POLYNOMIAL PROCESSES AND THEIR APPLICATIONS TO
MATHEMATICAL FINANCE

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ABSTRACT. We introduce a class of Markov stochastic processes called $m$-polynomial, for which the calculation of (mixed) moments up to order $m$ only requires the computation of matrix exponentials. This class contains affine processes, Feller processes with quadratic squared diffusion coefficient, as well as Lévy-driven SDEs with affine vector fields. Thus, many popular models such as the classical Black-Scholes, exponential Lévy or affine models are covered by this setting. The applications range from statistical GMM estimation to option pricing. For instance, the efficient and easy computation of moments can successfully be used for variance reduction techniques in Monte Carlo simulations.

1. Introduction

Pricing and hedging of contingent claims is the crucial computation done within every model in mathematical Finance. For European type claims this amounts to the computation of the expected value of a functional of the (discounted) price process under some martingale measure. Hedging portfolios are then constructed via appropriate derivatives of those expected values with respect to model parameters or to the prices, so called “Greeks”. Let us denote the (discounted) price process at time $T$, a vector in $\mathbb{R}^n$, by $X_T$. We can roughly distinguish three cases of complexity for the mentioned computations:

1. The probability distribution of $X_T$ is known analytically.
2. The characteristic function of $X_T$ is known analytically.
3. The local characteristics of $X_T$ are known analytically.

In the first case, a numerical quadrature algorithm is sufficient for the efficient computation of the contingent claim’s price $E[\phi(X_T)]$, where $\phi$ denotes some payoff function. In the second case, variants of Plancherel’s theorem are applied in order to evaluate the price $E[\phi(X_T)]$, for instance,

$$E[\phi(X_T)] = \int_{\mathbb{R}^n} \hat{\phi}(u) E[\exp(i\langle u, X_T \rangle)] du,$$

where $\hat{\phi}$ denotes the Fourier transform of the function $\phi$. Remark that often modifications of the original payoff function are used to make the Fourier methodology applicable. This is numerically efficient, even though its implementation (in particular the complex integration) can take some time (see, e.g., [3]). Also there are different levels of what it means to “know analytically” the characteristic function.
of \( X_T \). If one, e.g. in affine models, has to solve a high-dimensional Riccati equation for each \( u \in \mathbb{R}^n \) to calculate the characteristic function \( u \mapsto E[\exp(i(u, X_T))] \), the “analytic knowledge” means at least some pre-calculations which have to be performed efficiently, too. In other words, methods applying the characteristic function are efficient if the amount of pre-calculations is limited.

The third case is characterized by the use of Monte-Carlo simulation methods: one samples from the (unknown) distribution of \( X_T \) by generating, for instance through Euler schemes, approximate distributions for \( X_T \). From those approximate distributions, which should be easy to simulate, one draws a sufficient amount of samples. This two-step procedure is very robust, but takes a considerable amount of time.

In this article we would like to add a fourth case, which in the previous order would correspond to case 2\(^1\). We can describe a class of processes, called “polynomial processes”, where it is easy and efficient to compute moments of all orders of the random variable \( X_T \), even though neither the probability distribution nor its characteristic function need to be known. We shall analyze this class and show that exponential Lévy processes, affine processes or Jacobi-type processes belong to it. The method is best explained by an example: consider a stochastic volatility model of SVJJ-type \([9]\), i.e. both the logarithmic (discounted) price processes and the stochastic volatility can jump. Such models can be described by stochastic differential equations of the type

\[
\begin{align*}
dX_t &= (aX_t + bV_t + c)dt + \sqrt{V_t}dB_{t,1} + dZ_{t,1}, \\
dV_t &= (\alpha V_t + \beta)dt + \sqrt{V_t}dB_{t,2} + dZ_{t,2},
\end{align*}
\]

where \((B_1, B_2)\) are possibly correlated Brownian motions, and \((Z_1, Z_2)\) are Lévy processes, independent of \((B_1, B_2)\), where the second component \(Z_2\) is a subordinator. For such models there is no easy-to-implement (explicit) formula for the characteristic function, even though they are affine models. Assuming now appropriate moment conditions for the jump measures, the Markov process \((X, V)\) has the remarkable property that the expected value of any polynomial of the process \((X, V)\) is again a polynomial in \(X_0\) and \(V_0\). The coefficients of this polynomial can be calculated efficiently by exponentiating a matrix, which can be easily deduced from the generator. In other words, there is a dense subset of claims in the set of “all” claims, where the prices and hedging ratios are known explicitly (up to matrix exponentials). This explicit knowledge allows to compute prices of general claims by variance reduction techniques, which is more efficient (and easier to implement) than Fourier methods with “unknown” characteristic function.

The article is devoted to a study of polynomial processes in some depths together with their most obvious applications to mathematical Finance. As the most striking application we see the possibility to reduce the variance of Monte Carlo evaluations within a polynomial model by approximating the claim through elements of the dense subset of already priced claims. The remainder of our article is organized as follows: in Section 2\(^1\) we introduce formally the class of \(m\)-polynomial processes and draw several basic conclusions. In 3\(^1\) we analyze the conditions on the characteristics of a Feller semimartingale to be \(m\)-polynomial. Section 4\(^1\) deals with examples from the class of \(m\)-polynomial processes and Section 5\(^1\) with applications to pricing and hedging in mathematical Finance.
2. Polynomial Processes

We consider a time-homogeneous Markov process \( X := (X_t^x)_{t \geq 0, x \in S} \) with state space \( S \subseteq \mathbb{R}^n \), a closed subset of \( \mathbb{R}^n \), and semigroup \( (P_t)_{t \geq 0} \) defined by

\[
P_t f(x) := \mathbb{E}[f(X_t^x)] = \int_S f(\xi) p_t(x, d\xi), \quad x \in S,
\]

and acting on all functions \( f : S \to \mathbb{R} \) which are integrable with respect to the family of Markov kernels \( p_t(x, \cdot) \).

**Notation 2.1.** Let \( \text{Pol}_{\leq m}(S) \) denote the finite dimensional vector space of polynomials up to degree \( m \geq 0 \) on \( S \), i.e. the restriction of polynomials on \( \mathbb{R}^n \) to \( S \), defined by

\[
\text{Pol}_{\leq m}(S) := \left\{ \sum_{|k| \leq m} \alpha_k x^k \mid x \in S, \quad \alpha_k \in \mathbb{R} \right\}, \quad (2.1)
\]

where we use the multi-index notation \( k = (k_1, \ldots, k_n) \in \mathbb{N}_0^n \), \( |k| = k_1 + \cdots + k_n \)

and \( x^k = x_1^{k_1} \cdots x_n^{k_n} \). \( \text{Pol}_{\leq m}(S) \) is endowed with some norm \( \| \cdot \|_{\text{Pol}_{\leq m}} \) and its dimension is denoted by \( N < \infty \). Moreover, \( \text{Pol}_{m}(S) \) corresponds to the vector space of polynomials which are precisely of degree \( m \).

**Definition 2.2.** We call an \( S \)-valued time-homogeneous Markov process \( m \)-polynomial if

\[
P_t f(x) \in \text{Pol}_{\leq m}(S)
\]

holds true for all \( f \in \text{Pol}_{\leq m}(S) \) and \( t \geq 0 \). If \( X \) is \( m \)-polynomial for all \( m \geq 0 \), then it is called \( m \)-polynomial.

**Theorem 2.3.** Let \( X \) be a time-homogeneous Markov process with state space \( S \) and semigroup \( (P_t) \), pointwise continuous at \( t = 0 \), i.e. \( t \mapsto P_t f(x) \) is continuous at \( t = 0 \) for all \( x \in S \) and \( f : S \to \mathbb{R} \) where \( P_t f \) exists as finitely valued function on \( S \). Then the following assertions are equivalent:

(i) \( X \) is \( m \)-polynomial for some \( m \geq 0 \).

(ii) There exists a linear map \( A \) on \( \text{Pol}_{\leq m}(S) \), such that \( (P_t) \) restricted to \( \text{Pol}_{\leq m}(S) \) can be written as

\[
P_t|_{\text{Pol}_{\leq m}(S)} = e^{tA}
\]

for all \( t \geq 0 \).

(iii) The infinitesimal generator \( A \) is well defined on \( \text{Pol}_{\leq m}(S) \) and maps \( \text{Pol}_{\leq m}(S) \) to itself.

(iv) The Kolmogorov backward equation for an initial value \( f(\cdot, 0) \in \text{Pol}_{\leq m}(S) \)

\[
\frac{\partial f(x, t)}{\partial t} = Af(x, t)
\]

has a real analytic solution for all times \( t \in \mathbb{R} \). In particular, \( f(\cdot, t) \in \text{Pol}_{\leq m}(S) \).

