Contractively decomposable projections on noncommutative $L^p$-spaces

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Abstract

We describe and characterize the contractively decomposable projections on noncommutative $L^p$-spaces. Our result relies on a new lifting result for decomposable maps of independent interest and on some tools from ergodic theory. Our theorem is new even for finite-dimensional Schatten spaces. Our description allows us to connect this topic with $W^*$-ternary rings of operators and a slight generalization of our result for more general projections makes JBW$^*$-triples appear in this context. We also prove that all rectangular $L^p$-spaces associated with $W^*$-ternary rings of operators arise as contractively decomposable complemented subspaces of noncommutative $L^p$-spaces. Finally, we introduce a notion of $L^p$-space associated to each $\sigma$-finite JBW$^*$-triple and we explain the link with the context of this paper.

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1 Introduction

The investigation of the structure of projections and contractively complemented subspaces is a well-established topic of Banach space geometry. Recall that a bounded operator $P: X \to X$ on a Banach space $X$ is a projection if $P^2 = P$. If a subspace $Y$ of $X$ is the range of a contractive linear projection, we say that it is contractively complemented. See the surveys [Ran01] and [Mos06] for more information on the literature on the matter. Observe that by [CoS70, Theorem

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6], a subspace of a smooth Banach space $X$ can be the range of at most one projection of norm one.

Ando proved in [And66] (see also [Dou65] for the case $p = 1$) that the contractively complemented subspaces of a classical (=commutative) $L^p$-space $L^p(\Omega)$ for a finite measure space $\Omega$ are all isometrically isomorphic to an $L^p$-space where $1 \leq p < \infty$. Moreover, he described the structure of these projections with the help of conditional expectations and multiplication operators. We refer to the papers [Tza69], [BeL74], [Lac74, Theorem 3 p. 162] and references therein for the case of general measures.

On noncommutative $L^p$-spaces, we cannot expect anything so simple. Indeed, if $\sigma : S^p \to S^p$ denotes the transpose map on the Schatten space $S^p \overset{\text{def}}{=} \{ x : \ell^2 \to \ell^2 : \|x\|_p \overset{\text{def}}{=} (\text{Tr}(|x|^p))^\frac{1}{p} < \infty \}$, then the map $P \overset{\text{def}}{=} \frac{1}{2}(\text{Id}_{S^p} + \sigma) : S^p \to S^p$ is a contractive projection on the subspace of symmetric matrices of $S^p$, in sharp contrast with the classical setting of measure spaces.

However, Arazy and Friedman succeeded in [ArF78] and [ArF92] in describing the structure of contractively complemented subspaces of $S^p$ for any $1 \leq p \leq \infty$. Such a subspace is isometrically isomorphic to a $\ell^p$-sum of $S^p$-Cartan factors of type I-IV. Recall that these Cartan factors are rectangular spaces of operators, spaces of antisymmetric operators, spaces of symmetric operators and complex spin factors. Moreover, the authors [ArF92, p. 99] explicitly raise the problem of describing contractively complemented subspaces of general noncommutative $L^p$-spaces.

Actually, Friedman and Russo showed in [FrB85] that the range of a contractive projection on a noncommutative $L^1$-space (=predual of von Neumann algebra) is isometric to the predual of a JW$^*$-triple. Recall that a JW$^*$-triple is a weak* closed subspace of the space $B(H,K)$ of bounded operators between Hilbert spaces $H$ and $K$ which is closed under the triple product $(x,y,z) \mapsto xy^*z + zy^*x$. Furthermore, it is proved in [NO02] that completely contractively complemented subspaces of noncommutative $L^1$-spaces coincide isometrically with the preduals of $W^*$-ternary rings of operators. This concept has its roots in the paper [Hes62] of Hestenes and has been studied by many authors. Recall that a $W^*$-ternary ring of operators is a weak* closed subspace of the space $B(H,K)$ for some Hilbert spaces $H$ and $K$ which is also closed under the triple product $(x,y,z) \mapsto xy^*z$. Ruan introduced a type decomposition of these objects in [Rua04]. Finally, note that $W^*$-ternary rings of operators, like $C^*$-algebras, carry a natural operator space structure and injective dual operator spaces coincide with the $W^*$-ternary rings of operators, see [EOR01]. We refer to [BFT12], [BuT19], [BuT13a], [BuT13b], [DoR07], [Ham92], [Ham99], [KaR02], [NeR03], [Pir19], [SaS13], [SaS17] and [Zet83] for other papers on this topic.

Finally, in [LRR09, Theorem 1.1], it is shown that any completely 1-complemented subspace of $S^p$ is isometric to a direct sum of spaces of the form $S^p(H,K)$, where $H$ and $K$ are Hilbert spaces for $1 < p < \infty$. The proof relies on the explicit description of contractively complemented subspaces of $S^p$ of Arazy and Friedman.

The structure of 2-positive contractive projections on arbitrary noncommutative $L^p$-spaces was completely elucidated in the paper [ArR19], three decades after the publication of [ArF92]. The range of such a projection is completely order and completely isometrically isomorphic to some noncommutative $L^p$-space. Furthermore, a description of 2-positive contractive projections was provided which was new even for Schatten spaces. The approach relies on a symmetric two-sided change of density and a lifting argument of the projection at the level $p = \infty$.

In [Arh20], we investigated more generally the case of positive contractive projections. We highlighted the role of Jordan conditional expectations in this topic and we showed in a large number of cases that the range of a positive contractive projection is isometric to a nonassociative $L^p$-space associated to a JW$^*$-algebra, a notion that we defined in [Arh23]. A JW$^*$-algebra
can be viewed as a Jordan generalization of von Neumann algebras and this notion was introduced by Edwards [Edw80] (see also [You78]).

The goal of this paper is to investigate the case of contractively decomposable projections. Decomposable maps are a generalization of completely positive maps. Recall that a linear map $T: L^p(\mathcal{M}) \to L^p(\mathcal{N})$ between noncommutative $L^p$-spaces associated to von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ is decomposable [JuR04, (3.2)] if there exist bounded linear maps $v_1, v_2: L^p(\mathcal{M}) \to L^p(\mathcal{N})$ such that the linear map

(1.1) $\Phi \overset{\text{def}}{=} \begin{bmatrix} v_1 & T \\ T^* & v_2 \end{bmatrix} : S^p_2(L^p(\mathcal{M})) \to S^p_2(L^p(\mathcal{N})), \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} v_1(a) & T(b) \\ T^*(c) & v_2(d) \end{bmatrix}$

is completely positive, where $T^*(c) \overset{\text{def}}{=} T(c^*)^*$. In this case, we let

(1.2) $\|T\|_{\text{dec}, L^p(\mathcal{M}) \to L^p(\mathcal{N})} \overset{\text{def}}{=} \inf \max \{\|v_1\|, \|v_2\|\}$

where the infimum is taken over all maps $v_1$ and $v_2$. We say that $T$ is contractively decomposable if $\|T\|_{\text{dec}, L^p(\mathcal{M}) \to L^p(\mathcal{N})} \leq 1$. Note that if the von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ are hyperfinite, it is equivalent to saying that $T$ is contractively regular by [ArK23, Theorem 3.24], which means that for any operator space $E$, the map $T \otimes \text{Id}_E$ induces a contraction between the vector-valued noncommutative $L^p$-spaces $L^p(\mathcal{M}, E)$ and $L^p(\mathcal{N}, E)$.

Our main result is the following theorem which describes precisely the structure of contractively decomposable projections acting on noncommutative $L^p$-spaces. Roughly speaking, it says that such a projection is induced at the level $p = \infty$ by a contractively decomposable projection modulo a suitable nonsymmetric two-sided change of density.

**Theorem 1.1** Let $\mathcal{M}$ be a $\sigma$-finite von Neumann algebra equipped with a normal faithful state $\varphi$. Suppose $1 < p < \infty$. A bounded map $P: L^p(\mathcal{M}) \to L^p(\mathcal{M})$ is a contractively decomposable projection if and only if there exist a normal faithful positive linear form $\varphi$ and the range of this map admits a structure of a $\mathcal{W}^*$-ternary ring of operators.

In this case, the restriction of $w$ on $s(h)M_2(k)$ is a weak$^*$ continuous contractively decomposable projection and the range of this map admits a structure of a $\mathcal{W}^*$-ternary ring of operators.
In the spirit of the paper [Arh20], it is expected that the range of a contractively decomposable projection is some kind of $L^p$-space of a $W^*$-ternary ring of operators. We did not clarify this point in this paper. In the opposite direction, we observe in Section 5 that all rectangular $L^p$-spaces associated with $W^*$-ternary rings of operators arise as contractively decomposable complemented subspaces of noncommutative $L^p$-spaces.

Actually, our main result admits a generalization (see Theorem 7.3) to a more general class of contractive projections, the contractively $n$-pseudo-decomposable projections defined in Section 4. In this case, the range of $w$ admits a structure of a $JW^*$-triple. Note that $JW^*$-triples are particular cases of $JBW^*$-triples. It is well-known that the latter objects are strongly connected to bounded symmetric domains in complex Banach spaces [Isi19] [Kau83] and to physical systems [Fri05].

Finally, we construct $L^p$-spaces associated to suitable $JBW^*$-triples in Section 8 by using complex interpolation generalizing the construction of nonassociative $L^p$-spaces associated to $JBW^*$-algebras of our previous paper [Arh23]. Moreover, we conjecture in Conjecture 9.1 that a contractively complemented subspace of a noncommutative $L^p$-space is isometric to the $L^p$-space of a $JW^*$-triple.

**Approach of the paper** Suppose $1 < p < \infty$. Consider a contractively decomposable projection $P: L^p(M) \to L^p(M)$. The first step of our approach consists in showing that we can suppose in (1.1) with $T = P$ that the maps $v_1$ and $v_2$ are (completely positive) contractive projections $P_1$ and $P_2$ using ergodic theory (see Proposition 7.1). As a second step, we will show that a non-symmetric two-sided change of density allows to reduce the analysis to the case $p = \infty$ by some lifting argument (see Theorem 6.2) relying on the lifting result of [Arh20] achieved by a symmetric two-sided change of density (see Theorem 6.1). This lifting trick is of independent interest. Of course, a difficulty is the choice of suitable densities in order to perform the change of density. It should also be noted that the use of non-tracial Haagerup noncommutative $L^p$-spaces is essential in general for the case of tracial noncommutative $L^p$-spaces.

**Structure of the paper** The paper is organized as follows. Section 2 gives a brief presentation of Haagerup noncommutative $L^p$-spaces, followed by some preliminary facts that are at the root of our results. Section 3 contains background on TRO’s, Jordan algebras, $JBW^*$-triples which are necessary for our paper. In Section 4, we investigate properties of $n$-pseudo-decomposable maps which are a slight generalization of decomposable maps. The proofs in this part are (essentially) similar to the proofs of [ArK23]. In Section 5, we show that all rectangular $L^p$-spaces associated with $W^*$-ternary rings of operators arise as contractively decomposable complemented subspaces of noncommutative $L^p$-spaces. In Section 6, we present our new lifting result for ($n$-pseudo-)decomposable maps on noncommutative $L^p$-spaces. Section 7 contains a proof of (our main result) Theorem 1.1 which relies on our lifting result and ergodic theory. Indeed, we provide a more general statement for $n$-pseudo-decomposable maps which are defined in Section 4. In Section 8, we introduce $L^p$-spaces associated to $\sigma$-finite $JBW^*$-triples with the help of complex interpolation. Our construction generalizes the one of nonassociative $L^p$-spaces associated to $\sigma$-finite $JBW^*$-algebras introduced in [Arh23]. Finally, in Section 9, we describe open questions raised by the contents of this paper.
2 Preliminaries on Haagerup noncommutative $L^p$-spaces and Banach spaces

The readers are referred to the books [EfR00], [Pan02], [Pis03] and [Pis98] for details on operator spaces and completely bounded maps and to [Kos14], [PiX03], [Ray03] and [Ter81] for information on noncommutative $L^p$-spaces and references therein.

It is well-known that there are several equivalent constructions of noncommutative $L^p$-spaces associated with a von Neumann algebra. In Section 6 and Section 7, we will use Haagerup noncommutative $L^p$-spaces introduced in [Haa79] and described more precisely in [Ter81]. We denote by $s(x)$ the support of a positive operator $x$. If $M$ is a von Neumann algebra equipped with a normal semifinite faithful trace, then the topological $*$-algebra of all (unbounded) $\tau$-measurable operators $x$ affiliated with $M$ is denoted by $L^0(M, \tau)$.

In the sequel, we fix a normal semifinite faithful weight $\varphi$ on a von Neumann algebra $M$ acting on a Hilbert space $H$. The one-parameter modular automorphisms group associated with $\varphi$ is denoted by $\sigma^\varphi = (\sigma^\varphi_t)_{t \in \mathbb{R}}$ [Tak03, p. 92].

For $1 \leq p < \infty$, the spaces $L^p(M)$ are constructed as spaces of measurable operators with respect to some semifinite bigger von Neumann algebra, namely, the crossed product $\hat{M} \overset{\text{def}}{=} M \rtimes \sigma^\varphi \mathbb{R}$ of $M$ by the modular automorphisms group $\sigma^\varphi$, that is, the von Neumann subalgebra of $B(L^2(\mathbb{R}, H))$ generated by the operators $\pi(x)$ and $\lambda_s \otimes \text{Id}_H$, where $x \in M$ and $s \in \mathbb{R}$, defined by

\[(2.1) \quad (\pi(x)\xi)(t) \overset{\text{def}}{=} \sigma_{-t}^\varphi(x)(\xi(t)) \quad \text{and} \quad (\lambda_s \otimes \text{Id}_H)(\xi(t)) \overset{\text{def}}{=} \xi(t-s), \quad t \in \mathbb{R}, \ \xi \in L^2(\mathbb{R}, H).
\]

For any $s \in \mathbb{R}$, let $W(s)$ be the unitary operator on $L^2(\mathbb{R}, H)$ defined by

\[(2.2) \quad (W(s)\xi)(t) \overset{\text{def}}{=} e^{-ist}\xi(t), \quad \xi \in L^2(\mathbb{R}, H).
\]

The dual action $\tilde{\sigma} : \mathbb{R} \to B(\hat{M})$ on $M$ [Tak03, p. 260] is given by

\[(2.3) \quad \tilde{\sigma}_s(x) \overset{\text{def}}{=} W(s)xW(s)^*, \quad x \in \hat{M}, \ s \in \mathbb{R}.
\]

Then, by [Haa78a, Lemma 3.6] or [Tak03, p. 259], $\pi(M)$ is the fixed subalgebra of $\hat{M}$ under the family of automorphisms $\tilde{\sigma}_s$:

\[(2.4) \quad \pi(M) = \{ x \in \hat{M} : \tilde{\sigma}_s(x) = x \quad \text{for all} \ s \in \mathbb{R} \}.
\]

We identify $M$ with the subalgebra $\pi(M)$ in the crossed product $\hat{M}$. If $\psi$ is a normal semifinite weight on $M$, we denote by $\tilde{\psi}$ its Takesaki’s dual weight on the crossed product $\hat{M}$. We can give the following definition of [Haa78b] using the theory of operator valued weights. Indeed, Haagerup introduces an operator valued weight $T : M^+ \to M^+$ with values in the extended positive part $1^+ M^+$ of $M$, formally defined by

\[(2.5) \quad T(x) = \int_\mathbb{R} \tilde{\sigma}_s(x) \, ds
\]

and shows that for a normal semifinite weight $\psi$ on $M$, its dual weight is

\[(2.6) \quad \tilde{\psi} \overset{\text{def}}{=} \psi \circ T
\]

1. If $M = L^\infty(\Omega)$ then $M^+$ identifies to the set of equivalence classes of measurable functions $\Omega \to [0, \infty]$. 

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where \( \tilde{\psi} \) denotes the natural extension of the normal weight \( \psi \) to \( \hat{\mathcal{M}}^+ \). This dual weight satisfies the \( \hat{\sigma} \)-invariance relation \( \psi \circ \hat{\sigma} = \psi \), see [Ter81, (10)].

By [Str81, p. 301] [Haa78a, Theorem 3.7] [Ter81, Chap. II, Lemma 1], the map \( \psi \rightarrow \hat{\psi} \) is a bijection from the set of normal semifinite weights on \( \mathcal{M} \) onto the set of normal semifinite \( \hat{\sigma} \)-invariant weights on \( \hat{\mathcal{M}} \).

Recall that by [Haa78b, Lemma 5.2 and Remark p. 343] and [Haa78b, Theorem 1.1 (c)] the crossed product \( \hat{\mathcal{M}} \) is semifinite and there is a unique normal semifinite faithful trace \( \tau = \tau_\varphi \) on \( \hat{\mathcal{M}} \) satisfying

\[
(2.7) \quad (D_{\hat{\varphi}} : D\tau)_t = \lambda t \otimes \text{id}_H, \quad t \in \mathbb{R}
\]

where \( (D_{\hat{\varphi}} : D\tau)_t \) denotes the Connes cocycle [Str81, p. 48] [Tak03, p. 111] of the dual weight \( \hat{\varphi} \) with respect to \( \tau \). Moreover, \( \tau \) satisfies the relative invariance \( \tau \circ \hat{\sigma}_s = e^{-s}\tau \) for any \( s \in \mathbb{R} \) by [Haa781, Lemma 5.2].

If \( \psi \) is a normal semifinite weight on \( \mathcal{M} \), we denote by \( h_\psi \) the Pedersen-Takesaki derivative of the dual weight \( \hat{\psi} \) with respect to \( \tau \) given by [Str81, Theorem 4.10]. By [Str81, Corollary 4.8], note that the relation of \( h_\psi \) with the Radon-Nikodym cocycle of \( \hat{\psi} \) is

\[
(2.8) \quad (D_{\hat{\psi}} : D\tau)_t = h_\psi^t, \quad t \in \mathbb{R}.
\]

If \( \psi = \varphi \), we let \( D_{\varphi} \overset{\text{def}}{=} h_\varphi \) and we call it the density operator of \( \varphi \).

