ADDITIVE DIVISOR PROBLEM FOR MULTIPlicative FUNCTIONS

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Abstract. Let τ denote the divisor function, and f be any multiplicative function that satisfies some mild hypotheses. We establish the asymptotic formula or non-trivial upper bound for the shifted convolution sum \( \sum_{n \leq X} f(n)\tau(n-1) \). We also derive several applications to multiplicative functions in the automorphic context, including the functions \( \lambda_\pi(n) \), \( \mu(n)\lambda_\pi(n) \), and \( \lambda_\phi(n) \). Here \( \lambda_\pi(n) \) denotes the \( n \)-th Dirichlet coefficient of GL\(_m\) automorphic \( L \)-function \( L(s, \pi) \) for an automorphic irreducible cuspidal representation \( \pi \), \( \lambda_\phi(n) \) denotes the \( n \)-th Fourier coefficient of a holomorphic or Maass cusp form \( \phi \) on \( \text{SL}_2(\mathbb{Z}) \), and \( \mu(n) \) denotes the Möbius function.

We present two different arguments. The first one mainly relies on the uniform estimates for the binary additive divisor problem, while the second is based on the recent estimates of Bettin–Chandee for trilinear forms in Kloosterman fractions. In addition, the Bourgain–Kátai–Sarnak–Ziegler criterion and Linnik’s dispersion method are both employed in these two arguments.

1. Introduction

Let \( f(n) \) be a multiplicative function, and let \( \tau(n) \) be the number of divisors of the integer \( n \). We shall consider the problem to find the asymptotic behavior of the sum

\[
\sum_{n \leq X} f(n)\tau(n-1).
\]

One important example is the case of \( f(n) = \tau_k(n) \), which is the number of representations of \( n \) as a product of \( k \) factors. This corresponding problem is well known in number theory and is called the additive divisor problem. Its evaluation has a long and rich history. For \( k = 2 \), the sum was first estimated by Ingham, and the current best estimate that we know is

\[
\sum_{n \leq X} \tau(n)\tau(n-1) = XP_2(\log X) + O(X^{\frac{4}{3} + \varepsilon}),
\]

where \( P_k(x) \) is a polynomial of degree \( k \) (see [32] for more details). The case \( k = 3 \) was first settled by Hooley [15], whose method depends on the distribution of \( \tau(n) \) in arithmetic progressions and an identity for \( \tau_3(n) \). Later, Heath-Brown [16] gave an asymptotic formula with a power-saving error term. Recently, Topacogullari [44] subsequently improved their results and showed that

\[
\sum_{n \leq X} \tau_3(n)\tau(n-1) = XP_3(\log X) + O(X^{\frac{8}{3} + \varepsilon}).
\]
The first result for general $k$ was achieved by Linnik [28], who gave the asymptotic formula
\[ \sum_{n \leq X} \tau_k(n)\tau(n-1) = XP_k(\log X) + E_k(X) \]
with $E_k(X) \ll X(\log X)^{k-1}(\log \log X)^4$ by appealing to his own powerful dispersion method. This result was improved by Motohashi [31] to obtain $E_k(X) \ll X(\log \log X)^{c}(\log X)^{-1}$ for some constant $c = c(k)$. Here Motohashi applied a combination of uniform distribution of $\tau_k(n)$ in arithmetic progressions, Linnik’s dispersion method, and the Hardy–Littlewood circle method. Further, Fouvy–Tenenbaum [10], Drappeau [7] and Topacogullari [45] investigated the cancellations among the average of Kloosterman sums or Kloosterman fractions and made successive improvements on the error term $E_k(X)$.

In addition, various generalizations of the additive divisor problem were also studied extensively by many number theorists. These include some specific multiplicative functions such as $f(n) = \tau_k^m(n)$, and some classes of multiplicative functions which satisfy specific conditions. In 1973, Wolke [46] showed the following result: assume the multiplicative function $f$ satisfies
\[ |f(p^a)| \leq D_1a^{D_2} \quad \text{for all primes } p \text{ and integers } a \geq 1, \]
and
\[ \sum_{p \leq X} |f(p) - b| \ll \frac{X}{(\log X)^A} \quad \text{for all } A > 0, \]
where $b$ is a complex number and $D_1, D_2$ are positive real constants. Then one has
\[ \sum_{n \leq X} f(n)\tau(n-1) = C_fX(\log X)^b + O(X(\log X)^{Re(b-1)}(\log \log X)^c), \quad (1.1) \]
where the constants $C_f$ and $c$ depend on $f$. We also recall an interesting result [13] of Timofeev and Tulyaganov: for any non-negative multiplicative function $f(n)$ satisfying (i) $f(p^a) \leq A^\varepsilon$ for some $A > 0$, (ii) $f(n) \ll n^\varepsilon$ for any $\varepsilon > 0$, (iii) $\sum_{p \leq X} f(p) \ln p \geq \alpha X$ with some $\alpha > 0$, then one has
\[ \sum_{n \leq X} f(n)\tau(n-1) = (C_f + o(1)) \sum_{n \leq X} f(n) \log X \quad (1.2) \]
with an explicitly given constant $C_f$. What is more, Drappeau and Topacogullari [8] consider another class of multiplicative functions which satisfy (i) $f(p_1) = f(p_2)$ for any primes $p_1$ and $p_2$ with $p_1 \equiv p_2(\mod D)$, (ii) $|f(n)| \leq \tau_k(n)$ for some $k$, where $D$ is some fixed integer. They derived an asymptotic formula with a better error term
\[ \sum_{n \leq X} f(n)\tau(n-1) = 2 \sum_{\chi \text{ primitive}} \sum_{q \leq \sqrt{X}} \frac{\varphi(q)}{q^2} \sum_{n \leq X \atop (n,q)=1} f(n)\chi(n) + O\left(\frac{X}{(\log X)^A}\right), \quad (1.3) \]
where the implied constant depends only on $k, D$ and $A$. They also provided some interesting applications for $f$ being the arithmetic functions $\tau_z(n), b_\ell(n), z^{\omega(n)}$, where $z \in \mathbb{C}$, $\omega(n)$ is the number of distinct prime divisors of an integer $n$, and $\tau_z(n), b_\ell(n)$ are Dirichlet coefficients of $\zeta^z(s), \zeta^{1/[k,q]}(s)H(s)$ with some Euler product $H(s)$. Finally, it is mentioned that Fouvy and Tenenbaum [11] derives these applications from an alternative argument.

We find that the previously considered multiplicative functions $f$ should at least satisfy the condition $f(n) \ll n^\varepsilon$. It is of interest to know whether one can derive an asymptotic formula for sum $\sum_{n \leq X} f(n)\tau(n-1)$ without the restriction $f(n) \ll n^\varepsilon$. We further expect
that our result can apply to coefficients of GL\(_m\) automorphic L-functions for any fixed \(m\). Consequently, the purpose of this paper is to extend the previous results for a broader class of multiplicative functions. Let \(\mathcal{F}\) denote the class of all multiplicative functions \(f\) with the following hypotheses:

(i) The second moment of \(f\) is bounded by
\[
\sum_{n \leq X} |f(n)|^2 \ll X (\log X)^{c-1}
\]
for some constant \(c \geq 1\).

(ii) Let \(\varepsilon > 0\) denote an arbitrary small constant, and \(P\) denote the product of primes \(p\) which belong to the interval \([\exp((\log X)^{-\varepsilon/2}), \exp((\log X)^{1-\varepsilon/2})]\). Then \(f\) satisfies the following upper bound estimate condition in sieve theory
\[
\sum_{n \leq X} \sum_{(n,P)=1} |f(n)|^2 \ll \varepsilon X (\log X)^{1-\varepsilon}.
\]

(iii) The function \(f\) satisfies the Siegel–Walfisz criterion at primes, which says that for any fixed \(A > 0\),
\[
\sum_{p \leq X} f(p) - \frac{1}{\varphi(q)} \sum_{p \leq X} f(p) \ll_A \frac{X}{(\log X)^A}
\]
holds for all \((a,q) = 1\).

**Theorem 1.1.** Let \(f \in \mathcal{F}\). Then we have
\[
\sum_{n \leq X} f(n)\tau(n-1) = \sum_{n \leq X} f(n) \left( c(0,n)(\log n + 2\gamma) + 2c'(0,n) \right) + O(X(\log X)^{\frac{3}{2}+\varepsilon}),
\]
where \(c(s,n)\) is given by
\[
c(s,n) = \prod_{p|n} \left( 1 - \frac{1}{p^{s+1}} \right) \prod_{p|n} \left( 1 + \frac{1}{(p-1)p^{s+1}} \right),
\]
and the implied constant depends on \(\varepsilon\) and the implied constants in the hypotheses.

**Remark 1.1.** If there exists some constant \(k\) such that \(|f(n)| \leq \tau_k(n)\) for all integers \(n\), then the error term can be improved to \(O(X(\log X)^{\varepsilon})\). This can be seen from Remark 1.4 or Remark 6.1.

**Remark 1.2.** The main term in Theorem 1.1 can be evaluated by some standard methods in complex analysis, for instance Perron’s formula. First of all, by direct calculation, one has
\[
c(s,n) = \prod_{p|n} \left( 1 + \frac{1}{(p-1)p^{s+1}} \right) \prod_{p|n} \left( 1 - \frac{p}{(p-1)p^{s+1} + 1} \right) := h(s)g_s(n).
\]

It is worth pointing out that \(f(n)g_s(n)\) is multiplicative in variable \(n\). Consider the Dirichlet series
\[
D(w,s) = \sum_{n=1}^{\infty} f(n)g_s(n)n^{-w}
\]
for \(\text{Re}\, w > 1\) and \(\text{Re}\, s > -1\). Then we may obtain the asymptotic formula for
\[
\sum_{n \leq X} f(n)g_s(n)
\]
when \(\text{Re}\, s > -1\). Finally, the main term follows from partial summation and taking the derivative on variable \(s\). The detailed process will be seen in the applications.

Remark 1.3. This should be compared with the asymptotic formulae (1.1), (1.2) and (1.3) of Wolke, Timofeev–Tulyaganov and Drappeau–Topacogullari, respectively. It is known that the error term in the work of Drappeau–Topacogullari is \(\ll X(\log X)^{-A}\), which is stronger than our result. Note that Wolke’s result is better than ours only if \(\text{Re}\, b \leq 3/2\). Since the multiplicative functions that they consider are essentially periodic on the set of primes or close to a fixed number at primes on average, it is reasonable that these functions have better asymptotic behaviors. Moreover, our result is noticeably better than that of Timofeev–Tulyaganov.

Finally, we provide a brief overview of the proof of Theorem 1.1. We will give two different arguments, and they are both based on the generalized Bourgain–Kátai–Sarnak–Ziegler (BKSZ for short) criterion and Linnik’s dispersion method. The BKSZ criterion is actually a finite version of Vinogradov’s bilinear technique, which appeared in the works [4, 24]. It is very effective and widely applied to attack Sarnak’s disjointness conjecture. For our purpose, we have to generalize the BKSZ criterion to a greater extent, when compared with the works [5, 18] of Cafferata–Perelli–Zaccagnini and Jiang–Lü. It is worth illustrating that these two arguments have completely different starting points and key ingredients. We shall explain them below, respectively.

Our first approach is to use the idea of BKSZ at the beginning. Then the sum that we are concerned with
\[
\sum_{n \leq X} f(n)\tau(n - 1)
\]
can be rearranged as a bilinear sum
\[
\sum_{m \leq X/N} \sum_{p \sim N} f(m)f(p)\tau(pm - 1)
\]
and a sifted sum
\[
\sum_{n \leq X \atop (n, P) = 1} f(n)\tau(n - 1),
\]
where \(\exp((\log X)^{\varepsilon/2}) \ll N \ll \exp((\log X)^{1-\varepsilon/2})\). After applying the Cauchy–Schwarz inequality, the bilinear sum is generally reduced to estimate
\[
\sum_{p_1, p_2 \sim N} f(p_1)f(p_2) \sum_{m \leq X/N} \tau(p_1m - 1)\tau(p_2m - 1).
\]
Notice that the divisor function \(\tau\) is always positive. Though the innermost sum is the binary divisor problem, which has an asymptotic formula uniformly in parameters \(p_1, p_2\), we can not precisely calculate the main term of (1.4) in this way, due the application of Cauchy’s inequality. In order to overcome the positivity of \(\tau\), one may construct the approximation \(\tilde{\tau}\) of \(\tau\) such that the difference of \(\sum_{n \leq X} f(n)\tau(n - 1)\) and \(f(n)\tilde{\tau}(n - 1)\) is admissible, where \(\tilde{\tau}\) is also multiplicative. This may work if one refers to the idea in [30, Lemma 3.3].
Building on the works of Linnik [28, Chapter II(2)] and Motohashi [31], we here choose another approximation of $\tau(n - 1)$ by $\tau(n + h)$, where $h$ is any prime in $[X^{2/3}, X(\log X)^{-A}]$. The reason for our choice will be explained in Section 5. If the difference

$$
\sum_{n \leq X} f(n) \left( \tau(n - 1) - \tau(n + h) \right)
$$

(1.5)

is admissible for any prime $h \in [X^{2/3}, X(\log X)^{-A}]$, then the origin additive divisor problem is reduced to a ternary additive problem

$$
\sum_{h \leq X(\log X)^{-A}} \sum_{n \leq X} \Lambda(h) f(n) \tau(n + h),
$$

which can be solved easily by the Hardy–Litterwood circle method. Notice that the interval of $h$ is large enough. So we may study the summation over $h$

$$
\sum_{h \leq X(\log X)^{-A}} \Lambda(h) \tau(n + h),
$$

which is just the classical Titchmarsh problem with a large shifted parameter $n \in (0, X]$.

In order to prove that the sum (1.5) is admissible, we take the strategy discussed as before. We need to estimate the dispersion of difference

$$
\sum_{m \leq X/N} \left| \sum_{p \sim N} f(p) \left( \tau(pm - 1) - \tau(pm + h) \right) \right|^2
$$

(1.6)

and the sifted sum

$$
\sum_{n \leq X \atop (n,P)=1} f(n) \tau(n + \ell),
$$

(1.7)

where $\ell = -1$ or $\ell = h$ as above. The study of sum (1.6) then lies on the evaluation of four binary divisor sums

$$
\sum_{m \leq X/N} \tau(p1m + \overline{\omega}1) \tau(p2m + \overline{\omega}2),
$$

(1.8)

where $p1, p2 \sim N$, and $(\overline{\omega}1, \overline{\omega}2)$ is one of the pairs $(-1, -1), (1, h), (h, -1), (h, h)$. Moreover, the main term of (1.8) is required to be carefully handled. We hope that the contribution of four main terms can mutually cancel when summing over prime variables $p1, p2$ with the weight $f(p1)\overline{f(p2)}$. At this moment, the Siegel–Walfisz criterion in Hypothesis (iii) plays an important role. The sifted sum (1.7) is a standard sieve condition, and is estimated by Hypothesis (ii) and Lemma 3.8.

Our second approach is based on the hyperbola method, which means that $\tau(n)$ is written as a convolution $\sum_{ab=n} 1$. Thus, the problem is reduced to investigating the problem of Bombieri–Vinogradov type for $f(n)$ with a range of moduli at least $\sqrt{X}$. In fact, we can prove the following.

**Theorem 1.2.** Let $f \in \mathcal{F}$. For any $\varepsilon > 0$, we have

$$
\sum_{q \leq X^{1/3+\varepsilon}} \left| \sum_{n \leq X \atop n \equiv 1 (mod q)} f(n) - \frac{1}{\varphi(q)} \sum_{n \leq X \atop (n,q)=1} f(n) \right| \ll X(\log X)^{1/2+\varepsilon},
$$

where the implied constant depends on $\varepsilon$ and the implied constants in the hypotheses.
Remark 1.4. Let \( k \) be a fixed positive integer. Assume \( f \in \mathcal{F} \) satisfies \(|f(n)| \leq \tau_k(n)\) for all integers \( n \). By the work [9, Corollary] of Fouvry and Radziwiłł, one has

\[
\sum_{q \leq X^{\frac{17}{33} - \varepsilon}} \sum_{n \leq X \atop n \equiv 1 \pmod{q}} f(n) \bigg| \frac{1}{\varphi(q)} \sum_{n \leq X \atop (n,q)=1} f(n) \bigg| \ll \varepsilon X (\log X)^\varepsilon.
\]

Hence, the restriction \(|f(n)| \leq \tau_k(n)\) is removed at cost of the magnitude \((\log X)^\frac{1}{2}\). We also mention that Green [14] and Granville–Shao [13] considered the multiplicative functions with stronger restriction \(|f(n)| \leq 1\).

