Osserman Conjecture in dimension $n \neq 8, 16$

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Abstract

Let $M^n$ be a Riemannian manifold and $R$ its curvature tensor. For a point $p \in M^n$ and a unit vector $X \in T_p M^n$, the Jacobi operator is defined by $R_X = R(X, \cdot)X$. The manifold $M^n$ is called pointwise Osserman if, for every $p \in M^n$, the spectrum of the Jacobi operator does not depend of the choice of $X$, and is called globally Osserman if it depends neither of $X$, nor of $p$. Osserman conjectured that globally Osserman manifolds are two-point homogeneous. We prove the Osserman Conjecture for $n \neq 8, 16$, and its pointwise version for $n \neq 2, 4, 8, 16$. Partial result in the case $n = 16$ is also given.

1 Introduction

An algebraic curvature tensor $R$ in a Euclidean space $\mathbb{R}^n$ is a $(3, 1)$ tensor having the same symmetries as the curvature tensor of a Riemannian manifold. For $X \in \mathbb{R}^n$, the Jacobi operator $R_X : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $R_X Y = R(X, Y)X$. The Jacobi operator is symmetric and $R_X X = 0$ for all $X \in \mathbb{R}^n$. Throughout the paper, "eigenvalues of the Jacobi operator" refers to eigenvalues of the restriction of $R_X$, with $X$ a unit vector, to the subspace $X^\perp$.

Definition 1. An algebraic curvature tensor $R$ is called Osserman if the eigenvalues of the Jacobi operator $R_X$ do not depend of the choice of a unit vector $X \in \mathbb{R}^n$.

Definition 2. A Riemannian manifold $M^n$ is called pointwise Osserman if its curvature tensor is Osserman. If, in addition, the eigenvalues of the Jacobi operator are constant on $M^n$, the manifold $M^n$ is called globally Osserman.

Two-point homogeneous spaces ($\mathbb{R}^n$, $\mathbb{R}P^n$, $S^n$, $H^n$, $\mathbb{CP}^n$, $\mathbb{CH}^n$, $\mathbb{HP}^n$, $\mathbb{HH}^n$, $\mathbb{CayP^2}$, and $\mathbb{CayH^2}$) are globally Osserman, since the isometry group of each of this spaces is transitive on its unit sphere bundle. Osserman [13] conjectured that the converse is also true:

Osserman Conjecture. A globally Osserman manifold is two-point homogeneous.

For manifolds of dimension $n \neq 4k$, $k \geq 2$ the Osserman Conjecture is proved by Chi [5]. Further progress was made in [7, 9, 11, 12]. We refer to [6] for results on Osserman Conjecture in semi-Riemannian geometry. The characterization of $p$-Osserman manifolds (the averaging of the Jacobi operator over any $p$-plane has constant eigenvalues) was given by Gilkey in [8]: any $p$-Osserman Riemannian manifold with $2 \leq p \leq n - 2$ has constant sectional curvature.

Our main result is the following Theorem.
Theorem 1. A globally Osserman manifold of dimension \( n \neq 8, 16 \) is two-point homogeneous. A pointwise Osserman manifold of dimension \( n \neq 2, 4, 8, 16 \) is two-point homogeneous.

Note that in dimension two, any Riemannian manifold is pointwise Osserman, but globally Osserman manifolds are the ones having constant Gauss curvature.

In dimension four, the Osserman Conjecture is proved in [5]. However, there exist pointwise Osserman four-dimensional manifolds that are not two-point homogeneous (see [9, Corollary 2.7]).

A (pointwise or globally) Osserman manifold of dimension eight is known to be two-point homogeneous in each of the following cases: (i) the Jacobi operator has an eigenvalue of multiplicity at least 5 [9, Theorem 7.1], [11, Theorem 1.2]; (ii) the Jacobi operator has no more than two distinct eigenvalues [12, Theorem 2].

In dimensions sixteen, we have the following Theorem.

Theorem 2. A (pointwise or globally) Osserman manifold \( M^{16} \) is two-point homogeneous if the Jacobi operator has no eigenvalues of multiplicity 7, 8 and 9.

The paper is organized as follows. In Section 2, we consider algebraic curvature tensors with Clifford structure. All of them have the Osserman property, and, in the most cases, the converse is also true. This is the key statement of the paper (Proposition 1). Moreover, in the cases covered by the Theorems, the existence of the Clifford structure on a manifold implies that the manifold is two-point homogeneous (Proposition 2). Further in Section 2, we give the proof of the both Theorems assuming Proposition 1. Section 3 contains the proof of Proposition 1 modulo Propositions 3, 4 which are proved in Sections 4 and 5, respectively.

2 Manifolds with Clifford structure. Proof of the Theorems

Osserman algebraic curvature tensors with Clifford structure were introduced by Gilkey [7], Gilkey, Swann, Vanhecke [9]:

Definition 3. An algebraic curvature tensor \( R \) in \( \mathbb{R}^n \) has a Cliff(\( \nu \))-structure if

\[
R(X, Y)Z = \lambda_0(\langle X, Z \rangle Y - \langle Y, Z \rangle X) \\
+ \sum_{s=1}^{\nu} \frac{1}{2} (\mu_s - \lambda_0)(2\langle J_s X, Y \rangle J_s Z + \langle J_s Z, Y \rangle J_s X - \langle J_s Z, X \rangle J_s Y),
\]

(1)

where \( J_1, \ldots, J_\nu \) are skew-symmetric orthogonal operators satisfying the Hurwitz relations \( J_s J_q + J_q J_s = -2\delta_{qs} I_n \) and \( \mu_s \neq \lambda_0 \).

A Riemannian manifold \( M^n \) has a Cliff(\( \nu \))-structure if its curvature tensor does.

For skew-symmetric operators \( J_1, \ldots, J_\nu \) the Hurwitz relations are equivalent to the fact that \( \langle J_s X, J_q X \rangle = \delta_{sq} \| X \|^2 \) for all \( X \in \mathbb{R}^n \). Note that some of the \( \mu_s \)'s in (1) can be equal.
The Jacobi operator of the algebraic curvature tensor $R$ with the Clifford structure given by (1) has the form

$$R_X Y = \lambda_0 (\|X\|^2 Y - \langle Y, X \rangle X) + \sum_{s=1}^{\nu} (\mu_s - \lambda_0) \langle J_s X, Y \rangle J_s X,$$

and the tensor $R$ can be reconstructed from (2) using polarization and the first Bianchi identity.

For any unit vector $X$, the Jacobi operator $R_X$ given by (2) has constant eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_p$, where $\lambda_1, \ldots, \lambda_p$ are the $\mu_s$'s without repetitions. The eigenspace corresponding to the eigenvalue $\lambda_\alpha$, $\alpha \neq 0$ is $E_{\lambda_\alpha}(X) = \text{Span}(J_{\mu_1} X, \ldots, J_{\nu} X)$, and the $\lambda_0$-eigenspace is $E_{\lambda_0}(X) = (\text{Span}(X, J_1 X, \ldots, J_\nu X))^\perp$, provided $\nu < n - 1$. Hence a Cliff($\nu$) algebraic curvature tensor (manifold) is Osserman (pointwise Osserman, respectively).

We will show that, at the most cases, the converse is also true. Note however, that there exists at least one Osserman algebraic curvature tensor having no Clifford structure, namely the curvature tensor of the Cayley projective plane and, up to a sign, of its hyperbolic dual (see the Remark at the end of Section 3).

For a unit vector $X$, the vectors $J_1 X, \ldots, J_\nu X$ are linearly independent (even orthonormal) and are tangent to the unit sphere $S^{n-1} \subset \mathbb{R}^n$ at $X$. From the Adams Theorem [1] it follows that $\nu \leq \rho(n) - 1$, where $\rho(n)$ is the Radon number, defined as follows: for $n = 2^a + 3^b c$ with $c$ odd integer and $0 \leq b \leq 3$, $\rho(n) = 2^b + 8a$. Moreover, for every $\nu \leq \rho(n) - 1$, there exist the operators $J_1, \ldots, J_\nu$ with the required properties [3, 10], and so there exist algebraic curvature tensors in $\mathbb{R}^n$ having a Cliff($\nu$)-structure.

In [9] the following approach to the Osserman Conjecture was suggested:

(i) show that Osserman algebraic curvature tensors have Clifford structure;

(ii) classify Riemannian manifolds having curvature tensor as in (i).

