Geometry of WZW Orientifolds

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Abstract

We analyze unoriented Wess-Zumino-Witten models from a geometrical point of view. We show that the geometric interpretation of simple current crosscap states is as centre orientifold planes localized on conjugacy classes of the group manifold. We determine the locations and dimensions of these planes for arbitrary simply-connected groups and orbifolds thereof. The dimensions of the O-planes turn out to be given by the dimensions of symmetric coset manifolds based on regular embeddings. Furthermore, we give a geometrical interpretation of boundary conjugation in open unoriented WZW models; it yields D-branes together with their images under the orientifold projection. To find the agreement between O-planes and crosscap states, we find explicit answers for lattice extensions of Gaussian sums. These results allow us to express the modular $P$-matrix, which is directly related to the crosscap coefficient, in terms of characters of the horizontal subgroup of the affine Lie algebra. A corollary of this relation is that there exists a formal linear relation between the modular $P$- and the modular $S$-matrix.

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1 Introduction

There has been much progress in the study of interacting conformal field theories on open unoriented surfaces [1, 2, 3, 4]. Compared with the closed case, this requires the introduction of two new objects, boundary and crosscap states, which act as sources for closed strings. Starting with [3], there has been much interest in the geometrical interpretation of the first of these two: the boundary states. By extension of the toroidal results, they are believed to give a stringy description of D-branes [5] in curved backgrounds. Owing to the presence of localized Yang-Mills fields on their surface, their natural geometric interpretation is as vector-bundles wrapping closed submanifolds or more precisely coherent sheaves. And owing to the geometric interpretation the abstract CFT results may be extrapolated away from the special rational point to other areas of the moduli space of the theory.

In many realistic string compactifications however, the presence of boundary states leads to tadpoles, and we are forced to include crosscap states in CFT. In geometric language one needs to absorb the excess charges carried by the D-branes by the introduction of (some superposition of) orientifold planes [6]. Outside of toroidal or orbifold compactifications, little is known about the geometry of orientifolds (see [9] for a first attempt to study O-planes in a curved geometry). It is not clear whether orientifolds also should be interpreted as independent objects wrapping closed submanifolds. From a CFT point of view it is a priori not even clear that crosscap states yield properly localized charge distributions which may be interpreted as (lower-dimensional) orientifold planes. Known examples of toroidal or orbifold models are limited to special points in the stringy region of the geometric moduli-space where the target space is flat except for some singular points. Indeed from a perturbative string-theory point of view orientifolds seem little more than a bookkeeping device for excess curvature or charge, because they have no inherent moduli and are strictly rigid objects. On the other hand, non-perturbative studies do indicate that they may possess some properties which allow them to be seen as bound states of independent higher-dimensional objects in string theory. An example hereof is the splitting of O7-planes into D7-branes under splitting of the singularities of F-theory compactifications on K3 [9].

Here we wish to make a first step towards determining the geometrical characteristics of crosscap states. Understanding these would allow us to extend unoriented type I compactifications to regions away from rational points in the moduli space, and make contact with large volume geometry. We have to be careful, for it has not yet been established whether this is a truly sensible question to ask. The orientation reversal projection, characteristic of the crosscap state, may project out some blow-up modes which are necessary to reach the large volume point of the moduli space. Hence the target-space geometry, or more precisely its moduli space, of unoriented models is not directly equivalent to that of its oriented parent. In turn this means that we should also reconsider the geometry of boundary states in unoriented models, as it may need reinterpretation.

In this article, we will start to address these questions by studying orientifold planes on group manifolds. Strings on group manifolds are described by Wess-Zumino-Witten (WZW) models, which are a well-understood class of interacting CFT’s. In particular the crosscap
states which may arise in WZW-models are known. On the other hand, to consider the group manifold of a WZW-model as target spaces does not always agree with the conventional interpretation of the target space of string theories. Most importantly, in standard string scenarios we think of the string moving in a target space with dimension equal to the central charge of the bosonic CFT. With WZW models, on the other hand, one thinks of the string living on the group manifold with dimension that of the group itself. The word “geometry” depends on the setting in which it is discussed.

We will proceed with this caveat in mind. In section 2 we will use simple geometrical (group theoretical) techniques in the Lagrangian description of WZW models to classify the possible WZW orientifold planes by their locations and dimensions. For every element of the centre of the group, we find a configuration of O-planes that are localized on conjugacy classes. In section 3, we review the construction of abstract unoriented CFT’s in general and WZW-models in particular. We show that in order to determine the characteristics of WZW crosscap states, we need an expression for the pseudo-modular $P$-matrix in terms of characters of the horizontal subalgebra. This specific mathematical problem we answer in section 4, whereafter in section 5 we reach the objective of this article. We demonstrate that there is indeed a one-to-one correspondence between the so-called simple current crosscap states $[0]$ of Wess-Zumino-Witten models and the geometrical O-planes. We also show that the crosscap states of $[1]$ describe configurations of O-planes on non-simply connected group manifolds (orbifolds). In the final section, section 6, we describe the geometry of branes in unoriented WZW models, thereby completing the earlier study of branes on group manifolds $[2, 3, 4, 5]$.

The important calculation of section 4 that expresses the $P$-matrix of a WZW-model in terms of characters of the horizontal subalgebra is the heart of this paper. There are at least two reasons for us to surmise that it may have consequences beyond the immediate question posed here. In deriving the relation between the $P$-matrix and characters, one has to perform mathematically interesting ‘quadratic Gaussian sums’ on the weight lattices of the horizontal algebra. These are extensions of one-dimensional Gaussian sums and $d$-Cartesian dimensional Kloosterman sums, which play a role in analytic number theory. Physically interesting is the fact that the relation with characters of $G$ implies a linear relation between the modular $P$- and the modular $S$-matrix for WZW-models, albeit an analytic continuation of the latter, which we define in section (4.2). The construction of open-string theories relies heavily on the properties of both, and if the existence of such a linear relation were to hold for all abstract CFT’s, this could have important consequences, for instance for tadpole cancellation.

Concerning the notation: a tilde is used for the rightmoving sector, but is often omitted for the sake of clarity. The affine algebra will be denoted by $\mathfrak{g}$ and the horizontal subalgebra by $\bar{\mathfrak{g}}$. By extension all affine quantities are unbarred, for instance weights $\Lambda$ and characters $\chi$, and all horizontal quantities are barred. However, when the formulas are unambiguous, we will often drop the bars. Lie group characters are denoted by $X$.

Two final remarks: One, to prevent an overcluttered notation we have normalized the length squared of the highest root $\Psi$ for every group to two: $(\Psi, \Psi) = 2$. Two, in trying to
keep the presentation as simple as possible, we will assume that the group centres are cyclic. At the end of some sections, we will comment on the generalization to noncyclic centres, that occur for $D_r$, $r$ even, or tensor products of WZW models.

While we were applying the final touches, a paper [34] appeared which also finds some of the results discussed here.

## 2 Semiclassical analysis

WZW models, as do free CFT’s, have the advantage that there also exists a Lagrangian description of the theory. String theory on a group manifold $G$ can either be described by the abstract CFT built on an affine extension $\mathfrak{g}$ of the algebra $\bar{\mathfrak{g}}$ of $G$ or by the Wess-Zumino-Witten action [16, 17]

$$
S = -\frac{k}{16\pi} \int_{\Sigma} d^2x \text{Tr}(g^{-1}\partial_\mu g)^2 + \frac{k}{24\pi} \int_{B} d^3y \text{Tr}(g^{-1}dg)^3.
\tag{2.1}
$$

Here $g$ is a field that takes values in the group $G$ and lives on the string worldsheet $\Sigma$. The second term is topological; $B$ is a three-manifold whose boundary is $\Sigma$, and the level $k$ has to be integer. The Lagrangian description emphasizes the geometry and it is therefore the natural place to explore the geometrical consequences of orientation reversal.

The WZW theory possesses the chiral currents [16]

$$
W = g^{-1}\partial g, \quad \bar{W} = -\bar{\partial}gg^{-1},
\tag{2.2}
$$

whose modes span two commuting copies of the affine Lie algebra $\mathfrak{g}$. To construct an unoriented theory, consider the involutions

$$
\Omega_n := \Omega \cdot R_n, \quad (2.3)
$$

The worldsheet parity transformation $\Omega$ interchanges the chiral coordinates and $R_n$ acts on the coordinates as

$$
R_n : g \rightarrow \gamma^n g^{-1}, \quad n = 0,...,N - 1
\tag{2.4}
$$

where $\gamma$ generates the $\mathbb{Z}_N$ centre of $G$. $\Omega_n$ interchanges the chiral currents and is therefore a global invariance of this theory. When we gauge this symmetry, we obtain orientifold fixed planes at those points of the target space $G$ that are fixed under $\Omega_n$ [4]. Note that $\Omega$ by itself is not a symmetry. This tells us right away that WZW models do not possess space-time filling orientifold-planes, analogous to O9-planes in superstring theory. Instead, we only get lower dimensional planes at the points $g \in G$ for which

$$
g = \gamma^n g^{-1}.
\tag{2.5}
$$

Note that, if $g$ solves (2.3), so does every point in the set

$$
C(g) = \{hgh^{-1} | h \in G\},
\tag{2.6}
$$

4
so our planes are located at conjugacy classes of \( G \).

To solve (2.5) explicitly, recall that every conjugacy class contains at least one element of a maximal torus \( T \) of \( G \). So choose a Chevalley basis of the rank \( r \) algebra \( \hat{g} \) and let \( H^i, i = 1, \ldots, r \) be a basis for the Cartan subalgebra. In this basis their action on the weights returns the integral Dynkin labels. Then an element of the corresponding maximal torus can be written as \( g_t = \exp (2\pi i (t, H)) := \exp (2\pi i \sum_{i=1}^{r} t_i H^i) \). In this basis, the centre elements \( \gamma^n \) are given by

\[
\gamma^n = \exp\{2\pi i \frac{n}{N} \sum_i c_i H^i\},
\]

where \( C := \sum_i c_i H^i \) is tabulated in table 1. The eigenvalue of \( C \) modulo \( N \) is known as the conjugacy class of a representation \( \rho \) of the algebra \( \hat{g} \). It is constant modulo \( N \) within a representation, which means that \( C \) commutes modulo \( N \) with all other generators in that representation. This shows that \( \gamma^n \) indeed belongs to the centre (and, by turning the argument around, that the elements of the centre of \( G \) are in one-to-one correspondence with the conjugacy classes of \( \hat{g} \).)

It is now easy to find the solutions to (2.5). An element \( g_t \) is a fixed point of \( R_n \) when for all \( i \)

\[
2t_i = \frac{nc_i}{N} \mod 1 \rightarrow t_i = \frac{nc_i}{2N} - \frac{u_i}{2} \mod 1.
\]

(2.8)

Here \( u_i \) are vectors in a basis dual to the Dynkin basis, whose entries are either zero or one modulo even integers. We can phrase this in a basis-independent way, using that the dual of the weight space \( \mathfrak{L}_w \) is the coroot lattice \( \mathfrak{L}^\vee \). Eq. (2.8) is a vector equation on a basis of \( r \) simple coroots \( \check{\alpha}_i \), \( i = 1, \ldots, r \);

\[
2t = \frac{nc}{N} \mod \mathfrak{L}^\vee \rightarrow t = \frac{nc}{2N} - \frac{u}{2} \quad \text{with} \quad u \in \mathfrak{L}^\vee \mod 2\mathfrak{L}^\vee,
\]

(2.9)

with \( t = \sum_i t_i \check{\alpha}_i \), etc. The minus sign in front of \( u \) is chosen for later purposes (to make contact with (2.3)). So we have found \( 2^r \) solutions \( g_{n,u} \) to (2.5). In general, however, not all of them lie on different conjugacy classes. A quite non-trivial theorem states that when two elements of the maximal torus are conjugate, they are related by the action of the Weyl group \( W \). Now we can count how many and what type of O-planes we get.

Consider first the projection with \( n = 0 \) and define

\[
g_u := g_{0,u} = e^{-\pi i (u, H)}.
\]

(2.10)

Then, according to this theorem, \( g_u \) and \( g_{u'} \) lie on the same conjugacy class when \( u \) and \( u' \) belong to the same Weyl orbit, i.e. when there is a \( w \in W \) such that

\[
u' = w(u) \mod 2\mathfrak{L}^\vee.
\]

(2.11)

So the number of O-planes equals the number of Weyl orbits of \( \hat{g} \). We determined these numbers empirically and displayed them in table 1.

