Abstract

In the coded caching problem, as originally formulated by Maddah-Ali and Niesen, a server communicates via a noiseless broadcast link to multiple users that have local storage capability. In order for a user to decode the desired file from the coded multicast transmission, the demands of all the users must be globally known, which may violate the privacy of the users. To overcome this privacy problem, Wan and Caire recently proposed several schemes that attain coded multicasting gain while simultaneously guarantee information theoretic privacy of the users’ demands. In Device-to-Device (D2D) networks, the demand privacy problem is further exacerbated by the fact that each user is also a transmitter, which should know the files demanded by the remaining users in order to form its coded multicast transmissions. This paper solves this seemingly infeasible problem with the aid of a trusted server. Specifically, during the delivery phase, the trusted server collects the users’ demands and sends a query to each user, who then broadcasts coded multicast packets according to this query. The main contribution of this paper is the development of novel achievable schemes and converse bounds for

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D2D private caching with a trusted server, where users may be colluding (i.e., when some users share cached content and demanded file indices), that are to within a constant factor of one another.

First, a D2D private caching scheme is proposed, whose key feature is the addition of virtual users in the system in order to ‘hide’ the demands of the real users. By comparing the achievable load with an existing converse bound for shared-link caching without demand privacy constraint, the proposed scheme is shown to be order optimal, except for the very low memory size regime.

Second, for the D2D private caching problem with two users, a new scheme and a new converse bound under the constraint of uncoded cache placement (i.e., when each user stores directly a subset of the bits of the library) are developed, and shown to be to within a constant factor of one another for all memory regimes. To the best of our knowledge, this is the first converse bound for caching problems that genuinely accounts for the demand privacy constraint and is the key theoretical novelty of this work.

Finally, under the constraint of uncoded cache placement and user collusion, the two-user converse bound is extended to the general case of $K$ users, and by this resulting converse bound we are able to prove the order optimality within a constant factor gap in all memory regimes.

I. INTRODUCTION

Internet data traffic has grown dramatically in the last decade because of on-demand video streaming. From the fact that the users’ demands concentrate on a relatively limited number of files (e.g., latest films and shows) and that the price of memory components in the devices is usually negligible compared to the price of bandwidth, caching becomes an efficient and promising technique for future communication systems [3], which leverages the device memory to store data so that future requests for that data can be served faster.

Coded caching was originally proposed by Maddah-Ali and Niesen (MAN) for shared-link networks [4]. In the MAN model, a server has access to a library of $N$ equal-length files and is connected to $K$ users through an error-free broadcast link. Each user can store up to $M$ files in its cache. A caching scheme includes placement and delivery phases that are designed so as to minimize the worst-case load (i.e., the number of files sent on the shared link that suffices to satisfy every possible demand vector). In the original MAN model, no constraint is imposed in order to limit the amount of information that the delivery phase leaks to a user about the demands of the remaining users. Such a privacy constraint is critical in modern broadcast services, such as peer-to-peer networks, and is the focus of this paper.
In order to appreciate the main contributions of our work, in the next sub-section we briefly review the various models of coded caching studied in the literature, which will lead to the novel problem formulation in this paper.

A. Brief Review of Coded Caching Models

Table I shows relevant known results and new results for various coded caching models. The complete memory-load tradeoff is obtained as the lower convex envelope of the listed points. These results are valid for any system parameters \((N, K)\); other results that may lead to better tradeoffs but only apply to limited parameter regimes are not reported for sake of space.

TABLE I: Achievable worst-case loads for various coded caching models. \(d\{k\} \) is the resulting vector by removing the \(k\)th element from vector \(d\). \(N_e(d\{k\})\) represents the number of distinct elements in vector \(d\{k\}\).

| (M, R) | No Privacy | With Privacy |
|-------|------------|--------------|
| Shared-link | \(t = \frac{Nt}{K} + \sum_{k \in K} \min_1^{(K-1)} (\frac{K-t+1}{(K-t)}) \) | \(t = \frac{Nt}{K} + \sum_{k \in K} \min_1^{(K-1)} (\frac{K-t+1}{(K-t)}) \) |
| D2D | \(t = \frac{Nt}{K} + \sum_{k \in K} \min_1^{(K-1)} (\frac{K-t+1}{(K-t)}) \) | \(t = \frac{Nt}{K} + \sum_{k \in K} \min_1^{(K-1)} (\frac{K-t+1}{(K-t)}) \) |

1) Shared-link networks without privacy constraints: In the MAN placement phase, letting \(t = \frac{KM}{N} \in [0 : K]\) represent the number of times a file can be copied in the network’s aggregate memory (excluding the server), each file is partitioned into \(\frac{K}{t+1}\) equal-length subfiles, each of which is cached by a different \(t\)-subset of users. In the MAN delivery phase, each user demands one file. According to the users’ demands, the server sends \(\frac{K}{t+1}\) MAN multicast messages, each of which has the size of a subfile and is useful to \(t+1\) users simultaneously. The load of the MAN coded caching scheme is thus \(R = \frac{K}{t+1} = \frac{K-t+1}{t+1}\). The MAN scheme is said to achieve

\[R = \frac{K}{t+1} = \frac{K-t+1}{t+1}\]

In the MAN caching scheme, in order to allow the users to decode their demanded files, the composition of each multicast message sent by the server must be broadcasted along with the multicast message itself. This is akin to the “header” in linear network coding, that defines the structure of the linear combination coded messages to enable decoding. Such composition requires the broadcast of “metadata” along the coded messages. Since the size of the metadata is negligible with respect to the size of the messages in the information theoretic limit of large file size, the metadata overhead can be neglected when computing the achievable load.
a global coded caching gain, also referred to as multicasting gain, equal to $t + 1$ because the load with uncoded caching $R_{\text{uncoded}} = K - t = K(1 - M/N)$ is reduced by a factor $t + 1$. This gain scales linearly with network’s aggregate memory size. Yu, Maddah-Ali, and Avestimehr (YMA) in [5] proved that $\left(\frac{K - N_e(d)}{t+1}\right)$ of the MAN multicast messages are redundant when a file is requested simultaneously by multiple users, where $N_e(d) \in [\min(N, K)]$ is the number of distinct file requests in the demand vector $d \in [N]_K$. The YMA scheme is known to be exactly optimal under the constraint of uncoded cache placement [5], and order optimal to within a factor of 2 otherwise [8], for both worst-case load and average load when files are requested independently and equally likely. The converse bound under the constraint of uncoded cache placement for the worst-case load was first derived by a subset of the authors in [9], [10] by exploiting the index coding acyclic converse bound in [11]. For the case $N \geq K = 2$, the exact optimality was characterized in [12] by a non-trivial converse bound leveraging the symmetry in the coded caching problem.

2) Shared-link networks with privacy constraints: For the successful decoding of a MAN multicast message, the users need to know the composition of this message (i.e., which subfiles are coded together). As a consequence, users are aware of the demands of other users. In practice, schemes that leak information on the demand of a user to other users are highly undesirable. For example, this may reveal critical information on user behavior, and allow user profiling by discovering what types of content the users request. Shared-link coded caching with private demands, which aims to preserve the privacy of the users’ demands from other users, was originally discussed in [13] and recently analyzed information-theoretically by Wan and Caire (WC) in [6].

Relevant to this paper is the private coded caching scheme based on virtual user proposed in [6], which operates a MAN scheme as if there were KN users in total, i.e., NK − K virtual users in addition to the K real users, and the demands of the virtual users as set such that each of the N files is demanded exactly K times. This choice of demands for the virtual users is such that any real user “appears” to have requested equally likely any of the files from the view point of any other user, which guarantees the privacy of the demands. A straightforward improvement of the WC scheme is obtained by replacing the MAN delivery with the YMA delivery, as done in [14]. Compared to converse bounds for the shared-link model without privacy constraint, it can be shown that this scheme based on virtual users is order optimal in all regimes, except for $K < N$ and $M < \frac{N}{K}$ [6]; the problem in this regime can be intuitively understood as follows: for
$M = 0$ the WC achievable load is $N$ while the converse bound is $\min(K, N) = K$; the ratio of this two numbers can be unbounded.

To the best of our knowledge, the only converse bound that truly accounts for privacy constraints was proposed in [15] for the case $K = N = 2$. By combining the novel converse bound in [15] with existing bounds without privacy constraint, the exact optimality for $K = N = 2$ is characterized in [15].

3) **D2D networks without privacy constraints:** In practice, the content of the library may have been already distributed across the users’ local memories and thus can be delivered through peer-to-peer or Device-to-Device (D2D) communications. The shared-link coded caching model was extended to D2D networks by Ji, Caire, and Molisch (JCM) [16]. In the D2D delivery phase, each user broadcasts packets, as functions of its cached content and the users’ demands, to all other users. The D2D load is the sum of the bits sent by the all users normalized by the file length.

With the MAN cache placement, the JCM coded caching scheme further partitions each MAN subfile into $t$ equal-length sub-subfiles. Each user then acts as a shared-link server to convey its assigned sub-subfiles to the remaining users either with the MAN delivery [16] or the YMA delivery [7]. Yapar et al. (YWSC) in [7] proved that this scheme, which effectively splits the D2D network into $K$ parallel shared-link models, each having $N$ files and serving $K - 1$ users with memory size $t - 1$, is order optimal to within a factor of 4, and exactly optimal under the constraint of uncoded cache placement and one-shot delivery.

4) **D2D networks with privacy constraints: novel model including a trusted server:** In D2D networks, the demand privacy problem is further exacerbated by the fact that each user is also a transmitter, which must know the demands of the other users in order to form its coded multicast transmissions. This observation seems to suggest that privacy is impossible in D2D caching models. This paper solves this seemingly infeasible problem with the aid of a trusted server. This trusted server is connected to each user through an individual secure link and without access to the library, as illustrated in Fig. [1]. The placement phase is the same for the shared-link and D2D caching models. In the delivery phase, each user first informs the trusted server about the index of the demanded file. After collecting the information about the users’ demands and

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2 The delivery phase is called “one-shot” if any user can recover any requested bit from the content of its own cache and the transmitted messages by at most one other user.
Fig. 1: The formulated D2D private caching problem with a trusted server and $K = 3$ users.

the cached content, the trusted server sends a query to each user. Given the query, each user then broadcasts packets accordingly. *The trusted server acts only as a coordinator to warrant the demand privacy, but does not support any large load of communication.* The demands and the control commands to tell the users what to send can be seen as protocol information, requiring a communication load negligible with respect to the actual file transmission. Hence, the load of the system is still only supported by D2D communication. The objective of this paper is to design a two-phase D2D private caching scheme for $K$ users, $N$ files and memory size $M \geq N/K$ (so that the aggregate cache in the entire network suffices to store the entire library) with minimum number of transmitted bits by all users in the delivery phase, while preserving the privacy of the users’ demands against the other users.

The privacy of the user’s demand has been considered in the Private Information Retrieval (PIR) problem [17]. In the PIR setting, a user wants to retrieve a desired file from some distributed non-colluding databases (servers), and the objective is to prevent any server from retrieving any information about the index of the user’s demanded file. Recently, the authors in [18] characterized the information-theoretic capacity of the PIR problem by proposing a novel converse bound and a coded PIR scheme based on interference alignment. The $T$-privacy PIR problem with colluding servers were originally considered in [19], where it is imposed that any $T$-subset of queries sent from the user cannot reveal any information about the demand. The
$T$-robust PIR problem with at most $T$ colluding servers where each server has a local storage was considered in [20], [21]. Since D2D communications have not been considered in the PIR literature, the D2D caching problem with private demands treated here is not a special case of any existing PIR problem.

B. Contributions

We organize the main contributions of the paper into three logical classes.

a) Results for general $(N, K)$ from clever extensions of past works: we prove a constant gap result for all parameter regimes with the exception of the small memory regime.

b) Results specifically for the case $K = 2$ under uncoded cache placement: we prove the first known general converse bound under uncoded cache placement that accounts for privacy constraints and leads to a constant gap result for any number of files and any memory regime.

c) Results for general $(N, K)$ under uncoded cache placement and user collusion: we leverage the novel converse bound for the two-user case and prove a constant gap result for all parameter regime, while at the same time considering the setting of user collusion.

More specifically, our contributions are as follows.

a) General $(N, K)$ by extending past works: We start by giving the first known information-theoretic formulation of the D2D coded caching problem with demand privacy, for which we propose:

(a.1) Coded Scheme A (Theorem 1): We propose a scheme that carefully combines the idea of introducing virtual users [6] with that of splitting the D2D network into multiple parallel shared links [7].

(a.2) Optimality (Theorem 2): By comparing Scheme A with a converse bound for the shared-link model without the privacy constraint in [8], we prove that Scheme A is order optimal within a factor of 6 when $N \geq K$ and $MK/N \geq 2$, and within a factor of 12 when $N < K$ and $MK/N \geq 1$.

b) Case $K = 2$: novel converse bound to truly account for privacy constraints: At this point the regime $N > K$ and $MK/N \in [1, 2)$ is open, which motivates the in-depth study of the simplest open case, namely the two-user case. We propose:

(b.1) Coded Scheme B (Theorem 3): We propose a scheme that strictly outperforms Scheme A for the two-user case.
(b.2) **Novel Converse (Theorem 4):** Inspired by the converse bounds for non-private shared-link caching models under uncoded cache placement from [10] and for PIR systems from [18], we propose a novel converse bound under uncoded cache placement for the two-user case by fully considering the privacy constraint.

(b.3) **Optimality (Theorem 5):** With the novel converse bound, under the constraint of uncoded cache placement and \( N \geq K = 2 \), we show that Scheme B is exactly optimal when \( M \in [N/2,(N+1)/2] \) or \( M \in \left[ \frac{N(N-5)}{2(2N-3)}, N \right] \), and is order optimal within a factor of 3 (numerical simulations suggest 4/3) for the remaining memory size regime.

c) **General \((N,K)\): order optimality results for any system parameter when users may collude:**

With the above results for the two-user case, we can consider a more ‘robust’ notion of privacy that allows for colluding users. We propose:

(c.1) **Novel Converse (Theorem 6):** We extend the proposed two-user converse bound to the \( K \)-user system by dividing the \( K \) users into two groups, and derive a converse bound under uncoded cache placement and user collusion.

(c.2) **Optimality (Theorem 7):** Under the constraint of uncoded cache placement and user collusion, Scheme A is shown to be order optimal within a factor of 18 (numerical simulations suggest 27/2) when \( N > K \) and \( MK/N \in [1,2) \). This proves that Scheme A is order optimal in all memory regimes (that is, also in the regime that was open when we used as converse bound the one for the non-private shared-link model) and it is robust to colluding users.

**Remark 1** (The powerfulness of the two-user converse bound). *It is rather surprising and quite remarkable to see that in the considered problem, the converse for the case of \( K = 2 \) users with a clever cut-set like extension yields an order optimality result for any system parameters under the constraint of uncoded cache placement and user collusion. This is in stark contrast to plenty of famous multiuser information theory problems where the optimality results for the \( K = 2 \) case do not generalize, and give in fact little or no hint to the \( K > 2 \) case. Paramount examples include the general broadcast channel with degraded message sets [22], [23], the \( K \)-user Gaussian interference channel [24], [25], and the non-private shared-link coded caching [12].*

**Remark 2** (Cost of D2D). *By using the recent result in [26], one can immediately infer that, under the constraint of uncoded cache placement and without privacy constraint, the gap between*
the achieved loads in the shared-link and D2D scenarios is at most 2. This is no longer the case
when privacy is introduced, where the gap between the loads in private shared-link and private
D2D scenarios can be arbitrarily large (i.e., the gap is larger than $N/ \min(N, K)$ when $M = N/K$,
which can be unbounded). Similar observations were made in the context of secure shared-link
pliable index coding [27], where the authors showed that problems that are feasible without
security constraints became infeasible when security is considered (i.e., the gap is infinite).

C. Paper Organization

The rest of this paper is organized as follows. Section II formulates the D2D private caching
model with a trusted server. Section III lists all our technical results in this paper, and provides
some numerical evaluations. Sections IV and V provide proofs of the proposed achievable
schemes and converse bounds, respectively. Section VI concludes the paper. Some proofs (i.e.,
more technical lemmas and tedious gap derivations) may be found in the Appendices.

D. Notation Convention

Calligraphic symbols denote sets, bold symbols denote vectors, and sans-serif symbols denote
system parameters. In general, lower-case symbols denote realizations of random variables
indicated with upper-case symbols. We use $|\cdot|$ to represent the cardinality of a set or the
length of a vector. Sets of consecutive integers are denoted as $[a : b] := \{a, a + 1, \ldots, b\}$ and
$[n] := [1 : n]$. The symbol $\oplus$ represents bit-wise XOR. $a! = a \times (a - 1) \times \cdots \times 1$ represents the
factorial of $a$. We use the convention $\binom{x}{y} = 0$ if $x < 0$ or $y < 0$ or $x < y$.

II. System Model

A $(K, N)$ D2D private caching system with a trusted server is defined as follows. The library
contains $N$ independently generated files, denoted by $(F_1, F_2, \ldots, F_N)$, where each file is com-
posed of $B$ i.i.d. bits, where $B$ is assumed sufficiently large such that any subfile division of the
files is possible. There are $K$ users in the system, each of which is equipped with a local cache.
There is a trusted server without access to the library in the system. This server is connected
to each user through an individual secure link. In addition, there is also a broadcast link from
each user to all other users (e.g., a shared medium). We only consider the case $\min(K, N) \geq 2$.

3We assume a collision avoidance protocol for which when a user broadcasts, all the others stay quiet and listen (e.g., this
can be implemented in a practical wireless network using CSMA, as in the IEEE 802.11 standard).
since the case $K = 1$, a single node network, does not make sense in a D2D network and when $N = 1$ each user knows the demand of the other users.

Let $\epsilon_B \geq 0$ be a constant. The system operates in two phases.

**Placement Phase.** Each user $k \in [K]$ stores content in its cache without knowledge of later demand. We denote the cache of user $k \in [K]$ by

$$Z_k = (\mathcal{M}(C_k), C_k),$$

where $C_k$ represents the cached content, a function of the $N$ files, and $\mathcal{M}(C_k)$ represents the metadata/composition of $C_k$ (i.e., how $C_k$ is generated). For example, if $C_k$ is $(F_1, F_2)$, $\mathcal{M}(C_k)$ reveals that the first $B$ bits in $C_k$ are $F_1$ and second $B$ bits in $C_k$ are $F_2$. We have

$$H(C_k|\mathcal{M}(C_k), F_1, \ldots, F_N) = 0 \text{ (placement constraint)},$$

i.e., $C_k$ is a deterministic function of the library and of the metadata describing the cache encoding. Note that $\mathcal{M}(C_1), \ldots, \mathcal{M}(C_K)$ are random variables over $C_1, \ldots, C_K$, representing all types of cache placement which can be used by the $K$ users. In addition, for any $k \in [K]$, the realization of $\mathcal{M}(C_k)$ is known by user $k$ and the trusted server, and is not known by other users. As explained in Footnote [1], the size of the metadata is negligible with respect to the file size $B$. The cached content of user $k \in [K]$ in (1) is constrained by the cache size as

$$H(Z_k) \leq B(M + \epsilon_B) \text{ (cache size constraint)},$$

where $M \in \left[\frac{N}{K}, N\right]$ is the cache size. Note that for $M \geq N$ each user can cache the whole library, thus no delivery is needed; and for $MK < N$ there is not enough space in the overall network memory to store the whole library, thus the problem is not feasible.

**Delivery Phase.** During the delivery phase, each user $k \in [K]$ demands the file with index $d_k$, where $d_k$ is a realization of the random variable $D_k$ with range in $[N]$. The demand vector of the $K$ users, denoted by $D = (D_1, \ldots, D_K)$. The delivery phase contains the following steps:

- **Step 1:** Each user $k \in [K]$ sends the index of its demanded file to the trusted server.
- **Step 2:** According to the users’ demands and the metadata of cached content, the trusted server sends back the metadata $\mathcal{M}(P_k)$ to user $k$, where the metadata $\mathcal{M}(P_k)$ describes how the coded packets (i.e., payload) $P_k$, to be broadcasted by user $k \in [K]$, are composed. Note that $\mathcal{M}(P_1), \ldots, \mathcal{M}(P_K)$ are random variables over $P_1, \ldots, P_K$, representing all types of transmissions by the $K$ users.
\[ \text{Step 3: each user } k \in [K] \text{ broadcasts } X_k = (\mathcal{M}(P_k), P_k) \text{ to other users based only on the its local storage content } Z_k \text{ and the metadata } \mathcal{M}(P_k), \text{ that is} \]
\[ H(X_k | \mathcal{M}(P_k), Z_k) = 0 \text{ (encoding constraint).} \]  

Decoding. Let \( X := (X_j : j \in [K]) \) be the vector of all transmitted signals. To guarantee successful decoding at user \( k \in [K] \) it must hold that
\[ H(F_{D_k} | X, Z_k, D_k) \leq B \epsilon_B \text{ (decoding constraint),} \]
and to guarantee privacy it must hold
\[ I(D; X, Z_k | D_k) = 0 \text{ (privacy constraint).} \]

The privacy constraint in (6) corresponds to perfect secrecy in an information theoretic sense (see [23, Chapter 22]).

Objective. We say that load \( R \) is achievable if (recall that the size of the metadata is negligible with respect to the file size \( B \))
\[ \sum_{k \in [K]} H(X_k) \leq B(R + \epsilon_B) \text{ (load),} \]
while all the above constraints are satisfied and \( \lim_{B \to \infty} \epsilon_B = 0 \). The objective is to determine, for a fixed \( M \in \left[ \frac{N}{K}, N \right] \), the minimum achievable load, which is indicated by \( R^*(M) \).

Uncoded Cache Placement. If each user directly copies some bits of the files into its cache, the cache placement is said to be uncoded. The minimum load under the constraint of uncoded cache placement is denoted by \( R^*_u(M) \).

