Optimal constants for a mixed Littlewood type inequality

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Abstract For $p \in [2, \infty]$ a mixed Littlewood-type inequality asserts that there is a constant $C_{(m),p} \geq 1$ such that

$$\left( \sum_{i_1=1}^{\infty} \left( \sum_{i_2, \ldots, i_m=1}^{\infty} \left| T(e_{i_1}, \ldots, e_{i_m}) \right|^2 \right)^{\frac{1}{2} \frac{p-1}{p}} \right)^{\frac{p}{p-1}} \leq C_{(m),p} \|T\|$$

for all continuous real-valued $m$-linear forms on $\ell_p \times c_0 \times \cdots \times c_0$ (when $p = \infty$, $\ell_p$ is replaced by $c_0$). We prove that for $p > 2.18006$ the optimal constants $C_{(m),p}$ are $\left(2^{\frac{1}{p} - \frac{1}{2}}\right)^{m-1}$. When $p = \infty$, we recover the best constants of the mixed $(\ell_1, \ell_2)$-Littlewood inequality.

Keywords Absolutely summing operators · Hardy–Littlewood inequality · Bohnenblust–Hille inequality · Multiple summing operators

Mathematics Subject Classification 11Y60 · 46G25

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1 Introduction

The Hardy–Littlewood inequality [18, 1934] is a continuation of famous works of Littlewood [19, 1930] and Bohnenblust and Hille [9, 1931] and can be stated as follows:

- [18, Theorems 2 and 4] If \( p, q \geq 2 \) are such that
  \[ \frac{1}{2} < \frac{1}{p} + \frac{1}{q} < 1 \]
  then there is a constant \( D_{p,q} \geq 1 \) such that
  \[
  \left( \sum_{j,k=1}^{\infty} |T(e_j, e_k)|^{\frac{pq-q-p}{pq}} \right)^{\frac{pq-q-p}{pq}} \leq D_{p,q} \|T\| \quad (1)
  \]
  for all continuous bilinear forms \( T : \ell_p \times \ell_q \to \mathbb{R} \) (or \( \mathbb{C} \)). Moreover the exponent \( \frac{pq-q-p}{pq} \) is optimal.

- [18, Theorems 1 and 4] If \( p, q \geq 2 \) are such that
  \[ \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2} \]
  then there is a constant \( C_{p,q} \geq 1 \) such that
  \[
  \left( \sum_{j,k=1}^{\infty} |T(e_j, e_k)|^{\frac{3pq-2p-2q}{4pq}} \right)^{\frac{3pq-2p-2q}{4pq}} \leq C_{p,q} \|T\| \quad (2)
  \]
  for all continuous bilinear forms \( T : \ell_p \times \ell_q \to \mathbb{R} \) (or \( \mathbb{C} \)). Moreover the exponent \( \frac{3pq-2p-2q}{4pq} \) is optimal.

Above and henceforth, as usual in this field, when \( p \) and/or \( q \) is infinity, we consider \( c_0 \) instead of \( \ell_p \) and/or \( \ell_q \).

A unified version of the above two results of Hardy and Littlewood asserts that for \( \frac{1}{2} < \frac{1}{p} + \frac{1}{q} < 1 \) (and \( p, q \geq 2 \)) there is a constant \( K_{p,q} \geq 1 \) such that

\[
\left( \sum_{j=1}^{\infty} \left( \sum_{k=1}^{\infty} |T(e_j, e_k)|^{\frac{pq-2p-2q}{4pq}} \right)^{\frac{pq-2p-2q}{4pq}} \right)^{\frac{1}{\lambda}} \leq K_{p,q} \|T\| \quad (3)
\]

with \( \lambda = \frac{pq}{pq-p-q} \), for all continuous bilinear forms \( T : \ell_p \times \ell_q \to \mathbb{R} \) (see [21, Theorem 1] and [11] for a more general approach; moreover the exponents are optimal). Note that in fact (3) recovers (1) because for \( \frac{1}{2} < \frac{1}{p} + \frac{1}{q} \leq 1 \) we have \( 2 \leq \frac{pq}{pq-p-q} \) and by the canonical inclusion of sequence spaces we know that (3) implies (1). On the other hand, by symmetry we know from (3) that there is a constant \( K_{q,p} \geq 1 \) such that

