Homological invariants of the Stanley-Reisner ring of a $k$-decomposable simplicial complex

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**Definition [Woodroofe (2009)].** Let $\Delta$ be a simplicial complex on the vertex set $V$. A face $\sigma$ is called a shedding face if every face $\tau$ containing $\sigma$ satisfies the following exchange property: for every $\nu \in \sigma$ there is $w \in V \setminus \tau$ such that $(\tau \cup \{w\}) \setminus \{\nu\}$ is a face of $\Delta$.

**Definition [Woodroofe].** A simplicial complex $\Delta$ is recursively defined to be $k$-decomposable if either $\Delta$ is a simplex or else has a shedding face $\sigma$ with $\dim(\sigma) \leq k$ such that both $\Delta \setminus \sigma$ and $\text{lk}(\sigma)$ are $k$-decomposable.

The complexes $\{\}$ and $\{\emptyset\}$ are considered to be $k$-decomposable for all $k \geq -1$.

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$0$-decomposable simplicial complexes are precisely vertex decomposable simplicial complexes.
Let $K$ be a field, $R = K[V(\Delta)]$ and $I_\Delta$ be the Stanley-Reisner ideal of $\Delta$.

**Theorem [Khosh-Ahang, Moradi], [Ha, Woodroofe]** Let $\Delta$ be a vertex decomposable simplicial complex, $x$ a shedding vertex of $\Delta$ and $\Delta_1 = \text{del}_\Delta(x)$ and $\Delta_2 = \text{lk}_\Delta(x)$. Then

$$\text{pd}(R/I_\Delta) = \max\{\text{pd}(R/I_{\Delta_1}) + 1, \text{pd}(R/I_{\Delta_2})\},$$

$$\text{reg}(R/I_\Delta) = \max\{\text{reg}(R/I_{\Delta_1}), \text{reg}(R/I_{\Delta_2}) + 1\}.$$
Theorem [Ha]. Let $\Delta$ be a simplicial complex and let $\sigma$ be a face of dimension $d - 1$ in $\Delta$. Then

$$\text{reg}(R/I_\Delta) \leq \max\{\text{reg}(R/I_{\Delta \setminus \sigma}), \text{reg}(R/I_{\text{lk}(\sigma)}) + d\}.$$
For the monomial $u = x_1^{a_1} \cdots x_n^{a_n}$ in $R$, the support of $u$ denoted by $\text{supp}(u)$ is the set $\{x_i : a_i \neq 0\}$.

For the monomial $u$ and the monomial ideal $I$, set

$$I_u = \{ M \in \mathcal{G}(I) : x_i^{a_i} \nmid M \; \forall x_i \in \text{supp}(u) \}$$

and

$$I^u = \{ M \in \mathcal{G}(I) : M \not\in \mathcal{G}(I_u) \}$$
For a monomial ideal $I$ with $\mathcal{G}(I) = \{M_1, \ldots, M_r\}$, the monomial $u = x_1^{a_1} \cdots x_n^{a_n}$ is called a shedding monomial for $I$ if $I_u \neq 0$ and for each $M_i \in \mathcal{G}(I_u)$ and each $x_\ell \in \text{supp}(u)$ there exists $M_j \in \mathcal{G}(I^u)$ such that $M_j : M_i = x_\ell$.

Definition[Rahmati-Asghar, Yassemi]. A monomial ideal $I$ with $\mathcal{G}(I) = \{M_1, \ldots, M_r\}$ is called $k$-decomposable if $r = 1$ or else has a shedding monomial $u$ with $|\text{supp}(u)| \leq k + 1$ such that the ideals $I_u$ and $I^u$ are $k$-decomposable.
For a monomial ideal $I$ with $\mathcal{G}(I) = \{M_1, \ldots, M_r\}$, the monomial $u = x_1^{a_1} \cdots x_n^{a_n}$ is called a shedding monomial for $I$ if $I_u \neq 0$ and for each $M_i \in \mathcal{G}(I_u)$ and each $x_\ell \in \text{supp}(u)$ there exists $M_j \in \mathcal{G}(I^u)$ such that $M_j : M_i = x_\ell$.

