Structure resonance crossing in space charge dominated beams

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As an extension of previous theoretical study on the coherent structure resonance due to space charge effects [Chao Li and R. A. Jameson, Phys. Rev. Accel. Beams 21, 024204, 2018], this paper aims to demonstrate how the beam, as a whole, is spontaneously affected when the predicted mixed coherent 2nd/4th order structure resonance stop band around 90° phase advance is crossed. The beam characteristics during the structure resonance crossing, such as the rms emittance growth and the appearance of 2-fold/4-fold structure in phase space, are well explained by the mixture characteristic of different orders of structure resonance. The related “attracting” and “repulsive” effects in the structure resonance stop band from below and above crossing are considered as a natural beam reaction to the coherent structure resonance that the beam spontaneously moves to a structure resonance free region then the space charge takes a weaker importance. In the PIC simulation, it is found that the emittance growth is positively related to the time that the beam spends inside the structure resonance stop band. As a potential candidate mechanism, the incoherent particle-core resonance also has been checked. It is found that this incoherent resonance has basic difficulties in explaining the results obtained from the self-consistent PIC simulation. The incoherent particle-core resonance might lead to phase space distortion, emittance growth, or beam halo formation only on a long-time scale. This clarity of the discrepancies between coherent and incoherent particle-core resonance mechanisms will lead to a better understanding of the numerical study and experimental researches obtained recently. In addition, similar understanding can be extended to the study of higher order structure resonance.

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I. INTRODUCTION

The nonlinear beam dynamics in accelerators has been studied with analytical, numerical and experimental approaches for several decades [1–3]. Nowadays, the resonances driven by various nonlinear effects in accelerators are considered as the main sources leading to beam deterioration such as rms emittance growth, beam halo, and beam losses. Generally, the studies on beam dynamic behavior are in the frame of the Hamiltonian system and usually the nonlinear terms in the complex particle Hamiltonian are treated as perturbations. The resonance conditions are obtained with linearized perturbation techniques. Besides normal static external elements, the nonlinearity from the internal space charge effect is another main source that must be considered and treated carefully. Moreover, the nonlinear space charge couples the motions in different degrees of freedom in a self-consistent manner, especially in low energy and high current machines [4].

According to the beam motions to be focused on, the descriptions of nonlinear resonance are divided into single particle dynamics level (incoherent effect) and rms beam dynamics level (coherent effect). A typical example of single particle dynamics description is the tune diagram, which is filled with various resonance lines, \( n\nu_x + m\nu_y = l \), due to the multipoles and lattice imperfection [1]. It is widely used as a guidance on working point selection in the designs and operations. Considering the nonlinear effect, a wide tune spread will be formed [5]. When a beam is centered near the resonance lines in tune diagram, some particles can periodically cross the resonances if the synchronous motion is taken into account [6]. As to the coherent effect (rms level), nonlinearity from external elements and internal space charge effect plays a key role in beam collective instabilities [7, 8], which normally requires solving the perturbed Vlasov equation in a self-consistent manner [9–11]. However, the analytical solution of such system is not trivial, and the solvable problems are still limited to certain specific cases. Great efforts have been paid to extend these models to cover problems with various beam conditions.

As to the study of collective instability due to space charge effect with ion beams, one branch starts from the rms envelope equations [12], with which the 2nd order (envelope instability) structure resonance is well studied [13–15], while another branch is to solve the Vlasov-Poisson equations self-consistently [19–22]. Recently, a general theoretical study of the Vlasov-Poisson model in space charge physics is given in Ref. [19, 20], where the resonance phenomenon discussed in Ref. [21] has been extended to cover problems with various initial beam distributions and focusing.

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conditions. It is noteworthy that the driving force of the resonance derived from the Vlasov-Poisson does not come from the rms mismatch but from inner density mismatch. In Ref. [20], it states that the “structure resonance” can take place among these constructed eigenmodes if the lattice parameters are not elaborately optimized: it also proves the lower order structure resonance stop band can be naturally treated as one component of the higher order structure resonance stop bands; in addition, the 2nd order even structure resonance exactly describes the same coherent structure resonance as those from the envelope dynamics.

