STABILITY OF FIREBALLS AND $\gamma$-RAY BURSTS.

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Abstract

Fireballs are an essential part of any cosmological $\gamma$-ray burst. We derive here a stability criterion for fireballs and show that fireballs are Rayleigh-Taylor unstable in any region in which the entropy decreases outward. The instability begins to operate when the fireball becomes matter dominated. Among the possible implication of the instability are: (i) Conversion of a fraction of the radiation energy to a convective energy expressed in the motion of bubbles relative to each other. (ii) Penetration of fast bubbles through slower ones and creation of high $\gamma$ regimes which are essential for efficient conversion of the energy to $\gamma$-rays. (iii) Formation of rapid variation (of the scale of the bubbles) in the observed $\gamma$-rays.

Key Words: hydrodynamics–relativity–gamma rays: bursts
1. Introduction: Fireballs and Grbs

Regardless of the nature of their source, cosmological $\gamma$-ray bursts (GRBs) inevitably involve the creation of an expanding “fireball”: an extremely optically thick plasma of electrons and positrons. Even if the energy is initially entirely in the form of gamma-radiation, electrons and positrons will be spontaneously created by the reaction $\gamma + \gamma \rightarrow e^+ + e^-$ (Schmidt 1978, Cavallo & Rees 1978). The understanding of fireball evolution is therefore crucial to the understanding of the observed GRBs.

The current picture of fireball evolution is based on smooth spherical models (Goodman, 1986, Paczyński 1986, Shemi & Piran, 1990, Piran, Narayan & Shemi, 1993, Mészáros and Rees 1992, Rees & Mészáros, 1992) with a possible inclusion of shocks (Narayan, Paczyński & Piran, 1992; Rees & Mészáros 1994). In this letter we show that fireballs may be Rayleigh-Taylor unstable. The instability develops during and after the transition from radiation dominated to matter dominated fireball and therefore causes the evolution to differ from that predicted by a smooth spherical model. The existence of the instability may ease the constrains on the baryonic rest mass allowed to be entrained within the fireball, and may also have direct observational implications.

2. Fireball Evolution, Asymptotic Laws and Shocks Waves

Piran, Shemi & Narayan (1993) have analyzed the evolution of a fireball in terms of the evolution of its individual shells. This descriptions holds for an impulsive fireball, in which all the energy is released instantaneously (Goodman, 1986), as well as for a quasi-steady radiative wind, in which the energy is released over a period longer than the dynamical time (Paczyński, 1986). Under certain conditions, it is also valid for non spherical fragments of a spherical shell (Piran, 1994; Narayan & Piran, 1994).

Each shell is characterized by a radiation energy density $e_r$, a baryon number density $n$
and a relativistic Lorentz factor $\gamma$. The evolution may be divided into two phases. Initially the shell is radiation dominated, $\eta \equiv e_r/m_p n c^2 \gg 1$, and the evolution is described by: 

\[
e_r \propto R^{-4}, \quad n \propto R^{-3}, \quad \gamma \propto R \quad \text{and} \quad \eta \propto R^{-1} \quad (R \text{ is the fireball radius}).
\]

This description is valid after a rapid initial acceleration that gives the shell relativistic velocity. Since $\eta$ decreases the shell eventually becomes matter dominated. In the limit of $\eta \ll 1$ the baryons coast with: $\gamma \approx \eta_i$ (the index $i$ denotes initial quantities) and $n \propto R^{-2}$. The radiation continues to cool with $\eta \propto R^{-2/3}$. The transition between radiation and matter dominated evolution occurs roughly at $R = R_\eta \equiv R_i \eta_i$, where $\eta \approx 1$. $\eta$ and $\gamma$ are related by $\gamma(4\eta/3 + 1) - \eta/3\gamma = \eta_i + 1$ throughout the shell evolution.

The fireball becomes optically thin (at $R = R_\tau$) during the radiation dominated phase only if $\eta$ is extremely large (typically $> 10^5$). In this case the released radiation is thermal with $T \approx T_i(R/R_i)$ (Due to the relativistic expansion an observer at rest will see blue shifted radiation with $T_{obs} \approx \gamma T \approx T_i$). If $\eta_i$ is substantially smaller the shell becomes matter dominated ($\eta \approx 1$) before it becomes optically thin, and only an insignificant amount of radiation energy is left at $R \approx R_\tau$. However, not everything is lost. At a much larger radius the baryons may interact with the ISM to produce a SNR like shock (Mészáros & Rees 1993; Mészáros, Laguna, & Rees, 1993) The kinetic energy is then converted back to radiation at the shock and it produces a GRB if $\gamma \gtrsim 10^3$. The last condition poses a strong limit on the baryonic load: $M \lesssim 10^{-6} M_\odot (E/10^{51} \text{ergs})$, which might be difficult to satisfy. This stringent limit is the most serious open question that confronts cosmological GRB models today.

