The invariant factor of the chiral determinant

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The coupling of spin 0 and spin 1 external fields to Dirac fermions defines a theory which displays
gauge chiral symmetry. Quantum mechanically, functional integration of the fermions yields the
determinant of the Dirac operator, known as the chiral determinant. Its modulus is chiral invariant
but not so its phase, which carries the chiral anomaly through the Wess-Zumino-Witten term.
Here we find the remarkable result that, upon removal from the chiral determinant of this known
anomalous part, the remaining chiral invariant factor is just the square root of the determinant
of a local covariant operator of the Klein-Gordon type. This procedure bypasses the integrability
obstruction allowing to write down a functional that correctly reproduces both the modulus and
the phase of the chiral determinant. The technique is illustrated by computing the effective action
in two dimensions at leading order in the derivative expansion. The results previously obtained by
indirect methods are indeed reproduced.

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I. INTRODUCTION

In this work we consider even dimensional Dirac fermions which move in the presence of external fields of the
type scalar, pseudo-scalar, vector and axial-vector and with general non abelian internal degrees of freedom.
At the classical level the theory is invariant under chiral gauge transformations. Quantum-mechanically, Feynman’s
functional integral has to be evaluated, formally producing the determinant of the Dirac operator, the so called chiral
determinant. There is large amount of literature on this subject. A good recollection of it can be found in [1]. Much
work has been devoted to the purely gauge case (a single chirality and no zero spin external fields) regarding its
consistency and anomalies. Here we will discuss the full coupling case only. This kind of setting appears naturally in
low energy quark models of QCD [2, 3, 4, 5] and so has direct phenomenological implications. More generally it is of
immediate interest in models where fermions have to be integrated out. See e.g. [6] for an application to the study of
CP violation in early cosmology.

The effective action functional, the logarithm of the chiral determinant, is an extension of the thermodynamic
potentials where the Lagrangian multipliers are external background fields. This functional compactly embodies the
properties of the field-theoretic system and in particular its symmetries [7].

As is well-known chiral symmetry is not preserved after quantization, rather it develops an anomaly which survives
the process of regulating the ultraviolet divergencies [8, 9, 10]. The chiral anomaly does not affect the real part of
the effective action [11]; this quantity can be mapped to a bosonic theory and so can be computed using a plethora
of techniques since chiral invariance can be invoked to simplify the calculations.

The imaginary part of the effective action carries the chiral anomaly. For this reason it is more difficult to work
with but also more interesting. Most efforts have concentrated on the properties of the anomaly and of its generating
effective action, the gauged Wess-Zumino-Witten term, since it poses a theoretical challenge [11, 12, 13, 14, 15, 16,
17, 18, 19, 20, 21, 22, 23]. The computation of the effective action itself has received less attention. In some cases,
such as scalar and pseudo-scalar fields complying with the chiral circle constraint, the imaginary part of the effective
action is saturated by the Wess-Zumino-Witten term, at least to leading order, but in the general case, or at finite
temperature [24], there is a non trivial chiral invariant remainder.

The calculation of the imaginary part of the effective action is complicated by the lack of chiral invariance, both in
the functional itself and in the formalism used to compute it. Perturbative calculations or inverse mass expansions
hide the underlying chiral invariance of the remainder once the Wess-Zumino-Witten term has been separated. In
this view the derivative expansion method is advantageous. It is a non-perturbative approach which has the virtue
that different orders are not mixed by chiral transformations. In the imaginary part, the expansion starts with the
term with as many derivatives as the dimension of the space-time. This leading order term is the only one affected
by ultraviolet divergencies, and hence the only one with anomalous breaking of the chiral symmetry.

However, even if the expansion itself preserves chiral symmetry, this symmetry can be spoiled by the regularization.
This is the case of regularizations such as the heat kernel of the squared Dirac operator, \( \mathbf{D}^2 \) (a Klein-Gordon like
operator) [25] or direct \( \zeta \)-function regularization of the \( \mathbf{D} \) [26, 27] or \( \mathbf{D}^2 \). This is because chiral symmetry does not
act as a similarity transformation on the Dirac operator, and so powers of this operator do transform in a complicated way.

In order to have a proper construction of the effective action in the anomalous sector, it is natural to use the current, i.e., the variation of the effective action under a gauge field deformation. The current has a chiral covariant part which is therefore amenable to simple and direct computation. There is an integrability obstruction, however: as a consequence of the anomaly such current is not directly consistent \([1, 20]\); it differs from the covariant one by a known counter-term \([18]\).

Once the obstruction is bypassed and the consistent current is obtained, the effective action in the abnormal parity sector gets well defined \([1]\). This idea has been successfully implemented in \([28]\) where the method of covariant symbols \([29, 30]\) is used to obtain the covariant current. There the leading order term was obtained in two and four dimensions. The same approach has been applied (this time using the world-line method instead of covariant symbols) in \([31]\) to compute the next-to-leading order in two dimensions.

The motivation for this work is as follows: Upon removal from the fermionic effective action of its well understood anomalous contribution, a chiral invariant functional is obtained. However, this observation does not directly provide us with a set of “Feynman rules” to carry out manifestly chiral covariant calculations. The procedure based on the current, to bypass the integrability obstruction, is indirect. On the other hand, all ultraviolet finite contributions are free from any anomaly and so they would not be affected by any related obstruction. The latter should only affect the Wess-Zumino-Witten term. Therefore, if one wanted to compute the chiral invariant remainder at leading order in the derivative expansion, or the effective action beyond leading order, or finite temperature corrections (thermal corrections are ultraviolet and the anomaly is temperature independent \([32]\)), etc, there should be no obstruction from the anomaly. This suggests that such chiral invariant calculational scheme should exists. The Feynman rules-like scheme follows after the chiral invariant part of the effective action can be codified in the form Tr log \(K\). This is precisely what is achieve here. Namely, we construct a second order differential operator, \(K\) (cf. \((3.4)\)) that is manifestly chiral covariant and such that \((\text{Det} K)^{1/2}\) is just the chiral invariant part of the chiral determinant (modulus and phase). Since \(K\) is a standard Klein-Gordon operator, essentially the same methods available to the real part are applicable here. We illustrate our result by computing the effective action to two derivatives in a strict derivative expansion (no other approximation is involved) in two dimensions, and show that the correct result is reproduced.

Section II introduces definitions and summarizes results related to the chiral determinant. Section III describes the construction of the Klein-Gordon operator accounting for the chiral invariant part of the effective action and proves the main result. An application is also presented within the derivative expansion. Section IV presents the conclusions.

II. THE CHIRAL DETERMINANT

In this section we summarize some theoretical results in the literature regarding chiral fermions and their effective action.

A. The Dirac operator

We consider a Dirac operator \(D\) describing Dirac fermions coupled to spin 0 and spin 1 external fields with non-abelian degrees of freedom:

\[
D = \gamma_\mu (\partial_\mu + V_\mu) + \gamma_\mu \gamma_5 A_\mu + S + \gamma_5 P. \tag{2.1}
\]

The fermions live in a \(d\)-dimensional Euclidean space-time, and \(d\) is even. We will only consider in this work the case of zero temperature and flat space-time. Our conventions are:

\[
\gamma_\mu = \gamma_\mu^\dagger, \quad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \quad \gamma_5 = \gamma_5^\dagger = i^{d/2} \gamma_0 \cdots \gamma_{d-1}, \quad \text{tr}_{\text{Dirac}(1)} = 2^{d/2}. \tag{2.2}
\]

The external fields \(V_\mu(x), A_\mu(x), S(x)\) and \(P(x)\) are square matrices in the space of the fermionic internal degrees of freedom. Unitarity requires \(S(x)\) to be a hermitian matrix and \(V_\mu(x), A_\mu(x)\) and \(P(x)\) to be antihermitian.

