Newman-Janis Ansatz for rotating wormholes

A C Gutiérrez-Piñeres1, N H Beltrán2 and C S López-Monsalvo3

1Escuela de Física, Universidad Industrial de Santander, Santander, A. A. 678 Bucaramanga 680002, Colombia
2Instituto de Ciencias Nucleares, Universidad Autónoma de México, AP 70543, México, DF 04510, México
3Conacyt-Universidad Autónoma Metropolitana Azcapotzalco Avenida San Pablo Xalpa 180, Azcapotzalco, Reynosa Tamaulipas, 02200 Ciudad de México, México

1E-mail: acgutier@uis.edu.co
2E-mail: nicolas.hernandez7@correo.uis.edu.co
3E-mail: cslopezmo@conacyt.mx

Abstract. A central problem in General Relativity is obtaining a solution to describe the source’s interior counterpart for Kerr black hole. Besides, determining a method to match the interior and exterior solutions through a surface free of predefined coordinates remains an open problem. In this work, we present the ansatz formulated by the Newman-Janis to generate solutions to the Einstein field equation inspired by the mention problems. We present a collection of independent classes of exact interior solutions of the Einstein equation describing rotating fluids with anisotropic pressures. Furthermore, we will elaborate on some obtained solutions by alluding to rotating wormholes.

1. Introduction

In 1963 Roy Kerr [1] discovered their solution to the Einstein field equations describing a spinning black hole. One year later, Newman and Janis [2] demonstrated another form to derive this solution by making a complex transformation to the Schwarzschild solution. Since then, there is an increasing interest in generating new solutions from a prescribed seed using the original Newman and Janis ideas.

Nowadays, the discussion about how the Newman-Janis Ansatz (NJA) mathematical nature remains open to research. As far as we known, maybe the Drake and Szekeres work [3] and the set by Azreg [4] are currently the most telling about the mathematical view and the applicability of the NJA. In 2016, one of us and H. Quevedo [5] asked themselves if conformal symmetry prevails after using this formalism. The answer was no for a conformastatic seed. This paper shall continue this investigation and explore the possibility of finding new solutions to the Einstein equations describing self-rotating fluids.

Since the paper by Ellis in 1973 [6] and Morris and Thorne in 1988 [7] set up an aesthetical and conceptual renaissance of the notion of wormholes, abundant literature has grown up proposing new examples of non-traversable wormholes. Three among the many problems concerning the original idea remains open to debate: First, how to match the interior solution of the Einstein field equations, whose metric is given by a wormhole, to an exterior solution. Second, to provide wormholes formalism with arbitrary symmetries. Moreover, finally, to construct rotating wormholes with appropriate physical
behaviour. For a discussion on matching wormholes to their exterior space-time, see [8], whereas for discussion on rotating wormholes, see [9,10,11].

In section 2 of this paper, we present the ansatz formulated by Newman-Janis to generate solutions to the Einstein field equations for rotating space-times. We employ the differential geometry in the fashion that lets transparent the idea according to which the formalism abandons the transformation between metric spaces in favour of mappings between tetrads.

Section 3 presupposes that the rotating solution obtained using NJA describes a stationary interior space-time corresponding to an anisotropic fluid without heat. We implement this assumption in the Einstein field equations and achieve five independent classes of interior solutions.

Finally, in section 4, we present a short description of the mathematical aspects necessary to generate the rotating counterpart of static wormholes by employing the NJA. Next, we construct the Kerr-Newman wormholes using the Reissner-Nordström wormhole as a seed solution to illustrate the discussed ideas. We conclude this work with some remarks and open proposals for further investigations.

2. The Newman-Janis Ansatz
Let \((t, r, \theta, \phi)\) be the coordinates corresponding to point \(p\) in a manifold \(\mathcal{M}\), and \((\partial_t, \partial_r, \partial_\theta, \partial_\phi)\) the basis of tangent vectors of \(T_p(\mathcal{M})\). Thus, we have a general metric tensor \(g_{0} \in T^*_p(\mathcal{M})\) given in the manner by

\[
g_{0} = G(r) \, dt \otimes dt - \frac{1}{F(r)} \, dr \otimes dr - H(r) (d\theta \otimes d\theta + \sin^2 \theta \, d\phi \otimes d\phi). \tag{1}
\]

Below, we outline the procedure employed for the first time in 1964 by Newman and Janis [2] to derive the Kerr metric by performing a complex coordinate transformation on the Schwarzschild solution. We sketch it in modern fashion consisting of fourth steps and employ a static space-time sufficiently general to generate a master metric describing an axially rotating space-time.

