Local Boundedness and Harnack Inequality for Solutions of Linear Nonuniformly Elliptic Equations

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Abstract
We study local regularity properties for solutions of linear, nonuniformly elliptic equations. Assuming certain integrability conditions on the coefficient field, we prove local boundedness and Harnack inequality. The assumed integrability assumptions are essentially sharp and improve upon classical results by Trudinger. We then apply the deterministic regularity results to the corrector equation in stochastic homogenization and establish sublinearity of the corrector. © 2019 the Authors. Communications on Pure and Applied Mathematics is published by the Courant Institute of Mathematical Sciences and Wiley Periodicals, Inc.

1 Introduction and Main Results

We consider linear, second order, scalar elliptic equations in divergence form,

\[ -\nabla \cdot a \nabla u = 0, \]

where \( a: \Omega \rightarrow \mathbb{R}^{d \times d} \) is a measurable matrix field on a domain \( \Omega \subset \mathbb{R}^d, d \geq 2 \).

In order to measure ellipticity of \( a \), we introduce

\[ \lambda (x) := \inf_{\xi \in \mathbb{R}^d} \frac{\xi \cdot a(x) \xi}{|\xi|^2}, \quad \mu (x) := \sup_{\xi \in \mathbb{R}^d} \frac{|a(x) \xi|^2}{\xi \cdot a(x) \xi}, \]

and suppose that \( \lambda \) and \( \mu \) are measurable nonnegative functions. If \( \lambda^{-1} \) and \( \mu \) are essentially bounded (i.e., \( a \) is uniformly elliptic), the seminal contributions of De Giorgi [14] and Nash [28] ensure that weak solutions of \((1.1)\) are Hölder continuous. Moreover, Moser [25, 26] showed that weak solutions of \((1.1)\) satisfy the Harnack inequality, which then implies Hölder continuity. Here, we are interested in situations beyond uniform ellipticity.

In [32] Trudinger considered nonuniformly elliptic equations of the type \((1.1)\). Instead of essential boundedness, he assumed that \( \lambda^{-1} \in L^q(\Omega) \) and \( \mu \in L^p(\Omega) \) with \( \frac{1}{p} + \frac{1}{q} < \frac{2}{d} \) and proved that weak solutions to \((1.1)\) are locally bounded and satisfy the Harnack inequality. In this paper, we prove both results under the less...
restrictive and essentially optimal assumption \( \frac{1}{p} + \frac{1}{q} < \frac{2}{d-1} \). More precisely, we establish the following:

**Theorem 1.1.** Fix \( d \geq 2 \), a domain \( \Omega \subset \mathbb{R}^d \), and \( p, q \in (1, \infty] \) satisfying

\[
\frac{1}{p} + \frac{1}{q} < \frac{2}{d-1}.
\]

Let \( \alpha: \Omega \to \mathbb{R}^{d \times d} \) be such that \( \lambda \) and \( \mu \) given in (1.2) are nonnegative and satisfy

\[
\frac{1}{\lambda} \in L^q(\Omega), \quad \mu \in L^p(\Omega).
\]

Then any weak solution \( u \) of (1.1) in \( \Omega \) satisfies:

(i) **(Local boundedness)** For every \( \gamma > 0 \) there exists \( c = c(\gamma, d, p, q) \in [1, \infty) \) such that for any ball \( B_R \subset \Omega, R > 0 \), it holds that

\[
\|u\|_{L^\infty(B_{R/2})} \leq c \Lambda(B_R)^{(\frac{1}{p} + \frac{1}{q}) \gamma} \left( \int_{B_R} |u|^p \right)^{\frac{1}{p}},
\]

where \( \delta := \min\left\{ \frac{1}{d-1} - \frac{1}{2p}, \frac{1}{2q} \right\} > 0 \), \( p' := \frac{p}{p-1} \), and for every measurable set \( S \subset \Omega \)

\[
\Lambda(S) := \left( \int_S \mu^p \right)^{\frac{1}{p}} \left( \int_S \lambda^{-q} \right)^{\frac{1}{q}}.
\]

(ii) **(Harnack inequality)** If \( u \) is nonnegative in the ball \( B_R \subset \Omega \), then

\[
\sup_{B_{R/2}} u \leq c \inf_{B_{R/2}} u,
\]

where \( c = c(d, p, q, \Lambda(B_R)) \in [1, \infty) \).

**Remark 1.2.** As mentioned above, the conclusions of Theorem 1.1 are proven in the classical paper of Trudinger [32] under the more restrictive integrability condition

\[
\frac{1}{\lambda} \in L^q(\Omega), \quad \mu \in L^p(\Omega)
\]

with

\[
p, q \in (1, \infty], \quad \frac{1}{p} + \frac{1}{q} < \frac{2}{d-1};
\]

see also the paper by Murthy and Stampacchia [27] for related results. To the best of our knowledge, Theorem 1.1 contains the first improvements with respect to global integrability of \( \frac{1}{\lambda} \) and \( \mu \) compared to the corresponding results in [27, 32] (see [13] for a recent generalization of the findings in [27, 32] to nonlinear, nonuniformly elliptic equations under assumptions that match (1.7) in the linear case). Assumption (1.3) is essentially sharp in order to establish local boundedness (and thus also the validity of the Harnack inequality) for weak solutions of (1.1). Indeed, in view of a counterexample by Franchi, Serapioni, and Serra Cassano [20], the conclusion of Theorem 1.1 is false if condition (1.3) is replaced by \( \frac{1}{p} + \frac{1}{q} < \frac{2}{d-1} + \varepsilon \) for any \( \varepsilon > 0 \); see Remark 3.5 below. However, we emphasize that under an additional local assumption (e.g., that \( \lambda, \mu \) are in the Muckenhoupt class \( A_2 \))
stronger results are available under weaker global integrability assumptions; see, e.g., [11, 17].

Remark 1.3. Note that if \( \frac{1}{d-1} - \frac{1}{2p} - \frac{1}{2d} \) tends to 0 from above, the prefactor on the right-hand side in (1.4) blows up and we do not know if weak solutions of (1.1) are locally bounded in the borderline situation \( \frac{1}{p} + \frac{1}{q} = \frac{2}{d-1} \) in general. However, in the special case of two dimensions, we are able to show local boundedness of weak solutions under the minimal assumption \( p = q = 1 \) (and thus \( \frac{1}{p} + \frac{1}{q} = 2 = \frac{2}{d-1} \)); see Proposition 3.4.

As an application of Theorem 1.1 we consider the corrector equation in stochastic homogenization. Currently, homogenization and large-scale regularity for equations with random and degenerate coefficients is an active field of research; see e.g., [1–4, 6, 12, 15, 18, 19, 29]. Recently sublinearity (in \( L^\infty \)) of the corrector in stochastic homogenization was proven in [12] (see also [18]) under certain moment conditions that are comparable to (1.7) (see also [1, 15] for related results in the discrete setting). In [12, 15, 18], the \( L^\infty \)-sublinearity of the corrector is the key ingredient to prove quenched invariance principles for random walks [1, 15] or diffusion [12, 18] in a random environment with degenerate and/or unbounded coefficients. In this paper, we establish \( L^\infty \)-sublinearity of the corrector under relaxed moment conditions; see Proposition 5.3.

The paper is organized as follows: In Section 2 we present a technical lemma that implies an improved version of the Caccioppoli inequality. This lemma plays a prominent role in the proof of Theorem 1.1 and is the main source for the improvement compared to the previous results in [27, 32, 33]. In Section 3, we make precise the notion of weak solution and prove part (i) of Theorem 1.1 and local boundedness for weak subsolution of (1.1). Section 3 contains an improvement of part (i) of Theorem 1.1 valid only in two dimensions; see Proposition 3.4. In Section 4 we establish part (ii) of Theorem 1.1 as a consequence of a weak Harnack inequality for nonnegative weak supersolutions of (1.1) and the local boundedness. Moreover, we list in Section 4 several direct consequences of the Harnack inequality. In the final section, Section 5 we apply Theorem 1.1 to the corrector equation of stochastic homogenization and prove \( L^\infty \)-sublinearity of the corrector.

2 An Auxiliary Lemma

In this section, we provide a key estimate, formulated in Lemma 2.1 below, that is central in our proof of Theorem 1.1.

**Lemma 2.1.** Fix \( d \geq 2 \) and \( p \geq 1 \) satisfying \( p > \frac{d-1}{2} \) if \( d \geq 3 \). For \( 0 < \rho < \sigma < \infty \), let \( v \in W^{1,p^*}(B_\sigma) \) with \( \frac{1}{p^*} = \min \{ \frac{1}{2} - \frac{1}{2p} + \frac{1}{d-1}, 1 \} \) and \( \mu \in L^p(B_\sigma) \), \( \mu \geq 0 \) be such that \( \mu v^2 \in L^1(B_\sigma) \). Consider

\[
J(\rho, \sigma, v) := \inf \left\{ \int_{B_\sigma} \mu |\nabla \eta|^2 \, dx \mid \eta \in C_0^1(B_\sigma), \eta \geq 0, \eta = 1 \text{ in } B_\rho \right\}.
\]
Then there exists $\epsilon = \epsilon(d, p) \in [1, \infty)$ such that

\begin{equation}
J(\rho, \sigma, v) \leq \epsilon(\sigma - \rho)^{-\frac{2d}{\sigma + 1}} \|\mu\|_{L^p(B_\sigma \setminus B_\rho)} \cdot (\|\nabla v\|_{L^{p^*}(B_\sigma \setminus B_\rho)} + \rho^{\frac{2}{p - 2}} \|v\|_{L^{p^*}(B_\sigma \setminus B_\rho)}).
\end{equation}

**Proof of Lemma 2.1**

**Step 1.** We claim

\begin{equation}
J(\rho, \sigma, v) \leq (\sigma - \rho)^{-\frac{1}{2} + \epsilon} \left( \int_\rho^\sigma \left( \int_{S_r} \mu v^2 \right)^{\gamma} dr \right)^{\frac{1}{\gamma}} \quad \text{for every } \gamma > 0.
\end{equation}

Estimate (2.2) follows directly by minimizing among radially symmetric cutoff functions $\eta$. Indeed, for every $\epsilon \geq 0$ we have

\[
J(\rho, \sigma, v) \leq \inf \left\{ \int_\rho^\sigma \eta'(r)^2 \left( \int_{S_r} \mu v^2 + \epsilon \right) dr : \eta \in C^1(\rho, \sigma), \eta(\rho) = 1, \eta(\sigma) = 0 \right\}
\]

\[=: J_{1d, \epsilon}.\]

For $\epsilon > 0$, the one-dimensional minimization problem $J_{1d, \epsilon}$ can be solved explicitly and we obtain

\begin{equation}
J_{1d, \epsilon} = \left( \int_\rho^\sigma \left( \int_{S_r} \mu v^2 + \epsilon \right)^{-1} dr \right)^{-1}.
\end{equation}

