Abstract. We compute the number of irreducible rational curves of given degree with 1 tacnode or 1 triple point in $\mathbb{P}^2$ or 1 node in $\mathbb{P}^3$ meeting an appropriate generic collection of points and lines. As a byproduct, we also compute the number of rational plane curves of degree $d$ passing through $3d - 2$ given points and tangent to a given line. The method is 'classical', free of Quantum Cohomology.

In the past 15 years or so a number of classical problems in the enumerative geometry of curves in $\mathbb{P}^n$ were solved, first for $n = 2$, any genus [R1], then for any $n$, genus 0. The latter development was initiated by Kontsevich-Manin who developed and used the rather substantial machinery of Quantum Cohomology (cf. e.g. [FP]). Subsequently, in a series of papers [R2-R5] the author developed an elementary alternative approach, free of Quantum cohomology, and used it to solve a number of classical enumerative problems for rational, and sometimes elliptic, curves in $\mathbb{P}^n, n \geq 2$. The present paper continues this series. The object here is to enumerate the irreducible rational curves of given degree $d$ in $\mathbb{P}^2$ with one tacnode or one triple point passing through $3d - 2$ general points (see Theorems 2,5 below), as well as the irreducible rational curves in $\mathbb{P}^3$ with one ordinary node which contain a general points and are incident to $4d - 2a - 1$ general lines (see Thm 1 below). Note that the family of 1-tacnodal (resp. 1-triple point, resp. 1-nodal) curves in $\mathbb{P}^2$ (resp. $\mathbb{P}^2, \mathbb{P}^3$) is of codimension 1 in the family of all rational curves so that we are effectively computing the 'degree', in a sense, of certain natural divisors in the family of all rational curves. Indeed by a result of Diaz and Harris [DH] in the case of $\mathbb{P}^2$, the general member

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of any such divisor, if not nodal, is either 1-cuspidal (which case was enumerated in [R2]) or 1-tacnodal or has 1 triple point. It seems very likely, but doesn’t seem to be in the literature, that the natural analogue of this result also holds for any \( n \geq 3 \): i.e. that the general member of any divisor in the family of rational curves is either smooth, or 1-nodal reducible (any \( n \)), or 1-nodal irreducible (\( n=3 \)). As a byproduct of the proof of Theorem 2, we also obtain a formula for one of the ’characteristic numbers’ for rational plane curves, viz. the number of such curves of degree \( d \) passing through \( 3d – 2 \) general points and tangent to a given line (cf. Cor. 4 below).

Enumeration of rational plane curves with a tacnode or triple point appears to be well within the range of interest, at least, of classical geometers. For quartics, results of this kind, at least the enumeration of quartics with a triple point, were obtained by Zeuthen in 1882. Recently, enumerative results of this kind for curves with ‘few’ singularities on general surfaces were obtained by Kleiman and Piene [KP]. Enumeration of 1-nodal rational curves in \( \mathbb{P}^n \) for any \( n \) was recently announced by Zinger [Z], using Quantum Cohomology. As we shall see below, the 1-nodal case in \( \mathbb{P}^3 \) case is analogous to, but easier than the 1-tacnodal and 1 triple-point cases in \( \mathbb{P}^2 \).

Our proof is based on the intersection calculus on the nonsingular model of the surface swept out by the appropriate 1-parameter family of rational curves, developed in earlier papers [R2-R4], together with elementary residual-intersection (in particular, double-point) theory as in [F]

We begin by reviewing some qualitative results about families of rational curves in \( \mathbb{P}^n \), especially for \( n = 2 \) or \( 3 \). See [R2][R3] [R4] and references therein for details and proofs. In what follows we denote by \( \bar{V}_d \) the closure in the Chow variety of the locus of irreducible nonsingular rational curves of degree \( d \) in \( \mathbb{P}^n, n = 3 \), with the scheme structure as closure, i.e. the reduced structure (recall that the Chow form of a reduced 1-cycle \( Z \) is just the hypersurface in \( G(1, \mathbb{P}^3) \) consisting of all linear spaces meeting \( Z \)); if \( n = 2 \) \( \bar{V}_d \) is just the (closure of) the Severi variety. Thus \( \bar{V}_d \) is irreducible reduced of dimension

\[
\dim(\bar{V}_d) = \begin{cases} 
4d, \quad n = 3 \\
3d - 1, \quad n = 2.
\end{cases}
\]

