ASYMPTOTIC ANALYSIS ON POSITIVE SOLUTIONS OF THE LANE-EMDEN SYSTEM WITH NEARLY CRITICAL EXPONENTS

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Abstract. We concern a family \( \{ (u_\varepsilon, v_\varepsilon) \}_{\varepsilon > 0} \) of solutions of the Lane-Emden system on a smooth bounded convex domain \( \Omega \) in \( \mathbb{R}^N \)
\[
\begin{align*}
-\Delta u_\varepsilon &= v_\varepsilon^p \quad \text{in } \Omega, \\
-\Delta v_\varepsilon &= u_\varepsilon^q \quad \text{in } \Omega, \\
u_\varepsilon, v_\varepsilon &> 0 \quad \text{in } \Omega, \\
u_\varepsilon = v_\varepsilon = 0 \quad \text{on } \partial \Omega
\end{align*}
\]
for \( N \geq 4, \max\{1, \frac{N-2}{N-2}, \frac{N+2}{N-2}\} < p \leq q \) and small
\[
\varepsilon := \frac{N}{p+1} + \frac{N}{q+1} - (N-2) > 0.
\]
This system appears as the extremal equation of the Sobolev embedding \( W^{2,(p+1)/p}(\Omega) \hookrightarrow L^{q+1}(\Omega) \), and is also closely related to the Calderón-Zygmund estimate. Under the natural energy condition
\[
\sup_{\varepsilon > 0} \left( \| u_\varepsilon \|_{W^{2, \frac{N+2}{p+1}}(\Omega)} + \| v_\varepsilon \|_{W^{2, \frac{N+2}{q+1}}(\Omega)} \right) < \infty,
\]
we prove that the multiple bubbling phenomena may arise for the family \( \{ (u_\varepsilon, v_\varepsilon) \}_{\varepsilon > 0} \), and establish a detailed qualitative and quantitative description. If \( p < \frac{N}{N-2} \), the nonlinear structure of the system makes the interaction between bubbles so strong, so the determination process of the blow-up rates and locations is completely different from that of the classical Lane-Emden equation. If \( p \geq \frac{N}{N-2} \), the blow-up scenario is relatively close to (but not the same as) that of the classical Lane-Emden equation, and only one-bubble solutions can exist. Even in the latter case, the standard approach does not work well, which forces us to devise a new method. Using our analysis, we also deduce a general existence theorem valid on any smooth bounded domains.

1. Introduction

In this paper, we concern the Lane-Emden system, one of the simplest Hamiltonian-type elliptic systems. It is given as
\[
\begin{align*}
-\Delta u &= v^p \quad \text{in } \Omega, \\
-\Delta v &= u^q \quad \text{in } \Omega, \\
u, v &> 0 \quad \text{in } \Omega, \\
u = v = 0 \quad \text{on } \partial \Omega
\end{align*}
\] (1.1)
where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N \) for \( N \geq 3, p \in (\frac{2}{N-2}, \frac{N+2}{N-2}] \), and \( q \geq p \). As the classical (scalar) Lane-Emden equation
\[
\begin{align*}
-\Delta u &= u^p \quad \text{in } \Omega, \\
u, v &> 0 \quad \text{in } \Omega, \\
u = v = 0 \quad \text{on } \partial \Omega
\end{align*}
\] (1.2)
for $p \in [1, \frac{N+2}{N-2}]$ is the extremal equation of the Sobolev embedding $W^{1,2}_0(\Omega) \hookrightarrow L^{p+1}(\Omega)$, system (1.1) is that of the embedding $W^{2,(p+1)/p}(\Omega) \hookrightarrow L^{q+1}(\Omega)$ for a pair $(p, q)$ in the subcritical region

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N},$$

(1.3)
or on the (Sobolev) critical hyperbola

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}.$$  

(1.4)

System (1.1) is also closely related to the Calderón-Zygmund estimate in the sense that its solutions satisfy a circular relation

$$v \in L^{p+1}(\Omega) \Rightarrow -\Delta u = v^p \in L^{\frac{p+1}{p}}(\Omega) \Rightarrow u \in W^{2,\frac{p+1}{p}}(\Omega) \subset L^{q+1}(\Omega)$$

$$\Rightarrow -\Delta v = u^q \in L^{\frac{q+1}{q}}(\Omega) \Rightarrow v \in W^{2,\frac{q+1}{q}}(\Omega) \subset L^{p+1}(\Omega)$$

provided (1.3) or (1.4).

As the natural counterpart of (1.2), system (1.1) has received considerable attention for decades, and now abundant results on the existence theorem are available in the literature. However, as far as the authors know, only a few works have examined the qualitative properties of the system, such as symmetry, concentration, or multiplicity of solutions. The reader is advised to check the survey paper [3], which furnishes vast treatments on (1.1) known before the year 2014.

Our objective is to contribute to the latter direction by examining asymptotic behavior of solutions of (1.1) provided that the pair $(p, q)$ satisfies $p \leq q$, and (1.3), and is close to the critical hyperbola.

1.1. History and motivations of the problem. A solution of (1.1) is realized as a positive critical point of the energy functional

$$I_{p,q}(u,v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx - \frac{1}{p+1} \int_{\Omega} |v|^{p+1} \, dx - \frac{1}{q+1} \int_{\Omega} |u|^{q+1} \, dx$$

(1.5)

for $(u, v) \in W^{1,s}_0(\Omega) \times W^{1,\frac{1}{1-s}}_0(\Omega)$ for a suitable $s > 1$. Exploiting the strong indefiniteness of $E$, Hulshof and van der Vorst [19] and de Figueiredo and Felmer [12] proved independently that if $p, q > 1$ and (1.3) holds, then (1.1) has a solution. Moreover, by studying the associated minimization problem to the scalar equation, which is an equivalent formulation of (1.1),

$$\begin{cases}
-\Delta \left((-\Delta u)^{\frac{1}{p}}\right) = u^q & \text{in } \Omega, \\
u, -\Delta u > 0 & \text{in } \Omega, \\
u = \Delta u = 0 & \text{on } \partial \Omega,
\end{cases}$$

(1.6)

Bonheure et al. [2] extended the aforementioned existence results to cover any $p, q > 0$ satisfying $pq \neq 1$ and (1.3). On the other hand, a Pohozaev-type argument (refer to Mitidieri [22]) yields that if $\Omega$ is star-shaped and

$$\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{N-2}{N},$$

then (1.1) has no solution.

As shown in [2], (1.1) has a least energy solution (or a ground state), namely, a solution that attains the least value of $E$ among all nontrivial solutions, provided $pq \neq 1$ and (1.3). In fact, $u$ is a least energy solution of (1.6) if and only if $(u, ((-\Delta u)^{1/p})$ is a least energy solution of (1.1). In

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1The critical hyperbola, introduced by Mitidieri [22] and van der Vorst [28] (and soon treated by several researchers including Clément et al. [11]), has an essential role in determining the solution structure of (1.1).
Guerra investigated the asymptotic profile of least energy solutions of (1.1), assuming that $\Omega$ is convex and $p \geq 1$. Later, Choi and the first author of this paper [9] extended his results by covering $p < 1$ and arbitrary smooth bounded domains. Their works are natural generalizations of Han [18] and Rey [25] that studied least energy solutions of the scalar equation (1.2), and summarized as follows: Fix a number $p \in \left(\frac{N+2}{N-2}, \frac{N+2}{N-2}\right]$, and define $q_e$ by the relation
\[
\frac{1}{p+1} + \frac{1}{q_e+1} = \frac{N-2+\varepsilon}{N}
\] (1.7)
where $\varepsilon > 0$ is small. Clearly, $(p, q_e)$ is in the subcritical region (1.3) and approaches the critical hyperbola (1.4) as $\varepsilon \to 0$. Let $q_0$ be the limit of $q_e$ as $\varepsilon \to 0$ so that $(p, q_0)$ satisfies (1.4) and $p \leq q_0$. If $\{\{u_\varepsilon, v_\varepsilon\}\}_{\varepsilon\in(0,\varepsilon_0)}$ is a family of least energy solutions of (1.1) with $q = q_e$, then it blows up at an interior point $\xi_0$ of $\Omega$ as $\varepsilon \to 0$. In other words, there exists a family $\{x_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)} \subset \Omega$ such that
\[
u_\varepsilon(x_\varepsilon) = \|u_\varepsilon\|_{L^\infty(\Omega)} \to \infty, \quad x_\varepsilon \to \xi_0 \in \Omega, \quad \text{and} \quad u_\varepsilon, v_\varepsilon \to 0 \quad \text{in} \quad C^1_{\text{loc}}(\Omega \setminus \{\xi_0\})
\] (1.8)
as $\varepsilon \to 0$, after passing to a subsequence. Also, if we define
\[\alpha_\varepsilon = \frac{2(p+1)}{pq_e-1}, \quad \beta_\varepsilon = \frac{2(q_e+1)}{pq_e-1}, \quad \text{and} \quad \lambda_\varepsilon = u_\varepsilon^{\frac{1}{p}}(x_\varepsilon),
\] (1.9)
then, along a subsequence,
\[
\left(\lambda_\varepsilon^{-\alpha_\varepsilon} u_\varepsilon(\lambda_\varepsilon^{-1} \cdot +x_\varepsilon), \lambda_\varepsilon^{-\beta_\varepsilon} v_\varepsilon(\lambda_\varepsilon^{-1} \cdot +x_\varepsilon)\right) \to (U_{1,0}, V_{1,0}) \quad \text{in} \quad C^2_{\text{loc}}(\mathbb{R}^N) \quad \text{as} \quad \varepsilon \to 0.
\]
Here, $(U_{1,0}, V_{1,0})$ is the unique least energy solution of
\[
\begin{cases}
-\Delta U_{1,0} = V_{1,0}^{p-1} & \text{in} \quad \mathbb{R}^N, \\
-\Delta V_{1,0} = U_{1,0}^{q_0-1} & \text{in} \quad \mathbb{R}^N, \\
U_{1,0}, V_{1,0} > 0 \\
(U_{1,0}, V_{1,0}) \in W^{2,\frac{N+2}{N-2}}(\mathbb{R}^N) \times W^{2,\frac{N+2}{N-2}}(\mathbb{R}^N)
\end{cases}
\] (1.10)
such that
\[
U_{1,0}(0) = 1 = \max_{x \in \mathbb{R}^N} U_{1,0}(x). \quad \tag{1.11}
\]
Let $G$ and $\tau$ be the Green’s function and the Robin function of the Dirichlet Laplacian $-\Delta$ in $\Omega$, respectively. For $p \in (\frac{N+2}{N-2}, \frac{N+2}{N-2})$ and $\xi \in \Omega$, let $G(\cdot, \xi)$ be the unique solution of
\[
\begin{cases}
-\Delta_{x} \tilde{G}(x, \xi) = G^p(x, \xi) & \text{for} \quad x \in \Omega, \\
\tilde{G}(x, \xi) = 0 & \text{for} \quad x \in \partial \Omega
\end{cases}
\] (1.12)
$\tilde{H}$ be the $C^1$-regular part of $\tilde{G}$, and $\bar{\tau}(x) = \tilde{H}(x, x)$ for $x \in \Omega$; see Subsection 2.1. Then $\xi_0$ must be a critical point of the Robin function $\tau$ for $p \in [\frac{N+2}{N-2}, \frac{N+2}{N-2}]$, and that of $\bar{\tau}$ for $p \in (\frac{N+2}{N-2}, \frac{N+2}{N-2})$. Besides, the asymptotic behavior of $\|u_\varepsilon\|_{L^\infty(\Omega)}$ and a more exact asymptotic profile of $\{\{u_\varepsilon, v_\varepsilon\}\}_{\varepsilon\in(0,\varepsilon_0)}$ in $\Omega \setminus \{\xi_0\}$ than one in (1.8) can be given in terms of the quantities $N, (p, q_0), \varepsilon, (U_{1,0}, V_{1,0}), G(\cdot, \xi_0), \tau(\xi_0), \tilde{G}(\cdot, \xi_0)$, and $\bar{\tau}(\xi_0)$; refer to [17, Theorem 1.1], [9, Theorem B], and [21, Lemma 2.2]. Note that if $p > \frac{N+2}{N-2}$, one can exchange the roles of $p$ and $q$.

Recently, the first author and Pistoia [21] built multi-bubble solutions of (1.1) by employing the non-degeneracy result on the standard bubble $(U_{1,0}, V_{1,0})$ due to Frank and themselves [14]; refer to Lemma 2.4 below. Among others, they proved the following result: Assume that $N \geq 4$, $p \in (1, \frac{N-1}{N-2})$, and $(p, q) = (p, q_e)$ satisfies (1.7). Suppose also that $\Omega$ is a dumbbell-shaped domain.

\[2\] The unique existence of $(U_{1,0}, V_{1,0})$ was proved by Hulshof and Van der Vorst [20]. Throughout the paper, we call $(U_{1,0}, V_{1,0})$ the standard bubble.
with \( l - 1 \) necks for some \( l \in \mathbb{N} \).

Given any \( k \in \{1, \ldots, l\} \), there exists a small number \( \varepsilon_0 > 0 \) such that for each for all \( \varepsilon \in (0, \varepsilon_0) \), (1.1) has \( \binom{N}{k} \) solutions that blow up at \( k \) points as \( \varepsilon \to 0 \).

In the proof, they found a more exact asymptotic profile of the solutions, including their blow-up rate and location. Also, while they focused the case \( p \in (1, \frac{N-1}{N-2}) \), their method can also cover \( p \in \left[ \frac{N-1}{N-2}, \frac{N+2}{N-2} \right] \) as our analysis will indicate. See Theorem 1.5 below.

The result of [21] implies the existence of higher-energy solutions (or excited states) of (1.1) in some instances. Hence it is natural to ask how they behave as \((p, q)\) approaches the critical hyperbola. The primary purpose of this paper is to give a complete description of the asymptotic behavior of an arbitrary family of solutions provided that they satisfy a natural energy condition, \( \Omega \) is convex, and \( p > \max\{1, \frac{3}{N-2}\} \).

More precisely, we will see that the multiple bubbling phenomena may arise, while the cluster or tower phenomena cannot. Furthermore, performing a fine analysis, we will show that the asymptotic profile of the solutions depends on the value of \( p \). If \( p \) is less than the Serrin exponent \( \frac{N}{N-2} \), the interaction between bubbles turns out to be very strong. We have to reflect this fact in determining the blow-up rates and locations, so the analysis becomes difficult. If \( p \geq \frac{N}{N-2} \), the blow-up scenario is relatively close to that of the classical Lane-Emden equation. Nonetheless, we cannot follow the standard approach, because it only provides rough estimates that are unusable for \( p \) close to \( \frac{N}{N-2} \).

We will devise a new method to get over this technical issue.

Using our analysis, we will also obtain a general existence theorem that holds on arbitrary smooth bounded domains.

1.2. Main theorems. We now state the main theorems. The following is our general assumption for Theorems 1.2–1.4.

**Assumption 1.1.** We assume that \( N \geq 4 \), \( \Omega \) is a smooth bounded convex domain in \( \mathbb{R}^N \), \( p \in (\max\{1, \frac{3}{N-2}\}, \frac{N+2}{N-2}) \), \( \varepsilon_0 > 0 \) is a small number, and \( q_\varepsilon \) is the number determined by (1.7) for each \( \varepsilon \in (0, \varepsilon_0) \). Also, let \( \{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon \in (0, \varepsilon_0)} \) be a family of solutions of (1.1) with \( q = q_\varepsilon \) satisfying a natural energy condition

\[
\limsup_{\varepsilon \to 0} \left( \|u_\varepsilon\|_{W^{2, \frac{p+1}{p}}(\Omega)} + \|v_\varepsilon\|_{W^{2, \frac{q_\varepsilon+1}{q_\varepsilon}}(\Omega)} \right) \leq C
\]

for some constant \( C > 0 \).

The first theorem describes the multi-bubble phenomenon of (1.1).

**Theorem 1.2.** Suppose that Assumption 1.1 holds. Then, after passing to a subsequence, either

\[
(u_\varepsilon, v_\varepsilon) \to (u_0, v_0) \quad \text{in} \quad (C^2(\Omega))^2 \quad \text{as} \quad \varepsilon \to 0
\]

where \((u_0, v_0)\) is a solution of (1.1) with \( q = q_0 \), or the followings hold:

1. \((u_\varepsilon, v_\varepsilon) \rightharpoonup (0, 0)\) weakly and not strongly in \( W^{2, \frac{p+1}{p}}(\Omega) \times W^{2, \frac{q_\varepsilon+1}{q_\varepsilon}}(\Omega) \) as \( \varepsilon \to 0 \).

2. The family \( \{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon \in (0, \varepsilon_0)} \) blows up at \( k \in \mathbb{N} \) distinct points \( \xi_1, \ldots, \xi_k \in \Omega \) (apart from the boundary) as \( \varepsilon \to 0 \). In other words, there exist \( k \) families \( \{x_{i\varepsilon}\}_{\varepsilon \in (0, \varepsilon_0)} \subset \Omega \) for \( i = 1, \ldots, k \) such that

\[
u_{\varepsilon}(x_{i\varepsilon}) \to \infty, \quad x_{i\varepsilon} \to \xi_i \in \Omega, \quad \text{and} \quad u_\varepsilon, v_\varepsilon \to 0 \quad \text{in} \quad C^1_{\text{loc}}(\overline{\Omega} \setminus \{\xi_1, \ldots, \xi_k\})
\]

as \( \varepsilon \to 0 \).

3. Let \( \alpha_\varepsilon \) and \( \beta_\varepsilon \) be the numbers in (1.9). For a fixed index \( i = 1, \ldots, k \), we set \( \lambda_{i\varepsilon} = u_{i\varepsilon}^{1/\alpha_\varepsilon}(x_{i\varepsilon}) \). Then

\[
\left( \lambda_{i\varepsilon}^{\alpha_\varepsilon}u_{\varepsilon}(\lambda_{i\varepsilon}^{-1} + x_{i\varepsilon}), \lambda_{i\varepsilon}^{-\beta_\varepsilon}v_{\varepsilon}(\lambda_{i\varepsilon}^{-1} + x_{i\varepsilon}) \right) \to (U_{1,0}, V_{1,0}) \quad \text{in} \quad C^2_{\text{loc}}(\mathbb{R}^N) \quad \text{as} \quad \varepsilon \to 0
\]

\[3\] The precise definition of the dumbbell-shaped domain can be found in [21, Section 1].
where \((U_{1,0}, V_{1,0})\) is the standard bubble. In addition, there exists a constant \(C > 0\) such that
\[
\limsup_{\varepsilon \to 0} \frac{\lambda_{i\varepsilon}}{\lambda_{j\varepsilon}} \leq C \quad \text{for any } 1 \leq i \neq j \leq k. 
\] (1.17)

The second theorem depicts the exact asymptotic profile of solutions to (1.1) as \((p, q)\) approaches the critical hyperbola provided \(p < \frac{N}{N-2}\). As can be seen, the determination process of the blow-up rates and locations are much more cumbersome than that of the scalar equation (1.2) or the case \(p \geq \frac{N}{N-2}\).

**Theorem 1.3.** Suppose that Assumption 1.1 holds, \(p < \frac{N}{N-2}\), and Theorem 1.2 (1) is valid. By setting \(d_{i\varepsilon} = \frac{\lambda_{i\varepsilon}}{\lambda_{j\varepsilon}}\) and passing to a subsequence, we may assume that
\[
\delta_i = \lim_{\varepsilon \to 0} d_{i\varepsilon} \in (C^{-1}, C) \quad \text{for some } C > 1.
\]
Writing \((\delta, \xi) = (\delta_1, \ldots, \delta_k, \xi_1, \ldots, \xi_k) \in (0, \infty)^k \times \Omega^k\), let \(\tilde{G}_{\delta, \xi} : \Omega \to \mathbb{R}\) be the solution of
\[
\begin{align*}
-\Delta \tilde{G}_{\delta, \xi} &= \left( \sum_{i=1}^{k} \frac{N}{\delta_i^{N+1}} G(\cdot, \xi_i) \right)^p \quad \text{in } \Omega, \\
\tilde{G}_{\delta, \xi} &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\] (1.18)

For \(i = 1, \ldots, k\), let also \(\tilde{H}_{\delta, \xi, i} : \Omega \to \mathbb{R}\) be the local \(C^1\)-regularization of \(\tilde{G}_{\delta, \xi}\) around \(\xi_i\) whose precise definition is given in (2.6). Then the followings hold:

1. The \(i\)-th blow-up point \(\xi_i \in \Omega\) depicted in Theorem 1.2 (2) is a critical point of \(\tilde{H}_{\delta, \xi, i}\).
2. We have
\[
\lim_{\varepsilon \to 0} \varepsilon u_{\varepsilon}^{p+1}(x_{i\varepsilon}) = \frac{N}{q_0 + 1} S^{1-p/p_0} \|U_{1,0}\|^{p+1} \|U_{1,0}\|^{p/p_0} \sum_{i=1}^{k} \delta_i^{N/q_0+1} \tilde{H}_{\delta, \xi, i}(\xi_i)
\]
for all \(i = 1, \ldots, k\), where
\[
S := \inf_{U \in W^{2, \frac{pN}{p-2}}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\Delta U|^p}{\|U\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}} = \frac{\int_{\mathbb{R}^N} |\Delta U_{1,0}|^{p+1}}{\|U_{1,0}\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}} = \|U_{1,0}\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}.
\] (1.19)

In particular, \(\delta_i^{N/(q_0+1)} \tilde{H}_{\delta, \xi, i}(\xi_i) = \delta_j^{N/(q_0+1)} \tilde{H}_{\delta, \xi, j}(\xi_j)\) for all \(i, j = 1, \ldots, k\).
3. Without loss of generality, let us assume that \(\chi_{1\varepsilon} = \|u_{\varepsilon}\|_{L^\infty(\Omega)}\). Then
\[
\lim_{\varepsilon \to 0} \|u_{\varepsilon}\|_{L^\infty(\Omega)} u_{\varepsilon}(x) = \|U_{1,0}\|^{q_0} \delta_i^{N/q_0+1} G(x, \xi_i)
\]
and
\[
\lim_{\varepsilon \to 0} \|u_{\varepsilon}\|_{L^\infty(\Omega)}^{p} u_{\varepsilon}(x) = \|U_{1,0}\|^{q_0} \delta_i^{N/q_0+1} \tilde{G}_{\delta, \xi}(x)
\]
in \(C^1_{\text{loc}}(\Omega \setminus \{\xi_1, \ldots, \xi_k\})\)-sense.

The third theorem shows that if \(p \geq \frac{N}{N-2}\), then (1.1) behaves similarly to (1.2), and only one-bubble solutions exist for \(\Omega\) convex.

**Theorem 1.4.** Suppose that Assumption 1.1 holds, \(p \geq \frac{N}{N-2}\), and Theorem 1.2 (1) is valid. Then \(\{(u_{\varepsilon}, v_{\varepsilon})\}_{\varepsilon \in (0, \varepsilon_0)}\) blows up at only one point \(\xi_0 \in \Omega\) and
\[
\frac{\int_{\Omega} |\Delta u_{\varepsilon}|^{p+1}}{\|u_{\varepsilon}\|_{L^{p+1}(\Omega)}^{p+1}} \to S \quad \text{as } \varepsilon \to 0
\]
and $S > 0$ is the constant in (1.19). Also, the followings hold:

(1) The blow-up point $\xi_0$ is a critical point of the function $\tilde{\tau}$.

(2) We have

$$
\begin{align*}
\left\{ \begin{array}{ll}
\lim_{\varepsilon \to 0^+} \varepsilon \| u_\varepsilon \|_{L^\infty(\Omega)}^{N-1+p} = (p + 1) |\mathbb{S}^{N-1}| b_{N,p}^{N-1} & \text{ if } p = \frac{N}{N-2}, \\
\lim_{\varepsilon \to 0^+} \log \| u_\varepsilon \|_{L^\infty(\Omega)}^{N-1+p} = (N - 2) S_{N,p}^{N-1} |\mathbb{S}^{N-1}| b_{N,p}^{N-1} & \text{ if } p \in \left( \frac{N}{N-2}, \frac{N+2}{N-2} \right)
\end{array} \right.
\end{align*}
$$

where $b_{N,p} := \lim_{|y| \to \infty} |y|^{N-2} V_0(y) (0, \infty)$ and $|\mathbb{S}^{N-1}|$ is the surface measure of the unit sphere $S^{N-1}$ in $\mathbb{R}^N$.

(3) We have

$$
\lim_{\varepsilon \to 0} \| u_\varepsilon \|_{L^\infty(\Omega)} u_\varepsilon (x) = \| U_{1,0} \|_{L^0(\mathbb{R}^N)} G(x, \xi_0)
$$

and

$$
\begin{align*}
\left\{ \begin{array}{ll}
\lim_{\varepsilon \to 0} \frac{\| u_\varepsilon \|_{L^\infty(\Omega)}^{N-1+p}}{\log \| u_\varepsilon \|_{L^\infty(\Omega)}^{N-1+p}} u_\varepsilon (x) = \frac{p + 1}{N - 2} |\mathbb{S}^{N-1}| b_{N,p}^{N-1} G(x, \xi_0) & \text{ if } p = \frac{N}{N-2}, \\
\lim_{\varepsilon \to 0} \frac{\| u_\varepsilon \|_{L^\infty(\Omega)}^{N-2+p-2}}{\log \| u_\varepsilon \|_{L^\infty(\Omega)}^{N-2+p-2}} u_\varepsilon (x) = \| V_{1,0} \|_{L^p(\mathbb{R}^N)} G(x, \xi_0) & \text{ if } p \in \left( \frac{N}{N-2}, \frac{N+2}{N-2} \right)
\end{array} \right.
\end{align*}
$$

in $C^1_{loc}(\Omega \setminus \{ \xi_0 \})$-sense.

Finally, we deduce the following existence theorem extending the result in [21, Section 8], which does not require the convexity assumption on $\Omega$. Given any parameter $(\mu, \xi) \in (0, \infty) \times \mathbb{R}^N$, we set a bubble

$$(U_{\mu,\xi}(x), V_{\mu,\xi}(x)) = \left( \mu^{-\frac{N}{N+1}} U_{1,0}(x), \mu^{-\frac{N}{N+1}} V_{1,0}(x) \right)$$

for $x \in \mathbb{R}^N$, (1.20)

which solves (1.10), and its projection $(PU_{\mu,\xi}, PV_{\mu,\xi})$ on $W^{1,s}_0(\Omega) \times W^{1,s}_0(\Omega)$ for a suitable $s > 1$; refer to (4.2) for its precise definition.

**Theorem 1.5.** Assume that $N \geq 4$, $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, $k \in \mathbb{N}$, $p \in \left( \max\{ 1, \frac{3}{N-2} \}, \frac{N+2}{N-2} \right)$, and $q_k$ is the number determined by (1.7). Given any $(\tilde{d}, \tilde{x}) := (\tilde{d}_1, \ldots, \tilde{d}_k, \tilde{x}_1, \ldots, \tilde{x}_k) \in (0, \infty)^k \times \Omega^k$ and $i = 1, \ldots, k$, let $H_{\tilde{d},x,i} : \Omega \to \mathbb{R}$ be the local $C^1$-regularization of $G_{\tilde{d},x}$ around $x_i$ in (2.6). Then we set

$$
\nabla \Upsilon_{p,k}(\tilde{d}, \tilde{x}) = \left\{ \begin{array}{ll}
\sum_{i=1}^k d_i^{\frac{N}{N+1}} H_{\tilde{d},x,i}(x_i) - C_0 \log \left( \tilde{d}_1 \cdots \tilde{d}_k \right) & \text{if } p \in \left( \max\{ 1, \frac{3}{N-2} \}, \frac{N}{N-2} \right), \\
\sum_{i=1}^k d_i^{N-2} \tau(x_i) - \frac{1}{2} \sum_{i,j=1, i \neq j}^k \left( d_i^{N+1} d_j^{N+1} + d_i^{N+1} d_j^{N+1} \right) G(x_i, x_j) - C_0 \log \left( \tilde{d}_1 \cdots \tilde{d}_k \right) & \text{if } p \in \left( \frac{N}{N-2}, \frac{N+2}{N-2} \right)
\end{array} \right.
$$

(1.21)

where the value of $C_0 > 0$ is given in Section 6. Suppose that $(\tilde{d}, \xi) := (\tilde{d}_1, \ldots, \tilde{d}_k, \xi_1, \ldots, \xi_k) \in (0, \infty)^k \times \Omega^k$ is an isolated critical point of $\nabla \Upsilon_{p,k}$ and $\Lambda$ is an open neighborhood of $(\tilde{d}, \xi)$ such that

$$
\nabla \Upsilon_{p,k}(\tilde{d}, \tilde{x}) \neq 0 \quad \text{for all } (\tilde{d}, \tilde{x}) \in \partial \Lambda \quad \text{and} \quad \deg(\nabla \Upsilon_{p,k}, \Lambda, 0) \neq 0
$$

(1.22)

The well-definedness of the number $b_{N,p}$ was proved by Hulshof and Van der Vorst [20]; see Lemma 2.1 below.
where deg designates the Brouwer degree. Then there exist a small number $\varepsilon_0 > 0$ and a family $\{(u_{\varepsilon}, v_{\varepsilon})\}_{\varepsilon \in (0, \varepsilon_0)}$ of solutions of (1.1) with $q = q_{\varepsilon}$ such that

$$(u_{\varepsilon}, v_{\varepsilon}) \simeq \sum_{i=1}^{k} (PU_{\mu_{i\varepsilon}, x_{i\varepsilon}}, PV_{\mu_{i\varepsilon}, x_{i\varepsilon}}) \quad \text{in } \Omega$$

at the main order. Here, $\tilde{d}_{i\varepsilon} \to \tilde{d}_i$ and $x_{i\varepsilon} \to \xi_i$ as $\varepsilon \to 0$, and

$$\mu_{i\varepsilon} := \left\{ \begin{array}{ll}
\varepsilon \frac{(N-2)p-2}{2} \tilde{d}_{i\varepsilon} & \text{if } p \in (\max\{1, \frac{3}{N-1}\}, \frac{N-1}{2}), \\
\frac{-(N-2)\varepsilon}{W_{-1}(-(N-2)\varepsilon)} \frac{1}{\nu^2} \tilde{d}_{i\varepsilon} & \text{if } p = \frac{N}{N-2}, \\
\varepsilon \frac{1}{\nu^2} \tilde{d}_{i\varepsilon} & \text{if } p \in (\frac{N}{N-2}, \frac{N+2}{2})
\end{array} \right. \quad (1.23)$$

for each $i = 1, \ldots, k$, where $W_{-1}$ stands for the Lambert $W$ function.\(^5\) Particularly, the family $\{(u_{\varepsilon}, v_{\varepsilon})\}_{\varepsilon \in (0, \varepsilon_0)}$ satisfies the uniform energy bound (1.13).

Remark 1.6. A couple of remarks regarding the above theorems are in order.

(1) In Theorems 1.2 and 1.3, we use the convexity of the domain $\Omega$ only to show that the blow-up points are away from the boundary $\partial \Omega$.

Equation (1.2) is well-behaved under the Kelvin transform, and in particular, the moving plane method yields that the blow-up points for (1.2) are away from $\partial \Omega$ for any smooth bounded domain. Such an argument no longer works for system (1.1) in general, because it is rather ill-behaved under the Kelvin transform (see, e.g. [10, Proposition 4.4]).

In [9], Choi and the first author developed an alternative method to exclude the boundary blow-up behavior for the family of least energy solutions of (1.6), which mainly uses local Pohozaev identities. Unfortunately, their idea is not directly applicable to our general situation. Hence one has a challenging question: Is it possible to lift the convexity assumption on $\Omega$ in Theorems 1.2 and 1.3? If it is true, one will be also able to derive a general blow-up result for $p \in \left(\frac{N}{N-2}, \frac{N+2}{2}\right)$ owing to our proof.

(2) If $p = q_0 = \frac{N+2}{N-2}$, then system (1.1) with $q = q_0$ and $u = v$ is reduced to the critical Lane-Emden equation. By virtue of the invariance of (1.2) under the Kelvin transform, an energy condition corresponding to (1.13) automatically holds; refer to Li [23, Theorem 0.2]. The analogous results to Theorems 1.2 and 1.3 were established by Barhi et al. [1], and one to Theorem 1.4 was deduced by Grossi and Takahashi [16, Theorem 2.5].

