GLOBAL SOLVABILITY TO A SINGULAR CHEMOTAXIS-CONSUMPTION MODEL WITH FAST AND SLOW DIFFUSION AND LOGISTIC SOURCE

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Abstract. In this paper, we consider the following chemotaxis-consumption model with porous medium diffusion and singular sensitivity

$$\begin{cases}
    u_t = \Delta u^m - \chi \text{div}(\frac{u}{v} \nabla v) + \mu u(1 - u), \\
v_t = \Delta v - u^r v,
\end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) with zero-flux boundary conditions. It is shown that if $r < \frac{4}{N+2}$, for arbitrary case of fast diffusion ($m \leq 1$) and slow diffusion ($m > 1$), this problem admits a locally bounded global weak solution. It is worth mentioning that there are no smallness restrictions on the initial datum and chemotactic coefficient.

1. Introduction. In this paper, we investigate the global existence of weak solutions for the following chemotaxis model with singular sensitivity

$$\begin{cases}
    u_t = \Delta u^m - \chi \text{div}(\frac{u}{v} \nabla v) + \mu u(1 - u), \\
v_t = \Delta v - u^r v, \\
\frac{\partial u^m}{\partial n} \bigg|_{\partial \Omega} = \frac{\partial v}{\partial n} \bigg|_{\partial \Omega} = 0, \\
u(x, 0) = u_0(x), v(x, 0) = v_0(x),
\end{cases}$$

where $Q = \Omega \times \mathbb{R}^+$, $\Omega$ is a bounded domain of $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary, $m > 0$, $\chi > 0$, $\mu > 0$, $0 < r < \frac{4}{N+2}$. Here $u$, $v$ represent the cell density,

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signal concentration respectively, the chemotactic sensitivity is represented by $\chi$, the consumption of the signal is given by $-vu^r$, the proliferation or death of cells according to the logistic law $\mu u(1-u)$ with $\mu > 0$.

Throughout this paper, we assume that
\[
\begin{cases}
u_0^m \in L^\infty(\Omega) \cap W^{1,2}(\Omega), v_0 \in W^{2,p}(\Omega) & \text{for any } p > 1, \\
u_0 \geq 0, v_0 > 0 & \text{in } \Omega.
\end{cases}
\]

(2)

A classical chemotaxis model was proposed in 1971 by Keller and Segel [7],
\[
\begin{cases}
\frac{\partial u}{\partial t} = \Delta u - \chi \text{div}(\frac{u}{v} \nabla v), \\
\frac{\partial v}{\partial t} = \Delta v - vu.
\end{cases}
\]

(3)

This model has successfully been employed to find travelling wave solutions [11, 14, 22]. In fact, it had been observed that when bacteria are placed in one end of a capillary tube containing oxygen and an energy source, the motile Escherichia coli form a band [7], which travels at a constant speed due to chemotaxis, and the analysis shows that a traveling band will form only if chemotaxis is sufficiently strong. Recently, the global well-posedness of solutions for initial and boundary value problems also attracted a lot of people’s attention. More precisely, the global existence and large time behavior of solutions in one dimensional setting have been studied by some authors [13, 17]. Whereas in higher dimensional spaces, the global classical solutions and convergence near a constant steady state are established in [15], the global renormalized solutions are studied in [9, 20], recently, the global classical solution for the system (3) with a logistic growth term also be established in [10]. Besides the global solvability, the boundary layer theory of the system (3) also be studied by some authors, for more details, please refer to [3, 4]. On the other hand, the following chemotaxis model with singular sensitivity and logistic logistic growth term,
\[
\begin{cases}
\frac{\partial u}{\partial t} = \Delta u - \chi \text{div}(\frac{u}{v} \nabla v) + \kappa u - \mu u^2, \\
\frac{\partial v}{\partial t} = \Delta v - vu
\end{cases}
\]
or a generalized consumption term
\[
\begin{cases}
\frac{\partial u}{\partial t} = \Delta u - \chi \text{div}(\frac{u}{v} \nabla v), \\
\frac{\partial v}{\partial t} = \Delta v - vu^r
\end{cases}
\]
are also studied by some authors. The global classical solution is obtained for large $\chi$ and small $\mu$ in $n$-dimensional space $(n \geq 2)$, or for $r < 1$ with small $\chi$ in two dimensional space [12, 18].

