Minimum Scoring Rule Inference

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ABSTRACT. Proper scoring rules are devices for encouraging honest assessment of probability distributions. Just like log-likelihood, which is a special case, a proper scoring rule can be applied to supply an unbiased estimating equation for any statistical model, and the theory of such equations can be applied to understand the properties of the associated estimator. In this paper, we discuss some novel applications of scoring rules to parametric inference. In particular, we focus on scoring rule test statistics, and we propose suitable adjustments to allow reference to the usual asymptotic chi-squared distribution. We further explore robustness and interval estimation properties, by both theory and simulations.

Key words: B-robustness, Bregman estimate, composite score, Godambe information, M-estimator, pseudo-likelihood, Tsallis score, unbiased estimating equation

1. Introduction

Suppose we wish to fit a parametric statistical model \( \{ P_\theta : \theta \in \Theta \subseteq \mathbb{R}^p \} \), based on a random sample \( x = (x_1, \ldots, x_n) \) of size \( n \). The most popular tool for inference on the parameter \( \theta \) is the log-likelihood function, given by

\[
\ell(\theta) := \sum_{i=1}^{n} \log p_\theta(x_i),
\]

where \( p_\theta(x) \) is the density associated to \( P_\theta \). For instance, the maximum likelihood estimator is defined as \( \hat{\theta} = \arg \max_\theta \ell(\theta) \), and confidence regions with nominal coverage \( 1 - \alpha \) can be constructed as \( \{ \theta : W(\theta) \leq \chi^2_{p,1-\alpha} \} \), where \( W(\theta) = 2(\ell(\hat{\theta}) - \ell(\theta)) \) is the likelihood ratio statistic and \( \chi^2_{p,1-\alpha} \) is the \( (1 - \alpha) \)-quantile of the \( \chi^2_p \) distribution.

However, likelihood-based inference generally requires strict adherence to the model assumptions and can behave quite poorly under slight model misspecification. A possible solution is to resort to suitable pseudo-likelihood functions, which are intended as surrogates of the full likelihood. Useful examples are given by composite likelihoods (Cox & Reid, 2004; Varin et al., 2011), when the fully specified likelihood is computationally cumbersome or when a fully specified model is out of reach, and by quasi-likelihoods, which are derived from suitable unbiased estimating equations (see, among others, Adimari & Ventura, 2002).

Both full and pseudo-likelihood inference are special cases of a more general estimation technique based on proper scoring rules (e.g., Dawid, 1986), which are methods for encouraging honest assessment of probability distributions. In such a case, the log-likelihood function is replaced by the function

\[
S(x, \theta) := \sum_{i=1}^{n} S(x_i, P_\theta),
\]
where \( S(x, P) \) is a proper scoring rule, as described in the succeeding sections; this can be chosen to increase robustness, or for ease of computation. Minimizing (2) will yield an unbiased estimating equation, for any statistical model.

The appeal of scoring rule estimation lies in the potential for adaptation of the scoring rule to the problem at hand, and it forms a special case of \( M \)-estimation (e.g., Huber & Ronchetti, 2009). In view of this, under regularity conditions, asymptotic arguments indicate that the estimator \( \hat{\theta}_S := \arg \min_{\theta} S(x, \theta) \) is consistent and asymptotically normal, with asymptotic covariance matrix given by the inverse of the Godambe information. This allows the construction of Wald-type test statistics and confidence regions. However, as is well-known, Wald-type statistics force confidence regions to have an elliptical shape and may be less accurate for small sample sizes. On the other hand, the asymptotic distribution of the likelihood ratio-type statistics derived from (2) depart from the familiar likelihood result, involving instead a linear combination of independent chi-squared variates with coefficients given by the eigenvalues of a matrix related to Godambe information. As a consequence, most routine statistical analyses employ Wald-type statistics.

The aim of this paper is to discuss inference based on proper scoring rules. Because of the failure, in general, of the information identity, inference based on proper scoring rules requires suitable corrections. In particular, when considering the scoring rule ratio statistic for a parameter of interest, we discuss suitable adjustments that allow reference to the usual asymptotic chi-squared distribution. Particular focus is on robust proper scoring rules, that is, scoring rules that lead to estimators with bounded influence function. Indeed, in this case, the adjusted scoring rule ratio statistic can be used in the usual way to derive confidence regions for a multidimensional parameter of interest, while in general, a quasi-likelihood does not exist (McCullagh, 1991).

The paper is organized as follows. In Section 2, background theory and examples on proper scoring rules are given, while Section 3 focuses on proper scoring rule inference. Section 4 discusses asymptotic results on scoring rule procedures and introduces adjustments to the scoring rule ratio statistic that allow reference to the usual asymptotic chi-squared distribution. In Section 5, robustness properties of the scoring rules estimators are studied. In particular, conditions for robustness of the Bregman score are investigated in detail. Three examples dealing with confidence regions from the adjusted scoring rule ratio statistic are analysed in Section 6. Simulation results indicate that such adjustments allow accurate inferences, and it is argued that scoring rules have an important role to play in frequentist inference. Some concluding remarks are given in Section 7.

2. Proper scoring rules

Let \( X \) be a random variable taking values in a sample space \( \mathcal{X} \). A scoring rule (e.g., Dawid, 1986) is a loss function \( S(x, Q) \) measuring the quality of a quoted probability distribution \( Q \) for \( X \), in the light of the realized outcome \( x \) of \( X \). It is proper if, for any distribution \( P \) for \( X \), the expected score \( \mathbb{E}_{X \sim P} S(X, Q) \) is minimized by quoting \( Q = P \). Equivalently, the associated divergence or discrepancy function (Dawid, 1998), given by \( D(P, Q) := S(P, Q) - S(P, P) \), is always non-negative. There is a very wide variety of proper scoring rules: for general characterizations see, among others, McCarthy (1956) and Savage (1971), and for various special cases, see Dawid (1998, 2007), Dawid & Musio (2014), Gneiting & Raftery (2007) and Machete (2013). We now consider some of these in more detail.

