Fermion masses and mixings in an $SU(5)$ Grand Unified Model with an Extra Flavor Symmetry

Miguel D. Campos,1,∗ A. E. Cárcamo Hernández,1 S. Kovalenko,1 Iván Schmidt,1 and Erik Schumacher2,¶

1 Universidad Técnica Federico Santa María and Centro Científico-Tecnológico de Valparaíso
Casilla 110-V, Valparaíso, Chile
2 Fakultät für Physik, Technische Universität Dortmund
D-44221 Dortmund, Germany

We propose a model based on the $SU(5)$ Grand Unification with an extra $A_4 \otimes Z_2 \otimes Z_2' \otimes U(1)_f$ flavor symmetry, which accounts for the pattern of the SM fermion masses and mixings. The observed hierarchy of charged fermion masses and quark mixing matrix elements arises from a generalized Froggatt-Nielsen mechanism triggered by a scalar $24$ representation of $SU(5)$ charged under the global $U(1)_f$ and acquiring a VEV at the GUT scale. The light neutrino masses are generated via a radiative seesaw mechanism with a single heavy Majorana neutrino and neutral scalars running in the loops. The model predictions for both quark and lepton sectors are in good agreement with the experimental data. The model predicts an effective Majorana neutrino mass, relevant for neutrinoless double beta decay, with values $m_{\beta\beta} = 4$ meV and $50$ meV for the normal and the inverted neutrino spectrum, respectively. The model also features a suppression of CP violation in neutrino oscillations, a low scale for the heavy Majorana neutrino (few TeV) and, due to the unbroken $Z_2$ symmetry, a natural dark matter candidate.

I. INTRODUCTION

The great success of the Standard Model (SM) in the description of electroweak phenomena, recently concreted with the LHC discovery of the Higgs boson, leaves nevertheless many unresolved problems. Among the most pressing are the smallness of neutrino masses, the puzzling pattern of fermion masses and mixings, and the existence of the three families of quarks and leptons. In the search for a solution of these problems various extensions of the SM with additional flavor symmetries have been proposed in the literature (for a review see, e.g., Ref. [1–3]). Historically, some of these symmetries were hinted by the Tribimaximal (TBM) ansatz for the leptonic mixing matrix:

$$U_{\text{TBM}} = \begin{bmatrix} \sqrt{2}/3 & 1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}.$$  \hspace{1cm} (1)

leading to the neutrino mixing angles $(\sin^2 \theta_{12})_{\text{TBM}} = 1/3$, $(\sin^2 \theta_{23})_{\text{TBM}} = 1/2$ and $(\sin^2 \theta_{13})_{\text{TBM}} = 0$. However, recent measurements of a non-zero value of the reactor mixing angle $\theta_{13}$ by the Daya Bay [4], T2K [5], MINOS [6], Double CHOOZ [7] and RENO [8] have already ruled out the exact TBM pattern, as shown in Tables I and II (based on Ref. [9]) for the Normal (NH) and Inverted (IH) Hierarchies of the neutrino mass spectrum. Nevertheless, the smallness of the reactor angle still allows for the TBM to serve as a first order approximation in the construction of realistic models of lepton mixing based on flavor symmetries.

∗Electronic address: miguel.campos@postgrado.usm.cl
†Electronic address: antonio.carcamo@usm.cl
‡Electronic address: sergey.kovalenko@usm.cl
§Electronic address: ivan.schmidt@usm.cl
¶Electronic address: erik.schumacher@tu-dortmund.de

arXiv:1403.2525v1 [hep-ph]  11 Mar 2014
In this paper we propose a version of the $SU(5)$ GUT model with an additional global flavor symmetry group $A_4 \times Z_2 \times Z_2' \times Z_2'' \times U(1)_f$. It involves a horizontal symmetry $U_f(1)$, allowing to naturally introduce the fermion mass hierarchies through a generalized Froggatt-Nielsen mechanism [24]. The discrete symmetry groups $A_4$ and three different $Z_2$ are needed in order to reproduce the specific patterns of mass matrices in the quark and lepton sectors. The embedding of the model in a non-minimal $SU(5)$ GUT requires a significant extension of the scalar sector. The particular role of each additional scalar field and the corresponding particle assignments under the symmetry group of the model are explained in details in Section IV. On the other hand, in analogy to Ref. [29], we consider only one additional right-handed neutrino $N_R$ in order to explain the masses and mixings in the neutrino sector. The light neutrino masses are generated in our model through a radiative seesaw mechanism, in which neutrinos receive their masses only from radiative corrections at one-loop level. The smallness of the neutrino masses is a natural consequence of the small one-loop contributions and the quadratic dependence on the neutrino Yukawa couplings. In contrast to the regular seesaw Type I scenarios, the mass of the right-handed neutrino can therefore be kept at the TeV scale. For a general review of the radiative seesaw we refer readers, for example, to Ref. [27], and to Ref. [28] for its discussion in the context of flavor symmetries.

Our model describes a realistic pattern of the SM fermion masses and mixings. The model has 13 free effective parameters, which allow us to reproduce the experimental values of 18 observables, i.e., 9 charged fermion masses, 2 neutrino mass squared splittings, 3 lepton mixing parameters and 4 parameters of the Wolfenstein parametrization of the CKM quark mixing matrix. Let us note that the similar model of Ref. [19], with an $SU(5)$ GUT supersymmetric setup and flavor symmetries, has 14 free effective parameters aimed at reproducing the above mentioned 18 observables.

The paper is organized as follows. In section II we outline the proposed model. In section III we present our results regarding neutrino masses and mixing, which is followed by a numerical analysis. Our results for the quark sector, with the corresponding numerical analysis, are presented in section IV. We conclude with discussions and a summary in section V. Some necessary facts about the $A_4$ group are collected in appendix A.

II. THE MODEL

As is well known, the minimal $SU(5)$ GUT [29] with fermions in $\mathbf{5} + \mathbf{10}$ and the scalars in $\mathbf{5 + 24}$ representations of $SU(5)$, suffers from various problems. In particular, it predicts wrong relations between the down-type quark
and charged lepton masses, short proton life-time, and the unification of gauge couplings does not agree with the values of $\alpha_S$, $\sin \theta_W$ and $\alpha_{em}$ at the $M_Z$ scale. There is no place in the minimal model for a non-zero neutrino mass, in contradiction with the neutrino oscillation experiments. Some of these problems can be solved by an extension of the model field content including, in particular, a scalar $45$ representation of $SU(5)$ \cite{footnote}. However, in this next-to-minimal $SU(5)$ GUT the hierarchy among the fermion masses is not understood and translates to an unexplained hierarchy among the different Yukawa couplings. This motivates implementing a generalized Froggatt-Nielsen mechanism, where the fermion mass hierarchy is explained by a spontaneously broken group $U(1)_f$ with a special $U(1)_f$ charge assignment to the fields participating in the Yukawa terms. Our model is a multi-Higgs extension of the next-to-minimal $SU(5)$ GUT, with the full symmetry $G$ experiencing a two-step spontaneous breaking:

$$G = SU(5) \otimes A_4 \otimes Z_2 \otimes Z_2' \otimes Z_2'' \otimes U(1)_f$$

$$\downarrow \Lambda_{GUT}$$

$$SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \otimes Z_2$$

$$\downarrow \Lambda_{EW}$$

$$SU(3)_C \otimes U(1)_{em} \otimes Z_2$$

The discrete non-Abelian tetrahedral symmetry group $A_4$, the group of even permutations of four objects, is the smallest group with one three-dimensional and three distinct one-dimensional irreducible representations (irreps), naturally accommodating the three families of fermions. In the literature this group has been extensively studied in the context of the flavor problem and neutrino physics (cf. \cite{28, 43, 44}). The role of the other symmetry group factors of $G$ will be explained in what follows.