This simple description breaks down when the model predicts shell crossing. Consider two adjoint shells denoted $h$ and $l$ with $R_h < R_l$ separated initially by a distance $\Delta r$. When the shells become matter dominated they coast at $\gamma_h \approx \eta_{i_h}$ and $\gamma_l \approx \eta_{i_l}$. If $\eta_{i_l} < \eta_{i_h}$ the
inner shell moves faster and overtakes the outer one at \( R_{\eta l} + \Delta R_{sc} \approx R_{\eta l} + \eta_l^2 \Delta r \). \( \Delta R_{sc} \) is almost independent of \( \eta_h \) (provided that \( \eta_h \gg \eta_l \)). Shell crossing signifies the breakdown of the simple description given in the previous paragraphs (in fact it breaks down slightly earlier) and the formation of shock waves. In general, the expansion is smooth if \( d\eta_i/dr > 0 \) and shocks appear if \( d\eta_i/dr < 0 \). Narayan et. al. (1992) and recently Rees & Mészáros (1994) suggested that such shocks could convert a fraction of the kinetic energy back to radiation and, if this takes place at \( R > R_\tau \), produce GRBs even with a modest \( \eta \) fireballs.

3. Stability Analysis

3.1 Short Wavelength Limit and the Isobaric Approximation

Large scale deviations from spherical symmetry lead, as mentioned in §2, to quasi-spherical local expansion. We shall therefore be interested only in the short wave-length limit in which the transversial linear scale of the perturbations, \( L \), is small compared with the fireball’s thickness \( R/\gamma \).

The baryon’s thermal energy is always negligible in a fireball. Hence the pressure is radiation dominated, \( p = aT^4/3 \), and the proper energy density is: \( e = nmc^2 + aT^4 \). The speed of sound, \( c_s^2 = c^2/3(1 + 3/4\eta) \), is therefore comparable to the speed of light as long as \( \eta \) is not much smaller than unity. The time required to restore pressure equilibrium in a perturbed fluid element of size \( L \) (in the fluid’s rest frame) is \( \sim L/c_s \sim L/c \). In the observer frame it equals \( \gamma L/c \). For \( L < R/\gamma \) this is shorter than the time for changes in the unperturbed solution, \( \sim R/c \). Thus, for \( L < R/\gamma \) the perturbations are isobaric, i.e. we may assume that the pressure perturbation vanishes. (This assumption holds provided that the typical growth rate of the instability is not much faster than the typical rate of changes in the unperturbed solution).
3.2 Displacement Evolution

We analyze the time evolution of a small fluid element displaced from its original trajectory, $r_0(t)$. Quantities along $r_0(t)$ are denoted by 0 subscripts ($\gamma_0(t)$, $p_0(t)$, $e_0(t)$ etc...), while quantities along the perturbed trajectory $r'(t)$ are primed. The displacement of the element is defined as $\delta r \equiv r' - r_0$. Keeping only linear terms in $\delta r$ we obtain the acceleration of the displacement, $\delta \ddot{r}$:

$$\delta \ddot{r} = \frac{c^2}{\gamma_0^2 (e_0 + p_0)} \left\{ \left[ \left( 1 + \left( \frac{\partial e}{\partial p} \right)_S \right) \frac{1}{e_0 + p_0} \left( \frac{\partial p}{\partial r} \right)^2 - \frac{\partial^2 p}{\partial r^2} \right] \delta r + \frac{1}{c^2} \dot{r}_0 \left[ \left( 1 + \left( \frac{\partial e}{\partial p} \right)_S \right) \frac{1}{e_0 + p_0} \frac{\partial p}{\partial t} \frac{\partial^2 p}{\partial r \partial t} - \frac{\partial^2 p}{\partial r \partial t} \right] \delta r \right. \
\left. + \frac{2}{c^2} \frac{1}{\dot{r}_0 \gamma_0^2} \left( \frac{\partial p}{\partial r} + \frac{1}{c^2} \dot{r}_0 \frac{\partial p}{\partial t} \right) \delta \dot{r} - \frac{1}{c^2} \frac{\partial p}{\partial t} \delta \dot{r} \right\}. \tag{4}$$

Here, all pressure derivatives are evaluated along the unperturbed trajectory, $(r = r_0(t), t)$, and $S$ is the entropy.

