A robust control scheme for synchronizing fractional order disturbed chaotic systems with uncertainty and time-varying delay

Hai Gu\textsuperscript{a}, Jianhua Sun\textsuperscript{a} and Hadi Imani\textsuperscript{b}

\textsuperscript{a}School of Mechanical Engineering, Nantong Institute of Technology, Nantong, People’s Republic of China; \textsuperscript{b}Department of electrical Engineering, Ahar Branch, Islamic Azad University, Ahar, Iran

ABSTRACT

In this paper, a new method is presented for synchronization between two fractional order delayed chaotic systems, while there is uncertainty on the models and external disturbances enter the systems at the same time. The considered delay in the fractional order system is unspecified and varied with time, and of course it is present in different forms in master and slave systems. External disturbances enter the master–slave systems in a finite manner, albeit with an undetermined upper bound, and uncertainty is present in the nonlinear functions of chaotic systems. The goal of synchronizing a particular class of master–slave chaotic systems is achieved through a combination of adaptive and sliding mode techniques. The sliding mode method has been used to cover the effects of uncertainties and delay functions, and an adaptive method has been applied to ensure the stability of the proposed synchronization technique, disturbance upper bound estimation and overcoming the effects of delay variability. A practical example of the innovative method is simulated in MATLAB environment and the obtained results confirm the optimal efficiency of the proposed synchronization method.

1. Introduction

Most physical systems behave nonlinearly and exhibit complex dynamics. Chaotic systems are one of these nonlinear phenomena whose behaviours are strongly affected by the initial values and are presented in chemical reactions, lasers, electronic circuits as well as in natural phenomena such as the solar system, weather and so on (Malica et al., 2020; Zeebe & Lourens, 2019; Zhan et al., 2014). Because of the essence of chaotic systems, they are commonly used in numerous arenas for instance encryption, secure data transfer, and etc. (Chai et al., 2019; Liu et al., 2019).

Fractional order systems are one of the most important fields of research related to chaos theory (Yousefpour et al., 2020). In fact, well-known chaotic systems such as Lorenz, Chen, and etc. show a fractional order dynamic behaviour. Due to the ability of fractional order systems to express more accurately than the integer order type. They are able to describe and model application systems more precisely and are used in a wide range of applications, from signal and image processing to automation, robotic control and quantum (Ahmad et al., 2019; Khan et al., 2017; Sohail et al., 2018; Tripathi et al., 2010). Therefore, chaos theory requests additional evaluation and development in this area.

The issue of synchronization in fractional order delayed chaotic systems is important from several perspectives. First, synchronization offers remarkable potential for applications of chaotic systems (Moysis et al., 2020; Wang et al., 2019). Second, synchronization between fractional order systems is much more complex than integer order systems. Third, the existence of delay in most practical systems is obvious, and its considering is essential for the proper synchronization of chaotic systems. Given these important points, this paper addresses the issue of synchronization of fractional order delayed chaotic systems.

Among the various methods proposed to date for achieving synchronization between two chaotic systems, we can refer to full synchronization (Zhu, 2008), projective and lag (Chen et al., 2007; Wen & Xu, 2005), and different control methods such as adaptive (Hu et al., 2007), fuzzy (Boulkroune et al., 2016), active (Bhalekar & Daftardar-Gejji, 2010), passive (Kuntanapreeda, 2016), sliding mode (Li et al., 2020). However, the fundamental challenges of time delay, disturbance, and uncertainty are seldom considered simultaneously. Delays in chaotic systems occur due to the transfer of data, energy, or materials, and sometimes lead to plant instability. Despite the extensive studies in the field of stability analysis on the subject of synchronization in the presence of time delay,
its variability and unknownness is a challenge. Another challenge is the issue of external disturbance. Despite the variety of synchronization methods for integer order chaotic systems in the presence of disturbances, limited studies have been conducted for fractional order systems, and disturbances with unknown boundaries have rarely been considered. Given that the existence of any kind of uncertainty strongly affects the synchronization process, the problem of synchronization of fractional order systems in the simultaneous presence of variable unknown time delay, disturbance with unknown upper bound and uncertainty is highly challenging and appropriate measures need to be taken to ensure stability.

Therefore, by considering 1 – Fractional order system 2 – Existence of nonlinear terms 3 – Existence of disturbance with unknown boundary, 4 – Existence of uncertainty in system model and 5 – Existence of variable unspecified time delay, this paper presents a new method for synchronizing a particular class of chaotic systems. Uncertainty is a parametric type, and of course uncertainties are also considered on functions. In the proposed method, the upper bound of external disturbance is unknown. In addition, there are unknown variable time delays with different forms in the master and slave systems, which make it more difficult to achieve the synchronization goal, and are considered in this study along with external disturbance and uncertainty. To achieve the goal of synchronizing master and slave fraction systems, a combination method based on adaptive and robust sliding mode methods is proposed. The robust sliding mode method has been used to cover the effects of uncertainty, disturbance and delay, and because the delay is unknown and variable with time, the adaptive method has been applied to obtain the optimal coefficients of sliding mode as well as estimating the upper limit of disturbance.

