Computing Palindromic Trees for a Sliding Window and Its Applications

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Abstract

The palindromic tree (a.k.a. eertree) for a string $S$ of length $n$ is a tree-like data structure that represents the set of all distinct palindromic substrings of $S$, using $O(n)$ space [Rubinchik and Shur, 2018]. It is known that, when $S$ is over an alphabet of size $\sigma$ and is given in an online manner, then the palindromic tree of $S$ can be constructed in $O(n \log \sigma)$ time with $O(n)$ space. In this paper, we consider the sliding window version of the problem: For a fixed window length $d$, we propose two algorithms to maintain the palindromic tree of size $O(d)$ for every sliding window $S[i..i+d-1]$ over $S$, one running in $O(n \log \sigma')$ time with $O(d)$ space where $\sigma' \leq d$ is the maximum number of distinct characters in the windows, and the other running in $O(n + d\sigma)$ time with $d\sigma + O(d)$ space. We also present applications of our algorithms for computing minimal unique palindromic substrings (MUPS) and for computing minimal absent palindromic words (MAPW) for a sliding window.

1 Introduction

Palindromes. A palindrome is a string that reads the same forward and backward. Palindromic structures in strings have been heavily studied in the fields of string processing algorithms and combinatorics on strings [15, 10, 13, 19, 8, 1]. One of the most famous results related to palindromic structures is Manacher’s algorithm [15], which computes all maximal palindromes in a given string $S$. Manacher’s algorithm essentially computes all palindromes in $S$, since any palindromic substring of $S$ is a substring of some maximal palindrome in $S$. Another interesting topic is enumeration of distinct palindromes in a string. It is known that any string of length $n$ contains at most $n + 1$ distinct palindromes including the empty string [6]. Grout et al. [10] proposed an $O(n)$-time algorithm
which enumerate all distinct palindromes in a given string of length \( n \) over an integer alphabet of size \( \sigma = n^{O(1)} \). For the same problem in the online model, Kosolobov et al. [13] proposed an \( O(n \log \sigma) \)-time and \( O(n) \)-space algorithm for a general ordered alphabet. Kosolobov et al.’s algorithm is a combination of Manchek’s algorithm and Ukkonen’s online suffix tree construction algorithm [21].

Rubinchik and Shur [19] proposed a new data structure called certree, which permits efficient access to distinct palindromes in a string without storing the string itself. Eertrees can be utilized for solving problems related to palindromic structures, such as the palindromic counting problem and the palindromic factorization problem [19]. The size of the eertree of \( S \) is linear in the number \( p_S \) of distinct palindromes in \( S \) [19]. It is known that \( p_S \) is at most \( |S| \), and that it can be much smaller than the length \( |S| \) of the string, e.g., for \( S = \text{abc}^m \), \( p_S = 4 \) since all distinct palindromes in \( S \) are \( a, b, c, \) and the empty string. Thus, the size of the eertree of \( S \) can be much smaller than that of the suffix tree of \( S \) which is \( \Theta(n) \). Therefore, it is of significance if one can build eertrees without suffix trees. Rubinchik and Shur [19] indeed proposed an online eertree construction algorithm running in \( O(n \log \sigma) \) time without suffix trees.

Recently, a concept of palindromic structures called \textit{minimal unique palindromic substrings} (MUPS) is introduced by Inoue et al. [12]. A palindromic substring \( w = S[i..j] \) of a string \( S \) is called a MUPS of \( S \) if \( w \) occurs in \( S \) exactly once, and \( S[i+1..j-1] \) occurs at least twice in \( S \). MUPSs are utilized for solving the \textit{shortest unique palindromic substring} (SUPS) problem [12], which is motivated by an application in molecular biology. Watanabe et al. [22] proposed an algorithm to solve the SUPS problem based on the \textit{run-length encoding} (RLE) version of eertrees, named \texttt{eertre}^2.

**Our Contributions.** In this paper, we consider the problem of maintaining eertrees for the \textit{sliding window} model, that is, given a string \( S \) of length \( n \) and a \textit{window} of a fixed size \( d \), we maintain eertrees of substrings \( S[i..i+1..i+d-1] \) for incremental \( i = 0, 1, \ldots, n-d \). The sliding window model is a natural generalization of the online model, and the assumptions of this model are natural when we need to process a massive or a streaming string data with a limited memory space. Also, we consider the problem of maintaining MUPSs for a sliding window. In addition, we introduce a new concept of palindromic structures called \textit{minimal absent palindromic words} (MAPW), and consider the problem of maintaining MAPWs for a sliding window. A string \( w \) is called a MAPW of a string \( S \) if and only if \( w \) is a palindromic, \( w \) does not occur in \( S \), and \( w[1..|w|-2] \) occurs in \( S \). MAPWs can be seen as a palindromic version of the notion of \textit{minimal absent words} (MAW), which are extensively studied in the fields of string processing and bioinformatics [5, 17, 2, 18, 9].

