SINGULAR FOLD WITH REAL NOISE

PETER W. BATES
Department of Mathematics, Michigan State University
East Lansing, MI 48824, USA

JI LI
School of Mathematics and Statistics
Huazhong University of Science and Technology
Wuhan, Hubei 430074, China

MINGJI ZHANG
Department of Mathematics, New Mexico Institute of Mining and Technology
Socorro, NM 87801, USA

Abstract. We study the effect of small real noise on the jump behavior near a singular fold point, which is an important step in understanding the burst-spike behavior in many biological models. We show by the theory of center manifolds and random invariant manifolds that if the order of the noise is high enough, trajectories essentially pass the fold point in the manner as though there is no noise.

1. Introduction. Neurons and many other excitable cells exhibit burst-spike mode oscillations, which is characterized by slow passage near a silent phase and rapid, spike-like oscillations near an active phase. Among those cells is typically the pancreatic β-cell (see [12, 13]). Many such models contain multiple time scales and lead to very interesting issues related to the theory of singular perturbations.

Although a deterministic model is able to explain a few experiments, it fails in its accuracy. Moreover, noise may dramatically affect the dynamics of the deterministic model in some cases, through stochastic resonance for instance (see [2]). Furthermore, noise can add new dynamical scenarios, as in the case of [9]. Motivated by these facts, we extend the classical geometric singular perturbation theory to the real noisy case, hoping to lay the groundwork for further study of a noisy model of burst-spike behavior.

This paper, following our recent work [5], is devoted to the geometrical theory for slow-fast systems of ordinary differential equations driven by real noise near a fold point. By real noise, we mean the kind that can be studied in the pathwise sense (see page 57 in [1] for more details). To be explicit, we analyze trajectories

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* Corresponding author: Ji Li.
for systems of the form
\[
\begin{align*}
\dot{x} &= f(x, y, \epsilon) + \epsilon F(\theta_t \omega, x, y, \epsilon), \\
\dot{y} &= \epsilon g(x, y, \epsilon) + \epsilon^2 G(\theta_t \omega, x, y, \epsilon),
\end{align*}
\]
where \(f\), \(g\), \(F\), and \(G\) are smooth functions, \(\epsilon\) is a small parameter, and \(\theta_t \omega\) is a metric dynamical system over a probability space \((\Omega, \mathcal{F}, P)\), modeling the evolution of noise (see [1]).

In [5], we extended the classical geometric singular perturbation theory of Fenichel [7]: System (1.1) has a critical manifold of equilibria when \(\epsilon = 0\) given by \(f(x, y, 0) = 0\). The system may also be written in 'slow' time \(\tau = \epsilon t\):
\[
\begin{align*}
\epsilon x' &= f(x, y, \epsilon) + \epsilon F(\theta_\tau \omega, x, y, \epsilon), \\
y' &= g(x, y, \epsilon) + \epsilon G(\theta_\tau \omega, x, y, \epsilon),
\end{align*}
\]
giving, for \(\epsilon = 0\), \(y\)-dynamics on the critical manifold. We showed that the center-stable, center-unstable, and center manifolds of the critical manifold persist for small \(\epsilon \neq 0\), and that the dynamics thereon may be analyzed under the condition of normal hyperbolicity. In particular, the slow \(y\) equation is perturbed by small noise as a regular perturbation on the new random center manifold.

In this paper, we study one very common case called the fold point where normal hyperbolicity fails. At a fold point, one stable eigenvalue of the Jacobian \(\frac{\partial f}{\partial x}\) crosses the imaginary axis transversally at the origin. The well-known phenomenon of relaxation oscillation is related to this, in which solutions move slowly towards a fold point, jump from the fold point to another stable branch of the slow manifold, follow the slow dynamics again until reaching another fold point, and jump again, etc., with possibly consequent periodic solutions. Besides the classical relaxation oscillation, the burst-spike behavior is also related to the jumping behavior near a fold point. The geometric theory near fold points for deterministic systems was established by Szolnoky, Krupa and Wechselberger (see [8, 11]).

1.1. Generic fold. In a 2-dimensional slow-fast system
\[
\begin{align*}
\dot{x} &= f(x, y, \epsilon), \\
\dot{y} &= \epsilon g(x, y, \epsilon),
\end{align*}
\]
where \(x \in \mathbb{R}\), \(y \in \mathbb{R}\), and \(\epsilon > 0\) is very small, a point \((x_0, y_0)\) is called a generic fold if the following conditions hold:
\[
\begin{align*}
f(x_0, y_0, 0) &= 0, & \frac{\partial f}{\partial x}(x_0, y_0, 0) &= 0, \quad (1.4) \\
\frac{\partial^2 f}{\partial x^2}(x_0, y_0, 0) &\neq 0, & \frac{\partial f}{\partial y}(x_0, y_0, 0) &\neq 0, & g(x_0, y_0, 0) &\neq 0. \quad (1.5)
\end{align*}
\]
Assume, without loss of generality, that
\[
(x_0, y_0) = (0, 0), \quad \frac{\partial^2 f}{\partial x^2}(0, 0, 0) > 0, \quad \frac{\partial f}{\partial y}(0, 0, 0) < 0, \quad g(0, 0, 0) < 0. \quad (1.6)
\]
Let \(S = \{(x, y)|f(x, y, 0) = 0\}\) be the critical manifold. The above assumptions imply that there is a neighborhood \(U\) of the origin such that \((0, 0)\) is the only point in \(U \cap S\) where \(\frac{\partial f}{\partial x}\) vanishes. By the Implicit Function Theorem, \(S \cap U\) can be represented as \(y = \phi(x)\) and is approximately the parabola \(y = -\frac{f_y(0, 0, 0)}{f_x(0, 0, 0)} x^2\). The assumption \(\frac{\partial^2 f}{\partial x^2}(0, 0, 0) > 0\) implies that the left branch which we denote by \(S_a\) is
attracting and the right branch which we denote by $S_r$ is repelling. The reduced flow on the critical manifold is determined by

$$\phi'(x)x' = g(x, \phi(x), 0).$$

The assumption $g(0, 0, 0) < 0$ implies that restricted to a sufficiently small neighborhood $U$, the reduced flow on $S_a$ and $S_r$ is towards the fold point.

