Feynman diagrammatic approach to spinfoams

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Abstract

The spinfoams for people without the 3D/4D imagination could be an alternative title of our work. We derive spinfoams from operator spin-network diagrams. Our diagrams are the spin-network analog of the Feynman diagrams. Their framework is compatible with the framework of loop quantum gravity (LQG). For every operator spin-network diagram, we construct a corresponding operator spinfoam. Admitting all the spin-networks of LQG and all possible diagrams leads to a clearly defined large class of operator spinfoams. This way our framework provides a proposal for a class of 2-cell complexes that should be used in the spinfoam theories of LQG. Within this class, our diagrams are just equivalent to the spinfoams. The advantage, however, in the diagram framework is that it is self contained and all the amplitudes can be calculated directly from the diagrams without explicit visualization of the corresponding spinfoams. The spin-network diagram operators and amplitudes are consistently defined on their own. Each diagram encodes all the combinatorial information. We illustrate the applications of our diagrams: we introduce a diagram definition of Rovelli’s surface amplitudes as well as of the canonical transition amplitudes. Importantly, our operator spin-network diagrams are defined in a way general enough to accommodate all the versions of the Engle–Pereira–Rovelli–Livine or the Freidel–Krasnov model, as well as other possible models. The diagrams are also compatible with the structure of the LQG Hamiltonian operators, which is an additional advantage. Finally, a scheme for a complete definition of a spinfoam theory by declaring a set of interaction vertices emerges from the examples presented at the end of the paper.

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(Some figures may appear in colour only in the online journal)
1. Introduction

1.1. Motivation

The idea of spinfoam models is to define histories of the spin-networks by using 2-cell complexes colored by a given group $G$ representations and by intertwiners or equivalently by operators [1–8]. The Engle–Pereira–Rovelli–Livine (EPRL) model [10, 9] and the Freidel–Krasnov (FK) model [11] (combined with [12]) relate the spinfoams directly with the spin-network states, in particular with the spin-network states of LQG [6, 13, 14, 15–18]. What is still needed is a unique definition of a class of the 2-cell complexes that are taken into account. In the spinfoam literature, there are assumptions formulated consistently within a given framework, and the complexes are simplicial [9, 19, 20], cubular [21], linear [3], locally linear [12], combinatorially defined [22]; in some cases, other restrictions were made on the gluing of the 2-disks [8], or the spinfoams were derived as the Feynman diagrams from actions of group field theory models [24, 23, 25, 26].

On the one hand, the familiar simplicial 2-cell complexes are not sufficient because they do not apply to all the states of quantum geometry according to LQG. On the other hand, the classes of the linear and locally linear 2-complexes, as general as they are, unnecessarily invoke auxiliary affine spaces, affine structures which are not compatible with the diffeomorphism invariance of GR. Finally, general CW-complexes [27] admit a diversity that seems to go beyond the graphs and spin-networks.

1.2. Our goal—the spin-network diagrams

In this paper, we derive spinfoams from operator spin-network diagrams. Our diagrams are the spin-network analog of the Feynman diagrams. Their framework is compatible with the framework of loop quantum gravity (LQG). For every operator spin-network diagram, we construct a corresponding operator spinfoam. Admitting all the spin-networks of LQG and all possible diagrams leads to a clearly defined large class of operator spinfoams. Thus, our framework provides a proposal for a class of 2-cell complexes which can be used in the spinfoam theories. Within this class, our diagrams are just equivalent to the spinfoams.

The advantage of the diagram framework is that it is self-contained, and all the amplitudes can be calculated directly from the diagrams without explicit constructing the corresponding spinfoams. Indeed, the spin-network diagram operators and amplitudes can be consistently defined on their own. Given a diagram the reconstruction of an operator spinfoam itself no longer occurs because the diagram encodes all the information. It is also convenient because using the diagrams is much simpler than using the spinfoams. Therefore, one may call our framework the spin-networks for people without the three- and four-dimensional space imagination.

We illustrate the applications of our diagrams: we introduce a diagram definition of Rovelli’s surface amplitudes as well as the canonical transition amplitudes. Importantly, our operator spin-network diagrams are defined in a way general enough to accommodate all the versions of the EPRL or the FK model, as well other possible models. The diagrams are also compatible with the framework used in LQG to define the Hamiltonian operators, which is an additional advantage.

Our paper is organized as follows.

First, we illustrate our idea on a simple non-trivial example in the next subsection.

Next, in section 2, we introduce general definitions of graph diagrams and operator spin-network diagrams. For the reader’s convenience, we demonstrate how this framework can be
applied in a self-sufficient way, giving rise to operators and amplitudes calculated without explicit visualization of spinfoams.

On the other hand, we also examine in detail the transition from the diagrams to the spinfoams. We construct explicitly all the 2-cell complexes corresponding to our diagrams. Each of the 2-complexes is characterized by a diagram consisting of a set of graphs endowed with suitable relation in the set of vertices and links which we name the graph diagram. This is a generalization of diagrams defined by Frank Hellmann [28] for the simplicial triangulations. We introduce general graph diagrams (section 2) and present an exact, explicit construction of the corresponding 2-cell complexes (section 3). Next, we characterize the resulting 2-cell complexes and discuss their properties (section 4).

Operator spin-network diagrams are defined as suitably colored graph diagrams (section 5). For each given diagram, the coloring corresponds with the coloring of the appropriate 2-cell complex and makes it an operator spinfoam [29, 8] (due to the generalizations we include all the EPRL models).

In section 6, we show further examples of the diagrams and corresponding spinfoams. It is easy to identify the elements of the diagram corresponding to free propagation of the quantum state (the propagator of a spin-network) and to the interaction (nontrivial vertices of the spinfoam).

1.3. Example illustrating our idea

Before the systematic presentation of our construction and its results, let us illustrate our idea with a simple example. We will consider now an operator spinfoam defined on a familiar 2-cell complex, and introduce the corresponding diagram. General definitions of the graph diagram and operator spin-network diagram, respectively, will be formulated in the next section.

Consider a 2-cell complex $\kappa$ depicted in figure 1(a) whose boundary is marked by the brighter lines. We will construct the corresponding graph diagram. The 2-cell complex $\kappa$ has

![Figure 1. An example of a foam and its corresponding graphs. (a) The foam $\kappa$. (b) The foams $\kappa_1$ and $\kappa_2$ corresponding to the vertices $v_1$ and $v_2$. (c) The graphs corresponding to the foam $\kappa$.](image)
two internal vertices \(v_1\) and \(v_2\). First, we cut \(\kappa\) into two 2-cell complexes, say \(\kappa_1\) and \(\kappa_2\) as on figure 1(b). The complex \(\kappa_1\) (\(\kappa_2\)) is a neighborhood of the vertex \(v_1\) (\(v_2\)), and its boundary is the graph \(\Gamma_1\) (\(\Gamma_2\)) depicted in figure 1(c). Converely, given the graph \(\Gamma_1\) (\(\Gamma_2\)) in figure 2 (ignore the dashed lines for the present), in order to reconstruct the 2-cell complex \(\kappa_1\) (\(\kappa_2\)), one takes a homotopy of \(\Gamma_1\) (\(\Gamma_2\)) into a point, denotes the point \(v_1\) (\(v_2\)) and views the homotopy as in figure 1(b). The image of the homotopy is the complex \(\kappa_1\) (\(\kappa_2\)). However, in order to reconstruct the 2-cell complex \(\kappa\), one needs the information about gluing of \(\kappa_1\) and \(\kappa_2\). It is symbolically marked in figure 2 by the dashed curves connecting suitable nodes of the graphs.

The complete information can be encoded in the graphs \(\Gamma_1\) and \(\Gamma_2\) by indicating: (i) the pairs of nodes which should be glued with each other, that is, \((n_1, n'_1)\), \((n_2, n'_2)\), \((n_3, n'_3)\), and (ii) the links whose segments are glued at each node, that is at \((n_1, n'_1)\) we glue the segments of \((l_1, l_2, l_3)\) with the segments of \((l'_1, l'_2, l'_3)\), respectively, and similarly at \((n_2, n'_2)\), \((n_3, n'_3)\).

In conclusion, the gluing information can be encoded as the following relation:

\[
\nabla = \{ [(n_1; l_1, l_2, l_3), (n'_1; l'_1, l'_2, l'_3)], [(n_2; l_2, l_3, l_4), (n'_2; l'_2, l'_3, l'_4)], [(n_3; l_3, l_4, l_6), (n'_3; l'_3, l'_4, l'_6)] \}.
\]

The pair of graphs \(\Gamma_1\), \(\Gamma_2\) endowed with the relation \(\mathcal{R}\) is an example of a graph diagram (see the general definition below) equivalent to the 2-cell complex \(\kappa\).

Next we turn to an example of operator spin-network diagram equivalent to an operator spinfoam defined using the 2-cell complex \(\kappa\). An operator spinfoam can be defined by a
coloring \((\rho, P, A)\) of the elements of the 2-cell complex \(\kappa\) (given a Hilbert space \(\mathcal{H}\), by \(\mathcal{H}^*\) we denote the dual Hilbert space): (i) \(\rho\) is a coloring of its faces with irreducible representations of a given group \(G\), (ii) \(P\) is a coloring of its non-boundary edges with operators, and (iii) \(A\) is a coloring of its internal vertices with contractors, that is, tensors used to contract the operators assigned to the edges meeting at the vertex (the contractor is a new element which we introduce in order to generalize the notion of operator spinfoams [8] such that the EPRL model [9] (both Euclidean and Lorentzian) can be viewed as the SU(2) operator spinfoam model with the boundary Hilbert space consistent with the LQG kinematical Hilbert space).

To be specific, each (oriented) face \(f_i\) of \(\kappa\) is colored by a representation \(\rho_i\) of \(G\) in the Hilbert space \(\mathcal{H}_i\) (see figure 1—the faces are topological polygons, whose orientations are marked by blue semi-circles). To each internal edge \(e\) (unoriented), i.e. to each one of \(e_1,e_2,e_3,e_4,e_5\) (see figure 1(a)), and each of its end points \(w\), we assign a Hilbert space \(\mathcal{H}_{w,e}\), defined as follows. A face containing the edge \(e\) induces an orientation of \(e\). According to that orientation, the point \(v\) is either the beginning or the end of \(e\). The definition of the Hilbert space reads

\[
\mathcal{H}_{w,e} = \text{Inv}
\left(\bigotimes_i \mathcal{H}_{\rho_i} \otimes \bigotimes_j \mathcal{H}_{\rho_j}^*\right) \subset \bigotimes_i \mathcal{H}_{\rho_i} \otimes \bigotimes_j \mathcal{H}_{\rho_j}^*,
\]

where \(i\) \((j)\) label the faces such that \(v\) is the end (beginning) of \(e\) and Inv stands for the subspace of \(G\)-invariants. Elements of Inv\(\left(\bigotimes_i \mathcal{H}_{\rho_i} \otimes \bigotimes_j \mathcal{H}_{\rho_j}^*\right)\) inherit the index structure of the bigger space \(\bigotimes_i \mathcal{H}_{\rho_i} \otimes \bigotimes_j \mathcal{H}_{\rho_j}^*\). For example, given three representations of SU(2), \(\rho_1\) and \(\rho_2\) of spin \(\frac{1}{2}\) and \(\rho_3\) of spin 1, each element of Inv\(\left(\mathcal{H}_{\rho_1} \otimes \mathcal{H}_{\rho_2} \otimes \mathcal{H}_{\rho_3}\right)\) is of the form \(a \tau_{AB}^i\), where \(a \in \mathbb{C}\) and \(\tau\) is the invariant tensor obtained from the Pauli matrices. To the edge \(e\) itself, we assign the Hilbert space

\[
\mathcal{H}_e = \bigotimes_w \mathcal{H}_{e,w},
\]

where \(w\) runs through the set of the end points of \(e\). Note that \(\mathcal{H}_{e,w_1} = \mathcal{H}_{e,w_2}^*\), when \(w_1\) and \(w_2\) are different end points of the edge \(e\), so one may interpret every element of \(\mathcal{H}_e\) as an operator \(\mathcal{H}_{e,w_1} \rightarrow \mathcal{H}_{e,w_2}\) or \(\mathcal{H}_{e,w_2} \rightarrow \mathcal{H}_{e,w_1}\). Finally, the edge \(e\) is colored by an operator

\[
P_e \in \mathcal{H}_e.
\]

This way, the operator coloring \(e \mapsto P_e\) is defined for all internal edges of \(\kappa\).

The structure of the colorings \(\rho\) and \(P\) admits at each internal vertex \(v\) of \(\kappa\) a unique contraction of the operators \(P_e\) coloring the edges intersecting \(v\). Indeed, to each internal vertex \(v\), we assign the Hilbert space

\[
\mathcal{H}_v = \bigotimes_e \mathcal{H}_{e,v},
\]

where \(e\) ranges the set of edges meeting at \(v\). In other words,

\[
\mathcal{H}_v \equiv \bigotimes_i \mathcal{H}_{\rho_i} \otimes \mathcal{H}_{\rho_i}^*,
\]

where \(i\) labels the faces of \(\kappa\) intersecting \(v\). The natural contraction

\[
A^\text{Tr} : \mathcal{H}_v \rightarrow \mathbb{C}
\]

is the tensor product of the natural contractions

\[
\text{Tr}_i : \mathcal{H}_{\rho_i} \otimes \mathcal{H}_{\rho_i}^* \rightarrow \mathbb{C}.
\]

However, to accommodate the EPRL vertex amplitude defined on an SU(2) spinfoam, it is necessary to introduce general contractors at the internal vertices. Therefore, for the sake of
both naturalness and relevance, we admit all possible contractors, that is, we color the internal vertices $v$ by arbitrary elements

$$A_v \in \mathcal{H}_v^e.$$  (9)

After decomposing $A_v$ in the intertwiner basis, the coefficients become the so-called vertex amplitudes.

