BINOMIAL PREDICTORS

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Abstract. For a prime $p$ and nonnegative integers $n, k$, consider the set $A_{n, k}^{(p)} = \{ x \in [0, 1, ..., n] : p^k || \binom{n}{x} \}$. Let the expansion of $n + 1$ in base $p$ be: $n + 1 = \alpha_0 p^\nu + \alpha_1 p^{\nu-1} + ... + \alpha_\nu$, where $0 \leq \alpha_i \leq p - 1, \ i = 0, ..., \nu$. Then the number $n$ is called a binomial predictor in base $p$, if $|A_{n, k}^{(p)}| = \alpha_k p^{\nu-k}, \ k = 0, 1, ..., \nu$. We give a full description of the binomial predictors in base $p$.

1. Introduction

Let $p$ be a prime. For nonnegative integers $n, k$, consider the set

\begin{equation}
A_{n, k}^{(p)} = \{ x \in [0, 1, ..., n] : p^k || \binom{n}{x} \}. 
\end{equation}

Quite recently (2008), W.B. Everett [1] solved the following important problem.

Question 1. How, knowing $n$, to find the finite sequence

$|A_{n, 0}^{(p)}|, |A_{n, 1}^{(p)}|, ...$

(of course, without the direct calculations)?

Due to generality, the Everett’s formula is sufficiently complicated. In this paper we indicate an infinite set of $n$’s, for which Question 1 has especially simple solution which immediately follows from the expansion of $n + 1$ in base $p$. Conversely, in the limits of this set, knowing the sequence $\{|A_{n, k}^{(p)}|\}$, we can ”predict” the expansion of $n + 1$ in base $p$. In the connection with this, we introduce the following notion.

Definition 1. Let the expansion of $n + 1$ in base $p$ be: $n + 1 = \alpha_0 p^\nu + \alpha_1 p^{\nu-1} + ... + \alpha_\nu$, where $0 \leq \alpha_i \leq p - 1, \ i = 0, ..., \nu$. Then the number $n$ is called a binomial predictor in base $p$, if $|A_{n, k}^{(p)}| = \alpha_k p^{\nu-k}, \ k = 0, 1, ..., \nu$.

Example 1. It is easy to see that $n = 0$ is a binomial predictor in every base $p$.

Indeed, $|A_{0, 0}^{(p)}| = 1$, that is the binary expansion of 1 in every base.

Example 2. Let $p = 2, \ n = 11$.

1991 Mathematics Subject Classification. 11B37.
Then $n + 1 = 8 + 4$. The row of the binomial coefficients $\{ \binom{n}{x} : x = 0, 1, ..., 11 \}$ is:

$$1, 11, 55, 165, 330, 462, 462, 330, 165, 55, 11, 1.$$ 

Here $|A^{(2)}_{1,0}| = 8$, $|A^{(2)}_{1,1}| = 4$, $|A^{(2)}_{11,k}| = 0$, $k \geq 2$. Thus, by the definition, 11 is a binomial predictor in base 2.

**Example 3.** Let $p = 3$, $n = 23$. Then $n + 1 = 2 \cdot 3^2 + 2 \cdot 3$.

Here $|A^{(3)}_{23,0}| = 18$, $|A^{(3)}_{23,1}| = 6$, $|A^{(3)}_{23,k}| = 0$, $k \geq 2$. Thus 23 is a binomial predictor in base 3.

Our aim is to give a full description of binomial predictors in base $p$.

**Definition 2.** A nonnegative integer $n$ is called a Zumkeller’s number in base $p$ (in the case of $p = 2$ see sequence A089633 in [12]), if either it is 0 or its expansion in base $p$ has all digits $p - 1$, except, maybe, one; if the exceptional digit is the first (it can occur only in case of $p \geq 3$), then it could take an arbitrary value from 1, ..., $p - 2$; otherwise, it is only $p - 2$.

Our result is the following.

**Theorem 1.** $n \geq 0$ is a binomial predictor in base $p$ if and only if it is a Zumkeller’s number in the same base.

2. SOME CLASSICAL RESULTS ON BINOMIAL COEFFICIENTS

The binomial coefficients play a very important role in numerous questions of number theory. For example, it is very known proof of the beautiful Chebyshev’s theorem, using the binomial coefficients (see, for example, a Finsler’s version of the proof in [13]). A connection between some questions of divisibility of the binomial coefficients and the old conjecture of the infinity of tween primes is appeared in the author’s article [10].