By [Ter81, Chap. II, Prop. 4], the mapping \( \psi \rightarrow h_\psi \) gives a bijective correspondence between the set of all normal semifinite weights on \( \mathcal{M} \) and the set of positive selfadjoint operators \( h \) affiliated with \( \mathcal{M} \) satisfying

\[
(2.9) \quad \hat{\sigma}_s(h) = e^{-s}h, \quad s \in \mathbb{R}.
\]

Moreover, by [Ter81, Chap. II, Cor. 6], \( \omega \) belongs to \( \mathcal{M}_+ \) if and only if \( h_\omega \) belongs to \( L^0(\hat{\mathcal{M}}, \tau)_+ \). One may extend by linearity the map \( \omega \rightarrow h_\omega \) on \( \mathcal{M}_+ \). The Haagerup space \( L^1(\mathcal{M}, \varphi) \) is defined as the set \( \{ h_\omega : \omega \in \mathcal{M}_+ \} \), i.e. the range of the previous map.

By [Ter81, Chap. II, Th. 7], the mapping \( \omega \rightarrow h_\omega , \mathcal{M}_+ \rightarrow L^1(\mathcal{M}, \varphi) \) is a linear order isomorphism which preserves the conjugation, the module, and the left and right actions of \( \mathcal{M} \). Then \( L^1(\mathcal{M}, \varphi) \) may be equipped with a continuous linear functional \( \text{Tr}_\varphi : L^1(\mathcal{M}) \rightarrow \mathbb{C} \) defined by

\[
(2.10) \quad \text{Tr}_\varphi(h_\omega) \overset{\text{def}}{=} \omega(1), \quad \omega \in \mathcal{M}_+.
\]

[Ter81, Chap. II, Def. 13]. We also use the notation \( \text{Tr} \) instead of \( \text{Tr}_\varphi \). A norm on \( L^1(\mathcal{M}, \varphi) \) may be defined by

\[
\|h\|_1 \overset{\text{def}}{=} \text{Tr}(|h|) \quad \text{for every } h \in L^1(\mathcal{M}, \varphi). \quad \text{By [Ter81, Chap. II, Prop. 15], the map } \mathcal{M}_+ \rightarrow L^1(\mathcal{M}, \varphi), \omega \mapsto h_\omega \text{ is a surjective isometry.}
\]

More generally for \( 1 \leq p \leq \infty \), the Haagerup \( L^p \)-space \( L^p(\mathcal{M}, \varphi) \) associated with the normal faithful semifinite weight \( \varphi \) is defined [Ter81, Chap. II, Def. 9] as the subset of the topological \( * \)-algebra \( L^0(\mathcal{M}, \tau) \) of all (unbounded) \( \tau \)-measurable operators \( x \) affiliated with \( \mathcal{M} \) satisfying for any \( s \in \mathbb{R} \) the condition

\[
(2.11) \quad \hat{\sigma}_s(x) = e^{-s}\hat{\varphi}x \quad \text{if } p < \infty \quad \text{and} \quad \hat{\sigma}_s(x) = x \quad \text{if } p = \infty
\]

where \( \hat{\sigma}_s : L^0(\mathcal{M}, \tau) \rightarrow L^0(\mathcal{M}, \tau) \) is here the continuous \( * \)-automorphism obtained by a natural extension of the dual action (2.3) on \( \mathcal{M} \). By (2.4), the space \( L^\infty(\mathcal{M}, \varphi) \) coincides with \( \pi(\mathcal{M}) \) that we identify with \( \mathcal{M} \). The spaces \( L^p(\mathcal{M}, \varphi) \) are closed selfadjoint linear subspaces of \( L^0(\mathcal{M}, \tau) \).
They are closed under left and right multiplications by elements of \( \mathcal{M} \). If \( h = u|h| \) is the polar decomposition of \( h \in L^0(\mathcal{M}, \tau) \) then by [Ter81, Chap. II, Prop. 12] we have

\[
h \in L^p(\mathcal{M}, \varphi) \iff u \in \mathcal{M} \text{ and } |h| \in L^p(\mathcal{M}, \varphi).
\]

Suppose \( 1 \leq p < \infty \). By [Ter81, Chap. II, Prop. 12] and its proof, for any \( h \in L^0(\mathcal{M}, \tau)_+ \), we have \( h^p \in L^0(\mathcal{M}, \tau)_+ \). Moreover, an element \( h \in L^0(\mathcal{M}, \tau) \) belongs to \( L^p(\mathcal{M}, \varphi) \) if and only if \( |h|^p \) belongs to \( L^1(\mathcal{M}, \varphi) \). A norm on \( L^p(\mathcal{M}, \varphi) \) is then defined by the formula

\[
(2.12) \quad \|h\|_p \overset{\text{def}}{=} (\operatorname{Tr}|h|^p)^{\frac{1}{p}}
\]

if \( 1 \leq p < \infty \) and by \( \|h\|_\infty \overset{\text{def}}{=} \|h\|_M \), see [Ter81, Chap. II, Def. 14].

**Case of a normal faithful positive linear form** If \( \varphi \) is a normal faithful positive linear form on \( \mathcal{M} \) then by [HJX10, (1.13)] the density operator \( D_\varphi \) belongs to \( L^1(\mathcal{M}, \varphi) \) and

\[
(2.13) \quad \varphi(x) = \operatorname{Tr}_\varphi(D_\varphi x) = \operatorname{Tr}_\varphi(xD_\varphi), \quad x \in \mathcal{M}.
\]

**Duality** Let \( p, p^* \in [1, \infty) \) with \( \frac{1}{p} + \frac{1}{p^*} = 1 \). By [Ter81, Chap. II, Prop. 21], for any \( h \in L^p(\mathcal{M}, \varphi) \) and any \( k \in L^{p^*}(\mathcal{M}, \varphi) \) the elements \( hk \) and \( kh \) belong to the space \( L^1(\mathcal{M}, \varphi) \) and we have the tracial property \( \operatorname{Tr}(hk) = \operatorname{Tr}(kh) \).

If \( 1 \leq p < \infty \), by [Ter81, Ch. II, Th. 32] the bilinear form \( L^p(\mathcal{M}, \varphi) \times L^{p^*}(\mathcal{M}, \varphi) \to \mathbb{C} \), \((h, k) \mapsto \operatorname{Tr}(hk)\) defines a duality bracket between \( L^p(\mathcal{M}, \varphi) \) and \( L^{p^*}(\mathcal{M}, \varphi) \), for which \( L^{p^*}(\mathcal{M}, \varphi) \) is isometrically dual to \( L^p(\mathcal{M}, \varphi) \).

**Change of weight** It is essentially proved in [Ter81, p. 59] and also [Ray03, Theorem 5.1] that \( L^p(\mathcal{M}, \varphi) \) is independent of \( \varphi \) up to an isometric isomorphism preserving the order and bimodule structure of \( L^p(\mathcal{M}, \varphi) \), as well as the external products and Mazur maps. In fact given two normal semifinite faithful weights \( \varphi_1, \varphi_2 \) on \( \mathcal{M} \) there is an \( \ast \)-isomorphism \( \kappa: \mathcal{M}_1 \to \mathcal{M}_2 \) between the crossed products \( \mathcal{M}_1 \overset{\text{def}}{=} \mathcal{M} \rtimes_{\varphi_1} \mathbb{R} \) preserving \( \mathcal{M} \), as well as the dual actions and the traces of \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \), that is

\[
(2.14) \quad \pi_2 = \kappa \circ \pi_1, \quad \hat{\sigma}_2 \circ \kappa = \kappa \circ \hat{\sigma}_1 \quad \text{and} \quad \tau_2 = \tau_1 \circ \kappa^{-1}.
\]

Furthermore, \( \kappa \) extends naturally to a topological \( \ast \)-isomorphism \( \kappa: L^0(\mathcal{M}_1, \tau_1) \to L^0(\mathcal{M}_2, \tau_2) \) between the algebras of measurable operators, which restricts to isometric \( \ast \)-isomorphisms between the noncommutative \( L^p \)-spaces \( L^p(\mathcal{M}, \varphi_1) \) and \( L^p(\mathcal{M}, \varphi_2) \), preserving the \( \mathcal{M} \)-bimodule structures. Finally \( \kappa: L^1(\mathcal{M}, \varphi_1) \to L^1(\mathcal{M}, \varphi_2) \) preserves the traces:

\[
(2.15) \quad \operatorname{Tr}_{\varphi_1} = \operatorname{Tr}_{\varphi_2} \circ \kappa.
\]

Since \( \kappa \) preserves the \( p \)-powers operations, i.e. \( \kappa(h^p) = (\kappa(h))^p \) for any \( h \in L^0(\mathcal{M}) \), it induces an isometry from \( L^p(\mathcal{M}, \varphi_1) \) onto \( L^p(\mathcal{M}, \varphi_2) \). It is not hard to see that this isometry is a completely order isomorphism.

This independence allows us to consider \( L^p(\mathcal{M}, \varphi) \) as a particular realization of an abstract space \( L^p(\mathcal{M}) \).

**Centralizer of a normal faithful positive linear form** Recall that the centralizer [Str81, p. 38] of a normal faithful positive linear form is the von Neumann subalgebra \( \mathcal{M}^\varphi \overset{\text{def}}{=} \{ x \in \mathcal{M} : \sigma^\varphi_t(x) = x \text{ for all } t \in \mathbb{R} \} \). If \( x \in \mathcal{M} \), we have by [Str81, (2) p. 39]

\[
(2.16) \quad x \in \mathcal{M}^\varphi \iff \varphi(xy) = \varphi(yx) \text{ for any } y \in \mathcal{M}.
\]
Reduced noncommutative $L^p$-spaces Let $\varphi$ be a normal faithful positive linear form. If the orthogonal projection $e$ belongs to the centralizer of $\varphi$, we can consider the restriction $\varphi_e$ of $\varphi$ on $e\mathcal{M}e$. It is well-known that we can identify the Banach space $L^p(e\mathcal{M}e, \varphi_e)$ with the subspace $eL^p(\mathcal{M}, \varphi)e$ of $L^p(\mathcal{M}, \varphi)$, see [Wat88, p. 508]. Moreover, we have the following result.

Lemma 2.1 The Haagerup trace $\text{Tr}_\varphi$ restricts to $\text{Tr}_{\varphi_e}$ on $L^1(e\mathcal{M}e)$.

Let $e$ be an orthogonal projection of $\mathcal{M}$. Let us construct a normal faithful positive linear form with centralizer containing $e$. Consider two normal faithful positive linear forms $\varphi_1$ and $\varphi_2$ on $e\mathcal{M}e$ and $e^+\mathcal{M}e^+$. By [RaX03, p. 155], we can define a normal faithful positive linear form $\phi$ on $\mathcal{M}$ by

$$\phi(x) \overset{\text{def}}{=} \varphi_1(exe) + \varphi_2(e^+xe^{-1}), \quad x \in \mathcal{M}_+.$$  

Moreover, $e$ belongs to the centralizer of $\phi$ by (2.16) and we have $\phi_e = \varphi_1$.

Matrix order and norms We refer to [SkV19, Lemma A.2] and [Hia21, p. 155] for more information on the identification between $S^p_2(L^p(\mathcal{M}))$ and $L^p(M_n(\mathcal{M}))$. The next lemma is an easy generalization of [ArK23, Lemma 2.13] (for semifinite von Neumann algebras) left to the reader using reduction theory [HJX10].

Lemma 2.2 Let $\mathcal{M}$ be a von Neumann algebra. Suppose $1 \leq p \leq \infty$. Let $a, b$ and $c$ be elements of $L^p(\mathcal{M})$ such that the element $\begin{bmatrix} a & b \\ b^* & c \end{bmatrix}$ of $S^p_2(L^p(\mathcal{M}))$ is positive. Then we have

$$\|b\|_{L^p(\mathcal{M})} \leq \sqrt{\|a\|_{L^p(\mathcal{M})}\|c\|_{L^p(\mathcal{M})}}.$$  

The following is [Han09, Lemma 2.5].

Lemma 2.3 Suppose $\mathcal{M}$ is a von Neumann algebra equipped with a faithful semifinite normal weight $\varphi$. If $x$ is an element of the Haagerup noncommutative $L^p$-space $L^p(\mathcal{M})$, then we have

$$\begin{bmatrix} |x^*| & x \\ x^* & |x| \end{bmatrix} \in L^p(M_2(\mathcal{M})).$$  

Extension of maps on noncommutative $L^p$-spaces Let $\mathcal{N}$ be a von Neumann algebra equipped with a normal faithful linear form $\psi$. Consider a unital positive map $T: \mathcal{N} \to \mathcal{N}$ such that $\psi \circ T = \psi$. Given $1 \leq p < \infty$, the map

$$T_p: \quad D^{\frac{1}{p}}_\psi \mathcal{N} D^{\frac{1}{p}}_\psi \quad \mapsto \quad L^p(\mathcal{N})$$  

extends to a contractive map $T_p$ from $L^p(\mathcal{N})$ into $L^p(\mathcal{N})$. See [HJX10, Remark 5.6].

The following is a folklore observation, see [ArR19, Lemma 2.2] for a proof.

Lemma 2.4 Let $\mathcal{M}$ be a von Neumann algebra and $1 \leq p < \infty$. Let $h$ be a positive element of $L^p(\mathcal{M})$.

1. The map $s(h)\mathcal{M}s(h) \to L^p(\mathcal{M})$, $x \mapsto h^+ x h^+$ is injective.

2. Suppose $1 \leq p < \infty$. The subspace $h^+ \mathcal{M} h^+$ is dense in $s(h)L^p(\mathcal{M})s(h)$ for the topology of $L^p(\mathcal{M})$.  

8
**Normalized duality mappings** Recall that a normed linear space $X$ is said to be strictly convex (or rotund) if for any $x, y \in X$ the equalities $\frac{\|x+y\|}{2} = \|x\|, \|y\|$ imply $x = y$.

Let $X$ be a Banach space. For each $x \in X$, we can associate [Pat18, Definition 2.12] the subset

$$J_X(x) \overset{\text{def}}{=} \{ x^* \in X^* : \langle x, x^* \rangle_{X^*,X} = \|x\|_X^2 = \|x^*\|_{X^*}^2 \}$$

of the dual $X^*$.

The multivalued operator $J_X : X \to X^*$ is called the normalized duality mapping of $X$. From the Hahn-Banach theorem, for every $x \in X$, there exists $y^* \in X^*$ with $\|y^*\|_{X^*} = 1$ such that $\langle x, y^* \rangle_{X^*,X} = \|x\|_X$. Using $x^* = \|x\|_{X^*} y^*$, we conclude that $J_X(x) \neq \emptyset$ for each $x \in X$. If the dual space $X^*$ is strictly convex, $J_X$ is single-valued.

When $X$ is a reflexive strictly convex Banach space with a reflexive strictly convex dual space $X^*$, $J_X$ is a single-valued bijective map and its inverse $J_X^{-1} : X^* \to X^{**} = X$ is equal to $J_X : X^* \to X$.

If the Banach space $X$ is a noncommutative $L^p$-space, we have the following explicit description of the normalized duality mapping, see [ArR19] for a proof.

**Lemma 2.5** Suppose $1 < p < \infty$. If $h$ belongs to $L^p(M)$ with polar decomposition $h = u|h|$ then we have

$$J_{L^p(M)}(h) = \|h\|_p^{2-p}|h|^{p-1}u^*.$$ 

The next crucial result is proved in [Arh20].

**Lemma 2.6** Let $X$ be a smooth strictly convex reflexive Banach space. Let $P : X \to X$ be a contractive projection and $x$ be an element of $X$. Then $x$ belongs to $\text{Ran} P$ if and only if $J_X(x)$ belongs to $\text{Ran} P^*$.

**Projections and interpolation** We start by recalling some background on complex interpolation theory. We refer to the books [BelL76], [KPS82] and [Lun18] for more information. Let $Y_0, Y_1$ be two Banach spaces which embed into a topological vector space $\tilde{Y}$. We say that $(Y_0, Y_1)$ is an interpolation couple. Then the sum $Y_0 + Y_1 \overset{\text{def}}{=} \{ y \in \tilde{Y} : y = y_1 + y_2 \text{ for some } y_1 \in Y_1, y_2 \in Y_2 \}$ is well-defined and equipped with the norm

$$\|y\|_{Y_0 + Y_1} \overset{\text{def}}{=} \inf_{y = y_0 + y_1} (\|y_0\|_{Y_0} + \|y_1\|_{Y_1}).$$

The intersection $Y_0 \cap Y_1$ is equipped with the norm

$$\|y\|_{Y_0 \cap Y_1} \overset{\text{def}}{=} \max\{\|y\|_{Y_0}, \|y\|_{Y_1}\}, \quad y \in Y_0 \cap Y_1.$$ 

Consider the closed strip $S \overset{\text{def}}{=} \{ z \in \mathbb{C} : 0 \leq \text{Re} z \leq 1 \}$. Let us denote by $\mathcal{F}(Y_0, Y_1)$ the family of bounded continuous functions $f : S \to Y_0 + Y_1$, holomorphic on the open strip $S = \{ z \in \mathbb{C} : 0 < \text{Re} z < 1 \}$ inducing continuous functions $\mathbb{R} \to Y_0, t \mapsto f(it)$ and $\mathbb{R} \to Y_1, t \mapsto f(1 + it)$ which tend to 0 when $|t|$ goes to infinity. For any $f \in \mathcal{F}(Y_0, Y_1)$, we set

$$\|f\|_{\mathcal{F}(Y_0, Y_1)} \overset{\text{def}}{=} \max\left\{ \sup_{t \in \mathbb{R}} \|f(it)\|_{Y_0}, \sup_{t \in \mathbb{R}} \|f(1 + it)\|_{Y_1} \right\}.$$ 

If $0 \leq \theta \leq 1$, we define the subspace $(Y_0, Y_1)_{\theta} \overset{\text{def}}{=} \{ f(\theta) : f \in \mathcal{F}(Y_0, Y_1) \}$ of the Banach space $(Y_0 + Y_1)_{\theta}$. For any $y \in (Y_0, Y_1)_{\theta}$, we let

$$\|y\|_{(Y_0, Y_1)_{\theta}} \overset{\text{def}}{=} \inf \{ \|f\|_{\mathcal{F}(Y_0, Y_1)} : f \in \mathcal{F}(Y_0, Y_1), f(\theta) = y \).
Then by [Bel76, Theorem 4.1.2] \( (Y_0, Y_1)_{\theta} \) equipped with this norm is a Banach space.

We will use the following well-known result. See e.g. [ArK23, Lemma 4.8] for a slightly more general statement.