The proof of Theorem 1.2 starts with the generalized BKSZ criterion, which can decompose the sum above into two parts. One part need to estimate

\[
\sum_{q \leq X^{\frac{17}{33} - \varepsilon}} \sum_{n \leq X \atop (n,P)=1 \atop n \equiv 1 \pmod{q}} f(n) |f|.
\]

If the function \( f \) is divisor-bounded, the Titchmarsh–Brun inequality [40] can be used directly to estimate the inner sum (see Lemma 9.1 in [9]). However, it is not suitable for our situation. After exchanging the order of summations, the above double sum is transformed into the sifted sum similar to \((1.7)\), and can be treated in the same manner.

The other part can be rewritten as a combination of the sums of type

\[
\sum_{m \sim M} f(m) \left( \sum_{q \sim Q} \sum_{p \sim N} c_q f(p) - \sum_{q \sim Q} \sum_{p \in \mathcal{P}_u} c_q f(p) \frac{\varphi(q)}{\varphi(q)} \right),
\]

where \( \exp((\log X)^{\varepsilon/2}) \ll N \ll \exp((\log X)^{1-\varepsilon/2}) \), \( X^{3/4} \ll M \ll X/N \) and \( Q \ll X^{17/33 - \varepsilon} \).

After applying the Cauchy–Schwarz inequality, it reduces to bounding the dispersion

\[
\sum_{m \sim M} \psi \left( \frac{m}{M} \right) \left| \sum_{q \sim Q} \sum_{p \sim N} c_q f(p) - \sum_{q \sim Q} \sum_{p \in \mathcal{P}_u} c_q f(p) \frac{\varphi(q)}{\varphi(q)} \right|^2,
\]

where \( \psi \) is some compactly supported smooth function. We then follow the usual steps as in Linnik’s dispersion method. Opening the square makes the above sum split into three terms. The most difficult term is to estimate

\[
\sum_{q_1 \sim Q} \sum_{q_2 \sim Q} c_{q_1} c_{q_2} \sum_{p_1, p_2 \sim N} f(p_1) f(p_2) \sum_{p_1 m \equiv 1 \pmod{q_1}} \psi \left( \frac{m}{M} \right).
\]

Next, some familiar techniques and tools are used, including the Poisson summation formula, the Barban–Davenport–Halberstam theorem, Bezout’s identity and sums of Kloosterman fractions. Especially, we benefit from the key estimate of Bettin–Chandee for trilinear forms in Kloosterman fractions (see Lemma 3.5).

2. Applications

We now provide some examples, which fit into the framework of our theorem [1.1].
2.1. **Automorphic forms on** $\text{GL}_m$. Let $m \geq 2$, and let $\mathcal{A}(m)$ be the set of all cuspidal automorphic representations of $\text{GL}_m$ over $\mathbb{Q}$ with unitary central character. For each $\pi \in \mathcal{A}(m)$, the corresponding $L$-function is defined by absolutely convergent Dirichlet series as

$$L(s, \pi) = \sum_{n=1}^{\infty} \frac{\lambda_\pi(n)}{n^s}$$

for $\text{Re } s > 1$. By the Rankin–Selberg theory and the inequality $|\lambda_\pi(n)|^2 \leq \lambda_{\pi \times \tilde{\pi}}(n)$ for all positive integers $n$ (see [21, Lemma 3.1]), one has

$$\sum_{n \leq X} |\lambda_\pi(n)|^2 \leq \sum_{n \leq X} \lambda_{\pi \times \tilde{\pi}}(n) \ll_{\pi} X. \tag{2.1}$$

Furthermore, by utilizing the sieve technique, we have

$$\sum_{n \leq X} \frac{|\lambda_\pi(n)|^2}{(n, P(Y,Z)) = 1} \ll_{\pi} X \frac{\log Y}{\log Z}$$

for any $N_\pi < Y < Z \leq X^{\frac{1}{30m^2}}$, which has been shown in [19, Lemma 5.3]. Here $N_\pi$ is the arithmetic conductor of $\pi$. For any fixed $\pi \in \mathcal{A}(m)$, Hypotheses (i) and (ii) then hold with $c = 1$. We further assume that $\pi$ is self-dual and $\pi \neq \pi \otimes \chi$ for any quadratic primitive character $\chi$. We define the subset $\mathcal{A}^b(m)$ to be those $\pi \in \mathcal{A}(m)$ satisfying the above assumptions. Hypotheses (iii) directly follows from [23, Corollary 4.8].

Therefore, Theorem 1.1 can apply the multiplicative functions $\lambda_\pi(n)$ and $\mu(n)\lambda_\pi(n)$. Next, using the analytic properties of the twisted automorphic $L$-functions, one may show the quantities of main terms are admissible when compared to $O(X(\log X)^{\frac{1}{2}+\epsilon})$. However, we want to avoid calculating the main term in Theorem 1.1 for these two cases. We will provide another approach to get the following result.

**Theorem 2.1.** Fixed $\pi \in \mathcal{A}^b(m)$ with $m \geq 2$. Let $\lambda_\pi(n)$ denote the coefficients of $L(s, \pi)$. For any $\epsilon > 0$, we have

$$\sum_{n \leq X} \lambda_\pi(n)\tau(n-1) \ll X(\log X)^{\frac{1}{2}+\epsilon}$$

and

$$\sum_{n \leq X} \mu(n)\lambda_\pi(n)\tau(n-1) \ll X(\log X)^{\frac{1}{2}+\epsilon},$$

where the implied constant depends on $\pi, \epsilon$.

**Remark 2.1.** If $m \geq 5$, this result is completely new. The case of $m \leq 4$ has been treated in our previous work joint with Thorner and Wang [23] through the Bombieri–Vinogradov theorem on higher rank groups.

**Proof.** We only prove the second assertion, since it is a little more complicated. From the relation (7.1), we get

$$\sum_{n \leq X} \mu(n)\lambda_\pi(n)\tau(n-1) = \frac{(\log X)^A}{X} T_{\mu\lambda_\pi}(X) + O(X(\log X)^{\frac{1}{2}+\epsilon}), \tag{2.2}$$
where \( T_{\mu\lambda\pi}(X) \) is a ternary additive problem given by
\[
T_{\mu\lambda\pi}(X) = \sum_{h \leq X \log X^{-A}} \sum_{n \leq X} \Lambda(h) \mu(n) \lambda(n) \tau(n + h).
\]
If \( g \) is finitely supported, we define the exponential sum \( F_g : \mathbb{R}/\mathbb{Z} \to \mathbb{C} \) by the formula
\[
F_g(\alpha; X) = \sum_{n \leq X} g(n) e(n\alpha).
\]
Then we have
\[
T_{\mu\lambda\pi}(X) = \int_{0}^{1} F_{\Lambda}(\alpha; X \log X^{-A}) F_{\mu\lambda\pi}(\alpha; X) F_{\tau}(\alpha; 2X) d\alpha.
\]

In order to apply the circle method, we define the major arcs and the minor arcs as follows:
\[
\mathcal{M} = \bigcup_{1 \leq q \leq R} \bigcup_{1 \leq a \leq q \atop (a, q) = 1} \mathcal{M}(a, q)
\]
with \( \mathcal{M}(a, q) = \left[ \frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ} \right] \), and \( \mathfrak{m} \) is the complement of \( \mathcal{M} \) in \( \mathbb{T} \):
\[
\mathfrak{m} = \mathbb{T} \setminus \mathcal{M}.
\]
We take \( R = (\log X)^{10} \) and \( Q = X (\log X)^{-A - 10} \). Then \( T_{\mu\lambda\pi}(X) \) is divided into two parts as follows:
\[
T_{\mu\lambda\pi}(X) = \left( \int_{\mathcal{M}} + \int_{\mathfrak{m}} \right) F_{\Lambda}(\alpha; X \log X^{-A}) F_{\mu\lambda\pi}(\alpha; X) F_{\tau}(\alpha; 2X) d\alpha
\]
\[
:= T^{(1)}_{\mu\lambda\pi}(X) + T^{(2)}_{\mu\lambda\pi}(X).
\]
On the mirror arcs \( \mathfrak{m} \), it follows from the Cauchy–Schwarz inequality and the estimate (2.1) that
\[
T^{(2)}_{\mu\lambda\pi}(X) \ll \sup_{\alpha \in \mathfrak{m}} |F_{\Lambda}(\alpha; X \log X^{-A})| X (\log X)^{3/2}.
\]
By the classical estimate of exponential sum over primes \([17, \text{Theorem 13.6}]\), we have
\[
F_{\Lambda}(\alpha; X \log X^{-A}) \ll \left( Q^{\frac{1}{2}} X^{\frac{1}{2}} (\log X)^{-\frac{1}{2}} + R^{-\frac{1}{2}} X X (\log X)^{-A} + X^{\frac{1}{2}} \right) (\log X)^{3} \ll \frac{X}{(\log X)^{A + 2}}
\]
for any \( \alpha \in \mathfrak{m} \). Hence, the contribution of terms on the mirror arcs \( \mathfrak{m} \) satisfies
\[
T^{(2)}_{\mu\lambda\pi}(X) \ll \frac{X^2}{(\log X)^A}.
\]
(2.3)

Now we evaluate the contribution \( T^{(1)}_{\mu\lambda\pi}(X) \) of terms on the major arcs \( \mathcal{M} \). From now on we assume \( \alpha = \beta + a/q \in \mathcal{M}(a, q) \). It follows from \([22, \text{Section 5}]\) that
\[
F_{\mu\lambda\pi}(\alpha; X) \ll \left( 1 + \frac{X}{qQ} \right) \max_{1 \leq Y \leq X} \left| F_{\mu\lambda\pi}(\frac{a}{q}; Y) \right| \ll (\log X)^{A + 10} X \exp(-c \sqrt{\log X})
\]
\[
\ll X \exp \left( -\frac{c}{2} \sqrt{\log X} \right),
\]
(2.4)
where $c > 0$ is an ineffective constant due to the Siegel-type estimate of twisted automorphic $L$-functions [23, Theorem 4.1]. Similar to the treatment mirror arcs, we insert the bound (2.4) and then obtain that the contribution of terms on $\mathcal{M}$ satisfies

$$T^{(1)}_{\mu\lambda}(X) \ll X^2 \exp \left( -\frac{c}{3} \sqrt{\log X} \right).$$

Combining this with (2.3), we get $T_{\mu\lambda}(X) \ll X^2 (\log X)^{-A}$, which further yields from (2.2) that

$$\sum_{n \leq X} \mu(n)\lambda_{\pi}(n)\tau(n-1) \ll X(\log X)^{1/2+\varepsilon}.$$

This completes the proof of second assertion in Theorem 2.1.

In order to illustrate that Theorem 2.1 provides a non-trivial upper bound, we need to estimate the correct order of magnitude of

$$\sum_{n \leq X} |\lambda_{\pi}(n)|\tau(n-1).$$

We suppose that a cuspidal automorphic representation $\pi$ comes from certain symmetric power lift of $GL_2$, for simplicity. Let $\phi$ be any holomorphic cusp form of weight $k$ for $SL_2(\mathbb{Z})$, which is a normalized eigenform for the Hecke operators. Suppose that $\lambda_{\text{Sym}^r\phi}(n)$ are the Dirichlet coefficients of its $r$-th symmetric power $L$-function $L_r(s,\phi)$, where $r \geq 1$. Thanks to the Sato–Tate conjecture (which is now a theorem of Barnet-Lamb, Geraghty, Harris and Taylor [2]), it was proved by Lü [29, Lemma 3.4] and Tang and Wu [42, Theorem 1] that

$$\sum_{n \leq X} |\lambda_{\text{Sym}^r\phi}(n)| \sim c_r(\phi) \frac{X}{\log^{\delta_r} X},$$

where $\delta_r$ can be explicitly determined with $0 < \delta_r \leq 1 - 8/\pi^2 = 0.1849...$, and $c_r(\phi)$ is a positive constant depending on $\phi$ and $r$. According to the recent work [34] of Newton and Thorne, there exists a cuspidal automorphic representation $\pi$ on $GL_{r+1}$ such that

$$L(s,\pi) = L_r(s,\phi)$$

for all $r \geq 1$. Hence, we have

$$\sum_{n \leq X} |\lambda_{\pi}(n)| \gg \frac{X}{(\log X)^{1-8/\pi^2}},$$

which further yields from the result (1.2) of Timofeev and Tulyaganov that

$$\sum_{n \leq X} |\lambda_{\pi}(n)|\tau(n-1) \gg X(\log X)^{8/\pi^2}.$$

For this case, since $8/\pi^2 \geq 4/5 > 1/2$, it indicates that Theorem 2.1 does give a non-trivial saving.
2.2. Cusp forms on \( SL_2(\mathbb{Z}) \). Let \( k \) be an even positive integer, and let \( H_k \) denote the set of arithmetically normalized primitive cusp forms of weight \( k \) for \( SL_2(\mathbb{Z}) \) which are eigenfunctions of all the Hecke operators. Any \( \phi \in H_k \) has a Fourier expansion at infinity given by

\[
\phi(z) = \sum_{n=1}^{\infty} \lambda_\phi(n)n^{k-1}e(nz),
\]

where \( \lambda_\phi(1) = 1 \) and the eigenvalues \( \lambda_\phi(n) \in \mathbb{R} \). Deligne proved the Ramanujan conjecture, which asserts that

\[
|\lambda_\phi(n)| \leq \tau(n) \tag{2.5}
\]

for all \( n \geq 1 \). Similarly, let \( S_r \) be the set of arithmetically normalized Hecke–Maass cusp forms of eigenvalue \( \lambda = 1/4 + r^2 \) for \( SL_2(\mathbb{Z}) \). Then \( \phi \in S_r \) has the Fourier expansion at infinity given by

\[
\phi(z) = \sum_{n \neq 0} \lambda_\phi(n)\sqrt{y}K_{ir}(2\pi|n|y)e(nx),
\]

where \( K_{ir} \) is the \( K \)-Bessel function and \( \lambda_\phi(n) \) are eigenvalues of \( T_n \) with \( \lambda_\phi(n) = 1 \) and \( \lambda_\phi(n) \in \mathbb{R} \). Up to now, the Ramanujan conjecture on Maass cusp forms has not been proved. This is different from holomorphic cusp forms. The current best result is due to Kim and Sarnak, which states that

\[
|\lambda_\phi(n)| \leq \sqrt{\pi} \tau(n). \tag{2.6}
\]

Moreover, for any \( \phi \in H_k \cup S_r \), the eigenvalues \( \lambda_\phi(n) \) enjoy the multiplicative property

\[
\lambda_\phi(m)\lambda_\phi(n) = \sum_{d|(m,n)} \lambda_\phi\left(\frac{mn}{d^2}\right) \tag{2.7}
\]

for all integers \( m, n \geq 1 \). In particular, \( \lambda_\phi(n) \) are multiplicative. The Hecke \( L \)-function \( L(s, \phi) \) associated to \( \phi \) has the Euler product representation

\[
L(s, \phi) = \sum_{n \geq 1} \frac{\lambda_\phi(n)}{n^s} = \prod_p \left( 1 - \frac{\lambda_\phi(p)}{p^s} + \frac{1}{p^{2s}} \right)^{-1}.
\]

We rewrite the Euler product as

\[
L(s, \phi) = \prod_p \left( 1 - \frac{\alpha_\phi(p)}{p^s} \right)^{-1} \left( 1 - \frac{\beta_\phi(p)}{p^s} \right)^{-1},
\]

where \( \alpha_\phi(p), \beta_\phi(p) \) are complex numbers with \( \alpha_\phi(p) = \overline{\beta_\phi(p)} \), \( |\alpha_\phi(p)| = |\beta_\phi(p)| = 1 \). For each \( m \geq 1 \), we define the \( m \)-th symmetric power \( L \)-function by the degree \( m + 1 \) Euler product

\[
L(s, \text{sym}^m \phi) = \prod_p \prod_{0 \leq j \leq m} \left( 1 - \frac{\alpha_\phi(p)^{m-j}\beta_\phi(p)^j}{p^s} \right)^{-1}.
\]

Note that \( L(s, \text{sym}^0 \phi) = \zeta(s) \) and \( L(s, \text{sym}^1 \phi) = L(s, \phi) \). Recently, Newton and Thorne \[31\] Theorem B] proved that if \( \phi \in H_k \), then the \( m \)-th symmetric power lift \( \text{sym}^m \phi \) corresponds to a cuspidal automorphic representation of \( GL_{m+1}(\mathbb{A}) \) with trivial central character. This result implies that \( \text{sym}^m \phi \in \mathcal{A}^0(m+1) \) for all \( m \geq 1 \). If \( \phi \in S_r \), it is only known that \( \text{sym}^m \phi \in \mathcal{A}^0(m+1) \) for \( 2 \leq m \leq 4 \), due to the works \[12\,25\,26\] of Gelbert, Jacquet, Kim and Shahidi.
Another interesting example is the case \( f(n) = \lambda_\phi(n)^l \), where \( l \) is a positive integer. The problem that we are concerned with is the asymptotic behavior of the sum
\[
M_{\phi,l}(X) = \sum_{n \leq X} \lambda_\phi(n)^l \tau(n - 1).
\]
When \( l = 1 \), the spectral method or \( \delta \)-symbol method and its variants can give \( M_{\phi,l}(X) \ll X^{\frac{2}{3} + \varepsilon} \). Redmond [35,36] considered the case of \( l = 2 \). However, Redmond [37] then realized that his result for \( M_{\phi,2}(X) \) was not true, and said that he did not see how to do this so far. This is to say that any non-trivial upper bound of \( M_{\phi,l}(X) \) is not known if \( l \geq 2 \). Recently, the authors [20] investigated the levels of distributions of \( \lambda_\phi(n)^2 \) and \( \lambda_\phi(n)^3 \) in arithmetic progressions. When \( \phi \in H_k \), with the help of Brun–Titchmarsh inequality, we derived
\[
M_{\phi,2}(X) = c_\phi X \log X + O \left( X \log \log X \right)
\]
and
\[
M_{\phi,3}(X) \ll X (\log X)^{\frac{11}{13} - 1} \log \log X,
\]
where the constant \( c_\phi \) and the implied constants all depend only on \( \phi \). When \( \phi \in S_r \), with the help of Large sieve inequality, we derived
\[
M_{\phi,2}(X) = c_\phi X \log X + O \left( X \log \log X \right).
\]

In the following, we shall establish the asymptotic formulae of \( M_{\phi,l}(X) \) for more cases.