\[
\left( \sum_{k=1}^{\infty} \left( \sum_{j=1}^{\infty} |T(e_j, e_k)|^{\frac{pq-2p-2q}{4pq}} \right)^{\frac{pq-2p-2q}{4pq}} \right)^{\frac{1}{\lambda}} \leq K_{q,p} \|T\| .
\]
and if $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$ we have $\lambda = \frac{pq}{pq - p - q} \leq 2$; by a well known result sometimes credited to Minkowski (see [17, Corollary 5.4.2]), we know that

$$\left( \sum_{j=1}^{\infty} \left( \sum_{k=1}^{\infty} |T(e_j, e_k)|^p \right)^{\frac{1}{p}} \right)^{\frac{1}{2}} \leq \left( \sum_{k=1}^{\infty} \left( \sum_{j=1}^{\infty} |T(e_j, e_k)|^q \right)^{\frac{1}{q}} \right)^{\frac{1}{2}} \leq K_{q, p} \| T \|. \quad (4)$$

By (3), (4) combined with the Hölder inequality for mixed sums [8] (or interpolation in the sense of [2], if one prefers), since

$$\frac{1}{3pq} = \frac{1}{2} \left( \frac{1}{2} + \frac{1}{\lambda} \right),$$

we recover (2).

The recent years witnessed an increasing interest in the study of summability of multilinear operators (see, for instance, [10,24,25]) and in estimating constants of the multilinear and polynomial Hardy–Littlewood and related inequalities (see [2–4,6,14,15,27]). Perhaps the main motivations are potential applications (see, for instance, [20] for applications of the real-valued case of the estimates of the Bohnenblust–Hille inequality and [7,12] for applications of the complex-valued case).

One of the most far reaching generalizations of the Hardy–Littlewood inequality is the following theorem (see also [26]):

**Theorem 1.1** (See Albuquerque et al. [1]) Let $m \geq 2$ be a positive integer, $1 \leq k \leq m$ and $n_1, \ldots, n_k \geq 1$ be positive integers such that $n_1 + \cdots + n_k = m$. If $q_1, \ldots, q_k \in \left[ \frac{1}{1 - \left( \frac{1}{p_1} + \cdots + \frac{1}{p_m} \right)}, 2 \right]$ and $0 \leq \frac{1}{p_1} + \cdots + \frac{1}{p_m} \leq \frac{1}{2}$, then the following assertions are equivalent:

(a) There is a constant $C_k = C(k, p_1, \ldots, p_m, q_1, \ldots, q_k)$ such that

$$\left( \sum_{i_1=1}^{\infty} \cdots \left( \sum_{i_k=1}^{\infty} |T(e_{i_1}^{n_1}, \ldots, e_{i_k}^{n_k})|^{q_k} \right)^{\frac{1}{q_k}} \right)^{\frac{1}{q_k}} \leq C_k \| T \|$$

for all continuous $m$-linear forms $T : \ell_{p_1} \times \cdots \times \ell_{p_m} \to \mathbb{R}$.

(b) The numbers $q_1, \ldots, q_k$ satisfy

$$\frac{1}{q_1} + \cdots + \frac{1}{q_k} \leq \frac{k + 1}{2} - \left( \frac{1}{p_1} + \cdots + \frac{1}{p_m} \right).$$

Above, the notation $e_{j}^{n_j}$ represents the $n_j$-tuple $(e_j, \ldots, e_j)$. The optimal constants of the previous inequalities are essentially unknown. Recent works have shown that in general these constants have a sublinear growth (see [5–7], and references therein). One of the few cases in which the optimal constants are known for all $m$ is the case of mixed $(\ell_1, \ell_2)$-Littlewood inequality (see [22]):
• The optimal constants $C_{(m),\infty}$ satisfying

$$\sum_{i_1=1}^{\infty} \left( \sum_{i_2,\ldots,i_m=1}^{\infty} |T(e_{i_1},\ldots,e_{i_m})|^2 \right)^{\frac{1}{2}} \leq C_{(m),\infty} \|T\|$$

for all continuous real $m$-linear forms $T : c_0 \times \cdots \times c_0 \rightarrow \mathbb{R}$ are $2^{m-1}$.

