ON BOSE-CONDENSATION OF WAVE-PACKETS
IN HEAVY ION COLLISIONS

T. CSÖRGŐ, J. ZIMÁNYI
MTA KFKI RMKI,
H-1525 Budapest 114, POB 49, Hungary
E-mails: jzimanyi@sunserv.kfki.hu, csorgo@sunserv.kfki.hu

A recently obtained exact analytic solution to the wave-packet version of the pion-laser model is presented. In the rare gas limit, a thermal behaviour is found while the dense gas limiting case corresponds to Bose-Einstein condensation of wave-packets to the wave-packet mode with the smallest possible energy.

1 Introduction
The study of the statistical properties of quantum systems has a long history with important recent developments. In high energy physics quantum statistical correlations are studied in order to infer the space-time dimensions of the intermediate state formed in elementary particle reactions. In high energy heavy ion collisions hundreds of bosons are created. The correct theoretical description of their correlations is difficult. In this conference contribution we present a contradiction free treatment of multi-particle Bose-Einstein correlations for arbitrary large number of particles.

2 Model Assumptions:
A model system is described as

\[ \hat{\rho} = \sum_{n=0}^{\infty} p_n \hat{\rho}_n, \tag{1} \]

the density matrix of the whole system is normalized to one. Here \( \hat{\rho}_n \) is the density matrix for events with fixed particle number \( n \), which is normalized also to one. The probability for such an event is \( p_n \).

The multiplicity distribution is described by the set \( \{p_n\}_{n=0}^{\infty} \).

The density matrix of a system with a fixed number of boson wave packets has the form

\[ \hat{\rho}_n = \int d\alpha_1...d\alpha_n \rho_n(\alpha_1,...,\alpha_n) |\alpha_1,...,\alpha_n\rangle \langle \alpha_1,...,\alpha_n|, \tag{2} \]

where \( |\alpha_1,...,\alpha_n\rangle \) denote properly normalized \( n \)-particle wave-packet boson states.
The wave packet creation operator is given as
\[
\alpha_i^\dagger = \int \frac{d^3p}{(\pi \sigma^2)^{\frac{3}{4}}} e^{-\frac{(p-\pi_i)^2}{2\sigma^2}} e^{-i\pi_i(p-\pi_i) + i\omega(t-t_i)} \hat{a}^\dagger(p).
\]

The commutator
\[
[\alpha_i, \alpha_j^\dagger] = \langle \alpha_i | \alpha_j \rangle
\]
vanishes only in the case, when the wave packets do not overlap.

Here \( \alpha_i = (\xi_i, \pi_i, \sigma_i, t_i) \) refers to the center in space, in momentum space, the width in momentum space and the production time, respectively.

For simplicity we assume that \( \alpha_i = (\pi_i, \xi_i, \sigma, t_0) \).

We call the attention to the fact that although one cannot attribute exactly defined values for space and momentum at the same time, one can define precisely the \( \pi_i, \xi_i \) parameters.

The \( n \) boson states, normalized to unity, are given as
\[
| \alpha_1, \ldots, \alpha_n \rangle = \frac{1}{\sqrt{\sum_{\sigma(n)} \prod_{i=1}^n \langle \alpha_i | \alpha_{\sigma_i} \rangle}} \alpha_1^\dagger \ldots \alpha_n^\dagger | 0 \rangle.
\]

Here \( \sigma(n) \) denotes the set of all the permutations of the indexes \( \{1, 2, \ldots, n\} \) and the subscript \( \sigma_i \) denotes the index that replaces the index \( i \) in a given permutation from \( \sigma(n) \). The normalization factor contains a sum of \( n! \) term. These terms contain \( n \) different \( \alpha_i \) parameters. Thus the further calculation with these normalized states seems to be extremely difficult.

3 A New Type of Density Matrix

There is one special density matrix, however, for which one can overcome the difficulty, related to the non-vanishing overlap of many hundreds of wave-packets, even in an explicit analytical manner. This density matrix is the product uncorrelated single particle matrices multiplied with a correlation factor, related to the induced emission:
\[
\rho_n(\alpha_1, \ldots, \alpha_n) = \frac{1}{N(n)} \left( \prod_{i=1}^n \rho_1(\alpha_i) \right) \left( \sum_{\sigma(n)} \prod_{k=1}^n \langle \alpha_k | \alpha_{\sigma_k} \rangle \right).
\]
Normalization to one yields $\mathcal{N}(n)$. The weight factors describe induced emission:

$$\frac{\rho_n(\alpha_1, \ldots, \alpha_n)}{\prod_{j=1}^{n} \rho_1(\alpha_j)} = \frac{\sum \prod_{k=1}^{n} \langle \alpha_k | \sigma_k \rangle}{\mathcal{N}(n)} \quad (\text{overlap}).$$  \hspace{1cm} (7)