Let

\[A_1, \ldots, A_k \subset \mathbb{P}^n\]

be a generic collection of linear subspaces of respective codimensions \( a_1, \ldots, a_k \), \( 2 \leq a_i \leq n \) (so if \( n = 2 \) these are just points). We denote by

\[B = B_d = B_d(a_\cdot) = B_d(A_\cdot)\]

the normalization of the locus (with reduced structure)

\[
\{(C, P_1, \ldots, P_k) : C \in \bar{V}_d, P_i \in C \cap A_i, i = 1, \ldots, k\}
\]
which is also the normalization of its projection to $\tilde{V}_d$, i.e. the locus of degree-$d$ rational curves (and their specializations) meeting $A_1, \ldots, A_k$. We have

$$\dim B = (n + 1)d + n - 3 - \sum (a_i - 1).$$

When $\dim B = 0$ we set

$$N_d(a.) = \deg(B).$$

When $n = 2$, so all the $a_i = 2$ they will be dropped. The integer $k$ is called the length of the condition-vector $(a.)$. The numbers $N_d$ and $N_d(a.)$, first computed in general by Kontsevich and Manin (see for instance [FP] and references therein), are computed in [R2],[R3] by an elementary method based on recursion on $d$ and $k$.

Now suppose $\dim B = 1$ and let

$$\pi : X \to B$$

be the normalization of the tautological family of rational curves, and

$$f : X \to \mathbb{P}^n$$

the natural map. The following summarizes results from [R2][R3][R4] :

**Theorem 0.** (i) $X$ is smooth .

(ii) Each fibre $C$ of $\pi$ is either

(a) a $\mathbb{P}^1$ on which $f$ is either an immersion with at most one exception which maps to a cusp ($n = 2$) or an embedding ($n > 2$); or

(b) a pair of $\mathbb{P}^1$’s meeting transversely once, on which $f$ is an immersion with nodal image ($n = 2$) or an embedding ($n > 2$); or

(c) if $n = 3$, a $\mathbb{P}^1$ on which $f$ is a degree-1 immersion such that $f(\mathbb{P}^1)$ has a unique singular point which is an ordinary node.

(iii) If $n > 2$ then $\tilde{V}_{d,n}$ is smooth along the image $\tilde{B}$ of $B$, and $\tilde{B}$ is smooth except, in case some $a_i = 2$, for ordinary nodes corresponding to curves meeting some $A_i$ of codimension 2 twice. If $n = 2$ then $\tilde{V}_{d,n}$ is smooth in codimension 1 except for a cusp along the cuspidal locus and normal crossings along the reducible locus, and $\tilde{B}$ has the singularities induced from $\tilde{V}_{d,n}$ plus ordinary nodes corresponding to curves with a node at some $A_i$, and no other singularities.

Next, we review some of the enumerative apparatus introduced in [R3][R4] to study $X/B$. Set

$$m_i = m_i(a.) = -s_i^2, i = 1, \ldots, k.$$
Note that if \( a_i = a_j \) then \( m_i = m_j \); in particular for \( n = 2 \) they are all equal and will be denoted by \( m_d \). It is shown in [R2] [R3][R4] that these numbers can all be computed recursively in terms of data of lower degree \( d \) and lower length \( k \). For instance for \( n = 2 \) we have

\[
2m_d = \sum_{d_1 + d_2 = d} N_d N_d_1 d_1 d_2 \left( \frac{3d - 4}{3d_1 - 2} \right).
\]

For \( n > 2 \), note that

\[
s_i s_j = N_d(\ldots, a_i + a_j, \ldots, \hat{a}_j, \ldots), i \neq j.
\]

