An accelerated closed universe

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We study a model in which a closed universe with dust and quintessence matter components may look like an accelerated flat Friedmann-Robertson-Walker (FRW) universe at low redshifts. Several quantities relevant to the model are expressed in terms of observed density parameters, $\Omega_M$ and $\Omega_\Lambda$, and of the associated density parameter $\Omega_Q$ related to the quintessence scalar field $Q$.

I. INTRODUCTION

We still do not know the geometry of the universe. This question is intimately related to the amount of matter present in the universe. Observational evidences tell us that the measured matter density of baryonic and nonbaryonic components is less than one, i.e. its critical value. However, recent measurements of a type Ia distant supernova (SNe Ia)\textsuperscript{1, 2}, at redshift $z \sim 1$, indicate that in the universe there exists an important matter component that, in its most simple description, has the characteristic of the cosmological constant, i.e. a vacuum energy density which contributes to a large component of negative
pressure, and thus accelerates rather than decelerates the expansion of the universe. Other possible interpretations have been given for describing the astronomical data related to the accelerating expansion of the universe. We distinguish those related to topological defects from those which have received a great deal of attention today: the quintessence matter represented by a scalar field $Q^4$.

Various tests of the cosmological standard model, including spacetime geometry, peculiar galaxy velocities, structure formation, and very early universe descriptions (related to inflation and cosmic microwave background radiation), support a flat universe scenario. Specifically, the mentioned redshift-distance relation for supernova of type Ia, anisotropies in the cosmic microwave background radiation and gravitational lensing, all of them suggest that

$$\Omega_M + \Omega_\Lambda = 1.03^{+0.05}_{-0.04},$$

in which $\Omega_M = \left(\frac{8\pi G}{3H_0^2}\right) \rho_M^0$ and $\Omega_\Lambda = \frac{\Lambda}{3H_0^2}$, where $\rho_M^0$ and $H_0$ are the present values of the matter density and the Hubble parameter respectively. Here, $\Omega_\Lambda$ is the fraction of the critical energy density contained in a smoothly distributed vacuum energy referred to as a cosmological constant $\Lambda$, and $\Omega_M$ represents the matter density related to the baryonic and nobaryonic Cold Dark Matter (CDM) density. The constant $G$ represents the Newton constant and we have taken $c = 1$. In the following, all quantities that are evaluated at present time, i.e. at $t = t_0$, will be denoted by the subscript 0. Also, we will keep the value $c = 1$ for the speed of light throughout this paper.

On the light of these results an interesting question to ask is whether this flatness may be due to a sort of compensation among different components that enter into the dynamical equations. In this respect, our main goal in this paper is to address this sort of question by considering a simple model. In the literature we find some descriptions along these lines. For instance, a closed model has been studied with an important matter component whose equation of state is given by $p = -\rho/3$. Here, the universe expands at a constant speed. Other authors, while using the same astronomically observed properties for the universe, have added a nonrelativistic matter density in which the total matter density $\Omega_0$ is less than one, thus describing an open universe. Also, a flat decelerating universe model has been simulated. The common fact to all of these models is that, even though the starting geometry were other than that corresponding to the critical geometry, i.e. a flat geometry,
all these scenarios are, at low redshift, indistinguishable from a flat geometry, and none of them have included a cosmological constant.

In this paper we wish to consider a closed universe model composed of two matter components: one related to the usual dust matter and the other to quintessence matter. The geometry, together with these matter components, confabulates in such a special way that it gives rise to a flat accelerating universe scenario. The parameters of the resulting model could be fixed by using astronomical observations. Here the quintessence component is characterized by a scalar field $Q$ that satisfies the following equation of state:

$$P_Q = w_Q \rho_Q,$$

(2)

where, in general, the equation of state parameter $w_Q$ is assumed to be a time dependent quantity, and its present value runs in the range $-1 < w_Q < -1/3$. At this point we note that this range agrees with the values given to this parameter in the original quintessence model [4]. On the other hand, from the associated astronomical observations (related to the SNe Ia), consistence is obtained if the equation of state parameter $w_Q$ satisfies the bound $w_Q < -0.6$ at the present time [2]. However, new measurements may decrease the mathematical and/or statistical errors and thus this upper bound may increase (or decrease). In any case, in this paper we shall keep the range specified above, i.e., $-1 < w_Q < -1/3$.

