SOME RESULTS OF $p$-BIHARMONIC MAPS INTO A NON-POSITIVELY CURVED MANIFOLD

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Abstract. In this paper, we investigate $p$-biharmonic maps $u : (M, g) \to (N, h)$ from a Riemannian manifold into a Riemannian manifold with non-positive sectional curvature. We obtain that if
$$\int_M |\tau(u)|^{a+p} dv_g < \infty \text{ and } \int_M |d(u)|^2 dv_g < \infty,$$
then $u$ is harmonic, where $a \geq 0$ is a nonnegative constant and $p \geq 2$. We also obtain that any weakly convex $p$-biharmonic hypersurfaces in space form $N(c)$ with $c \leq 0$ is minimal. These results give affirmative partial answer to Conjecture 2 (generalized Chen’s conjecture for $p$-biharmonic submanifolds).

1. Introduction

Harmonic maps play a central roll in geometry. They are critical points of the energy $E(u) = \int_M \frac{|du|^2}{2} dv_g$ for smooth maps between manifolds $u : (M, g) \to (N, h)$ and the Euler-Lagrange equation is that tension field $\tau(u)$ vanishes. Extensions to the notions of $p$-harmonic maps, $F$-harmonic maps and $f$-harmonic maps were introduced and many results have been carried out (for instance, see [1, 2, 3, 8, 23]). In 1983, J. Eells and L. Lemaire [10] proposed the problem to consider the biharmonic maps: they are critical maps of the functional
$$E_2(u) = \int_M \frac{|\tau(u)|^2}{2} dv_g.$$
We see that harmonic maps are biharmonic maps and even more, minimizers of the bienergy functional. After G. Y. Jiang [15] studied the first and second variation formulas of the bienergy $E_2$, there have been extensive studies on biharmonic maps (for instance, see [9, 15, 16, 17, 21, 22, 24, 25]). Recently the first author and S. X. Feng in [13] introduced the following functional
$$E_{F,2}(u) = \int_M F(\frac{|\tau(u)|^2}{2}) dv_g,$$
where $\tau(u) = -\delta du = \text{trace}\tilde{\nabla}(du)$. The map $u$ is called an $F$-biharmonic map if it is a critical point of that $F$-bienergy $E_{F,2}(u)$, which is a generalization of biharmonic maps, $p$-biharmonic maps [14] or exponentially biharmonic maps. Notice that harmonic maps are always $F$-biharmonic by definition. When $F(t) = (2t)^{\frac{p}{2}}$, we have a $p$-bienergy functional

$$E_{p,2}(u) = \int_M |\tau(u)|^p dv_g,$$

where $p \geq 2$. The Euler-Lagrange equation of $E_{p,2}$ is $\tau_{p,2}(u) = 0$, where $\tau_{p,2}(u)$ is given by (13). A map $u : (M, g) \to (N, h)$ is called a $p$-biharmonic map if $\tau_{p,2}(u) = 0$. When $u : (M, g) \to (N, h)$ is a $p$-biharmonic isometric immersion, then $M$ is called a $p$-biharmonic submanifold in $N$.

Recently, N. Nakauchi, H. Urakawa and S. Gudmundsson [21] proved that biharmonic maps from a complete Riemannian manifold into a non-positive curved manifold with finite bienergy and energy are harmonic. S. Maeta [20] proved that biharmonic maps from a complete Riemannian manifold into a non-positive curved manifold with finite $(a + 2)$-bienergy $\int_M |\tau(u)|^{a+2} dv_g < \infty$ ($a \geq 0$) and energy are harmonic. In this paper, we first obtain the following result:

**Theorem 1.1** (cf. Theorem 3.1). Let $u : (M^m, g) \to (N^n, h)$ be a $p$-biharmonic map from a Riemannian manifold $(M, g)$ into a Riemannian manifold $(N, h)$ with non-positive sectional curvature and let $a \geq 0$ be a non-negative real constant.

(i) If

$$\int_M |\tau(u)|^{a+p} dv_g < \infty,$$

and the energy is finite, that is,

$$\int_M |du|^2 dv_g < \infty,$$

then $u$ is harmonic.

