AN INVESTIGATION INTO EXPLICIT VERSIONS OF BURGESS’ BOUND

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Abstract. Let \( \chi \) be a Dirichlet character modulo \( p \), a prime. In applications, one often needs estimates for short sums involving \( \chi \). One such estimate is the family of bounds known as Burgess’ bound. In this paper, we explore several minor adjustments one can make to the work of Enrique Treviño \[11\] on explicit versions of Burgess’ bound. For an application, we investigate the problem of the existence of a \( k \)th power non-residue modulo \( p \) which is less than \( p^\alpha \) for several fixed \( \alpha \). We also provide a quick improvement to the conductor bounds for norm-Euclidean cyclic fields found in \[7\].

1. Introduction

Let \( \chi \) be a Dirichlet character modulo \( q \). It is often useful to know the size of short character sums, i.e., sums of the form

\[
S_\chi(M, N) := \sum_{n=M+1}^{M+N} \chi(n)
\]

where \( M, N \) are real numbers with \( N < q \). A trivial bound for (1.1) is simply \( N \), since a Dirichlet character takes values which are either roots of unity or 0. On the other hand, we may estimate (1.1) entirely in terms of \( q \); the standard estimate in this direction is the Pólya–Vinogradov inequality. The following explicit version of this inequality is due to Frolenkov and Soundararajan \[5\].

**Theorem 1.1.** \[5, Theorem 2\] For a primitive Dirichlet character modulo \( q > q_0 \), we have

\[
|S_\chi(M, N)| \leq \alpha_1 \sqrt{q} \log q + \sqrt{q},
\]

where

\[
(\alpha_1, q_0) = \begin{cases} 
(\frac{2}{7}, 1200) & \text{if } \chi(-1) = 1 \\
(\frac{1}{2}, 40) & \text{if } \chi(-1) = -1.
\end{cases}
\]

In this work, we are concerned with characters of large modulus. Following the proof of Theorem 1.1 provided in \[5\, pg. 278\] closely, we may take a constant smaller than 1 in front of \( \sqrt{q} \) whenever \( q \) is bounded below. For an adjusted version of Theorem 1.1 that does not bound \( q \) below, see \[6, Lemma 3\].

**Corollary 1.2.** Let \( \chi \) be a primitive Dirichlet character modulo \( q > q_0 \), and \( \alpha_1 \) be the constant in Theorem 1.1. Then,

\[
|S_\chi(M, N)| \leq \alpha_1 \sqrt{q} \log q + \alpha_2 \sqrt{q},
\]

where

\[
\alpha_2 := \alpha_1 \cdot \begin{cases} 
\log \left( \frac{\pi^2}{10} + \frac{103 \cdot 0.0225}{q_0} + 2.8650 \right) & \text{if } \chi(-1) = 1 \\
\log \left( \pi^2 + \frac{20 \cdot 30 \pi}{\sqrt{400}} + \frac{103 \cdot 0.0225}{q_0} \right) & \text{if } \chi(-1) = 1.
\end{cases}
\]

For either parity, \( \alpha_2 \leq 1 \) for \( q_0 \geq 854 \). However, these savings are slight, even for very large \( q_0 \), since the limiting value of \( \alpha_2 \) is greater than 0.9466 (for even \( \chi \)) and 0.8203 (for odd \( \chi \)).

Between the trivial estimate, which is entirely in terms of \( N \), and the Pólya–Vinogradov inequality, which is entirely in terms of \( q \), we have a family of hybrid estimates due to D. A. Burgess (see, e.g. \[1\], \[2\], \[3\]) which take the following form if \( q = p \), a prime.

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Theorem 1.3. [2] Theorem 1 | Let $\chi$ be a non-principal character modulo $p$. Then, if $r$ is a positive integer,
\[ S_{\chi}(M,N) \ll N^{1-\frac{1}{p+\frac{1}{p}}} \log p, \]
for any non-negative integers $M, N$.

Proving Burgess’ bound requires the power of an estimate derived from the Weil bound [13]. For our purposes, we use an explicit variant of this estimate established by Treviño [11].

Theorem 1.4. [11] Theorem 1.2 | Let $p$ be a prime, $r$ a positive real number, and $B$ a positive real number satisfying $r \leq 9B$. Let $\chi$ be a non-principal character modulo $p$. Then
\[ \sum_{x \pmod{p}} \left| \sum_{1 \leq b \leq B} \chi(x+b)^{2r} \right| \leq (2r-1)! B^r p + (2r-1)B^{2r} \sqrt{p}, \]
where $(2r-1)! = (2r-1)(2r-3)\ldots(1)$.

This estimate is used in [11] to establish an explicit version of Burgess’ bound for prime moduli. In particular, the following is determined.

Theorem 1.5. [11] Theorem 1.4 | Let $p$ be a prime and $2 \leq r \leq 10$ be an integer. Let $\chi$ be a non-principal character modulo $p$. Let $M$ and $N$ be non-negative integers. Let $p_0$ be a positive real number. Then, for $p \geq p_0$, we can determine a constant $c(r)$ depending on $p_0$ and $r$ such that
\[ |S_{\chi}(M,N)| < c(r)N^{1-\frac{1}{p+\frac{1}{r}}} \log p)^{\frac{1}{r}}. \]

The exponent on the log $p$ can be improved by placing a mild condition on the size of $N$.

Theorem 1.6. [11] Theorem 1.6 | Let $p$ be a prime and $2 \leq r \leq 10$ be an integer. Let $\chi$ be a non-principal character modulo $p$. Let $M$ and $N$ be non-negative integers with $1 \leq N \leq 2p^{\frac{1}{2}+\frac{1}{r}}$. Let $p_0$ be a positive real number. Then, for $p \geq p_0$, we can determine a constant $C(r)$ depending on $p_0$ and $r$ such that
\[ |S_{\chi}(M,N)| < C(r)N^{1-\frac{1}{p+\frac{1}{r}}} \log p)^{\frac{1}{r}}. \]

By leveraging the $r = 2$ case of Theorem 1.5 against Theorem 1.6, Treviño notes that the restriction on $N$ in Theorem 1.6 can be omitted [11].

Corollary 1.7. | Let $p$ be a prime and $3 \leq r \leq 10$ be an integer. Let $\chi$ be a non-principal character modulo $p$. Let $M$ and $N$ be non-negative integers with $1 \leq N \leq 2p^{\frac{1}{2}+\frac{1}{r}}$. Let $p_0 \geq 10^10$ be a positive real number. Then, for $p \geq p_0$, the constant $C(r)$ in Theorem 1.6 is such that
\[ |S_{\chi}(M,N)| < C(r)N^{1-\frac{1}{p+\frac{1}{r}}} \log p)^{\frac{1}{r}}. \]

The constants $c(r)$ and $C(r)$ as provided by [11] are reproduced in Tables 1 and 2.

Table 1. Values for the constants $c(r)$ provided by Treviño.

| $r$ | $p_0 = 10^7$ | $p_0 = 10^{10}$ | $p_0 = 10^{20}$ |
|-----|--------------|----------------|---------------|
| 2   | 2.7381       | 2.5173         | 2.3549        |
| 3   | 2.0197       | 1.7385         | 1.3695        |
| 4   | 1.7308       | 1.5151         | 1.3104        |
| 5   | 1.6107       | 1.4572         | 1.2987        |
| 6   | 1.5482       | 1.4274         | 1.2901        |
| 7   | 1.5052       | 1.4042         | 1.2813        |
| 8   | 1.4703       | 1.3846         | 1.2729        |
| 9   | 1.4411       | 1.3662         | 1.2641        |
| 10  | 1.4160       | 1.3495         | 1.2562        |

The main aim of this paper is to obtain as many improvements to the size of the constants in Corollary 1.7 as possible. That is, we wish to prove the following.

