ESTIMATING THE FRACTAL DIMENSION OF SETS DETERMINED BY NONERGODIC PARAMETERS

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Abstract. Given fixed and irrational $0 < \alpha, \theta < 1$, consider the billiard table $B_\alpha$ formed by a $\frac{1}{2} \times 1$ rectangle with a horizontal barrier of length $\alpha$ emanating from the midpoint of a vertical side and a billiard flow with trajectory angle $\theta$. In 1969, Veech introduced two subsets $K_0(\theta)$ and $K_1(\theta)$ of $\mathbb{R}/\mathbb{Z}$ that are defined in terms of the continued fraction representation of $\theta \in \mathbb{R}/\mathbb{Z}$, and Veech showed that these sets have Hausdorff dimension 0 when $\theta$ is rational. Moreover, the set $K_1(\theta)$ describes the set of all $\alpha$ such that the billiard flow on $B_\alpha$ in direction $\theta$ is nonergodic. We show that the Hausdorff dimension of the sets $K_0(\theta)$ and $K_1(\theta)$ can attain any value in $[0, 1]$ by considering the continued fraction expansion of $\theta$. This result resolves an analogue of work completed by Cheung, Hubert, and Pascal in which they consider, for fixed $\alpha$, the set of $\theta$ such that the flow on $B_\alpha$ in direction $\theta$ is nonergodic.

1. Introduction. In 1969, Veech introduced two subsets (which we define below) $K_1(\theta) \subset K_0(\theta)$ of $\mathbb{R}/\mathbb{Z}$ that correspond to skew products over a rotation of the unit circle (see Section 1.1). Moreover, in [8] Veech showed that the sets $K_1(\theta)$ and $K_0(\theta)$ are countable when $\theta$ is rational; therefore, the Hausdorff dimension of these sets is zero when $\theta$ is rational. We study the dimension of the sets $K_0(\theta)$ and $K_1(\theta)$ by constructing upper and lower bound formulas for particular irrational values of $\theta$.

Any irrational element $\theta \in \mathbb{R}/\mathbb{Z}$ has an infinite continued fraction representation $[0; a_1, a_2, \ldots]$, where the positive integers $a_k$ are called the partial quotients or terms of $\theta$. Define the sequence $(\frac{m_k}{n_k})_{k=1}^{\infty}$ by $\frac{m_k}{n_k} := [0; a_1, a_2, \ldots, a_k]$ for each $k$. From continued fraction theory, if $k \in \mathbb{N}$, we have

$$n_{k+1} = a_{k+1}n_k + n_{k-1},$$

where $n_0 := 1$ and $n_1 := a_1$. Let $(b_k)_{k \in \mathbb{N}}$ be a sequence of even integers satisfying $|b_j| \leq a_{j+1}$ for each $j \in \mathbb{N}$. Given any $m \in \mathbb{Z}$,

$$\langle m; b_1, b_2, \ldots \rangle_{\theta} := \left(m\theta + \sum_{j=1}^{\infty} b_jn_j\theta\right) \mod 1,$$
defines a point of \( \{ x : 0 \leq x < 1 \} \). We are interested in dimension estimates for the sets

\[
K_0 (\theta) := \left\{ (m; b_1, b_2, \ldots)_\theta : b_j \text{ is eventually even and } \lim_{j \to \infty} b_j n_j \| n_j \theta \| = 0 \right\},
\]

\[
K_1 (\theta) := \left\{ (m; b_1, b_2, \ldots)_\theta : b_j \text{ is eventually even and } \sum_{j=1}^{\infty} |b_j| n_j \| n_j \theta \| < \infty \right\},
\]

where \( \| \cdot \| \) denotes the distance to the nearest integer. The fact that \( K_1 (\theta) \subset K_0 (\theta) \) is a consequence of the test for divergence from calculus.

Much can be said for a particular family of continued fractions \( \theta \) constructed in this paper. We say that \( \theta \) is divergent relative to some fixed positive integer \( M \) if the subsequence of terms \( a_k \) that are larger than \( M \) diverge to \( \infty \). For each \( A \subset \mathbb{R} \), denote by \( \text{Hdim} A \) the Hausdorff dimension of \( A \). The main result of this paper is the following theorem.

**Theorem.** For each \( \delta \in [0, 1] \) and \( M \in \mathbb{N}_{\geq 3} \) there exists a continued fraction \( \theta = [0; a_1, a_2, \ldots] \) divergent relative to \( M \) such that \( \text{Hdim} K_0 (\theta) = \text{Hdim} K_1 (\theta) = \delta \).

### 1.1. A geometric interpretation of Veech’s skew products and a physical interpretation of the set \( K_1 (\theta) \)

Veech [8] constructed examples of minimal and not uniquely ergodic dynamical systems as follows (see [8]). Take two copies of the unit circle and mark off a segment \( J \) of length \( 2\pi \alpha \) in the counterclockwise direction on each one with endpoint at 0. Now choose an irrational \( \theta \) and consider the following dynamical system. Start with a point \( p \) in the first circle. Rotate counterclockwise by \( 2\pi \theta \) repeatedly until the orbit lands in \( J \); then switch to the corresponding point in the second circle, rotate by \( 2\pi \theta \) until the first time the point lands in \( J \); switch back to the first circle and so forth. Veech showed there exists irrational \( \alpha \) for which this system is minimal and the Lebesgue measure is not uniquely ergodic. In this particular context, being not uniquely ergodic implies that there are open subsets of \( \mathbb{R}/\mathbb{Z} \) for which the amount of time the orbit spends in those open sets is not proportional to the sizes of the sets.

We may describe Veech’s dynamical system using a flow on a surface arising from a billiard. Consider billiards in the table formed by a \( \frac{1}{2} \times 1 \) rectangle with a horizontal barrier of length \( \alpha \) with one end touching at the midpoint of a vertical side. We can identify the top half of the table as the positive side and the bottom half as the negative side. A standard unfolding of this billiard table is shown in Figure 2. We can view the unfolded table as having two (identified) barriers of length \( 2\alpha \).

