Nondegeneracy of ground states for nonlinear scalar field equations involving the Sobolev-critical exponent at high frequencies in three and four dimensions

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Abstract

We consider nonlinear scalar field equations involving the Sobolev-critical exponent at high frequencies $\omega$. Since the limiting profile of the ground state as $\omega \to \infty$ is the Aubin-Talenti function and degenerate in a certain sense, from the point of view of perturbation methods, the nondegeneracy problem for the ground states at high frequencies is subtle. In addition, since the limiting profile (Aubin-Talenti function) fails to lie in $L^2(\mathbb{R}^d)$ for $d = 3, 4$, the nondegeneracy problem for $d = 3, 4$ is more difficult than that for $d \geq 5$ and an applicable methodology is not known. In this paper, we solve the nondegeneracy problem for $d = 3, 4$ by modifying the arguments in [2, 3]. We also show that the linearized operator around the ground state has exactly one negative eigenvalue.

Mathematics Subject Classification 35J20, 35B09, 35B33, 35Q55

1 Introduction

We consider the following nonlinear scalar field equation involving the Sobolev critical exponent:

$$-\Delta u + \omega u - |u|^\frac{4}{d-2} u - g(u) = 0, \quad u: \mathbb{R}^d \to \mathbb{C}, \tag{1.1}$$

where $d \geq 3$, $\omega > 0$, and $g$ is a complex-valued function on $\mathbb{C}$ to be specified shortly (see Assumption 1.1). It is worthwhile noting that the existence of the term $g$ is essential; Indeed, the Pohozaev’s identity (see (4.2)) shows that if $g$ is not placed, then (1.1) has no solution in $H^1(\mathbb{R}^d)$ for all $\omega > 0$.

Our aim is to prove the nondegeneracy of ground states to (1.1) under certain general assumptions about the function $g$ (see Theorem 1.1). Moreover, we show that the linearized operator around the ground state has exactly one negative eigenvalue (see Theorem 1.2). These results play important roles in the study of dynamics for the corresponding time-dependent nonlinear Schrödinger equation (see [4, 12]). Moreover, the nondegeneracy implies that there is no bifurcation branch from the ground state (see [5]).

Although the nondegeneracy of ground states has been studied for a large class of nonlinear elliptic equations, we have not found a methodology completely applicable to the equation (1.1) at high frequencies ($\omega \gg 1$) for $d = 3, 4$, even for the typical model $g(u) = |u|^{p-1}u$ with $1 < p < \frac{4}{d-2}$ (see Remark 1.3). The main difficulty is that the limiting profile of ground states as $\omega \to \infty$ is degenerate for all $d \geq 3$ and fails to lie in $L^2(\mathbb{R}^d)$ for $d = 3, 4$; Precisely, by an appropriate scaling, the equation (1.1) at high frequencies ($\omega \gg 1$) can be thought of as a perturbation of the equation

$$-\Delta u - |u|^{\frac{4}{d-2}} u = 0, \quad u: \mathbb{R}^d \to \mathbb{C}. \tag{1.2}$$
Note that the equation (1.2) is invariant under the scaling $u(x) \mapsto \lambda^{-1} u(\lambda^{-\frac{2}{d-2}} x)$. It is known that the “ground state” to (1.2) is the Aubin-Talenti function
\[
W(x) := \left( 1 + \frac{|x|^2}{d(d-2)} \right)^{-\frac{d-2}{2}}.
\] (1.3)

Hence, we can expect that the limiting profile of ground state $s$ to (1.1) as $\omega \to \infty$ is the Aubin-Talenti function $W$; Actually, we can prove this (see Lemma 4.2) and therefore see that the difficulty in the nondegeneracy problem arises from $W$ being degenerate in a certain sense for all $d \geq 3$ (see (2.2)) and not in $L^2(\mathbb{R}^d)$ for $d = 3, 4$.

Now, we state the assumptions about the function $g$:

**Assumption 1.1.** The function $g: \mathbb{C} \to \mathbb{C}$ in (1.1) is independent of the frequency $\omega$ and satisfies the following conditions:

1. The restriction of $g$ on the open interval $(0, \infty)$ is a real-valued $C^2$-function, and
   \[
g(0) = 0, \quad g(z) = \frac{z}{|z|} \arg g(|z|) \quad \text{for all } z \in \mathbb{C} \setminus \{0\}.
\] (1.4)
   Furthermore, $g$ satisfies the following condition:
   \[
g'(t)t - g(t) \geq 0 \quad \text{for all } t > 0.
\] (1.5)

We remark that the condition (1.5) is used mainly to ensure the existence of ground states to (1.1).

2. For $d \geq 3$, there exist $\max\left\{\frac{2}{d-2}, 1\right\} < p_1 < \frac{d+2}{d-2}$, $\max\left\{\frac{d+2}{d-2} - 2, p_1\right\} < p_2 < \frac{d+2}{d-2}$ and $C_1, C_2 > 0$ such that
   \[
   \limsup_{t \to 0} \frac{|g'(t)|}{t^{p_1-1}} \leq C_1, \quad \lim_{t \to \infty} \frac{g'(t)}{t^{p_2-1}} = C_2.
   \] (1.6)
   Note that the assumption about $P_1$ and $p_2$ reads: if $d = 3$, then $2 < p_1 < p_2$ and $3 < p_2 < 5$; and if $d \geq 4$, then $1 < p_1 < p_2 < \frac{d+2}{d-2}$. Moreover, the second condition in (1.6) implies
   \[
   \lim_{t \to \infty} \left| \frac{g(t)}{t^{p_2}} - \frac{C_2}{p_2} \right| = 0.
   \] (1.7)

3. Define\[
2^* := \frac{2d}{d-2}, \quad G(t) := \int_0^t g(s) \, ds.
\] (1.8)
   Then, for $d \geq 3$, there exists $C_3 > 0$ such that
   \[
   G(t) - \frac{1}{2^*} g(t)t \geq C_3 t^{p_2+1} \quad \text{for all } t \geq 0,
   \] (1.9)
   where $p_2$ is the same constant as in (1.6). We remark that the function on the left-hand side of (1.9) appears in the Pohozaev’s identity (see (4.2)).

We give some remarks about Assumption 1.1:

**Remark 1.1.** 1. The assumption $p_2 > \frac{d+2}{d-2} - 2$ is related to the existence of ground states; In fact, it is known (see (2) [12]) that when $d = 3$, $1 < p \leq 3$ and $g(u) = |u|^{p-1} u$, there exists $\omega_c > 0$ such that if $\omega > \omega_c$, then there is no ground state to (1.1).
2. The condition (1.5) implies
\[ g(t) t - 2G(t) \geq 0 \quad \text{for all } t > 0. \]  
(1.10)

Furthermore, it follows from (1.9) and (1.10) that
\[ \left( \frac{1}{2} - \frac{1}{2^p} \right) g(t) \geq \frac{1}{t} \left\{ G(t) - \frac{1}{2} g(t) t \right\} \geq C_3 t^{p_2} > 0 \quad \text{for all } t > 0. \]  
(1.11)

In particular, \( g \) is positive on \((0, \infty)\).

3. The assumption (1.6) together with \( g(0) = 0 \) shows that the following hold for all \( t \geq 0 \):
\[ |g'(t)| \lesssim t^{p_1 - 1} + t^{p_2 - 1}, \quad |g(t)| \lesssim t^{p_1} + t^{p_2}, \quad |G(t)| \lesssim t^{p_1 + 1} + t^{p_2 + 1}, \]  
where the implicit constants depend only on \( p_1, p_2, C_1 \) and \( C_2 \).

4. Let \( d \geq 3 \), \( \max\{ \frac{2}{d-2}, 1 \} < p_1 < \frac{d+2}{d-2} \), \( \max\{ \frac{d+2}{d-2} - 2, p_1 \} < p_2 < \frac{d+2}{d-2} \), and \( p_1 < q < p_2 \). Then, typical models of the function \( g \) satisfying all conditions in Assumption 1.7 are the following:
\[ g(t) = t^{p_2}, \quad g(t) = t^{p_1} + t^{p_2}, \quad g(t) = t^{p_1} + t^q + t^{p_2}, \]  
(1.13)
\[ g(t) = t^{p_1} - \gamma t^q + t^{p_2} \]  
with a small \( \gamma > 0 \) depending on \( d, p_1, p_2 \) and \( q \).

Next, we make it clear the meaning of ground state. In this paper, by a ground state to (1.1), we mean a minimizer of the following problem:
\[ m_\omega := \inf\{ S_\omega(u) : u \in H^1(\mathbb{R}^d) \setminus \{0\}, \mathcal{N}_\omega(u) = 0 \}, \]  
(1.15)
where \( S_\omega \) and \( \mathcal{N}_\omega \) denote the action and the Nehari functional associated with (1.1), namely,
\[ S_\omega(u) := \frac{1}{2} \| \nabla u \|_{L^2}^2 + \frac{\omega}{2} \| u \|_{L^2}^2 - \frac{1}{2} \| u \|_{L^{2^*}}^2 - \int_{\mathbb{R}^d} G(|u|), \]  
(1.16)
\[ \mathcal{N}_\omega(u) := S_\omega'(u)u = \| \nabla u \|_{L^2}^2 + \omega \| u \|_{L^2}^2 - \| u \|_{L^{2^*}}^2 - \int_{\mathbb{R}^d} g(|u|)|u|. \]  
(1.17)

Under Assumption 1.1 we can prove the existence of ground states to (1.1) (see, e.g., Section A of [3], Proposition 2.0.1 of [10] and Section 4.2 of [10]):

**Proposition 1.1.** Assume \( d \geq 3 \). Then, under Assumption 1.1, for any \( \omega > 0 \), there exists a ground state \( \Phi_\omega \) to (1.1) with the following properties: \( \Phi_\omega \) is a solution to (1.1); \( \Phi_\omega \in H^2(\mathbb{R}^d) \cap C^2(\mathbb{R}^d) \); \( \Phi_\omega \) is positive, radially symmetric about 0, and strictly decreasing as a function of \( |x| \) (hence \( \| \Phi_\omega \|_{L^\infty} = \Phi_\omega(0) \)).

In order to describe our main results (Theorem 1.1 and Theorem 1.2), we introduce several symbols:

**Notation 1.1.** Let \( d \geq 3 \).

1. For \( \omega > 0 \), we use \( \Phi_\omega \) to denote a ground state to (1.1) given in Proposition 1.1 (we do not need the uniqueness of ground states in this paper).
2. We use $H^1_{\text{rad}}(\mathbb{R}^d)$ to denote the set of functions in $H^1(\mathbb{R}^d)$ which is radially symmetric about 0.

3. For $f, g \in L^2(\mathbb{R}^d)$, we use $(f, g)$ to denote the inner product in the real Hilbert space $L^2_{\text{real}}(\mathbb{R}^d)$, namely,

$$
(f, g) := \Re \int_{\mathbb{R}^d} f(x)\overline{g(x)} \, dx.
$$

4. For $\omega > 0$, we use $L_{\omega,+}$ to denote (the real-part of) the linearized operator around $\Phi_\omega$, namely,

$$
L_{\omega,+} := -\Delta + V_{\omega,+} \quad \text{with} \quad V_{\omega,+} := \omega - \frac{d+2}{d-2} \Phi_{\omega}^{\frac{4}{d-2}} - g'(\Phi_\omega).
$$

Note that $L_{\omega,+}$ is a self-adjoint operator on the real Hilbert space $L^2_{\text{real}}(\mathbb{R}^d)$ with domain $H^2(\mathbb{R}^d)$. Furthermore, by a standard argument in the spectral theory, we see that the essential spectrum $\sigma_{\text{ess}}(L_{\omega,+})$ is identical to $[\omega, \infty)$, so that the discrete spectrum is included in $(-\infty, \omega)$:

$$
\sigma_{\text{disc}}(L_{\omega,+}) \subset (-\infty, \omega).
$$

5. We use $B_\omega : H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d) \to \mathbb{R}$ to denote the bilinear form associated with $L_{\omega,+}$, namely,

$$
B_\omega(u, v) := \langle \nabla u, \nabla v \rangle + \langle V_{\omega,+}u, v \rangle.
$$

Remark 1.2. 1. We can verify that the action $S_\omega$ and the Nehari functional $N_\omega$ (see (1.16) and (1.17)) are $C^2$ on the real Hilbert space $H^1_{\text{real}}(\mathbb{R}^d)$, and

$$
S_\omega'(u)v = (-\Delta u + \omega u - |u|^{\frac{4}{d-2}}u - g(u), v) \quad \text{for all } u, v \in H^1_d(\mathbb{R}^d),
$$

$$
[S_\omega''(\Phi_\omega)u]v = B_\omega(u, v) \quad \text{for all } u, v \in H^1_d(\mathbb{R}^d),
$$

$$
N_\omega''(\Phi_\omega)u = \langle L_{\omega,+}\Phi_\omega, u \rangle \quad \text{for all } u \in H^1_d(\mathbb{R}^d).
$$

2. Since $\Phi_\omega$ is a solution to (1.1), differentiating the equation, we see that the derivatives of $\Phi_\omega$ belong to the kernel of $L_{\omega,+}$:

$$
\partial_1 \Phi_\omega, \ldots, \partial_d \Phi_\omega \in \text{Ker } L_{\omega,+}.
$$

Now, we state the main results of this paper. The first one is concerning the nondegeneracy of ground states at high frequencies:

Theorem 1.1. Assume $d = 3, 4$. Under Assumption 1.1 and Notation 1.1, there exists $\omega_{\text{ng}} > 0$ such that the following hold for all $\omega > \omega_{\text{ng}}$:

(i) The ground state $\Phi_\omega$ to (1.1) is nondegenerate in $H^1_{\text{rad}}(\mathbb{R}^d)$; namely, if $u \in H^1_{\text{rad}}(\mathbb{R}^d)$ satisfies $B_\omega(u, v) = 0$ for all $v \in H^1_{\text{rad}}(\mathbb{R}^d)$, then $u$ must be trivial ($u \equiv 0$).

(ii) The kernel of $L_{\omega,+}$ consists of the derivatives of $\Phi_\omega$, namely,

$$
\text{Ker } L_{\omega,+} = \text{span} \{\partial_1 \Phi_\omega, \ldots, \partial_d \Phi_\omega\}.
$$

Remark 1.3. When $d = 3$ and $g(u) = |u|^{p-1}u$ with $3 < p < 5$, the nondegeneracy of ground states to (1.1) had been announced in [2]. However, there is a flaw in the proof. In this paper, we modify the argument in [2] and give a complete proof.
Remark 1.4. It is certain that the same argument as in [3] proves the nondegeneracy of
ground states to (1.1) at high frequencies ($\omega \gg 1$) for all $d \geq 5$.

The second result is concerning the negative eigenvalues of $L_{\omega,+}$:

Theorem 1.2. Assume $d \geq 3$. Then, under Assumption 1.1 and Notation 1.1 the following hold for all $\omega > 0$:

(i) Let $u \in H^1(\mathbb{R}^d)$. If $N_{\omega}(\Phi_\omega)u = 0$, then $B_\omega(u, u) \geq 0$.

(ii) $L_{\omega,+}$ has exactly one negative eigenvalue on the real Hilbert space $L^2_{\text{real}}(\mathbb{R}^d)$. Furthermore, the multiplicity of the negative eigenvalue is one.

The rest of the paper is organized as follows. In Section 2 we introduce the notation besides Notation 1.1. In Section 3 we give estimates for free and perturbed resolvents. In Section 4, we give properties of ground states to (1.1). In Section 5, we give a proof of Theorem 1.1. In Section 6, we give a proof of Theorem 1.2.

Acknowledgements

The authors would like to thank Professor Hiroaki Kikuchi for helpful discussion. This work was supported by JSPS KAKENHI Grant Number 20K03697.

2 Preliminaries

In this section, besides Notation 1.1 we introduce symbols used in this paper, with auxiliary results:

Notation 2.1. 1. Let $X$ and $Y$ be positive quantities. Then, we use $X \lesssim Y$ to indicate the inequalities $X \leq CY$ and $CX \leq Y$, respectively, where $C > 1$ is some constant independent of $\omega$; Note that the implicit constants in these notation are allowed to depend on the dimension $d$ and the constants $p_1$, $p_2$, $C_1$, $C_2$ and $C_3$ given in Assumption 1.1. Furthermore, we use $X \sim Y$ to indicate that $X \lesssim Y \lesssim X$.