However, considering the actual biological phenomenon, the migration rate of bacteria is related to the density. Thus some researchers are led to consider the following chemotaxis system with porous medium diffusion and singular sensitivity,
\[
\begin{cases}
\frac{\partial u}{\partial t} = \Delta u^m - \chi \text{div}(\frac{u}{v} \nabla v), \\
\frac{\partial v}{\partial t} = \Delta v - uv
\end{cases}
\]
for which, a global weak solution is established for the slow diffusion case $m > 1 + \frac{N}{4}$ with $N \geq 2$ [8], and a ‘very’ weak solution is obtained for $m > 1 + \frac{N-2}{2N}$ with $N \geq 2$ [23].
In addition to the chemotaxis-consumption model, there are also a lot of papers contributed to the chemotaxis-production system with singular sensitivity, i.e.,

$$
\begin{align*}
  u_t &= \Delta u - \chi \text{div}(\frac{u}{v} \nabla v), \\
  v_t &= \Delta v - v + u,
\end{align*}
$$

(4)

the global classical solution and uniform boundedness is established for small \( \chi \) in \([1, 21]\) for arbitrary dimensions. In addition, in \([2, 24]\), the authors also investigated the global existence of solutions for the system (4) with a logistic growth term.

To overcome the difficulty caused by the singularity of \( \chi \), we consider a regularized problem, and the first key step is to use the \( L^2 \)-norm estimate of cells to complete the lower bound estimate of chemoattractant. According to the lower bound estimation of chemoattractant, we then use the consumption term and the logistic growth term to control the chemotaxis term, and the local boundedness of the solutions is finally established.

The rest of this paper is organized as follows. In Section 2, we will give the definition of weak solution, and the lower bound estimation of chemoattractant is proved by logarithmic transformation. Then, we give the proof of main result in the Section 3.

2. Lower bound estimate of chemoattractant to the regularized problem.

The following notations will be used in the whole article.

**Notations:** \( Q_T = \Omega \times (0, T) \), \( \| \cdot \|_{L^p} = \| \cdot \|_{L^p(\Omega)} \).

Next, we give the definition of weak solutions to the problem (1).

**Definition 2.1.** \((u, v)\) is called a weak solution of (1), if \( u \in D_1, v \in D_2 \), such that

\[
- \int_{Q_T} u \varphi_t dxdt - \int_\Omega u(x, 0)\varphi(x, 0)dx + \int_{Q_T} (\nabla u^m - \chi \frac{u}{v} \nabla v) \nabla \varphi dxdt \\
= \mu \int_{Q_T} u(1 - u) \varphi dxdt,
\]

\[
- \int_{Q_T} v \varphi_t dxdt - \int_\Omega v(x, 0)\varphi(x, 0)dx + \int_{Q_T} \nabla v \nabla \varphi dxdt + \int_{Q_T} u^r v \varphi dxdt = 0
\]

for any \( \varphi \in C^\infty(Q_T) \) with \( \frac{\partial \varphi}{\partial n} \big|_{\partial \Omega} = 0, \varphi(x, T) = 0 \), where

\[
D_1 = \{ u \in L^\infty(Q_T); \nabla u^m \in L^\infty((0, T); L^2(\Omega)), \left(u^{r+2}\right)_t, \nabla u^{r+2} \in L^2(Q_T), \forall T > 0 \},
\]

\[
D_2 = \{ v \in L^\infty((0, T); W^{1, \infty}(\Omega)); v_0, D^2_v \in L^p(Q_T), \forall T > 0, p > 1 \}.
\]

In this sense, we get the following theorem.

**Theorem 2.2.** Assume that \( m > 0, r < \frac{4}{N+2}, \) and (2) holds. Then the problem (1) admits nonnegative global weak solutions \((u, v)\) with \( u \in D_1, v \in D_2 \), such that for any \( T > 0 \),

\[
\sup_{t \in (0, T)} \| u(\cdot, t) \|_{L^\infty} + \| v(\cdot, t) \|_{W^{1, \infty}} \leq M_1,
\]

(5)

\[
\sup_{t \in (0, T)} \int_\Omega |\nabla u^m|^2 dx + \int_0^T \int_\Omega \left( u^{m-1} \left| \frac{\partial u}{\partial t} \right|^2 + u^{m-2} |\nabla u|^2 \right) dxdt \leq M_2,
\]

(6)

\[
\| v \|_{W^{2,1}_p(Q_T)} \leq M_3, \text{ for any } p > 1,
\]

