1. Introduction

Recently [2,3] we found necessary and sufficient conditions for the convergence at a preassigned point of the spherical partial sums of the Fourier integral in a class of piecewise smooth functions in Euclidean space. These yield elementary examples of divergent Fourier integrals in three dimensions and higher. Meanwhile, several years ago Gottlieb and Orsag[1] observed that in two dimensions we may expect slower convergence at certain points, specifically for Fourier-Bessel series of radial functions.

In this paper we investigate the rate of convergence of the spherical partial sums of the Fourier integral for a class of piecewise smooth functions. The basic result is an asymptotic expansion which allows us to read off the rate of convergence at a pre-assigned point.

2. Statement of results

$\mathbb{R}^2$ denotes the Euclidean plane and $f \in L^1(\mathbb{R}^2)$ is an integrable function with Fourier transform and spherical partial sum denoted by

$$\hat{f}(\xi_1, \xi_2) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} f(x_1, x_2) e^{-i(\xi_1 x_1 + \xi_2 x_2)} \, dx_1 \, dx_2,$$

$$f_M(x_1, x_2) = \int_{\xi_1^2 + \xi_2^2 \leq M^2} \hat{f}(\xi_1, \xi_2) e^{i(\xi_1 x_1 + \xi_2 x_2)} \, d\xi_1 \, d\xi_2.$$

The spherical mean value with respect to $x = (x_1, x_2)$ is defined by

$$\bar{f}_x(r) = \frac{1}{2\pi} \int_0^{2\pi} f(x_1 + r \cos \theta, x_2 + r \sin \theta) \, d\theta.$$

We say that $f$ is piecewise smooth of class $C^k$ with respect to $x \in \mathbb{R}^2$ iff the mapping $r \to \bar{f}_x(r)$ is piecewise $C^k$ and vanishes for $r \geq r(x)$. Such a function has left and right limits at all points, together with the requisite derivatives. These will agree except at a finite set of points which we denote by $a_i, 1 \leq i \leq K$. The jumps are denoted by

$$\delta \bar{f}_x(a_i) := \bar{f}_x(a_i + 0) - \bar{f}_x(a_i - 0),$$

with a corresponding notation for the jumps of the derivatives.

**Theorem** Suppose that $f$ is piecewise smooth of class $C^2$ with respect to $x \in \mathbb{R}^2$. Then we have the asymptotic $(M \to \infty)$ formula
\[ f_M(x) - \bar{f}_x(0 + 0) = \sqrt{\frac{2}{M\pi}} \sum_{1 \leq i \leq K} \delta \bar{f}_x(a_i) \cos(Ma_i - \pi/4) \]
\[ + \frac{1}{M} \bar{f}'_x(0 + 0) + o(1/M). \]

**Corollary** If the spherical mean value with respect to \( x \) is continuous, then we have
\[ f_M(x) - \bar{f}_x(0 + 0) = O(1/M), \quad M \to \infty. \]

Otherwise
\[ -\infty < \liminf_M \sqrt{M}[f_M(x) - \bar{f}_x(0 + 0)] < \limsup_M \sqrt{M}[f_M(x) - \bar{f}_x(0 + 0)] < \infty. \]

**Example** If we choose \( f(x) = 1_{[0,R]}(|x|) \) as the indicator function of a disc of radius \( R \) centered at \((0,0)\), then \( \delta \bar{f}_x(r) = 0 \) unless \( x = 0, r = R \), where \( \delta \bar{f}_0(R) = -1 \); thus
\[ f_M(0,0) = 1 - \sqrt{\frac{2}{M\pi R}} \cos(MR - \pi/4) + O(1/M), \]
\[ f_M(x_1, x_2) = 1 + O(1/M) \quad 0 < x_1^2 + x_2^2 < R^2, \]
\[ f_M(x_1, x_2) = \frac{1}{2} + O(1/M) \quad x_1^2 + x_2^2 = R^2, \]
\[ f_M(x_1, x_2) = O(1/M) \quad x_1^2 + x_2^2 > R^2. \]

The speed of convergence at the center is strictly slower than at all other points.

3. **Proofs**

The spherical partial sum is written directly in terms of the spherical mean value by writing [3]
\[ f_M(x) = \int_0^a MJ_1(Mr) \bar{f}_x(r), dr \quad a = r(x). \]

From the identities for Bessel functions, we have
\[ MJ_1(Mr) = -\frac{d}{dr} J_0(Mr), \]
so that we may integrate by parts:
\[ f_M(x) = -\int_0^a \bar{f}_x(r) \frac{d}{dr} J_0(Mr) dr \]
\[ = \bar{f}_x(0 + 0) + \sum_{1 \leq i \leq K} \delta \bar{f}_x(a_i) J_0(Ma_i) + \int_0^a J_0(Mr) \bar{f}'_x(r) dr. \]
The Bessel function has the asymptotic behavior $J_0(x) = \sqrt{2/(\pi x)}[\cos(x - \pi/4) + O(1/x)]$, $x \to \infty$, which yields the first term. To handle the integral term, we define $K(x) = \int_0^x J_0(t) \, dt$. From the asymptotic behavior of $J_0$ it follows that $K(x)$ is bounded with a limit $K(\infty) = \int_0^\infty J_0(t) \, dt = 1$ (see the proof in the appendix below). Therefore we may integrate by parts once again to obtain

$$
\int_0^a J_0(Mr) \tilde{f}_x(r) \, dr = \frac{1}{M} \int_0^a \tilde{f}_x(r) \frac{d}{dr} K(Mr) \, dr
= -\frac{1}{M} \left( \sum_{1 \leq i \leq K} \delta \tilde{f}_x(a_i) K(Ma_i) + \int_0^a \tilde{f}_x''(r) K(Mr) \, dr \right).
$$

When $M \to \infty$ the first set of terms within the parenthesis clearly converges to $K(\infty) \sum_{1 \leq i \leq K} \delta \tilde{f}_x(a_i)$; the second term converges, by the dominated convergence theorem, to $K(\infty) \int_0^a \tilde{f}_x''(r) \, dr$. Applying the fundamental theorem of calculus on each sub-interval $(a_{i-1}, a_i)$ and collecting terms gives the stated form. The proof is complete.