2. For $d \geq 1$, we define the Fourier and inverse Fourier transformations to be that for $u \in L^1(\mathbb{R}^d)$,

$$\mathcal{F}[u](\xi) := \int_{\mathbb{R}^d} e^{-i\xi \cdot x} u(x) \, dx, \quad \mathcal{F}^{-1}[u](\xi) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\xi \cdot x} u(x) \, dx. \quad (2.4)$$

It is worthwhile noting that the Aubin-Talenti function $W$ (see (1.3)) is degenerate in the sense that

$$(-\Delta + V)W = 0 \quad \text{with} \quad V := -\frac{d+2}{d-2} W^{\frac{d}{d-2}}. \quad (2.2)$$

By a computation involving integration by parts, we can verify that the following holds for all $d \geq 3$ and $\max\{1, \frac{2}{d-2}\} < r \leq \frac{d+2}{d-2}$:

$$\langle W^r, \Lambda W \rangle = -\frac{\{4 - (d-2)(r-1)\}}{2(r+1)} \|W\|_{L^{r+1}}. \quad (2.3)$$

3. For $d \geq 1$, we define the Fourier and inverse Fourier transformations to be that for $u \in L^1(\mathbb{R}^d)$,
4. For $d \geq 3$ and $\lambda > 0$, we define the $H^1$-scaling operator $T_\lambda$ by

$$T_\lambda[v](x) := \lambda^{-1}v(\lambda^{-\frac{2}{d-2}}x).$$  \hspace{1cm} (2.5)

Observe from the fundamental theorem of calculus that for any $\mu > 0$,

$$T_\mu[W] - W = \int_1^\mu \frac{d}{d\lambda}T_\lambda[W]\,d\lambda = -\frac{2}{d-2}\int_1^\mu \lambda^{-1}T_\lambda[\Delta W]\,d\lambda. \hspace{1cm} (2.6)$$

5. For $d \geq 3$ and $\omega > 0$, we define $M_\omega$ and $\tilde{\Phi}_\omega$ as

$$M_\omega := \|\Phi_\omega\|_{L^\infty} = \Phi_\omega(0), \quad \tilde{\Phi}_\omega := T_{M_\omega}[\Phi_\omega] = M_\omega^{-1}\Phi_\omega(M_\omega^{-\frac{2}{d-2}}x). \hspace{1cm} (2.7)$$

It is worthwhile noting that

$$\|\tilde{\Phi}_\omega\|_{L^\infty} = \tilde{\Phi}_\omega(0) = 1, \quad (2.8)$$

$$-\Delta\tilde{\Phi}_\omega + M_\omega^{-\frac{4}{d-2}}\omega\Phi_\omega - \frac{d+2}{4}\omega - M_\omega^{-\frac{4}{d-2}}g(M_\omega\tilde{\Phi}_\omega) = 0. \hspace{1cm} (2.9)$$

6. We define the function $\delta$ on $(0, \infty)$ by

$$\delta(s) := \begin{cases} \frac{s^{\frac{1}{d-2}}}{\log(1+s^{-1})} & \text{if } d = 4, \\ \frac{1}{\log(1+s^{-1})} & \text{if } d = 4. \end{cases} \hspace{1cm} (2.10)$$

It is worthwhile noting (see (2.14) of [1]) that for $d = 3, 4$, there exists $C > 0$ depending only on $d$ such that

$$\left|\int_{|\xi| \leq 1} \frac{1}{(|\xi|^2 + s)|\xi|^2} \, d\xi - C\delta(s)^{-1}\right| \lesssim 1 \quad \text{for all } 0 < s < 1. \hspace{1cm} (2.11)$$

7. We define the function $\beta$ on $(0, \infty)$ by

$$\beta(s) := \delta(s)^{-1}s. \hspace{1cm} (2.12)$$

Note that $\beta$ is strictly increasing on $(0, \infty)$, so that the inverse exists.

8. We use $\alpha$ to denote the inverse function of $\beta$. When $d = 3$, $\beta(s) = s^{\frac{1}{d-2}}$ and the domain of $\alpha$ is $(0, \infty)$. When $d = 4$, the image of $(0, \infty)$ by $\beta$ is $(0, 1)$ (see [1]) and the domain of $\alpha$ is $(0, 1)$. Furthermore, we can verify the following (see [1]):

$$\alpha(t) \begin{cases} = t^2 & \text{for } d = 3 \text{ and } 0 < t < \infty, \\ \sim \delta(t)t & \text{for } d = 4 \text{ and } 0 < t \leq T_0, \end{cases} \hspace{1cm} (2.13)$$

where $T_0 > 0$ is some constant.

9. We use $S^{d-1}$ to denote the sphere in $\mathbb{R}^d$ of radius 1 centered at 0, namely $S^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$. Furthermore, $|S^{d-1}|$ denotes the area of $S^{d-1}$.

10. For $d \geq 1$ and $q \geq 1$, we use $L^q_{rad}(\mathbb{R}^d)$ to denote the set of functions in $L^q(\mathbb{R}^d)$ which is radially symmetric about 0.

11. For $d \geq 1$ and $q \geq 1$, the symbol $L^q_{weak}(\mathbb{R}^d)$ denotes the weak $L^q$ space (Marcinkiewicz space).
3 Estimates for free and perturbed resolvents

In this section, we give estimates for free and perturbed resolvents.

Let us begin by recalling several estimates given in [1]:

**Proposition 3.1** (Proposition 1.2 of [1]). Assume $d = 3, 4$. Let $\frac{d}{d-2} < q < \infty$. If $s > 0$ is sufficiently small dependently on $d$ and $q$, then the inverse of the operator $1 + (-\Delta + s)^{-1}V: L^q_{rad}(\mathbb{R}^d) \to L^q_{rad}(\mathbb{R}^d)$ exists; and the following estimates hold:

1. If $f \in L^q_{rad}(\mathbb{R}^d)$, then
   \[ \|1 + (-\Delta + s)^{-1}V\|_{L^q} \lesssim \delta(s)s^{-1}\|f\|_{L^q}, \tag{3.1} \]
   where the implicit constant depends only on $d$ and $q$.

2. If $f \in L^q_{rad}(\mathbb{R}^d)$ and $\langle f, V\Lambda W \rangle = 0$, then
   \[ \|1 + (-\Delta + s)^{-1}V\|_{L^q} \lesssim \|f\|_{L^q}, \tag{3.2} \]
   where the implicit constant depends only on $d$ and $q$.

**Lemma 3.1.** Assume $d \geq 1$.

1. Let $1 \leq q_1 \leq q_2 \leq \infty$ and $d(\frac{1}{q_1} - \frac{1}{q_2}) < 2$. Then, the following holds for all $s > 0$:
   \[ \|(-\Delta + s)^{-1}\|_{L^{q_1}(\mathbb{R}^d) \to L^{q_2}(\mathbb{R}^d)} \lesssim s^\frac{d}{2}(\frac{1}{q_1} - \frac{1}{q_2})^{-1}, \tag{3.3} \]
   where the implicit constant depends only on $d$, $q_1$, and $q_2$.

2. Let $1 < q_1 \leq q_2 < \infty$ and $d(\frac{1}{q_1} - \frac{1}{q_2}) < 2$. Then, the following holds for all $s > 0$:
   \[ \|(-\Delta + s)^{-1}\|_{L^{q_1}_{weak}(\mathbb{R}^d) \to L^{q_2}(\mathbb{R}^d)} \lesssim s^\frac{d}{2}(\frac{1}{q_1} - \frac{1}{q_2})^{-1}, \tag{3.4} \]
   where the implicit constant depends only on $d$, $q_1$, and $q_2$.

**Lemma 3.2.** Assume $d \geq 3$. Let $\frac{d}{d-2} < q < \infty$. Then, the following holds for all $s_0 \geq 0$:
\[ \|(-\Delta + s)^{-1}\|_{L^{d/q}_{rad}(\mathbb{R}^d) \to L^{s}(\mathbb{R}^d)} \lesssim 1, \tag{3.5} \]
where the implicit constant depends only on $d$ and $q$.

**Lemma 3.3** (Lemma 3.9 of [1]). Assume $d = 3, 4$. Then, the following holds for all $s > 0$:
\[ \|(-\Delta + s)^{-1}\Lambda W\|_{L^\infty} \lesssim 1 + \delta(s)^{-1}. \tag{3.6} \]

**Lemma 3.4** (Lemma 3.10 of [1]). Assume $d = 3, 4$. Then, there exists a constant $C > 0$ depending only on $d$ such that the following holds for all $\frac{d}{d-2} < q < \infty$, $f \in L^q_{rad}(\mathbb{R}^d)$ and $s > 0$:
\[ |\langle(-\Delta + s)^{-1}Vf, \Lambda W \rangle + C\Re\mathcal{F}[Vf](0)\delta(s)^{-1}| \lesssim \|f\|_{L^q}, \tag{3.7} \]
where the implicit constant depends only on $d$ and $q$.

Next, we give a variant of Lemma [3.4].
Lemma 3.5. Assume $d = 3, 4$. Then, there exists $C_0 > 0$ depending only on $d$ such that the following holds for all $s > 0$ and $f \in L^2(\mathbb{R}^d)$:

$$\left| \langle (-\Delta + s)^{-1} f, V\Lambda W \rangle - C_0 \left\{ (-\Delta + s)^{-1} f \right\}(0) \right| \lesssim \| f \|_{L^{2^*}},$$

(3.8)

where the implicit constant depends only on $d$.

Proof of Lemma 3.5. Let us begin by decomposing $\Lambda W$ as follows:

$$\Lambda W = -\left\{ \frac{d(d-2)}{2d} |x|^{-(d-2)} + Z \right\},$$

(3.9)

where

$$Z := \frac{d-2}{2} \left( 1 + \frac{|x|^2}{d(d-2)} \right)^{-\frac{d}{2}} - \frac{|x|^2}{2d} \left( 1 + \frac{|x|^2}{d(d-2)} \right)^{-\frac{d}{2}} - \left( \frac{|x|^2}{d(d-2)} \right)^{-\frac{d}{2}} \right\}.$$

(3.10)

Let $\chi_{\leq 1}$ be the indicator function of the set $\{|x| \leq 1\}$, namely $\chi_{\leq 1}(x) = 0$ if $|x| \leq 1$ and $\chi_{\leq 1}(x) = 0$ if $|x| > 1$. Then, by the fundamental theorem of calculus, and $(d - \frac{s^2}{2})\frac{2d}{d+2} < d$ for $d = 3, 4$, we see that

$$\|Z\|_{L^2} \lesssim \left\| (1 + |x|)^{-d} \right\|_{L^2} \|f\|_{L^{2^*}} \lesssim 1 + \|\chi_{\leq 1}\|_1 \int_0^1 \left( \theta^{\frac{d}{2} + |x|} \right)^{-d} \|f\|_{L^{2^*}} \lesssim 1.$$

(3.11)

Now, let $s > 0$ and $f \in L^2(\mathbb{R}^d)$. By $V\Lambda W = \Delta\Lambda W$ (see (2.22)), the decomposition (3.9), and $(-\Delta)|x|^{-(d-2)}$ being the Dirac’s delta measure at 0 with intensity $(d-2)|s|^{d-1}$ (see, e.g., Theorem 6.20 of [11]), we see that

$$\langle (-\Delta + s)^{-1} f, V\Lambda W \rangle = \langle (-\Delta + s)^{-1} f, \Delta\Lambda W \rangle$$

$$= \frac{d(d-2)}{2d} \langle (-\Delta + s)^{-1} f, (-\Delta)|x|^{-(d-2)} \rangle + \langle (-\Delta + s)^{-1} f, \Delta Z \rangle$$

(3.12)

$$= C_0 \left\{ (-\Delta + s)^{-1} f \right\}(0) + \langle (-\Delta + s)^{-1} f, \Delta Z \rangle,$$

where $C_0 > 0$ is some constant depending only on $d$. Consider the second term on the right-hand side of (3.12). By Hölder’s inequality, Lemma 3.1 and (3.11), we see that

$$\left| \langle (-\Delta + s)^{-1} f, \Delta Z \rangle \right| = \left| \langle (-\Delta + s)^{-1} f, \{ s - (-\Delta + s) \} Z \rangle \right|$$

$$\lesssim \|s(-\Delta + s)^{-1} f\|_{L^{2^*}} \|Z\|_{L^{2^*}} \|f\|_{L^{2^*}} \lesssim \| f \|_{L^{2^*}}.$$

(3.13)

Putting (3.12) and (3.13) together, we find that the claim (3.8) is true. 

We will apply Lemma 3.5 to $f = W$. In that case, the principal factor is the following:

$$A_0 := \left\{ \begin{array}{ll}
\sqrt{\pi} \int_{\mathbb{R}^d} e^{-|x|^2} \frac{d}{4\pi |x|^2} dx & \text{if } d = 3, \\
\frac{C}{(2\pi)^d} F[|W|^d] \{0\} = \frac{C}{(2\pi)^d} \int_{\mathbb{R}^d} W^d(x) dx & \text{if } d = 4,
\end{array} \right.$$

(3.14)

where $C$ is the same constant as in (2.11); Precisely, we have the following:
Lemma 3.6. Assume $d = 3, 4$. Then, the following holds for all $0 < s < 1$:

$$\delta(s)\{(−Δ + s)^{-1}W\}(0) = A_0 + o_s(1),$$

where $A_0$ is the same constant as in (3.14).

Proof of Lemma 3.6. Divide $W$ as follows

$$W = \{d(d-2)\}^{d-2 \over 2} |x|^{-(d-2)} + Y,$$

where

$$Y(x) := \left(1 + \frac{|x|^2}{d(d-2)}\right)^{-d \over 2} - \left(\frac{|x|^2}{d(d-2)}\right)^{-d \over 2}.$$ 

Note that

$$|Y(x)| \leq \left(\frac{|x|^2}{d(d-2)}\right)^{-d \over 2} \lesssim |x|^{-(d-2)}.$$  

Furthermore, by the fundamental theorem of calculus, $\frac{4d}{2d+3}(d-1) < d$ and $\frac{4d}{2d+3} > 1$ for $d = 3, 4$, we see that

$$\|Y\|_{L^{{d \over 2d+3}}} = \left\|\frac{d-2}{2} \int_0^1 \left(\theta + \frac{|x|^2}{d(d-2)}\right)^{-d \over 2} d\theta\right\|_{L^{{d \over 2d+3}}}$$

$$\lesssim \int_0^1 \theta^{-d \over 2} d\theta \| |x|^{-(d-1)}\|_{L^{{d \over 2d+3}}(|x| \leq 1)} + \| |x|^{-d} \|_{L^{{d \over 2d+3}}(|x| \leq 1)} \lesssim 1.$$  

Assume $d = 3$. Then, by (3.16), and the representation formula of the Green’s function for $(-Δ + s)^{-1}$ (see the remarks for Theorem 6.23 of [11]), we see that

$$(−Δ + s)^{-1}W\} = \{d(d-2)\}^{d-2 \over 2} \int_{\mathbb{R}^3} e^{-s \over 4\pi |x|} |x|^{-(d-2)} dx + \int_{\mathbb{R}^3} e^{-s \over 4\pi |x|} Y(x) dx.$$ 

Consider the first term on the right-hand side of (3.20). By the substitution of variables and $d = 3$, we see that

$$\{d(d-2)\}^{d-2 \over 2} \int_{\mathbb{R}^3} e^{-s \over 4\pi |x|} |x|^{-(d-2)} dx = s^{-{d \over 2}} \sqrt{3} \int_{\mathbb{R}^3} e^{-|x| \over 4\pi |x|^2} dx = \delta(s)^{-1} A_0.$$ 

Consider the second term on the right-hand side of (3.20). By (3.18), $d = 3$, Hölder’s inequality and (3.19), we see that

$$\left| \int_{\mathbb{R}^3} e^{-s \over 4\pi |x|} Y(x) dx \right| \lesssim \int_{|x| \leq 1} |x|^{-d} dx + \int_{1 \leq |x|} e^{-s \over 4\pi |x|} \| Y(x) \| dx$$

$$\lesssim 1 + s_n^{-{d \over 2}} \int_{\mathbb{R}^3} e^{-|x|} \| Y(s_n^{-{1 \over 2}} x) \| dx$$

$$\lesssim 1 + s_n^{-{d \over 2}} \| e^{-|x|} \|_{L^{{d \over 2d+3}}} \| Y(s_n^{-{1 \over 2}} \cdot) \|_{L^{{d \over 2d+3}}} \leq 1 + s_n^{-{d \over 2d+3}} \lesssim s^{-{d \over 2}}.$$  