In this paper, we propose an algorithm which maintains eertrees for a sliding window in a total of \( O(n \log \sigma') \) time using \( O(d) \) space where \( \sigma' \leq d \) is the maximum number of distinct characters in every window. We then give an alternative eertree construction algorithm for a sliding window which runs in \( O(n + d \sigma) \) time with \( d \sigma + O(d) \) space. As applications to the aforementioned result, we propose an algorithm which maintains MUPSs for a sliding window in a total of \( O(n \log \sigma') \) time using \( O(d) \) space, and an algorithm which maintains MAPWs for a sliding window in a total of \( O(n + d \sigma) \) time using \( O(d \sigma) \) space. We emphasize that our algorithms are stand-alone in the sense that they do not use
suffix trees, while the majority of existing efficient sliding window algorithms (see below) make heavy use of suffix trees.

**Related Work.** A typical and classical application to the sliding window model is data compression, such as Lempel-Ziv 77 (the original version) [23] and PPM [3]. Note that sliding-window Lempel-Ziv 77 is an immediate application of suffix trees for a sliding window, which can be maintained in $O(n\log \sigma')$ time using $O(d)$ space [7, 14, 20]. Recently, several algorithms for computing substrings for a sliding window with certain interesting properties are proposed: For instance, Crochemore et al. [4] introduced the problem of computing MAWs for a sliding window, and proposed an $O(n\sigma)$-time and $O(d\sigma)$-space algorithm using suffix trees for a sliding window. Mieno et al. [16] proposed an algorithm for computing minimal unique substrings (MUSs) [11] for a sliding window, in $O(n\log \sigma)$-time and $O(d)$ space, again based on suffix trees for a sliding window.

# 2 Preliminaries

## 2.1 Strings

Let $\Sigma$ be an alphabet of size $\sigma$. An element of $\Sigma$ is called a character. An element of $\Sigma^*$ is called a string. The length of a string $S$ is denoted by $|S|$. The empty string $\varepsilon$ is the string of length 0. If $S = xyz$, then $x$, $y$, and $z$ are called a prefix, substring, and suffix of $S$, respectively. They are called a proper prefix, proper substring, and proper suffix of $S$ if $x \neq S$, $y \neq S$, and $z \neq S$, respectively. For any $0 \leq i \leq |S| - 1$, $S[i]$ denotes the $i$-th character of $S$. For any $0 \leq i \leq j \leq |S| - 1$, $S[i..j]$ denotes the substring of $S$ starting at position $i$ and ending at position $j$, i.e., $S[i..j] = S[i]S[i+1]\ldots S[j]$. For convenience, $S[i..j] = \varepsilon$ for any $i > j$. A string $S$ is called a palindrome if $S[i] = S[|S| - i - 1]$ for every $0 \leq i \leq |S| - 1$. Note that the empty string is a palindrome. A substring $S[i..j]$ of $S$ is said to be a palindromic substring of $S$ if $S[i..j]$ is a palindrome. The center of a palindromic substring $S[i..j]$ of $S$ is $i + (j - i) / 2$. A palindromic substring $S[i..j]$ of $S$ is maximal if $i = 0$, $j = |S| - 1$, or $S[i-1..j+1]$ is not a palindrome. We denote the longest palindromic prefix (resp. suffix) of $S$ by $lpp(S)$ (resp. $lps(S)$). We denote the set of all distinct palindromes in $S$ by $DPal(S)$. It is known that $|DPal(S)| \leq |S| + 1$ [6]. For any non-empty strings $S$ and $w$, we denote $\#occ_S(w)$ the number of occurrences of $w$ in $S$. For convenience, we define $\#occ_S(\varepsilon) = |S| + 1$. String $w$ is said to be unique in $S$ if $\#occ_S(w) = 1$. Also, $w$ is said to be repeating in $S$ if $\#occ_S(w) \geq 2$. In what follows, we consider an arbitrary fixed string $S$ of length $n > 0$.

## 2.2 Eertrees (Palindromic Trees)

The eertree of $S$ denoted by $eertree(S)$ is a tree-like data structure that enables us to efficiently access each of the distinct palindromes in $S$ [19]. The eertree($S$) consists of $m$ normal nodes and two auxiliary nodes, denoted 0-node and -1-node, where $m = |DPal(S)| - 1$. Each normal node corresponds to each element of $DPal(S) \setminus \{\varepsilon\}$. For each normal node $v$, we denote the palindrome corresponding to $v$ by $pal(v)$, and its length by $len(v)$. For convenience, we define $pal(0\text{-node}) = pal(-1\text{-node}) = \varepsilon$, $len(0\text{-node}) = 0$, and
Figure 1: The eertree of $S = aaababababbabb$. The solid and broken arrows represent edges and suffix links, respectively. Note that $pal(v)$ is written inside each node $v$ in this figure, however, it is for only explanation. Namely, each node does not explicitly store the corresponding string.