Following this scheme, we derive Theorem 1 and Theorem 2 from two Propositions below. We show that by topological reasons, the Jacobi operator of an Osserman algebraic curvature tensor must always have an eigenvalue of multiplicity $m \geq n - \rho(n)$. Denote the sum of multiplicities of all the other eigenvalues by $\nu = n - 1 - m \leq \rho(n) - 1$ (we use the same notation $\nu$ as in Definition 3 aiming to find a Cliff($\nu$)-structure for $R$). Our proof works when the number $\nu$ is small enough compared to $n$. When $n \neq 8, 16$ this is guaranteed by the fact that $\rho(n)$ is small, but for $n = 16$ we need to impose extra conditions on the spectrum of the Jacobi operator, as in Theorem 2.

**Proposition 1.** Let $R$ be an Osserman algebraic curvature tensor in $\mathbb{R}^n$. Let $m$ be the maximal multiplicity of the eigenvalues of its Jacobi operator and $\nu = n - 1 - m$. If

$$n \geq 3\nu \quad \text{and} \quad n > \frac{(\nu + 1)^2}{4},$$

then $R$ has a Cliff($\nu$)-structure.

**Proposition 2 ([11], Theorem 1.2).** A Riemannian manifold $M^n$ with a Cliff($\nu$)-structure is two-point homogeneous, provided that

(a) $n \neq 2, 4, 8, 16$, or
(b) \( n = 8, \nu < 3, \) or
(c) \( n = 16, \nu \neq 8. \)

**Proof of Theorem 1 and Theorem 2.** Both Theorems follow from Propositions 1 and 2 directly, if we can show that the number \( \nu \) defined in Proposition 1 satisfies the inequalities \( n \geq 3\nu, \) \( n > \frac{(\nu+1)^2}{4}. \)

These inequalities follow from the fact that \( \nu \leq \rho(n) - 1. \) Indeed, as the formula for \( \rho(n) \) shows, for all \( n \neq 2, 4, 8, 16, \) \( n \geq 3(\rho(n) - 1), \) \( n > \frac{\rho(n)^2}{4}. \) For \( n = 16, \) we get \( \nu \leq 8. \) The hypothesis of Theorem 2 then implies that \( \nu \leq 5, \) and so \( 3\nu \leq 16, \) \( \frac{(\nu+1)^2}{4} < 16. \)

Hence it remains to show that \( \nu \leq \rho(n) - 1. \) Let \( M^n \) be a pointwise Osserman manifold. Locally, in a neighbourhood of a generic point \( x \in M^n, \) the Jacobi operator has a constant number of eigenvalues, with constant multiplicities. Let the Jacobi operator have \( p + 1 \) distinct eigenvalues, with multiplicities \( m_0, m_1, \ldots, m_p, \) respectively, \( m_0 + m_1 + \cdots + m_p = n - 1. \) Let \( m = m_0 \) be the maximal multiplicity and \( \nu = n - 1 - m. \)

For a vector unit \( X \in T_x M^n, \) the eigenspaces of the Jacobi operator are mutually orthogonal subspaces of \( TX S^{n-1}, \) of dimension \( m_0, m_1, \ldots, m_p. \) This gives \( p + 1 \) continuous plane fields in the tangent bundle \( TS^{n-1} \) of the unit sphere \( S^{n-1} \subset T_x M^n. \)

We follow the arguments of \([15, \text{p. 216}].\) Let \( f : S^{n-2} \to \text{SO}(n-1) \) be the clutching map for the tangent bundle \( TS^{n-1}, \) and \([f] \in \pi_{n-2}\text{SO}(n-1) \) its homotopy class. The bundle \( TS^{n-1} \) admits \( p + 1 \) continuous orthogonal plane fields of dimension \( m_0, m_1, \ldots, m_p, \) iff \([f] \) lies in the subgroup \( \pi_{n-2}(\text{SO}(m_0) \times \text{SO}(m_1) \times \cdots \times \text{SO}(m_p)) \) of \( \pi_{n-2}\text{SO}(n-1) \) defined by the inclusion map \( i : \text{SO}(m_0) \times \text{SO}(m_1) \times \cdots \times \text{SO}(m_p) \to \text{SO}(n-1). \) Similarly, the bundle \( TS^{n-1} \) admits \( \nu = n - m_0 - 1 \) continuous orthonormal vector fields, iff \([f] \) lies in the subgroup \( \pi_{n-2}\text{SO}(m_0) \) of \( \pi_{n-2}\text{SO}(n-1) \) defined by the inclusion map \( i' : \text{SO}(m_0) \to \text{SO}(n-1). \)

But the image of \( i_* \) lies in the image of \( i'_* \) since \( \text{SO}(m_0) \) is the largest of the \( \text{SO}(m_\alpha) \), and so every \( \text{SO}(m_\alpha) \) can be homotoped in \( \text{SO}(n-1) \) to lie inside \( \text{SO}(m_0). \)

Hence there exist \( \nu \) vector fields on \( S^{n-1}, \) and the Adams Theorem \([1]\) gives \( \nu \leq \rho(n) - 1. \) \( \square \)

The remaining part of the paper is devoted to the proof of Proposition 1.

**3 Proof of Proposition 1**

Let \( \tilde{R} \) be an Osserman algebraic curvature tensor in \( \mathbb{R}^n \) such that the corresponding Jacobi operator has \( p + 1 \) distinct eigenvalues \( \lambda_0, \lambda_1, \ldots, \lambda_p \) with multiplicities \( m_0, m_1, \ldots, m_p, \) respectively, \( m_0 + m_1 + \cdots + m_p = n - 1. \) Let \( m_0 \) be the maximal multiplicity and \( \nu = n - 1 - m_0, \) the sum of all the other multiplicities.

Consider an algebraic curvature tensor \( R = \tilde{R} - \tilde{\lambda}_0 R^1, \) where \( R^1 \) is the curvature tensor of the unit sphere. Then \( R \) is still Osserman, with the Jacobi operator having eigenvalues \( \lambda_\alpha = \tilde{\lambda}_\alpha - \tilde{\lambda}_0 \) with multiplicities \( m_\alpha \) for \( \alpha = 1, \ldots, p, \) respectively, and the eigenvalue 0 with multiplicity \( m_0. \) To prove Proposition 1 it is sufficient to show that \( R \) has a \( \text{Cliff}(\nu) \)-structure.
Let $\mu_1, \ldots, \mu_\nu$ be the $\lambda_\alpha$’s counting the multiplicities, that is, $\mu_1 = \cdots = \mu_m = \lambda_1, \mu_{m+1} = \cdots = \mu_{m+m_2} = \lambda_2, \ldots, \mu_m = \lambda_\nu$. In a Euclidean space $\mathbb{R}^\nu$, choose an orthonormal basis $e_1, \ldots, e_\nu$ and define the operator $\Lambda : \mathbb{R}^\nu \to \mathbb{R}^\nu$ by $\Lambda e_s = \mu_s e_s$, $s = 1, \ldots, \nu$. The matrix of $\Lambda$ is then given by $\Lambda = \diag(\mu_1, \ldots, \mu_\nu)$.

Proposition 1 follows from the two Propositions below.

**Proposition 3.** Let $R$ be an Osserman algebraic curvature tensor in $\mathbb{R}^n$. Let $0$ be the eigenvalue of its Jacobi operator with the maximal multiplicity $m$, and $\nu = n - 1 - m$.

Assume that $n \geq 3\nu$. Then there exists a linear map $M : \mathbb{R}^n \to \text{Hom}(\mathbb{R}^\nu, \mathbb{R}^n)$, $X \mapsto M_X$ such that the Jacobi operator admits the following linear decomposition:

$$R_X = M_X \Lambda M_X^t.$$  \hfill (3)

The map $X \mapsto M_X$ is determined uniquely up to a precomposition $X \mapsto M_X N$ with an element $N$ from the group $O_\Lambda = \{ N : N \Lambda N^t = \Lambda \}$.

**Proposition 4.** Let $R$ be an Osserman algebraic curvature tensor in $\mathbb{R}^n$ with the Jacobi operator having the form (3). Assume that $n > \frac{(\nu+1)^2}{4}$. Then $R$ has a Cliff($\nu$)-structure.

Proposition 3 is proved in Section 4. Using the Osserman property we successively show that for $k = 1, 2, 3, \ldots, n$ the following holds: for almost any set of $k$ orthonormal vectors $E_1, \ldots, E_k$ in $\mathbb{R}^n$, there exist linear operators $M_1, \ldots, M_k : \mathbb{R}^\nu \to \mathbb{R}^n$ such that $R_{x_1E_1+\cdots+x_kE_k} = (x_1M_1 + \cdots + x_kM_k) \Lambda (x_1M_1 + \cdots + x_kM_k)^t$ for all $x_1, \ldots, x_k$. Then for a vector $X = x_1E_1 + \cdots + x_nE_n$ we define $M_X = x_1M_1 + \cdots + x_nM_n$.