Fenichel’s theory implies that outside an arbitrarily small neighborhood $V$ of $(0, 0)$, for small $\epsilon > 0$, the manifolds $S_a$ and $S_r$ perturb smoothly to locally invariant slow manifolds $S_{a,\epsilon}$ and $S_{r,\epsilon}$, which under the current set-up are single solutions. For small $\rho > 0$ and an appropriate interval $J \subset \mathbb{R}$, let

$$\Delta^\text{in} = \{(x, \rho^2) | x \in J\}$$

be a section in $U$ transverse to $S_a$, and let

$$\Delta^\text{out} = \{\rho, y | y \in \mathbb{R}\}$$

be a section transverse to the fast fibers (See Figure 1). For the transition map $\pi : \Delta^\text{in} \to \Delta^\text{out}$, the following result is proved in [8]:

**Proposition 1.1.** There exists $\epsilon_0 > 0$ such that the following assertions hold for $\epsilon \in (0, \epsilon_0]$:

1. The manifold $S_{a,\epsilon}$ passes through $\Delta^\text{out}$ at a point $(\rho, h(\epsilon))$, where $h(\epsilon) = O(\epsilon^{\frac{2}{3}})$.

2. The transition map $\pi$ is a contraction with contraction rate $O(e^{-c\epsilon})$, where $c$ is a positive constant.

1.2. **Stochastic model.** When noise is taken into account, there are at least two fairly different ways to proceed: Either with a system of stochastic differential equations

$$\begin{align*}
    dx &= \frac{1}{\epsilon} f(x, y, \epsilon) d\tau + \frac{\sigma_1}{\sqrt{\epsilon}} F(x, y, \epsilon) dB_\tau, \\
    dy &= g(x, y, \epsilon) d\tau + \sigma_2 G(x, y, \epsilon) dB_\tau,
\end{align*}$$

(1.7)
where noise is related to Brownian motion and Itô integrals are involved, or with a system of random differential equations:

\[
\begin{align*}
\frac{dx}{dt} &= f(x, y, \epsilon) + \sigma_1 F(\theta^t \omega, x, y, \epsilon), \\
\frac{dy}{dt} &= \epsilon [g(x, y, \epsilon) + \sigma_2 G(\theta^t \omega, x, y, \epsilon)],
\end{align*}
\]

(1.8)

where \(\theta^t \omega\) is a metric dynamical system over a probability space, modeling the evolution of the noise.

In this paper, we study the random system (1.8) with the method of random dynamical systems. The noise in our case is assumed to be uniformly \(C^1\) small, allowing us to use the random invariant manifold and foliation theory developed in \([3, 4, 5]\).

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \(X\) be a Banach space. Let \(T = \mathbb{R}\) or \(\mathbb{Z}\) endowed with their Borel \(\sigma\)-algebra.

**Definition 1.** A family \((\theta^t)_{t \in T}\) of mappings from \(\Omega\) into itself is called a metric dynamical system if

1. \((\omega, t) \rightarrow \theta^t \omega\) is \(\mathcal{F} \otimes B(\mathbb{T})\) measurable;
2. \(\theta^0 = \text{id}_\Omega\), the identity on \(\Omega\), \(\theta^{t+s} = \theta^t \circ \theta^s\) for all \(t, s \in T\);
3. \(\theta^t\) preserves the probability measure \(\mathbb{P}\).

**Definition 2.** A map \(\phi : T \times \Omega \times X \rightarrow X, (t, \omega, x) \mapsto \phi(t, \omega, x)\), is called a random dynamical system (RDS) on the Banach space \(X\) over a metric dynamical system \((\Omega, \mathcal{F}, \mathbb{P}, \theta^t)_{t \in T}\) if

1. \(\phi\) is \(B(\mathbb{T}) \otimes \mathcal{F} \otimes B(X)\)-measurable;
2. The mappings \(\phi(t, \omega) := \phi(t, \omega, \cdot) : X \rightarrow X\) form a cocycle over \(\theta^t\):
   \[\phi(0, \omega) = 1d, \quad \text{for all } \omega \in \Omega,\]
   \[\phi(t + s, \omega) = \phi(t, \theta^s \omega) \circ \phi(s, \omega), \quad \text{for all } t, s \in T, \omega \in \Omega.\]

When \(\phi(\cdot, \omega, \cdot) : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n\) is continuous, \(\phi(t, \omega, x)\) is called a continuous random dynamical system. A continuous RDS \(\phi\) is called a smooth random dynamical system of class \(C^k, k \geq 1\), if for each \((t, \omega) \in \mathbb{R} \times \Omega\), \(\phi(t, \omega, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is \(C^k\)-smooth.

The concept of random dynamical system was formulated as a means to rigorously analyze systems modeling many phenomena that are subject to uncertainty or random influences in areas as widely ranging as physics, biology, climatology, economics, etc. Those influences may arise through stochastic forcing, uncertain parameters, random sources or inputs, and random boundary conditions. The canonical example is the solution operator for a random differential equation driven by real noise:

\[
\frac{dx}{dt} = f(\theta^t \omega, x),
\]

(1.9)

where \(x \in \mathbb{R}^d\), \(f : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d\) is a measurable function and \(f_\omega \in L_{loc}(\mathbb{R}, C^0_b(\mathbb{R}^d, \mathbb{R}^d))\), for \(f_\omega(t, \cdot) \equiv f(\theta^t \omega, \cdot)\) (see page 554 in [1]). Throughout this paper, we suppose these basic conditions hold whenever there is a random differential equation of the form (1.9). Here, \((\Omega, \mathcal{F}, \mathbb{P})\) is the classical Wiener space, i.e., \(\Omega = \{\omega : \omega(\cdot) \in C(\mathbb{R}, \mathbb{R}^d), \omega(0) = 0\}\) endowed with the open compact topology so that \(\Omega\) is a Polish
space, and \( P \) is the Wiener measure. The measurable dynamical system \( \theta^t \) on the probability space \((\Omega, F, P)\) is given by the Wiener shift \((\theta^t \omega)(\cdot) = \omega(t + \cdot) - \omega(t)\), for \( t > 0 \). It is well-known that \( P \) is invariant and ergodic under \( \theta^t \). This measurable dynamical system \( \theta^t \) is also called a metric dynamical system and models the noise in the system. Another important class of generators for random dynamical systems may be found in stochastic ordinary differential equations, for instance system (1.7).