If $p = 1$, then (1.1) is reduced to the biharmonic Lane-Emden equation. For this case, Geng [15] proved results corresponding to Theorems 1.2 and 1.3, assuming that $\Omega$ is convex and $n \geq 5$.

(3) The condition $p \geq 1$ is often used in Section 3: For instance, we need it when we apply the moving plane method to exclude the boundary blow-up. We also require it in the proof of Lemmas 3.2 and 3.3, and Propositions 3.10 and 3.12.

Moreover, the conditions $N \geq 4$ and $p > \frac{3}{N-2}$ are used in Section 5; see e.g. the proof of Lemmas 5.1 and 5.2. We wonder if these can be lifted in Theorems 1.2–1.4.

(4) In Theorems 1.3 and 1.4, we corrected some coefficients which were not properly given in [17, 9].

(5) For $p \in (1, \frac{N-1}{N-2})$, the function $\Upsilon_{p,k}$ in (1.21) coincides with the function $F_{1\varepsilon}$ in [21, (5.2)] up to a constant multiple.

\(^5\)If $x \in \left[-\frac{1}{2}, 0\right)$, then $y = W_{-1}(x)$ is the unique solution of $ye^y = x$ such that $y \in (-\infty, -1]$. It is known that $W_{-1}$ is a strictly deceasing function and $W_{-1}(x) \simeq \log(-x)$ for $x$ near $0$. In particular, $\frac{-(N-2)}{W_{-1}(-(N-2)\varepsilon)} \simeq \frac{(N-2)\varepsilon}{\log(-\varepsilon)}$ for $\varepsilon > 0$ small.
1.3. **Structure of the paper.** In Section 2, we provide several definitions and auxiliary lemmas needed throughout the paper. In Section 3, we carry out a blow-up analysis of \((u_\varepsilon, v_\varepsilon)\) by modifying the argument of Schoen [26] (see also, e.g., [15, 23, 24]) to be suitable for examining the coupled system (1.1). As a consequence, we show Theorem 1.2. In Section 4, we deduce various information on the shape of \((u_\varepsilon, v_\varepsilon)\). From it, we find necessary conditions for the blow-up rates and locations in Section 5, establishing Theorems 1.3 and 1.4. In Section 6, we prove the existence result stated in Theorem 1.5.

1.4. **Notations.** We list some notational conventions which will be used throughout the paper.

- Given \(N \in \mathbb{N}\), we write \(B(\xi, r) = \{x \in \mathbb{R}^N : |x - \xi| < r\}\) for each \(\xi \in \mathbb{R}^N\) and \(r > 0\).
- \(\mathbb{R}^N_+ := \mathbb{R}^{N-1} \times (0, \infty)\) is the upper-half space in \(\mathbb{R}^N\).
- The surface measure is denoted as \(dS\).
- Let \(D\) be a domain with boundary \(\partial D\). The outward unit normal vector on \(\partial D\) is written as \(\nu\).
- Let \((A)\) be a condition. We set \(1_{(A)} = 1\) if \((A)\) holds and 0 otherwise.
- For a function \(f : \mathbb{R}^N \to \mathbb{R}\) and \(l = 1, \ldots, N\), we write \(\partial_l f(x) = \partial_x^l f(x)\).
- \(C > 0\) is a generic constant independent of small numbers \(\varepsilon > 0\) that may vary from line to line.

## 2. Preliminary results

### 2.1. Green’s function \(G\) and its related functions.**

Let \(G\) be the Green’s function of the Laplacian \(-\Delta\) in \(\Omega\) with the Dirichlet boundary condition, and \(H\) the regular part of \(G\), that is,

\[
\begin{cases}
-\Delta_x H(x, \xi) = 0 & \text{for } x \in \Omega, \\
H(x, \xi) = \frac{\gamma_N}{|x - \xi|^{N-2}} & \text{for } x \in \partial\Omega \quad \text{where } \gamma_N := \frac{1}{(N-2)|S^{N-1}|}
\end{cases}
\]

for each \(\xi \in \Omega\). It is well-known that

\[
0 < G(x, \xi) = G(\xi, x) = \frac{\gamma_N}{|x - \xi|^{N-2}} - H(x, \xi) < \frac{\gamma_N}{|x - \xi|^{N-2}} \quad \text{for } (x, \xi) \in \Omega \times \Omega, \ x \neq \xi.
\]

The Robin function \(\tau\) is given by \(\tau(x) = H(x, x)\) for \(x \in \Omega\).

If \(p \in \left(\frac{2}{N-2}, \frac{N}{N-2}\right)\), then there is a number \(\varepsilon > 0\) depending only on \(N\) and \(p\) such that \(G^p(\cdot, \xi) \in L^{1+p}(\Omega)\) for each \(\xi \in \Omega\). Accordingly, (1.12) has the unique solution \(\widetilde{G}(\cdot, \xi) \in W^{2,1+p}(\Omega)\) that can be represented by an integral. Define the regular part \(\widetilde{H} : \Omega \times \Omega \to \mathbb{R}\) of \(\widetilde{G}\) by

\[
\widetilde{H}(x, \xi) = \begin{cases}
\frac{\gamma_{N,p,1}}{|x - \xi|^{(N-2)p-2}} - \tilde{G}(x, \xi) & \text{if } p \in \left(\frac{2}{N-2}, \frac{N}{N-2}\right), \\
\frac{\gamma_{N,p,1}}{|x - \xi|^{(N-2)p-2}} - \frac{\gamma_{N,p,2} H(x, \xi)}{|x - \xi|^{(N-2)p-N}} - \tilde{G}(x, \xi) & \text{if } p \in \left[\frac{N-1}{N-2}, \frac{N}{N-2}\right),
\end{cases}
\]

where

\[
\begin{cases}
\gamma_{N,p,1}^p = \gamma_{N,p,1}[(N-2)p - 2] & \text{for } p \in \left(\frac{2}{N-2}, \frac{N}{N-2}\right), \\
\gamma_{N,p,2}^p = \gamma_{N,p,2}[(N-2)p - 2(N-1)] & \text{for } p \in \left[\frac{N-1}{N-2}, \frac{N}{N-2}\right).
\end{cases}
\]

Then, an application of elliptic regularity theory to the equation of \(\widetilde{H}\) yields that the map \(x \in \Omega \mapsto \widetilde{H}(x, \xi)\) belongs to \(C^1_{\text{loc}}(\Omega)\) for each \(\xi \in \Omega\); refer to [9, Lemma 2.3] for the proof. Set \(\tilde{\tau}(x) = \widetilde{H}(x, x)\) for \(x \in \Omega\).
Given \( \delta = (\delta_1, \ldots, \delta_k) \in (0, \infty)^k \) and \( \xi = (\xi_1, \ldots, \xi_k) \in \Omega^k \) such that \( \xi_i \neq \xi_j \) for \( 1 \leq i \neq j \leq k \), let \( \tilde{G}_{\delta, \xi} : \Omega \to \mathbb{R} \) be the function defined by (1.18) and \( \tilde{H}_{\delta, \xi} : \Omega \to \mathbb{R} \) the global \( C^1 \)-regularization of \( \tilde{G}_{\delta, \xi} \) given as

\[
\tilde{H}_{\delta, \xi}(x) = \begin{cases} \sum_{i=1}^{k} \frac{\tilde{g}_{N,p,1} \delta_j^{\frac{N_p}{p+1}}}{|x - \xi_i|^{(N-2)p-2}} - \tilde{G}_{\delta, \xi}(x) & \text{if } p \in (\frac{2}{N-2}, \frac{N-1}{N-2}), \\
\sum_{i=1}^{k} \frac{\tilde{g}_{N,p,2} \delta_j^{\frac{N}{p+1}}}{|x - \xi_i|^{(N-2)p-2}} - \tilde{G}_{\delta, \xi}(x) & \text{if } p \in [\frac{N-1}{N-2}, \frac{N}{N-2}) \end{cases}
\]  

(2.4)

where

\[
A_{\delta, \xi, i} := \delta_i^{\frac{N}{p+1}} \tau(\xi_i) - \sum_{j=1, j \neq i}^{k} \delta_j^{\frac{N}{p+1}} G(\xi_i, \xi_j).
\]  

(2.5)

For each \( i = 1, \ldots, k \), we also define the local \( C^1 \)-regularization \( \tilde{H}_{\delta, \xi, i} : \Omega \to \mathbb{R} \) of \( \tilde{G}_{\delta, \xi} \) around \( \xi_i \) by

\[
\tilde{H}_{\delta, \xi, i}(x) = \begin{cases} \frac{\tilde{g}_{N,p,1} \delta_i^{\frac{N_p}{p+1}}}{|x - \xi_i|^{(N-2)p-2}} - \tilde{G}_{\delta, \xi}(x) & \text{if } p \in (\frac{2}{N-2}, \frac{N-1}{N-2}), \\
\frac{\tilde{g}_{N,p,2} A_{\delta, \xi, i} \delta_i^{\frac{N}{p+1}}}{|x - \xi_i|^{(N-2)p-2}} - \tilde{G}_{\delta, \xi}(x) & \text{if } p \in [\frac{N-1}{N-2}, \frac{N}{N-2}) \end{cases}
\]  

(2.6)

Reasoning as in the proof of [21, Lemma 2.11], we observe that \( \tilde{H}_{\delta, \xi} \in C^1(\overline{\Omega}) \) and \( \tilde{H}_{\delta, \xi, i} \) is continuously differentiable in a neighborhood of \( \xi_i \).

For future use, we further introduce the following functions: Let \( \tilde{H} : \Omega \times \Omega \to \mathbb{R} \) be the function such that

\[
-\Delta_x \tilde{H}(x, \xi) = 0 \quad \text{for } x \in \Omega
\]  

(2.7)

and

\[
\tilde{H}(x, \xi) = \begin{cases} \gamma_N \log \frac{|x - \xi|}{\xi_0} & \text{for } p = \frac{N}{N-2} \text{ and } x \in \partial \Omega, \\
\frac{|x - \xi|^{2p-2}}{\xi_0} & \text{for } p \in (\frac{2}{N-2}, \frac{N}{N-2}) \text{ and } x \in \partial \Omega \end{cases}
\]  

(2.8)

for each \( \xi \in \Omega \). Furthermore, we set

\[
\tilde{G}(x, \xi) = \frac{\gamma_N}{|x - \xi|^{(N-2)p-2}} - \tilde{H}(x, \xi) \quad \text{for } x, \xi \in \Omega, \ x \neq \xi.
\]  

(2.9)

Given \( p \in [\frac{N-1}{N-2}, \frac{N}{N-2}) \), let \( \overline{H} : \Omega \times \Omega \to \mathbb{R} \) be the function satisfying

\[
\begin{cases} -\Delta_x \overline{H}(x, \xi) = 0 & \text{for } x \in \Omega, \\
\overline{H}(x, \xi) = |x - \xi|^{N-(N-2)p} & \text{for } x \in \partial \Omega \end{cases}
\]  

(2.10)

for each \( \xi \in \Omega \).

### 2.2. Some results regarding the standard bubble \((U_{1,0}, V_{1,0})\).

Recall that the standard bubble \((U_{1,0}, V_{1,0})\) is the unique least energy solution of (1.10) satisfying (1.11). The first result on this pair is due to Hulshof and Van der Vorst [20].
Lemma 2.1. The pair \((U_{1,0}, V_{1,0})\) is unique, radially symmetric, and decreasing in the radial variable. Also, there exist numbers \(a_{N,p}, b_{N,p} > 0\) depending only on \(N\) and \(p\) such that

\[
\begin{aligned}
&\left\{ \begin{array}{ll}
\lim_{|y| \to \infty} |y|^{N-2}U_{1,0}(y) = a_{N,p} & \text{if } p \in (\frac{N-2}{N-2}, \frac{N+2}{N-2}], \\
\lim_{|y| \to \infty} |y|^{N-2}U_{1,0}(y) = a_{N,p} & \text{if } p = \frac{N}{N-2}, \\
\lim_{|y| \to \infty} \log |y|U_{1,0}(y) = a_{N,p} & \text{if } p = \frac{N}{N-2}, \\
\lim_{|y| \to \infty} |y|^{(N-2)p-2}U_{1,0}(y) = a_{N,p} & \text{if } p \in (\frac{2}{N-2}, \frac{N}{N-2}).
\end{array} \right.
\end{aligned}
\] (2.11)

If \(p \in (\frac{2}{N-2}, \frac{N}{N-2})\), we also have that

\[
b_{N,p}^p = a_{N,p}[(N-2)p - 2][N - (N-2)p].
\] (2.12)

In the next lemma, we present a refinement of (2.11), which generalizes [21, Corollaries 2.6 and 2.7]. Our proof is different from and simpler than one in [21]. For the sake of convenience, we write \(U_{1,0}(y) = U_{1,0}(|y|)\) and \(V_{1,0}(y) = V_{1,0}(|y|)\).

Lemma 2.2. Assume that \(r = |y| \geq 1\) and \((U'_{1,0}, V'_{1,0})\) denotes the derivative of \((U_{1,0}, V_{1,0})\) with respect to \(r\). There exists a constant \(C > 0\) depending only on \(N\) and \(p\) such that

\[
\left| V_{1,0}(r) - \frac{b_{N,p}}{r^{N-2}} \right| \leq \frac{C}{r^N} \quad \text{and} \quad \left| V'_{1,0}(r) + \frac{(N-2)b_{N,p}}{r^{N-1}} \right| \leq \frac{C}{r^{N-1}}.
\] (2.13)

Besides, if \(p \in (\frac{N}{N-2}, \frac{N+2}{N-2}]\), then

\[
\left| U_{1,0}(r) - \frac{a_{N,p}}{r^{N-2}} \right| \leq \frac{C}{r^N} \quad \text{and} \quad \left| U'_{1,0}(r) + \frac{(N-2)a_{N,p}}{r^{N-1}} \right| \leq \frac{C}{r^{N-1}}.
\] (2.14)

where \(\kappa_0 := (N-2)p - N > 0\). If \(p = \frac{N}{N-2}\), then

\[
\left| U_{1,0}(r) - \frac{a_{N,p} \log r}{r^{N-2}} \right| \leq \frac{C}{r^N} \quad \text{and} \quad \left| U'_{1,0}(r) + \frac{(N-2)a_{N,p} \log r}{r^{N-1}} \right| \leq \frac{C}{r^{N-1}}.
\] (2.15)

If \(p \in (\frac{2}{N-2}, \frac{N}{N-2})\), then

\[
\left| U_{1,0}(r) - \frac{a_{N,p}}{r^{(N-2)p-2}} \right| \leq \frac{C}{r^{(N-2)p-2} + \kappa_1} \quad \text{and} \quad \left| U'_{1,0}(r) + \frac{(N-2)p - 2}{r^{(N-2)p-1}} \right| \leq \frac{C}{r^{(N-2)p-1} + \kappa_1}
\] (2.16)

where \(\kappa_1 \in (0, \min\{(N - (N-2)p, ((N-2)p - 2)q_0 - N)\}).^6

Proof. In (3.22)–(3.24) of [20], it was found that

\[
\lim_{r \to \infty} \frac{rU'_{1,0}(r)}{U_{1,0}(r)} = \begin{cases} 2 - N & \text{if } p \in [\frac{N}{N-2}, \frac{N+2}{N-2}], \\
2 - (N-2)p & \text{if } p \in (\frac{2}{N-2}, \frac{N}{N-2}], \end{cases} \quad \text{and} \quad \lim_{r \to \infty} \frac{rV'_{1,0}(r)}{V_{1,0}(r)} = 2 - N.
\] (2.17)

Applying (2.11), (2.17), (1.10), and the radial symmetry of \((U_{1,0}, V_{1,0})\), we see

\[
(2 - N)b_{N,p} - r^{N-1}V'_{1,0}(r) = \int_{r}^{\infty} (t^{N-1}V'_{1,0}(t))' dt = -\int_{r}^{\infty} U'^{q_0}_{1,0}(t)t^{N-1} dt
\]

\[
= \begin{cases} O(r^{N-(N-2)q_0}) & \text{if } p \in [\frac{N}{N-2}, \frac{N+2}{N-2}], \\
O(r^{N-(N-2)q_0} \log r) & \text{if } p = \frac{N}{N-2}, \\
O(r^{N-((N-2)p-2)q_0}) & \text{if } p \in (\frac{2}{N-2}, \frac{N}{N-2}).
\end{cases}
\] (2.18)

as \(r \to \infty\), so the second inequality in (2.13) holds. By integrating it over \([r, \infty)\), we obtain the first inequality in (2.13).

\(^6\)We have that \(((N-2)p - 2)q_0 > N + 2\) as shown in [21, Lemma A.1].
To study the decay of $U_{1,0}(r)$ and $U'_{1,0}(r)$ as $r \to \infty$, we consider three different cases.

**Case 1:** $p = \left(\frac{N}{N-2}, \frac{N+2}{N-2}\right)$. In this case, arguing as in (2.18), we discover (2.14).

**Case 2:** $p = \frac{N}{N-2}$. By (2.13),

\[
\begin{align*}
N^{-1}U'_{1,0}(r) - U_{1,0}(1) &= - \int_1^r V_{1,0}'(t) t^{N-1} dt = - \int_1^r t^{N-1} (b_{N,p} t^{2-N} + O(t^{-N}))^p dt \\
&= - \int_1^r t^{N-1} \left(b_{N,p} t^{2-N} + O(t^{-N-2})\right) dt = -b_{N,p}^p \log r + O(1),
\end{align*}
\]

which implies that

\[U'_{1,0}(r) = -b_{N,p}^p \log r + O(1).\]

Integrating it over $[r, \infty)$, we obtain

\[U_{1,0}(r) = \frac{b_{N,p}^p}{N-2} \log r + O(r^{2-N}). \tag{2.19}\]

Comparing (2.19) with (2.11), we get that $b_{N,p}^p = (N-2)a_{N,p}$. From this, we deduce (2.15).

**Case 3:** $p \in (\frac{N}{N-2}, \frac{N}{N-2})$. From (2.18), we observe

\[
(r^{N-1}U'_{1,0}(r))' = -V_{1,0}'(r) r^{N-1} - r^{N-1} (b_{N,p} r^{2-N} + O(r^{2-((N-2)p-2)q_0}))^p
\]

\[
= -b_{N,p}^p r^{N-1} - r^{N-1} - O(r^{2-((N-2)p-2)q_0}).
\]

Let

\[U^*(r) = \frac{b_{N,p}^p}{[(N-2)p-2][N-(N-2)p]} r^{2-(N-2)p} = a_{N,p} r^{2-(N-2)p},\]

where the second equality comes from (2.12). Then

\[
(r^{N-1} U'_{1,0}(0)) = O(r^{N-1} \cdot r^{N-1} - O(r^{2-((N-2)p-2)q_0}) \cdot \log r).
\]

Integrating it over $[1, r]$ gives

\[U'_{1,0}(r) + a_{N,p}((N-2)p-2)r^{1-(N-2)p} = O(r^{1-N}) + O(r^{1-(N-2)p} \cdot r^{N-((N-2)p-2)q_0} \cdot \log r).
\]

This equality leads to (2.16).

Using the previous lemma, one can also relate the numbers $a_{N,p}$ and $b_{N,p}$ when $p \in \left[\frac{N}{N-2}, \frac{N+2}{N-2}\right]$; cf. (2.12).

**Lemma 2.3.** Let

\[A_1 = \int_{\mathbb{R}^N} U_{1,0}^{q_0} \quad \text{and} \quad A_2 = \int_{\mathbb{R}^N} V_{1,0}^p.
\]

If $p \in \left(\frac{N}{N-2}, \frac{N+2}{N-2}\right)$, then

\[a_{N,p} A_1 = b_{N,p} A_2. \tag{2.21}\]

If $p = \frac{N}{N-2}$, then

\[b_{N,p}^p = (N-2)a_{N,p} \quad \text{and} \quad a_{N,p} A_1 = a_{N,p} b_{N,p}^{-1} = b_{N,p}^{p+1} |\mathbb{S}^{N-1}|. \tag{2.22}\]

**Proof.** Suppose that $p \in \left(\frac{N}{N-2}, \frac{N+2}{N-2}\right)$. If $U^*_1$ denotes the Kelvin transform of $U_{1,0}$, then the representation formula gives

\[U^*_1(y) = \int_{\mathbb{R}^N} \frac{\gamma_N}{\gamma_N} \frac{y}{y-x} |x|^{-2} |\mathbb{S}^{N-2} V_{1,0}^p(x)} |x|^{-N-2} dx, \quad \text{and so} \quad U^*_1(0) = \gamma_N A_2 \]

where $\gamma_N > 0$ is the value in (2.1). We discover from (2.11) that

\[a_{N,p} = \lim_{|y| \to \infty} |y|^{N-2} U_{1,0}(y) = \lim_{|y| \to 0} U^*_1(0) = \gamma_N A_2. \tag{2.23}\]
Similarly, we have $b_{N,p} = \gamma_N A_1$, from which we obtain (2.21).

Assume that $p = \frac{N}{N-2}$. The first equality in (2.22) was shown in the proof of Lemma 2.2. Also, since the relation $b_{N,p} = \gamma_N A_1$ still holds, the second equality is true. The third equality is a consequence of $\gamma_N^{-1} = (N-2)|\mathbb{S}^{N-1}|$ and the first one.

Let

$$\left(\Psi^0_{1,0}(x), \Phi^0_{1,0}(x) \right) = \left( x \cdot \nabla U_{1,0}(x) + \frac{NU_{1,0}(x)}{q_0 + 1}, x \cdot \nabla V_{1,0}(x) + \frac{NV_{1,0}(x)}{p + 1} \right)$$

and

$$\left(\Psi^l_{1,0}(x), \Phi^l_{1,0}(x) \right) = (\partial_t U_{1,0}(x), \partial_t V_{1,0}(x))$$

for $x \in \mathbb{R}^N$ and $l = 1, \ldots, N$. Also, given any parameter $(\mu, \xi) \in (0, \infty) \times \Omega$, we set

$$\left(\Psi^0_{\mu,\xi}(x), \Phi^0_{\mu,\xi}(x) \right) = \left( \mu^{-\frac{N}{m+1}} \Psi^0_{1,0}(\mu^{-1}(x-\xi)), \mu^{-\frac{N}{p+1}} \Phi^0_{1,0}(\mu^{-1}(x-\xi)) \right)$$

and

$$\left(\Psi^l_{\mu,\xi}(x), \Phi^l_{\mu,\xi}(x) \right) = \left( \mu^{-\frac{N}{m+1}} \Psi^l_{1,0}(\mu^{-1}(x-\xi)), \mu^{-\frac{N}{p+1}} \Phi^l_{1,0}(\mu^{-1}(x-\xi)) \right)$$

for $x \in \mathbb{R}^N$ and $l = 1, \ldots, N$. In [14], Frank et al. proved a non-degeneracy result for the bubbles.

**Lemma 2.4.** Recall the pair $(U_{\mu,\xi}, V_{\mu,\xi})$ defined in (1.20). The space of solutions of the linearized system

$$\begin{cases}
-\Delta \Psi = pV_{\mu,\xi}^{p-1} \Phi & \text{in } \mathbb{R}^N, \\
-\Delta \Phi = q_0 U_{\mu,\xi}^{q_0-1} \Psi & \text{in } \mathbb{R}^N, \\
(\Psi, \Phi) \in \tilde{W}^{2, \frac{N+2}{m+1}}(\mathbb{R}^N) \times \tilde{W}^{2, \frac{N+2}{p+1}}(\mathbb{R}^N) & \text{or } \lim_{|x| \to \infty} (\Psi(x), \Phi(x)) = (0, 0)
\end{cases}$$

is spanned by

$$\left\{ \left(\Psi^0_{\mu,\xi}(x), \Phi^0_{\mu,\xi}(x) \right), \left(\Psi^1_{\mu,\xi}(x), \Phi^1_{\mu,\xi}(x) \right), \ldots, \left(\Psi^N_{\mu,\xi}(x), \Phi^N_{\mu,\xi}(x) \right) \right\}.$$

2.3. **Algebraic property of** $(p, q_\varepsilon)$. We shall need the following elementary lemma on $(p, q_\varepsilon)$.

**Lemma 2.5.** Let $(\alpha_\varepsilon, \beta_\varepsilon)$ be the pair in (1.9). If $\varepsilon > 0$ is small enough, then

$$\max\{\alpha_\varepsilon, \beta_\varepsilon\} < \min\{N-2, (N-2)p - 2\}$$

and

$$[(N-2)p - 2]\beta_\varepsilon > (N-2)\alpha_\varepsilon. \quad (2.27)$$

**Proof.** By (1.4), we have that $(N-2)(pq_0 - 1) = 2(p + q_0 + 2)$. Thus, if we let $\alpha_0$ and $\beta_0$ be the numbers given by (1.9) with $\varepsilon = 0$, then

$$\alpha_0 = \frac{2(p + 1)}{N - 2(p + 1) + \frac{2}{N-2}(q_0 + 1)} < N - 2$$

and

$$\beta_0 < N - 2.$$

Besides, because $\beta_0p - \alpha_0 = \alpha_0q_0 - \beta_0 = 2$, we see

$$\alpha_0q_0 - \beta_0(N-2) + 2q_0 = \beta_0 - \beta_0q_0(N-2) + 2(q_0 + 1) = \beta_0 - (N-2) - 2(p + 1) < 0,$$

which implies that $\alpha_0 < (N-2)p - 2$. Similarly, $\beta_0 < (N-2)p - 2$. Therefore (2.26) is true for every small $\varepsilon > 0$.

On the other hand, we have

$$[(N-2)p - 2](q_0 + 1) = N(p + 1) > (N-2)(p + 1),$$

so (2.27) is valid for $\varepsilon > 0$ small. \qed
2.4. Local Pohozaev-type identity. The following Pohozaev-type identity will be useful in the sequel.

Lemma 2.6. Let \( \{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon \in (0, \varepsilon_0)} \) be a family of solutions of (1.1) with \( q = q_\varepsilon \). If \( B(x_0, \rho) \subset \Omega \) for some \( x_0 \in \Omega \) and \( \rho > 0 \), then

\[
\varepsilon \int_{B(x_0, \rho)} \nabla u_\varepsilon \cdot \nabla v_\varepsilon = \rho \int_{\partial B(x_0, \rho)} \left( 2 \frac{\partial u_\varepsilon}{\partial \nu} \frac{\partial v_\varepsilon}{\partial \nu} - \nabla u_\varepsilon \cdot \nabla v_\varepsilon \right) dS \\
+ \rho \int_{\partial B(x_0, \rho)} \left( \frac{v_\varepsilon^{p+1}}{p+1} + \frac{u_\varepsilon^{q+1}}{q+1} \right) dS + N \int_{\partial B(x_0, \rho)} \left( \frac{v_\varepsilon}{p+1} \frac{\partial u_\varepsilon}{\partial \nu} + \frac{u_\varepsilon}{q+1} \frac{\partial v_\varepsilon}{\partial \nu} \right) dS. \tag{2.28}
\]

Proof. By multiplying the first equation of (1.1) by \((x - x_0) \cdot \nabla v_\varepsilon \) or \( v_\varepsilon \), and integrating the both sides over \( B(x_0, \rho) \), we obtain two identities. Also, by multiplying the second equation of (1.1) by \((x - x_0) \cdot \nabla u_\varepsilon \) or \( u_\varepsilon \), and integrating the both sides over \( B(x_0, \rho) \), we obtain two more identities. Combining them and applying (1.7), we deduce (2.28). \( \square \)

3. Estimates on distances between blow-up points

Throughout this section, we assume that \( N \geq 3 \), \( p \in \left( \frac{2}{N-2}, \frac{N+2}{N-2} \right) \), and \( p \geq 1 \).

Let \( \{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon \in (0, \varepsilon_0)} \) be a family of solutions of (1.1) with \( q = q_\varepsilon \). The convexity of the domain \( \Omega \), the moving plane method, and elliptic estimates guarantee the existence of a small number \( r_0 > 0 \) and a large number \( M_0 > 0 \) such that

\[ 0 < (u_\varepsilon + v_\varepsilon)(x) \leq M_0 \quad \text{for all } x \in \Omega \text{ such that } \text{dist}(x, \partial \Omega) \leq 2r_0; \]

refer to Page 188 in Guerra [17] for the proof. In particular, all blow-up points of \( \{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon \in (0, \varepsilon_0)} \) are away from the boundary \( \partial \Omega \).

3.1. Various types of blow-up points. We present the notion of a blow-up point, an isolated blow-up point, and an isolated simple blow-up point of solutions to system (1.1) by altering the original definition in [26].

Definition 3.1. Let \( \{\varepsilon_n\}_{n \in \mathbb{N}} \) be a sequence of positive small numbers such that \( \varepsilon_n \to 0 \) as \( n \to \infty \). Let also \( \{(u_n, v_n) = (u_{\varepsilon_n}, v_{\varepsilon_n})\}_{n \in \mathbb{N}} \) be a sequence of solutions of (1.1) with \( q = q_n := q_{\varepsilon_n} \).

(1) A point \( \xi \in \Omega \) is called a blow-up point of \( \{(u_n, v_n)\}_{n \in \mathbb{N}} \) if there exists a sequence \( \{x_n\}_{n \in \mathbb{N}} \subset \Omega \) such that \( x_n \) is a local maximum point of \( u_n \), \n
\[ u_n(x_n) \to \infty \quad \text{and} \quad x_n \to \xi \in \Omega \quad \text{as } n \to \infty. \]

For the sake of convenience, we will often say that \( x_n \to \xi \) is a blow-up point of \( \{(u_n, v_n)\}_{n \in \mathbb{N}} \).

(2) A point \( \xi \in \Omega \) is called an isolated blow-up point of \( \{(u_n, v_n)\}_{n \in \mathbb{N}} \) if \( \xi \) is a blow-up point and there exist numbers \( \rho_0 \in (0, r_0] \) small and \( C > 0 \) (independent of \( n \in \mathbb{N} \)) such that

\[ u_n(x) \leq C|x - x_n|^{-\alpha_n} \quad \text{and} \quad v_n(x) \leq C|x - x_n|^{-\beta_n} \quad \text{for all } x \in B(x_n, \rho_0) \setminus \{x_n\}. \tag{3.1}\]

Here, \( \alpha_n := \alpha_{\varepsilon_n} \) and \( \beta_n := \beta_{\varepsilon_n} \) are the numbers in (1.9).

(3) Define the spherical average of \( u_n \) and \( v_n \) by

\[
\bar{u}_n(r) = \frac{1}{|\partial B(x_n, r)|} \int_{\partial B(x_n, r)} u_n dS \quad \text{and} \quad \bar{v}_n(r) = \frac{1}{|\partial B(x_n, r)|} \int_{\partial B(x_n, r)} v_n dS
\]

for \( r \in (0, \rho_0) \). We say that an isolated blow-up point \( \xi \in \Omega \) of \( \{(u_n, v_n)\}_{n \in \mathbb{N}} \) is simple if \( \xi \) is an isolated blow-up point and each map \( r \mapsto \bar{r}^{\alpha_n} \bar{u}_n \) and \( r \mapsto \bar{r}^{\beta_n} \bar{v}_n \) has only one critical point in the interval \( (0, \rho_0) \) after the value of \( \rho_0 > 0 \) is suitably reduced.
3.2. Isolated blow-up points. As a preparation, we deduce an annular Harnack inequality.

Lemma 3.2. Let \( x_n \to \xi \) be an isolated blow-up point of \( \{(u_n, v_n)\}_{n \in \mathbb{N}} \). For any \( r \in (0, \frac{\rho}{2}) \), we have
\[
\max_{x \in B(x_n, 2r) \setminus B(x_n, \frac{\rho}{2})} u_n(x) \leq C \min_{x \in B(x_n, 2r) \setminus B(x_n, \frac{\rho}{2})} u_n(x)
\] (3.2)
and
\[
\max_{x \in B(x_n, 2r) \setminus B(x_n, \frac{\rho}{2})} v_n(x) \leq C \min_{x \in B(x_n, 2r) \setminus B(x_n, \frac{\rho}{2})} v_n(x)
\] (3.3)
for \( C > 0 \) independent of \( n \in \mathbb{N} \) and \( r \in (0, \frac{\rho}{2}) \).

