Continuous differentiability of a weak solution to very singular elliptic equations involving anisotropic diffusivity

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Abstract

In this paper we consider a very singular elliptic equation that involves an anisotropic diffusion operator, including one-Laplacian, and is perturbed by a $p$-Laplacian-type diffusion operator with $1 < p < \infty$. This equation seems analytically difficult to handle near a facet, the place where the gradient vanishes. Our main purpose is to prove that weak solutions are continuously differentiable even across the facet. Here it is of interest to know whether a gradient is continuous when it is truncated near a facet. To answer this affirmatively, we consider an approximation problem, and use standard methods including De Giorgi’s truncation and freezing coefficient methods.

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1 Introduction

This paper is concerned with continuous differentiability of a weak solution to a very singular elliptic equation of the form

$$L_{b,p}u := -b\Delta_1 u - \Delta_p u = f(x) \quad \text{for } x = (x_1, \ldots, x_n) \in \Omega \subset \mathbb{R}^n. \tag{1.1}$$

Here $b \in (0, \infty)$ and $p \in (1, \infty)$ are fixed constants, the dimension $n$ satisfies $n \geq 2$, $\Omega$ is a $n$-dimensional bounded domain with Lipschitz boundary, and $f$ is a given real-valued function in $\Omega$. The partial differential operators $\Delta_1$ and $\Delta_p$ often called, respectively, one-Laplacian and $p$-Laplacian, are in divergence forms given by

$$\Delta_1 u := \text{div} \left( \frac{\nabla u}{|\nabla u|} \right), \quad \Delta_p u := \text{div} \left( |\nabla u|^{p-2} \nabla u \right).$$

Here $\nabla u = (\partial_{x_1} u, \ldots, \partial_{x_n} u)$ with $\partial_{x_j} u = \partial u / \partial x_j$ for a function $u = u(x_1, \ldots, x_n)$, and $\text{div} X = \sum_{j=1}^n \partial_{x_j} X_j$ for a vector field $X = X(x_1, \ldots, x_n)$.

For the $p$-Poisson equations, that is, in the case $b = 0$, H"older continuity of a gradient was already established by many experts ([15], [20], [31], [32], [40], [43], [44]). These regularity results should not hold for the one-Poisson equations, even if $f$ is sufficiently smooth. In fact, in the one-dimensional case, a non-smooth function $u = u(x_1)$ may solve $-\Delta_1 u = 0$, as long as it is absolutely continuous and non-decreasing. Hence, it seems impossible to show continuous differentiability of weak solutions, even for the one-Laplace equation. This problem is due to the fact that the ellipticity of one-Laplacian

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degenerates in the direction $\nabla u$, which is substantially different from $p$-Laplacian. On the other hand, in the multi-dimensional case, the ellipticity of one-Laplacian is non-degenerate in directions that are orthogonal to $\nabla u$. It should be mentioned that this ellipticity becomes singular in a facet, the place where a gradient vanishes. By anisotropic diffusivity of one-Laplace operator $\Delta_1$, which consists of both degenerate and singular ellipticity, $\Delta_1$ seems, unlike $\Delta_p$, impossible to deal with in existing elliptic regularity theory.

For this reason, it will be interesting to consider a question whether a solution to (1.1) where $L_{b,p}$ contains both $\Delta_1$ and $\Delta_p$, is continuously differentiable ($C^1$). In the special case where the solution $u$ is convex, this $C^1$-regularity problem is already answered affirmatively [24, Theorem 1]. The aim of this paper is to establish continuity of derivatives without the convexity assumption.

The equation (1.1) is concerned with a minimizing problem of the energy functional

$$\mathcal{F}(u) := \int_\Omega (E(\nabla u) - fu) \, dx \quad \text{with} \quad E(z) = b|z| + \frac{|z|^p}{p}.$$  

Here, as the principal space, we often choose a closed affine space $u_0 + W^{1,p}_0(\Omega)$ for some fixed $u_0 \in W^{1,p}(\Omega)$. Formally considering the Euler–Lagrange equation corresponding to this energy minimizing problem, we can obtain (1.1). However, since the mapping $\mathbb{R}^n \ni z \mapsto |z| \in [0, \infty)$ is no longer differentiable at the origin, the term $\nabla u/|\nabla u|$ should be treated not in the classical sense on a place $\{\nabla u = 0\}$, often called a facet of $u$. This problem can be overcome by introducing a subdifferential, which is often useful for analysis of non-smooth convex functions. Also, since a solution $u$ is assumed to be in the first order Sobolev space, we should treat a weak solution in a distributional sense. Our definition of a weak solution is described in Section 1.3.

The energy density $E$ can be found in fields of materials science and fluid mechanics. We would like to mention some mathematical models for each field briefly.

A typical example of the former is the relaxation dynamics of a crystal surface below the roughening temperature, which was modeled by Spohn in [39]. There, evolution of $u = u(x,t)$, a height function of the crystal in a two-dimensional domain $\Omega$, is modeled to satisfy

$$\partial_t u = -\Delta \mu \quad \text{and} \quad \mu = -\frac{\delta \Phi}{\delta u}$$

from a thermodynamic viewpoint (see [29] and the references therein). Here $\mu$ is often called the chemical potential, and $\Phi$ denotes the crystal surface energy, given by

$$\Phi(u) := \int_\Omega |\nabla u| \, dx + \kappa \int_\Omega |\nabla u|^3 \, dx \quad \text{with a constant } \kappa > 0.$$  

Finally, evolution of a crystal surface results in a fourth order parabolic equation $b\partial_t u = \Delta L_{b,3}u$ with $b = 1/(3\kappa) > 0$. When $u$ is stationary, this model becomes a second order elliptic equation $L_{b,3}u = b\mu$. Here $\mu$ is sufficiently smooth since it is harmonic.

For the latter model, the density $E$ often appears with $(n, p) = (2, 2)$ when modeling an incompressible laminar flow of a Bingham fluid in a cylinder $\Omega \times \mathbb{R} \subset \mathbb{R}^3$. To be precise, we assume that the velocity field $U$ is of the form $U = (0, 0, u(x_1, x_3))$, and hence the incompressible condition $\text{div} U = 0$ clearly holds. It is also assumed that its speed is sufficiently slow enough to discard convection effects. Under these settings, the Euler–Lagrange equation for a Bingham fluid is given by

$$\frac{\delta \Psi}{\delta u} = -\partial_{x_3} \pi \quad \text{with} \quad \Psi(u) := \gamma \int_\Omega |\nabla u| \, dx + \frac{\mu}{2} \int_\Omega |\nabla u|^2 \, dx.$$  

Here $\gamma$ and $\mu$ are positive constants, and $\pi = \pi(x_3)$ denotes the pressure function, whose slope becomes constant under the laminar setting. Mathematical formulations on motion of Bingham fluids are found
Bingham fluids have two different aspects of plasticity and viscosity, which are respectively reflected by the diffusion operators $\Delta_1 u$ and $\Delta_2 u = \Delta u$ in the equation (1.1). It is worth mentioning that when $\gamma = 0$, this problem models motion of incompressible Newtonian fluids.

The density $E$ also arises in an optimal transport problem for a congested traffic dynamics ([11], [13]). There the optimal traffic flow is connected with a certain very degenerate elliptic problem, and higher regularity results related to this have been well-established. Our $C^1$-regularity result is inspired by a recent work by Bögelein–Duzaar–Giova–Passarelli di Napoli [10], which is also concerned with continuity of gradients on the very degenerate problem related to the optimal traffic flow. For further explanations and comparisons, see Section 1.2.

More generally, we will also consider

$$L u := L_1 u + L_p u = f \quad \text{in } \Omega \subset \mathbb{R}^n,$$

where the partial differential operators $L_1$ and $L_p$ generalize $-\Delta_1$ and $-\Delta_p$ respectively. The detailed conditions of $L_1$, $L_p$ will be described later in Section 1.3.

### 1.1 Our main results and strategy

For simplicity, we first consider the model case (1.1). Our result is

**Theorem 1.1.** Let $u$ be a weak solution to (1.1) with $f \in L^q(\Omega)$ for some $q \in (n, \infty)$. Then, $u$ is in $C^1(\Omega)$.

By Giga and the author [24], this type of $C^1$-regularity result was already established in the special case where $u$ is convex. The proof therein is substantially based on convex analysis and a strong maximum principle. In particular, the gradient $\nabla u$ is often considered not only as a distribution, but also as a subgradient, which made it successful to show $C^1$-regularity rather elementarily.

After that work was completed, the author has found it possible to prove continuity of derivatives of non-convex weak solutions. Instead of the strong maximum principle, we use standard methods from the Schauder theory and the De Giorgi–Nash–Moser theory to justify continuity of a gradient $\nabla u$ when it is truncated near a facet. Precisely speaking, we would like to show that for each small $\delta \in (0, 1)$, the truncated gradient

$$\mathcal{G}_\delta(\nabla u) := (|\nabla u| - \delta)^+ \frac{\nabla u}{|\nabla u|}$$

(1.3)
is locally Hölder continuous. This result can be explained by computing eigenvalues of the Hessian matrix $\nabla^2 E(\nabla u) = (\partial_{x_i x_j} E(\nabla u))_{i,j}$. In fact, $\nabla^2 E$ satisfies

$$\text{(ellipticity ratio of } \nabla^2 E(\nabla u)) = \frac{\text{(the largest eigenvalue of } \nabla^2 E(z_0))}{\text{(the lowest eigenvalue of } \nabla^2 E(z_0))} \leq C(p) \left(1 + b|z_0|^{1-p}\right)$$

for all $z_0 \in \mathbb{R}^n \setminus \{0\}$. In particular, for each fixed $\delta > 0$, the operator $L_{b,p}$ can be said to be *uniformly elliptic over a place* $\{|\nabla u| > \delta\}$. It should be emphasized that $\mathcal{G}_\delta(\nabla u)$ is supported in this place, and therefore this vector field will be Hölder continuous. Since it is easy to check that the continuous vector field $\mathcal{G}_\delta(\nabla u)$ uniformly converges to $\nabla u$, this implies that $\nabla u$ is also continuous. We should note that our analysis for $\mathcal{G}_\delta(\nabla u)$ substantially depends on the truncation parameter $\delta > 0$. Thus, our Hölder continuity estimates may blow up as $\delta$ tends to 0. In particular, on quantitative growth estimates of a gradient near a facet, less is known. This problem is essentially because that the equation (1.1) becomes non-uniformly elliptic near a facet. It should be emphasized that our problem substantially differs from what is often called the $(p, q)$-growth problem, where non-uniform ellipticity appears as a gradient blows up [34], [35].
The proof of Hölder continuity of $\mathcal{G}_{2,\delta}(\nabla u)$, where we have replaced $\delta$ by $2\delta$, broadly consists of two parts; relaxing the singular operator $L_{b, p}$, and establishing a priori Hölder estimates on relaxed vector fields. For relaxations, we will have to consider relaxed problems that are uniformly elliptic. In the special case (1.1), for an approximating parameter $\varepsilon \in (0, 1)$, the regularized equation is given by

$$L^e_{b, p} u_\varepsilon = f_\varepsilon.$$  

(1.4)

Here $f_\varepsilon$ converges to $f$ in a suitable sense, and the relaxed operator $L^e_{b, p}$ is given by

$$L^e_{b, p} u_\varepsilon := -\text{div} \left( \frac{b \nabla u_\varepsilon}{\sqrt{\varepsilon^2 + |\nabla u_\varepsilon|^2}} + \left( \varepsilon^2 + |\nabla u_\varepsilon|^2 \right)^{p/2-1} \nabla u_\varepsilon \right)$$

for each $\varepsilon \in (0, 1)$. This operator naturally appears when one approximates the density $E$ by

$$E_\varepsilon(z) := b \sqrt{\varepsilon^2 + |z|^2} + \frac{1}{p} \left( \varepsilon^2 + |z|^2 \right)^{p/2}.$$  

In particular, we regularize $|z|^s$ by $(\varepsilon^2 + |z|^2)^{s/p}$ for each $s \in \{1, p\}$, which allows for linearization. It should be noted that for each fixed $\varepsilon \in (0, 1)$, the equation (1.4) is uniformly elliptic, in the sense that there holds

$$\left( \text{the ellipticity ratio of } \nabla^2 E_\varepsilon(z_0) \right) \leq C(p) \left[ 1 + b \left( \varepsilon^2 + |z_0|^2 \right)^{\frac{1-p}{p}} \right] \leq C(b, p, \varepsilon)$$

for all $z_0 \in \mathbb{R}^n$. Moreover, sufficient smoothness for coefficients of the operator $L^e_{b, p}$ is also guaranteed. Therefore, from existing results on elliptic regularity theory, it is not restrictive to assume that a weak solution to (1.4) admits higher regularity $u_\varepsilon \in W^{1, \infty}_{\text{loc}} \cap W^{2, 2}_{\text{loc}}$. Moreover, when the external force term $f_\varepsilon$ is in $C^\infty$, it is possible to let $u_\varepsilon$ be even in $C^\infty$ with the aid of bootstrap arguments [30 Chapter V]. For each fixed $\delta \in (0, 1)$, we would like to establish an a priori Hölder continuity of a truncated gradient

$$\mathcal{G}_{2,\delta, \varepsilon}(\nabla u_\varepsilon) := \left( \sqrt{\varepsilon^2 + |\nabla u_\varepsilon|^2} - 2\delta \right)_+ \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|},$$  

(1.5)

whose estimate is independent of the approximation parameter $\varepsilon$ that is sufficiently smaller than $\delta$. Since the singular operator $L_{b, p}$ itself has to be relaxed by a non-degenerate one $L^e_{b, p}$, the truncating mapping $\mathcal{G}_{2,\delta}$ should also be replaced by another regularized one $\mathcal{G}_{2,\delta, \varepsilon}$. For this reason, we have to consider another vector field given by (1.5), which clearly differs from (1.3). The a priori Hölder estimate, which plays as a key lemma in this paper, enables us to apply the Arzelà–Ascoli theorem and to conclude that $\mathcal{G}_{2,\delta}(\nabla u)$ is also Hölder continuous. We will obtain the desired Hölder continuity estimates by standard methods, including De Giorgi’s truncation and a freezing coefficient argument. Our proofs of a priori Hölder continuity estimates are highly inspired by [10 §4–7].

For a general problem (1.2), we need to establish suitable approximation of the singular operator $\mathcal{L}$. There, for an approximation parameter $\varepsilon \in (0, 1)$, we have to introduce a uniformly elliptic operator $\mathcal{L}^e$ that is neither degenerate nor singular. A main problem herein is that we have to justify that the regularized solution $u_\varepsilon$ converges to the original solution $u$ in the distributional sense. In the special case $\mathcal{L} = L_{b, p}$, $\mathcal{L}^e = L^e_{b, p}$, and $f_\varepsilon \equiv f$, this justification is already discussed in [41] Lemma 4], which is based on arguments from [33 Theorem 6.1]. In this paper, we would like to give a more general approximating operator $\mathcal{L}^e$ that is based on the convolution by the Friedrichs mollifier. The novelty of our new approximation argument is that this works for a very singular operator $\mathcal{L}_1$ that differs from $b\Delta_1$. In particular, we are able to generalize the total variation energy functional in $\mathcal{F}$, as long as its corresponding
density $E_1$ is convex and positively one-homogeneous. Our setting concerning $E_1$ will be substantially different from previous works \cite{5, 6, 8, 27}, which are concerned with energy minimizers of purely linear growth, since ellipticity conditions therein appear to differ from ours.

It is worth mentioning that our strategy works even in the system case, and $C^1$-regularity on vector-valued problems has been established in another recent paper \cite{42} by the author. It should be emphasized that when one considers everywhere regularity for the system problem, it would be natural to let the divergence operator $\mathcal{L}$ have a symmetric structure, often called the Uhlenbeck structure (see \cite{11}, \cite{9}, \cite{43}). In particular, when it comes to everywhere $C^1$-regularity for vector-valued solutions, assumptions related to the Uhlenbeck structure may force us to restrict $\mathcal{L}_1 = bA_1$ (see Section 2.4 for details). In absence of this assumption, it is well-known that energy minimizers has partial regularity, and often lacks everywhere regularity. Both classical partial regularity results and counterexamples to full regularity are found in \cite[Chapter 4]{3}, \cite[Chapter 9]{25}. On partial regularity of local minimizers in BV spaces, both $p$-growth problems and purely linear growth problems are well-discussed in \cite{5, 6, 27, 38}. Although it will be interesting to consider partial regularity on non-Uhlenbeck-type problems instead of full regularity, we will not discuss this problem in the paper.

1.2 Comparisons to previous results on a very degenerate equation

Our proofs on continuity of derivatives are inspired by \cite{10}, which is concerned with everywhere regularity for gradients of solutions to very degenerate equations, or even systems, of the form

$$-\text{div} \left( (|\nabla v| - b)_+^{p'-1} \frac{\nabla v}{|\nabla v|} \right) = f \in L^q(\Omega) \quad \text{in } \Omega \subset \mathbb{R}^n. \quad (1.6)$$

Here $p'$ denotes the Hölder conjugate of $p \in (1, \infty)$, i.e., $p' = p/(p - 1)$. This equation is motivated by a mathematical model on congested traffic dynamics \cite{13}. There this model results in a minimizing problem of the form

$$\sigma_{opt} \in \arg \min \left\{ \int_{\Omega} E(\sigma) \, dx \right\} \quad \left( \begin{array}{l} \sigma \in L^p(\Omega; \mathbb{R}^n), \\ -\text{div} \sigma = f \text{ in } \Omega, \sigma \cdot v = 0 \text{ on } \partial \Omega \end{array} \right). \quad (1.7)$$

The equation (1.6) is connected with the minimizing problem (1.7). In fact, in a paper \cite{11}, it is proved that the optimal traffic flow $\sigma_{opt}$ of the problem (1.7) is uniquely given by $\nabla E^*(\nabla v)$, where $v$ solves (1.6) under a Neumann boundary condition. Also, it is worth mentioning that (1.6) can be written by $-\text{div}(\nabla E^*(\nabla v)) = f$, where

$$E^*(z) = \frac{1}{p'}(|z| - b)_+^{p'}$$

denotes the Legendre transform of $E$.

The question whether the flow $\sigma_{opt} = \nabla E^*(\nabla v)$ is continuous has been resolved affirmatively. There it has been to establish continuity of the vector field $\mathcal{G}_{b+\delta}(\nabla v)$ with $\delta > 0$. This expectation on continuity can be explained, similarly to (1.4), by calculating the ellipticity ratio of the Hessian matrix $\nabla^2 E^*(\nabla v)$. In fact, for every $z_0 \in \mathbb{R}^n$ with $|z_0| \geq \delta + b$, there holds

$$\left( \begin{array}{cc} \text{(ellipticity ratio of } \nabla^2 E^*(z_0)) \end{array} \right) = \frac{\text{(the largest eigenvalue of } \nabla^2 E^*(z_0))}{\text{(the lowest eigenvalue of } \nabla^2 E^*(z_0))} \leq (p - 1) \left( 1 + \delta^{-1} \right)$$
as \( \delta \to 0 \). This estimate suggests that for each fixed \( \delta > 0 \), the truncated gradient \( G_{b+\delta} (\nabla v) \) should be Hölder continuous. We note that it is possible to show \( G_{b+\delta} (\nabla v) \) uniformly converges to \( G_b (\nabla v) \) as \( \delta \to 0 \), and thus \( G_b (\nabla v) \) will be also continuous.

When \( v \) is a scalar function, this expectation on continuity of \( G_{b+\delta} (\nabla v) \) with \( \delta > 0 \) was first mathematically justified by Santambrogio–Vespri [37] in 2010 for the special case \( n = 2 \) with \( b = 1 \). The proof therein is based on oscillation estimates related to Dirichlet energy. For technical reasons related to this method, the condition \( n = 2 \) is essentially required. Later in 2014, Colombo–Figalli [14] established a more general proof that works for any \( n \geq 2 \) and any function \( E^* \) whose zero-levelset is sufficiently large enough to define a real-valued Minkowski gauge. This Minkowski gauge plays as a basic modulus to estimate ellipticity ratios on equations they treated. It should be mentioned that their strategy will not work for our problem \((1.1)\), since some structures of the density functions therein seems substantially different from ours. In fact, in our problem, the zero-levelset of \( E \) is only a singleton, and thus it is impossible to define the Minkowski gauge as a real-valued function. The recent work by Bögelein–Duzar–Gioga–Passarelli di Napoli [10] also focuses on proving Hölder continuity of \( G_{b+\delta} (\nabla v) \) for each \( \delta > 0 \) with \( b = 1 \). A novelty of this paper is that their strategy works even when \( v \) is a vector-valued function. There they considered an approximating problem of the form

\[
-\varepsilon \Delta v - \text{div} (\nabla E^*(\nabla v)) = f \quad \text{in } \Omega. \tag{1.8}
\]

The paper [10] establishes a priori Hölder continuity of \( G_{1+2\delta} (\nabla v) \) for each fixed \( \delta \in (0, 1) \), whose estimate is independent of an approximation parameter \( \varepsilon \in (0, 1) \).

Our proofs on a priori Hölder estimates are highly inspired by [10], §4–7]. There are three significant differences between their proofs and ours. The first is that we need to treat another relaxed vector field \( G_{2\delta, \varepsilon} (\nabla u) \) as in \((1.5)\), different from \( G_{2\delta} (\nabla v) \). This is essentially because our regularized problems as in \((1.4)\) are based on relaxing the very singular operator \( L_{b,p} \) (or the general operator \( L \)) itself, whereas their approximation given in \((1.8)\) does not change principal part at all. Hence, in the relaxed problem \((1.8)\), it is not necessary to adapt another mapping than \( G_{2\delta, \varepsilon} \). The second is some structural differences between density functions \( E \) and \( E^* \), which make our energy estimates in Section 3.5 different from [10], §4–5]. The third is that we have to make the approximation parameter \( \varepsilon \) sufficiently smaller than the truncation parameter \( \delta \). To be precise, when we prove a priori Hölder continuity of \( G_{2\delta, \varepsilon} (\nabla u) \), we carefully utilize the setting \( 0 < \varepsilon < \delta / 8 \). This condition will be required especially when two different moduli \( |\nabla u| \) and \( \sqrt{\varepsilon^2 + |\nabla u|^2} \) are both used. In the paper [10], no restrictions on \( \varepsilon \in (0, 1) \) are made, since their analysis for \((1.8)\) is based on a single modulus \( |\nabla v| \) only.

### 1.3 Some notations and our general results

In Section 1.3, we describe our settings on the equation (1.2) and describe the main result. Before stating our general result, we fix some notations.

We denote \( \mathbb{Z}_{\geq 0} := \{ 0, 1, 2, \ldots \} \) by the set of all non-negative integers, and \( \mathbb{N} := \mathbb{Z}_{\geq 0} \setminus \{ 0 \} \) by the set of all natural numbers. For real numbers \( a, b \in \mathbb{R} \), we often write \( a \vee b := \max\{ a, b \} \) and \( a \wedge b := \min\{ a, b \} \) for notational simplicity.

For \( n \times 1 \) column vectors \( x = (x_i)_{1 \leq i \leq n} \), \( y = (y_i)_{1 \leq i \leq n} \in \mathbb{R}^n \), we write

\[
\langle x \mid y \rangle := \sum_{i=1}^{n} x_i y_i, \quad |x| := \sqrt{\langle x \mid x \rangle} = \sqrt{\sum_{i=1}^{n} |x_i|^2}.
\]

In other words, we equip \( \mathbb{R}^n \) with the Euclidean inner product \( \langle \cdot \mid \cdot \rangle \) and the Euclidean norm \( |\cdot| \). We also
We also define the Frobenius norm $\|A\|$ as the operator norm of $A$ with respect to the canonical Euclidian norm,

$$\|A\| := \sup \left\{ \frac{|Ax|}{|x|} \mid x \in \mathbb{R}^n \setminus \{0\} \right\}.$$ 

We also define the Frobenius norm

$$|A| := \sqrt{\sum_{i,j=1}^{n} |a_{i,j}|^2}$$

for $A = (a_{i,j})_{1 \leq i,j \leq n}$. For $n \times n$ real symmetric matrices $A, B$, we write $A \leq B$ or $B \geq A$ when $B - A$ is non-negative.

For a scalar function $u = u(x_1, \ldots, x_n)$, we denote $\nabla u, \nabla^2 u, \nabla^3 u$ respectively by gradient vector, Hessian matrix, and third-order derivatives of $u$;

$$\|\nabla u\| := |(\partial_{x_i} u)_{1 \leq i \leq n}|, \quad \|\nabla^2 u\| := |(\partial_{x_i x_j} u)_{1 \leq i,j \leq n}|, \quad \|\nabla^3 u\| := |(\partial_{x_i x_j x_k} u)_{1 \leq i,j,k \leq n}|.$$ 

Similarly to $|\nabla u|$ and $|\nabla^2 u|$, we define

$$|\nabla^3 u| := \sqrt{\sum_{i,j,k=1}^n |\partial_{x_i x_j x_k} u|^2}.$$ 

For given $\mathbb{R}^n$-valued vector field $U = (U_1, \ldots, U_n)$ with $U_j = U_j(x_1, \ldots, x_n)$ for each $j \in \{1, \ldots, n\}$, we denote $DU = (\partial_{x_i} U_j)_{i,j}$ by the Jacobian matrix of $U$.

Let $U \subset \mathbb{R}^n$ be an open set. For given numbers $s \in [1, \infty], d \in \mathbb{N}$, and $k \in \mathbb{N}$, we denote $L^s(U; \mathbb{R}^d)$ and $W^{k,s}(U; \mathbb{R}^d)$ respectively by the Lebesgue space and the Sobolev space. To shorten the notations, we often write $L^s(U) := L^s(U; \mathbb{R})$ and $W^{k,s}(U) := W^{k,s}(U; \mathbb{R})$. It is mentioned that the Lebesgue space $L^s(U; \mathbb{R}^d)$ itself makes sense even when $U$ is only assumed to be Lebesgue measurable.

Throughout this paper, we let

$$\mathcal{L}_1 u := -\text{div}(\nabla E_1(\nabla u)), \quad \mathcal{L}_p u := -\text{div}(\nabla E_p(\nabla u)),$$

where $E_1$ and $E_p$ are non-negative convex functions in $\mathbb{R}^n$. For regularity of the densities, we require $E_1 \in C(\mathbb{R}^n) \cap C^2_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ and $E_p \in C^1(\mathbb{R}^n) \cap C^2_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ with $\beta_0 \in (0, 1]$. In this paper, we let $E_1$ be positively one-homogeneous. In other words, $E_1$ satisfies

$$E_1(\lambda z) = \lambda E_1(z) \quad \text{for all } z \in \mathbb{R}^n, \lambda > 0. \tag{1.9}$$

For $E_p$, we assume that there exists constants $0 < \lambda_0 \leq \Lambda_0 < \infty$ such that

$$|\nabla E_p(z)| \leq \Lambda_0 |z|^{p-1} \quad \text{for all } z \in \mathbb{R}^n, \tag{1.10}$$

$$\lambda_0 |z|^{p-2} \text{id} \leq \nabla^2 E_p(z) \leq \Lambda_0 |z|^{p-2} \text{id} \quad \text{for all } z \in \mathbb{R}^n \setminus \{0\}. \tag{1.11}$$
For $\beta_0$-Hölder continuity of Hessian matrices $\nabla^2 E_p$, it is assumed that
\[
\|\nabla^2 E_p(z_1) - \nabla^2 E_p(z_2)\| \leq \Lambda_0 \mu^{p-2-\beta_0} |z_1 - z_2|^{\beta_0}
\] (1.12)
holds for all $\mu \in (0, \infty)$, and $z_1, z_2 \in \mathbb{R}^n \setminus \{0\}$ satisfying
\[
\frac{\mu}{16} \leq |z_1| \leq 3\mu, \quad \text{and} \quad |z_1 - z_2| \leq \frac{\mu}{32}.
\] (1.13)

A typical example is
\[
E_1(z) = b|z|, \quad E_p(z) = \frac{1}{p}|z|^p,
\] (1.14)
It is easy to check that $E_1$ and $E_p$ defined as in (1.14) satisfy (1.9)–(1.11) with $\Lambda_0 := \min\{1, p-1\}$, $\Lambda_0 := \max\{1, p-1\}$. Moreover, by replacing $\Lambda_0 = \Lambda_0(n, p)$ sufficiently large, we may assume that (1.12) also holds with $\beta_0 = 1$ (see Section 2.4 for further explanations).

We give the definition of weak solutions to (1.2).

**Definition 1.2.** Let $p \in (1, \infty)$ and $q \in [1, \infty]$ satisfy
\[
\begin{cases}
np/np - n + p < q \leq \infty & (1 < p < n), \\
1 < q \leq \infty & (p = n), \\
1 \leq q \leq \infty & (n < p < \infty).
\end{cases}
\] (1.15)

For a given function $f \in L^q(\Omega)$, a function $u \in W^{1,p}(\Omega)$ is called a weak solution to (1.2) when there exists a vector field $Z \in L^{\infty}(\Omega; \mathbb{R}^n)$ such that the pair $(u, Z)$ satisfies
\[
\int_\Omega \langle Z + \nabla E_p(\nabla u) \rangle \nabla \phi \, dx = \int_\Omega f \phi \, dx \quad \text{for all } \phi \in W^{1,p}_0(\Omega),
\] (1.16)
and
\[
Z(x) \in \partial E_1(\nabla u(x)) \quad \text{for a.e. } x \in \Omega.
\] (1.17)

Here $\partial E_1$ denotes the subdifferential of $E_1$, defined by
\[
\partial E_1(z) := \{\zeta \in \mathbb{R}^n \mid E_1(w) \geq E_1(z) + \langle \zeta \mid w - z \rangle \text{ for all } w \in \mathbb{R}^n\}.
\]

It should be noted that (1.15) enables us to apply the compact embedding $W^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega)$ (see e.g., [2] Chapters 4 & 6), so that the weak formulation (1.16) makes sense. Clearly, the condition $q \in (n, \infty]$ implies (1.15).

Our main result is the following Theorem 1.3

**Theorem 1.3 (C^1-regularity theorem).** Let $p \in (1, \infty)$ and $\beta_0 \in (0, 1]$, and assume that the functions $E_1 \in C(\mathbb{R}^n) \cap C^{2,\beta_0}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ and $E_p \in C^1(\mathbb{R}^n) \cap C^{2,\beta_0}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ satisfy (1.9)–(1.12). If $u$ is a weak solution to (1.2) with $f \in L^q(\Omega)$ for some $q \in (n, \infty]$, then $u$ is in $C^1(\Omega)$.

In the case (1.14), the equation (1.2) becomes (1.1), and hence Theorem 1.3 clearly generalizes Theorem 1.1. Although we have succeeded in showing $C^1$-regularity of weak solutions, it should be mentioned that we have required $E_1, E_p \in C^{2,\beta_0}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ in Theorem 1.3. On the other hand, the previous proof of $C^1$-regularity of convex solutions [24, Theorem 4], where some tools from convex analysis are basically used, works under weaker assumptions that $E_1, E_p \in C^2(\mathbb{R}^n \setminus \{0\})$. Local $C^{2,\beta_0}_{\text{loc}}$ regularity of $E = E_1 + E_p$ outside the origin is used for technical reasons to make our perturbation arguments successful (see Remark 3.14 in Section 3.6).
1.4 Outlines of the paper

We conclude Section 1 by briefly describing outlines of this paper. Section 2 aims to give the proof of Theorem 1.3 by approximation arguments. For preliminaries, in Section 2.1 we give basic estimates for gradients and Hessian matrices of relaxed density functions. After stating some basic facts on the density \( E_1 \) in Section 2.2 we would like to check that the relaxation of the density \( E = E_1 + E_p \) convoluted by the Friedrichs mollifier satisfies good properties in Sections 2.3–2.4. Next in Sections 2.5–2.6 we consider an approximating problem for the equation (1.2). There we prove a convergence result on variational inequality problems (Proposition 2.11). As a corollary, we are able to deduce that a solution to an approximated equation surely converges to a weak solution to (1.2), under suitable Dirichlet boundary conditions (Corollary 2.12). Finally in Section 2.7 we would like to prove Theorem 1.3. There Theorem 2.14 is used without proofs, which states that a truncated gradient of an approximated solution, given by (1.5), is locally Hölder continuous, uniformly for an approximation parameter \( \varepsilon \in (0, \delta/8) \).

Section 3 is focused on giving the proof of Theorem 2.14. First in Section 3.1 we briefly describe inner regularity on generalized solutions, including a priori Lipschitz bounds (Proposition 3.1). In Section 3.2 we would like to give the proof of Theorem 1.3. There we use key Propositions 3.2–3.3, the proofs of which are given after deducing a weak formulation in Section 3.3. Throughout Sections 3.4–3.6 we would like to justify that the vector field \( g_{\delta, \varepsilon}(\nabla u_\varepsilon) \) satisfies a Campanato-type growth estimate under suitable conditions where a freezing coefficient method works (Proposition 3.2). In Section 3.7 we show a De Giorgi-type lemma on the scalar-valued function \( |g_{\delta, \varepsilon}(\nabla u_\varepsilon)| \) (Proposition 3.3). Some lemmata in Section 3.7 are easy to deduce, appealing to standard arguments given in [3, Chapter 3], [23, Chapter 5] and [17, Chapter 10, §4–5]. For the reader’s convenience, we provide detailed computations in the appendix (Section 4).

Finally, we would like to mention that our result is based on adaptations of [42], the author’s recent work that focuses on system problems. This paper also aims to provide full computations and proofs of some estimates that are omitted in [42]. Compared with [42], some computations herein might become somewhat complicated, since the density function \( E = E_1 + E_p \) is not assumed to be symmetric in scalar problems.

2 Approximation schemes and the proof of Theorem 1.3

The aim of Section 2 is to provide suitable approximation schemes for the equation (1.2). This approximation is necessary since the ellipticity ratio of (1.2) blows up near the facets, which makes it difficult to show local Sobolev regularity on second order derivatives. Therefore, we have to introduce relaxed equations whose ellipticity ratio is bounded even across the facets. When considering the operators \( \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_p \), we have to give a suitable relaxation of the anisotropic operator \( \mathcal{L}_1 \). This is possible by convolving the density function \( E_1 \) by the Friedrichs mollifier. The novelty concerning our relaxation is that this works as long as \( E_1 \) is positively one-homogeneous.

2.1 Preliminaries

First in Section 2.1 we prove quantitative estimates related to gradients or Hessian matrices for preliminaries. The results herein will play an important role in Sections 2.5.

First, we would like to prove Lemma 2.1 below.

