CONICAL METRICS WITH POSITIVE CONSTANT CURVATURE AND ALGEBRAIC NUMBERS, II

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ABSTRACT. In this paper, we consider the following curvature equation

\[
\begin{cases}
\Delta u + e^u = 4\pi \left(\alpha_0 \delta_0 + \alpha_1 \delta_1 + \sum_{j=2}^{n+m+1} \alpha_j \delta_j \right) & \text{on } \mathbb{C}, \\
u(x) = -2(2 + \alpha_\infty) \ln |x| + O(1) & \text{as } |x| \to \infty.
\end{cases}
\]

Geometrically, a solution \(u\) gives rise to a conical metric \(ds^2 = \frac{1}{2}e^u |dx|^2\) of curvature 1, with conic singularities at 0, 1, \(\infty\), and \(t_j, 1 \leq j \leq n + m\), with angles \(2\pi\theta_0, 2\pi\theta_1, 2\pi\theta_m\), and \(2\pi\theta_j\) at 0, 1, \(\infty\), and \(t_j\), respectively, where the angles \(2\pi\theta_p, p \in \{0, 1, \infty, t_1, \ldots, t_{n+m}\}\), are defined by \(\theta_p = \alpha_p + 1\).

Throughout this paper, we assume that \(\alpha_0, \alpha_1, \alpha_\infty, \alpha_k, 1 \leq k \leq n\), are non-integers and \(\alpha_k, n+1 \leq j \leq n+m\), are positive integers. The metric \(ds^2\) or the solution is called co-axial, which was introduced by Mondello and Panov, if there is a developing map \(h(x)\) of \(u\) such that the projective monodromy group is contained in the unit circle. In this paper, we are concerned with only the curvature equations possessing co-axial solutions.

Some sufficient and necessary conditions in terms of angles for the existence of such metrics were obtained by Mondello-Panov [13] and Eremenko [8]. However, these results did not discuss the locations of the singularities \(t_1, \ldots, t_{n+m}\), which is the main target of the present paper.

Given a set of angles \(\{2\pi\theta_0, 2\pi\theta_1, 2\pi\theta_\infty, 2\pi\theta_j, 1 \leq j \leq n+m\}\), we say an \((n+m)\)-tuple \((t_1, \ldots, t_{n+m})\) is an admissible solution if the curvature equation with the singular points \(t_i\) and \(\theta_{t_j} = \theta_j'\) has a co-axial solution. In this paper, we prove the following main results.

(i) Assume that \(m = 1\), i.e., there is only one integer \(\theta_{t_j}'\) among \(\theta_j\). Then (a) there are at most finitely admissible \((n+1)\)-tuples \((t_1, \ldots, t_{n+1})\). Moreover, let \(A\) be the smallest field containing \(Q\) and all \(\theta_j\)’s and \(Q(A)\) be the smallest field containing \(Q\) and all \(t_j, 1 \leq j \leq n+1\), where \((t_1, \ldots, t_{n+1}) \in A\). Then (b) \(Q(A)\) is a Galois extension of \(Q\) and the degree \([Q(A) : Q]\) is bounded by some constant \(M\) depending only on \(n\) and \(\theta_{n+1}'\). Furthermore, (c) for the case \(n = 0\), we obtain a sharp bound of the cardinality of the set \(A\). We apply this result due to Eremenko, Gabrielov, and Tarasov [10] and the monodromy of hypergeometric equations to obtain such a bound.

(ii) If an \((n+m)\)-tuple \((t_1, \ldots, t_{n+m})\) is admissible and \(m \geq 2\), then the transcendence degree of \(Q(t_1, \ldots, t_{n+m})\) over \(Q\) is less than or equal to \(m - 1\).

1. INTRODUCTION

Let \((S^2, g_0)\) be the 2-dimensional sphere with the standard smooth metric \(g_0\). In this paper, we consider the following classical problem in conformal geometry: Given \(t_1, \ldots, t_N \in S^2\) and a set of positive real numbers \(\theta_1, \ldots, \theta_N\), is there a metric of curvature 1 conformal to \(g_0\) such that the metric is smooth on \(S^2 \setminus \{t_1, \ldots, t_N\}\) and has conic singularities at \(t_j\) with the angle \(2\pi\theta_j\). This problem was solved in [8,2] in the cases where there are at most 2 non-integer angles among \(\{\theta_1, \ldots, \theta_N\}\) (We simply say that the angle is non-integer if \(\theta \not\in \mathbb{Z}\)). In this paper, we consider the case that there are at least 3 non-integer angles among \(\{\theta_1, \ldots, \theta_N\}\).
angles among \( \{ \theta_1, \ldots, \theta_n \} \). Without loss of generality, we may assume that three singularities with non-integer angles are 0, 1, and \( \infty \). Then it is well-known that this problem is equivalent to solving the following curvature equation

\[
\begin{align*}
\Delta u + e^u &= 4\pi(\alpha_0 \delta_0 + \alpha_1 \delta_1 + \sum_{j=1}^{n+m} \alpha_j \delta_j) \quad \text{in } \mathbb{C}, \\
u(x) &= -2(2 + \alpha_\infty) \ln |x| + O(1) \quad \text{as } |x| \to \infty,
\end{align*}
\]

where \( n \geq 0, m \geq 1, \alpha_0, \alpha_1, \alpha_\infty, \alpha_{t_j} \in \mathbb{R}_{>-1} \setminus \mathbb{Z} \) for all \( 1 \leq j \leq n \) and \( \alpha_{t_j} \in \mathbb{N} \) for all \( n + 1 \leq j \leq m \). Geometrically, a solution \( u(x) \) of (1.1) leads to a conformal metric \( ds^2 = \frac{1}{2} e^u |dx|^2 \) of curvature 1 on \( S^2 \) with angles \( 2\pi \theta_p \) at conic singularities \( p \in \{ 0, 1, \infty, t_j \} \), where the angles are given by

\[
\theta_p = \alpha_p + 1, \quad p \in \{ 0, 1, \infty, t_1, \ldots, t_{n+m} \},
\]

so

\[
\theta_p \in \mathbb{N}_{\geq 2}, \quad p \in \{ t_{n+1}, \ldots, t_{n+m} \}, \quad \theta_p \in \mathbb{R}_{>0} \setminus \mathbb{Z}, \quad p \in \{ 0, 1, \infty, t_1, \ldots, t_n \}.
\]

It is known that if (1.1) has a solution \( u(x) \), then there is a developing map \( h(x) \) such that \( h(x) \) is a meromorphic function on \( \mathbb{C} \) satisfying

\[
u(x) = \ln \frac{8|h'(x)|^2}{(1 + |h(x)|^2)^2}, \quad x \in \hat{\mathbb{C}} := \mathbb{C} \setminus \{ 0, 1, t_1, \ldots, t_{m+n} \}.
\]

This is known as the Liouville formula and this \( h(x) \) is multi-valued and has its projective monodromy group contained in \( \text{PSU}(2) \). More precisely, given any \( \ell \in \pi_1(\hat{\mathbb{C}}, q_0) \) where \( q_0 \in \hat{\mathbb{C}} \) is base point, the analytic continuation of \( h(x) \), denoted by \( \ell^* h(x) \), is also a developing map of \( u(x) \) and so \( \ell^* h(x) = \frac{a h(x) + b}{c h(x) + d} \) for some projective monodromy matrix \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{PSU}(2) \).

The curvature equation (1.1) has been extensively studied in the past several decades. For the recent development of this subject and its application, we refer [1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 15] to the interested reader. So far, the existence of solutions for (1.1) with general singular sources is challenging and seems to be out of reach. Mondello and Panov [15] studied a reduced problem: To describe possible angles (i.e. to describe the data \( \{ \theta_0, \theta_1, \theta_\infty, \theta_{t_1}, \ldots, \theta_{t_{n+m}} \} \)) such that (1.1) has solutions for some singular set \( \{ t_1, \ldots, t_{n+m} \} \). They introduced the measure \( d_1(Z^m_{\alpha_0+\alpha_1+\ldots+\alpha_{t_{n+m}}}, \theta - 1) = d_1(Z^m_{\alpha_0+\alpha_1+\ldots+\alpha_{t_{n+m}}}, \alpha) \) on

\[
\theta = (\theta_0, \theta_1, \theta_\infty, \theta_{t_1}, \ldots, \theta_{t_{n+m}}),
\]

i.e.

\[
\theta - 1 = \alpha = (\alpha_0, \alpha_1, \alpha_\infty, \alpha_{t_1}, \ldots, \alpha_{t_{n+m}}),
\]

where \( Z^m_{\alpha_0+\alpha_1+\ldots+\alpha_{t_{n+m}}} \) is the subset of \( Z^m_{\alpha_0+\alpha_1+\ldots+\alpha_{t_{n+m}}} \) consisting of vectors with odd sums of coordinates, and \( d_1 \) is the \( \ell_1 \)-distance. Then they proved the following result.

**Theorem A ([15]).** Suppose that the angles \( \theta_0, \theta_1, \theta_\infty \) and \( \theta_i, 1 \leq i \leq n + m, \) are fixed. Then the following hold.

(1) If (1.1) has a solution \( u(x) \) for some singular set \( \{ t_1, \ldots, t_{n+m} \} \), then

\[
d_1(Z^m_{\alpha_0+\alpha_1+\ldots+\alpha_{t_{n+m}}}, \theta - 1) \geq 1.
\]

Furthermore, if

\[
d_1(Z^m_{\alpha_0+\alpha_1+\ldots+\alpha_{t_{n+m}}}, \theta - 1) = 1,
\]

then there is a developing map \( h(x) \) such that the projective monodromy group of \( h(x) \) is contained in the unit circle, i.e. any projective monodromy matrix of \( h(x) \) is diagonal.
(2) Conversely, if \( d_1(Z_{n+m+n}^m, \theta - 1) > 1 \), then there exists some singular set \( \{ t_1, \ldots, t_{n+m} \} \) such that (1.1) has solutions.

**Definition 1.1.** In [15], the corresponding metric \( \frac{1}{2}e^{u(x)}|dx|^2 \) (if exists) is called co-axial if there is a developing map \( h(x) \) such that the projective monodromy group of \( h(x) \) is contained in the unit circle. In this paper, we also call this solution \( u(x) \) co-axial for convenience.

For the case (1.5), Theorem A shows that solutions must be co-axial if exist. However, the existence of co-axial metrics does not necessarily imply (1.5). Thus, there arises naturally the following question: Are there sufficient conditions on angles that guarantee a singular set \( \{ t_1, \ldots, t_{n+m} \} \) such that (1.1) has a co-axial solution? This question was completely solved by Eremenko [9].

**Theorem 1.2.** If (1.1) has a co-axial solution \( u(x) \) for some singular set \( \{ t_1, \ldots, t_{n+m} \} \), then there are \( \epsilon_p \in \{ \pm 1 \} \) for \( p \in \{ 0, 1, \infty, t_1, \ldots, t_n \} \) such that

\[
    k' := \sum_{p \in \{ 0, 1, \infty, t_1, \ldots, t_n \}} \epsilon_p \theta_p \in \mathbb{Z}_{\geq 0},
\]

\[
    k'' := \sum_{j=n+1}^{n+m} \theta_{t_j} - (n + m + 3) - k' + 2 \in 2\mathbb{Z}_{\geq 0}.
\]

Furthermore, if there is \( \eta \in \mathbb{R}_{>0} \) such that \( c = \eta b = \eta(b_1, \ldots, b_q) \) with \( b_j \in \mathbb{N} \) and \( \gcd(b_1, \ldots, b_q) = 1 \), where

\[
    c := (\theta_0, \theta_1, \theta_\infty, \theta_{t_1}, \ldots, \theta_{t_n}, 1, \ldots, 1),
\]

then we also have

\[
    2 \max_{n+1 \leq j \leq n+m} \theta_{t_j} \leq \sum_{j=1}^q b_j.
\]

Conversely, if there is no such \( \eta \) such that \( c = \eta b \), then (1.6)–(1.7) are also sufficient; if there is such \( \eta \) such that \( c = \eta b \), then (1.6)–(1.8) are also sufficient.

As we have remarked earlier, the existence of co-axial solutions does not necessarily imply (1.5). Let us recall Eremenko’s example [3]: \( (\theta_0, \theta_1, \theta_\infty, \theta_{t_1}, \theta_{t_2}) = (\beta, \beta, 2\beta, 2\beta, 3) \) for some \( \beta \in \mathbb{R}_{>0} \setminus \mathbb{N} \), for which it follows from Theorem B that co-axial solutions exist but it does not satisfy (1.5).

Here we have an interesting observation about Mondello-Panov’s condition (1.5) and Eremenko’s condition (1.8).

**Theorem 1.2.** Suppose that (1.5) holds. Then (1.6)–(1.7) imply (1.8) automatically, namely (1.1) has (co-axial) solutions for some singular set \( \{ t_1, \ldots, t_{n+m} \} \) if and only if (1.6)–(1.7) hold for some \( \epsilon_p \in \{ \pm 1 \} \).

**Theorem 1.2** will be proved in Section 4. Define

\[
    \Delta_{n+m} := \{ (t_1, \ldots, t_{n+m}) \in \mathbb{C}^{n+m} \mid t_j = 0, 1 \text{ or } t_j = t_k \text{ for some } j \neq k \}.
\]

Throughout the paper, we let \( \theta_0, \theta_1, \theta_\infty, \) and \( \theta_{j'}, j = 1, \ldots, n + m, \) be fixed real numbers such that \( \theta_0, \theta_1, \theta_\infty, \) and \( \theta_{j'}, j = 1, \ldots, n, \) are non-integers and \( \theta_{j'}, j = n + 1, \ldots, n + m, \)
are integers greater than 1. Let
\begin{equation}
A = A_\theta := \{(t_1, \cdots, t_{n+m}) \in \mathbb{C}^{n+m} \setminus \triangle_{n+m} \mid (1.1) \text{ with } \theta_{t_j} = \theta'_j, \quad j = 1, \ldots, n+m, \ \text{has co-axial solutions}\}.
\end{equation}

An \((n+m)\)-tuple \((t_1, \ldots, t_{n+m})\) in \(A\) will be called admissible for \((1.1)\). By Theorem 1.1 we know that \(A \neq \emptyset\) if and only if \((1.6)-(1.8)\) hold. In this case, a natural question that interest us is:

**What are the geometric and algebraic properties of the set \(A\), such as its dimension and so on?**

This paper aims to study this problem. Note that \((1.7)\) implies \(m \geq 1\), this is the reason why we assume \(m \geq 1\) in this paper.