Theorem 1.8. For $r = 2$, Theorem 1.4 holds with the constants provided in Table 1 and the condition that $1 \leq N < 2p^{\frac{1}{2}}$. For $3 \leq r \leq 6$ and $p \geq 10^8$, or $7 \leq r \leq 10$ and $p \geq 10^9$ holds with the constants provided in Table 2 and no restriction on $N$. 
Table 2. Values for the constants $C(r)$ provided by Treviño.

| $r$ | $p_0 = 10^{10}$ | $p_0 = 10^{15}$ | $p_0 = 10^{20}$ |
|-----|----------------|----------------|----------------|
| 2   | 3.6529         | 3.5851         | 3.5751         |
| 3   | 2.5888         | 2.5144         | 2.4945         |
| 4   | 2.1914         | 2.1258         | 2.1078         |
| 5   | 1.9841         | 1.9231         | 1.9043         |
| 6   | 1.8508         | 1.7959         | 1.7757         |
| 7   | 1.7586         | 1.7066         | 1.6854         |
| 8   | 1.6869         | 1.6384         | 1.6187         |
| 9   | 1.6283         | 1.5857         | 1.5654         |
| 10  | 1.5794         | 1.5410         | 1.5216         |

Table 3. Values for the constants $C(r)$.

| $r$ | $p_0 = 10^5$  | $p_0 = 10^6$  | $p_0 = 10^7$  | $p_0 = 10^8$  | $p_0 = 10^9$  | $p_0 = 10^{10}$ |
|-----|---------------|---------------|---------------|---------------|---------------|----------------|
| 2   | 3.7125        | 3.4682        | 3.3067        | 3.1980        | 3.1259        | 3.0679         |
| 3   | 2.7979        | 2.6371        | 2.5131        | 2.4318        | 2.3776        | 2.3358         |
| 4   | 2.4157        | 2.2980        | 2.2022        | 2.1513        | 2.0994        | 2.0613         |
| 5   | 2.1801        | 2.0981        | 2.0427        | 1.9755        | 1.9419        | 1.9084         |
| 6   | 2.0874        | 2.0037        | 1.9424        | 1.8962        | 1.8353        | 1.8054         |
| 7   | 1.8948        | 1.8454        | 1.8087        | 1.7820        | 1.7561        | 1.7291         |
| 8   | 1.7993        | 1.7609        | 1.7306        | 1.7093        | 1.6894        | 1.6696         |
| 9   | 1.7266        | 1.6963        | 1.6692        | 1.6492        | 1.6323        | 1.6186         |
| 10  | 1.6720        | 1.6411        | 1.6175        | 1.5991        | 1.5845        | 1.5727         |

| $r$ | $p_0 = 10^{11}$ | $p_0 = 10^{12}$ | $p_0 = 10^{13}$ | $p_0 = 10^{14}$ | $p_0 = 10^{15}$ | $p_0 = 10^{16}$ |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|
| 2   | 3.0280         | 2.9997         | 2.9790         | 2.9635         | 2.9515         | 2.9421         |
| 3   | 2.3025         | 2.2782         | 2.2600         | 2.2461         | 2.2351         | 2.2263         |
| 4   | 2.0329         | 2.0117         | 1.9956         | 1.9831         | 1.9733         | 1.9654         |
| 5   | 1.8831         | 1.8638         | 1.8487         | 1.8367         | 1.8272         | 1.8194         |
| 6   | 1.7825         | 1.7646         | 1.7503         | 1.7388         | 1.7294         | 1.7216         |
| 7   | 1.7081         | 1.6914         | 1.6779         | 1.6669         | 1.6577         | 1.6500         |
| 8   | 1.6501         | 1.6345         | 1.6219         | 1.6112         | 1.6023         | 1.5946         |
| 9   | 1.6029         | 1.5882         | 1.5762         | 1.5661         | 1.5575         | 1.5501         |
| 10  | 1.5629         | 1.5499         | 1.5384         | 1.5287         | 1.5205         | 1.5134         |

| $r$ | $p_0 = 10^{17}$ | $p_0 = 10^{18}$ | $p_0 = 10^{19}$ | $p_0 = 10^{20}$ | $p_0 = 10^{50}$ | $p_0 = 10^{75}$ |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|
| 2   | 2.9345         | 2.9282         | 2.9230         | 2.9185         | 2.8752         | 2.8658         |
| 3   | 2.2190         | 2.2128         | 2.2076         | 2.2029         | 2.1503         | 2.1368         |
| 4   | 1.9590         | 1.9537         | 1.9493         | 1.9455         | 1.9094         | 1.9011         |
| 5   | 1.8130         | 1.8077         | 1.8033         | 1.7996         | 1.7689         | 1.7630         |
| 6   | 1.7151         | 1.7097         | 1.7051         | 1.7012         | 1.6715         | 1.6668         |
| 7   | 1.6435         | 1.6380         | 1.6333         | 1.6292         | 1.5986         | 1.5947         |
| 8   | 1.5883         | 1.5828         | 1.5779         | 1.5738         | 1.5586         | 1.5532         |
| 9   | 1.5439         | 1.5384         | 1.5336         | 1.5294         | 1.4959         | 1.4925         |
| 10  | 1.5072         | 1.5019         | 1.4972         | 1.4930         | 1.4581         | 1.4548         |

In this and many other regards, the author is indebted to Treviño, since the method of proof will be essentially the same as his. There are two primary ways one could modify the arguments of [11] to obtain better constants. For one, Burgess’ bound is automatic when $N$ is large enough, since in such a scenario the Pólya–Vinogradov inequality is stronger. However, in [11], the simple estimate

$$|S_\chi(M, N)| \leq \sqrt{q} \log q,$$

is used. Here, we will use Theorem 1.2 instead. This has the effect of reducing the range of $N$ for which we need to establish Burgess’ bound, which in turn allows us to admit smaller constants in said bound. This alone yields a significant gains over the constants in [11].

The second technique we employ involves the following counting lemma.
Lemma 1.9. [11, Lemma 2.1] Let \( p \) be a prime and \( A \in \left[ 28, \frac{N}{p} \right) \) and \( N < p \) be positive integers. Then, we have
\[
V_2 := \sum_{x \pmod{p}} v^2(x) \leq 2AN \left( \frac{AN}{p} + \log 1.85A \right),
\]
where
\[
v(x) = |\{(a, n) \in \mathbb{N} | 1 \leq a \leq A, M < n \leq M + N \text{ and } n \equiv ax \pmod{p}\}|.
\]

In the next section, we relax the restrictions on \( A \) to extract some additional terms in this estimate. This will allow us to compute \( C(r) \) for smaller values of \( p_0 \). A bonus feature will be that the value of \( A \) has more influence on the size of the bound. However, in both our case and Treviño’s case, we note that this estimate seems to be a little less than twice as large as the actual value of \( V_2 \).

In several cases, determining improved constants \( c(r) \) for the case of \( r = 2 \) in Theorem 1.10 allows us to improve upon the constants \( C(r) \) across all \( r \). The values we determine for \( c(2) \) are provided in Table 6.