Given the operator spinfoam defined on the 2-cell complex $\kappa$ by any of the colorings $(\rho, P, A)$, there is a natural contraction

$$(A_{v_1} \otimes A_{v_2}) \cdot (P_{e_1} \otimes \ldots \otimes P_{e_5}) \in \mathcal{H}_{e_1,n} \otimes \mathcal{H}_{e_5,n'}$$  (10)

obtained by applying the contractor $A_{v_1}$ to the operators $P_{e_1}, P_{e_3}, P_{e_4}, P_{e_5}$ at $v_1$ and the contractor $A_{v_2}$ to operators $P_{e_1}, P_{e_3}, P_{e_4}, P_{e_5}$ at $v_2$. Here $n$ and $n'$ are the boundary vertices of $\kappa$, the ends of the internal edges $e_4$ and $e_5$, respectively, see figure 3(b).

The spinfoam operator $P(\kappa, \rho, P, A)$ usually involves an extra factor: the product of the so-called face amplitudes, i.e. numbers $A_f$ assigned to the faces $f$ (typically the dimension of the corresponding representation $[3, 4, 8, 31]$), and the so-called boundary edge amplitudes,
i.e. numbers $A_e$ assigned to boundary edges $e$ (typically the square root of the face amplitude inverse):

$$P(\kappa, \rho, P, A) = \left( \prod_e A_e \prod_f A_f \right) (A_{v_1} \otimes A_{v_2} \ldots (P_{e_1} \otimes \cdots \otimes P_{e_n})$$  \hspace{1cm} (11)

Let us now relate the current operator spinfoam definition with the natural operator spinfoam models defined in [8] and the EPRL model [9].

If the operator spinfoam introduced above comes from a natural operator spinfoam model, then $G$ is a compact group, each of the operators $P_{e_1}, ..., P_{e_n}$ (viewed as a map $\mathcal{H}_a \rightarrow \mathcal{H}_b$) is a projection and each of the contractors $A_{v_1}, A_{v_2}$ is the natural contractor $A^{T_{\mathcal{R}}}$.

On the other hand, if the operator spinfoam introduced above comes from the EPRL model, then $G = SU(2)$, each of the operators $P_{e_1}, ..., P_{e_n}$ (viewed as a map $\mathcal{H}_a \rightarrow \mathcal{H}_b$) is the identity map and each of the contractors $A_{v_1}, A_{v_2}$ is the EPRL contractor $A^{EPRL}$, directly given by the EPRL fusion map [9, 12, 36].

Now, after introducing the operator spinfoam $(\kappa, \rho, P, A)$, we are in the position to define an equivalent operator spin-network diagram. The idea is to encode in the graph diagram $(\Gamma_1, \Gamma_2, \mathcal{R})$ (see figure 2 and equation (1)) the data defining the operator spinfoam $(\kappa, \rho, P, A)$: (i) each link $l_1, ..., l_6$ ($l_1', ..., l_6'$) of the graph $\Gamma_1$ ($\Gamma_2$) (figure 2) corresponds to exactly one face of $\kappa$ (figure 1(a)) and inherits its orientation and the representation color $\rho_1, ..., \rho_6$ ($\rho_{1}', ..., \rho_{6}'$) (see figure 3), (ii) every node $n_1, n_2, n_3, n (n_1', n_2', n_3', n')$ of $\Gamma_1$ ($\Gamma_2$) (figure 2) corresponds to exactly one internal edge $e_1, e_2, e_3, e_4 (e_1', e_2', e_3', e_4')$ of $\kappa$ (figure 1(a)) and inherits the operator $P_{e_1', P_{e_2'}, P_{e_3'}, P_{e_4'}, P_{e_5}, P_{e_6}}$ (figure 3(a)), (iii) the graph $\Gamma_1$ ($\Gamma_2$) (figure 2) itself corresponds uniquely to the internal vertex $v_1 (v_2)$ (figure 1(a)) of $\kappa$ and inherits the contractor $A_1$ ($A_2$) (figure 3(a)). Every pair of nodes in relation $\mathcal{R}$ (equation (1)) is colored by a same operator, and every pair of links in relation $\mathcal{R}$ is colored by a same representation.

The resulting operator spin-network diagram equivalent to the operator spinfoam (figure 3(a)) is depicted in figure 3(b) and the information it contains is completed by the relation $\mathcal{R}$. The spinfoam operator $P(\kappa, \rho, P, A)$ is determined by the diagram itself: by taking one operator per each pair of the nodes $(n_i, n_i')$, $i = 1, 2, 3$, being in the relation $\mathcal{R}$, one operator per each free node $n$ and $n'$, multiplying all of them tensorially and contracting with the contractors $A_1$ and $A_2$. This defines the operator (10). The factor $\prod_e A_e \prod_f A_f$ present in (11) involves the reconstruction of the faces and the boundary edges of $\kappa$. Since $\kappa$ can be reconstructed from the diagram, so can be the face and the boundary amplitudes. However, the visualization of the 2-cell complex $\kappa$ is not necessary and the face and edge amplitudes may be read directly from the diagram. That observation motivates the framework we introduce in the next section.

2. Graph diagrams and operator spin-network diagrams

2.1. Graph diagrams

A general graph diagram ($\mathcal{G}, \mathcal{R}$) consists of a set $\mathcal{G}$ of oriented graphs $\{\Gamma_1, ..., \Gamma_N\}$ and a family $\mathcal{R}$ of relations defined as follows (see figure 4).

- $\mathcal{R}_{\text{node}}$: a symmetric relation in the set of nodes of the graphs which we call the node relation, such that each node $n$ is either in relation with precisely one $n' \neq n$ or is unrelated (in the latter case, it is called the boundary node).
- $\mathcal{R}_{\text{link}}$: a family of symmetric relations in the set of links of the graphs which we collectively call the link relation. If a node $n$ of a graph $\Gamma_1$ is in relation with a node $n'$ of a graph


Figure 4. A graph diagram. The thick dots represent the nodes of the graphs, the solid lines with arrows represent the oriented links of the graphs, the dashed lines illustrate the node relation $\mathcal{R}_{\text{node}}$ and the dotted lines illustrate the link relation $\mathcal{R}_{\text{link}}$.

$\Gamma_f$, then one defines a bijective map between incoming/outgoing links of $\Gamma_I$ at $n$, with outgoing/incoming links of $\Gamma_f$ at $n'$; no link is left free either at the node $n$ or at $n'$; two links identified with each other by the bijection are called to be in the relation $\mathcal{R}_{\text{link}}^{(n,n')}\arrowright$ at the pair of nodes $n, n'$; a link of $\Gamma_f/\Gamma_I$ which intersects $n/n'$ twice emerges in the relation twice: once as incoming and once as outgoing.

In order to be related, two nodes have to satisfy the consistency condition: the number of incoming/outgoing links in each of them has to coincide with the number of the outgoing/incoming links at the other one (with possible closed links counted twice). Note that two graphs can be treated as one disconnected graph. Thus to reduce that ambiguity we assume that all the graphs defining the diagram are connected.

2.2. Operator spin-network diagrams

An operator spin-network diagram $(G = \{\Gamma_1, \ldots, \Gamma_N\}, \mathcal{R}, \rho, P, A)$ is defined by coloring a graph diagram $(G, \mathcal{R})$ as follows:

- The coloring $\rho$ assigns to each link $\ell$ of each graph $\Gamma_I$, $I = 1, \ldots, N$, an irreducible representation of the group $G$:
  \[
  \ell \mapsto \rho_\ell. \tag{12}
  \]

It is assumed that whenever two links $\ell$ and $\ell'$ are mapped to each other by $\mathcal{R}_{\text{link}}$, then
  \[
  \rho_\ell = \rho_{\ell'} . \tag{13}
  \]
• The coloring $P$ assigns to each node $n$ an operator

$$n \mapsto P_n \in \mathcal{H}_n \otimes \mathcal{H}_n^*,$$

where $\mathcal{H}_n$ is defined at each node in the following way:

$$\mathcal{H}_n = \text{Inv} \left( \bigotimes_i H_{\rho_i} \otimes \bigotimes_j H_{\rho_j} \right) \subset \left( \bigotimes_i H_{\rho_i} \otimes \bigotimes_j H_{\rho_j} \right)^*,$$

where $i/j$ labels the links incoming/outgoing at $n$.

Whenever two nodes $n$ and $n'$ are related by $\mathcal{R}_{\text{node}}$, then (from (13) and (15)) it follows that $\mathcal{H}_n = \mathcal{H}_n^*$ and it is assumed about $P$ that

$$P_n = P_{n'}^*.$$  

• The coloring $A$ assigns to each graph $\Gamma$ a tensor

$$\Gamma \mapsto A_{\Gamma} \in \left( \bigotimes_n \mathcal{H}_n \right)^*$$

which we call a contractor, where $n$ runs through the nodes of $\Gamma$.

It is important that even though $\mathcal{H}_n$ consists only of the $G$-invariant elements of $\left( \bigotimes_i H_{\rho_i} \otimes \bigotimes_j H_{\rho_j} \right)$, we think of its elements as tensor products, elements of the big Hilbert space, possessing the index structure of $\left( \bigotimes_i H_{\rho_i} \otimes \bigotimes_j H_{\rho_j} \right)$.

If a node $n$ of one of the graphs $\Gamma$ is related by $\mathcal{R}_{\text{node}}$ with another node $n'$ (of the same or different graph), then $P_n$ and $P_{n'}$ are elements of the same Hilbert space $\bigotimes_{\alpha \in \{n,n'\}} \mathcal{H}_\alpha$, due to (16) they appear to be the same element

$$P_{\{n,n'\}} \in \bigotimes_{\alpha \in \{n,n'\}} \mathcal{H}_\alpha.$$  

A natural example of a contractor exists due to the fact that the Hilbert space $\bigotimes_n \mathcal{H}_n$ can be uniquely embedded into a space

$$\bigotimes_n \mathcal{H}_n \hookrightarrow \bigotimes_i H_{\rho_i} \otimes H_{\rho_i}^*,$$

where $i$ ranges the set of the links of $\Gamma$. Therefore, the distinguished element of $\left( \bigotimes_n \mathcal{H}_n \right)^*$ is

$$A_{\Gamma}^{\text{Tr}} = \bigotimes_i \text{Tr}_i,$$  

which is used e.g. in the $G$-BF theory. This contractor can also be used to define a version of the Euclidean EPRL model (viewed from the $\text{Spin}(4)$ spin-network perspective [12, 29, 8]). However, the $SU(2)$ spinfoam model constructed from the EPRL vertex amplitude defines a different contractor, which can be denoted by $A^{\text{EPRL}}$ [30, 31].

2.3. The spin-network diagram operator

Now, given an operator spin-network diagram $(G, \mathcal{R}; \rho, P, A)$, we would like to define an operator analogous to (11).

There is a canonical contraction

$$\tilde{P} = \left( \bigotimes_I A_I \right) \cdot \left( \bigotimes_n P_n \right),$$

where $I$ ranges over the set $G$ of the graphs, while $n$ labels the set of boundary nodes and the set of pairs of related nodes (with respect to $\mathcal{R}_{\text{node}}$). It is defined by contracting each $A_I$ with the
Each operator $P_n \in \mathcal{H}_n \otimes \mathcal{H}_n^*$ assigned to a boundary node $n$ and the $\mathcal{H}_n$ part of each operator $P_{\{n,n\} \in \bigotimes_{\tilde{n} \in \{n,n\}} \mathcal{H}_{\tilde{n}}}$ assigned to a node $n$ related to $n'$, where $n$ ranges the nodes of $\Gamma_f$. As a consequence, for each boundary node $n$ one index of $P_n$ remains uncontracted; thus, $\tilde{P}$ is an element of

$$\tilde{P} \in \bigotimes_{\text{boundary } n} \mathcal{H}_n^*.$$  

(22)

A comparison with (11) shows that we are still missing the face and the boundary amplitudes. Therefore, an operator defined by the operator spin-network diagram should have the following form:

$$P = (A_{\text{boundary}} A_{\text{face}}) (\bigotimes_{\ell} A_{\ell}) \bigotimes_{\text{boundary } n} P_n.$$  

(23)

where boundary and face amplitudes $A_{\text{boundary}}$ and $A_{\text{face}}$ have to be defined suitably as well as the boundary edges and faces themselves.

The boundary amplitude is given by the product over all the boundary nodes $n$ of the graph diagram of amplitudes assigned to the links $\ell$ intersecting these nodes:

$$A_{\text{boundary}} = \sqrt{\prod_{\text{boundary } n} \prod_{\ell} A_{\ell}}.$$  

(24)

The square root comes from the fact that each link is counted twice (once per each end).

The face amplitude is given by the product over equivalence classes $f$ of the face relation $R_{\text{face}}$ (which will be introduced in the subsequent subsection) of the face amplitude

$$A_{\text{face}} = \prod_{f} A_{f}.$$  

(25)

To specify the numbers $A_{\ell}$ and $A_{f}$, we need two functions defined on the space of irreducible representations of $G$:

$$A_{\ell} = f_{1}(\rho_{\ell}) \quad A_{f} = f_{2}(\rho_{f}),$$  

(26)

where $\rho_{f} = \rho_{\ell}$ for $\ell'$ being any representative of the equivalence class $f$ (we will see below that the labeling $\rho$ is consistent with the face relation).

