The Anticanonical Complex for Non-degenerate Toric Complete Intersections

Abstract. The anticanonical complex generalizes the Fano polytope from toric geometry and has been used to study Fano varieties with torus action so far. We provide an explicit description of the anticanonical complex for complete intersections in toric varieties defined by non-degenerate systems of Laurent polynomials. As an application, we classify the terminal Fano threefolds that are embedded into a fake weighted projective space via a general system of Laurent polynomials.

1. Introduction

The idea behind anticanonical complexes is to extend the features of the Fano polytopes from toric geometry to wider classes of varieties and thereby to provide combinatorial tools for the treatment of the singularities of the minimal model program. If $X$ is any $\mathbb{Q}$-Gorenstein variety, i.e. some positive multiple of a canonical divisor $K_X$ is Cartier, then these singularities are defined in terms of discrepancies that means the coefficients $a_E(X)$ of the exceptional divisors $E$ showing up in the ramification formula for a resolution $\pi : X' \rightarrow X$ of singularities:

$$K_{X'} = \pi^* K_X + \sum a_E(X) E.$$

The variety $X$ has at most terminal, canonical or $\varepsilon$-log terminal singularities if always $a_E(X) > 0$, $a_E(X) \geq 0$ or $a_E(X) > \varepsilon - 1$ for $0 \leq \varepsilon < 1$, where $\varepsilon = 0$ gives precisely the log terminal singularities. We briefly look at the toric case. For an $n$-dimensional toric Fano variety $Z$, one defines the Fano polytope to be the convex hull $A \subseteq \mathbb{Q}^n$ over the primitive ray generators of the describing fan of $Z$. For any toric resolution $\pi : Z' \rightarrow Z$ of singularities, the exceptional divisors $E_\varnothing$ are given by rays of the fan of $Z'$ and one obtains the discrepancies as

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\[ a_{E_\varrho}(Z) = \frac{\|v_\varrho\|}{\|v'_\varrho\|} - 1, \]

where \( v_\varrho \in \varrho \) is the shortest non-zero lattice vector and \( v'_\varrho \in \varrho \) is the intersection point of \( \varrho \) and the boundary \( \partial A \). In particular, a toric Fano variety \( Z \) with Fano polytope \( A \) has at most terminal (canonical, \( \epsilon \)-log terminal) singularities if and only if \( A \) contains no lattice points except the origin and its vertices (\( A \) contains no lattice points in its interior except the origin, \( \epsilon A \) contains no lattice points except the origin). This allows the use of lattice polytope methods in the study of singular toric Fano varieties; see [8, 22, 23] for work in this direction.

This principle has been extended by replacing the Fano polytope with a suitable polyhedral complex, named anticanonical complex in the setting of varieties with a torus action of complexity one, which encodes discrepancies in full analogy to the toric Fano polytope; see [6]. The more recent work [17] provides an existence result of anticanonical complexes for torus actions of higher complexity subject to conditions on a rational quotient. Applications to the study of singularities and Fano varieties can be found in [3, 9, 18].

In the present article, we provide an anticanonical complex for subvarieties of toric varieties arising from non-degenerate systems \( F = (f_1, \ldots, f_s) \) of Laurent polynomials in the sense of Khovanskii [25]; see also Definition 3.4. Even in the case \( s = 1 \), the resulting hypersurfaces form an interesting example class which is actively studied by several authors; see for instance [5, 14, 20].

We briefly summarize basic notions and first results, as provided in Sects. 3 and 4. Let \( F = (f_1, \ldots, f_s) \) be a non-degenerate system of Laurent polynomials in \( n \) variables and let \( \Sigma \) be any fan in \( \mathbb{Z}^n \) refining the normal fan of the Minkowski sum \( B_1 + \ldots + B_s \) of the Newton polytopes \( B_j \) of \( f_j \). Moreover, let \( Z \) be the toric variety associated with \( \Sigma \) and \( X_i \subseteq Z \) the closure of \( V(f_i) \subseteq \mathbb{T}^n \). The non-degenerate toric complete intersection defined by \( F \) and \( \Sigma \) is the subvariety

\[ X = X_1 \cap \ldots \cap X_s \subseteq Z. \]

By Theorems 3.9 and 4.3, the variety \( X \subseteq Z \) is a normal locally complete intersection, equals the closure of \( V(F) \subseteq \mathbb{T}^n \) and, in the Cox ring of \( Z \), the defining homogeneous equations of \( X \) generate a complete intersection ideal. Given a refinement \( \Sigma' \to \Sigma \) of fans, let \( Z' \to Z \) be the corresponding toric morphism and \( X' \subseteq Z \) the toric complete intersection defined by \( F \) and \( \Sigma' \). Then \( Z' \to Z \) is an ambient modification, meaning that we have a commutative diagram

\[
\begin{array}{ccc}
X' & \to & Z' \\
\downarrow & & \downarrow \\
X & \subseteq & Z
\end{array}
\]

with proper birational morphisms as downwards maps. It turns out that any exceptional divisor \( E_{X'} \subseteq X' \) has a unique host, that means an exceptional divisor \( E_{Z'} \subseteq Z' \) with \( E_{X'} \subseteq E_{Z'} \). If the fan \( \Sigma' \) is regular, then Theorem 4.3 tells us that \( X' \to X \) is a resolution of singularities; see also [25, Thm. 2.2]. Finally, the union \( Z_X \subseteq Z \) of all torus orbits intersecting \( X \) is open in \( Z \) and thus the corresponding
cones form a subfan $\Sigma_X \subseteq \Sigma$. Moreover, the support of $\Sigma_X$ equals the tropical variety of $V(F) \subseteq \mathbb{T}^n$; see Theorem 4.8.

We come to the first main result of the article. Suppose that $Z_X$ is $\mathbb{Q}$-Gorenstein. Then, for every $\sigma \in \Sigma_X$, we have a linear form $u_\sigma \in \mathbb{Q}^n$ evaluating to $-1$ on every primitive ray generator $v_\varrho$, where $\varrho$ is an extremal ray of $\sigma$. We set

$$A(\sigma) := \{ v \in \sigma; \ 0 \geq \langle u_\sigma, v \rangle \geq -1 \} \subseteq \sigma.$$ 

**Theorem 1.1.** Let $X \subseteq Z$ be an irreducible non-degenerate toric complete intersection defined by a system $F$ of Laurent polynomials in $n$ variables and a fan $\Sigma$ in $\mathbb{Z}^n$. If $Z_X$ is $\mathbb{Q}$-Gorenstein, then $X$ is $\mathbb{Q}$-Gorenstein and one obtains an anticanonical complex for $X$ by

$$A_X := \bigcup_{\sigma \in \Sigma_X} A(\sigma).$$

That means that for all ambient toric modifications $Z' \to Z$ the discrepancy of any exceptional divisor $E_{X'} \subseteq X'$ is given in terms of the defining ray $\varrho \in \Sigma'$ of its host $E_{Z'} \subseteq Z'$, the primitive generator $v_\varrho \in \varrho$ and the intersection point $v'_\varrho$ of $\varrho$ and the boundary $\partial A_X$ as

$$a_{E_{X'}}(X) = \frac{\| v_\varrho \|}{\| v'_\varrho \|} - 1.$$

Observe that in the above setting, each vertex of $A_X$ is a primitive ray generator of the fan $\Sigma$. Thus, in the non-degenerate complete toric intersection case, all vertices of the anticanonical complex are integral vectors; this does definitely not hold in other situations, see [6,17]. The following consequence of Theorem 1.1 yields in particular Bertini type statements on terminal and canonical singularities.

**Corollary 1.2.** Let $X \subseteq Z$ be an irreducible non-degenerate toric complete intersection given by a system $F$ of Laurent polynomials in $n$ variables and a fan $\Sigma$ in $\mathbb{Z}^n$. Let $Z_X$ be $\mathbb{Q}$-Gorenstein and $A_X$ the anticanonical complex from Theorem 1.1.

(i) $X$ has at most terminal singularities if and only if $A_X$ contains no lattice points except the origin and its vertices.

(ii) $X$ has at most canonical singularities if and only if $A_X$ contains no interior lattice points except the origin.

(iii) $X$ has at most $\varepsilon$-log terminal singularities if and only if $\varepsilon A_X$ contains no interior lattice points except the origin.

In particular, $X$ is log terminal. Moreover, $X$ has at most terminal (canonical, $\varepsilon$-log terminal) singularities if and only if its ambient toric variety $Z_X$ has at most terminal (canonical, $\varepsilon$-log terminal) singularities.

As an application of the first main result, we classify the general non-toric terminal Fano non-degenerate complete intersection threefolds sitting in fake weighted projective spaces; for the meaning of “general” in this context, see Definition 4.16. According to [25, Thm. 2.2], the general toric complete intersection is non-degenerate. Moreover, under suitable assumptions on the ambient toric variety,
we obtain the divisor class group and the Cox ring for free in the general case; see Corollary 4.20. This, by the way, allows us to construct many Mori dream spaces with prescribed properties; see for instance Example 4.21.

We turn to the second main result. Recall that a fake weighted projective space is an $n$-dimensional toric variety arising from a complete fan with $n + 1$ rays. Any fake weighted projective space $Z$ is uniquely determined up to isomorphism by its degree matrix $Q$, having as its columns the divisor classes $[D_i] \in \text{Cl}(Z)$ of the toric prime divisors $D_0, \ldots, D_n$ of $Z$; see Example 2.2. Moreover, for a toric complete intersection $X = X_1 \cap \ldots \cap X_s$ in $Z$, the relation degree matrix $\mu$ of $X$ has the divisor classes $[X_i] \in \text{Cl}(Z)$ as its columns.

**Theorem 1.3.** Let $X = X_1 \cap \ldots \cap X_s$ be a non-toric terminal Fano general toric complete intersection threefold in a fake weighted projective space $Z$. Then $\text{Cl}(X) = \text{Cl}(Z)$ and $X$ is a member of one of the following families, specified by the degree matrix $Q$ of $Z$ and the relation degree matrix $\mu$ of $X$; where we also list the anticanonical class $[-K_X]$ of $X$ and the numbers $-K_X^3$ and $h^0(-K_X)$:

| No. | $\text{Cl}(Z)$ | $Q$ | $\mu$ | $-K_X^3$ | $h^0(-K_X)$ |
|-----|----------------|-----|-------|----------|-------------|
| 1   | $\mathbb{Z}$  | $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ | $54$ | $30$ |
| 2   | $\mathbb{Z}$  | $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ | $24$ | $15$ |
| 3   | $\mathbb{Z}$  | $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ | $4$ | $5$ |
| 4   | $\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ | $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 2 \end{bmatrix}$ | $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ | $8$ | $5$ |
| 5   | $\mathbb{Z}$  | $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ | $27$ | $16$ |
| 6   | $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ | $27/2$ | $8$ |
| 7   | $\mathbb{Z}$  | $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$ | $1/2$ | $1$ |
| 8   | $\mathbb{Z}$  | $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 6 \\ 1 \end{bmatrix}$ | $1/2$ | $4$ |
| 9   | $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$ | $27/2$ | $8$ |
| 10  | $\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ | $\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 6 \\ 0 \end{bmatrix}$ | $1/2$ | $1$ |
| 11  | $\mathbb{Z}$  | $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 6 \\ 1 \end{bmatrix}$ | $2/3$ | $4$ |
| 12  | $\mathbb{Z}$  | $\begin{bmatrix} 1 & 1 & 1 & 2 & 3 \end{bmatrix}$ | $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$ | $8$ | $7$ |
| 13  | $\mathbb{Z}$  | $\begin{bmatrix} 1 & 1 & 2 & 2 & 3 \end{bmatrix}$ | $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$ | $27/2$ | $9$ |
| 14  | $\mathbb{Z}$  | $\begin{bmatrix} 1 & 1 & 2 & 3 & 3 \end{bmatrix}$ | $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$ | $64/3$ | $13$ |
| No. | Cl(Z)          | Q            | μ | $[-K_X]$ | $-K_X^3$ | $h^0(-K_X)$ |
|-----|----------------|--------------|---|----------|----------|-------------|
| 15  | Z              | [1 2 2 3 3]  | 6 | 5        | 125/6    | 12          |
| 16  | Z              | [1 1 1 2 4]  | 8 | 1        | 1        | 3           |
| 17  | $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} 1 1 1 2 4 \\ 0 0 1 1 1 \end{bmatrix}$ | 8 | $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ | 1/2 | 1 |
| 18  | Z              | [1 2 3 3 4]  | 12 | 1        | 1/3      | 2           |
| 19  | Z              | [1 1 3 4 4]  | 12 | 1        | 1/4      | 2           |
| 20  | Z              | [1 1 2 2 5]  | 10 | 1        | 1/2      | 2           |
| 21  | Z              | [1 1 2 3 6]  | 12 | 1        | 1/3      | 2           |
| 22  | $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} 1 1 2 3 6 \\ 0 1 1 0 1 \end{bmatrix}$ | 12 | $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ | 1/6 | 1 |
| 23  | Z              | [1 1 1 4 6]  | 12 | 1        | 1/2      | 3           |
| 24  | Z              | [1 1 2 6 9]  | 18 | 1        | 1/6      | 2           |
| 25  | Z              | [1 1 4 5 10] | 20 | 1        | 1/10     | 2           |
| 26  | Z              | [1 1 3 8 12] | 24 | 1        | 1/12     | 2           |
| 27  | Z              | [1 2 3 10 15]| 30 | 1        | 1/30     | 1           |
| 28  | Z              | [1 1 6 14 21]| 42 | 1        | 1/42     | 2           |
| 29  | Z              | [1 1 1 1 1 1] | 2 2 | 2        | 32       | 19          |
| 30  | Z              | [2 3]        | 1 | 6        | 6        |             |
| 31  | $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} 1 1 1 1 1 1 \\ 0 0 0 1 1 1 \end{bmatrix}$ | $\begin{bmatrix} 2 2 \\ 0 0 \end{bmatrix}$ | $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ | 16 | 9 |
| 32  | Z              | [1 1 1 2 2 2] | 4 4 | 1        | 2        | 3           |
| 33  | $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} 1 1 1 2 2 2 \\ 0 0 1 0 1 1 \end{bmatrix}$ | $\begin{bmatrix} 4 4 \\ 0 0 \end{bmatrix}$ | $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ | 1 | 1 |
| 34  | $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ | $\begin{bmatrix} 1 1 1 2 2 2 \\ 0 0 1 1 1 1 \end{bmatrix}$ | $\begin{bmatrix} 4 4 \\ 0 0 \end{bmatrix}$ | $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ | 1 | 2 |
| 35  | $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$ | $\begin{bmatrix} 1 1 1 2 2 2 \\ 0 0 1 0 1 1 \\ 0 1 0 1 0 1 \end{bmatrix}$ | $\begin{bmatrix} 4 4 \\ 0 0 \end{bmatrix}$ | $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ | 1/2 | 0 |
| 36  | Z              | [1 2 2 2 3 3] | 6 6 | 1        | 1/2      | 1           |
Each item of the table is realized by a family of general non-degenerate toric complete intersections \(X = X_1 \cap \ldots \cap X_s\) with at most terminal singularities inside a fake weighted projective space. Finally, the Cox ring of any such \(X\) is given by

\[
\mathcal{R}(X) = \mathbb{K}[T_0, \ldots, T_{s+3}] / \langle g_1, \ldots, g_s \rangle,
\]

\[
\deg(T_i) = [D_i] \in Cl(Z),
\]

\[
\deg(g_j) = [X_j] \in Cl(Z),
\]

where \(\mathbb{K}[T_0, \ldots, T_{s+3}] = \mathcal{R}(Z)\) is the Cox ring of \(Z\), the \(g_1, \ldots, g_s \in \mathcal{R}(Z)\) are defining \(Cl(Z)\)-homogeneous polynomials for \(X_1, \ldots, X_s \subseteq Z\) and \(T_0, \ldots, T_{s+3}\) define a minimal system of prime generators for \(\mathcal{R}(X)\).

We note some observations around this classification and link to the existing literature.

**Remark 1.4.** The toric terminal Fano complete intersection threefolds in a fake weighted projective space are precisely the three-dimensional terminal fake weighted projective spaces; up to isomorphy, there are eight of them [24].

**Remark 1.5.** For the \(X\) of Theorem 1.3 with \(Cl(Z)\) torsion free, the Fano index is given as \(q_X = \lceil -K_X \rceil\), regarding \(\lceil -K_X \rceil \in Cl(Z) = \mathbb{Z}\) as an integer. In the remaining cases, \(q_X\) is given by
Remark 1.6. Embeddings into weighted projective spaces have been intensely studied by several authors. Here is how Theorem 1.3 relates to well-known classifications in this case.