**Proof.** We start by proving (i) \( \Rightarrow \) (ii). Let \( \mathcal{L}(\text{Pol}_{\leq m}(S)) \) denote the space of all linear maps on \( \text{Pol}_{\leq m}(S) \). By the semigroup property,

\[
P_{(\cdot)}|_{\text{Pol}_{\leq m}(S)} : \mathbb{R}_+ \to \mathcal{L}(\text{Pol}_{\leq m}(S)) \quad (2.2)
\]
satisfies the Cauchy functional equation
\[
\begin{cases}
    P_{t+s} = P_tP_s & \text{for all } t, s \geq 0 \\
    P_0 = I
\end{cases}
\]  
(2.3)
Since \( \text{Pol}_{\leq m}(S) \) is finite dimensional, continuity of \( P_t \) at \( t = 0 \) already implies that there exists an \( A \in \mathcal{L}(\text{Pol}_{\leq m}(S)) \) such that \( P_t|_{\text{Pol}_{\leq m}(S)} = e^{tA} \).

Next, we show (ii) \( \Rightarrow \) (iii). By the definition of the infinitesimal generator, we have
\[
Af = \lim_{t \to 0} \frac{P_t f - f}{t} = \lim_{t \to 0} \frac{e^{tA}f - f}{t} = Af,
\]
which is obviously well defined with respect to \( \| \cdot \|_{\text{Pol}_{\leq m}} \) and in \( \text{Pol}_{\leq m}(S) \).

Assertion (iv) follows from (iii) since \( A \) maps \( \text{Pol}_{\leq m}(S) \) to itself. The Kolmogorov backward equation can therefore be understood as a linear ODE in the classical sense, whose solution is thus given by \( e^{tA}f(\cdot,0) \).

At last, we prove that (iv) implies (i). For any initial value \( f \) in an appropriate Banach space, \( P_t f \) is the unique solution of the Kolmogorov equation. On \( \text{Pol}_{\leq m}(S) \) it must therefore be equal to \( e^{tA}f \) and is hence a polynomial of degree smaller than or equal to \( m \) thereon. Thus, \( X \) is \( m \)-polynomial.

Remarks 2.4.
(i) There is no need in Theorem 2.3(ii) to restrict the time parameter \( t \) to \( \mathbb{R}_+ \). Actually, for \( t \in \mathbb{R} \), \( (e^{tA}) \) extends to a group generated by \( A \).

(ii) The assumption of pointwise continuity of \( P_t f \) at \( t = 0 \) can be replaced by different conditions. For example, stochastic continuity of \( X \) together with the existence of moments up to order \( m + \varepsilon \), for some \( \varepsilon > 0 \), is another sufficient hypothesis for Theorem 2.3.

(iii) It is certainly possible to consider other finite dimensional functional spaces instead of \( \text{Pol}_{\leq m}(S) \). For example, trigonometric polynomials are left invariant by Markov processes of the form \( X_t^x = x + L_t \), where \( L_t \) is a Lévy process. We have been choosing polynomials since we wanted to enlarge the class of affine processes. This statement is not true literally, since affine processes which do not admit first moments, can never be polynomial, however, if a regular affine process admits moments of order \( m \) it is \( m \)-polynomial. Therefore properties of \( m \)-polynomial processes apply to affine processes which admit moments up to order \( m \) (see Example 4.1).

Corollary 2.5. Let \( X \) be an \( m \)-polynomial process with semigroup \( (P_t) \), continuous at \( t = 0 \) and let \( f \in \text{Pol}_{\leq m}(S) \) be fixed. Then there exists a unique function \( Q : \mathbb{R} \times S \to \mathbb{R} \), being real analytic in time and \( Q(t, \cdot) \in \text{Pol}_{\leq m}(S) \) for all \( t \in \mathbb{R} \), such that
\[
Q(0, x) = f(x)
\]  
(2.4)
and
\[
Q(t - s, X_s) \text{ is a martingale for } s \geq 0.
\]  
(2.5)
Proof. By the martingale property of \( Q(t - s, X_s) \), we have for all \( s \geq 0 \)
\[
Q(t, x) = \mathbb{E}[Q(t - s, X_s^x)].
\]
Thus, by setting \( s = t \) we obtain
\[
Q(t, x) = \mathbb{E}[Q(0, X_t^x)] = \mathbb{E}[f(X_t^x)] = P_t f(x),
\]
since $Q(0, X_t^1) = f(X_t^1)$. As $X$ is $m$-polynomial, $Q(t, \cdot) \in \text{Pol}_{\leq m}(S)$ and by the previous Theorem it is a real analytic function. Clearly, $P_t f(x)$ satisfies (2.4) and (2.5).

We recall the following definition (see e.g. [17], [18]).

**Definition 2.6.** A real valued function $R : [0, \infty) \times S \to \mathbb{R}$ is said to be time-space harmonic for $X$ if $(R(s, X_s))_{s \geq 0}$ is a martingale. If, moreover, $R(s, x)$ is a polynomial in $s$ and $x$ that is time-space harmonic for $X$, it is called a time-space harmonic polynomial.

For $t = 0$, $Q(-s, X_s)$ as defined in Corollary 2.5 can be considered as a time-space harmonic function for the $m$-polynomial process $X$. If $\mathcal{A}(\text{Pol}_{\leq m}(S)) \subseteq \text{Pol}_{\leq (m-1)}(S)$, then it is a time-space harmonic polynomial, which is proved in the Corollary below (see also [1] for $S = \mathbb{R}$).

**Corollary 2.7.** If $\mathcal{A}(\text{Pol}_{\leq m}(S)) \subseteq \text{Pol}_{\leq (m-1)}(S)$, then $Q(-s, x)$ as defined in Corollary 2.5 is a time space harmonic polynomial.

**Proof.** For some fixed $f \in \text{Pol}_{\leq m}(S)$ we have

$$Q(-s, x) = P_{-s} f(x) = e^{-sA} f(x) = \sum_{n=0}^{\infty} \frac{(-s)^n}{n!} A^n f(x) = \sum_{n=0}^{m} \frac{(-s)^n}{n!} A^n f(x),$$

which is obviously a polynomial in $s$ and $x$ and a martingale due to Corollary 2.5.

3. **Polynomial Feller semimartingales**

For the class of Feller processes, we want to find sufficient conditions for $m$-polynomial processes in terms of the infinitesimal generator. We consider therefore a Feller semigroup $(P_t)$ on $S$. If $C_c^\infty(S)$ is contained in the domain of the infinitesimal generator $\mathcal{A}$, then it is well known that there exist real-valued functions $a_{kl}, b_k, c$ and a kernel $K(x, d\xi)$ on $S \times B(\mathbb{R}^n \setminus \{0\})$ such that for $u \in C_c^\infty(S)$, $\mathcal{A}$ is given by

$$\mathcal{A}u(x) = \frac{1}{2} \sum_{k,l=1}^{n} a_{kl} \frac{\partial^2 u(x)}{\partial x_k \partial x_l} + \sum_{k=1}^{n} b_k(x) \frac{\partial u(x)}{\partial x_k} - c(x)u(x)$$

$$+ \int_{\mathbb{R}^n \setminus \{0\}} \left( u(x + \xi) - u(x) - \sum_{k=1}^{n} \chi_k(\xi) \frac{\partial u(x)}{\partial x_k} \right) K(x, d\xi),$$

where $a(x) = (a_{kl}(x))_{k,l=1,\ldots,n}$ is a symmetric positive semidefinite matrix, $b(x) \in \mathbb{R}^n$, $c$ is non-negative, $K(x, \cdot)$ is a Radon measure on $S \times B(\mathbb{R}^n \setminus \{0\})$ and $\chi : \mathbb{R}^n \to \mathbb{R}^n$ some bounded continuous truncation function with $\chi(\xi) = \xi$ in a neighborhood of 0 (see for example [11] or [16]). Clearly, the above parameters have to satisfy certain admissibility conditions guaranteeing the existence of the process in $S$.

As the definition of $m$-polynomial processes requires the existence of moments up to order $m$ and thus the finiteness of $X_t$ a.s., we henceforth assume $X$ to be conservative, i.e. $c = 0$. If $X$ is additionally a semimartingale, which is automatically the case if $S = \mathbb{R}^n$ (see Theorem 7.16 in [1]), then its characteristics $(B, C, \nu)$ associated with the truncation function $\chi(\xi)$ are given by

$$B_t = \int_0^t b(X_s)ds, \quad C_t = \int_0^t a(X_s)ds, \quad \nu(dt, d\xi) = K(X_t, d\xi)dt.$$
We shall refer to \((b, a, K)\) as differential characteristics of \(X\) (see [14]).

In order to specify the form of \(a\), \(b\) and \(K\) such that \(\mathcal{A}\) generates an \(m\)-polynomial process, we start with the following Lemma.

**Lemma 3.1.** Let \(X_t^a\) be a conservative Feller process on \(S\). If for every \(f \in \text{Pol}_{\leq m}(S)\), there exists a polynomial \(g \in \text{Pol}_{\leq m}(S)\) and some \(t_0 > 0\) such that the process

\[
C_t^f = f(X_t^a) - f(x) - \int_0^t g(X_s^a)ds
\]

is a well defined martingale for \(t \in [0, t_0]\), then \(X\) is \(m\)-polynomial.