4. Comparison with one-dimensional convergence

In order to put the above in some perspective, we outline here a treatment of the corresponding questions for Fourier integrals in one variable, where we find a generic rate of convergence of $1/M$, even in the absence of continuity.

An integrable function $f(x), x \in \mathbb{R}^1$ has the Fourier transform

$$
\hat{f}(\xi) = (2\pi)^{-1} \int_{\mathbb{R}^1} f(x) e^{-i\xi x} \, dx
$$

with spherical partial sum $f_M(x) = \int_{-M}^M \hat{f}(\xi)e^{i\xi x} \, d\xi$. This can be expressed in terms of the symmetric average

$$
\bar{f}_M(x) = \frac{1}{2} [f(x + t) + f(x - t)]
$$

as

$$
f_M(x) = \frac{2}{\pi} \int_0^\infty \sin Mt \frac{\bar{f}_x(t)}{t} \, dt.
$$

Using this representation, we now state and prove

**Proposition.** Suppose that $t \to \bar{f}_x(t)$ is piecewise $C^2$ and zero for $t > a$. Then we have the asymptotic formula

$$
f_M(x) - \bar{f}_x(0 + 0) = \frac{2}{M\pi} [\bar{f}_x'(0 + 0) + \sum_{1 \leq i \leq K} \delta \tilde{f}_x(a_i) \frac{\cos Ma_i}{a_i}] + o(1/M).
$$

**Proof.** We write

$$
f_M(x) - \bar{f}_x(0 + 0) = \frac{2}{\pi} \int_0^a \frac{\bar{f}_x(t) - \bar{f}_x(0 + 0)}{t} \sin Mt \, dt - \frac{2}{\pi} \bar{f}_x(0 + 0) \int_a^\infty \frac{\sin Mt}{t} \, dt,
$$

$$
\int_0^a J_0(Mr) \tilde{f}_x(r) dr = \frac{1}{M} \int_0^a \tilde{f}_x(r) \frac{d}{dr} K(Mr) dr.
$$
where the last integral is conditionally convergent. In the first integral we write \( g(t) = \frac{\bar{f}_x(t) - \bar{f}_x(0+0)}{t} \) and integrate by parts in the form
\[
\int_0^a g(t) \sin Mt \, dt = -(1/M) \int_0^a g(t) d[\cos Mt]
\]
\[
= (1/M)[g(0) + \sum_{1 \leq i \leq K} \delta g(a_i) \cos Ma_i + \int_0^a g'(t) \cos Mt \, dt].
\]
But \( \delta g(a_i) = \frac{\delta \bar{f}_x(a_i)}{a_i} \).

The second integral can also be integrated by parts to obtain the asymptotic form
\[
\int_a^\infty \frac{\sin Mt}{t} \, dt = \frac{\cos Ma}{Ma} + O(1/M^2).
\]
Combining these produces the stated result.

**Remark.** A parallel result holds for Fourier series on the circle \(-\pi < x < \pi\). In this case the spherical partial sum of \( 2M = 2n + 1 \) terms is written
\[
f_M(x) = \frac{1}{\pi} \int_0^\pi \sin Mt \frac{\bar{f}_x(t)}{2 \sin(t/2)} \, dt.
\]
The above computations can be replicated with the result
\[
f_M(x) - \bar{f}_x(0 + 0) = \frac{2}{M\pi} [\bar{f}_x'(0 + 0) + \sum_{1 \leq i \leq K} \delta \bar{f}_x(a_i) \frac{\cos Ma_i}{2 \sin(a_i/2)}] + o(1/M)
\]

**Conclusion.** We note that the one-dimensional result (for both Fourier series and integrals) differs from the two-dimensional result in the appearance of the derivative term \( \bar{f}_x'(0 + 0) \) at the very first level of the asymptotic analysis.

5. **Appendix. Computation of \( K(\infty) \)**

The Bessel function can be written
\[
J_0(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{\cos xt}{\sqrt{(1 - t^2)}} \, dt.
\]
Thus by the Fubini theorem
\[
\int_0^M J_0(x) \, dx = \frac{1}{\pi} \int_{-1}^{1} \sin Mt \frac{1}{\sqrt{(1 - t^2)}} \, dt.
\]
This is the partial Fourier integral of the integrable function \((1 - t^2)^{-1/2}\) at \( t = 0 \).

When \( M \to \infty \) we can appeal to the convergence of one-dimensional Fourier integrals to conclude
\[
\lim_{M \to \infty} \int_0^M J_0(x) \, dx = 1.
\]
References

1. D. Gottlieb and S. Orsag, *Numerical Analysis of Spectral Methods*, SIAM, 1977.

2. M. Pinsky, Fourier inversion for piecewise smooth functions in several variables, Proceedings of the American Mathematical Society, 118(1993), 903-910.

3. M. Pinsky, Pointwise Fourier inversion and related eigenfunction expansions, Communications of Pure and Applied Mathematics, 47(1994), 653-681.