Estimating the mean of a heavy-tailed distribution under random censoring

Louiza Soltane, Djamal Meraghni, Abdelhakim Necir*

Laboratory of Applied Mathematics, Mohamed Khider University, Biskra, Algeria

Abstract

The central limit theorem introduced by Stute [The central limit theorem under random censorship. Ann. Statist. 1995; 23: 422-439] does not hold for some class of heavy-tailed distributions. In this paper, we make use of the extreme value theory to propose an alternative estimating approach of the mean ensuring the asymptotic normality property. A simulation study is carried out to evaluate the performance of this estimation procedure.

Keywords: Central limit theorem; Empirical process; Hill estimator; Kaplan-Meier estimator; Random censoring.

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*Corresponding author: necirabdelhakim@yahoo.fr

E-mail addresses:
louiza_stat@yahoo.com (L. Soltane)
djmeraghni@yahoo.com (D. Meraghni)
1. Introduction

Let \( X_1, \ldots, X_n \) be \( n \geq 1 \) independent copies of a non-negative random variable (rv) \( X \), defined over some probability space \((\Omega, \mathcal{A}, \mathbb{P})\), with absolutely continuous cumulative distribution function (cdf) \( F \). An independent sequence of independent rv’s \( Y_1, \ldots, Y_n \), with absolutely continuous cdf \( G \), censor them to the right, so that at each stage \( j \) we can only observe \( Z_j := \min(X_j, Y_j) \) and \( \delta_j := \mathbb{1}\{X_j \leq Y_j\} \), with \( \mathbb{1}\{\cdot\} \) denoting the indicator function. The rv \( \delta_j \) indicates whether or not there has been censorship. Throughout the paper, we use the notation \( S(x) := S(\infty) - S(x) \), for any \( S \). If \( H \) denotes the cdf of the observed \( Z \)'s, then, by the independence of \( X_1 \) and \( Y_1 \), we have \( H(z) = F(z) G(z) \). In our work, we assume that both \( F \) and \( G \) are heavy-tailed, this means that there exist to constants \( \gamma_1 > 0 \) and \( \gamma_2 > 0 \), called tail indices, such that

\[
\lim_{z \to \infty} F(xz) F(z) = x^{-1/\gamma_1} \quad \text{and} \quad \lim_{z \to \infty} G(xz) G(z) = x^{-1/\gamma_2},
\]

for any \( x > 0 \). Consequently, \( H \) is heavy-tailed too, with tail index \( \gamma := \gamma_1 \gamma_2 / (\gamma_1 + \gamma_2) \). The class of heavy-tailed distribution takes a significant role in extreme value theory. It includes distributions such as Pareto, Burr, Fréchet, \( \alpha \)-stable (\( 0 < \alpha < 2 \)) and log-gamma, known to be appropriate models for fitting large insurance claims, log-returns, large fluctuations of prices, etc. (see, e.g., Resnick, 2007). Examples of censored data with apparent heavy tails can be found in Gomes and Neves (2011).

The nonparametric maximum likelihood estimator of \( F \) is given by Kaplan and Meier (1958) as the product limit estimator

\[
F_n(x) := \begin{cases} 
1 - \prod_{Z_{j:n} \leq x} \left( \frac{n - j}{n - j + 1} \right)^{\delta_{[j:n]}} & \text{for } x < Z_{n:n} \\
1 & \text{for } x \geq Z_{n:n},
\end{cases}
\]

This estimator, known as Kaplan-Meier estimator of \( F \), may be expressed as follows

\[
F_n(x) := \sum_{i=2}^{n} W_{i,n} \mathbb{1}\{Z_{i:n} \leq x\},
\]

where, for \( 2 \leq i \leq n \),

\[
W_{i,n} := \frac{\delta_{[i:n]}}{n - i + 1} \prod_{j=1}^{i-1} \left( \frac{n - j}{n - j + 1} \right)^{\delta_{[j:n]}},
\]
(see, e.g., Reiss and Thomas, 2007, page 162). The aim of this paper is to propose an asymptotically normal estimator for the mean of $X$,

$$
\mu := \mathbb{E}[X] = \int_0^\infty F(x) dx,
$$

whose existence requires that $\gamma_1 < 1$. The sample mean for censored data is obtained by substituting, in the previous equation, the cdf $F$ by its estimator $F_n$ to have

$$
\tilde{\mu} := \sum_{i=2}^n \frac{\delta_{[i:n]}}{n - i + 1} \prod_{j=1}^{i-1} \left( \frac{n - j}{n - j + 1} \right)^{\delta_{[j:n]}} Z_{i:n}.
$$

The asymptotic normality of $\tilde{\mu}_n$ is established by Stute (1995), under the assumptions that the integrals

$$
I_1 := \int_0^\infty x^2 \Gamma_2^2(x) dH^{(1)}(x) \quad \text{and} \quad I_2 := \int_0^\infty x \left( \int_0^x \frac{dG(y)}{\overline{H}(y) \overline{G}(y)} \right)^{1/2} dF(x),
$$

be finite, where $\Gamma_0(x) := \exp \left\{ \int_0^x \frac{dH^{(0)}(z)}{\overline{H}(z)} \right\}$ with $H^{(j)}(v) := \mathbb{P}(Z \leq v, \delta = j)$, $j = 0, 1$. In the sequel, the latter functions will play a prominent role. However, when we deal with heavy-tailed distributions, the quantities $I_1$ and $I_2$ may be infinite. Indeed, suppose that both $F$ and $G$ are Pareto distributions, that is $\overline{F}(x) = x^{-1/\gamma_1}$ and $\overline{G}(x) = x^{-1/\gamma_2}$, for $x \geq 1$. This obviously gives $\overline{H}(x) = x^{-1/\gamma}$, $H^{(0)}(x) = \gamma (1 - x^{-1/\gamma})/\gamma_2$, $H^{(1)}(x) = \gamma (1 - x^{-1/\gamma})/\gamma_1$ and $\Gamma_0(x) = x^{1/\gamma_2}$. Whenever $(\gamma_1, \gamma_2)$ are such that $\gamma_1 > \gamma_2/(1 + 2 \gamma_2)$, we readily check that $I_1 = I_2 = \infty$. In other words, the range

$$
\mathcal{R} := \left\{ \gamma_1, \gamma_2 > 0 : \frac{\gamma_2}{1 + 2 \gamma_2} < \gamma_1 < 1 \right\},
$$

is not covered by the central limit theorem established by Stute (1995), and thus, another approach to handle this situation is needed. This problem was already addressed by Peng (2001) for sets of complete data from heavy-tailed distributions with tail indices lying between $1/2$ and $1$. Note that in the non censoring case, we have $\gamma_1 = \gamma$ meaning that $\gamma_2 = \infty$, consequently $\mathcal{R}$ reduces to Peng’s range. The consideration of the range $\mathcal{R}$ is motivated and supported from a practical point of view as well. Indeed, as an example Einmahl et al. (2008) analyzed the Australian AIDS survival dataset and found that $\gamma_1 = 0.14$ and $p = 0.28$ leading to $\gamma_2 = 0.05$. It is easily checked that these index values belong to $\mathcal{R}$ and therefore Stute’s result does not apply in this situation. To define our new estimator, we introduce an integer sequence $k = k_n$, representing a fraction of extreme order statistics, satisfying

$$
1 < k < n, \ k \to \infty \ \text{and} \ k/n \to 0 \ \text{as} \ n \to \infty,
$$

(1.3)
and we set \( h = h_n := H^{-1}(1-k/n) \), where \( K^{-1}(y) := \inf \{ x : K(x) \geq y \} \), \( 0 < y < 1 \), denotes the quantile function of a cdf \( K \). We start by decomposing \( \mu \) as the sum of two terms as follows:

\[
\mu = \int_0^h F(x)dx + \int_h^\infty F(x)dx =: \mu_1 + \mu_2,
\]

then we estimate each term separately. Integrating the first integral by parts and changing variables in the second respectively yield

\[
\mu_1 = hF(h) + \int_0^h xF(x)dx \quad \text{and} \quad \mu_2 = hF(h) \int_1^\infty \frac{F(hx)}{F(h)}dx.
\]

By replacing \( h \) and \( F(x) \) by \( Z_{n-k:n} \) and \( F_n(x) \) respectively and using formula (1.2), we get

\[
\hat{\mu}_1 := \prod_{j=1}^{n-k} \left( \frac{n-j}{n-j+1} \right)^{\delta_{[n-\gamma]}} Z_{n-k:n} + \sum_{i=2}^{n-k} \frac{\delta_{[n-\gamma]}}{n-i+1} \prod_{j=1}^{i-1} \left( \frac{n-j}{n-j+1} \right)^{\delta_{[j-\gamma]}} Z_{i:n}, \quad (1.4)
\]

as an estimator to \( \mu_1 \). Regarding \( \mu_2 \), we apply the well-known Karamata theorem (see, for instance, de Haan and Ferreira, 2006, page 363), to write

\[
\mu_2 \sim \frac{\gamma_1}{1-\gamma_1} hF(h), \quad \text{as} \ n \to \infty, \ 0 < \gamma_1 < 1.
\]