Let us give an argument for (2.3). First we observe that using the assumption $\mu v^2 \in L^1(B_\sigma)$ and a simple approximation argument, we can replace $\eta \in C^1(\rho, \sigma)$ with $\tilde{\eta} \in W^{1, \infty}(\rho, \sigma)$ in the definition of $J_{1d, \epsilon}$. Let $\tilde{\eta} : [\rho, \sigma] \to [0, \infty)$ be given by

\[\tilde{\eta}(r) := 1 - \left( \int_\rho^\sigma b(r)^{-1} dr \right)^{-1} \int_\rho^r b(r)^{-1} dr \quad \text{where } b(r) := \int_{S_r} \mu v^2 + \epsilon.
\]

Clearly, $\tilde{\eta} \in W^{1, \infty}(\rho, \sigma)$ (since $b \geq \epsilon > 0$), $\tilde{\eta}(\rho) = 1$, $\tilde{\eta}(\sigma) = 0$, and thus

\[J_{1d, \epsilon} \leq \int_\rho^\sigma \tilde{\eta}'(r)^2 b(r) dr = \left( \int_\rho^\sigma b(r)^{-1} dr \right)^{-1}.
\]

The reverse inequality follows by Hölder’s inequality: For every $\eta \in W^{1, \infty}(\rho, \sigma)$ satisfying $\eta(\rho) = 1$ and $\eta(\sigma) = 0$, we have

\[1 = \left( \int_\rho^\sigma \eta'(r) dr \right)^2 \leq \int_\rho^\sigma \eta'(r)^2 b(r) dr \int_\rho^\sigma b(r)^{-1} dr.
\]

Clearly, the last two displayed formulas imply (2.3).
Next, we deduce (2.2) from (2.3) by interpolation. For every $s > 1$, we obtain by the Hölder inequality

\[
\sigma - \rho = \int_\rho^\sigma \left( \frac{b}{b^s} \right)^{\frac{s-1}{s}} \leq \left( \int_\rho^\sigma b^{s-1} \right)^{\frac{1}{s}} \left( \int_\rho^\sigma 1 \right)^{\frac{s-1}{s}}
\]

with $b$ as above and by (2.3) that

\[
J_{1d, \varepsilon} \leq (\sigma - \rho)^{-\frac{s-1}{s}} \left( \int_\rho^\sigma \left( \int_{S_r} \mu v^2 + \varepsilon \right)^{s-1} dr \right)^{\frac{1}{s-1}}.
\]

Sending $\varepsilon$ to 0, we obtain (2.2) with $D_1 > 0$.

**Step 2.** Let us first assume $d \geq 3$. Note that $p > \frac{2}{d-1}$ implies $p^* \in [1, 2)$. To estimate the right-hand side of (2.2) we use the Sobolev inequality on spheres, which reads: $\forall s \in [1, d-1) \exists c = c(d, s) \in [1, \infty)$ such that for every $r > 0$

\[
(2.4) \quad \left( \int_{S_r} |\varphi|^{p^*} \right)^{\frac{1}{p^*}} \leq c \left( \left( \int_{S_r} |\nabla \varphi|^p \right)^{\frac{1}{p}} + \frac{1}{r} \left( \int_{S_r} |\varphi|^p \right)^{\frac{1}{p}} \right)
\]

where $\frac{1}{p^*} = \frac{1}{s} - \frac{1}{d-1}$. By (2.4) and the Hölder inequality, we find $c = c(p, d) \in [1, \infty)$ such that for every $\gamma > 0$

\[
(2.5) \quad \left( \int_\rho^\sigma \left( \int_{S_r} \mu v^2 \right)^{\gamma} dr \right)^{\frac{1}{\gamma}} \leq \left( \int_\rho^\sigma \left( \int_{S_r} \mu^{p^*} \right)^{\frac{\gamma}{p^*}} \left( \int_{S_r} |v|^{\frac{2p^*}{p^*+1}} \right)^{\frac{(p^*-1)\gamma}{p^*+1}} dr \right)^{\frac{1}{\gamma}}
\]

\[
\leq c \left( \int_\rho^\sigma \left( \int_{S_r} \mu^{p^*} \right)^{\frac{\gamma}{p^*}} \left( \int_{S_r} |\nabla v|^{p^*} \right)^{\frac{2\gamma}{2p^*}} + \frac{1}{r^{2\gamma}} \left( \int_{S_r} |v|^{p^*} \right)^{\frac{2\gamma}{2p^*}} \right)^{\frac{1}{\gamma}}
\]

where $\frac{p-1}{2p} = \frac{1}{p^*} - \frac{1}{d-1}$. The choice $\gamma = \frac{d-1}{d+1}$ yields $\frac{\gamma}{p} + 2\gamma = 1$, and we obtain from (2.2), (2.5), and the Hölder inequality

\[
J(\rho, \sigma, v) \leq \frac{c}{(\sigma - \rho)^{\frac{2d}{d-1}}} \left( \int_{B_\sigma \setminus B_\rho} \mu^{p^*} \right)^{\frac{1}{p^*}} \left( \left( \int_{B_\sigma \setminus B_\rho} |\nabla v|^{p^*} \right)^{\frac{2}{p^*}} + \frac{1}{p^2} \left( \int_{B_\sigma \setminus B_\rho} |v|^{p^*} \right)^{\frac{2}{p^*}} \right),
\]

which is the desired estimate.

Finally, we suppose $d = 2$. In this case we have $p^* = 1$. Instead of (2.4), we use the one-dimensional Sobolev inequality $\|\varphi\|_{L^\infty(S_1)} \leq c \|\varphi\|_{W^{1,1}(S_1)}$ to obtain the estimate (2.1) as above (but now also in the borderline case $p = 1$).
3 Local Boundedness Proof of Part (i) of Theorem 1.1

In this section we prove part (i) of Theorem 1.1 as a consequence of a local boundedness result for weak subsolutions of (1.1). Before we state the result, we first define the notion of weak solution to (1.1) that we consider here.

**Definition 3.1.** Fix a domain \( \Omega \subset \mathbb{R}^d \) and a coefficient field \( a: \Omega \to \mathbb{R}^{d \times d} \) such that \( \lambda, \mu \geq 0 \) given in (1.2) satisfy \( \frac{1}{\lambda} \in L^1(\Omega), \mu \in L^1(\Omega) \). The spaces \( H^1_0(\Omega, a) \) and \( H^1(\Omega, a) \) are, respectively, defined as the completion of \( C^1_0(\Omega) \) and \( C^1(\Omega) \) with respect to the norm \( \| \cdot \|_{H^1(\Omega, a)} := (A_1(\cdot, \cdot))^1/2 \), where

\[
A_1(u, v) := A(u, v) + \int_{\Omega} \mu uv \quad \text{with} \quad A(u, v) := \int_{\Omega} a \nabla u \cdot \nabla v.
\]

Moreover, we denote by \( H^1_{loc}(\Omega, a) \) the family of functions \( u \) satisfying \( u \in H^1(\Omega', a) \) for every bounded open set \( \Omega' \subset \Omega \).

We call \( u \) a weak solution (subsolution, supersolution) of (1.1) in \( \Omega \) if and only if \( u \in H^1(\Omega, a) \) and

\[
\forall \phi \in H^1_0(\Omega, a), \phi \geq 0 : A(u, \phi) = 0 \quad (\leq 0, \geq 0).
\]

Moreover, we call \( u \) a local weak solution of (1.1) in \( \Omega \) if and only if \( u \) is a weak solution of (1.1) in \( \Omega \) for every bounded open set \( \Omega' \subset \Omega \). Throughout the paper, we call a solution (subsolution, supersolution) of (1.1) in \( \Omega \) a-harmonic (a-subharmonic, a-supharmonic) in \( \Omega \).

For general properties of the spaces \( H^1(\Omega, a) \) and \( H^1_0(\Omega, a) \), we refer to [32, 33]. We only recall here the chain rule.

**Remark 3.2.** Let \( g: \mathbb{R} \to \mathbb{R} \) be uniformly Lipschitz-continuous with \( g(0) = 0 \), and consider the composition \( F := g(u) \). Then, \( u \in H^1_0(\Omega, a) \) (or \( u \in H^1(\Omega, a) \)) implies \( F \in H^1_0(\Omega, a) \) (or \( F \in H^1(\Omega, a) \)), and it holds that \( \nabla F = g'(u) \nabla u \) a.e. (see, e.g., [33, lemma 1.3]). In particular, if \( u \) satisfies \( u \in H^1(\Omega, a) \) (or \( u \in H^1(\Omega, a) \)), then also the truncations

\[
u_+ := \max\{u, 0\}, \quad u_- := \min\{u, 0\},
\]
satisfy \( u_+, u_- \in H^1(\Omega, a) \) (or \( u_+, u_- \in H^1_0(\Omega, a) \)).

Now we come to the local boundedness from above for weak subsolutions of (1.1).

**Theorem 3.3.** Fix \( d \geq 2 \), a domain \( \Omega \subset \mathbb{R}^d \), and \( p, q \in (1, \infty] \) satisfying (1.3). Let \( a: \Omega \to \mathbb{R}^{d \times d} \) be such that \( \lambda \) and \( \mu \) given in (1.2) are nonnegative and satisfy \( \frac{1}{\lambda} \in L^q(\Omega), \mu \in L^p(\Omega) \). Then every weak subsolution \( u \) of (1.1) in \( \Omega \) is locally bounded from above, and for every \( \gamma > 0 \) there exists \( c = c(\gamma, d, p, q) \in [1, \infty) \) such that for any ball \( B_R \subset \Omega \) and \( \theta \in (0, 1) \)

\[
\sup_{B_{\theta R}} u \leq c \frac{\Lambda(B_R)^{\frac{d}{\gamma}(1 + \frac{1}{\theta})}}{(1 - \theta)^{\frac{d}{p s}}} \left( \int_{B_R} u_+^{\gamma} \right)^{\frac{1}{\gamma}},
\]
where \( \delta = \min \left\{ \frac{1}{d-1} - \frac{1}{2p}, \frac{1}{d} \right\} - \frac{1}{2q} > 0, \ s := 1 + p' (1 + \frac{1}{s}) (\frac{1}{p} + \frac{1}{q}), \ p' = \frac{p}{p-1} \) and \( \Lambda(B_R) \) is defined in (1.3).

**Proof of Part (1) of Theorem 1.1.** Applying Theorem 3.3 to the subharmonic functions \( u \) and \( -u \) implies the desired statement. \( \square \)

As announced in Remark 1.3, in the case \( d = 2 \) we can relax the assumptions \( p, q > 1 \) and \( \frac{1}{p} + \frac{1}{q} < \frac{2}{d-1} \) of Theorem 1.1.