Then, (3.15) follows from (3.20), (3.21) and (3.22).
Next, assume $d = 4$. By the Fourier transform of the Green’s function for $(-\Delta + s)^{-1}$ (see Theorem 6.23 of [11]) and $-\Delta W = W^{\frac{d+2}{2}}$, we see that

$$\{(-\Delta + s)^{-1}W\}(0) = \mathcal{F}^{-1}[\mathcal{F}((-\Delta + s)^{-1}W)](0) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}[W](\xi) \frac{d\xi}{|\xi|^d + s}$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}[(-\Delta)W](\xi) \frac{d\xi}{|\xi|^d + s} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}[W^{\frac{d+2}{2}}](\xi) \frac{d\xi}{|\xi|^d + s}$$

$$= \frac{1}{(2\pi)^d} \int_{|\xi| \leq 1} \mathcal{F}[W^{\frac{d+2}{2}}](0) \frac{d\xi}{|\xi|^d + s} + \frac{1}{(2\pi)^d} \int_{|\xi| > 1} \mathcal{F}[W^{\frac{d+2}{2}}](\xi) \frac{d\xi}{|\xi|^d + s}$$

$$\approx \frac{1}{(2\pi)^d} \int_{|\xi| \leq 1} W^{\frac{d+2}{2}}(0) \frac{d\xi}{|\xi|^d + s}.$$  \hspace{1cm} (3.23)

It follows from (2.11) and $W(x) \sim (1 + |x|)^{-(d-2)}$ that

$$\int_{|\xi| \leq 1} \mathcal{F}[W^{\frac{d+2}{2}}](0) \frac{d\xi}{|\xi|^d + s} \sim \mathcal{C} \delta(s)^{-1} \|W^{\frac{d+2}{2}}\|_{L^1} \lesssim 1. \hspace{1cm} (3.24)$$

Consider the second term on the right-hand side of (3.23). By $W(x) \sim (1 + |x|)^{-(d-2)}$, we see that

$$\int_{|\xi| > 1} \mathcal{F}[W^{\frac{d+2}{2}}](\xi) \frac{d\xi}{|\xi|^d + s} \lesssim \int_{|\xi| > 1} \frac{1}{|\xi|^d + s} \left( \int_{\mathbb{R}^d} |e^{-ix\cdot\xi} - 1| W^{\frac{d+2}{2}}(x) \, dx \right) d\xi$$

$$\lesssim \int_{|\xi| > 1} \frac{1}{|\xi|^d + s} \left( \int_{\mathbb{R}^d} \left\{ \sin^2 \left( \frac{x \cdot \xi}{2} \right) + |\sin (x \cdot \xi)| \right\} W^{\frac{d+2}{2}}(x) \, dx \right) d\xi$$

$$\lesssim \int_{|\xi| > 1} \frac{1}{|\xi|^d} \left( \int_{\mathbb{R}^d} \min\{1, |\xi|^d (1 + |x|)^{-(d+2)} \} \, dx \right) d\xi$$

$$\lesssim \int_{|\xi| > 1} \frac{1}{|\xi|^d} \left( \int_{\mathbb{R}^d} |x|(1 + |x|)^{-(d+2)} \, dx \right) d\xi \lesssim 1.$$  \hspace{1cm} (3.25)

It remains to estimate the last term on the right-hand side of (3.23). By Cauchy-Schwarz inequality, Plancherel’s theorem and $W(x) \sim (1 + |x|)^{-(d-2)}$, we see that

$$\left| \int_{|\xi| \leq 1} \mathcal{F}[W^{\frac{d+2}{2}}](\xi) \frac{d\xi}{|\xi|^d + s} \right| \leq \left( \int_{|\xi| \leq 1} \frac{1}{|\xi|^d} \, d\xi \right)^{\frac{1}{2}} \|F^{\frac{d+2}{2}}\|_{L^2} \lesssim 1. \hspace{1cm} (3.26)$$

Then, (3.14) follows from (3.23), (3.25) and (3.26). \hfill \Box

The following lemma follows immediately from Lemma 3.3 and Lemma 3.6.

**Lemma 3.7.** Assume $d = 3, 4$. Then, the following holds for all $0 < s < 1$:

$$|\delta(s) \langle (-\Delta + s)^{-1}W, VAW \rangle - A_1| = o_1(1) \quad \text{with} \quad A_1 := \mathcal{C}_0 A_0, \hspace{1cm} (3.27)$$

where $\mathcal{C}_0$ and $A_0$ are the same constants as in Lemma 3.3 and 3.14, respectively.
In order to treat the term \( g \) in (1.1), we need the following lemma:

**Lemma 3.8.** Assume \( d = 3, 4 \). Under Assumptions (1.1) the following holds for all \( \omega > 0, 0 < s < 1, \mu > 0 \) and \( \max\{1, \frac{d}{(d-2)p_1}\} < r < \frac{4}{d} \):

\[
\left| \left( -\Delta + s \right)^{-1} g(\mu M_\omega W), V AW \right| + \left| g(\mu M_\omega W), AW \right| \leq \left\{ (\mu M_\omega)^{p_1} + (\mu M_\omega)^{p_2} \right\} s \frac{d}{p_1} - 1,
\]

(3.28)

where the implicit constant may depend on \( r \) (independent of \( \omega, s \) and \( \mu \)). Note that \( p_1 > \frac{d}{d-2} \) implies \( \frac{d}{(d-2)p_1} < \frac{4}{d} \).

**Proof of Lemma 3.8.** Let \( \omega > 0, 0 < s < 1 \) and \( \mu > 0 \). Observe from \( V AW = \Delta W \) (see (2.2)) that

\[
\left\{ \left( -\Delta + s \right)^{-1} g(\mu M_\omega W), V AW \right\} = \left\{ g(\mu M_\omega W), \left( -\Delta + s \right)^{-1} (\Delta - s + s) \Delta W \right\} = -\left\{ g(\mu M_\omega W), \Delta W \right\} + s \left( g(\mu M_\omega W), \left( -\Delta + s \right)^{-1} \Delta W \right).
\]

Let \( \max\{1, \frac{d}{(d-2)p_1}\} < r < \frac{4}{d} \). Then, by Hölder’s inequality, (1.12), \( W \leq 1, W \in L^{p_1r}(\mathbb{R}^d)\), and (3.22) in Lemma \( 3.1 \) we see that

\[
s \left| \left( -\Delta + s \right)^{-1} \Delta W \right| \leq s \left\{ (\mu M_\omega)^{p_1} + (\mu M_\omega)^{p_2} \right\} ||W||^2_{L^p} \left| \left( -\Delta + s \right)^{-1} \Delta W \right| \leq \left( \frac{1}{s} \right)^\frac{d-2}{p_1} ||\Delta W||^2_{L^\frac{2}{p_1 - 2}} \\
\]

(3.30)

where the implicit constant may depend on \( r \). Putting (3.29) and (3.30) together, we obtain the desired estimate (3.28).

\( \Box \)

### 4 Properties of ground states

In this section, we give the properties of ground states which are necessary in order to prove Theorem (1.1).

First, note that every \( H^1 \)-solution \( u \) to (1.1) obeys the following identity, as well as \( N_\omega(u) = 0 \) (see (2.1) of \( \text{[6]} \)):

\[
\mathcal{P}_\omega(u) := \frac{1}{2} \| \nabla u \|^2_{L^2} + \frac{\omega}{2} \| u \|^2_{L^2} - \frac{1}{2} \| u \|^2_{L^2} - \int_{\mathbb{R}^d} G(|u|) = 0.
\]

(4.1)

Computing \( \mathcal{P}_\omega - \frac{1}{2} \omega N_\omega \), we see that every \( H^1 \)-solution \( u \) to (1.1) obeys

\[
\omega \| u \|^2_{L^2} = d \int_{\mathbb{R}^d} \left\{ G(|u|) - \frac{1}{2} g(|u|) |u| \right\}.
\]

(4.2)

We give other basic properties of ground states in Section 4.1. Furthermore, we give key properties of ground states to proving the nondegeneracy in Section 4.2.
4.1 Basic properties of ground states

Under Assumption 1.1 we can prove several properties of ground states to (1.1). In particular, we can prove the following three lemmas (Lemma 4.1, Lemma 4.2 and Lemma 4.3) by the same argument as in [3]:

**Lemma 4.1** (cf. Lemma 2.3 of [3]). Assume $d \geq 3$. Under Assumption 1.1 the following hold:

\[
\lim_{\omega \to \infty} M_\omega = \infty, \quad \text{(4.3)}
\]
\[
\lim_{\omega \to \infty} M_\omega^{-\frac{4}{d-2}} \omega = 0. \quad \text{(4.4)}
\]

**Lemma 4.2** (cf. Proposition 2.1 [3]). Assume $d \geq 3$. Under Assumption 1.1 the following holds:

\[
\lim_{\omega \to \infty} \bar{\Phi}_\omega = W \quad \text{strongly in } \dot{H}^1(\mathbb{R}^d) \text{ and strongly in } C^2_{\text{loc}}(\mathbb{R}^d). \quad \text{(4.5)}
\]

**Lemma 4.3** (cf. Proposition 3.1 of [3]). Assume $d \geq 3$. Under Assumption 1.1 there exists $\omega_{\text{dec}} > 0$ such that for any $\omega > \omega_{\text{dec}}$ and $x \in \mathbb{R}^d$,

\[
\bar{\Phi}_\omega(x) \lesssim (1 + |x|)^{-(d-2)}. \quad \text{(4.6)}
\]

The following lemma follows from Hölder’s inequality, Sobolev’s one, Lemma 4.2 and Lemma 4.3:

**Lemma 4.4.** Assume $d \geq 3$. Under Assumption 1.1 the following holds:

\[
\lim_{\omega \to \infty} \|\tilde{\Phi}_\omega - W\|_{L^r} = 0 \quad \text{for all } \frac{d}{d-2} < r < \infty. \quad \text{(4.7)}
\]

Although the above lemmas give us helpful information about the ground states, they are insufficient to prove Theorem 1.1. We need a certain refinement of Lemma 4.4 (see Proposition 4.1 in Section 4.2). For the refinement, we prepare the following lemma:

**Lemma 4.5.** Assume $d = 3, 4$. Under Assumption 1.1 there exists $\omega_1 > 0$ such that for any $\omega > \omega_1$, there exists $\mu(\omega) = 1 + \alpha(1)$ such that

\[
\langle \{1 + (-\Delta + s(\omega))^{-1} \} \zeta_\omega, VAW \rangle = 0, \quad \text{(4.8)}
\]
\[
\lim_{\omega \to \infty} \|\nabla \zeta_\omega\|_{L^2} = 0, \quad \lim_{\omega \to \infty} \|\zeta_\omega\|_{L^r} = 0 \quad \text{for all } \frac{d}{d-2} < r < \infty, \quad \text{(4.9)}
\]

where

\[
s(\omega) := \{\mu(\omega)M_\omega\}^{-\frac{4}{d-2}}, \quad \zeta_\omega := T_{\mu(\omega)M_\omega}[\Phi_\omega] - W = T_{\mu(\omega)}[\bar{\Phi}_\omega] - W. \quad \text{(4.10)}
\]

Proof of Lemma 4.5. Let $\omega > 0$ and $\frac{3}{2} \leq \mu \leq \frac{3}{2}$, and define $s_\omega(\mu)$ and $\zeta_\omega(\mu)$ as

\[
s_\omega(\mu) := \{\mu M_\omega\}^{-\frac{4}{d-2}}, \quad \zeta_\omega(\mu) := T_{\mu M_\omega}[\Phi_\omega] - W = T_{\mu}[\bar{\Phi}_\omega] - W. \quad \text{(4.11)}
\]

By Lemma 4.1 we may assume that

\[
s_\omega(\mu) \leq 1. \quad \text{(4.12)}
\]
Observe from $V\Lambda W = \Delta W$ (see (2.2)) that
\[
\langle \{1 + (-\Delta + s_\omega(\mu))^{-1}V\} \zeta_\omega(\mu), V\Lambda W \rangle \\
= \langle \zeta_\omega(\mu), V\Lambda W \rangle + \langle (-\Delta + s_\omega(\mu))^{-1}V \zeta_\omega(\mu), s_\omega(\mu)\Lambda W - (-\Delta + s_\omega(\mu))\Lambda W \rangle \\
= \langle \zeta_\omega(\mu), s_\omega(\mu)V(-\Delta + s_\omega(\mu))^{-1}\Lambda W \rangle.
\]

Write $\zeta_\omega(\mu)$ as
\[
\zeta_\omega(\mu) = T_\mu[\tilde{\Phi}_\omega - W] + T_\mu[W] - W + \frac{2}{d-2}(\mu - 1)\Lambda W - \frac{2}{d-2}(\mu - 1)\Lambda W.
\]

Plugging (4.14) into the right-hand side of (4.13), and using (2.6) and the fundamental theorem of calculus, we see that
\[
\langle \{1 + (-\Delta + s_\omega(\mu))^{-1}V\} \zeta_\omega(\mu), V\Lambda W \rangle \\
= \langle T_\mu[\tilde{\Phi}_\omega - W], s_\omega(\mu)V(-\Delta + s_\omega(\mu))^{-1}\Lambda W \rangle \\
+ \frac{2}{d-2} \int_1^\mu \int_1^\nu \lambda^{-2}T_\lambda[2\Lambda W + \frac{2}{d-2}x \cdot \nabla \Lambda W] d\lambda d\nu, s_\omega(\mu)V(-\Delta + s_\omega(\mu))^{-1}\Lambda W \rangle \\
- \frac{2}{d-2} (\mu - 1) \langle \Lambda W, s_\omega(\mu)V(-\Delta + s_\omega(\mu))^{-1}\Lambda W \rangle.
\]

By the duality, Lemma 3.3 and $\delta(s_\omega(\mu))^{-1} \geq 1$ (see (2.10) and (4.12)), we see that
\[
\|s_\omega(\mu)V(-\Delta + s_\omega(\mu))^{-1}\Lambda W\|_{L^\infty} = \sup_{\|f\|_{L^2} = 1} |\langle s_\omega(\mu)V(-\Delta + s_\omega(\mu))^{-1}\Lambda W, f \rangle| \\
\leq s_\omega(\mu) \sup_{\|f\|_{L^2} = 1} \|(-\Delta + s_\omega(\mu))^{-1}\Lambda W\|_{L^\infty} \|Vf\|_{L^1} \\
\lesssim s_\omega(\mu) \sup_{\|f\|_{L^2} = 1} \{\delta(s_\omega(\mu))^{-1}\|V\|_{L^2}^2 \|f\|_{L^2} \} \lesssim s_\omega(\mu)\delta(s_\omega(\mu))^{-1}.
\]

Consider the first term on the right-hand side of (4.15). By Hölder’s inequality, (4.16), a computation involving the scaling, $\frac{1}{2} \leq \mu \leq \frac{3}{2}$ and Lemma 1.2 we see that
\[
\|T_\mu[\tilde{\Phi}_\omega - W], s_\omega(\mu)V(-\Delta + s_\omega(\mu))^{-1}\Lambda W\| \\
\lesssim s_\omega(\mu)\delta(s_\omega(\mu))^{-1}\|\tilde{\Phi}_\omega - W\|_{L^2} = a_\omega(1)s_\omega(\mu)\delta(s_\omega(\mu))^{-1}.
\]

Consider the second term on the right-hand side of (4.15). By Hölder’s inequality, (4.16), a computation involving the scaling and $\frac{1}{2} \leq \mu \leq \frac{3}{2}$, we see that
\[
\left| \int_1^\mu \int_1^\nu \lambda^{-2}T_\lambda[2\Lambda W + \frac{2}{d-2}x \cdot \nabla \Lambda W] d\lambda d\nu, s_\omega(\mu)V(-\Delta + s_\omega(\mu))^{-1}\Lambda W \right| \\
\lesssim s_\omega(\mu)\delta(s_\omega(\mu))^{-1} \int_1^\mu \int_1^\nu \lambda^{-2}\|2\Lambda W + \frac{2}{d-2}x \cdot \nabla \Lambda W\|_{L^2} d\lambda d\nu \\
\lesssim s_\omega(\mu)\delta(s_\omega(\mu))^{-1} \int_1^\mu \int_1^\nu \lambda^{-2}d\lambda d\nu \lesssim s_\omega(\mu)\delta(s_\omega(\mu))^{-1}|\mu - 1|^2.
\]
Consider the last term on the right-hand side of (4.15). Note that \( \mathcal{F}[V\Lambda W](0) = -\frac{d+2}{d-2} \langle W \frac{r^2}{d-2}, \Lambda W \rangle > 0 \) (see (2.2) and (2.3)). Then, Lemma 3.4 shows that there exists \( \mathcal{C} > 0 \) such that
\[
\left| \frac{2}{d-2} \langle \Lambda W, s_\omega(\mu) V(-\Delta + s_\omega(\mu))^{-1} \Lambda W \rangle + \mathcal{C}s_\omega(\mu)\delta(s_\omega(\mu))^{-1} \right| \lesssim s_\omega(\mu). \tag{4.19}
\]
Putting (4.15), (4.17), (4.18) and (4.19) together, we find that
\[
\left| \left\langle \left\{ 1 + (-\Delta + s_\omega(\mu))^{-1} V \right\} \zeta_\omega(\mu), V \Lambda W \right\rangle -(\mu - 1)\mathcal{C}s_\omega(\mu)\delta(s_\omega(\mu))^{-1} \right| \lesssim s_\omega(\mu)\delta(s_\omega(\mu))^{-1}\{o_\omega(1) + |\mu - 1| + 2\delta(s_\omega(\mu))\mu - 1\}. \tag{4.20}
\]
Furthermore, by (4.14) and the intermediate value theorem, we find that there exists \( \omega_1 > 0 \) such that if \( \omega > \omega_1 \), then there exists \( \mu(\omega) > 0 \) such that
\[
\mu(\omega) = 1 + o_\omega(1), \tag{4.21}
\]
\[
\left\langle \left\{ 1 + (-\Delta + s_\omega(\mu(\omega)))^{-1} V \right\} \zeta_\omega(\mu(\omega)), V \Lambda W \right\rangle = 0. \tag{4.22}
\]
Then, (4.22) corresponds to (4.8).