(so for \( n = 2 \) this is always 0). Also, letting \( R_i \) denote the sum of all fibre components not meeting \( s_i \), we have

\[
s_1 \cdot R_2 = \sum N_{d_1}(A_1, A_1, \mathbb{P}^{s_1}) N_{d_2}(A_2, A_2, \mathbb{P}^{s_2}).
\]

the summations being over all \( d_1 + d_2 = d, s_1 + s_2 = 3 \) and all decompositions \( A. = (A_1, A_2) \prod(A_1) \prod(A_2) \) (as unordered sequences or partitions); similarly for the other \( s_i R_j \). So all these numbers may be considered known. Then we have

\[
m_i = \frac{1}{2} (s_i R_j + s_i R_p - s_j R_p) - s_i s_j - s_i s_p + s_j s_p
\]

for any distinct \( i, j, p \), and the RHS here is an expression of lower degree and/or length, hence may be considered known.

Next, set

\[
L = f^*(\mathcal{O}(1)),
\]

and note that

\[
L^2 = N_d(2, a.), L s_i = N_d(a_1, \ldots a_{i+1} \ldots), i = 1, \ldots, k
\]

(in particular, \( L s_i = 0 \) if \( a_i = n. \) We computed in [R3] that, for any \( i \),

\[
L \sim d s_i - \sum_{F \in \mathcal{F}_i} \deg(F) F + \left( N_d(a_1, \ldots, a_i + 1, \ldots) + d m_i \right) F_0
\]

where \( F_0 \) is the class of a complete fibre and \( \mathcal{F}_i \) is the set of fibre components not meeting \( s_i \). Consequently we have

\[
N_d(2, a_1, \ldots) = 2d N_d(a_1 + 1, a_2, \ldots) + d^2 m_1(a.) - \sum_{F \in \mathcal{F}_1(a.)} (\deg F)^2
\]


and clearly the RHS is a lower degree/length expression, so all the $N_d(2, \ldots)$ are known. We also have for $n > 2$ that

$$N_d(a_1, a_2 + 1, \ldots) - N_d(a_1 + 1, a_2, \ldots) =$$

(8) \[dN_d(a_1 + a_2, \ldots) - \sum_{F \in (F_1 - F_2)(a.)} (\text{deg } F) + N_d(a_1 + 1, a_2, \ldots) + dm_1(a.)\]

and again the RHS here is 'known', hence so is the LHS, which allows us to 'shift weight' between the $a_i$'s till one of them becomes 2, so we may apply (7), and thus compute all of the $N_d(a.)$'s.

Next, it is easy to see as in [R3] that

(9) \[L.R_i = \sum_{d_1 + d_2 = d} \left(\frac{3d - 1}{3d_1 - 1}\right)d_1d_2N_{d_1}N_{d_2}, \quad n = 2\]

(in this case this is independent of $i$ and we will just write it as $L.R$);

(10) \[L.R_i = \sum d_2N_{d_1}(a^1, s_1)N_{d_2}(a^2, s_2), \quad n > 2\]

the summation for $n > 2$ being over all $d_1 + d_2 = d, s_1 + s_2 = 3$ and all decompositions

$$A. = (A^1) \coprod (A^2)$$

(as unordered sequences or partitions) such that $A_i \in (A^1)$.

Finally, the relative canonical class $K_{X/B} = K_X - \pi^*(K_B)$ was computed in [R3] as

(11) \[K_{X/B} = -2s_i - m_iF + R_i\]

for any $i$. Note that $-R_i^2$ equals the number of reducible fibres in the family $X/B$, a number we denote by $N^\text{red}_d(a.)$, and which is easily computable by recursion. From this we compute easily that

(12) \[L.K_{X/B} = -2N_d(...a_i + 1...) - dm_i + L.R_i,\]

(13) \[K^2_{X/B} = -N^\text{red}_d(a.).\]