**Proof.** Due to the martingale property, we obtain for \(t \leq t_0\)

\[
P_t f(x) - f(x) - \int_0^t P_s g(x)ds = E \left[ f(X_t^a) - f(x) - \int_0^t g(X_s^a)ds \right] = 0.
\]

Let us denote by \(S_\delta = S \cup \{\delta\}\), \((\delta \notin S)\) the Alexandrov one-point compactification of \(S\). By the Feller property of \((P_t)\), \(P_t f \in C(S_\delta)\) for all \(t \geq 0\) and \(\|\cdot\|_\infty = \sup_{x \in S_\delta} |\cdot|\) is well defined. Furthermore, for \(t \leq t_0\)

\[
\left\| \frac{1}{t} (P_t f - f) - g \right\|_\infty = \left\| \frac{1}{t} \int_0^t (P_s g - g)ds \right\|_\infty \leq \frac{1}{t} \int_0^t \|P_s g - g\|_\infty ds,
\]

which goes to 0 as \(t\) goes to 0 since by the Feller property \(\lim_{t \to 0} \|P_s g - g\|_\infty = 0\). Hence, with respect to \(\|\cdot\|_\infty\), \(\mathcal{A} f\) is well defined. Thus, it is also well defined with respect to \(\|\cdot\|_{\text{Pol}_{\leq m}(S)}\) and \(\mathcal{A} f = g \in \text{Pol}_{\leq m}(S)\). As \(P_t\) is continuous at \(t = 0\) (by the Feller property), we can apply Theorem [2.3] By assertion (iii) it then follows that \(X\) is \(m\)-polynomial.

\(\square\)

**Remark 3.2.** The operator \(\mathcal{G}\) satisfying \(g = \mathcal{G} f\) in equation (3.2) for \(f \in D(\mathcal{G})\) is usually called the extended infinitesimal generator.

In the next Lemma we establish a maximal inequality for semimartingales which lie - as we shall show in Theorem 3.6 below - in the class of \(m\)-polynomial processes. A similar statement and proof for the case of Lévy driven SDEs can be found in [12], see also [14]. Before stating the Lemma, let us define the following two conditions on the kernel \(K(x, d\xi)\):

**Condition A.** The kernel \(K(x, d\xi)\) is of the form

\[
K(x, d\xi) := \mu_{00}(d\xi) + \sum_{i \in I} X_{s, i} \mu_{i0}(d\xi) + \sum_{(i, j) \in J} X_{s, i} X_{s, j} \mu_{ij}(d\xi),
\]

where all \(\mu_{ij}\) are Lévy measures on \(\mathbb{R}^n\) with

\[
\int_{\|\xi\| > 1} \|\xi\|^m \mu_{ij}(d\xi) < \infty.
\]

The index sets \(I\) and \(J\) are defined by

\[
I = \{1 \leq i \leq n | S_i \subseteq \mathbb{R}_+\}
\]

and

\[
J = \{(i, j), i \leq j | S_i \times S_j \subseteq \mathbb{R}^2_+ \text{ or } S_i \times S_j \subseteq \mathbb{R}^2_-\},
\]

where \(S_i\) stands for the projection on the \(i\)th component.
Condition B. The kernel $K(x, dξ)$ satisfies
\[ K(X_t, dξ) := g^X_t(µ(dξ)), \]
where for each $x \in S$, $g^x$ denotes the pushforward of the measure $µ$ under the map $g^x$. Moreover, $g^x(y) = g(x, y)$ is affine in $x$, i.e.
\[ g : S \times \mathbb{R}^d \to \mathbb{R}^n, \quad (x, y) \mapsto H(y)x + h(y), \]
with $H : \mathbb{R}^d \to \mathbb{R}^{n \times n}$ and $h : \mathbb{R}^d \to \mathbb{R}^n$ some measurable functions. Finally, $µ$ is a Lévy measure on $\mathbb{R}^d$ integrating
\[ \int_{\mathbb{R}^d \setminus \{0\}} (\|H(y)\| + \|h(y)\|)|k| µ(dy) \text{ for all } 1 \leq k \leq m. \tag{3.4} \]

Remark 3.3. The integrability condition (3.4) (in particular for $k = 1$) can be weakened for specific choices of $H$ and $h$ (see Example 4.3).

Lemma 3.4. Let $m \geq 2$ and $X^c_t$ be a semimartingale with the following differential characteristics $(b, a, K)$ associated with the “truncation function” $χ(ξ) = ξ$: \footnote{The integrability assumptions on the Lévy measures constituting the compensator $ν$ allow the choice of this “truncation function”. Here, $\int_{\|ξ\| > 1} \|ξ\|µ(ξ) < \infty$ in case of Condition A and $\int_{\mathbb{R}^d \setminus \{0\}} (\|H(y)\| + \|h(y)\|)|k| µ(dy) < \infty$ in case of Condition B are sufficient.}
\[ b_t = b + \sum_{i=1}^{n} X_{t,i}β_i, \quad b, β_i \in \mathbb{R}^n, \]
\[ a_t = a + \sum_{i=1}^{n} X_{t,i}α_{i0} + \sum_{i,j} X_{t,i}X_{t,j}α_{ij}, \quad a, α_{ij} \in \mathbb{R}^{n \times n}, \]
and the kernel $K$ satisfying either Condition A or Condition B. Then, there exists a constant $C_m$ such that for every $t \in [0, 1],$
\[ \mathbb{E} \left[ \sup_{s \leq t} \|X^c_s\|^m \right] \leq C_m \left( \|x\|^m + 1 + \int_0^t \mathbb{E} \left[ \|X_s\|^m \right] ds \right) \tag{3.5} \]

Proof. Due to the choice of the “truncation function” $χ(ξ) = ξ$, the canonical representation of $X$ is given by
\[ X^c_t = x + B_t + X^c_t + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} ξ (µ^X(ds, dξ) − ν(ds, dξ)) \]
\[ = x + B_t + X^c_t + X^d_t, \]
where $µ^X$ denotes the random measure associated with the jumps of $X$, $X^c$ the continuous local martingale part and $X^d$ the purely discontinuous martingale part. Using the inequality
\[ \|x + y\|^m \leq 2^{m-1} (\|x\|^m + \|y\|^m) \tag{3.6} \]
for $m \geq 1$, we have
\[ \sup_{s \leq t} \|X^c_s\|^m \leq 2^{m-1} \left( \|x\|^m + \sup_{s \leq t} \|B_s\|^m + \sup_{s \leq t} \|X^c_s\|^m + \sup_{s \leq t} \|X^d_s\|^m \right). \]

Thus, in order to prove (3.5), it suffices to find estimates for $B$, $X^c$ and $X^d$. For the sake of notational simplicity we only consider the one dimensional case in the following computations. (We also write $\|\cdot\|$ instead of $\|\|\|$.) Moreover, the constant
we finally turn to the purely discontinuous martingale part \( X \).

\( \sup_{s \leq t} |B_s|^m = \sup_{s \leq t} \left| \int_0^s (b + X_u \beta_1) \, du \right|^m \leq \sup_{s \leq t} \left( \int_0^s |b + X_u \beta_1| \, du \right)^m \)

\[ \leq \left( \int_0^t |b + X_u \beta_1| \, du \right)^m = \left( \int_0^1 |b + X_u \beta_1| 1_{\{u \leq t\}} (u) \, du \right)^m \]

\[ \leq \int_0^1 |b + X_u \beta_1|^m (1_{\{u \leq t\}} (u))^m \, du \leq C_m \int_0^t (|b|^m + |X_u|^m |\beta_1|^m) \, du, \]

where the penultimate inequality is a consequence of Jensen’s inequality and the last one follows from (3.6). Fubini’s theorem finally leads to

\[ E \left[ \sup_{s \leq t} |B_s|^m \right] \leq C_m \int_0^t (1 + E [|X_s|^m]) \, ds. \]

In the case of \( X^c \), an application of the Burkholder-Davis-Gundy inequality yields

\[ E \left[ \sup_{s \leq t} |X_s^c|^m \right] \leq C_m E \left[ |X^c, X^d|_{\tilde{\tau}}^m \right]. \]

By our assumption, we have \( |X^c, X^c|_{\tilde{\tau}} = \int_0^t (a + X_s \alpha_{10} + X_s^2 \alpha_{11}) \, ds \geq 0 \). Hence, we obtain

\[ \left| \int_0^t (a + X_s \alpha_{10} + X_s^2 \alpha_{11}) \, ds \right|^m \leq \left( \int_0^t |a + X_s \alpha_{10} + X_s^2 \alpha_{11}| \, ds \right)^m \]

\[ \leq \int_0^t |a + X_s \alpha_{10} + X_s^2 \alpha_{11}|^{\frac{m}{2}} (1_{\{s \leq t\}} (s))^\frac{m}{2} \, ds \]

\[ \leq C_m \int_0^t (|a|^\frac{m}{2} + |X_s|^{\frac{m}{2}} |\alpha_{10}|^{\frac{m}{2}} + |X_s|^m |\alpha_{11}|^{\frac{m}{2}}) \, ds, \]

where the two last expressions follow again from (3.6) and Jensen’s inequality since we supposed \( m \geq 2 \). Thus, we have

\[ E \left[ \sup_{s \leq t} |X_s^c|^m \right] \leq C_m \int_0^t (1 + E [|X_s|^{\frac{m}{2}}] + E [|X_s|^m]) \, ds. \]

