LARGE GAPS BETWEEN CONSECUTIVE ZEROS OF THE RIEMANN
ZETA-FUNCTION. II

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Abstract. Assuming the Riemann Hypothesis we show that there exist infinitely many
consecutive zeros of the Riemann zeta-function whose gaps are greater than 2.9 times the
average spacing.

1. Introduction

Subject to the truth of the Riemann Hypothesis (RH), the nontrivial zeros of the Riemann
zeta-function can be written as $\rho = \frac{1}{2} + i\gamma$, where $\gamma \in \mathbb{R}$. Denote consecutive
ordinates of zeros by $0 < \gamma \leq \gamma'$, we define the normalized gap
$$\delta(\gamma) := (\gamma' - \gamma) \frac{\log \gamma}{2\pi}.$$ It is well-known that
$$N(T) := \sum_{0 < \gamma \leq T} 1 = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$ for $T \geq 10$. Hence $\delta(\gamma)$ is 1 on average. It is expected that there are arbitrarily large and
arbitrarily small (normalized) gaps between consecutive zeros of the Riemann zeta-function
on the critical line, i.e.
$$\lambda := \limsup_{\gamma} \delta(\gamma) = \infty \quad \text{and} \quad \mu := \liminf_{\gamma} \delta(\gamma) = 0.$$

In this article, we focus only on the large gaps, and prove the following theorem.

Theorem 1.1. Assuming RH. Then we have $\lambda > 2.9$.

Very little is known about $\lambda$ unconditionally. Selberg [16] remarked that he could prove
$\lambda > 1$. Conditionally, Bredberg [2] showed that $\lambda > 2.766$ under the assumption of RH
(see also [14,13,12,6,10] for work in this direction), and on the Generalized Riemann
Hypothesis (GRH) it is known that $\lambda > 3.072$ [11] (see also [8,15,13]). These results either
use Hall’s approach using Wirtinger’s inequality, or exploit the following idea of Mueller
[14].

Let $H : \mathbb{C} \to \mathbb{C}$ and consider the following functions
$$\mathcal{M}_1(H, T) = \int_0^T |H(\frac{1}{2} + it)|^2 dt$$ and
$$\mathcal{M}_2(H, T; c) = \int_{-c/L}^{c/L} \sum_{-c/L < \gamma \leq T} |H(\frac{1}{2} + i(\gamma + \alpha))|^2 d\alpha,$$
where $L = \log \frac{T}{2\pi}$. We note that if
$$h(c) := \frac{\mathcal{M}_2(H, T; c)}{\mathcal{M}_1(H, T)} < 1$$

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as \( T \to \infty \), then \( \lambda > c/\pi \), and if \( h(c) > 1 \) as \( T \to \infty \), then \( \mu < c/\pi \).

Mueller [14] applied this idea to \( H(s) = \zeta(s) \). Using \( H(s) = \sum_{n \leq T^{1-\varepsilon}} d_{2.2}(n)n^{-s} \), where the arithmetic function \( d_k(n) \) is defined in terms of the Dirichlet series

\[
\zeta(s)^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} \quad (\sigma > 1)
\]

for any real number \( k \), Conrey, Ghosh and Gonek [9] showed that \( \lambda > 2.337 \). Later, assuming GRH, they applied to \( H(s) = \zeta(s) \sum_{n \leq T^{1/2-\varepsilon}} n^{-s} \) and obtained \( \lambda > 2.68 \) [8]. By considering a more general choice

\[
H(s) = \zeta(s) \sum_{n \leq T^{1/2-\varepsilon}} d_r(n) P_{\log y/\log y}(n),
\]

where \( P(x) \) is a polynomial, Ng [15] improved that result to \( \lambda > 3 \) (using \( r = 2 \) and \( P(x) = (1 - x)^{30} \)). In the last two papers, GRH is needed to estimate some certain exponential sums resulting from the evaluation of the discrete mean value over the zeros in \( M_2(H, T; c) \). Recently, Bui and Heath-Brown [5] showed how one can use a generalization of the Vaughan identity and the hybrid large sieve inequality to circumvent the assumption of GRH for such exponential sums. Here we use that idea to obtain a weaker version of Ng’s result without provoking GRH. It is possible that Feng and Wu’s result \( \lambda > 3.072 \) can also be obtained just assuming RH by this method. However, we opt to work on Ng’s result for simplicity.

Instead of using the divisor function \( d(n) = d_2(n) \), we choose

\[
H(s) = \zeta(s) \sum_{n \leq y} h(n) P_{\log y/\log y}(n),
\]

where \( y = T^\vartheta \), \( P(x) \) is a polynomial and \( h(n) \) is a multiplicative function satisfying

\[
h(n) = \begin{cases} d(n) & \text{if } n \text{ is square-free}, \\ 0 & \text{otherwise}. \end{cases}
\]

(1)

In Section 3 and Section 4 we shall prove the following two key lemmas.

**Lemma 1.1.** Suppose \( 0 < \vartheta < \frac{1}{2} \). We have

\[
M_1(H, T) = \frac{AT(\log y)^9}{6} \int_0^1 (1 - x)^3 \left( \vartheta^{-1} P_1(x)^2 - 2 P_1(x) P_2(x) \right) dx + O(TL^8),
\]

where

\[
A = \prod_p \left( 1 + \frac{8}{p} \right) \left( 1 - \frac{1}{p} \right)^8
\]

and

\[
P_r(x) = \int_0^x t^r P(x - t) dt.
\]

**Lemma 1.2.** Suppose \( 0 < \vartheta < \frac{1}{2} \) and \( P(0) = P'(0) = 0 \). We have

\[
\sum_{0 < \gamma \leq T} H(\rho + i\alpha)H(1 - \rho - i\alpha) = \frac{ATL(\log y)^9}{6\pi} \int_0^1 (1 - x)^3 \text{Re} \left\{ \sum_{j=1}^{\infty} (i\alpha \log y)^j B(j; x) \right\} dx + O(TL^{9+\varepsilon})
\]
uniformly for \( \alpha \ll L^{-1} \), where
\[
B(j; u) = -\frac{2P_1(u)P_{j+2}(u)}{(j + 2)!} + \frac{2\vartheta P_2(u)P_{j+2}(u)}{(j + 2)!} + \frac{4\vartheta P_1(u)P_{j+3}(u)}{(j + 3)!}
- \frac{\vartheta}{(j + 2)!} \int_0^u t(\vartheta - t)^{j+2}P_1(u)P(u-t)dt
+ \frac{\vartheta}{(j + 1)!} \int_0^u t(\vartheta - t)^{j+1}P_2(u)P(u-t)dt - \frac{\vartheta}{6j!} \int_0^u t(\vartheta - t)^{j}P_3(u)P(u-t)dt.
\]