We extend the fermion sector of the SM by introducing only one additional field, an SM singlet Majorana neutrino, $N_R$. The three families of left-handed fermions, corresponding to the $\mathbf{5}$ irrep of $SU(5)$, are unified into an $A_4$ triplet in order to have one Yukawa term for the interaction with the right-handed neutrino $N_R$, analogously to Ref. \cite{29}. The remaining three families of left- and right-handed fermions are accommodated by the three different $A_4$ singlets $\mathbf{1}, \mathbf{1}', \mathbf{1}''$. The only right-handed SM singlet neutrino $N_R$ of our model is assigned to the $\mathbf{1}$ of $A_4$ in order for its Majorana mass term to be invariant under this symmetry. The presence of this term is crucial for our construction, as explained below. Note that neither the $\mathbf{1}'$ nor $\mathbf{1}''$ singlet representations of $A_4$ satisfy this condition, as can be seen from the multiplication rules in Eq. (A1). The fermion assignments under the group $G = SU(5) \otimes A_4 \otimes Z_2 \otimes Z_2' \otimes Z_2'' \otimes U(1)_f$ are:

$$\Psi^i = (\psi^{i(1)}, \psi^{i(2)}, \psi^{i(3)}) \sim (\mathbf{5}, 1, 1, 1, -1, Q_5^{(i)})$$

$$\psi^{i(1)}_i \sim (5, 1, 1, 1, Q_5^{(i)})$$

$$\psi^{i(2)}_i \sim (5, 1', 1, 1, Q_5^{(i)})$$

$$\psi^{i(3)}_i \sim (5, 1'', 1, 1, Q_5^{(i)})$$

$$\Psi^{ij}_i \sim (10, 1, 1, 1, Q_{10}^{(ij)})$$

$$\Psi^{ij}_j \sim (10, 1', 1, 1, Q_{10}^{(ij)})$$

$$\Psi^{ij}_j \sim (10, 1'', 1, 1, Q_{10}^{(ij)})$$

$$\Omega^{ij}_i \sim (10, 1, 1, 1, Q_{10}^{(ij)})$$

$$\Omega^{ij}_j \sim (10, 1', 1, 1, Q_{10}^{(ij)})$$

$$\Omega^{ij}_j \sim (10, 1'', 1, 1, Q_{10}^{(ij)})$$

$$N_R \sim (1, 1, 1, 1, -1, -1, 0).$$

More explicitly, the fermions are accommodated as:

$$\Psi^{(f)}_{ij} = \begin{pmatrix}
0 & u_3^{(f)} & -u_3^{(f)} & -u_1^{(f)} & -u_1^{(f)c} & -d_2^{(f)c} \\
-u_3^{(f)c} & 0 & u_1^{(f)c} & -u_1^{(f)c} & -d_2^{(f)c} & u_3^{(f)c} \\
u_2^{(f)} & -u_2^{(f)c} & 0 & u_3^{(f)c} & -u_3^{(f)} & -d_2^{(f)c} \\
-d_1^{(f)c} & u_2^{(f)} & u_3^{(f)c} & 0 & -l(f) & 0 \\
d_1^{(f)c} & -d_2^{(f)c} & -d_3^{(f)c} & l(f) & 0 & 0
\end{pmatrix}_L$$

$$f = 1, 2, 3$$

$$i, j = 1, 2, 3, 4, 5.$$
The scalar sector is composed of the following $SU(5)$ representations: one 24, one 45, six 5’s and eight 1’s. One set of three 5’s and the two sets of $SU(5)$ singlets are unified into three $A_4$ triplets. The remaining scalar fields, i.e., one 45, one 24, two 1’s and the remaining set of the three 5’s are accommodated by different $A_4$ singlets. Thus the $G$ assignments of the scalar fields of our model are:

$$
\chi = (\chi_1, \chi_2, \chi_3) \sim (1, 3, 1, -1, 1, 0), \quad \xi = (\xi_1, \xi_2, \xi_3) \sim \left(1, 3, 1, 1, 1, Q_1^{(5)}\right),
$$

$$
\Upsilon \sim \left(1, 1', 1, -1, 1, Q_1^{(1)}\right), \quad \Delta \sim \left(1, 1', 1, 1, 1, Q_1^{(2)}\right), \quad S_i = \left(S_i^{(1)}, S_i^{(2)}, S_i^{(3)}\right) \sim \left(5, 3, -1, 1, 1, Q_5^{(S)}\right),
$$

$$
H_i^{(1)} \sim \left(5, 1, 1, 1, -1, Q_5^{(H(1))}\right), \quad H_i^{(2)} \sim \left(5, 1', 1, 1, 1, Q_5^{(H(2))}\right), \quad H_i^{(3)} \sim \left(5, 1'', 1, -1, 1, Q_5^{(H(3))}\right),
$$

$$
\Sigma_j \sim \left(24, 1, 1, -1, 1, -1\right) = \frac{1}{2}, \quad \Phi_{jk}^{(1)} \sim \left(45, 1, 1, 1, -1, Q_4^{(1)}\right).
$$

We introduce two sets of $A_4$ triplets $SU(5)$ singlets in order to separate the interactions responsible for the light neutrino masses from those that generate the down-type quark and charged lepton masses. The $A_4$ triplet $SU(5)$ singlet $\chi$ is the only set of scalars which is neutral under the $U(1)_f$ symmetry, while the remaining scalars have non-trivial $U(1)_f$ charges. Notice that the two sets of 5’s, i.e., $H_i^{(f)}$ and $S_i^{(f)}$ ($f = 1, 2, 3$) have different $Z_2$ parities.

With respect to the fermion sector, only the three families of left-handed fermions, corresponding to the 5 irrep of $SU(5)$, are unified into an $A_4$ triplet. Then, in order to build the required Yukawa interactions for charged fermions, we need the following scalars: three 5’s assigned to different $A_4$ singlets, one 45 assumed to be a trivial $A_4$ singlet 1, the $SU(5)$ singlet $A_4$ triplet $\xi$, two $SU(5)$ singlets $\Upsilon$ and $\Delta$ transforming as non-trivial $A_4$ singlets $1'$, and the scalar field $\Sigma$ in the 24 representation of $SU(5)$. As previously mentioned, having scalar fields in the 45 representation of $SU(5)$ is crucial in order to get the correct mass relations of down-type quarks and charged leptons. Concerning the breakdown of the group $G$ in Eq. 2, the scalar field $\Sigma$ is needed to trigger the generalized Froggatt-Nielsen mechanism responsible for generating the masses of charged fermions via higher dimensional Yukawa terms. Besides that, the scalar field $\Sigma$ acquires a vacuum expectation value (VEV) at the GUT scale $\Lambda_{GUT} = 10^{16}$ GeV and triggers the first step of symmetry breaking in Eq. 2. This first step is also induced by the scalars $\xi$, $\Upsilon$ and $\Delta$ acquiring VEVs at the GUT scale. The second step of symmetry breaking, is due to the $H_i^{(f)}$, $\Phi_{jk}^{(f)}$ ($f = 1, 2, 3$) acquiring VEVs at the electroweak scale. The scalar $\chi$, being an $SU(5)$ singlet, may receive its VEV at any scale below $\Lambda_{GUT}$, in particular around TeV. The three 5’s $H_i^{(f)}$, which are assigned to different $A_4$ singlets, transform trivially under $Z_2$ and participate in the Yukawa interactions involving charged fermions. Since the remaining three 5’s $S_i$ are unified into an $A_4$ triplet and transform non-trivially under $Z_2$, they participate in the Yukawa interactions with the right-handed neutrino $N_R$. In analogy to Ref. 26, we assume that the $Z_2$ symmetry is not affected by the electroweak symmetry breaking. Therefore, the $A_4$ triplet $S_i$ does not acquire a VEV and consequently neutrinos do not receive masses at tree-level. The preserved $Z_2$ discrete symmetry also allows for stable dark matter candidates, as in Refs. 45, 46. In our model they are either the lightest neutral component of the $SU(2)$ doublet component of $S_i$ or the right-handed Majorana neutrino $N_R$. We do not address this question in the present paper. As in Ref. 26, the scalar $\chi$ generates a neutrino mass matrix texture compatible with the experimentally observed deviation from the TBM.
pattern. As we will explain in the following, the neutrino mass matrix texture generated via the one-loop seesaw mechanism is mainly due to the VEV of this scalar $\langle \chi \rangle = \Lambda_{\text{int}}$, which is assumed to be much larger than the scale of the electroweak symmetry breaking $\Lambda_{\text{EW}} \gg \Lambda_{\text{GUT}} = 246 \text{ GeV}$ and at the same time much lower than the GUT scale $\Lambda_{\text{int}} \ll \Lambda_{\text{GUT}} = 10^{16} \text{ GeV}$. This, along with the assumption that the scalars (excepting $\chi$) are charged under $U(1)_f$, leads to a mixing matrix that is TBM to a good approximation. The $Z_2^r$ discrete symmetry is also an important ingredient of our approach. Once it is imposed, it forbids the terms in the scalar potential involving odd powers of $\chi$. This results in a reduction of the number of free model parameters and selects a particular direction of symmetry breaking in the group space. Also, as it will be shown in section IV, due to the $Z_2^r$ symmetry, the top quark gets its mass mainly from $H^{(3)}$. The $Z_2^r$ symmetry is broken after the $\chi$, $\Upsilon$ and $H^{(3)}$ fields acquire non-vanishing VEVs. The symmetry $Z_2^r$ guaranties that the scalars giving the dominant contribution to the masses of the down-type quarks and the charged leptons are different from those providing masses to the up-type quarks. This is crucial for keeping realistic lepton mixing (sf. Ref [26]). The fact that down-type quarks and charged leptons are unified into $5$ and $\overline{5}$ irreps of $SU(5)$, will result in a trivial contribution to the quark mixing from the down-type quark sector. Thus, the quark mixing will arise solely from the up-type quark sector as shown in detail in section IV.

