Self-similar solutions of the compressible Navier–Stokes equations

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Abstract

We construct forward self-similar solutions (expanders) for the compressible Navier–Stokes equations. Some of these self-similar solutions are smooth, while others exhibit a singularity due to cavitation at the origin.

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1. Introduction

1.1. The model

We study self-similar solutions of the following compressible Navier–Stokes equations in $\mathbb{R}^d$ with $d \geq 1$:

$$
\begin{aligned}
\partial_t \rho + \text{div} (\rho u) &= 0, & t > 0, x \in \mathbb{R}^d, \\
\partial_t (\rho u) + \text{div} (\rho u \otimes u) + \nabla \pi &= \text{div} \tau, & t > 0, x \in \mathbb{R}^d, \\
\partial_t \left[ \rho \left( \frac{|u|^2}{2} + e \right) \right] + \text{div} \left[ u \left( \rho \left( \frac{|u|^2}{2} + e \right) + \pi \right) \right] - \text{div} q &= \text{div} (\tau \cdot u), & t > 0, x \in \mathbb{R}^d.
\end{aligned}
$$

(1.1)

Here $\rho(t, x)$ is the density of the fluid, $u(t, x)$ its velocity, $e(t, x)$ its internal energy, $\pi(t, x)$ its pressure, $\tau(t, x)$ its stress tensor, and finally $q(t, x)$ its internal energy flux. The fluid will furthermore be described by its temperature $\theta(t, x)$.

We assume the following constitutive relations:

- Joule’s first law:
  $$
e = C_V \theta,$$
  where $C_V > 0$ is the heat constant.

- Ideal gas law:
  $$\pi = \rho R \theta,$$
  where $R > 0$ is the ideal gas constant.

- Newtonian fluid: this implies that
  $$\tau := \lambda \text{div} u \text{Id} + 2\mu D(u), \quad D(u) = \frac{\nabla u + (\nabla u)^T}{2}, \quad \nabla u = (\partial_i u_j),$$
  where $\lambda$ and $\mu$ are the Lamé coefficients which satisfy
  $$\mu > 0 \quad \text{and} \quad 2\mu + d\lambda \geq 0.$$

- Fourier’s law:
  $$q = \kappa \nabla \theta,$$
  where $\kappa > 0$ is the thermal conductivity.

We refer to [15] for a more detailed discussion of these assumptions. The equations become

$$
\begin{aligned}
\partial_t \rho + \text{div} (\rho u) &= 0, \\
\partial_t (\rho u) + \text{div} (\rho u \otimes u) + \nabla (\rho R \theta) &= (\lambda + \mu) \nabla \text{div} u + \mu \Delta u, \\
\partial_t \left[ \rho \left( \frac{|u|^2}{2} + C_V \theta \right) \right] + \text{div} \left[ u \left( \rho \left( \frac{|u|^2}{2} + C_V \theta \right) + \rho R \theta \right) \right] - \kappa \Delta \theta &= \text{div} (\lambda (\text{div} u) u + 2\mu D(u) \cdot u).
\end{aligned}
$$

(cNS)
1.2. Forward self-similar solutions

The Eq. \((cNS)\) exhibit a scaling invariance; the set of solutions is left invariant by the transformation

\[
\begin{align*}
\rho(t,x) & \rightarrow \rho(\lambda^2 t, \lambda x), \\
\mathbf{u}(t,x) & \rightarrow \lambda \mathbf{u}(\lambda^2 t, \lambda x), \\
\theta(t,x) & \rightarrow \lambda^2 \theta(\lambda^2 t, \lambda x),
\end{align*}
\]

for \(\lambda > 0\).

This scaling invariance suggests looking for self-similar solutions of the form

\[
\begin{align*}
\rho(t,x) &= P\left(\frac{r}{\sqrt{t}}\right), \\
\mathbf{u}(t,x) &= \frac{1}{\sqrt{t}} U \left(\frac{r}{\sqrt{t}}\right) \frac{x}{r}, \\
\theta(t,x) &= \frac{1}{t} \Theta \left(\frac{r}{\sqrt{t}}\right),
\end{align*}
\]

where \(r = |x|\), \(P\), \(U\) and \(\Theta\) are scalar functions from \((0, \infty)\) to \(\mathbb{R}\).

It is natural to expect that there exist real numbers \(P_\infty\), \(U_\infty\), \(\Theta_\infty\) such that

\[
\begin{align*}
P(r) & \rightarrow P_\infty, \\
U(r) & \sim \frac{U_\infty}{r}, \\
\Theta(r) & \sim \frac{\Theta_\infty}{r^2}
\end{align*}
\]
as \(r \rightarrow \infty\), in which case this self-similar solution is associated to self-similar data

\[
(\rho, \mathbf{u}, \theta)(t = 0) = \left(P_\infty, \frac{U_\infty}{r^2}, \frac{\Theta_\infty}{r^2}\right).
\]

1.3. Known results

1.3.1. Weak and strong solutions of \((cNS)\)  

In the very rich existing literature, we mention weak, finite-energy solutions by Lions [16], variational solutions by Feireisl-Novotný-Petzeltová [6] (see also Feireisl [5]), classical solutions with finite energy by Matsumura-Nishida [17] (see also Huang-Li [9] with vacuum), solutions in Besov spaces with the interpolation index one by Danchin [4] (see also Chikami-Danchin [3]).

However, the initial velocity and temperature in (1.3) are homogeneous functions of degree 1 and 2, respectively, which therefore do not fit any of these frameworks. Indeed,

\[
\begin{align*}
u_0 & \not\in L^2(\mathbb{R}^d) \cup L^d(\mathbb{R}^d) \cup \dot{B}_{d-1}^d(\mathbb{R}^d), \\
\theta_0 & \not\in L^1(\mathbb{R}^d) \cup L^d(\mathbb{R}^d) \cup \dot{B}_{d-2}^d(\mathbb{R}^d).
\end{align*}
\]

Let us now try and be more specific, and explain why the classical construction methods cannot apply. Regarding weak solutions, the obstruction is obviously that the data (1.3) has infinite energy. Regarding strong solutions, the main obstacle is that the linear problem does not lead to \(\nabla \mathbf{u} \in L^1(0, T; L^\infty)\), which in turn prevents any control of the density in \(L^\infty\). Even worse, it is actually the case that

\[
\int_0^1 e^{(2\mu + \lambda) s} \text{div} \ u_0 \, ds \not\in L^\infty(\mathbb{R}^d).
\]
1.3.2. Self-similar solutions of (cNS) There are only few results in this direction. Under a different scaling property from the parabolic type (1.2), Qin-Su-Deng [18] proved the non-existence of forward and backward self-similar solutions to the compressible Navier–Stokes equations in one dimension. Local energy of forward and backward self-similar solutions was also investigated in [18] but the total energy blows up at $t = 0$ and $t = T$, respectively, where $T$ is the given time appearing in the definition of backward self-similar solutions. We also refer to related papers [7] by Guo-Jiang (isothermal compressible Navier–Stokes equations) and Li-Chen-Xie [14] (density-dependent viscosity).

1.3.3. The case of incompressible Navier–Stokes This case is different in two respects. First, the ansatz which we chose above (radial velocity) is incompatible with incompressibility, in fact, the velocity is irrotational; therefore, it is not possible to reduce the problem to a one-dimensional one, as we shall do in the present article. Second, the existence of forward self-similar solutions is known since strong solutions can be built up from small self-similar data: see for instance Cannone and Planchon [1], Chemin [2] and Koch and Tataru [13]. The case of large self-similar data was recently treated by Jia and Sverak [12], who could prove the existence of smooth self-similar solutions.

1.3.4. Vacuum state Few papers are known related to the vacuum. Xin [19] found the blow-up solutions for the initial density with the compact support. Hoff and Smoller [8] considered 1D barotropic Navier–Stokes equations and showed non-formulation of vacuum state due to the persistency of the almost everywhere positivity of the density. Jang and Masmoudi [10] proved local in time well-posedness of the 3D compressible Euler equations under the barotropic condition with a physical vacuum. We also refer to [11] for the overview about problems of vacuum state.

1.4. Obtained results

Below we only sketch below our two main results, and refer to Theorems 3.2 and 4.2 for complete statements.

**Theorem 1.1.** (Smooth self-similar solutions; simplified statement) Let $d \geq 3$. There exists a family of smooth self-similar solutions of the form (1.2) corresponding to data (1.3), where $P_\infty > 0$, $U_\infty < 0$, and $\Theta_\infty > 0$. The parameters $U_\infty$ and $\Theta_\infty$ have to be chosen sufficiently small, and the allowed values of $P_\infty$, $U_\infty$, $\Theta_\infty$ form a two-dimensional manifold. The profiles $(P(r), U(r), \Theta(r))$ are smooth functions of $[0, \infty)$ such that

$$
\inf_r P(r) > 0, \quad \sup_r P(r) < \infty,
$$

$$
|U(r)| \lesssim \frac{r}{(1 + r)^3}, \quad |U'(r)| \lesssim \frac{r}{(1 + r)^3},
$$

$$
|\Theta(r)| \lesssim \frac{1}{(1 + r)^2}, \quad |\Theta'(r)| \lesssim \frac{r}{(1 + r)^2},
$$
and furthermore,

\[ P(r) = P_{\infty} + O \left( \frac{1}{r^2} \right), \]

\[ U(r) = \frac{U_{\infty}}{r} + O \left( \frac{1}{r^3} \right), \]

\[ \Theta(r) = \frac{\Theta_{\infty}}{r^2} + O \left( \frac{1}{r^4} \right). \]

The previous theorem can be thought of as perturbative, around the trivial (self-similar) solution \((\rho, u, \theta) = (\text{Constant}, 0, 0)\).