**Theorem 2.2.** Let \( \phi \in H_k \) and \( l \geq 2 \) be any fixed integer. For any \( \varepsilon > 0 \), we have
\[
M_{\phi,l}(X) = XP_l(\log X) + O(X(\log X)^\varepsilon),
\]
where \( P_l(x) \) denotes a polynomial in \( x \) of degree \((2j)!/(j!(j + 1)!))\ for even \( l = 2j \), otherwise \( P_l(x) \equiv 0 \).

**Remark 2.2.** The conclusion of Timofeev and Tulyaganov [43], with the help of the Sato–Tate distribution, can yield
\[
\sum_{n \leq X} |\lambda_\phi(n)|^l \tau(n - 1) = (d_l(\phi) + o(1))X(\log X)^\eta,
\]
where \( \eta \) can be explicitly computed by
\[
\eta = \frac{2}{\pi} \int_0^\pi |2 \cos \theta|^l (\sin \theta)^2 \, d\theta = \frac{\Gamma(l + 1)}{\Gamma(l/2 + 2)\Gamma(l/2 + 1)}.
\]
Note that this result for even \( l \) has been achieved by L"u [29]. It is obvious that our theorem 2.2 refines the result above.

**Proof.** In order to apply Theorem 1.1 we need to verify these three hypotheses. By Deligne’s bound (2.5), we have
\[
\sum_{n \leq X} |\lambda_\phi(n)|^{2l} \leq \sum_{n \leq X} \tau(n)^{2l} \ll X(\log X)^{2l - 1}
\]
which implies Hypothesis (i) holds for \( c = 4^l \). Inserting the bound (2.5) again and applying the Brun–Titchmarsh inequality [40] Theorem 1], we obtain
\[
\sum_{\substack{n \leq X \ (n,P) = 1}} |\lambda_\phi(n)|^{2l} \leq \sum_{\substack{n \leq X \ (n,P) = 1}} \tau(n)^{2l} \ll \frac{X}{\log X} \exp \left( 4^l \sum_{p \leq X} \frac{1}{p} \right).
\]
Mertens’ theorem can yield
\[
\sum_{p \leq X} \frac{1}{p} = \sum_{p \leq X} \frac{1}{p} - \sum_{p \leq \exp((\log X)^{1-\varepsilon/2})} \frac{1}{p} + \sum_{p \leq \exp((\log X)^{\varepsilon/2})} \frac{1}{p} = \varepsilon \log \log X + O(1).
\]
Inserting this into (2.8) gives
\[
\sum_{n \leq X} \left| \lambda_\phi(n) \right|^2 \ll \frac{X}{(\log X)^{1-\varepsilon}}.
\]
which further yields Hypothesis (ii). For the last hypothesis, it is obvious from (2.7) that \( \lambda_\phi(p)^l \) is a linear combination of \( \lambda_{\text{sym}^j \phi}(p) \) with \( j \geq 0 \). Hence, Hypothesis (iii) follows from the estimates of Siegel–Walfisz type for Dirichlet \( L \)-functions and twisted symmetric power \( L \)-functions (see [23, Corollary 4.8]).

With the notation as in Theorem 1.1 and Remark 1.2 it follows easily from Remark 1.1 that
\[
M_{\phi,l}(X) = h(0) \sum_{n \leq X} \lambda_\phi(n)^l g_0(n) \log n + 2 \left( \gamma h(0) + h'(0) \right) \sum_{n \leq X} \lambda_\phi(n)^l g_0(n)
\]
\[
+ 2h(0) \sum_{n \leq X} \lambda_\phi(n)^l g'_0(n) + O(X(\log X)^\varepsilon).
\]
(2.9)
Following the idea in Remark 1.2 we consider the Dirichlet series
\[
F_l(w) = \sum_{n=1}^\infty \lambda_\phi(n)^l g_s(n) n^{-w},
\]
where \( g_s(n) \) is given by the product
\[
g_s(n) = \prod_{p \mid n} \left( 1 - \frac{p}{(p-1)p^{s+1}+1} \right).
\]
In view of the relation (2.7), the series \( D_l(w, s) \) can be decomposed as
\[
D_l(w, s) = F_l(w) U_l(w, s),
\]
where
\[
F_{2j}(w) = \zeta(w)^{A_j} L \left( \text{sym}^2 \phi, w \right) \prod_{1 \leq r \leq j-1} L \left( \text{sym}^2 \phi, s \right)^{C_j(r)} \quad (l = 2j),
\]
\[
F_{2j+1}(w) = L(\phi, w)^{B_j} L \left( \text{sym}^2+1 \phi, w \right) \prod_{1 \leq r \leq j-1} L \left( \text{sym}^2+1 \phi, s \right)^{D_j(r)} \quad (l = 2j + 1),
\]
and the constants \( A_j, B_j, C_j(r), D_j(r) \) are given by
\[
A_j = \frac{(2j)!}{j!(j+1)!}, \quad B_j = 2 \frac{(2j+1)!}{j!(j+2)!},
\]
\[
C_j(r) = \frac{(2j)! (2r+1)}{(j-r)! (j+r+1)!}, \quad D_j(r) = \frac{(2j+1)! (2r+2)}{(j-r)! (j+r+2)!}.
\]
The \( L \)-function \( F_l(w) \) is of degree \( 2^l \), and for even \( l = 2j \) all coefficients of \( F_{2j}(w) \) are nonnegative. Moreover, \( U_l(s, w) \) is a double Dirichlet series absolutely convergent in \( \text{Re}(w+s) > 0 \).
Let us use \( v_l(n) \) and \( u_{l,s}(n) \) to denote the Dirichlet coefficients of \( F_l(w) \) and \( U_{l,s}(w) \), respectively. By Perron’s formula with the analytic properties of symmetric power \( L \)-functions, a standard procedure yields

\[
\sum_{n \leq X} v_l(n) = XQ_l(\log X) + O(X^{1-\delta_l}), \tag{2.10}
\]

where \( Q_l(x) \) denotes a polynomial in \( x \), and satisfies that \( Q_{2j+1} \equiv 0 \) if \( l = 2j+1 \) is odd while \( \deg Q_{2j} = (2j)!/(j!(j+1)!)-1 \) for even \( l = 2j \), and \( \delta_l \in (0,1/3) \) is a constant depending \( l \). Suppose that \( \Re s > -1/2 \), then we have the convolution

\[
\lambda_\phi(n) f_s(n) = \sum_{n=ab} v_l(a) u_{l,s}(b)
\]

and

\[
\sum_{b=1}^{\infty} |u_{l,s}(b)| b^{-\sigma} \ll 1
\]

for any \( \sigma > 1/2 \). With the help of (2.10), we infer that

\[
\sum_{n \leq X} \lambda_\phi(n) f_s(n) = \sum_{b \leq X} u_{l,s}(b) \sum_{a \leq X/b} v_l(a) = X \sum_{b=1}^{\infty} \frac{u_{l,s}(b)}{b} Q_l \left( \log \frac{X}{b} \right) + O \left( X^{1+\epsilon} \sum_{b \geq X} |u_{l,s}(b)| b^{-1} \right) + O(X^{1-\delta_l})
\]

where \( P_{l,s}(x) \) is a polynomial in \( x \) with its coefficients depending on \( s \), and satisfies that \( P_{2j+1,s}(x) \equiv 0 \) if \( l = 2j+1 \) is odd while \( \deg P_{2j,s} = (2j)!/(j!(j+1)!)-1 \) for even \( l = 2j \). Subtracting this into (2.9), Theorem 2.2 then follows. \( \square \)

**Theorem 2.3.** Let \( \phi \in S_r \). For \( l = 3, 4 \), we have

\[
M_{\phi,l}(X) = XP_l(\log X) + O(X(\log X)^{\frac{2}{3}+\epsilon}),
\]

where \( P_4(x) \) denotes a polynomial in \( x \) of degree 2, and \( P_3(x) \equiv 0 \).

**Proof.** As the proof of Theorem 2.2, we first verify three hypotheses. By the fact \( \text{sym}^m \phi \in \mathcal{A}'(m+1) \) for \( 2 \leq m \leq 4 \) and the Rankin–Selberg theory, Landau’s lemma gives

\[
\sum_{n \leq X} \lambda_\phi(n)^6 = XQ_6(\log X) + O(X^{\frac{64}{63}+\epsilon}),
\]

\[
\sum_{n \leq X} \lambda_\phi(n)^8 = XQ_8(\log X) + O(X^{\frac{255}{254}+\epsilon}),
\]

where \( Q_6, Q_8 \) are polynomials with \( \deg Q_6 = 4 \) and \( \deg Q_8 = 14 \), respectively (see [27, Remark 1.7] for example). Thus, Hypothesis (i) holds for \( \lambda_\phi(n)^l \) with \( l = 3, 4 \). The last hypothesis also holds from the corresponding argument of Theorem 2.2. Now it remains to check the second one. Since the Ramanujan conjecture is still open for Maass cusp form, the Brun–Titchmarsh inequality can not be used as in (2.8). Fortunately, this barrier can be
overcome by applying a result [15, Theorem 01] of Hall and Tenenbaum, which states that if a non-negative multiplicative function \( g(n) \) satisfies
\[
\sum_{p \leq X} g(p) \log p \ll X \quad \text{and} \quad \sum_{p \leq X} \sum_{k \geq 2} \frac{g(p^k) \log p^k}{p^k} \ll 1,
\]
then one has
\[
\sum_{n \leq X} g(n) \ll \frac{X}{\log X} \sum_{n \leq X} \frac{g(n)}{n}.
\] (2.11)
We put
\[
g(n) = \begin{cases} 
\lambda_\phi(n)^8 & \text{if } (n, P) = 1, \\
0 & \text{if } (n, P) > 1.
\end{cases}
\]
Clearly, \( g(n) \) is multiplicative and non-negative. The multiplicative relation (2.7) gives
\[
\lambda_\phi(p)^8 = -8 + 8\lambda_{\phi x \phi}(p) + 7\lambda_{\text{sym}^2 \phi x \text{sym}^2 \phi}(p) + 6\lambda_{\text{sym}^3 \phi x \text{sym}^3 \phi}(p) + \lambda_{\text{sym}^4 \phi x \text{sym}^4 \phi}(p). \] (2.12)
Using Shahidi’s non-vanishing result [39] of Rankin–Selberg \( L \)-functions at \( \Re s = 1 \), we get
\[
\sum_{p \leq X} \lambda_{\text{sym}^j \phi x \text{sym}^j \phi}(p) \log p \ll X.
\]
Thus, it follows from the identity (2.12) that
\[
\sum_{p \leq X} \lambda_\phi(p)^8 \log p \ll X.
\]
Moreover, by the bound (2.6) of Kim and Sarnak, we find
\[
\sum_{p \leq X} \sum_{k \geq 2} \frac{\lambda_\phi(p^k)^8 \log p^k}{p^k} \ll \sum_{p \leq X} \frac{(|\alpha_\phi(p)|^8 + |\beta_\phi(p)|^8)}{p^{1+\delta+\epsilon}} \ll 1,
\] (2.13)
where the last step uses the convexity of \( L(s, \text{sym}^4 \phi \times \text{sym}^4 \phi) \). We then use the result (2.11) of Hall and Tenenbaum to get
\[
\sum_{n \leq X} \lambda_\phi(n)^8 \ll \frac{X}{\log X} \sum_{n \leq X} \frac{\lambda_\phi(n)^8}{n}.
\] (2.14)
By using the multiplicative property of \( \lambda_\phi(n) \) and the estimate (2.13), the logarithmic average of \( \lambda_\phi(n)^8 \) can be controlled by
\[
\sum_{\substack{n \leq X \\
(n, P) = 1}} \frac{\lambda_\phi(n)^8}{n} \ll \prod_{\substack{p \leq X \\
p \nmid P}} \left( 1 + \frac{\lambda_\phi(p)^8}{p} + \sum_{k \geq 2} \frac{\lambda_\phi(p^k)^8}{p^k} \right)
\ll \exp \left( \sum_{\substack{p \leq X \\
p \nmid P}} \left( \frac{\lambda_\phi(p)^8}{p} + \sum_{k \geq 2} \frac{\lambda_\phi(p^k)^8}{p^k} \right) \right)
\ll \exp \left( \sum_{\substack{p \leq X \\
p \nmid P}} \frac{\lambda_\phi(p)^8}{p} \right) \] (2.15).
By \([38]\) Proposition 2.3 of Rudnick and Sarnak, and summation by parts, we obtain
\[
\sum_{p \leq X} \frac{\lambda_{\text{sym}} \phi \times \text{sym} \phi(p)}{p} = \log \log X + O(1),
\]
(2.16)
where \(1 \leq j \leq 4\), and we use the fact that Hypothesis H of Rudnick and Sarnak holds for \(\text{sym} \phi\) with \(1 \leq j \leq 4\). By (2.12) and (2.16), we then derive
\[
\sum_{p \leq X} \frac{\lambda_{\phi}(p)^8}{p} = \sum_{p \leq X} \frac{\lambda_{\phi}(p)^8}{p} - \sum_{p \leq \exp((\log X)^{1-\epsilon/2})} \frac{\lambda_{\phi}(p)^8}{p} + \sum_{p \leq \exp((\log X)^{\epsilon/2})} \frac{\lambda_{\phi}(p)^8}{p} = 14\epsilon \log \log X + O(1).
\]
Combining this, (2.15) with (2.14), we have
\[
\sum_{n \leq X, (n,P)=1} \lambda_{\phi}(n)^8 \ll X \left(\frac{\log X}{1-14\epsilon}\right)^2.
\]
By the Hölder inequality and Mertens’ theorem, we also have
\[
\sum_{n \leq X, (n,P)=1} \lambda_{\phi}(n)^6 \ll \left(\sum_{n \leq X, (n,P)=1} \lambda_{\phi}(n)^8\right)^{\frac{2}{3}} \left(\sum_{n \leq X, (n,P)=1} 1\right)^{\frac{1}{3}} \ll \frac{X}{(\log X)^{1-11\epsilon}}.
\]
These complete the verification of Hypothesis (ii) for \(\lambda_{\phi}(n)^l\) with \(l = 3, 4\).

Now we can employ Theorem 1.1 and then get
\[
M_{\phi,l}(X) = h(0) \sum_{n \leq X} \lambda_{\phi}(n)^l g_0(n) \log n + 2(\gamma h(0) + h'(0)) \sum_{n \leq X} \lambda_{\phi}(n)^l g_0(n)
+ 2h(0) \sum_{n \leq X} \lambda_{\phi}(n)^l g_0(n) + O(X(\log X)^{\frac{2}{3}+\epsilon}),
\]
where \(l = 3, 4\). The main term can be computed as in the proof of Theorem 2.2, so we omit the details here.

\[\square\]

3. Some lemmas

3.1. Classical lemmas. In this section, we will state some results from the literature we need. The first lemma provides an upper bound on short sums of the divisor function in arithmetic progressions.

