Tauberian class estimates for vector-valued distributions

S. Pilipović and J. Vindas

Abstract. We study Tauberian properties of regularizing transforms of vector-valued tempered distributions. The transforms have the form\[ M^f(x, y) = (f * \varphi_y)(x), \]where the kernel \( \varphi \) is a test function and \( \varphi_y(\cdot) = y^{-n} \varphi(\cdot/y) \). We investigate conditions which ensure that a distribution that a priori takes values in a locally convex space actually takes values in a narrower Banach space. Our goal is to characterize spaces of Banach-space-valued tempered distributions in terms of so-called class estimates for the transform \( M^f(x, y) \). Our results generalize and improve earlier Tauberian theorems due to Drozhzhinov and Zav’yalov. Special attention is paid to finding the optimal class of kernels \( \varphi \) for which these Tauberian results hold.

Keywords: regularizing transforms, class estimates, Tauberian theorems, vector-valued distributions, wavelet transform.

§1. Introduction

Tauberian theorems are quite useful tools in several fields of mathematics, such as number theory, operator theory, differential equations, probability theory and mathematical physics. See Korevaar’s book [12] for an account of the one-dimensional theory. In the case of multidimensional Tauberian theorems, the subject has been deeply influenced by the extensive work of Drozhzhinov, Vladimirov and Zav’yalov. Their approach led to the incorporation of generalized functions in the area and resulted in a powerful Tauberian machinery for multidimensional Laplace transforms of distributions. We refer to the monographs [15] and [23] and the recent survey article [7], which give overviews of Tauberian theorems for generalized functions and their applications. Interestingly, distributional methods have been crucial to recent developments on complex Tauberian theorems [2]–[5] and [13].

The goal of this article is to generalize and improve various results due to Drozhzhinov and Zav’yalov given in [8] and [9]. This work might be regarded as a continuation of our previous paper [16], where we treated multidimensional Tauberian theorems for the quasi-asymptotics of vector-valued distributions (see also [21]). We shall deal here with characterizations of Banach-space-valued distributions in terms of so-called Tauberian class estimates.

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The central problem we consider can be formulated as follows. Fix a test function \( \varphi \in \mathcal{S}(\mathbb{R}^n) \). To a vector-valued tempered distribution \( f \) we associate the regularizing transform
\[
M^f_{\varphi}(x, y) = (f * \varphi_y)(x), \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}_+,
\]
where \( \varphi_y(t) = y^{-n} \varphi(t/y) \). We are interested in conditions in terms of \( M^f_{\varphi} \) that ensure that \( f \), which a priori takes values in a ‘broad’ (Hausdorff) locally convex space \( X \), actually takes values in a narrower Banach space \( E \), which is assumed to be continuously included in \( X \). If it is known a priori that \( f \) takes values in \( E \), then we can verify directly that it satisfies the norm estimate
\[
\|M^f_{\varphi}(x, y)\|_E \leq C \frac{(1 + y)^k(1 + |x|)^l}{y^k}
\]
for some \( k, l \) and \( C \). As in [8] and [9], we call (1.1) a class estimate. The problem of interest is thus the converse one: to what extent does a class estimate (1.1) allow us to conclude that \( f \) takes values in the Banach space \( E \)?

This Tauberian question was raised and studied by Drozhzhinov and Zav’yalov in one dimension in [8] and in several variables in [9], where they demonstrated that many Tauberian theorems for integral transforms become particular instances of this problem. (See also [10] for several other interesting applications.) In this article we revisit the problem and provide optimal results by finding the largest class of test functions \( \varphi \) for which the space of \( E \)-valued distributions (up to some correction terms) admits a characterization in terms of a class estimate.

Essentially, we will prove here that the desired optimal Tauberian kernels are given by the class of nondegenerate test functions that we introduced in [16] and which turns out to be much larger than that employed in [8] and [9] (in this article we call the latter one the class of strongly nondegenerate test functions; see Definition 5.1). Note that in Wiener Tauberian theory [12] the Tauberian kernels are those whose Fourier transforms do not vanish at any point. In our theory the Tauberian kernels will be those \( \varphi \) such that \( \hat{\varphi} \) does not identically vanish on any ray through the origin. Besides yielding more general results, we believe that our new approach, based on ideas from wavelet analysis, is much simpler than that in [8] and [9], as it totally avoids using ‘corona theorem’ type arguments and the structure of the Taylor polynomials of the Fourier transforms \( \hat{\varphi} \) of the kernels.

The article is organized as follows. Section 2 fixes the notation and collects some necessary background material on wavelet analysis of Banach-space-valued (Lizorkin) distributions. Section 3 also has a preparatory character: we discuss some basic properties and examples of regularizing transforms. The main body of the article is §§4–6. Section 4 deals with characterizations of \( \mathcal{S}'(\mathbb{R}^n, E) \) in terms of global and local class estimates, that is, when (1.1) holds for all \( (x, y) \) or when it is just assumed to hold for \( (x, y) \in \mathbb{R}^n \times (0, 1] \). In it we actually work with more general integral assumptions on \( M^f_{\varphi} \). In §5 we analyse the case of strongly nondegenerate test functions as kernels of the regularizing transform. Finally, in §6 we study Tauberian class estimate characterizations for distributions intertwining a representation of \( \mathbb{R}^n \) with the translation group; the underlying conditions here are given in terms of transforms with respect to (generalized) Littlewood-Paley pairs.
The paper is dedicated to the memory of Vasiliĭ Sergeevich Vladimirov and Boris Ivanovich Zav’yalov.

§ 2. Preliminaries

We use the notation $\mathbb{H}^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$ for the upper half-space. Locally convex spaces are always assumed to be Hausdorff. The space $E$ always denotes a fixed, but arbitrary, Banach space with norm $\| \cdot \|$. Measurability for $E$-valued functions is meant in the sense of Bochner (that is, almost everywhere pointwise limits of $E$-valued continuous functions); likewise, integrals for $E$-valued functions are taken in the Bochner sense. For test functions we set $\tilde{\phi}(t) = \phi(-t)$ and $\varphi_y(t) = y^{-n}\varphi(t/y)$.

2.1. Spaces of test functions. We use the standard notation for distributions, as explained in [15], [22] and [23], for instance. In particular, the Schwartz spaces of smooth test functions with compact support or which are rapidly decreasing are denoted by $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$, respectively. We choose the constants in the Fourier transform as $b_\varphi(u) = \int_{\mathbb{R}^n} \varphi(t) e^{-iu \cdot t} \, dt$.

Following [11], the space $\mathcal{S}_0(\mathbb{R}^n)$ of highly time-frequency localized functions over $\mathbb{R}^n$ is defined as the closed subspace of $\mathcal{S}(\mathbb{R}^n)$ consisting of those elements all of whose moments vanish, that is, $\eta \in \mathcal{S}_0(\mathbb{R}^n)$ if and only if $\int_{\mathbb{R}^n} t^m \eta(t) \, dt = 0$ for all $m \in \mathbb{N}^n$. This space is also known as the Lizorkin space of test functions. The corresponding space of highly localized functions over $\mathbb{H}^{n+1}$ is denoted by $\mathcal{S}(\mathbb{H}^{n+1})$. It consists of those $\Phi \in C^\infty(\mathbb{H}^{n+1})$ for which

$$\sup_{(x,y) \in \mathbb{H}^{n+1}} \left( y + \frac{1}{y} \right)^{k_1} (1 + |x|)^{k_2} \left| \frac{\partial^l}{\partial y^l} \frac{\partial^m}{\partial x^m} \Phi(x,y) \right| < \infty$$

for all $k_1, k_2, l \in \mathbb{N}$ and $m \in \mathbb{N}^n$. The canonical topology of this space is defined in the standard way [11].

We shall also employ an interesting class of subspaces of $\mathcal{S}(\mathbb{R}^n)$ introduced by Drozhzhinov and Zav’yalov in [9]. Let $I$ be an ideal of the ring $\mathbb{C}[t_1, t_2, \ldots, t_n]$ of (scalar-valued) polynomials over $\mathbb{C}$ in $n$ variables. Define $\mathcal{I}_I(\mathbb{R}^n)$ to be the subspace of $\mathcal{S}(\mathbb{R}^n)$ consisting of those $\phi$ such that all Taylor polynomials of their Fourier transforms $\widehat{\phi}$ at the origin belong to the ideal $I$. For example, we have $\mathcal{I}_0(\mathbb{R}^n) = \mathcal{S}_0(\mathbb{R}^n)$ if $I = \{0\}$ or $\mathcal{I}_I(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n)$ if $I = \mathbb{C}[t_1, t_2, \ldots, t_n]$.

Let $P_0, \ldots, P_q, \ldots$ be a system of homogeneous polynomials where each $P_q$ has degree $q$ (some may be identically 0). Consider the ideal $I = [P_0, P_1, \ldots, P_q, \ldots]$, namely, the ideal generated by the $P_q$; then we can show [9] that $\mathcal{I}_I(\mathbb{R}^n)$ is a closed subspace of $\mathcal{S}(\mathbb{R}^n)$ and in fact $\phi \in \mathcal{I}_I(\mathbb{R}^n)$ if and only if

$$\int_{\mathbb{R}^n} Q(t)\phi(t) \, dt = 0$$
for all polynomials $Q$ that satisfy the differential equations

$$P_q\left(\frac{\partial}{\partial t}\right)Q = 0, \quad q = 0, 1, \ldots .$$

When there is $d \in \mathbb{N}$ such that $P_q = 0$ for $q > d$, the previous requirement can be relaxed (see [9], Lemma A.5) by asking it to hold just for polynomials $Q$ with degree at most $d$.

We denote by $\mathbb{P}_d$ the ideal of (scalar-valued) polynomials of the form $Q(t) = \sum_{d \leq |m| < N} a_m t^m$, for some $N \in \mathbb{N}$.

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. We write $P^\varphi_q$ for the $q$-homogeneous term of the Taylor polynomial of its Fourier transform at 0, that is,

$$P^\varphi_q(u) = \sum_{|m| = q} \frac{\hat{\varphi}'(m)(0)}{m!} u^m, \quad q = 0, 1, 2, \ldots . \quad (2.1)$$

The ideal generated by these homogeneous polynomials is denoted by

$$I_\varphi = [P^\varphi_0, P^\varphi_1, \ldots, P^\varphi_q, \ldots]. \quad (2.2)$$

2.2. Spaces of vector-valued distributions. Let $\mathcal{A}(\Omega)$ be a topological vector space of test function over an open subset $\Omega \subseteq \mathbb{R}^n$ and let $X$ be a locally convex space. Let $\mathcal{A}'(\Omega, X) = L_b(\mathcal{A}(\Omega), X)$ denote the space of continuous linear mappings from $\mathcal{A}(\Omega)$ to $X$ with the topology of uniform convergence over bounded subsets of $\mathcal{A}(\Omega)$ [20]. We are mainly concerned with the spaces $\mathcal{D}'(\mathbb{R}^n, X), \mathcal{D}'(\mathbb{R}^n, X), \mathcal{D}'(\mathbb{H}^{n+1}, X)$ and $\mathcal{S}'_f(\mathbb{R}^n, X)$; see [18] and [19] for vector-valued distributions.

Observe that we have a well-defined continuous linear projection from $\mathcal{D}'(\mathbb{R}^n, X)$ onto $\mathcal{S}'_f(\mathbb{R}^n, X)$ as the restriction of $X$-valued tempered distributions to $\mathcal{S}'_f(\mathbb{R}^n)$. We do not want to introduce notation for this map, so if $f \in \mathcal{D}'(\mathbb{R}^n, X)$, we will simply denote again by $f$ its projection onto $\mathcal{S}'_f(\mathbb{R}^n, X)$.