The quantities \( h \) and \( F(h) \) are, as above, naturally estimated by \( Z_{n-k:n} \) and

\[
F(Z_{n-k:n}) = \prod_{j=1}^{n-k} \left( \frac{n-j}{n-j+1} \right)^{\delta_{[j-\gamma]}} ,
\]

respectively. Now, it is clear that to derive an estimator to \( \mu_2 \), one needs to estimate the tail index \( \gamma_1 \). The general existing method, which first appeared in Beirlant et al. (2007) and then developed in Einmahl et al. (2008), is to consider any consistent estimator of the extremal index \( \gamma \) based on the \( Z \)-sample and divide it by the proportion of non-censored observations in the tail. For instance, Einmahl et al. (2008) adapted Hill’s estimator to introduce an estimator \( \hat{\gamma}_1^{(H,c)} := \hat{\gamma}^H / \hat{p} \) to the tail index \( \gamma_1 \) under random right censorship, where

\[
\hat{\gamma}^H := \frac{1}{k} \sum_{i=1}^{k} \log Z_{n-i+1:n} / Z_{n-k:n}, \quad \text{and} \quad \hat{p} := \frac{1}{k} \sum_{i=1}^{k} \delta_{[n-i+1:n]},
\]

with \( k = k_n \) satisfying (1.3), are the classical Hill estimator and the proportion of upper non-censored observations respectively. It is proved in Brahim et al. (2015) that \( \hat{p} \) consistently estimates \( p := \gamma_2 / (\gamma_1 + \gamma_2) \), therefore \( \hat{\gamma}_1^{(H,c)} \) consistently estimates \( \gamma_1 = \gamma / p \). The authors of Brahim et al. (2015) provide a Gaussian approximation leading to the asymptotic normality of \( \hat{\gamma}_1^{(H,c)} \) by adopting a different approach from
that of Einmahl et al. (2008), who also showed that \( \hat{\gamma}_1^{(H,c)} \) is asymptotically normal. Consequently, we obtain

\[
\hat{\mu}_2 := \frac{\hat{\gamma}_1^{(H,c)}}{1 - \hat{\gamma}_1^{(H,c)}} Z_{n-k} \prod_{j=1}^{n-k} \left( \frac{n-j}{n-j+1} \right) \delta_{\{j\in[n]\}}, \quad \text{for } \hat{\gamma}_1^{(H,c)} < 1, \tag{1.5}
\]
as an estimator to \( \mu_2 \). Finally, with (1.4) and (1.5), we construct our estimator \( \hat{\mu} \) of the mean \( \mu \):

\[
\hat{\mu} := \sum_{i=2}^{n-k} \frac{\delta_{\{i\}}} {n-i+1} \prod_{j=1}^{i-1} \left( \frac{n-j}{n-j+1} \right) \delta_{\{j\in[n]\}} Z_{kn} + \prod_{j=1}^{n-k} \left( \frac{n-j}{n-j+1} \right) \frac{Z_{n-k} \delta_{\{j\in[n]\}}}{1 - \hat{\gamma}_1^{(H,c)}}.
\]

The rest of the paper is organized as follows. In Section 2, we state our main result which we prove in Section 4. Section 3 is devoted to a simulation study in which we investigate the finite sample behavior of the newly proposed estimator \( \hat{\mu} \). Finally, some results, that are instrumental to our needs, are gathered in the Appendix.

2. MAIN RESULTS

Our main result, established in the following theorem, consists in the asymptotic normality of the newly introduced estimator \( \hat{\mu} \). We notice that the asymptotic normality of extreme value theory based estimators is achieved in the second-order framework (see de Haan and Stadtmüller, 1996). Thus, it seems quite natural to suppose that cdf’s \( F \) and \( G \) satisfy the well-known second-order condition of regular variation. That is, we assume that there exist constants \( \tau_j < 0 \) and functions \( A_j, j = 1, 2 \) tending to zero, not changing sign near infinity and having regularly varying absolute values with indices \( \tau_j \), such that for any \( x > 0 \)

\[
\lim_{t \to \infty} \frac{F(tx)/F(t) - x^{-1/\tau_1}}{A_1(t)} = x^{-1/\tau_1} \frac{x^{\tau_1/\tau_1} - 1}{\tau_1^2}, \tag{2.6}
\]

\[
\lim_{t \to \infty} \frac{G(tx)/G(t) - x^{-1/\tau_2}}{A_2(t)} = x^{-1/\tau_2} \frac{x^{\tau_2/\tau_2} - 1}{\tau_2^2}.
\]

**Theorem 2.1.** Assume that the second-order conditions of regular variation (2.6) hold with \( \tau_2/(1 + 2\tau_2) < \tau_1 < 1 \). Let \( k = k_n \) be an integer sequence satisfying, in addition to (1.3), \( \lim_{n \to \infty} \sqrt{k} A_1(h) < \infty \) and \( \sqrt{k} h F(h) \to \infty \). Then there exist finite constants \( m \) and \( \sigma^2 > 0 \) such that

\[
\frac{\sqrt{k} (\hat{\mu} - \mu)}{Z_{n-k} F_n(Z_{n-k})} \xrightarrow{d} \mathcal{N}(m, \sigma^2), \text{ as } n \to \infty.
\]

**Remark 2.1.** We have

\[
m := \lambda_1 \left( \frac{1 - pr_1}{(1 - \tau_1)(1 - \gamma_1)} + \frac{\lambda_1}{(\gamma_1 + \tau_1 - 1)(1 - \gamma_1)} \right).
\]
with $\lambda_1 := \lim_{n \to \infty} \sqrt{k} A_1(h)$, whereas the computations of the asymptotic variance $\sigma^2$ are very tedious and result in an expression that is too complicated. However, the lack of a closed form for $\sigma^2$ could be overcome in applications, as both parameters are usually estimated by the respective sample mean and variance obtained by bootstrapping $\hat{\mu}$.

3. Simulation study

We carry out a simulation study to illustrate the performance of our estimator, through two sets of censored and censoring data, both drawn, in the first part, from Fréchet model

$$F(x) = \exp \left\{ -x^{-\gamma_1} \right\}, \quad G(x) = \exp \left\{ -x^{-\gamma_2} \right\}, \quad x \geq 0,$$

and, in the second part, from Burr model

$$F(x) = 1 - \left( 1 + x^{1/\eta} \right)^{-\eta/\gamma_1}, \quad G(x) = 1 - \left( 1 + x^{1/\eta} \right)^{-\eta/\gamma_2}, \quad x \geq 0,$$

where $\eta, \gamma_1, \gamma_2 > 0$. We fix $\eta = 1/4$ and choose the values 0.3, 0.4 and 0.5 for $\gamma_1$. For the proportion of the really observed extreme values, we take $p = 0.40, 0.50, 0.60$ and 0.70. For each couple $(\gamma_1, p)$, we solve the equation $p = \gamma_2 / (\gamma_1 + \gamma_2)$ to get the pertaining $\gamma_2$-value. We vary the common size $n$ of both samples $(X_1, ..., X_n)$ and $(Y_1, ..., Y_n)$, then for each size, we generate 1000 independent replicates. Our overall results are taken as the empirical means of the results obtained through the 1000 repetitions. To determine the optimal number (that we denote by $k^*$) of upper order statistics used in the computation of $\hat{\gamma}_1^{(H,c)}$, we apply the algorithm given in page 137 of Reiss and Thomas (2007). The performance of the newly defined estimator $\hat{\mu}$ is evaluated in terms of absolute bias (abs bias), mean squared error (mse) and confidence interval (conf int) accuracy via length and coverage probability (cov prob).

The results, summarized in Tables 3.1, 3.2 and 3.3 for Fréchet model and Table 3.4, 3.5 and 3.6 for Burr distribution, show that the same conclusions might be drawn in both cases. As expected, the sample size influences the estimation in the sense that the larger $n$ gets, the better the estimation is. On the other hand, it is clear that the estimation accuracy increases when the censoring percentage decreases, which seems logical. Moreover, the estimator performs best for the smaller value of the tail index, as we can see from Tables 3.1 and 3.4. Finally, many simulations realized with extreme value indices larger than 0.5, but whose results are not reported here,
Table 3.1. Absolute bias, mean squared error and 95%-confidence interval accuracy of the mean estimator based on 1000 right-censored samples from Frchet model with shape parameter 0.3

show that the estimator behaves poorly especially when the censorship proportion is high.
\[ \gamma_1 = 0.4 \rightarrow \mu = 1.489 \]

\[ p = 0.40 \]

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{n} & \hat{\mu} & \text{abs bias} & \text{mse} & \text{conf int} & \text{cov prob} & \text{length} \\
\hline
500 & 1.370 & 0.120 & 0.074 & 1.147 - 1.593 & 0.71 & 0.446 \\
1000 & 1.377 & 0.112 & 0.048 & 1.217 - 1.536 & 0.57 & 0.319 \\
1500 & 1.367 & 0.122 & 0.019 & 1.241 - 1.493 & 0.48 & 0.252 \\
2000 & 1.363 & 0.126 & 0.018 & 1.256 - 1.470 & 0.36 & 0.214 \\
\hline
\end{array}
\]

\[ p = 0.50 \]

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{n} & \hat{\mu} & \text{abs bias} & \text{mse} & \text{conf int} & \text{cov prob} & \text{length} \\
\hline
500 & 1.396 & 0.093 & 0.027 & 1.169 - 1.624 & 0.81 & 0.455 \\
1000 & 1.394 & 0.095 & 0.018 & 1.237 - 1.551 & 0.66 & 0.313 \\
1500 & 1.392 & 0.097 & 0.012 & 1.264 - 1.521 & 0.65 & 0.257 \\
2000 & 1.389 & 0.101 & 0.012 & 1.275 - 1.502 & 0.55 & 0.227 \\
\hline
\end{array}
\]

\[ p = 0.60 \]

