MATRIX FACTORIZATIONS FOR SELF-ORTHOGONAL CATEGORIES OF MODULES

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Abstract. For a commutative ring \( S \) and self-orthogonal subcategory \( C \) of \( \text{Mod}(S) \), we consider matrix factorizations whose modules belong to \( C \). Let \( f \in S \) be a regular element. If \( f \) is \( M \)-regular for every \( M \in C \), we show there is a natural embedding of the homotopy category of \( C \)-factorizations of \( f \) into a corresponding homotopy category of totally acyclic complexes. Moreover, we prove this is an equivalence if \( C \) is the category of projective or flat-cotorsion \( S \)-modules. Dually, using divisibility in place of regularity, we observe there is a parallel equivalence when \( C \) is the category of injective \( S \)-modules.

Introduction

Matrix factorizations of a nonzero element \( f \) in a regular local ring \( Q \) were introduced by Eisenbud [12] and shown to correspond to maximal Cohen-Macaulay \( Q/(f) \)-modules; in turn Buchweitz [5] gave a relation between these and totally acyclic complexes of finitely generated projective \( Q/(f) \)-modules. Indeed, this correspondence can be described as an equivalence of triangulated categories,

\[
\text{HMF}(Q,f) \cong \text{K}_{\text{tac}}(\text{prj}(Q/(f))),
\]

where \( \text{HMF}(Q,f) \) is the homotopy category of matrix factorizations of \( f \), and \( \text{K}_{\text{tac}}(\text{prj}(Q/(f))) \) is the homotopy category of totally acyclic complexes of finitely generated projective \( Q/(f) \)-modules. In part, our goal is to develop the notion of matrix factorizations more generally—relative to a self-orthogonal category of modules—with an emphasis on extending this equivalence.

Let \( S \) be a commutative ring, let \( f \in S \), and let \( C \) be an additive subcategory of \( \text{Mod}(S) \), the category of \( S \)-modules. A linear factorization of \( f \), defined by Dyckerhoff and Murfet [11], is a pair of \( S \)-modules \( M_0 \) and \( M_1 \) along with homomorphisms \( d_1 : M_1 \to M_0 \) and \( d_0 : M_0 \to M_1 \) satisfying \( d_1 d_0 = f1_{M_0} \) and \( d_0 d_1 = f1_{M_1} \). We define a \( C \)-factorization of \( f \) to be a linear factorization of \( f \) such that \( M_0, M_1 \in C \). The homotopy category of \( C \)-factorizations of \( f \), denoted \( \text{HF}(C,f) \), is the category whose objects are \( C \)-factorizations of \( f \) and whose morphisms are homotopy classes of the natural maps between \( C \)-factorizations; see Section 2. Taking \( C \) to be the category of finitely generated projective modules over a regular local ring, one obtains the usual notion of matrix factorizations in [12].

Set \( R = S/(f) \). To relate a \( C \)-factorization of \( f \) to a suitable type of totally acyclic complex of \( R \)-modules, a natural setting to consider is when \( C \) is self-orthogonal, that is, \( \text{Ext}_C^i(M,M') = 0 \) for every \( M, M' \in C \) and \( i \geq 1 \). If \( C \) is self-orthogonal

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and \( f \in S \) is \( S \)-regular and \( M \)-regular for every \( M \in C \), then the category \( R \otimes_S C \) is self-orthogonal—see Proposition 1.8—in which case there is a natural notion of total acyclicity. Proposition 2.5 thus relates \( C \)-factorizations of \( f \) to \( R \otimes_S C \)-totally acyclic complexes. Here, for a self-orthogonal category \( W \) in \( \text{Mod}(R) \), a \( W \)-totally acyclic complex is an acyclic complex of modules in \( W \) whose acyclicity is preserved by \( \text{Hom}_R(-, W) \) and \( \text{Hom}_R(W, -) \); this includes the usual notions of total acyclicity for complexes of projective or injective modules, and is a special case of that in [7].

In this setting, that is, if \( C \) is an additive self-orthogonal subcategory of \( \text{Mod}(S) \) and \( f \) is \( S \)-regular and \( M \)-regular for every \( M \in C \), then we prove in Theorem 3.5 that there is a full and faithful triangulated functor,

\[
T : \text{HF}(C, f) \longrightarrow K_{\text{tac}}(R \otimes_S C),
\]

where \( K_{\text{tac}}(R \otimes_S C) \) is the homotopy category of \( R \otimes_S C \)-totally acyclic complexes. This embedding extends work of Bergh and Jorgensen; indeed, its proof is closely modelled on that of [3, Theorem 3.5], which is recovered by setting \( C = \text{prj}(S) \).

The functor \( T \) sends a \( C \)-factorization of \( f \) to a 2-periodic complex, see Proposition 2.5, and so we do not expect it to be an equivalence without additional assumptions on \( S \) and \( C \). If \( S \) is a regular local ring and \( C = \text{Prj}(S) \) is the category of projective \( S \)-modules, then we show in Theorem 4.2 that there is a triangulated equivalence:

\[
\text{HF}(\text{Prj}(S), f) \overset{\cong}{\longrightarrow} K_{\text{tac}}(\text{Prj}(R)).
\]

Indeed, restricting to the subcategory of finitely generated projective modules, this is the equivalence due to Eisenbud [12] and Buchweitz [5] described above.

Parallel to this development, we consider a dual situation in terms of divisibility. If \( f \) is \( S \)-regular and \( M \)-divisible for every \( M \in C \), we observe in Theorem 3.6 that there is an embedding \( \text{HF}(C, f) \longrightarrow K_{\text{tac}}(\text{Hom}_S(R, C)) \). In particular, since injective \( S \)-modules are divisible, we obtain an equivalence for \( C = \text{Inj}(S) \), the category of injective \( S \)-modules, when \( S \) is a regular local ring; see Theorem 4.5.

Another natural (torsion-free) self-orthogonal category to consider is \( \text{FlatCot}(S) \), the category of flat-cotorsion \( S \)-modules; see Section 5. We prove in Theorem 5.4 that if \( S \) is a regular local ring, then there is a triangulated equivalence:

\[
\text{HF}(\text{FlatCot}(S), f) \overset{\cong}{\longrightarrow} K_{\text{tac}}(\text{FlatCot}(R)).
\]

Here \( K_{\text{tac}}(\text{FlatCot}(R)) \) is the homotopy category of acyclic complexes of flat-cotorsion \( R \)-modules such that for every flat-cotorsion \( R \)-module \( F \), application of \( \text{Hom}_R(F, -) \) and \( \text{Hom}_R(-, F) \) preserves acyclicity.

In addition to the classic equivalence described above, Buchweitz gave in [5] an equivalence, assuming \( S \) is a regular local ring, between the homotopy category of matrix factorizations of \( f \) and the singularity category of \( R \); this was proven explicitly by Orlov [21]. Along these lines, and as a consequence of the previous equivalence, we observe in Corollary 5.6 a triangulated equivalence,

\[
\text{HF}(\text{FlatCot}(S), f) \overset{\cong}{\longrightarrow} D_{\text{F-tac}}(\text{Flat}(R)),
\]

where \( D_{\text{F-tac}}(\text{Flat}(R)) \) is the subcategory of the pure derived category of flat \( R \)-modules consisting of \( F \)-totally acyclic complexes. This category plays the role of the singularity category in the context of the pure derived category, in that it vanishes if and only if \( R \) is regular; see [18, Proposition 9.7] and [19].
1. Self-orthogonal categories of modules

Throughout this paper, let $S$ be a commutative ring. The category of all $S$-modules is denoted $\text{Mod}(S)$. Tacitly, we assume all subcategories of $\text{Mod}(S)$ are full and closed under isomorphisms. We use standard homological notation throughout, and an $S$-complex means a chain complex of $S$-modules.

Let $\text{Prj}(S)$, $\text{Inj}(S)$, $\text{Flat}(S)$ denote the categories of projective, injective, and flat $S$-modules, respectively; $\text{prj}(S)$ denotes the category of finitely generated projective $S$-modules. Let $\text{Cot}(S)$ denote the category of cotorsion $S$-modules, that is, those $S$-modules $C$ such that $\text{Ext}^1_S(F,C) = 0$ for every flat $S$-module $F$. For brevity, write $\text{FlatCot}(S) = \text{Flat}(S) \cap \text{Cot}(S)$ for the category of flat-cotorsion $S$-modules.

**Definition 1.1.** Let $C$ be a subcategory of $\text{Mod}(S)$. The category $C$ is called self-orthogonal\(^1\) if $\text{Ext}^i_S(C,C') = 0$ for all $C, C' \in C$ and all $i \geq 1$.