In order to emphasize the chiral properties it will be convenient to work with fields with well defined transformation
under chiral rotations. To this end we express the Dirac operator in the form:

$$D = \slashed{D}_R P_R + \slashed{D}_L P_L + m_{LR} P_R + m_{RL} P_L,$$  \hspace{1cm} (2.3)

where

$$P_R = \frac{1}{2} (1 + \gamma_5), \quad P_L = \frac{1}{2} (1 - \gamma_5),$$  \hspace{1cm} (2.4)

and

$$D^{R,L}_\mu = \partial_\mu + v^{R,L}_\mu, \quad v^{R,L}_\mu = V_\mu \pm A_\mu, \quad m_{LR} = S + \mathcal{P}, \quad m_{RL} = S - \mathcal{P}.$$  \hspace{1cm} (2.5)

Unitarity then requires

$$(v^{R,L}_\mu)_\dagger = -v^{R,L}_\mu, \quad m^{\dagger}_{LR} = m_{RL}. \hspace{1cm} (2.6)$$

In addition, we assume the matrices $m_{LR}$, $m_{RL}$ to be nowhere singular. This excludes the much studied case of fermions with a single chirality, but avoids infrared singularities (in particular, in the derivative expansion to be considered below).

**B. The effective action**

The fermionic effective action $W$ is introduced through standard functional integration of the fermionic fields

$$e^{-W} = \int \mathcal{D}\bar{\psi} D\psi e^{-\int d^4 x \bar{\psi} D\psi} = \text{Det} D$$  \hspace{1cm} (2.7)

so formally

$$W = -\text{Tr} \log D$$  \hspace{1cm} (2.8)

modulo ultraviolet (UV) ambiguities. Tr denotes the functional trace and includes space-time, internal and Dirac degrees of freedom.

It is important to recall that the Dirac operator (or more generally an action) does not define a single quantum field theory but a whole class of them. The UV ambiguities affecting the effective action can be exposed, e.g., through its computation within perturbation theory. $W$ represents the sum of all one-loop Feynman diagrams, where the fermion runs over the loop and the external fields correspond to insertions in the loop. All diagrams with more than $d$ such insertions have more than $d$ fermion propagators and so are UV finite. These contributions are therefore independent of how the theory is regularized and renormalized. The UV ambiguity is thus a polynomial of the external fields of degree at most $d$.\footnote{Alternatively, taking $n$ successive variations of the effective action with respect the external fields in $D$, with $n > d$, yields an operator of the type $D^{-n}$ whose trace is UV convergent.}

Likewise, taking a derivative with respect the momentum of the insertion decreases the UV divergence degree by one. This implies that the UV ambiguity is a polynomial in the external momenta. In summary, the UV ambiguity in $W$ is just a local Lagrangian polynomial of the external fields and their derivatives, of mass dimension $d$, i.e., the standard counter-term allowed in a renormalizable theory. (Note that in principle one can choose to include in the counter-term new external fields not present in the original theory and this is advisable in some circumstances \cite{33,34}.) So the rule is to compute $W$ using any valid procedure (i.e., one that preserves all UV finite contributions) and then add the appropriate counter-term to obtain any of the several theories described by the same Dirac operator. That is, if $W_0(D)$ is a renormalized effective action, any other determination of the effective action is related to this one by the relation

$$W(D) = W_0(D) + W_{\text{ct}}(D)$$  \hspace{1cm} (2.9)

where $W_{\text{ct}} = \int d^4 x \mathcal{L}_{\text{ct}}(x)$ and $\mathcal{L}_{\text{ct}}(x)$ is a polynomial of degree at most $d$ in the “variables” $\partial_\mu$, $v_R$, $v_L$, $m_{LR}$, and $m_{RL}$.\footnote{\textit{R} and \textit{L} stand for right and left chirality respectively. We put them as sub- or super-indices indistinguishably.}
One of the ways to achieve a valid definition of $W(D)$ is through the \(\zeta\)-function technique \[33\]. The function
\[
\zeta(s,D) = \text{Tr}(D^s)
\] (2.10)
is UV finite for $\text{Re}\, s < -d$. Its analytical extension is a meromorphic function in the complex plane $s$ with simple poles at $s = -1, -2, \ldots, -d$ \[36\]. Because $s = 0$ is a regular point, the effective action can be defined as
\[
W(D) = -\frac{d}{ds}\text{Tr}(D^s)|_{s=0}.
\] (2.11)
Note that no further renormalization is needed, as the right-hand side is already UV finite. An interesting property of this renormalization is that it only depends on the spectrum of $D$: let \(\lambda_n\) be the spectrum of the Dirac operator, \(D\phi_n = \lambda_n\phi_n\);\(^3\)
\[
\text{Tr}(D^s) = \sum_n \lambda_n^s,
\] (2.12)
Now, it follows that two Dirac operators related by a similarity transformation $D' = S^{-1}DS$ have the same spectrum, and so the same effective action, within this regularization. This proves that all classical symmetries which are realized by similarity transformations can be preserved quantum-mechanically, if desired. That is, anomalies in this class of symmetries are not essential, in the sense that they can be removed by a suitable choice of counter-terms.

$W$ can be split into normal and abnormal parity components,
\[
W = W^+ + W^-.
\] (2.13)
\(W^\pm\) are the components which are even and odd, respectively, under the pseudo-parity (or intrinsic parity) transformation: $S \to +S$, $\gamma_\mu \to +\gamma_\mu$, $P \to -P$, $A_\mu \to -A_\mu$. Equivalently, $W^+$ is the component without Levi-Civita pseudo-tensor, is real (in Euclidean space) and even under the exchange $R \leftrightarrow L$, while $W^-$ is the component that contains the Levi-Civita pseudo-tensor, is purely imaginary and odd under under the exchange of chiral labels $R \leftrightarrow L$. Since $W^\pm$ are the real and imaginary parts, respectively, of $W$ one has, formally,
\[
W^+ = -\frac{1}{2}\text{Tr}\log(D^+D), \quad W^- = -\frac{1}{2}\text{Tr}\log(D^{-1}D^1).
\] (2.14)
$W^+$ is theoretically better understood than $W^-$ which is more challenging. Correspondingly, in this work our main focus will be on the abnormal parity component.

In a derivative expansion of $W$, the terms are classified by the number of covariant derivatives they carry. Due to Lorentz invariance, for $d$ even there are only terms of even order (since the only invariant tensors, the metric and the Levi-Civita pseudo-tensor, both have an even number of indices to be contracted.) All terms with more than $d$ derivatives in both $W^+$ and $W^-$ are UV finite. The expansion of $W^+$ starts at zero derivatives. The abnormal parity component starts at order $d$, due to the presence of the Levi-Civita pseudo-tensor. So in $W^-$ the leading order (LO) is the only term affected by UV ambiguities.

C. Chiral symmetry

The class of operators described in (2.3) is invariant under the group of local chiral transformations. Let \(\Omega_R(x)\) and \(\Omega_L(x)\) be matrices in internal space, assumed to be nowhere singular. (In fact unitary, in order to preserve the hermiticity properties of the external fields.) Then, the chirally rotated Dirac operator is
\[
D^\Omega = D^\Omega_R P_R + D^\Omega_L P_L + m_{LR}^\Omega P_R + m_{RL}^\Omega P_L
\] (2.15)
with
\[
(v_{\mu}^{R,L})^\Omega = \Omega_{R,L}^{-1} v_{\mu}^{R,L} \Omega_{R,L} + \Omega_{R,L}^{-1} [\partial_{\mu}, \Omega_{R,L}], \quad m_{LR}^\Omega = \Omega_{L}^{-1} m_{LR} \Omega_{R}, \quad m_{RL}^\Omega = \Omega_{R}^{-1} m_{RL} \Omega_{L}.
\] (2.16)

\(^3\) $D$ does not commute in general with $D^\dagger$ therefore it might not have a complete set of eigenvectors. This is no impediment for applying the $\zeta$-function method, which works for matrices of arbitrary Jordan form.
Also, \((D^{R,L})_\Omega = \Omega^{-1}_{R,L} D^{R,L} \Omega_{R,L}\).