2.4. Face and edge relations

The node relation $R_{\text{node}}$ and the link relation $R_{\text{link}}$ introduced with the definition of the graph diagram at the beginning of this section lead to the equivalence relation in the set of all links of all graphs. The resulting equivalence relation, which we call the face relation and denote by $R_{\text{face}}$, carries information about faces of the corresponding 2-complex and allows us to introduce face amplitude without explicit reference to the complex itself. Given a graph diagram $(G, R)$, the relation $R_{\text{face}}$ is defined as determined by the following properties:

- $R_{\text{face}}$ is an equivalence relation in the set of all the links $L$ of the graphs belonging to $G$,
- two different links $\ell$ and $\ell'$ are in relation $R_{\text{face}}$ when they are in the relation $R_{\text{link}}^{(n)}$ in some node $n$.

For further convenience, let us characterize possible equivalence classes of $R_{\text{face}}$.

(i) Each link unrelated to any other link by any of the $R_{\text{link}}^{(n,n')}$ relations sets a one-element equivalence class. We will call it an open equivalence class.

(ii) Each link $\ell$ related to itself by the link relation $R_{\text{link}}^{(n,n)}$ (at each of its ends $n = s(\ell), n' = t(\ell)$) sets a one-element equivalence class. We will call it a closed equivalence class.
(iii) Each pair of links \((\ell_1, \ell_2)\) such that \(\ell_1\) and \(\ell_2\) are related by the relation \(R^{(n,n')}_{\text{link}}\) and neither \(\ell_1\) nor \(\ell_2\) is related by relations from \(R_{\text{link}}\) with any other link sets a two-element equivalence class.

However, there are two subcases of such equivalence classes.

(a) If \(\ell_1\) and \(\ell_2\) are related by \(R^{(n,n')}_{\text{link}}\) only at one pair of their ends \(n, n'\), the equivalence class will be called \emph{open} and will refer to an external face.

(b) If \(\ell_1\) and \(\ell_2\) are related by \(R^{(n,n')}_{\text{link}}\) at both pairs of their ends, the equivalence class will be called \emph{cyclic} and will refer to an internal face.

(iv) Every \(k > 2\)-element sequence of links \((\ell_1, \ell_2, \ldots, \ell_k)\) such that \(\ell_i, \ell_{i+1}, i \in \{1, \ldots, k-1\}\), are in relation \(R^{(n,n')}_{\text{link}}\) and the links \(\ell_1, \ell_k\) are not in the relation \(R_{\text{link}}\) with any links not belonging to the sequence (i.e. the sequence is \emph{maximal}) sets a \(k\)-element equivalence class.

Again, there are two subcases of such equivalence classes:

(a) If \(\ell_1\) and \(\ell_k\) are not related by \(R^{(n,n')}_{\text{link}}\) at any pair of nodes, the equivalence class will be called \emph{open} and will refer to an external face.

(b) If \(\ell_1\) and \(\ell_k\) are related by \(R^{(n,n')}_{\text{link}}\) at some pair of nodes, the equivalence class will be called \emph{cyclic} and will refer to an internal face.

Each equivalence class \(f\) of \(R_{\text{face}}\) will be called the \emph{merged face}. Note that given an operator spin-network diagram \((G, R, \rho, P, A)\), two links which belong to the same merged face are colored by the same representation, which was used in the definition of the face amplitude.

For further convenience, we will also introduce the \emph{edge} relation \(R_{\text{edge}}\) determined by the following two properties:

- \(R_{\text{edge}}\) is an equivalence relation in the set of nodes \(N\) of all graphs belonging to \(G\),
- two different nodes \(n\) and \(n'\) are in relation \(R_{\text{edge}}\) when they are in the relation \(R_{\text{node}}\).

There are two types of equivalence classes:

(i) each node unrelated to any other node by \(R_{\text{node}}\) sets a one-element equivalence class,

(ii) each pair of nodes \(\{n, n'\}\) related by \(R_{\text{node}}\) sets a two-element equivalence class.

2.5. Boundary graph of the operator spin-network diagram

Given the operator spin-network diagram \((G, R, \rho, P, A)\), we now define its boundary. We will use a new operation defined on graphs—merging graphs—and naturally extend it to spin-networks.

Given two nodes \(n, n'\) in the graph diagram \((G, R)\), related by the relation \(R_{\text{node}}\) (they may be either nodes of a same graph or of two different graphs), the merging is defined in the following ways (figure 5).

(i) Remove these nodes from the graphs they belong to, together with segments of the links meeting at the nodes \(n, n'\). From each link we remove a segment containing the node \(n\) or \(n'\) respectively.

There are two degenerate cases requiring additional instructions:

(a) a link is a loop which begins and ends at \(n\) (or \(n'\)). In this case, we remove both the incoming segment and the outgoing segment,

(b) a link connects the nodes \(n\) and \(n'\). In this case, we remove the entire link (see figure 6).
(ii) We are left with a number of remaining open segments of links. We connect the links that started/ended at \( n \) with the links that ended/started at \( n' \) according to the relation \( R_{\text{link}} \).

The result of merging of the pair of nodes is a new graph diagram \( (G', R') \). We repeat the merging for another pair of related nodes. We go on until we reach the stage at which the resulting graph diagram has no pair of related nodes, that is, it consists of a set of graphs \( G_{\text{final}} \), with no relation. The graphs constitute the boundary graph (disconnected, if \( G_{\text{final}} \) contains more than one graph).

The resulting boundary graph does not depend on the order in which we merge the pairs of nodes. The proof of that fact, together with a more detailed construction, can be found in section 3. Equivalently, it is easy to perform the merging simultaneously at all the pairs of the related nodes. We illustrate it in figure 7.

The coloring \( \rho \) of a graph diagram introduced with the definition of the operator spin-network diagram is consistent with the merging of pairs of nodes. Indeed, each pair \( \ell, \ell' \) of merged links is labeled by the same representation \( \rho_\ell = \rho_{\ell'} \). Therefore, the boundary graph of the operator spin-network diagram inherits the labeling of links by the representations of \( G \).

At the beginning of this section, we defined the boundary nodes of the graph diagram. The nodes of the very boundary graph itself are precisely the boundary nodes defined before. Moreover, the links of the graph diagram meeting at the boundary nodes (the same that give contribution to the boundary amplitude) are precisely the links of the boundary graph.

2.6. The Euclidean EPRL model

There are two non-equivalent but mathematically natural proposals for constructing an operator spinfoam model using the EPRL constraints [9, 30, 31, 29, 8] (see also [11, 32, 33]). The first one is an SU(2) OS model, whereas the second one is a Spin(4) OS model. We incorporate now each of the approaches into the operator spin-network diagram framework.

2.6.1. The EPRL SU(2) OSD model

Given a value \( \gamma \neq \pm 1 \) of the Barbero–Immirzi parameter [34], the model proposed in [31] could be obtained from the SU(2) BF theory by replacing the natural contractors into suitable EPRL contractors [30], i.e. we use the following coloring.

- We color each link \( \ell \) with a unitary irreducible representation \( \rho_\ell \) of the SU(2) group of a dimension \( 2k_\ell + 1 \) (\( k_\ell \in \frac{1}{2} \mathbb{N} \)):

\[
\rho_\ell : \text{SU}(2) \rightarrow U(\mathcal{H}_\ell).
\] (27)
Figure 6. Merging of graphs in a graph diagram. (a) Two graphs in a graph diagram. (b) Removing the nodes in the relation. (c) Removing each link whose both endpoints were removed. (d) Merging the remaining links.

- We color each node \( n \) with an operator \( P_n \), which is the identity operator in the space \( \mathcal{H}_n \) of tensors invariant under the action of SU(2):

\[
\mathcal{H}_n = \text{Inv} \left( \bigotimes_i \mathcal{H}_{\ell_i}^* \otimes \bigotimes_j \mathcal{H}_{\ell_j} \right) \subset \left( \bigotimes_i \mathcal{H}_{\ell_i}^* \otimes \bigotimes_j \mathcal{H}_{\ell_j} \right),
\]  

(28)

where \( i/j \) labels the links incoming/outgoing at \( n \).
Figure 7. A graph diagram and the corresponding boundary graph. (a) A graph diagram. (b) The boundary graph is obtained by merging all the pairs of the related nodes.

- We color each graph $\Gamma$ with the EPRL contractor

$$A_{\Gamma}^{\text{EPRL}} \in \left( \bigotimes_{\mu} \mathcal{H}_\mu \right)^*$$

defined below.

Let $(j, k, l) \in \frac{1}{2} \mathbb{N} \times \frac{1}{2} \mathbb{N} \times \frac{1}{2} \mathbb{N}$ satisfy the triangle inequalities and $j + k + l \in \mathbb{N}$. For every such triple, we chose an invariant

$$C_{jkl}^{kl} \in \text{Inv}(\mathcal{H}_j \otimes \mathcal{H}_k \otimes \mathcal{H}_l)$$

normalized so that the corresponding map $C_{jkl}^{kl}$ is a unitary embedding into the corresponding subspace of the tensor product. By $C_{jkl}^{kl}$ we denote the adjoint operator. In the index notation, we omit $j, k, l$, e.g.,

$$CA_{1A_{2A_{3}}}: = (C_{jkl}^{kl})_{A_{1A_{2}}A_{3}}.$$  

Let $\rho_{j^{+}j^{-}} = \rho_j^+ \otimes \rho_j^-$ be a unitary irreducible representation of the group Spin(4) = SU(2)$^+ \times$SU(2)$^-$, and $(k, j^{+}, j^{-})$ satisfy triangle inequalities, and $k + j^{+} + j^{-} \in \mathbb{N}$. We define a function

$$\rho_{j^{+}j^{-}}^k: \quad \text{Spin}(4) \to \mathcal{H}_k \otimes \mathcal{H}_k^*$$

$$\left(\rho_{j^{+}j^{-}}^k (g^+, g^-) \right)_B^A := C_{A^*A}^j \rho_j^+ (g^+)^\dagger_{A} \rho_j^- (g^-)^{\dagger}_B C_{A^*A}^{B^*B^*}.$$  

Note that this definition is insensitive on the phase ambiguity in the choice of $C_{j^{+}j^{-}}^k$.

Given a graph $\Gamma$ and coloring of each link $\ell$ with SU(2) representations $\rho_{\ell}$ of dimension $2k_\ell + 1$ ($k_\ell \in \frac{1}{2} \mathbb{N}$), we introduce the EPRL contractor. First, we define an auxiliary coloring of the links of the graph $\Gamma$

$$j^{\pm}_\ell := \frac{|1 \pm \gamma|}{2} k_\ell.$$  

(32)
We color each graph \( \Gamma \) such that \( j^\pm_\ell \not\in \frac{1}{2}N \), we define the EPRL contractor to be zero:

\[
A^\text{EPRL}_\Gamma \equiv 0.
\]  

(33)

Otherwise, to each link \( \ell \) we assign a Spin(4)=SU(2)\(^+\)×SU(2)\(^-\) unitary irreducible representation \( \rho^\pm_\ell = \rho^+_\ell \otimes \rho^-_\ell \).

2.6.2. The EPRL Spin(4) OSD model. Given a value of the Barbero–Immirzi parameter \( \gamma \neq \pm 1 \), the model proposed in [8] is obtained from the Spin(4) BF theory by replacing the identity operators assigned to the diagram nodes by the projectors onto the subspaces of solutions to the EPRL constraints [29, 8]. The coloring is as follows.

- We color each link \( \ell \) with a unitary irreducible representation \( \rho^\pm_\ell \otimes \rho^-_\ell \) of Spin(4)=SU(2)\(^+\)×SU(2)\(^-\):

\[
(g^+, g^-) \mapsto \rho^+_\ell (g^+) \otimes \rho^-_\ell (g^-) : \mathcal{H}_\ell^+ \otimes \mathcal{H}_\ell^- \rightarrow \mathcal{H}_\ell^+ \otimes \mathcal{H}_\ell^- .
\]  

(35)

Denote by \((j^+_\ell, j^-_\ell)\), \(j^\pm_\ell \in \frac{1}{2}N\), the numbering of the classes of equivalent unitary irreducible representations, i.e. numbers defined by the following equation:

\[
2j^\pm_\ell + 1 = \dim \mathcal{H}_\ell^\pm .
\]  

(36)

- We color each node \( n \) with an operator \( \rho^\text{EPRL}_n \), which is the orthogonal projection onto the space \( \mathcal{H}_n^\text{EPRL} \) of solutions to EPRL constraints [29, 8]:

\[
\mathcal{H}_n^\text{EPRL} \subset \text{Inv} \left( \bigotimes_i \mathcal{H}^+_i \otimes \bigotimes_j \mathcal{H}^-_j \right) \otimes \text{Inv} \left( \bigotimes_i \mathcal{H}^-_i \otimes \bigotimes_j \mathcal{H}^+_j \right) ,
\]  

(37)

where \( i/j \) labels the links incoming/outgoing at \( n \). We recall the definition of this operator below.