**Example 1.2.** Evidently both $\text{Prj}(S)$ and $\text{Inj}(S)$ are self-orthogonal.

The category $\text{FlatCot}(S)$ is also self-orthogonal: Let $F$ and $F'$ be flat-cotorsion $S$-modules. If $P \to F$ is a projective resolution over $S$, then $\text{coker}(d_i^p)$ is a flat $S$-module for $i \geq 1$, hence $\text{Ext}^i_S(F,F') \cong \text{Ext}^1_S(\text{coker}(d_i^p),F') = 0$ for all $i \geq 1$.

**Definition 1.3.** Let $M$ be an $S$-module, $f \in S$, and $C$ be a subcategory of $\text{Mod}(S)$.

The element $f$ is $M$-regular if $fx = 0$ implies $x = 0$ for each $x \in M$; $f$ is $C$-regular if $f$ is $M$-regular for every $M \in C$.

The element $f$ is $M$-divisible if for every $x \in M$, there exists $y \in M$ with $fy = x$; $f$ is $C$-divisible if $f$ is $M$-divisible for every $M \in C$.

**Example 1.4.** Let $f \in S$ be an $S$-regular element.

If $C$ is a subcategory of $\text{Mod}(S)$ contained in the category of torsion-free $S$-modules, then $f$ is $C$-regular. In particular, $f$ is $\text{Flat}(S)$-regular, $\text{FlatCot}(S)$-regular, and $\text{Prj}(S)$-regular.

If $C$ is a subcategory of $\text{Mod}(S)$ contained in the category of divisible $S$-modules, then $f$ is $C$-divisible. In particular, $f$ is $\text{Inj}(S)$-divisible.

Let $S \to R$ be a ring homomorphism and let $C$ be a subcategory of $\text{Mod}(S)$. The following subcategories of $\text{Mod}(R)$ play a special role in this paper:

$$R \otimes_S C = \{W \in \text{Mod}(R) \mid W \cong R \otimes_S C, \text{ for some } C \in C\};$$

$$\text{Hom}_S(R,C) = \{W \in \text{Mod}(R) \mid W \cong \text{Hom}_S(R,C), \text{ for some } C \in C\}.$$

**Remark 1.5.** For any ring homomorphism $S \to R$, we have $R \otimes_S \text{Prj}(S) \subseteq \text{Prj}(R)$ and $\text{Hom}_S(R,\text{Inj}(S)) \subseteq \text{Inj}(R)$, see for example [9, Proposition 2.3]; the former is an equality if the homomorphism is local, the second is an equality if the homomorphism is a surjection. The equality for projective modules uses that projective modules over a local ring are free. We justify the equality for injective modules here: Let $I$ be an injective $R$-module and let $I \to E_S(I)$ be its injective envelope as an $S$-module. Since the natural map $\text{Hom}_S(R,I) \to I$ is an isomorphism, it follows that the induced injection $\text{Hom}_S(R,\text{Inj}(S)) \to \text{Hom}_S(R,E_S(I))$ of $R$-modules is essential and splits, thus is an isomorphism. It follows that $I \cong \text{Hom}_S(R,E_S(I))$.

For an $S$-module $M$, denote by $\text{pd}_S M$, $\text{id}_S M$, and $\text{fd}_S M$ the projective, injective, and flat dimensions of $M$ over $S$.

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\(^1\)This differs from [7], where the term was used to refer to $\text{Ext}^1$-orthogonality and is implied by the definition given here; our usage here agrees with what would be written as $C \perp \perp C$ in [23].
Remark 1.6. Let \( f \in S \) be an \( S \)-regular element, and set \( R = S/(f) \). If \( P \) is a projective \( R \)-module, then \( \text{pd}_S P = 1 \) (see [15, Part III, Theorem 3]); if \( I \) is an injective \( R \)-module, then \( \text{id}_S I = 1 \) (see [14, Theorem 202]). It thus follows that if \( F \) is a flat \( R \)-module, then \( \text{fd}_S F = 1 \); this uses the fact that an \( S \)-module \( M \) is flat if and only if its character dual \( \text{Hom}_E(M, \mathbb{Q}/\mathbb{Z}) \) is injective.

Part (i) of the next change of rings result is due to Rees [22]; part (iii) is dual. If \( M \) is an \( S \)-module, \( f \in S \), and \( R = S/(f) \), it is often convenient to identify \( R \otimes_S M \cong M/fM \) and \( \text{Hom}_S(R, M) \cong (0 :_M f) = \{ x \in M \mid fx = 0 \} \subseteq M \).

Lemma 1.7. Let \( f \in S \) be an \( S \)-regular element and set \( R = S/(f) \).

If \( M \) is an \( S \)-module such that \( f \) is \( M \)-regular and \( N \) is an \( R \)-module, then

(i) \( \text{Ext}^{i+1}_S(N, M) \cong \text{Ext}^i_R(N, R \otimes_S M) \) for all \( i \geq 0 \);
(ii) \( \text{Ext}^i_S(M, N) \cong \text{Ext}^i_R(R \otimes_S M, N) \) for all \( i \geq 0 \).

If \( M \) is an \( S \)-module such that \( f \) is \( M \)-divisible and \( N \) is an \( R \)-module, then

(iii) \( \text{Ext}^{i+1}_S(N, M) \cong \text{Ext}^i_R(\text{Hom}_S(R, M), N) \) for all \( i \geq 0 \);
(iv) \( \text{Ext}^i_S(N, M) \cong \text{Ext}^i_R(\text{Hom}_S(R, M)) \) for all \( i \geq 0 \).

Proof. (i) & (ii): See Matsumura [17, Lemma 2, p. 140] for a proof of these; (i) was originally shown by Rees [22, Theorem 2.1].

(iii): We give an argument dual to [22, Theorem 2.1], showing that the functor \( E^i(-) = \text{Ext}^{i+1}_S(-, -) \) is the \( i \)th right derived functor of \( \text{Hom}_R(\text{Hom}_S(R, M), -) \). Apply \( \text{Hom}_S(-, N) \) to the short exact sequence

\[
0 \longrightarrow \text{Hom}_S(R, M) \longrightarrow M \rightarrow M' \rightarrow 0
\]

to obtain the following exact sequence

\[
\text{Hom}_S(M, N) \rightarrow \text{Hom}_S(\text{Hom}_S(R, M), N) \rightarrow \text{Ext}^1_S(M, N) \rightarrow \text{Ext}^1_S(M, N) \rightarrow 0
\]

Since \( fN = 0 \), we obtain \( \text{Hom}_S(M, N) = 0 \). Additionally, multiplication by \( f \) on \( M \) or \( N \) induce the same map on \( \text{Ext}^1_S(M, N) \); also multiplication by \( f \). As \( fN = 0 \), this map must be 0, thus yielding

\[
\text{Ext}^1_S(M, N) \cong \text{Hom}_S(\text{Hom}_S(R, M), N) \cong \text{Hom}_R(\text{Hom}_S(R, M), N).
\]

Hence \( E^0(-) \cong \text{Hom}_R(\text{Hom}_S(R, M), -) \). For any injective \( R \)-module \( I \), we have \( \text{id}_S I = 1 \) by Remark 1.6, hence \( E^i(I) = 0 \) for \( i \geq 1 \). Finally, for a short exact sequence \( 0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0 \) of \( R \)-modules, \( \text{Hom}_S(M, N'') = 0 \) and so there is a long exact sequence

\[
0 \rightarrow E^0(N') \rightarrow E^0(N) \rightarrow E^0(N'') \rightarrow E^1(N') \rightarrow E^1(N) \rightarrow E^1(N'') \rightarrow \cdots,
\]

and it follows that \( E^i(-) \) is the \( i \)th right derived functor of \( \text{Hom}_R(\text{Hom}_S(R, M), -) \) and thus is isomorphic to \( \text{Ext}^i_R(\text{Hom}_S(R, M), -) \).

