An inverse problem for the differential operator on the graph with a cycle with different orders on different edges
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Abstract. We consider a variable order differential operator on a graph with a cycle. We study the inverse spectral problem for this operator by the system of spectra. The main results of the paper are the uniqueness theorem and the constructive procedure for the solution of the inverse problem.

Keywords: geometrical graphs, differential operators, inverse spectral problems, Weyl-type matrices, method of spectral mappings.

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1. Introduction

Differential operators on geometrical graphs (networks) play a fundamental role in many problems in science and engineering. Main results for second-order operators on graphs and their applications are described in [1–7]. In this paper, we focus our attention on inverse spectral problems that consist in recovering the coefficients of differential operators on graphs by their spectral characteristics. Thus we assume that the structure of the graph, boundary and matching conditions in the vertices are known a priori.

Although for second-order differential operators the inverse spectral theory has been developed fairly completely, there are only a few works for higher-order operators [8]. In paper [9], V.A. Yurko started to study inverse problems for various order differential operators, i.e. when the orders of differential equations are different on different edges of the graph. Papers [10, 11] describe some mechanical models with various order differential operators.

In work [9], an inverse problem is solved on a star-type graph. Now we plan to investigate an inverse problem for a various order operator on a graph with a cycle. We use the system of spectra, corresponding to different boundary and matching conditions, for recovering the potential of the differential operator. This problem statement is a natural generalization of the classical inverse Sturm-Liouville problem on a finite interval by two spectra (see monographs [12, 13]).

Let us come to the formulation of the problem. Consider a compact graph $G$ with the vertices $V = \{v_0, \ldots, v_m\}$ and the edges $E = \{e_0, \ldots, e_m\}$, where $e_j = [v_j, v_0]$, $j = 1, m$, and $e_0$ is a cycle containing only the vertex $v_0$. Thus $v_j$, $j = 1, m$, are boundary vertices and $v_0$ is the only internal vertex. Let $T_j$ be the length of the edge $e_j$. For each edge $e_j \in E$, we introduce the parameter $x_j \in [0, T_j]$ in such a way, that for $j = 1, m$, the end $x_j = 0$ corresponds to the vertex $v_j$, and the end $x_j = T_j$ corresponds to $v_0$. For $j = 0$, both ends correspond to the vertex $v_0$.

Fix the integers $2 = n_0 \leq n_1 \leq \cdots \leq n_m$. Consider the following differential equations of various orders:

$$y^{(n_j)}(x_j) + \sum_{\mu=0}^{n_j-2} q_{\mu j}(x_j) y^{(\mu)}(x_j) = \lambda y_j(x_j), \quad j = 0, m,$$

(1)

where $q_{\mu j} \in L[0, T_j]$. We call the collection $q := \{q_{\mu j}\}_{j=0, m; \mu=0, n_j-2}$ the potential on the graph $G$.

Now we are going to introduce matching conditions in the internal vertex $v_0$, that generalize Kirchhoff’s matching conditions for Sturm-Liouville operators on graphs [17] and matching
conditions for higher-order differential operators \[^3\]. Consider the linear forms

\[ U_{j\nu}(y_j) = \sum_{\mu=0}^{\nu} \gamma_{j\nu\mu} y_j^{(\mu)}(T_j), \quad \gamma_{j\nu\mu} := \gamma_{j\nu\mu} \neq 0, \quad j = 1, m, \nu = 0, n_j - 1, \]

\[ U_{0\nu}(y_0) = y_0^{(\nu)}(T_0), \quad \nu = 0, 1, \]

where \(\gamma_{j\nu\mu}\) are some complex numbers. Define the continuity condition Cont\((\nu)\) and the Kirchhoff’s condition Kirch\((\nu)\) of the \(\nu\)-th order:

\[
\text{Cont}(\nu): \begin{cases} 
U_{mn}(y_m) = U_{j\nu}(y_j), & j = 0, m - 1; \nu < n_j - 1, \\
y_0(0) = U_{0\nu}(y_0), & \text{if} \nu = 0;
\end{cases}
\]

\[
\text{Kirch}(\nu): \quad \sum_{j: \nu < n_j} U_{j\nu}(y_j) = \delta_{1\nu} y_0(0).
\]

Here and below \(\delta_{jk}\) is the Kronecker delta.