**Theorem 1.2.** (Cavitating self-similar solutions; simplified statement) Let \(d \geq 3\). There exists a family of self-similar solutions of the form (1.2) corresponding to data (1.3), where \(P_{\infty} > 0, U_{\infty} > 0, \Theta_{\infty} > 0\). The parameter \(P_{\infty}\) has to be chosen sufficiently small, and the allowed values of \(P_{\infty}, U_{\infty}, \Theta_{\infty}\) form a three-dimensional set.

The profiles \((P(r), U(r), \Theta(r))\) are smooth functions of \((0, \infty)\), which, for \(r \to 0\), behave as follows:

\[ P(r) = P_{\delta} \left( \frac{r}{\delta} \right)^{\frac{2d\alpha}{1-2\alpha}} + O \left( \frac{r}{\delta} \right)^{\frac{2d\alpha}{1-2\alpha} + 1 + d\alpha}, \]

\[ U(r) = \alpha r + O \left( r^{1 + \frac{2d\alpha}{1-2\alpha}} \right), \]

\[ \Theta(r) = \Theta_0 + O \left( r^2 \right). \]

Here, \(\alpha\) and \(P_{\delta}\) are small parameters.

The profiles \((P(r), U(r), \Theta(r))\) also satisfy the global bounds

\[ |P(r)| \lesssim P_{\delta} \min \left[ 1, \left( \frac{r}{\delta} \right)^{\frac{2d\alpha}{1-2\alpha}} \right], \]

\[ |U(r)| \lesssim \frac{\alpha r}{(1 + \sqrt{P_{\delta} r})^2}, \quad |U'(r)| \lesssim \frac{\alpha}{(1 + \sqrt{P_{\delta} r})^2}, \]

\[ \Theta(r) \lesssim \frac{1}{(1 + \sqrt{P_{\delta} r})^2}, \quad |\Theta'(r)| \lesssim \frac{\sqrt{P_{\delta} r}}{(1 + \sqrt{P_{\delta} r})^2}, \]

and finally

\[ P(r) = P_{\infty} + O \left( \frac{1}{r^2} \right), \]

\[ U(r) = \frac{U_{\infty}}{r} + O \left( \frac{1}{r^3} \right), \]

\[ \Theta(r) = \frac{\Theta_{\infty}}{r^2} + O \left( \frac{1}{r^4} \right). \]
Remark 1.3. • If $d = 1, 2$, solutions can be constructed in a very similar way to the above theorems. However, we excluded $d = 1, 2$ because an initial data of the type $\Theta_r^\infty$ is not locally integrable, and thus does not make sense in the sense of distributions. Furthermore, we could not ensure positivity of $\Theta_r^\infty$.

• Although $u(t, x) := t^{-1/2} U(t^{-1/2} |x|) x / |x| \notin L^1(0, 1; Lip(\mathbb{R}^d))$, one can define Lagrangian coordinates for the velocity fields defined in the two above theorems.

1.5. Organization of the paper

In Sect. 2, we derive the integro-differential equations which result from our ansatz. In Sect. 3, we state a complete version of the existence theorem in the smooth case, and proceed to prove it. In Sect. 4, we state a complete version of the existence theorem in the cavitating case, and proceed to prove it.

2. ODEs and integro-differential equations

2.1. Derivation of the system of ODEs

Consider solutions such that (1.2) is satisfied. Let us starting by proving that the partial differential equations (cNS) is equivalent to the following ordinary differential equations for any $r = |x| > 0$:

$$\begin{align*}
-\frac{1}{2} r P' + P' U + P \left( U' + \frac{d-1}{r} U \right) &= 0, \\
-\frac{1}{2} P U - \frac{1}{2} r (P U)' + \left( P U^2 \right)' + \frac{d-1}{r} P U^2 + (P R \Theta)' &= (2\mu + \lambda) \left( U'' + \frac{d-1}{r} U' - \frac{d-1}{r^2} U \right), \\
- P \left( \frac{U^2}{2} + C_V \Theta \right) - \frac{1}{2} r \left( P \left( \frac{U^2}{2} + C_V \Theta \right) \right)' + \left( U P \left( \frac{U^2}{2} + C_V \Theta \right) + U P R \Theta \right)' &= 2\mu \left( (U')^2 + \frac{d-1}{r^2} U^2 \right) + \lambda \left( U' + \frac{d-1}{r} U \right)^2, \\
&+ (2\mu + \lambda) \left( U'' + \frac{d-1}{r} U' - \frac{d-1}{r^2} U \right) U. 
\end{align*}$$

(2.1)

The above equations follow in a straightforward manner from the formulas ($f$ denoting a scalar function)

$$\nabla f(r) = f'(r) \frac{x}{r},$$

$$\text{div} \left( f(r) \frac{x}{r} \right) = f' + \frac{d-1}{r} f,$$

$$\text{div} \left( f(r) \frac{x}{r} \otimes \frac{x}{r} \right) = \left( f' + \frac{d-1}{r} f \right) \frac{x}{r},$$

$$\Delta \left( f(r) \frac{x}{r} \right) = \left( f'' + \frac{d-1}{r} f' - \frac{d-1}{r^2} f(r) \right) \frac{x}{r},$$

$$D \left( f(r) \frac{x}{r} \right) = \frac{f(r)}{r} \text{Id} + \left( f'(r) - \frac{f(r)}{r} \right) \frac{x}{r} \otimes \frac{x}{r},$$

where $\text{Id}$ is the identity matrix.
2.2. Integro-differential equations for smooth solutions

We next write the ordinary differential Eq. (2.1) as integral equations under the condition at \( r = 0 \) that

\[
P(0) = P_0 > 0, \quad P'(r) = O(1),
\]

\[
U(r) = O \left( r^2 \right), \quad U'(r) = O(r) \quad \text{with} \quad \frac{1}{2} r - U(r) > 0 \text{ for any } r > 0,
\]

\[
\Theta(0) = \Theta_0 > 0, \quad \Theta'(r) = O(r).
\]

We will obtain

\[
P(r) = e^{V(r)} P_0, \quad (2.2)
\]

\[
U(r) = \frac{r^{-d+1}}{2\mu + \lambda} \int_0^r r_1^{d-1} e^{-W(r) + W(r_1)} F_U(r_1) \, dr_1,
\]

\[
\Theta(r) = (d-2)r^{-d+2} \int_0^r r_1^{d-3} e^{-Z(r) + Z(r_1)} \, dr_1 \Theta_0 - \frac{U^2}{2C_V} + \frac{r^{-d+2}}{\kappa} \int_0^r r_1^{d-2} e^{-Z(r) + Z(r_1)} F_{\Theta}(r_1) \, dr_1,
\]

where

\[
F_U(r_1) := PU^2 + \int_0^{r_1} \frac{d - 1}{r_2} PU^2 \, dr_2 + PR\Theta - P_0 R\Theta_0,
\]

\[
F_{\Theta}(r_1) := UP \left( \frac{U^2}{2} + C_V \Theta \right) + UPR\Theta + \frac{d - 2}{r_1} \int_0^{r_1} \left( UP \left( \frac{U^2}{2} + C_V \Theta \right) + UPR\Theta \right) \, dr_2 + \left( \frac{\kappa}{C_V} - (2\mu + \lambda) \right) \left( \frac{U^2}{2} + \frac{d - 2}{2r_1} U^2 \right) - \lambda(d - 1) \left( \frac{U^2}{r_1} + \frac{d - 2}{r_1} \int_0^{r_1} U^2 \, dr_2 \right),
\]

and

\[
V(r) := \int_0^r U' \frac{d - 1}{2} U \, d\tilde{r}, \quad W(r) := \frac{1}{2(2\mu + \lambda)} \int_0^r \tilde{r} P(\tilde{r}) d\tilde{r}, \quad Z(r) := \frac{C_V}{2\kappa} \int_0^r \tilde{r} P(\tilde{r}) d\tilde{r}.
\]

**Proof of the formulas (2.2), (2.3) and (2.4).** It is easy to check that the first equation in (2.1) is equivalent to

\[
\frac{P'}{P} = \frac{U'}{\frac{1}{2} r - U} + \frac{d - 1}{r} U,
\]
which proves (2.2) by integrating.

The second equation in (2.1) is rewritten as

\[-\frac{1}{2}(rPU)' + \left( PU^2 \right)' + \frac{d-1}{r} PU^2 \]
\[-(2\mu + \lambda) \left( U' + \frac{d-1}{r} U \right)' + (PR\Theta)' = 0.\]

Integrating implies that

\[-\frac{1}{2} r PU + PU^2 + \int_0^r \frac{d-1}{r^2} P U^2 dr_2 - (2\mu + \lambda) \left( U' + \frac{d-1}{r} U \right) \]
\[+ PR\Theta - P_0 R\Theta_0 = 0,\]

and multiplying by \( r^{d-1} e^{W(r)} \) yields that

\[-(2\mu + \lambda) \left( r^{d-1} e^{W(r)} U \right)' \]
\[+ r^{d-1} e^{W(r)} \left( PU^2 + \int_0^r \frac{d-1}{r^2} P U^2 dr_2 + PR\Theta - P_0 R\Theta_0 \right) = 0.\]

Hence, we obtain (2.3) by integrating the above equation and multiplying by \( r^{-d+1} e^{-W(r)} \).