**Proposition 3.4.** Fix a domain \( \Omega \subset \mathbb{R}^2 \). Let \( a: \Omega \to \mathbb{R}^{2 \times 2} \) be measurable such that \( \lambda \) and \( \mu \) given in (1.2) are nonnegative and satisfy \( \frac{1}{2}, \mu \in L^1(\Omega) \). Then there exists \( C \in [1, \infty) \) such that for every weak solution \( u \) of (1.1) and for any ball \( B_R \subset \Omega \)

\[
\|u\|_{L^\infty(B_R)} \leq c \left( R \left( \int_{B_R} \lambda^{-1} \right)^{\frac{1}{2}} \left( \int_{B_R} a \nabla u \cdot \nabla u \right)^{\frac{1}{2}} + \int_{B_R} |u| \right). \tag{3.5}
\]

**Remark 3.5.** A version of Theorem 3.3 under the more restrictive integrability condition \( \frac{1}{p} + \frac{1}{q} < \frac{2}{d} \) can be found in [32, theorem 3.1]. In view of a counterexample presented in [20], assumption (1.3) \( \frac{1}{p} + \frac{1}{q} < \frac{2}{d-1} \) used in Theorem 1.1 and Theorem 3.3 is essentially optimal. Indeed, in [20, theorem 2] for every \( p, q > 1 \) satisfying \( \frac{1}{p} + \frac{1}{q} > \frac{2}{d-1} \) the authors construct a weight \( \omega \) with \( \omega^{-1} \in L^p(B_1) \) and \( \omega \in L^q(B_1) \), and an unbounded weak solution of \( -\nabla \cdot \omega \nabla u = 0 \) in \( B_1 \) provided \( d \geq 4 \). In fact, in [20, theorem 2] only the case \( d = 4 \) is considered but the extension to \( d \geq 5 \) is straightforward. In general, we are not able to say anything about the borderline situation \( \frac{1}{p} + \frac{1}{q} = \frac{2}{d-1} \) except for the special case \( d = 2 \).

Our proof of Theorem 3.3 is similar to that of [32, theorem 3.1] and relies on a modification of the Moser iteration method [25, 26]. Let us now briefly highlight the main difference of our approach compared to the arguments given in [32] and discuss where our improvement comes from. A simple consequence of the Hölder and Sobolev inequalities combined with the relation \( \frac{1}{p} + \frac{1}{q} < \frac{2}{d} \) is the following weighted Poincaré inequality: there exists \( \kappa = \kappa(p, q, d) > 1 \) (in fact, \( \frac{1}{\kappa} = \frac{p}{p-1} (1 + \frac{1}{q} - \frac{2}{d}) \)) such that for any ball \( B_R \) and \( u \) with compact support in \( B_R \)

\[
\left( \int_{B_R} |u|^{2\kappa} \right)^{\frac{1}{\kappa}} \leq c R^2 \left( \int_{B_R} \mu u \right)^{\frac{1}{p'}} \left( \int_{B_R} \lambda^{-q} \mu \right)^{\frac{1}{2}} \int_{B_R} a \nabla u \cdot \nabla u \tag{3.6}
\]

where \( c = c(d) \in [1, \infty) \). Inequality (3.6) and the Caccioppoli inequality are enough to use Moser’s iteration argument to prove local boundedness. In the situation of Theorem 3.3, i.e., with the relaxed assumption \( \frac{1}{p} + \frac{1}{q} < \frac{2}{d-1} \), we do not have a weighted Poincaré inequality in the form of (3.6) at hand. However, a version of (3.6) is valid if we replace the \( d \)-dimensional balls by \((d - 1)\)-dimensional spheres. In order to exploit this observation, we need an additional optimization...
step compared to the usual Caccioppoli inequality, which is gathered in Lemma 2.1. A similar idea, namely the use of the Sobolev inequality on spheres and a smart choice of certain cutoff functions, was previously used in [10] to obtain a new div-curl lemma and in [31] to show nonexistence of solutions for the Lane-Emden equation. The argument for Proposition 3.4 is different and in fact much simpler. It is mainly based on the maximum principle and Sobolev inequality in one dimension; see [20, prop. 1] for a similar argument.

PROOF OF THEOREM 3.3. Throughout the proof we write Ω if ≤ holds up to a positive constant that depends only on d, p, and q.

Step 1. We prove (3.4) for θ = \( \frac{1}{4} \), R = 2, and γ ≥ 2p'; i.e., for every γ ≥ 2p' there exists c = c(γ, d, p, q) ∈ [1, ∞) such that

\[
\| u^+ \|_{L_\infty(B_{1/2})} \leq c \Lambda(B_2) \left( \frac{p'}{p} (1 + \frac{1}{p}) \right) \| u^+ \|_{L^{p'}(B_2)}.
\]

For β ≥ 1 and N ∈ (0, ∞), we define

\[
F(u) := F_N^{\beta}(u) = \begin{cases} (u^+)^{\beta} & \text{for } u \leq N, \\ \beta N^{\beta - 1} u - (\beta - 1) N^{\beta} & \text{for } u \geq N. \end{cases}
\]

Set \( \phi := \eta^2 F(u) \) with \( \eta \geq 0, \eta \in C_0^1(B_2) \). By (3.2), we obtain

\[
\int_{\Omega} \eta^2 F'(u) a \nabla u \cdot \nabla u \leq -2 \int_{\Omega} \eta F(u) a \nabla u \cdot \nabla \eta.
\]

Definition (1.2) (in particular, \( |a \nabla u| \leq (\mu a \nabla u \cdot \nabla u)^{1/2} \)) and \( a \nabla u \cdot \nabla u \geq \lambda |\nabla u|^2 \)), Young’s inequality, and convexity of \( F \) in the form of \( F(u) \leq u F'(u) \) yield

\[
\int_{\Omega} \eta^2 F'(u) \lambda |\nabla u|^2 \leq 4 \int_{\Omega} F'(u) u^2 \mu |\nabla \eta|^2.
\]

We rewrite estimate (3.8) as

\[
\int_{\Omega} \eta^2 |\nabla G(u)|^2 \leq 4 \int_{\Omega} (u G'(u))^2 \mu |\nabla \eta|^2,
\]

where \( G(u) := \int_{0}^{u} |F'(t)|^{1/2} \, dt \).

Fix \( \frac{1}{2} \leq \rho < \sigma \leq 2 \). We optimize the right-hand side of (3.9) with respect to \( \eta \) satisfying \( \eta \in C_0^1(B_\sigma) \) and \( \eta = 1 \) in \( B_\rho \): we use Lemma 2.1 which by Hölder’s inequality implies

\[
\| \nabla G(u) \|_{L_\infty(B_\rho)}^2 \leq \Lambda(B_\sigma) (\sigma - \rho)^{-\frac{2d}{d+1}} \| u G'(u) \|_{W^{1,p_*(B_\sigma)}}^2.
\]

(3.10)
where $\frac{1}{p^*} = \min\{\frac{1}{2} - \frac{1}{2p} + \frac{1}{d-1}, 1\}$ as in Lemma 2.1. Notice that in order to apply Lemma 2.1 we used
\[ G'(u)u \in W^{1,p^*}(B_\sigma) \quad \text{and} \quad \mu(G'(u)u)^2 \in L^1(B_\sigma). \]
The first property is a consequence of $p^* < \frac{2q}{q+1}$ (by (1.3)), $u \in W^{1,\frac{2q}{q+1}}(\Omega)$ (by Hölder inequality and $u \in H^1(\Omega, \alpha)$), and the chain rule for Sobolev functions; see, e.g., [21, theorem 7.8]. For the second property, we use that $\mu u^2 \in L^1(\Omega)$ (since $u \in H^1(\Omega, \alpha)$) and $0 \leq G'(u) \leq \beta N^{N-1}$.

Next, we send the truncation parameter $N$ to infinity. Using
\[ \lim \inf_{N \to \infty} \frac{\|F_N'(u)\|}{\|u\|^{p^*}_{\sigma^*}(B_\sigma)} \geq \beta \|u\|^{\frac{\beta+1}{p}}_{\sigma^*}(B_\sigma), \]
and the definition of $G$, we obtain from (3.10)
\[ \|\nabla(u_+^{\alpha})\|_{L^{\frac{2q}{q+1}}(B_\rho)} \leq \Lambda(B_\sigma)^{\frac{1}{2}}(\sigma - \rho)^{-\frac{d}{q+1}} \alpha \|u_+^{\alpha}\|_{W^{1,p^*}(B_\sigma)} \]
with $\alpha = \frac{\beta+1}{2}$.

For future use, we note that if we choose $\eta \in C^1_0(B_\sigma)$ with $\eta = 1$ in $B_\rho$ and $\|\nabla \eta\|_{L^\infty} \leq 2(\sigma - \rho)^{-1}$, estimate (3.9), the Hölder inequality, and the choice of $G$ as above yield
\[ \|\nabla(u_+^{\alpha})\|_{L^{\frac{2q}{q+1}}(B_\rho)} \leq \Lambda(B_\sigma)^{\frac{1}{2}}(\sigma - \rho)^{-\frac{d}{q+1}} \alpha \|u_+^{\alpha}\|_{W^{1,p^*}(B_\sigma)}, \]
with $p' = \frac{p}{p-1}$. Let us now return to (3.11). Notice that condition (1.3) implies $\frac{2q}{q+1} > p^*$, and thus (3.11) contains an improvement in integrability of $\nabla u_+^{\alpha}$.