Colluding Users. We say that the users in the system collude if they exchange the indices of their demanded files and their cached content. Collusion is a natural consideration to increase the privacy level and is one of the most widely studied variants in the PIR problem [19], [28]–[30]. Privacy constraint against colluding users is a stronger notion than (6) and is defined as follows
\[ I(D; X, \{Z_k : k \in S\} | \{D_k : k \in S\}) = 0, \forall S \subseteq [K], S \neq \emptyset. \]

The optimal load under uncoded cache placement and the privacy constraint in (8) is denoted by \( R^*_{u,c}(M) \).

Remark 3. Obviously, for all cache sizes we have \( R^*_{u,c}(M) \geq R^*_u(M) \geq R^*(M) \). For \( K = 2 \), the privacy constraints in (6) and (8) are equivalent, and thus we have \( R^*_u(M) = R^*_a(M) \).
In the rest of the paper, to simplify the notations we use $R^*$, $R_u^*$, and $R_{u,c}^*$ to represent $R^*(M)$, $R_u^*(M)$, and $R_{u,c}^*(M)$, respectively.

III. MAIN RESULTS

A. Results for general $(N,K)$ by extending past works

Inspired by the virtual-user strategy in [6], [13], we propose a private coded caching scheme (referred to as Scheme A) with a novel cache placement. More precisely, We divide the D2D network into $K$ independent shared-link caching models, each of which serves

$$U := (K - 1)N$$

effective users, of which $(K - 1)(N - 1)$ users are virtual. In addition, instead of assigning one demand to each virtual user in the D2D scenario, we assign one demand to each virtual user for each of the $K$ divided shared-link models, such that each file is demanded by $K - 1$ effective users to be served in this shared-link model. The achieved load is given in the following theorem and the detailed description on the proposed scheme can be found in Section IV-A.

Theorem 1 (Scheme A). For the $(K,N)$ D2D private caching system with a trusted server, $R_{u,c}^*$ is upper bounded by the lower convex envelope of the following points

$$(M, R_A) = \left( \frac{N + t - 1}{K}, \frac{U \choose t} \right) - \left( \frac{U - N}{U \choose t} \right), \quad \forall t \in [U + 1], \ U = (K - 1)N. \quad (10)$$

By comparing Scheme A in Theorem 1 and the converse bound for the shared-link caching problem without privacy constraint in [8], we have the following order optimality results, whose proof can be found in Appendix D.

Theorem 2 (Order optimality of Scheme A). For the $(K,N)$ D2D private caching system with a trusted server, Scheme A in Theorem 1 is order optimal within a factor of $6$ if $N \geq K$ and $M \geq 2N/K$, and within a factor of $12$ if $N \leq K$. 

By comparing Scheme A in Theorem 1 and the converse bound for the shared-link caching problem without privacy constraint in [8], we have the following order optimality results, whose proof can be found in Appendix D.

Note that Scheme A satisfies the robust privacy constraint in (8) against colluding users.

Remark 4 (Reduction of Subpacketization for Scheme A). Scheme A in Theorem 1 divides each file into $K \left( \frac{U}{t-1} \right)$ equal-length subfiles, thus the subpacketization is $K \left( \frac{U}{t-1} \right) \approx K^2 \left( \frac{U}{t-1} \right)$, where
$H(p) = -p \log_2(p) - (1-p) \log_2(1-p)$ is the binary entropy function. Hence, the maximal subpacketization of the virtual-user scheme (when $t-1 \leq \frac{1}{2}$) is exponential in $U$, which is much higher than the maximal subpacketization of the $K$-user MAN coded caching scheme (which is exponential in $K$). Very recently, after the original submission of this paper, the authors in [31] proposed a shared-link private coded caching scheme based on the cache-aided linear function retrieval [32], which can significantly reduce the subpacketization of the shared-link virtual-user private caching scheme in [6]. In addition to the cached content by the MAN placement, the authors let each user privately cache some linear combinations of uncached subfiles in the MAN placement which are regarded as keys. In such way, the effective demand of each user in the delivery phase becomes the sum of these linear combinations and the subfiles of its desired file, such that the real remand is concealed. We can directly use the extension strategy in [16] to extend this shared-link private caching scheme to our D2D setting to obtain Scheme C, which achieves the lower convex envelope of $\left(\frac{K}{N}, N\right)$ and the following points

$$\left(M, R_C\right) = \left(\frac{t(N-1)}{K} + 1, \frac{(K-1)}{t} - \frac{(K-1-N)}{t} \left(\frac{K-1}{t-1}\right)\right), \forall t \in [K].$$

The subpacketization of the scheme in (11) is $K\left(\frac{K}{N}\right) \approx K2^{K\left(t/K\right)}$, which is the same as the $K$-user non-private D2D coded caching scheme in [16]. As the shared-link private caching scheme in [31], Scheme C also satisfies the robust privacy constraint in (8) against colluding users. □

B. Results for $K = 2$: novel converse bound to truly account for privacy constraints

The order optimality results in Theorem 2 is derived from an existing converse bound without privacy constraint and does not cover the regime $N > K$ and $M \in [N/K, 2N/K]$. Hence, we are motivated to derive a novel converse bound by fully incorporating the privacy constraint for the simplest open case, that is, for a two-user system.

When $K = 2$, we observe that in Scheme A some cached content is redundant; by removing those redundancies we derive a new scheme (referred to as Scheme B), whose detailed description can be found in Section IV-B.

**Theorem 3** (Scheme B). For the $(K, N) = (2, N)$ D2D private caching system with a trusted server, $R^*_u = R^*_{u,c}$ is upper bounded by the lower convex envelope of $(M, R_B) = (N, 0)$ and the following points

$$\left(M, R_B\right) = \left(\frac{N}{2} + \frac{Nt'}{2(N + t' - 1)}, \frac{N(N-1)}{(t' + 1)(N + t' - 1)}\right), \forall t' \in [0 : N - 1].$$
Remark 5. By comparing Scheme A for \( K = 2 \) and Scheme B, we find \( R_B \leq R_A \). The proof can be found in Appendix F.

The following converse bound is the key novelty of this paper. It truly accounts for the privacy constraint. The key is to derive several bounds that contain a ‘tricky’ entropy term that needs to be bounded in a non-trivial way; in some bounds this entropy term appears with a positive sign and in others with a negative sign; by linearly combining the bounds, the ‘tricky’ entropy term cancels out. Different from the converse bound in [15] for the shared-link caching with private demands for \( N = K = 2 \), our converse bound focuses on uncoded cache placement and works for any \( N \geq K = 2 \).

We start by introducing an example to illustrate in the simplest possible case the novel ideas needed to derive our novel converse bound.

Example 1 (D2D private caching system with \((K, N, M) = (2, 2, 6/5)\)). In this case, both Scheme A and Scheme B achieve load 7/5. The converse bound under the constraint of uncoded cache placement and one-shot delivery for D2D caching without privacy in [7] gives 4/5\(^4\). In the following, we prove that the load 7/5 is actually optimal for our D2D private caching problem under the constraint of uncoded cache placement.

Assume we have a working system, that is, a system where all encoding, decoding and privacy constraints listed in Section II are met. With a slight abuse of notation, a set operation over cache configurations is meant to represent the set operation over the cached information bits only, i.e., excluding metadata. In addition, each notation of a set or a vector of bits also includes the metadata for these bits. In the following, in order not to clutter the derivation with unnecessary ‘epsilons and deltas’, we shall neglect the terms (such as metadata, etc.) that contribute \( \epsilon_B = o(B) \) when \( B \to \infty \) to a bounds like the one in (15). Finally, without loss of generality (see Remark 7), each user caches a fraction \( M/N = 3/5 \) of each file and each bit in the library is cached by at least one user.

Assume that the cache configurations of the two users are \( Z_1^1 \) and \( Z_2^1 \), where \( Z_1^1 \cup Z_2^1 = \{F_1, F_2\} \). For the demand vector \((d_1, d_2) = (1, 1)\), any working scheme must produce transmitted signals \((X_1, X_2)\) such that the demand vector \((d_1, d_2) = (1, 1)\) can be satisfied. The following

\(^4\) For \( K = 2 \), any D2D caching scheme is one-shot.
observation is critical: because of the privacy constraint, from the viewpoint of user 1, there
must exist a cache configuration of user 2, denoted by \( Z_2^2 \), such that \( Z_1^1 \cup Z_2^2 = \{ F_1, F_2 \} \),
\[ H(X_2|Z_2^2, \mathcal{M}(P_2)) = 0 \], and \( F_2 \) can be decoded from \((X_1, Z_2^2)\). If such a cache configuration
\( Z_2^2 \) did not exist, then user 1 would know that the demand of user 2 is \( F_1 \) from \((Z_1^1, X_1, X_2, d_1)\),
which is impossible in a working private system. Similarly, from the viewpoint of user 2, there
must exist a cache configuration of user 1, denoted by \( Z_1^1 \), such that \( Z_1^1 \cup Z_2^1 = \{ F_1, F_2 \} \),
\[ H(X_1|Z_1^1, \mathcal{M}(P_1)) = 0 \], and \( F_2 \) can be decoded from \((X_2, Z_1^1)\).

From \((Z_1^1, Z_2^2)\), because of Remark 7\(^{\text{v}}\) for each file \( F_i, i \in \{1, 2\} \), we have
\[ |F_i \cap Z_1^1| = \frac{BM}{N} = \frac{3B}{5}, \quad (13a) \]
\[ |F_i \setminus Z_1^1| = |F_i \setminus Z_2^2| = B - \frac{3B}{5} = \frac{2B}{5}, \quad (13b) \]
\[ |F_i \cap Z_1^1 \cap Z_2^1| = \frac{B}{5}. \quad (13c) \]

Similarly, since \( Z_1^1 \cup Z_1^2 = Z_2^1 \cup Z_2^2 = \{ F_1, F_2 \} \), we also must have
\[ |F_i \cap Z_1^1 \cap Z_2^1| = \frac{B}{5}, \quad (13d) \]
\[ F_i \setminus Z_1^1 \subseteq F_i \cap Z_1^1 \cap Z_2^2. \quad (13e) \]

Inspired by the genie-aided converse bound for shared-link caching networks without privacy
in [5], [10], we construct a genie-aided super-user with cached content
\[ Z' = (Z_2^2 \setminus (F_1 \cup Z_1^2)), \quad (14) \]
who is able to recover the whole library from \((X_1, Z')\). Indeed, after file \( F_1 \) is reconstructed
from \((X_1, Z_1^1)\), the combination of \((F_1 \cup Z_1^2)\) and \( Z_2^2 \setminus (F_1 \cup Z_1^2)\) gives \( Z_2^2 \); now, file \( F_2 \) can be reconstructed from \((X_1, Z_2^2)\). Therefore, we have
\[ 2B = H(F_1, F_2) \leq H(X_1, Z') = H(X_1, Z_1^1, Z_2^2 \setminus (F_1 \cup Z_1^2)) \quad (15a) \]
\[ = H(X_1, Z_1^2) + H(Z_2^2 \setminus (F_1 \cup Z_1^2)|X_1, Z_1^2, F_1) \quad (15b) \]
\[ \leq H(X_1) + H(Z_2^2) + H(Z_2^2|Z_1^2, F_1) \quad (15c) \]

\(^{\text{v}}\)Intuitively, with uncoded cache placement, each file is split into disjoint pieces as \( F_i = (F_{i,(1)}, F_{i,(2)}, F_{i,(1,2)}), i \in [2]\),
and the users cache \( Z_1 = \bigcup_{i=1}^{2} F_{i,(1),1}, F_{i,(1,2)} \), \( Z_2 = \bigcup_{i=1}^{2} F_{i,(2),1}, F_{i,(1,2)} \); by symmetry, let \( x \in [0, 1] \) with \( |F_{i,(1)}| = |F_{i,(2)}| = Bx/2 \) and \( |F_{i,(1,2)}| = B(1-x) \) such that \( x/2 + 1 - x = M/N = 3/5 \rightarrow x = 2(1 - M/N) = 4/5 \). In the proof,
one can think of different cache configurations as different ways to split the files.
After considering \( Z_2 \), only the bits in \( F_2 \) are left, and because \( Z_2 \setminus Z_2 = (Z_2 \setminus Z_1) \setminus Z_2 \) following the reasoning leading to (13e); the last step in (15e) follows since the bits in a file are independent.

At this point, we need a bound that can be combined with the one in (15) such that it contains on the right hand side the term \( H(X_2) \), so that \( H(X_1) + H(X_2) \) can be bounded by \( BR_u \), and a term that allows one to get rid of the negative entropy of the random variable

\[
Q := F_2 \cap Z_1 \cap Z_2 \cap Z_2, \tag{16}
\]

which is illustrated in Fig. 2.

In the next step, we will introduce another approach to construct a genie-aided super-user, in order to derive an inequality eliminating \( Q \) in (15e). We then focus on cache configurations \( Z_1 \) and \( Z_2 \), and the transmitted packets \( X_2 \). Recall that \( F_1 \) can be reconstructed from \( (Z_1, X_2) \), and \( F_2 \) can be reconstructed from \( (Z_2, X_2) \). Furthermore, by recalling the definition of \( Q \) in (16), it
can be seen that the bits in \((F_2 \cap Z_1^2) \setminus Q\) are independent of \(X_2\). Hence, \(F_1\) can be reconstructed from \((Z_1^1 \cap F_1, Q, X_2)\). Hence, we can construct a super-user with cached content
\[
Z'' = (Z_1^1 \cap F_1, Z_1^2 \cap F_2, Q),
\]
who can decode both files. Thus
\[
2B = H(F_1, F_2) \leq H(X_2, Z'')
\]
\[
\leq H(X_2) + (Z_1^1 \cap F_1) + H(Z_1^2 \cap F_2) + H(Q).
\]
Finally, by summing (15e) and (18b), we have that any achievable rate under uncoded cache placement must satisfy
\[
R_u \geq \frac{H(X_1) + H(X_2)}{B} \geq \frac{7}{5}.
\]
The bound in (19) shows that Scheme A and Scheme B are indeed optimal for the considered memory point.

Remark 6 (A high-level explanation of Example [1].) The key take-away points in Example [1] are as follows:

- By exploiting the privacy constraints, we note that from the viewpoint of each user \(k\) (i.e., given cache \(Z_k\) and transmitted packets \((X_1, X_2)\)), any demand of the other user is equally possible. Hence, there must exist a cache configuration of the other user that allow for the decoding of any file using the same \((X_1, X_2)\).
- We introduce an auxiliary random variable \(Q\) to represents the set of bits \(F_2 \cap Z_1^1 \cap Z_2^1 \cap Z_2^2\). We then use two different approaches to construct genie-aided super-users to decode the whole library, in such a way that we can get rid of ‘tricky’ entropy term when the various bounds are summed together:

1) In the first approach, we focus on \((X_1, Z_2^1, Z_2^2)\) and construct a genie-aided super-user who can reconstruct the whole library by receiving \(X_1\). The bits in \(Q\) belong to the overlap of \(Z_2^1\) and \(Z_2^2\). Hence, the size of the genie-aided super-user’s cache decreases when \(|Q|\) increases. In other words, the needed transmitted load increases when \(|Q|\) increases (see (15e)).

2) In the second approach, we focus on \((X_2, Z_1^1, Z_1^2)\) and construct a genie-aided super-user who can reconstruct the whole library by receiving \(X_2\). Now the bits in \(Q\) are in the
cache of the super-user. Hence, the size of the genie-aided super-user’s cache increases when $|Q|$ increases. In other words, the needed transmitted load decreases when $|Q|$ increases (see (18b)).

Finally, by summing (15e) and (18b), the effect of $Q$ is fully cancelled, such that we derive (19).

In Section V-A we show how to generalize Example 1 to the case where $K = 2$ and $N \geq 2$, so as to arrive at the following theorem.

**Theorem 4** (Novel converse bound for the two-user system). For the $(K, N)$ D2D private caching system with a trusted server where $N \geq K = 2$, assuming $M = \frac{N}{2} + y$ where $y \in \left[0, \frac{N}{2}\right]$, we have the following bounds

$$R^{\star}_u \geq \left(1 - \frac{3y}{N}\right),$$

$$R^{\star}_u \geq K\left(\frac{1}{2} - \frac{y}{N}\right).$$

By comparing the novel converse bound in Theorem 4 and Scheme B in Theorem 3, we have the following performance guarantees under the constraint of uncoded cache placement (the proofs can be found in Appendix G).

**Theorem 5** (Optimality for the two-user system). For the $(K, N)$ D2D private caching system with a trusted server where $N \geq K = 2$, Scheme B is optimal under the constraint of uncoded cache placement when $\frac{N}{2} \leq M \leq \frac{N+1}{2}$ or $\frac{N(3N-5)}{2(2N-3)} \leq M \leq N$.

In general, under the constraint of uncoded cache placement, Scheme B is order optimal to within a factor of 3 (numerical simulations suggest $\frac{4}{3}$).

From Theorem 5, we can directly derive the following corollary.

**Corollary 1.** For the $(K, N)$ D2D private caching system with a trusted server where $K = 2$ and $N \in \{2, 3\}$, Scheme B is optimal under the constraint of uncoded cache placement.


C. Order optimality results for any system parameter when users may collude

In Section V-B we extend Theorem 4 to any $K \geq 2$ with the consideration of the privacy constraint against colluding users in (8). The main idea is to divide the users into two groups and generate a powerful aggregate user whose cache contains the caches of all users in each group (implying collusion). The derived converse bound is as follows.

**Theorem 6** (Novel converse bound for the $K$-user system). For the $(K, N)$ D2D private caching system with a trusted server where $N \geq K \geq 3$, assuming $M = \frac{N}{K} + \frac{2y}{K}$ where $y \in [0, \frac{N}{2}]$, we have

$$R_{u,c}^* \geq \frac{|K/2|}{|K/2|} \frac{2N/K}{2N/K} \times \text{RHS eq (20)}, \ h \in \{0 : [2N/K - 3]\}, \text{ only active for } N/K \geq 3/2,$$

(23)

$$R_{u,c}^* \geq \frac{|K/2|}{|K/2|} \times \text{RHS eq (21)},$$

(24)

$$R_{u,c}^* \geq \frac{|K/2|}{|K/2|} \times \text{RHS eq (22)}.$$  

(25)

By comparing Scheme A and the combination of the novel converse bound in Theorem 6 and the converse bound for shared-link caching without privacy in [10], we can characterize the order optimality of Scheme A under the constraint of uncoded cache placement and user collusion (the proof can be found in Appendix H).

**Theorem 7** (Order optimality for the $K$-user system). For the $(K, N)$ D2D private caching system with a trusted server where $N \geq K$, Scheme A is order optimal under the constraint of uncoded cache placement and user collusion, within a factor of 18 (numerical simulations suggest $27/2$).

□

Note that when $N < K$, Theorem 2 shows that Scheme A is generally order optimal within a factor of 12. Hence, from Theorems 2 and 7 we can directly have the following conclusion.

**Corollary 2.** For the $(K, N)$ D2D private caching system with a trusted server, Scheme A is order optimal under the constraint of uncoded cache placement and user collusion, within a factor of 18.

□
Remark 7 (On Optimality of Symmetric Placement). To derive the converse bound under the constraint of uncoded cache placement in Example 1, we assumed that every user caches a fraction $\frac{M}{N}$ of each file. This assumption is without loss of generality. Assume that there exists a caching scheme where users cache different fraction of the files. By taking a permutation of $[N]$ and by using the same strategy to fill the users’ caches, we can get another caching scheme. By symmetry, these two caching schemes have the same load. Hence, by considering all possible permutations and taking memory-sharing among all such cache schemes, we have constructed a scheme where every user caches the same fraction of each file, with the same achieved load as the original caching scheme.

In addition, in Example 1, we also assumed the total number of cached bits by each user is exactly $MB$, i.e., the cache of each user is full. Assume that the total number of cached bits by user $k$ is $M_k B$. By reasoning as above, we can prove that for any caching scheme, there must exist a caching scheme where $M_1 = \cdots = M_K$ and with the same load as the above scheme. Furthermore, the converse bounds in Theorem 4 and Theorem 6 derived under the assumption that $M_1 = \cdots = M_K = M$, are non-increasing with the increase of $M$. Hence, the assumption that the total number of cached bits by each user is exactly $MB$ bits, is also without loss of generality.

Hence, in the proof of our novel converse bounds, without loss of generality, we can assume each uses caches a fraction $\frac{M}{N}$ of each file. \hfill \square

Remark 8 (Coded vs Uncoded Cache Placement). For the non-private shared-link coded caching problem in [4], by comparing the optimal coded caching scheme with uncoded cache placement in [5] and the general converse bound in [8], it was proved that the gain of coded cache placement is at most 2. Similarly, for the non-private D2D coded caching problem in [16], by comparing the coded caching scheme with uncoded cache placement in [7] and the general converse bound in [8], it was proved that the gain of coded cache placement is at most 4. However, for the considered D2D private coded caching problem, by comparing the proposed converse bounds under uncoded cache placement and Scheme C (which is with coded cache placement), it is interesting to find that the gain of coded cache placement is not always within a constant gap. More precisely, let us focus on the two-user system and consider $M = \frac{N+1}{2}$. By letting $y = \frac{1}{2}$ and $h = 0$ in (20), we have $R_u^* \geq \frac{N-1}{2}$. By letting $t = 2$ in (11), Scheme C achieves the memory-load pair $(M, R_C) = (\frac{N+1}{2}, 1)$. Hence, we have $\frac{R_u^*}{R_C} \geq \frac{N-1}{2}$, which can be
unbounded.

D. Numerical Evaluations

We conclude the overview of our main results with some numerical evaluations. For the achievable schemes, we plot Scheme A in Theorem 1, Scheme B in Theorem 3 (for the two-user system), and Scheme C in Remark 4 (with coded cache placement). We also plot the converse bound under uncoded cache placement in Theorem 4 for \( K = 2 \) and the converse bound under uncoded cache placement and user collusion in Theorem 6 for \( K \geq 3 \). For sake of comparison, we also plot the converse bound in [8] and the converse bound under the constraint of uncoded cache placement in [10] for shared-link caching without privacy.