1Not to be confused with the conjugacy classes for groups used above.
Next, consider the orientifold projection by $\Omega_n$ where $n$ is even, i.e. $n = 2m$. Note that, when the order of the centre $N$ is odd, we can always bring $n$ to this form. It is easy to see that (2.5) is solved by
\[ g_{2m,u} = \gamma^m g_{0,u} \],

i.e. the locations of the $n = 2m$ planes are simply translations of the canonical $(n = 0)$ ones by centre elements $\gamma^m$. It is not hard to see that translation by an element of the centre is consistent with conjugation, i.e. that the conjugacy classes of $g_{0,u}$ map one-to-one to the conjugacy classes of $g_{2m,u}$. Hence the $n = 2m$ configuration of O-planes directly follows from the $n = 0$ result.

For $N$ even, we have the additional possibility that $n = 2m + 1$ odd. Then
\[ g_{2m+1,u} = \gamma^m g_{1,u} \],

solves (2.5) where $g_{1,u}$ is a solution for $n = 1$. So the locations of the $n = 2m + 1$ planes are translations of the $n = 1$ configuration by centre elements $\gamma^m$. The $n = 1$ and $n = 0$ configuration appear to be related by translations
\[ g_{n=1,u} = \sqrt{\gamma} g_{n=0,u} \],

where $\sqrt{\gamma}$ is one of the elements of the maximal torus that squares to $\gamma$. Of course $\sqrt{\gamma}$ itself is not an element of the centre. We must check anew how the $2^r$ solutions $g_{n=1,u}$ split into conjugacy classes. The total number of O-planes is now given by the number of Weyl orbits containing the $2^r$ solutions $g_{1,u}$. Again we determined these numbers empirically and they are also given in table 1.

Note that for $N$ even, a given configuration (both $n = 0, 1$) is invariant under translations by the $\mathbb{Z}_2$ subgroup of the centre. Indeed, from (2.5) it is easy to see that when $g$ is a solution, so is $-g$.

### 2.1 Dimensions of O-planes

To determine the dimension of the O-planes we need to know which elements of the group commute with $g_{n,u}$, the element of the maximal torus that defines the conjugacy class. This analysis can be done in the tangent space at the point $g_{n,u}$ of the group manifold, in other words we have to determine which elements of the Lie algebra commute with $g_{n,u}$. For a generic element of the maximal torus, this set (called the commutant $G_c$ henceforth) consists of only the Cartan sub-algebra, and then the dimension of the O-plane is $d - r$, where $d$ is the dimension of the group and $r$ its rank. For a centre element the commutant is of course the entire group, and therefore the O-plane has dimension 0 (i.e. it is an O0-plane) if and only if $g_{n,u}$ is an element of the centre. In all other case we have to examine the commutator of the root generators $E_\alpha$ with $g_{n,u}$.

\footnote{We may restrict our attention to elements of the maximal torus, since we are solving $g = \gamma^n g^{-1}$ only modulo conjugation. In fact eq. (2.14) would not be a solution otherwise.}
This can be done as follows. Note that

$$E^\alpha [(t, H)]^m = (t, H) [E^\alpha - [(t, H), E^\alpha]] [(t, H)]^{m-1} = [(t, H) - (\alpha, t)] E^\alpha [(t, H)]^{m-1} \quad (2.15)$$

Iterating this we get

$$E^\alpha [(t, H)]^m = [(t, H) - (\alpha, t)]^m E^\alpha \quad ,$$

and hence

$$E^\alpha e^{2\pi i (t, H)} = e^{2\pi i [(t, H) - (\alpha, t)]} E^\alpha \quad (2.16)$$

Therefore $E^\alpha$ commutes with $e^{2\pi i (t, H)}$ if and only if $(\alpha, t)$ is an integer. Obviously linear combinations of root generators are in the commutant if and only if every term separately is in the commutant. Hence the commutant is in fact a regular subalgebra. The dimension of the plane is $d - r - M$, where $M$ is the number of roots $\alpha$ with $(\alpha, t)$ integer. One may also write this as $d - d_c$, where $d_c$ is the dimension of the commutant $G_c$. In other words, the dimension of the O-plane is equal to the dimension of the coset space $G/G_c$.

We can say a little more about these coset spaces. Since $[e^{2\pi i (t, H)}]^2$ is an element of the centre, it must commute with all $E_\alpha$, and therefore $2(\alpha, t)$ must be an integer. Hence the set of root generators splits into two sets, one with $(\alpha, t)$ even that commutes with $e^{2\pi i (t, H)}$, and one with $(\alpha, t)$ odd that anti-commutes with it. This means that the Lie-algebra $\tilde{\mathfrak{g}}$ of $G$ has a direct sum decomposition $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_c + \tilde{\mathfrak{k}}$, where $\tilde{\mathfrak{g}}_c$ is the Lie algebra of the commutant and $\tilde{\mathfrak{k}}$ is the ‘odd’ part. Note that the commutator of two elements of $\tilde{\mathfrak{k}}$ commutes with $[e^{2\pi i (t, H)}]^2$, and hence is in $\tilde{\mathfrak{g}}_c$. This implies that the coset $G/G_c$ is a symmetric space.

The converse is also true: let $G_c \subset G$ be a regular embedding, such that $G/G_c$ is a symmetric space. Then the Lie algebra of $G$ can be decomposed as a direct sum $\mathfrak{g} = \tilde{\mathfrak{g}}_c + \tilde{\mathfrak{k}}$, and there exists an automorphism that fixes $\tilde{\mathfrak{g}}_c$ and sends $\tilde{\mathfrak{k}}$ to minus itself. If the embedding is regular, $\tilde{\mathfrak{g}}_c$ contains the entire Cartan subalgebra of $\mathfrak{g}$, and hence the Cartan subalgebra of $\tilde{\mathfrak{g}}$ is left unchanged by the automorphism. This implies that the automorphism is inner, i.e. there exists an element $x \in G$ such that $xhx^{-1} = h; xkx^{-1} = -k$ for $h \in \tilde{\mathfrak{g}}_c$ and $k \in \tilde{\mathfrak{k}}$. This element is unique up to multiplication by an element of the centre. Furthermore $x^2$ commutes with both $h$ and $k$, and hence it is an element of the centre. Therefore $x$ is a solution to (2.3).

Hence the classification of O-planes is now a matter of inspecting the table of symmetric spaces [19], and selecting the ones where the embedding is regular. The results are summarized in table [3]. In the table the trivial case, $G_c = G$, corresponding to O0-planes is omitted. This is always a solution for $n = 0$, and corresponds to $g = 1$. This table was actually obtained by counting the number of odd root generators in order to identify which conjugacy class belongs to which symmetric space. The relation between conjugacy classes and regular symmetric spaces is not one-to-one because of the possibility of multiplying $x$ with a center element. This may or may not change the conjugacy class to which $x$ belongs. For example, in the trivial case $G = G_c$ (yielding O0-planes) valid choices for $x$ are $1$ and $-1$ (if $-1$ is an element of $G$), but $1$ and $-1$ are obviously not conjugate. Up to this freedom of multiplying $x$ by elements of the centre, the number of different choices of $G_c$ for given $G$ and $n$ corresponds to the number of O-planes in the last column of [1]. In most cases where there
are more conjugacy classes per symmetric space this degeneracy is due to multiplication by 
−1, but for $D_r$, $r$ even, there are additional degeneracies.

This can be illustrated rather nicely for $G = SU(N)$, because in that case one can solve (2.4) explicitly in the group. For $n = 0$ the condition $g^2 = 1$ combined with $\det(g) = 1$ allows diagonal matrices with $p$ eigenvalues 1 and $q$ eigenvalues $-1$, with $N = p + q$ and $q$ even. This breaks $SU(N)$ to $S(U(p) \times U(q))$. Therefore the corresponding O-planes have dimension $2pq$.

For $n = 1$ we may take a square root of the center element $\text{diag}(\exp(2\pi i/N))$. If we take all $N$ entries of the square root equal to $\exp(\pi i/N)$ we would get determinant $-1$. Hence we have to flip an odd number of signs. The corresponding group element $g$ breaks $SU(N)$ to $S(U(p) \times U(q))$, but this time with $q$ odd. Note that if $N$ is odd either $p$ is odd or $q$ is odd, so that there is no real difference between $n = 0$ and $n = 1$, as expected.

Note that O-planes have smaller dimensions than D-branes except if $G_c$ is precisely the Cartan sub-algebra. This happens only for $SU(2)$, $n = 1$.

To summarize, simple geometric semiclassical analysis of the WZW-action tells us that the O-plane characteristics depend on the order of the centre. When the order of the centre $N$ is odd, there is only one configuration of O-planes modulo centre translations. This configuration has one O0-plane at the origin; the remaining O-planes are localized on conjugacy classes of various dimensions; their number is given in table I. For even $N$, there are two inequivalent configurations of O-planes modulo centre translations. The ‘even’ configuration has two O0-planes at $g = \pm 1$, and a remainder — see table I — of planes localized on conjugacy classes, whereas the ‘odd’ configuration consists completely of O-planes localized on certain conjugacy classes. This pattern generalizes to the non-cyclic centre of $D_r$, $r$ even, in a straightforward manner. When there are more cyclic factors in the centre, the number of inequivalent configurations is $2^s$, where $s$ is the number of even cyclic factors.

### 2.2 Planes in Orbifolds

As for free CFT’s, the WZW action (2.1) may also be used to describe orbifolds. For every subgroup $H$ of the centre, we can construct the orbifold $G/H$. Let $H = \mathbb{Z}_M$, with $M$ a divisor of $N$, be generated by $\xi := \gamma_{N/M}$. Then the orbifold consists of equivalence classes $(m = 0, \ldots, M - 1)$

\[
g \sim \xi^m g. \tag{2.18}
\]

The $n = 0$ O-planes in orbifolds are located at those points of $G/H$ for which

\[
g = g^{-1}. \tag{2.19}
\]

By (2.18), this relation only has to hold modulo translations by $\xi^m$. In terms of the covering group $G$, every O-plane configuration corresponding to $\xi^m$ appears in the orbifold. We will see in section 5.1 that the planes form $H$-translation invariant combinations. Obviously this generalizes directly to $n \neq 0$ planes.
2.3 An example: $SU(2)$

Let us illustrate these results with $SU(2)$. The group manifold is a three-sphere; the conjugacy classes are two-spheres of constant latitude and the centre elements are at the north and south poles. The standard orientifold symmetry $g \rightarrow g^{-1}$ is a reflection through the ‘axis of rotation’ with fixed points at the poles. In terms of the solutions (2.10) we simply have $g_{u=0} = 1$ and $g_{u=1} = -1$.

The action of the centre is to identify antipodal points on the sphere. So the combined action $g \rightarrow -g^{-1}$ leaves the equator fixed. The solution (2.14) indeed describes one O-plane at the equator, since $\pm \sqrt{\gamma} = \pm \text{diag}(i, -i)$.

When we identify $g \sim -g$, we obtain the group manifold of $SO(3)$. The orientifold projection $g \rightarrow g^{-1}$ gives the following configuration. The two $SU(2)$ O0-planes at the poles are identified by the orbifold symmetry, leaving one O0-plane at the pole for $SO(3)$. There is also one O2-plane at the equator, since this submanifold is now fixed under the combined action of the orientifold and orbifold symmetry.