2.3. Wavelet analysis on $\mathcal{S}'_0(\mathbb{R}^n, E)$. In this section we review some properties of the wavelet transform of (Lizorkin) distributions with values in the Banach space $E$. By a wavelet we simply mean an element of $\mathcal{S}'_0(\mathbb{R}^n)$. The wavelet transform of $f \in \mathcal{S}'_0(\mathbb{R}^n, E)$ with respect to the wavelet $\psi \in \mathcal{S}'_0(\mathbb{R}^n)$ is defined by

$$\mathcal{W}_\psi f(x, y) = \langle f(x + yt), \overline{\psi}(t) \rangle \in E, \quad (x, y) \in \mathbb{H}^{n+1}.$$  

It is then clear that $\mathcal{W}_\psi f \in C^\infty(\mathbb{H}^{n+1}, E)$.

Notice (see [11] and [14]) that $\mathcal{W}_\psi : \mathcal{S}'_0(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{H}^{n+1})$ is a continuous linear map. We are interested in those wavelets for which $\mathcal{W}_\psi$ admits a left inverse. For wavelet-based reconstruction, the so-called wavelet synthesis operator is used [11]. Given $\Phi \in \mathcal{S}(\mathbb{H}^{n+1})$, we define the wavelet synthesis operator with respect to the wavelet $\psi$ by

$$\mathcal{M}_\psi \Phi(t) = \int_0^\infty \int_{\mathbb{R}^n} \Phi(x, y) \frac{1}{y^n} \psi\left(\frac{t - x}{y}\right) \frac{dx}{y}, \quad t \in \mathbb{R}^n. \quad (2.3)$$

It can be shown that $\mathcal{M}_\psi : \mathcal{S}(\mathbb{H}^{n+1}) \rightarrow \mathcal{S}'_0(\mathbb{R}^n)$ is continuous (see [11] and [21]).
We say that the wavelet $\psi \in \mathcal{S}_0(\mathbb{R}^n)$ admits a reconstruction wavelet if there exists $\eta \in \mathcal{S}_0(\mathbb{R}^n)$ such that

$$c_{\psi,\eta}(\omega) = \int_0^\infty \frac{d\rho}{\rho} \psi(\rho\omega) \tilde{\eta}(\rho\omega) \neq 0, \quad \omega \in \mathbb{S}^{n-1},$$

(2.4)
is independent of the direction $\omega$; in this case we set $c_{\psi,\eta} := c_{\psi,\eta}(\omega)$. If $\psi$ admits the reconstruction wavelet $\eta$, there is a reconstruction formula for the wavelet transform on $\mathcal{S}_0(\mathbb{R}^n)$:

$$\text{Id}_{\mathcal{S}_0(\mathbb{R}^n)} = \frac{1}{c_{\psi,\eta}} \mathcal{M}_\eta \mathcal{W}_\psi.$$  (2.5)

In [16], Proposition 5.1, we have characterized those wavelets which have a reconstruction wavelet. They are in fact those elements of $\mathcal{S}_0(\mathbb{R}^n)$ that are nondegenerate in the sense of the following definition.

**Definition 2.1.** We say that the test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is nondegenerate if its Fourier transform does not vanish identically on any ray through the origin.

The wavelet synthesis operator (2.3) can be extended to $\mathcal{S}'_0(\mathbb{H}^{n+1}, E)$ as follows. Let $K \in \mathcal{S}'_0(\mathbb{H}^{n+1}, E)$. We define $\mathcal{M}_\psi : \mathcal{S}'_0(\mathbb{H}^{n+1}, E) \leftrightarrow \mathcal{S}'_0(\mathbb{R}^n, E)$, a continuous linear map, by

$$\langle \mathcal{M}_\psi K(t), \rho(t) \rangle = \langle K(x, y), \mathcal{W}_\psi \rho(x, y) \rangle, \quad \rho \in \mathcal{S}_0(\mathbb{R}^n).$$

(2.6)

We mention that our convention to identify a function $F$ of slow growth on $\mathbb{H}^{n+1}$ with an element of $\mathcal{S}'_0(\mathbb{R}^n, E)$, that is, one that satisfies a growth condition

$$\int_0^\infty \int_{\mathbb{R}^n} \left( \frac{1}{y} + y \right)^{-k} (1 + |x|)^{-l} \|F(x, y)\| \, dx \, dy < \infty,$$

is via the Bochner integral

$$\langle F(x, y), \Phi(x, y) \rangle = \int_0^\infty \int_{\mathbb{R}^n} \Phi(x, y) F(x, y) \frac{dx \, dy}{y}, \quad \Phi \in \mathcal{S}(\mathbb{H}^{n+1}).$$

If $\psi \in \mathcal{S}_0(\mathbb{R}^n)$ is nondegenerate and $\eta \in \mathcal{S}_0(\mathbb{R}^n)$ is a reconstruction wavelet for it, then (see [16], Proposition 5.3) the reconstruction formula

$$\text{Id}_{\mathcal{S}'_0(\mathbb{R}^n, E)} = \frac{1}{c_{\psi,\eta}} \mathcal{M}_\eta \mathcal{W}_\psi$$

(2.7)

holds; furthermore, we have the desingularization formula

$$\langle f(t), \rho(t) \rangle = \frac{1}{c_{\psi,\eta}} \int_0^\infty \int_{\mathbb{R}^n} \mathcal{W}_\psi f(x, y) \mathcal{W}_\eta \rho(x, y) \frac{dx \, dy}{y},$$

(2.8)

which is valid for all $f \in \mathcal{S}'_0(\mathbb{R}^n, E)$ and $\rho \in \mathcal{S}_0(\mathbb{R}^n)$. 


§ 3. The regularizing transform of \(E\)-valued distributions

We discuss in this section some basic properties of the regularizing transform of \(E\)-valued tempered distributions with respect to a test function \(\varphi \in \mathcal{S}(\mathbb{R}^n)\), which, as we said in the introduction, we define as the \(E\)-valued \(C^\infty\)-function

\[
M^f_\varphi(x, y) := (f * \varphi_y)(x), \quad (x, y) \in \mathbb{H}^{n+1},
\]

where \(f \in \mathcal{S}'(\mathbb{R}^n, E)\). If \(\psi \in \mathcal{S}_0(\mathbb{R}^n)\), we obviously have

\[
\mathcal{W}_\psi f = M^f_\psi.
\]

Naturally, the transform (3.1) perfectly makes sense for distributions with values in any arbitrary locally convex space.

The study of the (Tauberian) ‘converse’ of the bound on the regularizing transform of an \(E\)-valued distribution delivered by the next (Abelian) proposition is the core of this article.

**Proposition 3.1.** Let \(f \in \mathcal{S}'(\mathbb{R}^n, E)\) and \(\varphi \in \mathcal{S}(\mathbb{R}^n)\). Then, \(M^f_\varphi \in C^\infty(\mathbb{H}^{n+1}, E)\) is a function of slow growth on \(\mathbb{H}^{n+1}\). In addition, the linear map \(f \in \mathcal{S}'(\mathbb{R}^n, E) \mapsto M^f_\varphi \in \mathcal{S}'(\mathbb{H}^{n+1}, E)\) is continuous for the topologies of uniform convergence over bounded sets. Furthermore, if \(\mathcal{B} \subset \mathcal{S}'(\mathbb{R}^n, E)\) is bounded for the topology of pointwise convergence, then there exist \(k, l\) and \(C > 0\) such that

\[
\|M^f_\varphi(x, y)\| \leq C\left(\frac{1}{y} + y\right)^k (1 + |x|)^l, \quad (x, y) \in \mathbb{H}^{n+1},
\]

for all \(f \in \mathcal{B}\).

**Proof.** The proof is simple but we include it for the sake of completeness. The space \(\mathcal{S}'(\mathbb{R}^n, E)\) is bornological. Therefore, we have to show that this map takes bounded sets to bounded sets. Let \(\mathcal{B} \subset \mathcal{S}'(\mathbb{R}^n, E)\) be a bounded set. The Banach-Steinhaus theorem implies the existence of \(k_1 \in \mathbb{N}\) and \(C_1 > 0\) such that for all \(\rho \in \mathcal{S}(\mathbb{R}^n)\) and \(f \in \mathcal{B}\),

\[
\|\langle f, \rho \rangle\| \leq C_1 \sup_{t \in \mathbb{R}^n, |m| \leq k_1} (1 + |t|)^{k_1} |\rho^{(m)}(t)|.
\]

Consequently,

\[
\|M^f_\varphi(x, y)\| \leq C_1 \left(\frac{1}{y} + y\right)^{n + k_1} \sup_{u \in \mathbb{R}^n, |m| \leq k_1} (1 + |x| + y|u|)^{k_1} |\varphi^{(m)}(u)|
\]

\[
\leq C \left(\frac{1}{y} + y\right)^{n + 2k_1} (1 + |x|)^{k_1} \text{ for all } f \in \mathcal{B},
\]

where \(C = C_1 \sup_{u \in \mathbb{R}^n, |m| \leq k_1} (1 + |u|)^{k_1} |\varphi^{(m)}(u)|\). So, we obtain (3.2) with \(k = n + 2k_1\) and \(l = k_1\). If \(\mathcal{C} \subset \mathcal{S}'(\mathbb{H}^{n+1})\) is a bounded set of test functions, we have

\[
\|\langle M^f_\varphi(x, y), \Phi(x, y) \rangle\| = \left\| \int_0^\infty \int_{\mathbb{R}^n} M^f_\varphi(x, y)\Phi(x, y) \frac{dx \, dy}{y} \right\|
\]

\[
\leq C \int_0^\infty \int_{\mathbb{R}^n} \left(\frac{1}{y} + y\right)^k (1 + |x|)^l \Phi(x, y) \frac{dx \, dy}{y},
\]
which stays bounded for \( f \in \mathcal{B} \) and \( \Phi \in \mathcal{C} \). Therefore, the set

\[
\{ M^f_\varphi : f \in \mathcal{B} \} \subset \mathscr{S}'(\mathbb{H}^{n+1}, E)
\]

is bounded; hence the map is continuous.

We point out that the regularizing transforms enjoy excellent localization properties as shown by the following simple proposition.

**Proposition 3.2.** Let \( f \in \mathscr{S}'(\mathbb{R}^n, E) \) and let \( \varphi \in \mathscr{S}(\mathbb{R}^n) \). Suppose that \( K \subset \mathbb{R}^n \setminus \text{supp} f \) is a compact set. Then, for any positive integer \( k \in \mathbb{N} \) there exists \( C = C_k \) such that

\[
\sup_{x \in K} \| M^f_\varphi (x, y) \| \leq C y^k \quad \text{for all} \quad 0 < y < 1. \tag{3.3}
\]

**Proof.** Define the \( C(K, E) \)-valued tempered distribution whose evaluation at \( \rho \in \mathscr{S}(\mathbb{R}^n) \) is given by \( \langle G(t), \rho(t) \rangle(\xi) = (f * \rho)(\xi), \xi \in K \). Clearly, \( G \in \mathscr{S}'(\mathbb{R}^n, C(K, E)) \).

