THE LEAST H-EIGENVALUE OF SIGNLESS LAPLACIAN OF NON-ODD-BIPARTITE HYPERGRAPHS

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Abstract. Let $G$ be a connected non-odd-bipartite hypergraph with even uniformity. The least H-eigenvalue of the signless Laplacian tensor of $G$ is simply called the least eigenvalue of $G$ and the corresponding H-eigenvectors are called the first eigenvectors of $G$. In this paper we give some numerical and structural properties about the first eigenvectors of $G$ which contains an odd-bipartite branch, and investigate how the least eigenvalue of $G$ changes when an odd-bipartite branch attached at one vertex is relocated to another vertex. We characterize the hypergraph(s) whose least eigenvalue attains the minimum among a certain class of hypergraphs which contain a fixed non-odd-bipartite connected hypergraph. Finally we present some upper bounds of the least eigenvalue and prove that zero is the least limit point of the least eigenvalues of connected non-odd-bipartite hypergraphs.

1. INTRODUCTION

Since Lim [13] and Qi [16] independently introduced the eigenvalues of tensors or hypermatrices in 2005, the spectral theory of tensors developed rapidly, especially the well-known Perron-Frobenius theorem of nonnegative matrices was generalized to nonnegative tensors $[2, 6, 20, 21, 22]$. The signless Laplacian tensors $Q(G)$ [17] were introduced to investigating the structure of hypergraphs, just like signless Laplacian matrices to simple graphs. As $Q(G)$ is nonnegative, by using Perron-Frobenius theorem, many results about its spectral radius are presented $[9, 10, 12, 14, 23]$.

Let $G$ be a $k$-uniform connected hypergraph. Shao et al. [18] prove that zero is an H-eigenvalue of $Q(G)$ if and only if $k$ is even and $G$ is odd-bipartite. Some other equivalent conditions are summarized in [5]. Note that zero is an eigenvalue of $Q(G)$ if and only if $k$ is even and $G$ is odd-colorable [5]. So, there exist odd-colorable but non-odd-bipartite hypergraphs $[4, 15]$, for which zero is an N-eigenvalue. Hu and Qi [17] discuss the H-eigenvectors of zero eigenvalue of $Q(G)$ related to the odd-bipartitions of $G$, and use N-eigenvectors of zero eigenvalue of $Q(G)$ to discuss some kinds of partition of $G$, where an eigenvector is called H-(or N-)eigenvector if it can (or cannot) be scaled into a real vector.

Except the above work, the least H-eigenvalue of $Q(G)$ receives little attention. In this paper, we focus on the least H-eigenvalue of $Q(G)$. Qi [16] proved that each eigenvalue of $Q(G)$ of a connected $k$-uniform hypergraph $G$ has a nonnegative real part by using Gershgorin disks, which implies that the least H-eigenvalue of $Q(G)$

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is at least zero, and is zero if and only if \( k \) is even and \( G \) is odd-bipartite. If \( k \) is even, then \( Q(G) \) are positive semi-definite \( [17] \), and its least H-eigenvalue is a solution of minimum problem over a real unit sphere; see Eq. \( [23] \). So, throughout this paper, when discussing the least H-eigenvalue of \( Q(G) \), we always assume that \( G \) is connected non-odd-bipartite with even uniformity \( k \). For convenience, the least H-eigenvalue of \( Q(G) \) is simply called the least eigenvalue of \( G \) and the corresponding H-eigenvectors are called the first eigenvectors of \( G \).

In this paper we give some numerical and structural properties about the first eigenvectors of \( G \) which contains an odd-bipartite branch, and investigate how the least eigenvalue of \( G \) changes when an odd-bipartite branch attached at one vertex is relocated to another vertex. We characterize the hypergraph(s) whose least eigenvalue attains the minimum among a certain class of hypergraphs which contain a fixed non-odd-bipartite connected hypergraph. Finally we present some upper bounds of the least eigenvalue and prove that zero is the least limit point of the least eigenvalues of connected non-odd-bipartite hypergraphs. The perturbation result on the least eigenvalue in this paper is a generalization of that on the least eigenvalue of the signless Laplacian matrix of a simple graph in \([19]\).

2. Preliminaries

2.1. Eigenvalues of tensors. A real tensor (also called hypermatrix) \( A = (a_{i_1i_2...i_k}) \) of order \( k \) and dimension \( n \) refers to a multi-dimensional array with entries \( a_{i_1i_2...i_k} \in \mathbb{R} \) for all \( i_j \in [n] := \{1, 2, \ldots, n\} \) and \( j \in [k] \). Obviously, if \( k = 2 \), then \( A \) is a square matrix of dimension \( n \). The tensor \( A \) is called symmetric if its entries are invariant under any permutation of their indices.

Given a vector \( x \in \mathbb{C}^n \), \( Ax^k \in \mathbb{C} \) and \( Ax^{k-1} \in \mathbb{C}^n \), which are defined as follows:

\[
Ax^k = \sum_{i_1,i_2,...,i_k \in [n]} a_{i_1i_2...i_k} x_{i_1}x_{i_2}...x_{i_k},
\]

\[
(Ax^{k-1})_i = \sum_{i_2,...,i_k \in [n]} a_{i_1i_2...i_k} x_{i_2}...x_{i_k}, i \in [n].
\]

Let \( I = (i_1i_2...i_k) \) be the identity tensor of order \( k \) and dimension \( n \), that is, \( i_1i_2...i_k = 1 \) if \( i_1 = i_2 = \cdots = i_k \in [n] \) and \( i_1i_2...i_k = 0 \) otherwise.

**Definition 2.1 (\([13],[16]\))**. Let \( A \) be a real tensor of order \( k \) dimension \( n \). For some \( \lambda \in \mathbb{C} \), if the polynomial system \((\lambda I - A)x = 0\), or equivalently \( Ax^{k-1} = \lambda x^{k-1} \), has a solution \( x \in \mathbb{C}^n \setminus \{0\} \), then \( \lambda \) is called an eigenvalue of \( A \) and \( x \) is an eigenvector of \( A \) associated with \( \lambda \), where \( x^{k-1} := (x_1^{k-1}, x_2^{k-1}, \ldots, x_n^{k-1}) \).

In the above definition, \((\lambda, x)\) is called an eigenpair of \( A \). If \( x \) is a real eigenvector of \( A \), surely the corresponding eigenvalue \( \lambda \) is real. In this case, \( \lambda \) is called an H-eigenvalue of \( A \). Denote by \( \lambda_{\min}(A) \) the least H-eigenvalue of \( A \).