Let \( q(\cdot) \) denote the density of \( Q \) with respect to an underlying \( \sigma \)-finite measure \( \mu \) on \( \mathcal{X} \). Although greater generality is possible, in this paper, we will assume \( \mu \) is Lebesgue measure for \( \mathcal{X} \) a real interval, and counting measure for \( \mathcal{X} \) discrete. For a finite (especially binary) sample
space $\mathcal{X}$, a useful proper scoring rule is the Brier (Brier, 1950) or quadratic score $S(x, Q) = \{1 - q(x)\}^2 + \sum_{y \neq x} q(y)^2$, which is just the squared Euclidean distance between the vector $\mathbf{q} := (q(y) : y \in \mathcal{X})$ corresponding to $Q$, and the vector $\delta_x$ corresponding similarly to the one-point distribution at $x$. The associated discrepancy $D(P, Q)$ is the squared Euclidean distance between $\mathbf{p}$ (the vector corresponding to $P$) and $\mathbf{q}$. Another prominent proper scoring rule (Good, 1952) is the log score $S(x, Q) = -\log q(x)$, whose associated discrepancy is the Kullback–Leibler divergence $K(P, Q)$. These are both special cases (with, respectively, $\psi(t) \equiv t^2$ and $\psi(t) \equiv t \log t$) of a general separable Bregman score construction (e.g., Dawid, 2007, Equation (16)):

$$S(x, Q) = -\psi'(q(x)) - \int \left[\psi(q(y)) - q(y)\psi'(q(y))\right] d\mu(y),$$

where the defining function $\psi : \mathbb{R}^+ \to \mathbb{R}$ is convex and differentiable. The associated Bregman divergence is

$$D(P, Q) = \int \Delta \{p(y), q(y)\} d\mu(y),$$

where $\Delta(a, b) := \psi(a) - \psi(b) - \psi'(b)(a - b) \geq 0$ by convexity. Another important special case of this construction, the Tsallis score, arises on taking $\psi(t) \equiv t^\gamma$ ($\gamma > 1$). This yields

$$S(x, Q) = (\gamma - 1) \int q(y)^\gamma d\mu(y) - \gamma q(x)^{\gamma - 1},$$

with divergence function

$$D(P, Q) = \int p(y)^\gamma d\mu(y) + (\gamma - 1) \int q(y)^\gamma d\mu(y) - \gamma \int p(y)q(y)^{\gamma - 1} d\mu(y).$$

The density power divergence $d_\alpha$ of Basu et al. (1998) (see also Jones et al. (2001)) is just (6), with $\gamma = \alpha + 1$ and $\mu$ given by Lebesgue measure, multiplied by $1/\alpha$.

In order to evaluate the log score, we only need to know the value of the forecast density function, $q(\cdot)$, at the outcome $x$ of $X$ that nature in fact produces. So long as the size of $\mathcal{X}$ exceeds two, the log score is essentially the only proper scoring rule that is strictly local in this sense (Bernardo, 1979). However, we can weaken the locality requirement, and so admit further ‘local proper scoring rules’. For a sample space $\mathcal{X}$ that is an open subset of a Euclidean space, we ask that $S(x, Q)$ should depend on the density function $q(\cdot)$ only through its value and the value of a finite number of its derivatives at $x$. For the case that $\mathcal{X}$ is a real interval, Parry et al. (2012) show that any such local proper scoring rule is a linear combination of the log score and what they term a key local scoring rule, which they have characterized. A key local scoring rule has the convenient property that it can be computed without knowledge of the normalization constant of the density. The simplest key local scoring rule is based on the proposal by Hyvärinen (2005)

$$S_H(x, Q) = 2\Delta \log q(x) + |\nabla \log q(x)|^2,$$

where, in the case of a real sample space, $\nabla := \partial / \partial x$ and $\Delta := \partial^2 / \partial x^2$. Formula (7) can also be applied to the case of a multivariate observation $X = (X_1, \ldots, X_k)$, with $\nabla := (\partial / \partial x_j)$ and $\Delta := \sum_{j=1}^k \partial^2 / \partial x_j^2$. Further extensions to a general Riemannian sample space are possible: see Dawid & Lauritzen (2005).
2.1. Composite scores

In this section, we consider the case of a multidimensional variable $X$. Let $X^*$ be a subvector of (or, more generally, a function of) $X$, and let $S^*$ be a proper scoring rule for $X^*$. Then, we can define a proper scoring rule $S$ for $X$ as $S(x, Q) := S^*(x^*, Q^*)$, where $Q^*$ denotes the marginal distribution of $X^*$ when $X \sim Q$. Alternatively, let $X^\dagger$ denote another subvector or function of $X$. Then, a proper scoring rule can be generated as $S(x, Q) := S^*(x^*, Q^\dagger)$, where $Q^\dagger$ denotes the conditional distribution, when $X \sim Q$, of $X^*$, given $X^\dagger = x^\dagger$. By an abuse of language, we may refer to the specification of $(X^*, X^\dagger)$ as a conditional variable, $X_0$ say, and that of $Q^\dagger$, for every value $x^\dagger$ of $X^\dagger$, as its distribution, $Q_0$ say, and then we write $X_0 \sim Q_0$.

Now let $\{X_k\}$ be a collection of marginal and/or conditional variables, and let $S_k$ be a proper scoring rule for $X_k$. Then, we can construct a proper scoring rule for $X$ as

$$S(x, Q) := \sum_k S_k(x_k, Q_k),$$

where $X_k \sim Q_k$ when $X \sim Q$. The form (8) localizes the problem to the $\{X_k\}$, which can simplify the computation.

We term a scoring rule of the form (8) a composite scoring rule. In the special case that each $S_k$ is the log score, (8) becomes a (negative log) composite likelihood (e.g., Varin et al., 2011). Composite likelihood is often considered as a surrogate for the full likelihood function, useful in models with a complex dependence structure. The above reformulation allows us to treat composite likelihood in its own right, as supplying a proper scoring rule. And from this point of view, as we shall see, there is nothing special about composite likelihood: most of the existing results about it extend with very little change to the more general case of an arbitrary proper scoring rule (whether or not constructed as a composite score).

Example 1. Consider a spatial process $X = (X_v : v \in V)$, where $V$ is a set of lattice sites. For a joint distribution $Q$ for $X$, let $Q_v$ be the family of conditional distributions for $X_v$, given the values of $X_{\neq v}$, the variables at all other sites. If $Q$ is Markov, $Q_v$ depends only on $X_{\text{ne}(v)}$ (variables at sites neighbouring $v$). We can then construct a proper scoring rule $S(x, Q) := \sum_v S_0(x_v, Q_v)$, where $S_0$ is a proper scoring rule for the state at a single site. When $S_0$ is the log score, this is the (negative log) pseudo-likelihood of Besag (1975). For binary $X_v$ and $S_0$ the Brier score, it leads to the ratio matching method of Hyvärinen (2005). Some comparisons may be found in Dawid & Musio (2013).

3. Scoring rule inference

Let $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$, with $\Theta$ an open subset of $\mathbb{R}^p$, be a parametric family of distributions on $X$, and let $p_\theta(x)$ denote the probability density function of $P_\theta$. The validity of inference about $\theta$ using scoring rules can be justified by invoking the general theory of unbiased estimating functions.