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{n} & \hat{\mu} & \text{abs bias} & \text{mse} & \text{conf int} & \text{cov prob} & \text{length} \\
\hline
500 & 1.407 & 0.082 & 0.013 & 1.189 - 1.625 & 0.89 & 0.436 \\
1000 & 1.405 & 0.084 & 0.010 & 1.251 - 1.559 & 0.77 & 0.308 \\
1500 & 1.419 & 0.070 & 0.007 & 1.292 - 1.546 & 0.84 & 0.254 \\
2000 & 1.418 & 0.071 & 0.007 & 1.308 - 1.529 & 0.71 & 0.222 \\
\hline
\end{array}
\]

\[ p = 0.70 \]

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{n} & \hat{\mu} & \text{abs bias} & \text{mse} & \text{conf int} & \text{cov prob} & \text{length} \\
\hline
500 & 1.420 & 0.069 & 0.010 & 1.199 - 1.641 & 0.92 & 0.442 \\
1000 & 1.433 & 0.056 & 0.006 & 1.273 - 1.593 & 0.86 & 0.320 \\
1500 & 1.443 & 0.046 & 0.004 & 1.312 - 1.575 & 0.90 & 0.263 \\
2000 & 1.442 & 0.047 & 0.004 & 1.329 - 1.554 & 0.89 & 0.226 \\
\hline
\end{array}
\]

Table 3.2. Absolute bias, mean squared error and 95%-confidence interval accuracy of the mean estimator based on 1000 right-censored samples from Frchet model with shape parameter 0.4

4. Proofs

We begin by a brief introduction on some uniform empirical processes under random censoring. The empirical counterparts of \( H^{(j)} (j = 0, 1) \) are defined, for \( v \geq 0 \), by

\[
H^{(j)}_n(v) := \frac{1}{n} \sum_{i=1}^{n} 1 \{ Z_i \leq v, \delta_i = j \}, \ j = 0, 1.
\]
\[ \gamma_1 = 0.5 \rightarrow \mu = 1.772 \]

### $p = 0.40$

| $n$  | $\hat{\mu}$ | abs bias | mse   | conf int | cov prob | length |
|------|-------------|----------|-------|----------|----------|--------|
| 500  | 1.566       | 0.206    | 0.398 | 1.262 – 1.870 | 0.52     | 0.608  |
| 1000 | 1.550       | 0.223    | 0.176 | 1.372 – 1.727 | 0.28     | 0.355  |
| 1500 | 1.559       | 0.214    | 0.064 | 1.415 – 1.703 | 0.20     | 0.289  |
| 2000 | 1.549       | 0.224    | 0.061 | 1.426 – 1.671 | 0.13     | 0.245  |

### $p = 0.50$

| $n$  | $\hat{\mu}$ | abs bias | mse   | conf int | cov prob | length |
|------|-------------|----------|-------|----------|----------|--------|
| 500  | 1.577       | 0.195    | 0.180 | 1.309 – 1.846 | 0.53     | 0.537  |
| 1000 | 1.573       | 0.199    | 0.139 | 1.386 – 1.761 | 0.37     | 0.375  |
| 1500 | 1.578       | 0.195    | 0.051 | 1.430 – 1.725 | 0.20     | 0.294  |
| 2000 | 1.576       | 0.196    | 0.044 | 1.447 – 1.706 | 0.22     | 0.259  |

### $p = 0.60$

| $n$  | $\hat{\mu}$ | abs bias | mse   | conf int | cov prob | length |
|------|-------------|----------|-------|----------|----------|--------|
| 500  | 1.626       | 0.147    | 0.128 | 1.362 – 1.889 | 0.65     | 0.527  |
| 1000 | 1.617       | 0.155    | 0.034 | 1.430 – 1.805 | 0.56     | 0.375  |
| 1500 | 1.606       | 0.166    | 0.033 | 1.465 – 1.747 | 0.34     | 0.282  |
| 2000 | 1.622       | 0.150    | 0.029 | 1.494 – 1.751 | 0.34     | 0.258  |

### $p = 0.70$

| $n$  | $\hat{\mu}$ | abs bias | mse   | conf int | cov prob | length |
|------|-------------|----------|-------|----------|----------|--------|
| 500  | 1.632       | 0.141    | 0.046 | 1.375 – 1.888 | 0.72     | 0.513  |
| 1000 | 1.646       | 0.126    | 0.024 | 1.459 – 1.833 | 0.70     | 0.370  |
| 1500 | 1.668       | 0.104    | 0.017 | 1.516 – 1.821 | 0.68     | 0.305  |
| 2000 | 1.666       | 0.107    | 0.016 | 1.535 – 1.797 | 0.57     | 0.262  |

**Table 3.3.** Absolute bias, mean squared error and 95%-confidence interval accuracy of the mean estimator based on 1000 right-censored samples from Frchet model with shape parameter 0.5

In the sequel, we will use the following two empirical processes

\[
\sqrt{n} \left( \overline{H}_n^{(j)}(v) - \overline{H}^{(j)}(v) \right), \quad j = 0, 1; \quad v \geq 0,
\]

which may be represented, almost surely, by a uniform empirical process. Indeed, let us define, for each \( i = 1, \ldots, n \), the following rv

\[
U_i := \delta_i H^{(1)}(Z_i) + (1 - \delta_i)(\theta + H^{(0)}(Z_i)).
\]
\[ \gamma_1 = 0.3 \Rightarrow \mu = 1.228 \]

\[
p = 0.40
\]

| \( n \) | \( \tilde{\mu} \) | abs bias | mse | conf int | cov prob | length |
|-------|-------------|---------|-----|---------|---------|-------|
| 500   | 1.186       | 0.042   | 0.077 | 0.972 – 1.399 | 0.90 | 0.428 |
| 1000  | 1.179       | 0.049   | 0.019 | 1.038 – 1.32 | 0.8 | 0.282 |
| 1500  | 1.163       | 0.064   | 0.005 | 1.053 – 1.273 | 0.8 | 0.220 |
| 2000  | 1.164       | 0.063   | 0.005 | 1.068 – 1.261 | 0.72 | 0.193 |

\[
p = 0.50
\]

| \( n \) | \( \tilde{\mu} \) | abs bias | mse | conf int | cov prob | length |
|-------|-------------|---------|-----|---------|---------|-------|
| 500   | 1.186       | 0.042   | 0.009 | 0.991 – 1.380 | 0.94 | 0.388 |
| 1000  | 1.173       | 0.054   | 0.004 | 1.039 – 1.308 | 0.93 | 0.269 |
| 1500  | 1.068       | 0.047   | 0.003 | 1.180 – 1.292 | 0.88 | 0.224 |
| 2000  | 1.181       | 0.046   | 0.003 | 1.086 – 1.276 | 0.86 | 0.190 |

\[
p = 0.60
\]

| \( n \) | \( \tilde{\mu} \) | abs bias | mse | conf int | cov prob | length |
|-------|-------------|---------|-----|---------|---------|-------|
| 500   | 1.184       | 0.043   | 0.004 | 0.997 – 1.371 | 0.95 | 0.374 |
| 1000  | 1.192       | 0.036   | 0.002 | 1.058 – 1.326 | 0.96 | 0.268 |
| 1500  | 1.196       | 0.031   | 0.002 | 1.088 – 1.305 | 0.96 | 0.217 |
| 2000  | 1.194       | 0.034   | 0.002 | 1.099 – 1.288 | 0.92 | 0.190 |

\[
p = 0.70
\]

| \( n \) | \( \tilde{\mu} \) | abs bias | mse | conf int | cov prob | length |
|-------|-------------|---------|-----|---------|---------|-------|
| 500   | 1.198       | 0.029   | 0.003 | 1.012 – 1.384 | 0.97 | 0.373 |
| 1000  | 1.200       | 0.028   | 0.001 | 1.066 – 1.334 | 0.98 | 0.269 |
| 1500  | 1.208       | 0.020   | 0.001 | 1.098 – 1.317 | 0.98 | 0.219 |
| 2000  | 1.207       | 0.021   | 0.001 | 1.113 – 1.301 | 0.98 | 0.188 |

Table 3.4. Absolute bias, mean squared error and 95%-confidence interval accuracy of the mean estimator based on 1000 right-censored samples from Burr model with shape parameter 0.3

From Einmahl and Koning (1992), the rv’s \( U_1, ..., U_n \) are iid \((0, 1)\)-uniform. The empirical cdf and the uniform empirical process based upon \( U_1, ..., U_n \) are respectively denoted by

\[ U_n(s) := \frac{1}{n} \sum_{i=1}^{n} 1 \{ U_i \leq s \} \text{ and } \alpha_n(s) := \sqrt{n}(U_n(s) - s) \text{, } 0 \leq s \leq 1. \]

Deheuvels and Einmahl (1996) state that almost surely

\[ H_n^{(0)}(v) = U_n(H^{(0)}(v) + \theta) - U_n(\theta) \text{, for } 0 < H^{(0)}(v) < 1 - \theta, \]
\[ \gamma_1 = 0.4 \rightarrow \mu = 1.498 \]

