Field theory of low energy excitations of a mixture of two species of pseudospin-$\frac{1}{2}$ Bose gases with interspecies spin exchange

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Abstract. We develop a low energy effective field theory of a mixture of two species of pseudospin-$\frac{1}{2}$ atoms with interspecies spin exchange, in addition to density–density interaction. This approach is beyond the single-orbital-mode approximation. In a wide parameter regime, it indicates the existence of the four elementary excitations, especially the gapped mode due to interspecies spin exchange. The spectrum of the effective spin Hamiltonian yielded by the single-mode approximation can be obtained by quantizing the homogeneous excitation, which is spin excitation and is the long-wavelength limit of the gapped mode among the elementary excitations. These low energy excitations may be experimentally detected by using Bragg spectroscopy.

Keywords: Bose–Einstein condensation (theory), quantum gases, superfluidity
1. Introduction

Elementary excitations or collective modes are key properties of a Bose–Einstein condensate (BEC), and serve as probes of the ground states. The Bogoliubov theory of elementary excitations of BEC gives an elegant description of the Goldstone modes associated with the spontaneous breaking of $U(1)$ symmetry [1]–[3]. In recent years, BEC of ultra-cold dilute atomic gases has become one of the most active fields in physics. Among the most interesting topics are BEC of spinor atomic gases [3]–[7], for example, spin-1 and pseudospin-$\frac{1}{2}$ gases [8]–[12], as well as spinless mixtures [13]–[15]. It is interesting to study, as an extension of this topic, spinor mixtures with interspecies spin exchange. It has been theoretically found that a mixture of two distinct species of pseudospin-$\frac{1}{2}$ atoms with interspecies spin-exchange interaction exhibits interesting features beyond both those of spinor gases and those of a mixture of spinless gases; in particular, in a broad parameter regime, the ground state is entangled between the two species, rather than a mixture of BECs of individual species [16]–[21].

We expect our work to motivate more investigations along this line of research. Spin-exchange scattering between distinguishable atoms has been less studied, perhaps because of incomplete information on the inter-atomic potential. However, we note that interspecies spin-exchange interaction can be significant. There are calculations indicating significant scattering length for the spin-exchange scattering between distinguishable atoms [22]. Spin-changing scattering between distinguishable atoms has indeed been observed [23]. Experiments on multi-component Bose gases often had atom loss due to spin exchanges [9,14]. Significant differences between singlet and triplet scattering lengths have been observed in $^{41}$K–$^{87}$Rb, $^{40}$K–$^{87}$Rb and $^{6}$Li($^{7}$Li)–$^{23}$Na mixtures [15,24], implying significant interspecies spin exchanges. It is thus feasible to attempt experimental realization of the system studied here. One may use, for example, $^{85}$Rb and $^{87}$Rb, or $^{41}$K and $^{87}$Rb, as the two species, and $|F = 1, m_F = 2\rangle$ and $|F = 1, m_F = 1\rangle$ as the two pseudospin states [18].
In this paper, we treat a mixture of two distinct species of pseudospin-$\frac{1}{2}$ Bose gases by using a field theory approach beyond the single-orbital-mode approximation. We shall use the path integral formalism to develop a Bogoliubov-like mean field theory, in which each field has a specific value in the ground state. Excitations are then calculated as small deviations of the fields from those in the ground state.

Previously, elementary excitations in such a mixture have been studied as fluctuations of the single-particle orbital wavefunctions, and this has been restricted to a special parameter point, at which the many-body ground state is the so-called entangled BEC [19]. On the other hand, when the atoms are all condensed in the same orbital wavefunctions, there are spin excitations described by the effective spin Hamiltonian [17]. In the present approach, the spin excitations are obtained as due to spin flipping of the fields that remain spatially homogeneous, while the elementary excitations are plane-wave-like excitations of the phases of the fields. The former is the long-wavelength limit of the gapped mode among the elementary excitations. Furthermore, the low energy effective theory gives an excitation spectrum of the effective spin Hamiltonian that is the same as that obtained under the single-orbital-mode approach.