Proposition 4 is proved in Section 5. We show that, with an appropriate choice of the basis $e_1, \ldots, e_\nu$ in $\mathbb{R}^\nu$, the operators $J_s$ in $\mathbb{R}^n$ defined by $J_sX = M_X e_s$ give the Clifford structure for $R$.

**Remark.** The claim of Proposition 1 fails to be true at least in the case when $n = 16, \nu = 7$, since the curvature tensor of the Cayley projective plane $\text{Cay}P^2$ (and of its hyperbolic dual $\text{Cay}H^2$) has no Clifford structure.

This follows from the fact that, unlike the holonomy groups of $\mathbb{C}P^n$ and $\mathbb{H}P^n$, the holonomy group $\text{Spin}(9)$ of the Cayley projective plane has no proper normal subgroups [2, 4]. The nonexistence of the Clifford structure is also confirmed by the following octonionic computation based on the formula for the curvature tensor of $\text{Cay}P^2$ [4, Theorem 6.1].

Identify a tangent space to $\text{Cay}P^2$ with $\text{Cay} \oplus \text{Cay}$. Then for orthogonal vectors $X = (a, b)$, $Y = (c, d)$ the Jacobi operator has the form

$$R_XY = \frac{\alpha}{4}((4\|a\|^2 + \|b\|^2)c + 3(ab)d^*, (4\|b\|^2 + \|a\|^2)d + 3c^*(ab)),$$

where $*$ is the octonion conjugation and $\|a\|^2 = aa^*$, $\langle a, b \rangle = \frac{1}{2}(ab^* + ba^*)$.

It follows that for any unit vector $X$ the Jacobi operator $R_X$ has two eigenvalues: $\alpha$, of multiplicity 7, with the eigenspace

$$E_\alpha(X) = \{(c, d) : ad = cb, \langle a, c \rangle = \langle b, d \rangle = 0 \},$$

and $\frac{\alpha}{4}$, of multiplicity 8, with the eigenspace

$$E_{\frac{\alpha}{4}}(X) = \{(c, d) : a\|b\|^2d - \langle b, d \rangle b = (\|a\|^2c - \langle a, c \rangle a)b \}.$$
As it follows from (2), the existence of a Clifford structure would imply the existence of seven (respectively, eight) linear operators $J_i$ such that $E_8(X) = \text{Span}(I_1X, \ldots, I_6X)$ (respectively, $E_8(X) = \text{Span}(I_1X, \ldots, I_7X)$)

However, it is not difficult to see that there is no nonzero $\mathbb{R}$-linear operator $J : \text{Cay} \oplus \text{Cay} \rightarrow \text{Cay} \oplus \text{Cay}$ such that for all $X$, $JX \in E_8(X)$. With some calculation, one can show that the same is true for $E_8(X)$, as well.

Thus the curvature tensor of $\text{Cay}P^2$ admits neither Cliff(7)-, nor Cliff(8)-structure, hence no Clifford structure at all.

4 Proof of Proposition 3

The proof goes by the following plan. First, in Lemma 1, we show that for any vector $X$, there exists an operator $M_X$ satisfying (3). Next, in Lemma 3, we prove that for almost any two vectors $X, Y$, the operators $M_X$ and $M_Y$ can be chosen accordingly, that is, in such a way that $R_{X+Y} = (M_X x + M_Y y) \Lambda (M_X x + M_Y y)^t$ for all real $x, y$. In Lemma 4 we extend this result to the case of three vectors. Then we show that the existence of the linear decomposition of the form (3) for almost any three vectors already implies the existence of the global decomposition.

Let $R$ be an Osserman algebraic curvature tensor in $\mathbb{R}^n$, with the Jacobi operator having $p$ distinct nonzero eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$, of multiplicities $m_1, m_2, \ldots, m_p$, respectively, and the eigenvalue 0 of multiplicity $n - 1 - \nu$.

For a nonzero vector $X$, the subspaces Ker$R_X$ and Im$R_X$ are orthogonal and have dimension $n - \nu$ and $\nu$, respectively. By the Rakič duality principle [14], for two orthonormal vectors $X, Y$, the following holds: $Y$ is an eigenvector of $R_X$, if and only if $X$ is an eigenvector of $R_Y$ (with the same eigenvalue). For the eigenvalue 0, we get

$$Y \in \text{Ker}R_X \iff X \in \text{Ker}R_Y.$$  \hspace{1cm} (4)

We will use a slight modification of the duality principle, noting that for the eigenvalue 0 the assumption of orthonormality of $X$ and $Y$ can be dropped. Indeed, let $\psi \neq 0, \frac{\pi}{2}, \pi$ be the angle between unit vectors $X, Y$, and let $Z$ be a unit vector in $\text{Span}(X, Y)$ orthogonal to $X$ and such that $Y = \cos \psi X + \sin \psi Z$. Assume that $Y \in \text{Ker}R_X$. Since $X \in \text{Ker}R_X$, we have $Z \in \text{Ker}R_X$, and so $X \in \text{Ker}R_Z$. Then $R_y X = R_{\cos \psi X + \sin \psi Z} X = \cos \psi \sin \psi R(Z, X) X + \sin^2 \psi R_Z X = 0$, that is, $X \in \text{Ker}R_Y$.

We first show that the decomposition claimed in the Proposition exists for every single operator $R_X$.

**Lemma 1.** For any unit vector $X$, there exists a linear operator $M_X : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$R_X = M_X \Lambda M_X^t.$$  

The operator $M_X$ is determined uniquely up to a precomposition with an element $N \in O_{\Lambda} = \{ N \in O_n : N \Lambda N^t = \Lambda \}$. Moreover, $\text{Im}R_X = \text{Im}M_X$ and $M_X^t M_X = N^t N$ for some $N \in O_{\Lambda}$. 

**Proof.** Let $E_1, \ldots, E_p$ be an orthonormal basis of eigenvectors of $R_X$ with nonzero eigenvalues. For a vector $y = (y_1, \ldots, y_p) \in \mathbb{R}^p$ define $M_X y = y_1 E_1 + \ldots + y_p E_p$.
By the duality principle, for any \( \nu \) and let \( Y \) be the unit sphere in the subspace \( X^\perp \). We show that for an open dense set of vectors \( n \) have the maximal rank \( Y \). If \( M_X : \mathbb{R}^n \to \mathbb{R}^n \) is another operator such that \( R_X = M_X \), then \( \hat{M}_X \) is nonsingular since \( \text{rk} \hat{M}_X = \text{rk} R_X = \nu \). Therefore \( M_X = M_X N \), for some operator \( N \) in \( \mathbb{R}^n \) and \( M_X (A - \Lambda \Lambda^\top) \hat{M}_X = 0 \). Since \( \text{rk} M_X = \nu \), \( N \in O_\Lambda \).

Next we need the following generic position Lemma. Denote \( V_k(\mathbb{R}^n) \) the Stiefel manifold of \( k \)-tuples of orthonormal vectors in \( \mathbb{R}^n \).

**Lemma 2.** 1. Let \( n \geq 2\nu \). Then the set \( S_2 = \{(X, Y) \in V_2(\mathbb{R}^n) : \text{Im} R_X \cap \text{Im} R_Y = 0\} \) is open and dense in \( V_2(\mathbb{R}^n) \).

2. Let \( n \geq 3\nu \). Then the set \( S_3 = \{(X, Y, Z) \in V_3(\mathbb{R}^n) : \dim(\text{Im} R_X + \text{Im} R_Y + \text{Im} R_Z) = 3\nu \} \) is open and dense in \( V_3(\mathbb{R}^n) \).

3. If \((X, Y, Z) \in S_3 \), then for any unit vector \( U \in \text{Span}(Y, Z) \) the pair \((X, U)\) is in \( S_2 \).

**Proof.** The proof of 1. and 2. is quite similar and is based on the dimension count. Both \( S_2 \) and \( S_3 \) are open. We claim that they are also dense.

1. Let \( X \) be a unit vector in \( \mathbb{R}^n \), and \( S^{n-2} \) be the unit sphere in the subspace \( X^\perp \). We want to show that for an open dense set of vectors \( Y \in S^{n-2} \), \( \dim(\text{Ker} R_X \cap \text{Ker} R_Y) = n - 2\nu \).

