_1(x-y)\phi_1(x)dydx, \\
\phi_2(-\infty) &= \frac{a_2b_1 + a_1c_2}{b_1b_2}, \quad \phi_2(0) = 0, \quad c_2 = \mu_2 \int_0^\infty \int_{-\infty}^\infty J_2(x-y)\phi_2(x)dydx.
\end{align*}
\]

**Proof.** Denote the unique solution pairs of the above two problems by \((c_1, \phi_1)\) and \((c_2, \phi_2)\) respectively. For small \( \varepsilon > 0 \) and \( L > 0 \) to be determined later, we define

\[
\bar{h}(t) = (1 + \varepsilon)(c_1 + c_2)t + L, \quad \bar{u}_1 = (1 + \varepsilon)\phi_1(x - \bar{h}(t)), \quad \bar{u}_2 = (1 + \varepsilon)\phi_2(x - \bar{h}(t)).
\]

We show that there exists \( T > 0 \) such that \((\bar{u}_1, \bar{u}_2, \bar{h})\) satisfies

\[
\begin{align*}
\bar{u}_{1t} &\geq d_1 \int_0^{\bar{h}(t)} J_1(x-y)\bar{u}_1(t,y)dy - d_1 \bar{u}_1 + \bar{u}_1(a_1 - b_1 \bar{u}_1), \quad t > 0, \quad 0 \leq x < \bar{h}(t), \\
\bar{u}_{2t} &\geq d_2 \int_0^{\bar{h}(t)} J_2(x-y)\bar{u}_2(t,y)dy - d_2 \bar{u}_2 + \bar{u}_2(a_2 - b_2 \bar{u}_2 + c_2 \bar{u}_1), \quad t > 0, \quad 0 \leq x < \bar{h}(t), \\
u_i(t, \bar{h}(t)) &\geq 0, \quad t > 0, \tag{5.3}
\end{align*}
\]

\[
\bar{h}(0) \geq h(T), \quad \bar{u}_i(0, x) \geq u_i(T, x), \quad x \in [0, h(T)].
\]

Firstly, there is \( T > 0 \) such that

\[
u_i(t, x) \leq (1 + \frac{\varepsilon}{2}) \frac{a_1}{b_1}, \quad u_2(t, x) \leq (1 + \frac{\varepsilon}{2}) \frac{a_2b_1 + a_1c_2}{b_1b_2} \quad \text{for} \quad t \geq T, \quad x \in \mathbb{R}^+.
\]

We choose \( L > 0 \) large sufficiently so that

\[
\bar{u}_1(0, x) = (1 + \varepsilon)\phi_1(x - L) \geq (1 + \frac{\varepsilon}{2}) \frac{a_1}{b_1} \geq u_1(T, x) \quad \text{in} \quad [0, h(T)],
\]