2. The model

Consider a dilute gas of two species of bosonic atoms, with the number of atoms in each species conserved separately. Each atom possesses an internal degree of freedom represented as a pseudospin with $z$-component basis states $|\uparrow\rangle$ and $|\downarrow\rangle$, and can transit between the two. This system is described in terms of four interacting boson fields, with the Lagrangian density

$$L = \sum_{\alpha\sigma} i \bar{\psi}_{\alpha\sigma} \partial_t \psi_{\alpha\sigma} - \left( \mathcal{H} - \sum_{\alpha\sigma} \mu_{\alpha\sigma} \bar{\psi}_{\alpha\sigma} \psi_{\alpha\sigma} \right)$$

where $\alpha = a, b$ represents the two species and $\sigma = \uparrow, \downarrow$ represents the two basis states of pseudospin-$\frac{1}{2}$, $\psi_{\alpha\sigma} = \psi_{\alpha\sigma}(x)$ and $\mu_{\alpha\sigma}$ are the field and the chemical potential corresponding to the atoms of species $\alpha$ with pseudospin $\sigma$, respectively, $\mathcal{H}$ is the Hamiltonian density

$$\mathcal{H} = \sum_{\alpha\sigma} \psi_{\alpha\sigma}^\dagger \left( -\frac{1}{2m_\alpha} \nabla^2 + V \right) \psi_{\alpha\sigma} + \frac{1}{2} \sum_{\alpha\sigma\sigma'} g_{\sigma\sigma'}^{(\alpha\alpha)} |\psi_{\alpha\sigma}|^2 |\psi_{\alpha\sigma'}|^2$$

$$+ \sum_{\sigma\sigma'} g_{\sigma\sigma'}^{(ab)} |\psi_{\alpha\sigma}|^2 |\psi_{\alpha\sigma'}|^2 + g_e (\bar{\psi}_{a\uparrow} \psi_{a\downarrow} \bar{\psi}_{b\downarrow} \psi_{b\uparrow} + \bar{\psi}_{a\downarrow} \psi_{a\uparrow} \bar{\psi}_{b\uparrow} \psi_{b\downarrow}),$$

where $V = V(x)$ is the external potential, $g_{\sigma\sigma'}^{(\alpha\alpha)}$, $g_{\sigma\sigma'}^{(ab)}$ and $g_e$ are the interaction strengths for intraspecies scattering, interspecies scattering without spin exchange, and interspecies spin-exchange scattering respectively, proportional to the corresponding scattering length. For pseudospin-$\frac{1}{2}$ atoms, the intraspecies scattering strengths with and without spin exchange are the same [6].

If $g_e = 0$, the system is a mixture without interspecies spin exchange, equivalent to a mixture of four scalar Bose gases. Note that intraspecies spin exchange does not change the number of particles occupying each pseudospin state. The Hamiltonian would possess a symmetry of $U(1) \times U(1) \times U(1) \times U(1)$, corresponding to particle number conservation of all the four fields. With $g_e > 0$, however, the symmetry is lowered to $U(1) \times U(1) \times U(1)$,
corresponding to the conservations of $N_a$, $N_b$ as well as $N_{a\uparrow} - N_{a\downarrow} + N_{b\uparrow} - N_{b\downarrow} = 2S_z$ [19]. The essential feature of the system being determined by this symmetry, we can simplify the parameters without loss of essence, as the following. We set $V = 0$, and also assume $g_{aa} = g_\sigma$, $g_{a\downarrow} = g_\uparrow = g$, and $g_{ab} = g_{a\downarrow b\uparrow} = g_d$, such that $\mu_a = \mu_{a\downarrow} = \mu_a$ [18].

This simplification is also practically reasonable, as the difference between the spin-dependent parts of a scattering length is much smaller than that between the spin-independent parts.

We can define $\Psi_{\alpha}(x) = (\psi_{\uparrow}\psi(x), \psi_\downarrow(x))^T$, and the spin density operator $S_{\alpha\beta}(x) = \Psi_{\alpha}^\dagger s^i\Psi_{\alpha}$, $(i = x, y, z)$, where $s^i = \tau_i/2$ is the single-spin operator, $\tau_i$ being the Pauli matrix. Then the Hamiltonian density can be written as

$$H = \sum_\alpha \Psi_{\alpha}^\dagger \left( -\frac{1}{2m_\alpha} \nabla^2 + V \right) \Psi_\alpha + \frac{g_\alpha}{2} |\Psi_\alpha|^4 + \frac{g_b}{2} |\Psi_b|^4 + \frac{g_d}{2} |\Psi_a|^2 |\Psi_b|^2$$

$$+ 2g_c(S_{xx} S_{xx} + S_{yy} S_{yy}) + 2g_c S_{zz} S_{zz}, \quad (3)$$

where $g_{ab} \equiv g_s + g_d$, $g_z \equiv g_s - g_d$.