Then, since \( K \subset \mathbb{R}^n \setminus \text{supp} f \), for each \( \rho \in \mathscr{D}(\mathbb{R}^n) \), \( \langle G(\varepsilon t), \rho(t) \rangle = 0 \) for sufficiently small \( \varepsilon > 0 \). In particular, for a fixed \( k \in \mathbb{N} \),

\[
G(\varepsilon t) = O(\varepsilon^k) \quad \text{as} \quad \varepsilon \to 0^+ \quad \text{in} \quad \mathscr{D}'(\mathbb{R}^n, C(K, E)), \tag{3.4}
\]

where this relation is interpreted as a quasi-asymptotic bound in the sense of [15] and [16]. (The precise meaning of (3.4) is \( \| \langle G(\varepsilon t), \rho(t) \rangle \|_{C(K, E)} = O(\varepsilon^k) \) for each \( \rho \in \mathscr{D}(\mathbb{R}^n) \).) Now, it is well known (cf. Proposition 7.1 in [16] or Lemma 6 in [24]) that the quasi-asymptotic relation (3.4) remains valid in the space \( \mathscr{S}'(\mathbb{R}^n, C(K, E)) \). This means that we have the right to evaluate relation (3.4) at \( \varphi \in \mathscr{S}(\mathbb{R}^n) \), which immediately yields

\[
\sup_{x \in K} \| M^f_\varphi (x, y) \| = \| \langle G(yt), \varphi(t) \rangle \|_{C(K, E)} \leq C y^k
\]

for some \( C > 0 \), as claimed.

In the next sections we will focus our attention on the regularizing transform with respect to nondegenerate test functions in the sense of Definition 2.1. It is clear that if \( \int_{\mathbb{R}^n} \varphi(t) \, dt \neq 0 \), then \( \varphi \) is nondegenerate. We conclude this section by discussing two integral transforms which arise as regularizing transforms with respect to test functions satisfying the latter condition.

**Example 3.1** (The regularizing transform as a solution to Cauchy’s problem). When the test function is of a certain special form, the regularizing transform can become a solution to a PDE. We discuss a particular case in this example. Let the set \( \Gamma \subseteq \mathbb{R}^n \) be a closed convex cone with vertex at the origin. In particular, we may have \( \Gamma = \mathbb{R}^n \). Let \( P \) be a homogeneous polynomial of degree \( d \) such that \( \text{Re} \, P(iu) < 0 \) for all \( u \in \Gamma \setminus \{0\} \). We denote the subspace of distributions supported by \( \Gamma \) by \( \mathscr{S}'_\Gamma \subseteq \mathscr{S}'(\mathbb{R}^n) \) (see [22] and [23]). Consider the \( E \)-valued Cauchy problem

\[
\frac{\partial}{\partial t} U(x, t) = P \left( \frac{\partial}{\partial x} \right) U(x, t), \quad \lim_{t \to 0^+} U(x, t) = f(x) \quad \text{in} \quad \mathscr{S}'(\mathbb{R}^n_x), \tag{3.5}
\]

\[ \text{supp} \, \hat{f} \subseteq \Gamma, \quad (x, t) \in \mathbb{H}^{n+1}, \]
within the class of $E$-functions of slow growth over $\mathbb{H}^{n+1}$, that is,
\[
\sup_{(x,t)\in\mathbb{H}^{n+1}} \|U(x,t)\| \left( t + \frac{1}{t} \right)^{-k_1} (1 + |x|)^{-k_2} < \infty \quad \text{for some } k_1, k_2 \in \mathbb{N}.
\]
It is easy to verify that (3.5) has a unique solution satisfying this slow growth condition. Indeed,
\[
U(x,t) = \frac{1}{(2\pi)^n} \int f(u) e^{ix\cdot u} e^{iP(iu)} \, du = \frac{1}{(2\pi)^n} \int \hat{f}(u) e^{ix\cdot u} e^{P(it^{1/d}u)} \, du
\]
is the required solution. By [23], we can find a test function $\eta \in \mathcal{S}(\mathbb{R}^n)$ with the property $\eta(u) = e^{P(iu)}$, $u \in \Gamma$. Choosing $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat{\varphi} = \eta$, we express $U$ as
\[
U(x,t) = \left< f(\xi), \frac{1}{t^{n/d}} \varphi \left( \frac{x-\xi}{t^{1/d}} \right) \right> = M^f_\varphi(x,y) \quad \text{with } y = t^{1/d}.
\]
In particular, when $P(u) = |u|^2$, (3.5) is the Cauchy problem for the heat equation and $\varphi(\xi) = (2\sqrt{\pi})^{-n} e^{-\xi^2/4}$.

**Example 3.2** (Laplace transforms as regularizing transforms). Let $\Gamma$ be a closed convex acute cone (see [22] and [23]) with vertex at the origin. Its conjugate cone is denoted by $\Gamma^*$. The definition of an acute cone tells us that $\Gamma^*$ has nonempty interior. Set $C_\Gamma = \text{int } \Gamma^*$ and $T^{C_\Gamma} = \mathbb{R}^n + iC_\Gamma$. We denote the subspace of $E$-valued tempered distributions supported by $\Gamma$ by $\mathcal{S}_\Gamma(E)$. Given $h \in \mathcal{S}_\Gamma(E)$, its Laplace transform $[22]$ is
\[
\mathcal{L} \{ h; z \} = \langle h(u), e^{iz\cdot u} \rangle, \quad z \in T^{C_\Gamma};
\]
it is a holomorphic $E$-valued function on the tube domain $T^{C_\Gamma}$. Fix $\omega \in C_\Gamma$. We can write $\mathcal{L} \{ h; x + i\sigma \omega \}, x \in \mathbb{R}^n, \sigma > 0$, as a $\phi$-transform. In fact, choose $\eta_\omega \in \mathcal{S}(\mathbb{R}^n)$ such that $\eta_\omega(u) = e^{-\omega\cdot u}$, $u \in \Gamma$. Then, with $\hat{\varphi}_\omega = \eta_\omega$ and $\hat{f} = (2\pi)^n h$,
\[
\mathcal{L} \{ h; x + i\sigma \omega \} = M^f_{\eta_\omega}(x,\sigma).
\]
Notice that this is a particular case of Example 3.1 with $P_\omega(\xi) = i\omega \cdot \xi$.

**§ 4. Tauberian class estimates**

We now establish the Tauberian nature of the estimate
\[
\|M^f_\varphi(x,y)\| \leq C \frac{(1 + y)^k (1 + |x|)^j}{y^k}, \quad (x,y) \in \mathbb{H}^{n+1}. \tag{4.1}
\]
We call (4.1) a *global class estimate*. We will prove that if $f$ takes values in a ‘broad’ locally convex space which contains the narrower Banach space $E$, and if $f$ satisfies (4.1) for a nondegenerate test function $\varphi$, then there is a distribution $G$ with values in the broad space such that $\text{supp } \hat{G} \subseteq \{ 0 \}$ and $f - G \in \mathcal{S}'(\mathbb{R}^n, E)$. When the broad space is a normed space, $G$ simply reduces to a polynomial. This will be done in § 4.1.
In §4.2 we shall also investigate the consequences of (4.1) when it is only assumed to hold for \((x, y) \in \mathbb{R}^n \times (0, 1]\); we then call it a local class estimate. In this case the situation is slightly different and we obtain \(f - G \in \mathcal{S}'(\mathbb{R}^n, E)\), where \(\hat{G}\) has compact support but its support need not be the origin any longer. Nevertheless, the support of the Fourier transform of the correction term is still controlled by the so-called index of nondegenerateness of the test function, introduced below.

The correction terms, both in the case of global and of local estimates, can be eliminated if we augment the hypotheses by involving a convolution average with respect to another suitable test function.

Throughout this section, unless specified, \(X\) is assumed to be a locally convex topological vector space such that the Banach space \(E \subset X\) and the inclusion mapping \(E \rightarrow X\) is linear and continuous. Observe that the transform (3.1) makes sense for \(X\)-valued distributions as well. Furthermore, in order to gain generality, we will work with an integral version of the class estimate (4.1).

### 4.1. Global class estimates

We begin with a full characterization of \(\mathcal{S}'_0(\mathbb{R}^n, E)\) in terms of the wavelet transform.

**Proposition 4.1.** Let \(f \in \mathcal{S}'_0(\mathbb{R}^n, X)\) and let \(\psi \in \mathcal{S}_0(\mathbb{R}^n)\) be a nondegenerate wavelet. The following two conditions,

\[
\mathcal{W}_\psi f(x, y) \in E \text{ for almost all values of } (x, y) \in \mathbb{H}^{n+1},
\]

and there are constants \(k, l \in \mathbb{N}\) such that

\[
\int_0^\infty \int_{\mathbb{R}^n} \left( \frac{1}{y} + y \right)^{-k} (1 + |x|)^{-l} \| \mathcal{W}_\psi f(x, y) \| \, dx \, dy < \infty
\]

are necessary and sufficient for \(f \in \mathcal{S}'_0(\mathbb{R}^n, E)\).

**Proof.** The necessity is clear (Proposition 3.1). We prove the sufficiency. Let \(\eta\) be a reconstruction wavelet for \(\psi\). We apply the wavelet synthesis operator to the function \(K(x, y) = \mathcal{W}_\psi f(x, y)\); this is valid because our assumptions (4.2) and (4.3) ensure that \(K \in \mathcal{S}'(\mathbb{H}^{n+1}, E)\). So, set \(\tilde{f} := \mathcal{M}_\eta K \in \mathcal{S}'_0(\mathbb{R}^n, E) \subset \mathcal{S}'_0(\mathbb{R}^n, X)\).

We must therefore show that \(\tilde{f} = f\). Let \(\rho \in \mathcal{J}_0(\mathbb{R}^n)\). We have by definition (cf. (2.6)), (2.3) and (2.5)

\[
\langle \tilde{f}, \rho \rangle = \frac{1}{c_{\psi, \eta}} \int_0^\infty \int_{\mathbb{R}^n} f(t) \left( \frac{1}{y^n} \psi \left( \frac{t-x}{y} \right) \mathcal{W}_\eta \rho(x, y) \right) \frac{dx \, dy}{y}
\]

and

\[
\langle f, \rho \rangle = \frac{1}{c_{\psi, \eta}} \langle f, \mathcal{M}_\psi \mathcal{W}_\eta \rho \rangle = \frac{1}{c_{\psi, \eta}} \left( \int_0^\infty \int_{\mathbb{R}^n} f(t) \left( \frac{1}{y^n} \psi \left( \frac{t-x}{y} \right) \mathcal{W}_\eta \rho(x, y) \right) \frac{dx \, dy}{y} \right)
\]

Thus, with

\[
\Phi(x, y; t) = \frac{1}{y^{n+1}} \psi \left( \frac{t-x}{y} \right) \mathcal{W}_\eta \rho(x, y)
\]
our problem reduces to justifying interchanging the integrals with the dual pairing in
\[
\int_0^\infty \int_{\mathbb{R}^n} \langle f(t), \Phi(x,y;t) \rangle \, dx \, dy = \left\langle f(t), \int_0^\infty \int_{\mathbb{R}^n} \Phi(x,y;t) \, dx \, dy \right\rangle. \tag{4.4}
\]
To show (4.4), we verify that
\[
\left\langle w^*, \int_0^\infty \int_{\mathbb{R}^n} \langle f(t), \Phi(x,y;t) \rangle \, dx \, dy \right\rangle = \left\langle w^*, \left\langle f(t), \int_0^\infty \int_{\mathbb{R}^n} \Phi(x,y;t) \, dx \, dy \right\rangle \right\rangle \tag{4.5}
\]
for arbitrary \( w^* \in X' \) (this is where the local convexity and the Hausdorff property of \( X \) play a role). Since the integral involved in the left-hand side of (4.5) is a Bochner integral in \( E \) and the restriction of \( w^* \) to \( E \) belongs to \( E' \), we immediately obtain the interchange formula
\[
\left\langle w^*, \int_0^\infty \int_{\mathbb{R}^n} \langle f(t), \Phi(x,y;t) \rangle \, dx \, dy \right\rangle = \int_0^\infty \int_{\mathbb{R}^n} \langle w^*, \langle f(t), \Phi(x,y;t) \rangle \rangle \, dx \, dy. \tag{4.6}
\]
On the other hand, we can write \( \int_0^\infty \int_{\mathbb{R}^n} \Phi(x,y;t) \, dx \, dy \) as the limit of Riemann sums, convergent in \( J_0(\mathbb{R}^n) \). Then we easily justify the interchanges that yield
\[
\left\langle w^*, \left\langle f(t), \int_0^\infty \int_{\mathbb{R}^n} \Phi(x,y;t) \, dx \, dy \right\rangle \right\rangle = \int_0^\infty \int_{\mathbb{R}^n} \left\langle w^*, \langle f(t), \Phi(x,y;t) \rangle \right\rangle \, dx \, dy. \tag{4.7}
\]
Equality (4.5) now follows by comparing (4.6) and (4.7).