Thus, this paper has been organized as follows: In the second part, an introduction is given to fractional relations and calculations, and the third part expresses the chaotic system class and the novel synchronization method. In the fourth section, the simulation results are presented by applying the planned method to the fractional permanent magnet synchronous motor chaotic system. Finally, the conclusion is given in the fifth section.

2. Basics of fractional calculations

This section explains the definitions and relationships related to fractional calculations and essential lemmas used in the paper.

**Definition 2.1:** (Calderón et al., 2006): Equation (1) expresses fractional integral and derivative operators

\[
D_t^\alpha f(t) = \begin{cases} 
\frac{d^\alpha}{dt^\alpha} f(t), & \alpha > 0 \\
0, & \alpha = 0 \\
\int_t^a (t - \tau)^{-\alpha} f(\tau) d\tau, & \alpha < 0
\end{cases}
\]

(1)

where \( \alpha \) states the fractional order.

**Definition 2.2:** (Aghababa, 2013): The fractional order integral of the Riemann-Liouville type for the function \( f(t) \) of order \( \alpha \) is as follows

\[
t_0I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \frac{f(\tau)}{(t - \tau)^{1-\alpha}} d\tau
\]

(2)

The \( t_0 \) relates to the initial time and the Gamma function \( \Gamma(\alpha) \) is described in this way

\[
\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt
\]

(3)

where \( \alpha \) specifies the Gamma function operator.

**Definition 2.3:** (Aghababa, 2012): The fractional order derivative of the Riemann-Liouville type for the function \( f(t) \) of order \( \alpha (n - 1 < \alpha \leq n, n \in \mathbb{N}) \) is as follows

\[
t_0D_t^\alpha f(t) = \frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(n - \alpha)} d^n \int_{t_0}^t \frac{f(\tau)}{(t - \tau)^{n-\alpha+1}} d\tau
\]

(4)

**Definition 2.4:** (Sweilam et al., 2007): The fractional order derivative of the Caputo type for the function \( f(t) \) of order \( \alpha \) is as follows

\[
t_0D_t^\alpha f(t) = \begin{cases} 
\frac{1}{\Gamma(\alpha - m)} \int_{t_0}^t \frac{f^{(m)}(\tau)}{(t - \tau)^{\alpha-m+1}} d\tau, & m - 1 < \alpha < m \\
\frac{d^m f(t)}{dt^m}, & \alpha = m
\end{cases}
\]

(5)

where \( m \) specifies the first integer greater than \( \alpha \).

**Remark 2.1:** As stated by Definitions 2.3 and 2.4, the Caputo fractional derivative of a constant integer equals...
zero, while the Riemann-Leibnitz fractional derivative of a constant integer is not zero.

**Facts**

1. For $\alpha = m$, the function of $D_t^mf(t)$ and $\frac{d^m(t)}{dt}$ are the same.
2. For $\alpha = 0$, Equation (6) expresses the function of $D_t^f(t)$

$$D_t^f(t) = f(t) \quad (6)$$

3. Linear condition is established for all calculations based on the Caputo derivative.

$$D_t^f [f(t) + g(t)] = D_t^f f(t) + D_t^f g(t). \quad (7)$$

4. Equation (8) is established to multiply the Caputo fractional derivatives of the function $g(t)$

$$D_t^\alpha D_t^\beta g(t) = D_t^{\alpha+\beta} g(t) \quad (8)$$

The $\alpha$ and $\beta$ signify two fraction orders. Also, the following lemmas are used to design a novel synchronization method.

**Lemma 2.1: (Khamisuwan & Kuntanapreeda, 2016):**

Consider $x = 0$ as the equilibrium point of the non-autonomous fractional system (9).

$$D_t^\alpha x = f(x, t) \quad (9)$$

where $x = 0$ and $f(x, t)$ contents the Lipchitz condition with a factor of $l > 0$. Supposing positive gains for $\alpha_1, \alpha_2, \alpha_3$ and $\alpha$, there exists a Lyapunov function that contains the following

$$\alpha_1 x^\alpha \leq V(t, x) \leq \alpha_2 x \quad (10)$$

$$\dot{V}(t, x) \leq -\alpha_3 x \quad (11)$$

Then, the system is asymptotically stable and this is valid for both Caputo and Riemann-Leibnitz definitions.