### 3. The effective Lagrangian of low energy excitations

We focus on the parameter regime $g_c > g_z$. Other parameter regimes are considered elsewhere. Minimizing the energy, the mean field ground state is $\psi_{a\sigma} = \psi_{0\sigma} e^{i\Phi_{a\sigma}}$, where $\psi_{0\sigma}$ and $\Phi_{0\sigma}$ are constant; then the spin-exchange term becomes $2g_c \psi_{a\downarrow}^0 \psi_{a\uparrow}^0 \psi_{b\downarrow}^0 \psi_{b\uparrow}^0 \cos(\Phi_{a\downarrow}^0 - \Phi_{a\uparrow}^0 - \Phi_{b\downarrow}^0 + \Phi_{b\uparrow}^0)$. With the kinetic energy being zero because of time independence, minimizing the energy is equivalent to minimizing the potential part of the Lagrangian, which requires $\psi_{a\downarrow}^0 = \psi_{a\uparrow}^0 = \sqrt{\rho_a/2}$, $\psi_{b\downarrow}^0 = \psi_{b\uparrow}^0 = \sqrt{\rho_b/2}$ and $\Phi_{a\downarrow}^0 - \Phi_{a\uparrow}^0 - \Phi_{b\downarrow}^0 + \Phi_{b\uparrow}^0 = \pi$. We can arbitrarily choose the phases of the four fields under the above constraint to describe a ground state, as other choices are equivalent in the sense of spontaneous symmetry breaking. Therefore in the ground state, $\psi_{a\downarrow} = \psi_{a\uparrow} = \psi_{a} = \sqrt{\rho_a/2}$, $\psi_{b\downarrow} = -\psi_{b\uparrow} = \psi_{b} = \sqrt{\rho_b/2}$, where $\rho_a = N_a/\Omega$ is the number density of species $\alpha$, with $\Omega$ being the volume of the system. The chemical potential is evaluated to be $\mu_a = g_c \rho_a + \frac{i}{2} (g_b - g_c) \beta$, where $\beta \neq \alpha$.

We now study the elementary excitations. With a deviation from the mean field value, each field can be written as $\psi_{a\alpha}(r) = \psi_{a\alpha}(1 + \zeta_{\alpha\sigma}(r)) e^{i\Phi_{a\sigma}(r)}$, where $\zeta_{\alpha\sigma}(r)$ is a small quantity. Therefore

$$\mathcal{L} = \sum_{\alpha \sigma} \left[ -\frac{\rho_a}{2} \partial_t \Phi_{a\alpha} - \frac{\rho_a}{4m_a} (\nabla \Phi_{a\alpha})^2 - \rho_a \zeta_{a\sigma} \partial_t \Phi_{a\alpha} \right]$$

$$+ \frac{g_c}{2} \rho_a \rho_b \cos \left( \Phi_{a\downarrow} - \Phi_{a\uparrow} - \Phi_{b\downarrow} + \Phi_{b\uparrow} \right) - \frac{1}{2} \sum_{\alpha \beta, \sigma \sigma'} Q_{\alpha\sigma, \beta\sigma'} \zeta_{\alpha\sigma} \zeta_{\beta\sigma'}, \quad (4)$$

where higher order and constant terms have been neglected.

$$Q \equiv \begin{pmatrix} H_a & H_{ab} \\ H_{ab} & H_b \end{pmatrix}, \quad (5)$$

is a $4 \times 4$ matrix, with

$$H_a \equiv \rho_a \begin{pmatrix} g_a \rho_a + \frac{i}{2} g_c \rho_b & g_a \rho_a - \frac{i}{2} g_c \rho_b \\ g_a \rho_a - \frac{i}{2} g_c \rho_b & g_a \rho_a + \frac{i}{2} g_c \rho_b \end{pmatrix}, \quad (6)$$

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where $\alpha \neq \beta$, 
\[
H_{ab} \equiv \rho_\alpha \rho_b \left( \begin{array}{ccc}
g_s - \frac{1}{2}g_e & g_d - \frac{1}{2}g_e \\
g_d - \frac{1}{2}g_e & g_s - \frac{1}{2}g_e
\end{array} \right).
\]

Now consider the vacuum persistence amplitude $Z = \prod_{\alpha, \sigma} \int \mathcal{D}\Phi_{\alpha\sigma} \mathcal{D}\zeta_{\alpha\sigma} e^{i\int dt \int d^3x L}$, from which we obtain an effective Lagrangian as a function of $\Phi$ only, after dropping the total time derivative of $\Phi$, which does not affect the equation of motion, and integrating over $\zeta$:

\[
\mathcal{L}_{\text{eff}} = \frac{1}{2} \sum_{\alpha, \sigma, \sigma'} \rho_\alpha \rho_\beta (\partial_t \Phi_{\alpha\sigma})(Q)^{-1}(\alpha, \sigma, \beta, \sigma') - \sum_{\alpha} \frac{\rho_\alpha}{4m_\alpha} (\nabla \Phi_{\alpha\sigma})^2
- \frac{ge}{2} \rho_\alpha \rho_b \cos(\Phi_{a1} - \Phi_{a\downarrow} - \Phi_{b\downarrow} + \Phi_{b\uparrow})
\]