Let us consider the special case \(m = 1\) first. Denote
\[Q_\theta := \mathbb{Q}(\theta_0, \theta_1, \theta_\infty, \theta'_1, \ldots, \theta'_n) \subsetneq \mathbb{R}.\]

**Theorem 1.3.** Let \(m = 1\) and \((1.6)-(1.8)\) hold with \(\theta_{t_j} = \theta'_j\), i.e. \(A \neq \emptyset\). Then \(A\) is a finite set with \(#A \leq 2^{n+2} \times (\theta'_{n+1} - 1)!\). Furthermore, for each \(t = (t_1, \cdots, t_{n+1}) \in A\), all \(t_j\)'s are algebraic over \(Q_\theta\) and the field
\[
Q_\theta(A) := \mathbb{Q}_\theta(\{t_j|(t_1, \cdots, t_{n+1}) \in A\})
\]
is a Galois extension of \(Q_\theta\) and
\[
\left|Q_\theta(A) : Q_\theta\right| \leq M < \infty,
\]
where \(M\) is a constant depending on \(n\) and \(\theta'_{n+1}\).

In the special case when there are only three non-integer angles, i.e., when \(n = 0\), we can get a much sharper bound for the cardinality of \(A\). In the following, we will write \(t_1\) and \(\theta'_1\) simply by \(t\) and \(\theta\).

**Theorem 1.4.** Assume that \(n = 0\) and \(m = 1\). Assume that \(\epsilon_0, \epsilon_1 \in \{\pm 1\}\) are signs such that \(k := \theta_\infty + \epsilon_0 \theta_0 + \epsilon_1 \theta_1 \in \mathbb{Z}\) and \(\theta \equiv k \mod 2\). Then the cardinality of the set \(A\) is
\[
\begin{cases}
0, & \text{if } \theta \leq |k|, \\
(\theta^2 - k^2)/4, & \text{if } \theta > |k|,
\end{cases}
\]
counted with multiplicities (see Section 5 for an explanation of multiplicities). Moreover, if \(t \in A\), then the degree of \(t\) over \(Q_\theta\) is at most \((\theta^2 - k^2)/4\).

We remark that the condition \(\theta \equiv k \mod 2\) is necessary for \(A \neq \emptyset\), by \((1.7)\). Interestingly, a key ingredient in the proof of the theorem is the monodromy of the classical hypergeometric equations.

Clearly, Theorem 1.4 yields a sharp non-existence result for \((1.1)\) when \(n = 0\) and \(m = 1\). That is, when \(\theta \equiv k \mod 2\), if \(t \in \mathbb{C}\) is transcendental over \(Q_\theta\) or is algebraic of degree greater than \((\theta^2 - k^2)/4\) over \(Q_\theta\), then \((1.1)\) has no solutions. It would be an interesting problem to obtain a similar result for the case \(n > 0\).

For general cases \(m \geq 2\), we can not expect that \(A\) is a finite set.

**Theorem 1.5.** Let \(m \geq 2\) and \((1.6)-(1.8)\) hold, i.e. \(A \neq \emptyset\). Then \(A\) is an algebraic set of dimension \(\leq m - 1\). Furthermore, if we further assume
\begin{equation}
\theta_0, \theta_1, \theta_\infty, \theta'_1, \ldots, \theta'_n \in \overline{\mathbb{Q}}_{>0} \setminus \mathbb{Z},
\end{equation}
then for any \(t = (t_1, \ldots, t_{n+m}) \in A\), the transcendence degree of \(\overline{\mathbb{Q}}(t)\) over \(\overline{\mathbb{Q}}\) is \(\leq m - 1\).
One of the main ideas in our proofs of all the main results is the integrability of the curvature equation. This integrability property allows us to connect the partial differential equation with a second-order Fuchsian ordinary differential equation. In Section 2, we will derive those ODE’s. Applying integrability in terms of the developing maps, we will prove an equivalent condition for the existence of co-axial solutions. Based on this equivalent condition, we will prove Theorems 1.3 and 1.5 in Section 3. In Section 4, we give a proof of Theorem 1.2. In the final section, we present a proof of Theorem 1.4 using hypergeometric equations.

2. THE INTEGRABILITY OF CURVATURE EQUATIONS AND COMPLEX ODEs

In this section, we briefly review how to connect the integrability of the curvature equation (1.1) with a class of second order ODEs in complex variables; see e.g. [1, 9]. Given a solution \( u(x) \) of (1.1), we define

\[
Q(x) := -\frac{1}{2} \left( u_{xx}(x) - \frac{1}{2} u_x(x)^2 \right), \quad x \in \mathbb{C}.
\]

Then it is easy to see \( Q(x) = 0 \), so \( Q(x) \) is a meromorphic function in \( \mathbb{C} \) with poles belonging to \( \mathcal{I} := \{0, 1, t_1, \ldots, t_{n+m+1}\} \).

More precisely, at each \( p \in \mathcal{I} \), since (1.1) gives

\[
u(x) = 2 \alpha_p \ln |x - p| + O(1), \quad \text{for } x \text{ near } p,
\]

we have

\[
Q(x) = \frac{\alpha_p(\alpha_p + 2)}{4} (x - p)^{-2} + O((x - p)^{-1}), \quad \text{for } x \text{ near } p,
\]

and

\[
Q(x) = \frac{\alpha_{\infty}(\alpha_{\infty} + 2)}{4} x^{-2} + O(x^{-3}), \quad \text{for } x \to \infty.
\]

Thus,

\[
Q(x) = \sum_{p \in \mathcal{I}} \left[ \frac{\beta_p}{(x - p)^2} + \frac{d_p}{(x - p)} \right],
\]

where

\[
\beta_p := \frac{\alpha_p(\alpha_p + 2)}{4}, \quad p \in \{0, 1, t_1, \ldots, t_{n+m}, \infty\},
\]

and \( d_p \)'s are some constants satisfying

\[
\sum_{p \in \mathcal{I}} d_p = 0, \quad \sum_{p \in \mathcal{I}} (\beta_p + pd_p) = \beta_{\infty}.
\]

Thus, \( d_0 \) and \( d_1 \) can be uniquely determined by

\[
d := (d_{t_1}, \ldots, d_{t_{n+m}}), \quad t := (t_1, \ldots, t_{n+m})
\]

as

\[
d_1 = \beta_{\infty} - \sum_{p \in \mathcal{I} \setminus \{0, 1\}} (\beta_p + pd_p), \quad d_0 = \sum_{p \in \mathcal{I} \setminus \{0, 1\}} (\beta_p + (p - 1)d_p) - \beta_{\infty},
\]

while \( d \) could be considered as \( n + m \) unknown coefficients.
On the other hand, as mentioned in the introduction, the Liouville theorem (see e.g. [1]) says that for a solution \( u(x) \) of (1.1), there is a developing map \( h(x) \) such that

\[
u(x) = \ln \frac{8|h'(x)|^2}{(1 + |h(x)|^2)^2}, \quad x \in \hat{\mathbb{C}} := \mathbb{C} \setminus \mathcal{I}.
\]

The developing map \( h(x) \) is multi-valued in \( \mathbb{C} \) and due to (2.6), the projective monodromy group of \( h(x) \) is contained in \( \text{PSU}(2) \). More precisely, given any \( \ell \in \pi_1(\hat{\mathbb{C}}, q_0) \) where \( q_0 \in \mathbb{C} \) is a base point, the analytic continuation of \( h(x) \), denoted by \( \ell^* h(x) \), is also a developing map of \( u(x) \) and so \( \ell^* h(x) = \frac{ah(x) + b}{ch(x) + d} \) for some projective monodromy matrix \( \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{PSU}(2) \).

To connect \( Q(x) \) and \( h(x) \), we note that (2.6) implies

\[
\{ h(x), x \} := \left( \frac{h''(x)}{h'(x)} \right)' - \frac{1}{2} \left( \frac{h''(x)}{h'(x)} \right)^2 = u_{xx}(x) - \frac{1}{2}u_x(x)^2 = -2Q(x),
\]

where \( \{ h(x), x \} \) is the Schwarz derivative. We refer the reader to [11] for details of computation.

Consider the following second order linear ODE with the complex variable \( x \):

\[
y''(x) = Q(x)y(x), \quad x \in \mathbb{C}.
\]

Then a classical result (see [16]) says that for any basis \( \{ \hat{y}_1, \hat{y}_2 \} \) of (2.8), the Schwarz derivative \( \{ \hat{y}_2/\hat{y}_1, x \} \) always satisfies

\[
\{ \hat{y}_2/\hat{y}_1, x \} = -2Q(x).
\]

From this and (2.7), we conclude that if (2.8) is derived from a solution of (1.1), then there is a basis \( \{ y_1, y_2 \} \) of (2.8) such that

\[
h(x) = y_2(x)/y_1(x).
\]

Furthermore, the projective monodromy group of (2.8) with respect to \( y_1 \) and \( y_2 \) is the same as that of \( h(x) \), i.e. is contained in \( \text{PSU}(2) \).

Conversely, given \( Q(x) \) via (2.2)–(2.4), if the corresponding ODE (2.8) has a basis \( \{ y_1, y_2 \} \) such that the projective monodromy group of the ratio \( h(x) = y_2(x)/y_1(x) \) is contained in \( \text{PSU}(2) \), then \( u(x) := \ln \frac{8|h'(x)|^2}{(1 + |h(x)|^2)^2} \) is a solution of (1.1).

For the reader’s convenience, we now briefly recall some basic notions of ODEs like (2.8). We refer the reader to [13, 16] for a comprehensive introduction. Generally, (2.8) is called Fuchsian because the pole order of \( Q(x) \) is at most 2 at any singularities. Note that the Riemann scheme of (2.8) is given by

\[
\begin{pmatrix} 0 & 1 & t_1 & \cdots & t_{n+m} & \infty \\ -\frac{\alpha_0}{2} & -\frac{\alpha_1}{2} & -\frac{\alpha_1}{2} & \cdots & -\frac{\alpha_{n+m}}{2} & -(\frac{2n}{2} + 1) \\ -\frac{\alpha_0}{2} + 1 & -\frac{\alpha_1}{2} + 1 & -\frac{\alpha_1}{2} + 1 & \cdots & -\frac{\alpha_{n+m}}{2} + 1 & \frac{\alpha_m}{2} \end{pmatrix}.
\]

Observe that the exponent difference at \( t_j \) is an integer for \( n + 1 \leq j \leq n + m \). Since \( Q(x) \) comes from a solution \( u(x) \) of (1.1), it is well-known that (2.8) is apparent at \( t_j \) (i.e. no logarithmic singularity near \( t_j \)) and so the local monodromy matrix of (2.8) at \( t_j \) is \((-1)^{\alpha_j} I_2 \) for all \( n + 1 \leq j \leq n + m \). In this paper, we will also consider a Fuchsian differential equation (2.8) which may not come from a solution of (1.1). In this case, unless otherwise specified, we always assume that (2.8) with the Riemann scheme (2.11) is apparent at \( t_j \) for all \( n + 1 \leq j \leq n + m \).

It is well-known that the necessary and sufficient condition of (2.8) being apparent at \( t_j \) for all \( n + 1 \leq j \leq n + m \) can be derived from the Frobenius method.
There are polynomials

\[ \hat{P}_j(d, t) \in \mathbb{Q}_\theta(t)[d] = d_{t_j}^{\theta_j} + \text{l.o.t in } d \in \mathbb{Q}_\theta(t)[d] \]

of total degree \( \theta_j \) in \( d_j \), \( j = n + 1, \ldots, n + m \), such that (2.8) is apparent at \( t_j \) for all \( n + 1 \leq j \leq n + m \) if and only if \( d \) and \( t \) satisfy \( \hat{P}_j(d, t) = 0 \) for all \( j = n + 1, \ldots, n + m \).

Furthermore, \( \hat{P}_j(d, t) \) can be written as

\[ (2.12) \quad \hat{P}_j(d, t) = \prod_{p \in \mathcal{I}\{t_j\}} (p - t_j)^{N_j}, \]

for some \( N_j \in \mathbb{Z}_{\geq 0} \) and \( \mathcal{P}_j(d, t) \in \mathbb{Q}_\theta(d, t) \) such that \( \mathcal{P}_j(d, t) \) and \( \prod_{p \in \mathcal{I}\{t_j\}} (p - t_j)^{N_j} \) are coprime. Thus, (2.8) is apparent at \( t_j \) for all \( n + 1 \leq j \leq n + m \) if and only if \( \mathcal{P}_j(d, t) = 0 \) for all \( j = n + 1, \ldots, n + m \).

Proof. Let us take \( t_{n+m} \) for example and in the following proof, we write \( t_{n+m} = t \) for convenience. By the Frobenius method, (2.8) is apparent at the singularity \( t \) if and only if it has a solution of the form

\[ y(x) = \sum_{j=0}^{\infty} c_j u^{-\alpha_j/2}, \quad u = x - t, \quad c_0 = 1. \]

Observe that

\[ Q(x) = \sum_{p \in \mathcal{I}} \left[ \frac{\beta_p}{(x - p)^2} + \frac{d_p}{(x - p)} \right] \]

\[ = \frac{\beta_t}{u^2} + \frac{d_t}{u} + \sum_{p \in \mathcal{I}\{t\}} \left[ \frac{\beta_p}{(u + t - p)^2} + \frac{d_p}{u + t - p} \right] \]

\[ = \frac{\beta_t}{u^2} + \frac{d_t}{u} + \sum_{j=0}^{\infty} (-1)^j \sum_{p \in \mathcal{I}\{t\}} \frac{\beta_p (j + 1) + d_p (t - p)}{(t - p)^{j+2}} u^j \]

\[ = \sum_{j=0}^{\infty} \eta_j u^{j-2}, \]

where

\[ \eta_0 = \beta_t, \quad \eta_1 = d_t, \]

\[ \eta_j = (-1)^j \sum_{p \in \mathcal{I}\{t\}} \frac{\beta_p (j + 1) + d_p (t - p)}{(t - p)^{j+2}}, \quad j \geq 2, \]

i.e. \( \prod_{p \in \mathcal{I}\{t\}} (t - p)^j \eta_j \in \mathbb{Q}_0[d, t] \).