2.1 First step
We perform the mapping from spherical coordinates to outgoing Eddington-Finkelstein coordinates. In what follows, we shall use this metric tensor as a seed solution to implement the Newman Janis Ansatz (NJA). To this end, we first perform the transformation

\[
\Pi: (t, r, \theta, \phi) \rightarrow (u, r, \theta, \phi),
\]

which maps spherical coordinates to outgoing Eddington-Finkelstein coordinates through the relation

\[
u = t - \int \frac{dr}{\sqrt{FG}}.
\]

Consistently, the mapping \(\Pi\) induces the pullback \(\Pi^*\) which carries the metric \(g_{s}\) into the metric \(g_{0}\), i.e., \(\Pi^*g_{s} = g_{0}\). Then, the metric (1) can be getting through the action of the map \(\Pi^*\) on the metric \(g_{s} \in T^*_p(\mathcal{M})\) given by

\[
g_{s} = E^t \otimes E^t - E^r \otimes E^r - E^\theta \otimes E^\theta - E^\phi \otimes E^\phi \tag{2}
\]

where the basis vector \(E^t\) are given by the orthonormal tetrad

\[
E^t \equiv \sqrt{G} \, du + \frac{dr}{\sqrt{F}},
\]

\[
E^r \equiv \frac{dr}{\sqrt{F}},
\]

\[
E^\theta \equiv \sqrt{H} \, d\theta,
\]

\[
E^\phi \equiv \sin \theta \sqrt{H} \, d\phi.
\]

2.2 Second step
We can turn any real vector space \(V\) into a complex vector space \(V^C\) by forming the set \(V \times V\) of all pairs \((E_i, E_j)\) with \(E_i, E_j \in V\) and then writing \((E_i, E_j)\) as \(E_i + iE_j\). With this definition \(V^C\) becomes
complex vector space (For more details, see [12]). Hence, we can construct a set of vectors forming a basis of \( \{L, N, M, W\} \) in \( T^c_p(\mathcal{M}) \) defined by

\[
L \equiv \frac{1}{\sqrt{F}} (E_t + E_r) = \partial_r,
\]
\[
N \equiv \frac{\sqrt{F}}{2} (E_t - E_r) = \frac{\sqrt{F}}{2} \partial_u - \frac{1}{2} \partial_r,
\]
\[
M \equiv \frac{\sqrt{2}}{2} (E_\theta + i E_\phi) = \frac{\sqrt{2}}{2} \partial_\theta + \frac{\sqrt{2} i}{2 \sin \theta \sqrt{H}} \partial_\phi,
\]
\[
W \equiv \frac{\sqrt{2}}{2} (E_\theta - i E_\phi) = \frac{\sqrt{2}}{2} \partial_\theta - \frac{\sqrt{2} i}{2 \sin \theta \sqrt{H}} \partial_\phi.
\]

This basis constitutes a complex null tetrad, i.e., consists of two real null vectors \( L, N \) and two complex conjugate null vectors \( M, W \) thus, the scalar products of tetrads vectors satisfy: \( L \cdot L = M \cdot M = = N \cdot N = L \cdot M = M \cdot N = 0 \) and \( L \cdot N = -M \cdot W = 1 \).

2.3 Third step (a)

Next, we introduce the “rotated” Eddington-Finkelstein coordinates by performing the mapping

\[ \Pi_R: (u, r, \theta, \phi) \rightarrow (u_R, r_R, \theta_R, \phi_R) \]

through the transformation

\[
\begin{align*}
    u_R &= u - ia \cos \theta, \\
    r_R &= r + ia \cos \theta, \\
    \theta_R &= \theta, \\
    \phi_R &= \phi.
\end{align*}
\]

The map \( \Pi_R \) naturally induces the push-forward \( (\Pi_R)_* \),

\[ (\Pi_R)_*: T^c_p(\mathcal{M}) \rightarrow T^c_{\Pi_R(p)}(\mathcal{N}). \]