We now consider the general case of regularizing transforms with respect to nondegenerate kernels (see Definition 2.1).

**Theorem 4.1.** Let \( f \in \mathcal{S}'(\mathbb{R}^n, X) \) and let \( \varphi \in \mathcal{S}(\mathbb{R}^n) \) be nondegenerate. Sufficient conditions for the existence of an \( X \)-valued distribution \( G \in \mathcal{S}'(\mathbb{R}^n, X) \) such that \( f - G \in \mathcal{S}'(\mathbb{R}^n, E) \) and \( \text{supp} \, \hat{G} \subseteq \{0\} \) are:

(i) \( M^f_\varphi(x,y) \) takes values in \( E \) for almost all \( (x,y) \in \mathbb{H}^{n+1} \) and is measurable as an \( E \)-valued function;

(ii) there exist \( k, l \in \mathbb{N} \) such that

\[
\int_0^\infty \int_{\mathbb{R}^n} \left( \frac{1}{y} + y \right)^{-k} (1 + |x|)^{-l} \| M^f_\varphi(x,y) \| \, dx \, dy < \infty. \tag{4.8}
\]

In this case, we also have
\[
P^\varphi_q \left( \frac{\partial}{\partial t} \right) f \in \mathcal{S}'(\mathbb{R}^n, E) \quad \text{for all } q \in \mathbb{N}, \tag{4.9}
\]
where \( P^\varphi_q \) are the homogeneous terms of the Taylor polynomials of \( \varphi \) at the origin (cf. (2.1)).

**Proof.** Take the nondegenerate wavelet \( \psi_1 \in \mathcal{S}_0(\mathbb{R}^n) \) given by \( \hat{\psi}_1(u) = e^{-|u| - (1/|u|)} \).
Set \( \psi = \varphi * \psi_1 \); then \( \psi \in \mathcal{S}_0(\mathbb{R}^n) \) is also a nondegenerate wavelet. Using the same
argument as in the proof of Proposition 4.1, the interchange of integral and dual pairing performed in the following calculation is valid:
\[
\mathcal{W}_\psi f(x, y) = \langle f(x + yt), (\hat{\varphi} * \bar{\psi}_1)(t) \rangle = \left\langle f(x + yt), \int_{\mathbb{R}^n} \bar{\psi}_1(u) \varphi(u - t) \, du \right\rangle
\]
\[
= \int_{\mathbb{R}^n} \bar{\psi}_1(u) \langle f(x + yt), \varphi(u - t) \rangle \, du = \int_{\mathbb{R}^n} M^f_\varphi(x + yu, y) \bar{\psi}_1(u) \, du;
\]
where the integral is taken in the sense of Bochner. Thus, the restriction of \( f \) to \( \mathcal{S}_0(\mathbb{R}^n) \) is readily seen to satisfy the hypotheses of Proposition 4.1, and hence there exists \( g \in \mathcal{S}'(\mathbb{R}^n, E) \) such that \( \langle f - g, \rho \rangle = 0 \) for all \( \rho \in \mathcal{S}_0(\mathbb{R}^n) \). It follows directly that \( G = f - g \) satisfies \( \text{supp} \, \hat{G} \subseteq \{0\} \) and \( f - G \in \mathcal{S}'(\mathbb{R}^n, E) \).

Next, it is clear that \( G \) is an \( X \)-valued entire function with power series expansion, say, \( G(t) = \sum_{m \in \mathbb{N}^n} i^{m|} t^m w_m \), with \( w_m \in X \), so that \( \hat{G} \) is given by the multipole series
\[
\hat{G}(u) = \sum_{m \in \mathbb{N}^n} (-1)^{|m|} \delta^{(m)}(u) w_m,
\]
which converges in \( \mathcal{S}'(\mathbb{R}^n, X) \). The relation (4.9) will follow immediately if we can show that \( P_\varphi q(\partial/\partial t)G \) is \( E \)-valued. Observe that the hypotheses imply that \( M^G_\varphi(x, y) \in E \), for almost all \( (x, y) \). Hence, for almost all \( (x, y) \),
\[
M^G_\varphi(x, y) = \frac{1}{(2\pi)^n} \langle \hat{G}(u), e^{ix \cdot u} \hat{\varphi}(yu) \rangle = \sum_{m \in \mathbb{N}^n} \frac{\partial_{vm}}{\partial u m} (e^{ix \cdot u} \hat{\varphi}(yu))|_{u=0} w_m
\]
\[
= \sum_{q=0}^{\infty} y^q \sum_{|j|=q} \hat{\varphi}(j) \sum_{j \leq m} \binom{m}{j} (ix)^{m-j} w_m
\]
\[
= \sum_{q=0}^{\infty} (iy)^q \left( P_\varphi \left( \frac{\partial}{\partial x} \right) G \right)(x) \in E.
\]
But the latter readily implies that \( (P_\varphi q(\partial/\partial x)G)(x) \in E \), for all \( q \in \mathbb{N} \) and \( x \in \mathbb{R}^n \).

When \( X \) is a normed space, clearly the only \( X \)-valued distributions with support at the origin are precisely those having the form
\[
\sum_{|m| \leq N} \delta^{(m)} w_m, \quad w_m \in X.
\]
Thus, we have the following result.

**Corollary 4.1.** Let \( X \) be a normed space, \( f \in \mathcal{S}'(\mathbb{R}^n, X) \), and let \( \varphi \in \mathcal{S}(\mathbb{R}^n) \) be nondegenerate. Then conditions (i) and (ii) in Theorem 4.1 imply that there exists an \( X \)-valued polynomial \( P \) such that \( f - P \in \mathcal{S}'(\mathbb{R}^n, E) \). Furthermore,
\[
f \in \mathcal{I}_\varphi(\mathbb{R}^n, E),
\]
where \( I_\varphi \) is the ideal generated by the homogeneous terms of the Taylor polynomial of \( \hat{\varphi} \) at the origin (cf. (2.1) and (2.2)).
Proof. It remains to verify that $f \in \mathcal{S}_{I_\varphi}(\mathbb{R}^n, E)$. By Theorem 4.1 we see that if $P$ is an $X$-valued polynomial such that $f - P \in \mathcal{S}'(\mathbb{R}^n, E)$, then the polynomials $P_q(\partial/\partial t)P$ are $E$-valued for all $q \in \mathbb{N}$. We must show that if $\phi \in \mathcal{S}_{I_\varphi}(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} \phi(t)P(t)\,dt \in E$. Let $N$ be the degree of $P$. There are polynomials $Q_q$ such that
\[
T_N(u) = \sum_{|m| \leq N} \frac{\hat{\phi}(m)(0)}{m!} u^m = \sum_{q=0}^{N} Q_q(u)P_q^\varphi(u).
\]
So,
\[
\int_{\mathbb{R}^n} \phi(t)P(t)\,dt = \left\langle P(-i \frac{\partial}{\partial u})\delta(u), \tilde{\phi}(u) \right\rangle = \sum_{q=0}^{N} \left\langle P(-i \frac{\partial}{\partial u})\delta(u), Q_q(u)P_q^\varphi(u) \right\rangle = \sum_{q=0}^{N} (-i)^q Q_q \left( -i \frac{\partial}{\partial u} \right) \left( P_q^\varphi \left( \frac{\partial}{\partial u} \right) P \right)(0) \in E.
\]

Note that the degree of the polynomial $P$ occurring in Corollary 4.2 may depend only on $f$, and not on the test function. However, when the Taylor polynomials of $\tilde{\varphi}$ possess a rich algebraic structure, it is possible to say more about the degree of $P$. Recall that $P_d(\mathbb{R}^n)$ was defined in §2.1.

Corollary 4.2. Let the hypotheses of Corollary 4.1 be satisfied. If there exists $d \in \mathbb{N}$ such that $P_d(\mathbb{R}^n)$ is contained in the ideal generated by the polynomials $P_0^\varphi, \ldots, P_d^\varphi$, then there exists an $X$-valued polynomial $P$ of degree at most $d - 1$ such that $f - P \in \mathcal{S}'(\mathbb{R}^n, E)$. In particular, $f \in \mathcal{S}'_{P_d}(\mathbb{R}^n, E)$.

Proof. By Corollary 4.1, there exists an $X$-valued polynomial $\tilde{P}(t) = P(t) + \sum_{d \leq |m| \leq N} w_m t^m$ such that $f - \tilde{P} \in \mathcal{S}'(\mathbb{R}^n, E)$ and $P$ has degree at most $d - 1$. Then we must show that $w_m \in E$ for $d \leq |m| \leq N$. But Corollary 4.1 also implies that $Q(\partial/\partial t)\tilde{P}$ is an $E$-valued polynomial for any $Q \in I_\varphi$, and since $P_d(\mathbb{R}^n) \subseteq I_\varphi$, we see at once that $w_m = m!((\partial |m|/\partial t^n)\tilde{P})(0) \in E$, for $d \leq |m| \leq N$.

In general, it is not possible to replace the $X$-valued entire function $G$ by an $X$-valued polynomial in Theorem 4.1. However, we know some valuable information about $\hat{G}$. Since it is supported by the origin, we have already observed that
\[
\hat{G} = \sum_{m \in \mathbb{N}^n} \frac{(-1)^{|m|} \delta(m)}{m!} \mu_m(\hat{G}),
\]
where the vectors $\mu_m(\hat{G}) = \langle \hat{G}(u), u^m \rangle \in X$ are actually its moments and the series is convergent in $\mathcal{S}'(\mathbb{R}^n, X)$. This series is ‘weakly finite’, in the sense that for each $w^* \in X'$ there exists $N_{w^*} \in \mathbb{N}$ such that
\[
\langle w^*, \langle \hat{G}, \rho \rangle \rangle = \sum_{|m| \leq N_{w^*}} \frac{\rho(m)(0)}{m!} \langle w^*, \mu_m(\hat{G}) \rangle \quad \text{for all } \rho \in \mathcal{S}(\mathbb{R}^n).
\]
Furthermore, given any continuous seminorm $p$ on $X$, we can find an $N_p \in \mathbb{N}$ such that
\[ p\left( (\mathcal{G}, \rho) - \sum_{|m| \leq N_p} \frac{\rho^{(m)}(0)}{m!} \mu_m(\mathcal{G}) \right) = 0 \]
for all $\rho \in \mathcal{S}(\mathbb{R}^n)$. Finally, as we have also mentioned, its inverse Fourier transform $\mathcal{G}$ can be naturally identified with an entire $X$-valued function.