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
n & \hat{\mu} & \text{abs bias} & \text{mse} & \text{conf int} & \text{cov prob} & \text{length} \\
\hline
500 & 1.426 & 0.071 & 0.093 & 1.193 - 1.660 & 0.76 & 0.466 \\
1000 & 1.388 & 0.110 & 0.033 & 1.224 - 1.551 & 0.58 & 0.327 \\
1500 & 1.374 & 0.124 & 0.020 & 1.248 - 1.499 & 0.44 & 0.252 \\
2000 & 1.374 & 0.123 & 0.019 & 1.268 - 1.480 & 0.29 & 0.212 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
n & \hat{\mu} & \text{abs bias} & \text{mse} & \text{conf int} & \text{cov prob} & \text{length} \\
\hline
500 & 1.402 & 0.096 & 0.047 & 1.176 - 1.627 & 0.80 & 0.451 \\
1000 & 1.389 & 0.109 & 0.017 & 1.231 - 1.546 & 0.64 & 0.316 \\
1500 & 1.401 & 0.097 & 0.012 & 1.272 - 1.530 & 0.66 & 0.258 \\
2000 & 1.402 & 0.096 & 0.011 & 1.292 - 1.511 & 0.53 & 0.219 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
n & \hat{\mu} & \text{abs bias} & \text{mse} & \text{conf int} & \text{cov prob} & \text{length} \\
\hline
500 & 1.422 & 0.076 & 0.043 & 1.186 - 1.657 & 0.85 & 0.471 \\
1000 & 1.421 & 0.077 & 0.009 & 1.261 - 1.581 & 0.86 & 0.320 \\
1500 & 1.429 & 0.069 & 0.007 & 1.302 - 1.556 & 0.80 & 0.254 \\
2000 & 1.427 & 0.071 & 0.006 & 1.316 - 1.538 & 0.76 & 0.223 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
n & \hat{\mu} & \text{abs bias} & \text{mse} & \text{conf int} & \text{cov prob} & \text{length} \\
\hline
500 & 1.436 & 0.061 & 0.009 & 1.214 - 1.658 & 0.94 & 0.444 \\
1000 & 1.441 & 0.057 & 0.006 & 1.285 - 1.597 & 0.92 & 0.312 \\
1500 & 1.451 & 0.047 & 0.004 & 1.322 - 1.580 & 0.91 & 0.259 \\
2000 & 1.449 & 0.049 & 0.004 & 1.340 - 1.558 & 0.88 & 0.218 \\
\hline
\end{array}
\]

Table 3.5. Absolute bias, mean squared error and 95%-confidence interval accuracy of the mean estimator based on 1000 right-censored samples from Burr model with shape parameter 0.4

\[
H_n^{(1)}(v) = \mathbb{U}_n(H^{(1)}(v)), \quad \text{for } 0 < H^{(1)}(v) < \theta.
\]

It is easy to verify that almost surely
\[
\beta_n(v) := \sqrt{n} \left( \overline{H}_n^{(1)}(v) - \overline{H}^{(1)}(v) \right) = \alpha_n(\theta) - \alpha_n \left( \theta - \overline{H}^{(1)}(v) \right), \quad \text{for } 0 < \overline{H}^{(1)}(v) < \theta,
\]

(4.7)
\[ \gamma_1 = 0.5 \rightarrow \mu = 1.854 \]

\[ p = 0.40 \]

| \( n \) | \( \hat{\mu} \) | abs bias | mse | conf int | cov prob | length |
|-----|-------|--------|-----|---------|---------|-------|
| 500 | 1.654 | 0.200  | 0.760 | 1.330 – 1.978 | 0.50     | 0.649 |
| 1000| 1.648 | 0.206  | 0.114 | 1.460 – 1.836 | 0.26     | 0.375 |
| 1500| 1.630 | 0.224  | 0.098 | 1.478 – 1.782 | 0.14     | 0.304 |
| 2000| 1.621 | 0.233  | 0.090 | 1.491 – 1.752 | 0.14     | 0.260 |

\[ p = 0.50 \]

| \( n \) | \( \hat{\mu} \) | abs bias | mse | conf int | cov prob | length |
|-----|-------|--------|-----|---------|---------|-------|
| 500 | 1.603 | 0.252  | 0.554 | 1.253 – 1.952 | 0.67     | 0.700 |
| 1000| 1.658 | 0.196  | 0.090 | 1.470 – 1.847 | 0.34     | 0.378 |
| 1500| 1.653 | 0.202  | 0.049 | 1.501 – 1.804 | 0.25     | 0.303 |
| 2000| 1.656 | 0.198  | 0.045 | 1.530 – 1.782 | 0.22     | 0.252 |

\[ p = 0.60 \]

| \( n \) | \( \hat{\mu} \) | abs bias | mse | conf int | cov prob | length |
|-----|-------|--------|-----|---------|---------|-------|
| 500 | 1.688 | 0.166  | 0.066 | 1.417 – 1.959 | 0.67     | 0.542 |
| 1000| 1.693 | 0.161  | 0.036 | 1.508 – 1.879 | 0.54     | 0.371 |
| 1500| 1.695 | 0.159  | 0.031 | 1.544 – 1.846 | 0.39     | 0.301 |
| 2000| 1.705 | 0.149  | 0.027 | 1.576 – 1.834 | 0.34     | 0.258 |

\[ p = 0.70 \]

| \( n \) | \( \hat{\mu} \) | abs bias | mse | conf int | cov prob | length |
|-----|-------|--------|-----|---------|---------|-------|
| 500 | 1.737 | 0.117  | 0.060 | 1.462 – 2.012 | 0.77     | 0.550 |
| 1000| 1.737 | 0.117  | 0.036 | 1.547 – 1.927 | 0.74     | 0.380 |
| 1500| 1.749 | 0.105  | 0.016 | 1.593 – 1.904 | 0.70     | 0.311 |
| 2000| 1.753 | 0.101  | 0.014 | 1.621 – 1.885 | 0.60     | 0.264 |

Table 3.6. Absolute bias, mean squared error and 95%-confidence interval accuracy of the mean estimator based on 1000 right-censored samples from Burr model with shape parameter 0.5

\[ \tilde{\beta}_n(v) := \sqrt{n} \left( \overline{H}_n(0)(v) - \overline{H}(0)(v) \right) = -\alpha_n \left( 1 - \overline{H}(0)(v) \right), \text{ for } 0 < \overline{H}(0)(v) < 1 - \theta. \]  

(4.8)

Our methodology strongly relies on the well-known Gaussian approximation given in Corollary 2.1 by Csörgö et al. (1986). It says that: on the probability space \((\Omega, \mathcal{A}, \mathbb{P})\), there exists a sequence of Brownian bridges \(\{B_n(s); 0 \leq s \leq 1\}\) such that
for every $0 \leq \zeta < 1/4$,

$$\sup_{1/\varepsilon \leq s \leq 1} \frac{n^\zeta |\alpha_n(1-s) - B_n(1-s)|}{s^{1/2-\zeta}} = O_p(1).$$

(4.9)

For the increments $\alpha_n(\theta) - \alpha_n(\theta-s)$, we will need an approximation of the same type as (4.9). Following similar arguments, mutatis mutandis, as those used in the proofs of assertions (2.2) of Theorem 2.1 and (2.8) of Theorem 2.2 in Csörgö et al. (1986), we may show that, for every $0 < \theta < 1$ and $0 \leq \zeta < 1/4$, we have

$$\sup_{1/n \leq s \leq \theta} \frac{n^\zeta \{\alpha_n(\theta) - \alpha_n(\theta-s)\} - \{B_n(\theta) - B_n(\theta-s)\}}{s^{1/2-\zeta}} = O_p(1).$$

(4.10)

4.1. **Proof of Theorem 2.1.** Observe that $\hat{\mu} - \mu = (\hat{\mu}_1 - \mu_1) + (\hat{\mu}_2 - \mu_2)$, where

$$\hat{\mu}_1 - \mu_1 = \int_0^{Z_{n-k,n}} \overline{F}_n(x) dx - \int_0^h \overline{F}(x) dx,$$

and

$$\hat{\mu}_2 - \mu_2 = \prod_{j=1}^{n-k} \left(1 - \frac{\delta_{j:n}}{n-j+1} \right) \frac{\gamma(\mathcal{H},c)}{1-\gamma(\mathcal{H},c)} Z_{n-k,n} - \int_h^{\infty} \overline{F}(x) dx.$$

It is clear that

$$\hat{\mu}_1 - \mu_1 = \int_0^{Z_{n-k,n}} (\overline{F}_n(x) - \overline{F}(x)) dx - \int_{Z_{n-k,n}}^h \overline{F}(x) dx.$$

In view of Proposition 5 combined with equation (4.9) in Csörgö (1996), we have for any $x \leq Z_{n-k,n}$,

$$\frac{\overline{F}_n(x) - \overline{F}(x)}{\overline{F}(x)} = \int_0^x \frac{d(\overline{F}_n(1)(v) - \overline{F}(1)(v))}{\overline{H}(v)} - \int_0^x \frac{\overline{H}_n(v) - \overline{H}(v)}{\overline{H}^2(v)} d\overline{H}(1)(v) + O_p(1/k).$$

Integrating the first integral by parts yields

$$\frac{\overline{F}_n(x) - \overline{F}(x)}{\overline{F}(x)} = \frac{\overline{H}_n(1)(x) - \overline{H}(1)(x)}{\overline{H}(x)} - \left(\frac{\overline{H}_n(1)(0) - \overline{H}(1)(0)}{\overline{H}(0)} \right)$$

$$+ \int_0^x \frac{\overline{H}_n(1)(v) - \overline{H}(1)(v)}{\overline{H}^2(v)} d\overline{H}(v) - \int_0^x \frac{\overline{H}_n(v) - \overline{H}(v)}{\overline{H}^2(v)} d\overline{H}(1)(v) + O_p(1/k).$$