Consider a proper scoring rule $S$ on $X$, and write $S(x, \theta)$ for $S(x, P_\theta)$ and $s(x, \theta)$ for the gradient vector of $S(x, \theta)$ with respect to $\theta$, that is

$$s(x, \theta) = \nabla_\theta S(x, \theta) = \left( \frac{\partial S(x, \theta)}{\partial \theta_j} \right).$$

For $X \sim P$, where $P$ need not belong to $\mathcal{P}$, we can approximate $P$ within $\mathcal{P}$ by $P_{\theta_P}$, where

$$\theta_P := \arg \min_\theta D(P, P_\theta).$$

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$D$ being the discrepancy function associated with $S$. In particular, if $P = P_{\theta_0} \in \mathcal{P}$, where $\theta_0$ is the true value of the parameter, then $\theta_P = \theta_0$. Because $D(P, P_{\theta}) = S(P, P_{\theta}) - H(P)$, (10) is equivalent to

$$
\theta_P = \arg\min_{\theta} S(P, P_{\theta}).
$$

(11)

Now, let $x = (x_1, \ldots, x_n)$ be a random sample of size $n$ from $P$, and let $\hat{P}_n$ be the associated empirical distribution. Then, we can take $\hat{\theta}_S = \theta_{\hat{P}_n}$ as a point estimate of $\theta_P$, that is, $\hat{\theta}_S$ is the value of $\theta$ minimizing $S(\hat{P}_n, P_{\theta})$. Equivalently, it minimizes $nS(\hat{P}_n, P_{\theta})$, which is just the total empirical score

$$
S(x, \theta) := \sum_{i=1}^{n} S(x_i, \theta).
$$

Thus, the scoring rule estimate of $\theta_P$ is

$$
\hat{\theta}_S := \arg\min_{\theta} S(x, \theta) = \arg\min_{\theta} \sum_{i=1}^{n} S(x_i, \theta),
$$

which (under differentiability conditions) is the solution of the scoring rule estimating equation

$$
s(\theta) = 0.
$$

(12)

where $s(\theta) = s(x, \theta) := \sum_{i=1}^{n} s(x_i, \theta)$. Note that when $S$ is the log score, so that $S(x, \theta) = -\sum_{i=1}^{n} \log p_{\theta}(x_i)$, the scoring rule estimating equation (12) is just the (negative of) the likelihood equation, and the scoring rule estimate is just the maximum likelihood estimate.

For the special case that the discrepancy $D$ is the Tsallis/density power divergence, Basu et al. (1998) note that – unlike many other applications of minimum distance estimation (see for instance Cao et al., 1995) – this procedure does not require the preliminary construction of a continuous non-parametric density estimate of the true density $p(\cdot)$, so avoiding complications such as bandwidth selection. This pleasant property extends to all minimum discrepancy estimates based on proper scoring rules.

Generalizing a familiar property of the likelihood equation, the following theorem shows that, for any proper scoring rule and any family of distributions, the scoring rule estimating equation (12) is unbiased.

**Theorem 1.** For the scoring rule estimating function $s(x, \theta)$, it holds that

$$
E_P \{ s(X, \theta_P) \} = 0,
$$

where $E_P(\cdot)$ denotes expectation with respect to $P$.

**Proof.** For fixed $P$, $E_P S(X, \phi)$ is minimized at $\phi = \theta_P$. Thus, under sufficient regularity to allow interchange of expectation over $X$ and differentiation with respect to $\theta$, we have

$$
0 = \nabla_\phi E_P S(X, \phi)|_{\phi = \theta_P} = E_P \nabla_\phi S(X, \phi)|_{\phi = \theta_P} = E_P s(X, \theta_P).
$$

□

**Corollary 1.** (Dawid & Lauritzen, 2005; Dawid, 2007) For $P = P_{\theta} \in \mathcal{P}$

$$
E_{\theta} \{ s(X, \theta) \} = 0,
$$

where $E_{\theta}(\cdot)$ denotes expectation with respect to $P_{\theta}$.
As a consequence of Corollary 1, we have that Equation (12) delivers an unbiased estimating equation for the parameter \( \theta \), that is, the first Bartlett identity holds. The solution thus forms a special case of \( M \)-estimation (see, among others, Hampel et al., 1986 and Huber & Ronchetti, 2009). An important feature of this approach is that the choice of the scoring rule is entirely independent of the specific estimation problem under consideration. Any such choice supplies a universal \( M \)-estimation procedure, applying across all possible models in mutually consistent fashion. This extends the familiar universal applicability of maximum likelihood estimation to scoring rules other than the log score.

3.1. Example: Bregman estimation

Consider the separable Bregman score given by (3). We have

\[
-s(x, \theta) = \lambda(x, \theta) - E_\theta \lambda(X, \theta),
\]

with

\[
\lambda(x, \theta) = \nabla_\theta \psi'(p_\theta(x)) \quad (13)
\]

\[
= \psi''(p_\theta(x)) \nabla_\theta p_\theta(x). \quad (14)
\]

Because the function \( \psi \) was required to be convex, we have that \( \alpha := \psi'' \) must be non-negative. Any such choice of \( \alpha \) determines a suitable function \( \psi \), and hence a separable Bregman scoring rule. We term such a choice for \( \alpha \) a Bregman gauge.

Having fixed on a Bregman gauge function \( \alpha \), we can now solve any estimation problem, of any parametric dimensionality, based on observations on \( X \), by using the estimating function

\[
\lambda(x, \theta) = \alpha(p_\theta(x)) \nabla_\theta p_\theta(x). \quad (16)
\]

An unbiased estimating equation for \( \theta \), yielding an \( M \)-estimator, is obtained by equating the sample and population averages of \( \lambda \). The form (16) is, in this sense, a universal estimating function. For the special Bregman gauge \( \alpha(t) = 1/t \), we recover Fisher’s efficient score function and maximum likelihood estimation.

Location model. Bregman inference for a univariate location model is particularly straightforward. For such a model, we have

\[
p_\theta(x) = f(x - \theta), \quad (17)
\]

where \( f \) is a density on \( \mathbb{R} \) that we assume to be strictly positive everywhere and continuously differentiable. Using the separable Bregman score formula (3), we note that the integral term in \( S(x, \theta) \) does not depend on \( \theta \). Consequently, for the case of a location model, minimizing the empirical score is equivalent to maximizing

\[
\sum_{i=1}^n \xi\{f(x_i - \theta)\}, \quad (18)
\]

where \( \xi \equiv \psi' \) is a fixed increasing function (and \( \xi' \) is just the Bregman gauge \( \alpha \)). This generalizes maximum likelihood, for which \( \xi \equiv \log \).