- We color each graph \( \Gamma \) with the natural trace contractor (20)

\[
A^{\text{Tr}}_\Gamma = A^{\text{Tr}} .
\]  

(38)

Now we describe in detail the node operators. Given a node \( n \) and the corresponding Hilbert space \( \mathcal{H}_n \), the space of solutions to the EPRL constraints

\[
\mathcal{H}_n^\text{EPRL} \subset \mathcal{H}_n = \text{Inv} \left( \bigotimes_i \mathcal{H}^+_i \otimes \bigotimes_j \mathcal{H}^-_j \right) \otimes \text{Inv} \left( \bigotimes_i \mathcal{H}^-_i \otimes \bigotimes_j \mathcal{H}^+_j \right) \] 

(39)

is non-empty iff the following two conditions hold: (i) there exist half-integers \( k_\ell \) such that

\[
j^\pm_\ell = \left\lfloor \frac{1}{2} \pm \frac{\gamma}{2} \right\rfloor k_\ell
\]  

(40)

and (ii) provided the space

\[
\text{Inv} \left( \bigotimes_i \mathcal{H}^+_i \otimes \bigotimes_j \mathcal{H}^-_j \right) \otimes \text{Inv} \left( \bigotimes_i \mathcal{H}^-_i \otimes \bigotimes_j \mathcal{H}^+_j \right)
\]  

(41)

is non-trivial itself.

\[
\text{Inv} \left( \bigotimes_i \mathcal{H}^+_i \otimes \bigotimes_j \mathcal{H}^-_j \right) \otimes \text{Inv} \left( \bigotimes_i \mathcal{H}^-_i \otimes \bigotimes_j \mathcal{H}^+_j \right)
\]  

(41)
If \( |1\pm\gamma| j^{\pm} \not\in 1/2 \mathbb{N} \) or \( \mathcal{H}_n = \emptyset \), then
\[
\mathcal{H}_{n \text{EPRL}} = \emptyset.
\] (42)

In either case, we define
\[
P_{n \text{EPRL}} : \mathcal{H}_n \rightarrow \mathcal{H}_n
\] (43)
to be the orthogonal projection onto the subspace \( \mathcal{H}_{n \text{EPRL}} \).

If \( k_{\ell} \equiv \frac{2}{|1\pm\gamma| j^{\pm}} \in \frac{1}{2} \mathbb{N} \) and \( \mathcal{H}_{n \text{EPRL}} \neq \emptyset \), the space of EPRL solutions of the constraints is parametrized by SU(2) invariants \([35]\):
\[
\eta \in \text{Inv} \left( \bigotimes_i \mathcal{H}_i^{\ast} \otimes \bigotimes_j \mathcal{H}_j \right),
\] (44)
where each \( \mathcal{H}_\ell \) is the carrier space of the irreducible SU(2) representation \( \rho_\ell \) of the dimension \( 2k_{\ell} + 1 \) and \( i/j \) labels the links incoming/outgoing at \( n \). This parametrization is given by the EPRL map \([12]\)
\[
\iota_{\gamma}^{\text{EPRL}} : \text{Inv} \left( \bigotimes_i \mathcal{H}_i^{\ast} \otimes \bigotimes_j \mathcal{H}_j \right) \rightarrow \text{Inv} \left( \bigotimes_i \mathcal{H}_i^{\ast} \otimes \bigotimes_j \mathcal{H}_j \right) \otimes \text{Inv} \left( \bigotimes_i \mathcal{H}_i^{\ast} \otimes \bigotimes_j \mathcal{H}_j \right).
\] (45)

It is a good parametrization in the sense that when \( \frac{2}{|1\pm\gamma| j^{\pm}} \in \frac{1}{2} \mathbb{N} \) and \( \mathcal{H}_{n \text{EPRL}} \neq \emptyset \), the EPRL map is 1–1 \([12, 29, 35]\). The node operator \( P_{n \text{EPRL}} \) can be written in the form
\[
P_{n \text{EPRL}} = \sum_{\eta,\eta'} P_{n \text{EPRL}}^{\eta \eta'} (\iota_{\gamma}^{\text{EPRL}} (\eta'))^\dagger,
\] (46)
where the sum ranges over two bases of SU(2) intertwiners. The coefficients \( P_{n \text{EPRL}}^\eta \) are defined by the Hilbert product \( (\cdot|\cdot)_n \) in \( \mathcal{H}_n \), namely
\[
\sum_{\eta''} P_{n \text{EPRL}}^\eta (\iota_{\gamma}^{\text{EPRL}} (\eta'))^\dagger (\iota_{\gamma}^{\text{EPRL}} (\eta''))_n = \delta_{\eta\eta''}.
\] (47)

2.7. Other models

The EPRL SU(2) OSD model described in the previous section may be extended to the Lorentzian signature, when we restrict ourselves to 3-edge-connected graphs \([36]\). This formalism may also be applied to other spinfoam models, e.g. a class of natural operator spinfoam models \([8]\) described below. This model can be obtained by applying arbitrary (quantum) constraints on an arbitrary BF spinfoam model. The examples are the FK model \([11]\) and BC model \([37–40]\).

Given a natural operator spin-network model \([8]\), the group is an arbitrary compact \( G \), the coloring \( \rho \) takes values in the set of all the irreducible representations and the coloring \( A \) for every graph takes the value \( A_v = A^\dagger \). The coloring \( P \) takes values in the projection operators (including the zero operator) which are not specified. However, they are constrained by the naturalness conditions, the most important of which are as follows: each of the operators \( P_n \) and \( P_{n,w} \) is determined by the sequence (unordered, with repetitions) of the representation colors of the links intersecting a given node \( n \) (regardless of the structure of the other parts of the diagram) and in the case of a sequence \( \rho_1, \rho_1^\dagger \) the projection is not zero (see \([8]\) for details).

The operator spin-network diagrams allow us to control the diversity of possible spinfoams. They may be applied, for example, to the spinfoam cosmology \([41, 42]\), to list all the interactions in any order of vertex expansion.
2.8. Boundary functionals, transition amplitudes and spin-network evolution

Operator spin-network diagrams may be used to describe Rovelli’s boundary functionals [6, 40], as well as evolution of spin-network states and one-vertex interactions (OSD in the second case is called the simplified formalism).

2.8.1. From the diagrams to Rovelli’s boundary functionals. Consider a general spin-network operator diagram \((G, R; \rho, P, A)\). Define the Hilbert space of Rovelli’s boundary states to be the tensor product labeled by the boundary nodes:

\[
\mathcal{H}_b = \bigotimes_{\text{boundary } n} \mathcal{H}_n. \tag{48}
\]

The spin-network diagram operator \(P\) defined by formula (23) is an element of \(\mathcal{H}_b^\ast\).

Given a state \(\psi_b \in \mathcal{H}_b\), the Rovelli amplitude defined by the diagram is the natural contraction

\[
\langle W | \psi_b \rangle := \psi_b \ll P. \tag{49}
\]

2.8.2. Rovelli’s boundary transition amplitude as a spin-network operator diagram. Given an operator spin-network diagram \(D = (G, R; \rho, P, A)\) and a boundary state \(\psi_b\) (as defined in the previous example) construct an extended operator spin-network diagram \(D'\) defined as follows.

- The new set of graphs is \(G \sqcup \{\gamma_b\}\), where \(\gamma_b\) is the boundary graph of \(D\) with the reversed link orientation.
- The new relation \(R'\) is the relation \(R\) extended by the pairs \((n_b, n'_b)\), where \(n_b\) are the boundary nodes of \(D\) and \(n'_b\) are corresponding nodes of the graph \(\gamma_b\).
- The coloring \(\rho\) induces the coloring of the links of \(\gamma_b\) (see (13)).
- The coloring \(P\) induces the coloring of the nodes of \(\gamma_b\) (see (16)).
- The contractor labeling \(A\) is extended by \(A_{\gamma_b} = \psi\) (the contractor space \(\mathcal{H}_{\gamma_b}^\ast = \mathcal{H}_b^{**} = \mathcal{H}_b\), because links of \(\gamma_b\) are the reversed links of the boundary graph).

The spin-network diagram operator \(P'\) corresponding to \(D'\) is a complex number:

\[
P' = \psi_b \ll P = \langle W | \psi_b \rangle. \tag{50}
\]

2.8.3. Simplified formalism–one vertex interaction. Consider an operator spin-network diagram whose set \(G\) consists of two graphs: \(\Gamma_{\text{in}}\) and \(\Gamma_{\text{int}}\). The relation \(R\) does not relate any pair of nodes of \(\Gamma_{\text{in}}\), otherwise it is arbitrary. The coloring \(\rho\) is arbitrary, and the coloring \(P\) is restricted only by the condition that each \(P_n\) is a projection. The contractors are arbitrary. However, the contractor \(A_{\Gamma_{\text{in}}}\) is given a special meaning, i.e. it is considered to be the initial state \(\psi_{\text{in}} \in \mathcal{H}_{\Gamma_{\text{in}}}\).

The boundary graph of this operator spin-network diagram is interpreted as \(\Gamma_{\text{out}}\) (see figure 8(b)). The boundary Hilbert space is thought of as the space of final states \(\psi_{\text{out}}\). Given a state \(\psi_{\text{in}}\) (encoded in the contractor \(A_{\Gamma_{\text{in}}}\)), the spin-network diagram operator \(P\) is the final state \(\psi_{\text{out}}\) of the interaction described by \(A_{\Gamma_{\text{out}}}\) (see figure 9).
3. Construction of a 2-complex defined by a graph diagram

In the previous section, we introduced the operator spin-network diagrams. The example shown in section 1 explains how to obtain an operator spin-network diagram out of an operator spinfoam. In this section we pass from the graph diagrams to 2-complexes. Later on (in section 5), we will show how the coloring of a graph diagram induces the coloring of the 2-complex which will enable us to construct an operator spinfoam out of an arbitrary operator spin-network diagram.

It will be convenient to introduce the notion of a squid graph: a decomposition of a graph into a set of simpler (open) graphs, called squids (see section 3.1). Such a decomposition will make the relation \( R \) (introduced in the previous section) easier to deal with by making it a relation on a set of all squids of all graphs in the graph diagram. Thus, our initial data will
be a squid graph diagram \((G, R)\) being a set of squid graphs \(G\) and a set of pairs of squids \(R\) (together with appropriate maps \(\phi_r\) for each \(r \in R\)), such that each squid belongs to one pair \(r \in R\) at the most and whenever a pair of squids belongs to \(R\), then the two squids are homeomorphic.

The construction will be as follows: first we will show how to construct a 2-\(\Delta\)-complex out of an arbitrary squid graph. Each such complex refers to a spinfoam with one vertex. Then, we will see how to glue two such complexes along one pair of squids \(r \in R\) to obtain a 2-vertex foam. Finally, we will see that the gluing procedure does not require its objects to be 1-vertex foams and it has straightforward generalization to whatever foam, so that one can proceed gluing until all the set \(R\) was used.

3.1. The squid graph

We will use the following notation: an oriented graph \(\Gamma\) is a pair \((\mathcal{N}, \mathcal{L})\) where \(\mathcal{N} = \{n_1, \ldots, n_N\}\) is a set of nodes and \(\mathcal{L} = \{\ell_1, \ldots, \ell_L\}\) is a set of oriented links. For every link \(\ell\), we will denote its beginning (source) by \(s(\ell)\) and its end (target) by \(t(\ell)\).

Definition 1. A squid is an oriented graph \(\lambda = (\{n\} \cup \{x_1, \ldots, x_k\}, \{\ell_1, \ldots, \ell_k\})\) such that each link \(\ell_i\) satisfies the condition \(s(\ell_i) = n \land t(\ell_i) = x_i\). The node \(n\) is called the head of the squid. The links \(\ell_1, \ldots, \ell_k\) are called the legs of the squid and the nodes \(x_1, \ldots, x_k\) are called the leg-nodes of the squid (see figure 10).

For our applications, it is convenient to assume that the number of legs \(k\) is equal or greater than 2. This is because we will glue the squids in order to construct closed graphs out of them.

One can provide either a combinatorial definition of a squid graph as a set of squids with a rule of identifying of their boundaries or a geometrical one: a decomposition of an ordinary graph into several squids. The latter one reads as follows.

One can provide either a combinatorial definition of a squid graph as a set of squids with a rule of identifying of their boundaries or a geometrical one: a decomposition of an ordinary graph into several squids. The latter one reads as follows.

Given a graph \(\Gamma = (\mathcal{N}, \mathcal{L})\) split each link \(\ell_i\) into two links by introducing a new node \(x_i\). Then, reorient the new links in such a way that each of them begins at the old node and ends at the new node \(x_i\) (see figure 11). The resulting graph is

\[
\Gamma^{(i)} = (\tilde{\mathcal{N}} = \mathcal{N} \cup \{x\ell \in \mathcal{L} : \ell_i\})
\]

(51)
Figure 11. With each graph we associate a squid graph by subdivision and reorientation of the edges. (a) A graph $\Gamma$. (b) The corresponding squid graph $\Gamma^{(s)}$.

As a result, each old node $n \in N$ becomes a head of a squid $\lambda_n$, whose legs are the links of $\Gamma^{(s)}$ intersecting $n$ (when not needed, we will drop the subscript $n$).

**Definition 2.** Given a graph $\Gamma$ a squid graph corresponding to it is $\gamma = (\Gamma^{(s)}, S)$, where $\Gamma^{(s)}$ is the graph (51) and $S$ is the set of squids $S = \{\lambda_n, n \in N\}$.

Note that even though a squid is a graph, a squid graph is neither a squid nor a graph.

Given several squid-graphs $\{\gamma_I\}_{I \in \mathcal{I}}$, we will denote the disjoint sum of their squid-sets as

$$S = \bigsqcup_{I \in \mathcal{I}} S_I. \quad (52)$$

### 3.2. From a squid-graph to a 1-vertex foam

Consider a squid graph $\gamma = (\Gamma^{(s)}, S)$. We want to construct a 2-complex, with precisely one internal vertex, whose boundary is the graph $\Gamma^{(s)}$ (which will be called a 1-vertex foam). We will do it by formalizing the following shrinking procedure: draw the graph $\Gamma^{(s)}$ on a 3-sphere of radius 1 and then shrink the radius to 0. The track left by the graph defines the 2-complex $\kappa_\gamma$.