Let \( S \) be the unit sphere in the subspace \( \text{Ker} R_X \), \( \dim S = n - \nu - 1 \), and let \( E \) be a vector bundle with the base \( S \) and the fiber \( F_Z = \text{Ker} R_Z \cap X^\perp \) over a point \( Z \in S \) (\( \dim F_Z = n - \nu - 1 \) since \( X \in \text{Ker} R_Z \) by the duality principle (4)). Then \( SE \), the corresponding unit sphere bundle, is a compact analytic manifold of dimension \( 2n - 2\nu - 3 \).

Define the projection map \( \pi : SE \to S^{n-2} \) by \( \pi(Z, Y) = Y \). By the duality principle, for any \( Y \in S^{n-2} \), \( \pi^{-1}(Y) = \{(Z, Y) \in SE : Z \in \text{Ker} R_Y \} \). The map \( \pi \) is differentiable (even analytic) since the subspace \( \text{Ker} R_Z \) viewed as a point of the corresponding Grassmannian depends analytically on \( Z \).

Now if \( n = 2\nu \), then \( 2n - 2\nu - 3 < n - 2 \), and so for every \( Y \) from the open dense subset \( S^{n-2} \setminus \pi(\text{Ker} R_Y) \) of \( S^{n-2} \), \( \text{Ker} R_Y \cap \text{Ker} R_X = 0 \).

If \( n > 2\nu \), then \( \text{Ker} R_Y \cap \text{Ker} R_X \neq 0 \) for any \( Y \), hence the map \( \pi \) is surjective. By the Sard Theorem, for an open dense set of the \( Y \)’s in \( S^{n-2} \), \( d\pi \) has the maximal rank \( n - 2 \) at all the points of \( \pi^{-1}(Y) \). For such points \( \pi^{-1}(Y) = (S^{n-2\nu-1}_Y, Y) \), where \( S^{n-2\nu-1} \) is the unit sphere in \( \text{Ker} R_X \cap \text{Ker} R_Y \). So \( \dim(\text{Ker} R_X \cap \text{Ker} R_Y) = n - 2\nu \).

2. Let \((X, Y) \in S_2 \), and let \( S^{n-3} \) be the unit sphere in the subspace \( X^\perp \cap Y^\perp \). We show that for an open dense set of vectors \( Z \in S^{n-3} \), \( \dim(\text{Ker} R_X \cap \text{Ker} R_Y \cap \text{Ker} R_Z) = n - 3\nu \).

Let \( S \) be the unit sphere in the subspace \( \text{Ker} R_X \cap \text{Ker} R_Y \), \( \dim S = n - 2\nu - 1 \), and let \( E \) be a vector bundle with the base \( S \) and the fiber \( F_U = \text{Ker} R_U \cap (X^\perp \cap Y^\perp) \) over a point \( U \in S \) (\( \dim F_U = n - \nu - 2 \) since \( X, Y \in \text{Ker} R_U \)). Then \( SE \), the corresponding unit sphere bundle, is a compact analytic manifold of dimension \( 2n - 3\nu - 4 \).

The projection map \( \pi : SE \to S^{n-3} \) defined by \( \pi(U, Z) = Z \) is also analytic. By the duality principle, for any \( Z \in S^{n-3} \), \( \pi^{-1}(Z) = \{(U, Z) \in SE : U \in \text{Ker} R_Z \} \).
If \( n = 3\nu \), then the image of \( \pi \) does not cover \( S^{n-3} \), and we can take any \( Z \) from its complement.

Otherwise, \( \pi \) is surjective. Applying the Sard Theorem, we find an open dense set of points \( Z \in S^{n-3} \) such that \( \pi^{-1}(Z) = (S_{Z}^{n-3\nu-1}, Z) \), where \( S_{Z}^{n-3\nu-1} \) is the unit sphere in \( \text{Ker}R_{X} \cap \text{Ker}R_{Y} \cap \text{Ker}R_{Z} \). So \( \text{dim}(\text{Ker}R_{X} \cap \text{Ker}R_{Y} \cap \text{Ker}R_{Z}) = n - 3\nu \).

3. By the duality principle (4), \( \text{Ker}R_{U} \supset (\text{Ker}R_{Y} \cap \text{Ker}R_{Z}) \), and so \( \text{Im}R_{U} \supset (\text{Im}R_{Y} \cap \text{Im}R_{Z}) \). Then \( \text{Im}R_{U} \cap \text{Im}R_{X} \subset (\text{Im}R_{Y} + \text{Im}R_{Z}) \cap \text{Im}R_{X} = 0 \), since \( (X, Y, Z) \in S_{3} \).

We now show that the operator \( R_{X} \) admits the linear decomposition (3) on almost every two-plane in \( \mathbb{R}^{n} \). Define the symmetric operator \( R_{XY} \) by \( R_{XY}Z = \frac{1}{2}(R(X, Z)Y + R(Y, Z)X) \).

**Lemma 3.** Suppose that \( n \geq 2\nu \). Then for any pair of orthonormal vectors \( (X, Y) \in S_{2} \), there exist linear operators \( M_{1}, M_{2} : \mathbb{R}^{\nu} \to \mathbb{R}^{n} \) such that for all \( x, y \in \mathbb{R} \),

\[
R_{X+Y} = (M_{1}x + M_{2}y) \Lambda (M_{1}x + M_{2}y)^{t}.
\]

Moreover, the operators \( M_{1}, M_{2} \) are determined uniquely up to a precomposition \( M_{1}N, M_{2}N \) with an element \( N \in O_{\Lambda} \).

The uniqueness part can be rephrased as follows: once \( M_{1} \) with the property \( M_{1}\Lambda M_{1}^{t} = R_{X} \) is chosen, then there exists a unique \( M_{2} \) such that the pair \( M_{1}, M_{2} \) satisfies the equation of Lemma 3.

**Proof.** We have \( R_{X+Y} = R_{X}x^{2} + Cxy + R_{Y}y^{2} \), with a symmetric operator \( C = 2R_{XY} \). The claim is equivalent to the fact that

\[
R_{X} = M_{1}\Lambda M_{1}^{t}, \quad R_{Y} = M_{2}\Lambda M_{2}^{t}, \quad C = M_{1}\Lambda M_{1}^{t} + M_{2}\Lambda M_{2}^{t}.
\]

By Lemma 1 we can find two operators, \( M_{1}, M_{2} : \mathbb{R}^{\nu} \to \mathbb{R}^{n} \), such that \( R_{X} = M_{1}\Lambda M_{1}^{t} \), \( R_{Y} = M_{2}\Lambda M_{2}^{t} \), and so our goal is to show that they can be chosen in such a way that \( C = M_{1}\Lambda M_{1}^{t} + M_{2}\Lambda M_{2}^{t} \).

By the Osserman property, the operator \( R_{\cos \phi X + \sin \phi Y} \) is isospectral, for all \( \phi \in \mathbb{R} \). Its eigenspaces (viewed as the curves in the corresponding Grassmannians) are analytic with respect to \( \phi \). Locally, in a neighbourhood of the point \( \phi = 0 \), there exists an analytic orthogonal transformation \( U(\phi) \) such that \( U(0) = I_{n} \) and \( R_{\cos \phi X + \sin \phi Y} = U(\phi)RXU(\phi)^{t} \). Let \( U(\phi) = I_{n} + K\phi + \left(\frac{1}{2}K^{2} + K_{1}\right)\phi^{2} + o(\phi^{3}) \) be the Taylor expansion at \( \phi = 0 \), with \( K \) and \( K_{1} \) skew-symmetric operators.

Then we have

\[
C = [K, RX], \quad Ry = RX + \frac{1}{2}[K, C] + [K_{1}, RX],
\]

and so \( C = \tilde{M}_{2}\Lambda M_{2}^{t} + M_{1}\tilde{M}_{2}^{t} \), with \( \tilde{M}_{2} = KM_{1} \).