**Proof of Theorem 1.1.** We take \( \vartheta = \frac{1}{2} \). On RH we have
\[
\sum_{0 < \gamma \leq T} |H\left(\frac{1}{2} + i(\gamma + \alpha)\right)|^2 = \sum_{0 < \gamma \leq T} H(\rho + i\alpha)H(1 - \rho - i\alpha).
\]
Note that this is the only place we need to assume RH. Lemma 1.2 then implies that
\[
\int_{-c/L}^{c/L} \sum_{0 < \gamma \leq T} |H\left(\frac{1}{2} + i(\gamma + \alpha)\right)|^2 \, dx \sim \frac{AT(\log y)^9}{6\pi} \sum_{j=1}^{\infty} \frac{(-1)^j e^{2j+1}}{2j-1(2j + 1)} \int_0^1 (1 - x)^3 B(2j; x) \, dx.
\]
Hence
\[
h(c) = \frac{1}{2\pi} \sum_{j=1}^{\infty} \frac{(-1)^j e^{2j+1}}{2j-1(2j + 1)} \int_0^1 (1 - x)^3 B(2j; x) \, dx + o(1),
\]
as \( T \to \infty \). Consider the polynomial \( P(x) = \sum_{j=1}^{M} c_j x^j \). Choosing \( M = 6 \) and running Mathematica’s Minimize command, we obtain \( \lambda > 2.9 \). Precisely, with
\[
P(x) = 1000x^2 - 9332x^3 + 30134x^4 - 40475x^5 + 19292x^6,
\]
we have
\[
h(2.9\pi) = 0.99725 \ldots < 1,
\]
and this proves the theorem.

**Remark 1.1.** The above lemmas are unconditional. We note that in the case \( r = 2 \) apart from the arithmetical factor \( a_2 \) being replaced by \( A \), Lemma 1.1 is the same as what stated in [3] Lemma 2.1 (see also [3] Lemma 2.3), while Lemma 1.2, under the additional condition \( P(0) = P'(0) = 0 \), recovers Theorem 2 of Ng [15] (and also Lemma 2.6 of Bui [3]) without assuming GRH, though the latters are written in a slightly different and more complicated form. This is as expected because replacing the divisor function \( d(n) \) by the arithmetic function \( h(n) \) (as defined in [1]) in the definition of \( H(s) \) only changes the arithmetical factor in the resulting mean value estimates. This substitution, however, makes our subsequent calculations much easier. Our arguments also work if we set \( h(n) = d_r(n) \) when \( n \) is square-free for some \( r \in \mathbb{N} \) without much changes, but we choose \( r = 2 \) to simplify various statements and expressions in the paper.

**Remark 1.2.** In the course of evaluating \( M_2(H, T; c) \), we encounter an exponential sum of type (see Section 4.2)
\[
\sum_{n \leq y} h(n) P\left(\frac{\log y/n}{\log y}\right) \sum_{m \leq nT/2\pi} a(m)e\left(-\frac{m}{n}\right)
\]
for some arithmetic function \( a(m) \). At this point, assuming GRH, Ng [15] applied Perron’s formula to the sum over \( m \), and then moved the line of integration to \( \text{Re}(s) = 1/2 + \varepsilon \). The main term arises from the residue at \( s = 1 \) and the error terms in this case are easy
to handle. To avoid being subject to GRH, we instead use the ideas in [9] and [5]. That leads to a sum of type

$$\sum_{n \leq y} \mu(n)h(n)P\left(\frac{\log y/n}{\log y}\right).$$

This is essentially a variation of the prime number theorem, and here the polynomial \(P(x)\) is required to vanish with order at least 2 at \(x = 0\) (see Lemma 2.6). As a result, we cannot take the choice \(P(x) = (1 - x)^3\) as in [15]. Here it is not clear how to choose a “good” polynomial \(P(x)\). Our theorem is obtained by numerically optimizing over polynomials \(P(x)\) with degree less than 7. It is probable that by considering higher degree polynomials, we can establish Ng’s result \(\lambda > 3\) under only RH.

**Notation.** Throughout the paper, we denote 

\([n]_y := \log y/n \log y\).

For \(Q, R \in C^\infty([0, 1])\) we define 

\[Q_r(x) = \int_0^x t^r Q(x - t)dt \quad \text{and} \quad R_r(x) = \int_0^x t^r R(x - t)dt.\]

We let \(\varepsilon > 0\) be an arbitrarily small positive number, and can change from time to time.

**2. Various lemmas**

The following two lemmas are in [9] Lemma 2 and Lemma 3.

**Lemma 2.1.** Suppose that 

\[A(s) = \sum_{m=1}^{\infty} a(m)m^{-s}, \quad \text{where} \quad a(m) \ll_{\varepsilon} m^{\varepsilon}, \quad \text{and} \quad B(s) = \sum_{n \leq y} b(n)n^{-s}, \quad \text{where} \quad b(n) \ll_{\varepsilon} n^{\varepsilon}.\]

Then we have

\[\frac{1}{2\pi i} \int_{a+i}^{a+iT} \chi(1-s)A(s)B(1-s)ds = \sum_{n \leq y} \frac{b(n)}{n} \sum_{m \leq nT/2\pi} a(m)\left(-\frac{m}{n}\right) + O_{\varepsilon}(yT^{1/2+\varepsilon}),\]

where \(a = 1 + L^{-1}\).

**Lemma 2.2.** Suppose that 

\[A_j(s) = \sum_{n=1}^{\infty} a_j(n)n^{-s}\]

is absolutely convergent for \(\sigma > 1\), \(1 \leq j \leq k\), and that 

\[A(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_{j=1}^{k} A_j(s).\]

Then for any \(l \in \mathbb{N}\), we have

\[\sum_{n=1}^{\infty} \frac{a(ln)}{n^s} = \sum_{l_l = 1, \ldots, l_k = 1} \left( \sum_{n \geq 1 \left( n, \Pi_{i<j} l_i \right) = 1} \frac{a_j(l_j n)}{n^s} \right).\]

We shall need estimates for various divisor-like sums. Throughout the paper, we let 

\[F_\tau(n) = \prod_{p|n} \left( 1 + O(p^{-\tau}) \right),\]

for \(\tau > 0\) and the constant in the \(O\)-term is implicit and independent of \(\tau\).
Lemma 2.3. For any $Q \in C^\infty([0, 1])$, there exists an absolute constant $\tau_0 > 0$ such that

(i) $\sum_{an \leq y} \frac{h(an)Q([an]y)}{n} = C(\log y)^2 h(a) \prod_{p | a} \left(1 + \frac{2}{p}\right)^{-1} Q_1([a]y) + O(d(a)F_{\tau_0}(a)L)$,

(ii) $\sum_{an \leq y} \frac{h(an)Q([an]y) \log n}{n} = C(\log y)^3 h(a) \prod_{p | a} \left(1 + \frac{2}{p}\right)^{-1} Q_2([a]y) + O(d(a)F_{\tau_0}(a)L^2),$

where

$C = \prod_p \left(1 + \frac{2}{p}\right) \left(1 - \frac{1}{p}\right)^2.$

Proof. By a method of Selberg [16] we have

$\sum_{n \leq t} \frac{h(an)}{n} = \frac{C(\log t)^2}{2} h(a) \prod_{p | a} \left(1 + \frac{2}{p}\right)^{-1} + O(d(a)F_{\tau_0}(a)L)$

for any $t \leq T$. The first statement then follows from partial summation.