Since the $A_4$ triplet $S_i$ is assumed to participate in the Yukawa interactions with the right-handed neutrino $N_R$, we choose its $U(1)_f$ charge $Q_5^{(S)}$ to be:

$$Q_5^{(S)} = -Q_5^{(\chi)},$$  \hspace{1cm} (15)$$

We consider the following VEV pattern of the scalars fields of the model. The VEVs of the scalars $\Upsilon$, $\Delta$, $H_i^{(f)}$, $S_i^{(f)}$ ($f = 1, 2, 3$) and $\Sigma_i^f$ are:

$$\langle \Upsilon \rangle = v_\Upsilon, \quad \langle \Delta \rangle = v_\Delta, \quad \langle H_i^{(f)} \rangle = v_\Upsilon^{(f)} \delta_{i5}, \quad \langle S_i^{(f)} \rangle = v_\Sigma^{(f)} \delta_{i5}, \quad f = 1, 2, 3, \hspace{1cm} (16)$$

$$\langle \Sigma_i^f \rangle = v_\Sigma \text{diag} \left( 1, 1, 1, -\frac{3}{2}, -\frac{3}{2} \right), \quad i, j = 1, 2, 3, 4, 5. \hspace{1cm} (17)$$

It is worth mentioning that the VEV pattern for the $\Sigma$ field, which is consistent with the minimization conditions of the scalar potential, follows from the general group theory of spontaneous symmetry breakdown $[47]$. The requirement that $Z_2$ is preserved implies, according to the field assignment given above, that:

$$v_\Sigma^{(f)} = 0, \quad f = 1, 2, 3, \hspace{1cm} (18)$$

For the VEVs of the neutral components of the $A_4$ triplet scalars $\chi$ and $\xi$ we assume:

$$v_{\chi_1} = -v_{\chi_3} = \frac{v_\chi}{\sqrt{2}}, \quad v_{\chi_2} = 0, \quad v_{\xi_1} = v_{\xi_2} = v_{\xi_3} = \frac{v_\xi}{\sqrt{3}}. \hspace{1cm} (19)$$

Here $v = \Lambda_{\text{EW}}$ and $v_\chi = \Lambda_{\text{int}}$. We also assume $v_\xi = v_\Upsilon = v_\Delta = \Lambda_{\text{GUT}}$. The choice of directions in the $A_4$ space, given by Eq. (19), is justified by the observation that they describe a natural solution of the scalar potential minimization equations. Indeed, in the single-field case, $A_4$ invariance readily favors the $(1, 1, 1)$ direction over, e.g., the $(1, 0, 0)$ solution for large regions of parameter space. The vacuum $\langle \xi \rangle$ is a configuration that preserves a $Z_3$ subgroup of $A_4$, which has been extensively studied by many authors (see for example Refs. [26] [48] [50]).

On the other hand, the property of the $45$ dimensional irrep of $SU(5)$ implies that the $\Phi_{jk}^i$ satisfies the following relations $[30] [31]$:

$$\Phi_{jk}^i = -\Phi_{kj}^i, \quad \sum_{i=1}^{5} \Phi_{ij}^i = 0, \quad i, j, k = 1, 2, \cdots, 5. \hspace{1cm} (20)$$

Consequently, the only allowed non-zero VEVs of $\Phi_{jk}^i$ are:

$$\langle \Phi_{p5}^i \rangle = -\frac{1}{3} \langle \Phi_{45}^4 \rangle = v_\Phi, \quad \langle \Phi_{j5}^i \rangle = v_\Phi (\delta_{j5} - 4\delta_{j1}^i \delta_{51}^i), \quad i, j = 1, 2, 3, 4, 5, \quad p = 1, 2, 3, 5. \hspace{1cm} (21)$$

With the above particle content, the following renormalizable $\mathcal{L}_Y$ and higher dimensional $\mathcal{L}_Y^{(NR)}$ Yukawa terms arise:

$$\mathcal{L}_Y = \lambda_{\nu} (\psi^j S_i)_{1} N_R + M_N N_R N_R^c + h.c, \hspace{1cm} (22)$$
Furthermore, in order to relate quark masses with the quark mixing parameters, we set:

\[ \mathcal{L}^{(NR)}_{\nu} = \frac{\alpha_1}{\Lambda} \left( \frac{\Sigma_i^k \Sigma_j^k}{\Lambda^2} \right)^{a_1} (\psi_i^\dagger) H^{(1)}_i \Omega^{(1)}_{ij} + \frac{\alpha_2}{\Lambda} \left( \frac{\Sigma_i^k \Sigma_j^k}{\Lambda^2} \right)^{a_2} (\psi_i^\dagger) H^{(1)}_i \Omega^{(2)}_{ij} + \frac{\alpha_3}{\Lambda} \left( \frac{\Sigma_i^k \Sigma_j^k}{\Lambda^2} \right)^{a_3} (\psi_i^\dagger) H^{(1)}_i \Omega^{(3)}_{ij} \]

\[ + \frac{\beta_1}{\Lambda} \left( \frac{\Sigma_i^k \Sigma_j^k}{\Lambda^2} \right)^{b_1} (\psi_i^\dagger) \Phi_i^k \Omega^{(1)}_{ij} + \frac{\beta_2}{\Lambda} \left( \frac{\Sigma_i^k \Sigma_j^k}{\Lambda^2} \right)^{b_2} (\psi_i^\dagger) \Phi_i^k \Omega^{(2)}_{ij} + \frac{\beta_3}{\Lambda} \left( \frac{\Sigma_i^k \Sigma_j^k}{\Lambda^2} \right)^{b_3} (\psi_i^\dagger) \Phi_i^k \Omega^{(3)}_{ij} \]

\[ + \varepsilon_{ijklp} \left( \gamma_{12} \left( \frac{\Sigma_i m^{m}}{\Lambda^2} \right)^{x_{12}} (\psi_i^\dagger) H^{(2)}_i \Omega^{(2)}_{ij} \right) + \gamma_{21} \left( \frac{\Sigma_i m^{m}}{\Lambda^2} \right)^{x_{21}} (\psi_i^\dagger) H^{(2)}_j \Omega^{(1)}_{ij} \]

\[ + \gamma_{11} \left( \frac{\Sigma_i m^{m}}{\Lambda^2} \right)^{x_{11}} (\psi_i^\dagger) H^{(2)}_p \Omega^{(1)}_{ij} + \gamma_{22} \left( \frac{\Sigma_i m^{m}}{\Lambda^2} \right)^{x_{22}} (\psi_i^\dagger) H^{(2)}_p \Omega^{(2)}_{ij} \]

\[ + \gamma_{23} \left( \frac{\Sigma_i m^{m}}{\Lambda^2} \right)^{x_{23}} (\psi_i^\dagger) H^{(3)}_p \Omega^{(3)}_{ij} + \gamma_{31} \left( \frac{\Sigma_i m^{m}}{\Lambda^2} \right)^{x_{31}} (\psi_i^\dagger) H^{(3)}_p \Omega^{(1)}_{ij} \]

\[ + \gamma_{33} \left( \frac{\Sigma_i m^{m}}{\Lambda^2} \right)^{x_{33}} (\psi_i^\dagger) H^{(3)}_p \Omega^{(3)}_{ij} \right) + h.c \]