Recalling the caveat regarding differing views of target space between conventional string compactifications and WZW models, it is interesting to briefly compare the $SU(2)$ results to the well known results for $U(1)$; for, the $U(1)$ theory at the self-dual radius is equal to the $SU(2)$ level one model. In terms of $U(1)$ the standard orientifold symmetry $g \rightarrow g^{-1}$ is just the reflection $x \rightarrow -x$ modulo $2\pi R$, with fixed points $x = 0$ and $x = \pi R$ corresponding to $g = \pm 1$. The $n = 1$ orientifold symmetry $g \rightarrow -g^{-1}$ corresponds to the identification $x \simeq -x + \pi R$ modulo $2\pi R$, with fixed points at $x = \pm \pi R/2$. Within $SU(2)$ these solutions are conjugate, as are all solutions to the conditions $g^2 = \gamma, \gamma \in U(1)$: the center is the entire group. If we only consider the locations there is therefore just one solution in $U(1)$, up to equivalence. However, if we also consider the tensions there are two orientifold configurations for any radius $R$, one with O0-planes with opposite tensions and one with O0-planes with the same tension [21]. In the case of $U(1)$ at the self-dual radius or $SU(2)$ level 1, the underlying CFT interpretation of these two configurations is the same, and the difference corresponds in both cases to a simple current modification of the crosscap, as discussed in the next section, eqn. (3.8). In section 5 we will see that the two O0-planes in $SU(2)$ actually have opposite tension. Hence this configuration corresponds to the analogous one in the $U(1)$ picture. The other configuration is, in $SU(2)$ language, an O2-plane with non-zero tension. This corresponds in $U(1)$ language to two O0-planes with the same tension. In both cases the latter configuration is the one that one obtains from the simple current modified crosscap (3.8) with $n = 1$.

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3This example was studied prior to our work by C. Bachas et. al. [20].
3 Stringy analysis

The above geometric analysis is only valid in the large volume limit, i.e. at infinite level. At finite \( k \), we expect our results to acquire stringy corrections. Conformal field theory, being exact in the string tension, should provide these corrections.

We therefore consider the chiral algebra \( \mathcal{A} \) of a WZW model. It contains both the Virasoro algebra and an affine Lie algebra \( \mathfrak{g} \) spanned by the modes of a chiral current \( W \) given by \( (n, m \in \mathbb{Z} \text{ and } a, b = 1, \ldots, d) \)

\[
[W_n^a, W_m^b] = f_{c}^{ab}W_{n+m}^c + m\delta^{ab}\delta_{n+m-K}.
\] (3.1)

Here \( K \) is the central element whose eigenvalue is the level \( k \) and \( f_{c}^{ab} \) the structure constants of the horizontal Lie algebra \( \bar{\mathfrak{g}} \). At finite level, WZW models are rational (i.e. the number of unitary highest weight representations is finite). The unitary highest weight representations of \( \mathcal{A} \) are labeled by \( \Lambda = (\bar{\Lambda}, k) \) where \( \bar{\Lambda} \) is a highest weight representation of the horizontal subalgebra \( \bar{\mathfrak{g}} \) such that \( (\bar{\Lambda}, \Psi) \leq k \) with \( \Psi \) the highest root of \( \bar{\mathfrak{g}} \). The set of highest weights \( \bar{\Lambda} \) which obey this constraint, form the dominant affine Weyl chamber \( \mathcal{W}_k^+ \).

For closed oriented theories the chiral and anti-chiral algebras have to be combined in such a way that the torus partition function

\[
Z = \sum_{\Lambda,M} \chi_{\Lambda} Z^{\Lambda,M} \tilde{\chi}_{M}
\] (3.2)

is modular invariant. Here \( \chi_{\Lambda} \) are the affine \( \mathfrak{g} \)-characters of a representation \( \Lambda \). For modular invariance, it is necessary and sufficient to require the matrix \( Z \) to commute with the modular matrices \( S \) and \( T \); explicit expressions for these matrices are known in the case of WZW models and are given in the appendix. The invariant \( Z \) corresponds to the geometry of the space on which the strings live. When we take for \( Z \) the charge conjugation invariant, this space is the compact, simply-connected Lie group \( G \) whose Lie algebra is the compact real form of \( \bar{\mathfrak{g}} \). With the use of simple currents we can construct a large class of inequivalent invariants. Simple current invariants correspond geometrically to orbifolds of \( G \) by (subgroups of) the centre \([22, 23]\). For instance, for \( A_1 \) level \( k \), the modular invariant choices for \( Z \) come in an ADE classification. The A-series correspond to the charge conjugation invariant (which is just the diagonal invariant in this case); it describes an \( SU(2) \) manifold. The D-series are simple current invariants that describe \( SO(3) \). Finally, the E-series, that only exist for three values of \( k \), do not correspond to a simple current.

Recall that simple currents \([24]\) are primary fields whose fusion rules are such that fusion with \textit{any} other primary field yields exactly one other field and not several. The fusion rules of simple currents among themselves form an Abelian group. This group is isomorphic to the centre of \( G \)\(^4 \). We will denote the field that generates the simple current group \( H \) by \( J \), the elements by \( J^n \), \( n = 0, \ldots, N - 1 \) and the identity by 0, i.e. the vacuum representation.

\(^4\)Except for \( E_8 \) level 2 \([25]\).
An important quantity is the monodromy charge, defined by \((h_\Lambda)\) is the conformal weight of \(\Lambda\) and \(J_\Lambda\) is the unique fusion product of \(J\) and \(\Lambda\).)

\[
Q_{J}(\Lambda) := h_J + h_\Lambda - h_{J_\Lambda} \mod 1 .
\]

(3.3)

The charges of the other currents in \(H\) are related to that of the generator by \(Q_{J^n}(\Lambda) = nQ_{J}(\Lambda) \mod 1\). The monodromy charge is related to the conjugacy class \(C\) of the \(\mathfrak{g}\)-representation \(\Lambda\) as

\[
Q_{J}(\Lambda) = \frac{C(\Lambda)}{N} \mod 1 .
\]

(3.4)

To construct an unoriented theory, we project the spectrum described by (3.2) to states that are invariant under the interchange of left- and rightmovers by adding the Klein bottle partition function

\[
K = \sum_{\Lambda \in W^+} K^\Lambda_{\chi_\Lambda} .
\]

(3.5)

To ensure that the total closed unoriented spectrum, encoded in \((T + K)/2\), produces non-negative state multiplicities, \(K_\Lambda\) must be an integer that has the same parity and is bounded by \(Z_{\Lambda,\Lambda}\). The perturbative worldsheet expansion in this theory contains unoriented surfaces, i.e. surfaces with crosscaps. The worldsheet may also have boundaries. Their properties are encoded in the annulus and Möbius strip partition function:

\[
A_{ab} = \sum_{\Lambda \in W^+} A_{ab,\chi_\Lambda} , \quad M_a = \sum_{\Lambda \in W^+} M^\Lambda_{\chi_\Lambda} .
\]

(3.6)

The Chan-Paton labels \(a, b\) are used to distinguish the possible boundaries of the open strings. Open-closed string duality means that we may equivalently describe the three partition functions \(K\), \(A\) and \(M\) as the tree-level “exchange” of closed strings, propagating between boundary-states \(|B_a\rangle\) and crosscap states \(|\Gamma\rangle\):

\[
|B_a\rangle = \sum_{\Lambda \in W^+} B_{a,\Lambda}|\Lambda\rangle_B , \quad |\Gamma\rangle = \sum_{\Lambda \in W^+} \Gamma_{\Lambda}|\Lambda\rangle_\Gamma .
\]

(3.7)

We have expanded each onto a basis of Ishibashi boundary- and crosscap states respectively in order to make the conformal properties of these states manifest; \(B_{a,L}\) and \(\Gamma_{\Lambda}\) are known as the boundary and crosscap coefficients. These are not arbitrary but must obey various integrality constraints. For all simple current modifications of the charge conjugation invariants solutions to these constraints are known [11]. In case of the charge conjugation invariant, we have \(N\) solutions; one for every simple current \(J^n\):

\[
B^{(n)}_{a,\Lambda} = \frac{S_{a,\Lambda}}{\sqrt{S_{J^n,\Lambda}}} , \quad \Gamma^{(n)}_{\Lambda} = \frac{P^{J^n,\Lambda}}{\sqrt{S_{J^n,\Lambda}}} , \quad n = 0, \ldots, N - 1 ,
\]

(3.8)

where \(S_{a,b}\) is the modular \(S\)-matrix and \(P_{a,b}\) the ‘pseudo’-modular matrix equaling \(P := \sqrt{TST^2S}\sqrt{T}\). This matrix will form an important part of our story, and we will discuss
it in more detail below. Historically, the boundary state with the \( n = 0 \) coefficient is the famous Cardy state. The crosscap coefficient with \( n = 0 \) first appeared in [27]. In [10], it is shown that the above set of boundary and crosscap coefficients gives rise to a consistent unoriented spectrum for every simple current \( J^m \). Not all currents give rise to a different spectrum however. In [28], it is shown that two ‘Klein bottle currents’ \( J^m \) and \( J^m' \) produce the same spectrum if \( n + n' \) even modulo \( N \).

We claim that, just as the boundary states describe stringy D-branes, these crosscap states describe the orientifolds found in section 2 in the stringy regime. As a check, we would like to recover the semiclassical properties and positions in the large volume limit. In order to extract the position of the defects described by the crosscap state, we follow the method of [13, 29], by calculating the coupling of a ‘graviton’ to the crosscap in ‘momentum space’. Extrapolating from the \( U(1) \) free CFT and its chiral extension \( U(1)_k \), a ‘graviton’ is created by the vertex

\[
W^{a}_{-1} \tilde{W}^{b}_{-1} + W^{b}_{-1} \tilde{W}^{a}_{-1} |\mu, \tilde{\mu}\rangle_M .
\]

(3.9)

The weights \( \mu, \tilde{\mu} \) are the left- and right-‘momentum’, and required to be weights of \( M \) and \( M^c \) respectively (since we are discussing the charge conjugation modular invariant). Due to the multi-dimensionality of the ground-state of the Verma-module (by acting with the horizontal subalgebra), the range of \( \mu \) is not just the set of affine highest weight states \( W^{\pm}_{k} = \{ \Lambda \in L_{w} | (\Lambda, \Psi) < k \} \), but includes all weights \( \lambda \) found by acting with the horizontal subalgebra on \( W^{\pm}_{k} \). To find the location of the orientifold we compute the one-point function of a graviton, up to a normalization of the crosscap states which will be discussed in section 5.

\[
G^{ab}_{M}(\mu, \tilde{\mu}) = M \langle \mu, \tilde{\mu} | W^{a}_{1} \tilde{W}^{b}_{1} + \tilde{W}^{a}_{1} W^{b}_{1} | \Gamma \rangle .
\]

(3.10)

We will need the explicit form and reflection properties of the Ishibashi crosscap state

\[
W^{a}_{m} | \Lambda \rangle \Gamma = (-1)^{m} \tilde{W}^{a}_{-m} | \Lambda \rangle \Gamma , \quad | \Lambda \rangle \Gamma = \sum_{\text{levels } m} | \Lambda, m \rangle \otimes U(-1)^{m} | \tilde{\Lambda} = \Lambda, m \rangle ,
\]

(3.11)

where \( U \) is an anti-unitary operator that maps weights \( \mu \in M \) to their duals \( \mu^c \in M^c \). Here \( | \Lambda, m \rangle \) is a condensed notation for the complete set of weights of the horizontal algebra that appear at level \( m \) in the Verma module based on the highest weight \( \Lambda \). With the help of (3.11) one finds that

\[
G^{ab}_{M}(\mu, \tilde{\mu}) = -M \langle \mu, \tilde{\mu} | W^{a}_{1} W^{b}_{-1} + W^{b}_{1} W^{a}_{-1} | \Gamma \rangle
\]

\[
= -2k \delta^{ab} \sum_{\Lambda} \Gamma_{\Lambda} M \langle \mu, \tilde{\mu} | \Lambda \rangle \Gamma = -2k \delta^{ab} \delta_{\tilde{\mu}, \mu^c} \Gamma_{\Lambda} M .
\]

(3.12)

It is obvious from the computation that \( -2k \delta^{ab} \) just signifies the tensorial nature of the overlap, and that one may use any arbitrary closed string state to derive the momentum-space distribution (e.g. as in [15]). We will therefore drop this factor from now on, and call the remainder \( G^{ab}_{M}(\mu, \tilde{\mu}) \).