Finally, we consider the third equation in (2.1). By multiplying by \( r \) and, in a fashion similar to the argument for the second equation, we get that

\[-\frac{1}{2} \left( r^2 P \left( \frac{U^2}{2} + CV\Theta \right) \right)' \]
\[+ r^{2-d} \left( r^{d-1} \left( UP \left( \frac{U^2}{2} + CV\Theta \right) + UP R\Theta \right) \right)' - \kappa r^{-d+2} \left( r^{d-1} \Theta' \right)' \]
\[= r^{-d+2} \left( (2\mu + \lambda) \left( r^{d-1} U U' \right)' + \lambda (d-1) \left( r^{d-2} U^2 \right)' \right).\]

Integrating the above and performing integrations by parts give

\[-\frac{1}{2} r^2 P \left( \frac{U^2}{2} + CV\Theta \right) \]
\[+ r \left( UP \left( \frac{U^2}{2} + CV\Theta \right) + UPR\Theta \right) \]
\[+ (d-2) \int_0^r \left( UP \left( \frac{U^2}{2} + CV\Theta \right) + UPR\Theta \right) dr_2 \]
\[- \kappa \left( r\Theta' + (d-2) \left( \Theta(r) - \Theta_0 \right) \right) \]
\[= (2\mu + \lambda) \left( r U U' + (d-2) \int_0^r UU' dr_2 \right) \]
\[+ \lambda (d-1) \left( U^2 + (d-2) \int_0^r \frac{U^2}{r^2} dr_2 \right).\]
Dividing by $r$, and regarding this formula as an equation on the energy $U^2/2 + C_V \Theta$, we get

$$
- \frac{1}{2} r P \left( \frac{U^2}{2} + C_V \Theta \right) \\
+ U P \left( \frac{U^2}{2} + C_V \Theta \right) + U P R \Theta + \frac{d - 2}{r} \int_0^r \left( U P \left( \frac{U^2}{2} + C_V \Theta \right) + U P R \Theta \right) dr_2 \\
- \frac{\kappa}{C_V} \left( \frac{U^2}{2} + C_V \Theta \right)' + \frac{d - 2}{r} \left( \frac{U^2}{2} + C_V \Theta \right) \\
+ \frac{\kappa}{C_V} \left( \frac{(U^2)'}{2} + \frac{d - 2}{2r} U^2 \right) + \frac{d - 2}{r} \Theta_0 \\
= (2\mu + \lambda) \left( U U' + \frac{d - 2}{2r} U^2 \right) + \lambda (d - 1) \left( \frac{U^2}{r} + \frac{d - 2}{r} \int_0^r \frac{U^2}{r_2} dr_2 \right).
$$

Multiplying by $r^{d-2} e^{Z(r)}$ gives

$$
\frac{\kappa}{C_V} \left( r^{d-2} e^{Z(r)} \left( \frac{U^2}{2} + C_V \Theta \right) \right)' \\
= r^{d-2} e^{Z(r)} \left\{ U P \left( \frac{U^2}{2} + C_V \Theta \right) \\
+ U P R \Theta + \frac{d - 2}{r} \int_0^r \left( U P \left( \frac{U^2}{2} + C_V \Theta \right) + U P R \Theta \right) dr_2 \\
+ \left( \frac{\kappa}{C_V} - (2\mu + \lambda) \right) \left( \frac{(U^2)'}{2} + \frac{d - 2}{2r} U^2 \right) \\
- \lambda (d - 1) \left( \frac{U^2}{r} + \frac{d - 2}{r} \int_0^r \frac{U^2}{r_2} dr_2 \right) \\
+ \kappa \frac{d - 2}{r} \Theta_0 \right\} \\
=: r^{d-2} e^{Z(r)} \left( F_\Theta(r) + \kappa \frac{d - 2}{r} \Theta_0 \right),
$$

where $F_\Theta$ is also given in the formula (2.4). Integrating the above leads to

$$
\frac{\kappa}{C_V} r^{d-2} e^{Z(r)} \left( \frac{U^2}{2} + C_V \Theta \right) = \int_0^r r_1^{d-2} e^{Z(r_1)} \left( F_\Theta(r_1) + \kappa \frac{d - 2}{r_1} \Theta_0 \right) dr_1,
$$

which proves (2.4). □

2.3. Integro-differential equations for cavitating solutions

Let us consider the vacuum case at $x = 0$. Supposing the condition at $r = 0$ that, for $0 < \alpha < 1/2$,

$$
P(0) = 0, \quad U(0) = 0, \quad U'(0) = \alpha, \quad \Theta(0) = \Theta_0 > 0, \quad \Theta'(0) = 0,
$$

...
and that $P, U, \Theta$ are $C^1$, we get the integral equations

\begin{align}
P(r) &= e^{V(r) - V(\delta)} P(\delta) \text{ for given } \delta > 0, \\
U(r) &= \frac{r^{-d+1}}{2\mu + \lambda} \int_0^r r_1^{-d+1} e^{-W(r) + W(\delta)} \tilde{F}_U(r_1) \, dr_1, \\
\Theta(r) &= (d - 2)r^{-d+2} \int_0^r r_1^{-d+3} e^{-Z(r) + Z(\delta)} \, dr_1 \Theta_0 - \frac{U^2}{2C_V} \\
&\quad + \frac{r^{-d+2}}{\kappa} \int_0^r r_1^{-d+2} e^{-Z(r) + Z(\delta)} F(\Theta(r_1)) \, dr_1,
\end{align}

where $F_\Theta, V(r), W(r), Z(r)$ are same as in (2.2), (2.3), (2.4) and

\[
\tilde{F}_U(r_1) := PU^2 + \int_0^{r_1} \frac{d - 1}{r_2} PU^2 \, dr_2 + P \theta + d (2\mu + \lambda) \alpha.
\]

On the density $P(r)$, the exponent $V(r)$ itself diverges because of $V'(r) \simeq \frac{d\alpha}{2 - \alpha} \frac{1}{r}$, but we always regard $V(r) - V(\delta)$ as an integral from $\delta$ to $r$.

### 3. Smooth self-similar solutions

#### 3.1. Main result

**Notation 3.1.** In this section, we consider $R, \mu, \lambda, CV, P_0$ as fixed positive constants - the meaning of $P_0$ will soon be explained. We denote $C$ a constant which depends on $(R, \mu, \lambda, CV, P_0) \in (0, \infty)^5$; the implicit constant in the notations $\lesssim$ and $O(\cdot)$ has the same properties.

**Theorem 3.2.** For fixed $R, \mu, \lambda, CV, P_0 > 0$, if $\Theta_0$ is sufficiently small, there exists a unique continuously differentiable function $(P, U, \Theta) \in C^1([0, \infty))^3$ solving (2.1) such that, for $r$ small,

\[
P(0) = P_0, \\
U(r) = O \left( r^2 \right), \quad U'(r) = O(r), \\
\Theta(r) = \Theta_0, \quad \Theta'(r) = O(r).
\]

This satisfies the bounds

\[
|U(r)| \lesssim \frac{\Theta_0 r^2}{(1 + r)^3}, \quad |U'(r)| \lesssim \frac{\Theta_0 r}{(1 + r)^3}, \\
|\Theta(r)| \lesssim \frac{\Theta_0}{(1 + r)^2}, \quad |\Theta'(r)| \lesssim \frac{\Theta_0 r}{(1 + r)^2}.
\]

Furthermore, there exists $P_\infty > 0, U_\infty < 0, \Theta_\infty > 0$ such that

\[
P(r) = P_\infty + O \left( \frac{1}{r^2} \right)
\]
\[ U(r) = \frac{U_\infty}{r} + O \left( \frac{1}{r^3} \right), \]
\[ \Theta(r) = \frac{\Theta_\infty}{r^2} + O \left( \frac{1}{r^4} \right). \]

Finally,
\[ U_\infty = -2R \Theta_0 + O \left( \frac{\Theta_0^2}{r} \right), \]
\[ \Theta_\infty = \frac{2(d-2)\kappa}{CV P_0} \Theta_0 + O \left( \frac{\Theta_0^2}{r} \right). \]

### 3.2. Main steps of the proof

Let
\[ \| (U, \Theta) \|_\delta := \sup_{0 < r < \delta} \left[ r^{-2}|U(r)| + r^{-1}|U'(r)| + |\Theta(r)| + r^{-1}|\Theta'(r)| \right]. \]

To prove the existence of a local solution, let \( \Psi \) be the map which to \((U, \Theta)\) associates the right-hand side of (2.3) and (2.4):
\[ \Psi : (U, \Theta) \mapsto \text{RHS (2.3), RHS (2.4)}. \]

Define
\[ E_\delta = \left\{ (U, \Theta) \in C^1(0, \delta) \text{ such that } \Theta(0) = \Theta_0 \text{ and } \| (U, \Theta) \|_\delta < \infty \right\}. \]

Equipped with \( \| \cdot \|_\delta \), it is a Banach space (an affine Banach space to be precise).

**Lemma 3.3.** There exists \( C_0 > 0 \) such that: setting \( \epsilon = C_0 \Theta_0 \), if \( \delta \) and \( \epsilon \) are sufficiently small, then \( \Psi \) is a contraction on the ball \( B_{E_\delta}((0, \Theta_0), \epsilon) \).

By the Banach fixed point theorem, this lemma gives the local existence (close to zero) of solutions. They can then be prolonged for \( r > 0 \) as long as \( U, U', U_\infty, \frac{U}{U-\frac{1}{2}r}, \Theta, \Theta' \) are bounded. Let \([0, R)\) be the largest interval on which \( P, U, \Theta \) are well-defined. In other words, either \( R = \infty \), or
\[ \lim_{r \to R} |U(r)| + |U'(r)| + \frac{1}{|U - \frac{1}{2}r|} + |\Theta(r)| + |\Theta'(r)| = \infty. \]  

(3.1)

Define then, for constants \( M_1 \) and \( M_2 \), which are much smaller than 1,
\[ Z(s) = \sup_{0 < r < s} \left[ \frac{1}{M_1} r^{-(1+r)^3}|U(r)| + \frac{1}{M_1} r^{-1}(1+r)^3 \left| U'(r) + \frac{d-1}{r} U(r) \right| \right. \]
\[ \left. + \frac{1}{M_2} (1+r)^2|\Theta(r)| + \frac{1}{M_2} r^{-1}(1+r)^2|\Theta'(r)| \right]. \]

**Lemma 3.4.** If \( Z(r) \leq 1 \), then
\[ Z(r) \lesssim M_1 + \frac{\Theta_0}{M_1} + \frac{\Theta_0}{M_2} + \frac{M_1^2}{M_2}. \]
We now choose the constants $M_1$ and $M_2$ such that
\[
M_1^2 + M_1 M_2 \ll \Theta_0 \ll M_2 \ll M_1 \ll 1 \tag{3.2}
\]
(where the notation $A \ll B$ means $A \leq c B$, for a constant $c$ depending on the parameters of the problem $(R, \mu, \lambda, CV, P_0)$, which is chosen sufficiently small so that all arguments in the following apply). For instance, to achieve the above, we could choose
\[
M_2 = A \Theta_0, \quad M_1 = A^2 \Theta_0.
\]
Choosing $A$ sufficiently large ensures that $\Theta_0 \ll M_2 \ll M_1$; choosing then $\Theta_0 \ll \frac{1}{A^4}$ ensures that $M_1 \ll 1$ and $M_1^2 + M_1 M_2 \ll \Theta_0$.