In this paper, we study system (1.8), in which we take \( \sigma \) as functions of \( \epsilon \):

\[
\begin{align*}
\dot{x} &= f(x, y, \epsilon) + \epsilon^\alpha F(\theta^t \omega, x, y, \epsilon), \\
\dot{y} &= g(x, y, \epsilon) + \epsilon^3 G(\theta^t \omega, x, y, \epsilon),
\end{align*}
\]  

(1.10)

where \( \alpha > 0 \) and \( \beta > 1 \). We make the following:

**Assumptions.** \( f, g, F, \) and \( G \) are \( C^{r+1} \) in \((x, y, \epsilon)\) for some \( r \geq 2 \), and the \( C^1 \) norms \((x, y, \epsilon)\) of \( F \) and \( G \) are uniformly bounded. Moreover, \( F \) and \( G \) are measurable in \( \omega \) and \( C^0 \) in \( t \) for a.e. \( \omega \in \Omega \). Furthermore, \( f \) and \( g \) satisfy (1.4), (1.5), and (1.6).

Under the above conditions, the main theorem in [5] implies that outside an arbitrary small neighborhood \( V \) of \((0,0)\), the critical manifolds \( S_0 \) and \( S_r \) perturb smoothly to locally invariant random slow manifolds \( S_{a, \epsilon, \omega} \) and \( S_{r, \epsilon, \omega} \) for sufficiently small \( \epsilon \neq 0 \). Under the two-dimensional set-up, \( S_{a, \epsilon, \theta^t \omega} \) and \( S_{r, \epsilon, \theta^t \omega} \) are actually trajectories of (1.10). \( S_{a, \epsilon, \omega} \) and \( S_{r, \epsilon, \omega} \) are obtained as \( \epsilon \)-sections of two-dimensional, locally invariant random manifolds \( M_{a, \omega} \) and \( M_{r, \omega} \), respectively, of the extended system

\[
\begin{align*}
\dot{x} &= f(x, y, \epsilon) + \epsilon^\alpha F(\theta^t \omega, x, y, \epsilon), \\
\dot{y} &= g(x, y, \epsilon) + \epsilon^3 G(\theta^t \omega, x, y, \epsilon), \\
\dot{\epsilon} &= 0,
\end{align*}
\]  

(1.11)

for which \( S \times \{0\} \) is a critical manifold. Outside of a small neighborhood of the fold point \((0,0)\), there are two ‘normally hyperbolic’ parts: \( S_a \times \{0\} \) and \( S_r \times \{0\} \). (Though the \( \epsilon \)-direction is nonhyperbolic, its triviality enables dealing with the above parts as hyperbolic.) By the main theorem of [5], there exist attracting and repelling center-like locally invariant random manifolds \( M_{a, \omega} \) and \( M_{r, \omega} \), \( \epsilon \)-sections of which give \( S_{a, \epsilon, \omega} \) and \( S_{r, \epsilon, \omega} \).

Yet [5] does not tell anything about the behavior of \( M_{a, \omega} \) at the fold point because of the loss of normal hyperbolicity. This is the topic of the current paper. Of particular interests are to determine how the part \( S_{a, \epsilon, \omega} \) passes the fold point, and where are the nearby dynamics. We will show that for high enough order noise, the dynamics essentially persist as though there is no noise.

1.3. Main results. Recall that \( U \) is a neighborhood of \((0,0)\), in which \( S \cap U \) is approximately the parabola \( y = -\frac{f_y(0,0,0)}{f_x(0,0,0)} x^2 \). Also view \( U \) as its natural extension to include the \( \epsilon \)-direction in \( \mathbb{R}^3 \), thus being a small neighborhood of \((0,0,0)\). We choose \( U \) sufficiently small so that \( g(x, y, \epsilon) < 0 \) for \((x, y, \epsilon) \in U \). By (1.4), (1.5) and (1.6), a rescaling of \( x, y, \epsilon \) and \( t \) yields the following canonical form:

\[
\begin{align*}
\dot{x} &= -y + x^2 + O(\epsilon, xy, y^2, x^3) + \epsilon^\alpha F(\theta^t \omega, x, y, \epsilon), \\
\dot{y} &= -(-1 + O(x, y, \epsilon)) + \epsilon^3 G(\theta^t \omega, x, y, \epsilon),
\end{align*}
\]  

(1.12)

with \( F, G \) and \( \theta \) being modified in an obvious manner yet with the same properties as in Assumptions. Also modify the defining interval \( J \) in \( \Delta^0 \) so that it is a
cross section. We study the new transition map $\pi$ from $\Delta^{\text{in}}$ to $\Delta^{\text{out}}$ defined by the random flow of system (1.12). Our main result reads:

**Theorem 3.** For $\alpha \geq 1$ and $\beta \geq \frac{4}{3}$, there exists $\epsilon_0 > 0$ such that the following assertions hold for $\epsilon \in (0, \epsilon_0]$ and all $\omega$:

1. The manifold (trajectory) $S_{a,\epsilon,\theta t}^{\omega}$ passes through $\Delta^{\text{out}}$ at a point $(\rho, h(\epsilon, \omega))$, where $h(\epsilon, \omega) = O(\epsilon^{\frac{2}{3}})$.

2. The transition map $\pi$ is a contraction with contraction rate $O(e^{-c\epsilon})$, where $c$ is a positive constant.

Figure 2 may help.

