ON THE EXISTENCE OF FOLIATIONS BY SOLUTIONS TO THE EXTERIOR DIRICHLET PROBLEM FOR THE MINIMAL SURFACE EQUATION

ARI AIOLFI DANIEL BUSTOS JAIME RIPOLL

Abstract. Given an exterior domain $\Omega$ with $C^{2,\alpha}$ boundary in $\mathbb{R}^n$, $n \geq 3$, we obtain a 1-parameter family $u_\gamma \in C^\infty(\Omega)$, $|\gamma| \leq \pi/2$, of solutions of the minimal surface equation such that, if $|\gamma| < \pi/2$, $u_\gamma \in C^\infty(\Omega) \cap C^{2,\alpha}(\overline{\Omega})$, $u_\gamma|_{\partial \Omega} = 0$ with $\max_{\partial \Omega} \|\nabla u_\gamma\| = \tan \gamma$ and, if $|\gamma| = \pi/2$, the graph of $u_\gamma$ is contained in a $C^{1,1}$ manifold $M_\gamma \subset \overline{\Omega} \times \mathbb{R}$ with $\partial M_\gamma = \partial \Omega$. Each of these functions is bounded and asymptotic to a constant

$$c_\gamma = \lim_{\|x\| \to \infty} u_\gamma(x).$$

The mappings $\gamma \to u_\gamma(x)$ (for fixed $x \in \Omega$) and $\gamma \to c_\gamma$ are strictly increasing and bounded. The graphs of these functions foliate the open subset of $\mathbb{R}^{n+1}$

$$\{(x, z) \in \Omega \times \mathbb{R}, -u_{\pi/2}(x) < z < u_{\pi/2}(x)\}.$$

Moreover, if $\mathbb{R}^n \setminus \Omega$ satisfies the interior sphere condition of maximal radius $\rho$ and if $\partial \Omega$ is contained in a ball of minimal radius $\rho$, then

$$[0, \sigma_n \rho] \subset [0, c_{\pi/2}] \subset [0, \sigma_n \rho],$$

where

$$\sigma_n = \int_1^\infty \frac{dt}{\sqrt{t^{2(n-1)} - 1}}.$$

One of the above inclusions is an equality if and only if $\rho = \rho$, $\Omega$ is the exterior of a ball of radius $\rho$ and the solutions are radial.

These foliations were studied by E. Kuwert in [8] and our result answers a natural question about the existence of such foliations which was not touched in [8].

---

2020 Mathematics Subject Classification. 53A10, 53C42, 49Q05, 49Q20.

Key words and phrases. Exterior Dirichlet problem; Minimal surface equation; Minimal hypersurfaces foliations.
1. Introduction

The exterior Dirichlet problem (EDP) for the minimal surface equation consists on the study of existence/nonexistence and uniqueness of solutions of the PDE boundary problem

1. \[
\begin{align*}
\mathcal{M}(u) := \text{div} \left( \frac{\nabla u}{\sqrt{1+\|\nabla u\|^2}} \right) &= 0, \quad u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \\
\left. u \right|_{\partial \Omega} &= \varphi
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^n, \ n \geq 2, \) is an exterior domain that is, \( \Lambda := \mathbb{R}^n \setminus \overline{\Omega} \) is a relatively compact domain, and \( \varphi \in C^0(\partial \Omega) \) a given function. Additionally to existence or not of solutions of (1), one is also interested on global properties of their graphs in \( \mathbb{R}^{n+1}. \)

In \( \mathbb{R}^2 \) the EDP has a history which goes back to J. C. C. Nitsche who proved (Section 4 of [12]) that any solution of (1) has a \( C^1 \) expansion, for \( \|x\| \) big enough, of the form

2. \[
\begin{align*}
u(x_1, x_2) &= c_1 x_1 + c_2 x_2 + c \log \|x\| + O\left(\|x\|^{-1}\right).
\end{align*}
\]

Regarding the existence/non existence problem, R. Osserman [13] proved that there is a boundary data on the disk for which the EDP (1) on the complement of the disk has no bounded solution. R. Krust [7] proved that Osserman’s boundary data has no solution with horizontal end, that is, \( c_1 = c_2 = 0 \) in (2) or, equivalently, having vertical Gauss map at infinity, leaving opened the question about the existence or not of a boundary data for which the EDP has no solution at all that is, with no end type restriction. This was solved by N. Kutev and F. Tomi [9] who proved the existence of a boundary data, with arbitrarily small oscillation and with bounded \( C^{0,1} \) norm, for which (1) has no solution, irrespective of the asymptotic behavior. As to the existence problem, it is proved in [9] and [15] that (1) has a solution with horizontal end under conditions involving the curvature of the boundary of the domain, the Lipschitz constant and the oscillation of the boundary data.