(iv): Let \( P \) be a projective resolution of \( N \) over \( R \); standard tensor–Hom adjunction yields \( \text{Hom}_S(R \otimes_R P, M) \cong \text{Hom}_R(P, \text{Hom}_S(R, M)) \), and the desired isomorphism follows. \( \square \)

Proposition 1.8. Let \( C \) be a self-orthogonal subcategory of \( \text{Mod}(S) \), let \( f \in S \) be \( S \)-regular, and set \( R = S/(f) \). The following hold:

(i) If \( f \) is \( C \)-regular, then \( R \otimes_S C \) is self-orthogonal in \( \text{Mod}(R) \).
(ii) If \( f \) is \( C \)-divisible, then \( \text{Hom}_S(R, C) \) is self-orthogonal in \( \text{Mod}(R) \).
Proof. (i): For $S$-modules $C$, $C' \in \mathcal{C}$ and $i \geq 0$, Lemma 1.7(ii) yields that $\text{Ext}^i_S(R \otimes S C, R \otimes S C') \cong \text{Ext}^i_S(C, R \otimes S C')$. It will therefore be enough to show that $\text{Ext}^i_S(C, R \otimes S C') = 0$ for $i \geq 1$. As $f$ is $C$-regular, there is an exact sequence

$$0 \longrightarrow C' \overset{f}{\longrightarrow} C \longrightarrow R \otimes_S C' \longrightarrow 0.$$  

Application of the functor $\text{Hom}_S(C, -)$ yields a long exact sequence:

$$\cdots \longrightarrow \text{Ext}^i_S(C, C') \longrightarrow \text{Ext}^i_S(C, R \otimes_S C') \longrightarrow \text{Ext}^{i+1}_S(C, C') \longrightarrow \cdots$$

By assumption, $\text{Ext}^i_S(C, C') = 0 = \text{Ext}^{i+1}_S(C, C')$ for $i \geq 1$, and it follows that $\text{Ext}^i_S(C, R \otimes_S C') = 0$ for $i \geq 1$.

(ii): This is proved dually to part (i), using instead Lemma 1.7(iv) and the existence of an exact sequence

$$0 \longrightarrow \text{Hom}_S(R, C) \longrightarrow C \overset{f}{\longrightarrow} C \longrightarrow 0$$

for each $C \in \mathcal{C}$. 

\[ \square \]

2. $C$-factorizations and total acyclicity

Let $f \in S$. Extending the classic notion of matrix factorizations [12], Dyckerhoff and Murfet define [11] a linear factorization of $f$ to be a $\mathbb{Z}/2\mathbb{Z}$-graded $S$-module $M = M_0 \oplus M_1$ together with an $S$-linear differential $d : M \rightarrow M$ that is homogeneous of degree 1 and satisfies $d^2 = f 1^M$. We often write such a linear factorization as

$$(M, d) = (M_1 \xrightarrow{d_1} d_0 M_0),$$

where $d_1 d_0 = f 1^{M_0}$ and $d_0 d_1 = f 1^{M_1}$.

A morphism $\alpha : (M, d) \rightarrow (M', d')$ of linear factorizations of $f$ is a degree 0 map which commutes with the differentials on $M$ and $M'$; it consists of maps $\alpha_i : M_i \rightarrow M'_i$, for $i = 0, 1$, making the following diagram commute:

\[
\begin{array}{ccc}
M_1 & \overset{d_1}{\longrightarrow} & M_0 & \overset{d_0}{\longrightarrow} & M_1 \\
\downarrow{\alpha_1} & & \downarrow{\alpha_0} & & \downarrow{\alpha_1} \\
M'_1 & \overset{d'_1}{\longrightarrow} & M'_0 & \overset{d'_0}{\longrightarrow} & M'_1
\end{array}
\]

Definition 2.1. Let $\mathcal{C}$ be a subcategory of $\text{Mod}(S)$. A $\mathcal{C}$-factorization of $f$ is a linear factorization $(M, d)$ of $f$ such that $M_0, M_1 \in \mathcal{C}$.

Denote by $\mathcal{F}(C, f)$ the category whose objects are $\mathcal{C}$-factorizations of $f$ and whose morphisms are those described above.

In particular, if $\text{prj}(S)$ is the category of finitely generated projective $S$-modules, then a $\text{prj}(S)$-factorization of $f$ is the same as the usual notion of a matrix factorization of $f$, that is, $\mathcal{F}(\text{prj}(S), f) = \mathcal{M}_{\text{F}}(S, f)$.

We say two morphisms $\alpha, \beta : (M, d) \rightarrow (M', d')$ of linear factorizations are homotopic, and write $\alpha \sim \beta$, if there exists homomorphisms $h_0 : M_0 \rightarrow M'_0$ and $h_1 : M_1 \rightarrow M'_1$ satisfying the usual homotopy conditions:

$$\alpha_0 - \beta_0 = h_1 d_0 + d'_1 h_0 \quad \text{and} \quad \alpha_1 - \beta_1 = h_0 d_1 + d'_0 h_1.$$
From this, we define the associated homotopy category of $C$-factorizations of $f$, denoted $\text{HF}(C, f)$, to be the homotopy category whose objects are the same as $F(C, f)$ and whose morphisms are homotopy classes of morphisms of $C$-factorizations.

**Lemma 2.2.** Let $(M, d) \in F(C, f)$. If $f$ is $M$-regular, then $d_1$ and $d_0$ are injective. If $f$ is $M$-divisible, then $d_1$ and $d_0$ are surjective.

*Proof.* First assume $f$ is $M$-regular. The equality $d_0d_1 = f1_{M1}$ implies that for $x \in M_1$ with $d_1(x) = 0$, we have $0 = d_0d_1(x) = fx$. Since $f$ is $M$-regular, it follows that $x = 0$, hence $d_1$ is injective. Injectivity of $d_0$ is proved similarly.

Next assume $f$ is $M$-divisible. Let $x \in M_0$ be any element. Divisibility implies there exists $y \in M_0$ with $fy = x$. Since $d_1d_0 = f1_{M0}$, we have $d_1d_0(y) = fy = x$, hence $d_1$ is surjective. Surjectivity of $d_0$ is proved similarly. □

Given a category $C$ of $S$-modules, the notions of (left and right) $C$-totally acyclic complexes and (left and right) $C$-Gorenstein modules were defined in [7, Definition 1.1]; in the case where $C$ is self-orthogonal, these notions simplify to the following equivalent characterizations by [7, Propositions 1.3 and 1.5]. For an $S$-complex $T$, we set $Z_i(T) = \ker(d_i^T)$ for each $i \in \mathbb{Z}$.

**Definition 2.3.** Let $C$ be a self-orthogonal category of $S$-modules.

1. An $S$-complex $T$ is $C$-totally acyclic if $T$ is acyclic, $T_i \in C$ for $i \in \mathbb{Z}$, and for every $C \in C$, the complexes $\text{Hom}_S(T, C)$ and $\text{Hom}_S(C, T)$ are also acyclic.

2. An $S$-module $M$ is $C$-Gorenstein if $M = Z_0(T)$ for some $C$-totally acyclic complex $T$.

The homotopy category of $C$-totally acyclic complexes is denoted $K_{\text{tac}}(C)$. If $C$ is additive, then $K_{\text{tac}}(C)$ is triangulated.

A $\text{Prj}(S)$-Gorenstein module is called a Gorenstein projective module and an $\text{Inj}(S)$-Gorenstein module is called a Gorenstein injective module; these are the standard notions appearing in the literature.

The next lemma is used below to relate cokernel modules of $C$-factorizations to totally acyclic complexes.

**Lemma 2.4.** Let $C$ be a self-orthogonal subcategory of $\text{Mod}(S)$, let $f \in S$ be $S$-regular and $C$-regular, and set $R = S/(f)$. If $(M, d) \in F(C, f)$, then $\text{coker}(d_1)$ and $\text{coker}(d_0)$ are $R$-modules, and for any $C \in C$ and $i \geq 1$ the following hold:

1. $\text{Ext}_R^i(R \otimes_S C, \text{coker}(d_1)) = 0 = \text{Ext}_R^i(R \otimes_S C, \text{coker}(d_0))$,

2. $\text{Ext}_R^i(\text{coker}(d_1), R \otimes_S C) = 0 = \text{Ext}_R^i(\text{coker}(d_0), R \otimes_S C)$.

*Proof.* We prove the statements for $\text{coker}(d_1)$; proofs for $\text{coker}(d_0)$ are similar.

Note first that $\text{coker}(d_1)$ is an $R$-module, since $f \text{coker}(d_1) = 0$; indeed, we have $fM_0 \subseteq \text{im}(d_1)$ as $f1_{M0} = d_0$, and so $f1_{M0}$ induces the zero map on $\text{coker}(d_1)$.