This is a symmetry of the classical theory. Indeed, the action of \(D\) with the configurations \(\psi\) and \(\bar{\psi}\), is the same as that of \(D^\Omega\) with rotated configurations \(\psi^\Omega = (\Omega^{-1}_R P_R + \Omega^{-1}_L P_L)^\psi\), and \(\bar{\psi}^\Omega = \bar{\psi} (\Omega_L P_R + \Omega_R P_L)\).

Alternatively, let us note that the Dirac operator can be written as
\[
D = P_L \bar{\psi}_R P_R + P_R \bar{\psi}_L P_L + P_R m_{LR} P_R + P_L m_{RL} P_L,
\]
and so, in a convenient matrix form,
\[
D = \begin{pmatrix} m_{LR} & \bar{\psi}_L \\ \bar{\psi}_R & m_{RL} \end{pmatrix},
\]
the entries corresponding to the chiral subspaces \(\gamma_5 = \pm 1\). Likewise
\[
D^\Omega = \begin{pmatrix} \Omega^{-1}_L & 0 \\ 0 & \Omega^{-1}_R \end{pmatrix} \begin{pmatrix} m_{LR} & \bar{\psi}_L \\ \bar{\psi}_R & m_{RL} \end{pmatrix} \begin{pmatrix} \Omega_R & 0 \\ 0 & \Omega_L \end{pmatrix}.
\]

Then if \(\psi = (\psi_R, \psi_L)\) is a solution of \(D\psi = 0\), \(\psi^\Omega = (\Omega^{-1}_R \psi_R, \Omega^{-1}_L \psi_L)\) is a solution of \(D^\Omega \psi^\Omega = 0\).

As is well known classical symmetries may not survive quantum-mechanically. The property \(\text{det} AB = \text{det} A \text{det} B\), or \(\text{tr} \log(AB) = \text{tr} \log A + \text{tr} \log B\) holds for matrices. This property formally extends to operators, that is, it holds modulo UV ambiguities. In the chiral case this implies for the effective action
\[
W(D^\Omega) = W(D) + A(D, \Omega)
\]
where \(A(D, \Omega)\) is an \(\Omega\)-dependent polynomial counter-term (polynomial with respect to \(D\)) allowed by UV ambiguity in the definition of \(W\) (since \(W(D^\Omega)\) and \(W(D)\) qualify both as valid determinations of the effective action of \(D\) owing to the classical symmetry property).

\(A(D, \Omega)\) is a quantum-mechanical anomaly implying that \(W\) is not chirally invariant under local transformations. In the literature, the name anomaly, or more precisely consistent anomaly, refers to \(A(D, \Omega)\) for infinitesimal \(\Omega_{R,L}\).

Of course, part of the anomaly may come from a poor choice of \(W_{\text{ct}}\) in (2.13). Actually, this is the case for vector transformations. These are the transformations of the type \(\Omega_{R,L}(x) = \Omega_{R,L}(x)\). This follows from our previous observation that similarity transformations of \(D\) do not change the effective action if the \(\zeta\)-function prescription is adopted. Vector transformations are similarity transformations, \(D^{\Omega V} = \Omega^{-1}_V D \Omega_V\), therefore this symmetry needs not be spoiled at the quantum-mechanical level.

Full chiral symmetry is not protected by this mechanism. Because chiral rotations do not act as similarity transformations of \(D\), cf. (2.19), the spectrum is not preserved and an anomaly is introduced. Nevertheless, the anomaly can be restricted to the abnormal parity sector. Indeed, the adjoint Dirac operator
\[
D^\dagger = \begin{pmatrix} m_{RL} & -\bar{\psi}_R \\ -\bar{\psi}_L & m_{LR} \end{pmatrix}
\]
transforms as
\[
D^{\Omega \dagger} = \begin{pmatrix} \Omega^{-1}_L & 0 \\ 0 & \Omega^{-1}_R \end{pmatrix} \begin{pmatrix} m_{RL} & -\bar{\psi}_R \\ -\bar{\psi}_L & m_{LR} \end{pmatrix} \begin{pmatrix} \Omega_R & 0 \\ 0 & \Omega_L \end{pmatrix}.
\]

Therefore \(D^{\dagger} D\) does transform under a similarity transformation and hence, from (2.14), it follows that \(W^+(D)\) can be chosen to be chirally invariant. The remaining anomaly in \(W^-(D)\) cannot be completely removed. Its minimal form (applying counter-terms to remove non essential contributions) is the standard Bardeen’s form \([-10]\) if one chooses vector transformations to be non anomalous. This is the VA (vector-axial vector) form of the anomaly. Alternatively one can choose the LR form in which the anomaly is composed of two terms, one depending only on \(\psi^L_R\) and \(\Omega_R\) and another depending only on \(\psi^L_L\) and \(\Omega_L\). We choose the latter in this work and this choice fully fixes the LR form of the effective action in the abnormal parity sector.

For later reference, let us note that in the \(\zeta\)-function regularization the axial anomaly (there is no vector anomaly) takes the form \(\text{Tr} [\gamma_5 (\delta \Omega_L - \delta \Omega_R) D^\dagger]_{s=0}\). Formally \(D^{\dagger} |_{s=0}\) is the identity operator so there is a conflict of limits between the \(s\) of the trace of the multiplicative operator \(\delta \Omega_L - \delta \Omega_R\) and the \(0\) from the trace of \(\gamma_5\). A similar mechanism takes place for the chiral anomaly in any renormalization scheme.
D. WZW term and invariant remainder

The variation of the effective action under a finite chiral transformation can be obtained by integration of the infinitesimal variation (the consistent anomaly). More specifically, let \((m_{LR}, m_{RL}, v_R, v_L)\) be the field configuration obtained by applying the chiral rotation \((\Omega_R, \Omega_L)\) to the configuration \((\overline{m}_{LR}, \overline{m}_{RL}, \overline{v}_R, \overline{v}_L)\), then

\[
W(m, v) - W(\overline{m}, \overline{v}) = \Gamma(v_R, \Omega_R) - \Gamma(v_L, \Omega_L), \quad (v, m) = (\overline{m}, \overline{v})^\Omega, \tag{2.23}
\]

where \(W\) refers to the LR effective action. The function \(\Gamma(v, \Omega)\) verifies the obvious consistency condition

\[
\Gamma(v, \Omega) = -\Gamma(\overline{v}, \Omega^{-1}). \tag{2.24}
\]

Using \(\Gamma(v, \Omega)\) a functional saturating the chiral anomaly can be constructed, namely,

\[
\Gamma_{WZW}(v_R, v_L, U) = \Gamma(v_R, \Omega_R) - \Gamma(v_L, \Omega_L) + P_{ct}(\overline{v}_R, \overline{v}_L), \quad U = \Omega_L^{-1}\Omega_R. \tag{2.25}
\]

Here \(P_{ct}(v_R, v_L)\) is a polynomial known as Bardeen’s subtraction \([10]\). This is the counter-term needed to pass from the LR form of the effective action to its VA form. E.g., in two dimensions

\[
P_{ct}(v_R, v_L) = \frac{i}{4\pi} \int d^2x \epsilon_{\mu\nu} v^R_{\mu} v^L_{\nu}, \quad (d = 2). \tag{2.26}
\]

Of course, in any dimension \(P_{ct}(v_R, v_L) = \Gamma_{WZW}(v_R, v_L, U = 1)\).