It has been a difficult challenge in computing the Hausdorff dimension of the set \( NE \) all \((\alpha, \theta) \in \mathbb{R}^2\) such that the billiard flow is non-ergodic. In an elegant work of Cheung, Hubert and Masur [4], the Hausdorff dimension of the set of all \( \theta \) given \( \alpha \) (i.e., the vertical slices of the set of non-ergodic directions), was computed as either 0 or \( \frac{1}{2} \). Specifically, they showed that the dimension is \( \frac{1}{2} \) if and only if \( \sum_{k \in \mathbb{N}} \log \log q_{k+1} q_k \alpha < \infty \) and \( \alpha \) is irrational, where \( \alpha = [0; q_1, q_2, \ldots] \). The goal of this paper is to consider for the first time the Hausdorff dimension of the horizontal slices of the set \( NE \). In contrast to the dichotomy result for the vertical slices, we show that these Hausdorff dimensions can be any numbers between 0 and 1. Moreover, \( K_1 (\theta) \) is the set of all \( \alpha \) for which the flow in direction \( \theta \) is nonergodic (see [8]).

\[1\] From now on we omit the expression \( \text{mod} 1 \).
1.2. Motivation for considering \( \theta \) divergent relative to some fixed integer \( M \). In [8] Veech also showed that the sets \( K_i(\theta) \) have dimension 0 when \( \sup a_k < \infty \). Hence, if we wish to construct continued fractions \( \theta \) such that \( \text{Hdim} K_i(\theta) > 0 \), then we must consider those values of \( \theta \) satisfying \( \sup a_k = \infty \). Further, in unpublished work by Lothar Narins [7], it was conjectured that the sets \( K_i(\theta) \) have dimension 1 when the terms \( a_k \) satisfy \( n_k^\delta < a_{k+1} < 2n_k^\delta \) for some \( \delta \in (0,1) \) and sufficiently large \( k \). We prove a weaker version of this conjecture in Section 4.2 by imposing...
additional constraints on the sequence \((b_k)_{k \in \mathbb{N}}\). Motivated by these observations, we amalgamate these two approaches in order to construct a class of continued fractions for which we can find upper and lower bound estimates for the dimension of \(K_i(\theta)\) other than 1 and 0, respectively. Specifically, for any \(0 \leq \delta \leq 1\) and any integer \(M \geq 3\), we can construct a continued fraction \(\theta\) divergent relative to \(M\) such that \(\text{Hdim}K_0(\theta) \leq \delta\) and \(\delta \leq \text{Hdim}K_1(\theta)\) by applying upper and lower bound formulas constructed in Section 2 and Section 4, respectively.

1.3. Sketch of the derivation of the upper and lower bound formulas for \(\text{Hdim}K_1(\theta)\). In Section 2 we find an upper bound formula for \(\text{Hdim}K_0(\theta)\) for all continued fractions \(\theta\) that are divergent relative to some integer \(M\). This is achieved by applying a construction from [2] known as a self-similar covering to members of a family of subsets \((K_0^i(\theta))_{i \in \mathbb{N}}\) of \(K_0(\theta)\). The reason for reducing to a family of subsets is two-fold: first, it is much more convenient to apply the self-similar covering to the subsets \(K_0^i(\theta)\) than to \(K_0(\theta)\), and second, we will show that \(\text{Hdim}K_0(\theta) = \sup_{i} \text{Hdim}K_0^i(\theta)\) by showing that \(K_0(\theta) = \bigcup K_0^i(\theta)\) in Lemma 2.2 and exhibiting an upper bound for \(\text{Hdim} \bigcup K_0^i(\theta)\).

We construct a lower bound for \(\text{Hdim}K_1(\theta)\) on two separate occasions by applying a lower bound formula given by Falconer [3]. Falconer’s inequality applies to Cantor sets satisfying some mild conditions, which are stated in Section 4.1, so some of our work is dedicated to constructing Cantor sets contained in \(K_1(\theta)\) that satisfy the conditions needed to apply the lower bound formula. In each occasion for which Falconer’s formula is applied, we proceed by constructing an infinite family of Cantor sets contained in \(K_1(\theta)\) in a way that allows us to get lower bounds of the Cantor sets, and hence lower bounds of \(\text{Hdim}K_1(\theta)\), arbitrarily close to some specified value.

1.4. Overview of the paper. In Section 2 we give an upper bound for \(\text{Hdim}K_0(\theta)\) for \(\theta\) divergent relative to any fixed \(M \in \mathbb{N}\). Section 3 is devoted to an application of the upper bound formula in which we construct a continued fraction \(\theta\) such that \(\text{Hdim}K_0(\theta) = \text{Hdim}K_1(\theta) = 0\). In Section 4 we apply Falconer’s lower bound formula to give a lower bound for \(\text{Hdim}K_1(\theta)\), and we use this lower bound to construct a continued fraction \(\theta\) such that \(\text{Hdim}K_1(\theta) = 1\). We apply both upper and lower bound formulas in Section 5 to show that for any \(\delta \in (0, 1)\) and \(M \in \mathbb{N}_{\geq 3}\) we can construct a continued fraction such that \(\text{Hdim}K_0(\theta) = \text{Hdim}K_1(\theta) = \delta\).

2. Upper bound for Hausdorff dimension. In this section we give an upper bound formula for \(\text{Hdim}K_0(\theta)\) by applying a self-similar covering, introduced by Cheung [2], to subsets of \(K_0(\theta)\), which we call \(K_0^i(\theta)\) and define below. In particular, the self-similar covering will allow us to give an upper bound for \(\text{Hdim}K_0^i(\theta)\), and we show that this upper bound is also an upper bound for \(\text{Hdim}K_0(\theta)\).

Definition. Let \(M \in \mathbb{N}\). An irrational \(\theta\) with unbounded partial quotients is divergent relative to \(M\) if the subsequence of partial quotients formed by the terms that are greater than \(M\) diverges to \(\infty\).

If \(\theta\) is divergent relative to \(M\), define
\[
\kappa^\theta := \{k_i : a_{k_i+1} > M\},
\]
\[
k_0 := \min \kappa^\theta.
\]
The following theorem gives an upper bound formula for $\Hdim K_0(\theta)$ when $\theta$ is divergent relative to $M$.

**Theorem 2.1.** Let $\theta$ be divergent relative to $M$, and let $(k_i)_{i=0}^{\infty}$ enumerate the numbers $k$ satisfying $a_{k+1} > M$ in increasing order. Then

$$\Hdim K_0(\theta) \leq \limsup_{i \to \infty} \frac{\log a_{k+1}}{\log n_{k+1} - \log n_k}.$$  

We devote the remainder of Section 2 to proving Theorem 2.1.

2.1. **Reduction to a family of subsets of $K_0(\theta)$**. In this section we construct a family of subsets of $K_0(\theta)$. This will simplify our calculations and still allow us to obtain upper bounds of $\Hdim K_0(\theta)$.

**Definition.** Suppose $\theta$ is divergent relative to $M$. Define

$$K_0^i(\theta) := \{ (m; b_1, b_2, \ldots)_{\theta} : j > k_i \Rightarrow |b_j| n_j \|n_j\theta\| < \frac{1}{4M} \}.$$  

The sets $K_0^i(\theta)$ are nonempty for all sufficiently large $i$ since, by definition of $K_0(\theta)$, we have $\lim_{j \to \infty} |b_j| n_j \|n_j\theta\| = 0$.