The subscripts \( i, 1', 1'' \) denote projecting out the corresponding \( A_4 \) singlet in the product of the two triplets. The lightest of the physical neutral scalar states of \( H^{(1)}, H^{(2)}, H^{(3)} \) and \( \Phi \) should be interpreted as the SM-like 126 GeV Higgs observed at the LHC \cite{57}. Besides that, the low energy effective theory will correspond to a four Higgs doublet model with a light scalar color octet. The LHC signatures of TeV octet scalars in a SU(5) GUT model with a Higgs sector extended to include the 45 representation of SU(5) are studied in Ref. \cite{58}. As we will show in section \( IV \) the dominant contribution to the top quark mass mainly arises from \( H^{(3)} \). The SM-like 126 GeV Higgs also receives its main contributions from the CP even neutral state of the SU(2) doublet part of \( H^{(3)} \). The remaining scalars are heavy and outside the LHC reach. Our model is not predictive in the scalar sector, having numerous free uncorrelated parameters in the scalar potential that can be adjusted to get the required pattern of scalar masses. Therefore, the loop effects of the heavy scalars contributing to certain observables can be suppressed by the appropriate choice of the free parameters in the scalar potential. Fortunately, all these adjustments do not affect the charged fermion and neutrino sector, which is completely controlled by the fermion-Higgs Yukawa couplings and by certain combinations of \( U(1)_f \) charges \( Q_i^{(l)} \) appearing in the Yukawa terms of Eq. \cite{23}. The dimensionless couplings \( a_i, \beta_i, \gamma_{ij} \) \((i, j = 1, 2, 3)\), \( \zeta_1 \) and \( \zeta_2 \) in Eq. \cite{23} are \( \mathcal{O}(1) \) parameters and the following relations for the Froggat-Nielsen powers are fulfilled:

\[ a_i = Q^{(i)}_{10} + Q^{(j)}_{5} + Q^{(H^{(i)})}_{1} + Q^{(\psi)}_{1}, \quad b_i = Q^{(i)}_{10} + Q^{(j)}_{5} + Q^{(\Phi)}_{1} + Q^{(n)}_{1}, \quad i = 1, 2, 3 \]

\[ x_{11} = 2Q^{(i)}_{10} + 2Q^{(j)}_{5} + 2Q^{(X)}_{1}, \quad x_{12} = Q^{(i)}_{10} + Q^{(j)}_{5} + Q^{(2)}_{1} + Q^{(2)}_{1}, \quad x_{13} = Q^{(i)}_{10} + Q^{(j)}_{5} + Q^{(2)}_{1}, \quad i = 1, 2, 3 \]

\[ x_{22} = 2Q^{(i)}_{10} + 2Q^{(j)}_{5} + 2Q^{(2)}_{1}, \quad x_{23} = Q^{(i)}_{10} + Q^{(j)}_{5} + Q^{(2)}_{1}, \quad x_{23} = 2Q^{(2)}_{10} + Q^{(2)}_{5}, \quad x_{33} = Q^{(j)}_{10} + Q^{(j)}_{5} + Q^{(2)}_{1}, \quad x_{33} = 2Q^{(2)}_{10} + Q^{(2)}_{5}. \]

Furthermore, in order to relate quark masses with the quark mixing parameters, we set:

\[ \kappa = \frac{\Sigma_i^k \Sigma_j^k}{\Lambda^2} = \frac{\Sigma_{GUT}^{\lambda^2}}{\Lambda^2} = \lambda. \]

where \( \lambda = 0.225 \) is one of the parameters in the Wolfenstein parametrization. It is worth mentioning that the terms in the first and second lines of Eq. \cite{23} contribute to the masses of the down-type quarks and charged leptons, while the remaining terms give contributions to the up-type quark masses. We also assume \( Q^{(n)}_{1} \geq 1 \) and \( Q^{(X)}_{1} \geq 1 \). Therefore, the omitted terms in Eq. \cite{23} can be recovered by changing the \( A_4 \) singlet contractions in the terms of the first and second lines of Eq. \cite{23} and inserting the factors of \( \Delta/\Lambda \) and \( \Upsilon^2/\Lambda^2 \). Because of the preserved \( U(1)_f \) global symmetry, these insertions will generate an extra power of the form:

\[ \left( \frac{\Sigma_i^k \Sigma_j^k}{\Lambda^2} \right)^{Q^{(n)}_{1}} \text{ or } \left( \frac{\Sigma_i^k \Sigma_j^k}{\Lambda^2} \right)^{2Q^{(X)}_{1}}. \]

Therefore, the contributions to the down-type quark and charged lepton mass matrices due to these omitted terms will be suppressed by a factor which will be at most of the order of \( \Lambda^2 \sim 10^{-2} \) compared to the contributions coming from the first and second terms of Eq. \cite{23}. Note that in order to reproduce the non-trivial quark mixing consistent with experimental data, the up-type quark sector requires two \( 5' \)’s, i.e., \( H^{(3)}_1 \) and \( H^{(3)}_1 \), and two \( 1' \)’s, i.e., \( \Upsilon \) and \( \Delta \) irreps of \( SU(5) \) assigned to different \( A_4 \).
singlets. In the down-type quark sector, on the other hand, only one 5 irrep $H^{(1)}$ and three 1's, unified in the $A_4$ triplet $\xi = (\xi_1, \xi_2, \xi_3)$, are needed. As will be shown in the next sections the same set of irreps in the up-type quark sector would lead to the trivial Cabbibo-Kobayashi-Maskawa (CKM) mixing matrix.

III. LEPTON MASSES AND MIXING

The charged lepton mass matrix follows from Eq. (23) by using the product rules for the $A_4$ group given in Appendix A:

$$M_l = \frac{v_e}{\sqrt{3}} V_{lL}^\dagger\begin{pmatrix} \alpha_1 \kappa^{a_1} v_H^{(1)} - 6\beta_1 \kappa^{b_1} v_\Phi & 0 & 0 \\ 0 & \alpha_2 \kappa^{a_2} v_H^{(1)} - 6\beta_2 \kappa^{b_2} v_\Phi & 0 \\ 0 & 0 & \alpha_3 \kappa^{a_3} v_H^{(1)} - 6\beta_3 \kappa^{b_3} v_\Phi \end{pmatrix} = V_{lL}^\dagger \text{diag}(m_e, m_\mu, m_\tau),$$

(27)

with

$$V_{lL} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad \omega = e^{2\pi i/3}. \quad (28)$$

Since we assume that the dimensionless couplings $\alpha_i$ and $\beta_i$ ($i = 1, 2, 3$) are roughly of the same order of magnitude and we consider the VEVs $v_H^{(1)}$ and $v_\Phi$ of the order of the Electroweak scale $v \simeq 246$ GeV, the hierarchy among the charged lepton masses are explained by different combinations of $U(1)_f$ charges appearing in the Yukawa terms of Eq. (23).

Since the neutral components of the scalar fields $S_i$ have vanishing VEVs, the neutrino mass term does not appear at tree level, as in Ref. [26]. It arises at one-loop level in the form of a Majorana mass term

$$-\frac{1}{2} \bar{\nu} M_\nu \nu + \text{h.c.} \quad (29)$$

from radiative corrections involving the neutral components $H_i^0$ and $A_i^0$ of the $SU(2)$ doublet part of $S_i$ as well as the heavy Majorana neutrino $N_R$ running in the internal lines of the loops. The corresponding diagrams are shown in Fig. 1.

![Figure 1: One-loop Feynman diagrams contributing to the entries of the neutrino mass matrix.](image)

Due to the assumption $v_\chi >> v$, the quartic scalar interactions relevant for the computation of the neutrino mass matrix are given by the terms:

$$V(S, \chi) = \lambda_1^{(S\chi)} (S_i S_i^t)_1 (\chi \chi)_1 + \lambda_2^{(S\chi)} [(S_i S_i^t)_1 (\chi \chi)_1 + (S_i S_i^t)_1 (\chi \chi)_1] + \lambda_3^{(S\chi)} (S_i S_i^t)_{3s} (\chi \chi)_{3s} + \lambda_4^{(S\chi)} [e^{2\pi i/3} (S_i S_i^t)_{3s} (\chi \chi)_{3s} + \text{h.c}]. \quad (30)$$
Following Ref [29] we choose the quartic scalar couplings in the previous expression to be nearly universal, i.e.,

$$\lambda = \lambda^S \equiv \lambda^S_i = \lambda^S_i - \epsilon. \quad (31)$$

In practice, the coefficients need not be equal and indeed a non-zero \(\epsilon\) is required to generate two neutrino mass squared differences. In the approximation described above we obtain the one-loop neutrino mass matrix in the form

$$M_\nu \simeq \begin{pmatrix} A e^{2i\psi} & 0 & A \\ 0 & B & 0 \\ A & 0 & A e^{-2i\psi} \end{pmatrix}, \quad (32)$$

where:

$$A \simeq \frac{y_\nu^2}{16\pi^2 M_N} \left\{ \left( M_{A_1}^2 - M_{A_2}^2 + \frac{\epsilon v^2}{2} \right) \left[ D_0 \left( \frac{M_{H_i}^0}{M_N} \right) - D_0 \left( \frac{M_{A_1}^0}{M_N} \right) \right] \\ + \left( M_{A_3}^2 - M_{A_2}^2 + \frac{\epsilon v^2}{2} \right) \left[ D_0 \left( \frac{M_{A_2}^0}{M_N} \right) - D_0 \left( \frac{M_{A_3}^0}{M_N} \right) \right] \right\}, \quad (33)$$

$$B \simeq \frac{\epsilon y_\nu^2 v^2}{16\pi^2 M_N} \left[ D_0 \left( \frac{M_{H_i}^0}{M_N} \right) - D_0 \left( \frac{M_{A_2}^0}{M_N} \right) \right]. \quad (34)$$

$$\tan 2\psi \simeq \frac{1}{\sqrt{\frac{9}{4} \left( \frac{M_{A_3}^0 - M_{A_2}^0}{M_{A_2}^0 + M_{A_1}^0 - 2M_{A_0}^0} \right)^2 - 1}}. \quad (35)$$