Let $\tilde{R} = \sup \{ r \text{ such that } Z(r) \leq 1 \}$. By Lemma 3.3, $\tilde{R} \geq \delta$. Argue by contradiction and assume that $\tilde{R}$ is finite. Then, by (3.1), $\tilde{R} < R$. Furthermore, by Lemma 3.4 and the choice (3.2), $Z(\tilde{R}) < 1$. Then $(U, \Theta)$ can be prolonged over a short time interval where $Z < 1$. This contradicts the definition of $\tilde{R}$, and gives the desired result: $R = \tilde{R} = \infty$.

There remains to prove the asymptotic behavior of $P, U, \Theta$. This is achieved in the following lemma:

**Lemma 3.5.** There exists $P_\infty > 0, U_\infty \in \mathbb{R}, \Theta_\infty > 0$ such that
\[
P(r) = P_\infty + O \left( \frac{1}{r^2} \right), \quad U(r) = \frac{U_\infty}{r} + O \left( \frac{1}{r^3} \right), \quad \Theta(r) = \frac{\Theta_\infty}{r^2} + O \left( \frac{1}{r^4} \right).
\]
Furthermore, $\Theta(r) > 0$ for any $r$ and
\[
U_\infty = -2R \Theta_0 + O \left( \Theta_0^2 \right), \quad \Theta_\infty = \frac{2(d-2)\kappa}{CV P_0} \Theta_0 + O \left( \Theta_0^2 \right).
\]

3.3. Local existence: proof of Lemma 3.3

We assume here that $(U_1, \Theta_1), (U_2, \Theta_2) \in B_{E^\delta} (0, \epsilon)$, and let
\[
D = \| (U_1, \Theta_1) - (U_2, \Theta_2) \|^\delta.
\]
We aim to prove that
\[
\| \Psi (U_1, \Theta_1) - \Psi (U_2, \Theta_2) \|^\delta \lesssim (\epsilon + \delta) D. \tag{3.3}
\]
This proves that $\Psi$ acts as a contraction for $\epsilon, \delta$ sufficiently small. Furthermore, this proves that $\Psi$ stabilizes $B_{E^\delta} (0, \Theta_0), \epsilon)$. Indeed, choosing $C_0$ sufficiently large, this follows from (3.3) together with the observation that $(0, \tilde{\Theta}) := \Psi (0, \Theta_0)$ satisfies
\[
\tilde{\Theta}(r) = \Theta_0 \left( 1 + O \left( r^2 \right) \right) \quad \text{and} \quad |\tilde{\Theta}'(r)| \lesssim \Theta_0 r.
\]
It remains to prove (3.3)!
With the notation used in (2.3) and (2.4), it appears first that, as soon as 
\[\|(U, \Theta)\|_\delta < \epsilon,\]
\[
|V| \lesssim 1, \quad |W| + |Z| \lesssim r^2 \\
|V'| \lesssim \epsilon, \quad |W'| \lesssim r, \quad |Z'| \lesssim r \\
|P| \lesssim 1, \quad |P'| \lesssim \epsilon \\
|F_U(r)| \lesssim \epsilon r \\
|F_\Theta(r)| \lesssim \epsilon^2 r^2.
\]

We now turn to the difference between two solutions. We denote in the following 
\(\tilde{U}_1, \tilde{\Theta}_1 = \Psi(U_1, \Theta_1), \) etc... . By direct inspection,
\[
|V_1 - V_2| + |P_1 - P_2| \lesssim dr, \quad |W_1 - W_2| + |Z_1 - Z_2| \lesssim Dr^3.
\]
It follows that
\[
|\Theta_1(r) - \Theta_2(r)| \leq \int_0^r |\Theta'_1 - \Theta'_2| \, ds \lesssim \int_0^r Ds \, ds \lesssim dr^2 \\
|F_{U_1}(r) - F_{U_2}(r)| \lesssim D(\epsilon + \delta)r \\
|F_{\Theta_1}(r) - F_{\Theta_2}(r)| \lesssim D\epsilon r^2.
\]

Therefore,
\[
|\tilde{U}_1(r) - \tilde{U}_2(r)| \lesssim r^{1-d} \int_0^r r_1^{d-1} |e^{W_1(r_1) - W_1(r)} - e^{W_2(r_1) - W_2(r)}| F_{U_1}(r_1) \, dr_1 \\
+ r^{1-d} \int_0^r r_1^{d-1} e^{W_2(r_1) - W_2(r)} |F_{U_1}(r) - F_{U_2}(r_1)| \, dr_1 \\
\lesssim r^{1-d} \int_0^r r_1^{d-1} \left[D(3r^3)\epsilon r + D(\epsilon + \delta)r_1\right] \, dr_1 \lesssim D(\epsilon + \delta)r^2.
\]

Arguing similarly,
\[
|\tilde{U}'_1(r) - \tilde{U}'_2(r)| \lesssim D(\epsilon + \delta)r,
\]
and finally,
\[
|\tilde{\Theta}_1(r) - \tilde{\Theta}_2(r)| \lesssim D\epsilon r^3 \quad \text{and} \quad |\tilde{\Theta}'_1(r) - \tilde{\Theta}'_2(r)| \lesssim D\epsilon r^2,
\]
which concludes the proof.

### 3.4. Global existence: proof of Lemma 3.4

We assume here that we have a solution defined on \([0, R_0]\) for \(R_0 > 0\), such that 
\(Z(r) \leq 1\) for all \(r \in [0, R_0]\) which implies
\[
|U(r)| \leq M_1 r^2 (1 + r)^{-3}, \quad |U'(r)| \lesssim M_1 r (1 + r)^{-3}, \quad (3.4) \\
|\Theta(r)| \lesssim M_2 (1 + r)^{-2}, \quad |\Theta'(r)| \lesssim M_2 r (1 + r)^{-2}. \quad (3.5)
\]
Estimate of $P$. It follows from the definition of $V$ that

$$|V(r)| \leq \int_0^r \frac{s}{(1+s)^3} M_1 + \frac{d-1}{s} \frac{s^2}{(1+s)^3} M_1 \frac{1}{2s} \frac{s^2}{M_1} ds$$

$$\leq \int_0^r \frac{d-s}{s(1-\frac{1}{2})} ds \leq \frac{dM_1}{1-2M_1} =: M_0. \tag{3.6}$$

Hence we obtain that

$$e^{-M_0} P_0 \leq P(r) \leq e^{M_0} P_0, \quad r > 0, \tag{3.7}$$

$$|P'(r)| \leq |V'(r)| P(r) \leq \frac{d}{r(1-r)^3} e^{M_0} P_0 \leq \frac{2M_0}{(1+r)^3} e^{M_0} P_0, \quad r > 0. \tag{3.8}$$

Estimate of $U$. We have from (3.7) that

$$-e^{-M_0} P_0 \left(r^2 - r_1^2\right) \lesssim -W(r) + W(r_1) \lesssim e^{-M_0} P_0 \left(r^2 - r_1^2\right), \quad r \geq r_1. \tag{3.9}$$

Furthermore, by the above bound on $P$ and $P'$,

$$|RP(r) \Theta(r) - RP_0 \Theta_0| \lesssim M_2 \frac{r}{1+r}.$$ 

As a consequence,

$$F_U(r) \lesssim \left(M_1^2 + M_2\right) \frac{r}{1+r} \quad \text{and} \quad |F'_U(r)| \lesssim \frac{M_1^2 + M_2}{1+r}. \tag{3.10}$$

Therefore,

$$|U(r)| \lesssim \left(M_1^2 + M_2\right) r^{1-d} \int_0^r r_1^{d-1} e^{-C(r^2-r_1^2)} \frac{r_1}{1+r_1} dr_1.$$ 

Using the inequality (for $C > 0$)

$$\int_1^r s^\alpha e^{-C(r^2-s^2)} ds \lesssim r^{\alpha-1}, \tag{3.11}$$

we get that

$$|U(r)| \lesssim \left(M_1^2 + M_2\right) \frac{r^2}{(1+r)^3}.$$ 

As for the derivative of $U$, we can get the required estimates for small $r$ easily. In fact, we directly differentiate to estimate

$$|U'(r)| \lesssim \left(M_1^2 + M_2\right) r, \quad 0 < r \leq 1.$$
In order to deal with \( r > 1 \), we write

\[
U'(r) + \frac{d-1}{r} U(r) = \frac{F_{U}(r)}{2\mu + \lambda} - W'(r)U(r),
\]

(3.12)

In order to see that the right-hand side is decaying, we must take advantage of a cancellation between the two terms. It becomes apparent after integrating by parts that

\[
-W'(r)U(r) = -W'(r)\frac{r^{-d+1}}{2\mu + \lambda} \int_{0}^{r_{1}} r_{1}^{d-1} \partial_{r_{1}} e^{-W(r_{1})+W(r_{1})} F_{U}(r_{1}) \, dr_{1}
\]

\[
= -\frac{F_{U}(r)}{2\mu + \lambda} + W'(r)\frac{r^{-d+1}}{2\mu + \lambda} \int_{0}^{r} e^{-W(r_{1})+W(r_{1})} \partial_{r_{1}} \left( r_{1}^{d-1} F_{U}(r_{1}) \right) \, dr_{1},
\]

so that

\[
U'(r) + \frac{d-1}{r} U(r) = W'(r)\frac{r^{-d+1}}{2\mu + \lambda} \int_{0}^{r} e^{-W(r_{1})+W(r_{1})} \partial_{r_{1}} \left( r_{1}^{d-1} F_{U}(r_{1}) \right) \, dr_{1}.
\]