**Lemma 2.2: (Maligranda, 2008):** The following inequality holds for $x_i \in R, i = , n, 0 < q \leq 1$.

$$(|x_1| + |x_2| + \ldots + |x_n|)^q \leq |x_1|^q + |x_2|^q + \ldots + |x_n|^q. \quad (12)$$

3. Synchronization of two disturbed uncertain delayed chaotic systems of fractional order

In this section, the general forms of the two $n$-dimensional Disturbed Uncertain Fractional Order Chaotic Systems (DUFOCSs) are given. Also, the new robust synchronization method is fully explained. The master fractional order chaos system is defined as (Yu & Wang, 2012).

$$D_t^\alpha x_1 = f_{11}(x, t) + f_{12}(x, x - \tau^m_1(t), t) + \Delta f_1(x) + d_1^m(t)$$

$$D_t^\alpha x_2 = f_{21}(x, t) + f_{22}(x, x - \tau^m_2(t), t) + \Delta f_2(x) + d_2^m(t)$$

$$\vdots$$

$$D_t^\alpha x_n = f_{n1}(x, t) + f_{n2}(x, x - \tau^m_n(t), t) + \Delta f_n(x) + d_n^m(t) \quad (13)$$

where $\alpha \in (0, 1)$ is the fractional order of system, $x(t) = [x_1, x_2, \ldots, x_n]^T \in R^n$ denotes the states of the system, $f_i(x, t) \in R, i = 1, 2, \ldots, n$ expresses a definite nonlinear function of $t$ and $x$. The $f_{ij}(x, x - \tau^m_i(t), t) \in R, i = 1, 2, \ldots, n$ denotes a certain nonlinear function of the states $x$ and time $t$ and the delayed states in which $\tau^m_i(t)$ signifies the indefinite variable time delay. The $\Delta f_i(x) \in R, i = 1, 2, \ldots, n$ defines unknown parametric uncertainty and $d^m_i(t) \in R, i = 1, 2, \ldots, n$ states unknown bounded disturbance.

The following assumption applies to the master system.

**Assumption 3.1:** Suppose the disturbances to be in the form of Equation (14) in the master system.

$$|d_i^m(t)| \leq D_i^m, i = 1, \ldots, n \quad (14)$$

where $D_i^m, i = 1, \ldots, n$ indicates unspecified positive constant values. The slave system is also in the following form

$$D_t^\alpha y_1 = g_{11}(y, t) + g_{12}(y, y - \tau^m_1(t), t) + \Delta g_1(y) + d_1^m(t) + u_1(t)$$

$$D_t^\alpha y_2 = g_{21}(y, t) + g_{22}(y, y - \tau^m_2(t), t) + \Delta g_2(y) + d_2^m(t) + u_2(t)$$

$$\vdots$$

$$D_t^\alpha y_n = g_{n1}(y, t) + g_{n2}(y, y - \tau^m_n(t), t) + \Delta g_n(y) + d_n^m(t) + u_n(t) \quad (15)$$

where $y(t) = [y_1, y_2, \ldots, y_n]^T \in R^n$ indicates the states vector, $g_{ij}(y, t) \in R, i = 1, 2, \ldots, n$ denotes nonlinear function of $t$ and $y$ that is known. $g_{ij}(y, y - \tau^m_i(t), t) \in R, i = 1, 2, \ldots, n$ denotes a certain nonlinear function of the states $y$ and time $t$ and the delayed states where $\tau^m_i(t)$ signifies the indefinite variable time delay. The $\Delta g_i(y) \in R, i = 1, 2, \ldots, n$ specifies the unidentified parametric uncertainties, $u_i(t) \in R, i = 1, 2, \ldots, n$ states the control signal and $d_i^m(t) \in R, i = 1, 2, \ldots, n$ indicates the indefinite bounded disturbance.

The following assumption applies to the slave system.
**Assumption 3.2:** Suppose the disturbances to be in the form of Equation (16) in the slave system.

\[
|d_i^j(t)| \leq D_i^j, i = 1, \ldots, n
\]  

(16)

where \(D_i^j, i = 1, \ldots, n\) indicates unspecified positive constant values. According to the master and slave systems, the synchronization error vector is stated as

\[
e_i = y_i - x_i, i = 1, \ldots, n
\]  

(17)

Using dynamic Equations (13) and (15), the sync error is defined as follows

\[
\begin{align*}
D^e e_1 &= g_{11}(y, t) - f_{11}(x, t) + g_{12}(y, y - \tau_i^1(t), t) \\
&\quad - f_{12}(x, x - \tau_i^1(t), t) \\
&\quad + \Delta g_{1}(y) - \Delta f_{1}(x) + d_i^1(t) - d_i^m(t) + u_i(t) \\
D^e e_2 &= g_{21}(y, t) - f_{21}(x, t) + g_{22}(y, y - \tau_i^2(t), t) \\
&\quad - f_{22}(x, x - \tau_i^2(t), t) \\
&\quad + \Delta g_{2}(y) - \Delta f_{2}(x) \\
D^e e_n &= g_{n1}(y, t) - f_{n1}(x, t) + g_{n2}(y, y - \tau_i^1(t), t) \\
&\quad - f_{n2}(x, x - \tau_i^1(t), t) \\
&\quad + \Delta g_{n}(y) - \Delta f_{n}(x) + d_i^n(t) - d_i^m(t) + u_n(t)
\end{align*}
\]  