Proof. Given \( r \in (0, \frac{\rho}{2}) \), we set
\[
(U_{n,r}(y), V_{n,r}(y)) = \left( r^{\alpha_n} u_n(ry + x_n), r^{\beta_n} v_n(ry + x_n) \right) \quad \text{for} \ y \in B(0, 3).
\]
Since \( x_n \to \xi \) is an isolated blow-up point, we see from (3.1) that
\[
U_{n,r}(y) \leq C |y|^{-\alpha_n} \leq C \quad \text{and} \quad V_{n,r}(y) \leq C |y|^{-\beta_n} \leq C \quad \text{for all} \ y \in B \left( \frac{5}{2}, \frac{1}{4} \right).
\]
Also, \((U_{n,r}, V_{n,r})\) is a solution of a weakly coupled cooperative elliptic system
\[
\left( \Delta U_{n,r} \right) + \left( \frac{0}{U_{n,r}^{p-1}} V_{n,r}^{q-1} \right) \left( \frac{0}{V_{n,r}} \right) = \left( 0 \right) \quad \text{in} \ B \left( \frac{5}{2}, \frac{1}{4} \right).
\] (3.4)
Under the assumption that \( p \geq 1 \), system (3.4) satisfies all necessary conditions to apply Harnack’s inequality in [7, Theorem 1.1]. Consequently, (3.2)–(3.3) is valid. \( \square \)

We next show that in the case of an isolated blow-up point, a renormalization of \((u_n, v_n)\) tends to the standard bubble \((U_{1,0}, V_{1,0})\) as \( n \to \infty \).

Lemma 3.3. Let \( x_n \to \xi \) be an isolated blow-up point of \( \{(u_n, v_n)\}_{n \in \mathbb{N}} \) and \( \lambda_n = u_n^{1/\alpha_n}(x_n) \). Assume that \( \{R_n\}_{n \in \mathbb{N}} \) and \( \{\varsigma_n\}_{n \in \mathbb{N}} \) are families of numbers such that \( R_n \to \infty \) and \( \varsigma_n \to 0 \) as \( n \to \infty \). After possibly passing to a subsequence, we have
\[
\left\| \lambda_n^{-\alpha_n} u_n(\lambda_n^{-1} \cdot + x_n) - U_{1,0} \right\|_{C^2(B(0,2R_n^\varsigma))} + \left\| \lambda_n^{-\beta_n} v_n(\lambda_n^{-1} \cdot + x_n) - V_{1,0} \right\|_{C^2(B(0,2R_n^\varsigma))} \leq \varsigma_n \quad \text{(3.5)}
\]
and \( R_n \lambda_n^{-1} \to 0 \) as \( n \to \infty \).

Proof. The proof is similar to that of [23, Proposition 2.1] or [24, Lemma 3.2]. We sketch it to point out the necessary modifications.

Let \((U_n(y), V_n(y)) = \left( \lambda_n^{-\alpha_n} u_n(\lambda_n^{-1} y + x_n), \lambda_n^{-\beta_n} v_n(\lambda_n^{-1} y + x_n) \right)\) for \( y \in \lambda_n(\Omega - x_n) \).

By virtue of (1.1), Definition 3.1 (1), and (3.1), we have
\[
\begin{align*}
-\Delta U_n &= V_n^p, &-\Delta V_n &= U_n^{q_n}, &\text{in} \ B(0, \lambda_n \rho_0), \\
U_n(0) &= 1, &\nabla U_n(0) &= 0, \\
U_n(y) &\leq C |y|^{-\alpha_n} \quad \text{and} \quad V_n(y) \leq C |y|^{-\beta_n} \quad \text{for} \ y \in B(0, \lambda_n \rho_0).
\end{align*}
\] (3.6)
On the other hand, we infer from the maximum principle and (3.2) that \( U_n(y) \leq C \) for \( |y| \leq 1 \). Combining this with the last inequality of (3.6), we obtain that \( U_n(y) \leq C \) for \( |y| \leq \lambda_n \rho_0 \). Moreover, the standard elliptic regularity theory and (2.26) yield
\[
\|V_n\|_{L^\infty(B(y,1))} \leq C \left( \|V_n\|_{L^1(B(y,2))} + \|U_n^{q_n}\|_{L^\infty(B(y,2))} \right) \leq C \quad \text{for any} \ |y| \leq 1,
\]
which implies that $V_n(y) \leq C$ whenever $|y| \leq \lambda_n \rho_0$. A further application of elliptic regularity shows that there exists a pair $(U_0, V_0)$ of smooth positive functions in $\mathbb{R}^N$ such that

$$(U_n, V_n) \to (U_0, V_0) \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^N) \times C^2_{\text{loc}}(\mathbb{R}^N) \quad \text{as } n \to \infty,$$

along a subsequence. It holds that

$$
\begin{aligned}
-\Delta U_0 &= \psi'(0) = \Delta V_0 = \psi''(0) \quad \text{in } \mathbb{R}^N, \\
U_0(0) &= 1, \quad \nabla U_0(0) = 0, \\
U_0(y) &\leq \frac{C}{1 + |y|^\alpha} \quad \text{and} \quad V_0(y) \leq \frac{C}{1 + |y|^\beta} \quad \text{for } y \in \mathbb{R}^N.
\end{aligned}
$$

Using the fact that $\alpha \psi''(0) > 2$ and $\beta \psi'(0) > 2$, we conclude

$$U_0(x) = \int_{\mathbb{R}^N} \frac{\gamma_N}{|x - y|^{N-2}} \psi(x) dy \quad \text{and} \quad V_0(x) = \int_{\mathbb{R}^N} \frac{\gamma_N}{|x - y|^{N-2}} \psi'(x) dy \quad \text{for } x \in \mathbb{R}^N$$

where $\gamma_N > 0$ is the constant in (2.1). Thanks to the classification theorem of Chen et al. [6], it follows that $(U_0, V_0) = (U_{1,0}, V_{1,0})$ where $(U_{1,0}, V_{1,0})$ is the standard bubble provided $p \geq 1$.

**Remark 3.4.** The above lemma and (2.26) imply that each map $r \mapsto r^{\alpha} \tilde{u}_n$ and $r \mapsto r^{\beta} \tilde{v}_n$ has a critical point in $(0, R_n \lambda_n^{-1})$. In particular, if $\xi$ is isolated simple blow-up point of $\{(u_n, v_n)\}_{n \in \mathbb{N}}$, then we may assume that

$$(r^{\alpha} \tilde{u}_n(r))' < 0 \quad \text{and} \quad (r^{\beta} \tilde{v}_n(r))' < 0 \quad \text{for any } R_n \lambda_n^{-1} \leq r \leq \rho_0 \quad \text{and } n \in \mathbb{N}$$

by discarding a first few $(u_n, v_n)$'s.

### 3.3. Isolated simple blow-up points.

We concern a decay estimate of $(u_n, v_n)$ in a small ball (of a fixed radius) centered at an isolated simple blow-up point.

**Lemma 3.5.** Assume that Lemma 3.3 holds with $R_n \to \infty \quad \text{and} \quad 0 < \zeta_n < e^{-R_n}$. If $x_n \to \xi$ is an isolated simple blow-up point of $\{(u_n, v_n)\}_{n \in \mathbb{N}}$, then for a sufficiently small $\eta > 0$, there exist $C > 0$ and $\rho_1 \in (0, \frac{\rho_0}{2})$ independent of $n \in \mathbb{N}$ (but dependent on $\eta$) such that

$$v_n(x) \leq C \lambda_n^{\beta - (N-2)+\eta} |x - x_n|^{-(N-2)+\eta} \tag{3.8}$$

and

$$u_n(x) \leq \begin{cases}
C \lambda_n^{\alpha - (N-2)+\eta} |x - x_n|^{-(N-2)+\eta} & \text{if } p \in \left(\frac{N-2}{2}, \frac{N+2}{2}\right), \\
C \lambda_n^{\alpha - (N-2)+\eta} |x - x_n|^{-(N-2)+\eta} \log(\lambda_n |x - x_n|) & \text{if } p = \frac{N-2}{2}, \\
C \lambda_n^{\alpha+2-(N-2)+\eta} |x - x_n|^{-(N-2)+\eta} & \text{if } p \in \left(\frac{2}{N-2}, \frac{N}{2}\right),
\end{cases} \tag{3.9}$$

for $r_n := R_n \lambda_n^{-1} \leq |x - x_n| \leq \rho_1$.

**Proof.** Throughout the proof, we write $r = |x - x_n|$.

By (3.2), Definition 3.1 (3), (3.5), and (2.11),

$$r^{\alpha} u_n(x) \leq C r^{\alpha} \tilde{u}_n(r) \leq C r^{\alpha} \tilde{u}_n(r_n) \leq \begin{cases}
C r^{\alpha} \lambda_n^{R_n^{2-N}} = CR_n^{\alpha - (N-2)} & \text{if } p \in \left(\frac{N-2}{2}, \frac{N+2}{2}\right), \\
C r^{\alpha} \lambda_n^{R_n^{2-N}} \log R_n = CR_n^{\alpha - (N-2)} \log R_n & \text{if } p = \frac{N-2}{2}, \\
C r^{\alpha} \lambda_n^{R_n^{2-(N-2)p}} = CR_n^{\alpha+2-(N-2)p} & \text{if } p \in \left(\frac{2}{N-2}, \frac{N}{2}\right)
\end{cases} \quad \text{as } r_n \leq r \leq \rho_1 \tag{3.10}$$

for $r_n \leq r \leq \rho_1$. We have similar inequalities for $v_n$.

Define

$$L_n(\phi, \psi) = (\Delta \phi + v_n^{p-1} \eta_\psi, \Delta \psi + u_n^{q-1} \phi)$$

for a pair $(\phi, \psi)$ of functions on $\overline{B(x_n, R_0)} \setminus B(x_n, r_n)$. We know that $u_n$ and $v_n$ are positive on their domains, and $L_n(u_n, v_n) = 0$. Therefore a minor modification of the proof of [13, Theorem
1.2] shows that the maximum principle holds for \( L_n \). Let \( b_n = \beta_n(p - 1) \). Because of the condition 
\[ 1 \leq p < \frac{N + 2}{N - 2}, \]
we may assume that
\[ 0 \leq b_n = \frac{2(p - 1)(q_n + 1)}{pq_n - 1} < 2 \quad \text{and} \quad \alpha_n(q_n - 1) = 4 - b_n > 2 \quad \text{for all } n \in \mathbb{N}. \tag{3.11} \]

Also, if \( N = 4 \), then \( p > \frac{2}{N - 2} = 1 \) so \( b_n \) is away from 0. If \( N = 3 \), then \( p > 2 \) and so \( N - 4 + b_n = b_n - 1 \) is positive and away from 0.

Given a sufficiently small number \( \eta \in (0, 1) \), a direct calculation with (2.26) and (3.11) shows
\[
\Delta (r^{-\eta}) + v_n^{p-1} r^{-\eta - 2 + b_n} \leq -\eta (N - 2 - \eta) r^{-\eta - 2} + o(1) r^{-\beta_n(p-1)p-\eta - 2 + b_n}
\]
\[
\leq -\frac{\eta}{2} (N - 2 - \eta) r^{-\eta - 2} < 0
\]
and
\[
\Delta (r^{-\eta - 2 + b_n}) + u_n^{q-1} r^{-\eta} \leq -(\eta + 2 - b_n)(N - 4 - \eta + b_n) r^{-\eta - 4 + b_n} + o(1) r^{-\alpha_n(q_n - 1)p - \eta}
\]
\[
\leq -\frac{1}{2} (\eta + 2 - b_n)(N - 4 - \eta + b_n) r^{-\eta - 4 + b_n} < 0
\]
for \( r_n \leq r \leq \rho_1 \). Hence
\[
L_n \left( r^{-\eta}, r^{-\eta - 2 + b_n} \right) < 0 \quad \text{for } \eta \in (0, 1) \text{ small.}
\]

The above computations also yield
\[
L_n \left( r^{-N+4-b_n}, r^{-(N-2)+\eta} \right) < 0 \quad \text{for } \eta \in (0, 1) \text{ small.}
\]

At this moment, we divide the cases according to the value of \( p \).

**Case 1:** \( p \in \left( \frac{N}{N-2}, \frac{N+2}{N-2} \right) \). By the maximum principle, (3.10), and an analogous estimate for \( r^{\beta_n} v_n \),
\[
u_n(x) \leq M_n r^{-\eta} + Q_n r^{-N+4-b_n+\eta} \quad \text{and} \quad v_n(x) \leq M_n r^{-\eta - 2 + b_n} + Q_n r^{-(N-2)+\eta}
\tag{3.12}
\]
for \( x \) with \( r_n \leq r = |x - x_n| \leq \rho_1 \). Here,
\[
M_n := \left( \rho_1^\eta + \rho_1^{2-b_n+\eta} \right) \max \left\{ \max_{x \in \partial B(x_n, \rho_1)} u_n(x), \max_{x \in \partial B(x_n, \rho_1)} v_n(x) \right\}
\tag{3.13}
\]
and
\[
Q_n := C \lambda_n^{\beta_n-(N-2-\eta)} R_n^{-\eta} = C \max \left\{ \lambda_n^{\beta_n-(N-4+b_n-\eta)} R_n^{-2+b_n-\eta}, \lambda_n^{\beta_n-(N-2-\eta)} R_n^{-\eta} \right\}
\geq \max \left\{ r_n^{N-4+b_n-\eta} \max_{x \in \partial B(x_n, r_n)} u_n(x), r_n^{N-2-\eta} \max_{x \in \partial B(x_n, r_n)} v_n(x) \right\}
\tag{3.14}
\]
for a suitable \( C > 0 \), where the equality in (3.14) follows from (3.11) and
\[
\alpha_n - b_n = \alpha_n - p \beta_n + \beta_n = \beta_n - 2.
\tag{3.15}
\]

Furthermore, from (3.2)–(3.3), (3.7), and (3.12), we observe
\[
\begin{aligned}
\rho_1^{\alpha_n} \max_{x \in \partial B(x_n, \rho_1)} u_n(x) \leq C \theta_r^{\alpha_n} \left( M_n \theta^{-\eta} + Q_n \theta^{-N+4-b_n+\eta} \right), \\
\rho_1^{\beta_n} \max_{x \in \partial B(x_n, \rho_1)} v_n(x) \leq C \theta_r^{\beta_n} \left( M_n \theta^{-\eta - 2 + b_n} + Q_n \theta^{-(N-2)+\eta} \right) \quad \text{for all } r_n \leq \theta \leq \rho_1.
\end{aligned}
\]
Choosing small \( \theta \) and \( \eta \) (independent of \( m \)) and applying \( \beta_n p > 2 \), we get
\[
M_n \leq CQ_n = C \lambda_n^{\beta_n-(N-2)+\eta} R_n^{-\eta}.
\tag{3.16}
\]
By plugging (3.16) into (3.12), we derive

\[ v_n(x) \leq C \lambda_n^{\beta_n-(N-2)+\eta} R_n^{-\eta} \left( r^{-\eta-2+b_n} + r^{-(N-2)+\eta} \right) \]

for \( r_n \leq r \leq \rho_1 \),

which implies (3.8).

It remains to verify (3.9). By (3.8), the assumption \( p > \frac{N}{N-2} \), and (3.15),

\[ -\Delta u_n = v_n^p \leq C \lambda_n^{\beta_n-(N-2)+\eta} p r^{-(N-2)+\eta} \leq C \lambda_n^{\alpha_n-(N-2)+\eta} r^{-(N-2)+\eta} \]

for \( r_n \leq r \leq \rho_1 \).

Thus

\[ \Delta \left( u_n - C \lambda_n^{\alpha_n-(N-2)+\eta} r^{-(N-2)+\eta} \right) \geq 0 \quad \text{for } r_n \leq r \leq \rho_1. \]

Besides, (3.10) gives

\[ u_n \leq C \lambda_n^{\alpha_n-(N-2)+\eta} r^{2-N} \leq C \lambda_n^{\alpha_n-(N-2)+\eta} r^{-(N-2)+\eta}. \]

It follows from the maximum principle that

\[ u_n \leq C \left( \lambda_n^{\alpha_n-(N-2)+\eta} r^{-(N-2)+\eta} + \tilde{u}_n(\rho_1) \rho_1^\eta r^{-\eta} \right) \quad \text{for } r_n \leq r \leq \rho_1. \]

Arguing as in the derivation of (3.16) and applying (3.17) at this time, we see

\[ \tilde{u}_n(\rho_1) \leq C \lambda_n^{\alpha_n-(N-2)+\eta}. \]

Inserting (3.18) into (3.17), we deduce (3.9).

**Case 2:** \( p = \frac{N}{N-2} \). A slight modification of the argument in Case 1 gives (3.8)–(3.9).

**Case 3:** \( p \in \left( \frac{2}{N-2}, \frac{N}{N-2} \right) \). Arguing as in Case 1, together with (3.15) and \( b_n \leq (N-2)(p-1) \), we see that (3.12) holds with \( M_n \) in (3.13) and \( Q_n = C \lambda_n^{\beta_n-(N-2-\eta)} R_n^{-\eta} \) for some large \( C > 0 \). From this fact, we further deduce that

\[ u_n(x), \ v_n(x) \leq C \lambda_n^{\beta_n-(N-2)+\eta} R_n^{-\eta} r^{-(N-2)+\eta} \quad \text{for } r_n \leq r \leq \rho_1, \]

which in particular implies (3.8).

Let us check (3.9). By (3.15),

\[ -\Delta u_n = v_n^p \leq C \lambda_n^{\beta_n-(N-2)+\eta} p r^{-(N-2)+\eta} = C \lambda_n^{\alpha_n+2-(N-2)-\eta} p r^{-(N-2)+\eta} \]

for \( r_n \leq r \leq \rho_1 \). Also, since \( p \in \left( \frac{2}{N-2}, \frac{N}{N-2} \right) \), it holds that

\[ -\Delta \left( r^{2-(N-2-\eta)p} \right) = \left( (N-2)p - 2 - \eta p \right) (N - (N-2)p + \eta p) r^{-(N-2-\eta)p} \geq C r^{-(N-2-\eta)p}. \]

Therefore

\[ u_n(x) \leq C \left( \lambda_n^{\alpha_n+2-(N-2)-\eta} p r^{2-(N-2)p + \eta p} + \tilde{u}_n(\rho_1) \rho_1^\eta r^{-\eta} \right) \quad \text{for } r_n \leq r \leq \rho_1. \]

Furthermore, as in (3.18), we can show that

\[ \tilde{u}_n(\rho_1) \leq C \lambda_n^{\alpha_n+2-(N-2)-\eta} p + \eta p. \]

Combining (3.19) and (3.20), we conclude that (3.9) is true. \( \square \)

**Lemma 3.6.** In the setting of Lemma 3.5, we reduce the value of \( \rho_1 \) if needed. Then there exists a constant \( C = C(\rho) > 0 \) independent of \( n \in \mathbb{N} \) such that

\[ \limsup_{n \to \infty} \max_{x \in \partial B(x_n, \rho)} \lambda_n^{\alpha_n+2-(N-2)-\eta} v_n(x) \leq C(\rho) \]

for each \( \rho \in (0, \frac{\rho_1}{2}) \).
Proof. We infer from the comparison principle and (3.5) that
\[ v_n(x) \geq C\lambda_n^{\beta_n-(N-2)} \left( |x-x_n|^{2-N} - \rho_1^{2-N} \right) \text{ in } B(x_n, \rho_1) \setminus B(x_n, r_n). \] (3.22)

By (3.9), (3.22), (2.27), and (3.5),
\[ u_n^{\beta_n} \leq C v_n^{\alpha_n} \text{ in } B \left( x_n, \frac{\rho_1}{2} \right) \] (3.23)
for all \( n \in \mathbb{N} \).

Given a fixed \( \rho \in (0, \frac{\rho_1}{2}) \), we write
\[ (\hat{u}_n, \hat{v}_n) = (\lambda_n^{-\alpha_n} u_n, \lambda_n^{-\beta_n} v_n) \text{ where } \lambda_n := \max_{x \in \partial B(x_n, \rho)} \frac{1}{\lambda_n^{\beta_n}} v_n(x) \to 0 \text{ as } n \to \infty. \]

Then, by (3.15) and \( \alpha_n q_n - \beta_n = 2 \),
\[ -\Delta \hat{u}_n = \lambda_n^2 \hat{u}_n^p \text{ and } -\Delta \hat{v}_n = \lambda_n^2 \hat{v}_n^q \text{ in } B(x_n, \rho_1). \]

In light of the Harnack inequality (3.2)–(3.3) and (3.23), if \( K \) is a compact subset of \( B(x_n, \rho_1/2) \setminus \{ \xi \} \), there exists \( C = C(K) > 1 \) such that
\[ C(K)^{-1} \leq \hat{v}_n \leq C(K) \text{ and } \hat{u}_n \leq C\hat{v}_n^{\alpha_n} \leq C(K) \text{ on } K. \]

Hence standard elliptic estimates yield
\[ \hat{v}_n \to v_0 \text{ in } C^2_{\text{loc}} \left( B \left( x_n, \frac{\rho_1}{2} \right) \setminus \{ \xi \} \right) \]
passing to a subsequence, so that \( v_0 \) is harmonic and positive in \( B(x_n, \rho_1/2) \setminus \{ \xi \} \). Also, by (3.7), we have that \( (r^{\beta_0} \tilde{v}_0(r))' < 0 \) for \( 0 < r \leq \frac{\rho_1}{2} \), so \( v_0 \) is singular at 0. It follows from Bôcher’s theorem that
\[ -\int_{B(x_n, \rho)} \Delta \hat{v}_n = -\int_{\partial B(x_n, \rho)} \nabla \hat{v}_n \cdot \nu = -\int_{\partial B(\xi, \rho)} \nabla v_0 \cdot \nu + o(1) > c \]
for some number \( c > 0 \), and so
\[ c < -\int_{B(x_n, \rho)} \Delta \hat{v}_n = \int_{B(x_n, \rho)} \lambda_n^{-\beta_n} u_n^q \leq C \lambda_n^{-\beta_n} \lambda_n^{\alpha_n q_n - N} = C \lambda_n^{-\beta_n} \lambda_n^{\beta_n-(N-2)} \]
where we used (3.5) and (3.9) for the last inequality. As a result, (3.21) holds.

The following is the main result of this subsection.

**Proposition 3.7.** In the setting of Lemma 3.5, we reduce the value of \( \rho_1 \) if needed. Then there is a constant \( C > 0 \) independent of \( n \in \mathbb{N} \) such that
\[ v_n(x) \leq C\lambda_n^{\beta_n-(N-2)} |x-x_n|^{2-N} \] (3.24)
and
\[ u_n(x) \leq \begin{cases} 0 \end{cases} \lambda_n^{\alpha_n-(N-2)} |x-x_n|^{2-N} \quad & \text{if } p \in \left( \frac{N}{N-2}, \frac{N+2}{N-2} \right), \\
C \lambda_n^{\alpha_n-(N-2)} |x-x_n|^{2-N} \log(\lambda_n |x-x_n| + 2) \quad & \text{if } p = \frac{N}{N-2}, \\
C \lambda_n^{\alpha_n+2-(N-2)p} |x-x_n|^{2-(N-2)p} \quad & \text{if } p \in \left( \frac{2}{N-2}, \frac{N}{N-2} \right) \], (3.25)
for \( |x-x_n| \leq \frac{\rho_1}{3} \).

Furthermore, there exist numbers \( a, b > 0 \) and \( a' < 0 \) such that
\[ \begin{cases} \lambda_n^{N-2-\alpha_n} u_n \to a|\cdot -\xi|^{2-N} + h_1 \quad & \text{if } p \in \left( \frac{N}{N-2}, \frac{N+2}{N-2} \right), \\
\lambda_n^{N-2-\alpha_n} \log(\lambda_n)^{-1} u_n \to a|\cdot -\xi|^{2-N} + h_1 \quad & \text{if } p = \frac{N}{N-2}, \\
\lambda_n^{(N-2)p-2-\alpha_n} u_n \to a|\cdot -\xi|^{2-(N-2)p} + a'|\cdot -\xi|^{N-(N-2)p}h_2 + h_1 \quad & \text{if } p \in \left( \frac{N-1}{N-2}, \frac{N}{N-2} \right), \\
\lambda_n^{(N-2)p-2-\alpha_n} u_n \to a|\cdot -\xi|^{2-(N-2)p} + h_1 \quad & \text{if } p \in \left( \frac{2}{N-2}, \frac{N-1}{N-2} \right) \]. (3.26)
and
\[ \lambda_n^{N-2-\beta_n} v_n \to b \cdot -\xi^{2-N} + h_2 \quad (3.27) \]
in \( C^2_{\text{loc}} \left( B(\xi, \rho_n) \setminus \{\xi\} \right) \) as \( n \to \infty \), up to a subsequence. Here,
- if \( p \in \left( \frac{N}{N-2}, \frac{N+2}{N-2} \right) \), then \( h_1 \in C^\infty(B(\xi, \frac{\rho_n}{3})) \) is a harmonic function. If \( p \in \left( \frac{2}{N-2}, \frac{N}{N-2} \right) \), then \( h_1 \in C^{1,\sigma}(B(\xi, \frac{\rho_n}{3})) \) for some \( \sigma \in (0, 1) \);
- \( h_2 \in C^\infty(B(\xi, \frac{\rho_n}{3})) \) is a harmonic function;
- we have
\[
\begin{align*}
\begin{cases}
p b^{p-1} = a'([N-2)p - 2(N-1)] [N - (N-2)p] & \text{for } p \in \left( \frac{N-1}{N-2}, \frac{N}{N-2} \right), \\
b^p = a([N-2)p - 2][N - (N-2)p] & \text{for } p \in \left( \frac{2}{N-2}, \frac{N}{N-2} \right).
\end{cases}
\end{align*}
\]
(3.28)

Proof. First, we prove (3.24). If it is not true, then there exists \( \tilde{x}_n \in B(x_n, \frac{\rho_n}{3}) \) such that
\[ v_n(\tilde{x}_n) \lambda_n^{N-2-\beta_n} |\tilde{x}_n - x_n|^{N-2} \to \infty \quad \text{as } n \to \infty, \]
along a subsequence. By (3.5), we have that \( r_n < |\tilde{x}_n - x_n| \leq \frac{\rho_n}{3} \). If we set \( \tilde{r}_n = |\tilde{x}_n - x_n| \), then 0 is an isolated simple blow-up point of \( \{(\tilde{r}_n^\alpha u_n(\tilde{r}_n \cdot +x_n), \tilde{r}_n^\beta v_n(\tilde{r}_n \cdot +x_n))\} \). By (3.21),
\[ \max_{x \in \partial B(x_n, 1)} \lambda_n^{N-2-\beta_n} v_n(x) \to \infty, \]
which contradicts the assumption. Thus, the assertion follows.

In the rest of the proof, we will verify the claims (3.25), (3.26), and (3.27) by splitting the cases according to the value of \( p \).

Case 1: \( p \in \left( \frac{N}{N-2}, \frac{N+2}{N-2} \right) \). In this case, by arguing as in the proof of Lemma 3.6, we observe
\[ \limsup_{n \to \infty} \max_{x \in \partial B(x_n, \rho)} \lambda_n^{N-2-\alpha_n} u_n(x) \leq C(\rho) \]
for all \( \rho \in (0, \frac{\rho_n}{3}) \). Using this, we can imitate the proof of (3.24) to deduce (3.25).

We have
\[
\begin{align*}
\begin{cases}
-\Delta \left( \lambda_n^{N-2-\alpha_n} u_n \right) = \lambda_n^{N-2-\alpha_n} \left( \lambda_n^{N-2-\beta_n} v_n \right)^p, \\
-\Delta \left( \lambda_n^{N-2-\beta_n} v_n \right) = \lambda_n^{N-2-\beta_n} \left( \lambda_n^{N-2-\alpha_n} u_n \right)^q_n.
\end{cases}
\end{align*}
\]
From the above equations, (3.24)–(3.25), and elliptic regularity, we see that
\[ \lambda_n^{N-2-\alpha_n} u_n \to w_1 \quad \text{and} \quad \lambda_n^{N-2-\beta_n} v_n \to w_2 \quad \text{in } C^2_{\text{loc}} \left( B \left( \xi, \frac{\rho_n}{3} \right) \setminus \{\xi\} \right) \]
after passing to a subsequence, where \( w_1 \) and \( w_2 \) are harmonic in \( B(\xi, \frac{\rho_n}{3}) \setminus \{\xi\} \). By the comparison principle, we obtain (3.22) and
\[ u_n(x) \geq C \lambda_n^{\alpha_n-(N-2)} \left( |x - x_n|^{2-N} - \rho_1^{2-N} \right) \quad \text{in } B(x_n, \rho_1) \setminus B(x_n, r_n), \]
so \( w_1, w_2 > 0 \). It follows from Böcher’s theorem that
\[ w_1 = a \cdot -\xi^{2-N} + h_1 \quad \text{and} \quad w_2 = b \cdot -\xi^{2-N} + h_2 \quad \text{in } B \left( \xi, \frac{\rho_1}{3} \right) \setminus \{\xi\} \]
where \( h_1 \) and \( h_2 \) are harmonic in \( B(\xi, \frac{\rho_1}{3}) \). Also, by (3.7), \( w_1 \) and \( w_2 \) are singular at \( \xi \), which reads \( a, b > 0 \). As a consequence, (3.26) and (3.27) are true.

Case 2: \( p = \frac{N}{N-2} \). According to (3.5),
\[ u_n(x) \leq C \lambda_n^{\alpha_n-(N-2)} |x - x_n|^{2-N} \log(\lambda_n |x - x_n| + 2) \quad \text{for } x \in B(x_n, r_n). \]
Since \(-\Delta (r^{2-N} \log(\lambda_n r)) = (N-2) r^{-N} \), the comparison argument, (3.24), and (3.7) show
\[ u_n(x) \leq C \lambda_n^{\alpha_n-(N-2)} |x - x_n|^{2-N} \log(\lambda_n |x - x_n|) \quad \text{for } x \in B \left( x_n, \frac{\rho_1}{3} \right) \setminus B(x_n, r_n), \]
and we deduce (3.25).

We have
\[
\begin{align*}
-\Delta \left[ \lambda_n^{N-2-\alpha_n} (\log \lambda_n)^{-1} u_n \right] &= \lambda_n^{N-(N-2)p} (\log \lambda_n)^{-1} \left( \lambda_n^{N-2-\beta_n} v_n \right)^p, \\
-\Delta \left( \lambda_n^{N-2-\beta_n} v_n \right) &= \lambda_n^{N-(N-2)q_n} (\log \lambda_n)^{q_n} \left[ \lambda_n^{N-2-\alpha_n} (\log \lambda_n)^{-1} u_n \right]^{q_n}.
\end{align*}
\]
From the above equations, (3.24)–(3.25), and elliptic regularity, we see that
\[
\lambda_n^{N-2-\alpha_n} (\log \lambda_n)^{-1} u_n \to w_1 \quad \text{and} \quad \lambda_n^{N-2-\beta_n} v_n \to w_2 \quad \text{in} \quad C^2_{\text{loc}} \left( B \left( \xi, \frac{\rho_1}{3} \right) \setminus \{\xi\} \right)
\]
after passing to a subsequence, where \(w_1\) and \(w_2\) are harmonic in \(B(\xi, \frac{\rho_1}{3}) \setminus \{\xi\}\). By the comparison principle, we obtain (3.22) and
\[
\begin{align*}
\lambda_n^{N-2-\alpha_n} (\log \lambda_n)^{-1} u_n \to w_1 & \quad \text{and} \quad \lambda_n^{N-2-\beta_n} v_n \to w_2 \quad \text{in} \quad C^2_{\text{loc}} \left( B \left( \xi, \frac{\rho_1}{3} \right) \setminus \{\xi\} \right) \\
\end{align*}
\]
so \(w_1, w_2 > 0\). Reasoning as in Case 1, we conclude that (3.26) and (3.27) hold.

**Case 3:** \(p \in \left( \frac{2}{N-2}, \frac{N}{N-2} \right)\). According to (3.5),
\[
u_n(x) \leq C \lambda_n^{\alpha_n + 2-(N-2)p} \left| x - x_n \right|^{2-(N-2)p} \quad \text{for} \quad x \in B(x_n, r_n).
\]
Employing this inequality, the comparison argument, and (3.7), we deduce (3.25).