The maximum of (18) will be obtained by setting its derivative to 0, leading to the unbiased estimating equation

\[
\sum_{i=1}^n \lambda(x_i, \theta) = 0.
\]
where, in accordance with (16)
\[
\lambda(x, \theta) = -\alpha \{ f(x - \theta) \} f'(x - \theta).
\]
In this case, \( E_\theta \{ \lambda(X, \theta) \} \) is identically 0.

4. Asymptotics

Given a proper scoring rule, we can apply standard results on \( M \)-estimators to describe the properties of the scoring rule estimator \( \hat{\theta}_S \) defined by (12). Hereinafter, regularity conditions as detailed in, for example, Huber (1967), Barndorff-Nielsen & Cox (1994, Section 9.2) or Molenberghs & Verbeke (2005, Section 9.2.2), are assumed.

**Theorem 2.** Under suitable regularity conditions, the scoring rule estimator \( \hat{\theta}_S \) is consistent and asymptotically normal, with mean \( \theta_P \) and variance \( V_K(J_K)^{-1} / T \), where
\[
V = K^{-1} J(K^{-1})^T,
\]
with
\[
J = E_{\theta_P} \{ s(\theta_P) s(\theta_P)^T \} \quad (19)
\]
\[
K = E_{\theta} \left\{ \frac{\partial s(\theta)}{\partial \theta^T} \right\} \bigg|_{\theta = \theta_P}. \quad (20)
\]

When \( P = P_\theta \), then \( V = V(\theta) = K(\theta)^{-1} J(\theta) (K(\theta)^{-1})^T \), with \( J(\theta) = E_{\theta} \{ s(\theta) s(\theta)^T \} \) and \( K(\theta) = E_{\theta} \{ \frac{\partial s(\theta)}{\partial \theta^T} \} \).

The matrix \( G = V^{-1} \) is known as the Godambe information matrix Godambe (1960). Typically, the second Bartlett identity fails, that is, \( K \neq J \). This contrasts with the special case of the log score: when \( S(x, \theta) = -\sum_{i=1}^n \log p_{\theta_i}(x_i) \), and \( P = P_\theta \), we have \( G = K(\theta) = J(\theta) \), the Fisher information matrix.

4.1. Scoring rule test statistics

Hypothesis tests and confidence regions for \( \theta \) can be formed in the usual way by using a consistent estimate of the asymptotic variance \( V \). In particular, inference for \( \theta \) can be based on the scoring rule Wald-type statistic
\[
W^S_{W}(\theta) := (\hat{\theta}_S - \theta)^T V^{-1}(\hat{\theta}_S - \theta), \quad (21)
\]
which has an asymptotic chi-squared distribution on \( p \) degrees of freedom. The asymptotic \( \chi^2_p \) distributional result holds also for the scoring rule score-type statistic \( W^S_s(\theta) := s(\theta)^T J^{-1} s(\theta) \). A consistent estimate of \( V \) can be obtained by using estimates of the matrices \( J \) and \( K \):
\[
\hat{J} = \sum_{i=1}^n s(x_i, \hat{\theta}_S) s(x_i, \hat{\theta}_S)^T \quad \hat{K} = \sum_{i=1}^n \frac{\partial s(x_i, \theta)}{\partial \theta^T} \bigg|_{\theta = \hat{\theta}_S}.
\]

one can refer to Varin (2008) and Varin et al. (2011) for a detailed discussion of the issues related to the estimation of \( J \) and \( K \).

As is well-known, Wald-type statistics lack invariance under reparametrization and force confidence regions to have an elliptical shape. On the other hand, score-type statistics suffer
from numerical instability in many examples (e.g., Molenberghs & Verbeke, 2005, Chapter 9). In this respect, a scoring rule ratio statistic, of the form

\[ W^S(\theta) := 2 \{ S(x, \theta) - S(x, \widehat{\theta}_S) \} \]

would seem to form a more appealing basis for inference. However, the asymptotic distribution of (22) departs from the familiar likelihood result and involves a linear combination of independent chi-squared random variables with coefficients given by the eigenvalues of a matrix related to Godambe information (see, among others, Heritier & Ronchetti (1994) and Varin et al., 2011). More precisely

\[ W^S(\theta) \overset{D}{\to} \sum_{j=1}^p \mu_j Z_j^2, \]

where \( \mu_1, \ldots, \mu_p \) are the eigenvalues of \( JK^{-1} = KG^{-1} \) and \( Z_1, \ldots, Z_p \) are independent standard normal variates.

Analogous limiting results can be shown to hold for tests on subsets of \( \theta \). Let \( \theta \) be partitioned as \( \theta^T = (\psi^T, \lambda^T) \), where \( \psi \) is a \( p_0 \)-dimensional parameter of interest and \( \lambda \) is a \( (p - p_0) \)-dimensional nuisance parameter. Correspondingly, partition the scoring rule estimating function as \( s(\theta)^T = (s_\psi(\theta)^T, s_\lambda(\theta)^T) \), where \( s_\psi(\theta) = (\partial/\partial \psi)S(x, \theta) \) and \( s_\lambda(\theta) = (\partial/\partial \lambda)S(x, \theta) \), and

\[ K = \begin{pmatrix} \psi^T \psi & \psi^T \lambda \\ \lambda^T \psi & \lambda^T \lambda \end{pmatrix}, \quad K^{-1} = \begin{pmatrix} \psi^T \psi & \psi^T \lambda \\ \lambda^T \psi & \lambda^T \lambda \end{pmatrix}^{-1} \]

and similarly for \( G \) and \( G^{-1} \). Finally, let \( \widehat{\theta}_S \psi \) be the constrained scoring rule estimate of \( \theta \) for fixed \( \psi \), and let \( \widehat{\psi}_S \) be the \( \psi \)-component of \( \widehat{\theta}_S \).

A profile scoring rule Wald-type statistic for the \( \psi \) component may be defined as

\[ W_{wp}(\psi) := (\widehat{\psi}_S - \psi)^T (G_{\psi \psi})^{-1} (\widehat{\psi}_S - \psi), \]

and it has an asymptotic \( \chi^2_{p_0} \) null distribution. Moreover, using the asymptotic result (Rotnitzky & Jewell, 1990) \( s_\psi(\theta_S \psi)^T N_{p_0} (0, (K_{\psi \psi})^{-1} G_{\psi \psi} (K_{\psi \psi})^{-1}) \), the profile scoring rule score-type statistic \( W^S_{wp}(\psi) := s_\psi(\theta_S \psi)^T K_{\psi \psi} (G_{\psi \psi})^{-1} K_{\psi \psi} s_\psi(\theta_S \psi) \) has an asymptotic \( \chi^2_{p_0} \) null distribution. Finally, the profile scoring rule ratio statistic for \( \psi \), given by

\[ W^S_p(\psi) := 2 \{ S(x, \widehat{\theta}_S \psi) - S(x, \widehat{\theta}_S) \}, \]

is asymptotically distributed as \( \sum_{j=1}^{p_0} v_j Z_j^2 \), where \( v_1, \ldots, v_{p_0} \) are the eigenvalues of \( (K_{\psi \psi})^{-1} G_{\psi \psi} \). This result follows from Kent (1982, Theorem 3.1). When evaluating the eigenvalues of \( (K_{\psi \psi})^{-1} G_{\psi \psi} \), it is possible to replace \( \theta \) by \( \widehat{\theta}_S \psi \).