In fact, the external focusing strength usually changes as the beam being accelerated, which possibly results in the “structure resonance crossing”. This paper will focus on how the coherent and incoherent characteristics of a initial rms matched beam and inner particles are spontaneously affected during the beam crossing the structure resonance stop band. As an extension of the theoretical study of coherent structure resonance with Vlasov-Poisson model [19, 20], the 2nd order stop band around $90^\circ$ phase advance is chosen for detailed study to verify the validity of the theoretical prediction in the simulation and demonstrate the transient interaction between beam and structure resonances. These studies can be extended to higher order structure resonances with phase advance around $60^\circ$ (3rd order structure resonance) or $120^\circ$ (6th order structure resonance) [23, 24]. It is noteworthy that the fact that lower order coherent structure resonances are components of higher order structure resonances is the key to understand the results obtained from simulation and experiment reported in Ref. [25, 26].

The incoherent particle-core resonance, which is considered as one of the potential mechanism for the generation of particle tail in phase space, is intuitively used to explain the tail structure in phase space [27]. By simulating with initially rms matched beam conditions, we show that it has some difficulties to explain the basic phase space structures. The incoherent resonance may affect the beam but only on a long time scale.

In Section II, the model of structure resonance is briefly introduced. The terminologies used to describe the coherent and incoherent effect are introduced. In Section III, the equations to get the stop band are given explicitly. The phenomena when the beam crosses the coherent structure resonances are simulated with multi-particle PIC code and appropriately discussed in a transient sense. In Section IV, the incoherent particle characteristic is discussed. The conclusion and summary are given in Section V.

II. PHYSICAL MODEL AND TERMINOLOGIES

In this section, we briefly introduce the physical model and basic approach to deal with structure resonance. The solvable coupled Vlasov-Poisson equation is limited to the 4D KV distribution assumption [28]. In the following, the periodic FODO channel is used to model the evolution of coasting beam in accelerators. Considering the fact that the Hamiltonian of a matched beam in a periodic focusing channel is conserved, the equilibrium distribution function can be expressed as a function of generalized Hamiltonian:

$$f_0(x, p_x, y, p_y) = f(H_0),$$  \hspace{1cm} (1)

$$H_0 = k_x(s)x^2 + p_x^2 + k_y(s)y^2 + p_y^2 + V_{sc}(x, y),$$  \hspace{1cm} (2)

where $k_x(s)$ and $k_y(s)$ are the external focusing strength supplied by the quadrupoles, and $V_{sc}(x, y)$ is the space charge potential. The distribution function $f_0$ must meet the Vlasov equation and Poisson’s equation

$$\frac{\partial f_0}{\partial s} + [f_0, H_0] = 0,$$  \hspace{1cm} (3)

$$\Delta V_{sc}(x, y) = \frac{1}{\epsilon_0} \int \int f_0 dx dy,$$  \hspace{1cm} (4)

where $[\cdot, \cdot]$ is the Poisson bracket operator. Assuming that there exists a perturbation $f_1$ on the particle distribution function, it will lead to a perturbed space charge potential $V_1 = H_1$. Thus, the first order linearized Vlasov and Poisson equation can be obtained as:

$$\frac{\partial f_1}{\partial s} + [f_1, H_0] + [f_0, V_1] = 0,$$  \hspace{1cm} (5)

$$\Delta V_1(x, y) = \frac{1}{\epsilon_0} \int \int f_1 dx dy.$$  \hspace{1cm} (6)

The solvable sets of the equations are limited to the assumption of ideal KV distribution $f_0 = \delta(H_0)$. In general, the perturbed space charge potential can be expressed as the form of polynomial inside the beam

$$V_1 = \sum_{m=0}^{n} A_m(s)x^{n-m}y^m + \sum_{m=0}^{n-2} A_m^{(1)}(s)x^{n-m-2}y^m + \cdots.$$  \hspace{1cm} (7)

For a given order $n$ in Eq. (7), the even structure resonance and the odd structure resonance, which directly represent the tilts of the beam elliptical distribution in real space, could be treated separately on the basis of whether the index $m$ is restricted to even or odd integer values.