Lemma 2.1. Let \( s \in [1, \infty) \) and \( \sigma_1, \sigma_2, \sigma_3 \in (0, \infty) \) be fixed constants. Assume that a real-valued
function $H \in C(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\})$ satisfies

$$|\nabla H(z)| \leq L(\sigma_1 + |z|^{s-1}) \quad \text{for all } z \in \mathbb{R}^n \setminus \{0\},$$  \hspace{1cm} (2.1)$$

$$||\nabla^2 H(z)|| \leq L(\sigma_2 + |z|^{s-2}) \quad \text{for all } z \in \mathbb{R}^n \setminus \{0\},$$  \hspace{1cm} (2.2)$$

where $L \in (0, \infty)$ is constant. Then, for all $z_1, z_2 \in \mathbb{R}^n \setminus \{0\}$, there holds

$$|\nabla H(z_1) - \nabla H(z_2)| \leq C(s)L \cdot \varepsilon(\sigma_1, \sigma_2, |z_1|, |z_2|) \cdot \left( |z_1|^{-1} \wedge |z_2|^{-1} \right) |z_1 - z_2|,$$  \hspace{1cm} (2.3)$$

where $C(s) \in (0, \infty)$ is constant and

$$\varepsilon(\sigma_1, \sigma_2, t_1, t_2) := \left( t_1^{s-1} + t_2^{s-1} \right) + \sigma_1 + \sigma_2 \cdot (t_1 + t_2) \quad \text{for } t_1, t_2 > 0.$$ 

Moreover, if $H$ satisfies $H \in C^3(\mathbb{R}^n \setminus \{0\})$ and there holds

$$|\nabla^3 H(z)| \leq L(\sigma_3 + |z|^{s-3}) \quad \text{for all } z \in \mathbb{R}^n \setminus \{0\},$$  \hspace{1cm} (2.4)$$

then we have

$$|\nabla^2 H(z_1)(z_2 - z_1) - (\nabla H(z_2) - \nabla H(z_1))| \leq C(s)L \left[ \varepsilon(\sigma_1, \sigma_2, |z_1|, |z_2|) + \sigma_3 |z_1|^2 \right] \frac{|z_1 - z_2|^2}{|z_1|^2},$$  \hspace{1cm} (2.5)$$

for all $z_1, z_2 \in \mathbb{R}^n \setminus \{0\}$. Here $C(s) \in (0, \infty)$ is constant.

**Proof.** We distinguish (2.3) between two cases. In the case $|z_1 - z_2| \geq |z_1|/2$, we just use (2.1) and get

$$|\nabla H(z_1) - \nabla H(z_2)| \leq L \left[ 2\sigma_1 + \left( |z_1|^{s-1} + |z_2|^{s-1} \right) \right] \frac{|z_1 - z_2|}{|z_1|} \leq 4L \left[ \sigma_1 \left( |z_1|^{s-1} + |z_2|^{s-1} \right) \right] \frac{|z_1 - z_2|}{|z_1|} 
\leq 4L \cdot \varepsilon(\sigma_1, \sigma_2, |z_1|, |z_2|) \frac{|z_1 - z_2|}{|z_1|}.$$ 

In the remaining case $|z_1 - z_2| \leq |z_1|/2$, which yields $|z_2| \leq |z_2 - z_1| + |z_1| < 3|z_1|/2$, it is easy to check that

$$\frac{5}{2} |z_1| > |z_1| + |z_2| \geq |z_2 + t(z_1 - z_2)| \geq |z_1| - |z_1 - z_2| > \frac{1}{2} |z_1| > 0$$  \hspace{1cm} (2.6)$$

for all $t \in [0, 1]$. Hence, from (2.2), it follows that

$$||\nabla^2 H(z_1 + t(z_2 - z_1))|| \leq L \left[ \sigma_2 + 2^{-s+2} \left( S^{s-2} \vee 1 \right) |z_1|^{s-2} \right] \quad \text{for all } t \in [0, 1].$$

In particular, by $E_1 \in C^2(\mathbb{R}^2 \setminus \{0\})$, we can compute

$$|\nabla H(z_1) - \nabla H(z_2)| \leq \left[ \int_0^1 \nabla^2 H(z_2 + t(z_1 - z_2)) \cdot (z_1 - z_2) \, dt \right] |z_1 - z_2|
\leq \left( \int_0^1 ||\nabla^2 H(z_2 + t(z_1 - z_2))|| \, dt \right) |z_1 - z_2|$$
\[
\leq L \left[ \sigma_2 |z_1| + 2^{-s+2} \left( 5^{s-2} \vee 1 \right) |z_1|^{s-1} \right] \frac{|z_1 - z_2|}{|z_1|} \\
\leq C(s) L \cdot \mathcal{E}(\sigma_1, \sigma_2, |z_1|, |z_2|) \cdot \frac{|z_1 - z_2|}{|z_1|}.
\]

With \( z_1 \) and \( z_2 \) replaced by each other, we obtain (2.3).

We prove (2.5) by dividing into two cases. When \(|z_1 - z_2| \geq \frac{|z_1|}{2}\), we use (2.2) to get

\[
|\nabla^2 H(z_1)(z_2 - z_1) - (\nabla H(z_2) - \nabla H(z_1))| \\
\leq \left| \int_0^1 \nabla^2 H(z_1) \left[ z_2 - z_1 + |\nabla H(z_2) - \nabla H(z_1)| \right] dt \right| (z_2 - z_1) \\
\leq |z_2 - z_1| \int_0^1 \left| \nabla^2 H(z_1) \right| dt \\
\leq |z_2 - z_1| \int_0^1 \left( \int_0^1 \left| \nabla^3 H(z_1 + (1-s)t(z_2 - z_1)) \right|^2 ds \right)^{1/2} t|z_2 - z_1| dt \\
\leq C(s) L \left( \sigma_2 |z_1|^2 + |z_1|^{s-1} \right) \frac{|z_1 - z_2|^2}{|z_1|^2} \\
\leq C(s) L \left( \mathcal{E}(\sigma_1, \sigma_2, |z_1|, |z_2|) + \sigma_3 |z_1|^2 \right) \frac{|z_1 - z_2|^2}{|z_1|^2}.
\]

In the remaining case \(|z_1 - z_2| \leq \frac{|z_1|}{2}\), we recall (2.6), which yields

\[
|\nabla^3 H(z_1 + t(z_2 - z_1))| \leq L \left[ \sigma_3 + 2^{-s+3} \left( 5^{s-3} \vee 1 \right) |z_1|^{s-3} \right] \text{ for all } t \in [0, 1].
\]

By this and \( H \in C^3(\mathbb{R}^n \setminus \{0\}) \), we obtain

\[
|\nabla^2 H(z_1)(z_2 - z_1) - (\nabla H(z_2) - \nabla H(z_1))| \\
= \left| \int_0^1 \nabla^2 H(z_1) \left[ z_2 - z_1 + t(z_2 - z_1) \right] dt \right| (z_2 - z_1) \\
\leq |z_2 - z_1| \int_0^1 \left| \nabla^2 H(z_1) \right| dt \\
\leq |z_2 - z_1| \int_0^1 \left( \int_0^1 \left| \nabla^3 H(z_1 + (1-s)t(z_2 - z_1)) \right|^2 ds \right)^{1/2} t|z_2 - z_1| dt \\
\leq C(s) L \left( \sigma_2 |z_1|^2 + |z_1|^{s-1} \right) \frac{|z_1 - z_2|^2}{|z_1|^2} \\
\leq C(s) L \left( \mathcal{E}(\sigma_1, \sigma_2, |z_1|, |z_2|) + \sigma_3 |z_1|^2 \right) \frac{|z_1 - z_2|^2}{|z_1|^2}.
\]

In both cases, we are able to conclude (2.5). \(\square\)

Secondly, we would like to show that some growth estimates given in Lemma 2.1 will be kept under the convolution by mollifiers (Lemma 2.2). Also, we will check that ellipticity or monotonicity estimates can also be obtained (Lemma 2.3).

Let \( \{ j_\varepsilon \}_{0 < \varepsilon < 1} \subset C_c^\infty(\mathbb{R}^n) \) denote the Friedrichs mollifiers. That is, we fix a spherically symmetric real-valued function \( j \in C_c^\infty(\mathbb{R}^n) \) satisfying

\[
0 \leq j \text{ in } \mathbb{R}^n, \supp j \subset B_1(0), \int_{\mathbb{R}^n} j(x) dx = 1,
\]

and define

\[
0 \leq j_\varepsilon(x) := \varepsilon^{-n} j(x/\varepsilon) \quad \text{for } x \in \mathbb{R}^n, 0 < \varepsilon < 1.
\]

For a given function \( f \in L_1(\mathbb{R}^n, \mathbb{R}^m) \), we define \( j_\varepsilon \ast f \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^m) \) by

\[
(j_\varepsilon \ast f)(x) := \int_{\mathbb{R}^n} j_\varepsilon(y) f(x-y) dy \in \mathbb{R}^m.
\]
for each $x \in \mathbb{R}^n$. We note that when $f \in W^{1,s}_{\text{loc}}(\mathbb{R}^n)$ ($1 \leq s \leq \infty$), there holds $\nabla(j_\varepsilon \ast f) = j_\varepsilon \ast \nabla f$ by the definition of Sobolev derivatives. In Lemmata 2.2, 2.3 we show some quantitative estimates related to functions convoluted by mollifiers. Among them, the inequality 2.13 in Lemma 2.2 will become a key tool. Although some estimates in Lemmata 2.2–2.3 might be well-known in existing literatures (see e.g., [19], [22]), we would like to provide elementary proofs for the reader’s convenience.

Lemma 2.2 (Growth estimates). For a fixed constant $s \in [1, \infty)$, we have the following:

(1) Assume that real-valued functions $\{H_{\varepsilon}\}_{0<\varepsilon<1} \subset C^2(\mathbb{R}^n)$ satisfy

$$|\nabla H_{\varepsilon}(z)| \leq L_0 \left(\varepsilon^2 + |z|^2\right)^{(s-1)/2}$$

for all $z \in \mathbb{R}^n$, \hspace{1cm} (2.7)

$$\|\nabla^2 H_{\varepsilon}(z)\| \leq L_0 \left(\varepsilon^2 + |z|^2\right)^{s/2-1}$$

for all $z \in \mathbb{R}^n$, \hspace{1cm} (2.8)

where the constant $L_0 \in (0, \infty)$ is independent of $\varepsilon \in (0, 1)$. If $1 \leq s < 2$, then there holds

$$|\nabla H_{\varepsilon}(z_1) - \nabla H_{\varepsilon}(z_2)| \leq C \min\{ |z_1|^{s-2}, |z_2|^{s-2}\} |z_1 - z_2|$$

for all $(z_1, z_2) \in (\mathbb{R}^n \times \mathbb{R}^n) \setminus \{(0, 0)\}$. Here the constant $C \in (0, \infty)$ is given by $C := 2^{-s}L_0$.

(2) Assume that a real-valued function $H \in C(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\})$ satisfies (2.1)–(2.2) with $\sigma_1 = \sigma_2 = 0$ and $L_0 \leq (0, \infty)$. Then, there exists a constant $\Gamma_0 = \Gamma_0(n, s) \in (0, \infty)$ such that the function $H_{\varepsilon} := j_\varepsilon \ast H \in C^\infty(\mathbb{R}^n)$ ($0 < \varepsilon < 1$) satisfies (2.7)–(2.8) with $L_0 = \Gamma_0L_0 \in (0, \infty)$. In particular, $H_{\varepsilon}$ satisfies (2.9) with $C := 2^{-s}\Gamma_0L_0 \in (0, \infty)$, provided $1 \leq s < 2$.

(3) In addition to the assumptions in (2) let $H$ be in $C^{2,\beta_0}(\mathbb{R}^n \setminus \{0\})$, and satisfy

$$\|\nabla^2 H(z_1) - \nabla^2 H(z_2)\| \leq L_0 \mu^{s-2-\beta_0} |z_1 - z_2|^{\beta_0}$$

for all $z_1, z_2 \in \mathbb{R}^n$ enjoying (1.13) with $\mu \in (0, \infty)$. Also, assume that positive numbers $\delta, \varepsilon$ satisfy

$$0 < \varepsilon < \frac{\delta}{8^s} \quad \text{and} \quad 0 < \delta < 1.$$ \hspace{1cm} (2.10)

Then, for all $\mu \in (\delta, \infty)$, and $z_1, z_2 \in \mathbb{R}^n \setminus \{0\}$ satisfying

$$\delta + \frac{\mu}{4} \leq |z_1| \leq \delta + \mu, \quad |z_2| \leq \delta + \mu,$$ \hspace{1cm} (2.11)

we have

$$|\nabla^2 H_{\varepsilon}(z_1)(z_2 - z_1) - (\nabla H_{\varepsilon}(z_2) - \nabla H_{\varepsilon}(z_1))| \leq C_0 L_0 \mu^{s-2-\beta_0} |z_1 - z_2|^{1+\beta_0},$$ \hspace{1cm} (2.12)

where $C_0 \in (0, \infty)$ depends at most on $n$, $s$, and $\beta_0$.

Proof. \hspace{1cm} (1) By $s/2 - 1 < 0$ and (2.8), we compute

$$|\nabla H_{\varepsilon}(z_1) - \nabla H_{\varepsilon}(z_2)| = \left|\int_0^1 \nabla^2 H_{\varepsilon}(tz_1 + (1-t)z_2) \cdot (z_1 - z_2) \, dt\right|$$

$$\leq |z_1 - z_2| \int_0^1 \|\nabla^2 H_{\varepsilon}(tz_1 + (1-t)z_2)\| \, dt$$

$$\leq L |z_1 - z_2| \int_0^1 \left(\varepsilon^2 + |tz_1 + (1-t)z_2|^2\right)^{s/2-1} \, dt$$
We distinguish (2.16) between two cases. In the case $|z_1| < |z_2|$, then,

$$\int_0^1 |t z_1 + (1-t) z_2|^{s-2} \, dt.$$

It suffices to show

$$\int_0^1 |t z_1 + (1-t) z_2|^{s-2} \, dt \leq 2^{-s} \min\{ |z_1|^{s-2}, |z_2|^{s-2} \}$$

(2.14)

to prove (2.15). Without loss of generality we may assume that $|z_1| \leq |z_2|$. Then, $z_2 \neq 0$ clearly follows from $(z_1, z_2) \neq (0, 0)$. By the triangle inequality, we have

$$|t z_1 + (1-t) z_2| \geq (1-t) |z_2| - t |z_1| \geq (1-t) |z_2| \geq \frac{1}{2} |z_2| > 0 \quad \text{when } 0 \leq t \leq \frac{1}{4}.$$  

(2.15)

Again by $s-2 < 0$, we obtain

$$\int_0^1 |t z_1 + (1-t) z_2|^{s-2} \, dt \leq \frac{|z_2|^{s-2}}{2^{s-2}} \cdot \frac{1}{4} = \frac{|z_2|^{s-2}}{2^s},$$

which completes the proof of (2.14).

To find the constant $\Gamma_0(n, s) \in (0, \infty)$, for each fixed constant $\sigma \in [-1, \infty)$, we would like to show

$$h_{\sigma, \epsilon}(z) := (j_{\epsilon \cdot} \cdot | \cdot |^\sigma)(z) \leq C(n, \sigma) \left( e^2 + |z|^2 \right)^{\sigma/2}$$

for all $z \in \mathbb{R}^n, \epsilon \in (0, 1)$.

(2.16)

Here the constant $C(n, \sigma) \in (0, \infty)$ is independent of $\epsilon \in (0, 1)$. We note that the inclusion $| \cdot |^\sigma \in L^1_{\text{loc}}(\mathbb{R}^n)$ follows from $n \geq 2$ and $-1 \leq \sigma < \infty$. Hence, we are able to define $h_{\sigma} := j \cdot | \cdot |^\sigma \in C^\infty(\mathbb{R}^n)$. By change of variables, we have

$$h_{\sigma, \epsilon}(z) = \int_{B_{2\epsilon}(0)} j_{\epsilon}(y) |z - y|^{\sigma} \, dy$$

(2.17)

$$= \int_{B_{\epsilon}(0)} j_{\epsilon}(\tilde{y}) |z - \tilde{y} / \epsilon|^{\sigma} \, d\tilde{y}$$

$$= \epsilon^{\sigma} \int_{B_{1}(0)} j_{\epsilon}(\tilde{y}) |z / \epsilon - \tilde{y}|^{\sigma} \, d\tilde{y} = \epsilon^{\sigma} h_{\sigma}(z / \epsilon).$$

(2.18)

We distinguish (2.16) between two cases. In the case $|z| \leq 2\epsilon$, we use $\epsilon \leq \sqrt{e^2 + |z|^2} \leq \sqrt{5} \epsilon$ and (2.18).

Then, we have

$$h_{\sigma}(z) \leq C(n, \sigma) \left( 1 + 5^{\sigma/2} \right) \left( e^2 + |z|^2 \right)^{\sigma/2}$$

with $C(n, \sigma) := \max_{B_{2\epsilon}(0)} h_{\sigma} < \infty$.

In the remaining case $|z| > 2\epsilon$, we take the unique number $j \in \mathbb{N}$ such that $2^j \epsilon \leq |z| < 2^{j+1} \epsilon$. Then, for all $y \in B_{\epsilon}(0)$, we have

$$\frac{|z - y|^2}{e^2 + |z|^2} \geq \frac{2^j - 1}{4^j} \frac{2^j e^2}{4^j + 1} = \frac{(1-2^{-j})^2}{4^j + 1} \geq \frac{1}{17},$$

and

$$\frac{|z - y|^2}{e^2 + |z|^2} \leq 2 \cdot \frac{|y|^2 + |z|^2}{e^2 + |z|^2} \leq 2.$$

By these inequalities and (2.17), we have

$$h_{\sigma, \epsilon}(z) \leq \left[ 2^{\sigma/2} \sqrt{1 + 5^{\sigma/2}} \right] \left( e^2 + |z|^2 \right)^{\sigma/2}.$$
The estimate (2.7) follows from (2.16). In fact, there holds \( \nabla H_e = j_e * \nabla H \) a.e. in \( \mathbb{R}^n \), since \( H \in W^{1,\infty}_\text{loc}(\mathbb{R}^n) \) follows from (2.11) with \( \sigma_1 = 0 \). By applying (2.16) with \( \sigma = s - 1 \), (2.7) is easily obtained. To deduce (2.8), we recall that \( \nabla\nabla \leq 0 \) follows from (2.10) with \( \sigma_1 = \sigma_2 = 0 \), and therefore

\[
|\nabla H_e(z_1) - \nabla H_e(z_2)| = \left| \int_{B_{r_0}(0)} j_e(y) \frac{|z_1 - y|^{s-1} + |z_2 - y|^{s-1}}{|z_2 - y|} \, dy \right| 
\leq C(s) L \frac{|z_1 - z_2|}{\int_{B_{r_0}(0)} j_e(y) \, dy} \frac{|z_1 - y|^{s-1} + |z_2 - y|^{s-1}}{|z_2 - y|} \, dy 
\]

(2.19)

holds for all \( z_1, z_2 \in \mathbb{R}^n \). Fix \( z_0 \in \mathbb{R}^n \), and let \( \lambda \in \mathbb{R} \) be an arbitrary eigenvalue of the Hessian matrix \( \nabla^2 H_e(z_0) \). We take a unit eigenvector \( v_0 \) corresponding to this \( \lambda \). By \( H_e \in C^\infty(\mathbb{R}^n) \) and (2.19), we obtain

\[
|\lambda| = \left| \langle \nabla^2 H_e(z_0)v_0 | v_0 \rangle \right| 
= \lim_{t \to 0} \left| \frac{\langle \nabla^2 H_e(z_0 + tv_0) | v_0 \rangle - \langle \nabla^2 H_e(z_0) | v_0 \rangle}{t} \right| 
\leq \limsup_{t \to 0} \frac{\langle \nabla^2 H_e(z_0 + tv_0) - \nabla^2 H_e(z_0) | v_0 \rangle}{t} 
= C(s) L \left( h_{s-2, e}(z_0) + \limsup_{t \to 0} \int_{B_{r_0}(0)} \frac{j_e(y)}{|z_0 - y|} |z_0 + tv_0 - y|^{s-1} \, dy \right) 
= 2C(s) L h_{s-2, e}(z_0) \leq C(n, s) L \big( \epsilon^2 + |z_0|^2 \big)^{s/2-1}. 
\]

by (2.16) and Lebesgue’s dominated convergence theorem. Here we note again that the inclusion \( j_e|z_0 - \cdot|^{-1} \in L^1(B_{r_0}(0)) \) holds by \( n \geq 2 \). Also, the uniform convergence

\[
|z_0 + tv_0 - y|^{s-1} \to |z_0 - y|^{s-1} \quad \text{uniformly for } y \in B_{r_0}(0) 
\]

easily follows from \( s \geq 1 \). By these facts, we are able to apply Lebesgue’s dominated convergence theorem. This completes the proof of (2.3).

(3) Let \( z_1, z_2 \in \mathbb{R}^n \) satisfy (2.12). We first consider \( |z_1 - z_2| < \mu/32 \). In this case, it should be noted that for \( y \in B_{r_0}(0) \), there holds

\[
\delta + \frac{\mu}{16} \leq |z_1 - y| \leq 3\mu 
\]

by (2.11) and (2.12) and \( 0 < \delta < \mu \). Thus, we can apply (2.10) to obtain

\[
\left| \nabla^2 H_e(z_1)(z_2 - z_1) - (\nabla H_e(z_2) - \nabla H_e(z_1)) \right| 
= \left| \left( \int_0^1 \left[ \nabla^2 H_e(z_1) - \nabla^2 H_e(z_1 + t(z_2 - z_1)) \right] \, dt \right)(z_2 - z_1) \right| 
\leq |z_1 - z_2| \int_0^1 \left( \int_{B_{r_0}(0)} \left\| \nabla^2 H(z_1 - y) - \nabla^2 H(z_1 + t(z_2 - z_1) - y) \right\| \, j_e(y) \, dy \right) \, dt 
\leq L_0 \mu^{s-2-\beta_0} |z_1 - z_2|^{1+\beta_0}. 
\]

Here we have used an identity \( \nabla^2 H_e = j_e * \nabla^2 H \) in \( \mathbb{R}^n \setminus B_{r_0}(0) \). In fact, for each fixed \( \delta > 0 \), there holds \( H \in W^{2,\infty}_\text{loc}(\mathbb{R}^n \setminus B_\delta(0)) \) by (2.22), which yields \( \nabla^2 H_e = j_e * \nabla^2 H \) in \( \mathbb{R}^n \setminus B_{e+\delta}(0) \). By (2.11), we may take \( \delta := \delta - \epsilon > 0 \) to get this identity. In the remaining case \( |z_1 - z_2| > \mu/32 \), we use (2.3) and (2.19) to compute

\[
\left| \nabla^2 H_e(z_1)(z_2 - z_1) - (\nabla H_e(z_2) - \nabla H_e(z_1)) \right| 
\leq L_0 \mu^{s-2-\beta_0} |z_1 - z_2|^{1+\beta_0}. 
\]
\[
\begin{align*}
&\leq \|\nabla^2 \mathcal{H}_e(z_1)\| |z_1 - z_2| + C(s)L_0|z_1 - z_2| \int_{B_e(0)} J_e(y) \frac{|z_1 - y|^{s-1} + |z_2 - y|^{s-1}}{|z_1 - y|} \, dy \\
&\leq C(n, s)L_0|z_1 - z_2| \left( (\varepsilon^2 + |z_1|^2)^{s/2-1} + h_{s-2, e}(z_1) + h_{2, e}(z_1) \sup_{B_e(0)} |z_2 - y|^{s-1} \right) .
\end{align*}
\]

From (2.11)–(2.12), it is easy to get
\[
\frac{\mu^2}{16} \leq |z_1|^2 \leq \varepsilon^2 + |z_1|^2 \leq \delta^2 + (\delta + \mu)^2 \leq 5\mu^2 ,
\]
and
\[
|z_2 - y| \leq |z_2| + |y| \leq \delta + \mu \leq 2\mu \quad \text{for all } y \in B_e(0).
\]

By these estimates and (2.16), we have
\[
\mathcal{E} \leq C(n, s)\mu^{s-2} \cdot \left( \frac{32|z_1 - z_2|}{\mu} \right)^{\beta_0} = : C(n, s, \beta_0)\mu^{s-2-\beta_0}|z_1 - z_2|^{\beta_0} ,
\]
which yields (2.13). \hfill \Box

**Lemma 2.3** (Monotonicity estimates). For \( s \in (1, \infty) \) and \( \varepsilon \in (0, 1) \), we have the following:

1. Assume that \( \{H_e\}_{0 < e < 1} \subset C^{2}(\mathbb{R}^n) \)

   \[
   \nabla^2 H_e(z) \geq \gamma \left( \varepsilon^2 + |z|^2 \right)^{s/2-1} \text{id} \quad \text{for all } z \in \mathbb{R}^n
   \]

   for some constant \( \gamma \in (0, \infty) \). If \( s \in [2, \infty) \), then we have

   \[
   \langle \nabla H_e(z_1) - \nabla H_e(z_2) \mid z_1 - z_2 \rangle \geq C \max\{ |z_1|^{s-2}, |z_2|^{s-2} \} |z_1 - z_2|^2
   \]

   for all \( z_1, z_2 \in \mathbb{R}^n \). Here the constant \( C \in (0, \infty) \) is given by \( C(s, \gamma) := 2^s \gamma \).

2. Let a convex function \( H \in C^{1}(\mathbb{R}^n) \cap C^{2}(\mathbb{R}^n \setminus \{0\}) \)

   \[
   \nabla^2 H(z) \geq L_0 |z|^{s-2} \text{id} \quad \text{for all } z \in \mathbb{R}^n \setminus \{0\}
   \]

   for some constants \( L_0 \in (0, \infty), s \in (1, \infty) \). Then, there exists a constant \( \gamma_0 = \gamma_0(n, s) \in (0, 1) \)

   such that for every \( \varepsilon \in (0, 1) \), \( H_e := \varepsilon \cdot H \in C^{\infty}(\mathbb{R}^n) \) satisfies (2.20) with \( \gamma := \gamma_0 L_0 \in (0, \infty) \). In particular, \( H_e \) satisfies (2.21) with \( C := 2^{-s} \gamma_0 L_0 \in (0, \infty) \), provided \( 2 \leq s < \infty \).

**Proof.** As a preliminary, we check that when \( 2 \leq s < \infty \), there holds

\[
\int_0^1 \left| \varepsilon^2 + tz_2 + (1-t)z_1 \right|^{s/2-1} \, dt \geq 2^{-s} \left( \varepsilon^2 + \max\{ |z_1|^2, |z_2|^2 \} \right)^{s/2-1}
\]

for all \( z_1, z_2 \in \mathbb{R}^n \). Without loss of generality, we may let \( |z_1| \leq |z_2| \). Then, (2.15) yields

\[
\int_0^1 \left| \varepsilon^2 + tz_2 + (1-t)z_1 \right|^{s/2-1} \, dt \geq \frac{(\varepsilon^2 + |z_2|^2)^{s/2-1}}{2^{s-2}} \cdot \frac{1}{4} = 2^{-s} \left( \varepsilon^2 + |z_2|^2 \right)^{s/2-1}.
\]

(1) By (2.20) and (2.23), we have

\[
\langle \nabla H_e(z_1) - \nabla H_e(z_2) \mid z_1 - z_2 \rangle = \int_0^1 \langle \nabla^2 H_e(tz_1 + (1-t)z_2) (z_1 - z_2) \mid z_1 - z_2 \rangle \, dt
\]
\[ \geq y |z_1 - z_2|^2 \int_0^1 \left( \epsilon^2 + |tz_1 + (1-t)z_2|^2 \right)^{s/2-1} \, dt \]
\[ \geq 2^{-s} \lambda |z_1 - z_2|^2 \left( \epsilon^2 + \max \{ |z_1|^2, |z_2|^2 \} \right)^{s/2-1}. \]

From this and \( s/2 - 1 > 0 \), the estimate \( (2.22) \) immediately follows.

\[ (2) \] It suffices to find a constant \( \gamma_0 \in (0, 1) \) such that there holds
\[ \langle \nabla H_{\epsilon}(z_1) - \nabla H_{\epsilon}(z_2), z_1 - z_2 \rangle \geq \gamma_0 L_0 \left( \epsilon^2 + \max \{ |z_1|^2, |z_2|^2 \} \right)^{s/2-1} |z_1 - z_2|^2 \]
for all \( z_1, z_2 \in \mathbb{R}^n \). For fixed \( \epsilon \in (0, 1) \) and \( z_1, z_2 \in \mathbb{R}^n \), we set \( w(t) := z_1 + t(z_2 - z_1) \in \mathbb{R}^n \) for \( t \in [0, 1] \). We note that for a.e. \( y \in \mathcal{B}(0) \), there holds \( w(t) \neq \epsilon y \) for all \( t \in [0, 1] \). Hence by \( H \in C^2(\mathbb{R}^n \setminus \{0\}) \) and Fubini’s theorem, we are able to compute
\[ \langle \nabla H_{\epsilon}(z_1) - \nabla H_{\epsilon}(z_2), z_1 - z_2 \rangle \]
\[ = \int_{\mathcal{B}(0)} \langle \nabla H(z_1 - \epsilon y) - \nabla H(z_2 - \epsilon y), z_1 - z_2 \rangle \, j(y) \, dy \]
\[ = \int_{\mathcal{B}(0)} \left( \int_0^1 \nabla^2 H(z_2 - \epsilon y + t(z_1 - z_2))(z_1 - z_2) \, dt \right) \, j(y) \, dy \]
\[ \geq \int_{\mathcal{B}(0)} \left( \int_0^1 \max \{ |w(t)|, |z_1|^2, |z_2|^2 \} \, dt \right) \, j(y) \, dy \]
\[ \geq L_0 \int_{\mathcal{B}(0)} \left( \int_0^1 \max \{ |w(t)|, |z_1|^2, |z_2|^2 \} \, dt \right) \, j(y) \, dy \]
For each \( t \in [0, 1] \), we replace the integral domain \( \mathcal{B}(0) \) by a smaller one, either
\[ C^-_t := \{ y \in \mathcal{B}(0) \setminus \mathcal{B}_{1/2}(0) \mid \langle w(t) \mid y \rangle \leq 0 \} \quad \text{when } s \in [2, \infty), \]
or
\[ C^+_t := \{ y \in \mathcal{B}(0) \setminus \mathcal{B}_{1/2}(0) \mid \langle w(t) \mid y \rangle \geq 0 \} \quad \text{when } s \in (1, 2). \]

By the definitions, both of the sets \( C^+_t \) and \( C^-_t \) occupy at least one half of the domain \( \mathcal{B}(0) \setminus \mathcal{B}_{1/2}(0) \). Here we should recall that the function \( j \) is spherically symmetric. Hence, by direct computations, it is easy to check that
\[ \int_0^1 \left( \int_{\mathcal{B}(0)} |w(t) - \epsilon y|^2 (x, y) \, j(y) \, dy \right) \, dt \geq \gamma_1 \int_0^1 \left( \epsilon^2 + |tz_2 + (1-t)z_1|^2 \right)^{s/2-1} \, dt \]
for some constant \( \gamma_1 = \gamma_1(n, s) \in (0, 1) \). When \( s \in (1, 2) \), we simply use
\[ |tz_2 + (1-t)z_1| \leq \max \{ |z_1|, |z_2| \} \quad \text{for all } t \in [0, 1]. \]
Thus, we may choose \( \gamma_0 := \gamma_1. \) In the remaining case \( s \in [2, \infty) \), we may take \( \gamma_0 := 2^{-s} \gamma_1 \) by \( (2.22) \). \( \square \)

Finally, we conclude Section 2.1 by deducing continuity estimates of truncating mappings. For \( 0 < \delta < 1 \), we set
\[ \mathcal{G}_\delta(z) := (|z| - \delta) + \frac{\epsilon}{|z|} \quad \text{for } z \in \mathbb{R}^n. \]

Similarly, for \( 0 < \epsilon < \delta < 1 \), we set
\[ \mathcal{G}_{\delta, \epsilon}(z) := \left( \sqrt{\epsilon^2 + |z|^2} - \delta \right) + \frac{\epsilon}{|z|} \quad \text{for } z \in \mathbb{R}^n. \]
Lemma 2.4 below states that the mapping $G_{2\delta, \epsilon}$ has Lipschitz continuity, uniformly for sufficiently small $\epsilon > 0$. In the limiting case $\epsilon = 0$, Lipschitz continuity of $G_{\delta}$ is found in [10, Lemma 2.3]. Following the arguments therein, we would like to prove Lemma 2.4 by elementary computations. Before the proof, we recall an inequality

\[ \left| \frac{z_1}{|z_1|} - \frac{z_2}{|z_2|} \right| \leq 2 \min \{ |z_1|^{-1}, |z_2|^{-1} \} |z_1 - z_2| \quad \text{for all } z_1, z_2 \in \mathbb{R}^n \setminus \{0\}, \tag{2.24} \]

which is easy to deduce by the triangle inequality.

**Lemma 2.4** (Uniform Lipschitz continuity of $G_{2\delta, \epsilon}$). Let $\delta, \epsilon$ satisfy

\[ 0 < \epsilon < h\delta \quad \text{with } h \in (0, 2). \]

Then, the mapping $G_{2\delta, \epsilon}$ satisfies

\[ |G_{2\delta, \epsilon}(z_1) - G_{2\delta, \epsilon}(z_2)| \leq c_\tau |z_1 - z_2| \tag{2.25} \]

for all $z_1, z_2 \in \mathbb{R}^n$ with

\[ c_\tau(h) := 1 + \frac{8}{\sqrt{4 - h^2}}. \]

In particular, if $\delta, \epsilon$ satisfy (2.11), then (2.25) holds with $c_\tau = 1 + 64/\sqrt{255}$.

**Proof.** For preliminary, we introduce a function $\gamma_{\epsilon} \in C^\infty(\mathbb{R})$ by $\gamma_{\epsilon}(t) := \sqrt{\epsilon^2 + t^2}$ $(t \in \mathbb{R})$ for each fixed $0 < \epsilon < 1$. By direct calculations, we can easily check that $\|\gamma_{\epsilon}'\|_{L^\infty(\mathbb{R})} \leq 1$, and hence it follows that

\[ |\gamma_{\epsilon}(t_1) - \gamma_{\epsilon}(t_2)| \leq |t_1 - t_2| \quad \text{for all } t_1, t_2 \in \mathbb{R} \tag{2.26} \]

by the classical mean value theorem.