Consequently, \( y''(x) = Q(x) y(x) \) if and only if

\[ (2.13) \quad j(j - \theta_t) c_j = \sum_{k=0}^{j-1} \eta_j - k c_k, \quad \forall j \geq 0, \]

where \((j - \alpha_t/2)(j - \alpha_t/2 - 1) - \eta_0 = j(j - \alpha_t - 1) = j(j - \theta_t)\) is used. Clearly (2.13) holds automatically for \( j = 0 \), and (2.13) with \( j = 1 \) leads to \( c_1 = \frac{\sum_{k=0}^{j-1} \eta_j - k c_k}{j(j - \theta_t)} \). By an induction argument, for any \( 2 \leq j \leq \theta_t - 1 \), \( c_j \) can be uniquely solved by (2.13) as

\[ c_j = r_j \left[ d_t^{\theta_t} + \text{l.o.t in } d \right] \in \mathbb{Q}_\theta(t)[d] \]
with total degree $j$ in $d$, where $r_j \in \mathbb{Q}_\theta \setminus \{0\}$. Furthermore,
\[
\prod_{p \in \mathcal{T} \setminus \{t\}} (t - p)^j c_j \in \mathbb{Q}_\theta[d, t].
\]
Consequently, \((2.8)\) with $j = \theta_i$ leads to the existence of $r_{\theta_i} \in \mathbb{Q}_\theta \setminus \{0\}$ and a polynomial
\[
\hat{P}(d, t) = d_i^{\theta_i} + 1.0.1 \text{ in } d \in \mathbb{Q}_\theta(t)[d]
\]
with total degree $\theta_i$ in $d$ such that
\[
\theta_{i-1} \sum_{k=0}^{\theta_i-1} \eta_{j-k} c_k = r_{\theta_i} \hat{P}(d, t).
\]
Then it is standard by the Frobenius theory that \((2.8)\) is apparent at the singularity $t$ if and only if $\hat{P}(d, t) = 0$. Furthermore, the above argument also implies the existence of an integer $0 \leq N \leq \theta_i$ such that
\[
P(d, t) := \hat{P}(d, t) \prod_{p \in \mathcal{T} \setminus \{t\}} (t - p)^N \in \mathbb{Q}_\theta[d, t]
\]
and $P(d, t), \prod_{p \in \mathcal{T} \setminus \{t\}} (t - p)^N$ are coprime.

The case $n = 0$ is special and we plan to study the solvability of \((1.1)\) for this case in another paper. In this case, the ODE \((2.8)\) can be transformed into a Heun equation and this Heun equation has been well-studied in \([8, 11]\). In Section 5, we will use some results from \([8]\) and hypergeometric equations to prove Theorem 1.4.

### 3. Proofs of Theorems 1.3 and 1.5

This section is devoted to the proofs of Theorems 1.3 and 1.5. In the following we use notations
\[
\mathcal{I} := \{0, 1, t_1, \ldots, t_{n+m}\}, \quad \mathcal{I}_1 := \{0, 1, t_1, \ldots, t_n\}.
\]

#### 3.1. General setting

As mentioned in the introduction, following \([15]\), we call the metric $\frac{1}{4}e^{u(x)}|dx|^2$ and also the solution $u(x)$ of \((1.1)\) to be co-axial if there is a developing map $\hat{h}(x)$ of $u(x)$ such that the projective monodromy group of $h(x)$ is contained in the unit circle, that is, for any $\gamma \in \pi_1(\mathcal{C}, q_0)$, the analytic continuation of $h(x)$ along $\gamma$, denoted by $\gamma^* h(x)$, satisfies $\gamma^* h(x) = \lambda(\gamma) h(x)$ for some $\lambda(\gamma) \in \mathbb{C}$ satisfying $|\lambda(\gamma)| = 1$. Clearly this is equivalent to that the monodromy group of the associated ODE \((2.8)\) with $Q(x) = -\frac{1}{2}(u_{xx}(x) - \frac{1}{2}u_x(x)^2)$ is commutative. In Section 2, we have discussed the equivalence between the existence of a solution $u(x)$ of \((1.1)\) and a Fuchsian ODE \((2.8)\) with $Q(x)$ given by \((2.2)\) satisfying $P_j(d, t) = 0$ for all $n + 1 \leq j \leq n + m$ (see Theorem 2.4) such that the monodromy group is conjugate to a subgroup of SU(2). This equivalence can be further strengthened as follows when the solution $u(x)$ is co-axial.

In fact, given such an ODE \((2.8)\), take the basis of local solutions near $\infty$ as follows
\[
y_+(x) = x^{-\alpha} \sum_{j=0}^{\infty} c_+^j x^{-j}, \quad c_+, 0 = 1,
\]
\[
y_-(x) = x^{\frac{\alpha}{2}} + 1 \sum_{j=0}^{\infty} c_-^j x^{-j}, \quad c_-, 0 = 1.
\]
Suppose that (2.8) comes from a co-axial metric. By (2.10), we have \( h(x) = y_2(x)/y_1(x) \) for some solutions \( y_1(x), y_2(x) \) of (2.8). Consider a large circle \( |x| = R \) and let \( h(e^{2\pi i x}) \) be the analytic continuation of \( h(x) \) along the circle. Then by the co-axial condition, one has
\[
h(e^{2\pi i x}) = \lambda h(x)
\]
for some \( \lambda \neq 0 \). On the other hand, there is a matrix \( M_\infty = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C}) \) such that
\[
\left( \begin{array}{c} y_1(e^{2\pi i x}) \\ y_2(e^{2\pi i x}) \end{array} \right) = M_\infty \left( \begin{array}{c} y_1(x) \\ y_2(x) \end{array} \right).
\]
These two identities together imply that \( b = c = 0 \). Thus, after multiplying by some nonzero constants, we may assume that
\[
h(x) := y_+(x)/y_-(x),
\]
which satisfies
\[
h(x) = x^{-1-\alpha}(1 + O(x^{-1})) = x^{-\theta_\infty}(1 + O(x^{-1})), \quad \text{as } x \to \infty.
\]
Then under the basis \((y_+(x), y_-(x))\), we have \( M_\infty = \begin{pmatrix} e^{-\pi i \theta_\infty} & 0 \\ 0 & e^{\pi i \theta_\infty} \end{pmatrix} \) with \( e^{-\pi i \theta_\infty} \neq e^{\pi i \theta_\infty} \). Here for any \( p \in \mathcal{I} \cup \{\infty\}, \) we use \( M_p = \text{SL}(2, \mathbb{C}) \) to denote the monodromy matrix of ODE (2.8) with respect to a simple loop in \( \pi_1(\hat{\mathcal{C}}, q_0) \) that encircles \( p \) once. Now suppose that the monodromy group of (2.8) is commutative (this holds if (2.8) comes from a co-axial solution \( u(x) \)), then it follows that all the monodromy matrices under the basis \((y_+(x), y_-(x))\) are diagonal, i.e. \( M_p = \begin{pmatrix} e^{\pi i \alpha_p} & 0 \\ 0 & e^{-\pi i \alpha_p} \end{pmatrix} \) for \( p \in \mathcal{I}_1 = \{0, 1, t_1, \cdots, t_n\} \).

This implies that after analytic continuation,
\[
y_\pm(x) = (x - p)^{\rho_\pm} (c_\pm \theta_p + O(|x - p|)) \quad \text{near } p \in \mathcal{I}_1,
\]
where \( \{\rho^+_p, \rho^-_p\} = \{-\alpha_p/2, \alpha_p/2 + 1\} \) are the local exponents of (2.8) at \( p \) and \( c_\pm \neq 0 \). Thus there exists \( \epsilon_p \in \{\pm 1\} \) for \( p \in \mathcal{I}_1 \) such that
\[
h(x) = (x - p)^{\epsilon_p \theta_p} (c_p + O(x - p)), \quad \text{as } x \to p \in \mathcal{I}_1, \quad c_p \neq 0.
\]
From here and (3.3), we see that \( h(x) \) can be written as
\[
h(x) = \hat{h}(x) \prod_{p \in \mathcal{I}_1} (x - p)^{\epsilon_p \theta_p},
\]
where \( \hat{h}(x) \) is meromorphic in \( \mathbb{C} \) and satisfies \( \hat{h}(x) = x^{-\theta_\infty - c_0 \theta_0 - \epsilon_1 \theta_1 - \sum_{j=1}^{n} \epsilon_j \theta_j} (1 + O(x^{-1})) \) as \( x \to \infty \), so \( \hat{h}(x) \) is a rational function.

Conversely, given an ODE (2.8) with (2.11), which might not come from a solution of (1.1), we consider \( h(x) = y_+(x)/y_-(x) \) as in (5.1)-(5.2). If there are \( \epsilon_p \in \{\pm 1\} \) for \( p \in \mathcal{I}_1 \) such that \( h(x) \) has the expression (3.4) with \( \hat{h}(x) \) being a rational function, then for any \( \gamma \in \pi_1(\hat{\mathcal{C}}, q_0) \), \( \gamma^* h(x) = \lambda(\gamma) h(x) \) with \( |\lambda(\gamma)| = 1 \) (i.e. the projective monodromy group of \( h(x) \) is a subgroup of the unit circle, or equivalently the monodromy matrices of (2.8) are all diagonal under the basis \((y_+(x), y_-(x))\)). Thus
\[
u(x) := \ln \frac{|\hat{h}'(x)|^2}{(1 + |h(x)|^2)^2}
\]
is well-defined in \( \mathbb{C} \). Then a direct computation shows that \( u(x) \) is a co-axial solution of (1.1) with \( h(x) \) being a developing map. Therefore, the above argument proves the following result.
Lemma 3.1. The equation \((1.1)\) has co-axial solutions if and only if there is an ODE \((2.8)\) with the Riemann scheme \((2.11)\) such that \(h(x) = y_+(x)/y_-(x)\) has the expression \((3.4)\) for some \(\epsilon_p \in \{\pm 1\}\) for \(p \in \mathcal{I}_1\) and some rational function \(\hat{h}(x)\).

Remark that \(h(x)\) or equivalently \(\hat{h}(x)\) might have zeros or poles at \(t_{n+j}\) for some \(j \geq 1\). More precisely, the Riemann scheme \((2.11)\) indicates that the exponent difference is \(\alpha_j + 1 = \theta_j\) at \(t_j\), so near \(t_j\) with \(n + 1 \leq j \leq n + m\), we have

\[
h(x) = \frac{y_+(x)}{y_-(x)} = (x - t_j)\beta (c_t + O(x - t_j))
\]

for some \(\beta \in \{0, \pm \theta_j\}\), \(c_t \neq 0\),

and so does \(\hat{h}(x)\). Therefore, there is \(\mathcal{I}_1 \subset \mathcal{J}_1 \subset \mathcal{I}\) and \(\epsilon_p \in \{\pm 1\}\) for \(p \in \mathcal{J}_1 \setminus \mathcal{I}_1\) such that

\[
h(x) = \prod_{p \in \mathcal{J}_1} (x - p)^{\epsilon_p} \prod_{j=1}^{m_1} \frac{1}{x - a_j} \prod_{j=1}^{m_2} \frac{1}{x - b_j},
\]

where \(a_j, b_k\)’s satisfy

\[
a_j, b_k \in \mathbb{C} \setminus \mathcal{I}, \quad \{a_1, \ldots, a_{m_1}\} \cap \{b_1, \ldots, b_{m_2}\} = \emptyset.
\]

In other words,

\[
\mathcal{J}_1 \setminus \mathcal{I}_1 = \{t_{n+j} | h(t_{n+j}) \in \{0, \infty\}, 1 \leq j \leq m\}.
\]

Then by \(h(x) = y_+(x)/y_-(x)\), we see that each \(a_j\) (resp. \(b_j\)) is a zero of \(y_+(x)\) (resp. \(y_-(x)\)) and must be simple, namely each \(a_j\) is a simple zero of \(h(x)\) and each \(b_j\) is a simple pole of \(h(x)\), so \(a_j \neq a_k\) and \(b_j \neq b_k\) for any \(j \neq k\). Therefore,

\[
\text{Any two elements of the collection } [p \in \mathcal{I}; a_j, 1 \leq j \leq m_1; b_k, 1 \leq k \leq m_2]\text{ are distinct.}
\]

Clearly \((3.3)\) implies

\[
\sum_{p \in \mathcal{J}_1} \epsilon_p \theta_p + m_1 - m_2 = -\theta_\infty.
\]

By \((3.5)\) we have

\[
h'(x) = h(x) \left[ \sum_{p \in \mathcal{J}_1} \frac{\epsilon_p \theta_p}{x - p} + \sum_{j=1}^{m_1} \frac{1}{x - a_j} - \sum_{j=1}^{m_2} \frac{1}{x - b_j} \right]
\]

\[
= \prod_{p \in \mathcal{J}_1} (x - p)^{\epsilon_p} \prod_{j=1}^{m_1} \frac{1}{x - a_j} \prod_{j=1}^{m_2} \frac{1}{x - b_j}
\]

\[
= \prod_{p \in \mathcal{J}_1} (x - p)^{\epsilon_p \theta_p - 1} \frac{G(x)}{\prod_{j=1}^{m_2} (x - b_j)^2},
\]

where \(G(x)\) is a polynomial defined by

\[
G(x) := \prod_{j=1}^{m_1} (x - a_j) \prod_{j=1}^{m_2} (x - b_j) \sum_{p \in \mathcal{J}_1} \epsilon_p \theta_p \prod_{q \in \mathcal{J}_1 \setminus \{p\}} (x - q)
\]

\[+ \prod_{p \in \mathcal{J}_1} (x - p)^{m_2} \prod_{j=1}^{m_1} (x - b_j) \sum_{k=1}^{m_1} \prod_{j=1, j \neq k}^{m_1} (x - a_j)
\]

\[+ \prod_{p \in \mathcal{J}_1} (x - p)^{m_2} \prod_{j=1}^{m_1} (x - a_j) \sum_{k=1}^{m_2} \prod_{j=1, j \neq k}^{m_2} (x - b_j)
\]
Proof.

Note that (3.8) and (3.12) can be written as 
\[
\frac{1}{J} = \frac{\alpha_{\infty}}{\theta_{\infty}}x^L + \text{l.o.t},
\]
where \(|J| := \#J \geq n + 2\) and
\[
L := m_1 + m_2 + |J| - 1.
\]
(3.11)

Note that the last identity is due to (3.8).

Lemma 3.2. Suppose \(h(x)\) given by (3.5) is a developing map of a co-axial solution \(u(x)\) of (1.1), and let \(G(x)\) be defined by (3.10). Then
\[
L = m_1 + m_2 + |J| - 1 = \sum_{p \in \mathcal{I} \setminus \mathcal{J}_1} \alpha_p,
\]
(3.12)
\[
G(x) = -\theta_{\infty} \prod_{p \in \mathcal{I} \setminus \mathcal{J}_1} (x - p)^{\alpha_p}.
\]
(3.13)

Proof. Since \(h(x)\) is a developing map of \(u(x)\), it follows from (2.7) that
\[
2Q(x) = \left(\frac{h''(x)}{h'(x)}\right)' - \frac{1}{2} \left(\frac{h''(x)}{h'(x)}\right)^2.
\]
(3.14)

Let \(G(p) = 0\), first we claim that \(p \in \mathcal{I} \setminus \mathcal{J}_1\). Suppose not. Then
\[
p \notin \{0, 1, t_1, \ldots, t_{n+m}, a_1, \ldots, a_{m_1}, b_1, \ldots, b_m\}.
\]

Denote by \(\gamma \geq 1\) to be the zero order of \(G(x)\) at \(p\), then \(h'(x) = c(x - p)^\gamma (1 + O(x - p))\) with \(c \neq 0\) and so a direct computation via (3.14) implies
\[
-2Q(x) = -\gamma (\gamma + 2) 2(x-p)^{-2} + O((x-p)^{-1}),
\]
(3.15)
a contradiction with the fact that \(Q(x)\) is holomorphic in \(\mathbb{C} \setminus \{0, 1, t_1, \ldots, t_{n+m}\}\). This proves \(p \in \mathcal{I} \setminus \mathcal{J}_1\).