$S = aaababababbabb$

\[
\text{len}(-1\text{-node}) = -1.
\]

For any nodes $u, v$ in eertree($S$), there is an edge $(u, v)$ if and only if $\text{len}(u) + 2 = \text{len}(v)$ and $\text{pal}(u) = \text{pal}(v)[1..\text{len}(v) - 2]$. Each edge $(u, v)$ is labeled by a character $\text{pal}(v)[0]$. Also, each node $v$ in eertree($S$) has a suffix link denoted by slink($v$). For each node $v$ in eertree($S$) with $\text{len}(v) \geq 2$, slink($v$) points to the node corresponding to the longest palindromic proper suffix of $\text{pal}(v)$. For each node $v$ in eertree($S$) with $\text{len}(v) = 1$, slink($v$) points to the 0-node. Also, slink(0-node) = -1-node and slink(-1-node) = -1-node.

For each node $v$ in eertree($S$), $\text{inSL}(v) = |\{u \mid \text{slink}(u) = v\}|$ denotes the number of incoming suffix links of $v$. See Fig. 1 for an example of eertree($S$).

Note that each node $v$ does not store the string $\text{pal}(v)$ explicitly. Instead, we can obtain $\text{pal}(v)$ by traversing edges backward, from $v$ to the root, since $\text{pal}(u) = c \text{pal}(u')c$ for each node $u$ with $|\text{pal}(u)| \geq 2$ where $u'$ is the parent of $u$ and $c$ is the label of the edge $(u', u)$. Each node only stores pointers to its children and a constant number of integers. Thus, the size of eertree($S$) is linear in the number of nodes, i.e., $O(\text{|DPal}(S)|)$. It is known that eertree($S$) can be constructed in $O(n \log \sigma)$ time for any string $S$ given in an online manner [19].

3 Eertree for a Sliding Window

In this section, we show how to update a given eertree when sliding the window to the right by one character. Sliding a given window consists of two operations: deleting the leftmost character and appending a character to the right end. Namely, when the eertree of $S[i-1..j-1]$ is given, we first compute the eertree of $S[i..j-1]$ (deleting the leftmost character $S[i-1]$), and then, compute the eertree of $S[i..j]$ (appending a character $S[j]$). For updating the eertree when appending a character, we can apply the online algorithm [19] for constructing the eertree of a given string. On the other hand, for updating the eertree when
deleting the leftmost character, we propose new data structures in addition to the original eertree. Since the nodes of the eertree of a string represent all distinct palindromes in the string, we obtain the next lemma.

**Lemma 1.** There is a node \( \ell \) in eertree\((S[i..j-1])\) to be removed when deleting the leftmost character \( S[i-1] \) from \( S[i-1..j-1] \) if and only if (A) \( \text{pal}(\ell) \) is unique in \( S[i-1..j-1] \), (B) \( \text{pal}(\ell) = \text{lpp}(S[i-1..j-1]) \), and (C) \( \ell \) is a leaf node.

**Proof.** (\( \Rightarrow \)) (A) Since \( \ell \) is removed, \( \text{pal}(\ell) \) does not occur in \( S[i..j-1] \). Thus, \( \text{pal}(\ell) \) occurs in \( S[i-1..j-1] \) only as a prefix, i.e., \( \text{pal}(\ell) \) is unique in \( S[i-1..j-1] \). (B) Assume that \( \text{pal}(\ell) \) is shorter than \( \text{lpp}(S[i-1..j-1]) \). Then, \( \text{pal}(v) \) is a proper prefix of \( \text{lpp}(S[i-1..j-1]) \). Also, \( \text{pal}(v) \) is a proper suffix of \( \text{lpp}(S[i-1..j-1]) \) since \( \text{lpp}(S[i-1..j-1]) \) is a palindrome. This contradicts that \( \text{pal}(\ell) \) is unique in \( S[i-1..j-1] \). Thus, \( \text{pal}(\ell) = \text{lpp}(S[i-1..j-1]) \). (C) If we assume that \( \ell \) has a child, then \( \text{pal}(\ell) \) has an occurrence in \( S[i-1..j-1] \) that is not a prefix of \( S[i-1..j-1] \), a contradiction. (\( \Leftarrow \)) Since \( \text{pal}(\ell) \) is a palindromic prefix of \( S[i-1..j-1] \) and unique in \( S[i-1..j-1] \), \( \text{pal}(\ell) \) does not occur in \( S[i..j-1] \). Thus, \( \ell \) is removed when deleting \( S[i-1] \).

Namely, when the leftmost character of the window is deleted, at most one leaf will be removed from the eertree. Based on Lemma 1, we design an algorithm to detect the node to be removed, if it exists.

### 3.1 Determining Uniqueness of a Palindrome

Next, we show how to determine uniqueness of the palindrome corresponding to a specific node in a given eertree. First, we define two notions for each node in the eertree.