Once established, these constants can be applied to various number-theoretic problems. In particular, we establish the following improvement to an application, which is essentially [11, Theorem 1.10].

**Theorem 1.10.** Let \( p \) be a prime number and \( k > 1 \) be a positive divisor of \( p - 1 \). Let \( n_{p,k} \) be the least \( k \)th power non-residue modulo \( p \). Fix \( \alpha > \frac{1}{4} \). Then there is a computable \( Y \) (depending only on \( \alpha \)) for which \( n_{p,k} < p^\alpha \) whenever \( p \geq Y \).

In particular, Table 4 lists several such pairs of \( \alpha \) and \( Y \). Note that [11] established the pair \((\frac{1}{6}, 4732)\).

**Table 4.** Pairs \( \alpha, Y \) obtained in Theorem 1.10

| \( \alpha \) | \( Y \) |
|---|---|
| \( \frac{1}{4} \) | 83 |
| \( \frac{1}{5} \) | 334 |
| \( \frac{1}{6} \) | 3872 |

We also consider an application of Lezewski and McGown [7], which uses Burgess’ bound to bound the conductors of norm-Euclidean cyclic number fields with prime degree \( l \), \( 3 \leq l \leq 100 \). Using the improved constants we obtain, we may establish the following modest improvement to [7, Proposition 2.4].

**Proposition 1.11.** Table 5 provides unconditional bounds on the conductor \( f \) of norm-Euclidean cyclic number fields of odd prime degree \( 3 < l < 100 \).

**Table 5.** Unconditional bounds on the conductor \( f \) of norm-Euclidean cyclic number fields of odd prime degree \( 3 \leq l \leq 100 \). These bounds improve upon [7, Proposition 2.4] by no more than a factor of 100.

| \( l \) | \( f \) |
|---|---|
| 3 | 2.0 \cdot 10^{32} |
| 5 | 5.1 \cdot 10^{33} |
| 7 | 7.9 \cdot 10^{37} |
| 11 | 7.0 \cdot 10^{42} |
| 13 | 2.7 \cdot 10^{42} |
| 17 | 8.5 \cdot 10^{42} |
| 19 | 8.9 \cdot 10^{47} |
| 23 | 4.8 \cdot 10^{59} |
| 29 | 5.7 \cdot 10^{64} |
| 31 | 2.3 \cdot 10^{64} |
| 37 | 8.2 \cdot 10^{64} |
| 41 | 6.6 \cdot 10^{64} |
| 43 | 1.8 \cdot 10^{69} |
| 47 | 1.1 \cdot 10^{66} |
| 53 | 1.2 \cdot 10^{67} |
| 59 | 9.8 \cdot 10^{69} |
| 61 | 2.0 \cdot 10^{78} |
| 67 | 1.3 \cdot 10^{83} |
| 71 | 4.0 \cdot 10^{83} |
| 73 | 6.8 \cdot 10^{83} |
| 79 | 3.3 \cdot 10^{89} |
| 83 | 8.7 \cdot 10^{89} |
| 89 | 3.5 \cdot 10^{91} |
| 97 | 1.9 \cdot 10^{92} |

**Remark 1.12.** For the majority of this paper, it suffices to write \( C(r) \) or \( c(r) \) to represent the constants in Theorems 1.9 and 1.10. However, when it is necessary to highlight the dependence on \( p_0 \), we may also write \( C(r; p_0) \) or \( c(r; p_0) \).
2. Tighter Estimates for \( V_2 \)

In estimating \( V_2 \), we will make use of the following estimates for summatory functions.

**Lemma 2.1.** For \( x \geq 1 \),

\[
\sum_{n \leq x} \frac{1}{n} < \log x + \gamma + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{64x^4}
\]

*Proof.* The lemma can easily be verified for \( x = 1 \). Note that the approach here is substantially similar to \[8 Smith, Lemma 2.8\]. For \( x > 1 \), we know from Euler–Maclaurin summation \[9 Theorem B.5\] that for any integer \( K \)

\[
(2.1) \quad \sum_{1 < n \leq x} \frac{1}{n} - \log x = \sum_{i=1}^{K} \left( \frac{B_i}{i} - \frac{B_i(\{x\})}{ix^i} \right) - \int_{1}^{x} \frac{B_K(\{t\})}{t^{K+1}} \, dt,
\]

where \( B_i \) and \( B_i(x) \) are the \( i \)th Bernoulli number and polynomial, respectively. We may rewrite equation (2.1) to take advantage of the fact that we know \( \lim_{x \to \infty} \left( \sum_{n \leq x} \frac{1}{n} - \log x \right) = \gamma \), the Euler–Mascheroni constant. That is,

\[
\sum_{n \leq x} \frac{1}{n} - \log x = 1 - \sum_{i=1}^{K} \frac{B_i}{i} = \int_{1}^{x} \frac{B_K(t)}{t^{K+1}} \, dt + R_K(x) = \gamma + R_K(x),
\]

where

\[
R_K(x) = - \sum_{i=1}^{K} \frac{B_i(x)}{ix^i} + \int_{x}^{\infty} \frac{B_K(t)}{t^{K+1}} \, dt = O\left( \frac{1}{x} \right).
\]

Let \( K = 4 \). Then, after some rearranging,

\[
R_4(x) = \frac{1}{2x} + \frac{1}{12x^2} + \frac{1}{120x^4} = \frac{x}{2} \left( 1 - \frac{1}{2x} + \frac{1}{6x^2} + \frac{x}{2x^2} + \frac{x^2}{3x^3} - \frac{x}{2x^3} + \frac{x^3}{4x^4} + \frac{x^4}{4x^5} - \frac{x^2}{2x^3} \right)
\]

\[
+ \int_{x}^{\infty} \frac{1}{t^{5}} \, dt.
\]

The bracketed expression above is positive for \( x > 1 \), while the integral is \( O(x^{-4}) \), so with a tight enough estimate on the integral, we can improve our bound on \( \sum_{n \leq x} n^{-1} \) with two exact lower order terms \( \left( \frac{1}{120x^5} \text{ and } - \frac{1}{12x^4} \right) \) and one estimated lower order term \( \left( \frac{1}{120x^4} \right) \), with an adjustment no larger than \( \frac{1}{120x^4} \) itself. In particular, we have that

\[
\int_{x}^{\infty} \frac{1}{t^{5}} \, dt < - \frac{1}{120x^4} + \int_{x}^{\infty} \frac{1}{16t^5} \, dt = - \frac{1}{120x^4} + \frac{1}{64x^4}.
\]

Therefore,

\[
R_4(x) < \frac{1}{2x} + \frac{1}{12x^2} + \frac{1}{64x^4}.
\]

Hence, we have the proposition. 

**Lemma 2.2.** For \( x \geq 1 \),

\[
\sum_{n \leq x} \frac{\log n}{n^2} < -\zeta'(2) = \frac{\log x}{x} \frac{1}{x} - \frac{\log \log x}{x}.
\]

*Proof.* The proof is similar to that of Lemma 2.1. Here, we take \( K = 1 \) in \[8 Theorem B.5\] and observe that

\[
\lim_{x \to \infty} \left( \sum_{1 \leq n \leq x} \frac{\log n}{n^2} - \frac{\log x + 1}{x} \right) = -\zeta'(2).
\]

Thus,

\[
\sum_{1 \leq n \leq x} \frac{\log n}{n^2} = -\zeta'(2) - \frac{\log x}{x} \frac{1}{x} + R_1(x),
\]

where \( R_1(x) \) is small.
where
\[ R_1(x) = \frac{\log x}{2x^2} - \frac{\lfloor x \rfloor \log x}{x^2} - \int_{x}^{\infty} \left( t - \frac{1}{2} \right) \left( \frac{1}{t^3} - \frac{2 \log t}{t^3} \right) dt. \]
It can be verified that \( R_1(x) \) is bounded below by \( \frac{\log x}{x^2} \), establishing the result. \( \square \)

Alongside the two preceding estimates, we borrow the following lemmas directly from [11].

