Curvature Dependence of Running Gauge Coupling and Confinement in AdS/CFT Correspondence

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abstract

We construct IIB supergravity (viewed as dilatonic gravity) background with non-trivial dilaton and with curved four-dimensional space. Such a background may describe another vacuum of maximally supersymmetric Yang-Mills theory or strong coupling regime of (non)-supersymmetric gauge theory with (power-like) running gauge coupling which depends on curvature. Curvature dependent quark-antiquark potential is calculated where the geometry type of hyperbolic (or de Sitter universe) shows (or not) the tendency of the confinement. Generalization of IIB supergravity background with non-constant axion is presented. Quark-antiquark potential being again curvature-dependent has a possibility to produce the standard area law for large separations.

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1 Introduction

AdS/CFT correspondence [1] may provide new insights to the understanding of non-perturbative QCD. For example, in frames of Type 0 String Theory the attempts [2] have been done to reproduce such well-known QCD effects as running gauge coupling and possibly confinement. It is among the first problems to get the description of well-known QCD phenomena from bulk/boundary correspondence.

In another approach one can consider IIB supergravity (SG) vacuum which describes the strong coupling regime of a non-supersymmetric gauge theory. This can be achieved by the consideration of deformed IIB SG vacuums, for an example, with non-constant dilaton which breaks conformal invariance and supersymmetry (SUSY) of boundary supersymmetric Yang-Mills (YM) theory. Such a background will be the perturbation of AdS$_5 \times S_5$ vacuum. The background of such a sort (with non-trivial dilaton) which interpolates between AdS (UV) and flat space with singular dilaton (IR) has been found in ref. [3] where also conformal dimensions for (dilaton coupled) scalar have been found.

This solution of IIB SG [3] has been used in ref. [4] with the interpretation of it as the one describing the running gauge coupling (via exponent of dilaton). It has been shown that running gauge coupling has a power law behavior with ultraviolet (UV) stable fixed point and quark-antiquark potential [4] has been calculated. QCD properties of such a background have been discussed in detail in refs. [3]. Modifications of IIB SG solution with non-constant dilaton [3] due to presence of axion [7], constant self-dual vector [8] or world volume scalar [9] give further proof of the possible confinement and asymptotic freedom of the boundary non-SUSY gauge theory. Unfortunately, situation is very complicated here due to double role of IIB SG backgrounds. From one side they may indeed correspond to IR gauge theory (deformation of initial SUSY YM theory). On the same time such a background may simply describe another vacuum of the same maximally supersymmetric YM theory with non-zero vacuum expectation value (VEV) of some operator. Due to the fact that operators corresponding to deformation to another gauge theory are not known, it is unclear what is the case under discussion (interpretation of SG background). Only some indirect arguments as below may be given. As we see these arguments indicate that IIB SG background discussed in this work most probably corresponds to another vacuum of super YM theory under consideration. Then renormalization group (RG)
flow is induced in the theory via giving a non-zero VEV to some operator.

In the present paper, we continue the study of running dilaton and confinement from IIB supergravity backgrounds with non-trivial dilaton. We generalize the solution of ref. [3] for non-zero curvature of $d$-dimensional space. As a result, IIB supergravity background is changed drastically. The running dilaton (gauge coupling) depends explicitly on the four-dimensional curvature. The structure of quark-antiquark potential is modified. In a sense, confinement would become the characteristic of the Universe.

Let us make few remarks on AdS/CFT interpretation of IIB SG background. Choosing the coordinates in the asymptotically AdS$_5$ spacetime as

$$ds^2 = d\sigma^2 + S(\sigma) \sum_{i,j=0}^{3} \eta_{ij} dx^i dx^j ,$$

let assume a scalar field $\lambda$, e.g. dilaton, axion or other fields, obey the following equation:

$$\frac{d^2 \lambda}{d\sigma^2} + 4 \frac{d\lambda}{d\sigma} = M^2 \lambda$$

near the boundary. Here $M^2$ is the “mass” of $\lambda$ and $\sigma \to 0$ corresponds to the boundary of AdS. Then $\lambda$ is associated with the operator $O_\lambda$ with conformal dimension $\Delta = 2 + \sqrt{4 + M^2}$. The solution of (2) is given by

$$\lambda = Ae^{-(4-\Delta)\sigma} + Be^{-\Delta\sigma} .$$

The solution corresponding to $A$ is not normalizable but the solution to $B$ is normalizable. According to the argument in [19], the non-normalizable solution would be associated with the deformation of the $\mathcal{N} = 4$ theory by $O_\lambda$ but the normalizable solution would be associated with a different vacuum where $O_\lambda$ has a non-zero vacuum expectation value. The behavior of the dilaton found in this paper is normalizable and seems to be associated with the dimension 4 operator, say $tr F^2$. Then the argument in [19] would indicate that the solution found in this paper should correspond to another vacuum of $\mathcal{N} = 4$ theory. Nevertheless, there might be still possibility that the solution corresponds to non-supersymmetric gauge theory. Since there occurs the condensation of $tr F^2$ in the usual non-supersymmetric QCD, however, the solution given here would describe some features typical for the non-supersymmetric theory.

The situation is even more complicated due to limits of validity of dual SG description. In order that the classical supergravity description is valid,
the curvature should be small and the string coupling should be also small. If the curvature is large, the α’ corrections from string theory would appear. In the AdS/CFT correspondence, the radius $R_s$ of the curvature is given by

$$R_s = (4\pi g_s N)^{\frac{1}{4}}.$$  \hspace{1cm} (4)

Here $g_s$ is the string coupling and $N$ is the number of the coincident D-branes. Therefore we should require

$$g_s N \gg 1.$$ \hspace{1cm} (5)

On the other hand, the classical picture works when the string coupling is small:

$$g_s \ll 1.$$ \hspace{1cm} (6)

In the solution given in this paper, there appears the curvature singularity and $g_s$ depends on the coordinates since the dilaton is non-trivial. If we concentrate on the behavior near the boundary, which is asymptotically AdS and is far from the singularity, the solution would be reliable and SG description would be trusted.

The work is organized as follows. In the next section we give IIB supergravity background with non-constant dilaton and non-flat four-dimensional space. Via AdS/CFT it gives the curvature dependent (power-like) running gauge coupling and quark-antiquark potential where hyperbolic geometry seems to support confinement. In section 3 we generalize the background of section 2 for the case when axion presents. (Curvature dependent) quark-antiquark potential is found. It is shown that inflationary Universe (de Sitter) with axion might predict confinement. Some outlook is given in the last section. Additional solutions of IIB supergravity are presented in two Appendixes.

## 2 Solution, Running Gauge Coupling and Quark-Antiquark Potential

We start from the following action of dilatonic gravity in $d + 1$ dimensions:

$$S = -\frac{1}{16\pi G} \int d^{d+1}x \sqrt{-G} \left( R - \Lambda - \alpha G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right).$$ \hspace{1cm} (7)

In the following, we assume $\lambda^2 \equiv -\Lambda$ and $\alpha$ to be positive. The action \((7)\) is very general. It contains the effective action of type IIB string theory. The
type IIB supergravity, which is the low energy effective action of the type IIB string theory, has a vacuum with only non-zero metric and the anti-self-dual five-form. The latter is given by the Freund-Rubin-type ansatz:

\[
F_{\mu \nu \rho \kappa \lambda} = -\frac{\sqrt{\Lambda}}{2} \epsilon_{\mu \nu \rho \kappa \lambda}, \quad \mu, \nu, \cdots = 0, 1, \cdots, 4
\]

\[
F_{ijkpq} = -\frac{\sqrt{\Lambda}}{2} \epsilon_{ijkpq}, \quad i, j, \cdots = 5, \cdots, 9 .
\]

The vacuum has the topology of AdS$_5 \times$S$^5$. Since AdS$_5$ has a four dimensional Minkowski space as a subspace, especially on its boundary, AdS$_5$ has the four dimensional Poincaré symmetry ISO(1,3). S$^5$ has, of course, SO(6) symmetry.