**Definition 1.** For each node \( v \) in eertree\((S[i..j])\), we define

\[
\text{BegW}_{i,j}(v) = \{ t - \text{len}(v) + 1 \mid \text{lps}(S[i..t]) = \text{pal}(v) \text{ and } t \leq j \}, \quad \text{and} \\
\text{Beg}_{i,j}(v) = \text{BegW}_{0,j} = \{ t - \text{len}(v) + 1 \mid \text{lps}(S[0..t]) = \text{pal}(v) \text{ and } t \leq j \}.
\]

Intuitively, \( \text{BegW}_{i,j}(v) \) is the set of beginning positions of the palindrome \( \text{pal}(v) \) such that each of them corresponds to the longest palindromic suffix of some prefix of the window \( S[i..j] \). Also, \( \text{Beg}_{i,j}(v) \) is a variant of \( \text{BegW}_{i,j}(v) \) without the window constraint.

**Definition 2.** For each node \( v \) in eertree\((S[i..j])\), we define a pair of integers \( \text{BegPair}_{i,j}(v) \) such that

\[
\text{BegPair}_{i,j}(v).\text{first} = \begin{cases} 
\max \text{BegW}_{i,j}(v) & \text{if } \text{inSL}(v) = 0, \\
\max(\text{Beg}_{i,j}(v) \cap [i,j]) & \text{if } \text{inSL}(v) \geq 1, \\
some \text{position } x \text{ with } x < i & \text{otherwise}.
\end{cases}
\]


BegPair using stores elements in BegPair for an example of second largest positions in BegW and BegPair and since inSL where S(10 5 BegW pal eertree S before deleting the leftmost character BegPair position 6 is not palindromic suffix of and window [5 6]. Hence, its starting position 5 is an element of BegW pal S, the eertree S′ (11 6) = 5... Bega. Since inSL(u) = 0 and BegW 5,14 (u) = {5, 10}, BegPair 5,14 (u) = (10, 5). When deleting the leftmost character S[5] = c from the window S[5..14], BegW 5,14 (u) changes to BegW 6,14 (u) = {10}. However, we can set BegPair 6,14 (u) ← BegPair 5,14 (u) = (10, 5) since BegPair 6,14 (u).second = 5 < 6 is a valid value. Namely, we have already computed BegPair 6,14 (u) implicitly before deleting the leftmost character S[5] from S[5..14].

and

\[ \text{BegPair}_{i,j}(v).second = \begin{cases} \max \text{BegW}'_{i,j}(v) & \text{if } \text{inSL}(v) = 0 \\
 \max(\text{Beg}'_{i,j}(v) \cap [i,j]) & \text{if } \text{inSL}(v) \geq 1 \\
 \text{some position } y \text{ with } y < i & \text{otherwise.}
\end{cases} \]

where \( \text{BegW}'_{i,j}(v) = \text{BegW}_{i,j}(v) \setminus \{\text{BegPair}_{i,j}(v).first\} \)
and \( \text{Beg}'_{i,j}(v) = \text{Beg}_{i,j}(v) \setminus \{\text{BegPair}_{i,j}(v).first\} \).

The key point of the definition is that \( \text{BegPair}_{i,j}(v) \) stores the largest and second largest positions in \( \text{BegW}_{i,j}(v) \) when \( \text{inSL}(v) = 0 \). Otherwise, \( \text{BegPair}_{i,j}(v) \) stores elements in \( \text{Beg}_{i,j}(v) \) or some text position that is no longer inside the window, which are invariants of the algorithms we describe below. See Figure 2 for an example of \( \text{BegPair} \).

The next lemma states that we can determine if \( \text{pal}(v) \) is unique or not by using \( \text{BegPair}_{i,j}(v) \).
Algorithm 1 Update BegPair when deleting the leftmost character.

Require: \( v.BegPair = \text{BegPair}_{i-1,j-1}(v) \) for each node \( v \) in \( \text{eertree}(S[i-1..j-1]) \), and the node \( u \) in \( \text{eertree}(S[i..j-1]) \) corresponding to the longest palindromic prefix of \( S[i-1..j-1] \).

Ensure: \( v.BegPair = \text{BegPair}_{i,j-1}(v) \) for each node \( v \) in \( \text{eertree}(S[i..j-1]) \).

