A SURVEY ON THE AUTOMORPHISM GROUPS OF THE COMMUTING GRAPHS AND POWER GRAPHS

Mahsa Mirzargar

Abstract. Let \( G \) be a finite group. The power graph \( P(G) \) of a group \( G \) is the graph whose vertex set is the set of group elements where two elements are adjacent if one is a power of the other. The commuting graph \( \Delta(G) \) of a group \( G \), is the graph whose vertices are the group elements, two of them are joined if they commute. When the vertex set is \( G \setminus Z(G) \), this graph is denoted by \( \Gamma(G) \). Since the results based on the automorphism groups of these kinds of graphs are so sporadic, in this paper, we give a survey of all results on the automorphism groups of power graphs and commuting graphs obtained in the literature.

Keywords. Finite group; graph; vertex set; commuting graph; automorphism groups.

1. Introduction

There are many connections between graphs and groups. Generating graphs from semigroups and groups has a long history. In 1964, Bosak [6] studied a certain graph over semigroups. In [13], Zelinka studied the intersection graphs of nontrivial subgroups of finite Abelian groups. The well-known study of a directed graphs defined on the elements of a group is the Cayley digraph [7, 22, 40]. The investigation of graphs like these is very important, because they have valuable and numerous applications presented, for example, in the books [27], [28] and [29]. The directed power graph of a group was introduced by Kelarev and Quinn [24]. The definition was formulated so that it applied to semigroups as well. Accordingly, the power graphs of semigroups were first considered in [25], [23] and [26]. It is also explained in the survey [2] that the definition given in [24] covers all undirected graphs as well. This means that the undirected power graphs were also defined in [24] (see [2] for more detailed explanations). All of these papers used only the brief term ‘power graph’, even though they covered both directed and undirected power graphs. Kelarve and Quinn [23] defined another interesting classes of directed graphs, namely,
the divisibility graphs of semigroups. Let $S$ be a semigroup, the divisibility graph, $\text{Div}(S)$, of a semigroup $S$ is a directed graph with vertex set $S$ and there is an arc from $u$ to $v$ if and only if $u \neq v$ and $u\mid v$, i.e., the ideal generated by $v$ contains $u$. On the other hand, the power graph, $\overrightarrow{P}(S)$, of a semigroup $S$ is a directed graph in which the set of vertices is again $S$ and for $a, b \in S$ there is an arc from $a$ to $b$ if and only if $a \neq b$ and $b = a^m$ for some positive integer $m$.

![Diagram of the directed power graph of the dihedral group $D_8$.]

The undirected power graph $P(S)$ was also considered by Chakrabarty, Ghosh and Sen in [11]. Recall that $P(S)$ has vertex set $S$ and two vertices $a, b \in S$ are adjacent if and only if $a \neq b$ and $< a > \subseteq < b >$ or $< b > \subseteq < a >$ (which is equivalent to saying $a \neq b$ and $a^m = b$ or $b^m = a$ for some positive integer $m$). As a consequence, they proved that $P(G)$ is connected for any finite group $G$ and $P(G)$ is complete if and only if $G$ is a cyclic group of order 1 or $p^n$ [11].

![Diagram of the undirected power graph of the dihedral group $D_8$.]
The undirected power graphs became the main focus of study in [11] and in the subsequent papers by P. J. Cameron et al. [8, 9], which introduced the use of the brief term ‘power graph’ in the second meaning of an undirected power graph. For a group $G$, the digraph $\overrightarrow{P}(G)$ was considered in [37] as the main subject of study. The interested readers can be consulted [2, 32, 1] for more information about the power graphs. In this paper, we are also interested in the well-known commuting graphs and their automorphism groups. Let $G$ be a non-abelian group and let $Z(G)$ be the center of $G$. Associate a graph $\Gamma(G)$ with $G$ as follows: Take $G \setminus Z(G)$ as the vertices of $\Gamma(G)$ and join two distinct vertices $x$ and $y$, whenever $xy = yx$. The complement of the $\Gamma(G)$ is said to be the noncommuting graph. The noncommuting graph was first considered by Paul Erdos, when he posed the following problem in 1975 [36]: Let $G$ be a group whose noncommuting graph has no infinite complete subgraph. Is it true that there is a finite bound on the cardinalities of complete subgraphs of the noncommuting graph of $G$? B. H. Neumann [36] answered positively Erdos’ question. We refer the readers to [3, 4, 14, 35, 31] for more details about the noncommuting graph. In [1], authors related the power graph to the commuting graph and characterize when they are equal for finite groups. A new graph pops up while considering these graphs, a graph whose vertex set consists of all group elements, in which two vertices $x$ and $y$ are adjacent if they generate a cyclic group. They called this graph as the enhanced power graph of $G$. The enhanced power graph contains the power graph and is a subgraph of the commuting graph. We consider the commuting graph with vertex set $G$ and denoted it by $\Delta(G)$. 