**Definition**[Rahmati-Asghar, Yassemi]. A monomial ideal $I$ with $\mathcal{G}(I) = \{M_1, \ldots, M_r\}$ is called $k$-decomposable if $r = 1$ or else has a shedding monomial $u$ with $|\text{supp}(u)| \leq k + 1$ such that the ideals $I_u$ and $I^u$ are $k$-decomposable.
Theorem[Rahmati-Asghar, Yassemi]. A $d$-dimensional simplicial complex $\Delta$ is $k$-decomposable if and only if $I_{\Delta^\vee}$ is a squarefree $k$-decomposable ideal for any $k \leq d$. 
Theorem. Let $I$ be a $k$-decomposable ideal with the shedding monomial $u$. Then

$$\beta_{i,j}(I) = \beta_{i,j}(I^u) + \sum_{\ell=0}^{m} \binom{m}{\ell} \beta_{i-\ell,j-\ell}(I_u)$$

where $m = |\text{supp}(u)|$.

Outline of proof. If $f_1 < \cdots < f_t$ is an order of linear quotients on the minimal generators of $I^u$ and $g_{t+1} < \cdots < g_r$ is an order of linear quotients on the minimal generators of $I_u$, then

$$f_1 < \cdots < f_t < g_{t+1} < \cdots < g_r$$

is an order of linear quotients on the minimal generators of $I$.

$$\text{set}_I(f_i) = \text{set}_{I^u}(f_i), \quad \forall \ 1 \leq i \leq t$$

and

$$\text{set}_I(g_i) = \text{supp}(u) \cup \text{set}_{I^u}(g_i) \quad \forall \ t+1 \leq i \leq r$$
Also for any $t + 1 \leq i \leq r$, $\text{supp}(u) \cap \text{set}_{I_u}(g_i) = \emptyset$. Thus

$$|\text{set}_{I}(g_i)| = |\text{set}_{I_u}(g_i)| + m$$

$$\beta_{i,j}(I) = \sum_{\deg(f_k) = j - i} \left( |\text{set}_{I}(f_k)| \right) + \sum_{\deg(g_k) = j - i} \left( |\text{set}_{I}(g_k)| \right)$$

Applying the equality

$$\left( |\text{set}_{I_u}(g_k)| + m \right) = \sum_{\ell=0}^{m} \binom{m}{\ell} \left( |\text{set}_{I_u}(g_k)| \right)$$

we have

$$\beta_{i,j}(I) = \beta_{i,j}(I^u) + \sum_{\ell=0}^{m} \binom{m}{\ell} \beta_{i-\ell,j-\ell}(I_u)$$
Corollary. Let $I$ be a $k$-decomposable ideal with the shedding monomial $u$ and $m = |\text{supp}(u)|$. Then

- $\text{pd}(I) = \max\{\text{pd}(I^u), \text{pd}(I_u) + m\}$, and
- $\text{reg}(I) = \max\{\text{reg}(I^u), \text{reg}(I_u)\}$. 
**Theorem.** Let $\Delta$ be a $k$-decomposable simplicial complex on the vertex set $X$ with the shedding face $\sigma$. Then

- $\text{reg}(R/I_{\Delta}) = \max\{\text{reg}(R/I_{\Delta \setminus \sigma}), \text{reg}(R/I_{\text{lk}(\sigma)}) + |\sigma|\}$,
- $\text{pd}(R/I_{\Delta}) = \max\{\text{pd}(R/I_{\Delta \setminus \sigma}), \text{pd}(R/I_{\text{lk}(\sigma)})\}$,

where $I_{\Delta \setminus \sigma}$ and $I_{\text{lk}(\sigma)}$ are Stanley-Reisner ideals of $\Delta \setminus \sigma$ and $\text{lk}(\sigma)$ on the vertex sets $X$ and $X \setminus \sigma$, respectively.
Corollary. Let $\Delta$ be a shellable simplicial complex with the shelling order $F_1 < \cdots < F_k$ and $\dim(\Delta) = d$. For any $1 \leq i \leq k$, let $\Delta_i = \langle F_1, \ldots, F_i \rangle$ and $\mathcal{R}(F_i) = \{x \in F_i : F_i \setminus \{x\} \in \Delta_{i-1}\}$. Then

$$\text{reg}(R/I_\Delta) = \max\{|\mathcal{R}(F_1)|, \ldots, |\mathcal{R}(F_k)|\}.$$
**Definition.** Let $\mathcal{H}$ be a clutter. A vertex $v$ of $\mathcal{H}$ is **simplicial** if for every two edges $e_1$ and $e_2$ of $\mathcal{H}$ that contain $v$, there is a third edge $e_3$ such that $e_3 \subseteq (e_1 \cup e_2) \setminus \{v\}$.