2. Proof of the main theorem. Consider the extended system:

\begin{align*}
\dot{x} &= -y + x^2 + O(\epsilon, xy, y^2, x^3) + \epsilon^{\alpha} F(\theta^t \omega, x, y, \epsilon), \\
\dot{y} &= \epsilon(-1 + O(x, y, \epsilon)) + \epsilon^{\beta} G(\theta^t \omega, x, y, \epsilon), \\
\dot{\epsilon} &= 0.
\end{align*}

(2.1)

2.1. Blow-up. In [8], the authors apply the method of blow-up borrowed from [6] to overcome the difficulty caused by nonhyperbolicity. The blow-up is a coordinate transformation in which the nonhyperbolic fold point $(0, 0, 0)$ is blown-up to a two-sphere, which can be analyzed by standard methods of dynamical systems. We follow the same blow-up transformation. Letting $S^2$ be the two-sphere, we define $B = S^2 \times [0, \rho]$, where the constant $\rho > 0$ will be determined later and related to $\epsilon_0$ by $\epsilon_0 = \rho^3$. The blow-up transformation is a mapping

$\Phi : B \to \mathbb{R}^3$

satisfying

$\bar{x} = \bar{r} \bar{x}, \quad \bar{y} = \bar{r}^2 \bar{y}, \quad \bar{\epsilon} = \bar{r}^3 \bar{\epsilon}$,

(2.2)

with $(\bar{x}, \bar{y}, \bar{\epsilon}) \in S^2$, $\bar{r} \in [0, \rho]$. We choose $\rho$ so small that $\Phi(B) \subset U$. 
Denote by $X$ the random vector field corresponding to system (2.1). Since $X$ vanishes at the origin, and the blow-up transformation restricted to $S^2 \times (0,\rho]$ is a diffeomorphism, there exists a random vector field $\bar{X}$ on $B$ such that $D\Phi(\bar{X}) = X$. We analyze the vector field $\bar{X}$ on $B$ in three different charts: $K_1$, $K_2$ and $K_3$, which are obtained by setting $\bar{y} = 1$, $\bar{\epsilon} = 1$, and $\bar{x} = 1$, respectively, in the blow-up transformation (2.2).

The blow-up transformation in charts $K_1$, $K_2$, and $K_3$ are given by

$$x = r_1 x_1, \quad y = r_1^2, \quad \epsilon = r_1^3 \epsilon_1,$$

$$x = r_2 x_2, \quad y = r_2^2 y_2, \quad \epsilon = r_2^3,$$

$$x = r_3, \quad y = r_3^2 y_3, \quad \epsilon = r_3^3 \epsilon_3,$$

with coordinates $(x_k, r_k, \epsilon_k) \in \mathbb{R}^3$ for $k = 1, 2, 3$. We then have the following coordinate changes between these charts:

**Lemma 2.1.** (From [8]) Let $\kappa_{ij}$ be the change of coordinates from $K_i$ to $K_j$. One has

1. $\kappa_{12}$ is given by

$$x_2 = x_1 \epsilon_1^{-1/3}, \quad y_2 = \epsilon_1^{-2/3}, \quad r_2 = r_1 \epsilon_1^{1/3} \text{ for } \epsilon_1 > 0,$$

2. $\kappa_{21}$ is given by

$$x_1 = x_2 y_2^{-1/2}, \quad r_1 = r_2 y_2^{1/2}, \quad \epsilon_1 = y_2^{-3/2} \text{ for } y_2 > 0,$$

3. $\kappa_{23}$ is given by

$$r_3 = r_2 x_2, \quad y_3 = y_2 x_2^{-2}, \quad \epsilon_3 = x_2^{-3} \text{ for } x_2 > 0,$$

4. $\kappa_{32}$ is given by

$$x_2 = \epsilon_3^{-1/3}, \quad y_2 = y_3 \epsilon_3^{-2/3}, \quad r_2 = r_3 \epsilon_3^{1/3} \text{ for } \epsilon_3 > 0.$$

We borrow notation from [8]: $\bar{P}$ denotes an object in the blow-up which corresponds to an object $P$ in the original problem. If $\bar{P}$ is described in one of the charts, then $P_i$ denotes the object in chart $K_i$ for $i = 1, 2, 3$.

Let $\bar{X}_0 = \bar{X}|_{S^1 \times [0,\rho]}$. Figure 3 sketches the phase portrait of $\bar{X}_0$, which is deterministic and will be verified later. On the invariant circle $S^1$, there are four equilibria: $p_a, p_r, q_{in}, q_{out}$. These equilibria are hyperbolic for the flow on $S^1$, $p_a$ and $q_{out}$ being attracting, and $p_r$ and $q_{in}$ being repelling. The points $p_a$ and $p_r$ are end points of the blown-up critical manifolds $\bar{S}_a$ and $\bar{S}_r$, which are lines of equilibria for $X_0$. The existence of these four (and only these four) equilibria will be verified by analyzing in local charts.
2.2. **Dynamics in chart** $K_2$. The dynamics of the blown-up random vector field $\bar{X}$ in a neighborhood of the upper half sphere can be analyzed in chart $K_2$.

Inserting (2.4) into (2.1), we obtain in chart $K_2$ the following system:

\[
\begin{align*}
\dot{x}_2 &= r_2 (-y + x^2 + O(r_2)) + r_2^3 \alpha F_2(\theta_\omega, x_2, y_2, r_2), \\
\dot{y}_2 &= r_2 (-1 + O(r_2)) + r_2^3 \beta G_2(\theta_\omega, x_2, y_2, r_2), \\
\dot{r}_2 &= 0.
\end{align*}
\]

For $\alpha, \beta > \frac{2}{3}$, a rescaled time $t_2 = r_2 t$ yields:

\[
\begin{align*}
\dot{x}'_2 &= -y_2 + x^2 + O(r_2), \\
\dot{y}'_2 &= -1 + O(r_2), \\
\dot{r}_2 &= 0, \tag{2.10}
\end{align*}
\]

which can be viewed as a regular (random) perturbation of the following Riccati equation, for which we have dropped the subscript for readability:

\[
\begin{align*}
x' &= -y + x^2, \\
y' &= -1. \tag{2.11}
\end{align*}
\]

Note that, since we care only about the transition map for $r > 0$ ($\epsilon > 0$), it does not matter what time scale we use for the random flow. The following facts hold for system (2.11).

**Proposition 2.1.** (From [10]) The Riccati equation (2.11) has the following properties (see Figure 4):

1. Every orbit has a horizontal asymptote $y = y_r$, where $y_r$ depends on the orbit, such that $y$ approaches $y_r$ from above as $x \to \infty$.

2. There exists a unique orbit $\gamma_2$ with parametrization as $(x, s(x)), x \in \mathbb{R}$, which is asymptotic to the left branch of the parabola $x^2 - y = 0$ as $x \to -\infty$. The orbit $\gamma_2$ has a horizontal asymptote $y = -\Omega_0 < 0$ such that $y$ approaches $-\Omega_0$ from above as $x \to \infty$.