We prove (2.25) by considering three possible cases. If $|z_1|, |z_2| \leq \sqrt{(2\delta)^2 - \epsilon^2}$, then (2.25) is clear by $G_{2\delta, \epsilon}(z_1) = G_{2\delta, \epsilon}(z_2) = 0$. When $|z_1|, |z_2| > \sqrt{(2\delta)^2 - \epsilon^2}$, we use (2.24) and (2.26) to get

\[
\begin{align*}
|G_{2\delta, \epsilon}(z_1) - G_{2\delta, \epsilon}(z_2)| &= \left( \sqrt{\epsilon^2 + |z_1|^2} - 2\delta \right) \frac{z_1}{|z_1|} - \left( \sqrt{\epsilon^2 + |z_2|^2} - 2\delta \right) \frac{z_2}{|z_2|} \\
&\leq 2\delta \left| \frac{z_1}{|z_1|} - \frac{z_2}{|z_2|} \right| + \left( \gamma_{\epsilon}(|z_1|) - \gamma_{\epsilon}(|z_2|) \right) \frac{z_1}{|z_1|} \\
&\quad + \frac{4\delta}{|z_2|} + \frac{2\sqrt{\epsilon^2 + |z_2|^2}}{|z_2|} \frac{z_1}{|z_1|} - \frac{z_2}{|z_2|} \\
&\leq \left( \frac{4\delta}{|z_2|} + \frac{2\sqrt{\epsilon^2 + |z_2|^2}}{|z_2|} \right) |z_1 - z_2|.
\end{align*}
\]

Combining with $|z_2| > \sqrt{(2\delta)^2 - \epsilon^2} \geq \sqrt{4 - h^2}\delta \geq \sqrt{4 - h^2}(\epsilon/h)$, we obtain (2.25). In the remaining case, without loss of generality we may assume that $|z_1| > \sqrt{(2\delta)^2 - \epsilon^2} \geq |z_2|$. Then, $\gamma_{\epsilon}(|z_2|) \leq 2\delta$ is clear. Hence, (2.26) implies

\[
\begin{align*}
|G_{2\delta, \epsilon}(z_1) - G_{2\delta, \epsilon}(z_2)| &= \sqrt{\epsilon^2 + |z_1|^2} - 2\delta \\
&\leq \gamma_{\epsilon}(|z_1|) - \gamma_{\epsilon}(|z_2|) \\
&\leq |z_1| - |z_2| \leq |z_1 - z_2| \\
&\leq c_\tau |z_1 - z_2|,
\end{align*}
\]

which completes the proof. \(\square\)
2.2 Basic facts of positively one-homogeneous convex functions

In Section 2.2, we mention some basic properties of the positively one-homogeneous convex function $E_1$ without proofs. All of the proofs are already given in \[4, \text{Section 1.3}, \] \[36, \text{§13}, \] \[24, \text{Sections 6.1 & A.3} \]. Also, we briefly give some estimates that immediately follows from \[1.9 \] and Lemma 2.1.

We define a closed convex set $C_{E_1} \subset \mathbb{R}^n$ by

$$C_{E_1} := \{ z \in \mathbb{R}^n \mid E_1(z) \leq 1 \},$$

and a function $\tilde{E}_1 : \mathbb{R}^n \to [0, \infty)$ by

$$\tilde{E}_1(w) := \sup \{ \langle w \mid z \rangle \mid z \in C_{E_1} \} \quad \text{for } w \in \mathbb{R}^n,$$

often called the support function of the convex set $C_{E_1}$. The inequality $\tilde{E}_1 \geq 0$ is clear by the inclusion $0 \in C_{E_1}$. By \[1.9 \] and the definition of $\tilde{E}_1$, it is easy to check the Cauchy–Schwarz-type inequality:

$$\langle z \mid w \rangle \leq E_1(z) \tilde{E}_1(w) \quad \text{for all } z \in \mathbb{R}^n,$$

provided $w \in \mathbb{R}^n$ satisfies $\tilde{E}_1(w) < \infty$. Also, under the condition \[1.9 \], it is well-known that $\partial E_1$ is given by

$$\partial E_1(z) = \{ w \in \mathbb{R}^n \mid \tilde{E}_1(w) \leq 1 \text{ and } \langle z \mid w \rangle = E_1(z) \} \quad \text{for all } z \in \mathbb{R}^n.$$  \hfill (2.28)

In particular, $\partial E_1(0) = \{ w \in \mathbb{R}^n \mid \tilde{E}_1(w) \leq 1 \}$ holds. The identity

$$\langle z \mid w \rangle = E_1(z) \quad \text{for all } z \in \mathbb{R}^n, \ w \in \partial E_1(z) \quad \hfill (2.29)$$

is often called Euler’s identity.

We briefly give some estimates related to derivatives of $E_1$ outside the origin.

First, when $E_1 \in C^1(\mathbb{R}^n \setminus \{0\})$, it is easy to check

$$\nabla E_1(\lambda z) = \nabla E_1(z) \quad \text{for all } z \in \mathbb{R}^n \setminus \{0\}, \lambda > 0$$

by \[1.9 \]. In particular, we have $\nabla E_1 \in L^\infty(\mathbb{R}^n)$ and

$$\| \nabla E_1 \|_{L^\infty(\mathbb{R}^n)} = K_1 := \max \{ |\nabla E_1(z)| \mid z \in \mathbb{R}^n, |z| = 1 \} \in [0, \infty).$$

(2.31)

Moreover, from Euler’s identity \[2.29 \], it follows that

$$\partial E_1(z) \subset B_{K_1}(0) \quad \text{for all } z \in \mathbb{R}^n.$$ 

(2.32)

Also, by $E_1 \in C^1(\mathbb{R}^n \setminus \{0\})$, there holds $\partial E_1(z) = \{ \nabla E_1(z) \}$ for every $z \in \mathbb{R}^n \setminus \{0\}$.

Secondly, when $E_1$ satisfies $E_1 \in C^2(\mathbb{R}^n \setminus \{0\})$, from \[1.9 \] it follows that

$$\nabla^2 E_1(\lambda z) = \lambda^{-1} \nabla^2 E_1(z) \quad \text{for all } z_0 \in \mathbb{R}^n \setminus \{0\}, \lambda > 0.$$ 

(2.33)

Therefore, the constant

$$K_2 := \max \{ \| \nabla^2 E_1(z) \| \mid z \in \mathbb{R}^n, |z| = 1 \} \in [0, \infty),$$

satisfies

$$0 \leq \nabla^2 E_1(z) \leq \frac{K_2}{|z|} \text{id} \quad \text{for all } z \in \mathbb{R}^n \setminus \{0\}.$$ 

(2.34)

Finally, it is noted that when $E_1 \in C^2(\mathbb{R}^n \setminus \{0\})$ with $\beta_0 \in (0, 1]$, we are able to define a constant

$$K_{2,\beta_0} := \left\{ \frac{\| \nabla^2 E_1(z_1) - \nabla^2 E_1(z_2) \|}{|z_1 - z_2|^\beta_0} \mid \frac{16}{1} \leq |z_1| \leq 3 \text{ and } 0 < |z_1 - z_2| \leq \frac{1}{32} \right\} \in [0, \infty).$$

Then, by \[2.33 \], it is easy to check that

$$\| \nabla^2 E_1(z_1) - \nabla^2 E_1(z_2) \| \leq K_{2,\beta_0} \mu^{-1-\beta_0}|z_1 - z_2|^{\beta_0}.$$ 

(2.35)

for all $\mu \in (0, \infty)$ and $z_1, z_2 \in \mathbb{R}^n \setminus \{0\}$ enjoying \[1.13 \].
2.3 Approximation based on mollifiers

Section 2.3 is focused on the approximation of \( E = E_1 + E_p \), which is based on the convolution by the Friedrichs mollifier. We would like to check that the relaxed density \( E_\varepsilon = j_\varepsilon \ast E \) satisfies some quantitative estimates.

For the density \( E_p \in C^1(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\}) \) satisfying (1.10)--(1.11), we define an approximated function \( E_{p, \varepsilon} \in C^\infty(\mathbb{R}^n) \) by

\[
E_{p, \varepsilon} := j_\varepsilon \ast E_p, \quad \text{so that} \quad \nabla E_{p, \varepsilon} = j_\varepsilon \ast \nabla E_p
\]

for each \( \varepsilon \in (0, 1) \). By applying Lemmata 2.3, 4.5 (see also [19] Lemma 2, [22] Lemma 2.4), we can check that \( E_{p, \varepsilon} \) defined by (2.36) satisfies

\[
|\nabla E_{p, \varepsilon}(z_0)| \leq \Lambda_0'(\varepsilon^2 + |z_0|^2)^{(p-1)/2}, \quad (2.37)
\]

\[
\Lambda_0'(\varepsilon^2 + |z_0|^2)^{p/2-1} \leq \nabla^2 E_{p, \varepsilon}(z_0) \leq \Lambda_0'(\varepsilon^2 + |z_0|^2)^{p/2-1} \text{id} \quad (2.38)
\]

for all \( z_0 \in \mathbb{R}^n \) and \( \varepsilon \in (0, 1) \). Here the constants \( 0 < \Lambda_0' \leq \Lambda_0' < \infty \) may depend on \( n, p, \Lambda_0, \Lambda_0 \), but are independent of \( \varepsilon \in (0, 1) \). From (2.38), it is possible to get growth and monotonicity estimates for \( \nabla E_{p, \varepsilon} \) (see [19], [41] Lemma 3 for detailed computations). For growth estimates, for all \( z_1, z_2 \in \mathbb{R}^n \) and \( \varepsilon \in (0, 1) \), we have

\[
|\nabla E_{p, \varepsilon}(z_1) - \nabla E_{p, \varepsilon}(z_2)| \leq \Lambda_0' C(p)(\varepsilon^p - 2 + |z_1|^{p-2} + |z_2|^{p-2})|z_1 - z_2| \quad (2.39)
\]

provided \( 2 \leq p < \infty \), and

\[
|\nabla E_{p, \varepsilon}(z_1) - \nabla E_{p, \varepsilon}(z_2)| \leq \Lambda_0' C(p)|z_1 - z_2|^{p-1} \quad (2.40)
\]

provided \( 1 < p < 2 \). Hence, from (2.37) and (2.39)--(2.40), it follows that

\[
\nabla E_{p, \varepsilon}(z_0) \to \nabla E_p(z_0) \quad \text{as} \quad \varepsilon \to 0, \text{ locally uniformly for } z_0 \in \mathbb{R}^n. \quad (2.41)
\]

For monotonicity estimates, we can find a constant \( C = C(p) \in (0, \infty) \) such that

\[
\langle \nabla E_{p, \varepsilon}(z_1) - \nabla E_{p, \varepsilon}(z_2) \mid z_1 - z_2 \rangle \geq \Lambda C(p)|z_1 - z_2|^p \quad (2.42)
\]

holds for all \( z_1, z_2 \in \mathbb{R}^n, \varepsilon \in (0, 1) \) when \( 2 \leq p < \infty \). When \( 1 < p < 2 \), we have

\[
\langle \nabla E_{p, \varepsilon}(z_1) - \nabla E_{p, \varepsilon}(z_2) \mid z_1 - z_2 \rangle \geq \Lambda (\varepsilon^2 + |z_1|^2 + |z_2|^2)^{p/2-1}|z_1 - z_2|^2 \quad (2.43)
\]

for all \( z_1, z_2 \in \mathbb{R}^n, \varepsilon \in (0, 1) \).

For the positively one-homogeneous convex function \( E_1 \in C(\mathbb{R}) \cap C^2(\mathbb{R}^n \setminus \{0\}) \) and for each fixed \( \varepsilon \in (0, 1) \), we aim to set a smooth convex function \( \tilde{E}_{1, \varepsilon} \in C^\infty(\mathbb{R}^n) \) such that

\[
|\nabla E_{1, \varepsilon}(z_0)| \leq K_1 \quad \text{for all } z_0 \in \mathbb{R}^n, \varepsilon \in (0, 1); \quad (2.44)
\]

\[
\nabla E_{1, \varepsilon}(0) \text{ is independent of } \varepsilon \in (0, 1), \text{ written by } c_0 \in \mathbb{R}^n; \quad (2.45)
\]

\[
\tilde{E}_1(\nabla E_{1, \varepsilon}(z_0)) \leq 1 \quad \text{for all } z_0 \in \mathbb{R}^n, \varepsilon \in (0, 1); \quad \text{and} \quad (2.46)
\]

\[
O \leq \nabla^2 E_{1, \varepsilon}(z_0) \leq \frac{K_2'}{\sqrt{\varepsilon^2 + |z_0|^2}} \text{id} \quad \text{for all } z_0 \in \mathbb{R}^n, \varepsilon \in (0, 1). \quad (2.47)
\]
Here the constant $K'_2 \in (0, \infty)$ in (2.47) is independent of $\varepsilon \in (0, 1)$. This is possible by constructing a regularized function $E_{1, \varepsilon} \in C^\infty(\mathbb{R}^n)$ by

$$E_{1, \varepsilon} := j_\varepsilon \ast E_1,$$

so that $\nabla E_{1, \varepsilon} = j_\varepsilon \ast \nabla E_1$ (2.48)

for each $\varepsilon \in (0, 1)$. In Lemma 2.5 we check that this $E_{1, \varepsilon}$ surely satisfies the desired properties (2.44)–(2.47).

**Lemma 2.5.** Let $E_1 \in C(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\})$ be a positively one-homogeneous convex function. Then, for each $\varepsilon \in (0, 1)$, the relaxed function $E_{1, \varepsilon}$ defined by (2.48) satisfies (2.44)–(2.46). Moreover, if $E_1$ is in $E_1 \in C^2(\mathbb{R}^n \setminus \{0\})$, then there exists a constant $K'_2 = K'_2(K_1, K_2, n)$ such that (2.47) holds.

**Proof.** It is easy to get

$$\|\nabla (E_1 \ast j_\varepsilon)\|_{L^\infty(\mathbb{R}^n)} \leq \|\nabla E_1\|_{L^\infty(\mathbb{R}^n)} \|j_\varepsilon\|_{L^1(\mathbb{R}^n)} \leq K_1,$$

which yields (2.44).

By change of variables and (2.30), we can check that

$$(j_\varepsilon \ast \nabla E_1)(0) = \int_{\mathbb{R}^n} e^{-\varepsilon n j(-y/\varepsilon)} \nabla E_1(y) \, dy$$

$$= \int_{\mathbb{R}^n} f(-x) \nabla E_1(\varepsilon x) \, dx$$

$$= \int_{\mathbb{R}^n} f(-x) \nabla E_1(x) \, dx$$

$$= (j \ast \nabla E_1)(0) = c_0$$

for all $\varepsilon \in (0, 1)$. Clearly, this $c_0 \in \mathbb{R}^n$ is independent of $\varepsilon \in (0, 1)$, which yields (2.45).

We take arbitrary $w \in C_{E_1}$ and $z_0 \in \mathbb{R}^n$. We recall that there holds $\partial E_1(z_0 - y) = \{\nabla E_1(z_0 - y)\}$ for all $y \in \mathbb{R}^n \setminus \{z_0\}$ by $E_1 \in C^1(\mathbb{R}^n \setminus \{0\})$, and hence we obtain $\tilde{E}_1(\nabla E_1(z_0 - y)) \leq 1 < \infty$. Combining with (2.27), we compute

$$\langle \nabla (j_\varepsilon \ast E_1)(z_0) \mid w \rangle = \langle (j_\varepsilon \ast \nabla E_1)(z_0) \mid w \rangle$$

$$= \left(\int_{\mathbb{R}^n} j_\varepsilon(y) \nabla E_1(z_0 - y) \, dy \right) \langle w \rangle$$

$$= \int_{\mathbb{R}^n \setminus \{z_0\}} \langle \nabla E_1(z_0 - y) \mid w \rangle \cdot j_\varepsilon(y) \, dy$$

$$\leq E_1(w) \int_{\mathbb{R}^n \setminus \{z_0\}} j_\varepsilon(y) \, dy \leq 1.$$

Since $w \in C_{E_1}$ is arbitrary, this completes the proof of (2.46).

We let $E_1 \in C^2(\mathbb{R}^n \setminus \{0\})$, so that (2.34) holds. Let $z_0 \in \mathbb{R}^n$, and $\lambda$ be an arbitrary eigenvalue of $\nabla^2 E_{1, \varepsilon}(z_0)$. Then, $\lambda \geq 0$ is clear by convexity of $E_{1, \varepsilon} \in C^\infty(\mathbb{R}^n)$. Moreover, by (2.31)–(2.34), we are able to apply Lemma 2.2 to obtain

$$\lambda = |\lambda| \leq \|\nabla^2 E_{1, \varepsilon}(z_0)\| \leq K'_2 \left( \varepsilon^2 + |z_0|^2 \right)^{-1/2}$$

with $K'_2$ depending at most on $K_1, K_2, n$. This result yields (2.47).
It should be noted that (2.47) clearly yields
\[ \|\nabla^2 E_{1, \varepsilon}(z_0)\| \leq \frac{K'_2}{|z_0|} \quad \text{for all } \varepsilon \in (0, 1) \text{ and } z_0 \in \mathbb{R}^n \setminus \{0\}. \]

By this and (2.44), we are able to apply Lemma 2.1 with \( s = 1 \), \( \sigma_1 = \sigma_2 = 0 \). Then, there exists a constant \( C = C(K_1, K'_2) \) such that
\[ |\nabla E_{1, \varepsilon}(z_1) - \nabla E_{1, \varepsilon}(z_2)| \leq C \min\{ |z_1|^{-1}, |z_2|^{-1} \} |z_1 - z_2| \quad (2.49) \]
for all \( z_1, z_2 \in \mathbb{R}^n \setminus \{0\} \) and \( \varepsilon \in (0, 1) \). Hence from (2.44) and (2.49), it follows that
\[ \nabla E_{1, \varepsilon}(z_0) \to \nabla E_{1}(z_0) \quad \text{as } \varepsilon \to 0, \text{ locally uniformly for } z_0 \in \mathbb{R}^n \setminus \{0\}. \quad (2.50) \]

By \( E_{1, \varepsilon} \in C^1(\mathbb{R}^n) \), it should be noted that the inequality (2.49) is valid as long as \( (z_1, z_2) \neq (0, 0) \). Also, by convexity of \( E_{1, \varepsilon} \in C^1(\mathbb{R}^n) \), it is easy to get a monotonicity estimate
\[ \langle \nabla E_{1, \varepsilon}(z_1) - \nabla E_{1, \varepsilon}(z_2) \mid z_1 - z_2 \rangle \geq 0 \quad (2.51) \]
for all \( z_1, z_2 \in \mathbb{R}^n \).

Hereinafter we set constants \( 0 < \lambda \leq \Lambda < \infty, 0 < K < \infty \) by
\[ \lambda := \min\{\lambda_0, \lambda_0'\}, \quad \Lambda := \max\{\Lambda_0, \Lambda_0'\}, \quad K := \max\{K_1, K_2, K'_2, K_2, \beta_0\} \]
for notational simplicity. Finally, for an approximation parameter \( \varepsilon \in (0, 1) \), the regularized operator \( \mathcal{L}_\varepsilon \) is given by
\[ \mathcal{L}_\varepsilon u_\varepsilon := -\text{div}(\nabla E_{1, \varepsilon}(\nabla u_\varepsilon) + \nabla E_{\rho, \varepsilon}(\nabla u_\varepsilon)) = -\text{div}(\nabla E_\varepsilon(\nabla u_\varepsilon)), \quad (2.52) \]
with
\[ E_\varepsilon := E_{1, \varepsilon} + E_{\rho, \varepsilon} \in C^\infty(\mathbb{R}^n). \]

By (2.41), (2.45), and (2.50), we can easily check that
\[ \nabla E_\varepsilon(z_0) \to A_0(z_0) := \begin{cases} \nabla E_1(z_0) + \nabla E_{\rho}(z_0) & (z_0 \neq 0), \\ c_0 & (z_0 = 0) \end{cases} \quad (2.53) \]
for each fixed \( z_0 \in \mathbb{R}^n \). The limit vector field \( A_0 : \mathbb{R}^n \to \mathbb{R}^n \) is in general not continuous at the origin, but Borel measurable in \( \mathbb{R}^n \). In particular, for every \( \nu \in L^p(\Omega; \mathbb{R}^n) \), we can define a Lebesgue measurable vector field \( A_0(\nu) \), which is in \( L^p(\Omega; \mathbb{R}^n) \) by (1.10) and (2.31).

By (2.38) and (2.47), it is clear that \( E_\varepsilon \) satisfies
\[ \lambda (\varepsilon^2 + |z_0|^2)^{p/2-1} \text{id} \leq \nabla^2 E_\varepsilon(z_0) \leq \left[ \Lambda (\varepsilon^2 + |z_0|^2)^{p/2-1} + \frac{K}{\sqrt{\varepsilon^2 + |z_0|^2}} \right] \text{id} \quad (2.54) \]
for all \( z_0 \in \mathbb{R}^n, \varepsilon \in (0, 1) \). In particular, the ellipticity ratio of \( \nabla^2 E_\varepsilon(z_0) \) can be measured by \( \sqrt{\varepsilon^2 + |z_0|^2} \).

From the assumptions \( E_1, E_\rho \in C^2_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \) and Lemma 2.2, we would like to deduce an error estimate on the Hessian matrix \( \nabla^2 E_\varepsilon \). The following Lemma 2.6 will be applied in our freezing coefficient argument.

**Lemma 2.6.** Let positive numbers \( \delta, \varepsilon \) satisfy (2.71). Assume that the convex functions \( E_1 \in C^0(\mathbb{R}^n) \cap C^2_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \) and \( E_\rho \in C^1(\mathbb{R}^n) \cap C^2_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \) satisfy (1.9)–(1.12). Then, the relaxed function \( E_\varepsilon = E_{1, \varepsilon} + E_{\rho, \varepsilon} \), defined by (2.46) and (2.48) for each \( \varepsilon \in (0, 1) \), satisfies the following:
(1). Let $M \in (\delta, \infty)$ be a fixed constant. Then, for all $z_1, z_2 \in \mathbb{R}^n$ satisfying
\[
|z_1| \leq M \quad \text{and} \quad \delta \leq |z_2| \leq M,
\]
we have
\[
\langle \nabla E_\epsilon(z_1) - \nabla E_\epsilon(z_2) | z_1 - z_2 \rangle \geq C_1|z_1 - z_2|^2,
\]
and
\[
|\nabla E_\epsilon(z_1) - \nabla E_\epsilon(z_2)| \leq C_2|z_1 - z_2|.
\]
Here the constants $C_1, C_2 \in (0, \infty)$ depend at most on $p, \lambda, \Lambda, K, \delta, \epsilon,$ and $M.$

(2). For all $\mu \in (\delta, \infty),$ and $z_1, z_2 \in \mathbb{R}^n$ satisfying (2.12), we have
\[
|\nabla^2 E_\epsilon(z_1)(z_2 - z_1) - (\nabla E_\epsilon(z_2) - \nabla E_\epsilon(z_1))| |z_1 - z_2|^{1+\mu} \leq C\mu^{p-2-\beta_0} |z_1 - z_2|^{1+\mu},
\]
Here the constant $C \in (0, \infty)$ depends at most on $n, p, \beta_0, \lambda, \Lambda, K$ and $\delta.$

**Proof.**\footnote{1} It is easy to check
\[
\epsilon^l + |z_1|^l + |z_2|^l \leq 3M^l \quad \text{for every } l \in [0, \infty),
\]
and
\[
\begin{cases}
|z_2|^{p-2} \leq \delta^{p-2} & \text{for } 1 < p \leq 2, \\
|z_2|^{p-2} \geq \delta^{p-2} & \text{for } 2 < p \leq \infty,
\end{cases}
\]
by our settings of $z_1$ and $z_2.$ With these results in mind, we are able to deduce (2.55)–(2.56) by applying (2.9) or (2.21) with $H_\epsilon = E_{s, \epsilon}$ $(1 \leq s < \infty),$ or simply using (2.43) and (2.51).

\footnote{2} We apply Lemma (2.2) \footnote{3} to the functions $E_{1, \epsilon}$ and $E_{p, \epsilon}.$ It should be mentioned that the condition $0 < \delta < \mu$ clearly yields $\mu^{-1-\beta_0} = \mu^{1-\nu} \cdot \mu^{-2-\beta_0} \leq \delta^{1-p} \mu^{p-2-\beta_0},$ from which (2.57) easily follows.

\section{2.4 An alternative approximation scheme}

When the densities $E_1$ and $E_p$ have so called the Uhlenbeck structure, it is possible to give relaxed densities $E_{1, \epsilon}$ and $E_{p, \epsilon}$ without convolution by standard mollifiers. For this alternative approximation scheme, we are able to prove rather easily that the properties (2.37)–(2.38), (2.44)–(2.47) and (2.57), which are shown in Section 2.3, are valid. In Section 2.4, we would like to explain briefly how to justify these desired properties. For simplicity, we let the densities $E_1$ and $E_p$ be in $C^3$ outside the origin.

We consider the special case where the function $E_p$ is of the form
\[
E_p(z) = \frac{1}{2}g_p(|z|^2) \quad \text{and therefore} \quad \nabla E_p(z) = g_p'(z)z \quad \text{for } z \in \mathbb{R}^n \setminus \{0\}.
\]
Here $g_p : [0, \infty) \rightarrow [0, \infty)$ is a continuous function satisfying $g_p \in C^3((0, \infty)),$ and we let this $g_p$ admit positive constants $\gamma_p, \Gamma_p \in (0, \infty)$ such that
\[
\begin{cases}
|g_p'(\sigma)| \leq \Gamma_p \sigma^{p/2-1}, \\
|g''_p(\sigma)| \leq \gamma_p \sigma^{p/2-2}, \\
\gamma_p(\tau + \sigma)^{p/2-1} \leq g_p'(\tau + \sigma) + 2\sigma \min\{0, g''_p(\tau + \sigma)\}, \\
|g'''_p(\sigma)| \leq \Gamma_p \sigma^{p/2-3},
\end{cases}
\]
(2.58)
for all $\sigma \in (0, \infty)$, $\tau \in (0, 1)$. Then, for each $z = (z_1, \ldots, z_n) \in \mathbb{R}^n \setminus \{0\}$, we compute

\[
\begin{align*}
\nabla^2 E_p(z) &= g_p'(\|z\|^2)\text{id} + 2g''_p(\|z\|^2)z \otimes z, \\
\nabla^3 E_p(z) &= 4g'''_p(\|z\|^2)(z_iz_iz kz_{i,j,k} + 2g''_p(\|z\|^2)(z_j \delta_{kk} + z_k \delta_{ij} + z_{k,j} + z_{i,j,k}),
\end{align*}
\]

where $\delta_{ij}$ denotes Kronecker’s delta. From this result, we can check that $E_p = g_p(\|z\|^2)/2$ satisfies (1.10)–(1.11) with $0 < \lambda_0 = \gamma_p \leq \lambda_0 = \Lambda_0(\Gamma_p) < \infty$. Moreover, we have

\[
|\nabla^3 E_p(z)| \leq \Lambda_1|z_0|^{p-3} \quad \text{for all } z \in \mathbb{R}^n \setminus \{0\}.
\]

(2.59)

for some constant $\Lambda_1 \in (0, \infty)$ depending most on $\Gamma_p$ and $n$. From (2.59), it is easy to conclude that (1.12) holds for all $z_1, z_2 \in \mathbb{R}^n$ with (1.13) and $\beta_0 = 1$. Indeed, the triangle inequality and (1.13) imply

\[
0 < \frac{\mu}{32} \leq |z_1 + t(z_2 - z_1)| \leq 4\mu \quad \text{for all } t \in [0, 1].
\]

Hence by $E_p \in C^3(\mathbb{R}^n \setminus \{0\})$, (2.59) and the Cauchy–Schwarz inequality, we have

\[
|\nabla^3 E_p(z_2) - \nabla^3 E_p(z_1)| \leq |\nabla^3 E_p(z_2) - \nabla^3 E_p(z_1)|
\]

\[
\leq \left( \int_0^1 |\nabla^3 E_p(z_1 + t(z_2 - z_1))|^2 \, dt \right)^{1/2} |z_1 - z_2|
\]

\[
\leq \left( 32^{3-p} + 4^{p-3} \right) \Lambda_1 |z_1 - z_2|.
\]

We mention that this symmetric setting generalizes our model case of the $p$-Laplace operator. In fact, it is easy to check that a special function $g_p(\sigma) := 2\sigma^{p/2}/p$ satisfies (2.58) with

\[
\gamma_p := \min \{1, p-1\}, \quad \Gamma_p := \max \left\{1, \frac{|p-2|}{2}, \frac{|(p-2)(p-4)|}{4} \right\}
\]

In this case, the density $E_p$ and the operator $\mathcal{L}_p$ become $E_p(z) = \|z\|^p/p$ and $\mathcal{L}_p = -\Delta_p$ respectively. In particular, the density $E_p$ given by (1.14) surely satisfies (1.12) with $\beta_0 = 1$. In this setting, a relaxed density $E_{p, e}$ is alternatively given by

\[
E_{p, e}(z) := g_p(e^2 + \|z\|^2), \quad \text{so that} \quad \nabla E_{p, e}(z) = g_p'(e^2 + \|z\|^2)z
\]

for each $e \in (0, 1)$. By direct computations, we can check that this $E_{p, e}$ satisfies (2.59)–(2.60) with $\lambda_0' = \lambda_0, \lambda_0' = \Lambda_0$. Moreover, there holds

\[
|\nabla^3 E_{p, e}(z)| \leq \Lambda_0(e^2 + \|z\|^2)^{(p-3)/2} \quad \text{for all } z \in \mathbb{R}^n.
\]

(2.60)

Similarly, we let $E_1$ be of the form

\[
E_1(z) = \frac{1}{2} g_1(\|z\|^2) \quad \text{and therefore} \quad \nabla E_1(z) = g_1'(\|z\|^2)z \quad \text{for } z \in \mathbb{R}^n \setminus \{0\},
\]

where $g_1 : [0, \infty) \to [0, \infty)$ is a non-decreasing continuous function with $g_1 \in C^2((0, \infty))$. Since $E_1$ is positively one-homogeneous, this forces us to determine $g_1$ explicitly. Indeed, (2.30) and the Uhlenbeck structure imply that $|\nabla E_1(z)| = g_1'(\|z\|^2)|z|$ is constant for $z \in \mathbb{R}^n \setminus \{0\}$. For this reason, $g_1$ must satisfy $\sqrt{\sigma} g_1'(\sigma) = b$ for all $\sigma \in (0, \infty)$, where $b \in [0, \infty)$ is constant. Combining with $g_1(0) = E_1(0) = 0$, we
have $g_1(\sigma) = 2b\sigma^{1/2}$ ($\sigma \geq 0$), which yields $E_1(z) = b|z|$ and $\mathcal{L}_1 = -b\Delta_1$. In this special case, it is possible to give a relaxed density $E_{1,\varepsilon}$ by

$$E_{1,\varepsilon}(z) := b\sqrt{\varepsilon^2 + |z|^2},$$

so that $\nabla E_{1,\varepsilon}(z) = \frac{bz}{\sqrt{\varepsilon^2 + |z|^2}}$

for each $\varepsilon \in (0, 1)$, similarly to $E_{p,\varepsilon}$. By direct computations, it is easy to check that this $E_{1,\varepsilon}$ satisfies (2.44)–(2.47) with $c_0 = 0$, $K_1 = K_2 = K'_2 = b$. Moreover, there holds

$$|\nabla^3 E_{1,\varepsilon}(z)| \leq K_3\left(\varepsilon^2 + |z|^2\right)^{-1} \text{ for all } z \in \mathbb{R}^n.$$  (2.61)

Here the constant $K_3 \in (0, \infty)$ depends at most on $b$ and $n$.

It will be worth mentioning that under these settings, the inequality (2.57) in Lemma 2.6 can be deduced rather easily with $\beta_0 = 1$. This is possible by applying the following Lemma 2.7 with $H_\varepsilon = E_{s,\varepsilon}$ ($s \in \{1, p\}$). There it should be noted that these relaxed densities satisfy the assumption (2.64) by (2.60)–(2.61).

**Lemma 2.7.** Let $s \in (1, \infty)$, and positive numbers $\delta, \varepsilon$ satisfy (2.71). Assume that a real-valued function $H_\varepsilon \in C^3(\mathbb{R}^n)$ admits a constant $L \in (0, \infty)$, independent of $\varepsilon$, such that

$$|\nabla H_\varepsilon(z)| \leq L\left(\varepsilon^2 + |z|^2\right)^{(s-1)/2},$$  (2.62)

$$||\nabla^2 H_\varepsilon(z)|| \leq L\left(\varepsilon^2 + |z|^2\right)^{s/2 - 1},$$  (2.63)

$$|\nabla^3 H_\varepsilon(z)| \leq L\left(\varepsilon^2 + |z|^2\right)^{(s-3)/2},$$  (2.64)

for all $z \in \mathbb{R}^n$. Then, for all $z_1, z_2 \in \mathbb{R}^n$ satisfying (2.72), we have (2.73) with $\beta_0 = 1$.

**Proof.** We recall that there holds

$$(s+t)^\gamma \leq \begin{cases} \gamma t^{\gamma} & (\gamma \leq 0), \\ (1 + 2^{\gamma - 1})(s^{\gamma} + t^{\gamma}) & (0 < \gamma < \infty), \end{cases} \text{ for all } s \in [0, \infty), t \in (0, \infty).$$

Hence from (2.62)–(2.64), it follows that

$$\begin{align*}
|\nabla H_\varepsilon(z)| & \leq (1 + 2^{(s-3)/2})L \cdot (\sigma_1 + |z|^{s-1}), \\
||\nabla^2 H_\varepsilon(z)|| & \leq (1 + 2^{(s-4)/2})L \cdot (\sigma_2 + |z|^{s-2}), \text{ for all } z \in \mathbb{R}^n \setminus \{0\}, \\
|\nabla^3 H_\varepsilon(z)| & \leq (1 + 2^{(s-5)/2})L \cdot (\sigma_3 + |z|^{s-3}),
\end{align*}$$

where the constants $\sigma_1, \sigma_2, \sigma_3 \geq 0$ are defined by

$$\sigma_k := \begin{cases} 0 & (1 \leq s \leq k), \\
e^{s-k} & (k < s < \infty), \text{ for each } k \in \{1, 2, 3\}.
\end{cases}$$

By Lemma 2.1 and $|z| \geq \mu/4 > 0$, we are able to obtain

$$|\nabla^2 H_\varepsilon(z_1)(z_2 - z_1) - (\nabla H_\varepsilon(z_1) - \nabla H_\varepsilon(z_2))|$$
increasing sequence such that (2.12). By Lemma 2.8.