(ii) If $\text{Vol}(M, g) = \infty$, and

$$\int_M |\tau(u)|^{a+p} dv_g < \infty,$$

then $u$ is harmonic, where $p \geq 2$.

One of the most interesting problems in the biharmonic theory is Chen’s conjecture. In 1988, Chen raised the following problem:

**Conjecture 1** ([7]). Any biharmonic submanifold in $E^n$ is minimal.

There are many affirmative partial answers to Chen’s conjecture.

On the other hand, Chen’s conjecture was generalized as follows (cf. [6]): “Any biharmonic submanifolds in a Riemannian manifold with non-positive sectional curvature is minimal”. There are also many affirmative partial answers to this conjecture.
SOME RESULTS OF $p$-BIHARMONIC MAPS

(a) Any biharmonic submanifold in $H^3(-1)$ is minimal (cf. [5]).
(b) Any biharmonic hypersurfaces in $H^4(-1)$ is minimal (cf. [4]).
(c) Any weakly biharmonic hypersurfaces in space form $N^{m+1}(c)$ with $c \leq 0$ is minimal (cf. [18]).
(d) Any compact biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal (cf. [15]).
(e) Any compact $F$-biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal (cf. [13]).

For $p$-biharmonic submanifolds, it is natural to consider the following problem.

**Conjecture 2.** Any $p$-biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal.

For $p$-biharmonic submanifolds, we obtain the following result:

**Theorem 1.2** (cf. Theorem 4.1). Let $u : (M^m, g) \rightarrow (N^{m+1}, \langle \cdot, \cdot \rangle)$ be a weakly convex $p$-biharmonic hypersurface in a space form $N^{m+1}(c)$ with $c \leq 0$. Then $u$ is minimal, where $p \geq 2$.

These results give affirmative partial answers to the generalized Chen’s conjecture for $p$-biharmonic submanifold.

**2. Preliminaries**

In this section we give more details for the definitions of harmonic maps, biharmonic maps, $p$-biharmonic maps and $p$-biharmonic submanifolds.

Let $u : (M, g) \rightarrow (N, h)$ be a map from an $m$-dimensional Riemannian manifold $(M, g)$ to an $n$-dimensional Riemannian manifold $(N, h)$. The energy of $u$ is defined by

$$E(u) = \int_M |du|^2 dv_g.$$ 

The Euler-Lagrange equation of $E$ is

$$\tau(u) = \sum_{i=1}^m \{\nabla_{e_i} du(e_i) - du(\nabla_{e_i} e_i)\} = 0,$$

where we denote by $\nabla$ the Levi-Civita connection on $(M, g)$ and $\nabla$ the induced Levi-civita connection on $u^{-1}TN$ and $\{e_i\}_{i=1}^m$ is an orthonormal frame field on $(M, g)$. $\tau(u)$ is called the tension field of $u$. A map $u : (M, g) \rightarrow (N, h)$ is called a harmonic map if $\tau(u) = 0$.

To generalize the notion of harmonic maps, in 1983 J. Eells and L. Lemaire [10] proposed considering the bienergy functional

$$E_2(u) = \int_M |\tau(u)|^2 dv_g.$$
In 1986, G. Y. Jiang [15] studied the first and second variation formulas of the bienergy $E_2$. The Euler-Lagrange equation of $E_2$ is

$$\tau_2(u) = -\tilde{\Delta}(\tau(u)) - \sum_i R^N(\tau(u), du(e_i))du(e_i) = 0,$$

where $\tilde{\Delta} = \sum_i (\tilde{\nabla}_e \tilde{\nabla}_e \tau - \tilde{\nabla}_e \nabla_e \tau_e)$ is the rough Laplacian on the section of $u^{-1}TN$ and $R^N(X, Y) = [N \nabla_X, N \nabla_Y] - N \nabla_{[X,Y]}$ is the curvature operator on $N$. A map $u : (M, g) \to (N, h)$ is called a biharmonic map if $\tau_2(u) = 0$.