The second statement is an easy consequence of the first one. □

Lemma 2.4. For any $Q \in C^\infty([0, 1])$, we have

$\sum_{n \leq y} \frac{h(n)^2 \varphi(n)Q([n]y)}{n^2} \prod_{p | n} \left(1 + \frac{2}{p}\right)^{-2} = \frac{D(\log y)^4}{6} \int_0^1 (1 - x)^3 Q(x)dx + O(L^3),$

where

$D = \prod_p \left[1 + \frac{4(p-1)}{p^2} \left(1 + \frac{2}{p}\right)^{-2}\right] \left(1 - \frac{1}{p}\right)^4.$

Proof. The proof is similar to the above lemma. □

We need a lemma concerning the size of the function $F_{\tau_0}(n)$ on average.

Lemma 2.5. Suppose $-1 \leq \sigma \leq 0$. We have

$\sum_{n \leq y} \frac{d_k(n)F_{\tau_0}(n)}{n} \left(\frac{y}{n}\right)^\sigma \ll_k L^{k-1}\min\{|\sigma|^{-1}, L\}.$

Proof. We use Lemma 4.6 in [1] that

$\sum_{n \leq y} \frac{d_k(n)}{n} \left(\frac{y}{n}\right)^\sigma \ll_k L^{k-1}\min\{|\sigma|^{-1}, L\}.$

We have

$F_{\tau_0}(n) \leq \prod_{p | n} \left(1 + A p^{-\tau_0}\right) = \sum_{l | n} l^{-\tau_0} A^w(l)$

for some $A > 0$, where $w(n)$ is the number of prime factors of $n$. Hence

$\sum_{n \leq y} \frac{d_k(n)F_{\tau_0}(n)}{n} \left(\frac{y}{n}\right)^\sigma \ll \sum_{l \leq y} \frac{d_k(l) A^w(l)}{l^{1+\tau_0}} \sum_{n \leq y/l} \frac{d_k(n)}{n} \left(\frac{y/l}{n}\right)^\sigma \ll_k L^{k-1}\min\{|\sigma|^{-1}, L\},$

since $d_k(l) A^w(l) \ll l^{\tau_0/2}$ for sufficiently large $l$. □
Lemma 2.6. Let $F(n) = F(n, 0)$, where

$$F(n, \alpha) = \prod_{p|n} \left(1 - \frac{1}{p^{1+\alpha}}\right).$$

For any $Q \in C^\infty([0, 1])$ satisfying $Q(0) = Q'(0) = 0$, there exist an absolute constant $\tau_0 > 0$ and some $\nu \asymp (\log \log y)^{-1}$ such that

$$\mathcal{A}_1(y, Q; a, b, \alpha) = \sum_{an \leq y \atop (n, b) = 1} \frac{\mu(n)h(n)Q([an]_y)}{\varphi(n)n^{\alpha_1}} F(n, \alpha_2)F(n, \alpha_3)$$

$$= U_1V_1(b) \left(\frac{Q''([a]_y)}{(\log y)^2} + \frac{2\alpha_1 Q'([a]_y)}{\log y} + \alpha_2^2 Q([a]_y)\right)$$

$$+ O\left(F_0(b)L^{-3}\right) + O_{\varepsilon}\left(F_0(b)\left(\frac{y}{a}\right)^{-\nu} L^{-2+\varepsilon}\right)$$

uniformly for $\alpha_j \ll L^{-1}$, $1 \leq j \leq 3$, where $U_1 = U_1(0, \emptyset)$ and $V_1(n) = V_1(0, n, \emptyset)$, with

$$U_1(s, \alpha) = \prod_p \left[1 - \frac{2F(p, \alpha_2)F(p, \alpha_3)}{\varphi(p)p^{s+\alpha_1}}\right] \left(1 - \frac{1}{p^{1+s+\alpha_1}}\right)^{-2}$$

and

$$V_1(s, n, \alpha) = \prod_p \left[1 - \frac{2F(p, \alpha_2)F(p, \alpha_3)}{\varphi(p)p^{s+\alpha_1}}\right]^{-1}.$$

Proof. This is essentially a variation of the prime number theorem.

It suffices to consider $Q(x) = \sum_{j \geq 2} a_j x^j$. We have

$$\mathcal{A}_1(y, Q; a, b, \alpha) = \sum_{j \geq 2} \frac{a_j j!}{(\log y)^j} \sum_{(n, b) = 1} \frac{1}{2\pi i} \int_{(2)} \left(\frac{y}{a}\right)^s \frac{\mu(n)h(n)}{\varphi(n)n^{s+\alpha_1}} F(n, \alpha_2)F(n, \alpha_3) ds_{s+1}.$$

The sum over $n$ converges absolutely. Hence

$$\mathcal{A}_1(y, Q; a, b, \alpha) = \sum_{j \geq 2} \frac{a_j j!}{(\log y)^j} \frac{1}{2\pi i} \int_{(2)} \left(\frac{y}{a}\right)^s \sum_{(n, b) = 1} \frac{\mu(n)h(n)}{\varphi(n)n^{s+\alpha_1}} F(n, \alpha_2)F(n, \alpha_3) ds_{s+1}.$$

The sum in the integrand equals

$$\prod_{p|b} \left(1 - \frac{2F(p, \alpha_2)F(p, \alpha_3)}{\varphi(p)p^{s+\alpha_1}}\right) = U_1(s, \alpha) V_1(s, b, \alpha) \frac{1}{\zeta(1+s+\alpha_1)^2}.$$

Let $Y = o(T)$ be a large parameter to be chosen later. By Cauchy’s theorem, $\mathcal{A}_1(y, Q; a, b, \alpha)$ is equal to the residue at $s = 0$ plus integrals over the line segments $C_1 = \{s = it, t \in \mathbb{R}, |t| \geq Y\}$, $C_2 = \{s = \sigma \pm iY, -\frac{c}{\log Y} \leq \sigma \leq 0\}$, and $C_3 = \{s = -\frac{c}{\log Y} + it, |t| \leq Y\}$, where $c$ is some fixed positive constant such that $\zeta(1+s+\alpha_1)$ has no zeros in the region on the right hand side of the contour determined by the $C_j$’s. Furthermore, we require that for such $c$ we have $1/\zeta(\sigma + it) \ll \log (2 + |t|)$ in this region [see [17] Theorem 3.11]. Then the integral over $C_1$ is

$$\ll F_0(b)L^{-j} (\log Y)^2 / Y^j \ll F_0(b)L^{-2Y^{-2+\varepsilon}},$$

since $j \geq 2$. The integral over $C_2$ is

$$\ll F_0(b)L^{-j}(\log Y)/Y^{j+1} \ll F_0(b)L^{-2Y^{-3+\varepsilon}}.$$

Finally, the contribution from $C_3$ is

$$\ll F_0(b)L^{-j}(\log Y)^j \left(\frac{y}{a}\right)^{-c/\log Y} \ll F_0(b)\left(\frac{y}{a}\right)^{-c/\log Y} L^{-2+\varepsilon}.$$
Choosing $Y \asymp L$ gives an error so far of size $O_{\varepsilon}(F_{\tau_0}(b)(y/a)^{-\nu}L^{-2+\varepsilon}) + O_{\varepsilon}(F_{\tau_0}(b)L^{-4+\varepsilon})$.