(i) Numbers 1, 2, 3, 5, 11, 12, 29, 30 and 39 from Theorem 1.3 are smooth and thus appear in the classification of smooth Fano threefolds of Picard number one [21, 12.2].

(ii) Every variety $X$ from Theorem 1.3 with Fano index $q_X = 1$ defined by at most two equations in a weighted projective space $Z$ occurs in [19, Lists 16.6, 16.7].

(iii) The items from [19, Lists 16.6, 16.7] which don’t show up in Theorem 1.3 are note realizable as general complete intersections in a fake weighted projective space.

Recall that the Gorenstein index of a $\mathbb{Q}$-Gorenstein variety $X$ is the minimal positive integer $ı_X$ such that $ı_X K_X$ is a Cartier divisor. So, $ı_X = 1$ means that $X$ is Gorenstein.

Remark 1.7. The Gorenstein varieties in Theorem 1.3 are precisely the smooth ones. This is a direct application of Corollary 4.9 showing that $Z_X$ is the union of all torus orbits of dimension at least three and Proposition 4.13 which ensures that $X$ and $Z_X$ have the same Gorenstein index.

Remark 1.8. The anticanonical self intersection $-K_X^3$ together with the first coefficients of the Hilbert series of $X$ from Theorem 1.3 with $\text{Cl}(Z)$ having torsion occur in the Graded Ring Database [1,10]. Here are the corresponding ID’s:

| No. | 4 | 6 | 9 | 10 | 17 | 22 | 31 | 33 | 34 | 35 | 40 | 41 | 42 |
|-----|---|---|---|----|----|----|----|----|----|----|----|----|----|
| $q_X$ | 2 | 1 | 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

We observe that Numbers 17 and 36 from Theorem 1.3 both realise the numerical data from ID 3508 in the Graded Ring Database but the general members of the respective families are non-isomorphic.

Remark 1.9. For Numbers 35 and 42 from Theorem 1.3 the linear system $| - K_X |$ is empty. In particular these Fano threefolds $X$ do not admit an elephant, that means a member of $| - K_X |$ with at most canonical singularities. There appear to be only few known examples for this phenomenon, compare [19, List 16.7] and [30, Sec. 4].

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2. Background on Toric Varieties

In this section, we gather the necessary concepts and results from toric geometry and thereby fix our notation. We briefly touch some of the fundamental definitions but nevertheless assume the reader to be familiar with the foundations of the theory of toric varieties. We refer to [12,13,15] as introductory texts.

Our ground field $\mathbb{K}$ is algebraically closed and of characteristic zero. We write $\mathbb{T}^n$ for the standard $n$-torus, that means the $n$-fold direct product of the multiplicative group $\mathbb{K}^*$. By a torus we mean an affine algebraic group $\mathbb{T}$ isomorphic to some $\mathbb{T}^n$. A toric variety is a normal algebraic variety $Z$ containing a torus $\mathbb{T}$ as a dense open subset such that the multiplication on $\mathbb{T}$ extends to an action of $\mathbb{T}$ on $Z$.

Toric varieties are in covariant categorical equivalence with lattice fans. In this context, a lattice is a free $\mathbb{Z}$-module of finite dimension. Moreover, a quasifan (a fan) in a lattice $N$ is a finite collection $\Sigma$ of (pointed) convex polyhedral cones $\sigma$ in the rational vector space $N_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} N$ such that given $\sigma \in \Sigma$, we have $\tau \in \Sigma$ for all faces $\tau \preceq \sigma$ and for any two $\sigma, \sigma' \in \Sigma$, the intersection $\sigma \cap \sigma'$ is a face of both, $\sigma$ and $\sigma'$. The toric variety $Z$ and its acting torus $\mathbb{T}$ associated with a fan $\Sigma$ in $N$ are constructed as follows:

$$\mathbb{T} := \text{Spec } \mathbb{K}[M], \quad Z := \bigcup_{\sigma \in \Sigma} Z_{\sigma}, \quad Z_{\sigma} := \text{Spec } \mathbb{K}[\sigma^\vee \cap M],$$

where $M$ is the dual lattice of $N$ and $\sigma^\vee \subseteq M_{\mathbb{Q}}$ is the dual cone of $\sigma \subseteq N_{\mathbb{Q}}$. The inclusion $\mathbb{T} \subseteq Z$ of the acting torus is given by the inclusion of semigroup algebras arising from the inclusions $\sigma^\vee \cap M \subseteq M$ of additive semigroups. In practice, we will mostly deal with $N = \mathbb{Z}^n = M$, where $\mathbb{Z}^n$ is identified with its dual via the standard bilinear form $\langle u, v \rangle = u_1 v_1 + \ldots + u_n v_n$. In this setting, we have $N_{\mathbb{Q}} = \mathbb{Q}^n = M_{\mathbb{Q}}$. Moreover, given a lattice homomorphism $F: N \to N'$, we write as well $F: N_{\mathbb{Q}} \to N'_{\mathbb{Q}}$ for the associated vector space homomorphism.

We briefly recall Cox’s quotient construction $p: \hat{Z} \to Z$ of a toric variety $Z$ given by a fan $\Sigma$ in $\mathbb{Z}^n$ from [11]. We denote by $v_1, \ldots, v_r \in \mathbb{Z}^n$ the primitive generators of $\Sigma$, that means the shortest non-zero integral vectors of the rays $\varrho_1, \ldots, \varrho_r \in \Sigma$. We will always assume that $v_1, \ldots, v_r$ span $\mathbb{Q}^n$ as a vector space; geometrically this means that $Z$ has no torus factor. By $D_i \subseteq Z$ we denote the toric prime divisor corresponding to $\varrho_i \in \Sigma$. Throughout the article, we will make free use of the notation introduced around Cox’s quotient presentation.

**Construction 2.1.** Let $\Sigma$ be a fan in $\mathbb{Z}^n$ and $Z$ the associated toric variety. Consider the linear map $P: \mathbb{Z}^n \to \mathbb{Z}^n$ sending the $i$-th canonical basis vector $e_i \in \mathbb{Z}^n$ to the $i$-th primitive generator $v_i \in \mathbb{Z}^n$ of $\Sigma$, denote by $\delta = Q_{\mathbb{Z}^n}^r$ the positive orthant and define a fan $\hat{\Sigma}$ in $\mathbb{Z}^r$ by

$$\hat{\Sigma} := \{ \delta_0 \preceq \delta; \ P(\delta_0) \subseteq \sigma \text{ for some } \sigma \in \Sigma \}.$$ 

As $\hat{\Sigma}$ consists of faces of the orthant $\delta$, the toric variety $\hat{Z}$ defined by $\hat{\Sigma}$ is an open $\mathbb{T}^r$-invariant subset of $\hat{Z} = \mathbb{K}^r$. We also regard the linear map $P: \mathbb{Z}^r \to \mathbb{Z}^n$ as...
an \( n \times r \) matrix \( P = (p_{ij}) \) and then speak about the generator matrix of \( \Sigma \). The generator matrix \( P \) defines a homomorphism of tori:
\[
p: \mathbb{T}^r \to \mathbb{T}^n, \quad t \mapsto (t_1^{p_{11}} \cdots t_r^{p_{1r}}, \ldots, t_1^{p_{nr}} \cdots t_r^{p_{nr}}).
\]
This homomorphism extends to a morphism \( p: \hat{Z} \to Z \) of toric varieties, which in fact is a good quotient for the action of the quasitorus \( H = \ker(p) \) on \( \hat{Z} \). Let \( P^* \) be the transpose of \( P \), set \( K := \mathbb{Z}^r / \text{im}(P^*) \) and let \( Q: \mathbb{Z}^r \to K \) be the projection. Then \( \deg(T_i) := Q(e_i) \in K \) defines a \( K \)-graded polynomial ring
\[
\mathcal{R}(Z) := \bigoplus_{w \in K} \mathcal{R}(Z)_w := \bigoplus_{w \in K} \mathbb{K}[T_1, \ldots, T_r]_w = \mathbb{K}[T_1, \ldots, T_r].
\]
There is an isomorphism \( K \to \text{Cl}(Z) \) from the grading group \( K \) onto the divisor class group \( \text{Cl}(Z) \) sending \( Q(e_i) \in K \) to the class \([D_i]\) \( \in \text{Cl}(Z) \) of the toric prime divisor \( D_i \subseteq Z \) defined by the ray \( \rho_i \) through \( v_i \). Moreover, the \( K \)-graded polynomial ring \( \mathcal{R}(Z) \) is the Cox ring of \( Z \); see [4, Sec. 2.1.3].

**Example 2.2.** An \( n \)-dimensional *fake weighted projective space* is the (projective) toric variety \( Z \) arising from a fan \( \Sigma \) in \( \mathbb{Z}^n \) with generator matrix \( P \) and maximal cones \( \sigma_0, \ldots, \sigma_n \in \Sigma \) of the form
\[
P = [v_0, \ldots, v_n], \quad \sigma_j = \text{cone}(v_i; i \neq j)
\]
such that the cones of \( \Sigma \) cover the whole \( \mathbb{Q}^n \). Observe that in this setting, the fan \( \Sigma \) is uniquely determined by its generator matrix \( P \). The divisor class group of the fake weighted projective space \( Z \) is given by
\[
\text{Cl}(Z) \cong \mathbb{Z}^{n+1} / \text{im}(P^*) \cong \mathbb{Z} \times \Gamma
\]
with a finite abelian group \( \Gamma \). The classes \( w_i := [D_i] \in \text{Cl}(Z) \) of the torus invariant prime divisors \( D_i \subseteq Z \) corresponding to the primitive generators \( v_i \), are the columns of the degree matrix
\[
Q := [w_0, \ldots, w_n].
\]
By [4, Lemma 2.1.4.1], any \( n \) of the degrees \( w_i \) generate the divisor class group \( \text{Cl}(Z) \). Moreover, we can always assume the \( \mathbb{Z} \)-parts of the \( w_i \) to be positive. Regarding \( P^* \) and \( Q \) as homomorphisms, we obtain the exact sequence
\[
0 \to \mathbb{Z}^n \xrightarrow{P^*} \mathbb{Z}^{n+1} \xrightarrow{Q} \text{Cl}(Z) \to 0.
\]
The rows of \( P \) generate \( \ker(Q) \subseteq \mathbb{Z}^{n+1} \), which fixes \( P \) up to applying a unimodular matrix from the left. This allows us to reconstruct the fake weighted projective space \( Z \) just from its degree matrix \( Q \). The Cox ring of \( Z \) is given by
\[
\mathcal{R}(Z) = \mathbb{K}[T_0, \ldots, T_n], \quad \deg(T_i) = w_i = [D_i] \in \text{Cl}(Z).
\]
Moreover, the open subset \( \hat{Z} \subseteq \hat{Z} = \mathbb{K}^{n+1} \) from Construction 2.1 equals \( \mathbb{K}^{n+1} \setminus \{0\} \) and the quasitorus \( H = \ker(p) \) is isomorphic to \( \mathbb{K}^* \times \Gamma \). In particular, we see that \( Z \) is a usual weighted projective space if and only if \( \text{Cl}(Z) \) is torsion free.
We now explain the correspondence between effective Weil divisors on a toric variety $Z$ and the $K$-homogeneous elements in the polynomial ring $\mathcal{R}(Z)$. For any variety $X$, we denote by $X_{\text{reg}} \subseteq X$ the open subset of its smooth points and by $\text{WDiv}(X)$ its group of Weil divisors. We need the following pullback construction of Weil divisors with respect to morphisms $\varphi : X \to Y$: Given a Weil divisor $D$ having $\varphi(X)$ not inside its support, restrict $D$ to a Cartier divisor on $Y_{\text{reg}}$, apply the usual pullback and turn the result into a Weil divisor on $X$ by replacing its prime components with their closures in $X$.

**Definition 2.3.** Consider a toric variety $Z$ and its quotient presentation $p : \hat{Z} \to Z$. A *describing polynomial* of an effective divisor $D \in \text{WDiv}(Z)$ is a $K$-homogeneous polynomial $g \in \mathcal{R}(Z)$ with $\text{div}(g) = p^* D \in \text{WDiv}(\hat{Z})$.

**Example 2.4.** An effective toric divisor $a_1 D_1 + \ldots + a_r D_r$ on $Z$ has the monomial $T^{a_1} \ldots T^{a_r} \in \mathcal{R}(Z)$ as a describing polynomial. Moreover, in $K = \text{Cl}(X)$, we have

$$\deg(T^{a_1} \ldots T^{a_r}) = Q(a_1, \ldots, a_r) = [a_1 D_1 + \ldots + a_r D_r].$$

We list the basic properties of describing polynomials, which in fact hold in the much more general framework of Cox rings; see [4, Prop. 1.6.2.1 and Cor 1.6.4.6].

**Proposition 2.5.** Let $Z$ be a toric variety with quotient presentation $p : \hat{Z} \to Z$ as in Construction 2.1 and let $D$ be any effective Weil divisor on $Z$.

(i) There exist describing polynomials for $D$ and any two of them differ by a non-zero scalar factor.

(ii) If $g$ is a describing polynomial for $D$, then, identifying $K$ and $\text{Cl}(Z)$ under the isomorphism presented in Construction 2.1, we have

$$p^* (\text{div}(g)) = D, \quad \deg(g) = [D] \in \text{Cl}(Z) = K.$$

(iii) For every $K$-homogeneous element $g \in \mathcal{R}(Z)$, the divisor $p^* (\text{div}(g))$ is effective and has $g$ as a describing polynomial.

Let us see how base points of effective divisors on toric varieties are detected in terms of fans and homogeneous polynomials. Recall that each cone $\sigma \in \Sigma$ defines a distinguished point $z_\sigma \in Z$ and the toric variety $Z$ is the disjoint union over the orbits $\mathbb{T}^n \cdot z_\sigma$, where $\sigma \in \Sigma$.

**Proposition 2.6.** Let $Z$ be the toric variety arising from a fan $\Sigma$ in $\mathbb{Z}^n$ and $D$ an effective Weil divisor on $Z$. Then the base locus of $D$ is $\mathbb{T}^n$-invariant. Moreover, a point $z_\sigma \in Z$ is not a base point of $D$ if and only if $D$ is linearly equivalent to an effective toric divisor $a_1 D_1 + \ldots + a_r D_r$ with $a_i = 0$ whenever $v_i \in \sigma$.

**Proof.** The $\mathbb{T}^n$-invariance of the base locus is due to the facts that any divisor is linearly equivalent to a toric one and that the sections of a toric divisor are generated by character functions, that means, functions restricting to characters of $\mathbb{T}^n$. The second statement follows from the fact that the closure of $\mathbb{T}^n \cdot z_\sigma$ equals the intersection over all $D_i$ with $v_i \in \sigma$. \qed
In the later construction and study of non-degenerate subvarieties of toric varieties, we make essential use of the normal fan of a polytope and the correspondence between polytopes and divisors for toric varieties. Let us briefly recall the necessary background and notation.

**Proposition 2.7.** Let $B \subseteq \mathbb{Q}^n$ be a polytope, write $B' \preceq B$ for the faces of $B$ and consider the set

$$\Sigma(B) := \{ \sigma(B'); B' \preceq B \}, \quad \sigma(B') := \text{cone}(u - u'; u \in B, u' \in B')^\vee.$$ 

Then $\Sigma(B)$ is a quasifan in $\mathbb{Z}^n$, being a fan if and only if $\dim(B) = n$. Moreover, we have an inclusion-reversing bijection

$$\text{Faces}(B) \to \Sigma(B), \quad B' \mapsto \sigma(B').$$

**Proof.** This is standard convex geometry, see for instance [15, page 26, Proposition] or [12, Thm. 2.3.2, Prop. 2.3.7, Prop. 14.2.20]. □

The quasifan $\Sigma(B)$ is called the *normal fan* of the polytope $B \subseteq \mathbb{Q}^n$; note the slight abuse of language, as $\Sigma(B)$ is not a fan in the strict sense in general.

We gather the necessary facts on the normal fan of a *Minkowski sum* of polytopes $B_i \subseteq \mathbb{Q}^n$, defined as

$$B_1 + \ldots + B_s := \{ u_1 + \ldots + u_s; u_i \in B_i \} \subseteq \mathbb{Q}^n.$$

Given quasifans $\Sigma$ and $\Sigma'$ in $\mathbb{Z}^n$, we speak of a *refinement* $\Sigma' \to \Sigma$ if $\Sigma$ and $\Sigma'$ have the same support and every cone of $\Sigma'$ is contained in a cone of $\Sigma$.