Lemma 3.1. Let \(b\) be arbitrary positive integer and let \(q, l\) be integers. Suppose that \(Y \leq X, q \leq Y X^{-\epsilon}\). We have
\[
\sum_{X-Y < n \leq X, n \equiv l (\text{mod} q)} \tau(n)^b \ll \frac{\tau((l,q))^b X}{q} \left(\frac{\varphi(q)}{q} \log X\right)^{2^b-1}
\]
uniformly in \(l, q\), where the implied constant depends only on \(\epsilon\).

Proof. This lemma immediately follows from Shiu’s classical estimate \([40]\) Theorem 1] for multiplicative functions and Mertens’ theorem. \[\square\]
The next lemma gives an asymptotic formula for divisor function in arithmetic progression with an explicit main term.

**Lemma 3.2.** Denote $D(x; q, l)$ by

$$D(X; q, l) = \sum_{n \leq X \atop n \equiv l (\text{mod } q)} \tau(n).$$

Then we have

$$D(X; q, l) = Xq^{-1} \sum_{h \mid q} c_h(l)h^{-1}(\log X + 2 \gamma - 1 - 2 \log h) + O\left(\frac{X^{1+\varepsilon}}{\log X}\right)$$

provided $q \ll X^{\frac{2}{3} - \varepsilon}$, where $c_h(l)$ is the Ramanujan sum given by

$$c_h(l) = \sum_{h \mid (p, l)} d \mu \left(\frac{q}{d}\right).$$

**Proof.** This is a well known result when $(l, q) = 1$. Pongsriiam and Vaughan [33, Theorem 1.1] generalized it to the case of $(l, q) > 1$. □

We cite below a result of Motohashi [31, Lemma 8] for sums of Ramanujan sums.

**Lemma 3.3.** Let $m$ be an integer whose prime factors are all larger than $M$, and let $a$ be an arbitrary non-zero integer. We put

$$Y(a) = \sum_{h=1}^{\infty} |c_h(a)| h^{-2}, \quad Y^{(j)}(X; a, m) = \sum_{h \leq X \atop (h, m)=1} c_h(a)h^{-2}(\log h)^j.$$

Then we have

$$Y(a) = O\left(\sigma^{-1}_1(a)\right)$$

$$Y^{(j)}(x; a, m) = \sum_{r=0}^{j} \sigma^{(r)}_{-1}(a)\alpha_r(j) + O\left\{ \left(\frac{d(m)}{M} + \frac{d(a)}{X}\right)(\log aXM)^{j+1} \right\},$$

where the constant $\alpha_r(j)$ depends only on $r, j$, and the function $\sigma^{(r)}_{-1}(a)$ is defined by $\sigma^{(r)}_{-1}(a) = \sum_{d \mid a} (\log d)^{r}d^{-1}$.

Our next lemma is a truncated version of the Poisson summation formula in arithmetic progressions.

**Lemma 3.4.** Let $\psi : \mathbb{R} \longrightarrow [0, 1]$ be smooth and compactly supported in $[1/2, 5/2]$ such that $\psi(t) = 1$ for $1 \leq t \leq 2$. Then one has the equality

$$\sum_{m \equiv a (\text{mod } q)} \psi\left(\frac{m}{M}\right) = \mathring{\psi}(0)\frac{M}{q} + \frac{M}{q} \sum_{0 < |h| \leq H} \frac{1}{q} e\left(\frac{ah}{q}\right) \hat{\psi}\left(\frac{h}{q/M}\right) + O(M^{-1})$$

for any $H \geq (q/M) \log^4 2M$. Moreover, one has the equality

$$\sum_{(m, q) = 1} \psi\left(\frac{m}{M}\right) = \frac{\varphi(q)}{q} \psi(0)M + O(\tau(q)(\log 2M)^4),$$

where $\varphi(q)$ is the Euler totient function.
where $\hat{\psi}$ is the Fourier transform of $\psi$ defined by

$$\hat{\psi}(\xi) = \int_{-\infty}^{\infty} \psi(t)e(-\xi t) \, dt.$$  

**Proof.** See [9, Lemma 2.1]. \hfill \Box

The next lemma, due to Bettin and Chandee [3], provides a non-trivial bound for trilinear form with Kloosterman fractions.

**Lemma 3.5.** Let $\alpha = \{\alpha_m\}$, $\beta = \{\beta_n\}$, and $\nu = \{\nu_a\}$ be three sequences of complex numbers. For any non-zero integer $\vartheta$ and any $\varepsilon > 0$, we have,

$$\left| \sum_{a \sim A} \sum_{m \sim M} \sum_{n \sim N} \alpha(m)\beta(n)\nu(a)e\left(\vartheta \frac{am}{n}\right) \right| \ll \left( \sum_{a \sim A} |\nu(a)|^2 \right)^{\frac{1}{2}} \left( \sum_{m \sim M} |\alpha(m)|^2 \right)^{\frac{1}{2}} \left( \sum_{n \sim N} |\beta(n)|^2 \right)^{\frac{1}{2}} \times \left( 1 + \frac{\vartheta |A|}{MN} \right)^{\frac{1}{2}} \left( (AMN)^{\frac{2}{7} + \varepsilon} (M + N)^{\frac{1}{4}} + (AMN)^{\frac{3}{8} + \varepsilon} (AN + AM)^{\frac{1}{8}} \right).$$

**Proof.** See [3, Theorem 1]. \hfill \Box

The next lemma is related to the distribution of primes in arithmetic progressions. The key point here is that the moduli can be chosen larger than $\sqrt{X}$ and in particular that $a$ can be chosen even a little larger than $X$.

**Lemma 3.6.** Let $X \geq 2$ and $a \in \mathbb{Z}\setminus\{0\}$. There exists some positive constant $\delta$ such that

$$\sum_{q \leq x^{1/2+\delta} \atop (q,a) = 1} \left( \sum_{n \leq X \atop n \equiv a \mod q} \Lambda(n) - \frac{1}{\varphi(q)} \sum_{n \leq X} \Lambda(n) \right) \ll_A \frac{X}{(\log X)^A}$$

holds uniformly in $|a| \leq X^{1+\delta}$, where $A > 0$ is an arbitrary constant.

**Proof.** See [9, Theorem 2.1]. \hfill \Box

The next lemma is a simple estimate and is useful to deal with the main terms in our theorems.

**Lemma 3.7.** For $X \geq 2$ and $a \in \mathbb{Z}\setminus\{0\}$, we have

$$\sum_{m \leq x \atop (m,a) = 1} \frac{1}{\varphi(m)} = c(0,a)(\log x + \gamma) + c'(0,a) + O\left( \tau(a)x^{-1}\log x \right)$$

where $c(s,a)$ is given by $c(s,a) = \prod_{p|a} \left( 1 - \frac{1}{p^{s+1}} \right) \prod_{p \nmid a} \left( 1 + \frac{1}{(p-1)p^{s+1}} \right)$.

**Proof.** See [1, Lemma 5.1]. \hfill \Box

### 3.2. Sieve condition.

We denote by $P(Y,Z)$ the product of these primes $p$ which belong to the interval $[Y,Z)$, where $2 \leq Y < Z$. In this section, we shall seek an upper estimate for the sifted sum

$$\sum_{n \leq X \atop (n-a,P(Y,Z)) = 1} \tau(n)^2 \quad (3.3)$$

with $0 < |a| \leq X$ and $(a,P(Y,Z)) = 1$. It is clear that $\tau(n)^2 \leq \tau_4(n)$. Thus, it suffices to estimate the corresponding sum of $(3.3)$, in which $\tau_4(n)$ is instead of $\tau(n)^2$. For this purpose,
we require the distribution for \( \tau_4(n) \) in arithmetic progressions. Recall a result given in [41], which states that

\[
\sum_{n \leq X \atop n \equiv a (\text{mod } q)} \tau_4(n) = \frac{1}{\varphi(q)}XP_4(\log X) + O\left(X^{3+\varepsilon}\right) \tag{3.4}
\]

holds for any \((a, q) = 1\) and \(q \leq X^{\frac{2}{3}}\), where \(P_4(\log X)\) is a polynomial in \(\log X\) of degree 3 with real coefficients that depend on \(q\). More precisely, the polynomial \(P_4(\log X)\) is defined by

\[
P_4(\log X) = \sum_{j=0}^{3} \frac{1}{j!} B_{3-j}(\log X)\psi_q^{(j)}(1)
\]

with \(\psi_q(s) = \left(\sum_{k|q} k^{-s} \mu(k)\right)^4\).

Assume that \(2 \leq Y < Z \leq X^{\frac{2}{3}}\). We first remove the sieve condition \((n - a, P(Y, Z)) = 1\) by the Legendre formula

\[
\sum_{n \leq X \atop (n-a, P(Y, Z)) = 1} \tau_4(n) = \sum_{q|P(Y, Z)} \mu(q) \sum_{n \leq X \atop n \equiv a (\text{mod } q)} \tau_4(n).
\]

We keep the terms for \(q | P(Y, Z)\) with \(q \leq X^{\frac{1}{3}}\) and estimate the remaining ones. Applying Rankin’s trick and the trivial bound \(\tau_4(n) \ll n^\varepsilon\), we obtain

\[
\sum_{q|P(Y, Z) \atop q > X^{1/3}} |\mu(q)| \sum_{n \leq X \atop n \equiv a (\text{mod } q)} \tau_4(n) \ll X^{1+\varepsilon} \sum_{q|P(Y, Z) \atop q > X^{1/3}} |\mu(q)| q \ll X^{1-\varepsilon} \sum_{q|P(Y, Z) \atop q > X^{1/3}} |\mu(q)| q^{1-6\varepsilon} \ll X^{1-\varepsilon} \prod_{Y \leq p \leq Z} \left(1 + \frac{1}{p^{1-6\varepsilon}}\right) \ll X^{1-\frac{2\varepsilon}{3}}.
\]

To estimate the terms with \(q \leq X^{1/3}\), we use the asymptotic formula (3.4) and get

\[
\sum_{q|P(Y, Z) \atop q \leq X^{1/3}} \mu(q) \sum_{n \leq X \atop n \equiv a (\text{mod } q)} \tau_4(n) = X \sum_{j=0}^{3} \frac{1}{j!} B_{3-j}(\log X) \sum_{q|P(Y, Z) \atop q \leq X^{1/3}} \frac{\mu(q)\psi_q^{(j)}(1)}{\varphi(q)} + O\left(X^{1-\varepsilon}\right) \tag{3.5}
\]

After differentiating \(\psi_q(s)\), we find that \(\psi_q^{(j)}(1)/\varphi(q) \ll (\log \log q)^j/q\). Then applying Rankin’s trick again, the truncation \(q \leq X^{1/3}\) in the right-hand side of (3.5) can be removed up to error term \(X^{1-\frac{2\varepsilon}{3}}\). In addition, an elementary calculation arrives at

\[
\sum_{q|P(Y, Z)} \frac{\mu(q)\psi_q^{(j)}(1)}{\varphi(q)} \overset{\text{as } s=1}{=} \left(\sum_{q|P(Y, Z)} \frac{\mu(q)\psi_q(s)}{\varphi(q)}\right)^{(j)} \bigg|_{s=1} = \left(\prod_{Y \leq p \leq Z} \left(1 - \frac{\psi_p(s)}{p-1}\right)\right)^{(j)} \bigg|_{s=1} \ll \frac{\log^Y}{\log Z}.
\]

Combining all the above estimates, we can derive the following lemma.
Lemma 3.8. Suppose that $0 < Y < Z \leq X^{1/3}$. Let $P(Y,Z)$ denote the product of primes $p$ which belong to the interval $[Y, Z)$. Then we have

$$\sum_{n \leq X, (n - a, P(Y,Z)) = 1} \tau(n)^2 \ll X (\log X)^3 \frac{\log Y}{\log Z}$$

for any non-zero $|a| \leq X$ and $(a, P(Y,Z)) = 1$.

4. Reduction of Theorem 1.1

For convenience, we introduce a new notation

$$S_f(X, \ell) = \sum_{n \leq X} f(n) \tau(n + \ell),$$

where $\ell$ is a non-zero integer. First, we make a heuristic observation to illustrate the idea of Motohashi [31]. By Dirichlet’s hyperbola method, we may write

$$S_f(X, -1) \sim 2 \sum_{d \leq \sqrt{X}} \sum_{n \leq X} f(n). \tag{4.1}$$

From now on we assume that $h$ is arbitrary prime number in the interval $[X^{2/3}, X(\log X)^{-A}]$, where $A$ is any sufficiently large positive constant. Let us consider $S_f(X, h)$. We may also write

$$S_f(X, h) \sim 2 \sum_{d \leq \sqrt{X}} \sum_{n \leq X} f(n). \tag{4.2}$$

The inner sums of (4.1) and (4.2) can be put into the character sum

$$\varphi(d)^{-1} \sum_{n \leq X} f(n) + \varphi(d)^{-1} \sum_{\chi \pmod{d}} \bar{\chi}(-\ell) \sum_{n \leq X} f(n) \chi(n), \tag{4.3}$$

where $\ell = -1$ or $\ell = h$ as above. Note that $(\ell, d) = 1$ by the choice of $h$. Thus, they may well be approximately equal to the first term of (4.3). In other words, one may expect that $S_f(X, -1)$ differs little from $S_f(X, h)$ for each prime $h \in [X^{2/3}, X(\log X)^{-A}]$. If so, then $S_f(X, -1)$ will be approximately equal to

$$\pi(X(\log X)^{-A})^{-1} \sum_{X^{2/3} \leq h \leq X(\log X)^{-A}} S_f(X, h),$$

where $\pi(X(\log X)^{-A})$ denotes the number of primes less than $X(\log X)^{-A}$. This sum obviously belongs to the category of ternary problems, which can be addressed by the Hardy–Littlewood circle method.

In fact, after overcoming certain technical difficulties, it will turn out that $|S_f(X, -1) - S_f(X, h)|$ is relatively small, i.e. less than $X(\log X)^{1/2 + \varepsilon}$ uniformly for any prime $h \in [X^{2/3}, X(\log X)^{-A}]$. 
Now we begin to provide the rigorous procedure. Let $\varepsilon$ be any sufficiently small positive constant. Let $\Delta \in [(\log X)^{-A}, 2(\log X)^{-A}]$ be a real number such that the number

$$L := \frac{(\log X)^{1-\frac{\varepsilon}{2}} - (\log X)^{\frac{\varepsilon}{2}}}{\log(1 + \Delta)}$$

is an integer. It is clear that $L \ll (\log X)^{A+1}$. We then can partition the interval $[\exp((\log X)^{\varepsilon/2}), \exp((\log X)^{1-\varepsilon/2})]$ into at most $L$ intervals

$$I_\nu := [\exp((\log x)^{\varepsilon/2})(1 + \Delta)^{\nu}, \exp((\log x)^{\varepsilon/2})(1 + \Delta)^{\nu+1})] := [H_\nu, H_{\nu+1}]$$

with $0 \leq \nu \leq L$. Moreover, we set

$$P_\nu = \prod_{H_0 \leq p < H_{\nu+1}} p, \quad P_\nu = \{p \in I_\nu\}, \quad M_\nu = \left\{m \in [1, \frac{X}{H_{\nu+1}}] : (m, P_\nu) = 1\right\},$$

$$P_\nu M_\nu = \{pm : p \in P_\nu, m \in M_\nu\},$$

$$\mathcal{I} = \bigcup_{0 \leq \nu \leq L} P_\nu M_\nu \quad \text{and} \quad \mathcal{J} = [1, X] \setminus \mathcal{I}. \quad (4.4)$$

The intervals above are always meant as subsets of $\mathbb{N}$. Notice that each $n \in P_\nu M_\nu$ can be written in a unique way as $n = pm$ with $p \in P_\nu$ and $m \in M_\nu$. Thus, we get $|P_\nu M_\nu| = |P_\nu||M_\nu|$ and $P_\nu M_\nu \subset [1, X]$. Moreover, the sets $P_\nu M_\nu$ are pairwise disjoint for $0 \leq \nu \leq L$.

We are first concerned with the behavior of $S_f(X, \ell)$. By the decomposition above of $[1, X]$, the sum $S_f(X, \ell)$ is divided into two parts as follows

$$S_f(X, \ell) = \sum_{n \in \mathcal{I}} f(n) \tau(n + \ell) + \sum_{n \in \mathcal{J}} f(n) \tau(n + \ell)$$

$$:= S_f^\mathcal{I}(X, \ell) + S_f^\mathcal{J}(X, \ell).$$

The aim is to turn out that $|S_f(X, -1) - S_f(X, h)|$ is relatively small for any prime $h \in [X^{2/3}, X(\log X)^{-A}]$. So we shall estimate the difference of these two sums over $\mathcal{I}$ and $\mathcal{J}$, respectively.