4.2. Calibration of the scoring rule ratio statistic

Because the asymptotic null distribution of the scoring rule ratio statistic depends both on the statistical model and on the parameter of interest, adjustments to \( W^S(\theta) \) and \( W^S_p(\psi) \) are of interest. These aim for an asymptotic null distribution depending only on the dimension of the parameter of interest. Such adjustments have been discussed in the statistical literature for general pseudo-likelihood functions based on unbiased estimating equations: see, among others, Varin (2008), Pace et al. (2011, 2015), Varin et al. (2011) and references therein.

First, let us consider the scalar parameter case.
Theorem 3. For $p = 1$, the adjusted scoring rule ratio statistic satisfies

$$W_S^*(\theta)_{adj} := \frac{W_S^*(\theta)}{\mu_1} \rightarrow \chi^2_1,$$  \hspace{1cm} (23)

where $\mu_1 = J/K$.

The proof of theorem 3 is based on well-known results in Heritier & Ronchetti (1994) and Pace et al. (2011).

For $p > 1$, simple adjustments of the form (23) for $W_S^*(\theta)$ based on moment conditions can be considered as well. For instance, first-order moment matching (e.g., Rotnitzky & Jewell, 1990, Molenberghs & Verbeke, 2005, Section 9.3.3) gives the adjustment

$$W_S^*(\theta)_{m1} := \frac{W_S^*(\theta)}{\pi},$$  \hspace{1cm} (24)

where $\pi := \sum_{i=1}^p \mu_i / p = \text{tr}(JK^{-1})/p$. A $\chi^2_p$ approximation is used for the null distribution of $W_S^*(\theta)_{m1}$. Matching of moments up to higher order can also be considered, as in Satterthwaite (1946) and Wood (1989); see also Lindsay et al. (2000). Note, however, that the correction (24) to $W_S^*(\theta)$ might be inaccurate because it corrects only the first moment of the distribution and does not recover the full $\chi^2_p$ asymptotic distribution.

For $p > 1$, calibration of $W_S^*(\theta)$ can be based on the following theorem:

Theorem 4. Using the rescaling factor

$$A(\theta) = \frac{s(\theta)^T J^{-1} s(\theta)}{s(\theta)^T K^{-1} s(\theta)},$$  \hspace{1cm} (25)

we have

$$W_S^*(\theta)_{inv} = A(\theta) W_S^*(\theta) \rightarrow \chi^2_p.$$  \hspace{1cm} (26)

The proof of theorem 4 is based on formulae in Pace et al. (2011, 2015), who discuss alternatives to moment-based adjustments for likelihood-type ratio statistics, aiming to obtain a statistic with the usual $\chi^2_p$ asymptotic distribution. More details may be found in the online Supporting Information.

In the situation with nuisance parameters, adjustments to $W_P^*(\psi)$ of the form $W_P^*(\psi)_{m1}$ and $W_P^*(\psi)_{m2}$, analogous to $W_S^*(\theta)_{m1}$ and $W_S^*(\theta)_{m2}$, respectively, can be defined using the eigenvalues $\nu_1, \ldots, \nu_p$ of $(K^\psi\psi)^{-1} G^\psi\psi$, evaluated at $\hat{\theta}_S^\psi$. The extension of $W_S^*(\theta)_{inv}$ to the nuisance parameter case can be obtained following the results in Pace et al. (2011, 2015). We obtain

$$W_P^*(\psi)_{inv} = \frac{W_P^*(\psi)}{s(\hat{\theta}_S^\psi)^T K^\psi\psi (\hat{\theta}_S^\psi)s(\hat{\theta}_S^\psi)} W_P^*(\psi).$$

5. Robustness

The influence function (IF) (e.g., Hampel et al., 1986, Chapter 2) of an estimator measures the effect on it of a small contamination at the point $x$, standardized by the mass of that contamination. The supremum of the IF over the data-space measures the worst influence of such
contamination, so supplying a measure of gross-error sensitivity. A desirable robustness property for a statistical procedure is that the gross-error sensitivity be finite, that is, that the IF be bounded. This is termed \textit{B-robustness}.

From the general theory of \(M\)-estimators (e.g., Huber & Ronchetti, 2009), the IF of the estimator \(\hat{\theta}_S\), the solution of the unbiased estimating equation (12), is given by

\[
\text{IF}(x; s, P) \equiv K^{-1} s(x, \theta_P).
\]

Thus, if the function \(s(x, \theta)\) is, for each \(\theta\), bounded in \(x\), then the corresponding scoring rule estimator \(\hat{\theta}_S\) is \(B\)-robust. Note that, in general, the form of the function \(s(x, \theta)\) depends on the model \(P\) as well as the scoring rule \(S\). As an aside, we note that the IF can also be used to evaluate the asymptotic variance of \(\hat{\theta}_S\):

\[
V = \mathbb{E}_P \{\text{IF}(X; s, P) \text{IF}(X; s, P)^T\}.\]

5.1. Example: robustness of Bregman estimate

A necessary and sufficient condition for \(B\)-robustness of the Bregman estimate, where \(s\) is given by (13) with \(\lambda\) determined by (16), is as follows:

\textbf{Condition 1.} \textit{For all} \(\theta\), \(\lambda(x, \theta)\) is a bounded function of \(x\).

The aforementioned condition inextricably combines properties of the Bregman gauge function \(\alpha\) and the form of the model \(p_{\theta}\). We can also identify a useful set of sufficient conditions for \(B\)-robustness, which handles these ingredients separately. First, we introduce a definition.

\textbf{Definition 1.} We say that a function \(f: \mathbb{R}^+ \rightarrow \mathbb{R}^+\) is locally bounded if \(f(t)\) is bounded on each finite interval \(0 < t < M\).

In this case, \(f(0) = \lim_{t \downarrow 0} f(t)\) (if it exists) must be finite. For our applications, this condition will typically be sufficient.