**Proposition 2.8.** Let $B = B_1 + \ldots + B_s \subseteq \mathbb{Q}^n$ be the Minkowski sum of polytopes $B_1, \ldots, B_s \subseteq \mathbb{Q}^n$.

(i) For each face $B' \preceq B$, there are unique faces $B'_1 \preceq B_1, \ldots, B'_s \preceq B_s$ such that we have $B' = B'_1 + \ldots + B'_s$.

(ii) The cones of the normal fan $\Sigma(B)$ of $B$ are $\sigma(B') = \sigma(B'_1) \cap \ldots \cap \sigma(B'_s)$, where $B' \preceq B$ and $B' = B'_1 + \ldots + B'_s$ is the decomposition from (i).

(iii) The normal fan $\Sigma(B)$ of $B$ is the coarsest common refinement of the normal fans $\Sigma(B_i)$ of the $B_i$.

(iv) Given $B' \preceq B$ and $B'_1 \preceq B_i$ with $B' = B'_1 + \ldots + B'_s$ the cone $\sigma(B'_i) \in \Sigma(B_i)$ is minimal with the property that it contains $\sigma(B') \in \Sigma(B')$.

**Proof.** Again, this is standard convex geometry, see for instance [32, Thm. 3.1.2, Cor. 3.1.3]. □

The next aim is Proposition 2.10 providing base point freeness and ampleness criteria. By a *lattice polytope* we mean a polytope in $\mathbb{Q}^n$ such that each of its vertices belongs to $\mathbb{Z}^n$. 
Lemma 2.9. Let $B \subseteq \mathbb{Q}^n$ be an $n$-dimensional lattice polytope and $\Sigma$ a complete fan in $\mathbb{Z}^n$ with generator matrix $P = [v_1, \ldots, v_r]$. Define $a \in \mathbb{Z}^r$ by
\[
a := (a_1, \ldots, a_r), \quad a_i := -\min_{u' \in B} \langle u', v_i \rangle.
\]
For $u \in B$ consider $a(u) := P^* u + a$, the minimal face $B(u) \preceq B$ containing $u$ and the associated cone $\sigma(B(u)) \in \sigma(B)$. Then the entries of $a(u) \in \mathbb{Q}^r$ satisfy
\[
a(u)_i \geq 0, \text{ for } i = 1, \ldots, r, \quad a(u)_i = 0 \iff v_i \in \sigma(B(u)).
\]

Proof. First note that $a$ is indeed integral, because $B$ has integral vertices. For the $i$-th component of $a(u)$, we have
\[
a(u)_i = \langle a(u), e_i \rangle = \langle P^* u, e_i \rangle + \langle a, e_i \rangle = \langle u, v_i \rangle + a_i \geq 0.
\]
In particular, $a(u)_i = 0$ if and only if $\langle u, v_i \rangle = -a_i$ holds. By minimality of $B(u)$, the latter is equivalent to $\langle u'', v_i \rangle = -a_i$ for all $u'' \in B(u)$. We conclude
\[
v_i \in \sigma(B(u)) \iff \langle u, v_i \rangle = \min_{u' \in B} \langle u', v_i \rangle \iff a(u)_i = 0.
\]

\[
\Box
\]

Proposition 2.10. Consider a lattice polytope $B \subseteq \mathbb{Q}^n$, a complete fan $\Sigma$ in $\mathbb{Z}^n$ with generator matrix $P = [v_1, \ldots, v_r]$, the associated toric variety $Z$ and $a \in \mathbb{Z}^r$ as in Lemma 2.9. Define
\[
D := a_1 D_1 + \ldots + a_r D_r \in WDiv(Z).
\]
Moreover, for every vector $u \in B \cap \mathbb{Z}^n$, set $a(u) := P^* u + a \in \mathbb{Z}^r$ as in Lemma 2.9 and obtain effective divisors $D(u)$ on $Z$, all of the same class as $D$ by
\[
D(u) := a(u)_1 D_1 + \ldots + a(u)_r D_r \in WDiv(Z).
\]
If $\Sigma$ refines the normal fan $\Sigma(B)$, then $D$ and all $D(u)$ are base point free. If $\Sigma$ equals the normal fan $\Sigma(B)$, then the divisors $D$ and $D(u)$ are even ample.

Proof. As $P^* u$ lies in the kernel of $Q : \mathbb{Z}^r \to K = \text{Cl}(Z)$, all divisors $D(u)$ are in the class of $D$. Lemma 2.9 ensures that each $D(u)$ is effective and it gives us
\[
a(u)_i = 0 \iff v_i \in \sigma(B(u)).
\]
Thus, if $\Sigma$ refines $\Sigma(B)$, then $D$ and all $D(u)$ are base point free by Proposition 2.6. If $\Sigma$ equals $\Sigma(B)$, then $D$ and the $D(u)$ are ample due to [12, Thm. 6.1.14].
\[
\Box
\]
3. Laurent Systems and Their Newton Polytopes

We consider systems $F$ of Laurent polynomials in $n$ variables. Any such system $F$ defines a Newton polytope $B$ in $\mathbb{Q}^n$. The objects of interest are completions $X \subseteq Z$ of the zero set $V(F) \subseteq \mathbb{T}^n$ in the toric varieties $Z$ associated with refinements of the normal fan of $B$. In Proposition 3.7, we interpret Khovanskii’s non-degeneracy condition \[25\] in terms of Cox’s quotient presentation of $Z$. Theorem 3.9 gathers complete intersection properties of the embedded varieties $X \subseteq Z$ given by non-degenerate systems of Laurent polynomials.

We begin with recalling the basic notions around Laurent polynomials and Newton polytopes. Laurent polynomials are the elements of the Laurent polynomial algebra which we will use the short notation

$$\text{LP}(n) := \mathbb{K}[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]$$

**Definition 3.1.** Take any Laurent polynomial $f = \sum_{\nu \in \mathbb{Z}^n} \alpha_{\nu} T^\nu \in \text{LP}(n)$. The Newton polytope of $f$ is

$$B(f) := \text{conv}(\nu \in \mathbb{Z}^n; \alpha_{\nu} \neq 0) \subseteq \mathbb{Q}^n.$$ 

Given a face $B \preceq B(f)$ of the Newton polytope, the associated face polynomial is defined as

$$f_B = \sum_{\nu \in B \cap \mathbb{Z}^n} \alpha_{\nu} T^\nu \in \text{LP}(n).$$

**Construction 3.2.** Consider a Laurent polynomial $f \in \text{LP}(n)$ and a fan $\Sigma$ in $\mathbb{Z}^n$. The pullback of $f$ with respect to the homomorphism $p : \mathbb{T}^r \to \mathbb{T}^n$ defined by the generator matrix $P = (p_{ij})$ of $\Sigma$ has a unique presentation as

$$p^* f(T_1, \ldots, T_r) = f(T_1^{p_{11}} \cdots T_r^{p_{1r}}, \ldots, T_1^{p_{n1}} \cdots T_r^{p_{nr}}) = T^u g(T_1, \ldots, T_r)$$

with a Laurent monomial $T^u = T_1^{v_1} \cdots T_r^{v_r} \in \text{LP}(r)$ and a $K$-homogeneous polynomial $g \in \mathbb{K}[T_1, \ldots, T_r]$ being coprime to each of the variables $T_1, \ldots, T_r$. We call $g$ the $\Sigma$-homogenization of $f$.

**Lemma 3.3.** Consider a Laurent polynomial $f \in \text{LP}(n)$ with Newton polytope $B(f)$ and a fan $\Sigma$ in $\mathbb{Z}^n$ with generator matrix $P := \{v_1, \ldots, v_r\}$ and associated toric variety $Z$. Let $a := (a_1, \ldots, a_r)$ be as in Lemma 2.9 and $D \in \text{WDiv}(Z)$ the pushforward of $\text{div}(f) \in \text{WDiv}(\mathbb{T}^n)$.

(i) The $\Sigma$-homogenization $g$ of $f$ is a describing polynomial of $D$ and with the homomorphism $p : \mathbb{T}^r \to \mathbb{T}^n$ given by $P$, we have

$$g = T^a p^* f \in \mathcal{R}(Z), \quad T^a := T_1^{a_1} \cdots T_r^{a_r}.$$ 

(ii) The Newton polytope of $g$ equals the image of the Newton polytope of $f$ under the injection $\mathbb{Q}^r \to \mathbb{Q}^n$ sending $u$ to $a(u) := P^* u + a$, in other words

$$B(g) = P^* B(f) + a = \{a(u); \; u \in B(f)\}.$$
(iii) Consider a face $B \lesssim B(f)$ and the associated face polynomial $f_B$. Then the corresponding face $P^* B + a \lesssim B(g)$ has the face polynomial

$$g_{P^* B + a} = g(\tilde{T}_1, \ldots, \tilde{T}_r), \quad \tilde{T}_i := \begin{cases} 0 & v_i \in \sigma(B), \\ T_i & v_i \notin \sigma(B). \end{cases}$$

Moreover, for each monomial $T^v$ of $g - g_{P^* B + a}$ there is a proper face $\sigma < \sigma(B)$ such that every variable $T_i$ with $v_i \in \sigma(B) \setminus \sigma$ divides $T^v$.

(iv) The degree $\deg(g) \in K$ of the $\Sigma$-homogenization $g$ of $f$ and the divisor class $[D] \in Cl(Z)$ of $D \in WDiv(Z)$ are given by

$$\deg(g) = Q(a) = [a_1 D_1 + \ldots + a_r D_r] = [D].$$

(v) If $\Sigma$ is a refinement of the normal fan of $B(f)$, then the divisor $D \in WDiv(Z)$ is base point free on $Z$.

**Proof.** Assertions (i) to (iii) are direct consequences of Lemma 2.9. Assertion (iv) is clear by Proposition 2.5 and (v) follows from Proposition 2.10.

We turn to systems $F = (f_1, \ldots, f_s)$ of Laurent polynomials $f_j$ in $n$ variables over $\mathbb{K}$. We will also call such $F = (f_1, \ldots, f_s)$ for short a system in $LP(n)$ or just a *Laurent system* if no specification of $n$ is needed. The differential of a system $F = (f_1, \ldots, f_s)$ in $LP(n)$ at a point $z = (z_1, \ldots, z_n) \in \mathbb{T}^n$ is

$$\mathcal{D}F = \left( \frac{\partial f_i}{\partial z_j} (z) \right)_{1 \leq i \leq s, 1 \leq j \leq n}.$$ 

We will use this notation also for systems $G = (g_1, \ldots, g_s)$ of polynomials in $r$ variables and points $z \in \mathbb{K}^r$. Here are the basic notions around Laurent systems; observe that item (iii) is precisely Khovanskii’s non-degeneracy condition stated in [25, Sec. 2.1].

**Definition 3.4.** Consider a system $F = (f_1, \ldots, f_s)$ of Laurent polynomials $f_1, \ldots, f_s \in LP(n)$.

(i) Let $B_j := B(f_j) \subseteq \mathbb{Q}^n$ denote the Newton polytope of $f_j$. Then the *Newton polytope of $F$* is the Minkowski sum

$$B := B(F) := B_1 + \ldots + B_s \subseteq \mathbb{Q}^n.$$

(ii) The *face system $F'$ of $F$* associated with a face $B' \lesssim B$ of the Newton polytope $B$ of $F$ is the Laurent system

$$F' := (f_1', \ldots, f_s'),$$

where $f_j' = f_{B_j'}$ are the face polynomials associated with the faces $B_j' \lesssim B_j$ from the presentation $B' = B'_1 + \ldots + B'_s$.

(iii) We call $F$ *non-degenerate* if for every face $B' \lesssim B$, the differential $\mathcal{D}F'(z)$ is of rank $s$ for all $z \in V(F') \subseteq \mathbb{T}^n$. 
(iv) Let \( \Sigma \) be a fan in \( \mathbb{Z}^n \). The **\( \Sigma \)-homogenization** of \( F = (f_1, \ldots, f_s) \) is the system 
\[ G = (g_1, \ldots, g_s), \]
where \( g_j \) is the \( \Sigma \)-homogenization of \( f_j \).

(v) By an **\( F \)-fan** we mean a fan \( \Sigma \) in \( \mathbb{Z}^n \) that refines the normal fan \( \Sigma(B) \) of the Newton polytope \( B \) of \( F \).

Note that Khovanskii’s non-degeneracy Condition 3.4 (iii) is fulfilled for suitably general choices of \( F \). Even more, it is a concrete condition in the sense that for every explicitly given Laurent system \( F \), we can explicitly check non-degeneracy.

**Construction 3.5.** Consider a system \( F = (f_1, \ldots, f_s) \) in \( \text{LP}(n) \), a fan \( \Sigma \) in \( \mathbb{Z}^n \) and the \( \Sigma \)-homogenization \( G \) of \( F \). Define subvarieties 
\[ \hat{X} := V(G) := V(g_1, \ldots, g_s) \subseteq \hat{Z}, \quad X := \overline{V(f_1) \cap \ldots \cap V(f_s)} \subseteq Z, \]
where \( Z \) is the toric variety associated with \( \Sigma \) and \( \hat{Z} = \mathbb{K}^r \). The quotient presentation \( p : \hat{Z} \to Z \) gives rise to a commutative diagram
\[
\begin{array}{ccc}
\hat{X} & \xrightarrow{\hat{p}} & \hat{Z} \\
\downarrow \phi H & & \downarrow \phi H \\
X & \xrightarrow{p} & Z
\end{array}
\]
where \( \hat{X} := \hat{X} \cap \hat{Z} \subseteq \hat{Z} \) as well as \( X \subseteq Z \) are closed subvarieties and \( p : \hat{X} \to X \) is a good quotient for the induced \( H \)-action on \( \hat{X} \). In particular, \( X = p(\hat{X}) \).

In our study of \( \hat{X} \), \( \hat{X} \) and \( X \), the decompositions induced from the respective ambient toric orbit decompositions will play an important role. We work with distinguished points \( z_\sigma \in Z \). In terms of Cox’s quotient presentation, \( z_\sigma \in Z \) becomes explicit as \( z_\sigma = p(z_\hat{\sigma}) \), where \( \hat{\sigma} = \text{cone}(e_i; \; v_i \in \sigma) \in \hat{\Sigma} \) and the coordinates of the distinguished point \( z_\hat{\sigma} \in \hat{Z} \) are \( z_{\hat{\sigma},i} = 0 \) if \( v_i \in \sigma \) and \( z_{\hat{\sigma},i} = 1 \) otherwise.

**Construction 3.6.** Consider a system \( F = (f_1, \ldots, f_s) \) in \( \text{LP}(n) \), a fan \( \Sigma \) in \( \mathbb{Z}^n \) and the \( \Sigma \)-homogenization \( G = (g_1, \ldots, g_s) \) of \( F \). For every cone \( \sigma \in \Sigma \) define 
\[ g_j^\sigma := g_j(T_1^\sigma, \ldots, T_r^\sigma), \quad T_i^\sigma := \begin{cases} 0 & v_i \in \sigma, \\ T_i & v_i \notin \sigma. \end{cases} \]
This gives us a system \( G^\sigma := (g_1^\sigma, \ldots, g_s^\sigma) \) of polynomials in \( \mathbb{K}[T_i; \; v_i \notin \sigma] \). In the coordinate subspace \( \hat{Z}(\sigma) = V(T_i; \; v_i \in \sigma) \) of \( \mathbb{K}^r \), we have 
\[ \hat{X}(\sigma) := \hat{X} \cap \hat{Z}(\sigma) = V(G^\sigma) \subseteq \hat{Z}(\sigma). \]
Note that \( \hat{Z}(\sigma) \) is the closure of \( \mathbb{T}^r \cdot z_\hat{\sigma} \subseteq \mathbb{K}^r \), as \( z_{\hat{\sigma},i} = 0 \) if \( v_i \in \sigma \) and \( z_{\hat{\sigma},i} = 1 \) otherwise. Next consider \( \mathbb{T}^n \cdot z_\sigma \subseteq Z \) and define locally closed subsets 
\[ \hat{X}(\sigma) := \hat{X} \cap \mathbb{T}^r \cdot z_\hat{\sigma} \subseteq \hat{X}, \quad X(\sigma) := X \cap \mathbb{T}^n \cdot z_\sigma \subseteq X. \]
Then we have \( X(\sigma) = p(\hat{X}(\sigma)) \) and \( X \subseteq Z \) is the disjoint union of the subsets \( X(\sigma) \), where \( \sigma \in \Sigma \).
The key step for our investigation of varieties \( X \subseteq Z \) defined by Laurent systems is to interpret the non-degeneracy condition of a system \( F \) in terms of its \( \Sigma \)-homogenization \( G \).