A real tensor \( A \) of even order \( k \) is called positive semidefinite (or positive definite) if for any \( x \in \mathbb{R}^n \setminus \{0\} \), \( Ax^k \geq 0 \) (or \( Ax^k > 0 \)).

**Lemma 2.2 (\([16]\), Theorem 5)**. Let \( A \) be a real symmetric tensor of order \( k \) and dimension \( n \), where \( k \) is even. Then the following results hold.

1. \( A \) always has H-eigenvalues, and \( A \) is positive definite (or positive semidefinite) if and only if its least H-eigenvalue is positive (or nonnegative).
(2) \( \lambda_{\min}(A) = \min \{ Ax_k : x \in \mathbb{R}^n, \|x\|_k = 1 \} \), where \( \|x\|_k = (\sum_{i=1}^n |x_i|^k)^{\frac{1}{k}} \).

Furthermore, \( x \) is an optimal solution of the above optimization if and only if it is an eigenvector of \( A \) associated with \( \lambda_{\min}(A) \).

2.2. Uniform hypergraphs. A hypergraph \( G = (V(G), E(G)) \) is a pair consisting of a vertex set \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and an edge set \( E(G) = \{e_1, e_2, \ldots, e_m\} \), where \( e_j \subseteq V(G) \) for each \( j \in [m] \). If \( |e_j| = k \) for all \( j \in [m] \), then \( G \) is called a \( k \)-uniform hypergraph. The degree \( d_G(v) \) or simply \( d(v) \) of a vertex \( v \in V(G) \) is defined as \( d(v) = |\{e_j : v \in e_j\}| \). The order of \( G \) is the cardinality of \( V(G) \), denoted by \( \nu(G) \), and its size is the cardinality of \( E(G) \), denoted by \( \varepsilon(G) \). A walk in a \( G \) is a sequence of alternate vertices and edges: \( v_0e_1v_1e_2 \ldots e_lv_l \), where \( v_i, v_{i+1} \in e_i \) for \( i = 0, 1, \ldots, l - 1 \). A walk is called a path if all the vertices and edges appeared on the walk are distinct. A hypergraph \( G \) is called connected if any two vertices of \( G \) are connected by a walk or path.

If a hypergraph is both connected and acyclic, it is called a hypertree. The \( k \)-th power of a simple graph \( H \), denoted by \( H^k \), is obtained from \( H \) by replacing each edge (a 2-set) with a \( k \)-set by adding \((k - 2)\) additional vertices \([8]\). The \( k \)-th power of a tree is called power hypertree, which is surely a \( k \)-uniform hypertree. In particular, the \( k \)-th power of a path \( P_m \) (respectively, a star \( S_m \) (as a simple graph) with \( m \) edges is called a hyperpath (respectively, hyperstar), denote by \( P_m^k \) (respectively, \( S_m^k \)). In a \( k \)-th power hypertree \( T \), an edge is called a pendant edge of \( T \) if it contains \( k - 1 \) vertices of degree one, which are called the pendant vertices of \( T \).

Lemma 2.3 ([11]). Let \( G \) be a connected \( k \)-uniform hypergraph. Then \( G \) is a hypertree if and only if \( \varepsilon(G) = \frac{\nu(G) - 1}{k - 1} \).

The odd-bipartite hypergraphs was introduced by Hu and Qi [7], which is considered as a generalization of the ordinary bipartite graphs. The odd-bipartition is closely related to odd-traversal [15].

Definition 2.4 ([7]). Let \( k \) be even. A \( k \)-uniform hypergraph \( G = (V, E) \) is called odd-bipartite, if there exists a bipartition \( \{V_1, V_2\} \) of \( V \) such that each edge of \( G \) intersects \( V_1 \) (or \( V_2 \)) in an odd number of vertices (such bipartition is called an odd-bipartition of \( G \)); otherwise, \( G \) is called non-odd-bipartite.

Let \( G \) be a \( k \)-uniform hypergraph on \( n \) vertices \( v_1, v_2, \ldots, v_n \). The adjacency tensor of \( G \) \([12]\) is defined as \( A(G) = (a_{i_1i_2\ldots i_k}) \), an order \( k \) dimensional \( n \) tensor, where

\[
a_{i_1i_2\ldots i_k} = \begin{cases} \frac{1}{(k-1)!}, & \text{if } \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \in E(G); \\ 0, & \text{otherwise.} \end{cases}
\]

Let \( D(G) \) be a diagonal tensor of order \( k \) and dimension \( n \), where \( d_{i\ldots i} = d(v_i) \) for \( i \in [n] \). The tensor \( Q(G) = D(G) + A(G) \) is called the signless Laplacian tensor of \( G \) \([14]\). Observe that the adjacency (signless Laplacian) tensor of a hypergraph is symmetric.

Let \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n \). Then \( x \) can be considered as a function defined on the vertices of \( G \), that is, each vertex \( v_i \) is mapped to \( x_i = x_{v_i} \). If \( x \) is an eigenvector of \( Q(G) \), then it defines on \( G \) naturally, i.e., \( x_v \) is the entry of \( x \) corresponding to \( v \). If \( G_0 \) is a sub-hypergraph of \( G \), denote by \( x|_{G_0} \) the restriction of \( x \) on the vertices of \( G_0 \), or a subvector of \( x \) indexed by the vertices of \( G_0 \).
Denote by $E_G(v)$, or simply $E(v)$, the set of edges of $G$ containing $v$. For a subset $U$ of $V(G)$, denote $x^U := \Pi_{u \in U} x_u$, and $x^U_k := \sum_{u \in U} x^k_u$. Then we have
\begin{equation}
Q(G)x^k = \sum_{e \in E(G)} (x^k_e + kx^e),
\end{equation}
and for each $v \in V(G)$,
\[(Q(G)x^{k-1})_v = d(v)x^{k-1}_v + \sum_{e \in E(v)} x^{e \setminus \{v\}}.
\]
So the eigenvector equation $Q(G)x^{k-1} = \lambda x^{k-1}$ is equivalent to that for each $v \in V(G)$,
\begin{equation}
(\lambda - d(v))x^{k-1}_v = \sum_{e \in E(v)} x^{e \setminus \{v\}}.
\end{equation}
From Lemma 2.2(2), if $k$ is even, then $\lambda_{\min}(G) := \lambda_{\min}(Q(G))$ can be expressed as
\begin{equation}
\lambda_{\min}(G) = \min_{x \in \mathbb{R}^n, \|x\| = 1} \left\{ \sum_{e \in E(G)} (x^k_e + kx^e) \right\}.
\end{equation}
Note that if $k$ is odd, the Eq. (2.3) does not hold. The reason is as follows. If $G$ contains at least one edge, then by Perron-Frobenius theorem, the spectral radius $\rho(Q(G))$ of $Q(G)$ is positive associated with a unit nonnegative eigenvector $x$. Now $\lambda_{\min}(G) \leq Q(G)(-x)^k = -Q(G)x^k = -\rho(Q(G)) < 0$, a contradiction as $\lambda_{\min}(G) \geq 0$ (see [17, Theorem 3.1]).