To generalize the notion of biharmonic maps, the first author and S. X. Feng [13] introduced the $F$-bienergy functional

$$E_{F,2}(u) = \int_M F(\frac{\tau(u)^2}{2})dv_g,$$

where $F : [0, \infty) \to [0, \infty)$ is a $C^3$ function such that $F'' > 0$ on $(0, \infty)$. The Euler-Lagrange equation of $E_{F,2}$ is

$$\tau_{F,2}(u) = -\tilde{\Delta}(F'(\frac{\tau(u)^2}{2})\tau(u)) - \sum_i R^N(F'(\frac{\tau(u)^2}{2})\tau(u), du(e_i))du(e_i) = 0.$$

A map $u : (M, g) \to (N, h)$ is called a $F$-biharmonic map if $\tau_{F,2}(u) = 0$.

When $F(t) = (2t)^\frac{p}{2}$, we have a $p$-bienergy functional

$$E_{p,2}(u) = \int_M |\tau(u)|^p dv_g,$$

where $p \geq 2$. The Euler-Lagrange equation of $E_{p,2}$ is

$$\tau_{p,2}(u) = -\tilde{\Delta}(p|\tau(u)|^{p-2}\tau(u)) - \sum_i R^N(p|\tau(u)|^{p-2}\tau(u), du(e_i))du(e_i) = 0.$$

A map $u : (M, g) \to (N, h)$ is called a $p$-biharmonic map if $\tau_{p,2}(u) = 0$.

Now we recall the definition of $p$-biharmonic submanifolds (cf. [12]).

Let $u : (M, g) \to (N, h = \langle \cdot, \cdot \rangle)$ be an isometric immersion from an $m$-dimensional Riemannian manifold into an $m + \ell$-dimensional Riemannian manifold. We identify $du(X)$ with $X \in \Gamma(TM)$ for each $X \in M$. We also denote by $\langle \cdot, \cdot \rangle$ the induced metric $u^{-1}h$. The Gauss formula is given by

$$N\nabla_X Y = \nabla_X Y + B(X, Y), \quad X, Y \in \Gamma(TM),$$

where $B$ is the second fundamental form of $M$ in $N$. The Weingarten formula is given by

$$N\nabla_X \xi = -A_\xi X + \nabla^\perp_X \xi, \quad X \in \Gamma(TM), \xi \in \Gamma(T^\perp M),$$

where $A_\xi$ is the shape operator for a unit normal vector field $\xi$ on $M$, and $\nabla^\perp$ denotes the normal connection on the normal bundle of $M$ in $N$. For any $x \in M$, the mean curvature vector field $H$ of $M$ at $x$ is given by

$$H = \frac{1}{m} \sum_{i=1}^m B(e_i, e_i).$$
If an isometric \( u : (M, g) \rightarrow (N, h) \) is \( p \)-biharmonic, then \( M \) is called a \( p \)-biharmonic submanifold in \( N \). In this case, we remark that the tension field \( \tau(u) \) of \( u \) is written \( \tau(u) = mH \), where \( H \) is the mean curvature vector field of \( M \). The necessary and sufficient condition for \( M \) in \( N \) to be \( p \)-biharmonic is the following:

\[
-\tilde{\Delta}(|H|^{p-2}H) - \sum_i R^N(|H|^{p-2}H, e_i)e_i = 0.
\]

From (1), we obtain the necessary and sufficient condition for \( M \) in \( N \) to be \( p \)-biharmonic as follows:

\[
\Delta^\perp(|H|^{p-2}H) - \sum_{i=1}^m B(e_i, A|H|^{p-2}H(e_i)) + \sum_{i=1}^m R^N(|H|^{p-2}H, e_i)e_i = 0,
\]

\[
Tg(\nabla A|H|^{p-2}H) + Tr_g[\nabla^\perp([|H|^{p-2}H]Y)] - \sum_{i=1}^m R^N(|H|^{p-2}H, e_i)e_i \perp = 0,
\]

where \( \Delta^\perp = \sum_{i=1}^m (\nabla^\perp_{e_i}\nabla^\perp_{e_i} - \nabla^\perp_{\nabla^\perp_{e_i}e_i}) \) is the Laplace operator associated with the normal connection \( \nabla^\perp \).

We also need the following lemma.