(8)

In deriving this formula, we have neglected $(\nabla \zeta)^2$ and $\zeta^3$, $\zeta^4$ terms since only low energy dynamics is concerned. This Lagrangian has a cosine term similar to that in the sine–Gordon model. In 1+1D, this term leads to a solution of a topological soliton, which makes a very nontrivial contribution to the phase diagram, as discussed elsewhere. However, for such a system, the lower critical dimension, below which the fluctuations play important roles, is 2; therefore in 3 + 1D the fluctuations are small [1]. Thus we focus on the low energy limit in 3+1D, in which the fluctuation of $\Phi_{a\downarrow} - \Phi_{a\uparrow} - \Phi_{b\downarrow} + \Phi_{b\uparrow}$ is largely suppressed and we can make the approximation $\cos x \approx 1 - x^2/2$.

The conjugate relation between the phase $\Phi_{\alpha\sigma}$ and particle number $N_{\alpha\sigma}$, the conservation of $\mathcal{N} = \sum_{\alpha} N_{\alpha\sigma}$ and the fact that the mass term is proportional to $(\Phi_{a\downarrow} - \Phi_{a\uparrow} - \Phi_{b\downarrow} + \Phi_{b\uparrow})^2$ suggest a transformation

\[
\Gamma = U \Phi
\]

(9)

where $\Gamma \equiv (\gamma_1, \gamma_2, \gamma_3, \gamma_4)^T$, $\Phi \equiv (\Phi_{a\uparrow}, \Phi_{a\downarrow}, \Phi_{b\downarrow}, \Phi_{b\uparrow})^T$, and

\[
U \equiv \left( \begin{array}{cccc}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & 1 & 1 & 1
\end{array} \right),
\]

(10)

which is orthogonal, i.e. $U^{-1} = U^T$. Then the effective Lagrangian can be rewritten as

\[
\mathcal{L}_{\text{eff}} = \frac{1}{2}(\partial_t \Gamma)^T A^{-1}(\partial_t \Gamma) - \frac{1}{2}(\nabla \Gamma)^T M^{-1}(\nabla \Gamma) - \Gamma^T G \Gamma,
\]

(11)

where $G \equiv \text{diag}(0, 0, 0, g_e \rho_\alpha \rho_b)$,

\[
M^{-1} \equiv \frac{1}{2} \left( \begin{array}{cccc}
\rho_\alpha & 0 & 0 & 0 \\
0 & \rho_\beta & 0 & 0 \\
0 & 0 & \xi_+ & \xi_-
\end{array} \right),
\]

(12)
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with \( \xi_{\pm} = \frac{1}{2}(\rho_a/m_a \pm \rho_b/m_b) \), and \( A^{-1} = UD^T Q^{-1} D U^{-1} \), with \( D = D^T = \text{diag}(\rho_a, \rho_a, \rho_b, \rho_b) \). Hence

\[
A = UD^{-1}QD^{-1}U^T = \begin{pmatrix}
2g_a & g_{ab} - g_e & 0 & 0 \\
g_{ab} - g_e & 2g_b & 0 & 0 \\
0 & 0 & g_e \eta_+ + g_z & g_e \eta_- \\
0 & 0 & g_e \eta_- & g_e \eta_+ - g_z
\end{pmatrix},
\]

(13)

where \( \eta_{\pm} = \frac{1}{2}(\rho_b/\rho_a \pm \rho_a/\rho_b) \).

From the Euler–Lagrange equation

\[
\partial_t \left( \frac{\partial \mathcal{L}_{\text{eff}}}{\partial (\partial_t \Gamma)} \right) + \nabla \left( \frac{\partial \mathcal{L}_{\text{eff}}}{\partial \nabla \Gamma} \right) - \frac{\partial \mathcal{L}_{\text{eff}}}{\partial \Gamma} = 0,
\]

(14)

we obtain the equation of motion for \( \Gamma \):

\[
\partial_t^2 \Gamma - AM^{-1} \nabla^2 \Gamma + 2AG\Gamma = 0,
\]

(15)

4. Elementary excitations

For elementary excitations, as characterized by the frequency \( \omega \) and wavevector \( \mathbf{k} \), we seek solutions of the form

\[
\Gamma = \Gamma_0 \exp[-i(\omega t - \mathbf{k} \cdot \mathbf{r})],
\]

(16)

where \( \Gamma_0 \) is position independent. Hence we obtain

\[
(-\omega^2 + k^2 AM^{-1} + 2AG)\Gamma_0 = 0,
\]