Fix an edge number \(s = 1, m\) and orders \(k = 1, n_s - 1, \mu = k, n_s\). Let \(\Lambda_{sk\mu} = \{\lambda_{sk\mu}\}_{\nu \geq 1}\) be the spectrum of the boundary value problem \(L_{sk\mu}\) for the system \((1)\) under the boundary conditions

\[
y_j^{(\nu-1)}(0) = 0, \quad \nu = 1, k - 1, \mu, \]

\[
y_j^{(\nu-1)}(0) = 0, \quad \nu = 1, n_j - k, \quad j = 1, m \setminus s: n_j < k, \]

\[
y_j(0) = 0, \quad j = 1, m: n_j \leq k,
\]

and the matching conditions Cont\((\nu), \nu = 0, k - 1\), Kirch\((\nu), \nu = k, n_s - 1\), in the vertex \(v_0\). Note that the total number of the boundary conditions and the matching conditions equals \(\sum_{j=0}^{m} n_j\), i.e. the sum of the orders on the edges. In Section 3 we discuss the question of regularity for these conditions.

We will use the spectra \(\{\Lambda_{sk\mu}\}\) for recovering of the potential \(\{q_{j\nu}\}\), but this information is insufficient, and we need additional data related to the cycle. Let \(S_0(x_0, \lambda)\) and \(C_0(x_0, \lambda)\) be the solutions of the differential equation \((1)\) on the edge \(e_0\) \((n_0 = 2)\), satisfying the initial conditions

\[
S_0(0, \lambda) = C_0'(0, \lambda) = 0, \quad S_0'(0, \lambda) = C_0(0, \lambda) = 1.
\]

Denote

\[
h(\lambda) := S_0(T_0, \lambda), \quad H(\lambda) := C_0(T_0, \lambda) - S_0'(T_0, \lambda), \quad d(\lambda) := C_0(T_0, \lambda) + S_0'(T_0, \lambda). \quad (2)
\]

Note that the functions \(h(\lambda), H(\lambda)\) and \(d(\lambda)\) are entire in \(\lambda\) of order 1/2. Let \(\{\nu_n\}_{n \geq 1}\) be the zeros of \(h(\lambda)\), and \(\omega_n := \text{sign } H(\nu_n)\). Here we assume for the sake of simplicity, that the potential \(q_{00}(x_0)\) is real-valued. For the non-self-adjoint case, one can use the approach described in [14].

We study the following

**Inverse problem 1.** Given the spectra \(\Lambda_{sk\mu}\), \(s = 1, m, k = 1, n_s - 1, \mu = k, n_s\), and the signs \(\Omega := \{\omega_n\}_{n \geq 1}\), construct the potentials \(q_{j\nu}, j = 0, m, \mu = 0, n_j - 2\).

Our goal is to prove the unique solvability of Inverse problem 1 and develop a constructive procedure for its solution. Our approach is based on the method of spectral mappings [13][15] and some ideas of paper [16] concerning an inverse problem for Sturm-Liouville operator on a graph with a cycle. Our general strategy is to solve auxiliary inverse problems on the boundary edges. These problems are not local problems on intervals, since they use information from
the whole graph, but they are close to local problems by their properties. Then the problem is reduced to the well-studied Sturm-Liouville periodic inverse problem for the cycle [16,18].

The paper is organized as follows. In Section 2, we introduce so-called Weyl-type matrices for each of the boundary edges and show how to construct them by the given spectra. In Section 3, we study asymptotics of special solutions of system (1). In Section 4, we discuss auxiliary inverse problems on the boundary edges and on the cycle. In the Section 5, we arrive at the main results of our paper. We also provide Appendix with an example.

2. Weyl-type solution and Weyl-type matrices

In this section, we introduce some special solutions of system (1) and study their structural and analytical properties.