![Figure 2. The undirected power graph of the dihedral group $D_8$.](image1)

![Figure 3. The commuting graph $\Delta(D_8)$.](image2)
2. Preliminaries and background information

An action of a group $G$ on a set $X$ is the choice, for each $g \in G$ of a permutation $\pi_g : X \rightarrow X$ such that the following two conditions hold:

1. $\pi_e$ is the identity: $\pi_e(x) = x$ for each $x \in X$,
2. for every $g_1, g_2$ in $G$, $\pi_{g_1} \circ \pi_{g_2} = \pi_{g_1g_2}$.

For example, any group $G$ acts on itself $(X = G)$ by left multiplication functions. A group action of $G$ on $X$ is said to be faithful if different elements of $G$ act on $X$ in different ways: when $g_1 \neq g_2$ in $G$, there is an $x \in X$ such that $g_1x \neq g_2x$. For any graph $\Gamma$, we denote the sets of the vertices and the edges of $\Gamma$ by $V(\Gamma)$ and $E(\Gamma)$, respectively. Suppose $v \in V(\Gamma)$ and $V_1(\Gamma) \subseteq V(\Gamma)$, then $N(v)$ is the set of neighbours of $v$ and $(V_1(\Gamma))$ is the subgraph of $\Gamma$ induced by $V_1(\Gamma)$. The closed neighbourhood of a vertex $v$, denoted by $N[v]$, is the set of its neighbours and itself. The complement of $\Gamma$ is the graph $\overline{\Gamma}$ on the same vertices such that two vertices of $\overline{\Gamma}$ are adjacent if and only if they are not adjacent in $\Gamma$. For two graphs with disjoint vertex sets $V_1$ and $V_2$ their union is the graph $H$ in which $V(H) = V_1 \cup V_2$ and $E(H) = E_1 \cup E_2$. Define $nH$ to be the union of $n$ disjoint copies of $G$. The automorphism group of a graph $\Gamma$ is that set of all permutations on $V(\Gamma)$ that fix as a set the edges $E(\Gamma)$. The set of all automorphisms of a graph $\Gamma$ forms a permutation group, $Aut(\Gamma)$, acting on the object set $V(\Gamma)$. See [10] for the terminology and main results of permutation group theory. Let $A$ and $B$ be permutation groups acting on object sets $X$ and $Y$, respectively. Define $B \wr A = \{(a, f) \mid a \in A, f : X \rightarrow B\}$, $(a, f)(x, y) = (ax, fy)$ where $f(x) = b_x$. $B \wr A$ is said to be wreath product. It acts on $X \times Y$ as follows: for each $a \in A$ and any sequence $b_1, b_2, \ldots, b_n$ (where $n = |X|$) in $B$, there is a unique permutation in $A \wr B$ written $(a; b_1, \ldots, b_n)$, and $(a; b_1, \ldots, b_n)(x_1, y_1) = (ax_1, by_1)$. Suppose $S_n$ denotes the symmetric group on $\{1, 2, \ldots, n\}$, $\varphi$ is the Euler’s totient function. In what follows, we describe some important results relating the automorphism groups of a graph which are crucial in this paper. Frucht [18] described if $\Gamma$ is a connected graph, then $Aut(n\Gamma) \cong (Aut(\Gamma)) \wr S_n$, if no component of $\Gamma_1$ is isomorphic with a component of $\Gamma_2$, then $Aut(\Gamma_1 \cup \Gamma_2) \cong Aut(\Gamma_1) \times Aut(\Gamma_2)$ and applying the last two theorems we have the result: Let $\Gamma = n_1\Gamma_1 \cup n_2\Gamma_2 \cup \cdots \cup n_r\Gamma_r$, where $n_i$ is the number of components of $\Gamma$ isomorphic to $\Gamma_i$, then

$$Aut(\Gamma) \cong ((Aut(\Gamma_1)) \wr S_{n_1}) \times ((Aut(\Gamma_2)) \wr S_{n_2}) \times \cdots \times ((Aut(\Gamma_r)) \wr S_{n_r}).$$

An operation $\cdot$ on the set $S$ is associative if it satisfies the following associative law: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in S$. A semigroup is a set $S$ equipped with an associative binary operation $\cdot$. The set of the orders of all elements of $G$ is denoted by $\pi_e(G)$ and is said to be the spectrum of $G$. For $n \in N$, the cyclic group of order $n$ can be defined as the group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ of residues modulo $n$, the set $\langle g \rangle = \{g^n \mid n \in \mathbb{Z}\}$ is the cyclic group generated by $g$ in $G$. For a prime $p$, a group $G$ is said to be an elementary abelian $p$-group if $G$ is finite, abelian and
every nontrivial element of \( G \) has order \( p \). A group \( G \) is an AC-group, whenever the centralizers of non-central elements are abelian. The dihedral group \( D_{2n} \) is an example of an AC-group. The group \( G \) is said to be an EPPO-group, if all elements of \( G \) have prime power order.

3. Automorphism groups of power graphs

The first result about the automorphism groups of power graphs was obtained by P. Cameron in [8], where he explained that when the automorphism group and its graph are equal. P. Cameron proved the only finite group \( G \) for which \( \text{Aut}(G) = \text{Aut}(P(G)) \) is the Klein group \( Z_2 \times Z_2 \).