(7)

where \( M_i (i = 1, 2, 3) \) depend only on \( \chi, m, \mu, r, u_0, v_0 \) and \( T \), and \( M_5 \) also depends on \( p \).
To research the global solvability of weak solutions, we consider the following regularized problem

\[
\begin{aligned}
&u_{tt} = \Delta (u_t + \varepsilon) - \chi \nabla \cdot \left( \frac{u_t}{\rho_\varepsilon(v_\varepsilon)} \nabla v_\varepsilon \right) + \mu u_\varepsilon (1 - u_\varepsilon), \quad \text{in } Q, \\
v_{tt} = \Delta v_\varepsilon - u_\varepsilon^r v_\varepsilon, \quad &\text{in } Q, \\
\left. \frac{\partial u_\varepsilon}{\partial n} \right|_{\partial \Omega} = \left. \frac{\partial v_\varepsilon}{\partial n} \right|_{\partial \Omega} = 0, \\
u_\varepsilon(x, 0) = u_{\varepsilon 0}(x), v_\varepsilon(x, 0) = v_{\varepsilon 0}(x), \quad x \in \Omega,
\end{aligned}
\]

where \(u_{\varepsilon 0}, v_{\varepsilon 0} \in C^{2+\alpha}(\Omega)\) and \(v_{\varepsilon 0} > 0, \ u_{\varepsilon 0} \geq 0\) in \(\Omega\),

\[
\|u_{\varepsilon 0}\|_{L^\infty} + \|\nabla u_{\varepsilon 0}^\alpha\|_{L^2} + \|v_{\varepsilon 0}\|_{W^{2,p}} \leq 2\left(\|u_0\|_{L^\infty} + \|\nabla u_0^\alpha\|_{L^2} + \|v_0\|_{W^{2,p}}\right), \text{ for any } p > 1,
\]

\[
u_{\varepsilon 0} \to u_0 \text{ as } \varepsilon \to 0 \text{ in } L^p(\Omega), \text{ for any } p > 1,
\]

\[
v_{\varepsilon 0} \geq \varepsilon_0, \quad v_{\varepsilon 0} \to v_0 \text{ as } \varepsilon \to 0, \text{ uniformly in } \Omega,
\]

where \(\varepsilon \in (0, 1)\) and \(\rho_\varepsilon(s) = s + \varepsilon, \ s \geq 0\).

By the standard fixed point method, similar to [16, 19], we have the following local existence result to (8).

**Lemma 2.3.** Assume that \(u_{\varepsilon 0}, v_{\varepsilon 0}\) satisfy (9). Then for any \(\varepsilon > 0\), there exists \(T_{\text{max}} \in (0, +\infty]\) such that the problem (8) admits a unique classical solution \((u_\varepsilon, v_\varepsilon) \in C^{2+\alpha, 1+\alpha/2}(\Omega \times (0, T_{\text{max}}))\) with

\[
u_\varepsilon \geq 0, \quad v_\varepsilon \geq 0, \text{ for all } (x, t) \in \Omega \times (0, T_{\text{max}}),
\]

such that either \(T_{\text{max}} = \infty\), or

\[
\limsup_{t \nearrow T_{\text{max}}} (\|u_\varepsilon(\cdot, t)\|_{L^\infty} + \|v_\varepsilon\|_{W^{1,\infty}}) = \infty.
\]

To show the existence of solutions to the problem (1), we need to know about some a priori estimates on \((u_\varepsilon, v_\varepsilon)\). In what follows, we use \(C, C_i, M\) to denote some constants independent of \(\varepsilon\), if no special explanations, which depend at most on \(\chi, m, \mu, r, u_0\) and \(v_0\).

**Lemma 2.4.** Assume that \(m > 0, \ 0 < r < 1, \ u_{\varepsilon 0}, \ v_{\varepsilon 0}\) satisfy (9). If \((u_\varepsilon, v_\varepsilon)\) is a classical solution of (8) in \(Q_T\) for some \(T > 0\), then there exist constants \(C_i\) (\(i=1,2,3,4,5\)) independent of \(\varepsilon\) and \(T\), such that

\[
\sup_{t \in (0, T)} (\|v_\varepsilon(\cdot, t)\|_{L^\infty} + \|u_\varepsilon(\cdot, t)\|_{L^1}) \leq C_1,
\]

\[
\int_0^T \int_\Omega u_\varepsilon^2 dxdt \leq C_2 T + C_3,
\]

\[
\|v_\varepsilon\|_{W^{2,1}(Q_T)} \leq C_4 T + C_5.
\]

**Proof.** By maximum principle, it is easy to obtain that

\[
\|v_\varepsilon\|_{L^\infty} \leq \|v_{\varepsilon 0}\|_{L^\infty} \leq C.
\]
Integrating the two sides of the first PDE of (8) over $\Omega$ yields
\[
\frac{d}{dt} \int_{\Omega} u_\varepsilon dx + \mu \int_{\Omega} u_\varepsilon^2 dx = \mu \int_{\Omega} u_\varepsilon dx \leq \frac{\mu}{2} \int_{\Omega} u_\varepsilon^2 dx + C.
\]
From the above inequality, and combining with the $L^\infty$-estimate above obtained, (10) and (11) can be derived. By (10) and (11), it is easy to see that
\[
\int_0^T \| u_\varepsilon^r v_\varepsilon \|_{L^2}^2 dt \leq C T + C'
\]
since $r < 1$. By $L^2$-theory of linear parabolic equations, (12) can be proven. \(\square\)

Lemma 2.5. Assume that $m > 0$, $r < \frac{4}{N+2}$ and $(u_{e0}, v_{e0})$ satisfy (9). If $(u_e, v_e)$ is a classical solution of (8) in $Q_T$ for some $T > 0$, then there exists a constant $\lambda > 0$ such that
\[
\inf_{x \in \Omega} v_e(x,t) \geq \lambda > 0, \quad \text{for all} \quad t \in [0,T],
\]
where $\lambda$ depends on $T$.