Hence, it is very easy to verify that the basis vector \( \xi \equiv \{L, N, M, W\} \in T^c_p(\mathcal{M}) \) is mapped by \( \Pi_R \) into the basis \( \xi_R \equiv \{L_R, N_R, M_R, W_R\} \in T^c_{\Pi_R(p)}(\mathcal{N}) \) and then

\[
\begin{align*}
L_R &= \partial_{r_R}, \\
N_R &= \frac{\sqrt{F}}{\sqrt{A}} \partial_u - \frac{B}{2} \partial_{r_R}, \\
M_R &= \frac{\sqrt{2}}{2 \sqrt{\Psi}} \left[ ia \sin \theta_R (\partial_{u_R} - \partial_{r_R}) + \partial_{\theta_R} + \frac{i}{\sin \theta_R} \partial_{\phi_R} \right], \\
W_R &= \frac{\sqrt{2}}{2 \sqrt{\Psi}} \left[ ia \sin \theta_R (\partial_{r_R} - \partial_{u_R}) + \partial_{\theta_R} - \frac{i}{\sin \theta_R} \partial_{\phi_R} \right],
\end{align*}
\]

where \( A = A(r, \theta), B = B(r, \theta), \Psi = \Psi(r, \theta) \) are functions to be known.

2.4 Third step (b)

As it is evident from (4), \( \xi_R \) is a complex null consisting of two real null vectors \( L_R, N_R \) and two complex conjugate vectors \( M_R, W_R \), hence, the scalar products of tetrads vectors satisfy:

\[
L_R \cdot L_R = M_R \cdot M_R = N_R \cdot N_R = L_R \cdot M_R = M_R \cdot N_R = 0 \text{ and } L_R \cdot N_R = -M_R \cdot W_R = 1.
\]

Moreover,

\[
\eta_{ij}^{j_R} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0
\end{pmatrix}
\]

are the components of a specific metric tensor \( G_R \) respect to the complex null tetrad, i.e.,

\[
(G_R)_{ij} = (L_R)_j(N_R)_i + (N_R)_j(L_R)_i - (M_R)_j(W_R)_i - (W_R)_j(M_R)_i.
\]

Hence, the metric tensor \( G_R \) can be expressed in terms of the real basis \( \partial_R \) as:
\[ \mathcal{G}_R = A \ l_{uR} \l_{duR} + 2 \sqrt{\frac{\mathcal{A}}{B}} \ l_{uR} \l_{drR} + 2 \sin^2 \theta_R \left( \sqrt{\frac{\mathcal{A}}{B}} - A \right) \ l_{uR} \l_{d\phi_R} - 2 \sin^2 \theta_R \sqrt{\frac{\mathcal{A}}{B}} \ l_{drR} \l_{d\phi_R} - \Psi \ l_{d\theta} \l_{d\theta} + \frac{\sin^2 \theta_R}{B} \left[ a^2 \sin^2 \theta_R (AB - 2 \sqrt{AB}) - B \Psi \right] \ l_{d\phi_R} \l_{d\phi_R}. \]  

(6)

2.5 Fourth step

We introduce the Boyer Lindquist coordinates by implementing a map

\[ \Pi_{BL}: (u_R, r_R, \theta_R, \phi_R) \rightarrow (T, R, \Theta, \Phi) \]

from the coordinates \((u_R, r_R, \theta_R, \phi_R)\) into the \((T, R, \Theta, \Phi)\) through the following transformation:

\[ u_R = T - \frac{2}{a^2 + FH} R, \quad \phi_R = \Phi + \frac{a}{a^2 + FH} R. \]