Hölder’s inequality with exponent $\frac{2q}{(q+1)p^*}$ yields with $\delta = \min\{\frac{1}{d-1} - \frac{1}{2p} \cdot \frac{1}{2}, \frac{1}{2q} \} > 0$ (notice that (1.3) and $q > 1$ imply $\delta > 0$)
\[ \left( \int_{B_\rho} |\nabla(u_+^{\alpha(1+\delta)})|^{p^*} \right)^{\frac{1}{p^*}} = \alpha (1+\delta) \left( \int_{B_\rho} |\nabla u_+|^{p^*} u_+^{(\alpha(1+\delta))} \right)^{\frac{1}{p^*}} \]
\[ \leq (1+\delta) \left( \int_{B_\rho} |\nabla u_+|^{\frac{2q}{q+1}} \right)^{\frac{q+1}{2q}} \left( \int_{B_\rho} u_+^{\alpha\delta} \right)^{\delta}. \]

Hence, by (3.11) with $\chi := 1+\delta > 1$,
\[ \|\nabla(u_+^{\alpha\chi})\|_{L^{p^*}(B_\rho)} \leq (\sigma - \rho)^{-\frac{d}{q+1}} \Lambda(B_\sigma)^{\frac{1}{2}} \alpha \chi \|u_+^{\alpha}\|_{W^{1,p^*}(B_\sigma)} \]
By the Sobolev inequality (using $p^* \geq 1$ and $\chi \leq \frac{d}{d-1}$), we get that
\[ \|u_+^{\alpha\chi}\|_{L^{p^*}(B_\rho)} \leq \|u_+^{\alpha}\|_{W^{1,p^*}(B_\rho)}, \]
and thus there exists \( c = c(d, p, q) \in [1, \infty) \) such that for every \( \alpha \geq 1 \)

\[
(3.16) \quad \| u_+^{\alpha \chi} \|_{W^{1, p_\ast}(B_\rho)} \lesssim \left( \frac{c \Lambda(B_\rho)^{\frac{1}{2} \alpha \chi}}{(\alpha - \rho)^{\frac{d}{2 - \gamma}}} \right)^{\frac{1}{\alpha \chi}} \| u_+^{\frac{1}{p_\ast}} \|_{W^{1, p_\ast}(B_\rho)}.
\]

Estimate (3.16) can be iterated in the usual way: Fix \( \alpha \geq 1 \) and for \( v \in \mathbb{N} \), set 
\( \alpha_v = \alpha \chi^{v-1}, \rho_v = \frac{1}{2} + \frac{1}{2^{v+1}}, \sigma_v := \rho_v + \frac{1}{2^v} = \rho_{v-1} \) (where \( \rho_0 := 1 \)), and (3.16) reads

\[
\left\| \frac{1}{\alpha} \frac{1}{\chi^v} \| u_+^{\alpha \chi} \|_{W^{1, p_\ast}(B_{\rho_v})} \right\| \left( \frac{c \Lambda(B_1)^{\frac{1}{2} \alpha_v \chi}}{(\alpha_v - \rho_v)^{\frac{d}{2 - \gamma}}} \right)^{\frac{1}{\alpha_v \chi^v}} \left\| u_+^{\frac{1}{p_\ast}} \|_{W^{1, p_\ast}(B_{\rho_v})} \right\|
\]

and thus

\[
(3.17) \quad \| u_+ \|_{L^\infty(B_{\frac{1}{2}})} \leq \prod_{v=1}^{\infty} \left( \frac{c \alpha \Lambda(B_1)^{\frac{1}{2} \alpha \chi}}{\alpha_v^\frac{1}{\chi^v}} \right)^{\frac{1}{\alpha \chi}} \| u_+^{\frac{1}{p_\ast}} \|_{W^{1, p_\ast}(B_1)}
\]

To estimate the right-hand side of (3.17), we use (3.12) and the fact that \( p_\ast \leq \frac{2q}{q + 1} \leq 2 \leq 2p' \):

\[
\| \nabla (u_+^\alpha) \|_{L^{p_\ast}(B_1)} \lesssim \| \nabla (u_+^\alpha) \|_{L^{2p'}(B_1)} \overset{(3.12)}{\lesssim} \Lambda(B_2)^{\frac{1}{2} \alpha} \| u_+^{\frac{1}{p_\ast}} \|_{L^{2p'}(B_2)},
\]

\[
\| u_+^\alpha \|_{L^{p_\ast}(B_1)} \lesssim \| u_+^{\frac{1}{p_\ast}} \|_{L^{2p'}(B_2)}.
\]

Since \( \alpha, \Lambda \geq 1 \) and \( \sum_{v=1}^{\infty} v \chi^{-v} \leq 1 \), we get that there exists \( c = c(d, p, q, \alpha) \in [1, \infty) \) such that

\[
\| u_+ \|_{L^\infty(B_{1/2})} \lesssim c \Lambda(B_1)^{\frac{1}{2} \alpha} \left( \frac{1}{\chi^{v+1}} \right) \| u_+^{\frac{1}{p_\ast}} \|_{W^{1, p_\ast}(B_1)} \lesssim c \Lambda(B_2)^{\frac{1}{2} \alpha} \| u_+ \|_{L^{2p'}(B_2)},
\]

which proves the claim by setting \( \gamma = 2p' \alpha \geq 2p' \) (recall \( \chi = 1 + \delta \)).

**Step 2.** The general case. It is well-known how to lift the result of Step 1 to prove the claim. For the convenience of the reader, we provide the arguments following the presentation in [22]. First, by scaling we deduce from (3.7) that for \( \gamma \geq 2p' \) and \( R > 0 \)

\[
(3.18) \quad \| u_+ \|_{L^\infty(B_{R/4})} \leq c \Lambda(B_R)^{\frac{1}{2} \gamma} \left( \frac{1}{\gamma + 1} \right)^{\frac{d}{2 - \gamma}} \| u_+ \|_{L^\gamma(B_R)},
\]

where \( c = c(\gamma, d, p, q) \in [1, \infty) \) is the same as in (3.7). Now the statement for \( \gamma \geq 2p' \) follows by applying for every \( y \in B_{\theta R} \) estimate (3.18) with \( B_R \) replaced
by \( B_{(1-\theta)R}(y) \), i.e.,
\[
\|u^+\|_{L^\infty(B_{(1-\theta)R}(y))} \leq \frac{c\Lambda(B_{(1-\theta)R}(y))^{\frac{p'}{p}(1+\frac{1}{s})}}{((1-\theta)R)^{\frac{d}{p'}}} \|u^+\|_{L^{p'}(B_{(1-\theta)R}(y))}
\]
\[
\leq \frac{c\Lambda(B_R)^{\frac{p'}{p}(1+\frac{1}{s})}}{(1-\theta)^{\frac{d}{p'}} R^{\frac{d}{p'}}} \|u^+\|_{L^{p'}(B_R)},
\]
where \( s = 1 + p'(1 + \frac{1}{\theta})(\frac{1}{p} + \frac{1}{q}) \) and thus
\[
(3.19) \quad \|u^+\|_{L^\infty(B_0)} \leq \frac{c\Lambda(B_R)^{\frac{p'}{p}(1+\frac{1}{s})}}{(1-\theta)^{\frac{d}{p'}} R^{\frac{d}{p'}}} \|u^+\|_{L^{p'}(B_R)} \quad \text{for every } \gamma \geq 2p'.
\]

Hence, it remains to prove estimate (3.4) for \( \gamma \in (0, 2p') \). For given \( \gamma \in (0, 2p') \), we first observe that
\[
\|u^+\|_{L^2p'(B_R)} \leq \|u^+\|_{L^{\infty}(B_R)} \|u^+\|_{L^{p'}(B_R)}
\]
and thus by (3.19) (with \( \gamma = 2p' \)) and Young’s inequality
\[
(3.20) \quad \|u^+\|_{L^\infty(B_0)} \leq \frac{c\Lambda(B_R)^{\frac{1}{2}(1+\frac{1}{s})}}{(1-\theta)^{\frac{d}{p'}} R^{\frac{d}{p'}}} \|u^+\|_{L^\infty(B_R)} \|u^+\|_{L^{2p'}(B_R)}
\]
\[
\leq \frac{c\Lambda(B_R)^{\frac{1}{2}(1+\frac{1}{s})}}{(1-\theta)^{\frac{d}{p'}} R^{\frac{d}{p'}}} \|u^+\|_{L^\infty(B_R)} \|u^+\|_{L^{2p'}(B_R)}
\]
\[
\leq \frac{1}{2} \|u^+\|_{L^\infty(B_0)} + (2c)^{\frac{2p'}{2p'}} \Lambda(B_R)^{\frac{p'}{p}(1+\frac{1}{s})} \|u^+\|_{L^{p'}(B_R)},
\]
where \( c = c(d, p, q) \in [1, \infty) \). Set \( f(t) := \|u^+\|_{L^\infty(B_t)} \), \( t \in (0, 1] \). The estimate (3.20) implies that there exists \( c' = c(\gamma, d, p, q) \in [1, \infty) \) such that for all \( 0 < r < R \leq 1 \)
\[
f(r) \leq \frac{1}{2} f(R) + \frac{c\Lambda(B_R)^{\frac{p'}{p}(1+\frac{1}{s})}}{(1-\frac{r}{R})^{\frac{d}{p'}} R^{\frac{d}{p'}}} \|u^+\|_{L^{p'}(B_R)}
\]
\[
\leq \frac{1}{2} f(R) + \frac{c\Lambda(B_1)^{\frac{p'}{p}(1+\frac{1}{s})}}{(R-r)^{\frac{d}{p'}}} \|u^+\|_{L^{p'}(B_1)}.
\]
Hence, by [22, lemma 4.3], we find \( c = c(\gamma, d, p, q) \in [1, \infty) \) such that for all \( 0 < r < R < 1 \),
\[
\|u^+\|_{L^\infty(B_r)} \leq \frac{c\Lambda(B_1)^{\frac{p'}{p}(1+\frac{1}{s})}}{(R-r)^{\frac{d}{p'}}} \|u^+\|_{L^{p'}(B_1)},
\]
and the claim (3.4) (with \( \theta = r \) and \( R = 1 \)) follows.
PROOF OF PROPOSITION 3.4. Clearly, it suffices to show that every weak sub-
solution $u$ of (1.1) is locally bounded from above and there exists $c \in [1, \infty)$ such
that for any ball $B_R \subset \Omega$ it holds that

$$
\sup_{B_{R/2}} u \leq c \left( R \left( \int_{B_R} \lambda^{-1} \right)^{1/2} \left( \int_{B_R} a \nabla u_+ \cdot \nabla u_+ \right)^{1/2} + \int_{B_R} u_+ \right).
$$

The maximum principle (see [33, theorem 3.1]) yields

$$
\sup_{B_R} u \leq \sup_{\partial B_R} u_+ \text{ for every } B_R \subset \Omega.
$$

In [33], the maximum principle (3.22) is proven for much more general equations.
For the convenience of the reader we recall the argument for the specific situation
considered here at the end of the proof. Next, we prove (3.21) for $\mathcal{D}$
the general case follows by scaling). In view of Fubini’s theorem, we find

$$
\int_{\partial B_1} u_+ + |\nabla u_+| \leq 2 \int_{B_1 \setminus B_1/2} u_+ + |\nabla u_+|.
$$

Hence, by the Sobolev inequality in one dimension,

$$
\sup_{B_1/2} u \leq \sup_{B_{1/2^\rho}} u_+ \sup_{\partial B_{1/2^\rho}} u_+ \leq \int_{\partial B_1} u_+ + |\nabla u_+| \leq \int_{B_1} u_+ + |\nabla u_+| \leq 2\int_{B_1} u_+ + |\nabla u_+| \leq 2\|u_+\|_{L^1(B_1)} + 2\|\lambda^{-1}\|_{L^1(B_1)} \left( \int_{B_1} a \nabla u_+ \cdot \nabla u_+ \right)^{1/2},
$$

where the last inequality follows by Hölder’s inequality and (1.2) (in the form
$\lambda |\nabla u_+|^2 \leq a \nabla u_+ \cdot \nabla u_+$).