As $f$ is $C$-regular, Lemma 2.2 yields an exact sequence

$$\begin{array}{ccc}
0 & \longrightarrow & M_1 \longrightarrow M_0 \longrightarrow \text{coker}(d_1) \longrightarrow 0.
\end{array}$$

(1)

Let $C \in C$. Application of $\text{Hom}_S(C, -)$ to the exact sequence (1) yields a long exact sequence:

$$\cdots \longrightarrow \text{Ext}_S^i(C, M_0) \longrightarrow \text{Ext}_S^i(C, \text{coker}(d_1)) \longrightarrow \text{Ext}_S^{i+1}(C, M_1) \longrightarrow \cdots$$
As $M_0$ and $M_1$ are in $C$, we obtain that $\text{Ext}_S^i(C, M_0) = 0 = \text{Ext}_S^{i+1}(C, M_1)$ for $i \geq 1$, and hence $\text{Ext}_S^i(C, \text{coker}(d_1)) = 0$ for $i \geq 1$. Since $\text{coker}(d_1)$ is an $R$-module, Lemma 1.7(ii) now yields $\text{Ext}_S^i(R \otimes_S C, \text{coker}(d_1)) \cong \text{Ext}_S^i(C, \text{coker}(d_1)) = 0$ for $i \geq 1$. This gives (i).

For (ii), instead apply $\text{Hom}_S(-, C)$ to the exact sequence (1) to obtain a long exact sequence for $i \geq 1$:

$$\cdots \to \text{Ext}_S^i(M_1, C) \to \text{Ext}_S^{i+1}(\text{coker}(d_1), C) \to \text{Ext}_S^{i+1}(M_0, C) \to \cdots$$

As $M_0$ and $M_1$ are in $C$, we obtain that $\text{Ext}_S^i(M_1, C) = 0 = \text{Ext}_S^{i+1}(M_0, C)$ for $i \geq 1$. It follows that $\text{Ext}_S^{i+1}(\text{coker}(d_1), C) = 0$ for $i \geq 1$. Employing Lemma 1.7(i), we obtain $\text{Ext}_S^i(\text{coker}(d_1), R \otimes_S C) \cong \text{Ext}_S^{i+1}(\text{coker}(d_1), C) = 0$ for all $i \geq 1$. □

If $M$ is an $S$-module, $\alpha$ is an $S$-homomorphism, and $R = S/(f)$, then we set $\overline{M} = R \otimes_S M$ and $\overline{\alpha} = R \otimes_S \alpha$; context should make this clear.

**Proposition 2.5.** Let $C$ be a subcategory of $\text{Mod}(S)$, let $f \in S$ be $S$-regular and $C$-regular, and set $R = S/(f)$. Let $(M, d) \in F(C, f)$. The $R$-sequence

$$T^M := \cdots \to \overline{d_0} \to \overline{M}_0 \xrightarrow{\overline{d}_1} \overline{M}_1 \to \overline{d}_0 \to \overline{M}_0 \to \cdots$$

is acyclic. If $C$ is self-orthogonal, then $T^M$ is $R \otimes_S C$-totally acyclic.

**Proof.** First, as $d_1d_0 = f1M_0$ and $d_0d_1 = f1M_1$, we have $\overline{d_1} \overline{d}_0 = 0 = \overline{d}_0 \overline{d}_1$ and so the sequence $T^M$ is a complex of $R$-modules.

We now show $T^M$ is acyclic. Let $x \in M_1$ such that $\overline{x} \in \ker(\overline{d}_1)$. It follows that $d_1(x) \in fM_0$, whence there exists $y \in M_0$ such that $d_1(x) = fy$. As $fy = d_1d_0(y)$, it follows that $d_1(x) = d_1d_0(y)$, hence $d_1(x - d_0(y)) = 0$. Injectivity of $d_1$, see Lemma 2.2, implies that $x = d_0(y)$. Hence $\overline{d}_0(y) = \overline{x}$, and so $H_{2i+1}(T^M) = 0$ for every $i \in \mathbb{Z}$. A similar argument (using injectivity of $d_0$) yields $H_{2i}(T^M) = 0$ for every $i \in \mathbb{Z}$, thus proving the complex $T^M$ is acyclic.

Multiplication by $f$ on the exact sequence $0 \to M_1 \xrightarrow{d_1} M_0 \to \text{coker}(d_1) \to 0$, along with the snake lemma, yields an exact sequence

$$\text{coker}(d_1) \xrightarrow{f} \text{coker}(d_1) \to \text{coker}(\overline{d}_1) \to 0.$$ 

Since $\text{coker}(d_1)$ is an $R$-module (see Lemma 2.4), this implies $\text{coker}(\overline{d}_1) \cong \text{coker}(d_1)$; similarly, $\text{coker}(\overline{d}_0) \cong \text{coker}(d_0)$. Acyclicity of $T^M$ gives $Z_{2i}(T^M) \cong \text{coker}(d_0)$ and $Z_{2i+1}(T^M) \cong \text{coker}(d_1)$ for every $i \in \mathbb{Z}$.

Fix $C \in C$. To verify the complexes $\text{Hom}_R(T^M, R \otimes_S C)$ and $\text{Hom}_R(R \otimes_S C, T^M)$ are acyclic, it suffices to show that the exact sequences

$$0 \to \text{coker}(d_0) \to \overline{M}_0 \to \text{coker}(d_1) \to 0$$

and

$$0 \to \text{coker}(d_1) \to \overline{M}_1 \to \text{coker}(d_0) \to 0$$

remain exact upon application of $\text{Hom}_R(R \otimes_S C, -)$ and $\text{Hom}_R(-, R \otimes_S C)$. This follows from Lemma 2.4. Therefore, as $R \otimes_S C$ is self-orthogonal by Proposition 1.8, we obtain that $T^M$ is $R \otimes_S C$-totally acyclic. □

We have the next dual results involving divisibility:
Lemma 2.6. Let \( C \) be a self-orthogonal subcategory of \( \text{Mod}(S) \), let \( f \in S \) be \( S \)-regular and \( C \)-divisible, and set \( R = S/(f) \). If \((M,d) \in F(C,f)\), then \( \ker(d_1) \) and \( \ker(d_0) \) are \( R \)-modules, and for any \( C \subseteq C \) and \( i \in \mathbb{N} \) the following hold:

(i) \( \text{Ext}^i_R(\text{Hom}_S(R,C),\ker(d_1)) = 0 = \text{Ext}^i_R(\text{Hom}_S(R,C),\ker(d_0)) \),

(ii) \( \text{Ext}_R^i(\ker(d_1),\text{Hom}_S(R,C)) = 0 = \text{Ext}_R^i(\ker(d_0),\text{Hom}_S(R,C)) \).

Proof. Dual to the proof of Lemma 2.4; use instead Lemma 1.7(iii,iv).

Proposition 2.7. Let \( C \) be a subcategory of \( \text{Mod}(S) \), let \( f \in S \) be \( S \)-regular and \( C \)-divisible, and set \( R = S/(f) \). Let \((M,d) \in F(C,f)\). The \( R \)-sequence

\[
\tilde{T}^M := \cdots \xrightarrow{(d_0)_*} \text{Hom}_S(R,M_1) \xrightarrow{(d_1)_*} \text{Hom}_S(R,M_0) \xrightarrow{(d_0)_*} \cdots
\]

is acyclic. If \( C \) is self-orthogonal, then \( \tilde{T}^M \) is \( \text{Hom}_S(R,C) \)-totally acyclic.

Proof. Dual to the proof of Proposition 2.5; use instead Lemma 2.6.

3. A full and faithful functor

Let \( C \) be a self-orthogonal subcategory of \( \text{Mod}(S) \). We denote by \( K(C) \) the homotopy category of \( C \), whose objects are complexes of modules in \( C \) and morphisms are homotopy classes of degree zero chain maps. Further, we consider the full subcategory \( K_{\text{tac}}(C) \) whose objects are the \( C \)-totally acyclic complexes in \( K(C) \).

Proposition 3.1. Let \( C \) be an additive self-orthogonal subcategory of \( \text{Mod}(S) \), let \( f \in S \) be \( S \)-regular and \( C \)-regular, and set \( R = S/(f) \). There is a triangulated functor

\[
\begin{array}{ccc}
\text{T} : \text{HF}(C,f) & \longrightarrow & K_{\text{tac}}(R \otimes_S C)
\end{array}
\]

defined, in notation from Proposition 2.5, as \( T(M,d) = T^M \) and \( T([\alpha]) = [\overline{\alpha}] \).

Proof. Let \( [\alpha], [\beta] : (M,d) \rightarrow (M',d') \) be morphisms in \( \text{HF}(C,f) \). Set \( T^M \) and \( T'^M \) as the complexes constructed in Proposition 2.5 and associated to \( M \) and \( M' \), respectively. Define \( \overline{\alpha}, \overline{\beta} : T^M \rightarrow T'^M \) as the evident 2-periodic chain maps induced by \( \alpha \) and \( \beta \). If \( [\alpha] = [\beta] \), then there is a homotopy \( h \) from \( \alpha \) to \( \beta \); this induces a 2-periodic homotopy \( T_h \) from \( \overline{\alpha} \) to \( \overline{\beta} \), implying that \( [\overline{\alpha}] = [\overline{\beta}] \) in \( K_{\text{tac}}(R \otimes_S C) \). Notice that as \( 1^M = 1^{'M} \) if \( [\alpha] = [1^M] \), then \( [\overline{\alpha}] = [1^{'M}] \).