Proof. Similar to [8], we make a logarithmic transformation to $v_e$, that is let
\[
\omega_e(x,t) = -\ln \left( \frac{v_e(x,t)}{\|v_{e0}\|_{L^\infty(\Omega)}} \right), \quad (x,t) \in \Omega \times [0,T).
\]
We rewrite the equation about chemoattractant as
\[
\begin{aligned}
\omega_{et} &= \Delta \omega_e - |\nabla \omega_e|^2 + u_e^r, & \text{in} \quad Q_T, \\
\frac{\partial \omega_e}{\partial n} \bigg|_{\partial \Omega} &= 0, \\
\omega_e(x,0) &= \omega_{e0} = -\ln \left( \frac{v_{e0}}{\|v_{e0}\|_{L^\infty}} \right), & \text{in} \quad \Omega.
\end{aligned}
\] (13)

By comparison principle, we have $0 \leq \omega_e \leq \overline{\omega}_e$, where
\[
\begin{aligned}
\overline{\omega}_{et} &= \Delta \overline{\omega}_e + u_e^r, & \text{in} \quad Q_T, \\
\frac{\partial \overline{\omega}_e}{\partial n} \bigg|_{\partial \Omega} &= 0, \\
\overline{\omega}_e(x,0) &= \omega_{e0}, & \text{in} \quad \Omega.
\end{aligned}
\] (14)

We use the Neumann heat semigroup theory to obtain
\[
\overline{\omega}_e(\cdot,t) = e^{t \Delta} \omega_{e0} + \int_0^t e^{(t-s) \Delta} u_e^r(\cdot,s) ds.
\]
Then
\[
\|\omega_\varepsilon(\cdot, t)\|_{L^\infty} \leq \|\omega_{0\varepsilon}\|_{L^\infty} + \int_0^t \|e^{(t-s)\Delta} u_\varepsilon^r(\cdot, s)\|_{L^\infty} ds
\]
\[
\leq \|\omega_{0\varepsilon}\|_{L^\infty} + C_1 \int_0^t \left(1 + (t-s)^{\frac{N_r}{2}}\right) \|u_\varepsilon^r(\cdot, s)\|_{L^2}, ds
\]
\[
\leq \|\omega_{0\varepsilon}\|_{L^\infty} + C_1 \int_0^t \left(1 + (t-s)^{\frac{N_r}{2}}\right) s^{\frac{2}{2-r}} ds \left(\int_0^t \|u_\varepsilon^r(\cdot, s)\|^2_{L^2} ds\right)^{\frac{r}{2}}
\]
\[
\leq \|\omega_{0\varepsilon}\|_{L^\infty} + C_2 \left(t + \frac{1}{1 - \frac{N_r}{2(2-r)}} t^{1 - \frac{N_r}{2(2-r)}}\right) \left(\int_0^t \|u_\varepsilon^r(\cdot, s)\|^2_{L^2} ds\right)^{\frac{r}{2}}
\]
(15)
since \(r < \frac{4}{N+2} \left(\frac{N_r}{2(2-r)}\right) < 1\). Noticing that \(0 \leq \omega_\varepsilon \leq \omega_\varepsilon\) and using inequalities (11) and (15), we have
\[
\sup_{t < T} \|\omega_\varepsilon(\cdot, t)\|_{L^\infty} \leq \sup_{t < T} \|\omega_\varepsilon(\cdot, t)\|_{L^\infty} \leq \|\omega_{0\varepsilon}\|_{L^\infty} + CT^{1 + \frac{r}{2}}
\]
as well as
\[
\inf_{x \in \Omega} v_\varepsilon(x, t) \geq \|v_{0\varepsilon}\|_{L^\infty} e^{-\|\omega_{0\varepsilon}\|_{L^\infty} - CT^{1 + \frac{r}{2}}} \quad \text{for all } t \leq T.
\]
Noticing that \(v_0 > 0\) on \(\Omega\), it implies that there exists \(l > 0\) such that \(v_0 > l\). Then \(v_{0\varepsilon} \geq v_0 > l\) on \(\Omega\), thus \(\|\omega_{0\varepsilon}\|_{L^\infty}\) is bounded. Then for any \(T > 0\), there exists a constant \(\lambda > 0\) such that
\[
\|v_{0\varepsilon}\|_{L^\infty} e^{-\|\omega_{0\varepsilon}\|_{L^\infty} - CT^{1 + \frac{r}{2}}} > \lambda,
\]
and we complete the proof.