**Lemma 2.3.** [11] Lemma 2.2 | For \( x > 1 \) real, we have
\[
\sum_{n \leq x} \frac{\phi(n)}{n} \leq \frac{6}{\pi^2} x + \log x + 1.
\]

**Lemma 2.4.** [11] Lemma 2.3 | For \( x \geq 1 \),
\[
\sum_{n \leq x} \phi(n)n \leq \frac{2}{\pi^2} x^3 + \frac{1}{2} x^2 \log x + x^2.
\]

Note that the leading constants in these results are correct (see [10], for example). Any savings that could potentially be made here would come from improving the lower order terms, perhaps as in [12] Hilfssatz 1.

**Lemma 2.5.** [11] Lemma 2.4 | For \( x \geq 1 \),
\[
\sum_{n \leq x} \log \left( \frac{x}{n} \right) \leq x - 1.
\]

From Lemma 1.9 recall that \( v(x) = \left\{ (a, n) \in \mathbb{N} \mid 1 \leq a \leq A, M < n \leq M + N \text{ and } n \equiv ax \pmod{p} \right\} \).

The following estimate essentially comes from [11] Lemma 2.1. However, some care has been taken to minimize the reliance on a lower bound for the parameter \( A \). This will allow us to determine Burgess constants for much smaller \( p \).

**Lemma 2.6.** Let \( p \) be a prime and \( N \) be a positive integer. Let \( A > 1 \) be an integer satisfying \( 11A < N \). Then,
\[
V_2 \leq 2AN \left( 0.83575 \frac{AN}{p} + \frac{6}{\pi^2} \log(e^{\gamma + 1} + \frac{1}{\pi^2 \sqrt{1 + \frac{1}{\sqrt{\pi A}}}} A) + \frac{3}{2} + A - \frac{1}{2N} - \frac{1}{A} \right).
\]

**Proof.** Note that in [11], \( V_2 \) counts quadruples \( (a_1, a_2, n_1, n_2) \) satisfying \( 1 \leq a_1, a_2 \leq A \) and \( M < n_1, n_2 \leq M + N \) where \( a_1 n_2 \equiv a_2 n_1 \pmod{p} \). Treviño concludes that we must have [11] formula (2.16)
\[
V_2 \leq AN + \frac{2N^2}{p} S_1 + \frac{2N}{p} S_2 + 2NS_3 + A^2 - A,
\]
where
\[
(2.2) \quad S_1 = \sum_{1 \leq a_2 \leq A} \sum_{1 \leq a_1 < a_2} \frac{a_1 + a_2}{\gcd(a_1, a_2)} = \frac{3}{4} A^2 - \frac{3}{4} A,
\]
\[
S_2 = \sum_{1 \leq a_2 \leq A} \sum_{1 \leq a_1 < a_2} \frac{a_1 + a_2}{\gcd(a_1, a_2)} = \frac{3}{2} \sum_{1 \leq d \leq A} \sum_{2 \leq b_2 \leq \frac{A}{d}} \phi(b_2) b_2,
\]
and
\[
S_3 = \sum_{1 \leq a_2 \leq A} \sum_{1 \leq a_1 < a_2} \frac{\gcd(a_1, a_2)}{\max(a_1, a_2)} = \sum_{1 \leq d \leq A} \sum_{2 \leq b_2 \leq \frac{A}{d}} \phi(b_2) b_2.
\]

Using Lemma 2.4 on \( S_2 \), we see that
\[
S_2 \leq \frac{3(3)}{\pi^2} A^3 + \frac{3(2)}{4} A^2 \log A - \frac{3A^2}{4} \sum_{d \leq A} \frac{\log d}{d^2} - \frac{3}{2} A^2 \zeta(2).
\]

Applying Lemma 2.2 to the above sum,
\[
(2.3) \quad S_2 \leq \frac{3(3)}{\pi^2} A^3 + \frac{3(2)}{4} A^2 \log A - \frac{3A^2}{4} \left( -\zeta'(2) - \frac{\log A}{A} - \frac{1}{A} - \frac{\log(A)}{A^2} \right) + \frac{3}{2} A^2 \zeta(2),
\]
for \( A > 1 \). Using Lemma 2.3 on \( S_3 \), we see that
\[
S_3 \leq \frac{6}{\pi^2} A \sum_{d \leq A} \frac{1}{d} + \sum_{d \leq A} \frac{\log A}{d}.
\]
Applying Lemmas 2.1 and 2.5 to the relevant sums yields, for $A > 1$,

\( S_3 \leq \frac{6}{\pi^2} A \log(e^{\gamma + \frac{1}{12\pi}} - \frac{1}{12\pi} A) + A - 1. \)

Now, if we combine equation (2) with (2.2), (2.3), and (2.4), we determine that

\[
V_2 \leq 2AN \left( \frac{3A^2\zeta(3)}{\pi^2 p} + \frac{\pi^2 A \log A}{8p} + \frac{3A\zeta'(2)}{4p} + \frac{3 \log A}{4p} \right) + \left( \frac{A\pi^2}{4p} - \frac{3N}{4p} \right) + \left( \frac{3AN}{4p} + \frac{3 \log A}{4p} \right) + \left( \frac{6}{\pi^2} \log(e^{\gamma + \frac{1}{12\pi}} - \frac{1}{12\pi} A) + \frac{3}{2} + \frac{A - 1}{2N} - \frac{1}{A} \right).
\]

With the conditions on $A$ as stated, we can verify that

\[
\frac{3A^2\zeta(3)}{\pi^2 p} + \frac{\pi^2 A \log A}{8p} + \frac{3A\zeta'(2)}{4p} + \frac{3 \log A}{4p} < \frac{11A^2}{16} \leq \frac{4N}{16p}
\]

and

\[
\left( \frac{A\pi^2}{4p} - \frac{3N}{4p} \right) < 0
\]

because $N \geq 11A$. We also have $A \geq 2$, so that

\[
\left( \frac{3AN}{4p} + \frac{3 \log A}{4p} \right) = \frac{3AN}{4p} \left( 1 + \frac{1}{AN} + \frac{\log A}{A^2 N} \right) \leq \frac{6AN}{8p} \left( 1 + \frac{1}{11A^2} + \frac{\ln A}{11A^2} \right) \leq \frac{3(1.031)AN}{4p}.
\]

Combining these estimates in (2.5) establishes the result. 

\[\square\]

If we restrict $A$ and $N$ in terms of $p$, we can obtain a better estimate for $V_2$, which will help us reduce the power on the logarithm in Theorem 1.6.