(17)

from which it is found that the four excitations are given by

\[
\omega_{1,II}^2 = \frac{k^2}{2} \left[ \frac{g_a \rho_a}{m_a} + \frac{g_b \rho_b}{m_b} \pm \sqrt{\left( \frac{g_a \rho_a}{m_a} - \frac{g_b \rho_b}{m_b} \right)^2 + \left( g_{ab} - g_e \right)^2 \rho_a \rho_b} \right],
\]

(18)

\[
\omega_{III,IV}^2 = \frac{1}{2} \left[ Bk^2 + \Delta^2 + \sqrt{Ck^4 + Dk^2 + \Delta^2} \right],
\]

(19)

where

\[
\Delta^2 = \left| g_e \left( \frac{\rho_b}{\rho_a} \rho_b \right) - 2g_e g_z \right| \rho_a \rho_b,
\]

(20)

\[
B \equiv (g_e/2)(\rho_b/m_a + \rho_a/m_b), \quad C \equiv (g_e^2/4)(\rho_b/m_a - \rho_a/m_b)^2 + g_z^2 \rho_a \rho_b / m_a m_b, \quad \text{and} \quad D \equiv g_e \rho_a \rho_b / m_a m_b + \rho_a / m_a - \rho_a / m_b - 2g_e g_z (\rho_b/m_a + \rho_a/m_b) + 2g_z^2 (\rho_a/m_a + \rho_b/m_b)\).
\]

It can be seen that \( \omega_{IV} \) has a gap \( \Delta \), due to the nonvanishing \( g_e \), while the other three excitations, as Goldstone modes, are gapless. As \( k \to 0 \),

\[
\omega_{III}^2 \approx \frac{1}{2} \left( B - \frac{D}{2\Delta^2} \right) k^2 - \frac{C}{4\Delta^2} k^4,
\]

(21)

\[
\omega_{IV}^2 \approx \Delta^2 + \frac{1}{2} \left( B + \frac{D}{2\Delta^2} \right) k^2 + \frac{C}{4\Delta^2} k^4.
\]

(22)
When \( \rho_a = \rho_b = \rho \), we have \( B = \frac{1}{2} g_e (1/m_a + 1/m_b) \rho \), \( D = -2 g_e g_z (g_e - g_z) (1/m_a + 1/m_b) \rho^3 \), and \( \Delta^2 = 2 g_e (g_e - g_z) \rho^2 \). Note that all our calculations are made on the assumption that \( g_e > g_z > 0 \).

\( \omega^2 \) may be negative in some cases, which means that the mean field ground state with \( \psi_{a1} = \psi_{a1} = \sqrt{\rho_a / 2} \), \( \psi_{b1} = -\psi_{b1} = \sqrt{\rho_b / 2} \) is unstable and a phase transition occurs. From the secular equation we see that \( \omega^2 \geq 0 \) is satisfied for any \( k \) if and only if the matrix \( A \) is positive definite, as the matrix \( M^{-1} \) is positive definite while \( G \) is semi-positive definite. This means \( g_a > 0 \), \( g_b > 0 \), \( 4 g_a g_b > (g_{ab} - g_e)^2 \) and \( g_e > g_z \). The first three conditions can be naturally satisfied. If \( g_e < g_z \), we have \( B^2 - C = (g_e^2 - g_z^2) \rho_a \rho_b / m_a m_b < 0 \) and \( 2 B \Delta^2 - D = 2 (g_e^2 - g_z^2) (\rho_a / m_a + \rho_b / m_b) < 0 \); then \( \omega_{IV}^2 \) becomes negative for any \( k \) and fluctuations will destroy the mean field ground state to form a new phase.

The parameter point of \( g_e = g_z \) is a point of quantum phase transition. The gap \( \Delta \) calculated above vanishes at this point, signaling the inappropriateness of the present mean field theory for this phase. Indeed the phase at \( g_e = g_z \) is the so-called entangled BEC discussed previously using the single-orbital-mode approximation, in which the two species are maximally entangled in their collective spins, and BEC occurs in an interspecies two-particle singlet state. At \( g_e = g_z \), the gap calculated in a single-orbital-mode approximation does not vanish, and is on the contrary maximal \([19,21]\). An appropriate field theory for this phase is under development.