Regarding the behavior in \( \mathbb{R}^{n+1}, \ n \geq 2, \) of the graphs of the solutions of (1), we remark that the fundamental solutions (see next section) on the exterior of any given open ball \( B \) of \( \mathbb{R}^n, \) provide examples of foliations with horizontal ends of the open subset of \( \mathbb{R}^n \)

\[ \left\{ (x, z) \in \mathbb{R}^n \setminus \overline{B} \times \mathbb{R} \text{ such that } -v(x) < z < v(x) \right\} \]

where the graph of \( v \) is the top of a generalized catenoid with neck size determined by \( B. \) This foliation is parametrized by the angle that the Gauss map of the graph of the fundamental solution at the boundary of the domain makes with the positive vertical axis (note that if \( \gamma \) is such angle relatively to a fundamental solution \( u \in C^2(\mathbb{R}^n \setminus B), \) then \( \tan \gamma = \)
A question that arises is if such a similar phenomenon happens with an arbitrary exterior domain.

This question was partially answered by the third author in $\mathbb{R}^2$ (Theorem 1 of [14]). A complete answer in the two dimensional case was obtained in $\mathbb{R}^2$ where the authors prove that the limit of the leaves in Theorem 1 of [14] can be included in the foliation.

We recall that R. Krust proved in [7] that if there are two different solutions in $\mathbb{R}^3$ with the same Gauss map at infinity then there is a continuum of solutions foliating the space in between.

The case $\mathbb{R}^n$ for $n \geq 3$, to the authors’ knowledge, was investigated only in the work of E. Kuwert [8] where it is proved that the Krust foliation theorem [7] is true in any dimension, leaving open however the problem of existence or not of such foliations.

In the present paper we investigate the existence of foliations to the EDP in $\mathbb{R}^n$ for $n \geq 3$ in arbitrary exterior domains of $\mathbb{R}^n$ but in the special case that the boundary data $\varphi$ in (1) is zero. We use in part the technique of [14] for proving that an exterior domain $\Omega$ of $\mathbb{R}^n$, $n \geq 3$, determines a non trivial foliation of minimal hypersurfaces in $\Omega \times \mathbb{R} \subset \mathbb{R}^{n+1}$ containing the trivial solution as a leaf. As it happens in the 2–dimensional case, this foliation has horizontal ends and is parametrized by the maximal angle that the Gauss map of the leaves in $\mathbb{R}^{n+1}$ make with the positive vertical axis at $\partial \Omega$. Moreover, any leaf has a limit height at infinity which can be estimated by the geometry of the domain (see Theorem 1 for a precise statement).

A natural problem is to extend our result to more general boundary data. To succeed, applying the technique used here (or of [15]), one needs to guarantee the existence of at least one solution with the given boundary data. However, although not having a counter example, we do believe, as it happens in the 2–dimensional case, that without hypothesis on the boundary data such a solution may not exist. And even if one solution exists, it can possibly be the only one. This happens in the 2–dimensional case on the exterior of a disk for certain boundary data, as proved in Theorem 2.9 of [15]. Even though, it seems to us that a more difficult part on the nonzero boundary data case is to estimate the values at infinity of the solutions: as done here, one needs the fundamental solutions as barriers and the way they are used applies, in principle, only for zero boundary data.
2. Fundamental solutions

Given $\lambda > 0$ and $p \in \mathbb{R}^n$ let $B_\lambda (p)$ be the ball centered at $p$ and with radius $\lambda$, $n \geq 2$. The radial function

$$v_\lambda (x) = \lambda \int_1^r \frac{dt}{\sqrt{t^{2(n-1)} - 1}}, \quad r = \|x - p\|, \quad x \in \mathbb{R}^n \setminus B_\lambda (p),$$

is a solution of (1) in $\mathbb{R}^n \setminus B_\lambda (p)$ vanishing at $\partial B_\lambda (p)$. We call $v_\lambda$, or any vertical translation of $v_\lambda$, a fundamental solution. The graph of $v_\lambda$ is half of an $n-$dimensional catenoid. By using isometries and homotheties one obtains a family of radial solutions, which we also call fundamental solutions, defined in the exterior of any fixed ball which gradient at the boundary of the ball varies from 0 to $\infty$.

In this paper we are interested only when $n \geq 3$. In this case we then have

$$0 < \sigma_n := \int_1^\infty \frac{dt}{\sqrt{t^{2(n-1)} - 1}} < \infty$$

so that, from (3), $v_\lambda (x)$ has a limit as $\|x\| \to \infty$ not depending on $p$, which we denote by $v_\lambda (\infty)$ and which is given by

$$v_\lambda (\infty) = \sigma_n \lambda.$$

Fundamental Solutions with $\lambda = 1$

3. The result and its proof

A fundamental tool in PDE, used several times in the proof of Theorem [1] is the comparison principle. In our case it states that if $\Omega$ is a bounded domain in $\mathbb{R}^n$, $u, v \in C^2 (\Omega)$ satisfies $\mathcal{M} (u) = \mathcal{M} (v) = 0$ and $u \leq v$ at $\partial \Omega$ that is,

$$\limsup_k (u(x_k) - v(x_k)) \leq 0$$
for any sequence $x_k$ in $\Omega$ which leaves any compact subset of $\Omega$, then $u \leq v$ in $\Omega$ (Proposition 3.1 of [16]). An easy consequence of the comparison principle is the maximum principle which asserts that if $u,v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy $\mathcal{M}(u) = \mathcal{M}(v) = 0$ in $\Omega$ then

$$\max_{\Omega} |u - v| = \max_{\partial\Omega} |u - v|$$

(Proposition 3.2 of [16]).