Now for each \(p \in \mathcal{I} \setminus \mathcal{J}_1\), we claim that \(p\) is a zero of \(G(x)\) with order \(\alpha_p\). Suppose \(h'(x) = c(x - p)^\gamma (1 + O(x - p))\) with \(c \neq 0\) and \(\gamma \in \mathbb{Z}_{\geq 0}\). Again a direct computation via (3.14) implies (3.15), so \(\gamma (\gamma + 2)/4 = \beta_p = \alpha_p(\alpha_p + 2)/4\), namely \(\gamma = \alpha_p\). This proves (3.12) - (3.13).

Corollary 3.3. Suppose that \(h(x)\) given by (3.5) is a developing map of a co-axial solution \(u(x)\) of (1.1). Then \(m_1, m_2 \in \mathbb{Z}_{\geq 0}\) are determined by
\[
2m_1 = \sum_{p \in \mathcal{I} \setminus \mathcal{J}_1} \theta_p - \sum_{p \in \mathcal{J}} \epsilon_p \theta_p - (n + m + 1),
\]
(3.16)
\[
2m_2 = \sum_{p \in \mathcal{I} \setminus \mathcal{J}_1} \theta_p + \sum_{p \in \mathcal{J}} \epsilon_p \theta_p - (n + m + 1),
\]
(3.17)
where \(\mathcal{J} := \mathcal{J}_1 \cup \{\infty\}\) and \(\epsilon_\infty = 1\). In particular, \(\mathcal{I} \setminus \mathcal{J}_1 \neq \emptyset\).

Proof. Note that (3.3) and (3.12) can be written as
\[
m_1 - m_2 = -\sum_{p \in \mathcal{J}_1} \epsilon_p \theta_p - \theta_\infty = -\sum_{p \in \mathcal{J}} \epsilon_p \theta_p,
\]
Comparing (3.18) and (3.19), we conclude that (3.13) holds if and only if

We say that a common zero of the polynomials

where we use (3.12) and

then

(3.20)

is of total degree \(L\) and

\[
C_j(t_2) = \sum_{j=1}^{L} C_j(t_2)x^{L-j},
\]

where we use (3.12) and \(C_j(t_2) \in \mathbb{Q}[t_2]\) are symmetric polynomials of \(t_2\) of degree \(j\). Comparing (3.18) and (3.19), we conclude that (3.13) holds if and only if \((a, b, t)\) is a common zero of the polynomials

(3.20)

\[B_j(a, b, t) := R_j(a, b, t_1) - C_j(t_2), \quad 1 \leq j \leq L.\]

Remark that \(B_j(a, b, t)\) is of total degree \(j\).

Definition 3.4. We say that a common zero \((a, b, t) \in \mathbb{C}^{m_1+m_2+n+m}\) of the polynomials \(B_j\)'s in (3.20) is admissible if any two elements of \((a, b, t_1, \ldots, t_{n+m})\) do not equal and none of them equals to 0, 1.

Then the above argument shows that once (2.8) comes from a co-axial solution \(u(x)\) of (1.1), then there exist \(I_1 \subset J_1 \subsetneq I\) and \(\epsilon_p \in \{\pm 1\}\) for \(p \in J_1\) such that \(m_1, m_2\) defined by (3.16)–(3.17) are non-negative integers, (3.12) holds and the corresponding polynomials \(B_j\)'s in (3.20) has an admissible zero \((a, b, t)\). The following result shows that the converse statement also holds.

Lemma 3.5. Suppose there exist \(I_1 \subset J_1 \subsetneq I\) and \(\epsilon_p \in \{\pm 1\}\) for \(p \in J_1\) such that \(m_1, m_2\) defined by (3.16)–(3.17) are non-negative integers, i.e. (3.12) holds and the corresponding polynomials \(B_j\)'s in (3.20) are well-defined. If \(B_j\)'s have an admissible zero \((a, b, t)\), then (1.1) has a co-axial solution \(u(x)\) with its developing map given by (3.5).
Proof. Under the above assumptions, we consider the function \( h(x) \) given by (3.5). Then \( h'(x) \) is given by (3.9) with \( G(x) \) given by (3.10). Since \((a, b, t)\) is an admissible zero of polynomials \( B_j\)’s, we see that \( G(x) \) satisfies (3.13). Inserting (3.13) into (3.9), we obtain

\[
(3.21) \quad h'(x) = -\theta_{\infty} \frac{\prod_{p \in \mathfrak{I}} (x - p)^{e_p \theta_{p - 1}} \prod_{p \in \mathfrak{J}} (x - p)^{a_p}}{\prod_{j=1}^{m_2} (x - b_j)^2}.
\]

Furthermore, (3.16) and (3.17) imply (3.8), so

\[
(3.22) \quad h(x) = x^{-\theta_{\infty}} (1 + O(x^{-1})), \quad \text{as } x \to \infty.
\]

It follows that

\[
(3.23) \quad h'(x) = -\theta_{\infty} x^{-\theta_{\infty} - 1} (1 + O(x^{-1})), \quad \text{as } x \to \infty.
\]

Define \( Q(x) \) via this \( h(x) \) by (3.14), i.e. \( Q(x) = -\frac{1}{2} \{ h(x), x \} \). Then by (3.21)–(3.23), a direct computation shows that

\[
(3.24) \quad Q(x) = \frac{\theta_{\infty}^2 - 1}{4(x - p)^2} + \frac{d_p}{x - p} + O(1), \quad \text{near } p \in \mathfrak{I} = \{ 0, 1, t_1, \ldots, t_{m+n} \},
\]

\[
(3.25) \quad Q(x) = \frac{\theta_{\infty}^2 - 1}{4x^2} + O(x^{-3}), \quad \text{near } x = \infty,
\]

\[
(3.26) \quad Q(x) = \frac{d_{b_j}}{x - b_j} + O(1), \quad \text{near } x = b_j,
\]

for some \( d_p \in \mathbb{C} \), and \( Q(x) \) is holomorphic at elsewhere. We need to prove \( d_{b_j} = 0 \).

Consider the corresponding ODE (2.3) with this \( Q(x) \). Again by \( Q(x) = -\frac{1}{2} \{ h(x), x \} \), a classical result says that there is a basis \( (y_1(x), y_2(x)) \) of solutions of (2.8) such that \( h(x) = y_1(x)/y_2(x) \). From here and the expression (3.5) of \( h(x) \), we conclude that any solution of (2.8) has no logarithmic singularities at any singularities. On the other hand, by (3.26) we know that the local exponents of (2.8) at \( b_j \) are 0, 1, so there is a local solution of the following form

\[
\tilde{y}(x) = 1 + c_1(x - b_j) + c_2(x - b_j)^2 + \cdots, \quad \text{near } b_j.
\]

Inserting this and (3.26) into \( \tilde{y}''(x) = Q(x)\tilde{y}(x) \) we immediately obtain \( d_{b_j} = 0 \), i.e. \( Q(x) \) is holomorphic at any \( b_j \). Thus it follows from (3.24)–(3.25) that the Riemann scheme of (2.8) is (2.11), and (2.8) is apparent at \( t_j \) for any \( n + 1 \leq j \leq n + m \).

Let \((y_+(x), y_-(x))\) be the basis of local solutions of (2.8) given by (3.1). Then \((y_1(x), y_2(x)) = (y_+(x), y_-(x)) (a b c d)^{-1} \) for some \( ad - bc \neq 0 \), so \( h(x) = \frac{ay_+(x) + cy_-(x)}{by_+(x) + dy_-(x)} \). Since the under the basis \((y_+(x), y_-(x))\), we have \( M_{\infty} = \left( \begin{array}{cc} e^{-2\pi i \alpha_{\infty}} & 0 \\ 0 & e^{2\pi i \alpha_{\infty}} \end{array} \right) \), so we see from (3.22) that

\[
e^{-2\pi i \alpha_{\infty}} \frac{ay_+(x) + cy_-(x)}{by_+(x) + dy_-(x)} = e^{-2\pi \theta_{\infty}} h(x).
\]

From here and \( e^{-2\pi i \alpha_{\infty}} \neq 1 \), we easily obtain \( b = c = 0 \) and so \( h(x) = ay_+(x)/dy_-(x) = y_+(x)/y_-(x) \), where \( a/d = 1 \) follows from (3.22) and (3.1). Then by Lemma 3.1 we conclude that (1.1) has a co-axial solution \( u(x) \) with \( h(x) \) being its developing map. \( \square \)

Therefore, the problem turns to study the admissible zeros of the polynomials \( B_j\)’s. The number of the polynomials is \( L = m_1 + m_2 + |\mathfrak{J}_1| - 1 \) with \( |\mathfrak{J}_1| \leq |\mathfrak{I}| - 1 = n + m + 1 \), and the number of variables is \( m_1 + m_2 + n + m \geq L \). Thus there are two different cases: (i) \( m_1 + m_2 + n + m = L \) and (ii) \( m_1 + m_2 + n + m > L \). For the case (i), we may
expect that the number of common zeros of the polynomials $B_j$'s is finite, but we cannot expect the validity of this assertion for the case (ii).

Therefore, let us study the case $m_1 + m_2 + n + m = L$ first. In this case, $|J_1| = |Z| - 1 = n + m + 1$, so $\mathcal{I} \setminus J_1$ consists of a single point belonging to $\{t_{n+j} | 1 \leq j \leq m\}$. Without loss of generality, we may assume

\begin{equation}
(3.27)
J_1 = \{0, 1, t_1, \ldots, t_{n+m-1}\}, \quad \mathcal{I} \setminus J_1 = \{t_{n+m}\}.
\end{equation}

Then $t_1 = (t_1, \ldots, t_{n+m-1})$, (3.19) becomes

\begin{equation}
(3.28)
\prod_{p \in \mathcal{I} \setminus J_1} (x - p)^{\alpha_p} = (x - t_{n+m})^{\alpha_{n+m}} = x^L + \sum_{j=1}^{L} (-1)^j \binom{L}{j} t_{n+m}^j x^{L-j},
\end{equation}

with

\begin{equation}
(3.29)
L = m_1 + m_2 + n + m = \alpha_{n+m}
\end{equation}

and so the polynomials $B_j$'s become

\begin{equation}
(3.30)
B_j(a, b, t) := R_j(a, b, t_1) - \sum_{j=1}^{L} (-1)^j \binom{L}{j} t_{n+m}^j, \quad 1 \leq j \leq L.
\end{equation}

Define

\[ A = A_{(\epsilon_p)} := \left\{ t \in \mathbb{C}^{n+m} \mid \text{there is } a, b \text{ such that } (a, b, t) \text{ is an admissible zero of the polynomials } B_j \text{'s in (3.30)} \right\}. \]

Then Lemma 3.5 shows $A \subset A$, where $A$ is defined by (1.9). The first main result is as follows.

**Theorem 3.6.** Suppose that for $J_1$ given by (3.27), there exist $\epsilon_p \in \{\pm 1\}$ for $p \in J_1$ such that $m_1, m_2$ defined by (3.16)–(3.17) are non-negative integers, i.e. (3.29) holds and the corresponding polynomials $B_j$'s in (3.30) are well-defined. Suppose that $A \neq \emptyset$. Then $A$ is a finite set with $\# A \leq (\theta_{n+m} - 1)! / m_1! m_2!$.

Furthermore, for each $t = (t_1, \ldots, t_{n+m}) \in A$, all $t_j$'s are algebraic over $\mathbb{Q}_0$, and the field

\[ \mathbb{Q}_0(A) := \mathbb{Q}_0(\{t_j | (t_1, \ldots, t_{n+m}) \in A\}) \]

is a Galois extension of $\mathbb{Q}_0$ and

\[ [\mathbb{Q}_0(A) : \mathbb{Q}_0] \leq M < \infty, \]

where $M$ is a constant depending on the integer angles $\theta_j$'s.

**Proof.** Given any $t_0 = (t_{0,1}, \ldots, t_{0,n+m}) \in A$. Then the polynomials $B_j$'s in (3.30) have an admissible zero $(a, b, t)$ such that $t = t_0$. We claim that

\begin{equation}
(3.31)
The \text{ polynomial system } B_j(a, b, t) = 0 \text{ with } 1 \leq j \leq L \text{ has at most } L! = (\theta_{n+m} - 1)! \text{ solutions.}
\end{equation}

Once this claim is proved, we can conclude that $A$ is a finite set with $\# A \leq (\theta_{n+m} - 1)! / m_1! m_2!$ due to the fact that $B_j$ is invariant under any permutation of $a_1, \ldots, a_{m_1}$ and any permutation of $b_1, \ldots, b_{m_2}$ for all $j$.

To prove the claim (3.31), we consider the homogenization of this polynomial system

\begin{equation}
(3.32)
\widetilde{B}_j(a, b, t, \varepsilon) := \varepsilon^j B_j(a/\varepsilon, b/\varepsilon, t/\varepsilon) = 0, \quad j = 1, \ldots, L.
\end{equation}

We show that the solution of (3.32) with $\varepsilon = 0$ must be $0$, i.e. $(a, b, t) = 0$. 
First of all, since $0, 1 \in J_1$, we see from (3.10) and (3.30) that

$$(3.33) \quad B_L(a, b, t) = \frac{\epsilon_0 \theta_0}{\theta_\infty} \prod_{j=1}^{n+m-1} t_j \prod_{j=1}^{m} a_j \prod_{j=1}^{m_2} b_j - (-1)^L t_{n+m}^L,$$

and so

$$\tilde{B}_m(a, b, t, \varepsilon) = \frac{\epsilon_0 \theta_0}{\theta_\infty} \prod_{j=1}^{n+m-1} t_j \prod_{j=1}^{m} a_j \prod_{j=1}^{m_2} b_j - (-1)^L t_{n+m}^L.$$

Thus $\tilde{B}_L(a, b, t, 0) = 0$ implies $t_{n+m} = 0$.