**Lemma 2.7** Let \( (Y_0, Y_1) \) be an interpolation couple and let \( C \) be a contractively complemented subspace of \( Y_0 + Y_1 \). We assume that the corresponding contractive projection \( P : Y_0 + Y_1 \to Y_0 + Y_1 \) satisfies \( P(Y_i) \subset Y_i \) and that the restriction \( P : Y_i \to Y_i \) is contractive for \( i = 0, 1 \). Then \( (Y_0 \cap C, Y_1 \cap C) \) is an interpolation couple and the canonical inclusion \( J : C \to Y_0 + Y_1 \) induces an isometric isomorphism \( J \) from \( (Y_0 \cap C, Y_1 \cap C)_{\theta} \) onto the subspace \( P((Y_0, Y_1)_{\theta}) = (Y_0, Y_1)_{\theta} \cap C \) of \( (Y_0, Y_1)_{\theta} \).

### Strictly monotone ordered Banach spaces

An ordered Banach space \( X \) (or its norm) is said to be monotone if \( 0 \leq x \leq y \) implies \( \|x\| \leq \|y\| \) for any \( x, y \in X \). A monotone ordered Banach space \( X \) is said to be strictly monotone if \( 0 \leq x \leq y \) and \( x \neq y \) implies \( \|x\|_X < \|y\|_X \).

We will use the following crucial lemma of [Arch20]. For the sake of completeness, we give the following short argument.

**Lemma 2.8** Let \( X \) be a strictly monotone ordered Banach space. Let \( T : X \to X \) be a contraction. Consider \( x, y \in X \) such that \( 0 \leq x \leq y \) with \( T(y) = y \) and \( T(x) = 0 \). Then \( x = 0 \).

**Proof** : We have

\[
y = T(y) - T(x) = T(y - x).
\]

Since \( T \) is contractive, we deduce that

\[
\|y\|_X = \|T(y) - T(x)\|_X = \|T(y - x)\|_X \leq \|y - x\|_X.
\]

Since \( 0 \leq y - x \leq y \) we infer that

\[
\|y\|_X = \|y - x\|_X
\]

and finally \( x = 0 \) by strict monotonicity of the norm.

---

### 3 Preliminaries on TRO’s, Jordan algebras and JBW*-triples

We start by providing a little overview of the theory of ternary rings of operators.

**Ternary rings of operators** A ternary ring of operators (or simply TRO) is a norm closed subspace \( V \) of the space \( B(H, K) \) for some Hilbert spaces \( H \) and \( K \) which is closed under the triple product \( (x, y, z) \mapsto xy^*z \), see e.g. [BLM04, 4.4.1 p. 161]. A TRO \( V \) is called a \( W^* \)-TRO if it is weak* closed in the dual Banach space \( B(H, K) \). A sub-TRO of a TRO \( V \) is a closed subspace \( W \) of \( V \) satisfying \( WW^*W \subset W \). We refer to the references given in Section 1 for more information on TROs.

**Example 3.1** A basic example of \( W^* \)-TRO is given by \( eMf \) where \( e, f \) are orthogonal projections of a von Neumann algebra \( \mathcal{M} \).

**Operator space structures of TROs** We also note that every TRO admits an operator space structure. Indeed, let us assume that \( V \) is a TRO contained in \( B(H, K) \). Then for each \( n \in \mathbb{N} \), the matrix space \( M_n(V) \) can be identified with a TRO contained in \( M_n(B(H, K)) = B(H^n, K^n) \). This provides a canonical operator space matrix norm on \( V \) such that each \( M_n(V) \) is again a TRO. By [KaR02, Proposition 2.1], the TRO-matrix norms are uniquely determined on each TRO and does not depend on the choice of the representing Hilbert spaces.
TRO-homomorphisms  A TRO-homomorphism (or triple morphism) is a linear map \( T : V \to W \) between two TROs respecting the ternary product:

\[
T(xy^*z) = T(x)T(y)^*T(z), \quad x, y, z \in V.
\]

If in addition, \( T \) is an injection from \( V \) onto \( W \), we call \( T \) a TRO-isomorphism from \( V \) onto \( W \). By [EOR01, Proposition 2.4], a TRO-isomorphism between \( W^*-\)TROs is necessarily weak*-continuous. By [EOR01, Proposition 2.1], every TRO-homomorphism is completely contractive, and every injective TRO-homomorphism is completely isometric. Finally, if \( T : V \to W \) is a linear map between TRO’s then by [Ham99, Proposition 2.1] (see also [BLM04, Corollary 4.4.6 p. 163]) \( T \) is a surjective complete isometry if and only if \( T \) is a surjective 2-isometry if and only if \( T \) is a TRO-isomorphism.

Example 3.2 Every finite-dimensional TRO \( V \) is completely isometric (i.e. triple isomorphic) to a finite direct sum of rectangular matrix algebras, i.e. it has the form

\[
V = M_{m_1,n_1} \oplus \cdots \oplus M_{m_k,n_k},
\]

see [EOR01], [Kan13, Corollary A.2] and [Smi00].

Linking algebra  If \( \mathcal{M} \) is a von Neumann algebra and \( e \) is an orthogonal projection in \( \mathcal{M} \), then \( e \mathcal{M}(1-e) \) is a \( W^* \)-TRO by Example 3.1. Conversely, if the subspace \( V \) of \( B(H,K) \) is a \( W^* \)-TRO, then we can consider the subspace \( V^* \) of \( B(K,H) \), the von Neumann subalgebras \( M(V) = \overline{V^*V}^* \), \( N(V) = \overline{VV^*}^* \) of \( B(K) \) and \( B(H) \) and the von Neumann algebra

\[
R(V) = \begin{bmatrix} M(V) & V^* \\ V & N(V) \end{bmatrix} \subset B(K \oplus H)
\]

which is called the linking von Neumann algebra of \( V \), see [Rua04, p. 846]. Then there exists a TRO-isomorphism \( V = eR(V)e^\perp \), where \( e \) and \( e^\perp \) are defined as follows:

\[
e = \begin{bmatrix} \text{Id}_K & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad e^\perp = \begin{bmatrix} 0 & 0 \\ 0 & \text{Id}_H \end{bmatrix}.
\]

Completely contractive projections on TROs  We will use the following result [BLM04, Theorem 4.4.9 p. 165] (it is essentially a slight generalization of a result of Youngson [You83]).

Theorem 3.3 Let \( V \) be a TRO and \( P : V \to V \) be a completely contractive projection. For any \( x,y,z \in V \), we have

\[
P(P(x)P(y)^*P(z)) = P(xP(y)^*P(z)) = P(P(x)y^*P(z)) = P(P(x)P(y)^*z).
\]

Furthermore, the range \( \text{Ran}(P) \) of \( P \) is a TRO with triple product \( (x,y,z) \mapsto P(xy^*z) \).

Let \( W \) be a sub-TRO of a TRO \( V \). A projection \( P : V \to V \) on a sub-TRO \( W \) is a TRO-conditional expectation on \( W \) [EOR01, p. 499] if we have

\[
P(zx^*y) = P(z)x^*y, \quad P(xz^*y) = xP(z)^*y, \quad P(xy^*z) = xy^*P(z), \quad z \in V, x,y \in W.
\]

Remark 3.4 Let \( P : V \to V \) be a projection from \( V \) onto a sub-TRO \( W \). Then by [EOR01, Theorem 2.3] (see also [BLM04, Corollary 4.4.10 p. 166]) the following properties are equivalent:

1. \( P \) is completely contractive,
2. $P$ is contractive,

3. $P$ is a contractive conditional expectation.

Now, we give a brief presentation of JBW$^\ast$-algebras, since these algebras provide examples of JBW$^\ast$-triples (see Example 3.7) and since the $L^p$-spaces introduced in Section 8 generalize the nonassociative $L^p$-spaces of JBW$^\ast$-algebras defined in the paper [Arh23].

**Various Jordan algebras** A Jordan algebra $A$ over a field $K$ is a vector space $A$ over $K$ equipped with a commutative bilinear product that satisfies $(x^2 \circ y) \circ x = x^2 \circ (y \circ x)$ for any $x, y \in A$. A Jordan algebra $A$ over $\mathbb{R}$ is called formally real [HOS84, p. 69] if for any $x_1, \ldots, x_n \in A$ the relation $x_1^2 + \cdots + x_n^2 = 0$ implies $x_1 = \cdots = x_n = 0$. Following [AlS03, Definition 1.5 p. 5], a JB-algebra is a Jordan algebra over $\mathbb{R}$ with identity element $1$ equipped with a complete norm satisfying the properties $\|x \circ y\| \leq \|x\| \|y\|$, $\|x^2\| = \|x\|^2$, $\|x^2\| \leq \|x^2 + y^2\|$ for any $x, y \in A$. A JBW-algebra is a JB-algebra which is a dual Banach space [HOS84, p. 111]. In this case, the predual is unique. Recall that a JB-algebra is always unital by [HOS84, Lemma 4.1.7].

**Example 3.5** If $\mathbb{O}$ is the algebra of octonions, then the space $H_3(\mathbb{O}) = \left\{ \begin{bmatrix} \alpha & \alpha & \beta \\ \bar{\alpha} & b & \gamma \\ \beta & \bar{\gamma} & c \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{O}, a, b, c \in \mathbb{R} \right\}$ of hermitian 3x3 matrices with entries in $\mathbb{O}$ equipped with the product $(x, y) \mapsto x \circ y = \frac{1}{2}(xy + yx)$ is a unital formally real Jordan algebra by [HOS84, Proposition 2.9.2 p. 69] of dimension 27. By [HOS84, Corollary 3.1.7 p. 77] and its proof, we can equip $H_3(\mathbb{O})$ with a norm that makes it a JB-algebra. With this structure, $H_3(\mathbb{O})$ is a JBW-algebra.

A JB$^\ast$-algebra [HOS84, p. 91] [CGRP14, Definition 3.3.1] is a complex Banach space $A$ which is a complex Jordan algebra equipped with an involution satisfying

\[(3.3) \quad \|x \circ y\| \leq \|x\|||y||, \quad \|x^\ast\| = \|x\| \quad \text{and} \quad \|\{x, x^\ast, x\}\| = \|x\|^3\]

for any $x, y \in A$, where we use the Jordan triple product

\[\left\{x, y, z\right\} \overset{\text{def}}{=} (x \circ y) \circ z + (y \circ z) \circ x - (x \circ z) \circ y.\]

A JBW$^\ast$-algebra [CGRP18, p. 4] is a JB$^\ast$-algebra which is a dual Banach space. For the links between JBW$^\ast$-algebras and JBW-algebras, we refer to [BHK17, pp. 4-5] and [CGRP18, Corollary 5.1.29 p. 9 and Corollary 5.1.41 p. 15]. In short, the JBW-algebras are exactly the selfadjoint parts of JBW$^\ast$-algebras.

**Example 3.6** A von Neumann algebra $\mathcal{M}$ equipped with the Jordan product

\[(3.4) \quad x \circ y \overset{\text{def}}{=} \frac{1}{2}(xy + yx), \quad x, y \in \mathcal{M}\]

is a JBW$^\ast$-algebra.
The last axiom is a Jordan analogue of the Gelfand-Naimark axiom of $C^*$-algebras. Moreover, if $X$ is a dual Banach space, then it is called a JBW$^*$-triple [Chu12, Definition 2.5.30] [Hor87b].

In this case, the predual is unique by [CGRP14, Theorem 5.7.38 p. 233]. We refer to the books [Chu12], [CGRP14], [Isi19] and to the important papers [Hor87b] for more information on JB$^*$-triples.

**Example 3.7** A JBW$^*$-algebra $\mathcal{M}$ admits a structure of JBW$^*$-triple with triple product
\[(x,y,z) \mapsto \{x,y,z\} \overset{\text{def}}{=} (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*.\]

See [Chu12, Lemma 3.1.6 p. 174] and [CGRP18, p. 224].

**Triple homomorphisms** A linear map $T: X \to Y$ between two JB$^*$-triples $X$ and $Y$ is called a triple homomorphism if it preserves the triple product:
\[T(\{x,y,z\}) = \{T(x),T(y),T(z)\}, \quad x, y, z \in X.\]

A bijective triple homomorphism is called a triple isomorphism. In JB$^*$-triples, the metric structure and the algebraic structure determine each other. Indeed, by [Chu12, Theorem 3.1.7 p. 175] and [Chu12, Theorem 3.1.20 p. 183] a linear bijection between two JB$^*$-triples is an isometry if and only if it is a triple-isomorphism. In this case, this bijection is necessarily unique and continuous, see [CGRP18, Corollary 5.7.39 p. 234].

**JC$^*$-triples and JW$^*$-triples** A JC$^*$-triple (or $J^*$-algebra) [Chu12, Definition 2.5.34 p. 169] [CGRP18, p. 440] [Isi19, Definition 22.7.1 p. 460] is a norm closed subspace of $B(H,K)$ which is also closed under the map $x \mapsto xx^*x$. By a polarization argument, any JC$^*$-triple is closed under the triple product $(x,y,z) \mapsto \{x,y,z\} \overset{\text{def}}{=} \frac{1}{2}(xy^*z + zy^*x)$. A JC$^*$-triple is a JB$^*$-triple. Finally, a JC$^*$-triple is called a JW$^*$-triple if it is a dual Banach space. Consequently, a JW$^*$-triple is a JBW$^*$-triple. We shall often say that a JBW$^*$-triple is a JW$^*$-triple if it is isometrically isomorphic to a JW$^*$-triple.

**Example 3.8** A W$^*$-TRO is a JW$^*$-triple with triple product $(x,y,z) \mapsto \{x,y,z\} \overset{\text{def}}{=} xy^*z$.

**Example 3.9** By [CGRP18, Proposition 5.4.1 p. 362], the complexification $H_\mathbb{C}(\mathbb{O}) = H_\mathbb{C}(\mathbb{O})$ of the JBW-factor $H_\mathbb{O}$ is equipped with a structure of JBW$^*$-factor. The underlying JBW$^*$-triple is called a Cartan factor of type VI.

**Example 3.10** By [Isi19, p. 140], the space $\text{Asym}_n \overset{\text{def}}{=} \{x \in M_n : x^t = -x\}$ of skew-symmetric complex matrices of $M_n$ is a JBW$^*$-triple (called a Cartan factor of type $\text{III}_0$). If $n$ is even, it is even equipped with a structure of reversible JW$^*$-algebra by [Isi19, Proposition 25.2.2 p. 513].
Projections on JBW*-triples  We will use the following, which is [CGRP18, Theorem 5.6.59 p. 199] (see also [Isi19, Theorem 14.4.1 p. 252] and [Chu12, Theorem 3.3.1 p. 202]). This is a fundamental result of Kaup [Kau84] and Stachó [Sta82]. The last sentence is from [FrB84] and [FrB85]. See also [ChE77], [EIS79] and [RoY82] for particular cases.

Theorem 3.11 If \( P : X \to X \) is a weak* continuous contractive projection on a JBW*-triple \( X \), then the range \( P(X) \) is a JBW*-triple with the triple product given by \( \{x,y,z\}_P = P(\{x,y,z\}) \) where \( x,y,z \in P(X) \). Moreover, we have

\[
P\{P(x), y, P(z)\} = P\{P(x), P(y), P(z)\}, \quad x,y,z \in X.
\]

Finally, if \( X \) is a JW*-triple then \( P(X) \) is a JW*-triple.

In general, the range \( P(X) \) is not a subtriple of \( X \) (see [FrB82, Example 1 p. 66] or [Kau84, Example 3 p. 99]). But note that if \( P(X) \) is known to be a subtriple then the triple product \( \{ \cdot, \cdot, \cdot \}_P \) coincides with the original triple product of \( X \) because in JB*-triples norm and triple product determine each other (see e.g. [Chu12, Theorem 3.1.7 p. 175 and Theorem 3.1.20 p. 183]).

Tripotents and Peirce projections  An element \( u \) in a JBW*-triple \( X \) satisfying \( \{u,u,u\} = u \) is called a tripotent. When \( X \) is a JW*-triple, these elements are precisely the partial isometries of \( X \). With a tripotent \( u \) and \( 0 \leq i \leq 2 \), we can introduce the Peirce projections \( P_i(u) : X \to X \) with range \( X_i(u) \). For any \( x \in X \), we have

\[
P_2(u)(x) \overset{\text{def}}{=} \{u, \{u,x,u\}, u\}, \quad P_1(u) \overset{\text{def}}{=} 2(D(u,u) - P_2(u))
\]

and

\[
P_0(u) + P_1(u) + P_2(u) = \text{Id}_X.
\]

These maps are contractive linear projections by [FrB85b, Corollary 1.2]. A crucial property of JBW*-triples is that for a tripotent \( u \) of \( X \) the Peirce-2 subspace \( X_2(u) \) is a JBW*-algebra with product \( (a,b) \mapsto a \circ b \overset{\text{def}}{=} \{a,u,b\} \), involution \( a^* \overset{\text{def}}{=} \{u,a,u\} \) where \( a,b \in X_2(u) \), and unit \( u \).

Example 3.12 In the case where \( X \) is a JBW*-algebra and \( e \in X \) is a projection, i.e. a selfadjoint idempotent, the Peirce projections are given by the following expressions (essentially [HOS84, p. 48])

\[
P_2(e)x = \{e,x^*, e\}, \quad P_1(e)x = 2\{e,x^*, 1-e\}, \quad P_0(e)x = \{1-e, x^*, 1-e\},
\]

where \( x \in X \).

A tripotent \( u \) in a JBW*-triple \( X \) is called complete if \( X_0(u) = \{0\} \) [HKPP20, p. 16] [CGRP14, p. 517] and unitary if \( X = X_2(u) \).

Orthogonality and order  Two tripotents \( u \) and \( v \) are said orthogonal if \( v \) belongs to \( X_0(u) \). This relation is symmetric on the set of tripotents. For two tripotents \( u \) and \( v \), we write \( u \leq v \) if \( v - u \) is a tripotent orthogonal to \( u \). The relation \( \leq \) is a partial order on the set of tripotents by [EdR98a, Theorem 2.1].

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Example 3.13 Let $\mathcal{M}$ be a JBW*-algebra. Each projection (i.e., selfadjoint idempotent) $p$ of $\mathcal{M}$ is a tripotent. Moreover, for any projections $e, f$ in $\mathcal{M}$, we have $e \leq f$ if and only if $e \circ f = e$ and $e \perp f$ if and only if $e \circ f = 0$.

We have the following characterization of triple order ([BHKPP18, Proposition 6.9]).

Proposition 3.14 Let $X$ be a JBW*-triple and let $u, v$ be two tripotents of $X$. Then the following assertions are equivalent:

1. $u \leq v$,
2. $u = \{u, v, u\}$,
3. $u$ is a projection in the JBW*-algebra $X_2(v)$.