In Fig. 3, we consider the case where \( K = 2 \) and \( N = 8 \). Here the converse bounds in [10] and [8] are the same. It can be seen in Fig. 3 that, Scheme B and the proposed converse bound meet for all memories except \( 4.5 \leq M \leq 6 \). When \( 4 < M \leq 5.8 \), Scheme C, with coded cache placement, achieves a lower load than the converse bound under uncoded cache placement in Theorem 4.

In Fig. 4, we consider the case where \( K = 10 \) and \( N = 40 \). It can be seen in Fig. 4 that compared to the converse bound in [10], the proposed converse bound is tighter when \( M \) is small. This is mainly because in the proposed converse bound we treat \( K/2 = 5 \) users as a powerful super-user, which loosens the converse bound when \( M \) grows. However, for the low memory size regime, this strategy performs well and gives the order optimality result of Scheme A, while the gap between the converse bound in [10] and Scheme A is not a constant. Hence, combining the proposed converse bound and the converse bound in [10], we can obtain the order optimality results of Scheme A for any memory size.

In Fig. 5, we consider the case where \( K = 40 \) and \( N = 10 \). It can be seen that the multiplicative gap between Scheme A and the converse bounds for non-private shared-link coded caching problem is within a constant. In addition, Scheme A outperforms Scheme C for any \( M \in [0, N] \).

IV. Achievable Schemes

Note that in the rest of the paper, when we introduce achievable schemes, we directly provide the construction of the payloads and skip the description on their metadata.
A. Proof of Theorem 1: Description of Scheme A

Recall $U = (K - 1)N$ as defined in (9). The main idea is to generate $(K - 1)(N - 1)$ virtual users that are labelled as users $\{K + 1, \ldots, (K - 1)(N - 1) + K\}$.

**Placement Phase.** Each file $F_i$, where $i \in [N]$, is partitioned into $K \binom{U}{i-1}$ equal-length subfiles, denoted by $S_{i,1}, \ldots, S_{i,K \binom{U}{i-1}}$, where each piece has $\frac{B}{K \binom{U}{i-1}}$ bits. For each user $k \in [K]$, we aim to generate the subfiles to be delivered in the $k^{th}$ shared-link model, in which user $k$ broadcasts packets as the server and there are $K - 1$ real user and $(K - 1)(N - 1)$ virtual users to be served.
In other words, there are in total $(K-1)(N-1) + K - 1 = U$ effective users to be served, whose union set is $[(K-1)(N-1) + K] \setminus \{k\}$.

We randomly generate a permutation of $[(k-1)\binom{U}{t-1} + 1 : k\binom{U}{t-1}]$, denoted by

$$P_{i,k} = (p_{i,k}[1], \ldots, p_{i,k}\left[\binom{U}{t-1}\right]),$$

independently and uniformly over the set of all possible permutations. We sort all sets $\mathcal{W} \subseteq [(K-1)(N-1)+K]\setminus\{k\}$ where $|\mathcal{W}| = t-1$, in a lexicographic order, denoted by $\mathcal{W}(1), \ldots, \mathcal{W}\left(\binom{U}{t-1}\right)$.

For each $j \in \left(\binom{U}{t-1}\right)$, we generate a subfile

$$f_{i,j}^k = S_{i,p_{i,k}[j]},$$

which is cached by users in $\{k\} \cup \mathcal{W}(j) \cap [K]$.

Each real user $k \in [K]$ caches all $\binom{U}{t-1}$ subfiles with superscript $k$, and $\binom{U-1}{t-2}$ subfiles with superscript $k'$ for each $k' \in [K] \setminus \{k\}$. Hence, each user caches $\binom{U}{t-1} + (K-1)\binom{U-1}{t-2}$ subfiles, each of which has $\frac{B}{K(t-1)}$ bits, requiring memory size

$$M = \frac{\binom{U}{t-1} + (K-1)\binom{U-1}{t-2}}{K\binom{U}{t-1}} = \frac{(K-1)(t-1) + U}{KU} = \frac{N + t - 1}{K}. \quad (27)$$

Moreover, for each file $i \in [N]$, the random permutations $p_{i,j}$ where $j \in [K]$ are unknown to user $k \in [K]$. Hence, from the viewpoint of user $k$, each of its cached subfiles of $F_i$ with the
same superscript is equivalent from the viewpoint of user \( k \), while each of its uncached subfiles of \( F_i \) with the same superscript is also equivalent.\(^6\)

**Delivery Phase.** Consider any demand vector \((D_1, \ldots, D_K) = (d_1, \ldots, d_K) \in [N]^K\). We divide the transmissions from the \( K \) users into \( K \) shared-link transmissions. Let us focus on the \( k \)th, \( k \in [K] \), shared-link transmission, where user \( k \) serves as the transmitter.

We first assign one demanded file to each virtual user such that each file in the library is demanded by \( K - 1 \) effective users to be served in the \( K \) shared-link transmission (i.e., the effective users in \([ (K - 1)(N - 1) + K \) \) \{\( k \}\)). For each \( k' \in [ (K - 1)(N - 1) + K \) \{\( k \}\), we denote the demanded file of effective user \( k' \) in the \( K \) shared-link transmission by \( d_{k'}^k \).

Hence, for each real user \( k' \in [K] \) \{\( k \}\), let
\[
d_{k'}^k = d_{k'}.
\]

We then define
\[
n_{i,k} := |\{ k' \in [K] \setminus \{ k \} : d_{k'} = i \}|, \quad \forall i \in [N],
\]
which represents the number of real users in \([K] \setminus \{ k \}\) demanding \( F_i \). One file is assigned to each of the \((K - 1)(N - 1)\) virtual users as follows. For each file \( i \in [N] \), we let
\[
d_{1 + K + (i-1)(K-1) - \sum_{q \in [i-1]} n_{q,k}}^k = \cdots = d_{K+i(K-1)-\sum_{q \in [i]} n_{q,k}}^k = i.
\]

For example, when \( i = 1 \), we let
\[
d_{K+1}^k = \cdots = d_{2K-K-1}^k = 1,
\]
when \( i = 2 \), we let
\[
d_{2K-K-1}^k = \cdots = d_{3K-K-2}^k = 2,
\]
and so on. Hence, each file is requested by \( K - 1 \) effective users in the user set \([ (K - 1)(N - 1) + K \) \{\( k \}\). For each file, we randomly and uniformly choose an effective user demanding this file as a leader user. The leader set is denoted by \( L_k \).

We generate a random permutation of \([ (K - 1)(N - 1) + K \) \{\( k \}\), denoted by \( q_k = (q_{k,1}, \ldots, q_{k,U}) \), independently and uniformly over the set of all possible permutations. \( q_k \) is unknown to each real user.

\(^6\) In our paper, the statement that from the viewpoint of a user \( A \) and \( B \) are equivalent, means that given the known information of this user, \( A \) and \( B \) are identically distributed.
For each set $S \subseteq [U]$ where $|S| = t$, by computing $S' = \bigcup_{j' \in S} \{q_{k,j'}\}$, we generate the multicast message

$$W^k_S = \bigoplus_{j \in S} f^k_{d_{q_{k,j}} \setminus \{q_{k,j}\}}$$

The trusted server asks user $k$ to broadcast $X_k$ to other users, where

$$X_k = \left( W^k_S : (\bigcup_{j' \in S} \{q_{k,j'}\}) \cap L_k \neq \emptyset \right) ,$$

Note that in the metadata of $W^k_S$, the set $S$ is revealed.

**Decodability.** We focus on user $k \in [K]$. In the $j^{th}$ transmission where $j \in [K] \setminus \{k\}$, it was shown in [5, Lemma 1], user $k$ can reconstruct each multicast message $W^j_S$ where $S \subseteq [U]$ and $|S| = t$. User $k$ then checks the metadata of each $W^j_S$ where $S \subseteq [U]$ and $|S| = t$. If $W^j_S$ contains $t - 1$ cached subfiles, and one uncached subfile from $F_{d_k}$, user $k$ knows this message is useful to it and decodes the uncached subfile.

It is obvious that each subfile of $F_{d_k}$ which is not cached by user $k$, appears in one multicast message. More precisely, for each $j \in ([K] \setminus \{k\})$ and each $W \subseteq ([K - 1](N - 1) + K \setminus \{k, j\}$ where $|W| = t - 1$, user $k$ can recover the subfile $f^j_{d_k, W}$ from the multicast message $W^j_{W \cup \{k\}}$. Hence, after considering all transmitted packets in the delivery phase, user $k \in [K]$ can recover

$$(K - 1)\binom{(K - 1)(N - 1) + K - 2}{t - 1} = (K - 1)\binom{U - 1}{t - 1}$$

subfiles of $F_{d_k}$ which it has not cached. Recall that user $k$ has cached $\binom{U}{t - 1} + (K - 1)\binom{U - 1}{t - 2}$ subfiles of $F_{d_k}$. Hence, it can recover all the

$$(K - 1)\binom{U - 1}{t - 1} + \binom{U}{t - 1} + (K - 1)\binom{U - 1}{t - 2} = (K - 1)\binom{U}{t - 1} + \binom{U}{t - 1} = K\binom{U}{t - 1}$$

subfiles of $F_{d_k}$.

**Privacy.** In the $k^{th}$ shared-link transmission, since $q_k$ is unknown to each real user, this user cannot know the exact users to whom each multicast message is useful. By the same reason as the shared-link private caching scheme in [6] which is based on virtual users, our proposed D2D private caching scheme satisfies the privacy constraints in (6) and (8). Because of the existence of the $(K - 1)(N - 1)$ virtual users, each file is demanded by exactly $K - 1$ effective users to be served in the $k^{th}$ shared-link system, and from the viewpoint of each real user (including user $k$), all the effective users (except itself) are equivalent.

**Performance.** Each user $k \in [K]$ broadcasts $\binom{U}{t} - \binom{U - N}{t}$ multicast messages, each of which contains $\frac{B}{K\binom{U}{t - 1}}$ bits. Hence, the achieved load coincides with (10).
B. Proof of Theorem 3: Description of Scheme B

We now focus on the two-user system (i.e., $K = 2$) and propose an improved scheme that does not introduce virtual users and removes the redundancy in the placement of Scheme A. Let us first focus on a simple example to illustrate the key insights.

Example 2 ($K = 2$, $N = 3$, and $t = 3$). Scheme A. Each file $F_i$, where $i \in [3]$, is partitioned into 6 subfiles as $F_i = \{S^1_{i,1}, S^1_{i,2}, S^1_{i,3}, S^2_{i,1}, S^2_{i,2}, S^2_{i,3}\}$. User 1 caches $Z_1 = (S^1_{i,1}, S^1_{i,2}, S^1_{i,3}, S^1_{i,1}, S^2_{i,1})$, and user 2 caches $Z_2 = (S^1_{i,1}, S^1_{i,2}, S^2_{i,1}, S^2_{i,2}, S^2_{i,3})$.

In the delivery phase, we assume that the demand vector is $(1, 1)$. The trusted server asks user 1 to transmit

$$X_1 = S^1_{1,3} \oplus S^1_{2,1} \oplus S^1_{3,1}, \quad (33)$$

and user 2 to transmit

$$X_2 = S^2_{1,3} \oplus S^2_{2,1} \oplus S^2_{3,1}. \quad (34)$$

It can be seen that Scheme A is decodable and private. Note that user 2 caches $(S^1_{2,1}, S^1_{2,2})$ but only uses $S^1_{2,1}$ in the decoding procedure. Similarly, user 2 caches $(S^1_{3,1}, S^1_{3,2})$ but only uses $S^1_{3,1}$ in the decoding procedure. In other words, the cached subfiles $S^1_{2,2}$ and $S^1_{3,2}$ are redundant for user 2. Similarly, the cached subfiles $S^2_{2,2}$ and $S^2_{3,2}$ are redundant for user 1. The same is true for any demand vector.

We propose to remove this cache redundancy as follows.

Scheme B. We partition each file $F_i$, where $i \in [3]$, into 4 subfiles as $F_i = \{S^1_{i,1}, S^1_{i,2}, S^2_{i,1}, S^2_{i,2}\}$. User 1 caches $(S^1_{i,1}, S^1_{i,2}, S^2_{i,1})$, and user 2 caches $(S^1_{i,1}, S^2_{i,1}, S^2_{i,2})$.

In the delivery phase, we assume that the demand vector is $(1, 1)$. The trusted server asks user 1 to transmit

$$X_1 = S^1_{1,2} \oplus S^1_{2,1} \oplus S^1_{3,1}, \quad (35)$$

and user 2 to transmit

$$X_2 = S^2_{1,2} \oplus S^2_{2,1} \oplus S^2_{3,1}. \quad (36)$$

Similar to the analysis of Scheme A, Scheme B is decodable and private. The same can be done for any demand vector.
In this example, Scheme B achieves the memory-load pair \((\frac{9}{4}, \frac{1}{2})\). Note that Scheme A achieves the memory-load pairs \((2, 1)\) for \(t = 1\), and \((\frac{5}{2}, \frac{1}{3})\) for \(t = 2\). Hence, by memory-sharing between these two memory-load pairs, Scheme A achieves the load \(\frac{2}{3}\) when \(M = \frac{9}{4}\).

We now ready to provide the general description of Scheme B.

**Placement Phase.** Each file \(F_i\), where \(i \in [N]\), is partitioned in two equal-length parts, denoted as \(F_i = F_{i1} \cup F_{i2}\) where \(|F_{i1}| = |F_{i2}| = B/2\). For each \(k \in [2]\), we further partition \(F_{i}^k\) into \((\binom{N-1}{t'} + \binom{N-2}{t'-1})\) equal-length subfiles, denoted by \(S_{i,1}^k, \ldots, S_{i,N-1}^k\), where each subfile has \(\frac{B}{2((\binom{N-1}{t'}) + (\binom{N-2}{t'-1}))}\) bits. We randomly generate a permutation of \([(\binom{N-1}{t'}) + (\binom{N-2}{t'-1})]\), denoted by \(p_{i,k} = (p_{i,k}[1], \ldots, p_{i,k}[\binom{N-1}{t'} + (\binom{N-2}{t'-1})])\), independently and uniformly over the set of all possible permutations. We let user \(k\) cache all subfiles of \(F_i^k\). In addition, we let the other user \(i\), the user in \([2] \setminus \{k\}\) cache \(S_{i,\{1\},k}[1], \ldots, S_{i,\{N-2\},k}[\binom{N-2}{t'-1}]\).

Considering all the files, each user in total caches \((\binom{N-1}{t'} + 2\binom{N-2}{t'-1})N\) subfiles, requiring memory

\[
M = \frac{\binom{N-1}{t'} + 2\binom{N-2}{t'-1}N}{2((\binom{N-1}{t'}) + (\binom{N-2}{t'-1}))} = \frac{N}{2} + \frac{Nt'}{2(N + t' - 1)}.
\]

**Delivery Phase.** Consider any demand vector \((D_1, \ldots, D_K) = (d_1, \ldots, d_K) \in [N]^K\). We first focus on the transmission by user 1, in charge of delivering the subfiles with superscript 1. For each subset \(S \subseteq [N]\) where \(|S| = t' + 1\), we generate an XOR message containing exactly one subfile of each file in \(S\). More precisely, for each subset \(S \subseteq [N]\) where \(|S| = t' + 1\),

- If \(d_2 \in S\), we pick a non-picked subfile among \(S_{d_2,p_{d_2},1}[\binom{N-2}{t'-1} + 1], \ldots, S_{d_2,p_{d_2},1}[\binom{N-2}{t'-1} + (\binom{N-2}{t'-1})]\). In addition, for each \(i \in S \setminus \{d_2\}\), we pick a non-picked subfile among \(S_{i,\{1\},i}[\binom{N-2}{t'-1} + 1], \ldots, S_{i,\{1\},i}[\binom{N-2}{t'-1} + (\binom{N-2}{t'-1})]\).
- If \(d_2 \notin S\), for each \(i \in S\), we pick a non-picked subfile among \(S_{i,\{1\},i}[\binom{N-2}{t'-1} + 1], \ldots, S_{i,\{1\},i}[\binom{N-2}{t'-1} + (\binom{N-2}{t'-1})]\).

We let \(W_{S}^1\) be the XOR of the picked \(t' + 1\) subfiles, where \(|W_{S}^1| = 2(\binom{N-1}{t'} + (\binom{N-2}{t'-1}))/2(\binom{N-1}{t'} + (\binom{N-2}{t'-1}))\).

We proceed similarly for user 2. We let \(W_{S}^2\) be the binary sum of the picked \(t' + 1\) subfiles, where \(|W_{S}^2| = 2(\binom{N-1}{t'} + (\binom{N-2}{t'-1}))/2(\binom{N-1}{t'} + (\binom{N-2}{t'-1}))\).

Finally, the trusted server asks users 1 and 2 to transmit \(X_1 = (W_3^1 : S \subseteq [N], |S| = t' + 1)\) and \(X_2 = (W_3^2 : S \subseteq [N], |S| = t' + 1)\), respectively.

**Decodability.** We focus on user 1. In each message \(W_3^2\) where \(S \subseteq [N], |S| = t' + 1\), and \(d_1 \in S\), user 1 caches all subfiles except one subfile from \(F_{d_1}\), so user 1 can recover this subfile. Hence, user 1 in total recovers \(\binom{N-1}{t'}\) uncached subfiles of \(F_{d_1}\), and thus can recover \(F_{d_1}\). Similarly, user 2 can also recover \(F_{d_2}\).
Privacy. Let us focus on user 1. Since user 1 does not know the random permutations generated in the placement phase, from its viewpoint, all subfiles in $F^1_i$ where $i \in [N]$ are equivalent. $X_1$ contains $^{(N)}_i$ messages, each of which corresponds to a different $(t'+1)$-subset of $[N]$ and contains exactly one subfile of each file in the subset. Hence, the compositions of $X_1$ for different demands of user 2 are equivalent from the viewpoint of user 1. In addition, $X_2$ is generated independent of $D_2$, and thus $X_2$ cannot reveal any information of $D_2$. As a result, the demand of user 2 is private against user 1. Similarly, the demand of user 1 is private against user 2.

Performance. Each user broadcasts $^N_{(t'+1)}$ messages, each of which contains $B_2^{(N-1)t'+(N-2)t'-1}$ bits. Hence, the achieved load is

$$R = \frac{2^{(t'+1)}}{2^{(N-1)t'+(N-2)t'-1}} = \frac{N(N-1)}{(t'+1)(N+1)}. \quad (38)$$

V. Novel Converse Bounds Under the Constraint of Uncoded Cache Placement

In this section, we provide the proofs of our novel converse bounds in Theorems 4 and 6. We first introduce the proposed converse bound for the two-user system by generalizing Example 1, and then extend it to the $K$-user system.

A. Proof of Theorem 4 Two-user system

We focus on uncoded cache placement. Without loss of generality, each uses caches a fraction $M/M$ of each file (as explained in Remark 7). Let

$$M = \frac{N}{2} + y, \quad (39)$$

where $y \in [0, \frac{N}{2}]$.

Assume the cache configurations of the two users are $(Z^1_1, Z^1_2)$, where $Z^1_1 \cup Z^1_2 = \{F_1, \ldots, F_N\}$. For the demand vector $(d_1, d_2) = (1, 1)$, any achievable scheme must produce transmitted packets $(X_1, X_2)$, such that the demand vector $(d_1, d_2) = (1, 1)$ can be satisfied. By the privacy constraint in (6), from the viewpoint of user 1 with cache configuration $Z^1_1$, there must exist some cache configuration $Z^2_j$ such that $Z^1_1 \cup Z^2_j = \{F_1, \ldots, F_N\}$, $H(X_2|Z^2_j, M(P_2)) = 0$, and $H(X_j|X_1, Z^2_j) = 0$, for any $j \in [N]$; otherwise, user 1 will know that the demand of user 2 is not $F_j$. Similarly, we have the following lemmas.
Lemma 1. For any $i \in [N]$ and $j \in [N]$, there must exist some cache configuration $Z_1^i$ and $Z_2^j$, such that

\[ Z_1^i \cup Z_2^j = Z_1^i \cup Z_2^j = \{F_1, \ldots, F_N\}; \quad (40a) \]

\[ H(X_1|Z_1^i, \mathcal{M}(P_1)) = H(X_2|Z_2^j, \mathcal{M}(P_2)) = 0; \quad (40b) \]

\[ H(F_i|X_2, Z_1^i) = H(F_j|X_1, Z_2^j) = 0. \quad (40c) \]

Lemma 2. From $Z_1^i$ and $Z_2^j$ where $i, j \in [N]$ as in Lemma 1, it must hold

- consider $Z_1^i$ where $i \in [N]$. For any $j' \in [N]$, there must exist a cache configuration denoted by $Z_2^{(i,j')}$ such that $Z_1^i \cup Z_2^{(i,j')} = \{F_1, \ldots, F_N\}$, $H(X_2|Z_2^{(i,j')}, \mathcal{M}(P_2)) = 0$, and $H(F_j'|X_1, Z_2^{(i,j')}) = 0$; and
- consider $Z_2^j$ where $j \in [N]$. For any $i' \in [N]$, there must exist a cache configuration denoted by $Z_1^{(i',j)}$ such that $Z_1^{(i',j)} \cup Z_2^j = \{F_1, \ldots, F_N\}$, $H(X_1|Z_1^{(i',j)}, \mathcal{M}(P_1)) = 0$, and $H(F_i|X_2, Z_2^j) = 0$.

In addition, by definition of Lemma 1

- when $i = 1$, we have $Z_2^{(1,j')} = Z_2^j$ for each $j' \in [N]$; when $j = 1$, we have $Z_1^{(i,1)} = Z_1^i$ for each $i' \in [N]$; and
- when $j' = 1$, we have $Z_2^{(i,1)} = Z_2^j$ for each $i \in [N]$; when $i' = 1$, we have $Z_1^{(1,j)} = Z_1^i$ for each $j \in [N]$.