Now in case $n = 3$, for any condition-vector $(a.)$ of weight

$$\sum (a_i - 1) = 4d - 1,$$

we denote by $N^\times_d(a.)$ the number of 1-nodal irreducible rational curves in $P^3$ meeting a generic collection of linear spaces of respective codimensions $(a_1, \ldots, a_k)$. 
Theorem 1. We have for any $i = 1, \ldots, k$,

\begin{equation}
(14) \quad N_d^\times(a.) = \frac{(d - 3)N_d(2, a.) - N_d(...a_i + 1...)}{(4d + 2)m_i + 2L,R_i + N_d^{\text{red}}(a.)}.
\end{equation}

**proof.** We set things up so as to apply the **double point formula** (in a relative form [F]). Let

$$X_B^2$$

denote the fibre square of $X/B$, and

$$\Delta \simeq X \subset X_B^2$$

the diagonal. Then $X_B^2$ is smooth except at points $(p, p)$, where $p$ is a singular point of a fibre of $X/B$. Locally at such points, if $X/B$ is given locally at $p$ by

$$xy = t,$$

then $X_B^2$ is given by

$$x_1y_1 = x_2y_2 = t,$$

and therefore has an ordinary 3-fold double point. Moreover the diagonal $\Delta$ is defined by

$$x_1 = x_2, y_1 = y_2$$

and in particular is a non-Cartier divisor. Let

\begin{equation}
(15) \quad b : Y \to X_B^2
\end{equation}

denote the blow-up of $\Delta$, with exceptional divisor

$$\Delta' = b^*(\Delta).$$

Then it is easy to see that $b$ is a small resolution of $X_B^2$ with exceptional locus

$$E = \sum E_p$$

consisting of a $\mathbb{P}^1$ for each relatively singular point $p$. Moreover the fibre of $Y$ over $B$ corresponding to a reducible fibre $C_1 \cup_p C_2$ of $X/B$ is of a ‘honeycomb’ shape

$$B_{(p, p)}C_1^2 \cup C_1 \times C_2$$

$$C_2 \times C_1 \cup B_{(p, p)}C_2^2$$

where

$$B_{(p, p)}C_1^2 \cap B_{(p, p)}C_2^2 = E_p, C_1 \times C_2 \cap C_2 \times C_1 = \emptyset.$$
Also $\Delta' \to \Delta$ is just the blowing up of all the points $p$. Furthermore, the identity of rational functions, locally at $p$,
\[
\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1}{x_2} = \frac{y_2}{x_1}
\]
shows that there $b : Y \to X_B^2$ coincides with the blowup of the ideal $(y_1, x_2)$, and with that of the ideal $(y_2, x_1)$.

We will need to know the normal bundle $\nu = N_{\Delta'/Y}$.

To this end, note that, clearly
\[
\omega_{\Delta'} = b^* \omega_X (E).
\]

On the other hand, the map (15) is small, hence crepant, so
\[
\omega_Y = b^* \omega_{X_B^2} = b_1^* \omega_X \otimes b_2^* \omega_X \otimes c^* \omega_B^{-1}
\]

where we use the evident maps
\[
b_i : Y \to X, i = 1, 2, c : Y \to B.
\]

Therefore by the adjunction formula we conclude
\[
\nu = b^* \omega_{X/B}^{-1}(E).
\]

Now we are ready to apply the double-point formula to the map
\[
f^2 = (f_1, f_2) : X_B^2 \to \mathbb{P}^3 \times \mathbb{P}^3.
\]

This shows that the cycle of ordered pairs $(x_1, x_2)$ in $X/B^2$ such that $f(x_1) = f(x_2)$ is residual to $\Delta'$ in $f^2*(\Delta_{\mathbb{P}^3})$. Therefore, as in ([CR], Thm 3) we find that
\[
N_d^X(a.) = \frac{1}{2} c_3(f_1^*(\mathcal{O}(1)) \otimes f_2^*(\mathcal{Q}) \otimes \mathcal{O}(-\Delta')),
\]

where $\mathcal{Q}$ denotes the universal quotient bundle on $\mathbb{P}^3$. Then a straightforward calculation, based on the intersection calculus reviewed above, yields the formula (14). □