**Proposition 3.7.** Let \( F = (f_1, \ldots, f_s) \) be a non-degenerate system in \( LP(n) \) and let \( \Sigma \) be an \( F \)-fan in \( \mathbb{Z}^n \).

(i) The differential \( DG(\hat{z}) \) of the \( \Sigma \)-homogenization \( G \) of \( F \) is of full rank \( s \) at every point \( \hat{z} \in \hat{X} \).

(ii) For each cone \( \sigma \in \Sigma \), the differential \( DG^\sigma(\hat{z}) \) of the system \( G^\sigma \) is of full rank \( s \) at every point \( \hat{z} \in \hat{X}(\sigma) \).

(iii) For every \( \sigma \in \Sigma \), the scheme \( \hat{X}(\sigma) := \hat{X} \cap \mathbb{T}^r \cdot z_\sigma \), provided it is non-empty, is a closed subvariety of pure codimension \( s \) in \( \mathbb{T}^r \cdot z_\sigma \).

**Proof.** We care about (i) and on the way also prove (ii). Since \( g_1, \ldots, g_s \) are \( H \)-homogeneous, the set of points \( \hat{z} \in \hat{Z} \) with \( DG(\hat{z}) \) of rank strictly less than \( s \) is \( H \)-invariant and closed in \( \hat{Z} \). Thus, as \( p: \hat{Z} \to Z \) is a good quotient for the \( H \)-action, it suffices to show that for the points \( \hat{z} \in \hat{X} \) with a closed \( H \)-orbit in \( \hat{Z} \), the differential \( DG(\hat{z}) \) is of rank \( s \). That means that we only have to deal with the points \( \hat{z} \in \hat{X} \cap \mathbb{T}^r \cdot z_\sigma \), where \( \sigma \in \Sigma \).

So, consider a point \( \hat{z} \in \hat{X} \cap \mathbb{T}^r \cdot z_\sigma \), let \( \sigma' \in \Sigma(B) \) be the minimal cone with \( \sigma \subseteq \sigma' \) and let \( B' \preceq B \) be the face corresponding to \( \sigma' \in \Sigma(B) \). Then we have the Minkowski decomposition

\[
B' = B'_1 + \ldots + B'_r, \quad B'_j \preceq B_j = B(f_j).
\]

From Proposition 2.8 we infer that \( \sigma'_j = \sigma(B'_j) \) is the minimal cone of the normal fan \( \Sigma(B'_j) \) with \( \sigma \subseteq \sigma'_j \). Let \( F' \) be the face system of \( F \) given by \( B' \preceq B \). Define \( G' = (g'_1, \ldots, g'_s) \), where \( g'_j \) is the face polynomial of \( g_j \) defined by

\[
P'^*B'_j + a_j \preceq P'^*B_j + a_j = B(g_j), \quad g_j = T'^*_j p'^* f_j.
\]

According to Lemma 3.3 (iii), the polynomials \( g'_j \) only depend on the variables \( T_i \) with \( v_i \notin \sigma(B'_j) \). Moreover, we have

\[
g'_j = g'^\sigma_j, \quad j = 1, \ldots, s,
\]

because due to the minimality of \( \sigma'_j = \sigma(B'_j) \) each monomial of \( g_j - g'_j \) is a multiple of some \( T_i \) with \( v_i \in \sigma \). Thus, \( G' = G'^\sigma \). Using the fact that \( \hat{z}_i = 0 \) if and only if \( v_i \in \sigma \), we observe

\[
g'^\sigma_j(\hat{z}) = g_j(\hat{z}) = 0, \quad \text{rank } DG'^\sigma(\hat{z}) = s \Rightarrow \text{rank } DG(\hat{z}) = s.
\]

This reduces the proof of (i) to showing that \( DG'^\sigma(\hat{z}) \) is of full rank \( s \), and the latter proves (ii). Choose \( \tilde{z} \in \mathbb{T}^r \) such that \( \tilde{z}_i = \hat{z}_i \) for all \( i \) with \( v_i \notin \sigma \). Using again that the polynomials \( g'_j \) only depend on \( T_i \) with \( v_i \notin \sigma \), we see

\[
g'^\sigma_j(\tilde{z}) = g'^\sigma_j(\hat{z}) = 0, \quad j = 1, \ldots, s, \quad DG'^\sigma(\tilde{z}) = DG'^\sigma(\hat{z}).
\]
We conclude that $F'(p(\tilde{z})) = 0$ holds. Thus, the non-degeneracy condition on the Laurent system $F$ ensures that $DF'(p(\tilde{z}))$ is of full rank $s$. Moreover, we have

$$DG^\sigma(\tilde{z}) = DG^\sigma(\tilde{z}) = (T^{a_1}, \ldots, T^{a_s})(\tilde{z}) \cdot DF'(p(\tilde{z})) \circ Dp(\tilde{z}).$$

Since $T^{a_j}(\tilde{z}) \neq 0$ holds for $j = 1, \ldots, s$ and $p : \mathbb{T}^r \to \mathbb{T}^n$ is a submersion, we finally obtain that $DG^\sigma(\tilde{z})$ is of full rank $s$, which proves (i) and (ii). Assertion (iii) follows from (ii) and the Jacobian criterion for complete intersections.

**Corollary of proof 3.8.** Consider a system $F$ in $LP(n)$, an $F$-fan $\Sigma$ in $\mathbb{Z}^n$ and the $\Sigma$-homogenization $G$ of $F$. Then $F$ is non-degenerate if and only if all $DG^\sigma(\hat{z})$, where $\sigma \in \Sigma$ and $\hat{z} \in \hat{X}(\sigma)$, are of full rank.

A first application gathers complete intersection properties for the varieties defined by a non-degenerate Laurent system. Note that the codimension condition imposed on $\tilde{X} \setminus \hat{X}$ in the fourth assertion below allows computational verification for explicitly given systems of Laurent polynomials.

**Theorem 3.9.** Consider a non-degenerate system $F = (f_1, \ldots, f_s)$ in $LP(n)$, an $F$-fan $\Sigma$ in $\mathbb{Z}^n$ and the $\Sigma$-homogenization $G = (g_1, \ldots, g_s)$ of $F$.

(i) The variety $\tilde{X} = V(G)$ in $\tilde{Z} = \mathbb{K}^r$ is a complete intersection of pure dimension $r - s$ with vanishing ideal

$$I(\tilde{X}) = \langle g_1, \ldots, g_r \rangle \subseteq \mathbb{K}[T_1, \ldots, T_r].$$

(ii) With the zero sets $V(F) \subseteq \mathbb{T}^n$ and $V(G) \subseteq \mathbb{K}^r$ and the notation of Construction 3.5, we have

$$\hat{X} = \overline{V(G)} \cap \mathbb{T}^r \subseteq \hat{Z}, \quad X = \overline{V(F)} \subseteq Z.$$

In particular, the irreducible components of $X \subseteq Z$ are the closures of the irreducible components of $V(F) \subseteq \mathbb{T}^n$.

(iii) The closed hypersurfaces $X_j = \overline{V(f_j)} \subseteq Z$, where $j = 1, \ldots, s$, represent $X$ as a scheme-theoretically locally complete intersection

$$X = X_1 \cap \ldots \cap X_s \subseteq Z.$$

(iv) If $\tilde{X} \setminus \hat{X}$ is of codimension at least two in $\tilde{X}$, then $\tilde{X}$ is irreducible and normal and, moreover, $X$ is irreducible.

**Proof.** Assertion (i) is clear by Proposition 3.7 (i) and the Jacobian criterion for complete intersections. For (ii), consider a nonempty $\hat{X}(\sigma) = \hat{X} \cap \mathbb{T}^r \cdot z_\sigma$. Then

$$\dim(\hat{X}(\sigma)) = \dim(\mathbb{T}^r \cdot z_\sigma) - s = r - \dim(\hat{\sigma}) - s < r - s = \dim(\tilde{X}),$$

where we use Proposition 3.7 (iii) for the first equality. In particular, no irreducible component of $\tilde{X}$ is contained in $\hat{X} \setminus \mathbb{T}^r$. The assertions follow.

We prove (iii). Each $f_j$ defines a divisor on $Z$ having support $X_j$ and according to Lemma 3.3 (v) this divisor is base point free on $Z$. Thus, for every $\sigma \in \Sigma$, we find a monomial $h_{\sigma,j}$ of the same $K$-degree as $g_j$ without zeroes on the affine chart $\hat{Z}_\sigma \subseteq$
\[ \hat{Z} \text{ defined by } \hat{\sigma} \in \hat{\Sigma}. \] We conclude that the invariant functions \( g_1/h_{\sigma,1}, \ldots, g_s/h_{\sigma,s} \) generate the vanishing ideal of \( X \) on the affine toric chart \( Z_\sigma \subseteq Z \).

We turn to (iv). Proposition 3.7 and the assumption that \( \bar{X} \setminus \hat{X} \) is of codimension at least two in \( \bar{X} \) allow us to apply Serre’s criterion and we obtain that \( \bar{X} \) is normal. In order to see that \( \bar{X} \) is irreducible, note that \( H \) acts on \( \hat{Z} \) with attractive fixed point \( 0 \in \hat{Z} \). This implies \( 0 \in \bar{X} \), Hence \( \bar{X} \) is connected and thus, by normality, irreducible.

\[ \mathbf{Corollary 3.10.} \text{ For a non-degenerate system } F = (f_1, \ldots, f_s) \text{ in } LP(n), \text{ an } F\text{-fan } \Sigma \text{ in } \mathbb{Z}^n \text{ with generator matrix } P = [v_1, \ldots, v_r] \text{ and associated toric variety } Z, \text{ set } \]
\[ X := V(F) \subseteq Z. \]

If \( \hat{Z} \setminus \hat{Z} \) is of dimension at most \( r - s - 2 \), then \( X \) is irreducible. In particular, if \( Z \) is a fake weighted projective space and \( s < n \), then \( X \) is irreducible.

\[ \mathbf{Proof.} \text{ Let } \bar{X} = V(G) \subseteq K^r \text{ be the subvariety defined by the } \Sigma\text{-homogenization of } F. \text{ Theorem 3.9 (i) says that we have } \bar{X} \text{ is of dimension } r - s. \text{ By definition, } \bar{X} \setminus \hat{X} \text{ is contained in } \hat{Z} \setminus \hat{Z}. \text{ Thus, } \bar{X} \setminus \hat{X} \text{ is of codimension at least two in } \bar{X} \text{ and Theorem 3.9 (iv) applies. This proves the first statement. The second one follows from the facts that for an } n\text{-dimensional fake weighted projective space } Z, \text{ we have } r = n + 1 \text{ and } \hat{Z} \setminus \hat{Z} \text{ consists of the origin of } \hat{Z} = K^{n+1}; \text{ see Example 2.2.} \]

The statements (i) and (iv) of Theorem 3.9 extend in the following way to the pieces cut out from \( \hat{X} \) by the closures of the \( \mathbb{T}^r \)-orbits of \( \hat{Z} = K^r \).

\[ \mathbf{Proposition 3.11.} \text{ Consider a non-degenerate system } F = (f_1, \ldots, f_s) \text{ in } LP(n), \text{ an } F\text{-fan } \Sigma \text{ in } \mathbb{Z}^n, \text{ the } \Sigma\text{-homogenization } G = (g_1, \ldots, g_s) \text{ of } F, \text{ a cone } \sigma \in \Sigma \text{ and } \]
\[ \bar{Z}(\sigma) = V(T_i; v_i \in \sigma), \quad \bar{X}(\sigma) = \bar{X} \cap \bar{Z}(\sigma). \]

If \( \hat{X}(\sigma) \setminus \hat{X}(\sigma) \) is of codimension least one in \( \bar{X}(\sigma) \), then \( \bar{X}(\sigma) = \bar{X} \cap \bar{Z}(\sigma) \) is a subvariety of pure codimension \( s \) in \( \bar{Z}(\sigma) \) with vanishing ideal
\[ I(\bar{X}(\sigma)) = \langle g_1^\sigma, \ldots, g_s^\sigma \rangle \subseteq K[T_i; v_i \notin \sigma]. \]

If \( \hat{X}(\sigma) \setminus \hat{X}(\sigma) \) is of codimension at least two in \( \bar{X}(\sigma) \), then the variety \( \hat{X}(\sigma) \) is irreducible and normal.

\[ \mathbf{Proof.} \text{ If } \hat{X}(\sigma) \setminus \hat{X}(\sigma) \text{ is of codimension least one in } \bar{X}(\sigma), \text{ then Proposition 3.7 and the Jacobian criterion ensure that } \bar{X}(\sigma) \text{ is a complete intersection in } K^r \text{ with the equations } g_j = 0, j = 1, \ldots, s, \text{ and } T_i = 0, v_i \in \sigma. \text{ This gives the first statement. If } \hat{X}(\sigma) \setminus \hat{X}(\sigma) \text{ is of codimension at least two in } \bar{X}(\sigma), \text{ then we obtain irreducibility and normality as in the proof of (iv) of Theorem 3.9, replacing } \hat{X} \text{ with } \hat{X}(\sigma). \]

4. Non-degenerate Toric Complete Intersections

We take a closer look at the geometry of the varieties $X \subseteq Z$ arising from non-degenerate Laurent systems. The main statements of the section are Theorem 4.3, showing that $X \subseteq Z$ is always quasi-smooth and Theorem 4.8 giving details on how $X$ sits inside $Z$. Using these, we can prove Theorem 1.1 which describes the anticanonical complex. First we give a name to our varieties $X \subseteq Z$, motivated by Theorem 4.8. Finally, we see that for a general choice of the defining Laurent system and an easy-to-check assumption on the ambient toric variety $Z$, we obtain divisor class group and Cox ring of $X$ for free, see Corollary 4.20.

**Definition 4.1.** By a non-degenerate toric complete intersection we mean a subvariety $X \subseteq Z$ defined by a non-degenerate system $F$ in $\text{LP}(n)$ and an $F$-fan $\Sigma$ in $\mathbb{Z}^n$.

We first take a look at the possible singularities of toric complete intersections and global resolutions of these singularities. The class of singularities we will have to deal with is the following; see also [2].

**Definition 4.2.** We call a variety $X$ quasi-smooth if it is covered by affine open sets $U_i$ being the good quotient of a smooth affine variety $\tilde{U}_i$ by a quasitorus $H_i$.

According to Cox’s quotient construction, every toric variety is quasi-smooth in the above sense. Note that quasi-smooth singularities need not be orbifold singularities. For instance, $V(T_1 T_2 - T_3 T_4) \subseteq \mathbb{K}^4$ is not $\mathbb{Q}$-factorial but quasi-smooth, as it is the quotient of $\mathbb{K}^4$ by $\mathbb{K}^*$ acting via $t \cdot (z_1, z_2, z_3, z_4) = (t z_1, t^{-1} z_2, t z_3, t^{-1} z_4)$.

We are ready for our statement about the singularities of toric complete intersections. The second part of the theorem below is Khovanskii’s resolution of singularities [25, Thm. 2.2]; observe that our proof works without any ingredients from the theory of holomorphic functions.

**Theorem 4.3.** Let $X' \subseteq Z$ and $X \subseteq Z$ be non-degenerate toric complete intersections given by a system $F$ in $\text{LP}(n)$ and $F$-fans $\Sigma'$, $\Sigma$ in $\mathbb{Z}^n$, where $\Sigma'$ refines $\Sigma$.

(i) The variety $X$ is normal, quasi-smooth and we have $X \cap Z_{\text{reg}} \subseteq X_{\text{reg}}$.

(ii) If $\Sigma'$ regular, then $X' \to X$ is a resolution of singularities.

In particular, every non-degenerate toric complete intersections given by a system $F$ in $\text{LP}(n)$ and a regular $F$-fan $\Sigma$ in $\mathbb{Z}^n$ is smooth.