**Lemma 2.5.** Let $G$ be a $k$-uniform hypergraph, and $(\lambda, x)$ be an eigenpair of $Q(G)$. If $E(u) = E(v)$ and $\lambda \neq d(u)$, then $x^{k}_u = x^{k}_v$.

**Proof.** Consider the eigenvector equation of $x$ at $u$ and $v$ respectively,
\[
(\lambda - d(u))x^k_u = \sum_{e \in E(u)} x^e, \quad (\lambda - d(v))x^k_v = \sum_{e \in E(v)} x^e.
\]
As $E[u] = E[v]$, $d(u) = d(v)$ and $\sum_{e \in E(u)} x^e = \sum_{e \in E(v)} x^e$. The result follows. \hfill \Box

**Lemma 2.6** ([11]). Let $G$ be a $k$-uniform hypergraph with the minimum degree $\delta(G) > 0$, where $k$ is even. Then $\lambda_{\min}(G) < \delta(G)$.

### 3. Properties of the first eigenvectors

Let $G_1$, $G_2$ be two vertex-disjoint hypergraphs, and let $v_1 \in V(G_1), v_2 \in V(G_2)$. The *coalescence* of $G_1$, $G_2$ with respect to $v_1, v_2$, denoted by $G_1(v_1) \circ G_2(v_2)$, is obtained from $G_1, G_2$ by identifying $v_1$ with $v_2$ and forming a new vertex $u$. The graph $G_1(v_1) \circ G_2(v_2)$ is also written as $G_1(u) \circ G_2(u)$. If a connected graph $G$ can be expressed in the form $G = G_1(u) \circ G_2(u)$, where $G_1, G_2$ are both nontrivial and connected, then $G_1$ is called a branch of $G$ with root $u$. Clearly $G_2$ is also a branch of $G$ with root $u$ in the above definition.

We will give some properties of the first eigenvectors of a connected $k$-uniform $G$ which contains an odd-bipartite branch. We stress that $k$ is even in this and the following sections.

**Lemma 3.1.** Let $G = G_0(u) \circ H(u)$ be a connected $k$-uniform hypergraph, where $H$ is odd-bipartite. Let $x$ be a first eigenvector of $G$. Then the following results hold.
Let \( H \) be an odd-bipartite \( k \)-uniform hypergraph, where \( H \) is odd-bipartite. Then

\[
\lambda_{\min}(G_0) \geq \lambda_{\min}(G),
\]

with equality if and only if for any first eigenvector \( y \) of \( G_0 \), \( y_u = 0 \) and \( \tilde{y} \) is a first eigenvector of \( G \), where \( \tilde{y} \) is defined by

\[
\tilde{y}_v = \begin{cases} 
y_v, & \text{if } v \in V(G_0); \\
0, & \text{otherwise}. \end{cases}
\]
Proof. Suppose that \( y \) is a first eigenvector of \( G_0 \), \( \|y\|_k = 1 \), and \( y_u \geq 0 \). Let \( \{U, W\} \) be an odd-bipartition of \( H \), where \( u \in U \). Define \( \bar{y} \) by

\[
\bar{y}_u = \begin{cases} 
  y_v, & \text{if } v \in V(G_0); \\
  y_u, & \text{if } v \in U \setminus \{u\}; \\
  -y_u, & \text{if } v \in W.
\end{cases}
\]

Then \( \|\bar{y}\|_k = 1 + (\nu(H) - 1)y_u^k \), and

\[
Q(G)\bar{y}^k = \sum_{e \in E(G)} (\bar{y}_e^k + k\bar{y}_e^c) = \sum_{e \in E(G_0)} (\bar{y}_e^k + k\bar{y}_e^c) + \sum_{e \in E(H)} (\bar{y}_e^k + k\bar{y}_e^c)
= Q(G_0)y^k + \sum_{e \in E(H)} (ky_u^k - ky_u^k)
= \lambda_{\min}(G_0).
\]

By Eq. (2.3), we have

\[
\lambda_{\min}(G) \leq \frac{Q\bar{y}^k}{\|\bar{y}\|_k}^k = \frac{\lambda_{\min}(G_0)}{1 + (\nu(H) - 1)y_u^k} \leq \lambda_{\min}(G_0),
\]

where the first equality holds if and only if \( \bar{y} \) is also a first eigenvector of \( G \), and the second equality holds if and only if \( y_u = 0 \) (Note that \( \lambda_{\min}(G_0) > 0 \) as \( G_0 \) is connected and non-odd-bipartite). The result now follows.

Theorem 3.3. Let \( G = G_0(u) \circ H(u) \) be a connected non-odd-bipartite \( k \)-uniform hypergraph, where \( H \) is odd-bipartite.

1. If \( y \) is a first eigenvector of \( G_0 \) with \( y_u \neq 0 \), then

\[ \lambda_{\min}(G_0) > \lambda_{\min}(G). \]

2. If \( x \) is a first eigenvector of \( G \) such that \( x_u = 0 \) and \( x|_{G_0} \neq 0 \), then

\[ \lambda_{\min}(G_0) = \lambda_{\min}(G). \]

Proof. By Lemma 3.2, we can get the assertion (1) immediately. Let \( x \) be a first eigenvector of \( G \) as in (2). By Lemma 3.1(2), \( \sum_{e \in E(G_0)} x^c \setminus \{u\} = 0 \). Considering the eigenvector equation (2.2) of \( x \) at each vertex of \( V(G_0) \), we have

\[ Q(G_0)(x|_{G_0})^{k-1} = \lambda_{\min}(G)(x|_{G_0})^{k-1}. \]

So \( x|_{G_0} \) is an eigenvector of \( Q(G_0) \) associated with the eigenvalue \( \lambda_{\min}(G) \). The result follows by Lemma 3.2.