Fix \( j = 0, m \). Let \( \{ C_{kj}(x_j, \lambda) \}_{k=1}^{n} \) be a fundamental system of solutions of equation (1) on the edge \( e_j \) under initial conditions \( C_{kj}^{\mu-1}(0, \lambda) = \delta_{k\mu}, \ k, \mu = 1, n_j \). For each fixed \( x_j \in [0, T_j] \), the functions \( C_{kj}^{\mu-1}(x_j, \lambda) \) are entire in \( \lambda \)-plane of order \( 1/n_j \). We also have

\[
\det \left[ C_{kj}^{(\mu-1)}(x_j, \lambda) \right]_{k,\mu} = 1, \quad j = 0, m. \tag{3}
\]

Fix \( s = 1, m \) and \( k = 1, n_s - 1 \). Let \( \Psi_{sk} = \{ \psi_{skj} \}_{j=1,m} \) be the solutions of system (1) satisfying the conditions

\[
\begin{align*}
\psi_{skj}^{(\nu-1)}(0) &= \delta_{k\nu}, \quad \nu = 1, k, \\
\psi_{skj}^{(\xi-1)}(0) &= 0, \quad \xi = 1, n_j - 1, \quad j = 1, m \setminus s: k < n_j, \\
\psi_{skj}(0) &= 0, \quad j = 1, m: k \geq n_j,
\end{align*}
\tag{4}
\]

\[
\text{Cont}(\nu), \quad \nu = 0, k - 1, \quad \text{Kirch}(\nu), \quad \nu = k, n_s - 1. \tag{5}
\]

The vector-function \( \Psi_{sk} \) is called the Weyl-type solution of order \( k \) for the boundary vertex \( v_s \). Additionally define \( \psi_{sk, s}(x, \lambda) = C_{k,s}(x, \lambda), \ s = 1, m. \)

Let \( M_{sk, \mu}(\lambda) := \psi_{sk}^{(\mu-1)}(0, \lambda) \). For each fixed \( s = 1, m \), the matrix \( M_s(\lambda) := [M_{sk, \mu}(\lambda)]_{k,\mu=1}^{n_s} \) is called the Weyl-type matrix with respect to the boundary vertex \( v_s \). The notion of the Weyl-type matrices is a generalization of the notion of the Weyl function (\( m \)-function) for the classical Sturm-Liouville operator (see [13,19]) and the notion of Weyl matrices for higher-order differential operators (see [8,9,15]).

It follows from (1), that \( M_{sk, \mu}(\lambda) = \delta_{k\mu} \) for \( k \geq \mu \). Moreover,

\[
\psi_{sk}(x, \lambda) = C_{ks}(x, \lambda) + \sum_{\mu=1}^{n_s} M_{sk, \mu}(\lambda) C_{\mu s}(x, \lambda), \quad s = 1, m, \quad k = 1, n_s. \tag{6}
\]

Now we plan to study the connection between the Weyl-type matrices, the spectra \( \Lambda_{sk, \mu} \) and the functions, defined in (2). For this purpose, one can easily expand the functions \( \psi_{skj}(x_j, \lambda) \) by the fundamental systems \( C_{kj}^{\mu}(x_j, \lambda) \) and substitute these expansions into the matching conditions (5). Solving the resulting linear system \( E_{sk} \), one gets for \( s = 1, m, 1 \leq k < \mu \leq n_s \):

\[
M_{sk, \mu}(\lambda) = -\frac{\Delta_{sk, \mu}(\lambda)}{\Delta_{sk, \mu}(\lambda)} \tag{7}
\]

Here \( \Delta_{sk, \mu}(\lambda), k \leq \mu, \) is the characteristic function for the boundary value problem \( L_{sk, \mu} \), and its zeros coincide with the eigenvalues \( \Lambda_{sk, \mu} \). The functions \( \Delta_{sk, \mu} \) are entire in \( \lambda \) and, consequently, \( M_{sk, \mu}(\lambda) \) are meromorphic in \( \lambda \). Similarly to [9], one can easily prove the following fact.
Lemma 1. Each characteristic function $\Delta_{sk\mu}(\lambda)$ can be determined uniquely by its zeros $\Lambda_{sk\mu} = \{\lambda_{sk\mu}\}_{\ell \geq 1}$.