In 2013, Doostabadi, Erfanian and Jafarzadeh asserted that the full automorphism group of the power graph of the cyclic group \( Z_n \) is isomorphic to the direct product of some symmetry groups.

**Conjecture 3.1.** [16] For every positive integer \( n \),

\[
\text{Aut}(P(Z_n)) \cong S_{\varphi(n)+1} \times \prod_{d \in D(n) \setminus \{1,n\}} S_{\varphi(d)}
\]

where \( D(n) \) is the set of positive divisors of \( n \), and \( \varphi \) is the Euler’s totient function.

In fact, if \( n \) is a prime power, then \( P(Z_n) \) is a complete graph by [11] which implies that \( \text{Aut}(P(Z_n)) \cong S_n \). Hence, the conjecture does not hold if \( n = p^m \) for any prime \( p \) and integer \( m > 2 \). In [17], proved that this conjecture holds for the remaining case. Feng, Ma and Wang [17], describe the full automorphism group of the power (di)graph of an arbitrary finite group. As an application, this conjecture is valid if \( n \) is not a prime power. Denote by \( C(G) \) the set of all cyclic subgroups of \( G \). For \( C \in C(G) \), let \([C]\) denote the set of all generators of \( C \). Write

\[
C(G) = \{ C_1, \cdots, C_k \} \text{ and } [C_i] = \{ [C_{i1}], \cdots, [C_{is_i}] \}.
\]

Define \( P(G) \) as the set of permutations \( \sigma \) on \( C(G) \) preserving order, inclusion and noninclusion, i.e., \([C_i^\sigma] = [C_i]\) for each \( i \in \{1, \cdots, k\} \) and \( C_i \subseteq C_j \) if and only if \( C_i^\sigma \subseteq C_j^\sigma \). Note that \( P(G) \) is a permutation group on \( C(G) \). This group induces the faithful action on the set \( G \):

\[
(3.1) \quad G \times P(G) \rightarrow G, \quad ([C_i], \sigma) \mapsto [C_i^\sigma].
\]

For \( \Omega \subseteq G \), let \( S_{\Omega} \) denote the symmetric group on \( \Omega \). Since \( G \) is the disjoint union of \([C_1], \cdots, [C_k]\), we get the faithful group action on the set \( G \):

\[
(3.2) \quad G \times \prod_{i=1}^{k} S_{[C_i]} \rightarrow G, \quad ([C_i], (\xi_1, \cdots, \xi_k)) \mapsto ([C_i]^\xi). \]

By using the above-mentioned symbols we have:
Theorem 3.1. [17] Let $G$ be a finite group. Then

$$\text{Aut}(\overrightarrow{P}(G)) = (\prod_{i=1}^{k} S_{\{C_i\}}) \times P(G),$$

where $P(G)$ and $\prod_{i=1}^{k} S_{\{C_i\}}$ act on $G$ as in (3.1) and (3.2), respectively.

In the power graph $P(G)$, for $x, y \in G$, define $x \equiv y$ if $N(x) = N(y)$. Observe that $\equiv$ is an equivalence relation. Let $\bar{x}$ denote the equivalence class containing $x$. Write

$$\mathcal{U}(G) = \{\bar{x} | x \in G\} = \{\bar{u}_1, \ldots, \bar{u}_l\}.$$

Since $G$ is the disjoint union of $u_1, \ldots, u_l$, the following is a faithful group action on the set $G$:

$$G \times \prod_{i=1}^{l} S_{\bar{u}_i} \rightarrow G, \quad (x, (\tau_1, \tau_2, \ldots, \tau_l)) \mapsto x^{\tau_i}, \text{ where } x \in \bar{u}_i.$$

Similar to the last theorem, for the automorphism groups of undirected power graphs we have:

Theorem 3.2. [17] Let $G$ be a finite group. Then

$$\text{Aut}(P(G)) = (\prod_{i=1}^{l} S_{\bar{u}_i}) \times P(G),$$

where $P(G)$ and $\prod_{i=1}^{l} S_{\bar{u}_i}$ act on $G$ as in (3.1) and (3.3), respectively.

By combining Theorems 3.1 and 3.2, the authors in [17], obtained that $\text{Aut}(\overrightarrow{P}(G)) = \text{Aut}(P(G))$ if and only if $x = [x]$ for each $x \in G$. Indeed, this result demonstrates relationship between power graphs and directed power graphs.

A graph $\Gamma$ is said to be a subgraph of another graph $\Delta$ (or $\Delta$ is a supergraph of $\Gamma$), if $V(\Gamma) \subseteq V(\Delta)$ and $E(\Gamma) \subseteq E(\Delta)$. Hamze and Ashrafi [19] defined the main supergraph $\mathcal{S}(G)$ of $P(G)$ with the vertex set $G$ and two elements $x, y \in G$ are adjacent if and only if $o(x)|o(y)$ or $o(y)|o(x)$ and proved that there is not a group $G$, such that $\text{Aut}(\mathcal{S}(G)) = \text{Aut}(G)$. In what follows, $\Omega_{\alpha_i}(G) = |\{y|o(y) = \alpha_i\}|$.