4 $n$-pseudo-decomposable maps on noncommutative $L^p$-spaces

In this section, we define and give several properties of $n$-pseudo-decomposable maps on Haagerup noncommutative $L^p$-spaces. The proofs are (essentially) similar to the ones of [ArK23] provided for results on decomposable maps on tracial noncommutative $L^p$-spaces. The author hesitated to choose the terminology «n-decomposable operator». Since there are already two different notions for the name «decomposable operator» (the one of [Haa85] [JuR04] [ArK23] and the one of [Sto13, Definition 1.2.8 p. 7]), we chose «n-pseudo-decomposable».

Definition 4.1 Suppose $1 \leq p \leq \infty$. Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. Consider some $n \in \{1, 2, \ldots, \infty\}$. A linear map $T: L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is $n$-pseudo-decomposable if there exist some bounded linear maps $v_1, v_2: L^p(\mathcal{M}) \to L^p(\mathcal{N})$ such that the linear map

\[
\Phi \overset{\text{def}}{=} \begin{bmatrix} v_1 & T \end{bmatrix}^T : S^p_n(L^p(\mathcal{M})) \to S^p_n(L^p(\mathcal{N})), \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} v_1(a) & T(b) \\ T^*(c) & v_2(d) \end{bmatrix}
\]

is $n$-positive, that means that $\text{Id}_{S^p_n} \otimes \Phi: S^p_n(L^p(\mathcal{M})) \to S^p_n(L^p(\mathcal{M})) \otimes S^p_n(L^p(\mathcal{N}))$ is a positive map, where $T^*(c) \overset{\text{def}}{=} T(c^*)^*$. In this case, we let

\[
\|T\|_{n-\text{pdec}, L^p(\mathcal{M}) \to L^p(\mathcal{N})} \overset{\text{def}}{=} \inf \max\{\|v_1\|, \|v_2\|\}
\]

where the infimum is taken over all maps $v_1$ and $v_2$. We say that $T$ is contractively $n$-pseudo-decomposable if $\|T\|_{n-\text{pdec}, L^p(\mathcal{M}) \to L^p(\mathcal{N})} \leq 1$.

If $n \in \{1, 2, \ldots, \infty\}$, we denote by $\text{PDec}_n(L^p(\mathcal{M}), L^p(\mathcal{N}))$ the set of $n$-pseudo-decomposable operators between two noncommutative $L^p$-spaces.

If $n = \infty$, we recover the decomposable maps of [JuR04] and of the memoir [ArK23] (and the decomposable maps of [Haa78b] if in addition $p = \infty$) and we have $\|T\|_{\infty-\text{pdec}, L^p(\mathcal{M}) \to L^p(\mathcal{N})} = \|T\|_{\text{dec}, L^p(\mathcal{M}) \to L^p(\mathcal{N})}$. By [ArK23, Theorem 3.23], if the von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ are approximately finite-dimensional and equipped with normal semifinite faithful traces, the decomposable norm $\|\cdot\|_{\text{dec}, L^p(\mathcal{M}) \to L^p(\mathcal{N})}$ and the regular norm $\|\cdot\|_{\text{reg}, L^p(\mathcal{M}) \to L^p(\mathcal{N})}$ of [Pi95] are identical. We will show that the infimum (4.2) is a minimum (see Proposition 4.3). Note that in the conditions of Definition 4.1 the maps $v_1$ and $v_2$ are $n$-positive.
Let $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ be von Neumann algebras. Suppose $1 \leq p \leq \infty$. Let $T_1 : L^p(\mathcal{M}_1) \to L^p(\mathcal{M}_2)$ and $T_2 : L^p(\mathcal{M}_2) \to L^p(\mathcal{M}_3)$ be some $n$-pseudo-decomposable maps. It is easy to see that the composition $T_2 \circ T_1$ is $n$-pseudo-decomposable and that

\[(4.3) \quad \|T_2 \circ T_1\|_{n\text{-pdec}} \leq \|T_2\|_{n\text{-pdec}} \|T_1\|_{n\text{-pdec}}.\]

Recall that for any matrix $\alpha \in M_{m,n}$, the map

\[(4.4) \quad L^p(M_n(\mathcal{M})) \to L^p(M_n(\mathcal{M})), \quad x \mapsto \alpha^* x \alpha\]

is completely positive.

**Proposition 4.2** Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. Suppose $1 \leq p \leq \infty$. Consider some $n \in \{1, 2, \ldots, \infty\}$. If $\lambda \in \mathbb{C}$ and if $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is $n$-pseudo-decomposable then the map $\lambda T$ is $n$-pseudo-decomposable and $\|\lambda T\|_{n\text{-pdec},L^p(\mathcal{M})\to L^p(\mathcal{N})} = |\lambda| \|T\|_{n\text{-pdec},L^p(\mathcal{M})\to L^p(\mathcal{N})}$.

**Proof** : By symmetry, it suffices to prove $\|\lambda T\|_{n\text{-pdec}} \leq |\lambda| \|T\|_{n\text{-pdec}}$, since then $\|T\|_{n\text{-pdec}} = \frac{1}{|\lambda|} \|\lambda T\|_{n\text{-pdec}} \leq \frac{1}{|\lambda|} \|\lambda T\|_{n\text{-pdec}}$. We can write $\lambda = |\lambda| \theta$ where $\theta$ is a complex number such that $|\theta| = 1$. Assume that $v_1, v_2 : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ are linear maps such that the map

$$
\begin{bmatrix}
1 & 0 \\
0 & \theta
\end{bmatrix}
\begin{bmatrix}
v_1(\cdot) \\
v_2(\cdot)
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & \theta
\end{bmatrix}
$$

is also $n$-positive on $S^2_p(L^p(\mathcal{M}))$ by composition of $n$-positive maps. But it is not difficult to check that the latter operator equals $\left[ \begin{array}{cc} \frac{|\lambda|v_1}{\theta} & \frac{\lambda T}{\theta} \\
\frac{|\lambda|v_2}{\theta} & \frac{\lambda T}{\theta} \end{array} \right]$. Consequently the map $|\lambda| : \left[ \begin{array}{cc} \frac{v_1}{\theta} & \frac{T}{\theta} \\
\frac{v_2}{\theta} & \frac{T}{\theta} \end{array} \right] = \left[ \begin{array}{cc} \frac{|\lambda|v_1}{\theta} & \frac{\lambda T}{\theta} \\
\frac{|\lambda|v_2}{\theta} & \frac{\lambda T}{\theta} \end{array} \right]$ is also $n$-positive. We deduce that $\lambda T$ is $n$-pseudo-decomposable and that $\|\lambda T\|_{n\text{-pdec}} \leq \max \{\|\lambda v_1\|, \|\lambda v_2\|\} = |\lambda| \max \{\|v_1\|, \|v_2\|\}$. Passing to the infimum yields the desired inequality $\|\lambda T\|_{n\text{-pdec}} \leq |\lambda| \|T\|_{n\text{-pdec}}$. \[\square\]

**Proposition 4.3** Let $\mathcal{M}$ and $\mathcal{N}$ be two von Neumann algebras. Suppose $1 < p < \infty$. Consider some $n \in \{1, 2, \ldots, \infty\}$. Let $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be an $n$-pseudo-decomposable map. Then the infimum in the definition of $\|T\|_{n\text{-pdec}}$ is actually a minimum i.e. we can choose $v_1$ and $v_2$ in (4.2) such that

$\|T\|_{n\text{-pdec},L^p(\mathcal{M})\to L^p(\mathcal{N})} = \max \{\|v_1\|, \|v_2\|\}$.

**Proof** : For any integer $n$, let $v_k, w_k : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ be bounded maps such that the map

$$
\begin{bmatrix}
v_k \\
w_k
\end{bmatrix}
\begin{bmatrix}
v_1(\cdot) \\
v_2(\cdot)
\end{bmatrix}
\begin{bmatrix}
w_k(\cdot) \\
v_1(\cdot)
\end{bmatrix}
$$

is $n$-positive with $\max \{\|v_k\|, \|w_k\|\} \leq \|T\|_{n\text{-pdec}} + \frac{1}{n}$. Note that since the Banach space $L^p(\mathcal{N})$ is reflexive, the closed unit ball of the space $B(L^p(\mathcal{M}), L^p(\mathcal{N}))$ of bounded operators in the weak operator topology is compact by [Alb11, Exercise 3.7 (iii) p. 122]. Hence the bounded sequences $(v_k)$ and $(w_k)$ admit convergent subnets $(v_{\alpha})$ and $(w_{\alpha})$ in the weak operator topology which converge to some $v, w \in B(L^p(\mathcal{M}), L^p(\mathcal{N}))$.

Now, it is easy to see that $\begin{bmatrix} v \\ w \end{bmatrix} \begin{bmatrix} T \\ T \end{bmatrix} = v_{\alpha} \begin{bmatrix} T \\ T \end{bmatrix} w_{\alpha}$ in the weak operator topology of $B(S^2_p(L^p(\mathcal{M})), S^2_p(L^p(\mathcal{N})))$. By [ArK23, Lemma 2.8], the operator on the left hand side is $n$-positive as a weak limit of $n$-positive operators. Moreover, using the weak lower semicontinuity of the norm, we see that $\|v\| \leq \liminf \|v_{\alpha}\| \leq \|T\|_{n\text{-pdec}}$ and $\|w\| \leq \liminf \|w_{\alpha}\| \leq \|T\|_{n\text{-pdec}}$. Hence, we have $\max \{\|v\|, \|w\|\} = \|T\|_{n\text{-pdec}}$. \[\square\]
Proposition 4.4 Let \( \mathcal{M} \) and \( \mathcal{N} \) be two von Neumann algebras. Suppose \( 1 \leq p \leq \infty \). Consider some \( n \in \{1, 2, \ldots, \infty\} \). Then the set \( \mathrm{PDec}_n(L^p(\mathcal{M}), L^p(\mathcal{N})) \) of \( n \)-pseudo-decomposable operators is a vector space and \( \| \cdot \|_{n-\mathrm{pdec},L^p(\mathcal{M})\rightarrow L^p(\mathcal{N})} \) is a norm on \( \mathrm{PDec}_n(L^p(\mathcal{M}), L^p(\mathcal{N})) \).

Proof: Let \( T_1, T_2: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N}) \) be \( n \)-pseudo-decomposable maps. There exist some bounded linear maps \( v_1, v_2, w_1, w_2: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N}) \) such that
\[
\begin{pmatrix} v_1 & T_1 \\ T_1^* & v_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} w_1 & T_2 \\ T_2^* & w_2 \end{pmatrix}
\]
are \( n \)-positive. We can write
\[
\begin{pmatrix} v_1 & T_1 \\ T_1^* & v_2 \end{pmatrix} + \begin{pmatrix} w_1 & T_2 \\ T_2^* & w_2 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 & T_1 + T_2 \\ T_1^* + T_2^* & v_2 + w_2 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 & T_1 + T_2 \\ (T_1 + T_2)^* & v_2 + w_2 \end{pmatrix}.
\]
Moreover, this map is \( n \)-positive. Hence \( T_1 + T_2 \) is \( n \)-pseudo-decomposable. Furthermore, we deduce that
\[
\|T_1 + T_2\|_{n-\mathrm{pdec}} \leq \max \{\|v_1 + w_1\|, \|v_2 + w_2\|\}
\leq \max \{\|v_1\| + \|w_1\|, \|v_2\| + \|w_2\|\} \leq \max \{\|v_1\|, \|v_2\|\} + \max \{\|w_1\|, \|w_2\|\}.
\]
Passing to the infimum, we conclude that the sum \( T_1 + T_2 \) is \( n \)-pseudo-decomposable and that \( \|T_1 + T_2\|_{n-\mathrm{pdec}} \leq \|T_1\|_{n-\mathrm{pdec}} + \|T_2\|_{n-\mathrm{pdec}} \). The absolute homogeneity is Proposition 4.2.

Suppose \( \|T\|_{n-\mathrm{pdec}} = 0 \). By Proposition 4.3, the map \( \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}: S^2_n(L^p(\mathcal{M})) \rightarrow S^2_n(L^p(\mathcal{N})) \) is \( n \)-positive. Now, let \( x \in L^p(\mathcal{M}) \). By Lemma 2.3, the element \( \begin{pmatrix} |x|^2 & x \\ x^* & |x| \end{pmatrix} \) of \( S^2_n(L^p(\mathcal{M})) \) is positive. We deduce that the element \( \begin{pmatrix} 0 & T(x) \\ T(x)^* & 0 \end{pmatrix} \) is also positive. Using Lemma 2.2, we infer that \( T(x) = 0 \). We conclude that \( T = 0 \).

Now, we give an example.

Proposition 4.5 Let \( \mathcal{M} \) and \( \mathcal{N} \) be two von Neumann algebras. Suppose \( 1 \leq p \leq \infty \). Consider some \( n \in \{2, \ldots, \infty\} \). Let \( T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N}) \) be an \( n \)-positive map. Then \( T \) is \( \left[ \frac{n}{2} \right] \)-pseudo-decomposable and
\[
\|T\|_{n-\mathrm{pdec},L^p(\mathcal{M})\rightarrow L^p(\mathcal{N})} \leq \|T\|_{L^p(\mathcal{M})\rightarrow L^p(\mathcal{N})}.
\]

Proof: We see that the map \( \begin{pmatrix} T & T \\ T & T \end{pmatrix}: S^2_n(L^p(\mathcal{M})) \rightarrow S^2_n(L^p(\mathcal{N})) \) is \( \left[ \frac{n}{2} \right] \)-positive. We infer that the map \( T \) is \( \left[ \frac{n}{2} \right] \)-pseudo-decomposable and that the inequality is true.

Proposition 4.6 Let \( \mathcal{M} \) and \( \mathcal{N} \) be two von Neumann algebras. Suppose \( 1 \leq p \leq \infty \). Consider some \( n \in \{1, \ldots, \infty\} \). An \( n \)-pseudo-decomposable \( T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N}) \) is a linear combination of \( n \)-positive maps.

Proof: Suppose that the map \( T \) is \( n \)-pseudo-decomposable. There exist some linear maps \( v_1, v_2: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{N}) \) such that \( \Phi = \begin{pmatrix} v_1 & T \\ T^* & v_2 \end{pmatrix} \) is \( n \)-positive. By (4.4), the maps \( T_1 = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ i & i \end{pmatrix} \Phi \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, T_2 = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -i & i \end{pmatrix} \Phi \begin{pmatrix} 1 & i \\ -1 & -i \end{pmatrix} \) and \( T_3 = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ -i & -i \end{pmatrix} \Phi \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \) are \( n \)-positive from \( L^p(\mathcal{M}) \) into \( L^p(\mathcal{N}) \) and it is easy to check that \( T = T_1 - T_2 + i(T_3 - T_4) \).

This result allows us to describe the \( n \)-pseudo-decomposable maps on classical \( L^p \)-spaces.

Example 4.7 Let \( \Omega \) and \( \Omega' \) be \( \sigma \)-finite measure spaces. If \( n \geq 1 \), any \( n \)-pseudo-decomposable map \( T: L^p(\Omega) \rightarrow L^p(\Omega) \) is necessarily regular (= decomposable by [ArK23, Theorem 3.24]). The converse is obviously true.

Remark 4.8 A similar notion of \( n \)-pseudo-decomposable map could be defined between ternary rings of operators, generalizing the decomposable maps of [KalR02, Section 7].

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5 Complementation of rectangular $L^p$-spaces in noncommutative $L^p$-spaces

We start by giving examples of decomposable maps on noncommutative $L^p$-spaces. We will also use this observation in the proof of Theorem 1.1. In the two following results, we can use Haagerup noncommutative $L^p$-spaces or Kosaki noncommutative $L^p$-spaces [Kos84] (see also [Ray03] for the construction of the positive cone).

**Lemma 5.1.** Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful weight $\varphi$. Suppose $1 \leq p \leq \infty$. Let $a, b \in \mathcal{M}$. Then the two-sided multiplication operator $M_{a,b} : L^p(\mathcal{M}) \to L^p(\mathcal{M})$, $x \mapsto axb$ is decomposable and we have

$$\|M_{a,b}\|_{\text{dec},L^p(\mathcal{M}) \to L^p(\mathcal{M})} \leq \|a\|_{L^\infty(\mathcal{M})} \|b\|_{L^\infty(\mathcal{M})}. \tag{5.1}$$

More precisely, the map $\Phi \overset{\text{def}}{=} [M_{a,b}, M_{a,b}^*] : S^p_2(L^p(\mathcal{M})) \to S^p_2(L^p(\mathcal{M}))$ is completely positive.

**Proof:** For any $z \in \mathcal{M}$, we see that $M_{a,b}^*(z) = (M_{a,b}(z^*))^* = (az^*)^* = b^*za^*$. Consequently, for any element $\begin{bmatrix} x & y \\ z & t \end{bmatrix}$ of the space $S^p_2(L^p(\mathcal{M}))$, we have

$$\Phi \left( \begin{bmatrix} x & y \\ z & t \end{bmatrix} \right) = \begin{bmatrix} axa^* & ayb \\ b^*za^* & b^*tb \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b^* \end{bmatrix} \begin{bmatrix} a^* & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^*.$$

We deduce that $\Phi$ is completely positive. By (1.2), we infer that $\|M_{a,b}\|_{\text{dec},L^p(\mathcal{M}) \to L^p(\mathcal{M})} \leq \max\{\|M_{a,a^*}\|, \|M_{b^*,b}\|\}$. By Hölder’s inequality, we see that for any $x \in L^p(\mathcal{M})$ we have $\|M_{a,a^*}(x)\|_{L^p(\mathcal{M})} = \|axa^*\|_{L^p(\mathcal{M})} \leq \|x\|_{L^p(\mathcal{M})} \|a\|_{L^\infty(\mathcal{M})}$. So $\|M_{a,a^*}\| \leq \|a\|_{L^\infty(\mathcal{M})}$ and similarly $\|M_{b,b^*}\| \leq \|b\|_{L^\infty(\mathcal{M})}$. We obtain that

$$\|M_{a,b}\|_{\text{dec},L^p(\mathcal{M}) \to L^p(\mathcal{M})} \leq \max\{\|a\|_{L^\infty(\mathcal{M})}^2, \|b\|_{L^\infty(\mathcal{M})}^2\}.$$

Replacing $a$ by $\frac{1}{\|a\|}a$ and $b$ by $\frac{1}{\|b\|}b$, we conclude that the inequality (5.1) is true. \hfill \blacksquare

**Proposition 5.2.** Let $e, f$ be some orthogonal projections of a von Neumann algebra $\mathcal{M}$ equipped with a normal semifinite faithful weight $\varphi$. Suppose $1 \leq p \leq \infty$. The linear map $P : L^p(\mathcal{M}) \to L^p(\mathcal{M})$, $x \mapsto exf$ is a contractively decomposable projection whose range is the closed subspace $eL^p(\mathcal{M})f$.