In the limit of a non relativistic flow with a non relativistic equation of state Eq. (4) reduces to the equation derived using a somewhat different method by Goodman (1990). For an incompressible fluid, $c_s \to \infty$, in a steady state where pressure gradient supports the fluid against gravity, $\rho^{-1} \partial p/\partial r = -g$, Eq. (4) gives the classical Rayleigh-Taylor stability criterion, $\partial \rho/\partial r < 0$.

3.3 A Qualitative Criterion for Local Convective Stability

Eq. (4) describes the behavior of the Lagrangian displacement of a perturbed fluid element. The growth of the displacement does not indicate, however, that a convective instability exists. Consider, for example, the flow of an expanding sphere where the velocity increases outward. If a fluid element is displaced from its original shell to some outer shell and continues to move with the outer shell, the displacement will grow although the fluid element does not overtake outer shells, i.e. although the instability is not convective.
Convective instabilities exist only if the acceleration experienced by the displaced fluid element is larger than the unperturbed acceleration of the fluid at the new position of the fluid element (rather than that at the original position of the element). The difference between the acceleration of a displaced fluid element and the unperturbed acceleration of the fluid in the displaced position at the rest frame of the shell into which the element had been displaced is

\[
\delta a = \frac{c^2}{e + p} \frac{1}{\partial r} \left[ \left( \frac{\partial e}{\partial p} \right)_S \frac{\partial p}{\partial r} - \frac{\partial e}{\partial r} \right] \delta r = \frac{c^2}{(e + p)^2} \left( \frac{\partial p}{\partial r} \right)_S^2 \left[ \left( \frac{\partial e}{\partial p} \right)_S - \frac{\partial e}{\partial p} \right] \delta r, \tag{5}
\]

where the derivatives are evaluated at the shell rest frame and \( \delta r \) and \( \delta a \) are evaluated at rest.

Using \( e = nmc^2 + aT^4 \) and \( p = aT^4/3 \), eq. (5) may be written in the form

\[
\delta a = \frac{c^2}{(e + p)^2} \frac{3}{\eta} p \left[ \frac{\partial S}{\partial r} \frac{\partial p}{\partial r} \right]_{RF} \delta r, \tag{6}
\]

where \( S \), the entropy per baryon, is \( \log T^3/n \). The subscript \( RF \) denotes that the derivatives are to be taken at the shell rest frame.

Eq. (6) implies that convective instability will occur whenever the rest frame pressure and entropy gradients are parallel. This criterion is similar to the Schwarzschild criterion for convection in stars. However, at the case at hand, the fluid background flow is not stationary. Thus, in order for the instability to be of importance it is necessary that the time scale for instability growth be comparable to or shorter than the flow dynamical time scale. Only than a true dynamical instability will exist. We therefore compare the typical time in the observer frame for instability growth, \( \tau_i \sim \sqrt{\delta r/\delta a} \), with the typical time for shell acceleration, the dynamical time \( \tau_\gamma \),

\[
\tau_\gamma \equiv \frac{\gamma}{\dot{\gamma}} = \frac{\gamma}{\gamma^3 v \dot{v}/c^2} = \frac{\gamma (e + p)}{c} \left( \frac{\partial p}{\partial r} \right)_{RF}^{-1}.
\]

The ratio of these times is

\[
\Theta \equiv \frac{\tau_i}{\tau_\gamma} \sim \sqrt{\frac{\eta}{3} \left[ \partial \log p/\partial r \right]_{RF}^{1/2}}. \tag{7}
\]
We conclude that: (i) A fireball flow is convectively unstable in regions where the entropy per baryon decreases outward (the pressure is assumed to decrease outward). (ii) As long as the fireball is highly dominated by radiation, $\eta \gg 1$, the typical time for convective instability development is large compared to the dynamical time, $\Theta \gg 1$. However, as the shell approaches the matter dominated phase, i.e. when $\eta \sim \text{few}$, convective instabilities will develop rapidly. A rapid instability development may be obtained for larger $\eta$ values if the entropy scale length is considerably smaller than the pressure scale length.