As an extension, we can consider the solution where the dilaton is non-trivial but the anti-self-dual five-form is the same as in (8). Furthermore if we require the solution has the symmetry of ISO(1,3) $\times$ SO(6), the metric should have the following form:

\[
ds^2 = G_{\mu \nu} dx^\mu dx^\nu + g_{mn} dx^m dx^n
\]

where $g_{mn}$ is the metric of S$^5$ and ref.[3]

\[
G_{\mu \nu} dx^\mu dx^\nu = f(y) dy^2 + y \sum_{i,j=0}^{d-1} \eta_{ij} dx^i dx^j .
\]

In order to keep the symmetry of ISO(1,3) $\times$ SO(6), the dilaton field $\phi$ can only depend on $y$. Then by integrating five coordinates on S$^5$, we obtain the effective five dimensional theory, which corresponds to $d = 4$ and $\alpha = \frac{1}{2}$ case in (8). We keep working with above dilatonic gravity as it will be easy to come to IIB supergravity ($d = 4, \alpha = \frac{1}{2}$) at any step.

From the variation of the action (8) with respect to the metric $G_{\mu \nu}$, we obtain\footnote{The conventions of curvatures are given by}

\[
0 = R_{\mu \nu} - \frac{1}{2} G_{\mu \nu} R + \frac{\Lambda}{2} G_{\mu \nu} - \alpha \left( \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} G_{\mu \nu} G^{\rho \sigma} \partial_\rho \phi \partial_\sigma \phi \right)
\]
and from that of dilaton $\phi$

$$0 = \partial_\mu \left( \sqrt{-G} G^\mu{}_{\nu} \partial_\nu \phi \right). \tag{12}$$

We assume that $\phi$ depends only on one of the coordinate, say $y \equiv x^d$ as in type IIB supergravity solution with the symmetry of $ISO(1, 3) \times SO(6)$ and we also assume, as a generalization of (10), that $G_{\mu\nu}$ has the following form

$$ds^2_{d+1} = \sum_{\mu, \nu=0}^{d} G_{\mu\nu} dx^\mu dx^\nu = f(y)dy^2 + y \sum_{i,j=0}^{d-1} g_{ij} dx^i dx^j \tag{13}$$

Here $g_{ij}$ is the metric in the Einstein manifold, which is defined by

$$r_{ij} = k g_{ij}. \tag{14}$$

Here $r_{ij}$ is the Ricci tensor given by $g_{ij}$ and $k$ is a constant, especially $k > 0$ for sphere and $k = 0$ for the flat Minkowski space and $k < 0$ for hyperboloid. Such a solution generalizes the previous solution of ref. [3] (where $k = 0$) as boundary gauge QFT lives now in four-dimensional curved spacetime. The case of $k = 1$ is especially interesting as it corresponds to gauge theory in de Sitter (inflationary) Universe.

The equations of motion (11) and (12) take the following forms:

0 = \frac{1}{2} \frac{rf}{y} - \frac{d}{8} \frac{(d-1)}{y^2} \frac{\lambda^2}{2} f + \frac{\alpha}{2} (\phi')^2 \tag{15}

0 = - \left( r_{ij} - \frac{1}{2} g_{ij} \right) \frac{f}{y} + \frac{d-1}{4} \frac{f'}{fy} - \frac{(d-1)(d-4)}{8} \frac{\lambda^2}{2} f - \frac{\alpha}{2} (\phi')^2 \tag{16}

0 = \left( \sqrt{\frac{\lambda}{f} \phi} \right)' \tag{17}

Here $'$ expresses the derivative with respect to $y$ and $r \equiv g^{ij} r_{ij} = kd$. Eq.(13) corresponds to $(\mu, \nu) = (d,d)$ in (11) and Eq.(16) to $(\mu, \nu) = (i,j)$. The case of $(\mu, \nu) = (0,i)$ or $(i,0)$ is identically satisfied. Integrating (17), we find

$$\phi' = c \sqrt{\frac{f}{y^d}}. \tag{18}$$
Substituting (18) into (15), we can solve it algebraically with respect to \( f \):

\[
f = \frac{d(d - 1)}{4y^2 \lambda^2 \left( 1 + \frac{\alpha c^2}{\lambda^2 y^d} + \frac{k d}{\lambda^2 y^d} \right)}.
\]  \hfill (19)

Then we find from (18) and (19),

\[
\phi = c \int dy \sqrt{\frac{d(d - 1)}{4y^{d+2} \lambda^2 \left( 1 + \frac{\alpha c^2}{\lambda^2 y^d} + \frac{k d}{\lambda^2 y^d} \right)}}.
\]  \hfill (20)

When \( y \) is small, \( f(y) \) in (19) behaves as

\[
f(y) \sim \frac{d(d - 1)y^{d-2}}{4\alpha c^2}, \quad \hfill (21)
\]

which makes a curvature singularity at \( y = 0 \). The scalar curvature behaves when \( y \sim 0 \) as

\[
R \sim \alpha c^2 y^{-d}. \quad \hfill (22)
\]

The curvature singularity would be generated by the singular behavior of the dilaton \( \phi \) when \( y \sim 0 \):

\[
\phi(y) \sim \text{sgn}(c) \sqrt{\frac{d(d - 1)}{4\alpha}} \ln y. \quad \hfill (23)
\]

Here \( \text{sgn}(c) \) expresses the sign of \( c \):

\[
\text{sgn}(c) = \begin{cases} 
+1 & \text{if } c > 0 \\
-1 & \text{if } c < 0
\end{cases}. \hfill (24)
\]

The curvature singularity tells there should appear the \( \alpha' \) correction from the string theory and the supergravity description would break down when \( y \sim 0 \). Conversely and hopefully, the curvature singularity might be apparent and vanish when we can include full string corrections. In any case, the solution would be valid if we investigate the behavior near the boundary \( (y \to +\infty) \).

We also note that the dilaton field behaves near the boundary \( (y \to +\infty) \) as

\[
\phi \sim \phi_0 - c \sqrt{\frac{d}{d\lambda^2 y^d}} + \cdots. \quad \hfill (25)
\]
The term of \( \mathcal{O} \left( \frac{1}{y^2} \right) \) might tell that the solution given here would correspond to the condensation of the dimension \( d \) operator, say, \( \text{tr} F^2 \). In the usual non-(or lower-)supersymmetric QCD, it is widely believed that there would occur the condensation of \( \text{tr} F^2 \). Therefore not depending on that the solution given here corresponds the real deformation from the \( \mathcal{N} = 4 \) theory or the deformation of the vacuum, the solution would possibly reflect the structure of non-supersymmetric QCD.