Authors in [19] also define the graph $\Delta$ with vertex set $V(\delta) = \pi_e(G)$ and two vertices $a_i$ and $a_j$ are adjacent if and only if $a_i|a_j$ or $a_j|a_i$.

Theorem 3.3. [19] Let $G$ be a finite group with spectrum $\pi_e(G) = \{a_1, \ldots, a_k\}$ and choose a representative set $\{t_1, t_2, \ldots, t_k\}$, where for each $i$, $1 \leq i \leq k, ti \in K_{\Omega_{\alpha_i}}(G)$. Then,

1. If $\deg(t_i)$’s are distinct then $\text{Aut}(\mathcal{S}(G)) = S_{\Omega_{\alpha_1}}(G) \times \cdots \times S_{\Omega_{\alpha_k}}(G)$.
2. If \( \text{deg}(t_{1}) = \cdots = \text{deg}(t_{k}) \), any two distinct vertices of \( K_{\Omega_{a_{1}}} (G), \cdots, K_{\Omega_{a_{r}}} (G) \) are adjacent and \( N_{\Delta}[a_{i}] = \cdots = N_{\Delta}[a_{r}] \) then \( \text{Aut}(S(G)) \) has a subgroup isomorphic to \( S_{\Omega_{a_{1}}} (G) \times \cdots \times S_{\Omega_{a_{r}}} (G) \).

3. If \( \text{deg}(t_{1}) = \cdots = \text{deg}(t_{k}) \), all vertices of \( K_{\Omega_{a_{1}}} (G), \cdots, K_{\Omega_{a_{r}}} (G) \) are adjacent and \( N_{\Delta}[a_{i}] \)'s are distinct then \( \text{Aut}(S(G)) \) has a subgroup isomorphic to \( S_{\Omega_{a_{1}}} (G) \times \cdots \times S_{\Omega_{a_{r}}} (G) \).

4. If \( \text{deg}(t_{1}) = \cdots = \text{deg}(t_{k}) \), \( N_{\Delta}[a_{i}] = \cdots = N_{\Delta}[a_{r}] \) and for each two \( m, n, 1 \leq m, n \leq r \), \( K_{\Omega_{a_{m}}} (G) \) and \( K_{\Omega_{a_{n}}} (G) \) are disjoint then \( \text{Aut}(S(G)) \) has a subgroup isomorphic to \( S_{\Omega_{a_{1}}} (G) \times \cdots \times S_{\Omega_{a_{r}}} (G) \).

5. If \( \text{deg}(t_{1}) = \cdots = \text{deg}(t_{k}) \), \( N_{\Delta}[a_{i}] \)'s are distinct and for each \( m, n, 1 \leq m, n \leq r \), \( K_{\Omega_{a_{m}}} (G) \) and \( K_{\Omega_{a_{n}}} (G) \) are disjoint then \( \text{Aut}(S(G)) \) has a subgroup isomorphic to \( S_{\Omega_{a_{1}}} (G) \times \cdots \times S_{\Omega_{a_{r}}} (G) \).

6. \( \text{Aut}(S(G)) = A_{1} \times \cdots \times A_{q} \), where \( A_{i}, 1 \leq i \leq q \), are subgroups appeared in Cases (2–5).

In [20], Theorem 2.8, it is proved that if \( G \) is an EPPO-group of order \( p_{1}^{{n}_{1}} \cdots p_{k}^{{n}_{k}} \) and \( V_{i} = \{ 1 \neq g \in G \mid o(g)^{p_{i}^{n_{i}}} \} \) then \( S(G) = K_{1} + (\bigcup_{i=1}^{k} K_{|V_{i}|}) \). The authors applied the structure of \( S(G) \) to determine its automorphism.

**Theorem 3.4.** [19] Let \( G \) be a finite group and \( e_{1}, \cdots, e_{l} \) are distinct values of \( |V_{1}|, \cdots, |V_{k}| \). Define \( B_{i} = \{|V_{j}| \mid |V_{j}| = e_{i} \} \). Then,

\[
\text{Aut}(S(G)) = (S_{|V_{1}|} l S_{B_{1}}) \times \cdots \times (S_{|V_{k}|} l S_{B_{k}}).
\]

Suppose \( G \) is a finite group and \( C(G) = \{ C_{1}, \cdots, C_{k} \} \) is the set of all cyclic subgroups of \( G \). Define \( L_{G} \) to be the graph with vertex set \( C(G) \) in which two cyclic subgroups \( C_{i} \) and \( C_{j} \) are adjacent if one is contained in the other or there is a cyclic subgroup \( C_{k} \) such that \( C_{i} \subseteq C_{k} \) and \( C_{j} \subseteq C_{k} \). It is clear that the subgraphs of \( P(G) \) induced by a cyclic subgroup are complete. So, \( P(G) = W_{G}[K_{b_{1}}, K_{b_{2}}, \cdots, K_{b_{k}}] \) with \( b_{i} = \varphi(|C_{i}|) \).