We turn next to the enumeration of rational plane curves with a tacnode. Denote by $N_d^{\text{tac}}$ (resp. $\kappa_d$) the number of rational curves with a tacnode (resp. cusp) passing through $3d - 2$ generic points in $\mathbb{P}^2$, and recall that $\kappa_d$ was computed in [R3].
Theorem 2. We have

\begin{equation}
N^\text{tac}_d = (d - 4)N_d - (d - 7)(d^2 + 7d - 4)m + \frac{d - 7}{2}L.R + 2N^\text{red}_d + \frac{1}{2}\kappa_d
\end{equation}

**proof.** We use an analogous setup and notation as above, this time for a pencil of rational curves in \(\mathbb{P}^2\) through \(3d - 2\) points. To begin with, we describe an algebraic setup for the (relative) Gauss mapping. Consider the relative principal parts sheaf for \(L\) on \(X/B\), which fits in an exact sequence

\begin{equation}
0 \rightarrow \Omega_{X/B}(L) \rightarrow P_{X/B}(L) \rightarrow L \rightarrow 0.
\end{equation}

Setting \(V = H^0(\mathcal{O}_{\mathbb{P}^2}(1))\), we have a natural map \(V \rightarrow P_{X/B}(L)\) which combines with the Euler sequence to yield an exact diagram

\begin{align*}
0 & \rightarrow f^*\mathcal{O}_{\mathbb{P}^2}(1) \rightarrow V \otimes \mathcal{O}_X \rightarrow f^*\mathcal{O}_{\mathbb{P}^2}(1) \rightarrow 0 \\
0 & \rightarrow \Omega_{X/B}(L) \rightarrow P_{X/B}(L) \rightarrow L \rightarrow 0.
\end{align*}

Let \(P'\) denote the pushout of \(P_{X/B}(L)\) by the natural map

\(\Omega_{X/B}(L) \rightarrow \omega_{X/B}(L)\).

Now let \(b : X'/B \rightarrow X/B\) denote the blowup of all singular points of fibres and all 'cuspidal' points, i.e. all points \((x, b) \in X\) such that \(f(x)\) is a cusp on \(f(X_b)\) (see [R3] for a discussion and enumeration of these). Let \(E = \sum E_p\) be the exceptional divisor over the singular points of fibres and \(U = \sum U_q\) be the exceptional divisor over the cuspidal points. Set \(f' = b \circ f\). Then it is easy to see by a local computation that

**Lemma 3.** The image of the natural map

\(f'^*\mathcal{O}_{\mathbb{P}^2} \rightarrow b^*\omega_{X/B}\)

coincides with \(b^*\omega_{X/B}(\sim E - U)\)

**proof.** It suffices to prove locally at each fibre node or cuspidal point \(p\) that

\(\text{im}(f'^*\mathcal{O}_{\mathbb{P}^2} \rightarrow \omega_{X/B}) = \omega_{X/B}.I_p\).

In local coordinates at \(p\), the family \(X/B\) is given by

\(xy = t\)
with \( f = (x, y) \) and \( \omega_{X/B} \) is generated by \( dx \wedge dy.dt^{-1} \). Since
\[
dx \wedge dt = xdx \wedge dy, dy \wedge dt = -ydx \wedge dy,
\]
we have that \( f^*\Omega_{\mathbb{P}^2} \) is generated by \( xdx \wedge dy.dt^{-1}, ydx \wedge dy.dt^{-1} \), as claimed.

In the cuspidal case, our family has local coordinates \( u, t \) with
\[
f = (u^2, (u^2 - t)u) = (x, y).
\]
Here \( \omega_{X/B} \) is generated by \( du \) and its subsheaf generated by
\[
dx = 2udu, dy = (3u^2 - t)du
\]
clearly coincides with the subsheaf generated by \( udu \) and \( tdu \), as claimed \( \square \).