We have
\[
\begin{align*}
-\Delta \left( \lambda_n^{(N-2)p-2-\alpha_n} u_n \right) &= \left( \lambda_n^{N-2-\beta_n} v_n \right)^p, \\
-\Delta \left( \lambda_n^{N-2-\beta_n} v_n \right) &= \lambda_n^{N-(N-2)p q_n + 2 q_n} \left( \lambda_n^{N-2-\alpha_n} \right)^{q_n}. \tag{3.29}
\end{align*}
\]
Note that
\[
N - (N - 2)p q_n + 2 q_n = -2p - 2 + \varepsilon_n(p + 1)(q_n + 1) < 0.
\]
Therefore, arguing as in Case 1, we obtain (3.27). Also, by (3.29), (3.24)–(3.25), elliptic regularity, and the assumption \(p < \frac{N}{N-2}\),
\[
\lambda_n^{N-2-\alpha_n} u_n \to w_1 \quad \text{in} \quad C^2_{\text{loc}} \left( B \left( \xi, \frac{5 \rho_1}{12} \right) \setminus \{\xi\} \right), \tag{3.30}
\]
up to a subsequence, and \(w_1\) satisfies
\[
-\Delta w_1 = \left( \frac{b}{|x - \xi|^{N-2} + h_2} \right)^p \quad \text{in} \quad B \left( \xi, \frac{5 \rho_1}{12} \right).
\]
From (3.28), we see that
\[
-\Delta \left( w_1 - \frac{a}{|x - \xi|^{(N-2)p-2}} \right) = \left( \frac{b}{|x - \xi|^{N-2} + h_2} \right)^p - \frac{b^p}{|x - \xi|^{(N-2)p}} \quad \text{in} \quad B \left( \xi, \frac{5 \rho_1}{12} \right). \tag{3.31}
\]
Since \(p < \frac{N-1}{N-2}\), the right-hand side of (3.31) belongs to \(L^{N+\sigma'}(B(\xi, \frac{5 \rho_1}{12}))\) for some \(\sigma' \in (0, 1)\). By elliptic regularity, (3.26) holds.

**Case 4:** \(p \in \left( \frac{N}{N-2}, \frac{N}{N-2} \right)\). In this case, the proof goes along the same lines as that of Case 3 until the derivation of (3.30). If we set \(a, a'\) as in (3.28), then
\[
-\Delta \left( w_1(x) - \frac{a}{|x - \xi|^{(N-2)p-2}} \right) = \frac{b^p}{|x - \xi|^{(N-2)p}} - \frac{bb^{p-1} h_2(x)}{|x - \xi|^{(N-2)(p-1)}} \quad \text{for} \quad x \in B \left( \xi, \frac{5 \rho_1}{12} \right) \tag{3.32}
\]
and
where we used the fact that \( h_2 \) is harmonic in \( B(\xi, \frac{5\rho}{12}) \). Since \( p < \frac{N}{N-2} \), the right-hand side of (3.32) belongs to \( L^{N+\sigma'}(B(\xi, \frac{5\rho}{12})) \) for some \( \sigma' \in (0, 1) \). By elliptic regularity, (3.26) holds.

3.4. Exclusion of bubble accumulation. We will prove that each blow-up point of \( \{(u_n, v_n)\}_{n \in \mathbb{N}} \) is isolated simple and distant from the other ones. The following result is a key ingredient of the proof.

**Lemma 3.8.** Let \( x_n \to \xi \) be an isolated simple blow-up point of \( \{(u_n, v_n)\}_{n \in \mathbb{N}} \), and \( h_1 \) and \( h_2 \) be the functions in Proposition 3.7. Assume that \( h_1(x) = A + O(|x - \xi|) \) and \( h_2(x) = B + O(|x - \xi|) \) as \( x \to \xi \).

1. If \( p \in \left( \frac{N}{N-2}, \frac{N+2}{N-2} \right) \), then
   \[
   \frac{aB}{p+1} + \frac{bA}{q_0 + 1} \leq 0. \tag{3.33}
   \]

2. If \( p \in \left( 0, \frac{2}{N-2} \right) \), then
   \[
   A \leq 0. \tag{3.34}
   \]

**Proof.** (1) One can prove it by following the argument in [24, Proposition 3.2].

(2) Fix \( \rho \in (0, \frac{2}{N-2}) \). By Pohozaev identity (2.28) and (1.1), we have

\[
\rho \int_{\partial B(x_n, \rho)} \left( 2 \frac{\partial u_n}{\partial \nu} \frac{\partial v_n}{\partial \nu} - \nabla u_n \cdot \nabla v_n \right) dS + \rho \int_{\partial B(x_n, \rho)} \left( \frac{v_n^{p+1}}{p+1} + \frac{u_n^{q+1}}{q+1} \right) dS
\]

\[
+ N \int_{\partial B(x_n, \rho)} \left( \frac{1}{p+1} - \frac{\varepsilon_n}{N} \right) v_n \frac{\partial u_n}{\partial \nu} + \frac{u_n}{q+1} \frac{\partial v_n}{\partial \nu} \right) dS = \varepsilon_n \int_{B(x_n, \rho)} v_n^{p+1} \geq 0. \tag{3.35}
\]

We will derive (3.34) by plugging the estimates in Proposition 3.7 into (3.35).

First, applying (3.26)–(3.27), we compute

\[
\lim_{n \to \infty} \lambda_n^{(N-2)(p+1)-2-(\alpha_n+\beta_n)} \rho \int_{\partial B(x_n, \rho)} \left( 2 \frac{\partial u_n}{\partial \nu} \frac{\partial v_n}{\partial \nu} - \nabla u_n \cdot \nabla v_n \right) dS
\]

\[
= ab((N-2)p-2)(N-2) |S|^{N-1} \rho^{2-(N-2)p} + O \left( \rho^{N-(N-2)p} \right) \tag{3.36}
\]

as \( \rho \to 0 \). Second, from (2.27) and (3.15), we obtain

\[
\lim_{n \to \infty} \lambda_n^{(N-2)(p+1)-2-(\alpha_n+\beta_n)} \rho \int_{\partial B(x_n, \rho)} \left( \frac{v_n^{p+1}}{p+1} + \frac{u_n^{q+1}}{q+1} \right) dS
\]

\[
= \left[ \frac{b^{p+1}}{p+1} |S|^{N-1} \rho^{2-(N-2)p} + O \left( \rho^{N-(N-2)p} \right) \right] + 0. \tag{3.37}
\]

Lastly, we see

\[
\lim_{n \to \infty} \lambda_n^{(N-2)(p+1)-2-(\alpha_n+\beta_n)} N \int_{\partial B(x_n, \rho)} \left[ \left( \frac{1}{p+1} - \frac{\varepsilon_n}{N} \right) v_n \frac{\partial u_n}{\partial \nu} + \frac{u_n}{q+1} \frac{\partial v_n}{\partial \nu} \right] dS
\]

\[
= - |S|^{N-1} \left( \frac{abN}{p+1} ((N-2)p-2) \rho^{2-(N-2)p} \right) \tag{3.38}
\]

\[
- |S|^{N-1} \left[ \frac{abN}{q_0 + 1} ((N-2)\rho^{2-(N-2)p} + \frac{bN}{q_0 + 1} (N-2)A \right] + O \left( \rho^{N-(N-2)p} \right) .
\]

By (1.4) and (3.28),

\[
ab((N-2)p-2)(N-2) + \frac{b^{p+1}}{p+1} - \left[ \frac{abN}{p+1} ((N-2)p-2) + \frac{abN}{q_0 + 1} (N-2) \right] = 0. \tag{3.39}
\]
Putting (3.36)–(3.39) into (3.35), we derive
\[
O \left( \rho^{N-(N-2)p} + \rho \right) = \frac{bN}{q_0 + 1} (N - 2) \left| S^{N-1} \right| A \geq 0. 
\] (3.40)
Finally, invoking \( p < \frac{N}{N-2} \) and taking \( \rho \to 0 \) in (3.40), we establish (3.34). \( \square \)

**Remark 3.9.** Using (3.35) with \( \rho = \frac{\rho_1}{3} \) and
\[
\varepsilon_n \int_{B(x_n, \rho_n^p)} r_n^{p+1} \geq C \varepsilon_n \lambda_{\alpha_n}^\beta \gamma_{(p+1)-N} \int_{B(0, \lambda_{\alpha_n}^p)} V_{1,0}^{p+1} \geq C \varepsilon_n \lambda_{\alpha_n}^\beta \gamma_{(p+1)-N},
\]
one can derive
\[
\varepsilon_n \leq \begin{cases} 
C \lambda_{\alpha_n}^2 \gamma_{(p+1)-N} & \text{if } p \in (\frac{N-2}{N-1}, \frac{N}{N-2}), \\
C \lambda_{\alpha_n}^2 \gamma_{(p+1)-N} \log \lambda_{\alpha_n} & \text{if } p = \frac{N}{N-2}, \\
C \lambda_{\alpha_n}^2 \gamma_{(p+1)-N} \beta & \text{if } p \in (\frac{2}{N-2}, \frac{N}{N-2}).
\end{cases} 
\] (3.41)
In particular, \( \lambda_{\alpha_n} \to 1 \) as \( n \to \infty \), and one can replace each \( \alpha_n \) and \( \beta_n \) in (3.24)–(3.27) with \( \alpha_0 \) and \( \beta_0 \), respectively.

**Proposition 3.10.** We reduce the value of \( \rho_1 \) if needed. Then every isolated blow-up point of \( \{(u_n, v_n)\}_{n \in \mathbb{N}} \) is in fact isolated simple.

**Proof.** Let \( x_n \to \xi \) be an isolated blow-up point of \( \{(u_n, v_n)\} \). We assert that each map \( r \mapsto r^\alpha \tilde{u}_n \) and \( r \mapsto r^\beta \tilde{v}_n \) has only one critical point in the interval \( (0, \rho_2) \) for some \( \rho_2 \in (0, \rho_1) \).

Suppose not. In view of Remark 3.4, after passing to a subsequence of \( (u_n, v_n) \), either the map \( r \mapsto r^\alpha \tilde{u}_n \) has the second critical point \( \zeta_1 \geq R_n \lambda_{\alpha_n}^{-1} \) or the map \( r \mapsto r^\beta \tilde{v}_n \) has the second critical point \( \zeta_2 \geq R_n \lambda_{\beta_n}^{-1} \), and \( \zeta_n := \min \{ \zeta_1, \zeta_2 \} \to 0 \) as \( n \to \infty \).\(^7\) Let
\[
(w_n, z_n)(y) = \left( \zeta_n^\alpha u_n(\zeta_n y + x_n), \zeta_n^\beta v_n(\zeta_n y + x_n) \right) \quad \text{for } |y| \leq \frac{\rho_1}{3 \zeta_n}.
\]
It satisfies
\[
\begin{align*}
-\Delta w_n &= z_n^p, \\
w_n, z_n &> 0 \\
w_n(y) &\leq C |y|^{-\alpha_n}, z_n(y) \leq C |y|^{-\beta_n} \quad \text{for } y \in B \left( 0, \frac{\rho_1}{3 \zeta_n} \right).
\end{align*}
\]
Furthermore, each map \( r \mapsto r^\alpha \tilde{u}_n \) and \( r \mapsto r^\beta \tilde{z}_n \) has precisely one critical point in the interval \( (0, 1) \), and
\[
\begin{align*}
either \frac{d}{dr} [r^\alpha w_n(r)] \bigg|_{r=1} &= 0 \quad \text{or} \quad &\frac{d}{dr} [r^\beta z_n(r)] \bigg|_{r=1} &= 0.
\end{align*}
\] (3.42)
In particular, \( 0 \in \mathbb{R}^N \) is an isolated simple blow-up point of \( \{(w_n, z_n)\}_{n \in \mathbb{N}} \).

By Proposition 3.7, (3.26) and (3.27) hold in \( C^2_{\text{loc}}(\mathbb{R}^N \setminus \{0\}) \) if \( (u_n, v_n) \), \( \lambda_n \), and \( \xi \) are replaced with \( (w_n, z_n) \), \( \tilde{\lambda}_n := u_n^{1/\alpha_n} (x_n) \), and \( 0 \), respectively. In addition,
\[
\begin{align*}
w_n(0) &= \zeta_n^\alpha u_n(x_n) = (\zeta_n \lambda_n)^\alpha \geq R_n^{\alpha_n} \to \infty, \\
z_n(0) &= \zeta_n^\beta v_n(x_n) \geq C \zeta_n^\beta \lambda_n^{\beta_n} (x_n) = C (\zeta_n \lambda_n)^\beta \geq R_n^{\beta_n} \to \infty \quad \text{(by (3.23))}
\end{align*}
\]
as \( n \to \infty \). Therefore the numbers \( a \) and \( b \) in (3.26)–(3.27) are positive and \( \lim \inf_{|x| \to \infty} h_2(x) \geq 0 \).

Since \( h_2 \) is harmonic in \( \mathbb{R}^N \), it follows that \( h_2(x) = B \geq 0 \) for all \( x \in \mathbb{R}^N \); see Lemma 3.8.

At this moment, we divide the cases according to the value of \( p \).

**Case 1:** \( p \in [\frac{N}{N-2}, \frac{N+2}{N-2}) \). Since \( h_1 \) is harmonic in \( \mathbb{R}^N \), we have that \( h_1(x) = A \geq 0 \) for all \( x \in \mathbb{R}^N \); see Lemma 3.8.\(^7\)

\(^7\)If \( r \mapsto r^\alpha \tilde{u}_n \) has no critical point on \( [R_n \lambda_{\alpha_n}^{-1}, \rho_1] \), then we set \( \zeta_1 = \infty \). Similarly, if \( r \mapsto r^\beta \tilde{v}_n \) has no critical point on \( [R_n \lambda_{\alpha_n}^{-1}, \rho_1] \), then we set \( \zeta_2 = \infty \).
Suppose that \( \zeta_n = \zeta_{1n} \) for infinitely many \( n \in \mathbb{N} \). Then the first equality of (3.42) holds for such \( m \)'s, and so
\[
0 = \frac{d}{dr} \left[ r^{\alpha_0} (ar^{2-N} + h_1) \right]_{r=1} = -a(N - 2 - \alpha_0) + A\alpha_0.
\]
Owing to (2.26), it holds that \( A > 0 \). From (3.33), it follows that
\[
0 < \frac{aB}{p+1} + \frac{bA}{q_0+1} \leq 0,
\]
a contradiction. Consequently, \( \zeta_n = \zeta_{2n} \) for all but finitely many \( n \in \mathbb{N} \). Using the second equality of (3.42), we see that this cannot happen either. The assertion must be true.

**Case 2:** \( p \in (\frac{2}{N-2}, \frac{N-1}{N}) \). By the first equation of (3.29) (with a suitable change of notations) and (3.28),
\[
-\Delta h_1(x) = (b|x|^{2-N} + B)^p - b|x|^{-(N-2)p} \geq 0 \quad \text{for } x \in \mathbb{R}^N
\]
and \( \liminf_{|x| \to \infty} h_1(x) \geq 0 \). Thus the strong maximum principle yields that \( h_1 \geq 0 \) in \( \mathbb{R}^N \), and either \( A = h_1(0) > 0 \) or \( h_1 = 0 \) in \( \mathbb{R}^N \). In light of (3.34), the first possibility cannot happen, and so \( h_1 = 0 \) in \( \mathbb{R}^N \). If the first equality of (3.42) holds for infinitely many \( n \in \mathbb{N} \), then
\[
0 = \frac{d}{dr} \left[ r^{\alpha_0} (ar^{2-(N-2)p} + 0) \right]_{r=1} = -a((N - 2)p - 2 - \alpha_0) < 0 \quad \text{(by (2.26))},
\]
a contradiction. If the second equality of (3.42) holds for infinitely many \( n \in \mathbb{N} \), then
\[
0 = \frac{d}{dr} \left[ r^{\beta_0} (br^{2-N} + B) \right]_{r=1} = -b(N - 2 - \beta_0) + \beta_0 B,
\]
from which we obtain that \( B > 0 \). However, it is again absurd, because the left-hand side of (3.43) is 0, while the right-hand side is nonzero. As a result, the assertion holds.

**Case 3:** \( p \in (\frac{N-1}{N-2}, \frac{N}{N}) \). We have
\[
-\Delta h_1(x) = (b|x|^{2-N} + B)^p - b|x|^{-(N-2)p} - \frac{1}{p}t^{p-1}|x|^{-(N-2)(p-1)}B \geq 0 \quad \text{for } x \in \mathbb{R}^N
\]
where we used \( p \geq 1 \) to get the inequality. Moreover, since \( a > 0, a' < 0 \), and \( h_2 = B \geq 0 \) in \( \mathbb{R}^N \),
\[
\text{either } B > 0 \text{ and } \liminf_{|x| \to \infty} h_1(x) = +\infty \text{ or } B = 0.
\]
The former situation cannot happen in view of (3.44) and the maximum principle. The latter situation is also impossible because of the argument in Case 2. The assertion now follows.

**Lemma 3.11.** Let \( \{(u_n, v_n)\}_{n \in \mathbb{N}} \) be a sequence of solutions of (1.1) with \( q = q_n \). Given any \( \eta > 0 \) small and \( R > 0 \) large, there exists a constant \( C > 0 \) depending on \( \eta \) and \( R \) such that if
\[
\max_{x \in \Omega} \max \left\{ u_n^{1/\alpha}(x), v_n^{1/\beta}(x) \right\} \geq C,
\]
then the followings hold for some \( k \in \mathbb{N} \) determined by the energy condition (1.13), up to a subsequence:

(1) There exists a set \( \{x_{n,1}, \ldots, x_{n,k}\} \) of local maxima of \( u_n \) such that \( \{B(x_{n,i}, R\lambda_{n,i}^{-1})\}_{i=1}^k \) is a disjoint collection of subsets in \( \Omega \). Here, \( \lambda_{n,i} := u_n^{1/\alpha}(x_{n,i}) \).

(2) For \( i = 1, \ldots, k \), it holds that
\[
\left\| \lambda_{n,i}^{-\alpha} u_n(\lambda_{n,i}^{-1} \cdot + x_{n,i}) - U_{1,0} \right\|_{C^2(B(0,R))} + \left\| \lambda_{n,i}^{-\beta} v_n(\lambda_{n,i}^{-1} \cdot + x_{n,i}) - V_{1,0} \right\|_{C^2(B(0,R))} \leq \eta.
\]
(3) We have that
\[
\left( \min_{i=1,...,k} |x-x_{n,i}|^{\alpha} \right) u_n(x) + \left( \min_{i=1,...,k} |x-x_{n,i}|^{\beta} \right) v_n(x) \leq C \quad \text{for all } x \in \Omega. \tag{3.45}
\]
(4) We have that
\[
\frac{u_n^{1/\alpha_n}(x_{n,i})}{v_n^{1/\beta_n}(x_{n,i})} \to U_{1,0}^{1/\alpha_0}(0) \text{ or } V_{1,0}^{-1/\beta_0}(0) \quad \text{as } n \to \infty \text{ for each } i = 1, \ldots, k. \tag{3.46}
\]

Proof. By suitably modifying the argument in the proof of [24, Lemma 5.1] and applying the energy condition (1.13), one can prove the next claim: Given any \( \eta > 0 \) small and \( R > 0 \) large, there exists a constant \( C > 0 \) depending on \( \eta \) and \( R \) such that if \( \{K_n\}_{n \in \mathbb{N}} \) is a sequence of compact subsets of \( \Omega \) and \( \{(u_n, v_n)\}_{n \in \mathbb{N}} \) is a sequence of solutions of (1.1) with \( q = q_n \) satisfying
\[
\max_{x \in \Omega \setminus K_n} \max \left\{ \text{dist}(x, K_n) u_n^{1/\alpha_n}(x), \text{dist}(x, K_n) v_n^{1/\beta_n}(x) \right\} \geq C,
\]
then, along a subsequence, either
\[
\|\lambda_n^{\alpha_n} u_n(\lambda_n^{-1} \cdot + x_{1n}) - U_{1,0} \|_{C^2(B(0,R))} + \|\lambda_n^{-\beta_n} v_n(\lambda_n^{-1} \cdot + x_{1n}) - V_{1,0} \|_{C^2(B(0,R))} \leq \eta
\]
for a local maximum point \( x_{1n} \) of \( u_n \) in \( \Omega \setminus K_n \), or
\[
\|\lambda_n^{\alpha_n} u_n(\lambda_n^{-1} \cdot + x_{2n}) - U_{1,0} \|_{C^2(B(0,R))} + \|\lambda_n^{-\beta_n} v_n(\lambda_n^{-1} \cdot + x_{2n}) - V_{1,0} \|_{C^2(B(0,R))} \leq \eta
\]
for a local maximum point \( x_{2n} \) of \( v_n \) in \( \Omega \setminus K_n \). Here, \( \lambda_n := u_n^{1/\alpha_n}(x_{1n}), \lambda_n := v_n^{1/\beta_n}(x_{2n}), \) and \( \text{dist}(x, \emptyset) := 1 \) where \( \emptyset \) denotes the empty set.\(^8\)

Applying the above claim, we follow the proof of [24, Proposition 5.1]. Then we can obtain the desired result. \( \square \)

**Proposition 3.12.** Let \( x_{n,i} \to \xi_i \) be blow-up points of \( \{(u_n, v_n)\}_{n \in \mathbb{N}} \) for \( i = 1, \ldots, k \) where \( k \in \mathbb{N} \). Then there exists \( \eta_0 > 0 \) independent of \( n \in \mathbb{N} \) such that \( |x_{n,i} - x_{n,j}| \geq \eta_0 \) for all \( 1 \leq i \neq j \leq k \). In particular, every blow-up point is isolated, and by Proposition 3.10, it is isolated simple.

**Proof.** Suppose not. Without loss of generality, we may assume that
\[
\zeta_n := |x_{n,1} - x_{n,2}| = \min_{1 \leq i \neq j \leq k} |x_{n,i} - x_{n,j}| \to 0 \quad \text{as } n \to \infty.
\]

Let
\[
(w_n, z_n)(y) = \left( \zeta_n^{\alpha_n} u_n(\zeta_n y + x_{n,1}), \zeta_n^{\beta_n} v_n(\zeta_n y + x_{n,1}) \right) \quad \text{for } y \in (\Omega - x_{n,1})/\zeta_n.
\]

It satisfies
\[
\begin{align*}
-\Delta w_n &= z_n^p \quad \text{in } (\Omega - x_{n,1})/\zeta_n, \\
-\Delta z_n &= w_n^q \quad \text{in } (\Omega - x_{n,1})/\zeta_n, \\
w_n, z_n &> 0 \quad \text{in } (\Omega - x_{n,1})/\zeta_n.
\end{align*}
\]

Set \( y_{n,j} = \zeta_n^{-1}(x_{n,2} - x_{n,1}) \). By employing (3.46) and arguing as in the proof of [23, Proposition 4.2], we see that
\[
w_n(0), w_n(y_{n,2}), z_n(0), z_n(y_{n,2}) \to \infty \quad \text{as } n \to \infty.
\]

Let \( \bar{y} \in \mathbb{R}^N \) be such that \( y_{n,2} \to \bar{y} \in S^{N-1} \) as \( n \to \infty \), passing to a subsequence. Also, by (3.45), 0 and \( \bar{y} \) are isolated blow-up points of \( \{(w_n, z_n)\}_{n \in \mathbb{N}} \). Thanks to Proposition 3.10, they are isolated simple.

\( ^8 \)If a classification theorem [6, 5] for the critical Lane-Emden system in \( \mathbb{R}^N \) continues to hold under a local integrability condition, we will not need (1.13) in Theorems 1.2-1.4. However, because system (1.10) is not invariant under the Kelvin transform, deducing it is a very challenging problem. For related results, see the references [27, 5, 8] and others devoted to the Lane-Emden conjecture.
Let $\bar{S}$ be the set of blow-up points of $\{(w_n, z_n)\}_{n \in \mathbb{N}}$. Clearly, $0, \bar{y} \in \bar{S}$ and $\inf\{|\bar{y}_1 - \bar{y}_2|: \bar{y}_1, \bar{y}_2 \in \bar{S}, \bar{y}_1 \neq \bar{y}_2\} \geq 1$.

At this moment, we divide the cases according to the value of $p$.

**Case 1:** $p \in (\frac{N}{N-2}, \frac{N+2}{N-2})$. Denote $\tilde{\lambda}_n = w^1_{\alpha_n}(0)$. As in the proof of Proposition 3.7, one can find constants $a_1, a_2, b_1, b_2 > 0$ and nonnegative harmonic functions $h_1^*$ and $h_2^*$ in $\mathbb{R}^N \setminus (\bar{S} \setminus \{0, \bar{y}\})$ such that

$$\tilde{\lambda}_n^{N-2-\alpha} w_n \to a_1 |2^{N-2} + a_2 | - \bar{y} |2^{N-2} + h_1^* \text{ in } C^2_{\text{loc}} \left( \mathbb{R}^N \setminus \bar{S} \right)$$

and

$$\tilde{\lambda}_n^{N-2-\beta} z_n \to b_1 |2^{N} + b_2 | - \bar{y} |2^{N} + h_2^* \text{ in } C^2_{\text{loc}} \left( \mathbb{R}^N \setminus \bar{S} \right)$$

(3.47) as $n \to \infty$, up to a subsequence. In particular, the functions $a_2 | - \bar{y} |2^{N} + h_1^*$ and $b_2 | - \bar{y} |2^{N} + h_2^*$ are positive near 0, which contradicts (3.33).

**Case 2:** $p = \frac{N}{N-2}$. The proof is similar to Case 1, so we omit it.

**Case 3:** $p \in (\frac{2}{N-2}, \frac{N-1}{N-2})$. The proof of Proposition 3.7 shows that (3.47) holds and

$$\tilde{\lambda}_n^{N-2-\alpha} w_n \to a_1 |2^{(N-2)p} + a_2 | - \bar{y} |2^{(N-2)p} + h_1^* \text{ in } C^2_{\text{loc}} \left( \mathbb{R}^N \setminus \bar{S} \right)$$

up to a subsequence, where $(a, b) = (a_1, b_1)$ or $(a_2, b_2)$ satisfies (3.28), and $h_1^* \in C^1, \sigma > 0, (\mathbb{R}^N \setminus (\bar{S} \setminus \{0, \bar{y}\}))$ for some $\sigma \in (0, 1)$. By the maximum principle and the inequality

$$(1 + x + y)p \geq 1 + x^p + y^p \text{ for } x, y \geq 0 \text{ and } p \geq 1,$$

$h_1^*$ is nonnegative. In particular, $a_2 | - \bar{y} |2^{(N-2)p} + h_1^* > 0$ near 0, which contradicts (3.34).

**Case 4:** $p \in [\frac{N-1}{N-2}, \frac{N}{N-2})$. Arguing as in Case 3, we deduce (3.47) and

$$\tilde{\lambda}_n^{N-2-\alpha} w_n \to a_1 |2^{-(N-2)p} + a_2 | - \bar{y} |2^{-(N-2)p} + h_3^* \text{ in } C^2_{\text{loc}} \left( \mathbb{R}^N \setminus \bar{S} \right)$$

up to a subsequence, where

$$0 \leq h_3^* := a'_3 |N-(N-2)p h_2^* + a'_2 | - \bar{y} |N-(N-2)p h_2^* + h_1^* \in C^0, \sigma \left( \mathbb{R}^N \setminus (\bar{S} \setminus \{0, \bar{y}\}) \right)$$

for some $\sigma \in (0, 1)$. In particular, $a_2 | - \bar{y} |2^{-(N-2)p} + h_3^* > 0$ near 0, which contradicts (3.34). \qed

3.5. **Proof of Theorem 1.2.** If $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon \in (0, \varepsilon_0]}$ is bounded in $(L^\infty(\Omega))^2$, then elliptic regularity readily implies (1.14). In the following, we prove Theorem 1.2 (1)–(3) after assuming that $(u_\varepsilon, v_\varepsilon)$ blows up at $k \in \mathbb{N}$ points $\xi_1, \ldots, \xi_k \in \Omega$ as $\varepsilon \to 0$.

**Proof of Theorem 1.2 (2).** As already remarked, the convexity assumption on $\Omega$ leads us that each blow-up point $\xi_i$ is away from $\partial \Omega$. Moreover, Proposition 3.7 implies that

$$(u_\varepsilon, v_\varepsilon) \to (0, 0) \text{ in } C^1_{\text{loc}} \left( B \left( \xi_i, \frac{\rho_2}{3} \right) \setminus \{\xi_i\} \right) \text{ as } \varepsilon \to 0$$

(3.48)

for each index $i = 1, \ldots, k$ and a small number $\rho_2 > 0$. Then Harnack’s inequality in [7, Theorem 1.1], (3.48), and elliptic regularity yield

$$(u_\varepsilon, v_\varepsilon) \to (0, 0) \text{ in } C^1_{\text{loc}} \left( \Omega \setminus \bigcup_{i=1}^k B \left( \xi_i, \frac{\rho_2}{6} \right) \right) \text{ as } \varepsilon \to 0$$

(3.49)

Combining (3.48) and (3.49), we obtain (1.15). \qed
Proof of Theorem 1.2 (3). The assertion (1.16) follows from the proof of Lemma 3.3.

Let us prove (1.17). By (3.24), there is a constant $C > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$ such that

$$v_\varepsilon \leq C\lambda_\varepsilon^{\beta_\varepsilon-(N-2)} \quad \text{in } B\left(x_{i\varepsilon}, \frac{\rho_0}{3}\right) \setminus B\left(x_{i\varepsilon}, \frac{\rho_0}{6}\right)$$

(3.50)

for any $i = 1, \ldots, k$ and $\varepsilon \in (0, \varepsilon_0)$ small. On the other hand, by (1.16) and $\alpha_\varepsilon q_\varepsilon - \beta_\varepsilon = 2$,

$$v_\varepsilon(x) = \int_{\Omega} G(x, y) u_{i\varepsilon}^q(y) dy \geq C \int_{B(x_{i\varepsilon}, \frac{\rho_0}{4})} u_{i\varepsilon}^q$$

$$\geq C\lambda_\varepsilon^{\alpha_\varepsilon q_\varepsilon - N} \left( \int_{B^N} U_{i1,0}^{q_\varepsilon} + o(1) \right) \geq C\lambda_\varepsilon^{\beta_\varepsilon-(N-2)}$$

(3.51)

for any $x \in B(x_{i\varepsilon}, \frac{\rho_0}{4})$ and $1 \leq j \neq i \leq k$. From (3.50) and (3.51), we discover (1.17). □

Proof of Theorem 1.2 (1). By virtue of (1.13), we may assume that

$$\begin{cases} (u_\varepsilon, v_\varepsilon) \to (u_0, v_0) & \text{weakly in } W^{2, \frac{p+1}{p}}(\Omega) \times W^{2, \frac{q_0+1}{q_0}}(\Omega) \text{ as } \varepsilon \to 0, \\ (u_\varepsilon, v_\varepsilon) \text{ pointwise in } \Omega \end{cases}$$

up to a subsequence, and $(u_0, v_0)$ is a solution of (1.1) in which the positivity condition is replaced with $u, v \geq 0$ in $\Omega$. On the other hand, thanks to (3.48) or (3.49), $(u_0, v_0)$ vanishes at an interior point of $\Omega$. By the strong maximum principle, $(u_0, v_0) = (0, 0)$ in $\Omega$.

Furthermore, by (1.1), Theorem 1.2 (3), Remark 3.9, and Fatou’s lemma,

$$\liminf_{\varepsilon \to 0} \left( \| u_\varepsilon \|_{W^{2, \frac{p+1}{p}}(\Omega)}^{\frac{p+1}{p}} + \| v_\varepsilon \|_{W^{2, \frac{q_0+1}{q_0}}(\Omega)}^{\frac{q_0+1}{q_0}} \right) \geq \liminf_{\varepsilon \to 0} \left[ \int_{B(x_{i\varepsilon}, \lambda_{i\varepsilon}^{-1})} v_{i\varepsilon}^{p+1} + \int_{B(x_{i\varepsilon}, \lambda_{i\varepsilon}^{-1})} u_{i\varepsilon}^{q_\varepsilon(q_0+1)} \right]$$

\[ \geq \int_{B(0,1)} V_{1,0}^{p+1} + \int_{B(0,1)} U_{1,0}^{q_0+1} > 0, \]

which implies that there is no subsequence of $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon \in (0, \varepsilon_0)}$ converging to $(0, 0)$ strongly in $W^{2, (p+1)/p}(\Omega) \times W^{2, (q_0+1)/q_0}(\Omega)$. □

4. POINTWISE ESTIMATES FOR $(u_\varepsilon, v_\varepsilon)$

Throughout this section, we assume that $N \geq 3$, $p \in \left( \frac{2}{N-2}, \frac{N+2}{N-2} \right)$, and $p \geq 1$.