**Definition.** A clutter $\mathcal{H}$ is **chordal** if every minor of $\mathcal{H}$ has a simplicial vertex.
**Corollary.** Let $\mathcal{H}$ be a chordal clutter, $x \in V(\mathcal{H})$ be a simplicial vertex for $\mathcal{H}$ and $e = \{x, x_1, \ldots, x_d\}$ be an edge of $\mathcal{H}$ containing $x$. Then

\[
\text{reg}(R/I(\mathcal{H})) = \max\{\text{reg}(R/I(\mathcal{H}')), \text{reg}(R/I(\mathcal{H}/\{x_1, \ldots, x_d\})) + d\}
\]

where

\[
E(\mathcal{H}') = \{e \in E(\mathcal{H}) : \{x_1, \ldots, x_d\} \not\subseteq e\} \cup \\{\{x_1, \ldots, x_d\}\}.
\]

\[
\text{reg}(R/I(\mathcal{H})) \leq \max\left\{\sum_{i=1}^{d} \text{reg}(R/I(\mathcal{H}\setminus x_i)) + (d-1), \text{reg}(R/I(\mathcal{H}/\{x_1, \ldots, x_d\})) + d\right\}
\]
For a graph $G$, let $J_m(G)$ be the ideal generated by all squarefree monomials $u$ of degree $m$, such that $\text{supp}(u)$ is an independent set of $G$.

**Theorem.** Let $G$ be a chordal graph and $x$ be a simplicial vertex of $G$. Set $I = J_m(G)$, $J = J_m(G \setminus x)$ and $K = J_{m-1}(G \setminus N_G[x])$. Then $I = J + xK$ is a 0-decomposable ideal. Moreover, if $I \neq 0$, then

(i) $\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i-1,j-1}(J) + \beta_{i,j-1}(K)$

(ii) If $J \neq 0$, then $\text{pd}(I) = \max\{\text{pd}(J) + 1, \text{pd}(K)\}$

(iii) $I$ has a $m$-linear resolution.
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(iii) $I$ has a $m$-linear resolution.
Definition. [Khosh-Ahang, Moradi] Let $\Delta = \langle F_1, \ldots, F_m \rangle$ be a simplicial complex with the vertex set $V(\Delta) = \{x_1, \ldots, x_n\}$ and $s_1, \ldots, s_n \in \mathbb{N}$ be arbitrary integers. We define the $(s_1, \ldots, s_n)$-expansion of $\Delta$ to be a simplicial complex with the vertex set $\{x_{11}, \ldots, x_{1s_1}, x_{21}, \ldots, x_{2s_2}, \ldots, x_{n1}, \ldots, x_{ns_n}\}$ and the facets

$$\{\{x_{i_1r_1}, \ldots, x_{i_kr_k}\} : \{x_{i_1}, \ldots, x_{i_k}\} \in F(\Delta), (r_1, \ldots, r_k) \in [s_{i_1}] \times \cdots \times [s_{i_k}]\}$$

We denote this simplicial complex by $\Delta^{(s_1, \ldots, s_n)}$

Theorem. [Moradi, Rahmati-Asghar] Let $\Delta$ be a simplicial complex on $\{x_1, \ldots, x_n\}$ and $\alpha \in \mathbb{N}^n$.

$$\Delta \text{ is } k\text{-decomposable} \iff \Delta^\alpha \text{ is } k\text{-decomposable}$$
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$$\{\{x_{i1}r_1, \ldots, x_{ik_i}r_{k_i}\} : \{x_{i1}, \ldots, x_{ik_i}\} \in \mathcal{F}(\Delta), (r_1, \ldots, r_{k_i}) \in [s_{i1}] \times \cdots \times [s_{ik_i}]\}$$

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Theorem. [Moradi, Rahmati-Asghar] Let $\Delta$ be a simplicial complex on $\{x_1, \ldots, x_n\}$ and $\alpha \in \mathbb{N}^n$.

$$\Delta \text{ is } k\text{-decomposable} \iff \Delta^\alpha \text{ is } k\text{-decomposable}$$
Outline of proof: Let $\Delta$ be $k$-decomposable, $X$ be the vertex set of $\Delta$, $x_i \in X$ and

$$\Delta' = \Delta \cup \langle (F \setminus \{x_i\}) \cup \{x'_i\} : F \in \mathcal{F}(\Delta), x_i \in F \rangle$$

Also $(l_{\Delta'})_{x'_i} = x_i l_{(\text{lk}_{\Delta}(x_i))'\vee}$ and $(l_{\Delta'})_{x'_i} = x_i' l_{\Delta'\vee}$. 