5. Evaluation of $S_f^\mathcal{J}(X, -1) - S_f^\mathcal{J}(X, h)$

Since $P_\nu M_\nu$ are pairwise disjoint for $0 \leq \nu \leq L$, we obtain that

$$S_f^\mathcal{J}(X, -1) - S_f^\mathcal{J}(X, h) \ll \sum_{0 \leq \nu \leq L} \left| \sum_{pm \in P_\nu M_\nu} f(pm)(\tau(pm - 1) - \tau(pm + h)) \right|. \quad (5.1)$$

For $pm \in P_\nu M_\nu$, we have $p \in P_\nu$ and $m \in M_\nu$. The $f(pm)$ in (5.1) can be factored as $f(p)f(m)$ by its multiplicativity. Thus, we get

$$S_f^\mathcal{J}(X, -1) - S_f^\mathcal{J}(X, h) \ll \sum_{0 \leq \nu \leq L} \sum_{m \in M_\nu} |f(m)| \left| \sum_{p \in P_\nu} f(p)(\tau(pm - 1) - \tau(pm + h)) \right|. $$
Using the Cauchy–Schwarz inequality and exchanging the order of summations, the inner sum can be performed as follows

\[
\sum_{m \in M_{\nu}} |f(m)| \sum_{p \in P_{\nu}} f(p)(\tau(pm - 1) - \tau(pm + h)) \leq \left( \sum_{m \in M_{\nu}} |f(m)|^2 \right)^{\frac{1}{2}} \left( \sum_{p \in P_{\nu}} \left| \sum_{m \leq X/H_{\nu+1}} f(p)(\tau(pm - 1) - \tau(pm + h)) \right|^2 \right)^{\frac{1}{2}}
\]

\[
\leq \left( \sum_{m \in M_{\nu}} |f(m)|^2 \right)^{\frac{1}{2}} \left( \sum_{m \leq X/H_{\nu+1}} \left| \sum_{p \in P_{\nu}} f(p)(\tau(pm - 1) - \tau(pm + h)) \right|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{m \in M_{\nu}} |f(m)|^2 \right)^{\frac{1}{2}} \left( \sum_{p_1, p_2 \in P_{\nu}} f(p_1)\overline{f(p_2)}(V_1 - V_2 - V_3 + V_4) \right)^\frac{1}{2},
\]

where the four terms \(V_j (j = 1, 2, 3, 4)\) are given by

\[
V_1 = \sum_{m \leq X/H_{\nu+1}} \tau(p_1 m - 1)\tau(p_2 m - 1), \quad V_2 = \sum_{m \leq X/H_{\nu+1}} \tau(p_1 m - 1)\tau(p_2 m + h),
\]

\[
V_3 = \sum_{m \leq X/H_{\nu+1}} \tau(p_1 m + h)\tau(p_2 m - 1), \quad V_4 = \sum_{m \leq X/H_{\nu+1}} \tau(p_1 m + h)\tau(p_2 m + h).
\]

The diagonal contribution in the last line of (5.2), that is \(p_1 = p_2\) for each \(\nu\) yields at most

\[
\left( \sum_{m \in M_{\nu}} |f(m)|^2 \right)^{\frac{1}{2}} \left( \sum_{p \in P_{\nu}} |f(p)|^2 \sum_{n \leq X} |\tau(n)|^2 \right)^{\frac{1}{2}} \leq \frac{X^{\frac{1}{2}}(\log X)^2}{H_{\nu+1}^{\frac{1}{2}}} \left( \sum_{p \in P_{\nu}, m \in M_{\nu}} |f(pm)|^2 \right)^{\frac{1}{2}},
\]

by Lemma 3.1 and the multiplicative property of \(f(n)\). Hence, by summing over \(\nu\) and the Cauchy–Schwarz inequality, the diagonal contribution to \(S_f^T(X, -1) - S_f^T(X, h)\) is less than

\[
X^{\frac{1}{2}}(\log X)^2 \left( \sum_{0 \leq \nu \leq L} H_{\nu}^{-1} \right)^{\frac{1}{2}} \left( \sum_{0 \leq \nu \leq L} \sum_{p \in P_{\nu}, m \in M_{\nu}} |f(pm)|^2 \right)^{\frac{1}{2}} \leq X^{\frac{1}{2}} \exp\left(- (\log x)^{\frac{1}{2}}\right) \left( \sum_{n \leq X} |f(n)|^2 \right)^{\frac{1}{2}} \leq X \exp\left(- (\log x)^{\frac{1}{2}}\right) \leq X \exp\left(- (\log x)^{\frac{1}{2}}\right).
\]

Now we turn our attention to the off-diagonal terms. For \(p_1 \neq p_2\), the sums \(V_i\) for \(1 \leq i \leq 4\) are shifted convolution sums associated to the divisor function. Therefore, we have to estimate

\[
V(\varpi_1, \varpi_2) = \sum_{m \leq X/H_{\nu+1}} \tau(p_1 m + \varpi_1)\tau(p_2 m + \varpi_2),
\]
where \( p_1, p_2 \in \mathcal{P}_\nu \) with \( p_1 \neq p_2 \), and \((\varpi_1, \varpi_2)\) is one of the pairs \((-1, -1), (-1, h), (h, -1), (h, h)\).

Putting \( X_1 = p_1X/H_{\nu+1} - \varpi_1 \) and using the notation of Lemma 3.2, we have

\[
V(\varpi_1, \varpi_2) = \sum_{d \leq \sqrt{X_1} \atop (d, p_1) = 1} \left( 2D(p_2X/H_{\nu+1} - \varpi_2; dp_2, \varpi_2 - \varpi_1p_1p_2) 
- D(d\sqrt{X_1}/p_2/p_1 + (\varpi_2p_1 - \varpi_1p_2)/p_1; dp_2, \varpi_2 - \varpi_1p_1p_2) \right)
:= 2V^{(1)}(\varpi_1, \varpi_2) - V^{(2)}(\varpi_1, \varpi_2),
\]

where \( p\bar{p} \equiv 1(\mod d) \). By Lemma 3.1, one has

\[
D(p_2X/H_{\nu+1} - \varpi_2; dp_2, \varpi_2 - \varpi_1p_1p_2) = D(X; dp_2, \varpi_2 - \varpi_1p_1p_2) + O\left( \frac{\tau(d)X}{dp_2(\log X)^{A-2}} \right).
\]

Moreover, the terms corresponding to \( d \) such that \( p_2|d \) in \( V^{(1)}(\varpi_1, \varpi_2) \) contribute by the amount \( O(X(\log X)^4/H_{\nu}^2) \), in which Lemma 3.1 is used again. Noticing that \( X_1 = X(1 + O((\log X)^{-4})) \), we further have

\[
V^{(1)}(N; v_1, v_2) = \sum_{d \leq \sqrt{X} \atop (d, p_1p_2) = 1} D(X; dp_2, \varpi_2 - \varpi_1p_1p_2) + O\left( \frac{X}{H_{\nu}(\log X)^{A-4}} \right).
\]

Before appealing to Lemma 3.2, on the condition \((d, p_1p_2) = 1\), we have

\[
c_r(\varpi_2 - \varpi_1p_1p_2) = \begin{cases} 
  c_r(\varpi_2p_1 - \varpi_1p_2) & \text{if } r \mid d, \\
  -c_r(\varpi_2p_1 - \varpi_1p_2) & \text{if } r = p_2r_1, r_1 \mid d.
\end{cases}
\]

Hence, we get from Lemma 3.2 that

\[
D(X; dp_2, \varpi_2 - \varpi_1p_1p_2) = \frac{X}{dp_2} \sum_{r \mid d} \frac{c_r(\varpi_2p_1 - \varpi_1p_2)}{r} (\log X + 2\gamma - 1 - 2\log r)
+ O\left( \frac{X \log X}{dH_{\nu}^2} \sum_{r \mid d} \frac{|c_r(\varpi_2p_1 - \varpi_1p_2)|}{r} \right) + O\left( X^{\frac{1}{2} + \epsilon} \right).
\]

Then, after some elementary computations, we see that, using the notation of Lemma 3.3

\[
V^{(1)}(\varpi_1, \varpi_2) = \frac{X}{2p_2} (\log X + 2\gamma - 1)(\log X + 2\gamma) Y^{(0)}(\sqrt{X}; \varpi_2p_1 - \varpi_1p_2, p_1p_2)
- \frac{X}{p_2} (\log X + \log X + 4\gamma - 1) Y^{(1)}(\sqrt{X}; \varpi_2p_1 - \varpi_1p_2, p_1p_2)
+ \frac{2X}{p_2} Y^{(2)}(\sqrt{X}; \varpi_2p_1 - \varpi_1p_2, p_1p_2)
+ O\left( \frac{X \log X}{H_{\nu}} Y(\varpi_2p_1 - \varpi_1p_2) \right) + O\left( \frac{X}{H_{\nu}(\log X)^{A-4}} \right). \tag{5.5}
\]
In the same way, we just get
\[ V^{(2)}(\varpi_1, \varpi_2) = \frac{X}{p_2} \left( \log X + 2\gamma - 2 \right) Y^{(0)}(\sqrt{X}; \varpi_2 p_1 - \varpi_1 p_2, p_1 p_2) \]
\[- \frac{3X}{p_2} Y^{(1)}(\sqrt{X}; \varpi_2 p_1 - \varpi_1 p_2, p_1 p_2) + O \left( \frac{X \log X}{H_\nu} Y(\varpi_2 p_1 - \varpi_1 p_2) \right) + O \left( \frac{X}{H_\nu (\log X)^{A-4}} \right). \]

Inserting (5.5) and (5.6) into (5.4), we have
\[ V(\varpi_1, \varpi_2) = \frac{X}{p_2} \left( (\log X + 2\gamma - 1)(\log X + 2\gamma - 1) + 1 \right) Y^{(0)}(\sqrt{X}; \varpi_2 p_1 - \varpi_1 p_2, p_1 p_2) \]
\[- \frac{2X}{p_2} \left( \log X + 4\gamma - \frac{5}{2} \right) Y^{(1)}(\sqrt{X}; \varpi_2 p_1 - \varpi_1 p_2, p_1 p_2) \]
\[ + \frac{4X}{p_2} Y^{(2)}(\sqrt{X}; \varpi_2 p_1 - \varpi_1 p_2, p_1 p_2) \]
\[ + O \left( \frac{X \log X}{H_\nu} Y(\varpi_2 p_1 - \varpi_1 p_2) \right) + O \left( \frac{X}{H_\nu (\log X)^{A-4}} \right). \]

Applying Lemma 3.3, we derive that there exist three polynomials \( w_j(x) \) of degree \( j \) with numerical coefficients such that
\[ V(\varpi_1, \varpi_2) = \frac{X}{H_\nu} \sum_{j=0}^{2} \sigma_{-1}^{(2-j)}(\varpi_2 p_1 - \varpi_1 p_2) w_j(\log X) + O \left( \frac{X}{H_\nu (\log X)^{A-4}} \sigma_{-1}^{(0)}(\varpi_2 p_1 - \varpi_1 p_2) \right). \]

We emphasize that the coefficients of \( w_j \) do not depend on \( p_1, p_2, \varpi_1, \varpi_2 \). Then we obtain
\[ \sum_{p_1, p_2 \in \mathcal{P}_\nu \atop p_1 \neq p_2} f(p_1) f(p_2) (V_1 - V_2 - V_3 + V_4) \]
\[ \ll \frac{X (\log X)^2}{H_\nu} \sum_{j=0}^{2} \left| \sum_{p_1, p_2 \in \mathcal{P}_\nu \atop p_1 \neq p_2} f(p_1) f(p_2) \left( \sigma_{-1}^{(2-j)}(p_2 - p_1) - \sigma_{-1}^{(j)}(h_2 - p_1) - \sigma_{-1}^{(j)}(p_2 - h_1) \right) \right| \]
\[ + \sigma_{-1}^{(j)}(h_2 - p_1)) \right| + \frac{X}{H_\nu (\log X)^{A-4}} \max_{\varpi_2 = -1, h_1 \in \mathcal{P}_\nu} \sum_{p_2 \in \mathcal{P}_\nu} |f(p)|^2 \sum_{p_2 \in \mathcal{P}_\nu} \sigma_{-1}^{(0)}(\varpi_2 p_1 - \varpi_1 p_2). \]

By the definition of \( \sigma_{-1}^{(j)} \) and exchanging the order of summations, we have
\[ \sum_{p_1, p_2 \in \mathcal{P}_\nu \atop p_1 \neq p_2} f(p_1) f(p_2) \sigma_{-1}^{(j)}(p_2 - p_1) \]
\[ = \sum_{p_1, p_2 \in \mathcal{P}_\nu \atop p_1 \neq p_2} f(p_1) f(p_2) \sum_{d|(p_2 - p_1)} \frac{\log d)^j}{d} + O \left( \frac{H_\nu}{(\log X)^A (\log H_\nu)^{B-1}} \sum_{p \in \mathcal{P}_\nu} |f(p)|^2 \right) \]
\[ = \sum_{p_1 \in \mathcal{P}_\nu} f(p_1) \sum_{d \leq (\log H_\nu)^B} \frac{(\log d)^j}{d} \sum_{p_2 \in \mathcal{P}_\nu \atop p_2 \equiv p_1 (\text{mod } d)} f(p_2) + O \left( \frac{H_\nu}{(\log X)^A (\log H_\nu)^{B-1}} \sum_{p \in \mathcal{P}_\nu} |f(p)|^2 \right). \]
The contributions of \( \sigma_j(hp_1 + p_2), \sigma_j(p_1 + hp_2), \sigma_j(h(p_1 - p_2)) \) can be computed in the same manner. Moreover, the last term in (5.7) is less than \( \frac{X}{H \nu (\log X)^{2A - 4}} \sum_{p \in P} |f(p)|^2 \). Accordingly, we can obtain from Hypothesis (iii) that

\[
\sum_{p_1, p_2 \in P, p_1 \neq p_2} f(p_1)\overline{f(p_2)}(V_1 - V_2 - V_3 + V_4) \ll \frac{X(\log X)^2}{(\log H \nu)^{B - 1}} \sum_{p \in P} (1 + |f(p)|^2).
\]

Here the \( B \) is an arbitrarily large constant. By inserting this into (5.2), summing over \( \nu \) and using the Cauchy–Schwarz inequality, the off-diagonal contribution to \( S_f(X, -1) - S_f(X, h) \) is bounded by

\[
\frac{X^{A+1}}{\log H^{B - 1}} \left( \sum_{0 \leq \nu \leq L} \frac{1}{(\log H \nu)^{B - 1}} \left( \sum_{0 \leq \nu \leq L} \sum_{p \in P, m \in M \nu} (|f(m)|^2 + |f(pm)|^2) \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}
\]

\[
\ll \frac{X(\log X)}{(\log H)^{B - 1}} \left( \sum_{0 \leq \nu \leq L} \sum_{n \leq X} |f(n)|^2 \right)^{\frac{1}{2}}
\]

\[
\ll \frac{X(\log X)^{A+1}}{(\log H)^{B - 1}}.
\]

Therefore, by taking \( B = (3A + c)/\varepsilon + 1 \) for any \( A > 0 \), we obtain the following proposition from (5.3) and (5.8).

Proposition 5.1. Let the notation be as above. Then we have

\[
S_f(X, -1) - S_f(X, h) \ll \frac{X}{(\log X)^A}
\]

uniformly for any prime \( h \in [X^{2/3}, X(\log X)^{-A}] \) and any \( A > 0 \).