The first important contributions into theory of binomial coefficients belong to Legendre (1830), Kummer (1852) and Lucas (1878). Let $p$ a prime and $a_p(n)$ be such exponent that

$$p^{a_p(n)} \| n.$$ 

(2)

Let, furthermore,

$$n = n_0 p^m + n_1 p^{m-1} + ... + n_m, \ 0 \leq t_i \leq p - 1.$$ 

(3)

be the expansion of $n$ in base $p$. Denote
A.-M. Legendre [7, p.12] empirically noticed that (in our notations)
\[ a_p(n!) = (n - s_p(n))/(p - 1). \]
A proof see, e.g., in [9]. From (5) we immediately obtain:
\[ a_p\left(\binom{n}{x}\right) = a_p(n!) - a_p(x!) - a_p((n - x)!)) = ((n - s_p(n)) - (x - s_p(x)) - (n - x - s_p(n - x))/(p - 1) =
\]
\[ (s_p(x) + s_p(n - x) - s_p(n))/(p - 1). \]

**Example 4.**

Since \( s_2(2n) = s_2(n) \), then for the central binomial coefficients we find
\[ a_2\left(\binom{2n}{n}\right) = s_2(n); \ a_2\left(\binom{2n + 1}{n}\right) = s_2(n + 1) - 1. \]
Notice that in [11] was posed a question which remains open up to now.

**Question 2.** Does the diophantine equation \( s_p(n) = s_q(n) \), where \( p \neq q \) are fixed primes, have infinitely many solutions?

It easy to see that, by (6), the following equalities are equivalent:
\[ s_p((p - 1)n) = s_q((q - 1)m), \]
\[ (p - 1)a_p\left(\binom{pm}{p}\right) = (q - 1)a_q\left(\binom{qm}{q}\right). \]
Furthermore, note that (6) implies the following simple corollary.

**Corollary 1.** For every lattice pair \((x, y) \geq (0, 0)\), we have the triangle inequality:
\[ s_p(x + y) \leq s_p(x) + s_p(y). \]
The equality attains if and only if \( \binom{x+y}{x} \) is not multiple of \( p \).

Now we can treat of Question 1 in a different foreshortening. Consider the equation
\[ s_p(x) + s_p(n - x) - s_p(n) = k(p - 1), \ x \in [0, 1, \ldots, n]. \]
For \( k \geq 0 \), denote \( \lambda_p^{(k)}(n) \) the number of solutions of (8). Thus we see that
\[ |A_n^{(p)}| = \lambda_p^{(k)}(n), \ k = 0, 1, \ldots \]
In 1852, Kummer [6] made an important observation (a proof one can find, e.g., in [3]):
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\(a_p\left(\binom{n}{m}\right)\) is the number of “carries” which appear in adding \(x\) and \(n-x\) in base \(p\).

This statement plays a large role in the Everett’s matrix method which was used for receiving his general formula.

Another important result was obtained by Lucas [8]. He proved that if together with (3)
\[ t = t_0p^m + t_1p^{m-1} + ... + t_m, \quad 0 \leq t_i \leq p - 1. \]
then
\[ \binom{n}{t} \equiv \prod_{i=0}^{m} \binom{n_i}{t_i} \pmod{p}. \]

From (11) immediately follows the next corollary ([2]).

**Corollary 2.**
\[ \lambda_p^{(0)}(n) = (n_0 + 1)(n_1 + 1)...(n_m + 1). \]

**Proof.** Indeed, (12) gives the number of all nonzero products in (11), when \(0 \leq t_i \leq n_i, \quad i = 0, ..., m\). Since \(t_i, n_i \in [0, ..., p - 1]\), then none of the considered products is divisible of \(p\).

Note that in the binary case in (12) sufficiently consider only factors with \(n_i = 1\). Therefore, we have
\[ \lambda_2^{(0)}(n) = 2^{s_2(n)}. \]

The latter is known result of J.Glaisher (1899; in [4] A.Granville gives a new elegant proof; generalizations in other direct see in [4]-[5]).