This is maximal if all are emitted with same $\alpha_1$:

$$\frac{\rho_n(\alpha_1, \ldots, \alpha_1)}{[\rho_1(\alpha_1)]^n} = \frac{n!}{\mathcal{N}(n)}, \quad (\text{full overlap}),$$  \hspace{1cm} (8)

and minimal if the overlap is negligible,

$$\frac{\rho_n(\alpha_1, \ldots, \alpha_n)}{\prod_{j=1}^{n} \rho_1(\alpha_j)} = \frac{1}{\mathcal{N}(n)} \quad (\text{no overlap}).$$  \hspace{1cm} (9)

Thus we have wildly fluctuating weight. E.g. for 800 pions the induced emission weight fluctuates between $[1, 800!] \simeq [1, 10^{1977}]$. With this special density matrix one can proceed with the calculations.

3.1 Single-Particle Density Matrix:

For the sake of simplicity we assume a factorizable Gaussian form for the distribution of the parameters of the single-particle states:

$$\rho_1(\alpha) = \rho_x(\xi) \rho_p(\pi) \delta(t - t_0),$$

$$\rho_x(\xi) = \frac{1}{(2\pi R^2)^{\frac{3}{2}}} \exp(-\xi^2/(2R^2)),$$

$$\rho_p(\pi) = \frac{1}{(2\pi mT)^{\frac{3}{2}}} \exp(-\pi^2/(2mT)).$$  \hspace{1cm} (10)

These expressions are given in the frame where we have a non-expanding static source at rest.

A multiplicity distribution when Bose-Einstein effects are switched off (denoted by $p_n^{(0)}$), is a FREE choice in the model. We assume independent emission,

$$p_n^{(0)} = \frac{n_0^n}{n!} \exp(-n_0).$$  \hspace{1cm} (11)

This completes the specification of the model.
4 Solution of Recurrences:

The probability of finding events with multiplicity \( n \), as well as the single-particle and the two-particle momentum distribution in such events is given as

\[
N^{(n)}_1(k) = \sum_{i=1}^{n} \frac{p_{n-i}}{p_n} G_i(k, k),
\]

(12)

\[
N^{(n)}_2(k_1, k_2) = \sum_{i=2}^{n} \sum_{m=1}^{i-1} \frac{p_{n-i}}{p_n} \left[ G_m(k_1, k_1) G_{i-m}(k_2, k_2) + G_m(k_1, k_2) G_{i-m}(k_2, k_1) \right].
\]

(13)

In ref\(^3\) recurrence relations were given for the construction of the functions \( G_m(k_1, k_2) \). In refs\(^1, 2\) an explicit analytic form was obtained for these functions. We present now this method.

To arrive to this solution we introduce the following auxiliary quantities:

\[
\gamma_\pm = \frac{1}{2} \left( 1 + x \pm \sqrt{1 + 2x} \right), \quad x = R^2 \sigma^2.
\]

(14)

Using the notation

\[
\sigma_T = 2mT_p, \quad T_p = T + \frac{\sigma^2}{2m}, \quad R_p^2 = R^2 + \frac{mT}{\sigma^2 \sigma_T^2},
\]

(15)

we arrive to the formulae of the plane-wave pion laser model of Pratt, ref\(^3\), if we replace \( R \) in ref\(^3\) with \( R_p \) and \( T \) in ref\(^3\) with \( T_p \). The general analytical solution of the model is given through the generating function of \( p_n \),

\[
G(z) = \sum_{n=0}^{\infty} p_n z^n = \exp \left( \sum_{n=1}^{\infty} C_n (z^n - 1) \right),
\]

(16)

where \( C_n \) is the combinant\(^4, 5\) of order \( n \),

\[
C_n = \frac{n_0^n}{n} \left[ \gamma_+^n - \gamma_-^n \right]^{-3},
\]

(17)

and the general analytic solution for the functions \( G_n(k_1, k_2) \) reads as:

\[
G_n(k_1, k_2) = j_n e^{-\frac{b_n}{2}} \left[ \left( \gamma_+^n k_1 - \gamma_-^n k_2 \right)^2 + \left( \gamma_+^n k_2 - \gamma_-^n k_1 \right)^2 \right],
\]

(18)

\[
j_n = n_0^n \left[ \frac{b_n}{n} \right]^{\frac{1}{2}}, \quad b_n = \frac{1}{\sigma^2_T} \gamma_+^n - \gamma_-^n.
\]

(19)
The detailed proof that the analytic solution to the multi-particle wave-packet model is indeed given by the above equations is described in refs. 1, 2.

