On the bulk block expansion
for a monodromy defect

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For a free–field flat monodromy defect, a formula for the finite part
of the correlator is obtained as a double power series in (1 − x) and (1 − \bar{x})
where x and \bar{x} are lightcone coordinates. It takes the particular form
of a series in (1 − x) with coefficients finite sums of hypergeometric
functions of 1 − \bar{x} and is identified with a bulk block expansion. A
simple expression for the coefficient of the (1 − x)^n(1 − \bar{x})^m term is
thereby found as an explicit function of the flux and dimension. Some
typical examples are presented.
1. Introduction

This brief communication is a strictly technical addendum to two previous reports, [1], [2], concerning basic CFT quantities for the simplest codimension 2 monodromy defect. The set-up and basic structures have been detailed in these reports, and references therein, and so, to save space, I will assume that all the equations in [1] and [2] are available.

The question addressed here is the form of the bulk block expansion of the free–field correlator (Green function), G. The defect block expansion is relatively straightforward, amounting to the easily effected Fourier decomposition of G. The bulk block case is not so simple even for free fields. Its structure is outlined in the next section where the precise objective of the present calculation is explained.

2. The bulk block expansion

The relevant bulk block expansion for free fields has been given in a certain form in [3] where other references are given. It is algebraically convenient to use, as there, the (independent) lightcone Lorentzian coordinates x and \( \overline{\tau} \) on the two-dimensional space orthogonal to the planar defect and the bulk expansion reads, [3],

\[
G(x, \overline{\tau}) = \left( \frac{\sqrt{x\overline{\tau}}}{(1-x)(1-\overline{\tau})} \right)^\Delta \left( 1 + \sum_{l=0}^{\infty} c_l f_{2\Delta+l,l}(x, \overline{\tau}) \right). \tag{1}
\]

The bulk blocks take the form, in this case,\(^2\)

\[
(1-x)(1-\overline{\tau})^{-\Delta} f_{2\Delta+l,l}(x, \overline{\tau}) \equiv \overline{f}_{2\Delta+l,l}(x, \overline{\tau}) = \\
\sum_{n=0}^{\infty} \sum_{j=-n}^{n} A_{n,j}(\Delta, l) (1-x)^n (1-\overline{\tau})^{l+j} 2F_1(\Delta + l + j, \Delta + l + j; 2(\Delta + l + j); 1-\overline{\tau}), \tag{2}
\]

with the constants \( A_{n,j} \) separately determined by recursion. One objective is to compute the coefficients \( c_l \) as functions of \( \Delta \) and \( \delta \), although I will not accomplish this here. Rather, I will recover the composition (1) just in the sense of being a double power series in \( 1-x \) and \( 1-\overline{\tau} \). This is the limited objective of the present note. The basic formulae now follow.

I remark that (1) has been written using the CFT correlator normalisation adopted in [3]. The normalisation used in [1,2], and here, is the standard QFT one,

\(^2\) \( \Delta \) here stands for \( \Delta^\phi \) which equals \( d/2 - 1 \) for standard fields.
in terms of which (1) reads,
\[ G(x, \overline{x}, \delta) = G(x, \overline{x}, 0) \left( 1 + \sum_{l=0}^{\infty} c_l \int_{2\Delta+l,l}(x, \overline{x}) \right), \] (3)
which can be rearranged to the form,
\[ G_{\text{sub}}(x, \overline{x}, \delta) = N(\sqrt{x\overline{x}})^\Delta \sum_{l=0}^{\infty} c_l \int_{2\Delta+l,l}(x, \overline{x}), \] (4)
where \( G_{\text{sub}} \) is the subtracted ‘Green function’ \( (G(\delta) - G(0)) \) derived in [2] in terms of the Appell \( F_1 \) function, viz,
\[ G_{\text{sub}} = NC x^\delta \sqrt{x\overline{x}} F_1(\Delta + \delta, \Delta, 1, 2\Delta + 1, 1 - x\overline{x}, 1 - x). \] (5)
\( N \) is a convention dependent normalisation, and cancels from (4). \( C \) is a constant that results from the calculation of \( G_{\text{sub}} \) as the integral of a cut discontinuity and equals, \(^3\)
\[ C = \frac{\Gamma(\Delta - \delta + 1) \Gamma(\Delta + \delta) \sin \pi \delta}{\pi \Gamma(2\Delta + 1)}. \] (6)
I leave this factor understood and put it back at the end.