Recall that

$$\sqrt{n} (\overline{H}_n(v) - \overline{H}(v)) = \sqrt{n} \left(\overline{H}_n^{(1)}(v) - \overline{H}^{(1)}(v)\right) + \sqrt{n} \left(\overline{H}_n^{(0)}(v) - \overline{H}^{(0)}(v)\right),$$

which by representations (4.7) and (4.8) becomes

$$\sqrt{n} (\overline{H}_n(v) - \overline{H}(v)) = \alpha_n(\theta) - \alpha_n \left(\theta - \overline{H}^{(1)}(v)\right) - \alpha_n \left(1 - \overline{H}^{(0)}(v)\right).$$
Furthermore, from the classical central limit theorem, we have $\overline{P}_n^{(1)} (0) - \overline{P}^{(1)} (0) = O_p \left( n^{-1/2} \right)$. Therefore, we have

$$
\frac{\overline{T}_n (x) - \overline{T} (x)}{\overline{F} (x)} = \frac{1}{\sqrt{n}} \beta_n (x) + \frac{1}{\sqrt{n}} \int_0^x \beta_n (v) d\overline{H} (v) + O_p \left( \frac{1}{k} \right) + O_p \left( \frac{1}{\sqrt{n}} \right). \tag{4.11}
$$

By letting $a_n := (k/n)^{1/2} / (h \overline{F} (h))$, it is easy to verify that

$$
\frac{\sqrt{k} (\hat{\mu}_1 - \mu_1)}{h \overline{F} (h)} = \sum_{i=1}^6 T_{ni},
$$

where

$$
T_{n1} := a_n \int_0^{Z_{n-k:n}} \frac{\beta_n (x)}{\overline{H} (x)} \overline{F} (x) dx,
$$

$$
T_{n2} := a_n \int_0^{Z_{n-k:n}} \left\{ \int_0^x \frac{\beta_n (v)}{\overline{H}^2 (v)} d\overline{H} (v) \right\} \overline{F} (x) dx,
$$

$$
T_{n3} := -a_n \int_0^{Z_{n-k:n}} \left\{ \int_0^x \frac{\beta_n (v) + \tilde{\beta}_n (v)}{\overline{H}^2 (v)} d\overline{H}^{(1)} (v) \right\} \overline{F} (x) dx,
$$

$$
T_{n4} := a_n O_p \left( \sqrt{n} / k \right) \int_0^{Z_{n-k:n}} \overline{F} (x) dx,
$$

$$
T_{n5} := -a_n \sqrt{n} \int_{Z_{n-k:n}}^h \overline{F} (x) dx \quad \text{and} \quad T_{n6} := O_p \left( a_n \right).
$$

By using the Gaussian approximation (4.10), we obtain

$$
T_{n1} = a_n \int_0^{Z_{n-k:n}} \frac{\overline{F} (x)}{\overline{H} (x)} B_n (x) dx
\quad + \quad o_p \left( 1 \right) a_n \int_0^{Z_{n-k:n}} \left( \overline{H}^{(1)} (x) \right)^{1/2} \overline{F} (x) dx,
$$

where

$$
B_n (x) := B_n (\theta) - B_n \left( \theta - \overline{H}^{(1)} (x) \right), \quad \text{for} \ 0 < \overline{H}^{(1)} (x) < \theta. \tag{4.12}
$$

Next, we show that the second term of $T_{n1}$ tends to zero in probability, leading to

$$
T_{n1} = a_n \int_0^{Z_{n-k:n}} \frac{\overline{F} (x)}{\overline{H} (x)} B_n (x) dx + o_p \left( 1 \right).
$$
Let $0 \leq \zeta < 1/4$ and note that since $\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}^{(1)}$, then $\mathcal{H}^{(1)} \leq \mathcal{H}$ and

$$O_p(\bar{n}^{-\zeta}) a_n \int_0^{Z_{n-k:n}} \frac{\mathcal{F}(x)}{(\mathcal{H}(x))^{1/2\zeta}} dx \leq O_p(1) \bar{n}^{-\zeta} a_n \int_0^{Z_{n-k:n}} \frac{\mathcal{F}(x)}{(\mathcal{H}(x))^{1/2\zeta}} dx.$$

We show that

$$n^{-\zeta} a_n \int_0^{Z_{n-k:n}} \frac{\mathcal{F}(x)}{(\mathcal{H}(x))^{1/2\zeta}} dx = n^{-\zeta} a_n \int_0^{h} \frac{\mathcal{F}(x)}{(\mathcal{H}(x))^{1/2\zeta}} dx + o_p(1).$$

Indeed, we have

$$\left| \int_0^{Z_{n-k:n}} \frac{\mathcal{F}(x)}{(\mathcal{H}(x))^{1/2\zeta}} dx - \int_0^{h} \frac{\mathcal{F}(x)}{(\mathcal{H}(x))^{1/2\zeta}} dx \right| = \left| \int_{h}^{\max(h,Z_{n-k:n})} \frac{\mathcal{F}(x)}{(\mathcal{H}(x))^{1/2\zeta}} dx \right|$$

By using Potter’s inequalities, given in assertion 5 of Proposition B.1.9 in de Haan and Ferreira (2006), we write for $\epsilon > 0$,

$$n^{-\zeta} a_n \int_{\max(h,Z_{n-k:n})}^{\min(h,Z_{n-k:n})} \frac{\mathcal{F}(x)}{(\mathcal{H}(x))^{1/2\zeta}} dx \leq k^{-\zeta} \left[ x^{-1/\gamma_1+(1/2\zeta)} / \gamma_2 \right] \max(1,Z_{n-k:n}/h) \min(1,Z_{n-k:n}/h).$$

On the other hand, combining Corollary 2.2.2 with Potter’s inequalities given in Proposition B.1.9 (5) in de Haan and Ferreira (2006), yields that $Z_{n-k:n}/h \rightarrow 1$ in probability. Therefore, the right-hand side of the previous inequality tends to zero, as sought. Now, we show that $T_{n1}$ may be rewritten into

$$T_{n1} = a_n \int_0^{h} \frac{\mathcal{F}(x)}{(\mathcal{H}(x))} B_n(x) dx + o_p(1).$$

Observe that

$$T_{n1} = a_n \int_0^{h} \frac{\mathcal{F}(x)}{(\mathcal{H}(x))} B_n(x) dx + a_n \int_h^{Z_{n-k:n}} \frac{\mathcal{F}(x)}{(\mathcal{H}(x))} B_n(x) dx + o_p(1),$$

with the second term in the right-hand side tending to zero in probability. Indeed, for fixed $0 < \eta, \epsilon < 1$, we have

$$\mathbb{P} \left( \left| a_n \int_h^{Z_{n-k:n}} \frac{\mathcal{F}(v)}{(\mathcal{H}(v))} B_n(v) dv \right| > \eta \right)$$

$$\leq \mathbb{P} \left( \left| \frac{Z_{n-k:n}}{h} - 1 \right| > \epsilon \right) + \mathbb{P} \left( \left| a_n \int_h^{(1+\epsilon)h} \frac{\mathcal{F}(v)}{(\mathcal{H}(v))} B_n(v) dv \right| > \eta \right),$$

where, in virtue of the fact that $Z_{n-k:n}/h \overset{P}{\rightarrow} 1$, the first term tends to zero. It remains to show that the second term in the right-hand side is also asymptotically
negligible. We have \( \overline{\mathcal{H}}^{(1)} \leq \overline{\mathcal{P}} \), then
\[
E \left| a_n \int_{h}^{(1+\varepsilon)h} \frac{F(v)}{H(v)} \mathcal{B}_n(v) dv \right| \leq a_n \int_{h}^{(1+\varepsilon)h} \frac{F(v)}{H^{(1)}(v)} dv.
\]

Changing variables and applying Potter’s inequalities to the regularly varying function \( \overline{\mathcal{F}}(x)/\sqrt{\overline{\mathcal{H}}(x)} \), yield that, for all large \( n \) and \( \xi > 0 \), we have
\[
E \left| a_n \int_{h}^{(1+\varepsilon)h} \frac{F(v)}{H(v)} \mathcal{B}_n(v) dv \right| \leq a_n \int_{h}^{(1+\varepsilon)h} \frac{hF(h)}{\sqrt{H(h)}} \int_1^{1+\varepsilon} v^{-1/\gamma(1)/(2\gamma)\xi} dv.
\]

The latter integral is clearly finite and tends to zero as \( \varepsilon \downarrow 0 \). By similar arguments using approximations \((4.9)\) and \((4.10)\), we also show that
\[
T_{n2} = a_n \int_0^h \left\{ \int_0^x \frac{B_n(v)}{H^2(v)} dv \right\} \overline{\mathcal{F}}(x) dx + o_P(1) \tag{4.14}
\]
and
\[
T_{n3} = -a_n \int_0^h \left\{ \int_0^x \frac{B_n^*(v)}{H^2(v)} dv \right\} \overline{\mathcal{F}}(x) dx + o_P(1), \tag{4.15}
\]
where
\[
B_n^*(x) := B_n(x) - B_n \left( 1 - \overline{\mathcal{H}}^{(0)}(x) \right), \text{ for } 0 < \overline{\mathcal{H}}^{(0)}(x) < 1 - \theta. \tag{4.16}
\]