We fix $k \in \mathbb{N}$. Recalling $(\lambda_{i\varepsilon}, x_{i\varepsilon}) \in (0, \infty) \times \Omega$ in Theorem 1.2 and the functions in (1.20) and (2.24)–(2.25), we write

$$\mu_{i\varepsilon} = \lambda_{i\varepsilon}^{-1}, \quad (U_{i\varepsilon}, V_{i\varepsilon}) = (U_{\mu_{i\varepsilon}x_{i\varepsilon}, x_{i\varepsilon}}, V_{\mu_{i\varepsilon}, x_{i\varepsilon}}), \quad \text{and} \quad (\Psi_{i\varepsilon}^l, \Phi_{i\varepsilon}^l) = (\Psi_{\mu_{i\varepsilon}, x_{i\varepsilon}}, \Phi_{\mu_{i\varepsilon}, x_{i\varepsilon}})$$

(4.1)

for $i = 1, \ldots, k$ and $l = 0, \ldots, N$. Let also $(PU_{i\varepsilon}, PV_{i\varepsilon})$ be the unique solution of the system

$$\begin{cases} -\Delta PU_{i\varepsilon} = V_{i\varepsilon}^p & \text{in } \Omega, \\ -\Delta PV_{i\varepsilon} = U_{i\varepsilon}^{q_\varepsilon} & \text{in } \Omega, \\ PU_{i\varepsilon} = PV_{i\varepsilon} = 0 & \text{on } \partial \Omega, \end{cases}$$

(4.2)

and $(P\Psi_{i\varepsilon}^l, P\Phi_{i\varepsilon}^l)$ be the unique solution of the system

$$\begin{cases} -\Delta P\Psi_{i\varepsilon}^l = pV_{i\varepsilon}^{p-1}\Phi_{i\varepsilon}^l & \text{in } \Omega, \\ -\Delta P\Phi_{i\varepsilon}^l = q_\varepsilon U_{i\varepsilon}^{q_\varepsilon-1}\Psi_{i\varepsilon}^l & \text{in } \Omega, \\ P\Psi_{i\varepsilon}^l = P\Phi_{i\varepsilon}^l = 0 & \text{on } \partial \Omega, \end{cases}$$

(4.3)

Henceforth, we will often drop the subscript $\varepsilon$ for brevity, writing $\mu_i = \mu_{i\varepsilon}, U_i = U_{i\varepsilon}, \Psi_i^l = \Psi_{i\varepsilon}^l, PU_i = PU_{i\varepsilon}$, and so on.
By a standard comparison argument, we have

\[
PU_i(x) = \begin{cases}
U_i(x) - \frac{aN_p}{\gamma N} \mu_i^{\frac{Np}{\gamma p + 1}} H(x, \xi_i) + o\left(\mu_i^{\frac{Np}{\gamma p + 1}}\right) & \text{if } p \in \left(\frac{N}{N-2}, \frac{N+2}{N-2}\right), \\
U_i(x) - \frac{aN_p}{\gamma N} \mu_i^{\frac{Np}{\gamma p + 1}} \left[\hat{H}(x, \xi_i) - \log \mu_i H(x, \xi_i)\right] + o\left(\mu_i^{\frac{Np}{\gamma p + 1}}\right) & \text{if } p = \frac{N}{N-2}, \\
U_i(x) - \frac{aN_p}{\gamma N} \mu_i^{\frac{Np}{\gamma p + 1}} \hat{H}(x, \xi_i) + o\left(\mu_i^{\frac{Np}{\gamma p + 1}}\right) & \text{if } p \in \left(\frac{2}{N-2}, \frac{N}{N-2}\right),
\end{cases}
\]

and

\[
PV_i(x) = V_i(x) - \frac{bN_p}{\gamma N} \mu_i^{\frac{Np}{\gamma p + 1}} H(x, \xi_i) + o\left(\mu_i^{\frac{Np}{\gamma p + 1}}\right)
\]

in $C^1(\Omega)$. Here, $H$ and $\hat{H}$ are the functions satisfying (2.1) and (2.7)–(2.8), respectively. Similarly,

\[
P\Psi_i^0(x) = \begin{cases}
\Psi_i^0(x) + \frac{N \mu_i^{\frac{Np}{\gamma p + 1}}}{\gamma N} \mu_i^{\frac{Np}{\gamma p + 1}} H(x, \xi_i) + o\left(\mu_i^{\frac{Np}{\gamma p + 1}}\right) & \text{if } p \in \left(\frac{N}{N-2}, \frac{N+2}{N-2}\right), \\
\Psi_i^0(x) - \frac{N \mu_i^{\frac{Np}{\gamma p + 1}}}{\gamma N} \mu_i^{\frac{Np}{\gamma p + 1}} \log \mu_i H(x, \xi_i) + o\left(\mu_i^{\frac{Np}{\gamma p + 1}} \log \mu_i\right) & \text{if } p = \frac{N}{N-2}, \\
\Psi_i^0(x) + \frac{N \mu_i^{\frac{Np}{\gamma p + 1}}}{\gamma N} \mu_i^{\frac{Np}{\gamma p + 1}} \hat{H}(x, \xi_i) + o\left(\mu_i^{\frac{Np}{\gamma p + 1}}\right) & \text{if } p \in \left(\frac{2}{N-2}, \frac{N}{N-2}\right),
\end{cases}
\]

\[
P\Psi_i(x) = \begin{cases}
\Psi_i(x) + \frac{aN_p}{\gamma N} \mu_i^{\frac{Np}{\gamma p + 1}} \partial_{\xi_1} H(x, \xi_i) + o\left(\mu_i^{\frac{Np}{\gamma p + 1}}\right) & \text{if } p \in \left(\frac{N}{N-2}, \frac{N+2}{N-2}\right), \\
\Psi_i(x) - \frac{aN_p}{\gamma N} \mu_i^{\frac{Np}{\gamma p + 1}} \log \mu_i \partial_{\xi_1} H(x, \xi_i) + o\left(\mu_i^{\frac{Np}{\gamma p + 1}} \log \mu_i\right) & \text{if } p = \frac{N}{N-2}, \\
\Psi_i(x) + \frac{aN_p}{\gamma N} \mu_i^{\frac{Np}{\gamma p + 1}} \partial_{\xi_1} \hat{H}(x, \xi_i) + o\left(\mu_i^{\frac{Np}{\gamma p + 1}}\right) & \text{if } p \in \left(\frac{2}{N-2}, \frac{N}{N-2}\right),
\end{cases}
\]

and

\[
P\Phi_i(x) = \begin{cases}
\Phi_i(x) + \frac{N}{\gamma N} \frac{bN_p}{\gamma N} \mu_i^{\frac{Np}{\gamma p + 1}} H(x, \xi_i) + o\left(\mu_i^{\frac{Np}{\gamma p + 1}}\right) & \text{if } l = 0, \\
\Phi_i(x) + \frac{bN_p}{\gamma N} \mu_i^{\frac{Np}{\gamma p + 1}} \partial_{\xi_1} H(x, \xi_i) + o\left(\mu_i^{\frac{Np}{\gamma p + 1}}\right) & \text{if } l = 1, \ldots, N
\end{cases}
\]

in $C^1(\Omega)$. Here, $\partial_{\xi_1} H(x, \xi)$ and $\partial_{\xi_1} \hat{H}(x, \xi)$ stand for the $l$-th components of $\nabla_{\xi} H(x, \xi)$ and $\nabla_{\xi} \hat{H}(x, \xi)$, respectively.

Setting $\mu_\varepsilon = (\mu_{\varepsilon 1}, \ldots, \mu_{\varepsilon k}) \in (0, \infty)^k$ and $x_\varepsilon = (x_{\varepsilon 1}, \ldots, x_{\varepsilon k}) \in \Omega^k$, we define $PU_{\mu_\varepsilon, x_\varepsilon}$ as the solution of the equation

\[
\begin{cases}
-\Delta PU_{\mu_\varepsilon, x_\varepsilon} = \left(\sum_{i=1}^{k} PV_{i\varepsilon}\right)^p & \text{in } \Omega, \\
PU_{\mu_\varepsilon, x_\varepsilon} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(4.9)

We will often drop the subscript $\varepsilon$, writing $\mu = \mu_\varepsilon$, $PU_{\mu, x} = PU_{\mu_{\varepsilon}, x_\varepsilon}$, and so on.

**Lemma 4.1.** There exists a constant $C > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$ such that

\[
\left\|PU_{\mu_\varepsilon, x_\varepsilon} - \sum_{i=1}^{k} PU_{i\varepsilon}\right\|_{L^\infty(\Omega)} \leq \begin{cases}
C \mu_{\varepsilon}^{\frac{N}{\gamma p + 1}} & \text{if } p \in \left(\frac{N}{N-2}, \frac{N+2}{N-2}\right), \\
C \mu_{\varepsilon}^{\frac{N}{\gamma p + 1}} |\log \mu_{1\varepsilon}| & \text{if } p = \frac{N}{N-2}, \\
C \mu_{1\varepsilon}^{\frac{N}{\gamma p + 1}} & \text{if } p \in \left(\frac{2}{N-2}, \frac{N}{N-2}\right),
\end{cases}
\]

(4.10)

for all $\varepsilon \in (0, \varepsilon_0)$.

For the proof of the above lemma, we need the following integral estimates.
Lemma 4.2. Let \( x \in B(0, \lambda) \) for sufficiently large \( \lambda > 0 \). We have
\[
\int_{B(0, \lambda)} \frac{1}{|x - y|^{N-2}} \frac{1}{(1 + |y|)^a} dy \leq \begin{cases} 
C(1 + |x|)^{2-N} & \text{if } a > N, \\
C(1 + |x|)^{2-a} & \text{if } 2 < a < N, \\
C\lambda^{2-a} & \text{if } 0 < a < 2.
\end{cases}
\]

Proof. The proof is similar to that of [29, Lemma B.2]. We omit the details. \( \square \)

Proof of Lemma 4.1. Fix a small number \( \rho > 0 \). By virtue of (1.17), (1.20), and (4.5),
\[
|V_i(y)| + |PV_i(y)| = O \left( \mu_1^{N+i} \right)
\]
as \( \varepsilon \to 0 \). Hence
\[
P_U\mu(x) - \sum_{i=1}^k P_U(x) = \sum_{i=1}^k \int_{B(x, \rho)} G(x, y) \left[ \left( \sum_{i=1}^k PV_i(x) \right)^p - \sum_{i=1}^k V_i^p \right] (y) dy + O \left( \mu_1^{N+i} \right)
\]
for \( x \in \Omega \). Besides, by applying Lemma 4.2 and the fact that
\[
\frac{Np}{q_0 + 1} > \frac{N}{p + 1} \quad \text{if and only if} \quad p > \frac{N}{N-2},
\]
one can show that the integral in the right-hand side of (4.11) is bounded by
\[
C \int_{B(x, \rho)} \frac{1}{|x - y|^{N-2}} \left( \mu_i^{N+i} V_i^{p-1}(y) + \mu_i^{N+i} \right) dy \leq \begin{cases} 
C\mu_1^{N+i} & \text{if } p \in (\frac{N}{N-2}, \frac{N+2}{N-2}], \\
C\mu_1^{N+i} \log \mu_1 & \text{if } p = \frac{N}{N-2}, \\
C\mu_1^{N+i} & \text{if } p \in (\frac{2}{N-2}, \frac{N}{N-2}]
\end{cases}
\]
refer to the proof of [21, Appendix A.4]. Therefore (4.10) holds. \( \square \)

In the rest of this section, we study the pointwise behavior of a family \( \{ (u, v) \} \in (0, \varepsilon_0) \) of solutions of (1.1) with \( q = q_\varepsilon \). For simplicity, we assume that the results in Sections 3 and 3.5 hold for the whole family \( \{ (u, v) \} \in (0, \varepsilon_0) \), not only for its subsequence.

First, we examine \( (u, v) \) near each blow-up point \( \xi_i \) for \( i = 1, \ldots, k \).

Lemma 4.3. Given \( i = 1, \ldots, k \), let
\[
(w_i(y), z_i(y)) = \left( \lambda_i^{-\alpha_\varepsilon} u_{i\varepsilon}(\lambda_i^{-1} y + x_{i\varepsilon}), \lambda_i^{-\beta_\varepsilon} v_{i\varepsilon}(\lambda_i^{-1} y + x_{i\varepsilon}) \right) \quad \text{for } y \in B(0, \lambda_{i\varepsilon} \rho_2)
\]
where \( \rho_2 > 0 \) is a small number determined in the proof of Proposition 3.10. Then there exists a constant \( C > 0 \) independent of \( \varepsilon \in (0, \varepsilon_0) \) such that
\[
\|w_i - U_{i,0}\|_{L^\infty(B(0, \lambda_{i\varepsilon} \rho_2))} + \|z_i - V_{i,0}\|_{L^\infty(B(0, \lambda_{i\varepsilon} \rho_2))} \leq \begin{cases} 
C\mu_1^{N-2} & \text{if } p \in (\frac{N}{N-2}, \frac{N+2}{N-2}], \\
C\mu_1^{N-2} \log \mu_1 & \text{if } p = \frac{N}{N-2}, \\
C\mu_1^{(N-2)p-2} & \text{if } p \in (\frac{2}{N-2}, \frac{N}{N-2}]
\end{cases}
\]
for all \( \varepsilon \in (0, \varepsilon_0) \).

Proof. Here, we deal with the case \( p < \frac{N}{N-2} \) only. The remaining cases can be treated similarly.

Assume that there exists a decreasing sequence \( \{ \varepsilon_n \} \in (0, \varepsilon_0) \) such that \( \varepsilon_n \to 0 \) as \( n \to \infty \) and
\[
\|w_{in} - U_{i,0}\|_{L^\infty(B(0, \lambda_{i\varepsilon} \rho_2))} \geq \|z_{in} - V_{i,0}\|_{L^\infty(B(0, \lambda_{i\varepsilon} \rho_2))} \quad \text{for all } n \in \mathbb{N}
\]
where \((w_{in}, z_{in}) := (w_{in}, z_{in})\) and \(\lambda_{in} := \lambda_{in}^{1}\); if no such a sequence exists, we change the role of \(w_{in} - U_{1,0}\) and \(z_{in} - V_{1,0}\) in the following argument. Let \(\Lambda_{n} \geq 0\) and \(y_{n} \in B(0, \lambda_{in} \rho_{2})\) satisfy
\[
\Lambda_{n} = \|w_{in} - U_{1,0}\|_{L^{\infty}(B(0, \lambda_{in} \rho_{2}))}.
\]
Suppose that (4.14) does not hold. By (1.17) and (3.41), we have
\[
\Lambda_{n}^{-1} \lambda_{in}^{2-(N-2)p} \to 0 \quad \text{and} \quad \Lambda_{n}^{-1} \varepsilon_{n} \to 0 \quad \text{as} \quad n \to \infty.
\tag{4.15}
\]
Also, we may assume that \(y_{n} \in B(0, \frac{\lambda_{in} \rho_{2}}{2})\) for all \(n \in \mathbb{N}\). If we set
\[
(W_{in}, Z_{in}) = (\Lambda_{n}^{-1}(w_{in} - U_{1,0}), \Lambda_{n}^{-1}(z_{in} - V_{1,0})) \quad \text{in} \quad B(0, \lambda_{in} \rho_{2}),
\]
it satisfies
\[
\begin{align*}
-\Delta W_{in} &= \frac{z_{in} - V_{1,0}}{w_{in} - U_{1,0}} Z_{in} \quad \text{in} \quad B(0, \lambda_{in} \rho_{2}), \\
-\Delta Z_{in} &= \frac{w_{in} - U_{1,0}}{w_{in} - U_{1,0}} W_{in} + \Lambda_{n}^{-1}(U_{1,0}^{q_{n}} - U_{1,0}^{q_{0}}) \quad \text{in} \quad B(0, \lambda_{in} \rho_{2}), \\
|W_{in}(y_{n})| &= 1 \quad \text{and} \quad \nabla W_{in}(y_{n}) = 0
\end{align*}
\]
where \(q_{n} := q_{n}^{\varepsilon_{n}}\). By (3.5), and Propositions 3.12 and 3.7, it holds that
\[
w_{in}(y) \leq C U_{1,0}(y) \quad \text{and} \quad z_{in}(y) \leq C V_{1,0}(y) \quad \text{for} \quad y \in B(0, \lambda_{in} \rho_{2}).
\tag{4.16}
\]
Employing the representation formula, (4.16), and Lemma 4.2, we get
\[
|Z_{in}(x)| \leq C \int_{B(0, \lambda_{in} \rho_{2})} \frac{1}{|x - y|^{N-2}} \left(1 + |y|\right)^{(N-2)p-(N-2)q_{n}} dy + C \Lambda_{n}^{-1} \varepsilon_{n} + O\left(\Lambda_{n}^{-1} \lambda_{in}^{2-N}\right)
\]
for all \(x \in B(0, \frac{\lambda_{in} \rho_{2}}{2})\), as \(n \to \infty\). From this, we compute
\[
|W_{in}(x)| \leq C \int_{B(0, \lambda_{in} \rho_{2})} \frac{1}{|x - y|^{N-2}} V_{1,0}^{p-1}(y)|Z_{in}(y)| dy + O\left(\Lambda_{n}^{-1} \lambda_{in}^{2-(N-2)p}\right)
\]
for all \(x \in B(0, \frac{\lambda_{in} \rho_{2}}{2})\). Therefore, together with (4.15), we conclude that \(|y_{n}|\) is bounded and there exists \(y_{0} \in \mathbb{R}^{N}\) such that \(y_{n} \to y_{0}\) as \(n \to \infty\), along a subsequence. Furthermore, \((W_{in}, Z_{in})\) converges to a pair \((W_{i0}, Z_{i0})\), up to a subsequence, which satisfies
\[
\begin{align*}
-\Delta W_{i0} &= p V_{1,0}^{p-1} Z_{i0} \quad \text{in} \quad \mathbb{R}^{N}, \\
-\Delta Z_{i0} &= q_{0} U_{1,0}^{q_{0} - 1} W_{i0} \quad \text{in} \quad \mathbb{R}^{N}, \\
|W_{i0}(x)| + |Z_{i0}(x)| &\to 0 \quad \text{as} \quad |x| \to \infty.
\end{align*}
\]
Lemma 2.4 and the conditions \(W_{i0}(0) = \nabla W_{i0}(0) = 0\) show that \((W_{i0}, Z_{i0}) = (0, 0)\) in \(\mathbb{R}^{N}\). This is absurd because we also have that \(|W_{i0}(y_{0})| = 1\), so (4.14) is true. \(\Box\)

Combining (1.17), (3.41), (4.4)-(4.5), (4.10), and (4.14), we obtain

**Corollary 4.4.** Set
\[
(\psi_{\varepsilon}, \phi_{\varepsilon}) = \left( u_{\varepsilon} - P U_{\mu_{\varepsilon}, \chi_{\varepsilon}}, v_{\varepsilon} - \sum_{i=1}^{k} P V_{i\varepsilon} \right) \quad \text{in} \quad \Omega.
\tag{4.17}
\]
Let $\mathcal{A}_{\varepsilon, \rho_2} = \Omega \setminus \bigcup_{i=1}^{k} B(x_i, \rho_2)$ where $\rho_2 > 0$ is determined in the proof of Proposition 3.10. Then there exists a constant $C > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$ such that

\[
\|u_{\varepsilon}\|_{L^\infty(\mathcal{A}_{\varepsilon, \rho_2})} \leq \begin{cases} 
C \mu_{1\varepsilon}^{N-1} & \text{if } p \in \left(\frac{N}{N-2}, \frac{N+2}{N-2}\right), \\
C \mu_{1\varepsilon}^{N-1} \log \mu_{1\varepsilon} & \text{if } p = \frac{N}{N-2}, \\
C \mu_{1\varepsilon}^{q_0+1} & \text{if } p \in \left(\frac{2}{N-2}, \frac{N}{N-2}\right)
\end{cases}
\]  

(4.20)

and

\[
\|
\|v_{\varepsilon}\|_{L^\infty(\mathcal{A}_{\varepsilon, \rho_2})} \leq C \mu_{1\varepsilon}^{q_0+1}
\]  

(4.21)

for all $\varepsilon \in (0, \varepsilon_0)$.

**Proof.** Here, we deal with the case $p < \frac{N}{N-2}$ only. The remaining case can be treated similarly.

By the representation formula and (4.16),

\[
u_{\varepsilon}(x) = \int_{\Omega} G(x, y) v_{\varepsilon}^p(y) dy \leq C \sum_{i=1}^{k} \int_{B(x_i, \frac{\rho_2}{2})} v_{\varepsilon}^p + C\|v_{\varepsilon}\|_{L^\infty(\mathcal{A}_{\varepsilon, \rho_2/2})}^p
\]

(4.22)

and

\[
v_{\varepsilon}(x) \leq C \sum_{i=1}^{k} \int_{B(x_i, \frac{\rho_2}{2})} u_{\varepsilon}^{q_0} + C\|u_{\varepsilon}\|_{L^\infty(\mathcal{A}_{\varepsilon, \rho_2/2})}^{q_0} \leq C \mu_{1\varepsilon}^{q_0+1} + C\|u_{\varepsilon}\|_{L^\infty(\mathcal{A}_{\varepsilon, \rho_2/2})}^{q_0}
\]

(4.23)

for any $x \in \mathcal{A}_{\varepsilon, \rho_2}$. Applying (4.22), (4.23), (1.15), (4.12), and the inequality $pq_0 > 1$, we obtain (4.20)–(4.21).

If $p \in \left(\frac{2}{N-2}, \frac{N}{N-2}\right]$, we have to improve the estimate of $\phi_{\varepsilon}$ which was given in (4.19).

**Lemma 4.6.** Let $a_0 = ((N-2)p-2)(q_0-1) > 4$. Let $\eta$ be 0 for $a_0 < N$ and any small positive number for $a_0 = N$.

(1) Suppose that $p \in \left(\frac{N}{N-2}, \frac{N}{N-2}\right)$. Then there exists a constant $C > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$ such that

\[
|\phi_{\varepsilon}(x)| \leq C \left[ \sum_{i=1}^{Np_0} \mu_{\varepsilon}^{q_0+1} + \mu_{\varepsilon}^{q_0+1} \|v_{\varepsilon}\|_{L^\infty(\mathcal{A}_{\varepsilon, \rho_2/2})} \right] \sum_{i=1}^{k} (\mu_{1\varepsilon} + |x-x_i\varepsilon|)^{2-N}
\]

(4.24)
for all \( x \in \Omega \) and \( \varepsilon \in (0, \varepsilon_0) \).

(2) Suppose that \( p = \frac{N}{N-2} \). Then there exists a constant \( C > 0 \) independent of \( \varepsilon \in (0, \varepsilon_0) \) such that

\[
|\phi_\varepsilon(x)| \leq C \left[ \frac{N_q}{\mu_1^{q_0+1}} \log \mu_1 |\phi_\varepsilon|^{q_0} + C \frac{N_{(p+2)}}{\mu_1^{q_0+1}} \log \mu_1 |\phi_\varepsilon|^{q_0+1} \sum_{i=1}^{k} (\mu_1 + |x - x_i|)^{2-N} \right. \\
+ \left. C \mu_1^{q_0+1-N} \log \mu_1 |\phi_\varepsilon|^{q_0} \sum_{i=1}^{k} (\mu_1 + |x - x_i|)^{2-a_0+\eta} I_{a_0 \leq N} \right] (4.25)
\]

for all \( x \in \Omega \) and \( \varepsilon \in (0, \varepsilon_0) \).

Proof. (1) By (4.10), (4.4), (4.18), (4.20), (3.41), and the relation

\[
\frac{N(p+2)}{q_0+1} - (N - a_0) = \frac{Npq_0}{q_0+1},
\]

we have

\[
|\phi_\varepsilon(x)| = \sum_{i=1}^{k} \int_{B(x_i, \varepsilon_2)} G(x, y) \left[ U_i(y) + O \left( \frac{N_{(p+2)}}{\mu_1^{q_0+1}} \right) \right] \frac{N_q}{\mu_1^{q_0+1}} \log \mu_1 |\phi_\varepsilon|^{q_0} - U_i^{q_0}(y) dy + O \left( \frac{N_{pq_0}}{\mu_1^{q_0+1}} \right)
\]

\[
\leq C \sum_{i=1}^{k} \int_{B(x_i, \varepsilon_2)} \frac{1}{|x - y|^{N-2}} \left[ \varepsilon U_i^{q_0} |\phi_\varepsilon| \log \mu_1 |\phi_\varepsilon|^{q_0} + \mu_1^{q_0+1} U_i^{q_0-1} |\phi_\varepsilon|^{q_0+1} \right] dy + O \left( \frac{N_{pq_0}}{\mu_1^{q_0+1}} \right)
\]

\[
\leq C \left[ \frac{N_{(p+2)}}{\mu_1^{q_0+1}} \log \mu_1 \sum_{i=1}^{k} (\mu_1 + |x - x_i|)^{2-N} \right. \\
+ \left. \mu_1^{q_0+1-N} \sum_{i=1}^{k} (\mu_1 + |x - x_i|)^{2-a_0+\eta} I_{a_0 \leq N} \right] + O \left( \frac{N_{pq_0}}{\mu_1^{q_0+1}} \right)
\]

for all \( x \in \Omega \) and \( \varepsilon \in (0, \varepsilon_0) \). Here, the last inequality can be estimated as in (4.13).

(2) Using

\[
U_i(x) \leq C \mu_i^{q_0+1} |\phi_\varepsilon| (\mu_i + |x - x_i|)^{2-N}
\]

for all \( x \in \Omega \) and \( i = 1, \ldots, k \), we can compute as in (4.26).

\[\square\]

5. Determining the blow-up locations

Letting \( I_{p,q} \) be the energy functional (1.5) associated to (1.1), we write \( I_\varepsilon = I_{p,q,\varepsilon} \). In this section, we will derive necessary conditions for the parameters \( (\mu, \mathbf{v}) = (\mu, \mathbf{v}_\varepsilon) \in (0, \infty)^k \times \Omega^k \) from the identities

\[
I_\varepsilon' \left( PU_{\mu,\mathbf{v}} + \psi_\varepsilon \sum_{i=1}^{k} PV_i + \phi_\varepsilon \right) \left( P\Psi_j, P\Phi_j \right) = 0 \quad \text{for} \quad j = 1, \ldots, k \quad \text{and} \quad l = 0, \ldots, N \quad (5.1)
\]

where \( (\psi_\varepsilon, \phi_\varepsilon) \) is the pair in (4.17). Note that the left-hand side of (5.1) equals

\[
J_1(\mu, \mathbf{v}) + J_2(\mu, \mathbf{v}) + J_3(\mu, \mathbf{v})
\]

\[
:= \left[ \int_{\Omega} \nabla PU_{\mu,\mathbf{v}} \cdot \nabla P\Phi_j + \sum_{i=1}^{k} \int_{\Omega} \nabla PV_i \cdot \nabla P\Phi_j - \int_{\Omega} \left( \sum_{i=1}^{k} PV_i \right)^p P\Phi_j - \int_{\Omega} (PU_{\mu,\mathbf{v}})^q P\Psi_j \right]
\]
\[ + \int_{\Omega} (P U_{\mu,x})^q P \Psi_j - \int_{\Omega} (P U_{\mu,x})^{q_e} P \Psi_j \] (5.2)

\[ + \left[ I'_{\varepsilon} \left( P U_{\mu,x} + \psi_{e} \sum_{i=1}^{k} PV_i + \phi_{\varepsilon} \right) \left( P \Psi_j, P \Psi_j \right) - I'_{\varepsilon} \left( P U_{\mu,x}, \sum_{i=1}^{k} PV_i \right) \left( P \Psi_j, P \Psi_j \right) \right]. \]

A direct computation with (4.2), (4.3), (4.6)–(4.8), and (4.9) reveals that

\[ J_{1\varepsilon}(\mu, x) = \int_{\Omega} \left[ \sum_{i=1}^{k} U_{i}^{q_0} - (P U_{\mu,x})^{q_0} \right] P \Psi_j \] (5.3)

and

\[ J_{3\varepsilon}(\mu, x) = -\int_{\Omega} \left[ (P U_{\mu,x} + \psi_{e})^{q_e} - (P U_{\mu,x})^{q_e} \right] \Psi_j \]

\[ - \int_{\Omega} \left[ \left( \sum_{i=1}^{k} PV_i + \phi_{\varepsilon} \right)^{p} - \left( \sum_{i=1}^{k} PV_i \right)^{p} \right] \Phi_j \]

\[ + O \left( \mu_{1}^{N_{q_0}+1} \right) \int_{\Omega} \left| \sum_{i=1}^{k} PV_i + \phi_{\varepsilon} \right|^p - \left( \sum_{i=1}^{k} PV_i \right)^{p} \Phi_j \]

\[ + \begin{cases} O \left( \mu_{1}^{N_{q_0}+1} \right) \int_{\Omega} |(P U_{\mu,x} + \psi_{e})^{q_e} - (P U_{\mu,x})^{q_e}| & \text{if } p \in (\frac{N}{N-2}, \frac{N+2}{N-2}), \\
O \left( \mu_{1}^{N_{q_e}+1} \right) \int_{\Omega} |(P U_{\mu,x} + \psi_{e})^{q_e} - (P U_{\mu,x})^{q_e}| & \text{if } p = \frac{N}{N-2}, \\
O \left( \mu_{1}^{N_{q_e}+1} \right) \int_{\Omega} |(P U_{\mu,x} + \psi_{e})^{q_e} - (P U_{\mu,x})^{q_e}| & \text{if } p \in (\frac{2}{N-2}, \frac{N}{N-2}) \end{cases} \] (5.4)

as \( \varepsilon \to 0 \). From now on, we study each of \( J_{1\varepsilon}, J_{2\varepsilon}, \) and \( J_{3\varepsilon} \). There are four mutually exclusive cases according to the value of \( p \).

5.1. **The case that** \( p \in (\frac{N}{N-2}, \frac{N+2}{N-2}) \). We start with the following lemma.

**Lemma 5.1.** If \( N \geq 3 \) and \( p \in (\frac{N}{N-2}, \frac{N+2}{N-2}) \), then there exists a constant \( C > 0 \) independent of \( \varepsilon \in (0, \varepsilon_0) \) such that

\[ \mu_{i}^{2N_{q_e}+1} \int_{B(x,i,\rho)} U_{i}^{q_e-1} + U_{i}^{q_e-2} |\Psi_i \|^2 \leq C \left( \mu_{1}^{2N_{q_e}+1} \right), \]

\[ \mu_{i}^{2N_{q_e}+1} \int_{B(x,i,\rho)} U_{i}^{q_e-2} |\Phi_i \|^2 \leq C \left( \mu_{1}^{2N_{q_e}+1} \right), \]

\[ \mu_{i}^{2N_{q_0}+1} \int_{B(x,i,\rho)} V_{i}^{p-1} + V_{i}^{p-2} |\Phi_i \|^2 \leq C \left( \mu_{1}^{2N_{q_0}+1} \right), \]

for \( i = 1, \ldots, k \) and \( l = 1, \ldots, N \). In particular, all the integrals above are of order \( o(\mu_{1}^{N-2}) \) for \( N \geq 4 \).

**Proof.** Using (1.20), (2.24)–(2.25), and (2.13)–(2.14), one can compute directly to derive the above results. \( \square \)

In the rest of this subsection, we assume that \( N \geq 4 \).