**Lemma 2.2.**

$$K_0(\theta) \subset \bigcup_{i=0}^{\infty} K_0^i(\theta).$$  

**Proof.** If $y \in K_0(\theta)$, then $y = (m; b_1, b_2, \ldots)_{\theta}$ for some sequence $m, b_1, b_2, \ldots$ satisfying $b_j n_j \|n_j\theta\| \to 0$. Hence, there is an $i$ such that $j > k_i$ implies $b_j n_j \|n_j\theta\| < \frac{1}{4M}$. Therefore $y \in K_0^i(\theta)$.  

**Lemma 2.3.**

$$\Hdim \bigcup_{i=0}^{\infty} K_0^i(\theta) = \sup \{ \Hdim K_0^i(\theta) : i \geq 0 \}.$$  

**Proof.** For any set $F \subset \mathbb{R}$ and $s \geq 0$, denote by $\mathcal{H}^s(F)$ the $s$-dimensional Hausdorff measure of $F$. Then $\Hdim K_0^i(\theta) \leq \Hdim \bigcup_i K_0^i(\theta)$ for each $i$, so $\sup_i \Hdim K_0^i(\theta) \leq \Hdim \bigcup_i K_0^i(\theta)$.

Conversely, let $s = \sup_i \Hdim K_0^i(\theta)$. It suffices to show $\mathcal{H}^{s+\varepsilon} \bigcup_i K_0^i(\theta) = 0$ when $\varepsilon > 0$. Let $\delta > 0$. For each $i$ we can cover $K_0^i(\theta)$ by intervals $A_{ij}$ such that the sum of their radii by the power $s + \varepsilon$ is less than $2^{-i}\delta$. The union of all intervals $A_{ij}$, over $i$ and $j$, covers $\bigcup_i K_0^i(\theta)$ and the sum of their radii raised by the power $s + \varepsilon$ is less than $\delta$. Therefore, $\mathcal{H}^{s+\varepsilon} \bigcup_i K_0^i(\theta) = 0$.  

A consequence of the following lemma is that each $x \in K_0^i(\theta)$ can be expressed as $(m; b_1, b_2, \ldots)_{\theta}$ such that for all $j > k_{i_0}$ either $a_{j+1} > M$ or $b_j = 0$. This phenomenon was the motivation for defining the sets $K_0^i(\theta)$.

**Lemma 2.4.** There exists an integer $i_0$ such that if $j \geq k_{i_0}, j \notin \kappa^0$ and $|b_j| n_j \|n_j\theta\| < \frac{1}{4M}$, then $b_j = 0$.

**Proof.** Take $i_0$ sufficiently large so that $k_{i_0} \geq \min \kappa^0$. Since $j \notin \kappa^0$, $a_{j+1} \leq M$. Hence,

$$\frac{1}{4M} > |b_j| n_j \|n_j\theta\|.$$
Thus, \( a_{j+1} > M |b_j| \). Since \( a_{j+1} < M \), it follows that \( b_j = 0 \). \( \square \)

2.2. Specification of self-similar covering. Given \( |b_i| \leq a_{i+1} \) and an even \( b \) satisfying \( |b| \leq a_{k+1} \), define by

\[
I(m; b_1, b_2, \ldots, b_{k-1}, b)
\]

the interval of length \( \frac{8}{n_k} \) centered at

\[
m\theta + \sum_{j=1}^{k-1} b_j ||n_j\theta|| + b||n_k\theta||.
\]

Lemma 2.5. If \( x \in K_0(\theta) \), then \( x \in I(m; b_1, \ldots, b_k) \).

Proof. Let \( x = \langle m; b_1, \ldots \rangle_\theta \in K_0(\theta) \). Since \( \theta \) is between \( m_k n_k \) and \( m_k n_k + 1 \), it follows that \( \left| \frac{m_k}{n_k} - \theta \right| \leq \frac{m_k + 1}{n_k + 1} \). Multiplying by \( n_k \) gives \( |m_k - n_k\theta| \leq |m_k - n_k \frac{m_k + 1}{n_k + 1}| \).

Therefore,

\[
\|n_k\theta\| \leq |m_k - n_k\theta|
\]

\[
\leq \frac{m_k - n_k}{n_k + 1}
\]

\[
= \frac{1}{n_k + 1}.
\]

Since \( n_{k+2} > 2n_k \), we have \( \frac{1}{n_{k+1}} > \frac{2}{n_{k+2}} \) and \( \frac{1}{n_{k+2}} > \frac{2}{n_{k+2}} \). Therefore,

\[
\sum_{i=0}^{\infty} \frac{1}{n_{k+2}^{(i+1)}} < \sum_{i=0}^{\infty} \frac{1}{2n_{k+2}^{(i+1)}} \]

which is a geometric series that simplifies to \( \frac{2}{n_{k+2}} \).

Similarly, \( \sum_{i=1}^{\infty} \frac{1}{n_{k+2}^{(i+1)}} \). Without loss of generality, suppose \( m = 0 \). Then

\[
\left| x - \sum_{i=1}^{k} b_i \|n_i\theta\| \right| = \sum_{i=k+1}^{\infty} b_i \|n_i\theta\|
\]

\[
\leq \sum_{i=k+1}^{\infty} |b_i| \|n_i\theta\|
\]

\[
\leq \sum_{i=k+1}^{\infty} \frac{|b_i|}{n_i+1}
\]

\[
\leq \sum_{i=k+1}^{\infty} \frac{a_{i+1}}{n_{i+1}}
\]

\[
\leq \sum_{i=k+1}^{\infty} \frac{1}{n_i + 1}
\]

\[
= \sum_{i=0}^{\infty} \frac{1}{n_{k+2i+1}} + \sum_{i=1}^{\infty} \frac{1}{n_{k+2i}}
\]
\begin{align*}
\lesssim \frac{2}{n_k+1} + \frac{2}{n_k+2} \\
\lesssim \frac{4}{n_k+1}.
\end{align*}