**Lemma 2.7.** Let $p$ a prime and $N$ be a positive integer. Let $A > 2$ be an integer such that $2AN < p$. Then

\[
V_2 \leq 2AN \left( \frac{3}{2} + \frac{6}{\pi^2} \log(e^{\gamma + \frac{1}{12\pi}} - \frac{1}{12\pi} A) + \frac{A - 1}{2N} - \frac{1}{A} \right).
\]

**Proof.** Under the condition $2AN < p$, [11] Lemma 4.1 establishes that

\[
V_2 \leq AN + 2NS_3 + A^2 - A.
\]

The proof follows by using (2.4) and factoring out $2AN$. 

\[\square\]

### 3. Main Theorems

We will begin by reproducing the proof of Theorem 1.6 with modifications according to Lemma 2.7.

**Proof of Theorem 1.6.** Let $C(r)$ be a parameter chosen so that it satisfies $C(r) < C(r)$. Then, we may use the trivial bound and our assumption on $N$, to establish the result for $N$ outside the ranges

\[
C(2)^{\frac{3}{2}}p^{\frac{3}{4} + \frac{1}{4} \sqrt{\log p}} < N < 2p^{\frac{3}{4}}.
\]

when $r = 2$ or, using Burgess for $r - 1$,

\[
C(r)^{\frac{3}{2} + \frac{1}{4} \sqrt{\log p}} < N < \min \left( 2p^{\frac{3}{4} + \frac{1}{4} \sqrt{\log p}}, \left( \frac{C(r - 1)}{C(r)} \right)^{\frac{3}{2} + \frac{1}{4} \sqrt{\log p}} \right)
\]

for $r \geq 3$. Now, we may proceed by induction, assuming that for all $h < N$, we have

\[
|S_h(M, N)| \leq C(r)h^{1 - \frac{1}{2}p^{\frac{1}{4} + \frac{1}{4} \sqrt{\log p}}}.
\]
Note that we have already established the result for \( h \leq C(r)^{r^2 + \frac{r}{p} + \sqrt{\log p}} \). Then, assume that for all \( h < N \), we have \(|S_h(M, N)| \leq C(r)h^{1 + \frac{1}{r^2}}(\log p)^{\frac{1}{p}}\). For such an \( h \), we can shift our character sum to yield

\[
S_h(M, N) = \sum_{n=M+1}^{M+N} \chi(n + h) + \sum_{n=M+1}^{M+N} \chi(n) - \sum_{n=M+N+1}^{M+N+h} \chi(n).
\]

In anticipation of using our inductive hypothesis on the last two sums in the above equation, we may write

\[
S_h(M, N) = \sum_{n=M+1}^{M+N} \chi(n + h) + 2\theta(h)E(h),
\]

where \( |\theta(h)| \leq 1 \) and \( E(h) = \max_K|S_h(K, h)| \). Now, let \( A \) and \( B \) be real numbers and average over all the shifts of length \( h = ab \) where \( a \leq A, b \leq B \). Doing so establishes

\[
S_h(M, N) = \frac{1}{|A||B|} \sum_{a \leq A} \sum_{b \leq B} \chi(n + ab) + \frac{2}{|A||B|} \sum_{a \leq A} \sum_{b \leq B} \theta(ab)E(ab).
\]

Let

\[
V := \sum_{x \text{ (mod } p)} v(x) \left| \sum_{b \leq B} \chi(x + b) \right|^r.
\]

Then we have

\[
|S_h(M, N)| \leq \frac{V}{|A||B|} + \frac{2}{|A||B|} \sum_{a \leq A} \sum_{b \leq B} E(ab).
\]

Define

\[
V_1 := \sum_{x \text{ (mod } p)} v(x), \quad V_2 := \sum_{x \text{ (mod } p)} v^2(x), \quad \text{and } W := \sum_{x \text{ (mod } p)} \left| \sum_{1 \leq k \leq B} \chi(x + b) \right|^{2r}.
\]

and apply Hölder’s inequality to \( V \), with conjugates \( \frac{2r-1}{2r} \) and \( 2r \) to get,

\[
V \leq \left( \sum_{x \text{ (mod } p)} v(x)^{\frac{2r-2}{2r}} v(x)^{\frac{1}{2r}} \right)^{\frac{2r}{2r-1}} W^{\frac{1}{p'}}.
\]

We apply Hölder’s inequality a second time to the first sum, with conjugates \( \frac{2r-1}{2r-2} \) and \( 2r-2 \), resulting in

\[
V \leq V_1^{1 - \frac{1}{2r}} V_2^{\frac{1}{2r}} W^{\frac{1}{p'}}.
\]

We already have bounds for each of \( V_1, V_2, \) and \( W \). Trivially, \( V_1 = |A||N| \leq AN \). Using Lemma 2.7 we bound \( V_2 \) (meaning we insist that \( A > 2 \) and \( 2AN < p \)), and, using Theorem 1.4 we bound \( W \) for \( r \leq 9B \). With these bounds in hand, and letting \( k = AB/N \), we have

\[
\frac{V}{|A||B|} \leq \frac{V_1^{1 - \frac{1}{2r}} V_2^{\frac{1}{2r}} W^{\frac{1}{p'}}}{|A||B|} \leq \frac{A}{A - 1} \cdot \frac{B}{B - 1} \cdot \frac{N^{1 - \frac{1}{2r}}}{k^{\frac{1}{2r}}} \cdot \frac{2WB^{\frac{1}{p'}}}{B} \cdot \left( \frac{6}{\pi^2} \log(e^{1 + \frac{1}{r}}) \right)^{\frac{1}{2r}} \cdot \left( \frac{e^{1 + \frac{1}{r}}}{1 + \log(e^{1 + \frac{1}{r}})} \right)^{\frac{1}{r}} \cdot \frac{3}{2} + \frac{k}{2B} - \frac{1}{2\nu_r(p)} - \frac{1}{A}.
\]
We wish to minimize the right-hand side of (3.3). We can start by fixing $B$ so that \( \frac{(2WB)^{\frac{1}{2}}}{B} \) is minimized. One sees that such a $B$ is

\[
(2r - 3)!(r - 1))^{\frac{1}{2}}p^{\frac{1}{2}}.
\]

Making the choice (3.4) for $B$, we determine that

\[
(2r - 3)!(r - 1))^{\frac{1}{2}}p^{\frac{1}{2}}.
\]

One may note that this exact expression is an improvement upon \[11, formula (3.9)\]. For example, with \( r = 2 \), we have \( 8^{\frac{1}{2}}p^{\frac{1}{2}} \) in place of \( 12^{\frac{1}{2}}p^{\frac{1}{2}} \).

For the error term, recall that the induction hypothesis implies

\[
E(ab) \leq C(r)(ab)^{1 - \frac{1}{2}}p^{\frac{1}{2}}(\log p)^{\frac{1}{2}}
\]

and thus,

\[
\frac{1}{p^{\frac{1}{2} - 2}}(\log p)^{\frac{1}{2}} \left( \frac{2}{|A||B|} \sum_{a \leq A, b \leq B} \theta(ab)E(ab) \leq \frac{2C(r)}{AB} \sum_{a \leq A, b \leq B} (ab)^{1 - \frac{1}{2}} \right.
\]

\[
\leq \frac{2C(r)}{AB} \left( \int_1^{A + 1} t^{\frac{1}{2}} dt \right) \left( \int_1^{B + 1} t^{\frac{1}{2}} dt \right) \frac{AB}{(A - 1)(B - 1)}
\]

\[
\leq C(r)(AB)^{1 - \frac{1}{2}} \left( \frac{2}{(2 - \frac{1}{2})^2} \right) \frac{(A + 1)(B + 1)}{AB} \frac{AB}{(A - 1)(B - 1)}
\]

\[
= \frac{2r^2}{(2r - 1)^2}C(r)(B + 1)^{1 - \frac{1}{2}}N^{1 - \frac{1}{2}} \left( \frac{(A + 1)(B + 1)}{AB} \right)^{1 - \frac{1}{2}} \frac{AB}{(A - 1)(B - 1)}.
\]