**Theorem 3.5.** [19] Let \( G \) be a finite group with \( C(G) = \{ C_{1}, \cdots, C_{k} \} \) and choose a representative set \( \{ t_{1}, t_{2}, \cdots, t_{k} \} \), where for each \( i, 1 \leq i \leq k, t_{i} \in K_{b_{i}} \). Then,

1. If \( \text{deg}(t_{i}) \)'s are distinct then \( \text{Aut}(P(G)) = S_{b_{1}} \times \cdots \times S_{b_{k}} \).

2. If \( \text{deg}(t_{i}) = \cdots = \text{deg}(t_{i}) \), any two distinct vertices of \( K_{b_{1}}, \cdots, K_{b_{k}} \) are adjacent and \( N_{W_{G}}[C_{i}] = \cdots = N_{W_{G}}[C_{i}] \) then \( \text{Aut}(P(G)) \) has a subgroup isomorphic to \( S_{b_{a_{1}}} \times \cdots \times S_{b_{a_{r}}} \).
3. If \( \text{deg}(t_{i_1}) = \cdots = \text{deg}(t_{i_r}) \), all vertices of \( K_{b_{i_1}}, \ldots, K_{b_{i_r}} \) are adjacent and \( N_{W_G}[C_{i_1}], \ldots, N_{W_G}[C_{i_r}] \)'s are distinct then \( \text{Aut}(P(G)) \) has a subgroup isomorphic to \( S_{b_{i_1}} \times \cdots \times S_{b_{i_r}} \).

4. If \( \text{deg}(t_{i_1}) = \cdots = \text{deg}(t_{i_r}) \), \( N_{W_G}[C_{i_1}] = \cdots = N_{W_G}[C_{i_r}] \) and for each two \( m, n, 1 \leq m, n \leq r, K_{b_{i_m}} \) and \( K_{b_{i_n}} \) are disjoint then \( \text{Aut}(P(G)) \) has a subgroup isomorphic to \( S_{b_{i_1}} \cdot \cdots \cdot S_{b_{i_r}} \).

5. If \( \text{deg}(t_{i_1}) = \cdots = \text{deg}(t_{i_r}) \), \( N_{W_G}[C_{i_1}] \)'s are distinct and for each \( m, n, 1 \leq m, n \leq r, K_{b_{i_m}} \) and \( K_{b_{i_n}} \) are disjoint then \( \text{Aut}(P(G)) \) has a subgroup isomorphic to \( S_{b_{i_1}} \times \cdots \times S_{b_{i_r}} \).

6. \( \text{Aut}(P(G)) = A_1 \times \cdots \times A_q \), where \( A_i, 1 \leq i \leq q, \) are subgroups appeared in Cases (2–5).

### 3.1. Examples

In this section, we present \( \text{Aut}(P(G)) \) and \( \text{Aut}(\overline{P}(G)) \) for some families of finite groups such as \( Z_n, Z_{p^n}, D_{2n}, Q_{4n}, U_{6n}, V_{8n} \) and so on. These results obtained in several papers in different ways. In [5], the authors used the graph structure from [30] and computed the automorphism groups of \( P(G) \) for the above groups. In [17], the authors by using Theorem 3.1 and Theorem 3.2, computed the automorphism groups of \( P(G) \) and \( \overline{P}(G) \) for these groups. In [19], authors obtained these results from Theorem 3.3.

**Example 3.1.** [17] If \( n \) be a positive integer then,

\[
\text{Aut}(\overline{P}(Z_n)) \cong \prod_{d \in D(n)} S_{\varphi(d)},
\]

\[
\text{Aut}(P(Z_n)) \cong \begin{cases} S_n & n \text{ is a prime power} \\ S_{\varphi(n)+1} \times \prod_{d \in D(n) \setminus \{1,n\}} S_{\varphi(d)} & \text{otherwise} \end{cases},
\]

and if \( n \geq 2 \) then,

\[
\text{Aut}(P(Z^n_p)) = \text{Aut}(\overline{P}(Z^n_p)) \cong S_{p-1} \cdot S_m,
\]

where \( m = \frac{n^p-1}{p-1} \) and \( Z^n_p \) denote the elementary abelian \( p \)-group.