**Example 4.1.** We consider $X = C(\mathbb{R})$ and $E = C_b(\mathbb{R})$, the space of continuous bounded functions. Let $\chi_q \in C(\mathbb{R})$ be nontrivial and such that $\text{supp} \chi_q \subset (q, q+1)$, $q \in \mathbb{N}$. Furthermore, for each $q \in \mathbb{N}$ find a harmonic homogeneous polynomial $Q_q$ of degree $q$, that is, $\Delta Q_q = 0$. Consider the $E$-valued distribution
\[ \mathcal{G}(t, \xi) = \sum_{q=0}^{\infty} Q_q(t) \chi_q(\xi) \in \mathcal{S}'(\mathbb{R}^n_+; C(\mathbb{R}_\xi)) \setminus \mathcal{S}'(\mathbb{R}^n_+; C_b(\mathbb{R}_\xi)). \]
Its Fourier transform is given by an infinite multipole series supported at the origin, that is,
\[ \hat{\mathcal{G}}(u, \xi) = (2\pi)^n \sum_{q=0}^{\infty} \left( Q_q \left( i \frac{\partial}{\partial u} \right) \delta \right)(u) \chi_q(\xi). \]
Let $h \in \mathcal{S}'(\mathbb{R}^n, C_b(\mathbb{R}))$ and let $\varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \mathcal{S}_0(\mathbb{R}^n)$ be a nondegenerate test function such that, for all $N > 2$, its Fourier transform satisfies $\hat{\varphi}(u) = |u|^2 + O(|u|^N)$ as $u \to 0$. If $f = h + \mathcal{G} \in \mathcal{S}'(\mathbb{R}^n, C(\mathbb{R}))$, then $M^f_{\varphi}(x, y) = M^h_{\varphi}(x, y)$ for all $(x, y) \in \mathbb{H}^{n+1}$. Thus, $f$ satisfies all the hypotheses of Theorem 4.1; however, there is no $C(\mathbb{R})$-valued polynomial $P$ such that $f - P \in \mathcal{S}'(\mathbb{R}^n, C_b(\mathbb{R}))$.

The occurrence of the correction term $G$ in Theorem 4.1 can be eliminated if we augment the hypotheses by involving a convolution average of $f$ with respect to another test function as follows. We then obtain a characterization of $\mathcal{S}'(\mathbb{R}^n, E)$.

**Theorem 4.2.** Let $f \in \mathcal{S}'(\mathbb{R}^n, X)$, let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be nondegenerate, and let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \varphi_0(t) \, dt \neq 0$. Then $f \in \mathcal{S}'(\mathbb{R}^n, E)$ if and only if the following three conditions hold.

(i) $M^f_{\varphi}(x, y)$ takes values in $E$ for almost all $(x, y) \in \mathbb{H}^{n+1}$ and is measurable as an $E$-valued function.

(ii) There exist $k, l \in \mathbb{N}$ such that (4.8) holds.

(iii) There is $l \in \mathbb{N}$ such that $(f * \varphi_0)(x) \in E$ a.e., it is measurable, and satisfies
\[ \int_{\mathbb{R}^n} (1 + |x|)^{-l} ||(f * \varphi_0)(x)|| \, dx < \infty. \] (4.10)

**Proof.** We apply Theorem 4.1 to obtain $G \in \mathcal{S}'(\mathbb{R}^n, X)$ such that $f - G \in \mathcal{S}'(\mathbb{R}^n, E)$ and $\text{supp} \hat{G} \subset \{0\}$. Using (iii), we conclude that, almost everywhere, $(G * \varphi_0)(x) \in E$ and that this function defines an element of $\mathcal{S}'(\mathbb{R}^n, E)$ as a tempered $E$-valued function. Let $\sigma > 0$ be sufficiently small so that $|\hat{\varphi}(u)| > 0$.
for $|u| < \sigma$. Since $\text{supp} \, \widehat{G} \subset \{0\}$, it suffices to show that $\langle G, \rho \rangle \in E$ for each $\rho \in \mathcal{D}(B(0, \sigma))$, in such a case we have $G \in \mathcal{S}'(\mathbb{R}^n, E)$. Now, setting $\hat{\chi} = \rho/\hat{\varphi}_0$,

$$\langle G, \rho \rangle = \langle \hat{G}, \hat{\chi} \cdot \varphi_0 \rangle = (2\pi)^n \left\langle \hat{G}(t), \int_{\mathbb{R}^n} \chi(-\xi) \varphi_0(\xi - t) \, d\xi \right\rangle$$

$$= (2\pi)^n \int_{\mathbb{R}^n} \chi(-\xi) (G * \varphi_0)(\xi) \, d\xi \in E,$$

where the interchange with the integral sign can be established as in the proof of Proposition 4.1 and the very last integral is taken in the Bochner sense.

Note if $\varphi$ is such that $\int_{\mathbb{R}^n} \varphi(t) \, dt \neq 0$, then we can use $\varphi = \varphi_0$ in Theorem 4.2 and condition (iii) becomes part of (ii); a stronger result is, however, stated below in Corollary 4.3.

4.2. Local class estimates. We now proceed to study local class estimates, namely, the case when (4.1) is only assumed to hold for $(x, y) \in \mathbb{R}^n \times (0, 1]$. We again work with an integral condition instead of a pointwise bound in order to gain generality. We start by pointing out that $M^f_\varphi(x, y)$ may sometimes be trivial for $y \in (0, 1)$, this may happen even when $\varphi$ is nondegenerate.

Example 4.2. Let $\omega \in S^{n-1}$, $r \in \mathbb{R}_+$; set $[0, r\omega] = \{\sigma \omega : \sigma \in [0, r]\}$. Suppose that $f \in \mathcal{S}'(\mathbb{R}^n, X)$ is such that $\text{supp} \, \hat{f} \subset [0, r\omega]$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is any test function satisfying $\text{supp} \, \hat{\varphi} \subset \mathbb{R}^n \setminus [0, r\omega]$; then

$$M^f_\varphi(x, y) = \frac{1}{(2\pi)^n} \left\langle \hat{f}(u), e^{izu} \varphi(yu) \right\rangle = 0 \quad \text{for all } y \in (0, 1).$$

Fortunately, it turns out that the only distributions $f \in \mathcal{S}'(\mathbb{R}^n, X)$ that can satisfy a local class estimate with respect to a nondegenerate test function are, modulo elements of $\mathcal{S}'(\mathbb{R}^n, E)$, those whose Fourier transforms are compactly supported. We will prove this below.

We need to introduce some terminology in order to move further on. We will make use of weak integrals (Pettis integrals) for $X$-valued functions as defined, for example, in [17], Ch. 3. We say that a tempered $X$-valued distribution $g \in \mathcal{S}'(\mathbb{R}^n, X)$ is weakly regular if there exists an $X$-valued function $\tilde{g}$ such that $\rho \tilde{g}$ is weakly integrable over $\mathbb{R}^n$ for all $\rho \in \mathcal{S}(\mathbb{R}^n)$ and

$$\langle g, \rho \rangle = \int_{\mathbb{R}^n} \rho(t) \tilde{g}(t) \, dt \in X,$$

where the last integral is taken in the weak sense. We identify $g$ with $\tilde{g}$, so, as usual, we write $g = \tilde{g}$.

We recall some facts about (vector-valued) compactly supported distributions. Let $g \in \mathcal{S}'(\mathbb{R}^n, X)$ have support in $B(0, r)$, the closed ball of radius $r$. Then the following version of the Schwartz-Paley-Wiener theorem holds:

$$G(z) = \langle g(u), e^{-iz \cdot u} \rangle, \quad z \in \mathbb{C}^n,$$
is an $X$-valued entire function which defines a weakly-regular tempered distribution, and $G(\xi) = \hat{g}(\xi)$, $\xi \in \mathbb{R}^n$; moreover, $G$ is of weakly exponential type, that is, for all $w^* \in X'$ we can find constants $C_{w^*} > 0$ and $N_{w^*} \in \mathbb{N}$ with
\[
|\langle w^*, G(z) \rangle| \leq C_{w^*}(1 + |z|)^{N_{w^*}}e^{r|\text{Im} z|}, \quad z \in \mathbb{C}^n.
\] (4.11)

Conversely, if $G$ is an $X$-valued entire function which defines a weakly regular tempered distribution and for all $w^* \in X'$ there exist $C_{w^*} > 0$ and $N_{w^*} \in \mathbb{N}$ such that (4.11) holds, then $\hat{G} = g$, where $g \in \mathcal{S}'(\mathbb{R}^n, X)$ and supp $g \subseteq \overline{B}(0, r)$.

The following concept for nondegenerate test functions is of much relevance for the problem under consideration.

**Definition 4.1.** Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be nondegenerate. Given $\omega \in S^{n-1}$, consider the function of one variable $R_\omega(r) = \varphi(r\omega) \in C^\infty[0, \infty)$. We define the index of nondegenerateness of $\varphi$ as the (finite) number
\[
\tau = \inf\{r \in \mathbb{R}_+: \text{supp} R_\omega \cap [0, r] \neq \emptyset \forall \omega \in S^{n-1}\}.
\]

We are ready to state and prove the main Tauberian result in this subsection. We shall consider slightly more general norm estimates for the regularizing transform $M^f_\varphi$, in terms of functions $\Psi : \mathbb{R}^n \times (0, 1] \rightarrow \mathbb{R}_+$ which, for all $x, \xi \in \mathbb{R}^n$ and $y \in (0, 1]$, satisfy
\[
\Psi(0, y) \geq C_1 y^k \quad \text{and} \quad \Psi(x + \xi, y) \leq C_2 \Psi(x, y)(1 + |\xi|)^l
\] (4.12)
for some constants $C_1, C_2 > 0$ and $k, l \in \mathbb{N}$. In the next theorem $L^{p, p'}_{\varphi}(\{0, 1\} \times \mathbb{R}^n, E)$ stands for the mixed $L^{p, p'}$-space of $E$-valued functions on $(0, 1] \times \mathbb{R}^n$ with respect to the weight $\Psi$, that is, the space of $E$-valued measurable functions $F$ such that
\[
\int_0^1 \left( \int_{\mathbb{R}^n} (\|F(x, y)\|\Psi(x, y))^p \, dx \right)^{p'/p} \frac{dy}{y} < \infty.
\]
The parameters are assumed to satisfy $p, p' \in [1, \infty]$.

**Theorem 4.3.** Let $f \in \mathcal{S}'(\mathbb{R}^n, X)$ and let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a nondegenerate test function with index of nondegenerateness $\tau$. Assume that:

(i) $M^f_\varphi(x, y)$ takes values in $E$ for almost all $(x, y) \in \mathbb{R}^n \times (0, 1]$ and is measurable as an $E$-valued function on $\mathbb{R}^n \times (0, 1]$;

(ii) there is a function $\Psi : \mathbb{R}^n \times (0, 1]$ that satisfies (4.12) and such that $M^f_\varphi \in L^{p, p'}_{\Psi}((0, 1] \times \mathbb{R}^n, E)$.

Then, for any $r > \tau$, there exists an $X$-valued entire function $G$, which defines a weakly-regular tempered $X$-valued distribution and satisfies (4.11), such that $\text{F} - G \in \mathcal{S}'(\mathbb{R}^n, E)$.