Before we examine \( T_{n4} \), we provide an approximation to \( T_{n5} \), for which a change of variables yields
\[
T_{n5} = -\sqrt{k} \int_{Z_{n-k,n}/h}^{1} \frac{\overline{\mathcal{F}}(hx)}{\overline{\mathcal{F}}(h)} dx.
\]

For the purpose of using the second-order condition of regular variation \((2.6)\) of \( \overline{\mathcal{F}} \), we write
\[
T_{n5} = -\sqrt{k} A_1(h) \int_{Z_{n-k,n}/h}^{1} \left( \frac{\overline{\mathcal{F}}(hx)/\overline{\mathcal{F}}(h) - x^{-1/\gamma}}{A_1(h)} \right) dx - \sqrt{k} \int_{Z_{n-k,n}/h}^{1} x^{-1/\gamma} dx. \tag{4.17}
\]

From the inequality \((2.3.23)\) of Theorem 2.3.9 in de Haan and Ferreira (2006), page 48, we infer that the first integral in \((4.17)\) is equal to
\[
(1 + o_P(1)) \int_{Z_{n-k,n}/h}^{1} x^{-1/\gamma} (x^{\tau_1/\gamma} - 1) / \gamma_1 \tau_1 dx,
\]
which tends to zero in probability due to the fact that $Z_{n-k:n}/h \xrightarrow{p} 1$. Moreover, the term $\sqrt{k}A_1(h)$ has, by assumption, a finite limit. Consequently, the first term in the right-hand side of (4.17) is asymptotically negligible. We develop the second integral and make a Taylor’s expansion. Knowing, once again, that $Z_{n-k:n}/h \xrightarrow{p} 1$ ultimately yields that

$$T_{n5} = (1 + o_p (1)) \sqrt{k} \left( \frac{Z_{n-k:n}}{h} - 1 \right).$$

By using result (2.7) of Theorem 2.1 in Brahimi et al. (2015), we get

$$T_{n5} = \gamma \sqrt{\frac{n}{k}} B^*_n (h) + o_p (1). \quad (4.18)$$

Next, we readily check that the fourth term $T_{n4}$ tends to zero in probability. Indeed, we have $\int_0^{Z_{n-k:n}} F(x) dx < \mu$ and by assumption $\sqrt{k}hF(h) \to \infty$. Finally, for the last term $T_{n6}$ we use the second-order regular variation of the tails $\overline{F}$ and $\overline{G}$. From Lemma 3 in Hua and Joe (2011), there exist two positive constants $c_1$ and $c_2$ such that $h = (1 + o(1)) c_1 (k/n)^{-\gamma}$ and $\overline{F}(h) = (1 + o(1)) c_2 (k/n)^{\gamma/\gamma_1}$, thus $a_n = (1 + o(1)) c_1 c_2 (k/n)^{1/2 + \gamma - \gamma/\gamma_1}$. But the indices $\gamma_1$ and $\gamma_2$ belong to $\mathcal{R}$, hence $1/2 + \gamma - \gamma/\gamma_1$ are positive. Therefore, $a_n \to 0$ and $T_{n6} = o_p (1)$. The four approximations (4.13), (4.14), (4.15) and (4.18) together with the asymptotic negligibility of both $T_{n4}$ and $T_{n6}$ give

$$\frac{\sqrt{k} (\hat{\mu}_1 - \mu_1)}{\overline{F}(h)} = a_n \int_0^h \frac{B_n(x)}{\overline{H}(x)} \overline{F}(x) dx + a_n \int_0^h \left\{ \int_0^x \frac{B_n(v)}{\overline{H}(v)} d\overline{H}(v) \right\} \overline{F}(x) dx \right.$$

$$- a_n \int_0^h \left\{ \int_0^x \frac{B_n^*(v)}{\overline{H}(v)} d\overline{H}(v) \right\} \overline{F}(x) dx + \gamma \sqrt{\frac{n}{k}} B^*_n (h) + o_p (1). \quad (4.19)$$

Let us now treat the term $\sqrt{k} (\hat{\mu}_2 - \mu_2) / (\overline{hF}(h))$. Consider the following forms of $\mu_2$ and $\hat{\mu}_2$:

$$\mu_2 = h \overline{F}(h) \int_1^\infty \frac{\overline{F}(h x)}{\overline{F}(h)} dx \quad \text{and} \quad \hat{\mu}_2 = \frac{\hat{\gamma}^{(H,c)}_{1}}{1 - \hat{\gamma}^{(H,c)}_{1}} Z_{n-k:n} \overline{F}(Z_{n-k:n}) \frac{\overline{F}_n (Z_{n-k:n})}{\overline{F}(Z_{n-k:n})},$$
and decompose $\sqrt{k} (\hat{\mu}_2 - \mu_2) / (h\overline{F}(h))$ into the sum of

$$S_{n1} := \sqrt{k} \frac{\hat{\gamma}_1}{1 - \hat{\gamma}_1} \overline{F}(Z_{n-k:n}) \overline{F}_n(Z_{n-k:n}) \left\{ \frac{Z_{n-k:n}}{h} - 1 \right\},$$

$$S_{n2} := \sqrt{k} \frac{\overline{F}(Z_{n-k:n}) \overline{F}_n(Z_{n-k:n})}{\overline{F}(Z_{n-k:n})} \left\{ \frac{\overline{F}_n(Z_{n-k:n})}{\overline{F}(Z_{n-k:n})} - 1 \right\},$$

$$S_{n3} := \sqrt{k} \frac{\hat{\gamma}_1}{1 - \hat{\gamma}_1} \left\{ \frac{\overline{F}_n(Z_{n-k:n})}{\overline{F}(h)} - \left( \frac{Z_{n-k:n}}{h} \right)^{-1/\gamma_1} \right\},$$

$$S_{n4} := \sqrt{k} \frac{\gamma_1}{1 - \gamma_1} \left\{ \left( \frac{Z_{n-k:n}}{h} \right)^{-1/\gamma_1} - 1 \right\},$$

$$S_{n5} := \sqrt{k} \frac{\gamma_1}{1 - \gamma_1} \left\{ 1 - \int_1^{\infty} \frac{\overline{F}(hx)}{\overline{F}(h)} dx \right\}.$$  

For the first term, we have $\hat{\gamma}_1 \xrightarrow{p} \gamma_1$ and $Z_{n-k:n}/h \xrightarrow{p} 1$, which, in view of the regular variation of $\overline{F}$, implies that $\overline{F}(Z_{n-k:n}) = (1 + o_p(1)) \overline{F}(h)$. Moreover, from (5.29) we infer that $\overline{F}_n(Z_{n-k:n}) = (1 + o_p(1)) \overline{F}(Z_{n-k:n})$. It follows that

$$S_{n1} = (1 + o_p(1)) \frac{\gamma_1}{1 - \gamma_1} \sqrt{k} \left( \frac{Z_{n-k:n}}{h} - 1 \right),$$

which, by applying result (2.7) of Theorem 2.1 in Brahimi et al. (2015), is approximated as follows:

$$S_{n1} = (1 + o_p(1)) \frac{\gamma_1 \gamma_1}{1 - \gamma_1} \sqrt{n} K_n(h).$$  \hspace{1cm} (4.20)$$

By using similar arguments, we easily show that

$$S_{n2} = (1 + o_p(1)) \frac{1}{(1 - \gamma_1)^2} \sqrt{k} \left( \frac{\gamma_1}{\gamma_1} - \gamma_1 \right),$$

which, by applying result (2.9) (after a change of variables) of Theorem 2.1 in Brahimi et al. (2015), becomes

$$S_{n2} = \frac{1 + o_p(1)}{\gamma_1 \gamma_1} \left( \frac{\sqrt{n} K_n(h)}{1 - \gamma_1} \right).$$  \hspace{1cm} (4.21)$$

For $S_{n3}$, we have

$$S_{n3} = (1 + o_p(1)) \frac{\gamma_1}{1 - \gamma_1} \sqrt{k} \left( \frac{\overline{F}_n(Z_{n-k:n})}{\overline{F}(Z_{n-k:n})} - 1 \right).$$
Using Proposition 5.1, we have
\[ S_{n3} = (1 + o_p(1)) \sqrt{\frac{k}{n}} \frac{\gamma_1}{1 - \gamma_1} \left( \int_0^h \frac{B_n(v)}{H(v)} d\bar{H}(v) - \int_0^h \frac{B^*_n(v)}{\bar{H}(v)} d\bar{H}(1)(v) \right) \] (4.22)
\[ + (1 + o_p(1)) \frac{\gamma_1}{1 - \gamma_1} \sqrt{\frac{n}{k}} B_n(h) + o_p(1). \]

For the fourth term, we use the second-order condition (2.6) of \( \bar{F} \) and the fact that \( Z_{n-k:n}/h \overset{p}{\to} 1 \) to get
\[ S_{n4} = o_p \left( \sqrt{k} A_1(h) \right) = o_p(1), \text{ as } n \to \infty. \] (4.23)

For \( S_{n5} \), we apply the mean value theorem with the fact \( Z_{n-k:n}/h \overset{p}{\to} 1 \) to have
\[ S_{n5} = -(1 + o_p(1)) \frac{1}{1 - \gamma_1} \sqrt{k} \left( \frac{Z_{n-k:n}}{h} - 1 \right). \]
Using, once again, result (2.7) of Theorem 2.1 in Brahimi et al. (2015) yields
\[ S_{n5} = -(1 + o_p(1)) \frac{\gamma}{1 - \gamma_1} \sqrt{\frac{n}{k}} B_n^*(h). \] (4.24)