We can also obtain the correlation function \( \langle T[\gamma_\mu(t,x)\gamma_\nu(0,0)] \rangle \), \( \mu, \nu = 1, 2, 3, 4 \). What interests us most is \( \langle T[\gamma_4(t,x)\gamma_4(0,0)] \rangle \). In momentum space,

\[
G_4(k, \omega) = -i \langle [\gamma_4(k, \omega) \gamma_4(-k, -\omega)] \rangle = \frac{g_e \eta_+ - g_z}{\omega^2 - \omega_{IV}^2 + i0^+}.
\]

By neglecting the \( k^4 \) term in \( \omega_{IV} \) we obtain

\[
iG_4(x, t) = i \int \frac{d^3k \, d\omega}{(2\pi)^4} \frac{g_e \eta_+ - g_z}{\omega^2 - \omega_{IV}^2 + i0^+} e^{i(k \cdot x - \omega t)}
= -\frac{g_e \eta_+ - g_z}{4\pi^2 v} \partial_\nu K_0(\nu \sqrt{(r^2 - v^2 t)^2} \Delta)
\]

(23)

where \( r = |x|, \nu = \sqrt{B / 2 + D / 4 \Delta^2} \), and \( K_0(z) \) is the modified Bessel function of the second kind which has the following asymptotic behavior:

\[
K_0(z) = \begin{cases} 
-\ln z & z \ll 1 \\
\frac{\pi}{2z} e^{-z} & z \gg 1.
\end{cases}
\]

From \( G_4(x, t) \), we can also obtain the correlation function of the spin-exchange operator \( S_e \equiv \psi_{a1}^\dagger \psi_{a1} \psi_{b1}^\dagger \psi_{b1} \).

\[
\langle S_e^\dagger(x, 0) S_e(0, 0) \rangle \propto \langle e^{-i\gamma_4(x, 0)} e^{i\gamma_4(0, 0)} \rangle = e^{iG_4(x, 0)} e^{-iG_4(1, 0)},
\]

(24)

where \( 1 \) is a vector of short-range cut-off length. According to (23), with \( g_e > g_z \), \( \langle S_e^\dagger(x, 0) S_e(0, 0) \rangle \) decreases with the increase of \( r \).
5. Homogeneous excitations

We now consider a homogeneous excitation $\Gamma(x, t) = \Gamma(t)$, i.e. fluctuations purely caused by spin flipping while the orbital wavefunctions remain homogeneous. In this case the effective Lagrangian is 

$$L_{\text{homo}} = \int d^3 x \mathcal{L}_{\text{homo}} = \Omega [\frac{1}{2} (\partial_t \Gamma)^T A^{-1} (\partial_t \Gamma) - g_x p_a p_b \gamma_4^2].$$

The canonical momentum conjugate with $\gamma_\mu$ is

$$p_\mu = \frac{\partial L}{\partial \dot{\gamma}_\mu} = \Omega A_{\mu \nu}^{-1} \gamma_\nu$$

($\mu, \nu = 1, 4$). The effective Hamiltonian is thus

$$H_{\text{homo}} = p_\mu \dot{\gamma}_\mu - L = \frac{1}{2\Omega} p_\mu A_{\mu \nu} p_\nu + \Omega g_x p_a p_b \gamma_4^2.$$  \hfill (26)

Quantization of these excitations is carried out by imposing the commutation relation 

$$[\gamma_\mu, p_\nu] = i \delta_{\mu \nu}.$$  

Recalling that $\gamma_\mu = U_{\mu \nu} \Phi_\rho$ and that the $N_{a \sigma}$ and $\Phi_{a \sigma}$ are conjugate variables with $[N_{a \sigma}, \Phi_{b \sigma}] = i \delta_{a b, \sigma \sigma'}$, we obtain

$$p_\nu = - U_{\nu \rho} N_{\rho},$$  \hfill (27)

where $N_{\rho}$ represents $N_1 = N_{a 1}$, $N_2 = N_{a 1} - N_{b 1}$ and $N_4 = N_{b 1}$. That is,

$$p_1 = - \frac{1}{\sqrt{2}} (N_{a 1} + N_{a 1}) = - \frac{1}{\sqrt{2}} N_a,$$
$$p_2 = - \frac{1}{\sqrt{2}} (N_{b 1} + N_{b 1}) = - \frac{1}{\sqrt{2}} N_b,$$
$$p_3 = - \frac{1}{2} (N_{a 1} - N_{a 1} + N_{b 1} - N_{b 1}) = - S_z,$$
$$p_4 = - \frac{1}{2} (N_{a 1} - N_{b 1} - N_{b 1} + N_{b 1}).$$  \hfill (28)