To find the spatial-distribution we need to “Fourier” transform. The analogue of Fourier modes for group-manifolds are the matrix elements \( \langle \lambda | R_{\Lambda}(g) | \lambda' \rangle \), where \( R_{\Lambda}(g) \) stands for the
group element $g$ in the representation $\Lambda$ and $\lambda, \lambda'$ are weights $\lambda \in \Lambda$ and $\lambda' \in \Lambda^c$. The Fourier transformation to find the distribution amounts to computing

$$\hat{G}(g) = \sum_{\Lambda} \sum_{\lambda, \lambda'} G_{\Lambda}(\lambda, \lambda') \langle \lambda | R_{\Lambda}(g) | \lambda' \rangle = \sum_{\Lambda \in \mathcal{W}_k^+} \chi_{\Lambda}(g) \Gamma_{\Lambda}.$$  (3.13)

The last step follows from the fact that highest weights $\Lambda \notin \mathcal{W}_k^+$ do not appear in the theory.

The above is simply repeating what has been done for the boundary states in [13]. There on the rhs of (3.13) the boundary coefficient, and thus the modular $S$-matrix appears instead of the crosscap coefficient which depends on the modular $P$-matrix. Mathematically it was known that the modular $S$-matrix is expressible in terms of a character of the horizontal Lie group. This meant that the boundary state analogue of (3.13) was easily evaluated by using completeness of the group characters, and the geometric analysis could proceed straightaway. Our aim in the next section will be to derive an expression for the $P$-matrix in terms of group characters of $G$ as well.

4 The $P$-matrix, characters and Gaussian sums

4.1 Properties of the $P$-matrix

Before we turn to this calculation of the $P$-matrix in WZW models, we briefly discuss some of its properties that hold for arbitrary CFT’s. As we indicated, the $P$-matrix plays an important role in the construction of open unoriented string theories [3]. The $P$-matrix encodes the channel transformation for the M"{o}bius strip, i.e. it relates the open and closed string channels of this diagram [4]. The action $P$ may be represented in terms of the modular $T$- and $S$-matrix as $P = \sqrt{TST^2S\sqrt{T}}$, with $T$ diagonal and $\sqrt{T}$ defined as $\exp[i\pi(h_a-c/24)]$.

When the CFT has a $\mathbb{Z}_N$ simple current group generated by $J$, the matrix elements of $S$ are related as follows ($n = 0, ..., N - 1$) [24, 25]:

$$S_{J^n i, j} = e^{2\pi i Q J^n} S_{i, j},$$  (4.1)

where $Q$ is the monodromy charge (3.3). A similar relation can be derived for the matrix elements of the $P$-matrix [11]

$$P_{J^{2m} i, j} = \phi(2m, i) e^{2\pi i Q J^n} P_{i, j},$$  (4.2)

where

$$\phi(2m, i) := \exp \left[ \pi i \left( h_i - h_{J^{2m}} - 2Q J^n (J^m i) \right) \right].$$  (4.3)

In the application in the main text, we have $i = 0$ or $i = J$ and $J$ integer spin, in which case the phases $\sigma(2m) := \phi(2m, 0)$, $\sigma(2m + 1) := \phi(2m, J)$ are just signs.

Note that there is no simple relation for matrix elements that are related by odd currents. This paper basically explains why: the corresponding O-plane configurations are not related by a centre translation.
4.2 The $P$-matrix in WZW models and $G$-characters

The objective of this section, to which we now return, is to find an expression for the $P$-matrix of WZW models in terms of group characters, since this enables us to find the location of the O-planes by using orthogonality of these characters.

For WZW models we know the explicit expressions for both the modular $S$- and the modular $T$-matrix. We have given them in the appendix. The $P$-matrix is therefore explicitly known. Some properties of Lie algebra characters and the different forms in which they may be expressed are also given in the appendix. Inspection shows that to find $P$ in terms of characters we must perform the matrix multiplication inherent in the definition of $P$. After substituting the formulas for the $S$ and $T$ matrix we therefore have to compute

$$P_{\Lambda,N} = N_S^2 \exp \left[ -\pi i \left( \frac{kd}{4h} \right) \right] \exp \left[ \frac{\pi i}{2h} \left( (\Lambda, \Lambda + 2\rho) + (N, N + 2\rho) \right) \right] \times \sum_{R \in \mathcal{W}_k^+} \sum_{w,v \in \mathcal{W}} \operatorname{sign}(wv) \exp \left[ -\frac{2\pi i}{h} \left( w(\Lambda + \rho) + (R, R) - (R, R + 2\rho) + (v(N + \rho), R + \rho) \right) \right].$$ (4.4)

Here $\mathcal{W}_k^+$ is the set of affine dominant integral weights, i.e. the set of weights $R$ of $G$ with nonnegative Dynkin labels such that $(R, \Psi) \leq k$, with $\Psi$ the highest root. The vector $\rho$ is the Weyl vector, $h := k + g^\vee$ the height, $g^\vee$ the dual Coxeter number, $W$ the Weyl group of $G$ and $\operatorname{sign}(w)$ equals $(-1)^{\# \text{ generators of } w}. N_S$ is the normalization of the $S$-matrix given in eq. (A.12).

We can simplify this expression by shifting the sum over $R$ to a sum over $M = R + \rho$. When $(R, \Psi) \leq k$ then $(R + \rho, \Psi) \leq k + (\rho, \Psi) = h - 1$. Furthermore, as the Dynkin labels $R_i \geq 0$, it follows from the definition of the Weyl vector that $R_i + \rho_i = R_i + 1 \geq 1$. So this shifted weight $M$ lives in the interior of the strictly dominant affine Weyl chamber $\mathcal{W}_h^{++}$. However, since the $S$-matrix vanishes at the boundaries of this chamber, we can extend the sum back to the dominant Weyl chamber $\mathcal{W}_h^+$.

$$P_{\Lambda,N} = N_S^2 \exp \left[ -\pi i \left( \frac{kd}{4h} \right) \right] \exp \left[ \frac{\pi i}{2h} \left( (\Lambda, \Lambda + 2\rho) + (N, N + 2\rho) - 4(\rho, \rho) \right) \right] \times \sum_{M \in \mathcal{W}_k^+} \sum_{w,v \in \mathcal{W}} \operatorname{sign}(wv) \exp \left[ -\frac{2\pi i}{h} \left( w(\Lambda + \rho), M \right) + (M, M) + (v(N + \rho), M) \right].$$ (4.5)

Next, for every $v$ we reorder the sum over $w$ to $vw$. Then, since Weyl reflections preserve the inner product, we can combine the sums over $M$ and $v$ to a sum over weights $M$ that live in all Weyl reflections of the dominant Weyl chamber, i.e. the set $\mathcal{W}_h$. It may appear that the fixed points of the Weyl group at the boundaries are now undercounted. However, as we have seen before, they don’t contribute to the $P$-matrix, so we can add or subtract them at will.

There is, however, a more natural range for $M$. Note that the summand is invariant under a translation $M \rightarrow M + h\alpha$ with $\alpha$ an arbitrary simple coroot. Recall that for every
element of the weight lattice $L_w$ that does not lie on the boundary of some Weyl chamber, there is a unique element of the affine Weyl group (at ‘level’ $h$) that maps it to the dominant Weyl chamber $W_h^+$. The affine Weyl group is the (semi-direct) product of the horizontal Weyl group $W$ and the group of translations by $hL^\vee$, with $L^\vee$ the coroot lattice. So every weight in $W_h = W(W_h^+)$ that does not lie at the boundary of some chamber, represents one of the elements in $L_w/hL^\vee$, where $L_w$ is the weight lattice where all boundaries are removed. So instead of summing over all elements in $W_h$, we may sum over cosets $M \in L_w/hL^\vee$, where we included the boundaries again since they do not contribute to $P$ anyway. So

$$P_{\Lambda,N} = N^2_\mathcal{S} \exp \left\{ -\frac{\pi i}{4h} \left( kd + 12(\rho,\rho) \right) \right\} \times \sum_{M \in L_w/hL^\vee \atop w \in W} \operatorname{sign}(w) \exp \left\{ -\frac{2\pi i}{h} \left[ (w(\Lambda + \rho), M) - (M, M) + (N + \rho, M) \right] \right\} .$$

We can now complete squares in the second line:

$$P_{\Lambda,N} = N^2_\mathcal{S} \exp \left\{ -\frac{\pi i}{4h} \left( kd + 12(\rho,\rho) \right) \right\} \times \sum_{M \in L_w/hL^\vee \atop w \in W} \operatorname{sign}(w) \exp \left\{ -\frac{\pi i}{h} \left( (w(\Lambda + \rho), N + \rho) \right) \exp \left\{ \frac{2\pi i}{h} \left( M - \frac{w(\Lambda + \rho) + N + \rho}{2} \right)^2 \right\} .$$

Upon using the strange formula for the length of $\rho$ (recall that we are using the normalization $(\Psi,\Psi) = 2$)

$$(\rho,\rho) = \frac{1}{12}dg^\vee,$$

we can write this as

$$P_{\Lambda,N} = N^2_\mathcal{S} \exp \left\{ -\frac{\pi i}{4h} \left( kd + 12(\rho,\rho) \right) \right\} \sum_{w \in W} \operatorname{sign}(w) \exp \left\{ -\frac{\pi i}{h} \left( (w(\Lambda + \rho), N + \rho) \right) \sum_{M \in L_w/hL^\vee} \exp \left\{ \frac{2\pi i}{h} \left( M - \frac{X}{2} \right)^2 \right\} ,$$

where we introduced the lattice Gaussian sum on $L_w$ with metric $C^{-1}$.

$$\mathcal{G}_h(C^{-1}, X) := \sum_{M \in L_w/hL^\vee} \exp \left\{ \frac{2\pi i}{h} \left( M - \frac{X}{2} \right)^2 \right\} .$$

In the next section, see equation (4.32), we prove that this sum equals

$$\mathcal{G}_h(C^{-1}, X) = \left\| \frac{L_w}{L^\vee} \right\|^{1/2} \left( \frac{hi}{2} \right)^{r/2} \sum_{u \in L^{\vee}/2L^{\vee}} (-1)^{\hat{u}(u,u)+(u,X)},$$

where $u$ is the set of all coroots modulo even coroots. In other words, $u$ is the set of vectors $u = \sum_i u_i\tilde{\alpha}^i$ on the coroot lattice $L^\vee$ whose entries $u_i$ are either 0 or 1. Hence
\[(u, X) = \sum_i u_i X^i \text{ where } X^i \text{ are the Dynkin labels, and } (u, u) = \sum_{j=1}^r u_i C^{ij} u_j \text{ where } C \text{ is the inner product matrix of the coroots } C^{ij} = (\tilde{\alpha}^i, \tilde{\alpha}^j).\]

Taking everything together, we find

\[P_{\Lambda,N} = \mathcal{N}_P \sum_{u \in L^\vee/2L^\vee} (-1)^{\frac{h}{2}(u,u)+(u,N+\rho)} \sum_{w \in W} \text{sign}(w) \exp \left[ -\frac{\pi i}{h} (w(\Lambda + \rho), N + \rho + hu) \right] \]

with

\[\mathcal{N}_P := 2^{-r/2} \mathcal{N}_S = i^{(d-r)/2} |L_w/L^\vee|^{-1/2} \frac{1}{\sqrt{(2h)^r}}. \quad (4.13)\]

Two results immediately follow from eq. (4.12). Using the Weyl character formula (A.3) \( P \) can be seen to equal the sum of characters \( X_\Lambda \) of the horizontal subgroup,

\[P_{\Lambda,N} = \sum_u \mathcal{N}_u^{(N)} X_\Lambda (g_u^N) \quad (4.14)\]

evaluated at the group element

\[g_u^N = \exp \left\{ -\frac{\pi i}{h} (N + \rho + hu, H) \right\}, \quad (4.15)\]

where \( \mathcal{N}_u^{(N)} \) equals

\[\mathcal{N}_u^{(N)} := \mathcal{N}_P (-1)^{\frac{h}{2}(u,u)+(u,N+\rho)} \sum_{w \in W} \text{sign}(w) \exp \left\{ -\frac{\pi i}{h} (w(\rho), N + \rho + hu) \right\}. \quad (4.16)\]

And, due to the formal equivalence between characters \( X_\Lambda \) and the modular \( S \) matrix, one derives a linear relation between \( S \) and \( P \),

\[P_{\Lambda,N} = 2^{-r/2} \sum_u (-1)^{\frac{h}{2}(u,u)+(u,N+\rho)} S_{\Lambda,N+\rho+hu} \quad (4.17)\]

This relation exists only in the formal sense, as the \( S \)-matrix for non-integer weights is only defined by analytic continuation through its equivalence with characters. Nonetheless the very existence of these relations is remarkable. Both, however, hinge on the proof of eq. (4.11) with which we now proceed.