(3.13)

By (3.7), (3.8) and (3.10),

\[
\left| \partial_{r_{1}} \left( \frac{r_{1}^{d-1} F_{U}(r_{1})}{W'(r_{1})} \right) \right| = 2(2\mu + \lambda) \left| (d-1) r_{1}^{d-2} F_{U}(r_{1}) + r_{1}^{d-1} \frac{F_{U}(r_{1})}{r_{1} P(r_{1})} \right|
\]

\[
+ r_{1}^{d-1} F_{U}(r_{1}) \left( -r_{1}^{-2} P(r_{1})^{-1} - r_{1}^{-1} P'(r_{1}) P(r_{1})^{-2} \right)
\]

\[
\lesssim r_{1}^{d-3} \left( M_{1}^{2} + M_{2} \right)
\]

(3.14)

Therefore, for \( r > 1 \),

\[
\left| U'(r) + \frac{d-1}{r} U(r) \right| \lesssim \left( M_{1}^{2} + M_{2} \right) r r^{-d+1} \int_{0}^{r} e^{-C(r^{2}-r_{1}^{2})} r_{1}^{d-3} \, dr_{1} \lesssim \left( M_{1}^{2} + M_{2} \right) r^{-2},
\]

where the last inequality follows from (3.11). Summarizing, we obtain the estimate

\[
\left| U'(r) + \frac{d-1}{r} U(r) \right| \lesssim \left( M_{1}^{2} + M_{2} \right) \frac{r}{(1+r)^{2}}.
\]

(3.15)

**Estimate of \( \Theta \).** Observe that

\[
|F_{\Theta}(s)| \lesssim (M_{1}^{2} + M_{1} M_{2}) \frac{s}{(1+s)^{2}},
\]

while, if \( 0 \leq r_{1} \leq r \),

\[
-Z(r) + Z(r_{1}) \leq C \left( r_{1}^{2} - r^{2} \right).
\]
It follows then from (3.11) that

$$|\Theta(r)| \lesssim \frac{\Theta_0 + M_1^2 + M_1 M_2}{(1 + r)^2}.$$ 

Finally, noticing that

$$(d - 2)r^{-d+2} \int_0^r r_1^{d-3} e^{-Z(r)+Z(r_1)} \, dr_1$$

$$= 1 + (d - 2)r^{-d+2} \int_0^r r_1^{d-3} \left[ e^{-Z(r)+Z(r_1)} - 1 \right] \, dr_1, \quad (3.16)$$

one can prove that

$$|\Theta'(r)| \lesssim \left( \Theta_0 + M_1^2 + M_1 M_2 \right) \frac{r}{(1 + r)^2}.$$ 

3.5. Asymptotic behavior: proof of Lemma 3.5

Asymptotic behavior of \( P \). It follows from the estimate (3.8) that

$$\int_0^\infty |P'(r)| \, dr < \infty \quad \text{and} \quad P(r) = P_\infty + O \left( \frac{1}{r^2} \right) \quad \text{with} \quad P_\infty = \int_0^\infty P'(r) \, dr.$$

Asymptotic behavior of \( U \). The terms \( P U^2 + P R \Theta \), which are parts of \( F_U \), can be estimated by

$$\frac{r^{-d+1}}{2\mu + \lambda} \int_0^r \frac{r_1^{d-1} e^{-W(r)+W(r_1)}}{r_1 P(r_1)} \left( P U^2 + P R \Theta \right) \, d\tau$$

$$\lesssim r^{-d+1} \int_0^r r_1^{d-1} e^{-c(r^2-r_1^2)} \frac{1}{(1 + r_1)^2} \, dr_1 \lesssim r^{-3}.$$ 

The other terms can be written after integration by parts

$$\frac{r^{-d+1}}{2\mu + \lambda} \int_0^r \frac{r_1^{d-1} 2(2\mu + \lambda)}{r_1 P(r_1)} \left( \partial_r e^{-W(r)+W(r_1)} \right) \left( \int_0^{r_1} \frac{d - 1}{r_2} P U^2 \, dr_2 - P_0 R \Theta_0 \right) \, d\tau$$

$$= \frac{2r^{-1}}{P(r)} \left( \int_0^r \frac{d - 1}{r_2} P U^2 \, dr_2 - P_0 R \Theta_0 \right) + A(r), \quad (3.17)$$

where

$$A(r) = 2r^{-d+1} \int_0^r e^{-W(r)+W(r_1)} \partial_r \left( \frac{r_1^{d-2}}{P(r_1)} \left( \int_0^{r_1} \frac{d - 1}{r_2} P U^2 \, dr_2 - P_0 R \Theta_0 \right) \right) \, d\tau.$$ 

Noting that

$$\int_0^\infty \left| \frac{d - 1}{r_2} P U^2 \right| \, dr_2 < \infty \quad \text{and} \quad \int_0^r \frac{d - 1}{r_2} P U^2 \, dr_2$$
Self-similar solutions of the compressible Navier–Stokes equations

\[ = \int_0^\infty \frac{d-1}{r^2} P U^2 \, dr_2 + O \left( \frac{1}{r^2} \right), \]

we see that, for a constant \( U_\infty \),

\[ \frac{2 r^{-1}}{P(r)} \left( \int_0^{r_1} \frac{d-1}{r_2} P U^2 \, dr_2 - P_0 R \Theta_0 \right) = \frac{U_\infty}{r} + O \left( \frac{1}{r^3} \right). \]

Finally, it follows from (3.11) that \( A(r) \lesssim r^{-3} \).

Asymptotic behavior of \( \Theta \). In a manner similar to the above argument for \( U \), we deduce that \( \Theta \) can be written as

\[ \Theta(r) = (d - 2) r^{-d+2} \int_0^r r_1^{d-3} e^{-Z(r) + Z(r_1)} \, dr_1 \Theta_0 - \frac{U^2}{2 C_V} \]

\[ + \frac{r^{-d+2}}{\kappa} \int_0^r r_1^{d-2} e^{-Z(r) + Z(r_1)} \tilde{F}_\Theta(r_1) \, dr_1 + O \left( \frac{1}{r^4} \right), \]

where

\[ \tilde{F}_\Theta(r_1) := \frac{d - 2}{r_1} \int_0^{r_1} \left( U P \left( \frac{U^2}{2} + C_V \Theta \right) + U P R \Theta \right) \, dr_2 \]

\[ - \frac{\lambda (d - 1)(d - 2)}{r_1} \int_0^{r_1} \frac{U^2}{r_2} \, dr_2. \]

The integration by parts allows us to handle these terms in a similar way to that for \( U \) and we obtain that there exists \( C_2 \) such that

\[ \tilde{\Theta}(r) = C_2 r^{-2} + O \left( r^{-4} \right) \text{ as } r \to \infty. \]

Positivity of temperature for all \( r > 0 \). It is easy to check that there exists \( c > 0 \) such that

\[ (d - 2) r^{-d+2} \int_0^r r_1^{d-3} e^{-Z(r) + Z(r_1)} \, dr_1 \Theta_0 \geq c \Theta_0 (1 + r)^{-2} \text{ for any } r \geq (3.18) \]

On the other hand, we can estimate the others as

\[ \left| \Theta(r) - (d - 2) r^{-d+2} \int_0^r r_1^{d-3} e^{-Z(r) + Z(r_1)} \, dr_1 \Theta_0 \right| \]

\[ \lesssim \left( M_1^2 + M_1 M_2 \right) (1 + r)^{-2} \ll \Theta_0 (1 + r)^{-2}, \]

(3.19)

which proves the positivity of \( \Theta \) provided that \( \Theta_0 \ll 1 \).

Expansion of \( \Theta_\infty \) and \( U_\infty \). Observe first that

\[ \Theta(r) = (d - 2) r^{-d+2} \int_0^r r_1^{d-3} e^{-Z(r) + Z(r_1)} \, dr_1 \Theta_0 + O \left( \frac{\Theta_0^2}{r^2} \right) + O \left( \frac{1}{r^3} \right). \]
Since $P_\infty = P_0 + O(\Theta_0)$, we get $Z(r) = \frac{CVP_0}{4\kappa} r^2 + O(\Theta_0 r^2)$, and therefore, as $r \to \infty$,

$$(d - 2)r^{-d+2} \int_0^r r_1^{d-3} e^{-Z(r) + Z(r_1)} \, dr_1 \Theta_0 = \left[ \frac{2(d-2)\kappa}{CVP_0} \Theta_0 + O(\Theta_0^2) \right] \frac{1}{r^2}.$$ 

This means that

$$\Theta_\infty = \frac{2(d-2)\kappa}{CVP_0} \Theta_0 + O(\Theta_0^2).$$

Arguing similarly, for $U$, one finds that $U_\infty = -2R\Theta_0 + O(\Theta_0^2)$.

### 4. Cavitating self-similar solutions

#### 4.1. Main result

**Notation 4.1.** In this section, we consider $R, \mu, \lambda, C_V$ as fixed positive constants. We denote $C$ a constant which depends on $(R, \mu, \lambda, C_V) \in (0, \infty)^4$; the implicit constant in the notations $\lesssim$ and $O(\cdot)$ has the same properties.

Furthermore, for given quantities $A$ and $B$, we denote $A \ll B$ to mean that $A \leq cB$ for a constant $c = c(R, \mu, \lambda, C_V)$ which is sufficiently small so that all the needed arguments apply.