Combining (3.3), (3.5), and (3.6) in (3.2), we determine that

\[
\left| S_x(M, N) \right| \leq \frac{2B}{(A - 1)(B - 1)} \left( (2r - 3)!(r - 1) \right)^{\frac{1}{2}} \frac{AB}{(2r - 3)(r - 1)} \left( \frac{r - 3)(2r - 1)(r - 1) + 1}{(2r - 3)(r - 1)} \right)^{\frac{1}{2}}
\]

\[
\cdot \left( \frac{6}{\pi^2} \log(e^{\gamma + \frac{1}{2} - \frac{1}{12A} - \frac{1}{144A} + \frac{1}{2688A} - \log(\log(A))} + \frac{3}{2} + \frac{k}{2B} - \frac{1}{2B} - \frac{1}{2\nu_r(p)} \right)^{\frac{1}{2}}
\]

\[
+ \frac{2r^2}{(2r - 1)^2}C(r)(B + 1)^{1 - \frac{1}{2}} \left( \frac{(A + 1)(B + 1)}{AB} \right)^{1 - \frac{1}{2}} \frac{AB}{(A - 1)(B - 1)}.
\]

If we set the right hand side of (3.7) equal to $C(r)$ and solve, we find that

\[
C(r) = \frac{2B}{(A - 1)(B - 1)} \left( (2r - 3)!(r - 1) \right)^{\frac{1}{2}} \frac{AB}{(2r - 3)(r - 1)} \left( \frac{r - 3)(2r - 1)(r - 1) + 1}{(2r - 3)(r - 1)} \right)^{\frac{1}{2}}
\]

\[
\cdot \left( \frac{6}{\pi^2} \log(e^{\gamma + \frac{1}{2} - \frac{1}{12A} - \frac{1}{144A} + \frac{1}{2688A} - \log(\log(A))} + \frac{3}{2} + \frac{k}{2B} - \frac{1}{2B} - \frac{1}{2\nu_r(p)} \right)^{\frac{1}{2}}
\]

\[
1 - \frac{2r^2}{(2r - 1)^2}C(r)(B + 1)^{1 - \frac{1}{2}} \left( \frac{(A + 1)(B + 1)}{AB} \right)^{1 - \frac{1}{2}} \frac{AB}{(A - 1)(B - 1)}.
\]

Up to the issue of minimizing $C(r)$, this proves the result.

Say we have chosen an $r$ and fixed a lower bound $p_0$ for $p$. Note that this fixes $B$ in (3.4). To have used Lemma 2.7 we must have had $2 < A < \frac{r}{r - 1}$ and $2AN < p$, and we know that $A = \frac{4N}{B}$. Initially, one may take a poor guess for $C(r)$, but better guesses yield better constants, so one should iterate this process to determine optimal choices for $C(r)$. Having chosen $C(r)$, and using (3.1), we can pick $k$ such that

\[
\frac{2B}{C(r)^2p^{\frac{1}{2}}(\log p)^{\frac{1}{2}}} < k < \frac{Bp}{2\nu_r(p)^2},
\]

which is contained in

\[
\frac{2B}{N} < k < \frac{Bp}{2N^2}.
\]
The value of $C(r)$ decreases in the parameter $A$. Noting that each choice of $k$ fixes a lower bound on $A$, say $A_0$, we can vary $k$ over (3.8), evaluating $C(r)$ at $A = A_0$. Taking the value of $k$ which produces the smallest value for $C(r)$, we determine the constants given in Table 3.

The proof of Theorem 1.6 establishes explicit Burgess constants for a limited range of $N$. If we have access to a version of Theorem 1.5 with $r = 2$ and $c(2)$ small enough, we can extend this range for $r > 2$. Here, we prove such a result.

**Proof of Theorem 1.5 (for $r = 2$).** The argument will effectively be the same as in the proof of Theorem 1.6. Again the proof is by induction, where, in light of Theorem 1.2 and the trivial bound on character sums, we only need to consider $N$ in the range

$$c(2)^2 p^{2/3} \log p < N < \frac{(\alpha_1 \sqrt{p} \log p + \alpha_2 \sqrt{p})^2}{c(2)^2 p^{2/3} \log p},$$

where $c(2) < c(2)$. Now, our inductive step is to assume that for all $h < N$, we have $|S_h(M, N)| \leq c(2) h^{2/3} p^{2/3} (\log p)^{2/3}$. Therefore, the error term will be

$$\frac{1}{p^{1/3} \log p} \left[ \frac{2}{(2r-1)} \sum_{a \leq A} \sum_{b \leq B} \theta(ab) E(ab) \right]$$

(3.9)

$$\leq \frac{2 r^2}{(2r-1)^2} c(r) k^{-1/2} N^{1/2} \frac{(A+1)(B+1)}{AB} \frac{2^{-1}}{AB} \frac{AB}{(A-1)(B-1)}.$$

In light of Lemma 2.6 (thereby insisting that $2 < A < N$), we may bound the main term as

$$\frac{V}{[A][B]} \leq \frac{A}{A-1} \cdot \frac{B}{B-1} \cdot \frac{N^{1/2}}{k^{1/2}} \cdot \frac{(2WB)^{1/2}}{B} \cdot \left( \frac{0.83575 k \nu_2(p)}{pB} + \frac{6}{\pi^2} \log(e^{\gamma} + \frac{1}{12\pi^2} + \frac{1}{3\pi^2} \cdot \frac{k \nu_2(p)}{B}) + \frac{3}{2} + \frac{k}{2B} - \frac{1}{2 r \nu_2(p)} - \frac{1}{4} \right)^{1/2}.$$

Combining (3.10) and (3.9) in (3.2) implies, for $r = 2$,

$$\frac{|S_h(M, N)|}{N^{3/4} p^{2/3} (\log p)^{2/3}} \leq \frac{AB}{(A-1)(B-1)} \left( \frac{8 c(2)^2}{(A+1)(B+1)} \right)^{1/2} \left( \frac{(A+1)(B+1)}{AB} \right)^{1/2} \cdot \left( \frac{0.83575 k \nu_2(p)}{pB} + \frac{6}{\pi^2} \log(e^{\gamma} + \frac{1}{12\pi^2} + \frac{1}{3\pi^2} \cdot \frac{k \nu_2(p)}{B}) + \frac{3}{2} + \frac{k}{2B} - \frac{1}{2 r \nu_2(p)} - \frac{1}{2} \right)^{1/2}.$$ 

If we set the right hand side of (3.7) equal to $c(2)$ and solve, we find that

$$c(2) = \frac{AB}{(A-1)(B-1)} \left( \frac{0.83575 \frac{k \nu_2(p)}{pB} + \frac{6}{\pi^2} \log(e^{\gamma} + \frac{1}{12\pi^2} + \frac{1}{3\pi^2} \cdot \frac{k \nu_2(p)}{B}) + \frac{3}{2} + \frac{k}{2B} - \frac{1}{2 r \nu_2(p)} - \frac{1}{2} \right)^{1/2}.$$

We may minimize $c(2)$ as we did with $C(r)$, noting that the conditions on $A$ in Lemma 2.6 require that we have $2 < A < \frac{N}{11}$, or rather that we vary $k$ so that

$$\frac{2B}{c(2)^2 p^{2/3} (\log p)} < k < \frac{B}{11}.$$

In light of (3.4), one notes that $B = p^{2/3}$ in this setting. Choosing the $k$ which optimizes $c(2)$ gives us the constants in Table 3.