The maximum principle has an useful application on Differential Geometry, known as the tangency principle. In our case it says that if $M_1$ and $M_2$ are minimal hypersurfaces of $\mathbb{R}^n$ (with or without boundary and not necessarily graphs) that have a tangency at some interior or boundary point $p \in M_1 \cap M_2$, and if $M_1$ is in one side of $M_2$ in a neighborhood $p$, then $M_1$ coincides with $M_2$ in a neighborhood of $p$ [4].

We also remark that once we have a priori $C^1$ estimates for the solutions of the minimal surface equation (or to more general quasi linear elliptic PDEs) we also have $C^{1,\alpha}$ a priori estimates from the Hölder theory (Ch 13 of [5]). Then well known arguments (see, for example, Section 2.1 of [16]) allow to reduce the $C^{2,\alpha}$ a priori estimates and the regularity of solutions of quasi linear elliptic PDEs to a priori estimates and regularity theory of linear elliptic PDEs (Ch 6 of [5]).

In the statement of Theorem 1 we set, for convenience, $s := \tan \gamma$, $|\gamma| \leq \pi/2$ and, we use $u_\gamma$ and $c_\gamma$ as in the Abstract.

**Theorem 1.** Assume that $\Omega$ is an exterior domain of $C^{2,\alpha}$ class such that $\Lambda := \mathbb{R}^n \setminus \overline{\Omega}$, $n \geq 3$, satisfies the interior sphere condition with maximal radius $\rho$, namely: Given $p \in \partial \Lambda$, there is a $(n-1)$-dimensional sphere $S_p$ of radius $\rho$ such that $p \in S_p$, $S_p \subset \overline{\Lambda}$ and $\rho$ is maximal under these conditions. Let $\varrho$ be the radius of the smallest open ball $B_\varrho$ of $\mathbb{R}^n$ such that $\partial \Omega \subset B_\varrho$. Given $s \in [-\infty, \infty]$ there is a bounded function $u_s \in C^\infty(\Omega)$ satisfying $\mathcal{M}(u_s) = 0$ in $\Omega$, $u_{-s} = -u_s$ and such that:

If $-\infty < s < \infty$ then $u_s \in C^\infty(\Omega) \cap C^{2,\alpha}(\overline{\Omega})$,

$$u_s|_{\partial\Omega} = 0$$

and

$$\max_{\partial\Omega} ||\nabla u_s|| = \max_{\Omega} ||\nabla u_s|| = |s|.$$  

The graph of $u_{\infty}$ is contained in a $C^{1,1}$-manifold $M \subset \overline{\Omega} \times \mathbb{R}$ with boundary $\partial M = \partial \Omega$.

For any $s \in [-\infty, \infty]$ there exists the limit

$$u_s(\infty) := \lim_{\|x\| \to \infty} u_s(x),$$
and
\[ \lim_{\|x\| \to \infty} \| \nabla u_s(x) \| = 0. \]

Moreover, the maps \( s \mapsto u_s(x) \), for fixed \( x \in \Omega \) and \( s \mapsto u_s(\infty) \) are strictly increasing, bounded and we have the inclusions
\[ [-\sigma_n \rho, \sigma_n \rho] \subset [-u_\infty(\infty), u_\infty(\infty)] \subset [-\sigma_n \varrho, \sigma_n \varrho] \]
where \( \sigma_n \) is given by (4). If one of the inclusions is an equality then \( \rho = \varrho \), \( \Omega \) is the exterior of a ball of radius \( \rho \) and the \( u_s \) are the fundamental solutions.

Finally, the graphs of the solutions of \( u_s, s \in (-\infty, \infty) \) foliate the open subset of \( \mathbb{R}^{n+1} \)
\[ O := \{(x, z) \in \mathbb{R}^{n+1} \setminus \overline{\Omega} \times \mathbb{R} \text{ such that } u_{-\infty}(x) < z < u_\infty(x)\}. \]

**Proof.** We first consider the case that \( -\infty < s < \infty \). Since the case \( s = 0 \) is trivial and, since if \( u_s \) is a solution satisfying (6), (7) and (9) then \( u_{-s} = -u_s \) is a solution also satisfying these conditions, we may assume \( s > 0 \). Let \( a \in \mathbb{R} \) be such that \( B_a = B_a(0) \), the open ball in \( \mathbb{R}^n \) of radius \( a \) centered at origin, contains \( \overline{\Omega} \). Let \( v_a \in C^0(\mathbb{R}^n \setminus B_a) \) be given by (3) with \( p = 0 \). We see that \( \| \nabla v_a(x) \| \to 0 \) as \( \|x\| \to \infty \) and we may then choose \( k \in \mathbb{N}, k > a + 1 \), large enough such that
\[ \| \nabla v_a \|_{\partial B_k} \leq \frac{s}{2}. \]