**Theorem 4.2.** There exists a constant $\epsilon > 0$ such that, if

$$P_\delta + \delta + \Theta_0 + \alpha + \frac{P_\delta \Theta_0}{\alpha} + \frac{\alpha^2}{P_\delta \Theta_0} + \alpha \log \frac{1}{P_\delta \delta^2} < \epsilon, \quad (4.1)$$

then there exists a solution $(P, U, \Theta) \in C^1([0, \infty)) \times C^1([0, \infty)) \times C^{\frac{2d\alpha}{1-2\alpha}}$ solving (2.1) such that $P(\delta) = P_\delta$, $U(0) = 0$, $U'(0) = \alpha$, $\Theta(0) = \Theta_0$, $\Theta'(0) = 0$. For $r$ small, this satisfies

$$P(r) = P_\delta \left( \frac{r}{\delta} \right)^\frac{2d\alpha}{1-2\alpha} + O \left( \frac{r}{\delta} \right)^{\frac{2d\alpha}{1-2\alpha} + 1 + d\alpha},$$

$$U(r) = \alpha r + O \left( r^{1 + \frac{2d\alpha}{1-2\alpha}} \right),$$

$$\Theta(r) = \Theta_0 + O \left( r^2 \right).$$

It also satisfies the global bounds

$$|P(r)| \lesssim P_\delta \min \left[ 1, \left( \frac{r}{\delta} \right)^\frac{2d\alpha}{1-2\alpha} \right],$$

$$|U(r)| \lesssim \frac{\alpha r}{(1 + \sqrt{P_\delta r})^2}, \quad |U'(r)| \lesssim \frac{\alpha}{(1 + \sqrt{P_\delta r})^2},$$

$$|\Theta(r)| \lesssim \frac{1}{(1 + \sqrt{P_\delta r})^2}, \quad |\Theta'(r)| \lesssim \frac{\sqrt{P_\delta r}}{(1 + \sqrt{P_\delta r})^2}.$$
Furthermore, there exists $P_\infty > 0$, $U_\infty < 0$, $\Theta_\infty > 0$ such that

\[
P(r) = P_\infty + O\left(\frac{1}{r^2}\right),
\]

\[
U(r) = \frac{U_\infty}{r} + O\left(\frac{1}{r^3}\right),
\]

\[
\Theta(r) = \frac{\Theta_\infty}{r^2} + O\left(\frac{1}{r^4}\right).
\]

Finally,

\[
U_\infty = \frac{2d(2\mu + \lambda)}{P_\delta} \alpha \left[1 + O\left(\frac{\alpha \log \frac{1}{P_\delta \delta^2}}{P_\delta \delta^2}\right)\right],
\]

\[
\Theta_\infty = \frac{2(d - 2)\kappa}{C_V P_\delta} \Theta_0 \left[1 + O\left(\frac{\alpha \log \frac{1}{P_\delta \delta^2}}{P_\delta \delta^2}\right) + O\left(\frac{\alpha^2}{P_\delta \Theta_0}\right)\right].
\]

Remark 4.3. The above result remains true if $\Theta_0$ is not assumed to be small as in (4.1), but only $O(1)$.

Remark 4.4. To clarify the meaning of (4.1), we can scale $P_\delta$ and $\Theta_0$ in terms of $\alpha$

\[
P_\delta = \alpha^{\nu_1}, \quad \Theta_0 = \alpha^{\nu_2},
\]

while keeping $\delta \ll 1$ fixed. Then (4.1) amounts to requiring that $\alpha \ll 1$, together with

\[
\nu_1 > 0, \quad \nu_2 > 0, \quad \text{and} \quad 1 < \nu_1 + \nu_2 < 2.
\]

4.2. Main steps of the proof

Let

\[
\|(U, \Theta)\|_\delta^{\delta} := \sup_{0 < r < \delta} \left[r^{-1}|U(r)| + |U'(r)| + |\Theta(r)| + r^{-1}|\Theta'(r)|\right].
\]

To prove the existence of a local solution, let $\Psi$ be the map which to $(U, \Theta)$ associates the right-hand side of (2.3) and (2.4):

\[
\Psi : (U, \Theta) \mapsto (\text{RHS}(2.7), \text{RHS}(2.8)).
\]

Define

\[
E_\delta = \{(U, \Theta) \in C^1(0, \delta) \text{ such that } \Theta(0) = \Theta_0 \text{ and } \|(U, \Theta)\|_\delta < \infty\}.
\]

This is an affine Banach space.

Lemma 4.5. If $\alpha + \delta + P_\delta + \Theta_0 + \frac{P_0 \Theta_0}{\alpha}$ is sufficiently small, $\Psi$ is a contraction on $B_{E_\delta}(\Theta_0, \alpha r, \alpha/2)$. 


By the Banach fixed point theorem, this lemma gives the local existence (close to zero) of solutions. In order to prolong them, we will argue as in the case of smooth self-similar solutions and define a \( Z \) function.

Observing that

\[
\begin{align*}
& r^{-d+1} \int_0^r r_1^{d-1} e^{-C P_\delta(r^2-r_1^2)} \, dr_1 \simeq \frac{r}{(1 + \sqrt{P_\delta} r)^2}, \\
& r^{-d+2} \int_0^r r_1^{d-3} e^{-C P_\delta(r^2-r_1^2)} \, dr_1 \simeq \frac{1}{(1 + \sqrt{P_\delta} r)^2}, \\
& \left| \partial_r \left[ r^{-d+2} \int_0^r r_1^{d-3} e^{-C P_\delta(r^2-r_1^2)} \, dr_1 \right] \right| \lesssim \frac{P_\delta r}{1 + P_\delta r^2},
\end{align*}
\]

we are led to defining

\[
Z(s) := \sup_{0 < r < s} \frac{1}{M_1} r^{-1} \left( 1 + \sqrt{P_\delta} r \right)^2 |U(r)| + \frac{1}{M_1'} \left( 1 + \sqrt{P_\delta} r \right)^2 |U'(r)| + \frac{d - 1}{r} U(r)
\]

\[
+ \frac{1}{M_2} \left( 1 + \sqrt{P_\delta} r \right)^2 |\Theta(r)| + \frac{1}{M_2} (P_\delta)^{-1} \left( 1 + \sqrt{P_\delta} r \right)^2 |\Theta'(r)|.
\]

In the above, the constants are chosen such that

\[
M_1 + M_1' + M_2 \ll 1 \quad \text{and} \quad M_1 < M_1';
\]

these constants will be determined more precisely shortly.

**Lemma 4.6.** Assume that \( M_1' \log \frac{1}{P_\delta \delta^2} \) is sufficiently small. Then \( Z(\delta) < 1 \) provided that

\[
\alpha \ll M_1, \quad \alpha + \Theta_0 \ll M_2, \quad \alpha^2 \ll M_2 P_\delta.
\]

Furthermore, if \( Z(r) \leq 1 \) for some \( r > 0 \), then

\[
Z(r) \lesssim M_1 + \frac{P_\delta M_2}{M_1} + \frac{M_1}{M_1'} + \frac{\alpha}{M_1} + \frac{\Theta_0}{M_2} + \frac{M_1 M_1'}{P_\delta M_2}.
\]

Finally, the asymptotic behavior of \( U, \Theta, \) and \( P \) is established in the following lemma:

**Lemma 4.7.** There exists \( P_\infty > 0, U_\infty \in \mathbb{R}, \Theta_\infty > 0 \) such that

\[
P(r) = P_\infty + O \left( \frac{1}{r^2} \right), \quad U(r) = \frac{U_\infty}{r} + O \left( \frac{1}{r^3} \right), \quad \Theta(r) = \frac{\Theta_\infty}{r^2} + O \left( \frac{1}{r^4} \right).
\]

Furthermore, \( \Theta(r) > 0 \) for any \( r \), provided that

\[
M_1 M_2 + \frac{M_1 M_1'}{P_\delta} \ll \Theta_0.
\]

Finally, \( \Theta_\infty \) and \( U_\infty \) can be expanded as

\[
U_\infty = \frac{2d(2\mu + \lambda)}{P_\delta} \alpha \left[ 1 + O \left( \alpha \log \frac{1}{P_\delta \delta^2} \right) \right],
\]

\[
\Theta_\infty = \frac{2(d - 2)\kappa}{C_V \bar{P_\delta}} \Theta_0 \left[ 1 + O \left( \alpha \log \frac{1}{P_\delta \delta^2} \right) + O \left( \frac{\alpha^2}{P_\delta \Theta_0} \right) \right].
\]
It remains to choose the constants $M_1, M'_1, M_2$, and to understand which range is allowed for parameters $\alpha, \delta, P_\delta, \Theta_0$, so that lemmas 4.5, 4.6 and 4.7 apply. Let us summarize the requirements:

- We are interested in the perturbative regime where $\alpha, \delta, P_\delta, \Theta_0, M_1, M'_1, M_2 \ll 1$.
- Local well posedness: requires $P_\delta \Theta_0 \ll \alpha$.
- Control of $P$: requires $M'_1 \log \frac{1}{P_\delta^2} \ll 1$.
- $Z(\delta) < 1$: requires $\alpha \ll M_1, \alpha + \Theta_0 \ll M_2, \alpha^2 \ll M_2 P_\delta$.
- Bootstrap on $Z$: requires $M_1 P_\delta + M_2 M'_1 + \alpha M_1 + \frac{\Theta_0}{M_2} + M_2 M'_1 \ll 1$.
- Positivity of $\Theta$: requires $M_1 M_2 + M_1 M'_1 P_\delta \ll \Theta_0$.

First, we define $M_1, M'_1, M_2$ as follows:

$$M_1 = \Lambda \alpha, \quad M'_1 = \Lambda^2 \alpha, \quad M_2 = \frac{\alpha}{P_\delta}.$$ 

Here $\Lambda > 0$ is taken so big that $\frac{P_\delta M_2}{M'_1} + \frac{M_1}{M'_1} + \frac{\alpha}{M_1} = \frac{3}{\Lambda} \ll 1$. Assuming that $\alpha, P_\delta, \Theta_0 \ll 1$, the remaining conditions reduce to the requirement that

$$\alpha^2 \ll P_\delta \Theta_0 \ll \alpha, \quad \text{and} \quad \alpha \log \frac{1}{P_\delta^2} \ll 1,$$

which completes the proof.