In the [21, 15], the dihedral group \( D_{2n} \), semi-dihedral group \( SD_{2n} \), generalized quaternion group of \( Q_{4n} \), semidihedral groups \( SD_{8n} \) are defined by the following presentations:

\[
D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle,
\]

\[
SD_{2n} = \langle a, b \mid a^{2n} = b^2 = 1, b^{-1}ab = a^{-1} \rangle,
\]

\[
Q_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle,
\]

\[
U_{6n} = \langle a, b \mid a^{2n} = b^3 = 1, a^{-1}ba = b^{-1} \rangle,
\]

\[
V_{8n} = \langle a, b \mid a^{2n} = b^4 = 1, ba = a^{-1}b^{-1}, b^{-1}a = a^{-1}b \rangle.
\]

Now, we are ready to state next example.
Example 3.2. [17] For \( n \geq 3 \),

\[
\text{Aut}(P(D_{2n})) \cong \prod_{d \in D(n)} S_{\varphi(d)} \times S_n,
\]

\[
\text{Aut}(P(D_{2n})) \cong \begin{cases} 
S_{n-1} \times S_n, & n \text{ is a prime power} \\
S_n \times \prod_{d \in D(n)} S_{\varphi(d)} & \text{otherwise}
\end{cases},
\]

and let \( n \geq 3 \) then,

\[
\text{Aut}(P(Q_{4n})) \cong \prod_{d \in D(2n)} S_{\varphi(d)} \times (S_2 \wr S_n),
\]

\[
\text{Aut}(P(Q_{4n})) \cong \begin{cases} 
S_2 \times S_{2n-2} \times (S_2 \wr S_n), & n \text{ is a power of } 2 \\
\prod_{d \in D(2n)} S_{\varphi(d)} \times (S_2 \wr S_n) & \text{otherwise}
\end{cases}.
\]

Example 3.3. [5] If \( k \) is nonnegative integer and satisfies \( n = 3^k t \) for some positive integer \( t \) such that \( 3 \nmid t \) then,

\[
\text{Aut}(P(U_{6n})) \cong \begin{cases} 
\prod_{d \mid 3n} S_{\varphi(d)} \times \prod_{d \mid 2n,d \nmid n} S_{\varphi(d)} \times S_3 & k = 0 \\
\prod_{d \mid 2n,d \nmid n} S_{\varphi(d)} \times S_3 \times \prod_{d \mid n} S_{\varphi(d)} \times \prod_{d \mid n,d \nmid t} S_{\varphi(d)} \times S_3 & k = 1 \\
\times \prod_{d \mid n,d \nmid t} S_{\varphi(d)} \times S_2 & k \geq 2
\end{cases},
\]

if \( n = 2^k t \) for a nonnegative \( k \) and some positive odd integer \( t \) then,

\[
\text{Aut}(P(V_{6n})) \cong \begin{cases} 
S_{2n} \times S_2 \times S_n \times \prod_{d \mid 2n,d \nmid n} S_{\varphi(d)} \times S_2 \times \prod_{d \mid 2n} S_{\varphi(d)} & k = 0 \\
S_{2n+1} \times S_2 \times S_n \times \prod_{d \mid n} S^2 \times S_2 \times S_2 & t = 1, k \geq 1 \\
S_{2n} \times S_2 \times S_n \times \prod_{d \mid t} S_{\varphi(d)} \times \prod_{d \mid 2^{t-1},d \not| t} S_{\varphi(d)} & t > 1, k \geq 1
\end{cases},
\]

also,

\[
\text{Aut}(P(SD_{6n})) \cong \begin{cases} 
S_{n-2} \times S_{2n} \times (S_2 \wr S_n), & n \text{ is a power of } 2 \\
\prod_{d \mid 4n} S_{\varphi(d)} \times S_{2n} \times (S_2 \wr S_n) & \text{otherwise}
\end{cases}.
\]

The smallest sporadic group is the first Mathieu group \( M_{11} \), it has order 7920. There are many presentations for the group \( M_{11} \), we give two of its known presentation, [39].

\[
M_{11} \cong \langle a, b, c \mid a^{11} = b^5 = c^4 = (ac)^3 = 1, b^4ab = a^4, c^3bc = b^2 \rangle >.
\]

\[
\cong \langle a, b, c \mid a^2 = b^2 = c^2 = d^2 = (ab)^2 = (bc)^3 = (bd)^4 = (cd)^5 = (abdbd)^3 = 1 \rangle >.
\]

The paper by Armond (1960) increased the interest to finite simple groups, as Janko in Australia found (1965) the first new sporadic group \( J_1 \) a century later after Mathieu’s. It turns out that \( J_1 \) had order 175560. A presentation for \( J_1 \) in terms of its standard generators is given below [12]:

\[
J_1 \cong \langle a, b \mid a^2 = b^3 = (ab)^7 = (abab^{-1})^5 = (abab^{-1})^6abab(ab^{-1})^2 = 1 \rangle >.
\]

The automorphism groups of \( M_{11} \) and \( J_1 \) are determined as follows:
Example 3.4. [5] Let $M_{11}$ be the first Mathieu group and $J_1$ be the first Janko group, then,

$\text{Aut}(P(M_{11})) \cong \langle S_{10} \wr S_{144} \rangle \times \langle S_4 \wr S_{596} \rangle \times \langle S_2 \wr S_{55} \rangle \times \langle (S_6 \wr S_3) \times (S_2 \wr S_4) \times S_2 \rangle \wr S_{165}$,

$\text{Aut}(P(J_1)) \cong \langle S_{10} \wr S_{596} \rangle \times \langle S_6 \wr S_{4180} \rangle \times \langle S_{18} \wr S_{154n} \rangle$

$\times \langle (S_2 \times S_8) \times (S_4 \wr S_3) \rangle \times \langle (S_2 \wr S_3) \rangle \wr S_{1463}$.