To prove that $(a, b, t_1, \ldots, t_{n+m-1}) = 0$, we define

$$(3.34) \quad \tilde{h}(x) = x^{\epsilon_0 \theta_0 + \epsilon_1 \theta_1} \prod_{j=1}^{n+m-1} (x - t_j)^{\epsilon_1 \theta_1} \prod_{j=1}^{m} (x - a_j) \prod_{j=1}^{m_2} (x - b_j),$$

i.e. replace the non-homogenous factor $x - 1$ of $h(x)$ in (3.5) with $x$. Then

$$\tilde{h}'(x) = x^{\epsilon_0 \theta_0 + \epsilon_1 \theta_1 - 2} \prod_{j=1}^{n+m-1} (x - t_j)^{\epsilon_1 \theta_1 - 1} \tilde{G}(x) \prod_{j=1}^{m} (x - a_j) \prod_{j=1}^{m_2} (x - b_j)^2,$$

where the expression of $\tilde{G}(x)$ is obtained from that of $G(x)$ in (3.10) by replacing the term $x - 1$ with $x$. In other words,

$$\tilde{G}(x) = -\theta_\infty \left(x^L + \sum_{j=1}^{L} \tilde{R}_j(a, b, t_1)x^{L-j}\right),$$

where $\tilde{R}_j(a, b, t_1)$ is the homogeneous part of degree $j$ of $R_j(a, b, t)$, and so

$$\tilde{B}_j(a, b, t_1, t_{n+m}, 0) = \tilde{R}_j(a, b, t_1) - (-1)^j \binom{L}{j} t_{n+m}^j, \quad 1 \leq j \leq L.$$

Thus $\tilde{B}_j(a, b, t_1, 0, 0) = 0$ for any $j$ yields $\tilde{R}_j(a, b, t_1) = 0$ for any $j$ and so

$$\tilde{G}(x) = -\theta_\infty x^L = -\theta_\infty x^{m_1 + m_2 + n + m}.$$

Suppose the number of those $t_j$’s being 0 is $n_1$, the number of those $b_j$’s being 0 is $n_2$ and denote

$$I_2 = \{j \in \{1, \ldots, n + m - 1\} \mid t_j = 0\}.$$

Then

$$\tilde{h}(x) = x^{\epsilon_0 \theta_0 + \epsilon_1 \theta_1 + \sum_{j \in I_2} \epsilon_1 \theta_1} x^{n_1 - n_2} (c + o(x))$$

and so

$$\tilde{h}'(x) = x^{\epsilon_0 \theta_0 + \epsilon_1 \theta_1 + \sum_{j \in I_2} \epsilon_1 \theta_1} x^{n_1 - n_2 - 1} (c' + o(x))$$

for some $c, c' \neq 0$. Therefore,

$$\epsilon_0 \theta_0 + \epsilon_1 \theta_1 + \sum_{j \in I_2} \epsilon_1 \theta_1 + n_1 - n_2 - 1$$

$$= \epsilon_0 \theta_0 + \epsilon_1 \theta_1 - 2 + \sum_{j \in I_2} (\epsilon_1 \theta_1) + m_1 + m_2 + n + m - 2n_2,$$

namely $n_1 + n_2 + \# I_2 = m_1 + m_2 + n + m - 1$. Since $n_1 \leq m_1$ and $\# I_2 \leq n + m - 1$, we obtain $n_1 = m_1$, $n_2 = m_2$ and $\# I_2 = n + m - 1$, i.e. $(a, b, t_1) = 0$.

Therefore, the homogenized system (3.32) has no solutions at infinity. Then by the Bezout theorem, the polynomial system $B_j(a, b, t) = 0$ with $1 \leq j \leq L$ has exactly $L!$
solutions by counting multiplicity. This proves the claim (3.31). Furthermore, it follows from Lemma 3.7 below that for any solution \((a, b, t)\), every element of \((a, b, t)\) is algebraic over \(\mathbb{Q}_\theta\), i.e. every element belongs to \(\overline{\mathbb{Q}_\theta}\), where \(\overline{\mathbb{Q}_\theta}\) denotes the algebraic closure of \(\mathbb{Q}_\theta\).

Let \((a, b, t)\) be an admissible zero of the polynomials \(B_j\)’s. Let \(\sigma : \overline{\mathbb{Q}_\theta} \to \overline{\mathbb{Q}_\theta}\) be any automorphism of \(\overline{\mathbb{Q}_\theta}\) such that \(\sigma(x) = x\) for any \(x \in \mathbb{Q}_\theta\). Then it follows from \(B_j(a, b, t) \in \mathbb{Q}_\theta[a, b, t]\) that

\[
(\sigma(a_1), \ldots, \sigma(a_{m_1}), \sigma(b_1), \ldots, \sigma(b_{m_2}), \sigma(t_1), \ldots, \sigma(t_{n+m}))
\]

is also an admissible zero of the polynomials \(B_j\)’s, namely \((\sigma(t_1), \ldots, \sigma(t_{n+m})) \in A\) as long as \((t_1, \ldots, t_{n+m}) \in A\). This proves that \(\mathbb{Q}_\theta(A)\) is a Galois extension of \(\mathbb{Q}_\theta\), and \([\mathbb{Q}_\theta(A) : \mathbb{Q}_\theta]\) is bounded by a constant depending on the integer angles \(\theta_j\)’s, because the degree of the minimal polynomial of each \(t_j\) is bounded by \((\theta_{t_n+m} - 1)!\) and \(#A \leq (\theta_{t_n+m} - 1)!\). The proof is complete. □

**Lemma 3.7.** Given complex numbers \(y_1, \ldots, y_k\), let \(K\) be the algebraic closure of the field \(\mathbb{Q}(y_1, \ldots, y_k)\). Let \(H_j(x_1, \ldots, x_\ell) \in K[x_1, \ldots, x_\ell]\) for \(j = 1, \ldots, \ell\). Suppose that the polynomial system

\[
(3.35) \quad H_j(x_1, \ldots, x_\ell) = 0, \quad 1 \leq j \leq \ell,
\]

has only finitely many solutions. Then the coordinates of any solution \((t_1, \ldots, t_\ell)\) of (3.35) are algebraic over \(\mathbb{Q}(y_1, \ldots, y_\ell)\).

**Proof.** Let \(I\) be the ideal of \(K[x_1, \ldots, x_\ell]\) generated by the polynomials \(H_j\). For \(i = 1, \ldots, \ell\), consider the elimination ideal \(I_i = I \cap K[x_i]\). Under the assumption that (3.35) has finitely many solutions, \(I_i\) is generated by some polynomial \(f_i(x_i) \in K[x_i]\) whose roots are precisely those \(t_i\) such that \((t_1, \ldots, t_\ell)\) is a solution of (3.35) for some \((t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_\ell)\), by the closure theorem in the elimination theory (see [7, Theorem 3.6, Section 3.3]). Since all roots of \(f_i\) are in \(\overline{\mathbb{Q}_\theta}\), this shows that the coordinates of any solution \((t_1, \ldots, t_\ell)\) of (3.35) are algebraic over \(\mathbb{Q}(y_1, \ldots, y_\ell)\). □

### 3.2. The case \(m = 1\)

**This section is devoted to the proof of Theorem 1.3.**

**Proof of Theorem 1.3** Let \(m = 1\) and (1.6)–(1.8) with \(\theta_{t_j} = \theta_j^0\) hold, i.e., the set \(A\) in (1.9) is non-empty. Given any \(t_0 = (t_{0,1}, \ldots, t_{0,n+1}) \in A\). Then there exists \(\mathcal{I}_1 \subset \mathcal{I} \subset \mathcal{I}\) and \(\epsilon_p \in \{\pm 1\}\) for \(p \in \mathcal{I}_1\) such that \(m_1, m_2\) defined by (3.16)–(3.17) are non-negative integers, (3.29) holds and the corresponding polynomials \(B_j\)’s in (3.20) has an admissible zero \((a, b, t)\) with \(t = t_0\). Since \(m = 1\) implies \(\mathcal{I}_1 \cup \{t_{n+m}\} = \mathcal{I}\), we always have

\[
\mathcal{J}_1 = \mathcal{I}_1 = \{0, 1, t_1, \ldots, t_n\} \quad \text{and} \quad \mathcal{I} \setminus \mathcal{I}_1 = \{t_{n+1}\},
\]

i.e. it satisfies (3.27) and so the polynomials \(B_j\)’s become (3.30). Then by Theorem 3.6 we have \(t_0 \in A_{(p)}\) and \(#A_{(p)} \leq \frac{m_1m_2}{m_1!m_2!}\) (3.30).

Remark that in general, the set

\[
\Lambda := \{(\epsilon_p)_{p \in \mathcal{I}_1} \mid \epsilon_p \in \{\pm 1\}, m_1, m_2\ \text{defined by (3.16)–(3.17) are non-negative integers and so (3.29) holds}\}
\]

might contain multiple elements. Of course \(#\Lambda \leq 2^{n+2}\). Since the above argument implies

\[
A = \bigcup_{(\epsilon_p)_{p \in \Lambda}} A_{(\epsilon_p)}
\]
we conclude that $A$ is a finite set with $\#A \leq 2n+1(\theta_{n+1}' - 1)!$ and $\mathbb{Q}_p(A)$ is a finite Galois extension of $\mathbb{Q}_p$ with $[\mathbb{Q}_p(A) : \mathbb{Q}_p] \leq M'$, where $M'$ is a constant depending on $n$ and the integer angle $\theta_{n+1}'$. The proof is complete.

3.3. The general case $m \geq 2$. Since $n + 2 \leq |J_i| \leq n + m + 1$, we have $m_1 + m_2 + n + m \geq L = m_1 + m_2 + |J_i| - 1$, and in particular, the case $m_1 + m_2 + n + m > L$ appears generally, so we can not expect that the corresponding polynomial system

\[(3.39)\]

has only finitely many solutions. This is the difference from the $m = 1$ case. Our key observation is following.

**Lemma 3.8.** The solution set $W$ of the polynomial system (3.36) in $\mathbb{C}^{m_1+m_2+n+m}$ is an affine algebraic set of dimension $n + m + 1 - |J_i| \leq m - 1$.

Consequently, the dimension of the set of admissible zeros of the polynomials $B_j$'s is $\leq m - 1$.

**Proof.** Note that

\[W = \{(a, b, t_1, t_2) \in \mathbb{C}^{m_1+m_2+n+m} \mid B_j(a, b, t_1, t_2) = 0, \quad \forall j\},\]

and

\[\widetilde{W} := W \setminus \{(a, b, t_1, t_2) \in \mathbb{C}^{m_1+m_2+n+m} \mid t_j = 0 \text{ for some } t_j \text{ in } t_2\}\]

is an open subset of $W$. Fix any $(n + m + 2 - |J_i|)$-tupe

\[t_{2,0} = (t,0) \in \{t \in \{1\} \times \mathbb{C}^{n+m+1-|J_i|} \mid t_j \neq 0 \text{ for any } t_j \text{ in } t_2\},\]

(i.e. the first component of $t_{2,0}$ is 1.) we claim that

\[(3.37)\]

there are $L!$ points (by counting multiplicities) in $W$ satisfying $t_2 = t t_{2,0}$ for some $t \in \mathbb{C}$.

Once this claim is proved, then the dimension of $W$ is $n + m + 1 - |J_i| \leq m - 1$.

To prove this claim, we insert $t_2 = t t_{2,0}$ into (3.19), we obtain

\[(3.38)\]

\[\prod_{t_j \in \mathbb{F}, \neq \mathfrak{q}_i} (x - t_j x - t_{2,0})^{\alpha_{ij}} = x^L + \sum_{j=1}^L C_j(t_{2,0}) t^j x^{L-j}.\]

Then the polynomial system (3.36) becomes

\[(3.39)\]

\[\tilde{B}_j(a, b, t_1, t) := R_j(a, b, t_1) - C_j(t_{2,0}) t^j = 0, \quad j = 1, \ldots, L.\]

The advantage of this new polynomial system (3.39) is that the numbers of unknowns and equations are the same. Note that $C_L(t_{2,0}) = (-1)^L \prod t_{2,0}^{\alpha_{ij}} \neq 0$. Then the same proof as (3.32) shows that the homogenized system of (3.39) has no solutions at infinity, so it follows from the Bezout theorem that the polynomial system (3.39) has exactly $L!$ solutions by counting multiplicity. This proves the claim (3.37).

**Proof of Theorems 1.5.** Due to the finite choices of $J_i$ and $(\epsilon_p)_{p \in J_i}$, the assertion that the dimension of $A$ is $\leq m - 1$ follows from Lemma 3.8.

Now we further assume (1.10). Then

\[\mathbb{Q}_p = \mathbb{Q}(\theta_0, \theta_1, \theta_\infty, \theta'_1, \ldots, \theta'_n) \subset \mathbb{Q}.\]
Given any \( t_0 = (\tilde{t}_1, \ldots, \tilde{t}_{n+m}) \in A \), i.e. (1.1) has a co-axial solution for some \( t_0 = (\tilde{t}_1, \ldots, \tilde{t}_{n+m}) \) satisfying \( \tilde{t}_j \neq 0, 1 \) for any \( j \) and \( \tilde{t}_j \neq \tilde{t}_k \) for any \( j \neq k \). Without loss of generality, we may assume the corresponding

\[
J_1 = \{0, 1, t_1, \ldots, t_{n+i}\}, \quad |J_1| = n + i + 2,
\]

\[
I \setminus J_1 = \{t_{n+i+1}, \ldots, t_{n+m}\},
\]

for some \( 0 \leq i \leq m - 1 \). Denote

\[
t_{1,0} = (\tilde{t}_1, \ldots, \tilde{t}_{n+i}), \quad t_{2,0} = (1, \frac{\tilde{t}_{n+i+2}}{\tilde{t}_{n+i+1}}, \ldots, \frac{\tilde{t}_{n+m}}{\tilde{t}_{n+i+1}}).
\]

Then there is \((a_0, b_0)\) such that \((a, b, t_1, t) = (a_0, b_0, t_{1,0}, \tilde{t}_{n+i+1}) \in \mathbb{C}^{m_1 + m_2 + |J_1| - 1}\) is a solution of (3.39). Since \( \mathbb{Q}_\theta \subset \mathbb{C} \) implies

\[
\tilde{B}_j(a, b, t_2, t) \in \mathbb{Q}[\frac{\tilde{t}_{n+i+2}}{\tilde{t}_{n+i+1}}, \ldots, \frac{\tilde{t}_{n+m}}{\tilde{t}_{n+i+1}}][a, b, t_2, t],
\]

it follows from Lemma 3.7 that \((t_{1,0}, \tilde{t}_{n+i+1})\) is algebraic over \( \mathbb{Q}[\frac{\tilde{t}_{n+i+2}}{\tilde{t}_{n+i+1}}, \ldots, \frac{\tilde{t}_{n+m}}{\tilde{t}_{n+i+1}}]\), namely, the transcendence degree of \( \mathbb{Q}(\tilde{t}_1, \ldots, \tilde{t}_{n+m}) \) over \( \mathbb{Q} \) is \( \leq m - 1 \). The proof is complete. \( \square \)

4. EREMENKO’S THEOREM

Eremenko’s Theorem B is a deep result, and in this section, we would like to make some discussions about it and prove Theorem 1.2. By using the notions in Section 3, we suppose that there exist \( I_1 \subset J_1 \subset I \) and \( \epsilon_p \in \{\pm 1\} \) for \( p \in J := J_1 \cup \{\infty\} \) such that

\[
k' := \sum_{p \in J} \epsilon_p \theta_p \in \mathbb{Z}_{\geq 0},
\]

\[
k'' := \sum_{p \in I \setminus J_1} \theta_p - k' - (n + m + 1) \in 2\mathbb{Z}_{\geq 0}.
\]

Set

\[
c := (\theta_0, \theta_1, \theta_\infty, \theta_{\infty 1}, \ldots, \theta_p, 1, \ldots, 1).
\]

Suppose that

\[
\text{Either } c \text{ is incommensurable or there is } \eta \neq 0 \text{ such that } c = \eta b = \eta(b_1, \ldots, b_q) \text{ with } b_j \in \mathbb{N}, \gcd(b_1, \ldots, b_q) = 1 \text{ and }
\] \[
2 \max_{j \in I \setminus J_1} \theta_j \leq \sum_{j=1}^q b_j.
\]

Then the proof of Theorem B in [8] actually yields the following result, a more precise form of Theorem B.