Similar arguments applied at the point \( \phi = \pi/2 \) show that \( C = \tilde{M}_{1}\Lambda M_{1}^{t} + M_{2}\tilde{M}_{1}^{t} \) for some operator \( \tilde{M}_{1} : \mathbb{R}^{\nu} \to \mathbb{R}^{n} \). Equating the expressions for \( C \) we get

\[
\tilde{M}_{1}\Lambda M_{1}^{t} + M_{2}\Lambda M_{2}^{t} = \tilde{M}_{2}\Lambda M_{2}^{t} + M_{1}\tilde{M}_{2}^{t}. \tag{6}
\]

Let \( Z \in \text{Ker}R_{X} \cap \text{Ker}R_{Y} \). Then \( M_{1}^{t}Z = M_{2}^{t}Z = 0 \). Acting on the vector \( Z \) by the both sides of (6) we get \( M_{1}(\Lambda M_{1}^{t}Z) = M_{2}(\Lambda M_{2}^{t}Z) \). The pair \((X, Y)\) was chosen in \( S_{2} \), so the subspaces \( \text{Im}M_{1} = \text{Im}R_{X} \) and \( \text{Im}M_{2} = \text{Im}R_{Y} \) have zero
intersection in $\mathbb{R}^n$. It follows that $\tilde{M}_1^2 Z = \tilde{M}_2^2 Z = 0$. Hence $\text{Im} \tilde{M}_1, \text{Im} \tilde{M}_2 \subset \text{Im} M_1 \oplus \text{Im} M_2$. In other words, there exist linear operators $S_1, S_2, S_3, S_4 : \mathbb{R}^\nu \to \mathbb{R}^\nu$ such that

$$\tilde{M}_1 = M_1 S_1 + M_2 S_3, \quad \tilde{M}_2 = M_1 S_2 + M_2 S_4.$$ Substituting this back to (6) we get

$$M_2(S_3 \Lambda + \Lambda S_3^t)M_2^t + M_2(\Lambda S_4^t - S_4 \Lambda)M_1^t - M_1(S_2 \Lambda + \Lambda S_2^t)M_1^t + M_1(S_1 \Lambda - \Lambda S_1^t)M_2^t = 0.$$ Using again the fact that $\text{Im} M_1 \cap \text{Im} M_2 = 0$ we find that $S_2 \Lambda, S_3 \Lambda$ are skew-symmetric operators in $\mathbb{R}^\nu$, and $\Lambda S_4^t = S_4 \Lambda$. Then

$$C = M_2 S_4 \Lambda M_4^t + M_1 \Lambda S_3^t M_3^t.$$ Take a vector $Z \in \text{Ker} R_X$. Then $M_1^t Z = 0$, and $M_2^t KZ = -(S_3^t M_1^t + S_3^t M_1^t)Z = -S_3^t M_1^t Z$ since $M_2 = K M_1 = M_1 S_2 + M_2 S_4$. Acting on $Z$ by both sides of the second equation of (5), and then taking the inner product with $Z$, we obtain $\langle R_Y Z, Z \rangle = \frac{1}{\nu}[(K, C)Z, Z] = -\langle C K Z, Z \rangle$ since $R_X Z = 0$. Substituting $R_Y = M_2 A M_2^t$, $C = M_2 S_4 A M_4^t + M_1 \Lambda S_3^t M_3^t$ we find

$$\langle M_2 A M_2^t Z, Z \rangle = -\langle (M_2 S_4 \Lambda M_4^t + M_1 \Lambda S_3^t M_3^t) K Z, Z \rangle = -(M_2 S_4 \Lambda M_4^t K Z, Z) - \langle \Lambda S_3^t M_3^t K Z, M_1^t Z \rangle = \langle M_2 S_4 \Lambda S_3^t M_3^t Z, Z \rangle,$$

and so $\langle (S_3^t - \Lambda) M_1^t Z, (S_3^t - \Lambda) M_1^t Z \rangle = \langle M_2 S_4 \Lambda S_3^t M_3^t Z, Z \rangle = 0$ for all $Z \in \text{Ker} R_X$.

The restriction of the operator $M_2^t$ to $\text{Ker} R_X$ is epimorphic (otherwise the images of $R_X$ and $R_Y$ would have a nonzero intersection), so $M_2^t(\text{Ker} R_X) = \mathbb{R}^\nu$. Hence the symmetric operator $S_4 \Lambda S_3^t - \Lambda$ vanishes. It follows that $S_4 \in O_\Lambda$.

Now replace $M_2$ by $\tilde{M}_2 = M_2 S_4$. Then $M_2 A M_2^t$ is still $R_Y$, and $C = M_1 A M_1^t + M_2 A M_2^t$. So $M_1, M_2$ is the sought pair of operators.

To finish the proof it remains to show the uniqueness. Suppose that for operators $M_1, M_2, M_3, M_4 : \mathbb{R}^\nu \to \mathbb{R}^n$,

$$R_{x+y} = (M_1 x + M_2 y) \Lambda (M_1 x + M_2 y)^t = (M_3 x + M_4 y) \Lambda (M_3 x + M_4 y)^t.$$ By Lemma 1, there exist $N_1, N_2 \in O_\Lambda$ such that $M_3 = M_1 N_1, M_4 = M_2 N_2$. Equating the terms with $xy$ we then obtain

$$M_1 (\Lambda - N_1 \Lambda N_1^t) M_2^t + M_2 (\Lambda - N_2 \Lambda N_2^t) M_1 = 0.$$ Since $\text{Im} M_1 \cap \text{Im} M_2 = 0$, it follows that $N_1 \Lambda N_1^t = \Lambda$, hence $N_1 = N_2$. □

The next step is to show that $R_X$ admits the linear decomposition (3) on almost every three-space in $\mathbb{R}^n$.

**Lemma 4.** Suppose that $n \geq 3\nu$. Then for any triple of orthonormal vectors $(X, Y, Z) \in S_3$, there exist linear operators $M_1, M_2, M_3 : \mathbb{R}^\nu \to \mathbb{R}^n$ such that for all $x, y, z \in \mathbb{R}$,

$$R_{x+y+z} = (M_1 x + M_2 y + M_3 z) \Lambda (M_1 x + M_2 y + M_3 z)^t.$$
Proof. Since the triple \((X, Y, Z)\) is in \(S_3\), every pair \((X, \cos \phi Y + \sin \phi Z)\) must be in \(S_2\) by 3. of Lemma 2. Then for any \(\phi \in \mathbb{R}\), we can find the operators \(M_1, M_2(\phi)\) such that

\[
R_{x, x+y(z)} = (M_1 x + M_2(\phi) y) \Lambda (M_1 x + M_2(\phi) y)^t, \tag{7}
\]
(the fact that \(M_1\) can be chosen independent of \(\phi\) follows from Lemma 1).

Denote \(M_2 = M_2(0), M_3 = M_2(\pi/2)\). Then

\[
R_X = M_1 \Lambda M_1', \quad R_Y = M_2 \Lambda M_2', \quad R_Z = M_3 \Lambda M_3'.
\]

Let \(P(\phi) = M_2(\phi) = \cos \phi M_2 - \sin \phi M_3\). We want to show that \(P(\phi) = 0\).

The terms with \(xy\) of (7) give \(M_1 \Lambda P(\phi)^t + P(\phi) \Lambda M_1' = 0\), and so \(P(\phi) = M_1 S(\phi)\), with \(S(\phi) = -\Lambda P(\phi)^t M_1(\Lambda M_1' M_1)^{-1}\) linear operator in \(\mathbb{R}^n\). Substituting \(P(\phi) = M_1 S(\phi)\) in the terms of (7) with \(y^2\), and dividing by \(\cos \phi \sin \phi\) we obtain

\[
M_1 (S(\phi) \Lambda M_2(\phi)^t (\cos \phi \sin \phi)^{-1}) + M_2 (\Lambda P(\phi)^t / \sin \phi) + M_3 (\Lambda P(\phi)^t / \cos \phi) = 2R_{YZ} - 2M_2 M_3 - M_2 M_2',
\]

The fact that \((X, Y, Z) \in S_3\) means that the images \(\text{Im} M_1 = \text{Im} R_X, \text{Im} M_2 = \text{Im} R_Y\), and \(\text{Im} M_3 = \text{Im} R_Z\) span a subspace of dimension \(3\nu\) in \(\mathbb{R}^n\). It follows that the operator \(M_1 \oplus M_2 \oplus M_3 : \mathbb{R}^\nu \to \mathbb{R}^n\) is one-to-one and so operators \(S(\phi) \Lambda M_2(\phi)^t (\cos \phi \sin \phi), \Lambda P(\phi)^t / \sin \phi\) and \(\Lambda P(\phi)^t / \cos \phi\) are independent of \(\phi\). In particular, both \(P(\phi)/\sin \phi\) and \(P(\phi)/\cos \phi\) must be constant. This is only possible when \(P(\phi)\) vanishes identically.

Then \(M_2(\phi) = \cos \phi M_2 + \sin \phi M_3\) and the claim follows from (7). \qed

With Lemma 4, we can finish the proof of the Proposition as follows.