**Remark 3.1.** One can make some additional improvements to $C(r)$ using $c(2)$. Observe that, if

$$C(2; p_1) \leq c(2; p_0)(\log p_1)^{2/3} \leq C(2; p_0),$$

$$2B/c(2)^2 p^{2/3} (\log p) < k < B/11.$$
then, in light of Theorem 1.5 we may replace $C(2; p_0)$ with $c(2)(\log p_1)^{\frac{1}{2}}$ for any $p \in [p_0, \infty)$. Checking this for $r = 2$ and $p_1 = 10p_0$ (using the $c(2)$ corresponding to even $\chi$, since they are larger in all cases), we may adjust the constants in Theorem 1.8 to those in Table 3 for $p_0 \leq 10^5$. In order to minimize the upper bound in (3.11) in the proof of Theorem 1.6, we should use these adjusted constants when stepping from $r = 2$ to $r = 3$. Using the adjusted constants for $r = 2$, we determine the constants for $r = 3$, as provided below. One may wish to verify that establishing better constants in Theorem 1.5 for $p = 3$ would not admit the same adjustments.

\[\begin{array}{cccccccc}
\chi \text{ even} & p_0 = 10^5 & p_0 = 10^6 & p_0 = 10^7 & p_0 = 10^8 & p_0 = 10^9 & p_0 = 10^{10} \\
& 1.9256 & 1.7309 & 1.5962 & 1.4989 & 1.4276 & 1.3732 \\
\chi \text{ odd} & & 1.8779 & 1.6918 & 1.5734 & 1.4786 & 1.4092 & 1.3563 \\
\end{array}\]

\[\begin{array}{cccccccc}
\chi \text{ even} & p_0 = 10^{11} & p_0 = 10^{12} & p_0 = 10^{13} & p_0 = 10^{14} & p_0 = 10^{15} & p_0 = 10^{16} \\
& 1.3732 & 1.3299 & 1.2943 & 1.2641 & 1.2381 & 1.2151 \\
\chi \text{ odd} & & 1.3563 & 1.3141 & 1.2795 & 1.2501 & 1.2246 & 1.2021 \\
\end{array}\]

\[\begin{array}{cccccccc}
\chi \text{ even} & p_0 = 10^{17} & p_0 = 10^{18} & p_0 = 10^{19} & p_0 = 10^{20} & p_0 = 10^{20} & p_0 = 10^{75} \\
& 1.1945 & 1.1759 & 1.1589 & 1.1433 & 1.1288 & 0.9178 \\
\chi \text{ odd} & & 1.1819 & 1.1635 & 1.1467 & 1.1312 & 1.1167 & 0.8961 \\
\end{array}\]

As in [11, Corollary 1.8], we can omit the condition on $N$ in Theorem 1.6 by using the Burgess bound in Theorem 1.4 with $r = 2$. The advantage of having a smaller constant is that our results are now valid for primes as small as $10^8$ (previously we could only take primes as small as $10^{10}$).

**Proof of Theorem 1.8.** We need to establish that if $N \geq 2p^{\frac{1}{2}+\frac{1}{2}}$, then Theorem 1.5 implies the inequality in Theorem 1.8. For $p \geq p_0$, Corollary 1.5 implies

\[|S_\chi(M, N)| < c(2; p_0)N^{\frac{1}{2}}p^{\frac{1}{2}}(\log p)^{\frac{1}{2}}.\]

For $r \geq 3$, this inequality implies

\[|S_\chi(M, N)| < C(r; p_0)N^{1-\frac{1}{2}}p^{\frac{r-1}{2r}}(\log p)^{\frac{1}{2}}\]

whenever

\[N > \left(\frac{C(r; p_0)}{c(2; p_0)}\right)^{\frac{2r}{2r-1}}p^{\frac{2r}{2r-1}}(\log p)^{\frac{1}{2}}.\]

Now, if

\[N \leq \left(\frac{C(r; p_0)}{c(2; p_0)}\right)^{\frac{2r}{2r-1}}p^{\frac{2r}{2r-1}}\]

then

\[N < 2p^{\frac{1}{2}+\frac{1}{2}}\]

whenever

\[(3.11) \quad \left(\frac{c(2; p_0)}{C(r; p_0)}\right)^{\frac{2r}{2r-1}} < \frac{2p^{\frac{1}{2}}}{(\log p)^{\frac{1}{2}}}\]

Taking all combinations of $p_0$ and $r$ available in Tables 3 and 6, one can verify that (3.11) holds for $p > p_0 \geq 10^8$ when $3 \leq r \leq 6$ or $p > p_0 \geq 10^9$ when $7 \leq r \leq 10$.

**Remark 3.2.** One could improve the ranges on $p_0$ in Theorem 1.8 by making the constant 2 in the condition on $N$ worse. Since this would be a less restrictive condition, it would result in larger $C(r)$. However, it appears the adjustment that would be necessary to extend Theorem 1.8 to $p_0 \geq 10^5$ would cause the constants to be much larger than desirable. For this reason, we make no adjustment and accept $p_0 \geq 10^9$ in Theorem 1.8.
4. Least $k$th Power Non-Residues

We wish to improve upon the result of Treviño [11, Theorem 1.10], which established that there is a $k$th power non-residue $(\mod p)$ smaller than $p^k$ for all primes greater than $10^{4732}$. The approach used by Treviño could be used to establish a result for $p^n$ up to the bound $\alpha > \frac{1}{\log x}$. We apply our constants in Treviño’s proof to establish a similar result for $p^n$ where $\alpha = \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$. The proof starts with the following lower bound for character sums.

**Lemma 4.1.** [11, Lemma 5.3] Let $x \geq 286$ be a real number, and let $y = x^{\frac{1}{k} + \frac{1}{\alpha}}$ for some $\delta > 0$. Let $\chi$ be a non-principal character $(\mod p)$ for some prime $p$. If $\chi(n) = 1$ for all $n \leq y$, then

$$\sum_{n \leq x} \chi(n) \geq x \left( 2 \log(\delta \sqrt{c} + 1) - \frac{1}{\log^2 x} - \frac{1}{\log^2 y} - \frac{1}{x} \right).$$

We are now in a position to prove Theorem 1.10.