When the scalar field $Q$ component is added to the relevant baryons (assumed to be described by $\Omega_b = 0.04 \pm 0.01$) and cold dark matter (contributing to the total mass with $\Omega_{CDM} = 0.30 \pm 0.10$) components, all added together these give a value for $\Omega_M = 0.34 \pm 0.11$ [11], and the resulting scenario is called a QCDM model, different from $\Lambda$CDM where, in place of the quintessence scalar field $Q$, is placed the cosmological constant $\Lambda$.

One of the goals that we have in mind in the present paper is to investigate the conditions under which a scenario with positive curvature may mimic an accelerating flat universe at low redshifts. This idea allows us to determinate the exact contribution of the scalar field $Q$ (together with the curvature term) that gives rise to an effective cosmological constant term. In this way we obtain an effective cosmological scenario that coincides with the accepted $\Lambda$CDM model.
II. THE FIELD EQUATIONS

In order to write the corresponding field equations we use the following effective Einstein action:

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} R + \frac{1}{2} (\partial_\mu Q)^2 - V(Q) + L_M \right]. \]  

(3)

Here, \( R \) is the scalar curvature, \( V(Q) \) is the scalar potential associated with the quintessence scalar field and \( L_M \) represents the matter components other than the \( Q \)-component.

Let us considering a FRW metric

\[ ds^2 = dt^2 - a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \]  

(4)

where \( a(t) \) represents the scale factor, and the curvature parameter \( k \) takes the values \( k = -1, 0, 1 \) corresponding to an open, flat and closed three-geometry, respectively. We shall assume that the \( Q \) field is homogeneous, i.e. is only a time-dependent quantity, and the geometry of the universe is taken to be closed, i.e. \( k = 1 \). With these assumptions we obtain from the action (3) the following Einstein field equations:

\[ H^2 = \frac{8\pi G}{3} \left( \rho_M + \rho_Q \right) - \frac{1}{a^2}, \]  

(5)

\[ \dot{H} + H^2 = -\frac{4\pi G}{3} (\rho_M + \rho_Q + 3P_M + 3P_Q), \]  

(6)

and the evolution equation for the scalar field \( Q \):

\[ \ddot{Q} + 3H \dot{Q} = -\frac{\partial V(Q)}{\partial Q}. \]  

(7)

Here the overdots stand for derivatives with respect to the time \( t \), \( H = \frac{\dot{a}}{a} \) defines the Hubble expansion rate, \( \rho_M \) is the effective matter energy density, and \( P_M \) is the pressure associated with this matter. \( \rho_Q \) and \( P_Q \) are the average energy density and average pressure related to \( Q \) which we define to be given by \( \rho_Q = \frac{1}{2} \dot{Q}^2 + V(Q) \) and \( P_Q = \frac{1}{2} \dot{Q}^2 - V(Q) \).

These two quantities are related by the equation of state, eq. (2).

This set of equations reveals a combination of two non-interacting matter components that are represented by perfect fluids.

In the next section we study the consequences that arise when the basic set of equations together with the corresponding equations of states for the matter and the scalar field components (which relate \( p_M \) to \( \rho_M \) and \( p_Q \) to \( \rho_Q \), respectively), are used for describing a model which resembles the standard \( \Lambda \)CDM model.
III. CHARACTERISTICS OF THE MODEL

In this section we shall impose conditions under which closed universes may look similar to a flat universe at low redshift.

We start by considering a closed FRW model which has two matter components. One of these components is related to a nonrelativistic dust (i.e. matter whose equation of state is \( P_M = 0 \)), and the other, the quintessence component whose equation of state is expressed by eq. (2). In the following, we shall assume that the quintessence component together with the curvature term combine in such a way that they give rise to a cosmological constant term in a flat universe model. In this scenario, the cold dark matter (assumed to be described by dust), together with a cosmological constant, form the main matter ingredients of the model. To this goal we impose the following condition:

\[
\frac{8\pi G}{3} \rho_Q(t) - \frac{1}{a^2(t)} = \frac{\Lambda}{3},
\]

from which we could get an explicit expression for \( \rho_Q \) as a function of time if the time dependence of the scale factor \( a(t) \) is known.