We finally turn to the purely discontinuous martingale part \( X^d \). A further application of the Burkholder-Davis-Gundy inequality yields

\[ E \left[ \sup_{s \leq t} |X_s^d|^m \right] \leq C_m E \left[ |X^d, X^d|_{\tilde{\tau}}^m \right], \]

where \( |X^d, X^d|_{\tilde{\tau}} = \sum_{0 \leq s \leq t} |\Delta X_s|^2 \) (see for example [13], I.4.51). Since \( B \) and \( X^c \) are both continuous, we therefore have

\[ Y_t := [X^d, X^d]_t = \sum_{0 \leq s \leq t} |\Delta X_s|^2 = \int_0^t \int_{\mathbb{R} \setminus \{0\}} |\xi|^2 \mu X (ds, d\xi). \]
Following the approach of [12], let us define $T_n = \inf\{ t \mid Y_t \geq n \}$. Then,

$$Y_{t \wedge T_n} = \sum_{s \leq t \wedge T_n} (Y_{s-} + \Delta Y_s)^{\bar{n}} - (Y_{s-})^{\bar{n}}$$

$$= \int_0^{t \wedge T_n} \int_{\mathbb{R} \setminus \{0\}} (Y_{s-} + |\xi|^2)^{\bar{n}} - (Y_{s-})^{\bar{n}} \mu^X(ds,d\xi),$$

which is due to the fact that $Y$ is purely discontinuous, non-decreasing and $\Delta Y_s = |\Delta X_s|^2$. Furthermore, since $\nu(ds,d\xi)$ is the predictable compensator of $\mu^X$

$$\mathbb{E} \left[ Y_{t \wedge T_n} \right] = \mathbb{E} \left[ \int_0^{t \wedge T_n} \int_{\mathbb{R} \setminus \{0\}} (Y_{s-} + |\xi|^2)^{\bar{n}} - (Y_{s-})^{\bar{n}} \nu(ds,d\xi) \right]. \quad (3.7)$$

The inequalities (see [12])

$$(y+z)^p - y^p \leq 2^{p-1}(y^{p-1}z + z^p), \quad (3.8)$$

$$(y^{p-1}z \leq \varepsilon y^p + \frac{z^p}{\varepsilon^{p-1}}, \quad (3.9)$$

for $x, y, z \geq 0, \varepsilon > 0$ and $p \geq 1$, then yield

$$(Y_{s-} + |\xi|^2)^{\bar{n}} - (Y_{s-})^{\bar{n}} \leq 2^{\bar{n}-1} \left( \varepsilon Y_{s-}^{\bar{m}} |\xi|^2 + \frac{1}{\varepsilon^{\bar{m}-1}} |\xi|^2 + |\xi|^m \right),$$

as well as

$$((Y_{s-} + |\xi|^2)^{\bar{n}} - (Y_{s-})^{\bar{n}}) X_s \leq 2^{\bar{n}-1} \left( \varepsilon Y_{s-}^{\bar{m}} |\xi|^2 + \frac{1}{\varepsilon^{\bar{m}-1}} |\xi|^2 |X_s|^{\bar{m}} + |\xi|^m |X_s| \right)$$

for $X_s \geq 0$. In the first case, inequality (3.9) is applied to $(Y_{s-})^{\frac{p}{2}-1}$, i.e. $x = 1$ and in the second case to $(Y_{s-})^{\frac{p}{2}-1} X_s$. If the kernel satisfies Condition A, (3.7)

can therefore be dominated by

$$\mathbb{E} \left[ Y_{t \wedge T_n} \right] \leq 2^{\bar{n}-1} \mathbb{E} \left[ \int_0^{t \wedge T_n} \int_{\mathbb{R} \setminus \{0\}} \left( \varepsilon Y_{s-}^{\bar{m}} |\xi|^2 + \frac{1}{\varepsilon^{\bar{m}-1}} |\xi|^2 + |\xi|^m \right) \mu_{00}(d\xi) ds \right]$$

$$+ \int_0^{t \wedge T_n} \int_{\mathbb{R} \setminus \{0\}} \left( \varepsilon Y_{s-}^{\bar{m}} |\xi|^2 + \frac{1}{\varepsilon^{\bar{m}-1}} |\xi|^2 |X_s|^{\bar{m}} + |\xi|^m |X_s| \right) \mu_{10}(d\xi) ds$$

$$+ \int_0^{t \wedge T_n} \int_{\mathbb{R} \setminus \{0\}} \left( \varepsilon Y_{s-}^{\bar{m}} |\xi|^2 + \frac{1}{\varepsilon^{\bar{m}-1}} |\xi|^2 |X_s|^m + |\xi|^m |X_s|^2 \right) \mu_{11}(d\xi) ds$$

$$\leq 2^{\bar{n}-1} \varepsilon \left( \int_{\mathbb{R} \setminus \{0\}} |\xi|^2 (\mu_{00}(d\xi) + \mu_{10}(d\xi) + \mu_{11}(d\xi)) \right) \mathbb{E} \left[ n^{\frac{p}{2}} \wedge Y_{t \wedge T_n}^{\frac{p}{2}} \right]$$

$$+ 2^{\bar{n}-1} (t \wedge T_n) \int_{\mathbb{R} \setminus \{0\}} \left( \frac{1}{\varepsilon^{\bar{m}-1}} |\xi|^2 + |\xi|^m \right) \mu_{00}(d\xi)$$

$$+ 2^{\bar{n}-1} \int_0^{t \wedge T_n} \int_{\mathbb{R} \setminus \{0\}} \left( \frac{1}{\varepsilon^{\bar{m}-1}} |\xi|^2 \mathbb{E} [ |X_s|^{\bar{m}}] + |\xi|^m \mathbb{E} [ |X_s|] \right) \mu_{10}(d\xi) ds$$

$$+ 2^{\bar{n}-1} \int_0^{t \wedge T_n} \int_{\mathbb{R} \setminus \{0\}} \left( \frac{1}{\varepsilon^{\bar{m}-1}} |\xi|^2 \mathbb{E} [ |X_s|^m] + |\xi|^m \mathbb{E} [ |X_s|^2] \right) \mu_{11}(d\xi) ds.$$
Again, by (3.8) and (3.9), the integrand can be dominated by
\[\varepsilon = 2^{-\frac{m}{2}} \left( \int_{\mathbb{R} \setminus \{0\}} |\xi|^2 (\mu_{00}(d\xi) + \mu_{10}(d\xi) + \mu_{11}(d\xi)) \right)^{-1},\]
leads to
\[
\frac{1}{2} \mathbb{E} \left[ Y_{t \wedge T_n}^2 \right] \leq \mathbb{E} \left[ Y_{t \wedge T_n}^2 \right] - \frac{1}{2} \mathbb{E} \left[ \mathbb{E} \left[ X_s^m \right] \right] ds.
\]
Since \(Y_{t \wedge T_n}^2 \leq Y_{t \wedge T_{n+1}}^2\), we can apply the monotone convergence theorem to obtain
\[
\mathbb{E} \left[ Y_{t \wedge T_n}^2 \right] \leq C_m \int_0^T \left( 1 + \mathbb{E} \left[ X_s^2 \right] + \mathbb{E} \left[ X_s^m \right] \right) ds.
\]
In the case of Condition B, (3.7) becomes
\[
\mathbb{E} \left[ Y_{t \wedge T_n}^2 \right] = \mathbb{E} \left[ \int_{0}^{t \wedge T_n} \int_{\mathbb{R} \setminus \{0\}} (Y_{s-} + |H(z)X_s + h(z)|^2)^{\frac{m}{2}} - (Y_{s-}^2)^{\frac{m}{2}} \right] \mu(dz)ds.
\]
Again, by (3.8) and (3.9), the integrand can be dominated by
\[
(2^{-\frac{m}{2} - 1} (Y_{s-}^2)^{-1} |H(z)X_s + h(z)|^2 + |H(z)X_s + h(z)|^m) \leq
2^{\frac{m}{2}} (Y_{s-}^2)^{-1} |H(z)|^2 |X_s|^2 + Y_{s-}^2 |h(z)|^2 + \frac{1}{2} |H(z)X_s + h(z)|^m \leq
2^{\frac{m}{2}} (\varepsilon Y_{s-}^2 (|H(z)|^2 + |h(z)|^2) + \frac{1}{\varepsilon - 2} (|X_s|^m |H(z)|^2 + |h(z)|^2) + 2^{m-2} |X_s|^m |H(z)|^m + 2^{m-2} |h(z)|^m).
\]
Note, that the integrability assumptions on \(\mu\) guarantee
\[
\int_{\mathbb{R} \setminus \{0\}} (|H(z)|^2 + |h(z)|^2) \mu(dz) < \infty, \quad (3.10)
\]
\[
\int_{\mathbb{R} \setminus \{0\}} (|H(z)|^m + |h(z)|^m) \mu(dz) < \infty.
\]
By applying the same arguments as before and by choosing
\[
\varepsilon = 2^{-(\frac{m}{2} + 1)} \left( \int_{\mathbb{R} \setminus \{0\}} (|H(z)|^2 + |h(z)|^2) \mu(dz) \right)^{-1},
\]
we obtain
\[
\mathbb{E} \left[ Y_{t \wedge T_n}^2 \right] \leq C_m \int_0^T (1 + \mathbb{E} \left[ X_s^m \right]) ds < \infty.
\]
Combining all these estimates, we finally have for \( t \in [0, 1] \)