**Proof:** By Lemma 5.1, the map $P$ is contractively decomposable. For any $x \in L^p(\mathcal{M})$, we have $P^2(x) = P(exf) = e^2xf^2 = exf = P(x)$. So $P^2 = P$, i.e. $P$ is a projection. The last assertion is clear. \hfill \blacksquare

2. Indeed, we have

$$\frac{1}{\|a\|_\infty \|b\|_\infty} \|M_{a,b}\|_{\text{dec},L^p(\mathcal{M}) \to L^p(\mathcal{M})} = \left\| M_{\frac{1}{\|a\|_\infty}a, \frac{1}{\|b\|_\infty}b} \right\|_{\text{dec},L^p(\mathcal{M}) \to L^p(\mathcal{M})} \leq \max\left\{ \left\| \frac{1}{\|a\|_\infty}a \right\|^2, \left\| \frac{1}{\|b\|_\infty}b \right\|^2 \right\}.$$
Remark 5.3 Recall that a linear projection on a JBW* -triple $X$ is said structural if for any elements $x,y,z$ of $X$ we have

$$P(\{x, P(y), z\}) = \{P(x), y, P(z)\}.$$  

By [ECR96, Theorem 5.3], such a projection is necessarily contractive and weak* continuous. Moreover, it is proved in [ECR96] that the map $P \mapsto P(X)$ is a bijection from the set of structural projections on $X$ onto the set of weak* closed inner ideals of $X$ (i.e. the set of weak* closed subspaces $J$ of $X$ satisfying $\{J, X, J\} \subset J$). If $p = \infty$, note that by [ECR96, Theorem 6.1] the projections $P : L^\infty(M) \to L^\infty(M)$ of Proposition 5.2 are exactly the structural projections of $M$. We refer to [EHR03] for another characterization of structural projections on JBW* -triples and to [EdR89] and [EdR98b] for more information on weak* closed inner ideals.

In the spirit of [KaR02, p. 869], we define the rectangular $L^p$-spaces of $W^*$-TROs using Kosaki noncommutative $L^p$-spaces of [Kos84] [Ray03]. Indeed, here we need an inclusion of the linking von Neumann algebra in its noncommutative $L^p$-spaces. So we cannot use Haagerup noncommutative $L^p$-spaces in this definition.

Definition 5.4 Let $V$ be a $W^*$-TRO with linking von Neumann algebra $R(V)$. Suppose that the von Neumann algebra $R(V)$ is $\sigma$-finite equipped with a normal faithful state $\varphi$. Suppose $1 \leq p < \infty$. We define the rectangular $L^p$-space $L^p(V, \varphi)$ to be the norm closure of $V = eR(V)e^\perp$ in the Kosaki noncommutative $L^p$-space $L^p(R(V), \varphi)$.

It is easy to check that $L^p(V, \varphi) = eL^p(R(V), \varphi)e^\perp$ since $R(V)$ is dense in the Banach space $L^p(R(V), \varphi)$. We let $L^\infty(V, \varphi) \overset{\text{def}}{=} V$. An immediate use of Proposition 5.2 gives the following result.

Corollary 5.5 Let $V$ be a $W^*$-TRO such that the linking von Neumann algebra $R(V)$ is $\sigma$-finite and equipped with a normal faithful state $\varphi$. Suppose $1 \leq p \leq \infty$. Then the rectangular $L^p$-space $L^p(V, \varphi)$ is a contractively complemented subspace of the Kosaki noncommutative $L^p$-space $L^p(R(V), \varphi)$ by a contractively decomposable projection.

Remark 5.6 Recall that Kosaki noncommutative $L^p$-spaces are defined by interpolation. Using Lemma 2.7 combined with Proposition 5.2, it is immediate that we have the complex interpolation formula $L^p(V, \varphi) = (L^\infty(V, \varphi), L^1(V, \varphi))_\lambda$.

The paper [GJL18, Appendix 6] contains related results. However, note that in [GJL18, Appendix 6], it is written that for any $W^*$-TRO $V$ of $B(H)$ we have an isometry $V_p = (V, V_1)_p$ where $V_p$ is the closure of the intersection $X \cap S^p(H)$ in the Schatten space $S^p(H)$, relying on the existence of a couple of compatible contractive projections $(P_\infty, P_1)$ on $V$ and $V_1$ (and Lemma 2.7). However, this existence seems unclear in the general case where $V$ is not a weak* closed inner ideal of $B(H)$. The concrete representation of $V$ in $B(H)$ seems to be crucial.

Remark 5.7 Using the construction of noncommutative $L^p$-spaces of [Ter82], it is left to the interested reader to generalize the previous corollary to arbitrary $W^*$-TROs using weights instead states. In this situation, a von Neumann algebra is not a subset of its noncommutative $L^p$-spaces.
6 A lifting for \( n \)-pseudo-decomposable maps on noncommutative \( L^p \)-spaces

Our main tool will be the following result [ArR19, Theorem 2.7] which is a lifting theorem for positive maps acting on noncommutative \( L^p \)-spaces which has its roots in [JRX05, Theorem 3.1].

**Theorem 6.1** Let \( \mathcal{M} \) and \( \mathcal{N} \) be von Neumann algebras and \( n \in \{1, 2, \ldots, \infty \} \). Suppose \( 1 \leq p < \infty \). Let \( T: L^p(\mathcal{M}) \to L^p(\mathcal{N}) \) be an \( n \)-positive linear map. Let \( h \) be a positive element of \( L^p(\mathcal{M}) \). Then there exists a unique linear map \( v: \mathcal{M} \to s(\mathcal{T}(h))\mathcal{N}s(\mathcal{T}(h)) \) such that

\[
T(h^{\frac{1}{p}}xh^{\frac{1}{p}}) = T(h)^{\frac{1}{p}}v(x)T(h)^{\frac{1}{p}}, \quad x \in \mathcal{M}.
\]

Moreover, this map \( v \) is unital, \( n \)-positive, contractive and normal.

Now, in the same spirit we prove a lifting theorem for \( n \)-pseudo-decomposable maps.

**Theorem 6.2** Let \( \mathcal{M} \) and \( \mathcal{N} \) be von Neumann algebras and \( n \in \{1, 2, \ldots, \infty \} \). Suppose \( 1 \leq p < \infty \). Let \( T: L^p(\mathcal{M}) \to L^p(\mathcal{N}) \) be an \( n \)-pseudo-decomposable map and \( v_1, v_2: L^p(\mathcal{M}) \to L^p(\mathcal{N}) \) some maps such that the operator \( \Phi \) of (4.1) is \( n \)-positive. Let \( h \) and \( k \) be positive elements of \( L^p(\mathcal{M}) \). Then there exists a unique linear map \( w: \mathcal{M} \to s(v_1(h))\mathcal{N}s(v_2(k)) \) such that

\[
T(h^{\frac{1}{p}}xk^{\frac{1}{p}}) = v_1(h)^{\frac{1}{p}}w(x)v_2(k)^{\frac{1}{p}}
\]

for any \( x \in \mathcal{M} \). Moreover, \( w \) is normal and contractively \( n \)-pseudo-decomposable.

**Proof**: Consider the positive element \( H \overset{\text{def}}{=} \begin{bmatrix} h & 0 \\ 0 & k \end{bmatrix} \) of the space \( S_p^2(L^p(\mathcal{M})) \). The support of the element \( \Phi(H) \) of \( S_p^2(L^p(\mathcal{M})) \) is given by

\[
s(\Phi(H)) = s(\Phi \left( \begin{bmatrix} h & 0 \\ 0 & k \end{bmatrix} \right)) = s(\begin{bmatrix} v_1(h) & 0 \\ 0 & v_2(k) \end{bmatrix}) = \begin{bmatrix} s(v_1(h)) & 0 \\ 0 & s(v_2(k)) \end{bmatrix}.
\]

Using Theorem 6.1 with the \( n \)-positive operator (4.1) and the positive element \( H \) instead of \( h \), we see that there exists a unique linear map \( v: M_2(\mathcal{M}) \to s(\Phi(H))M_2(\mathcal{N})s(\Phi(H)) \) such that for any element \( X \overset{\text{def}}{=} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \) of \( M_2(\mathcal{M}) \) we have

\[
\Phi \left( \begin{bmatrix} h & 0 \\ 0 & k \end{bmatrix}^{\frac{1}{p}} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} h & 0 \\ 0 & k \end{bmatrix}^{\frac{1}{p}} \right) = \Phi \left( \begin{bmatrix} h & 0 \\ 0 & k \end{bmatrix} \right)^{\frac{1}{p}} v(X) \Phi \left( \begin{bmatrix} h & 0 \\ 0 & k \end{bmatrix} \right)^{\frac{1}{p}}.
\]

Moreover, the map \( v \) is a normal \( n \)-positive contraction. On the other hand, we have

\[
\Phi \left( \begin{bmatrix} h & 0 \\ 0 & k \end{bmatrix}^{\frac{1}{p}} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} h & 0 \\ 0 & k \end{bmatrix}^{\frac{1}{p}} \right) = \Phi \left( \begin{bmatrix} h^{\frac{1}{p}}x_{11}h^{\frac{1}{p}} & h^{\frac{1}{p}}x_{12}k^{\frac{1}{p}} \\ k^{\frac{1}{p}}x_{21}h^{\frac{1}{p}} & k^{\frac{1}{p}}x_{22}k^{\frac{1}{p}} \end{bmatrix} \right)
\]

\[
\overset{(4.1)}{=} \begin{bmatrix} v_1(h^{\frac{1}{p}}x_{11}h^{\frac{1}{p}}) & T(h^{\frac{1}{p}}x_{12}k^{\frac{1}{p}}) \\ T^\circ(k^{\frac{1}{p}}x_{21}h^{\frac{1}{p}}) & v_2(k^{\frac{1}{p}}x_{22}k^{\frac{1}{p}}) \end{bmatrix}.
\]
On the other hand, we have

\[(6.6) \quad \Phi \left( \begin{bmatrix} h & 0 \\ 0 & k \end{bmatrix} \right)^* v(\lambda) \Phi \left( \begin{bmatrix} h & 0 \\ 0 & k \end{bmatrix} \right)^* = \begin{bmatrix} v_1(h)^* & 0 \\ 0 & v_2(k)^* \end{bmatrix} v(\lambda) \begin{bmatrix} v_1(h)^* & 0 \\ 0 & v_2(k)^* \end{bmatrix}. \]

Consequently, for any element \( X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \) of \( M_2(M) \), we have

\[(6.7) \quad \begin{bmatrix} v_1(h)^* x_{11} h^+ & T(h^+ h) x_{12} k^+ \\ T^0 (k^+ h) x_{21} h^+ & v_2(k^+ h) x_{22} k^+ \end{bmatrix} = \begin{bmatrix} v_1(h)^* & 0 \\ 0 & v_2(k)^* \end{bmatrix} v(X) \begin{bmatrix} v_1(h)^* & 0 \\ 0 & v_2(k)^* \end{bmatrix}. \]

Now, we introduce the map

\[ u: M \to s(\Phi(H)) M_2(N) s(\Phi(H)) \]

defined by \( u(x) \) for some normal maps \( u_1: M \to s(\Phi(H)) M_2(N) s(\Phi(H)) \), \( u_2: M \to s(\Phi(H)) M_2(N) s(\Phi(H)) \), and \( w: M \to s(\Phi(H)) M_2(N) s(\Phi(H)) \). The map \( u \) is completely positive and obviously normal. By composition, the map \( u \) is normal and \( p \)-positive.

We can write \( u = \begin{bmatrix} u_1 & w \\ u_2 & u_3 \end{bmatrix} \) for some normal maps \( u_1: M \to s(\Phi(H)) M_2(N) s(\Phi(H)) \), \( u_2: M \to s(\Phi(H)) M_2(N) s(\Phi(H)) \), and \( u_3: M \to s(\Phi(H)) M_2(N) s(\Phi(H)) \). The map \( u \) is positive hence \( * \)-preserving. Thus, for any \( x \in M \) we have

\[ \begin{bmatrix} u_1(x)^* & u_3(x)^* \\ u_2(x)^* & u_2(x)^* \end{bmatrix} = \begin{bmatrix} u_1(x)^* & u_2(x)^* \\ u_3(x)^* & u_2(x)^* \end{bmatrix} = u(x)^* = u(x^*) = \begin{bmatrix} u_1(x)^* & u_3(x)^* \\ u_2(x)^* & u_2(x)^* \end{bmatrix}. \]

In particular, we deduce that \( w(x^*) = u_3(x)^* \) for any \( x \in M \). Hence \( u_3 = w^p \). So \( u = \begin{bmatrix} u_1 & w \\ u_2 & u_3 \end{bmatrix} \).

This implies that \( w \) is \( n \)-pseudo-decomposable and that \( u_1 \) and \( u_2 \) are \( n \)-positive. Moreover, for any \( x \in M \), we see that

\[ \begin{bmatrix} v_1(h)^+ x_{11} h^+ & T(h^+ h) x_{12} k^+ \\ T^0 (k^+ h) x_{21} h^+ & v_2(k^+ h) x_{22} k^+ \end{bmatrix} = \begin{bmatrix} v_1(h)^+ & 0 \\ 0 & v_2(k)^+ \end{bmatrix} v \begin{bmatrix} x & x \\ x & x \end{bmatrix} \begin{bmatrix} x & x \\ x & x \end{bmatrix} \begin{bmatrix} v_1(h)^+ & 0 \\ 0 & v_2(k)^+ \end{bmatrix}. \]

Looking at the \((1,2)\)-entry, we obtain \((6.2)\). Looking at the other entries, we deduce by uniqueness that \( u_1 \) and \( u_2 \) are the liftings given by Theorem 6.1 of the \( n \)-positive maps \( v_1, v_2: L^p(M) \to L^p(N) \). Consequently, we have the inequalities \( \|u_1\| \leq 1 \) and \( \|u_2\| \leq 1 \) and thus \( \|w\|_{\text{n-dec}, M \to N} \leq 1 \). The uniqueness is obvious.

The case \( n = \infty \) gives the following particular case.

**Corollary 6.3** Let \( M \) and \( N \) be von Neumann algebras. Suppose \( 1 \leq p < \infty \). Let \( T: L^p(M) \to L^p(N) \) be a decomposable map and \( v_1, v_2: L^p(M) \to L^p(N) \) be some maps such that the operator...
(4.1) is completely positive. Let $h$ and $k$ be positive elements of $L^p(M)$. Then there exists a unique linear map $w: M \to s(v_1(h))N s(v_2(k))$ such that

$$T(\text{h}_k^x w_k^x) = v_1(h) w(v_2(k))$$

for any $x \in M$. Moreover, $w$ is normal and contractively decomposable.

**Remark 6.4** Let $M$ be a von Neumann algebra. By [Han85, Theorem 1.6], if the von Neumann algebra $N$ is injective, we have a canonical isomorphism $\text{CB}(M, N) = \text{Dec}(M, N)$, that means that any completely bounded map $T: M \to N$ is necessary decomposable with $\|T\|_{\text{cb}, M \to N} = \|T\|_{\text{Dec}, M \to N}^\ast$.

## 7 Contractively $n$-pseudo-decomposable projections

Let $X$ be a Banach space and $T: X \to X$ be a bounded operator. For any integer $m \geq 1$, we define the average $A_{m,T} \overset{\text{def}}{=} \frac{1}{m} \sum_{k=1}^m T^k$ of the first $m$ iterates of $T$. Now, we use ergodic theory to obtain information on contractively $n$-pseudo-decomposable projections on noncommutative $L^p$-spaces. It is important to note in the following result that the map $[P_1 \ P \ P_2]$ is (a priori) not necessarily contractive. It is related to Question 9.9.

**Proposition 7.1** Let $M$ be a von Neumann algebra and $n \in \{1, 2, \ldots, \infty\}$. Suppose $1 < p < \infty$. Let $P: L^p(M) \to L^p(M)$ be a contractively $n$-pseudo-decomposable projection. There exist (n-positive) contractive projections $P_1, P_2: L^p(M) \to L^p(M)$ such that the linear map $[P_1 \ P \ P_2 : S_n^p(L^p(M)) \to S_n^p(L^p(M))$ is an $n$-positive projection.

**Proof**: By Proposition 4.3, there exist bounded linear maps $v_1, v_2: L^p(M) \to L^p(M)$ such that the linear map $\Phi = \begin{bmatrix} v_1 & P \ P_2 \end{bmatrix} : S_n^p(L^p(M)) \to S_n^p(L^p(M))$ is $n$-positive with $\max\{\|v_1\|, \|v_2\|\} = 1$. By composition, note that the map $\Phi^k$ is $n$-positive for any integer $k \geq 1$. Since $P^2 = P$, we have $\Phi^k = \begin{bmatrix} v_1 & P_2 \ P_2 \end{bmatrix}^k = \begin{bmatrix} v_1^k & P_2^k \ (P_2^k) \end{bmatrix} = \begin{bmatrix} v_1^k & P_2^k \ (P_2^k) \end{bmatrix} = \begin{bmatrix} v_1^k & P_2^k \ P_2^k \end{bmatrix}$ for any $k \geq 1$. For any integer $m \geq 1$, we infer that the average

$$A_{m,\Phi} = \frac{1}{m} \sum_{k=1}^m \Phi^k = \frac{1}{m} \sum_{k=1}^m \begin{bmatrix} v_1^k & P_2^k \ (P_2^k) \end{bmatrix} = \frac{1}{m} \sum_{k=1}^m v_1^k \ P_2^k \ (P_2^k) = \begin{bmatrix} A_{m,v_1} \ P_1 \ P_2 \end{bmatrix}$$

is $n$-positive. The maps $v_1$ and $v_2$ are contractions, hence power-bounded. By [EFHN15, Theorem 8.22 p. 149]3, since the Banach space $L^p(M)$ is reflexive ($1 < p < \infty$), we deduce that the maps $v_1$ and $v_2$ are mean ergodic. This means that the sequences $(A_{m,v_1})$ and $(A_{m,v_2})$ converge for the strong operator topology of the space $\text{B}(L^p(M))$ to some bounded operators $P_1, P_2: L^p(M) \to L^p(M)$. By [EFHN15, Lemma 8.3 p. 137], the operators $P_1$ and $P_2$ are projections. Since each average $A_{m,v_i}$ ($i = 1, 2$) is clearly contractive, by the strong lower semicontinuity of the norm, the maps $P_1$ and $P_2$ are also contractive. It is obvious that the sequence $(A_{m,\Phi})$ of $n$-positive maps converges strongly (hence weakly) to the bounded operator $[P_1 \ P \ P_2]$. By [ArK23, Lemma 2.10], we conclude that the map $[P_1 \ P \ P_2]$ is $n$-positive. \[\square\]

3. In [EFHN15], the averages are defined with a sum $\sum_{k=0}^{m-1}$. However, it is obvious that the result is also true with a sum $\sum_{k=1}^m$. 22
The particular case $n = \infty$ gives the following.