Furthermore, analysing the structure of determinants in the systems $E_{sk}$ (see the example in Appendix for clarity), we obtain the relations

$$\Delta_{sk\mu}(\lambda) = (d(\lambda) - 2)F_{sp}(\lambda) + h(\lambda)G_{sk\mu}(\lambda), \quad k = 1,$$

$$\Delta_{sk\mu}(\lambda) = h(\lambda)G_{sk\mu}(\lambda), \quad k > 1,$$

where $F_{sp}(\lambda), G_{sp}(\lambda), G_{sk\mu}(\lambda)$ are some combinations of $C_{ij}^{(\nu)}(x_j, T_j)$. We will use formulas (8), (9) to find the data, associated with the cycle, from the characteristic determinants $\Delta_{sk\mu}(\lambda)$.

3. Asymptotic behavior of the Weyl-type solution

Fix $j = 0, m$. Let $\lambda = \rho_j^{n_j}$. The $\rho$-plane can be partitioned into sectors of angle $\frac{\pi}{n_j}$:

$$S_{\nu j} = \arg \rho_j \in \left(\frac{\nu \pi}{n_j}, \frac{(\nu + 1)\pi}{n_j}\right), \quad \nu = 0, 2n_j - 1.$$ 

Let us fix one of them and call it simply $S_j$. Then the roots $R_{1j}, R_{2j}, \ldots, R_{n_{\nu j}}$ of the equation $R^{n_j - 1} = 0$ can be numbered in such a way that

$$\Re(\rho_j R_{1j}) < \Re(\rho_j R_{2j}) < \cdots < \Re(\rho_j R_{n_{\nu j}}), \quad \rho_j \in S_j. \quad (10)$$

Denote

$$\Omega_{0j} := 1, \quad \Omega_{kj} := \det \left[P_{\xi_{\nu j}}^{(\nu - 1)}\right]_{\nu = 1}^{n_j}, \quad \omega_{kj} := \frac{\Omega_{k-1,j}}{\Omega_{kj}}, \quad j = 0, m, k = 1, n_j,$$

$$[1]_j := 1 + O(\rho_j^{-1}), \quad |\rho_j| \to \infty.$$ 

The following Lemma has been proved in [9]:

Lemma 2. Fix $j = 0, m$ and a sector $S_j$ with property (10). Let $k = 1, n_j - 1$ and let $y_j(x_j, \lambda)$ and $z_j(x_j, \lambda)$ be solutions of equation (11) on the edge $e_j$ under the initial conditions

$$y_j(0) = y'_j(0) = \cdots = y^{(k-1)}_j(0) = 0,$$

$$z_j(0) = z'_j(0) = \cdots = z^{(k-2)}_j(0) = 0, \quad z^{(k-1)}_j(0) = 1.$$ 

Then for $x_j \in (0, T_j]$, $\nu = 0, n_j - 1, \rho_j \in S_j$, $|\rho_j| \to \infty$,

$$y_j^{(\nu)}(x_j, \lambda) = \sum_{\mu = k+1}^{n_j} A_{\mu j}(\rho_j) (\rho_j R_{\mu j})^\nu \exp(\rho_j R_{\mu j} x_j)[1]_j,$$

$$z_j^{(\nu)}(x_j, \lambda) = \frac{\omega_{kj}}{\rho_j^{n_j}} (\rho_j R_{kj})^\nu \exp(\rho_j R_{kj} x_j)[1]_j + \sum_{\mu = k+1}^{n_j} B_{\mu j}(\rho_j) (\rho_j R_{\mu j})^\nu \exp(\rho_j R_{\mu j} x_j)[1]_j,$$

where the coefficients $A_{\mu j}(\rho_j), B_{\mu j}(\rho_j)$ do not depend on $x_j$. Here and below we assume that arg $\rho_j =$ const, as $|\rho_j| \to \infty$. 