Lemma 3.4. Let \( G = G_0(u) \circ H(u) \) be a connected non-odd-bipartite \( k \)-uniform hypergraph, where \( H \) is odd-bipartite. If \( x \) is a first eigenvector of \( G \), then

\[ \beta_H(x) := d_H(u)x_u^k + \sum_{e \in E_H(u)} x^c \leq 0. \]

Furthermore, if \( \beta_H(x) = 0 \) and \( x|_{G_0} \neq 0 \), then \( x_u = 0 \) and \( \lambda_{\min}(G_0) = \lambda_{\min}(G) \); or equivalently if \( x_u \neq 0 \), then \( \beta_H(x) < 0 \).
Proof. Let $\lambda := \lambda_{\min}(G)$. By Eq. (2.2), for each $v \in V(G_0) \setminus \{u\}$,
\begin{equation}
((Q(G) - \lambda I)x^{k-1})_v = ((Q(G_0) - \lambda I)(x|_{G_0})^{k-1})_v = 0.
\end{equation}
For the vertex $u$,
\begin{align*}
\lambda x^{k-1}_u &= (Q(G)x^{k-1})_u = d_G(u)x^{k-1}_u + \sum_{e \in E_G(u)} x^{e\{u\}}, \\
&= d_{G_0}(u)x^{k-1}_u + \sum_{e \in E_{G_0}(u)} x^{e\{u\}} + d_H(u)x^{k-1}_u + \sum_{e \in E_H(u)} x^{e\{u\}} \\
&= (Q(G_0)(x|_{G_0})^{k-1})_u + d_H(u)x^{k-1}_u + \sum_{e \in E_H(u)} x^{e\{u\}}.
\end{align*}
So,
\begin{equation}
((Q(G_0) - \lambda I)(x|_{G_0})^{k-1})_u = -\left( d_H(u)x^{k-1}_u + \sum_{e \in E_H(u)} x^{e\{u\}} \right).
\end{equation}

By Lemma 3.2 and Lemma 2.2(1), $Q(G_0) - \lambda I$ is positive semidefinite. Then $(Q(G_0) - \lambda I)y^k \geq 0$ for any real and nonzero $y$. So, by Eq. (3.2) and Eq. (3.3), we have
\begin{align*}
0 \leq (Q(G_0) - \lambda I)(x|_{G_0})^k &= (x|_{G_0})^T((Q(G_0) - \lambda I)(x|_{G_0})^{k-1} \\
&= -x_u \left( d_H(u)x^{k-1}_u + \sum_{e \in E_H(u)} x^{e\{u\}} \right) \\
&= -\beta_H(x).
\end{align*}
So we have $\beta_H(x) \leq 0$.

Suppose that $\beta_H = 0$ and $x|_{G_0} \neq 0$. If $x_u = 0$, by Corollary 3.3(2), $\lambda_{\min}(G_0) = \lambda_{\min}(G)$. If $x_u \neq 0$, then $d_H(u)x^{k-1}_u + \sum_{e \in E_H(u)} x^{e\{u\}} = 0$. By Eq. (3.2) and Eq. (3.3), $(\lambda_{\min}(G), x|_{G_0})$ is an eigenpair of $Q(G_0)$, implying that $\lambda_{\min}(G) = \lambda_{\min}(G_0)$ by Lemma 3.2. However, $(x|_{G_0})_u = x_u \neq 0$, a contradiction to Corollary 3.3(1). \hfill $\Box$

**Lemma 3.5.** Let $G = G_0(r) \circ T(r)$ be a connected non-odd-bipartite $k$-uniform hypergraph, where $T$ is a power hypertree. If $x$ is a first eigenvector of $G$ and $x_p \neq 0$ for some $p \in V(T)$, then $x_q \neq 0$ whenever $q$ is a vertex of $T$ such that $p$ lies on the unique path from $r$ to $q$.

**Proof.** It suffices to consider three vertices $u, v, w$ in a common edge $e \in E(T)$, where $d(u) \geq 2$, $d(v) = 1$, $d(w) \geq 2$, and $u$ lies on the path from $r$ to $w$. We will show $x_u \neq 0 \Rightarrow x_v \neq 0 \Rightarrow x_w \neq 0$. Write $G = G_0(u) \circ \bar{T}(u)$, where $G_0$ contains $G_0$ as a sub-hypergraph, and $\bar{T}$ is a sub-hypergraph of $T$ such that $e$ is the only edge of $\bar{T}$ containing $u$. Suppose that $x_u \neq 0$. If $x_v = 0$, by Lemma 3.4,
\begin{equation}
\beta_{\bar{T}}(x) = x^{k}_u + x^e = x^{k}_u \leq 0,
\end{equation}
a contradiction. So $x_v \neq 0$. If $x_w = 0$, then by Eq. (2.2), $x_v = 0$ as $\lambda_{\min}(G) < \delta(G) = 1$ by Lemma 2.1A also a contradiction. So $x_w \neq 0$. \hfill $\Box$

**Lemma 3.6.** Let $G = G_0(r) \circ T(r)$ be a connected non-odd-bipartite $k$-uniform hypergraph, where $T$ is a power hypertree. If $x$ is a first eigenvector of $G$ and $x_r \neq 0$, then $|x_u| < |x_w|$ whenever $u, w$ are two vertices of $T$ such that $u$ lies on the unique path from $r$ to $w$, $d(u) \geq 2$ and $d(w) \geq 2$. 


Proof. By Lemma 3.1(1), we have
\[ (1 - \lambda)|x_v|^k = |x^r| = |x_u||x_v|^{k-1}|x_w|, \]
which implies that
\[ |x_v| = \left( \frac{|x_u||x_v|}{1 - \lambda} \right)^{\frac{1}{k}}. \]