3. Energy estimates and global existence to the slow and fast diffusion cases. We first consider the slow diffusion case. By Lemma 2.5, for any given \(T > 0\),
\[
\frac{1}{\rho_\varepsilon(v_\varepsilon)} = \frac{1}{v_\varepsilon + \varepsilon} \leq \frac{1}{\lambda}, \quad (x, t) \in Q_T.
\]
(16)
Here \(\lambda\) is independent of \(\varepsilon\), and it depends on \(T\). Then completely similar to the proof of [5], we have

**Proposition 1.** Assume that \(m > 1, r < \frac{4}{N+2}\), and (9) holds. If \((u_\varepsilon, v_\varepsilon)\) is a classical solution of (8) in \(Q_T\) for some \(T > 0\), then for any \(\varepsilon > 0\),
\[
\sup_{t \in (0, T)} (\|u_\varepsilon(\cdot, t)\|_{L^\infty} + \|v_\varepsilon(\cdot, t)\|_{W^{1, \infty}}) \leq M_1,
\]
(17)
\[
\sup_{t \in (0, T)} \int_\Omega |\nabla u_\varepsilon|^m dx + \int_0^T \int_\Omega \left|u_\varepsilon^{m-1} |\frac{\partial u_\varepsilon}{\partial t}|^2 + u_\varepsilon^{m-2} |\nabla u_\varepsilon|^2\right| dx dt \leq M_2,
\]
(18)
\[
\|v_\varepsilon\|_{W^{2, 1}(Q_T)} \leq M_3 \quad \text{for any } p > 1,
\]
(19)
where \(M_i (i = 1, 2, 3)\) are independent of \(\varepsilon\), which depend only on \(\chi, m, \mu, r, u_0, v_0\) and \(T\).

Next, we pay our attention to the fast diffusion case, that is \(m \leq 1\).
Lemma 3.1. Assume that $0 < m \leq 1$, $r < \frac{4}{N+2}$ and $(u_{\varepsilon}, v_{\varepsilon})$ is a classical solution of (8) in $Q_T$ for some $T > 0$, then for any $q > 0$,

$$\sup_{0 < t < T} \int_{\Omega} u_{\varepsilon}^{q+1} dx + \int_{0}^{T} \int_{\Omega} u_{\varepsilon}^{q-1}(u_{\varepsilon} + \varepsilon)^{m-1}|\nabla u_{\varepsilon}|^2 + u_{\varepsilon}^{q+2} dx dt \leq C,$$  

(20)

where $C$ is independent of $\varepsilon$, and it depends only on $\chi$, $m$, $\mu$, $r$, $u_0$, $v_0$, $q$ and $T$.

Proof. Multiplying the first PDE of (8) by $1 + \ln u_{\varepsilon}$, and integrating it over $\Omega$ yields

$$\frac{d}{dt} \int_{\Omega} u_{\varepsilon} \ln u_{\varepsilon} dx + m \int_{\Omega} (u_{\varepsilon} + \varepsilon)^{m-1}\frac{|\nabla u_{\varepsilon}|^2}{u_{\varepsilon}} dx + \mu \int_{\Omega} u_{\varepsilon}^2 \ln u_{\varepsilon} dx$$

$$\leq \chi \int_{\Omega} \frac{1}{v_{\varepsilon} + \varepsilon} \nabla v_{\varepsilon} \nabla u_{\varepsilon} dx + \mu \int_{\Omega} u_{\varepsilon} (1 + \ln u_{\varepsilon}) dx$$

$$= -\chi \int_{\Omega} u_{\varepsilon} \frac{\Delta v_{\varepsilon}}{v_{\varepsilon} + \varepsilon} dx + \chi \int_{\Omega} u_{\varepsilon} \frac{\nabla v_{\varepsilon}}{v_{\varepsilon} + \varepsilon} dx + \mu \int_{\Omega} u_{\varepsilon} (1 + \ln u_{\varepsilon}) dx$$

$$\leq \int_{\Omega} u_{\varepsilon}^{q+2} dx + \frac{\chi^2}{\lambda^2} \int_{\Omega} |\Delta v_{\varepsilon}|^2 dx + \frac{\chi^2}{\lambda^2} \int_{\Omega} |\nabla v_{\varepsilon}|^4 dx + C$$

for $t \leq T$. From the Gagliardo-Nirenberg interpolation inequality, we see that

$$\|\nabla v_{\varepsilon}\|_{L^p}^p \leq C_1 \|\nabla^2 v_{\varepsilon}\|_{L^2}^2 \|v_{\varepsilon}\|_{L^\infty}^2 + C_2 \|v_{\varepsilon}\|_{L^\infty}.$$