The \( \int \) have the general power series expansion (see [3] equn.(2.30)),
\[ \int_{2\Delta+l,l}(x, \overline{x}) = \sum_{n,m \geq 0} k_{n,m}(\Delta, l) (1 - x)^n (1 - \overline{x})^m, \]
so that (4) is,
\[ G_{\text{sub}}(x, \overline{x}, \delta) = N(\sqrt{x\overline{x}})^\Delta \sum_{n,m \geq 0} C_{n,m} (1 - x)^n (1 - \overline{x})^m, \] (7)
where,
\[ C_{n,m}(\Delta, \delta) = \sum_{l=0}^{\infty} c_l(\Delta, \delta) k_{n,m}(\Delta, l), \] (8)
and the restricted aim here, as mentioned, is the determination of the ‘total’ coefficients, \( C_{n,m} \). From (7) and (5) this amounts to finding the double series expansion of the quantity,
\[ \mathcal{F}(1 - x, 1 - \overline{x}) \equiv x^\delta F_1(\Delta + \delta, \Delta, 1, 2\Delta + 1, 1 - x\overline{x}, 1 - x) \equiv x^\delta F_1(a, b, 1, c, 1 - x\overline{x}, 1 - x). \] (9)

\(^3\) This constant is denoted by \( -C_{\text{free}} \) in [3].
First, the coefficient of $(1-x)^n$ is sought, and then that of $(1-x)^m$ in this is to be found.

Expanding (9) in powers of $(1-x)$ (which is now denoted $\mu$) reduces to the expansion of,

$$(1-\mu)^\delta F_1(a, b, 1; c; \mu + \overline{\mu} - \mu \overline{\mu}, \mu),$$

in powers of $\mu$ which is a slightly awkward process, as it stands.

However, the calculation is eased considerably by separating the $\mu$ and $\overline{\mu}$ dependencies in the $F_1$ through applying the transformation, [4] p.30, eqn.(54),

$$F_1(a, b, b'; c; x, y) = (1 - y)^{-a}F_1(a, b, c - b - b', c; \frac{y-x}{y-1}, \frac{y}{y-1}),$$

which yields a handier form for $G_{\text{sub}}$ than (5) and gives for the right-hand side of (9),

$$(1-\mu)^\delta F_1(a, b, 1, c; \mu + \overline{\mu} - \mu \overline{\mu}, \mu)
= (1-\mu)^{\delta-a}F_1(a, b, c - 1 - b, c; \frac{\overline{\mu} \mu - \overline{\mu}}{\mu - 1}, \frac{\mu}{\mu - 1})$$

which is simpler to expand in $\mu$.

The outside factor can be treated by binomial expansion so I just concentrate on the $r$th derivative of the $F_1$ in (10) with respect to $\mu$.

The derivatives of any function of $\frac{\mu}{\mu - 1}$ with respect to $\mu$, evaluated at 0, can be expressed formally in terms of the generalised Laguerre polynomial, $L_n^{-1}(x)$, by,

$$\left.\frac{1}{r!} \partial^r \mu F\left(\frac{\mu}{\mu - 1}\right)\right|_{\mu=0} = L_r^{-1}(\partial_\mu F(\mu))|_{\mu=0}$$

$$= \sum_{j=0}^{r} \frac{(-1)^j}{j!} \left(\frac{r - 1}{r - j}\right) \partial_\mu^j F(\mu)|_{\mu=0}.$$  \hspace{1cm} (11)