With our analytic solution we decreased the algorithmic complexity of the problem of $n$ symmetrized boson system from the original $n!$ to 1.

5 Multiplicity Distribution:

The mean and the second factorial moments are defined as follows

$$\langle n \rangle = \sum_n n p_n = \sum_{i=1}^{\infty} iC_i, \quad (20)$$

$$\langle n(n-1) \rangle = \sum_n n(n-1) p_n = \langle n \rangle^2 + \sum_{i=2}^{\infty} i(i-1) C_i. \quad (21)$$

The large $n$ behavior of the multiplicity distribution, $p_n$, depends on $n_0/\gamma^2$. This multiplicity distribution has an interesting property. To see this, we define the quantity $n_c$ as

$$n_c = \gamma^2 = \left( \frac{1 + x + \sqrt{1 + 2x}}{2} \right)^{\frac{3}{2}}. \quad (22)$$

One can show that $\langle n \rangle$ is finite, if $n_0 < n_c$ and $\langle n \rangle$ is infinite, if $n_0 > n_c$. Hence $n_c$ is a critical value for the parameter $n_0$.

6 Solution for the Inclusive Distributions:

In the previous sections we obtained results the multiplicity distribution and for exclusive momentum distributions. To obtain the inclusive distribution we introduce the auxiliary quantity

$$G(k_1, k_2) = \sum_{i=1}^{\infty} G_i(k_1, k_2). \quad (23)$$

Higher order Bose-Einstein symmetrization effects are negligible, if the first term dominates the above infinite sum, i.e. if $G(k_1, k_2) = G_1(k_1, k_2)$.)

Now the averaging over the exclusive distributions can be performed,

$$N_1(k) = \sum_{n=1}^{\infty} p_n N_1^{(n)}(k) = G(k, k), \quad (24)$$
and the two-particle inclusive correlation functions can be evaluated as

\[ C_2(k_1, k_2) = \frac{N_2(k_1, k_2)}{N_1(k_1) N_1(k_2)} = 1 + \frac{G(k_1, k_2) G(k_2, k_1)}{G(k_1, k_1) G(k_2, k_2)}. \]  

(25)

This is an exact result, obtained without the random phase approximation and without assuming that the number of sources is large. However, this result is valid only when Bose-Einstein condensation does not play a role, i.e. when \( n_0 < n_c = \gamma_3^{3/2} \).

7 Rare gas limiting case:

For large source sizes or large effective temperatures, \( x \gg 1 \) we have:

\[ G_n(k_1, k_2) \propto \left( \frac{2}{x} \right)^{\frac{3}{2}(n-1)} \exp \left[ -\frac{n}{2\sigma_T^2} (k_1^2 + k_2^2) - \frac{R_p^2}{2n} (k_1 - k_2)^2 \right], \]  

(26)

\[ C_n = \frac{n_0^n}{n^3} \left( \frac{2}{x} \right)^{\frac{3}{2}(1-n)}. \]  

(27)

From this equation we can see that the effective temperatures and the effective radii are decreased by a factor of \( 1/n \) for \( n \)-th order symmetrization effects.

The multiplicity distribution is shifted to high multiplicities:

\[ p_n = \frac{n_0^n}{n!} \exp(-n_0) \left[ 1 + \frac{n(n-1) - n_0^2}{2(2x)^{3/2}} \right]. \]  

(28)

The momentum distribution is obtained in this approximation by an expansion in the small parameter \( \epsilon = (2/x)^{3/2} \). In this approximation we get

\[ N_1(k) = \frac{n_0}{(\pi\sigma_T^2)^{3/2}} \exp \left( -\frac{k^2}{\sigma_T^2} \right) + \frac{n_0^2}{(\pi\sigma_T^2 x)^{3/2}} \exp \left( -\frac{2}{\sigma_T^2} k^2 \right) \]  

(29)

\[ N_1^{(n)}(k) = \frac{n}{(\pi\sigma_T^2)^{3/2}} \exp \left( -\frac{k^2}{\sigma_T^2} \right) \left[ 1 + \frac{(n-1)}{(2x)^{3/2}} \left( 2^\frac{3}{2} \exp \left( -\frac{k^2}{\sigma_T^2} \right) - 1 \right) \right]. \]  

(30)

The single-particle inclusive and exclusive momentum distributions are enhanced at low momentum. The influence of the wave packet size on the pion multiplicity distribution is shown in Fig.1. The effect of the wave packet width on the momentum distribution is displayed on Fig.2.
As the size of the wave packets is changed, the overlap changes and stimulated emission results in larger multiplicities.