If we change the coordinate \( y \) by \( \rho \), which is defined by

\[
\rho \equiv - \int dy \frac{f(y)}{y} = - \int dy \frac{d(d-1)}{4y^3\lambda^2 \left( 1 + \frac{\alpha c^2}{\lambda^2 y^2} + \frac{kd}{\lambda^2 y^2} \right)},
\]

the metric in (13) has the following form

\[
G_{\mu\nu} dx^\mu dx^\nu = \Omega^2(\rho) \left( d\rho^2 + \sum_{i,j=0}^{d-1} g_{ij} dx^i dx^j \right).
\]

Here \( \Omega^2(\rho) \) is given by solving \( y \) in (26) with respect to \( \rho \): \( \Omega^2(\rho) = y(\rho) \). When \( \rho \) is small, \( y \) is large and the structure of the spacetime becomes AdS asymptotically. From (26), we find

\[
\rho = \frac{\sqrt{d(d-1)}}{\lambda y^\frac{1}{2}} \left( 1 + \mathcal{O} \left( y^{-1} \right) \right).
\]

Therefore we find

\[
\Omega^2(\rho) = y(\rho) = \frac{R_s^2}{\rho^2} \left( 1 + \mathcal{O} \left( \rho^2 \right) \right),
\]

\[
R_s \equiv \frac{\sqrt{d(d-1)}}{\lambda}.
\]

We can compare the above behavior with that of the previous \( \text{AdS}_5 \times \mathbb{S}^5 \) solution in type IIB supergravity [4]. The \( \text{AdS}_5 \) part in the solution has the form of

\[
ds^2_{\text{AdS}_5} = (4\pi g_s N)^{\frac{1}{2}} \cdot \frac{1}{\rho^2} \left( d\rho^2 + \sum_{i,j=0}^{d-1} \eta_{ij} dx^i dx^j \right).
\]

Therefore we find

\[
R_s = (4\pi g_s N)^{\frac{1}{2}},
\]
where \( g_s \) is the string coupling and \( N \) is the flux of the five-form \( F \) in (8) through \( S^5 \), which is produced by the \( N \) coincident D3-branes. Using the definition of \( R_s \) in (29), the solution (19) and (20) has the following form:

\[
\begin{aligned}
f &= \frac{R_s^2}{4y^2 \left( 1 + \frac{c^2 R_s^2}{2d(d-1)y^2} + \frac{k}{(d-1)y} \right) }, \\
\phi &= c \int dy \sqrt{\frac{R_s^2}{4y^{d+2} \left( 1 + \frac{c^2 R_s^2}{2d(d-1)y^2} + \frac{k}{(d-1)y} \right) }}.
\end{aligned}
\]

(32)

Here we put \( \alpha = \frac{1}{2} \) and \( d = 4 \) in order to get explicitly IIB supergravity background. On the other hand, if we change the coordinate by

\[
\sigma = \int dy \sqrt{f(y)},
\]

(33)

the metric in (13) has the following form

\[
G_{\mu\nu}dx^\mu dx^\nu = d\sigma^2 + S(\sigma) \sum_{i,j=0}^{d-1} g_{ij} dx^i dx^j,
\]

(34)

where \( S(\sigma) \) is given by solving \( y \) in (33) with respect to \( \sigma \): \( S(\sigma) = y(\sigma) \).

We now consider the case \( k < 0 \). First let the dilaton field to be constant or small. Then from Eq.(19), when \( y \) decreases from the positive infinity, the function \( f \) increases and diverges at a finite value of \( y : y = y_0 \) and after that the signature of the metric seems to change. This is not, however, real but apparent. Near \( y = y_0 \), the function \( f(y) \) behaves as

\[
f(y) \sim \frac{f_0}{y - y_0},
\]

(35)

where \( f_0 \) is a constant. When we introduce a new coordinate \( u \) by

\[
y - y_0 = u^2,
\]

(36)

the metric has the following form when \( y \sim y_0 \),

\[
ds_{d+1}^2 \sim 4f_0 du^2 + y_0 \sum_{i,j=0}^{d-1} \eta_{ij} dx^i dx^j.
\]

(37)

The metric in (37) is regular even when \( u \sim 0 \) \((y \sim y_0)\) and there is no curvature singularity. The change of coordinates in (36) tells that \( y \) increases
again as \( u \) increases when \( u > 0 \). Then when we write the solution by the coordinate \( u \), the solution connects two boundaries at \( u = -\infty \) and \( u = +\infty \). The structure of the spacetime, however, changes when the dilaton becomes large. Let us write \( f(y) \) in the following form:

\[
f(y) = \frac{d(d-1)}{4y^2\lambda^2 h(y)}, \quad h(y) \equiv 1 + \frac{\alpha c^2}{\lambda^2 y^d} + \frac{kd}{\lambda^2 y}.
\]  

(38)

We now investigate the condition \( h(y) \) vanishes or \( f(y) \) diverges and changes its sign. The minimum \( h_{\text{min}} \) of \( h(y) \) can be found by the equation \( \frac{dh(y)}{dy} = 0 \), which can be solved as follows:

\[
y = y_0 \equiv \left( -\frac{\alpha c^2}{k} \right)^{\frac{1}{d-1}}
\]  

(39)

and we find

\[
h_{\text{min}} = 1 + \frac{k(d-1)}{\lambda^2} \left( -\frac{\alpha c^2}{k} \right)^{-\frac{1}{d-1}}.
\]  

(40)

Therefore \( h(y) \) does not vanish if \( h_{\text{min}} > 0 \), that is

\[
c^2 > c_0^2 \equiv -\frac{k}{\alpha} \left( -\frac{\lambda^2}{k(d-1)} \right)^{1-d}.
\]  

(41)

When \( c^2 > c_0^2 \), the solution connects the boundary at \( y = \infty \) with the singular boundary at \( y = 0 \) as in the \( k = 0 \) and \( k > 0 \) cases.

We now consider the running of the gauge coupling. Usually the AdS string coupling, which is the square of the coupling in \( \mathcal{N} = 4 \) \( SU(N) \) super-Yang-Mills when \( d = 4 \), is proportional to an exponential of the dilaton field \( \phi \), which we assume in the following. From (20), when \( y \) is large and \( d > 2 \), we find that the dilaton field behaves as

\[
\phi = \phi_0 + \frac{c}{2\lambda} \sqrt{\frac{d(d-1)}{d-1}} \left\{ -\frac{2y^{-\frac{d-2}{2}}}{d} + \frac{2}{d+2} \cdot \frac{kd}{2\lambda^2 y^{-\frac{d-2}{2}} - 1} + \ldots \right\}.
\]  

(42)

Here \( \cdots \) expresses the higher order terms of \( \frac{1}{y} \). We now assume the gauge coupling has the following form \( [4, 5, 6, 7, 8, 9] \) (of course, other ways to define running gauge coupling might be possible)

\[
g = g_s c^{2\beta} \sqrt{\frac{\alpha}{d(d-1)}} (\phi - \phi_0)
\]

\[
= g_s \left\{ 1 - \frac{2\beta c \sqrt{\alpha}}{d\lambda} y^{-\frac{d-2}{2}} + \frac{kd\beta c \sqrt{\alpha}}{(d+2)\lambda^3} y^{-\frac{d-2}{2} - 1} + \ldots \right\}
\]  

(43)
In case of type IIB supergravity \((\alpha = \frac{1}{2})\),
\[
\beta = \sqrt{\frac{d(d-1)}{2}}
\]  
and using the definition of \(R_s\) in (29), we find
\[
g = g_s \left\{ 1 - \frac{c R_s}{d} y^{-\frac{d}{2}} + \frac{kc R_s^3}{2(d+2)(d-1)} y^{-\frac{d}{2}-1} + \cdots \right\}
\]  
The next-to-leading order term is proportional to \(k\) if \(k \neq 0\). This changes the renormalization group equations drastically. If we multiply \(N^{\frac{1}{2}}\) with \(g\), we obtain the 't Hooft coupling \(g_H = g N^{\frac{1}{2}}\). If we define a new coordinate \(U\) by
\[
y = U^2,
\]
\(U\) expresses the scale on the (boundary) \(d\) dimensional space (due to holography [10]). Following the correspondence between long-distances/high-energy in the AdS/CFT scheme, \(U\) can be regarded as the energy scale of the boundary field theory. Then from (43), we obtain the following renormalization group equation
\[
\beta(U) \equiv U \frac{dg}{dU} = -d(g - g_s) - \frac{2kd\beta c \sqrt{\alpha}}{(d+2)\lambda^3} \left( -\frac{d\lambda}{2\beta c \sqrt{\alpha}} \right)^{\frac{d}{2}+1} \left( \frac{g - g_s}{g_s^{\frac{d}{2}}} \right)^{\frac{d}{2}+1} + \cdots.
\]

The leading behavior is identical with the previous works [4, 6, 7, 8] but the next to leading term contains the fractional power of \((g - g_s)\) although the square of \((g - g_s)\) appears for \(k = 0\) case. We should note that the qualitative behavior does not depend on \(\beta\) which appears in the coupling (43).