We can represent the construction of the cache configurations in Lemmas 1 and 2 by an N-ary tree, as illustrated in Fig. 6

- Two vertices (assumed to be represented by cache configurations $Z_1^i$ and $Z_2^j$) are connected by an edge with superscript $(i,j)$, if $Z_1^i \cup Z_2^j = \{F_1, \ldots, F_N\}$, $H(X_1|Z_1^i, \mathcal{M}(P_1)) = H(X_2|Z_2^j, \mathcal{M}(P_2)) = 0$, and $H(F_i|X_2, Z_1^i) = H(F_j|X_1, Z_2^j) = 0$.
- For each $i \in [N]$, $Z_1^i$ is connected to exactly N vertices, which are $Z_2^{(i,j')}$ where $j' \in [N]$.
- For each $j \in [N]$, $Z_2^j$ is connected to exactly N vertices, which are $Z_1^{(i',j)}$ where $i' \in [N]$.

Consider $Z_i^i$ where $i \in [N]$. Recall that $M = N/2 + y$, and that for each $j' \in [N]$, we have $Z_1^i \cup Z_2^{(i,j')} = \{F_1, \ldots, F_N\}$. For each file $F_p$ where $p \in [N]$, by defining

\[ Z_{1,p}^i := Z_1^i \cap F_p, \quad Z_{2,p}^{(i,j')} := Z_2^{(i,j')} \cap F_p, \quad \forall j' \in [N], \quad (41a) \]

we have

\[ |F_p \setminus Z_{1,p}^i| = |F_p \setminus Z_{2,p}^{(i,1)}| = \cdots = |F_p \setminus Z_{2,p}^{(i,N)}| = \frac{B}{2} - \frac{yB}{N}; \quad (41b) \]
For each file $p \in [N]$, we define that

$$Q_{1,p}^i = Z_{1,p}^i \cap Z_{2,p}^{(i,j')} \cap \cdots \cap Z_{N,p}^{(i,N)},$$

and that $q_{1,p}^i = |Q_{1,p}^i|$.

Similarly, focus on $Z_2^j$ where $j \in [N]$, and we have

$$Z_{2,p}^j := Z_2^j \cap F_p, \ Z_{1,p}^{(i,j)} := Z_{1,p}^{(i,j)} \cap F_p, \ \forall i' \in [N];$$

$$|F_p \setminus Z_{2,p}^j| = |F_p \setminus Z_{1,p}^{(1,j)}| = \cdots = |F_p \setminus Z_{1,p}^{(N,j)}| = \frac{B}{2} - \frac{yB}{N};$$

$$|Z_{2,p}^j \cap Z_{1,p}^{(i',j)}| = \frac{2yB}{N}, \ \forall i' \in [N];$$

$$(F_p \setminus Z_{2,p}^j) \subseteq Z_{1,p}^{(i',j)}, \ \forall i' \in [N].$$

For each file $p \in [N]$, we define that

$$Q_{2,p}^j = Z_{2,p}^j \cap Z_{2,p}^{(1,j)} \cap \cdots \cap Z_{2,p}^{(N,j)},$$
and that \( q_{2,p}^2 = |Q_{2,p}^2| \).

After the above definitions, we are ready to prove Theorem 4. As illustrated in Example 1, we will use two different approaches to construct powerful super-users.

**First approach:** Consider \( Z_i^j \) where \( i \in [N] \). We then focus the connected vertices of \( Z_i^j \) in Fig. 6, i.e., \( Z_{2}^{(i,j')} \) where \( j' \in [N] \). By the construction, from \((X_1, Z_2^{(i,j')})\), we can reconstruct \( F_{j'} \). The first approach is inspired by the acyclic index coding converse bound in [5], [10] for shared-link caching without privacy. We pick a permutation of \([N]\), assumed to be \( u = (u_1, \ldots, u_N) \), where \( u_1 = i \). We can construct a genie-aided super-user with the cache

\[
\bigcup_{p \in [N]} Z_{2}^{(i,u_p)} \setminus \big(F_{u_1} \cup \cdots \cup F_{u_{p-1}} \cup Z_{2}^{(i,u_1)} \cup \cdots \cup Z_{2}^{(i,u_{p-1})}\big). \tag{45}
\]

The genie-aided super-user can successively decode the whole library from its cache and \( X_1 \).

More precisely, it can first decode \( F_{u_1} \) from \((X_1, Z_2^{(i,u_1)})\). From \((X_1, F_{u_1}, Z_2^{(i,u_1)} \cup \cdots \cup Z_2^{(i,u_{p-1})})\), then it can decode \( F_{u_2} \). By this way, the genie-aided super-user can decode the whole library. Hence, we have

\[
\begin{align*}
H(F_1, \ldots, F_N) \\
\leq H(X_1) + H \left( \bigcup_{p \in [N]} Z_{2}^{(i,u_p)} \setminus \big(F_{u_1} \cup \cdots \cup F_{u_{p-1}} \cup Z_{2}^{(i,u_1)} \cup \cdots \cup Z_{2}^{(i,u_{p-1})}\big) \right) \\
\leq H(X_1) + H(Z_{2}^{(i,u_1)}) + H(Z_{2}^{(i,u_2)}|F_{u_1}, Z_{2}^{(i,u_1)}) + \cdots + H \left( Z_{2}^{(i,u_{p-1})}|F_{u_1}, \ldots, F_{u_{p-1}}, Z_{2}^{(i,u_1)} \cup \cdots \cup Z_{2}^{(i,u_{p-1})} \right) \\
= H(X_1) + H(Z_{2}^{(i,i)}) + H(Z_{2}^{(i,u_2)}|F_{i}, Z_{2}^{(i,i)}) + \cdots + H \left( Z_{2}^{(i,u_{p-1})}|F_{i}, \ldots, F_{u_{p-1}}, Z_{2}^{(i,i)} \cup \cdots \cup Z_{2}^{(i,u_{p-1})} \right) \\
= H(X_1) + H(Z_{2}^{(i,i)}) + H(Z_{2}^{(i,u_2)}|Z_{2}^{(i,i)}) + \cdots + H \left( Z_{2}^{(i,u_{p-1})}|Z_{2}^{(i,i)} \cup \cdots \cup Z_{2}^{(i,u_{p-1})} \right) \\
= H(X_1) + H(Z_{2}^{(i,i)}) + H(Z_{2}^{(i,u_2)}|Z_{2}^{(i,i)}) + H(Z_{2}^{(i,u_3)}|Z_{2}^{(i,u_2)} \cup Z_{2}^{(i,i)}) + \cdots + H \left( Z_{2}^{(i,u_{p-1})}|\cdots Z_{2}^{(i,u_{p-1})}ight)
\end{align*}
\]

where (46d) comes from that \( u_1 = i \), (46e) comes from that all bits in the library are independent, (46f) comes from the chain rule of the entropy.

From (46f), it will be proved in Appendix A-A and Appendix A-B that (recall \( y = M - N/2 \)),

\[
\begin{align*}
H(X_1) & \geq \frac{B}{2} - \frac{yB}{N}; \\
H(X_1) & \geq \frac{B}{2} - \frac{4yB}{N} + q_{1,u_2}.
\end{align*}
\]
In addition, by considering all permutations of \([N]\) where the first element is \(i\), we can list all \((N - 1)!\) inequalities as in (46f). By summing all these \((N - 1)!\) inequalities, we can obtain the following inequality, which will be proved in Appendix A-C.

\[
H(X_1) \geq \frac{NB}{2} - yB - \frac{4(N - 1)yB}{(h + 2)N} + \frac{2}{h + 2} \sum_{p \in [N] \setminus \{i\}} q^i_{1,p} \tag{49}
\]

\[
- \sum_{i \in [N]} \sum_{p \in [N] \setminus \{i\}} \left\{ \frac{N - 2}{(h + 1)(h + 2)} \left( \frac{2yB}{N} - q_{1,p}^i \right) + \frac{h}{(h + 2)N} \left( \frac{B}{2} - \frac{yB}{N} \right) \right\}, \forall h \in [0 : N - 3].
\]

By considering all \(i \in [N]\), we can list all \(N\) inequalities as in (49). By summing all these \(N\) inequalities, we obtain

\[
H(X_1) \geq \frac{NB}{2} - yB - \frac{4(N - 1)yB}{(h + 2)N} + \frac{2}{(h + 2)N} \sum_{j \in [N]} \sum_{p \in [N] \setminus \{j\}} q^j_{2,p} \tag{50a}
\]

\[
- \sum_{j \in [N]} \sum_{p \in [N] \setminus \{j\}} \left\{ \frac{N - 2}{(h + 1)(h + 2)N} \left( \frac{2yB}{N} - q_{2,p}^j \right) + \frac{h}{(h + 2)N} \left( \frac{B}{2} - \frac{yB}{N} \right) \right\}, \forall h \in [0 : N - 3].
\]

We now consider \(Z_2^j\) where \(j \in [N]\). By the similar step as above to derive (50a), we obtain

\[
H(X_2) \geq \frac{NB}{2} - yB - \frac{4(N - 1)yB}{(h + 2)N} + \frac{2}{(h + 2)N} \sum_{j \in [N]} \sum_{p \in [N] \setminus \{j\}} q^j_{2,p} \tag{51}
\]

\[
- \sum_{j \in [N]} \sum_{p \in [N] \setminus \{j\}} \left\{ \frac{N - 2}{(h + 1)(h + 2)N} \left( \frac{2yB}{N} - q_{2,p}^j \right) + \frac{h}{(h + 2)N} \left( \frac{B}{2} - \frac{yB}{N} \right) \right\}, \forall h \in [0 : N - 3].
\]

By summing (50a) and (51), we obtain

\[
R^*_uB \geq H(X_1) + H(X_2) \geq NB - 2yB - \frac{8(N - 1)yB}{(h + 2)N} - \frac{N - 2}{(h + 1)(h + 2)N} 4y(N - 1)B \]

\[
- \frac{h(N - 1)}{(h + 2)} \left( \frac{B}{2} - \frac{2yB}{N} \right) + \left( \frac{2}{(h + 2)N} + \frac{N - 2}{(h + 1)(h + 2)N} \right) \left( \sum_{i \in [N]} \sum_{p \in [N] \setminus \{i\}} q^i_{1,p} + \sum_{j \in [N]} \sum_{p \in [N] \setminus \{j\}} q^j_{2,p} \right), \forall h \in [0 : N - 3]. \tag{52}
\]

**Second approach:** We then use the second approach to construct genie-aided super-users.

We first consider \(X_2\). By the construction, from \((X_2, Z_1^i)\) where \(i \in [N]\), we can reconstruct \(F_i\).

Now we fix an integer \(i \in [N]\). We pick a permutation of \([N]\), assumed to be \(u = (u_1, \ldots, u_N)\), where \(u_1 = i\). We can construct a genie-aided super-user with the cache

\[
\bigcup_{p \in [N]} \left( Z_{1,up}^{u_1} \cup Q_{1,up}^{u_1} \cup \cdots \cup Q_{1,up}^{u_{p-1}} \right). \tag{53}
\]
Now we prove that the genie-aided super-user can successively decode the whole library from its cache and $X_2$. Note that from $(Z_{u_1}^{u_1}, X_2)$, we can reconstruct $F_{u_1}$. Furthermore, for each file $F_{p_1}$ where $p_1 \in [N] \setminus \{u_1\}$, by recalling the definition of $Q_{1,p_1}^{u_1}$ in (42), it can be seen that the bits in $Z_{1,p_1}^{u_1} \setminus Q_{1,p_1}^{u_1}$ are independent of $X_2$. Hence, it is enough to reconstruct $F_{u_1}$ from $(X_2, Z_{1,u_1}^{u_1}, Q_{1,u_1}^{u_1}, \ldots, Q_{1,u_N}^{u_1})$, and thus the super-user can reconstruct $F_{u_1}$. After recovering $F_{u_1}$, the super-user can reconstruct $F_{u_2}$ from $(X_2, F_{u_1}, Z_{1,u_2}^{u_2}, Q_{1,u_2}^{u_1}, \ldots, Q_{1,u_N}^{u_2})$. By this way, the genie-aided super-user can decode the whole library. Hence, we have

$$H(X_2) \geq H(F_1, \ldots, F_N) - H(\bigcup_{p \in [N]} (Z_{1,u_p}^{u_1} \cup Q_{1,u_p}^{u_1} \cup \cdots \cup Q_{1,u_p}^{u_{p-1}}))$$

(54a)

$$\geq (H(F_{u_1}) - H(Z_{1,u_1}^{u_1})) + (H(F_{u_2}) - H(Z_{1,u_2}^{u_2}, Q_{1,u_2}^{u_1})) + \ldots + (H(F_{u_N}) - H(Z_{1,u_N}^{u_N}, Q_{1,u_N}^{u_1}, \ldots, Q_{1,u_N}^{u_{N-1}})).$$

(54b)

From (54b), it will be proved in Appendix B-A and Appendix B-B that,

$$H(X_2) \geq \frac{B}{2} - \frac{yB}{N};$$

(55)

$$H(X_2) \geq B - \frac{2yB}{N} - q_{1,u_2}^j;$$

(56)

By letting the two permutations to derive (46f) and (54b) be the same, we now sum (47) and (55) to obtain

$$R^*_u B \geq H(X_1) + H(X_2) \geq B - \frac{2yB}{N},$$

(57)

which coincides with the proposed converse bound in (22). Similarly, by summing (48) and (56), we obtain

$$R^*_u B \geq H(X_1) + H(X_2) \geq 2B - \frac{6yB}{N},$$

(58)

which coincides with the proposed converse bound in (21).

In addition, by considering all permutations of $[N]$ where the first element is $i$, we can list all $(N-1)!$ inequalities as in (54b). By summing all these $(N-1)!$ inequalities, we can obtain the following inequalities, which will be proved in Appendix B-C

$$H(X_2) \geq \frac{NB}{2} - yB - \frac{2}{h+2} \sum_{p \in [N] \setminus \{i\}} q_{1,p}^i - \sum_{p \in [N] \setminus \{i\}} \left\{ \frac{\sum_{n \in [N] \setminus \{i,p\}} q_{n,p}^i}{(h+1)(h+2)} + \frac{h}{h+2} \left( \frac{B}{2} - \frac{yB}{N} \right) \right\}, \forall h \in [0 : N-3].$$

(59)
By considering all $i \in [N]$, we can list all $N$ inequalities as in (59). By summing all these $N$ inequalities, we obtain

$$H(X_2) \geq \frac{NB}{2} - yB - \frac{2}{(h + 2)N} \sum_{i \in [N]} \sum_{p \in [N] \setminus \{i\}} q_{1,p}^i$$

$$- \sum_{i \in [N]} \sum_{p \in [N] \setminus \{i\}} \left\{ \frac{\sum_{n \in [N] \setminus \{i,p\}} q_{1,p}^n}{(h + 1)(h + 2)N} + \frac{h}{(h + 2)N} \left( \frac{B}{2} - \frac{yB}{N} \right) \right\}, \forall h \in [0 : N - 3]. \quad (60)$$

We now consider $X_1$. By the similar steps as above to derive (60), we obtain

$$H(X_1) \geq \frac{NB}{2} - yB - \frac{2}{(h + 2)N} \sum_{i \in [N]} \sum_{p \in [N] \setminus \{i\}} q_{2,p}^j$$

$$- \sum_{j \in [N]} \sum_{p \in [N] \setminus \{j\}} \left\{ \frac{\sum_{n \in [N] \setminus \{j,p\}} q_{2,p}^n}{(h + 1)(h + 2)N} + \frac{h}{(h + 2)N} \left( \frac{B}{2} - \frac{yB}{N} \right) \right\}, \forall h \in [0 : N - 3]. \quad (61)$$

By summing (60) and (61), we obtain

$$R_u^a \geq H(X_1) + H(X_2) \geq NB - 2yB - \frac{2}{(h + 2)N} \left( \sum_{j \in [N]} \sum_{p \in [N] \setminus \{j\}} q_{2,p}^j + \sum_{i \in [N]} \sum_{p \in [N] \setminus \{i\}} q_{1,p}^i \right)$$

$$- \frac{1}{(h + 1)(h + 2)N} \left( \sum_{j \in [N]} \sum_{p \in [N] \setminus \{j\}} \sum_{n \in [N] \setminus \{j,p\}} q_{2,p}^n + \sum_{i \in [N]} \sum_{p \in [N] \setminus \{i\}} \sum_{n \in [N] \setminus \{i,p\}} q_{1,p}^n \right)$$

$$- h(N - 1) \left( \frac{B - \frac{2yB}{N}}{h + 2} \right), \forall h \in [0 : N - 3]. \quad (62a)$$

where (62b) comes from that

$$\sum_{j \in [N]} \sum_{p \in [N] \setminus \{j\}} \sum_{n \in [N] \setminus \{j,p\}} q_{2,p}^n = \sum_{j1 \in [N]} \sum_{p1 \in [N] \setminus \{j1\}} q_{2,p1}^{j1}$$

and that

$$\sum_{i \in [N]} \sum_{p \in [N] \setminus \{i\}} \sum_{n \in [N] \setminus \{i,p\}} q_{1,p}^n = (N - 2) \sum_{i1 \in [N]} \sum_{p1 \in [N] \setminus \{i1\}} q_{1,p1}^{i1}.$$ 

Finally, by summing (52) and (62b), we obtain $\forall h \in [0 : N - 3]$

$$R_u^a \geq \frac{1}{2} \left\{ N - 2y - \frac{8(N - 1)y}{(h + 2)N} - \frac{(N - 2)(N - 1)y}{(h + 1)(h + 2)N} - \frac{h(N - 1)}{(h + 2)} \left( 1 - \frac{2y}{N} \right) \right\}$$

$^7$ In the sum $\sum_{j \in [N]} \sum_{p \in [N] \setminus \{j\}} \sum_{n \in [N] \setminus \{j,p\}} q_{2,p}^n$, let us compute the coefficient of term $q_{2,p1}^{j1}$ where $j1 \neq p1$. $q_{2,p1}^{j1}$ appears in the sum when $p = p1$ and $n = j1$. Hence, there are $N - 2$ possibilities of $j$, which are $[N] \setminus \{p1,j1\}$. So the coefficient of $q_{2,p1}^{j1}$ in the sum is $N - 2$. 
\[ + \frac{1}{2} \left\{ N - 2yN - \frac{h(N - 1)}{(h + 2)} \left( 1 - \frac{2y}{N} \right) \right\} \]
\[ = N - 2y - \frac{4y + (N - 1)h}{h + 2} + \frac{h^2(n - 1) - N(N - 3) + h(N + 1)2y}{N(h + 1)(h + 2)}. \]

which coincides with the proposed converse bound in (20).

**B. Proof of Theorem 6: K-user System**

We extend the proposed converse bound for the two-user system to K-user system and consider the privacy constraint against colluding users in (8). In the following, we consider the case where K/2 is an integer and 2N/K is also an integer. In Appendix C we generalize the proof to any K and N.

Let \( M = \frac{N}{K} + \frac{2y}{K} \), where \( y \in [0, \frac{N}{2}] \). We use a genie-aided proof by generating two aggregate users, denoted by \( k_1 \) and \( k_2 \). We assume the cache size of each aggregate user is \( MB \times \frac{K}{2} = \frac{NB}{2} + yB \), i.e., the cache size of each aggregate user is the total cache sizes of K/2 users. In addition, the demanded files of aggregate users \( k_1 \) and \( k_2 \) are the union sets of the demanded files of users in \([K/2]\) and of users in \([K/2 + 1 : K]\), respectively. The objective is to design a two-user D2D private caching scheme with minimum load \( R^*_g \), such that each aggregate user can decode its demanded files without knowing anything about the demand of the other aggregate user.

Obviously, for any K-user D2D private caching satisfying the encoding (4), decoding (5), and privacy constraints (8), it must be an achievable scheme for the above genie system. In other words, \( R^*_{u,c} \geq R^*_g \). Hence, in the following we characterize a converse bound for \( R^*_g \), which is also a converse bound for \( R^*_{u,c} \).

We partition the N files into 2N/K equal-size groups, each of which contains K/2 files. Each aggregate user demands one group of files. Hence, it is equivalent to the two-user D2D private caching problem with 2N/K files, each of which has KB/2 bits, and each of the two users caches \((\frac{NB}{2} + yB)\) bits in its cache and demands one file.

We assume the caches of aggregate users \( k_1 \) and \( k_2 \) are \( A^1_1 \) and \( A^1_2 \). The transmitted packets by aggregate users \( k_1 \) and \( k_2 \) are denoted by \( X'_1 \) and \( X'_2 \), such that from \((X'_2, A^1_1)\) aggregate user \( k_1 \) can decode the files in group 1 and from \((X'_1, A^1_2)\) aggregate user \( k_2 \) can also decode the files in group 1. We then also construct the cache configurations of aggregate users \( k_1 \) and \( k_2 \) by a 2N/K-ary tree, as we did in Section V-A.
By the first approach of constructing converse bound described in Section V-A, when we consider $A_1$ where $i \in \left[\frac{2N}{K}\right]$ (cache of aggregate user 1 from which and $X_2'$, the files in group $i$ can be reconstructed), with a permutation of $[2N/K]$ denoted by $u = (u_1, \ldots, u_{2N/K})$ where $u_1 = i$, we obtain (from the similar derivations of (47) and (48)),

$$H(X_1') \geq \left(\frac{B}{2} - \frac{yB}{N}\right) \frac{K}{2}; \quad (64)$$

$$H(X_1') \geq \frac{K}{2}B - \frac{K}{2} \frac{4yB}{N} + q_{1,u_2}^i, \quad (65)$$

where $q_{1,u_2}^i$ represent the number of bits in $A_1^i \cap A_2^{(i,1)} \cap \cdots \cap A_2^{(i,2N/K)}$, which are from the files in group $u_2$.