Define a functor \( \text{T} : \text{HF}(C,f) \rightarrow K_{\text{tac}}(R \otimes_S C) \) as follows: For an object \((M,d)\), set \( T(M,d) = T^M \), and for a morphism \([\alpha] : (M,d) \rightarrow (M',d')\), set \( T([\alpha]) = [\overline{\alpha}] \).

The above remarks justify that \( \text{T} \) is well-defined on both objects and morphisms, that \( \text{T} \) preserves identities, and that \( \text{T} \) preserves compositions by the following equalities:

\[
\text{T}([\alpha]) \text{T}([\beta]) = [\overline{\alpha}] [\overline{\beta}] = [(\overline{\alpha}) (\overline{\beta})] = [\overline{\alpha \beta}] = \text{T}([\alpha \beta]).
\]

Moreover, the functor \( \text{T} \) respects the triangulated structures, that is, \( \text{T} \) is additive, \( \text{T}((M,d)[1]) = T^{M[1]} = T^M[1] = \text{T}((M,d))[1] \), and \( \text{T} \) preserves exact triangles.

Lemma 3.2. Let \( C \) be a self-orthogonal subcategory of \( \text{Mod}(S) \), let \( f \in S \) be \( C \)-regular, and set \( R = S/(f) \). If \( M, M' \in C \) and \( \varphi \in \text{Hom}_R(M,M') \), then there exists \( \psi \in \text{Hom}_S(M,M') \) such that \( \psi = \varphi \).
Proof. Let \( \varphi : \overline{M} \to \overline{M}' \) be an \( R \)-homomorphism. There is an exact sequence
\[
0 \longrightarrow M' \xrightarrow{f} M' \xrightarrow{\pi'} \overline{M}' \longrightarrow 0.
\]
As \( \text{Ext}^1_S(M, M') = 0 \), we obtain an exact sequence
\[
0 \longrightarrow \text{Hom}_S(M, M') \longrightarrow \text{Hom}_S(M, M') \longrightarrow \text{Hom}_S(M, \overline{M}') \longrightarrow 0.
\]
Let \( \pi : M \to \overline{M} \) be the canonical quotient map. The map \( \varphi \pi \in \text{Hom}_S(M, \overline{M}') \) lifts to a map \( \psi \in \text{Hom}_S(M, M') \) such that \( \pi' \psi = \varphi \pi \), that is, \( \psi = \varphi \).

The following arguments to show that \( T \) is full and faithful closely follow those given in [3], put into the more general setting of totally acyclic complexes from [7].

**Proposition 3.3.** The functor \( T \) in Proposition 3.1 is faithful.

**Proof.** Set \( W = R \otimes_S C \). Let \( [\alpha] : M \to M' \) be a morphism in \( \text{HF}(C, f) \) such that \( T([\alpha]) \) is the zero morphism in \( \text{K}_{\text{ac}}(W) \). Our goal is to show \( [\alpha] = [0] \), that is, \( \alpha \) is null homotopic in \( F(C, f) \). Write \( \alpha : M \to M' \) as:
\[
\begin{array}{cccc}
M_1 & \xrightarrow{d_1} & M_0 & \xrightarrow{d_0} & M_1 \\
\downarrow{\alpha_1} & & \downarrow{\alpha_0} & & \downarrow{\alpha_1} \\
M'_1 & \xrightarrow{d'_1} & M'_0 & \xrightarrow{d'_0} & M'_1
\end{array}
\]

Let \( \overline{\alpha} : T(M, d) \to T(M', d') \) denote the 2-periodic chain map induced by \( \alpha \). The assumption \( T([\alpha]) = [0] \) in \( \text{K}_{\text{ac}}(W) \) implies that \( \overline{\alpha} \) is null homotopic (i.e., \( \overline{\alpha} \sim 0 \)). Let \( \sigma \) be a null homotopy for \( \overline{\alpha} \); notice, however, that \( \sigma \) need not be 2-periodic.

We have the following diagram:
\[
\begin{array}{cccc}
\cdots & \xrightarrow{\overline{\alpha_1}} & M_1 & \xrightarrow{\overline{d_1}} & M_0 & \xrightarrow{\overline{d_0}} & M_1 & \xrightarrow{\overline{\alpha_1}} & \cdots \\
\downarrow{\overline{\sigma_1}} & & \downarrow{\overline{\sigma_0}} & & \downarrow{\overline{\sigma_1}} & & \downarrow{\overline{\sigma_0}} & & \downarrow{\overline{\sigma_1}} & & \cdots \\
\cdots & \xrightarrow{\overline{\sigma_1}} & M'_1 & \xrightarrow{\overline{d'_1}} & M'_0 & \xrightarrow{\overline{d'_0}} & M'_1 & \xrightarrow{\overline{\sigma_1}} & \cdots
\end{array}
\]

In particular, we have the following relations (coming from degrees 1 and 2):
\[
\begin{align*}
(2) \quad & \overline{\alpha_1} = \overline{d'_0} \sigma_1 + \sigma_0 \overline{d_1}, \\
(3) \quad & \overline{\alpha_0} = \overline{d'_1} \sigma_2 + \sigma_1 \overline{d_0}.
\end{align*}
\]

Lemma 3.2 yields \( S \)-module homomorphism liftings \( h_{2i} : M_0 \to M'_1 \) of \( \sigma_{2i} \) and \( h_{2i+1} : M_1 \to M'_0 \) of \( \sigma_{2i+1} \) for \( i \in \mathbb{Z} \). The exact sequence \( 0 \to M'_1 \xrightarrow{f} M'_1 \xrightarrow{\pi} \overline{M}'_1 \to 0 \) induces an exact sequence:
\[
0 \longrightarrow \text{Hom}_S(M_1, M'_1) \xrightarrow{f} \text{Hom}_S(M_1, M'_1) \xrightarrow{\pi_*} \text{Hom}_S(M_1, \overline{M}'_1) \longrightarrow 0,
\]
where \( \pi_* = \text{Hom}_S(M_1, \pi) \). Since \( \alpha_1 - \overline{d'_0} h_1 - h_0 d_1 \in \ker(\pi_*) \) by (2), one obtains a map \( \beta_1 \in \text{Hom}_S(M_1, M'_1) \) such that \( f \beta_1 = \alpha_1 - \overline{d'_0} h_1 - h_0 d_1 \). Similarly, using instead (3), one obtains \( \beta_2 \in \text{Hom}_S(M_0, M'_0) \) such that \( f \beta_2 = \alpha_0 - \overline{d'_1} h_2 - h_1 d_0 \).
Define $s_1 = h_1 + d'_1 \beta_1$. We claim that $(h_0, s_1)$ is a null homotopy of $\alpha : M \to M'$.

First, we have:
\[
d_0 s_1 + h_0 d_1 = d_0'(h_1 + d'_1 \beta_1) + h_0 d_1
\]
\[
= d_0' h_1 + d_0' d'_1 \beta_1 + h_0 d_1
\]
\[
= d_0' h_1 + f \beta_1 + h_0 d_1
\]
\[
= d_0' h_1 + \alpha_1 - d_0' h_1 - h_0 d_1 + h_0 d_1
\]
\[
= \alpha_1.
\]

Next, precomposing the equality $f \beta_1 = \alpha_1 - d_0' h_1 - h_0 d_1$ with $d_0$ gives:
\[
f \beta_1 d_0 = (\alpha_1 - d_0' h_1 - h_0 d_1) d_0
\]
\[
= \alpha_1 d_0 - d_0' h_1 d_0 - h_0 f
\]
\[
= d_0' \alpha_0 - d_0' h_1 d_0 - h_0 f
\]
\[
= d_0'(\alpha_0 - h_1 d_0) - h_0 f
\]
\[
= d_0'(f \beta_2 + d'_1 h_2) - h_0 f
\]
\[
= f d_0' \beta_2 + d_0' d'_1 h_2 - f h_0
\]
\[
= f (d_0' \beta_2 + h_2 - h_0).
\]

As $f$ is $M'_1$-regular, this yields
\[
(4) \quad \beta_1 d_0 = d_0' \beta_2 + h_2 - h_0.
\]