Set \( \Omega_k = B_k \cap \Omega \) and
\[ T_k = \left\{ t \geq 0 ; \exists w_t \in C^{2,\alpha}(\overline{\Omega_k}) \text{ s.t. } M(w_t) = 0, \sup_{\Omega_k} \| \nabla w_t \| \leq s, w_t|_{\partial \Omega} = 0, w_t|_{\partial B_k} = t \right\}. \]

The set \( T_k \) is not empty since \( 0 \in T_k \). Moreover, \( \sup T_k < \infty \) since
\[ \sup_{\Omega_k} \| \nabla w_t \| \leq s \]
for all \( t \in T_k \). We will prove that
\[ t_k := \sup T_k \in T_k \]
and that
\[ \sup_{\Omega_k} \| \nabla w_{t_k} \| = \sup_{\partial \Omega_k} \| \nabla w_{t_k} \| = s. \]

Taking a sequence \( (t_{m_k}^k) \) in \( T_k \) converging to \( t_k \) as \( m \to \infty \) the corresponding functions \( w_{t_{m_k}^k} \) have uniformly bounded \( C^1 \) norm. By elliptic PDE theory ([5], [16]) there is a subsequence of \( w_{t_{m_k}^k} \) converging on the \( C^2 \) norm on \( \overline{\Omega_k} \) to a function \( w_k \in C^{2,\alpha}(\overline{\Omega_k}) \) which satisfies \( M(w_k) = 0 \).
in \( \Omega_k \). Clearly \( w_k|_{\partial \Omega} = 0 \), \( w_k|_{\partial B_k} = t_k \) and \( \sup_{\Omega_k} \| \nabla w_k \| \leq s \). It follows that \( t_k \in T_k \) and that \( w_k = w_{t_k} \).

From the maximality of \( t_k \) we claim that we cannot have \( \sup_{\Omega_k} \| \nabla w_k \| < s \). Indeed: Consider a function \( \phi \in C^{2,\alpha}(\mathbb{R}^n) \) such that \( \phi|_{B_k-1} = 0 \) and \( \phi|_{\mathbb{R}^n \setminus B_k} = 1 \), set

\[
C^{2,\alpha}_0(\Omega_k) = \{ \omega \in C^{2,\alpha}(\Omega_k) \mid \omega|_{\partial \Omega_k} = 0 \},
\]

and define \( T: [-1,1] \times C^{2,\alpha}_0(\Omega_k) \to C^{\alpha}(\Omega_k) \) by

\[
T(t,\omega) = M(\omega + w_k + t\phi).
\]

Then \( T(0,0) = 0 \). One may see that the Fréchet derivative \( \partial_2 T(0,\omega_k) = dM_{w_k} \) is invertible (Theorem 3.3 of [5]) so that, from the implicit function theorem on Banach spaces (Theorem 17.6 of [5]), there exists a continuous function \( t \mapsto \omega(t) \in C^{2,\alpha}_0(\Omega_k) \) (continuous on the \( C^{2,\alpha} \) topology) such that \( T(t,\omega(t)) = 0 \), \( t \in (-\varepsilon,\varepsilon) \). Therefore, since \( \| \nabla w_k \|_{\Omega_k} < s \) there exists \( t \in (0,\varepsilon) \) such that

\[
\sup_{\Omega_k} \| \nabla (\omega(t) + w_k + t\phi) \| < s.
\]

Since

\[
M(\omega(t) + w_k + t\phi) = T(t,\omega(t)) = 0,
\]

\( \omega(t) + w_k + t\phi = 0 \) at \( \partial \Omega \) and \( \omega(t) + w_k + t\phi = t_k + t \) at \( \partial B_k \), it follows that \( t_k + t \in T_k \), contradiction since \( t_k = \sup T_k \). We then have \( \sup_{\Omega_k} \| \nabla w_k \| = s \). We claim that

\[
(15) \quad \sup_{\partial B_k} \| \nabla w_k \| \leq s/2.
\]

Indeed: Since the graph of \( v_a \) is vertical at \( \partial B_a \) it follows from the comparison principle (see [5], Ch 10, or Proposition 3.1 of [16]) that

\[
(16) \quad v_a + t_k - v_a(x_0) \leq w_k \leq t_k
\]

where \( x_0 \) is any but fixed point of \( \partial B_k \). From (12) and (16) we get (15). By the gradient maximum principle ([5], Ch 15) we obtain

\[
\sup_{\Omega_k} \| \nabla w_k \| = \sup_{\partial \Omega_k} \| \nabla w_k \| = s.
\]

Letting \( k \to \infty \) and using the diagonal method we obtain a subsequence of \( w_k \) converging uniformly \( C^2 \) on compact subsets of \( \Omega \) to a function \( u_s \in C^{2,\alpha}(\overline{\Omega}) \) satisfying \( M(u_s) = 0 \) in \( \Omega \), (3) and (7). From elliptic PDE regularity ([5], u_s \in C^\infty(\Omega).