For the residue at $s = 0$, write this as

$$\sum_{j \geq 2} \frac{a_j}{\log y} \frac{1}{2\pi i} \int \frac{1}{\zeta(1+s+\alpha_1)} \frac{1}{s^{j+1}},$$

where the contour is a circle of radius $\asymp L^{-1}$ around the origin. This integral is trivially bounded by $O(L^{-2})$ so that taking the first term in the Taylor series of $\zeta(1+s+\alpha_1)$ finishes the proof.

\begin{lemma}
For any $Q, R \in \mathbb{C}^\infty([0,1])$, there exists an absolute constant $\tau_0 > 0$ such that

$$A_2(y, Q, R; a_1, a_2, \alpha_1) = \sum_{a_1, a_2 \leq y} \frac{h(a_1 a_2) h(a_1 m) Q([a_1 m]_y) R([a_1 a_2]_y) V_1(a_1 a_2)}{l m^{1+\alpha_1}},$$

$$= U_2 \frac{\log y}{p} \left[ 1 + \frac{2V_1(p)}{p} \left( 1 + \frac{2}{p} \right) \left( 1 + \frac{2V_1(p)}{p} \right)^{-1} \right] \left( 1 - \frac{1}{p} \right)^4,$$

$$V_2(n) = \prod_{p | n} \left( 1 + \frac{2V_1(p)}{p} \right)^{-1}, \quad V_3(n) = \prod_{p | n} \left( 1 + \frac{2}{p} \right) \left( 1 + \frac{2V_1(p)}{p} \right)^{-1},$$

and

$$V_4(n) = \prod_{p | n} \left[ 1 + \frac{2V_1(p)}{p} \left( 1 + \frac{2}{p} \right) \left( 1 + \frac{2V_1(p)}{p} \right)^{-1} \right]^{-1}. $$

\end{lemma}

\begin{proof}
The proof uses Selberg’s method [16] similarly to Lemma 2.3. One first executes the sum over $m$, and then the sum over $l$.

\end{proof}

\begin{lemma}
For any $Q, R \in \mathbb{C}^\infty([0,1])$, we have

(i) $$\sum_{l_1 l_2 \leq y} \frac{h(l_1 l_2) h(l_1) Q([l_1]_y) R([l_1 l_2]_y)}{l_1^{1+\alpha_1} l_2^{1+\alpha_1}} F(l_1, \alpha_2) F(l_1 l_2, \alpha_3) V_1(l_1 l_2) F(l_1 l_2) V_2(l_1 l_2) V_3(l_1 l_2) V_4(l_1 l_2) = \frac{W(\log y)^6}{6} \int_0^1 \int_0^x (1-x)^3 y^{-\alpha_1 t_1} Q(x) R(x-t_1) dt_1 dx + O(L^5),$$

(ii) $$\sum_{pl_1 l_2 \leq y} \frac{\log p}{(p^{1+\alpha_3} - 1)p^{\alpha_3}} \frac{h(pl_1 l_2) h(l_1) Q([l_1]_y) R([pl_1 l_2]_y)}{l_1^{1+\alpha_1} l_2^{1+\alpha_1}} F(pl_1, \alpha_2) F(pl_1 l_2, \alpha_3) V_1(pl_1 l_2) F(pl_1 l_2) V_2(pl_1 l_2) V_3(pl_1 l_2) V_4(pl_1 l_2) = \frac{W(\log y)^7}{3} \int_0^1 \int_{t_1 \geq 0} (1-x)^3 y^{-\alpha_1 t_1} R(x-t_1 - t_2) dt_1 dt_2 dx + O(L^6)$$

uniformly for $\alpha \leq L^{-1}, 1 \leq j \leq 5$, where

$$W = \prod_p \left( 1 + \frac{2F(p) V_1(p) V_2(p) V_4(p)}{p} + \frac{4F(p)^2 V_1(p) V_2(p) V_4(p)}{p} \right) \left( 1 - \frac{1}{p} \right)^6.$$

\end{lemma}
Proof. We consider the first statement. We start with the sum over $l_2$ on the left hand side of (i), which is
\[
\sum_{l_2 \leq y/l_1} \frac{h(l_2)}{l_2^{1+\alpha_1}} R([l_1/l_2]_y) F(l_2, \alpha_3) V_1(l_2) V_2(l_2) V_4(l_2).
\]
As in how we prove Lemma 2.3, this equals
\[
\Pi_p \left\{ W_1(p)^{-1} \left( 1 - \frac{1}{p} \right)^2 \right\} \left( \log y \right)^2 \Pi_{l_1 \leq y} \frac{h(l_1)^2 Q([l_1]_y)}{l_1} F(l_1, \alpha_3) V_1(l_1) V_2(l_1) V_4(l_1) W_1(l_1) \right) t_1 R([l_1]_y - t_1) dt_1 + O(L), \tag{2}
\]
where
\[
W_1(n) = \Pi_{p|n} \left( 1 + \frac{2F(p)V_1(p)V_3(p)V_4(p)}{p} \right)^{-1}.
\]
Hence the required expression is
\[
\Pi_p \left\{ W_1(p)^{-1} \left( 1 - \frac{1}{p} \right)^2 \right\} \left( \log y \right)^2 \sum_{l_1 \leq y} \frac{h(l_1)^2 Q([l_1]_y)}{l_1} F(l_1, \alpha_3) V_1(l_1) V_2(l_1) V_4(l_1) W_1(l_1) \right) t_1 R([l_1]_y - t_1) dt_1 + O(L^5).
\]
Using Selberg’s method \[16\] again we have
\[
\sum_{l_1 \leq t} \frac{h(l_1)^2}{l_1} F(l_1, \alpha_2) F(l_1, \alpha_3) V_1(l_1) V_2(l_1) V_4(l_1) W_1(l_1)
\]
\[
= \Pi_p \left\{ W_2(p)^{-1} \left( 1 - \frac{1}{p} \right)^4 \right\} \frac{(\log t)^4}{24} + O(L^3)
\]
for any $t \leq T$, where
\[
W_2(n) = \Pi_{p|n} \left\{ 1 + \frac{4F(p)^2 V_1(p) V_2(p) V_4(p) W_1(p)}{p} \right\}^{-1}.
\]
Partial summation then implies that (3) is equal to
\[
\Pi_p \left\{ W_1(p)^{-1} W_2(p)^{-1} \left( 1 - \frac{1}{p} \right)^6 \right\} \left( \log y \right)^6 \Pi \left[ \int_0^1 \left( 1 - x \right)^3 y^{-\alpha_1 t_1} t_1 Q(x) R(x - t_1) dt \right] dx + O(L^5).
\]
It is easy to check that the arithmetical factor is $W$, and we obtain the first statement.