Here \(M_{H_i}^0\) and \(M_{A_i}^0\) \((i = 1, 2, 3)\) are the masses of the CP even and CP odd neutral scalars contained in the \(SU(2)\) doublet component of the \(S_i\). We introduced the function [29]:

$$D_0(x) = \frac{-1 + x^2 - \ln x^2}{(1 - x^2)^2}. \quad (36)$$

The neutrino mass matrix given in Eq. (32) depends effectively only on three parameters: \(A\), \(B\), and \(\psi\). As shown in Eqs. (33), (34), the parameters \(A\) and \(B\) depend on various model parameters. It is important that \(A\) and \(B\) are loop-suppressed and approximately inversely proportional to \(M_N\). The Eqs. (33) and (34) also demonstrate that a non-vanishing mass splitting between the CP even \(H_1^0\) and CP odd \(A_1^0\) neutral scalars is crucial. Its absence would lead to massless neutrinos at one-loop level. Note also that universality in the quartic couplings of the scalar potential, which correspond to \(\epsilon = 0\), would imply \(B \sim 0\) and therefore lead to only one massive neutrino. For simplicity, we parametrize the non-universality of the relevant couplings through the parameter \(\varepsilon\), defined in Eq. (31). As will be shown below, the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix depends only on the parameter \(\psi\), while the neutrino mass squared splittings are controlled by the parameters \(A\) and \(B\).

A complex symmetric Majorana mass matrix \(M_\nu\), as in Eq. (29), can be diagonalized by a unitary rotation of the neutrino fields so that

$$\nu' = V_\nu \cdot \nu \quad \rightarrow \quad V_\nu^T M_\nu (V_\nu^T)^T = diag (m_{\nu_1}, m_{\nu_2}, m_{\nu_3}) \quad \text{with} \quad V_\nu V_\nu^T = 1,$$

where \(m_{1,2,3}\) are real and positive. The rotation matrix has the form

$$V_\nu = \begin{pmatrix} \cos \theta & 0 & \sin \theta e^{-i\phi} \\ 0 & 1 & 0 \\ -\sin \theta e^{i\phi} & 0 & \cos \theta \end{pmatrix} P_\nu, \quad \text{with} \quad P_\nu = diag \left( e^{i\alpha_1/2}, e^{i\alpha_2/2}, e^{i\alpha_3/2} \right), \quad \theta = \pm \frac{\pi}{4}, \quad \phi = -2\psi. \quad (37)$$

We identify the Majorana neutrino masses and Majorana phases \(\alpha_i\) for the two possible solutions with \(\theta = \pi/4, -\pi/4\) with NH and IH, respectively. They are

$$\text{NH : } \theta = +\frac{\pi}{4}: \quad m_{\nu_1} = 0, \quad m_{\nu_2} = B, \quad m_{\nu_3} = 2A, \quad \alpha_1 = \alpha_2 = 0, \quad \alpha_3 = \phi, \quad (38)$$

$$\text{IH : } \theta = -\frac{\pi}{4}: \quad m_{\nu_1} = 2A, \quad m_{\nu_2} = B, \quad m_{\nu_3} = 0, \quad \alpha_2 = \alpha_3 = 0, \quad \alpha_1 = -\phi. \quad (39)$$
Note that the non-vanishing Majorana phases are \( \phi \) and \(-\phi\) for NH and IH, respectively.

With the rotation matrices in the charged lepton sector \( V_{lL} \), given in Eq. (28), and in the neutrino sector \( V_\nu \), given in Eq. (37), we find the PMNS mixing matrix:

\[
U = V_{lL}^\dagger V_\nu \simeq \begin{pmatrix}
\frac{\cos \theta}{\sqrt{3}} - \frac{e^{i\phi} \sin \theta}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{\cos \theta}{\sqrt{3}} + \frac{e^{-i\phi} \sin \theta}{\sqrt{3}} \\
\frac{\cos \theta}{\sqrt{3}} - \frac{e^{i\phi} \sin \theta}{\sqrt{3}} & \frac{2\pi}{\sqrt{3}} & \frac{2\pi}{\sqrt{3}} \\
\frac{\cos \theta}{\sqrt{3}} - \frac{e^{i\phi} \sin \theta}{\sqrt{3}} & \frac{-2\pi}{\sqrt{3}} & \frac{-2\pi}{\sqrt{3}} + \frac{e^{-i\phi} \sin \theta}{\sqrt{3}}
\end{pmatrix} \mathcal{P}_\nu,
\]

(40)

It follows from the standard parametrization of the leptonic mixing matrix that the lepton mixing angles are [60]:

\[
\sin^2 \theta_{12} = \frac{|U_{e2}|^2}{1 - |U_{e3}|^2} = \frac{1}{2 + \cos \phi}, \quad \sin^2 \theta_{13} = |U_{e3}|^2 = \frac{1}{3} (1 \pm \cos \phi),
\]

(41)

\[
\sin^2 \theta_{23} = \frac{|U_{\mu3}|^2}{1 - |U_{e3}|^2} = \frac{2 \mp (\cos \phi + \sqrt{3} \sin \phi)}{4 \mp 2 \cos \phi},
\]

where the upper sign corresponds to NH \((\theta = +\pi/4)\) and the lower one to IH \((\theta = -\pi/4)\). The PMNS matrix (40) of our model reproduces the magnitudes of the corresponding matrix elements of the TBM ansatz [1] in the limit \(\phi = 0\) (IH) and \(\phi = \pi\) (NH) respectively. In both cases the special value for \(\phi\) implies that the physical neutral scalars \(H_0^i\) and \(A_0^i\) are degenerate in mass. Notice that the lepton mixing angles are solely controlled by the Majorana phases \(\pm \phi\), where the plus and minus signs again correspond to NH and IH, respectively.

The Jarlskog invariant \(J\) and the CP violating phase \(\delta\) are given by [60]:

\[
J = \text{Im} \left( U_{e1} U_{\mu2} U_{e2}^* U_{\mu1}^* \right) \simeq -\frac{1}{6\sqrt{3}} \cos 2\theta, \quad \sin \delta = \frac{8J}{\cos \theta_{13} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13}}.
\]

(42)

Since \(\theta = \pm \frac{\pi}{4}\), we predict \(J \simeq 0\) and \(\delta \simeq 0\) for \(v_x \gg v\), implying that in our model CP violation is suppressed in neutrino oscillations.

In the following we adjust the free parameters of our model to reproduce the experimental values given in the Tables I-II and discuss some implications of this choice of the parameters.