**Estimate of** \( J_{3\varepsilon} \). According to (5.4), (4.10), and Lemma 4.2 and 5.1, we have
\[
|J_{3\varepsilon}(\mu, x)| \leq \sum_{m=1}^{k} \int_{B(x_m, \rho_2)} \left| (PU_{\mu, x} + \psi_{\varepsilon})^{q_{\varepsilon}} - (PU_{\mu, x})^{q_{\varepsilon}} - q_0 U_j^{q_{0} - 1} \psi_{\varepsilon} \right| |\Psi_j^f|
+ \sum_{m=1}^{k} \int_{B(x_m, \rho_2)} \left| \left( \sum_{i=1}^{k} PV_i + \phi_{\varepsilon} \right)^p - \left( \sum_{i=1}^{k} PV_i \right)^p - p V_j^{p - 1} \phi_{\varepsilon} \right| |\Phi_j^f| + o \left( \mu_1^{N - 2} \right). \tag{5.5}
\]

By (4.18), it holds that
\[
\left| (PU_{\mu, x} + \psi_{\varepsilon})^{q_{\varepsilon}} - (PU_{\mu, x})^{q_{\varepsilon}} - q_0 U_j^{q_{0} - 1} \psi_{\varepsilon} \right| |\Psi_j^f| \leq C \left( \frac{2N}{\mu_1^{p+1}} U_j^{q_{0} - 1} + \frac{N(q_0 + 1)}{\mu_1^{p+1}} \right) \text{ in } B(x_m, \rho_2) \tag{5.6}
\]
for \( m \neq j \), and
\[
\left| (PU_{\mu, x} + \psi_{\varepsilon})^{q_{\varepsilon}} - (PU_{\mu, x})^{q_{\varepsilon}} - q_0 U_j^{q_{0} - 1} \psi_{\varepsilon} \right| \leq C \left[ \left( PU_{\mu, x} \right)^{q_{\varepsilon} - 2} \psi_{\varepsilon}^2 1_{q_{\varepsilon} \geq 2} + |\psi_{\varepsilon}|^{q_{\varepsilon}} + \left| q_{\varepsilon} (PU_{\mu, x})^{q_{\varepsilon} - 1} - q_0 U_j^{q_{0} - 1} \right| |\psi_{\varepsilon}| \right] \text{ in } B(x_j, \rho_2). \tag{5.7}
\]
Thus we infer from Lemma 5.1, (5.6)–(5.7), and (3.41) that
\[
\sum_{m=1}^{k} \int_{B(x_m, \rho_2)} \left| (PU_{\mu, x} + \psi_{\varepsilon})^{q_{\varepsilon}} - (PU_{\mu, x})^{q_{\varepsilon}} - q_0 U_j^{q_{0} - 1} \psi_{\varepsilon} \right| |\Psi_j^f| = o \left( \mu_1^{N - 2} \right). \tag{5.8}
\]
Similarly, we find
\[
\sum_{m=1}^{k} \int_{B(x_m, \rho_2)} \left| \left( \sum_{i=1}^{k} PV_i + \phi_{\varepsilon} \right)^p - \left( \sum_{i=1}^{k} PV_i \right)^p - p V_j^{p - 1} \phi_{\varepsilon} \right| |\Phi_j^f| = o \left( \mu_1^{N - 2} \right). \tag{5.9}
\]
As a result,
\[
J_{3\varepsilon}(\mu, x) = o \left( \mu_1^{N - 2} \right). \tag{5.10}
\]

**Estimate of \( J_{1\varepsilon} \).** We observe from (5.3), (4.4), (4.6)–(4.7), (4.10), and Lemma 4.2 that
\[
J_{1\varepsilon}(\mu, x) = - \sum_{m=1}^{k} \int_{B(x_m, \rho_2)} \left[ (PU_{\mu, x})^{q_{0}} - \sum_{i=1}^{k} U_j^{q_{0}} \right] P \Psi_j^f + o \left( \mu_1^{N - 2} \right). \tag{5.11}
\]
Define
\[
\hat{\psi}_{\mu, x} = PU_{\mu, x} - \sum_{i=1}^{k} PU_i \text{ in } \Omega. \tag{5.12}
\]
Then, (4.10) yields that \( \hat{\psi}_{\mu, x} = O(\mu_1^{N}) \), and (4.4) leads to
\[
PU_{\mu, x} = U_m + \sum_{i=1, i \neq m}^{k} U_i - \frac{\alpha_N \mu_{\mu}}{\gamma N} \sum_{i=1}^{k} \mu_i^{p+1} H(\xi_i) + \hat{\psi}_{\mu, x} + \mathcal{R}_1 \text{ in } B(x_m, \rho_2) \tag{5.13}
\]
for each \( m = 1, \ldots, k \). Here, \( \mathcal{R}_1 \) is a remainder term satisfying
\[
|\mathcal{R}_1| + |\nabla \mathcal{R}_1| = o \left( \mu_1^{N} \right). \tag{5.14}
\]
From (5.13), we obtain
\[(PU_{\mu,x})^{q_0} - \sum_{i=1}^{k} U_i^{q_0} = q_0 U_m^{q_0-1} \left\{ \sum_{i=1, i \neq m}^{k} U_i - \frac{a_{N,p}}{\gamma_N} \sum_{i=1}^{k} \mu_i^{N+1} H(\cdot, \xi_i) + \hat{\psi}_{\mu,x} + R_1 \right\} + O(\mu_1^{N+1}) \text{ in } B(x_m, \rho_2). \quad (5.15)\]

Also, using (2.14) and (5.14), we see

\[U_i(x) = U_i(x_m) + \nabla U_i(x_m) \cdot (x - x_m) + O(\mu_1^{N+1}|x - x_m|^2) \quad \text{for } x \in B(x_m, \rho_2),\]
\[\nabla U_i(x) = -\mu_i^{N+1} (N - 2) a_{N,p} \frac{x - x_i}{|x - x_i|^N} + o(\mu_1^{N+1}) \quad \text{for } x \in B(x_m, \rho_2) \text{ and } i \neq m,\]

and

\[R_1(x) = R_1(x_m) + o(\mu_1^{N+1}|x - x_m|) \quad \text{for } x \in B(x_m, \rho_2).\]

For the moment, we assume that \(l = 1, \ldots, N\). Applying the above estimates, (5.15), Lemma 5.1, and the oddness of \(\Psi_{l,0}^1\) in the \(l\)-th variable, we deduce

\[\sum_{m=1}^{k} \int_{B(x_m, \rho_2)} \left[ (PU_{\mu,x})^{q_0} - \sum_{i=1}^{k} U_i^{q_0} \right] P\Psi_j^l\]
\[= q_0 \sum_{m=1}^{k} \int_{B(x_m, \rho_2)} U_m^{q_0-1}(x) \left[ \sum_{i=1, i \neq m}^{k} U_i(x) - \frac{a_{N,p}}{\gamma_N} \sum_{i=1}^{k} \mu_i^{N+1} H(x, \xi_i) + \hat{\psi}_{\mu,x}(x) + R_1(x) \right] \]
\[\times \left[ \Psi_j^l(x) + \frac{a_{N,p}}{\gamma_N} \mu_j^{N+1} \partial_{\xi_j} H(x, \xi_j) + o(\mu_j^{N+1}) \right] dx + o(\mu_1^{N-2})\]
\[= q_0 \int_{B(x_j, \rho_2)} U_j^{q_0-1}(x) \left[ \sum_{i=1, i \neq j}^{k} U_i(x) - \frac{a_{N,p}}{\gamma_N} \sum_{i=1}^{k} \mu_i^{N+1} H(x, \xi_i) + \hat{\psi}_{\mu,x}(x) + R_1(x) \right] \Psi_j^l(x) dx\]
\[+ o(\mu_1^{N-2}) \quad (5.16)\]
\[- q_0 (N - 2) a_{N,p} \sum_{i=1, i \neq j}^{k} \mu_i^{N+1} \int_{B(\xi_j, \rho_2)} U_j^{q_0-1}(x) \frac{(\xi_j - \xi_i) \cdot (x - \xi_j)}{|\xi_j - \xi_i|^N} \Psi_j^l(x) dx\]
\[- q_0 \frac{a_{N,p}}{\gamma_N} \sum_{i=1}^{k} \mu_i^{N+1} \int_{B(\xi_j, \rho_2)} U_j^{q_0-1}(x) \nabla_x H(\xi_j, \xi_i) \cdot (x - \xi_j) \Psi_j^l(x) dx\]
\[+ q_0 \int_{B(x_j, \rho_2)} U_j^{q_0-1} \hat{\psi}_{\mu,x} \Psi_j^l + o\left(\mu_1^{N+1} \int_{B(x_j, \rho_2)} U_j^{q_0-1}(x)|x - x_j| \Psi_j^l(x) dx\right)\]
\[+ O\left(\mu_1^{N+1} \int_{B(x_j, \rho_2)} U_j^{q_0-1}(x)|x - x_j|^2 \Psi_j^l(x) dx\right) + o(\mu_1^{N-2})\]

where \(\nabla_x H(\xi_j, \xi_i) := \nabla_x H(x, \xi_i)|_{x=\xi_j}\). By reminding (2.20) and using integration by parts, it is plain to check the identities

\[A_1 = -q_0 \int_{\mathbb{R}^N} U_{1,0}^{q_0-1} \Psi_{1,0}^1 dx_1 = \cdots = -q_0 \int_{\mathbb{R}^N} U_{1,0}^{q_0-1} \Psi_{1,0}^N dx_N = -\frac{q_0 (q_0 + 1)}{N} \int_{\mathbb{R}^N} U_{1,0}^{q_0-1} \Psi_{1,0}^0 \quad (5.17)\]
and

\[ A_2 = -p \int_{\mathbb{R}^N} V_{1,0}^{p-1} \Phi_{1,0}^0 x_1 = \cdots = -p \int_{\mathbb{R}^N} V_{1,0}^{p-1} \Phi_{1,0}^N x_N = -\frac{p(p+1)}{N} \int_{\mathbb{R}^N} V_{1,0}^{p-1} \Phi_{1,0}^0. \] (5.18)

Using (5.17), we evaluate each term of the rightmost side of (5.16). Then we get

\[
\sum_{m=1}^{k} \int_{B(x_m, \rho_2)} \left[ (PU_{\mu,x})^{q_0} - \sum_{i=1}^{k} U_i^{q_0} \right] P\Phi_j^l = \mu_j^{N-2} \frac{a_{N,p}}{\gamma_N} A_1 \partial_l H(\xi_j, \xi_j) \\
- \frac{a_{N,p}}{\gamma_N} A_1 \sum_{i=1, i \neq j}^{k} \mu_i^{N-1} \mu_j^{N-1} \partial_l G(\xi_j, \xi_i) + q_0 \int_{B(x_j, \rho_2)} U_j^{q_0-1} \psi_{\mu,x} \Phi_j^l + o \left( \mu_1^{N-2} \right) \] (5.19)

where \( \partial_l H(\xi_j, \xi_j) := \partial_l H(x, \xi_j) \bigr|_{x=\xi_j} \) and \( \partial_l G(\xi_j, \xi_i) := \partial_l G(x, \xi_i) \bigr|_{x=\xi_j} \), respectively. On the other hand, by the representation formula for \( \hat{\psi}_{\mu,x} \), Fubini’s theorem and the bound

\[
|U_j^{q_0-1} \Phi_j^l| = O \left( \frac{N^{q_0}}{\mu_1^{q_0+1}} \right) = o \left( \frac{N}{\mu_1^{q_0+1}} \right) \text{ in } C^1(\overline{\Omega} \setminus B(x_j, \rho_2)),
\]

we have

\[
q_0 \int_{B(x_j, \rho_2)} U_j^{q_0-1} \hat{\psi}_{\mu,x} \Phi_j^l \\
= q_0 \int_{\Omega} \left[ \left( \sum_{i=1}^{k} PV_i \right)^p - \sum_{i=1}^{k} V_i^p \right] (y) \int_{B(x_j, \rho_2)} G(y, x) U_j^{q_0-1}(x) \Phi_j^l(x) dx dy \\
= \int_{\Omega} \left[ \left( \sum_{i=1}^{k} PV_i \right)^p - \sum_{i=1}^{k} V_i^p \right] \left( P\Phi_j^l + R_2 \right)
\]

where \( R_2 \) is a remainder term such that \( |R_2| + |\nabla R_2| = o(\mu_1^{q_0+1}) \). Applying (5.18) and arguing as in (5.16) and (5.19), we see that the rightmost side of (5.20) is equal to

\[
p \int_{B(x_j, \rho_2)} V_j^{p-1}(x) \left[ \sum_{i=1, i \neq j}^{k} V_i \left( \frac{b_{N,p}}{\gamma_N} \sum_{i=1}^{k} \mu_i^{N-1} H(x, \xi_i) + R'_2(x) \right) \right] \Phi_j^l(x) dx + o \left( \mu_1^{N-2} \right) \] (5.21)

where \( R'_2 \) is a term satisfying \( |R_2| + |\nabla R_2| = o(\mu_1^{q_0+1}) \). Consequently, by combining (5.19) and (5.20)–(5.21), and employing (2.21), we obtain

\[
\sum_{m=1}^{k} \int_{B(x_m, \rho_2)} \left[ (PU_{\mu,x})^{q_0} - \sum_{i=1}^{k} U_i^{q_0} \right] P\Phi_j^l = 2\mu_j^{N-2} \frac{a_{N,p}}{\gamma_N} A_1 \partial_l H(\xi_j, \xi_j) \\
- \frac{a_{N,p}}{\gamma_N} A_1 \sum_{i=1, i \neq j}^{k} \left( \mu_i^{N-1} \mu_j^{N-1} + \mu_i^{q_0+1} \mu_j^{q_0+1} \right) \partial_l G(\xi_j, \xi_i) + o \left( \mu_1^{N-2} \right). \] (5.22)

Analogously, if \( l = 0 \), then using the evenness of \( \Phi^0_{m, \mu} \), (5.17)–(5.18), and (1.4), we derive

\[
\sum_{m=1}^{k} \int_{B(x_m, \rho_2)} \left[ (PU_{\mu,x})^{q_0} - \sum_{i=1}^{k} U_i^{q_0} \right] P\Phi_j^0 = \mu_j^{N-2} \frac{a_{N,p}}{\gamma_N} (N-2)A_1 H(\xi_j, \xi_j) 
\]
Conclusion. We note that where (5.26) SEUNGHYEOK KIM AND SANG-HYUCK MOON

\[
- \frac{a_{N,p}}{\gamma_N} N A_1 \sum_{i=1, i \neq j}^k \left( \mu_j^{\frac{N}{0}} \frac{\mu_j^{\frac{N}{0}+1}}{q_0 + 1} + \mu_j^{\frac{N}{0}+1} \mu_j^{\frac{N}{0}+1} \frac{N}{p + 1} \right) G(\xi_j, \xi_i) + o\left(\mu_1^{N-2}\right). \tag{5.23}
\]

Estimate of $J_{2x}$. Assume that $l = 1, \ldots, N$. Using Taylor’s theorem, the relation that $|q_0 - q_0| = O(\varepsilon)$, (3.41), $N \geq 4$, and the oddness of $\Psi^l_{1,0}$ in the $l$-th variable, we get

\[
\int_{\Omega} \left[ (PU_{\mu,x})^{q_0} - (PU_{\mu,x})^{q_0} \right] P \Psi_j^0
\]

\[= (q_0 - q_0) \int_{\Omega} (PU_{\mu,x})^{q_0} \log(PU_{\mu,x}) P \Psi_j^0 + O\left( \varepsilon^2 \int_{\Omega} (PU_{\mu,x})^{q_0+O(\varepsilon)} P \Psi_j^0 \right) \]

\[= (q_0 - q_0) \int_{B(x_0, \rho_2)} U_j^{q_0} \log U_j \left[ \Psi_j^0(x) + \frac{a_{N,p}}{\gamma_N} \mu_j^{\frac{N}{p+1}} \partial_{\xi,j} H(x, \xi_j) \right] + o\left(\mu_1^{N-2}\right) \]

\[= o\left(\mu_1^{N-2}\right). \tag{5.24}
\]

For all $\mu > 0$. Differentiating the both equalities with respect to $\mu$ and putting $\mu = 1$ on the results, we obtain

\[
\int_{\mathbb{R}^N} U_j^{q_0+1} \Psi_j^0 = 0 \quad \text{and} \quad A_3 := \int_{\mathbb{R}^N} U_j^{q_0+1} \log U_j = \frac{N}{(q_0 + 1)^2} \int_{\mathbb{R}^N} U_j^{q_0+1} dx > 0. \tag{5.25}
\]

Using (5.25) and computing as in (5.24), we have

\[
\int_{\Omega} \left[ (PU_{\mu,x})^{q_0} - (PU_{\mu,x})^{q_0} \right] P \Psi_j^0
\]

\[= (q_0 - q_0) \int_{B(x_0, \rho_2)} U_j^{q_0} \log U_j \left[ \Psi_j^0(x) + \frac{a_{N,p}}{\gamma_N} \mu_j^{\frac{N}{p+1}} H(x, \xi_j) \right] + o\left(\mu_1^{N-2}\right) \]

\[= o\left(\mu_1^{N-2}\right). \tag{5.26}
\]

Conclusion. Recalling (1.17), we write $d_i = d_i = \mu_i \mu_1^{\frac{1}{i+1}} \in (0, \infty)$ for $i = 1, \ldots, k$ and $\varepsilon \in (0, \varepsilon_0)$. From (5.1)–(5.2), (5.10), (5.11), (5.22)–(5.23), (5.24), and (5.26), we see

\[
2d_j^{N-2} (\nabla \xi H) (\xi_j, \xi_j) - \sum_{i=1, i \neq j}^k \left( d_j^{\frac{N}{0}} d_j^{\frac{N}{0}+1} + d_j^{\frac{N}{0}+1} d_j^{\frac{N}{0}+1} \right) (\nabla \xi G) (\xi_j, \xi_j) = o(1) \tag{5.27}
\]

where $(\nabla \xi H) (\xi_j, \xi_j) := \nabla \xi H (\xi_j, \xi_j) |_{\xi = \xi_j}$ and $(\nabla \xi G) (\xi_j, \xi_j) := \nabla \xi G (\xi_j, \xi_j) |_{\xi = \xi_j}$, respectively, and

\[
(N - 2)d_j^{N-2} H(\xi_j, \xi_j) - N \sum_{i=1, i \neq j}^k \left( d_j^{\frac{N}{0}+1} d_j^{\frac{N}{0}+1} + d_j^{\frac{N}{0}+1} d_j^{\frac{N}{0}+1} \right) G(\xi_j, \xi_j)
\]

\[= \frac{A_3 + o(1)}{\gamma_N q_0 - q_0} \frac{1}{\mu_1^{N-2}} + o(1) \tag{5.28}
\]

for $j = 1, \ldots, k$. 
5.2. The case that \( p = \frac{N}{N-2} \). It turns out that analysis of this threshold situation resembles to that of the previous case, albeit we also have to exploit a refined estimate of \( \phi \) given in (4.25). In this subsection, we will briefly explain how the computations need to be changed.

Estimate of \( J_{3\varepsilon} \). A slight alteration of the argument in the previous subsection yield (5.5) and (5.8). However, an analogous argument no more yields (5.9). Instead, one can deduce it by applying (4.25) as a substitute and conducting tedious but straightforward computations. In conclusion,

\[
J_{3\varepsilon}(\mu, x) = o \left( \mu_1^{N-2} \right)
\]

(5.29)

for \( N \geq 4 \).

Estimate of \( J_{1\varepsilon} \). We argue as in the previous subsection, suitably caring for the modified terms in the expansion of \( PU_i \). Since the integral \( A_2 \) in (2.20) is divergent when \( p = \frac{N}{N-2} \), we instead work with the quantity

\[
A_2[R] = \int_{B(x, \rho)} V_{i, 0}^{p-1} = -p \int_{B(x, \rho)} V_{i, 0}^{p-1} \Phi_{i, 0}^l dx = -\frac{p(p+1)}{N} \int_{B(x, \rho)} V_{i, 0}^{p-1} \Phi_{i, 0}^{l} dx
\]

where \( R > 0 \) and \( l = 1, \ldots, N \). By (2.15),

\[
A_2[R] = \frac{p}{\gamma N} |S^{N-1}| \log R (1 + o_R(1) \text{ as } R \to \infty}
\]

where \( o_R(1) \to 0 \) as \( R \to \infty \). In particular, we have a formula analogous to (5.21):

\[
p \int_{B(x, \rho)} V_{i, 0}^{p-1} \sum_{i=1, i \neq j}^k V_i(x) - \frac{\mu_1^{N-1}}{\gamma N} \int_{B(x, \rho)} \sum_{i=1}^k \frac{\mu_i^{N-1}}{\gamma N} H(x, \xi_i) + R_2(x) \right) \Phi_j(x) dx + O \left( \mu_1^{N-2} \right)
\]

\[
= \mu_1^{N-2} \frac{b_{N, p}}{\gamma N} A_2 \left[ \rho_2 \mu_1^j \right] \partial_t H(\xi, \xi_j) - \frac{b_{N, p}}{\gamma N} A_2 \left[ \rho_2 \mu_1^j \right] \sum_{i=1, i \neq j}^k \frac{\mu_i^{N-1}}{\gamma N} \mu_j^{N-1} \partial_t G(\xi, \xi_j) + o \left( \mu_1^{N-2} \log \mu_1 \right)
\]

\[
= \mu_1^{N-2} \log \mu_j \frac{b_{N, p}}{\gamma N} |S^{N-1}| \partial_t H(\xi, \xi_j) - \frac{b_{N, p}}{\gamma N} |S^{N-1}| \sum_{i=1, i \neq j}^k \frac{\mu_i^{N-1}}{\gamma N} \mu_j^{N-1} \log \mu_j \partial_t G(\xi, \xi_j) + o \left( \mu_1^{N-2} \log \mu_j \right)
\]

(5.30)

From (5.30) and (2.22), we deduce (5.11),

\[
\sum_{m=1}^k \int_{B(x, \rho_2)} \left[ (PU_{\mu, x})_i^0 - \sum_{i=1}^k U_{i, 0}^0 \right] P \Psi_j^l = 2 \mu_j^{N-2} \log \mu_1 \frac{a_{N, p}}{\gamma N} A_1 \partial_t H(\xi_j, \xi_j)
\]

\[
- \frac{a_{N, p}}{\gamma N} A_1 \sum_{i=1, i \neq j}^k \left( \frac{\mu_i^{N-1}}{\gamma N} + \frac{\mu_j^{N-1}}{\gamma N} \right) \left[ \log \mu_1 \partial_t G(\xi, \xi_j) + o \left( \mu_1^{N-2} \log \mu_1 \right) \right]
\]

(5.31)

for \( l = 1, \ldots, N \), and

\[
\sum_{m=1}^k \int_{B(x, \rho_2)} \left[ (PU_{\mu, x})_i^0 - \sum_{i=1}^k U_{i, 0}^0 \right] P \Psi_j^0 = \mu_j^{N-2} \log \mu_1 \frac{a_{N, p}}{\gamma N} (N - 2) A_1 H(\xi_j, \xi_j)
\]

\[
- \frac{a_{N, p}}{\gamma N} A_1 \sum_{i=1, i \neq j}^k \left( \frac{\mu_i^{N-1}}{\gamma N} + \frac{\mu_j^{N-1}}{\gamma N} \right) \left[ \log \mu_1 \partial_t G(\xi, \xi_j) + o \left( \mu_1^{N-2} \log \mu_1 \right) \right].
\]

(5.32)
Estimate of $J_{2\varepsilon}$. Arguing as in (5.24), we get
\[ \int_{\Omega} \left[ (PU_{\mu,x})^{q_0} - (PU_{\mu,x})^{q_\varepsilon} \right] P\Psi_j^l = o \left( \mu_1^{N-2} \right) \quad (5.33) \]
for $l = 1, \ldots, N$. Moreover, as in (5.26), we have
\[ \int_{\Omega} \left[ (PU_{\mu,x})^{q_0} - (PU_{\mu,x})^{q_\varepsilon} \right] P\Psi_j^0 \]
\[ = (q_0 - q_\varepsilon) q_0 \int_{B(x_j, r_j)} U_j^{q_0} \log U_j \left[ \Psi_j^{q_0}(x) + \frac{N}{p+1} a_{N,p} \mu_j^N \left| \log \mu_j \right| H(x, \xi_j) + o \left( \mu_j^N \left| \log \mu_j \right| \right) \right] \, dx \]
\[ + o \left( \mu_1^{N-2} \right) \quad (5.34) \]
\[ = (q_0 - q_\varepsilon) (A_3 + o(1)) + o \left( \mu_1^{N-2} \left| \log \mu_1 \right| \right). \]

Conclusion. As before, we write $d_i = d_i = \mu_i \varepsilon \mu_i^{-1} \in (0, \infty)$ for $i = 1, \ldots, k$ and $\varepsilon \in (0, \varepsilon_0)$. From (5.1)–(5.2), (5.29), (5.11), (5.31)–(5.32), and (5.33)–(5.34), we see
\[ 2d_j^{N-2} (\nabla \xi H)(\xi_j, \xi_j) - \sum_{i=1, i \neq j}^k \left( \frac{d_j^{N-1} d_i^{N-1}}{q_0 + 1} + \frac{d_j^{N-1} d_i^{N-1}}{p+1} \right) G(\xi_i, \xi_j) \]
\[ = \frac{(A_3 + o(1)) \gamma_N}{a_{N,p} A_1 \mu_1^{N-2} \left| \log \mu_1 \right|} + o(1) \quad (5.35) \]
and
\[ (N - 2) d_j^{N-2} H(\xi_j, \xi_j) - N \sum_{i=1, i \neq j}^k \left( \frac{d_j^{N-1} d_i^{N-1}}{q_0 + 1} + \frac{d_j^{N-1} d_i^{N-1}}{p+1} \right) G(\xi_i, \xi_j) \]
\[ = \frac{(A_3 + o(1)) \gamma_N}{a_{N,p} A_1 \mu_1^{N-2} \left| \log \mu_1 \right|} + o(1) \]
for $j = 1, \ldots, k$.

5.3. The case that $p \in \left( \frac{N-1}{N}, \frac{N-2}{N-2} \right) \cap \left( \frac{2}{N-2}, \frac{N}{N-2} \right)$. We start with the following lemma that will be used in this and the next subsections. Unlike the previous subsections, we need an auxiliary parameter $\kappa > 0$ to make the ‘remainder’ terms truly small.

Lemma 5.2. Let $N \geq 3$, $p \in \left( \frac{2}{N-2}, \frac{N}{N-2} \right)$, $a_0 = ((N - 2)p - 2)(q_0 - 1) > 4$, and $\kappa \in (0, 1)$ be a sufficiently small number. Then there exists a constant $C > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$ such that
\[ \mu_{i, N-1}^{2pN} \int_{B(x_i, \mu_{i, N-1}^\varepsilon)} \left( U_i^{q_\varepsilon - 1} + U_i^{q_\varepsilon - 2} \right) \Psi_i^{\varepsilon} \left| 1_{q_\varepsilon \geq 2} \right| \leq C \left( \mu_1^{2((N-2)p-2)} + \mu_1^{pN} \kappa(N-a_0) \left| \log \mu_1 \right| \right), \]
\[ \mu_{i, N-1}^{2pN} \int_{B(x_i, \mu_{i, N-1}^\varepsilon)} U_i^{q_\varepsilon - 2} \Psi_i^{\varepsilon} \left| 1_{q_\varepsilon \geq 2} \right| \leq C \left( \mu_1^{2((N-2)p-2)-1} + \mu_1^{pN} \kappa(N-a_0) \left| \log \mu_1 \right| \right), \]
\[ \mu_{i, N-1}^{pN} \int_{B(x_i, \mu_{i, N-1}^\varepsilon)} V_i^{p-2} \phi_i^{2} \left| \Psi_i^{\varepsilon} \right| \leq C \left( \mu_1^{(N-2)p-2} \kappa(N-(N-2)p) \right) = o \left( \mu_1^{(N-2)p-2} \right), \]
\[ \mu_{i, N-1}^{pN} \int_{B(x_i, \mu_{i, N-1}^\varepsilon)} V_i^{p-2} \phi_i^{2} \left| \Psi_i^{\varepsilon} \right| \leq C \left( \mu_1^{2((N-2)p-2)-1} \left| \log \mu_1 \right| \right) + o \left( \mu_1^{(N-2)p-2} \right), \]
\[ \mu_{i, N-1}^{pN} \int_{B(x_i, \mu_{i, N-1}^\varepsilon)} V_i^{p-2} \phi_i^{2} \left| \Psi_i^{\varepsilon} \right| \leq C \left( \mu_1^{(N-2)p-2} \left| \log \mu_1 \right| \right) + o \left( \mu_1^{(N-2)p-2} \right), \]
for $i = 1, \ldots, k$ and $l = 1, \ldots, N$. In particular, all the integrals above are of order $o(\mu_1^{(N-2)p-2})$ provided $N \geq 4$ and $p > \frac{3}{N-2}$. 

Proof. We can obtain the first, second, and third estimates as in Lemma 5.1. To derive the fourth and fifth estimates, we utilize (4.24). For instance, applying \( p q_0 > 1 \) and \( \frac{N(p+1)}{q_0+1} = (N - 2)p - 2 \) also, we compute

\[
\int_{B(x_1, \mu_{1}^{\kappa})} V^{p-2}_i \phi^2 \Phi_i^0
\]

\[
\leq C \left[ \mu_{1}^{N(p+1)} + \mu_{1}^{q_0+1} \right] \left( \log \mu_1 + \mu_{1}^{2N(p+1)-2q_0} \right) \int_{B(0, \mu_{1}^{\kappa-1})} \frac{dy}{(1 + |y|)^{(N-2)(p-1)}}
\]

\[
+ \mu_{1}^{N(p+1)q_0} \int_{B(0, \mu_{1}^{\kappa-1})} \frac{dy}{(1 + |y|)^{(N-2)(p+1)}}
\]

\[+ 1_{a_0 \leq N \mu_1^{N(p+1)q_0}} \int_{B(0, \mu_{1}^{\kappa-1})} \frac{dy}{(1 + |y|)^{(N-2)(p+1)+(2a_0-2)}}\]

\[= o \left( \mu_{1}^{(N-2)p-2} \right) + O \left( \mu_1^{2(N-2)p-2} \right) + O \left( \mu_1^{2(N-2)p-2} \right).
\]

The other estimates can be deduced similarly. □

Estimate of \( J_{3\kappa} \). In this estimate, we assume that \( N \geq 4 \) and \( p \in \{ \max\{1, \frac{3}{N-2}\}, \frac{N}{N-2} \} \) so that it can be also applied in the next subsection.

According to (5.4), (4.10), (4.24), and Lemmas 4.2 and 5.2, we have

\[
|J_{3\kappa}(\mu, x)| \leq \sum_{m=1}^{k} \int_{B(x_1, \mu_{1}^{\kappa})} \left| (PU_{\mu,x} + \psi \epsilon) U_{q_0}^{q_0-1} - q_0 U_{j}^{q_0-1} \psi \epsilon \right| \Phi_j^0
\]

\[+ \sum_{m=1}^{k} \int_{B(x_1, \mu_{1}^{\kappa})} \left| \left( \sum_{i=1}^{k} PV_i + \phi \epsilon \right)^p - \left( \sum_{i=1}^{k} PV_i \right)^p \right| \Phi_j^0 + o \left( \mu_{1}^{(N-2)p-2} \right)\]

for \( \kappa \in (0, 1) \) small enough. By (4.18), it holds that

\[
\left| (PU_{\mu,x} + \psi \epsilon)^{q_0} - (PU_{\mu,x})^{q_0} - q_0 U_{j}^{q_0-1} \psi \epsilon \right| \Phi_j^0 \leq C \left( \frac{2Np}{\mu_{1}^{N(p+1)}} U_{q_0}^{q_0-1} + \mu_{1}^{Np} \right) \text{ in } B(x_1, \mu_{1}^{\kappa}) \] (5.36)

for \( m \neq j \), and

\[
\left| (PU_{\mu,x} + \psi \epsilon)^{q_0} - (PU_{\mu,x})^{q_0} - q_0 U_{j}^{q_0-1} \psi \epsilon \right| \leq C \left[ \mu_{1}^{\frac{Np}{q_0+1}} + \mu_{1}^{\frac{Np}{q_0+1}} U_{j}^{q_0-1} 1_{\epsilon_0 \geq 2} + \epsilon \mu_{1}^{\frac{Np}{q_0+1}} U_{j}^{q_0-1} \left( 1 + U_{j}^{O(\epsilon)} \right) \log U_{j} \right] \text{ in } B(x_1, \mu_{1}^{\kappa}). \] (5.37)

Thus we infer from Lemma 5.2, (5.36)–(5.37), and (3.41) that

\[
\sum_{m=1}^{k} \int_{B(x_1, \mu_{1}^{\kappa})} \left| (PU_{\mu,x} + \psi \epsilon)^{q_0} - (PU_{\mu,x})^{q_0} - q_0 U_{j}^{q_0-1} \psi \epsilon \right| \Phi_j^0 = o \left( \mu_{1}^{(N-2)p-2} \right).
\]

Moreover, since \( p < \frac{N}{N-2} \leq 2 \), we know

\[
\left| \left( \sum_{i=1}^{k} PV_i + \phi \epsilon \right)^p - \left( \sum_{i=1}^{k} PV_i \right)^p - pV_j^{p-1} \phi \epsilon \right| \leq CV_{m}^p \text{ in } B(x_1, \mu_{1}^{\kappa}) \] (5.38)
for $m \neq j$, and

\[
\left| \left( \sum_{i=1}^{k} PV_i + \phi_\varepsilon \right)^p - \left( \sum_{i=1}^{k} PV_i \right)^p - pV_j^{p-1} \phi_\varepsilon \right| \leq C \left[ \left( \sum_{i=1}^{k} PV_i \right)^{p-2} \phi_\varepsilon^2 + \left( \sum_{i=1}^{k} PV_i \right)^{p-1} - V_j^{p-1} \right] |\phi_\varepsilon| \quad \text{in } B(x_j, \mu_1^{\varepsilon k}).
\]  

(5.39)

By Lemmas 5.1 and 5.2, and (5.38)–(5.39),

\[
\sum_{m=1}^{k} \int_{B(x_m, \mu_1^{\varepsilon k})} \left| \left( \sum_{i=1}^{k} PV_i + \phi_\varepsilon \right)^p - \left( \sum_{i=1}^{k} PV_i \right)^p - pV_j^{p-1} \phi_\varepsilon \right| |\Phi_j^l| = o \left( \mu_1^{(N-2)p-2} \right).
\]

Therefore

\[
J_{3\varepsilon} (\mu, x) = o \left( \mu_1^{(N-2)p-2} \right).
\]

(5.40)

Estimate of $J_{1\varepsilon}$. In the remaining part of this subsection, we assume that $N \geq 4$ and $p \in \left[ \frac{N-1}{N-2}, \frac{N}{N-2} \right] \cap \left( \frac{3}{N-2}, \frac{N}{N-2} \right)$.