The constraint equation (8) may be written as

\[
\Omega_Q \left( \frac{\rho_Q(t)}{\rho_0^Q} \right) + \Omega_k \left( \frac{a_0}{a(t)} \right)^2 = \Omega_\Lambda,
\]

where the curvature density parameter \( \Omega_k \) and the quintessence density parameter \( \Omega_Q \) are defined by \( \Omega_k = - \left( \frac{1}{a_0 H_0} \right)^2 \) \((< 0)) \) and \( \Omega_Q = \left( \frac{8\pi G}{3 H_0^2} \right) \rho_0^Q \) \((> 0)) \) respectively. When we evaluate eq. (9) at the present epoch, we get a relation among the density parameters

\[
\Omega_Q + \Omega_k = \Omega_\Lambda.
\]

Since \( \Omega_k < 0 \), we get that \( \Omega_Q \) must be greater than \( \Omega_\Lambda \).

Under the condition (8), the time-time component of Einstein equations, eq. (5), becomes analogous to that of a flat universe where the matter density \( \Omega_M \) and the cosmological constant density \( \Omega_\Lambda \) form the main matter components today. Thus, this equation reads

\[
H^2(t) = H_0^2 \left[ \Omega_\Lambda + \Omega_M \frac{\rho_M(t)}{\rho_0^M} \right].
\]

Notice that, when this expression is evaluated at present time, i.e. \( t = t_0 \), we get \( \Omega_M + \Omega_\Lambda = 1 \), which lies within the observational range as is seen from eq. (1). Therefore, we may
associate with this scenario the so called ΛCDM-model. For numerical computations we shall take $\Omega_M = 0.3$ and $\Omega_\Lambda = 0.7$. The latter choice agrees with the value of a cosmological constant which is constrained to be $\Omega_\Lambda \leq 0.7$ by QSO lensing surveys\[12\].

Thus our basic equations are (7) and (11), together with the constraint equations (9). Notice that this set of equations should not be confused with the one specified by Starobinski\[13, 14\]. They describe dust and a scalar field in a flat FRW model. In our case, we take dust and scalar field in a close FRW universe. Our scalar field is set up in such a way that we constraint the Einstein field equations to have a flat form.

It is well known that eq. (11) can be solve exactly for a non-relativistic perfect fluid (dust)\[15, 16, 17\]:

$$a(t) = a_0 \left( \frac{\Omega_M}{\Omega_\Lambda} \right)^{1/3} \sinh^{2/3} \left( \beta t \right),$$  (12)

where $\beta = \frac{3}{2} \sqrt{\Omega_\Lambda} H_0$.

This solution allows us to write explicit expressions for $\rho_Q$ and $\rho_M$:

$$\rho_Q(t) = \rho_Q^0 \left[ \Omega_\Lambda + (\Omega_Q - \Omega_\Lambda) \left( \frac{\Omega_\Lambda}{\Omega_M} \right)^{2/3} \sinh^{-4/3} \left( \frac{3}{2} \sqrt{\Omega_\Lambda} H_0 t \right) \right],$$  (13)

and

$$\rho_M(t) = \rho_M^0 \left( \frac{\Omega_\Lambda}{\Omega_M} \right) \sinh^{-2} \left( \frac{3}{2} \sqrt{\Omega_\Lambda} H_0 t \right).$$  (14)

We observe that the energy density $\rho_M$ becomes dominant for $t \ll t_{eq}$, where $t_{eq}$ is the time when the two fluids, the quintessence scalar field $Q$ and the CDM components, become identical. This time is given by

$$t_{eq} = \frac{2}{3 H_0 \sqrt{\Omega_\Lambda}} \sinh^{-1} \left\{ \left[ (1 + \sqrt{1 + D})^{2/3} - D^{1/3} \right]^{3/2} \right\},$$  (15)

where the parameter $D$ is defined by $D = \frac{4}{27} \frac{(\Omega_Q - \Omega_\Lambda)^3}{\Omega_\Lambda^2 \Omega_M^2}$.