In this section, we prove Lemma 2.8 below, which plays important roles in a mathematical justification |

Here we have used $|z_1|, |z_2| \leq 2\mu$ and $\sigma_k \leq \mu^{s-k}$ ($k < s < \infty$), which immediately follow from (2.11) and (2.12). By $H_\varepsilon \in C^3(\mathbb{R}^n)$, the estimate above is valid even for $z_2 = 0$, and this completes the proof. \hfill $\square$

Finally, when considering the special case where the operators are of the form

$$L_1 u = -\text{div} \left( \frac{\nabla u}{|\nabla u|} \right), \quad L_{\mu} u = -\text{div} \left( g_\mu'(|\nabla u|^2) \nabla u \right),$$

where $g_\mu$ satisfies (2.58), we may introduce relaxed operators alternatively by

$$L_{\mu} u := \text{div} \left( \frac{\nabla u}{\sqrt{\varepsilon^2 + |\nabla u|^2}} \right), \quad L_{\mu} u := \text{div} \left( g_\mu'((\varepsilon^2 + |\nabla u|^2) \nabla u \right).$$

We note that all of the proofs after Section 2.4 work for this different approximation scheme. Also, it should be worth mentioning that, under this setting, everywhere $C^1$-regularity of weak solutions for elliptic system problems have been established in another recent paper [42] by the author. There our method works under the assumption $g_\mu \in C^{2,\beta_0}_\text{loc}((0, \infty))$ with $\beta_0 \in (0, 1]$.

### 2.5 A fundamental result on convergences of vector fields

In this section, we prove Lemma 2.8 below, which plays important roles in a mathematical justification of convergence for approximated solutions.

**Lemma 2.8.** Assume that $E_1 \in C^0(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\})$ is a positively one-homogeneous convex function, and $E_\mu \in C^1(\mathbb{R}^n)$ satisfies (1.10). Let $U \subset \mathbb{R}^n$ be a measurable set, and let $\{\varepsilon_k\}_{k=1}^\infty \subset (0, 1)$ be a non-increasing sequence such that $\varepsilon_k \to 0$ as $k \to \infty$. Assume that relaxed functions $E_1, \varepsilon, E_{\mu, \varepsilon} \in C^1(\mathbb{R}^n)$ satisfy (2.37), (2.41), (2.46), and (2.50). Then, we have the following:

1. For each fixed $v \in L^p(U; \mathbb{R}^n)$, we have

   $$\nabla E_{\varepsilon_k}(v) \to A_0(v) \quad \text{in} \quad L^p(U; \mathbb{R}^n),$$

   where $A_0$ is defined by (2.53).

2. Assume that a vector field $v_0 \in L^p(U; \mathbb{R}^n)$ and a sequence of vector fields $\{v_{\varepsilon_k}\}_{k=1}^\infty \subset L^p(U; \mathbb{R}^n)$ satisfy $v_{\varepsilon_k} \to v_0$ in $L^p(U; \mathbb{R}^n)$ as $k \to \infty$. Then, there exists a subsequence $\{\varepsilon_{k_j}\}_{j=1}^\infty$ such that

   $$\nabla E_{\mu, \varepsilon_{k_j}}(v_{\varepsilon_{k_j}}) \to \nabla E_{\mu}(v_0) \quad \text{in} \quad L^p(U; \mathbb{R}^n).$$

3. Assume that a sequence of measurable vector fields $\{v_{\varepsilon_k} : U \to \mathbb{R}^n\}_{k=1}^\infty$ satisfies

   $$v_{\varepsilon_k}(x) \to v_0(x) \quad \text{for a.e.} \quad x \in U$$

for some measurable vector field $v_0 : U \to \mathbb{R}^n$. Then, we have

   $$Z_k(x) := \nabla E_{1, \varepsilon_k}(v_{\varepsilon_k}(x)) \to \nabla E_1(v_0(x)) =: Z_0 \quad \text{in} \quad L^\infty(D; \mathbb{R}^n),$$
where \( D := \{ x \in U \mid v_0(x) \neq 0 \} \). Moreover, there exists a vector fields \( Z \in L^\infty(U, \mathbb{R}^n) \) and a subsequence \( \{ \varepsilon_{k_j} \}_{j=1}^\infty \) such that
\[
Z_{k_j} \rightharpoonup Z \quad \text{in} \quad L^\infty(U; \mathbb{R}^n),
\]
(2.69)
\[
Z(x) \in \partial E_1(v_0(x)) \quad \text{for a.e.} \quad x \in U.
\]
(2.70)
In particular, if a sequence of vector fields \( \{ v_{\varepsilon_k} \}_{k=1}^\infty \subset L^s(U; \mathbb{R}^n) \) \((1 \leq s < \infty)\) converges to \( v_0 \in L^s(U; \mathbb{R}^n) \) with respect to the strong topology in \( L^s(U; \mathbb{R}^n) \), then there exist a subsequence \( \{ \varepsilon_{k_j} \}_{j=1}^\infty \) and a vector field \( Z \in L^\infty(U; \mathbb{R}^n) \) satisfying \(2.69) \) and \(2.70)\).

**Proof.** \([1]\) We fix \( v \in L^p(U; \mathbb{R}^n) \). We note that for all \( \varepsilon \in (0, 1) \),
\[
|\nabla E_\varepsilon(z) - A_0(z)| \leq |\nabla E_\varepsilon(z)| + |A_0(z)| \leq C(1 + |z|^{p-1}) \quad \text{for all} \quad z \in \mathbb{R}^n.
\]
Here the constant \( C = C(p, \Lambda, K) \) is independent of \( \varepsilon \). This clearly yields
\[
|\nabla E_{\varepsilon_k}((v(x)) - A_0(v(x))|' \leq C(p, \Lambda, K)(1 + |v(x)|^p) =: g(x) \quad \text{for a.e.} \quad x \in U,
\]
where \( g \in L^1(U) \) is independent of \( k \in \mathbb{N} \). From \(2.53) \), we have already known that \( \nabla E_{\varepsilon_k}((v(x)) \to A_0(v(x)) \) for a.e. \( x \in U \). Therefore by Lebesgue’s dominated convergence theorem, we obtain \(2.65) \).

\([2]\) By \([12, \text{Theorem 4.9}] \), we may take a subsequence \( \{ v_{\varepsilon_k} \}_{k=1}^\infty \) such that
\[
v_{\varepsilon_{k_j}}(x) \to v_0(x) \quad \text{for a.e.} \quad x \in U,
\]
(2.71)
\[
|v_{\varepsilon_{k_j}}(x)| \leq h(x) \quad \text{for a.e.} \quad x \in U,
\]
(2.72)
where \( h \in L^p(U) \) is independent of \( j \in \mathbb{N} \). From \(2.41) \) and \(2.71 \), we can check that \( \nabla E_{p, \varepsilon_{k_j}}(v_{\varepsilon_{k_j}}(x)) \to \nabla E_p(v_0(x)) \) for a.e. \( x \in U \). Also, as in the proof of \([1]) \) we can check that for a.e. \( x \in U \) there holds
\[
|\nabla E_{p, \varepsilon_{k_j}}(v_{\varepsilon_{k_j}}(x)) - \nabla E_p(v_0(x))|' \leq C(1 + |v_0(x)|^p + h(x)^p) =: g(x).
\]
Here the constant \( C = C(p, \Lambda, K) > 0 \) is independent of \( j \in \mathbb{N} \), and so is \( g \in L^1(U) \). We can conclude \(2.66) \) from Lebesgue’s dominated convergence theorem.

\([3]\) By \(2.53) \) and \(2.67) \), we can check that \( Z_k(x) \to Z_0(x) \) for a.e. \( x \in D \). By \(2.44) \) the vector field \( Z_k \) satisfies \( \|Z_k\|_{L^\infty(U)} \leq K \) for all \( k \in \mathbb{N} \). Hence, the weak convergence \(2.68) \),
\[
i.e., \quad \int_D \langle Z_k(x) | \phi(x) \rangle \, dx \to \int_D \langle Z_0(x) | \phi(x) \rangle \, dx \quad \text{for all} \quad \phi \in L^1(D; \mathbb{R}^n)
\]
easily follows from Lebesgue’s dominated convergence theorem.

We should recall that the sequence \( \{ Z_k \}_k \subset L^\infty(U; \mathbb{R}^n) \) is bounded, and that the function space \( L^1(U; \mathbb{R}^n) \), which is the predual of \( L^\infty(U; \mathbb{R}^n) \), is separable. Hence by \([12, \text{Corollary 3.30}] \), we may take a vector field \( Z \in L^\infty(U; \mathbb{R}^n) \) and a subsequence \( \{ \varepsilon_{k_j} \}_j \) such that \(2.69) \) holds. We are left to show that this weak* limit \( Z \) satisfies \(2.70) \). Since it is clear that \( Z_{k_j} \rightharpoonup Z \) in \( L^\infty(D, \mathbb{R}^n) \), we have already known that \( Z(x) = Z_0(x) \in \partial E_1(v_0(x)) \) for a.e. \( x \in D \). From \(2.69) \), it follows that
\[
\text{ess sup}_{x \in U} \hat{E}_1(Z(x)) \leq \lim \inf_{j \to \infty} \text{ess sup}_{x \in U} \hat{E}_1(Z_{k_j}(x)).
\]
(2.73)
For the proof of \(2.73) \), see \([24, \text{Lemma 4}] \). Recall \(2.46) \), and we have
\[
Z(x) \in \{ w \in \mathbb{R}^n \mid \hat{E}_1(w) \leq 1 \} = \partial E_1(0)
\]
(2.74)
for a.e. \( x \in U \). This result implies that \(2.70) \) holds for a.e. \( x \in U \setminus D \). The last statement is a consequence from \([12, \text{Theorem 4.9}] \).
Remark 2.9. In the proof of Lemma 2.8 (3), it is also possible to check that \( Z \in L^\infty(\Omega; \mathbb{R}^n) \) satisfies the Euler’s identity;
\[
\langle v_0(x) \mid Z(x) \rangle = E_1(v_0(x))
\]  
(2.75)
for a.e. \( x \in U \). In fact, for a.e. \( x \in U \setminus D \), (2.76) is clear by \( v_0(x) = 0 \) and \( E_1(0) = 0 \). In the remaining case \( x \in D \), we have for all \( k \in \mathbb{N} \)
\[
E_{1, \varepsilon_k}(v_{\varepsilon_k}(x)) = E_{1, \varepsilon_k}(0) + \int_0^1 \langle \nabla E_{1, \varepsilon_k}(tv_{\varepsilon_k}(x)) \mid v_{\varepsilon_k}(x) \rangle \, dt,
\]
applying the fundamental theorem of calculus for \( E_{1, \varepsilon_k} \in C^\infty(\mathbb{R}^n) \). Since it is clear that \( E_{1, \varepsilon_k} \) converges to \( E_1 \) locally uniformly in \( \mathbb{R}^n \), it follows that \( E_{1, \varepsilon_k}(v_{\varepsilon_k}(x)) \to E_1(v_0(x)) \) for a.e. \( x \in U \). Also, for a.e. \( 0 < t < 1 \) and a.e. \( x \in D \), we have
\[
\left\{ \begin{array}{ll}
\langle \nabla E_{1, \varepsilon_k}(tv_{\varepsilon_k}(x)) \mid v_{\varepsilon_k}(x) \rangle & \to \langle \nabla E_1(tv_0(x)) \mid v_0(x) \rangle \quad \text{as } k \to \infty, \\
|\langle \nabla E_{1, \varepsilon_k}(tv_{\varepsilon_k}(x)) \mid v_{\varepsilon_k}(x) \rangle | & \leq K \sup_k |v_{\varepsilon_k}(x)| =: C_x \quad \text{for all } k \in \mathbb{N}.
\end{array} \right.
\]
Here we have used (2.30), (2.50), (2.67). Since \( C_x \) is a finite constant for a.e. \( x \in D \), which is independent of \( k \in \mathbb{N} \), we can apply Lebesgue’s dominated convergence theorem. Letting \( k \to \infty \) and noting that \( \langle \nabla E_1(tv_0(x)) \mid v_0(x) \rangle = \langle Z_0(x) \mid v_0(x) \rangle = \langle Z(x) \mid v_0(x) \rangle \) for a.e. \( x \in D \) and a.e. \( t \in (0, 1) \), we are able to obtain
\[
E_1(v_0(x)) = E_1(0) + \int_0^1 \langle Z(x) \mid v_0(x) \rangle \, dt = \langle Z(x) \mid v_0(x) \rangle \quad \text{for a.e. } x \in D.
\]
From (2.28) and (2.74)–(2.75), it follows that \( Z \) satisfies (2.70).

2.6 Convergence results on approximated variational problems

Only in Section 2.6 the exponent \( q \) is assumed to satisfy (1.15). Similarly to Definition 1.2, we define a weak solution of the Dirichlet boundary value problem
\[
\begin{aligned}
\mathcal{L}u &= f \quad \text{in } \Omega, \\
u &= u_0 \quad \text{on } \partial \Omega.
\end{aligned}
\]
(2.76)

Definition 2.10. Let \( p \in (1, \infty) \) and \( q \in [1, \infty] \) satisfy (1.15). Let functions \( f \in L^q(\Omega) \) \( u_0 \in W^{1,p}(\Omega) \) be given. A function \( u \in u_0W^{1,p}(\Omega) \) is called the weak solution of the Dirichlet boundary value problem (2.76) when there exists a vector field \( Z \in L^\infty(\Omega; \mathbb{R}^n) \) such that the pair \((u, Z)\) satisfies (1.16)–(1.17).

The main aim of Section 2.6 is to prove that there uniquely exists a solution of the Dirichlet problem (2.76), and this solution can be obtained as a limit function of the approximation problem
\[
\begin{aligned}
\mathcal{L}^\varepsilon u_\varepsilon &= f_\varepsilon \quad \text{in } \Omega, \\
u_\varepsilon &= u_0 \quad \text{on } \partial \Omega.
\end{aligned}
\]
(2.77)
Here for each \( \varepsilon \in (0, 1) \), the divergence operator \( \mathcal{L}^\varepsilon \) is given by (2.52), and the function \( f_\varepsilon \in L^q(\Omega) \) satisfies
\[
f_\varepsilon \rightharpoonup f \quad \text{in } \sigma(L^q(\Omega), L^q(\Omega)) \quad \text{as } \varepsilon \to 0.
\]
(2.78)
In other words, we assume that the approximated external force term \( f_\varepsilon \in L^q(\Omega) \) converges to \( f \) with respect to the weak and the weak” topology of \( L^q(\Omega) \), respectively when \( q < \infty \) and \( q = \infty \).

We set
\[
\mathcal{F}_0(v) := \int_\Omega E_1(\nabla v) \, dx + \int_\Omega E_p(\nabla v) \, dx - \int_\Omega f v \, dx
\]
for each \( v \in W^{1,p}(\Omega) \). We also set
\[
\mathcal{F}_\varepsilon(v) := \int_\Omega E_{1,\varepsilon}(\nabla v) \, dx + \int_\Omega E_{p,\varepsilon}(\nabla v) \, dx - \int_\Omega f_{\varepsilon} v \, dx
\]
for each \( v \in W^{1,p}(\Omega) \), \( \varepsilon \in (0, 1) \). As mentioned in Section [1], the equation \([1, 2]\) is connected with a minimizing problem of the functional \( \mathcal{F}_0 \) given by \((2.79)\). However, it should be noted that this functional is, in general, neither Gâteaux differentiable nor Fréchet differentiable. This is substantially due to non-smoothness of the density function \( E_1 \) at the origin. Therefore, it is natural to regard the term \( \nabla E_1(\nabla u) \) as a vector field that satisfies \((1.17)\), as in Definition \(2.10\). There we face to check whether it is possible to construct a vector field \( Z \) satisfying \((1.17)\). We can overcome this problem by introducing regularized functionals \( \mathcal{F}_\varepsilon(0 < \varepsilon < 1) \) given by \((2.80)\), and by applying Lemma \(2.8\). Our strategy works even for a variational inequality problem (Proposition \(2.11\)).

**Proposition 2.11.** Let \( p \in (1, \infty) \) and \( q \in [1, \infty) \) satisfy \((7.15)\). Assume that \( f \in L^q(\Omega) \) and \( \{f_{\varepsilon}\}_{0 < \varepsilon < 1} \subset L^q(\Omega) \) satisfy \((2.78)\). We define functionals in \( \mathcal{F}_\varepsilon(0 < \varepsilon < 1) \) by \((2.79)-(2.80)\), where the density \( E_{\varepsilon} \) is given by \( E_{\varepsilon} = f_{\varepsilon} + \tilde{E} \) with \( E = E_1 + E_p \) satisfying \((1.9)-(1.11)\). Assume that \( \mathcal{K} \subset W^{1,p}(\Omega) \) is a non-empty closed convex set, and there exists a positive constant \( C_{\mathcal{K}} \in (0, \infty) \) such that
\[
\|v_1 - v_2\|_{L^p(\Omega)} \leq C_{\mathcal{K}}\|\nabla v_1 - \nabla v_2\|_{L^p(\Omega)} \quad \text{for all} v_1, v_2 \in \mathcal{K}.
\]
Then, for a function \( u \in \mathcal{K} \), these following are equivalent:

(1). The function \( u \) satisfies
\[
u = \arg \min \{\mathcal{F}_0(v) \mid v \in \mathcal{K}\}.
\]
In other words, \( u \) is the unique minimizer of the functional \( \mathcal{F}_0 : \mathcal{K} \to \mathbb{R} \).

(2). There exists \( Z \in L^\infty(\Omega; \mathbb{R}^n) \) such that the pair \((u, Z)\) satisfies \((1.17)\), and
\[
\int_\Omega \langle Z, \nabla(\psi - u) \rangle \, dx + \int_\Omega \langle \nabla E_p(\nabla u), \nabla(\psi - u) \rangle \, dx \geq \int_\Omega f(\psi - u) \, dx
\]
for all \( \psi \in \mathcal{K} \).

Moreover, there exists a unique function \( u \in \mathcal{K} \) satisfying these equivalent properties \((1) \sim (2)\) and there holds \( u_{\varepsilon} \to u \) in \( W^{1,p}(\Omega) \) up to a subsequence. Here \( u_{\varepsilon} \) is a function defined by
\[
u_{\varepsilon} := \arg \min \{\mathcal{F}_\varepsilon(v) \mid v \in \mathcal{K}\} \in \mathcal{K}
\]
for each \( \varepsilon \in (0, 1) \).

**Proof.** We first mention that the right hand sides of \((2.82)\) and \((2.84)\) are well-defined. To prove this, we will check coerciveness of \( \mathcal{F}_\varepsilon \). For each fixed \( \varepsilon \in (0, 1) \), \( E_{p,\varepsilon} \) satisfies
\[
E_{p,\varepsilon}(z) \geq E_{p,\varepsilon}(0) + \langle \nabla E_{p,\varepsilon}(0), z \rangle + \lambda C(p)(|z|^p - \varepsilon^p) \quad \text{for all} \ z \in \mathbb{R}^n.
\]
In fact, by \((2.42) - (2.43)\) we can compute

\[
E_{p, e}(z) - E_{p, e}(0) - \langle \nabla E_{p, e}(0) | z \rangle = \int_0^1 \langle \nabla E_{p, e}(tz) - \nabla E_{p, e}(0) | tz \rangle \frac{dt}{t} \geq AC(p) \left| z \right|^p \int_0^1 t^p \, dt - \varepsilon^p.
\]

It is easy to get the last inequality in the case \(2 \leq p < \infty\). When \(1 < p < 2\), we should note that

\[
\left( \varepsilon^2 + |tz|^2 \right)^{p/2} - \left( \varepsilon^2 + |tz|^2 \right)^{p/2 - 1} \geq \left( \varepsilon^2 + |tz|^2 \right)^{p/2 - 1}
\]

for all \(t \in [0, 1]\), which yields the desired inequality. Hence, for each \(v \in W^{1, p}(\Omega)\), \(\mathcal{F}_e\) satisfies

\[
\mathcal{F}_e(v) \geq \int_\Omega \langle \nabla E_{p, e}(0) | \nabla v \rangle \, dx - \int_\Omega f_e \, v \, dx \geq \int_\Omega \langle \nabla E_{p, e}(0) | \nabla v \rangle \, dx + \lambda C(p) \left( \| \nabla v \|_{L^p(\Omega)}^p - |\Omega| \varepsilon^p \right) - C(n, p, q, \Omega) \| f_e \|_{L^q(\Omega)} \| v \|_{W^{1, p}(\Omega)}
\]

by \(E_{p, e}(0) \geq 0\), Hölder’s inequality and the continuous embedding \(W^{1, p}(\Omega) \hookrightarrow L^p(\Omega)\). We may take and fix a function \(v_0 \in \mathcal{K}\). Then, by \((2.81)\), we have

\[
\| v \|_{W^{1, p}(\Omega)} \leq \| v - v_0 \|_{W^{1, p}(\Omega)} + \| v_0 \|_{W^{1, p}(\Omega)} \leq C \cdot \left( \| \nabla v \|_{L^p(\Omega)} + \| v_0 \|_{W^{1, p}(\Omega)} \right)
\]

with a new constant \(C\) depending on \(C_\mathcal{K}\). With this in mind, by applying Young’s inequality, we are able to find constants \(\gamma = \gamma(\lambda, p) \in (0, 1)\) and \(\Gamma = \Gamma(n, p, q, \lambda, \Omega, C_\mathcal{K}) \in (1, \infty)\) such that

\[
\mathcal{F}_e(v) \geq \gamma \| \nabla v \|_{L^p(\Omega)}^p - \Gamma \cdot \left( 1 + \| \nabla E_{p, e}(0) \|_{L^p(\Omega)}^p + \| v_0 \|_{W^{1, p}(\Omega)} \| f_e \|_{L^q(\Omega)} + \| f_e \|_{L^q(\Omega)}^p \right)
\]

(2.85)

for all \(v \in \mathcal{K}\). By the estimate \((2.85)\), which implies that \(\mathcal{F}_e\) is a coercive functional in \(\mathcal{K}\), it is easy to construct a minimizer of \(\mathcal{F}_e\) in \(\mathcal{K}\) by direct methods. Here we note that the convex functional \(\mathcal{F}_e\) is continuous with respect the strong topology of \(W^{1, p}(\Omega)\), and hence sequentially lower-semicontinuous with respect to the weak topology of \(W^{1, p}(\Omega)\). We note that the density \(E_{p, e}\) is strictly convex by \((2.54)\). From this fact and \((2.81)\), we can easily conclude uniqueness of a minimizer of the functional \(\mathcal{F}_e\). Therefore, the right hand side of \((2.84)\) is well-defined. This result similarly holds for \(\mathcal{F}_0\). In fact, \((2.85)\) is valid even when \(\varepsilon = 0\), and strict convexity of \(E\) also follows from \((1.1)\). For detailed arguments and computations omitted in the proof, see e.g., [21 Chapter 8], [25 Chapter 4], [41] §3.

Let \((u, Z) \in \mathcal{K} \times L^{\infty}(\Omega; \mathbb{R}^n)\) satisfy \((1.17)\) and \((2.83)\). Then, for each \(\psi \in \mathcal{K}\), it is easy to get

\[
\langle Z | \nabla (\psi - u) \rangle \leq E_1(\nabla \psi) - E_1(\nabla u) \quad \text{a.e. in } \Omega,
\]

which is so called a subgradient inequality. Similarly, for each \(\psi \in \mathcal{K}\), we have another subgradient inequality

\[
\langle \nabla E_{p, e}(\nabla u) | \nabla (\psi - u) \rangle \leq E_{p, e}(\nabla \psi) - E_{p, e}(\nabla u) \quad \text{a.e. in } \Omega
\]

by \(\partial E_{p, e}(z) = \{ \nabla E_{p, e}(z) \}\) for all \(z \in \mathbb{R}^n\). From these, we can easily check \(0 \leq \mathcal{F}(\psi) - \mathcal{F}(u) \) for all \(\psi \in \mathcal{K}\), which implies that \(u\) satisfies \((2.82)\).

We have already proved the implication \((2.7) \Rightarrow (1.1)\) and unique existence of the minimizer \((2.82)\). Hence, in order to complete the proof, it suffices to show the following two claims. The first is that the function \(u_\varepsilon \in \mathcal{K}(0 < \varepsilon < 1)\) defined by \((2.84)\) converges to a certain function \(u \in \mathcal{K}\) strongly in \(W^{1, p}(\Omega)\).
up to a sequence. The second is that this limit function $u$ satisfies (2) We would like to prove these assertions by applying Lemma 2.8.

For each $\varepsilon \in (0, 1)$, we set a function $u_{\varepsilon} \in \mathcal{K}$ by (2.84). Then, for each $\psi \in \mathcal{K}$, we have

$$0 \leq \frac{\mathcal{F}_\varepsilon(u_{\varepsilon} + t(\psi - u_{\varepsilon})) - \mathcal{F}_\varepsilon(u_{\varepsilon})}{t} \quad \text{for all } t \in (0, 1).$$

Letting $t \to 0$ and noting $E_\varepsilon \in C^\infty(\mathbb{R}^n)$, we are able to obtain

$$\int_\Omega \langle \nabla E_{1, \varepsilon}(\nabla u_{\varepsilon}) \mid \nabla (\psi - u_{\varepsilon}) \rangle \, dx + \int_\Omega \langle \nabla E_{p, \varepsilon}(\nabla u_{\varepsilon}) \mid \nabla (\psi - u_{\varepsilon}) \rangle \, dx$$

$$\geq \int_\Omega f_\varepsilon(\psi - u_{\varepsilon}) \, dx \quad \text{(2.86)}$$

for all $\psi \in \mathcal{K}$.

We choose the number $\varepsilon_0 \in (0, 1)$ satisfying

$$\sup_{0 < \varepsilon \leq \varepsilon_0} |\nabla E_{p, \varepsilon}(0)| \leq 1 \quad \text{and} \quad \sup_{0 < \varepsilon \leq \varepsilon_0} \|f_\varepsilon\|_{L^2(\Omega)} \leq F_*$$

for some constant $F_* \in (0, \infty)$. This is possible by (1.10), (2.41) and (2.78). We will show that the net $(u_{\varepsilon})_{0 < \varepsilon \leq \varepsilon_0} \subset H^{1, p}(\Omega)$ is bounded. To show this, by (2.81), it suffices to find a constant $C \in (0, \infty)$, independent of $\varepsilon \in (0, \varepsilon_0]$, such that there holds

$$\sup_{0 < \varepsilon \leq \varepsilon_0} \|\nabla v_\varepsilon\|_{L^p(\Omega)} \leq C < \infty. \quad \text{(2.87)}$$

Without loss of generality, we may let $E_{p}(0) = 0$, then we get

$$|E_p(z)| \leq C(p, \Lambda)|z|^p \quad \text{for all } z \in \mathbb{R}^n$$

by elementary computations based on $E_{p} \in C^1(\mathbb{R}^n)$ and (1.10) (see e.g., [41, Lemma 3]). On the other hand, by Euler’s identity (2.29), it is clear that $|E_1(z)| \leq K|z|$ for all $z \in \mathbb{R}^n$. We apply (2.16) with $\sigma = 1$ and $\sigma = p$, so that we have

$$|E_{\varepsilon}(z)| \leq C(n, p, \Lambda, K)(1 + |z|^p) \quad \text{for all } z \in \mathbb{R}^n$$

by Young’s inequality. Since $v_\varepsilon$ satisfies (2.84), we have

$$\mathcal{F}_\varepsilon(v_\varepsilon) \leq \mathcal{F}_\varepsilon(v_0) \leq C(n, p, q, \Lambda, K, \Omega) \left(1 + \|\nabla v_0\|_{L^p(\Omega)}^p + \|f_\varepsilon\|_{L^q(\Omega)} \|v_0\|_{W^{1, p}(\Omega)}\right)$$

by Hölder’s inequality and the continuous embedding $W^{1, p}(\Omega) \hookrightarrow L^q(\Omega)$. By applying (2.85) with $v = v_\varepsilon$, we can find a finite constant $C \in (0, \infty)$, depending at most on $n, p, q, \Lambda, \Omega, C_\mathcal{K}$, and $F_*$, such that (2.87) holds.

Hence by a weak compactness argument, we may take a sequence $(\varepsilon_N)_{N=1}^\infty \subset (0, \varepsilon_0)$ and a function $u \in \mathcal{K}$ such that $\varepsilon_N \to 0$ and $u_{\varepsilon_N} \to u$ in $W^{1, p}(\Omega)$. By the compact embedding $W^{1, p}(\Omega) \hookrightarrow L^q(\Omega)$, we may also assume that $u_{\varepsilon_N} \to u$ in $L^q(\Omega)$ by taking a subsequence. For this relabeled sequence, which we write again as $(u_{\varepsilon_N})_{N=1}^\infty$, we will show that $\nabla u_{\varepsilon_N} \to \nabla u$ in $L^p(\Omega; \mathbb{R}^n)$. Here we set and compute an integral $I(\varepsilon)$ as following:

$$I(\varepsilon) := \int_\Omega \langle \nabla E_{\varepsilon}(\nabla u) - \nabla E_{\varepsilon}(\nabla u_{\varepsilon}) \mid \nabla u - \nabla u_{\varepsilon} \rangle \, dx$$
We note that \(I(\varepsilon) \geq 0\) follows from (2.42)–(2.43) and (2.51). For \(I_1(\varepsilon_N)\), by testing \(\psi := u \in K\) in (2.86), we obtain

\[
I_1(\varepsilon_N) \leq \int f_{EN}(u_{EN} - u) \, dx \leq F_s\|u_{EN} - u\|_{L^p(\Omega)} \to 0
\]
as \(N \to \infty\). Here we have used Hölder’s inequality and \(u_{EN} \to u\) in \(L^q' (\Omega)\). For \(I_2(\varepsilon_N)\), we have already known that \(\nabla E_{EN}(\nabla u) \to A_0(\nabla u)\) in \(L^p'(\Omega; \mathbb{R}^n)\) from Lemma 2.8. Combining this result with \(\nabla u_{EN} \to \nabla u\) in \(L^p(\Omega; \mathbb{R}^n)\), we have \(I_2(\varepsilon_N) \to 0\) as \(N \to \infty\). Finally, we obtain

\[
0 \leq \liminf_{N \to \infty} I(\varepsilon_N) \leq \limsup_{N \to \infty} I(\varepsilon_N) \leq 0,
\]

which yields \(I(\varepsilon_N) \to 0\) as \(N \to \infty\). From this convergence result, we are able to conclude that \(\nabla u_{EN} \to \nabla u\) in \(L^p(\Omega; \mathbb{R}^n)\). In fact, when \(1 < p < 2\), we use Hölder’s inequality, (2.43) and (2.51) to compute

\[
\|\nabla u_{EN} - \nabla u\|_{L^p(\Omega)}^p \leq \left( \int_{\Omega} (\varepsilon_N^2 + |\nabla u_{EN}|^2 + |\nabla u|^2)^{p/2} \right) \cdot \sup_{N} \left( \int_{\Omega} (\varepsilon_N^2 + |\nabla u_{EN}|^2 + |\nabla u|^2)^{p/2} \right)^{(2-p)/p}
\]

\[
\leq C \varepsilon^{-2/p} \cdot I(\varepsilon_N)^{2/p} \to 0 \quad \text{as} \quad N \to \infty.
\]

Here we note that \(C\) is finite by \(\nabla u_{EN} \to \nabla u\) in \(L^p(\Omega; \mathbb{R}^n)\). Similarly, when \(2 \leq p < \infty\), we use (2.42) and (2.51) to get

\[
\|\nabla u_{EN} - \nabla u\|_{L^p(\Omega)}^p \leq (AC(p))^{-1} \cdot I(\varepsilon_N) \to 0 \quad \text{as} \quad N \to \infty.
\]

Therefore, we are able to apply Lemma 2.8 \((2)\) \((3)\) By taking a subsequence, we may assume that

\[
\begin{align*}
\nabla E_{1,EN}(\nabla u_{EN}) &\to Z \quad \text{in} \quad L^\infty(\Omega; \mathbb{R}^n), \\
\nabla E_{p,EN}(\nabla u_{EN}) &\to \nabla E_p(\nabla u) \quad \text{in} \quad L^p'(\Omega; \mathbb{R}^n),
\end{align*}
\]

for some \(Z \in L^\infty(\Omega; \mathbb{R}^n)\) satisfying (1.17). Combining these results with (2.86) and

\[
\begin{align*}
\nabla (\psi - u_{EN}) &\to \nabla (\psi - u) \quad \text{in} \quad L^p(\Omega; \mathbb{R}^n), \\
\psi - u_{EN} &\to \psi - u \quad \text{in} \quad L^q(\Omega),
\end{align*}
\]

for each fixed \(\psi \in K\), we are able to check that the pair \((u, Z) \in K \times L^\infty(\Omega; \mathbb{R}^n)\) satisfies (2.83) for all \(\psi \in K\). This completes the proof.

From Proposition 2.11 we are able to show unique existence of a solution to the Dirichlet boundary value problem, as stated in Corollary 2.12 below.
Corollary 2.12. Let \( p \in (1, \infty) \) and \( q \in [1, \infty) \) satisfy (1.15). Assume that \( f \in L^q(\Omega) \) and \( \{f_\varepsilon\}_{0 < \varepsilon < 1} \subset L^q(\Omega) \) satisfy (2.78). We define functionals in \( \mathcal{F}_\varepsilon(0 \leq \varepsilon < 1) \) by (2.79)–(2.80), where the density \( E_\varepsilon \) is given by \( E_\varepsilon = f_\varepsilon + E \) with \( E = E_1 + E_p \) satisfying (1.9)–(1.11). Then, the Dirichlet problem (2.76) has a unique solution for each \( u_0 \in W^{1,p}(\Omega) \). Moreover, this solution \( u \) in \( u_0 + W^{1,p}_0(\Omega) \) is characterized by the minimizing property

\[
u = \arg \min \left\{ \mathcal{F}_\varepsilon(v) \mid v \in u_0 + W^{1,p}_0(\Omega) \right\},\]

and there holds \( u_\varepsilon \rightarrow u \) in \( W^{1,p}(\Omega) \) up to a subsequence. Here \( u_\varepsilon \) is the unique solution of (2.77). In other words, \( u_\varepsilon \) satisfies \( u_\varepsilon \in u_0 + W^{1,p}_0(\Omega) \) and

\[
\int_\Omega \langle \nabla E_{1,\varepsilon}(\nabla u_\varepsilon) \mid \nabla \phi \rangle \, dx + \int_\Omega \langle \nabla E_{p,\varepsilon}(\nabla u_\varepsilon) \mid \nabla \phi \rangle \, dx = \int_\Omega f_\varepsilon \phi \, dx
\]

for all \( \phi \in W^{1,p}_0(\Omega) \).