Finally, we recall the argument of [33] for (3.22). Set $\varphi := (u - \sup_{\partial B_R} u_+)_+$. Since $\varphi \in H_0^1(B_R, a)$ and $\varphi \geq 0$, we can use $\varphi$ as a test function in (3.2) and obtain

$$
0 \geq \int_{B_R} a \nabla u \cdot \nabla \varphi \geq \int_{B_R} \lambda |\nabla \varphi|^2 \geq \|\lambda^{-1}\|_{L^1(B_R)} \|\nabla \varphi\|_{L^2(B_R)^2}^2,
$$

and thus, by the Poincaré inequality and $\varphi = 0$ on $\partial B_R$, we obtain $\varphi = 0$ and consequently (3.22). □

4 Harnack Inequality—Proof of Part (ii) of Theorem 1.1
and Some Applications

The main result of this section is the following:

THEOREM 4.1 (Weak Harnack inequality). Fix $d \geq 2$, a domain $\Omega \subset \mathbb{R}^d$, and
$p,q \in (1, \infty]$ satisfying (1.3). Let $a: \Omega \to \mathbb{R}^{d \times d}$ be such that $\lambda$ and $\mu$ given in
(1.2) are nonnegative and satisfy $\frac{1}{\lambda} \in L^q(\Omega)$, $\mu \in L^p(\Omega)$. Let $u$ be a nonnegative
weak supersolution of (1.1) in $\Omega$. Then, for every $0 < \theta < \tau < 1$, $\gamma \in (0, \frac{p}{2})$,
where $\frac{1}{q_*} = \frac{1}{2} + \frac{1}{2q} - \frac{1}{d}$ if $1 + \frac{1}{q} > \frac{2}{d}$ and $q_* = +\infty$ otherwise (i.e., if $d = 2$ and $q = \infty$), and any $B_R \subset \Omega$, there exists $C = C(d, p, q, \theta, \gamma, \Lambda(B_R)) \in [1, \infty)$ such that

$$ (\frac{1}{R^d} \int_{B_{\tau R}} u^\gamma) \frac{1}{2} \leq C \inf_{B_{\tau R}} u. \tag{4.1} $$

In fact, the constant $C$ in (4.1) satisfies $C = c_1 e^{\Lambda(B_R)c_2}$ with

$$ c_1 = c_1(\gamma, d, p, q, \tau, \theta) \in [1, \infty) \quad \text{and} \quad c_2 = c_2(\gamma, d, p, q) > 0. $$

Proof of Part (ii) of Theorem 1.1. Notice $q_* > 1$ for every $d \geq 2$. Hence, combining the local boundedness estimate (3.4) with $D \subset 2$ and Theorem 4.1 ($q_* > 1$ allows $\gamma = \frac{1}{2}$ and Theorem 4.1 ($q_* > 1$), we obtain

$$ \sup_{B_{R/2}} u \leq c \Lambda(B_R)^{2p'(1+\frac{1}{2})} \left(\frac{1}{R^d} \int_{B_{3/4R}} u^\frac{1}{2} \right)^2 \leq c \Lambda(B_R)^{2p'(1+\frac{1}{2})} \inf_{B_{R/2}} u, $$

with $c = c(d, p, q) \in [1, \infty)$ and $C = C(d, p, q, \Lambda(B_R)) \in [1, \infty)$, which proves the claim. \hfill $\Box$

In [32], Trudinger proved the conclusion of Theorem 4.1 under the more restrictive assumption $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$. We prove Theorem 4.1 by combining the strategy of Trudinger in the proof of [32, theorem 4.1] with the local boundedness result Theorem 3.3 and an improved Caccioppoli inequality due to Lemma 2.1. Even though experts might already anticipate how to adapt the arguments of [32], we give a detailed proof at the end of this section.

Before proving Theorem 4.1 we list some consequences of Theorem 1.1, which are by now standard and thus we only give the statements without proofs. In the uniformly elliptic setting, the Harnack inequality implies Hölder continuity of weak solutions to (1.1). As observed in [32], due to the explicit dependence of the constant $c$ in (1.6) on $\Lambda(B_R)$, this is in general not true anymore in the nonuniformly elliptic setting. However, Theorem 1.1 yields the following large-scale Hölder continuity:

Corollary 4.2 (Hölder continuity “on large scales”). Consider the situation of Theorem 1.1. For $R > 0$ set $\overline{\Lambda}_R := \sup_{R' \geq R} \Lambda(B_R)$. Suppose that $u$ is a weak solution of (1.1) in $B_{R_0} \subset \Omega$ and $\overline{\Lambda}_{R_1} < \infty$ for some $0 < R_1 < \frac{1}{4} R_0$. Then, for all $R \in [R_1, \frac{1}{2} R_0]$

$$ \text{osc}_{B_R} u \leq c \left( \frac{R}{R_0} \right)^\theta \left( \frac{1}{B_{R_0}} |u| \right), $$

where $c$ and $\theta$ are positive constants depending on $d$, $p$, $q$, and $\overline{\Lambda}_{R_1}$.

Remark 4.3. If $\Lambda(B)$ is bounded uniformly for all balls $B \subset \Omega$, Corollary 4.2 implies usual Hölder continuity of $a$-harmonic functions in $\Omega$. This improves [32, theorem 5.1] since it relaxes the integrability assumptions on $\lambda^{-1}$ and $\mu$. However,
under such local assumptions Hölder regularity of $a$-harmonic functions is proven under much weaker integrability conditions; see, for example, [17].

A direct consequence of Corollary 4.2 is the following zero-order Liouville property:

**Corollary 4.4 (Liouville Theorem).** Fix $d \geq 2$ and $p, q \in (1, \infty]$ satisfying (1.3). Let $a: \mathbb{R}^d \to \mathbb{R}^{d \times d}$ be a measurable coefficient field such that $\lambda, \mu$ given by (1.2) are nonnegative and satisfy $\frac{1}{\lambda} \in L^q_{\text{loc}}(\mathbb{R}^d)$, $\mu \in L^p_{\text{loc}}(\mathbb{R}^d)$. Moreover, suppose that $\limsup_{R \to \infty} \Lambda(B_R) < \infty$. Then every bounded local weak solution $u$ of (1.1) in $\mathbb{R}^d$, in the sense of Definition 3.1, is constant.

In [16, theorem 3], the conclusion of Corollary 4.4 is proven (relying on the results of Trudinger in [32, 33]) under the more restrictive assumption $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$ (notice that [16, theorem 3] also applies in situations with additional lower-order terms that are not considered here).

Finally, we provide the proof of Theorem 4.1.

**Proof of Theorem 4.1.** Throughout the proof we write $\lesssim$ if $\leq$ holds up to a positive constant that depends only on $d$, $p$, and $q$. Without loss of generality we set $R = 1$ and suppose that $u \geq \varepsilon > 0$. In what follows, we suppose $1 + \frac{1}{q} > \frac{2}{d}$.

The remaining case $d = 2$ with $q = \infty$ can be done with no additional difficulties by appealing to corresponding versions of the Sobolev inequality.

**Step 1.** Fix $0 < \theta < \tau < 1$. We claim that

$$
(4.2) \quad \exp\left(\int_{B_{\tau}} \log(u)\right) \leq C \inf_{B_{\theta}} u,
$$

where $C = c_1 e^{\Lambda(B_1)\v^2}$ with

$$
c_1 = c_1(d, p, q, \tau, \theta) \in [1, \infty) \quad \text{and} \quad c_2 = c_2(d, p, q) \in [1, \infty).
$$

Testing (3.2) with $\phi = \eta u^{-1}$ with $\eta \geq 0$ and $\eta \in H^1_0(B_1, a)$, we obtain

$$
(4.3) \quad \int_{B_1} \frac{1}{u} a \nabla u \cdot \nabla \eta \, dx - \int_{B_1} \frac{\eta}{u^2} a \nabla u \cdot \nabla \eta \, dx \geq 0.
$$

Setting $v := \log(\frac{k}{u})$, $k > 0$, we see $a \nabla v \cdot \nabla \eta = -\frac{1}{u} a \nabla u \cdot \nabla \eta$; hence (4.3) and $u > 0$ imply

$$
\int_{B_1} a \nabla v \cdot \nabla \eta \, dx \leq 0.
$$

The arbitrariness of $\eta$ implies that $v$ is $a$-subharmonic in the sense of (3.2). Hence Theorem 3.3 with $\gamma = q_*$, where $\frac{1}{q_*} = \frac{1}{2} + \frac{1}{2q} - \frac{1}{d}$ (recall the assumption
$1 + \frac{1}{q} > \frac{2}{d}$), yields

$$\sup_{B_\theta} v \lesssim \Lambda(B_\tau) \frac{C'}{\alpha} \left(1 + \frac{1}{\theta} + \frac{1}{\tau} \right) \left(\int_{B_\tau} |v|^{q_*} \right)^{\frac{1}{q_*}}$$

(4.4)

$$\leq \Lambda(B_1) \frac{C}{\alpha} \left(1 + \frac{1}{\theta} + \frac{1}{\tau} \right) \left(\int_{B_\tau} |v|^{q_*} \right)^{\frac{1}{q_*}},$$

with $s = 1 + p'(1 + \frac{1}{\theta}) \left(\frac{1}{p} + \frac{1}{\tau} \right)$. Next, we replace $\eta$ in (4.3) by $\eta^2$ with $\eta \in C_0^1(B_1)$ and obtain (using (1.2) and applying Young's inequality)

$$\int_{B_1} \eta^2 \lambda |
abla u|^2 \leq 4 \int_{B_1} \mu |\nabla \eta|^2.$$

Choosing $\eta$ such that $\eta = 1$ in $B_\tau$ and $|\nabla \eta| \leq 2(1 - \tau)^{-1}$, we obtain

(4.5)

$$\int_{B_\tau} \lambda |\nabla u|^2 \leq 4^2 (1 - \tau)^{-2} \int_{B_1} \mu.$$

Finally, we choose $k > 0$ such that $\int_{B_\tau} \eta = 0$, i.e., $k := \exp(\int_{B_\tau} \log(u))$, and thus by a combination of the Hölder and Sobolev inequalities (note that $q_*$ is the Sobolev exponent for $2\eta+1$)

(4.6) $$\left(\int_{B_\tau} |v|^{q_*} \right)^{\frac{1}{q_*}} \lesssim \left(\int_{B_\tau} |\nabla u|^{2\eta+1} \right)^{\frac{q_*+1}{2\eta+1}} \leq \|\lambda^{-1}\|_{L^\eta(B_\tau)} \left(\int_{B_\tau} \lambda |\nabla u|^2 \right)^{\frac{1}{2}}.$$

Combining (4.4)–(4.6), we obtain

$$\sup_{B_\theta} v \lesssim \Lambda(B_1) \frac{C'}{\alpha} \left(1 + \frac{1}{\theta} + \frac{1}{\tau} \right) \left(\int_{B_\tau} |v|^{q_*} \right)^{\frac{1}{q_*}}$$

(4.7)

$$\approx \Lambda(B_1) \frac{C}{\alpha} \left(1 + \frac{1}{\theta} + \frac{1}{\tau} \right) \left(\int_{B_\tau} |v|^{q_*} \right)^{\frac{1}{q_*}} \approx \frac{\Lambda(B_1)}{\alpha} \left(1 + \frac{1}{\theta} + \frac{1}{\tau} \right) \left(\int_{B_\tau} |v|^{q_*} \right)^{\frac{1}{q_*}}.$$

Finally, the definitions of $v$ and $k$ yield the claimed estimate (4.2).