**Proof.** By Proposition 3.7 (i), the variety $\hat{X}$ is smooth. By Construction 3.5, we have the good quotient $p : \hat{X} \to X$ by the quasitorus $H = \ker(p)$. Thus, $X$ is quasi-smooth. Being a smooth variety $\hat{X}$ is also normal. As good quotients preserve normality, we see that $X$ is normal. Moreover, the quasitorus $H = \ker(p)$ acts freely on $p^{-1}(Z_{\text{reg}})$, hence on $\hat{X} \cap p^{-1}(Z_{\text{reg}})$ and thus $p : \hat{X} \to X$ preserves smoothness over $X \cap Z_{\text{reg}}$. This proves (i). For (ii), just observe that we have $Z_{\text{reg}} = Z'$.

Consequently, $X'_{\text{reg}} = X'$ holds due to (i). $\square$
The next aim is to provide details on the position of $X$ inside the toric variety $Z$. Our considerations elaborate the transversality statement on $X$ and the torus orbits of $Z$ made in [25] for the smooth case. Here, tropical varieties enter the game. We briefly recall the necessary background from this topic, where, due to our needs, we can restrict to the case of a trivially valued algebraically closed ground field of characteristic zero.

**Construction 4.4.** For $f \in \text{LP}(n)$ let $S^{n-1}$ be the set of $(n-1)$-dimensional cones of the normal fan of its Newton polytope. The *tropical hypersurface* of $f$ is

$$\text{trop}(f) := \bigcup_{\sigma \subseteq Q^n} \sigma.$$

Given a closed subvariety $X \subseteq \mathbb{T}^n$, let $I(K) \subseteq \text{LP}(n)$ be the associated vanishing ideal. Then the *tropical variety* associated with $X$ is

$$\text{trop}(X) := \bigcap_{I(X)} \text{trop}(f) \subseteq Q^n.$$

We will make use of the structure theorem for tropical varieties, going back to Bieri and Groves [7]. In order to state it, we need further notions from convex geometry. Consider a fan $\Delta$ in $\mathbb{Z}^n$ of pure dimension $d$, that means that all maximal cones of $\Delta$ are of dimension $d$. One calls $\Delta$ connected in codimension one, if any two points of its support can be connected via a path leading only through cones of dimension at least $d-1$. Moreover, a *balancing* of $\Delta$ is a map assigning to every maximal cone $\delta \in \Delta$ a weight $w_\delta \in \mathbb{Z}_{>0}$ such that for each $(d-1)$-dimensional cone $\tau \in \Delta$, the primitive generators $v_\delta$ of the one-dimensional fan $\{\delta + \text{lin}(\tau); \tau \lesssim \delta\}$ in $\mathbb{Z}^n/(\text{lin}(\tau) \cap \mathbb{Z}^n)$ satisfy $\sum w_\delta v_\delta = 0$. The structure theorem for tropical varieties then reads as follows; see [26, Thm. 3.3.6, Cor. 3.5.5].

**Theorem 4.5.** Let $X \subseteq \mathbb{T}^n$ be an irreducible closed subvariety of dimension $d$. Then $\text{trop}(X)$ is the support of a fan $\Delta$ in $\mathbb{Z}^n$, which is of pure dimension $d$, connected in codimension one and admits a balancing.

Another important fact is Tevelev’s Lemma; see [31, Lemma 2.2] and also [26, Thm. 6.3.4]. We reformulate it using the following notation.

**Notation 4.6.** Let $Z$ be the toric variety arising from a fan $\Sigma$ in $\mathbb{Z}^n$. Recall that for $\sigma \in \Sigma$ we set $X(\sigma) = X \cap \mathbb{T}^n \cdot \sigma$. Given a closed subvariety $X \subseteq Z$, we write

$$\Sigma_X := \{ \sigma \in \Sigma; X(\sigma) \neq \emptyset \}, \quad \text{trop}(X) := \text{trop}(X \cap \mathbb{T}^n).$$

**Lemma 4.7.** (Tevelev) Consider a fan $\Sigma$ in $\mathbb{Z}^n$, its associated toric variety $Z$ and a closed subvariety $X \subseteq Z$. Then $\sigma \in \Sigma$ belongs to $\Sigma_X$ if and only if $\text{trop}(X)$ intersects the relative interior of $\sigma$.

The set of cones $\Sigma_X$ plays a central role in combinatorial Cox ring theory [4, Chap. 3]. In general, $\Sigma_X \subseteq \Sigma$ is far from being nicely structured, for instance, from being a subfan. Our next result describes the situation for non-degenerate toric complete intersections.
Theorem 4.8. Consider an irreducible non-degenerate toric complete intersection $X \subseteq Z$ given by a system $F = (f_1, \ldots, f_s)$ in LP($n$) and an $F$-fan $\Sigma$ in $\mathbb{Z}^n$.

(i) For every $\sigma \in \Sigma_X$, the scheme $X(\sigma) \cap \mathbb{T}^n \cdot z_\sigma$ is a closed subvariety of pure codimension $s$ in $\mathbb{T}^n \cdot z_\sigma$.

(ii) The subset $\Sigma_X \subseteq \Sigma$ is a subfan and the subset $Z_X := \mathbb{T}^n \cdot X \subseteq Z$ is an open toric subvariety.

(iii) All maximal cones of $\Sigma_X$ are of dimension $n - s$ and the support of $\Sigma_X$ equals $\text{trop}(X)$.

Proof. We prove (i). Given a cone $\sigma \in \Sigma_X$ consider $\hat{\delta} \in \hat{\Sigma}$ and the corresponding affine toric charts and the restricted quotient map:

$$
\begin{align*}
\hat{X} \cap \hat{Z}_{\hat{\delta}} &= \hat{X}_{\hat{\delta}} \subseteq \hat{Z}_{\hat{\delta}} = p^{-1}(Z_\sigma) \\
X \cap Z_\sigma &= X_\sigma \subseteq Z_\sigma
\end{align*}
$$

From Proposition 3.7 we infer that $\hat{X}(\hat{\delta}) = \mathbb{T}^r \cdot z_{\hat{\delta}} \cap \hat{X}$ is a reduced subscheme of pure codimension $s$ in $\mathbb{T}^r \cdot z_{\hat{\delta}}$. The involved vanishing ideals on $Z_\sigma$ and $\hat{Z}_{\hat{\delta}}$ satisfy

$$I(X_\sigma) + I(\mathbb{T}^n \cdot z_\sigma) = I(\hat{X}_{\hat{\delta}})^H + I(\mathbb{T}^r \cdot z_{\hat{\delta}})^H = \left( I(\hat{X}_{\hat{\delta}}) + I(\mathbb{T}^r \cdot z_{\hat{\delta}}) \right)^H.$$

We conclude that the left hand side ideal is radical. In order to see that $X(\sigma)$ is of codimension $s$ in $\mathbb{T}^n \cdot z_\sigma$, look at the restriction

$$p : \mathbb{T}^r \cdot z_{\hat{\delta}} \rightarrow \mathbb{T}^n \cdot z_\sigma.$$

This is a geometric quotient for the $H$-action, it maps $\hat{X}(\hat{\delta})$ onto $X(\sigma)$ and, as $\hat{X}(\hat{\delta})$ is $H$-invariant, it preserves codimensions.

We prove (ii) and (iii). First note that, due to (i), every cone $\sigma \in \Sigma_X$ satisfies $\dim(\sigma) \leq n - s$. In a first step, we show that $\text{trop}(X)$ is contained in the support of $\Sigma_X$. Since $\Sigma$ is complete, every point of $\text{trop}(X)$ lies in the relative interior of some cone $\sigma \in \Sigma$. According to Tevelev’s Lemma, this cone $\sigma$ belongs to $\Sigma_X$. We conclude that $\text{trop}(X)$ is covered by the cones of $\Sigma_X$.

From now on, we make use of the structure theorem for tropical varieties. It provides us with a balanced fan structure $\Delta$ on $\text{trop}(X)$ such that all maximal cones of $\Delta$ are of dimension $n - s$.

The next step is to show that any given maximal cone $\sigma \in \Sigma_X$ is of dimension $n - s$. Tevelev’s Lemma says that the relative interior of $\sigma$ intersects a cone $\delta$ from $\Delta$, which we may assume to be maximal and hence of dimension $n - s$. As already seen, $\delta$ is covered by cones $\sigma_1, \ldots, \sigma_k \in \Sigma_X$, each of which we may assume to be of dimension $n - s$. As $\sigma$ and the $\sigma_i$ all belong to the fan $\Sigma$, we must have $\sigma \subseteq \sigma_i$ for some $i$. By maximality of $\sigma$, this means $\sigma = \sigma_i$ and hence $\dim(\sigma) = n - s$.

We show that every maximal cone $\sigma \in \Sigma_X$ is covered by maximal cones from $\Delta$. As just seen, $\dim(\sigma) = n - s$. Using again Tevelev’s Lemma, we find maximal cones $\delta \in \Delta$ that intersect the relative interior of $\sigma$. Let $\delta_1, \ldots, \delta_k \in \Delta$ be
all maximal cones with this property. We claim that \( \delta_1, \ldots, \delta_k \) cover \( \sigma \). Otherwise, we find inside the relative interior \( \sigma^\circ \) of \( \sigma \) a boundary point \( v \) of \( \sigma^\circ \cap (\delta_1, \ldots, \delta_k) \), sitting on a facet \( \delta'_i \) of one of the \( \delta_i \). The balance condition implies that any cone of dimension \( n - s - 1 \) of \( \Delta \) is a facet of at least two maximal cones of \( \Delta \). Thus, there must be another cone \( \delta \in \Delta \) having \( \delta'_i \) as a facet. As \( \delta \) intersects \( \sigma^\circ \), it must be some \( \delta_j \). This contradicts to the fact that \( v \in \delta'_i \) is a boundary point.

Knowing that \( \text{trop}(X) \) is precisely the union of the cones of \( \Sigma_X \), we directly see that \( \Sigma_X \) is a fan: Given \( \sigma \in \Sigma_X \), every face \( \tau \preccurlyeq \sigma \) is contained in \( \text{trop}(X) \). In particular, the relative interior of \( \tau \) intersects \( \text{trop}(X) \). Using once more Tevelev’s Lemma, we obtain \( \tau \in \Sigma_X \).

**Corollary 4.9.** Let \( X \subseteq Z \) be a non-degenerate toric complete intersection given by \( F = (f_1, \ldots, f_s) \) in \( \mathbb{LP}(n) \) and a simplicial \( F \)-fan \( \Sigma \). If \( \hat{X} \backslash \hat{X} \) is of dimension strictly less than \( r - n \), then we have

\[
\Sigma_X = \{ \sigma \in \Sigma ; \dim(\sigma) \leq n - s \}.
\]

**Proof.** Assume that \( \sigma \in \Sigma \) is of dimension \( n - s \) but does not belong to \( \Sigma_X \). Then \( X(\sigma) = \emptyset \) and hence \( \hat{X}(\sigma) = \emptyset \). This implies

\[
\hat{X}(\sigma) = V(g_1, \ldots, g_s) \cap V(T_i; \; v_i \in \sigma) = \bigcup_{\hat{\tau} < \tau} \hat{X} \cap \mathbb{T}^r \cdot z_\tau.
\]

As \( \Sigma \) is simplicial, \( P \) defines a bijection from \( \hat{\Sigma} \) onto \( \Sigma \). Moreover, \( \hat{\sigma} \) and \( \sigma \) both have \( n - s \) rays and we can estimate the dimension of \( \hat{X}(\sigma) \) as

\[
\dim(\hat{X}(\sigma)) \geq r - (n - s) \geq r - n.
\]

Due to \( \dim(\hat{X} \backslash \hat{X}) < r - n \), we have \( \hat{X} \cap \mathbb{T}^r \cdot z_\tau \subseteq \hat{X} \) for some \( \hat{\sigma} < \tau \in \hat{\Sigma} \). Thus, \( \sigma \) is a proper face of \( P(\tau) \in \Sigma_X \). This contradicts to Theorem 4.8 (iii).

**Example 4.10.** Let \( f = S_1 + S_2 + 1 \in \mathbb{K}[S_1, S_2, S_3] \) and \( \Sigma \) the fan in \( \mathbb{Z}^3 \) given via its generator matrix \( P = [v_1, \ldots, v_5] \) and maximal cones \( \sigma_{ijk} = \text{cone}(v_i, v_j, v_k) \):

\[
P = \begin{bmatrix}
-2 & 2 & 0 & 0 & 0 \\
-2 & 0 & 2 & 0 & 0 \\
1 & 1 & 1 & 1 & -1
\end{bmatrix}, \quad \Sigma_{\text{max}} = \{\sigma_{124}, \sigma_{134}, \sigma_{234}, \sigma_{125}, \sigma_{135}, \sigma_{235}\}.
\]

Then \( f \) is non-degenerate in \( \mathbb{LP}(3) \) and \( \Sigma \) is an \( F \)-fan. Thus, we obtain a non-degenerate toric hypersurface \( X \subseteq Z \). The \( \Sigma \)-homogenization of \( f \) is

\[
g = T_1^2 + T_2^2 + T_3^2.
\]

The minimal ambient toric variety \( Z_X \subseteq Z \) is the open toric subvariety given by the fan \( \Sigma_X \) with the maximal cones \( \sigma_{ij} = \text{cone}(v_i, v_j) \) given as follows

\[
\Sigma_{X_{\text{max}}} = \{\sigma_{14}, \sigma_{15}, \sigma_{24}, \sigma_{25}, \sigma_{34}, \sigma_{35}\}.
\]

In particular, the fan \( \Sigma_X \) is a proper subset of the set of all cones of dimension at most two of the fan \( \Sigma \).
Remark 4.11. The variety $X$ from Example 4.10 is a rational $\mathbb{K}^*$-surface as constructed in [4, Sec. 5.4]. More generally, every weakly tropical general arrangement variety in the sense of [16, Sec. 5] is an example of a non-degenerate complete toric intersection.

We approach the proof of Theorem 1.1. The following pullback construction relates divisors of $Z$ to divisors on $X$.

Construction 4.12. Let $X \subseteq Z$ be an irreducible non-degenerate toric complete intersection. Denote by $i : X \cap Z_{\text{reg}} \to X$ and $j : X \cap Z_{\text{reg}} \to Z_{\text{reg}}$ the inclusions. Then Theorems 4.3 and 4.8 (ii) yield a well defined pullback homomorphism

$$\text{WDiv}^\mathbb{T}(Z) = \text{WDiv}^\mathbb{T}(Z_{\text{reg}}) \to \text{WDiv}(X), \quad D \mapsto D|_X = i_* j^* D,$$

where we set $\mathbb{T} = \mathbb{T}^n$ for short. By Theorem 4.8 (i), this pullback sends any invariant prime divisor on $Z$ to a sum of distinct prime divisors on $X$. Moreover, we obtain a well defined induced pullback homomorphism for divisor classes

$$\text{Cl}(Z) \to \text{Cl}(X), \quad [D] \mapsto [D]|_X.$$

The remaining ingredients are the adjunction formula given in Proposition 4.13 and Proposition 4.14 providing canonical divisors which are suitable for the ramification formula.

Proposition 4.13. Let $X \subseteq Z$ be an irreducible non-degenerate toric complete intersection given by a system $F = (f_1, \ldots, f_s)$ in $LP(n)$.

(i) Let $C_j \in \text{WDiv}(Z)$ be the pushforward of $\text{div}(f_j)$ and $K_Z$ an invariant canonical divisor on $Z$. Then the canonical class of $X$ is given by

$$[K_X] = [K_Z + C_1 + \ldots + C_s]|_X \in \text{Cl}(X).$$

(ii) The variety $X$ is $\mathbb{Q}$-Gorenstein if and only if $Z_X$ is $\mathbb{Q}$-Gorenstein. If one of these statements holds, then $X$ and $Z_X$ have the same Gorenstein index.

(iii) Consider the boundary divisor $D_X := -K_Z|_X$ on $X$. Then the pair $(X, D_X)$ is canonical.