Proof. We apply Proposition 2.11 with \( \mathcal{K} = u_0 + W^{1,p}_0(\Omega) \). Under this setting, it is easy to check that for each \( \varepsilon \in (0, 1) \), the unique minimizer \( u_\varepsilon \) of the functional \( \mathcal{F}_\varepsilon : u_0 + W^{1,p}_0(\Omega) \rightarrow \mathbb{R} \) is characterized by the weak formulation (2.88). Also, it should be mentioned that when a pair \((u, Z) \in W^{1,p}(\Omega) \times L^\infty(\Omega; \mathbb{R}^n)\) satisfies (2.83) for all \( \psi \in u_0 + W^{1,p}_0(\Omega) \), then it also satisfies (1.16). In fact, for each \( \phi \in W^{1,p}_0(\Omega) \), we can test \( \psi := u_0 + \phi \) into (2.83), from which (1.16) is easy to deduce. \( \square \)

Remark 2.13. The proofs of Proposition 2.11 and Corollary 2.12 work even when \( E_1 \in C^0(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\}) \), \( E_p \in C^1(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\}) \). In fact, all of the inequalities applied in the proof of Proposition 2.11 follow from these regularity assumptions. It should be noted that (2.51) follows from convexity of \( E_{1,\varepsilon} = f_\varepsilon + E_1 \).

2.7 Proof of main theorem

Corollary 2.12 in the previous Section 2.6 implies that a weak solution to (1.2) can be approximated by a weak solution to

\[
-\text{div}(\nabla E_\varepsilon(\nabla u_\varepsilon)) = f_\varepsilon \quad \text{with} \quad E_\varepsilon = f_\varepsilon + (E_1 + E_p),
\]

under a suitable Dirichlet boundary condition, where \( f_\varepsilon \) satisfies (2.79). In the proof of Theorem 1.3, we aim to prove \( \mathcal{G}_\delta(\nabla u) \) is Hölder continuous. There Theorem 2.14 below plays an important role.

Theorem 2.14. In addition to assumptions of Theorem 1.3, we let positive numbers \( \delta, \varepsilon \) satisfy (2.77), and a function \( u_\varepsilon \in W^{1,p}(\Omega) \) be a weak solution to (2.89) in \( \Omega \) with

\[
\|f_\varepsilon\|_{L^q(\Omega)} \leq F,
\]

and

\[
\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq L
\]

for some constants \( F, L \in (0, \infty) \). Then, for each fixed \( x_0 \in \Omega \), there exist a sufficiently small ball \( B_{r_0}(x_0) \subset \Omega \) and a sufficiently small number \( \alpha \in (0, 1) \), such that \( \mathcal{G}_{2\delta,\varepsilon}(\nabla u_\varepsilon) \) is in \( C^\alpha(B_{r_0/2}(x_0); \mathbb{R}^n) \). Moreover, we have

\[
|\mathcal{G}_{2\delta,\varepsilon}(\nabla u_\varepsilon(x))| \leq \mu_0 \quad \text{for all} \ x \in B_{r_0/2}(x_0)
\]

and

\[
|\mathcal{G}_{2\delta,\varepsilon}(\nabla u_\varepsilon(x_1)) - \mathcal{G}_{2\delta,\varepsilon}(\nabla u_\varepsilon(x_2))| \leq C|x_1 - x_2|^{\alpha} \quad \text{for all} \ x_1, x_2 \in B_{r_0/2}(x_0).
\]

Here the exponent \( \alpha \in (0, 1) \), the radius \( r_0 \in (0, 1) \) and the constants \( C, \mu_0 \in (0, \infty) \) depend at most on \( n, p, q, \beta_0, \lambda, K, F, L, \) dist \((x_0, \partial \Omega)\), and \( \delta \), but are independent of \( \varepsilon \). Moreover, the constant \( \mu_0 \) does not depend on \( \delta \).
Theorem 2.14 is proved in Section 3. We would like to conclude Section 2 by giving the proof of Theorem 1.3.

Proof. Let \( u \in W^{1,p}(\Omega) \) be a weak solution to (1.2). For each \( \varepsilon \in (0, 1) \), we put the unique function \( u_\varepsilon \in u + W^{1,p}_0(\Omega) \) that solves the Dirichlet boundary value problem

\[
\begin{aligned}
\mathcal{L}^\varepsilon u_\varepsilon &= f_\varepsilon \quad \text{in } \Omega, \\
u_\varepsilon &= u \quad \text{on } \partial\Omega.
\end{aligned}
\]

By Corollary 2.12 we may take a subsequence \( \{u_{\varepsilon_j}\}_{j=1}^\infty \subset u + W^{1,p}_0(\Omega) \) such that \( u_{\varepsilon_j} \to u \) in \( W^{1,p}(\Omega) \). Hence, we may take finite constants \( F, L \in (0, \infty) \), which are independent of \( j \in \mathbb{N} \), such that (2.90) holds for every \( \varepsilon = \varepsilon_j \). Moreover, by taking a subsequence if necessary, we may assume that

\[
\nabla u_{\varepsilon_j}(x) \to \nabla u(x) \quad \text{for a.e. } x \in \Omega.
\]

We fix arbitrary \( \delta \in (0, 1) \). Without loss of generality we let \( \varepsilon_j \in (0, \delta/8) \) for all \( j \in \mathbb{N} \). By Theorem 2.14 we can choose a sufficiently small ball \( B_{\rho_\delta}(x_\delta) \subset \Omega \) such that the sequence \( \{\mathcal{G}_{\delta_j} u_{\varepsilon_j} \}_{j=1}^\infty \subset C^\alpha(B_{\rho_\delta/2}(x_\delta), \mathbb{R}^n) \) is bounded. Here the Hölder exponent \( \alpha \in (0, 1) \) depends on \( \delta \) but is independent of \( \varepsilon_j \). Hence, by taking a subsequence, we can find a continuous vector field \( v_\delta \in C^\alpha(B_{\rho_\delta/2}(x_\delta), \mathbb{R}^n) \) such that \( \mathcal{G}_{\delta_j} u_{\varepsilon_j} \to v_\delta \) uniformly in \( B_{\rho_\delta/2}(x_\delta) \). On the other hand, from (2.94) we have already known that \( \mathcal{G}_{\delta_j} u_{\varepsilon_j} \to \mathcal{G}_\delta(\nabla u) \) a.e. in \( \Omega \) as \( j \to \infty \). This implies that \( v_\delta = \mathcal{G}_\delta(\nabla u) \in C^\alpha(B_{\rho_\delta/2}(x_\delta), \mathbb{R}^n) \) holds for each fixed \( \delta \in (0, 1) \) with \( \alpha = \alpha(\delta) \in (0, 1) \). Since \( x_\delta \in \Omega \) is arbitrary, this completes the proof of \( \mathcal{G}_\delta(\nabla u) \in C^0(\Omega; \mathbb{R}^n) \).

By the definition of \( \mathcal{G}_\delta \), it is clear that

\[
\sup_\Omega |\mathcal{G}_{\delta_1}(\nabla u) - \mathcal{G}_{\delta_2}(\nabla u)| \leq |\delta_1 - \delta_2| \quad \text{for all } \delta_1, \delta_2 \in (0, 1).
\]

In particular, there exists a continuous vector field \( v_0 \in C^0(\Omega; \mathbb{R}^n) \) such that \( \mathcal{G}_\delta(\nabla u) \to v_0 \) uniformly in \( \Omega \). On the other hand, there clearly holds \( \mathcal{G}_\delta(\nabla u) \to \nabla u \) a.e. in \( \Omega \) as \( \delta \to 0^+ \). Thus, \( \nabla u = v_0 \in C^0(\Omega; \mathbb{R}^n) \) is realized, and this completes the proof of Theorem 1.3. \( \square \)

3 A priori Hölder estimates of the mapping \( \mathcal{G}_{\delta_j}(\nabla u_{\varepsilon_j}) \)

In Section 3, we consider weak solutions to the regularized equation (2.89), and show a priori Hölder bounds of truncated gradients (Theorem 2.14).

3.1 Higher regularity on approximated solutions

In Section 3.1, we briefly describe inner regularity on weak solutions to (2.89).

We first note that it is not restrictive to assume that \( u_\varepsilon \in W^{1,p}(\Omega) \), a weak solution to (2.89) in \( \Omega \), satisfies \( u_\varepsilon \in W^{1,\infty}_{\text{loc}}(\Omega) \cap W^{2,2}_{\text{loc}}(\Omega) \) for each \( \varepsilon \in (0, 1) \). This is possible by appealing to existing elliptic regularity theory, since the ellipticity ratio of (2.89) is bounded for each fixed \( \varepsilon \in (0, 1) \). In fact, (2.84) implies that the regularized density \( E_\varepsilon \in C^\infty(\Omega) \) satisfies

\[
c_\varepsilon \left(1 + |x|^2\right)^{p/2-1} \leq \nabla^2 E_\varepsilon(z) \leq C_\varepsilon \left(1 + |x|^2\right)^{p/2-1} \quad \text{id} \quad \text{for all } z \in \mathbb{R}^n
\]

for some constants \( 0 < c_\varepsilon < C_\varepsilon < \infty \) that may depend on an approximation parameter \( \varepsilon \). Moreover, if \( f_\varepsilon \in C^\infty(\Omega) \), then \( u_\varepsilon \in C^\infty(\Omega) \) follows from bootstrap arguments [30, Chapters 4–5]. We recall
that our approximation arguments work as long as (2.78) holds. Under this setting, we may choose $f_\varepsilon \in C^\infty(\Omega)$, so that the solution $u_\varepsilon$ satisfies (2.39) even in the classical sense. Also, even when $f_\varepsilon$ is not smooth but in $L^q(\Omega)$ ($n < q \leq \infty$), by standard arguments as in [25] Chapter 8, it is possible to get $u_\varepsilon \in C^{1,\alpha}_{\text{loc}}(\Omega) \cap W^{2,2}_{\text{loc}}(\Omega)$ for some $\alpha = \alpha(n, p, q, c_\varepsilon, C_\varepsilon) \in (0, 1)$. Here we note that the ratio $C_\varepsilon/c_\varepsilon$ substantially depends on $\varepsilon$, and hence the exponent $\alpha$ may tend to 0 as $\varepsilon \to 0$.

Since our arguments are local, we often let $u_\varepsilon \in W^{1,\infty}(B_\varepsilon(x_0)) \cap W^{2,2}(B_\varepsilon(x_0))$ for some fixed open ball $B_\rho(x_0) \subseteq \Omega$. Under this setting, integrating by parts, we are able to deduce a weak formulation

$$
\int_{B_\rho(x_0)} \langle \nabla^2 E_\varepsilon(\nabla u_\varepsilon) \nabla \partial_j u_\varepsilon, \nabla \phi \rangle \, dx = - \int_{B_\rho(x_0)} f \partial_j \phi \, dx
$$

(3.1)

for all $j \in \{1, \ldots, n\}$ and $\phi \in W^{1,2}_0(B_\rho(x_0))$. Here we mention that the inclusion $\nabla^2 E_\varepsilon(\nabla u_\varepsilon) \in L^2(B_\rho(x_0); \mathbb{R}^n)$ follows from $u_\varepsilon \in W^{1,\infty}(B_\rho(x_0)) \cap W^{2,2}(B_\rho(x_0))$.

Without proofs, we use local Lipschitz bounds of $u_\varepsilon$, uniformly for an approximation parameter $\varepsilon \in (0, 1)$ (Proposition 3.1). Based on Moser’s iterations, these estimates were already shown in the author’s work [41] Proposition 4] by choosing test functions whose supports never intersect the facets of regularized solutions. There higher regularity assumptions $E_\varepsilon, f_\varepsilon \in C^\infty(\Omega)$ are imposed, so that $u_\varepsilon \in C^\infty(\Omega)$. However, the proof therein works as long as $u_\varepsilon \in W^{1,\infty}_{\text{loc}}(\Omega) \cap W^{2,2}_{\text{loc}}(\Omega)$, since the test functions chosen in [41] Proposition 4] become admissible under this regularity.

**Proposition 3.1.** Let $u_\varepsilon \in W^{1,p}(\Omega)$ be a weak solution to (2.39) in $\Omega \setminus \Omega_\varepsilon$ with $\varepsilon \in (0, 1)$. Fix an open ball $B_r \subseteq \Omega$ with $r \in (0, 1]$. Then, for $n \geq 3$, we have

\[
\text{ess sup}_{B_{2r}} |\nabla u_\varepsilon| \leq \frac{C(n, p, q, \lambda, \Lambda, K)}{(1 - \theta)^{r/p}} \left(1 + \|f_\varepsilon\|_{L^p(B_r)}^{1/p - 1} + r^{-n/p} \|\nabla u_\varepsilon\|_{L^p(B_r)}\right)
\]

for all $\theta \in (0, 1)$. For $n = 2$, we have

\[
\text{ess sup}_{B_{2r}} |\nabla u_\varepsilon| \leq \frac{C(n, p, q, \lambda, \Lambda, \chi)}{(1 - \theta)^{\chi/p - 1/\chi}} \left(1 + \|f_\varepsilon\|_{L^{\chi}(B_r)}^{1/\chi - 1} + r^{-2/p} \|\nabla u_\varepsilon\|_{L^p(B_r)}\right)
\]

for all $\theta \in (0, 1), \chi \in (1, \infty)$.

It is noted that even when the external force term $f_\varepsilon$ is less regular than $f_\varepsilon \in L^q$, local Lipschitz estimates of $u_\varepsilon$ can be obtained by the recent result of [7] Theorems 1.9 & 1.11]. There, external force terms are assumed to be in a Lorentz space or an Orlicz space, and the computations therein are based on De Giorgi’s truncation.

### 3.2 Three basic propositions and the proof of Theorem 2.14

We first fix some notations in Section 3. Throughout Section 3 we assume that the positive numbers $\delta, \varepsilon$ satisfy (2.11). For each $u_\varepsilon \in W^{1,\infty}_{\text{loc}}(\Omega) \cap W^{2,2}_{\text{loc}}(\Omega)$, a weak solution to the regularized equation (2.39) in $\Omega$, we define

\[V_\varepsilon := \sqrt{\varepsilon^2 + |\nabla u_\varepsilon|^2} \in L^\infty_{\text{loc}}(\Omega) \cap W^{1,2}_{\text{loc}}(\Omega),\]

and

\[U_{\delta, \varepsilon} := (V_\varepsilon - \delta)^2 \in L^\infty_{\text{loc}}(\Omega) \cap W^{1,2}_{\text{loc}}(\Omega).\]

For given numbers $\mu \in (0, \infty), \nu \in (0, 1)$ and an open ball $B_{\rho}(x_0) \subseteq \Omega$, we define a superlevel set

\[S_{\mu, \nu}(x_0) := \{x \in B_{\rho}(x_0) \mid V_\varepsilon(x) - \delta > (1 - \nu)\mu\}.
]
For $f \in L^1(B_{\rho}(x_0); \mathbb{R}^m)$, we define an average integral
\[
\int_{B_{\rho}(x_0)} f \, dx := \frac{1}{|B_{\rho}(x_0)|} \int_{B_{\rho}(x_0)} f \, dx \in \mathbb{R}^m,
\]
which is often written by $(f)_{x_0, \rho}$ for notational simplicity.

We set an exponent $\beta \in (0, 1)$ by
\[
\beta := \begin{cases} 
1 - n/q & (n < q < \infty), \\
\beta_0 & (q = \infty),
\end{cases}
\]
where $\beta_0$ is an arbitrary number satisfying $0 < \beta_0 < 1$. The number $\beta$ often appears when one considers regularity of weak solutions to the Poisson equation $-\Delta v = f \in L^q$. It is well-known that this weak solution $v$ is locally $\beta$-Hölder continuous, which can be proved by the standard freezing coefficient method (see e.g., [28, Theorem 3.13]; see also [3, Chapter 3] and [23, Chapter 5] as related items).

To show local a priori Hölder estimates (Theorem 2.14), we apply Propositions 3.2–3.3 below.

**Proposition 3.2.** Let $u_\varepsilon$ be a weak solution to (2.89) in $\Omega$. Assume that positive numbers $\delta, \varepsilon, \mu, F, M$, and an open ball $B_{\rho}(x_0) \Subset \Omega$ satisfy (2.11), (2.90).

\[
0 < \delta < \mu,
\]
and
\[
\text{ess sup}_{B_{\rho}(x_0)} V_\varepsilon \leq \delta + \mu \leq M.
\]

Then, there exist sufficiently small numbers $\nu \in (0, 1/6)$, $\rho_* \in (0, 1)$, which depend at most on $n, p, q, \beta_0, \Lambda, \Lambda, K, F, M$, and $\delta$, but are independent of $\varepsilon$, such that the following statement holds true. If there hold $0 < \rho \leq \rho_*$ and
\[
|S_{\rho, \mu, \nu}(x_0)| > (1-\nu)|B_{\rho}(x_0)|,
\]
then the limit
\[
\Gamma_{2\delta, \varepsilon}(x_0) := \lim_{r \to 0} \{G_{2\delta, \varepsilon}(\nabla u_\varepsilon)\}_{x_0, r} \in \mathbb{R}^n
\]
exists. Moreover, this limit satisfies
\[
|\Gamma_{2\delta, \varepsilon}(x_0)| \leq \mu,
\]
and we have the following Campanato-type growth estimate
\[
\int_{B_{\rho}(x_0)} \left|G_{2\delta, \varepsilon}(\nabla u_\varepsilon) - \Gamma_{2\delta, \varepsilon}(x_0)\right|^2 \, dx \leq \mu^2 r^{-2\beta} \quad \text{for all } r \in (0, \rho].
\]
Here the exponent $\beta \in (0, 1)$ is defined by (3.2).

**Proposition 3.3.** Let $u_\varepsilon$ be a weak solution to (2.89) in $\Omega$. Assume that positive numbers $\delta, \varepsilon, \mu, F, M$, and an open ball $B_{\rho}(x_0) \Subset \Omega$ satisfy $0 < \rho \leq 1$, (2.11), (2.90).

\[
\text{ess sup}_{B_{\rho}(x_0)} |G(\nabla u_\varepsilon)| \leq \mu \leq \mu + \delta \leq M,
\]
and
\[
|S_{\rho/2, \mu, \nu}(x_0)| \leq (1-\nu)|B_{\rho/2}(x_0)|
\]

\[
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\]
for some constant \( \nu \in (0, 1/6) \). Then, there exist constants \( \kappa \in (2^{-\beta}, 1) \) and \( C_* \in [1, \infty) \), which depend at most on \( n, p, q, F, \lambda, \Lambda, K, M, \delta, \) and \( \nu \), but are independent of \( \epsilon \), such that we have either

\[
\mu^2 < C_* \rho^\beta, \tag{3.10}
\]

or

\[
\text{ess sup}_{B_{\rho/2}(x_0)} |\mathcal{J}_{\delta, \epsilon}(\nabla u_\epsilon)| \leq \kappa \mu. \tag{3.11}
\]

Here the exponent \( \beta \in (0, 1) \) is defined by (3.2).

Our analysis broadly depends on whether a scalar function \( V_\epsilon \) degenerates or not, which can be judged by measure assumptions as in (3.5) or (3.9). We would like to describe each individual part.

The assumption (3.5) in Proposition 3.2 suggests that \( V_\epsilon \) should be non-degenerate near the point \( x_0 \). This expectation enables us to apply freezing coefficient methods to show Campanato-type growth estimates as in (3.7). In other words, Proposition 3.2 is based on analysis over non-degenerate points of a scalar function \( V_\epsilon \) or a gradient \( \nabla u_\epsilon \). To justify this, however, we will face to check that the average integral \( (\nabla u_\epsilon)_{x_0, r} \in \mathbb{R}^n \) never vanishes even when the radius \( r \) tends to 0. To answer this affirmatively, we will like to find the suitable numbers \( \nu \) and \( \rho_* \) as in Proposition 3.2. These numbers are the key criteria for judging whether it is possible to separate degenerate and non-degenerate points of a gradient \( \nabla u_\epsilon \). When choosing these important numbers, we will need a variety of energy estimates. There, the assumptions (3.3) and (3.5) are used to show these energy estimates.

When (3.5) fails but instead a reversed estimate like (3.9) holds, we will have to consider a case where \( V_\epsilon \) may degenerate. There we appeal to a truncation method to deduce a De Giorgi-type oscillation lemma (Proposition 3.3). There we have to check that the scalar function \( U_{\delta, \epsilon} \) is a subsolution to a certain uniformly elliptic equation. This is possible since \( U_{\delta, \epsilon} \) is supported in \( \{ V_\epsilon > \delta \} \), where the ellipticity ratio of \( \nabla^2 E_\epsilon(\nabla u_\epsilon) \) is bounded.

**Remark 3.4.** Let \( \delta, \epsilon \) satisfy (2.11). For \( z \in \mathbb{R}^n \) and \( \mu \in (0, \infty) \), the inequalities \( |\mathcal{J}_{\delta, \epsilon}(z)| \leq \mu \) and \( \epsilon^2 + |z|^2 \leq (\delta + \mu)^2 \) are equivalent. In particular, (3.4) is equivalent to (3.8). Also, under these equivalent conditions, we can easily check that \( u_\epsilon \) satisfies

\[
\text{ess sup}_{B_{\mu}(x_0)} |\mathcal{J}_{2\delta, \epsilon}(\nabla u_\epsilon)| \leq (\mu - \delta) \leq \mu. \tag{3.12}
\]

From Propositions 3.1–3.3, we would like to give the proof of Theorem 2.14.

**Proof.** For each fixed \( x_s \in \Omega \), we first fix

\[
R := \min \left\{ \frac{1}{2}, \frac{1}{3} \text{ dist} (x_s, \partial \Omega) \right\} > 0,
\]

so that \( B_{2R}(x_s) \subset \Omega \) holds. By Proposition 3.1 we may take a finite constant \( \mu_0 \in (1, \infty) \), depending at most on \( n, p, q, \lambda, \Lambda, K, F, L \) and \( R \), such that we have

\[
\text{ess sup}_{B_{\mu}(x_s)} V_\epsilon \leq \mu_0. \tag{3.13}
\]

We set \( M := 1 + \mu_0 \), so that \( \mu_0 + \delta \leq M \) clearly holds.

We choose and fix the numbers \( \nu \in (0, 1/6) \), \( \rho_* \in (0, 1) \) as in Proposition 3.2, which depend at most on \( n, p, q, \beta_0, \lambda, \Lambda, K, F, M, \) and \( \delta \). Corresponding to this \( \nu \), we choose finite constants
We define the desired Hölder exponent \( \alpha \in (0, \beta/2) \) by \( \alpha := -\log \kappa/\log 4 \), so that the identity \( 4^{-\alpha} = \kappa \) holds. We also put the radius \( \rho_0 \) such that it satisfies

\[
0 < \rho_0 \leq \min \left\{ \frac{R}{2}, \rho_* \right\} < 1 \quad \text{and} \quad C_* \rho_0^2 \leq \kappa^2 \mu_0^2,
\]

(3.14) which depends at most on \( n, p, q, \beta_0, \Lambda, K, F, M, \) and \( \delta \). We set non-negative decreasing sequences \( \{\rho_k\}_k^{\infty}, \{\mu_k\}_k^{\infty} \) by \( \rho_k := 4^{-k} \rho_0, \mu_k := \kappa^{-k} \mu_0 \) for \( k \in \mathbb{N} \). By \( 2^{-\beta} < \kappa = 4^{-\alpha} < 1 \) and (3.14), we can easily check that

\[
C_* \rho_k^2 \leq 2^{-\beta k + \kappa} \mu_0 \leq \kappa^{k+1} \mu_0 = \mu_{k+1},
\]

(3.15) and

\[
\mu_k = 4^{-\alpha k} \mu_0 = \left( \frac{\rho_k}{\rho_0} \right)^\alpha \mu_0
\]

(3.16) for every \( k \in \mathbb{Z}_{\geq 0} \).

We claim that for every \( x_0 \in B_{\rho_0}(x_*) \), the limit

\[
\Gamma_{\delta, e} (x_0) := \lim_{r \to 0} \left( \mathcal{S}_{\delta, e} (\nabla u_e) \right)_{x_0, r} \in \mathbb{R}^n
\]

exists, and this limit satisfies

\[
\int_{B_r (x_0)} \left| \mathcal{S}_{\delta, e} (\nabla u_e) - \Gamma_{\delta, e} (x_0) \right|^2 \, dx \leq 4^{2\alpha + 1} \left( \frac{r}{\rho_0} \right)^{2\alpha} \mu_0^2 \quad \text{for all } r \in (0, \rho_0].
\]

(3.17) In the proof of (3.17), we introduce a set

\[
\mathcal{N} := \left\{ k \in \mathbb{Z}_{\geq 0} \mid |S_{\rho_k/2, \mu_k, \nu} (x_0)| > (1 - \nu)|B_{\rho_k/2} (x_0)| \right\},
\]

and define a number \( k_* := \min \mathcal{N} \in \mathbb{Z}_{\geq 0} \) when \( \mathcal{N} \neq \emptyset \). To show (3.17), we consider the three possible cases:

1(1) \( \mathcal{N} \neq \emptyset \) and \( \mu_{k_*} > \delta \). 1(2) \( \mathcal{N} \neq \emptyset \) and \( \mu_{k_*} < \delta \). 1(3) \( \mathcal{N} = \emptyset \).

It should be mentioned that when \( \mathcal{N} \neq \emptyset \), there clearly holds \( |S_{\rho_k/2, \mu_k, \nu} (x_0)| \leq (1 - \nu)|B_{\rho_k/2} (x_0)| \) for every \( k \in \{ 0, 1, \ldots, k_* - 1 \} \), since \( k_* \) is the minimum number of \( \mathcal{N} \). Thus, in the cases (1)(2) we are able to obtain

\[
\text{ess sup}_{B_k} \left| S_{\delta, e} (\nabla u_e) \right| \leq \mu_k \quad \text{for every } k \in \{ 0, 1, \ldots, k_* \}
\]

(3.18) by repeatedly applying (3.15) and Proposition 3.3 with \( (\rho, \mu) = (\rho_k, \mu_k) \) for \( k \in \{ 0, 1, \ldots, k_* - 1 \} \). Here we write \( B_k := B_{\rho_k} (x_0) \) \( (k \in \mathbb{Z}_{\geq 0}) \) for notational simplicity.

By \( k_* \in \mathcal{N} \), we are able to apply Proposition 3.2 in the open ball \( B_{\rho_{k_*}/2} (x_0) \) with \( \mu = \mu_{k_*} \) (see Remark 3.4). In particular, the limit \( \Gamma_{\delta, e} (x_0) \) exists and it satisfies

\[
\int_{B_r (x_0)} \left| \mathcal{S}_{\delta, e} (\nabla u_e) - \Gamma_{\delta, e} (x_0) \right|^2 \, dx \leq \left( \frac{2r}{\rho_{k_*}} \right)^{2\beta} \mu_{k_*}^2 \quad \text{for all } r \in \left( 0, \frac{\rho_{k_*}}{2} \right],
\]

(3.19) and

\[
\left| \Gamma_{\delta, e} (x_0) \right| \leq \mu_{k_*}.
\]

(3.20) When \( 0 < r \leq \rho_{k_*}/2 \), we use (3.16), (3.19) and \( \alpha < \beta \) to get

\[
\int_{B_r (x_0)} \left| \mathcal{S}_{\delta, e} (\nabla u_e) - \Gamma_{\delta, e} (x_0) \right|^2 \, dx \leq \left( \frac{2r}{\rho_{k_*}} \right)^{2\alpha} \left( \frac{\rho_{k_*}}{\rho_0} \right)^{2\alpha} \mu_0^2 = 4^\alpha \left( \frac{r}{\rho_0} \right)^{2\alpha} \mu_0^2.
\]

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To every \( r \in (\rho_{k+1}/2, \rho_0] \), there corresponds a unique integer \( k \in \{0, \ldots, k_{\ast}\} \) such that \( \rho_{k+1} < r \leq \rho_k \). By (3.13), we compute
\[
\int_{B_r(x_0)} |\mathcal{G}_{2\delta, \varepsilon}(\nabla u_\varepsilon) - \Gamma_{2\delta, \varepsilon}(x_0)|^2 \, dx \leq 2 \int_{B_r(x_0)} \left( |\mathcal{G}_{2\delta, \varepsilon}(\nabla u_\varepsilon)|^2 + |\Gamma_{2\delta, \varepsilon}(x_0)|^2 \right) \, dx
\]
\[
\leq 2 \left( \text{ess sup}_{B_k} |\mathcal{G}_{\delta, \varepsilon}(\nabla u_\varepsilon)|^2 + |\Gamma_{\delta, \varepsilon}(x_0)|^2 \right)
\]
\[
\leq 4 \mu_k^2 \leq 4 \left( \frac{\rho_k}{\rho_0} \right)^{2\alpha} \mu_0^2 \leq 4 \left( \frac{4}{\rho_0} \right)^{2\alpha} \mu_0^2.
\]

(2) We recall (3.12) in Remark 3.4 which yields \( |\mathcal{G}_{2\delta, \varepsilon}(\nabla u_\varepsilon)| = 0 \) a.e. in \( B_{k_{\ast}} \), and
\[
|\mathcal{G}_{2\delta, \varepsilon}(\nabla u_\varepsilon)| \leq |\mathcal{G}_{\delta, \varepsilon}(\nabla u_\varepsilon)| \quad \text{a.e. in } \Omega.
\]
Combining with (3.18), we have
\[
\text{ess sup}_{B_k} |\mathcal{G}_{2\delta, \varepsilon}(\nabla u_\varepsilon)| \leq \mu_k \quad \text{for every } k \in \mathbb{Z}_{\geq 0}.
\]
This clearly yields \( \Gamma_{2\delta, \varepsilon}(x_0) = 0 \). To every \( r \in (0, \rho_0] \), there corresponds a unique \( k \in \mathbb{Z}_{\geq 0} \) such that \( \rho_{k+1} < r \leq \rho_k \). By (3.22) and \( \kappa = 4^{-\alpha} \), we have
\[
\int_{B_r(x_0)} |\mathcal{G}_{2\delta, \varepsilon}(\nabla u_\varepsilon) - \Gamma_{2\delta, \varepsilon}(x_0)|^2 \, dx = \int_{B_r(x_0)} |\mathcal{G}_{2\delta, \varepsilon}(\nabla u_\varepsilon)|^2 \, dx
\]
\[
\leq \text{ess sup}_{B_k} |\mathcal{G}_{2\delta, \varepsilon}(\nabla u_\varepsilon)|^2 \leq \mu_k^2 = 4^{-2\alpha} \mu_0^2
\]
\[
\leq \left[ 4 \left( \frac{r}{\rho_0} \right) \right]^{2\alpha} \mu_0^2 = 16 \left( \frac{r}{\rho_0} \right)^{2\alpha} \mu_0^2.
\]

(3) There clearly holds \( |S_{pk/2, \mu_0, \varepsilon}| \leq (1 - \nu)|B_{pk/2}(x_0)| \) for every \( k \in \mathbb{Z}_{\geq 0} \). Applying (3.15) and Proposition 3.3 with \( (\rho, \mu) = (\rho_k, \mu_k) \), \( k \in \mathbb{Z}_{\geq 0} \) repeatedly, we can easily check that
\[
\text{ess sup}_{B_k} |\mathcal{G}_{\delta, \varepsilon}(\nabla u_\varepsilon)| \leq \mu_k \quad \text{for every } k \in \mathbb{Z}_{\geq 0}.
\]
In particular, (3.22) clearly follows from this result and (3.24), and therefore the proof of (3.17) can be accomplished, similarly to (2).

In all possible cases, \( \Gamma_{2\delta, \varepsilon}(x_0) \) exists and satisfies (3.17). Here it should be noted that the limit \( \Gamma_{2\delta, \varepsilon} \) satisfies
\[
|\Gamma_{2\delta, \varepsilon}(x_0)| \leq \mu_0 \quad \text{for all } x_0 \in B_{\rho_0}(x_\ast)
\]
by (3.13) and (3.21). From (3.17) and (3.23), we would like to show that
\[
|\Gamma_{2\delta, \varepsilon}(x_1) - \Gamma_{2\delta, \varepsilon}(x_2)| \leq \left( \frac{2^{2\alpha + 2n/2}}{\rho_0^{2\alpha}} \mu_0 \right) |x_1 - x_2|^\alpha
\]
for all \( x_1, x_2 \in B_{\rho_0/2}(x_\ast) \). We prove (3.24) by dividing into the two cases. In the case \( r := |x_1 - x_2| \leq \rho_0/2 \), we set a point \( x_3 := (x_1 + x_2)/2 \) in \( B_{\rho_0}(x_\ast) \). Noting the inclusions \( B_{r/2}(x_3) \subset B_r(x_j) \subset B_{\rho_0/2}(x_j) \subset B_{\rho_0}(x_\ast) \) for each \( j \in \{1, 2\} \), we use (3.17) to obtain
\[
|\Gamma_{2\delta, \varepsilon}(x_1) - \Gamma_{2\delta, \varepsilon}(x_2)|^2
\]
Then, we have a non-decreasing Lipschitz function whose non-differentiable points are finitely many. For any non-negative $u_1 \leq u_2 \leq \Gamma = \sum_{j=1}^{n} \int_{B_{\rho}(x_0)} |V_2 \Delta \psi(V_2)| V_2 dx$, we set

\[ \left( \Gamma_{2,\delta,\varepsilon}(x_1) - \Gamma_{2,\delta,\varepsilon}(x_2) \right)^2 dx \]

\[ \leq 2 \left( \int_{B_{\rho}(x_1)} |G_{2,\delta,\varepsilon}(\nabla u_2) - G_{2,\delta,\varepsilon}(\nabla u_1)|^2 dx + \int_{B_{\rho}(x_2)} |G_{2,\delta,\varepsilon}(\nabla u_2) - G_{2,\delta,\varepsilon}(\nabla u_1)|^2 dx \right) \]

\[ \leq 2^{n+1} \left( \int_{B_{\rho}(x_1)} |G_{2,\delta,\varepsilon}(\nabla u_2) - G_{2,\delta,\varepsilon}(\nabla u_1)|^2 dx + \int_{B_{\rho}(x_2)} |G_{2,\delta,\varepsilon}(\nabla u_2) - G_{2,\delta,\varepsilon}(\nabla u_1)|^2 dx \right) \]

\[ \leq 2^{n+1} \cdot 16 \left( \frac{r}{\rho_0} \right)^{2\alpha} \mu_0^2 = \left( \frac{2^{\alpha+2+n/2}}{\rho_0} \mu_0 \right)^2 \left| x_1 - x_2 \right|^2 \alpha, \]

which yields (3.24). In the remaining case $|x_1 - x_2| > \rho_0/2$, we simply use (3.24) to get

\[ \left| \Gamma_{2,\delta,\varepsilon}(x_1) - \Gamma_{2,\delta,\varepsilon}(x_2) \right| \leq 2\mu_0 \leq 2 \cdot \frac{2^{\alpha}|x_1 - x_2|^\alpha}{\rho_0^\alpha} \mu_0, \]

which completes the proof of (3.24). Finally, we mention that the mapping $\Gamma_{2,\delta,\varepsilon}$ is a Lebesgue representative of $G_{2,\delta,\varepsilon} \in L^p(\Omega, \mathbb{R}^n)$ by Lebesgue’s differentiation theorem, and therefore the desired estimates (2.92) (2.93) immediately follow from (3.23) (3.24) with $C = C(n, \alpha, \rho_0) \in (0, \infty)$. □

### 3.3 A weak formulation

In Section 3.3, we deduce a weak formulation on regularized solutions (Lemma 3.5).