and

\[
\bar{u}_2(0, x) = (1 + \varepsilon)\phi_2(x - L) \geq (1 + \frac{\varepsilon}{2}) \frac{a_2b_1 + a_1c_2}{b_1b_2} \geq u_2(T, x) \quad \text{in} \quad [0, h(T)].
\]
Moreover, by denoting \( z := z(t, x) = x - \bar{h}(t) \), we have
\[
\bar{u}_{1t} = -(1 + \varepsilon)^2(c_1 + c_2)\phi'_1(z) \geq -(1 + \varepsilon)c_1\phi'_1(z) \\
= (1 + \varepsilon) \left( d_1 \int_{-\infty}^{\bar{h}(t)} J_1(x-y)\phi_1(z(t,y))dy - d_1\phi_1(z) + \phi(z)(a_1 - b_1\phi_1(z)) \right) \\
\geq d_1 \int_{0}^{\bar{h}(t)} J_1(x-y)\bar{u}_1(t,y)dy - d_1\bar{u}_1 + \bar{u}_1 (a_1 - b_1\bar{u}_1),
\]
and
\[
\bar{u}_{2t} = -(1 + \varepsilon)^2(c_1 + c_2)\phi'_2(z) \geq -(1 + \varepsilon)c_2\phi'_2(z) \\
\geq (1 + \varepsilon) \left( d_2 \int_{-\infty}^{\bar{h}(t)} J_2(x-y)\phi_2(z(t,y))dy - d_2\phi_2(z) + \phi_2(z)(a_2 + c_2\phi_1(z) - b_2\phi_2(z)) \right) \\
\geq d_2 \int_{0}^{\bar{h}(t)} J_2(x-y)\bar{u}_2(t,y)dy - d_2\bar{u}_2 + \bar{u}_2 (a_2 - b_2\bar{u}_2 + c_2\bar{u}_1).
\]
Clearly it remains to show fourth inequality in (5.3). In fact,
\[
\sum_{i=1}^{2} \mu_i \int_{0}^{\bar{h}(t)} \int_{\bar{h}(t)}^{\infty} J_i(x-y)\bar{u}_i(t,x)dydx = (1 + \varepsilon) \sum_{i=1}^{2} \mu_i \int_{0}^{\bar{h}(t)} \int_{\bar{h}(t)}^{\infty} J_i(x-y)\phi_i(x - \bar{h}(t))dydx \\
= (1 + \varepsilon) \sum_{i=1}^{2} \mu_i \int_{-\infty}^{\bar{h}(t)} \int_{0}^{\infty} J_i(x-y)\phi_i(x)dydx \\
\leq (1 + \varepsilon) \sum_{i=1}^{2} \mu_i \int_{-\infty}^{\bar{h}(t)} \int_{0}^{\infty} J_i(x-y)\phi_i(x)dydx \\
= (1 + \varepsilon)(c_1 + c_2) = \bar{h}'(t).
\]
Therefore, (5.3) holds. By Theorem 5.5, \( \bar{h}(t) \geq h(t + T) \). Hence, \( \limsup_{t \to \infty} \frac{\bar{h}(t)}{t} \leq \liminf_{t \to \infty} \frac{\bar{h}(t)}{t} = (1 + \varepsilon)(c_1 + c_2) \). The arbitrariness of \( \varepsilon \) implies the limit (5.2).

6 Discussion

In this paper, we study three nonlocal diffusion models with a free boundary in one dimension space. Compared with the existing related works [11 7 8 9], our models only have a free boundary, and the other one is fixed. More precisely, we assume that species can only expand their habitat through one side. On the other side, we suppose that once species cross the boundary, they will die promptly (models (1.2) and (1.4)), or that they can not jump through the boundary (model (1.3)).

For models (1.2) and (1.3), we first establish the well-posedness and spreading-vanishing dichotomy, and then some criteria governing spreading and vanishing are given. In contrast to [11], the solution component \( u(t, x) \) of (1.2) will converge to a bounded positive function \( U(x) \) as \( t \to \infty \) locally uniformly in \([0, \infty)\) if \( h_{\infty} = \infty \). Also we find that if \( f'(0) \geq d/2 \), then spreading happens for (1.3) which is different from the criteria of model (1.2) and that in [11].

Moreover, by a approximate method we prove that (1.3) has a finite spreading speed if and only if (J1) holds, which is analogous to that in [8]. However, when (J1) is satisfied by \( J \), only lower and upper bounds for spreading speed are obtained for model (1.2). For the exact spreading speed, we leave it as a future work. Accelerated spreading occurs for (1.2) if (J1) is violated.
At last, we extend partial conclusions in [6] to the nonlocal diffusion version (1.4) by considering a steady state problem on half space and a technical result. Besides, we prove that the accelerated spreading will happen if

1. the diffusion kernel function for prey violates (J1) and the weakly hunting condition holds, i.e. $a_1b_1b_2 > a_2b_1c_1 + a_1c_1c_2$, or

2. the kernel function of predator does not satisfy (J1).

If kernel functions for prey and predator both satisfy (J1), then we obtain that accelerated spreading cannot happen. These results indicate that spreading speed of (1.4) is influenced by the kernel functions of two species, which is very different from local diffusion version in [6].

For the classical Lotka-Volterra competition model with nonlocal diffusions and a free boundary, we expect that the parallel conclusions hold.

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