Stimulated emission results in enhancement of pions in the low momentum modes.
8 Correlation functions:

In the highly condensed $x << 1$ and $n_0 >> n_c$ Bose gas limit a kind of lasing behaviour and an optically coherent behaviour is obtained, which is characterized by the disappearance of the bump in the correlation function:

$$C_2(k_1, k_2) = 1.$$ 

On the other hand, in the rare gas limit we get

$$C_2(k_1, k_2) = 1 + \exp \left(-R_p^2 \Delta k^2 \right).$$ (31)

8.1 Exclusive Correlation Functions:

In a highly condensed limiting case, $x << 1$ and $n_0 >> n_c$, the exclusive and inclusive correlation functions coincide, $C_2(k_1, k_2) = 1$.

In the rare gas limit, $x >> 1$, the exclusive and inclusive correlations are different, and the exclusive correlation has the form

$$C_2^{(n)}(k_1, k_2) = \frac{n^2}{n(n-1)} \frac{N_2^{(n)}(k_1, k_2)}{N_1^{(n)}(k_1) N_1^{(n)}(k_2)} = 1 + \lambda_k \exp \left(-R_{K,s}^2 \Delta k_s^2 - R_{K,o}^2 \Delta k_o^2 \right),$$ (32)

where $K = 0.5(k_1 + k_2)$, $\Delta k_s = \Delta k - K(\Delta k \cdot K)/(K \cdot K)$ and $\Delta k_o = K(\Delta k \cdot K)/(K \cdot K)$.

The momentum dependent parameters are given as follows:

$$\lambda_k = 1 + \frac{2}{(2x)^\frac{5}{2}} \left[1 - 2^{(5/2)} \exp \left(-\frac{K^2}{\sigma^2_T} \right) \right],$$ (33)

$$R_{K,s}^2 = R_p^2 + \frac{1}{(2x)^\frac{5}{2}} \left[R_p^2 - \sqrt{2} \exp \left(-\frac{K^2}{\sigma^2_T} \right) \left((n+2)R_p^2 + \frac{2}{\sigma^2_T} \right) \right],$$ (34)

$$R_{K,o}^2 = R_{K,s}^2 + \frac{n}{x^2} \frac{K^2}{\sigma^2_T} \exp \left(-\frac{K^2}{\sigma^2_T} \right).$$ (35)

The dependence of the parameters of the correlation functions on the mean momentum of the two pions, $K$, is shown on Fig.2. The influence of the wave packet size on the two particle correlation is illustrated on Fig.3. (These limiting cases are discussed in more details in ref[1]).
9 Highlights:

In this paper we presented a consequent quantum mechanical description of the correlations caused by the multi-particle Bose-Einstein symmetrization of a system of large number of bosons. We introduced the overlapping multi-particle wave-packet density matrix describing stimulated emission. We reduced the algorithmic complexity of the description of the \( n \) boson states from \( n! \) to 1 by obtaining an explicit analytical solution for the problem.

We have shown, that the radius and intercept parameters depend on the mean momentum of the two pions even for static sources, due to multi-particle symmetrization effects.

We have found an enhancement of the wave-packets in the low momentum modes, due to multi-particle Bose-Einstein symmetrization. When all pions are in the lowest momentum state, the system may be interpreted as a pion-laser.
Multi-Particle Symmetrization Effects

![Graph showing momentum-dependent reduction of the intercept parameter $\lambda_K$, the side-wards and the outwards radius parameters, $R_{K,s}$ and $R_{K,o}$ from their static values of 1 and $R_p$, respectively.](image)

Figure 4: Momentum-dependent reduction of the intercept parameter $\lambda_K$, the side-wards and the outwards radius parameters, $R_{K,s}$ and $R_{K,o}$ from their static values of 1 and $R_p$, respectively.

Acknowledgments

The present work was supported in part by the US - Hungarian Joint Fund MAKA 652/1998, by the National Scientific Research Fund (OTKA, Hungary) Grant 024094, and by NWO (Netherland) - OTKA grant N25487. The authors thank these sponsoring organizations for their aid.

References

1. J. Zimányi and T. Csörgő, [http://xxx.lanl.gov/hep-ph/9705432](http://xxx.lanl.gov/hep-ph/9705432)
2. T. Csörgő and J. Zimányi, Phys. Rev. Lett. **80** (1998) 916 - 918
3. S. Pratt, Phys. Lett. **B301** (1993) 159 ; W. Q. Chao, C. S. Gao and Q. H. Zhang, J. Phys. G. Nucl. Part. Phys. **21** (1995) 847
4. M. Gyulassy and S. K. Kaufmann, Phys. Rev. Lett. **40** (1978) 298
5. S. Hegyi, Phys. Lett. **B309** (1993) 443-450