Hence, we found that beta-function explicitly depends on the curvature of four-dimensional manifold. Of course, curvature dependence is not yet logarithmic as it happens with usual quantum field theories (QFTs) (perturbative consideration) in curved spacetime [11]. The power-like running of gauge coupling is much stronger than in \(k = 0\) case. Note that previous discussion of power-like running includes GUTs with large internal dimensions [12]. In the case under investigation we get the gauge coupling beta-function as an expansion on fractional powers of gauge coupling.

We now consider the static potential between “quark” and “antiquark” [5]. We evaluate the following Nambu-Goto action
\[
S = \frac{1}{2\pi} \int d\tau d\sigma \sqrt{\det \left( g_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu \right)}.
\]
with the “string” metric $g_{\mu \nu}$, which could be given by multiplying a dilaton function $k(\phi)$ to the metric tensor in (4). Especially we choose $k(\phi)$ by

$$k(\phi) = e^{2\gamma \sqrt{\alpha/(d-1)(\phi-\phi_0)}} = 1 - \frac{2\gamma c}{d\lambda y^2} + \cdots .$$

(49)

In case of type IIB supergravity,

$$\gamma = \beta = \sqrt{d/(d-1)} .$$

(50)

We consider the static configuration $x^0 = \tau$, $x^1 = x = \sigma$, $x^2 = x^3 = \cdots = x^{d-1} = 0$ and $y = y(x)$. We also choose the coordinates on the boundary manifold so that the line given by $x^0 =$constant, $x^1 \equiv x$ and $x^2 = x^3 = \cdots = x^{d-1} = 0$ is geodesic and $g_{11} = 1$ on the line. Substituting the configuration into (48), we find

$$S = \frac{T}{2\pi} \int dx k(\phi(y)) y \sqrt{f(y)/y} (\partial_x y)^2 + 1 .$$

(51)

Here $T$ is the length of the region of the definition of $\tau$. The orbit of $y$ can be obtained by minimizing the action $S$ or solving the Euler-Lagrange equation $\delta S/\delta y - \partial_x (\delta S/\delta (\partial_x y)) = 0$. The Euler-Lagrange equation tells that

$$E_0 = \frac{k(\phi(y)) y}{\sqrt{f(y)/y} (\partial_x y)^2 + 1}$$

(52)

is a constant. If we assume $y$ has a finite minimum $y_0$, where $\partial_x y|_{y=y_0} = 0$, $E_0$ is given by

$$E_0 = k(\phi(y_0)) y_0 .$$

(53)

Introducing a parameter $t$, we parameterize $y$ by

$$y = y_0 \cosh t .$$

(54)

Then we find

$$\frac{dx}{dt} = \frac{y_0^{-\frac{1}{2}}}{A} \cosh^{-\frac{1}{2}} t \left\{1 + B \cosh^{-1} y_0^{-1} + \cdots \right\},$$

$$A \equiv \frac{2\lambda}{\sqrt{d(d-1)}}, \quad B \equiv -\frac{kd}{2\lambda^2} .$$

(55)
Here we assume that $y_0$ is large enough and the orbit of the string does not approach to the singularity at $y = 0$, where the supergravity description breaks down. Taking $t \to +\infty$, we find the distance $L$ between "quark" and "anti-quark" is given by

$$L = \frac{C \frac{3}{2} y_0^{-\frac{1}{2}}}{A} + \frac{B C \frac{3}{2} y_0^{-\frac{1}{2}}}{A} + \cdots$$

$$C_a \equiv \int_{-\infty}^{\infty} dt \cosh^{-a} t = \frac{2^{(a-1)} \Gamma \left( \frac{a}{2} \right)^2}{\Gamma(a)}. \quad (56)$$

We should note that the large $y_0$ corresponds to small $L$. As one sees the next-to-leading correction to distance depends on the curvature of spacetime.

Eq. (56) can be solved with respect to $y_0$ and we find

$$y_0 = \left( \frac{C \frac{3}{2}}{A L} \right)^2 \left\{ 1 + \frac{2 B C \frac{3}{2}}{C \frac{3}{2}} \left( \frac{A L}{C \frac{3}{2}} \right)^2 + \cdots \right\}. \quad (57)$$

Using (52), (54) and (56), we find the following expression for the action $S$

$$S = \frac{T}{2\pi} E(L)$$

$$E(L) = \int_{-\infty}^{\infty} dt \cosh^{-a} t \frac{k(\phi(y(t)))^2 y(t)^2}{k(\phi(y_0)) y_0}. \quad (58)$$

Here $E(L)$ expresses the total energy of the "quark"-"anti-quark" system. The energy $E(L)$ in (58), however, contains the divergence due to the self energies of the infinitely heavy "quark" and "anti-quark". The sum of their self energies can be estimated by considering the configuration $x^0 = \tau$, $x^1 = x^2 = x^3 = \cdots = x^{d-1} = 0$ and $y = y(\sigma)$ (note that $x_1$ vanishes here) and the minimum of $y$ is $y_D$ where brane would lies. We divide the region for $y$ to two ones, $\infty > y > y_0$ and $y_0 > y > y_D$. Using the parameterization of (54) and identifying $t$ with $\sigma$ ($t = \sigma$) for the region $\infty > y > y_0$, we find the following expression of the sum of self energies:

$$E_{self} = \int_{-\infty}^{\infty} dt k(\phi(y(t))) y(t) \sqrt{\frac{f(\phi(y(t))) (\partial_y y(t))^2}{y}} + 2 \int_{y_0}^{y_D} dy k(\phi(y)) \sqrt{y f(y)}. \quad (59)$$
Then the finite potential between “quark” and “anti-quark” is given by
\[ E_{qq}(L) \equiv E(L) - E_{\text{self}} = \frac{1}{A} \left( \frac{C_3}{AL} \right) \left\{ D_0 + B \left( \frac{C_5 D_0}{C_4} + D_2 \right) \left( \frac{AL}{C_3} \right)^2 + \cdots \right\} \]

\[ D_d \equiv 2 \int_0^\infty dt \cosh^{\frac{d+1}{2}} t e^{-t} + \frac{4}{d-1} = \frac{2^{\frac{d-1}{2}} \Gamma \left( \frac{d-1}{4} \right)^2}{\Gamma \left( \frac{d-1}{2} \right)} . \quad (60) \]

Here we neglected the \( L \) independent terms. Note that leading and next-to-leading term does not depend on the parameter \( \gamma \) in (19). The leading behavior is consistent with the previous works and attractive since \( D_0 = -2.39628... \) but we should note that next-to-leading term is linear in \( L \) (for \( k = 0 \) it was cubic), which does not depend on the dimension \( d \). Since \( \frac{C_5 D_0}{C_4} + D_2 = 3.49608 > 0 \) and \( B \) is negative if \( k \) is positive and vice versa from (53). Therefore the linear potential term in (60) is repulsive if \( k > 0 \) (sphere, i.e. gauge theory in de Sitter Universe) and attractive if \( k < 0 \) (hyperboloid).