A definition of the 2-cell complex $\kappa_\gamma$ follows quite clearly from figure 12. Nonetheless we will spell out now a full rigorous definition consistent with the theory of 2-cell complexes [27] by declaring its sets of faces, edges and vertices together with the gluing functions.

The graph $\Gamma^{(s)}$ viewed as a $\Delta$-complex is a triple $\left(\mathcal{L}, \mathcal{N}; f_{1\rightarrow 0}^{(b)}\right)$ where $f_{1\rightarrow 0}^{(b)} : \partial \mathcal{L} \rightarrow \mathcal{N}$ is a function from boundaries of links to nodes (see the appendix for notation details). To form a 2-complex, we need to add the following: one extra 0-simplex $v$ (the middle point of the sphere), a set of 1-simplexes which will be tracks of nodes $\mathcal{E}_N := \{I_n : n \in N\}$ and a set of 2-simplexes (faces) which will be tracks of links $\mathcal{F}_{\mathcal{L}} := \{\Delta_\ell : \ell \in \mathcal{L}\}$ (each $\Delta_\ell$ is a triangle).
The 2-complex we are constructing is given by a 5-ple $\kappa = (F, \mathcal{E}, V; f_{2\rightarrow 1}, f_{1\rightarrow 0})$. The sets of 2-, 1- and 0-cells are as follows:

$$F = F_L, \quad \mathcal{E} = L \cup \mathcal{E}_N, \quad V = \{v\} \cup N.$$  \hspace{1cm} (53)

What remains to be defined are the functions $f_{1\rightarrow 0} : \partial \mathcal{E} \to V$ and $f_{2\rightarrow 1} : \partial F \to \bigsqcup \mathcal{E} / \sim_1$ (where $\sim_1$ is the relation connecting preimages of the function $f_{1\rightarrow 0}$).

It is obvious that $f_{1\rightarrow 0}$ restricted to $L$ is just $f_{1\rightarrow 0}^{(0)}$, so let us take a look at the added edges. Each of $e \in \mathcal{E}_N$ is labeled by a node $n \in \mathcal{N}$ and all of them meet at the central point $v$. We orient them to be outgoing from $v$, so the function $f_{1\rightarrow 0}$ acts like

$$f_{1\rightarrow 0} : \begin{cases} \partial L \ni x & \mapsto f_{1\rightarrow 0}^{(0)}(x) \\ \partial \mathcal{E}_N \ni L_a(0) & \mapsto v \\ \partial \mathcal{E}_N \ni L_a(1) & \mapsto n, \end{cases}$$  \hspace{1cm} (54)

where $e : t \to e(t), (t \in [0, 1])$ is any parametrization of an oriented 1-cell $e$.

The function $f_{2\rightarrow 1}$ acts on a sum of boundaries of triangles. First, let us take a look on a CW structure of a boundary of a single triangle $\Delta_{VNX}$. It is a 1-complex $([VN, NX, VX], \{V, N, X\}; f)$ (action of $f$ is obvious). The triangle $\Delta_{VNX}$ is a triangle with ordered vertices (we assume $V$ to be the first vertex, $N$ the second and $X$ the third one), so there is a natural orientation of its edges: from the earlier end to the later one (in the sense that $s(VN) = V, s(NX) = N, s(VX) = V$). We introduce the $VNX$ structure at each of the triangles $\Delta_\ell$.

Given a link $\ell$ of the boundary graph $\Gamma^{(\ell)}$, we glue it to the $NX$-edge of the corresponding triangle $\Delta_{\ell}$ in a such way that $N$ is always the starting point of the link and $X$ is its ending...
point. On the other hand, the edges $VN$ and $VX$ are glued to the internal edges, i.e. the ones from the set $\mathcal{E}_X$. The $V$ point will be the starting point of these edges which, according to the $f_{1\to0}$ function, appear to be the $v$ vertex.

To be specific, the function $f_{2\to1}$ act as

$$f_{2\to1} : \partial \Delta_\ell \ni \begin{cases} N_X(t) & \mapsto [\ell(t)] \\ V_N(t) & \mapsto [\ell(t)] \\ V_X(t) & \mapsto [\ell(t)] \end{cases}$$

(55)

where square brackets stands for equivalence classes of $\sim_1$.

A few words of comment about the equivalence classes of $\sim_1$ are as follows. Since the relation $\sim_1$ is the identification of ending points of edges, the equivalence class of $x \in \text{Int}(\ell)$ is just $[x]$. When one considers the equivalence class of one of the endpoints of edge $x \in \partial \mathcal{E}$, it turns out to be the set $(f_{1\to0})^{-1}(f_{1\to0}(x))$ (for more details about $\sim_1$ see the appendix). To make the formulas more transparent, sometimes we will drop this relation.

Note that since all link $\ell$ starts at the heads of the squids, each head is the $N$ vertex and each leg-node is an $X$ vertex. Thus, two triangles can meet either by their head (i.e. they have the same number of legs). We will define now the gluing of $\kappa_{y_1}$ and $\kappa_{y_2}$ along the squids $\lambda$ and $\lambda'$. The operation will be denoted by $\kappa = \kappa_{y_1} \sqcup_{\lambda \to \lambda'} \kappa_{y_2}$. To be more specific, we denote the duality map by $\phi : \lambda \to \lambda'$ defining the morphism of these squids (it needs to be a bijection).

The 2-complex $\kappa = \kappa_{y_1} \sqcup_{\lambda \to \lambda'} \kappa_{y_2}$ is just the union of the complexes $\kappa_1$ and $\kappa_2$ with certain pairs of edges identified\(^4\). Figure 13 shows an example of the gluing. To be more specific,

- let us take a disjoint union of the complexes $\tilde{\kappa} = \kappa_{y_1} \sqcup \kappa_{y_2}$;
- the map $\phi$ between $\lambda$ and $\lambda'$ gives a set of pairs of edges

$$R_\phi = \{ \alpha = (\ell, \ell') : \ell \in \mathcal{L}_\lambda, \ell' \in \mathcal{L}_{\lambda'}, \ell = \phi(\ell') \};$$

(57)

- we identify each pair of edges $\alpha \in R_\phi$:

$$\tilde{\kappa} \mapsto \tilde{\kappa}/\alpha_1 \mapsto (\tilde{\kappa}/\alpha_1)/\alpha_2 \mapsto \cdots \mapsto \kappa.$$ 

(58)

Identifying a pair of edges is a procedure after which the resulting 2-complex differs from the original one only by the fact that the two edges became one edge (and by its all topological implications). The detailed definition of this operation is given in appendix A.2 together with the theorem stating that the result of such gluing does not depend on the order of gluings (appendix A.3). Thanks to that theorem our operation is well defined.

\(^4\) We refer to the links of the squids as the edges of the foams.
Figure 13. The two initial graphs $\Gamma_1$ and $\Gamma_2$ are glued along the squids $\lambda$ and $\lambda'$ into the complex $\kappa_1 \cup_{(\lambda, \lambda')} \kappa_2$. (a) The boundary squid graphs $\gamma_1$ and $\gamma_2$ of two 1-vertex spinfoams $\kappa_{\gamma_1}$ and $\kappa_{\gamma_2}$. The dashed lines mean that only a part of each graph is depicted, $\lambda$ and $\lambda'$ denote squids. (b) The foam $\kappa_{\gamma_1}$ bounded by the squid graph $\gamma_1$. (c) Gluing of the two foams along the squids $\lambda$ and $\lambda'$.

The resulting 2-complex will be denoted as

$$\kappa = \kappa_{\gamma_1} \cup_{(\lambda, \lambda')} \kappa_{\gamma_2} = (F, E, V; f_{1 \rightarrow 0}, f_{1 \rightarrow 0})$$

and its components are as follows.

- The set of faces is just the union $F = F_1 \cup F_2$.
- The set of edges is the union divided by a relation $E = E_1 \cup E_2 / \sim_{\phi,0}$, where two edges $e, e'$ are in the relation $\sim_{\phi,0}$ if and only if $e \in \lambda$, $e' \in \lambda'$ (or opposite) and $\phi(e) = e'$ (or $\phi^{-1}(e) = e'$ in the opposite case).
- The set of vertices is defined in an analogous way: $V = V_1 \cup V_2 / \sim_{\phi,0}$ where $v \sim_{\phi,0} v'$ if and only if $v \in \lambda_1$ and $v' \in \lambda_2$ (or opposite) and $\phi(v) = v'$ (or $\phi^{-1}(v) = v'$ in the opposite case).
- The function $f_{1 \rightarrow 0}$ coincides with the functions $f_{1 \rightarrow 0}^{(1)}$ and $f_{1 \rightarrow 0}^{(2)}$ at their domains, followed by the projection $\pi_{\sim_{\phi,0}}$ onto the equivalence classes of the relation $\sim_{\phi,0}$:

$$f_{1 \rightarrow 0} : \begin{cases} \frac{\partial \tilde{\epsilon}_1}{\partial x} \ni x \mapsto \pi_{\sim_{\phi,0}} \circ f_{1 \rightarrow 0}^{(1)}(x) \\ \frac{\partial \tilde{\epsilon}_2}{\partial x} \ni x \mapsto \pi_{\sim_{\phi,0}} \circ f_{1 \rightarrow 0}^{(2)}(x). \end{cases}$$
However, one needs to perform the consistency check with the relation ~_{φ,1}, i.e. check if \( x \sim_{φ,1} x' \) implies \( f_{1\rightarrow0}(x) = f_{1\rightarrow0}(x') \).

Outside the glued squids it is obviously satisfied, since in this regime equivalence classes of ~_{φ,1} are one-element sets. Assume therefore that \( x \in \partial e \) for \( e \in λ \) and we have \( x' \neq x \) such that \( x' \sim_{φ,1} x \). If it is so, \( x' \) must be in \( λ' \), and \( φ(x) = x' \). We have \( f_{1\rightarrow0}(x') = π_{φ-0} f_{1\rightarrow0}^{(1)}(x) \) and \( f_{1\rightarrow0}(x) = π_{φ-0} f_{1\rightarrow0}^{(2)}(x') \). However, since \( φ \) is a morphism of \( Δ \)-complexes, the condition \( φ(x) = x' \) must follow \( φ(f_{1\rightarrow0}^{(1)}(x)) = f_{1\rightarrow0}^{(2)}(x') \); thus, \( f_{1\rightarrow0}(x) = f_{1\rightarrow0}(x') \), what ends the proof.

• The function \( f_{2\rightarrow1} \) coincides with the functions \( f_{2}^{(1)}_{1\rightarrow1} \) and \( f_{2}^{(2)}_{1\rightarrow1} \) at their domains, followed by the projection \( π_{φ-0} \):

\[
 f_{2\rightarrow1} : \begin{cases}
 \mathcal{∂F}_1 \ni x \mapsto π_{φ-0} f_{2}^{(1)}_{1\rightarrow1}(x) \\
 \mathcal{∂F}_2 \ni x \mapsto π_{φ-0} f_{2}^{(2)}_{1\rightarrow1}(x).
\end{cases}
\] (61)

The new set of boundary squids of \( κ_γ \cup κ_{λ'} \) is

\[
 S = (S_1 \cup S_2) \setminus \{λ, λ'\}.
\] (62)

Thus, we have just obtained a foam with the squid structure on its boundary \((κ, S)\) being the gluing of two 1-vertex foams \((κ_{λ'}, S_1)\) and \((κ_{λ'}, S_2)\).

3.4. Continuation of the gluing procedure to more general cases

What needs to be done now is to show that the same step we have just done from \( n = 1 \) to \( n + 1 = 2 \) can be done from arbitrary \( n \) to \( n + 1 \) in an inductive way.

The key step in the construction is noting that all we needed during our construction so far was the knowledge that the foam we glue is the proper spinfoam, with squids drawn on its boundary. Indeed, none of the steps in subsection 3.3 requires that the complex \( κ \), whose cells were glued, must have the form \( κ_γ \cup κ_{λ'} \). The only thing that is needed are general properties of \( 2 - Δ \)-complex and the squid structure we introduced on the boundary, i.e. decomposition into squids. Thus, any result of the gluing procedure may be a starting point for another gluing procedure of this type. Moreover, the gluing procedure is independent from the order in which pairs of squids are glued, which is an obvious implication of the commutativity of gluing of pairs of edges (appendix A.3).

So, we are finally able to complete the construction.

(i) Let \((G, R)\) be a graph diagram. Let \( G \) be a set of squid graphs constructed from graphs being elements of \( G \). Let \( R \) be the relation on the set of all squids defined as follows: \( λ \) is in relation with \( λ' \) iff the head of \( λ \) is in relation \( R_{node} \) with the head of \( λ' \). For each pair \((λ, λ') \in R\), the link relation \( R_{link} \) induces naturally a morphism of 1-complexes \( φ : λ \to λ' \) which identifies each leg of \( λ \) with a leg of \( λ' \) in a 1–1 way.

(ii) We construct a family of 1-vertex foams \( κ_γ \) by creating one from each squid graph \( γ \in G \) (i.e. from each \( Γ ∈ \mathcal{Γ} (G) \)).

(iii) We take the disjoint union of all these 1-vertex foams:

\[
 κ = \bigcup_{γ ∈ G} κ_γ.
\] (63)

(iv) We denote \( κ_0 = \tilde{κ} \), and then we order pairs of squids \( r \in R \) by numbers from \( n = 1 \) to \( n \# R \) and perform gluings by saying \( κ_n = κ_{n-1}/~_n \), i.e. we glue the complex along the pairs of squids one after the other.