1: if \( \text{pal}(u) = \text{lpp}(S[i-1..j-1]) \) is unique in \( S[i-1..j-1] \) then
2:   \( u' \leftarrow \text{slink}(u) \)
3:   \( \text{inSL}(u') \leftarrow \text{inSL}(u') - 1 \)
4:   if \( \text{inSL}(u') = 0 \) then
5:     \( x \leftarrow i-1+\text{len}(u)-\text{len}(u') \) \( \text{\backslash \ backslash \ x \ is \ a \ starting \ position \ of \ \text{pal}(u') \)  
6:     if \( x > u'.\text{BegPair.first} \) then
7:       \( u'.\text{BegPair.second} \leftarrow u'.\text{BegPair.first} \)
8:     \( u'.\text{BegPair.first} \leftarrow x \)
9:   else if \( x > u'.\text{BegPair.second} \) then
10:  \( u'.\text{BegPair.second} \leftarrow x \)
11: end if
12: end if
13: end if

Lemma 2. For each node \( v \) in \( \text{eertree}(S[i..j]) \), \( \text{pal}(v) \) is unique in \( S[i..j] \) if and only if \( \text{inSL}(v) = 0 \) and \( \text{BegPair}_{i,j}(v).\text{second} < i \).

Proof. \( (\Rightarrow) \) We show the contraposition. If there is a node \( u \) with \( \text{slink}(u) = v \), then \( \text{pal}(v) \) occurs as a suffix of \( \text{pal}(u) \). Also, \( \text{pal}(v) \) occurs as a prefix of \( \text{pal}(u) \) since they are palindromes. Thus, \( \text{pal}(v) \) is repeating in \( S[i..j] \). On the other hand, if \( \text{BegPair}_{i,j}(v).\text{second} \in [i,j] \), then \( \text{pal}(v) \) occurs at positions \( \text{BegPair}_{i,j}(v).\text{second} \) and \( \text{BegPair}_{i,j}(v).\text{first} \). Thus, \( \text{pal}(v) \) is repeating in \( S[i..j] \).

\( (\Leftarrow) \) Since \( \text{inSL}(v) = 0 \), each occurrence of \( \text{pal}(v) \) in \( S[i..j] \) is the longest palindromic suffix of some prefix of \( S[i..j] \). Also, since \( \text{BegPair}_{i,j}(v).\text{second} < i \), \( \text{pal}(v) \) occurs exactly once in \( S[i..j] \).

Next, we introduce our algorithms to update \( \text{BegPair}_{i,j}(\cdot) \) for a sliding window.

3.1.1 Updating BegPair When Deleting the Leftmost Character.

When deleting the leftmost character \( S[i-1] \) from \( S[i-1..j-1] \), each \( \text{BegPair}_{i,j-1}(v) \) is computed by Algorithm 1. Assume that each node \( v \) stores a variable \( v.BegPair \), which is equal to \( \text{BegPair}_{i,j-1}(v) \) at the beginning of the algorithm. In line 1, we can determine the uniqueness of \( \text{pal}(u) \) by using Lemma 2 in constant time since we know the location of the node \( u \) in \( \text{eertree}(S[i-1..j-1]) \). All the other lines can be processed in constant time, clearly. Thus, the total running time of Algorithm 1 is constant.

Next, in order to prove the correctness of the algorithm, we focus on the number of incoming suffix links of each node. When deleting the leftmost character, the value \( \text{inSL}(v) \) does not increase for any node \( v \). By the definition of \( \text{BegPair} \), we do not need to update \( v.BegPair \) for each \( v \) such that \( \text{inSL}(v) \) does not change when deleting the character. Thus, we only consider \( v.BegPair \) for a node \( v \) such that \( \text{inSL}(v) \) decreases and becomes zero when deleting the
character. Also, the candidate for such a node is only \( u' = \text{slink}(u) \) where \( u \) is the node corresponding to \( \text{lpp}(S[i..j-1]) \), since from Lemma 1, \( u \) is the only candidate for a node to be removed. Hence, we update \( u'.\text{BeginPair} \) only if \( \text{pal}(u) = \text{lpp}(S[i..j-1]) \) is unique. If \( \text{pal}(u) \) is unique, we check whether \( \text{insL}(u') \) has become zero or not, and naively update \( u'.\text{BeginPair} \) if needed.

3.1.2 Update \( \text{BeginPair} \) When Appending a Character.

When appending a character \( S[j] \) to the right end of \( S[i..j-1] \), each \( \text{BeginPair}_{i,j}(v) \) is computed by Algorithm 2. As in Algorithm 1, we assume that each \( v.\text{BeginPair} \) stores \( \text{BeginPair}_{i,j-1}(v) \) at the beginning of the algorithm. Clearly, the running time of Algorithm 2 is constant. Next, we show the correctness. Recall that \( \text{BeginW}_{i,j-1}(v) \) for all nodes \( v \) in \( \text{eertree}(S[i..j-1]) \) represent the starting positions of the longest palindromic suffixes of all prefixes of \( S[i..j-1] \). Thus, when appending a character \( S[j] \) to \( S[i..j-1] \), it is enough to consider only the longest palindromic suffix of \( S[i..j] \) for updating \( \text{BeginPair} \). Hence, in Algorithm 2, we update \( v.\text{BeginPair} \) only for the node \( u \) that corresponds to \( \text{lps}(S[i..j]) \).