The (gauged) Wess-Zumino-Witten (WZW) functional \(\Gamma_{WZW}(v_R, v_L, U)\) depends on \(v_{RL}\) and a field \(U\) and saturates the anomaly by construction provided only that \(U\) transforms as \(U \rightarrow \Omega_R^{-1}U\Omega_L\). (Note that although the WZW term depends on \(U\), this dependence cancels in the anomaly.) For instance, in two dimensions

\[
\Gamma_{WZW}(v_R, v_L, U) = -\frac{i}{12\pi} \epsilon_{\mu\nu\sigma} \text{tr} (U^{-1} \partial_{\mu} U^{-1} \partial_{\nu} U^{-1} \partial_{\sigma} U) d^3x + \frac{i}{4\pi} \epsilon_{\mu\nu} \left( -\partial_{\mu} U^{-1} v^R_{\nu} - U^{-1} \partial_{\nu} U v^R_{\mu} + U v^R_{\mu} U^{-1} v^L_{\nu} \right) d^2x. \tag{2.27}
\]

Since \(m_{LR}\) and \(m_{RL}^{-1}\) both transform as required for \(U\), the anomaly is saturated by the following effective action

\[
\Gamma_{WZW} = \frac{1}{2} \left( \Gamma_{WZW}(v_R, v_L, m_{LR}) + \Gamma_{WZW}(v_R, v_L, m_{RL}^{-1}) \right) = \frac{1}{2} \left( \Gamma_{WZW}(v_R, v_L, m_{LR}) - \Gamma_{WZW}(v_L, v_R, m_{RL}) \right), \tag{2.28}
\]

which is odd under the exchange \(L \leftrightarrow R\). That is,

\[
\Gamma_{WZW}(m, v) - \Gamma_{WZW}(\overline{m}, \overline{v}) = \Gamma(v_R, \Omega_R) - \Gamma(v_L, \Omega_L), \quad (v, m) = (\overline{m}, \overline{v})^\Omega. \tag{2.29}
\]

Explicitly, in two dimensions,

\[
\Gamma_{WZW} = -\frac{i}{24\pi} \int \epsilon_{\mu\nu\sigma} \text{tr} \left( m_{LR}^{-1} \partial_{\mu} m_{LR} m_{LR}^{-1} \partial_{\nu} m_{LR} m_{LR}^{-1} \partial_{\sigma} m_{LR} - m_{RL}^{-1} \partial_{\mu} m_{RL} m_{RL}^{-1} \partial_{\nu} m_{RL} m_{RL}^{-1} \partial_{\sigma} m_{RL} \right) d^3x + \frac{i}{5\pi} \epsilon_{\mu\nu} \left( \partial_{\mu} m_{RL} m_{RL}^{-1} v^R_{\nu} - \partial_{\nu} m_{RL} m_{RL}^{-1} v^L_{\nu} - m_{LR}^{-1} \partial_{\mu} m_{RL} v^R_{\nu} + m_{RL}^{-1} \partial_{\nu} m_{RL} v^L_{\nu} \right. \left. - m_{RL} v^L_{\mu} m_{RL}^{-1} v^R_{\nu} + m_{LR} v^R_{\mu} m_{RL}^{-1} v^L_{\nu} \right) d^2x. \tag{2.30}
\]

Actually this is a generalized Wess-Zumino-Witten term since \(m_{RL}, m_{LR}\) are not restricted to lie on the chiral circle.\(^5\)

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\(^4\) And conversely \([28]\), \(\Gamma(v, \Omega) = \Gamma_{WZW}(v_R = v, v_L = 0, U = \Omega)\).

\(^5\) The chiral circle constraint corresponds to \(m_{LR} = MU\), \(m_{RL} = MU^{-1}\), where \(U\) is unitary and \(M\) is a constituent mass that cancels in \(\Gamma_{WZW}\).
Therefore one can write
\[ W^- = W_c^- + \Gamma_{gWZW}, \]  
(2.31)
where \( W_c^- \) is chirally invariant and \( \Gamma_{gWZW} \) reproduces the anomaly. Of course, one could transfer contributions from \( W_c^- \) to \( \Gamma_{gWZW} \) and the latter would still saturate the anomaly, however, our choice of \( \Gamma_{gWZW} \) is distinguished in the sense that it is composed of terms depending only on \( m_{LR} \) plus terms depending only on \( m_{RL} \) and contains just LO terms in the derivative expansion.

We will refer to \( W_c^- \) as the chiral remainder, that is, the chiral invariant terms left after the anomaly saturating part \( \Gamma_{gWZW} \) has been subtracted. At leading order in a derivative expansion, the remainder \( W_c^- \) vanishes identically when the scalar and pseudo-scalar fields satisfy a generalized chiral circle constraint (namely, when \( m_{RL} m_{LR} \) is a c-number) but \( W_c^- \) is a non trivial functional outside the chiral circle \([28]\) or beyond LO \([31]\).

E. Computation of \( W^- \) from the current

The operator \( D^1D \) is of the Klein-Gordon type therefore there are several techniques to address the computation of \( W^+ \). This is further simplified by the fact that chiral symmetry is preserved in the normal parity sector. This helps to reduce the number of allowed structures.

The situation in the abnormal parity sector is quite different since chiral symmetry is not preserved there. This means that \( W^- \) cannot be written using simple chiral covariant blocks like \( F_{\mu \nu}^{R,L}, m_{RL}, m_{LR} \) and their chiral covariant derivatives and this complicates considerably its calculation and even its proper mathematical definition beyond perturbation theory \([1]\). In addition, the operator \( D^{1^{-1}}D \) in (2.14) is not of the Klein-Gordon type, in fact, it is not even local. The obvious method is to use \( D^2 \) as Klein-Gordon operator (to obtain \( W = -\frac{1}{2} \text{Tr} \log(D^2) \)). The computation through the heat kernel of \( D^2 \) or its \( \zeta \)-function becomes quite involved due to the lack of full chiral symmetry (vector symmetry is preserved).

Since exploiting full chiral symmetry is essential, the route to attack the problem has been to use the current. The current is defined as the variation of the effective action with respect to the gauge fields:
\[ \delta_e W = \int d^d x \text{tr} \left( J^R_{\mu}(x) \delta u^R_\mu(x) + J^L_{\mu}(x) \delta u^L_\mu(x) \right). \]  
(2.32)
Here the trace includes only internal degrees of freedom (not Dirac ones). Formally,
\[ \delta_e W = -\text{Tr} \left( \frac{1}{D} \delta_e D \right), \quad \delta_e D = \delta \phi_R P_R + \delta \phi_L P_L. \]  
(2.33)
Although this quantity is more UV convergent than the effective action, a direct computation of the current using this expression is still subject to UV ambiguities and different determinations of the current differ by local polynomial counter-terms of mass dimension \( d-1 \). Unlike \( W^- \), and this is the key point of using the current to compute the effective action, the current in the abnormal parity sector can be computed preserving local chiral covariance. This is known as the covariant version of the current, \( J_{c,\mu}^{R,L} \). Unfortunately, as a consequence of the chiral anomaly, the covariant current (of the abnormal parity sector) is not consistent, that is, it is not a true variation of any effective action. The consistent current is obtained by adding the appropriate counter-term \([18]\):
\[ J_{\mu}^{R,L} = J_{c,\mu}^{R,L} + P_{\mu}^{R,L} \quad (W^- \text{ sector}). \]  
(2.34)
The current counter-term is fully fixed by the chiral anomaly (once, e.g., the LR version has been adopted). For instance, for \( d = 2 \), one finds \( P_{\mu}^{R,L} = \frac{i \epsilon_{\mu \nu} v^{R,L}_{\nu}}{4 \pi} \). This term is just the polynomial non covariant part of the current derived from \( \Gamma_{gWZW} \) in (2.30).

Let us remark that the current coming from \( W_c^- \) is both covariant and consistent, and does not coincide with the covariant current \( J_{c,\mu}^{R,L} \), which is not consistent. The covariant current picks up some covariant terms from \( \Gamma_{gWZW} \).

Once the consistent current is obtained the effective action can be reconstructed from it. This method is the basis of \([1]\) to achieve a suitable definition of \( W^- \) beyond perturbation theory.

The approach based on the current has the virtue of being fully chiral covariant. It has been used in the derivative expansion calculations of \([28]\) and \([31]\), where the unknown \( W_c^- \) is determined so that when added to the known WZW term the consistent current is reproduced.
III. THE INVARIANT PART OF THE EFFECTIVE ACTION

In this section the main result of the paper is presented, namely, we show how, upon separation of the anomalous WZW contribution, the effective action can also be expressed as the Tr log of a local operator of the Klein-Gordon type which, in addition, is manifestly chiral covariant.