As seen from Eqs. (38), (39) and (40), (41) we have only three effective free parameters to fit: \(\phi\), \(A\) and \(B\). It is noteworthy that in our model a single parameter \(\phi\) determines all three neutrino mixing parameters \(\sin^2 \theta_{13}\), \(\sin^2 \theta_{12}\) and \(\sin^2 \theta_{23}\), as well as the Majorana phases \(\alpha_i\). The parameters \(A\) and \(B\) control the two mass squared splittings \(\Delta m^2_{21}\). Therefore we actually fit only \(\phi\) to adjust the values of \(\sin^2 \theta_{ij}\), while \(A\) and \(B\) for the NH and the IH hierarchies are simply

\[
\text{NH} : \quad m_{\nu_1} = 0, \quad m_{\nu_2} = B = \sqrt{\Delta m^2_{21}} \approx 9\text{meV}, \quad m_{\nu_3} = 2A = \sqrt{\Delta m^2_{31}} \approx 51\text{meV};
\]

\[
\text{IH} : \quad m_{\nu_2} = B = \sqrt{\Delta m^2_{21} + \Delta m^2_{13}} \approx 50\text{meV}, \quad m_{\nu_1} = 2A = \sqrt{\Delta m^2_{31}} \approx 49\text{meV}, \quad m_{\nu_3} = 0,
\]

(43)

(44)

as follows from Eqs. (38), (39) and the definition \(\Delta m^2_{ij} = m_i^2 - m_j^2\). In Eqs. (43), (44) we assumed the best fit values of \(\Delta m^2_{ij}\) from the Tables I-II. In Eqs. (43), (44) we have fitted the \(\sin^2 \theta_{ij}\) to the experimental values in Tables I-II. The best fit result is:

\[
\text{NH} : \quad \phi = -0.877\pi, \quad \sin^2 \theta_{12} \approx 0.34, \quad \sin^2 \theta_{23} \approx 0.61, \quad \sin^2 \theta_{13} \approx 0.0246;
\]

\[
\text{IH} : \quad \phi = 0.12\pi, \quad \sin^2 \theta_{12} \approx 0.34, \quad \sin^2 \theta_{23} \approx 0.6, \quad \sin^2 \theta_{13} \approx 0.025.
\]

(45)

(46)

Comparing Eqs. (45), (46) with Tables I-II we see that \(\sin^2 \theta_{13}\) and \(\sin^2 \theta_{23}\) are in excellent agreement with the experimental data, for both NH and IH, with \(\sin^2 \theta_{12}\) within a 2\(\sigma\) deviation from its best fit values. It has been shown in Ref. [20] that the solution in Eqs. (45), (46) does imply neither fine-tuning nor very large values of dimensionful parameters.
With the values of the model parameters given in Eqs. (43)-[46], derived from the oscillation experiments, we can predict the amplitude for neutrinoless double beta (0νββ) decay, which is proportional to the effective Majorana neutrino mass

\[ m_{\beta\beta} = \sum_j u_{ej}^2 m_{\nu_j}, \]  

where \( u_{ej} \) and \( m_{\nu_j} \) are the PMNS mixing matrix elements and the Majorana neutrino masses, respectively. Then, from Eqs. (37)-(40) and (43)-(46), we predict the following effective neutrino masses for both hierarchies:

\[ m_{\beta\beta} = \frac{1}{3} \left( B + 4A \cos^2 \frac{\phi}{2} \right) = \begin{cases} 
4 \text{ meV} & \text{for NH} \\
50 \text{ meV} & \text{for IH} 
\end{cases} \]  

This is beyond the reach of the present and forthcoming 0νββ decay experiments. The presently best upper limit on this parameter \( m_{\beta\beta} \leq 160 \text{ meV} \) comes from the recently quoted EXO-200 experiment \[ T_{1/2}^{0\nu\beta\beta} (^{136}\text{Xe}) \geq 1.6 \times 10^{25} \text{ yr at 90 \% CL}. \] This limit will be improved within a not too distant future. The GERDA experiment \[ \text{[62, 63]} \] is currently moving to “phase-II”, at the end of which it is expected to reach \( T_{1/2}^{0\nu\beta\beta} (^{76}\text{Ge}) \geq 2 \times 10^{26} \text{ yr, corresponding to} \) \( m_{\beta\beta} \leq 100 \text{ meV}. \) A bolometric CUORE experiment, using \( ^{130}\text{Te} [64] \), is currently under construction. Its estimated sensitivity is around \( T_{1/2}^{0\nu\beta\beta} (^{130}\text{Te}) \sim 10^{26} \text{ yr corresponding to} \) \( m_{\beta\beta} \sim 50 \text{ meV}. \) There are also proposals for ton-scale next-to-next generation 0νββ experiments with \( ^{136}\text{Xe} [65, 66] \text{ and} \) \( ^{76}\text{Ge} [62, 67] \text{ claiming sensitivities over} \) \( T_{1/2}^{0\nu\beta\beta} \sim 10^{27} \text{ yr, corresponding to} \) \( m_{\beta\beta} \sim 12 - 30 \text{ meV}. \) For recent experimental reviews, see for example Ref. \[ \text{[68]} \text{ and references therein. Thus, according to Eq. [48] our model predicts} \) \( T_{1/2}^{0\nu\beta\beta} \text{ at the level of sensitivities of the next generation or next-to-next generation 0\nu\beta\beta experiments.} \]

IV. QUARK MASSES AND MIXING

Using Eq. [23] and the product rules for the \( A_4 \) group listed in Appendix \[ \text{A} \text{ we find the mass matrices for up- and down-type quarks in the form:} \]

\[ M_U = \begin{pmatrix} C & F & G \\ F & D & H \\ G & H & E \end{pmatrix}, \]  
\[ M_D = \frac{v_X}{3\Lambda} \begin{pmatrix} \alpha_1 K^{x_1} v_H^{(1)} + 2\beta_1 K^{b_1} v_\Phi \\ 0 \\ 0 \end{pmatrix}, \]

\[ = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \]

\[ = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \]

\[ \alpha_3 K^{x_3} v_H^{(1)} + 2\beta_3 K^{b_3} v_\Phi \]  

where:

\[ C = 8 \gamma_{11} \left( \kappa^{x_{11}} \frac{v_X^2}{\Lambda^2} + \zeta_1 \kappa^{y_{11}} \frac{v_X^2}{\Lambda^2} \right) v_H^{(2)}, \]  

\[ F = 4 \left( \gamma_{12} + \gamma_{21} \right) \kappa^{x_{12}} \frac{v_X}{\Lambda} v_H^{(2)}, \]  

\[ D = 8 \gamma_{22} \kappa^{x_{22}} v_H^{(2)}, \]  

\[ G = 4 \left[ \gamma_{13} \kappa^{x_{13}} \frac{v_Y}{\Lambda} + \zeta_2 \gamma_{31} \kappa^{y_{31}} \frac{v_Y^2}{\Lambda^2} + \gamma_{31} \kappa^{y_{31}} \frac{v_Y^2}{\Lambda^2} \right] v_H^{(3)}, \]  

\[ H = 4 \left( \gamma_{23} \kappa^{x_{23}} \frac{v_Y}{\Lambda} + \gamma_{32} \kappa^{x_{32}} \frac{v_X}{\Lambda} \right) v_H^{(3)}, \]  

\[ E = 8 \gamma_{33} \kappa^{x_{33}} v_H^{(3)}. \]  

In analogy to the leptonic sector we assume that the dimensionless couplings \( \alpha_i, \beta_i, \gamma_{ij} (i, j = 1, 2, 3) \text{ and} \) \( \zeta_i \) are roughly of the same order of magnitude, with the VEVs \( v_H^{(f)} (f = 1, 2, 3) \) and \( v_\Phi \) being at the Electroweak scale \( v \simeq 246 \text{ GeV}. \)
Then, the hierarchy among the quark masses can be explained by different combinations of $U(1)$ charges shown in Eq. (24).

The well known hierarchy among the down type quark masses is approximately described by:

$$m_d : m_s : m_b \approx \lambda^4 : \lambda^2 : 1.$$  

(52)

with $m_b \approx \lambda^3 m_t$.

To fulfill the above hierarchy, we set:

$$a_1 = b_1 = 6, \quad a_2 = b_2 = 4, \quad a_3 = b_3 = 2, \quad v_{H}^{(1)} \sim v \sim \frac{v}{\sqrt{2}}.$$  

(53)

Here we have taken into account our previous assumption $v_\xi = \lambda \Lambda$ where $\lambda = 0.225$ (see Eqs. (19), (25)).

Assuming that the hierarchy of charged fermion masses and quark mixing matrix elements are explained by the Froggatt-Nielsen mechanism we adopt an approximate universality of the dimensionless Yukawa couplings in Eq. (23). Specifically, we set:

$$\beta_1 = \beta_3 = -\beta_2,$$  

(54)

so that the down type quark and charged lepton masses will be determined by four dimensionless parameters, i.e, $\alpha_1$, $\alpha_2$, $\alpha_3$ and $\beta_1$. We fit these parameters to reproduce the experimental values of the down type quarks and charged leptons. The results are shown in Table III for the following best fit values of the model parameters:

$$\alpha_1 = 1.36, \quad \alpha_2 = 2.06, \quad \alpha_3 = 3.77, \quad \beta_1 = 0.18.$$  

(55)

| Observable | Model Value | Experimental Value |
|------------|-------------|--------------------|
| $m_d$ (MeV) | 2.91        | $2.9^{+0.5}_{-0.4}$ |
| $m_s$ (MeV) | 57.1        | $57.7^{+16.8}_{-15.7}$ |
| $m_b$ (GeV) | 2.73        | $2.82^{+0.09}_{-0.04}$ |
| $m_e$ (MeV) | 0.487       | 0.487               |
| $m_\mu$ (MeV) | 102.8       | $102.8 \pm 0.0003$ |
| $m_\tau$ (GeV) | 1.75        | $1.75 \pm 0.0003$ |

Table III: Model and experimental values of the down type quark and charged lepton masses (at the $M_Z$ scale).