(18)

By rewriting system error (18) as follows:

\[
\begin{align*}
\dot{D}^e e_1 &= h_{11}(x, y, t) + h_{12}(x, y, y - \tau_i^1(t), t) \\
&\quad + \Delta h_{1}(x, y) + d_i^1(t) + u_i(t) \\
\dot{D}^e e_2 &= h_{21}(x, y, t) + h_{22}(x, y, y - \tau_i^2(t), t) \\
&\quad - f_{21}(x, x - \tau_i^1(t), t) \\
&\quad + \Delta h_{2}(x, y) - \Delta f_{2}(x) \\
\dot{D}^e e_n &= h_{n1}(x, y, t) + h_{n2}(x, y, y - \tau_i^1(t), t) \\
&\quad - f_{n1}(x, x - \tau_i^1(t), t) \\
&\quad + \Delta h_{n}(x, y) - \Delta f_{n}(x) + d_i^n(t) - d_i^m(t) + u_n(t)
\end{align*}
\]  

(19)

\[
\begin{align*}
\dot{D}^e &= \sum_{i=1}^{n} e_i^2 \\
\dot{V}_i &= \sum_{i=1}^{n} \frac{1}{2} D_i e_i^2 + \frac{1}{2} \frac{1}{\lambda_i} \hat{\Psi}_i + \frac{1}{2} \frac{1}{\mu_i} \hat{D}_i^2
\end{align*}
\]  

(20)

**Theorem 3.1:** Consider error system (19). This system can be stabilized with the control signal (22) as follows

\[
u_i = -h_{11}(x, y, t) - k_i e_i - \hat{\Psi}_i \text{sign}(e_i) - \hat{D}_i \text{sign}(e_i)\]  

(22)

Accordingly, the two systems (13) and (15) are synchronized and the error will be zero between the two master and slave systems.

\[
e_i = y_i - x_i = 0
\]  

(23)

\[
\begin{align*}
\dot{V} &= \sum_{i=1}^{n} V_i(t) \\
\dot{V}_i &= \dot{D}^e e_i^2 - \frac{1}{\lambda_i} \hat{\Psi}_i \dot{\hat{\Psi}}_i - \frac{1}{\mu_i} \hat{D}_i \dot{\hat{D}}_i \\
\dot{V}_i &= e_i h_{11}(x, y, t) \\
&\quad + h_{21}(x, y, y - \tau_i^1(t), t) \\
&\quad - f_{21}(x, x - \tau_i^1(t), t) \\
&\quad + \Delta h_{2}(x, y) + d_i^1(t) + u_i(t) \\
&\quad - \frac{1}{\lambda_i} \hat{\Psi}_i \dot{\hat{\Psi}}_i - \frac{1}{\mu_i} \hat{D}_i \dot{\hat{D}}_i
\end{align*}
\]  

(24)

By defining the following function that expresses the effects of the presence of variable unknown time delay and also uncertainty in the error system dynamics for
synchronization,

\[ \psi_i(x, y, t) = h_i(x, y, x - \tau_{i}^m(t), y - \tau_{i}^s(t), t) + \Delta h_i(x, y) \tag{27} \]

\[ |\psi_i(x, y, t)| \leq \Psi_i, i = 1, \ldots, n \]

Equation (26) can be rewritten as follows

\[ \dot{V}_i(t) = e_i(h_i(x, y, t) + \psi_i(x, y, t) + d_i(t)) \]
\[ + u_i(t) - \frac{1}{\lambda_i} \dot{\Psi}_i \dot{\Psi}_i - \frac{1}{\mu_i} \dot{D}_i \dot{D}_i \tag{28} \]

To achieve synchronization and also to ensure stability, the control signal is selected as follows.

\[ u_i(t) = -h_i(x, y, t) - k_i e_i - \dot{\Psi}_i \text{sign}(e_i) - \dot{D}_i \text{sign}(e_i) \tag{29} \]

The control signal consists of four expressions. The expression \( h_i(x, y, t) \) is used to cover the difference between specific nonlinear functions in master and slave systems. The expression \( k_i e_i \) is a normal state feedback rule to ensure stronger stability and \( k_i \) are the controller gains. The first \( \text{sign}(e_i) \) is used to cover the effects of delay and uncertainty, and the second \( \text{sign}(e_i) \) overcomes the effects of disturbance, while the coefficients of both \( (\dot{\Psi}_i \) and \( \dot{D}_i \) are obtained online by the adaptive method.