A. Covariant Klein-Gordon operator

In order to construct such a covariant operator of the Klein-Gordon type, we will use the convenient matrix notation of (2.18). That is, provided that the operators \( A \) and \( D \) commute with \( B \) and \( C \) anticommute with \( \gamma_5 \), we can use \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) to represent \( P_R A P_R + P_R B P_L + P_L C P_R + P_L D P_L \). These matrices multiply as usual. The only thing to be noted is the relation

\[
\text{Tr} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{2} \text{Tr}(A + D) + \frac{1}{2} \text{Tr}(\gamma_5 (A - D)).
\]

In this notation

\[
D = \begin{pmatrix} m_{LR} & \bar{\psi}_L \\ \bar{\psi}_R & m_{RL} \end{pmatrix}.
\]

Let us introduce the two new operators

\[
D' = \begin{pmatrix} m_{LR} & -\bar{\psi}_L \\ -\bar{\psi}_R & m_{RL} \end{pmatrix},
\]

\[
D^m = \begin{pmatrix} m_{LR}^{-1} & 0 \\ 0 & m_{RL}^{-1} \end{pmatrix} D \begin{pmatrix} m_{LR} & 0 \\ 0 & m_{RL} \end{pmatrix} = \begin{pmatrix} m_{RL}^{-1} \bar{\psi}_R m_{RL} & \bar{\psi}_L m_{LR} \\ m_{RL} \bar{\psi}_R m_{RL} & m_{RL} \bar{\psi}_L m_{RL} \end{pmatrix},
\]

and their product

\[
K = D'D^m = \begin{pmatrix} K_L & 0 \\ 0 & K_R \end{pmatrix},
\]

\[
K_L = m_{RL} m_{LR} - \bar{\psi}_R m_{LR}^{-1} \bar{\psi}_L m_{RL}, \quad K_R = m_{RL} m_{LR} - \bar{\psi}_L m_{RL}^{-1} \bar{\psi}_R m_{RL}.
\]

These operators are related by \( K_R = m_{RL} K_L m_{RL}^{-1} \) and \( K_L = m_{RL} K_R m_{RL}^{-1} \). \(^6\)

Chirally, \( D' \) transforms as \( D \) while \( D^m \) transforms as \( D \). As a consequence \( K \) is a manifestly covariant operator (i.e., chiral rotations act as similarity transformations on it) which is local and of the Klein-Gordon type. This is the operator we were looking for. A formal hand-waving argument shows that \( (\text{Det} K)^{1/2} \) is the chiral invariant part of the chiral determinant \( \text{Det} D \). This is as follows:

\[
W = -\text{Tr} \log D = -\frac{1}{2} \text{Tr} \log D'D
\]

\[
= -\frac{1}{2} \text{Tr} \log D'D^m \begin{pmatrix} m_{RL}^{-1} m_{LR} & 0 \\ 0 & m_{RL}^{-1} m_{LR} \end{pmatrix}
\]

\[
= -\frac{1}{2} \text{Tr} \log K + \frac{1}{2} \text{Tr} (\gamma_5 \log(m_{RL}^{-1} m_{LR})).
\]

In the second equality we have used that \( D \) and \( D' \) have the same effective action. This is because the trace of products involving an odd number of \( \gamma_\mu \) (with or without \( \gamma_5 \)) vanish, and so only terms with an even number of \( \bar{\psi}_{R,L} \)

---

\(^6\) It might look awkward that \( K_L \) (a chirally left operator) appears in the subspace \( \gamma_5 = +1 \) instead of \( \gamma_5 = -1 \) (and similarly for \( K_R \)). The natural assignments would be achieved by using instead \( D^{m'} = \begin{pmatrix} m_{RL} & 0 \\ 0 & m_{LR} \end{pmatrix} D \begin{pmatrix} m_{LR}^{-1} & 0 \\ 0 & m_{RL}^{-1} \end{pmatrix} \) and \( K' = D^{m'} D' = \begin{pmatrix} K_R & 0 \\ 0 & K_L \end{pmatrix} \). Both treatments are equivalent (the same inversion takes place if one uses \( \text{Tr} \log(D'^{-1} D) \) instead of \( \text{Tr} \log(DD'^{-1}) \)). We favor \( K_{R,L} \) since we prefer the structure \( U^{-1} dU \) over \( dUU^{-1} \). The same effect would be obtained by changing \( \gamma_5 \to -\gamma_5 \), so that (2.3) becomes

\[
D = P_R \bar{\psi}_R + P_L \bar{\psi}_L + P_L m_{LR} + P_L m_{RL}.
\]
will give a contribution to $W$. (In other way, in even dimensions the representations $\gamma_\mu$ and $-\gamma_\mu$ are equivalent.) Therefore, we can symmetrize with respect to $\mathcal{D}_{R,L} \rightarrow -\mathcal{D}_{R,L}$. In the third and fourth equalities we make use of the formal identity

$$\text{Tr} \log(AB) = \text{Tr} \log A + \text{Tr} \log B. \quad (3.6)$$

It implies that (formally) operators commute inside $\text{Tr} \log$ and so the factors can be rearranged at will.

In the right-hand side of (3.5) $-\frac{1}{2} \text{Tr} \log K$ is a chiral invariant contribution whereas the second term $\frac{1}{2} \text{Tr} (\gamma_5 \log(m^{-1}_{LR} m_{RL}))$ represents the anomalous part. This latter term has three conspicuous features: first, just as the anomaly, it presents an indetermination of the type $0 \times \infty$ (0 from $\text{tr} \gamma_5 = 0$ and $\infty$ from the trace of the multiplicative operator $m^{-1}_{LR} m_{RL}$). This is typical of anomalous contributions. Second, although $\text{Tr} (\gamma_5 \log(m^{-1}_{LR} m_{RL}))$ is not chiral invariant, its variation is invariant. Indeed,

$$\delta \text{Tr} (\gamma_5 \log(m^{-1}_{LR} m_{RL})) = -\text{Tr} (\gamma_5 m_{LR}^{-1} \delta m_{LR}) + \text{Tr} (\gamma_5 m_{RL}^{-1} \delta m_{RL}). \quad (3.7)$$

This is also characteristic of the anomalous WZW term: its variation is covariant up to polynomial counter-terms (the latter are missed by our formal manipulations). Finally, a third feature is that this last term is separable in two contributions, one depending only on $m_{LR}$ and one depending only on $m_{RL}$, just as $\Gamma_{gWZW}$.

In view of this, comparison of (3.5) with (2.31) suggests identifying the covariant part with $W^+ + W^-$, while the non covariant part would represent the anomalous contribution, $\Gamma_{gWZW}$. The latter cannot be recovered from the formal expression. That is,

$$W = -\frac{1}{2} \text{Tr} \log K + \Gamma_{gWZW}, \quad (3.8)$$

or

$$W_c(D) = -\frac{1}{2} \text{Tr} \log K. \quad (3.9)$$

It will be important to realize that within the derivative expansion all orders beyond the LO are UV convergent. This LO contains the UV ambiguities and because different orders are not mixed by chiral rotations, it carries all the anomalous contributions. This implies that the formal manipulations used to arrive to (3.5) are correct beyond LO in the abnormal parity sector and so (3.8) certainly holds to all orders beyond the lowest one. In fact we would expect it to hold to all orders, included the LO one. The reason is that in most expansions of interest (such as perturbation theory or inverse mass expansions) higher orders are increasingly UV convergent and so (3.8) should be fulfilled to all UV convergent orders of all those expansions. This covers many contributions which belong to the LO from the point of view of the derivative expansion. This expectation is indeed correct. As we show subsequently, the relation (3.8) holds for all (even) space-time dimensions.

**B. Proof of the main result**

In order to prove the identification in (3.8) we should make the statement precise.