From now on $p_0 \approx 1.84742$ is the unique real number satisfying

$$\Gamma\left(\frac{p_0 + 1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$  \hspace{1cm} (6)

Our main result provides the optimal constants of a Hardy–Littlewood-type inequality that encompasses (5); as far as we know this is the first time in which a Hardy–Littlewood type inequality (except for the case of mixed $(\ell_1, \ell_2)$-Littlewood inequality) is proved to have optimal constants with exponential growth:

**Theorem 1.2** Let $m \geq 2$ be a positive integer and $p \geq \frac{p_0}{p_0-1} \approx 2.18006$. The optimal constant $C_{(m),p}$ such that

$$\left( \sum_{i_1=1}^{\infty} \left( \sum_{i_2,\ldots,i_m=1}^{\infty} |T(e_{i_1},\ldots,e_{i_m})|^2 \right)^{\frac{1}{2}} \right)^{\frac{p}{p-1}} \leq C_{(m),p} \|T\|,$$

for all continuous $m$-linear forms $T : \ell_p \times c_0 \times \cdots \times c_0 \rightarrow \mathbb{R}$ is \(2^{\frac{1}{p} - \frac{1}{p-1}}\).\(^{m-1}\).

Note that the above Hardy–Littlewood type inequality holds for $p \geq 2$ (see Theorem 1.1). When $p = 2$ it is simple to prove that the optimal constants are $C_{(m),p} = 1$. As a consequence of the arguments of our proof of Theorem 1.2 we remark that for $2 < p < \frac{p_0}{p_0-1}$ the optimal constants still have exponential growth; so an eventual decrease on the order of the growth when $p \rightarrow 2$ does not happen. Moreover, for $2 < p < \frac{p_0}{p_0-1} \approx 2.18006$, the difference between the bases in the exponential upper and lower estimates of $C_{(m),p}$ is not bigger than $4 \cdot 10^{-4}$ (see the Figs. 1, 2).

In the final section we also provide upper and lower estimates for the sharp constants $C_{p,\infty}$ of the real case of (2), showing that

$$2^{\frac{1}{p} - \frac{1}{p-1}} \leq C_{p,\infty} \leq 2^{\frac{1}{p} - \frac{1}{2p}}$$

for all $p \geq \frac{p_0}{p_0-1} \approx 2.18006$. This result recovers, in particular, the optimality of the constant $\sqrt{2}$ of the real case of the Littlewood’s $4/3$ inequality obtained in [15].

**2 The proof of Theorem 1.2**

The Khinchine inequality (see [13]) asserts that, for any $0 < q < \infty$, there are positive constants $A_q, B_q$ such that regardless of the scalar sequence $(a_j)_{j=1}^n$ we have

$$A_q \left( \sum_{j=1}^{n} |a_j|^2 \right)^{\frac{1}{2}} \leq \left( \int_0^1 \left( \sum_{j=1}^{n} a_j r_j(t) \right)^q dt \right)^{\frac{1}{q}} \leq B_q \left( \sum_{j=1}^{n} |a_j|^2 \right)^{\frac{1}{2}},$$
Fig. 1 Plots of the functions $A^{-\frac{1}{x-1}}$ and $2^{\frac{1}{x}} - \frac{1}{x}$, for $x \in [2, \frac{p_0}{p_0-1}]$

Fig. 2 Plot of the function $\left(A^{-\frac{1}{x-1}} - 2^{\frac{1}{x}} - \frac{1}{x}\right)$, for $x \in [2, \frac{p_0}{p_0-1}]$

where $r_j$ are the Rademacher functions. For real scalars, U. Haagerup [16] proved that if $p_0$ is the number defined in (6) then

$$A_q = \sqrt{2} \left( \frac{\Gamma\left(\frac{q+1}{2}\right)}{\sqrt{\pi}} \right)^{\frac{1}{q}}, \quad \text{for } 1.84742 \approx p_0 < q < 2$$