Choose an orthonormal basis \(E_1, \ldots, E_n\) in \(\mathbb{R}^n\) in such a way that every triple \((E_i, E_j, E_k)\) is in \(S_3\) and every pair \((E_i, E_j)\) is in \(S_2\). The set of such bases is open and dense in the Stiefel manifold \(V_n(\mathbb{R}^n) = O(n)\).

For every \(i = 2, \ldots, n\), let \(M_i\), \(\tilde{M}_i\) be the operators constructed as in Lemma 3 on the vectors \(E_i, E_i\) (by Lemma 1 we can take \(M_1\) the same for all the \(i\)'s). Then for \(i = 2, \ldots, n\)

\[
M_1 \Lambda M_i = R_1, \quad M_1 \Lambda M_i = R_i, \quad M_1 \Lambda M_i + M_1 \Lambda M_i = 2R_{1i},
\]

where \(R_k = R_{E_k}, R_{kl} = R_{E_k E_l}\).

By Lemma 4, for any pair \(i \neq j, i, j \geq 2\) there exist operators \(\tilde{M}_i, \tilde{M}_j\) satisfying

\[
R_{x, E_i + y E_i + z E_j} = (M_1 x + \tilde{M}_i y + \tilde{M}_j z) \Lambda (M_1 x + \tilde{M}_i y + \tilde{M}_j z)^t. \tag{8}
\]

In particular, \(\tilde{M}_1 \Lambda \tilde{M}_j + \tilde{M}_j \Lambda \tilde{M}_j = 2R_{ij}\). On the other hand, taking \(z = 0\) in (8) and applying the uniqueness part of Lemma 3 we get \(\tilde{M}_j = M_j\). Similarly, \(\tilde{M}_j = M_j\). It follows that \(M_1 \Lambda M_1 + M_1 \Lambda M_1 = 2R_{1j}\).

Now for an arbitrary vector \(X = x_1 E_1 + \cdots + x_n E_n\), define the operator \(M_X : \mathbb{R}^\nu \to \mathbb{R}^n\) by

\[
M_X = x_1 M_1 + \cdots + x_n M_n.
\]

Then \(M_X \Lambda M_X = \sum_{i=1}^n R_{ix_i}^2 + \sum_{i<j} 2R_{ij} x_i x_j = R_X\).

The fact that the map \(X \mapsto M_X\) is determined uniquely up to a precomposition with a fixed element from \(O_\Lambda\) follows from Lemma 2.
5 Proof of Proposition 4

We are given an Osserman algebraic curvature tensor $R$ in $\mathbb{R}^n$ with the Jacobi operator having $p + 1$ distinct eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_p$, and 0, of multiplicities $m_1, m_2, \ldots, m_p$, and $n - 1 - \nu$, respectively. The number $\nu$ satisfies the inequality

$$n > \frac{(\nu + 1)^2}{4}. \quad (9)$$

In the Euclidean space $\mathbb{R}^\nu$, with the fixed orthonormal basis $e_1, \ldots, e_\nu$, the linear operator $\Lambda$ is defined by $\Lambda e_s = \mu_s e_s$, $s = 1, \ldots, \nu$, where $\mu_1 = \cdots = \mu_{m_1} = \lambda_1$, $\mu_{m_1+1} = \cdots = \mu_{m_1+m_2} = \lambda_2$, \ldots, $\mu_{\nu-m_\nu+1} = \cdots = \mu_\nu = \lambda_p$.

The Jacobi operator of $R$ has the form

$$R_X = M_X \Lambda M_X^t, \quad (10)$$

where $M : \mathbb{R}^n \to \text{Hom}(\mathbb{R}^\nu, \mathbb{R}^n), \ X \mapsto M_X$ is a linear map determined uniquely up to a precomposition with an element $N$ from the group $O_\Lambda = \{ N : N \Lambda N^t = \Lambda \}$.

The central role in the proof is played by a quadratic map $\Phi : \mathbb{R}^n \to \text{Hom}(\mathbb{R}^\nu, \mathbb{R}^n)$ defined by

$$\Phi(X) = M_X^t M_X, \quad X \in \mathbb{R}^n.$$  

In terms of the map $\Phi$, the Osserman property of $R$ has the following form.

**Lemma 5.** For every unit vector $X$,

1. the operator $\Lambda \Phi(X) : \mathbb{R}^\nu \to \mathbb{R}^\nu$ is similar to $\Lambda$; in particular, it has the same spectrum as $\Lambda$;
2. there exists $N \in O_\Lambda$ such that $N^t \Phi(X) N = I_\nu$.

**Proof.** By Lemma 1, for every unit vector $X$, there exists an element $N$ (depending on $X$) in the group $O_\Lambda$ such that $\Phi(X) = N^t N$. Then $\Lambda \Phi(X) = \Lambda N^t N = N^{-1} \Lambda N$.

The proof of the Proposition goes by induction by $p$, the number of distinct nonzero eigenvalues of the Jacobi operator.

**Base.** Let $p = 1$, that is, the Jacobi operator has only two eigenvalues: $\lambda_1$ with multiplicity $\nu$, and 0 with multiplicity $n - 1 - \nu$. Then $\Lambda = \lambda_1 I_\nu$ and by 1. of Lemma 5

$$M_X^t M_X = \Phi(X) = \|X\|^2 I_\nu.$$  

Define the operators $J_s : \mathbb{R}^n \to \mathbb{R}^n, \ s = 1, \ldots, \nu$ by $J_s X = M_X e_s$. Then for all $X, Y \in \mathbb{R}^n$,

$$R_X Y = \lambda_1 M_X^t M_X Y = \lambda_1 \sum_{s=1}^\nu (M_X^t Y, e_s) M_X e_s = \lambda_1 \sum_{s=1}^\nu (J_s X, Y) J_s X,$$

that is, the Jacobi operator has the required form ((2), with $\lambda_0 = 0$).

Moreover, for any nonzero $X$, the vectors $J_1 X, \ldots, J_\nu X$ are linearly independent (otherwise $\text{rk} R_X < \nu$), and so all the operators $J_s$ are skew-symmetric since $R_X X = 0$.

We also have $J_s J_q + J_q J_s = -2 \delta_{qs} I_\nu$ for all $1 \leq q, s \leq \nu$, since for any vector $X$, $(J_s X, J_q X) = (M_X e_s, M_X e_q) = (M_X^t M_X e_s, e_q) = \|X\|^2 \delta_{qs}$.
Thus the Jacobi operator has the form (2), with the skew-symmetric orthogonal operators $J_1, \ldots, J_r$, satisfying the Hurwitz relations.

**Step.** The plan of proof is the following. Suppose that we already know that for $p = k \geq 1$ any Osserman algebraic curvature tensor has a Clifford structure. Let $R$ be an Osserman curvature tensor with $p = k + 1$ distinct nonzero eigenvalues of the Jacobi operator.

For every unit vector $X$, the $\lambda_\alpha$-eigenspace of $R_X$ is $E_{\lambda_\alpha}(X) = \{M_X u : u \in \mathbb{R}^\nu, R_X(M_X u) = \lambda_\alpha(M_X u)\}$. This defines a subspace $S_{\lambda_\alpha}(X) \subset \mathbb{R}^\nu$ of dimension $m_\alpha$, the multiplicity of the eigenvalue $\lambda_\alpha$, consisting of vectors $u \in \mathbb{R}^\nu$ satisfying $R_X(M_X u) = \lambda_\alpha(M_X u)$.

The key step is to show that, with a particular choice of $\lambda_\alpha$, the subspace $S_{\lambda_\alpha}(X)$ is independent of $X$ (Lemma 7), that is, there exists a fixed subspace $S \subset \mathbb{R}^\nu$, $\dim S = m_\alpha$ such that $E_{\lambda_\alpha}(X) = M_X S$ for all unit vectors $X \in \mathbb{R}^n$.

We then choose a basis $u_1, \ldots, u_{m_\alpha}$ in $S$ and define the operators $J_s : \mathbb{R}^n \to \mathbb{R}^n, \ s = 1, \ldots, m_\alpha$ by

$$J_s X = M_X u_s.$$  

For every unit vector $X$ and for every $s$, $J_s X$ is an eigenvector of $R_X$ with the eigenvalue $\lambda_\alpha$. In particular, $\langle J_s X, X \rangle = 0$ and so all the $J_s$’s are skew-symmetric. In Lemma 8 we show that the basis $u_1, \ldots, u_{m_\alpha}$ can be chosen in such a way that the operators $J_s$ are also orthogonal and satisfy the Hurwitz relations.