### 4.3. Local existence: proof of Lemma 4.5

Estimates on auxiliary functions. Consider $(U, \Theta) \in B_{E^3}((\alpha r, \Theta_0), \frac{\alpha}{2})$, which implies in particular that $U(r) \geq \frac{\alpha}{2}r$. We derive various estimates on $V, W, Z, P, F_U$ and $F_\Theta$. First observe that, for $r < \delta$,

$$V(\delta) - V(r) = \int_r^\delta \frac{U'}{\frac{1}{2}r - U} \frac{d}{dr} \log \left( \frac{\delta}{r} \right).$$

As a consequence,

$$P(r) \leq P_\delta \left( \frac{r}{\delta} \right)^{\frac{\alpha}{r}}.$$

It is then easy to see that

$$|W| + |Z| \lesssim P_\delta \left( \frac{r}{\delta} \right)^{\frac{\alpha}{r}} r^2 \quad \text{and} \quad |W'| + |Z'| \lesssim P_\delta \left( \frac{r}{\delta} \right)^{\frac{\alpha}{r}} r.$$

Furthermore,

$$|V'(r)| \lesssim \frac{\alpha}{r} \quad \text{and} \quad |P'(r)| \lesssim P_\delta \frac{\alpha}{r} \left( \frac{r}{\delta} \right)^{\frac{\alpha}{r}}.$$

We deduce from the above that

$$|\tilde{F}_U - d (2\mu + \lambda) \alpha| \lesssim P_\delta \left( \alpha + \Theta_0 \right) \quad \text{and} \quad |F_\Theta(r)| \lesssim \left( P_\delta \alpha \Theta_0 + \alpha^2 \right) r.$$
Estimates on differences of auxiliary functions. Consider two elements \((U_1, \Theta_1)\) and \((U_2, \Theta_2)\) of \(B_{E^s((\alpha r, \Theta_0), \frac{\alpha}{r})}\), with associated functions \(V_1, V_2, \text{etc... .}\) We will denote \(D\) as the distance as follows:

\[
D = \| (U_1, \Theta_1) - (U_2, \Theta_2) \|_\delta.
\]

To start with, it is obvious that

\[
|\Theta_1(r) - \Theta_2(r)| \lesssim Dr^2.
\]

Next observe that \(|V_1' - V_2'| \lesssim \frac{D}{r}\), which implies \(|[V_1(\delta) - V_1(r)] - [V_2(\delta) - V_2(r)]| \lesssim D \log \left(\frac{\delta}{r}\right)\). Using successively, the inequalities \(|e^x - e^y| \leq \max(e^x, e^y)|x-y|\) and \(\sup_{0 < t < 1} r^\sigma \log \frac{1}{r} \sim \frac{1}{\sigma}\), we get

\[
|P_1(r) - P_2(r)| \lesssim P_\delta \left(\frac{r}{\delta}\right)^{d\alpha} D \log \left(\frac{\delta}{r}\right) \lesssim D \frac{P_\delta}{\alpha}.
\]

This leads to

\[
|W_1(r) - W_2(r)| + |Z_1(r) - Z_2(r)| \lesssim D \frac{P_\delta r^2}{\alpha},
\]

\[
|\tilde{F}_{U_1}(r) - \tilde{F}_{U_2}(r)| \lesssim D \left[ P_\delta + \frac{P_\delta \Theta_0}{\alpha} \right],
\]

\[
|F_{\Theta_1}(r) - F_{\Theta_2}(r)| \lesssim D \left[ P_\delta \Theta_0 + \alpha \right] r.
\]

**Estimate on \(\Psi(U_1, \Theta_1) - \Psi(U_2, \Theta_2)\).** Let \((\tilde{U}_i, \tilde{\Theta}_i) := \Psi(U_i, \Theta_i)\). Then

\[
|\tilde{U}_1(r) - \tilde{U}_2(r)| \lesssim r^{1-d} \int_0^r r_{1-d}^{d-1} |e^{W_1(r_1)} - W_1(r_1)| |\tilde{F}_{U_1}(r_1)| dr_1
\]

\[
\quad + r^{1-d} \int_0^r r_{1-d}^{d-1} e^{W_1(r_1)-W_2(r_1)} |\tilde{F}_{U_1}(r_1) - \tilde{F}_{U_2}(r_1)| dr_1
\]

\[
\lesssim r^{1-d} \int_0^r r_{1-d}^{d-1} \left[ P_\delta + \frac{P_\delta \Theta_0}{\alpha} \right] D dr_1
\]

\[
\lesssim \left[ P_\delta + \frac{P_\delta \Theta_0}{\alpha} \right] D r.
\]

Similarly,

\[
|\tilde{U}_1'(r) - \tilde{U}_2'(r)| \lesssim \left[ P_\delta + \frac{P_\delta \Theta_0}{\alpha} \right] D,
\]

and finally,

\[
|\Theta_1(r) - \Theta_2(r)| \lesssim D \left[ \frac{P_\delta \Theta_0}{\alpha} + \alpha \right] r^2,
\]

\[
|\Theta_1'(r) - \Theta_2'(r)| \lesssim D \left[ \frac{P_\delta \Theta_0}{\alpha} + \alpha \right] r.
\]
As a conclusion,
\[ \| \Psi(U_1, \Theta_1) - \Psi(U_2, \Theta_2) \|_\delta \lesssim \left[ P_\delta + \alpha + \frac{P_\delta \Theta_0}{\alpha} \right] \| (U_1, \Theta_1) - (U_2, \Theta_2) \|_\delta. \]

Therefore, \( \Psi \) acts as a contraction provided \( P_\delta + \alpha + \frac{P_\delta \Theta_0}{\alpha} \ll 1 \). It remains to see that this stabilizes \( B_{E \delta}((\alpha r, \Theta_0), \alpha/2) \), which will be achieved in the next point.

The first iterate. Denoting \( (\hat{U}, \hat{\Theta}) = \Psi(\alpha r, \Theta_0) \), it follows that
\[
\begin{align*}
|\hat{U} - \alpha r| &\lesssim r P_\delta (\alpha + \Theta_0), \\
|\hat{U}' - \alpha| &\lesssim P_\delta (\alpha + \Theta_0), \\
|\hat{\Theta} - \Theta_0| &\lesssim (\alpha^2 + \Theta_0 P_\delta) r^2, \\
|\hat{\Theta}'| &\lesssim (\alpha^2 + \Theta_0 P_\delta) r.
\end{align*}
\]

Therefore,
\[ \| \psi(\alpha r, \Theta_0) - (\alpha r, \Theta_0) \|_\delta \lesssim P_\delta \Theta_0 + \alpha^2 + \alpha P_\delta. \]

This is \( \ll \alpha \), provided that \( \alpha + P_\delta \Theta_0 \alpha \) is sufficiently small.

Hölder continuity of \( P \). We discuss around \( r = 0 \) only. Observe that
\[
\begin{align*}
&|U'(r) + \frac{d - 1}{r} U(r)| = \left| \frac{1}{2\mu + \lambda} \tilde{T}_U(r) - W'(r)U(r) \right|, \\
&\frac{1}{2\mu + \lambda} \tilde{T}_U(r) = d\alpha + O(\alpha^d), \quad W'(r)U(r) = O(r^2), \\
&\frac{1}{2} r - U(r) = \left( \frac{1}{2} - \alpha \right) r + \alpha r - U(r) = \left( \frac{1}{2} - \alpha \right) r + O\left( r^{1+d\alpha} \right).
\end{align*}
\]

These give that, for \( r \leq \delta \),
\[
e^{\frac{V(\delta) - V(r)}{\delta}} = \exp \left( \int_\delta^r \frac{d\alpha}{1 - \alpha r_1} dr_1 + O \left( r^{1+d\alpha} \right) \right) = \left( \frac{r}{\delta} \right)^{\frac{2d\alpha}{1-d\alpha}} \left( 1 + O \left( r^{1+d\alpha} \right) \right),
\]

which proves \( P(\cdot) \in C^{\frac{2d\alpha}{1-d\alpha}}(0, \delta) \).

### 4.4. Global existence: proof of Lemma 4.6

Estimate of \( P \). For \( r \leq \delta \), it follows from the boundedness of \( U \) that
\[
V(\delta) - V(r) \geq \int_r^\delta \frac{\alpha}{2} + \frac{d-1}{s} \frac{\alpha^2 s}{2} ds \geq \frac{d\alpha}{1 - \alpha} \log \frac{\delta}{r} \geq d\alpha \log \frac{\delta}{r},
\]

and similarly, using \( \alpha \ll 1 \),
\[
V(\delta) - V(r) \leq 4\alpha d \log \frac{\delta}{r}.
\]
Therefore,
\[
\left(\frac{r}{\delta}\right)^{4d\alpha} P_\delta \leq P(r) \leq \left(\frac{r}{\delta}\right)^{d\alpha} P_\delta \quad \text{for any } r \in [0, \delta].
\]

As for \( r \geq \delta \),

\[
|V(r) - V(\delta)| = \left| \int_\delta^r \frac{U' + \frac{d-1}{s}U}{\frac{1}{2}s - U} \, ds \right| \lesssim \int_\delta^r \frac{M'_1}{s (1 + \sqrt{P_\delta s})^2} \, ds
\]

\[
\lesssim M'_1 \left[ \log \frac{1}{P_\delta \delta^2} + 1 \right] \lesssim 1,
\]

under the assumption that \( M'_1 \log \frac{1}{P_\delta \delta^2} \) is \( O(1) \). This implies that, for \( r \geq \delta \),

\[
P_\delta \lesssim P(r) \lesssim P_\delta.
\]

Finally, we record that, for any \( r > 0 \),

\[
V'(r) \lesssim \frac{M'_1}{r (1 + \sqrt{P_\delta r})^2},
\]

\[
W(r) + Z(r) \lesssim P_\delta r^2,
\]

\[
W'(r) + Z'(r) \lesssim P_\delta r.
\]

The starting point: \( r = \delta \). We start by checking that \( Z(\delta) \ll 1 \). Regarding the first three summands in the definition of \( Z \), it is clear from the fixed point argument as soon as we choose

\[
\alpha \ll M_1 \quad \text{and} \quad \alpha + \Theta_0 \ll M_2.
\]

Turning to the fourth and last summand, we need to show that \( |\Theta'(\delta)| \ll M_2 P_\delta r \).