Furthermore, $M^f_\varphi - G \in L^{p, p'}_{\Psi}((0, 1] \times \mathbb{R}^n, E)$ and we can choose $G$ so that $\hat{G} = \chi \hat{f}$, where $\chi \in \mathcal{D}(\mathbb{R}^n)$ is an arbitrary test function that satisfies $\chi(t) = 1$ for $|t| \leq \tau$ and has support contained in the ball of radius $r$ and centre at the origin.
Proof. Let \( r_1 \) be such that \( \tau < r_1 < r \). It is easy to find a reconstruction wavelet \( \eta \in \mathcal{S}_0(\mathbb{R}^n) \) for \( \varphi \), in the sense that
\[
1 = \int_0^\infty \hat{\varphi}(r\omega)\hat{\eta}(r\omega) \frac{dr}{r} \quad \text{for every } \omega \in \mathbb{S}^{n-1},
\]
with the property \( \text{supp } \hat{\eta} \subset B(0,r_1) \). Indeed, if we choose a nonnegative \( \kappa \in \mathcal{S}(\mathbb{R}^n) \) with support in \( B(0,r_1) \setminus \{0\} \) and equal to 1 in a neighbourhood of the sphere \( \tau \mathbb{S}^{n-1} = \{ u \in \mathbb{R}^n : |u| = \tau \} \), then the same argument given in the proof of Proposition 5.1 in [16] shows that
\[
\hat{\eta}(x) = \frac{\kappa(x)\hat{\varphi}(x)}{\int_0^\infty \kappa(rx)|\hat{\varphi}(rx)|^2 \, dr/r}
\]
fulfills the requirements. The usual calculation (see [11], Ch. 1, §14) is valid and so, for \( \rho \in \mathcal{S}_0(\mathbb{R}^n) \),
\[
\rho(t) = \int_0^\infty \int_{\mathbb{R}^n} \frac{1}{y^n} \varphi \left( \frac{x-t}{y} \right) W_{\pi}\rho(x,y) \, dx \, dy.
\]
Now observe that if \( \text{supp } \hat{\rho} \subset \mathbb{R}^n \setminus B(0,r_1) \), then
\[
W_{\pi}\rho(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot u} \hat{\rho}(u)\hat{\eta}(-yu) \, du = 0 \quad \text{for all } y \in [1, \infty).
\]
Thus, we can apply the argument employed in Proposition 4.1 to show that
\[
\langle f, \rho \rangle = \int_0^1 \int_{\mathbb{R}^n} M^f_{\varphi}(x,y)W_{\pi}\rho(x,y) \, dx \, dy \quad (4.13)
\]
for all \( \rho \in \mathcal{S}(\mathbb{R}^n) \) with \( \text{supp } \hat{\rho} \subset \mathbb{R}^n \setminus B(0,r_1) \). Choose \( \chi_1 \in C^\infty(\mathbb{R}^n) \) such that \( \chi_1(u) = 1 \) for all \( u \in \mathbb{R}^n \setminus B(0,r) \) and \( \text{supp } \chi_1 \subset \mathbb{R}^n \setminus B(0,r_1) \). Now, \( \chi_1 * f \) is well defined since \( \chi_1 \in \mathcal{S}'(\mathbb{R}^n) \) (the space of convolutors), and actually (4.13) and the continuity of \( W_{\pi} \) imply that \( (2\pi)^{-n} \chi_1 * f \in \mathcal{S}'(\mathbb{R}^n, E) \). Therefore, \( G = f - (2\pi)^{-n} \chi_1 * f \) satisfies the requirements because \( \hat{G} = \chi \hat{f} \), where \( \chi(\xi) = 1 - \chi_1(-\xi) \), and so \( \text{supp } \hat{G} \subset B(0,r) \). Since \( \hat{\chi}_1(\xi) = (2\pi)^n \delta(\xi) - \hat{\varphi}(-\xi) \), and so
\[
M^f_{\varphi} - G(x,y) = M^f_{\varphi}(x,y) - \frac{1}{(2\pi)^n} \left( f(\xi), \frac{1}{y^n} \varphi \left( \frac{x+t-\xi}{y} \right), \hat{\chi}(t) \right)
\]
\[
= M^f_{\varphi}(x,y) - \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} M^f_{\varphi}(x+t,y)\hat{\chi}(t) \, dt,
\]
we readily obtain the \( L^{p,p'} \)-norm estimate for \( M^f_{\varphi} - G \).

It is tempting to think that in Theorem 4.3 we could take \( G \) with support in \( B(0,\tau) \); however, this is not true in general, as the following counterexample shows.

Example 4.3. Let \( X, E \) and the sequence \( \{\chi_q\}_{q=1}^\infty \) be as in Example 4.1. We work in dimension \( n = 1 \). We assume additionally that \( \sup_{\xi} |\chi_q(\xi)| = 1 \) for all \( q \in \mathbb{N} \). Let
The wavelet $\psi$, given by $\hat{\psi}(u) = e^{-|u|^{-(1/(1-\tau))}}$ for $|u| > \tau$ and $\hat{\psi}(u) = 0$ for $|u| \leq \tau$, has index of nondegenerateness $\tau$. Consider the $C(\mathbb{R})$-valued distribution

$$f(t, \xi) = \sum_{q=1}^{\infty} e^{q+i(\tau+1/q)t} \chi_q(\xi) \in \mathcal{S}'(\mathbb{R}^t, C(\mathbb{R}) \setminus \mathcal{S}'(\mathbb{R}^t, C^b(\mathbb{R}))$$

Then

$$\mathcal{W}_\psi f(x, y)(\xi) = \sum_{1 \leq q < y/(\tau(1-y))} \exp\left(q + (ix - y)\left(\tau + \frac{1}{q}\right) - \frac{q}{y - q\tau(1-y)}\right) \chi_q(\xi), \quad 0 < y < 1,$$

and hence $\|\mathcal{W}_\psi f(x, y)\|_{C^b(\mathbb{R})} \leq 1$ for all $0 < y < 1$. Therefore, the hypotheses of Theorem 4.3 are fully satisfied, however, $f - G \notin \mathcal{S}'(\mathbb{R}, C^b(\mathbb{R}))$ for any $G \in \mathcal{S}'(\mathbb{R}, C(\mathbb{R}))$ with supp $G \subseteq [-\tau, \tau]$.

We obtain a local class estimate characterization of $\mathcal{S}'(\mathbb{R}^n, E)$ if we combine Theorem 4.3 with the same argument as we employed in the proof of Theorem 4.2.

**Theorem 4.4.** Let $f \in \mathcal{S}'(\mathbb{R}^n, X)$, let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be nondegenerate with index of nondegenerateness $\tau$, and let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ be such that $\varphi_0(u) \neq 0$ for all $|u| \leq \tau$. Then, $f \in \mathcal{S}'(\mathbb{R}^n, E)$ if and only if the following three conditions hold.

(i) $M^f_\varphi(x, y)$ takes values in $E$ for almost all $(x, y) \in \mathbb{H}^{n+1}$ and is measurable as an $E$-valued function.

(ii) There exist $k, l \in \mathbb{N}$ such that

$$\int_0^1 y^k (1 + |x|)^{-l} \|M^f_\varphi(x, y)\| \, dx \, dy < \infty. \quad (4.14)$$

(iii) There is $l \in \mathbb{N}$ such that $(f * \varphi_0)(x) \in E$ almost everywhere, is measurable and condition (4.10) holds.

In particular, when $\int_\mathbb{R} \varphi(t) \, dt \neq 0$ the third hypothesis in Theorem 4.4 becomes superfluous and we therefore have the following corollary.

**Corollary 4.3.** Let $f \in \mathcal{S}'(\mathbb{R}^n, X)$ and let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\int_\mathbb{R} \varphi(t) \, dt \neq 0$. Then $f \in \mathcal{S}'(\mathbb{R}^n, E)$ if and only if conditions (i) and (ii) in Theorem 4.4 are satisfied.

§ 5. Strongly nondegenerate test functions

A strengthened version of both Theorem 4.1 and Theorem 4.3 holds if we restrict the nondegenerate test functions to those fulfilling the requirements of the following definition.

**Definition 5.1.** Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. We call $\varphi$ strongly nondegenerate if there exist constants $N \in \mathbb{N}$, $r > 0$ and $C > 0$ such that

$$C |u|^N \leq |\widehat{\varphi}(u)| \quad \text{for all } |u| \leq r. \quad (5.1)$$
The class of test functions in Definition 5.1 turns out to be the same as that employed by Drozhzhinov and Zav’yalov in [9], but they formulated their notion of nondegenerateness in terms of the Taylor polynomials of $\tilde{\varphi}$. We say that a polynomial $P$ is nondegenerate (at the origin, in the Drozhzhinov-Zav’yalov sense) if for each $\omega \in S^{n-1}$

$$P(r\omega) \neq 0, \quad r \in \mathbb{R}_+.$$ 

Drozhzhinov and Zav’yalov then considered the class of test functions $\varphi \in \mathcal{S}(\mathbb{R}^n)$ for which there exists $N \in \mathbb{N}$ such that

$$\sum_{|m| \leq N} \frac{\tilde{\varphi}(m)(0)u^m}{m!}$$

is a nondegenerate polynomial. It is easy to see that the latter property introduced by Drozhzhinov and Zav’yalov is equivalent to strong nondegenerateness in the sense of Definition 5.1. It should also be noticed that strong nondegenerateness is included in Definition 2.1, but, of course, Definition 2.1 gives many more test functions (cf. [16], Remark 4.3).

We can now state our first result concerning strongly nondegenerate test functions.

**Theorem 5.1.** Let $f \in \mathcal{S}'(\mathbb{R}^n, X)$ and let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be strongly nondegenerate. Assume that:

(i) $M^f(x,y)$ takes values in $E$ for almost all $(x,y) \in \mathbb{R}^n \times (0,1]$ and is measurable as an $E$-valued function on $\mathbb{R}^n \times (0,1]$;

(ii) there are $k,l \in \mathbb{N}$ such that (4.14) holds.

Then there exists $G \in \mathcal{S}'(\mathbb{R}^n, X)$ such that $f - G \in \mathcal{S}'(\mathbb{R}^n, E)$ and $\supp \tilde{G} \subseteq \{0\}$. Moreover, relations (4.9) hold.

If, in addition, condition (iii) in Theorem 4.4 holds for some test function $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \varphi_0(t) \, dt \neq 0$, then $f \in \mathcal{S}'(\mathbb{R}^n, E)$.