Inheriting the terminologies defined in the former researches [19, 20], we briefly introduce the physical meaning of the notations used in the following study. The collective modes $I_{j,k,l}(s)$ physically represents the integral of the surface electric field discontinuity from period to period. Noting $S$ as the length of one focusing period, $I_{j,k,l}(s)$ will meet $I'(s) = M(S)I(s)$, which is exactly the Mathieu Equation. The stability of the system is decided by the eigenvalues $\lambda$ of the Jacobi matrix $M(S)$. The phase advance of $I_{j,k,l}(s)$ is noted as $\Phi_{j,k,l}$; $\Phi_e$ is used to depict the phase advance of the rms matched envelope oscillation characteristics in one period, which is always $360^\circ$; $\Phi_e$ naturally
represents the periodicity of the lattice, and the words “envelope oscillation period” and “lattice period” are considered to be equivalent. \( \sigma_s \) is used to describe the single particle phase advance. The nonlinear effects from external elements and internal space charge cause different particles to have different particle phase advance \( \sigma_s \), leading to a beam phase advance spread. \( \sigma_0 \) and \( \sigma \) are standard notations used to evaluate the average focusing strength in one focusing period without and with space charge. The coherent structure resonance conditions are expressed explicitly as \( \Phi_{j,k,l}^{(1)} + \Phi_{j,k,l}^{(2)} = n \times 360^\circ \) and \( \Phi_{j,k,l}^{(1)}/\Phi_{c} = n/m \). It will be shown in the following that the structure resonance takes place accompanied by the eigenphase breaking. The parameter space where the structure resonance takes place is named as unstable stop bands. The incoherent particle-core resonance is express as \( \sigma_s/\Phi_{c} = n/m \). More mathematical details can be found in Ref. [19–22].

III. THE STRUCTURE RESONANCE STOP BAND CROSSING – COHERENT EFFECT

In the accelerator designs and operations, the principle “the resonance should be passed through as fast as possible if it cannot be avoided” is widely used in accelerator physics design [20–30]. In the following, the 2nd order structure resonance around the 90° phase advance is used to demonstrates the interaction between beam and structure resonance when the 2nd order stop band is crossed on purpose. As discussed, the 2nd order structure resonance is actually one component of the 4th order collective structure resonance [20]. In the following, the 2nd order structure resonance particularly refers to this mixed stop band. For simplicity, the beam with equal emittance in two degrees of freedom is adopted to model the beam behavior in symmetric periodic FODO channels \((|k_x| = |k_y|)\).

A. The 2nd order collective structure resonance

For the 2nd order structure resonance, the related perturbed space charge potential inside near the beam boundary is \( V_{2e} = A_0(s)x^2 + A_2(s)y^2 \) for the even structure resonance and \( V_{2o} = A_1(s)xy \) for the odd structure resonance. Inheriting the notation in Ref. [19–21], dynamic system \( I'(s) = M(s)I(s) \) is constructed with \((I_{0,2,0}, I_{2,0,2})\) and \((I_{1,1,1}, I_{1,1,-1})\), representing the 2nd order even and odd structure resonance respectively. The explicit forms of the eigenphases with and without beam current, structure resonance driving terms, and the related forms of the perturbed space charge potential can be found in Tab. I in Ref. [20]. The Jacobi matrix \( M(s) \) is with the form

\[
M(s) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
J_{21}(s) & J_{22}(s) & J_{23}(s) & 0 \\
J_{41}(s) & 0 & J_{43}(s) & J_{44}(s)
\end{pmatrix},
\]

(8)

The explicit form of each element \( J_{ij}(s) \) in above equation can be found in the Appendix.

It is noteworthy that the well-known envelope instability, starting from the rms envelope model, describes the same physics as the 2nd order even structure resonance [19–21]. Recently, the sum and difference instabilities are studied in Ref. [17] with Chernin’s model [18], which gives exactly the same result as that from the 2nd order odd structure resonance. From the experiment point of view, great efforts have been put into the spectrum measurement of the \( I_{0,2,0} \) and \( I_{2,0,2} \) in GSI [31] and CERN [32].