Since we have fixed the values of $\Omega_M$ and $\Omega_\Lambda$ with present astronomical observations, the time $t_{eq}$ will depend on the free parameter $\Omega_Q$. For instance, if we take $\Omega_Q = 0.75$, this time is quite close to the present time. In fact, in this case, $t_{eq}$ represents approximately 99% of $t_0$. For $\Omega_Q = 0.95$, this corresponds to 78 percent approximately. For the period $t > t_{eq}$ it is the scalar field $Q$ that dominates. In particular, at present time we find $\rho_Q^0 > \rho_M^0$ or, equivalently, $\Omega_Q > \Omega_M$. And, due to the expansion acceleration measured for the universe
which implies that $\Omega_\Lambda > \Omega_M$, our model satisfies the following inequalities for the density parameters: $\Omega_Q > \Omega_\Lambda > \Omega_M$.

Continuing with our analysis, we would like to obtain the explicit time dependence for the equation of state parameter $w_Q$. In order to do so, we take into account eq. (2) and from the constraint eq. (9) we obtain that

$$w_Q(t) = -\frac{1}{3} \left[ \frac{(\Omega_Q - \Omega_\Lambda) (a_0/a(t))^2 + 3 \Omega_\Lambda}{(\Omega_Q - \Omega_\Lambda) (a_0/a(t))^2 + \Omega_\Lambda} \right],$$

(16)

where $a(t)$ is given by eq. (12). Notice that the case $\Lambda = 0$ gives $w_Q(t) = -1/3 = \text{const.}$, situation that was studied in ref. [10]. For $\Lambda \neq 0$, this parameter is always negative, since for $a \to 0$, the parameter $w_Q \to -1/3$ and, when $a \to \infty$, we get $w_Q \to -1$. Thus, we find that the parameter $w_Q$ lies in the range $-1 < w_Q < -1/3$. We also should note that if we would have considered the open model, i.e. the case with $k = -1$ in the FRW metric, the equation of state parameter would have an unattractive characteristic, since this would lie in the range $-\infty < w_Q < -1$ violating the dominant energy condition, i.e. $|P_Q| \leq \rho_Q$.\[18\]

Another interesting characteristic of the quintessence scalar field to be determined is the form of the scalar potential, $V(Q)$. In order to do this, we notice from definitions $\rho_Q$ and $P_Q$ that the scalar potential becomes given by $V(Q) = (1/2)(1-w_Q)\rho_Q$ where we have used the equation of state (2). If in this expression we substitute the corresponding expressions (9) and (16), we obtain that

$$V(t) = \frac{1}{3} \rho_Q^0 \left[ 3 \left( \frac{\Omega_\Lambda}{\Omega_Q} \right) + 2 \left( 1 - \frac{\Omega_\Lambda}{\Omega_Q} \right) \left( \frac{a_0}{a(t)} \right)^2 \right].$$

(17)

On the other hand, from the same definitions for $\rho_Q$ and $P_Q$ we get that $\dot{Q} = \sqrt{\rho_Q \left[ 1 + w_Q \right]}$ and, after substituting the corresponding expression for $\rho_Q$ and $w_Q$, we get an explicit equation for $\dot{Q}$, as a function of the scale factor, which can be integrated and obtain

$$Q(t) = \frac{3\alpha}{\beta} \left( \frac{\Omega_\Lambda}{\Omega_M} \right)^{1/2} \left( \frac{a(t)}{a_0} \right)^{(1/2)} \, _2F_1 \left( \frac{1}{2} \frac{1}{6}; \frac{1}{6}; - \left( \frac{\Omega_\Lambda}{\Omega_M} \right) \left( \frac{a(t)}{a_0} \right)^3 \right),$$

(18)

where $\, _2F_1$ is the generalized hypergeometric function and $\alpha = \sqrt{\frac{2}{3} \rho_Q^0 \left( 1 - \frac{\Omega_\Lambda}{\Omega_M} \right)}$. This expression has been obtained by using MAPLE.

As we have mentioned below eq. (10) the range $\Omega_\Lambda < \Omega_Q$ has to be satisfied. On the other hand, an upper limit for $\Omega_Q$ could be restricted considering the data specified in ref. [19],
where the range for $\Omega_k$ is established, i.e. $-0.15 < \Omega_k < -0.02$, in which case we obtain $\Omega_Q < 0.75 - 0.85$. Thus, considering that $\Omega_\Lambda$ is in the range $\Omega_\Lambda \sim 0.6 - 0.7$ we may write for $\Omega_Q$ the following range: $0.6 - 0.7 < \Omega_Q < 0.75 - 0.85$.