**Lemma 3.5.** Let $u_\varepsilon$ be a weak solution to (2.52) in $\Omega$ with $0 < \varepsilon < 1$. Assume that $\psi : [0, \infty) \to [0, \infty)$ is a non-decreasing Lipschitz function whose non-differentiable points are finitely many. For any non-negative function $\xi \in W^{1,\infty}(B_{\rho}(x_0))$ that is compactly supported in an open ball $B_{\rho}(x_0) \subseteq \Omega$, we set

\[
\begin{align*}
J_1 &:= \int_{B_{\rho}(x_0)} \langle \nabla^2 E_\varepsilon(\nabla u_\varepsilon) \nabla \xi, \nabla \xi \rangle \psi(V_\varepsilon) V_\varepsilon \, dx, \\
J_2 &:= \int_{B_{\rho}(x_0)} \langle \nabla^2 E_\varepsilon(\nabla u_\varepsilon) \nabla \xi, \nabla V_\varepsilon \rangle \xi \psi'(V_\varepsilon) V_\varepsilon \, dx, \\
J_3 &:= \sum_{j=1}^{n} \int_{B_{\rho}(x_0)} \langle \nabla^2 E_\varepsilon(\nabla u_\varepsilon) \nabla \partial_{x_j} u_\varepsilon, \nabla \partial_{x_j} u_\varepsilon \rangle \xi \psi(V_\varepsilon) \, dx, \\
J_4 &:= \int_{B_{\rho}(x_0)} |f_\varepsilon|^2 \psi(V_\varepsilon) V_\varepsilon^{2-p} \xi \, dx, \\
J_5 &:= \int_{B_{\rho}(x_0)} |f_\varepsilon|^2 \psi'(V_\varepsilon) V_\varepsilon^{2-p} \xi \, dx, \\
J_6 &:= \int_{B_{\rho}(x_0)} |f_\varepsilon| \|
abla \xi \| \psi(V_\varepsilon) V_\varepsilon \, dx.
\end{align*}
\]

Then, we have

\[ 2J_1 + J_2 + J_3 \leq \frac{n}{\lambda} (J_4 + J_5) + 2J_6. \]

The resulting weak formulation (3.26) is fully used in Sections 3.5 and 3.7.

**Proof.** For each $j \in \{1, \ldots, n\}$, we test $\phi := \xi \psi(V_\varepsilon) \partial_{x_j} u_\varepsilon \in W^{1,2}_0(B)$ into (3.1). By summing with respect to $j \in \{1, \ldots, n\}$ and using

\[ \sum_{j=1}^{n} \partial_{x_j} u_\varepsilon \nabla \partial_{x_j} u_\varepsilon = \frac{1}{2} \sum_{j=1}^{n} \nabla \left( \partial_{x_j} u_\varepsilon \right)^2 = \frac{1}{2} \nabla V_\varepsilon^2 = V_\varepsilon \nabla V_\varepsilon, \]

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Lemma 3.6

we get

\[ J_1 + J_2 + J_3 = - \int_B f \sum_{j=1}^n \partial x_j \left( \zeta \psi (V_e) \partial x_j u_e \right) \, dx \]

\[ = - \int_B f \zeta \psi (V_e) \left( \nabla u_e \cdot \nabla \zeta \right) \, dx - \int_B f \zeta \psi' (V_e) \left( \nabla V_e \cdot \nabla u_e \right) \, dx \]

\[ - \int_B f \zeta \psi (V_e) \Delta u_e \, dx \]

\[ = -(J_7 + J_8 + J_9). \]

For \( J_2, J_3 \), we use (2.54) to get

\[
\begin{aligned}
J_2 & \geq \lambda \int_B V_e^{p-1} |\nabla V_e|^2 \zeta \psi'(V_e) \, dx, \\
J_3 & \geq \lambda \int_B V_e^{p-1} |\nabla^2 u_e|^2 \zeta \psi (V_e) \, dx.
\end{aligned}
\]

(3.27)

With this in mind, we compute

\[ |J_8| \leq \frac{1}{2} \int_B V_e^{p-1} |\nabla V_e|^2 \zeta \psi'(V_e) \, dx + \frac{1}{2\lambda} \int_B |f_e|^{2} |V_e|^{1-p} |\nabla u_e|^2 \zeta \psi'(V_e) \, dx \leq \frac{J_2}{2} + \frac{J_3}{2\lambda}, \]

and

\[
\begin{aligned}
|J_9| & \leq \sqrt{m} \int_B |f_e| \zeta \psi (V_e) |\nabla^2 u_e| \, dx \\
& \leq \frac{\lambda}{2} \int_B V_e^{p-2} |\nabla^2 u_e|^2 \zeta \psi (V_e) \, dx + \frac{n}{2\lambda} \int_B |f_e|^2 \psi (V_e) V_e^{2-p} \, dx \\
& \leq \frac{J_2}{2} + \frac{n}{2\lambda} J_4.
\end{aligned}
\]

by Young’s inequality. Combining these results with \( |J_7| \leq J_6 \), we easily conclude (3.26). \( \square \)

3.4 Perturbation results from a higher integrability lemma

For a given ball \( B_\rho (x_0) \subset \Omega \), we consider an \( L^2 \) -mean oscillation of the gradient \( \nabla u_e \), which is given by

\[ \Phi (x_0, \rho) := \int_{B_\rho (x_0)} \left| \nabla u_e - (\nabla u_e)_{x_0, \rho} \right|^2 \, dx. \]

Sections 3.4–3.6 are devoted to prove Proposition 3.2 by showing a variety of quantitative estimates related to this \( \Phi \). Among them, comparisons between weak solutions to (2.52) and harmonic functions play an important role.

Before making a comparison argument, we have to verify higher integrability estimates for \( |\nabla u_e - (\nabla u_e)_{x_0, \rho}| \). This is possible by showing Lemma 3.6 below.

Lemma 3.6 (Higher integrability lemma). Let \( u_e \) be a weak solution to (2.52) in \( \Omega \). Assume that positive numbers \( \delta, \varepsilon, \mu, F, M \), and an open ball \( B_\rho (x_0) \subset \Omega \) satisfy (2.71), (2.90) and (3.3)–(3.4). Then, for any vector \( \zeta \subset \mathbb{R}^n \) satisfying

\[ \delta + \frac{\mu}{4} \leq |\zeta| \leq \delta + \mu, \]

(3.28)
there exists a constant $\theta = \theta(n, p, q, \lambda, \Lambda, K, M, \delta)$ such that
\[ 0 < \theta \leq \min\{\beta, \beta_0\} < 1, \] (3.29)
and
\[ \int_{B_{\rho/2}(x_0)} |\nabla u_x - \xi|^2(1+\theta) \, dx \leq C \left[ \int_{B_{\rho}(x_0)} |\nabla u_x - \xi|^2 \, dx \right]^{1+\theta} + F^2(1+\theta) \rho^{2\theta(1+\theta)} \]. (3.30)

Here the constant $C \in (0, \infty)$ depends at most on $n, p, q, \beta_0, \lambda, \Lambda, K, M, \delta$.

In the proof of Lemma 3.6 we use so called Gehring’s lemma without proofs.

**Lemma 3.7** (Gehring’s lemma). Let $B = B_R(x_0) \subset \mathbb{R}^n$ be an open ball, and non-negative function $g, h$ satisfy $g \in L^s(B), h \in L^2(B)$ with $1 < s < \tilde{s} \leq \infty$. Suppose that there holds
\[ \int_{B_{R/2}(x_0)} g^s \, dx \leq \tilde{C} \left[ \left( \int_{B_{2R}(x_0)} g \, dx \right)^s + \int_{B_{2R}(x_0)} h^s \, dx \right] \]
for all $B_{2R}(z_0) \subset B$. Here $\tilde{C} \in (0, \infty)$ is a constant independent of $z_0$ and $r > 0$. Then, there exists a sufficiently small positive number $\tilde{\theta} = \tilde{\theta}(s, \tilde{s}, n, \tilde{C})$ such that $g \in L^{\sigma_0\tilde{\theta}}(B)$ with $\sigma_0 := s(1+\theta) \in (s, \tilde{s})$. Moreover, for each $\sigma \in [s, \sigma_0]$, we have
\[ \left( \int_{B_{R/2}(x_0)} g^\sigma \, dx \right)^{1/\sigma} \leq C \left[ \left( \int_{B_{R}(x_0)} g^s \, dx \right)^{1/s} + \left( \int_{B_{R}(x_0)} h^\sigma \, dx \right)^{1/\sigma} \right], \]
where the constant $\tilde{C} \in (0, \infty)$ depends at most on $\sigma, s, \tilde{s}$, and $\tilde{C}$.

The proof of Gehring’s lemma is found in [45] Theorem 3.3], which is based on ball decompositions [45] Lemma 3.1] and generally works for a metric space with a doubling measure. As related items, see also [25] Section 6.4], where the classical Calderon–Zygmund cube decomposition is fully applied to prove Gehring’s lemma.

**Proof.** We later prove that there holds
\[ \int_{B_{\rho/2}(x_0)} |\nabla u_x - \xi|^2 \, dx \leq \hat{C} \left[ \left( \int_{B_{\rho}(x_0)} |\nabla u_x - \xi|^{2n/n+2} \right)^{n+2} + \int_{B_{\rho}(x_0)} |\rho f_x|^2 \, dx \right] \] (3.31)
for any open ball $B_{\rho}(x_0) \subset B := B_{\rho}(x_0)$. Here $\hat{C} \in (0, 1)$ is a constant depending on $n, p, \lambda, \Lambda, K, M,$ and $\delta$. Then, by applying Lemma 3.7 with $(s, \tilde{s}) := (1+2/n, q(n+2)/2n), g := |\nabla u_x - \xi|^{2n/n+2} \in L^s(B), h := |\rho f_x|^{2n/n+2} \in L^\tilde{s}(B),$ we are able to find a small exponent $\tilde{\theta} = \tilde{\theta}(\hat{C}, n, q) > 0$ and a constant $C = C(n, q, \hat{C}, \tilde{\theta}) > 0$ such that there hold (3.29) and
\[ \int_{B_{\rho/2}(x_0)} |\nabla u_x - \xi|^{2(1+\theta)} \, dx \leq C \left[ \left( \int_{B_{\rho}(x_0)} |\nabla u_x - \xi|^2 \, dx \right)^{1+\theta} + \int_{B_{\rho}(x_0)} |\rho f_x|^{2(1+\theta)} \, dx \right]. \]

We note that (3.29) yields $2(1+\theta) \leq q,$ and therefore the function $|\rho f_x|^{2(1+\theta)}$ is integrable in $B_{\rho}(x_0).$ Moreover, by Hölder’s inequality, we have
\[ \int_{B_{\rho}(x_0)} |\rho f_x|^{2(1+\theta)} \, dx \leq C(n, \tilde{\theta}) F^2(1+\theta) \rho^{2\theta(1+\theta)}, \]
from which (3.30) follows.

We take and fix an arbitrary open ball $B_r(z_0) \subset B$. To show (3.31), we set a function $w_x \in W^{1,\infty}(B_r(z_0))$ by

$$w_x(x) := u_x(x) - (u_x)_{z_0} - \langle \zeta \mid x - z_0 \rangle$$

for $x \in B_r(z_0)$, so that $\nabla w_x = \nabla u_x - \zeta$ holds. We choose a cutoff function $\eta \in C^1_c(B_r(z_0))$ satisfying

$$\eta \equiv 1 \quad \text{on } B_{r/2}(z_0) \quad \text{and} \quad |\nabla \eta| \leq \frac{4}{r} \quad \text{in } B_r(z_0),$$

and test $\phi := \eta^2 w_x \in W^{1,\rho}_0(B_r(z_0))$ into (2.89). Then, we obtain

$$0 = \int_{B_r(z_0)} \langle \nabla E_x(\nabla u_x) - \nabla E_x(\zeta) \mid \nabla \phi \rangle \, dx - \int_{B_r(z_0)} f_x \phi \, dx$$

$$= \int_{B_r(z_0)} \eta^2 \langle \nabla E_x(\nabla u_x) - \nabla E_x(\zeta) \mid \nabla w_x \rangle \, dx$$

$$+ 2 \int_{B_r(z_0)} \eta w_x \langle \nabla E_x(\nabla u_x) - \nabla E_x(\zeta) \mid \nabla \eta \rangle \, dx - \int_{B_r(z_0)} \eta^2 f_x w_x \, dx$$

$$=: J_1 + J_2 + J_3.$$ 

The assumptions (3.3)–(3.4) and (3.28) enables us to apply Lemma 2.27 (1). As a result, we have

$$J_1 \geq C_1 \int_{B_r(z_0)} \eta^2 |\nabla w_x|^2 \, dx$$

by (2.55). Similarly for $J_2$, we can use (2.56) and Young’s inequality to obtain

$$|J_2| \leq C_2 \int_{B_r(z_0)} \eta |w_x| |\nabla w_x| |\nabla \eta| \, dx$$

$$\leq \frac{C_1}{2} \int_{B_r(z_0)} \eta^2 |\nabla w_x|^2 \, dx + \frac{C_2^2}{2C_1} \int_{B_r(z_0)} |\nabla \eta|^2 w_x^2 \, dx.$$ 

For $J_3$, we use Young’s inequality to obtain

$$|J_3| \leq \frac{1}{r^2} \int_{B_r(z_0)} \eta^2 w_x^2 \, dx + \frac{r^2}{4} \int_{B_r(z_0)} \eta^2 f_x^2 \, dx.$$

By these estimates and our choice of $\eta$, we are able to compute

$$\int_{B_{r/2}(z_0)} |\nabla w_x|^2 \, dx \leq 2^n \int_{B_r(z_0)} |\nabla (\eta w_x)|^2 \, dx$$

$$\leq C \left[ \int_{B_r(z_0)} |\nabla \eta|^2 + \frac{\eta^2}{r^2} \right] w_x^2 \, dx + r^2 \int_{B_r(z_0)} \eta^2 |f_x|^2 \, dx$$

$$\leq C \left[ r^{-2} \int_{B_r(z_0)} w_x^2 \, dx + \int_{B_r(z_0)} |\rho f_x|^2 \, dx \right]$$

with $C \in (0, \infty)$ depending on $n, C_1, C_2$. By the definition, $w_x$ satisfies

$$\int_{B_r(z_0)} w_x \, dx = 0,$$

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which enables us to use the Poincaré–Sobolev inequality [16, Chapter IX, Theorem 10.1];

\[ \int_{B_{r}(z_0)} w^2 \, dx \leq C(n) r^2 \left( \int_{B_{r}(z_0)} |\nabla w|^2 \, dx \right)^{\frac{\mu^2}{n}}. \]

Recall \( \nabla w_e = \nabla u_e - \zeta \), and this completes the proof of (3.31). \( \square \)

Now we would like to give perturbation estimates based on comparisons to harmonic functions (Lemma 3.8). This result can be obtained by the regularity assumption \( C_{\text{loc}}^{2,\beta_0}(\mathbb{R}^n \setminus \{0\}) \), and plays an important role in the proof of Proposition 3.2.

**Lemma 3.8.** Let \( u_e \) be a weak solution to (2.52) in \( \Omega \). Assume that positive numbers \( \delta, \varepsilon, \mu, F, M, \) and \( \varepsilon \) satisfy \( 0 < \rho \leq 1, (2.71), (2.90) \) and (3.3)–(3.4). In addition, we let

\[ |(\nabla u_e)_{x_0, \rho}| \geq \delta + \frac{\mu}{4}, \tag{3.32} \]

and consider the Dirichlet boundary problem

\[
\begin{aligned}
-\text{div} \left( \nabla^2 E_e \left( (\nabla u_e)_{x_0, \rho} \right) \nabla v_e \right) &= 0 \quad \text{in} \quad B_{\rho/2}(x_0), \\
v_e &= u_e \quad \text{on} \quad \partial B_{\rho/2}(x_0).
\end{aligned}
\tag{3.33}
\]

Then, there uniquely exists a function \( v_e \) that solves (3.33). Moreover, we have

\[ \int_{B_{\rho/2}(x_0)} |\nabla u_e - \nabla v_e|^2 \, dx \leq C \left\{ \left[ \frac{\Phi(x_0, \rho)}{\mu^2} \right]^\beta \Phi(x_0, \rho) + \left( F^2 + F^{2(1+\beta)} \right) \right\}^2, \tag{3.34} \]

where \( \beta \) is the positive constant given in Lemma 3.6 and

\[ \int_{B_{\rho/2}(x_0)} |\nabla v_e - \nabla (v_e)_{x_0, \tau \rho}|^2 \, dx \leq C \tau^2 \int_{B_{\rho/2}(x_0)} |\nabla v_e - (\nabla v_e)_{x_0, \rho/2}|^2 \, dx \tag{3.35} \]

for all \( \tau \in (0, 1/2] \). Here the exponent \( \beta \) is defined by (3.2), and the constants \( C \in (0, \infty) \) in (3.34)–(3.35) depend at most on \( n, p, q, \beta_0, \lambda, K, M, \) and \( \delta \).

**Proof.** For notational simplicity, we write \( \zeta := (\nabla u_e)_{x_0, \tau \rho} \in \mathbb{R}^n \) and \( B := B_{\rho/2}(x_0) \). Then by (3.4), it is easy to check that

\[ |\zeta| \leq \int_{B_{\rho/2}(x_0)} V_e \, dx \leq \delta + \mu. \tag{3.36} \]

Combining with (2.11), (3.3) and (3.32), we have already known that

\[ 0 < \frac{\delta^2}{16} \leq \frac{\mu^2}{16} \leq \varepsilon^2 + |\zeta|^2 \leq \delta^2 + (2\mu)^2 \leq 5\mu^2 \leq 5M^2. \]

Hence by (2.54), there exist a constant \( l_0 = l_0(p) \in (0, 1) \) and another constant \( m \in (0, 1) \), which depends at most on \( p, \lambda, K, M, \delta \), such that

\[ mid \leq l_0 \lambda \mu^{p-2} \id \leq \nabla^2 E_e(\zeta) \leq m^{-1} \id. \]

In other words, \( \nabla^2 E_e(\zeta) \) is uniformly elliptic, which yields unique existence of the solution \( v_e \in u_e + W_{0,1}^{1,2}(B) \) of the problem (3.33). Moreover, since the coefficient matrix \( \nabla^2 E(\zeta) \) is constant, we are
able to find a constant $C = C(m, n) \in (0, \infty)$ such that (3.35) holds (see e.g., [3] Lemma 2.17, [23] Proposition 5.8).

We note that functions $u_\epsilon \in W^{1, \infty}(B)$ and $v_\epsilon \in u_\epsilon + W^{1, 2}_0(B)$ respectively satisfy

$$
\int_B \langle \nabla E_\epsilon(\nabla u_\epsilon) \rangle \nabla \phi \, dx = \int_B f_\epsilon \phi \, dx
$$

for all $\phi \in W^{1, 1}_0(B)$, and

$$
\int_B \langle \nabla^2 E_\epsilon(\zeta) \nabla v_\epsilon \rangle \nabla \phi \, dx = 0
$$

for all $\phi \in W^{1, 2}_0(B)$. In particular, for all $\phi \in W^{1, 2}_0(B)$, we have

$$
\int_B \langle \nabla^2 E_\epsilon(\zeta) (\nabla u_\epsilon - \nabla v_\epsilon) \rangle \nabla \phi \, dx
$$

$$
= \int_B \langle \nabla^2 E_\epsilon(\zeta) \nabla u_\epsilon - \nabla^2 E_\epsilon(\nabla u_\epsilon) \rangle \nabla \phi \, dx + \int_B f_\epsilon \phi \, dx
$$

$$
= \int_B \langle \nabla^2 E_\epsilon(\zeta) \nabla u_\epsilon - \zeta - (\nabla E_\epsilon(\nabla u_\epsilon) - \nabla E_\epsilon(\zeta)) \rangle \nabla \phi \, dx + \int_B f_\epsilon \phi \, dx,
$$

where it is noted that the vectors $\nabla^2 E_\epsilon(\zeta) \zeta$, $\nabla E_\epsilon(\zeta) \in \mathbb{R}^n$ are constant. By (3.4), (3.32) and (3.36), we are able to apply Lemma 2.6. As a result, there exists a constant $C \in (0, \infty)$, depending at most on $n$, $p$, $\beta_0$, $\Lambda$, $K$ and $\delta$, such that a weak formulation

$$
\int_B \langle \nabla^2 E_\epsilon(\zeta) \nabla u_\epsilon - \nabla v_\epsilon \rangle \nabla \phi \, dx \leq C \mu^{p-2-\beta_0} \int_B |\nabla u_\epsilon - \zeta|^{1+\beta_0} |\nabla \phi| \, dx + \int_B |f_\epsilon| |\phi| \, dx
$$

holds for all $\phi \in W^{1, 2}_0(B)$. Now we test $\phi := u_\epsilon - v_\epsilon \in W^{1, 2}_0(B)$ into this weak formulation. By Hölder’s inequality and the Poincaré inequality, we have

$$
|_{l_1, l_2} \mu^{p-2} \int_B |\nabla u_\epsilon - \nabla v_\epsilon|^2 \, dx
$$

$$
\leq C(n, p, \beta_0, \Lambda, K, \delta) \mu^{p-2-\beta_0} \int_B |\nabla u_\epsilon - \zeta|^{1+\beta_0} |\nabla u_\epsilon - \nabla v_\epsilon| \, dx + \int_B |f_\epsilon| |u_\epsilon - v_\epsilon| \, dx
$$

$$
\leq C(n, p, \beta_0, \Lambda, K, \delta) \mu^{p-2-\beta_0} \left( \int_B |\nabla u_\epsilon - \zeta|^{2(1+\beta_0)} \, dx \right)^{1/2} \left( \int_B |\nabla u_\epsilon - \nabla v_\epsilon|^2 \, dx \right)^{1/2}
$$

$$
+ C(n, q) F \mu^{\beta+\frac{q}{2}} \left( \int_B |\nabla u_\epsilon - \nabla v_\epsilon|^2 \, dx \right)^{1/2}.
$$

As a result, we obtain

$$
\int_B |\nabla u_\epsilon - \nabla v_\epsilon|^2 \, dx \leq C \left[ \frac{1}{\mu^{2\beta_0}} \int_B |\nabla u_\epsilon - \zeta|^{2(1+\beta_0)} \, dx + F^2 \mu^{2(2-p)} \rho^{2\beta} \right],
$$

where the constant $C \in (0, \infty)$ depends at most on $n$, $p$, $q$, $\beta_0$, $\lambda$, $K$, $M$, and $\delta$. By (3.3) and (3.36), it is easy to get

$$
|\nabla u_\epsilon - \zeta| \leq 2(\delta + \mu) \leq 4\mu \quad \text{a.e. in } B_\rho(x_0).
$$

Since $\zeta$ clearly satisfies (3.23) by (3.32) and (3.36), we are able to apply Lemma 3.6. Noting $\bar{\theta} \leq \beta_0$, we obtain

$$
\int_B |\nabla u_\epsilon - \nabla v_\epsilon|^2 \, dx \leq C \left[ \frac{C(n, \bar{\theta})}{\mu^{2\bar{\theta}}} \int_B |\nabla u_\epsilon - \zeta|^{2(1+\bar{\theta})} \, dx + F^2 \mu^{2(2-p)} \rho^{2\bar{\theta}} \right].
$$
for each \( \frac{1}{\mu^2} \) depend at most on \( n, p, q, \beta_0, \lambda, K, M, \) and \( \delta \). Recalling \( 0 < \rho \leq 1 \) and \( \delta < \mu < M \) by (3.3)–(3.4), we finally obtain (3.34).

### 3.5 Energy estimates

We go back to the weak formulation given in Lemma 3.5 and deduce some energy estimates under suitable assumptions as in Proposition 3.2.

In Section 3.5, we introduce a vector field \( G_{p,e} \in C^\infty(\mathbb{R}^n; \mathbb{R}^n) \) by

\[
G_{p,e}(z) := g_{p,e}(|z|^2)z \quad \text{for} \quad z \in \mathbb{R}^n \quad \text{with} \quad g_{p,e}(t) := \left( \epsilon^2 + t \right)^{(p-1)/2} \quad (0 \leq t < \infty)
\]

for each \( \epsilon \in (0, 1) \). By definition, it is clear that the mapping \( G_{p,e} : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is bijective. By direct computations, it is easy to notice that the Jacobian matrix \( DG_{p,e} \) satisfies

\[
\left( \epsilon^2 + |z|^2 \right)^{(p-1)/2} \text{id} \leq DG_{p,e}(z) \leq p\left( \epsilon^2 + |z|^2 \right)^{(p-1)/2} \text{id}
\]

for all \( \epsilon \in (0, 1) \) and \( z \in \mathbb{R}^n \). In particular, similarly to (2.42), we can find a constant \( c = c(p) \in (0, \infty) \) such that there holds

\[
\langle G_{p,e}(z_1) - G_{p,e}(z_2), z_1 - z_2 \rangle \geq c(p)|z_1 - z_2|^{p+1} \quad \text{for all} \quad z_1, z_2 \in \mathbb{R}^n,
\]

which clearly yields

\[
|G_{p,e}(z_1) - G_{p,e}(z_2)| \geq c(p)|z_1 - z_2|^p \quad \text{for all} \quad z_1, z_2 \in \mathbb{R}^n.
\]

By this estimate and \( G_{p,e}(0) = 0 \), the inverse mapping \( G_{p,e}^{-1} \) satisfies

\[
|G_{p,e}^{-1}(w)| \leq C(p)|w|^{1/p} \quad \text{for all} \quad w \in \mathbb{R}^n
\]

with \( C = c(p)^{-1} \in (0, \infty) \). Also, similarly to Lemma 3.3(1), it is possible to find a constant \( C = C(p) \in (0, \infty) \) such that there holds

\[
\langle G_{p,e}(z_1) - G_{p,e}(z_2), z_1 - z_2 \rangle \geq C(p) \max \{|z_1|^{p-1}, |z_2|^{p-1}\}|z_1 - z_2|^2,
\]

and therefore

\[
|G_{p,e}(z_1) - G_{p,e}(z_2)| \geq C(p) \max \{|z_1|^{p-1}, |z_2|^{p-1}\}|z_1 - z_2| \quad (3.39)
\]

for all \( z_1, z_2 \in \mathbb{R}^n \). The estimates (3.38)–(3.39) are used in Section 3.5.

**Lemma 3.9.** Let \( u_\epsilon \) be a weak solution to (2.52) in \( \Omega \). Assume that positive numbers \( \delta, \epsilon, \mu, F, M, \) and an open ball \( B_p(x_0) \Subset \Omega \) satisfy (2.7), (2.90) and (3.3)–(3.4). Then, for a fixed constant \( \nu \in (0, 1) \), we have

\[
\int_{B_{\tau p}(x_0)} |D[G_{p,e}(\nabla u_\epsilon)]|^2 \, dx \leq C \frac{1}{\tau^n} \left[ \frac{1}{(1-\tau)^2 \rho^2} + F^2 \frac{\rho^{-\frac{n}{2}}}{\nu} \right] \mu^{2p},
\]

and

\[
\frac{1}{|B_{\tau p}(x_0)|} \int_{S_{\tau p, \rho}(x_0)} |D[G_{p,e}(\nabla u_\epsilon)]|^2 \, dx \leq C \frac{\nu}{\tau^n} \left[ \frac{F^2 \rho^{-\frac{n}{2}}}{\nu} \right] \mu^{2p}
\]

for all \( \tau \in (0, 1) \). Here \( G_{p,e} \) is the mapping defined by (3.37), and the constants \( C \in (0, \infty) \) in (3.40)–(3.41) depend at most on \( n, p, q, \lambda, K, \) and \( \delta \).
Proof. We often write $B = B_\rho(x_0)$ for notational simplicity. By (2.54) and direct calculations, we have

$$
|D[G_{\rho, \varepsilon}(\nabla u_\varepsilon)]|^2
= \left| g_{\rho, \varepsilon} \left( |\nabla u_\varepsilon|^2 \right) \nabla^2 u_\varepsilon + 2 g_{\rho, \varepsilon}' \left( |\nabla u_\varepsilon|^2 \right) (\nabla u_\varepsilon \otimes \nabla^2 u_\varepsilon \nabla u_\varepsilon) \right|^2
\leq 2 V_{\varepsilon}^{2(p-1)} |\nabla^2 u_\varepsilon|^2 + 2 (p - 1)^2 V_{\varepsilon}^{2(p-3)} |\nabla u_\varepsilon \otimes \nabla^2 u_\varepsilon |^2
\leq \gamma_\rho V_{\varepsilon}^{2(p-1)} |\nabla^2 u_\varepsilon|^2
$$

with $\gamma_\rho := 2(p^2 - 2p + 2) > 0$. Noting this, we apply Lemma 3.5 with

$$
\psi(t) = t^p \tilde{\psi}(t) \quad (0 \leq t < \infty).
$$

Here $\tilde{\psi}$ is a convex function that is to be chosen later as either of the following;

$$
\tilde{\psi}(t) \equiv 1, \quad (t - \delta - k)^2 \quad (0 \leq t < \infty) \quad \text{for some constant } k > 0. \tag{3.43}
$$

We choose a cutoff function $\eta \in C^1_c(B_\rho(x_0))$ such that

$$
\eta \equiv 1 \quad \text{on } B_{\tau \rho}(x_0) \quad \text{and} \quad |\nabla \eta| \leq \frac{2}{(1 - \tau) \rho} \quad \text{in } B_\rho(x_0), \tag{3.44}
$$

and set $\zeta := \eta^2$. Then, we have

$$
L_1 + L_2
\begin{align*}
&:= \int_B \sum_{j=1}^n \left\langle \nabla^2 E_\varepsilon(\nabla u_\varepsilon) \nabla \partial_{x_j} u_\varepsilon \mid \nabla \partial_{x_j} u_\varepsilon \right\rangle V_{\varepsilon}^p \psi(V_{\varepsilon}) \eta^2 \, dx
\quad + \int_B \left\langle \nabla^2 E_\varepsilon(\nabla u_\varepsilon) \nabla V_{\varepsilon} \mid \nabla V_{\varepsilon} \right\rangle \psi'(V_{\varepsilon}) V_{\varepsilon} \eta^2 \, dx
\leq 4 \int_B \left| \left\langle \nabla^2 E_\varepsilon(\nabla u_\varepsilon) \nabla V_{\varepsilon} \mid \nabla \eta \right\rangle \right| V_{\varepsilon} \psi(V_{\varepsilon}) \eta \, dx
\quad + \frac{n}{\lambda} \int_B |f_\varepsilon|^2 V_{\varepsilon}^{2(p-3)} (\psi(V_{\varepsilon}) + V_{\varepsilon} \psi'(V_{\varepsilon})) \eta^2 \, dx
\quad + 4 \int_B |f_\varepsilon| |\nabla \eta| V_{\varepsilon} \psi(V_{\varepsilon}) \, dx
\quad =: R_1 + R_2 + R_3.
\end{align*}
$$

For $R_1$, we use the Cauchy–Schwarz inequality

$$
\left| \left\langle \nabla^2 E_\varepsilon(\nabla u_\varepsilon) \nabla V_{\varepsilon} \mid \nabla \eta \right\rangle \right| \leq \sqrt{\left\langle \nabla^2 E_\varepsilon(\nabla u_\varepsilon) \nabla V_{\varepsilon} \mid \nabla V_{\varepsilon} \right\rangle} \cdot \sqrt{\left\langle \nabla^2 E_\varepsilon(\nabla u_\varepsilon) \nabla \eta \mid \nabla \eta \right\rangle},
$$

which holds true by (2.54). Combining with Young’s inequality, we have

$$
R_1 \leq L_2 + 4 \int_B \left| \left\langle \nabla^2 E_\varepsilon(\nabla u_\varepsilon) \nabla \eta \mid \nabla \eta \right\rangle \right| V_{\varepsilon} \psi(V_{\varepsilon})^2 \, dx
$$

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For $\mathbf{R}_3$, we apply Young’s inequality to get

$$\mathbf{R}_3 \leq 2 \int_B |f_\varepsilon|^2 \psi'(V_\varepsilon)V_\varepsilon^{3-p} \eta^2 \, dx + 2 \int_B |\nabla \eta|^2 \frac{V_\varepsilon \psi'(V_\varepsilon)^2}{\psi'(V_\varepsilon)} V_\varepsilon^{p-2} \, dx.$$  

Therefore, by (2.54), we obtain

$$L_1 \leq \int_B |\nabla \eta|^2 \left[ (4\Lambda + 2)V_\varepsilon^{p-2} + 4K V_\varepsilon^{-1} \right] \frac{V_\varepsilon \psi'(V_\varepsilon)^2}{\psi'(V_\varepsilon)} \, dx$$

$$+ \left( 2 + \frac{m}{\Lambda} \right) \int_B |f_\varepsilon|^2 \eta^2 \psi'(V_\varepsilon) + V_\varepsilon \psi'(V_\varepsilon) \, dx.$$  

(3.46)

Here we note that the assumptions (3.3)–(3.4) enable us to have $V_\varepsilon \leq \delta + \mu \leq 2\mu$ a.e. in $B = B_\delta(x_0)$, and $\mu' = \mu^{1-2p} \cdot \mu^{2p} \leq \delta^{1-2p} \mu^{2p}$ for every $0 < l < 2p$. Hence it follows that

$$\left[ (4\Lambda + 2)V_\varepsilon^{p-2} + 4K V_\varepsilon^{-1} \right] \frac{V_\varepsilon \psi'(V_\varepsilon)^2}{\psi'(V_\varepsilon)} \leq C(\Lambda, K) \frac{V_\varepsilon^{p-2} + V_\varepsilon^{-1}}{p\tilde{\psi}(V_\varepsilon) + V_\varepsilon \tilde{\psi}'(V_\varepsilon)}$$

$$\leq C(\Lambda, K) \frac{(1 + \delta^{1-2p})\mu^{2p} \tilde{\psi}(V_\varepsilon)^2}{p\tilde{\psi}(V_\varepsilon) + V_\varepsilon \tilde{\psi}'(V_\varepsilon)} \quad \text{a.e. in } B,$$  

(3.47)

and

$$\frac{\psi(V_\varepsilon) + V_\varepsilon \psi'(V_\varepsilon)}{V_\varepsilon^{p-2}} = (p + 1)V_\varepsilon^{p-2} \tilde{\psi}(V_\varepsilon) + V_\varepsilon \tilde{\psi}'(V_\varepsilon)$$

$$\leq 4(p + 1)\delta^{2(1-p)} \mu^{2p} \left[ \tilde{\psi}(V_\varepsilon) + V_\varepsilon \tilde{\psi}'(V_\varepsilon) \right] \quad \text{a.e. in } B.$$  

(3.48)

By (3.42) and (3.46)–(3.48), we obtain

$$\begin{align*}
\int_B D \left[ G_{p, \varepsilon}(\nabla u_\varepsilon) \right] \eta^2 \tilde{\psi}(V_\varepsilon) \, dx & \leq C_p \Lambda L_1 \\
& \leq C\mu^{2p} \int_B |\nabla \tilde{\psi}(V_\varepsilon)|^2 \frac{\tilde{\psi}(V_\varepsilon)^2}{p\tilde{\psi}(V_\varepsilon) + V_\varepsilon \tilde{\psi}'(V_\varepsilon)} \, dx + \int_B |f_\varepsilon|^2 \eta^2 \left[ \tilde{\psi}(V_\varepsilon) + V_\varepsilon \tilde{\psi}'(V_\varepsilon) \right] \, dx 
\end{align*}$$

(3.49)

with $C = C(n, p, \Lambda, K, \delta) \in (0, \infty)$. From (3.49) we will deduce (3.40)–(3.41).