Since the $j$th derivative at 0 of the $F_1$ in (10) with respect to the second variable slot reduces to a hypergeometric function via, [4] p.15 and p.19 eqn.(19),

$$F_1^{(j)}(\overline{\mu}, 0) = \frac{(a)_j(c')_j}{(c)_j} F_1(a + j, b, c' + j, c + j; \overline{\mu}, 0) = \frac{(a)_j(c')_j}{(c)_j} 2 F_1(a + j, b, c + j; \overline{\mu}),$$

4 This follows from a fractional coordinate transformation of the Picard integral representation of $F_1$.  

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the required \( r \)th derivative of \( F_1 \) is, from (11),
\[
\frac{\partial^r}{\partial \mu^r} F_1(a, b, c', c; \mu, \frac{\mu}{\mu - 1}) \bigg|_{\mu=0} = r! \sum_{j=0}^r \frac{(-1)^j}{j!} \binom{r - 1}{r - j} \frac{(a)_j(c')_j}{(c)_j} \binom{a + j, b, c + j; \mu}{2F_1(a + j, b, c + j; \mu)}. \tag{12}
\]

The next step is to combine (12) with the factor \((1 - \mu)^{-\Delta}\) in (10) to give the \( n \)th derivative, at \( \mu = 0 \). This produces,
\[
\frac{\partial^n}{\partial \mu^n} F(\mu, \mu) \bigg|_{\mu=0} = \sum_{r=0}^n \binom{n}{r} (\Delta)_{n-r} \sum_{j=0}^r \frac{(-1)^j}{j!} \binom{r - 1}{r - j} \frac{(a)_j(c')_j}{(c)_j} \binom{a + j, b, c + j; \mu}{2F_1(a + j, b, c + j; \mu)}. \tag{13}
\]

I note that the same equation holds when \( \mu \) and \( \bar{\mu} \) are interchanged, if, at the same time, \( \delta \) is replaced by \( 1 - \delta \).

Swapping round summations in (13) allows one to define ‘universal’ coefficients, polynomials in \( \Delta \), by,
\[
U_{n,j}(\Delta) \equiv \sum_{r=j}^n \binom{n}{r} (\Delta)_{n-r} \sum_{j=0}^r \frac{(-1)^j}{j!} \binom{r - 1}{r - j} \frac{(a)_j(c')_j}{(c)_j} \binom{a + j, b, c + j; \mu}{2F_1(a + j, b, c + j; \mu)},
\]
and (13) takes the more compact form,
\[
\frac{\partial^n}{\partial \mu^n} F(\mu, \mu) \bigg|_{\mu=0} = \sum_{j=0}^n (-1)^j U_{n,j}(\Delta) \frac{(a)_j(c')_j}{j!(c)_j} \binom{a + j, b, c + j; \mu}{2F_1(a + j, b, c + j; \mu)}. \tag{14}
\]

To a factor of \( n! \), (14) is the coefficient of \((1 - x)^n\) in the bulk block expansion and one sees that it is given by a \textit{finite} sum of hypergeometric functions of \((1 - \bar{x})\) which seems to be a different organisation compared to existing formulations.

The coefficient of \( \bar{\mu}^m \) in (14) can now be found using the definition of the hypergeometric function. This quickly leads to the coefficient of the term \( \mu^n \bar{\mu}^m / n!m! \) in the bulk block expansion which was my ultimate objective,
\[
C_{n,m}(\Delta, \delta) \equiv C(\Delta, \delta) \sum_{j=0}^n (-1)^j U_{n,j}(\Delta) \frac{(a)_j(c')_j}{j!(c)_j} \frac{(a + j)(b)_m}{(c + j)_m} \binom{a + j, b, c + j; \mu}{2F_1(a + j, b, c + j; \mu)} \tag{15}
\]
where the constant \( C \), (6), has been reinstated and the parameters \( a = \Delta + \delta, b = \Delta, b' = 1, c' = \Delta, c = 2\Delta + 1 \) inserted resulting in a final, easily calculable formula.
3. A few examples and properties of the coefficients

It is possible that further algebraic reductions could be found for (15) but, for the present, just some particular, lower order cases will be given in order to exhibit the general structure.