As customary, we use the quark and charged lepton masses evaluated at the $M_Z$ scale [69]. As seen from Table III there is good agreement of the model values for these masses with the experimental ones.

The CKM quark mixing matrix is defined as [60]:

$$K = R_D^U R_D = \begin{pmatrix} K_{ud} & K_{us} & K_{ub} \\ K_{cd} & K_{cs} & K_{cb} \\ K_{td} & K_{ts} & K_{tb} \end{pmatrix},$$  

(56)

where the rotation matrices $R_D$ and $R_U$ are derived from

$$R_D^U M_U M_D^U R_U = diag \left( m_u^2, m_c^2, m_t^2 \right), \quad R_D^U M_D M_D^U R_D = diag \left( m_d^2, m_s^2, m_b^2 \right).$$  

(57)

From Eq. (56) it follows that

$$M_D M_D^U = diag \left( m_d^2, m_s^2, m_b^2 \right).$$  

(58)

Thus $R_D = 1_{3 \times 3}$ and, consequently, the CKM quark mixing matrix does not receive contributions from the down-type quark sector, meaning that quark mixing arises solely from the up-type quark sector. Thus, the CKM matrix satisfies the following relation:

$$M_U M_U^U = K^\dagger diag \left( m_u^2, m_c^2, m_t^2 \right) K.$$  

(59)
Now we determine relations between the entries of the up-type quark mass matrix, following from the observed pattern of up-type quark masses and mixings. As the first order approximation we decouple the top quark from the up and charm quarks, so that the mixing between the top quark and the light up-type quarks can be neglected. In this approximation, the Cabbibo angle as well as the masses of the up-type quarks can be successfully reproduced as $|D| \approx m_c$, $|F| \approx \lambda m_c$, $|C| \approx (x \lambda^4 + \lambda^2)m_c$ and $|E| = m_t$, where $x$ is an $O(1)$ parameter and $\lambda = 0.225$. When the mixing between the top quark and the light up and charm quarks is considered, one has to reproduce all the CKM matrix elements. In our model it is fully determined by the rotation matrix of the up-type quark sector.

It is well known that the mixing angles $\theta_{13}$ and $\theta_{23}$ of the standard parametrization of the CKM matrix satisfy $\sin \theta_{13} \approx \lambda^3$ and $\sin \theta_{23} \approx \lambda^2$. Therefore, the realistic pattern of the quark masses and mixings implies that the entries of the up-type quark mass matrix should satisfy:

$$
|C| \approx \lambda |F| \approx \lambda^2 |D|, \quad |G| \approx \lambda |H| \approx \lambda^3 |E|, \quad |D| \approx \lambda^4 |E|, \quad |E| \approx m_t.
$$

(60)

To fulfill these relations, we set:

$$
x_{11} = x_{12} = x_{22} = 4, \quad x_{13} = 3, \quad x_{31} = y_2 = x_{23} = x_{32} = 1, \quad y_1 = 6, \quad x_{33} = 0, \quad v^{(2)}_H \sim v^{(3)}_H \sim \frac{v}{\sqrt{2}}, \quad \zeta_1 \sim \zeta_2 \sim 1, \quad |\gamma_{ij}| \sim \frac{1}{8}, \quad i, j = 1, 2, 3.
$$

(61)

Recall that $\Lambda = \lambda^{-1} \Lambda_{GUT}$, $\kappa = \lambda$, $v_\Delta = v_T = \Lambda_{GUT}$ (see Eqs. (49), (25)). Furthermore, in order to obtain a realistic quark mixing, we require that the $(2, 2)$, $(3, 3)$, $(2, 3)$ and $(3, 2)$ entries of the up-type quark mass matrix are the only complex entries with the same complex phase.

Hence, the mass matrix for up-type quarks takes the following form:

$$
M_U = m_t \begin{pmatrix}
(y + z \lambda^2)\lambda^6 & f \lambda^5 & b \lambda^3 \\
\frac{b}{f} \lambda^5 & a \lambda^4 e^{i \sigma} & c \lambda^2 e^{i \sigma} \\
b \lambda^3 & c \lambda^2 e^{i \sigma} & d \lambda^2
\end{pmatrix},
$$

(62)

where $y, z, a, b, c, d$ and $f$ are $O(1)$ parameters and we have taken into account that $m_c \approx \lambda^4 m_t$.

Therefore, the mass matrix for up-type quarks satisfies the following relation:

$$
M_U M_U^\dagger = m_t^2 \begin{pmatrix}
b^2 \lambda^6 & b c \lambda^5 & b d \lambda^3 e^{-i \sigma} \\
\frac{b}{c} \lambda^5 & c^2 \lambda^4 & c d \lambda^2 \\
b d \lambda^3 & c d \lambda^2 & d^2
\end{pmatrix} + m_t^2 \begin{pmatrix}
X & S & T \\
S^\dagger & Y & U \\
T^\dagger & U & Z
\end{pmatrix},
$$

(63)

where:

$$
X = \lambda^6 \left( f^2 \lambda^4 + y^2 \lambda^6 + 2 y z \lambda^8 + z^2 \lambda^{10} \right), \\
Y = a^2 \lambda^8 + f^2 \lambda^{10}, \\
Z = c^2 \lambda^4 + b^2 \lambda^6, \\
S = a f \lambda^9 e^{-i \sigma} + f y \lambda^{11} + f z \lambda^{13}, \\
T = c f \lambda^7 e^{-i \sigma} + b y \lambda^{11} + b z \lambda^{11}, \\
U = a c \lambda^6 + b f \lambda^8.
$$

(64)

Notice that the first term in Eq. (63) gives the leading contribution to $M_U M_U^\dagger$, while the second one is crucial to generate the up and charm quark masses.

From Eq. (59) it follows that:

$$
M_U M_U^\dagger \simeq m_t^2 \begin{pmatrix}
|K_{td}|^2 & K_{td}^\dagger K_{ts} & K_{td}^\dagger K_{tb} \\
K_{td} K_{td}^\dagger & |K_{ts}|^2 & K_{ts}^\dagger K_{tb} \\
K_{tb}^\dagger K_{td} & K_{ts}^\dagger K_{ts} & |K_{tb}|^2
\end{pmatrix}.
$$

(65)

Therefore we can write down

$$
M_U M_U^\dagger \simeq m_t^2 \begin{pmatrix}
W^2 \lambda^6 \left[ y^2 + (\rho - 1)^2 \right] & W^2 \lambda^5 (-i \eta + \rho - 1) & W \lambda^3 (i \eta - \rho + 1) \\
W^2 \lambda^5 (i \eta + \rho - 1) & W^2 \lambda^4 & -W \lambda^2 \\
W \lambda^3 (-i \eta - \rho + 1) & -W \lambda^2 & 1
\end{pmatrix}
$$

(66)
using the Wolfenstein parameterization of the CKM matrix \cite{60,70}:

\[
K \simeq \begin{pmatrix}
1 - \frac{\lambda^2}{2} & \frac{\lambda}{2} W \lambda^3 (\rho - i \eta) \\
-\lambda & 1 - \frac{\lambda^2}{2} W \lambda^2 \\
W \lambda^3 (1 - \rho - i \eta) & -W \lambda^2
\end{pmatrix},
\]

(67)

with the Wolfenstein parameters given by \cite{60}:

\[
\lambda = 0.22535 \pm 0.00065, \quad W = 0.811^{+0.022}_{-0.012}, \quad \overline{\rho} = 0.131^{+0.026}_{-0.013}, \quad \overline{\eta} = 0.345^{+0.013}_{-0.014}
\]

(68)

\[
\overline{\rho} \simeq \rho \left(1 - \frac{\lambda^2}{2}\right), \quad \overline{\eta} \simeq \eta \left(1 - \frac{\lambda^2}{2}\right).
\]

(69)

Comparing Eqs. (63) and (66) we find the following relations:

\[
b \simeq W \sqrt{\eta^2 + (\rho - 1)^2}, \quad c \simeq -W, \quad d \simeq 1, \quad \sigma \simeq \arctan \left(-\frac{\eta}{1 - \rho}\right).
\]

(70)

Since \(d \simeq 1\), it follows from the previous relations that the quark mixing in our model is described by four effective dimensionless parameters, i.e., \(b, c, \sigma\) and \(\lambda\). The \(\lambda\) parameter in the Wolfenstein parametrization is fixed by the ratio between the Grand Unification scale \(\Lambda_{GUT}\) and the cutoff \(\Lambda\) of our model.