**Lemma 2.1** (Gaffney, [11]). Let \((M, g)\) be a complete Riemannian manifold. If a \( C^1 \) a-form \( \alpha \) satisfies that \( \int_M |\alpha|dv < \infty \) and \( \int_M (\delta \alpha)dv < \infty \), or equivalently, a \( C^1 \) vector \( X \) defined by \( \alpha(Y) = \langle X, Y \rangle \ (\forall Y \in \Gamma(TM)) \) satisfies that \( \int_M |X|dv < \infty \) and \( \int_M \text{div}(X)dv < \infty \), then

\[
\int_M (-\delta \alpha)dv = \int_M \text{div}(X)dv = 0.
\]

### 3. Main results of \( p \)-biharmonic maps

In this section, we obtain the following result.

**Theorem 3.1.** Let \( u : (M^m, g) \rightarrow (N^n, h) \) be a \( p \)-biharmonic map from a Riemannian manifold \((M, g)\) into a Riemannian manifold \((N, h)\) with non-positive sectional curvature and let \( a \geq 0 \) be a non-negative real constant.

(i) If

\[
\int_M |	au(u)|^{a+p}dv < \infty,
\]

and the energy is finite, that is,

\[
\int_M |du|^2dv < \infty,
\]

then \( u \) is harmonic.

(ii) If \( \text{Vol}(M, g) = \infty \), and

\[
\int_M |	au(u)|^{a+p}dv < \infty,
\]

then \( u \) is harmonic.
then \( u \) is harmonic, where \( p \geq 2 \).

**Proof.** Take a fixed point \( x_0 \in M \) and for every \( r > 0 \), let us consider the following cut off function \( \lambda(x) \) on \( M \):

\[
0 \leq \lambda(x) \leq 1, \quad x \in M,
\]

\[
\lambda(x) = 1, \quad x \in B_r(x_0),
\]

\[
\lambda(x) = 0, \quad x \in M - B_{2r}(x_0),
\]

\[
|\nabla \lambda| \leq \frac{C}{r},
\]

where \( B_r(x_0) = \{ x \in M : d(x, x_0) < r \} \), \( C \) is a positive constant and \( d \) is the distance of \( (M, g) \). From (13), we have

\[
0 \geq \int_M \langle -\tilde{\Delta}(|\tau(u)|^{p-2} \tau(u)), \lambda^2 |\tau(u)|^a \tau(u) \rangle \, dv_g
\]

\[
= \int_M (\nabla(|\tau(u)|^{p-2} \tau(u)), \tilde{\nabla}(\lambda^2 |\tau(u)|^a \tau(u))) \, dv_g
\]

\[
= \int_M \sum_{i=1}^m (\tilde{\nabla}_{e_i}(\lambda^2 |\tau(u)|^a \tau(u))), \tilde{\nabla}_{e_i}(\lambda^2 |\tau(u)|^a \tau(u))) \, dv_g
\]

\[
= \int_M \sum_{i=1}^m (\tilde{\nabla}_{e_i}(\lambda^2 |\tau(u)|^a \tau(u))), \tilde{\nabla}_{e_i}(\lambda^2 |\tau(u)|^a \tau(u))) \, dv_g
\]

\[
= \int_M \sum_{i=1}^m \left[ (|\tau(u)|^{p-2} \tilde{\nabla}_{e_i} \tau(u) + (p - 2)|\tau(u)|^{p-4}(\tilde{\nabla}_{e_i} \tau(u)), \tau(u)) \tau(u), \right.
\]

\[
\left. + 2\lambda e_i(\lambda)|\tau(u)|^a \tau(u) + \lambda^2 e_i(\lambda)|\tau(u)|^a \tau(u) \right] \, dv_g
\]

\[
= \int_M \sum_{i=1}^m \left[ (|\tau(u)|^{p-2} \tilde{\nabla}_{e_i} \tau(u) + (p - 2)|\tau(u)|^{p-4}(\tilde{\nabla}_{e_i} \tau(u)), \tau(u)) \tau(u), \right.
\]

\[
\left. + 2\lambda e_i(\lambda)|\tau(u)|^a \tau(u) + a\lambda^2 |\tau(u)|^{a-2} \tilde{\nabla}_{e_i} \tau(u), \tau(u)) \tau(u) \right. \]

\[
+ \lambda^2 |\tau(u)|^a \tilde{\nabla}_{e_i} \tau(u)) \right] \, dv_g
\]