Of course, the confinement depends on the large \( L \) behavior of the potential. When \( L \) is large, however, the orbit of the string would approach to the curvature singularity at \( y = 0 \), where the supergravity description would break down. Despite of this, it might be interesting to investigate the large \( L \) behavior. Since the behavior of \( f(y) \) and the dilaton \( \phi \) when \( y \) is small is given by (21) and (23), the integrand in (51) behaves as

\[ k \left( \phi(y) \right) y \sqrt{\frac{f(y)}{y}} \left( \partial_x y \right)^2 + 1 \]

\[ \sim y^{\text{sgn}(c)} \sqrt{\frac{d(d-1)}{2} + 1} \sqrt{\frac{d(d-1)}{4 \alpha c^2}} y^{d-3} \left( \partial_x y \right)^2 + 1 \]

\[ = \frac{d(d-1)}{4 \alpha c^2 \left( \text{sgn}(c) \sqrt{\frac{d(d-1)}{2} + \frac{d+1}{2}} \right)^2} \left( \partial_x \tilde{U} \right)^2 + \left( \tilde{U} \right)^{\gamma_0} . \quad (61) \]

Here

\[ \tilde{U} \equiv y^{\text{sgn}(c)} \sqrt{\frac{d(d-1)}{2} + \frac{d+1}{2}} , \quad \gamma_0 \equiv \frac{2\text{sgn}(c) \sqrt{\frac{d(d-1)}{2} + \frac{d+1}{2}}}{\text{sgn}(c) \sqrt{\frac{d(d-1)}{2} + \frac{d+1}{2}}} . \quad (62) \]
When $d = 4$, $0 < \gamma_0 < 2$ when $c > 0$ and $\gamma < 0$ when $c < 0$. According to the analysis in [19], the orbit of the string goes straight to the region $y \sim 0$ when $c > 0$ ($0 < \gamma_0 < 2$) and the potential becomes independent of $L$. In this case, however, the potential would receive the $\alpha'$ correction from the string theory. On the other hand, when $c < 0$ ($\gamma < 0$), there is an effective barrier which prevents the orbit of string from approaching into the curvature singularity and the potential would not receive the $\alpha'$ correction so much and the supergravity description would be reliable. Furthermore $c < 0$ ($\gamma < 0$) case predicts the confinement.

We can also evaluate the potential between monopole and anti-monopole by using the Nambu-Goto action for $D$-string instead of (48) (cf. ref. [13]):

$$S = \frac{1}{2\pi} \int d\tau d\sigma \frac{1}{k(\phi)^2} \sqrt{\det \left( g_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu \right)} .$$

(63)

For the static configuration $x^0 = \tau$, $x^1 \equiv x = \sigma$, $x^2 = x^3 = \cdots = x^{d-1} = 0$ and $y = y(x)$, we find, instead of (61)

$$S = \frac{T}{2\pi} \int dx y \left( \frac{f(y)}{y} \right) \left( \frac{\partial_x y}{y} \right)^2 + 1 .$$

(64)

We should note that $k(\phi)$ is replaced by $\frac{1}{k(\phi)}$ compared with quark anti-quark case (61), which corresponds the replacement of $\gamma \to -\gamma$. As the potential in (60) does not depend on $\gamma$ in the given order, we find the monopole anti-monopole potential $E_{m\bar{m}}$ is identical with $E_{q\bar{q}}$ when $L$ is small:

$$E_{m\bar{m}}(L) = E_{q\bar{q}}(L) .$$

(65)

If we consider, however, large $L$ behavior as in (61), we find

$$\frac{y}{k(\phi(y))} \sqrt{\frac{f(y)}{y} \left( \frac{\partial_x y}{y} \right)^2 + 1} \sim y^{-\text{sgn}(c)} \sqrt{\frac{d(d-1)}{2} + 1} \sqrt{\frac{d(d-1)}{4\alpha c^2} y^{d-3} \left( \frac{\partial_x y}{y} \right)^2 + 1}$$

$$= \sqrt{\frac{d(d-1)}{4\alpha c^2} \left( -\text{sgn}(c) \sqrt{\frac{d(d-1)}{2} + \frac{d+1}{2}} \right)^2 \left( \partial_x \tilde{U}(m) \right)^2 + \left( \tilde{U}(m) \right)^{\gamma_0(m)}} .$$

(66)
Here

$$\bar{U}^{(m)} \equiv y^{-\text{sgn}(c)} \sqrt{\frac{d(d-1)}{2} + \frac{d+1}{2}}, \quad \gamma^{(m)} \equiv \frac{-2\text{sgn}(c) \sqrt{\frac{d(d-1)}{2} + 2}}{-\text{sgn}(c) \sqrt{\frac{d(d-1)}{2} + \frac{d+1}{2}}}. \quad (67)$$

we should note that $\text{sgn}(c)$ in (61) and (62) is replaced by $\text{sgn}(c)$ in (66) and (67). Therefore the behavior of the potential between monopole and anti-monopole for large $L$ is changed from that of the potential between quark and anti-quark, that is, monopole and anti-monopole would be confined for $c > 0$ but would not be confined for $c < 0$.

It is not difficult to study the curvature dependence in more detail, for example, numerically for different choices of parameters and regions. Nevertheless, we do not do this as most qualitative features are clear.

3 **Axionic background with non-zero curvature and non-constant dilaton**

Let us present now the generalization of the above IIB SG background with non-trivial dilaton when non-constant axion is included into the action. Such a study for the case of flat four-dimensional space has been presented earlier in ref.[7] (for the effects of additional scalars, see also ref.[9]).

We include the axion field $\chi$ into the action of type IIB supergravity ($\alpha = \frac{1}{2}$) in (7), following ref.[14]

$$S = -\frac{1}{16\pi G} \int d^{d+1}x \sqrt{-G} \left( R + \lambda^2 - \frac{1}{2} G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} e^{2\phi} G^{\mu\nu} \partial_\mu \chi \partial_\nu \chi \right). \quad (68)$$

We work in the coordinate choice (13) and we assume that the $d$-dimensional manifold is curved (14) and $\chi$ only depends on $y$. Then, instead of (15-17), we obtain

$$0 = \frac{1}{2} \frac{r f}{f} \frac{y}{f} - \frac{d(d-1)}{8} \frac{1}{y^2} + \frac{\lambda^2}{2} f + \frac{1}{4} (\phi')^2 - \frac{1}{4} e^{2\phi} (\chi')^2 \quad (69)$$

$$0 = - \left( r_{ij} - \frac{1}{2} r g_{ij} \right) \frac{f}{f} + \frac{d-1}{4} \frac{f'}{f y} - \frac{(d-1)(d-4)}{8} \frac{1}{y^2} + \frac{\lambda^2}{2} f \quad (69)$$
\[-\frac{1}{4}(\phi')^2 + \frac{1}{4}e^{2\phi}(\chi')^2\} g_{ij} \tag{70}\]

\[0 = \left(\sqrt{\frac{y^d}{f}} \phi'\right)' + \sqrt{\frac{y^d}{f}} e^{2\phi}(\chi')^2 \tag{71}\]

\[0 = \left(\sqrt{\frac{y^d}{f}} e^{2\phi} \chi'\right)' \tag{72}\]

Eq.(72) can be integrated to give

\[\sqrt{\frac{y^d}{f}} e^{2\phi} \chi' = c_\chi \tag{73}\]