(v) The resulting 2-complex is

\[
 κ = κ_{#R}.
\] (64)
4. Properties

In the previous section we have constructed a 2-complex $\kappa$ corresponding to a squid graph diagram $D = (G, R)$. Now, we will analyze the structure of this foam. We will discuss the way faces may intersect at edges, edges may intersect at vertices and we will discuss possible topologies of the faces.

4.1. Types of edges

Using the notation introduced in section 3.2 (see (55)), one may distinguish three types of edges: $VN$, $VX$ and $NX$; however, the $NX$ edges may be of two subtypes: internal or external. All these types are presented in figure 14. Properties of the edges are as follows.

(i) The $VN$-type edge is a history of the head of a squid (the head itself corresponds to the point $N$). It is always sheared by at least two faces (actually: the number of faces corresponds to the number of legs of the squid it was build from). The faces are always consistently oriented with the edge.

(ii) The $VX$-type edge is a history of the leg node of a squid (the leg node itself corresponds to the point $X$). It is always shared by precisely two faces (coming from the links that were meeting at the node). Both these faces are oriented opposite to the edge.

(iii) The $NX$-type edge is a leg of a squid. It is a boundary edge if and only if the squid it belongs to was not glued to another squid.

(iv) The $NX$-type edge is an internal edge of a complex if and only if the squid it belongs to was glued to another squid. In such a case, this edge is sheared by precisely two faces. The orientation of the faces is consistent with the orientation of the edge (by definition, see 3.2).
Figure 15. The spinfoam (a) arises from the foam of figure 14 by gluing the squids of heads $N_2$ and, respectively, $N_3$. We focus on the foam vertices $N_2$ and $X$. Their neighborhoods (b) are bounded by a $\theta$-like graph ($N_2$) and, respectively, a loop-like graph ($X$). The neighborhood of $X$ is a disk. (a) A fragment of a spinfoam containing internal vertices $N_2$ and $X$. (b) Neighborhoods of the vertices $N_2$ and, respectively, $X$. The neighborhoods are bounded by the depicted graphs.

4.2. Types of vertices

There are three main types of vertices in the 2-complex of our construction: $V$, $N$ and $X$. However, the $N$ and $X$ type split into two and three subtypes, respectively, so we have six classes of vertices to describe.

(i) Vertices of type $V$. They are always internal vertices. There is one such vertex for each squid graph $\gamma \in G$.

(ii) Vertices of type $N$ are the heads of squids. Such a vertex is an internal vertex if and only if the squid it came from was glued with another squid. In such a case, the $N$-type vertex looks like the vertex of type $V$ coming from the $\theta$-graph (see figure 15–of course the similarity is only local).

There are always two edges of type $VN$ ending at such a vertex and a number of $NX$ type edges starting at this vertex. Since $NX$ edges are removable, after removing them the $N$ type vertex becomes a bivalent vertex in the middle of a $VV$ edge, and therefore it is also called removable.

(iii) The vertex of type $N$ coming from the non-glued squid is the boundary vertex. It is then a node of the boundary graph.

(iv) Vertices of type $X$ are the leg nodes of the squids. Such a vertex is an internal vertex if and only if the squid leg it came from is half of a link that belongs to a cyclic equivalence class of the face relation $R_{\text{face}}$ or a 1-element equivalence class of $R_{\text{link}}$ (i.e. it is an element of an equivalence class of $R_{\text{face}}$ relation of type 2, 3(b) or 4(b), see section 2.4). Locally it looks like a $V$-type vertex for a loop graph (see figure figure 15).

Since all edges ending at such $X$-type vertex are removable (i.e. $VX$ and internal $NX$), the vertex itself will also be called removable.

(v) If none of the squids which a vertex of type $X$ belongs to is glued to any other squid, it is a simple boundary vertex (it is a leg node in the middle point of a link being unrelated to
any other link by $R_{\text{link}}$ relations, i.e. belonging to a type 1 equivalence class of the face relation $R_{\text{face}}$.

Removing the $VX$ edge ending at such a vertex makes it a boundary bivalent node. It will be called removable.

(vi) The last possibility for the $X$-type vertex is that it is the middle of a link $\ell$ being an element of an open equivalence class of the face relation $R_{\text{face}}$ (i.e. an equivalence class of type $3(a)$ or $4(a)$—see section 2.4).

Such a vertex is also removable because the internal edges ending at it are $VX$ or internal $NX$ type and, after removing them, the vertex becomes a bivalent boundary node, like in the previous case.

Thus, the only non-removable vertices are type $V$ and boundary type $N$ vertices.

It is worth noting that, topologically, a neighborhood of each vertex of type $X$ is a disk. There are two possibilities: for a boundary vertex $X$ and for an internal vertex $X$ (see figure 16). In both cases, the edges meeting at $X$ form a sequence of $NX$- and $VX$-type edges alternately, i.e. an $NX$ edge is followed by a $VX$ edge and a $VX$ edge is followed by an $NX$ edge. In the case of a boundary vertex $X$, the sequence starts and ends with two (different) boundary $NX$ edges. In the case of an internal vertex $X$, we can choose any $NX$ edge as a starting one and the last $VX$ edge in a sequence is followed by the beginning $NX$ edge.

4.3. Description of faces

The set of faces does not change during the gluing procedure, so the final set $F$ is just the union of original sets $F_{\gamma}$ for all initial squid graphs $\gamma \in \mathcal{G}$. The faces are all triangular. Each face has two internal edges ($VN$ and $VX$) and the third edge ($NX$ type) is also internal if the squid the face came from was glued to another squid, and it is a boundary edge, if the squid was not glued to anything. Topologically, each face is a disk, placed onto some skeleton. Given a face, none of its edges are glued with other edges of the same face (see section 4.1).
5. The final operator spinfoam corresponding to an operator spin-network diagram

5.1. Removing the redundant edges and vertices

In the previous section, some edges and vertices have been marked as removable. These were the $VX$ edges, the internal $NX$ edges, $X$ vertices and internal $N$ vertices. They were auxiliary in our construction, while the other edges and vertices have a direct correspondence with elements of a graph diagram. We will remove them now from the 2-complex $\kappa$ of equation (64) by merging the higher dimension cells sharing the removable ones.

The resulting 2-complex $\kappa_D$ can be characterized in terms of the corresponding graph diagram $D = (G, R)$:

- for each graph $\gamma \in G$, there is one internal vertex $v_\gamma$,
- for each boundary node $n$ of the graph diagram (i.e. a node that is unrelated by the node relation), there is a boundary vertex of the 2-complex (denoted also by $n$),
- for each equivalence class of the edge relation $R_{\text{edge}}$ there is an internal edge of the 2-complex. If the equivalence class is one-element $\{n\}$, then the edge meets the boundary (at the boundary vertex corresponding to the node $n$) and ends at the internal vertex corresponding to the graph that $n$ belongs to. If the equivalence class is two-element $\{n, n'\}$, then the edge connects the internal vertices corresponding to the graphs that $n$ and $n'$ belong to,
- for each link of the boundary graph of the graph diagram there is a boundary edge of the 2-complex. The edge connects the boundary vertices of the 2-complex that correspond to the same nodes of the boundary graph that the link connects,
- for each equivalence class of the face relation $R_{\text{face}}$, there is an oriented face of the 2-complex. It is oriented and glued to the skeleton of the 2-complex in a way which will be described shortly (see section 5.2).

5.2. Properties of the final 2-complex

In section 4, we have discussed the properties of vertices, edges and faces of the 2-complex obtained out of squid graphs, before removing the extra cells. Now we will characterize possible classes of cells resulting from this removal.

5.2.1. Vertices. Each internal vertex of the resulting 2-complex comes from a vertex of type $V$. Its structure is completely characterized by the graph $\Gamma \in G$, to which it corresponds (the graph is the boundary of a sufficiently small neighborhood of the vertex).

Each boundary vertex of the resulting 2-complex comes from a node of the graph diagram unrelated to any other nodes by $R_{\text{node}}$ (i.e. from a boundary vertex of type $N$). Its structure is given by the structure of the node of the boundary graph it corresponds to (a sufficiently small neighborhood of a boundary vertex $N_n$ is a Cartesian product of a squid $\lambda_n$ and an interval $[0, 1]$).

5.2.2. Edges. There are three types of edges: the boundary edges, the internal edges with one end on the boundary and the internal edges with no end at the boundary (there are no internal edges with both ends at the boundary).

Each boundary edge results from the merging of two boundary $NX$ edges sharing the $X$ vertex.
The internal edges with one end at the boundary correspond to one-element equivalence classes of the edge relation $R_{\text{edge}}$. Each of them comes from a single $VN$ edge, where $N$ is on a boundary.

Each internal edge with no end at the boundary comes from a pair of $VN$ edges sharing the $N$ vertex (thus, the $N$ vertex was removed). It is possible that both merged $VN$ edges started at the same $V$ vertex. In such a case, the obtained internal edge is a loop starting and ending at one internal vertex. However, the internal edges generally connect pairs of internal vertices.

5.2.3. Faces. Each face of the final 2-complex corresponds to one equivalence class of the face relation $R_{\text{face}}$. Thanks to the structure of the $X$ vertices, each face is a union of the triangular faces of the complex $\kappa$ (64) sharing one $X$-type vertex. Therefore, types of faces correspond to the types of equivalence classes of $R_{\text{face}}$ and to the types of (removed) $X$-vertices (see section 2.4 and, respectively, section 4.2).

There are two types of faces: faces which overlap the boundary edges and faces which overlap only the internal edges.

- Each face which overlaps the boundary edges corresponds to an open equivalence class of $R_{\text{face}}$ (i.e. the equivalence class of type 1, 3(a) or 4(a)). The $X$-vertex it came from was a boundary vertex (i.e. $X$ vertex of type 5 or 6), so the face contains precisely one boundary edge (being the boundary $N_1N_2$ link that came from the same $X$ vertex). We orient this face in agreement with the boundary link it contains. Other edges of that face are in order: the $X_2V_1$ (where $X_2$ is the ending of the boundary edge), then possibly some sequence of edges $V_1V_2, \ldots, V_{k-1}V_k$ (however, $k$ may be equal to 1), and then $V_kN_1$.

Some of the $V_i$s may be equal, in which case it affects the topology of the face. Moreover, it may happen that $N_1 = N_2$ (the boundary edge is a loop) and thus $V_1 = V_k$. In such a case, all the edges $N_1V_1$ and $N_2V_k$ are equal—with all the consequences for the topology (i.e. the face is either a cone or a cylinder).

It is impossible to obtain a face that contains more than one boundary edge.

- Each face which overlaps no boundary edges corresponds to a closed equivalence class of $R_{\text{face}}$ (i.e. the equivalence class of type 2, 3(b) or 4(b)). The $X$-vertex it came from was an internal vertex (i.e. $X$ vertex of type 4). All edges of this face are internal $VV$ edges.

To orient such a face, recall the structure of the $X$ type vertex. In the previous section, we have not used the orientation of the links of the unsquided graphs but we will invoke it now (as in the previous item). Each $VNX$ triangle meeting at the considered $X$ vertex inherits an orientation from the unsquided graph. One can check that for each two triangles neighboring at this $X$ vertex, their orientations agree. Therefore, the face obtained by removing the $VX$ and $NX$ edges also inherits that orientation.

In other words, we orient the faces in such a way, that if one considers a neighborhood of any internal vertex $v_\Gamma$, then its boundary agrees with the graph $\Gamma$, including the orientation. The edges of such a face form a sequence. Some elements of this sequence (edges or vertices) may appear more than once.

Note that even though the interior of each face is a disk, its boundary may be glued in a topologically nontrivial way. A suitable example is shown and explained in figure 17.

5.3. The coloring

Having defined the 2-complex $\kappa_\mathcal{D}$ in section 5.1 for the graph diagram $\mathcal{D} = (\mathcal{G}, \mathcal{R})$, we will now define the operator spinfoam $(\kappa_\mathcal{D}, \rho, P, A)$ for the operator spin-network diagram
Figure 17. An example of a face having the projective plane topology. At each step, primes show objects that will be identified in later steps. (a) The fragments of squid graphs to be glued. Dashed lines coming from the nodes express that only fragment of graphs are shown. Darker dashed binding the nodes show the node relation, dotted lines together with small letter \( a, a', \ldots, d, d' \) show the link relation. (b) The segments of the 1-vertex foams bounded by the squids. For simplification, they have been cut along the edges \( V_1 X' \) and \( V_2 X'' \). (c) The previous picture without the simplification. The orientations of the edges show the way they will be glued with the primed ones. (d) The result of the gluing. The arrows define the way the points on the sides of the square are identified. The only edges of the 2-complex are the two \( V_1 V_2 \) edges forming the ‘equator’ of the projective plane. (The points \( N_1, N_2 \) and \( X \) are no longer vertices of the 2-complex.)

\[(G, \mathcal{R}; \rho, P, A)\]. To define it, we need to define the coloring of \( \kappa_D \), which will be induced by the coloring of the diagram in a straightforward way.

- Each face \( f \) corresponding to the equivalence class \( [\ell_i] = \{\ell_1, \ldots, \ell_k\} \) of the relation \( \mathcal{R}_{\text{face}} \) is colored by the representation \( \rho_f := \rho_{\ell_i} \) for an (arbitrary) representative of the equivalence class (because the coloring \( \rho_{\ell_i} \) is constant on the equivalence classes). The corresponding carrier Hilbert spaces will be denoted by \( H_{\rho_f} \).

This coloring induces the coloring of the boundary edges in a way consistent with the coloring of the boundary graph of the operator spin-network diagram.
• Each edge \( e \) which has one end on the boundary corresponds to a boundary node \( n \) of the diagram. Each such node is colored by an operator \( P_n \) (see (14)), which induces a coloring of the edge \( P_e := P_n \).