We therefore obtain:
\[
d'_1 h_0 + s_1 d_0 = d'_1 h_0 + (h_1 + d'_1 \beta_1) d_0
\]
\[
= d'_1 h_0 + h_1 d_0 + d'_1 \beta_1 d_0
\]
\[
= d'_1 h_0 + h_1 d_0 + d'_1 (d_0' \beta_2 + h_2 - h_0), \text{ by (4)},
\]
\[
= d'_1 h_0 + h_1 d_0 + f \beta_2 + d'_1 h_2 - d'_1 h_0
\]
\[
= h_1 d_0 + \alpha_0 - d'_1 h_2 - h_1 d_0 + d'_1 h_2
\]
\[
= \alpha_0.
\]

Hence $\alpha : M \to M'$ is homotopic to 0, that is, $[\alpha] = [0]$ in $HF(C, f)$. \hfill \Box

**Proposition 3.4.** The functor $T$ in Proposition 3.1 is full.

**Proof.** Set $W = R \otimes_S C$. Let $(M, d)$ and $(M', d')$ be objects in $HF(C, f)$ and suppose $\overline{\pi} : T(M, d) \to T(M', d')$ is a degree 0 chain map, not necessarily 2-periodic, that represents a morphism $[\overline{\pi}]$ in $K_{\text{tac}}(W)$; in particular, we have a commutative diagram:

\[
\begin{array}{cccccccc}
\cdots & \longrightarrow & \overline{M}_1 & \xrightarrow{\overline{d}_1} & \overline{M}_0 & \xrightarrow{\overline{d}_0} & \overline{M}_1 & \xrightarrow{\overline{d}_1} & \overline{M}_0 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \overline{M}'_1 & \xrightarrow{\overline{d}'_1} & \overline{M}'_0 & \xrightarrow{\overline{d}'_0} & \overline{M}'_1 & \xrightarrow{\overline{d}'_1} & \overline{M}'_0 & \longrightarrow & \cdots 
\end{array}
\]

By Lemma 3.2, for $i \in \mathbb{Z}$ we can lift $\overline{\pi}_{2i}$ to $\alpha_{2i} : M_0 \to M'_0$ and $\overline{\pi}_{2i+1}$ to $\alpha_{2i+1} : M_1 \to M'_1$. In particular, we obtain the following diagram that commutes...
modulo $f$:

\[
\begin{array}{cccc}
M_0 & \overset{d_0}{\rightarrow} & M_1 & \overset{d_1}{\rightarrow} M_0 \\
\downarrow{\alpha_2} & & \downarrow{\alpha_1} & \downarrow{\alpha_0} \\
M_0' & \overset{d_0'}{\rightarrow} & M_1' & \overset{d_1'}{\rightarrow} M_0'
\end{array}
\]

The exact sequence $0 \rightarrow M_1' \xrightarrow{\alpha_1 d_0 - d_0'\alpha_2} \rightarrow M_1' \xrightarrow{\pi} M_1' \rightarrow 0$ induces an exact sequence

\[
0 \rightarrow \text{Hom}_S(M_0, M_1') \xrightarrow{f} \text{Hom}_S(M_0, M_1') \xrightarrow{\pi^*} \text{Hom}_S(M_0, M_1') \rightarrow 0.
\]

Since $\alpha_1 d_0 - d_0'\alpha_2 \in \ker(\pi)$, there exists a map $\sigma_0 \in \text{Hom}_S(M_0, M_1')$ such that

\[
(5) \quad \alpha_1 d_0 - d_0'\alpha_2 = f\sigma_0.
\]

Similarly, there exists $\sigma_1 \in \text{Hom}_S(M_1, M_0')$ such that

\[
(6) \quad \alpha_0 d_1 - d_1'\alpha_1 = f\sigma_1.
\]

We now define new maps in order to construct a morphism in $F(C, f)$; define

\[
\begin{align*}
\gamma_0 &= \alpha_0 + d_0'\sigma_0, \\
\gamma_1 &= \alpha_1 + d_1'\sigma_1 + \sigma_0 d_1.
\end{align*}
\]

We aim to verify that the following diagram is commutative:

\[
\begin{array}{cccc}
M_0 & \overset{d_0}{\rightarrow} & M_1 & \overset{d_1}{\rightarrow} M_0 \\
\downarrow{\gamma_0} & & \downarrow{\gamma_1} & \downarrow{\gamma_0} \\
M_0' & \overset{d_0'}{\rightarrow} & M_1' & \overset{d_1'}{\rightarrow} M_0'
\end{array}
\]

The equality (6), along with $d_1 d_0 = f1^{M_0}$ and $d_0' d_1' = f1^{M_1'}$, imply

\[
fd_0'\sigma_1 d_0 = d_0' (\alpha_0 d_1 - d_1'\alpha_0) d_0 = f d_0'\sigma_0 - f\alpha_0 d_0,
\]

and so as $f$ is $M_1'$-regular, we have

\[
(8) \quad d_0'\sigma_1 d_0 = d_0'\alpha_0 - \alpha_1 d_0.
\]

First we verify the left square of (7) commutes:

\[
\begin{align*}
\gamma_1 d_0 &= (\alpha_1 + d_0'\sigma_1 + \sigma_0 d_1) d_0 \\
&= \alpha_1 d_0 + d_0'\sigma_1 d_0 + f\sigma_0 \\
&= \alpha_1 d_0 + (d_0'\alpha_0 - \alpha_1 d_0) + f\sigma_0, \quad \text{by } (8), \\
&= d_0'\alpha_0 + f\sigma_0 \\
&= d_0'\sigma_0 \\
&= d_0'(\alpha_0 + d_1'\sigma_0) \\
&= d_0'\gamma_0.
\end{align*}
\]
Next we verify the right square of (7) commutes:
\[
\begin{align*}
d_1\gamma_1 &= d_1'(\alpha_1 + d_0\sigma_1 + \sigma_0d_1) \\
&= d_1'\alpha_1 + f\sigma_1 + d_1'\sigma_0d_1 \\
&= d_1'\alpha_1 + \alpha_0d_1 - d_1'\alpha_1 + d_1'\sigma_0d_1, \quad \text{by (6)}, \\
&= \alpha_0d_1 + d_1'\sigma_0d_1 \\
&= (\alpha_0 + d_1'\sigma_0)d_1 \\
&= \gamma_0d_1.
\end{align*}
\]

Thus \(\gamma = (\gamma_0, \gamma_1)\) is a morphism \((M, d) \to (M', d')\) in \(F(C, f)\).

We next claim \(\overline{\sigma} \sim \overline{\tau}\), i.e., that \(T(\overline{\gamma}) = [\overline{\sigma}]\). We start with the following diagram (displaying homological degrees 3 to \(-1\)):

\[
\begin{array}{cccccccc}
\cdots & \rightarrow & M_1 & \stackrel{d_1}{\rightarrow} & M_0 & \stackrel{d_0}{\rightarrow} & M_1 & \cdots \\
\uparrow{\overline{\tau} - \overline{\sigma}_1} & & \uparrow{\overline{\tau} - \overline{\sigma}_0} & & \uparrow{\overline{\tau} - \overline{\sigma}_0} & & \uparrow{\overline{\tau} - \overline{\sigma}_1} & \\
\cdots & \rightarrow & M'_1 & \stackrel{\overline{d}_1}{\rightarrow} & M'_0 & \stackrel{\overline{d}_0}{\rightarrow} & M'_1 & \cdots \\
\end{array}
\]

Evidently, \(\overline{\sigma}_1\) and \(\overline{\sigma}_0\) give the start of a homotopy in degree 1:
\[
\overline{\sigma}_1 - \overline{\sigma}_1 = \overline{\sigma}_0 + \overline{d}_0\overline{\sigma}_1 + \sigma_0\overline{d}_1 = \overline{d}_0\overline{\sigma}_1 + \sigma_0\overline{d}_1.
\]

Note that the subcategory \(W\) is self-orthogonal by Proposition 1.8. As \(T(M, d)\) and \(T(M', d')\) are \(W\)-totally acyclic complexes, the arguments in [10, Appendix] show that we may extend \(\overline{\sigma}_1\) and \(\overline{\sigma}_0\) to a null homotopy of the displayed morphism, giving \(\overline{\tau} \sim \overline{\sigma}\). Indeed, extending the homotopy to the left is done by the proof of [10, Proposition A.3], with \(W\)-total acyclicity of \(T(M', d')\) standing in for the assumptions in loc. cit. (and [8, Lemma 2.4] in place of [8, Lemma 2.5]): extending the homotopy to the right uses the dual proof for [10, Proposition A.1]. It follows that \(T(\overline{\gamma}) = [\overline{\sigma}]\) hence \(T\) is full. \(\square\)

The following recovers [3, Theorem 3.5] when one takes \(C = \text{prj}(S)\).