**Theorem 4.1 ([8]).** Equation (1.1) admits a co-axial solution for some singular set \( \{t_1, \ldots, t_{n+m}\} \) with the developing map

\[
h(x) = \prod_{p \in J_1} (x - p)^{c \theta_p} \prod_{j=1}^{n_1} (x - a_j) \prod_{j=1}^{n_2} (x - b_j)
\]

if and only if (4.1)–(4.3) hold.

It is not difficult to see that the conditions (1.6)–(1.8) in Theorem B imply (4.1)–(4.3) in Theorem 4.1. Theorem 4.1 shows that under the conditions (4.1)–(4.3), the polynomials \( B_j \)'s in (3.20) always have an admissible zero. This is really a remarkable result.
Example 4.2. Consider the case \( \{\theta_0, \theta_1, \theta_\infty, \theta_1\} = \{\frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \theta_1\} \) with \( \theta_1 \in \mathbb{N}_{\geq 2} \). If \( \theta_1 \) is small, then it is easy to calculate those \( t \)'s such that (1.1) has co-axial solutions. For example, if \( \theta = 2 \), then \( t = \frac{2}{3} \); if \( \theta = 3 \), then \( t = \frac{2+2}{3} \). When \( \theta \geq 4 \), the polynomial system \( B_j = 0 \) becomes very complicated. In Section 5 we will give more examples.

Example 4.3. Consider the case \( \{\theta_0, \theta_1, \theta_\infty, \theta_1, \theta_{t_2}\} = \{\frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \theta_{t_2}\} \) with \( \theta_{t_2} \in \mathbb{N}_{\geq 2} \).

If \( \theta_{t_2} = 3 \), then \( t_2 = \frac{1+\sqrt{3}}{2} \).

Now we turn back to the condition (1.8) in Theorem B if the vector \( c \) is commensurable. Under the conditions (1.6)–(1.7), it is not difficult to see that (1.8) holds automatically provided \( k' + k'' \geq 1 \) and \( \theta_{t_j} \geq 1 \) for all \( j \). A more interesting thing is the assertion of Theorem 1.2 which says that (1.8) holds provided that Mondello-Panov’s condition (1.5) holds.

Proof of Theorem 1.2. Here we prove a more general result than Theorem 1.2. To simplify the notations, we reformulate this problem. Given \( n \geq 1 \) positive non-integer numbers \( \theta_1, \ldots, \theta_n \) and \( m - n \geq 0 \) positive integer numbers \( \theta_{n+1}, \ldots, \theta_m \in \mathbb{N}_{\geq 2} \). Suppose

(4.4)

\[ d_1(\mathbb{Z}_o^m, \theta - 1) = 1, \]

and there are \( \epsilon_j \in \{\pm 1\} \) for \( 1 \leq j \leq n \) such that

(4.5)

\[ k':= \sum_{j=1}^n \epsilon_j \theta_j \in \mathbb{Z}_{\geq 0}, \]

(4.6)

\[ k'':= \sum_{j=n+1}^m \theta_j - m - k' + 2 \in 2\mathbb{Z}_{\geq 0}. \]

Let

\[ c:= (\theta_1, \ldots, \theta_n, 1, \ldots, 1), \]

and suppose \( c \) is commensurable, i.e. there is \( \eta \in \mathbb{R}_{>0} \) such that \( c = \eta b = \eta (b_1, \ldots, b_q) \) with \( b_j \in \mathbb{N} \) and \( \gcd(b_1, \ldots, b_q) = 1 \). We want to prove that

(4.7)

\[ 2 \max_{n+1 \leq j \leq m} \theta_j \leq \sum_{j=1}^q b_j = \sum_{j=1}^n b_j + (k' + k'') \eta^{-1}. \]

Then Theorem 1.2 is equivalent to (4.7) with the case \( n \geq 3 \).

Clearly (4.4) implies \( n \geq 2 \) and (4.5), (4.6) imply \( m - n \geq 1 \). By renaming \( \theta_j \)'s if necessary, we may always assume

\[ 0 < \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n, \quad 2 \leq \theta_{n+1} \leq \theta_{n+2} \leq \cdots \leq \theta_m. \]

By (4.6) we have

\[ k'' + k' = \sum_{j=n+1}^m \theta_j - m + 2 \geq \theta_m + 2(m - n - 1) - m + 2, \]

i.e.

\[ \max_{n+1 \leq j \leq m} \theta_j = \theta_m \leq k' + k'' - m + 2n \leq k' + k'' + n - 1, \]

so to prove (4.7), it suffices to prove

(4.8)

\[ 2n - 2 \leq \sum_{j=1}^n b_j + (\eta^{-1} - 2)(k' + k''). \]
On the other hand, since \( c = \eta b = \eta(b_1, \ldots, b_q) \) with \( b_j \in \mathbb{N} \), we have \( \eta \neq 1 \), which implies \( \eta^{-1} \in \mathbb{N}_{\geq 2} \) if \( k' + k'' \geq 1 \). Thus we always have

\[
(\eta^{-1} - 2)(k' + k'') \geq 0 \quad \text{and} \quad \sum_{j=1}^{n} b_j \geq n.
\]

This implies that (4.8) holds if \( n = 2 \). Thus it remains to consider the case \( n \geq 3 \). We consider two cases separately.

**Case 1.** \( k' + k'' \geq 1 \). This is a simple case.

Then \( N := \eta^{-1} \in \mathbb{N}_{\geq 2} \) and for \( 1 \leq j \leq n \), \( \theta_j = b_j/N \notin \mathbb{Z} \). Consequently, \( \text{dist}(\theta_j, \mathbb{Z}) := \min_{k \in \mathbb{Z}} |\theta_j - k| \geq 1/N \) for all \( 1 \leq j \leq n \), so

\[
1 = d_1(\mathbb{Z}_m, \theta - 1) \geq \sum_{j=1}^{n} \text{dist}(\theta_j, \mathbb{Z}) \geq n/N,
\]

i.e. \( N \geq n \). Thus

\[
\sum_{j=1}^{n} b_j + (\eta^{-1} - 2)(k' + k'') = \sum_{j=1}^{n} b_j + (N - 2)(k' + k'') \geq n + n - 2.
\]

This proves (4.8).

**Case 2.** \( k' + k'' = 0 \). This case is not trivial.

Note that (4.8) is equivalent to

\[
2n - 2 \leq \sum_{j=1}^{n} b_j = \eta^{-1} \sum_{j=1}^{n} \theta_j.
\]

Note that \( 1 \leq b_1 \leq b_2 \leq \cdots \leq b_n \). If \( b_3 \geq 2 \) then we have

\[
\sum_{j=1}^{n} b_j \geq 1 + 1 + 2(n - 2) = 2n - 2,
\]

i.e. (4.9) holds. So we only need to consider the remaining case that \( b_1 = b_2 = b_3 = 1 \), i.e. \( \eta = \theta_1 = \theta_2 = \theta_3 \). Let \( \ell \geq 3 \) such that

\[
(4.10) \quad b_1 = \cdots = b_\ell = 1, \quad b_{\ell+1} \geq 2.
\]

Now we claim that \( b_n \geq \ell \). Once this claim is proved, then

\[
\sum_{j=1}^{n} b_j \geq \ell + 2(n - \ell - 1) + \ell = 2n - 2,
\]

i.e. (4.9) holds and the proof is complete.

Assume by contradiction that \( b_n \leq \ell - 1 \). Since \( k' = k'' = 0 \), we have

\[
\sum_{j=\ell+1}^{m} \theta_j = m - 2,
\]

and there is \( J_1 \subset \{1, \ldots, n\} \) such that \( \sum_{j \in J_1} \theta_j = \sum_{j \in J_2} \theta_j \), where \( J_2 = \{1, \ldots, n\} \setminus J_1 \).

This implies \( \sum_{j \in J_1} b_j = \sum_{j \in J_2} b_j \), so

\[
(4.11) \quad \sum_{j=1}^{n} b_j \text{ is even}.
\]
On the other hand, by (4.4) there is \((k_1, \ldots, k_m) \in \mathbb{Z}^m\) such that \(\sum_{j=1}^m k_j\) is odd and
\[
\sum_{j=1}^m |\theta_j - 1 - k_j| = 1.
\]
Since \(\theta_j \in \mathbb{Z}\) for \(j \geq n + 1\), we have \(k_j = \theta_j - 1\) for \(j \geq n + 1\) and so \(\sum_{j=n+1}^m (\theta_j - 1) = m - 2 - (m - n) = n - 2\). Denote \(\tilde{k}_j = k_j + 1\), then we obtain
\[
\sum_{j=1}^n |\theta_j - \tilde{k}_j| = 1 \quad \text{and} \quad \sum_{j=1}^n \tilde{k}_j = \sum_{j=1}^m k_j + 2 \quad \text{is odd.}
\]
Note that \(\theta_j = b_j \eta = b_j \theta_1\). Let \(\theta_1 = a + r\) with \(a \in \mathbb{Z}\) and \(r \in (0, 1)\).

**Case 2-1.** \(0 < r \leq 1/2\). Then
\[
1 = \sum_{j=1}^n |\theta_j - \tilde{k}_j| \geq \sum_{j=1}^\ell |\theta_1 - \tilde{k}_j| \geq \ell r,
\]
so \(r \leq 1/\ell \leq 1/3\). Consequently, for any \(1 \leq j \leq n\), \(b_j \leq b_n \leq \ell - 1\) implies \(\theta_j = b_j \theta_1 = b_j a + b_j r\) with \(b_j r \in (0, 1)\). From here and \(1 = \sum_{j=1}^n |\theta_j - \tilde{k}_j|\), we claim that
\[
|\theta_j - \tilde{k}_j| = b_j r \quad \text{and so} \quad \tilde{k}_j = b_j a \quad \forall j.
\]
Indeed, we have \(|\theta_j - \tilde{k}_j| \in \{b_j r, 1 - b_j r\}\) for all \(j\). Recalling (4.10), if there is \(1 \leq j_0 \leq \ell\) such that \(|\theta_{j_0} - \tilde{k}_{j_0}| = 1 - r\), then
\[
1 \geq \sum_{j=1}^\ell |\theta_1 - \tilde{k}_j| \geq (\ell - 1)r + 1 - r > 1,
\]
a contradiction. Thus \(|\theta_j - \tilde{k}_j| = b_j r = r\) for all \(j \leq \ell\). Consequently, if \(|\theta_{j_0} - \tilde{k}_{j_0}| = 1 - b_{j_0} r\) for some \(\ell + 1 \leq j_0 \leq n\), then it follows from \(b_{j_0} \leq \ell - 1\) that
\[
1 = \sum_{j=1}^n |\theta_j - \tilde{k}_j| \geq \ell r + 1 - b_{j_0} r > 1,
\]
a contradiction. This proves the claim (4.12) and so \(a \sum_{j=1}^n b_j = \sum_{j=1}^n \tilde{k}_j\) is odd, a contradiction with (4.11).

**Case 2-2.** \(1/2 < r < 1\). Then \(\theta_1 = a + 1 - (1 - r)\) and
\[
1 = \sum_{j=1}^n |\theta_j - \tilde{k}_j| \geq \sum_{j=1}^\ell |\theta_1 - \tilde{k}_j| \geq \ell (1 - r),
\]
so \(1 - r \leq 1/\ell \leq 1/3\). Consequently, for any \(1 \leq j \leq n\), \(b_j \leq b_n \leq \ell - 1\) implies \(\theta_j = b_j \theta_1 = b_j (a + 1) - b_j (1 - r)\) with \(b_j (1 - r) \in (0, 1)\). Then as in Case 2-1, we can prove that
\[
|\theta_j - \tilde{k}_j| = b_j (1 - r) \quad \text{and so} \quad \tilde{k}_j = b_j (a + 1) \quad \forall j,
\]
which implies so \((a + 1) \sum_{j=1}^n b_j = \sum_{j=1}^n \tilde{k}_j\) is odd, again a contradiction with (4.11).

The proof is complete. \(\square\)

As mentioned in Section 1, the converse statement of Theorem 1.2 cannot hold in general, i.e. (1.6)–(1.8) can not imply Mondello-Panov’s condition (1.5). For example, let \(\{\theta_0, \theta_1, \theta_\infty, \theta_{t_1}, \theta_{t_2}\} = \{\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\}\) with \(\theta_{t_2} \in 2\mathbb{N} + 1\). Then it does not satisfy (1.5) but satisfy (1.6)–(1.8), so co-axial solutions exist for some singular set \(\{t_1, t_2\}\).
The case \( n = 0 \) (i.e. exactly three non-integer angles) is very special. In this case, we can prove that the converse of Theorem 1.2 also holds. The proof will be presented in another paper.