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Now we are going to apply Lemma 2 to the Weyl-type solution, in order to study its asymptotic behavior. Fix an edge \( s = \frac{1}{m} \) and an order \( k = \frac{1}{n_s} - 1 \). For brevity, further we omit the indices \( s \), \( k \) if they are fixed, \( \psi_j(x, \lambda) := \psi_{skj}(x, \lambda) \). Fix a ray \( \{ \lambda: \arg \lambda = \theta \} \), \( \theta \neq 0, \pi \), which belongs to some sectors \( S_j \) with property (10) for each \( j = 0, m \). It follows from (4) and Lemma 2 that

\[
\psi_s^{(\nu)}(x, \lambda) = \frac{\omega_{ks}}{\rho_s^{k+1}}(\rho_s R_{ks})^\nu \exp(\rho_s R_{ks}x_s)[1]_s + \sum_{\mu=k+1}^{n_s} A_{\mu s}(\rho_s)(\rho_s R_{\mu s})^\nu \exp(\rho_s R_{\mu s}x_s)[1]_s, \quad \nu = 0, n_s - 1,
\]

\[
\psi_j^{(\nu)}(x, \lambda) = \sum_{\mu=1}^{n_j} A_{\mu j}(\rho_j)(\rho_j R_{\mu j})^\nu \exp(\rho_j R_{\mu j}x_j)[1]_j, \quad j = 1, m \setminus s, \quad \nu = 0, n_s - 1,
\]

\[
\psi_0^{(\nu)}(x, \lambda) = \sum_{\mu=1}^{n_j} A_{\mu 0}(\rho_0)(\rho_0 R_{\mu 0})^\nu \exp(\rho_0 R_{\mu 0}x_0)[1]_0, \quad \nu = 0, 1.
\]

Substitution of these representations into the matching conditions (5) gives a linear system \( D_{sk} \) with respect to the coefficients \( A_{\mu j}(\rho_j) \). Since each \( A_{\mu j}(\rho_j) \) in this system is multiplied by the corresponding exponent \( \exp(\rho_j R_{\mu j}T_j) \) and \( [1]_j = 1 + o(\lambda), |\lambda| \to \infty \), we obtain the following asymptotics for the determinant of \( D_{sk} \):

\[
d_{sk}(\lambda) = d_{sk}^0 \lambda^{\nu_{sk}} \exp(P_{sk}(\lambda))(1 + o(\lambda)), \quad |\lambda| \to \infty,
\]

where

\[
P_{sk}(\lambda) = \rho_s \left( \sum_{\mu=k+1}^{n_s} R_{\mu s} \right) T_s + \sum_{j=1, m \setminus s} A_{\mu j}(\rho_j) \left( \sum_{\mu=\max(n_j-k,1)+1}^{n_j} R_{\mu j} \right) T_j + \rho_0 R_{20} T_0,
\]

and \( \nu_{sk} \) is a rational power of \( \lambda \). In order to have the main term of the asymptotics (11) distinct from zero, we impose the requirement

\[
d_{sk}^0 \neq 0, \quad s = \frac{1}{m}, \quad k = \frac{1}{n_s} - 1.
\]

The matching conditions (5), satisfying (12), are called regular.

One can easily show that the determinant \( \Delta_{sk\mu}(\lambda) \) asymptotically equals

\[
d_{sk}^0(\rho_s R_{ks})^{\mu} \lambda^{\nu_{sk}} \exp(P_{sk}(\lambda))
\]

up to a nonzero constant (under the current assumptions on \( \lambda \)). Consequently, if the matching conditions are regular, then \( \Delta_{sk\mu}(\lambda) \neq 0 \). Hence the boundary value problems \( L_{sk\mu} \) have only discrete spectra.

Solving the system \( D_{sk} \) by the Cramer’s rule, we obtain, in particular

\[
A_{\mu s}(\rho_s) = O(\rho_s^{1-k} \exp(\rho_s(R_{ks} - R_{\mu s})T_s)), \quad \mu = k + 1, n_s,
\]

and finally arrive at the following assertion.