Using (73), we can delete \(\chi\) in (71) and obtain

\[0 = \sqrt{\frac{y^d}{f}} \left(\sqrt{\frac{y^d}{f}} \phi'\right)' + e^{-2\phi} c_\chi^2 \tag{74}\]

Eq.(74) gives another integral:

\[c_\phi = \frac{y^d}{f}(\phi')^2 - c_\chi^2 e^{-2\phi} \tag{75}\]

By using (73) and (75), we can delete \(\chi'\) and \(\phi'\) in (69):

\[0 = \frac{1}{2} r f \frac{y}{y^d} - \frac{d(d-1)}{8} \frac{1}{y^2} + \frac{\chi^2}{2} f + \frac{c_\phi f}{4y^d} \tag{76}\]

which can be solved algebraically with respect to \(f(y)\):

\[f = \frac{d(d-1)}{4y^2 \chi^2 \left(1 + \frac{c_\phi}{2\chi^2 y^2} + \frac{kd}{\chi^2 y}\right)} \tag{77}\]

The obtained metric is identical to that in (19), where the axion vanishes, if we replace \(c_\phi\) in (77) with \(\frac{a^2}{2}\). Therefore there appears the curvature singularity at \(y = 0\) again and the supergravity description would breaks down when \(y \sim 0\). Note that as we work with IIB SG we assume that \(d = 4\).

We now introduce a new coordinate \(\eta\) by

\[\eta = -\int dy \sqrt{\frac{f}{y^d}} = \int dy \sqrt{\frac{d(d-1)}{4y^d+2\chi^2 \left(1 + \frac{c_\phi}{2\chi^2 y^2} + \frac{kd}{\chi^2 y}\right)}} \tag{78}\]
Eqs. (73) and (75) can be written as follows:

\[ c_\chi = e^{2\phi} \frac{d\chi}{d\eta} \]  

(79)

\[ c_\phi = \left( \frac{d\phi}{d\eta} \right)^2 - c_\chi^2 e^{-2\phi} . \]  

(80)

Eq. (80) can be integrated to give

\[ e^\phi = \frac{c_\chi}{\sqrt{c_\phi}} \sinh \left( \sqrt{c_\phi} (\eta - \eta_0) \right) . \]  

(81)

Here \( \eta_0 \) is a constant of the integration. Substituting (81) into (79) and integrating it, we find

\[ \chi = \chi_0 - \frac{\sqrt{c_\phi}}{c_\chi} \coth \left( \sqrt{c_\phi} (\eta - \eta_0) \right) . \]  

(82)

Here \( \chi_0 \) is a constant of the integration. Axion describes the running theta angle.

When \( y \to +\infty \), the geometry of the spacetime approaches to AdS_5 asymptotically. Then Eq. (78) can be integrated perturbatively

\[ \eta = \frac{1}{\lambda} \sqrt{1 - \frac{d-1}{d} \left( \frac{1}{y^2} - \frac{kd}{2(d+2)\lambda^2 y^{d+1}} + \cdots \right)} . \]  

(83)

Here we have chosen the constant of the integration so that \( \eta \) vanishes when \( y \) goes to positive infinity. When \( \eta \) vanishes, \( \phi \) and \( \chi \) behave as,

\[ e^\phi \to - \frac{c_\chi}{\sqrt{c_\phi}} \sinh \left( \eta_0 \sqrt{c_\phi} \right) \]

\[ \chi \to \chi_0 + \frac{\sqrt{c_\phi}}{c_\chi} \coth \left( \eta_0 \sqrt{c_\phi} \right) . \]  

(84)

We should note that \( k \)-dependence does not appear in \( \phi \) and \( \chi \) if we use the coordinate \( \eta \) because it is hidden in this coordinate. If we choose \( \eta_0 = 0 \), \( e^\phi \to 0 \). Since \( 4\pi e^\phi \) can be regarded as the Yang-Mills coupling constant and \( \rho \to 0 \) \((y \to +\infty)\) corresponds to the ultraviolet fixed point from the viewpoint of AdS/CFT correspondence, the theory can be regarded as asymptotically free.
We now compare the above results with those in \[7\] for \( k = 0 \) and \( d = 4 \). We introduce a new coordinate \( r \) by

\[
e^{-\eta \sqrt{\frac{2c_\phi}{3}}} = \tanh \left( \frac{\lambda}{\sqrt{3}} (r - r_0) \right) .
\]

The coordinate transformation (85) can be given in terms of \( y \) when \( k = 0 \) and \( d = 4 \) by using (78) and (83),

\[
y^2 = K^4(r) \equiv \sqrt{\frac{c_\phi}{2\lambda^2}} \sinh \left( \frac{2\lambda}{\sqrt{3}} (r - r_0) \right) .
\]

Then the metric in (13) for \( k = 0 \) has the following form

\[
ds_{d+1}^2 = dr^2 + K^2(r) \sum_{i,j=0}^{d-1} \eta_{ij} dx^i dx^j .
\]

By using (85), the dilaton and axion fields in (81) and (82) can be rewritten as follows

\[
e^\phi = \frac{c_\chi}{2 \sqrt{c_\phi}} \left\{ \left( \coth \left( \frac{\lambda}{\sqrt{3}} (r - r_0) \right) \right)^{\sqrt{3}} e^{-\eta_0 \sqrt{c_\phi}} - \left( \tanh \left( \frac{\lambda}{\sqrt{3}} (r - r_0) \right) \right)^{\sqrt{3}} e^{-\eta_0 \sqrt{c_\phi}} \right\}
\]

\[
\chi = \chi_0 - \frac{\sqrt{c_\phi}}{c_\chi} \left( \frac{\coth \left( \frac{\lambda}{\sqrt{3}} (r - r_0) \right)}{\coth \left( \frac{\lambda}{\sqrt{3}} (r - r_0) \right)} \right)^{\sqrt{3}} + 1 .
\]

Then the solution in \[7\] seems to be a special case corresponding to \( \eta_0 = \chi_0 = 0 \).

Let us consider the potential between quark and anti-quark. As we are interested in the case of asymptotically free theory, we put \( \eta_0 = 0 \) in (81) and \( d = 4 \). Then we find

\[
k(\phi(y)) \equiv e^\phi = \frac{c_\phi R_s}{4c_\chi y^2} \left( 1 - \frac{k}{3\lambda^2 y} + \cdots \right)
\]

\[
f(y) = \frac{R_s^2}{4y^2} \left( 1 - \frac{4k}{\lambda^2 y} + \cdots \right) .
\]
Then in a way similar to the discussion in the second section where axion is not present instead of (55) and (56), we find

\[
\frac{dx}{dt} = R_s \sqrt{2y_0} \frac{\cosh^{-\frac{3}{2}} t}{\lambda^2 y_0} \left\{ 1 + \frac{2k}{\lambda^2 y_0} \left( -\frac{1}{\cosh t} - \frac{1}{3 (\cosh t + 1)} \right) + \cdots \right\} \tag{90}
\]

\[
L = R_s \sqrt{2y_0} \left\{ C_\frac{3}{2} + \frac{2k}{\lambda^2 y_0} \left( -C_\frac{7}{2} - \frac{E_\frac{1}{2}}{3} \right) + \cdots \right\} \tag{91}
\]

\[
E_a \equiv \int_{-\infty}^{\infty} dt \frac{\cosh^{-a} t}{\cosh t + 1}.
\]

Eq. (91) can be solved with respect to \(y_0\) as follows

\[
y_0 = \frac{1}{2} \left( \frac{C_\frac{3}{2}}{R_s L} \right)^2 \left\{ 1 + \frac{8k}{\lambda^2 C_\frac{3}{2}} \left( -C_\frac{7}{2} - \frac{E_\frac{1}{2}}{3} \right) \left( \frac{C_\frac{3}{2}}{R_s L} \right)^{-2} + \cdots \right\}. \tag{92}
\]