By considering all permutations of $[2N/K]$ whose first element is $i$, we obtain (from the similar derivation of (49)),

$$H(X_1') \geq \frac{NB}{2} - yB - \frac{2}{h+2} \left\{ \left(\frac{2N}{K} - 1\right) \frac{2yB}{N} \frac{K}{2} \right\} + \frac{2}{h+2} \sum_{p \in \left[\frac{2N}{K}\right] \setminus \{i\}} q_{1,p}^i$$

$$- \sum_{p \in \left[\frac{2N}{K}\right] \setminus \{i\}} \left\{ \frac{2N}{K} - 2 \left(\frac{2yB}{N} \frac{K}{2} - q_{1,p}^i\right) + \frac{h}{h+2} \left(\frac{B}{2} - \frac{yB}{N}\right) \frac{K}{2} \right\}, \forall h \in \left[0 : \frac{2N}{K} - 3\right]. \quad (66)$$

By considering all $i \in \left[\frac{2N}{K}\right]$ to bound $H(X_1')$ and all $j \in \left[\frac{2N}{K}\right]$ to bound $H(X_2')$, we sum all inequalities as in (66) to obtain (from the similar derivation of (52)),

$$R_g^* B \geq NB - 2yB - \frac{4}{h+2} \left\{ \left(\frac{2N}{K} - 1\right) \frac{2yB}{N} \frac{K}{2} \right\} - \frac{2N}{K} - 2 \left(\frac{2N}{K} - 2 \left(\frac{2yB}{N} \frac{K}{2} - q_{1,p}^i\right) + \frac{h}{h+2} \left(\frac{B}{2} - \frac{yB}{N}\right) \frac{K}{2} \right\}$$

$$- \left( \sum_{i \in \left[\frac{2N}{K}\right]} \sum_{p \in \left[\frac{2N}{K}\right] \setminus \{i\}} q_{1,p}^i + \sum_{j \in \left[\frac{2N}{K}\right]} \sum_{p \in \left[\frac{2N}{K}\right] \setminus \{j\}} q_{2,p}^j \right), \forall h \in \left[0 : \frac{2N}{K} - 3\right]. \quad (67)$$

Similarly, by the second approach of constructing converse bound described in Section V-A, when we consider $X_2'$ and the same permutation as the one to derive (64) and (65), we obtain (from the similar derivations of (55) and (56)),

$$H(X_2') \geq \left(\frac{B}{2} - \frac{yB}{N}\right) \frac{K}{2}; \quad (68)$$

$$H(X_2') \geq \frac{K}{2}B - \frac{K}{2} \frac{2yB}{N} - q_{1,u_2}^i, \quad (69)$$
By summing (64) and (68), we prove (25). By summing (65) and (69), we prove (24).

In addition, by the second approach of constructing converse bound described in Section V-A, after considering all permutations to bound \( H(X'_1) \) and all permutations to bound \( H(X'_2) \), we obtain (from the similar derivation of (62b)),

\[
R^*B \geq NB - 2yB - \frac{h \left( \frac{2N}{K} - 1 \right)}{(h + 2)} \left( B - \frac{2yB}{N} \right) \frac{K}{2} - \left( \frac{2}{(h + 2)} \frac{2N}{K} + \frac{2N/K - 2}{(h + 1)(h + 2) \frac{2N}{K}} \right)
\]

\[
\left( \sum_{i \in \left[ \frac{2N}{K} \right]} \sum_{p \in \left[ \frac{2N}{K} \right] \setminus \{i\}} q^{i}_{1,p} + \sum_{j \in \left[ \frac{2N}{K} \right]} \sum_{p \in \left[ \frac{2N}{K} \right] \setminus \{j\}} q^{j}_{2,p}, \forall h \in \left[ 0 : \frac{2N}{K} - 3 \right]. \right)
\]

(70)

By summing (67) and (70), we prove (23).

VI. CONCLUSIONS

We introduced a novel D2D private caching model with a trusted server, which aims to preserve the privacy of the users’ demands. We proposed novel D2D private coded caching schemes, which were proved to be order optimal by matching a novel converse bound under the constraint of uncoded cache placement and user collusion. Further works include proving even tighter converse bounds than the extension from the two-user system, novel converse bounds for any cache placement, and the research on decentralized D2D private coded caching problem with a trusted server.

APPENDIX A

PROOFS OF EQUATIONS (47) (48) (49)

Recall that by considering a permutation of \([N]\), assumed to be \(u = (u_1, \ldots, u_N)\), where \(u_1 = i\), we can derive (46f),

\[
H(F_1, \ldots, F_N) \leq H(X_1) + H(Z^{(i,i)}_2) + \sum_{p \in [2:N]} H\left(Z^{(i,u_p)}_{2,u_p}, \ldots, Z^{(i,u_p)}_{2,u_p} | Z^{(i,i)}_2 \right). \quad (71)
\]

For each \(p \in [2 : N]\), since \(Z^{(i,i)}_{2,u_p} = \frac{B}{2} + \frac{yB}{N}\), we have

\[
H\left(Z^{(i,u_p)}_{2,u_p}, \ldots, Z^{(i,u_p)}_{2,u_p} | Z^{(i,i)}_2 \right) \leq H(F_p | Z^{(i,i)}_2) = \frac{B}{2} - \frac{yB}{N}. \quad (72)
\]
A. Proof of (47)

Now we bound each term \( H(Z_{2,u_p}^{(i,u_2)}, \ldots, Z_{2,u_p}^{(i,u_p)} | Z_{2,u_p}^{(i,i)}) \) where \( p \in [2:N] \) in (71) by \( \frac{B}{2} - \frac{yB}{N} \), to obtain

\[
H(F_1, \ldots, F_N) \leq H(X_1) + H(Z_2^{(i,i)}) + \sum_{p \in [2:N]} H(Z_{2,u_p}^{(i,u_2)}, \ldots, Z_{2,u_p}^{(i,u_p)} | Z_{2,u_p}^{(i,i)}) \tag{73a}
\]

\[
\leq H(X_1) + H(Z_2^{(i,i)}) + (N - 1) \left( \frac{B}{2} - \frac{yB}{N} \right) \tag{73b}
\]

\[
= H(X_1) + \frac{NB}{2} + yB + (N - 1) \left( \frac{B}{2} - \frac{yB}{N} \right). \tag{73c}
\]

Hence, we have

\[
H(X_1) \geq \frac{B}{2} - \frac{yB}{N}, \tag{74}
\]

which proves (47).

B. Proof of (48)

We first prove for each \( i \in [N] \) and \( n, p \in [N] \setminus \{i\} \), we have

\[
H \left( Z_{2,p}^{(i,n)} | Z_{2,p}^{(i,i)} \right) = H \left( Z_{2,p}^{(i,n)} | Z_{2,p}^{(i,i)}, F_p \setminus Z_{1,p}^{i} \right) \tag{75a}
\]

\[
= H \left( Z_{2,p}^{(i,n)} \cap Z_{1,p}^{i} | Z_{2,p}^{(i,i)}, F_p \setminus Z_{1,p}^{i} \right) \tag{75b}
\]

\[
= H \left( Z_{2,p}^{(i,n)} \cap Z_{1,p}^{i} | Z_{2,p}^{(i,i)} \right) \tag{75c}
\]

\[
\leq H \left( Z_{2,p}^{(i,n)} \cap Z_{1,p}^{i} \right) - q_{1,p} \tag{75d}
\]

\[
= 2yB \frac{N}{N} - q_{1,p}, \tag{75e}
\]

where (75a) comes from that \( Z_{2,p}^{(i,i)} \cap Z_{1,p}^{i} = Z_{2,p}^{(i,n)} \cap Z_{1,p}^{i} = F_p \) and thus \( (F_p \setminus Z_{1,p}^{i}) \subseteq Z_{2,p}^{(i,i)} \), (75b) and (75c) come from that all bits in the library are independent, (75d) comes from (42), (75e) comes from (41c).

Now we bound each term \( H \left( Z_{2,u_p}^{(i,u_2)}, \ldots, Z_{2,u_p}^{(i,u_p)} | Z_{2,u_p}^{(i,i)} \right) \) where \( p \in [3:N] \) in (71) by \( \frac{B}{2} - \frac{yB}{N} \), to obtain

\[
H(F_1, \ldots, F_N) \leq H(X_1) + H(Z_2^{(i,i)}) + \sum_{p \in [2:N]} H(Z_{2,u_p}^{(i,u_2)}, \ldots, Z_{2,u_p}^{(i,u_p)} | Z_{2,u_p}^{(i,i)}) \tag{66a}
\]

\[
\leq H(X_1) + \frac{NB}{2} + yB + H(Z_{2,u_p}^{(i,u_2)} | Z_{2,u_p}^{(i,i)}) + (N - 2) \left( \frac{B}{2} - \frac{yB}{N} \right) \tag{76b}
\]
\[ H(X_1) + \frac{NB}{\sqrt{2}} + yB + \frac{2yB}{\sqrt{N}} - q_{1,u_2}^i + (N - 2) \left( \frac{B}{\sqrt{2}} - \frac{yB}{\sqrt{N}} \right), \]  
(76c)

where (76c) comes from (75c).

Hence, we have

\[ H(X_1) \geq \frac{NB}{\sqrt{2}} - yB - \frac{2yB}{\sqrt{N}} + q_{1,u_2}^i - (N - 2) \left( \frac{B}{\sqrt{2}} - \frac{yB}{\sqrt{N}} \right) \]  
(77a)

\[ = B - \frac{4yB}{\sqrt{N}} + q_{1,u_2}^i, \]  
(77b)

which proves (48).

C. Proof of (49)

Note that when \( N = 2 \), (49) does not exist. Hence, in the following we consider \( N \geq 3 \) to prove (49).

From (71), we have

\[ H(F_1, \ldots, F_N) \leq H(X_1) + H(Z_{2}^{(i,i)}) + \sum_{p \in [2:N]} H \left( Z_{2,up}^{(i,u)}, \ldots, Z_{2,up}^{(i,u)} \mid Z_{2,up}^{(i,i)} \right) \]  
(78a)

\[ = H(X_1) + H(Z_{2}^{(i,i)}) + \sum_{p \in [2:N]} \left\{ H \left( Z_{2,up}^{(i,u)} \mid Z_{2,up}^{(i,i)} \right) + H \left( Z_{2,up}^{(i,u)}, \ldots, Z_{2,up}^{(i,u_{p-1})} \mid Z_{2,up}^{(i,i)}, Z_{2,up}^{(i,u_{p})} \right) \right\} \]  
(78b)

\[ = H(X_1) + \frac{NB}{\sqrt{2}} + yB + \sum_{p \in [2:N]} H \left( Z_{2,up}^{(i,u)} \mid Z_{2,up}^{(i,i)} \right) + \sum_{p \in [2:N]} H \left( Z_{2,up}^{(i,u)}, \ldots, Z_{2,up}^{(i,u_{p-1})} \mid Z_{2,up}^{(i,i)}, Z_{2,up}^{(i,u_{p})} \right). \]  
(78c)

By considering all permutations of \([N]\) where the first element is \( i \) and summing all inequalities as (78c), we have

\[ H(X_1) \geq \frac{NB}{\sqrt{2}} - yB - \frac{1}{(N - 1)!} \sum_{u:u_1 = i} \sum_{p \in [2:N]} H \left( Z_{2,up}^{(i,u)} \mid Z_{2,up}^{(i,i)} \right) \]
\[ - \frac{1}{(N - 1)!} \sum_{u:u_1 = i} \sum_{p \in [2:N]} H \left( Z_{2,up}^{(i,u_2)}, \ldots, Z_{2,up}^{(i,u_{p-1})} \mid Z_{2,up}^{(i,i)}, Z_{2,up}^{(i,u_{p})} \right), \]  
(79a)

\[ = \frac{NB}{\sqrt{2}} - yB - \sum_{p \in [N] \backslash \{i\}} H \left( Z_{2,p}^{(i,p)} \mid Z_{2,p}^{(i,i)} \right) - \frac{1}{(N - 1)!} \sum_{p \in [N] \backslash \{i\}} \sum_{u:u_1 = i, u_2 = p} H \left( Z_{2,p}^{(i,u_2)}, \ldots, Z_{2,p}^{(i,u_{r-1})} \mid Z_{2,p}^{(i,i)}, Z_{2,p}^{(i,u)} \right), \]  
(79b)

where (79b) comes from the re-arrangements on the summations.
To bound the last term in (79b), we now focus on one file $F_p$ where $p \in [N] \setminus \{i\}$ and bound the following term

$$
\sum_{r \in [2:N]} \sum_{u:u = i, u_r = p} H \left( Z_{2,p}^{(i,u_2)}, \ldots, Z_{2,p}^{(i,u_{r-1})}, Z_{2,p}^{(i,i)}, Z_{2,p}^{(i,p)} \right). \tag{80}
$$

Note that the conditional entropies in (80) are conditioned on the same term, which is $Z_{2,p}^{(i,i)} \cup Z_{2,p}^{(i,p)}$. In addition, for any $n \in [N] \setminus \{i, p\}$, we have

$$
Z_{2,p}^{(i,n)} \setminus (Z_{2,p}^{(i,i)} \cup Z_{2,p}^{(i,p)}) \subseteq F_p \setminus (Z_{2,p}^{(i,i)} \cup Z_{2,p}^{(i,p)}).
$$

Hence, we divide the bits in $F_p \setminus (Z_{2,p}^{(i,i)} \cup Z_{2,p}^{(i,p)})$ into sub-pieces, and denote (with a slight abuse of notation)

$$
F_p \setminus (Z_{2,p}^{(i,i)} \cup Z_{2,p}^{(i,p)}) = \{F_{p,S} : S \subseteq ([N] \setminus \{i,p\})\}, \tag{81a}
$$

where

$$
F_{p,S} = \left( F_p \setminus (Z_{2,p}^{(i,i)} \cup Z_{2,p}^{(i,p)}) \right) \cap \left( \bigcap_{n \in S} Z_{2,p}^{(i,n)} \right) \setminus \left( \bigcup_{n \notin S} Z_{2,p}^{(i,n)} \right). \tag{81b}
$$

In other words, $F_{p,S}$ represents the bits in $F_p \setminus (Z_{2,p}^{(i,i)} \cup Z_{2,p}^{(i,p)})$ which are exclusively in $Z_{2,p}^{(i,n)}$, where $n \in S$.

We then define

$$
f_t := \sum_{S \subseteq ([N] \setminus \{i,p\}) : |S| = t} |F_{p,S}|, \; \forall t \in [0 : N - 2], \tag{82}
$$

as the total length of sub-pieces $F_{p,S}$ where $|S| = t$.

In (75e), we proved that for each $n \in [N] \setminus \{i,p\}$, we have $H(Z_{2,p}^{(i,n)} | Z_{2,p}^{(i,i)}) \leq \frac{2yB}{N} - q_{1,p}^i$. Hence, we also have

$$
H(Z_{2,p}^{(i,n)} | Z_{2,p}^{(i,i)} \cup Z_{2,p}^{(i,p)}) \leq H(Z_{2,p}^{(i,n)} | Z_{2,p}^{(i,i)}) \leq \frac{2yB}{N} - q_{1,p}^i. \tag{83}
$$

By summing (83) over all $n \in [N] \setminus \{i, p\}$, we have

$$
\sum_{t \in [0 : N - 2]} t f_t = \sum_{n \in [N] \setminus \{i, p\}} \sum_{S \subseteq [N] \setminus \{i,p\} : n \in S} |F_{p,S}| \leq (N - 2) \left( \frac{2yB}{N} - q_{1,p}^i \right). \tag{84}
$$

In addition, since $F_p \setminus (Z_{2,p}^{(i,i)} \cup Z_{2,p}^{(i,p)}) = (F_p \setminus Z_{2,p}^{(i,i)}) \setminus (Z_{2,p}^{(i,p)} \setminus Z_{2,p}^{(i,i)})$, we have

$$
|F_p \setminus (Z_{2,p}^{(i,i)} \cup Z_{2,p}^{(i,p)})| = |F_p \setminus Z_{2,p}^{(i,i)}| - |Z_{2,p}^{(i,p)} \setminus Z_{2,p}^{(i,i)}| = \frac{B}{2} - \frac{yB}{N} - H(Z_{2,p}^{(i,p)} | Z_{2,p}^{(i,i)}).
$$

Hence, we have

$$
\sum_{t \in [0 : N - 2]} f_t = \sum_{S \subseteq [N] \setminus \{i,p\}} |F_{p,S}| = \frac{B}{2} - \frac{yB}{N} - H(Z_{2,p}^{(i,p)} | Z_{2,p}^{(i,i)}). \tag{85}
$$
From the above definitions, we can re-write (80) as follows,

\[
\sum_{r \in [2: N]} \sum_{u_1 = i, u_r = p} H \left( Z_{2,p}^{(i,u_2)}, \ldots, Z_{2,p}^{(i,u_{r-1})} | Z_{2,p}^{(i,t)}, Z_{2,p}^{(i,p)} \right) = \sum_{r \in [2: N]} \sum_{u_1 = i, u_r = p} \sum_{S \subseteq ([N]\{i,p\}): S \cap \{u_2, \ldots, u_{r-1}\} \neq \emptyset} |\mathcal{F}_{p,S}|. \tag{86}
\]

In (86), for each \( r \in [2 : N] \), we can compute

\[
\sum_{u_1 = i, u_r = p} \sum_{S \subseteq ([N]\{i,p\}): S \cap \{u_2, \ldots, u_{r-1}\} \neq \emptyset} |\mathcal{F}_{p,S}| = \sum_{t \in [0: N-2]} (N - 2)! \frac{\binom{N-2}{t} - \binom{N-r-1}{t}}{\binom{N-2}{t}} f_t. \tag{87}
\]

This is because in \( \sum_{S \subseteq ([N]\{i,p\}): S \cap \{u_2, \ldots, u_{r-1}\} \neq \emptyset} |\mathcal{F}_{p,S}| \), there are \( \binom{N-2}{t} - \binom{N-r-1}{t} \) sub-pieces whose \( S \) has \( t \) elements. Considering all permutations \( u \) where \( u_1 = i \) and \( u_r = p \), by the symmetry, the coefficient of each \( |\mathcal{F}_{p,S}| \) where \( S = t \) should be the same. In addition, there are in total \( \binom{N-2}{t} \) sub-pieces whose \( S \) has \( t \) elements. Hence, we obtain (87).

Considering all \( r \in [2 : N-2] \), from (87) we have

\[
\sum_{r \in [2: N]} \sum_{u_1 = i, u_r = p} H \left( Z_{2,p}^{(i,u_2)}, \ldots, Z_{2,p}^{(i,u_{r-1})} | Z_{2,p}^{(i,t)}, Z_{2,p}^{(i,p)} \right) = \sum_{r \in [2: N]} \sum_{t \in [0: N-2]} (N - 2)! \frac{\binom{N-2}{t} - \binom{N-r-1}{t}}{\binom{N-2}{t}} f_t \tag{88a}
\]

\[
= (N - 2)! \sum_{t \in [0: N-2]} \sum_{r \in [2: N]} \frac{\binom{N-2}{t} - \binom{N-r-1}{t}}{\binom{N-2}{t}} f_t \tag{88b}
\]

\[
= (N - 2)! \sum_{t \in [0: N-2]} \left( \frac{(N-2)(N-2)}{\binom{N-2}{t}} - \frac{(N-2)(N-2)}{\binom{N-2}{t+1}} \right) f_t \tag{88c}
\]

\[
= (N - 1)! \sum_{t \in [0: N-2]} \frac{t}{t+1} f_t, \tag{88d}
\]

where (88c) comes from the Pascal’s Triangle, \( \binom{N-3}{t} + \cdots + \binom{i}{t} = \binom{N-2}{t+1} \).

The next step is to use Fourier-Motzkin elimination on \( f_t \) where \( t \in [0 : N-2] \) in (88d) (as we did in [10]) with the help of (84) and (85). More precisely, we fix one integer \( h \in [0 : N-3] \). We multiply (84) by \( \frac{(N-1)!}{(h+1)(h+2)} \) and multiply (85) by \( \frac{(N-1)!h}{h+2} \), and sum them to obtain

\[
\sum_{t \in [0: N-2]} \left( t \frac{(N-1)!}{(h+1)(h+2)} + \frac{(N-1)!h}{h+2} \right) f_t \leq \frac{(N-1)!}{(h+1)(h+2)} \left( 2yB - q^i_{1,p} \right) + \frac{(N-1)!h}{h+2} \left( \frac{B}{2} - yB - H(Z_{2,p}^{(i,p)} | Z_{2,p}^{(i,i)}) \right). \tag{89}
\]

From (89), we have

\[
\frac{(N-1)!h}{h+1} f_h + \frac{(N-1)!}{(h+1)} f_{h+1}
\]
We then take (90) into (88d) to obtain,

\[
\sum_{t \in [0:N-2]} \sum_{t \in [h,h+1]} H \left( Z_{2,p}^{(i,u_2)}, \ldots, Z_{2,p}^{(i,u_{r-1})}, Z_{2,p}^{(i,i)}, Z_{2,p}^{(i,p)} \right)
\]

\[
\leq \frac{(N-1)!}{(h+1)(h+2)} \left( \frac{2yB}{N} - q_{1,p} \right) + \frac{(N-1)!}{h+2} \left( \frac{B}{2} - \frac{yB}{N} - H(Z_{2,p}^{(i,p)} | Z_{2,p}^{(i,i)}) \right)
\]

\[
- \sum_{t \in [0:N-2]} (N-1)! \left( \frac{(N-1)!}{(h+1)(h+2)(t+1)} f_t \right)
\]

(91a)

\[
\leq \frac{(N-1)!}{(h+1)(h+2)} \left( \frac{2yB}{N} - q_{1,p} \right) + \frac{(N-1)!}{h+2} \left( \frac{B}{2} - \frac{yB}{N} - H(Z_{2,p}^{(i,p)} | Z_{2,p}^{(i,i)}) \right)
\]

(91b)

Finally, we take (91b) into (79b) to obtain, for each \( h \in [0:N-3] \),

\[
H(X_1) \geq \frac{NB}{2} - yB - \sum_{p \in [N\setminus i]} H \left( Z_{2,p}^{(i,p)} | Z_{2,p}^{(i,i)} \right)
\]

(92a)

\[
- \frac{1}{(N-1)!} \sum_{p \in [N\setminus i]} \sum_{r \in [2:N]} \sum_{u_1 = u, u_r = p} H \left( Z_{2,p}^{(i,u_2)}, \ldots, Z_{2,p}^{(i,u_{r-1})}, Z_{2,p}^{(i,i)}, Z_{2,p}^{(i,p)} \right)
\]

(92b)

\[
\geq \frac{NB}{2} - yB - \sum_{p \in [N\setminus i]} H \left( Z_{2,p}^{(i,p)} | Z_{2,p}^{(i,i)} \right)
\]

\[
- \frac{1}{(N-1)!} \sum_{p \in [N\setminus i]} \left\{ \frac{(N-1)!}{(h+1)(h+2)} \left( \frac{2yB}{N} - q_{1,p} \right) + \frac{(N-1)!h}{h+2} \left( \frac{B}{2} - \frac{yB}{N} - H(Z_{2,p}^{(i,p)} | Z_{2,p}^{(i,i)}) \right) \right\}
\]

(92c)

\[
= \frac{NB}{2} - yB - \frac{2}{h+2} \sum_{p \in [N\setminus i]} H \left( Z_{2,p}^{(i,p)} | Z_{2,p}^{(i,i)} \right) - \sum_{p \in [N\setminus i]} \left\{ \frac{(N-2)}{(h+1)(h+2)} \left( \frac{2yB}{N} - q_{1,p} \right) + \frac{h}{h+2} \left( \frac{B}{2} - \frac{yB}{N} \right) \right\}
\]

(92d)

where (92d) comes from (75e). Hence, we prove (49).
APPENDIX B
PROOFS OF EQUATIONS (55) (56) (59)

The proofs of (55) (56) (59) come from a similar strategy used in Appendix A. Hence, in the following, we briefly describe the proofs of (55) (56) (59).