With the help of Eq. (8), Fig. 1 shows the eigenvalues \( \text{Abs} [\lambda_{j,k,l}] \) and eigenphases \( \Phi_{j,k,l}^{(1)} \) of the 2nd order even and odd structure resonance when the beam current is fixed and the depressed current phase advance \( \sigma \) varies from 100° to 70°. Correspondingly, the zero-current phase advance \( \sigma_0 \) varies roughly from 110° to 80°. Since the symmetric condition chosen here, the phase advances are equal in two degrees of freedom with and without space charge. The results are shown in the Fig. 1. For the 2nd order even structure resonance, whenever the eigenphases merging (eigenphases locking between \( \Phi_{2,0,2} = \Phi_{0,2,0} \), and it is also termed as confuent resonance [20]) takes place, as region \((84°, 89°)\), the absolute eigenvalue \( \text{Abs} [\lambda] \) leaves the unit 1, representing collective instabilities and resulting in rms emittance growth. The 2nd order odd structure resonance does not lead to any instability since the symmetric condition is chosen here.

B. The 2nd order structure resonance crossing study

In the following, the Paritice-In-Cell (PIC) code PTOPO [33, 34] is used for numerical simulations to study the structure resonance crossing. The well matched WaterBag (WB) beam composed of 50000 macro-particles is used as initial beam distribution. Two approaches are applied for the rectangular meshes generation in the Poisson solver. One is static case that the mesh size keeps the same as the aperture size; the other is the dynamics case that the mesh size is as large as 6 times the rms beam size and varies in each time step. In both cases, the Dirichlet boundary condition is adopted. There is no qualitative difference between the results of these two cases, and in the following study only the results from the dynamical case are given. The pipe size is large enough to ensure there is no beam loss during the whole calculation. The lattice made up of 400 FODO periods with the focusing strength varying linearly, which
is long enough in cases discussed here to ensure no further emittance growth in simulations, is used to imitate the structure resonance crossing process.

With a fixed beam current and the adiabatically varying quadrupole focusing strength, two cases of the beam evolution are studied when the 2nd order structure resonance stop is crossed. One is from the below (depressed phase advance $\sigma$ varies from 75° to 95° during 400 periods linearly), and the other is inversely from the above. Fig. 2a shows the evolution of phase advances along the lattice when the structure resonance stop band – grey region roughly with $84^\circ < \sigma < 89^\circ$ – is crossed from below and above. The blue and red curves represent the periodic phase advances in x and y direction respectively, which is obtained with the help of the equation $\sigma = \int_{s_0}^{s} 1/\beta(s)ds$. The black curves are the centers of the red and blue curves. In principle, the simulated phase advances are limited in the region bounded by the yellow and green lines which are the designed depressed phase advance $\sigma$ and zero-current phase advance $\sigma_0$. Fig. 3a shows the rms emittance evolution correspondingly.

In Fig. 2a, when the structure resonance is crossed from below, the simulated beam phase advance obeys the designed values spontaneously until the beam gets into the stop band around period 200. Thereafter the speed of beam structure resonance crossing gets “faster” than it is supposed to be (indicated by the cross point between the green line and the boundary of the stop band) and the beam gets out of the stop band round period 250, after which it attains a local equilibrium state finally. Fig. 3a indicates that the rms emittance growth takes place once the beam gets into the structure resonance stop band and no further growth appears after passing through it, till period 265, which will be explained later. Inversely, Fig. 2b and Fig. 3b show the phase advance and emittance evolution when the structure resonance stop band is crossed from above. In Fig. 2b it can be verified that the beam stays for a longer time than it is supposed to be in the stop band, which means the speed of resonance crossing is “slower”. The beam gets out of the resonance stop band at around period 370. Compared with the case of structure resonance crossing from below,
The beam has a larger emittance growth finally.