We can write \( G = \bar{G}_0(w) \circ \bar{T}(w) \), where \( \bar{G}_0 \) contains \( G_0 \) and the edge \( e \) as the only one containing \( w \). Then by Lemma 3.4, noting \( x_w \neq 0 \),
\[ \beta_{\bar{T}}(x) = (d_T(w) - 1)x_w^k + \sum_{\bar{\varepsilon} \in E_T(w) \setminus \{e\}} x_{\bar{\varepsilon}} < 0. \]
By Lemma 3.3(1), we have
\[ (d_T(w) - 1)|x_w|^k < \sum_{\bar{\varepsilon} \in E_T(w) \setminus \{e\}} |x_{\bar{\varepsilon}}|. \]
Considering the eigenvector equation of \( x \) at \( w \), by Lemma 2.5, Eq. (3.4) and Eq. (3.5), we have
\[ (d_T(w) - \lambda)|x_w|^k = \sum_{\bar{\varepsilon} \in E_T(w) \setminus \{e\}} |x_{\bar{\varepsilon}}| + |x_u||x_v|^{k-1}|x_w| \]
\[ > (d_T(w) - 1)|x_w|^k + |x_u| \left( \frac{|x_u||x_v|}{1 - \lambda} \right)^{\frac{1}{k-1}} |x_w| \]
\[ = (d_T(w) - 1)|x_w|^k + (1 - \lambda)^{1 - \frac{1}{k}} |x_u|^\frac{1}{k} |x_w|^\frac{k}{k}. \]
So
\[ (1 - \lambda)|x_w|^k > (1 - \lambda)^{1 - \frac{1}{k}} |x_u|^\frac{1}{k} |x_w|^\frac{k}{k}, \]
and hence
\[ |x_w|^\frac{k}{k} > (1 - \lambda)^{\frac{k}{k}} |x_w|^\frac{k}{k} > |x_u|^\frac{1}{k}. \]

Denote by \( G = G_0(r) \circ P_m^k(r) \) the coalescence of \( G_0 \) and \( P_m^k \) by identifying one vertex of \( G_0 \) and one pendent vertex of \( P_m^k \) and forming a new vertex \( r \).

Lemma 3.7. Let \( G = G_0(r) \circ P_m^k(r) \) be a connected non-odd-bipartite \( k \)-uniform hypergraph, where \( P_m^k \) is a hyperpath with \( m \) edges. Starting from the root \( r \), label edges of \( P_m^k \) as \( e_m, e_{m-1}, \ldots, e_1 \), and some vertices of those edges as
\[ r = 2m, 2m - 1, 2m - 2, \ldots, 2i - 1, 2i - 2, \ldots, 2, 1, 0, \]
where \( \{2i, 2i - 1, 2i - 2\} \subseteq e_i \) for \( i \in [m] \), \( d_{P_m}(2i) = 1 \) and \( d_{P_m}(2i - 1) = 1 \) for \( i \in [m - 1] \), \( d_{P_m}(2m) = d_{P_m}(0) = 1 \). If \( x \) is a first eigenvector of \( G \) and \( x_r \neq 0 \), then
\[ |x_2| = f_i(\lambda_{\min}(G))^\frac{2}{k} |x_0|, \quad i \in [m], \]
where \( f_i(x) \) is defined recursively as \( f_0(x) = 1 \), \( f_1(x) = (1 - x)^\frac{1}{k} \),
\[ f_{i+1}(x) = (2 - x)(1 - x)^{\frac{k}{k} - 1} f_i(x) - f_{i-1}(x), \quad i \in [m - 1]. \]
Furthermore, \(0 < f_i(\lambda_{\text{min}}(G)) < 1\), and \(f_i(\lambda_{\text{min}}(G))\) is strictly decreasing in \(i\).

**Proof.** By Lemma 3.5, \(x_v \neq 0\) for each \(v \in V(P_m)\). Let \(\lambda := \lambda_{\text{min}}(G)\). Then \(0 < \lambda < 1\) by Lemma 2.6. By Lemma 2.5 and Eq. (2.2),

\[
|x_0| = |x_1|, \quad |x_2| = (1 - \lambda)|x_0|.
\]

So, by Lemma 2.5, Lemma 3.1(1) and Eq. (3.3), considering the eigenvector equation of \(x\) at the vertex \(2i\) (\(1 \leq i \leq m - 1\)), we have

\[
(2 - \lambda)|x_{2i}|^{k-1} = |x_{2i-1}|^{k-2}|x_{2i-2}| + |x_{2i+1}|^{k-2}|x_{2i+2}|
\]

\[
= \left(\frac{|x_{2i}|}{1 - \lambda}\right)^{\frac{k}{2} - 1} |x_{2i-2}| + \left(\frac{|x_{2i+1}|}{1 - \lambda}\right)^{\frac{k}{2} - 1} |x_{2i+2}|
\]

\[
= (1 - \lambda)^{1 - \frac{k}{2}} |x_{2i}|^{\frac{k}{2} - 1} \left(\left|x_{2i-2}\right|^\frac{k}{2} + \left|x_{2i+2}\right|^\frac{k}{2}\right).
\]

Thus, for \(i = 1, \ldots, m - 1\),

\[
|x_{2i+2}|^{\frac{k}{2}} = (2 - \lambda)(1 - \lambda)^{\frac{k}{2} - 1}|x_{2i}|^{\frac{k}{2}} - |x_{2i-2}|^{\frac{k}{2}}.
\]

It is easy to verify that for \(i \in [m]\),

\[
|x_{2i}|^{\frac{k}{2}} = f_i(\lambda)|x_0|^{\frac{k}{2}}.
\]

By Lemma 3.6 for \(i \in [m]\), \(0 < f_i(\lambda) < 1\), and \(f_i(\lambda)\) is strictly decreasing in \(i\). \(\square\)

### 4. Perturbation of the least eigenvalues

We first give a perturbation result on the least eigenvalues under relocating an odd-bipartite branch. Let \(G_0, H\) be two vertex-disjoint hypergraphs, where \(v_1, v_2\) are two distinct vertices of \(G_0\), and \(u\) is a vertex of \(H\) (called the root of \(H\)). Let \(G = G_0(v_2) \circ H(u)\) and \(\tilde{G} = G_0(v_1) \circ H(u)\). We say that \(\tilde{G}\) is obtained from \(G\) by relocating \(H\) rooted at \(u\) from \(v_2\) to \(v_1\); see Fig. 4.1.

![Figure 4.1](image)

**Lemma 4.1.** Let \(G = G_0(v_2) \circ H(u)\) and \(\tilde{G} = G_0(v_1) \circ H(u)\) be connected non-odd-bipartite \(k\)-uniform hypergraphs, where \(H\) is odd-bipartite. If \(x\) is a first eigenvector of \(G\) such that \(|x_{v_1}| \geq |x_{v_2}|\), then

\[
\lambda_{\text{min}}(\tilde{G}) \leq \lambda_{\text{min}}(G),
\]

with equality if and only if \(x_{v_1} = x_{v_2} = 0\), and \(\bar{x}\) defined in (4.4) is a first eigenvector of \(\tilde{G}\).
Proof. Let \( x \) be a first eigenvector of \( G \) such that \( \|x\|_k = 1 \) and \( x_{v_1} \geq 0 \). We divide the discussion into three cases. Denote \( \lambda := \lambda_{\min}(G) \).