\[
\mathbb{E} \left[ \sup_{s \leq t} |X_s|^m \right] \leq \tilde{C}_m \left( |x|^m + \int_0^t (1 + \mathbb{E} [ |X_s|^2] + \mathbb{E} [ |X_s|^m] + \mathbb{E} [ |X_s|^m] ) \, ds \right) \leq C_m \left( |x|^m + \int_0^t (1 + \mathbb{E} [ |X_s|^m] ) \, ds \right).
\]

\[ \square \]

**Remark 3.5.** The proof of the above Lemma in the multidimensional case is completely identical. One only has to make use of the equivalence of the \( p \)-norms and the fact that \( \sum_{i,j} x_i x_j |^m \leq C \| x \|^m \).

**Theorem 3.6.** Let \( m \geq 2 \) and \( X_t^x \) be a conservative Feller semimartingale on \( S \) whose infinitesimal generator on \( C_c^\infty(S) \) is of form (3.1) with \( c = 0 \). Assume furthermore that \( \mathbb{E} |(X_t^x)|^m| < \infty \) for all \( t \in [0, 1] \) (of course \([0, 1]\) could be replaced by \([0, \varepsilon]\) for any \( \varepsilon > 0 \)). Then, \( X \) is \( m \)-polynomial if its differential characteristics \((b, a, K)\) associated with the “truncation function” \( \chi(\xi) = \xi \) are as in Lemma 3.4 with \( K \) satisfying either Condition A or B.

**Proof.** Let us denote the right side of (3.1) literally by \( A^\# \). In view of Lemma 3.1 we show that for every \( f \in \text{Pol}_{\leq m}(S) \), \( A^\# f \in \text{Pol}_{\leq m}(S) \) and that

\[ M_t^f := f(X_t^x) - f(x) - \int_0^t A^\# f(X_s) \, ds \quad (3.11) \]

is a well defined martingale for \( t \leq 1 \). Indeed, our assumptions on \( a, b \) and \( K \), imply \( A^\# (\text{Pol}_{\leq m}(S)) \subseteq \text{Pol}_{\leq m}(S) \). This is obvious for the differential operator part of \( A^\# \). For the integral part, which we denote by \( A^\#_i \), we have in the case of Condition A

\[
A^\#_i f(x) = \int_{\mathbb{R}^n \setminus \{0\}} \left( f(x + \xi) - f(x) - \sum_{k=1}^n \xi_k \frac{\partial f(x)}{\partial x_k} \right) \mu_0(d\xi) + \sum_{i \in I} \int_{\mathbb{R}^n \setminus \{0\}} \left( f(x + \xi) - f(x) - \sum_{k=1}^n \xi_k \frac{\partial f(x)}{\partial x_k} \right) x_i \mu_i(d\xi) + \sum_{(i,j) \in I} \int_{\mathbb{R}^n \setminus \{0\}} \left( f(x + \xi) - f(x) - \sum_{k=1}^n \xi_k \frac{\partial f(x)}{\partial x_k} \right) x_i x_j \mu_{ij}(d\xi),
\]

which lies in \( \text{Pol}_{\leq m}(S) \), since \( f(x + \xi) - f(x) - \sum_{k=1}^n \xi_k \frac{\partial f(x)}{\partial x_k} \) is in \( \text{Pol}_{\leq m-2}(S) \).

By the integrability conditions on the Lévy measures \( \mu_{ij} \) the integral is also well defined and the truncation function \( \chi \) can be chosen to be the identity. In the case
of Condition B, $A^\#_1$ is given by

$$A^\#_1 f(x) = \int_{\mathbb{R}^n \setminus \{0\}} \left( f(x + \xi) - f(x) - \sum_{k=1}^n \xi_k \frac{\partial f(x)}{\partial x_k} \right) K(x, d\xi) =$$

$$\int_{\mathbb{R}^n \setminus \{0\}} \left( f(x + g(x, y)) - f(x) - \sum_{k=1}^n (g(x, y)k) \frac{\partial f(x)}{\partial x_k} \right) \mu(dy) =$$

$$\int_{\mathbb{R}^n \setminus \{0\}} \left( f(x + H(y)x + h(y)) - f(x) - \sum_{k=1}^n (H(y)x + h(y))k \frac{\partial f(x)}{\partial x_k} \right) \mu(dy).$$

This is clearly in $\text{Pol}_{\leq m}(S)$ and by the assumptions on the Lévy measure $\mu$, namely

$$\int_{\mathbb{R}^n \setminus \{0\}} (\|H(y)\|^k + \|h(y)\|k) \mu(dy) < \infty, \quad \text{for all } 1 \leq k \leq m,$$

the integral is also well defined. Hence, only the martingale property of (3.11) remains to be shown. By applying Itô’s formula it follows that $M^f_t$ is a local martingale. However, since we have $A^\# f \in \text{Pol}_{\leq m}(S)$ for $f \in \text{Pol}_{\leq m}(S)$, we can dominate $\sup_{s \leq t} \|M^f_s\|$ for $t \leq 1$ with the following expression:

$$\sup_{s \leq t} \|M^f_s\| = \sup_{s \leq t} \left\| f(X^x_s) - f(x) - \int_0^s A^\# f(X^x_u) du \right\| \leq \sup_{s \leq t} \left\| f(X^x_s) \right\| + \|f(x)| + \sup_{s \leq t} \left\| \int_0^s A^\# f(X^x_u) du \right\| \leq C_1 (1 + \sup_{s \leq t} \|X^x_s\|^m) + C_2 (1 + \|x\|^m) + \int_0^t C_3 (1 + \|X^x_s\|^m) ds \leq C \left( 1 + \|x\|^m + \sup_{s \leq t} \|X^x_s\|^m \right),$$

with $C = C_1 + C_2 + C_3$ for some constants $C_1, C_2, C_3$. By Lemma 3.4 and our moment assumption on $X$, we know that $E \left[ \sup_{s \leq t} \|X^x_s\|^m \right]$ is finite for every $t \in [0, 1]$. This implies that

$$E \left[ \sup_{s \leq t} \|M^f_s\| \right] < \infty$$

for every $t \leq 1$. Thus, $M^f_t$ is a martingale for $t \in [0, 1]$ and Lemma 3.1 yields the assertion.

\[\square\]

\textbf{Remark 3.7.} It is important to note that the differential characteristics of $X$ in Theorem 3.6 are specified with respect to the “truncation function” $\chi(\xi) = \xi$. While $a$ and $K$ do not depend on the choice of $\chi$, the characteristic $b = b(\chi)$ does. So, if one chooses another truncation function $\tilde{\chi}$ instead of $\chi$, then $b(\tilde{\chi})$ transforms as follows

$$b_t(\tilde{\chi}(\xi)) = b_t(\xi) + \int_{\mathbb{R}^n \setminus \{0\}} (\tilde{\chi}(\xi) - \xi) K(X_t, d\xi).$$
Thus, the requirement that $a$ and $K$ are as in Theorem 3.6 and
\[
\left( b_t(\tilde{\chi}(\xi)) + \int_{\mathbb{R}^n \setminus \{0\}} (\xi - \tilde{\chi}(\xi)) K(X_t, d\xi) \right) \in \text{Pol}_{\leq 1}(S)
\]
is an equivalent condition guaranteeing that $X$ is $m$-polynomial.

**Remark 3.8.** Note that an operator of form (3.1) with $\chi(\xi) = \xi$ can satisfy
\[
\mathcal{A}(\text{Pol}_{\leq m}(S)) \subseteq \text{Pol}_{\leq m}(S),
\] (3.12)
even though the conditions of Theorem 3.6 are not fulfilled. In the case of $m = 2$, consider for example the one dimensional process with $b$ as in Theorem 3.6 Then, in order to satisfy (3.12), $K(\cdot, d\xi)$ can be any function as long as
\[
\left( a(x) + \int_{\mathbb{R} \setminus \{0\}} \xi^2 K(x, d\xi) \right) \in \text{Pol}_{\leq 2}(S).
\] (3.13)
If $m \geq 3$ and (3.13) is satisfied, then (3.12) implies $K(\cdot, d\xi) \in \text{Pol}_{\leq 3}(S)$.