The effective Hamiltonian (26) can be solved easily. It can be seen that $p_1$, $p_2$ and $p_3$ are conserved quantities, because $N_a$, $N_b$ and $S_z$ are conserved. The effective Hamiltonian can be rewritten as

$$H_{\text{homo}} = \frac{1}{2\Omega} \sum_{i,j=1,2} p_i A_{ij} p_j + \frac{(g_x^2 - g_z^2)}{2\Omega (g_x \eta_+ - g_z)} p_3^2 + \frac{g_x \eta_+ - g_z}{2\Omega} p_4^2 + \Omega g_x p_a p_b \gamma_4^2,$$  \hfill (29)

where $p_4' \equiv p_4 - g_x \eta_- / g_x \eta_+ + g_z p_3$. $p_1 = - N_a / \sqrt{2}$, $p_2 = - N_b / \sqrt{2}$ and $p_3 = - S_z$ are all conserved, while the part depending of $p_4'$ and $\gamma_4$ is like the Hamiltonian of a harmonic oscillator. Therefore the spectrum of $H$ is

$$E_{\text{homo}}(p_1, p_2, p_3, n) = \frac{1}{2\Omega} \sum_{i,j=1,2} p_i A_{ij} p_j + \frac{(g_x^2 - g_z^2)}{2\Omega (g_x \eta_+ - g_z)} p_3^2 + \left( n + \frac{1}{2} \right) \Delta$$
$$= E_0 + \frac{(g_x^2 - g_z^2)}{2\Omega (g_x \eta_+ - g_z)} S_z^2 + \left( n + \frac{1}{2} \right) \Delta$$  \hfill (30)

where

$$E_0 = \frac{1}{2\Omega} \left[ g_x N_a^2 + g_z N_b^2 + (g_{ab} - g_x) N_a N_b \right].$$  \hfill (31)
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is fixed, and Δ is nothing but the energy gap in (20). In the ground state, \( S_z = 0, n = 0 \), and thus the energy is

\[
E(S_z = 0, n = 0) = E_0 + \frac{1}{2} \Delta,
\]

where \( \Delta/2 \) is the zero-point energy of homogeneous fluctuation. It gives the energy of the elementary excitations with \( k = 0 \), which is absent in (18) and (19)) as it cannot be obtained in the approach in section 4. The excitation energy of the homogeneous excitation is

\[
E_{\text{homo}}(S_z, n) - E(S_z = 0, n = 0) = \frac{(g_e^2 - g_z^2)}{2\Omega(g_e\eta_+ - g_z)}S_z^2 + n\sqrt{2g_e(g_e\eta_+ - g_z)}\rho_a\rho_b,
\]

where \( n = 0, 1, \ldots \).

It can be seen that the fourth elementary excitation, discussed in section 4, reduces to the homogeneous excitation as \( k \to 0 \).

6. The single-orbital-mode approximation

For the ground state of a Bose gas, usually making the approximation of the single orbital mode works very well and it is common practice. For our system, this approximation means that only one orbital mode is contained in each field, that is, \( \psi_{\alpha\sigma} \approx \alpha_{\sigma}\phi_{\alpha\sigma} \), where \( \alpha_{\sigma} \) denotes the annihilation operator of the orbital mode function \( \phi_{\alpha\sigma} \). Then the spin operator for species \( \alpha \) is \( S_\alpha \equiv \int d^3x \psi_\alpha(x) = \alpha_{\sigma}^\dagger \mathbf{s}_{\sigma\sigma}^\alpha \alpha_{\sigma} \), and thus the Hamiltonian \( H = \int d^3x \mathcal{H}(x) \) becomes, up to a constant,

\[
H_s = K_e(S_{az}S_{bz} + S_{ay}S_{by}) + J_zS_{az}S_{bz},
\]

where \( K_e \) and \( J_z \) are effective parameters determined by the interaction strengths as well as single-particle orbital wavefunctions and energies. Then the total spin \( S \) of the system is conserved. In the uniform case, \( V = 0 \), it can be found that \( K_e = 2g_e/\Omega \), and \( J_z = 2g_z/\Omega \), where \( \Omega \) is the volume of the system.

Under the single-orbital-mode approximation, \( \psi_{\alpha\sigma} \approx \alpha_{\sigma}\phi_{\alpha\sigma} \), the elementary excitation, that is, the fluctuated phase factor \( e^{i\Phi_{\alpha\sigma}} \) with a wave-like dependence on \( x \) and \( t \), can only be attributed to the fluctuation of the orbital wavefunction \( \phi_{\alpha\sigma} \). This verifies the previous treatment of elementary excitation using the Gross–Pitaevskii-like equation governing the single-particle orbital wavefunctions. Only in the long-wavelength limit does the gapped elementary excitation reduce the homogeneous excitation.

For a homogeneous excitation, the phase factor \( e^{i\Phi_{\alpha\sigma}} \) is position independent, and is attributed to a spin degree of freedom. Hence a homogeneous excitation is a spin excitation, with the orbital degree of freedom remaining the same as those in the ground state. Therefore, these excitations should be the same as those of the effective spin Hamiltonian \( H_s \) obtained under the single-orbital-mode approximation.