**Proof.** By Theorem 4.3, we can assume that $\sup \tilde{f} \subseteq B(0,1)$. Let $\rho \in \mathcal{S}(\mathbb{R}^n)$ such that $\rho(u) = 1$ for $u \in B(0,3/2)$ and $\sup \rho \subseteq B(0,2)$. We can find $\sigma, C_1 > 0$ and $N \in \mathbb{N}$ such that $2\sigma \leq 1$ and $C_1 |u|^N \leq |\tilde{\varphi}(u)|$ for all $u \in B(0,2\sigma)$. Given $\tilde{n} \in \mathcal{S}_0(\mathbb{R}^n)$, then $\tilde{\chi}(u) = \tilde{\chi}_n(u) = \rho(u)\eta(u)/\tilde{\varphi}(\sigma u)$ defines an element of $\mathcal{S}(\mathbb{R}^n)$ in a continuous fashion; consequently, the mapping $\gamma: \mathcal{S}_0(\mathbb{R}^n) \to [0,\infty)$ given by

$$\gamma(\tilde{n}) = (2\pi)^n \int_{\mathbb{R}^n} (1 + |\xi|^l)|\chi(\xi)| \, d\xi$$

is a continuous seminorm over $\mathcal{S}_0(\mathbb{R}^n)$. Now, for any $\tilde{n} \in \mathcal{S}_0(\mathbb{R}^n)$,

$$\langle f, \tilde{n} \rangle = \langle \tilde{f}(u), \tilde{\chi}(u)\tilde{\varphi}(\sigma u) \rangle = (2\pi)^n \int_{\mathbb{R}^n} \chi(-\xi)M^f(\xi, \sigma) \, d\xi.$$ 

Therefore, $\|\langle f, \eta \rangle\| \leq (C/\sigma^k)\gamma(\tilde{n})$ for all $\tilde{n} \in \mathcal{S}_0(\mathbb{R}^n)$, and this implies that the restriction of $f$ to $\mathcal{S}_0(\mathbb{R}^n)$ belongs to $\mathcal{S}'_0(\mathbb{R}^n, E)$. The argument we gave in the proof of Theorem 4.1 yields the existence of $G$ satisfying all the requirements. That (4.9) holds for each $q$ is shown exactly as in the proof of Theorem 4.1. Finally, the proof of the last assertion of the theorem is the same as that of Theorem 4.2.
In dimension \( n = 1 \) there is no distinction between nondegenerateness and strong nondegenerateness, provided we consider test functions from \( \mathcal{S}(\mathbb{R}) \setminus \mathcal{S}_0(\mathbb{R}) \). Actually, a stronger result than Theorem 5.1 holds in the one-dimensional case.

**Proposition 5.1.** Suppose that \( f \in \mathcal{S}'(\mathbb{R}, X) \) and let \( \varphi \in \mathcal{S}(\mathbb{R}) \) be such that
\[
\int_{-\infty}^{\infty} t^d \varphi(t) \, dt \neq 0, \text{ for some } d \in \mathbb{N}.
\]
If conditions (i) and (ii) in Theorem 5.1 are satisfied, then there exists an \( X \)-valued polynomial \( P \) of degree at most \( d - 1 \) such that \( f - P \in \mathcal{S}'(\mathbb{R}, E) \).

**Proof.** We assume that \( d \) is the smallest integer with the assumed property. There exists \( \phi \in \mathcal{S}(\mathbb{R}) \) such that \( \phi^{(d)} = (-1)^d \varphi \) and
\[
\int_{-\infty}^{\infty} \phi(t) \, dt \neq 0.
\]
Then \( M_{\phi}^{(d)}(x, y) = y^{-d} M_{\phi}(x, y) \). Hence an application of Corollary 4.3 gives that \( f^{(d)} \in \mathcal{S}'(\mathbb{R}, E) \), and this clearly implies the existence of a polynomial \( P \) with the desired properties.

Observe that the conclusion of Proposition 5.1 does not hold for multidimensional regularizing transforms in general, even if the kernels \( \varphi \) are strongly nondegenerate. This is shown by Example 4.1. Of course, if \( X \) in Theorem 5.1 is a normed space, then \( G \) must be an \( X \)-valued polynomial; this fact is stated in the next corollary. Corollary 5.1 extends an important result of Drozhzhinov and Zav’yalov’s (see [9], Theorem 2.1).

**Corollary 5.1.** Let the hypotheses of Theorem 5.1 be satisfied. If \( X \) is a normed space, then there is an \( X \)-valued polynomial \( P \) such that \( f - P \in \mathcal{S}'(\mathbb{R}, E) \). Furthermore, \( f \in \mathcal{S}'_f(\mathbb{R}^n, E) \).

As in Corollary 4.2, we can recover the following result of Drozhzhinov and Zav’yalov’s (cf. Theorem 2.2 in [9]).

**Corollary 5.2.** Let the hypotheses of Corollary 5.1 be satisfied. If there is \( d \in \mathbb{N} \) such that \( \mathbb{P}_d(\mathbb{R}^n) \) is contained in the ideal generated by the polynomials \( P_0^\varphi, P_1^\varphi, P_2^\varphi, \ldots, P_d^\varphi \), then there exists an \( X \)-valued polynomial \( P \) of degree at most \( d - 1 \) such that \( f - P \in \mathcal{S}'(\mathbb{R}^n, E) \). In particular, \( f \in \mathcal{S}'_{P_d}(\mathbb{R}^n, E) \).

§ 6. Vector-valued distributions intertwining representations of \( \mathbb{R}^n \)

As an application of our ideas, we extend the one-dimensional considerations from [8], § 4, to the multidimensional case. Throughout this section we suppose that the Banach space \( E \) is continuously and linearly embedded into the locally convex space \( X \) and that both carry a representation of \( (\mathbb{R}^n, +) \), that is, there is a mapping \( \pi : \mathbb{R}^n \to L_b(X) \) such that
\[
\begin{align*}
\pi(x + h) &= \pi(x)\pi(h), \text{ for all } x, h \in \mathbb{R}^n; \\
\pi(x)\mathbf{v} &\in E \text{ for every } \mathbf{v} \text{ and } x \in \mathbb{R}^n.
\end{align*}
\]
It follows from (b) and the closed graph theorem that, in fact, (the restriction of) \( \pi(x) \in L_b(E) \) and thus \( \pi \) induces a representation of \( \mathbb{R}^n \) on \( E \) as well. We further assume that \( \pi \) is a tempered \( C_0 \)-group of operators on \( E \), namely,
\[
\begin{align*}
\text{(c)} \text{ there exist } l \text{ and } C \text{ such that } ||\pi(x)||_{L_b(E)} &\leq C(1 + |x|)^l, \ x \in \mathbb{R}^n; \\
\text{(d)} \text{ } \text{lim}_{x \to 0} ||\pi(x)\mathbf{v} - \mathbf{v}|| &\to 0 \text{ for each } \mathbf{v} \in E.
\end{align*}
\]
We denote the translation operators on $\mathbb{R}^n$ by $T_h$, so that their actions on functions (and vector-valued distributions) are given by $(T_h \phi)(x) = \phi(x - h)$. We then say that $f \in \mathcal{S}'(\mathbb{R}^n, X)$ intertwines $\pi$ and $T$ if $\pi(x) \circ f = T_{-x} f = f \circ T_x$ for each $x \in \mathbb{R}^n$, that is, if

$$\pi(x)((f, \varphi)) = (f, T_x \varphi) \quad \text{for every } \varphi \in \mathcal{S}(\mathbb{R}^n) \text{ and } x \in \mathbb{R}^n.$$ 

We shall also need the notion of regularly varying functionals, introduced and studied by Drozhzhinov and Zav’yalov in [8], §2. Let $\mathcal{J}$ be a nonnegative functional acting on nonnegative measurable functions $g: (0, 1] \to [0, \infty]$. It is notationally convenient to employ a dummy variable of evaluation and write $\mathcal{J}(g) = \mathcal{J}_y(g(y))$. The functional is called regularly varying of index $\alpha$ if the following five conditions are satisfied.

(I) $\mathcal{J}_y \left( \int g(y, \xi) \, d\xi \right) \leq \int \mathcal{J}_y(g(y, \xi)) \, d\xi$ for all nonnegative measurable function $g(y, \xi)$.

(II) $\mathcal{J}$ is monotone, $\mathcal{J}(g_1) \leq \mathcal{J}(g_2)$ whenever $g_1(y) \leq g_2(y)$ a.e.

(III) $\mathcal{J}$ is homogeneous, $\mathcal{J}(\lambda g) = \lambda \mathcal{J}(g)$, $\lambda \geq 0$.

(IV) $\mathcal{J}$ has the monotone convergence property, that is, $\mathcal{J}(g_k) \to \mathcal{J}(g)$ as $k \to \infty$, whenever $g_k(y) \nearrow g(y)$ almost everywhere as $k \to \infty$.

(V) For every $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$\mathcal{J}_y(g(ay)) \leq \begin{cases} C_\varepsilon a^{\alpha + \varepsilon} \mathcal{J}_y(g(y)) & \text{if } a \geq 1, \\ C_\varepsilon a^{\alpha - \varepsilon} \mathcal{J}_y(g(y)) & \text{if } a \leq 1 \end{cases}$$

for every nonnegative measurable function with support in $(0, 1]$.

**Example 6.1.** A typical example of such a $\mathcal{J}$ is given by a weighted $L^q$ norm $(1 \leq q \leq \infty)$ with respect to a Karamata regularly varying function. Let $c \in L^\infty_{\text{loc}}(0, 1]$ be regularly varying at 0 of index $\alpha$, that is, a positive measurable function that satisfies

$$\lim_{y \to 0^+} \frac{c(ay)}{c(y)} = a^\alpha, \quad a > 0.$$ 

In view of Potter’s estimates (see [1], Ch. 1), the functional

$$\mathcal{J}^{q, c}(g) = \left( \int_0^1 \left( \frac{g(y)}{c(y)} \right)^q \frac{dy}{y} \right)^{1/q}$$

(with the obvious adjustments when $p = \infty$) defines a regularly varying functional of index $\alpha$. When $c(y) = y^\alpha$, we simply write $\mathcal{J}^{q, c} = \mathcal{J}_q^{q, \alpha}$.

We also need the following definition.

**Definition 6.1.** Let $\alpha \in \mathbb{R}$, $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$, and let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a nondegenerate test function with index of nondegenerateness $\tau$. The pair $(\varphi_0, \varphi)$ is said to be a Littlewood-Paley pair (LP-pair) of order $\alpha$ if $\hat{\varphi}_0(u) \neq 0$ for $|u| \leq \tau$ and $\varphi \in \mathcal{F}_{|\alpha|}(\mathbb{R}^n)$, that is, $\int_{\mathbb{R}^n} t^m \varphi(t) \, dt = 0$ for all multi-indices $m \in \mathbb{N}^n$ with $|m| \leq |\alpha|$. (Note that if $\alpha < 0$, the latter condition on the moments of $\varphi$ is void and thus simply becomes $\varphi \in \mathcal{S}(\mathbb{R}^n)$.)
We also point out that every regularly varying functional $\mathcal{J}$ of index $\alpha$ satisfies inequalities
\[ \mathcal{J}^{1,\beta}(g) \leq C_\beta \mathcal{J}(g) \]  
for some $C_\beta > 0$ if $\beta < \alpha$ (see [8], Lemma 2.5).

**Theorem 6.1.** Let $f \in \mathcal{S}'(\mathbb{R}^n, X)$ intertwine the representation $\pi$ and the translation group $T$, let $\mathcal{J}$ be a regularly varying functional of index $\alpha \in \mathbb{R}$, and let $(\varphi_0, \varphi)$ be an LP-pair of order $\alpha$. Assume that $\langle f, \varphi_0 \rangle \in E$ and that $M^f_\varphi(0, y) \in E$ for almost all $y \in (0, 1]$ and is measurable as an $E$-valued function on $(0, 1]$, and
\[ \mathcal{J}_y(\|M^f_\varphi(0, y)\|) < \infty. \]  

Then, $f \in \mathcal{S}'(\mathbb{R}^n, E)$ and there is a continuous seminorm $\gamma$ on $\mathcal{S}(\mathbb{R}^n)$, independent of $f$, such that
\[ \|\langle f, \rho \rangle\| + \mathcal{J}_y(\|M^f_\varphi(0, y)\|) \leq (\|\langle f, \varphi_0 \rangle\| + \mathcal{J}_y(\|M^f_\varphi(0, y)\|))(\gamma(\rho) + \gamma(\theta)) \]  
for every $\rho \in \mathcal{S}(\mathbb{R}^n)$ and $\theta \in \mathcal{S}_{\mathbb{P}^\alpha}(\mathbb{R}^n)$.