Recall that by considering a permutation of $[N]$, assumed to be $u = (u_1, \ldots, u_N)$, where $u_1 = i$, we can derive (54b),

\[
H(X_2) \geq (H(F_i) - H(Z_{1,i}^i)) + \sum_{p \in [2:N]} \left( H(F_{u_p}) - H(Z_{1,u_p}^{u_p}, Q_{1,u_p}^{u_2}, \ldots, Q_{1,u_p}^{u_{p-1}}) \right). \tag{93}
\]

For each $p \in [2:N]$, we have

\[
H(F_{u_p}) - H(Z_{1,u_p}^{u_p}, Q_{1,u_p}^{u_2}, \ldots, Q_{1,u_p}^{u_{p-1}}) \geq 0. \tag{94}
\]

A. Proof of (55)

Now we bound each term $H(F_{u_p}) - H(Z_{1,u_p}^{u_p}, Q_{1,u_p}^{u_2}, \ldots, Q_{1,u_p}^{u_{p-1}})$ where $p \in [2 : N]$ in (93) by 0, to obtain

\[
H(X_2) \geq H(F_i) - H(Z_{1,i}^i) = \frac{B}{2} - \frac{yB}{N}, \tag{95}
\]

which proves (55).

B. Proof of (56)

Now we bound each term $H(F_{u_p}) - H(Z_{1,u_p}^{u_p}, Q_{1,u_p}^{u_2}, \ldots, Q_{1,u_p}^{u_{p-1}})$ where $p \in [3 : N]$ in (93) by 0, to obtain

\[
H(X_2) \geq \left( H(F_i) - H(Z_{1,i}^i) \right) + \sum_{p \in [2:N]} \left( H(F_{u_p}) - H(Z_{1,u_p}^{u_p}, Q_{1,u_p}^{u_2}, \ldots, Q_{1,u_p}^{u_{p-1}}) \right) \tag{96a}
\]

\[
\geq \left( H(F_i) - H(Z_{1,i}^i) \right) + \left( H(F_{u_2}) - H(Z_{1,u_2}^{u_2}, Q_{1,u_2}^i) \right) \tag{96b}
\]

\[
\geq H(F_i) - H(Z_{1,i}^i) + H(F_{u_2}) - H(Z_{1,u_2}^{u_2}) - H(Q_{1,u_2}^i) \tag{96c}
\]

\[
= B - \frac{2yB}{N} - q_{1,u_2}^i. \tag{96d}
\]

which proves (56).
C. Proof of (59)

Note that when \( N = 2 \), (59) does not exist. Hence, in the following we consider \( N \geq 3 \) to prove (59).

From (93), we have

\[
H(X_2) \geq (H(F_i) - H(Z_{1,i}^1)) + \sum_{p \in [2:N]} \left[ H(F_{1,p}) - H(Z_{1,u_p}^{u_p}, Q_{1,u_p}^{u_2}, \ldots, Q_{1,u_p}^{u_{p-1}}) \right] \quad (97a)
\]

\[
= (H(F_i) - H(Z_{1,i}^1)) + \sum_{p \in [2:N]} \left( H(F_{1,p}) - H(Z_{1,u_p}^{u_p}) - H(Q_{1,u_p}^{u_p} | Z_{1,u_p}^{u_p}) - H(Q_{1,u_p}^{u_2} \ldots, Q_{1,u_p}^{u_{p-1}} | Z_{1,u_p}^{u_p}, Q_{1,u_p}^{u_1}) \right) \quad (97b)
\]

\[
= N \left( \frac{B}{2} - \frac{yB}{N} \right) - \sum_{p \in [2:N]} H(Q_{1,u_p}^{u_p} | Z_{1,u_p}^{u_p}) - \sum_{p \in [2:N]} H(Q_{1,u_p}^{u_2} \ldots, Q_{1,u_p}^{u_{p-1}} | Z_{1,u_p}^{u_p}, Q_{1,u_p}^{u_1}). \quad (97c)
\]

By considering all permutations of [\( N \)] where the first element is \( i \) and summing all inequalities as (97c), we can obtain

\[
H(X_2) \geq N \left( \frac{B}{2} - \frac{yB}{N} \right) - \frac{1}{(N - 1)!} \sum_{u_1 = i} \sum_{p \in [2:N]} H(Q_{1,u_p}^{u_p} | Z_{1,u_p}^{u_p})
\]

\[
- \frac{1}{(N - 1)!} \sum_{u_1 = i} \sum_{p \in [2:N]} H(Q_{1,u_p}^{u_2} \ldots, Q_{1,u_p}^{u_{p-1}} | Z_{1,u_p}^{u_p}, Q_{1,u_p}^{u_1}) \quad (98a)
\]

\[
= N \left( \frac{B}{2} - \frac{yB}{N} \right) - \sum_{p \in [N] \setminus \{i\}} H(Q_{1,p}^{p} | Z_{1,p}^{p}) - \frac{1}{(N - 1)!} \sum_{p \in [N] \setminus \{i\}} \sum_{r \in [2:N]} \sum_{u_1 = i, u_r = p} H(Q_{1,u_r}^{u_r} \ldots, Q_{1,u_r}^{u_{r-1}} | Z_{1,u_r}^{u_r}, Q_{1,u_r}^{u_1}) \quad (98b)
\]

where (98b) comes from the re-arrangements on the summations.

To bound the last term in (98b), we now focus on one file \( F_p \) where \( p \in [N] \setminus \{i\} \) and bound the following term

\[
\sum_{r \in [2:N]} \sum_{u_1 = i, u_r = p} H(Q_{1,u_r}^{u_r} \ldots, Q_{1,u_r}^{u_{r-1}} | Z_{1,u_r}^{u_r}, Q_{1,u_r}^{u_1}). \quad (99)
\]

We divide the bits in \( F_p \setminus (Z_{1,p}^p \cup Q_{1,p}^i) \) into sub-pieces, and denote

\[
F_p \setminus (Z_{1,p}^p \cup Q_{1,p}^i) = \{ G_p,S : S \subseteq ([N] \setminus \{i,p\}) \}, \quad (100a)
\]

where \( G_p,S = (F_p \setminus (Z_{1,p}^p \cup Q_{1,p}^i)) \cap (\cap_{n \in S} Q_{1,p}^n) \setminus (\cup_{n \in S} Q_{1,p}^n) \). \quad (100b)

We then define

\[
g_t := \sum_{S \subseteq ([N] \setminus \{i,p\}) : |S| = t} |G_p,S|, \quad \forall t \in [0 : N - 2]. \quad (101)
\]
For each \( n \in [N] \setminus \{i, p\} \), we have \( H(Q^n_{1,p} | Z^n_{1,p}, Q^i_{1,p}) \leq H(Q^n_{1,p}) \). Hence, we have

\[
\sum_{t \in [0: N-2]} t g_t \leq \sum_{n \in [N] \setminus \{i, p\}} q^n_{1,p}.
\]  

(102)

In addition, since \( F_p \setminus (Z^n_{1,p} \cup Q^i_{1,p}) = (F_p \setminus Z^n_{1,p}) \setminus (Q^i_{1,p} \setminus Z^n_{1,p}) \), we have

\[
\sum_{t \in [0: N-2]} g_t = \sum_{S \subseteq [N] \setminus \{i, p\}} |G_{p, S}| = \frac{B}{2} - \frac{yB}{N} - H(Q^i_{1,p} | Z^i_{1,p}).
\]  

(103)

From the above definitions, we can re-write (98b) (as we did to obtain (88d)),

\[
\sum_{r \in [2:N]} \sum_{u_1 = i, u_r = p} H(Q^{u_2}_{1,p}, \ldots, Q^{u_{r-1}}_{1,p} | Z^p_{1,p}, Q^i_{1,p}) = \sum_{r \in [2:N]} \sum_{u_1 = i, u_r = p} \sum_{S \subseteq (N) \setminus \{i, p\}} |G_{p, S}|.
\]  

(104a)

\[
= \sum_{r \in [2:N]} \sum_{t \in [0: N-2]} (N-2)! \frac{(N-2)}{2} \frac{(N-2)}{t} g_t.
\]  

(104b)

\[
= (N-1)! \sum_{t \in [0: N-2]} \frac{t}{t+1} g_t.
\]  

(104c)

By Fourier-Motzkin elimination on \( g_t \) where \( t \in [0: N-2] \) in (104c) with the help of (102) and (103), we obtain for each \( h \in [0: N-3] \),

\[
\sum_{r \in [2:N]} \sum_{u_1 = i, u_r = p} H(Q^{u_2}_{1,p}, \ldots, Q^{u_{r-1}}_{1,p} | Z^p_{1,p}, Q^i_{1,p}) \leq \frac{(N-1)!}{(h+1)(h+2)} \sum_{n \in [N] \setminus \{p, i\}} q^n_{1,p} + \frac{(N-1)! h}{h+2} \left( \frac{B}{2} - \frac{yB}{N} - H(Q^i_{1,p} | Z^i_{1,p}) \right).
\]  

(105)

Finally, by taking (105) into (98b), we obtain for each \( h \in [0: N-3] \),

\[
H(X_2) \geq N \left( \frac{B}{2} - \frac{yB}{N} \right) - \frac{2}{h+2} \sum_{p \in [N] \setminus \{i\}} H(Q^i_{1,p} | Z^p_{1,p}) - \sum_{p \in [N] \setminus \{i\}} \left\{ \sum_{n \in [N] \setminus \{p, i\}} q^n_{1,p} \frac{h}{(h+1)(h+2)} - \frac{h}{h+2} \left( \frac{B}{2} - \frac{yB}{N} \right) \right\}
\]  

(106a)

\[
\geq N \left( \frac{B}{2} - \frac{yB}{N} \right) - \frac{2}{h+2} \sum_{p \in [N] \setminus \{i\}} q^i_{1,p} - \sum_{p \in [N] \setminus \{i\}} \left\{ \sum_{n \in [N] \setminus \{p, i\}} q^n_{1,p} \frac{h}{(h+1)(h+2)} - \frac{h}{h+2} \left( \frac{B}{2} - \frac{yB}{N} \right) \right\},
\]  

(106b)

where (106b) comes from that \( H(Q^i_{1,p} | Z^p_{1,p}) \leq H(Q^i_{1,p}) = q^i_{1,p} \). Hence, we prove (59).
APPENDIX C

GENERALIZATION OF THE PROOF IN SECTION V-B

In Section V-B we prove Theorem 6 for the case where \( K/2 \) is an integer and \( 2N/K \) is also an integer. In the following, we only consider the case where \( K/2 \) is not integer and \( \frac{N}{[K/2]} \) is not an integer neither. The proof for the case where \( K/2 \) is an integer and \( 2N/K \) is not an integer, or \( K/2 \) is not an integer and \( \frac{N}{[K/2]} \) is an integer, can be directly derived from the following proof.

Recall \( M = \frac{N}{K} + \frac{2y}{K} \), where \( y \in [0, \frac{N}{2}] \). We first fix one user \( k \in [K] \) (assuming now \( k = K \)). We can divide the users in \([K] \setminus \{k\}\) into two groups, and generate an aggregate user for each group. Denoted the two aggregate users by \( k_1 \) and \( k_2 \), respectively. The cache size of each aggregate user is \( MB \times \frac{K_1}{2} \). In addition, the demanded files of aggregate users \( k_1 \) and \( k_2 \) are the union sets of the demanded files of users in \([(K - 1)/2\] and of users in \([(K + 1)/2 : K - 1\] respectively.

By denoting \( N_1 := \lfloor 2N/K \rfloor \lfloor K/2 \rfloor \), we divide files in \([N_1] \) into \([2N/K] \) non-overlapping groups, each of which contains \([K/2] \) files. Each aggregate user demands one group of files.

We assume that the caches of aggregate users \( k_1 \) and \( k_2 \) are \( A_1 \) and \( A_2 \). The transmitted packets by aggregate users \( k_1 \) and \( k_2 \) are denoted by \( X_1 \) and \( X_2 \), and the transmitted packets by user \( k = K \) are denoted by \( X_k \), such that from \((X_2, X_k, A_1)\) aggregate user \( k_1 \) can decode the files in group 1 and from \((X_1, X_k, A_2)\) aggregate user \( k_2 \) can also decode the files in group 1. We then construct the cache configurations of aggregate users \( k_1 \) and \( k_2 \) by a \([2N/K]\)-ary tree, as we did in Section V-A.

In the first approach, when we consider \( A_i \) where \( i \in \lfloor 2N/K \rfloor \) (cache of aggregate user \( k_1 \) where from \((X_2, X_k, A_i)\), the files in group \( i \) can be decoded), by constructing a genie-aided super-user as in (45) (the cache of this super-user is denoted by \( A \)), by Fano’s inequality,

\[
H(F_1, \ldots, F_N|\{F_\ell : \ell \in [N_1 + 1 : N]\}) \leq H(X_1') + H(X_k) + H(A|\{F_\ell : \ell \in [N_1 + 1 : N]\})
\]

(107)

By considering one permutation of \([2N/K]\), denoted by \( u = (u_1, \ldots, u_{2N/K}) \) where \( u_1 = i \), from the similar derivations of (64) and (65), we obtain

\[
H(X_1') + H(X_k) \geq \left( \frac{B}{2} - \frac{yB}{N} \right) [K/2] ;
\]

(108)

\[
H(X_1') + H(X_k) \geq [K/2] B - [K/2] \frac{4yB}{N} + q_{1,u_2}^i.
\]

(109)
By considering all permutations of \([\lceil 2N/K \rceil] - 1\) where the first element is \(i\), and sum all inequalities as in (107). From the similar derivation of (66), we obtain
\[
H(X'_1) + H(X_k) \geq \left( \frac{B}{2} - \frac{yB}{N} \right) N_1 - \frac{2}{h+2} \left( \left\lfloor \frac{2N/K}{2} \right\rfloor \frac{2yB}{N} \frac{K}{2} \right) + \frac{2}{h+2} \sum_{p \in \lceil 2N/K \rceil \setminus \{i\}} q^i_p \\
- \frac{(\lceil 2N/K \rceil - 1)(\lceil 2N/K \rceil - 2) 2yB}{N} \frac{K}{2} - \frac{(\lceil 2N/K \rceil - 1)h}{h+2} \left( \frac{B}{2} - \frac{yB}{N} \right) \frac{K}{2} \\
+ \frac{|2N/K| - 2}{(h+1)(h+2)} \sum_{p \in \lceil 2N/K \rceil \setminus \{i\}} q^i_p,
\]
where (110a) comes from that
\[
\frac{N}{N_1} = \frac{\lceil 2N/K \rceil - 1}{\lceil 2N/K \rceil} \frac{K}{2} = N - \frac{N}{\lceil 2N/K \rceil} \leq (\frac{2N/K - 1}{2}) \frac{K}{2}.
\]

By considering all \(i \in \lceil 2N/K \rceil\) to bound \(H(X'_1) + H(X_k)\), and all \(j \in \lceil 2N/K \rceil\) to bound \(H(X'_2) + H(X_k)\), we sum all inequalities as (110a) to obtain (from the similar derivation of (67)),
\[
R^*_u, c_B + H(X_k) \geq \left( \frac{B}{2} - \frac{yB}{N} \right) N - \frac{4}{h+2} \left( \frac{2N/K - 1}{2} \frac{2yB}{N} \right) \\
- \frac{(2N/K - 1)(2N/K - 2) 4yB}{N} \frac{K}{2} - \frac{h(2N/K - 1)}{(h+1)(h+2)} \left( \frac{B}{2} - \frac{yB}{N} \right) \frac{K}{2} \\
+ \left( \frac{2}{(h+2) \lceil 2N/K \rceil} + \frac{|2N/K| - 2}{(\lceil 2N/K \rceil)(h+1)(h+2)} \right) \left( \sum_{i \in \lceil 2N/K \rceil} \sum_{j \in \lceil 2N/K \rceil \setminus \{i\}} q^i_p + \sum_{j \in \lceil 2N/K \rceil} \sum_{p \in \lceil 2N/K \rceil \setminus \{j\}} q^j_p \right),
\]
\forall h \in [0 : \lceil 2N/K \rceil - 3].

Similarly, in the second approach, when we consider \((X'_2, X_k)\) and the same permutation as the one to derive (108) and (109), by constructing a genie-aided super-user as in (53) (the cache of this super-user is denoted by \(A'\)), by Fano’s inequality,
\[
H(F_1, \ldots, F_N|\{F_\ell : \ell \in [N_1 + 1 : N]\}) \leq H(X'_2) + H(X_k) + H(A'|\{F_\ell : \ell \in [N_1 + 1 : N]\}).
\]

From the similar derivations of (68) and (69), we obtain
\[
H(X'_2) + H(X_k) \geq \left( \frac{B}{2} - \frac{yB}{N} \right) \frac{K}{2};
\]
\[ H(X'_2) + H(X_k) \geq \frac{[K/2]}{B} - \frac{2yB}{N} - q_{1,u_2}^i. \] (115)

In addition, by considering all permutations to bound \( H(X'_1) + H(X_k) \) and all permutations to bound \( H(X'_2) + H(X_k) \), we sum all inequalities as in (113) to obtain (from the similar derivation of (70)),

\[
R_{u,c}^* B + H(X_k) \geq \frac{N_1}{N} \left\{ NB - 2yB - \frac{h}{(h+2)} \left( B - \frac{2yB}{N} \right) - \frac{2}{(h+2) \left[ 2N/K \right]} \right. \\
+ \frac{[2N/K] - 2}{(h+1)(h+2) \left[ 2N/K \right]} \left( \sum_{j \in \left[ \frac{2N}{K} \right]} \sum_{p \in \left[ \frac{2N}{K} \right] \setminus \{j\}} q_{2,p}^j + \sum_{i \in \left[ \frac{2N}{K} \right]} \sum_{p \in \left[ \frac{2N}{K} \right] \setminus \{i\}} q_{1,p}^i \right), \forall h \in [0 : [2N/K] - 3].
\] (116)

By summing (108) and (114), summing (109) and (115), and summing (112) and (116), we obtain

\[
R_{u,c}^* B + H(X_k) \geq \left( B - \frac{2yB}{N} \right) \frac{[K/2]}{B}; \tag{117a}
\]

\[
R_{u,c}^* B + H(X_k) \geq \left( 2B - \frac{6yB}{N} \right) \frac{[K/2]}{B}; \tag{117b}
\]

\[
R_{u,c}^* B + H(X_k) \geq \frac{N_1}{N} \left\{ NB - 2yB - \frac{2}{h+2} \left( \frac{2N/K}{h+2} \right) \right. \\
- \frac{2N/K - 1}{h+1}(h+2) \frac{2yB K}{2} - \frac{h(2N/K - 1)}{h+2} \left( B - \frac{2yB}{N} \right) \frac{K}{2}, \forall h \in [0 : [2N/K] - 3]. \tag{117c}
\]

Finally we consider all \( k \in [K] \) and sum inequalities as (117), to obtain (recall that \( R_{u,c}^* B \geq \sum_{k \in [K]} H(X_k) \)),

\[
R_{u,c}^* B \geq \frac{K}{2 \left[ \frac{K}{2} \right]} \left( B - \frac{2yB}{N} \right) \frac{[K/2]}{B} = \frac{[K/2]}{[K/2]} \left( B - \frac{2yB}{N} \right) \frac{K}{2}; \tag{118a}
\]

\[
R_{u,c}^* B \geq \frac{K}{2 \left[ \frac{K}{2} \right]} \left( 2B - \frac{6yB}{N} \right) \frac{[K/2]}{B} = \frac{[K/2]}{[K/2]} \left( 2B - \frac{6yB}{N} \right) \frac{K}{2}; \tag{118b}
\]

\[
R_{u,c}^* B \geq \frac{K}{2 \left[ \frac{K}{2} \right]} \frac{N_1}{N} \left\{ NB - 2yB - \frac{2}{h+2} \left( \frac{2N/K - 1}{h+2} \right) \right. \\
- \frac{2N/K - 1}{h+1}(h+2) \frac{2yB K}{2} - \frac{h(2N/K - 1)}{h+2} \left( B - \frac{2yB}{N} \right) \frac{K}{2} \right\} \\
= \frac{[K/2]}{[K/2]} \frac{2N/K}{2N/K} \left\{ NB - 2yB - \frac{2}{h+2} \left( \frac{2N/K - 1}{h+2} \right) \frac{2yB K}{N} \right\}. \tag{118c}
\]
\[ -\frac{(2N/K - 1)(2N/K - 2)}{(h+1)(h+2)} 2yB K \frac{2h}{N} \frac{N}{2} \frac{(2N/K - 1)}{(h+2)} \left( \frac{B - 2yB}{N} \right) \frac{K}{2}, \quad \forall h \in [0 : \lceil 2N/K \rceil - 3], \]

(118c)

where (118c) comes from (recall that \( N_1 := \lceil 2N/K \rceil \lfloor K/2 \rfloor \)).

\[ \frac{K}{2 \lfloor K/2 \rfloor} \frac{N_1}{N} = \frac{K}{2 \lfloor K/2 \rfloor} \frac{\lceil 2N/K \rceil \lfloor K/2 \rfloor}{\lfloor K/2 \rfloor \lceil 2N/K \rceil}. \quad (119) \]

Hence, we prove Theorem 6

**Appendix D**

**Proof of Theorem 2**

We first provide a direct upper bound of the achieved load of Scheme A in Theorem 1, since

\[ \frac{(U_t - 1) - (U_{t-1})}{(U_t - 1)} \leq \frac{(U_t)}{(U_{t-1})} = \frac{U_t - 1}{U_{t-1}}. \]

**Lemma 3.** The achieved load of Scheme A in Theorem 1 is upper bound by the lower convex envelop of \((N/K, N)\) and

\[ \left( \frac{N + t - 1}{K}, \frac{U - t + 1}{t} \right), \quad \forall t \in [U + 1]. \quad (120) \]

We then introduce the following lemma, whose proof is in Appendix E.