It follows from Lemma 3 that if we let \( P \) denote the image of the natural map
\[
V \otimes \mathcal{O}_X \to b^* P',
\]
then we get a diagram with exact rows and columns
\[
\begin{array}{ccccccc}
0 & \to & f'^*\Omega_{\mathbb{P}^2}(1) & \to & V \otimes \mathcal{O}_X & \to & f^*\mathcal{O}_{\mathbb{P}^2}(1) & \to & 0 \\
0 & \to & b^*\omega_{X/B}(L - E - U) & \to & P & \to & b^*L & \to & 0.
\end{array}
\]
(19)

Now let
\[
\phi : I \to \mathbb{P}^2
\]
be the incidence (or flag) variety, with universal flag
\[
V \otimes \mathcal{O}_I = F^0 \supset F^1 = \phi^*\Omega_{\mathbb{P}^2}(1) \supset F^2 \supset (0),
\]
\[
F^0/F^1 = \phi^*\mathcal{O}(1).
\]

Now the diagram (19) gives rise to a lifting of \( f' \) to a morphism
\[
g : X' \to I
\]
with
\[
g^*(F^0/F^1) = L' := b^*(L), g^*(F^1/F^2) = b^*(\omega_{X/B})(-E - U).
\]
(20)

Clearly, the value of \( g \) at a point
\[
(x, b) \in X' \setminus E \setminus U
\]
mapping to 

\[ y \in \mathbb{P}^2 \]

is the pair 

\[ (y, T_y(f(X_b))) \]

consisting of \( y \) and the tangent line to \( f(X_b) \) at \( f(y) \); in particular, the tacnodes in the family \( X/B \) correspond (1:2) to the double points of \( g \), which we propose to count as in the proof of Theorem 1. To this end, the construction of a 'good' desingularization of \( X' \times_B X' \) must be modified to take into account the fact that each \( E_p \) appears in its fibre over \( B \) with multiplicity 2.

Consider then the fibre square

\[ \tilde{Y} = X' \times_B X'. \]

Its singular points are as follows. First, for each cuspidal point \( q \) on \( X \), we get a 3-fold ODP \( (y, y) \) where \( y \) is the intersection of \( U_q \) with the proper transform of the fibre of \( X/B \) containing \( q \). We desingularize this by blowing up \( U_q \times U_q \) (which is the same locally as blowing up the diagonal). Here the construction is exactly as in the proof of Thm 1.

Additionally, \( \tilde{Y} \) is singular along each \( E_p \times E_p \), where it has a local equation of the form

\[ x_1^2 u_1 = x_2^2 u_2 \]

where \( x = 0 \) is the equation of \( E_p \), \( u \) is a coordinate on \( E \) and \( x_i, u_i \) are the pullbacks of \( x, u \). This can be desingularized by blowing up the locus

\[ x_1 = x_2 = 0, \]

i.e. \( E_p \times E_p \). The exceptional divisor maps to \( E_p \times E_p \) generically with degree 2, ramified over the 4 rulings

\[ E_p \times \{0, \infty\} \cup \{0, \infty\} \times E_p, \]

where \( 0, \infty \) are the 2 intersections of \( E_p \) with the other 2 components of its fibre, and having \( \mathbb{P}^1 \) fibres over \( \{0, \infty\}^2 \).

Let

\[ b : Y \to \tilde{Y} \]

be the global desingularization thus obtained, and \( \Delta'' \subset Y \) the proper (=total) transform of the diagonal. It is easy to see that the map

\[ b : \Delta'' \to X' \]
identifies $\Delta''$ with the blowup $X''$ of all singular points of set-theoretic fibres of $X/B$ (i.e. all the points of the form $(y, y)$ or $(0, 0)$ or $(\infty, \infty)$ in the above notation.

The normal bundle

$$\nu = N_{\Delta''/Y}$$

can be computed much like before: first,

$$\omega_{\Delta''} = b^*\omega_{X'}(E'')$$

where $E'' = \sum E''_r$ is the exceptional divisor of $b|_{\Delta''}$. Next, as before,

$$\omega_{\bar{Y}} = b^1_1\omega_X \otimes b^*_2\omega_X \otimes \omega^{−1}_B.$$ 

As $\bar{Y}$ has an ordinary double surface generically along $\bigcup_p E_p$, it follows that

$$\omega_Y = b^*(\omega_{\bar{Y}})(−\bigcup_p E_p).$$

Putting these together we see that, using divisor notation and setting $K = K_X - K_B$,