Therefore, \( x \in I(m; b_1, \ldots, b_k) \). \qedhere

\textbf{Definition.} (Section 3 of \cite{2}) Let \( X \) be a metric space and \( J \) a countable set. Given \( \sigma \subset J \times J \) and \( \alpha \in J \) we let \( \sigma(\alpha) \) denote the set of all \( \alpha' \in J \) such that \( (\alpha, \alpha') \in \sigma \). We say a sequence \( (\alpha_j)_{j \in \mathbb{N}} \) of elements in \( J \) is \( \sigma \)-admissible if \( \alpha_{j+1} \in \sigma(\alpha_j) \) for all \( j \in \mathbb{N} \); and we let \( J^\sigma \) denote the set of all \( \sigma \)-admissible sequences in \( J \). By a self-similar covering of \( X \) we mean a triple \( (\mathcal{B}, J, \sigma) \) where \( \mathcal{B} \) is a collection of bounded subsets of \( X \), \( J \) a countable index set for \( \mathcal{B} \), and \( \sigma \subset J \times J \) such that there is a map \( \mathcal{E} : K_0(\theta) \to J^\sigma \) that assigns to each \( x \in X \) a \( \sigma \)-admissible sequence \( (\alpha_j^x)_{j \in \mathbb{N}} \) such that for all \( x \in X \) we have the following:
\begin{enumerate}[(i)]
\item \( \bigcap_{j=1}^\infty B(\alpha_j^x) = \{x\} \), and
\item \( \text{diam } B(\alpha_j^x) \to 0 \) as \( j \to \infty \), where \( B(\alpha) \) denotes the element of \( \mathcal{B} \) indexed by \( \alpha \).
\end{enumerate}

\textbf{2.3. A self-similar covering of } \( K_0^\text{ad} (\theta) \). \textbf{We have access to a self-similar covering of } \( K_0^\text{ad} (\theta) \). Define
\[ J := \left\{ (m; b_1, b_2, \ldots, b_{k-1}) : m \in \mathbb{Z}, k \in \kappa^\theta, b_j \in \mathbb{Z} (1 \leq j \leq k) , \right. \]
\[ \left. |b_j| \leq a_{j+1}, \text{ and } b_j = 0 \text{ if both } k_{i_0} < j < k, j \notin \kappa^\theta \right\}, \]
\[ \mathcal{B} := \{ I(\beta) : \beta \in J \}, \]
and define \( \sigma \subset J \times J \) such that for each \( \alpha_{k_i} = (m; b_1, b_2, \ldots, b_{k_i-1}) \in J \) we have\footnote{As mentioned after Theorem 3.1 of \cite{2}, we can take elements of \( \mathcal{B} \) to be subsets of the ambient space \( X \).}
\[ \sigma(\alpha_{k_i}) = \{ (m; b_1, b_2, \ldots, b_{k_i-2}, b) : |b| \leq a_{k_i+1} \}. \]
Let \( J^\sigma \) denote the set of all \( \sigma \)-admissible sequences in \( J \). By Lemma 2.5 we can define
\[ \mathcal{E} : K_0(\theta) \to J^\sigma \]
\[ x \mapsto (\alpha_j^x)_{j=0}^\infty, \]
where \( x \in I(\alpha_j) = I(m; b_1, \ldots, b_{j-1}) \in J_0 \) for each \( \alpha_j \in (\alpha_j^x) \). Suppose \( x \in K_0^\text{ad} (\theta) \). Our triple \( (\mathcal{B}, J, \sigma) \) satisfies (i) of the definition of a self-similar covering; apply Lemma 2.5 to show \( x \in I(\beta_j) \) for all \( \beta_j \in J_0 \). Since \( \lim_{j \to \infty} \text{diam } I(\alpha_j^x) = 0 \), \( \bigcap_{j=0}^\infty I(\alpha_j^x) = \{x\} \); (ii) is also satisfied.
Define
\[ E_i := \bigcup I(m; b_1, \ldots, b_{k_i-1}) , \]
where the union is over all finite sequences \( m, b_1, \ldots, b_{k_i-1} \) with \( k_i \in \kappa^\theta \) and \( |b_j| \leq a_{j+1} \) for each \( j \).
2.4. Calculation. In this section we give an upper bound on $\text{Hdim}K_0^\theta(\theta)$. The following lemma is a direct consequence of the definition of $\sigma$.

**Lemma 2.6.** Let $\alpha \in J$ and $k = |\alpha|$. Then

$$
\# \sigma(\alpha) \leq a_{k+1}.
$$

(3)

**Proof.** Let $\alpha$ be a sequence in $J$ of length $k_i \in \kappa^\theta$. Then $\alpha = (m; b_1, \ldots, b_{k_i-1})$ where both $j > k_i$ and $j \not\in \kappa^\theta$ imply $b_j = 0$. Since $|\alpha| = k$, $\alpha$ corresponds to an interval belonging to $E_i$. Given an interval centered at $m\theta + \sum_{i=1}^j b_i \|n_i\theta\|$, let us call the intervals centered at $(m\theta + \sum_{i=1}^j b_i \|n_i\theta\|) + b_{j+1} \|n_{j+1}\theta\|$ with $|b_j| \leq a_{j+1}$ the children intervals of the original interval. Hence, the centers determined by the children of the intervals comprising $E_i$ are determined by all even integers $b_k$ satisfying $b_k \leq a_{k+1}$. □

Since the value of $|I(\beta)|$ does not depend on the choice of $\beta \in \sigma(\alpha)$, a direct consequence of inequality (3) is that for each $s \geq 0$ we have

$$
\sum_{\beta \in \sigma(\alpha)} \left( \frac{|I(\beta)|}{|I(\alpha)|} \right)^s = \# \sigma(\alpha) \left( \frac{|I(\beta)|}{|I(\alpha)|} \right)^s.
$$

(4)

Further, if $\beta \in \sigma(\alpha)$, then

$$
\# \sigma(\alpha) \left( \frac{|I(\beta)|}{|I(\alpha)|} \right)^s \leq 1 \iff \# \sigma(\alpha) \left( \frac{nk_i}{nk_{i+1}} \right)^s \leq 1.
$$