• Each edge \( e \) which has no end on the boundary corresponds to a pair of related nodes \( \{n, n'\} \) in the diagram. Each such pair is colored by \( P_{\{n,n'\}} \) (see (18)), which induces a coloring of the edge \( P_e := P_{\{n,n'\}} \).

• Each internal vertex \( v \) (that is a vertex of the type \( V' \)) corresponds to a graph \( \Gamma \in \mathcal{G} \), which is colored by a contractor \( A_\Gamma \) (see (17)). This induces the coloring of the vertex: \( A_v := A_\Gamma \).

This completes the definition of the coloring.

6. Examples of diagrams

6.1. The very first example

The very first example of the operator spin-network diagram has already been presented in section 1.3 and motivated our definitions.

6.2. The trivial (static) spinfoams

A trivial operator spinfoam is, briefly speaking, defined by the histories of constant in time spin-networks. It is natural to ask which operator spin-network diagram gives a trivial spinfoam as a result. The question is somewhat tricky because the way our framework was introduced was motivated by decomposing a foam into neighborhoods of internal vertices. The trivial spinfoams, on the other hand, have no internal vertices. Therefore, the answer will not be completely trivial. This example teaches us which elements of the diagrams should be thought of as the trivial evolution (nothing happening, no ‘interaction’).

Given \( (\Gamma, \rho) \), which is an oriented graph labeled by representations, consider the operator spinfoam representing the trivial evolution. The corresponding foam has the topology \( \kappa = \Gamma \times [0,1] \). The boundary graph is \( \Gamma_{\text{in}} \cup \Gamma_{\text{out}} \), where \( \Gamma_{\text{in}} = \Gamma \) and \( \Gamma_{\text{out}} \) is obtained from \( \Gamma \) by switching the orientations of all the links. For each link \( \ell \) of \( \Gamma_{\text{in}} \), the face \( \ell \times [0,1] \) of \( \kappa \) is oriented in the agreement with \( \ell \) and colored by \( \rho_\ell \). For each node \( n \) of \( \Gamma_{\text{in}} \), the corresponding internal edge \( n \times [0,1] \) of the foam is colored by the operator \( P_n \in \mathcal{H}_n \otimes \mathcal{H}_n^* \) defined by the natural contraction (that is, \( P_n \) defines the operator \( \text{id} : \mathcal{H}_n \to \mathcal{H}_n \)). Those data define an operator spinfoam \( (\kappa, \rho, P) \) (due to the absence of internal vertices, no vertex contractors are needed). An example of a foam \( \kappa \) is shown in figure 18.

We now give a receipt for an operator spin-network diagram which gives an equivalent operator spinfoam. The diagram will consist of the so-called generalized \( \theta \)-graphs.

• For each node \( n \) of the graph \( \Gamma_{\text{in}} \), we introduce one graph \( \tilde{\theta}_n \) in the following way (see also figure 19).
  (i) The graph \( \theta_n \) is defined as follows. It has two nodes \( n_{\text{in}} \) and \( n_{\text{out}} \). For each outgoing link \( \ell \) at the node \( n \) in \( \Gamma \) there is one link \( \ell^{(o)} \) at \( n_{\text{in}} \) to \( n_{\text{out}} \) in \( \theta_n \). For each incoming link \( \ell \) at the node \( n \) in \( \Gamma \), there is one link \( \ell^{(i)} \) going from \( n_{\text{out}} \) to \( n_{\text{in}} \) in \( \theta_n \).
  (ii) We construct the graph \( \tilde{\theta}_n \) by adding a node at each link of \( \theta_n \) (and splitting the link into two new links). Each new node will be denoted either by \( s_\ell \) if it is on the link \( \ell^{(i)} \) or by \( t_\ell \) if it is on the link \( \ell^{(o)} \). The new links will be denoted by \( \ell^{(i)}_{\text{in/out}} \) (see figure 19). The new links inherit the orientation of the links of \( \theta_n \).

• For each link \( \ell \) of the initial graph \( \Gamma \), the node relation \( \mathcal{R}_{\text{node}} \) is defined to relate the node \( s_\ell \) of \( \tilde{\theta}_{\ell^{(i)}} \) and the node \( t_\ell \) of \( \tilde{\theta}_{\ell^{(o)}} \).
At each pair \((s_\ell, t_\ell)\), the link relation \(R_{\text{link}}^{(s_\ell, t_\ell)}\) is defined to relate the link \(\ell_{\text{in}}^{(s)}\) with \(\ell_{\text{in}}^{(t)}\) and, consequently, the link \(\ell_{\text{out}}^{(s)}\) with \(\ell_{\text{out}}^{(t)}\) (i.e. it does not mix in- and out-links).

Note that no node of type \(s_\ell\) or \(t_\ell\) is left unrelated and all nodes \(n_{\text{in}}\) and \(n_{\text{out}}\) are unrelated (i.e. they are boundary nodes).

- We set the following coloring.
  
  (i) Each link of each \(\tilde{\theta}\) graph is colored by the representation of the link of \(\Gamma\) it comes from.
  
  (ii) Each boundary node \(n\) and each pair of the related internal nodes \(\{n', n''\}\) is colored by the identity operator, the canonical element of the corresponding space \(H_n \otimes H_{n'}^*\) and, respectively, of \(H_n \otimes H_{n''}^*\).
  
  (iii) Each graph in the diagram is colored by the natural contractor \(A^{\text{Tr}}\).

Figure 20 shows the resulting graph diagram (the natural colorings are described above). Now, the foam defined by this graph diagram is not exactly the trivial one (figure 18). Instead, we have obtained the spinfoam presented in figure 20(c). It is obtained by dividing each of the faces of the original foam by a horizontal edge and extending the colorings in such a way that the resulting operator is unchanged. Hence, the foam we have obtained is equivalent to the trivial one.
Figure 20. The trivial graph diagram and reconstruction of the corresponding 2-complex. The node relation is not drawn for the simplicity of the figure but it can be read from the link relation at the pairs of blue nodes. (a) The graph diagram $D$ corresponding to the trivial spinfoam. The dotted lines show the link relation. The node relation is omitted. (b) The thinner (disjoint) graph is the boundary graph of $D$ (dashed lines show the correspondence between nodes of the diagram and nodes of the boundary graph). (c) The spinfoam constructed from the diagram $D$. The horizontal internal edges are all bivalent.

What we learn from this example is that the $\theta$ graphs colored by the canonical trace contractors and identity operators, accompanied by suitable node relations, play the role of identities (no interaction) in the spin-network diagrams.

6.3. 1-interaction vertex spinfoams

Now we will use our formalism to describe a simple non-trivial evolution of a spin-network. First we test the formalism on a very well-known example of a foam. Next, we show a quite simple diagram whose corresponding foam exceeds our graphical skills.

Consider a one internal vertex operator spinfoam defining the evolution of the spin-network states on a graph $\Gamma_{in}$ in whose links are colored by $\rho_{in}$ (with representations of a group $G$) into the spin-networks on a graph $\Gamma_{out}$ whose links are colored by $\rho_{out}$. Suppose for simplicity that all the operators coloring the internal edges are the identities and the internal vertex is colored by a contractor $A_v$.

The neighborhood of the vertex is bounded by a graph $\Gamma_{int}$ (see figure 21(b)) endowed with the induced link coloring $\rho_{int}$, node coloring $P_{int}$, the contractor $A_{int} = A_v$, and relating some of its nodes with the initial graph and the other nodes with the final graph. This information defines the nontrivial evolution. The quadruple referred to as the interaction operator spin-network $(\Gamma_{int}, \rho_{int}, P_{int}, A_{int})$ becomes an element of the corresponding operator spin-network diagram (figure 22(a)).

To construct an operator spin-network diagram representing this operator spinfoam, we perform at first the construction of the previous example to the initial data $(\Gamma_{in}, \rho_{in})$. The result of this intermediate step is the operator spin-network diagram of the previous example. Next, we extend it by the interaction operator spin-network $(\Gamma_{int}, \rho_{int}, P_{int}, A_{int})$. The relation $R$ is extended in the way depicted in figure 22(a).

The above example uses a very simple form of spinfoam. We choose it because it is easy to draw the corresponding 2-complex explaining the construction. However, the power of diagrammatic formalism resides in more complicated diagrams, when drawing
the spinfoam on a two-dimensional sheet of paper is difficult or even impossible. Consider a graph diagram shown in figure 23(a). For every coloring turning this graph into an operator spin-network diagram, the calculation of the corresponding operator defined for the boundary graph (figure 23(b)) is quite simple.

7. Summary, conclusions and outlook

The operator spin-network diagrams and their framework are suited to play the analogous role in the covariant formulations of LQG to the Feynman diagrams in QFT. Our diagram description provides an itemization of the operator spinfoams in terms of simpler elements: graphs, node/link relations and colors. Similar ideas were introduced before by Hellmann in his PhD Thesis [28].
The diagram framework introduced in this paper includes the EPRL spinfoam model defining dynamics of all LQG states. The signature may be either Euclidean or Lorentzian. The natural operator spinfoam models introduced in [8] can also be described by another class of operator spin-network diagrams.

There are two ways of thinking of the spinfoam models of gravity.

The first one is orthodox covariant, in which the states of the theory are defined on spinfoam boundaries. It admits a natural formulation in terms of the operator spin-network diagrams presented in section 2.8.

The second one splits the boundary into the initial and final parts supporting the initial and, respectively, final states. The application of the diagram framework to the initial/final state transition amplitudes was addressed in section 6. What emerges from those examples is the scheme of a theory defined by the operator spin-network diagrams. A specific theory can be defined by using the following elements:

• a fixed set of the interaction graphs of the links colored by representations, nodes by operators and themselves colored by contractors,

• the set of the ‘propagators’ (the trivial interaction graphs) that is the generalized theta graphs constructed in section 6.2 of the links colored by the group representations, nodes colored by the identity operators and themselves colored by the natural trace contractors.

With these blocks, we first construct all the possible 1-interaction vertex diagrams, and then all their compositions.
The fixing of a set of interaction graphs mounts to a truncation of the graph diagram framework, in the sense that only a special class of graphs is allowed. Whether and in what way it should be done is an open physical question. For example, the OSD formulation of a group field theory model would allow only interaction graphs derived from the potential of the theory. The Hamiltonians of the gravitational field defined in the canonical LQG would lead to a different class of interaction graphs. In any case, a suitable truncation of the OSD framework is technically very easy to implement.

Suppose that the operators coloring the nodes of the diagrams are restricted to be only projections. Then, each colored graph in an operator spin-network diagram can be assigned an operator on its own in such a way that the spin-network diagram operator becomes the composition of the vertex and propagator operators. That simplifies the framework even further.

There are several technical problems we have not addressed in this paper but we will do it elsewhere. We briefly discuss them now.

We claim that the 2-complexes obtained from the graph diagrams set the right class of the 2-complexes for the spinfoams models of LQG to be defined on. The first question is whether there are foams that cannot be obtained this way. More exactly, what are the CW-complexes that are out of range by composing the graph diagrams? There are obvious degenerate examples in which a vertex or an edge is intersected by no face but those are not used for foams. Are there any proper examples?

The second question concerns the equivalence between different diagrams. Certainly, there are differently looking diagrams which define the same operators. For example, the diagrams in section 6.3 are written in a way breaking the time symmetry. It is not difficult to first restore the symmetry by adding on the top one more diagram representing the static foam of the final state. Next, the lower static foam diagram (corresponding to the initial state) can be removed in a suitable way. The resulting diagram is equivalent but looks differently. Another source of the equivalent diagrams is the spin-network cylindrical consistency equivalence.

In the technical part of the construction of a 2-complex from a graph diagram, the squid graphs were introduced. Their usefulness suggests that they may play a more important role than an auxiliary tool. Do they play a fundamental role by any chance?

One of the open problems of the spinfoam approaches to the 4D gravity is the definition of the total amplitude that takes into account all the foams. A recent breakthrough in this issue is Rovelli–Smerlak’s projective limit definition [22]. How do our diagrams fit in this limit?

Another open issue of the spinfoam models are bubble divergences. The question arises: does our graph diagram framework help in dealing with bubbles? The work on a suitable graph diagram algorithm is in progress. Hopefully, results will be helpful.

Those questions will be answered soon either by us or by the readers.

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Appendix. Δ-complexes

To make the paper self-contained, we will provide here a definition of the Δ-complex. We will also give the strict definition in terms of 2-Δ-complexes of the edge-gluing procedure, which is the base of the gluing procedure used in this paper. Finally, we will prove the theorem saying that (under some assumptions) gluing along two pairs of edges commute (we are not sure whether it is the strongest version of the theorem. However, it is sufficient for our needs).

In our considerations n-simplex will always mean n-simplex with ordered vertices. A n-simplex will be denoted by Δ1n. While considering two-dimensional complexes we will use Δ1 without a superscript to denote a two-simplex. One-simplexes will be called intervals when considered separately (and denoted then by I), edges when embedded into a 2-Δ-complex, and links when considered as elements of the boundary complex (i.e. graph). The zero-simplexes will be called vertices, when considered as elements of the 2-complex, and nodes, when they are elements of graph, and denoted by v and n, respectively.

A.1. The definition

Consider a number of sets Cm, where m ∈ {0, 1, ..., n}, each of them containing m-simplexes: Cm = {Δ1m1, ..., Δ1Nm} (number Nm is not necessarily finite). For each of them, one can define a boundary set ∂Cm = ⋃m i=1 ∂Δ1m. The boundary of a m-simplex is always the union of (m − 1)-simplexes.