(21) $$\nu = -(b^*(\omega_{X'} - \omega_B - E) - E'') = -(K + U - E'').$$

Consequently, we have

(22) $$\nu^2 = K^2 - 2N^\text{red}_d - 2\kappa_d$$

where $\kappa_d$ is the number of cuspidal rational plane curves of degree $d$ through $3d - 2$ generic points, computed in [R3]. Of course, we also have

(23) $$L\nu = -L.K = 2dm - L.R$$

Now, to use double-point theory consider the cartesian product $I^2 = I \times I$ with projections

$$p_i : I^2 \to I, i = 1, 2, \phi^2 : I^2 \to (\mathbb{P}^2)^2.$$ 

Let $F^\cdot$ be the dual filtration to $F^\cdot$, defined by

$$F^i = (F^0/F^i)^* \subset (F^0)^*.$$ 

In $I^2$, $\phi^{2*}\Delta_{\mathbb{P}^2}$ is defined as the zero-locus of a map

$$p_1^*F_1 \to p_2^*(F_2/F_1).$$
and inside this, $\Delta_I$ is defined as the zero-locus of a map

$$p_1^*(F_2/F_1) \to p_2^*(F_3/F_2).$$

Considering as before the map

$$f^2 = (f_1, f_2) : Y \to I^2,$$

double-point (or more precisely, residual-intersection) theory shows that

$$2N_d^{tac} = c_2(f_1^*(F_1^*) \otimes f_2^*((V^* \otimes \mathcal{O})/F_1) \otimes \mathcal{O}(-\Delta'))c_1(f_1^*((F_2/F_1)^*) \otimes f_2^*(F_3/F_2) \otimes \mathcal{O}(-\Delta')).$$

Using (20), the $c_2$ factor can be identified, writing $L_i = p_i^* L, K_i = p_i^*(\omega_{X/B})$, as

$$L_2^2 + L_1 L_2 - L_2 \Delta' + (L_1 - \Delta')^2$$

while the $c_1$ factor is

$$(K_1 - E - U) + L_2 - \Delta'.$$

Then a routine computation, using as above the necessary intersection theory on $Y$ and $X$ yields the formula (18). □

As a bonus for the construction of the lift $g$ above- and without using any double-point theory, we get a formula for one of the ‘characteristic numbers’, namely the number $N_d^t$ of rational plane curves through $3d - 2$ points tangent to a given line. Thus let

$$p_2 : I \to \mathbb{P}^{2v}$$

be the natural map of the incidence variety to the dual projective plane, and set

$$f^v = p_2 \circ g : X' \to \mathbb{P}^{2v}, L^v = f^v \ast \mathcal{O}(1).$$

As above, we can write

$$L^v \sim (2d - 2)s_1 - \sum_{F \in F_1} (\deg(F) - 1)F' - \sum E_p - \sum U_q + xF_0$$

where the first sum is over all fibre components of $X/B$ not meeting $s_1$, $F'$ denotes the proper transform of $F$ on $X'$, the second and third sums cover over all exceptional divisors of $X'/X$, $F_0$ denotes a fibre of $X'/B$, and we
have simply used the fact that the dual to a rational curve of degree $d$ has
degree $2d - 2$. Now intersect this expression with $s_1$ and note that

$$L^v.s_1 = N_{d \to},$$

i.e. the number of degree-$d$ rational curves through $A_1, ..., A_{3d-2}$ with a
given tangent direction at $A_1$. This yields

$$x = N_{d \to} + (2d - 2)m_d.$$ 

Since $N^t_d = L^v^2$, this number can now be easily computed from our inter-
section calculus, yielding the following

**Corollary 3.** We have

$$N^t_d = 4(d - 1)^2m_d + d(d - 1)N_{d \to} + N^\text{red}_d - \kappa_d - 2 \sum_{F \cap s_1 = \emptyset} (\deg(F) - 1)^2.$$ 