Here we assume again that \(y_0\) is large and \(L\) is large and not to break the supergravity description. Then using (58), we obtain the following expression for \(E(L)\)

\[
E(L) = \frac{R_s}{2} \left( \frac{C_\frac{3}{2}}{R_s L} \right) \left\{ C_\frac{3}{2} + \frac{k}{\lambda^2} \left( \frac{C_\frac{3}{2}}{R_s L} \right)^{-2} \left( - \frac{16}{3} C_\frac{7}{2} + \frac{4}{3} E_\frac{1}{2} \right) - \frac{4}{C_\frac{3}{2}} \left( C_\frac{1}{2} + \frac{E_\frac{1}{2}}{3} \right) \right\} + \cdots. \tag{93}
\]

Note that the integral is finite before subtraction the self energy of quark and anti-quark. We should note that the linear potential appears in the next-to-leading term. The coefficient \(\left( - \frac{16}{3} C_\frac{7}{2} + \frac{4}{3} E_\frac{1}{2} \right)\) of the next-to-leading term is negative, since \(C_\frac{7}{2}, C_\frac{3}{2}\) and \(E_\frac{1}{2}\) are positive and \(-\frac{16}{3} C_\frac{7}{2} + \frac{4}{3} E_\frac{1}{2}\) is negative, what can be easily found

\[
- \frac{16}{3} C_\frac{7}{2} + \frac{4}{3} E_\frac{1}{2} = - \frac{4}{3} \int_{-\infty}^{\infty} dt \cosh^{-\frac{3}{2}} t \left( 4 - \frac{\cosh t}{\cosh t + 1} \right) < - \frac{4}{3} \int_{-\infty}^{\infty} dt \cosh^{-\frac{3}{2}} t (4 - 1) < 0. \tag{94}
\]
Therefore the linear potential in the next-to-leading term becomes attractive if \( k < 0 \) and repulsive if \( k > 0 \). The result is consistent to the potential without axion in (60). We should note that Eqs. (78) and (81) tell that the dilaton field behaves as

\[
\phi \sim -\sqrt{\frac{d(d-1)}{2}} \ln y ,
\]

which corresponds \( c < 0 \) case in the pure dilaton case in (23). Since the behavior of \( f(y) \) in (74) is essential identical with the pure dilaton case in (19), the supergravity description would be valid even for large \( L \) and the confinement for quarks would be predicted (and monopoles would not be confined).

We now investigate the supersymmetric background. For \( k = 0 \) it has been found in ref. [7]. We look for its \( k \)-dependent generalization. Since we consider the background where the fermion fields, that is, dilatino \( \xi \) and gravitino \( \psi_{\mu} \) vanish, if the variation under some of the supersymmetry transformations of these fermionic fields vanishes, the corresponding supersymmetries are preserved. The supersymmetry transformations of these fields are given by [14]

\[
\begin{align*}
\delta \xi &= -\frac{1}{2} \left( e^{\phi} \partial_{\mu} \chi - \partial_{\mu} \phi \right) \gamma^\mu e^* , \\
\delta \xi^* &= -\frac{1}{2} \left( e^{\phi} \partial_{\mu} \chi + \partial_{\mu} \phi \right) \gamma^\mu \epsilon , \\
\delta \psi_{\mu} &= \left( \nabla_{\mu} + \frac{1}{4} e^{\phi} \partial_{\mu} \chi - \frac{\lambda}{4\sqrt{3}} \gamma_{\mu} \right) \epsilon , \\
\delta \psi^*_{\mu} &= \left( \nabla_{\mu} - \frac{1}{4} e^{\phi} \partial_{\mu} \chi - \frac{\lambda}{4\sqrt{3}} \gamma_{\mu} \right) \epsilon^* .
\end{align*}
\]

When substituting the solution in (81) and (82) into \( \delta \xi \) and \( \delta \chi^* \), we find

\[
\begin{align*}
\delta \xi &= -\frac{\sqrt{c_{\phi}}}{2} \left( \frac{1 - \cosh \left( \sqrt{c_{\phi}} (\eta - \eta_0) \right)}{\sinh \left( \sqrt{c_{\phi}} (\eta - \eta_0) \right)} \right) \gamma^\eta e^* , \\
\delta \xi^* &= -\frac{\sqrt{c_{\phi}}}{2} \left( \frac{1 + \cosh \left( \sqrt{c_{\phi}} (\eta - \eta_0) \right)}{\sinh \left( \sqrt{c_{\phi}} (\eta - \eta_0) \right)} \right) \gamma^\eta \epsilon .
\end{align*}
\]
Therefore all the supersymmetries break down in general since $\delta \chi$ and $\delta \chi^*$ do not vanish. In the limit of $c_\phi \to 0$, however, we find

$$\delta \xi \to 0,$$

$$\delta \xi^* = -\frac{1}{\eta - \eta_0} \gamma \eta \epsilon. \quad (98)$$

Therefore there is a possibility that half of the supersymmetries corresponding to $\epsilon^*$ survives in this limit. It should be noted that, in the limit, $f(y)$ in (77) becomes

$$f = \frac{d(d-1)}{4y^2\lambda^2 \left(1 + \frac{k\lambda}{\lambda_y}\right)}, \quad (99)$$

which tells that the metric of the spacetime becomes nothing but the metric of $\text{AdS}_5 \times S^5$ although the dilaton and the axion fields are non-trivial. Then if we choose the spinor parameter $\epsilon^*$ by using the Killing spinor $\zeta$ in $\text{AdS}_5 \times S^5$ as follows [14, 7]

$$\epsilon^* = e^{\frac{d}{2}} \zeta \to c^\frac{1}{2} \chi (\eta - \eta_0)^{\frac{1}{2}} \zeta, \quad (100)$$

$\delta \psi^\mu$ vanishes in the limit of $c_\phi \to 0$, which tells that half of the supersymmetry corresponding to $\epsilon^*$, in fact, survives in this limit. This situation does not depend on $k$. Such a solution corresponds to some vacuum of maximally supersymmetric YM theory where supersymmetry is broken to $\mathcal{N} = 2$. (Note that deformations of $\mathcal{N} = 4$ super YM theory which flow to fixed points like in refs. [15, 16] may also define running gauge coupling). In the limit of $c_\phi \to 0$, the solution in (81) and (82) has the following form:

$$e^\phi \to c_\chi (\eta - \eta_0)$$

$$\chi \to \chi_0 - \frac{1}{c_\chi (\eta - \eta_0)}. \quad (101)$$

Even in the limit, the theory becomes asymptotically free when $\eta_0 = 0$ since the coupling is assumed to be given by $e^\phi$ vanishes in the ultraviolet limit corresponding to $\eta = 0$. We should also note that the potential ($\eta_0 = 0$ case) between quark and anti-quark in (93) is not changed in the leading and next-to-leading orders since $c_\phi$ is not included to the corresponding expression.
4 Discussion

In summary, we found the background of IIB supergravity with non-constant dilaton, non-zero curvature of four-dimensional space-time and with (or without) non-trivial axion. By assuming the coupling is given by the exponential of the dilaton field $\phi$, AdS/CFT interpretation of such a solution gives the (power-like) running gauge coupling and predicts its curvature dependence. In the presence of axion, background may have half of supersymmetries unbroken. In all cases, we calculated quark-antiquark potential and showed that the term linear on distance $L$ explicitly depends on the curvature. Hence, there is the possibility that curvature of Universe might predict the confinement.