For the order \(O(1)\) parameters in Eq. (62) it is natural to set \(f = y = z\) so that the light quark masses are determined by two dimensionless parameters, i.e., \(a\) and \(z\), while the top quark mass is predicted in our model since \(d \simeq 1\) and \(v_3^{(3)} \sim \frac{v}{\sqrt{2}}\). We fit these parameters to reproduce the up-type quark mass spectrum. The results are shown Table IV for the following best fit values

\[
z = \frac{1}{2}, \quad a = 1.85.
\]

(71)

The CKM matrix in our model is approximately equal to the Wolfenstein parametrization, as seen Eq. (68), and thus consistent with the experimental data. The agreement of our model with the experimental data is as good as in the models of Refs. \cite{71,76} and better than, for example, those in Refs. \cite{77,83}.

The Jarlskog invariant \(J\) and the CP violating phase \(\delta\), respectively, are given by \cite{84,85}:

\[
J = \text{Im} \left[ K_{us} K_{cb} K_{ub}^* K_{cs}^* \right] = A^2 \eta \lambda^6 - \frac{1}{2} A^2 \eta \lambda^8, \quad \delta = \arctan \left(\frac{\eta}{\rho}\right).
\]

(72)

The values of these observables as well as the up-type quark masses are juxtaposed together with the experimental data in Table IV. The experimental values of the quark masses, which are given at the \(M_Z\) scale, have been taken from Ref. \cite{69}, whereas the experimental values of the CKM matrix elements and the Jarlskog invariant \(J\) are taken from Ref. \cite{68}. As seen from Table IV, most of the analyzed physical parameters are in rather good agreement with the experimental data, except for \(m_u, V_{ub}\) and \(V_{cd}\), which reproduce the corresponding experimental values only with order of magnitude accuracy.

V. CONCLUSIONS

We proposed a model based on the group \(SU(5) \otimes A_4 \otimes Z_2 \otimes Z'_2 \otimes Z''_2 \otimes U(1)_f\), which is an extension of the model of Ref. \cite{20}. The model has in total 13 effective free parameters, which allowed us to reproduce 18 observables, i.e., 9 charged fermion masses, 2 neutrino mass squared splittings, 3 lepton mixing parameters and the 4 parameters of the Wolfenstein parametrization of the CKM quark mixing matrix. The observed hierarchy of the charged fermion masses arises from a generalized Froggatt-Nielsen mechanism where the charged fermions get masses via non-renormalizable operators invariant under the gauge and flavor symmetries. It is triggered by a scalar field \(\Sigma\) in the \(24\) representation of \(SU(5)\) charged under the global \(U(1)_f\) symmetry and acquiring a VEV at the GUT scale. Thus, the hierarchy of the charged fermion masses and the quark mixing matrix elements arises as a consequence of the power dependence of the charged fermion mass matrix elements on particular combinations of the \(U(1)_f\) charges.
The predicted values of the effective Majorana neutrino mass $m_{\beta\beta}$ for $0\nu\beta\beta$ decay are 4 meV and 50 meV for the normal and the inverted neutrino spectrum, respectively.

An unbroken $Z_2$ discrete symmetry of our model also allows for stable dark matter candidates, as in Refs. [45, 46]. They could be either the lightest neutral component of the $SU(5)$ 5-plet $S_i$ or the right-handed Majorana neutrino $N_R$. We do not address this subject in the present paper.

**Acknowledgments**

A.E.C.H. was partially supported by Fondecyt (Chile), Grants No. 11130115, 1100582 and 1140390. E.S. was supported by DFG CONICYT Grant No. PA 803/7-1. E.S. acknowledges hospitality at UTFSM during part of this collaboration.

**Appendix A: The product rules for $A_4$**

The following product rules for the $A_4$ group were used in the construction of our model Lagrangian:

$\mathbf{3} \otimes \mathbf{3} = \mathbf{3}_s \oplus \mathbf{3}_a \oplus \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}''$, \hspace{1cm} (A1)

$1 \otimes 1 = 1$, \hspace{1cm} $1' \otimes 1'' = 1$, \hspace{1cm} $1' \otimes 1' = 1''$, \hspace{1cm} $1'' \otimes 1'' = 1'$, \hspace{1cm} (A2)

Denoting $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ as the basis vectors for two $A_4$-triplets $\mathbf{3}$, one finds:

$$(\mathbf{3} \otimes \mathbf{3})_1 = x_1 y_1 + x_2 y_2 + x_3 y_3,$$

$$(\mathbf{3} \otimes \mathbf{3})_{3_s} = (x_2 y_3 + x_3 y_2, x_3 y_1 + x_1 y_3, x_1 y_2 + x_2 y_1),$$

$$(\mathbf{3} \otimes \mathbf{3})_{3_a} = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1),$$

$$(\mathbf{3} \otimes \mathbf{3})_{1'} = x_1 y_1 + \omega x_2 y_2 + \omega^2 x_3 y_3,$$

$$(\mathbf{3} \otimes \mathbf{3})_{1''} = x_1 y_1 + \omega^2 x_2 y_2 + \omega x_3 y_3,$$
where $\omega = e^{i2\pi a}$. The representation $1$ is trivial, while the non-trivial $1'$ and $1''$ are complex conjugate to each other. Comprehensive reviews of discrete symmetries in particle physics can be found in Refs. [24, 86–88].
[54] Y. H. Ahn and S. K. Kang, Phys. Rev. D 86 093003 (2012) [arXiv:1203.4185 [hep-ph]].

[55] R. N. Mohapatra and C. C. Nishi, Phys. Rev. D 86 073007 (2012) [arXiv:1208.2875 [hep-ph]].

[56] M. -C. Chen, J. Huang, J. -M. O'Bryan, A. M. Wijangco and F. Yu, JHEP 1302 021 (2013) [arXiv:1210.6982 [hep-ph]].

[57] G. Aad et al. [ATLAS Collaboration], Phys. Lett. B 716, 1 (2012); S. Chatrchyan et al. [CMS Collaboration], Phys. Lett. B 716, 30 (2012).

[58] S. Khalil, S. Salem and M. Allam, arXiv:1401.1482 [hep-ph].

[59] A. E. C´arcamo Hern´andez, R. Mart´ınez and F. Ochoa, Phys. Rev. D 87 075009 (2013) [arXiv:1302.1757 [hep-ph]].

[60] J. Beringer et al. (Particle Data Group), Phys. Rev. D 86 010001 (2012).

[61] EXO Collaboration, M. Auger et al., Phys.Rev.Lett. 109, 032505 (2012), arXiv:1205.5608.

[62] GERDA Collaboration, K.-H. Ackermann et al., (2012), arXiv:1212.4067.

[63] GERDA Collaboration, I. Abt et al., (2004), arXiv:hep-ex/0404039.

[64] F. Alessandria et al., (2011), arXiv:1109.0494.

[65] KamLAND-Zen Collaboration, A. Gando et al., Phys.Rev. C85, 045504 (2012), arXiv:1201.4664.

[66] EXO-200 Collaboration, D. Auty, Recontres de Moriond, http://moriond.in2p3.fr/ (2013).

[67] Majorana Collaboration, C. Aalseth et al., Nucl.Phys.Proc.Suppl. 217, 44 (2011), arXiv:1101.0119.

[68] A. S. Barabash, arXiv:1209.4241 [nucl-ex].

[69] K. Bora, arXiv:1206.3909 [hep-ph].

[70] L. Wolfenstein, Phys. Rev. Lett. 51, 1945 (1983).

[71] S. F. King, S. Morisi, E. Peinado and J. W. F. Valle, Phys. Lett. B 724, 68 (2013) [arXiv:1301.7065 [hep-ph]].

[72] A. E. C´arcamo Hern´andez and R. Rahman, arXiv:1007.0447 [hep-ph].

[73] H. Fritzsch and J. Planck, Phys. Lett. B 237, 451 (1990).

[74] A. E. C´arcamo Hern´andez and I. de Medeiros Varzielas, to appear.

[75] H. Fritzsch and Z. z. Xing, Phys. Rev. D 48, 2349 (1993); G. Altarelli and F. Feruglio, Rev. Mod. Phys. 82 2701 (2010) [arXiv:1002.0211 [hep-ph]].

[76] P. Ramond, Group Theory: A Physicist’s Survey, Cambridge University Press, UK (2010).