For the second statement, we first notice that the contribution of the terms involving $p^{-s}$ with $\text{Re}(s) > 1$ is $O(L^6)$. Hence the left hand side of (ii) is
\[
2 \sum_{l_1 l_2 \leq y} \frac{h(l_1 l_2) h(l_1) Q([l_1]_y)}{l_1^{1+\alpha_1}} F(l_1, \alpha_2) F(l_1 l_2, \alpha_3) V_1(l_1 l_2) V_2(l_1 l_2) V_3(l_1 l_2) V_4(l_1 l_2)
\]
\[
= \sum_{p \leq y} \frac{h(l_1 l_2) h(l_1) Q([l_1]_y)}{p^{1+\alpha_4 + \alpha_5}} \frac{R([pl_1 l_2]_y)}{p^{1+\alpha_4 + \alpha_5}} + O(L^6).
\]
The same argument shows that we can include the terms $p|l_1 l_2$ in the innermost sum with an admissible error $O(L^6)$, so that the above expression is equal to
\[
2 \sum_{p \leq y} \frac{\log p}{p^{1+\alpha_4 + \alpha_5}} \sum_{l_1 l_2 \leq y/p} \frac{h(l_1 l_2) h(l_1) Q([l_1]_y)}{l_1^{1+\alpha_1}} F(l_1, \alpha_2) F(l_1 l_2, \alpha_3) V_1(l_1 l_2) V_2(l_1 l_2) V_3(l_1 l_2) V_4(l_1 l_2) + O(L^6).
\]
We have
\[ \sum_{p \leq t} \log \frac{p}{p} = \log t + O(1) \]
for any \( t \leq T \). The result follows by using Part (i) and partial summation. \( \square \)

3. Proof of Lemma 1.1

To evaluate \( M_1(H, T) \), we first appeal to Theorem 1 of \[1\] and obtain
\[ M_1(H, T) = T \sum_{m,n \leq y} \frac{h(m)h(n)P([m]_y)P([n]_y)(m,n)}{mn} \left( \log \frac{T(m,n)^2}{2\pi mn} + 2\gamma - 1 \right) + O_B(TL^{-B}) + O_\epsilon(y^{2T^\epsilon}) \]
for any \( B > 0 \), where \( \gamma \) is the Euler constant. Using the Möbius inversion formula
\[ f((m,n)) = \sum_{\frac{|m|}{|n|} \leq l} \mu(d) f\left( \frac{l}{d} \right), \]
we can write the above as
\[ T \sum_{l \leq y} \sum_{d|l} \mu(d) \frac{d(l)}{l} \sum_{m,n \leq y/l} \frac{h(ln)h(ln)P([ln]_y)P([ln]_y)}{mn} \left( \log \frac{T}{2\pi d^2 mn} + 2\gamma - 1 \right) + O_B(TL^{-B}). \]
We next replace the term in the bracket by \( \log \frac{T}{2\pi mn} \). This produces an error of size
\[ \ll T \sum_{l \leq y} \frac{d(l)^2}{l} \left( \sum_{n \leq y/l} \frac{d(n)}{n} \right)^2 \sum_{d|l} \frac{\log d}{d} \ll TL^8. \]
Hence
\[ M_1(H, T) = T \sum_{l \leq y} \frac{\varphi(l)}{l^2} \sum_{m,n \leq y/l} \frac{h(ln)h(ln)P([ln]_y)P([ln]_y)}{mn} (L - \log m - \log n) + O(TL^8) \]
\[ = TL \sum_{l \leq y} \frac{\varphi(l)}{l^2} \left( \sum_{n \leq y/l} \frac{h(ln)P([ln]_y)}{n} \right)^2 - 2T \sum_{l \leq y} \frac{\varphi(l)}{l^2} \sum_{m,n \leq y/l} \frac{h(ln)h(ln)P([ln]_y)P([ln]_y)\log n}{mn} + O(TL^8). \]
The result follows by using Lemma 2.3, Lemma 2.4 and Lemma 2.5. Here we use a fact which is easy to verify that \( C^2D = A \).

4. Proof of Lemma 1.2

We denote \( H(s) = \zeta(s)G(s) \), i.e.
\[ G(s) = \sum_{n \leq y} \frac{h(n)P([n]_y)}{n^s}. \]
By Cauchy’s theorem we have
\[ \sum_{0 < \gamma \leq T} H(\rho + i\alpha)H(1 - \rho - i\alpha) = \frac{1}{2\pi i} \int_{C} \frac{\zeta'}{\zeta}(s)(s + i\alpha)\zeta(1 - s - i\alpha)G(s + i\alpha)G(1 - s - i\alpha)ds, \]
where \( C \) is the positively oriented rectangle with vertices at \( 1 - a + i, a + i, a + iT \) and \( 1 - a + iT \). Here \( a = 1 + L^{-1} \) and \( T \) is chosen so that the distance from \( T \) to the nearest \( \gamma \).
is $\gg L^{-1}$. It is standard that the contribution from the horizontal segments of the contour is $O_\varepsilon(yT^{1/2+\varepsilon})$.

We denote the contribution from the right edge by $N_1$, where
\[ N_1 = \frac{1}{2\pi i} \int_{a+i}^{a+iT} \chi(1 - s - ia) \frac{\zeta'}{\zeta}(s) \zeta(s + ia)^2 G(s + ia) G(1 - s - ia) ds. \] (4)

From the functional equation we have
\[ \frac{\zeta'}{\zeta}(1 - s) = \frac{\chi'}{\chi}(1 - s) - \frac{\zeta'}{\zeta}(s). \]

Hence the contribution from the left edge, by substituting $s$ by $1 - s$, is
\[ \frac{1}{2\pi i} \int_{a-i}^{a-iT} \frac{\zeta'}{\zeta}(1 - s) \zeta(1 - s + ia) \zeta(s - ia) G(1 - s + ia) G(s - ia) ds \]
\[ = \frac{1}{2\pi i} \int_{a-i}^{a-iT} \left( \frac{\chi'}{\chi}(1 - s) - \frac{\zeta'}{\zeta}(s) \right) \zeta(1 - s + ia) \zeta(s - ia) G(1 - s + ia) G(s - ia) ds \]
\[ = -N_2 + N_1 + O_\varepsilon(yT^{1/2+\varepsilon}), \]
where
\[ N_2(\beta, \gamma) = \frac{1}{2\pi i} \int_{a+i}^{T-a} \frac{\chi'}{\chi} \left( \frac{1}{2} - it \right) \zeta(1 - s + ia) \zeta(s - ia) G(1 - s + ia) G(s - ia) ds. \] (5)