### 4.3 Gaussian sums for Lattices

The Gaussian sum \( g_k(a,0) \) is the discrete analog of the Gaussian integral:

\[g_k(a,0) = \sum_{n=0}^{k-1} e^{\frac{2\pi i a n^2}{k}}. \quad (4.18)\]
Gaussian sums play a role in analytic number theory (for a review, see [30]). Their main application is to count the number of solutions to the congruence \( x^2 \equiv a \pmod{p} \); the number \( a \) is called a quadratic residue mod \( p \).

This sum has been evaluated to
\[
g_{mk}(a, 0) = mg_k(a, 0), \quad \gcd(ma, mk) = m\epsilon \mathbb{Z},
\]
\[
g_k(a, 0) = \left( \frac{a}{k} \right) g(1, k), \quad \gcd(a, k) = 1,
\]
\[
g_k(1, 0) = \frac{1}{2}(1 + i)(1 + i^{-k})\sqrt{k}, \quad (4.19)
\]
where \( \left( \frac{a}{k} \right) \) is the Legendre symbol. Decomposing \( k \) into primes \( k = \prod p_i \), it equals the product of Jacobi symbols,
\[
\left( \frac{a}{k} \right) = \prod_{i; k = \prod p_i} \left( \frac{a}{p_i} \right),
\]
\[
\left( \frac{a}{p} \right) = \begin{cases} 1 & \text{if } x^2 = a \pmod{p} \text{ has a solution for } x \in \mathbb{Z} \\ -1 & \text{if not} \end{cases}. \quad (4.20)
\]

For purposes of comparison, we also give the Gaussian sum over half integers
\[
g_k(a, \frac{1}{2}) = \sum_{n=\frac{1}{2}}^{k-\frac{1}{2}} e^{\frac{2\pi i}{k} a n^2} = \frac{1}{2} g(a, 4k) - g(a, k). \quad (4.21)
\]
For \( a = 1 \) this is simply
\[
g_k(1, \frac{1}{2}) = \frac{1}{2}(1 + i)(1 - i^{-k})\sqrt{k}. \quad (4.22)
\]
For \( SU(2) \) the weight lattice is one-dimensional and the sum in (4.3) equals a regular Gaussian sum. For higher rank groups we will need a lattice generalization of this sum.

Consider an even lattice \( L^\vee \) with volume \( |L^\vee| \) and dual lattice \( L_w \). In the application we have in mind, \( L^\vee \) is the coroot lattice of a Lie algebra \( \bar{g} \) and \( L_w \) the weight lattice, but the following discussion applies to arbitrary even lattices. The restriction to even lattices is important, since only then is \( L^\vee \) (and every integer multiple \( hL^\vee \)) a sublattice of \( L_w \). The quadratic Gaussian sum on \( L_w/hL^\vee \) is defined as
\[
G_h(C^{-1}, X) := \sum_{\mu \in L_w/hL^\vee} \exp \left\{ \frac{2\pi i}{h} \left( \frac{1}{2} \right)^2 \right\}
\]
\[
= \sum_{m'} \exp \left[ \frac{2\pi i}{h} (m_i - \frac{X_i}{2})C^{-1}_{i,j} (m_j - \frac{X_j}{2}) \right], \quad (4.23)
\]
where \( X \) is an integral weight and \( |y|^2 \) is the length squared of \( y \), computed with the lattice metric \( C^{-1} \) of \( L_w \). The precise summation range of \( m_i \) follows from the first form of the
expression. In order to evaluate this sum, it is convenient to define ($\epsilon > 0$)

$$\tau := h - i\epsilon \rightarrow \frac{1}{\tau} = \frac{1}{h} + \frac{i\epsilon}{h^2} + \mathcal{O}(\epsilon^2).$$  \hspace{1cm} (4.25)

Consider then the sum over the full weight lattice

$$\Xi(\tau) := \sum_{\lambda \in L} \exp \left\{ \frac{2\pi i}{\tau} \left| \frac{\mu - X}{2} \right|^2 \right\},$$

which due to the small regulating parameter introduced in (4.25) is now finite and well defined (it is a theta function). We can write $\Xi$ in two different ways. In the first, we split the sum over $\lambda$ in a $\epsilon$-dependent sum over $h\beta$, $\beta \in L^\vee$ and a $\epsilon$-independent sum over $5\mu \in L_w/hL^\vee$:

$$\Xi(\tau) = \sum_{\mu \in L_w/hL^\vee} \exp \left[ \frac{2\pi i}{h} \left| \frac{\mu - X}{2} \right|^2 \right] \sum_{\beta \in L^\vee} \exp \left[ -2\pi \epsilon \left| \beta + \frac{\mu - X/2}{h} \right|^2 + \mathcal{O}(\epsilon^2) \right].$$  \hspace{1cm} (4.27)

The last sum can be approximated by a Gaussian integral that to leading order in $\epsilon$ does not depend on $\mu - X/2$, and we recognize the Gaussian sum in the prefactor:

$$\Xi(\tau) = G_h(C^{-1}, X) \left[ \frac{1}{\sqrt{\epsilon}} \int dx e^{-2\pi x^2} + \mathcal{O}(1) \right].$$  \hspace{1cm} (4.28)

On the other hand, we can first use Poisson’s resummation formula (which is allowed since $\epsilon > 0$) to write the sum (4.26) as

$$\Xi(\tau) = |L_w|^{-1} \left( \frac{i\tau}{2} \right)^{r/2} \sum_{\alpha \in L^\vee} \exp \left[ -\frac{\pi i}{2} |\alpha|^2 \right] \exp \left[ 2\pi i \left( \frac{X}{2}, \alpha \right) \right],$$

and then split the sum over $\alpha$ in a $\epsilon$-dependent sum over $2\beta$, $\beta \in L^\vee$ and a $\epsilon$-independent sum over $u \in L^\vee/2L^\vee$

$$\Xi(\tau) = |L_w|^{-1} \left( \frac{i\tau}{2} \right)^{r/2} \sum_{u \in L^\vee/2L^\vee} (-1)^{(u,X)+h(u,u)} \sum_{\beta \in L^\vee} \exp \left[ -2\pi \epsilon |\beta - u/2|^2 \right].$$  \hspace{1cm} (4.30)

As before, we can approximate the sum over $\beta$ by an integral that does not depend on $u/2$ to leading order in $\epsilon$:

$$\Xi(\tau) = |L_w|^{-1} \left( \frac{hi}{2} \right)^{r/2} \sum_{u \in L^\vee/2L^\vee} (-1)^{(u,X)+h(u,u)} \left[ \frac{1}{\sqrt{\epsilon}} \int dx e^{-2\pi x^2} + \mathcal{O}(1) \right].$$  \hspace{1cm} (4.31)

\footnote{We have to be careful here, since the summand is not independent of the representative of the coset $\mu \in L_w/hL^\vee$. We must therefore choose a particular set of representatives and this is implicit in the notation. In the final answer, equation (4.32), this ambiguity disappears.}

\footnote{Again, at this stage we must choose a representative for $u$ in order for this expression to be well-defined.}
We can now equate the two results (4.28) and (4.31). Comparing the $O(1/\sqrt{\epsilon})$ terms, we find for the quadratic Gauss sum

$$G_h(C^{-1}, X) = \left| \frac{L_w}{L^\vee} \right|^{1/2} \left( \frac{ih}{2} \right)^{r/2} \sum_u (-1)^{(u,X)+\frac{1}{2}(u,u)} ,$$

(4.32)

where we used $|L_w| = |L^\vee|^{-1}$ in obtaining the final equation. For rank one $SU(2)$ this indeed reduces to the known results (4.20) and (4.22).

To appreciate how remarkable this result is note that we started with a summation of gaussians involving the integrable weights at a certain level. Yet the result depends on the level in a very simple way, namely only through the volume of the affine Weyl chamber. Although not shown here explicitly, the dependence on $X$ is also very simple. A priori it is clear that the result only depends on the classes $L_w/2L_w$, and is constant on the Weyl orbits of these classes (the number of distinct classes is in fact precisely given by the entries in the last column of table II). If $h$ is even, the summation over $u$ yields $2^r$ for $X=0$, and zero if $X$ belongs to a non-trivial class. For odd $h$ the result is slightly more complicated. The summation can take three values on the classes, either 0 or $\pm 2^l$, with $l \leq r$. There are many more interesting features we could mention, and undoubtedly more still to be discovered, but let us return now to the main subject of this paper.

5 The semiclassical limit

We are now in a position to show that there is a one-to-one correspondence between the O-planes constructed in section II and the crosscap states of section III. We will do this by demonstrating that the distribution (3.13), that gives the location of the defects described by the crosscap states in the large volume limit, is peaked at the solutions $g_{n,u}$ of equation (2.5). Let us first consider the charge conjugation invariant. There is a normalization issue we have suppressed up till now. In the more extensively studied case of D-branes, the graviton-crosscap coupling is not directly given by the boundary coefficients, but by a reflection coefficient $R_{\Lambda,a}$ that differs from it by a factor $[31, 32]$. By “boundary coefficient” we concretely mean here the coefficients $B$ that appear in the expression of the annulus as

$$A_{iab} = \sum_\Lambda S_{i\Lambda}B_{\Lambda a}B_{\Lambda b},$$

and that are determined by integrality. These are the coefficients presented in (3.8). The additional factors come from the definition of the annulus amplitude and the normalization of the identity near a boundary. In the Cardy case, the reflection coefficients are $S_{\Lambda a}/S_{0a}$, and the boundary coefficients $S_{\Lambda a}/\sqrt{S_{\Lambda 0}}$. We should expect similar factors to appear between the crosscap states $\Gamma_{\Lambda}$ (B.8) and the O-plane reflection coefficients. The latter we will denote as $U_{\Lambda}$. An additional complication is the fact that we have to know these factors for non-zero values of $J$. Our proposal is to use the following definitions

$$R_{a,\Lambda}^{(n)} = \frac{S_{a,\Lambda}}{S_{0a}}, \quad U_{\Lambda}^{(n)} = \frac{P_{n,\Lambda}}{S_{00}} , \quad n = 0, \ldots, N - 1 ,$$

(5.1)

It may appear counter-intuitive that for $n \neq 0$ the phases from the denominator in (3.8) were dropped. The rationale behind this is that the D-brane reflection coefficients should
not change if we modify the O-plane reflection coefficients, by going from $P_{0\Lambda}$ to $P_{J^n,\Lambda}$. On the other hand, at the one-loop level, the Möbius strip amplitude does change, and this in its turn enforces (through open sector integrality) a change in the annulus amplitude. The latter change corresponds always to a different choice of the boundary conjugation matrix $A^0_{ab}$. This can be taken into account by extra phases in the boundary coefficients. In other words, the phases in the denominators in (3.8) belong naturally in the boundary and crosscap coefficients that enter in the definition of the one-loop partition functions, but not in the reflection coefficients. Obviously this is a conjecture, and not a proof. The fact that this definition will turn out to make sense geometrically gives additional support for this conjecture, and we hope that it may help resolve a long-standing debate on the proper description and the origin of these phases. In fact, we will see that the geometric picture gives a nice way of understanding boundary conjugation.