For the last term, we first note that
\[ \frac{S_{n6}}{\sqrt{k}} = \int_1^\infty x^{-1/\gamma_1} dx - \int_1^\infty \frac{F(hx)}{F(h)} dx. \]
Then, by applying the uniform inequality of regularly varying functions (see, e.g., Theorem 2.3.9 in de Haan and Ferreira, 2006, page 48) together with the regular variation of \( |A_1| \), we show that
\[ S_{n6} \sim \frac{\sqrt{k} A_1(h)}{(\gamma_1 + \tau_1 - 1)(1 - \gamma_1)}. \] (4.25)

By gathering (4.20), (4.21), (4.22), (4.23), (4.24) and (4.25) we end up with
\[ \frac{\sqrt{k} (\hat{\mu}_2 - \mu_2)}{h F(h)} = \frac{\gamma_1}{1 - \gamma_1} \sqrt{\frac{k}{n}} \left\{ \int_0^h \frac{B_n(v)}{H(v)} d\bar{H}(v) - \int_0^h \frac{B^*_n(v)}{\bar{H}(v)} d\bar{H}(1)(v) \right\} \]
\[ + \sqrt{\frac{n}{k}} \left\{ - \frac{\gamma_1 B_n(h)}{p (1 - \gamma_1)^2} - \gamma B^*_n(h) + \int_1^\infty \frac{v^{-1} B^*_n(hv) dv}{p (1 - \gamma_1)^2} \right\} \]
\[ + R_{n1} + o_p(1), \]
where
\[ R_{n1} := \frac{\sqrt{k} A_1(h)}{(1 - \gamma_1)} \left\{ \frac{1}{(1 - p \gamma_1)(1 - \gamma_1)} + \frac{1}{(\gamma_1 + \tau_1 - 1)} \right\}. \]
Finally, by summing up equations (4.19) and (4.26) we obtain

\[ \frac{\sqrt{k}(\hat{\mu} - \mu)}{hF(h)} = \sum_{i=1}^{5} D_{ni} + R_{n1} + o_P(1), \]

where

\[ D_{n1} := a_n \int_0^h \frac{B_n(v)}{H(v)} F(v) dv, \quad D_{n2} := a_n \int_0^h \left\{ \int_0^x \frac{B_n(u)}{H(u)} d\bar{H}(u) \right\} \bar{F}(x) dx, \]

\[ D_{n3} := -a_n \int_0^h \left\{ \int_0^x \frac{B_n(u)}{H^2(v)} d\bar{H}(1)(v) \right\} \bar{F}(x) dx, \]

\[ D_{n4} := \frac{\gamma_1}{1 - \gamma_1} \sqrt{\frac{k}{n}} \left( \int_0^h \frac{B_n(v)}{H^2(v)} d\bar{H}(v) - \int_0^h \frac{B_n^*(v)}{H^2(v)} d\bar{H}(1)(v) \right), \]

\[ D_{n5} := \sqrt{\frac{n}{k}} \left( -\frac{\gamma_1}{p(1 - \gamma_1)^2} B_n(h) + \frac{1}{p(1 - \gamma_1)^2} \int_1^\infty v^{-1} B_n^*(hv) dv \right). \]

Note that \( D_{n2} \) is of the form \(-a_n \int_0^h \psi(x) d\varphi(x)\), where \( \varphi(x) := \int_x^\infty \bar{F}(u) du \) and \( \psi(x) := \int_0^x B_n(v) / \bar{H}^2(v) d\bar{H}(v) \). Integrating by parts yields

\[ D_{n2} = a_n \int_0^h \varphi(v) \frac{B_n(v)}{H^2(v)} d\bar{H}(v) - \sqrt{\frac{k}{n}} \int_0^h \frac{\bar{F}(x) dx}{hF(h)} \int_0^h \frac{B_n(v)}{H^2(v)} d\bar{H}(v) \]

Equation (B.1.9) in Theorem B.1.5 (Karamata’s theorem) in de Haan and Ferreira (2006) yields that \( \int_h^\infty \bar{F}(x) dx / (hF(h)) \to \gamma_1 / (1 - \gamma_1) \). We apply the same technique to \( D_{n3} \) and get

\[ D_{n2} + D_{n3} + D_{n4} = L_{n2} + L_{n3} + R_{n2}, \]

where \( R_{n2} := o_P(D_{n4}) \) and

\[ L_{n2} := a_n \int_0^h \frac{B_n(v)}{H^2(v)} \varphi(v) d\bar{H}(v) \] and \( L_{n3} := -a_n \int_0^h \frac{B_n^*(v)}{H^2(v)} \varphi(v) d\bar{H}(1)(v). \)

This yields the following new decomposition:

\[ \frac{\sqrt{k}(\hat{\mu} - \mu)}{hF(h)} = \sum_{i=1}^{4} L_{ni} + R_{n1} + R_{n2} + o_P(1), \]

with \( L_{n1} := D_{n1} \) and \( L_{n4} := D_{n5} \). The four \( L_{ni} \) are centred Gaussian rv’s whose asymptotic second moments are finite, as we will see thereafter. Indeed, \( L_{n4} \) is the Gaussian approximation to Hill’s estimator given by result (2.9) of Theorem 2.1 in Brahimi et al. (2015), hence we have \( \lim_{n \to \infty} E[L_{n4}^2] < \infty \). For the three
others, we literally compute the asymptotic moments of order two. Note that from
the covariance structure in Csörgö (1996), page 2768, we have the following useful
formulas:

\[
\begin{align*}
\mathbb{E} [B_n (u) B_n (v)] &= \min \left( \overline{H}^{(1)} (u), \overline{H}^{(1)} (v) \right) - \overline{H}^{(1)} (u) \overline{H}^{(1)} (v), \\
\mathbb{E} [B_n^* (u) B_n^* (v)] &= \min \left( \overline{H} (u), \overline{H} (v) \right) - \overline{H} (u) \overline{H} (v), \\
\mathbb{E} [B_n (u) B_n^* (v)] &= \min \left( \overline{H}^{(1)} (u), \overline{H}^{(1)} (v) \right) - \overline{H}^{(1)} (u) \overline{H} (v).
\end{align*}
\]

(4.27)

After elementary but very tedious calculations, using these formulas with l’Hôpital’s
rule, we get as \( n \to \infty \),

\[
\begin{align*}
\frac{k}{n} \int_0^h \int_0^h \frac{\mathbb{E} [B_n (u) B_n (v)]}{\overline{H}^2 (u) \overline{H}^2 (v)} \, d\overline{H} (u) \, d\overline{H} (v) \to p, \\
\frac{k}{n} \int_0^h \int_0^h \frac{\mathbb{E} [B_n^* (u) B_n^* (v)]}{\overline{H}^2 (u) \overline{H}^2 (v)} \, d\overline{H} (u) \, d\overline{H}^2 (v) \to p^2, \\
\frac{k}{n} \int_0^h \int_0^h \frac{\mathbb{E} [B_n (u) B_n^* (v)]}{\overline{H}^2 (u) \overline{H}^2 (v)} \, d\overline{H} (u) \, d\overline{H}^2 (v) \to p^2.
\end{align*}
\]

(4.28)

By using the results above, we obtain

\[
\mathbb{E} [L_{n1}]^2 \to \frac{2\gamma^2 \gamma_1^2}{(\gamma_1 - \gamma + \gamma \gamma_1)(\gamma_1 - 2\gamma + 2\gamma \gamma_1)},
\]

\[
\mathbb{E} [L_{n2}]^2 \to \frac{2p\gamma_1^4}{(\gamma_1 - 1)^2(\gamma_1 - \gamma + \gamma \gamma_1)(\gamma_1 - 2\gamma + 2\gamma \gamma_1)},
\]

and

\[
\mathbb{E} [L_{n3}]^2 \to \frac{2p^2 \gamma_1^4}{(\gamma_1 - 1)^2(\gamma_1 - \gamma + \gamma \gamma_1)(\gamma_1 - 2\gamma + 2\gamma \gamma_1)}.
\]

As a consequence, we conclude that

\[
\sqrt{k \frac{\hat{\mu} - \mu}{h F(h)}} \xrightarrow{d} \mathcal{N} (m, \sigma^2), \quad \text{as} \quad n \to \infty,
\]

where \( m := \lim_{n \to \infty} R_{n1} \) and \( \sigma^2 := \lim_{n \to \infty} \mathbb{E} \left[ \sum_{i=1}^{4} L_{ni} \right]^2 \). The expression of \( m \)
is simple and easily obtainable whilst that of \( \sigma^2 \) is very complicated and requires
extremely laborious computations. However, we readily check that, it is finite.
Indeed, in addition to the finiteness of the asymptotic second moments \( \mathbb{E} [L_{ni}]^2 \), the
asymptotic covariances \( \mathbb{E} [L_{ni} L_{nj}] \) are, in virtue of Cauchy-Schwarz inequality, finite
as well. Finally, we use the facts that \( Z_{n-k:n}/h \) and \( \overline{F}(Z_{n-k:n})/F(h) \) tend to 1 in
probability to achieve the proof. \( \square \)
5. Appendix

In the following basic proposition, we give an asymptotic representation to the Kaplan-Meier product limit estimator (in $Z_{n-k:n}$). This result will of prime importance in the study of the limiting behaviors of many statistics based on censored data exhibiting extreme values.