**Proof.** Note that
\[ M^f_\varphi(x, y) = \pi(x)(M^f_\varphi(0, y)), \]  
and using this we can check that condition (i) in Theorem 4.4 is satisfied, as implied by the assumption (d) on $\pi$. Using (6.2), property (c) of $\pi$ and (6.1) we see that hypothesis (ii) of Theorem 4.4 holds. Furthermore, the function $f \ast \varphi_0 \in C(\mathbb{R}^n, E)$ and satisfies condition (iii) in Theorem 4.4. Consequently, $f \in \mathcal{S}'(\mathbb{R}^n, E)$.

In order to find $\gamma$ such that (6.3) holds, we first consider
\[ L^\mathcal{J}((0, 1], E) = \{g: (0, 1] \rightarrow E: \|g(\lambda)\|_E \text{ is measurable and } \mathcal{J}_\lambda(\|g(\lambda)\|_E) < \infty\} \]  
(and analogously for other regularly varying functionals); it is a Banach space (cf. [8], Remark 2.1). Obviously, there exists $\beta \in \mathbb{R}$ such that
\[ \int_0^1 \|\langle f(\lambda t), \phi(t) \rangle\| \frac{d\lambda}{\lambda^{\beta+1}} < \infty \text{ for each } \phi \in \mathcal{S}(\mathbb{R}^n). \]  
This means that the vector-valued distribution $F$ given by
\[ (F(t), \phi(t))(\lambda) = \langle f(\lambda t), \phi(t) \rangle \]  
takes values in the Banach space $L^{1,\beta}(\mathbb{R}^{+}, E)$, with $L^{1,\beta}$ as defined in Example 6.1. We can assume that $\beta < \alpha$ so that $L^\mathcal{J}((0, 1], E)$ is continuously embedded into $L^{1,\beta}(\mathbb{R}^{+}, E)$, as follows from (6.1). Since
\[ \|M^F_\varphi(x, y)\|_{L^\mathcal{J}((0, 1], E)} = \mathcal{J}_\lambda(\|M^F_\varphi(\lambda x, \lambda y)\|) \leq (1 + |x|)^\beta \mathcal{J}_\lambda(\|M^F_\varphi(0, \lambda y)\|) \leq C_\varepsilon y^\beta \left( \frac{y^\varepsilon}{1 + y^\varepsilon} \right) (1 + |x|)^\beta \mathcal{J}_\lambda(\|M^F_\varphi(0, \lambda)\|), \]  
for $x, y \in \mathbb{H}^n$, Corollary 4.1 applies and we can deduce the existence of functions $c_m \in L^{1,\beta}((0, 1], E)$, $|m| \leq N$, such that
\[ \mathcal{J}_\lambda \left( \left\| \langle f(\lambda t), \phi(t) \rangle \right\| - \sum_{|m| \leq N} c_m(\lambda) \int_{\mathbb{R}^n} t^m \phi(t) \, dt \right) < \infty, \quad \phi \in \mathcal{S}(\mathbb{R}^n). \]  
(6.4)
We may also assume that each $c_m$ is bounded on $[1/2, 1]$. Fix $|m| \leq N$ and pick $\phi \in \mathcal{S}(\mathbb{R}^n)$ with \[ \int_{\mathbb{R}^n} t^j \phi(t) \, dt = \delta_{j,m}. \] Then, for each fixed $a > 0$

\[
\mathbf{c}_m(a) - a^{|m|} \mathbf{c}_m = \left( \left\langle \mathbf{f}(\lambda t), a^{-|n|} \phi \left( \frac{t}{a} \right) \right\rangle - \sum_{|m| \leq N} \mathbf{c}_m(a^{-|n|} \int_{\mathbb{R}^n} t^m \phi \left( \frac{t}{a} \right) \, dt) \right) - \left( \left\langle \mathbf{f}(a \lambda t), \phi(t) \right\rangle - \sum_{|m| \leq N} \mathbf{c}_m(a) \int_{\mathbb{R}^n} t^m \phi(t) \, dt \right) \in L^\mathcal{J}((0,1], E).
\]

We use these relations with a fixed multi-index $\alpha \leq |m| \leq N$ and $a = 1/2$, and write

\[
\mathbf{c}_m \left( \frac{\lambda}{2} \right) - 2^{-|m|} \mathbf{c}_m(\lambda) = \mathbf{b}(\lambda) \in L^\mathcal{J}((0,1], E). \tag{6.5}
\]

Iterating (6.5) $\nu$ times, we deduce that for each $\nu \in \mathbb{Z}_+$

\[
\mathbf{c}_m(\lambda) = 2^{|m|} \left( 2^{-\nu|m|} \mathbf{c}_m(2^\nu \lambda) + \sum_{j=1}^\nu 2^{-|m|j} \mathbf{b}(2^j \lambda) \right), \quad 0 < \lambda \leq 2^{-\nu}.
\]

Denote the characteristic function of a set $A$ by $\chi_A$ and pick $0 < \varepsilon < |m| - \alpha$. Then

\[
\mathcal{J}(\|\mathbf{c}_m\|) \leq \mathcal{J}(\chi(1/2,1]\|\mathbf{c}_m\|) + 2^{|m|} \sum_{\nu=1}^\infty 2^{-\nu|m|} \mathcal{J}(\chi(1/2,1]\|\mathbf{c}_m(2^\nu \lambda)\|) + 2^{|m|} \mathcal{J}(\chi) \left( \left\| \sum_{j=1}^\infty 2^{-j|m|} \mathbf{b}(2^j \lambda) \chi(0,2^{-j} \|\lambda\|) \right\| \right) \leq C \varepsilon \mathcal{J}(\chi(1/2,1]\|\mathbf{c}_m\|) + \mathcal{J}(\chi(0,1/2]\|\mathbf{b}\|) \sum_{j=1}^\infty 2^{-j(m-\alpha-\varepsilon)} < \infty,
\]

so that

\[
\mathbf{c}_m \in L^\mathcal{J}((0,1], E) \quad \text{for } \alpha < |m| \leq N. \tag{6.6}
\]

Combining (6.4) with (6.6), we now conclude that

\[
\mathcal{J}_y(\|M_\psi^\mathcal{F}(0,y)\|) < \infty \quad \text{for every } \psi \in \mathcal{F}_{\mathcal{F}[\alpha]}(\mathbb{R}^n). \tag{6.7}
\]

Our final step is to use (6.7) to infer the existence of the required seminorm $\gamma$. We define two normed spaces of $E$-valued distributions intertwining $\pi$ and $T$. The first of them is the space $Y$ consisting of all those $g$ such that

\[
\|g\|_Y = \|\langle g, \varphi_0 \rangle\| + \mathcal{J}_y(M_\psi^g(0,y)) < \infty. \tag{6.8}
\]

For the second space, we consider a fixed but arbitrary bounded set $B \subset \mathcal{S}(\mathbb{R}^n) \times \mathcal{F}_{\mathcal{F}[\alpha]}(\mathbb{R}^n)$ with the only requirement that $B$ contains the LP-pair $(\varphi_0, \varphi)$ and define $\tilde{Y}$ to be the space of those $g$ such that

\[
\|g\|_{\tilde{Y}} = \sup_{(\rho, \theta) \in B} (\|\langle g, \rho \rangle\| + \mathcal{J}_y(M_\psi^g(0,y)))) < \infty. \tag{6.9}
\]
A standard argument (see, for example, Assertion 6.2 in [8] or Proposition 5.4 in [14]) shows that both \( Y \) and \( \bar{Y} \) are Banach spaces. Relation (6.7) applied to an arbitrary element of \( Y \) and the Banach-Steinhaus theorem imply that \( Y = \bar{Y} \) as vector spaces. Since the identity mapping \( (Y, \| \cdot \|_Y) \to (Y, \| \cdot \|_Y) \) is obviously continuous, the open mapping theorem gives that the norms (6.8) and (6.9) are equivalent. Thus, again applying the Banach-Steinhaus theorem, we find that the bilinear mapping

\[
Y \times (\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}^1_0(\mathbb{R}^n)) \ni (g, (\rho, \theta)) \mapsto \langle (g, \rho), M^\gamma_0(0, \cdot) \rangle \in E \times L^2((0, 1], E)
\]

is continuous. This yields the existence of \( \gamma \) with the desired properties. The proof is complete.

**Example 6.2** (Besov spaces). Let \( (\varphi_0, \varphi) \) be an LP-pair of order \( s \), let \( p, q \in [1, \infty] \), and let \( c \in L^\infty_{\text{loc}}(0, 1] \) be a regularly varying function (at 0) of index \( s \). We define the Besov space \( B^{c}_{p,q}(\mathbb{R}^n) \) as the Banach space of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that \( f * \varphi_0 \in L^p(\mathbb{R}^n) \) and \( M^f_\varphi(\cdot, y) \in L^p(\mathbb{R}^n) \) for each \((0, 1] \) and

\[
\|f\|_{B^{c}_{p,q}(\mathbb{R}^n)} = \|f * \varphi_0\|_{L^p(\mathbb{R}^n)} + \mathbb{J}^q_{y,c}(\|M^f_\varphi(x, y)\|_{L^p(\mathbb{R}^n)} < \infty.
\]

When \( c(y) = y^s \), we recover the classical Besov space \( B^{s}_{p,q}(\mathbb{R}^n) = B^{c}_{p,q}(\mathbb{R}^n) \).

If we consider \( E = L^p(\mathbb{R}^n) \) and the vector-valued distribution \( f \) given by \( (f, \rho) = f * \hat{\rho} \), which takes values in \( X = L^p(\mathbb{R}^n, (1 + |x|^N) \, dx) \) for some sufficiently large \( N \), Theorem 6.1 yields that the definition of \( B^{c}_{p,q}(\mathbb{R}^n) \) is independent of \( \varphi_0, \varphi \) and that different choices of LP-pairs of order \( s \) lead to equivalent norms. Moreover, given arbitrary \( \rho \in \mathcal{S}(\mathbb{R}^n) \) and \( \varphi \in \mathcal{S}^1_0(\mathbb{R}^n) \), there are constants \( C_1 \) and \( C_2 \) such that

\[
\|f * \rho\|_{L^p(\mathbb{R}^n)} \leq C_1\|f\|_{B^{c}_{p,q}(\mathbb{R}^n)} \tag{6.10}
\]

and

\[
\mathbb{J}^q_{y,c}(\|M^f_\varphi(x, y)\|_{L^p(\mathbb{R}^n)}) \leq C_2\|f\|_{B^{c}_{p,q}(\mathbb{R}^n)} \tag{6.11}
\]

for all \( f \in B^{c}_{p,q}(\mathbb{R}^n) \). The constants \( C_1 \) and \( C_2 \) in these inequalities can be chosen to be the same if we let \( \rho \) and \( \varphi \) vary on bounded subsets of \( \mathcal{S}(\mathbb{R}^n) \). If \( s < 0 \), here we can take any \( \varphi \in \mathcal{S}(\mathbb{R}^n) \) without any assumption on its moments.

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\(^1\)When \( p = \infty \), we apply our arguments with \( E = UC(\mathbb{R}^n) \), the space of uniformly continuous functions; the \( L^\infty \) assumption itself implies \( f * \varphi_0 \in UC(\mathbb{R}^n) \) and \( M^f_\varphi(\cdot, y) \in UC(\mathbb{R}^n) \) for each \( y > 0 \) (see [6], Theorem 3).
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