**Case 1:** \( x_{v_2} > 0 \). Write \( x_{v_1} = \delta x_{v_2} \), where \( \delta \geq 1 \). Define \( \tilde{x} \) on \( \tilde{G} \) by

\[
(4.1) \quad \tilde{x}_v = \begin{cases} 
    x_v, & \text{if } v \in V(G_0); \\
    \delta x_v, & \text{if } v \in V(H) \setminus \{u\}.
\end{cases}
\]

Then \( \|\tilde{x}\|_k^k = 1 + (\delta^k - 1) \sum_{v \in V(H) \setminus \{u\}} x_v^k \), and

\[
(4.2) \quad Q(\tilde{G})\tilde{x}^k = \sum_{e \in E(\tilde{G})} (\tilde{x}_e + k\tilde{x})
\]

\[
= Q(G)x^k + (\delta^k - 1) \sum_{e \in E(H)} (x_e^k + kx^e)
\]

\[
= \lambda + (\delta^k - 1) \sum_{e \in E(H)} (x_e^k + kx^e).
\]

By the eigenvector equation of \( x \) at each vertex \( v \in V(H) \setminus \{u\} \),

\[
(4.3) \quad d(v)x_v^k + \sum_{e \in E_H(v)} x^e = \lambda x_v^k.
\]

By the eigenvector equation of \( x \) at \( u \),

\[
d(u)x_u^k + \sum_{e \in E_G(u)} x^e = \alpha_{G_0}(x) + \beta_H(x) = \lambda x_u^k,
\]

where \( \alpha_{G_0}(x) := d_{G_0}(u)x_u^k + \sum_{e \in E_{G_0}(u)} x^e \). By Eq. (4.2) and Eq. (4.3), we have

\[
\alpha_{G_0} + \sum_{e \in E(H)} (x_e^k + kx^e) = \lambda \sum_{v \in V(H)} x_v^k.
\]

So

\[
\sum_{e \in E(H)} (x_e^k + kx^e) = \lambda \sum_{v \in V(H)} x_v^k - \alpha_{G_0}(x)
\]

\[
= \lambda \sum_{v \in V(H)} x_v^k - (\lambda x_u^k - \beta_H(x))
\]

\[
= \lambda \sum_{v \in V(H) \setminus \{u\}} x_v^k + \beta_H(x).
\]

Thus

\[
Q(\tilde{G})\tilde{x}^k = \lambda + (\delta^k - 1) \left( \lambda \sum_{v \in V(H) \setminus \{u\}} x_v^k + \beta_H(x) \right)
\]

\[
= \lambda \left( 1 + (\delta^k - 1) \sum_{v \in V(H) \setminus \{u\}} x_v^k \right) + (\delta^k - 1)\beta_H(x)
\]

As \( x_{v_2} \neq 0 \), \( \beta_H(x) < 0 \) by Lemma 3.4

\[
\lambda_{\min}(\tilde{G}) \leq \frac{Q(\tilde{G})\tilde{x}^k}{\|\tilde{x}\|_k^k} = \lambda + \frac{(\delta^k - 1)\beta_H(x)}{\|\tilde{x}\|_k^k} \leq \lambda = \lambda_{\min}(G),
\]
where the first equality holds if and only if \( \tilde{x} \) is a first eigenvector of \( \tilde{G} \), and the second equality holds if and only if \( \delta = 1 \), i.e., \( x_{v_1} = x_{v_2} \). By the eigenvector equations of \( x \) and \( \tilde{x} \) at \( v_2 \) respectively, we will get \( \beta_H(x) = 0 \), a contradiction. So, in this case, \( \lambda_{\min}(\tilde{G}) < \lambda_{\min}(G) \).

**Case 2:** \( x_{v_2} = 0 \). First assume \( x_{v_1} = 0 \). Define \( \tilde{x} \) on \( \tilde{G} \) by

\[
(4.4) \quad \tilde{x}_v = \begin{cases} 
  x_v, & \text{if } v \in V(G_0); \\
  x_v, & \text{if } v \in V(H) \setminus \{u\}.
\end{cases}
\]

Then \( \|\tilde{x}\|_k^k = 1 \), and

\[
\lambda_{\min}(\tilde{G}) \leq Q(\tilde{G})\tilde{x}^k = Q(G)x^k = \lambda_{\min}(G),
\]

with equality if and only if \( \tilde{x} \) is a first eigenvector of \( \tilde{G} \).

Now assume that \( x_{v_1} > 0 \). By Corollary 3.3(2) and its proof, \( \lambda_{\min}(G) = \lambda_{\min}(G_0) \) as \( x_u = 0 \) and \( x|_{G_0} \neq 0 \); furthermore, \( x|_{G_0} \) is a first eigenvector of \( G_0 \). By Corollary 3.3(1), \( \lambda_{\min}(G_0) > \lambda_{\min}(\tilde{G}) \) as \( (x|_{G_0})_{v_1} \neq 0 \), thinking of \( v_1 \) a coalescence vertex between \( G_0 \) and \( H \) in \( G \). So \( \lambda_{\min}(\tilde{G}) < \lambda_{\min}(G) \).

**Case 3:** \( x_{v_2} < 0 \). Write \( x_{v_1} = -\delta x_{v_2} \), where \( \delta \geq 1 \). Define \( \tilde{x} \) on \( \tilde{G} \) by

\[
(4.5) \quad \tilde{x}_v = \begin{cases} 
  x_v, & \text{if } v \in V(G_0); \\
  -\delta x_v, & \text{if } v \in V(H) \setminus \{u\}.
\end{cases}
\]

By a similar discussion to Case 1 by replacing \( \delta \) by \( -\delta \), we also have \( \lambda_{\min}(\tilde{G}) < \lambda_{\min}(G) \). \( \square \)

**Corollary 4.2.** Let \( G \) be a connected non-odd-bipartite \( k \)-uniform hypergraph, and \( G_{s,t} \) be the hypergraph obtained by coalescing \( G \) with two hyperpaths \( P^k_s \) and \( P^k_t \) by identifying a pendent vertex of \( P^k_s \) and a pendent vertex of \( P^k_t \) both with a vertex \( u \) of \( G \), where \( s \geq t \geq 1 \). If \( x \) is a first \( Q \)-eigenvector of \( G_{s,t} \) and \( x_u \neq 0 \), then

\[
\lambda_{\min}(G_{s,t}) > \lambda_{\min}(G_{s+1,t-1}).
\]