Now, for any \( s \in [0,\infty) \), the graph \( G_s \) of \( u_s \) is by construction of (uniform) bounded slope (see [17]). It follows from Proposition 3 of
that $G_s$ is regular at infinity that is, $u_s$ has a twice differentiable expansion

$$u_s(x) = c_s + a_s \|x\|^{2-n} + \sum_{j=1}^{n} c_{s,j} x_j \|x\|^{-n} + O(\|x\|^{-n})$$

from which it follows that

$$u_s(\infty) := \lim_{\|x\| \to \infty} u_s(x) = c_s.$$

It also follows from (17) that

$$\lim_{\|x\| \to \infty} \|\nabla u_s\|(x) = 0$$

which implies that $G_s$ is horizontal at infinity that is, (9) is satisfied. This proves that (8) and (9) are satisfied for $s \in [0, \infty)$. Let $v_\varphi$ be the fundamental solution on $\mathbb{R}^n \setminus B_\varphi$ which gradient infinity at $\partial B_\varphi$. Given $s \in [0, \infty)$ we claim that $u_s(\infty) < v_\varphi(\infty)$. Indeed, coming from $-\infty$ with the graph $G_\varphi$ of $v_\varphi$ using vertical translations, since the gradient of $v_\varphi$ at the boundary of $B_\varphi$ is infinity, it follows from the tangency principle that the first contact between $G_\varphi$ and the graph of $u_s$ has to be at infinity and with the boundary of $G_\varphi$ strictly below the level $x_n+1=0$. Hence, at the level $x_n+1=0$ one necessarily has $u_s(\infty) < v_\varphi(\infty)$. It follows from the claim and from (5) that $u_s$ is bounded by $\sigma_\varphi$ for all $s \in [0, \infty)$. Clearly we have $u_s \leq u_t$ and also $u_s(\infty) \leq u_t(\infty)$ if $s \leq t$. Hence, for any increasing sequence $s_m \to \infty$ the sequence $u_{s_m}$ converges uniformly on compact subsets of $\Omega$ to a $C^1$ function $u_\infty$ in $\Omega$ satisfying $\mathcal{M}(u_\infty) = 0$.

For proving that the graph $G_\infty$ of $u_\infty$ is contained in a $C^{1,1}$ manifold with boundary $\partial \Omega$ consider a fixed ball $B_a$ with $a > \varphi$. By [11], given $s \in [0, \infty]$ there is a minimizer $v_s$ on the space $BV(\Omega_a)$ of bounded variation functions on $\Omega_a$ (see [6]), for the functional

$$\mathcal{F}_s(w) = \int_{\Omega_a} \sqrt{1 + \|\nabla w\|^2} + \int_{\partial \Omega_a} \|w - \phi_s\|, \quad w \in BV(\Omega_a),$$

where $\phi_s \in C^\infty(\partial \Omega_a)$ satisfies $\phi_s|_{\partial \Omega} = 0$, $\phi_s|_{\partial B_a} = u_s|_{\partial B_a}$. Since $u_s$ is also a minimizer for $\mathcal{F}_s$ for $0 \leq s < \infty$, we have $u_s|_{\Omega_a} = v_s$ by uniqueness [11] (the equality is in $BV(\Omega_a)$). Noting that

$$\lim_{s \to \infty} \mathcal{F}_s(w) = \mathcal{F}_\infty(w), \quad w \in BV(\Omega_a),$$

$$\lim_{s \to \infty} \mathcal{F}_s(u_s) = \lim_{s \to \infty} \mathcal{F}_\infty(u_s),$$
we have (writing only \( u_s \) instead of \( u_s|_{\Omega_s} \))

\[
\mathcal{F}_\infty(v_\infty) = \lim_{s \to \infty} \mathcal{F}_s(v_\infty) \geq \lim_{s \to \infty} \mathcal{F}_s(v_s) = \lim_{s \to \infty} \mathcal{F}_s(u_s) \geq \lim_{s \to \infty} \mathcal{F}_\infty(u_s) = \mathcal{F}_\infty(u_\infty),
\]

where, in the last inequality, we used that \( \mathcal{F}_\infty \) is lower semicontinuous. It follows that \( \mathcal{F}_\infty(v_\infty) = \mathcal{F}_\infty(u_\infty) \) and hence, by uniqueness, \( v_\infty = u_\infty \) in \( \Omega_a \). From Theorem 4.2 of [2] applied to the functional \( \mathcal{F}_\infty \), by choosing \( \Phi = \partial \Omega \), \( \phi_i \equiv 0 \), and using also Theorem 4.7, we conclude that the graph of \( u_\infty \) is contained in a \( C^{1,1} \) manifold \( M \) with boundary which boundary is \( \partial \Omega \).