**Step 2.** Fix $0 < \theta < \tau < 1$. We claim that there exist

$$s_0 = s_0(d, p, q, \theta, \tau, \Lambda(B_1)) > 0 \quad \text{and} \quad C = C(d, p, q, \theta, \tau, \Lambda(B_1)) < \infty$$

such that

(4.7) $$\left(\int_{B_\theta} |u|^{s_0} \right)^{\frac{1}{s_0}} \leq C \exp \left(\int_{B_\tau} \log(u) \right).$$
In fact, it holds that \( s_0^{-1} \leq c_1 e^{\Lambda(B_1) x^2} \) with \( c_1 = c_1(d, p, q, \tau, \theta) \in [1, \infty) \) and \( c_2 = c_2(d, p, q) \). Set \( w := v_- = (\log u_K)^+ \). For given \( \beta \geq 1 \) and \( \eta \geq 0 \in C^1_0(B_1) \), we consider the test function
\[
\phi(x) = \eta^2(x) u^{-1}(x)(w^\beta(x) + (2\beta)^\beta).
\]
The fact that \( u \) is \( a \)-superharmonic (see (3.2)), \( w \geq 0 \), and the elementary inequality (coming from Young’s inequality)
\[
(4.8) \quad \beta w^{\beta-1} \leq \frac{1}{2}(w^\beta + (2\beta)^\beta)
\]
yield (using \( \nabla \phi = \frac{2n}{u^2}(w^\beta + (2\beta)^\beta) \nabla u + \frac{n_2}{u^2}(\beta w^{\beta-1} - w^\beta - (2\beta)^\beta) \nabla u \))
\[
\frac{1}{2} \int \frac{n_2}{u^2} (\beta w^{\beta-1} + \frac{1}{2}(2\beta)^\beta) a \nabla u \cdot \nabla u \leq \frac{1}{2} \int \frac{n_2}{u^2} (w^\beta + (2\beta)^\beta) a \nabla u \cdot \nabla u
\]
\[
\leq 2 \int \frac{n_2}{u} (w^\beta + (2\beta)^\beta) a \nabla u \cdot \nabla \eta.
\]
Appealing to (1.2) and Young’s inequality, we estimate the right-hand side of (4.9):
\[
2 \int \frac{n_2}{u} (w^\beta + (2\beta)^\beta) a \nabla u \cdot \nabla \eta
\]
\[
\leq \frac{1}{4} \int \frac{n_2}{u^2} (\beta w^{\beta-1} + (2\beta)^\beta) a \nabla u \cdot \nabla u
\]
\[
+ 4 \int \left( \frac{1}{\beta} w^{\beta+1} + (2\beta)^\beta \right) \mu |\nabla \eta|^2.
\]
Note that the first term on the right-hand side in (4.10) can be absorbed into the left-hand side of (4.9) and we obtain, using \( \beta \geq 1 \) and the definition of \( w \),
\[
(4.11) \quad \beta \int n_2 w^{\beta-1} a \nabla w \cdot \nabla w \leq 16 \int (w^{\beta+1} + (2\beta)^\beta) \mu |\nabla \eta|^2.
\]
Fix \( 0 < \theta < \rho < \sigma < \tau < 1 \). Let \( \eta \in C^1_0(B_\rho) \) be such that \( \eta = 1 \) in \( B_\rho \). Minimizing the right-hand side of (4.11) among such cutoff functions, we obtain with the help of Lemma (2.1) (1.2) and \( \beta \geq 1 \)
\[
\frac{2}{\beta + 1} \int_{B_\rho} \lambda |\nabla w|^{\frac{\beta+1}{2}}^2
\]
\[
\leq c(\sigma - \rho)^{-\frac{2d}{d+\alpha}} \|\mu\|_{L^p(B_\rho)} \left( \|w^{\beta+1}\|_{W^{1,p}(B_\rho)}^2 + (2\beta)^\beta \right).
\]
where \( c = c(d, p, \theta) \in [1, \infty) \). Hence, setting \( \alpha := \frac{\beta+1}{2} \), we obtain by Hölder inequality and (1.5)
\[
(4.12) \quad \|\nabla (w^\alpha)\|_{L^{\frac{2d}{\alpha+d}}(B_\rho)} \leq \frac{c\Lambda(B_1)^{1/2}}{(\sigma - \rho)^{\frac{d}{d+\alpha}}} \|w^\alpha\|_{W^{1,p}(B_\rho)} + (4\alpha)^\alpha).
\]
Using (3.13)–(3.15) (with $u$ replaced by $w$), we derive from (4.12) an analogue of (3.16):

$$\|w^{\alpha}x\|_{W^{1,p_*}(B_0)} \leq \left( \frac{c \Lambda(B_1) \alpha \chi}{(\sigma - \rho) \frac{q}{2} \frac{q}{2}} \right) \left( \|w^{\alpha}\|_{W^{1,p_*}(B_0)}^{1/2} + 4\alpha \right),$$

where $\chi = 1 + \delta$, with $\delta = \min\left\{ \frac{1}{\sigma - 1} - \frac{1}{2\rho}, \frac{1}{2} \right\} > 0$ and $c = c(d, p, q, \theta, \tau) \in [1, \infty)$. For $\nu \in \mathbb{N}$ set

$$\alpha_{\nu} = \chi^{-1}, \quad \rho_{\nu} = \theta + 2^{-\nu}(\tau - \theta), \quad \text{and} \quad \sigma_{\nu} := \rho_{\nu} + 2^{-\nu}(\tau - \theta) = \rho_{\nu - 1}.$$  

Then

$$\|w^{\nu}\|_{W^{1,p_*}(B_{\rho_{\nu}})} \leq \left( \frac{c \Lambda(B_1)(4\nu^2 \chi \tau)^{\nu}}{(\tau - \theta) \frac{q}{2} \frac{q}{2}} \right) \left( \|w^{\nu-1}\|_{W^{1,p_*}(B_{\rho_{\nu - 1}})}^{1/2} + 4\nu^{\nu-1} \right).$$

Estimate (4.14) can be iterated and we find that there exists $c_1 = c_1(d, p, q, \theta, \tau) \in [1, \infty)$ such that for every $s \geq 1$,

$$\|w\|_{L^s(B_0)} \leq c_1 \Lambda(B_1)^{c_2^2} (\|w\|_{W^{1,p_*}(B_{\tau})} + s).$$

Recalling the fact $w = v_-$, estimates (4.5) and (4.6) and the fact that $p_* \leq \frac{2q}{q+1}$ (by (1.3)), we obtain for $s \geq 1$

$$\|w\|_{L^s(B_0)} \leq c'_1 \Lambda(B_1)^{c'_2} s,$$

where $c'_1$ and $c'_2$ have the same dependencies as $c_1$ and $c_2$ in (4.15), respectively. Estimate (4.16) and the choice $s_0 := (2c'_1 \Lambda(B_1)^{c'_2} e)^{-1}$ yield for every $j \in \mathbb{N}$

$$\frac{s_0^j \|w\|^j_{L^j(B_0)}}{j!} \leq \frac{1}{2^j e^j j!} \leq \frac{1}{2^j},$$

and thus

$$\int_{B_0} \exp(s_0 w) \leq \sum_{j=0}^{\infty} \frac{s_0^j \|w\|^j_{L^j(B_0)}}{j!} \leq \sum_{j=0}^{\infty} \frac{1}{2^j} = 2.$$

Recall that $w = (\log \frac{u}{k})^+$, with $k = \exp\left(\int_{B_{\tau}} \log(u)\right)$, and thus

$$\left( \int_{B_0} \left( \frac{u}{k} \right)^{\infty} \right)^{\frac{1}{s_0}} \leq (2 + |B_1|)^{\frac{1}{s_0}}$$

implies

$$\left( \int_{B_0} u^{s_0} \right)^{\frac{1}{s_0}} \leq (2 + |B_1|)^{2e c'_1 \Lambda(B_1)^{c'_2}} \exp\left(\int_{B_{\tau}} \log(u)\right),$$

which proves the claim.
Step 3. Fix \( \varepsilon \in (0, \frac{d}{2p}) \) and \( 0 < \tau < \tau' < 1 \). We claim that for every \( \gamma \in (\varepsilon, \frac{d}{2}) \) there exists \( C = C(\gamma, d, \varepsilon, p, q, \Lambda(B_1), \tau, \tau') \in [1, \infty) \) such that

\[
(4.17) \quad \left( \int_{B_{\tau'}} u^\gamma \right)^{\frac{1}{\gamma}} \leq C \left( \int_{B_{\tau'}} u^\varepsilon \right)^{\frac{1}{\varepsilon}},
\]

where \( C \leq c_1 \Lambda^{c_2 \frac{\gamma}{\varepsilon}} \) with

\[
c_1 = c_1(\gamma, d, p, q, \tau, \tau') \in [1, \infty) \quad \text{and} \quad c_2 = c_2(\gamma, d, p, q) \in [1, \infty).
\]

Recall \( u > 0 \). Testing \( (3.2) \) with \( \phi := \eta^2 u^\beta \) where \( \beta \in (-1, 0) \) and \( \eta \geq 0 \), \( \eta \in C_0^1(B_1) \), we obtain

\[
\int \eta^2 u^\beta a \nabla u \cdot \nabla \eta \leq \frac{2}{|\beta|} \int \eta u^\beta a \nabla u \cdot \nabla \eta.
\]

Young’s inequality and \( (1.2) \) (in the form \( |a \nabla u| \leq (\mu a \nabla u \cdot \nabla u)^{1/2} \) and \( a \nabla u \cdot \nabla u \geq \lambda |\nabla u|^2 \)) yield

\[
(4.18) \quad \int \eta^2 \lambda |\nabla (u^{\beta + \frac{1}{2}})|^2 \leq \left( \frac{\beta + 1}{|\beta|^2} \right) \int u^{\beta + 1} \mu |\nabla \eta|^2.
\]