\[
= \int_M \sum_{i=1}^m \left[ (p - 1)\lambda e_i(\lambda)|\tau(u)|^a \tilde{\nabla}_{e_i} \tau(u), \tau(u)) \right] \, dv_g
\]

\[
+ \int_M \sum_{i=1}^m (a(p - 1) + (p - 2)|\tau(u)|^{a-4} \tilde{\nabla}_{e_i} \tau(u), \tau(u))^2 \, dv_g
\]

\[
+ \int_M \sum_{i=1}^m \lambda^2 |\tau(u)|^{a-2} \tilde{\nabla}_{e_i} \tau(u), \tilde{\nabla}_{e_i} \tau(u)) \, dv_g
\]
By assumption (10), we have

\[ \int_M \sum_{i=1}^{m} 2(p-1)\lambda e_i(\lambda) \cdot |\tau(u)|^{a+p-2} \langle \tilde{\nabla}_e \tau(u), \tau(u) \rangle \, dv_g \]

where the inequality follows from \(|a(p-1) + (p-2)|\lambda^2|\tau(u)|^{a+p-4} \langle \tilde{\nabla}_e \tau(u), \tau(u) \rangle^2 \geq 0.\) From (7), we have

\[ \int_M \sum_{i=1}^{m} \lambda^2 |\tau(u)|^{a+p-2} \langle \tilde{\nabla}_e \tau(u), \tilde{\nabla}_e \tau(u) \rangle \, dv_g, \]

By using Young’s inequality, we have

\[ -2(p-1) \int_M \sum_{i=1}^{m} \langle \tilde{\nabla}_e \tau(u), \lambda e_i(\lambda) \rangle |\tau(u)|^{a+p-2} \tau(u) \rangle \, dv_g. \]

From (8) and (9), we have

\[ \int_M \sum_{i=1}^{m} \lambda^2 |\tau(u)|^{a+p-2} \langle \tilde{\nabla}_e \tau(u), \tilde{\nabla}_e \tau(u) \rangle \, dv_g \]

|\tau(u)|^{a+p} \, dv_g \]

\[ \leq 4(p-1)^2 \int_M |\nabla \lambda|^2 |\tau(u)|^{a+p} \, dv_g \]

\[ \leq 4(p-1)^2 C^2 \int_M |\tau(u)|^{a+p} \, dv_g \]

\[ \leq \frac{4(p-1)^2 C^2}{r^2} \int_M |\tau(u)|^{a+p} \, dv_g. \]

By assumption \(\int_M |\tau(u)|^{a+p} \, dv_g < \infty,\) letting \(r \to \infty\) in (10), we have

\[ \int_M \sum_{i=1}^{m} |\tau(u)|^{a+p-2} \langle \tilde{\nabla}_e \tau(u), \tilde{\nabla}_e \tau(u) \rangle \, dv_g = 0. \]

Therefore, we obtain that \(|\tau(u)|\) is constant and \(\tilde{\nabla}_X \tau(u) = 0\) for any vector field \(X\) on \(M.\)

Therefore, if \(\text{Vol}(M) = \infty\) and \(|\tau(u)| \neq 0,\) then

\[ \int_M |\tau(u)|^{a+p} \, dv_g = |\tau(u)|^{a+p} \text{Vol}(M) = \infty, \]

which yields a contradiction. Thus, we have \(|\tau(u)| = 0,\) i.e., \(u\) is harmonic. We have (ii).
For (i), assume both $\int_M |\tau(u)|^{a+p}dv_g < \infty$ and $\int_M |du|^2dv_g < \infty$. Define a 1-from $\alpha$ on $M$ defined by

$$\alpha(X) = |\tau(u)|^{\frac{a+p}{2p+1}}\langle du(X), \tau(u) \rangle$$

for any vector $X \in \Gamma(TM)$.