We have seen that \( s \to u_s(\infty) \) is increasing and bounded by \( \sigma \rho \). If \( c := \lim_{s \to \infty} u_s(\infty) \) then we have \( u_s \leq u_\infty \leq c \), \( s \in [0, \infty) \), by the comparison principle, and hence there is the limit \( u_\infty(\infty) \) of \( u_\infty(x) \) as \( \|x\| \to \infty \) and \( u_\infty(\infty) = c \), proving the second inclusion of (10). We shall prove now (9) for \( s = \infty \).

By the way \( u_\infty \) is obtained we can not conclude directly that the graph of \( u_\infty \) is of (uniform) bounded slope and hence we don’t know if \( u_\infty \) is regular at infinity and admits an expansion as (17). But this is actually the case, indeed: Since \( u_\infty(\infty) = c \) the tangent cone to the graph of \( u_\infty \) at infinity is the hyperplane \( \mathbb{R}^n = \{ x_{n+1} = 0 \} \) of \( \mathbb{R}^{n+1} \) (see [18]) and hence, from Theorem 1 of [18] it follows that \( \nabla u_\infty \) has a limit at infinity and \( \| \nabla u_\infty \| \) is bounded outside some compact. Since \( u_\infty \) is bounded this limit has to be zero and this proves (9) for \( s = \infty \).

Let \( c \in [0, \sigma \rho] \) be given. We prove that there is a non negative solution \( w_c \in C^0(\overline{\Omega}) \cap C^\infty(\Omega) \) of \( (1) \) such that \( w_c|_{\partial \Omega} = 0 \) and

\[
\lim_{\|x\| \to \infty} w_c(x) = c.
\]

Define

\[
\mathcal{F} = \left\{ f \in C^0(\overline{\Omega}) : \ f \text{ is a subsolution of } \mathcal{M} \text{ in } \Omega, \ f = 0 \text{ in } \partial \Omega \text{ and } \limsup_{\|x\| \to \infty} f(x) \leq c \right\}.
\]

Clearly \( \mathcal{F} \neq \emptyset \) and it follows from the the comparison principle that \( f \leq c \) for all \( f \in \mathcal{F} \). We may then apply Perron’s method ([5], Section 2.8) to conclude that

\[
w_c(x) = \sup \{ f(x) ; \ f \in \mathcal{F} \}, \ x \in \overline{\Omega},
\]

is \( C^\infty \) and satisfies \( \mathcal{M}(w_c) = 0 \) in \( \Omega \). For proving that

\[
\lim_{\|x\| \to \infty} w_c(x) = c
\]
take $a > 0$ large enough, such that $\Lambda \subset B_a$ satisfies $v_a(\infty) > c$. We have that $f \in C^0(\overline{\Omega})$ given by

$$f(x) = \begin{cases} 0 & \text{if } x \in \overline{\Omega} \cap B_a \\ \max\{0, v_a(x) - (v_a(\infty) - c)\} & \text{if } x \in \mathbb{R}^n \setminus B_a\end{cases}$$

is a subsolution relatively to the (1) satisfying $f|_{\partial \Omega} = 0$ and

$$\lim_{\|x\| \to \infty} f(x) = c.$$  

It follows that $f \in \mathcal{I}$ and then $f \leq w_c \leq c$, which proves (20).

It remains to prove that $w_c$ extends $C^0$ to $\overline{\Omega}$ and that $w_c|_{\partial \Omega} = 0$. Given $p \in \partial \Omega$, by hypothesis there is an open ball $B_\rho$ contained in $\Lambda$ such that $\partial B_\rho$ is tangent to $\partial \Omega (= \partial \Lambda)$ at $p$. Since $c \leq \sigma_n \rho = v_\rho(\infty)$ and $v_\rho = 0$ at $\partial B_\rho$ it follows from the comparison principle that $0 \leq w_c \leq v_\rho$. Since $p$ is arbitrary this proves the claim that is, $w_c$ extends $C^0$ to $\overline{\Omega}$ and $w_c|_{\partial \Omega} = 0$.

Now, assume that $0 \leq c < \sigma_n \rho$. Then we may find a fundamental solution $\tilde{v}$ defined on the exterior of a ball of radius $\rho$, contained in $\Lambda$, tangent to $\partial \Omega$ with bounded gradient at the boundary of the ball and such that

$$\tilde{v}(\infty) = \frac{c + \sigma_n \rho}{2}.$$  

By the comparison principle it follows that $0 \leq w_c \leq \tilde{v}$. This proves that $w_c$ extends $C^1$ to $\overline{\Omega}$ and, by PDE regularity [5], $w_c \in C^{2,\alpha}(\overline{\Omega}) \cap C^\infty(\Omega) \cap C^\infty(\Omega)$.