Then we further obtain that

$$\frac{d}{dt} \int_{\Omega} u_{\varepsilon} \ln u_{\varepsilon} dx + m \int_{\Omega} (u_{\varepsilon} + \varepsilon)^{m-1}\frac{|\nabla u_{\varepsilon}|^2}{u_{\varepsilon}} dx + \mu \int_{\Omega} u_{\varepsilon}^2 \ln u_{\varepsilon} dx$$

$$\leq \int_{\Omega} u_{\varepsilon}^{q+2} dx + \frac{\chi^2}{\lambda^2} \int_{\Omega} |\nabla v_{\varepsilon}|^2 dx + C$$

for $t \leq T$. Recalling (11) and (12), we obtain that

$$\int_{0}^{T} \int_{\Omega} (u_{\varepsilon} + \varepsilon)^{m-1}\frac{|\nabla u_{\varepsilon}|^2}{u_{\varepsilon}} dx dt + \mu \int_{0}^{T} \int_{\Omega} u_{\varepsilon}^{q+2} dx dt \leq C.$$

(21)

Multiplying the first PDE of (8) by $u_{\varepsilon}^q$ for any $q > 0$, and integrating it over $\Omega$, we obtain the following inequality

$$\frac{1}{q+1} \frac{d}{dt} \int_{\Omega} u_{\varepsilon}^{q+1} dx + m q \int_{\Omega} u_{\varepsilon}^{q-1}(u_{\varepsilon} + \varepsilon)^{m-1}\frac{|\nabla u_{\varepsilon}|^2}{u_{\varepsilon}} dx + \mu \int_{\Omega} u_{\varepsilon}^{q+2} dx$$

$$\leq q \chi \int_{\Omega} v_{\varepsilon} \frac{1}{v_{\varepsilon} + \varepsilon} \nabla v_{\varepsilon} \nabla u_{\varepsilon} dx + \mu \int_{\Omega} u_{\varepsilon}^{q+1} dx$$

$$= -\frac{q \chi}{q + 1} \int_{\Omega} u_{\varepsilon}^{q+1} \frac{\Delta v_{\varepsilon}}{v_{\varepsilon} + \varepsilon} dx + \frac{q \chi}{q + 1} \int_{\Omega} u_{\varepsilon}^{q+1} \frac{|\nabla v_{\varepsilon}|^2}{v_{\varepsilon} + \varepsilon^2} dx + \mu \int_{\Omega} u_{\varepsilon}^{q+1} dx$$

$$\leq \frac{3 \mu}{4} \int_{\Omega} u_{\varepsilon}^{q+2} dx + \frac{4}{\mu} \left( \frac{q \chi}{(q+1)\lambda} \right)^{q+2} \int_{\Omega} |\Delta v_{\varepsilon}|^{q+2} dx$$

$$+ \left( \frac{4}{\mu} \right)^{q+2} \int_{\Omega} |\nabla v_{\varepsilon}|^{2q+4} dx + 4^{q+1}\mu |\Omega|$$

for $t \leq T$. By the Gagliardo-Nirenberg interpolation inequality, we see that

$$\|\nabla v_{\varepsilon}\|_{L^{2q+4}}^{2q+4} \leq C_1 \|\nabla^2 v_{\varepsilon}\|_{L^2}^{2q+2} \|v_{\varepsilon}\|_{L^\infty}^{q+2} + C_2 \|v_{\varepsilon}\|_{L^\infty}^{2q+4}.$$
Combining the above two inequality, and using (10), we infer that for $t \leq T$,
\[
\frac{1}{q+1} \int_{\Omega} u^{q+1}_t dx + mq \int_{\Omega} u^q (u_\varepsilon + \varepsilon)^{m-1} |\nabla u_\varepsilon|^2 dx + \frac{\mu}{4} \int_{\Omega} u^{q+2}_\varepsilon dx \\
\leq \frac{C}{\mu^{q+1} \lambda^{2(q+1)}} \int_{\Omega} |\nabla^2 v_\varepsilon|^{q+2} dx + C,
\]
which implies that
\[
\sup_{0 < t < T} \int_{\Omega} u^{q+1}_\varepsilon dx + mq \int_{\Omega} \int_{0}^{T} u^q (u_\varepsilon + \varepsilon)^{m-1} |\nabla u_\varepsilon|^2 dx dt + \frac{\mu}{4} \int_{0}^{T} \int_{\Omega} u^{q+2}_\varepsilon dx dt \\
\leq \frac{C}{\mu^{q+1} \lambda^{2(q+1)}} \int_{0}^{T} \int_{\Omega} |\nabla^2 v_\varepsilon|^{q+2} dx dt + C,
\] 
where $C$ depend on $q$. Recalling the second equation of (8), and noticing that (10), then by $L^p$-theory of linear parabolic equations, we obtain that
\[
\int_{0}^{T} \int_{\Omega} |\nabla^2 v_\varepsilon|^{q+2} dx dt \leq C_1 \int_{0}^{T} \int_{\Omega} |u_\varepsilon|^{q+2} dx dt + C_2 \leq \eta \int_{0}^{T} \int_{\Omega} |u_\varepsilon|^{q+2} dx dt + C_\eta
\]
for any small $\eta > 0$. Substituting (24) into (23) with $\eta$ appropriately small, and combining with (21), we finally obtain that (20).