Proof. Due to Theorem 4.3 and Theorem 4.8 (ii) it suffices to have the desired canonical divisor on $Z_{\text{reg}} \cap X \subseteq X_{\text{reg}}$. By Theorem 3.9, the classical adjunction formula applies, proving (i). For (ii), note that the divisors $C_j$ on $Z$ are base point free by Lemma 3.3 (v) and hence Cartier. The assertions of (ii) follow. For (iii) we just note that $K_X + D_X$ equals $C_1 + \ldots + C_s$ and thus is Cartier as just observed. $\square$

Proposition 4.14. Consider an irreducible non-degenerate system $F$ in $LP(n)$, a refinement $\Sigma' \to \Sigma$ of $F$-fans and the corresponding ambient modification

$$X' \subseteq Z'$$

$$\begin{array}{ccc}
X' & \subseteq & Z' \\
\pi & & \pi \\
\downarrow & & \downarrow \\
X & \subseteq & Z
\end{array}$$

of toric complete intersections. Then, for every $\sigma \in \Sigma_X$, there are canonical divisors $K_X(\sigma)$ on $X$ and $K_{X'}(\sigma)$ on $X'$ such that
\[(i) \ K_{X'}(\sigma) = \pi^*K_X(\sigma) \text{ holds on } X' \setminus Y', \text{ where } Y' \subseteq Z' \text{ is the exceptional locus of the toric modification } \pi : Z' \rightarrow Z, \]

\[(ii) \ K_{X'}(\sigma) - \pi^*K_X(\sigma) = K_{Z'}|_{X'} - \pi^*K_Z|_{X'} \text{ holds on } \pi^{-1}(Z_\sigma) \cap X', \text{ where } Z_\sigma \subseteq Z_X \text{ is the affine toric chart defined by } \sigma \in \Sigma_X. \]

**Proof.** Fix \( \sigma \in \Sigma_X \). Then there is a vertex \( u \in B \) of the Newton polytope \( B = B(F) \) such that the maximal cone \( \sigma(u) \in \Sigma(B) \) contains \( \sigma \). Write \( u = u_1 + \ldots + u_s \) with vertices \( u_j \in B(f_j) \). With the corresponding vertices \( a(u_j) = P^*u_j + a_j \) of the Newton polytopes \( B(g_j) \), we define

\[D(\sigma, j) := a(u_j)_1D_1 + \ldots + a(u_j)_rD_r \in \text{WDiv}(Z).\]

Let \( C_j \in \text{WDiv}(Z) \) be the pushforward of \( \text{div}(f_j) \). Propositions 2.6 and 2.10 together with Lemma 3.3 (v) tell us

\[[D(\sigma, j)] = [C_j] = \deg(g_j) \in K = \text{Cl}(Z), \quad \text{supp}(D(\sigma, j)) \cap Z_\sigma = \emptyset.\]

Also for the \( \Sigma' \)-homogenization \( G' \) of \( F \), the vertices \( u_j \in B(f_j) \) yield corresponding vertices \( a'(u_j) \in B(g'_j) \) and define divisors

\[D'(\sigma, j) := a'(u_j)_1D_1 + \ldots + a'(u_j)_rD_r + l \in \text{WDiv}(Z').\]

As above we have the pushforwards \( C'_j \in \text{WDiv}(Z') \) of \( \text{div}(f_j) \) and, by the same arguments, we obtain

\[[D'(\sigma, j)] = \deg(g'_j) \in K' = \text{Cl}(Z'), \quad \text{supp}(D'(\sigma, j)) \cap \pi^{-1}(Z_\sigma) = \emptyset.\]

Take the invariant canonical divisors \( K_Z \) on \( Z \) and \( K_{Z'} \) in \( Z' \) with multiplicity \(-1\) along all invariant prime divisors and set

\[K_X(\sigma) := (K_Z + \sum_{j=1}^{s} D(\sigma, j))|_X, \quad K_{X'}(\sigma) := (K_{Z'} + \sum_{j=1}^{s} D'(\sigma, j))|_{X'}.\]

According to Proposition 4.13, these are canonical divisors on \( X \) and \( X' \) respectively. Properties (i) and (ii) are then clear by construction. \( \square \)

**Proof of Theorem 1.1.** First observe that \( A_X \) is an anticanonical complex for the toric variety \( Z_X \). Now, choose any regular refinement \( \Sigma' \rightarrow \Sigma \) of the defining \( F \)-fan \( \Sigma \) of the irreducible non-degenerate toric complete intersection \( X \subseteq Z \). This gives us modifications \( \pi : Z' \rightarrow Z \) and \( \pi : X' \rightarrow X \). Standard toric geometry and Theorem 4.3 yield that both are resolutions of singularities.

Proposition 4.14 provides us with canonical divisors on \( X' \) and \( X \). We use them to compute discrepancies. Over each \( X \cap Z_\sigma \), where \( \sigma \in \Sigma_X \), we obtain the discrepancy divisor as

\[K_{X'}(\sigma) - \pi^*K_X(\sigma) = K_{Z'}|_X - \pi^*K_Z|_X.\]

By Theorem 4.8 (i), every exceptional prime divisor \( E'_X \subseteq X' \) admits a unique exceptional prime divisor \( E'_Z \subseteq Z' \) with \( E'_X \subsetneq E'_Z \). Construction 4.12 guarantees that the discrepancy of \( E'_X \) with respect to \( \pi : X' \rightarrow X \) and that of \( E'_Z \) with respect to \( \pi : Z' \rightarrow Z_X \) coincide. \( \square \)
We conclude the section by discussing the divisor class group and the Cox ring of a non-degenerate toric complete intersection and the effect of a general choice of the defining Laurent system.

**Proposition 4.15.** Consider a non-degenerate toric complete intersection $X \subseteq \mathbb{Z}$ given by a system $F = (f_1, \ldots, f_s)$ in $\text{LP}(n)$ and an $F$-fan $\Sigma$ in $\mathbb{Z}^n$. If $\hat{X} \setminus \hat{X}$ is of codimension at least two in $\hat{X}$ and the pullback $\text{Cl}(\mathbb{Z}) \to \text{Cl}(X)$ is an isomorphism, then the Cox ring of $X$ is given by

$$\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_r]/(g_1, \ldots, g_s), \quad \text{deg}(T_i) = [X_i] \in \text{Cl}(X),$$

where $G = (g_1, \ldots, g_s)$ is the $\Sigma$-homogenization of $F$. In this situation, we have moreover the following statements.

(i) If $\hat{X} \cap V(T_i) \setminus \hat{X}$ is of codimension at least two in $\hat{X} \cap V(T_i)$, then $T_i$ defines a prime element in $\mathcal{R}(X)$.

(ii) If $\text{deg}(g_j) \neq \text{deg}(T_i)$ holds for all $i, j$, then the variables $T_1, \ldots, T_r$ define a minimal system of generators for $\mathcal{R}(X)$.

**Proof.** Theorem 3.9 (iv) ensures that $\hat{X}$ is normal. This allows us to apply [4, Cor. 4.1.1.5], which shows that the Cox ring $\mathcal{R}(X)$ is as claimed. The supplementary assertion (i) is a consequence of Proposition 3.11 and (ii) is clear. \qed

In this proposition, the assumption of $\text{Cl}(\mathbb{Z}) \to \text{Cl}(X)$ being an isomorphism might be difficult to check in practice. We can get rid of it by considering general Laurent systems and using the Grothendieck-Lefschetz Theorem. Let us first make precise what we mean by “general”.

**Definition 4.16.** Let $B_1, \ldots, B_s \subseteq \mathbb{Q}^n$ be lattice polytopes. Then each of the $B_j$ defines a finite dimensional vector subspace

$$V(B_j) := \bigoplus_{v \in B_j \cap \mathbb{Z}^n} \mathbb{K} \cdot T_1^{v_1} \cdots T_n^{v_n} \subseteq \text{LP}(n).$$

The *Laurent space* associated with the polytopes $B_1, \ldots, B_s \subseteq \mathbb{Q}^n$ is the finite-dimensional vector space

$$V(B_1, \ldots, B_s) := V(B_1) \oplus \cdots \oplus V(B_s).$$

By a *general Laurent system* we mean a non-empty open subset $\mathcal{F} \subseteq V(B_1, \ldots, B_s)$ such that every $F = (f_1, \ldots, f_s) \in \mathcal{F}$ satisfies

$$B(f_1) = B_1, \ldots, B(f_s) = B_s.$$

**Definition 4.17.** Let $B_1, \ldots, B_s \subseteq \mathbb{Q}^n$ be lattice polytopes, $\mathcal{F} \subseteq V(B_1, \ldots, B_s)$ a general Laurent system and $\Sigma$ a fan in $\mathbb{Z}^n$.

(i) We call $\mathcal{F}$ *non-degenerate* if every Laurent system $F \in \mathcal{F}$ is so.

(ii) We call $\Sigma$ an $\mathcal{F}$-*fan* if it refines the normal fan of $B_1 + \ldots + B_s$. 
Let $\mathcal{F}$ be non-degenerate, $\Sigma$ an $\mathcal{F}$-fan and $Z$ the associated toric variety. Then every $F \in \mathcal{F}$ defines a non-degenerate toric complete intersection

$$X = \overline{V(F)} \subseteq Z.$$  

The general toric complete intersection given by $F$ is the set $\mathcal{X}$ of all varieties $X = V(F) \subseteq Z$, where $F \in \mathcal{F}$. We call $\mathcal{X}$ Fano, smooth, etc. if all its elements are so.

Let us reformulate in this context Khovanskii’s statement from [25, Thm. 2.2], saying that almost all Laurent system are non-degenerate; for a proof of the following, we refer to [27, Prop. 1.5.7].

**Proposition 4.18.** Let $B_1, \ldots, B_s \subseteq \mathbb{Q}^n$ be lattice polytopes. Then the set of all non-degenerate systems $F = (f_1, \ldots, f_s)$ in $LP(n)$ such that $f_i$ has $B_i$ as its Newton polytope for $i = 1, \ldots, s$ is non-empty and open in $V(B_1, \ldots, B_s)$. In particular, $V(B_1, \ldots, B_s)$ admits a non-degenerate general Laurent system.

Here comes our statement on the divisor class group and the Cox ring of a toric complete intersection using generality with respect to the Laurent space.

**Proposition 4.19.** Consider lattice polytopes $B_1, \ldots, B_s \subseteq \mathbb{Q}^n$, where $n - s \geq 3$, a non-degenerate general Laurent system $F = (f_1, \ldots, f_s)$ in $\mathcal{F} \subseteq V(B_1, \ldots, B_s)$ and an $\mathcal{F}$-fan $\Sigma$ in $\mathbb{Z}^n$ with generator matrix $P = [v_1, \ldots, v_r]$. Assume that all members $X = X_1 \cap \ldots \cap X_n$ of the associated toric complete intersection $\mathcal{X}$ satisfy

(\*) for $X'_0 := Z$ and $X'_j := X_j \cap X'_{j-1}$, $j = 1, \ldots, s$, each $X_{j+1}|X'_j$ is ample and each $\hat{X}'_j \setminus \hat{X}'_{j-1}$ is of codimension at least two in $\hat{X}'_j$.

Then every $X \in \mathcal{X}$ is irreducible and normal, the pullback $\text{Cl}(Z) \rightarrow \text{Cl}(X)$ is an isomorphism and the Cox ring of $X$ is given as

$$\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_r]/\langle g_1, \ldots, g_s \rangle, \quad \deg(T_i) = [D_i] \in \text{Cl}(X) = \text{Cl}(Z),$$

where $G = (g_1, \ldots, g_s)$ is the $\Sigma$-homogenization of $F = (f_1, \ldots, f_s) \in \mathcal{F}$ defining $X \in \mathcal{X}$ and $D_i \subseteq Z$ the prime divisor corresponding to $T_i \in \mathcal{R}(Z) = \mathbb{K}[T_1, \ldots, T_r]$.

**Proof.** First observe that all the $X'_j$ stemming from $\mathcal{F}$ are non-degenerate toric complete intersections of dimension at least three. By assumption (\*), they are all irreducible, $X_j|X'_{j-1}$ is ample and Lemma 3.3 yields that $X_j|X'_{j-1}$ is base point free. Thus, the Grothendieck-Lefschetz Theorem [29] yields pullback isomorphisms

$$\psi_j : \text{Cl}(X'_{j-1}) \rightarrow \text{Cl}(X'_j), \quad j = 1, \ldots, s,$$

provided the divisor $X_j|X'_{j-1}$ on the variety $X'_{j-1}$ is chosen generally in its linear system. In the initial step, the linear system of $X_1$ is just the projective space over the corresponding homogeneous component of the Cox ring, which due to Lemma 3.3 (ii) is given by

$$\mathcal{R}(Z)[X_1] \cong V(B_1).$$
Shrinking $\mathcal{F}$ suitably, we achieve that all the $X_1$ are general in their linear systems and thus come with pullback isomorphisms $\psi$. We proceed inductively. Assumption (⋆) ensures that $X_j^\prime \setminus \hat{X}_j^\prime$ is of codimension at least two in $X_j^\prime$. Applying Proposition 4.15 to $X_{j-1}^\prime$, we see

$$\mathcal{R}(X_{j-1}^\prime)[x_j|x_{j-1}^\prime] = \mathcal{R}(Z)[x_j]/\langle g_1, \ldots, g_{j-1} \rangle[x_j],$$

where $\mathcal{R}(Z)[x_j] = V(B_j)$, again by Lemma 3.3 (ii). Shrinking $\mathcal{F}$ again, we can ensure the necessary general choice for all $X_j|_{X_j^\prime}$ stemming from $\mathcal{F}$ and obtain the pullback isomorphisms $\psi_j: \text{Cl}(X_{j-1}^\prime) \to \text{Cl}(X_j^\prime)$. Composing all $\psi_j$ gives a pullback isomorphism $\text{Cl}(Z) \to \text{Cl}(X)$ and we can apply Proposition 4.15. □

**Corollary 4.20.** Consider lattice polytopes $B_1, \ldots, B_s \subseteq \mathbb{Q}^n$, where $n - s \geq 3$, a non-degenerate general Laurent system $\mathcal{F} \subseteq V(B_1, \ldots, B_s)$ and an $\mathcal{F}$-fan $\Sigma$ in $\mathbb{Z}^n$ with generator matrix $P = [v_1, \ldots, v_r]$. Consider the associated toric complete intersection $X$ and its members

$$X = X_1 \cap \ldots \cap X_s \subseteq Z.$$

Assume $\dim(\hat{Z} \setminus \hat{Z}) \leq r - s - 2$ and that each $X_j$ is an ample divisor on $Z$. Then $\text{Cl}(X) = \text{Cl}(Z)$ and the Cox ring of $X$ is given as

$$\mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_r]/\langle g_1, \ldots, g_s \rangle, \quad \deg(T_i) = [D_i] \in \text{Cl}(X) = \text{Cl}(Z),$$

where $G = (g_1, \ldots, g_s)$ is the $\Sigma$-homogenization of $F = (f_1, \ldots, f_s)$ and $D_i \subseteq Z$ the prime divisor corresponding to $T_i \in \mathcal{R}(Z) = \mathbb{K}[T_1, \ldots, T_r]$.

**Example 4.21.** Corollary 4.20 enables us to produce Mori dream spaces with prescribed properties. For instance, consider general toric hypersurfaces

$$X = V(f) \subseteq \mathbb{P}_{1,1,2} \times \mathbb{P}_{1,1,2} = Z,$$

where $f$ is $\mathbb{Z}^2$-homogeneous of bidegree $(d_1, d_2)$ with $d_1, d_2 \in \mathbb{Z}_{\geq 1}$. Corollary 4.20 directly yields $\text{Cl}(X) = \mathbb{Z}^2$ and delivers the Cox ring as

$$\mathcal{R}(X) = \mathbb{K}[T_0, T_1, T_2, S_0, S_1, S_2]/\langle f \rangle, \quad w_0 = w_1 = (1, 0), \quad w_2 = (2, 0), \quad u_0 = u_1 = (0, 1), \quad u_2 = (0, 2),$$

where $w_i = \deg(T_i)$ and $u_i = \deg(S_i)$. Corollary 1.2 tells us that $X$ has worst canonical singularities. Moreover, if for instance $d_1 = d_2 = d$, then in the cases

$$d > 4, \quad d = 4, \quad d < 4,$$

the Mori dream space $X$ has an ample canonical class, trivial canonical class or is Fano, accordingly; use Proposition 4.13.
5. Fake Weighted Terminal Fano Threefolds

Here we prove Theorem 1.3. The first and major part uses the whole theory developed so far to establish suitable upper bounds on the specifying data. Having reduced the problem to working out a manageable number of cases, we proceed computationally, which involves besides a huge number of divisibility checks the search for lattice points inside polytopes tracing back to the terminality criterion provided in Corollary 1.2. A second and minor part concerns the verifying and distinguishing items listed in Theorem 1.3, where we succeed with Corollary 4.20 and the computation of suitable invariants.