3. The function $s(x)$ has the asymptotic expansions

\[
\begin{align*}
s(x) &= x^2 + \frac{1}{2x} + O \left( \frac{1}{x^4} \right), \quad x \to -\infty, \\
s(x) &= -\Omega_0 + \frac{1}{x} + O \left( \frac{1}{x^3} \right), \quad x \to \infty.
\end{align*}
\]

4. All orbits to the right of $\gamma_2$ are backward asymptotic to the right branch of the parabola $x^2 - y = 0$.
5. Every orbit to the left of \( \gamma_2 \) has a horizontal asymptote \( y = y_l > y_r \), where \( y_l \) depends on the orbit, such that \( y \) approaches \( y_l \) from below as \( x \to -\infty \).

![Figure 4. Solutions of the Riccati equation](image)

(Refer to the notation of \( P \), \( \bar{P} \), and \( P_i \).) It will turn out that the corresponding orbit \( \tilde{\gamma} \) to \( \gamma_2 \) is backward asymptotic to the equilibrium \( p_a \) and forward asymptotic to the equilibrium \( p_{out} \) on the equator of \( S^2 \). Moreover, close to \( \tilde{\gamma} \), the trajectory \( S_{a,e,\theta,\omega} \) leaves a neighborhood of \( p_a \), passes near the upper half of the two-sphere and enters into a neighborhood of \( q_{out} \).

For small \( \delta > 0 \), we define

\[
\Sigma^{in}_2 = \{(x_2, y_2, r_2) | y_2 = \delta^{-2/3}\}, \quad \Sigma^{out}_2 = \{(x_2, y_2, r_2) | x_2 = \delta^{-1/3}\}.
\]

Let \( \pi_2 \) be the transition map from \( \Sigma^{in}_2 \) to \( \Sigma^{out}_2 \) determined by (2.10). Let \( q_0 \) be the intersection of \( \gamma_2 \) with \( \Sigma^{in}_2 \).

**Proposition 2.2.** The transition map \( \pi_2 \) has the following properties:

(i) \( \pi_2(q_0) = (\delta^{-1/3}, -\Omega_0 + \delta^{1/3} + O(\delta), 0) \).

(ii) A neighborhood of \( q_0 \) is mapped diffeomorphically into a neighborhood of \( \pi_2(q_0) \).

**Remark 2.1.** In the second part in Proposition 2.2, we mean that for each fixed \( \omega \), a fixed neighborhood of \( q_0 \) is mapped diffeomorphically onto a neighborhood of \( \pi_2(q_0) \) and all of these images, created by varying \( \omega \), are contained in a fixed small neighborhood of \( \pi_2(q_0) \).

### 2.3. Dynamics in chart \( K_1 \).

Chart \( K_1 \) is used to analyze the dynamics in a neighborhood containing \( p_a \) and \( p_r \). Inserting (2.3) into system (2.1) and rescaling with \( t_1 = r_1 t \), we obtain the following system in chart \( K_1 \):

\[
\begin{align*}
x'_1 &= -1 + x_1^2 + \frac{1}{2} \epsilon_1 x_1 + O(r_1) + r_1^{3\alpha - 2} \epsilon_1^\alpha F_1(\theta^1 \omega, x_1, r_1, \epsilon_1) \\
&\quad - \frac{1}{2} r_1^{3\beta - 3} \epsilon_1^\beta x_1 G_1(\theta^1 \omega, x_1, r_1, \epsilon_1), \\
r'_1 &= \frac{1}{2} r_1 \epsilon_1 (-1 + O(r_1)) + \frac{1}{2} r_1^{3\beta - 2} \epsilon_1^\beta G_1(\theta^1 \omega, x_1, r_1, \epsilon_1).
\end{align*}
\]
\[ \epsilon_1' = \frac{3}{2} \epsilon_1^2 (1 + O(r_1)) - \frac{3}{2} r_1^{3\beta - 3} \epsilon_1^{\beta+1} G_1(\theta^t \omega, x_1, r_1, \epsilon_1). \] (2.12)

This system has two (deterministic) invariant subsystems: on the invariant subspace \( \epsilon_1 = 0 \), the system reads

\[
\begin{align*}
x_1' &= -1 + x_1^2 + O(r_1), \\
r_1' &= 0,
\end{align*}
\] (2.13)

and on the invariant subspace \( r_1 = 0 \), the system reads

\[
\begin{align*}
x_1' &= -1 + x_1^2 + \frac{1}{2} \epsilon_1 x_1, \\
\epsilon_1' &= \frac{3}{2} \epsilon_1^2.
\end{align*}
\] (2.14)

The intersection of these two subspaces is the \( x_1 \)-axis, restricted to which there are two hyperbolic equilibria: \( x_1 = \pm 1 \), corresponding to \( p_r \) and \( p_a \), respectively, on the equator \( S^1 \). We focus on the more interesting dynamics near \( p_a (x_1 = -1) \).

For the subsystem (2.13), the nonhyperbolic equilibrium \( (x_1, r_1) = (-1, 0) \) has a one-dimensional attracting center manifold \( S_{a,1} \), corresponding to the critical manifold \( S_a \) of the original system.

For the subsystem (2.14), the nonhyperbolic equilibrium \( (x_1, \epsilon_1) = (-1, 0) \) has a one-dimensional attracting center manifold \( N_{a,1} \), along which \( \epsilon_1 \) increases if \( \epsilon_1 > 0 \). So the part of \( N_{a,1} \) in the half plane \( \epsilon_1 > 0 \) is unique.

For system (2.12), the nonhyperbolic equilibrium \( (x_1, r_1, \epsilon_1) = (-1, 0, 0) \) has eigenvectors: \( <1, 0, 0>, <0, 1, 0>, \) and \( <-1, 0, 4> \), corresponding to one stable and two center directions. When \( \alpha \geq 1 \) and \( \beta \geq \frac{4}{3} \), system (2.12) has a two-dimensional random center manifold to the equilibrium \(( -1, 0, 0) \), which intersects \{ \( r_1 = 0 \) \} and \{ \( \epsilon_1 = 0 \) \} at \( N_{a,1} \) and \( S_{a,1} \), respectively.