It now follows that a sufficient condition for \(B\)-robustness of the Bregman estimate is as follows:

\textbf{Condition 2}

\begin{enumerate}
\item The Bregman gauge \(\alpha = \psi''\) is locally bounded, and
\item both \(p_{\theta}(x)\) and \(\nabla_{\theta} p_{\theta}(x)\) are bounded in \(x\), for each \(\theta\).
\end{enumerate}

Note that if Condition 1 or Condition 2 (ii) holds for one parametrization, they equally hold for any other.

The Brier score, with \(\psi(t) \equiv t^2\), satisfies Condition 2 (i) – indeed, \(\alpha(t) \equiv 2\) is bounded on the whole of \((0, \infty)\). Other such ‘totally bounded’ examples include \(\psi(t) \equiv 2t \tan^{-1}(t) - \log(1 + t^2), \text{with} \alpha(t) \equiv 2/(1 + t^2), \text{and} \psi(t) \equiv (1 + t) \log(1 + t), \text{with} \alpha(t) \equiv 1/(1 + t)\). The Tsallis/density power score, with \(\psi(t) \propto t^\gamma \text{ and} \alpha(t) \propto t^{\gamma-2}\) is locally bounded but not totally bounded for \(\gamma > 2\). However, for the log score, with \(\psi(t) \equiv t \log(t), \alpha(t) \equiv 1/t\) is not bounded at 0, so this particular Bregman scoring rule violates the local boundedness Condition 2 (i). And this is reflected in the fact that the maximum likelihood estimator is typically not \(B\)-robust.
For a real location model, with $p_\theta(x) = f(x - \theta)$, the Bregman score will yield a $B$-robust estimator if and only if

**Condition 3** $(d/du)\psi'(f(u)) = \psi''(f(u)) f'(u)$ is bounded.

In particular (cf. Basu et al., 1998), for a real location model, the necessary and sufficient condition that the Tsallis/density power score supply a $B$-robust estimator is that $f(u)^{\gamma-2} f'(u)$ be a bounded function of $u$.

A sufficient condition for Condition 1 to hold is as follows:

**Condition 4**

(i) $\alpha$ is locally bounded, and
(ii) $f'(u)$ is bounded.

Condition 4 (ii) implies Condition 2 (ii), because boundedness of $f'$ implies boundedness of $f$ (a proof of this is given in the online Supporting Information). For instance, Condition 4 (ii) holds for the normal, logistic, Cauchy and extreme value distributions.

For a real scale model, with $p_\theta(x) = \theta f(\theta x) (x > 0)$, the Bregman score yields a $B$-robust estimator if and only if

**Condition 5** $\alpha \{\theta f'(\theta x)\} \{f(\theta x) + \theta x f'(\theta x)\}$ is bounded in $x$ for all $\theta$.

We have the following sufficient condition:

**Condition 6**

(i) $\alpha$ is locally bounded, and
(ii) $f(u)$ and $uf'(u)$ are bounded on $\mathbb{R}^+$.

We again remark that Condition 6 (i) holds for the Brier and Tsallis score, but not for the log score. The log-normal, exponential and Gamma (with $\alpha \geq 1$) densities satisfy Condition 6 (ii). For a general location-scale model, and more generally for a regression-scale model, a sufficient condition for Condition 1 to hold is (i) $\alpha$ is locally bounded and (ii) $f(u)$, $f'(u)$ and $uf'(u)$ are bounded on $\mathbb{R}^+$.

For other studies of robust estimation based on Bregman divergence and on proper scoring rules, see Fujisawa & Eguchi (2008) and Kanamori & Fujisawa (2013).

**6. Examples**

In this section, we provide simulation results to assess coverage probabilities of confidence regions based on the adjustments of the scoring rule ratio statistic $W_S(\theta)$. Three examples are described. The first deals with a multivariate normal distribution, the second with a location-scale model and the third with a linear regression model. The examples are chosen so that we can easily do closed form calculations for both the Tsallis score (5) and the log score. In the two last examples, the focus is on showing the accuracy of the calibration of the scoring rule ratio statistic and on studying the robustness properties of the Tsallis score with respect to classical robust procedures based on $M$-estimators.

Several values of the parameter $\gamma$ of the Tsallis score are explored in order to investigate the interplay between robustness and efficiency. We find that, for small values of $\gamma$, the procedures have strong robustness properties with little loss of asymptotic efficiency relative to maximum likelihood.
Example 2. Equicorrelated normal model. We discuss inference on the correlation coefficient $\rho$ of an equicorrelated multivariate normal distribution. This illustrative example is considered by Cox & Reid (2004).

Let $(X_i : i = 1, \ldots, n)$ be independent realizations of a $q$-variate normal random variable, with standard margins and with $\text{corr}(X_{ir}, X_{is}) = \rho (r, s = 1, \ldots, q, r \neq s)$. Thus, the density function of $X_i$ is

$$p(x_i; \rho) = \frac{\exp \left\{ \frac{-1}{2(1-\rho^2)} \left( \sum_{r=1}^{q} x_{ir}^2 - \frac{\rho^2}{1-\rho^2} \bar{x}_i^2 \right) \right\}}{\sqrt{(2\pi)^q (1-\rho) \{1 + \rho(q-1)\}}} ,$$

where $\bar{x}_i := \sum_{r=1}^{q} x_{ir}/\sqrt{q}$.

Straightforward calculations show that the Tsallis empirical score is $S(x_i, \rho) = \sum_{i=1}^{n} S(x_i, \rho)$, with

$$S(x_i, \rho) = -\gamma p(x_i; \rho)^{\gamma-1} + \frac{(\gamma - 1)}{\gamma q (2\pi)^q (1-\rho)^{\gamma-1} (1-q)(1-q-1)} .$$

In order to assess the quality of the proposed adjustment $W^S(\rho)_{\text{adj}}$ (see Theorem 3) of the scoring rule ratio statistic based on $S(\rho)$, we ran a simulation experiment with $n = 30$, $q = 10$ and $\rho = 0.5$. For comparison, we also consider the pairwise log-likelihood, given by

$$\ell^P(\rho) := -\frac{nq(\gamma - 1)}{4} \log(1-\rho^2) - \frac{q - 1 + \rho}{2(1-\rho^2)} SS_W - \frac{(q - 1)(1 - \rho)}{2(1-\rho^2)} SS_B/q ,$$

where $SS_W := \sum_{i=1}^{n} (x_{ir} - \bar{x}_i)^2$ and $SS_B := q^2 \sum_{i=1}^{n} \bar{x}_i^2$: see Cox & Reid (2004) and Pace et al. (2011), who find that the adjustment of the pairwise likelihood ratio statistics has reasonable coverage properties. Note that the pairwise log-likelihood, as an example of composite log-likelihood, is a special case of a proper scoring rule.