The structure resonance crossing must be studied in a transient sense [20]. The down-threshold and up-threshold of the stop band vary with the beam rms emittance growing during the crossing. As a result, in the case of the below crossing, the up-threshold moves to near 90°, and beam roughly suffers 15 more periods from structure resonance, until around period 265; Similarly, for the crossing from above, the down-threshold also increases a little bit, and actually beam gets out of stop band earlier than it is supposed to be, roughly at period 300. Another characteristic of these two sets of studies is the “faster” and “slower” resonance crossing speed. It seems that the resonance stop band has an “attracting” effect if the beam crosses from below and an “repulsive” effect if it crosses from above. The reason is, in a general sense, under the comprehensive influence of an external field and internal space charge field, if there is any instability taking place, the beam always spontaneously tries to get rid of this imbalance force which normally
leads to rms emittance or beam halo growth. This process can also be viewed as a kind of “relief” from the energy point of view. It is a natural result that the beam itself always tries to get to a state where the space charge takes a weaker influence [16]. Here, it is reflected by the fact that the transient beam tune depression $\eta = \sigma / \sigma_0$ turns to be larger than designed values once beam is affected by structure resonance.

Fig. [H] shows the phase space distribution evolution along the FODO lattice during the structure resonance crossing from below (top) and above (bottom). Clearly, the 4-fold phase space structures – the evidence of the 4th order structure resonance – start to develop since the beam gets into the resonance stop band both from the below crossing and above crossing. Question might be asked why the 4th order structure resonance appears in the 2nd order structure resonance stop bands. Actually, the reason is already explained above that the lower

FIG. 4. The phase space $(x - p_x)$ profiles during structure resonance crossing from below (top) and above (bottom). The locations for each plot are chosen at period 50, 150, 250, 350.

FIG. 5. a): The final emittance growth ratio after 400 FODO periods versus the number of FODO periods used in resonance crossing speed control within 400 FODO periods; b): The final emittance growth ratio after 400 FODO periods versus the effective number of periods where the beam is in the structure resonance stop band. The abscissas in above two figures are plotted on a logarithmic scale.
order stop bands are components of the higher order structure resonance stop bands \[19, 20\]. In a general sense, the appearance of different orders of structure resonances requires appropriate driving force (Eq. [7]). Starting from a WB initial distribution, the 2nd and 4th order perturbed potential exists from the very beginning and will evolve self-consistently. Thus, both the 2-fold and the 4-fold structure phase space are supposed to appear in this 2nd order structure resonance stop band (Another example will be shown in Sec. [IV]). Finally, the significant beam halo is formed due to these mixed structure resonance effect.

To ensure generality of the above understanding, various initial beam distribution, like Parabolic distribution and truncated Gaussian distribution, are used for similar study. The patterns of emittance distribution and truncated Gaussian distribution, structure resonance effect.

The structure resonance crossing speed and rms discusse here since the rms equivalence [12].

C. The structure resonance crossing speed and rms emittance growth

If resonance crossing cannot be avoided, it is now still under discussion how it should be manipulated: cross it adiabatically or as fast as possible. Here we mainly pay attention to the relationship between beam rms emittance and the structure resonance crossing speed. With the same method to control the beam phase advance as discussed in the above subsection, here, within total 400 FODO periods, the first N (N=10, 20, 50, 100, 200, 400) periods are adjusted to make the beam phase advances change linearly from \(75^\circ\) to \(95^\circ\) (below crossing) and from \(95^\circ\) to \(75^\circ\) (above crossing). Again, the final rms emittance growth after 400 periods is used as the evaluation of the beam quality.

Fig. 6a shows the final emittance growth after 400 FODO periods as function of the number of periods (N) used for resonance crossing speed control. The red and blue dots respectively represent the resonance crossings from above and below. The different subscript \(n\) represents different resonance crossing speed in which the focusing strength are adjusted in the first N (N=10, 20, 50, 100, 200, 400) periods. With the same resonance crossing speed, the emittance growth from the above crossing is much larger than that from the below crossing, which indicates the same physics as we explained above in Fig. 3. For different resonance crossing speed, both below crossing and above crossing indicate that the structure resonance stop bands should be crossed as quickly as possible to avoid significant emittance growth. Fig. 3 shows the emittance growth as a function of the time that the beam spends within resonance stop band. As expected, the emittance growth is positively related to the time that the beam stays in the structure resonance stop band (the horizontal coordinate in Fig. 5 is with logarithmic scale).