It remains to prove (4.19). By (4.11), (2.6), Lemma 4.2 and (4.21), we see that
\[
\|\nabla\zeta_\omega\|_{L^2} \leq \|\nabla T_{\mu(\omega)}[\tilde{\Phi}_\omega - W]\|_{L^2} + \|\nabla\{T_{\mu(\omega)}[W] - W\}\|_{L^2}
\]
\[
\lesssim \|\nabla\{\tilde{\Phi}_\omega - W\}\|_{L^2} + \int_1^{\mu(\omega)} \lambda^{-1} \|\nabla\Lambda W\|_{L^2} d\lambda = o_\omega(1). \tag{4.23}
\]
Similarly, we can verify that for any \( \frac{d}{d+2} < r < \infty \),
\[
\|\zeta_\omega\|_{L^r} \lesssim \mu(\omega)^{\frac{d-2}{d+2}} \|\tilde{\Phi}_\omega - W\|_{L^r} + \int_1^{\mu(\omega)} \lambda^{-1 + \frac{2}{d+2}} \|\Lambda W\|_{L^r} d\lambda = o_\omega(1). \tag{4.24}
\]
Thus, we have completed the proof of the lemma. \( \square \)

### 4.2 Key properties of ground states

The aim of this subsection is to give key properties of ground states to proving Theorem 1.1 (see Proposition 4.1).

Let us begin by introducing the notation used in this subsection:

**Notation 4.1.**

1. We use \( \omega_1 \) to denote the constant given by Lemma 4.5.
2. For \( \omega > \omega_1 \), we use \( \mu(\omega) \) to denote the constant given by Lemma 4.5.
3. We define
\[
s(\omega) := \{\mu(\omega) M_\omega\}^{-\frac{d}{d-2}} \omega, \tag{4.25}
\]
\[
\kappa(\omega) := -\{g(\mu(\omega) M_\omega W), \Lambda W\}, \tag{4.26}
\]
\[
t(\omega) := \{\mu(\omega) M_\omega\}^{-\frac{d+2}{d-2}} \kappa(\omega), \tag{4.27}
\]
\[
Q_\omega := T_{\mu(\omega)}[\tilde{\Phi}_\omega], \quad \zeta_\omega := Q_\omega - W. \tag{4.28}
\]

In addition, we define
\[
h_\omega := Q_\omega^{\frac{d+2}{d-2}} - W^{\frac{d+2}{d-2}} - \frac{d+2}{d-2} \frac{W^{\frac{1}{d-2}} \zeta_\omega}{\zeta_\omega} + t(\omega)\kappa(\omega)^{-1}\{g(\mu(\omega) M_\omega Q_\omega) - g(\mu(\omega) M_\omega W)\}. \tag{4.29}
\]
By Lemma 4.3 and \( \mu(\omega) = 1 + o(1) \), we see that
\[
\|Q_\omega\|_{L^r} \sim \|W\|_{L^r} \sim 1 \quad \text{for all } \frac{d}{d-2} < r \leq \infty,
\]
where the implicit constants may depend on \( r \).

Observe from (2.9) that \( Q_\omega \) obeys
\[
- \Delta Q_\omega + s(\omega)Q_\omega - Q_\omega^{\frac{d+2}{d-2}} - t(\omega)\kappa(\omega)^{-1}g(\mu(\omega)M_\omega Q_\omega) = 0.
\]
Furthermore, it follows from (4.31) and \( W \) being a solution to (1.2) that
\[
(-\Delta + s(\omega) + V)\zeta_\omega = -s(\omega)W + t(\omega)\kappa(\omega)^{-1}g(\mu(\omega)M_\omega W) + h_\omega.
\]

The following lemma gives us the limiting profile of \( \kappa(\omega) \):

**Lemma 4.6.** Assume \( d = 3, 4 \). Under Assumption \([\mathcal{J}]\) and Notation \([\mathcal{J}]\) the following holds:
\[
\lim_{\omega \to \infty} \frac{\kappa(\omega)}{\{\mu(\omega)M_\omega\}^{p_2}} = \frac{C_2}{p_2} \langle W^{p_2}, AW \rangle.
\]

**Remark 4.1.** Note that \( \langle W^{p_2}, AW \rangle < 0 \) (see (2.3)). Furthermore, (4.33) implies that
\[
\kappa(\omega) \sim M_\omega^{p_2}.
\]

Now, we state key properties of ground states to proving the nondegeneracy (Theorem 1.1):

**Proposition 4.1.** Assume \( d = 3, 4 \). Under Assumption \([\mathcal{J}]\) and Notation \([\mathcal{J}]\) the following hold:

1. \[
\lim_{\omega \to \infty} s(\omega) = 0, \quad \lim_{\omega \to \infty} t(\omega) = 0.
\]

2. \[
\|\zeta_\omega\|_{L^r} \lesssim s(\omega)^{\frac{d+2}{d-2} - \frac{d}{p} - \varepsilon} \quad \text{for all } \omega > \omega_1, \quad \frac{d}{d-2} < r \leq \infty \text{ and } \varepsilon > 0,
\]
   where the implicit constant may depend on \( r \) and \( \varepsilon \) (independent of \( \omega \)).

3. \[
|A_1\beta(s(\omega)) - t(\omega)| = o(1)\beta(s(\omega)),
\]
   where \( A_1 \) is the same constant as in Lemma 3.7.

### 4.3 Proofs of Lemma 4.6 and Proposition 4.1

**Proof of Lemma 4.6.** By (1.7), (4.3) and \( \mu(\omega) = 1 + o(1) \), we see that
\[
\lim_{\omega \to \infty} \frac{g(\mu(\omega)M_\omega W)}{\{\mu(\omega)M_\omega W\}^{p_2}} - \frac{C_2}{p_2} |W^{p_2}AW| = 0 \quad \text{almost everywhere in } \mathbb{R}^d.
\]
Furthermore, by (1.12) and \( \frac{2}{d-2} < p_1 \leq p_2 \), we see that
\[
\frac{g(\mu(\omega)M_\omega W)}{\{\mu(\omega)M_\omega W\}^{p_2}} - \frac{C_2}{p_2} |W^{p_2}AW| \lesssim \{W^{p_1} + W^{p_2}\} AW \in L^1(\mathbb{R}^d).
Then, the Lebesgue’s dominated convergence theorem together with (4.38) and (4.39) shows that
\[
\left| \frac{\kappa(\omega)}{\{\mu(\omega)M_{\omega}\}^p_2} + \frac{C_2}{p_2}(W^{p_2}, AW) \right| \leq \int_{\mathbb{R}^d} \frac{g(\mu(\omega)M_{\omega}W)}{\{\mu(\omega)M_{\omega}W\}^p_2} - \frac{C_2}{p_2} |W^{p_2}AW| \to 0 \quad \text{as } \omega \to \infty.
\] (4.40)

Thus, we have proved the lemma.

In order to prove Proposition 4.1, we need the following lemma:

**Lemma 4.7.** Assume \(d = 3, 4\). Under Assumption 1.1 and Notation 4.1 the following holds for all sufficiently large \(\omega > \omega_1\):
\[
t(\omega) \lesssim s(\omega)^{\frac{d-2}{2} - \varepsilon} \quad \text{for all } 0 < \varepsilon < 1,
\] (4.41)
where the implicit constant may depend on \(\varepsilon\).

**Remark 4.2.** The estimate (4.41) is improved to \(t(\omega) \sim \beta(s(\omega))\) in (4.37) of Proposition 4.1.

**Proof of Lemma 4.7** By (4.2), (1.9), (4.30) and \(t(\omega) \sim M_{\omega}^{\frac{d+2}{2} + p_2^2}\) (see (4.27) and (4.34)), we see that if \(\omega\) is sufficiently large (hence Lemma 3.1 implies \(\mu(\omega)M_{\omega} \gg 1\)), then
\[
\|Q_{\omega}\|^2_{L^2} = s(\omega)^{-1}\{\mu(\omega)M_{\omega}\}^{-2}d \int_{\mathbb{R}^d} \left\{ G(\mu(\omega)M_{\omega}Q_{\omega}) - \frac{1}{2}g(\mu(\omega)M_{\omega}Q_{\omega})M_{\omega}Q_{\omega} \right\}
\geq s(\omega)^{-1}\{\mu(\omega)M_{\omega}\}^{-2}dC_3\{\mu(\omega)M_{\omega}\}^{p_2+1}\|Q_{\omega}\|_{L^{p_2+1}}^{p_2+1}
\gtrsim s(\omega)^{-1}M_{\omega}^{-\frac{d+2}{2} + p_2} \sim s(\omega)^{-1}t(\omega).
\] (4.42)

Hence, the claim (4.41) follows from (4.42) and the following estimate:
\[
\|Q_{\omega}\|^2_{L^2} \lesssim s(\omega)^{\frac{d-2}{2} - 1 - \varepsilon} \quad \text{for all } 0 < \varepsilon < 1.
\] (4.43)

We shall prove (4.43). Let \(0 < \varepsilon < 1\). We allow the implicit constants to depend on \(\varepsilon\). Observe from (1.31) that
\[
\|Q_{\omega}\|^2_{L^2} = \langle (-\Delta + s(\omega))^{-1}(-\Delta + s(\omega))Q_{\omega}, Q_{\omega} \rangle
\[
= \langle (-\Delta + s(\omega))^{-1}Q_{\omega}^{\frac{d+2}{\varepsilon}}, Q_{\omega} \rangle
\]
\[
+ t(\omega)\langle (-\Delta + s(\omega))^{-1}\kappa(\omega)^{-1}g(\mu(\omega)M_{\omega}Q_{\omega}), Q_{\omega} \rangle.
\] (4.44)

Consider the first term on the right-hand side of (4.44). By Cauchy-Schwarz inequality, Lemma 5.1 and (1.30), we see that
\[
\left| \langle (-\Delta + s(\omega))^{-1}Q_{\omega}^{\frac{d+2}{\varepsilon}}, Q_{\omega} \rangle \right| \leq \|(-\Delta + s(\omega))^{-1}Q_{\omega}^{\frac{d+2}{\varepsilon}}\|_{L^2} \|Q_{\omega}\|_{L^2}
\]
\[
\lesssim s(\omega)^{\frac{d}{2} - 1 - \varepsilon} \|Q_{\omega}^{\frac{d+2}{\varepsilon}}\|_{L^\infty} \|Q_{\omega}\|_{L^2} \lesssim s(\omega)^{\frac{d}{2} - 1 - \varepsilon} \|Q_{\omega}\|_{L^2}.
\] (4.45)
Move on to the second term on the right-hand side of (4.47). Fix $0 < \varepsilon < \frac{(d-2)p_1 - 2}{d}$. Then, by (4.34), Hölder’s inequality, Lemma 3.1, (4.32), and (4.42), we see that

$$t(\omega)\left|(-\Delta + s(\omega))^{-1}\kappa(\omega)^{-1}g(\mu(\omega)M_\omega W, Q_\omega)\right|$$

$$\lesssim t(\omega)M_\omega^{-p_2}\|(-\Delta + s(\omega))^{-1}g(\mu(\omega)M_\omega W)\|_{L_{\frac{p_1}{p_1+1}}}\|Q_\omega\|_{L_{\frac{p_1+1}{p_1}}}$$

$$\lesssim t(\omega)M_\omega^{-p_2}s(\omega)^{\frac{d\omega}{2(p_1+1)}}\|g(\mu(\omega)M_\omega W)\|_{L_{\frac{p_1}{p_1}}}$$

$$\lesssim t(\omega)M_\omega^{-p_2}s(\omega)^{\frac{d\omega}{2(p_1+1)}}M_{p_2}^s \lesssim s(\omega)^{\frac{d\omega}{2(p_1+1)}}\|Q_\omega\|^2_{L_2} = o(1)\|Q_\omega\|^2_{L_2}.$$

Putting (4.44), (4.45) and (4.46) together, we find that the claim (4.43) is true. Thus, we have proved the lemma.

Now, we are in a position to prove Proposition 4.1.

**Proof of Proposition 4.1.** We use the symbols in Notation 1.1.

The claim (4.33) follows immediately from Lemma 4.1 and (4.34).

We shall prove (4.36). Let $\omega > \omega_1, \frac{d}{1+\varepsilon} < r < \infty$ and $0 < \varepsilon < 1$. We allow the implicit constants to depend on $r$ and $\varepsilon$. By (4.32) in Proposition 3.1 together with (4.38) in Lemma 4.1 and (4.32), we see that

$$\|\zeta_\omega\|_{L^r} \lesssim \|\{1 + (-\Delta + s(\omega))^{-1}V\}\zeta_\omega\|_{L^r}$$

$$= \|(-\Delta + s(\omega))^{-1}(-\Delta + s(\omega) + V)\zeta_\omega\|_{L^r}$$

$$\lesssim s(\omega)\|(-\Delta + s(\omega))^{-1}W\|_{L^r} + t(\omega)\kappa(\omega)^{-1}\|(-\Delta + s(\omega))^{-1}g(\mu(\omega)M_\omega W)\|_{L^r}$$

$$+ \|(-\Delta + s(\omega))^{-1}h_\omega\|_{L^r}.$$

(4.47)

Consider the first term on the right-hand side of (4.47). By Lemma 3.1 we see that

$$s(\omega)\|(-\Delta + s(\omega))^{-1}W\|_{L^r} \lesssim s(\omega)^{\frac{d\omega}{2} - \frac{d}{r}}\|W\|_{L_{\frac{d-2}{d-2}}^r} \lesssim s(\omega)^{\frac{d\omega}{2} - \frac{d}{r}}.$$

(4.48)

Move on to the second term on the right-hand side of (4.47). By Lemma 4.1, (4.34), Lemma 3.1 and (4.12), we see that

$$t(\omega)\kappa(\omega)^{-1}\|(-\Delta + s(\omega))^{-1}g(\mu(\omega)M_\omega W)\|_{L^r}$$

$$\lesssim s(\omega)^{\frac{d\omega}{2} - \frac{d}{r} - \varepsilon}M_\omega^{-p_2}s(\omega)^{-\frac{d}{r}}\|g(\mu(\omega)M_\omega W)\|_{L_{\frac{4}{2}}}$$

$$\lesssim s(\omega)^{\frac{d\omega}{2} - \frac{d}{r} - \varepsilon}M_\omega^{-p_2}\sum_{j=1,2} M_{\omega}^{p_j}\|W\|_{L_{\frac{d}{p_j}}} \lesssim s(\omega)^{\frac{d\omega}{2} - \frac{d}{r} - \varepsilon}.$$

(4.49)