Observe that \( \Theta \) has the same derivative as

\[
(d - 2) r^{-d+2} \int_0^r r_1^{-d+3} \left[ e^{-Z(r) + Z(r_1)} - 1 \right] \, dr_1 \Theta_0 - \frac{U^2}{2C_V}
\]

\[+ \frac{r^{-d+2}}{\kappa} \int_0^r r_1^{-d+2} e^{-Z(r) + Z(r_1)} F(\Theta(r_1)) \, dr_1,
\]

which we write \( I + II + III \). One can then check that, on \([0, \delta]\),

\[
|I'| \lesssim P_\delta \Theta_0 r,
\]

\[
|II'| \lesssim \alpha^2 r,
\]

\[
|III'| \lesssim \left( P_\delta \alpha \Theta_0 + \alpha^2 \right) r.
\]

Therefore, \( Z(\delta) \ll 1 \), provided that

\[
\alpha^2 \ll M_2 P_\delta.
\]
Estimate of $U$. From a similar argument to that in (3.9),
\[ e^{-C P_\delta (r^2 - r_1^2)} \lesssim e^{-W(r) + W(r_1)} \lesssim e^{-C^{-1} P_\delta (r^2 - r_1^2)} \quad \text{if } 0 \leq r_1 \leq r. \]

Furthermore,
\[
\tilde{F}_U(r) \lesssim \left( \frac{P_\delta r^2}{4 (1 + \sqrt{P_\delta} r)} + 1 \right) M_1^2 + \frac{P_\delta M_2}{(1 + \sqrt{P_\delta} r)^2} + \alpha \lesssim M_1^2 + P_\delta M_2 + \alpha,
\]
\[
\tilde{F}_U'(r) \lesssim \frac{M_1'M_1 + P_\delta M_2}{r}.
\]

It follows by (4.2) that
\[
|U(r)| \lesssim \frac{r}{(1 + \sqrt{P_\delta} r)^2} \left( M_1^2 + P_\delta M_2 + \alpha \right).
\]

As for the derivative, the required estimate for small $r$ is easy, since differentiating directly gives
\[
\left| U'(r) + \frac{d-1}{r} U(r) \right| = \left| \frac{1}{2\mu + \lambda} \tilde{F}_U(r) - W'(r) U(r) \right| \lesssim M_1^2 + P_\delta M_2 + M_1 + \alpha \quad \text{for any } r \leq \frac{1}{\sqrt{P_\delta}}.
\]

In order to investigate for large $r \geq 1/\sqrt{P_\delta}$, first notice that
\[
\left| \partial_r \left( \frac{r^{d-1} \tilde{F}_U(r)}{W'(r)} \right) \right| \lesssim r^{d-3} \alpha + M_1 + P_\delta M_2 \frac{1}{P_\delta} \left( 1 + \left( \frac{r}{\delta} \right)^{-4d} \alpha \right),
\]
where there appears a slight singularity $r^{-4d} \alpha$ around $r = 0$ because of $P$ in the denominator, compared with (3.14), while it does not change the behavior. Using the inequality $r^{1-d} \int_0^r e^{-C P_\delta (r^2 - r_1^2)} r^{d-3} \, dr_1 \lesssim \frac{1}{P_\delta r^3}$ for $r \geq \frac{1}{\sqrt{P_\delta}}$, we get, by (3.13), that
\[
\left| U'(r) + \frac{d-1}{r} U(r) \right| = \left| W'(r) \left\{ \frac{r^{d-1}}{2\mu + \lambda} \right\} \right| \int_0^r e^{-W(r) + W(r_1)} \, dr_1 \left( \frac{r^{d-1} F_U(r_1)}{W'(r_1)} \right) \, dr_1 \lesssim P_\delta r^{1-P_\delta r^3} \left( \frac{\alpha + M_1 + P_\delta M_2}{P_\delta} \right) \lesssim \frac{1}{P_\delta r^2} (\alpha + M_1 + P_\delta M_2) \quad \text{for } r \geq \frac{1}{\sqrt{P_\delta}}.
\]

Estimate of $\Theta$. Observe that, for $r \leq 1/\sqrt{P_\delta}$,
\[
F_{\Theta}(r) \lesssim (M_1 M_1' + P_\delta M_1 M_2) r,
\]
which proves that
\[
|\Theta(r)| \lesssim \Theta_0 + \frac{M_1 M_1'}{P_\delta} + M_1 M_2 \quad \text{for } r \leq \frac{1}{\sqrt{P_\delta}}.
\]
Furthermore, for any $r > 0$,

$$F_\Theta(r) \lesssim \frac{1}{r} \left( M_1 M_2 + \frac{M_1 M'_1}{P_\delta} \right),$$

which gives, by (4.2),

$$|\Theta(r)| \lesssim \frac{1}{P_\delta r^2} \left( \Theta_0 + M_1 M_2 + \frac{M_1 M'_1}{P_\delta} \right).$$

Finally, if $r \leq \frac{1}{\sqrt{P_\delta}}$,

$$|\Theta'(r)| \lesssim P_\delta r \left( \Theta_0 + M_1 M_2 + \frac{M_1 M'_1}{P_\delta} \right),$$

while if $r \geq \frac{1}{\sqrt{P_\delta}}$,

$$|\Theta'(r)| \lesssim \frac{1}{r} \left( \Theta_0 + M_1 M_2 + \frac{M_1 M'_1}{P_\delta} \right).$$

This completes the proof of the desired estimate on $Z$.

4.5. Asymptotic behavior: proof of Lemma 4.7

The fact that $P$ converges, while $\Theta \sim \frac{\Theta_\infty}{r^2}$ and $U \sim \frac{U_\infty}{r}$, can be proved as in the smooth case and will not be developed here.

Turning to the positivity of $\Theta$, we observe first that, by the estimates above, there exists

$$\Theta(r) = (d-2)r^{-d+2} \int_0^r r_1^{d-3} e^{-Z(r_1)} dr_1 \Theta_0
+ O \left( \left( M_1 M_2 + \frac{M_1 M'_1}{P_\delta} \right) \frac{1}{(1 + \sqrt{P_\delta r})^2} \right).$$

Since, by (4.2),

$$(d-2)r^{-d+2} \int_0^r r_1^{d-3} e^{-Z(r_1)} dr_1 \Theta_0 \gtrsim \frac{\Theta_0}{(1 + \sqrt{P_\delta r})^2},$$

this means that $\Theta > 0$ provided

$$M_1 M_2 + \frac{M_1 M'_1}{P_\delta} \ll \Theta_0.$$ 

Expansion of $\Theta_\infty$ and $U_\infty$. Observe that

$$P_\infty = P_\delta \left[ 1 + O \left( V(\infty) - V(\delta) \right) \right] = P_\delta \left[ 1 + O \left( \alpha \log \frac{1}{P_\delta \delta^2} \right) \right].$$
Since \( W(r) = \frac{P_\delta}{4(2\mu + \lambda)} r^2 \left[ 1 + O \left( \alpha \log \frac{1}{P_\delta \delta^2} \right) \right] \), as \( r \to \infty \),

\[
\frac{r^{-d+1}}{2\mu + \lambda} \int_0^r r_1^{d-1} e^{-W(r)+W(r_1)} d(2\mu + \lambda) \alpha dr_1 = \frac{2d(2\mu + \lambda)}{P_\delta} \alpha \left[ 1 + O \left( \alpha \log \frac{1}{P_\delta \delta^2} \right) \right] \frac{1}{r},
\]

and

\[
\frac{r^{-d+1}}{2\mu + \lambda} \int_0^r r_1^{d-1} e^{-W(r)+W(r_1)} \left( \int_0^{r_1} \frac{d-1}{r_2} PU^2 dr_2 \right) dr_1 = O \left( \frac{\alpha^2}{P_\delta r} \right).
\]

These prove that

\[
U(r) = \frac{U_\infty}{r} + O \left( \frac{1}{r^3} \right) \quad \text{with} \quad U_\infty = \frac{2d(2\mu + \lambda)}{P_\delta} \alpha \left[ 1 + O \left( \alpha \log \frac{1}{P_\delta \delta^2} \right) \right].
\]

Similarly, since \( Z(r) = \frac{C_V P_\delta}{4\kappa} r^2 \left[ 1 + O \left( \alpha \log \frac{1}{P_\delta \delta^2} \right) \right] \), as \( r \to \infty \),

\[
(d - 2)r^{-d+2} \int_0^r r_1^{d-3} e^{-Z(r)+Z(r_1)} dr_1 \Theta_0 = \frac{2(d - 2)\kappa}{C_V P_\delta} \Theta_0 \left[ 1 + O \left( \alpha \log \frac{1}{P_\delta \delta^2} \right) \right] \frac{1}{r^2},
\]

and

\[
\frac{r^{-d+2}}{\kappa} \int_0^r r_1^{d-3} e^{-Z(r)+Z(r_1)} F_\Theta dr_1 = O \left( \frac{\alpha \Theta_0}{P_\delta r^2} + \frac{\alpha^2}{P_\delta^2 r^2} \right).
\]

These give that

\[
\Theta_\infty = \frac{2(d - 2)\kappa}{C_V P_\delta} \Theta_0 \left[ 1 + O \left( \alpha \log \frac{1}{P_\delta \delta^2} \right) + O \left( \frac{\alpha^2}{P_\delta^2 \Theta_0} \right) \right].
\]

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