**Corollary 7.2** Let $\mathcal{M}$ be a von Neumann algebra. Suppose $1 < p < \infty$. Let $P: L^p(\mathcal{M}) \to L^p(\mathcal{M})$ be a contractively decomposable projection. There exist contractive (completely positive) projections $P_1, P_2: L^p(\mathcal{M}) \to L^p(\mathcal{M})$ such that the map \[
\begin{bmatrix}
P_1 \\
P_2
\end{bmatrix} = S^p_\phi(L^p(\mathcal{M})) \to S^p_\phi(L^p(\mathcal{M}))
\] is a completely positive projection.

Suppose $1 \leq p < \infty$. Let $\mathcal{M}$ be a $\sigma$-finite (= countably decomposable) von Neumann algebra and $P: L^p(\mathcal{M}) \to L^p(\mathcal{M})$ be a positive contractive projection. In [ArR19, Section 6], we introduced the support $s(P)$ of $\operatorname{Ran} P$ as the supremum in $\mathcal{M}$ of the supports of the positive elements in $\operatorname{Ran} P$:

\begin{equation}
(7.1) \quad s(P) \overset{\text{def}}{=} \bigvee_{h \in \operatorname{Ran} P, h \geq 0} s(h).
\end{equation}

Under these assumptions, by [ArR19, Proposition 6.1], there exists a positive element $h$ of $\operatorname{Ran} P$ such that

\begin{equation}
(7.2) \quad s(P) = s(h).
\end{equation}

We have $P(h) = h$. In this case, by [ArR19, Proposition 6.4], we have

\begin{equation}
(7.3) \quad P(y) = P(s(h)ys(h)), \quad y \in L^p(\mathcal{M}).
\end{equation}

The following theorem is the main result of this paper.

**Theorem 7.3** Let $\mathcal{M}$ be a $\sigma$-finite von Neumann algebra equipped with a normal faithful state $\varphi$. Suppose $1 < p < \infty$ and $n \in \{1, 2, \ldots, \infty\}$. A bounded map $P: L^p(\mathcal{M}) \to L^p(\mathcal{M})$ is a contractively $n$-pseudo-decomposable projection if and only if there exist a normal faithful positive linear form $\phi$ on the von Neumann algebra $M_2(\mathcal{M})$, two positive elements $h, k \in L^p(\mathcal{M})$ with support projection $s(h)$ and $s(k)$, some linear maps $w: \mathcal{M} \to s(h)Ms(h), u_1: \mathcal{M} \to s(h)Ms(h), u_2: \mathcal{M} \to s(k)Ms(k)$ such that the map $Q = \begin{bmatrix} u_1 & w \\ w^* & u_2 \end{bmatrix}$ and the element $H = \begin{bmatrix} h & 0 \\ 0 & k \end{bmatrix}$ of $S^p_\phi(L^p(\mathcal{M}))$ satisfy the following properties:

1. for any $x \in L^p(\mathcal{M})$ we have $P(x) = P(s(h)xs(h))$, 
2. $Q$ is a weak* continuous unital contractive $n$-positive map and its restriction on the von Neumann algebra $s(H)M_2(\mathcal{M})s(H)$ is a faithful projection, 
3. $s(H)$ belongs to the centralizer of $\phi$ and $\phi_{s(H)} = (\operatorname{Tr} \otimes \varphi)_{s(H)}$, 
4. for any $y \in s(H)M_2(\mathcal{M})s(H)$ we have $(\operatorname{Tr} \otimes \operatorname{Tr}_\varphi)(H^pQ(y)) = (\operatorname{Tr} \otimes \operatorname{Tr}_\varphi)(H^py)$, 
5. for any $x \in s(h)Ms(k)$, we have

\begin{equation}
(7.4) \quad P(h^xk^*xk^*) = h^xw(x)k^*, \quad x \in \mathcal{M}.
\end{equation}

In this case, the restriction of $w$ is a weak* continuous $n$-pseudo decomposable projection and the range of this map admits a structure of JW*-triple. Moreover, if $n = \infty$ then the range of $w$ admits a structure of $W^*\text{-TRO}$.
Proof : By Proposition 7.1, there exist contractive (n-positive) projections \( P_1, P_2 : L^p(M) \rightarrow L^p(M) \) such that the map

\[
\Phi \overset{\text{def}}{=} \begin{bmatrix} P_1 & P \\ P^* & P_2 \end{bmatrix} : S^0_2(L^p(M)) \rightarrow S^0_2(L^p(M))
\]

is an \( n \)-positive projection. We fix a normal faithful state \( \varphi \) on \( M \) such that \( L^p(M) = L^p(M, \varphi) \). Considering the projection \( e \overset{\text{def}}{=} \begin{bmatrix} s(P_1) & 0 \\ 0 & s(P_2) \end{bmatrix} \), we can define with the construction (2.17) the normal faithful positive linear form \( \phi \) on the von Neumann algebra \( M_2(M) \) by

\[
\phi(x) \overset{\text{def}}{=} (\text{Tr} \otimes \varphi)_e(exe) + (\text{Tr} \otimes \varphi)_e(e^\perp x e^\perp), \quad x \in M_2(M).
\]

The projection \( e \) belongs to the centralizer of \( \phi \) and \( \phi_e = (\text{Tr} \otimes \varphi)_e \). By (7.2), there exist positive elements \( h \) and \( k \) of Ran \( P_1 \) and Ran \( P_2 \) respectively such that \( s(h) = s(P_1) \) and \( s(k) = s(P_2) \).

We let \( H \overset{\text{def}}{=} \begin{bmatrix} h & 0 \\ 0 & k \end{bmatrix} \) and we introduce the algebra

\[
\mathcal{N} \overset{\text{def}}{=} s(H)M_2(M)s(H) = \begin{bmatrix} s(h) & 0 \\ 0 & s(k) \end{bmatrix} M_2(M) \begin{bmatrix} s(h) & 0 \\ 0 & s(k) \end{bmatrix}^* = \begin{bmatrix} s(h)Ms(h) & s(h)Ms(k) \\ s(k)Ms(h) & s(k)Ms(k) \end{bmatrix}.
\]

Lemma 7.4 For any \( x \in L^p(M) \), we have \( P(s(h)x s(k)) = P(x) \).

Proof : If \( x \in s(h)^{-1}L^p(M) \) then we will show that \( P(x) = 0 \). By Lemma 2.3, the element \( \begin{bmatrix} |x^+| \\ x \end{bmatrix} \) of \( S^0_2(L^p(M)) \) is positive. Note that \( |x^+| \) belongs to \( s(h)^{-1}L^p(M) \). Hence, we have

\[
\begin{bmatrix} 0 & P(x) \\ P^*(x) & P_2(|x|) \end{bmatrix} \overset{(7.3)}{=} \begin{bmatrix} P_1(|x^+|) & P(x) \\ P^*(x) & P_2(|x|) \end{bmatrix} = \Phi \begin{bmatrix} |x^+| & x \\ x^+ & |x| \end{bmatrix} \overset{(7.5)}{=} \begin{bmatrix} 0 & P(x) \\ P^*(x) & P_2(|x|) \end{bmatrix}.
\]

Since \( \Phi \) is positive, this element is positive. Using Lemma 2.2, we infer that \( P(x) = 0 \). If \( x \in L^p(M)s(k) \) then similarly we have \( P(x) = 0 \). Now, if \( x \in L^p(M) \) we have

\[
P(x) = P((s(h) + s(h)^{-1})x(s(k) + s(k)^{-1})))
\]

\[
= P(s(h)x s(k)) + P(s(h)x s(k)^{-1}) + P(s(h)^{-1}x s(k)) + P(s(h)^{-1}x s(k)^{-1})
\]

\[
= P(s(h)x s(k)).
\]

Remark 7.5 For any \( x \in S^0_2(L^p(M)) \), we deduce from Lemma 7.4 combined with (7.3) that

\[
\Phi(s(H)x s(H)) = \Phi(x).
\]

By applying Theorem 6.2 to the \( n \)-pseudo-decomposable map \( P : L^p(M) \rightarrow L^p(M) \), we see that there exists a linear map \( w : M \rightarrow s(P_1(h))Ms(P_2(k)) = s(h)Ms(k) \) such that

\[
(P(h^\perp x k^\perp)) \overset{(6.2)}{=} h^\perp w(x)k^\perp, \quad x \in M.
\]

Moreover, this map \( w \) is normal and contractively \( n \)-pseudo-decomposable. Let \( u_1 : M \rightarrow s(h)Ms(h) \) and \( u_2 : M \rightarrow s(k)Ms(k) \) be the liftings of \( n \)-positive maps \( P_1 \) and \( P_2 \) provided
by Theorem 6.1 with respect to the elements $h$ and $k$. We introduce the linear map $Q \overset{\text{def}}{=} [u_1 \ w^\ast \ w_2] : M_2(\mathcal{M}) \to \mathcal{N}$. For any element $[x_{11} \ x_{12} \ x_{21} \ x_{22}]$ of $M_2(\mathcal{M})$, we have

$$
(7.7) \quad \Phi \left( H^\dagger \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} H^\dagger \right) = \Phi \left( \begin{bmatrix} h^\dagger \ x_{11} & h^\dagger \ x_{12} \\ k^\dagger \ x_{21} & k^\dagger \ x_{22} \end{bmatrix} \right) 
$$

$$
(7.5) \quad P_1(h^\dagger \ x_{11}) = P(h^\dagger \ x_{12} k^\dagger) \quad P(h^\dagger \ x_{21} k^\dagger) = \Phi \left( \begin{bmatrix} h^\dagger \ u_1(x_{11}) & h^\dagger \ u_2(x_{22}) \\ k^\dagger \ w^\ast(x_{21}) & k^\dagger \ w^\ast(x_{22}) \end{bmatrix} \right) 
$$

Consequently, the map $Q$ is the lifting of the $n$-positive map $\Phi$ provided by Theorem 6.1 with respect to the positive element $H$. Hence this map $Q$ is unital, contractive, normal and $n$-positive.

**Lemma 7.6** The restriction $Q|\mathcal{N} : \mathcal{N} \to \mathcal{N}$ is a projection.

**Proof**: For any $x \in \mathcal{N}$, we have

$$
H^\dagger Q(x) H^\dagger \overset{(7.7)}{=} \Phi(\Phi(x) H^\dagger) = \Phi^2(\Phi(x) H^\dagger) \overset{(7.7)}{=} \Phi(\Phi(x) H^\dagger) \overset{(7.7)}{=} \Phi(x) H^\dagger = H^\dagger Q(x) H^\dagger. 
$$

Using Lemma 2.4, we obtain the conclusion. 

**Lemma 7.7** The projection $Q|\mathcal{N} : \mathcal{N} \to \mathcal{N}$ is faithful.

**Proof**: If $x = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ is an element of $\mathcal{N}$ satisfying $Q(x) = 0$, we have

$$
\Phi(\Phi(x) H^\dagger) \overset{(7.7)}{=} H^\dagger Q(x) H^\dagger = 0. 
$$

So by (7.7), we have $P_1(h^\dagger \ x_{11}) = 0$ and $P_2(k^\dagger \ x_{22}) = 0$. By [Dix77, 1.6.9], we have $0 \leq x_{11} \leq \|x_{11}\|_\infty$ so

$$
0 \leq h^\dagger \ x_{11} h^\dagger \leq \|x_{11}\|_\infty h. 
$$

Recall that the norm of a noncommutative $L^p$-space is strictly monotone. By Lemma 2.8, we see that $h^\dagger \ x_{11} h^\dagger = 0$. Since $x_{11}$ belongs to $\mathcal{N}$, we conclude by Lemma 2.4 that $x_{11} = 0$. Similarly, we show that $x_{22} = 0$. By Lemma 2.2 (see also [Bha07, Proposition 1.3.2]), we conclude that $x = 0$.

Now, we consider the normal faithful positive linear form $\psi$ on the von Neumann algebra $\mathcal{N}$ defined by

$$
\psi(x) \overset{\text{def}}{=} (\text{Tr} \otimes \text{Tr}_\varphi)(H^p x), \quad x \in \mathcal{N}. 
$$

**Lemma 7.8** The restriction $Q|\mathcal{N}$ preserves $\psi$ i.e. we have $\psi \circ Q = \psi$.

**Proof**: Since $h$ is positive, by Lemma 2.5, we have $J_{L^p(\mathcal{M})}(h) = \|h\|_p^{2-p} h^{p-1}$. By [PiX03, Corollary 5.2], the Banach space $L^p(\mathcal{M})$ is smooth and strictly convex. Using the contractive dual map $P_1^* : L^p(\mathcal{M}) \to L^p(\mathcal{M})$ and Lemma 2.6, we see that $P_1^*(\|h\|_p^{2-p} h^{p-1}) = \|h\|_p^{2-p} h^{p-1}$, that is

$$
(7.8) \quad P_1^*(h^{p-1}) = h^{p-1}. 
$$

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Similarly, we have $P_2^*(k^{p-1}) = k^{p-1}$. Using [ArK23, Lemma 3.3] in the second equality, we conclude that
\[
\Phi^*((H^{p-1})^\dagger) = \left[ P_1 P_2^* \right] \begin{bmatrix} h & 0 \\ 0 & k \end{bmatrix}^{p-1} \left[ P_1 P_2^* \right] = \left[ P_1 P_2^* \right]^{p-1} \begin{bmatrix} h^{p-1} & 0 \\ 0 & k^{p-1} \end{bmatrix}.
\]

For any $z \in L^p(\mathcal{N})$, it follows that
\[
(7.9) \quad (\text{Tr} \otimes \text{Tr}_\phi)(H^{p-1} \Phi(z)) = (\text{Tr} \otimes \text{Tr}_\phi)(\Phi^*(H^{p-1})z) = (\text{Tr} \otimes \text{Tr}_\phi)(H^{p-1}z).
\]

In particular, for any $x \in \mathcal{N}$, we have
\[
(\text{Tr} \otimes \text{Tr}_\phi)(\Phi^*(H^{p-1})x) = (\text{Tr} \otimes \text{Tr}_\phi)(H^{p-1}\Phi(x)).
\]

That is $(\text{Tr} \otimes \text{Tr}_\phi)(H^{p}Q(x)) = (\text{Tr} \otimes \text{Tr}_\phi)(H^{p}x)$. Hence $\psi(Q(x)) = \psi(x)$.

Note that $s(h), Ms(k)$ is a W*-TRO (hence a JBW*-triple). So we can use Theorem 3.3 and Theorem 3.11 for the description of the range of the restriction of the map $\psi$. Consequently we have proved the "only if" part of Theorem 7.3.

Conversely, suppose that the conditions of Theorem 7.3 are satisfied. We introduce the normal faithful positive linear form $\psi$ on the von Neumann algebra $\mathcal{N} \defeq s(H)M_\mathcal{M}s(H)$ defined as the restriction of $(\text{Tr} \otimes \text{Tr}_\phi)(H^{p}x)$. Recall that $D_\psi \in L^1(\mathcal{N}, \psi)$ denotes the density operator associated with $\psi$. From (2.14), we have a canonical map $\kappa: L^p(\mathcal{N}, \psi) \to L^p(\mathcal{N}, (\text{Tr} \otimes \text{Tr}_\phi)s(H))$ which induces a completely order and isometric identification. If $x \in \mathcal{N}$, using Lemma 2.1 in the third equality, we see that
\[
\text{Tr}_\phi(D_\psi x) = (\text{Tr} \otimes \text{Tr}_\phi)(H^{p}x) = (\text{Tr} \otimes \text{Tr}_\phi)(H^{p}x) = \text{Tr}_\phi(\kappa^{-1}(H^{p}x)).
\]

We conclude that
\[
(7.10) \quad D_\psi = \kappa^{-1}(H)^p.
\]

With the fourth condition of Theorem 7.3, we can consider by (2.18) with $\mathcal{N}$ instead of $\mathcal{M}$ the contractive $n$-positive operator $Q_p: L^p(\mathcal{N}, \psi) \to L^p(\mathcal{N}, \psi)$ induced by the restriction $Q|\mathcal{N}: \mathcal{N} \to \mathcal{N}$ of $Q$ and defined by
\[
(7.11) \quad Q_p(D_\psi^\frac{1}{p} x D_\psi^\frac{1}{p}) = D_\psi^\frac{1}{p} Q(x) D_\psi^\frac{1}{p}, \quad x \in \mathcal{N}.
\]

For any $x \in \mathcal{N}$, note that
\[
Q_p^2(D_\psi^\frac{1}{p} x D_\psi^\frac{1}{p}) = D_\psi^\frac{1}{p} Q(x) D_\psi^\frac{1}{p} = D_\psi^\frac{1}{p} Q^2(x) D_\psi^\frac{1}{p} = D_\psi^\frac{1}{p} Q(x) D_\psi^\frac{1}{p}.
\]

We deduce that $Q_p^2 = Q_p$, i.e. that the map $Q_p$ is a projection.
We also consider the restrictions $\psi_1$ and $\psi_2$ of the normal positive linear forms $\text{Tr}_\varphi(h^p)$ and $\text{Tr}_\varphi(k^q)$ on the von Neumann algebras $s(h)\mathcal{M}(s(h))$ and $s(k)\mathcal{M}(s(k))$. With the canonical identifications $\kappa_1 : L^p(s(h)\mathcal{M}(s(h)), \psi_1) \to L^p(s(h)\mathcal{M}(s(h)), \varphi_{s(h)})$ and $\kappa_2 : L^p(s(k)\mathcal{M}(s(k)), \psi_2) \to L^p(s(k)\mathcal{M}(s(k)), \varphi_{s(k)})$, we have similarly to (7.10) the equalities

\begin{equation}
D_{\psi_1} = \kappa_1^{-1}(h)^p \quad \text{and} \quad D_{\psi_2} = \kappa_2^{-1}(k)^p.
\end{equation}