IV. THE COMOVING VOLUME ELEMENT

In this section we would like to obtain some observational consequences for our model. One important issue in astronomy is that referred to the number count-redshift relation. The number of galaxies in a comoving element in a solid angular area $d\Omega$ with redshift between $z$ and $z + dz$ is sensitive to the comoving volume element $dV_C$ which we define as following for the closed FRW metric

$$dV_C = a_0^3 \frac{r^2}{\sqrt{1 - r^2}} dr d\Omega.$$  

This expression gives rise to the comoving volume element as

$$\frac{dV_C}{dz d\Omega} = \frac{D_m^2(z)}{\sqrt{1 + \Omega_k H_0^2 D_m^2(z)}} \frac{dD_m(z)}{dz},$$

where the proper motion distance $D_m = a_0 r$ was introduced.

In order to get an explicit expression for the observable comoving volume element per solid angle and per redshift interval, we need $D_m$ as a function of the redshift $z$. Using $D_m(z) = D_L(z)/(1 + z)$, where $D_L$ represent the luminosity distance defined by $D_L(z) = \left(\frac{L}{4 \pi F}\right)^{1/2}$, where $L$ is the rest-frame luminosity and $F$ is an apparent flux, we get that

$$D_m(z) = \frac{1}{H_0} \frac{1}{\sqrt{\Omega_Q - \Omega_\Lambda}} \sin \left[ \sqrt{\Omega_Q - \Omega_\Lambda} \int_0^z \frac{dx}{\sqrt{\Omega_\Lambda + \Omega_M (1 + x)^3}} \right].$$

With this latter expression we obtain for the comoving volume element

$$\frac{dV_C}{dz d\Omega} = \frac{1}{H_0^3} \frac{[H_0 D_m(z)]^2}{\sqrt{\Omega_\Lambda + \Omega_M (1 + z)^3}} \sqrt{\frac{1 - (\Omega_Q - \Omega_\Lambda) [H_0 D_m(z)]^2}{1 + (\Omega_Q - \Omega_\Lambda) [H_0 D_m(z)]^2}}.$$  

Expression (21) has the consequence that, given a population of objects of constant density and determinable distance measures, we can in principle constrain the value of the $\Omega_Q$ parameter and determine whether the universe may be considered to be closed, even though it looks quite flat at low redshift. At low redshift ($z < 1$) all the models become indistinguishable one to another. This could be seen from the fact that at low redshift we may expand eq. (21) and obtain, by keeping the first term of the expansion as a leader term

$$\frac{dV_C}{dz d\Omega} \sim \frac{1}{H_0^2} z^2,$$  

(22)
which becomes completely $\Omega$-parameters independent. But, at high enough redshift, i.e. $z > 1$, the flat model shows a volume-per-redshift larger than those related to closed models.

V. CONCLUSIONS

We have studied a closed universe model in which, apart from the usual CDM component, we have included a quintessence scalar field. At low redshift it looks flat. This means that we have fine tuned the quintessence component together with the curvature term for getting a flat model in which an effective cosmological constant $\Lambda$ is the main matter component in agreement with the observed acceleration of the universe.

We have found the intrinsic properties of the model and especially, the characteristics of the quintessence scalar field $Q$. For instance, the scalar potential $V(Q)$ appears to follow an almost inverse power-law expression coincident with those usually evoked in models where quintessence has been taken into account. In a similar way, another property of this field is obtained when we impose an equation of state of the form $P_Q = w_Q \rho_Q$ where, in agreement with SNe Ia astronomical measures, it is found that this parameter lies in the range $-1 < w_Q < -1/3$. However, we should notice that this range appears as a condition for mimicking a flat model, and not as an imposition coming from observational constraints.

Finally, we have described a kinematical property for our model. Specifically, we have determined in the last section the comoving volume per solid angle per redshift interval as a function of the redshift. We have found that the different models (including the flat one) become indistinguishable at low enough redshift ($z \ll 1$). However, at high redshift, i.e. at $z \geq 1$, we have found that the closed models become distinguishable from the flat model. Perhaps the coming astronomical programs will decide which of the considered models describes more properly the dynamics of our observed universe.

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