**Theorem 3.5.** Let \(C\) be an additive self-orthogonal subcategory of \(\text{Mod}(S)\), let \(f \in S\) be \(S\)-regular and \(C\)-regular, and set \(R = S/(f)\). The triangulated functor \(T : \text{HF}(C, f) \to K_{\text{tac}}(R \otimes_S C)\) is full and faithful.

**Proof.** Combine Propositions 3.1, 3.3, and 3.4. \(\square\)

In fact, the results of this section have dual statements involving divisibility. In summary, one can show the following:

**Theorem 3.6.** Let \(C\) be an additive self-orthogonal subcategory of \(\text{Mod}(S)\), let \(f \in S\) be \(S\)-regular and \(C\)-divisible, and set \(R = S/(f)\). There is a triangulated functor \(\overline{T} : \text{HF}(C, f) \to K_{\text{tac}}(\text{Hom}_S(R, C))\) that is full and faithful.

**Proof.** One first notices that a version of Proposition 3.1 holds, by defining a functor \(\overline{T}\) using Proposition 2.7. Then using a dual version of Lemma 3.2, one can establish analogues of Propositions 3.3 and 3.4. \(\square\)
4. Equivalences for projective and injective factorizations

In this section, we consider \( \text{Prj}(S) \)- and \( \text{Inj}(S) \)-factorizations, referred to as projective and injective factorizations, respectively. Our goal here is to show that if \( S \) is a regular local ring, \( f \in S \) is nonzero, and \( R = S/(f) \), then projective factorizations of \( f \) correspond to Gorenstein projective \( R \)-modules; this can be considered as an extension of the classic bijection [12, Corollary 6.3] between matrix factorizations (having no trivial direct summand) and maximal Cohen-Macaulay \( R \)-modules (having no free direct summand). Dually, we observe a correspondence between injective factorizations of \( f \) and Gorenstein injective \( R \)-modules.

If one considers \( \text{prj}(S) \) in place of \( \text{Prj}(S) \) in the next result, then the classic proof, as in [12], uses the Auslander–Buchsbaum formula. However, we use an approach here that does not require the modules to be finitely generated.

**Proposition 4.1.** Assume \( S \) is a regular local ring, let \( f \in S \) be nonzero, and set \( R = S/(f) \). If \( M \) is a Gorenstein projective \( R \)-module, then there exists a projective factorization \( (P, d) \in F(\text{Prj}(S), f) \) with \( \text{coker}(d_1) = M \).

**Proof.** Let \( M \) be a Gorenstein projective \( R \)-module. As \( fM = 0 \), a result of Bennis and Mahdou [2, Theorem 4.1] yields \( \text{Gpd}_S M = \text{Gpd}_R M + 1 = 1 \), where \( \text{Gpd} \) denotes Gorenstein projective dimension. As \( S \) is regular, \( M \) has finite projective dimension over \( S \), thus by [6, 4.4.7] we have \( \text{pd}_S M = \text{Gpd}_S M = 1 \).

Now a standard construction yields a projective factorization of \( f \) which corresponds to \( M \): First choose a projective resolution \( P \) of \( M \) over \( S \) having the form \( 0 \to P_1 \xrightarrow{d_1} P_0 \to M \to 0 \). Application of \( \text{Hom}_S(P_0, -) \) to this sequence gives an exact sequence:

\[
0 \to \text{Hom}(P_0, P_1) \to \text{Hom}(P_0, P_0) \to \text{Hom}(P_0, M) \to 0.
\]

As \( fM = 0 \), the map \( f1^{P_0} \) is sent to 0, hence this sequence shows there exists a map \( d_0 : P_0 \to P_1 \) such that \( d_1 d_0 = f1^{P_0} \). Further, \( d_1(d_0 d_1) = f1^{P_0} d_1 = d_1(f1^{P_1}) \), and since \( d_1 \) is injective this implies that \( d_0 d_1 = f1^{P_1} \). It follows that \( (P, d) \) is a projective factorization of \( f \) such that \( \text{coker}(d_1) = M \). \qed

**Theorem 4.2.** Assume \( S \) is a regular local ring, let \( f \in S \) be nonzero, and set \( R = S/(f) \). There is a triangulated equivalence

\[
\mathbb{T} : \text{HF}(\text{Prj}(S), f) \to \text{K}_{\text{tac}}(\text{Prj}(R))
\]

given by the functor from Proposition 3.1.

**Proof.** The triangulated functor \( \mathbb{T} \) given in Proposition 3.1, applied to \( C = \text{Prj}(S) \), is full and faithful by Theorem 3.5. Also note that \( R \otimes_S \text{Prj}(S) = \text{Prj}(R) \) (see Remark 1.5) and so the functor \( \mathbb{T} \) has the claimed codomain. It remains to show that \( \mathbb{T} \) is essentially surjective. Let \( T \in \text{K}_{\text{tac}}(\text{Prj}(R)) \). Then \( Z_0(T) \) is a Gorenstein projective \( R \)-module. By Proposition 4.1 there is a \( \text{Prj}(S) \)-factorization \( (P, d) \) such that \( \text{coker}(d_1) = Z_0(T) \).

We argue that \( \mathbb{T}(P, d) \) is homotopic to \( T \). Notice that \( Z_0(\mathbb{T}(P, d)) = Z_0(T) \) by construction. There exists a degree 0 chain map \( \phi : \mathbb{T}(P, d) \to T \) that lifts the identity map \( Z_0(\mathbb{T}(P, d)) \to Z_0(T) \) by [7, Lemma 3.1]; the lifting \( \phi \) is a homotopy equivalence by [7, Proposition 3.3(b)]. \qed
Corollary 4.3. Assume $S$ is a regular local ring, let $f \in S$ be nonzero, and set $R = S/(f)$. There is a triangulated equivalence between $HF(\text{Prj}(S), f)$ and the stable category of Gorenstein projective $R$-modules.

Proof. Combine Theorem 4.2 with the equivalence between $K_{\text{tac}}(\text{Prj}(R))$ and the stable category of Gorenstein projective $R$-modules; see e.g., [7, Example 3.10]. □

There are dual results for injective factorizations:

Proposition 4.4. Assume $S$ is a regular local ring, let $f \in S$ be nonzero, and set $R = S/(f)$. If $M$ is a Gorenstein injective $R$-module, then there exists an injective factorization $(I, d) \in F(\text{Inj}(S), f)$ with $\ker(d_1) = M$.

Proof. Dual to the proof of Proposition 4.1, where one instead uses [2, Theorem 4.2] in place of [2, Theorem 4.1] and [6, 6.2.6] in place of [6, 4.4.7]. □

Theorem 4.5. Assume $S$ is a regular local ring, let $f \in S$ be nonzero, and set $R = S/(f)$. There is a triangulated equivalence

$$\overline{T} : HF(\text{Inj}(S), f) \xrightarrow{\simeq} K_{\text{tac}}(\text{Inj}(R))$$

given by the functor from Theorem 3.6.

Proof. Similar to the proof of Theorem 4.2; appeal instead to Theorem 3.6 and Proposition 4.4. □

Corollary 4.6. Assume $S$ is a regular local ring, let $f \in S$ be nonzero, and set $R = S/(f)$. There is a triangulated equivalence between $HF(\text{Inj}(S), f)$ and the stable category of Gorenstein injective $R$-modules.

Proof. Combine Theorem 4.5 and the equivalence between $K_{\text{tac}}(\text{Inj}(R))$ and the stable category of Gorenstein injective $R$-modules; see [16, Proposition 7.2]. □

5. AN EQUIVALENCE FOR FLAT-COTORSION FACTORIZATIONS

In this section, assume $S$ is a commutative noetherian ring. We give an equivalence in the case of the self-orthogonal category $\text{FlatCot}(S)$ of flat-cotorsion $S$-modules (that is, the category of $S$-modules that are both flat and cotorsion). The approach is similar to the previous section, but requires some extra care; in particular, we must establish a fact corresponding to the one from [2] used above.

Denote by $M_p^\wedge = \varprojlim(S/p^n \otimes_S M)$ the $p$-adic completion of an $S$-module $M$. By [13], an $S$-module $M$ is flat-cotorsion if and only if it is isomorphic to a product over $p \in \text{Spec } S$ of completions of free $S_p$-modules, that is, $M \cong \prod_{p \in \text{Spec } S} (\bigoplus_{B(p)} S_p)_{\wedge}$ for some sets $B(p)$.

Lemma 5.1. Let $\pi : S \rightarrow R$ be a surjective ring homomorphism. Then we have an equality $R \otimes_S \text{FlatCot}(S) = \text{FlatCot}(R)$.