Using again the fourth condition of Theorem 7.3, we see that for any $x \in \mathcal{N}$

\[ \psi_1(u_1(x)) = \text{Tr}_\varphi(h^p x) = (\text{Tr} \otimes \text{Tr}_\varphi) \left( H^p \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \right) = (\text{Tr} \otimes \text{Tr}_\varphi) \left( H^p \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \right) = \psi_1(x). \]

We deduce that $u_1|s(h)\mathcal{M}(s(h)) : s(h)\mathcal{M}(s(h)) \to s(h)\mathcal{M}(s(h))$ preserves $\psi_1$. Similarly, the linear map $u_2|s(k)\mathcal{M}(s(k)) : s(k)\mathcal{M}(s(k)) \to s(k)\mathcal{M}(s(k))$ preserves $\psi_2$. By (2.18), these maps admit contractive extensions $\tilde{P}_1 : L^p(s(h)\mathcal{M}(s(h)), \psi_1) \to L^p(s(h)\mathcal{M}(s(h)), \psi_1)$ and $\tilde{P}_2 : L^p(s(k)\mathcal{M}(s(k)), \psi_2) \to L^p(s(k)\mathcal{M}(s(k)), \psi_2)$ such that

\begin{equation}
\tilde{P}_1(D^{\frac{1}{2}}_\psi x D^{\frac{1}{2}}_\psi) = D^{\frac{1}{2}}_\psi u_1(x) D^{\frac{1}{2}}_\psi \quad \text{and} \quad \tilde{P}_2(D^{\frac{1}{2}}_\psi x_k D^{\frac{1}{2}}_\psi) = D^{\frac{1}{2}}_\psi u_2(x) D^{\frac{1}{2}}_\psi.
\end{equation}

We let $P_1 \overset{\text{def}}{=} \kappa_1 P_1\kappa_1^{-1}$ and $P_2 \overset{\text{def}}{=} \kappa_2 P_2\kappa_2^{-1}$. We have

\begin{equation}
P_1(h^{\frac{1}{2}} x h^{\frac{1}{2}}) = \kappa_1 P_1\kappa_1^{-1}(h^{\frac{1}{2}} x h^{\frac{1}{2}}) = \kappa_1 \tilde{P}_1(D^{\frac{1}{2}}_\psi x D^{\frac{1}{2}}_\psi) \overset{(7.14)}{=} \kappa_1(D^{\frac{1}{2}}_\psi u_1(x) D^{\frac{1}{2}}_\psi) \overset{(7.12)}{=} h^{\frac{1}{2}} u_1(x) h^{\frac{1}{2}}
\end{equation}

and a similar equality for $P_2$. For any $x \in \mathcal{N}$, we obtain

\begin{equation}
\kappa Q_p\kappa^{-1}(H^{\frac{1}{2}} x H^{\frac{1}{2}}) = \kappa Q_p(\kappa^{-1}(H)^{\frac{1}{2}} x \kappa^{-1}(H)^{\frac{1}{2}}) \overset{(7.10)}{=} \kappa Q_p(D^{\frac{1}{2}}_\psi x D^{\frac{1}{2}}_\psi)
\end{equation}

\begin{equation}
= \kappa(D^{\frac{1}{2}}_\psi Q(x) D^{\frac{1}{2}}_\psi) \overset{(7.16)}{=} H^{\frac{1}{2}} Q(x) H^{\frac{1}{2}} = \begin{bmatrix} h^{\frac{1}{2}} u_1(x_{11}) & h^{\frac{1}{2}} & h^{\frac{1}{2}} w(x_{12}) k^{\frac{1}{2}} \\ k^{\frac{1}{2}} w(x_{21}) h^{\frac{1}{2}} & k^{\frac{1}{2}} u_2(x_{22}) k^{\frac{1}{2}} \end{bmatrix}
\end{equation}

\begin{equation}
= \begin{bmatrix} P_1(h_{\frac{1}{2}} x_{11} h_{\frac{1}{2}}) & P(h_{\frac{1}{2}} x_{12} k_{\frac{1}{2}}) \\ P^o(k_{\frac{1}{2}} x_{21} h_{\frac{1}{2}}) & P_2(k_{\frac{1}{2}} x_{22} k_{\frac{1}{2}}) \end{bmatrix} = \begin{bmatrix} P_1 & P \\ P^o & P_2 \end{bmatrix} (H^{\frac{1}{2}} x H^{\frac{1}{2}}).
\end{equation}

By density, with Lemma 2.4, we conclude that

\[ \kappa Q_p\kappa^{-1} = \begin{bmatrix} P_1 |s(h)\mathcal{M}(s(h))\to s(h)\mathcal{M}(s(h)) \\ P_2 |s(h)\mathcal{M}(s(h))\to s(h)\mathcal{M}(s(h)) \end{bmatrix}. \]

Since $s(h)$ and $s(k)$ belong to the centralizer of the projection $s(H)$ belongs to the centralizer of $\text{Tr} \otimes \varphi$. So we have an order isometric identification of $L^p(\mathcal{N}, (\text{Tr} \otimes \varphi)_{s(H)})$ as a subspace of the Banach space $L^p(M_2(\mathcal{M}), \psi)$. By Lemma 5.1, the linear map

\[ \Phi : S^2_2(L^p(\mathcal{M})) \to L^p(\mathcal{N}, (\text{Tr} \otimes \varphi)_{s(H)}), \quad \begin{bmatrix} x & y \\ z & t \end{bmatrix} \mapsto \begin{bmatrix} s(h)x s(h) & s(h) y s(k) \\ s(k) z s(h) & s(k) t s(k) \end{bmatrix}. \]

is completely positive. Using the first point of Theorem 7.3 in the second equality, we obtain that

\[ \kappa Q_p\kappa^{-1} \Phi = \begin{bmatrix} P_1 |s(h)\mathcal{M}(s(h))\to s(h)\mathcal{M}(s(h)) \\ P_2 |s(h)\mathcal{M}(s(h))\to s(h)\mathcal{M}(s(h)) \end{bmatrix} \Phi = \begin{bmatrix} P_1(s(h) \cdot s(h)) & P \\ P_2(s(k) \cdot s(k)) \end{bmatrix} \]

By composition, this map is $n$-positive. Furthermore, we have $\|P_1(s(h) \cdot s(h))\|_{L^p(\mathcal{M}) \to L^p(\mathcal{M})} \leq 1$ and $\|P_2(s(k) \cdot s(k))\|_{L^p(\mathcal{M}) \to L^p(\mathcal{M})} \leq 1$. We deduce by (4.2) that the map $P$ is a contractively $n$-pseudo-decomposable projection. The proof is complete. \hfill \Box
Remark 7.9 The case where $\mathcal{M}$ is not $\sigma$-finite is beyond the scope of this paper. Probably, it suffices to adapt the painful method of [Arh19].

Remark 7.10 The author believes that with slight modifications, we can prove the case $p = 1$ as in [Arh19].

Remark 7.11 If $n = \infty$, we do not know if $w(s(h)Ms(k))$ is a sub-TRO of the $W^*$-TRO $s(h)Ms(k)$. In this case, $w$ would be a TRO-conditional expectation, see Remark 3.4. More generally, if $n \geq 1$, we do not know if $w(s(h)Ms(k))$ is a sub-triple of $s(h)Ms(k)$. However, it is written in [BuP02, Lemma 4.1] [EdR96, Lemma 5.3] that there exists a JW$^*$-subtriple $X$ of $s(h)Ms(k)$ such that $X$ is linearly isometric to $w(s(h)Ms(k))$ and such that $X$ is the range of a weak*-continuous projection on $s(h)Ms(k)$.

8 \(L^p\)-spaces associated to $\sigma$-finite JBW$^*$-triples

In this section, we introduce $L^p$-spaces associated to some suitable JBW$^*$-triples and we connect these spaces to the nonassociative $L^p$-spaces introduced in our previous paper [Arh19]. It is important to note that a JBW$^*$-triple does not admit an involution, contrarily to the case of JBW$^*$-algebras.

We will use the following result [Wer18, Corollary 5.3] on complex interpolation. See [CoS98], [HaP89], [LiP64], [Wat95] and [Wat00] for variants of this result. Here, we use the notation $\overline{Y}$ for the complex conjugate of a Banach space $Y$, that is the same set, with the same addition and norm but with the conjugate multiplication by a scalar. The elements of the Banach space $\overline{Y}$ are denoted by $\overline{y}$ where $y \in Y$. If $u: Y \rightarrow Y$ is a linear map, we can define a linear map $\overline{u}: \overline{Y} \rightarrow \overline{Y}$ by $\overline{u}(\overline{y}) \overset{\text{def}}{=} u(y)$.

**Theorem 8.1** Let $Y$ be a Banach space. Let $v: \mathcal{H} \rightarrow Y$ be a contractive injective map from a Hilbert space $\mathcal{H}$ into $Y$ with dense range. The composition $v \circ \overline{v^*}: \overline{\mathcal{H}} \rightarrow Y$ defines an interpolation couple and we have $(\overline{\mathcal{H}}, Y)_\theta = \mathcal{H}$. Moreover, for any $0 < \theta < 1$ we have $(\overline{\mathcal{H}}, Y)_\theta = (\overline{\mathcal{H}}^*, Y)_{1-\theta}$.

**Support projection of a linear functional** Consider a JBW$^*$-algebra $\mathcal{M}$ equipped with a normal positive linear functional $\varphi$. The support idempotent $e$ of $\varphi$ is the unique projection $e$ of $\mathcal{M}$ such that $\{ x \in \mathcal{M} : \varphi(x^* \circ x) = 0 \} = U_{1-e}(\mathcal{M})$, where

\[
U_\delta(a) \overset{\text{def}}{=} \{ a, x, a \}, \quad a \in \mathcal{M}
\]

see [CGRP18, Definition 5.10.18 p. 284]. We have $\varphi(e) = \| \varphi \|$ by [CGRP18, Proposition 5.10.20 p. 284].

**Support tripotent of a linear functional** Assume that $X$ is a JBW$^*$-triple and consider some linear functional $\varphi \in X_*$. By [CGRP18, Definition 5.10.58 p. 300] [FrB85b, Proposition 2] there exists a unique tripotent $s(\varphi) \in X$, called the support tripotent of $\varphi$, such that

1. $\varphi = \varphi \circ P_2(s(\varphi))$,
2. $\varphi |_{X_2(s(\varphi))}$ is a faithful weak* continuous positive functional on the JBW$^*$-algebra $X_2(s(\varphi))$, 

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where $P_2(s(\varphi)) = X \rightarrow X$ is the Peirce projection with range $X_2(s(\varphi))$ defined in (3.6). Furthermore, we have

\begin{equation}
\|\varphi\| = \varphi(s(\varphi)) = \|\varphi|X_2(s(\varphi))\|
\end{equation}

and $s(\varphi)$ is the support idempotent of $\varphi|X_2(s(\varphi))$ in the JBW*-algebra $X_2(s(\varphi))$.

It is essentially shown in [FrBS85b, part (b) in the proof of Proposition 2] and [CGRP18, p. 300] that

\begin{equation}
\text{if } u \text{ is a tripotent in } X \text{ with } \|\varphi\| = \varphi(u), \text{ then } s(\varphi) \leq u.
\end{equation}

**Example 8.2** Consider a JBW*-algebra $\mathcal{M}$ equipped with a normal positive linear functional $\varphi$. The support tripotent $s(\varphi)$ of $\varphi$ is equal to the support idempotent $e$ of $\varphi$. Moreover, the support tripotent $s(\varphi) = e$ is complete if and only if $e = 1$. By [CGRP18, Proposition 5.10.20 (ii) p. 284], this is equivalent to the faithfulness of the functional $\varphi$.

**Proof** : We can suppose that $\varphi \neq 0$. By [CGRP18, Proposition 5.10.20 (ii) p. 284], we have $\varphi(e) = \|\varphi\|$. Consequently by (8.3), we deduce that $s(\varphi) \leq e$. By Lemma 3.14, we see in particular that $s(\varphi)$ is a projection of the JBW*-algebra $X_2(e)$. We have $\varphi(s(\varphi)) = \|\varphi\| = \|\varphi\| \|s(\varphi)\|$. By [CGRP18, Proposition 5.10.20 (iii) p. 284], we infer that

$$\{e, s(\varphi), e\}^{8.1} = U_e(s(\varphi)) = \|s(\varphi)\|e = e.$$ 

Using again Lemma 3.14, we deduce that $e \leq s(\varphi)$. We conclude that $e = s(\varphi)$.

If $e = 1$ then $e$ is a unitary. So it is complete by [HoN88, (1.21) (ii)]. In the opposite direction, if the projection $s(\varphi) = e$ is complete then we have $P_0\{1-e\}^{13.8} = \{1-e, 1-e, 1-e\} = 1-e$ since $1-e$ is a projection, hence a tripotent. Consequently, $1-e$ belongs to the Peirce-0 subspace $\mathcal{M}_0(e)$ which is equal to $\{0\}$ since $e$ is complete. We conclude that $e = 1$. ■

**Sesquilinear form associated to a functional** In [BaF87, Proposition 1.2] [CGRP18, Proposition 5.10.60 p. 301], the authors showed that given an element $\varphi \in X$, and an element $z \in X$ such that $\varphi(z) = \|\varphi\| = \|z\| = 1$, we have a positive hermitian sesquilinear form

\begin{equation}
\langle x, y \rangle_{\mathcal{H}_\varphi} \overset{\text{def}}{=} \varphi(\{x, y, z\}), \quad x, y \in X
\end{equation}

whose associated seminorm

\begin{equation}
\|x\|_{\mathcal{H}_\varphi} \overset{\text{def}}{=} \sqrt{\varphi(\{x, x, z\})}, \quad x \in X
\end{equation}

is independent of $z$. Moreover, by [EdR98a, Lemma 4.1] the kernel of the seminorm $\|\cdot\|_{\mathcal{H}_\varphi}$ is precisely $X_0(s(\varphi))$, that is

\begin{equation}
\{x \in X : \|x\|_{\mathcal{H}_\varphi} = 0\} = X_0(s(\varphi)).
\end{equation}

**$\sigma$-finite JBW*-triples** Following [EdR98a, p. 293], we say that a tripotent $u$ in a JBW*-triple is $\sigma$-finite if any family of pairwise orthogonal nonzero smaller tripotents is at most countable. Clearly, 0 is a $\sigma$-finite tripotent. A JBW*-triple $X$ is said to be $\sigma$-finite if every tripotent $u$ in $X$ is $\sigma$-finite. By [EdR98a, Theorem 3.2], this is is equivalent to saying that every orthogonal subset of tripotents in $X$ is at most countable.

We have the following characterization [EdR98a, Theorem 3.2] of $\sigma$-finiteness.
Lemma 8.3 A tripotent $u$ of a JBW*-triple $X$ is $\sigma$-finite if and only if it is equal to the support tripotent $s(\varphi)$ for some linear functional $\varphi \in X_{**}$.

Example 8.4 Recall that an element $e$ of a JBW*-algebra is said to be a projection if $e^* = e$ and $e \circ e = e$ and that projections $e$ and $f$ are called orthogonal if $e \circ f = 0$. By [HKPP20, Lemma 7.2 (b)], a projection $e$ is a $\sigma$-finite (i.e. any family of pairwise orthogonal smaller nonzero projections is at most countable) if and only if it is a $\sigma$-finite tripotent.

We will use the following result [EdR98a, Lemma 4.2]. Recall that $X_2(u)$ and $X_0(v)$ are the ranges of the Peirce projections $P_2(u)$ and $P_0(v)$ defined in (3.6) and (3.7).

Lemma 8.5 Let $X$ be a JBW*-triple and let $v$ be a $\sigma$-finite tripotent in $X$. Let $u$ be a tripotent in $X$ such that $X_2(u)_{sa} \cap X_0(v) = \{0\}$. Then $u$ is $\sigma$-finite.

Now, we give the following characterization of $\sigma$-finite JBW*-triples. We refer to [EdR98a, Theorem 4.4] for a lot of other characterizations of $\sigma$-finite JBW*-triples.

Proposition 8.6 A JBW*-triple $X$ is $\sigma$-finite if and only if $X$ admits a linear functional $\varphi \in X_*$ such that the support tripotent $s(\varphi)$ is complete.

Proof : $\Rightarrow$: By [CGRP18, Fact 5.7.25 p. 226], there exists a complete tripotent $u$, which is $\sigma$-finite since $X$ is $\sigma$-finite. With Lemma 8.3, we deduce that $u$ is equal to the support tripotent $s(\varphi)$ for some linear functional $\varphi \in X_{**}$.

$\Leftarrow$: Suppose that $X$ admits a functional $\varphi \in X_*$ such that the support tripotent $s(\varphi)$ is complete, i.e. $X_0(s(\varphi)) = \{0\}$. Lemma 8.3 says that $s(\varphi)$ is $\sigma$-finite. Now, if $u$ is a tripotent in $X$ then $X_2(u)_{sa} \cap X_0(s(\varphi)) = \{0\}$. We deduce with Lemma 8.5 that $u$ is $\sigma$-finite. We conclude that $X$ is $\sigma$-finite.

Example 8.7 Recall that a JBW*-algebra $M$ is $\sigma$-finite if any orthogonal family of non-zero projections is countable. By [Arh23], a JBW*-algebra is $\sigma$-finite if and only if there exists a normal faithful state $\varphi$. A JBW*-algebra $M$ is $\sigma$-finite as a JBW*-algebra if and only if it is $\sigma$-finite as a JBW*-triple.

Proof : $\Rightarrow$: If $M$ is $\sigma$-finite as a JBW*-algebra then there exists a normal faithful state. By Example 8.2, its support tripotent is complete. Consequently, $M$ admits a linear functional $\varphi \in X_*$, such that the support tripotent $s(\varphi)$ is complete. We conclude with Proposition 8.6.

$\Leftarrow$: Suppose that $M$ is $\sigma$-finite as a JBW*-triple. Every orthogonal subset of tripotents in $A$ is at most countable. Hence every orthogonal subset of projections of $M$ is at most countable by Example 3.13.