Figure 5. Geometry and dynamics in chart \( K_1 \).
The (extended) original section $\Delta^\text{in}$ is transformed to a subset of $\{(x_1, r_1, \epsilon_1)|r_1 = \rho\}$. We restrict our discussion to the set

$$D_1 := \{(x_1, r_1, \epsilon_1)|0 \leq r_1 \leq \rho, 0 \leq \epsilon_1 \leq \delta\},$$

where $\delta > 0$ is the small constant used earlier in defining $\Sigma_2^\text{in}$ and $\Sigma_2^\text{out}$. Define

$$\Sigma_1^\text{in} = \{(x_1, r_1, \epsilon_1) \in D_1|r_1 = \rho\}, \quad \Sigma_1^\text{out} = \{(x_1, r_1, \epsilon_1) \in D_1|\epsilon_1 = \delta\}.$$

We study the transition map $\pi_1$ from $\Sigma_1^\text{in}$ to $\Sigma_1^\text{out}$. Note that $\Sigma_1^\text{out}$ is transformed by $\kappa_{12}$ to $\Sigma_2^\text{in}$.

From the theory of random center manifolds for a fixed point, we have

**Proposition 2.3.** For $\alpha \geq 1$, $\beta \geq \frac{1}{\epsilon}$ and sufficiently small $\rho, \delta$, the following hold for system (2.12):

1. There exists an attracting two-dimensional local random center manifold $M_{a,1,\omega}$ which contains $S_{a,1}$ and $N_{a,1}$. In $D_1$, $M_{a,1,\omega}$ is given as a graph $x_1 = b(r_1, \epsilon_1, \omega)$. The branch $N_{a,1}$ in $\{r_1 = 0, \epsilon_1 > 0\}$ is unique.

2. There exist stable random invariant foliations of the phase space with base $M_{a,1,\omega}$ and one-dimensional fibers. For any $\epsilon < 2$, one can choose $\rho, \delta$ small enough such that the contraction along each fiber during a time interval $[0, T]$ is stronger than $e^{-\epsilon T}$.

3. The special orbit $\gamma_2$ in chart $K_2$ is transformed by $\kappa_{21}$ to $N_{a,1}$.

**Proof.** 1 and 2 are standard. To prove part 3, we use the expansion in part 3 of Proposition 2.1. As $x_2 \to -\infty$, $\gamma_2$ has the expansion

$$\left(x_2, x_2^2 + \frac{1}{2x_2} + O(\frac{1}{x_2^2}), 0\right),$$

which is transformed by $\kappa_{21}$ to

$$\left(x_2 \left(x_2^2 + \frac{1}{2x_2} + O(\frac{1}{x_2^2})\right)^{-\frac{1}{2}}, 0, \left(x_2^2 + \frac{1}{2x_2} + O(\frac{1}{x_2^2})\right)^{-\frac{3}{2}}\right) \to (-1, 0, 0).$$

Since

$$\left(x_2 \left(x_2^2 + \frac{1}{2x_2} + O(\frac{1}{x_2^2})\right)^{-\frac{1}{2}} + 1\right) \left(x_2^2 + \frac{1}{2x_2} + O(\frac{1}{x_2^2})\right)^{-\frac{3}{2}} \to -\frac{1}{4},$$

it follows that $\kappa_{12}^{-1}(\gamma_2) \to (-1, 0, 0)$ along the eigenvector $<-1, 0, 4>$. Since the branch $N_{a,1}$ in $\{r_1 = 0, \epsilon_1 > 0\}$ is unique, it must be the same as $\kappa_{21}(\gamma_2).$ \qed

Let $R_1$ be the rectangle in $\Sigma_1^\text{in}$ defined by $|1 + x_1| \leq \beta_1$, where $\beta_1$ is small. The constants $\rho, \delta$ and $\beta_1$ are chosen such that $M_{a,1,\omega} \cap \Sigma_1^\text{in} \subset R_1$ for any $\omega$. For $0 < \epsilon < \delta$, let $I_{a}(\epsilon)$ be the line $R_1 \cap \{\epsilon_1 = \epsilon\}$.

In order to study the transition map $\pi_1$, we need first to estimate how long a trajectory stays in $D_1$. From the third equation of system (2.12), we have

$$\left(\frac{1}{\epsilon_1}\right)' = -\frac{3}{2}(1 + O(r_1)).$$

This establishes:
Lemma 2.2. The transition time $T$ of a solution of system (2.12) from a point $p = (x_1, \rho, \epsilon_1) \in \Sigma_{1}^{\omega}$ to the point $\pi_1(p) \in \Sigma_{1}^{out}$ satisfies

$$T = \frac{2}{3} \left( \frac{1}{\epsilon_1} - \frac{1}{3} \right) (1 + O(\rho))$$

uniformly for any $\omega$.

This, with part 2 of Proposition 2.3, then gives:

Proposition 2.4. For $\rho, \delta$ and $\beta_1$ sufficiently small, $\pi_1(R_1)$ is a wedge-like region in $\Sigma_{1}^{out}$: For fixed $c < 2$, there exists a constant $K > 0$ depending on $c, \rho, \delta$, and $\beta_1$ such that for $\epsilon_1 \in (0, \delta]$ the map $\pi_1|_{I_{a}(\epsilon_1)}$ is a contraction stronger than $K e^{\frac{2}{3} \left( \frac{1}{\epsilon_1} - \frac{1}{3} \right)}$.

Remark 2.2. Note that, for different $\omega$, $R_1$ is mapped to different wedge-like regions in $\Sigma_{1}^{out}$, but all these are contained in a fixed wedge-like region (Figure 5 should help).

2.4. Dynamics in chart $K_3$. Chart $K_3$ is used to analyze the dynamics in a neighborhood containing $q_{out}$. Inserting (2.5) into system (2.1) and rescaling with $t_3 = r_3 t$, we obtain the following system in chart $K_3$:

$$
\begin{align*}
    r_3' &= r_3 \tilde{F}_3(\theta^t \omega, r_3, y_3, \epsilon_3), \\
    y_3' &= -2 y_3 \tilde{F}_3(\theta^t \omega, r_3, y_3, \epsilon_3) + \epsilon_3(-1 + O(r_3)) + r_3^{\beta_3 - 3} \epsilon_3^2 G_3(\theta^t \omega, r_3, y_3, \epsilon_3), \\
    \epsilon_3' &= -3 \epsilon_3 \tilde{F}_3(\theta^t \omega, r_3, y_3, \epsilon_3),
\end{align*}
$$

(2.16)

where $\tilde{F}_3(\theta^t \omega, r_3, y_3, \epsilon_3) = 1 - y_3 + O(r_3)$, $O(r_3)$ being uniform for $\omega$.