**Lemma 4.** The multiplicative gap between the lower convex envelop of the memory-load tradeoff \( \left( \frac{N_t - t - 1}{K}, \frac{U_t - t + 1}{t} \right) \) where \( t \in [U] \), and the lower convex envelop of the memory-load tradeoff \( \left( \frac{N_t}{K}, \frac{K-t}{t+1} \right) \) where \( t \in [2 : K] \), is at most 3 when \( M \geq \frac{2N}{K} \).

We then prove the two cases in Theorem 2 where \( N \geq K \) and \( N < K \).

A. \( N \geq K \)

**Converse.** It was proved in [8] that for the shared-link caching model with \( N \geq K \), the lower convex envelope of the corner points \( \left( \frac{N_t}{K}, \frac{K-t}{t+1} \right) \), where \( t \in [0 : K] \), achieved by the MAN caching scheme in [4] is order optimal within a factor of 2. In addition, it was proved in [10] that these corner points are successively convex. Hence, when \( M \geq \frac{2N}{K} \), the lower convex envelop of \( \left( \frac{N_t}{K}, \frac{K-t}{t+1} \right) \), where \( t \in [2 : K] \) is order optimal within a factor of 2. We will also use this converse in our model. Hence, for \( M \in [2N/K, N] \), \( R^* \) is lower bounded by the lower convex envelope \( \left( \frac{N_t}{K}, \frac{K-t}{2(t+1)} \right) \), where \( t \in [2 : K] \).

**Achievability.** From Lemma 4 it can be seen that from the proposed scheme in Theorem 1 we can achieve the lower convex envelop of the memory-load tradeoff \( \left( \frac{N_t}{K}, \frac{3(K-t)}{t+1} \right) \) where \( t \in [2 : K] \).
As a result, the proposed scheme in Theorem 1 is order optimal within a factor of 6 when \( N \geq K \) and \( M \geq \frac{2N}{K} \).

B. \( N < K \)

Converse. It was proved in [33] that for the shared-link caching model with \( N < K \), the lower convex envelope of the corner points \((0,N)\) and \(\left(\frac{Nt}{K}, \frac{K-t}{t+1}\right)\), where \( t \in [K] \), achieved by the MAN caching scheme in [4] is order optimal within a factor of 4.

Since the corner points \(\left(\frac{Nt}{K}, \frac{K-t}{t+1}\right)\) where \( t \in [K] \), are successively convex, the lower convex envelop of the MAN caching scheme for \( N < K \) is as follows. There exists one \( t_2 \in [K] \), such that the lower convex envelop of the MAN caching scheme for \( M \in [0, Nt_2/K] \) is the memory-sharing between \((0,N)\) and \(\left(\frac{Nt_2}{K}, \frac{K-t_2}{t_2+1}\right)\), while the lower convex envelop for \( M \in [Nt_2/K, N] \) is the lower convex envelop of the successive corner points \(\left(\frac{Nt}{K}, \frac{K-t}{t+1}\right)\) where \( t \in [t_2 : K] \). In addition, it is obvious that \( t_2 \) is the maximum value among \( x \in [K] \) such that the memory-sharing between \((0,N)\) and \(\left(\frac{Nx}{K}, \frac{K-x}{x+1}\right)\) at the memory \( M' = \frac{N(x-1)}{K} \) leads to a lower load than \( \frac{K-x+1}{x} \). More precisely, if we interpolate \((0,N)\) and \(\left(\frac{Nx}{K}, \frac{K-x}{x+1}\right)\) where \( x \in [K] \) to match \( M' = \frac{N(x-1)}{K} \), the achieved load is

\[
-\frac{N - \frac{K-x}{x+1}}{x} \frac{N(x-1)}{K} + N = \frac{(K-x)(x-1)}{x(x+1)} + \frac{N}{x}.
\]

Hence, we have

\[
t_2 := \arg \max_{x \in [K]} \left\{ \frac{(K-x)(x-1)}{x(x+1)} + \frac{N}{x} \leq \frac{K-x+1}{x} \right\} = \left\lfloor \frac{2K-N+1}{N+1} \right\rfloor.
\]

We then interpolate \((0,N)\) and \(\left(\frac{Nt_2}{K}, \frac{K-t_2}{t_2+1}\right)\) to match \( M_1 = N/K \), to get the memory-load tradeoff

\[
(M_1, R_1) = \left(\frac{N}{K}, N - \frac{N - \frac{K-t_2}{t_2+1}}{t_2} \right).
\]

Hence, it is equivalent to say the lower convex envelop of the achieved memory-load tradeoffs by the MAN caching scheme for \( M \geq N/K \) also has two regimes.

1) \( M \in \left[\frac{N}{K}, \frac{Nt_2}{K}\right] \). The lower convex envelop is the memory-sharing between \((M_1, R_1)\) and \(\left(\frac{Nt_2}{K}, \frac{K-t_2}{t_2+1}\right)\).

2) \( M \in \left[\frac{Nt_2}{K}, N\right] \). The lower convex envelop of the MAN scheme is the lower convex envelop of the corner points \(\left(\frac{Nt}{K}, \frac{K-t}{t+1}\right)\), where \( t \in [t_2 : K] \).

Since the MAN scheme is order optimal within a factor of 4, \( R^* \) is lower bounded by the lower convex envelope of the corner points \((M_1, \frac{R_1}{4})\) and \(\left(\frac{Nt}{K}, \frac{K-t}{4(t+1)}\right)\), where \( t \in [t_2 : K] \).
Achievability. Let us first focus on $M = N/K$. The achieved load by the proposed scheme in Theorem 1 is $N$. In the following, we will prove $N \leq 2R_1$. More precisely,

$$N - 2R_1 = 2 \frac{N - \frac{K-t_2}{t_2+1}}{t_2} - N$$

$$= \frac{2N(t_2 + 1) - 2(K - t_2) - Nt_2(t_2 + 1)}{t_2(t_2 + 1)}$$

$$= \frac{-Nt_2^2 + (N + 2)t_2 - 2(K - N)}{t_2(t_2 + 1)}$$

$$= \frac{-t_2(Nt_2 - N - 2) - 2(K - N)}{t_2(t_2 + 1)}$$

$$= \frac{-(Nt_2 - N - 2) - \frac{2(K - N)}{t_2}}{(t_2 + 1)}. \quad (123)$$

We consider the following two cases.

1) $t_2 = 1$. From (123), we have

$$N - 2R_1 = \frac{2 - 2(K - N)}{2} \leq 0, \quad (124)$$

which follows $K > N$.

2) $t_2 > 1$. From (123), we have

$$N - 2R_1 \leq \frac{-(2N - N - 2) - \frac{2(K - N)}{t_2}}{1} < 0, \quad (125)$$

which follows $N \geq 2$ and $K > N$.

Hence, from the proposed scheme in Theorem 1 we can achieve $(M_1, 2R_1)$. In addition, from Lemma 4 it can be seen that from the proposed scheme in Theorem 1 we can achieve the lower convex envelop of the memory-load tradeoff $(\frac{Nt}{K}, 3\frac{(K-t)}{t+1})$ where $t \in [t_2 : K]$.

As a result, the proposed scheme in Theorem 1 is order optimal within a factor of 12 when $N < K$.

**Appendix E**

**Proof of Lemma 4**

It was proved in [10] that the corner points $(\frac{Nt}{K}, \frac{K-t}{t+1})$ where $t \in [0 : K]$ are successively convex, i.e., for each memory size $M \in \left[\frac{Nt}{K}, \frac{N(t+1)}{K}\right]$ where $t \in [0 : K - 1]$, the lower convex envelop is obtained by memory-sharing between $(\frac{Nt}{K}, \frac{K-t}{t+1})$ and $(\frac{N(t+1)}{K}, \frac{K-t-1}{t+2})$. Hence, in order to prove Lemma 4 in the following we prove from $(\frac{N+1-t_1}{K}, \frac{U-t_1+1}{t_1})$ where $t_1 \in [U]$, we can achieve $(\frac{Nt}{K}, 3\frac{(K-t)}{t+1})$ for each $t \in [2 : K]$. 
We now focus on one \( t \in [2 : K] \). We let \( t_1 = N(t - 1) + 1 \) such that the memory size is
\[
\frac{N + t_1 - 1}{K} = \frac{N + N(t - 1) + 1 - 1}{K} = \frac{Nt}{K}. \tag{126}
\]
The achieved load is
\[
\frac{U - t_1 + 1}{t_1} = \frac{U - \frac{U(t - 1)}{K - 1}}{\frac{U(t - 1)}{K - 1} + 1}
= \frac{U(K - 1) - U(t - 1)}{U(t - 1) + (K - 1)}
= \frac{K - t}{t - 1 + \frac{K - 1}{N}}
\leq \frac{K - t}{t - 1}
\leq 3\frac{K - t}{t + 1}, \tag{127}
\]
where (127) comes from \( t \geq 2 \). Hence, we prove the proof of Lemma 4.

**APPENDIX F**

**PROOF OF REMARK 5**

Recall that for the two-user system, the achieved corner points of Scheme A are \( \left( \frac{N + t - 1}{2}, \frac{N - t + 1}{t} \right) \), where \( t \in [N + 1] \). The achieved corner points of Scheme B are \( \left( \frac{N}{2} + \frac{Nt'}{2(N + t' - 1)}, \frac{N(N-1)}{(t' + 1)(N + t' - 1)} \right) \) and \( (N, 0) \), where \( t' \in [0 : N - 1] \).

To prove Scheme B is strictly better than Scheme A for the two-user system, we prove that for each \( t \in [N] \), by memory-sharing between \( \left( \frac{N}{2} + \frac{Nt'}{2(N + t' - 1)}, \frac{N(N-1)}{(t' + 1)(N + t' - 1)} \right) \) and \( (N, 0) \), where \( t' = t - 1 \), we can obtain \( \left( \frac{N + t - 1}{2}, \frac{N - t + 1}{t} \right) \). More precisely, we let \( \alpha = \frac{(N+1)(t+1)}{N(N-1)} \). We have
\[
\alpha \left( \frac{N}{2} + \frac{Nt'}{2(N + t' - 1)} \right) + (1 - \alpha)N = \frac{(N + t' - 1)(N - t')}{N(N - 1)} \cdot \frac{N(N + 2t' - 1)}{2(N + t' - 1)} + \frac{t'(t' - 1)}{N(N - 1)}
= \frac{(N - 1)(N - t')}{2(N - 1)}
= \frac{N - t + 1}{2}; \tag{128}
\]
\[
\alpha \frac{N(N-1)}{(t' + 1)(N + t' - 1)} + (1 - \alpha) \times 0 = \frac{N - t'}{t' + 1} = \frac{N - t + 1}{t}. \tag{129}
\]
APPENDIX G

PROOF OF THEOREM 5

A. Optimality in Theorem 5

When \( N = 2 \), it can be easily checked that the converse bound in Theorem 4 is a piecewise curve with corner points \( \left( \frac{N}{2}, N \right), \left( \frac{3N}{4}, \frac{1}{2} \right) \), and \( (N, 0) \), which can be achieved by Scheme B in (12). Hence, in the following, we focus on \( N > 2 \).

Recall that \( M = \frac{N}{2} + y \). For \( 0 \leq y \leq \frac{1}{2} \), from the converse bound in (20) with \( h = 0 \), we have
\[
R_u^* \geq N - 2y - \frac{4y + (N - 1)h}{h + 2} + \frac{h^2(N - 1) - N(N - 3) + h(N + 1)2y}{(h + 1)(h + 2)} \frac{2y}{N}
\]
\[
= N - 2y - 2y - y(N - 3)
\]
\[
= N - y(N + 1).
\]
(130)

In other words, when \( \frac{N}{2} \leq M \leq \frac{N + 1}{2} \), the converse bound on \( R_u^* \) in (130) is a straight line between \( \left( \frac{N}{2}, N \right) \) and \( \left( \frac{N + 1}{2}, \frac{N - 1}{2} \right) \). In addition, Scheme B in (12) achieves \( \left( \frac{N}{2}, N \right) \) with \( t' = 0 \), and \( \left( \frac{N + 1}{2}, \frac{N - 1}{2} \right) \) with \( t' = 1 \). Hence, we prove Scheme B is optimal under the constraint of uncoded cache placement when \( \frac{N}{2} \leq M \leq \frac{N + 1}{2} \).

For \( \frac{2N}{3} \leq M \leq \frac{3N}{4} \) (i.e., \( \frac{N}{6} \leq y \leq \frac{N}{4} \)), from the converse bound in (21), we have
\[
R_u^* \geq 2 - \frac{6y}{N} = 5 - \frac{6M}{N}.
\]
(131)

By noticing that \( \frac{N(3N-5)}{2(2N-3)} \geq \frac{2N}{3} \) when \( N \geq 3 \), from (131), it can be seen that when \( M = \frac{N(3N-5)}{2(2N-3)} \), \( R_u^* \geq \frac{N}{2N-3} \), coinciding with Scheme B in (12) with \( t' = N - 2 \). When \( M = \frac{3N}{4} \), \( R_u^* \geq \frac{1}{2} \), coinciding with Scheme B in (12) with \( t' = N - 1 \). Hence, we prove Scheme B is optimal under the constraint of uncoded cache placement when \( \frac{N(3N-5)}{2(2N-3)} \leq M \leq \frac{3N}{4} \).

Finally, for \( \frac{3N}{4} \leq M \leq N \) (i.e., \( \frac{N}{4} \leq y \leq \frac{N}{2} \)), from the converse bound in (22), we have
\[
R_u^* \geq 1 - \frac{2y}{N} = 2 - \frac{2M}{N}.
\]
(132)

From (132), it can be seen that when \( M = \frac{3N}{4} \), \( R_u^* \geq \frac{1}{2} \), coinciding with Scheme B in (12) with \( t' = N - 1 \). When \( M = N \), \( R_u^* \geq 0 \), which can be also achieved by Scheme B. Hence, we prove Scheme B is optimal under the constraint of uncoded cache placement when \( \frac{3N}{4} \leq M \leq N \).
B. Order Optimality in Theorem 5

From Theorem 4, we can compute the proposed converse bound is a piecewise curve with the corner points

\[
\left( \frac{N}{2} + \frac{Nh'}{2(N + 2h' - 2)}, \frac{(h' - 1)(N + h') + (N - 1)N}{(h' + 1)(N + 2h' - 2)} \right), \quad \forall h' \in [0 : N - 2],
\]

(133)

\((\frac{3N}{4}, \frac{1}{2})\), and \((N, 0)\)\(^8\). Note that the proposed converse bound is a piecewise linear curve with the above corner points, and that the straight line in the memory-load tradeoff between two achievable points is also achievable by memory-sharing. Hence, in the following, we focus on each corner point of the converse bound, and characterize the multiplicative gap between Scheme B and the converse bound.

Note that in (133), when \(h' = 0\), we have \((\frac{N}{2}, N)\); when \(h' = 1\), we have \((\frac{N+1}{2}, \frac{N-1}{2})\); when \(h' = N - 2\), we have \((\frac{2N}{3}, 1)\). In addition, in Appendix G-A we proved the optimality of Scheme B under the constraint of uncoded cache placement when \(M \leq \frac{N+1}{2}\) or when \(M \geq \frac{3N}{4}\). Hence, in the following, we only need to compare Scheme B and the corner points in (133) where \(h' \in [2 : N - 2]\) and \(N \geq 4\).

In Remark 5, we show that Scheme B is strictly better than Scheme A. We will prove the multiplicative gap between Scheme A and the corner points in (133) where \(h' \in [2 : N - 2]\) and \(N \geq 4\), is no more than 3.

Recall that the achieved points of Scheme A for the two-user system are

\[
\left( \frac{N + t - 1}{2}, \frac{N - t + 1}{t} \right), \quad \forall t \in [N + 1].
\]

(134)

We want to interpolate the achieved points of Scheme A to match the converse bound at at the memory size \(M = \frac{N}{2} + \frac{Nh'}{2(N + 2h' - 2)}\) where \(h' \in [2 : N - 2]\). By computing

\[
\frac{N + t - 1}{2} = \frac{N}{2} + \frac{Nh'}{2(N + 2h' - 2)} \quad \iff \quad t = \frac{Nh'}{N + 2h' - 2} + 1,
\]

(135)

\(^8\) The first corner point in (133) is \((\frac{N}{2}, N)\) with \(h' = 0\), and the last corner point is \((N, 0)\). For each \(h' \in [N - 3]\), we obtain the corner point in (133) by taking the intersection between the converse bounds in (20) with \(h = h' - 1\) and \(h = h'\). The corner point in (133) with \(h' = N - 2\), is obtained by taking the intersection between the converse bounds in (20) with \(h = N - 3\) and the converse bound in (21). The corner point \((\frac{3N}{4}, \frac{1}{2})\) is obtained by taking the intersection between the converse bounds in (21) and (22).
and observing \(\frac{N-t+1}{t}\) is non-increasing with \(t\), it can be seen that the achieved load of Scheme A at \(M = \frac{N}{2} + \frac{Nh'}{2(N+2h'-2)}\) is lower than

\[
R' = \frac{N - \frac{Nh'}{N+2h'-2} + 1}{Nh'} = \frac{N^2 + (N + 2)(h' - 1)}{Nh'}.
\]

(136)

By comparing \(R'\) and \(\frac{(h'-1)(N+h')+(N-1)N}{(h'+1)(N+2h'-2)}\), we have

\[
R' = \frac{(N^2 + (N + 2)(h' - 1))(h' + 1)(N + 2h' - 2)}{Nh'((h'+1)(N+h')+(N-1)N)}.
\]

(137)

In addition, we compute

\[
3Nh'((h'-1)(N+h')+(N-1)N) - (N^2 + (N + 2)(h' - 1))(h' + 1)(N + 2h' - 2)
= 2N^3h' - N^3 - 6N^2h' - 3Nh'^2 + (N - 4)h'^3 + 3N^2 + 2Nh' + 4h'(h'-1) - 4.
\]

(138)

Now we want to prove the RHS of (138) is larger than 0 for \(N \geq 4\) and \(h' \in [2 : N - 2]\). More precisely, when \(N = 4\) and \(h' = 2\), we can compute the RHS of (138) is equal to 36; when \(N = 5\) and \(h' = 2\), the RHS of (138) is equal to 138; when \(N = 5\) and \(h' = 3\), the RHS of (138) is equal to 216. Now we only need to consider \(N \geq 6\) and \(h' \in [2 : N - 2]\).

When \(N \geq 6\) and \(h' \in [2 : N - 2]\), we have

\[
2N^3h' - N^3 - 6N^2h' - 3Nh'^2 + (N - 4)h'^3 + 3N^2 + 2Nh' + 4h'(h'-1) - 4
> 2N^3h' - N^3 - 6N^2h' - 3Nh'^2
= (N^3h' - 6N^2h') + (0.5N^3h' - 3Nh'^2) + (0.5N^3h' - N^3)
\geq 0.
\]

(139)

Hence, we prove

\[
3Nh'((h'-1)(N+h')+(N-1)N) - (N^2 + (N + 2)(h' - 1))(h' + 1)(N + 2h' - 2) > 0.
\]

(140)

By taking (140) into (137), we prove that the multiplicative gap between Scheme A and the corner points in (133) where \(h' \in [2 : N - 2]\) and \(N \geq 4\), is less than 3.

In conclusion, we prove that Scheme B is order optimal under the constraint of uncoded cache placement within a factor of 3.
Appendix H

Proof of Theorem 7

In this proof, for the achievability, we consider the load in Lemma 3 which is an upper bound of the achieved load of Scheme A.