IV. SINGLE PARTICLE DYNAMICS – INCOHERENT EFFECT

From the incoherent single particle dynamics point of view, it is intuitive to attribute the 4-fold structure resonance to the 4th order particle-core resonance \((90^\circ/360^\circ = 1 : 4)\) since the phase advance discussed in this paper is mainly around \(\sigma \sim 90^\circ\). Thus, the direction of the particle-core resonance islands moving towards or outward to the core during structure resonance crossing is adopted to interpret the 4-fold phase space structure [27] and how severe of beam halo when structure resonance stop band is crossed from below and above.

Using the particle-core resonance model [35, 36], Fig. 6 shows the Poincaré section plot of the single particle motion in phase space. Clearly, when the structure resonance is crossed from above, four particle-core resonance islands are generated in the origin point by a bifurcation process [37, 38] and the formed islands move outward the core during the structure resonance crossing process (Fig. 6a → Fig. 6b). In contrast, for the crossing from below, the particle-core resonance islands move from outside into the inner core (Fig. 6d → Fig. 6a). It is intuitive if one believes that the 4-fold phase space structure in phase space is related to this particle-core resonance, since this particle-core resonance can gradually bring particles in the core and form the stable 4-fold phase space structure; In another case, if particle core resonance islands move from outside towards the inner core, since no particle is located outside initially, less particles will occupy the 4-fold islands and less halo particles are generated. However, this is misleading, because the fold structures are mainly related to the coherent collective effect rather than the incoherent particle-core resonance if the space charge is the only nonlinearity source in the beam system. For instance, this incoherent particle-core resonance cannot predict a two-fold phase space structure which actually appears in the simulations of both below crossing and above crossing, as shown in Fig. 7. It is exactly the evidence of the 2nd order structure resonance – coherent effect. As pointed out by the former study [16], we argue that the incoherent particle-core resonance might lead to the 4-fold phase space structure, but only on a long-time scale. Here, we believe that the formed 4-fold phase space structure attained in the simulation corresponds to the mixed 4th/2nd collective structure resonance.

V. SUMMARY AND ACKNOWLEDGEMENTS

As a follow-up of the previous analytical study of the structure resonance [19, 20], this paper studies how the beam is spontaneously affected by the structure resonance. Since the analytical studies are based on the KV beam distribution assumption and the linearized perturbation theory, the nonlinear resonance damping effect and the resonance saturation effect
are not included. The study in this paper clearly shows the transient behavior of the beam when the structure resonance stop band is crossed. The mixed 2nd/4th order coherent structure resonance gives quite reasonable explanation to the results obtained from the PIC simulations. It must be emphasized again that the interaction between the beam and the resonance needs to be studied in a transient sense. The “attracting” and “repulsive” effect of the stop band is a spontaneous beam reaction since the rms characteristics are modified by the structure resonance. It is also found that the beam emittance growth is positively related to the time that the beam is affected by the structure resonance. The final beam equilibrium status is a comprehensive result as a comprise of the structure resonance and the nonlinear resonance damping. The incoherent particle-core resonance can cause phase space distortion, emittance growth, and beam halo formation, but only on a long-time scale.

Another importance in this study is that the simulation clearly proves the conclusion from previous theoretical prediction, that the lower order stop bands are naturally included in higher order stop bands \[19, 20\]. The study of higher order of structure resonance can be extended and understood in the same frame discussed here. One example is the 3rd/6th mixed structure resonance around 60° phase advance \(3 \times 60° = 180°, 6 \times 60° = 360°[16] \) and 120° phase advance \(3 \times 120° = 360°, 6 \times 120° = 720° [19, 23] \). The understanding of the coherent and incoherent resonance in space charge physics leads us to a better understanding of recent experimental results \[23, 26, 39\] . However, despite the progress in interpretation of the nonlinear phenomena, there is still a large gap between analytical prediction, numerical simulation, and the experiment result. In the near future, further study will be extended to the real 6D particle dynamics.