Set \( \alpha = \frac{\beta + 1}{2} \in (0, \frac{1}{2}) \). Fix \( \tau \leq \rho < \sigma \leq \tau' \) and consider \( \eta \in C_0^1(B_\sigma) \) satisfying \( \eta = 1 \) in \( B_\rho \). We estimate the right-hand side of \( (4.18) \) by either applying Lemma 2.1 or choosing a linear cutoff function \( \eta \). In combination with the Hölder inequality, we get \( c = c(p, d, \tau) \in [1, \infty) \) such that

\[
(4.19) \quad \| \nabla (u^\alpha) \|_{L^\frac{2q}{q - 1}(B_\rho)} \leq c \Lambda^{\frac{1}{2}}(\sigma - \rho)^{-\frac{d}{2} \tau},
\]

\[
(4.20) \quad \| \nabla (u^\sigma) \|_{L^\frac{2q}{q - 1}(B_\rho)} \leq c \Lambda^{\frac{1}{2}}(\sigma - \rho)^{-\frac{d}{2} \tau}.
\]

Using \( (3.13)-(3.15) \), we derive from \( (4.19) \)

\[
(4.21) \quad \| u^{\alpha \frac{1}{2}} \|^2 \omega_{W^{1, p}_\sigma(B_\tau)} \leq \left( \frac{c \Lambda^{\frac{1}{2}}}{(\sigma - \rho)^{-\frac{d}{2} \tau}} \right) \| u^\alpha \|^2 \omega_{W^{1, p}_\sigma(B_\tau)},
\]

where \( \chi = 1 + \delta \), with \( \delta = \min\{\frac{1}{\sigma - 1}, \frac{1}{\frac{d}{2p} - \frac{1}{2}, \frac{1}{2q}} \} > 0 \). Fix \( \kappa \in (0, \frac{1}{2}) \). We set \( \alpha_0 := \kappa \chi \in (0, \frac{1}{2}) \) and \( \alpha_i = \frac{\kappa \chi}{1 + \delta} \) for \( i \in \mathbb{N} \). Fix \( n \in \mathbb{N} \) such that \( \alpha_n < \frac{\kappa}{2p} \leq \alpha_{n - 1} \), \( \tau_1 := \tau + \frac{\kappa}{4}(\tau' - \tau) \) and \( \tau_2 := \tau + \frac{3}{4}(\tau' - \tau) \). Iterating \( (4.21) \) \( n \)-times, we find \( C = C(\kappa, d, \varepsilon, p, q, \tau, \tau', \Lambda(B_1)) \in [1, \infty) \) such that

\[
(4.22) \quad \| u^\kappa \|_{W^{1, p}_\sigma(B_{\tau})} \leq C \| u^{\alpha_n} \|^\frac{1}{\omega_{W^{1, p}_\sigma(B_{\tau})}}.
\]

Observe that the assumption \( \varepsilon \frac{2p - 1}{2p} \leq \alpha_{n - 1} \) yields \( \chi_n \leq \frac{\kappa}{\rho} \frac{2p}{\rho - 1} \) and thus \( C \leq c_1 \Lambda^{c_2 \frac{\kappa}{\rho}} \) with \( c_1 = c_1(\kappa, d, \varepsilon, p, q, \tau, \tau') \in [1, \infty) \) and \( c_2 = c_2(d, p, q) \in [1, \infty) \).
Using \( p_* \leq \frac{2q}{q + 1} \) and the choice of \( \alpha_\eta \) (i.e., \( \alpha_\eta \leq \epsilon \frac{p - 1}{2p} \)), we estimate the right-hand side of (4.22)

\[
\|u^{\partial \alpha_\eta}\|_{W^{1,p_*}(B_{r_2})} \leq C \|u\|_{L^{\frac{2p}{p - \alpha_\eta}}(B_{r_2})} \leq C \|u\|_{L^\infty(B_{r_2})}.
\]

Using the Sobolev inequality and (4.19), we get for the left-hand side of (4.22)

\[
\|u^K\|_{L^{\frac{2q}{q_1}}(B_{r})} \leq \|\nabla(u^K)\|_{L^{\frac{2q}{q_1}}(B_{r})} + |B_{r}|^{\frac{1}{p} - \frac{1}{p_*}} \|u^K\|_{L^{p_*}(B_{r})} \leq C \|u^K\|_{W^{1,p_*}(B_{r})},
\]

where \( \frac{1}{q_*} = \frac{1}{2} + \frac{1}{2q} - \frac{1}{d} \). Then a combination of (4.22)–(4.24) yields the desired claim (4.17) for \( \gamma = \kappa q_* \in (0, \frac{1}{2} q_*) \).

**Step 4. Conclusion.** Fix \( 0 < \theta < \tau < 1 \). Combining Steps 1, 2, and 3 (with \( \varepsilon = s_0 \)), we obtain

\[
\left( \int_{B_{r}} u^\gamma \right)^{\frac{1}{\gamma}} \leq C_1 \left( \int_{B_{r+1}} u^{s_0} \right)^{\frac{1}{s_0}} \leq C_1 C_2 \exp \left( \int_{B_{r+1}} \log(u) \right) \leq C_1 C_2 C_3 \inf_{B_{s_0}} u,
\]

where \( C_1, C_2, C_3 \in [1, \infty) \) satisfy the desired dependencies. \( \square \)

## 5 Sublinear Corrector in Random Homogenization with Degenerate Coefficients

In this section we apply Theorem 1.1 in the context of stochastic homogenization. Stochastic homogenization for uniformly elliptic equations dates back to the classical papers [24, 30]. Currently, stochastic homogenization beyond uniform ellipticity is an active field of research; see, e.g., [1, 4, 6, 12, 15, 18, 19, 29].

A central object in the homogenization of linear elliptic equations is the so-called corrector: For \( \xi \in \mathbb{R}^d \), the corrector \( \phi_\xi \) is characterized almost surely by solving

\[
\nabla \cdot a^\omega (\nabla \phi_\xi + \xi) = 0 \quad \text{in } \mathbb{R}^d \quad \text{and} \quad \lim_{R \to \infty} \frac{1}{R} \int_{B_R} |\phi_\xi| = 0.
\]

Here, we assume that the coefficient fields \( \{a^\omega(x)\}_{x \in \mathbb{R}^d} \subset \mathbb{R}^{d \times d} \) are statistically homogeneous and ergodic, and nonuniformly elliptic (see below for the precise assumptions). In [1, 12, 15, 18], the corrector \( \phi \) is used prominently to prove quenched invariance principles for the random walk [1, 15] or diffusion [12, 18] in a random environment with degenerate and/or unbounded coefficients. The key ingredient in [1, 12, 15, 18] is to upgrade the \( L^1 \)-sublinearity into \( L^\infty \)-sublinearity, i.e., to show \( \frac{1}{R} \|\phi_\xi\|_{L^\infty(B_R)} \to 0 \) as \( R \to \infty \). In this section, we show that the results of Section 3 can be used to weaken the assumption of [12, 18] in order to establish \( L^\infty \)-sublinearity of the corrector. The application to the quenched invariance
principle for the random walk in a random degenerate environment can be found in [7].

Let us now be more precise and phrase the assumptions on the coefficient fields by appealing to the language of ergodic, measure-preserving dynamical systems (which is a standard in the theory of stochastic homogenization; see, e.g., the seminal paper [30]): Let \((\Omega, \mathcal{F}, \mathbb{P})\) denote a probability space and \(\tau = (\tau_x)_{x \in \mathbb{R}^d}\) a family of measurable mappings \(\tau_x : \Omega \to \Omega\) satisfying

- (group property) \(\tau_0 \omega = \omega\) for all \(\omega \in \Omega\) and \(\tau_{x+y} = \tau_x \tau_y\) for all \(x, y \in \mathbb{R}^d\).
- (stationarity) For every \(x \in \mathbb{R}^d\) and \(B \in \mathcal{F}\) it holds that \(\mathbb{P}(\tau_x B) = \mathbb{P}(B)\).
- (ergodicity) All \(B \in \mathcal{F}\) with \(\tau_x B = B\) for all \(x \in \mathbb{R}^d\) satisfy \(\mathbb{P}(B) \in \{0, 1\}\).

For a random field \(a : \Omega \to \mathbb{R}^{d \times d}\) and \(\omega \in \Omega\), we denote by \(a_\omega : \mathbb{R}^d \to \mathbb{R}^{d \times d}\) its stationary extension given by \(a_\omega(x) := a(\tau_x \omega)\).

**Assumption 5.1.** There exists exponents \(p, q \in [1, \infty]\) satisfying \(\frac{1}{p} + \frac{1}{q} < \frac{2}{d-1}\) if \(d \geq 3\) such that the following is true: The random variables \(\lambda, \mu\) given by

\[
(5.2) \quad \lambda(\omega) := \inf_{\xi \in \mathbb{R}^d} \frac{\xi \cdot a(\omega) \xi}{|\xi|^2}, \quad \mu(\omega) := \sup_{\xi \in \mathbb{R}^d} \frac{|a(\omega) \xi|^2}{\xi \cdot a(\omega) \xi},
\]

are nonnegative and satisfy the moment condition

\[
(5.3) \quad \mathbb{E}[\lambda^{-q}] < \infty, \quad \mathbb{E}[\mu^p] < \infty,
\]

where \(\mathbb{E}\) denotes the expected value.

Assumption [5.1] ensures the existence of a well-defined corrector, which is the subject of the following:

**Lemma 5.2.** Suppose that Assumption [5.1] is satisfied. Then there exists \(\Omega_1 \subset \Omega\) with \(\mathbb{P}(\Omega_1) = 1\) such that the following is true: For every \(\omega \in \Omega_1\) and all \(i = 1, \ldots, d\) there exists a weak solution \(\phi_i \in H^1_{\text{loc}}(\mathbb{R}^d, a^\omega)\) of

\[
(5.4) \quad \nabla \cdot a^\omega(e_i + \nabla \phi_i) = 0 \quad \text{in} \quad \mathbb{R}^d,
\]

with \(a^\omega(x) = a(\tau_x \omega)\), which is sublinear in \(L^1\) in the sense

\[
(5.5) \quad \limsup_{R \to \infty} R^{-1} \left( \int_{B_R} |\phi_i| \right) = 0,
\]

and for every \(z \in \mathbb{Q}^d\) it holds that

\[
(5.6) \quad \limsup_{R \to \infty} \left( \int_{B_R(Rz)} a(\nabla \phi_i + e_i) \cdot (\nabla \phi_i + e_i) \right) \leq \mathbb{E}[\mu].
\]

We omit the proof of Lemma 5.2 since it is by now standard. In fact, if \(a\) is supposed to be symmetric, a stronger statement can be found in [12, sec. 4].
Appealing to an additional truncation argument as e.g., in [5,18] similar arguments as in [12, sec. 4] can also be used to cover the nonsymmetric case.