The precise distribution that we therefore wish to evaluate is

$$\tilde{G}(n)(g) = \sum_{\Lambda \in W^+_g} U^{(n)}_{\Lambda} \mathcal{X}(g)$$  \hspace{1cm} (5.2)$$

rather than eq. (3.13). In the infinite volume limit, $k \to \infty$, the sum is over all highest weight representations of $\tilde{g}$. We can then use completeness of the group characters, provided we have an expression for $P_{J^n,\Lambda}$ as a sum of $G$-characters. This is the very fact we accomplished in (4.14),

$$P_{J^n,\Lambda} = \sum_u \mathcal{N}_u^{(n)} \mathcal{X}(g_u^{(n)})$$,  \hspace{1cm} (5.3)$$

where

$$\mathcal{N}_u^{(n)} := \mathcal{N}_P(-1)^{\frac{h}{2}(u, u)+(u, J^n + \rho)} \sum_{w \in W} \text{sign}(w) \exp \left\{-\frac{\pi i}{h} (w(\rho), J^n + \rho + hu)\right\}$$,  \hspace{1cm} (5.4)$$

and the character in the $P$-matrix is evaluated at the group element

$$g_u^{(n)} := \exp \left\{-\frac{\pi i}{h} (J^n + \rho, H)\right\} \exp \left\{-\pi i (u, H)\right\}.$$  \hspace{1cm} (5.5)$$

We will start with $n = 0$. In the infinite volume limit, the argument of the characters in (5.3) becomes (we ignore $\rho/h$)

$$g_u^{(0)} \xrightarrow{h \to \infty} e^{-\pi i (u, H)} = g_u,$$  \hspace{1cm} (5.6)$$

where $g_u$ is the solution to (2.7) for $n = 0$. To work out the normalizations we make use of the denominator identity,

$$\sum_{w \in W} \text{sign}(w)e^{(w(\rho), \lambda)} = \prod_{\text{positive roots } \alpha} \left(2 \sinh \left[\frac{(\alpha, \lambda)}{2}\right]\right),$$  \hspace{1cm} (5.7)$$

applied to the ratio

$$\frac{\sum_{w \in W} \text{sign}(w) \exp \left\{-\frac{\pi i}{h} (w(\rho), \rho + hu)\right\}}{\sum_{w \in W} \text{sign}(w) \exp \left\{-\frac{2\pi i}{h} (w(\rho), \rho)\right\}}.$$  \hspace{1cm} (5.8)$$
The result is
\[
\prod_{\text{positive roots } \alpha} \frac{\sin \left( \frac{\pi}{2} (\alpha, \rho + h u) \right)}{\sin \left( \frac{\pi}{h} (\alpha, \rho) \right)} .
\] (5.9)

If \((\alpha, u)\) is even, the \(u\) term may be dropped, and in the limit \(h \to \infty\) the corresponding factor in the product contributes a factor \(\frac{1}{2}\). This is obviously the correct answer if \(u = 0\):
\[
\frac{N_0^{(0)}}{S_{0,0}} \to Q_0^{(0)} := 2^{-r/2} 2^{-(d-r)/2} = 2^{-d/2} ,
\] (5.10)

If on the other hand \((\alpha, u)\) is odd (since \(u\) is a co-root it is integer) the numerator contributes \(\sin(\frac{1}{2} \pi (\alpha, u))\), which remains finite in the limit \(h \to \infty\). We can write the limit of the ratio as follows
\[
\frac{\sin \left( \frac{\pi}{2} (\alpha, \rho + h u) \right)}{\sin \left( \frac{\pi}{h} (\alpha, \rho) \right)} \to e^{\frac{1}{2} \pi i (\alpha, u)} \times \left\{ \begin{array}{ll}
\frac{1}{2} & \text{for } (\alpha, u) \text{ even} \\
\frac{h}{\pi i (\alpha, \rho)} & \text{for } (\alpha, u) \text{ odd}
\end{array} \right.
\] (5.11)

The result is then
\[
\frac{N_u^{(0)}}{S_{0,0}} \to Q_u^{(0)} := 2^{D/2} (-1)^{\frac{1}{2} (\alpha, u)} \left[ \frac{h}{\pi i} \right]^{D/2} \prod_{\alpha, \text{odd}} \left( \frac{1}{(\alpha, \rho)} \right) Q_0^{(0)} ,
\] (5.12)

where the product is over all positive roots with \((\alpha, u)\) odd, and where \(D\) is the dimension of the O-plane. This is equal to \(D = d - r - M\), where \(M\) is the number of roots with \((\alpha, u)\) even. The number of “odd” positive roots is equal to \(\frac{1}{2} (d - r) - \frac{1}{2} M = \frac{1}{2} D\). Since we started from a real expression, the result is of course real. This implies that \(D\) must be a multiple of four, as indeed it is according to table 2 (with \(n = 0!\)). Note that the result diverges for large \(h\) as \(h^{D/2}\). Obviously this is a volume factor. At finite \(h\), previous results on D-branes suggests that the branes are fuzzy. Hence even an O0 plane, a point, has then a finite volume. But as \(h\) tends to infinity, the ratio of the volume of a higher dimensional object and that of a point goes to infinity. This phenomenon has not been observed for WZW D-branes for the simple reason that they all occupy regular conjugacy classes of the same dimension \(d - r\) [13]. Although we believe we correctly understand the relative normalization, the overall normalization is another matter, as is the relative normalization between D-branes and O-planes. Overall normalizations of amplitudes are tricky in general, and even more so in our case where one does not consider a genuine string theory, so that ‘graviton scattering’ is only called that way by analogy. We will not try to address this issue here and turn instead to the main goal, the localization of the planes, for which the precise normalization is of little relevance.

The distribution (5.2) can now easily be computed using completeness of the group characters:
\[
\tilde{G}^{(0)}(g) = \sum_u Q_u^{(0)} \sum_{\Lambda} \mathcal{X}_\Lambda(g_u) \mathcal{X}_\Lambda(g)
= \sum_u Q_u^{(0)} \delta(g_u - g) .
\] (5.13)
This shows that the crosscap state $|\Gamma^{(0)}\rangle$ corresponds to a localized charge distribution in curved geometries and that it provides a conformal field theory description of the $n = 0$ orientifold planes of section 2. Note that, since the Weyl group preserves inner products, the contributions to the distribution from characters $\chi_\Lambda$ evaluated at $g_u$ and $g_{w(u)}$ have the same coefficient $Q_u^{(0)}$. This is what one should expect, since this factor can be interpreted as a charge (in fact a tension) and $g_u$ and $g_{w(u)}$ describe the same plane, as we discussed in section 2. The delta function is actually a $\delta$-function on conjugacy classes, not on individual group elements. The sum on $u$ usually contains several members of the same conjugacy class, thus producing a multiplicity factor which one may absorb in the definition of $Q_u^{(0)}$. Note that the distribution we get is peaked at all locations of O-planes corresponding to $n = 0$, in contrast to the situation for D-branes, which live on a single conjugacy class. This reflects the well-known fact that D-branes come with Chan-Paton multiplicities which can be switched on or off at will, whereas there is no such freedom for O-planes: one can either choose to have one or not to have one, but one cannot distribute them freely over the available locations.

In a similar way, the crosscap states $|\Gamma^{(n\not=0)}\rangle$ describe O-planes that correspond to a non-trivial element of the centre. First take $n$ even, $n = 2m$. Now (note that $\sigma(2m)$ is a sign, see the discussion after equation (4.3))

$$P_{J^{2m},A} = \sigma(2m)e^{2\pi i Q_{Jm}(\Lambda)} P_{0,\Lambda} = \sigma(2m) \sum_u N_u^{(0)} e^{2\pi i \frac{ma(\Lambda)}{N}} \chi_\Lambda(g_u^{(0)}) = \sigma(2m) \sum_u N_u^{(0)} \chi_\Lambda(\gamma^m g_u^{(0)}),$$  

(5.14)

where we used (4.4) and (A.5). In the large volume limit, $g_u^{(0)} \to g_u$ and the distribution becomes

$$\tilde{G}^{(2m)}(g) = \sigma(2m) \sum_u Q_u^{(0)} \delta(\gamma^m g_u - g).$$  

(5.15)

So the corresponding O-plane configurations are translations of the canonical ones by $\gamma^m$, up to an overall sign. When we compare this with the discussion in section 2, we conclude that the crosscap state $|\Gamma^{(2m)}\rangle$ describes the O-plane configuration with centre element $\gamma^{2m}$ in the defining orientifold group.

Now the computation for $n = 1, N$ even, remains. Following the same steps in eq. (5.14), we can show that for $n = 2m + 1$ the crosscap reflection coefficient is

$$P_{J^{2m+1},A} = \sigma(2m + 1) \sum_u N_u^{(1)} \chi_\Lambda(\gamma^m g_u^{(1)}).$$  

(5.16)

Note that in the large $h$ limit, $J^n/h$ is finite. Hence in the large volume limit, the group element (5.5) approaches, for $n = 1$,

$$g_u^{(1)} \to \exp \left\{ -\pi i \frac{1}{N} \sum_{i=1}^r c_i H^i \right\} \exp \left\{ -\pi i (u, H) \right\} = \sqrt{\gamma} g_u,$$  

(5.17)
where $\sqrt{\gamma}$ is a group element that squares to the generator of the centre $\gamma$. In obtaining this result, we have written $(J, H)$ in Dynkin basis as $\sum_{ji} J_j C_{ji}^{-1} H_i$. The Dynkin labels of a simple current are always of the form $J_j = k \delta_{qj}$ for some $q$. Now we observe the following property of the quadratic form matrix $C^{-1}$: for all simple Lie algebra's the column vector $C_{iq}^{-1}$ equals

$$C_{iq}^{-1} = \frac{c_i}{N} \mod 1,$$

as can be shown by simple inspection of the tables in for instance [18]. From this relation between the quadratic form matrix and the conjugacy classes, it follows that $\sqrt{\gamma}$ indeed squares to $\gamma$ and that $g^{(1)}_u$ approaches the solutions (2.14) in the large volume limit. To compute the tension in the large volume limit we need to consider

$$Q_u^{(1)} := \lim_{h \to \infty} \frac{N_u^{(1)}}{S_{0,0}} = \lim_{h \to \infty} \frac{N_u^{(1)}}{S_{0,j}} = \lim_{h \to \infty} 2^{-r/2} (-1)^{h(u,u)+(u,J+J)} \prod_{\alpha > 0} \frac{\sin \left[ \frac{\pi}{2} \gamma(\alpha, \rho + J + hu) \right]}{\sin \left[ \frac{\pi}{2} \gamma(\alpha, \rho + J) \right]}$$

The $J$ in the denominator is introduced just for convenience. At level $k$ the simple currents can be written as $k J_1 = (h - g^\vee) J_1$, where $J_1$ is a fundamental weight. The ratio of the sines can then be written as follows

$$\frac{\sin \left[ \frac{\pi}{2} \gamma(\alpha, \rho + J + hu) \right]}{\sin \left[ \frac{\pi}{2} \gamma(\alpha, \rho + J) \right]} = \frac{\exp \left( \frac{1}{2} \pi i (\alpha, J_1 + u) \right) [\exp(\frac{\pi i y}{2}) - \exp(\pi i (\alpha, J_1 + u))] \exp(-\frac{\pi i y}{2})}{\exp(\pi i (\alpha, J_1)) [\exp(\frac{\pi i y}{2}) - \exp(-\frac{\pi i y}{2})]}$$

where $y = (\alpha, \rho - g^\vee J_1)$. Using (5.18) the inner product $(\alpha, J_1)$ equals

$$(\alpha, J_1) = \sum_{i,j} \alpha_i C_{ij}^{-1}(J_1) = \sum_i \alpha_i c_i / N \mod 1$$

Shifting the values of $c_i$ by $N$ does not affect the corresponding centre element. Therefore it is possible to define $c_i$ in such a way that the last relation holds exactly, not just modulo integers. Then we get, using (2.8), $(\alpha, J_1 + u) = 2(\alpha, t)$. For the ratio we find then

$$e^{\frac{1}{2} \pi i (\alpha, J_1 + u)} e^{-\pi i (\alpha, J_1)} \times \begin{cases} \frac{1}{2} \frac{h}{\pi i (\alpha, y)} & \text{for } (2(\alpha, t) \text{ even)} \\ \frac{1}{\pi i (\alpha, y)} & \text{for } (2(\alpha, t) \text{ odd)} \\ 
$$

Putting everything together and using $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$ we find

$$Q_u^{(1)} = 2^{D/2} 2^{-d/2}(-1)^{h(u,u)+(u,J)} e^{-\pi i (\rho, J_1)} \left[ \frac{h}{\pi i} \right]^{D/2} \prod_{\alpha, \text{odd}} \left[ \frac{1}{(\alpha, \rho - g^\vee J_1)} \right],$$

where $\alpha, \text{odd}$ means “all positive roots with $2(\alpha, t) \text{ odd}”$. As in the case $n = 0$ the tension diverges as $h^{D/2}$. In this case $D$ is not always a multiple of 4, but the factor $e^{-\pi i (\rho, J_1)}$ is purely imaginary precisely if $D = 2 \mod 4$, so that the final result is real.
The distribution becomes
\[ \tilde{G}(g)_{2m+1} = \sigma(2m + 1) \sum_u Q_u^{(1)} \delta(\gamma^m \sqrt{\gamma} g_u - g). \] (5.24)

This shows that \(|\Gamma(2m+1)|\) describes the O-planes with a nontrivial centre element \(\gamma^{2m+1}\) in the defining orientifold group. We have thus succeeded in showing that the geometric interpretation of simple current crosscap states for the charge conjugation invariant of a WZW model are the centre O-plane configurations on a compact, simply-connected Lie group manifold. The exact CFT computation has also given us the (relative) charges of each.