Fix any $l = 0, \ldots, N$. Arguing as in Subsection 5.1, we obtain equalities analogous to (5.11) and (5.19):

\[
J_{1\varepsilon} (\mu, x) = -\sum_{m=1}^{k} \int_{B(x_m, \mu_1^{\varepsilon k})} \left[ (PU_{\mu, x})^{q_0} - \sum_{i=1}^{k} U_i^{q_0} \right] P\Psi_j^l + o \left( \mu_1^{(N-2)p-2} \right),
\]

(5.41)

and

\[
\sum_{m=1}^{k} \int_{B(x_m, \mu_1^{\varepsilon k})} \left[ (PU_{\mu, x})^{q_0} - \sum_{i=1}^{k} U_i^{q_0} \right] P\Psi_j^l = \frac{a_{N,p}}{\gamma_N} A_1 \mu_1^{(N-2)p-2} \partial_1 \tilde{H} (\xi_j, \xi_j)
\]

\[-\frac{a_{N,p}}{\gamma_N} A_1 \sum_{i=1, i \neq j}^{k} \mu_i^{q_0+1} \mu_j^{q_0+1} \partial_1 \tilde{G} (\xi_j, \xi_i) + q_0 \int_{B(x_m, \mu_1^{\varepsilon k})} U_i^{q_0-1} \psi_{\mu, x} \Phi_j^l + o \left( \mu_1^{(N-2)p-2} \right) \quad \text{for } x \in \Omega.
\]

(5.42)

where $\tilde{\psi}_{\mu, x}$, $\tilde{H}$, and $\tilde{G}$ are the functions defined by (5.12), (2.7)–(2.8), and (2.9), respectively. Unfortunately, analogous computations to (5.20)–(5.21) only give a crude estimate on the integral in the right-hand side of (5.42). We will overcome this technical issue by performing a more delicate analysis. Define the regular part $\tilde{\psi}_{\mu, x}$ of $\psi_{\mu, x}$ by

\[
\tilde{\psi}_{\mu, x} (x) = \int_{\mathcal{A}} G (x, y) \left[ \left( \sum_{i=1}^{k} PV_i \right)^p - \sum_{i=1}^{k} V_i^p + \frac{pb_{N,p}}{\gamma_N} \mu_1^{q_0+1} \sum_{i=1}^{k} A_{d, \xi, i} V_i^{p-1} \right] (y) dy \quad \text{for } x \in \Omega.
\]

Here, $(\mu, x) = (\mu_\varepsilon, x_\varepsilon) = (\mu_1^{\varepsilon k}, \ldots, \mu_k^{\varepsilon k}, x_1, \ldots, x_k) \in (0, \infty)^k \times \Omega^k$, $x_\varepsilon \to \xi = (\xi_1, \ldots, \xi_k) \in \Omega^k$ and $d = d_\varepsilon := \mu_1^{\varepsilon k} \mu_\varepsilon \to \delta \in (0, \infty)^k$ as $\varepsilon \to 0$, $A_{d, \xi, i}$ is the quantity in (2.5) (where $\delta$ is replaced with $d$), and

\[
\mathcal{A} = \mathcal{A}_\varepsilon := \Omega \setminus \cup_{i=1}^{k} B(x_i, \xi_i) \quad \text{where } g_i = g_{i\varepsilon} := \mu_1^{\varepsilon k} + \rho_2 |x_{i\varepsilon} - \xi_i|^{\varepsilon k},
\]

$\kappa \in (0, \frac{1}{(N-2)p+1})$ is a number small enough, and $\rho_2 > 0$ is one in the proof of Proposition 3.10.
Lemma 5.3. Assume that $N \geq 4$ and $p \in \left(\frac{N-2}{N-1}, \frac{N}{N-2}\right)$. Recall the functions $\tilde{H}_{d, \xi}$ in (2.4) (where $\delta$ is replaced with $d$) and $\mathcal{P}$ in (2.10). Then

$$\tilde{\psi}_{\mu, x}(x) = \left(\frac{b_{N,p}}{\gamma_N}\right)^p \left[-\frac{N_p}{\mu_1^{N_{0+1}}} \tilde{H}_{d, \xi}(x) + \frac{\tilde{\gamma}_{N,p,1}}{\gamma_N} \sum_{i=1}^k \mu_i^{N_{0+1}} \tilde{H}(x, \xi_i) - \frac{\tilde{\gamma}_{N,p,2}}{\mu_1^{N_{0+1}}} \sum_{i=1}^{N_{(p-1)}} A_{d, \xi,i} \tilde{H}(x, \xi_i)\right] + \mathcal{R}_3(x) \quad \text{for } x \in \Omega$$

(5.43)

where $\mathcal{R}_3$ is a remainder term satisfying $|\mathcal{R}_3(x)| + |\nabla \mathcal{R}_3(x)| = o(\mu_1^{N_{0+1}})$ and the values of the constants $b_{N,p}$, $\gamma_N$, $\tilde{\gamma}_{N,p,1}$, and $\tilde{\gamma}_{N,p,2}$ are found in (2.11), (2.1), and (2.3).

Proof. By virtue of (2.13) and (4.5), we have

$$\left|V_i(x) - \mu_i^{N_{0+1}} \frac{b_{N,p}}{\gamma_N} \right| \leq C \mu_1^{N_{0+1}} \frac{\mu_i + |x_i - \xi_i|}{|x - \xi_i|^{N-2}}$$

and

$$\left|PV_i(x) - \mu_i^{N_{0+1}} \frac{b_{N,p}}{\gamma_N} G(x, \xi_i) \right| \leq C \mu_1^{N_{0+1}} \frac{\mu_i + |x_i - \xi_i|}{|x - \xi_i|^{N-1}}$$

for $x \in \Omega \setminus B(x_i, \rho_i)$. Thus

$$\left[\left(\sum_{i=1}^k PV_i\right)^p - \sum_{i=1}^k V_i^p + \frac{ppb_{N,p}}{\gamma_N} \mu_1^{N_{0+1}} \sum_{i=1}^k A_{d, \xi,i} V_i^{p-1}\right](y)$$

$$\left(\frac{b_{N,p}}{\gamma_N}\right)^p \mu_1^{N_{0+1}} \left[\left(\sum_{i=1}^k \frac{d_i^{N_{0+1}} G(y, \xi_i)}{\gamma_N} \right)^p - \sum_{i=1}^k \left(\frac{d_i^{N_{0+1}} \gamma_N}{|y - \xi_i|^{N-2}}\right)^p\right]$$

$$+ p \left(\frac{b_{N,p}}{\gamma_N}\right)^p \mu_1^{N_{0+1}} \sum_{i=1}^k A_{d, \xi,i} \left(\frac{\gamma_N}{|y - \xi_i|^{N-1}}\right)^p + \sum_{i=1}^k \mu_i^{N_{0+1}} O\left(\frac{\mu_i + |x_i - \xi_i|}{|y - \xi_i|^{(N-2)p+1}}\right)$$

(5.44)

for $y \in A$ where $d_i = \mu_i \mu_1^{-1}$. Besides,

$$\left|\sum_{i=1}^k \frac{d_i^{N_{0+1}} G(y, \xi_i)}{\gamma_N} \right|^p - \left(\frac{d_i^{N_{0+1}} \gamma_N}{|y - \xi_i|^{N-2}}\right)^p + pA_{d, \xi,i} \left(\frac{d_i^{N_{0+1}} \gamma_N}{|y - \xi_i|^{N-2}}\right)^{p-1}$$

$$\leq \frac{C}{|y - \xi_i|^{(N-2)(p-1)-1}}$$

for $y \in B(x_i, \rho_2)$. (5.45)

From (5.44), (5.45), and the dominated convergence theorem, it follows that

$$\tilde{\psi}_{\mu, x}(x) = \left(\frac{b_{N,p}}{\gamma_N}\right)^p \mu_1^{N_{0+1}} \int_A G(x, y) \left[\left(\sum_{i=1}^k \frac{d_i^{N_{0+1}} G(y, \xi_i)}{\gamma_N}\right)^p - \sum_{i=1}^k \left(\frac{d_i^{N_{0+1}} \gamma_N}{|y - \xi_i|^{N-2}}\right)^p\right]$$

$$+ p \sum_{i=1}^k A_{d, \xi,i} \left(\frac{d_i^{N_{0+1}} \gamma_N}{|y - \xi_i|^{N-2}}\right)^{p-1} dy + \mathcal{R}_3(x)$$

$$= \left(\frac{b_{N,p}}{\gamma_N}\right)^p \mu_1^{N_{0+1}} \left[\tilde{G}_{d, \xi}(x) - \tilde{\gamma}_{N,p,1} \sum_{i=1}^k \frac{d_i^{N_{0+1}}}{|x - \xi_i|^{(N-2)p-2}} + \tilde{\gamma}_{N,p,2} \sum_{i=1}^k A_{d, \xi,i} \frac{d_i^{N_{0+1}}}{|x - \xi_i|^{(N-2)p-N}}\right]$$

for $x \in \Omega \setminus B(x_i, \rho_i)$. (5.46)
for } x \in \Omega \text{ provided } \kappa \in (0, \frac{1}{(N-2)p+1}) \text{ small. Here, } R'_3 \text{ is a function such that } |R'_3(x)| + |\nabla R'_3(x)| = o(\mu_1^{\frac{N}{Q+1}}). \text{ This and (2.4) give us the desired equality (5.43).} \square

By the definition of } \tilde{\psi}_{\mu,x}, \text{ we have }
\tilde{\psi}_{\mu,x}(x) = \tilde{\psi}_{\mu,x}(x) - \frac{pb_{N,p}^{N\mu_1^{\frac{N}{Q+1}}}}{\gamma_N} \sum_{i=1}^{k} A_{d,\xi,i} \int_{A} G(x, y) V_i^{p-1}(y) dy

\quad + \sum_{i=1}^{k} \int_{B(x_i, \xi_i)} G(x, y) \left[ \left( \sum_{m=1}^{k} PV_m \right)^p - \sum_{m=1}^{k} V_m^{p} \right] (y) dy \quad \text{for } x \in \Omega.

Hence
\begin{align}
q_0 \int_{B(x_j, \mu_1^\gamma)} U_j^{p-1} \tilde{\psi}_{\mu,x} \Psi_j - q_0 \int_{B(x_j, \mu_1^\gamma)} U_j^{p-1} \tilde{\psi}_{\mu,x} \Psi_j

\quad = \sum_{i=1}^{k} \int_{B(x_i, \xi_i)} \int_{B(x_i, \xi_i)} \left[ \left( \sum_{m=1}^{k} PV_m \right)^p - \sum_{m=1}^{k} V_m^{p} \right] (y) G(y, x) \left( q_0 U_j^{p-1} \Psi_j \right) (x) dy dx

\quad - \frac{b_{N,p}}{\gamma_N} \sum_{i=1}^{k} A_{d,\xi,i} \int_{A} G(x, y) pV_i^{p-1}(y) \left( q_0 U_j^{p-1} \Psi_j \right) (x) dy dx

\quad =: I_1 - \frac{b_{N,p}}{\gamma_N} \sum_{i=1}^{k} A_{d,\xi,i} I_{2i}.
\end{align}

Applying Fubini's theorem, we compute
\begin{align}
I_1 = \sum_{i=1}^{k} \int_{B(x_i, \xi_i)} \left[ \left( \sum_{m=1}^{k} PV_m \right)^p - \sum_{m=1}^{k} V_m^{p} \right] (y) \left[ P\Phi_j^I(y) + O \left( \mu_1^{\frac{N\mu_0}{Q+1}} \mu_1^{N-[(N-2)p-2]q_0} \right) \right] dy

\quad = o \left( \mu_1^{((N-2)p-2)} \right).
\end{align}

Let us calculate } I_{2i}. \text{ Because of (2.13),}
\begin{align}
\left| V_i^{p-1}(y) - \mu_i^{\frac{N(p-1)}{Q+1} + 1} \frac{\tilde{\psi}_{N,p}^{p-1}}{|y - x_i|^{(N-2)(p-1)}} \right| \leq C \mu_1^{\frac{N(p-1)}{Q+1} + 1} \frac{1}{|y - x_i|^{(N-2)(p-1)+1}} \quad \text{for } y \in A.
\end{align}

Accordingly,
\begin{align}
\int_{A} G(x, y) pV_i^{p-1}(y) dy = \mu_i^{\frac{N(p-1)}{Q+1} + 1} \int_{A} G(x, y) \frac{pb_{N,p}^{p-1}}{|y - x_i|^{(N-2)(p-1)}} dy + R_4(x)

\quad = \left( \frac{b_{N,p}}{\gamma_N} \right)^{p-1} \gamma_{N,p} \int_{A} \left[ |x - x_i|^{N-(N-2)p} - \Phi(x, x_i) \right] \left( \sum_{m=1}^{k} G(x, y) \frac{pb_{N,p}^{p-1}}{|y - x_i|^{(N-2)(p-1)}} dy + R_4(x)

\quad = \frac{N(p-1)}{Q+1} \sum_{m=1}^{k} \int_{B(x_m, \xi_m)} G(x, y) \frac{pb_{N,p}^{p-1}}{|y - x_i|^{(N-2)(p-1)}} dy + R_4(x)
\end{align}

where } R_4 \text{ is a function satisfying } |R_4(x)| + |\nabla R_4(x)| = o(\mu_1^{\frac{N}{Q+1}}). \text{ On the other hand, integration by parts and the oddness of } \Phi^I_{1,0} \text{ in the } l-\text{th variable yield}
\[
\begin{align*}
\mu_i^{\alpha_{m+1}} & := A_1 \mu_i^{\alpha_{m+1}} + \sum_{i=1}^{N} \mu_i^{\alpha_{m+1}} \left[ \partial_i \overline{H}(x_j, x_i) - (1 - \delta_{ij})(N - (N - 2)p)(x_j - x_i)_l \right] + o \left( \frac{N}{\mu_1^{\alpha_{m+1}}} \right) \\
\mu_i^{\alpha_{m+1}} & := A_1 \mu_i^{\alpha_{m+1}} + \sum_{i=1}^{N} \mu_i^{\alpha_{m+1}} \left[ \partial_i \overline{H}(x_j, x_i) - (1 - \delta_{ij})(N - (N - 2)p)(x_j - x_i)_l \right] + o \left( \frac{N}{\mu_1^{\alpha_{m+1}}} \right)
\end{align*}
\]

and

\[
\begin{align*}
\mu_i^{\alpha_{m+1}} & := A_1 \mu_i^{\alpha_{m+1}} + \sum_{i=1}^{N} \mu_i^{\alpha_{m+1}} \left[ \partial_i \overline{H}(x_j, x_i) - (1 - \delta_{ij})(N - (N - 2)p)(x_j - x_i)_l \right] + o \left( \frac{N}{\mu_1^{\alpha_{m+1}}} \right)
\end{align*}
\]

where \( A_1 > 0 \) is the constant in (2.20) and \( \delta_{ij} \) stands for the Kronecker delta. By integrating (5.48) over \( B(x_j, \mu_1^{\alpha_{m+1}}) \), applying (5.49), (5.50), and the oddness of \( \Psi^l \) in the \( l \)-th variable, we obtain

\[
I_{2i} = A_1 \left( \frac{b_{N,p}}{\gamma N} \right)^{\alpha_{m+1}} \tilde{\gamma}_{N,2} \mu_i^{\alpha_{m+1}} \left[ \partial_i \overline{H}(x_j, x_i) - (1 - \delta_{ij})(N - (N - 2)p)(x_j - x_i)_l \right] + o \left( \frac{N}{\mu_1^{\alpha_{m+1}}} \right)
\]

for \( l = 1, \ldots, N \). Combining (5.42), (5.46), (5.47), (5.51), (2.8), the identities

\[
b_{N,p} = \gamma N A_1 \quad \text{and} \quad \left( \frac{b_{N,p}}{\gamma N} \right)^{\alpha_{m+1}} \tilde{\gamma}_{N,1} = a_{N,p},
\]

(see (2.3) and (2.12) for the second identity), and the estimate

\[
q_0 \int_{B(x_j, \mu_1^{\alpha_{m+1}})} U_j^{\alpha_{m+1}} \tilde{\psi}_{\mu,x} \Psi_j = -A_1 \mu_j^{\alpha_{m+1}} \partial_i \tilde{\psi}_{\mu,x}(\xi_j) + o \left( \mu_1^{(N-2)p-2} \right),
\]

we establish

\[
\sum_{i=1}^{k} \int_{B(x_j, \mu_1^{\alpha_{m+1}})} \left( PU_{\mu,x} \right)^{\alpha_{m+1}} - \sum_{i=1}^{k} U_i^{\alpha_{m+1}} \right] P \Psi_j
\]

\[
= \frac{a_{N,p}}{\gamma N} A_1 \mu_j^{(N-2)p-2} \partial_i \overline{H}(\xi_j, \xi_j) - \frac{a_{N,p}}{\gamma N} A_1 \mu_j^{(N-2)p-2} \partial_i \overline{H}(\xi_j, \xi_j) - A_1 \mu_j^{(N-2)p-2} \partial_i \tilde{\psi}_{\mu,x}(\xi_j)
\]

\[
= A^{p+1} \tilde{\gamma}_{N,2} \mu_j^{(N-2)p-2} \partial_i \overline{H}(x_j, x_i) + \sum_{i=1}^{k} \mu_j^{(N-2)p-2} \partial_i \overline{H}(x_j, x_i) + o \left( \mu_1^{(N-2)p-2} \right)
\]

\[
= A^{p+1} \tilde{\gamma}_{N,2} \mu_j^{(N-2)p-2} \partial_i \overline{H}(x_j, x_i) + \sum_{i=1}^{k} \mu_j^{(N-2)p-2} \partial_i \overline{H}(x_j, x_i) + o \left( \mu_1^{(N-2)p-2} \right)
\]

\[
= A^{p+1} \tilde{\gamma}_{N,2} \mu_j^{(N-2)p-2} \partial_i \overline{H}(x_j, x_i) + \sum_{i=1}^{k} \mu_j^{(N-2)p-2} \partial_i \overline{H}(x_j, x_i) + o \left( \mu_1^{(N-2)p-2} \right)
\]
for \( l = 1, \ldots, N \). Similarly, we discover

\[
\sum_{m=1}^{k} \int_{B(x_m, \mu^1)} \left[ (PU_{\mu,x})^{q_0} - \sum_{i=1}^{k} U_i^{q_0} \right] P\Psi_j^0
\]

\[
= NA_1^{p+1} q_0 + 1 \mu_j^1 \left[ \frac{N_p}{0+1} \tilde{H}_{\delta, \xi}(\epsilon_j) - \sum_{i=1, i \neq j}^{k} \frac{\gamma_{N,p,1} \mu_{1}^{q_0+1} \gamma_{N,p,1} \mu_{1}^{q_0+1}}{\xi_j - \xi_i} \right]
\]

\[
+ o \left( \mu_1^{(N-2)p-2} \right).
\]  

(5.53)

**Estimate of \( J_{2e} \).** Arguing as in (5.24) and using \((N-2)p - 2 > 1\), we get

\[
\int_{\Omega} \left[ (PU_{\mu,x})^{q_0} - (PU_{\mu,x})^{q_0} \right] P\Psi_j^0 = o \left( \mu_1^{(N-2)p-2} \right)
\]  

(5.54)

for \( l = 1, \ldots, N \). Moreover, as in (5.26), we have

\[
\int_{\Omega} \left[ (PU_{\mu,x})^{q_0} - (PU_{\mu,x})^{q_0} \right] P\Psi_j^0 = (q_0 - q_e) (A_3 + o(1)) + o \left( \mu_1^{(N-2)p-2} \right).
\]  

(5.55)

**Conclusion.** From (5.1)–(5.2), (5.40), (5.41), (5.52)–(5.53), and (5.54)–(5.55), we see

\[
(\nabla \tilde{H}_{\delta, \xi}(\epsilon_j) + ((N-2)p - 2) \gamma_{N,p,1} \sum_{i=1, i \neq j}^{k} d_i^{q_0+1} \frac{\xi_j - \xi_i}{|\xi_j - \xi_i|^{(N-2)p}}
\]

\[
+ (N - (N-2)p) \gamma_{N,p,2} \sum_{i=1, i \neq j}^{k} A_{d, \xi, i} d_i^{q_0+1} \frac{\xi_j - \xi_i}{|\xi_j - \xi_i|^{(N-2)p}} = o(1)
\]  

(5.56)

and

\[
NA_1^{p+1} q_0 + 1 \mu_j^1 \left[ \tilde{H}_{\delta, \xi}(\epsilon_j) - \frac{N_p}{0+1} \gamma_{N,p,1} \frac{d_i^{q_0+1}}{|\xi_j - \xi_i|^{(N-2)p-2}} + \gamma_{N,p,2} \sum_{i=1, i \neq j}^{k} A_{d, \xi, i} d_i^{q_0+1} \frac{\xi_j - \xi_i}{|\xi_j - \xi_i|^{(N-2)p-N}} \right]
\]

\[
= (A_3 + o(1)) \frac{q_0 - q_e}{\mu_1^{(N-2)p-2}} + o(1)
\]  

(5.57)

for \( j = 1, \ldots, k \).

**5.4. The case that \( p \in (\max\{1, \frac{3}{N-2}\}, \frac{N-1}{N-2}) \).** Compared to the previous case \( p \in [\frac{N-1}{N-2}, \frac{N}{N-2}) \), the analysis here is more straightforward. Arguing as in the previous subsection, we observe that (5.40), (5.54), and (5.55) are still true. Furthermore, we have the equalities that are analogous to (5.52) and (5.53):

\[
\int_{\Omega} \left[ (PU_{\mu,x})^{q_0} - \sum_{i=1}^{k} U_i^{q_0} \right] P\Psi_j^0
\]

\[
= A_1^{p+1} \mu_j^1 \left[ \frac{N_p}{0+1} \gamma_{N,p,1} \frac{d_i^{q_0+1}}{|\xi_j - \xi_i|^{(N-2)p}} + \gamma_{N,p,2} \sum_{i=1, i \neq j}^{k} A_{d, \xi, i} d_i^{q_0+1} \frac{\xi_j - \xi_i}{|\xi_j - \xi_i|^{(N-2)p-N}} \right] + o \left( \mu_1^{(N-2)p-2} \right)
\]  

for \( l = 1, \ldots, N \), and

\[
\int_{\Omega} \left[ (PU_{\mu,x})^{q_0} - \sum_{i=1}^{k} U_i^{q_0} \right] P\Psi_j^0
\]

\[
= A_1^{p+1} \mu_j^1 \left[ \frac{N_p}{0+1} \gamma_{N,p,1} \frac{d_i^{q_0+1}}{|\xi_j - \xi_i|^{(N-2)p}} + \gamma_{N,p,2} \sum_{i=1, i \neq j}^{k} A_{d, \xi, i} d_i^{q_0+1} \frac{\xi_j - \xi_i}{|\xi_j - \xi_i|^{(N-2)p-N}} \right] + o \left( \mu_1^{(N-2)p-2} \right)
\]  

for \( l = 1, \ldots, N \), and

\[
\int_{\Omega} \left[ (PU_{\mu,x})^{q_0} - \sum_{i=1}^{k} U_i^{q_0} \right] P\Psi_j^0
\]
Note that for \( j \) imply
\[
\gamma_{N,p,1} \frac{d_{i}^{Np}}{|x - \xi_{i}|^{(N-2)p-2}} = o(1)
\]
and
\[
\gamma_{N,p,2} \frac{d_{i}^{Np}}{|x - \xi_{i}|^{(N-2)p-2}} = o(1)
\]
for \( j = 1, \ldots, k \).

### 5.5. Proof of Theorems 1.3 and 1.4.

We are ready to establish Theorems 1.3 and 1.4.

**Proof of Theorem 1.3.** We assume that \( N \geq 4 \) and \( p \in (\max\{1, \frac{3}{N-2}, \frac{N}{N-2}\}) \).

1. Fix any \( j = 1, \ldots, k \) and \( l = 1, \ldots, N \). By (2.4) and (2.6),
\[
\tilde{H}_{d,\xi}(x) = \begin{cases} 
\tilde{H}_{d,\xi,j}(x) + \sum_{i=1, i \neq j}^{k} \gamma_{N,p,1} \frac{d_{i}^{Np}}{|x - \xi_{i}|^{(N-2)p-2}} & \text{if } p \in (\frac{2}{N-2}, \frac{N-1}{N-2}), \\
\tilde{H}_{d,\xi,j}(x) + \sum_{i=1, i \neq j}^{k} \gamma_{N,p,1} \frac{d_{i}^{Np}}{|x - \xi_{i}|^{(N-2)p-2}} - \gamma_{N,p,2} A_{d,\xi} \frac{d_{i}^{Np}}{|x - \xi_{i}|^{(N-2)p-2}} & \text{if } p \in [\frac{N-1}{N-2}, \frac{N}{N-2}]
\end{cases}
\]
for \( x \in \Omega \). Therefore (5.56) and (5.58) are equivalent to \( \partial_{l} \tilde{H}_{d,\xi,j}(\xi_{j}) = o(1) \). Taking \( \varepsilon \to 0 \) leads to \( \partial_{l} \tilde{H}_{d,\xi,j}(\xi_{j}) = 0 \).

2. By (5.57), (5.59), and (5.60),
\[
\frac{N A_{1}^{p+1}}{q_{0} + 1} \frac{d_{j}^{N}}{d_{j}^{N+1}} \tilde{H}_{d,\xi,j}(\xi_{j}) = (A_{3} + o(1)) \frac{q_{0} - q_{\varepsilon}}{\mu_{1,\varepsilon}^{(N-2)p-2}} + o(1).
\]
Note that \( \mu_{1,\varepsilon} = u_{\varepsilon}^{-1/\alpha_{\varepsilon}}(x_{\varepsilon}) \), \( \lim_{\varepsilon \to 0} \alpha_{\varepsilon} = \frac{N}{q_{0} + 1} \), \( \lim_{\varepsilon \to 0} \mu_{1,\varepsilon} = 1 \), and
\[
q_{0} - q_{\varepsilon} = \frac{\varepsilon(q_{0} + 1)^{2}}{N + \varepsilon(q_{0} + 1)}.
\]
From these observations, (1.19), (2.20), and (2.25), we see
\[
\lim_{\varepsilon \to 0} u_{\varepsilon}^{p+1}(x_{\varepsilon}) = \frac{N^{2} A_{1}^{p+1}}{A_{3}(q_{0} + 1)^{3}} \frac{d_{j}^{N}}{d_{j}^{N+1}} \tilde{H}_{d,\xi,j}(\xi_{j}) = \frac{N}{q_{0} + 1} S \frac{p_{(q_{0} + 1)^{p+1}}}{p_{(q_{0} + 1)^{p+1}}} \frac{N}{q_{0} + 1} \delta_{j}^{N+1} \tilde{H}_{d,\xi,j}(\xi_{j})
\]
for \( j = 1, \ldots, k \). From this, the assertions in the statement easily follow.

3. Fixing a small number \( \eta \in (0, \rho_{2}) \), we assume that \( x \in \Omega \setminus \bigcup_{i=1}^{k} B(\xi_{i}, \eta) \). Then (3.25) and (4.20) imply
\[
\lambda_{1,\varepsilon}^{\alpha_{\varepsilon}} u_{\varepsilon}^{q_{\varepsilon}}(x) \leq C \sum_{i=1}^{k} \lambda_{i,\varepsilon}^{\alpha_{\varepsilon}(q_{\varepsilon} + 1)} \lambda_{i,\varepsilon}^{q_{\varepsilon}(N-2)p-2} \leq C \eta^{-q_{\varepsilon}(N-2)p-2} \lambda_{1,\varepsilon}^{-2(p+1)+O(\varepsilon)} \to 0 \quad \text{as } \varepsilon \to 0.
\]
Using this, (1.16), (4.16), and the dominated convergence theorem, we conclude
\[
\|u_\varepsilon\|_{L^\infty(\Omega)} v_\varepsilon(x) = \sum_{i=1}^{k} G(x, \xi_i) \int_{B(x_i, \frac{\varepsilon}{2})} \lambda_{1\varepsilon}^{\varepsilon} u_\varepsilon(y) dy + \sum_{i=1}^{k} \int_{B(x_i, \frac{\varepsilon}{2})} [G(x, y) - G(x, \xi_i)] \lambda_{1\varepsilon}^{\varepsilon} u_\varepsilon(y) dy + \int_{\Omega \setminus \bigcup_{i=1}^{k} B(x_i, \frac{\varepsilon}{2})} G(x, y) \lambda_{1\varepsilon}^{\varepsilon} u_\varepsilon(y) dy \rightarrow \|U_{1,0}\|^{p_0}_{L^{p_0}(\mathbb{R}^N)} \sum_{i=1}^{k} \delta_i^{\varepsilon_0+1} G(x, \xi_i) + O(0) + 0 + 0
\]
in \(C^1(\Omega \setminus \bigcup_{i=1}^{k} B(\xi_i, \eta))\) as \(\varepsilon \to 0\).

Besides, because
\[-\Delta (\lambda_{1\varepsilon}^{\varepsilon} u_\varepsilon) = (\lambda_{1\varepsilon}^{\varepsilon} v_\varepsilon)^p \rightarrow \|U_{1,0}\|^{p_0}_{L^{p_0}(\mathbb{R}^N)} \left(\sum_{i=1}^{k} \delta_i^{\varepsilon_0+1} G(\cdot, \xi_i)\right)^p,\]
we know
\[\|u_\varepsilon\|_{L^\infty(\Omega)} v_\varepsilon(x) = \lambda_{1\varepsilon}^{\varepsilon} u_\varepsilon(x) \rightarrow \|U_{1,0}\|^{p_0}_{L^{p_0}(\mathbb{R}^N)} \tilde{G}_{\delta, \xi}(x) \text{ in } C^1_{\text{loc}}(\Omega \setminus \{\xi_1, \ldots, \xi_k\})\]
as \(\varepsilon \to 0\). This completes the proof. \(\square\)

Proof of Theorem 1.4. In light of [17, Theorems 1.1-1.3], it suffices to show that \(\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon \in (0, \varepsilon_0)}\) has only one blow-up point. Recall from (5.27) and (5.35) that
\[\delta_j^{N-2} \nabla \tau(\xi_j) - \sum_{i=1, i \neq j}^{k} \left(\delta_i^{\varepsilon_0+1} \delta_j^{\varepsilon_0+1} + \delta_i^{\varepsilon_0+1} \delta_j^{\varepsilon_0+1}\right) (\nabla \xi G)(\xi_i, \xi_j) = 0 \quad (5.62)\]
for \(j = 1, \ldots, k\), provided \(p \in \left[\frac{N}{N-2}, \frac{N+2}{N-2}\right]\). According to [4, Corollary 3.2], the Robin function \(\tau\) is strictly convex on bounded convex domains in \(\mathbb{R}^N\) for \(N \geq 3\). Therefore, by slightly modifying the proof of [16, Theorem 3.2], we conclude that there does not exist \(\xi = (\xi_1, \ldots, \xi_k)\) satisfying (5.62) for \(k \geq 2\).