Finally, we take up the problem of enumerating the triple points, i.e.
of computing the number $N^\text{tri}_d$ of irreducible rational curves with a triple
point passing through $3d-2$ generic points. Again our tool will be a suitable
residual-intersection computation. Using notations as above, take 3 copies

$$Y_{12}, Y_{13}, Y_{23}$$

of the small resolution of $X^2_B$ constructed above. We have evident 'first
projection' maps

$$Y_{12} \to X, Y_{13} \to X$$

which allows us to construct

$$Y_{12} \times_X Y_{13}$$

together with a map

$$p_{23} : Y_{12} \times_X Y_{13} \to X^2_B.$$ 

Let $Z$ be the unique component of $Y_{12} \times_X Y_{13}$ which dominates $X^2_B$. Define

$$Y^*$$

as the blowup of $Z$ in the proper transform of the diagonal in $X^2_B$. Then $Y^*$
comes equipped with projections

$$p_{ij} : Y^* \to Y_{ij}.$$
a map\[ f^3 = (f_1, f_2, f_3) : Y^* \to (P^2)^3 \]
as well as with a birational map\[ b^* : Y^* \to X^3_B. \]

An elementary analysis shows that $Y^*$ has exactly 2 singular points, each isomorphic to a cone over a Segre variety $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$, hence also to the germ at the origin of the determinantal variety $M_{2 \times 3}^1$ of $(2 \times 3)$ matrices of rank at most 1, and consequently each of these points admits a small 'determinantal' resolution with $\mathbb{P}^1$ as exceptional locus. It can be shown that this resolution admits a natural morphism to the relative Hilbert scheme $\mathcal{Hilb}_3(X/B)$. Though we could replace $Y^*$ by this resolution, this will not be necessary since by definition the big diagonals are already Cartier on $Y^*$.

Now consider on $Y^*$ the locus\[ D_{12} = p_{12}^* D \]
where $D \subset Y$ is the double locus of $f^2$, i.e. the closure of the locus of distinct point $(y_1, y_2)$ in the same fibre which map to a node, i.e. such that $f(y_1) = f(y_2)$. Then clearly\[ [D_{12}] = p_{12}^*[D] = c_2(Q_2(L_1 - \Delta_{12}) \]
where\[ Q_i = f_i^* Q, L_i = f_i^* L, \Delta_{12} = p_{12}^*(\Delta'). \]

Now consider the intersection $D_{12}.D_{13}$. This consists of the locus we want, i.e. that of ordered distinct triples $(y_1, y_2, y_3)$ on the same fibre such that $f(y_1) = f(y_2) = f(y_3)$, plus the locus $D_{12} \Delta_{23}$. Then residual-intersection theory tells us that\[ 6N_{d} = c_2(Q_2(L_1 - \Delta_{12}))c_2(Q_3(L_1 - \Delta_{13} - \Delta_{23})). \]

This expression can be computed by routine calculations, using the intersection calculus developed above; the only possibly nonobvious terms are\[ \Delta_{12}^2 \Delta_{13}^2 = \Delta_{12}^2 \Delta_{23}^2 = K^2, \]
\[ \Delta_{12}^2 \Delta_{13} \Delta_{23} = (K - E)^2. \]

The former follows from the fact that\[ p_{12*}(\Delta_{13}^2) = -p_{12*}(p_{13}^*(K_1 - E)) = -K_1, \]
and that $\Delta_{12}^2 = -(K_1 - E)$ (cf. (17)). The latter follows from the fact that\[ p_{12*}(\Delta_{13} \Delta_{23}) = \Delta_{12}. \]

This yields
Theorem 5. We have

\[ 6N_d^{\text{tri}} = (3d^2 - 18d + 30)N_d + (3d - 18)(dm_d - L.R) - 6N_d^{\text{red}} \]

Check. For \( d = 4 \), we have \( m_d = 428, N_d = 620, L.R = 3276, N_d^{\text{red}} = 2124 \), therefore

\[ N_4^{\text{tri}} = 60, \]

a number computed classically by Zeuthen and recently, with modern methods, by Kleiman and Piene [KP]. I am grateful to Steve Kleiman for this information, which led to a correction of an error in an earlier statement of the formula.

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