**Remark 3.9.** As an assumption in Theorem 3.6, we required $E[\|X_t\|^m] < \infty$ for $t \in [0, 1]$. In the case of Lévy SDEs of the form
\[
X_t = x + \int_0^t \left( b + \sum_{i=1}^n X_{s,i}\beta_t \right) ds
\]
\[
+ \int_0^t \left( a + \sum_{i=1}^n X_{s,i}\alpha_0 + \sum_{i \leq j} X_{s,i}X_{s,j}\alpha_{ij} \right) dB_s
\]
\[
+ \int_0^t \int_{\mathbb{R} \setminus \{0\}} (H(y)X_s + h(y))(\mu^X(ds, dy) - \mu(dy)ds),
\]
where $B_t$ is standard Brownian motion in $\mathbb{R}^n$ and $\mu^X(ds, dy)$ a Poisson random measure with compensator $\mu(dy)ds$ satisfying (3.4), this condition is automatically fulfilled if
\[
\left( a + \sum_{i=1}^n X_{s,i}\alpha_0 + \sum_{i \leq j} X_{s,i}X_{s,j}\alpha_{ij} \right) \frac{1}{2}
\]
is Lipschitz continuous on $S$. This can be shown by proving that the map $\Lambda_x(Z)_t$
\[
\Lambda_x(Z)_t = x + \int_0^t \left( b + \sum_{i=1}^n Z_{s,i}\beta_t \right) ds
\]
\[
+ \int_0^t \left( a + \sum_{i=1}^n Z_{s,i}\alpha_0 + \sum_{i \leq j} Z_{s,i}Z_{s,j}\alpha_{ij} \right) dB_s
\]
\[
+ \int_0^t \int_{\mathbb{R} \setminus \{0\}} (H(y)Z_s + h(y))(\mu^X(ds, dy) - \mu(dy)ds),
\]
is a contraction on the Banach space $C_{pr}([0,t]; L^m(\Omega, \mathbb{R}^n))$ (see [7] for details). The proof can be based on the same arguments as those used in Lemma (3.4). A similar result is also stated in [12].
In the case of affine characteristics, the existence of the moments of the Lévy measure in the sense of (3.3) might also be sufficient for $E[\|X_t^x\|^m]$ to be finite (see Lemma 5.3 and 6.5 in [5]).

4. Examples

In order to apply Theorem 3.6 to the following examples, we assume throughout this section $m \geq 2$, i.e. the process $X$ will admit moments up to order at least 2.

Example 4.1 (Affine processes). Every conservative, regular affine process $X$ on $S = \mathbb{R}^p_+ \times \mathbb{R}^{n-p}$ is $m$-polynomial, if $E[\|X_t^x\|^m] < \infty$ for $t \in [0,1]$. For details on affine processes see [5].

Proof. On $C_c^2$, the generator of an affine process is given by

$$A u(x) = \frac{1}{2} \sum_{k,l=1}^n \left( a_{kl} + \sum_{i=1}^p x_i \alpha_{i0,kl} \right) \frac{\partial^2 u(x)}{\partial x_k \partial x_l} + \left( b + \sum_{i=1}^n x_i \beta_i, \nabla u(x) \right)$$

$$+ \int_{S \setminus \{0\}} \left( u(x + \xi) - u(x) - \langle \chi(\xi), \nabla u(x) \rangle \right) \left( \mu_{00}(d\xi) + \sum_{i=1}^p x_i \mu_{i0}(d\xi) \right),$$

where the parameters have to satisfy certain admissibility conditions (see [5]). By Theorem 3.6 and Remark 3.7 it then follows directly that $X$ is $m$-polynomial.

If a conservative affine process has constant drift and exhibits only state independent jumps, $A(\text{Pol}_{\leq m}(S)) \subseteq \text{Pol}_{\leq (m-1)}(S)$ holds true. In this case $Q(-s,x) = P^{-s}f(x)$ is a time space harmonic polynomial by Corollary 2.7

Example 4.2 (Lévy processes). Let $L$ be a Lévy process on $\mathbb{R}^n$ with triplet $(b,a,\mu)$ satisfying $\int_{\|\xi\|>1} \|\xi\|^m \mu(d\xi) < \infty$. Then, the Markov process $X_t^x = x + L_t$ is $m$-polynomial.

Proof. This can immediately be seen from Theorem 3.6 since $b$, $a$ and $\mu$ are simply constant functions.

Example 4.3 (Exponential Lévy models). Exponential Lévy models are of the form

$$X_t^x = xe^{L_t},$$

(4.1)

where $L_t$ is a Lévy process on $\mathbb{R}$ with triplet $(b,a,\mu)$. $X_t$ usually corresponds to the price process under a martingale measure. The absence of arbitrage then imposes conditions on $\mu$ and $b$ since the discounted price process $e^{-rt}X_t$ must be a martingale:

$$\int_{|y|>1} e^y \mu(dy) < \infty \text{ and } b = r - \frac{1}{2} a - \int_\mathbb{R} \left( e^y - 1 - y 1_{|y| \leq 1} \right) \mu(dy).$$

The infinitesimal generator of (4.1) is given by

$$A u(x) = \frac{ax^2}{2} \frac{d^2 u(x)}{dx^2} + rx \frac{du(x)}{dx} + \int_\mathbb{R} \left( u(xe^y) - u(x) - x(e^y - 1) \frac{du(x)}{dx} \right) \mu(dy).$$

In terms of Theorem 3.6, we are in the situation of Condition B with $g(x,y) = H(y)x = (e^y - 1)x$. Note that the integral part of $A$ is well defined for $u \in \text{Pol}_{\leq m}(\mathbb{R})$.
if
\[ \int_{|y| > 1} e^{my} \mu(dy) < \infty. \] (4.2)

Indeed, by Taylor’s theorem
\[
\left| \int_{\mathbb{R}} \left( u(xe^y) - u(x) - x(e^y - 1) \frac{du(x)}{dx} \right) \mu(dy) \right| \leq \\
\int_{|y| \leq 1} \sup_{z \leq \sup_{|y| \leq 1} (e^y - 1)x} \left| \frac{d^2 u(z)}{dz^2} \right| \mu(dy) + \\
\int_{|y| > 1} |C(x)|(1 + |e^y - 1|^m)\mu(dy),
\]
where \( C(x) \in \text{Pol}_{\leq m}(\mathbb{R}) \). Since \( |(e^y - 1)|^2 \leq \lambda |y|^2 \) for \( |y| \leq 1 \) and some constant \( \lambda \) and since \( |(e^y - 1)|^m \leq 2^{m-1}(e^m + 1) \), we obtain that (4.2) is sufficient for the existence of the integral. Condition (3.10) in the proof of Lemma 3.4 is then automatically fulfilled. Furthermore, (4.2) guarantees also the existence of \( \mathbb{E} \| (X_t^x) \|^m \) \( < \infty \), whence by Theorem 3.6 exponential models are \( m \)-polynomial if (4.2) holds true. Note that the Black-Scholes model falls into the realm of this example.

**Example 4.4** (Lévy driven SDEs). Let \( L_t \) denote a Lévy process on \( \mathbb{R}^d \) with generating triplet \( (b, a, \mu) \). Suppose furthermore that \( V_1, \ldots, V_d \) are affine functions, i.e. we have
\[ V_i : S \to \mathbb{R}^n, \quad x \mapsto H_i x + h_i, \]
where \( H_i \in \mathbb{R}^{n \times n} \) and \( h_i \in \mathbb{R}^n \). A process \( X \) which solves the stochastic differential equation of type
\[ dX_t = \sum_{i=1}^d V_i(X_{t-})dL_{t,i}, \quad X_0 = x \in S, \] (4.3)
in \( S \) and which leaves \( S \) invariant is \( m \)-polynomial if \( \int_{\|\xi\| > 1} \|\xi\|^m \mu(d\xi) < \infty \). If \( V_1, \ldots, V_d \) are linear, then we even have \( \mathcal{A}(\text{Pol}_k(S)) \subseteq \text{Pol}_k(S) \) for all \( 0 \leq k \leq m \) (in contrast to \( \mathcal{A}(\text{Pol}_{\leq k}(S)) \subseteq \text{Pol}_{\leq k}(S) \)).