5.1 Planes in Orbifolds

We will now consider a WZW model with a simple current modular invariant. To be more precise, we will only consider invariants that are extensions of the chiral algebra [24]. Such an invariant can be built with the use of an integer spin (conformal weight) simple current group \(H\), that is a subgroup of the centre. We take \(H = \mathbb{Z}_M\), where \(M\) a divisor of \(N\) and denote the generator by \(L := J^{N/M}\). Only fields that have monodromy charge zero with respect to all currents in \(H\) contribute to the torus partition function. The geometric interpretation of this invariant is that of strings living on an orbifold \(G/H\). Only representations of \(G\) that are not-faithful with respect to the conjugacy classes corresponding to \(H\) appear as representations of this coset. Given the relation between the simple current charges and conjugacy classes, equation (3.4), it follows that all representations of \(G/H\) appear in the torus partition function.

For simple current modifications of the charge conjugation invariant, the crosscap coefficients are [11]
\[ \Gamma_{\Lambda} = \frac{1}{\sqrt{|H|}} \sum_{n=0}^{M-1} \eta(n) \frac{P_{L^n, \Lambda}}{\sqrt{S_{0, \Lambda}}} . \] (5.25)

The signs \(\eta(n)\) are constrained by the requirement that \(\Gamma_{\Lambda}\) vanishes for charged fields:
\[ \eta(n) = \eta(n + 2m) \exp[\pi i (h_{L^n} - h_{L^{n+2m}})] , \] (5.26)
for all \(m = 0, ..., M\).

We now want to give a geometrical interpretation of the crosscap states for this theory, based on the crosscap coefficients in equation (5.25) and the solutions of the sign-rule (5.26). First note that the distribution corresponding to this crosscap state is still given by (5.2), simply because the crosscap coefficient vanishes for those \(\Lambda\) that are not representations of the orbifold when the \(\eta\) satisfy the sign-rule. Again, we must make a distinction between the crosscap coefficient and the graviton-crosscap coupling. Following our previous argument, we presume the last is given by:
\[ U_{\Lambda} = \frac{1}{\sqrt{|H|}} \sum_{n=0}^{M-1} \eta(n) \frac{P_{L^n, \Lambda}}{S_{00}} . \] (5.27)
Let us first consider $M$ odd. We can rewrite the sum over $P$-matrix elements as follows:

\[
\sum_{n=0}^{M-1} \eta(n) P_{L^n,\Lambda} = \sum_{m=0}^{M-1} \eta(2m) P_{L^{2m},\Lambda} = \eta(0) \sum_{m=0}^{M-1} e^{2\pi i Q_L^m(\Lambda)} P_{0,\Lambda} = \eta(0) \sum_{m=0}^{M-1} \sum_u N_u^0 \chi_\Lambda(\gamma^m g_u^{(0)}) .
\]

(5.28)

In the first step we used that all currents in $\mathbb{Z}_{\text{odd}}$ groups are even. In the second step, we combined the signrule (5.26) and the identity (4.2). In the last step, we used our character formula (4.12) for the $P$-matrix and used equation (A.5) from the appendix. In the limit $h \to \infty$, the group element $g_u^{(0)} \to g_u$, so the distribution $\tilde{G}$ becomes

\[
\tilde{G}(g) = \frac{\eta(0)}{\sqrt{|H|}} \sum_{m=0}^{M-1} \sum_u Q_u^{(0)} \delta(\gamma^m g_u - g) .
\]

(5.29)

Hence this crosscap state describes an $H$-translation invariant combination that survives the orbifold projection. The overall sign $\eta(0)$ can be interpreted as the overall sign of the charges (tensions) of the O-planes.

For $M$ even, we have to make a distinction between even and odd currents:

\[
\sum_{n=0}^{M-1} \eta(n) P_{L^n,\Lambda} = \sum_{m=0}^{M/2-1} \eta(2m) P_{L^{2m},\Lambda} + \sum_{m=0}^{M/2-1} \eta(2m+1) P_{L^{2m+1},\Lambda} = \eta(0) \sum_{m=0}^{M/2-1} \sum_u N_u^{(0)} \chi_\Lambda(\gamma^m g_u^{(0)}) + \eta(1) \sum_{m=0}^{M/2-1} \sum_u N_u^{(1)} \chi_\Lambda(\gamma^m g_u^{(1)}) .
\]

(5.30)

In the large volume limit, this crosscap state describes a configuration of O-planes given by

\[
\tilde{G}(g) = \frac{\eta(0)}{\sqrt{|H|}} \sum_{m=0}^{M/2} \sum_u Q_u^{(0)} \delta(\gamma^m g_u - g) + \frac{\eta(1)}{\sqrt{|H|}} \sum_{m=0}^{M/2} \sum_u Q_u^{(1)} \delta(\gamma^m \sqrt{g_u} - g) .
\]

(5.31)

This describes two $H$-invariant combinations, if we take into account that every $m$ in the sum describes a $\mathbb{Z}_2$ translation-invariant configuration. The relative sign is actually the relative sign of the tensions of the two $H$-invariant configurations, and is therefore relevant.

Similar remarks apply for general simple current invariants. The crosscap state is always of the form (3.8). The number of free signs equals the number of even cyclic factors $p$ in the simple current group, so we have $2^p$ different crosscaps. In geometric terms, this is the number of relative tensions between the $1+p$ translation invariant combinations of O-plane configurations.
6 Branes and Planes

The geometry of D-branes in open oriented WZW models has been discussed in [12, 13, 14, 15]. What we would like to stress here, is that this geometry differs in the unoriented open string. As reviewed in section 3, boundaries in CFT are described by a boundary state $|B_a\rangle$. The boundary and crosscap states must be such that the open unoriented string partition function $(\mathcal{A} + \mathcal{M})/2$ produces nonnegative integers. Due to the Möbius projection, the set of boundary labels $a$ is reduced; some boundaries are identified and some boundaries are projected. More precisely, define the boundary conjugate $a^*$ of $a$ by

$$A^0_{ab} = \begin{cases} 1 & \text{when } b = a^* \\ 0 & \text{when } b \neq a^* \end{cases},$$ (6.1)

then complex boundaries $a \neq a^*$ are identified and real boundaries are projected. In terms of Chan-Paton gauge groups, on a complex pair of boundaries lives a unitary group. Real boundaries carry a symplectic or orthogonal gauge group, depending on whether the Möbius coefficient $M^0_a$ is plus or minus one.

In [11], the boundary and crosscap coefficients (and therefore the annulus and Möbius strip coefficients) for arbitrary simple current invariants are presented (see [28, 33] for proofs). We will discuss the charge conjugation invariant first. The boundary labels $a$ are in one-to-one correspondence with the primary fields (so we label them by $A$). Recalling the boundary coefficients corresponding to the crosscap coefficients (3.8)

$$B_{\Lambda,A}^{(n)} = \frac{S_{\Lambda,A}}{\sqrt{S_{\Lambda,J^n}}},$$ (6.2)

we can calculate the boundary conjugation matrix

$$A^* = J^n A^c .$$ (6.3)

The geometric interpretation of these branes can be derived in a similar way as was done in [13]: calculate the brane-graviton coupling and Fourier transform this to a distribution on group space. Again the coupling of a graviton to a brane is given by the boundary reflection coefficient [13] $R_{\Lambda,A}$, which differs from the boundary coefficient by a normalization. The reflection coefficient is exactly equal to a character of the Lie group $G$

$$R_{\Lambda,A} = \frac{S_{\Lambda,A}}{S_{0,A}} = \chi_{\Lambda}(g_A)$$ (6.4)

evaluated at

$$g_A = \exp\left\{\frac{-2\pi i}{\hbar} \sum_{i=1}^{r} ((A + \rho), H)\right\} .$$ (6.5)

For a boundary $A$ the distribution that gives the localization of the defect in the limit of infinite level is given by

$$\tilde{G}_A(g) = \sum_{\Lambda} R_{\Lambda,A} \chi_{\Lambda}(g)$$
\[ \sum_{\Lambda} \mathcal{X}_{\Lambda}(g_A) \mathcal{X}_{\Lambda}(g) = \delta(g_A - g) \quad (6.6) \]

where \( g_A \) is the group element in the \( h \to \infty \) limit:

\[ g_A \to \exp \left\{ 2\pi i \sum_{i=1}^{r} (\alpha, H) \right\}, \quad (6.7) \]

where we keep the relative weight \( \alpha^i = A^i / h \in \mathbb{R} \) fixed in taking the limit. To obtain the location of the conjugate boundary, note that

\[ S_{\Lambda, R_n A^c} = e^{2\pi i Q_{r_n A^c}} S_{\Lambda, A}^* = e^{2\pi i \sum_{\Lambda} C(\Lambda)} S_{0; A^c} \mathcal{X}_{\Lambda}(g_A^{-1}) = S_{0; A^c} \mathcal{X}_{\Lambda}(\gamma^n g_A^{-1}) \quad (6.8) \]

where we used two well-known properties of the \( S \)-matrix, \( S_{ij} = S_{ji}^* \) and equations (4.1) in the first step, the correspondence between monodromy charges and conjugacy classes (3.4) in the second and the property (A.5) in the last. The location of the conjugate boundary is therefore

\[ \tilde{G}_{A^c}(g) = \delta(\gamma^n g_A^{-1} - g) \quad (6.9) \]

So the boundary conjugation (6.3) has a nice geometric interpretation: given a boundary \( A \) at \( g_A \), its charge conjugate is located at \( R_n(g_A) = \gamma^n g_A^{-1} \). Real boundaries therefore correspond to branes that are left fixed (not necessarily pointwise) by the orientifold group, whereas complex boundaries form an \( R_n \) invariant pair.

Take again \( SU(2) \). In the oriented theory, the boundaries are located at circles of constant latitude. When we perform the standard orientifold projection, i.e. reflection through the axis of rotation, all these circles are left fixed. The branes either have an \( SO \) or \( Sp \) gauge group, depending on the Frobenius-Schur indicator of the boundary label \( A \). When \( A \) is a vector representation of \( SU(2) \), the projection is orthogonal and branes labelled by a spinor representation are symplectic. When we gauge the orientifold group with a non-trivial element of the centre, a brane is identified with its image obtained by reflection through the equator. These branes carry unitary gauge theories on their worldvolumes. Only for even level we have a real boundary with an orthogonal gauge group, labelled by \( A = k/2 \), that lies on top of the orientifold plane.

### 6.1 Branes in orbifolds

Boundary states for a (symmetric) simple current invariant are presented in [11] (see [28] for a proof of integrality of the annulus coefficients). We will again consider an extension invariant by a group \( H = \mathbb{Z}_M \) and generator \( L = J^{N/M} \). We will assume that the simple currents do not have fixed points (for a definition, see [24]). Then the boundary states are

\[ \text{Here, and in the following, we made the choice } -1 \text{ for the overall sign of the Möbius strip.} \]
labelled by $H$-orbits $[A] := (A, LA, L^2A, ..., L^{M-1}A)$ and the Ishibashi states only exist for chargeless fields $\Lambda$. The reflection coefficients are given by

$$R_{\Lambda,[A]} = \frac{\sqrt{|H|} S_{\Lambda,A}}{S_{0,A}} . \quad (6.10)$$

The distribution to be calculated is

$$\tilde{G}_{[A]} = \sqrt{|H|} \sum_{\Lambda, C_H(\Lambda)=0} \mathcal{X}_\Lambda(g_A)\mathcal{X}_\Lambda(g) . \quad (6.11)$$

We restricted the sum over $\Lambda$ to non-faithful $G$ representations with respect to the subgroup $H$, since the boundary coefficient is not automatically zero for chargeless fields. We can extend the sum to all $\Lambda$ by inserting a projector

$$\delta_M^{C(\Lambda),0} = \frac{1}{|H|} \sum_{m=0}^{M-1} e^{2\pi imC(\Lambda)/M} ,$$

where the superscript $M$ on the delta-function means modulo $M$. Then

$$\tilde{G}_{[A]} = \frac{1}{\sqrt{|H|}} \sum_{m=0}^{M-1} \sum_{\Lambda} \mathcal{X}_\Lambda(\gamma^m g_A)\mathcal{X}_\Lambda(g)$$

$$= \frac{1}{\sqrt{|H|}} \sum_{m=0}^{M-1} \delta(\gamma^m g_A - g) . \quad (6.12)$$

We used again the equivalence between characters of the Lie group and the $S$-matrix, the relation between conjugacy classes and monodromy charges and, in the last step, we inserted a projector on the trivial class and used (A.5). From this result we immediately see that the boundary states in the orbifold theory describe $H$-invariant combinations.