Thus
\[ \sum_{0<\gamma\leq T} H(\rho + ia) H(1 - \rho - ia) = 2\text{Re}(N_1) - N_2 + O_\varepsilon(yT^{1/2+\varepsilon}). \] (6)

4.1. **Evaluate $N_2$.** We move the line of integration in (5) to the $\frac{1}{2}$-line. As before, this produces an error of size $O_\varepsilon(yT^{1/2+\varepsilon})$. Hence we get
\[ N_2 = \frac{1}{2\pi} \int_{1-a}^{T-a} \frac{\chi'}{\chi} \left( \frac{1}{2} - it \right) |H(\frac{1}{2} + it)|^2 dt + O_\varepsilon(yT^{1/2+\varepsilon}). \]

From Stirling’s approximation we have
\[ \frac{\chi'}{\chi} \left( \frac{1}{2} - it \right) = -\log \frac{t}{2\pi} + O(t^{-1}) \quad (t \geq 1). \]

Combining this with Lemma 1.1 and integration by parts, we easily obtain
\[ N_2 = -\frac{ATL(\log y)^9}{12\pi} \int_0^1 (1 - x)^3 \left( \theta^{-1} P_1(x)^2 - 2P_1(x)P_2(x) \right) dx + O(TL^9). \] (7)

4.2. **Evaluate $N_1$.** It is easier to start with a more general sum
\[ N_1(\beta, \gamma) = \frac{1}{2\pi i} \int_{a+i(1+\alpha)}^{a+i(T+\alpha)} \chi(1 - s) \left( \frac{\zeta'}{\zeta}(s + \beta) \zeta(s + \gamma) \zeta(s) \sum_{m \leq y} \frac{h(m)P([m]y)}{m^s} \right) \]
\[ \left( \sum_{n \leq y} \frac{h(n)P([n]y)}{n^{1-s}} \right) ds, \]
so that $N_1 = N_1(-ia, 0)$. From Lemma 2.1, we obtain
\[ N_1(\beta, \gamma) = \sum_{n \leq y} \frac{h(n)P([n]y)}{n} \sum_{m \leq nT/2\pi} a(m) e \left( -\frac{m}{n} \right) + O_\varepsilon(yT^{1/2+\varepsilon}), \]
where the arithmetic function \( a(m) \) is defined by

\[
\frac{\zeta'}{\zeta}(s + \beta)\zeta(s + \gamma)\zeta(s) \sum_{m \leq y} \frac{h(m)P([m]_y)}{m^s} = \sum_{m=1}^{\infty} \frac{a(m)}{m^s}. \quad (8)
\]

By the work of Conrey, Ghosh and Gonek [9 Sections 5–6 and (8.2)], and the work of Bui and Heath-Brown [5], we can write

\[
N_1(\beta, \gamma) = Q(\beta, \gamma) + E + O_\varepsilon(yT^{1/2+\varepsilon}),
\]

where

\[
Q(\beta, \gamma) = \sum_{l \leq y} \frac{h(l)P([l]_y)}{l} \mu(n) \sum_{m \leq nT/2\pi} \frac{a(lm)}{\varphi(n)}
\]

and

\[
E \ll_B T^{\varepsilon - B} + y^{1/3}T^{5/6+\varepsilon}
\]

for any \( B > 0 \).

Let

\[
\frac{\zeta'}{\zeta}(s + \beta)\zeta(s + \gamma)\zeta(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}.
\]

From [8] and Lemma 2.2 we have

\[
a(lm) = \sum_{l=1/l_1}^{l_1 \leq y} \sum_{m=1/m_1}^{m_1 \leq y} \frac{h(l_1m_1)P([l_1m_1]_y)g(l_2m_2)}{\varphi(n)}.
\]

Hence

\[
Q(\beta, \gamma) = \sum_{l_1l_2 \leq y} \frac{h(l_1l_2n)P([l_1l_2]_y)}{l_1l_2n} \mu(n) \sum_{m_1 \leq y} \frac{h(l_1m_1)P([l_1m_1]_y)}{\varphi(n)} g(l_2m_2). \quad (11)
\]

**Lemma 4.1.** Suppose \( a \) and \( b \) are coprime, squarefree integers. Then we have

\[
G(x; a, b) := \sum_{\substack{n \leq x \atop (n, b) = 1}} g(an)
\]

\[
= -\frac{x^{1-\beta}}{1-\beta} \sum_{a=2a_3} \frac{1}{a_2} \left( \frac{\zeta'}{\zeta}(1-\beta+\gamma)\zeta(1-\beta)F(b, -\beta + \gamma)F(a_2b, -\beta)ight.
\]

\[
+ \frac{x^{1-\gamma}}{1-\gamma} \sum_{a=2a_3} \frac{1}{a_2} \left( \frac{\zeta'}{\zeta}(1+\beta-\gamma) + \sum_{p\mid b} \frac{\log p}{p^{1+\beta-\gamma} - 1} \right) \zeta(1-\gamma)F(b)F(a_2b, -\gamma)
\]

\[
- \frac{x^{1-\gamma}}{1-\gamma} \sum_{p\mid a_3} \frac{1}{p^3a_2} \log p \frac{1}{1-p^{-(1+\beta-\gamma)}} \zeta(1-\gamma)F(pb)F(pa_2b, -\gamma)
\]

\[
+ x \sum_{a=2a_3} \frac{1}{a_2} \left( \frac{\zeta'}{\zeta}(1+\beta) + \sum_{p\mid b} \frac{\log p}{p^{1+\beta} - 1} \right) \zeta(1+\gamma)F(b, \gamma)F(a_2b)
\]

\[
- x \sum_{p\mid a_3} \frac{1}{p^3a_2} \log p \frac{1}{1-p^{-(1+\beta)}} \zeta(1+\gamma)F(pb, \gamma)F(pa_2b)
\]

\[
+ O_{B,\varepsilon}(\log ab)^{1+\varepsilon}(\log x)^{-B}.
\]
Proof. It is standard that up to an error term of size $O_{\varepsilon}(\log ab)^{1+\varepsilon}x(\log x)^{-B}$ for any $B > 0$, $G(x; a, b)$ is the sum of the residues at $s = 1 - \beta$, $s = 1 - \gamma$ and $s = 1$ of

$$\frac{x^s}{s} \sum_{(n,b)=1} \frac{g(an)}{n^s}.$$ 

Combining (10) and Lemma 2.2, the above expression is

$$\frac{x^s}{s} \sum_{a=a_1a_2a_3} \left( - \sum_{(n,b)=1} \frac{\Lambda(a_1n)}{(a_1n)^\beta} \right) \left( \sum_{(n,a_1b)=1} \frac{1}{(a_2n)^\gamma} \right) \left( \sum_{(n,a_1a_2b)=1} \frac{1}{n^\delta} \right)$$

$$= \frac{x^s}{s} \sum_{a=a_1a_2a_3} \frac{1}{a_1^\beta a_2^\gamma} \left( - \sum_{(n,b)=1} \frac{\Lambda(a_1n)}{n^{\beta+\gamma}} \right) \zeta(s + \gamma) \zeta(s) F(a_1b, s + \gamma - 1) F(a_1a_2b, s - 1).$$

We have

$$- \sum_{(n,b)=1} \frac{\Lambda(a_1n)}{n^{\beta+\gamma}} = \begin{cases} \zeta(s + \beta) + \sum_{p\mid b} \frac{\log p}{p^{\beta+\gamma} - 1} & \text{if } a_1 = 1, \\ \frac{\log p}{1 - p^{s+\beta}} & \text{if } a_1 = p, \\ 0 & \text{otherwise.} \end{cases}$$

The result follows. \hfill \square

In view of the above definition, the innermost sum in (11) is

$$G(nT/2\pi m_1; l_2, l_1 n).$$

We then write

$$Q(\beta, \gamma) = \sum_{j=1}^6 Q_j(\beta, \gamma)$$

corresponding to the decomposition of $G(x; a, b)$ in Lemma 4.1.