We fix the notation around a non-degenerate complete intersection $X$ in an $n$-dimensional fake weighted projective space $Z$. The defining fan of $\Sigma$ in $\mathbb{Z}^n$ is simplicial, complete and we denote its primitive generators by $v_0, \ldots, v_n$. From Example 2.2 we know that the divisor class group $\text{Cl}(Z)$ is of the form

$$\text{Cl}(Z) = \mathbb{Z} \times \mathbb{Z}/t_1 \mathbb{Z} \times \ldots \times \mathbb{Z}/t_q \mathbb{Z}$$

and, in concrete cases, we write its elements as $(x, \overline{e}_1, \ldots, \overline{e}_q)$, where $x \in \mathbb{Z}$ and $e_i = 0, 1, \ldots, t_i - 1$. By $w_i = (x_i, \eta_{i1}, \ldots, \eta_{iq}) \in \text{Cl}(Z)$ we denote the classes of the torus invariant prime divisors $D_i$ on $Z$. Recall, also from Example 2.2, that any $n$ of $w_0, \ldots, w_n$ generate the divisor class group $\text{Cl}(Z)$ and that we may assume

$$0 < x_0 \leq \ldots \leq x_n.$$

As before, $X \subseteq Z$ arises from a Laurent system $F$ in $\text{LP}(n)$ and $\Sigma$ is an $F$-fan. We denote by $G = (g_1, \ldots, g_s)$ the $\Sigma$-homogenization of $F = (f_1, \ldots, f_s)$. Recall that the $\text{Cl}(Z)$-degree $\mu_j = (u_j, \zeta_{j1}, \ldots, \zeta_{jq})$ of $g_j$ is base point free.

**Lemma 5.1.** A divisor class $[D] \in \text{Cl}(Z)$ is base point free if and only if for any $i = 0, \ldots, n$ there exists an $l_i \in \mathbb{Z}_{\geq 1}$ with $[D] = l_i w_i \in \text{Cl}(Z)$.

**Proof.** This is a direct consequence of Proposition 2.6 and the fact that the maximal cones of $\Sigma$ are given by $\text{cone}(v_j; j \neq i)$ for $i = 0, \ldots, n$. \hfill $\square$

The following lemma provides effective bounds on the orders $t_1, \ldots, t_q$ of the finite cyclic components of $\text{Cl}(Z)$ in terms of the $\mathbb{Z}$-parts $x_i$ of the generator degrees $w_0, \ldots, w_n$ and $u_j$ of the relation degrees $\mu_1, \ldots, \mu_s$ for the case of a toric complete intersection $X$ in a weighted projective space $Z$ with $x_0 = 1$.

**Lemma 5.2.** Consider the case $x_0 = 1$. If $\mu = (u, \zeta_1, \ldots, \zeta_q) \in \text{Cl}(Z)$ be a base point free divisor class, then, for any $k = 1, \ldots, q$ and $j = 1, \ldots, n$, we have

$$t_k \mid \text{lcm} \left( \frac{u}{x_i}; i = 1, \ldots, n, \ i \neq j \right).$$

In particular all $t_k$ divide $u$. Moreover, for the $\mathbb{Z}$-parts $u_j$ of the relation degrees $\mu_j$, we see that each of $t_1, \ldots, t_q$ divides $\gcd(u_1, \ldots, u_s)$. 
Proof. Due to \( x_0 = 1 \), we may assume \( \eta_{11} = \ldots = \eta_{1q} = 0 \). Lemma 5.1 delivers \( l_i \in \mathbb{Z}_{\geq 1} \) with \( \mu = l_i w_i \). For \( i = 0, \ldots, n \) that means

\[
(l_0, 0, \ldots, 0) = l_1 w_1 = \mu = l_i w_i = (l_i x_i, l_i \eta_{i1}, \ldots, l_i \eta_{iq}).
\]

Thus, we always have \( u = l_i x_i \) and \( l_i \eta_{ik} = 0 \). Now, fix \( 1 \leq j \leq n \). As any \( n \) of the \( w_i \) generate \( \text{Cl}(Z) \), we find \( \alpha \in \mathbb{Z}^{n+1} \) with \( \alpha_j = 0 \) and

\[
\alpha_0 w_0 + \ldots + \alpha_n w_n = (1, \bar{1}, \ldots, \bar{1}).
\]

Scalar multiplication of both sides with \( \text{lcm}(l_i; 1 \leq i \leq n, i \neq j) \) gives the first claim. The second one is clear. \( \square \)

The next bounding lemma uses terminality. Given \( \sigma \in \Sigma \), let \( I(\sigma) \) be the set of indices such that the \( v_i \) with \( i \in I(\sigma) \) are precisely the primitive ray generators of \( \sigma \) and \( u_\sigma \in \mathbb{Q}^n \) a linear form evaluating to \(-1\) on each \( v_i \) with \( i \in I(\sigma) \). As before, we look at \( A(\sigma) := \{ v \in \sigma; 0 \geq \langle u_\sigma, v \rangle \geq -1 \} \subseteq \sigma \).

The point \( z_\sigma \in Z \) is at most a terminal singularity of \( Z \) if and only if \( 0 \) and the \( v_i \) with \( i \in I(\sigma) \) are the only lattice points in \( A(\sigma) \). According to Theorem 1.1, the analogous statement holds for the points \( x \in X \) with \( x \in T^n \cdot z_\sigma \).

**Lemma 5.3.** Consider \( \sigma \in \Sigma \) such that \( z_\sigma \in Z \) is at most a terminal singularity of \( Z \).

(i) If \( \sigma \) is of dimension two, then \( \sigma \) is a regular cone and \( \text{Cl}(Z) \) is generated by the \( w_i \) with \( i \notin I(\sigma) \). In particular, \( \gcd(x_i; i \notin I(\sigma)) = 1 \) holds.

(ii) If \( \sigma \) is of dimension at least two, then \( \gcd(x_i; i \notin I(\sigma)) \) is strictly less than the sum over all \( x_i \) with \( i \in I(\sigma) \).

**Proof.** The first assertion can easily be verified directly. We turn to the second one. Using \( x_i \in \mathbb{Z}_{\geq 1} \) and \( x_0 v_0 + \ldots + x_n v_n = 0 \), we obtain

\[
v' := - \sum_{i \notin I(\sigma)} x_i v_i = \sum_{i \in I(\sigma)} x_i v_i \in \sigma^o \cap \mathbb{Z}^n.
\]

Write \( v' = \gcd(x_i; i \notin I(\sigma)) v \) with \( v \in \sigma^o \cap \mathbb{Z}^n \). Due to \( \text{dim}(\sigma) \geq 2 \), the vector \( v \) does not occur among \( v_0, \ldots, v_n \). Evaluating \( u_\sigma \) yields

\[
0 \geq \langle u_\sigma, v \rangle = \gcd(x_i; i \notin I(\sigma))^{-1} \langle u_\sigma, v' \rangle = - \gcd(x_i; i \notin I(\sigma))^{-1} \sum_{i \in I(\sigma)} x_i.
\]

By assumption, we have \( v \notin A(\sigma) \). Consequently, the right hand side term is strictly less than \(-1\). This gives us the desired estimate. \( \square \)
We turn to bounds relying on the Fano property of our toric complete intersection threefold $X$ in a fake weighted projective space $Z$. A tuple $\xi = (x_0, \ldots, x_n)$ of positive integers is ordered if $x_0 \leq \ldots \leq x_n$ holds and well-formed if any $n$ of its entries are coprime. For an ordered tuple $\xi$, we define

$$m(\xi) := \text{lcm}(x_0, \ldots, x_n), \quad M(\xi) := \begin{cases} 2m(\xi), & x_n = m(\xi), \\ m(\xi), & x_n \neq m(\xi). \end{cases}$$

We deal with well-formed ordered tuples $\xi = (x_0, \ldots, x_n)$ with $n \geq 4$. It will be an essential step in the proof of Theorem 1.3 to show that the Fano property forces the inequality

$$(n - 3)M(\xi) < x_0 + \ldots + x_n. \quad (5.3.1)$$

**Lemma 5.4.** Consider an ordered $\xi = (x_0, \ldots, x_4)$ such that any three of $x_0, \ldots, x_4$ are coprime and condition (5.3.1) is satisfied. Then $x_4 \leq 41$ holds or we have $1 \leq x_0, x_1, x_2 \leq 2$ and $x_3 = x_4$.

**Proof.** We first settle the case $x_4 = m(\xi)$. Then $x_4$ is divided by each of $x_0, \ldots, x_3$. This implies

$$\gcd(x_i, x_j) = \gcd(x_i, x_j, x_4) = 1, \quad 0 \leq i < j \leq 3.$$

Consequently, $x_0 \cdots x_3$ divides $x_4$. Subtracting $x_4$ from both sides of the inequality (5.3.1) leads to

$$x_0 \cdots x_3 \leq x_4 < x_0 + \ldots + x_3.$$

Using $1 \leq x_0 \leq \ldots \leq x_3$ and pairwise coprimeness of $x_0, \ldots, x_3$, we conclude that the tuple $(x_0, x_1, x_2, x_3)$ is one of

$$(1, 1, 2, 3), \quad (1, 1, 1, x_3).$$

In the first case, we arrive at $x_4 < x_0 + \ldots + x_3 = 7$. In the second one, $x_4 = dx_3$ holds with $d \in \mathbb{Z}_{\geq 1}$. Observe

$$dx_3 = x_4 < x_0 + \ldots + x_3 = 3 + x_3.$$

Thus, we have to deal with $d = 1, 2, 3$. For $d = 1$, we arrive at $x_3 = x_4$ and the cases $d = 2, 3$ lead to $x_3 \leq 2$ which means $x_4 < 5$.

Now we consider the case $x_4 < m(\xi)$. Then $m(\xi) = lx_4$ with $l \in \mathbb{Z}_{\geq 2}$. From inequality (5.3.1) we infer $l \leq 4$ as follows:

$$lx_4 = m(\xi) < x_0 + \ldots + x_4 \leq 5x_4.$$

We first treat the case $x_3 = x_4$. Using the assumption that any three of $x_0, \ldots, x_4$ are coprime, we obtain

$$\gcd(x_i, x_4) = \gcd(x_i, x_3, x_4) = 1, \quad i = 0, 1, 2.$$
Consequently, \( x_2 x_4 \leq m(\xi) = lx_4 \) and \( x_2 \leq l \leq 4 \). For \( l = 2 \) this means \( 1 \leq x_0, x_1, x_2 \leq 2 \). For \( l = 3, 4 \), we use again (5.3.1) and obtain

\[
x_4 < \frac{1}{l-2}(x_0 + x_1 + x_2) \leq 12.
\]

Now we turn to the case \( x_3 < x_4 \). Set for short \( d_i := \gcd(x_i, x_4) \). Then, for all \( 0 \leq i < j \leq 3 \), we observe

\[
\gcd(d_i, d_j) = \gcd(x_i, x_j, x_4) = 1.
\]

Consequently \( d_0 \cdots d_3 \mid x_4 \). For \( i = 0, \ldots, 3 \), write \( x_i = f_i d_i \) with \( f_i \in \mathbb{Z}_{\geq 1} \). Then \( f_i \) divides \( lx_4 \) and hence \( l \). Fix \( i_0, \ldots, i_3 \) pairwise distinct with \( d_{i_0} \leq \ldots \leq d_{i_3} \).

Using (5.3.1), we obtain

\[
(l-1)d_{i_0} \cdots d_{i_3} \leq (l-1)x_4 < f_{i_0}d_{i_0} + \ldots + f_{i_3}d_{i_3} \leq (2+2l)d_{i_3}.
\]

For the last estimate, observe that due to \( l = 2, 3, 4 \), all \( f_i \neq 1 \) have a common factor 2 or 3. Thus, as any three of \( x_0, \ldots, x_3 \) are coprime, we have \( f_i = 1 \) for at least two \( i \). We further conclude

\[
d_{i_0}d_{i_1}d_{i_2} < \frac{(2+2l)}{l-1} \leq 6.
\]

This implies \( d_{i_0} = d_{i_1} = 1 \) and \( d_{i_2} \leq 5 \). We discuss the case \( f_{i_3} = 1 \). There, we have \( x_{i_3} = d_{i_3} \), hence \( x_{i_3} \mid x_4 \). By assumption, \( x_0 \leq \ldots \leq x_3 < x_4 \) and thus \( x_{i_3} < x_4 \). We conclude \( d_{i_3} = x_{i_3} \leq x_4/2 \). From above we infer

\[
(l-1)x_4 < f_{i_0}d_{i_0} + \ldots + f_{i_3}d_{i_3} \leq l(2+d_{i_2}) + \frac{x_4}{2}.
\]

Together with \( l = 2, 3, 4 \) and \( d_{i_2} \leq 5 \) as observed before, this enables us to estimate \( x_4 \) as follows:

\[
x_4 < 2l\frac{2+d_{i_3}}{2l-3} \leq 28.
\]

Now let \( f_{i_3} > 1 \). Then \( 2d_{i_3} \leq f_{i_3}d_{i_3} = x_{i_3} < x_4 \) holds. This gives \( d_{i_3} < x_4/2 \).

Using \( d_{i_3} \mid x_4 \) we conclude \( d_{i_3} \leq x_4/3 \). Similarly as before, we proceed by

\[
(l-1)x_4 < f_{i_0}d_{i_0} + \ldots + f_{i_3}d_{i_3} \leq l(2+d_{i_2}) + ld_{i_3} \leq l(2+d_{i_2}) + l\frac{x_4}{3}.
\]

Inserting \( l = 2, 3, 4 \) and the bound \( d_{i_2} \leq 5 \) into the estimate just obtained, finally leads to the desired estimate

\[
x_4 < 3l\frac{2+d_{i_2}}{2l-3} \leq 42.
\]

\(\Box\)

**Lemma 5.5.** Consider a well-formed ordered \( \xi = (x_0, \ldots, x_5) \) satisfying (5.3.1). Then \( x_5 \leq 21 \) holds or we have \( 1 \leq x_0, x_1 \leq 2 \) and \( x_2 = x_3 = x_4 = x_5 \).
Proof. Let $x_5 \geq 22$. We have $M(\xi) = lx_5$ with $l \geq 2$. From (5.3.1) we infer $2lx_5 < 6x_5$, hence $l = 2$. Thus, we can reformulate (5.3.1) as

$$3x_5 < x_0 + \ldots + x_4.$$ 

Moreover, $M(\xi) = 2x_5$ implies $a_i x_i = 2x_5$ with suitable $a_i \in \mathbb{Z}_{\geq 2}$ for $i = 0, \ldots, 4$. In particular, the possible values of $x_0, \ldots, x_4$ are given as

$$x_5, \quad \frac{2}{3}x_5, \quad \frac{1}{2}x_5, \quad \frac{2}{5}x_5, \quad \frac{1}{3}x_5, \quad \frac{2}{7}x_5, \quad \ldots.$$ 

We show $x_4 = x_5$. Suppose $x_4 < x_5$. Then $x_4 \leq 2x_5/3$. We have $x_1 \geq 2x_5/3$, because otherwise $x_1 \leq x_5/2$ and thus

$$3x_5 < x_0 + \ldots + x_4 \leq \frac{1}{2}x_5 + \frac{1}{2}x_5 + \frac{2}{3}x_5 + \frac{2}{3}x_5 + \frac{2}{3}x_5 = 3x_5,$$

a contradiction. We conclude $x_1 = \ldots = x_4 = 2x_5/3$. By well-formedness, the integers $x_1, \ldots, x_5$ are coprime. Combining this with

$$3x_1 = \ldots = 3x_4 = 2x_5$$

yields $x_5 = 3$, contradicting $x_5 \geq 22$. Thus, $x_4 = x_5$, and we can update the previous reformulation of (5.3.1) as

$$2x_5 < x_0 + \ldots + x_3.$$ 

We show $x_3 = x_5$. Suppose $x_3 < x_5$. Then, by the limited stock of possible values for the $x_i$, the displayed inequality forces $x_3 = 2x_5/3$ and one of the following

$$x_2 = \frac{2}{3}x_5, \quad x_1 = \frac{2}{3}x_5, \quad \frac{1}{2}x_5, \quad \frac{2}{5}x_5, \quad x_2 = \frac{1}{2}x_5, \quad x_1 = \frac{1}{2}x_5.$$ 

By well-formedness, $x_1, \ldots, x_5$ are coprime. Depending on the constellation, this leads to $x_5 = 3, 6, 15$, contradicting $x_5 \geq 22$. Thus, $x_3 = x_5$. Observe

$$x_5 < x_0 + x_1 + x_2, \quad \gcd(x_1, x_j, x_5) = 1, \quad 0 \leq i < j \leq 2.$$ 