Using this lemma, we further have

**Lemma 3.2.** Assume that $0 < m \leq 1$, $r < \frac{4}{N+2}$ and (9) holds. If $(u_\varepsilon, v_\varepsilon)$ is a classical solution of (8) in $Q_T$ for some $T > 0$, then
\[
\sup_{t \in (0, T)} \|u_\varepsilon(\cdot, t)\|_{L^\infty} + \sup_{t \in (0, T)} \|v_\varepsilon(\cdot, t)\|_{W^{1, \infty}} \leq C_T,
\]
where $C_T$ depends on $T$, and it is independent of $\varepsilon$.

**Proof.** We choose $q$ such that $q > N$ in Lemma 3.1, then by Neumann heat semigroup theory, we have
\[
\sup_{t \in (0, T)} \|v_\varepsilon(\cdot, t)\|_{W^{1, \infty}} \leq C.
\]
Next, we establish the boundedness of $u_\varepsilon$ by classical Moser’s iterative technique similar to [5, 6], and the result can be proven.

By Lemma 2.3, Proposition 1, and Lemma 3.2, we have the global existence of classical solutions for the problem (8). Next, we also prove the following Lemma.

**Lemma 3.3.** Assume that $0 < m \leq 1$, $r < \frac{4}{N+2}$ and (9) holds. Then (8) admits a unique global classical solution $(u_\varepsilon, v_\varepsilon)$ such that for any $T > 0$,
\[
\|v_\varepsilon\|_{W^{2,1}_{p,1}(Q_T)} \leq \tilde{M}_1, \text{ for any } p > 1, \tag{26}
\]
\[
\sup_{t \in (0, T)} \int_{\Omega} |\nabla((\varepsilon + u_\varepsilon)^{m-1}u_\varepsilon)|^2 dx + \int_{0}^{T} \int_{\Omega} (\varepsilon + u_\varepsilon)^{m-1} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 dx dt \leq \tilde{M}_2, \tag{27}
\]
where $\tilde{M}_1$ and $\tilde{M}_2$ depend on $T$, and both of them are independent of $\varepsilon$. 


Proof. By $L^p$-theory of linear parabolic equations, it is easy to obtain (26) by (25). Multiplying the first equation of (8) by $\frac{\partial (\varepsilon + u_\varepsilon)^{m-1} u_\varepsilon}{\partial t}$, and integrating it over $\Omega$ gives

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla (\varepsilon + u_\varepsilon)^{m-1} u_\varepsilon|^2 dx + m \int_{\Omega} (\varepsilon + u_\varepsilon)^{m-1} |\partial u_\varepsilon/\partial t|^2 dx 
\leq -\chi \int_{\Omega} \nabla \cdot \left( \frac{u_\varepsilon}{v_\varepsilon + \varepsilon} \nabla v_\varepsilon \right) \frac{\partial (\varepsilon + u_\varepsilon)^{m-1} u_\varepsilon}{\partial t} dx + \mu \int_{\Omega} u_\varepsilon (1 - u_\varepsilon) \frac{\partial (\varepsilon + u_\varepsilon)^{m-1} u_\varepsilon}{\partial t} dx 
\leq C_\chi^2 \int_{\Omega} \left| \nabla \cdot \left( \frac{u_\varepsilon}{v_\varepsilon + \varepsilon} \nabla v_\varepsilon \right) \right|^2 (\varepsilon + u_\varepsilon)^{m-1} dx + \frac{m}{2} \int_{\Omega} (\varepsilon + u_\varepsilon)^{m-1} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 dx + C 
\leq C_1(T) \left( 1 + \int_{\Omega} (\nabla v_\varepsilon|^2 + (\varepsilon + u_\varepsilon)^{m-1} |\nabla u_\varepsilon|^2) dx \right) + \frac{m}{2} \int_{\Omega} (\varepsilon + u_\varepsilon)^{m-1} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 dx.
$$