Consider the last term on the right-hand side of (4.47). By Lemma 3.2 Lemma 3.1, a computation involving the fundamental theorem of calculus, Hölder’s inequality, (4.34),
Lemma 4.7 and (1.12), we see that

\[ \|(-(\Delta + s(\omega))^{-1} h_\omega\|_{L^r} \]

\[ \lesssim \|Q(\omega) - W\|_{L^{4r/3}} - \frac{d+2}{d-2} W_{\frac{d-2}{2}} \|\xi_\omega\|_{L^{4r/(3r)}} \]

\[ + t(\omega)\kappa_1(\omega)^{-1}s(\omega)^{-\frac{d}{d-2}} \|g(\mu(\omega)M_\omega Q_\omega) - g(\mu(\omega)M_\omega W)\|_{L^{\frac{2r}{r-1}}} \]

\[ \lesssim \|W|\|_2 \|\xi_\omega\|_2^2 + \|\xi_\omega\|_{L^{4r/(3r)}}^2 \]

\[ + t(\omega)\kappa_1(\omega)_2^{-1}s(\omega)_2^{-\frac{d}{d-2}} \int_0^1 g'(\mu(\omega)M_\omega \{W + \theta \xi_\omega\})d\theta \mu(\omega)M_\omega \xi_\omega\|_{L^{\frac{2r}{r-1}}} \]

\[ \lesssim \|\xi_\omega\|_{L^{2r}} \|\xi_\omega\|_{L^r} + \|\xi_\omega\|_{L^{4r/(3r)}} \|\xi_\omega\|_{L^r} \]

\[ + s(\omega)^{-\frac{d-2}{2}} - \frac{d}{2} M_{\omega, p_2}^j \sum_{j=1,2} M_{\omega, p_}^j \{\|W_{\frac{d-2}{2}} \xi_\omega\|_{L^{\frac{2r}{(r-1)}}} + \|\xi_\omega\|_{L^{\frac{2r}{(r-1)}}} \}. \]

Here, for \( j = 1, 2 \), define \( r(p_j) \) as

\[ r(p_j) := \begin{cases} \frac{d-2}{2d} & \text{if } \frac{2}{d} < p_j < \frac{d}{d-2}, \\ \infty & \text{if } \frac{d}{d-2} \leq p_j < \frac{4d}{4d-2}. \end{cases} \]

(4.51)

Note that \( \frac{d}{d-2} < r(p_j) \). Fix \( \frac{d}{d-2} < r < r(p_j) \). Then, by Hölder’s inequality, we see that if \( d = 3, 4 \) (hence \( \frac{d}{4} \leq \frac{d}{d-2} \)), then

\[ \|W_{\frac{d-2}{2}} \xi_\omega\|_{L^{\frac{2r}{(r-1)}}} \lesssim \|\xi_\omega\|_{L^{\frac{2r}{(r-1)}}}. \]

(4.52)

Plugging (4.52) into (4.50), and using (4.9) in Lemma 4.5, we see that

\[ \|(-(\Delta + s(\omega))^{-1} h_\omega\|_{L^r} \lesssim \|\xi_\omega\|_{L^r} + \|\xi_\omega\|_{L^{4r/(3r)}} \lesssim \|\xi_\omega\|_{L^r}. \]

(4.53)

Furthermore, putting (4.47), (4.48), (4.49) and (4.52) together, we find that (4.36) holds, namely, it holds that

\[ \|\xi_\omega\|_{L^r} \lesssim s(\omega)^{\frac{d-2}{2} - \frac{d}{2} - \frac{d}{r} - \varepsilon} \text{ for all } \frac{d}{d-2} < r < \infty \text{ and } \varepsilon > 0, \]

(4.54)

where the implicit constant may depend on \( r \) and \( \varepsilon \).

We shall prove the last claim (4.37). To this end, we introduce the following symbols:

\[ \mathcal{X}(\omega) := \delta(s(\omega))\langle(-\Delta + s(\omega))^{-1} W, V \Lambda W\rangle, \]

(4.55)

\[ \mathcal{X}(\omega) := \kappa_1(\omega)^{-1}\langle(-\Delta + s(\omega))^{-1} g(\mu(\omega)M_\omega W), V \Lambda W\rangle, \]

(4.56)

\[ \mathcal{X}(\omega) := \langle(-\Delta + s(\omega))^{-1} h_\omega, V \Lambda W\rangle. \]

(4.57)

Observe from \( \beta(s) = \delta(\omega)^{-1}s \) (see (2.12), (1.12), and (1.8) in Lemma 1.5, that

\[ \beta(s(\omega))\mathcal{X}(\omega) - t(\omega)\mathcal{X}(\omega) = \mathcal{X}(\omega). \]

(4.58)
Furthermore, by (4.58), Lemma 3.7, (4.59) and Lemma 4.7, we see that
\[ |\mathcal{H}(\omega) - 1| \]
(4.60)
can be made acceptable to (4.37). Hence, it suffices to show that there exists \( \beta \) such that
\[ \beta(s) = \delta(s)^{-1} \] (4.61)
holds. Thus, we have completed the proof of Proposition 4.1.

5 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We give a proof of the claim (i) in Section 5.1 and the claim (ii) in Section 5.2.

5.1 Nondegeneracy of ground states

In this section, we prove the claim (i) of Theorem 1.1. We will use the symbols in Notation 4.1.
Proof of Theorem 1.1. Suppose for contradiction that the claim (i) of Theorem 1.1 is false. Then, we can take a sequence \( \{ \omega_n \} \) in \((\omega_1, \infty)\) with the following properties: \( \lim_{n \to \infty} \omega_n = \infty \); and for any \( n \geq 1 \), there exists a nontrivial real-valued radial function \( z_n \in H^1_{\text{rad}}(\mathbb{R}^d) \) such that

\[
\{- \Delta + \omega_n - \frac{d+2}{d-2} \Phi_{\omega_n}^\frac{1}{d-2} - g'(\Phi_{\omega_n})\}z_n = 0 \quad \text{in the weak sense.} \tag{5.1}
\]

We see that \( z_n \in C^2(\mathbb{R}^d) \) (see, e.g., Theorem 11.7 of [11]), which together with \( z_n \) being nontrivial and the Strauss’ radial lemma (see Lemma 1 of [14]) shows that \( 0 < \|z_n\|_{L^\infty} < \infty \). Thus, we may assume that

\[
\|z_n\|_{L^\infty} = 1 \quad \text{for all } n \geq 1. \tag{5.2}
\]

Furthermore, by Lemma 4.3, Lemma 4.6 and Proposition 4.1, we may assume that for any \( n \geq 1 \), there exists \( \mu_n = 1 + o_n(1) \) such that, defining \( s_n, \kappa_n, t_n, Q_n \) and \( \zeta_n \) as

\[
s_n := \{ \mu_n M_{\omega_n} \}^{-\frac{1}{2}} \omega_n, \tag{5.3}
\]

\[
\kappa_n := -(g(\mu_n M_{\omega_n} W), \Lambda W), \quad t_n := \{ \mu_n M_{\omega_n} \}^{-\frac{d+2}{d-2}} \kappa_n, \tag{5.4}
\]

\[
Q_n := T_{\mu_n} [\tilde{\Phi}_{\omega_n}], \quad \zeta_n := Q_n - W, \tag{5.5}
\]

we have the following:

1. \( \lim_{n \to \infty} s_n = 0, \quad \lim_{n \to \infty} t_n = 0. \tag{5.6} \)

2. \( |Q_n(x)| \lesssim (1 + |x|)^{-\frac{d-2}{2}}, \quad \|Q_n\|_{L^r} \sim 1 \quad \text{for all } \frac{d}{d-2} < r \leq \infty, \tag{5.7} \)

where the implicit constant in the second claim may depend on \( r \).

3. \( \|\zeta_n\|_{L^r} \lesssim s_n^{\frac{4-2r}{d-2}} \frac{d}{d-2} - \varepsilon \quad \text{for all } \frac{d}{d-2} < r < \infty \text{ and } \varepsilon > 0, \tag{5.8} \)

where the implicit constant may depend on \( r \) and \( \varepsilon \).

4. There exists \( \varepsilon_0 > 0 \) such that

\[
|A_1 \beta(s_n) - t_n| \lesssim o_n(1) \beta(s_n), \tag{5.9} \]

where \( A_1 \) is the same constant as in Lemma 3.7. We may write (5.9) as

\[
|A_1 \delta(s_n)^{-1} s_n - t_n| = o_n(1) t_n. \tag{5.10} \]

5. \( \lim_{n \to \infty} \frac{\kappa_n}{\{ \mu_n M_{\omega_n} \}^{p_2}} = -\frac{C_2}{p_2} \langle W^{p_2}, \Lambda W \rangle, \quad \kappa_n \sim M_{\omega_n}^{p_2}. \tag{5.11} \)

We define \( \tilde{z}_n \) by

\[
\tilde{z}_n(x) := z_n \{ \mu_n M_{\omega_n} \}^{-\frac{2}{d-2}} x. \tag{5.12} \]

This transformation preserves the norm of \( L^\infty(\mathbb{R}^d) \). Hence, (5.2) shows that

\[
\|\tilde{z}_n\|_{L^\infty} = 1 \quad \text{for all } n \geq 1. \tag{5.13} \]
Observe from (4.31) and (5.1) that
\[ -\Delta Q_n + s_n Q_n - \frac{d+2}{2} g(\mu_n M_{\omega_n} Q_n) = 0, \] (5.14)
\[ \{ -\Delta + s_n - \frac{d+2}{d-2} Q_n^\frac{4}{d-2} - \{ \mu_n M_{\omega_n} \}^{-\frac{1}{2}} g'(\mu_n M_{\omega_n} Q_n) \} \tilde{z}_n = 0. \] (5.15)

We give an outline of how to derive a contradiction:

**Overview of the proof.** We break the proof into small claims:

**Claim 1.** There exist a subsequence of \{\tilde{z}_n\} (which we continue to denote by \{\tilde{z}_n\}) and a non-zero constant \(c_\infty \neq 0\) such that
\[ \lim_{n \to \infty} \tilde{z}_n = c_\infty AW \] weakly in \(\dot{H}^1(\mathbb{R}^d)\) and strongly in \(C^1_{\text{loc}}(\mathbb{R}^d)\). (5.16)

**Claim 2.** The following identity holds for all \(n \geq 1\):
\[ \frac{s_n}{\tau_n}(Q_n, \tilde{z}_n) = \frac{\{\mu_n M_{\omega_n}\}^p}{\kappa_n} \rho_n, \] (5.17)
where
\[ \rho_n := \{\mu_n M_{\omega_n}\}^{-p} \frac{d+2}{4} g(\mu_n M_{\omega_n} Q_n) - \frac{d-2}{4} g'(\mu_n M_{\omega_n} Q_n) \mu_n M_{\omega_n} Q_n, \tilde{z}_n). \] (5.18)

**Claim 3.** The left-hand side of (5.17) obeys
\[ c_\infty \lim_{n \to \infty} \frac{s_n}{\tau_n}(Q_n, \tilde{z}_n) = -\frac{(d+2) - (d-2)p}{4} c_\infty^2, \] (5.19)
where \(c_\infty \neq 0\) is the same constant as in (5.10). We remark that the factor \(c_\infty\) on the left-hand side of (5.19) is necessary to fix the sign of the right-hand side.

Before stating the other claims, we introduce the function \(Q_n^{\perp}\) as
\[ Q_n^{\perp} := Q_n - \tau_n VW \quad \text{with} \quad \tau_n := \frac{\langle (-\Delta + s_n)^{-1} Q_n, VW \rangle}{\langle (-\Delta + s_n)^{-1} VW, VW \rangle}. \] (5.20)

Note that
\[ \langle (-\Delta + s_n)^{-1} Q_n^{\perp}, VW \rangle = 0. \] (5.21)
Furthermore, observe from \(\frac{s_n}{\tau_n} = (1 + o_n(1)) \frac{\delta(s_n)}{A_1}\) (see (5.10)) that
\[ \frac{s_n}{\tau_n}(Q_n, \tilde{z}_n) = \frac{1}{A_1} \{ 1 + o_n(1) \} \{ \delta(s_n) \langle \tau_n VW, \tilde{z}_n \rangle + \delta(s_n) \langle Q_n^{\perp}, \tilde{z}_n \rangle \}. \] (5.22)

The rest of the claims are the following:

**Claim 4.**
\[ \lim_{n \to \infty} \langle (-\Delta + s_n)^{-1} VW, VW \rangle = -\langle AW, VW \rangle > 0. \] (5.23)

**Claim 5.** Assume \(d = 3\). Then, for each \(0 < \varepsilon < 1\), the following holds:
\[ \lim_{n \to \infty} \delta(s_n) \langle (-\Delta + s_n)^{-1} Q_n, VW \rangle \geq \frac{1-\varepsilon}{2} A_1. \] (5.24)

**Claim 6.** Assume \(d = 4\). Then, the following holds:
\[
\lim_{n \to \infty} \delta(s_n) \langle (-\Delta + \tilde{s}_n)^{-1}Q_n, VAW \rangle = A_1. \tag{5.25}
\]

Claim 7.
\[
\lim_{n \to \infty} \delta(s_n) \langle Q_n^1, \tilde{z}_n \rangle = 0. \tag{5.26}
\]

Now, accepting the above claims for the time being, we derive a contradiction:

Completion of the proof. Assume \(d = 3\). Let \(\{\tilde{z}_n\}\) be the same sequence as in (5.16). Then, by (5.23) and (5.24), we see that for each \(0 < \varepsilon < 1\),
\[
\lim_{n \to \infty} \delta(s_n) \tau_n \langle VAW, \Lambda W \rangle \leq -\frac{1 - \varepsilon}{2} A_1. \tag{5.27}
\]

Then, by (5.22), (5.16), (5.26) and (5.27), we see that
\[
c_\infty \lim_{n \to \infty} \frac{s_n}{t_n} \langle Q_n, \tilde{z}_n \rangle = c_\infty. \tag{5.30}
\]

Furthermore, by (5.19), (5.28), \(p_2 < \frac{d+2}{d-2}\) and \(c_\infty \neq 0\), we see that
\[
\frac{(d + 2) - (d - 2)p_2}{4} \geq 1 - \frac{\varepsilon}{2}. \tag{5.29}
\]

However, taking sufficiently small \(\varepsilon\), we see that (5.29) contradicts \(p_2 > 3\). Thus, the claim (i) of Theorem 1.1 must be true in the case \(d = 3\).

Next, assume \(d = 4\). Then, by (5.22), (5.16), (5.26) and (5.25), we see that
\[
\lim_{n \to \infty} \frac{s_n}{t_n} \langle Q_n, \tilde{z}_n \rangle = -c_\infty. \tag{5.31}
\]

Furthermore, by (5.19), (5.30) and \(c_\infty \neq 0\), we see that
\[
\frac{(d + 2) - (d - 2)p_2}{4} = 1. \tag{5.31}
\]

However, (5.31) contradicts \(p_2 > 1\). Thus, the claim (i) of Theorem 1.1 must be true in the case \(d = 4\).

It remains to prove the above claims.

Preliminaries. By (1.12), \(\mu_n = 1 + o_n(1)\), \(\|Q_n\|_{L^\infty} \lesssim 1\) (see (5.7)), and (4.3) in Lemma 4.1, we see that
\[
\{\mu_n M_{\omega_n}^{-\frac{d-2}{d+2}} |g'(\mu_n M_{\omega_n} Q_n)|\} \lesssim M_{\omega_n}^{-\frac{p_1}{p+1}} \left\{ (M_{\omega_n} Q_n)^{p_1-1} + (M_{\omega_n} Q_n)^{p_2-1} \right\} \tag{5.32}
\]
\[
\lesssim M_{\omega_n}^{p_1 - \frac{d+2}{d-2}} Q_n^{p_1-1} = o_n(1).\]

By the Pohozaev's identity (1.12) (see also the first line of (1.12)), (1.12), \(M_{\omega_n}^{-\frac{d+2}{d-2} + p_2 - 2} \sim t_n\) (see (5.4) and (6.11), (5.11) and (5.10), we see that
\[
\|Q_n\|_{L^2}^2 \lesssim s_n^{-1} M_{\omega_n}^{p_1 + 2} \sum_{j=1}^{2} M_{\omega_n}^{p_1 + 1} \|Q_n\|_{L^{p_1+1}}^{p_1+1} \lesssim s_n^{-1} t_n \lesssim \delta(s_n)^{-1}. \tag{5.33}
\]
Moreover, by (5.11), (5.12) and \( \|Q_n\|_{L^\infty} \lesssim 1 \) (see (5.7)), we see that
\[
t_n \kappa_n^{-1} |g(\mu_n M_w, Q_n)| \lesssim t_n \{ M_w^{(p_2-p_1)} Q_n^{p_1} + Q_n^{p_2} \} \lesssim t_n Q_n^{p_1}. \tag{5.34}
\]

We shall show that for \( d = 3, 4 \),
\[
s_n \|Q_n\|_{L^1} \lesssim 1, \tag{5.35}
\]
\[
\|\Delta Q_n\|_{L^1} \lesssim 1. \tag{5.36}
\]

By (5.34), Hölder’s inequality, (5.7), (5.33), \( t_n \sim \delta(s_n)^{-1} s_n \) (see (5.10)), and (2.10), we see that: if \( 1 < p_1 \leq 2 \), then
\[
\|Q_n\|_{L^{\frac{d+2}{d-2}}} \lesssim \|Q_n\|_{L^{\frac{d+2}{d-2}}}^2 + \int_{\mathbb{R}^d} t_n \kappa_n^{-1} |g(\mu_n M_w, Q_n)| \lesssim 1 + t_n \|Q_n\|_{L^1}^2 \|Q_n\|_{L^2}^{2(p_1-1)} \tag{5.37}
\]
\[
\lesssim 1 + t_n \|Q_n\|_{L^1}^2 \|Q_n\|_{L^2}^{2(p_1-1)} \lesssim 1 + s_n \delta(s_n)^{-p_1} \|Q_n\|_{L^1}^2 \|Q_n\|_{L^2}^{2(p_1-1)} \lesssim 1 + \|Q_n\|_{L^1}^{2-p_1},
\]

whereas if \( 2 < p_1 < \frac{d+2}{d-2} \), then
\[
\|Q_n\|_{L^{\frac{d+2}{d-2}}} \lesssim \|Q_n\|_{L^{\frac{d+2}{d-2}}}^2 + \int_{\mathbb{R}^d} t_n \kappa_n^{-1} |g(\mu_n M_w, Q_n)| \lesssim 1 + \delta(s_n)^{-2} s_n \lesssim 1. \tag{5.38}
\]

Moreover, by the positivity of \( Q_n \), (5.14), the divergence theorem, and the exponential decay of \( |\nabla Q_n| \) (see, e.g., Lemma 2 of [6]), we see that
\[
s_n \|Q_n\|_{L^1} = \lim_{R \to \infty} \int_{|x| = R} \nabla Q_n \cdot \frac{x}{|x|} \sigma + \int_{\mathbb{R}^d} \Delta Q_n + \int_{\mathbb{R}^d} t_n \kappa_n^{-1} g(\mu_n M_w, Q_n) \tag{5.39}
\]
\[
\leq \|Q_n\|_{L^{\frac{d+2}{d-2}}} + \int_{\mathbb{R}^d} t_n \kappa_n^{-1} |g(\mu_n M_w, Q_n)|.
\]

Then, (5.39) follows from (5.37), (5.38) and (5.39). Furthermore, (5.36) follows from (5.35), (5.14), (5.37) and (5.38).