Moreover, in [30] the automorphism groups of $P(Z_{pq})$, $P(Z_{pqr})$ and $P(Z_{p^2q^2})$ are calculated as follows:

$\text{Aut}(P(Z_{pq})) \cong S_{\phi(pq)+1} \times S_{p-1} \times S_{q-1}$,

$\text{Aut}(P(Z_{pqr})) \cong S_{\phi(pqr)} \times S_{p-1} \times S_{q-1} \times S_{r-1} \times S_{\phi(pq)} \times S_{\phi(pr)} \times S_{\phi(qr)}$,

$\text{Aut}(P(Z_{p^2q^2})) \cong S_{\phi(p^2q^2)+1} \times S_{p-1} \times S_{\phi(p^2)} \times S_{q-1} \times S_{\phi(q^2)} \times S_{\phi(pq)} \times S_{\phi(p^2q^2)}$.

As we mentioned in above Theorem 3.4 is playing a main role in finding automorphism group of power graphs. In [19], the authors obtained the following results from Theorem 3.3.

Example 3.5. [19] If $n$ is odd, then

$\text{Aut}(S(D_{2n})) = \begin{cases} S_{n-1} \times S_n & \text{n is a prime power}, \\ S_n \times \prod_{d|n} S_{\phi(d)} & \text{otherwise}, \end{cases}$

and if $n$ is even then

$\text{Aut}(S(D_{2n})) = \begin{cases} S_{2n} & \text{n is a power of 2}, \\ S_{\phi(n)+1} \times S_{n+1} \prod_{1,n,2} S_{\phi(d)} & \text{otherwise}, \end{cases}$

if $n$ is odd, then

$\text{Aut}(S(T_{4n})) = S_{2n} \times \prod_{d|2n} S_{\phi(d)}$,

and if $n$ is even then

$\text{Aut}(S(T_{4n})) = \begin{cases} S_{4n} & \text{n is a power of 2}, \\ S_{\phi(2n)+1} \times S_{2n+2} \prod_{1,2,n,4} S_{\phi(d)} & \text{otherwise}, \end{cases}$

for arbitrary $n$,

$\text{Aut}(S(\text{SD}_{2n})) = \begin{cases} S_{8n} & \text{n is a power of 2}, \\ S_{\phi(4n)+1} \times S_{2n+1} \times S_{2n+2} \prod_{1,4,n,2,4} S_{\phi(d)} & \text{otherwise}, \end{cases}$

if $n = 2^k$ then $\text{Aut}(S(V_{8n})) \cong S_{8n}$, and if $n$ is an odd prime then $\text{Aut}(S(V_{8n})) = S_{2n+3} \times S_{2n} \times S_{3\phi(n)} \times \prod_{1,2,n} S_{\phi(d)}$. 
4. Automorphism groups of commuting graphs

The commuting graphs $\Delta(G)$ and $\Gamma(G)$ of a group $G$ are defined in the introduction. The following theorem established the relation between $\text{Aut}(G)$, $\text{Aut}(\Delta(G))$ and $\text{Aut}(\Gamma(G))$.

**Theorem 4.1.** [33] Let $G$ be a finite group, then

1. $\text{Aut}(G) = \text{Aut}(\Delta(G))$ if and only if $|G| = 1$.
2. $\text{Aut}(\Delta(G)) \cong \text{Aut}(\Gamma(G)) \times S_{Z(G)}$.

Mirzargar, Pach and Ashrafi studied the subgroups of $\text{Aut}(\Delta(G))$ in [33, 34]. The first subgroups are $\text{Aut}(\Gamma(G))$ and $\text{Aut}(G)$, then they added some automorphisms of graph to $\text{Aut}(G)$ and constructed bigger subgroups. Define two permutations $\Phi_{x,y}, \phi : G \to G$ as follows: $\Phi_{x,y}$ fixed each element $a \in G \setminus \{x, y\}$ and maps $x$ into $y$ and vice-versa; and, the permutation $\phi$ is defined by $x \to x^{-1}$ for each element $x \in G$. They also defined $\text{Aut}^*(G) = \langle \text{Aut}(G), \phi \rangle$ and considered to the equality of the subgroups and the main group.

**Theorem 4.2.** [33] $\text{Aut}^*(G) = \text{Aut}(\Delta(G))$ if and only if $G \cong S_3$.

Let the cosets $Z(G)x_1, Z(G)x_2, \ldots, Z(G)x_{m-1}$ of the group $G/Z(G)$ and define a new graph $\Delta^u(G)$ with $V(\Delta^u(G)) = \{x_0 = 1, x_1, \ldots, x_{m-1}\}$ and $E(\Delta^u(G)) = \{(x_ix_j | x_ix_j = x_jx_i, 0 \leq i < j \leq m - 1\}$. Notice when $|Z(G)| = 1$ then $\Delta(G) \cong \Delta^u(G)$. It is clear that every two elements in one of these cosets commute. Hence we have a complete graph in any of these cosets. On the other hand, if there exists $x_i \in Z(G)x_1, x_j \in Z(G)x_j$ satisfying $x_ix_j = x_jx_i$, then for every $y_i \in Z(G)x_i, y_j \in Z(G)x_j$ we have $y_iy_j = y_jy_i$. Finally, the set of all $\phi \in \text{Aut}(\Delta(G))$ such that for $a, b \in G$ if $ab^{-1} \in Z(G)$, then $\phi(a)\phi(b)^{-1} \in Z(G)$ is denoted by $T$. These notations are applied in [33] to prove two following theorems.