**Lemma 3.** Fix \( s = \frac{1}{m} \) and a sector \( S_s \) with property (10). For \( x_s \in (0, T_s) \), \( \nu = 0, n_s - 1 \), \( k = \frac{1}{n_s} \), the following asymptotic formula holds

\[
\psi_{sk\lambda}^{(\nu)}(x, \lambda) = \frac{\omega_{ks}}{\rho_s^{k+1}}(\rho_s R_{ks})^\nu \exp(\rho_s R_{ks}x_s)[1]_s, \quad \rho_s \in S_s, \quad |\rho_s| \to \infty.
\]
4. Auxiliary inverse problems

In this section we consider auxiliary inverse problems of recovering the differential operator on each fixed edge. We start from the boundary edges. Fix $s = 1, m$ and consider the following inverse problems of the edge $e_s$.

**IP(s).** Given the Weyl-type matrix $M_s$, construct the potential $q_s := \{q_{n\alpha}\}_{\alpha=0}^{m-2}$ on the edge $e_s$.

In IP(s) we construct the potential on the single edge $e_s$, but the Weyl-type matrix $M_s$ brings global information from the whole graph. In other words, IP(s) is not a local inverse problem related only to the edge $e_s$.

Let us prove the uniqueness theorem for the solution of IP(s). For this purpose together with $q$ we consider a potential $\tilde{q}$. Everywhere below if a symbol $\alpha$ denotes an object related to $q$ then $\tilde{\alpha}$ will denote the analogous object related to $\tilde{q}$.

**Theorem 1.** Fix $s = 1, m$. If $M_s = \tilde{M}_s$, then $q_s = \tilde{q}_s$. Thus, the specification of the Weyl-type matrix $M_s$ uniquely determines the potential $q_s$ on the edge $e_s$.

**Proof.** Denote $\psi_s(x_s, \lambda) := [\psi^{(n-1)}_{sk}(x_s, \lambda)]_{k,\nu=1}^{n_s}$, $C_s(x_s, \lambda) := [C^{(n-1)}_{ks}(x_s, \lambda)]_{k,\nu=1}^{n_s}$. Then by (3) we get

$$\psi_s(x_s, \lambda) = C_s(x_s, \lambda)M_s^T(\lambda), \quad (13)$$

where $T$ is the sign for the trasposition. Define the matrix $P_s(x_s, \lambda) = [P_{sjk}(x_s, \lambda)]_{j,k=1}^{n_s}$ by the formula

$$P_s(x_s, \lambda) = \psi_s(x_s, \lambda)(\tilde{\psi}_s(x_s, \lambda))^{-1}.$$  

Applying Lemma 3, we get

$$P_{s1k}(x_s, \lambda) = O(\rho_s^{-1}), \quad k = 1, n_s, \quad x_s \in (0, T_s), \quad \arg \lambda \neq 0, \pi, \quad |\lambda| \to \infty. \quad (14)$$

Transform the matrix $P_s(x_s, \lambda)$, using (13) and $M_s = \tilde{M}_s$:

$$P_s(x_s, \lambda) = C_s(x_s, \lambda)(\tilde{C}_s(x_s, \lambda))^{-1}.$$  

Taking (3) into account, we conclude that for each fixed $x_s$, the matrix-valued function $P_s(x_s, \lambda)$ is an entire analytic function in $\lambda$ of order $1/n_s$. Together with (13), this yields $P_{s1k}(x_s, \lambda) = 1$, $P_{s1k}(x_s, \lambda) = 0$, $k = 1, n_s$. Consequently, $\psi_{sk}(x_s, \lambda) = \tilde{\psi}_{sk}(x_s, \lambda)$ and $q_s = \tilde{q}_s$.

Using the method of spectral mappings, one can get a constructive procedure for the solution of IP(s). It can be obtained by the same arguments as for $n$-th order differential operators on a finite interval (see [13], Ch. 2) for details).

For the Sturm-Liouville operator on the cycle $e_0$, we consider the following auxiliary inverse problem.