**Proof.** For \( u \in C^2(S) \subseteq \mathcal{D}(\mathcal{A}) \) and general Lipschitz continuous functions \( V_1, \ldots, V_d \) the infinitesimal generator of \( X \) is given by
\[
\mathcal{A}u(x) = \frac{1}{2} \sum_{k,l=1}^n \left( (V_1(x) \ldots V_d(x))a(V_1(x) \ldots V_d(x))' \right)_{kl} \frac{\partial^2 u(x)}{\partial x_k \partial x_l} + \\
\sum_{i=1}^d V_i(x) b_i, \nabla u(x) + \\
\int \left( u \left( x + \sum_{i=1}^d V_i(x) y_i \right) - u(x) - \sum_{i=1}^d V_i(x) y_i, \nabla u(x) \right) \mu(dy).
\]
This corresponds again to the situation of Condition B with \( q(x, y) = H(y)x + h(y) = \sum_{i=1}^d H_i y_i x + h_i y_i \). Theorems 3.6 and Remark 3.9, which actually treat a slight generalization of (4.3), then yield the assertion.
If \( V_1, \ldots, V_d \) are linear, the entries of the diffusion matrix are quadratic and the
drift is linear. This compensates the decreasing degree of the second and the first derivatives respectively. In the jump part we only get terms of order \( m \).

Thus, in this case \( A \) maps \( \text{Pol}_m(S) \) into itself. \( \Box \)

**Example 4.5** (Jacobi process). Another example of a polynomial process is the Jacobi process (see [10]) which is the solution of the stochastic differential equation

\[
dX_t = -\beta(X_t - \theta)dt + \sigma \sqrt{X_t(1 - X_t)} dB_t, \quad X_0 = x \in [0, 1],
\]

on \( S = [0, 1] \), where \( \theta \in [0, 1] \) and \( \beta, \sigma > 0 \). Its generator is a self-adjoint operator satisfying the following eigenvalue equation

\[
AQ_n(x) = \lambda_n Q_n(x), \quad n \in \mathbb{N},
\]

where the eigenfunctions \( Q_n \) are the Jacobi polynomials and the eigenvalues \( \lambda_n \) are given by \(-\frac{\pi^2}{n(n-1+\frac{2i\theta}{\pi})}\).

This example can be extended by adding jumps, where the jump times correspond to those of a Poisson process \( N \) with intensity \( \lambda \) and the jump size is a function of the process level. Indeed, if a jump occurs, then the process is reflected at \( \frac{1}{2} \) so that it remains in the interval \([0, 1]\), i.e. we have

\[
dX_t = -\beta(X_t - \theta)dt + \sigma \sqrt{X_t(1 - X_t)} dB_t + (1 - 2X_t) dN_t, \quad X_0 = x \in [0, 1],
\]

whose generator is given by

\[
Au = \frac{1}{2} \sigma^2 (x(1-x)) \frac{d^2 u(x)}{dx^2} - \beta(x-\theta) \frac{du(x)}{dx} + \lambda(u(1-x) - u(x)).
\]

In terms of Condition B, we have here \( g(x,y) = -2yx + y \) and \( \mu(dy) = \lambda \delta_1(dy) \).

**Example 4.6** (Pearson diffusions). The above example 4.5 (without jumps) as well as Ornstein-Uhlenbeck and Cox-Ingersoll-Ross processes, all of them with mean-reverting drift, can be subsumed under the class of so called Pearson diffusions which are the solutions to SDEs of the form

\[
dX_t = -\beta(X_t - \theta)dt + \sqrt{\alpha_{11}X_t^2 + \alpha_{10}X_t + a} dB_t, \quad X_0 = x,
\]

where \( \beta > 0 \) and \( \alpha_{11}, \alpha_{10} \) and \( a \) are specified such as the square root is well defined. Forman and Sørensen [8] give a complete classification of the different types of the Pearson diffusion in terms of their invariant distributions.

5. Applications

By Theorem 2.3 we know that there exists a linear map \( A \) such that moments of \( m \)-polynomial processes can simply be calculated by computing \( e^{tA} \). Indeed, by choosing a basis \( \langle e_1, \ldots, e_N \rangle \) of \( \text{Pol}_{\leq m}(S) \) the matrix corresponding to this linear map which we also denote by \( A = (A_{ij})_{i,j=1,\ldots,N} \) can be obtained through

\[
Ae_i = \sum_{j=1}^N A_{ij} e_j.
\]

Writing \( f \) as \( \sum_{k=1}^N \alpha_k e_k \), we then have

\[
P_t f = e^{tA} \left( \sum_{k=1}^N \alpha_k e_k \right) = \sum_{k=1}^N \alpha_k e_k = (\alpha_1, \ldots, \alpha_N) e^{tA}(e_1, \ldots, e_N)', \quad (5.1)
\]
which means that moments of polynomial processes can be evaluated simply by computing matrix exponentials.

By means of the one-dimensional Cox-Ingersoll-Ross process
\[
dX_t = (b + \beta X_t)dt + \sigma \sqrt{X_t}dW_t, \quad b, \sigma \in \mathbb{R}_+, \quad \beta \in \mathbb{R},
\]
we exemplify how moments of order \(m\) can be calculated. The generator is given by
\[
A u(x) = \frac{1}{2} \sigma^2 x \frac{d^2 u(x)}{dx^2} + (b + \beta x) \frac{du(x)}{dx}.
\]
Applying \(A\) to \((x^0, x^1, \ldots, x^m)\) yields the following \((m+1) \times (m+1)\) matrix
\[
A = \begin{pmatrix}
0 & \cdots & b & \beta & 0 & \cdots & 0
0 & \cdots & 2b + \sigma^2 & 2\beta & 0 & \cdots & 0
0 & \cdots & 0 & 3b + 3\sigma^2 & 3\beta & 0 & \cdots & 0
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & mb + \frac{m(m-1)}{2} \sigma^2 & m\beta
\end{pmatrix}.
\]
Hence, \(E[(X_t^x)^k] = P_t x^k = (0, \ldots, 1, \ldots, 0)e^{tA}(x^0, \ldots, x^k, \ldots, x^m)\).

**Remark 5.1.** Note that \(A\) is a lower triangular matrix, whose eigenvalues are the diagonal elements. Since in this case they are all distinct, the matrix is diagonalizable. This holds true for all one-dimensional affine processes since the \((k+1)\)th diagonal element is given by
\[
k \left( \beta + \int_{\mathbb{R}_+} (\xi - \chi(\xi)) d\mu_{10}(d\xi) \right),
\]
where the integral part can only appear if the process is supported on \(\mathbb{R}_+\). Recall the notation \(\mu_{10}\) from Condition A. Of course there are many efficient algorithms to evaluate such matrix exponentials.

**Remark 5.2.** If \(n > 1\) one has to apply well-known techniques from linear algebra (such as in FEM) in order to enumerate efficiently basis of \(\text{Pol}_{m}(S)\) and to exploit sparsity properties of \(A\).

**5.1. Moment calculation - Generalized Method of Moments (GMM).** In view of this easy and fast technique of moment calculation for polynomial processes, the Generalized Method of Moments (GMM) qualifies for parameter estimation and thus, for model calibration. The implementation of a typical moment condition of the type
\[
f(X_t, \theta) = \begin{pmatrix}
X_t^{n_1} X_{t+s}^{m_1} - \mathbb{E}[X_t^{n_1} X_{t+s}^{m_1}]
\vdots
X_t^{n_q} X_{t+s}^{m_q} - \mathbb{E}[X_t^{n_q} X_{t+s}^{m_q}]
\end{pmatrix}, \quad n_i, m_i \in \mathbb{N}, \quad 1 \leq i \leq q,
\]
where \(\theta\) is the set of parameters to be estimated, is simple since
\[
\mathbb{E}[X_t^n X_{t+s}^m] = \mathbb{E}[X_t^n \mathbb{E}[X_{t+s}^m|X_t]]
\]
can also be computed easily. In the case of one-dimensional jump-diffusions, Zhou [19] already uses this method for GMM estimation.
5.2. Pricing - Variance reduction. The fact that moments of polynomial processes are analytically known also gives rise to new and efficient techniques for pricing and hedging issues. Let $X$ be an $m$-polynomial process and $G : S \to \mathbb{R}^n$ a deterministic bi-measurable map such that the price processes are given through

$$S_t = G(X_t)$$

under a martingale measure. Typically $G = \exp$, if $X$ are log-prices. We denote by $F = \phi(S_T)$ a bounded measurable European claim for some maturity $T > 0$, whose price at $t \geq 0$ is given by the risk neutral evaluation formula

$$p_t^F = \mathbb{E}[\phi(S_T) | \mathcal{F}_t] = \mathbb{E}[(\phi \circ G)(X_T) | \mathcal{F}_t].$$

Obviously, claims of the form

$$F = f \circ G^{-1}(S_T)$$

for $f \in \text{Pol}_{\leq m}(S)$ are analytically tractable, as we have

$$p_t^F = \mathbb{E}[(f \circ G^{-1})(S_T) | \mathcal{F}_t] = P_{T-t}f(G^{-1}(S_t)) = e^{(T-t)A}f(G^{-1}(S_t))$$

for $0 \leq t \leq T$, where $A$ is the previously defined linear operator on $\text{Pol}_{\leq m}(S)$. The sensitivities of the price process with respect to the factors of $X$ can then be calculated by

$$\nabla p_t^F = \nabla P_{T-t}f(G^{-1}(S_t))\nabla G^{-1}(S_t).$$

(5.3)

Although claims are in practice not of form $[5.2]$, the explicit knowledge of the price of polynomial claims can be used for variance reduction techniques based on control variates. Instead of using the estimator

$$\hat{\pi}_0^F = \frac{1}{L} \sum_{i=1}^{L} (\phi \circ G)(X_T^i)$$

in a Monte-Carlo simulation, where $X_T^1, \ldots, X_T^L$ are $L$ samples of $X_T$, we can use

$$\hat{\pi}_0^F = \frac{1}{L} \sum_{i=1}^{L} ((\phi \circ G)(X_T^i) - (f(X_T^i) - \mathbb{E}[f(X_T)])),$$

where $f \in \text{Pol}_{\leq m}(S)$ is an approximation of $\phi \circ G$ and serves as control variate. Both estimators are unbiased and the second clearly outperforms the first since $\text{Var} (\hat{\pi}_0^F) < \text{Var} (\hat{\pi}_0^F)$, where the ratio of the variances depends on the accuracy of the polynomial approximation.