Now we state the main result of this section, namely the almost sure $L^\infty$-sublinearity of the corrector.

**Proposition 5.3.** Suppose that Assumption 5.1 is satisfied. Then there exists $\Omega_2 \subset \Omega_1$ with $P(\Omega_2) = 1$ such that the following is true: For every $\omega \in \Omega_2$ and all $i = 1, \ldots, d$, the functions $\phi_i \in H^1_{\text{loc}}(\mathbb{R}^d, \mathcal{E}_\omega)$ satisfying (5.4) and (5.5) are sublinear in $L^\infty$ in the sense

\begin{equation}
\limsup_{R \to \infty} R^{-1} \|\phi_i\|_{L^\infty(B_R)} = 0.
\end{equation}

**Remark 5.4.** In [12], the sublinearity of the corrector in the form (5.7) is shown under moment conditions (5.3) with the more restrictive relation $\frac{1}{p} + \frac{1}{q} < \frac{d}{2}$ (see also [1] for a similar result in the discrete setting and [18] for a corresponding statement with strictly elliptic, unbounded coefficients). In two dimensions, Proposition 5.3 might be not surprising since an analogous statement in a discrete setting was already proven by Biskup in [8]. For completeness and since the argument in [8] uses the discrete structure, we include this case here. The counterexample to local boundedness in [20] suggests that Assumption 5.1 should be almost optimal for the conclusion of Proposition 5.3. In fact, it was recently shown by Biskup and Kumagai [9] in a discrete setting that the corresponding statement of Proposition 5.3 fails if (5.3) only holds for $p, q$ satisfying $\frac{1}{p} + \frac{1}{q} > \frac{2}{d-1}$ provided $d \geq 4$.

**Proof of Proposition 5.3.** Throughout the proof we write $\preceq$ if $\leq$ holds up to a positive constant that depends only on $d$, $p$, and $q$. Before we give the details of the proof, we briefly explain the idea. There are two obstructions to deducing the statement directly from Theorem 1.1: First, we are not able to prove local boundedness of the corrector by considering (5.4) as an equation for $\phi_i$ with the right-hand side $\nabla \alpha e_i$ as it is, e.g., done in [12]. Second, and more severe, the right-hand side $\nabla \cdot a e_i$ is not small in general. We overcome these issues by appealing to a two-scale argument: We introduce an additional length scale $\rho R$ with $0 < \rho \ll 1$ and compare $\phi_i$ on balls with radius $\sim \rho R$ with $a$-harmonic functions $\phi_i + e_i \cdot x - c$ with a suitable chosen $c \in \mathbb{R}$. Using the $L^1$-sublinearity of $\phi_i$ and the fact that the linear part coming from $e_i \cdot x$ can be controlled by $\rho > 0$ on each ball of radius $\sim \rho R$, we obtain the desired claim.

**Step 1.** As a preliminary step, we recall the needed input from ergodic theory. In view of the spatial ergodic theorem, we obtain from the moment condition (5.3) that there exists $\Omega' \subset \Omega$ with $P(\Omega') = 1$ such that for $\omega \in \Omega'$ it holds...
\[ \lambda(\tau(\omega))^{-1} \in L^q_{\text{loc}}(\mathbb{R}^d), \mu(\tau(\omega)) \in L^p_{\text{loc}}(\mathbb{R}^d), \text{and for every } z \in \mathbb{Q}^d, \]
\[
\lim_{R \to \infty} \int_{B_R(0)} \lambda(\tau(\omega))^{-q} \, dx = \mathbb{E}[\lambda^{-q}] \\
\lim_{R \to \infty} \int_{B_R(0)} \mu(\tau(\omega))^p \, dx = \mathbb{E}[\mu^p];
\]
(5.8)

see, e.g., [23, theorem 11.18].

Step 2. Conclusion for \( p, q > 1 \). We set \( \Omega_2 := \Omega_1 \cap \Omega' \), where \( \Omega' \) is given as in Step 1. Clearly \( \Omega_2 \) has full measure. From now on we fix \( \omega \in \Omega_2 \).

Fix \( \rho \in (0, \frac{\delta}{2}) \) and cover \( B_R \) with finitely many balls \( B_{\rho R}(z) \), \( z \in \rho \mathbb{Z}^d \cap B_1 =: \mathcal{Z}_\rho \). For \( z \in \mathcal{Z}_\rho \), set \( u^i_\omega(z) := \phi_i(x) + e_i \cdot (x - z) \). Obviously, (5.4) implies that \( u^i_\omega \) is \( a^\omega \)-harmonic. Hence, (1.4) (with \( \gamma = 1 \)) yields

\[
\| u^i_\omega \|_{L^\infty(B_{2\rho R}(z))} \leq \Lambda(B_{2\rho R}(z)) \sup_{x \in B_{\rho R}(z)} |u^i_\omega(x)| \, dx + \rho R
\]

(5.9)

where \( p = \frac{p}{p-1}, \delta = \min\left\{ \frac{1}{2p} - \frac{1}{2q}, \frac{1}{2q} \right\} > 0 \). Estimate (5.9) implies the following \( L^\infty \) estimate on \( \phi_i \):

\[
\| \phi_i \|_{L^\infty(B_R)} \leq \sup_{z \in \mathcal{Z}_\rho} \| \phi_i \|_{L^\infty(B_{\rho R}(z))} \leq \sup_{z \in \mathcal{Z}_\rho} \| u_i \|_{L^\infty(B_{2\rho R}(z))} + d \rho R
\]

\[
\leq \sup_{z \in \mathcal{Z}_\rho} \Lambda(B_{2\rho R}(z)) \left( \sup_{z \in \mathcal{Z}_\rho} \| u_i \|_{L^\infty(B_{2\rho R}(z))} \right) + \rho R
\]

(5.10)

Clearly for every \( z \in \mathcal{Z}_\rho \), it holds that \( \Lambda(B_{2\rho R}(z)) = \Lambda(B_R(R'z')) \) with \( R' = 2d \rho R \) and \( z' = (2d \rho)^{-1}z \in (2d)^{-1} \mathbb{Z}^d \cap B_{(2d \rho)^{-1}} =: \mathcal{Z}'_\rho \). Using (5.8) and the fact that \( \mathcal{Z}_\rho \) (and thus \( \mathcal{Z}'_\rho \)) is a finite set, we obtain

\[
\lim_{R \to \infty} \sup_{z \in \mathcal{Z}_\rho} \Lambda(B_{2\rho R}(z)) = \lim_{R' \to \infty} \sup_{z' \in \mathcal{Z}'_\rho} \left( \int_{B_{R}(z')} \lambda(\tau(\omega))^{-q} \, dx \right)^{\frac{1}{q}} \left( \int_{B_{R}(z')} \mu(\tau(\omega))^p \, dx \right)^{\frac{1}{p}} \]

(5.11)

\[
\leq \mathbb{E}[\lambda^{-q}]^{\frac{1}{q}} \mathbb{E}[\mu^p]^{\frac{1}{p}}.
\]

Finally, we combine (5.10) and (5.11) with the \( L^1 \)-sublinearity of \( \phi_i \), i.e., (5.5), to obtain

\[
\lim_{R \to \infty} R^{-1} \| \phi_i \|_{L^\infty(B_R)} \leq \rho \left( \mathbb{E}[\lambda^{-q}]^{\frac{1}{q}} \mathbb{E}[\mu^p]^{\frac{1}{p}} \right)^{p' \left( 1 + \frac{1}{p} \right)} + \rho.
\]

The arbitrariness of \( \rho > 0 \) implies (5.7) and finishes the proof.
Step 3. The remaining case: \( d = 2 \) and \( p = q = 1 \). Let \( \Omega_2 \) be as in Step 2. From now on we fix \( \omega \in \Omega_2 \) and use the same notation as in Step 2.

Using estimate (3.5) instead of (1.4), we obtain

\[
\| u_i \|_{L^\infty(B_{2\rho R}(R^d))} \lesssim \int_{B_{4\rho R}(R^d)} |\phi_i(x)| \, dx + \rho R
\]

\[
+ \rho R \left( \int_{B_{4\rho R}(R^d)} \lambda^{-1} \left( \int_{B_{4\rho R}(R^d)} a(\nabla \phi_i + e_i) \cdot (\nabla \phi_i + e_i) \, dx \right)^{\frac{1}{2}} \right),
\]

instead of (5.9). Estimate (5.12) implies the following \( L^\infty \) estimate on \( \phi_i \):

\[
\| \phi_i \|_{L^\infty(B_R)} \lesssim \sup_{z \in \mathbb{Z}_R} \| \phi_i \|_{L^\infty(B_{2\rho R}(R^d))}
\]

\[
\lesssim \sup_{z \in \mathbb{Z}_R} \| u_i \|_{L^\infty(B_{2\rho R}(R^d))} + 2\rho R
\]

\[
\lesssim \sup_{z \in \mathbb{Z}_R} \left( \int_{B_{4\rho R}(R^d)} |\phi_i| + \rho R
\right.
\]

\[
+ \rho R \left( \int_{B_{4\rho R}(R^d)} \lambda^{-1} \left( \int_{B_{4\rho R}(R^d)} a(\nabla \phi_i + e_i) \cdot (\nabla \phi_i + e_i) \, dx \right)^{\frac{1}{2}} \right)\]

\[
\lesssim \rho^{-d} \int_{B_{2\rho R}} |\phi_i| + \rho R
\]

\[
+ \rho R \sup_{z \in \mathbb{Z}_R} \left( \int_{B_{4\rho R}(R^d)} \lambda^{-1} \left( \int_{B_{4\rho R}(R^d)} a(\nabla \phi_i + e_i) \cdot (\nabla \phi_i + e_i) \, dx \right)^{\frac{1}{2}} \right).\]

Using (5.6) and (5.8), we obtain similar to (5.11)

\[
\limsup_{R \to \infty} \sup_{z \in \mathbb{Z}_R} \left( \int_{B_{4\rho R}(R^d)} \lambda^{-1} \left( \int_{B_{4\rho R}(R^d)} a(\nabla \phi_i + e_i) \cdot (\nabla \phi_i + e_i) \, dx \right)^{\frac{1}{2}} \right)
\]

\[
\leq E[\lambda^{-1}] \frac{1}{2} \mathbb{E}[\mu]^{\frac{1}{2}}.
\]

Finally, we combine (5.13) and (5.14) with the \( L^1 \)-sublinearity of \( \phi_i \), i.e., (5.5), to obtain

\[
\limsup_{R \to \infty} R^{-1} \| \phi_i \|_{L^\infty(B_R)} \lesssim \rho \mathbb{E}[\lambda^{-1}] \frac{1}{2} \mathbb{E}[\mu]^{\frac{1}{2}} + \rho.
\]

The arbitrariness of \( \rho > 0 \) implies (5.7) and finishes the proof.

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