Thus, the metric (6) admits the form

\[ \mathcal{G} = \left( \frac{a^2 \cos^2 \Theta + FH}{a^2 \cos^2 \Theta \sqrt{\sqrt{\frac{\mathcal{A}}{B}}}} \right) dT \l_{dT} - \frac{\Psi}{\sqrt{\frac{\mathcal{A}}{B} + a^2}} \ l_{dR} \l_{dR} - \Psi \ l_{d\theta} \l_{d\theta} - \Psi \sin^2 \Theta \left[ 1 + \frac{a^2 \sin^2 \Theta \sqrt{2(FH - \sqrt{\frac{\mathcal{A}}{B} + a^2 \cos^2 \Theta \sqrt{\sqrt{\frac{\mathcal{A}}{B}}}})}}{(a^2 \cos^2 \Theta \sqrt{\sqrt{\frac{\mathcal{A}}{B}}})} \right] \ l_{d\phi} \l_{d\phi} - 2 \sqrt{\frac{\mathcal{A}}{B}} \ l_{dR} \l_{dR} \ l_{d\theta} \l_{d\theta} - 2 \sqrt{\frac{\mathcal{A}}{B}} \ l_{dR} \l_{d\phi} \ l_{d\theta} \l_{d\phi}. \]

(7)

This result shows that, by implementing the former successive step, the Newman-Janis Ansatz generates a stationary axially symmetric space-time from static spherically symmetric.

3. Rotating anisotropic fluids from the NJA

To interpret the metric tensor (7), we presuppose that it describes a stationary interior space-time corresponding to an anisotropic fluid without heat. This kind of fluids admits a representation in terms of a tensor’s components in the following system of independent equations for the energy density and the anisotropic pressures

\[ (T)^{ab} = \mu(V)^a(V)^b + P_R (e_R)^a (e_R)^b + P_\Theta (e_\Theta)^a (e_\Theta)^b + P_\Phi (e_\Phi)^a (e_\Phi)^b, \]

(8)

provieded that the space-time endowed with some Lorentzian pseudo-Riemannian metric with signature \((+2)\).

\[ (\mathcal{G})^{ab} = -(V)^a(V)^b + (e_R)^a (e_R)^b + (e_\Theta)^a (e_\Theta)^b + (e_\Phi)^a (e_\Phi)^b, \]

(9)

where \(\mu(R, \Theta)\) is the energy density of the fluid and \(P_R(R, \Theta), P_\Theta(R, \Theta)\) and \(P_\Phi(R, \Theta)\) are the pressure along the corresponding directions. The word-lines of the fluids are integral curves of the velocity vector \(V\), which satisfies the condition \(\mathcal{G}(V, V) = -1\). Here, \((V)^a, (e_R)^a, (e_\Theta)^a\) and \((e_\Phi)^a\) are the components of the vectors of the orthonormal tetrada given by

\[ V = -\frac{1}{\sqrt{\Psi(FH + a^2)}} \left( \frac{\sqrt{\mathcal{A}}}{\sqrt{\frac{\mathcal{A}}{B}}} + a^2 \right) \partial_T + a \partial_\phi, \]

\[ e_R = \frac{\sqrt{\Psi(FH + a^2)}}{\Psi} \partial_R, \]

\[ e_\Theta = \frac{1}{\sqrt{\Psi}} \partial_\Theta, \]

\[ e_\Phi = -\frac{1}{\sqrt{\Psi \sin^2 \Theta}} (a \sin^2 \Theta \partial_T + \partial_\phi). \]

(10)

Thus, the Einstein field equations \(\mathbf{G} = \delta \mathbf{G} \) with the metric tensor (7) and the energy-momentum tensor given by (8) yields the following system of independent equations for the energy density and the anisotropic pressures.

1 We developed most of the following calculations using a computational routine in Maple 2020. To preserve the consistency with the Differential Geometry package, we changed the signature (-2) to (2) in this section and the rest.
\[\mu = \frac{1}{32\pi \rho \Psi^3} \{3\rho^4(\Psi^2_\theta + \Delta \Psi^2_R) - 4\rho^4\Psi(\Psi_\theta\theta + \Delta \Psi_{RR}) - \rho^2\Psi^2_\theta (\rho^2\Delta_R - a^2\sin^2\Theta K_R) + 2a^2\sin^2\Theta \rho^2\Psi^2 K_{RR} - 2\cot\Theta \rho^2\Psi^2_\theta (K + a^2) + \Psi^2 [3a^2\sin^2\Theta K^2_R - 4(\rho^2 + 4a^2\cos^2\Theta)(K + +a^2)] + 12\cos^2\Theta \Delta^2]\},
\]