It is obtained by placing Equation (29) in Equation (28)

\[ \dot{V}_i(t) = -k_i e_i^2 + e_i \psi_i(x, y, t) - \dot{\Psi}_i e_i \text{sign}(e_i) + e_i d_i(t) \]
\[ - \dot{D}_i e_i \text{sign}(e_i) - \frac{1}{\lambda_i} \dot{\Psi}_i \dot{\Psi}_i - \frac{1}{\mu_i} \dot{D}_i \dot{D}_i \tag{30} \]

By mathematical simplification, Equation (31) is obtained

\[ \dot{V}_i(t) \leq -k_i e_i^2 + |e_i| \psi_i - |e_i| \dot{\Psi}_i + |e_i| d_i \]
\[ - |e_i| \dot{D}_i - \frac{1}{\lambda_i} \dot{\Psi}_i \dot{\Psi}_i - \frac{1}{\mu_i} \dot{D}_i \dot{D}_i \]
\[ \leq -k_i e_i^2 + |e_i| \psi_i - |e_i| \dot{\Psi}_i - \frac{1}{\lambda_i} \dot{\Psi}_i - \frac{1}{\mu_i} \dot{D}_i \dot{D}_i \]
\[ \leq -k_i e_i^2 + \dot{\Psi}_i \left( |e_i| - \frac{1}{\lambda_i} \dot{\Psi}_i \right) + \dot{D}_i \left( |e_i| - \frac{1}{\mu_i} \dot{D}_i \right) \tag{31} \]

By choosing the adaptive rules as follows

\[ \dot{\Psi}_i = \lambda_i |e_i| \]
\[ \dot{D}_i = \mu_i |e_i| \tag{32} \]

It is obtained by placing Equation (32) in Equation (31), as well as applying the superposition principle

\[ \dot{V}_i(t) \leq -k_i e_i^2 \tag{33} \]

According to the relation (33), it is clear that by applying the control law (29) and the adaptive laws (32), the closed-loop system is asymptotically stable. As a result, synchronization of the two DUFOCSs is guaranteed and the error between the master and slave systems will be zero.

4. Simulation results

Simulation in MATLAB environment is performed to show the capability of the proposed method. One of the most important industrial turbulence systems is the Permanent Magnet Synchronous Motor (PMSM). The PMSM is widely used in many high performance applications such as electric vehicles, robotics and wind turbines (Eriksson, 2019) and its motor drive system exhibits chaotic dynamic behaviour which can destroy the stability of the system and even cause to collapse it. For this reason, in this section, a practical example is used to evaluate the proposed synchronization method. The transformed model of PMSM with the smooth air gap is as follows (Hu et al., 2015):

\[
\begin{align*}
\dot{x} &= a(y - x) - \bar{T}_L \\
\dot{y} &= (b - z)x - y + \bar{u}_q, \\
\dot{z} &= -z + xy + \bar{u}_d
\end{align*}
\tag{34}
\]

where

\[
\begin{align*}
x &= \frac{R}{L_q} \dot{x} - \frac{p_n L_q \psi_{ld}}{RB_{equ}} \dot{y}, \\
y &= \frac{z}{J_{equ}}, \\
b &= \frac{p_n L_q \psi_{ld}}{B_{equ}} \bar{u}_d, \\
\bar{u}_d &= \frac{p_n L_q \psi_{ld}}{B_{equ}} \bar{u}_d, \\
\bar{u}_q &= \frac{p_n L_q \psi_{ld}}{B_{equ}} \bar{u}_d, \\
\bar{L}_L &= \frac{L_d^2}{2 J_{equ}} \bar{T}_L
\end{align*}
\]

Definitions and descriptions of PMSM system parameters are all given in (Hu et al., 2015). Considering \( x_1 = x, x_2 = y, x_3 = z \), the state space equations of the chaotic PMSM and its fractional order explanation are in the form of Equations (35) and (36), respectively.

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) - \bar{T}_L \\
\dot{x}_2 &= -x_2 - x_1 x_3 + b x_1 + \bar{u}_q, \\
\dot{x}_3 &= -x_3 - x_1 x_3 + \bar{u}_d
\end{align*}
\tag{35}
\]

\[
\begin{align*}
\frac{d^\alpha x_1}{dt^\alpha} &= a(x_2 - x_1) \\
\frac{d^\alpha x_2}{dt^\alpha} &= -x_2 - x_1 x_3 + b x_1 \\
\frac{d^\alpha x_3}{dt^\alpha} &= -x_3 - x_1 x_3
\end{align*}
\tag{36}
\]

where \( \alpha \) as the fractional order and the phase portrait representation of the system (35) is shown in Figure 1.
As uncertainties, disturbances, and delays enter the system (35), the master system takes the form of Equation (37).

\[
\begin{align*}
\frac{d^\alpha x_1}{dt^\alpha} &= a(x_2 - x_1(t - \tau^m_1(t))) + \Delta f_1(x) + d^m_1(t) \\
\frac{d^\alpha x_2}{dt^\alpha} &= -x_2(t - \tau^m_2(t)) - x_1 x_3 + bx_1 + \Delta f_2(x) + d^m_2(t) \\
\frac{d^\alpha x_3}{dt^\alpha} &= -x_3(t - \tau^m_3(t)) - x_1 x_3 + \Delta f_3(x) + d^m_3(t)
\end{align*}
\] (37)

where parametric uncertainties and external disturbances for the master system are as follows.