3.2 Maintaining the Longest Palindromic Prefix

Next, we show how to maintain the longest palindromic prefix on the eertree for a sliding window. In what follows, we denote the size of the window by \( d \), i.e., \( d = j - i + 1 \). We define a cyclic array \( L \) of size \( d \) defined as follows: Let \( p \) be the starting position of \( \text{lps}(S[i..i+d-1]) \) for window \( S[i..i+d-1] \). For each \( i \leq k \leq p \), we keep an invariant such that \( L[k \mod d] \) stores the length of the longest maximal palindromic substring of \( S[0..i+d-1] \) starting at position \( k \). For each \( p < k' \leq i+d-1 \), \( L[k' \mod d] \) stores an arbitrary value. The next lemma states that we can maintain \( L \) for a sliding window efficiently.

Lemma 3. We can update \( L \) for all consecutive sliding windows over \( S \) in a total of \( O(n) \) time using \( O(d) \) space.

Proof. When appending a character to the right end, we run Manacher’s algorithm [15] and compute the longest palindromic suffix of the window. While running Manacher’s algorithm, we can obtain all maximal palindromic substrings of \( S[0..i+d-1] \) that starts before the position \( p \). Thus, it is easy to compute the longest maximal palindromic substring starting at each position.

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**Algorithm 2** Update \( \text{BeginPair} \) when appending a character.

**Require:** \( v.\text{BeginPair} = \text{BeginPair}_{i,j-1}(v) \) for each node \( v \) in \( \text{eertree}(S[i..j-1]) \), and the node \( u \) in \( \text{eertree}(S[i..j]) \) corresponding to the longest palindromic suffix of \( S[i..j] \).

**Ensure:** \( v.\text{BeginPair} = \text{BeginPair}_{i,j}(v) \) for each node \( v \) in \( \text{eertree}(S[i..j]) \).

1. if \( u \) exists in \( \text{eertree}(S[i..j-1]) \) then
2. \( u.\text{BeginPair}.\text{second} \leftarrow u.\text{BeginPair}.\text{first} \)
3. else
4. \( u.\text{BeginPair}.\text{second} \leftarrow -1 \)
5. end if
6. \( u.\text{BeginPair}.\text{first} \leftarrow j - \text{len}(v) + 1 \)
We show how to compute \( \ell \) window. Given a traver-
sing of \( S \), we can find a palindrome of length \( \ell \) in \( d \) time..

Figure 3: Examples for Lemma 5. On the left diagram, the can-
didates for the longest palindromic prefix of \( S[i..i + d - 2] \) are \( \text{ababa} \) that is the prefix of \( S[i..i + d - 2] \) of length \( \ell_i - 1 \), and \( \text{aba} \) that is the longest maximal palindrome.

For each \( 1 \leq i \leq n - d \), let \( \ell_i \) be the length of \( lpp(S[i..i + d - 1]) \). Clearly, \( \ell_0 = L[0] \). Also, let \( \ell_i' \) be the length of \( lpp(S[i..i + d - 2]) \).

**Lemma 4.** For each \( 1 \leq i \leq n - d \), \( \ell_i' = \max\{\ell_i - 1 - 2 \leq L[i \mod d]\} \).

**Proof.** For the sake of contradiction, we assume that there is a palindrome prefix \( S[i..q] \) of \( S[i..i + d - 2] \) that is longer than \( \max\{\ell_i - 1 - 2 \leq L[i \mod d]\} \). Then, palindrome \( S[i..q] \) is not maximal, and thus, \( S[i..i + d - 2] \) is also a palindrome. Hence, \( \ell_i - 1 \geq |S[i..q]| = q - i + 1 > \ell_i - 1 - 2 \).

**Lemma 5.** We can compute the longest palindromic prefix of each window in amortized \( O(1) \) time using \( O(d) \) space.

**Proof.** By Lemma 3, we can maintain \( L \) in amortized constant time for each window. Given \( \ell_i - 1 \), we can compute \( \ell_i' \) in constant time by Lemma 4. Finally, we show how to compute \( \ell_i \) by using \( \ell_i' \). If \( S[i..i + d - 1] \) is a palindrome, \( L[i \mod d] = d \), then \( \ell_i = d \). Otherwise, the length of the longest palindromic prefix does not change by appending a character, i.e., \( \ell_i = \ell_i' \). The time complexity is clearly amortized constant.