We first apply (3.49) with $\tilde{\psi}$ and $\eta$ given by (3.43) and (3.45) respectively. Then, we have

$$\begin{align*}
\int_{B_{r_\varepsilon}(x_0)} D \left[ G_{p, \varepsilon}(\nabla u_\varepsilon) \right]^2 \, dx & \leq C(n, p, \lambda, \Lambda, K, \delta) \mu^{2p} \left[ \frac{1}{\rho} \int_B |\nabla \eta|^2 \, dx + \int_B |f_\varepsilon|^2 \eta^2 \, dx \right] \\
& \leq C(n, p, q, \lambda, \Lambda, K, \delta) \mu^{2p} \left[ \frac{1}{(1-\tau)^2 \rho^2 + F^2 \rho^{-\frac{2m}{q}}} \right] |B_\varepsilon(x_0) |
\end{align*}$$

by Hölder’s inequality. We note $|B_{r_\varepsilon}(x_0)| = \tau^n |B_\varepsilon(x_0)|$, and this yields (3.40).

Next, we let $\tilde{\psi}$ and $\eta$ satisfy (3.44)–(3.45). Here we determine the constant $k > 0$ by

$$k := (1 - 2\tau)\mu \geq \frac{\mu}{2} > 0.$$  

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Then, from (3.4), it follows that \((V_e - \delta - k)_+ \leq \mu - k = 2\nu\mu\) a.e. in \(B = B_{\rho}(x_0)\). Hence by direct computations, we can check that
\[
\begin{cases}
\frac{\psi(V_e)^2}{p\psi(V_e) + V_e\psi'(V_e)} = \frac{(V_e - \delta - k)^2}{p(V_e - \delta - k)_+ + 2V_e} \leq \frac{(2\nu\mu)^3}{0 + 2(\delta + k)} \leq 8\nu^3\mu^2,
\psi(V_e) + V_e\psi'(V_e) \leq 3V_e(V_e - \delta - k)_+ \leq 3 \cdot 2\mu \cdot 2\nu = 12\nu^2,
\end{cases}
\]
a.e. in \(B\). Also, there holds \(V_e - \delta - k > (1 - \nu)\mu - k = \nu\mu > 0\) in \(S_{\tau\rho,\mu,\nu}(x_0)\). These results and (3.49) enable us to compute
\[
\nu^2\mu^2 \int_{S_{\tau\rho,\mu,\nu}(x_0)} |D[G_{p,e}(\nabla u_e)]|^2 \, dx
\leq \int_B \eta^2 |D[G_{p,e}(\nabla u_e)]|^2 \psi(V_e) \, dx
\leq C\mu^{2p} \left[ 8\nu^3\mu^2 \int_B |\nabla \eta|^2 \, dx + 12\nu\mu^2 \int_B |f_e|^2 \eta^2 \, dx \right]
\leq C\nu^2\mu^{2p+2} \left[ \frac{\nu}{(1 - \tau)^2} + \frac{F^2\rho^{-2n/q}}{\nu} \right] |B_{\rho}(x_0)|
\]
for some constant \(C = C(n, p, q, \lambda, \Lambda, K, \delta) \in (0, \infty)\). From this and \(|B_{\tau\rho}(x_0)| = \tau^n|B_{\rho}(x_0)|\), (3.41) follows.

From Lemma 3.9 and (3.38)–(3.39), we prove Lemma 3.10 below. This result is used later in the next Section 3.6.

**Lemma 3.10.** Let \(u_e\) be a weak solution to (2.52) in \(\Omega\). Assume that positive numbers \(\delta, \varepsilon, \mu, F, M, \) and an open ball \(B_{\rho}(x_0) \Subset \Omega\) satisfy (2.71), (2.90) and (3.3)–(3.4). If (3.5) holds for some \(\nu \in (0, 1/6)\), then for all \(\tau \in (0, 1)\), we have
\[
\Phi(x_0, \tau \rho) \leq \frac{C_{\gamma} \mu^2}{\tau^n} \left[ \frac{\nu^{2n}}{(1 - \tau)^2} + \frac{F^2\rho^{-2n/q}}{\nu} \right],
\]
(5.0)
where the constant \(C_{\gamma} \in (0, \infty)\) depends at most on \(n, p, q, \lambda, \Lambda, K, \) and \(\delta\).

**Proof.** We first set \(\xi := G_{p,e}^{-1}(G_{p,e}(\nabla u_e))_{\tau\rho}, \) so that there holds
\[
\int_{B_{\tau\rho}} [G_{p,e}(\nabla u_e) - G_{p,e}(\xi)] \, dx = 0.
\]
In particular, we can use the Poincaré–Sobolev inequality
\[
\int_{B_{\tau\rho}(x_0)} |G_{p,e}(\nabla u_e) - G_{p,e}(\xi)|^2 \, dx \leq C(n)(\tau \rho)^2 \left( \int_{B_{\tau\rho}(x_0)} |D[G_{p,e}(\nabla u_e)]|^{2n} \, dx \right)^{\frac{2n}{2n+2}}.
\]
By (3.3)–(3.4), it is easy to get
\[
|G_{p,e}(\nabla u_e)| = V_e^{p-1} |\nabla u_e| \leq V_e^p \leq (\delta + \mu)^p \leq (2\mu)^p \quad \text{a.e. in } B_{\rho}(x_0)
\]
This inequality and (3.38) yield
\[
|\xi| \leq C(p)|G_{p,e}(\xi)|^{1/p} \leq C(p) \sup_{B_{\rho}(x_0)} |G_{p,e}(\nabla u_e)|^{1/p} \leq C(p)\mu.
\]

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To prove (3.50), we fix arbitrary $\tau \in (0, 1)$ and set
\[
\begin{align*}
\mathbf{I} &:= \frac{1}{|B_{\tau p}(x_0)|} \int_{B_{\tau p}(x_0) \setminus S_{p, \mu, \nu}(x_0)} \|\nabla u_\tau - \zeta\|^2 \, dx, \\
\mathbf{II} &:= \frac{1}{|B_{\tau p}(x_0)|} \int_{B_{\tau p}(x_0) \cap S_{p, \mu, \nu}(x_0)} \|\nabla u_\tau - \zeta\|^2 \, dx.
\end{align*}
\]
For $\mathbf{I}$, we use (3.39) and the Poincaré–Sobolev inequality to compute
\[
\frac{|\nabla u_\tau - \zeta|}{\mu} \leq (\delta + \mu) + C(p) \mu \leq C(p) \mu \quad \text{a.e. in } B_\rho(x_0).
\]
Combining with (3.54), we get
\[
\mathbf{I} \leq \frac{|B_{\tau p}(x_0) \setminus S_{p, \mu, \nu}(x_0)|}{|B_{\tau p}(x_0)|} \cdot \frac{C(p)}{\mu^2} \mu^2 \leq \frac{C(p) \nu^2}{\tau^n}.
\]
Here we have used $|B_{\tau p}(x_0)| = \tau^n |B_\rho(x_0)|$. Before estimating $\mathbf{II}$, we compute
\[
(1 - \nu)\mu < V_\rho - \delta \leq |\nabla u_\tau| + e - \delta \leq |\nabla u_\tau| - \frac{7}{8} \delta \quad \text{a.e. in } S_{p, \mu, \nu}(x_0), \quad (3.51)
\]
which immediately follows from (2.11) and the definition of $S_{p, \mu, \nu}(x_0)$. The assumption $\nu \in (0, 1/6)$ and (3.51) clearly yield
\[
\frac{5}{6} \mu \leq |\nabla u_\tau| \quad \text{a.e. in } S_{p, \mu, \nu}(x_0).
\]
With this in mind, we use (3.39) and the Poincaré–Sobolev inequality to compute
\[
\mathbf{II} \leq \frac{C(p)}{\mu^{2(p-1)}|B_{\tau p}(x_0)|} \int_{B_{\tau p}(x_0) \cap S_{p, \mu, \nu}(x_0)} |G_{p, \nu}(\nabla u_\tau) - G_{p, \nu}(\zeta)|^2 \, dx \leq \frac{C(p)}{\mu^{2(p-1)}(\tau p)^2} \int \left[ D \left[ G_{p, \nu}(\nabla u_\tau) \right] \right]^{\frac{2n}{n+2}} \, dx \overset{\text{a.e.}}{=} \frac{C(p)}{\mu^{2(p-1)}(\tau p)^2} \cdot \mathbf{III}^{1 + 2/n}. \]
Noting that there holds $B_{\tau p}(x_0) \setminus S_{p, \mu, \nu}(x_0) = B_{\tau p}(x_0) \setminus S_{p, \mu, \nu}(x_0)$ as measurable sets, we decompose $\mathbf{III} = \mathbf{III}_1 + \mathbf{III}_2$ with
\[
\begin{align*}
\mathbf{III}_1 &:= \frac{1}{|B_{\tau p}(x_0)|} \int_{B_{\tau p}(x_0) \setminus S_{p, \mu, \nu}(x_0)} D \left[ G_{p, \nu}(\nabla u_\tau) \right] \frac{2n}{n+2} \, dx, \\
\mathbf{III}_2 &:= \frac{1}{|B_{\tau p}(x_0)|} \int_{S_{p, \mu, \nu}(x_0)} D \left[ G_{p, \nu}(\nabla u_\tau) \right] \frac{2n}{n+2} \, dx.
\end{align*}
\]
For $\mathbf{III}_1$, by applying Hölder’s inequality and (3.40), we compute
\[
\mathbf{III}_1^{1 + 2/n} \leq \left[ \frac{|B_{\tau p}(x_0) \setminus S_{p, \mu, \nu}(x_0)|}{|B_{\tau p}(x_0)|} \right]^{2/n} \int_{B_{\tau p}} \left[ D \left[ G_{p, \nu}(\nabla u_\tau) \right] \right]^2 \, dx \leq C(n, p, q, \lambda, \Lambda, K, \delta) \left( \frac{1}{1 - \tau} \right)^{2/n} + E^2 \rho^{-\frac{2n}{n+2}}.
\]
For the remained integral $\mathbf{III}_2$, we similarly use Hölder’s inequality and (3.41) to have
\[
\mathbf{III}_2^{1 + 2/n} \leq \left[ \frac{|S_{p, \mu, \nu}(x_0)|}{|B_{\tau p}(x_0)|} \right]^{2/n} \frac{1}{|B_{\tau p}(x_0)|} \int_{E_{p, \mu, \nu}(x_0)} \left[ D \left[ G_{p, \nu}(\nabla u_\tau) \right] \right]^2 \, dx.
\]
\[ C(n, p, q, \lambda, K, \delta)^{\frac{2p}{\tau n}} \left[ \frac{\nu}{(1-\tau)^2 \rho^2} + \frac{F^2 \rho^{2\beta}}{\nu} \right]. \]

Finally, we apply (3.54) to obtain
\[ \Phi(x_0, \tau \rho) \leq \int_{B_{\tau \rho}(x_0)} |\nabla u_\epsilon - \zeta|^2 \, dx = I + II \]
\[ \leq \frac{C(p)\nu \mu^2}{\tau^n} + \frac{C(n, p)}{\mu^{2p-2}} (\tau \rho)^2 \left( III_1^{1+2/n} + III_2^{1+2/n} \right) \]
\[ \leq \frac{C(n, p, q, \lambda, K, \delta)\mu^2}{\tau^n} \left[ \nu \frac{\nu r^2 + \nu^2/n}{(1-\tau)^2 \rho^2} + \frac{\tau^2 + \nu^{1+2/n}}{\nu} F^2 \rho^{2\beta} \right]. \]

Recalling \( 0 < \tau < 1 \) and \( 0 < \nu < 1/6 \), we are able to find \( C_\tau \in (0, \infty) \) such that (3.50) holds. \( \square \)

### 3.6 Campanato-type decay estimates by shrinking arguments

The aim of Section 3.6 is to give the proof of Proposition 3.2 by standard shrinking methods. A significant point therein is to verify that the average integral \( (\nabla u_\epsilon)_{x_0, r} \) never vanishes even when the radius \( r \) tends to zero. We will justify this expectation by similar computations as in Lemma 3.12.

For preliminaries, we deduce Lemmata 3.11, 3.12, which are key tools to make our shrinking arguments successful. There we use some energy estimates from Lemmata 3.8 and 3.10 in Sections 3.4, 3.5.

**Lemma 3.11.** Let \( u_\epsilon \) be a weak solution to (2.52) in \( \Omega \). Assume that positive numbers \( \delta, \varepsilon, \mu, F, M, \) and an open ball \( B_\rho(x_0) \Subset \Omega \) satisfy (2.11), (2.90) and (3.3)-(3.4). Assume that there hold (3.32) and
\[ \Phi(x_0, \rho) \leq \frac{\mu^2}{\tau^n} \] for some \( \tau \in (0, 1/2) \). Here \( \Theta \) is the constant given in Lemma 3.6. Then, we have
\[ \Phi(x_0, \tau \rho) \leq C \left[ \tau^2 \Phi(x_0, \rho) + \frac{\rho^{2\beta}}{\tau^n} \mu^2 \right]. \] Here the constant \( C_\tau \in (0, \infty) \) depends at most on \( n, p, q, \beta_0, \lambda, K, F, M, \) and \( \delta \).

Before the proof, we recall that there holds
\[ \int_{B_r(x_0)} |f(x) - (f)_{x_0, r}|^2 \, dx = \min_{\xi \in \mathbb{R}^m} \int_{B_r(x_0)} |f(x) - \xi|^2 \, dx \] (3.54)

for all \( f \in L^2(B_r(x_0); \mathbb{R}^m) \). This is easy to deduce by considering a smooth convex function \( g \in C^\infty(\mathbb{R}^m, \mathbb{R}) \) defined by
\[ g(\xi) := \int_{B_r(x_0)} |f(x) - \xi|^2 \, dx \]
\[ = \|f\|^2_{L^2(B_r(x_0))} - 2|B_r(x_0)||\langle f, \xi \rangle| + |B_r(x_0)||\xi|^2, \]
which clearly satisfies \( \nabla g((f)_{x_0, r}) = 0 \).
Proof. Let \( v \in u + W^{1,2}_0(B_{\rho/2}(x_0)) \) be the unique solution of (3.33). We first apply (3.54) to get
\[
\Phi(x_0, \tau \rho) \leq \int_{B_{\rho/2}(x_0)} |\nabla u - (\nabla v_E)_{x_0, \tau \rho}|^2 \, dx
\]
\[
\leq \frac{2}{(2\pi)^n} \int_{B_{\rho/2}(x_0)} |\nabla u_E - \nabla v_E|^2 \, dx + 2 \int_{B_{\rho/2}(x_0)} |\nabla v_E - (\nabla v_E)_{x_0, \tau \rho}|^2 \, dx
\]
For the second average integral, we use (3.34) and (3.54) to compute
\[
\int_{B_{\rho/2}(x_0)} |\nabla v_E - (\nabla v_E)_{x_0, \tau \rho}|^2 \, dx
\]
\[
\leq C \tau^2 \int_{B_{\rho/2}(x_0)} |\nabla v_E - (\nabla v_E)_{x_0, \tau \rho}|^2 \, dx \leq C \tau^2 \int_{B_{\rho/2}(x_0)} |\nabla v_E - (\nabla v_E)_{x_0, \rho}|^2 \, dx
\]
\[
\leq C \left[ \tau^2 \int_{B_{\rho/2}(x_0)} |\nabla u - \nabla v_E|^2 \, dx + 2^n \tau^2 \int_{B_{\rho/2}(x_0)} |\nabla u - (\nabla u)_{x_0, \rho}|^2 \, dx \right]
\]
\[
\leq C \left[ \tau^2 \int_{B_{\rho/2}(x_0)} |\nabla u - \nabla v_E|^2 \, dx + \tau^2 \Phi(x_0, \rho) \right]
\]
with the constant \( C \in (0, \infty) \) depending at most on \( n, p, q, \beta_0, \lambda, K, M, \) and \( \delta \). We use (3.34) and (3.52) to compute
\[
\Phi(x_0, \tau \rho) \leq C \left[ \frac{1}{\tau^n} \int_{B_{\rho/2}(x_0)} |\nabla u - \nabla v_E|^2 \, dx + \tau^2 \Phi(x_0, \rho) \right]
\]
\[
\leq C \left[ \frac{\Phi(x_0, \rho)}{\mu^2} + \frac{F^2 + F^{2(1+\theta)}}{\tau^n} \rho^{2\beta} + \tau^2 \Phi(x_0, \rho) \right]
\]
\[
\leq C \left[ \tau^2 \Phi(x_0, \rho) + \left( F^2 + F^{2(1+\theta)} \right) \rho^{2\beta} \frac{\mu^{2}}{\tau^n} \left( \frac{\mu}{\delta} \right)^2 \right]
\]
Here we have used (3.3) to obtain (3.53). \( \square \)

Lemma 3.12. Let \( u \), be a weak solution to (2.22) in \( \Omega \). Assume that positive numbers \( \delta, \epsilon, \mu, F, M \), and an open ball \( B_{\rho}(x_0) \) \( \subseteq \Omega \) satisfy (2.7), (2.9), and (3.3). Then, for each fixed \( \theta \in (0, 9/64) \), there exist numbers \( \nu \in (0, 1/6) \) and \( \hat{\rho} \in (0, 1) \), which depend at most on \( n, p, q, \lambda, K, F, M, \delta, \) and \( \theta \), but are independent of \( \epsilon \), such that when \( 0 < \rho < \hat{\rho} \) and (3.5) hold, we have
\[
|\nabla u_E (x_0, \rho)| \geq \delta + \frac{\mu}{2}, \tag{3.55}
\]
and
\[
\Phi(x_0, \rho) \leq \theta \mu^2. \tag{3.56}
\]
Proof. We will later choose constants \( \tau \in (0, 1), \nu \in (0, 1/6) \), and \( \hat{\rho} \in (0, 1) \). Let the radius \( \rho \) satisfy \( 0 < \rho < \hat{\rho} \), and assume that there holds (3.5). By (3.54), we have
\[
\Phi(x_0, \rho) \leq \int_{B_{\rho}(x_0)} |\nabla u - (\nabla u)_{x_0, \tau \rho}|^2 \, dx = J_1 + J_2
\]
with
\[
\begin{align*}
\mathbf{J}_1 & := \frac{1}{|B_{\rho}(x_0)|} \int_{B_{\rho}(x_0)} |\nabla u_\varepsilon - (\nabla u_\varepsilon)_{x_0, \tau \rho}|^2 \, dx, \\
\mathbf{J}_2 & := \frac{1}{|B_{\rho}(x_0)|} \int_{B_{\rho}(x_0) \setminus B_{\tau \rho}(x_0)} |\nabla u_\varepsilon - (\nabla u_\varepsilon)_{x_0, \tau \rho}|^2 \, dx.
\end{align*}
\]

For \( \mathbf{J}_1 \), we use (3.50) to obtain
\[
\mathbf{J}_1 = \tau^n \Phi(x_0, \tau \rho) \leq C_\tau \mu^2 \left[ \frac{\nu^{2/n}}{(1-\tau)^2} + \frac{F^2}{\nu} \rho^{2\beta/q} \right]
\]
with \( C_\tau = C_\tau(n, \rho, \lambda, \Lambda, K, M, \delta) \in (0, \infty) \). For \( \mathbf{J}_2 \), we use (3.3) to have \(|\nabla u_\varepsilon| \leq V_e \leq \delta + \mu \leq 2\mu \) a.e. in \( B_\rho(x_0) \), and hence \(|(\nabla u_\varepsilon)_{x_0, \tau \rho}| \leq 2\mu \). This yields
\[
\mathbf{J}_2 \leq 8\mu^2 \cdot \frac{|B_{\rho}(x_0) \setminus B_{\tau \rho}(x_0)|}{|B_{\rho}(x_0)|} = 8\mu^2 (1 - \tau^n) \leq 8n\mu^2 (1 - \tau),
\]
where we have used \( 1 - \tau^n = (1 + \tau + \cdots + \tau^{n-1})(1 - \tau) \leq n(1 - \tau) \). As a result, we obtain
\[
\Phi(x_0, \rho) \leq C_\tau \mu^2 \left[ \frac{\nu^{2/n}}{(1-\tau)^2} + \frac{F^2}{\nu} \rho^{2\beta} \right] + 8n(1 - \tau)\mu^2.
\]

We first fix
\[
\tau := 1 - \frac{\theta}{24n} \in (0, 1), \quad \text{so that there holds} \quad 8n(1 - \tau) = \frac{\theta}{3}.
\]

Next we choose \( \nu \in (0, 1/6) \) sufficiently small that
\[
\nu \leq \min \left\{ \frac{(\theta(1-\tau)^2)^{n/2}}{3C_\tau}, \frac{3 - 8\sqrt{\theta}}{23} \right\},
\]
so that we have
\[
\frac{C_\tau \nu^{2/n}}{(1-\tau)^2} \leq \frac{\theta}{3} \quad \text{and} \quad \sqrt{\theta} \leq \frac{3 - 23\nu}{8}.
\]

Corresponding to this \( \nu \), we choose and fix sufficiently small \( \hat{\rho} \in (0, 1) \) satisfying
\[
\hat{\rho}^{2\beta} \leq \frac{\nu \theta}{3C_\tau (1 + F^2)},
\]
which yields \( C_\tau F^2 \hat{\rho}^{2\beta} / \nu \leq \theta / 3 \). Our settings of \( \tau, \nu, \hat{\rho} \) clearly yield (3.56).

We are left to show (3.55). By (3.5) and (3.51), we compute
\[
\int_{B_{\rho}(x_0)} |\nabla u_\varepsilon| \, dx \geq \frac{|S_{\rho, \mu, \nu}(x_0)|}{|B_{\rho}(x_0)|} \cdot \essinf_{S_{\rho, \mu, \nu}(x_0)} |\nabla u_\varepsilon| \geq (1 - \nu) \cdot \left( (1 - \nu)\mu + \frac{7}{8} \delta \right) > 0.
\]

On the other hand, by applying the triangle inequality, the Cauchy–Schwarz inequality and (3.56), it is easy to get
\[
\int_{B_{\rho}(x_0)} |\nabla u_\varepsilon| \, dx - |(\nabla u_\varepsilon)_{x_0, \rho}| = \int_{B_{\rho}(x_0)} |\nabla u_\varepsilon| - |(\nabla u_\varepsilon)_{x_0, \rho}| \, dx.
\]
\begin{align*}
\leq & \int_{B_\rho(x_0)} |\nabla u_e - (\nabla u_e)_{x_0,\rho}| \, dx \\
\leq & \sqrt{\Phi(x_0, \rho)} \leq \sqrt{\theta} \mu.
\end{align*}

Again by the triangle inequality, we obtain

\[|(\nabla u_e)_{x_0,\rho}| \geq \left| \int_{B_\rho(x_0)} |\nabla u_e| \, dx \right| - \left| \int_{B_\rho(x_0)} |\nabla u_e| \, dx - |(\nabla u_e)_{x_0,\rho}| \right| \]

\[\geq \left( (1 - \nu)^2 - \sqrt{\theta} \right) \mu + \frac{7}{8} (1 - \nu) \delta.
\]

By our setting of \(\nu\) and \((3.3)\), we can check that

\[\left( (1 - \nu)^2 - \sqrt{\theta} \right) \mu + \frac{7}{8} (1 - \nu) \delta - \left( \delta + \frac{\mu}{2} \right) = \left( \frac{1}{2} - 2\nu + \nu^2 - \sqrt{\theta} \right) \mu - \left( \frac{1}{8} + \frac{7}{8} \nu \right) \delta
\]

\[\geq \left( \frac{3 - 23\nu}{8} - \sqrt{\theta} \right) \mu \geq 0,
\]

which completes the proof of \((3.56)\).

\[\square\]

We infer to an elementary lemma on Campanato-type growth estimates (Lemma 3.13).

**Lemma 3.13.** Fix an open ball \(B_{\rho}(x_0) \subset \mathbb{R}^n\), and let \(A \in (0, \infty), \gamma \in (0, 1)\) be given constants. Assume that a function \(f \in L^2(B_{\rho}(x_0); \mathbb{R}^m)\) satisfy

\[\int_{B_{\rho}(x_0)} |f(x) - (f)_{x_0,r}|^2 \, dx \leq A^2 \left( \frac{r}{\rho} \right)^{2\gamma}\]

for all \(r \in (0, \rho]\). Then, the limit

\[F(x_0) := \lim_{r \to 0} (f)_{x_0,r} \in \mathbb{R}^m\]

exists, and there holds

\[\int_{B_{\rho}(x_0)} |f(x) - F(x_0)|^2 \, dx \leq c_\tau A^2 \left( \frac{\rho}{r} \right)^{2\gamma}\]

for all \(r \in (0, \rho]\). Here the constant \(c_\tau \in (2, \infty)\) depends at most on \(n\) and \(\gamma\).

Lemma 3.13 can be proved in a straightforward way. More precisely, we fix \(\tau \in (0, 1)\) arbitrarily, and define \(\rho_k := \tau^k \rho \in (0, \rho]\) and \(F_k := (f)_{x_0,\rho_k} \in \mathbb{R}^m\) for each \(k \in \mathbb{Z}_{\geq 0}\). Then, by the Cauchy–Schwarz inequality and \((3.57)\), it is easy to check that \(\{F_k\}_{k=0}^\infty \subset \mathbb{R}^m\) is a Cauchy sequence. Moreover, there exists a constant \(c_\tau = c_\tau(n, \gamma, \tau) \in (1, \infty)\) such that the limit \(F_\infty := \lim_{k \to \infty} F_k \in \mathbb{R}^m\) satisfies

\[\int_{B_{\rho}(x_0)} |f(x) - F_\infty|^2 \, dx \leq c_\tau A^2 \left( \frac{\rho}{r} \right)^{2\gamma}\]

for all \(r \in (0, \rho]\), from which we are able to conclude that the limit \(F(x_0)\) exists and coincides with \(F_\infty\).

Finally we give the proof of Proposition 3.2.

**Proof.** Let \(\beta = \beta(n, q)\) and \(\theta = \theta(n, p, q, \beta_0, \lambda, K, M, \delta)\) be positive constants given by \((3.2)\) and Lemma 3.6 respectively. We will later determine a sufficiently small constant \(\tau \in (0, 1/2)\), and corresponding to this \(\tau\), we will put the desired constants \(\rho_* \in (0, 1)\) and \(\nu \in (0, 1/6)\). We first assume that

\[0 < \tau < \tau_1 = \frac{1}{16}, \quad \text{and therefore} \quad \theta := \tau^{\frac{n\beta}{2}} \in \left(0, \frac{9}{64}\right)\]

\[(3.58)\]
holds. Throughout the proof, we let \( \nu \in (0, 1/6) \) and \( \hat{\rho} \in (0, 1) \) be sufficiently small constants satisfying Lemma 3.12 with \( \theta \in (0, 9/64) \) given by (3.58). Corresponding to this \( \hat{\rho} \), we assume that \( \rho_* \) is so small that there holds

\[
0 < \rho_* \leq \hat{\rho}(n, p, q, \beta_0, \lambda, K, F, M, \delta, \tau) < 1, \tag{3.59}
\]

so that Lemma 3.12 can be applied for an open ball whose radius is less than \( \rho_* \).

Let the ball \( B_{\rho}(x_0) \) satisfy \( 0 < \rho < \rho_* \), and let (3.5) hold for the constant \( \nu \in (0, 1/6) \). We set a non-negative decreasing sequence \( \{\rho_k\}_{k=0}^\infty \) by \( \rho_k := \tau^k \rho \) for \( k \in \mathbb{Z}_{\geq 0} \). We will choose suitable \( \tau \) and \( \rho_* \) such that there hold

\[
\Phi(x_0, \rho_k) \leq \tau^{2\beta k} \frac{\mu^2}{\mu^2} \tag{3.60}
\]

and

\[
|\nabla u(x_0, \rho_k)| \geq \delta + \left[ \frac{1}{2} - \frac{1}{8} \sum_{j=0}^{k-1} 2^{-j} \right] \mu \geq \delta + \frac{\mu}{4} \tag{3.61}
\]

for all \( k \in \mathbb{Z}_{\geq 0} \), which will be proved by mathematical induction. For \( k = 0, 1 \), we apply Lemma 3.12 to deduce (3.55)–(3.56) with \( \tau = \frac{\mu}{4} \). In particular, we have

\[
\Phi(x_0, \rho) \leq \tau^{\frac{\mu^2}{\mu^2}} \mu^2, \tag{3.62}
\]

and hence (3.60) is obvious when \( k = 0 \). From (3.55), we have already known that (3.61) holds for \( k = 0 \). Also, the results (3.55) and (3.62) enable us to apply Lemma 3.11 to obtain

\[
\Phi(x_0, \rho_1) \leq C_\tau \left[ \tau^2 \Phi(x_0, \rho) + \frac{L_{2\beta}}{\tau^\mu} \mu^2 \right]
\]

\[
\leq C_\tau \tau^{2(1-\beta)} \frac{\mu^2}{\mu^2} + C_\tau \frac{\mu^2}{\tau^\mu} \mu^2,
\]

where \( C_\tau = C_\tau(n, p, q, \beta_0, \lambda, K, F, M, \delta) \in (0, \infty) \) is a constant given in Lemma 3.11. Now we assume that \( \tau \) and \( \rho_* \) satisfy

\[
C_\tau \tau^{2(1-\beta)} \leq \frac{1}{3}, \tag{3.63}
\]

and

\[
C_\tau \frac{\mu^2}{\mu^2} \leq \frac{1}{3} \tau^{\tau_{\mu^2}+2\mu^2}, \tag{3.64}
\]

so that (3.60) is satisfied for \( k = 1 \). In particular, by (3.58), (3.62) and the Cauchy–Schwarz inequality, we obtain

\[
\left| (\nabla u(x_0, \rho_1) - (\nabla u(x_0, \rho_0) \right| \leq \int_{B_{\rho_1}(x_0)} |\nabla u(x_0, \rho_1) - (\nabla u(x_0, \rho_0)| \, dx
\]

\[
\leq \left( \int_{B_{\rho_1}(x_0)} |\nabla u(x_0, \rho_1) - (\nabla u(x_0, \rho_0)|^2 \, dx \right)^{1/2} = \tau \frac{\mu^2}{\mu^2} \Phi(x_0, \rho)^{1/2}
\]

\[
\leq \tau \frac{\mu^2}{\mu^2} \mu \leq \tau \frac{\mu^2}{\mu^2} \leq \frac{1}{8} \mu,
\]

where we have used (3.58) and \( 0 < \theta < 1 \). Combining this result with (3.55), we use the triangle inequality to get

\[
|(|\nabla u(x_0, \rho_1)| \geq |(|\nabla u(x_0, \rho_0)| - |(\nabla u(x_0, \rho_1) - (\nabla u(x_0, \rho_0)| \geq \left( \delta + \frac{\mu}{8} \right) - \frac{\mu}{8},
\]

which yields (3.61) for \( k = 1 \). Next, we assume that the claims (3.60)–(3.61) are valid for an integer \( k \geq 1 \). Then, the estimate \( \Phi(x_0, \rho_k) \leq \tau \frac{\mu^2}{\mu^2} \mu^2 \) clearly holds. Combining this result with (3.4) and the
induction hypothesis (3.61), we have clarified that the solution \( u_\varepsilon \) satisfies all the assumptions of Lemma 3.11 in a smaller ball \( B_{\rho_k}(x_0) \subset B_{\rho_k}(x_0) \). By Lemma 3.11 (3.63), and the induction hypothesis (3.60), we compute

\[
\Phi(x_0, \rho_{k+1}) \leq C \left[ \tau^2 \Phi(x_0, \rho_k) + \frac{\rho_k^{2\beta}}{\tau^n} \mu^2 \right] \\
= C_1 \tau^{2(1-\beta)} \cdot \tau^{2\beta k} \cdot \frac{a_{2\delta}}{\tau^n} \mu^2 + C_1 \tau^{2\beta} \cdot \frac{\rho_k^{2\beta}}{\tau^n} \mu^2 \\
\leq \tau^{2\beta(k+1)} \cdot \frac{a_{2\delta}}{\tau^n} \mu^2,
\]

which implies that (3.60) holds for \( k+1 \). By the Cauchy–Schwarz inequality and the induction hypothesis (3.61), we have

\[
\left| (\nabla u_\varepsilon)_{x_0, \rho_{k+1}} - (\nabla u_\varepsilon)_{x_0, \rho_k} \right| \leq \int_{B_{\rho_{k+1}}(x_0)} |\nabla u_\varepsilon - (\nabla u_\varepsilon)_{x_0, \rho_k}| \, dx \\
\leq \left( \int_{B_{\rho_{k+1}}(x_0)} |\nabla u_\varepsilon - (\nabla u_\varepsilon)_{x_0, \rho_k}|^2 \, dx \right)^{1/2} = \tau^{-n/2} \Phi(x_0, \rho_k)^{1/2} \\
\leq \tau^{\beta k} \cdot \frac{a_{2\delta}}{\tau^n} \mu \leq 2^{-k} \cdot \frac{1}{8} \mu,
\]

where we have used (3.58) and \( 0 < \theta < 1 \). Therefore, by the induction hypothesis (3.61) and the triangle inequality, we finally get

\[
\left| (\nabla u_\varepsilon)_{x_0, \rho_{k+1}} \right| \geq \left| (\nabla u_\varepsilon)_{x_0, \rho_k} \right| - \left| (\nabla u_\varepsilon)_{x_0, \rho_{k+1}} - (\nabla u_\varepsilon)_{x_0, \rho_k} \right| \\
\geq \delta + \left[ \frac{1}{2} - \frac{1}{8} \sum_{j=0}^{k-1} 2^{-j} \right] \mu - \frac{1}{8} \cdot 2^{-k} \mu,
\]

which implies that (3.61) is valid for \( k+1 \). This completes the proof of (3.60)–(3.61).