We shall show that
\[
\|\nabla \tilde{z}_n\|_{L^2} \lesssim 1. \tag{5.40}
\]

Multiplying both sides of (5.15) by \( \tilde{z}_n \), integrating the resulting equation, and using the first inequality in (5.32), we see that
\[
\|\nabla \tilde{z}_n\|_{L^2}^2 + s_n \|\tilde{z}_n\|_{L^2}^2 \lesssim \int_{\mathbb{R}^d} Q_n^{\frac{d+2}{d-2}} |\tilde{z}_n|^2 + \int_{\mathbb{R}^d} \left\{ M_n^{p_1-\frac{d+2}{d-2}} Q_n^{p_1-1} + M_n^{p_2-\frac{d+2}{d-2}} Q_n^{p_2-1} \right\} |\tilde{z}_n|^2. \tag{5.41}
\]

Consider the first term on the right-hand of (5.41). When \( d = 3 \), it follows from (5.7) and (5.13) that
\[
\int_{\mathbb{R}^3} Q_n^{\frac{d}{d-2}} |\tilde{z}_n|^2 \leq \|Q_n^1\|_{L^1} \|\tilde{z}_n\|_{L^\infty}^2 \lesssim 1, \tag{5.42}
\]
Furthermore, by (5.47), we may assume that

$$\int_{\mathbb{R}^d} Q_n^{q-1} |\tilde{z}_n|^2 \leq \|Q_n^2\|_{L^4} \|\tilde{z}_n\|_{L^\infty} \|\tilde{z}_n\|_{L^4} \lesssim \|\nabla \tilde{z}_n\|_{L^2}. \quad (5.43)$$

Consider the second term on the right-hand of (5.41). Assume $d = 4$. Note that $1 < p_1 < p_2 < 3$. By $M_{\omega_n}^{\frac{d-2}{2}} \sim \omega_n^{-1} s_n$ (see (5.3)), (5.7), the Gagliardo-Nirenberg inequality and Young’s one, the following holds for all $1 < q < 3$:

$$M_{\omega_n}^{q-\frac{d+2}{2}} \int_{\mathbb{R}^d} Q_n^{q-1} |\tilde{z}_n|^2 \lesssim (\omega_n^{-1} s_n)^{\frac{3-q}{2}} \|Q_n^q\|_{L^{q+1}} \|\tilde{z}_n\|^2_{L^{q+1}} \lesssim o_n(1) \{ s_n^{\frac{q+1}{q-1}} \|\tilde{z}_n\|^2_{L^2} + \|\nabla \tilde{z}_n\|^2_{L^2} \}, \quad (5.44)$$

where the implicit constants may depend on $q$. Next, assume $d = 3$. Note that $2 < p_1 < p_2 < 5$. By $M_{\omega_n}^{\frac{d-2}{2}} \sim \omega_n^{-1} s_n$, Hölder’s inequality, (5.7) and Sobolev’s inequality, we see that: for $3 < q < 5$,

$$M_{\omega_n}^{q-\frac{d+2}{2}} \int_{\mathbb{R}^d} Q_n^{q-1} |\tilde{z}_n|^2 \lesssim (\omega_n^{-1} s_n)^{\frac{5-q}{2}} \|Q_n^q\|_{L^5} \|\tilde{z}_n\|_{L^5} \|\tilde{z}_n\|_{L^6} \lesssim o_n(1) \|\nabla \tilde{z}_n\|_{L^2}, \quad (5.45)$$

and for $2 < q \leq 3$,

$$M_{\omega_n}^{q-\frac{d+2}{2}} \int_{\mathbb{R}^d} Q_n^{q-1} |\tilde{z}_n|^2 \lesssim (\omega_n^{-1} s_n)^{\frac{5-q}{2}} \|Q_n^q\|_{L^5} \|\tilde{z}_n\|_{L^5} \|\tilde{z}_n\|_{L^6} \lesssim o_n(1) \{ s_n^{\frac{q+1}{q-1}} \|\tilde{z}_n\|^2_{L^2} + \|\nabla \tilde{z}_n\|^2_{L^2} \}, \quad (5.46)$$

where the implicit constants in (5.44) and (5.46) may depend on $q$. Plugging the estimates (5.42) through (5.46) into (5.41), we obtain (5.40).

We can prove the following uniform estimates in a way similar to Lemma 4.3 of [3]:

$$|\tilde{z}_n(x)| \lesssim (1 + |x|)^{-(d-2)}, \quad \|\tilde{z}_n\|_{L^r} \lesssim 1 \quad \text{for all } \frac{d}{d-2} < r \leq \infty. \quad (5.47)$$

Now, we give the proofs of the above claims:

**Proof of Claim 1.** We shall prove (5.40).

By (5.47), (5.13), (5.32) and (5.7), we see that

$$\|\tilde{z}_n\|_{W^{2,d}} \sim \|\tilde{z}_n\|_{L^{2d}} + \|\Delta \tilde{z}_n\|_{L^{2d}} \lesssim 1. \quad (5.48)$$

Then, it follows from $\tilde{z}_n$ being radial, (5.40), (5.48) and the Rellich-Kondrashov compactness theorem that there exist a subsequence of $\{\tilde{z}_n\}$ (which we continue to denote by $\{\tilde{z}_n\}$) and a real-valued radial function $\tilde{z}_\infty \in \dot{H}^1_{\text{rad}}(\mathbb{R}^d) \cap C^1_{\text{loc}}(\mathbb{R}^d)$ such that

$$\lim_{n \to \infty} \tilde{z}_n = \tilde{z}_\infty \quad \text{weakly in } \dot{H}^1(\mathbb{R}^d) \text{ and strongly in } C^1_{\text{loc}}(\mathbb{R}^d). \quad (5.49)$$

Furthermore, by (5.47), we may assume that

$$\lim_{n \to \infty} \tilde{z}_n = \tilde{z}_\infty \quad \text{weakly in } L^q(\mathbb{R}^d) \text{ for all } \frac{d}{d-2} < q < \infty. \quad (5.50)$$
Now, let $\phi \in C_c^\infty(\mathbb{R}^d)$ be a test function. Then, by (5.15), the fundamental theorem of calculus, (5.49), (5.6), (5.13), (5.32), (5.7) and (5.8), we see that

$$\left| \langle (-\Delta + V)\tilde{z}_\infty, \phi \rangle \right| \lesssim \left| \langle \Delta (\tilde{z}_n - \tilde{z}_\infty), \phi \rangle \right| + s_n(\langle \tilde{z}_n, \phi \rangle + \{\mu_n M_{\omega_n}\}^{-\frac{d}{d-2}} |\langle g'(\mu_n M_{\omega_n} Q_n)\tilde{z}_n, \phi \rangle| \right.$$

$$+ \left| \langle Q_n^{-\frac{d}{d-2}} (\tilde{z}_n - \tilde{z}_\infty), \phi \rangle \right| + \left| \langle W + \theta \zeta_n \rangle^{\frac{d+2}{d-2}} \zeta_n d\theta \tilde{z}_\infty, \phi \rangle \right|

\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \quad (5.51)$$

Thus, it holds that $(-\Delta + V)\tilde{z}_\infty = 0$ in the distribution sense. Then, we can verify that $\tilde{z}_\infty \in W^{2,2} \cap L^\infty(\mathbb{R}^d)$, which further implies $\tilde{z}_\infty \in W^{2,q}(\mathbb{R}^d)$ for all $q > 2^*$. Then, the Sobolev’s inequality shows $\tilde{z}_\infty \in C^1(\mathbb{R}^d)$. Furthermore, by the same argument as in the proof of Lemma 1 of [4], we see that $\tilde{z}_\infty \in C^2(\mathbb{R}^d)$.

**Proof of Claim 2.** We shall prove (5.17).

Put $\tilde{w}_n := x \cdot \nabla Q_n$. By an elementary computation, (5.14) and $t_n\kappa_n^{-1} = \{\mu_n M_{\omega_n}\}^{-\frac{d+2}{d-2}}$ (see (5.4)), we can verify that

\[-\Delta \tilde{w}_n + s_n \tilde{w}_n = x \cdot \nabla (-\Delta Q_n + s_n Q_n) - 2\Delta Q_n\]

$$= \left\{ \frac{d+2}{d-2} Q_n^{\frac{d}{d-2}} + \{\mu_n M_{\omega_n}\}^{-\frac{d}{d-2}} g'(\mu_n M_{\omega_n} Q_n) \right\} \tilde{w}_n - 2 \{ s_n Q_n - Q_n^{\frac{d}{d-2}} - t_n \kappa_n^{-1} g(\mu_n M_{\omega_n} Q_n) \}. \quad (5.53)$$

Multiplying both sides of (5.15) by $\tilde{w}_n$, and (5.53) by $\tilde{z}_n$, and integrating the difference of the resulting equations, we see that

$$s_n(\langle Q_n, \tilde{z}_n \rangle) = \langle Q_n^{\frac{d}{d-2}}, \tilde{z}_n \rangle + t_n \kappa_n^{-1} (g(\mu_n M_{\omega_n} Q_n), \tilde{z}_n) \quad (5.54)$$

Moreover, multiplying both sides of (5.14) by $\tilde{z}_n$, and (5.15) by $Q_n$, integrating the difference of the resulting equations, and using $t_n \kappa_n^{-1} = \{\mu_n M_{\omega_n}\}^{-\frac{d+2}{d-2}}$, we see that

$$\frac{4}{d-2} \langle Q_n^{\frac{d+2}{d-2}}, \tilde{z}_n \rangle = t_n \kappa_n^{-1} (g(\mu_n M_{\omega_n} Q_n) - \mu_n M_{\omega_n} g'(\mu_n M_{\omega_n} Q_n) Q_n, \tilde{z}_n) \quad (5.55)$$

Plugging (5.55) divided by $\frac{4}{d-2}$ into (5.54), we obtain (5.17).

**Proof of Claim 3.** We shall prove (5.19). To this end, it suffices to show that

$$\lim_{n \to \infty} \rho_n = \frac{(d+2) - (d-2)p_2 C_2}{4 p_2} (W^{p_2}, AW)c_\infty. \quad (5.56)$$
Indeed, it follows from (5.17) and (5.11) that
\[ c_{\infty} \frac{s_n}{t_n} \lim_{n \to \infty} \langle Q_n, \tilde{z}_n \rangle = c_{\infty} \lim_{n \to \infty} \frac{\{\mu_n M_{\omega_n}\}^{p_2}}{\kappa_n} \lim_{n \to \infty} \rho_n = - \frac{c_{\infty}^{p_2}}{C_2(W^{p_2}, \Lambda W)} \lim_{n \to \infty} \rho_n, \tag{5.57} \]
which together with (5.56) implies (5.19).

Let us prove (5.56). Observe from the definition of \( \{ \infty \} \) (see (5.13)), we see that
\[ \{ \infty \} \leq \langle \rho \rangle_{L^2} \leq 1 \] (see (5.13)). Then, we see from (5.50) and (5.52) that
\[ \lim_{n \to \infty} \langle W^{p_2}, \tilde{z}_n - c_{\infty} \Lambda W \rangle = 0. \tag{5.59} \]

Consider the second term on the right-hand side of (5.58). We shall show that
\[ \lim_{n \to \infty} \langle g(\mu_n M_{\omega_n} Q_n) - \frac{C_2}{p_2} W^{p_2}, \tilde{z}_n \rangle = 0. \tag{5.60} \]

First, observe that
\[ \left| \frac{g(\mu_n M_{\omega_n} Q_n)}{\{\mu_n M_{\omega_n}\}^{p_2}} - \frac{C_2}{p_2} W^{p_2}, \tilde{z}_n \right| \leq \int_{\mathbb{R}^d} \left| \frac{g(\mu_n M_{\omega_n} Q_n)}{\{\mu_n M_{\omega_n}\}^{p_2}} - \frac{C_2}{p_2} Q^{p_2}_n \tilde{z}_n \right| dx + \frac{C_2}{p_2} |Q^{p_2}_n - W^{p_2}, \tilde{z}_n|. \tag{5.61} \]

By (1.7), (5.1), Lemma 4.2 \( \|Q_n\|_{L^\infty} \lesssim 1 \) and \( \|\tilde{z}_n\|_{L^\infty} = 1 \) (see (5.13)), we see that
\[ \lim_{n \to \infty} \left| \frac{g(\mu_n M_{\omega_n} Q_n)}{\{\mu_n M_{\omega_n}\}^{p_2}} - \frac{C_2}{p_2} Q^{p_2}_n \tilde{z}_n \right| = 0 \quad \text{almost everywhere in } \mathbb{R}^d. \tag{5.62} \]

Furthermore, by (1.12), (5.7), (5.44) and \( \frac{2}{d-2} < p_1 < p_2 \), we see that
\[ \left| \frac{g(\mu_n M_{\omega_n} Q_n)}{\{\mu_n M_{\omega_n}\}^{p_2}} - \frac{C_2}{p_2} Q^{p_2}_n \tilde{z}_n \right| \lesssim \{Q^{p_1}_n + Q^{p_2}_n\} |\tilde{z}_n| \lesssim (1 + |x|)^{-2(d-2)(p_1+1)} \in L^1(\mathbb{R}^d). \tag{5.63} \]
Hence, the Lebesgue’s dominated convergence theorem shows that the first term on the right-hand side of (5.61) obeys
\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} \frac{g(\mu_n M^{(1)} Q_n)}{\mu_n M^{(1)} Q_n} - \frac{C_2}{p_2} |Q_n^{p_2} \tilde{z}_n| \, dx = 0. \tag{5.64}
\]
Move on to the second term on the right-hand side of (5.61). Fix \(0 < \varepsilon < 1\) with \(2\varepsilon < (d - 2)p_2 - 2\). Note that \(\frac{d(p_2 - 1)}{4 - d + 2\varepsilon} > \frac{d}{d - 2}\). Then, by the fundamental theorem of calculus, Hölder’s inequality, (6.8) and (5.37), we see that
\[
\lim_{n \to \infty} |\langle Q_n^{p_2} - W^{p_2}, \tilde{z}_n \rangle| \lesssim \lim_{n \to \infty} \|W\| \|\zeta_n\|\|p_2^{\frac{d(p_2 - 1)}{4 - d + 2\varepsilon}} \zeta_n\| \|\tilde{z}_n\| \frac{d}{d - 2} = 0. \tag{5.65}
\]
Then, (5.60) follows from (5.61), (5.64) and (5.65). Similarly, we can prove that the last term on the right-hand side of (5.58) obeys
\[
\lim_{n \to \infty} \left| \langle \mu_n M^{(1)} Q_n \rangle^{p_2 - 1} Q_n - C_2 W^{p_2} \tilde{z}_n \rangle \right| = 0. \tag{5.66}
\]
Then, (5.60) follows from (5.58), (5.59), (5.60) and (5.66).