By (20) and (26), we obtain

$$
\sup_{t \in (0,T)} \int_{\Omega} |\nabla (\varepsilon + u_\varepsilon)^{m-1} u_\varepsilon|^2 dx + \int_0^T \int_{\Omega} (\varepsilon + u_\varepsilon)^{m-1} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 dx dt \leq C_2(T),
$$

and (27) is obtained. \hfill \Box

**Proof of main result.** By Proposition 1, Lemma 3.1–Lemma 3.3, for any $T > 0$, we have

$$
u_\varepsilon \rightarrow u \text{ in } L^\infty(Q_T),
$$

$$
(u_\varepsilon + \varepsilon)^{\frac{m+1}{m}} \rightarrow u^{\frac{m+1}{m}} \text{ in } W^{1,1}_2(Q_T),
$$

$$
v_\varepsilon \rightarrow v \text{ in } W^{2,1}_p(Q_T) \text{ for any } p \in (1, \infty),
$$

$$
u_\varepsilon \rightarrow u \text{ in } L^p(Q_T),
$$

$$
v_\varepsilon \rightarrow v \text{ uniformly in } Q_T.
$$

From Sobolev imbedding theorem we infer that

$$
u_\varepsilon \rightarrow u \text{ in } L^p(Q_T).$$

By (19), (26) and $t$-Anisotropic Embedding Theorem, we also have

$$
v_\varepsilon \rightarrow v, \quad \frac{1}{v_\varepsilon + \varepsilon} \rightarrow \frac{1}{v}, \text{ uniformly.}
$$

Noticing that for $t < T$,

$$
|\varepsilon + u_\varepsilon|^{m-1} u_\varepsilon - u^m| \leq |\varepsilon + u_\varepsilon|^m - u^m| + |\varepsilon + u_\varepsilon|^{m-1}|
\leq \begin{cases} 
|\varepsilon + u_\varepsilon - u|^m + \varepsilon^m, & \text{if } m < 1, \\
C|\varepsilon + u_\varepsilon - u| + C\varepsilon, & \text{if } m \geq 1,
\end{cases}
$$

then

$$
(\varepsilon + u_\varepsilon)^{m-1} u_\varepsilon \rightarrow u^m \text{ in } L^p(Q_T) \text{ for any } p \in (1, +\infty).
$$
Noticing that for any $\varphi \in C^\infty(\overline{Q_T})$ with $\frac{\partial \varphi}{\partial n}|_{\partial \Omega} = 0$, $\varphi(x,T) = 0$, we arrive at
\[
- \int_{Q_T} u \varphi_t \, dx \, dt - \int_{\Omega} u_0 \varphi(x,0) \, dx - \int_{Q_T} (u + \varepsilon)^{m-1} u \Delta \varphi \, dx \, dt
\]
\[
= \chi \int_{Q_T} \frac{u}{\varepsilon + u} \nabla v \nabla \varphi \, dx \, dt + \int_{Q_T} \mu u (1 - u) \varphi \, dx \, dt
\]
\[
- \int_{Q_T} v \varphi_t \, dx \, dt - \int_{\Omega} v_0 \varphi(x,0) \, dx + \int_{Q_T} \nabla v \nabla \varphi \, dx \, dt + \int_{Q_T} u \nabla v \nabla \varphi \, dx \, dt = 0.
\]
Letting $\varepsilon \to 0$ yields
\[
- \int_{Q_T} u \varphi_t \, dx \, dt - \int_{\Omega} u_0 \varphi(x,0) \, dx - \int_{Q_T} u^m \Delta \varphi \, dx \, dt
\]
\[
= \chi \int_{Q_T} \frac{u}{v} \nabla v \nabla \varphi \, dx \, dt + \int_{Q_T} \mu u (1 - u) \varphi \, dx \, dt
\]
\[
- \int_{Q_T} v \varphi_t \, dx \, dt - \int_{\Omega} v_0 \varphi(x,0) \, dx + \int_{Q_T} \nabla v \nabla \varphi \, dx \, dt + \int_{Q_T} u \nabla v \nabla \varphi \, dx \, dt = 0.
\]
Noticing that $\nabla u^m \in L^2(Q_T)$, we further obtain that
\[
- \int_{Q_T} u \varphi_t \, dx \, dt - \int_{\Omega} u_0 \varphi(x,0) \, dx + \int_{Q_T} (\nabla u^m - \chi \frac{u}{v} \nabla v) \nabla \varphi \, dx \, dt
\]
\[
= \mu \int_{Q_T} u (1 - u) \varphi \, dx \, dt.
\]
By the uniform estimates in Proposition 1, Lemma 3.1, Lemma 3.2 and Lemma 3.3, we complete the proof of Theorem 2.2. \qed

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