The critical value of $s$ is $s = \frac{\log \# \sigma(\alpha)}{\log nk_{i+1} - \log nk_i}$. Let $\varepsilon > 0$, and let $s' = s + \varepsilon$. Then

$$
\sum_{\beta \in \sigma(\alpha)} \left( \frac{|I(\beta)|}{|I(\alpha)|} \right)^{s'} \leq 1 \iff \left( \frac{nk_i}{nk_{i+1}} \right)^s \# \sigma(\alpha) \leq 1
$$

$$
\iff \left( \frac{nk_i}{nk_{i+1}} \right)^s \leq 1.
$$

Theorem 5.3 of [1] implies $\text{Hdim}K_0(\theta) \leq s'$. Since $\varepsilon$ can be arbitrarily small, we have $\text{Hdim}K_0(\theta) \leq s$. Thus,

$$
\text{Hdim}K_0(\theta) \leq \limsup_{i \to \infty} \frac{\log \# \sigma(\alpha)}{\log nk_{i+1} - \log nk_i} 
$$

$$
\leq \limsup_{i \to \infty} \frac{\log a_{k+1}}{\log nk_{i+1} - \log nk_i},
$$

and this proves Theorem 2.1. □

3. A nontrivial example of $\theta$ satisfying $\text{Hdim}K_0(\theta) = 0$. In [8] Veech showed that the Hausdorff dimension of $K_1(\theta)$ vanishes when $\sup_k a_k < \infty$. Using our upper bound formula (2) we give an example for which $\sup_j a_j = \infty$ and $\text{Hdim}K_0(\theta) = 0$. We define $|x| := \max \{n \in \mathbb{Z} : n \leq x\}$, $[x] := \min \{n \in \mathbb{Z} : n \geq x\}$, and denote by $[x]$ the nearest integer to $x$ (to avoid ambiguity when rounding half-integers, we round to the nearest even integer). Let $0 < \delta \leq 1$ be given and fix $M = \left[ \frac{2^7}{\delta} \right]$. We specify the continued fraction representation of $\theta$ recursively. We choose $a_1, a_2, \ldots, a_{k_0} = M$, where $k_0$ is the smallest index for which $n_{k_0}^\delta > \max \left\{ 2M, \frac{1}{\delta - 1} \right\}$. 

Note that there exists an integer between $n^δ_{k_0}$ and $2^δ n^δ_{k_0}$ since $n^δ_{k_0} (2^δ - 1) > 1$. Choose $a_{k_0+1} \in \mathbb{Z}$ such that $n^δ_{k_0} < a_{k_0+1} < 2n^δ_{k_0}$. Recursively, given $k_i$, define

$$k_{i+1} := k_i + 1 + \left\lceil \frac{\delta i \log n_{k_i}}{\log M} \right\rceil,$$

(5)

set $a_{k_i+2} = a_{k_i+3} = \cdots = a_{k_{i+1}} = M$, and choose $a_{k_{i+1}}$ such that $n^δ_{k_i} < a_{k_{i+1}} < 2n^δ_{k_i}$. This completes the recursive definition of the sequence $(a_k)_{i \geq 0}$. Moreover, a direct consequence of equation (5) is

$$M^{k_{i+1} - k_i - 1} \geq n^δ_{k_i}.$$  

(6)

**Proposition.** Under the above construction, $\theta$ is divergent relative to $M$.

**Proof.** By construction of $(a_k)_{i \geq 0}$, if $i \geq 0$, then $a_{k_{i+1}} > M$ and $n^δ_{k_i} < a_{k_{i+1}} < 2n^δ_{k_i}$. Therefore, $a_{k_{i+1}}$ diverges to $\infty$. \hfill \Box

**Theorem 3.1.** If $\theta$ is constructed as above, then $\text{Hdim}_0(\theta) = 0$.

**Proof.** For any integer $i \geq 0$,

$$n_{k_{i+1}} > a_{k_{i+1}} n_{k_{i+1} - 1} = M n_{k_{i+1} - 1} > M a_{k_{i+1} - 1} n_{k_{i+1} - 2} = M^2 n_{k_{i+1} - 2} \quad \vdots$$

$$= M^{k_{i+1} - k_i - 1} n_{k_i} \geq n^δ_{k_i} n_{k_i} = n^δ_{k_i+1}.$$  

Therefore, $\log n_{k_{i+1}} > (\delta i + 1) \log n_{k_i}$, and this implies

$$\lim_{i \to \infty} \frac{\log n_{k_{i+1}}}{\log n_{k_i}} = \infty.$$  

(7)

Since $\theta$ is divergent relative to $M$,

$$\text{Hdim}_0(\theta) \leq \limsup_{i \to \infty} \frac{\log a_{k_{i+1}}}{\log n_{k_{i+1}} - \log n_{k_i}} \leq \limsup_{i \to \infty} \frac{\log 2 n^δ_{k_i}}{\log n_{k_i} - \log n_{k_i}} \leq \limsup_{i \to \infty} \frac{\delta \log n_{k_i} + \log 2}{\log n_{k_{i+1}} - \log n_{k_i}}$$

$$= \limsup_{i \to \infty} \frac{\delta}{\log n_{k_{i+1}} - \log n_{k_i}} > 0.$$  

\hfill \Box
4. A Lower bound for $\text{Hdim} K_1(\theta)$. In Section 4.1 we state a lower bound formula given by Falconer \cite{5} for a particular class of Cantor sets. We can approximate $K_1(\theta)$ by a family of Cantor sets, so we show how Falconer’s formula can be used to give a lower bound for $K_1(\theta)$. In Section 4.2 we use Falconer’s formula to provide an example of a continued fraction $\theta$ such that $\text{Hdim} K_1(\theta) = 1$.

4.1. Falconer’s lower bound formula. The following construction is from \cite{5}. Let $[0,1] = F_0 \supset F_1 \supset F_2 \supset \cdots$ be a decreasing sequence of sets, with each $F_k$ a union of a finite number of disjoint closed intervals (called $k$th level basic intervals), with each interval of $F_k$ containing at least two intervals of $F_{k+1}$, and the maximum length of $k$th level intervals tending to 0 as $k \to \infty$. Then the set

$$F := \bigcap_{k=0}^{\infty} F_k$$

is a totally disconnected subset of $[0,1]$. The condition needed to apply Falconer’s lower bound formula is

$$(*) \text{ For } j \in \mathbb{N}, \text{ each } (j-1)\text{th level interval contains at least } m_j \geq 2 j \text{th level intervals that are separated by gaps of at least } \gamma_j, \text{ where } 0 < \gamma_{j+1} < \gamma_j \text{ for each } j.$$

Falconer’s lower bound formula is

$$\text{Hdim} F \geq \liminf_{j \to \infty} \frac{\log (m_0 m_1 \cdots m_j)}{-\log (m_{j+1} \gamma_{j+1})}.$$  \hspace{1cm} (8)$$

4.2. An example of $\theta$ such that $\text{Hdim} K_1(\theta) = 1$. In this section we apply inequality (8) to construct a continued fraction $\theta$ such that $\text{Hdim} K_1(\theta) = 1$. Fix $\delta \in (0,1)$ and $\varepsilon \in (0,\delta)$. For the remainder of Section 4, let $\theta$ be an element such that there exists a $k_0$ sufficiently large so that $n_{k_0}^{\delta-\varepsilon} \geq 3$, $n_{b_0}^\varepsilon \geq 9$, and for all $k \geq k_0$, $n^{\delta}_k < a_{k+1} < 2 n^\varepsilon_k$, $b_k$ is even and $|b_k| < \lfloor n_k^{\delta-\varepsilon} \rfloor$. Our strategy is to construct an infinite family of Cantor sets contained in $K_1(\theta)$, each of which satisfies the conditions $(*)$ needed to apply Falconer’s lower bound formula, allowing us to give lower bounds of $\text{Hdim} K_1(\theta)$ arbitrarily close to 1.