**Proof of Claim 4.** We shall prove (5.23). By \(V A W = -(-\Delta) A W\), the second resolvent equation and Lemma 3.3, we see that
\[
|\langle(-\Delta + s_n)^{-1}A W, V A W \rangle + \langle A W, V A W \rangle|
= |\langle(-\Delta + s_n)^{-1}A W, V A W \rangle - \langle(-\Delta)^{-1}A W, V A W \rangle|
= s_n|\langle(-\Delta + s_n)^{-1}(-\Delta)^{-1}A W, V A W \rangle| = s_n|\langle(-\Delta + s_n)^{-1}A W, V A W \rangle|
\leq s_n\|(-\Delta + s_n)^{-1}A W\|_{L^\infty} \|V A W\|_{L^1} \lesssim s_n \delta(s_n)^{-1},
\]
which together with \(V := -\frac{d + 2}{d - 2} W^{-\frac{1}{2}} < 0\) (see (2.22)) implies (5.23).

**Proof of Claim 5.** We shall prove (5.24). To this end, it suffices to show that for each \(0 < \varepsilon < 1\),
\[
\liminf_{n \to \infty} \delta(s_n) \{(-\Delta + s_n)^{-1} Q_n \}(0) \geq \frac{1 - \varepsilon}{2} A_0. \tag{5.68}
\]
Indeed, by Lemma 3.5, (5.7), (5.68) and \(A_1 := \mathfrak{c}_0 A_0\) (see (3.27)), we see that
\[
\delta(s_n) \{(-\Delta + s)^{-1} Q_n, V A W \} = \delta(s_n) \mathfrak{c}_0 \{(-\Delta + s)^{-1} Q_n \}(0) + \delta(s_n) Q_n \|_{L^2}
\geq \mathfrak{c}_0 \frac{1 - \varepsilon}{2} A_0 + o_n(1) = \frac{1 - \varepsilon}{2} A_1 + o_n(1), \tag{5.69}
\]
which implies (5.24). Let us prove (5.68). Assume \(d = 3\). Put \(Y_n := \sqrt{3} e^{-\sqrt{3}|x|} |x|^{-1}\). Note that
\[
\Delta Y_n - s_n Y_n = 0 \quad \text{in } \mathbb{R}^3 \setminus \{0\}. \tag{5.70}
\]
Let \(\varepsilon_0 > 0\) and \(R > 0\). Then, the convergence of \(\{\hat{\Phi}_n\}\) to \(W\) in \(C^2_{\text{loc}}(\mathbb{R}^d)\) (see Lemma 4.2) together with \(Q_n = T_{\mu_n} \{\tilde{\Phi}_n\}\) and \(\mu_n = 1 + o_n\) shows that there exists a number \(n(\varepsilon_0, R)\) such that if \(n \geq n(\varepsilon_0, R)\), then
\[
Q_n(R) \geq (1 - \varepsilon_0)W(R) \geq \frac{(1 - \varepsilon_0)R}{\sqrt{3 + R^2}} Y_n(R). \tag{5.71}
\]
where \( \chi_n \) being positive, we see that if in three dimensions (see, e.g., Theorem 6.23 of [11]), the substitution of variables, and we want. By (5.70), (5.14), the positivity of \( \mu_n \) and the definition of \( t_n \) (see [5.4]), we see that

\[
\Delta \{(1 - \varepsilon)Y_n - Q_n\} - s_n\{(1 - \varepsilon)Y_n - Q_n\} = Q_n^5 + \{\mu_n M_n\}^{-5}g(\mu_n M_n, Q_n) \geq 0. \quad (5.73)
\]

Then, the maximum principle together with (5.71) and (5.73) shows that

\[
\chi_{\geq R} Q_n \geq (1 - \varepsilon)\chi_{\geq R} Y_n \quad \text{for all } n \geq n(\varepsilon_0, R), \quad (5.74)
\]

where \( \chi_{\geq R} \) is the indicator function of \( \{|x| \geq R\} \), namely \( \chi_{\geq R}(x) = 0 \) if \( |x| \leq R \) and \( \chi_{\geq R}(x) = 1 \) if \( |x| \geq R \). By (5.74), \( 4\pi e^{-\sqrt{s_n}|x|/|x|^{-1}} \) being the Green’s function for \( -\Delta + s_n \) in three dimensions (see, e.g., Theorem 6.23 of [11]), the substitution of variables, and \( Q_n \) being positive, we see that if \( n \geq n(\varepsilon_0, R) \), then

\[
\delta(s_n)((-\Delta + s_n)^{-1}Q_n)(0) = \delta(s_n)((-\Delta + s_n)^{-1}(1 - \chi_{\geq R})Q_n)(0) + \delta(s_n)((-\Delta + s_n)^{-1}\chi_{\geq R}Y_n)(0)
\]

\[
\geq (1 - \varepsilon)\delta(s_n)((-\Delta + s_n)^{-1}\chi_{\geq R}Y_n)(0)
\]

\[
= (1 - \varepsilon)\delta(s_n) \int_{|x| \geq R} \frac{e^{-\sqrt{s_n}|x|}}{4\pi|x|} \sqrt{3}e^{-\sqrt{s_n}|x|/|x|^{-1}} \, dx
\]

\[
= (1 - \varepsilon)\sqrt{3} \int_{|x| \geq \sqrt{s_n}R} \frac{e^{-2|x|}}{4\pi|x|^2} \, dx = (1 - \varepsilon) \frac{\sqrt{3}}{2} \int_{|x| \geq 2\sqrt{s_n}R} \frac{e^{-|x|}}{4\pi|x|^2} \, dx
\]

It follows from (5.75) that

\[
\lim_{n \to \infty} \delta(s_n)\{(\Delta + s_n)^{-1}Q_n\}(0) \geq \frac{1 - \varepsilon}{2} \frac{\sqrt{3}}{4\pi |x|^2} \int_{R^3} \frac{e^{-|x|}}{4\pi |x|^2} \, dx = \frac{1 - \varepsilon}{2} A_0. \quad (5.76)
\]

Thus, we have proved (5.68) and therefore the claim (5.21) is true.

**Proof of Claim 6.** We shall prove (5.25). By \( Q_n = W + \zeta_n \) and Lemma 3.4, we see that

\[
|\delta(s_n)(\{(-\Delta + s_n)^{-1}Q_n, V\Lambda W\} - A_1| \leq o_n(1) + |\delta(s_n)(\{(-\Delta + s_n)^{-1}\zeta_n, V\Lambda W\}|. \quad (5.77)
\]

Hence, for (5.25), it suffices to show that

\[
\lim_{n \to \infty} |\delta(s_n)(\{(-\Delta + s_n)^{-1}\zeta_n, V\Lambda W\}| = 0. \quad (5.78)
\]

Assume \( d = 4 \). Let \( 0 < \varepsilon_1 < \min\{(d - 2)p_1, 2, 1\} \). Then, by \( V\Lambda W = \Delta\Lambda W \) (see [22]), Hölder’s inequality, and (3.4) in Lemma 3.1, we see that

\[
\delta(s_n)\{(-\Delta + s_n)^{-1}\zeta_n, V\Lambda W\} = \delta(s_n)\{\Delta\zeta_n, (-\Delta + s_n)^{-1}\Lambda W\}
\]

\[
\leq \delta(s_n)\|\Delta\zeta_n\|_{L^{q_1}}\|\{(-\Delta + s_n)^{-1}\Lambda W\|_{L^{q_2}}
\]

\[
\leq \|\Delta\zeta_n\|_{L^{q_1}}\|\{(-\Delta + s_n)^{-1}\Lambda W\|_{L^{q_2}} \leq \delta(s_n)\frac{s_n^{-1}}{s_n^{-1}} \|\Delta\zeta_n\|_{L^{q_1}} \leq \delta(s_n) s_n^{-1} \|\Delta\zeta_n\|_{L^{q_1}}. \quad (5.79)
\]
where the implicit constants may depend on \( \varepsilon_1 \). Note here that \( \frac{d p_1}{2 + \frac{1}{p_1}} > \frac{d}{d - \frac{1}{2}} \) and \( \frac{16 d}{(d-2)(8 + 3 \varepsilon_1)} > \frac{d}{d - \frac{1}{2}} \). Hence, by (5.14), (5.31), the fundamental theorem of calculus, Hölder’s inequality, (5.17), (5.8) with \( \varepsilon = \frac{d}{8} \), and \( t_n \sim \delta(s_n)^{-1}s_n \) (see (5.10)), we see that

\[
\| \Delta \zeta_n \|_{L^\frac{d}{d-\varepsilon_1}} = \| \Delta \{ Q_n - W \} \|_{L^\frac{d}{d-\varepsilon_1}} = \| \Delta Q_n + W \|_{L^\frac{d}{d-\varepsilon_1}} \\
\leq s_n \| Q_n \|_{L^\frac{d}{d-\varepsilon_1}} + \| (W + |\zeta_n|)^{\frac{d}{d-\varepsilon_1}} \zeta_n \|_{L^\frac{d}{d-\varepsilon_1}} + t_n \| Q_n^\dag \|_{L^\frac{d}{d-\varepsilon_1}} \\
\leq s_n \| Q_n \|_{L^\frac{d}{d-\varepsilon_1}} + \| W + |\zeta_n| \|^{\frac{d}{d-\varepsilon_1}} \| \zeta_n \|_{L^\frac{d}{d}} + t_n \| Q_n^\dag \|_{L^\frac{d}{d-\varepsilon_1}} \\
\leq \delta(s_n)^{-1} s_n,
\]

where the implicit constants may depend on \( \varepsilon_1 \). Furthermore, by the interpolation inequality (Hölder’s inequality), (5.33) and (5.35), we see that

\[
(5.80)
\]

\[
(5.81)
\]

where the implicit constant may depend on \( \varepsilon_1 \). Plugging (5.81) into (5.80), we obtain

\[
\| \Delta \zeta_n \|_{L^\frac{d}{d-\varepsilon_1}} \lesssim \delta(s_n)^{-\frac{(d-2-\varepsilon_1)}{d}} s_n^\frac{2(d-2-\varepsilon_1)}{d} + s_n^\frac{d-\varepsilon_1}{d} + \delta(s_n)^{-1} s_n,
\]

where the implicit constant may depend on \( \varepsilon_1 \). Then, it follows from (5.79), (5.82), the definition of \( \delta(s) \) (see (2.10)) and \( d = 4 \) that

\[
\delta(s_n)|\langle (-\Delta + s_n)^{-1} \zeta_n, VAW \rangle| \lesssim s_n^{-\frac{1}{2} + \frac{1}{4} - \frac{1}{4} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}} + \frac{1}{4} + \frac{1}{4} \to 0 \quad \text{as } n \to \infty.
\]

Thus, we have proved (5.78) and therefore the claim (5.25) is true.

**Proof of Claim 7.** We shall prove (5.20).

It follows from \(-\Delta + s_n + V = \{ 1 + V(-\Delta + s_n)^{-1} \}(-\Delta + s_n) \) that (5.15) is written as

\[
\tilde{z}_n = (-\Delta + s_n)^{-1} \{ 1 + V(-\Delta + s_n)^{-1} \}^{-1} \tilde{V}_n \tilde{z}_n,
\]

where

\[
\tilde{V}_n := V + \frac{d}{d-2} \frac{\phi_{n+1}}{\phi_{n+1}^\frac{4}{d-2}} + \mu_\omega M_\omega \frac{4}{d-2} g'(\mu_\omega M_\omega Q_n)
\]

\[
= \frac{d}{d-2} \left( \frac{4}{d-2} \phi_{n+1}^\frac{4}{d-2} - W \right) + \mu_\omega M_\omega \frac{4}{d-2} g'(\mu_\omega M_\omega Q_n).
\]

By (5.84), and the adjoint operator of \( \{ 1 + V(-\Delta + s_n)^{-1} \}^{-1} \) on \( L^2_{\text{real}}(\mathbb{R}^d) \) being \( \{ 1 + (-\Delta + s_n)^{-1} V \}^{-1} \), we see that

\[
\langle Q_n^\dag, \tilde{z}_n \rangle = \langle \{ 1 + (-\Delta + s_n)^{-1} V \}^{-1} \rangle \langle \{ 1 + (-\Delta + s_n)^{-1} V \}^{-1} Q_n^\dag, \tilde{V}_n \tilde{z}_n \rangle.
\]

Furthermore, by (5.86), Hölder’s inequality, (5.2) in Proposition 5.1, (5.24) and \( p_1 > \frac{d}{d-2} \) (see Assumption 1.1), we see that

\[
(5.86)
\]

\[
(5.87)
\]

| (5.87) | (5.87) |
|---|---|

29
Consider the first factor on the right-hand side of (5.87). First, observe from (5.23), $V A W = \Delta A W$, Lemma 3.3 and (5.36) that

$$|\tau_n| \lesssim \|(-\Delta + s_n)^{-1}Q_n, V A W\| = \|\Delta Q_n, (-\Delta + s_n)^{-1}A W\|$$

$$\leq \|\Delta Q_n\|_{L^1} \|(-\Delta + s_n)^{-1}A W\|_{L^\infty} \lesssim (s_n)^{-1}. \tag{5.88}$$

Then, by (5.20), (3.4) in Lemma 3.1 Lemma 3.2 together with $p_1 + 1 > \frac{d}{d-2}$, (5.88) and (5.7), we see that

$$\|(\Delta + s_n)^{-1}Q_n\|_{L^{p_1+1}} = \|(\Delta + s_n)^{-1}\{Q_n - \tau_n V A W\}\|_{L^{p_1+1}}$$

$$\leq \|(\Delta + s_n)^{-1}Q_n\|_{L^{p_1+1}} + \|\tau_n\|(\Delta + s_n)^{-1}V A W\|_{L^{p_1+1}}$$

$$\lesssim s_n^{-\frac{d-2}{2}} \frac{d}{2(p_1+1)} \|Q_n\|_{L^{d}} + \delta(s_n)^{-1}\|V A W\|_{L^{\frac{d(p_1+1)}{d-2}}}, \tag{5.89}$$

$$\lesssim s_n^{-\frac{d-2}{2}} \frac{d}{2(p_1+1)} + \delta(s_n)^{-1}. \tag{5.89}$$

Consider the second factor on the right-hand side of (5.87). Let $0 < \varepsilon < \frac{d}{2(p_1+1)}$. Note that $W^\frac{6-d}{2} \in L^{p_1+1}(\mathbb{R}^d)$ for $d = 3, 4$, and $\frac{d(p_1+1)}{d-2(p_1+1)} > \frac{d}{d-2}$. Then, by the definition of $\tilde{V}_n$ (see (5.83)), the second inequality in (5.32), Hölder’s inequality, (5.7), (5.8), (5.47) and $M_{\omega_n}^p \frac{d+2}{d-2} \sim t_n \sim (s_n)^{-1} s_n$ (see (5.4), (5.11) and (5.10)), we see that

$$\|\tilde{V}_n \tilde{z}_n\|_{L^{p_1+1}} \lesssim \|(W + |\zeta_n|)^{\frac{6-d}{2}} \zeta_n \tilde{z}_n\|_{L^{p_1+1}} + M_{\omega_n}^p \frac{d+2}{d-2} \|Q_n^{-1} \tilde{z}_n\|_{L^{p_1+1}}$$

$$\leq \|(W + |\zeta_n|)^{\frac{6-d}{2}}\|_{L^{p_1+1}} \|\zeta_n\|_{L^\infty} \|\tilde{z}_n\|_{L^{d}} + M_{\omega_n}^p \frac{d+2}{d-2} \|Q_n^{-1}\|_{L^{p_1+1}} \|\tilde{z}_n\|_{L^{p_1+1}}$$

$$\lesssim \frac{d-2}{2} \varepsilon + \delta(s_n)^{-1} s_n, \tag{5.90}$$

where the implicit constants may depend on $\varepsilon$. Putting (5.87), (5.89) and (5.90) together, and taking a sufficiently small $\varepsilon$ depending on $d$ and $p_1$, we see from (2.10) that

$$\delta(s_n)\|Q_n^{-1} \tilde{z}_n\| \lesssim \delta(s_n)\left\{s_n^{-\frac{d-2}{2}} \frac{d}{2(p_1+1)} + \delta(s_n)^{-1}\right\}\left\{s_n^{-\frac{d-2}{2}} + \delta(s_n)^{-1} s_n\right\}$$

$$= \left\{\delta(s_n) s_n^{-\frac{d-2}{2}} \frac{d}{2(p_1+1)} + 1\right\}\left\{s_n^{-\frac{d-2}{2}} + \delta(s_n)^{-1} s_n\right\} \tag{5.91}$$

$$= \delta(s_n)^{-d-2(\frac{d}{2(p_1+1)} - \varepsilon)} + s_n^{-\frac{d-2}{2}} \frac{d}{2(p_1+1)} + s_n^{-\frac{d-2}{2}} + \delta(s_n)^{-1} s_n \to 0 \text{ as } n \to \infty.$$
By (1.25), what we need to prove is that
\[ \dim \ker L_{\omega,+} = d. \] (5.92)

Although we can prove (5.92) by an argument similar to the proof of Theorem 0.3 in [10], we give a proof as the kernel contains the components arising from the translation invariance of (1.1).