Hence, the energy spectrum of a homogeneous excitation (33) can be approximately equalized with the spectrum of \( H_s \), that is

\[
E_s = E_{\text{homo}} = \frac{(K_e - J_z)^2}{4(K_e\eta_+ - J_z)}S_z^2 + \frac{1}{2}n\sqrt{2K_e(K_e\eta_+ - J_z)}\rho_a\rho_b.
\]
Figure 1. $E$ as a function of $S_z$ with $n = 0$. $x = K_e/J_z$ and $N$ is the number of particles. The + signs show the numerical solution of $H_s$, and the solid line is the plot of $E = (K_e + J_z)S_z^2/4$. They fit extremely well.

Figure 2. $E$ as a function of $n$ with $S_z = 0$. $x = K_e/J_z$ and $N$ is the number of particles. The dashed line represents the numerical solution of $H_s$ and the solid line is the plot of $E = \frac{1}{2} n \sqrt{2K_e(K_e - J_z)} N$. Note that the low-lying excited states correspond to small values of $n$, for which the low energy field theory and the single-orbital-mode approximation fit well.

which, for $N_a = N_b = N$, reduces to

$$E_s = \frac{1}{4}(K_e + J_z)S_z^2 + \frac{1}{2} n \sqrt{2K_e(K_e - J_z)} N. \quad (36)$$

We have numerically diagonalized the effective spin Hamiltonian $H_s$ and compared the result with the above expression for spectrum (36). As shown in figures 1 and 2, they fit very well for small $n$. For a Bose gas in the absence of a magnetic field, $N$ is very large.
while $S_z$ is very small. Hence in (36), unless $n = 0$, the first term is much smaller than the second term. Therefore, the low-lying states must be those with a certain small $S_z$ and with $n = 0$. This firmly indicates that our field theory and the single-orbital-mode approximation fit very well for low energy excitations.

This result also confirms a previous perturbative treatment of the anisotropic coupling between the collective spins [18]. Both the unperturbed isotropic Hamiltonian and the anisotropic perturbation conserve $S_z$; hence the eigenstate is a superposition of states $|S, S_z\rangle$ with the same value of $S_z$ and different values of $S$. The expansion coefficients turn out to be the ‘wavefunctions’ of a harmonic oscillator in coordinate $S$, also giving the spectrum (36). The total spin $S$ is indeed equivalent to $\gamma_4 \equiv (\Phi_{a\uparrow} - \Phi_{a\downarrow} - \Phi_{b\uparrow} + \Phi_{b\downarrow})/2$, as the angle between the collective spins of the two species is just $2\gamma_4 - \pi$ [19].

7. Summary and discussion

We have described the low energy excitations of a mixture of two species of pseudospin-$\frac{1}{2}$ Bose gases with interspecies spin exchanges, which entangle the two species of atoms when the system undergoes BEC. We developed a low energy effective field theory, which can describe various low energy excitations very well in a unified framework. As an interesting generalization of the usual Bogoliubov theory to the present multi-component Bose gas with a spin degree of freedom, this theory gives four elementary excitations. The most interesting aspect is the gap in one of the four excitations. On the other hand, quantizing homogeneous excitations yields the excitation spectrum which can be attributed to a spin degree of freedom and gives the zero-point energy of the elementary excitations in the long-wavelength limit. Interestingly, this leads to an analytical solution of the effective spin Hamiltonian obtained under the single-orbital-mode approximation.

Notice that in a realistic system in a trapping potential, there is a cut-off of Goldstone modes due to the trap; thus the low energy excitations become discrete collective modes [5, 25].

The elementary excitations or collective modes can be measured by using Bragg spectroscopy, on the basis of two-photon Bragg scattering [26]. In particular, several modes coexisting at a given value of the momentum transfer can be excited and measured [27]. For a trapped gas, the collective modes can also be measured by perturbing the trapping potential [28]. Similarly, the excitations discussed in our work can be experimentally measured by using the above method. The homogeneous excitations can also be measured, in a similar way to the collective modes in a trap, which are not plane waves [26].

The gap in a collective mode is a feature that does not arise for the usual mixtures, where the particle number of each spin state is conserved [15, 24]. The nonvanishing value of $g_{e\downarrow}$, which accounts for the gap, as well as $g_{e\uparrow}$, which characterizes the difference of scattering lengths of like-spin and unlike-spin scattering processes, both originate from the interspecies spin-exchange interaction. Therefore $g_e$ and $g_z$ are of roughly the same order of magnitude. Many experiments have been carried out to measure this interaction [24], indicating a considerable value of the scattering length, which is about $100 a_B$, where $a_B$ is the Bohr radius. Therefore we expect that experimentally this system could be realized and that the gapped mode could be detected.
Field theory of low energy excitations of a mixture of two species of Bose gases

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