We show $x_2 = x_5$ by excluding all values $x_2 < x_5$. First note $x_2 > x_5/3$. Assume $x_2 = 2x_5/5$. Then, by the above inequality, $x_1 = 2x_5/5$. We obtain

$$5x_1 = 5x_2 = 2x_5,$$

thus $\gcd(x_1, x_2, x_5) = 1$ implies $x_5 = 5$, a contradiction to $x_5 \geq 22$. Next assume $x_2 = x_5/2$. The inequality leaves us with

$$x_1 = \frac{1}{2}x_5, \quad \frac{2}{5}x_5, \quad \frac{1}{3}x_5, \quad \frac{2}{7}x_5.$$ 

Thus, using $\gcd(x_1, x_2, x_5) = 1$ we arrive at $x_5 = 2, 10, 6, 14$ respectively, contradicting $x_5 \geq 22$. Finally, let $x_2 = 2x_5/3$. Then we have to deal with

$$x_1 = \frac{2}{3}x_5, \quad \frac{1}{2}x_5, \quad \frac{2}{5}x_5, \quad \frac{1}{3}x_5, \quad \frac{2}{7}x_5, \quad \frac{1}{4}x_5, \quad \frac{2}{9}x_5, \quad \frac{1}{5}x_5, \quad \frac{2}{11}x_5.$$
Using \( \gcd(x_1, x_2, x_5) = 1 \) gives \( x_5 = 3, 6, 15, 3, 21, 12, 9, 15 \) in the first eight cases, excluding those. Thus, we are left with the three cases

\[
x_2 = \frac{2}{3}x_5, \quad x_1 = \frac{2}{11}x_5, \quad x_0 = \frac{2}{11}x_5, \quad \frac{1}{5}x_5, \quad \frac{2}{13}x_5.
\]

In the first one, coprimeness of \( x_0, x_1, x_5 \) gives \( x_5 = 11 \) and in the second one coprimeness of \( x_0, x_2, x_5 \) implies \( x_5 = 6 \). The third case is excluded by

\[
\gcd(x_1, x_2, x_5) = 1 \Rightarrow x_5 = 33, \quad \gcd(x_0, x_2, x_5) = 1 \Rightarrow x_5 = 39.
\]

Thus, \( x_2 = x_5 \). We care about \( x_0 \) and \( x_1 \). Well-formedness and \( x_2 = \ldots = x_5 \) yield that \( x_0, x_5 \) as well as \( x_1, x_5 \) are coprime. Thus, we infer \( 1 \leq x_0, x_1 \leq 2 \) from

\[
a_0x_0 = 2x_5, \quad a_1x_1 = 2x_5.
\]

\[\square\]

**Lemma 5.6.** There exist only two ordered well-formed septuples \( (x_0, \ldots, x_6) \) satisfying (5.3.1), namely \( (1, 1, 1, 1, 1, 1, 1) \) and \( (2, 2, 3, 3, 3, 3, 3) \).

**Proof.** The case \( x_6 = 1 \) gives the first tuple. Let \( x_6 > 1 \). Then \( M(\xi) = lx_6 \) holds with \( l \geq 2 \). Using (5.3.1), we see \( 3lx_6 < 7x_6 \) which means \( l = 2 \). We obtain

\[
5x_6 < x_0 + \ldots + x_5
\]

by adapting the inequality (5.3.1) to the present setting. Similar to the preceding proof, \( M(\xi) = 2x_6 \) leads to presentations

\[
x_i = \frac{2}{a_i}x_6, \quad a_i \in \mathbb{Z}_{\geq 2}, \quad i = 0, \ldots, 5.
\]

Now, pick the unique \( j \) with \( x_0 \leq \ldots \leq x_{j-1} < x_j = \ldots = x_6 \). Well-formedness implies \( j \geq 2 \). Moreover \( x_{j-1} \leq 2x_6/3 \) holds and thus

\[
5x_6 < \frac{2}{3}jx_6 + (6 - j)x_6 = \frac{18 - j}{3}x_6.
\]

This implies \( j < 3 \). Thus \( j = 2 \), which means \( x_0 \leq x_1 < x_2 = \ldots = x_6 \). Adapting the inequality (5.3.1) accordingly gives

\[
x_6 < x_0 + x_1.
\]

Moreover, by well-formedness, \( x_0, x_6 \) as well as \( x_1, x_6 \) are coprime. Consequently, we can deduce \( 1 \leq x_0 \leq x_1 \leq 2 \) from

\[
a_0x_0 = 2x_6, \quad a_1x_1 = 2x_6.
\]

Now, \( x_6 > 1 \) excludes \( x_1 = 1 \). Next, \( x_0 = 1 \) would force \( x_6 = 2 = x_1 \), contradicting the choice of \( j \). Thus, we arrive at \( x_0 = x_1 = 2 \) and \( x_2 = \ldots = x_6 = 3 \). \[\square\]
The last tool package for the proof of Theorem 1.3 supports the verification of candidates in the sense that it allows us to show that each of the specifying data in the list do indeed stem from a toric complete intersection. The first statement is standard toric geometry; see for instance [12, Thm. 6.1.7, Thm. 6.1.14].

**Proposition 5.7.** Let Z be a complete toric variety arising from a lattice fan $\Sigma$ in $\mathbb{Z}^n$. Given an invariant Weil divisor $C = a_1 D_1 + \ldots + a_r D_r$ on Z consider the associated divisorial polytope
\[ B(C) = \{ u \in \mathbb{Q}^n; \langle u, v_i \rangle \geq -a_i, \ i = 1, \ldots, r \} \subseteq \mathbb{Q}^n. \]
If C is base point free, then $B(C)$ has integral vertices and $\Sigma$ refines the normal fan of $B(C)$. If in addition C is ample, then $B(C)$ is a full-dimensional lattice polytope having $\Sigma$ as its normal fan.

Given base point free classes $\mu_1, \ldots, \mu_s$ on a toric variety Z, the question is whether or not these are the relation degrees of a (general) toric complete intersection.

**Proposition 5.8.** Let Z be a complete toric variety arising from a lattice fan $\Sigma$ in $\mathbb{Z}^n$ and let $\mu_1, \ldots, \mu_s \in \text{Cl}(Z)$ admit base point free ample invariant representatives $C_1, \ldots, C_s$ on Z, respectively. Moreover, assume
\[ n - s \geq 3, \quad \text{and} \quad \text{dim}(\bar{Z} \setminus \hat{Z}) \leq r - s - 2. \]
Then there is a non-degenerate general Laurent system $\mathcal{F} \subseteq V(B(C_1), \ldots, B(C_s))$, having $\Sigma$ as an $\mathcal{F}$-fan. For each $(f_1, \ldots, f_s) \in \mathcal{F}$, the associated toric complete intersection $X \subseteq Z$ satisfies $\text{Cl}(X) = \text{Cl}(Z)$ and its Cox ring is given as
\[ \mathcal{R}(X) = \mathbb{K}[T_1, \ldots, T_r]/\langle g_1, \ldots, g_s \rangle, \]
where $(g_1, \ldots, g_s)$ is the $\Sigma$-homogenization of system $(f_1, \ldots, f_s)$ and $\text{deg}(T_i) \in \text{Cl}(Z)$ is given as the class $[D_i] \in \text{Cl}(Z)$ of the prime divisor $D_i \subseteq Z$ corresponding to the variable $T_i \in \mathbb{K}[T_1, \ldots, T_r]$.

**Proof.** By assumption, each $C_j$ is ample and thus Proposition 5.7 yields that each of the $B(C_j)$ and hence also $B(C_1) + \ldots + B(C_s)$ have $\Sigma$ as their normal fan. Moreover, the non-degenerate general Laurent system $\mathcal{F}$ exists by Proposition 4.18. Finally, Corollary 4.20 delivers the statement on the Cox ring. $\square$

**Proof of Theorem 1.3.** Let Z be a fake weighted projective space arising from a fan $\Sigma$ in $\mathbb{Z}^n$ and let $X = X_1 \cap \ldots \cap X_s \subseteq Z$ be a member of a terminal Fano non-degenerate general complete intersection threefold. Write $G = (g_1, \ldots, g_s)$ for the $\Sigma$-homogenization of the defining Laurent system $F = (f_1, \ldots, f_s)$ of $X \subseteq Z$. We have
\[ \text{Cl}(Z) = \mathbb{Z} \times \mathbb{Z}/t_1 \mathbb{Z} \times \ldots \times \mathbb{Z}/t_q \mathbb{Z}, \]
for the divisor class group of Z. As before, the generator degrees $w_i = \text{deg}(T_i)$ and the relation degrees $\mu_j = \text{deg}(g_j)$ in Cl(Z) are given as
\[ w_i = [D_i] = (x_i, \eta_{i1}, \ldots, \eta_{iq}), \quad \mu_j = [X_j] = (u_j, \xi_{j1}, \ldots, \xi_{jq}). \]
We assume that the presentation \( X \subseteq Z \) is irredundant in the sense that no \( g_i \) has a monomial \( T_i \); otherwise, as the \( \text{Cl}(Z) \)-grading is pointed, we may write \( g_j = T_i + h_j \) with \( h_j \) not depending on \( T_i \) and, eliminating \( T_i \), we realize \( X \) in a smaller fake weighted projective space. Moreover, suitably renumbering, we achieve

\[
x_0 \leq \ldots \leq x_n, \quad u_1 \leq \ldots \leq u_s.
\]

According to the generality condition, we may assume that every monomial of degree \( \mu_j \) shows up in the relation \( g_j \), where \( j = 1, \ldots, s \). In particular, as Lemma 5.1 shows \( \mu_j = l_{ji} w_i \) with \( l_{ji} \in \mathbb{Z}_{\geq 1} \), we see that each power \( T_i^{l_{ji}} \) is a monomial of \( g_j \). By irredundance of the presentation, we have \( l_{ji} \geq 2 \) for all \( i \) and \( j \).

We will now establish effective bounds on the \( w_i \) and \( \mu_j \) that finally allow a computational treatment of the remaining cases. The following first constraints are caused by terminality. By Corollary 4.9, all two-dimensional cones of \( \Sigma \) belong to \( \Sigma_X \) and by Corollary 1.2, the toric orbits corresponding to these cones host at most terminal singularities of \( Z \). Thus, Lemma 5.3 (i) tells us that \( \text{Cl}(Z) \) is generated by any \( n-1 \) of \( w_0, \ldots, w_n \). In particular, any \( n-1 \) of \( x_0, \ldots, x_n \) are coprime and, choosing suitable generators for \( \text{Cl}(Z) \), we achieve

\[
\text{Cl}(Z) = \mathbb{Z} \times \mathbb{Z}/t_1 \mathbb{Z} \times \ldots \times \mathbb{Z}/t_q \mathbb{Z}, \quad q \leq n - 1.
\]

Next, we see how the Fano property of \( X \) contributes to bounding conditions. Generality and Corollary 4.20 ensure that \( X \) inherits its divisor class group from the ambient fake weighted projective space \( Z \). Moreover, by Proposition 4.13, the anticanonical class \(-K_X\) of \( X \) is given in terms of the generator degrees \( w_i = \deg(T_i) \), the relation degrees \( \mu_j = \deg(g_j) \) and \( n = s + 3 \) as

\[
[-K_X] = w_0 + \ldots + w_n - \mu_1 - \ldots - \mu_s \in \text{Cl}(Z) = \text{Cl}(X).
\]

Now, consider the tuples \( \xi = (x_0, \ldots, x_n) \) and \( (u_1, \ldots, u_s) \) of \( \mathbb{Z} \)-parts of the generator and relation degrees. As seen above, we have \( u_j = l_{ji} x_i \) with \( l_{ji} \in \mathbb{Z}_{\geq 2} \) for all \( i \) and \( j \). Thus, \( m(\xi) = \text{lcm}(x_0, \ldots, x_n) \) divides all \( u_j \), in particular \( m(\xi) \leq u_j \). Moreover, if \( m(\xi) \neq x_n \), then we even have \( 2m(\xi) \leq u_j \). Altogether, with \( M(\xi) := 2m(\xi) \) if \( m(\xi) \neq x_n \) and \( M(\xi) := m(\xi) \) else, we arrive in particular at the inequality (5.3.1):

\[
(n-3)M(\xi) = sM(\xi) \leq u_1 + \ldots + u_s < x_0 + \ldots + x_n.
\]

This allows us to conclude that the number \( s \) of defining equations for our \( X \subseteq Z \) is at most three. Indeed, inserting \( 2x_n \leq u_j \) and \( x_i \leq x_n \), we see that \( 2sx_n \) is strictly less than \( (n+1)x_n = (s+4)x_n \). We go through the cases \( s = 1, 2, 3 \) and provide upper bounds on the generator degrees \( x_0, \ldots, x_n \).

Let \( s = 1 \). Then \( n = 4 \). We will show \( x_4 \leq 41 \). As noted above any three of \( x_0, \ldots, x_4 \) are coprime. Thus, Lemma 5.4 applies, showing that we have \( x_4 \leq 41 \) or the tuple \( (x_0, \ldots, x_4) \) is one of

\[
(1, 1, 1, x_4, x_4), \quad (1, 1, 2, x_4, x_4), \quad (1, 2, 2, x_4, x_4).
\]
In the latter case, consider $\sigma = \text{cone}(v_0, v_1, v_2) \in \Sigma$. Corollary 4.9 ensures $\sigma \in \Sigma_X$. Due to by Corollary 1.2, we may apply Lemma 5.3 (ii), telling us

$$x_4 = \gcd(x_3, x_4) < x_0 + x_1 + x_2 \leq 5.$$  

Let $s = 2$. Then $n = 5$. We will show $x_5 \leq 21$. According to Lemma 5.5, we only have to treat the case $x_2 = \ldots = x_5$. As noted above, we have

$$x_5 = \gcd(x_2, \ldots, x_5) = 1.$$  

Let $s = 3$. Then $n = 6$. Lemma 5.6 leaves us with $(x_0, \ldots, x_6)$ being one of the tuples $(1, 1, 1, 1, 1, 1)$ and $(2, 2, 3, 3, 3, 3, 3)$. As before, we can exclude the second configuration.

Next, we perform a computational step. Subject to the bounds just found, we determine all ordered, well formed tuples $\xi = (x_0, \ldots, x_n)$, where $n = s + 3$ and $s = 1, 2, 3$, that admit an ordered tuple $(u_1, \ldots, u_s)$ such that

$$u_1 + \ldots + u_s > x_0 + \ldots + x_n,$$

$l_{ji} := \frac{u_j}{x_i} \in \mathbb{Z}_{\geq 2}, \quad j = 1, \ldots, s, \quad i = 0, \ldots, n$

holds and any $n - 1$ of $x_0, \ldots, x_n$ are coprime. This is an elementary computation leaving us with about a hundred tuples $\xi = (x_0, \ldots, x_n)$, each of which satisfies $x_0 = 1$.

As a consequence, we can bound the data of the divisor class group $\text{Cl}(\mathbb{Z})$. As noted, we have $q \leq n - 1$ and Lemma 5.2 now provides upper bounds on the orders $t_k$ of the finite cyclic factors. This allows us to compute a list of specifying data $(Q, \mu_1, \ldots, \mu_s)$ of candidates for $X \subseteq \mathbb{Z}$ by building up degree maps

$$Q: \mathbb{Z}^{n+2} \to \text{Cl}(\mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}/t_1\mathbb{Z} \times \cdots \times \mathbb{Z}/t_q\mathbb{Z}, \quad e_i \mapsto w_i$$

and pick out those that satisfy the constraints established so far. In a further step, we check the candidates for terminality using the criterion provided Corollary 1.2; computationally, this amounts to a search of lattice points in integral polytopes. The affirmatively tested candidates form the list of Theorem 1.3. All the computations have been performed with Magma are available; see [28].

Proposition 5.8 shows that each specifying data $(Q, \mu)$ in the list of Theorem 1.3 stems indeed from a general toric complete intersection $X$ in the fake weighted projective space $\mathbb{Z}$. Finally, Corollary 4.20 ensures that the Cox ring of all listed $X$ is as claimed. In particular, none of the $X$ is toric. Most of the listed families can be distinguished via the divisor class group $\text{Cl}(X)$, the anticanonical self intersection $-K_X^3$ and $h^0(-K_X)$. For Numbers 12 and 39, observe that their Cox rings have non-isomorphic configurations of generator degrees, which also distinguishes the members of these families.

\[ \Box \]

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