We begin with $Q_1(\beta, \gamma)$. Writing $l_2l_3$ for $l_2$, and $m$ for $m_1$, we have $Q_1(\beta, \gamma)$ equals

$$- \frac{(T/2\pi)^{1-\beta}}{1-\beta} \zeta(1 - \beta + \gamma) \zeta(1 - \beta) \sum_{l_1 l_2 l_3 \leq y} \frac{h(l_1 l_2 l_3) h(l_1 m) P([l_1 m]_y)}{l_1^{1-\gamma} l_2^{1-\gamma} l_3^{1-\beta} m^{1-\beta}} F(l_1, -\beta + \gamma) F(l_1 l_2, -\beta)$$

$$+ \sum_{n \leq y/2} \frac{\mu(n) \varphi(n) P([l_1 l_2 l_3 m]_y)}{\varphi(n) n^{\gamma}} F(n, -\beta + \gamma) F(n, -\beta).$$

From Lemma 2.6, the innermost sum is

$$U_1 V_1 (l_1 l_2 l_3 m) \left( P''([l_1 l_2 l_3]_y) \log y \right)^2 + \frac{2\beta P'([l_1 l_2 l_3]_y)}{\log y} + \frac{\beta^2 P([l_1 l_2 l_3]_y)}{\log y}$$

$$+ O_\varepsilon(F_{\gamma_0}(l_1 l_2 l_3 m) L^{-\gamma}) + O_\varepsilon \left( F_{\gamma_0}(l_1 l_2 l_3 m) \left( \frac{y}{l_1 l_2 l_3} \right)^{-\nu} L^{-2+\varepsilon} \right).$$

By Lemma 2.5, the contributions of the $O$-terms to $Q_1(\beta, \gamma)$ is $O_\varepsilon(TL^{\beta+\varepsilon})$. Hence

$$Q_1(\beta, \gamma) = -U_1 (T/2\pi)^{1-\beta} \zeta(1 - \beta + \gamma) \zeta(1 - \beta) \sum_{l_1 l_2 \leq y} \frac{F(l_1, -\beta + \gamma) F(l_1 l_2, -\beta)}{l_1^{1-\gamma} l_2^{1-\gamma}}$$

$$\left( \frac{A_2(y, P, P''; l_1, l_2, -\beta)}{\log y} + \frac{2\beta A_2(y, P, P'; l_1, l_2, -\beta)}{\log y} + \frac{\beta^2 A_2(y, P, P; l_1, l_2, -\beta)}{\log y} \right)$$

$$+ O_\varepsilon(TL^{\beta+\varepsilon}).$$
Using Lemmas 2.7–2.8 we obtain

\[
Q_1(\beta, \gamma) = -\frac{A(T/2\pi)^{1-\beta}(\log y)^{10}}{6} \zeta(1-\beta+\gamma) \zeta(1-\beta) \int_0^1 \int_0^x \int_0^{x} (1-x)^3 y^{\beta-t}\gamma t \, dt \, dx + O(TL^{9+\varepsilon}).
\]

Here we have used a fact which is easy to verify that \(U_1U_2W = A\).

For \(Q_2(\beta, \gamma)\), we write the sum \(\sum_{p\mid l_1} \sum_{\substack{n, l_2 \leq y \\mid \phi(l_2)}} F(l_1)F(l_1l_2, -\gamma) \sum_{n, m \leq y} \frac{\mu(n)h(n)P([l_1l_2l_3]_y)}{\varphi(n)n^\gamma} F(n)F(n, -\gamma)\). The remaining terms are \(\sum_{p\mid l_1} \sum_{\substack{n, \ l_2 \leq y \\mid \phi(l_2)}} F(l_1)F(l_1l_2, -\gamma) \sum_{n, m \leq y} \frac{\mu(n)h(n)P([l_1l_2l_3]_y)}{\varphi(n)n^\gamma} F(n)F(n, -\gamma)\). The same argument shows that the last term in (13) is also \(O(1L^{9+\varepsilon})\).
Similarly to $Q_1(\beta, \gamma)$, we thus obtain

$$Q_2(\beta, \gamma) = \frac{A(T/2\pi)^{1-\gamma}(\log y)^{10}}{6} \frac{\zeta'}{(1 + \beta - 1)\zeta(1 - \gamma)} \int_0^1 \int_0^x (1 - x)^{\gamma(t-t_1)} t t_1$$

$$\left( \frac{P(x-t_1)}{(\log y)^2} + \frac{2\gamma P_0(x-t_1)}{\log y} + \gamma^2 P_1(x-t_1) \right) dt dt_1 dx + O_{\varepsilon}(TL^{9+\varepsilon}).$$

(14)

The fourth term $Q_4(\beta, \gamma)$ is in the same form as $Q_2(\beta, \gamma)$. The same calculations yield

$$Q_4(\beta, \gamma) = \frac{A(T/2\pi)(\log y)^{8}}{6} \frac{\zeta'}{(1 + \beta - 1)\zeta(1 + \gamma)} \int_0^1 \int_0^x (1 - x)^{\gamma^{-1}t_1 t_1}$$

$$P_1(x)P(x-t_1)dt_1 dx + O_{\varepsilon}(TL^{9+\varepsilon}).$$

(15)

To evaluate $Q_3(\beta, \gamma)$, we rearrange the sums and write $Q_3(\beta, \gamma)$ in the form

$$-\frac{(T/2\pi)^{1-\gamma}}{1 - \gamma} \sum_{pl_1 l_2 l_3 \leq y} \frac{\log p}{(p^{1+\beta - 1})p^{\gamma}} \frac{h(p)[l_1 l_2 l_3]h(l_1 m)P([l_1 m]_y)}{l_1 l_2^{1+\gamma}l_3 m^{1-\gamma}}$$

$$F(pl_1)F(pl_1 l_2, -\gamma) \sum_{n \leq y/pl_1 l_2 (n, pl_1 l_2 l_3) = 1} \frac{\mu(n)h(n)P([pl_1 l_2 l_3 n]_y)}{\varphi(n) n^{\gamma}} F(n)F(n, -\gamma).$$