1. The standard LR effective action of $D$

The functional $W^-(D)$ is perfectly well defined once the two following conditions are met, first, the LR version of the anomaly is chosen, and second, $W^-(D)$ depends only on $D$ (no new field absent from $D$ is introduced in the functional). This is unique because the allowed ambiguity would be an abnormal parity and chiral invariant polynomial composed of $F_{\mu\nu}^{RL}$ and $m_{LR}, m_{RL}$ and their covariant derivatives, but no such polynomial exists. (We disregard topological contributions, such as $\int d^2x \epsilon_{\mu\nu} \text{tr} (F_{\mu\nu}^{RL} - F_{\mu\nu}^{LR})$ in two dimensions.) Since $\Gamma_{gWZW}$ is also well defined, $W^-(D)$ is unique too. On the other hand $W^+(D)$ is not unique. A basic definition would be, e.g., its $\zeta$-function determination from $D^1 D$. New chiral invariant determinations are then obtained by adding arbitrary normal parity and chiral invariant polynomials of $F_{\mu\nu}^{RL}$ and $m_{LR}, m_{RL}$ and their covariant derivatives. These do not vanish identically, e.g., $\int d^2x \epsilon_{\mu\nu} \text{tr} (m_{LR} m_{RL})$ in two dimensions. For concreteness we assume that the above mentioned basic determination has been taken for $W^+(D)$. We will use the notation $W_s(D)$ (with subindex $s$ from “standard”) to denote this specific determination of the effective action:

$$W_s(D) = W^-_{LR}(D) - \frac{1}{2} \frac{d}{ds} \text{Tr}((D^1 D)^s) \Big|_{s=0}. \quad (3.10)$$
2. Natural and standard effective action from $K$

Let us now consider the functional

$$W(K) = -\frac{1}{2} \text{Tr} \log K. \tag{3.11}$$

As noted before, given a differential operator, such as $D$ or $K$, the logarithm of its determinant is unique modulo a counter-term action which is polynomial regarding its dependence on the fields present in the game and their derivatives. A basic definition of $W(K)$ follows from its $\zeta$-function determination. Two nice properties of this determination are i) it does not introduce new fields in the game, and ii) it preserves all symmetries realized as similarity transformations of $K$ (including chiral symmetry). We will refer to determinations with these two properties as “natural” determinations. All determinations of the type $\text{Tr} f(K,s)$ with $f(x,s) \to \log(x)$ as $s \to 0$, are of natural type. All natural determinations differ from the $\zeta$-function one by a chiral invariant polynomial constructed with the fields in $K$ and their derivatives. We will take the $\zeta$-function determination as the standard one, to be denoted $W_s(K)$:

$$W_s(K) = -\left. \frac{1}{d} \frac{d}{ds} \text{Tr}(K^s) \right|_{s=0}. \tag{3.12}$$

It order to analyze this point further, let us introduce the chiral covariant quantities

$$M_R = m_{RL} m_{LR}, \quad M_L = m_{LR} m_{RL}, \quad Q_{\mu}^R = m_{LR}^1 (D_{\mu} m)_{LR}, \quad Q_{\mu}^L = m_{RL}^1 (D_{\mu} m)_{RL}, \tag{3.13}$$

where

$$(D_{\mu} m)_{LR} = D_{\mu}L m_{LR} - m_{LR} D_{\mu}^R, \quad (D_{\mu} m)_{RL} = D_{\mu}R m_{RL} - m_{RL} D_{\mu}^L. \tag{3.14}$$

In terms of these fields

$$D^m = \left( \begin{array}{cc} m_{RL} & \partial_R + \partial_L \\ \partial_L - \partial_L^2 & m_{LR} \end{array} \right), \tag{3.15}$$

and

$$K = \left( \begin{array}{ccc} M_L - \partial_L^2 - \partial_L \partial_L & 0 \\ 0 & M_R - \partial_R^2 - \partial_R \partial_R \end{array} \right). \tag{3.16}$$

Therefore, within the class of natural determinations, the ambiguity in $W(K)$ is a polynomial in $M_{LR}, D_{\mu}^{LR},$ and $Q_{\mu}^{LR}$. Unfortunately, $Q_{\mu}^{R,L}$ is not a polynomial with respect the original fields in $D$. This implies that, in general, an UV ambiguity that would be admissible from the point of view of $W(K)$, will not be admissible for $W(D)$, when inserted in (3.8). That is, even if $W(K)$ is obtained through a natural determination, one must still allow for removal by counter-terms of contributions which are not polynomials with respect to $D$ (and so are incorrect) but polynomials with respect to $K$. Such polynomials are absent in two dimensions in the abnormal parity sector,\footnote{The possible candidates, $\int d^2 x \varepsilon_{\mu} \varepsilon_{\nu} \text{tr} (Q_{\mu}^R Q_{\nu}^R - Q_{\mu}^L Q_{\nu}^L)$, and $\int d^2 x \varepsilon_{\mu} \varepsilon_{\nu} \text{tr} (|D_{\mu}^R Q_{\nu}^R| - |D_{\mu}^L Q_{\nu}^L|)$, vanish.} But they exists in four or more dimensions. They also exist in two dimensions in the normal parity sector, e.g., $\int d^2 x \text{tr} (Q_{\mu}^R Q_{\mu}^R + Q_{\mu}^L Q_{\mu}^L)$.

3. Proof of the statement

The statement to be proven is then that for a certain natural determination of $W(K)$ the relation (3.8) holds. Or equivalently,

$$W_s(D) - \Gamma_{zwzw}(D) - W_s(K) = \text{“Chiral invariant polynomial of } m_{LR}, m_{RL}, D_{\mu}^{LR}, \text{and } Q_{\mu}^{LR,R\nu}. \tag{3.17}$$
Note that in the abnormal parity sector the polynomial will depend only on $D^{L,R}_{\mu}$, and $Q^{L,R}_{\mu}$; but not explicitly on $m_{LR}, m_{RL}$ due to dimensional reasons: there should be precisely $d$ fields carrying a Lorentz index and they already saturate the mass dimension of the counter-term. (As always, the polynomial has degree at most $d$ and this holds too for similar polynomials below.)

As we argued before $W_s(D') = W_s(D)$. On the other hand, from the relation $K = D'D^m$ we can conclude that $\text{Tr} \log K$ can be computed through $\text{Tr} \log D' + \text{Tr} \log D^m$, modulo UV ambiguities, that is

$$\frac{1}{2} (W_s(D) + W_s(D^m)) - W_s(K) = \text{``Polynomial of } m_{LR}, m_{RL}, \partial_\mu, v^{L,R}_\mu, \text{ and } Q^{L,R}_\mu \text{''}.$$ 

(3.18)

In this polynomial we have to allow for the variables $Q^{L,R}_\mu$, since they appear in $D^m$, cf. (3.15). Indeed, if we represent the Dirac operator as

$$D = D(m_{LR}, m_{RL}, v_R, v_L)$$

(3.19)

then

$$D^m = D\big|_{v \rightarrow v + Q,R \rightarrow L} = D(m_{RL}, m_{LR}, v_L + Q_L, v_R + Q_R).$$

(3.20)

From its definition, (3.3), it is clear that $D^m$ is related to $D$ by a (generalized) chiral rotation (cf. (2.19)) with $\Omega_R = m_{RL}$ and $\Omega_L = m_{LR}$. Therefore, using (2.23) and (2.24),

$$W_s(D^m) = W_s(D) - \Gamma(v_R, m^{-1}_{RL}) + \Gamma(v_L, m^{-1}_{LR}).$$

(3.21)

Substituting in (3.18) we find

$$W_s(D) - W_s(K) - \frac{1}{2} (\Gamma(v_R, m^{-1}_{RL}) - \Gamma(v_L, m^{-1}_{LR})) = \text{``Polynomial of } m_{LR}, m_{RL}, \partial_\mu, v^{L,R}_\mu, \text{ and } Q^{L,R}_\mu \text{''}. $$

(3.22)

This can be rewritten as

$$W_s(D) - \Gamma_{gWZW}(D) - W_s(K) = P(D) + \text{``Polynomial of } m_{LR}, m_{RL}, \partial_\mu, v^{L,R}_\mu, \text{ and } Q^{L,R}_\mu \text{''},$$

(3.23)

where we have defined

$$P(D) = \frac{1}{2} (\Gamma(v_R, m^{-1}_{RL}) - \Gamma(v_L, m^{-1}_{LR})) - \Gamma_{gWZW}(D).$$

(3.24)