$l_{\Delta'}$ is $k$-decomposable

$\downarrow$

$x_i' l_{\Delta'\vee}$ and $x_i l_{(\text{lk}_{\Delta}(x_i))'\vee}$ are $k$-decomposable ideals

Also for any minimal generator $x_i x^{X \setminus F} \in (l_{\Delta'})_{x'_i}$,

$$(x'_i x^{X \setminus F} : x_i x^{X \setminus F}) = (x'_i)$$
Let $\Delta' = \Delta \cup \langle (F \setminus \{x_i\}) \cup \{x'_i\} : F \in \mathcal{F}(\Delta), x_i \in F \rangle$ be $k$-decomposable.

If $\mathcal{F}(\Delta) = \{F\}$, then clearly it is $k$-decomposable.

Suppose that $\Delta$ has more than one facet and $\sigma$ be a shedding face of $\Delta'$ and let $\text{lk}_{\Delta'} \sigma$ and $\Delta' \setminus \sigma$ are $k$-decomposable. We have two cases:

Case 1. $x'_i \in \sigma$ or $x_i \in \sigma$. Then

$$\Delta = \Delta' \setminus \sigma$$

and so $\Delta$ is $k$-decomposable.

Case 2. $x_i \not\in \sigma$ and $x'_i \not\in \sigma$.

$\text{lk}_{\Delta'} \sigma$ and $\Delta' \setminus \sigma$ are, respectively, some expansions of $\text{lk}_{\Delta} \sigma$ and $\Delta \setminus \sigma$.

So by induction $\text{lk}_{\Delta} \sigma$ and $\Delta \setminus \sigma$ are $k$-decomposable.
Theorem. [Moradi, Rahmati-Asghar] Let $\Delta$ be a $k$-decomposable simplicial complex on $[n]$ and $\alpha = (s_1, \ldots, s_n)$. Then

- $\text{pd}(S^\alpha/I_{\Delta^\alpha}) = \text{pd}(S/I_{\Delta}) + s_1 + \cdots + s_n - n$
- $\text{depth}(S^\alpha/I_{\Delta^\alpha}) = \text{depth}(S/I_{\Delta})$

Theorem. [Moradi, Rahmati-Asghar] Let $\Delta$ be a simplicial complex on $[n]$ and $\alpha = (s_1, \ldots, s_n)$. Then

$$\text{reg}(I_{\Delta^\alpha}) \leq \text{reg}(I_{\Delta}) + r$$

where $r = |\{ i : s_i > 1 \}|$. 
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$$\text{reg}(I_{\Delta^\alpha}) \leq \text{reg}(I_\Delta) + r$$

where $r = |\{ i : s_i > 1 \}|$.

Outline of proof. First by induction on $s_i$ we show that

$$\text{reg}(I_{\Delta(1, \ldots, 1, s_i, 1, \ldots, 1)}) \leq \text{reg}(I_\Delta) + 1$$
Theorem. [Moradi,Rahmati-Asghar] Let $\Delta$ be a $k$-decomposable simplicial complex on $[n]$ and $\alpha = (s_1, \ldots, s_n)$. Then

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Outline of proof. First by induction on $s_i$ we show that

$$\text{reg}(I_{\Delta^{(1,\ldots,1,s_i,1,\ldots,1)}}) \leq \text{reg}(I_\Delta) + 1$$
\[
\text{reg}(I_{\Delta^{(1,\ldots,1,s_i,1,\ldots,1)}}) \leq \max\{\text{reg}(I_{\text{lk}_{\Delta}(x_i)}) + 1, \text{reg}(I_{\Delta^{(1,\ldots,1,s_i-1,1,\ldots,1)}})\}.
\]

Then from the equality

\[
\Delta^{(s_1,\ldots,s_n)} = (\Delta^{(s_1,\ldots,s_{n-1},1)})(1,\ldots,1,s_n)
\]

we have

\[
\text{reg}(I_{\Delta^{(s_1,\ldots,s_n)}}) \leq \text{reg}(I_{\Delta^{(s_1,\ldots,s_{n-1},1)}}) + 1
\]

and one can get the result by induction on \( n \).
Thanks!