**Proposition 5.1.** Assume that the second-order conditions (2.6) hold. Let $k = k_n$ be an integer sequence satisfying, in addition to (1.3), $\sqrt{k}A_j(h) = O(1)$, for $j = 1, 2$, as $n \to \infty$. Then there exists a sequence of Brownian bridges $\{B_n(s); 0 \leq s \leq 1\}$ such that
\[
\sqrt{k} \left( \frac{F_n(Z_{n-k:n})}{F(Z_{n-k:n})} - 1 \right) = \sqrt{\frac{n}{k}} B_n(h) + \sqrt{\frac{k}{n}} \left( \int_0^h B_n(v) \frac{d\bar{H}}{H^2}(v) - \int_0^h B_n^*(v) \frac{d\bar{H}^{(1)}}{H^2}(v) \right) + o_p(1),
\]
where $B_n(v)$ and $B_n^*(v)$ are respectively defined in (4.12) and (4.16). Consequently,
\[
\sqrt{k} \left( \frac{F_n(Z_{n-k:n})}{F(Z_{n-k:n})} - 1 \right) \xrightarrow{d} N(0, p), \text{ as } n \to \infty. \tag{5.29}
\]

**Proof.** Multiplying (4.11) by $\sqrt{k}$ yields
\[
\sqrt{k} \frac{\bar{F}_n(x) - \bar{F}(x)}{\bar{F}(x)} = \sqrt{\frac{k}{n}} \alpha_n(\theta) - \alpha_n \left( \frac{\theta - \bar{H}^{(1)}(x)}{\bar{H}(x)} \right)
\]
\[
+ \sqrt{\frac{k}{n}} \int_0^x \alpha_n(\theta) - \alpha_n \left( \frac{\theta - \bar{H}^{(1)}(v)}{\bar{H}^2(v)} \right) d\bar{H}(v)
\]
\[
- \sqrt{\frac{k}{n}} \int_0^x \alpha_n(\theta) - \alpha_n \left( \frac{\theta - \bar{H}^{(1)}(v)}{\bar{H}^2(v)} \right) - \alpha_n \left( \frac{1 - \bar{H}^{(0)}(v)}{\bar{H}^2(v)} \right) d\bar{H}^{(1)}(v)
\]
\[
+ O_p \left( \frac{1}{\sqrt{k}} \right) + O_p \left( \sqrt{\frac{k}{n}} \right).
\]

The Gaussian approximations (4.9) and (4.10), in $x = Z_{n-k:n}$, and the facts that $\sqrt{k/n}$ and $1/\sqrt{k}$ tend to zero as $n \to \infty$, lead to
\[
\sqrt{k} \frac{\bar{F}_n(Z_{n-k:n}) - \bar{F}(Z_{n-k:n})}{\bar{F}(Z_{n-k:n})} = \sqrt{\frac{n}{k}} B_n(Z_{n-k:n}) + \sqrt{\frac{k}{n}} \int_0^{Z_{n-k:n}} B_n(v) \frac{d\bar{H}^{(1)}}{H^2}(v) + o_p(1).
\]
Applying Lemma 5.1 completes the proof. The asymptotic normality property is straightforward. For the variance computation, we use, in addition to (4.28), the following results:

\[
\int_0^h \frac{\mathbf{E} [B_n(u) B_n(h)]}{H^2(u)} d\Pi(u) \to -p \quad \text{and} \quad \int_0^h \frac{\mathbf{E} [B_n^*(u) B_n(h)]}{H^2(u)} d\Pi^*(u) \to -p^2,
\]

similarly obtained as (4.28).

\[\square\]

**Lemma 5.1.** Assume that the second-order conditions of regular variation (2.6) and let \( k := k_n \) be an integer sequence satisfying (1.3). Then

\[\begin{align*}
(i) & \quad \sqrt{k/n} \int_{\frac{Z_{n-k:n}}{h}} B_n(v) \frac{d\Pi(v)}{H^2(v)} = o_\mathbb{P}(1). \\
(ii) & \quad \sqrt{k/n} \int_{\frac{Z_{n-k:n}}{h}} B_n^*(v) \frac{d\Pi^*(v)}{H^2(v)} = o_\mathbb{P}(1). \\
(iii) & \quad \sqrt{n/k} \{B_n(Z_{n-k:n}) - B_n(h)\} = o_\mathbb{P}(1). \\
(iv) & \quad \sqrt{n/k} \{B_n^*(Z_{n-k:n}) - B_n^*(h)\} = o_\mathbb{P}(1).
\end{align*}\]

**Proof.** We begin by proving the first assertion. For fixed \( 0 < \eta, \epsilon < 1 \), we have

\[
\mathbb{P} \left( \left| \sqrt{k/n} \int_{\frac{Z_{n-k:n}}{h}} B_n(v) \frac{d\Pi(v)}{H^2(v)} \right| > \eta \right) \\
\leq \mathbb{P} \left( \left| \frac{Z_{n-k:n}}{h} - 1 \right| > \epsilon \right) + \mathbb{P} \left( \left| \sqrt{k/n} \int_{h}^{(1+\epsilon)h} B_n(v) \frac{d\Pi(v)}{H^2(v)} \right| > \eta \right).
\]

It is clear that the first term in the right-hand side tends to zero as \( n \to \infty \). Then, it remains to show that the second one goes to zero as well. Indeed, observe that

\[
\mathbf{E} \left| \sqrt{k/n} \int_{h}^{(1+\epsilon)h} B_n(v) \frac{d\Pi(v)}{H^2(v)} \right| \leq -\sqrt{k/n} \int_{h}^{(1+\epsilon)h} \mathbf{E} |B_n(v)| \frac{d\Pi(v)}{H^2(v)}.
\]

From the first result of (4.27), we have \( \mathbf{E} |B_n(v)| \leq \sqrt{\Pi^*(v)} \). Then

\[
\mathbf{E} \left| \sqrt{k/n} \int_{h}^{(1+\epsilon)h} B_n(v) \frac{d\Pi(v)}{H^2(v)} \right| \leq -\sqrt{k/n} \int_{h}^{(1+\epsilon)h} \sqrt{\Pi^*(v)} \frac{d\Pi(v)}{H^2(v)} \frac{d\Pi(v)}{H^2(v)},
\]

which, in turn, is less than or equal to

\[
\sqrt{k/n} \sqrt{\Pi^*(h)} \left( \frac{1}{H((1+\epsilon)h)} - \frac{1}{H(h)} \right).
\]
Since $\bar{\Pi}(h) = k/n$, then this may be rewritten into
\[
\sqrt{\frac{\bar{\Pi}^{-1}(h)}{\bar{\Pi}(h)}} \left( \frac{\bar{\Pi}(h)}{\bar{\Pi}(1 + \epsilon)h} - 1 \right). 
\]
Since $\bar{\Pi}^{-1}(h) \sim p \bar{\Pi}(h)$ and $\bar{\Pi} \in \mathcal{RV}_{(-1/\gamma)}$, then the previous quantity tends to $p^{1/2} \left((1 + \epsilon)^{1/\gamma} - 1\right)$ as $n \to \infty$. Being arbitrary, $\epsilon$ may be chosen small enough so that this limit be zero. By similar arguments, we also show assertion (ii), therefore we omit the details. The last two assertions are shown following the same technique, that we use to prove (iv). Notice that, from the definition of $B^*_n(v)$ and the second covariance formula in (4.27), we have
\[
\{B^*_n(v) ; v \geq 0\} \overset{d}{=} \{B_n(\bar{\Pi}(v)) ; v \geq 0\},
\]
where $\{B_n(s) ; 0 \leq s \leq 1\}$ is a sequence of standard Brownian bridges. Hence
\[
\sqrt{\frac{n}{k}} \{B^*_n(Z_{n-k:n}) - B^*_n(h)\} \overset{d}{=} \sqrt{\frac{n}{k}} \{B_n(\bar{\Pi}(Z_{n-k:n})) - B_n(\bar{\Pi}(h))\}.
\]
Let $\{W_n(t) ; 0 \leq s \leq 1\}$ be a sequence of standard Wiener processes such that $B_n(t) = W_n(t) - tW_n(1)$. Then $\sqrt{n/k} \{B^*_n(Z_{n-k:n}) - B^*_n(h)\}$ equals in distribution to
\[
\sqrt{\frac{n}{k}} \{W_n(\bar{\Pi}(Z_{n-k:n})) - W_n(\bar{\Pi}(h))\} \overset{d}{=} \{\bar{\Pi}(Z_{n-k:n}) - \bar{\Pi}(h)\} W_n(1). 
\]
Using the facts that $\bar{\Pi}(h) = k/n$ and $\bar{\Pi}(Z_{n-k:n})/\bar{\Pi}(h) = (1 + o_p(1))$, we get
\[
\sqrt{\frac{n}{k}} \{\bar{\Pi}(Z_{n-k:n}) - \bar{\Pi}(h)\} = \sqrt{\frac{k}{n}} \left( \frac{\bar{\Pi}(Z_{n-k:n})}{\bar{\Pi}(h)} - 1 \right) = o_p(1).
\]
On the other hand $\sqrt{n/k} \{W_n(\bar{\Pi}(Z_{n-k:n})) - W_n(\bar{\Pi}(h))\}$ is a sequence of Gaussian rv’s with mean zero and variance
\[
\frac{n}{k} \{\bar{\Pi}(Z_{n-k:n}) - \bar{\Pi}(h)\} = \frac{\bar{\Pi}(Z_{n-k:n})}{\bar{\Pi}(h)} - 1,
\]
which tends to zero (in probability), as $n \to \infty$. This achieves the proof. \qed

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