The planes $\{ \epsilon_3 = 0 \}$, $\{ r_3 = 0 \}$ and the $y_3$-axis are invariant under the flow of (2.16). For (2.16), $q_{out} = (0, 0, 0)$ is a hyperbolic equilibrium, one eigenvalue of which is $\lambda = -2$ with eigenvector $(0, 1, 0)$. Let $\gamma_3 := \kappa_2(\gamma_2)$. By (2.8) and part 3 of Proposition 2.1, $\gamma_3$ has the expansion

$$(0, -\Omega_0 \epsilon_3^2 + \epsilon_3 + O(\epsilon_3), \epsilon_3)$$

as $\epsilon_3 \to 0^+$.

This establishes:

Lemma 2.3. The orbit $\gamma_3$ lies in the plane $\{ r_3 = 0 \}$, converges to $q_{out}$ as $\epsilon_3 \to 0^+$, and is tangent at $q_{out}$ to the vector $(0, 1, 0)$.

Figure 6. Geometry and dynamics in chart $K_3$. 

See Figure 6 for geometric illustrations. To analyze the flow near \( q_{\text{out}} \), let \( \beta_3 > 0 \) be small. We define

\[
\Sigma_{3}^{\text{in}} = \{(r_3, y_3, \epsilon_3) | r_3 \in [0, \rho], y_3 \in [-\beta_3, \beta_3], \epsilon_3 = \delta \},
\]

\[
\Sigma_{3}^{\text{out}} = \{(r_3, y_3, \epsilon_3) | r_3 = \rho, y_3 \in [-\beta_3, \beta_3], \epsilon_3 \in [0, \delta] \},
\]

and let \( \pi_3 \) be the transition map from \( \Sigma_{3}^{\text{in}} \) to \( \Sigma_{3}^{\text{out}} \) defined by the flow of system (2.16).

We rescale system (2.16) dividing by \( \tilde{F}_3(\theta^t \omega, r_3, y_3, \epsilon_3) \):

\[
\begin{align*}
    r_3' &= r_3, \\
    y_3' &= -2y_3 - \frac{\epsilon_3}{1-y_3} + \epsilon_3 r_3 \tilde{G}_3(\theta^t \omega, r_3, y_3, \epsilon_3), \\
    \epsilon_3' &= -3\epsilon_3,
\end{align*}
\]

(2.17)

where for \( \omega \in \Omega \), the \( C^1 \) norm of \( \tilde{G} \) in \((r_3, y_3, \epsilon_3)\) is uniformly bounded. This system cannot be \( C^1 \) linearized because of resonance. Consider instead the subsystem on \( \{r_3 = 0\} \):

\[
\begin{align*}
    y_3' &= -2y_3 - \frac{\epsilon_3}{1-y_3}, \\
    \epsilon_3' &= -3\epsilon_3,
\end{align*}
\]

(2.18)

which can be \( C^1 \) linearized by the transformation:

\[
\tilde{y}_3 = \tilde{\psi}(y_3, \epsilon_3) = y_3 + O(\epsilon_3(\epsilon_3 + y_3))
\]

to

\[
\begin{align*}
    \tilde{y}_3' &= -2\tilde{y}_3 - \epsilon_3, \\
    \epsilon_3' &= -3\epsilon_3.
\end{align*}
\]

(2.19)

Under this transform, system (2.17) is transformed to:

\[
\begin{align*}
    r_3' &= r_3, \\
    \tilde{y}_3' &= -2\tilde{y}_3 - \epsilon_3 + \epsilon_3 r_3 \tilde{G}_3(\theta^t \omega, r_3, \tilde{y}_3, \epsilon_3), \\
    \epsilon_3' &= -3\epsilon_3.
\end{align*}
\]

(2.20)

A corresponding inverse transformation is denoted by

\[
y_3 = \psi(\tilde{y}_3, \epsilon_3) = \tilde{y}_3 + O(\epsilon_3(\tilde{y}_3 + \epsilon_3)).
\]

**Proposition 2.5.** The transition map \( \pi_3 \) has the form:

\[
\pi_3(r_3, y_3, \delta) = \left( \begin{array}{c}
\pi_{31} \\
\pi_{32} \\
\pi_{33}
\end{array} \right) = \frac{\rho}{(\psi(y_3, \delta) - \delta)(\frac{\rho}{\rho})^2 + O(r_3^3 \log r_3)}
\]

(2.21)

\[
\frac{(\frac{\rho}{\rho})^3 \delta}{\left( \begin{array}{c}
\psi(y_3, \delta) - \delta
\end{array} \right)}.
\]

**Proof.** Consider system (2.20). The \( r_3 \) and \( \epsilon_3 \) equations imply

\[
r_3(t) = r_3(0)e^t, \quad \epsilon_3(t) = \epsilon_3(0)e^{-3t}.
\]

Let \((r_3, y_3, \delta)\) be a point on \( \Sigma_{3}^{\text{in}} \), and \( T \) be the transition time under system (2.20). From the above,

\[
\rho = r_3 e^T,
\]
which implies
\[ T = \log \frac{\rho}{r_3}. \] (2.21)

Then
\[ \pi_{33} = \delta e^{-3T} = \delta \left( \frac{r_3}{\rho} \right)^3. \]

The equation of \( \dot{y}_3 \) implies:
\[ \dot{y}_3 = -2\dot{y}_3 - \delta e^{-3t} + r_3 e^t \delta e^{-3t} O(1). \]

Hence,
\[
\dot{y}_3(T) = e^{-2T} \dot{y}_3(0) + \int_0^T e^{2(t-T)} (-\delta e^{-3t} + r_3 \delta e^{-2t} O(1)) \, dt \\
= e^{-2T} \left( \tilde{\psi}(y_3, \delta) - \delta \right) + \delta e^{-3T} + r_3 \delta O (e^{-2T}T) \\
= \left( \frac{r_3}{\rho} \right)^2 \left( \tilde{\psi}(y_3, \delta) - \delta \right) + \delta \left( \frac{r_3}{\rho} \right)^3 + r_3 \delta O \left( \frac{r_3}{\rho} \right)^2 \log r_3 \\
= \left( \tilde{\psi}(y_3, \delta) - \delta \right) \left( \frac{r_3}{\rho} \right)^2 + O \left( r_3^3 \log r_3 \right),
\]

\[
\pi_{32} = \psi(\dot{y}_3(T), \epsilon_3) = \dot{y}_3(T) + O \left( \frac{r_3}{\rho} \right)^3 \delta = \left( \tilde{\psi}(y_3, \delta) - \delta \right) \left( \frac{r_3}{\rho} \right)^2 + O \left( r_3^3 \log r_3 \right).
\]

With the understanding of dynamics in charts \( K_1, K_2, \) and \( K_3, \) we are able to make the following two graphs:

Figure 7. Dynamics on \( S^2. \)
Figure 7 sketches the dynamics on $S^2$ for the blown-up vector field $\bar{X}$, and Figure 8 exhibits a typical path $S_{a,\epsilon,\theta;\omega}$.