Introduce an algebraic curvature tensor $\hat{R}$ defined by its Jacobi operator as

$$\hat{R}_X Y = \lambda_\alpha \sum_{s=1}^{m_\alpha} \langle J_s X, Y \rangle J_s X.$$  

(11)

Then $\hat{R}$ is Osserman, with the Jacobi operator having two eigenvalues, $\lambda_\alpha$ and 0.

Moreover, for every unit vector $X$, the $\lambda_\alpha$-eigenspace of $R_X$ and $\hat{R}_X$ is the same: Span($J_1 X, \ldots, J_{m_\alpha} X$). It follows that the algebraic curvature tensor $R - \hat{R}$ is also Osserman. Its Jacobi operator $R_X - \hat{R}_X$, for any unit vector $X$, has constant eigenvalues $\lambda_1, \ldots, \lambda_{\alpha-1}, \lambda_{\alpha+1}, \ldots, \lambda_{k+1}, 0$ with constant multiplicities (in fact, the $\lambda_\beta$-eigenspaces of $R_X - \hat{R}_X$ are the same as that of $R_X$, and $\ker(R_X - \hat{R}_X) = \ker R_X \oplus E_{\lambda_\alpha}(X)$).

The number of nonzero eigenvalues of the Jacobi operator of $R - \hat{R}$ is one less than that for $R$, and so by the induction assumption the algebraic curvature tensor $R - \hat{R}$ has a Clifford structure:

$$(R_X - \hat{R}_X) Y = \sum_{i=1}^{\nu - m_\alpha} \mu_i \langle J_i X, Y \rangle J_i X$$  

(12)

for all $X, Y \in \mathbb{R}^n$, with skew-symmetric orthogonal operators $J_i$ satisfying the Hurwitz relations. Together with (11) this gives a Cliff($\nu$)-structure for $R$, provided the operators $J_s$ in (11) and the operators $J_i$ in (12) satisfy the Hurwitz relations $J_s J_s + J_s J_i = 0$. This is indeed the case, since for any $X \in \mathbb{R}^n$, $J_s X$ and $J_i X$ are eigenvectors of $R_X$ which correspond to different eigenvalues, and so are orthogonal.

This proves the inductive step and hence Proposition 4.

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Following the above plan, we choose an eigenvalue $\lambda_\alpha$ such that $\lambda_\alpha^{-1}$ is the smallest from among $\lambda_\beta^{-1}$, $\beta = 1, \ldots, p$. Then for every unit vector $X$, the symmetric operator $\lambda_\alpha A^{-1} - \Phi(X)$ is semidefinite. Indeed, from Lemma 5, $\Phi(X) = N^t N$ for $N \in O_\alpha = \{N : N AN^t = \Lambda\}$. Then $\lambda_\alpha A^{-1} - \Phi(X) = \lambda_\alpha A^{-1} - N^t N = N^t (\lambda_\alpha A - I) N = N^t (\lambda_\alpha A - I) N$. The operator $\lambda_\alpha A - I$ is diagonal in the basis $e_1, \ldots, e_v$, with diagonal entries $\frac{\lambda_\alpha}{\lambda_\beta} - 1 = \lambda_\alpha (\frac{\lambda_\beta}{\lambda_\alpha} - 1)$. All these numbers have the same sign for $\beta \neq \alpha$.

The $\lambda_\alpha$-eigenspace of the Jacobi operator $R_X$ is $E_{\lambda_\alpha}(X) = \{M_X u : u \in \mathbb{R}^n, R_X (M_X u) = \lambda_\alpha (M_X u)\}$. In view of (10) the condition $R_X (M_X u) = \lambda_\alpha (M_X u)$ is equivalent to $\lambda \Phi(X) u = \lambda_\alpha u$, that is, to the fact that $u$ is a $\lambda_\alpha$-eigenvector of the operator $\Lambda \Phi(X)$.

Introduce an algebraic set $\mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^v$ as follows:

$$\mathcal{U} = \{(X, u) \in \mathbb{R}^n \times \mathbb{R}^v : R_X (M_X u) = \lambda_\alpha (M_X u)\} = \{(X, u) \in \mathbb{R}^n \times \mathbb{R}^v : \lambda \Phi(X) u = \lambda_\alpha u\}.$$ 

Let $p_1 : \mathcal{U} \to \mathbb{R}^n$, $p_2 : \mathcal{U} \to \mathbb{R}^v$ be the projections, $p_1((X, u)) = X, p_2((X, u)) = u$. For every $X \neq 0$ the set $p_2 p_1^{-1}(X)$ is the set of vectors $u \in \mathbb{R}^v$ satisfying $\lambda \Phi(X) u = \lambda_\alpha u$, that is, a linear space of dimension $m_\alpha$, the multiplicity of the eigenvalue $\lambda$. So $\dim \mathcal{U} = n + m_\alpha$. It appears that, with our choice of $\lambda_\alpha$, for every $u \in \mathbb{R}^v$ the subset $p_1 p_2^{-1}(u) \subset \mathbb{R}^n$ is also a linear subspace.

**Lemma 6.** Let $\lambda_\alpha$ be the eigenvalue such that $\lambda_\alpha^{-1} = \min\{\lambda_1^{-1}, \ldots, \lambda_p^{-1}\}$. Then for every $u \in \mathbb{R}^v$ the set

$$p_1 p_2^{-1}(u) = \{X \in \mathbb{R}^n : \lambda \Phi(X) u = \lambda_\alpha u\}$$

is a linear subspace.

**Proof.** The claim is trivial if $u = 0$. Let $u \neq 0$. The set $p_1 p_2^{-1}(u)$ is a cone and so it is sufficient to prove that for any two unit nonparallel vectors $X, Y$, the set $p_1 p_2^{-1}(u)$ contains the unit circle in the two-plane $\text{Span}(X, Y)$. Let $Z$ be a unit vector in $\text{Span}(X, Y)$ orthogonal to $X$, and $\theta$ be the angle between $X$ and $Y$, so that $Y = X \cos \theta + Z \sin \theta$.

Introduce a unit vector function $X_t = \cos t X + \sin t Z$. Then $X = X_0, Y = X_\theta$ and we have $\Lambda \Phi(X) u = \lambda_\alpha u$. As $\Phi(X_\theta) = \cos^2 \theta \Phi(X) + \sin^2 \theta \Phi(Z) + \sin \theta \cos \theta (M_X M_X + M_M X)$ we find

$$\Lambda \Phi(Z) u = \lambda_\alpha u - \cos \theta \Lambda (M_X M_Z + M_M X) u.$$  \hspace{1cm} (13)

Let $u_t$ be a $\lambda_\alpha$-eigenvector of the operator $\Lambda \Phi(X_t)$ twice differentiable at $t = 0$ and such that $u_0 = u$. Denote $\dot{u} = du_t/\text{dt}|_{t=0}$, $\ddot{u} = d^2 u_t/\text{dt}^2|_{t=0}$. Differentiating

$$\Lambda \Phi(X_t) u_t = (\cos^2 t \Lambda \Phi(X) + \sin t \cos t \Lambda (M_X M_Z + M_M) X)$$

$$+ \sin^2 t \Lambda \Phi(Z) u_t = \lambda_\alpha u_t$$ \hspace{1cm} (14)

at $t = 0$ we get $\Lambda (M_X M_Z + M_M X) u + \Lambda \Phi(X) \dot{u} = \lambda_\alpha \ddot{u}$ and so

$$(M_X M_Z + M_M X) u = (\lambda_\alpha \Lambda^{-1} - \Phi(X)) \ddot{u}.$$ \hspace{1cm} (15)