\[P_R = \frac{1}{32\pi \rho \Psi^3} \{4\rho^4\Psi_\theta\theta + 3\rho^4(\Psi^2_R - \Psi^2_\theta) + \rho^2\Psi^2_\theta (\rho^2\Delta_R - 2\Delta K_R) + \rho^2\Psi^2_\theta [a^2\sin^2\Theta + 2\cot\Theta (K + a^2)] - 4K(K + 2a^2) + \Psi^2 [a^2\sin^2\Theta K^2_R + 4a^2\cos^2\Theta (6\rho^2 + a^2\sin^2\Theta) - -(\Delta + a^2\cos^2\Theta)]\},
\]

\[P_\theta = \frac{1}{32\pi \rho \Psi^3} \{4\rho^2\Delta\Psi (\rho^2\Psi_{RR} - \Psi K_{RR}) + 2\rho^4\Psi^2 (F_{RRH} + FH_{RR}) + 2\rho^2 [\cot\Theta \Psi^2_\theta (K + a^2(1 + \sin^2\Theta)) + \Psi^2_\theta (\rho^2\Delta_R - \Delta K_R)] + 3\rho^4 (\Psi^2_\theta - \Delta \Psi^2_\theta) + \Psi^2 [K^2_R (8\Delta - a^2\sin^2\Theta) - -6\rho^2 K_R\Delta_R + 4 (\rho^4 F_{RHR} + a^2\cos^2\Theta \Delta)]\},
\]

where \(\Psi\) denotes derivative and we used the simple notation \(\rho^2 \equiv K + a^2\cos^2\Theta, K \equiv H\sqrt{F/G}\) and \(\Delta \equiv FH + a^2\). In addition to the above system, the Einstein equations give us the following two nontrivial independent equations:

\[3\Psi R \Psi_{,\theta} - 2\Psi_{,\theta} \rho^4 + 3a^2\sin 2\Theta K_R \Psi^2 = 0,
\]

\[\left(\Psi K_{,R} + 2\Psi \cot\theta\right) (K + a^2\cos^2\Theta)^2 - \Psi (2K - a^2\cos^2\Theta K^2_R) = 0,
\]

which we must solve to determine the geometric and physics conditions on \(K\) and \(\Psi\). As we know, there at least five classes of solutions of the above equation system. They are:

Class 1: \(K = c_1, \Psi = \frac{a^2\cos^2\Theta + c_1}{\cos\theta}\),

Class 2: \(K = R^2, \Psi = \frac{a^2}{(\sqrt{2} c_2 \cos\Theta R - c_2)^2}\),

Class 3: \(K = (R - c_1)^2, \Psi = \frac{-a^2 [\cos^2\Theta + (R - c_1)^2]}{(R - c_1)^2 \cos^2\Theta}\),

Class 4: \(K = R^2 + c_1 R + c_2, \Psi = \frac{a^2 \cos^2 \Theta + R^2 + c_1 R + c_2}{(R - c_1)^2 \cos^2 \Theta}\),

Class 5: \(K = \frac{\exp(c_2 + R) + 2}{2 \exp(c_2 + R)}, \Psi = \frac{2a^2 \cos^2 \Theta \exp(c_2 + R) - \exp(c_2 + R) + 2}{\cos^2 \Theta \exp(c_2 + R) + 2}\),

c_1, c_2 and c_3 are arbitrary constants, and \(K^2 = H^2 F/G\) is a relation between the coefficients of the second static metric. As we can see, each class defines a kind of specific rotating fluid. In particular, if we introduce \(c_1 = c_2\) in the Class 4, we get a solution of the form \(\Psi = a^2\cos^2\Theta + R^2\), and \(K = R^2\) form. Additionally, if we choose \(K = \sqrt{F/G}\) determining the Schwarzschild solution (i.e., \(H = R^2\) and \(F = G = 1 - \alpha / R\), with \(\alpha\) an arbitrary constant), we achieve a rotating fluid without heat flux with all pressures equal to zero. That case corresponds to Kerr’s solution. We will generate rotating wormholes form statics via the NJA by imposing some geometric and physics conditions on \(K\) and \(\Psi\) functions; it will be the objective of the following section.