and

$$A_q = 2^{\frac{1}{x}} - \frac{1}{x}, \quad \text{for } 1 \leq q \leq p_0 \approx 1.84742.$$
\[
\left( \sum_{i_1=1}^{\infty} \left( \sum_{i_2, \ldots, i_m=1}^{\infty} |T(e_{i_1}, \ldots, e_{i_m})|^2 \right)^{\frac{p}{2}} \right)^{\frac{p-1}{p}} \leq (A_{\frac{p}{p-1}}^{-1})^{m-1} \left( \sum_{i_1=1}^{\infty} \int \left| \sum_{i_2, \ldots, i_m=1}^{\infty} r_{i_2}(t_2) \cdots r_{i_m}(t_m) T(e_{i_1}, \ldots, e_{i_m}) \right|^p dt_2 \cdots dt_m \right)^{\frac{p-1}{p}} \\
= (A_{\frac{p}{p-1}}^{-1})^{m-1} \left( \int \left( \sum_{i_1=1}^{\infty} r_{i_2}(t_2) e_{i_2}, \ldots, \sum_{i_m=1}^{\infty} r_{i_m}(t_m) e_{i_m} \right)^p dt_2 \cdots dt_m \right)^{\frac{p-1}{p}} \\
\leq (A_{\frac{p}{p-1}}^{-1})^{m-1} \sup_{\sum_{i_1=1}^{\infty} r_{i_2}(t_2) e_{i_2}, \ldots, \sum_{i_m=1}^{\infty} r_{i_m}(t_m) e_{i_m}} \left( T \left( \sum_{i_1=1}^{\infty} r_{i_2}(t_2) e_{i_2}, \ldots, \sum_{i_m=1}^{\infty} r_{i_m}(t_m) e_{i_m} \right) \right) \right)^p dt_2 \cdots dt_m \right)^{\frac{p-1}{p}} \\
\leq (A_{\frac{p}{p-1}}^{-1})^{m-1} \frac{1}{\|T\|} (2^{1 - \frac{1}{p}})^{m-1} \|T\| 
\]
whenever \( p \geq \frac{p_0}{p_0 - 1} \approx 2.18006 \). Now let us show that \((2^{1 - \frac{1}{p}})^{m-1} \|T\|\) is the best possible constant. Let

\[
T_2 : \ell_p \times c_0 \to \mathbb{R} \n\]

and \(T_2^{x_2} : \ell_p \to \mathbb{R}\) be given, for \(x_i = (x_1^i, x_2^i, \ldots)\), \(i = 1, 2, \ldots\) by

\[
T_2(x_1, x_2) = (x_1^2 + x_2^2) x_1^1 + (x_1^2 - x_2^2) x_1^2, 
\]

and

\[
T_2^{x_2} (x_1) = T_2 (x_1, x_2), \n\]

for each \(x_2 \in c_0\). Observe that

\[
\|T_2\| = \sup \left\{ \|T_2^{x_2}\| : \|x_2\|_{c_0} = 1 \right\}. 
\]

Let us estimate (9). Since \((\ell_p)^* = \ell_{\frac{p}{p-1}}\), we have

\[
\|T_2\| = \sup \{ \|T_2^{x_2}\| : \|x_2\|_{c_0} = 1 \} \\
= \sup_{x_1 \in \mathcal{B}_{\ell_p}} \|T_2^{x_2} (x_1)\| : \|x_2\|_{c_0} = 1 \} \\
= \sup_{x_1 \in \mathcal{B}_{\ell_p}} \|(x_1^2 + x_2^2) x_1^1 + (x_1^2 - x_2^2) x_1^2\| : \|x_2\|_{c_0} = 1 \} \\
= \sup_{x_1 \in \mathcal{B}_{\ell_p}} \|(x_1^2 + x_2^2, x_1^1 - x_2^2, 0, 0, \ldots)\|_{\ell_p} : \|x_2\|_{c_0} = 1 \} \\
= \sup_{x_1 \in \mathcal{B}_{\ell_p}} \left\{ |1 + x|^{\frac{p}{p-1}} + |1 - x|^{\frac{p}{p-1}} \right\}^{\frac{1}{p-1}} : x \in [-1, 1], \right\} = 2. 
\]