Given $|b_i| \leq a_{i+1}$, define by

$$L(m; b_1, b_2, \ldots, b_{k-1})$$

the interval of length $||n_{k-1} \theta||$ concentric with $I(m; b_1, \ldots, b_{k-1})$. The following lemma is a consequence of our choice of the length of $L(m; b_1, b_2, \ldots, b_{k-1})$.

**Lemma 4.1.** The gaps between consecutive intervals of the form $L(m; b_1, \ldots, b_{k+1})$ are of length $||n_k \theta||$.

**Proof.** The distance between the centers of adjacent intervals $L(m; b_1, b_2, \ldots, b_{k+1})$ and $L(m; b_1, b_2, \ldots, b_{k+1} + 2)$ is $2 ||n_k \theta||$. Between these centers is the gap between the intervals as well as two half intervals (specifically, the right half of $L(m; b_1, b_2, \ldots, b_{k+1})$ and the left half of $L(m; b_1, b_2, \ldots, b_{k+1} + 2)$). Hence, the size of the gap is $||n_k \theta||$. \hspace{1cm} $\Box$

The following lemma gives a sufficient condition for $L(m; b_1, b_2, \ldots, b_{k+1}) \subset L(m; b_1, b_2, \ldots, b_k)$.

**Lemma 4.2.** Suppose we are given $m, b_1, \ldots, b_{k-1}, b$. If $|b| \leq \frac{a_{k+1}}{8}$, then

$$L(m; b_1, \ldots, b_{k-1}, b) \subset L(m; b_1, \ldots, b_{k-1}).$$
Proof. We may suppose $a_{k+1} \geq 9$ since the claim is vacuously true otherwise. Let $y \in L(m; b_1, \ldots, b_{k-1}, b)$, and let $x$ be the center of the corresponding parent interval $L(m; b_1, \ldots, b_k)$. Let us call $x'$ the center of the interval $L(m; b_1, \ldots, b_{k-1}, b)$.

Then

$$|x - y| \leq |x' - x| + |x' - y|$$

$$\leq |b| \|n_k \theta\| + \frac{1}{2} \|n_k \theta\|$$

$$\leq \frac{a_{k+1}}{n_{k+1}} + \frac{1}{2}$$

$$< \left(\frac{a_{k+1}}{4} - \frac{1}{2}\right) + \frac{1}{2}$$

$$= \frac{a_{k+1}}{4n_{k+1}}$$

$$< \frac{a_{k+1}}{4a_{k+1}n_k}$$

$$= \frac{1}{4n_k}$$

$$< \frac{\|n_{k-1} \theta\|}{2}.$$

Theorem 4.3. The particular $\theta$ that we have described in this section satisfies $\overline{\text{Hdim}} K_1(\theta) = 1$.

Proof. Define

$$F_j := \bigcup L(m; b_1, \ldots, b_{k_0 + j - 1}),$$

where the union is over all finite sequences such that $m = b_1 = \cdots = b_{k_0 - 1} = 0$. If $k \geq k_0$, then

$$|b_k| \leq \left\lfloor n_k^{\delta - \varepsilon} \right\rfloor$$

$$\leq n_k^{\delta - \varepsilon}$$

$$= \frac{n_\delta}{n_k^{\varepsilon}}$$

$$< \frac{a_{k+1}}{8}.$$

Thus, by Lemma 4.2, if $j \geq 0$, then

$$F_{j+1} \subset F_j.$$

Define

$$F(k_0, \varepsilon) := \bigcap_{j=0}^{\infty} F_j.$$

Lemma 4.4. $F(k_0, \varepsilon) \subset K_1(\theta)$.

Proof of Lemma 4.4. If $y \in F(k_0, \varepsilon)$, then we have a sequence of the form $b_1, b_2, \ldots$ such that $y = \langle m; b_1, b_2, \ldots, b_{k_0}, b_{k_0+1}, \ldots, \rangle$, where $b_1 = \cdots = b_{k_0-1} = 0$ and, by
construction, each $b_k$ is even. We show that $y$ satisfies the constraints imposed on elements of $K_1(\theta)$:

$$
\sum_{j=1}^{\infty} |b_j| n_j \|n_j \theta\| \leq \sum_{j=1}^{\infty} \frac{n_j^\delta \epsilon + 1}{n_j + 1} \leq \sum_{j=1}^{\infty} \frac{n_j^\delta \epsilon + 1}{n_j + 1} \leq \sum_{j=1}^{\infty} \frac{1}{n_j^\epsilon}.
$$

Using the ratio test on the latter series, we have

$$
\lim_{j \to \infty} \frac{n_j^\epsilon}{n_j^\delta + 1} \leq \lim_{j \to \infty} \left( \frac{n_j}{a_j + 1 n_j} \right)^\epsilon \leq \lim_{j \to \infty} \frac{1}{a_j + 1} = 0.
$$

Therefore, $\sum_{j=1}^{\infty} |b_j| n_j \|n_j \theta\| < \infty$, so $y \in K_1(\theta)$. This proves the lemma. 

For convenience, define $k_j := k_0 + j$. We show that $F(k_0, \epsilon)$ satisfies $(\ast)$. Denote by $m_j$ the number of children intervals $L(m; b_1, b_2, \ldots, b_{k_j})$ of $F$ so that $m_j$ counts the number of even integers $b$ satisfying $|b| < \left\lfloor n_k^{\delta - \epsilon} \right\rfloor$: i.e.,

$$
m_j = \left\lfloor n_k^{\delta - \epsilon} \right\rfloor.
$$

Denote by $\gamma_j$ the length of the gaps between children intervals of $L(m; b_1, b_2, \ldots, b_{k_j})$ so that, by Lemma 4.1,

$$
\gamma_j = \|n_{k_j} \theta\|.
$$

From continued fraction theory,

$$
\frac{1}{2n_{k+1}} < \|n_k \theta\| < \frac{1}{n_{k+1}}.
$$

Since $\frac{1}{n_{k+j+1}} < \frac{1}{n_{k+j}}$ for each $j$, we have $0 < \gamma_{k+j+1} < \gamma_{k+j}$. By inequality \ref{eq:ratio_test}

$$
\lim_{j \to \infty} \gamma_j = 0.
$$

Each interval $L(m; b_1, \ldots, b_{k_j-1})$ contains at least 3 intervals of $L(m; b_1, b_2, \ldots, b_{k_j})$, and the gaps $\gamma_j$ decrease monotonically to 0 as $j \to \infty$. Hence, the conditions $(\ast)$ for Falconer’s lower bound formula are satisfied. A useful consequence of equation \ref{eq:ratio_test} is

$$
n_{k+1} < 2a_{k+1} n_k.
$$

To simplify our calculation of the lower bound for $\text{Hdim} F(k_0, \epsilon)$, we use the fact that $k \geq k_0$ implies

$$
n_{k+1} < (a_{k+1} + 1) n_k < 3n_k^{1+\delta}.
$$

Taking log on both sides of the expression $n_{k+1} < 3n_k^{1+\delta}$ gives

$$
\log n_{k+1} < \left( 1 + \delta + \frac{\log 3}{\log n_{k_0}} \right) \log n_k.
$$