Table 1 reports the empirical coverages of confidence intervals based on several statistics: the full likelihood ratio $W(\rho)$, the Wald statistic from the full model $W_w(\rho)$, the Tsallis Wald statistic $W^S_w(\rho)$ and the adjustment (23) of the Tsallis empirical score likelihood ratio statistic $W^S(\rho)_{\text{adj}}$ for three values of $\gamma$. Finally, the pairwise Wald statistic $W^P_w(\rho)$ and the adjustment (23) of the pairwise likelihood ratio statistic $W^P(\rho)_{\text{adj}}$ are also given. We note that the proposed adjustment (23) of $W^S(\rho)$ shows a reasonable performance in terms of coverage. In particular, when $\gamma$ is small, it proves to be a good competitor of the pairwise likelihood ratio statistic $W^P(\rho)_{\text{adj}}$, with the advantage of using the full likelihood. However, the Tsallis Wald statistic $W^S_w(\rho)$ appears useless.

| $1 - \alpha$ | 0.90 | 0.95 | 0.99 |
|-------------|------|------|------|
| $W(\rho)$   | 0.903 | 0.942 | 0.993 |
| $W_w(\rho)$ | 0.903 | 0.939 | 0.994 |
| $W^S_w(\rho)$, $\gamma = 2$ | 1.000 | 1.000 | 1.000 |
| $W^S_w(\rho)$, $\gamma = 1.5$ | 1.000 | 1.000 | 1.000 |
| $W^S_w(\rho)$, $\gamma = 1.25$ | 1.000 | 1.000 | 1.000 |
| $W^S(\rho)_{\text{adj}}, \gamma = 2$ | 0.789 | 0.830 | 0.883 |
| $W^S(\rho)_{\text{adj}}, \gamma = 1.5$ | 0.849 | 0.906 | 0.958 |
| $W^S(\rho)_{\text{adj}}, \gamma = 1.25$ | 0.886 | 0.937 | 0.982 |
| $W^P(\rho)_{\text{adj}}$ | 0.895 | 0.945 | 0.991 |
| $W^P_w(\rho)$ | 0.892 | 0.938 | 0.989 |
Table 2. Scale and location model. Empirical coverages (based on 5000 replications) of 0.95 confidence regions based on different statistics, under the $N(0, 1)$ model and the $0.95 \cdot N(0, 1) + 0.05 \cdot N(0, 10^2)$ contaminated model, with $\gamma = 2, 1.5, 1.25$

|                   | $N(0, 1)$ |                  | $n = 10$ | $n = 20$ | $n = 30$ | cont. $N(0, 1)$ | $n = 10$ | $n = 20$ | $n = 30$ |
|-------------------|-----------|------------------|----------|----------|----------|-----------------|----------|----------|----------|
| $W(\theta)$      | 0.934     | 0.938            | 0.942    | 0.652    | 0.475    | 0.357           |          |          |          |
| $W^S(\theta)$, $\gamma = 2$ | 0.914     | 0.926            | 0.937    | 0.913    | 0.926    | 0.931           |          |          |          |
| $W^S(\theta)$, $\gamma = 1.5$ | 0.928     | 0.939            | 0.942    | 0.914    | 0.924    | 0.926           |          |          |          |
| $W^S(\theta)$, $\gamma = 1.25$ | 0.948     | 0.947            | 0.945    | 0.898    | 0.908    | 0.908           |          |          |          |
| $W^S(\theta)_{inv}$, $\gamma = 2$ | 0.872     | 0.918            | 0.931    | 0.886    | 0.931    | 0.939           |          |          |          |
| $W^S(\theta)_{inv}$, $\gamma = 1.5$ | 0.912     | 0.936            | 0.942    | 0.914    | 0.937    | 0.931           |          |          |          |
| $W^S(\theta)_{inv}$, $\gamma = 1.25$ | 0.925     | 0.940            | 0.942    | 0.916    | 0.938    | 0.935           |          |          |          |
| $W^S(\theta)_{m1}$, $\gamma = 2$ | 0.981     | 0.967            | 0.962    | 0.978    | 0.959    | 0.952           |          |          |          |
| $W^S(\theta)_{m1}$, $\gamma = 1.5$ | 0.954     | 0.953            | 0.953    | 0.948    | 0.945    | 0.944           |          |          |          |
| $W^S(\theta)_{m1}$, $\gamma = 1.25$ | 0.942     | 0.947            | 0.943    | 0.925    | 0.937    | 0.934           |          |          |          |
| $W^H(\theta)$    | 0.966     | 0.953            | 0.954    | 0.925    | 0.912    | 0.915           |          |          |          |

Example 3. Scale and location model. Let $\theta = (\mu, \sigma)$, where $\mu \in \mathbb{R}$ is a location parameter and $\sigma > 0$ a scale parameter. In this case, we have $p(x; \theta) = p_0((x - \mu)/\sigma)/\sigma$, where $p_0()$ is the standard distribution. The Tsallis empirical score is $S(x, \theta) = \sum_{i=1}^{n} S(x_i, \theta)$, with

$$S(x_i, \theta) = -\gamma p(x_i; \theta)^{(\gamma-1)} + \frac{(\gamma - 1)}{\sigma^{\gamma-1}} \int p_0(x)^\gamma dx,$$

for $i = 1, \ldots, n$.

We ran a simulation experiment, for several values of $n$ and with $\gamma = 2, 1.5, 1.25$, in order to assess the quality of the proposed adjustments of the Tsallis scoring rule ratio statistic based on $S(x, \theta)$. For comparison, we considered also the Wald-type statistic $W^H(\theta)$ from the well-known Huber location-scale $M$-estimator (Hampel et al., 1986, Section 4.2).

Table 2 gives the results of a Monte Carlo experiment that compares confidence regions for $\theta$ based on the full likelihood ratio $W(\theta)$, the Tsallis Wald statistic $W^S(\theta)$ and the adjustments (23) and (24) of the Tsallis empirical score likelihood ratio statistic, and the Huber Wald statistic $W^H(\theta)$, when the central model is normal. Data are generated from two different distributions: the $N(0, 1)$ model and the contaminated model $0.95 \cdot N(0, 1) + 0.05 \cdot N(0, 10^2)$. The value $\gamma = 1.25$ gives approximately 0.95 efficiency under the normal model (Basu et al., 1998). We note that the proposed adjustments of $W^S(\theta)$ show fairly good performance in terms of coverage, both under the central model and under the contaminated model. However, the Tsallis Wald statistic $W^S(\theta)$ and the Huber Wald statistic $W^H(\theta)$ exhibit poor coverage under the contaminated model.