Clearly, (3.23) will be equivalent to (3.17) provided that $P(D)$ is also a polynomial. This is the case as we now show: using (2.28) and (2.25)

$$\Gamma_{gWZW}(D) = \frac{1}{2} \left[ \Gamma_{WZW}(v_R, v_L, m_{LR}) - \Gamma_{WZW}(v_L, v_R, m_{RL}) \right] = \frac{1}{2} \left[ \left( \Gamma(v_R, \Omega_R) - \Gamma(v_L, \Omega_L) + P_{ct} (v^{\Omega^{-1}_R}_{R}, v^{\Omega^{-1}_L}_{L}) \right) \bigg|_{\Omega_R = 1, \Omega_L = m^{-1}_{LR}} \right. \\
- \left. \left( \Gamma(v_L, \Omega_R) - \Gamma(v_R, \Omega_L) + P_{ct} (v^{\Omega^{-1}_L}_{L}, v^{\Omega^{-1}_R}_{R}) \right) \bigg|_{\Omega_R = 1, \Omega_L = m^{-1}_{RL}} \right] = \frac{1}{2} \left[ \Gamma(v_R, m^{-1}_{RL}) - \Gamma(v_L, m^{-1}_{LR}) + P_{ct}(v_R, v^m_{LR}) - P_{ct}(v_L, v^m_{RL}) \right].$$

(3.25)

In this expression

$$(v^m_{LR})_\mu = (v^\partial_{L})_\mu \bigg|_{\Omega = m_{LR}} = (\Omega^{-1}_R \partial_\mu \Omega + \Omega^{-1}_L v^{L}_\mu \partial_\Omega) \bigg|_{\Omega = m_{LR}} = v^R_\mu + Q^R_\mu, \quad (v^m_{RL})_\mu = v^L_\mu + Q^L_\mu.$$ 

(3.26)

This implies

$$P(D) = -\frac{1}{2} \left[ P_{ct}(v_R, v_R + Q_R) - P_{ct}(v_L, v_L + Q_L) \right].$$

(3.27)

This is a polynomial constructed with $v^{R,L}_\mu$ and $Q^{R,L}_\mu$ as advertised. Because the left-hand side of (3.23) is chiral invariant, so is the polynomial on the right. This proves (3.17).
C. Explicit Klein-Gordon form

The operators $K_{R,L}$ can be brought to a manifest Klein-Gordon form if desired. Indeed,

$$K_R = m_{RL} m_{LR} - \Psi_R m_{LR}^{-1} \Psi_L m_{LR}$$
$$= m_{RL} m_{LR} - \Psi_R^2 - \Psi_R m_{LR}^{-1}(\Psi_L m_{LR} - m_{LR} \Psi_R)$$
$$= m_{RL} m_{LR} - (D_R^\mu)^2 - \frac{1}{2} \sigma_{\mu \nu} F^R_{\mu \nu} - D_R^\mu m_{LR}^{-1}(D^R \gamma_{\mu \nu} \gamma_{\nu})$$
$$= N_R - (\tilde{D}_R^\mu)^2$$

(3.28)

with

$$\tilde{M}_R = m_{RL} m_{LR} - \frac{1}{2} \sigma_{\mu \nu} F^R_{\mu \nu} + (B_R^\mu)^2 - [D_R^\mu, B_R^\mu],$$
$$\tilde{D}_R^\mu = D_R^\mu + B_R^\mu,$$
$$B_R^\mu = \frac{1}{2} \gamma_{\mu \nu} m_{LR}^{-1}(D^R \gamma_{\mu \nu}) = \frac{1}{2} \gamma_{\mu \nu} Q_{\nu}.$$  

(3.29)

In these expressions we have used the relations

$$\gamma_{\mu \nu} = \delta_{\mu \nu} + \sigma_{\mu \nu}, \quad F^R_{\mu \nu} = [D_R^\mu, D_R^\nu], \quad (D_R \gamma_{\mu \nu})_{LR} = D_{\mu}^R m_{LR} - m_{LR} D_{\mu}^R.$$  

(3.30)

Of course, there is an entirely analogous expression for $K_L$. This gives (using (3.1))

$$W^+ = -\frac{1}{4} \text{Tr} \log(\tilde{M}_R - (\tilde{D}_R^\mu)^2) - \frac{1}{4} \text{Tr} \log(\tilde{M}_L - (\tilde{D}_L^\mu)^2),$$
$$W^- = -\frac{1}{4} \text{Tr} \left(\gamma_5 \log(\tilde{M}_R - (\tilde{D}_R^\mu)^2)\right) - \frac{1}{4} \text{Tr} \left(\gamma_5 \log(\tilde{M}_L - (\tilde{D}_L^\mu)^2)\right).$$  

(3.31)

Note that $\tilde{M}_{R,L}$ and $\tilde{D}_{R,L}^\mu$ commute with $\gamma_5$. The second relation is quite remarkable: it allows to treat the (chiral invariant part of the) imaginary part of the fermionic effective action, $W^-$, with the same techniques available to attack the real part, e.g., the heat kernel expansion, preserving manifest chiral invariance throughout without any obstruction from the chiral anomaly.

D. Direct computation of the leading order in two dimensions

In this section we carry out explicitly the computation of $W^-$ in two dimensions to LO in the derivative expansion. This verifies the identification (3.8) in this case and illustrates its use.

To this end we apply the convenient method of Chan [37] to compute the trace of the logarithm of $K_R$. This takes the form

$$\text{Tr} \left(\gamma_5 \log(\tilde{M}_R - (\tilde{D}_R^\mu)^2)\right) = \int \frac{d^4 x}{(2\pi)^4} \text{Tr} \gamma_5 \left(\tilde{N}_R + \frac{p^2}{2} [\tilde{D}_R^\mu, \tilde{N}_R]^2 + \cdots\right),$$  

(3.32)

where $\tilde{N}_R = 1/(p^2 + \tilde{M}_R)$ and the dots refer to higher orders in the derivative expansion with respect to $\tilde{D}_R^\mu$.

Actually we want to expand with respect to the original covariant derivatives $D_{\mu}^{R,L}$ (rather than $\tilde{D}_R^\mu$). Noting that $B_{\mu}^R$ is of first order (contains exactly one covariant derivative) and $\tilde{M}_R$ is $m_{RL} m_{LR}$ plus terms of second order, it follows that each given order in the derivative expansion of $\text{Tr} (\gamma_5 \log K_R)$ gets contributions from a finite number of terms of the Chan expansion, therefore the computation is feasible order by order using this scheme.

To proceed to LO in two dimensions requires the second order (two derivative) contributions from the first two Chan terms $- \text{log} \tilde{N}_R$ and $[\tilde{D}_R^\mu, \tilde{N}_R]^2$. Introducing

$$N_R = (p^2 + m_{RL} m_{LR})^{-1}$$

(3.33)

one finds

$$-\text{Tr} \left(\gamma_5 \log(N_R)\right) = -\text{Tr} \left(\gamma_5 \log N_R\right) + \text{Tr} \left(\gamma_5 N_R \left(-\frac{1}{2} \sigma_{\mu \nu} F^R_{\mu \nu} + (B_R^\mu)^2 - [D_R^\mu, B_R^\mu]\right)\right) + O(D^4),$$
$$[\tilde{D}_R^\mu, \tilde{N}_R]^2 = [D_R^\mu + B_R^\mu, N_R]^2 + O(D^4).$$  

(3.34)
Introducing these expressions in Chan’s formula (3.32) and taking the Dirac trace using the two dimensional identity
\[ \text{tr}(\gamma_5 \log K_R)_{d=2, LO} = 2i \int \frac{d^2xdp}{(2\pi)^d} \epsilon_{\mu\nu} \text{tr} \left( \frac{1}{2} N_R[D^R_\mu Q^R_\nu] + \frac{1}{2} N_R F^R_{\mu\nu} - \frac{p^2}{d} [D^R_\mu, N_R] [Q^R_\nu, N_R] \right), \] (3.35)
where we have used \( Q^R_\mu = m^{-1}_{LR} (D_\mu m)_{LR} \).