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Appendix: The explicit form of the 2nd order Jacobi Matrix for collective structure resonance analysis.

Note \(C_{i,j}(s) = i/\beta_x(s) + j/\beta_y(s)\) for simplicity, where \(\beta(s)\) is the betatron function, for the 2nd order even structure resonance:

\[
\begin{align*}
J_{21} &= -C_{2,0}^2 - C_{2,0} \frac{2K a(2a + b)}{\epsilon_x} \frac{1}{2(a + b)^2} \\
J_{22} &= C_{2,0}^2 \\
J_{23} &= -C_{2,0} \frac{2K ab}{\epsilon_x} \frac{1}{2(a + b)^2} \\
J_{41} &= -C_{0,2} \frac{2K ab}{\epsilon_x} \frac{1}{2\Gamma(a + b)^2} \\
J_{43} &= -C_{0,2}^2 - C_{0,2} \frac{2K b(a + b)}{\epsilon_x} \frac{1}{2\Gamma(a + b)^2} \\
J_{44} &= C_{0,2}^2 \\
\end{align*}
\]

(A.1)
For the 2nd order odd structure resonance,

\[
J_{21} = -C_{1,1}' - C_{1,1} \frac{2K ab(1 + \Gamma)}{\epsilon_x 2\Gamma(a + b)^2}
\]

\[
J_{22} = \frac{C_{1,1}'}{C_{1,1}}
\]

\[
J_{23} = -C_{1,1} \frac{2K ab(-1 + \Gamma)}{\epsilon_x 2\Gamma(a + b)^2}
\]

\[
J_{41} = -C_{1,-1} \frac{2K ab(1 + \Gamma)}{\epsilon_x 2\Gamma(a + b)^2}
\]

\[
J_{43} = -C_{1,-1}' - C_{1,-1} \frac{2K ab(-1 + \Gamma)}{\epsilon_x 2\Gamma(a + b)^2}
\]

\[
J_{44} = -\frac{C_{1,-1}'}{C_{1,-1}}
\]

(A.2)

Here \( K \) is the generalized perveance, \( a(s) \) and \( b(s) \) are the beam size during one period, \( \Gamma = \epsilon_x/\epsilon_y \) is the emittance ratio between different degrees of freedom. In this paper, \( \Gamma = 1 \) is selected for analysis.