(11)
Further, since
\[ m_{j+1} = \left\lfloor \frac{n_{k_{j+1}}^{\delta - \varepsilon}}{2} \right\rfloor \]
and
\[ \gamma_{j+1} = \|n_{k_{j+1}}\theta\| > \frac{1}{2n_{k_{j+1}} + 1} \]
\[ > \frac{1}{4a_{k_{j+1}} + 1} \]
\[ > \frac{1}{8n_{k_{j+1}}^{1+\delta}}, \]
we have
\[ m_{j+1}\gamma_{j+1} > \frac{1}{16n_{k_{j+1}}^{1+\varepsilon}}. \] (12)

Using Falconer’s lower bound inequality gives
\[ \text{Hdim}F(k_0, \varepsilon) \geq \liminf_{j \to \infty} \frac{\log (m_0 \cdots m_j)}{-\log (m_{j+1}\gamma_{j+1})} \]
\[ \geq \liminf_{j \to \infty} \frac{\delta - \varepsilon}{(1+\varepsilon)\log n_{k_{j+1}}} \]
\[ \geq \liminf_{j \to \infty} \frac{\delta - \varepsilon}{1+\varepsilon} \left( \frac{1}{1+\delta + \frac{\log 3}{\log n_{k_0}}} + \cdots + \frac{1}{1+\delta + \frac{\log 3}{\log n_{k_0}}^{j+1}} \right) \]
\[ = \liminf_{j \to \infty} \frac{\delta - \varepsilon}{1+\varepsilon} \left( \frac{1}{1+\delta + \frac{\log 3}{\log n_{k_0}}} \right) \left( 1 - \left( \frac{1}{1+\delta + \frac{\log 3}{\log n_{k_0}}} \right)^{j+1} \right) \]
\[ = \frac{\delta - \varepsilon}{(1+\varepsilon)\left( \delta + \frac{\log 3}{\log n_{k_0}} \right)}. \]

If \( \varepsilon \to 0 \) and \( k_0 \to \infty \), then \( \text{Hdim}F(k_0, \varepsilon) \to 1 \). Therefore, \( \text{Hdim}K_1(\theta) = 1 \).

5. \( \text{Hdim}K_i(\theta) \) attains any value in \([0, 1]\).

**Theorem 5.1.** For any \( \delta \in [0, 1] \), there is a \( \theta \) satisfying \( \text{Hdim}K_1(\theta) = \delta \).

The cases \( \delta = 0 \) and \( \delta = 1 \) are handled by Theorem 3.1 and Theorem 4.3, respectively. Let us choose \( \delta \in (0, 1) \). Our strategy is to construct \( \theta \) in terms of \( \delta \) in a particular way that gives
\[ \delta \leq \text{Hdim}K_1(\theta) \leq \text{Hdim}K_0(\theta) \leq \delta. \]

**Proof.** Suppose \( M \in \mathbb{N}_{\geq 3} \). We construct a continued fraction \( \theta \) divergent relative to \( M \). In what follows we construct integers \( a_k \) in terms of the indices \( k_i \) by taking
Therefore, $H_{k+1}$ is defined according to (13). Choose $k_{i+1}$ to be the smallest $k \geq k_i + 1$ such that
\[
\begin{cases}
    n_k < n^2_{k_i} & \text{if } k < k_{i+1}, \\
    n_k \geq n^2_{k_i} & \text{if } k = k_{i+1}.
\end{cases}
\]
By this recursive definition, $n_{k_{i+1}} \geq n^2_{k_i}$ and $n_{k_{i+1}-1} < n^2_{k_i}$. If $k_{i+1} = k_i + 1$, then
\[
\begin{align*}
    n_{k_{i+1}} &= a_{k_{i+1}}n_{k_i} + n_{k_i-1} \\
    &< 2a_{k_{i+1}}n_{k_i} \\
    &< 4n^1_{k_i} \\
    &< (M + 1)n^2_{k_i}.
\end{align*}
\]
If $k_{i+1} > k_i + 1$, then $k_{i+1}$ is not of the form $j + 1$ for any $j > i$; for if $j > i$, then $k_{j+1} > k_{i+1}$. Therefore, in this case, we have $a_{k_{i+1}} \in [3, M]$, so
\[
\begin{align*}
    n_{k_{i+1}} &= a_{k_{i+1}}n_{k_{i+1}-1} + n_{k_{i+1}-2} \\
    &< (a_{k_{i+1}} + 1)n_{k_{i+1}-1} \\
    &< (M + 1)n^2_{k_i}.
\end{align*}
\]
Therefore,
\[
    n^2_{k_i} \leq n_{k_{i+1}} < (M + 1)n^2_{k_i}
\]
and
\[
    n^\delta_{k_i} < a_{k_i+1} < 2n^\delta_{k_i}.
\]
Further, it will be used in the upper and lower bound calculations that inequality (14) implies
\[
\lim_{j \to \infty} \log n_{k_j}/\log n_{k_{j+1}} = \frac{1}{2}.
\]