Setting

$$s_c = \max_{\partial \Omega} \|\nabla w_c\|,$$

we prove that $u_{s_c} = w_c$. By contradiction, assume the opposite. Then, setting

$$d := \lim_{\|x\| \to \infty} u_{s_c}$$

we cannot have $d > c$ or $d < c$. Indeed: Assume, by contradiction, that $d > c$. Let $p \in \partial \Omega$ be such that $\|\nabla w_c\|(p) = s_c$. If $\|\nabla u_{s_c}\|(p) = s_c$ we cannot have $w_c(x) \leq u_{s_c}(x)$ for all $x \in \overline{\Omega}$ because of the boundary tangency principle. But if have $w_c > u_{s_c}$ this inequality must hold only on a bounded open subset of $\Omega$ since $c < d$. One can then make a vertical translation of the graph of one of the solutions to get a tangency between their graphs, with one of them in one side of the other, contradicting the tangency principle.

The remaining possibility

$$\|\nabla u_{s_c}\|(p) < s_c = \|\nabla w_c\|(p)$$
also implies that \( w_c > u_{s_c} \) must hold on a bounded open subset of \( \Omega \) leading, as before, to a contradiction with the tangency principle. The case that \( d < c \) cannot happen by the same arguments. This proves that \( c = d \) and, arguing with the tangency principle again, that \( w_c = u_{s_c} \).

Finally, take an increase sequence \( c_m \in [0, \sigma_n \rho) \) converging to \( \sigma_n \rho \) as \( m \to \infty \). The sequence \( s_{c_m} \) is increasing and then has a limit \( s \in [0, \infty] \).

The sequence \( (u_{s_{c_m}}) \) converges uniformly \( C^2 \) on compact subsets of \( \Omega \) to a solution \( u_s \in C^0 (\overline{\Omega}) \cap C^\infty (\Omega) \), \( u_s |_{\partial \Omega} = 0 \) and \( \sup_{\partial \Omega} \| \nabla u_s \| = s \). As before we obtain \( u_s = w_{\sigma_n \rho} \), proving that

\[ [0, \sigma_n \rho] \subset [0, u_\infty (\infty)] . \]

This concludes the proof of (10).

If one of the inclusions in (10) is an equality and the corresponding graphs of the solutions with infinite gradient at \( \partial \Omega \) are not the same, then either one is below the other or they intersect in interior points. The first case cannot occur because of the boundary tangency principle. The second case neither because otherwise one can make a vertical translation of one of them to get a tangency between the graphs, with one in one side of the other, contradicting the tangency principle. Hence, in case of equality in some of the inclusions (10), \( \Omega \) is the exterior of a ball of radius \( \rho = \varrho \). Is a particular consequence of the proof of the foliation property, given below, that the solutions \( u_s \) are necessarily the fundamental solutions.

For proving that the graphs of the solutions \( u_s \), \( s \in (-\infty, \infty) \), foliate the open subset \( O \) of \( \mathbb{R}^n \) (defined in (11)) we apply Theorem 2 of [8]. It is enough to prove that any solution \( u \in C^0 (\overline{\Omega}) \) of the minimal surface equation in \( \Omega \) with horizontal end and such that \( u |_{\partial \Omega} = 0 \) coincides with \( u_s \) for some \( s \in [-\infty, \infty] \).

By using Theorems 4.2 and 4.7 of [2], as above, we may conclude that the graph of \( u \) is a \( C^{1,1} \) manifold \( M \) with boundary and, since \( u \in C^0 (\overline{\Omega}) \) is a solution of the minimal surface equation in \( \Omega \), \( M \) is a minimal hypersurface with boundary \( \partial \Omega \) of \( \mathbb{R}^n \). Representing \( M \), locally, as a graph near any given point of \( \partial \Omega \) (= \( \partial M \)), we may use PDE regularity theory to conclude that, indeed, \( M \) is a \( C^{2,\alpha} \) manifold. Moreover, the assumption that \( u \) has horizontal end implies, as already argued before, that \( u \) is bounded and that there exists the limit

\[ d := \lim_{\| x \| \to \infty} u (x) . \]
If $M$ has no vertical tangent space at any point of $\partial \Omega$ then it follows by PDE regularity that $u \in C^{2,\alpha}(\Omega) \cap C^\infty(\Omega)$. Setting $s = \max_{\partial \Omega} \| \nabla u \|$, we can argue as before to prove that $u = u_s$.

Assume that $M$ has a vertical tangent space at some point of $\partial \Omega$. We claim then that $u = u_\infty$ or $u = u_{-\infty}$. We first prove that $d = u_\infty(\infty)$ or $d = u_{-\infty}(\infty)$. By contradiction, first assume that $0 < u_\infty(\infty) < d$.

Arguing with the tangency principle it is easy to see then that $u_\infty \leq u$. But then $u_\infty \in C^0(\overline{\Omega})$ and the graph $G$ of $u_\infty$ is a minimal hypersurface of $C^{2,\alpha}$ class with boundary $\partial \Omega$ which has a vertical tangent space at some point $p \in \partial \Omega$. The hypersurfaces $G$ and $M$ then must have a tangency at $p$. By the boundary tangency principle it follows that $G = M$, contradiction!