[1] S.-Y. Lee, *Accelerator physics* (World Scientific Publishing Company, 2011).
[2] M. Reiser, *Theory and design of charged particle beams* (John Wiley & Sons, 2008).
[3] I. Hofmann, "Space Charge Physics for Particle Accelerators" (Springer International Publishing, 2017).
[4] A. W. Chao, K. H. Mess, M. Tigner, and F. Zimmermann, *Handbook of accelerator physics and engineering* (World scientific, 2013).
[5] S. Cousineau, *Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment* 561, 297 (2006).
[6] G. Franchetti and I. Hofmann, *Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment* 561, 195 (2006).
[7] F. J. Sacherer, *Transverse space charge effects in circular accelerators*, Ph.D. thesis (1968).
[8] F. Sacherer, IEEE Transactions on Nuclear Science 20, 825 (1973).
[9] A. W. Chao, *Physics of collective beam instabilities in high energy accelerators* (Wiley, 1993).
[10] R. L. Gluckstern, *Proceedings, 1970 Proton Linear Accelerator Conference*, Conf. Proc. C700928, 811 (1970).
[11] R. L. Gluckstern, R. Chasman, and K. Crandall, *Proceedings, 1970 Proton Linear Accelerator Conference*, Conf. Proc. C700928, 823 (1970).
[12] F. J. Sacherer, IEEE Transactions on Nuclear Science 18, 1105 (1971).
[13] J. Struckmeier and M. Reiser, Part. Accel. 14, 227 (1983).
[14] S. M. Lund and B. Bukh, Physical Review Special Topics-Accelerators and Beams 7, 024801 (2004).
[15] C. Li and Q. Qin, *Physics of Plasmas* 22, 023108 (2015).
[16] C. Li and Y. L. Zhao, Phys. Rev. ST Accel. Beams 17, 124202 (2014).
[17] Y.-S. Yuan, O. Boine-Frankenheim, and I. Hofmann, Phys. Rev. Accel. Beams 20, 104201 (2017).
[18] D. Chernin, Part. Accel. 24, 29 (1988).
[19] C. Li and Q. Qin, in *Proc. of ICFA Advanced Beam Dynamics Workshop on High-Intensity and High-Brightness Hadron Beams (HB’16)* (JACoW).
[20] C. Li and R. A. Jameson, Phys. Rev. Accel. Beams 21, 024801 (2013).
[21] I. Hofmann, L. J. Laslett, L. Smith, and I. Haber, Particle Accelerators 13, 145 (1983).
[22] I. Hofmann, Physical Review E 57, 4713 (1998).
[23] D.-O. Jeon, K. R. Hwang, J.-H. Jang, H. Jin, and H. Jang, Phys. Rev. Lett. 114, 184802 (2015).
[24] I. Hofmann and O. Boine-Frankenheim, Phys. Rev. Lett. 115, 204802 (2015).
[25] D.-O. Jeon, Physical Review Accelerators and Beams 19, 064201 (2016).
[26] K. Ito, H. Okamoto, Y. Tokashiki, and K. Fukushima, Physical Review Accelerators and Beams 20, 064201 (2017).
[27] D. Jeon, L. Groening, and G. Franchetti, Phys. Rev. ST Accel. Beams 12, 054204 (2009).
[28] I. M. Kapchinskij and V. V. Vladimirskij, Proceedings of the 9th International Conference on High Energy Accelerators, (CERN, Geneva, 1959), P. 274 (1959).
[29] Z. Li, P. Cheng, H. Geng, Z. Guo, Y. He, C. Meng, H. Ouyang, S. Pei, B. Sun, J. Sun, et al., Physical Review Special Topics-Accelerators and Beams 16, 080101 (2013).
[30] S. Henderson, W. Abraham, A. Aleksandrov, C. Allen, J. Alonso, D. Anderson, D. Arenius, T. Arthur, S. Assadi, J. Ayers, et al., Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment 763, 610 (2014).
[31] R. Singh, P. Forck, P. Kowina, W. Müller, J. T. Kamga, T. Weiland, and M. Gasior, IBIC Proceedings (2014).
[32] A. Oeftiger, “Quadrupolar pick-ups to measure space charge tune spreads of bunched beams,” Invited talk in Space Charg 2017 (2017).

[33] L. Chao, Z. Zhi-Lei, Q. Xin, X. Xian-Bo, H. Yuan, and Y. Lei, Chinese physics C 38, 037005 (2014).

[34] Z. Liu, C. Li, Q. Qin, Y. Zhao, and F. Yan, in 57th ICFA Advanced Beam Dynamics Workshop on High-Intensity and High-Brightness Hadron Beams (HB’16), Malmö, Sweden, July 3-8, 2016 (JACOW, Geneva, Switzerland, 2016) pp. 195–198.

[35] R. A. Jameson (World Scientific, ISBN 981-02-2537-7, pp.530-560. LA-UR-94-3753, Los Alamos National Laboratory. AIP Proceedings of the 1994 Joint US-CERN-Japan International School on Frontiers of Accelerator Technology, Maui, Hawaii, USA, 3-9 November, 1994).

[36] C. Li, Q. Xin, H. Yuan, Y. Lei, and Y. Batygin, Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment 770, 169 (2015).

[37] R. Dilão, Nonlinear dynamics in particle accelerators, Vol. 23 (World Scientific, 1996).

[38] C. Li, Z. Liu, Y. Zhao, and Q. Qin, Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment 813, 13 (2016).

[39] L. Groening, I. Hofmann, W. Barth, W. Bayer, G. Clemente, L. Dahl, P. Forck, P. Gerhard, M. Kaiser, M. Maier, et al., Physical review letters 103, 224801 (2009).