6. Evaluation of \( S_f(X, -1) - S_f(X, h) \)

In order to estimate the contribution of corresponding sums over the set \( J \), we first define the following subsets of \([1, X] \):

\[
J_1^{(\nu)} = \left\{ n \in [1, X] : n \text{ has exactly one divisor in } P_\nu \text{ and none in } \bigcup_{0 \leq h < \nu} P_h \right\},
\]

\[
J_1 = \bigcup_{0 \leq \nu \leq L} J_1^{(\nu)},
\]

\[
J_2 = \left\{ n \in [1, X] : n \text{ has at least one prime factor in } \bigcup_{0 \leq \nu \leq L} P_\nu \right\},
\]

\[
J_3 = \left\{ n \in [1, X] : n \text{ has no prime factors in } \bigcup_{0 \leq \nu \leq L} P_\nu \right\}.
\]

By the definitions of these subsets of \([1, X] \), it is clear that \( P_\nu M_\nu \subset J_1^{(\nu)} \). Thus we have \( \mathcal{I} \subset J_1 \). Moreover, \( J_2 \cup J_3 = [1, X] \) and \( J_2 \cap J_3 = \emptyset \). Thus, we get

\[
J \subset (J_1 \setminus \mathcal{I}) \cup (J_2 \setminus J_1) \cup J_3.
\]
As a consequence, it follows from Hypothesis (i) that

\[ S_T^J(X, \ell) \ll \sum_{n \in J_1 \setminus \mathcal{I}} |f(n)| \tau(n + \ell) + \sum_{n \in J_2 \setminus J_1} |f(n)| \tau(n + \ell) + \left| \sum_{n \in J_3} f(n) \tau(n + \ell) \right| \]

\[ \ll X \log X \frac{1}{\nu} \left( \sum_{n \in J_1 \setminus \mathcal{I}} |\tau(n + \ell)|^2 + \sum_{n \in J_2 \setminus J_1} |\tau(n + \ell)|^2 \right)^{\frac{1}{2}} + \left| \sum_{n \in J_3} f(n) \tau(n + \ell) \right|. \tag{6.1} \]

For each \( \nu \in [0, L] \), it is obvious that

\[ \mathcal{J}^{(\nu)} \setminus \mathcal{P}_\nu \mathcal{M}_\nu \subset \mathcal{P}_\nu \left( \frac{X}{H_{\nu + 1}}, \frac{X}{H_{\nu}} \right). \]

Hence, we have

\[ \sum_{n \in J_1 \setminus \mathcal{I}} |\tau(n + \ell)|^2 \ll \sum_{0 \leq \nu \leq L} \sum_{p \in \mathcal{P}_\nu} \sum_{n \in X - \ell \leq n \leq X - \ell (\nu) \leq X (\log X)^{-A}} |\tau(n)|^2. \]

The innermost sum is actually related to the coefficients \( |\tau(n)|^2 \) in arithmetic progression over a short interval. By the inequality in Lemma 3.1, we deduce that

\[ \sum_{n \in J_1 \setminus \mathcal{I}} |\tau(n)|^2 \ll \frac{X}{\log X^A - 3}. \]

Further, we obtain

\[ \sum_{n \in J_1 \setminus \mathcal{I}} |\tau(n + \ell)|^2 \ll \frac{X}{(\log X)^{A - 4}}. \tag{6.2} \]

Moreover, on account of

\[ \mathcal{J}_2 \setminus \mathcal{J}_1 \subset \bigcup_{0 \leq \nu \leq L} \{ n \in [1, X] : n \text{ has at least two prime factors in } \mathcal{P}_\nu \}, \]

we get

\[ \sum_{n \in J_2 \setminus J_1} |\tau(n + \ell)|^2 \ll \sum_{0 \leq \nu \leq L} \sum_{p_1, p_2 \in \mathcal{P}_\nu} \sum_{n \leq X - \ell \leq n \equiv - \ell (\mod p)} |\tau(n)|^2. \]

Similar to the argument of (6.2), we use Lemma 3.1 again and then obtain

\[ \sum_{n \in J_2 \setminus J_1} |\tau(n + \ell)|^2 \ll X (\log X)^3 \sum_{0 \leq \nu \leq L} \left( \frac{|\mathcal{P}_\nu|}{H_{\nu}} \right)^2 \ll \frac{X}{(\log X)^{A - 4}}. \tag{6.3} \]

Moreover, it follows from the Cauchy–Schwarz inequality that

\[ \sum_{n \in J_3} f(n) \tau(n + \ell) \ll \left( \sum_{n \leq X (n, P) = 1} |f(n)|^2 \right)^{\frac{1}{2}} \left( \sum_{n \leq X \not\equiv - \ell (P)} \tau(n)^2 \right)^{\frac{1}{2}}, \]

where \( P = P(\exp((\log X)^{e/2}), \exp((\log X)^{1-e/2})) \). Note that \( \ell = -1 \) or \( \ell = h \in [X^{2/3}, X (\log X)^{-A}] \) is a prime, so \( (\ell, P) = 1 \). Directly applying Lemma 3.8 with \( Y = \exp ((\log X)^{e/2}) \), \( Z = \)
\[ \exp \left( (\log X)^{1 - \varepsilon/2} \right), a = \ell, \text{ and inserting Hypothesis (ii), we then get} \]
\[ \sum_{n \in J_3} f(n) \tau(n + \ell) \leq X (\log X)^{\frac{1}{2} + \varepsilon}. \] (6.4)

Combining (6.2)–(6.4) with (6.1), we finally conclude the following proposition.

**Proposition 6.1.** Let the notation be as above. For any prime \( h \in [X^{2/3}, X(\log X)^{-A}] \), we have
\[ S_f^J(X, -1) - S_f^J(X, h) \ll X (\log X)^{\frac{1}{2} + \varepsilon}. \]

**Remark 6.1.** Assume the Ramanujan conjecture holds for \( f(n) \), which means there exists some positive integer \( k \) such that |\( f(n) \)| \( \leq \tau_k(n) \) for all \( n \in \mathbb{N} \). Opening the divisor function and using Dirichlet’s hyperbola method, it reduces
\[ \sum_{n \in J_3} f(n) \tau(n + \ell) = 2 \sum_{d \leq \sqrt{X-\ell}} \sum_{\substack{n \leq X, \, (n, P) = 1 \\ n \equiv \ell \ (\text{mod} \ d)}} f(n).
\]

Note that \( \ell = 1 \) or \( \ell = h \in [X^{2/3}, X(\log X)^{-A}] \) is a prime, so \((d, \ell) = 1\). The Brun–Titchmarsh inequality [40, Theorem 1] gives
\[ \sum_{\substack{n \leq X, \, (n, P) = 1 \\ n \equiv \ell \ (\text{mod} \ d)}} f(n) \ll X (\log X)^{1-k\varepsilon}. \]

A slight estimate of Titchmarsh [42, Equation (3.2)] states that
\[ \sum_{d \leq X} \frac{1}{\varphi(d)} = \zeta(2)\zeta(3)\zeta(6) \log X + O(1). \]

Hence, we have, under the Ramanujan conjecture for \( f(n) \),
\[ \sum_{n \in J_3} f(n) \tau(n + \ell) \leq X (\log X)^{k\varepsilon}. \]

This yields that \( S_f^J(X, -1) - S_f^J(X, h) \ll X (\log X)^{k\varepsilon} \).

7. **Proof of Theorem 1.1**

By Proposition 5.1 and Proposition 6.1 we obtain
\[ |S_f(X, -1) - S_f(X, h)| \ll X (\log X)^{\frac{1}{2} + \varepsilon} \]
uniformly for any prime \( h \in [X^{2/3}, X(\log X)^{-A}] \). Thus, we have
\[ S_f(X, -1) = \left( \frac{\log X}{X} \right)^A T_f(X) + O(X (\log X)^{\frac{1}{2} + \varepsilon}), \] (7.1)
where \( T_f(X) \) is a ternary additive problem given by
\[ T_f(X) = \sum_{h \leq X(\log X)^{-A}} \sum_{n \leq X} \Lambda(h) f(n) \tau(n + h). \]

Hence, the additive divisor problem that we are considering is reduced to the study of a ternary additive problem \( T_f(X) \). The Hardy–Littlewood circle method can be applied. Here we provide another direct argument. Notice that the interval of summation over \( h \) is large.
where the ranges of all parameters have been considered. Now we turn to estimate the original divisor problem of Titchmarsh

\[ T(Y, n) = \sum_{h \leq Y} \Lambda(h) \tau(h + n), \]

where the shift parameter \( n \) is in the interval \( 0 < n \ll Y(\log Y)^A \).

By the definition of divisor function, we have

\[ \tau(m) = 2 \sum_{q|m \atop q < \sqrt{m}} 1 + \delta_\Box(m), \quad \delta_\Box(m) := \begin{cases} 1 & \text{if } m \text{ is a perfect square}, \\ 0 & \text{otherwise}. \end{cases} \] (7.2)

Therefore, we deduce that

\[ T(Y, n) = 2 \left( \sum_{q \leq \sqrt{Y+n} \atop (q,n)=1} \sum_{h \leq Y} \Lambda(h) - \sum_{\sqrt{m}<q\leq\sqrt{Y+n} \atop h \equiv -n \pmod{q}} \sum_{h \leq q^2-n \atop h \equiv -n \pmod{q}} \Lambda(h) \right) + O(Y^{1/2+\varepsilon}) \] (7.3)

\[ := 2(S_1(Y, n) - S_2(Y, n)) + O(Y^{1/2+\varepsilon}). \]

It follows from Lemma 3.6 that

\[ S_1(Y, n) = \sum_{q \leq \sqrt{Y+n} \atop (q,n)=1} \frac{1}{\varphi(q)} \sum_{h \leq Y} \Lambda(h) + O\left(\frac{Y}{(\log Y)^A}\right), \] (7.4)

where the ranges of all parameters have been considered. Now we turn to estimate \( S_2(Y, n) \), and hope that \( S_2(Y, n) \) can be approximated by

\[ \sum_{\sqrt{m}<q<\sqrt{Y+n} \atop (q,n)=1} \frac{1}{\varphi(q)} \sum_{h \leq q^2-n} \Lambda(h). \]

In fact, we shall show that the difference of \( S_2(Y, n) \) and its approximation satisfies

\[ U(Y, n) := \sum_{\sqrt{m}<q<\sqrt{Y+n} \atop (q,n)=1} \left( \sum_{h \leq q^2-n \atop h \equiv -n \pmod{q}} \Lambda(h) - \frac{1}{\varphi(q)} \sum_{h \leq q^2-n} \Lambda(h) \right) \ll \frac{Y}{(\log Y)^A}. \] (7.5)

Since the condition \( h \leq q^2 - n \) makes \( q \) and \( h \) interconnected, Lemma 3.6 can not be directly used. To relax this condition, we shall decompose the sums over \( q \) and \( h \) into short intervals \((Q(1 + \Delta_1)^{-1}, Q]\) and \((H(1 + \Delta_1)^{-1}, H]\), where \( \Delta_1 = (\log Y)^{-B} \) with \( B > A + 3 \). Then we have

\[ U(Y, n) \ll \sum_{j_1, j_2 \geq 0} \left| \sum_{Q(1+\Delta_1)^{-1}<q\leq Q \atop (q,n)=1} \left( \sum_{H(1+\Delta_1)^{-1}<h\leq H \atop h \equiv -n \pmod{q} \atop h \leq q^2-n \atop (h,q)=1} \Lambda(h) - \frac{1}{\varphi(q)} \sum_{h \leq q^2-n \atop h \equiv -n \pmod{q}} \Lambda(h) \right) \right|, \]

where \( Q = \sqrt{Y+n(1+\Delta_1)^{-j_1}}, H = Y(1+\Delta_1)^{-j_2} \). Note that the condition \( n \leq q^2 \) can be dropped as soon as \( H + n \leq Q^2(1+\Delta_1)^{-2} \). The contribution of \( j_1, j_2 \) such that \( H + n >
\[ Q^2(1 + \Delta_1)^{-2} \text{ is at most} \]
\[
\sum_{\sqrt{n} < q \leq \sqrt{Y+n}} \sum_{(q,n) = 1} \Lambda(h) \ll (\log Y) \sum_{\sqrt{n} < q \leq \sqrt{Y+n}} \frac{(\Delta_1(q^2 - n) + 1)}{q} \ll \Delta_1(Y \log Y)^2.
\]

Therefore, we obtain
\[
U(Y, n) \ll \Delta_1(Y \log Y)^2 + \Delta_1^{-2}(\log Y)^2 \times \max_{Q \leq \sqrt{Y+n}} \left| \sum_{(q,n) = 1} \sum_{\sqrt{Y+n} < q \leq Q} \Lambda(h) - \frac{1}{\varphi(q)} \sum_{H(1 + \Delta_1)^{-1} < h \leq H} \Lambda(h) \right|.
\]

Let \( \delta > 0 \) be the real number given in Lemma 3.6. It is clear that
\[
\sum_{Q(1 + \Delta_1)^{-1} < q \leq Q} \left( \sum_{H(1 + \Delta_1)^{-1} < h \leq H} \Lambda(h) - \frac{1}{\varphi(q)} \sum_{H(1 + \Delta_1)^{-1} < h \leq H} \Lambda(h) \right) \ll \Delta_1 H(\log H),
\]

which is acceptable if \( H \leq \Delta_1^2 Y \). Suppose \( H > \Delta_1^2 Y \), then Lemma 3.6 can be applied and yields
\[
\sum_{Q(1 + \Delta_1)^{-1} < q \leq Q} \left( \sum_{H(1 + \Delta_1)^{-1} < h \leq H} \Lambda(h) - \frac{1}{\varphi(q)} \sum_{H(1 + \Delta_1)^{-1} < h \leq H} \Lambda(h) \right) \ll \frac{H}{(\log H)^{3A}}.
\]

In summary, we have the claimed estimate (7.5), which means
\[
S_2(Y, n) = \sum_{\sqrt{n} < q \leq \sqrt{Y+n}} \varphi(q) \sum_{q \leq \sqrt{h+n}} \Lambda(h) + O\left(\frac{Y}{(\log Y)^4}\right).
\]

Inserting (7.4) and (7.6) into (7.3), and applying Lemma 3.7 and partial summation, we have
\[
T(Y, n) = 2 \sum_{h \leq Y} \Lambda(h) \sum_{q \leq \sqrt{h+n}} \varphi(q) + O\left(\frac{Y}{(\log Y)^4}\right)
\]
\[
= c_0(n) \left( (Y+n) \log (Y+n) - n \log n + (2\gamma - 1)Y + 2c_0'(n)Y \right) + O\left(\frac{Y}{(\log Y)^A}\right),
\]

where \( Y = X/(\log X)^A \). Substituting this asymptotic formula into (7.1), we then get
\[
S_f(X, -1) = \sum_{n \leq X} f(n) \left( c_0(n) (\log n + 2\gamma) + 2c_0'(n) \right) + E_f(X, -1) + O\left( X(\log X)^{\frac{5}{2} + \varepsilon} \right),
\]

where the term \( E_f(X, -1) \) satisfies
\[
E_f(X, -1) \ll \sum_{n \leq X} |f(n)c_0(n)| \log \left( 1 + \frac{Y}{n} \right) + \sum_{n \leq X} |f(n)c_0(n)| \left| \frac{n}{Y} \log \left( 1 + \frac{Y}{n} \right) - 1 \right|.
\]

Notice that \( c_0(n) \ll 1 \). It follows from partial summation and Hypothesis (i) that
\[
E_f(X, -1) \ll \frac{X}{(\log X)^{A - \frac{11}{4}}},
\]
Hence, taking $A > c/2$ gives

$$S_f(X, -1) = \sum_{n \leq X} f(n)(c_0(n)(\log n + 2\gamma) + 2c'_0(n)) + O(X(\log X)^{3/4 + \varepsilon}).$$

This completes the proof of Theorem 1.1.

8. PROOF OF THEOREM 1.2

For convenience, we put

$$E(f, X; q) := \sum_{n \leq X} f(n) - \sum_{n \leq X} f(n).$$

Let $c_q$ be given by $c_q = \text{sgn} \ E(f, X; q)$. With the notation as in (4.4), we divide $E(f, X; q)$ into two parts $E^I(f, X; q)$ and $E^{J}(f, X; q)$, which are the corresponding sums with additional restrictions $n \in I$ and $n \in J$, respectively.