5. Proof of Theorem 1.4

The goal of this section is to prove Theorem 1.4. Write \( t_1 \) and \( \theta_1 \), simply by \( t \) and \( \theta \). The function \( Q(x) \) in the differential equation (2.8) associated to the curvature equation (1.1) in the case under consideration is

\[
Q(x) = \frac{\beta_0}{x^2} + \frac{d_0}{x} + \frac{\beta_1}{(x-1)^2} + \frac{d_1}{x-1} + \frac{\beta_1}{(x-t)^2} + \frac{d_1}{x-t},
\]

where \( \beta_p = \alpha_p (\alpha_p + 2)/4 = (\theta_p^2 - 1)/4 \) and \( d \)'s satisfy (2.4), i.e.,

\[
d_0 + d_1 + d_\infty = 0, \quad \beta_0 + \beta_1 + \beta_\infty + d_1 + td_\infty = \beta_\infty.
\]

Before we proceed further, we note that the set \( \{\alpha_p/2 + 1, -\alpha_p/2\} \) is invariant under the substitution \( \alpha_p \mapsto -2 - \alpha_p = -\theta_p - 1 \). For simplicity of discussion later on, we assume that \( \alpha_0 \) and \( \alpha_1 \) satisfy

\[
\alpha_0 = \epsilon_0 \theta_0 - 1, \quad \alpha_1 = \epsilon_1 \theta_1 - 1,
\]

where \( \epsilon_0, \epsilon_1 \) are given in the assumption of Theorem 1.4. Thus, \( \alpha_0 + \alpha_1 + \alpha_\infty = \epsilon_0 \theta_0 + \epsilon_1 \theta_1 + \theta_\infty - 3 \) is an integer.

In view of (5.2), we regard \( d_0 \) and \( d_1 \) as functions of \( d := d_1 \) and let \( Q_{d,t}(x) \) denote the rational function \( Q(t) \) in (5.1). By Theorem 2.1 there exists a polynomial \( P(d,t) \in Q[d,t] \) of degree \( \theta \) in \( d \) such that the differential equation

\[
y''(x) = Q_{d,t}(x)y(x)
\]

is apparent at \( t \) if and only if \( P(d,t) = 0 \), where \( Q_\theta = Q(\theta_0, \theta_1, \theta_\infty) \). The degree of \( P(d,t) \) in \( d \) is \( \theta \), but its total degree is in general strictly larger than \( \theta \) (see (2.12)). In our proof of Theorem 1.4 we shall introduce another pair \( (\lambda, t) \) of unknowns in place of \( (d,t) \).

Let

\[
\lambda := td_0 + t \frac{\alpha_0 \alpha_1}{2} + \frac{\alpha_0 \alpha_\infty}{2} = t(t-1)d_1 + t(\beta_0 + \beta_1 + \beta_\infty) + t \frac{\alpha_0 \alpha_1}{2} + \frac{\alpha_0 \alpha_\infty}{2},
\]

where the second equality follows from (5.2). Indeed, the parameter \( \lambda \) is used as the accessory parameter of a certain Heun equation considered in [10], where for the simplicity of computation, the apparent singularity was put at \( x = 0 \) instead of \( x = t \) in this paper, and the condition for apparentness at \( x = 0 \) was shown to be equivalent to the vanishing of the characteristic polynomial of a finite Jacobi matrix (see Proposition 2.4 and Equation (2.8) of [10]). For the convenience of the reader, we recall the equivalence, stated in our setting, as follows (see also Lemma B.8 of [9]): \( (5.4) \) is apparent at \( t \) if and only if \( \lambda \) is an eigenvalue of the \( \theta \times \theta \) matrix

\[
M = M(t) := \begin{pmatrix}
B_1 & A_1 & A_2 & A_3 & \cdots & A_{\theta-1} \\
D_2 & B_2 & A_2 & & & B_{\theta-1} \\
D_3 & B_3 & & & & B_{\theta-1} \\
& & \ddots & \ddots & \ddots & \ddots \\
& & & \ddots & B_{\theta-1} & A_{\theta-1} \\
& & & & D_{\theta-1} & B_{\theta-1} \\
& & & & & D_{\theta}
\end{pmatrix}
\]
We now apply Parts (i) and (ii) to (5.8). Let
\[
A_j = A_j(t) := t(t-1)j(j-\theta),
\]
\[
B_j = B_j(t) := (2t-1)(j-1)(j-2) - (j-1)[(t-1)\alpha_0 + t\alpha_1 + (2t-1)\alpha_t] + t\alpha'\alpha'',
\]
\[
D_j = D_j(t) := (j-2)(j-3) - (j-2)(\alpha_0 + \alpha_1 + \alpha_t) + \alpha'\alpha'',
\]
with
\[
\alpha' := -\frac{\alpha_0 + \alpha_1 + \alpha_t + \alpha_\infty}{2} - 1, \quad \alpha'' := \frac{\alpha_\infty - \alpha_0 - \alpha_1 - \alpha_t}{2}.
\]
That is, (5.4) is apparent at \(t\) if and only if \(\lambda\) is an eigenvalue of \(M(t)\), i.e., a zero of the polynomial
\[
P(\lambda, t) := \det(\lambda I_\theta - M(t)).
\]
The polynomial \(P(\lambda, t)\) has the following properties.

Lemma 5.1. 
(i) The total degree and the degree in \(\lambda\) of \(P(\lambda, t)\) are both \(\theta\).
(ii) The roots of \(P(\lambda, 0)\) are \((j-1)(\alpha_0 + \alpha_t - j + 2), j = 1, \ldots, \theta\).
(iii) Let \(P_\infty(\lambda) := \lim_{t\to\infty} P(\lambda, t)/t^\theta\). Then the roots of \(P_\infty(\lambda)\) are
\[
(5.7) \quad \lambda_j := (j-1)(\alpha_\infty + \alpha_t - j + 2) - \beta_\infty - \beta_t + \beta_0 + \beta_1 + \frac{\alpha_\infty\alpha_1}{2} - \frac{\alpha_t\alpha_\infty}{2},
\]
\(j = 1, \ldots, \theta\).

Proof. The first statement follows easily from the observations that \(\text{deg}_t A_j(t) = 2\), \(\text{deg}_t B_j(t) = 1\), and \(D_j(t)\) are constant polynomials.

The second property follows immediately from the fact that \(A_j(0) = 0\) and hence the roots of \(P(\lambda, 0)\) are simply \(B_j(0)\). The third property is proved in Theorem B.3 of [6]. Since the conclusions are stated in different ways, here we provide a sketch of proof.

Let \(u = 1/x\). We check directly that \(y(x)\) is a solution of (5.4) if and only if \(\tilde{y}(u) := uy(1/u)\) satisfies
\[
(5.8) \quad \frac{d^2}{du^2} \tilde{y}(u) = \tilde{Q}(u)\tilde{y}(u),
\]
where \(\tilde{Q}(u) = u^{-\lambda}Q(1/u)\). Clearly, this is a Fuchsian differential equation with Riemann scheme
\[
\left(\begin{array}{cccc}
0 & 1 & 1/t & \infty \\
-\alpha_\infty/2 & -\alpha_1/2 & -\alpha_t/2 & -(\alpha_0/2 + 1) \\
\alpha_\infty/2 + 1 & \alpha_1/2 + 1 & \alpha_t/2 + 1 & \alpha_0/2
\end{array}\right).
\]
Thus,
\[
\tilde{Q}(u) = \frac{\beta_\infty}{u^2} + \frac{\beta_t}{u} + \frac{\beta_1}{(u-1)^2} + \frac{\beta_t}{u-1} + \frac{\beta_1}{(u-1/t)^2} + \frac{\beta_t}{u-1/t}
\]
for some complex numbers \(\tilde{d}_0, \tilde{d}_1, \text{and} \tilde{d}_{1/t}\) satisfying
\[
\tilde{d}_0 + \tilde{d}_1 + \tilde{d}_{1/t} = 0, \quad \tilde{d}_1 + \frac{\tilde{d}_{1/t}}{t} + \beta_\infty + \beta_1 + \beta_t = \beta_0.
\]
In fact, computing the partial fraction decomposition of \(\tilde{Q}(u)\), we find that
\[
\tilde{d}_0 = 2\beta_1 + 2t\beta_t + d_1 + t^2 d_t, \quad \tilde{d}_1 = -2\beta_1 - d_1, \quad \tilde{d}_{1/t} = -2t\beta_t - t^2 d_t.
\]
We now apply Parts (i) and (ii) to (5.8). Let
\[
\lambda = \frac{\tilde{d}_0}{t} + \frac{\alpha_\infty\alpha_1}{2t} + \frac{\alpha_t\alpha_\infty}{2}.
\]
By Parts (i) and (ii), there is a polynomial \( \tilde{P}(\tilde{\lambda}, 1/t) \) such that (5.8) is apparent at \( 1/t \) if and only if \( \tilde{P}(\tilde{\lambda}, 1/t) = 0 \) and the roots of \( \tilde{P}(\tilde{\lambda}, 0) \) are

\[
(j - 1)(\alpha_\infty + \alpha_t - j + 2), \quad j = 1, \ldots, \theta.
\]

On the other hand, we see from (5.2) and (5.5) that

\[
\tilde{d}_0 = t(t - 1)d_t + 2t\beta_t + \beta_\infty + \beta_1 - \beta_0 - \beta_t
\]

\[
= \lambda + t(\beta_\infty + \beta_t - \beta_0 - \beta_1) - t\frac{\alpha_0\alpha_1}{2} - \frac{\alpha_0\alpha_t}{2} + \beta_\infty + \beta_1 - \beta_0 - \beta_t,
\]

so \( \tilde{\lambda} \) and \( \lambda \) are related by

\[
\tilde{\lambda} = \frac{\lambda}{t} + \beta_\infty + \beta_t - \beta_0 - \beta_1 - \frac{\alpha_0\alpha_1}{2} + \frac{\alpha_\infty\alpha_t}{2} + \frac{1}{t}(\beta_\infty + \beta_1 - \beta_0 - \beta_t) + \frac{1}{2t}(\alpha_1\alpha_\infty - \alpha_0\alpha_t).
\]

Letting \( t \to \infty \), we conclude that the roots of \( P_\infty(\lambda) \) are given by (5.7). \( \square \)

**Corollary 5.2.** Let \( \lambda \) and \( \hat{\lambda}_j \) be given by (5.3) and (5.7), respectively. Let \( y_+(x; \lambda, t) \) and \( y_-(x; \lambda, t) \) be solutions of (5.3) of the form

\[
y_+(x; \lambda, t) = x^{\alpha_0/2+1} \sum_{j=0}^{\infty} c_{+, j}(\lambda, t)x^j, \quad c_{+, 0}(\lambda, t) = 1
\]

and

\[
y_-(x; \lambda, t) = x^{-\alpha_0/2} \sum_{j=0}^{\infty} c_{-, j}(\lambda, t)x^j, \quad c_{-, 0}(\lambda, t) = 1.
\]

Then as \( t \to \infty \) with \( \lambda/t \to \hat{\lambda}_j \), \( y_\pm(x; \lambda, t) \) converge to solutions of the Fuchsian differential equation with Riemann scheme

\[
\begin{pmatrix}
0 & 1 & \alpha_\infty \\
-\alpha_0/2 & -\alpha_1/2 & -(\hat{\alpha}_\infty/2 + 1) \\
\alpha_0/2 + 1 & \alpha_1/2 + 1 & \hat{\alpha}_\infty/2
\end{pmatrix},
\]

where \( \hat{\alpha}_\infty = \alpha_\infty + \alpha_t - 2j + 2 \).

**Proof.** Let \( \hat{d}_0, \hat{d}_1, \hat{d}_t \) be the limits of \( d_0, d_1, \) and \( d_t \) as \( t \to \infty \) and \( \lambda/t \to \hat{\lambda}_j \). From (5.5), we know that

\[
\hat{d}_0 = (j - 1)(\alpha_\infty + \alpha_t - j + 2) - \beta_\infty - \beta_t + \beta_0 + \beta_1 - \frac{\alpha_0\alpha_\infty}{2}
\]

and \( (t - 1)d_t \) converge to finite numbers as \( t \to \infty \) and \( \lambda/t \to \hat{\lambda}_j \). The latter implies that \( \hat{d}_t = 0 \). Thus, we have

\[
\hat{Q}_j(x) := \lim_{t \to \infty, \lambda/t \to \hat{\lambda}_j} Q_{d,t}(x) = \frac{\beta_0}{x^2} + \frac{\hat{d}_0}{x} + \frac{\beta_1}{(x - 1)^2} + \frac{\hat{d}_1}{x - 1}.
\]

From the relation \( d_0 + d_1 + d_t = 0 \), we see that \( \hat{d}_1 = -\hat{d}_0 \) and

\[
\hat{d}_1 + \beta_0 + \beta_1 = -(j - 1)(\alpha_\infty + \alpha_t - j + 2) + \beta_\infty + \beta_t + \frac{\alpha_0\alpha_\infty}{2}
\]

\[
= \frac{1}{4} (\alpha_\infty + \alpha_t - 2j + 2)(\alpha_\infty + \alpha_t - 2j + 4).
\]

It follows that

\[
y''(x) = \hat{Q}_j(x)y(x)
\]
is a Fuchsian differential equation with Riemann scheme given by (5.9). The convergence of \( Q_{d,t}(x) \) to \( \hat{Q}_j(x) \) is clearly uniform on any compact subset of \( \mathbb{C} \setminus \{0, 1\} \). Therefore, \( y_\pm(x; \lambda, t) \) converge to solutions of \( y''(x) = \hat{Q}_j(x)y(x) \). This completes the proof. \( \square \)

Now for \( p \in \{0, 1, \infty, t\} \), let \( M_p(\lambda, t) \) be the monodromy matrices of (5.4) around \( x = p \) with respect to the basis \((y_+(x; \lambda, t), y_-(x; \lambda, t))^t\). Recalling that the Riemann scheme of (5.4) is
\[
\begin{pmatrix}
0 & 1 & t & \infty \\
-\alpha_0/2 & -\alpha_1/2 & -\alpha_2/2 & -(\alpha_\infty/2 + 1) \\
\alpha_0/2 + 1 & \alpha_1/2 + 1 & \alpha_2/2 + 1 & \alpha_\infty/2 \\
\end{pmatrix},
\]
we know that
\[
M_0(\lambda, t) = \begin{pmatrix}
e^{\pi i \alpha_0} & 0 \\
0 & e^{-\pi i \alpha_0}
\end{pmatrix}, \quad M_\infty(\lambda, t) = (-1)^{\delta-1}I_2.
\]
Also, the monodromy matrices satisfy
\[
M_0M_1M_\infty = I_2.
\]

**Lemma 5.3.** We have
\[
M_1(\lambda, t) = \begin{pmatrix}
e^{\pi i \alpha_1} & b(\lambda, t) \\
c(\lambda, t) & e^{-\pi i \alpha_1}
\end{pmatrix}
\]
for some complex numbers \( b(\lambda, t) \) and \( c(\lambda, t) \) satisfying \( b(\lambda, t)c(\lambda, t) = 0 \).