**IP(0).** Given $d(\lambda)$, $h(\lambda)$ and $\Omega$, construct $q_{00}(x_0)$.

This inverse problem was studied in [17, 18] and other papers. In fact, one can easily construct Dirichlet spectral data $\{\nu_n, \alpha_n\}_{n \geq 1}$ by the data $\{d(\lambda), h(\lambda), \Omega\}$, and reduce IP(0) to the classical Sturm-Liouville problem [12, 13, 19, 20]. Thus, IP(0) has a unique solution which can be found by the following algorithm.

**Algorithm 1.** (16) Given $d(\lambda)$, $h(\lambda)$ and $\Omega$.

1. Find the zeros of $h(\lambda)$, $\{\nu_n\}_{n \geq 1}$. 

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2. Calculate \( H(\nu_n) := \omega_n \sqrt{d^2(\nu_n)} - 4 \).

3. Find \( S'_0(T_0, \nu_n) := (d(\nu_n) - H(\nu_n))/2 \).

4. Calculate \( \alpha_n := \hat{h}(\nu_n)S'_0(T_0, \nu_n), \hat{h}(\lambda) := \frac{dh(\lambda)}{d\lambda} \).

5. Construct \( q_0 \) from the given spectral data \( \{\nu_n, \alpha_n\}_{n \geq 1} \) by solving the classical Sturm-Liouville problem.

5. Solution of Inverse Problem 1

Now we are ready to present a constructive procedure for the solution of Inverse Problem 1.

Algorithm 2. Given the spectra \( \Lambda_{sk\mu}, s = 1, m, k = 1, n_s - 1, \mu = k, n_s \), and the signs \( \Omega \).

1. Construct the characteristic functions \( \Delta_{sk\mu}(\lambda) \) by their zeros \( \Lambda_{sk\mu} \).

2. Find the Weyl-type matrices \( M_s(\lambda), s = 1, m \), via (7).

3. For each \( s = 1, m \), solve the inverse problem \( IP(s) \) and find the potential \( q_s \) on the edge \( e_s \).

4. Construct the solutions \( C_{ks}(x_s, \lambda), s = 1, m, k = 1, n_s \).

5. Find \( h(\lambda), d(\lambda) \) from (8), (9).

6. Solve \( IP(0) \) by \( d(\lambda), h(\lambda) \) and \( \Omega \), using Algorithm 1, and and construct the potential on the cycle \( e_0 \).

On step 5, we assume that there exist at least one edge with the order \( n_s > 2 \). The case of all \( n_s = 2 \) was considered in [16]. Then \( h(\lambda) \) can be easily determined from (9), and then \( d(\lambda) \) from (8).

Remark. Note that there are often considered inverse problems by the Weyl functions and their generalizations. But in the present case, the functions \( d(\lambda) - 2 \) and \( h(\lambda) \) can not be uniquely recovered from the Weyl matrices \( M_s \), if these functions have common zeros.

Theorem 1 together with the uniqueness of the solution for \( IP(0) \) yields the following result.

Theorem 2. The spectra \( \Lambda_{sk\mu}, k = 1, n_s - 1, \mu = k, n_s \), and the signs \( \Omega \) determine the potential \( q \) on the graph \( G \) uniquely.

Appendix. Example

In this section, we consider an example, that illustrates how the entries of the Weyl-type matrices can be found from the linear system \( E_{sk} \) (see Section 2) and shows the structure of determinants.

Let \( s = 1, m \) and \( k = 1, n_s - 1 \) be fixed. For brevity, in this section we omit the indices \( s \) and \( k \), when they are fixed. So we write \( \psi_j(x_j, \lambda) \) instead of \( \psi_{skj}(x_j, \lambda) \). We substitute the following expansions

\[
\psi_j(x_j, \lambda) = \sum_{\mu=1}^{n_j} M_j^\mu(\lambda)C_{\mu j}(x_j, \lambda), \quad j = 1, m.
\]
into the matching conditions (3) and obtain the coefficients $M_k^\mu(\lambda)$ from a linear system by the Cramer’s rule: 

$$M_k^\mu(\lambda) = -\frac{\Delta_k(\lambda)}{\Delta_0(\lambda)}.$$ 

Let $m = 2$, $n_1 = 3$, $n_2 = 4$. Fix $s = 1$, $k = 1$. Then the boundary conditions (4) take the form:

$$\psi_1(0, \lambda) = 1, \quad \psi_2(0, \lambda) = \psi'_2(0, \lambda) = \psi''_2(0, \lambda) = 0.$$ 