The ratio $C'_{n,m} = C_{n,m}/C$ is listed for a typical set of coefficients,

$$C'_{0,1} = \frac{\Delta(\delta + \Delta)}{1 + 2\Delta}, \quad C'_{0,2} = \frac{\Delta(\delta + \Delta)(1 + \delta + \Delta)}{2(1 + 2\Delta)},$$

$$C'_{4,0} = \frac{\Delta(3 + \Delta)(4 - \delta + \Delta)(4 - \delta + \Delta)(2 - \delta + \Delta)(1 - \delta + \Delta)}{4(3 + 4\Delta(2 + \Delta))},$$

$$C'_{3,1} = \frac{\Delta(3 - \delta + \Delta)(2 - \delta + \Delta)(1 - \delta + \Delta)(\delta + \Delta)}{4(3 + 4\Delta(2 + \Delta))},$$

$$C'_{2,2} = \frac{\Delta^2 (1 + \Delta)(2 - \delta + \Delta)(1 - \delta + \Delta)(\delta + \Delta)(1 + \delta + \Delta)}{4(2 + \Delta)(1 + 2\Delta)(3 + 2\Delta)}.$$

The expressions simplify at the midpoint $\delta = 1/2$ ($\mathbb{Z}_2$ monodromy). For example, at this point,

$$C'_{6,0} = \frac{1}{512} \Delta(4 + \Delta)(5 + \Delta)(7 + 2\Delta)(9 + 2\Delta)(11 + 2\Delta),$$

$$C'_{5,1} = \frac{1}{512} \Delta^2(4 + \Delta)(1 + 2\Delta)(7 + 2\Delta)(9 + 2\Delta),$$

$$C'_{4,2} = \frac{1}{512} \Delta^2(1 + \Delta)(7 + 2\Delta)(3 + 4\Delta(2 + \Delta)),$$

$$C'_{3,3} = \frac{1}{512(3 + \Delta)} \Delta^2(1 + \Delta)(2 + \Delta)(1 + 2\Delta)(3 + 2\Delta)(5 + 2\Delta).$$

The coefficients satisfy the exchange requirement, $C_{n,m}(\Delta, \delta) = C_{m,n}(\Delta, 1-\delta)$. This is not obvious from (15) and so provides a useful algebraic check.

The typical variation of a coefficient with flux and dimension is shown below in Fig.1. Physical quantities have to be made periodic in $\delta$ with period 1. Since there is some significance in the value at $\delta + 1$, the range of $\delta$ has been doubled.

As functions of $\Delta$, the coefficients vanish at $\Delta = 0$, i.e. in dimension $d = 2$, then rise to a maximum (if $0 < \delta < 1$) followed by a monotonic decrease to zero as $\Delta \to \infty$. 5
Fig1. Variation of the coefficient $C_{3,1}(\Delta, \delta)$ with flux, $\delta$, and dimension, $\Delta$. The range of $\delta$ has been extended to twice its unit cell.

4. Comments and conclusion

From existing formulae in the literature, e.g. [3], the coefficient of $\mu^n$, is, (see (2)),

$$\sum_{j=-n}^{n} A_{n,j}(\Delta, l) \sum_{l=0}^{\infty} c_l (1 - \overline{x})^{l+j} {\text{2}_F_1}(\Delta + l + j, \Delta + l + j, 2(\Delta + l + j); 1 - \overline{x}),$$

(16)

and it would seem a difficult task to reconcile this with (14) in general although individual coefficients could be compared. At the moment, I am not able to do this and there remains the problem of extracting the $c_l$ constants from (8).

The fact that the interchange symmetry, $C_{n,m}(\delta) = C_{m,n}(1 - \delta)$, is not immediately apparent from (15) suggests that there is a significant rearrangement that would make this so.

I note that physical quantities show a discontinuity in the $\delta$-derivative at the unit cell boundary.

The use of the subtracted $G_{\text{sub}}$, rather than the complete $G$, [1], [5], bypasses any divergence issues.
References.

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