**Proof.** Write \( M_1(\lambda, t) = \begin{pmatrix}a & b \\
c & d\end{pmatrix} \). Note that the eigenvalues of \( M_1 \) are \( e^{\pi i \alpha_1} \) and \( e^{-\pi i \alpha_1} \) and those of \( M_\infty \) are \( e^{\pi i \alpha_\infty} \) and \( e^{-\pi i \alpha_\infty} \). Together with the relation \( M_0M_1M_\infty = I_2 \), these informations yield
\[
a + d = e^{\pi i \alpha_1} + e^{-\pi i \alpha_1} \\
e^{\pi i \alpha_0} + de^{-\pi i \alpha_0} = (-1)^{\delta-1}(e^{\pi i \alpha_\infty} + e^{-\pi i \alpha_\infty}).
\]
Eliminating \( d \), we get
\[
(e^{\pi i \alpha_0} - e^{-\pi i \alpha_0})a = (-1)^{\delta-1}(e^{\pi i \alpha_\infty} + e^{-\pi i \alpha_\infty}) - (e^{\pi i(\alpha_1 - \alpha_0)} + e^{\pi i(-\alpha_1 - \alpha_0)}).
\]
Now we have \( \alpha_0 + \alpha_1 + \alpha_\infty = \epsilon_0 \theta_0 + \epsilon_1 \theta_1 + \theta_\infty - 3 = k - 3 \). Thus, the condition \( \theta \equiv k \mod 2 \) implies that
\[
e^{\pi i(\alpha_1 - \alpha_0)} = e^{\pi i(3-k+\alpha_\infty)} = (-1)^{\delta-1}e^{\pi i \alpha_\infty}
\]
and hence \((-1)^{\delta-1}e^{-\pi i \alpha_\infty} = e^{\pi i(\alpha_0+\alpha_1)}\). From these, we see that \( a = e^{\pi i \alpha_1} \). It follows that \( d = e^{-\pi i \alpha_1} \) and \( bc = 0 \). This proves the lemma. \( \square \)

Let \( P(\lambda, t) = P_1(\lambda, t) \ldots P_m(\lambda, t) \) be the factorization of \( P(\lambda, t) \) into a product of irreducible polynomials over \( \mathbb{C} \). It is easy to see that for each \( j \), we have either \( b(\lambda, t) \equiv 0 \) or \( c(\lambda, t) \equiv 0 \) on \( \mathbb{A}_j := \{(\lambda, t) \in \mathbb{C}^2 : P_j(\lambda, t) = 0\} \). Thus, if we set
\[
P_b(\lambda, t) := \prod_{b(\lambda, t)\equiv 0 \text{ on } \mathbb{A}_j} P_j(\lambda, t), \quad P_c(\lambda, t) := \prod_{c(\lambda, t)\equiv 0 \text{ on } \mathbb{A}_j} P_j(\lambda, t),
\]
then the cardinality of the set \( \mathbb{A} \) is equal to the number of intersections of the two curves \( P_b(\lambda, t) = 0 \) and \( P_c(\lambda, t) = 0 \) in the affine plane \( \mathbb{C}^2 \), with the multiplicity of a point in \( \mathbb{A} \) defined to be that of the corresponding intersection point of \( P_b \) and \( P_c \). Furthermore, by Part (ii) of Lemma 5.1 the two curves \( P_b(\lambda, t) = 0 \) and \( P_c(\lambda, t) = 0 \) do not intersect at infinity. Therefore, by Bezout’s theorem, the number of intersections of the two curves in the affine plane is \( (\deg P_b)(\deg P_c) \). Hence, to prove Theorem 1.4 we only need to determine the degrees of \( P_b \) and \( P_c \).
Proposition 5.4. Let \( k := \theta_\infty + \epsilon_0 \theta_0 + \epsilon_1 \theta_1 \) be given as in the statement of Theorem 1.4.
If \( \theta \leq |k| \), then
\[
\begin{align*}
\deg P_b = 0 & \text{ and } \deg P_c = \theta, \quad \text{if } k > 0, \\
\deg P_b = \theta & \text{ and } \deg P_c = 0, \quad \text{if } k < 0.
\end{align*}
\]
If \( \theta > |k| \), then \( \deg P_b = (\theta - k)/2 \) and \( \deg P_c = (\theta + k)/2 \).

To prove Proposition 5.4, we recall that, by Lemma 5.1(i), the total degree and the degree in \( \lambda \) of \( P(\lambda, t) \) are equal. It is easy to see that every factor of \( P(\lambda, t) \) has the same property. It follows that the degree of \( P_b(\lambda, t) \) is equal to the degree of the polynomial \( P_{b,\infty}(\lambda) := \lim_{t \to \infty} P_b(\lambda t, t/t^{\deg P_b}) \), which in turn is equal to the number of \( \hat{\lambda}_j \) that are roots of \( P_{b,\infty}(\lambda) \), where \( \hat{\lambda}_j \) are given by (5.7). In other words, to determine the degree of \( P_b(\lambda, t) \), we shall count how many \( \hat{\lambda}_j \) such that \( \lim_{t \to \infty, \lambda/t \to \hat{\lambda}_j} c(\lambda, t) \neq 0 \) there are.

Lemma 5.5. Let \( \hat{\lambda}_j \) be given by (5.7) and
\[
\hat{b} = \lim_{t \to \infty, \lambda/t \to \hat{\lambda}_j} b(r, t), \quad \hat{c} = \lim_{t \to \infty, \lambda/t \to \hat{\lambda}_j} c(r, t).
\]
Then
\[
\begin{align*}
\hat{b} = 0, \hat{c} \neq 0, \quad & \text{if } j > (k + \theta)/2, \\
\hat{b} \neq 0, \hat{c} = 0, \quad & \text{if } j \leq (k + \theta)/2.
\end{align*}
\]

Proof. By Corollary 5.2, \( y_\pm(x; \lambda, t) \) converge to solutions \( \hat{y}_\pm(x) \) of the Fuchsian differential equation with Riemann scheme (5.9), where \( \hat{y}_\pm(x) \) are local solutions near 0 of the form \( \hat{y}_+(x) = x^{\alpha_0/2 + 1}(1 + O(x)) \) and \( \hat{y}_-(x) = x^{-\alpha_0/2 + 1} + O(x) \). Then \( w_{0,\pm}(x) := x^{\alpha_0/2}(x - 1)^{1/2} \hat{y}(x) \) are solutions of the hypergeometric differential equation with Riemann scheme
\[
\begin{pmatrix}
0 & 1 & \infty \\
0 & 0 & \alpha \\
1 - \gamma & \gamma - \alpha - \beta & \beta
\end{pmatrix},
\]
where
\[
\gamma = -\alpha_0, \quad \alpha = -\left(\frac{\alpha_0 + \alpha_1 + \alpha_\infty}{2} + 1\right), \quad \beta = \frac{\alpha_\infty - \alpha_0 - \alpha_1}{2}.
\]
Let \( w_{1,\pm}(x) \) be the local solutions near \( x = 1 \) of the same hypergeometric differential equation of the form \( w_{1,+}(x) = (x - 1)^{-\alpha - \beta}(1 + O(x - 1)) \) and \( w_{1,-}(x) = 1 + O(x - 1) \). By Theorem 4.7.1 of [13, Chapter 2], the connection matrix \( P \) such that \( (w_{0,+}(x), w_{0,-}(x))^t = P(w_{1,+}(x), w_{1,-}(x))^t \) is
\[
P = \begin{pmatrix}
\frac{(2 - \gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(1 + \alpha - \gamma)\Gamma(1 + \beta - \gamma)\Gamma(\gamma - \alpha - \beta)} & \frac{(2 - \gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(1 + \alpha)\Gamma(1 + \beta - \gamma)\Gamma(\gamma - \alpha - \beta)} \\
\frac{(1 + \alpha - \gamma)\Gamma(1 + \beta - \gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha - \beta)} & \frac{(1 + \alpha - \gamma)\Gamma(1 + \beta - \gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha - \beta)}
\end{pmatrix}.
\]
Then the monodromy matrix of the hypergeometric differential equation around \( x = 1 \) with respect to the basis \( (w_{0,+}(x), w_{0,-}(x))^t \) is
\[
P e^{2\pi i(\gamma - \alpha - \beta)} = \begin{pmatrix}
0 & 1 \\
0 & 1
\end{pmatrix} P^{-1}.
\]
Now according to our choice of $\alpha_0$ and $\alpha_1$ in (5.5), we have
\[
\alpha = \left( \frac{\alpha_0 + \alpha_1 + \hat{\alpha}_\infty}{2} + 1 \right)
\]
\[
= \left( \frac{\epsilon_0 \theta_0 - 1}{2} + \frac{\epsilon_1 \theta_1 - 1}{2} + \frac{\theta_\infty - 1 + \theta - 1 - 2j + 2}{2} \right) - 1
\]
\[
= j - k + \frac{\theta}{2},
\]
which is an integer by the assumption $\theta \equiv k \mod 2$. Thus, $P$ is upper-triangular when $j \leq (k + \theta)/2$ and is lower-triangular when $j > (k + \theta)/2$. This implies that $\hat{b} \neq 0$ and $\hat{c} = 0$ if $j \leq (k + \theta)/2$, and $\hat{b} = 0$ and $\hat{c} \neq 0$ if $j > (k + \theta)/2$.

\begin{proof}[Proof of Proposition 5.4 and Theorem 1.4] Since $j = 1, 2, \cdots, \theta$, Proposition 5.4 follows from Lemma 5.5. Consequently, Theorem 1.4 holds.
\end{proof}

**Remark 5.6.** The entry $D_j$ in (5.6) can be expressed as
\[
D_j = \frac{1}{4}(2j - k - \theta - 2)(2j - k - \theta + 2\theta_\infty - 2).
\]
Thus, one has $D_j = 0$ when $j = (k + \theta)/2 + 1$. It follows that when $|k| < \theta$, the matrix $M$ in (5.6) is of the form
\[
M = \begin{pmatrix} M_1 & * \\ 0 & M_2 \end{pmatrix},
\]
where $M_1 = M_1(t)$ and $M_2 = M_2(t)$ are square matrices of size $(\theta + k)/2$ and $(\theta - k)/2$, respectively. Hence $P(\lambda, t) := \det(\lambda I - M) = \det(\lambda I - M_1) \det(\lambda I - M_2)$. One would expect that $P_c(\lambda, t) = \det(\lambda I - M_1)$ and $P_b(\lambda, t) = \det(\lambda I - M_2)$. Indeed, one can prove that this is the case by considering the roots of $P_b(\lambda, 0)$ and $P_c(\lambda, 0)$. We will not give the details here.

For small $\theta$, it is possible to completely write down the set $A$. We give some examples below.

**Example 5.7.** Assume that $\theta = 2$. Then the set $A$ is unempty only when $k = 0$. In this case, we compute that
\[
P(\lambda, t) = \lambda(\lambda - \theta_\infty t - \epsilon_0 \theta_0).
\]
The intersection of the two lines $\lambda = 0$ and $\lambda - \theta_\infty t - \epsilon_0 \theta_0 = 0$ is $(\lambda, t) = (0, -\epsilon_0 \theta_0/\theta_\infty)$. Therefore, the set $A$ consists of a single point $-\epsilon_0 \theta_0/\theta_\infty$.

Note that in Example 5.5 of [6], we have shown that for the case $\theta_0 = 1/3, \theta_1 = 1/6$, and $\theta_\infty = 1/2$ with $\epsilon_0 = \epsilon_1 = -1$, the set $A$ is $\{2/3\}$. This agrees with the general result above.

**Example 5.8.** Assume that $\theta = 3$ and $|k| = 1$. To simplify notations, we will write $\epsilon_0 \theta_0$ and $\epsilon_1 \theta_1$ simply by $\theta_0$ and $\theta_1$. We compute that when $k = 1$,
\[
P(\lambda, t) = (\lambda + (\theta_0 + \theta_1)t - 2\theta_0)
\times (\lambda^2 - ((\theta_0 + \theta_1 - 2)t + \theta_0 + 1)\lambda + (\theta_0 - 1)(\theta_0 + \theta_1)t),
\]
and when $k = -1$,
\[
P(\lambda, t) = \lambda(\lambda^2 + ((3\theta_0 + 3\theta_1 + 2)t - 3\theta_0 - 1)\lambda
+ 2(\theta_0 + \theta_1)(\theta_0 + \theta_1 + 1)\theta_0\theta_0 + \theta_1 + 1)t + \theta_0(\theta_0 + 1)).
\]
We find that the set $A$ consists of

$$\frac{\theta_0 \theta_\infty \pm \sqrt{-k \theta_0 \theta_1 \theta_\infty}}{\theta_0 + \theta_1 \theta_\infty}.$$ 

Note that the case $\theta_0 = 1/3$, $\theta_1 = 1/6$, and $\theta_\infty = 1/2$ was considered in Example 5.6 of [6], where we found that the set $A$ is $\{2(1 \pm i)/3\}$, as the result above says.

**Example 5.9.** Let $\theta = 4$. Again, we write $\epsilon_0 \theta_0$ and $\epsilon_1 \theta_1$ simply by $\theta_0$ and $\theta_1$, respectively.

When $k = 2$, we find that the set $A$ consists of the three roots of

$$\theta_\infty (\theta_\infty - 1)(\theta_\infty - 2)t^3 + 3\theta_0 \theta_\infty (\theta_\infty - 1)t^2 + 3\theta_\infty \theta_0 (\theta_0 - 1)t + \theta_0 (\theta_0 - 1)(\theta_0 - 2).$$

When $k = -2$, they are the roots of

$$\theta_\infty (\theta_\infty + 1)(\theta_\infty + 2)t^3 + 3\theta_0 \theta_\infty (\theta_\infty + 1)t^2 + 3\theta_\infty \theta_0 (\theta_0 + 1)t + \theta_0 (\theta_0 + 1)(\theta_0 + 2).$$

When $k = 0$, they are the roots of

$$\theta_\infty^2 (\theta_\infty^2 - 1)t^4 + 4\theta_0 \theta_\infty^2 (\theta_\infty^2 - 1)t^3 + 6\theta_0 \theta_\infty (\theta_0 \theta_\infty + 1)t^2 + 4\theta_\infty \theta_0 (\theta_0^2 - 1)t + \theta_0^2 (\theta_0^2 - 1).$$

(The polynomials $P(\lambda, t)$ themselves are too complicated to be displayed here.)

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