**Theorem 4.3.** [33] Let $G$ be a group. Then,

1. $\text{Aut}(\Delta^u(G))$ is a subgroup of $\text{Aut}(\Delta(G))$. Moreover, $\text{Aut}(\Delta^u(G)) = \text{Aut}(\Delta(G))$ if and only if $|Z(G)| = 1$.
2. If $G$ is not centerless then $T$ is a subgroup of $\text{Aut}(\Delta(G))$, and $\text{Aut}(\Delta(G)) = T$ if and only if for each pair $a, b$ of elements of $G$ with $C_G(a) = C_G(b)$, we have $ab^{-1} \in Z(G)$.

**Theorem 4.4.** [33] Let $|Z(G)| \geq 2$, where $G$ be a nonabelian group. If $T = \text{Aut}(\Delta(G))$ then $G/Z(G)$ is an elementary abelian 2-group.
For a finite group $G$ define a labelled graph $\Delta^v(G)$ as follows. For $a, b \in G$ let $a \sim b$ if $C_G(a) = C_G(b)$. Clearly, $\sim$ is an equivalence relation, the equivalence class of $a \in G$ is $A(a) = \{x | C_G(x) = C_G(a)\}$. Let us denote the equivalence classes by $A_1, \ldots, A_k$, these are the vertices of $\Delta^v(G)$. Two vertices $A_i$ and $A_j$ are connected if and only if $a_i a_j = a_j a_i$, for some $a_i \in A_i, a_j \in A_j$. At first, we note that if there exists $a_i \in A_i, a_j \in A_j$ satisfying $a_i a_j = a_j a_i$, then for every $b_i \in A_i, b_j \in A_j$ we have $a_i C_G(a_i) = C_G(b_i)$. So, $b_i \in C_G(a_j) = C_G(b_j)$ implies that $b_i b_j = b_j b_i$. Each equivalence class is the union of some sets of the form $c_i a$ where $i \in \mathbb{N}$ with all vertices adjacent to the elements of $G$. Therefore, the intersection of two proper element centralizers of an AC-group is the nonabelian group.

Theorem 4.5. [33] There is a subgroup $A$ of $Aut(\Delta(G))$ such that $A \cong Aut(\Delta^v(G))$ and $Aut(\Delta(G)) = \langle S_{A_1}, \cdots, S_{A_k} \rangle \times A$.

In [38], Rocke proved that the following are equivalent:

1. $G$ has abelian centralizers;
2. If $xy = yx$, then $C_G(x) = C_G(y)$ whenever $x, y \notin Z(G)$;
3. If $xy = yx$ and $xz = zx$, then $yz = zy$ whenever $x \notin Z(G)$;
4. If $U$ and $B$ are subgroups of $G$ and $Z(G) < C_G(U) \leq C_G(B) < G$ then $C_G(U) = C_G(B)$.

Therefore, the intersection of two proper element centralizers of an AC-group is the center of $G$. If $G$ is an AC-group, then $\Delta(G)$ is a union of some complete graphs with all vertices adjacent to the elements of $Z(G)$. So, $\Delta(G)$ is $n_1 C_G(x_1) \cap Z(G) \cup n_2 C_G(x_2) \cap Z(G) \cup \cdots \cup n_r C_G(x_r) \cap Z(G)$ and also every element of $Z(G)$ is adjacent to all elements of $G$, such that for each $i, 1 \leq i \leq r$, we have $n_i$ isomorphic components with complete graph of size $|C_G(x_i) \cap Z(G)|$. In [33], the authors proved that if $G$ is an AC-group with the above notations then,

$$Aut(\Delta(G)) \cong ((S_{|C_G(x_1)|} l S_{n_1}) \times ((S_{|C_G(x_2)|} l S_{n_2}) \times \cdots \times ((S_{|C_G(x_r)|} l S_{n_r}) = S_2(G).$$

Finally, from [33], $|Aut(\Delta(G))|$ can not be a prime power or a square-free number. Moreover, $|Aut(\Delta(G))| = 1$ if and only if $G$ is trivial, $Aut(\Gamma(G))$ is abelian if and only if $G$ is a group of order 1 or 2. Also if $|G| > 2$ then $Aut(\Delta(G))$ is a nonabelian group.
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Mahsa Mirzargar
Faculty of Science
Mahallat institute of higher education
Mahallat, I. R. IRAN

m.mirzargar@gmail.com