By Lemma 2.5, the innermost sum is

$$U_1 V_1(\beta, \gamma) \frac{P''([pl_1 l_2 l_3]_y)}{(\log y)^2} + \frac{2\gamma P'([pl_1 l_2 l_3]_y)}{\log y} + \gamma^2 P([pl_1 l_2 l_3]_y)$$

$$+ O(F_{\tau}(pl_1 l_2 l_3)M^{-3}) + O_{\varepsilon} \left( F_{\tau}(pl_1 l_2 l_3) \left( \frac{y}{pl_1 l_2 l_3} \right)^{-\nu} L^{-2-\varepsilon} \right).$$

The contribution of the $O$-terms, using Lemma 2.5, is $O_{\varepsilon}(TL^{9+\varepsilon})$. The remaining terms contribute

$$-\frac{U_1(T/2\pi)^{1-\gamma}}{(1 - \gamma)} \sum_{ pl_1 l_2 \leq y } \frac{\log p}{(p^{1+\beta - 1})p^{\gamma}} \frac{F(pl_1)F(pl_1 l_2, -\gamma)}{l_1 l_2^{1+\gamma}}$$

$$\left( \frac{\phi_2(y, P, P'': l_1, pl_1 l_2, -\gamma)}{(\log y)^2} + \frac{2\gamma \phi_2(y, P, P': l_1, pl_1 l_2, -\gamma)}{\log y} + \gamma^2 \phi_2(y, P, P; l_1, pl_1 l_2, -\gamma) \right).$$

In view of Lemma 2.7, this equals

$$-U_1 U_2(T/2\pi)^{1-\gamma}(\log y)^4 \frac{1}{(1 - \gamma)} \sum_{ pl_1 l_2 \leq y } \frac{\log p}{(p^{1+\beta - 1})p^{\gamma}} \frac{h(pl_1 l_2)h(l_1)}{l_1 l_2^{1+\gamma}}$$

$$F(pl_1)F(pl_1 l_2, -\gamma) V_1(pl_1 l_2) V_2(l_1) V_3(pl_2) V_4(pl_1 l_2) \int_0^{[l_1]_y} y^\gamma t P([l_1]_y - t)$$

$$\left( \frac{P([pl_1 l_2]_y)}{(\log y)^2} + \frac{2\gamma P_0([pl_1 l_2]_y)}{\log y} + \gamma^2 P_1([pl_1 l_2]_y) \right) dt + O(TL^{9}).$$

From Lemma 2.8(ii) we obtain

$$Q_3(\beta, \gamma) = -\frac{A(T/2\pi)^{1-\gamma}(\log y)^{11}}{3} \frac{\zeta'(1 - \gamma) \int_0^1 \int_{t_1 t_2 \geq 0} (1 - x)^{\gamma(t - t_1) - \beta t_1 t_2} t t_1 P(x - t)$$

$$\left( \frac{P(x-t_1-t_2)}{(\log y)^2} + \frac{2\gamma P_0(x-t_1-t_2)}{\log y} + \gamma^2 P_1(x - t_1 - t_2) \right) dt dt_1 dt_2 dx + O_{\varepsilon}(TL^{9+\varepsilon}).$$

(16)
The term $Q_5(\beta, \gamma)$ is in the same form as $Q_3(\beta, \gamma)$. The same calculations give

$$Q_5(\beta, \gamma) = -\frac{A(T/2\pi)(\log y)^9}{3} \zeta(1 + \gamma) \int_0^1 \int_{t_1 + t_2 \leq x} (1 - x)^3 y^{-\gamma t_1 - \beta t_2} t_1 \ t_2^3 P_1(x) P(x - t_1 - t_2) \ dt_1 \ dt_2 \ dx + O_\varepsilon(TL^{9+\varepsilon}).$$

(17)

Finally, we have $Q_6(\beta, \gamma) = O_B(TL^{-B})$ for any $B > 0$.

Collecting the estimates (6), (7), (12), (14)–(17), and letting $\beta = -i\alpha$, $\gamma \to 0$ we easily obtain Lemma 1.2.

REFERENCES

[1] R. Balasubramanian, J. B. Conrey and D. R. Heath-Brown, Asymptotic mean square of the product of the Riemann zeta-function and a Dirichlet polynomial, J. reine angew. Math. 357 (1985), 161–181.
[2] J. Bredberg, Large gaps between consecutive zeros, on the critical line, of the Riemann zeta-function, preprint, available on arXiv at [http://arxiv.org/abs/1101.3197](http://arxiv.org/abs/1101.3197)
[3] H. M. Bui, Large gaps between consecutive zeros of the Riemann zeta-function, J. Number Theory 131 (2011), 67-95.
[4] H. M. Bui, J. B. Conrey, M. P. Young, More than 41% of the zeros of the zeta function are on the critical line, Acta Arith. 150 (2011), 35–64.
[5] H. M. Bui, D. R. Heath-Brown, On simple zeros of the Riemann zeta-function, to appear in Bull. London Math. Soc., available on arXiv at [http://arxiv.org/abs/1302.5018](http://arxiv.org/abs/1302.5018)
[6] H. M. Bui, M. B. Milinovich, Nathan Ng, A note on the gaps between consecutive zeros of the Riemann zeta-function, Proc. Amer. Math. Soc. 138 (2010), 4167–4175 .
[7] J. B. Conrey, A. Ghosh, S. M. Gonek, A note on gaps between zeros of the Riemann zeta-function, Bull. London Math. Soc. 16 (1984), 421–424.
[8] J. B. Conrey, A. Ghosh, S. M. Gonek, Large gaps between zeros of the zeta-function, Mathematika 33 (1986), 212–238.
[9] J. B. Conrey, A. Ghosh, S. M. Gonek, Simple zeros of the Riemann zeta function, Proc. London Math. Soc 76 (1998), 497–522.
[10] S. Feng, X. Wu, On gaps between zeros of the Riemann zeta-function, J. Number Theory 132 (2012), 1385–1397.
[11] S. Feng, X. Wu, On large spacing of the zeros of the Riemann zeta-function, J. Number Theory 133 (2013), 2538–2566.
[12] R. R. Hall, A new unconditional result about large spaces between zeta zeros, Mathematika 52 (2005), 101–113.
[13] H. L. Montgomery, A. M. Odlyzko, Gaps between zeros of the zeta function, Topics in Classical Number Theory, Coll. Math. Soc. Janos Bolyai 34, North-Holland (1984), 1079–1106.
[14] J. Mueller, On the difference between consecutive zeros of the Riemann zeta function, J. Number Theory 14 (1982), 327–331.
[15] Nathan Ng, Large gaps between the zeros of the Riemann zeta function, J. Number Theory 128 (2008), 509–556.
[16] A. Selberg, The zeta-function and the Riemann hypothesis, C. R. Dixième Congrès Math. Scandinaves (1946), 187–200.
[17] E. C. Titchmarsh, The theory of the Riemann zeta-function, revised by D. R. Heath-Brown, Clarendon Press, second edition, 1986.

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