If $0 \leq d < u_\infty(\infty)$, since $u_s$ converges uniformly on compacts of $\Omega$ to $u_\infty$, as $s \to \infty$, there is $s$ large enough such that $u_s(\infty) > d$. By using the tangency principle one may see that this leads to a contradiction. For similar reasons one excludes the case $d < u_{-\infty}(\infty)$ and $u_{-\infty}(\infty) < d \leq 0$.

It then follows that $d = u_\infty(\infty)$ or $d = u_{-\infty}(\infty)$ from what one easily obtains, from the tangency principle once more, that $u = u_\infty$ or $u = u_{-\infty}$. This concludes with the proof of the theorem. \(\square\)

Remarks.

(a) It is true that the graph of the limit solution $u_\infty$ of the EDP in $\mathbb{R}^2$ is a $C^{1,\alpha}$ surface with boundary. Moreover, it holds $u_\infty \in C^0(\overline{\Omega})$ in this case \[15\]. In higher dimensions, as proved in the Theorem \[1\] the graph of the solution $u_\infty$ is part of a $C^{1,1}$ manifold with boundary $\partial \Omega$. However, we do not know if $u_\infty \in C^0(\overline{\Omega})$. The $2$–dimensional case is studied in \[15\] using classical Plateau’s problem technique which is typically $2$–dimensional.
(b) The EDP for the minimal surface equation is studied in the Riemannian setting in [1] and [3].

REFERENCES

[1] A. Aiolfi, J. Ripoll, M. Soret: The Dirichlet problem for the minimal hypersurface equation on arbitrary domains of a Riemannian manifold, Manuscripta Mathematica, Vol. 149, 2016, 71–81.
[2] T. Bourni: $C^{1,\alpha}$ Theory for Prescribed Mean Curvature Equation with Dirichlet Data, J Geom Anal, Vol. 21, 2011, 982–1035.
[3] N. do Espirito-Santo, J. Ripoll: Some existence results on the exterior Dirichlet problem for the minimal hypersurface equation, Ann. I. H. Poincaré/An Non Lin, Vol. 28, 2011, 385–393.
[4] F. Fontanele, S. Silva: A Tangency principle and applications, Illinois Journal of Mathematics, Vol 45, N 1, 213–228, 2001
[5] D. Gilbarg, N. Trudinger: Elliptic Partial Differential Equations of Second Order, Springer, Berlin, 1998.
[6] E. Giusti: Minimal Surfaces and Functions of Bounded Variation, Monographs in Mathematics, Birkhäuser, 1984
[7] R. Krust: Remarques sur le problème extérieur de Plateau, Duke Math. J., Vol. 59, 1989, 161-173.
[8] E. Kuwert: On solutions of the exterior Dirichlet problem for the minimal surface equation, Ann. Inst. Henry Poincaré, Vol. 10, N. 4, 1993, 445–451.
[9] N. Kutev, F. Tomi: Existence and non existence for the exterior Dirichlet problem for the minimal surface equation in the plane, Journal of Differential and Integral Equations, Vol. 11, N. 6, 1998, 917 – 92822.
[10] R. López: Constant mean curvature surfaces with boundary, Springer, Berlin, 2013.
[11] M. Miranda: Dirichlet problem with $L^1$ data for the non-homogeneous minimal surface equation, Indiana Univ. Math. J., Vol. 24, 1974/1975, 227–241.
[12] J. C. C. Nitsche: Vorlesungen über Minimalflächen, Grundlagen der mathematischen Wissenschaften 199, Springer-Verlag, Berlin, 1975.
[13] R. Osserman, A Survey of Minimal Surfaces, Van Nostrand Reinhold Math. Studies 25, New York, 1969.
[14] J. Ripoll: Some characterization, uniqueness and existence results for euclidean graphs of constant mean curvature with planar boundary, Pacific Journal of Mathematics, Vol. 198, N. 1, 2001, 175-196.
[15] J. Ripoll, F. Tomi: On solutions to the exterior Dirichlet problem for the minimal surface equation with catenoidal ends, Advances in Calculus of Variations, Vol. 7, N. 2, 2014, 205 - 226.
[16] J. Ripoll, F. Tomi: Notes on the Dirichlet problem of a class of second order elliptic partial differential equations on a Riemannian manifolds, Ensaios Matemáticos, Brazilian Math Soc, Vol. 32, 2018, 1 - 64.
[17] R. Shoem: Uniqueness, symmetry and embeddedness of minimal surfaces, J. Diff. Geom, Vol. 18, 1983, 577 - 591.
[18] L. Simon: Asymptotic behaviour of minimal graphs over exterior domains, Annales de l’I. H. P., section C, tome 4, N. 3, 1987, 231-242.
Ari Aiolfi
Universidade Federal de Santa Maria
ari.aiolfi@ufsm.br
Brazil

Daniel Bustos
Universidad del Tolima
dfbustosr@ut.edu.co
Colombia

Jaime Ripoll
Universidade Federal do Rio Grande do Sul
jaime.ripoll@ufrgs.br
Brazil