Next, we show how to maintain the node corresponding to the longest palindromic prefix of the sliding window. Let \( u, v \) be the nodes in \( \text{eertree}(S[i-1..i+d-2]) \) corresponding to \( lpp(S[i..i+d-2]) \) and \( lpp(S[i..i+d-2]) \), respectively. If \( u \) is given, then we can detect \( v \) by traversing the eertree as follows: We consider two cases depending on the length of \( lpp(S[i..i + d - 2]) \), that is, \( \ell_i' \), based on Lemma 4. In the case where \( \ell_i' = \ell_i - 1 - 2 \), it is easy to see that \( v \) is the parent of \( u \). In the other case, i.e., when \( \ell_i' = L[i \mod d] \), then we further consider two sub-cases. Let \( e_i = i + \ell_i - 1 \) be the ending position of \( lpp(S[i..i + d - 1]) \). Let \( c_i \) be the center of the palindrome \( lpp(S[i..i + d - 2]) \). If \( c_i \leq e_i - 1 \), then we traverse the suffix links from \( u \) until the center of its corresponding palindrome matches \( c_i \). Then, \( v \) is a descendant of the last node we traversed above, and hence, we can find \( v \) by going down edges from the node according to characters.
\[ S[c_i + 1], \ldots, S[i + \ell'_{i} - 1] \]. Otherwise, we simply go down the edges from the 0-node or the \(-1\)-node according to the characters \( S[c_i + 1], \ldots, S[i + \ell'_{i} - 1] \).

Next, we analyze the time complexity by counting the total number of edges and suffix links traversed. The total number of times we go up to the parent of a node is at most \( n \) since it occurs at most once for each deleting step. On the other hand, traversing a suffix link and going down to a child node can occur twice or more for a single deleting step. The total number of suffix links traversed is equal to the total number of times \( c_i \) increases. It is clearly \( O(n) \) since \( c_i \) is non-decreasing during the above procedures for all \( i \). Finally, the total number of times we go down to children is at most the number of times we go up to parents, plus \( n \), i.e., \( O(n) \). Also, traversing a suffix link and going up to the parent takes \( O(1) \) time. In addition, going down to a child node takes \( O(\log \sigma') \) time, where \( \sigma' \) is the maximum number of distinct characters in every window. Thus, the above procedures for all \( i = 0, \ldots, n - d \) require \( O(n \log \sigma') \) total time.

To summarize this section, we obtain the following theorem.

**Theorem 1.** We can maintain eertrees for a sliding window in total of \( O(n \log \sigma') \) time using \( O(d) \) space.

By applying a little modification to the above algorithms, we obtain a variant of our algorithm that can be faster than Theorem 1 when \( d\sigma < n \log \sigma' \), by using additional \( d\sigma \) space.

**Corollary 1.** We can maintain eertrees for a sliding window in total of \( O(n + d\sigma) \) time using \( d\sigma + O(d) \) space.

**Proof.** In the original eertrees, each node stores a binary search tree to maintain branches dynamically. Instead, we use an array of integers of size \( \sigma \), which allows to add, delete, and search for a node pointer (i.e., edge) labeled by a given character in constant time. Thus, the \( \log \sigma' \) factor in our time complexity can be removed. On the other hand, we need \( \sigma + O(1) \) space to represent each node object, and \( \Theta(\sigma) \) time to initialize it. If we naively initialize such a node object when adding a new node, the total time complexity increases to \( O(n\sigma) \). However, we can reuse node objects that had been removed when deleting a character since such removed nodes and new nodes to be added are leaves, i.e., they do not have any child (Lemma 1). Thus, by reusing node objects, we do not need to initialize an array of size \( \sigma \) when adding a new leaf node. The total number of node objects to initialize is at most \( d + 2 \), and it costs \( O(d\sigma) \) total time to initialize them.

## 4 Applications of Eertrees for a Sliding Window

In this section, we apply our sliding-window eertree algorithm of Section 3 to computing minimal unique palindromic substrings and minimal absent palindromic words for a sliding window.

### 4.1 Computing Minimal Unique Palindromic Substrings for a Sliding Window

A substring \( S[i..j] \) of \( S \) is called a **minimal unique palindromic substring (MUPS)** of \( S \) if and only if \( S[i..j] \) is a palindrome, \( S[i..j] \) is unique in \( S \), and \( S[i+1..j-1] \)
is repeating in \( S \). We denote \( \text{MUPS}(S) \) the set of intervals corresponding to MUPSs of \( S \), i.e., \( \text{MUPS}(S) = \{[i,j] \mid S[i..j] \text{ is a MUPS of } S\} \). For example, palindromic substring \( S[9..13] \) of string \( S = \text{aaababababbab} \) is a MUPS of \( S \) since \( S[9..13] = \text{bbab} \) is repeating in \( S \).

We first show the next lemma which states a relationship between eertrees and MUPSs.

**Lemma 6.** A string \( w \) is a MUPS of \( S \) if and only if there is a node \( v \) in eertree(\( S \)) such that \( \text{pal}(v) = w \), \( \text{pal}(v) \) is unique in \( S \) and \( \text{pal}(u) \) is repeating in \( S \), where \( u \) is the parent of \( v \).