The displaced fluid elements are accelerated by radiation pressure. The energy driving the instability is therefore the radiation energy. At the stage when the instability is expected to develop rapidly, i.e. when $\eta \sim \text{few}$, most of the shell energy is in the radiation field and had not been transformed to baryon kinetic energy. This energy is therefore available for driving the instability. In the limit of $\eta \ll 1$ most of the radiation energy had already been converted to baryon energy, and only a small fraction of it is available for the instability. At this stage the instability will continue to develop due to the energy it acquired during the transition phase from radiation to matter dominated evolution. However, the energy associated with the instability will not increase substantially during this phase (The displacement will continue to grow, however without substantial acceleration).

4. The Competition between Instability Growth and Shock Formation

Shock waves are formed in the (Lagrangian) fireball regions where $\partial \eta_i / \partial r < 0$. Unless the energy density increases steeply with $\eta$, such regions will also have $\partial S / \partial r < 0$. This implies that regions where shocks will develop, i.e. where $\partial \eta_i / \partial r < 0$, are quite generally also convectively unstable. As explained in §2, a shock wave is formed at $R_{\text{sc}} = R_\eta + \eta^2 \Delta r$ as the inner high $\eta$ shells overtake the outer low $\eta$ shells ($\Delta r$ is the initial separation of the shells). The instability, on the other hand, begins to operate at $R \sim R_\eta$. Thus, if
\[ \Delta r \gg R_i/\eta \] then \[ R_{sc} - R_\eta \gg R_\eta \] and the shock forms long after the instability started developing. If, however, the inverse condition \[ \Delta r \lesssim R_i/\eta \] holds, instability development and shock formation are competing processes. The following numerical example demonstrates that even in this case the instability develops substantially prior to shock formation.

We have calculated numerically the evolution of a fireball with an initial radius \( R_i = 10^7 \text{cm} \), initial temperature \( T_i = 1\text{MeV} \), and \( \eta_i \) profile decreasing outward from \( \eta_h = 50 \) to \( \eta_l = 5 \) over \( \Delta r \sim 7 \cdot 10^5 \text{cm} \). Fig. 1 presents the solution of (4) for the evolution of the displacement of a fluid element initially lying at the \( \eta_i \) transition layer (we have chosen \( \delta r > 0 \) and \( \delta \dot{r} = 0 \) at \( t = t_i \)). The displacement grows by the time of shock formation, \( r_0/R_i \approx 30 \), by two orders of magnitude. The instability is indeed convective, as seen by comparing the displacement evolution to the evolution of the distance between two adjacent shells (Note that the initial rapid displacement growth, obtained for \( r_0/R_i \lesssim 1 \), is not convective and is due to the differential expansion). The results presented in fig. 1 are consistent with the qualitative analysis of §3.3, according to which fireball regions where \( \partial S/\partial r < 0 \) are subject to substantial convective instability growth during the transition from radiation to matter dominated evolution. A shock wave forms only at the end of the computation shown in fig. 1. This demonstrates that the instability indeed won over the shock formation. It should be noted that as the inner shell overtakes the outer shell some of the baryon kinetic energy is converted back to radiation energy as the pressure in the shells interface region increases. This energy is then available for driving the instability.

5. Conclusions and Applications to Astrophysical Fireballs and Grbs

We have seen that a convective instability develops whenever \( \partial S/\partial r < 0 \). This includes, quite generally, the important case of \( \partial \eta/\partial r < 0 \), which appears whenever an inner high \( \eta \) shell accelerates an external low \( \eta \) shell. The instability operates from the stage of
transition from radiation to matter dominated evolution, $\eta \sim \text{few and } R \sim R_\eta$, until the fireball becomes optically thin at $R_\tau$ (or at $\eta^{1/2}R_\tau$ where the electrons decouple from the photons). The instability competes with shocks that form at shell crossing within $\Delta R_{sc}$ after $R_\eta$.