Some remarks are in order. In the right-hand side of (3.35) \( \text{tr} \) no longer includes Dirac space. Also, \( d \) refers to the dimension of momentum space in the sense of dimensional regularization. This is a device related to Chan’s formula which applies to bosonic (Klein-Gordon) theories with arbitrary internal degrees of freedom. The Dirac gaumas are all the time in two dimensions (they are not dimensionally extended) and in particular \( \gamma_5 = i \gamma_0 \gamma_1 \) anticommutes with all \( \gamma_\mu \).

The first term inside the trace in (3.35) is not directly UV convergent but it is so upon integration by parts. Similarly, the identity
\[ \int d^2x \epsilon_{\mu\nu} \text{tr} \left( \frac{1}{2} N_R F^R_{\mu\nu} - \frac{1}{2} N_L F^L_{\mu\nu} \right) = \int d^2x \epsilon_{\mu\nu} \text{tr} \left( -N_R(D_\mu m)_{RL} N_L(D_\nu m)_{LR} \right) \] (3.36)
allows to bring the second term to an UV convergent form (up to terms which are symmetric under the exchange \( L \leftrightarrow R \) and so cancel in (3.31)). After those replacements \( d \) can be set to two:

\[ \text{Tr}(\gamma_5 \log K_R)_{d=2, LO} = -i \int \frac{d^2xdp}{(2\pi)^2} \epsilon_{\mu\nu} \text{tr} \left( [D^R_\mu, N_R] Q^R_\nu + N_R(D_\mu m)_{RL} N_L(D_\nu m)_{LR} + p^2 [D^R_\mu, N_R] [Q^R_\nu, N_R] \right). \] (3.37)

This can be worked out by using standard manipulations:
\[ [D^R_\mu, N_R] = -N_R[D^R_\mu, m_{RL} m_{LR}] N_R, \quad [Q^R_\mu, N_R] = -N_R[Q^R_\mu, m_{RL} m_{LR}] N_R, \] (3.38)
plus integration by parts in momentum space to bring the expression to a simpler form. Then the effective action can be written as
\[ W_{c, d=2, LO} = -i \frac{1}{2} \int \frac{d^2xdp}{(2\pi)^2} \epsilon_{\mu\nu} \text{tr} \left( N^2_R m_{RL}(D_\mu m)_{LR} N_R m_{RL}(D_\nu m)_{LR} \right. \]
\[ \left. - N^2_L m_{LR}(D_\mu m)_{RL} N_{LR} m_{LR}(D_\nu m)_{RL} \right) \] (3.39)

Because the momentum appears in matricial expressions it is not possible to carry out the momentum integral directly (except for particular abelian configurations). The obvious approach is then to take matrix elements in an eigenbasis of the matrices involved. In the present case one can define two (local) orthonormal basis of eigenvectors in internal space, by the relations
\[ m_{LR}|j, R\rangle = m_j|j, L\rangle, \quad m_{RL}|j, L\rangle = m_j|j, R\rangle, \quad m_j > 0. \] (3.40)

In fact the \( |j, R\rangle \) are just eigenvectors of \( m_{RL} m_{LR} \) with eigenvalues \( m_j^2 \) while the \( |j, L\rangle \) are eigenvectors of \( m_{LR} m_{RL} \) with the same eigenvalues \( m_j^2 \). One can now take matrix elements to compute the trace in (3.39). In doing so \( m_{RL} m_{LR} \) and \( m_{LR} m_{RL} \) are replaced by their eigenvalues and the momentum integral becomes straightforward. This gives\(^8\)
\[ W_{c, d=2, LO} = \frac{i}{8\pi} \int d^2x \epsilon_{\mu\nu} \sum_{i,j} \frac{1}{m_i^2 - m_j^2} \left[ \frac{1}{2} \left( \frac{1}{m_i^2} + \frac{1}{m_j^2} \right) - \frac{\log(m_j^2/m_i^2)}{m_i^2 - m_j^2} \right] \]
\[ \times \left( \langle i, R|m_{RL}(D_\mu m)_{LR}|j, R\rangle \langle j, R|m_{RL}(D_\nu m)_{LR}|i, R\rangle \right. \]
\[ - \langle i, L|m_{LR}(D_\mu m)_{RL}|j, L\rangle \langle j, L|m_{LR}(D_\nu m)_{RL}|i, L\rangle \] (3.41)

\(^8\) The limit \( m_j^2 \to m_i^2 \) is finite and it correctly reproduces the contribution from diagonal matrix elements.
This result is that already obtained in [28] where the effective action is obtained from the current, a result later verified in [31] using the world-line approach, and should also be identical to that quoted in [1] (p.146).

A similar calculation can be done for $W^+$. Here we find that the result of [38] is also reproduced, after removal of a spurious counter-term $1/(16\pi)\int d^2x \text{ tr} (Q^R\mu Q^R\mu + Q^L\mu Q^L\mu)$. The calculation of more complicated cases get quite involved using this direct Chan’s approach. This is due to the fact that $\gamma_\mu$ appears (twice) in the effective connection $B_{R,L}^\mu$. In [39] we have set up a method of Chan type specific for $K_{R,L}$ and have calculated $W^-_{c}$ to four derivatives in two and four dimensions. It is found that the results quoted in [28] for the LO in four dimensions and in [31] for the NLO in two dimensions are correctly reproduced.

IV. CONCLUSIONS

In this work we have found a remarkable result, namely, there is a local operator, $K$, which correctly reproduces the non-trivial chiral invariant factor of $\text{Det} D$. Moreover, such operator is explicitly constructed. This allows to address a direct non perturbative definition of the chiral determinant as well as it evaluation with explicit chiral invariance throughout.

As nice feature is that, unlike other approaches, we have not introduced chirally rotated fields to carry out our construction. We work all the time with the original LR variables. Another nice feature is that the calculation through $\text{Tr log } K$ allows to go directly to the terms one needs without reconstruction from the current.

The derivation has been carried out for flat space-time, but at present there is no indication that it cannot be extended to the curved case, so we can conjecture that a similar operator $K$ exists for curved space-times. This would allow to address the computation of the effective action of fermions in the presence of external gravitational fields, beyond the known WZW-like contributions. Likewise, the construction quite probably admits an extension to manifolds with generic topology, and in particular those corresponding to thermal compactification needed in the imaginary time approach to finite temperature.

It is also interesting that bypassing the chiral obstruction suggests a direct way to put the chiral invariant part of the chiral determinant in the lattice. The geometrical (rather than dynamical) anomalous WZW term can then be added.

Finally, the construction presented here could be translatable to other actions afflicted by anomalies. In particular, in the operator formulation of non commutative field theory [41] it would seem that $K$ would take the same form as given here. This is because $x$ appears always in non abelian fields for which we do not assume any particular commutation properties. This observation is confirmed by the fact that the heat-kernel expansion in non commutative field theory takes the same form as in the ordinary one [41].

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[1] R. D. Ball, Phys. Rept. 182, 1 (1989).
[2] J. Gasser and H. Leutwyler, Ann. Phys. 158, 142 (1984).
[3] D. Espriu, E. de Rafael and J. Taron, Nucl. Phys. B345, 22 (1990).
[4] C. V. Christov et al., Prog. Part. Nucl. Phys. 37, 91 (1996), [hep-ph/9604441].
[5] J. Bijnens, Phys. Rept. 265, 369 (1996), [hep-ph/9502335].
[6] J. Smit, JHEP 09, 067 (2004), [hep-ph/0407161].
[7] C. Itzykson and J.-B. Zuber, Quantum field theory (McGraw-Hill, New York, 1980).
[8] S. L. Adler, Phys. Rev. 177, 2426 (1969).
[9] J. S. Bell and R. Jackiw, Nuovo Cim. A60, 47 (1969).

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9 In these two references the more efficient technique of labeled operators is used, instead of the eigenbasis method.
10 This is relevant since, as shown in [28], the number of chiral invariant functionals depending analytically on the original variables that one can write down is much less than the total number of chiral invariant functionals (using, e.g., rotated variables).