We first focus on the case where \( N \leq 6K \), and compare Scheme A with the shared-link caching converse bound under the constraint of uncoded cache placement (without privacy) in [10]. Recall that when \( M \in \left[ \frac{N}{K}, N \right] \), the converse bound in [10] is a piecewise curve with corner points \( \left( \frac{Nt}{K}, \frac{Kt}{t+1} \right) \), where \( t \in [K] \). It was proved in Appendix D-A that Scheme A can achieve the corner points \( \left( \frac{Nt}{K}, 3\frac{Kt}{t+1} \right) \), where \( t \in [2 : K] \). In addition, when \( M = \frac{N}{K} \), the converse bound in [10] is \( R^*_u \geq \frac{K-1}{2} \), while the achieved load of Scheme A is

\[
N \leq 6K \leq 9(K - 1), \quad \text{when } K \geq 3.
\]

Hence, the multiplicative gap between Scheme A and the converse bound in [10] at \( M = \frac{N}{K} \) is no more than 18. So we prove that \( N \leq 6K \), Scheme A is order optimal under the constraint of uncoded cache placement within a factor of 18.

In the rest of the proof, we focus on the case where \( N > 6K \). It was proved in Theorem 2 that when \( N \geq K \) and \( M \geq \frac{2N}{K} \), Scheme A is order optimal within a factor of 6. Hence, in the following we consider \( \frac{N}{K} \leq M \leq \frac{2N}{K} \), which is then divided into three memory size regimes, and prove the order optimality of Scheme A separately.

Regime 1: \( \frac{N}{K} \leq M \leq \frac{N}{K} + \frac{Nh_1}{2(N + Kh_1 - K)} \), where \( h_1 := \left\lfloor \frac{4(K - 2)(N - K)}{K(N - 4K + 8)} \right\rfloor \); (141a)

Regime 2: \( \frac{N}{K} + \frac{Nh_1}{2(N + Kh_1 - K)} \leq M \leq \frac{N}{K} + \frac{Nh_2}{2(N + Kh_2 - K)} \), where \( h_2 := \left\lfloor \frac{2N}{K} - 2 \right\rfloor \); (141b)

Regime 3: \( \frac{N}{K} + \frac{Nh_2}{2(N + Kh_2 - K)} \leq M \leq \frac{2N}{K} \). (141c)

Note that when \( N > 6K \), we have \( h_1 := \left\lfloor \frac{4(K - 2)(N - K)}{K(N - 4K + 8)} \right\rfloor < 10 \) and \( h_2 := \left\lfloor \frac{2N}{K} - 2 \right\rfloor \geq 10 \). Thus we have \( h_1 < h_2 \). In addition, we have

\[
\frac{N}{K} + \frac{Nh_2}{2(N + Kh_2 - K)} \leq \frac{N}{K} + \frac{\frac{2N}{K} - 2}{2(N + Kh_2 - K)} = \frac{4N}{3K}.
\]

Hence, the above memory regime division is possible.

From the converse bound in (23), for each \( h \in [0 : \left\lfloor 2N/K - 3 \right\rfloor] \) we have,

\[
R^*_u \geq \frac{\left\lfloor K/2 \right\rfloor \cdot 2N/K}{\left\lfloor K/2 \right\rfloor \cdot 2N/K} \left\{ N - 2y - \frac{8y + h(2N - K)}{2h + 4} + \frac{h^2K(2N - K) - 2N(2N - 3K) + hK(K + 2N)}{(h + 1)(h + 2)KN} \right\}
\]
\[ \geq \frac{6}{13} \left\{ N - 2y - \frac{8y + h(2N - K)}{2h + 4} + \frac{h^2K(2N - K) - 2N(2N - 3K) + hK(2N)}{(h + 1)(h + 2)KN} y \right\}, \quad (143) \]

where (143) comes from that \( K \geq 3 \) and \( N > 6K \).

In Regimes 1 and 2, we will use (143) as the converse bound. In Regime 3, we use the shared-link caching converse bound under the constraint of uncoded cache placement in [10].

A. Regime 1

It can be computed that the converse bound in (143) for \( \frac{N}{K} \leq M \leq \frac{N}{K} + \frac{Nh_1}{2(N + Kh_1 - K)} \) is a piecewise curve with the corner points

\[ \left( \frac{N}{K} + \frac{Nh' K}{2(N + Kh' - K)}, \frac{6 K(h' - 1)(2N + Kh') + 2N(2N - K)}{13(4h' + 1)(N + Kh' - K)} \right), \forall h' \in [0 : h_1], \quad (144) \]

where \( h' = 0 \) represents the first corner point where \( M = N/2 \), and each corner point in (144) with \( h' \) is obtained by taking the intersection of the converse bounds in (143) between \( h = h' - 1 \) and \( h = h' \).

For the achievability, we take the memory-sharing between \( \left( \frac{N}{K}, N \right) \) and \( \left( \frac{N + t_3 - 1}{K}, \frac{U - t_3 + 1}{t_3} \right) \), where \( t_3 = 2K - 3 \). Notice that

\[ \frac{N + t_3 - 1}{K} = \frac{N + 2K - 4}{K} = \frac{N}{K} + \frac{2K - 4}{K}. \]

(145)

In addition, we have

\[ \frac{N}{K} + \frac{Nh_1}{2(N + Kh_1 - K)} = \frac{N}{K} + \frac{Nh_1}{2(N + Kh_1)} \quad (146a) \]

\[ \leq \frac{N}{K} + \frac{N(4(K-2)(N-K))}{2(4(K-2)(N-K) - K)} \quad (146b) \]

\[ = \frac{N}{K} + \frac{4N(K-2)(N-K)}{2((N-K)K(N-4K+8) + 4K(K-2)(N-K))} \quad (146c) \]

\[ = \frac{N}{K} + \frac{4N(K-2)(N-K)}{2KN(N-K)} \quad (146d) \]

\[ = \frac{N}{K} + \frac{2K - 4}{K}, \quad (146e) \]

where (146b) comes from \( \frac{Nh_1}{2(N + Kh_1 - K)} \) is increasing with \( h_1 \) and \( h_1 \leq \frac{4(K-2)(N-K)}{K(N-4K+8)} \). From (145) and (146e), we can see that this memory-sharing can cover all memory sizes in regime 1.

When \( h' = 0 \), we have the corner point in (144) is \( \left( \frac{N}{2}, \frac{6N}{13} \right) \), while Scheme A achieves \( \left( \frac{N}{2}, N \right) \). Hence, the multiplicative gap between Scheme A and the converse is \( \frac{13}{6} \).
For each $h' \in [h_1]$, we now interpolate Scheme A between $(M_1, R_1) = \left( \frac{N}{R}, N \right)$ and $(M_2, R_2) = \left( \frac{N + h' - 1}{R}, \frac{N + h' + 1}{t_3} \right)$ to match the corner point in the converse bound

$$(M_3, R_3) = \left( \frac{N}{R} + \frac{N h'}{2(N + h' - K)}, \frac{6}{13} \frac{K(h' - 1)(2N + Kh') + 2N(2N - K)}{4(h' + 1)(N + Kh' - K)} \right).$$

More precisely, by memory-sharing between $(M_1, R_1)$ and $(M_2, R_2)$ with coefficient

$$\alpha = \frac{M_2 - M_3}{M_2 - M} = \frac{N(4K - h'K - 8) + 4K(h' - 1)(K - 2)}{4(K - 2)(N + h'K - K)}$$

such that $\alpha M_1 + (1 - \alpha)M_2 = M_3$, we get at $M_3$ Scheme A can achieve,

$$R' = \alpha R_1 + (1 - \alpha)R_2 = N\frac{-12N + 8K^2(h' - 1) + K(N(8 - h') - 14h' + 12)}{4(2K - 3)(N + h'K - K)}.$$  

In the following, we compare $R'$ and $R_3$ to obtain

$$\frac{R'}{R_3} = \frac{13(N(h' + 1)\left\{-12N + 8K^2(h' - 1) + K(N(8 - h') - 14h' + 12)\right\}}{6(2K - 3)(K(h' - 1)(2N + Kh') + 2N(2N - K))}.$$  

Finally, we will prove

$$\frac{6R'}{13R_3} = \frac{N(h' + 1)\left\{-12N + 8K^2(h' - 1) + K(N(8 - h') - 14h' + 12)\right\}}{(2K - 3)(K(h' - 1)(2N + Kh') + 2N(2N - K))} < 8.$$  

We can compute that

$$8(2K - 3)(K(h' - 1)(2N + Kh') + 2N(2N - K)) - N(h' + 1)\left\{-12N + 8K^2(h' - 1) + K(N(8 - h') - 14h' + 12)\right\}$$

$$\geq 8(2K - 3)(K(h' - 1)(2N + Kh') + 2N(2N - K)) - N(h' + 1)\left(-12N + 8K^2(h' - 1) + KN(8 - h')\right)$$

$$= (32(2K - 3) + 12(h' + 1) - K(8 - h')(h' + 1))N^2 - (8K(h' + 1)(h' - 1) - 16(2K - 3)(h' - 2))KN$$

$$+ 8(2K - 3)K^2h'(h' - 1)$$

$$\geq (32(2K - 3) + 12(h' + 1) - K(8 - h')(h' + 1))N^2 - (8(h' + 1)(h' - 1) - 16(h' - 2))K^2N$$

$$+ 8(2K - 3)K^2h'(h' - 1),$$

where (151a) comes from $h' \geq 1$ and (151b) comes from $K \geq 3$.

Recall that $N > 6K$, and that $h' \leq h_1 = \left\lfloor \frac{4(K - 2)(N - K)}{K(N - 4K + 8)} \right\rfloor < 10$.

We first focus on $h' = 9$. If $h' = 9$, it can be seen that $6K < N < \frac{32}{5} K$. Hence, we have

$$8(2K - 3)K^2h'(h' - 1) > \frac{5}{4}(2K - 3)KNh'(h' - 1) \geq \frac{5}{4}K^2Nh'(h' - 1) = 90K^2N.$$  

We take $h' = 9$ and (152) into (151c) to obtain

$$8(2K - 3)(K(h' - 1)(2N + Kh') + 2N(2N - K)) - N(h' + 1)\left\{-12N + 8K^2(h' - 1) + K(N(8 - h') - 14h' + 12)\right\}$$
\[ (74K + 24)N^2 - (640 - 112 - 90)K^2N \]  
\[ 74KN^2 - 438K^2N \]  
\[ > 0, \]  
\[ \text{where (153c) comes from } N > 6K. \]

We then focus on \( h' = 8 \). If \( K = 3 \), from (151c), we have the RHS of (151c) becomes \[ 204N(N - 18) + 12096 \], which is larger than 0 since \( N > 6K \geq 18 \). Now we consider \( K \geq 4 \). From (151b), we have

\[
8(2K-3)(K(h' - 1)(2N+Kh') + 2N(2N - K)) - N(h' + 1)\{ -12N + 8K^2(h' - 1) + K(N(8 - h') - 14h' + 12) \} \\
> (32(2K - 3) + 12(h' + 1) - K(8 - h')(h' + 1))N^2 - (8K(h' + 1)(h' - 1) - 16(2K - 3)(h' - 2))KN \\
\geq (32(2K - 3) + 12(h' + 1) - K(8 - h')(h' + 1))N^2 - (8K(h' + 1)(h' - 1) - 20K(h' - 2))KN \\
= ((56 + h'^2 - 7h')K + 12h' - 84)N^2 - (32 + 8h'^2 - 20h')K^2N \\
\geq (56 + h'^2 - 7h')KN^2 - (32 + 8h'^2 - 20h')K^2N \\
> 6(56 + h'^2 - 7h')K^2N - (32 + 8h'^2 - 20h')K^2N \\
= 0,
\]

where (154b) comes from \( K \geq 4 \) and thus \( \frac{2K-3}{K} \geq \frac{5}{4} \), and (154c) comes from \( N > 6K \).

Lastly, we consider \( h' \in [7] \). From (151c), we have

\[
8(2K-3)(K(h' - 1)(2N+Kh') + 2N(2N - K)) - N(h' + 1)\{ -12N + 8K^2(h' - 1) + K(N(8 - h') - 14h' + 12) \} \\
> (32(2K - 3) + 12(h' + 1) - K(8 - h')(h' + 1))N^2 - (8(h' + 1)(h' - 1) - 16(h' - 2))K^2N \\
= ((56 + h'^2 - 7h')K + 12h' - 84)N^2 - (24 + 8h'^2 - 16h')K^2N \\
\geq (56 + h'^2 - 7h' + 4h' - 28)KN^2 - (24 + 8h'^2 - 16h')K^2N \\
> 6(28 + h'^2 - 3h')K^2N - (24 + 8h'^2 - 16h')K^2N \\
= (144 - 2h'^2 - 2h')K^2N \\
> 0
\]
where (155c) comes from \( h' \leq 7 \) and \( K \geq 3 \), which lead to \( 12h' - 84 \geq (4h' - 28)K \), and (155d) comes from \( N > 6K \), and (155f) comes from \( h' \in [7] \).

In conclusion, we prove (150). In other words, under the constraint of uncoded cache placement and user collusion, Scheme A is order optimal within a factor of \( \frac{13}{6} \times 8 < 18 \) for the memory size Regime 1.

**B. Regime 2**

Similar to the converse bound for Regime 1, it can be computed that the converse bound in (143) for \( \frac{N}{K} + \frac{Nh'}{2(N+Kh'-K)} \leq M \leq \frac{N}{K} + \frac{Nh}{2(N+Kh'-K)} \) is a piecewise curve with the corner points

\[
\left( \frac{N}{K} + \frac{Nh'}{2(N+Kh'-K)}, \frac{6K(h' - 1)(2N + Kh') + 2N(2N - K)}{13(4(h' + 1)(N + Kh' - K))} \right), \forall h' \in [h_1 : h_2]. \tag{156}
\]

For the achievability, we take the memory-sharing among the achieved points in (120), \( \frac{N+t-1}{K}, \frac{U-t+1}{t} \), where \( t \in [U+1] \). We want to interpolate the achieved points of Scheme A to match the converse bound at the memory size \( M = \frac{N}{K} + \frac{Nh'}{2(N+Kh'-K)} \) where \( h' \in [h_1 : h_2] \). By computing

\[
\frac{N+t-1}{K} = \frac{N}{K} + \frac{Nh'}{2(N+Kh'-K)} \iff t = \frac{N}{2(N+Kh'-K)} + 1, \tag{157}
\]

and observing \( \frac{U-t+1}{t} \) is non-increasing with \( t \), it can be seen that the achieved load of Scheme A at \( M = \frac{N}{K} + \frac{Nh'}{2(N+Kh'-K)} \) is lower than

\[
R' = \frac{U - \frac{Nh'/K}{2(N+Kh'-K)}}{\frac{Nh'/K}{2(N+Kh'-K)}} + 1. \tag{158}
\]

By comparing \( R' \) and \( \frac{6K(h' - 1)(2N + Kh') + 2N(2N - K)}{4(h' + 1)(N + Kh' - K)} \), we have

\[
\frac{R'}{R_3} = \frac{13}{6} \frac{4(N + Kh' - K)(h' + 1)(2K^2N(h' - 1) + K(2N^2 + 2N + 2h' - 3Nh' - 2) - 2N(N - 1))}{KNh'(K(h' - 1)(2N + Kh') + 2N(2N - K))} \tag{159}
\]

Since \( K \geq 3 \), we have

\[
h' \geq h_1 = \left\lfloor \frac{4(K-2)(N-K)}{K(N-4K+8)} \right\rfloor \geq \left\lfloor \frac{2(N-K)}{N-4K+8} \right\rfloor > 2; \tag{160a}
\]

\[
h' \leq h_2 = \left\lfloor \frac{2N}{K} - 2 \right\rfloor < \frac{2N}{K}. \tag{160b}
\]
In the following, we will use (160) and $N > 6K \geq 18$ to prove
\[
\frac{6R'}{13R_3} = \frac{4(N + Kh' - K)(h' + 1)(2K^2N(h' - 1) + K(2N^2 + 2N + 2h' - 3Nh' - 2) - 2N(N - 1))}{KNh'(K(h' - 1)(2N + Kh') + 2N(2N - K))} < 8.
\]  
(161)

We can compute that
\[
8KNh'(K(h' - 1)(2N + Kh') + 2N(2N - K)) \\
- 4(N + Kh' - K)(h' + 1)(2K^2N(h' - 1) + K(2N^2 + 2N + 2h' - 3Nh' - 2) - 2N(N - 1)) \\
\geq 8KNh'(K(h' - 1)(2N + Kh') + 2N(2N - K)) \\
- 4(N + Kh' - K)(h' + 1)(2K^2N(h' - 1) + K(2N^2 + 2N + 2h' - 3Nh' - 2)) \\
= 8K(N - K) + 8K^3N(h' - 1) + 4KN^2(h' - 2) + 8KN(3N^2h' - 4KNh' - N^2) \\
+ 4Kh'(3KNh'^2 - 3KN - 2Kh') + 4KNh'^2(3N - 2K - 2) + 8K^2N + 16KN^2 + 8K^2h' + 8K^2h'^2
\]  
(162a)
\[
> 8K(N - K) + 8K^3N(h' - 1) + 4KN^2(h' - 2) + 8KN(3N^2h' - 4KNh' - N^2) \\
+ 4Kh'(3KNh'^2 - 3KN - 2Kh') + 4KNh'^2(3N - 2K - 2)
\]  
(162b)
\[
> 8KN(3N^2h' - 4KNh' - N^2) + 4Kh'(3KNh'^2 - 3KN - 2Kh')
\]  
(162c)
\[
= 8KN(N^2h' - N^2) + 8KN(2N^2h' - 4KNh') + 4Kh'(KNh'^2 - 3KN) + 4Kh'(2KNh'^2 - 2Kh'^2)
\]  
(162d)
\[
> 0,
\]  
(162e)

where (162d) and (162f) come from $N > 6K$ and $h' > 2$.

In conclusion, we prove (161). In other words, under the constraint of uncoded cache placement and user collusion, Scheme A is order optimal within a factor of $\frac{13}{6} \times 8 < 18$ for the memory size Regime 2.

C. Regime 3

When $\frac{N}{R} \leq M \leq \frac{2N}{K}$, the converse bound in [10] is a straight line between $(\frac{N}{R}, \frac{K-1}{2})$ and $(\frac{2N}{K}, \frac{K-2}{3})$, which is denoted by $R_{[10]}(M)$. Hence, the converse bound in [10] for Regime 3 where $\frac{N}{R} + \frac{Nh_2}{2(N + Kh_2 - K)} \leq M \leq \frac{2N}{K}$ is a straight line. When $M = \frac{2N}{K}$, we proved in Appendix D-A that the multiplicative gap between Scheme A and the converse bound in [10] is no more than 6. Hence, in the rest of this proof, we focus on the memory size $M = \frac{N}{R} + \frac{Nh_2}{2(N + Kh_2 - K)} \leq M \leq \frac{2N}{K}$.
Recall that \( h_2 := \lfloor \frac{2N}{K} - 2 \rfloor \leq \frac{2N}{K} - 2 \), we note that

\[
\frac{N}{K} + \frac{Nh_2}{2(N + Kh_2 - K)} \leq \frac{N}{K} + \frac{N \left( \frac{2N}{K} - 2 \right)}{2 \left( N + K \left( \frac{2N}{K} - 2 \right) - K \right)} = \frac{4N}{3K}.
\]  

(163)

Hence, the load of the converse bound in [10] at \( M = \frac{N}{K} + \frac{Nh_2}{2(N + Kh_2 - K)} \) is strictly higher than the one at \( M' = \frac{4N}{3K} \). By computing the converse bound in [10] at \( M' = \frac{4N}{3K} \) is

\[
R_{[10]}(M) = \frac{2K - 1}{3} + \frac{1K - 2}{3} = \frac{4K - 5}{9},
\]

(164)

at \( M = \frac{N}{K} + \frac{Nh_2}{2(N + Kh_2 - K)} \), we have

\[
R^*_{u,c} \geq R_{[10]}(M) > R_{[10]}(M') = \frac{4K - 5}{9}.
\]

(165)

For the achievability, it was proved in (158) that the achieved load of Scheme A at \( M = \frac{N}{K} + \frac{Nh_2}{2(N + Kh_2 - K)} \) is lower than

\[
R' = \frac{U - \frac{Nh_2K}{2(N + Kh_2 - K)} + 1}{\frac{Nh_2K}{2(N + Kh_2 - K)}} \leq \frac{U - \frac{N(2N/K - 3)K}{2(N + K(2N/K - 3) - K)} + 1}{\frac{N(2N/K - 3)K}{2(N + K(2N/K - 3) - K)}} = \frac{(6K - 8)N^2 - (8K - 11)KN + 6N - 8K}{2N^2 - 3KN},
\]

(166a)

(166b)

(166c)

where (166b) comes that \( \frac{U - \frac{Nh_2K}{2(N + Kh_2 - K)} + 1}{\frac{Nh_2K}{2(N + Kh_2 - K)}} \) is non-increasing with \( t \), and that \( h_2 \leq 2N/K - 3 \).

Finally, we compare \( R' \) and \( \frac{4K - 5}{9} \) to obtain,

\[
\frac{R'}{\frac{4K - 5}{9}} = \frac{9(6K - 8)N^2 - (8K - 11)KN + 6N - 8K}{(2N^2 - 3KN)(4K - 5)}.
\]

(167)

In addition, we compute

\[
2(2N^2 - 3KN)(4K - 5) - ((6K - 8)N^2 - (8K - 11)KN + 6N - 8K)
\]

\[
= 2N(5KN - 6N - 8K^2) + (19KN - 6N) + 8K
\]

\[
> 2N(5KN - 6N - 8K^2)
\]

\[
\geq 2N(3KN - 8K^2)
\]

\[
\geq 0,
\]

(168a)

(168b)

(168c)

(168d)

where (168b) and (168c) come from \( K \geq 3 \), and (168d) comes from \( N > 6K \). By taking (168d) into (167), it can be seen that the multiplicative gap between Scheme A and the converse bound in [10] at \( M = \frac{N}{K} + \frac{Nh_2}{2(N + Kh_2 - K)} \) is less than 18.
In conclusion, we prove that under the constraint of uncoded cache placement and user collusion, Scheme A is order optimal within a factor of 18 for the memory size Regime 3.

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