\[
\begin{align*}
\Delta f_1(x) &= -0.2 \sin(3t)x_1 \\
\Delta f_2(x) &= 0.2 \sin(t)x_2 \\
\Delta f_3(x) &= 0.1 \cos(2t)x_3 \\
d^m_1(t) &= -0.15 \cos(t) \\
d^m_2(t) &= 0.1 \cos(3t) \\
d^m_3(t) &= 0.2 \sin(5t)
\end{align*}
\] (38)

And the slave system is as follows

\[
\begin{align*}
\frac{d^\alpha y_1}{dt^\alpha} &= a(y_2 - y_1(t - \tau^s_1(t))) + \Delta g_1(y) + d^s_1(t) + u_1(t) \\
\frac{d^\alpha y_2}{dt^\alpha} &= -y_2(t - \tau^s_2(t)) - y_1 y_3 + \Delta g_2(y) + d^s_2(t) + u_2(t) \\
\frac{d^\alpha y_3}{dt^\alpha} &= -y_3(t - \tau^s_3(t)) - y_1 y_3 + \Delta g_3(y) + d^s_3(t) + u_3(t)
\end{align*}
\] (40)

where parametric uncertainties and external disturbances for the slave system are as follows.

\[
\begin{align*}
\Delta g_1(y) &= -0.2 \cos(3t)y_1 \\
\Delta g_2(y) &= 0.15 \sin(2t)y_2 \\
\Delta g_3(y) &= 0.2 \cos(t)y_3 \\
d^s_1(t) &= 0.1 \sin(5t) \\
d^s_2(t) &= 0.2 \cos(t) \\
d^s_3(t) &= -0.1 \sin(3t)
\end{align*}
\] (41)

\[\text{Figure 1. Phase portraits of chaotic system (35).}\]
Figure 2. Profiles of time delays. (a) The time delay ($\tau^m_1(t)$). (b) The time delay ($\tau^m_2(t)$). (c) The time delay ($\tau^m_3(t)$). (d) The time delay ($\tau^s_1(t)$). (e) The time delay ($\tau^s_2(t)$). (f) The time delay ($\tau^s_3(t)$).

Figure 3. The display of synchronization errors ($e_x, e_y, e_z$) between master and slave systems.
Figure 4. The time-domain display of signal \((x_1, y_1)\) for master and slave systems.

Figure 5. The time-domain display of signal \((x_2, y_2)\) for master and slave systems.
And \( u_1(t), u_2(t) \) and \( u_3(t) \) signify the control signals obtained from Equation (29) to realize synchronization between the master and slave systems. The error equations are defined as:

\[
\begin{align*}
    e_1 &= y_1 - x_1 \\
    e_2 &= y_2 - x_2 \\
    e_3 &= y_3 - x_3
\end{align*}
\]

(43)

Then, the error dynamics between the two chaotic systems is found as

\[
\begin{align*}
    \frac{d^\alpha e_1}{dt^\alpha} &= a(e_2) \\
    &\quad + y_1(t - \tau_1^m(t)) - x_1(t - \tau_1^m(t)) - 0.2 \cos(3t)y_1 \\
    &\quad + 0.2 \sin(3t)x_1 + 0.1 \sin(5t) + 0.15 \cos(t) + u_1 \\
    \frac{d^\alpha e_2}{dt^\alpha} &= -(y_2(t - \tau_2^m(t)) - x_2(t - \tau_2^m(t))) \\
    &\quad - (y_1y_3 - x_1x_3) + be_1 + 0.15 \sin(2t)y_2 \\
    &\quad - 0.2 \sin(t)x_2 + 0.2 \cos(t) - 0.1 \cos(3t) + u_2 \\
    \frac{d^\alpha e_3}{dt^\alpha} &= -(y_3(t - \tau_3^m(t)) - x_3(t - \tau_3^m(t))) \\
    &\quad - (y_1y_3 - x_1x_3) + 0.2 \cos(t)y_3 \\
    &\quad - 0.1 \cos(2t)x_3 = -0.1 \sin(3t) - 0.2 \sin(5t) + u_3
\end{align*}
\]

(44)

To evaluate the ability of the proposed method in more depth, some different profiles are considered for time delays \( \tau_1^m(t), \tau_2^m(t), \tau_3^m(t) \), \( \tau_1^s(t) \), \( \tau_2^s(t) \) and \( \tau_3^s(t) \) as shown in Figure 2. By setting initial values as \( x_1(0) = 3, y_1(0) = 1, z_1(0) = 4 \) for the master system and \( x_2(0) = 6.2, y_2(0) = -1.4, z_2(0) = 2 \) for the slave system and choosing the controller parameters as

\[
\begin{align*}
    k_1 &= 50, k_2 = 50, k_3 = 50 \\
    \mu_1 &= 0.01, \mu_2 = 0.01, \mu_3 = 0.01 \\
    \lambda_1 &= 0.01, \lambda_2 = 0.01, \lambda_3 = 0.01
\end{align*}
\]