**Proof of Theorem 1.10.** Let $k$ be a divisor of $p - 1$ and $\chi$ be a non-principal character modulo $p$ of order $k$. Then, if $\chi(n) \neq 1$, $n$ is a $k$th power non-residue modulo $p$. Fix $\alpha$ to be one of $\frac{1}{4}, \frac{1}{5}, \frac{1}{6}$. Let $x \geq 286$ and $y = x^{\frac{1}{k} + \frac{1}{\alpha}} = p^\delta$, where $\delta > 0$ will be a constant that is determined when $x$ is fixed later in the proof. If we suppose that for all $n \leq y$ we have $\chi(n) = 1$, then by comparing Lemma 4.1 with Theorem 1.8 we have that

$$C(r)x^{1 - \frac{1}{r}}(\log p)^{\frac{1}{r}} \geq x \left( 2 \log(\delta \sqrt{c} + 1) - \frac{1}{\log^2 x} - \frac{1}{\log^2 y} - \frac{1}{x} \right).$$

If we take $x = p^{\frac{1}{k} + \frac{1}{\alpha}}$, then we have

$$(4.1) \quad C(r)p^{\log \log p - \frac{1}{r}} \geq 2 \log(\delta \sqrt{c} + 1) - \frac{1}{\log^2 x} - \frac{1}{\log^2 y} - \frac{1}{x}.$$

One observes that, as $p$ increases, the left-hand side of (4.1) eventually decreases to 0, while the right-hand side eventually increases to $2 \log(\delta \sqrt{c} + 1)$. Therefore, we can obtain a contradiction for $p$ large enough. For each choice of $\alpha$, we will need to take $r$ large enough so that $\delta > 0$. The cases $\alpha = \frac{1}{4}$ and $\alpha = \frac{1}{5}$ are easier, since $\delta > 0$ for $r \geq 4$ or $r \geq 7$, respectively. Taking (4.1) with $\alpha = \frac{1}{4}$ and $r = 4$ gives us $\delta = 0.060136 \ldots$ and $C(4; 10^{75}) = 1.9011$. With these values, we determine that (4.1) fails when $p \geq 10^{83}$. Taking (4.1) with $\alpha = \frac{1}{5}$ and $r = 7$ gives us $\delta = 0.015691 \ldots$ and $C(7; 10^{75}) = 1.5947$. With these values, we determine that (4.1) fails when $p \geq 10^{843}$.

In the case of $\alpha = \frac{1}{6}$, we take a little more care to ensure a good result. For this case, $\delta$ is only positive once $r \geq 21$, so we must compute $C(r)$ for larger $r$ than were given in Theorem 1.8. Treviño gives us a rough sense of how large the primes need to be for Theorem 1.10 to hold when $\alpha = \frac{1}{6}$. That is, he shows $p \geq 10^{4732}$, which suggests that computing $C(r; 10^{4732})$ should be suitable for our purposes. We have done so for $2 \leq r \leq 25$ and compiled these results in Table 7.

| $r$ | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|---|
| $C(r)$ | 2.8470 | 2.1051 | 1.8821 | 1.7492 | 1.6561 | 1.5859 |

| $r$ | 8 | 9 | 10 | 11 | 12 | 13 |
|-----|---|---|---|---|---|---|
| $C(r)$ | 1.5308 | 1.4862 | 1.4492 | 1.4180 | 1.3913 | 1.3681 |

| $r$ | 14 | 15 | 16 | 17 | 18 | 19 |
|-----|---|---|---|---|---|---|
| $C(r)$ | 1.3478 | 1.3298 | 1.3138 | 1.2995 | 1.2865 | 1.2747 |

| $r$ | 20 | 21 | 22 | 23 | 24 | 25 |
|-----|---|---|---|---|---|---|
| $C(r)$ | 1.2640 | 1.2541 | 1.2450 | 1.2367 | 1.2289 | 1.2217 |

For each $r \geq 21$ in Table 7, we can check inequality (4.1) using $C(r)$ and the appropriate $\delta$ to determine which $r$ gives us the best possible contradiction. It happens that this is when $r = 23$, where we have $\delta = 0.006802 \ldots$ and (4.1) is false for all $p \geq 10^{3872}$. \square
5. Norm-Euclidean Cyclic Fields

In [7, Proposition 2.4], Burgess’ bound is used to provide unconditional upper bounds on the size of the conductor of norm-Euclidean cyclic fields with prime degree $3 \leq \ell \leq 100$. Here, we update these bounds using the improved Burgess constants.

**Proof of Theorem 1.11** Following the proof of [8, Proposition 5.7], let $100 < q_1 < q_2$ be primes and define $D_2(r)$ by

$$D_2(r) \geq \frac{K_1}{K_2} \frac{(1 + C(r)^{-1})}{C(r)},$$

where

$$K_1 = \left(1 + q_1^{-\frac{1}{r}}\right) \left(1 + q_2^{-\frac{1}{r}}\right) \text{ and } K_2 = (1 - q_1^{-1}) (1 - q_2^{-1}).$$

In the proof in [7, pp. 2547-48], we may take $C(r) = C(r; 10^{40})$, where $r = 4$ for $\ell = 5, 7$ and $r = 3$ otherwise. Then, by inequality (8.1) in [7], the bound on the conductor for $\ell > 3$ is given by the smallest $f$ for which

$$f \geq 2.7 D_2(r)^r (l - 1)^r f^{\frac{3l+1}{r^2}} (\log f)^{\frac{5}{2}}.$$

For $\ell = 3$, we use $r = 4$ and

$$f \geq 8 D_2(4)^r (l - 1)^r f^{\frac{3l+1}{r^2}} (\log f)^{\frac{5}{2}}.$$

We computed $C(3; 10^{40}) = 2.1590344\ldots$ and $C(4; 10^{40}) = 1.9146092\ldots$, which yields $D_2(3) = 3.5239$ and $D_2(4) = 3.1608$ (rounded up). Then we determined where (5.1) was true to establish the bounds in Table 5. □

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References

[1] D. A. Burgess, On character sums and L-series, Proc. London Math. Soc. (3) 12 (1962), 193–206. MR 0132733
[2] ______, On character sums and L-series. II, Proc. London Math. Soc. (3) 13 (1963), 524–536. MR 0148626
[3] ______, The character sum estimate with \( r = 3 \), J. London Math. Soc. (2) 33 (1986), no. 2, 219–226. MR 838632
[4] Daniele Dona, Harald A. Helfgott, and Sebastian Żuniga Alterman, Explicit \( L^2 \) bounds for the Riemann \( \zeta \) function, arXiv e-prints (2019), arXiv:1906.01097.
[5] D. A. Frolenkov and K. Soundararajan, A generalization of the Pólya-Vinogradov inequality, Ramanujan J. 31 (2013), no. 3, 271–279. MR 3081668
[6] Kostadinka Lapkova, Correction to: Explicit upper bound for the average number of divisors of irreducible quadratic polynomials, Monatsh. Math. 186 (2018), no. 4, 675–678. MR 3829217
[7] Pierre Lezowski and Kevin J. McGown, The Euclidean algorithm in quintic and septic cyclic fields, Math. Comp. 86 (2017), no. 307, 2535–2549. MR 3647971
[8] Kevin J. McGown, Norm-Euclidean cyclic fields of prime degree, Int. J. Number Theory 8 (2012), no. 1, 227–254. MR 2887892
[9] Hugh L. Montgomery and Robert C. Vaughan, Multiplicative number theory. I. Classical theory, Cambridge Studies in Advanced Mathematics, vol. 97, Cambridge University Press, Cambridge, 2007. MR 2378655
[10] S. S. Pillai and S. D. Chowla, On the Error Terms in some Asymptotic Formulae in the Theory of Numbers (1), J. London Math. Soc. 5 (1930), no. 2, 95–101. MR 1574229
[11] Enrique Treviño, The Burgess inequality and the least \( k \)th power non-residue, Int. J. Number Theory 11 (2015), no. 5, 1653–1678. MR 3376232
[12] Arnold Walfisz, Weylsche Exponentialsummen in der neueren Zahlentheorie, Mathematische Forschungsberichte, XV, VEB Deutscher Verlag der Wissenschaften, Berlin, 1963. MR 0220685
[13] André Weil, On some exponential sums, Proc. Nat. Acad. Sci. U. S. A. 34 (1948), 204–207. MR 0027006

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