Now, consider functions fm→m−1 for m = 0, ..., n − 1 and set of relations ∼0, ..., ∼n−1, such that

- each relation ∼m is defined on the set Cm,
- the relation ∼0 is the identity relation,
- each function fm→m−1 is a map fm→m−1 : ∂Cm ∈ Cm−1/∼m−1,
- the next relation ∼m is defined by the function fm→m−1 by

\[ x \sim_m y \Leftrightarrow f_{m \rightarrow m-1}(x) = f_{m \rightarrow m-1}(y). \]  \hspace{1cm} \text{(A.1)}

The function fm→m−1 are called the boundary functions and they define the way in which higher-dimension simplexes are glued onto the lower-dimension skeleton. It is worth noting that since the first relation, ∼0, is a trivial relation, it can be omitted in the construction. Then, any other relation is inductively constructed from functions fm→m−1. Thus, what is essential in the construction of the Δ-complex are the boundary functions, not the relations (however, they are very useful in geometrical interpretation).

Having these notions, we may define a Δ-complex.

Definition 3. Δ-complex is a collection of sets Ci, where i = 0, 1, ..., n together with the functions fi→i−1 for i = 1, ..., n defined as above:

\[ \kappa = (C_0, \ldots, C_n ; f_{n \rightarrow n-1}, \ldots, f_{1 \rightarrow 0}). \]  \hspace{1cm} \text{(A.2)}

When considering 2-Δ-complexes, we will use the notation

\[ \kappa = (\mathcal{F}, \mathcal{E}, \mathcal{V} ; f_{2 \rightarrow 1}, f_{1 \rightarrow 0}). \]  \hspace{1cm} \text{(A.3)}
A.2. How to glue a 2-\(\Delta\)-complex along a pair of edges?

We will define now the procedure of identifying two edges in a 2-\(\Delta\)-complex. The definition is a special case of such a procedure, which can be given for arbitrary dimension of both the complex and the simplexes to glue.

**Definition 4. Gluing along two edges.** Given a 2-\(\Delta\)-complex \(\kappa = (F, E, V; f_{2\to 1}, f_{1\to 0})\) and a pair of (different) edges \(\alpha = (e_A, e_B)\) of \(E\) one may define a 2-\(\Delta\)-complex \(\kappa/\alpha\) being the \(\kappa\) with the edges \(e_A\) and \(e_B\) glued together. The resulting complex has the form

\[
\kappa/\alpha = (F, E/\alpha_1, V/\alpha_0; \pi_{\alpha_1} \circ f_{2\to 1}, \pi_{\alpha_0} \circ f_{1\to 0} \circ (\pi_{\alpha_1})^{-1}).
\] (A.4)

To make the definition complete, we have to specify the symbols that appear in the above formula.

The set \(E/\alpha_1\) is simply the set \(E\) with edges \(e_A\) identified with \(e_B\). Formally it can be written as

\[
E/\alpha_1 \ni [e] = \begin{cases} e & \Leftrightarrow e \not\in \{e_A, e_B\} \\ [e_A] & \Leftrightarrow e \in \{e_A, e_B\}, \end{cases}
\] (A.5)

where \([e_A]\) when considered combinatorially is a single element labeled by such a label and when considered topologically (as an edge) acts just as its representant (i.e. \([e_A](t) = e_A(t))\).

The projection map \(\pi_{\alpha_1} : E \to E/\alpha_1\) is obvious.

The set of vertices \(V/\alpha_0\) is the set \(V\) with ends of edges \(e_A\) and \(e_B\) appropriately identified. This procedure is intuitively obvious. However, it needs some care when being defined formally.

Let us name the beginning vertex of \(e_A\) by \(v_{A0}\), its ending vertex by \(v_{A1}\), and respectively \(v_{B0}\) and \(v_{B1}\) for \(e_B\) (i.e. \(f_{1\to 0}(s(e_A)) =: v_{A0}\), etc). If each of \(v_{A0}, v_{A1}, v_{B0}, v_{B1}\) is a different vertex, then the quotient space \(V/\alpha_0\) is as easy to construct as in the case of \(E/\alpha_1\). However, it is possible that some (or even all) of the \(v_{A0}, \ldots, v_{B1}\) vertices are the same. We will consider two cases: first, when in the resulting quotient space there is one equivalence class for all of that points, and second, when there are two equivalence classes for them (only the latter one were used in this paper).

The first case arises when at least one of the following equalities holds:

\[
v_{A0} = v_{A1} \quad \text{or} \quad v_{A1} = v_{B0}
\] (A.6)

or any of those two with \(A\) and \(B\) replaced\(^5\). In such a case, the two edges are mapped to a circle with one vertex on it and the quotient vertex space is

\[
V/\alpha_0 \ni [v] = \begin{cases} v & \Leftrightarrow v \not\in \{v_{A0}, v_{A1}, v_{B0}, v_{B1}\} \\ [v_{A0}] & \Leftrightarrow v \not\in \{v_{A0}, v_{A1}, v_{B0}, v_{B1}\}. \end{cases}
\] (A.7)

If none of conditions (A.6) is satisfied (i.e edges either do not intersect or intersect at their beginnings or endings, or both, but ending with ending and beginning with beginning), the result of gluing is not a circle but an interval, and the quotient vertex space is

\[
V/\alpha_0 \ni [v] = \begin{cases} v & \Leftrightarrow v \not\in \{v_{A0}, v_{A1}, v_{B0}, v_{B1}\} \\ [v_{A0}] & \Leftrightarrow v \not\in \{v_{A0}, v_{B0}\} \\ [v_{A1}] & \Leftrightarrow v \not\in \{v_{A1}, v_{B1}\}. \end{cases}
\] (A.8)

The action of the projection map \(\pi_{\alpha_0}\) in both cases is obvious.

What one should note is that in spite of the presence of \((\pi_{\alpha_0})^{-1}\) in the boundary function \(\pi_{\alpha_0} \circ f_{1\to 0} \circ (\pi_{\alpha_1})^{-1}\), the boundary function is well defined, there is only one case, when \((\pi_{\alpha_1})^{-1}\) is multivalued \([e_A]\) and in that case \(\pi_{\alpha_0} \circ f_{1\to 0}\) gives the same result for both \(e_A\) and \(e_B\).

\(^5\) Since all the procedures are symmetric with respect to the change of \(e_A\) and \(e_B\), any consequent change of \(A\) and \(B\) makes all the statements valid.
A.3. Theorem of commutativity

In our paper, a certain special case of the gluing procedure is performed. All the edges we glue are boundary edges. And since the boundary of one-vertex spinfoams are squid graphs (see section 3.2), they have some very useful feature: all the boundary vertices may be divided into two types: (i) those which have only outgoing boundary edges (heads of the squids), and (ii) those which have only ingoing boundary edges (leg-nodes).

Since only the boundary edges are glued, this feature provides that only gluing of the second type appears, i.e. it is not possible to glue two edges such that ending of one of them is the beginning of another. The feature holds during gluing of boundary edges because after each gluing the boundary of the new complex becomes the subgraph of the original boundary.

Thanks to that fact, it is sufficient for our use to state and prove the theorem of commutativity under the following assumption: consider four different edges grouped in two pairs \( \alpha = (e_A, e_B) \) and \( \beta = (e_C, e_D) \) such that

\[
\forall_{i,j=A,B,C,D} s(e_i) \neq t(e_j).
\]  
\( (A.9) \)

The commutativity theorem states

**Theorem 1.** For any 2-\( \Delta \)-complex \( \kappa \) and any four different edges \( e_A, \ldots, e_D \) such that \( (A.9) \) holds, the following identity is true:

\[
(\kappa / \alpha) / \beta = (\kappa / \beta) / \alpha,
\]  
\( (A.10) \)

where \( \alpha = (e_A, e_B) \) and \( \beta = (e_C, e_D) \).

**Proof.**

One should prove that each part of the 2-\( \Delta \)-complexes is equal.

The regime of faces is trivial, since the gluing does not effect the set \( \mathcal{F} \).

The regime of edges is not trivial but it is obvious. Since \( \mathcal{E}/\alpha = (\mathcal{E} \setminus \{ e_A, e_B \}) \cup \{ e_A \} \) and since \( \{ e_A, e_B \} \cap \{ e_C, e_D \} = \emptyset \), we have

\[
(\mathcal{E}/\alpha) / \beta = ((\mathcal{E} \setminus \{ e_A, e_B \}) \cup \{ e_A \}) \setminus \{ e_C, e_D \} \cup \{ e_C \}
\]  
\( (A.11) \)

which is symmetric with respect to the change of the order of \( \alpha \) and \( \beta \).

Having the set equality \( (\mathcal{E}/\alpha) / \beta = (\mathcal{E}/\beta) / \alpha \), one may consider the action of the projection maps \( \pi_{\alpha_1} \circ \pi_{\alpha_2} \) and \( \pi_{\beta_1} \circ \pi_{\beta_2} \), which is obviously the same.

Now we may go to the vertices regime.

Thanks to assumption \( (A.9) \), we may decompose the set \( \mathcal{V} \) into a disjoint sum

\[
\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_1 \cup \mathcal{V}_{\text{rest}}.
\]  
\( (A.12) \)

where \( \mathcal{V}_0 = \{ v_{A0}, v_{B0}, v_{C0}, v_{D0} \} \) are the starting points of the glued edges, \( \mathcal{V}_1 \) are their ending points and \( \mathcal{V}_{\text{rest}} = \mathcal{V} \setminus (\mathcal{V}_0 \cup \mathcal{V}_1) \). None of gluing act on \( \mathcal{V}_{\text{rest}} \) and the action of the gluing procedure on \( \mathcal{V}_0 \) and \( \mathcal{V}_1 \) is independent and may be considered separately.

Let us take a look on \( \mathcal{V}_0 \). The first quotient can be noted as

\[
\mathcal{V}_0 / \alpha = (\mathcal{V}_0 \setminus \{ v_{A0}, v_{B0} \}) \cup \{ \{ v_{A0} \} \} = ((\mathcal{V}_0 \setminus \{ v_{A0}, v_{B0} \}) \setminus \{ v_{A0}, v_{B0} \}) \cup \{ \{ v_{A0} \} \}
\]  
\( (A.13) \)

where one cannot omit the subtraction in the \( (\{ v_{C0}, v_{D0} \} \setminus \{ v_{A0}, v_{B0} \}) \) term because we do not know whether the two sets intersect or not.
Now let us take the second quotient. Note that now one does not identify points \( v_{C0} \) and \( v_{D0} \) but their equivalence classes \([v_{C0}]\) and \([v_{D0}]\) with respect to the relation \( \sim_\alpha \). The quotient is

\[
(V_0/\alpha)/\beta = \left( (\{v_{C0} \setminus \{v_{A0}, v_{B0}\}\} \cup \{[v_{C0}], [v_{D0}]\}) \cup \{[[v_{C0}]]\} \right) /
\left( (\{v_{C0} \setminus \{v_{A0}, v_{B0}\}\} \cup \{[v_{C0}], [v_{D0}]\}) \cup \{[[v_{C0}]]\} \right).
\]

(A.14)

Now if \([v_{C0}, v_{D0}] \cap \{v_{A0}, v_{B0}\} = \emptyset\), then the equivalence classes \([v_{C0}]\), \([v_{D0}]\) are just the elements \( v_{C0} \) and \( v_{D0} \). So in this case the first term gives the empty set, while in the second term the subtraction gives just \([v_{A0}],[v_{C0}]\), so finally the result set is \([v_{A0}],[v_{C0}]\). However, if at least one of the latter points (say \( v_{C0} \)) belongs to \( [v_{A0}, v_{B0}] \), then \([v_{C0}] = [v_{A0}]\), so the second term vanishes, and the first term is \((v_{D0}) \setminus [v_{A0}, v_{B0}]\) \cup \{[[v_{C0}]]\}, which also vanishes: either because \( [v_{D0}] = v_{D0} \) (which occurs for \( v_{D0} \notin \{v_{A0}, v_{B0}\}\) or because \( v_{D0} \in [v_{A0}, v_{B0}]\). So, finally, the result set in the second case is \((V_0/\alpha)/\beta = \{[[v_{C0}]]\}\), which is equal to \([[[v_{A0}]]]\) (because \([v_{C0}] = [v_{A0}]\)).

In both cases, the set

\[
(V_0/\alpha)/\beta = \begin{cases}
[[v_{C0}]] & \text{for } [v_{A0}, v_{B0}] \cap [v_{C0}, v_{D0}] = \emptyset, \\
[[[v_{C0}]]] & \text{for } [v_{A0}, v_{B0}] \cap [v_{C0}, v_{D0}] \neq \emptyset,
\end{cases}
\]

(A.15)

is insensitive for the change of the order \( \alpha \) and \( \beta \), which is the object of the proof. \( \Box \)

The same reasoning goes for the set \( V_1 \) and thus for all the set \( V \).

Since then, the set \((V/\alpha)/\beta\) being the image of \( \pi_{\alpha} \circ \pi_\beta \), is the same as the image of \( \pi_{\alpha_0} \circ \pi_\beta \), and it is reasonable to ask whether they are the same maps. The answer is in the affirmative what obviously follows from formula (A.15) describing the set \( V \).

Quod erat demonstrandum. For our use, the following further consideration is needed: since we glue the series of pairs of edges \( \alpha_1, \ldots, \alpha_k \), we need to know whether any reordering of this series is equivalent. However, since every permutation can be composed out of transpositions of neighbor elements, the theorem of this section implies that any permutation of \( \alpha \)'s gives the same quotient complex.

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