It is worth mentioning that the previous pricing algorithm has also important consequences for hedging, since the greeks for "polynomial claims" $F = f(X_T)$ can be explicitly and efficiently calculated, again by matrix exponentials: the matrix exponential calculates the coefficients of the polynomial $x \mapsto E[f(X_T)]$, calculating a derivative of this polynomial is then an algebraic question. These considerations are then applied in equation $[5.3]$. Now assuming a complete market situation for the real-world claim $\phi(S_T) = \phi \circ G(X_T)$, i.e. there is a trading strategy $\eta$ such that

$$\phi(S_T) = E[\phi(S_T)] + \int_0^T \eta_t \cdot dS_t,$$
we can conclude that
\[
\phi(S_T) - f(X_T) = E[\phi(S_T)] - E[f(X_T)] + \int_0^T (\eta - \nabla p^T_t) \bullet dS_t.
\]
Therefore, if we assume that \( \phi(S_T) - f(X_T) \) has a small variance, then also the stochastic integral representing the difference of the cumulative gains and losses of the two hedging portfolios, namely the one built by the unknown strategy \( \eta \) and the one built by the known strategy \( \nabla p^T_t \), is small.

Of course, this method of variance reduction can be applied to all polynomial processes, including all kinds of examples mentioned in section 3. However, affine models for which the generalized Riccati ODEs (see [5]) cannot be explicitly solved are of particular interest. This is illustrated in the example below.

**Example 5.3.** The following affine stochastic volatility model comprising two volatility factors is a modification of a model initially proposed by Bates [2]. The price process is specified as \( S_t = S_0 e^{U_t} \) with dynamics
\[
d\left( \begin{array}{c}
x_t \\ u_t \\ v_t \\
\end{array} \right) = \left( \begin{array}{c}
\frac{r - u_t}{2} - \lambda_1 u_t f(\xi - 1)F_1(d\xi) - \lambda_2 v_t f(\xi - 1)F_2(d\xi) \\
\frac{b_1 - \beta_1 u_t + \beta_2 v_t}{2} \\
\frac{b_2 - \beta_2 u_t + \beta_1 v_t}{2} \\
\end{array} \right) dt
\]
\[
+ \left( \begin{array}{ccc}
\sigma_1 \sqrt{\mathcal{U}_t} & 0 & 0 \\
0 & \sigma_2 \sqrt{\mathcal{V}_t} & 0 \\
0 & 0 & \sigma_3 \sqrt{\mathcal{W}_t} \\
\end{array} \right) \left( \begin{array}{c}
dB_{1,t} \\
dB_{2,t} \\
dB_{3,t} \\
\end{array} \right)
\]
\[
+ dZ_{t,1} + dZ_{t,2},
\]
where for \( k = 1, 2, Z_k \) are pure jump processes in \( \mathbb{R} \times \mathbb{R}_+^2 \) with linear jump arrival intensity \( \lambda_1 u \), \( \lambda_2 v \) and trivariate jump size distribution \( F_k \). The usual way to obtain the price of European options is to solve the following Riccati equations
\[
\frac{\partial \phi(t,x,u,v)}{\partial t} = rx + b_1 \psi_1(t,x,u,v) + b_2 \psi_2(t,x,u,v),
\]
\[
\frac{\partial \psi_1(t,x,u,v)}{\partial t} = \frac{1}{2}(x^2 - x) - \beta_1 \psi_1(t,x,u,v) + \beta_2 \psi_2(t,x,u,v)
\]
\[
+ \frac{1}{2} \sigma_1^2 \psi_1^2(t,x,u,v) + \rho_1 \sigma_1 x \psi_1(t,x,u,v)
\]
\[
+ \lambda_1 \left( \int_{\mathbb{R} \times \mathbb{R}^2_+} e^{x_1 + \psi_2(t,x,u,v) \xi_2 + \psi_2(t,x,u,v) \xi_3} - 1 \right) F_1(d\xi)
\]
\[
- x \int_{\mathbb{R} \times \mathbb{R}^2_+} (e^{\xi_1} - 1) F_1(d\xi),
\]
\[
\frac{\partial \psi_2(t,x,u,v)}{\partial t} = \frac{1}{2}(x^2 - x) - \beta_2 \psi_2(t,x,u,v) + \beta_1 \psi_1(t,x,u,v)
\]
\[
+ \frac{1}{2} \sigma_2^2 \psi_2^2(t,x,u,v) + \rho_2 \sigma_2 x \psi_2(t,x,u,v)
\]
\[
+ \lambda_2 \left( \int_{\mathbb{R} \times \mathbb{R}^2_+} e^{x_1 + \psi_2(t,x,u,v) \xi_2 + \psi_2(t,x,u,v) \xi_3} - 1 \right) F_2(d\xi)
\]
\[
- x \int_{\mathbb{R} \times \mathbb{R}^2_+} (e^{\xi_1} - 1) F_2(d\xi).
\]
and to apply Fourier pricing methods as suggested in [6]. As in this case explicit solutions of the Riccati equations are not available, our approach to use an approximating polynomial as control variate in the Monte-Carlo simulation is particularly expedient. For the calculation of the matrix exponential yielding the price of the polynomial claim, we only need to apply the infinitesimal generator

\[
Af(x, u, v) = \left( r - \frac{u}{2} \right) \frac{\partial f}{\partial x} + (b_1 - \beta_{11} u + \beta_{12} v) \frac{\partial f}{\partial u} + (b_2 - \beta_{22} v + \beta_{21} u) \frac{\partial f}{\partial v} + \left( \frac{u}{2} + \frac{v}{2} \right) \frac{\partial^2 f}{\partial x^2} + \left( \frac{u}{2} \right) \frac{\partial^2 f}{\partial u^2} + \left( \frac{v}{2} \right) \frac{\partial^2 f}{\partial v^2} + \sigma_1 u \frac{\partial f}{\partial u} + \sigma_2 v \frac{\partial f}{\partial v} + \lambda_1 u \int (f(x + \xi_1, u + \xi_2, v + \xi_3) - f(x, u, v) - (e^{\xi_1} - 1) \frac{\partial f}{\partial x}) F_1(d\xi) + \lambda_2 v \int (f(x + \xi_1, u + \xi_2, v + \xi_3) - f(x, u, v) - (e^{\xi_1} - 1) \frac{\partial f}{\partial x}) F_2(d\xi)
\]

to polynomials of the form \(x^{k_1} y^{k_2} z^{k_3}\), \(k_1 + k_2 + k_3 \leq m\) in order to determine the matrix \(A\). The price of the approximating polynomial claim is then simply calculated by means of (5.1).

In order to present the positive impact of our variance reduction method graphically, we implemented the following simplified version of the above model (indeed the original Bates model [2]) with dynamics of the form

\[
d \begin{pmatrix} X_t \\ V_t \end{pmatrix} = \begin{pmatrix} \left( r - \frac{V_t}{2} - AV_t f_1(e^{\xi} - 1)F(\xi) \right) dt + \begin{pmatrix} \sqrt{V_t} \\ \frac{\rho \sqrt{V_t}}{\sqrt{1 - \rho^2 \sqrt{V_t}}} \end{pmatrix} \begin{pmatrix} dB_{t,1} \\ d\beta_{1,2} \end{pmatrix} + \begin{pmatrix} dZ_t \\ 0 \end{pmatrix} \end{pmatrix}
\]

where \(Z\) is a pure jump process in \(\mathbb{R}\) with jump intensity \(\lambda V\) and jump size distribution \(F\). Figure 5.2 illustrates the comparison between the Monte Carlo simulation for European call prices with and without variance reduction.

Our variance reductions technique can also be successfully applied to derivatives involving several assets, provided that their dynamics are described by a polynomial process. This can be done simply by approximating European payoff functions depending on several variables with multivariate polynomials.

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Figure 1. Comparison: Monte Carlo simulation for European Call prices with and without variance reduction.

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