8.1. Contribution of $E^I(f, X; q)$. Considering the $q$ in dyadic ranges, it suffices to estimate $\sum_{q \sim Q} c_q E^I(f, X; q)$ with $Q \leq X^{17/33 - \varepsilon}$. By the definition of interval $I$ and using the Cauchy–Schwarz inequality, we have

$$\sum_{q \sim Q} c_q E^I(f, X; q) \approx \sum_{0 \leq \nu \leq L} \left( \sum_{m \in \mathcal{M}_\nu} |f(m)|^2 \right)^{1/2} T_\nu(Q)^{1/2},$$

where $T_\nu(Q)$ is defined by

$$T_\nu(Q) = \sum_{m \leq X/H_{\nu+1}} \left| \sum_{q \sim Q} \sum_{p \in \mathcal{P}_\nu} c_q f(p) - \sum_{q \sim Q} \sum_{p \in \mathcal{P}_\nu} c_q f(p) \right|^2.$$

By smooth dyadic subdivisions and squaring out $T_\nu(Q)$, we then write

$$T_\nu(Q) \approx (\log X) \max_{M \leq X/H_{\nu+1}} \sum_m \psi\left(\frac{m}{M}\right) \left| \sum_{q \sim Q} \sum_{p \in \mathcal{P}_\nu} c_q f(p) - \sum_{q \sim Q} \sum_{p \in \mathcal{P}_\nu} c_q f(p) \right|^2$$

$$= (\log X) \max_{M \in [X^{3/4}, X/H_{\nu+1}]} \left( W_\nu(Q, M) - 2 \text{Re} V_\nu(Q, M) + U_\nu(Q, M) \right) + O(X^{3/4 + \varepsilon}),$$

(8.2)
where \( \psi \) is the smooth function as in Lemma 3.4 and \( W_\nu(Q, M), V_\nu(Q, M), U_\nu(Q, M) \) are defined by

\[
W_\nu(Q, M) = \sum_{m} \psi \left( \frac{m}{M} \right) \left| \sum_{q \sim Q} \sum_{p \in P_\nu} c_q f(p) \right|^2,
\]

\[
V_\nu(Q, M) = \sum_{m} \psi \left( \frac{m}{M} \right) \left( \sum_{q \sim Q} \sum_{p \in P_\nu} c_q f(p) \right) \left( \sum_{q \sim Q} \sum_{p \in P_\nu} \frac{c_q f(p)}{\varphi(q)} \right),
\]

\[
U_\nu(Q, M) = \sum_{m} \psi \left( \frac{m}{M} \right) \left| \sum_{q \sim Q} \sum_{p \in P_\nu} \frac{c_q f(p)}{\varphi(q)} \right|^2.
\]

Our goal is to evaluate each term individually.

8.2. **Evaluation of** \( U_\nu(Q, M) \). Let us begin with the simplest term

\[
U_\nu(Q, M) = \sum_{q_1 \sim Q, q_2 \sim Q} \frac{c_{q_1} c_{q_2}}{\varphi(q_1) \varphi(q_2)} \sum_{p_1 \in P_\nu} \sum_{p_2 \in P_\nu} f(p_1) \overline{f(p_2)} \sum_{(m,q_1,q_2)=1} \psi \left( \frac{m}{M} \right).
\]

By Poisson’s formula (3.2) in Lemma 3.4 the innermost sum over \( m \) is

\[
\sum_{(m,q_1,q_2)=1} \psi \left( \frac{m}{M} \right) = \frac{\varphi(q_1 q_2)}{q_1 q_2} \hat{\psi}(0)M + O(\tau(q_1) \tau(q_2)(\log X)^4).
\]

This yields

\[
U_\nu(Q, M) = \hat{\psi}(0)M \sum_{q_1 \sim Q, q_2 \sim Q} \frac{c_{q_1} c_{q_2}}{\varphi(q_1) \varphi(q_2)} \frac{\varphi(q_1 q_2)}{q_1 q_2} \sum_{p_1 \in P_\nu} \sum_{p_2 \in P_\nu} f(p_1) \overline{f(p_2)} + O\left( \frac{H^2_\nu}{(\log X)^{4 \cdot 8}} \right).
\]

We remove the conditions \((p_1,q_1)=1\) and \((p_2,q_2)=1\) at the cost of the admissible error term \( O(MH_\nu) \). Moreover, we denote \( \delta = (q_1,q_2) \), and then get

\[
U_\nu(Q, M) = \hat{\psi}(0)M \sum_{\delta} \frac{1}{\delta \varphi(\delta)} \sum_{k_1k_2 \sim Q/\delta} \sum_{(k_1,k_2)=1} c_{q_1 k_1} c_{q_2 k_2} \left( \sum_{p_1 \in P_\nu} f(p_1) \right) \left( \sum_{p_2 \in P_\nu} \overline{f(p_2)} \right)
\]

\[+ O(MH_\nu) + O\left( \frac{H^2_\nu}{(\log X)^{4 \cdot 8}} \right).
\]

8.3. **Evaluation of** \( V_\nu(Q, M) \). By the definition of \( V_\nu(Q, M) \) and exchanging the order of summations, we have

\[
V_\nu(Q, M) = \sum_{q_1 \sim Q, q_2 \sim Q} \sum_{p_1 \in P_\nu} \sum_{p_2 \in P_\nu} f(p_1) \overline{f(p_2)} \sum_{m \equiv t_1(\text{mod } q_1)} \psi \left( \frac{m}{M} \right).
\]

where \( \overline{p} \) denotes the multiplicative inverse of \( p_1 \) modulo \( q_1 \). By applying the Möbius inversion formula and inserting Poisson’s formula (3.1) with \( H = dq_1(\log X)^4/M \), the innermost sum
is equal to
\[
\sum_{m \equiv \ell_1(\text{mod } q_1) \atop (m, q_2) = 1} \psi\left(\frac{m}{M}\right) = \sum_{d \mid q_2 \atop (d, q_1) = 1} \mu(d) \sum_{m \equiv \eta(\text{mod } q_1)} \psi\left(\frac{m}{M}\right)
\]
\[
= \hat{\psi}(0) M \sum_{d \mid q_2 \atop (d, q_1) = 1} \frac{\mu(d)}{d} + O(\tau(q_2)(\log X)^4),
\]
where \(\eta\) is a common solution of the congruences \(\eta \equiv \ell_1(\text{mod } q_1)\) and \(\eta \equiv 0(\text{mod } d)\). Inserting this into \(8.4\), we deduce that
\[
V_\nu(Q, M) = \hat{\psi}(0) M \sum_{\delta} \frac{1}{\delta \varphi(\delta)} \sum_{r_1, r_2 \sim Q / \delta \atop (r_1, r_2) = 1} \sum_{\nu_1, \nu_2} \left( \sum_{p_1 \in P_\nu \atop (p_1, q_1) = 1} f(p_1) \right) \left( \sum_{p_2 \in P_\nu \atop (p_2, q_2) = 1} f(p_2) \right)
\]
\[
+ O\left(\frac{Q H_\nu^2}{(\log X)^{\frac{A}{2}-6}}\right).
\]
Comparing this with \(8.3\), we obtain the relation
\[
V_\nu(Q, M) = U_\nu(Q, M) + O(M H_\nu) + O\left(\frac{Q H_\nu^2}{(\log X)^{\frac{A}{2}-8}}\right). \tag{8.5}
\]

8.4. Evaluation of \(W_\nu(Q, M)\). The evaluation of \(W_\nu(Q, M)\) is the most difficult and it involves the key arguments. Before applying them, in this section we reduce the range of the summation by elementary estimates. By definition, we have
\[
W_\nu(Q, M) = \sum_{q_1 \sim Q \atop q_2 \sim Q} \sum_{p_1, p_2 \in P_\nu \atop p_1 \neq p_2} \sum_{m \equiv 1(\text{mod } q_1) \atop (m, q_2) = 1} \psi\left(\frac{m}{M}\right) + O(H_\nu M \log X), \tag{8.6}
\]
where the error term comes from the diagonal contribution for \(p_1 = p_2\). For the off-diagonal terms, we first factorize each of the variables \(q_1\) and \(q_2\) to control the contributions of their greatest common divisor in some ranges. For notational conventions, we decompose the variables \(q_1, q_2\) in a unique way as follows:

\[
\begin{cases}
\delta = (q_1, q_2), \\
q_1 = \delta k_1, q_2 = \delta k_2, \\
k_1 = \delta_1 k_1' \text{ with } \delta_1 | \delta^\infty \text{ and } (k_1', \delta) = 1, \\
k_2 = \delta_2 k_2' \text{ with } \delta_2 | \delta^\infty \text{ and } (k_2', \delta) = 1.
\end{cases}
\]

Note that the conditions of summation over \(m\) in \(8.6\) imply that we necessarily have
\(p_1 \equiv p_2(\text{mod } \delta)\).

Denote \(W_\nu^{(1)}(Q, M)\) to be the sum of the terms in \(8.6\) with \(\delta > D := \exp\left((\log X)^{\varepsilon/4}\right)\). Thus, we obtain from Lemma 3.1 that
\[
W_\nu^{(1)}(Q, M) \ll \sum_{\delta > X^\varepsilon \atop k_2 \sim Q / \delta} \sum_{p_1, p_2 \in P_\nu \atop p_1 \neq p_2(\text{mod } \delta)} \sum_{|f(p_1)|^2} \sum_{\delta / 2 < m \leq 5 M / 2 \atop m \equiv \ell_2(\text{mod } \delta k_2)} \tau(|p_1 m - a|)
\]
\[
\ll H_\nu^2 M (\log X)^4 \frac{\tau^2}{D}. \tag{8.7}
\]
Denote \( W'(Q, M) \) to be the sum of terms in (8.6) with \( \delta \leq D \) and \( \delta_1 > D \). Then we obtain
\[
W'_\nu(Q, M) \ll \sum_{\delta \leq D} \sum_{\delta_1 \equiv \delta (\mod \delta_1)} \sum_{\nu} |f(p_1)|^2 \sum_{p_1 \equiv p_2 (\mod \delta_1)} \tau(|p_2 m - a|) \sum_{m=\mathbb{P}^2(\mod \delta_1 k'_1)} \sum_{m=\mathbb{P}^2(\mod \delta_2 k'_2)} \sum_{\delta, \delta_1 \leq D} \sum_{\delta_2 \equiv \delta (\mod \delta_2)} \sum_{\nu}
\]
where \( W'_\nu(Q, M) \) can be written as
\[
\sum_{\delta, \delta_1 \leq D} \sum_{\delta_2 \equiv \delta (\mod \delta_2)} \sum_{\nu} c_{\delta \delta_1} k'_1 \delta_{\delta_2} k'_2 \sum_{p_1 \equiv p_2 (\mod \delta_1)} f(p_1) f(p_2) \sum_{m=\mathbb{P}^2(\mod \delta_1 k'_1)} \sum_{m=\mathbb{P}^2(\mod \delta_2 k'_2)} \psi\left(\frac{m}{M}\right).
\]
The congruence conditions in (8.9) are equivalent to the four congruences
\[
m \equiv \overline{p_1} (\mod \delta_1), \quad m \equiv \overline{p_2} (\mod \delta_2),
\]
\[
m \equiv \overline{p_1} (\mod k'_1), \quad m \equiv \overline{p_2} (\mod k'_2).
\]
The first two equations are equivalent to \( m \equiv a \lambda (\mod \delta_1 \delta_2) \), where \( \lambda(p_1, p_2) \) is some congruence class modulo \( \delta \delta_1 \delta_2 \), only depending on the congruence classes of \( p_1 \mod \delta_1 \) and \( p_2 \mod \delta_2 \). Finally, we see that \( m \) satisfies the single congruence
\[
m \equiv m_0 (\mod \eta k'_1 k'_2),
\]
with
\[
\eta = \delta \delta_1 \delta_2
\]
and
\[
m_0 = \lambda k'_1 k'_2 \overline{p_1} \overline{p_2} + \eta \overline{p_1} k'_2 \overline{p_2} + \eta \overline{p_2} k'_1 \overline{p_1} (\mod \eta k'_1 k'_2),
\]
where the \( \overline{\ } \)–symbol respectively means the inverse of \( x \) modulo \( \eta, k'_1 \) and \( k'_2 \).

We then apply Poisson’s formula in Lemma 3.4 and deduce
\[
\psi\left(\frac{m}{M}\right) = \hat{\psi}(0) \frac{M}{\eta k'_1 k'_2} + \frac{M}{\eta k'_1 k'_2} \sum_{1 \leq |h| \leq H} e\left(\frac{hm_0}{\eta k'_1 k'_2}\right) \hat{\psi}\left(\frac{hM}{\eta k'_1 k'_2}\right) + O(M^{-1})
\]
with
\[
H = M^{-1} Q^2 (\log X)^4.
\]
Inserting (8.10) into (8.9), we obtain
\[
W'_\nu(Q, M) = \hat{\psi}(0) M R + M R_1 + O\left(\frac{H^2 Q^2}{M}\right),
\]
where
\[
R = \sum_{\delta, \delta_1 \leq D} \sum_{\delta_2 \equiv \delta (\mod \delta_2)} \sum_{\nu} c_{\delta \delta_1} k'_1 \delta_{\delta_2} k'_2 \sum_{p_1 \equiv p_2 (\mod \delta_1)} f(p_1) f(p_2),
\]
\[ \mathcal{R}_1 = \sum_{\delta, \delta_1, \delta_2 \leq D} \sum_{k_1' \sim Q/\delta_1} \sum_{k_2' \sim Q/\delta_2} \sum_{\eta k_1' k_2'} \frac{c_{\delta \delta_1 k_1'} c_{\delta \delta_2 k_2'}}{\eta k_1' k_2'} \sum_{p_1, p_2 \in \mathcal{P}_\nu} f(p_1) \overline{f(p_2)} \]

and the error term comes from the contribution of the term \( O(M^{-1}) \) in (8.10). Now it remains to evaluate \( \mathcal{R} \) and \( \mathcal{R}_1 \).

We first treat the term \( \mathcal{R} \), which can be rewritten as

\[ \mathcal{R} = \sum_{\delta, \delta_1, \delta_2 \leq D} \sum_{k_1' \sim Q/\delta_1} \sum_{k_2' \sim Q/\delta_2} \frac{c_{\delta \delta_1 k_1'} c_{\delta \delta_2 k_2'}}{\delta \delta_1 \delta_2 k_1' k_2'} \sum_{\alpha \equiv (\mod \delta)} \sum_{p \in \mathcal{P}_\nu} f(p_1) \left( \sum_{p_2 \equiv \alpha (\mod \delta)} f(p_2) \right). \]

By Hypotheses (ii) and (iii), the Barban–Davenport–Halberstam theorem (see [17, Theorem 17.5]) yields

\[ \sum_{k \leq K} \sum_{\alpha (\mod k)} \left| \sum_{p \in \mathcal{P}_\nu} f(p) - \frac{1}{\varphi(k)} \sum_{p \equiv \alpha (\mod k)} \phi \right|^2 \ll \frac{H^2}{(\log X)^{16A}} \tag{8.12} \]

provided that \( K \leq \exp \left( (\log X)^{\varepsilon/2} \right) \). Then it follows from (8.12) that

\[ \mathcal{R} = \mathcal{R}_0 + O \left( \frac{H^2}{(\log X)^{8A}} \right), \]

where \( \mathcal{R}_0 \) is given by

\[ \mathcal{R}_0 = \sum_{\delta, \delta_1, \delta_2 \leq D} \sum_{k_1' \sim Q/\delta_1} \sum_{k_2' \sim Q/\delta_2} \sum_{\alpha \equiv (\mod \delta)} \frac{c_{\delta \delta_1 k_1'} c_{\delta \delta_2 k_2'}}{\delta \delta_1 \delta_2 k_1' k_2'} \left( \sum_{p_1 \in \mathcal{P}_\nu} f(p_1) \right) \left( \sum_{p_2 \in \mathcal{P}_\nu} f(p_2) \right). \]

We next extend the summation over all \( \delta, \delta_1, \delta_2 \) to get

\[ \mathcal{R}_0 = \mathcal{R}^* + O \left( D^{-\frac{1}{2}} H^2 \right) \]

the error term being estimated by similar arguments as those for (8.7) and (8.8), where \( \mathcal{R}^* \) is given by

\[ \mathcal{R}^* = \sum_{\delta, \delta_1, \delta_2} \sum_{k_1' \sim Q/\delta_1 \mod \delta_1} \sum_{k_2' \sim Q/\delta_2 \mod \delta_2} \frac{c_{\delta \delta_1 k_1'} c_{\delta \delta_2 k_2'}}{\delta \delta_1 \delta_2 k_1' k_2'} \left( \sum_{p_1 \in \mathcal{P}_\nu} f(p_1) \right) \left( \sum_{p_2 \in \mathcal{P}_\nu} f(p_2) \right) \]

\[ = \sum_{\delta} \frac{1}{\delta} \phi(\delta) \sum_{k_1, k_2 \sim Q/\delta} \frac{c_{\delta k_1} c_{\delta k_2}}{\delta k_1 k_2} \left( \sum_{p_1 \in \mathcal{P}_\nu} f(p_1) \right) \left( \sum_{p_2 \in \mathcal{P}_\nu} f(p_2) \right). \]

So we can conclude that

\[ \mathcal{R} = \mathcal{R}^* + O \left( \frac{H^2}{(\log X)^{16A}} \right). \tag{8.13} \]