For \( d \geq 2 \) and \( k \geq 0 \), let \( N_{d,k} \) denote the dimension of the space of spherical harmonics of degree \( k \). It is known that \( N_{d,0} = 1 \), \( N_{d,1} = d \), and \( N_{d,k} \) is finite for all \( k \geq 2 \).

Furthermore, for each \( k \geq 0 \), let \( Y_k^1, \ldots, Y_k^{N_{d,k}} \) be mutually orthogonal spherical harmonics of degree \( k \). Note that \( Y_k^m \) and \( Y_k^\ell \) are orthogonal if either \( k \neq \ell \) or \( m \neq n \), and
\[ -\Delta_{S^{d-1}} Y_k^m = k(d + k - 2)Y_k^m \quad \text{for all } k \geq 0 \text{ and } 1 \leq m \leq N_{d,k}, \] (5.93)

where \( \Delta_{S^{d-1}} \) is the Laplace-Beltrami operator on the sphere \( S^{d-1} \).

Now, suppose for contradiction that (5.92) is false. Then, we can take a nontrivial function \( f_0 \in H^2(\mathbb{R}^d) \) such that
\[ L_{\omega,+} f_0 = 0, \quad \langle f_0, \partial_j \Phi_\omega \rangle = 0 \quad \text{for all } 1 \leq j \leq d. \] (5.94)

We will use the polar coordinates, namely \( r = |x| \), \( \sigma = \frac{x}{|x|} \), and \( \sigma_j := \frac{x_j}{|x|} \) (1 \( \leq j \leq d \)) for \( x \in \mathbb{R}^d \setminus \{0\} \). Note that the spherical harmonics are functions of \( \sigma \). We may take \( Y_0^1 = 1 \) and \( Y_1^j = \sigma_j \) for all \( 1 \leq j \leq d \). Then, we may write \( f_0 \) as follows:
\[ f_0(x) = c_0(r) + \sum_{j=1}^{d} c_0^j(r) \sigma_j + \sum_{k=2}^{\infty} \sum_{m=1}^{N_{d,k}} c_k^m(r) Y_k^m(\sigma), \] (5.95)

where
\[ c_0(r) := \int_{S^{d-1}} f_0(r\sigma) d\sigma, \]
\[ c_0^j(r) := \int_{S^{d-1}} \sigma_j f_0(r\sigma) d\sigma \quad \text{for all } 1 \leq j \leq N_{d,1} = d, \] (5.96)
\[ c_k^m(r) := \int_{S^{d-1}} f_0(r\sigma) Y_k^m(\sigma) d\sigma \quad \text{for all } k \geq 2 \text{ and } 1 \leq m \leq N_{d,k}. \]

We may write \( L_{\omega,+} \) as
\[ L_{\omega,+} = -\frac{\partial^2}{\partial r^2} - \frac{d-1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \Delta_{S^{d-1}} + V_{\omega,+} \quad \text{with } V_{\omega,+} := \omega - \frac{d+2}{d-2} \Phi_\omega + g'(\Phi_\omega). \] (5.97)

Let \( k \geq 0 \) and \( 1 \leq m \leq N_{d,k} \). By \( L_{\omega,+} f_0 = 0 \) (see (5.94), (5.95), (5.97) and (5.93)), we see that
\[ 0 = \int_{S^{d-1}} L_{\omega,+} f_0(r\sigma) Y_k^m(\sigma) d\sigma = A_{\omega,k} c_k^m(r), \] (5.98)

where
\[ A_{\omega,k} := -\frac{d^2}{dr^2} - \frac{d-1}{r} \frac{d}{dr} + \frac{k(d+k-2)}{r^2} + V_{\omega,+}. \] (5.99)

Note that \( A_{\omega,0} = L_{\omega,+}|_{H^2_\text{rad}(\mathbb{R}^d)}. \) Hence, the nondegeneracy of \( \Phi_\omega \) in \( H^1_\text{rad}(\mathbb{R}^d) \) (see the claim (i) of Theorem 1.1) shows that \( c_0 \) must be trivial.
We shall show that $c_k^m$ are trivial for all $k \geq 1$ and $1 \leq m \leq N_{d,k}$. To this end, put $\gamma_k^m := r^{d-2} c_k^m$. Then, we may write (5.98) as

$$\frac{d^2 \gamma_k^m}{dr^2} = \left\{ \frac{(d-1)(d-3)}{4r^2} + \frac{k(d+k-2)}{r^2} + V_{\omega,+}(r) \right\} \gamma_k^m. \quad (5.100)$$

Note that $\gamma_k^m(0) = 0$, and $\gamma_k^m(r) \to 0$ exponentially as $r \to \infty$ (see, e.g., the proof of Lemma 2 of [6]). Furthermore, the standard theory of linear ODEs shows that the zeros of $\gamma_k^m$ are isolated.

In order to derive a contradiction, it suffices to show that $\gamma_k^m$ is trivial for all $k \geq 1$. Let $k \geq 1$ and $1 \leq m \leq N_{d,k}$. Furthermore, let $a_{k,m} > 0$ be the second zero of $\gamma_k^m$; The first zero is 0, and we regard $a_{k,m} = \infty$ if $\gamma_k^m$ has no zero in $(0, \infty)$.

Suppose for contradiction that $\gamma_k^m$ is nontrivial. Then, considering $-\gamma_k^m$ instead of $\gamma_k^m$ if necessary, we may assume that

$$\gamma_k^m(r) > 0 \quad \text{for all } 0 < r < a_k^m, \quad \frac{d\gamma_k^m}{dr}(0) \geq 0, \quad \frac{d\gamma_k^m}{dr}(a_k^m) < 0 \text{ if } a_k^m \neq \infty. \quad (5.101)$$

Note that $\frac{d\gamma_k^m}{dr}(a_k^m) \neq 0$, as otherwise $\gamma_k^m$ becomes trivial.

Since $\Phi_\omega$ is strictly decreasing as a function of $|x|$, we see that

$$\frac{d\Phi_\omega}{dr}(r) < 0 \quad \text{for all } r > 0. \quad (5.102)$$

Observe that

$$\partial_j \Phi_\omega(x) = \sigma_j \frac{d\Phi_\omega}{dr}(r). \quad (5.103)$$

By (1.25), (5.97), (5.103) and (5.93), we see that

$$A_{\omega,1} \frac{d\Phi_\omega}{dr} = -\frac{d^3\Phi_\omega}{dr^3} \frac{d-1}{r} \frac{d^2\Phi_\omega}{dr^2} + \frac{d-1}{r^2} \frac{d\Phi_\omega}{dr} + V_{\omega,+} \frac{d\Phi_\omega}{dr} = 0. \quad (5.104)$$

By (5.100), integration by parts and (5.104), we see that

$$\int_0^{a_k^m} \left\{ \frac{(d-1)(d-3)}{4r^2} + \frac{k(d+k-2)}{r^2} + V_{\omega,+}(r) \right\} \gamma_k^m \frac{d\Phi_\omega}{dr} r^{d-1} \, dr$$

$$= \xi(a_k^m) - \int_0^{a_k^m} \frac{d^2\gamma_k^m}{dr^2} \frac{d\Phi_\omega}{dr} r^{d-1} \, dr - \int_0^{a_k^m} \frac{d\gamma_k^m}{dr} d_{k,\omega} d_{1-r^2} \, dr$$

$$= \xi(a_k^m) + \int_0^{a_k^m} \gamma_k^m d_{k,\omega} d_{1-r^2} \, dr + \int_0^{a_k^m} \gamma_k^m d_{k,\omega} d_{1-r^2} \, dr$$

$$+ \int_0^{a_k^m} \gamma_k^m d_{k,\omega} d_{1-r^2} \, dr$$

$$= \xi(a_k^m) + \int_0^{a_k^m} \left\{ \frac{d-1}{r^2} + \frac{(d-1)(d-3)}{4r^2} + V_{\omega,+} \right\} \gamma_k^m \frac{d\Phi_\omega}{dr} r^{d-1} \, dr,$$

where

$$\xi(a_k^m) := \left. \frac{d\gamma_k^m}{dr}(r) \frac{d\Phi_\omega}{dr}(r) r^{d-1} \right|_{r=a_k^m}. \quad (5.106)$$
Hence, we see that
\[ \int_0^{a_k^n} k(d + k - 2) - (d - 1) \frac{\gamma_k m \Phi_{\omega}}{dr} \frac{dr}{r^{d-1}} dr = \xi(a_k^n). \] (5.107)

Note that
\[ \frac{k(d + k - 2) - (d - 1)}{r^2} > 0 \quad \text{for all } d \geq 3 \text{ and } k \geq 2. \] (5.108)

Observe from (5.101) and (5.102) that
\[ \gamma_k^m(r) \frac{d\Phi_{\omega}}{dr}(r) \frac{dr}{r^{d-1}} < 0 \quad \text{for all } 0 < r < a_k^m, \] (5.109)
\[ \xi(a_k^m) > 0 \quad \text{if } a_k^m \neq \infty, \quad \xi(a_k^m) = 0 \quad \text{if } a_k^m = \infty. \] (5.110)

Assume either \( k \geq 1 \) and \( a_k^m \neq \infty \), or else \( k \geq 2 \). Then, (5.107) together with (5.108), (5.109) and (5.110) shows a contradiction. It remains the case where \( k = 1 \) and \( a_k^m = \infty \) for some \( 1 \leq m \leq d \). Note that \( a_k^m = \infty \) implies that \( \gamma_1^m > 0 \) on \((0, \infty)\). Moreover, by \( \langle f_0, \partial_m \Phi_{\omega} \rangle = 0 \) (see (5.94)), (5.103) and the definition of \( c_1^m(r) \) (see (5.96)), we see that
\[ 0 = \int_0^\infty \int_{S^d-1} f_0(r\sigma) \frac{d\Phi_{\omega}}{dr}(r) r^{d-1} d\sigma dr = \int_0^\infty c_1^m(r) \frac{d\Phi_{\omega}}{dr}(r) r^{d-1} dr. \] (5.111)

Then, by \( \gamma_1^m \) being positive, \( \frac{d\Phi_{\omega}}{dr} < 0 \) (see (5.102)), \( \gamma_1^m(r) := r^{\frac{d-1}{2}} c_1^m(r) \) and (5.111), we see that
\[ 0 > \int_0^\infty \gamma_1^m(r) \frac{d\Phi_{\omega}}{dr}(r) r^{d-1} dr = 0. \] (5.112)

This is a contradiction. Thus, we have proved that \( c_k^m \) is trivial for all \( k \geq 0 \) and \( 1 \leq m \leq N_{d,k} \), which implies that \( f_0 \) is trivial. However, this contradicts \( f_0 \) being nontrivial. Thus, the second claim \((ii)\) of Theorem 1.2 must be true.

6 Proof of Theorem 1.2

In this section, we give a proof of Theorem 1.2.

Proof of Theorem 1.2. Let us begin with the proof of the claim \((i)\). Suppose for contradiction that there exists a real-valued function \( f_* \in H^1(\mathbb{R}^d) \) such that
\[ N_{\omega}(\Phi_{\omega}) f_* = 0, \] (6.1)
\[ B_{\omega}(f_*, f_*) = -1. \] (6.2)

Then, define the function \( Z : \mathbb{R}^2 \to \mathbb{R} \) by
\[ Z(a, b) := \Phi_{\omega} + a f_* + b \Phi_{\omega}. \] (6.3)

Note that \( N_{\omega}(Z(0, 0)) = N_{\omega}(\Phi_{\omega}) = 0 \). Moreover, by (1.24), \( \Phi_{\omega} \) being a positive solution to (1.1), and the condition (1.3), we see that
\[ \frac{\partial}{\partial b} N_{\omega}(Z(a, b)) \big|_{(a, b) = (0, 0)} = N_{\omega}'(\Phi_{\omega}) \Phi_{\omega} = \langle L_{\omega} \Phi_{\omega}, \Phi_{\omega} \rangle \] (6.4)
\[ = - \frac{4}{d-2} \langle \Phi_{\omega} \Phi_{\omega}, \Phi_{\omega} \rangle - \langle g' \Phi_{\omega}, \Phi_{\omega} \rangle < 0. \]
Hence, the implicit function theorem together with (6.1) shows that there exists $a_0 > 0$ and a $C^2$-function $h: (-a_0, a_0) \to \mathbb{R}$ such that

\[
N_\omega(Z(a, h(a))) = 0 \quad \text{for all } a \in (-a_0, a_0),
\]

(6.5)

\[
h(0) = 0, \quad \frac{dh}{da}(0) = -\frac{N_\omega'(\Phi_\omega)f_s}{N_\omega''(\Phi_\omega)\Phi_\omega} = 0.
\]

(6.6)

Since $\Phi_\omega$ is a ground state to (1.1), (6.5) shows that

\[
S_\omega(Z(a, h(a))) \geq S_\omega(\Phi_\omega) \quad \text{for all } a \in (-a_0, a_0).
\]

(6.7)

Observe from $S_\omega'(\Phi_\omega) = 0$ (see (1.22)) that

\[
\left| \frac{d}{da}S_\omega(Z(a, h(a))) \right|_{a=0} = S_\omega'(\Phi_\omega)\{f_s + \frac{dh}{da}(0)\Phi_\omega\} = 0.
\]

(6.8)

Furthermore, by (6.6), (1.23) and (6.2), we see that

\[
\left. \frac{d^2}{da^2}S_\omega(Z(a, h(a))) \right|_{a=0} = \left[ S_\omega''(\Phi_\omega)\{f_s + \frac{dh}{da}(0)\Phi_\omega\} \right] \{f_s + \frac{dh}{da}(0)\Phi_\omega\} = -1,
\]

(6.9)

which together with $S_\omega$ and $h$ being $C^2$ implies that there exists $0 < a_1 < a_0$ such that

\[
\left. \frac{d^2}{da^2}S_\omega(Z(a, h(a))) \right|_{a=0} < 0 \quad \text{for all } a \in (-a_1, a_1).
\]

(6.10)

Thus, Taylor’s expansion of $S_\omega(Z(a, h(a)))$ around $a = 0$ together with (6.8) and (6.10) shows that

\[
S_\omega(Z(a, h(a))) - S_\omega(\Phi_\omega) = \frac{a^2}{2} \left. \frac{d^2}{da^2}S_\omega(Z(a, h(a))) \right|_{a=0} < 0 \quad \text{for all } a \in (-a_1, a_1),
\]

(6.11)

where $\theta \in (0, 1)$ is some constant. However, this contradicts (6.7). Thus, we have proved the claim (i).

We move on to the proof of the claim (ii). It follows from $\sigma_{\text{disc}}(L_{\omega, +}) \subset (-\infty, \omega)$ (see (1.20)) and $\{L_{\omega, +}\Phi_\omega, \Phi_\omega\} < 0$ (see (6.4)) that $L_{\omega, +}$ has at least one negative eigenvalue. Let $E_1 < 0$ be the first eigenvalue, and let $E_2$ be the second one counting multiplicity (possibly, $E_1 = E_2$). Since $0 \in \sigma_{\text{disc}}(L_{\omega, +})$ (see (1.23)), we see that $E_2 \leq 0$. Moreover, the min-max principle (see, e.g., Theorem XIII.2 of [13]) together with (1.24) and the first claim (i) shows that

\[
E_2 = \sup_{\phi \in H^1(\mathbb{R}^d)} \inf_{u \in H^1(\mathbb{R}^d), \|u\|_{L^2} = 1} B_\omega(u, u) \geq \inf_{u \in H^1(\mathbb{R}^d), \|u\|_{L^2} = 1} B_\omega(u, u) \geq 0.
\]

(6.12)

Thus, we see that $E_2 = 0$ and the claim (ii) is true. \qed

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