4. NJA for rotating wormholes

4.1 A class of static wormholes fixed by the Newman-Janis Ansatz

To construct static wormholes form solutions of the Einstein field equations for an interior region of the space-time, we first bring the metric (1) to the form:

\[G_0 = -e^{2\phi(R)} dT \otimes dT + \frac{\text{d}R}{1 - R^2} + R^2 (d\theta \otimes d\theta + \sin^2\theta d\Phi \otimes d\Phi).
\]

Hence, we introduced the usual spherical coordinates (\(R, \Theta, \Phi\)) and performed the identification
\[ G(R) = e^{2\varphi(R)}, \quad F(R) = 1 - \frac{b(R)}{R}, \quad H(R) = R^2. \] (18)

The functions \( \varphi \) and \( b \) determine the gravitational redshift and the spatial shape of the wormhole [7]. Naturally, the match of wormhole to an exterior space-time must occur on a determined value \( R_s \) to the radial coordinate \( R \). To guarantee that the Birkhoff theorem is satisfied in the wormhole exterior is mandatory to impose the conditions:

(i) \( b(R) = b(R_s) = \text{constant} \equiv b_0, \quad R > R_s, \)

(ii) \( \varphi(R) = \frac{1}{2} \ln \left( 1 - \frac{b_0}{R} \right), \quad R > R_s. \)

Additionally, to ensure that we get asymptotically flat solutions, we demand that:

(i) \( \lim_{R \to \infty} b(R) \rightarrow 0, \)

(ii) \( \lim_{R \to \infty} \varphi \rightarrow 0. \)

To embed, in the three-dimensional Euclidean space, a two-dimensional surface with the same geometry we set up \( \left( \Theta = \frac{\pi}{2}, T = \text{constant} \right) \) in the metric (17)

\[ S = \frac{1}{1-b(R)/R} \text{d}R \otimes \text{d}R + R^2 \text{d}\Phi \otimes \text{d}\Phi. \] (19)

Then, the surface of rotation

\[ \chi(R, \Phi) = \left( R \cos \Theta, R \sin \Theta, Z(R) \right) \] (20)

determines the spatial geometry of the wormhole spacetime through the single function \( Z(R) \), which satisfy the equation

\[ \frac{dZ(R)}{dR} = \pm \left( \frac{R}{b(R)} - 1 \right)^{1/2}. \] (21)

If we assume that the metric (17) determines the existence of a static wormhole, their rotating counterpart generated by the NJA presents a restriction given by the relation between the metric functions provided by \( K^2 = (F/G)R^4 \). Hence, the unique solutions to the Einstein field equations describing static wormholes suitable to generate rotating wormholes by using NJA satisfy the relation:

\[ e^{2\varphi(R)} = \frac{R^4}{K^2} \left[ 1 - \frac{b(R)}{R} \right] \] (22)

with \( K \) determined by an arbitrary static solution of the Einstein field equations, consistent with the five Classes in the former section.

4.2 Space-time for rotating wormholes

The procedure employed here to construct rotating wormholes starts by bringing the stationary axially symmetric space-time previously generated via the NJA to the following form:

\[ \mathcal{G} = -e^{2\nu(R, \Theta)} \text{d}T \otimes \text{d}T + e^{2\mu_2(R, \Theta)} \text{d}R \otimes \text{d}R + e^{2\mu_3(R, \Theta)} d\Theta \otimes d\Theta + e^{2\lambda(R, \Theta)} (d\Phi - \omega \text{d}T) \otimes (d\Phi - \omega \text{d}T), \] (23)

where

\[ e^{2\nu} = \frac{\Delta \Psi}{\rho^4} \left[ (a^2 + K - a^2 \sin^2 \Theta) \Delta \right], \quad e^{2\mu_2} = \frac{\Psi}{\Delta}, \quad e^{2\mu_3} = \Psi, \]

\[ e^{2\lambda} = \sin^2 \Theta \frac{\Psi}{\rho^4} \left[ (a^2 + K) - a^2 \sin^2 \Theta \Delta \right], \quad \omega = \frac{a(a^2 + K - \Delta)}{(a^2 + K)^2 - a^2 \sin^2 \Theta \Delta}, \]

\[ K^2 = \frac{F}{G} H^2, \quad \rho^2 = K + a^2 \cos^2 \Theta, \quad \Delta = FH + a^2, \]

where, as we known, each couple \((K, \Psi)\) determine a new rotating solution of the Einstein field equations provided a static solution is known. Next, we suppose that there exists a function \( B = B(R, \Theta) \) such that \([9,10,11]::\)

\[ e^{2\mu_2} = \mathcal{G}_{RR} = \frac{1}{1 - \frac{B}{R}}, \quad \lim_{R \to \infty} \frac{B}{R} = 0, \quad \lim_{R \to \infty} \mathcal{G}_{TT} = 1. \]