We are left to estimate \( \mathcal{R}_1 \). By definition of \( m_0 \), we know

\[ e \left( \frac{hm_0}{\eta k_1' k_2'} \right) = e \left( \lambda h \frac{k_1' k_2'}{\eta} + h \frac{\bar{p}_1 k_2'}{k_1'} + h \frac{\bar{p}_2 k_1'}{k_2'} \right). \]
In order to transform the exponent, we apply Bezout’s relation twice to write
\[
\frac{\eta p_1 k_1'}{k_1} - \frac{1}{\eta p_1 k_1' k_2'} = \frac{1}{\eta p_1 k_1' k_2'} - \frac{k_1'}{\eta p_1 k_2'} \mod 1
\]
\[
= \frac{1}{\eta p_1 k_1' k_2'} - \frac{\eta k_1'}{p_1 k_2'} - \frac{p_1 k_1'}{\eta} \mod 1.
\]
Moreover, we have
\[
\frac{\eta p_2 k_1'}{k_2'} - \frac{\eta k_1'}{p_1 k_2'} = \frac{\eta(p_1 - p_2)p_2 k_1'}{p_1 k_2'} \mod 1.
\]
Thus, these relations yield
\[
e\left(\frac{hm\nu_0}{\eta k_1' k_2'}\right) = e\left(\frac{\lambda h k_1' k_2'}{\eta} - \frac{h p_1 k_1' k_2'}{\eta} + \frac{h}{\eta p_1 k_1' k_2'} + h\frac{\eta(p_1 - p_2)p_2 k_1'}{p_1 k_2'}\right).
\]
Next we split up the summation over \(p_1, p_2, h, k_1', k_2'\) into arithmetic progressions \(\alpha_i (\mod \eta), 1 \leq i \leq 5\). If we fix the congruence class \(\alpha_i (\mod \eta)\), then the exponents \(\lambda h k_1' k_2'\) and \(h\frac{p_1 k_1' k_2'}{\eta}\) are fixed. Thus, we further have
\[
R_1 \ll D^{18} \max_{\delta, \delta_1, \delta_2 \leq D}\sum_{p_1, p_2 \in \mathcal{P}_\nu} \sum_{1 \leq |h| \leq H} \sum_{k_1' k_2' \sim Q / (\delta \delta_1 \delta_2)} \frac{c_{\delta, \delta_1, \delta_2} k_1' k_2'}{k_1' k_2'} f(p_1) f(p_2)
\]
\[
\times \hat{\psi}\left(\frac{h M}{\eta k_1' k_2'}\right) e\left(\frac{h}{\eta p_1 k_1' k_2'}\right) e\left(h\frac{\eta(p_1 - p_2)p_2 k_1'}{p_1 k_2'}\right),
\]
where the variables \(p_1, p_2\) satisfy the congruence condition \(p_1 \equiv (p_2 (\mod \delta))\), and \(p_1, p_2, h, k_1', k_2'\) are in congruence classes \(\alpha_i (\mod \eta), 1 \leq i \leq 5\), respectively.

A straightforward calculation gives
\[
\frac{\partial^{a_1 + a_2 + a_3 + a_4}}{\partial h^{a_1} \partial p_1^{a_2} \partial k_1'^{a_3} \partial k_2'^{a_4}} \left(e\left(\frac{h}{\eta p_1 k_1' k_2'}\right) \hat{\psi}\left(\frac{h M}{\gamma k_1' k_2'}\right)\right) \ll (1 + |h|)^{-a_1} p_1^{-a_2} k_1'^{-a_3} k_2'^{-a_4}
\]
for integers \(0 \leq a_1, a_2, a_3, a_4 \leq 1\), \(p_1 \in \mathcal{P}_\nu, k_1', k_2' \in \mathcal{Q} / D^2, 2Q\) and \(h \in [-H, H]\). By partial summations over variables \(p_1, h, k_1', k_2'\), we obtain
\[
R_1 \ll \frac{D^{22} \log X}{Q^2} \max_{\delta, \delta_1, \delta_2 \leq D} \sum_{p_1, p_2 \in \mathcal{P}_\nu} \sum_{1 \leq |h| \leq H} \left|f(p_1) f(p_2)\right|
\]
\[
\times \sum_{1 \leq |h| \leq H} \sum_{k_1' k_2' \leq 2Q} \xi_1(h) \xi_2(k_1') \xi_3(k_2') e\left(h\frac{\eta(p_1 - p_2)p_2 k_1'}{p_1 k_2'}\right),
\]
where \(\xi_1, \xi_2, \xi_3\) are three sequences of complex numbers with \(|\xi_1(n)|, |\xi_2(n)|, |\xi_3(n)| \leq 1\). Notice that the symmetry of variables \(p_1, p_2\) allows us to without loss of generality replace the condition \(1 \leq |h| \leq H\) by \(1 \leq h \leq H\).

We shall apply Lemma 3.5 to bound the exponential sum on the second line of (8.14). So we localize each of the variables \(h, k_1', k_2'\) dyadically around powers of two that we denote respectively by \(H_1, K_1, K_2\). For each such dyadic partition and fixed variables \(p_1, p_2 \in \mathcal{P}_\nu, \eta \leq D^3\), we apply Lemma 3.5 with the following choice of variables:
\[
\vartheta \to p_1 - p_2, a \to h, m \to \eta p_2 k_1', n \to p_1 k_2',
\]
where the left side of $\rightarrow$ corresponds to notations of Lemma 3.5 while the right side of $\rightarrow$ corresponds to our current notation) and parameters

$$|\vartheta| \ll H_\nu, \ A \to H_1, \ M \to \eta p_2 K_1, \ N \to p_1 K_2.$$  

Note that $\eta p_2 K_1 \ll D^3 H_\nu K_1$ and that $p_1 K_2 \ll H_\nu K_2$. Hence, we appeal to Lemma 3.5 to infer

$$\sum \sum \xi_1(h)\xi_2(k'_1)\xi_3(k'_2)e\left(h\frac{(p_1-\eta p_2)\overline{p_2 k'_1}}{p_1 k'_2}\right) \ll D^2 (H_1 K_1 K_2) \frac{1}{\frac{H_1}{H_\nu K_1 K_2}}$$

$$\times \left((H_1 H_\nu^2 K_1 K_2) \frac{1}{\hat{\nu}^2} (H_\nu (K_1 + K_2)) \frac{1}{\hat{\nu}^2} + (H_1 H_\nu^2 K_1 K_2) \frac{1}{\hat{\nu}^2} (H_1 H_\nu (K_1 + K_2)) \frac{1}{\hat{\nu}^2}\right).$$

Summing over all the dyadic partitions and applying Hypothesis (ii), we then get

$$R_1 \ll D^{25} \left(M^{-\frac{15}{20}} H_\nu \frac{M^2}{Q} + M^{-1} H_\nu \frac{M^2}{Q} \frac{15}{8}\right). \tag{8.15}$$

Finally, inserting the estimates (8.13), (8.15) of $R$ and $R_1$ into (8.11), we have

$$W_\nu(Q, M) = \hat{\psi}(0) M R^* + O \left(M^{\frac{5}{20} + \epsilon} H_\nu \frac{M^2}{Q} + M^{\epsilon} H_\nu \frac{M^2}{Q} \frac{15}{8} + \frac{M H_\nu^2}{(\log X)^{8A}}\right).$$

Comparing this with (8.3), we also have the relation

$$W_\nu(Q, M) = U_\nu(Q, M) + O \left(M^{\frac{5}{20} + \epsilon} H_\nu \frac{M^2}{Q} + M^{\epsilon} H_\nu \frac{M^2}{Q} \frac{15}{8} + \frac{M H_\nu^2}{(\log X)^{8A}}\right). \tag{8.16}$$

### 8.5. Return Section 8.1

We recall that $Q \leq X^{17/33-\epsilon}$. Substituting (8.3), (8.5) and (8.16) into (8.2), we see

$$T_\nu(Q) \ll \frac{X H_\nu}{(\log X)^{4A}},$$

which further yields from (8.11) that

$$\sum c_q E_J(f, X; q) \ll \frac{X}{(\log X)^{A-2}} \tag{8.17}$$

holds for any $A \geq 2$.

### 8.6. Contribution of $E_J(f, X; q)$

With the notation as in Section 6, we have

$$\sum c_q E_J(f, X; q) \ll \sum_{n \in J} |f(n)| \sum_{n \equiv 1 \pmod{q}} |c_q| + (\log Q) \sum_{n \in J} |f(n)|$$

$$\ll \sum_{n \in J} |f(n)| \tau(n - 1) + (\log X) \sum_{n \in J} |f(n)| \tag{8.18}$$

The first term on the second line of (8.18) has been estimated in Section 6, which gives

$$\sum_{n \in J} f(n) \tau(n - 1) \ll X (\log X)^{\frac{1}{2} + \epsilon}. \tag{8.19}$$
Now we treat the second term on the second line of (8.18). By the decomposition of \( J \) and Hypothesis (i), we have

\[
\sum_{n \in J} |f(n)| \ll \sum_{n \in J_1 \setminus J} |f(n)| + \sum_{n \in J_2 \setminus J_1} |f(n)| + \sum_{n \in J_3} |f(n)|
\]

\[
\ll X (\log X)^{\frac{1}{2}} (|J_1 \setminus I|^\frac{1}{2} + |J_2 \setminus J_1|^\frac{1}{2}) + \left( \sum_{n \in J_3} |f(n)|^2 \right)^\frac{1}{2} |J_3|^\frac{1}{2}.
\]

(8.20)

Moreover, it follows from the definitions of these sets that

\[
|J_1 \setminus I| \ll \sum_{0 \leq \nu \leq \lambda} \sum_{p \in \mathcal{P}_\nu} \frac{X}{H_\nu (\log X)^A} \ll \frac{X}{(\log X)^{A-1}},
\]

\[
|J_2 \setminus J_1| \ll \sum_{0 \leq \nu \leq \lambda} \sum_{p_1, p_2 \in \mathcal{P}_\nu} \frac{X}{p_1 p_2} \ll \frac{X}{(\log X)^{A-1}}.
\]

and

\[
|J_3| = \sum_{n \leq X \atop (n, 1) = 1} 1 \ll \frac{X}{(\log X)^{1-\varepsilon}}.
\]

where \( P = P \left( \exp \left( (\log X)^{\varepsilon/2} \right), \exp \left( (\log X)^{1-\varepsilon/2} \right) \right) \), and \( A \) is an arbitrarily constant with \( A > c + 3 \). Inserting these estimates and Hypothesis (ii) into (8.20), we get

\[
\sum_{n \in J} |f(n)| \ll \frac{X}{(\log X)^{1-\varepsilon}}.
\]

Substituting this and (8.19) into (8.18), we finally derive

\[
\sum_{q \leq \sqrt{X}} c_q E(f, X; q) \ll X (\log X)^{\frac{1}{2} + \varepsilon}.
\]

(8.21)

This completes the proof of Theorem 1.2, in view of (8.17).

9. Second proof of Theorem 1.1

We shall first deduce the following estimate from Section 8

\[
\sum_{q \leq \sqrt{X}} \left( \sum_{n \equiv 1(\mod q)} f(n) - \frac{1}{\varphi(q)} \sum_{n \equiv 1(\mod q)} f(n) \right) \ll \varepsilon X (\log X)^{\frac{1}{2} + \varepsilon}.
\]

(9.1)

With notation \( I, J \) as in Section 8, it is easily seen from the argument in (8.18)-(8.21) that

\[
\sum_{q \leq \sqrt{X}} \left( \sum_{n \equiv 1(\mod q)} f(n) - \frac{1}{\varphi(q)} \sum_{n \equiv 1(\mod q)} f(n) \right) \ll \varepsilon X (\log X)^{\frac{1}{2} + \varepsilon}.
\]

Then it suffices to estimate the contribution of terms from \( n \in I \). Similar to the argument in Section 7, we shall decompose the sums over \( q \) and \( n \) into short intervals \( (Q(1 + \Delta_2)^{-1}, Q] \)
and \((N(1 + \Delta_2)^{-1}, N]\). Then we have
\[
T^I := \sum_{q \leq \sqrt{X}} \left( \sum_{n \leq q^2, n \in \mathcal{I}} f(n) - \frac{1}{\varphi(q)} \sum_{n \leq q^2, n \equiv 1 \mod{q}} f(n) \right)
\]
\[
\ll \sum_{j_1, j_2 \geq 0} \left| \sum_{Q(1+\Delta_2)^{-1}<q\leq Q} \left( \sum_{n \leq q^2, n \in \mathcal{I}} f(n) - \frac{1}{\varphi(q)} \sum_{n \leq q^2, n \in \mathcal{I}} f(n) \right) \right| \]
\[
:= \sum_{j_1, j_2 \geq 0} |T^I_{j_1, j_2}|,
\]
where \(Q = \sqrt{X}(1 + \Delta_2)^{-j_1}, N = X(1 + \Delta_2)^{-j_2}\). Since \(n < N\) and \(q > Q(1 + \Delta_2)^{-1}\) in each sum \(T^I_{j_1, j_2}\), the condition \(n \leq q^2\) can be dropped as soon as
\[
N \leq Q^2(1 + \Delta_2)^{-2}. \tag{9.2}
\]
If the condition (9.2) is satisfied, the variables \(q, n\) are independent. Thus, we can apply the estimate (8.17) for \(N \geq X^{1-\varepsilon}\) and the large sieve inequality for \(N < X^{1-\varepsilon}\). These give
\[
\sum_{N \leq q \leq Q^2(1 + \Delta_2)^{-2}} |T^I_{j_1, j_2}| \ll \Delta_2^{-2} (\log X)^2 \frac{X}{(\log X)^A}
\]
for any \(A > 0\). It remains to estimate the contribution of \(j_1, j_2\) when (9.2) is not satisfied. The treatment here is different from the corresponding in Section 7, since there is no good upper bound for the individual \(f(n)\). From \(n(1 + \Delta_2) > N\) and \(q \leq Q\), we deduce that \(n > q^2(1 + \Delta_2)^{-3}\). So the contribution of \(j_1, j_2\) such that \(N > Q^2(1 + \Delta_2)^{-2}\) is at most
\[
\sum_{q \leq \sqrt{X}} \left( \sum_{q^2(1+\Delta_1)^{-3}<n\leq q^2} |f(n)| + \frac{1}{\varphi(q)} \sum_{q^2(1+\Delta_1)^{-3}<n\leq q^2} |f(n)| \right)
\]
\[
\ll X^{\frac{1}{2}} (\log X)^{\frac{1}{2}} \left( \sum_{n \leq X} \left( \sum_{\sqrt{n} \leq q \leq \sqrt{X(1+\Delta_1)^{3/2}}} 1 \right)^2 \right)^{\frac{1}{2}} + \Delta_1^{\frac{1}{2}} X (\log X)^{\frac{1}{2}} \tag{9.3},
\]
where we exchange the order of summations over \(q\) and \(n\), and use the Cauchy–Schwarz inequality and Hypothesis (i). We next use the trivial bound \(\tau(n-1)\) for the sum over \(q\) once and exchange the order of summations again. This gives
\[
\sum_{n \leq X} \left( \sum_{\sqrt{n} \leq q \leq \sqrt{X(1+\Delta_1)^{3/2}}} 1 \right)^2 \ll \sum_{q \leq \sqrt{X(1+\Delta_1)^{3/2}}} \sum_{q^2(1+\Delta_1)^{-3}<n\leq q^2} \tau(n-1)
\]
\[
\ll \sum_{q \leq \sqrt{X(1+\Delta_1)^{3/2}}} \tau(q) \sum_{q^2(1+\Delta_1)^{-3}<n\leq q} \tau(n)
\]
\[
\ll \Delta_1 X (\log X)^2.
\]
Inserting this into (9.3), we obtain that the contribution of \(j_1, j_2\) such that \(N \geq X^{1-\varepsilon}\) is bounded by \(O\left(\Delta_1^{\frac{1}{2}} X (\log X)^{\frac{1}{2}}\right)\). On taking \(\Delta_1 = (\log X)^{-A/3}\) for some sufficiently large \(A\),
we get
\[ T^2 \ll \frac{X}{(\log X)^{A/4}}, \]
which further implies the estimate (9.1).

By the identity (7.2) and Hypothesis (i), we have
\[
\sum_{n \leq X} f(n) \tau(n - 1) = 2 \sum_{q \leq \sqrt{X}} \sum_{q^2 n \leq X \atop n \equiv 1 (\text{mod} \ q)} f(n) + O\left( X^{3/4 + \varepsilon} \right).
\]

According to Theorem 1.2 and the estimate (9.1), we get
\[
\sum_{n \leq X} f(n) \tau(n - 1) = 2 \sum_{q \leq \sqrt{X}} ^\frac{1}{\varphi(q)} \sum_{q^2 n \leq X \atop (n,q)=1} f(n) + O\left( X(\log X)^{1/2 + \varepsilon} \right).
\]

Finally, Theorem 1.1 follows from exchanging the order of summations and inserting the estimate in Lemma 3.7.

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