**Proof.** (\( \Rightarrow \)) Since \( w \) is a MUPS of \( S \), it is clear that there is a node \( v \) such that \( \text{pal}(v) = w \) and it is unique in \( S \). Also, since \( \text{pal}(v) = w \neq \varepsilon \), \( v \) has the parent \( u \), which represents the string \( w[1..|w| - 2] \). By the definition of MUPS, \( \text{pal}(u) = w[1..|w| - 2] \) is repeating in \( S \). (\( \Leftarrow \)) Since the palindrome \( \text{pal}(v) = w \) is unique in \( S \) and \( \text{pal}(u) = w[1..|w| - 2] \) is repeating in \( S \), \( w \) is a MUPS of \( S \).

Using Lemma 6, we can efficiently compute MUPSs with eertrees. Next, we consider how to compute MUPSs from a given eertree in an offline manner.

**Lemma 7.** Given eertree(\( S \)), we can compute \( \text{MUPS}(S) \) in \( O(\text{DPal}(S)) \) time.

**Proof.** For each node \( v \), we can detect whether \( \text{pal}(v) \) is a MUPS or not in \( O(1) \) time by combining Lemma 2 and Lemma 6. The starting position of a palindrome \( \text{pal}(v) \) which is unique in \( S \) is stored in \( v.\text{BegPair}_{0,n-1}(v).\text{first} \). Therefore, we can compute \( \text{MUPS}(S) \) by a single traversal on eertree(\( S \)).

Also, we can efficiently maintain MUPSs for a sliding window.

**Theorem 2.** We can maintain the set of MUPSs for a sliding window in a total of \( O(n \log \sigma') \) time using \( O(d) \) space.

**Proof.** In addition to the eertree data structure described in Section 3, we add 1-bit information ismups into each node. This bit ismups is set to 1 if the node corresponds to a MUPS and to 0 otherwise. We first consider to delete the leftmost character \( S[i-1] \) from \( S[i-1..j-1] \). In this case, only prefixes of \( S[i-1..j-1] \) are those whose number of occurrences in the sliding window change. We check the nodes corresponding to the longest and the second longest palindromic prefixes, and update ismups of them accordingly. We do not need to care about other palindromic prefixes since they must be repeating in \( S[i..j-1] \).

Symmetrically, we can maintain ismups in the case when appending a character \( S[j] \) to \( S[i..j-1] \).

### 4.2 Computing Minimal Absent Palindromic Words for a Sliding Window

A string \( w \) is called a minimal absent palindromic word (MAPW) of string \( S \) if and only if \( w \) is a palindrome, \( w \) does not occur in \( S \), and \( w[1..|w| - 2] \) occurs in \( S \). For example, palindrome \( w = \text{aabb} \) is a MAPW of string \( S = \text{aaababababab} \) since \( w \) does not occur in \( S \) and \( w[1..|w| - 2] = \text{abba} \) occurs in \( S \) at position 8. For a relation between MAPWs and eertrees, the next lemma holds.
Lemma 8. For any non-empty string \( w \in \Sigma^* \), \( w \) is a MAPW of a string \( S \) if and only if there is a node \( u \) in \text{eertree}(S) \) such that \( \text{pal}(u) = w[1..|w| - 2] \), \( \text{len}(u) = |w| - 2 \), and \( u \) does not have an edge labeled by \( w[0] \).

Proof. (\( \Rightarrow \)) Since \( w \) is a MAPW, \( w[1..|w| - 2] \) is a palindromic substring of \( S \), and thus, there is a node \( u \) with \( \text{pal}(u) = w[1..|w| - 2] \) and \( \text{len}(u) = |w| - 2 \). Also, since the palindrome \( w \) does not occur in \( S \), \( u \) does not have an edge labeled by \( w[0] \). (\( \Leftarrow \)) Since \( u \) is a node in \text{eertree}(S), the string \( \text{pal}(u) = w[1..|w| - 2] \) occurs in \( S \). Also, since \( u \) does not have an edge labeled by \( w[0] \), the string \( w \) does not occur in \( S \). Thus, \( w \) is a MAPW of \( S \).

In order to maintain the set of MAPWs on top of \text{eertree}(S), we store an array \( M_v \) of size \( \sigma \) to each node \( v \) in \text{eertree}(S) where \( M_v[c] = 0 \) if \( v \) has an edge labeled by \( c \) and \( M_v[c] = 1 \) otherwise. By Lemma 8, \( M_v[c] = 1 \) iff \( c \text{pal}(v)c \) is a MAPW of \( S \). It is easy to see that \( M_v \) for all nodes \( v \) (i.e., all MAPWs of \( S \)) can be computed by traversing \text{eertree}(S) only once. Thus, the next corollary holds.

Corollary 2. The number of MAPWs of \( S \) is at most \((|\text{DPal}(S)| + 1)\sigma \). Also, given \text{eertree}(S), the set of MAPWs of \( S \) can be computed in \( O(|\text{DPal}(S)|\sigma) \) time.

Also, we can maintain MAPWs for a sliding window by applying Corollary 1.

Theorem 3. We can maintain the set of MAPWs for a sliding window in a total of \( O(n + d\sigma) \) time using \( O(d\sigma) \) space.

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