Additionally, to avoid conic singularities we assume that, on the axis of rotation \( \Theta = 0 \) or \( \Theta = \pi \), the following relation it is always possible:
\[ G_\Theta = \frac{G_{\Phi \Phi}}{\sin^2 \Theta} \]

and, that the near the throat:

\[ R \, \partial_R B < B. \]

Furthermore, to ensure the applicability of the NJA, the choices of the metric functions corresponding to the static space-time and the corresponding to the rotating space-time are restricted by the following constrain:

\[ F = 1 - \frac{b(R)}{R}, \quad e^{2\varphi(R)} = \frac{R^2}{K^2} \left[ 1 - \frac{b(R)}{R} \right]^\frac{1}{2}, \]

where, as is known, exist a unique correspondence between each choice of \( K \) and \( \Psi \) determined by the classes of rotating solutions discussed above. Then, finally, the weakest condition to be satisfied by the rotating wormhole is given by the following expression

\[ \frac{b(R, \Theta)}{R} = 1 - \frac{R^2}{\Psi} \left( 1 - \frac{b(R)}{R} + \frac{a^2}{R} \right), \]

provided the \( b(R) \) function defines a static wormhole and \( B(R, \Theta) \) satisfies all the conditions previously discussed.

4.3 Example of Rotating wormholes from the NJA

As a straightforward example, let us consider the static solution

\[ b(R) = R_S - q^2 R, \]  

where \( R_S \) and \( q \) are real constants. This solution determines the Reissner-Nordström wormhole, the spatial geometry of the wormhole spacetime through the single function \( Z(R) \) satisfying

\[ dZ(R) = \pm \left( \frac{R}{R_S - q^2 R} - 1 \right)^\frac{1}{2}, \]

and energy-momentum tensor given by [7]

\[ \sigma(R) = p(R) = \tau(R) = \frac{q^2}{8\pi R^5}. \]

Accordingly, if we choose, for example \( K \) and \( \Psi \) as given by the Class 2 of rotating solutions obtained by the NJA, then we achieve a Kerr-Newman wormhole:

\[ \frac{B(R, \Theta)}{R} = 1 - \frac{R^2}{R^2 + a^2 \cos^2 \Theta} \left( 1 - \frac{q^2}{R} + \frac{a^2}{R} \right). \]

The wormhole rotates with angular velocity given by

\[ \omega = \frac{q^2}{(a^2 + R^2)^2 - a^2 \sin^2 \Theta \left( a^2 + R^2 + q^2 - R_S R \right)}, \]

and their corresponding energy-momentum tensor given by the dynamical quantities:

\[ \sigma = \tau = p = \frac{q^2}{8\pi \left( a^2 \cos^2 \Theta + R^2 \right)^2}. \]

In other words, the rotating counterpart of the Reissner-Nordström wormhole obtained by the NJA corresponds to the Kerr-Newman wormhole.

5. Conclude and remark

This paper presented the ansatz formulated by Newman-Janis to generate solutions to the Einstein field equations for rotating space-times. We performed the algorithm’s structure based on modern differential geometry to keep open the discussion about their nature. Actually, we discuss the algorithm step setting successive maps on the tangent and cotangent spaces, and compile some independent classes of solutions of the Einstein field equations for rotating space-times, all obtained from prescribed static and spherically space-times.

We have elaborated on the possibility of constructing static wormholes and their rotating counterpart by choosing any of the classes of solutions generated by the Newman-Janis Ansatz. These classes restrict the nature of both static and rotating wormholes. To illustrate the discussed ideas, we generated the
Kerr-Newman wormhole using the Reissner-Nordström wormhole as a seed solution. It will be fructiferous to produce many examples of rotating wormholes operating with the ideas presented in this work. The problem of match wormholes to their exterior space-time remains open. One emergent here is studying the wormholes’ asymptotically flat characteristics and using a $C^3$ junction technique [13,14]. We will explore that in subsequent work.

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