We are ready to prove our main theorem.

2.5. **Proof of the main theorem.** Consider first the map $\Pi : \Sigma^\text{in}_1 \to \Sigma^\text{out}_3$:

$$\Pi := \pi_3 \circ \kappa_{23} \circ \pi_2 \circ \kappa_{12} \circ \pi_1.$$  

It follows from Proposition 2.4 that $\pi_1(R_1 \cap M_{a,1,\omega})$ is a smooth curve transverse to the plane $\{r_1 = 0\}$. Then $\kappa_{12}(\pi_1(R_1 \cap M_{a,1,\omega}))$ is a smooth curve transverse to the plane $\{r_2 = 0\}$. By Proposition 2.2, the image of the curve under $\pi_2$ has the form

$$\left\{(\delta^{-\frac{1}{3}}, h_{\text{out}}^3(r_2, \omega), r_2) \mid r_2 \in [0, \rho \delta^{\frac{1}{3}}]\right\},$$

where $h_{\text{out}}^3(r_2, \omega)$ is a smooth function. Under the transformation $\kappa_{23}$, this curve is then transformed to a smooth curve of the form

$$\left\{(r_3, h_{\text{in}}^3(r_3, \omega), \delta) \mid r_3 \in [0, \rho]\right\}$$

with $(0, h_{\text{in}}^3(0, \omega), \delta) = \kappa_{23}(\gamma_2 \cap \Sigma^\text{out}_2)$.

Proposition 2.5 then implies that $\Pi(R_1 \cap M_{a,1,\omega})$ has the form

$$\left\{(\rho, h_{\text{out}}^3(\epsilon_3, \omega), \epsilon_3) \mid \epsilon_3 \in [0, \delta]\right\},$$

where $h_{\text{out}}^3(\epsilon_3, \omega) = O\left(\epsilon_3^{2/3}\right)$.

We can show that $h_{\text{out}}^3$ has the asymptotic expansion:

$$h_{\text{out}}^3(\epsilon_3, \omega) = -\Omega_0 \epsilon_3^2 + O(\epsilon_3 \log \epsilon_3).$$ (2.22)

Let $y_3^*$ be the $y_3$ coordinate where $\gamma_3$ intersects $\Sigma^\text{in}_3$. By (2.8) and part 3 of Proposition 2.1,

$$y_3^* = \delta^{2/3} \epsilon_3^{-1/3} = -\Omega_0 \delta^{2/3} + \delta + O(\delta^{5/3})$$

for small $\delta$. 

**Figure 8. Path of $S_{a,\epsilon,\theta;\omega}$.**
On the other hand, by system (2.19),
\[ \tilde{y}_3^*(\delta) = \delta + a\delta^{2/3} \]
for some constant \(a\). But
\[ \tilde{y}_3^*(\delta) = \tilde{\psi}(y^*_3, \delta) = y^*_3 + O(y^*_3(y^*_3 + \delta)) = \delta - \Omega_0\delta^{2/3} + O(\delta^{5/3}). \]
It follows that \(a = -\Omega_0\), and \(\tilde{y}_3^*(\delta) = \delta - \Omega_0\delta^{2/3}\). Then (2.22) follows from Proposition 2.5.

Part 1 of Theorem 3 follows from (2.5) and (2.22).

Now we prove part 2. From Proposition 2.4, \(\pi_1(R_1)\) is a wedge-like region around \(\pi_1(R_1 \cap M_{a,1,\omega})\) of width \(O(e^{-c/\epsilon})\) for some constant \(c > 0\). Since \(\kappa_{12}, \kappa_2\) and \(\kappa_{23}\) are diffeomorphisms restricted to correspondingly small regions, one has \(\kappa_{23} \circ \pi_2 \circ \kappa_{12} \circ \pi_1(R_1)\) is also a wedge-like region of width \(O(e^{-c/\epsilon})\) around \(\Pi_1(R_1 \cap M_{a,1,\omega})\). Then Proposition 2.5 implies that \(\Pi(R_1)\) is a wedge-like region of width \(O(e^{-c/\epsilon})\) around \(\Pi(R_1 \cap M_{a,1,\omega})\). For fixed \(\epsilon, \rho\), because \(\epsilon = \epsilon_1 r_1^3 = \epsilon_3 r_3^3\) is a constant of motion, line \(\epsilon_1 = \text{constant} \) in \(\Sigma^m_1\), where \(r_1 = \rho\), is mapped to line \(\epsilon_3 = \text{constant} \) in \(\Sigma^m_3\), where \(r_3 = \rho\). Restricted to these lines, the map \(\Pi\) is a contraction with rate \(O(e^{-c/\epsilon})\) for some \(c > 0\). Correspondingly, the original transition map \(\pi\) is a contraction with rate \(O(e^{-c/\epsilon})\).

3. Closing remarks. In this paper, we considered the two-dimensional case and showed that when noise is of high enough order, deterministic dynamics essentially persist. For the three-dimensional case, there may be another slow direction or another fast direction. We believe that for the former case there is a similar result near a generic fold for high enough order noise. The second case could be more subtle: Even if there is a center manifold reduction for the deterministic system, the persistence of this reduction is under investigation. We leave these nontrivial questions along with the study of a noisy burst-spike model to a follow-up paper.

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E-mail address: bates@math.msu.edu
E-mail address: liji@hust.edu.cn
E-mail address: mzhang@nmt.edu