Example 4. Linear regression model. Consider the linear regression model

$$y = X \beta + \sigma \varepsilon,$$

where $X$ is a fixed $n \times p$ matrix, $\beta \in \mathbb{R}^p$ ($p \geq 1$) an unknown regression coefficient, $\sigma > 0$ a scale parameter and $\varepsilon$ an $n$-dimensional vector of random errors from a standard normal distribution. We take $\sigma = 1$ as known. The Tsallis empirical score is $S(y, \beta) = \sum_{i=1}^{n} S(y_i, \beta)$, with

$$S(y_i, \beta) = -\frac{\gamma}{(\sqrt{2\pi})^{\gamma-1}} \exp \left\{ -\frac{\gamma - 1}{2} (y_i - x_i^T \beta)^2 \right\} + (\gamma - 1) \int \phi(x)^\gamma dx,$$

where $x_i^T$ is the $i$-th row of $X$ and $\phi(\cdot)$ is the standard normal density.
Table 3. Linear regression model. Empirical coverages (based on 5000 replications) of 0.95 confidence regions based on different statistics, under the $N(0,1)$ and the $0.95 \cdot N(0,1) + 0.05 \cdot N(0,10^2)$ models, with $\gamma = 2, 1.5, 1.25$

|        | $N(0,1)$ |           |           | cont. $N(0,1)$ |
|--------|----------|-----------|-----------|---------------|
|        | $n = 15$ | $n = 30$  | $n = 50$  | $n = 15$      | $n = 30$ | $n = 50$ |
| $W(\beta)$ | 0.950    | 0.954     | 0.951     | 0.605         | 0.518   | 0.417   |
| $W^S_2(\beta), \gamma = 2$ | 1.000    | 1.000     | 1.000     | 1.000         | 1.000   | 1.000   |
| $W^S_2(\beta), \gamma = 1.5$ | 0.999    | 0.998     | 0.998     | 0.988         | 0.998   | 1.000   |
| $W^S_2(\beta), \gamma = 1.25$ | 0.966    | 0.927     | 0.975     | 0.884         | 0.948   | 0.976   |
| $W^S_2(\beta), \gamma = 2$ | 0.958    | 0.955     | 0.955     | 0.963         | 0.943   | 0.940   |
| $W^S_2(\beta), \gamma = 1.5$ | 0.949    | 0.954     | 0.952     | 0.948         | 0.948   | 0.939   |
| $W^S_2(\beta), \gamma = 1.25$ | 0.949    | 0.952     | 0.952     | 0.940         | 0.941   | 0.934   |
| $W^S_2(\beta), \gamma = 2$ | 0.958    | 0.955     | 0.955     | 0.963         | 0.943   | 0.940   |
| $W^S_2(\beta), \gamma = 1.5$ | 0.949    | 0.954     | 0.952     | 0.948         | 0.948   | 0.940   |
| $W^S_2(\beta), \gamma = 1.25$ | 0.949    | 0.952     | 0.952     | 0.940         | 0.941   | 0.934   |
| $W^H(\beta)$ | 0.944    | 0.954     | 0.952     | 0.876         | 0.910   | 0.899   |

In order to assess the quality of the proposed adjustments of the Tsallis scoring rule ratio statistic based on $S(y, \beta)$, we ran a simulation experiment with $p = 3$ and for several values of $n$, with $\lambda = 2, 1.5, 1.25$. For comparison, we considered also the well-known Huber regression $M$-estimator (Hampel et al., 1986). As in the previous example, for this estimator we consider the Wald-type statistic $W^H_w(\beta)$.

Our specific model is as follows. In (28), all entries of the first column of $X$ are 1, those of the second column are generated as independent standard normal variables, $z_1, \ldots, z_n$, while the third column consists of the integers from 1 to $n$. The model is $y_i = \beta_1 + \beta_2 z_i + \beta_3 i + \varepsilon_i$, and the true parameter is $\beta = (1, 2, 3)$. As for Example 3, $\varepsilon_1, \ldots, \varepsilon_n$ were generated from one of two distributions: the $N(0,1)$ model or the contaminated model $0.95 \cdot N(0,1) + 0.05 \cdot N(0,10^2)$.

Table 3 compares confidence regions for $\beta$ based on the full likelihood ratio $W(\beta)$, the Tsallis Wald statistic $W^S_2(\beta)$ and the adjustments (23) and (24) of the Tsallis empirical score likelihood ratio statistic, and the Huber Wald statistic $W^H_w(\beta)$, when the central model is normal. We note that the proposed adjustments of $W^S(\beta)$ show a satisfactory performance in terms of coverage, in particular, when $\gamma$ is small, both under the central model and under the contaminated model. However, the Tsallis Wald statistic $W^S_w(\beta)$ and the Huber Wald statistic $W^H_w(\beta)$ have poor coverage under the contaminated model.

7. Concluding remarks

We have presented a general approach to parametric estimation theory, based on replacing the full log-likelihood by a proper scoring rule. This includes well-studied cases such as full, pseudo, composite, pairwise log-likelihoods, as well as a very wide variety of other cases, not directly or indirectly related to likelihood at all. Under smoothness conditions, any proper scoring rule can be applied to any statistical model and delivers an associated $M$-estimator. While this may lose efficiency in comparison with full likelihood methods, it can exhibit improved robustness, or computational advantages. In Section 5, we identified some common situations where use of an appropriate scoring rule achieves $B$-robustness.

We can use a scoring rule estimator to construct hypothesis tests and confidence intervals. In addition to obtaining analogues of the Wald and score test statistics, which are available for general $M$-estimators, when basing inference on a scoring rule we also have an analogue of the Wilks (log-likelihood ratio) statistic. The distributions of these analogues differ from those...
based on the full likelihood, and we have considered adjustments to bring them more into line. The simulation studies in Section 6 indicate that adjusted scoring rule likelihood ratio-type statistics yield confidence regions whose coverage properties are satisfactory. Both the moment-matching correction and the correction given in Theorem 4 perform well and are preferred to the use of Wald-type statistics.

Other approaches, invoking the theory of unbiased estimating functions, have been suggested to produce likelihood ratio-type statistics that allow reference to the usual asymptotic chi-squared distribution (Lunardon et al., 2013; Lunardon & Ronchetti, 2014). However, in contrast to the proposed likelihood ratio-type statistic, which directly adjusts the scoring rule itself, these tests are derived from an unbiased estimating function for which a convex function to be used in the likelihood ratio-type statistics is usually unavailable. Thus, they may be more cumbersome to compute.

In more realistic applications, analytic expressions for the required terms $K$ and $J$ may be unavailable, and numerical evaluation would then seem to offer the most straightforward solution. This issue is under investigation.

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