(10)
In order to verify the last equality, note that since
\[
\sup \left\{ (|1 + x|^1 + |1 - x|^1)^{1 \over 2} : x \in [-1, 1] \right\} = 2,
\]
by the norm inclusion \( \ell_1 \subseteq \ell_{\infty}^{p \mapsto 1} \) for \( p \in [2, \infty) \), we have \( \|\cdot\|_{\ell_{\infty}^{p \mapsto 1}} \leq \|\cdot\|_{\ell_1} \). Therefore, for \( p \in [2, \infty) \) we have
\[
\sup \left\{ (|1 + x|^{p \mapsto 1} + |1 - x|^{p \mapsto 1})^{p \mapsto 1 \over p} : x \in [-1, 1] \right\} 
\leq \sup \left\{ (|1 + x|^1 + |1 - x|^1)^{1 \over 2} : x \in [-1, 1] \right\} = 2.
\]

On the other hand, it is obvious that
\[
\sup \left\{ (|1 + x|^{p \mapsto 1} + |1 - x|^{p \mapsto 1})^{p \mapsto 1 \over p} : x \in [-1, 1] \right\} \geq \left( |1 + 1|^{p \mapsto 1} + |1 - 1|^{p \mapsto 1} \right)^{p \mapsto 1 \over p} = 2.
\]

In order to show that \( (2^{1 \over 2 - 1 \over p})^{m-1} \) is the best possible constant satisfying (7), let
\[
S_p : \ell_p \to \ell_p \\
S_p(x_1) = (x_1^2, x_1^3, \ldots)
\]
and
\[
S_0 : c_0 \to c_0 \\
S_0(x_2) = (x_2^2, x_2^3, \ldots)
\]
be the backward shifts (and let \( S_p^r \) and \( S_0^r \) denote \( S_p \circ \cdots \circ S_p \) and \( S_0 \circ \cdots \circ S_0 \) composed \( r \) times, respectively). Let \( T_2 : \ell_p \times c_0 \to \mathbb{R} \) be as in (8) and define \( T_3 : \ell_p \times c_0 \times c_0 \to \mathbb{R} \)
\[
T_3(x_1, x_2, x_3) = (x_1^3 + x_3^3) \left[ (x_1^2 + x_2^2)x_1^1 + (x_2^2 - x_2^2)x_1^2 \right] + (x_3^3 - x_3^2)
\times \left[ (x_2^3 + x_4^3)x_3^1 + (x_3^2 - x_3^2)x_3^3 \right]
\]
\[
= (x_1^3 + x_3^3) T_2(x_1, x_2) + (x_3^3 - x_3^2) T_2 \left( S_p^2(x_1), S_0^2(x_2) \right),
\]
and \( T_4 : \ell_p \times c_0 \times c_0 \times c_0 \to \mathbb{R} \) by
\[
T_4(x_1, x_2, x_3, x_4) = (x_4^4 + x_4^2) T_3(x_1, x_2, x_3) + (x_4^4 - x_4^2) T_3 \left( S_p^4(x_1), S_0^4(x_2), S_0^4(x_3) \right)
\]
and so on. Inductively, for all \( m \geq 3 \) the \( m \)-linear operator \( T_m : \ell_p \times c_0 \times \cdots \times c_0 \to \mathbb{R} \) is defined by
\[
T_m(x_1, \ldots, x_m) = (x_m^1 + x_m^2) T_{m-1}(x_1, \ldots, x_{m-1}) \\
+ (x_m^1 - x_m^2) T_{m-1} \left( S_p^{2m-2}(x_1), S_0^{2m-2}(x_2), S_0^{2m-2}(x_3), \ldots, S_0^{2m-2}(x_{m-1}) \right),
\]
By induction on \( m \geq 2 \) we shall show that
\[
\| T_m \| = 2^{m-1}.
\]
The case $m = 2$ is already done in (10). Let us suppose that $\|T_{m-1}\| = 2^{(m-1)-1}$. Therefore,

$$|T_m(x_1, \ldots, x_m)| \leq |x_m^1 + x_m^2||T_{m-1}(x_1, \ldots, x_{m-1})| + |x_m^1 - x_m^2||T_{m-1}(S_p^{-2m-2}(x_1), S_0^{-2m-2}(x_2), \ldots, S_0^{-2m-3}(x_3), \ldots, S_0^{-2m-2}(x_{m-1}))| \leq 2^{m-2}||x_m^1 + x_m^2||x_1||x_p^1\cdots||x_{m-1}||c_0 + |x_m^1 - x_m^2||