We turn now to an astrophysical fireball which appears in a cosmological $\gamma$-ray burst. The energy release is $10^{51}$ ergs and the rising time indicates that the initial radius $R_i$ is around $10^7$ cm (or less). With these parameters we have: $R_\eta = R_i \eta_i = 10^{11} \text{cm} R_i^{7} \eta_i^{4}$ (where $R_i^{7} = R_i / 10^7$ cm and $\eta_i^{4} = \eta_i / 10^4$), $R_\tau = (\sigma_T E / \eta_i m_p c^2)^{1/2} = 6 \cdot 10^{12} \text{cm} E_{51}^{1/2} \eta_i^{4}^{1/2}$ (where $E_{51} = E / 10^{51}$ ergs) and $\Delta R_{sc} \approx \Delta r \eta_i^2 = 10^{14} \text{cm} \Delta r_6 \eta_i^2 = 3^3 R_\eta (\Delta r_6 / R_i^{7}) \eta_i^4$. Thus, for $\eta < 3.3 \cdot 10^4 E_{51}^{1/3} R_i^{17}^{-2/3}$ the instability will operate during a period when the fireball expands by a factor of ten or more. This gives ample time for the growth of the instability. Shell crossing and shocks occur within $\Delta R_{sc}$ from $R_\eta$. If $\Delta r$ is not too small then $\Delta R_{sc} \gg R_\eta$. However, even when this condition is not satisfied we have demonstrated that the instability operates rapidly enough before the shock forms. We conclude that generally, the instability has enough time to develop substantially whenever the conditions for its existence are fulfilled.

We stress that our stability analysis presented is valid for a quasi-steady wind flow generated when the energy is released over a time scale larger than the light crossing time of the source, as well as for a fireball flow where the energy release is instantaneous. The (Lagrangian) regions of a wind created during periods when the emitted entropy increases with time, leading to a spatial profile with $\partial S / \partial r < 0$, will be unstable. This might have important implications to the recent model, suggested by Rees & Mészáros (1994), according to which GRBs are produced due to the formation of shocks between adjoint shells in the fireball. The formation of such shocks requires a wind where $\eta$ increases

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with time, resulting in regions with $\partial \eta/\partial r < 0$. Such regions are likely to satisfy $\partial S/\partial r < 0$ (unless the source luminosity varies enormously with $\eta$) and are, therefore, unstable. This implies that the effects of the instability must be taken into account within the framework of this model.

The convective fireball instability described here will give rise to convection zones where high $S$ bubbles penetrate into low $S$ shells and vice versa. It is difficult to estimate, at present, the full implications of the instability to fireballs and to the observation of $\gamma$-ray bursts, since for such an estimate it is necessary to consider the non-linear regime of instability development, where non-linear dissipation affects bubble motion. In the following we list several possible effects. First and most important is the fact that a situation where a high $\eta$ shell accelerates a low $\eta$ shell is unstable. The instability may therefore provide a mechanism for allowing high $\eta$ shells to penetrate through low $\eta$ ones, thus allowing to achieve Lorentz factors higher than that corresponding to the average $\eta$. This may ease the stringent limit on the allowed baryonic load.

The mixing induced by the instability of high and low $\eta_i$ shells will give rise to regions where the baryon density $n$ fluctuates over length scales much shorter than both the fireball thickness and the source size, which are comparable and of the order of $R_i$. If the burst is produced by a collision with the ISM or by collisions of sub-shells within the fireball, this small scale structure may account for the rapid temporal variations of the observed pulse. This effect is especially important for the scenario where the burst is produced by a collision with the ISM. In this case, the duration over which an observer receives the radiation from a single sub-shell of the fireball is (Katz 1994) $R_c/\eta_i c$, where the collision radius $R_c$ is (Piran 1994) $R_c = 1.3 \cdot 10^{15} \text{cm} E_{51}^{1/3} \eta_{4}^{-2/3} n_{I}^{-1/3}$ ($n_I$ is the ISM number density in units of $\text{cm}^{-3}$). This duration is much larger than the source dynamical
time, \( R_c/\eta_i R_i = 1.3 \cdot 10^4 E_51^{1/3} \eta_{4}^{-5/3} n_i^{-1/3} \). Thus, it would be hard to explain the rapid temporal variations of the pulse as resulting from temporal variations of the source.

Finally, the generation of local random motions within the fireball shell will cause the fireball evolution to differ from that of a spherical smooth model due to the mixing induced by these motions and due to the kinetic energy associated with them. Such motions may further contribute to the build up of a magnetic field, which is necessary for reconverting the baryon kinetic energy to radiation energy.

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**Figure Captions**

Fig. 1: The evolution of a fireball sub-shell initially lying at the $\eta_i$ transition layer. The horizontal axis measures the distance which the shell moved, which is proportional to the time that passes from the beginning. Solid line: The displacement $\delta r$ normalized to its initial value $\delta r_i$; Dashed line: $\eta_i$; Dash-dot line: $\Theta$; Dotted line: The distance between the shell and an outer adjoint shell normalized to the distance at $t_i$. Note that a shock wave forms only at the end of this computation.
This figure "fig1-1.png" is available in "png" format from:

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