(45)

The synchronization results of the implementation of the proposed robust technique of the chaotic fractional system are shown in Figures 3–6. Figure 3 displays the synchronization error resulting from the execution of the proposed robust technique. As it turns out, after a finite time around 0.002 s, the two master and slave systems demonstrate exactly the same behaviour. By presenting the performance of each state in both master and slave systems, the synchronization is displayed in more detail in Figures 4–6. Figure 4 presents the performance of \( x_1, y_1 \), Figure 5 displays the activities of \( x_2, y_2 \), and Figure 6 presents the manners of \( x_3, y_3 \) in both master and slave systems. The tracking quality of each master state by the slave state is quite evident in these figures. In general, the

![Figure 6](image)

Figure 6. The time-domain display of signal \((x_3, y_3)\) for master and slave systems.
simulation results indicate that the proposed robust technique is well able to synchronize two chaotic systems of fractional order in the presence of external disturbances, uncertainties and unknown variable time delays.

5. Conclusion

This paper presented a new robust method for synchronizing fractional order chaotic systems. Despite the nonlinear terms in the fractional model of the chaotic system, three issues of external disturbances, uncertainties and time-varying delays were considered simultaneously in synchronization. The time-varying delay as well as the upper bound of the disturbance were unspecified and the uncertainty could be even non-parametric. Delays could also take various forms in master and slave systems. To achieve synchronization, a combined control method proposed that included adaptive and sliding mode techniques. The sliding mode method used to overcome the effects of time-varying delay, disturbance and uncertainty, and the adaptive method was used to estimate the upper bounds of disturbance as well as to obtain online gains for the sliding mode method. Simulation in MATLAB environment showed the ability of the proposed method to achieve synchronization of two PMSM fractional order chaotic systems in the shortest time. Considering the saturation constraint on the control signal can be a great way to complete and develop this study.

Disclosure statement

No potential conflict of interest was reported by the author(s).

References

Aghababa, M. P. (2012). Finite-time chaos control and synchronization of fractional-order nonautonomous chaotic (hyperchaotic) systems using fractional nonsingular terminal sliding mode technique. *Nonlinear Dynamics, 69*(1), 247–261. https://doi.org/10.1007/s11071-011-0261-6

Aghababa, M. P. (2013). Design of a chatter-free terminal sliding mode controller for nonlinear fractional-order dynamical systems. *International Journal of Control, 86*(10), 1744–1756. https://doi.org/10.1080/00207179.2013.796068

Ahmad, B., Alghanmi, M., Alsaeedi, A., & Agarwal, R. P. (2019). Nonlinear impulsive multi-order Caputo-type generalized fractional differential equations with infinite delay. *Mathematics, 7*(11), 1108. https://doi.org/10.3390/math7111108

Bhalekar, S., & Daftardar-Gejji, V. (2010). Synchronization of different fractional order chaotic systems using active control. *Communications in Nonlinear Science and Numerical Simulation, 15*(11), 3536–3546. https://doi.org/10.1016/j.cnsns.2009.12.016

Boulkroune, A., Bouzeriba, A., Bouden, T., & Azar, A. T. (2016). Fuzzy adaptive synchronization of uncertain fractional-order chaotic systems. In A. Azar & S. Vaidyanathan (Eds.), *Advances in chaos theory and intelligent control* (pp. 681–697). Cham: Springer. https://doi.org/10.1007/978-3-319-30340-6_28

Calderón, A. J., Vinagre, B. M., & Feliu, V. (2006). Fractional order control strategies for power electronic buck converters. *Signal Processing, 86*(10), 2803–2819. https://doi.org/10.1016/j.sigpro.2006.02.022

Chai, X., Fu, X., Gan, Z., Lu, Y., & Chen, Y. (2019). A color image cryptosystem based on dynamic DNA encryption and chaos. *Signal Processing, 155*, 44–62. https://doi.org/10.1016/j.sigpro.2018.09.029

Chen, Y., Chen, X., & Gu, S. (2007). Lag synchronization of structurally nonequivalent chaotic systems with time delays. *Nonlinear Analysis: Theory, Methods & Applications, 66*(9), 1929–1937. https://doi.org/10.1016/j.na.2006.02.033

Eriksson, S. (2019). *Permanent magnet synchronous machines*. Ha, J., Liu, L., Ma, D. W., & Ullah, N. (2015). Adaptive nonlinear feedback control of chaos in permanent-magnet synchronous motor system with parametric uncertainty. *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science, 229*(12), 2314–2323. https://doi.org/10.1177/0954406214557344

Hu, M., Xu, Z., Zhang, R., & Hu, A. (2007). Adaptive full state hybrid projective synchronization of chaotic systems with the same and different order. *Physics Letters A, 365*(4), 315–327. https://doi.org/10.1016/j.physleta.2007.01.038