3. PROOF OF NECESSITY

A proof of necessity we easily derive from only the first condition of Definition 1, i.e. from the equality \(|A_{n,0}^{(p)}| = \alpha_0p^\nu\). In connection with (3), note that \(\nu \neq m\) only when \(n_0 = n_1 = ... = n_m = p - 1\). This a Zumkeller’s number in base \(p\). In other cases we consider the equality \(|A_{n,0}^{(p)}| = \alpha_0p^m\). In its turn,\(\alpha_0 \neq n_0 \leq p - 2\) only when \(n_1 = ... = n_m = p - 1\). In this case we also have a Zumkeller’s number in base \(p\). Let now \(n\) be a binomial predictor in base \(p\), when \(n_0 = p - 1\), but not all \(n_i\) equal \(p - 1\). Thus, we have
\[ |A_{n,0}^{(p)}| = (p - 1)p^m. \]

From this equality and Corollary 2 we conclude that exactly \(m\) from \(m+1\) brackets in product (12) equal \(p\), while some one bracket equals \(p - 1\). This means that exactly \(m-1\) digits from \(n_1, ..., n_m\) equal to \(p - 1\), while some one digit equals \(p-2\). Thus in this case we again have a Zumkeller’s number in base \(p\).
4. Proof of sufficiency

Here we use the Kummer’s theorem in the following equivalent form: $a_p\binom{n}{m}$ is the number of "carries" which appear in subtracting $x$ from $n$ in base $p$.

Let $n$ be a Zumkeller’s number. Evidently, in the trivial case when $n = p^{m+1} - 1 = (p-1) \lor (p-1) \lor ... \lor (p-1)$ we have a binomial predictor in base $p$ (we use $\lor$ as operator of concatenation). Indeed, here $n + 1 = p^{m+1}$ and, by (9) and (12), $|A_{n,0}^{(p)}| = p^{m+1}$. On the other hand, for every $x \leq n$, in the subtracting $x$ from $n$ in base $p$ we have no any "carries", i.e. $|A_{n,k}^{(p)}| = 0$, $k \geq 1$. Analogously, we have a binomial predictor in case $n = n_0 \lor (p-1) \lor (p-1) \lor ... \lor (p-1)$, $n_0 \leq p-2$. Indeed, here $n + 1 = (n_0 + 1)p^m$ and, by (9) and (12), $|A_{n_0,0}^{(p)}| = (n_0 + 1)p^m$. and again we have no any "carries" during the subtracting $x$ from $n$ in base $p$. Let now $n$ have a unique digit $p-1$ in its expansion in base $p$. Consider such $n$ of the general form:

$$(15) \quad n = \underbrace{(p-1) \lor ... \lor (p-1)}_{t} \lor (p-2) \lor \underbrace{(p-1) \lor ... \lor (p-1)}_{m-t}.$$ 

Then

$$(16) \quad n + 1 = \underbrace{(p-1) \lor ... \lor (p-1) \lor 0 \lor ... \lor 0}_{t+1} \lor \underbrace{0 \lor ... \lor 0}_{m-t} = (p-1)(p^{m} + p^{m-1} + ... + p^{m-t}).$$

Let the $x$ in base $p$ has the form:

$$(17) \quad x_0 \lor ... \lor x_{t-1} \lor x_t \lor x_{t+1} \lor ... \lor x_m.$$ 

If $x_t \leq p-2$, then in subtracting $x$ from $n$ in base $p$ we have not "carries". Evidently, that the number of such $x$'s is

$$(18) \quad |A_{n,0}^{(p)}| = p^t(p-1)p^{m-t} = (p-1)p^m.$$ 

If $x_t = p-1$, such that also

$$(19) \quad x_t = x_{t-1} = x_{t-2} = ... = x_r = p-1,$$

then, in view of $x \leq n$, the length of this chain is not more than $t$, i.e.

$$(20) \quad 1 \leq r \leq t.$$ 

In this case the number of "carries" is equal to the length of the chain, i.e. $t - r + 1$, and, by Kummer’s theorem, we have

$$(21) \quad x \in A_{n,t-r+1}^{(p)}.$$
Putting

\[ t - r + 1 = k, \]

we easily calculate \(|A_{n,k}^{(p)}|\) :

\[ |A_{n,k}^{(p)}| = p^{m+1-k} \cdot \frac{p-1}{p} = (p-1)p^{m-k}, \quad k = 1, \ldots, t, \]

where the factor \( \frac{p-1}{p} \) corresponds to the digit \( x_{r-1} \), i.e. the place after the end of chain (19). Comparison of (18) and (23) with (16) shows that \( n \) is a binomial predictor. This completes the proof of Theorem 1. ■

**Acknowledgment.** The author is grateful to J.-P. Allouche for sending a scanned copy of pp. 10-12 of Legendre book [7], and to D. Berend for important advises.

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