Proof. First notice that for a flat-cotorsion $S$-module $\prod_{p \in \text{Spec } S} (\bigoplus_{B(p)} S_p)_{\wedge}$, there is an isomorphism

$$R \otimes_S \left( \prod_{p \in \text{Spec } S} (\bigoplus_{B(p)} S_p)_{\wedge} \right) \cong \prod_{p \in \text{Spec } S} (\bigoplus_{B(p)} R_{\pi(p)})_{\wedge},$$

since $R$ is finitely presented as an $S$-module. It is now immediate that there is an inclusion $R \otimes_S \text{FlatCot}(S) \subseteq \text{FlatCot}(R)$. The other inclusion follows by observing that every flat-cotorsion $R$-module can be expressed in a form given by the right side of this isomorphism, since $\text{Spec } R = \pi(\text{Spec } S)$. □
The next lemma is needed in place of the change of rings facts for Gorenstein projective and Gorenstein injective dimensions from \[2\]. As in \[7, \text{Definition 4.3}\], refer to a FlatCot\((S)\)-totally acyclic complex as a totally acyclic complex of flat-cotorsion \(S\)-modules and a FlatCot\((S)\)-Gorenstein module as a Gorenstein flat-cotorsion \(S\)-module; see Definition 2.3. Gorenstein flat-cotorsion \(S\)-modules are by \[7, \text{Theorem 5.2}\] precisely the modules that are both Gorenstein flat—that is, isomorphic to \(Z_0(F)\) for some \(F\)-totally acyclic complex \(F\) of flat \(S\)-modules—and cotorsion.

**Lemma 5.2.** Let \(f \in S\) be a regular element and set \(R = S/(f)\). Let \(M\) be a Gorenstein flat-cotorsion \(R\)-module. There is an exact sequence of \(S\)-modules

\[
0 \longrightarrow M' \longrightarrow F \longrightarrow M \longrightarrow 0,
\]

with \(M'\) a Gorenstein flat-cotorsion \(S\)-module and \(F\) a flat-cotorsion \(S\)-module.

**Proof.** As \(M\) is a Gorenstein flat-cotorsion \(R\)-module, there is a totally acyclic complex \(T\) of flat-cotorsion \(R\)-modules such that \(Z_0(T) = M\). For each \(i \in \mathbb{Z}\), we may find—because flat covers exist for all modules \[4\]—a surjective flat cover \(F_i \to Z_i(T)\) over \(S\); the kernel \(K_i = \ker(F_i \to Z_i(T))\) is cotorsion by Wakamatsu’s Lemma \[24, \text{Lemma 2.1.1}\]. In fact, since \(Z_i(T)\) is a cotorsion \(R\)-module, it is also a cotorsion \(S\)-module for each \(i \in \mathbb{Z}\) by \[24, \text{Proposition 3.3.3}\], hence \(F_i\) is flat-cotorsion for each \(i \in \mathbb{Z}\). Indeed, \(Z_i(T)\) being a cotorsion \(S\)-module also yields \(\text{Ext}^1_S(F_{i-1}, Z_i(T)) = 0\); from this and the snake lemma we obtain, for each \(i \in \mathbb{Z}\), the following commutative diagram with exact rows and columns:

\[
\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & & 0 \\
0 & & K_i & & T_i' & & K_{i-1} & & 0 \\
0 & & F_i & & F_i \oplus F_{i-1} & & F_{i-1} & & 0 \\
0 & & Z_i(T) & & T_i & & Z_{i-1}(T) & & 0 \\
0 & & 0 & & 0 & & 0 & & 0 \\
\end{array}
\]

As \(K_i\) and \(K_{i-1}\) are cotorsion \(S\)-modules, so is \(T_i'\). Additionally, as \(T_i\) is a flat \(R\)-module, \(\text{fd}_S T_i = 1\); see Remark 1.6. From \[1, \text{2.4.F}\], we obtain that \(T_i'\) is a flat \(S\)-module.

Now glue together the short exact sequences from the top rows of these diagrams to obtain an acyclic complex \(T'\) of flat-cotorsion \(S\)-modules with \(Z_i(T') = K_i\) for each \(i \in \mathbb{Z}\). Fix a flat-cotorsion \(S\)-module \(N\). Evidently, as each \(K_i\) is cotorsion, we obtain \(\text{Hom}_S(N, T')\) is acyclic. Moreover, for each \(i \in \mathbb{Z}\),

\[
\text{Ext}^1_S(K_i, N) \cong \text{Ext}^2_S(Z_i(T), N) \cong \text{Ext}^1_R(Z_i(T), R \otimes_S N) = 0,
\]

where the first isomorphism follows from the left vertical exact sequence in the diagram and the second follows from Lemma 1.7(i). For the last equality, note that as \(N\) is a flat-cotorsion \(S\)-module, we have \(R \otimes_S N\) is a flat-cotorsion \(R\)-module.
by Lemma 5.1. It now follows that $\text{Hom}_S(T',N)$ is also acyclic. Thus $T'$ is a totally acyclic complex of flat-cotorsion $S$-modules. In particular, $Z_0(T') = K_0$ is a Gorenstein flat-cotorsion $S$-module, and the claim follows.

**Proposition 5.3.** Assume $S$ is a regular local ring, let $f \in S$ be nonzero, and set $R = S/(f)$. If $M$ is a Gorenstein flat-cotorsion $R$-module, then there exists a flat-cotorsion factorization $(F,d) \in F(\text{FlatCot}(S),f)$ with $\text{coker}(d_1) = M$.

**Proof.** Let $M$ be a Gorenstein flat-cotorsion $R$-module. Lemma 5.2 yields an exact sequence

$$0 \rightarrow F_1 \xrightarrow{d_1} F_0 \rightarrow M \rightarrow 0,$$

with $F_0$ a flat-cotorsion $S$-module and $F_1$ a Gorenstein flat-cotorsion $S$-module. By [7, Theorem 5.2], $F_1$ is cotorsion and Gorenstein flat. As $S$ is regular, we have $\text{fd}_S M < \infty$, hence $\text{fd}_S M = \text{Gfd}_S M \leq 1$ by [6, 5.2.8]. Thus $F_1$ is also flat [1, 2.4.F], hence flat-cotorsion.

As in Proposition 4.1, a standard construction applied to the sequence above provides a flat-cotorsion factorization $(F,d)$ of $f$ with $\text{coker}(d_1) = M$. □

**Theorem 5.4.** Assume $S$ is a regular local ring, let $f \in S$ be nonzero, and set $R = S/(f)$. There is a triangulated equivalence $T : \text{HF}(\text{FlatCot}(S),f) \simeq \text{K}_{\text{tac}}(\text{FlatCot}(R))$ given by the functor in Proposition 3.1.

**Proof.** Similar to the proof of Theorem 4.2, using Proposition 5.3 in place of Proposition 4.1, and Lemma 5.1 in place of Remark 1.5. □

**Corollary 5.5.** Assume $S$ is a regular local ring, let $f \in S$ be nonzero, and set $R = S/(f)$. There is a triangulated equivalence between $\text{HF}(\text{FlatCot}(S),f)$ and the stable category of Gorenstein flat-cotorsion $R$-modules.

**Proof.** This equivalence follows from Theorem 5.4 and the triangulated equivalence between $\text{K}_{\text{tac}}(\text{FlatCot}(R))$ and the stable category of Gorenstein flat-cotorsion $R$-modules given in [7, Summary 5.7]. □

One motivation for considering totally acyclic complexes of flat-cotorsion $R$-modules is their relation to the next analogue of the singularity category as described by Murfet and Salarian [19].

The pure derived category of flat $S$-modules is defined as the Verdier quotient $\text{D(Flat}(S)) = \text{K(Flat}(S))/\text{K}_{\text{tac}}(\text{Flat}(S))$ of the homotopy category of flat $S$-modules by its subcategory of pure acyclic complexes of flat $S$-modules. Neeman proves in [20, Theorem 1.2] that $\text{D(Flat}(S))$ is equivalent to $\text{K(Prj}(S))$, and moreover, Murfet and Salarian show [19, Lemma 4.22] that $\text{D}_{F_{\text{tac}}}(\text{Flat}(S))$, the subcategory of $\text{D(Flat}(S))$ of $F$-totally acyclic complexes, is equivalent to $\text{K}_{\text{tac}}(\text{Prj}(S))$, assuming that $S$ is a commutative noetherian ring having finite Krull dimension.

**Corollary 5.6.** Assume $S$ is a regular local ring, let $f \in S$ be nonzero, and set $R = S/(f)$. There is a triangulated equivalence $\text{HF}(\text{FlatCot}(S),f) \simeq \text{D}_{F_{\text{tac}}}(\text{Flat}(R))$.

**Proof.** Combine Theorem 5.4 and [7, Summary 5.7]. □
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