$$

$$\times S_p^{-2m-2}(x_1)||x_p^1\cdots||x_1||x_p^2\cdots||x_{m-1}||c_0

$$

$$\leq 2^{m-2}||x_m^1 + x_m^2|| + |x_m^1 - x_m^2||x_1||x_p^1\cdots||x_{m-1}||c_0

$$

$$= 2^{m-1}||x_1||x_p^1\cdots||x_{m-1}||c_0 \max\{|x_m^1, |x_m^2|\}

$$

$$\leq 2^{m-1}||x_1||x_p\cdots||x_{m-1}||c_0.

$$

We thus have $\|T_m\| \leq 2^{m-1}$. Now consider $a_m = e_1 + e_2$ and note that

$$\|T_m\| \geq \sup \{|T_m(x_1, \ldots, x_{m-1}, a_m)| : x_1 \in B_{\ell_p}, x_2 \in B_{c_0}, \ldots, x_{m-1} \in B_{c_0}\}

$$

$$= 2\|T_{m-1}\| = 2^{m-1}

$$

and hence $\|T_m\| = 2^{m-1}$.

Since

$$\left(\sum_{i_1} \sum_{i_2, \ldots, i_m} |T_m(e_{i_1}, \ldots, e_{i_m})|^2 \right)^{\frac{1}{2}} \leq \left(\frac{p-1}{p} \left(2^{-\frac{1}{p}}\right) \right)^{m-1}

$$

the proof is done.

### 3 Final remarks

The same argument used in the proof of Theorem 1.2 shows that for $2 < p < \frac{p_0}{p_0 - 1} \approx 2.18006$ the optimal constants also have exponential growth; curiously, for $p = 2$ the situation is quite different and the optimal constants are 1. In fact, note that the second part of the proof (the optimality proof) holds for all $p \geq 2$. Moreover, the first part of the proof gives us the estimate $C_{m,p} \leq (A^{-1}_{p-1})^{m-1}$. We thus have, for $2 \leq p < \frac{p_0}{p_0 - 1} \approx 2.18006$, the following inequalities

$$\left(2^{-\frac{1}{p}}\right)^{m-1} \leq C_{m,p} \leq \left(\frac{1}{\sqrt{2}} \left(\frac{\Gamma\left(\frac{2p-1}{2p-2}\right)^{\frac{1-p}{p}}\right)^{m-1}

$$

For $p \geq 2$, we know that

$$\left(\sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} |T(e_j, e_k)|^2\right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \leq \sqrt{2} \|T\| \quad (11)

$$

with $\lambda = \frac{p}{p-1}$, for all continuous bilinear forms $T : \ell_p \times c_0 \rightarrow \mathbb{R}$ (see, for instance, [2, Theorem 1.2 and Remark 5.1]). By interpolating (11) and the result of Theorem 1.2 for
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$m = 2$ in the sense of [2] or using the Hölder inequality for mixed sums [8] we obtain, for $p ≥ \frac{p_0}{p_0 - 1} ≈ 2.18006$,

$$\left( \sum_{j,k=1}^{\infty} \left| T(e_j, e_k) \right|^{4p} \right)^{\frac{3p-2}{4p}} \leq \left( \sqrt{2} \| T \| \right)^{1/2} \left( 2^{\frac{p-2}{2p}} \| T \| \right)^{1/2} = 2^{\frac{1}{2} - \frac{1}{p}} \| T \| .$$

Using the approach of the previous section we obtain the lower estimate

$$C_{p, \infty} ≥ \frac{\left( \sum_{j,k=1}^{2} \left| T_2(e_j, e_k) \right|^{4p} \right)^{\frac{3p-2}{4p}}}{\| T_2 \|} = \frac{4^{\frac{3p-2}{4p}}}{2} = 2^{\frac{1}{2} - \frac{1}{p}}$$

and thus

$$2^{\frac{1}{2} - \frac{1}{p}} ≤ C_{p, \infty} ≤ 2^{\frac{1}{2} - \frac{1}{p}} .$$

When $p = \infty$ we recover the well known optimal estimate of the famous Littlewood’s 4/3 that can be found in [15].

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