Consequently,

$$\psi_0(x_0, \lambda) = M_0^1(\lambda)C_0(x_0, \lambda) + M_0^2(\lambda)S_0(x_0, \lambda),$$

$$\psi_1(x_1, \lambda) = C_{11}(x_1, \lambda) + M_1^2(\lambda)C_{21}(x_1, \lambda) + M_1^3(\lambda)C_{31}(x_1, \lambda),$$

$$\psi_2(x, \lambda) = M_2^1(\lambda)C_{42}(x_2, \lambda).$$

Let $U_{j\mu}(\psi_j) = \psi_j^{(\mu)}(T_j, \lambda)$. Then the matching conditions (5) give the following system (we omit the arguments $(T_j, \lambda)$ of $C_{\mu j}$):

$$
\begin{bmatrix}
-1 + C_0 & S_0 & 0 & 0 & 0 \\
-1 & 0 & C_{21} & C_{31} & 0 \\
C'_0 & -1 + S'_0 & C''_{21} & C''_{31} & C''_{42} \\
0 & 0 & C''_{21} & C''_{31} & C''_{42} \\
-1 & 0 & 0 & 0 & C_{24}
\end{bmatrix}
\begin{bmatrix}
M_0^1 \\
M_0^2 \\
M_1^2 \\
M_1^3 \\
M_2^1
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} = 0.
$$

The determinant of this system equals

$$\Delta_0(\lambda) = -(d(\lambda) - 2) \cdot \begin{vmatrix} C_{21} & C_{31} \\ C''_{21} & C''_{31} \end{vmatrix} \cdot C_{42} + h(\lambda) \cdot \begin{vmatrix} C''_{21} & C''_{31} \\ C''_{21} & C''_{31} \end{vmatrix} C_{42} - \begin{vmatrix} C_{21} & C_{31} \\ C''_{21} & C''_{31} \end{vmatrix} C_{42}$$

and $M_k^\mu(\lambda) = -\frac{\Delta_k(\lambda)}{\Delta_0(\lambda)}$, $k = 2, 3$, where $\Delta_k(\lambda)$ can be obtained from $\Delta_0(\lambda)$ by change of $C_{k1}$ to $C_{11}$.

For $s = 1$, $k = 2$ we have

$$\psi_1(0, \lambda) = 0, \quad \psi'_1(0, \lambda) = 1, \quad \psi_2(0, \lambda) = \psi'_2(0, \lambda) = 0,$$

$$\psi_1(x_1, \lambda) = C_{21}(x_1, \lambda) + M_1^2(\lambda)C_{31}(x_1, \lambda),$$

$$\psi_2(x_2, \lambda) = M_2^2(\lambda)C_{32}(x_2, \lambda) + M_2^3(\lambda)C_{42}(x_2, \lambda).$$

The determinant of the system is

$$
\begin{bmatrix}
-1 + C_0 & S_0 & 0 & 0 & 0 \\
-1 & 0 & C_{31} & 0 & 0 \\
0 & 0 & -C''_{31} & C''_{32} & C''_{32} \\
0 & 0 & C''_{31} & C''_{32} & C''_{32} \\
-1 & 0 & 0 & C_{32} & C_{42}
\end{bmatrix}
= h(\lambda) \begin{vmatrix} C_{31} & C_{32} & C_{42} \\ C''_{31} & C''_{32} & C''_{32} \\ 0 & C_{32} & C_{42} \end{vmatrix} - C_{31} \begin{vmatrix} C_{32} & C_{42} \\ C''_{32} & C''_{42} \end{vmatrix}.
$$

Thus, formulas (5), (6) are valid in this case.

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