Khamsuwan, P., & Kuntanapreeda, S. (2016, May 24). A linear matrix inequality approach to output feedback control of fractional-order unified chaotic systems with one control input. *Journal of Computational and Nonlinear Dynamics, 11*(5), 051021. https://doi.org/10.1115/1.4033384

Khan, U., Ellahi, R., Khan, R., & Mohyud-Din, S. T. (2017). Extracting new solitary wave solutions of Benny–Luke equation and Phi-4 equation of fractional order by using (G'/G)-expansion method. *Optical and Quantum Electronics, 49*(11), 1–14. https://doi.org/10.1007/s11082-017-1191-4

Kuntanapreeda, S. (2016). Adaptive control of fractional-order unified chaotic systems using a passivity-based control approach. *Nonlinear Dynamics, 84*(4), 2505–2515. https://doi.org/10.1007/s11071-016-2661-0

Li, W., Bai, G., & Marrani, H. I. (2020). A new robust finite-time synchronization and anti-synchronization method for uncertain chaotic systems by using adaptive estimator and terminal sliding mode approaches. *Journal of Control, Automation and Electrical Systems, 31*(6), 1375–1385. https://doi.org/10.1007/s40313-020-00650-4

Liu, X., Hou, B., & Zhao, Q. (2019). Monitoring data encryption method for howitzer shell transfer arm using chaos and compressive sensing. *Journal of Algorithms & Computational Technology, 13*, 1–13. https://doi.org/10.1017/jact.2017.873598

Malica, T., Bouchez, G., Woltersberger, D., & Sciannama, M. (2020). Spatiotemporal complexity of chaos in a phase-conjugate feedback laser system. *Optics Letters, 45*(4), 819–822. https://doi.org/10.1364/OL.383557

Maligranda, L. (2008). Some remarks on the triangle inequality norms. *Banach Journal of Mathematical Analysis, 2*(2), 31–41. https://doi.org/10.15352/bjma/1240336290

Moysis, L., Petavratzi, E., Marwan, M., Volos, C., Nistazakis, H., & Ahmad, S. (2020). Analysis, synchronization, and robotic application of a modified hyperjerk chaotic system. *Complexity, 2020*, 1–15. https://doi.org/10.1155/2020/82826850
Sohail, A., Maqbool, K., & Ellahi, R. (2018). Stability analysis for fractional-order partial differential equations by means of space spectral time Adams-Bashforth Moulton method. Numerical Methods for Partial Differential Equations, 34(1), 19–29. https://doi.org/10.1002/num.22171

Sweilam, N. H., Khader, M. M., & Al-Bar, R. F. (2007). Numerical studies for a multi-order fractional differential equation. Physics Letters A, 371(1-2), 26–33. https://doi.org/10.1016/j.physleta.2007.06.016

Tripathi, D., Pandey, S. K., & Das, S. (2010). Peristaltic flow of viscoelastic fluid with fractional Maxwell model through a channel. Applied Mathematics and Computation, 215(10), 3645–3654. https://doi.org/10.1016/j.amc.2009.11.002

Wang, L., Dong, T., & Ge, M. F. (2019). Finite-time synchronization of memristor chaotic systems and its application in image encryption. Applied Mathematics and Computation, 347, 293–305. https://doi.org/10.1016/j.amc.2018.11.017

Wen, G., & Xu, D. (2005). Nonlinear observer control for full-state projective synchronization in chaotic continuous-time systems. Chaos, Solitons & Fractals, 26(1), 71–77. https://doi.org/10.1016/j.chaos.2004.09.117

Yousefpour, A., Jahanshahi, H., Munoz-Pacheco, J. M., Bekiros, S., & Wei, Z. (2020). A fractional-order hyper-chaotic economic system with transient chaos. Chaos, Solitons & Fractals, 130, 109400. https://doi.org/10.1016/j.chaos.2019.109400

Yu, F., & Wang, C. (2012). A novel three dimension autonomous chaotic system with a quadratic exponential nonlinear term. Engineering, Technology & Applied Science Research, 2(2), 209–215. https://doi.org/10.48084/etasr.86

Zeebe, R. E., & Lourens, L. J. (2019). Solar system chaos and the Paleocene–Eocene boundary age constrained by geology and astronomy. Science, 365(6456), 926–929. https://doi.org/10.1126/science.aax0612

Zhan, T. S., Chen, J. L., Chen, S. J., Huang, C. H., & Lin, C. H. (2014). Design of a chaos synchronisation-based maximum power tracking controller for a wind-energy-conversion system. IET Renewable Power Generation, 8(6), 590–597. https://doi.org/10.1049/iet-rpg.2013.0268

Zhu, F. (2008). Full-order and reduced-order observer-based synchronization for chaotic systems with unknown disturbances and parameters. Physics Letters A, 372(3), 223–232. https://doi.org/10.1016/j.physleta.2007.06.081