**Proof.** Using the method of labeling vertices as in Eq. (3.6), we label some of the vertices \( P^k_s \) as

\[
 u = 2s, 2s - 1, 2s - 2, \ldots, 2, 1, 0,
\]

and label some of the vertices of \( P^k_t \) as

\[
 u = 2t, 2t - 1, 2t - 2, \ldots, 2, 1, 0.
\]

Then, by Lemma 3.7

\[
 |x_u| = f_s(\lambda)^\frac{k}{2} |x_0| = f_t(\lambda)^\frac{k}{2} |x_0|,
\]

where \( \lambda := \lambda_{\min}(G_{s,t}) \), and \( f_t(x) \) is defined as in Lemma 3.7. As \( s \geq t, 0 < f_s(\lambda) \leq f_t(\lambda) \) also by Lemma 3.7. So, combining the eigenvector equation on \( 0, |x_0| \geq |x_0| > |x_2| > 0 \). Now relocating the pendent edge of \( P^k_t \) rooted at \( \bar{2} \) and attaching to the pendent vertex \( 0 \) of \( P^k_s \), we arrive at the hypergraph \( G_{s+1,t-1} \). The result follows by Lemma 3.1. \( \square \)

**Lemma 4.3.** Let \( G \) and \( G_{s,t} \) be as defined in Corollary 4.2. Then

\[
\lambda_{\min}(G_{s,t}) \geq \lambda_{\min}(G_{s+t,0}).
\]

Furthermore, if \( x \) is a first \( Q \)-eigenvector of \( G_{s,t} \) and \( x_u \neq 0 \), then

\[
\lambda_{\min}(G_{s,t}) > \lambda_{\min}(G_{s+t,0}).
\]
Proof. Suppose that the labeling of some vertices of $P_k$ and $P^k_t$ is as in the proof of Corollary 4.2. Let $x$ be a first eigenvector of $G_{s,t}$. If $x_0 = 0$, then $|x_0| \geq |x_u|$; otherwise $|x_0| > |x_u|$ by Lemma 4.6. Relocating $P^k_t$ rooted at $u$ and attaching to the pendent vertex 0 of $P_k$, we arrive at the hypergraph $G_{s+t,0}$. The result follows by Lemma 4.1.

A hypergraph is called a minimizing hypergraph in a certain class of hypergraphs if its least eigenvalue attains the minimum among all hypergraphs in the class. Denoted by $T_m(G_0)$ the class of hypergraphs with each obtained from a fixed connected non-odd-bipartite hypergraph $G_0$ by attaching some hypertrees at some vertices of $G_0$ respectively (i.e., identifying a vertex of a hypertree with some vertex of $G_0$ each time) such that the number of its edges equals $\varepsilon(G_0) + 1$. We will characterize the minimizing hypergraph(s) in $T_m(G_0)$.

**Theorem 4.4.** Let $G_0$ be a connected non-odd-bipartite $k$-uniform hypergraph. If $G$ is a minimizing hypergraph in $T_m(G_0)$, then $G = G_0(u) \circ P_m(u)$ for some vertex $u$ of $G_0$.

Proof. Suppose that $G$ is a minimizing hypergraph in $T_m(G_0)$, and $G$ has no the structure as desired in the theorem. We will get a contradiction by the following three cases.

**Case 1:** $G$ contains hypertrees attached at two or more vertices of $G_0$. Let $T_1, T_2$ be two hypertrees attached at $v_1, v_2$ of $G_0$ respectively. Let $x$ be a first eigenvector of $G$, and assume that $|x_{v_1}| \geq |x_{v_2}|$. Relocating $T_2$ rooted at $v_2$ and attaching to $v_1$, we will get a hypergraph $\hat{G} \in T_m(G_0)$ such that $\lambda_{\min}(\hat{G}) \leq \lambda_{\min}(G)$ by Lemma 4.1. Repeating the above operation, we finally arrive at a hypergraph $G^{(1)}$ with only one hypertree $T^{(1)}$ attached at one vertex $u_0$ of $G_0$ such that $\lambda_{\min}(G^{(1)}) \leq \lambda_{\min}(G)$.

**Case 2:** $T^{(1)}$ contains edges with three or more vertices of degree greater than one, i.e. $T^{(1)}$ is not a power hypertree. Let $e$ be one of such edges containing $u, v, w$ with $d(u), d(v), d(w)$ all greater than one. Let $x$ be a first eigenvector of $G^{(1)}$, and assume that $|x_u| \geq |x_w|$. Relocating the hypertree rooted at $w$ and attaching to $u$, we will get a hypergraph $\bar{G} \in T_m(G_0)$ such that $\lambda_{\min}(\bar{G}) \leq \lambda_{\min}(G^{(1)})$ by Lemma 4.1. Repeating the above operation on the edge $e$ until $e$ contains exactly 2 vertices of degree greater than one, and on each other edges like $e$, we finally arrive at a hypergraph $G^{(2)}$ such that the unique hypertree $T^{(2)}$ attached at $u_0$ is a power hypertree, and $\lambda_{\min}(G^{(2)}) \leq \lambda_{\min}(G^{(1)})$.

**Case 3:** $T^{(2)}$ contains more than one pendent edges except the edge(s) containing $u_0$. Let $x$ be a first eigenvector of $G^{(2)}$. We assert that $|x_{u_0}| = \max_{v \in V(G_0)} |x_v|$. Otherwise, there exists a vertex $v_0$ of $G$ such that $|x_{v_0}| > |x_{u_0}|$. Relocating $T^{(2)}$ rooted at $u_0$ and attaching to $v_0$, we will get a hypergraph $\bar{G} \in T_m(G_0)$ such that $\lambda_{\min}(\bar{G}) < \lambda_{\min}(G^{(2)})$ by Lemma 4.1. Then $\lambda_{\min}(\bar{G}) < \lambda_{\min}(G)$, a contradiction to $G$ being minimizing. We also assert that there exists one pendent vertex $v_0$ of $T^{(2)}$ such that $x_{v_0} \neq 0$. Otherwise by Lemma 3.3, $x_{T^{(2)}} = 0$, in particular $x_{u_0} = 0$, and hence $x = 0$ by the first assertion, a contradiction.