We would like to complete the proof of Proposition 3.2. For each \( r \in (0, \rho] \), there uniquely exists \( k \in \mathbb{Z}_{\geq 0} \) such that \( \rho_{k+1} < r \leq \rho_k \). Then, by (3.54), (3.60) and \( \tau^k \leq \tau^{-1} (r/\rho) \), we have

\[
\Phi(x_0, r) \leq \tau^{-n} \Phi(x_0, \rho_k) \leq \tau^{\frac{a_{2\delta}}{\tau^n} - n - 2\beta} \mu^2 \left( \frac{r}{\rho} \right)^{2\beta} (3.65)
\]

for all \( r \in (0, \rho] \). Recalling Lemma 2.3 and using (3.54) again, we compute

\[
\int_{B_r(x_0)} \left| G_{2\delta, \varepsilon}(\nabla u_\varepsilon) - G_{2\delta, \varepsilon}(\nabla u_\varepsilon)_{x_0, r} \right|^2 \, dx \\
\leq \int_{B_r(x_0)} \left| G_{2\delta, \varepsilon}(\nabla u_\varepsilon) - G_{2\delta, \varepsilon}(\nabla u_\varepsilon)_{x_0, r} \right|^2 \, dx \\
\leq C_1^2 \Phi(x_0, r)
\]

with \( c_1 = 1 + 64 / \sqrt{255} \). Combining this result with (3.65), by Lemma 3.13, we are able to conclude that the limit \( \Gamma_{2\delta, \varepsilon}(x_0) \in \mathbb{R}^n \) exists, and there holds

\[
\int_{B_r(x_0)} \left| G_{2\delta, \varepsilon}(\nabla u_\varepsilon) - \Gamma_{2\delta, \varepsilon}(x_0) \right|^2 \, dx \leq c_1 c_1^2 \frac{a_{2\delta}}{\tau^n} - n - 2\beta \mu^2 \left( \frac{r}{\rho} \right)^{2\beta} (3.66)
\]

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for all \( r \in (0, \rho] \). Here the constant \( c_\pm \in (2, \infty) \) depends at most on \( n \) and \( q \). Finally, we let
\[
\tau^{2(1-\theta)} \leq c_\pm^{-1} \tau^2. \tag{3.67}
\]
Then, the desired estimate (3.7) clearly follows from (3.66)–(3.67) and \( 0 < \theta < 1 \). We note that (2.11) and (3.4) imply \( \|2\phi_{\delta,e}(\nabla u_\varepsilon)\| \leq \mu \) a.e. in \( B_{\rho}(x_0) \), and therefore (3.6) is obvious.

Finally, we mention that we may choose a sufficiently small constant \( \tau = \tau(C_*, \beta) \in (0,1) \) enjoying (3.58), (3.63), and (3.67). For this fixed \( \tau \), we take sufficient small numbers \( \nu \in (0,1/6), \tilde{\beta} \in (0,1) \) as in Lemma 3.11 depending at most on \( n, p, q, \lambda, K, F, M, \delta \), and \( \theta = \tau \hat{\beta}^2 \). Then, we are able to determine a sufficiently small radius \( \rho_* = \rho_*(C_*, \tilde{\beta}, \tilde{\rho}) \in (0,1) \) satisfying (3.59) and (3.64), and this completes the proof. \( \square \)

**Remark 3.14.** An induction claim (3.61) tells us that the gradient \( \nabla u_\varepsilon \) itself may not vanish at the point \( x_0 \), and thus \( \nabla u_\varepsilon \) will satisfy the Campanato-type growth estimate (3.65) at \( x_0 \). To justify these, as well as energy estimates in Section 3.5, we have appealed to freezing coefficient arguments in Section 3.4. There the regularity assumption \( E \in C^{2,\beta}_{\operatorname{loc}}(\mathbb{R}^n \setminus \{0\}) \) is used. Also, it should be noted that the condition \( \beta < 1 \) is substantially used in (3.65), which makes our recursive proof of (3.66)–(3.67) successful. Hence, in the case \( q = \infty \), we are not allowed to let \( \beta = 1 \). Here it is worth recalling that it is generally impossible to get \( C^{1,1} \)-regularity for a weak solution to the Poisson equation \( -\Delta v = f \in L^\infty \).

### 3.7 Caccioppoli-type energy bounds

In Section 3.7, we prove Proposition 3.3 by De Giorgi’s truncation. Here we would like to show a Caccioppoli-type estimate (Lemma 3.15).

**Lemma 3.15.** Assume that \( u_\varepsilon \) is a weak solution to (2.32) in \( \Omega \). Assume that positive numbers \( \delta, \varepsilon, \mu, M, \) and an open ball \( B_\rho(x_0) \subset \Omega \) satisfy (2.11) and (3.8). Then, the scalar function \( U_{\delta, \varepsilon} \in L^\infty(B_\rho(x_0)) \cap W^{1,2}(B_\rho(x_0)) \) satisfies
\[
\int_{B_{\rho}(x_0)} (A_{\delta, \varepsilon}(\nabla U_{\delta, \varepsilon}) \nabla U_{\delta, \varepsilon} \cdot \nabla \zeta) \, dx \leq C_0 \left[ \int_{B_{\rho}(x_0)} |f_\varepsilon|^2 \zeta \, dx + \mu \int_{B_{\rho}(x_0)} |f_\varepsilon| |\nabla \zeta| \, dx \right] \tag{3.68}
\]
for any non-negative function \( \zeta \in W^{1,\infty}(B_\rho(x_0)) \) that is compactly supported in \( B_\rho(x_0) \). Here the constant \( C_0 \in (0, \infty) \) depends at most on \( n, p, \lambda, \Lambda, M, \) and \( \delta \). The matrix-valued function \( A_{\delta, \varepsilon}(\nabla U_{\delta, \varepsilon}) \) satisfies
\[
\lambda_* \mathrm{id} \leq A_{\delta, \varepsilon}(\nabla U_{\delta, \varepsilon}) \leq \Lambda_* \mathrm{id} \quad \text{in } B_\rho(x_0), \tag{3.69}
\]
where the constants \( 0 < \lambda_* \leq \Lambda_* < \infty \) depend at most on \( p, \lambda, \Lambda, K, M, \) and \( \delta \), but are independent of \( \varepsilon \). In particular, we have
\[
\int_{B_{\rho}(x_0)} |\nabla \eta(U_{\delta, \varepsilon} - k, \varepsilon)|^2 \, dx \leq C \left[ \int_{B_{\rho}(x_0)} |\nabla \eta|^2(U_{\delta, \varepsilon} - k)^2 \, dx + \mu^2 \int_{A_{k, \rho}(x_0)} |f_\varepsilon|^2 \eta^2 \, dx \right] \tag{3.70}
\]
for all \( k \in (0, \infty) \) and for any non-negative function \( \eta \in C^1_\varepsilon(B_{\rho}(x_0)) \). Here \( A_{k, \rho}(x_0) := \{ x \in B_{\rho}(x_0) \mid U_{\delta, \varepsilon}(x) > k \} \), and the constant \( C \in (0, \infty) \) depends on \( \lambda_*, \Lambda_*, \) and \( C_0 \).

Lemma 3.15 implies that a function \( U_{\delta, \varepsilon} \) is a subsolution in the sense that the inequality (3.68) always hold true for any non-negative test function \( \zeta \). The key point is to make sure that the coefficient matrix \( A_{\delta, \varepsilon}(\nabla U_{\delta, \varepsilon}) \) in (3.66) is uniformly elliptic. This is possible because the function \( U_{\delta, \varepsilon} \) is supported in \( \{ \varepsilon > \delta \} \), where the ellipticity ratio of \( \nabla^2 U_{\delta, \varepsilon}(\nabla U_{\delta, \varepsilon}) \) becomes bonded, uniformly for \( \varepsilon \in (0, \delta/8) \). It should be noted again that this uniform ellipticity is substantially dependent on \( \delta \in (0,1) \).
Proof. We write \( B = B_\rho(x_0) \), \( A_k = A_{k,\rho}(x_0) \) for notational simplicity. The assumption \((3.3)\) clearly yields
\[
U_{\delta, \varepsilon} = |\varphi_{\delta, \varepsilon}(\nabla u_\varepsilon)|^2 \leq \mu^2 \quad \text{a.e. in } B_\rho(x_0).
\]
We define a set \( A := \{ x \in B \mid V_\varepsilon(x) > \delta \} \subset B \). We claim that the desired matrix \( A_{\delta, \varepsilon}(\nabla u_\varepsilon) \) is given by
\[
A_{\delta, \varepsilon}(\nabla u_\varepsilon) := \chi_{A} V_\varepsilon \cdot \nabla^2 E_\varepsilon(\nabla u_\varepsilon) + \chi_{B \setminus A} \cdot \text{id}.
\]
Here for a measurable set \( E \subset B \), the function \( \chi_{E} : B \to \{ 0, 1 \} \) denotes the characteristic function of \( E \), i.e.,
\[
\chi_{E}(x) := \begin{cases} 
1 & (x \in E), \\
0 & (x \in B \setminus E).
\end{cases}
\]
We prove this claim by applying Lemma \((3.5)\) with \( \psi(t) := (t-\delta)_+ (0 \leq t < \infty) \). Under this setting, there holds \( \psi(V_\varepsilon) = |\varphi_{\delta, \varepsilon}(\nabla u_\varepsilon)| \). Therefore, we recall Remark \((3.4)\) and use \((3.8)\) to get
\[
\begin{aligned}
\psi(V_\varepsilon)V_\varepsilon = & \frac{|\varphi_{\delta, \varepsilon}(\nabla u_\varepsilon)||V_\varepsilon|}{V_\varepsilon^{p-1}} \leq \frac{\mu M}{\delta^{p-1}}, \\
\frac{V_\varepsilon^2}{V_\varepsilon^{p-1} \cdot \psi'(V_\varepsilon)} & \leq M^{\frac{2}{p-1}} \cdot \chi_{A} \leq \frac{M^2}{\delta^{p-1}}, \\
\psi(V_\varepsilon)V_\varepsilon & \leq \mu M,
\end{aligned}
\]
a.e. in \( B \). Hence, we obtain
\[
J_4 + J_5 + J_6 \leq C(n, p, M, \delta) \left( \int_{B} |f_\varepsilon|^2 \zeta \, dx + \mu \int_{B} |f_\varepsilon| |\nabla \zeta| \, dx \right).
\]
Since the function \( U_{\delta, \varepsilon} \) vanishes on \( B \setminus A \) and the identity \( \nabla U_{\delta, \varepsilon} = 2\psi(V_\varepsilon)\nabla V_\varepsilon \) holds, we obtain
\[
J_1 = \int_{A} \left( \nabla^2 E_\varepsilon(\nabla u_\varepsilon) \nabla V_\varepsilon \mid \nabla \zeta \right) \psi(V_\varepsilon)V_\varepsilon \, dx = \frac{1}{2} \int_{B} \left( A_{\delta, \varepsilon}(\nabla u_\varepsilon) \nabla U_{\delta, \varepsilon} \mid \nabla \zeta \right) \, dx.
\]
We also note that the integrals \( J_2, J_3 \) defined by \((3.25)\) are non-negative by \((3.27)\). Hence by discarding these integrals, we are able to conclude \((3.68)\). Over the set \( A \subset B \), we have
\[
V_\varepsilon \cdot \nabla^2 E_\varepsilon(\nabla u_\varepsilon) \geq \Lambda V_\varepsilon^{p-1} \cdot \text{id} \geq \lambda \delta^{p-1} \cdot \text{id}, \quad \text{and}
\]
\[
V_\varepsilon \cdot \nabla^2 E_\varepsilon(z) \leq \left( \Lambda V_\varepsilon^{p-1} + K \right) \cdot \text{id} \leq \left( \Lambda M^{p-1} + K \right) \cdot \text{id}.
\]
Here we have used \((3.4)\). Therefore, by setting
\[
\begin{aligned}
\lambda_*(p, \lambda, \delta) & := \min\{ 1, \lambda \delta^{p-1} \}, \\
\Lambda_*(p, \Lambda, K, M) & := \max\{ 1, \Lambda M^{p-1} + K \},
\end{aligned}
\]
we have \((3.69)\).

By approximation, we may test non-negative function \( \zeta := \eta^2 (U_{\delta, \varepsilon} - k)_+ \in W^{1,2}_0(B) \) into \((3.68)\), since it is compactly supported. Here we note that \( \zeta \) vanishes in \( B \setminus A_k \), and there holds \( (U_{\delta, \varepsilon} - k)_+ \leq U_{\delta, \varepsilon} \leq \mu^2 \) in \( B \). Hence by \((3.69)\) and Young’s inequality, we have
\[
\lambda_* \int_{B} |\nabla (U_{\delta, \varepsilon} - k)_+|^2 \eta^2 \, dx
\]

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We choose the constants $\hat{\varepsilon}$

**Proof.**

Lemma 3.16. Let $u_\varepsilon$ be a weak solution to (2.89) in $\Omega$. Assume that positive numbers $\delta, \varepsilon, \mu, F, M,$ and an open ball $B_\rho(x_0) \subseteq \Omega$ satisfy (2.71), (2.90) and (3.8). Then, there exists a number $\hat{\varepsilon} \in (0, 1)$, depending at most on $n, p, q, \lambda, K, M,$ and $\delta$, such that if there holds

$$\frac{|\{x \in B_{\rho/2}(x_0) \mid U_{\delta, \varepsilon}(x) > (1 - \theta)\mu^2\}|}{|B_{\rho/2}(x_0)|} \leq \hat{\varepsilon},$$

(3.72)

for some $\theta \in (0, 1)$, then we have either

$$\mu^2 < \frac{\rho^\beta}{\theta} \quad \text{or} \quad \text{ess sup}_{B_{\rho/2}(x_0)} U_{\delta, \varepsilon} \leq \left(1 - \frac{\theta}{2}\right)\mu^2.$$

Lemma 3.17. Under the assumptions of Proposition 3.3 for every integer $i_* \in \mathbb{N}$, we have either

$$\mu^2 < \frac{2i_\varepsilon \rho^\beta}{\nu} \quad \text{or} \quad \frac{|\{x \in B_{\rho/2}(x_0) \mid U_{\delta, \varepsilon}(x) > (1 - 2^{-i_*}\nu)\mu^2\}|}{|B_{\rho/2}(x_0)|} < \frac{C_\nu}{\nu \sqrt{i_*}}.$$

Here the constant $C_\nu \in (0, \infty)$ depends at most on $n, p, q, \lambda, K, F, M,$ and $\delta$.

Following standard arguments as in [17, Chapter 10, Propositions 4.1 & 5.1], it is easy to prove Lemmata 3.16, 3.17. For the reader’s convenience, we would like to provide the proofs in the appendix (Section 4).

Combining these lemmata, we give the proof of Proposition 3.3.

**Proof.** We choose the constants $\hat{\varepsilon} \in (0, 1)$ and $C_\hat{\varepsilon} \in (0, \infty)$, depending at most on $n, p, q, \lambda, K, F, M,$ and $\delta$, as in Lemma 3.16, 3.17. Corresponding to these numbers, we fix a sufficiently large number $i_* = i_*(C_\nu, \nu, \hat{\varepsilon}) \in \mathbb{N}$ enough to verify

$$\frac{C_\nu}{\nu \sqrt{i_*}} \leq \hat{\varepsilon}, \quad \text{and} \quad 0 < 2^{-(i_*+1)} < 1 - 2^{-2\beta}.$$

(3.73)

Then, the desired constants are given by $C_\nu := 2i_\varepsilon / \nu \in [1, \infty), \kappa := \sqrt{1 - 2^{-(i_*+1)}} \in (2^{-\beta}, 1)$. In fact, by Lemma 3.17 and (3.73), we have either $\mu^2 < 2i_\varepsilon \rho^\beta / \nu = C_\nu \rho^\beta$ or (3.72) with $\theta := 2^{-i_*} \nu$. The first
case clearly yields \(3.10\). The second case enables us to apply Lemma \(3.16\) and hence we have either \(\mu^2 < \rho^2 / \theta = C \rho^2\) or
\[
\text{ess sup}_{B_{\rho_i}(x_0)} [S_{\delta, \epsilon} (\nabla u_\epsilon)]^2 \leq \text{ess sup}_{B_{\rho_i}(x_0)} U_{\delta, \epsilon} \leq \left(1 - \frac{\theta}{2}\right) \mu^2 = (\kappa \mu)^2.
\]
In all the possible cases, we obtain either \(3.10\) or \(3.11\).

\section*{4 Appendix: Proofs for some basic estimates}

In Section \(4\) we would like to provide the proofs Lemmata \(3.16\)–\(3.17\) in Section \(3.7\) for the reader’s convenience. These results can be deduced by the fact that the function \(U_{\delta, \epsilon}\) satisfies Caccioppoli-type energy bounds since it is a weak subsolution to a uniformly elliptic problem (Lemma \(3.15\)). We mention that the strategy herein are based on De Giorgi’s levelset argument given in \(17\) Chapter 10, §4–5. See also \(10\) §7 as a related item.

The proof of Lemma \(3.16\) is substantially found in \(17\) Chapter 10, Proposition 4.1).

\begin{proof}
For every \(i \in \mathbb{Z}_{\geq 0}\), we set
\[
\rho_i := \frac{\rho}{4} (1 + 2^i), \quad k_i := \left(1 - \frac{1}{2} (1 + 2^i \theta)\right) \mu^2,
\]
and define measurable sets \(B_i := B_{\rho_i}(x_0), A_i := \{x \in B_i \mid U_{\delta, \epsilon} (x) > k_i\}\).

To show Lemma \(3.16\) we will find constants \(C = C(n, p, q, \lambda, \Lambda, K, M, \delta, \theta) \in [1, \infty)\) and \(\zeta = \zeta(n, q) \in (0, 2\beta/n]\) such that there holds
\[
R_{i+1} \leq \frac{C}{2} \cdot 4^{2i} \left[1 + \frac{L^2 \mu}{\theta^2 \mu^4}\right] R_i^{1+\zeta}, \quad \text{where } R_i := \frac{|A_i|}{|B_i|}, \tag{4.1}
\]
for every \(i \in \mathbb{Z}_{\geq 0}\). For fixed \(i \in \mathbb{Z}_{\geq 0}\), we choose a cutoff function \(\eta \in C_0^1(B_i)\) satisfying
\[
\eta \equiv 1 \text{ on } B_{i+1}, \quad |\nabla \eta| \leq \frac{2}{\rho_i - \rho_{i+1}} = \frac{2^{i+4}}{\rho}, \tag{4.2}
\]
We also note that
\[
\begin{cases}
(U_{\delta, \epsilon} - k_i)_{+} \geq k_{i+1} - k_i = 2^{-i+2} \theta \mu^2 & \text{a.e. in } A_{i+1}, \tag{4.3} \\
(U_{\delta, \epsilon} - k_i)_{+} \leq \mu^2 - k_i \leq \theta \mu^2 & \text{a.e. in } B_{\rho_i}(x_0),
\end{cases}
\]
and
\[
\frac{|B_i|}{|B_{i+1}|} = \left(\frac{\rho_i}{\rho_{i+1}}\right)^n = \left(\frac{1 + 2^{-i}}{1 + 2^{-(i+1)}}\right)^n = \left(\frac{1}{1 + 2^{-i}}\right)^n \leq 2^n \tag{4.4}
\]
for every \(i \in \mathbb{Z}_{\geq 0}\). We first consider the case \(n \geq 3\). In this case, we may apply the Sobolev embedding \(W_0^{1,2}(B_i) \hookrightarrow L^{2n/3}(B_i)\) to the function \(\eta(U_{\delta, \epsilon} - k_i)_{+} \in W_0^{1,2}(B_i)\). By \(4.2\)–\(4.3\), Hölder’s inequality and the Caccioppoli-type estimate \(3.70\), we have
\[
\begin{aligned}
|A_{i+1}| \cdot 4^{-(i+2)} \theta^2 \mu^4 & \leq \int_{A_{i+1}} [\eta(U_{\delta, \epsilon} - k_i)_{+}]^2 \, dx \\
& \leq C(n) |A_i|^{\frac{1}{2}} \int_{B_i} |\nabla [\eta(U_{\delta, \epsilon} - k_i)_{+}^2] |^2 \, dx
\end{aligned}
\]

\end{proof}
\[
\begin{align*}
&\leq C |A_i|^{\frac{2}{n}} \int_{B_i} |\nabla \eta|^2 (U_{\delta, e} - k)_e^2 \, dx + \mu^2 \int_{A_i} |f_e|^2 \eta^2 \, dx \\
&\leq C |A_i|^{\frac{2}{n}} \frac{4^{i+4}}{\rho^2} \cdot (\theta \mu)^2 \cdot |A_i| + F^2 |A_i|^{1 - \frac{2}{\rho^2}} \\
&\leq C \cdot 4^i \theta^2 \mu^4 |A_i|^{1 + \frac{2}{n}} \left[ \frac{|A_i|^{\frac{2}{n}(1 - \beta)}}{\rho^2} + \frac{1}{\theta^2 \mu^2} \right],
\end{align*}
\]

which yields
\[
|A_{i+1}| \leq C \cdot 4^i |A_i|^{1 + \frac{2}{n}} \left[ \frac{|A_i|^{\frac{2}{n}(1 - \beta)}}{\rho^2} + \frac{1}{\theta^2 \mu^2} \right]
\]

for some constant \( C = C(n, p, q, \lambda, \Lambda, K, F, M, \delta) \in (0, \infty) \). From this estimate and (4.4), the desired inequality (4.1) follows with \( \varsigma = 2\beta/n \). In fact, we obtain
\[
R_{i+1} \leq 2^n \cdot \frac{|A_{i+1}|}{|B_i|^{1 + \varsigma}} \cdot |B_i|^{\varsigma} \\
\leq C \cdot 4^i \cdot \left[ \frac{|A_i|^{\frac{2}{n}(1 - \beta)}}{\rho^2} + \frac{1}{\theta^2 \mu^2} \right] |B|^\varsigma \cdot \left( \frac{|A_i|}{|B_i|} \right)^{1 + \varsigma} \\
\leq C \cdot 4^i \cdot \left[ 1 + \frac{\rho^{2\beta}}{\theta^2 \mu^2} \right] R_i^{1 + \varsigma},
\]

for every \( i \in \mathbb{Z}_{\geq 0} \),

where we have used \( |A_i| \leq |B_i| = C(n) \rho^n \leq C(n) \rho^n \). In the remaining case \( n = 2 \), we choose a sufficiently large constant \( \sigma \in (1, \infty) \) satisfying
\[
\varsigma := \beta - \frac{1}{\sigma} = 1 - 2 \frac{1}{q} - \frac{1}{\sigma} > 0,
\]

and apply the alternative Sobolev embedding \( W_0^{1,2}(B_i) \hookrightarrow L^{2\sigma}(B_i) \) to the function \( \eta(U_{\delta, e} - k)_e \in W_0^{1,2}(B_i) \). Then, we similarly obtain
\[
|A_{i+1}| \cdot 4^{-(i+2)} \theta^2 \mu^4 \leq \int_{A_{i+1}} \left( \eta(U_{\delta, e} - k)_e \right)^2 \, dx \\
\leq |A_i|^{1 - \frac{2}{n}} \int_{B_i} \left( \eta(U_{\delta, e} - k)_e \right)^{2\sigma} \, dx \quad \begin{aligned}
&\leq C(n, \sigma) |A_i|^{1 - \frac{2}{n}} \cdot \rho_i^{\frac{2}{\sigma}} \int_{B_i} \left| \nabla \eta \right| (U_{\delta, e} - k)_e^2 \, dx \\
&\leq C |A_i|^{1 - \frac{2}{n}} \cdot \rho_i^{\frac{2}{\sigma}} \int_{B_i} \left[ \left| \nabla \eta \right|^2 (U_{\delta, e} - k)_e^2 \, dx + \mu^2 \int_{A_i} |f_e|^2 \eta^2 \, dx \right] \\
&\leq C |A_i|^{1 - \frac{2}{n}} \cdot \rho_i^{\frac{2}{\sigma}} \left[ \frac{4^{i+4}}{\rho^2} \cdot \left( \theta \mu \right)^2 \cdot |A_i| + F^2 |A_i|^{1 - \frac{2}{\rho^2}} \right] \\
&\leq C \cdot 4^i \theta^2 \mu^4 |A_i|^{2 - \frac{2}{\sigma} - \frac{2}{\rho^2}} \cdot \rho_i^{\frac{2}{\sigma}} \left[ \frac{|A_i|^{\frac{2}{n}(1 - \beta)}}{\rho^2} + \frac{1}{\theta^2 \mu^2} \right].
\end{aligned}
\]

Recall (4.5), and we get
\[
|A_{i+1}| \leq C \cdot 4^i |A_i|^{1 + \varsigma} \cdot \rho_i^{\frac{2}{\sigma}} \left[ \frac{|A_i|^{\frac{2}{n}(1 - \beta)}}{\rho^2} + \frac{1}{\theta^2 \mu^2} \right].
\]
for some \( C = C(n, p, q, \lambda, \Lambda, K, M, \delta) \in (0, \infty) \). The desired estimate (4.1) follows from this estimate with \( \varsigma \) given by (4.5). In fact, by (4.4)–(4.5) we obtain

\[
R_{i+1} \leq 2^2 \left| \frac{A_{i+1}}{|B_i|^{1+\varsigma}} \cdot |B_i|^\varsigma \right|
\leq C \cdot 2^{2i} \left[ \frac{|A_i|^{1-\mu}}{\rho^2 \mu^4} + 1 \right] \left( \rho_i \rho_i^2 |B|^\varsigma \cdot \frac{|A_i|}{|B_i|} \right)^{1+\varsigma}
\leq C_s \cdot \frac{2}{2^{2i}} \left[ 1 + \frac{\rho_i^2 \mu^2}{\theta^2 \mu^4} \right] R_i^{1+\varsigma}
\]

for every \( i \in \mathbb{Z}_0 \).

where we have used \( |A_i| \leq |B_i| = C \rho_i^n \leq C \rho^2 \) and \( \rho_i \rho_i^2 |B|^\varsigma \leq C \rho_i^{2\theta} \).

Now we determine the constant \( \hat{\nu} = \hat{\nu}(n, p, q, \lambda, \Lambda, K, F, M, \delta) \in (0, 1) \) by

\[
\hat{\nu} := C_s^{-\frac{1}{\gamma}} 16^{-\frac{1}{\gamma}}.
\]

Let \( \mu \) satisfy \( \mu \geq \rho^\mu / \theta \). Then, by (3.7) and (4.1), we have already known that

\[
R_{i+1} \leq C_s \cdot 16^i \cdot R_i^{1+\varsigma}
\]

for every \( k \in \mathbb{Z}_0 \), and \( R_0 \leq \hat{\nu} \). By applying [30] Chapter 2, Lemma 4.7 to the sequence \( \{R_i\}_{i=0}^\infty \), we are able to conclude \( R_i \to 0 \) as \( i \to \infty \). From this, it follows that

\[
0 \leq \left| \left\{ x \in B_{\rho/4}(x_0) \mid U_{\delta, \epsilon}(x) > (1-\theta/2)\mu^2 \right\} \right| \leq \liminf_{i \to \infty} |A_i| = 0.
\]

Hence we have

\[
\esssup_{B_{\rho/4}(x_0)} U_{\delta, \epsilon} \leq \left( 1 - \frac{\theta}{2} \right) \mu^2.
\]

which completes the proof. \( \square \)

The proof of Lemma 3.17 is substantially based on [17] Chapter 10, Proposition 5.1. Here we use a well-known inequality;

\[
(l-k) \cdot |B_{\rho} \cap \{v < k\}| \leq \frac{C(n) \rho^{n+1}}{|B_{\rho} \cap \{v > l\}|} \int_{B_{\rho} \cap \{k \leq v < l\}} |\nabla v| \, dx
\]

(4.6)

for all \( v \in W^{1,1}(B_{\rho}) \) and \(-\infty < k < l < \infty \) (see [17] Chapter 10, §5.1 for the proof).

**Proof.** For each \( i \in \mathbb{Z}_0 \), we set a measurable set

\[
A_i := \left\{ x \in B_{\rho/2}(x_0) \mid U_{\delta, \epsilon}(x) > k_i \right\}
\]

with \( k_i := (1-2^{-i} \nu) \mu^2 \).

By (3.9), it is easy to check that

\[
|B_{\rho/2}(x_0) \setminus A_i| \geq |B_{\rho/2}(x_0) \setminus A_0| \geq v |B_{\rho/2}(x_0)| = C(n) \nu \rho^n
\]

(4.7)

for every \( i \in \mathbb{Z}_0 \). We apply (4.6) to the function \( U_{\delta, \epsilon} \in W^{1,1}(B_{\rho/2}(x_0)) \) with \( (k, l) = (k_i, k_{i+1}) \). By the Cauchy–Schwarz inequality and (4.7), we have

\[
\nu \mu^2 2^{i+1} |A_{i+1}| \leq \frac{C(n) \rho^{n+1}}{|B_{\rho/2}(x_0) \setminus A_i|} \int_{A_i \setminus A_{i+1}} |\nabla U_{\delta, \epsilon}| \, dx.
\]
In particular, by (4.8)–(4.9), we are able to find a constant \( \Lambda \)
\[
\therefore \quad |A_{i+1}|^2 \leq \frac{C(n)\rho^2}{v^2 \mu^2} \left( |A_i| - |A_{i+1}| \right) \int_{B_{\rho/2}(x_0)} |\nabla(U_{\delta,e} - k_i)_+|^2 \, dx
\]
for every \( i \in \mathbb{Z}_{\geq 0} \). We choose a cutoff function \( \eta \in C_c^1(B_{\rho}(x_0)) \) such that
\[
\eta \equiv 1 \text{ on } B_{\rho/2}(x_0), \quad |\nabla \eta| \leq \frac{4}{\rho},
\]
and apply Lemma 5.15. Then, for every \( i \in \mathbb{Z}_{\geq 0} \), we have
\[
\int_{B_{\rho/2}(x_0)} |\nabla(U_{\delta,e} - k_i)_+|^2 \, dx
\]
\[
\leq \int_{B_{\rho}(x_0)} |\nabla \eta(U_{\delta,e} - k_i)_+|^2 \, dx
\]
\[
\leq C \left[ \int_{B_{\rho}(x_0)} |\nabla \eta|^2 (U_{\delta,e} - k_i)_+^2 \, dx + \mu^2 \int_{A_i} |f_{\varepsilon}|^2 \eta^2 \, dx \right]
\]
\[
\leq \frac{C}{\rho^2} \left[ \frac{v^2 \mu^4}{4^2} |B_{\rho}(x_0)| + F^2 \rho^{n+2}\beta \right] \leq \frac{C v^2 \mu^4}{4^2 \rho^2} \left[ \frac{2^i \rho^\beta}{v \mu^2} \right]^2 + 1 \cdot |B_{\rho/2}(x_0)|
\]
for some constant \( C = C(n, p, q, \lambda, \Lambda, K, F, M, \delta) \in (0, \infty) \). Here we have used Hölder’s inequality and 
\[
(U_{\delta,e} - k_i)_+ \leq \mu^2 - k_i = 2^{-i} v \mu^2 \quad \text{a.e. in } B_{\rho}(x_0).
\]
Take and fix \( i_* \in \mathbb{N} \) arbitrarily, and assume that \( \mu \) satisfies \( \mu^2 \geq 2^{i_*} \rho^\beta / v \). Then, for every \( i \in \{0, \ldots, i_* - 1\} \), there clearly holds
\[
\frac{2^i \rho^\beta}{v \mu^2} \leq \frac{2^{i_*} \rho^\beta}{v \mu^2} \leq 1.
\]
In particular, by (4.8)–(4.9), we are able to find a constant \( C_\perp \in (1, \infty) \), depending at most on \( n, p, q, \lambda, \Lambda, K, F, M, \) and \( \delta \), such that
\[
|A_{i+1}|^2 \leq \frac{C_\perp^2}{v^2} |B_{\rho/2}(x_0)| \cdot (|A_i| - |A_{i+1}|)
\]
holds for each \( i \in \{0, \ldots, i_* - 1\} \). This estimate yields
\[
i_* |A_{i_*}|^2 \leq \sum_{i=0}^{i_*-1} |A_{i+1}|^2
\]
\[
\leq \frac{C_\perp^2}{v^2} |B_{\rho/2}(x_0)| \sum_{i=0}^{i_*-1} (|A_i| - |A_{i+1}|)
\]
\[
\leq \frac{C_\perp^2}{v^2} |B_{\rho/2}(x_0)| \cdot |A_0| \leq \frac{C_\perp^2}{v^2} |B_{\rho/2}(x_0)|^2.
\]
Hence we obtain
\[
|A_{i_*}| \leq \frac{C_\perp}{v \sqrt{i_*}} |B_{\rho/2}(x_0)|,
\]
which completes the proof. \( \square \)
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