The second derivative of (14) at $t = 0$ has the form

$$-2 \Lambda \Phi(X) + 2 \Lambda \Phi(Z) u + 2 \Lambda (M_X M_Z + M_M X) \ddot{u} + \Lambda \Phi(X) \dddot{u} = \lambda_\alpha \dddot{u}.$$
Substituting the expression for $\Lambda \Phi(Z)u$ from (13) we obtain
\[
2\Lambda(M_X^t M_Z + M_Z^t M_X)(\hat{u} - \cot \theta u) + \Lambda \Phi(X)\hat{u} = \lambda_\alpha \hat{u}.
\]
Acting on both sides by $\Lambda^{-1}$ and taking the inner product with $u$ we get
\[
2\langle (M_X^t M_Z + M_Z^t M_X)u, (\hat{u} - \cot \theta u) \rangle + \langle (\Phi(X) - \lambda_\alpha \Lambda^{-1})u, \hat{u} \rangle = 0.
\]
Substituting $(M_X^t M_Z + M_Z^t M_X)u$ from (15) we obtain
\[
2\langle (\lambda_\alpha \Lambda^{-1} - \Phi(X))\hat{u}, \hat{u} \rangle + \langle (\Phi(X) - \lambda_\alpha \Lambda^{-1})u, \hat{u} + 2 \cot \theta \hat{u} \rangle = 0.
\]
The second term on the left hand side vanishes since $\Lambda \Phi(X)u = \lambda_\alpha u$, hence we get
\[
\langle (\lambda_\alpha \Lambda^{-1} - \Phi(X))\hat{u}, \hat{u} \rangle = 0.
\]
With our choice of $\lambda_\alpha$, the symmetric operator $\lambda_\alpha \Lambda^{-1} - \Phi(X)$ is semidefinite. It follows that $(\lambda_\alpha \Lambda^{-1} - \Phi(X))\hat{u} = 0$, which implies $(M_X^t M_Z + M_Z^t M_X)u = 0$ by (15). Then by (13) $\Lambda \Phi(Z)u = \lambda_\alpha u$, and so
\[
\Lambda \Phi(X_t)u = (\cos^2 t \Lambda \Phi(X) + \sin^2 t \Lambda (M_X^t M_Z + M_Z^t M_X))u
\]
for all $t \in \mathbb{R}$. It follows that $X_t = \cos t X + \sin t Z \in p_1p_2^{-1}(u)$ for all $t$, that is, $p_1p_2^{-1}(u)$ contains the unit circle in the two-plane $\text{Span}(X, Y)$.

\begin{lemma}
The subspace $p_2p_1^{-1}(X) \subset \mathbb{R}^\nu$ is the same for all $X \neq 0$. In other words, the $\lambda_\alpha$-eigenspace of the operator $\Lambda \Phi(X)$ does not depend on the choice of a unit vector $X \in \mathbb{R}^n$.
\end{lemma}

\begin{proof}
The proof is based on the dimension count. For every point $u \in \mathbb{R}^\nu$ let $d(u)$ be the dimension of the linear space $p_1p_2^{-1}(u)$.

For every set of $m_\alpha + 1$ linearly independent vectors $u_1, \ldots, u_{m_\alpha+1}$ in $\mathbb{R}^\nu$, we must have
\[
d(u_1) + \cdots + d(u_{m_\alpha+1}) \leq m_\alpha n.
\]
Indeed, if the inequality (16) is violated, then the subspaces $p_1p_2^{-1}(u_1), \ldots, p_1p_2^{-1}(u_{m_\alpha+1})$ have a nonzero intersection in $\mathbb{R}^n$. It follows that for a unit vector $X$ from this intersection, the equation $\Lambda \Phi(X)u = \lambda_\alpha u$ has at least $m_\alpha + 1$ linearly independent solutions, while the $\lambda_\alpha$-eigenspace of the operator $\Lambda \Phi(X)$ has dimension $m_\alpha$ by 1. of Lemma 5.

Now let $u_1, \ldots, u_{m_\alpha}$ be a set of $m_\alpha$ linearly independent vectors in $\mathbb{R}^\nu$ such that $d(u_1) + \cdots + d(u_{m_\alpha})$ takes the maximal possible value.

Let $S = \text{Span}(u_1, \ldots, u_{m_\alpha})$, $\dim S = m_\alpha$. For every point $u \notin S$, we have $d(u) + \sum_{i=1}^{m_\alpha} d(u_i) \leq m_\alpha n$ by (16), and $d(u) \leq d(u_i)$ for all $i = 1, \ldots, m_\alpha$ by the construction of the set $u_1, \ldots, u_{m_\alpha}$. So $d(u) \leq \frac{m_\alpha}{m_\alpha + 1} n$.

Now the subset $p_2^{-1}(\mathbb{R}^\nu \setminus S) \subset \mathcal{U}$ is projected by $p_2$ to a subset of $\mathbb{R}^\nu \setminus S$, of dimension not greater than $\nu$, with the fibers being linear subspaces of dimension not greater than $\frac{m_\alpha}{m_\alpha + 1} n$. If $p_2^{-1}(\mathbb{R}^\nu \setminus S)$ has a nonempty interior in $\mathcal{U}$, then $\frac{m_\alpha}{m_\alpha + 1} n + \nu \geq \dim \mathcal{U} = n + m_\alpha$, and so
\[
n \leq (\nu - m_\alpha)(m_\alpha + 1).
\]
Proof. Since the operators \( s, q \) for all \( J \) defined by \( u \)

For any find an element \( N \)

Lemma 8. There exist a basis \( \lambda \)

Then \( \lambda_{\alpha} \parallel X \parallel^{2} \)-eigenspace of the operator \( \Lambda(X) \) contains an open subset of \( S \), hence coincides with \( S \).

In Lemma 7, we constructed an \( m_{\alpha} \)-dimensional subspace \( S \subset \mathbb{R}^{\nu} \) such that for all \( X \in \mathbb{R}^{\nu} \) and all \( u \in S \)

Following our plan, we take a basis \( u_{1}, \ldots, u_{m_{\alpha}} \) in \( S \) and define linear operators \( J_{s} : \mathbb{R}^{n} \to \mathbb{R}^{n} \), \( s = 1, \ldots, m_{\alpha} \) by

For every unit vector \( X \), the \( J_{s}X \)'s span \( E_{\lambda_{\alpha}}(X) \), the \( \lambda_{\alpha} \)-eigenspace of \( R_{X} \). Then \( \langle J_{s}X, X \rangle = 0 \) and so all the operators \( J_{s} \) are skew-symmetric.

To prove the induction step (and hence to prove the Proposition) it remains to show that, with an appropriate choice of the basis \( u_{1}, \ldots, u_{m_{\alpha}} \), the operators \( J_{s} \) are also orthogonal and satisfy the Hurwitz relations.

Lemma 8. There exist a basis \( u_{1}, \ldots, u_{m_{\alpha}} \) in \( S \) such that the operators \( J_{s} \)

defined by \( J_{s}X = M_{X}u_{s} \) satisfy

for all \( s, q = 1, \ldots, m_{\alpha} \).

Proof. Since the operators \( J_{s} \) are skew-symmetric, the condition \( J_{s}J_{q} + J_{q}J_{s} = -2\delta_{qs}I_{n} \) is equivalent to the fact that for all \( X \in \mathbb{R}^{n} \),

\[
\langle J_{s}X, J_{q}X \rangle = \delta_{qs}\|X\|^{2}.
\] (17)

To construct the required basis we pick an arbitrary unit vector \( X_{0} \) and find an element \( N_{0} \in O_{\Lambda} \) such that \( N_{0}^{2} \Phi(X_{0})N_{0} = I_{s} \) according to Lemma 5. For any \( u \in S \) we have \( \lambda_{\alpha}u = \Lambda \Phi(X_{0})u = \Lambda N_{0}^{-1}N_{0}^{-1}u = N_{0}\Lambda N_{0}^{-1}u \), since \( N_{0}\Lambda N_{0}^{-1} = \Lambda \). It follows that \( \Lambda(N_{0}^{-1}u) = \lambda_{\alpha}(N_{0}^{-1}u) \), that is, \( N_{0}^{-1}u \) lies in the \( \lambda_{\alpha} \)-eigenspace of \( \Lambda \), the coordinate subspace \( \text{Span}(e_{m'+1}, \ldots, e_{m'+m_{\alpha}}) \subset \mathbb{R}^{\nu} \), \( m' = m_{1} + \cdots + m_{\alpha} - 1 \). Define

\[
u_{s} = N_{0}e_{m'+s}, \quad s = 1, \ldots, m_{\alpha}.
\]

The operators \( J_{s} \) constructed from this basis satisfy (17). Indeed, for any unit vector \( X \), we have

\[
\langle J_{s}X, J_{q}X \rangle = \langle M_{X}u_{s}, M_{X}u_{q} \rangle = \langle \Phi(X)u_{s}, u_{q} \rangle = \langle \lambda_{\alpha}\Lambda^{-1}u_{s}, u_{q} \rangle
\]

\[
= \lambda_{\alpha}\langle \Lambda^{-1}N_{0}e_{m'+s}, N_{0}e_{m'+q} \rangle = \lambda_{\alpha}\langle N_{0}^{2}\Lambda^{-1}N_{0}e_{m'+s}, e_{m'+q} \rangle
\]

\[
= \lambda_{\alpha}\langle \Lambda^{-1}e_{m'+s}, e_{m'+q} \rangle = \langle e_{m'+s}, e_{m'+q} \rangle = \delta_{sq}.
\]
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