$\ell^p$-spaces of a $\sigma$-finite JBW*-triple Let $X$ be a JBW*-triple equipped with some functional $\varphi \in X_*$ such that $\|\varphi\|_{X_*} = 1$. We will introduce some embedding of $X$ in its predual $X_*$. For that, we define for any $y \in X$ the linear functional $\varphi_y : X \to \mathbb{C}$ by

$$\varphi_y(x) \overset{\text{def}}{=} \varphi(\{x, y, s(\varphi)\}), \quad x \in X.$$ (8.7)

Proposition 8.8 Suppose that the support tripotent $s(\varphi)$ is complete. For any $y \in X$, the conjugate linear map $i : X \to X_*$, $y \mapsto \varphi_y$ is well-defined, injective and contractive. Moreover, its range is dense in the Banach space $X_*$.  

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Proof: Recall that the triple product of a JBW$^*$-triple is separately weak$^*$ continuous, see e.g. [CGRP18, Theorem 5.7.20 p. 224]. Hence for any $y \in X$ the map $x \mapsto \{x, y, s(\varphi)\}$ is weak$^*$ continuous. So by composition the functional $\varphi_y$ is weak$^*$ continuous. In a JB$^*$-triple, the triple product is contractive, see [CGRP18, Corollary 7.1.7 p. 440], [FrB86, Corollary 3] and [Chu12, p. 215]. Consequently for any $x, y \in X$, we have

$$|\varphi_y(x)| \overset{(8.7)}{=} |\varphi(\{x, y, s(\varphi)\})| \leq \|\{x, y, s(\varphi)\}\|_X \leq \|x\|_X \|y\|_X \|s(\varphi)\|_X \leq \|x\|_X \|y\|_X.$$  

We deduce that the map $i$ is contractive. Let $y \in X$. Suppose $\varphi_y = 0$. We have

$$\|y\|_{H_{\varphi}}^2 \overset{(8.5)}{=} \varphi(\{y, y, s(\varphi)\}) \overset{(8.7)}{=} \varphi_y(y) = 0.$$  

Since the tripotent $s(\varphi)$ is complete, we have $X_0(s(\varphi)) = \{0\}$. By (8.6), we see that $\|\cdot\|_{H_{\varphi}}$ is a norm on $X$. We obtain that $y = 0$. We conclude that the linear map $i$ is injective.

If $x \in X$ belongs the annihilator $i(X)^\perp$ of $i(X)$, we have

$$\|x\|_{H_{\varphi}}^2 \overset{(8.5)}{=} \varphi(\{x, x, s(\varphi)\}) \overset{(8.7)}{=} \varphi_x(x) = \langle i(x), x \rangle = 0.$$  

Using again the fact that $\|\cdot\|_{H_{\varphi}}$ is a norm, we infer that $x = 0$. We obtain that $i(X)^\perp = \{0\}$. By [Meg98, Proposition 1.10.15 (c) p. 93], we conclude that the range $i(X)$ of $i$ is dense in the space $X_*$.

If $X$ is a complex Banach space then the map $j: \overline{X} \to X_*$, $y \mapsto \varphi_y$ is linear. Hence $(\overline{X}, X_*)$ is an interpolation couple of complex Banach spaces. In the spirit of the construction of noncommutative $L^p$-spaces of Kosaki [Kos84] and nonassociative $L^p$-spaces of [Arh23], we introduce the following definition by using the complex interpolation method.

Definition 8.9 Let $X$ be a JBW$^*$-triple equipped with a functional $\varphi \in X_*$ with $\|\varphi\|_{X_*} = 1$ such that the support tripotent $s(\varphi)$ is complete. For any $1 < p < \infty$, we let

$$L^p(X, \varphi) \overset{\text{def}}{=} (\overline{X}, X_*)_{\frac{1}{p}}.$$  

We say that this space is the $L^p$-space associated with $X$ and $\varphi$.

We shall write simply $L^p(X)$ when there is no ambiguity on the functional $\varphi$. Note that the sum $\overline{X} + X_*$ is isometric to $X_*$ and that the intersection $\overline{X} \cap X_* = j(X)$, whose norm is defined by (2.21), is isometric to $X$. By [Lun18, Proposition 2.4 p. 50], we have the contractive inclusions $\overline{X} \subset L^p(X) \subset X_*$. Moreover, by [BeL76, Theorem 4.2.2 p. 91] the subspace $\overline{X}$ is dense in the Banach space $L^p(X)$. We let $L^1(X, \varphi) \overset{\text{def}}{=} X_*$.

Example 8.10 Let $\mathcal{M}$ be a JBW$^*$-algebra with product $\circ$ equipped with a normal faithful state $\varphi$. Recall that we can see $\mathcal{M}$ as a JBW$^*$-triple, see Example 3.7. In [Arh23], we introduced the embedding $i: \mathcal{M} \to \mathcal{M}_*$, $y \mapsto \varphi_y(y \circ \cdot)$ for defining nonassociative $L^p$-spaces associated to $(\mathcal{M}, \varphi)$. We will describe the link between these spaces and the ones of Definition 8.9.

By Example 8.2, the support idempotent of the functional $\varphi$ is equal to the unit $1$ of the JBW$^*$-algebra $\mathcal{M}$ and equal to its support tripotent $s(\varphi)$. Now, for any $x, y \in \mathcal{M}$, we have

(8.8) $$\varphi_y(x) \overset{(8.7)}{=} \varphi(\{x, y, s(\varphi)\}) = \varphi(\{x, y, 1\}) \overset{(3.5)}{=} \varphi(y^* \circ x).$$  

Moreover, the linear map $\Phi: \overline{\mathcal{M}} \to \mathcal{M}$, $\overline{y} \mapsto y^*$ is an isometric isomorphism by (3.3). Furthermore, for any $y \in \mathcal{M}$, observe that

$$i(\Phi(\overline{y})) = i(y^*) = \varphi(y^* \circ \cdot) \overset{(8.8)}{=} \varphi_y = j(\overline{y}).$$  

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We deduce the following commutative diagram.

\[
\begin{array}{ccc}
M & \xrightarrow{i} & M_* \\
\phi \downarrow & & \downarrow j \\
M & \xrightarrow{j} & M_*
\end{array}
\]

By [Pis03, Remark 2.7.8 p. 59], we conclude that there exist isometric isomorphisms between the spaces of Definition 8.9 and the ones of [Arch23]. In other words, the spaces defined in Definition 8.9 generalize the nonassociative Lp-spaces defined in [Arch23].

Let X be a JBW∗-triple equipped with a functional ϕ ∈ X∗ such that s(ϕ) is complete. The formula (8.4) defines a complex scalar product. We denote by Hϕ the associated Hilbert space, that is the completion of X for the associated norm (8.5). In the next result, we will identify the space L2(X) to this Hilbert space and will describe the dual of the Banach space Lp(X).

**Proposition 8.11** Let X be a JBW∗-triple equipped with a functional ϕ ∈ X∗ with ||ϕ||X, = 1 such that its support tripotent s(ϕ) is complete.

1. The Banach space L2(X) is linearly isometric to the Hilbert space Hϕ. More precisely, the norms of L2(X) and of Hϕ coincide on their common linear subspace X.

2. Suppose 1 < p < ∞. We have (Lp(X))∗ = Lp′(X) isometrically.

**Proof**: Since the triple product is contractive [CGRP18, Corollary 7.1.7 p. 440] [FrB86, Corollary 3], we have for any x ∈ X

\[
\|x\|_{H_ϕ} \overset{(8.5)}{=} \sqrt{ϕ(\{x, x, s(ϕ)\})} \leq \sqrt{||ϕ||X \|\{x, x, s(ϕ)\||} \leq \sqrt{\|x\|X \|x\|X s(ϕ)\|X} = \|x\|_X.
\]

So, the canonical map u: X → Hϕ is contractive and injective with dense range, hence with weak∗ dense range. It is easy to check that u is weak∗ continuous. Indeed, consider a bounded net (xi) of X such that xi → x for the weak topology of the dual Banach space X. Since the triple product of a JBW∗-triple is separately weak∗ continuous [CGRP18, Theorem 5.7.20 p. 224], we have for any y ∈ X

\[
\langle x_i, y \rangle_{H_ϕ} \overset{(8.4)}{=} ϕ(\{x_i, y, s(ϕ)\}) \rightarrow ϕ(\{x, y, s(ϕ)\}) \overset{(8.4)}{=} \langle x, y \rangle_{H_ϕ}.
\]

Since X is dense in the space Hϕ and since the net is bounded, it converges to x for the weak∗ topology of Hϕ by [Meg98, Exercise 2.71 p. 234]. With [CGRP18, Corollary 5.1.20 p. 7], we conclude that u is weak∗ continuous. By [Meg98, Theorem 3.1.17 p. 290], its preadjoint v def = u∗: (Hϕ)∗ → X∗ is injective and has dense range. Note that (Hϕ)∗ = Hϕ. For any x, y ∈ X, we have

\[
ϕ(\{x, y, s(ϕ)\}) = \langle u(x), \overline{y}\rangle_{H_ϕ, H_ϕ} = \langle x, u_*(\overline{y})X, X^* \rangle.
\]

Hence v(\overline{y}) = u_*(\overline{y}) \overset{(8.9)}{=} ϕ(\{x, y, s(ϕ)\}) \overset{(8.7)}{=} ϕ_y. So

\[
v \circ v(\overline{y}) = v \circ v(\overline{y}) = v \circ u(\overline{y}) = v(\overline{y}) = ϕ_y.
\]

Now, it suffices to use Theorem 8.1 with Y = X∗ since the composition v ∘ v: Y∗ → Y is equal to j: X → X∗, y → ϕ_y.

Now, we give some geometric properties of these Lp-spaces.
Proposition 8.12 Let $X$ be a JBW$^*$-triple equipped with a functional $\varphi \in X_*$ with $\|\varphi\|_{X_*} = 1$ such that $s(\varphi)$ is complete. Suppose $1 < p < \infty$. The Banach space $L^p(X)$ is uniformly convex (hence reflexive) and uniformly smooth.

Proof : By the reiteration theorem [Bel76, Theorem 4.6.1 p. 101], the Banach space $L^p(X)$, $1 < p < 2$ is a complex interpolation space between the predual $X_*$ and the space $L^2(X)$. Since the latter is an Hilbert space by Proposition 8.11, it results from [Cwr82] that $L^p(X)$ is uniformly convex. The same argument works for $2 < p < \infty$, since then the space $L^p(X)$ is a complex interpolation space between $X$ and $L^2(X)$. The last assertion is a consequence of [Meg98, Theorem 5.5.12 p. 500] which says that a normed space is uniformly convex if and only if its dual space is uniformly smooth.

We finish this section with the following result which will be used in Example 9.7.

Proposition 8.13 Let $X$ be a JBW$^*$-triple equipped with a functional $\varphi \in X_*$ with $\|\varphi\|_{X_*} = 1$ such that $s(\varphi)$ is complete. Let $\alpha$ be an automorphism of $X$ satisfying $\varphi \circ \alpha = \varphi$. Then, for each $1 \leq p < \infty$, the map $\alpha$ induces a surjective isometry $\alpha_p : L^p(X) \to L^p(X)$.

Proof : Recall that a triple automorphism is an isometry, see e.g. [Chu12, Theorem 3.1.20 p. 183]. Hence $\|\alpha^{-1}(s(\varphi))\|_X = \|s(\varphi)\|_X = 1$ and $\varphi(\alpha^{-1}(s(\varphi))) = \varphi(s(\varphi)) = 1$. Consequently $\varphi(\{x, y, \alpha^{-1}(s(\varphi))\}) \overset{(8.4)}{=} \langle x, y \rangle_{\mathcal{H}_\varphi}$ for any $x, y \in X$. Using the invariance $\varphi \circ \alpha = \varphi$, we see that for any $x, y \in X$

$$\varphi_{\alpha(y)}(\alpha(x)) \overset{(8.7)}{=} \varphi(\{\alpha(x), \alpha(y), s(\varphi)\}) = \varphi(\{x, y, \alpha^{-1}(s(\varphi))\})$$

$$= \varphi(\{x, y, \alpha^{-1}(s(\varphi))\}) = \langle x, y \rangle_{\mathcal{H}_\varphi} \overset{(8.4)}{=} \varphi(\{x, y, s(\varphi)\}) \overset{(8.7)}{=} \varphi_{\alpha(y)}(x).$$

We infer that the map $j(X) \to j(X)$, $\varphi \mapsto \varphi_{\alpha(y)}$, induces a surjective isometry $\alpha_1 : L^1(X) \to L^1(X)$, which sends $j(X)$ into itself isometrically since $\alpha$ is an isometry. Consequently, we obtain the result by interpolation (here we use the fact that a linear contraction with contractive linear inverse is necessarily isometric).

Remark 8.14 If $X$ is a $W^*$-TRO, the link between these $L^p$-spaces and the rectangular $L^p$-spaces of Section 5 is unclear.

9 Open questions

The following naive conjecture tries to clarify the open question of [ArF78, p. 99] on contractively complemented subspaces of noncommutative $L^p$-spaces. Moreover, a clarification of Remark 8.14 is necessary in order to see if it is a good statement.

Conjecture 9.1 Suppose $1 < p < \infty$ with $p \neq 2$. Let $Y$ be a Banach space. Then $Y$ is isometric to a contractively complemented subspace of a Haagerup noncommutative $L^p$-space $L^p(M, \psi)$ where $M$ is a $\sigma$-finite von Neumann algebra equipped with a normal faithful state $\psi$ if and only if $Y$ is isometric to the $L^p$-space $L^p(X, \varphi)$ of a JW$^*$-triple $X$ equipped with a linear functional $\varphi \in X_*$ with $\|\varphi\|_{X_*} = 1$ such that the support tripotent $s(\varphi)$ is complete.

It is not clear if the following question has an affirmative answer. A suitable notion of trace exists on JBW$^*$-algebras.

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Question 9.2 Does there exist a useful notion of trace on $\text{JBW}^*$-triples which allows us to define $L^p$-spaces associated to $\text{JBW}^*$-triples?

We continue with the following precise question which asks if the category of $L^p$-spaces of $\text{JBW}^*$-triples and contractions is projectively stable [NeR11, p. 295].

Question 9.3 Suppose $1 < p < \infty$ with $p \neq 2$. Is it true that each contractively complemented subspace of an $\text{JBW}^*$-triple is isometric to another $L^p$-space of a $\text{JBW}^*$-triple?

Recall that a contractive projection $P: X \to X$ on a Banach space $X$ is said to be bi-contractive if the projection $\text{Id} - P$ is also contractive and that an isometry $T: X \to X$ such that $T^2 = \text{Id}$ is called a symmetry (or involutive isometry). It is easy to see and well-known that every symmetry $T: X \to X$ gives rise to a bi-contractive projection $P = \frac{1}{2}(I + T)$ with complement $\text{Id} - P = \frac{1}{2}(I - T)$. Nevertheless, if $P: X \to X$ is a bi-contractive projection on $X$ then in general $T = 2P - \text{Id}$ need not be a symmetry. It is a classical topic to determine if a Banach space (or a class of Banach spaces) has the property that each bi-contractive projection $P: X \to X$ has the form $P = \frac{1}{2}(I + T)$ for some symmetry $T: X \to X$. This problem is explicitly written in [Rus94, Problem 2 p. 261].

It is known that $L^p$-spaces of measures spaces and $\text{JB}^*$-triples have this property, see [ByS72], [BeL77] and [FeB87]. Moreover, if $1 \leq p < \infty$ it is proved in [ArF92, Theorem 9.4] that each Schatten space $S^p$ has also this property. Consequently, the following problem is natural.

Problem 9.4 Suppose $1 < p < \infty$ with $p \neq 2$. Describe arbitrary bi-contractive projections acting on noncommutative $L^p$-spaces.

It is easy to give examples of bi-contractive projections acting on noncommutative $L^p$-spaces.

Example 9.5 Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal faithful state $\varphi$. Consider a $*$-automorphism $\alpha$ of $\mathcal{M}$ satisfying $\alpha^2 = \text{Id}_\mathcal{M}$ and preserving the state, i.e. $\varphi \circ \alpha = \varphi$. Then $\frac{1}{2}(\text{Id} + \alpha_p): L^p(\mathcal{M}) \to L^p(\mathcal{M})$ is a bicontractive projection.

Now, we state a problem similar to Problem 9.4.

Problem 9.6 Suppose $1 < p < \infty$ with $p \neq 2$. Describe the bi-contractive projections acting on the $L^p$-spaces $L^p(X)$ of a $\text{JBW}^*$-triple equipped with a linear functional $\varphi \in X_*$ with $\|\varphi\|_{X_*} = 1$ such that the support tripotent $s(\varphi)$ is complete.

Similarly to Example 9.7, we can give examples of bi-contractive projections on the $L^p$-spaces of a $\text{JBW}^*$-triple.

Example 9.7 Let $X$ be a $\text{JBW}^*$-triple equipped with a linear functional $\varphi \in X_*$ with $\|\varphi\|_{X_*} = 1$ such that the support tripotent $s(\varphi)$ is complete. If $\alpha$ is an automorphism of $X$ satisfying $\alpha^2 = \text{Id}_X$ and $\varphi \circ \alpha = \varphi$ then with Proposition 8.13, we see that $\frac{1}{2}(\text{Id} + \alpha_p): L^p(X) \to L^p(X)$ is a bi-contractive projection. Note that isometries of Cartan factors are described in [Isi19, Chapter 19].

Let $\mathcal{M}$ be a von Neumann algebra. Recall that the noncommutative $L^p$-space $L^p(\mathcal{M}, \varphi)$ is independent of the normal semifinite faithful weight $\varphi$ up to an isometric isomorphism, see [Ter81, p. 59] and [Ray03, Theorem 5.1]. So the next question is natural.

Question 9.8 Suppose $1 < p < \infty$. Let $X$ be a $\text{JBW}^*$-triple equipped with a functional $\varphi \in X_*$ with $\|\varphi\|_{X_*} = 1$ such that $s(\varphi)$ is complete. Is it true that the Banach space $L^p(X, \varphi)$ is independent from $\varphi$?
We finish with the following question which generalizes the question of [ArK23, Remark 3.6]. A positive answer would allow us to improve Proposition 7.1.

**Question 9.9** Suppose $1 < p < \infty$. Let $M$ and $N$ be von Neumann algebras. Consider some $n \in \{1, 2, \ldots, \infty\}$. If $T': L^p(M) \to L^p(M)$ is a contractively $n$-pseudo-decomposable map, does there exist some linear maps $v_1, v_2$ such that the map $\Phi$ of (4.1) is $n$-positive and contractive?

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