The complete interpretation of IIB SG background via AdS/CFT correspondence is not yet clear. We gave the arguments that most probably our background corresponds to another vacuum of maximally supersymmetric YM theory with some non-zero VEV operator. However, the possibility that it may be deformation of theory to another less symmetric (super) YM theory is not yet completely ruled out. The only possibility to understand it now is to investigate all properties of SG background and compare it with properties of corresponding QFT.

For example, it would be really interesting to find further development of such a scenario so that to present more realistic (logarithmic) behavior for running gauge coupling. Clearly, major modifications of background are necessary. Note in that respect the recent paper \[17\] where it was shown that AdS orbifolds may describe the running gauge coupling.

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Appendix A

In this appendix, we point out that there are many kinds of Einstein manifolds which satisfy Eq.\[14\]. The Einstein equations are given by,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{1}{2}\Lambda g_{\mu\nu} = T^{\text{matter}}_{\mu\nu}. \quad (102)$$
Here $T_{\mu\nu}^{\text{matter}}$ is the energy-momentum tensor of the matter fields. If we consider the vacuum solution where $T_{\mu\nu}^{\text{matter}} = 0$, Eq. (102) can be rewritten as

$$R_{\mu\nu} = \frac{\Lambda}{2} g_{\mu\nu} .$$

(103)

If we put $\Lambda = 2k$, Eq. (103) is nothing but the equation for the Einstein manifold (14). The Einstein manifolds are not always homogeneous manifolds like flat Minkowski, (anti-)de Sitter space or Nariai space but they can be some black hole solutions like Schwarzschild black hole,

$$ds^2 \equiv \sum_{\mu,\nu=0}^3 g_{\mu\nu} dx^\mu dx^\nu = - \left( 1 - \frac{r_0}{r} \right) dt^2 + \frac{dr^2}{\left( 1 - \frac{r_0}{r} \right)} + r^2 d\Omega^2 ,$$

(104)

or Kerr one for $k = 0$ or Schwarzschild (anti-) de Sitter black hole

$$ds^2 = - \left( 1 - \frac{\mu}{x} - \frac{2k}{3} x^2 \right) dt^2 + \frac{dr^2}{\left( 1 - \frac{\mu}{x} - \frac{2k}{3} x^2 \right)} + r^2 d\Omega^2 ,$$

(105)

for $k \neq 0$. In these solutions, the curvature singularity at $r = 0$ has a form of line penetrating AdS$_5$ and the horizon makes a tube surrounding the singularity. This configuration seems to express D-string whose boundary lies on the boundary of AdS$_5$ or possibly D3-brane. Especially in case of Kerr or Kerr-(anti-)de Sitter solution, the object corresponding to the singularity has an angular momentum.

We should note that the dilaton depends on the geometry of the boundary manifold only through $k$ as in (20). Therefore the behavior of the running coupling or renormalization group equation is irrelevant with the existence of the black hole singularity.

Appendix B

In this Appendix we present one more solution of IIB supergravity with two time like signatures of metric. The physical interpretation of this solution is not quite clear as well as its dual interpretation.

It was already few times mentioned that AdS radial coordinate plays the role of energy coordinate via holographic correspondence. It is also known

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4This type of solutions for $k = 0$ case has been considered in ref. [18]
that in general relativity there were attempts to identify the energy with
time flow. Then the following interesting question appears: Can the same
sort AdS solution be re-interpreted as the one depending from extra time
coordinates? In a sense one has then new IIB SG solution with a few time-
like signatures. There was some discussion of solutions with a few time-
signatures in various gravitational theories.

In order to get the time dependent solution and consider a kind of AdS
cosmology, we perform the analytic continuation in the solution in (19) and
(20) with $k = 0$ as follows:

$$c^2 \rightarrow -c^2, \quad \phi_0 \rightarrow \phi_0 - \frac{1}{2} \sqrt{\frac{(d - 1)}{d\alpha}} \ln(-1).$$

Then we obtain the following metric and the dilaton field:

$$ds^2_{d+1} = f(y) dy^2 + y \sum_{i,j=0}^{d-1} \eta_{ij} dx^i dx^j$$

$$f = \frac{d(d - 1)}{4y^2 (\lambda^2 - \alpha c^2 y^d)}$$

$$\phi = \phi_0 + \frac{1}{2} \sqrt{\frac{(d - 1)}{d\alpha}} \ln \left\{ \frac{-2\alpha c^2}{\lambda^2 y^d} + 1 \mp \sqrt{\left(\frac{2\alpha c^2}{\lambda^2 y^d} + 1\right)^2 - 1} \right\}$$

We can directly check that the solution (107-109) satisfies (11) and (12).
When $\lambda^2 - \alpha c^2 y^d < 0$, dilaton field $\phi$ is real and $f(y)$ becomes negative, which
tells that $y$ can be regarded as another time coordinate (AdS time) besides
the physical time coordinate in $d$-dimensional Minkowski space corresponding
to $\eta_{ij}$ in (107). We have unusual signature of the metric with two time-like
coordinates. Changing the coordinate $y$ by

$$y = \left(\frac{\alpha c^2}{\lambda^2}\right)^{\frac{1}{d}} \sin^{\frac{2}{d}} t,$$

we obtain the following metric and the dilaton field

$$ds^2_{d+1} = -\frac{d - 1}{d\lambda^2} dt^2 + \left(\frac{\alpha c^2}{\lambda^2}\right)^{\frac{1}{d}} \sin^{\frac{2}{d}} t \sum_{i,j=0}^{d-1} \eta_{ij} dx^i dx^j$$

$$\phi = \phi_0 + \frac{1}{2} \sqrt{\frac{(d - 1)}{d\alpha}} \ln \left(\frac{1 \mp \cos t}{\sin t}\right)^2.$$
Note that $t = 0, \pi$ corresponds to $y = 0$. Therefore there is a curvature singularity there. This tells that $\alpha'$ expansion in string theories becomes unreliable and we need to exclude the region $t \sim 0, \pi$. Eq. (112) tells that the coupling becomes $t$-dependent, especially in case of type IIB supergravity case we find

$$g = g_s e^{\phi - \phi_0} = g_s \left( \frac{1 + \cos t}{\sin t} \right)^{\sqrt{2-\pi}}. \quad (113)$$

If we change the coordinate $t$ by $\tau$ as

$$\tau = \left( \frac{d-1}{d c \sqrt{\alpha}} \right) \int \frac{dt}{\sin^{d-1} t}, \quad (114)$$

we have the metric in the following form

$$ds^2_{d+1} = \Theta(\tau) \left( -d\tau^2 + \sin^{2\frac{d}{2}} t \sum_{i,j=0}^{d-1} \eta_{ij} dx^i dx^j \right), \quad (115)$$

where

$$\Theta(\tau) \equiv \left( \frac{\alpha c^2}{\lambda^2} \right)^{\frac{1}{2}} \sin^{\frac{d}{2}} t(\tau). \quad (116)$$

Note that $t$ is solved with respect to $\tau$ by using (114). It follows from the above speculation that one can understand running of gauge coupling also as dependence on "second time" (AdS time). It would be interesting to understand if such a picture may have any physical meaning.

The conclusion drawn from such an interpretation is that AdS solution may contain a few times. Then the possibility of a kind of phase transition between these times should be considered (this is, of course, highly speculative). The physical time should be naturally defined by observer living in such a world. One possibility may be to introduce potential depending on angles defining the sort of signature of any particular dimension. Then the minimum of this potential may probably define the real physical time. In any case, the interpretation of IIB SG solution considered in this appendix could be understood simply as one more IIB SG solution.

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