Note here that

$$\int_M |\alpha|^2 dv_g = \int_M \left[ \sum_{i=1}^{m} |\alpha(e_i)|^2 \right]^{\frac{1}{2}} dv_g$$

$$= \int_M \left[ \sum_{i=1}^{m} |\tau(u)|^{\frac{a+p}{2p+1}}\langle du(e_i), \tau(u) \rangle \right]^{\frac{1}{2}} dv_g$$

$$\leq \int_M |\tau(u)|^{\frac{a+p}{2p+1}}|du|dv_g$$

$$\leq \left[ \int_M |\tau(u)|^{a+p}dv_g \right]^{\frac{1}{2}} \left[ \int_M |du|^2dv_g \right]^{\frac{1}{2}} < \infty. \tag{12}$$

Now we compute

$$-\delta \alpha = \sum_{i=1}^{m} (\nabla_{e_i} \alpha)(e_i) = \sum_{i=1}^{m} [\nabla_{e_i} \alpha(e_i) - \alpha(\nabla_{e_i} e_i)]$$

$$= \sum_{i=1}^{m} \nabla_{e_i} [|\tau(u)|^{\frac{a+p}{2p+1}}\langle du(e_i), \tau(u) \rangle]$$

$$- \sum_{i=1}^{m} |\tau(u)|^{\frac{a+p}{2p+1}}\langle du(\nabla_{e_i} e_i), \tau(u) \rangle$$

$$= \sum_{i=1}^{m} |\tau(u)|^{\frac{a+p}{2p+1}}\langle \nabla_{e_i} du(e_i) - du(\nabla_{e_i} e_i), \tau(u) \rangle$$

$$= |\tau(u)|^{\frac{a+p}{2p+1}+1},$$

where the fourth equality follows from that $|\tau(u)|$ is constant and $\tilde{\nabla}_X \tau(u) = 0$ for $X \in \Gamma(TM)$. Since $\int_M |\tau(u)|^{a+p}dv_g < \infty$ and $|\tau(u)|$ is constant, the function $-\delta \alpha$ is also integrable over $M$. From this and (12), we can apply Lemma 2.1 for the 1-from $\alpha$. Therefore we have

$$0 = \int_M (-\delta \alpha) dv_g = \int_M |\tau(u)|^{\frac{a+p}{2p+1}+1} dv_g,$$

so we have $\tau(u) = 0$, that is, $u$ is harmonic. \qed

4. Main results of $p$-biharmonic hypersurfaces

In this section, we obtain the following result.
Theorem 4.1. Let \( u : (M^m, g) \to (N^{m+1}, \langle \cdot, \cdot \rangle) \) be a weakly convex \( p \)-biharmonic hypersurface in a space form \( N^{m+1}(c) \) with \( c \leq 0 \). Then \( u \) is minimal, where \( p \geq 2 \).

Proof. Assume that \( H = h \nu \), where \( \nu \) is the unit normal vector field on \( M \). Since \( M \) is weakly convex, we have \( h \geq 0 \). Set \( C = \{ q \in M : h(q) > 0 \} \). We will prove that \( A \) is an empty set.

If \( C \) is not empty, we see that \( C \) is an open subset of \( M \). We assume that \( C_1 \) is a nonempty connect component of \( C \). We will prove that \( h \equiv 0 \) in \( C_1 \), thus a contradiction.

Firstly, we prove that \( h \) is a constant in \( C_1 \).

Let \( q \in C_1 \) be a point. Choose a local orthonormal frame \( \{e_i, i = 1, \ldots, m\} \) around \( q \) such that \( \langle B, \nu \rangle \) is a diagonal matrix and \( \nabla e_i, e_j |_q = 0 \).

From equation (3), we have at \( q \)

\[
0 = \sum_{i=1}^{m} \langle (\nabla_{e_i} A_{(h^{p-2}H)}) (e_i), e_k \rangle + \sum_{i=1}^{m} \langle A_{(h^{p-2}H)^{\perp}} (e_i), e_k \rangle \\
= \sum_{i=1}^{m} e_i \langle A_{(h^{p-2}H)} (e_i), e_k \rangle + \sum_{i=1}^{m} \langle B(e_i, e_k), \nabla_{e_i} (h^{p-2}H) \rangle \\
= \sum_{i=1}^{m} e_i \langle h^{p-2}H, B(e_i, e_k) \rangle + \sum_{i=1}^{m} \langle B(e_i, e_k), \nabla_{e_i} (h^{p-2}H) \rangle \\
= \sum_{i=1}^{m} \langle h^{p-2}H, \nabla_{e_i} B(e_i, e_i) \rangle + 2 \sum_{i=1}^{m} \langle B(e_i, e_k), \nabla_{e_i} (h^{p-2}H) \rangle \\
= m \langle h^{p-2}H, \nabla_{e_k} H \rangle + 2 \langle \lambda_k \nu, \nabla_{e_i} (h^{p-2}H) \rangle \\
= mh^{p-1}e_k (h) + 2(p-1)h^{p-2} \lambda_k e_k (h) \\
= (mh + 2(p-1)\lambda_k)h^{p-2}e_k (h),
\]