5.1. **Upper bound.** As constructed in Section 5, $\theta$ is divergent relative to $M$ since the subsequence $(a_{k_i+1})_{i \geq 0}$ of terms of $(a_k)_{k \geq 1}$ which are larger than $M$ also satisfy $n^\delta_{k_i} < a_{k_i+1} < 2n^\delta_{k_i}$. Therefore, $\lim_{i \to \infty} a_{k_{i+1}} = \infty$. Hence,
\[
\begin{align*}
    HdimK_0(\theta) &\geq \limsup_{j \to \infty} \frac{\log a_{k_{j+1}}}{\log n_{k_{j+1}} - \log n_{k_j}} \\
    &\leq \limsup_{j \to \infty} \frac{\log n^\delta_{k_{j+1}} + \log 2}{\log n_{k_{j+1}} - \log n_{k_j}} \\
    &= \limsup_{j \to \infty} \frac{\delta \log n_{k_{j+1}}}{1 - \frac{\log n_{k_j}}{\log n_{k_{j+1}}}} \\
    &= \delta.
\end{align*}
\]
Therefore, $HdimK_0(\theta) \leq \delta$. 
5.2. **Lower bound.** Let \( \varepsilon \in (0, 1) \). Choose \( k'_0 \geq k_0 \) sufficiently large so that \( k_i \geq k'_0 \) implies \( a_{k_i+1}^\varepsilon < \frac{a_{k_i+1}}{8} \). Define

\[
F_j := \bigcup L \left( m; b_1, b_2, \ldots, b_{k'_0-1}, \ldots, b_{k_j-1} \right)
\]

to be the union over all finite sequences of even terms \( b_1, \ldots, b_{k_j-1} \) such that if \( k = k_i \geq k'_0 \) then \( b_k \) is even and satisfies \( |b_k| < \left[ a_{k_i+1}^\varepsilon \right] \), otherwise \( b_k = 0 \). Without loss of generality, let \( m = 0 \). Define

\[
F_\varepsilon := F((k'_0, \varepsilon)) := \bigcap_{j=0}^{\infty} F_j.
\]

Lemma 4.2 implies that \( F_{j+1} \subset F_j \) for \( j \in \mathbb{N} \) since if \( k_i \geq k'_0 \), then

\[
|b_{k_i}| < \left[ a_{k_i+1}^\varepsilon \right] \leq a_{k_i+1}^\varepsilon < a_{k_i+1}.
\]

**Lemma 5.2.** \( F_\varepsilon \subset K_1(\theta) \).

**Proof of Lemma 5.2.** Let \( y \in F_\varepsilon \). Then we are given a sequence of the form \( b_1, b_2, \ldots \) such that \( y = \langle m; b_1, b_2, \ldots, b_{k'_0}; b_{k'_0+1}, \ldots \rangle \), \( |b_k| \) is even and satisfies \( |b_k| < \left[ a_{k'_0+1}^\varepsilon \right] \) if \( k_k > k'_0 \) and \( b_k = 0 \) otherwise. Further,

\[
\sum_{j=1}^{\infty} |b_j| n_j \|n_j\theta\| \leq \sum_{j=1}^{\infty} a_{j+1}^{\varepsilon+1} n_j \frac{n_j}{n_{j+1}}
\]

\[
\leq \sum_{j=1}^{\infty} a_{j+1}^{\varepsilon+1} n_j \frac{n_j}{n_{j+1}}
\]

\[
= \sum_{j=1}^{\infty} \frac{1}{a_j^{\varepsilon+1}}
\]

\[
\leq \sum_{j=1}^{\infty} \frac{1}{n_j^{1(1-\varepsilon))}}.
\]

Using the ratio test on the latter series, we have

\[
\lim_{j \to \infty} \frac{n_j^{(1-\varepsilon)\delta}}{n_{j+1}^{(1-\varepsilon)\delta}} \leq \lim_{j \to \infty} \frac{1}{a_{j+1}^{(1-\varepsilon)\delta}} = 0.
\]

Therefore, \( \sum_{j=1}^{\infty} |b_j| n_j \|n_j\theta\| < \infty \). Hence \( y \in K_1(\theta) \), and this proves the lemma.

Define

\[
M_j := \left[ a_{k_j+1}^\varepsilon \right],
\]

so that \( M_j \) is a lower bound on the number of intervals in \( F_{j+1} \) contained in intervals of \( F_j \). Define

\[
\Gamma_j := \|n_k\theta\|,
\]

so that, by Lemma 4.1, \( \Gamma_j \) is a lower bound on the gaps between the intervals of \( F_{j+1} \). Since \( \frac{1}{n_{j+2}} < \frac{1}{2n_{j+1}} \) for each \( j \), we have \( 0 < \Gamma_{j+1} < \Gamma_j \). Inequality (9)
implies that \( \frac{1}{2^{n_{k_j}+1}} < \Gamma_j < \frac{1}{n_{k_j}+1} \) for each \( j \), so \( \lim_{j \to \infty} \Gamma_j = 0 \). It is the case that each interval \( L(m; b_1, \ldots, b_{k_j-1}) \) contains at least 3 intervals of \( F_{k_j+1} \) and the gaps \( \Gamma_j \) decrease monotonically to 0 as \( j \to \infty \). The conditions (*) for Falconer’s lower bound formula are satisfied.

To simplify our calculation of the lower bound of \( H_{\dim} F \), we use the fact that

\[
\Gamma_{j+1} = \left\| n_{k_{j+1}} \theta \right\|
\]

\[
> \frac{1}{2n_{k_{j+1}}+1}
\]

\[
= \frac{1}{4a_{k_{j+1}}+1n_{k_{j+1}}}
\]

\[
> \frac{1}{8n_{k_{j+1}}}
\]

and

\[
M_{j+1} = \left\lfloor a_{k_{j+1}}^\varepsilon \right\rfloor
\]

\[
\geq \frac{a_{k_{j+1}}^\varepsilon}{2}
\]

\[
\geq \frac{n_{k_{j+1}}^\varepsilon}{2}
\]

imply

\[
M_{j+1} \Gamma_{j+1} > \frac{1}{16n_{k_{j+1}}^\varepsilon}
\]

Using Falconer’s lower bound inequality gives

\[
H_{\dim} F \geq \liminf_{j \to \infty} \frac{\log(M_0 \cdots M_j)}{\log(M_{j+1} \Gamma_{j+1})}
\]

\[
\geq \liminf_{j \to \infty} \frac{\varepsilon \log (a_{k_0+1} + \log a_{k_{j+1}} + \cdots + \log a_{k_j+1})}{(1 + \delta - \delta \varepsilon) \log n_{k_{j+1}}}
\]

\[
\geq \frac{\delta \varepsilon}{1 + \delta - \delta \varepsilon} \liminf_{j \to \infty} \left( \frac{\log n_{k_0} + \log n_{k_{j+1}} + \cdots + \log n_{k_j}}{\log n_{k_{j+1}}} \right)
\]

\[
= \frac{\delta \varepsilon}{1 + \delta - \delta \varepsilon} \liminf_{j \to \infty} \left( \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{j+1}} \right)
\]

so \( H_{\dim} F (k_0, \varepsilon) \to \delta \) as \( \varepsilon \to 1 \). Therefore, \( H_{\dim} K_1 (\theta) \geq \delta \), and this proves Theorem 5.1.

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