When the simple currents that define the modular invariant partition function do have fixed points, some boundary and Ishibashi labels may have multiplicities. In the presence of O-planes, i.e. the crosscap given by (5.25), the boundary coefficients have an extra phase that depends on the multiplicity label and the relative O-plane charge (the signs $\eta$). (In [11] this phase is $\alpha_J$.) Although it would be interesting to give a geometrical interpretation of the multiplicities and this phase, we will leave this for future research.

## 7 Conclusion

We have seen that crosscap states in WZW models and orbifolds thereof correspond to localized orientifolds, as they do in free CFT. The centre of the horizontal subalgebra plays an important classifying role in the possible orientifold projections; this is the geometrical counterpart of the role of simple currents in the exact CFT description. To compute the
various numbers, locations, dimensions and charges of the possible geometric O-planes, we
needed to prove mathematically important lattice generalization of Gaussian sums. The
consequence of this is a highly remarkable relation between the pseudo-modular $P$-matrix
and characters of the horizontal subalgebra.

This in turn implies a linear relation between the $P$- and the $S$-matrix. It is intriguing
that linear relations between the two matrices is precisely what is needed in order to cancel
tadpoles in open string theories. It would be very interesting to know if such a relation also
exists in other CFT’s. Although on the one hand there is no obvious generalization of the
analytic continuation of the labels of $S$, on the other hand most CFT’s are WZW-based.
It is therefore conceivable that such a relation might be derivable at least for coset CFT’s.
Whether relations of this kind actually do help with tadpole cancellation is an interesting
question which we hope to address in the future.

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sums are considered. Although the method used to derive the sum is essentially the same as
ours, the gaussian sum considered in [35] is different; there the range is over $L_w/hL_w$ instead
of $L_w/hL^\vee$.

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A Characters

In this appendix, we list some formulas that are used in the main text. For an extensive
review on Lie algebras and groups, we refer to [18].

A character maps the information of a highest weight representation $R_\Lambda$ of the Lie algebra
$\hat{\mathfrak{g}}$ to the complex numbers:

$$\bar{\chi}_\Lambda(h_t) = \text{Tr} \exp\{2\pi i R_\Lambda(h_t)\} , \quad (A.1)$$

where $h_t = \sum t_i H^i \equiv (t, H)$ is an element of the Cartan subalgebra, whose generators we
denote as $H^i$. The trace is over all weight vectors $\lambda$ in the representation with highest
weight $\Lambda$ and $R_\Lambda(h_t)$ is the $\Lambda$ representation of $h_t$. As weights are eigenstates of the Cartan
subalgebra, $R_\Lambda(H^i)|\lambda\rangle = \lambda^i|\lambda\rangle$, we have

$$\bar{\chi}_\Lambda(h_t) = \sum_\lambda \text{mult}_\Lambda(\lambda) \exp\{2\pi i (\lambda, t)\} , \quad (A.2)$$

where $\text{mult}_\Lambda(\lambda)$ denotes the multiplicity of the weight $\lambda$ in the representation $R_\Lambda$. A second,
more powerful, expression for the character is given by the Weyl character formula,

$$\bar{\chi}_\Lambda(h_t) = \frac{\sum_{w \in W} \text{sign}(w) \exp[2\pi i (w(\Lambda + \rho), t)]}{\sum_{w \in W} \text{sign}(w) \exp[2\pi i (w(\rho), t)]} . \quad (A.3)$$

The sum is over the Weyl group $W$ weighted by the sign of each element defined as $\text{sign}(w) = (-1)^\# \text{ of generators of } W$. The vector $\rho$ is the Weyl vector and equals half the sum of positive
roots, or, equivalently, the sum of the fundamental weights $\rho = \sum_{i=1}^{r} \Lambda_i$. These expressions are basis-independent. It is often convenient to choose the Dynkin basis in weight space and to choose a basis in the Lie-algebra (Chevalley basis) so that the eigenvalues of $H^i$ are the Dynkin labels. The inner product in weight space is then $(\lambda, \mu) = \sum_{i,j} \lambda_i C^{-1}_{ij} \mu_j$, where $C^{-1}$ is the inverse of the symmetrized Cartan matrix.

**Group characters**

Eq. (A.1) may also be read as a character of the Lie Group $G$,

$$X_\Lambda(g) = \text{Tr}_\Lambda(g) , \quad (A.4)$$

evaluated at the group element $g_t = \exp\{2\pi i \sum_i t_i H^i\}$. In the main text we use two properties. One is that characters shift by a phase under transformations $g \rightarrow \gamma g$ where $\gamma$ is the generator of the centre of $G$:

$$X_\Lambda(\gamma^n g) = \exp\{2\pi i \frac{n}{N} C(\Lambda)\} X_\Lambda(g) . \quad (A.5)$$

What is really important to us, however, is that characters serve as the analogue of Fourier modes for group manifolds. They are orthogonal and complete

$$\int \mu_G(g) X^*_\Lambda(g) X_M(g) = \delta_{\Lambda,M} , \quad \sum_{\Lambda} X_\Lambda(g) X_\Lambda(h) = \delta(g-h) \quad (A.6)$$

The sum is over dominant weights only and the delta-function $\delta(g-h)$ is defined with respect to the Haar measure $\int d\mu_G(a) F(a) \delta(a-b) = F(b)$. The $\delta$-function is defined on conjugacy classes, i.e. it vanishes unless $g$ and $h$ belong to the same conjugacy class.

**A.1 The modular $S$ and $T$-matrix**

The characters that appear in the partition functions, i.e. the characters of the chiral algebra $A$, form a representation of the modular group, generated by $T: \tau \rightarrow \tau+1$ and $S: \tau \rightarrow -1/\tau$:

$$\chi_\Lambda(\tau + 1) = \sum_M T^M_\Lambda \chi_M(\tau) , \quad (A.7)$$

$$\chi_\Lambda(-1/\tau) = \sum_M S^M_\Lambda \chi_M(\tau) . \quad (A.8)$$

The $T$ matrix is diagonal and equals

$$T_{L,M} = \exp \left[ 2\pi i \left( h_L - \frac{c}{24} \right) \right] \delta_{\Lambda,M} \quad (A.9)$$

$$= \exp \left[ 2\pi i \left( \frac{(\Lambda, \Lambda + 2\rho)}{2h} - \frac{1}{24} kd \right) \right] \delta_{\Lambda,M} . \quad (A.10)$$
The modular $S$-matrix is unitary and symmetric and equals

$$S_{\Lambda,M} = N_S \sum_{w \in W} \text{sign}(w) \exp \left[ \frac{-2\pi i}{h} (w(\Lambda + \rho), M + \rho) \right], \quad (A.11)$$

with normalization

$$N_S = e^{2\pi i \left( \frac{d-r}{8} \right)} \left| \frac{L_w}{L^\vee} \right|^{-1/2} \frac{1}{\sqrt{h^r}}. \quad (A.12)$$

Here $|L_w/L^\vee|$ is the number of weights inside the unit cell of the coroot lattice.

Comparison with the Weyl character formula eq. (A.3) shows that the modular $S$-matrix is proportional to a character of the horizontal sub-algebra

$$S_{L,M} = N_S \left( \sum_{w \in W} \text{sign}(w) \exp \left[ \frac{-2\pi i}{h} (w(\rho), M + \rho) \right] \right) X_{\Lambda} \left( -\frac{(M + \rho, H)}{h} \right) = S_{0,M+\rho} X_{\Lambda} \left( -\frac{(M + \rho, H)}{h} \right). \quad (A.13)$$

Note that if we express $M + \rho$ in Dynkin basis and $H$ in Chevalley basis, then $(M + \rho, H) \equiv \sum_i (M + \rho)_i C^{-1}_{ij} H^j$.

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\[
\begin{array}{|c|c|c|c|}
\hline
\mathfrak{g} & \text{Current } J & C & \# \text{ O-planes} \\
\hline
A_r, r \text{ even} & - & 0 & 1 + \frac{r}{2} \\
A_r, r \text{ even} (k,0,\ldots,0) & \sum_{i=1}^{r} iH^i & 1 + \frac{r}{2} (\ast) \\
A_r, r \text{ odd} & - & 0 & 1 + \frac{(r+1)}{2} \\
A_r, r \text{ odd} (k,0,\ldots,0) & \sum_{i=1}^{r} iH^i & 1 + \frac{(r-1)}{2} \\
B_r, r \text{ even} & - & 0 & 2 + \frac{r}{2} \\
B_r, r \text{ even} (k,0,\ldots,0) & H^r & \frac{r}{2} \\
B_r, r \text{ odd} & - & 0 & 1 + \frac{(r+1)}{2} \\
B_r, r \text{ odd} (k,0,\ldots,0) & H^r & \frac{(r+1)}{2} \\
C_r & (0,\ldots,0,k) & \sum_{i=1}^{\lfloor (r+1)/2 \rfloor} H^{2i-1} & 1 + r \\
D_r, r \text{ even} & - & 0 & 3 + \frac{r}{2} \\
D_r, r \text{ even} (0,\ldots,0,k) & 2\sum_{i=1}^{r/2-1} H^{2i-1} + (2-r)H^{r-1} - rH^r & 2 \\
D_r, r \text{ even} (k,0,\ldots,0) & 2H^{r-1} + 2H^r & \frac{r}{2} \\
D_r, r \text{ even} (0,\ldots,0,k,0) & 2\sum_{i=1}^{r/2-1} H^{2i-1} - rH^{r-1} + (2-r)H^r & 2 \\
D_r, r \text{ odd} & - & 0 & 1 + \frac{r+1}{2} \\
D_r, r \text{ odd} (0,\ldots,0,k) & 2\sum_{i=1}^{(r-1)/2} H^{2i-1} + (2-r)H^{r-1} - rH^r & 2 \\
D_r, r \text{ odd} (k,0,\ldots,0) & 2H^{r-1} + 2H^r & 1 + \frac{r+1}{2} (\ast) \\
D_r, r \text{ odd} (0,\ldots,0,k,0) & 2\sum_{i=1}^{(r-1)/2} H^{2i-1} - rH^{r-1} + (2-r)H^r & 2 (\ast) \\
G_2 & - & 0 & 2 \\
F_4 & - & 0 & 3 \\
E_6 & - & 0 & 3 \\
E_6 (0,0,0,0,k,0) & H^4 - H^2 + H^4 - H^5 & 3 (\ast) \\
E_7 & - & 0 & 4 \\
E_7 (0,0,0,0,0,k,0) & H^4 + H^6 + H^4 & 2 \\
E_8 & - & 0 & 3 \\
\hline
\end{array}
\]

Table 1: Currents, Conjugacy classes and the number of O-planes. Column two gives the Dynkin labels of the current (at level \(k\)) that corresponds to the conjugacy class generator in column three according to (3.4). An asterisk indicates orientifold configurations related to a previous one by an even current, in which case they are just centre translations of each other.
Table 2: Dimensions of O-planes. The first column indicates the group manifold, the second the power $n$ of the current, and the third the subgroup $G_c$ that commutes with the orientifold map. The dimension of the O-plane is equal to the dimension of the tangent space of $G/G_c$ given in column 4. In all cases $p + q = N$. For $SO(N)$, $N$ even, we indicate in column 2 the conjugacy class under consideration.