where \( \lambda_k \) is the \( k \)th principle curvature of \( M \) at \( q \), which is nonnegative by the assumption that \( M \) is weakly convex. Since \( (mh + 2(p-1)\lambda_k)h^{p-2} > 0 \) at \( q \), we have \( e_k (h) = 0 \) at \( q \), for \( k = 1, \ldots, m \), which implies that \( \nabla h = 0 \) at \( q \). Because \( q \) is an arbitrary point in \( C_1 \), we have \( \nabla h = 0 \) in \( C_1 \). Therefore we obtain that \( h \) is constant in \( C_1 \).

Secondly, we prove that \( h \) is zero in \( C_1 \).

From the equation (2), we have

\[
\Delta h^{2p-2} = \Delta \langle h^{p-2}H, h^{p-2}H \rangle \\
= 2 \langle \Delta^\perp (h^{p-2}H), h^{p-2}H \rangle + 2 \langle \nabla (h^{p-2}H) \rangle^2
\]
\begin{equation}
2|\nabla^\perp (h^{p-2}H)|^2 + 2 \sum_{i=1}^{m} \langle B(A_{h^{p-2}H} e_i, e_i), h^{p-2}H \rangle \\
- \sum_{i=1}^{m} \langle R^N (h^{p-2}H, e_i)e_i, h^{p-2}H \rangle \\
\geq |\nabla^\perp (h^{p-2}H)|^2 + 2 \sum_{i=1}^{m} \langle B(A_{h^{p-2}H} e_i, e_i), h^{p-2}H \rangle,
\end{equation}

where the inequality follows from the sectional curvature of $(N, h)$ is non-positive. Now we state two inequalities:

\begin{equation}
|\nabla^\perp (h^{p-2}H)|^2 \geq h^2 p - 4|\nabla^\perp H|^2
\end{equation}

and

\begin{equation}
\sum_{i=1}^{m} \langle B(A_{h^{p-2}H} e_i, e_i), h^{p-2}H \rangle \geq mh^2 p.
\end{equation}

In fact,

\begin{equation}
|\nabla^\perp (h^{p-2}H)|^2 = |(p-2)h^{p-4}\langle \nabla^\perp H, H \rangle H + h^{p-2}\nabla^\perp H|^2 \\
= (p-2)^2 h^{2p-6} \langle \nabla^\perp H, H \rangle^2 + h^{2p-4}|\nabla^\perp H|^2 \\
+ 2(p-2)h^{3p-6} \langle \nabla^\perp H, H \rangle^2 \\
\geq h^{2p-4}|\nabla^\perp H|^2,
\end{equation}

and

\begin{equation}
\sum_{i=1}^{m} \langle B(A_{h^{p-2}H} e_i, e_i), h^{p-2}H \rangle \\
= \sum_{i=1}^{m} h^{2p-2} \langle B(A_{e_i} e_i, e_i), \nu \rangle \\
= \sum_{i=1}^{m} h^{2p-2} \langle A_{e_i} e_i, A_{e_i} e_i \rangle \\
= \sum_{i,j=1}^{m} h^{2p-2} |\langle B(e_i, e_j), \nu \rangle|^2 \\
\geq mh^2 p.
\end{equation}

From (13), (14) and (15), we have

\begin{equation}
\triangle h^{2p-2} \geq 2h^{2p-4}|\nabla^\perp H|^2 + 2mh^2 p.
\end{equation}

So we have

\begin{equation}
\triangle h^{2p-2} \geq 2mh^2 p.
\end{equation}
From equation (16), we have in $C_1$

$$0 = \Delta h^{2p-2} \geq 2mh^{2p}.$$ 

We know that $h \equiv 0$ in $C_1$. This is a contradiction. □

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