6. Existence results

Let \(X_{p,q_0} = W^{2,(p+1)/p}(\Omega) \times W^{2,(q_0+1)/q_0}(\Omega)\). Given \(\tilde{\rho} \in (0, 1)\) small, we define the configuration set
\[\Lambda_{\tilde{\rho}} = \left\{(\tilde{d}, x) = (\tilde{d}_1, \ldots, \tilde{d}_k, x_1, \ldots, x_k) \in (\tilde{\rho}, \tilde{\rho}^{-1})^k \times \Omega^k : \text{dist}(x_i, \partial \Omega) \geq \tilde{\rho} \text{ for } 1 \leq i \leq k, \text{dist}(x_i, x_j) \geq \tilde{\rho} \text{ for } 1 \leq i \neq j \leq k\right\}. \quad (6.1)\]
We write \(\mu_\varepsilon = (\mu_{1\varepsilon}, \ldots, \mu_{k\varepsilon})\) where \(\mu_{i\varepsilon}\) is the parameter defined by (1.23). Using this \(\mu_\varepsilon\) and \(x\) in (6.1), we also set \((U_{i\varepsilon}, V_{i\varepsilon})\) and \((\psi_{i\varepsilon}, \phi_{i\varepsilon})\) by (4.1) (replacing \(x_i\) for \(x_{i\varepsilon}\)), and \(P U_{\mu_\varepsilon, x}\) by (4.9). As before, we will often drop the subscript \(\varepsilon\).

The argument in [21], based on the classical Lyapunov-Schmidt reduction, gives the following

Proposition 6.1. Suppose that the assumptions in Theorem 1.5 hold. Then, for each \(\varepsilon \in (0, \varepsilon_0)\) and \((\tilde{d}, x) \in \Lambda_{\rho}\), there exist a unique pair \((\psi_{\mu, x}, \phi_{\mu, x}) \in X_{p,q_0}\) and numbers \(\{e_{it}\}_{i=1,\ldots,k, t=0,\ldots,N} \subset\)
We consider $\mathbb{R}^{(N+1)k}$ such that

\begin{align*}
\begin{cases}
-\Delta (PU_{\mu,x} + \psi_{\mu,x}) = \left| \sum_{i=1}^{k} PV_i + \phi_{\mu,x} \right|^{p-1} \left( \sum_{i=1}^{k} PV_i + \phi_{\mu,x} \right) + p \sum_{i=1}^{k} \sum_{l=0}^{N} c_{il} V_i^{p-1} \Phi_i^l \quad \text{in } \Omega, \\
-\Delta \left( \sum_{i=1}^{k} PV_i + \phi_{\mu,x} \right) = \left| PU_{\mu,x} + \psi_{\mu,x} \right|^{q-1} (PU_{\mu,x} + \psi_{\mu,x}) + q_0 \sum_{i=1}^{k} \sum_{l=0}^{N} c_{il} U_i^{q-1} \Psi_i^l \quad \text{in } \Omega, \\
\psi_{\mu,x} = \phi_{\mu,x} = 0
\end{cases}
\end{align*}

on $\partial \Omega$ \quad (6.2)

satisfying

\begin{align*}
\int_{\Omega} \left( PV_i^{p-1} \Phi_i^l \phi_{\mu,x} + q_0 U_i^{q_0-1} \Psi_i^l \psi_{\mu,x} \right) = 0 \quad \text{for } i = 1, \ldots, k \text{ and } l = 0, \ldots, N \quad (6.3)
\end{align*}

and

\begin{align*}
\|\Delta \psi_{\mu,x}\|_{L^2}^{q_0 \frac{q_0+1}{q_0}} (\Omega) + \|\Delta \phi_{\mu,x}\|_{L^2}^{q_0 \frac{q_0+1}{q_0}} (\Omega) \leq C \left\| \sum_{i=1}^{k} U_i^{q_0} - (PU_{\mu,x})^{q_0} \right\|_{L^{q_0 \frac{q_0+1}{q_0}} (\Omega)} \quad (6.4)
\end{align*}

for some $C > 0$ depending only on $N$, $p$, $\Omega$, $\varepsilon_0$, and $\bar{\rho}$.

Proof. One can argue as in [21, Proposition 4.6, Lemma 3.3].

Employing the above proposition, we will prove Theorem 1.5. To find the desired solution of (1.1), it is sufficient to find $(d, x) \in A_{\bar{\rho}}$ such that

\begin{align*}
c_{jl} = 0 \quad \text{for } j = 1, \ldots, k \text{ and } l = 0, \ldots, N. \quad (6.5)
\end{align*}

Owing to (6.2), equation (6.5) is equivalent to

\begin{align*}
I_{\varepsilon}' \left( PU_{\mu,x} + \psi_{\mu,x}, \sum_{i=1}^{k} PV_i + \phi_{\mu,x} \right) \left( P\Psi_j, P\Phi_j \right) = 0 \quad \text{for } j = 1, \ldots, k \text{ and } l = 0, \ldots, N; \quad (6.6)
\end{align*}

cf. (5.1).\(^9\) In the following, we will express (6.6) in terms of the function $\Upsilon_{p,k}$ in (1.21). As in the previous section, we will split the cases according to the value of $p$.

6.1. The case $p \in (\frac{N}{N-2}, \frac{N+2}{N-2})$. A direct computation using (6.2)–(6.4), Hölder’s inequality, and the Sobolev inequality shows

\begin{align*}
I_{\varepsilon}' \left( PU_{\mu,x} + \psi_{\mu,x}, \sum_{i=1}^{k} PV_i + \phi_{\mu,x} \right) \left( P\Psi_j, P\Phi_j \right) - I_{\varepsilon}' \left( PU_{\mu,x}, \sum_{i=1}^{k} PV_i \right) \left( P\Psi_j, P\Phi_j \right)
\quad = \int_{\Omega} \left( (PU_{\mu,x})^{q_0} - \sum_{i=1}^{k} U_i^{q_0} \right) \psi_{\mu,x} - 2 \int_{\Omega} \nabla \psi_{\mu,x} \cdot \nabla \phi_{\mu,x} = o (\mu_1^{N-2}) \quad (6.7)
\end{align*}

\(^9\)To guarantee the positivity of the solution of (1.1), we work with the energy functional (1.5) whose nonlinear terms $|\cdot|^{p+1}$ and $|\cdot|^{q+1}$ are substituted with $v_i^{p+1}$ and $w_i^{q+1}$, respectively. Thanks to the maximum principle for cooperative elliptic systems in [13], all components of the solution are positive.
uniformly in \((\bar{d}, x) \in \Lambda_p\) as \(\varepsilon \to 0\); cf. (5.10). By Estimates of \(J_{1\varepsilon}\) and \(J_{2\varepsilon}\) in Subsection 5.1, (6.7), (5.61), (1.19), (2.20), (5.25), and (2.23), equality (6.6) reads
\[
d_j^{N-2}\nabla \tau(x_j) - \sum_{i=1, i \neq j}^k \left( d_i^{j+1} d_j^{N} + d_i^{N} d_j^{p+1} \right) (\nabla \xi G)(x_i, x_j) = o(1) \tag{6.8}
\]
and
\[
(N - 2)d_j^{N-2}\tau(x_j) - N \sum_{i=1, i \neq j}^k \left( \frac{d_i^{j+1}}{q_0 + 1} + \frac{d_i^{N}}{p + 1} \right) G(x_i, x_j) = \frac{1}{A_1 A_2} S^{\frac{p(q_0+1)}{p^0q_0-1}} + o(1) \tag{6.9}
\]
for \(j = 1, \ldots, k\); cf. (5.27) and (5.28). We set \(C_0 = (A_1 A_2)^{-1} S^{p(q_0+1)/(p^0q_0-1)} > 0\) in the definition of \(\Upsilon_{p,k}\) in (1.21). Then it is easy to see that (6.8) and (6.9) are equivalent to
\[
\nabla x_j \Upsilon_{p,k}(\bar{d}, x) + o(1) = 0 \quad \text{and} \quad \frac{\partial \Upsilon_{p,k}}{\partial d_j} (\bar{d}, x) + o(1) = 0 \quad \text{for } j = 1, \ldots, k, \tag{6.10}
\]
respectively.

If \((\delta, \xi) \in (0, \infty)^k \times \Omega^k\) is an isolated critical point of \(\Upsilon_{p,k}\) for which (1.22) holds, then properties of the Brouwer degree yield a solution \((\bar{d}_\varepsilon, x_\varepsilon) \in \Lambda\) of (6.10) for \(\varepsilon > 0\) small such that \((\bar{d}_\varepsilon, x_\varepsilon) \to (\delta, \xi)\) as \(\varepsilon \to 0\). The proof is completed.

6.2. **The case that** \(p = \frac{N}{N-2}\). Let \(\varepsilon_p = \left[\frac{-((N-2)\varepsilon)}{W_1(1-(N-2)\varepsilon)}\right]^{\frac{1}{N-2}}\), which is a small positive number for \(\varepsilon > 0\) small. By the definition of the function \(W_{-1}\), it holds that
\[
-(N - 2)\varepsilon \mu_p^2 - N = W_{-1}(-(N - 2)\varepsilon) \Rightarrow -(N - 2)\varepsilon = -(N - 2)\varepsilon \mu_p^2 - N e^{-(N-2)\varepsilon \mu_p^2 - N} \]
\[
\Leftrightarrow \mu_p = e^{-\varepsilon \mu_p^2 - N} \Leftrightarrow \mu_p^{N-2} \log \mu_p = -\varepsilon.
\]
Using these relations and arguing as in the previous subsection, we see that (6.6) is equivalent to (6.8) and
\[
(N - 2)d_j^{N-2}\tau(x_j) - N \sum_{i=1, i \neq j}^k \left( \frac{d_i^{N}}{q_0 + 1} + \frac{d_i^{p+1}}{p + 1} \right) G(x_i, x_j)
\]
\[
= \frac{(A_3 + o(1)) \gamma_N}{a_{N,p} A_1} \frac{q_0 - q_\varepsilon}{\mu_1^{N-2} \log \mu_1^{-1}} + o(1) = \frac{1}{S^{N-1}\left|S_p^{p+1}\right|A_1} S^{\frac{p(q_0+1)}{p^0q_0-1}} + o(1). \tag{6.11}
\]
If we set \(C_0 = (|S^{N-1}|b_{N,p} A_1)^{-1} S^{p(q_0+1)/(p^0q_0-1)} > 0\), then (6.8) and (6.11) are reduced to (6.10). Thus the desired conclusion follows from the preceding discussion.

6.3. **The case that** \(p \in \left[\frac{N-1}{N-2}, \frac{N}{N-2}\right] \cap \left(\frac{3}{N-2}, \frac{N}{N-2}\right)\). Reasoning as in (6.7), we obtain
\[
I_{\varepsilon}' \left( P_{\mu,x} + \psi_{\mu,x}, \sum_{i=1}^k PV_i + \phi_{\mu,x} \right) \left( P\Phi^l_{ij}, P\Phi^l_{ij} \right) - I_{\varepsilon}' \left( P_{\mu,x}, \sum_{i=1}^k PV_i \right) \left( P\Phi^l_{ij}, P\Phi^l_{ij} \right)
\]
\[
= o \left( \mu_1^{N-2(p-2)} \right) \tag{6.12}
\]
uniformly in $(\mathbf{d}, x) \in \Lambda_\varepsilon$ as $\varepsilon \to 0$; cf. (5.40). By Estimates of $J_{1\varepsilon}$ and $J_{2\varepsilon}$ in Subsection 5.3, (6.12), (5.61), (1.19), (5.25), and the identity

\[
\begin{align*}
\begin{cases}
\frac{\partial^{N_p}}{\partial x^{N_p}} \tilde{H}_{d,x}(x) = \tilde{H}_{d,x}(x), \\
\frac{\partial^{N_p}}{\partial x^{N_p}} A_{d,x,i} = A_{d,x,i}
\end{cases}
\end{align*}
\]

for $\mathbf{d} = d_1 d_1, \ldots, d_1 d_k$, $x \in \Omega$, $i = 1, \ldots, k$,

equality (6.6) reads

\[
\begin{align*}
\nabla \tilde{H}_{d,x}(x_j) + ((N - 2)p - 2)\tilde{\gamma}_{N,p,1} \sum_{i=1, i \neq j}^k \frac{\tilde{\gamma}_{N,p,1} d_1^{N_p} x_j - x_i}{|x_j - x_i|^{(N-2)p}} & \\
+ (N - (N - 2)p)\tilde{\gamma}_{N,p,2} \sum_{i=1, i \neq j}^k A_{d,x,i} \frac{\tilde{\gamma}_{N,p,1} d_1^{N_p} x_j - x_i}{|x_j - x_i|^{(N-2)(p-1)}} &= o(1) \tag{6.13}
\end{align*}
\]

and

\[
\begin{align*}
\frac{N}{q_0 + 1} d_1^{N_p} \sum_{i=1, i \neq j}^k \frac{\tilde{\gamma}_{N,p,1} d_1^{N_p} x_j - x_i}{|x_j - x_i|^{(N-2)p-2}} & \\
+ \sum_{i=1, i \neq j}^k \tilde{\gamma}_{N,p,2} A_{d,x,i} d_1^{N_p} & = o(1) \tag{6.14}
\end{align*}
\]

for $j = 1, \ldots, k$; cf. (5.56) and (5.57). To conclude the proof, it is enough to prove the following.

**Lemma 6.2.** Let $I$ and $II$ be the left-hand sides of (6.13) and (6.14), respectively. We set $C_0 = (p + 1)A_1^{-(p+1)} S^{p(q_0+1)/(pq_0-1)} > 0$ in the definition of $\tilde{\gamma}_{p,k}$ in (1.21). For $j = 1, \ldots, k$, we have

\[
\nabla x_j \tilde{\gamma}_{p,k}(\tilde{d}, x) = (p + 1)d_1^{N_p} \quad \text{and} \quad \tilde{d}_j \frac{\partial \tilde{\gamma}_{p,k}}{\partial \tilde{d}_j}(\tilde{d}, x) = (p + 1)II. \tag{6.15}
\]

**Proof.** 1. We set a function

\[
P(x, \xi) = - (\nabla \xi G)(x, \xi) \cdot \nu(\xi) \quad \text{for} \ x \in \Omega \ \text{and} \ \xi \in \partial \Omega,
\]

and fix $j = 1, \ldots, k$. By (2.6) and (1.18),

\[
\tilde{H}_{d,x,j}(x) = \int_{\Omega} G(x, z) \left[ \frac{\tilde{\gamma}_{N,p,1} d_1^{N_p}}{|z - x_j|^{N-2}} \right]^p \left[ \sum_{m=1}^k d_1^{N_p} G(z, x_m) \right] dz
\tag{6.16}
\]

\[
+ \int_{\partial \Omega} P(x, z) \frac{\tilde{\gamma}_{N,p,1} d_1^{N_p}}{|z - x_j|^{(N-2)p-2}} dS_z - \tilde{\gamma}_{N,p,2} A_{d,x,j} d_1^{N_p} |x - x_j|^{N-(N-2)p}
\]

for $x \in \Omega$, where the subscript $z$ in $dS_z$ refers to the variable of integration. Also, the representation formula tells us that

\[
\tilde{\gamma}_{N,p,2} |x - x_j|^{N-(N-2)p} = p \int_{\Omega} G(x, z) \left( \frac{\tilde{\gamma}_{N,p,1}}{|z - x_j|^{N-2}} \right)^{p-1} dz
\tag{6.17}
\]

\[
+ \tilde{\gamma}_{N,p,2} \int_{\partial \Omega} P(x, z) |z - x_j|^{N-(N-2)p} dS_z.
\]
Plugging (6.17) into (6.16) and differentiating the result in the $x$-variable, we see
\[
\frac{d_j^{N_{q_0+1}}}{d_j^{N_{q_0+1}}} \left( \nabla_x \tilde{H}_{d,x,j} \right) (x_j) = d_j^{N_{q_0+1}} \int_{\Omega} \left( \nabla_x G(x_j, z) \right) \left[ \left( \frac{\gamma_N d_j^{N_{q_0+1}}}{|z - x_j|^{N_{q_0+1}}} \right)^p - \sum_{m=1}^k \frac{d_j^{N_{q_0+1}}}{d_m^{N_{q_0+1}}} G(z, x_m) \right]^p 
\]
\[
- pA_{d,x,j} \left( \frac{\gamma_N d_j^{N_{q_0+1}}}{|z - x_j|^{N_{q_0+1}}} \right)^{p-1} dz
\]
\[
+ \tilde{\gamma}_{N,p,1} \int_{\partial\Omega} (\nabla_x P)(x_j, z) \frac{d_j^{N_{q_0+1}}}{|z - x_j|^{(N-2)p-2}} dS_z
\]
\[
- \tilde{\gamma}_{N,p,2} A_{d,x,j} d_j^{N_{q_0+1}} \int_{\partial\Omega} (\nabla_x P)(x_j, z) |z - x_j|^{-(N-2)p} dS_z.
\]

(6.18)

Note that the first term on the right-hand side of (6.18) is well-defined for $(N-2)p < N$, because its integrand is bounded by
\[
C \frac{1}{|z - x_j|^{N-1}} \left( \frac{1}{|z - x_j|^{(N-2)p-2}} + \frac{1}{|z - x_j|^{(N-2)(p-1)-1}} \right) \text{ for } z \in \Omega \text{ near } x_j.
\]
Moreover, rewriting (6.16) as
\[
\tilde{H}_{d,x,i}(x) = \int_{\Omega} G(x, z) \left[ \sum_{m=1}^k \left( \frac{\gamma_N d_j^{N_{q_0+1}}}{|z - x_m|^{N_{q_0+1}}} \right)^p - \sum_{m=1}^k \frac{d_j^{N_{q_0+1}}}{d_m^{N_{q_0+1}}} G(z, x_m) \right]^p dz
\]
\[
+ \tilde{\gamma}_{N,p,1} \int_{\partial\Omega} P(x, z) \frac{d_j^{N_{q_0+1}}}{|z - x_m|^{(N-2)p-2}} dS_z - \tilde{\gamma}_{N,p,2} A_{d,x,i} d_j^{N_{q_0+1}} |x - x_i|^{-(N-2)p} dS_z
\]
\[
- \tilde{\gamma}_{N,p,1} \sum_{m=1, m \neq i}^k \frac{d_j^{N_{q_0+1}}}{d_m^{N_{q_0+1}}} |x - x_m|^{-(N-2)p-2}
\]

(6.19)

for $x \in \Omega$ and $i = 1, \ldots, k$, and then differentiating it with respect to the parameter $x_j$, we observe
\[
\left( \nabla_{x_j} \tilde{H}_{d,x,i} \right) (x) = p \int_{\Omega} G(x, z) \left[ (N-2) \gamma_N d_j^{N_{q_0+1}} \frac{z - x_j}{|z - x_j|^{(N-2)p+2}} \nabla_{x_j} G(z, x_j) \right] dz
\]
\[
+ ((N-2)p - 2) \tilde{\gamma}_{N,p,1} d_j^{N_{q_0+1}} \int_{\partial\Omega} P(x, z) \frac{z - x_j}{|z - x_j|^{(N-2)p}} dS_z
\]
\[
- \tilde{\gamma}_{N,p,2} \left( \nabla_{x_j} A_{d,x,i} \right) d_j^{N_{q_0+1}} |x - x_i|^{-(N-2)p}
\]
\[
- ((N-2)p - 2) \tilde{\gamma}_{N,p,1} d_j^{N_{q_0+1}} |x - x_j|^{-(N-2)p}
\]

(6.20)

for $i \neq j$ and $x \in \Omega$ near $x_i$. Putting (6.17) into (6.19) and differentiating the result, we also obtain
\[
\left( \nabla_{x_j} \tilde{H}_{d,x,j} \right) (x) = p \int_{\Omega} G(x, z) \left[ (N-2) \gamma_N d_j^{N_{q_0+1}} \frac{z - x_j}{|z - x_j|^{(N-2)p+2}} \nabla_{x_j} G(z, x_j) \right] dz
\]
\[ \left( \sum_{m=1}^{k} \frac{N_{m+1}^N}{d_m^{90+1}} G(z, x_m) \right)^{p-1} \frac{N_{(p+1)}}{d_j^{90+1}} (\nabla_x G)(z, x_j) \]

\[ -(N - 2)(p - 1) \gamma_N^{p-1} A_{d,x,j} \frac{N_{(p+1)}}{d_j^{90+1}} \frac{z - x_j}{|z - x_j|((N-2)(p-1)+2)} dz \]

\[ + ((N - 2)p - 2) \tilde{\gamma}_{N,p,1} d_j^{90+1} \int_{\partial\Omega} P(x, z) \frac{z - x_j}{|z - x_j|((N-2)p+2)} dS_z \]

\[ + (N - (N - 2)p) \gamma_{N,p,2} A_{d,x,j} \frac{N_{(p+1)}}{d_j^{90+1}} \int_{\partial\Omega} P(x, z) \frac{z - x_j}{|z - x_j|((N-2)(p-1)+2)} dS_z \]

\[ - \gamma_{N,p,2} \left( \nabla_x A_{d,x,j} \right) \frac{N_{(p+1)}}{d_j^{90+1}} |x - x_j|^{N-(N-2)p} \]

for \( x \in \Omega \) near \( x_j \). Combining (6.20) and (6.21), we find

\[ \sum_{i=1}^{k} \frac{N_{i+1}^N}{d_i^{90+1}} \left( \nabla_{x_j} \tilde{H}_{d,x,i} \right) (x_i) \]

\[ = pd_j^{90+1} \int_{\Omega} G(x_i, z) \left[ (N - 2) \gamma_N^{p} d_j^{90+1} \frac{N_{p+1}}{d_j^{90+1}} \frac{z - x_j}{|z - x_j|((N-2)p+2)} \right. \]

\[ - \left( \sum_{m=1}^{k} \frac{N_{m+1}^N}{d_m^{90+1}} G(z, x_m) \right)^{p-1} \frac{N_{(p+1)}}{d_j^{90+1}} (\nabla_x G)(x_j, z) \]

\[ -(N - 2)(p - 1) \gamma_N^{p-1} A_{d,x,j} \frac{N_{(p+1)}}{d_j^{90+1}} \frac{z - x_j}{|z - x_j|((N-2)(p-1)+2)} dz \]

\[ + ((N - 2)p - 2) \tilde{\gamma}_{N,p,1} d_j^{90+1} \int_{\partial\Omega} P(x_j, z) \frac{z - x_j}{|z - x_j|((N-2)p+2)} dS_z \]

\[ + (N - (N - 2)p) \gamma_{N,p,2} A_{d,x,j} \frac{N_{(p+1)}}{d_j^{90+1}} \int_{\partial\Omega} P(x_j, z) \frac{z - x_j}{|z - x_j|((N-2)(p-1)+2)} dS_z \]

\[ + p \sum_{i=1, i \neq j}^{k} \frac{N_{i+1}^N}{d_i^{90+1}} \int_{\Omega} G(x_i, z) \left[ (N - 2) \gamma_N^{p} d_j^{90+1} \frac{N_{p+1}}{d_j^{90+1}} \frac{z - x_j}{|z - x_j|((N-2)p+2)} \right. \]

\[ - \left( \sum_{m=1}^{k} \frac{N_{m+1}^N}{d_m^{90+1}} G(z, x_m) \right)^{p-1} \frac{N_{(p+1)}}{d_j^{90+1}} (\nabla_x G)(x_j, z) \]

\[ + ((N - 2)p - 2) \tilde{\gamma}_{N,p,1} \sum_{i=1, i \neq j}^{k} \frac{N_{i+1}^N}{d_i^{90+1}} \frac{N_{p+1}}{d_j^{90+1}} \int_{\partial\Omega} P(x_i, z) \frac{z - x_j}{|z - x_j|((N-2)p+2)} dS_z - \frac{x_i - x_j}{|x_i - x_j|((N-2)p)} \]
\( f, g \in C^2(D), \quad \int_D (g \Delta f - f \Delta g)(z)dz = \int_{\partial D} \left( \frac{\partial f}{\partial \nu} g - \frac{\partial g}{\partial \nu} f \right) (z)dS_z. \) (6.24)

For simplicity, let us assume that \( x_j = 0 \) and write \( \Omega_r = \Omega \setminus B(0, r) \) for \( r > 0 \) small. If

\[
\begin{align*}
\{ f_1(z) := G(0, z), \\
g_1(z) := ((N - 2)p - 2) \tilde{\gamma}_{N,p,1} d_j^{N(p+1)} \frac{z}{|z|^{(N-2)p}} + (N - (N - 2)p) \tilde{\gamma}_{N,p,2} A_{d,x,j} \frac{N_p}{|z|^{(N-2)(p-1)}}
\end{align*}
\]

to \( z \in \Omega_r \), then (6.24) with \( (f, g) = (f_1, g_1) \) and \( D = \Omega_r \) is reduced to

\[
p \int_{\Omega_r} G(0, z) \left[ (N - 2) \tilde{\gamma}_N d_j^{N(p+1)} \frac{z}{|z|^{(N-2)p+2}} - (N - 2)(p - 1) \tilde{\gamma}_N A_{d,x,j} \frac{N_p}{|z|^{(N-2)(p-1)+2}} \right] dz
\]

\[
= - \int_{\partial \Omega} P(0, z) g_1(z)dS_z + o_r(1) \quad (6.25)
\]

where \( o_r(1) \to 0 \) as \( r \to 0 \). Indeed, it is a consequence of (2.2), the symmetry of \( \tilde{x}_i \) on \( \partial B(0, r) \) with respect to the origin, and the relations

\[
\Delta f_1(z) = 0 \quad \text{for } z \in \Omega_r \quad \text{and} \quad \frac{\partial f_1}{\partial \nu}(z) = -P(0, z) \quad \text{for } z \in \partial \Omega.
\]

Also, if

\[
\begin{align*}
f_2(z) := G(x_i, z) \quad \text{(for any } i \neq j, \\
g_2(z) := ((N - 2)p - 2) \tilde{\gamma}_{N,p,1} \frac{z}{|z|^{(N-2)p}}
\end{align*}
\]

to \( z \in \Omega_r \setminus B(x_i, s) \), then plugging \( (f, g) = (f_2, g_2) \) and \( D = \Omega_r \setminus B(x_i, s) \) into (6.24), taking \( s \to 0 \), and using the estimates

\[
\int_{\partial B(x_i,s)} \left( \frac{\partial f_2}{\partial \nu} g_2 \right)(z)dS_z = ((N - 2)p - 2) \tilde{\gamma}_{N,p,1} \int_{\partial B(x_i,s)} \frac{(N - 2) \tilde{\gamma}_N}{|z - x_i|^{N-1}} \frac{z}{|z|^{(N-2)p}} dS_z + o_s(1)
\]

\[
= ((N - 2)p - 2) \tilde{\gamma}_{N,p,1} \frac{x_i}{|x_i|^{(N-2)p}} + o_s(1) \quad \text{(by (2.1))}
\]

give

\[
p(N - 2) \tilde{\gamma}_N \int_{\Omega_r} G(x_i, z) \frac{z}{|z|^{(N-2)p+2}} dz
\]

\[
= ((N - 2)p - 2) \tilde{\gamma}_{N,p,1} \left[ - \int_{\partial \Omega} P(x_i, z) \frac{z}{|z|^{(N-2)p}} dS_z + \frac{x_i}{|x_i|^{(N-2)p}} \right]. \quad (6.26)
\]

Finally, if

\[
\begin{align*}
f_3(z) := (\nabla_x G)(0, z), \\
g_3(z) := \tilde{\gamma}_{N,p,1} d_j^{N(p+1)} \frac{1}{|z|^{(N-2)p-2}} - \tilde{\gamma}_{N,p,2} A_{d,x,j} \frac{N_p}{|z|^{N-(N-2)p}}
\end{align*}
\]

to \( z \in \Omega_r \), then (6.24) with \( (f, g) = (f_3, g_3) \) and \( D = \Omega_r \) reads

\[
\int_{\Omega} (\nabla_x G)(0, z) \left[ \tilde{\gamma}_N d_j^{N(p+1)} \frac{1}{|z|^{(N-2)p}} - p \tilde{\gamma}_N A_{d,x,j} \frac{N_p}{|z|^{(N-2)(p-1)}} \right] dz
\]

\[
= - \int_{\partial \Omega} (\nabla_x P)(0, z) g_3(z)dS_z + o(1). \quad (6.27)
\]

In view of (6.23), (6.18), (6.22), (6.25)–(6.27), and the dominated convergence theorem, we have

\[
\nabla_{x_j} \Upsilon_{p,k}(d, x) = (p + 1) d_j^{N(p+1)} I
\]
Consequently, the second equality in (6.15) is true.

(2) By (2.6), (1.18), and the assumption that \((N - 2)p < N\), we have

\[
\sum_{i=1}^{k} d_i^{\frac{N(p+1)}{2}} H_{\bar{d},x,i}(x_i)
\]

\[
= \sum_{i=1}^{k} \int_{\Omega} \left[ \left( \frac{\gamma_N d_i^{\frac{N}{2}}}{|z - x_i|^{N-2}} \right)^{p+1} - d_i^{\frac{N}{2}} G(x_i, z) \left( \sum_{m=1}^{k} d_m^{\frac{N}{2}} G(z, x_m) \right)^p \right] dz
\]

\[
+ \gamma_N^{p+1} \sum_{i=1}^{k} \int_{\mathbb{R}^N \setminus \Omega} \frac{d_i^{\frac{N(p+1)}{2}}}{|z - x_i|^{(N-2)(p+1)}} dz - \gamma_N p, 2 \sum_{i=1}^{k} A_{\bar{d},x,i} d_i^{\frac{Np}{2}} |x_i - x_i|^{N-(N-2)p} = 0
\]

where the first term on the right-hand side is well-defined. It follows from (1.21), (2.4), and (6.14) that

\[
\frac{q_0 + 1}{N(p+1)} \left[ d_j \frac{\partial Y_{p,k}}{\partial d_j}(\bar{d}, x) - (p + 1) II \right]
\]

\[
= \int_{\Omega} \left[ \left( \frac{\gamma_N d_j^{\frac{N}{2}}}{|z - x_j|^{N-2}} \right)^{p+1} - d_j^{\frac{N}{2}} G(z, x_j) \left( \sum_{m=1}^{k} d_m^{\frac{N}{2}} G(z, x_m) \right)^p \right] dz
\]

\[
+ \gamma_N^{p+1} \int_{\mathbb{R}^N \setminus \Omega} \frac{d_j^{\frac{N(p+1)}{2}}}{|z - x_j|^{(N-2)(p+1)}} dz - \frac{q_0 + 1}{N(p+1)} C_0
\]

\[
- \int_{\Omega} \left[ \gamma_N^{p+1} \sum_{i=1}^{k} \frac{d_j^{\frac{N}{2}}}{|z - x_j|^{N-2}} \frac{d_i^{\frac{Np}{2}}}{|z - x_i|^{(N-2)p}} - d_j^{\frac{N}{2}} G(z, x_j) \left( \sum_{m=1}^{k} d_m^{\frac{N}{2}} G(z, x_m) \right)^p \right] dz
\]

\[
+ \frac{q_0 + 1}{N(p+1)} \left[ \gamma_N p, 2 \sum_{i=1}^{k} A_{\bar{d},x,i} d_i^{\frac{Np}{2}} |x_j - x_i|^{N-(N-2)p} + \gamma_N p, 2 \sum_{i=1}^{k} A_{\bar{d},x,i} d_i^{\frac{Np}{2}} |x_j - x_i|^{N-(N-2)p-N} \right] + \frac{q_0 + 1}{N} S_{\frac{q_0+1}{N}}^{\frac{p}{p+1}} = 0.
\]

Consequently, the second equality in (6.15) is true. \(\square\)
6.4. **The case that** $p \in (\max\{1, \frac{3}{N-2}\}, \frac{N-1}{N-2})$. This case is easier to handle than the previous one, and the value of $C_0 > 0$ remains the same. We skip the details.

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