Note that $T^{(2)}$ consists of $d_{T^{(2)}}(u_0)$ sub-hypertrees sharing a common vertex $u_0$. Let $T^{(2)}_p$ be the sub-hypertrees of $T^{(2)}$ attached at $u_0$ which contains $w_0$. If $d_{T^{(2)}}(u_0) = 1$, let $p$ be the furthest vertex of degree greater 2 on the path starting from $v_0$ to $w_0$, and let $T_p$ be the hypertree attached to $p$ which contains no vertices of the path except $p$. Relocating $T_p$ rooted at $p$ and attaching to $w_0$, we will arrive at
a hypergraph still in $T_m(G_0)$ but with a smaller least eigenvalue by Lemma 3.6 and Lemma 4.1 regardless of $x_p$ being zero or not, a contradiction. If $d_{T^2(u_0)} > 1$, let $T_2^{(2)}$ be the sub-hypertree of $T^{(2)}$ attached at $u_0$ which contains no $w_0$. Relocating $T_2^{(2)}$ from $u_0$ and attaching to $w_0$, we still arrive at a hypergraph in $T_m(G_0)$ but with a smaller least eigenvalue, also a contradiction. The result now follows. □

5. Least limit point of the least eigenvalues

In this section we will investigate the upper bounds of the least eigenvalues, from which we show that the least limit point of the least eigenvalues of connected non-odd-bipartite hypergraphs is zero.

Lemma 5.1. Let $G$ be a non-odd-bipartite $k$-uniform hypergraph. Then $G$ contains an odd-bipartite sub-hypergraph with at least $\frac{\varepsilon(G)}{2}$ edges.

Proof. Let $T \subseteq V(G)$ be a random subset given by $\Pr[v \in T] = \frac{1}{2}$, these choices being mutually independent. Set $B = V(G) \setminus T$. Call an edge $e$ odd-transversal if exactly the cardinality of $e \cap T$ is odd. Let $X$ be the number of odd-transversal edges. We decompose

$$X = \sum_{e \in E(G)} X_e,$$

where $X_e$ is the indicator random variable for $e$ being odd-transversal, i.e, $X_e = 1$ if $e$ is odd-transversal, and $X_e = 0$ otherwise. Then the expectation

$$E[X_e] = \Pr[X_e] = \sum_{i \text{ is odd}, i \in [k]} \binom{k}{i} \left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{k-i} = \frac{1}{2}$$

So $E(X) = \sum_{e \in E(G)} E[X_e] = \frac{\varepsilon(G)}{2}$. Thus $X \geq \frac{\varepsilon(G)}{2}$ for some choice of $T$, and the set of those odd-transversal edges forms an odd-bipartite sub-hypergraph. □

Theorem 5.2. Let $G = G_0(u) \diamond H(u)$ be a connected non-odd-bipartite $k$-uniform hypergraph, where $H$ is odd-bipartite. Then

$$\lambda_{\min}(G) \leq \frac{k\varepsilon(G_0)}{\nu(G)}$$

Proof. By Lemma 5.1, there is a proper subset $T$ of $V(G_0)$ such that the number of odd transversal edges of $G_0$ respect to $T$ is at least $\frac{\varepsilon(G_0)}{2}$. Let $\{U, W\}$ be an odd-bipartition of $H$, where $u \in U$. Define $x$ by

$$x_v = \begin{cases} 1, & \text{if } v \in T \cup U; \\ -1, & \text{otherwise}. \end{cases}$$
Then $\|x\|_k = \nu(G)$. We write $e \sim T$ (or $e \sim T$) to denote that $e$ is odd-transversal (or not odd-transversal) respect to $T$. By Eq. (2.3) and Lemma 5.1,

$$\lambda_{\min}(G) \leq \frac{Qx^k}{\|x\|_k} = \frac{1}{\nu(G)} \left( \sum_{e \in E(G_0), e \in T} (x_e^k + kx_e^k) + \sum_{e \in E(G_0), e \sim T} (x_e^k + kx_e^k) + \sum_{e \in E(H)} (x_e^k + kx_e^k) \right).$$

$$= \frac{1}{\nu(G)} \sum_{e \in E(G_0), e \sim T} 2k \leq \frac{k\varepsilon(G_0)}{\nu(G)}.$$

**Theorem 5.3.** Let $G = G_0(u) \circ H(u)$ be a connected non-odd-bipartite $k$-uniform hypergraph, where $H$ is odd-bipartite. Then

$$\lambda_{\min}(G) \leq \frac{d_{G_0}(u)}{\nu(H)}.$$

**Proof.** Let $\{U, W\}$ be an odd-bipartition of $H$, where $u \in U$. Define $x$ by

$$x_v = \begin{cases} 1, & \text{if } v \in U; \\ -1, & \text{if } v \in W; \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|x\|_k = \nu(H)$, and

$$\lambda_{\min}(G) \leq \frac{Qx^k}{\|x\|_k} = \frac{1}{\nu(H)} \left( \sum_{e \in E(G_0)} (x_e^k + kx_e^k) \right) + \frac{1}{\nu(H)} \left( \sum_{e \in E(H)} (x_e^k + kx_e^k) \right) = \frac{d_{G_0}(u)}{\nu(H)}.$$

**Remark 5.4.** In Theorem 5.2, the upper bound

$$\frac{k\varepsilon(G_0)}{\nu(G)} = \frac{k\varepsilon(G_0)}{\nu(G)} \frac{\nu(G_0)}{\nu(G)} = d(G_0) \frac{\nu(G_0)}{\nu(G)},$$

where $d(G_0)$ is the average degree of the vertices of $G_0$. So, if fixing $G_0$, and letting $H$ have enough vertices, then the bounds in Theorems 5.2 and 5.3 will be much smaller than $\delta(G)$, the upper bound in Lemma 2.6.

By Theorem 5.3 and Lemma 2.3, we get the following result immediately.

**Corollary 5.5.** Let $G = G_0(u) \circ T_m(u)$ be a connected non-odd-bipartite $k$-uniform hypergraph, where $T_m(u)$ is a hypertree with $m$ edges. Then

$$\lambda_{\min}(G) \leq \frac{d_{G_0}(u)}{(k-1)m + 1}.$$
By Lemma 3.2, $\lambda_{\min}(G_0^0(r) \diamond P_m^k(r))$ and $\lambda_{\min}(G_0^0(r) \diamond S_m^k(r))$ are both decreasing in $m$, which implies that they have limits. By Corollary 5.5, those two limits are both 0. As for a connected non-odd-bipartite hypergraph, its least eigenvalue is greater than 0. So we get the following result.

**Corollary 5.6.** Zero is the least limit point of the least eigenvalues of connected non-odd-bipartite hypergraphs.

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