Isotropic tensor-valued polynomial functions of fourth-order tensors

B. A. Younis & G. F. Smith

Abstract

Fourth-order tensor-valued functions appear in numerous fields of study. The formulation of practical models for these complex functions often requires their representation in terms of tensors of order two. In this paper, we develop an appropriate representation formula by assuming that the isotropic fourth-order tensor–valued function is a polynomial function in the components of two symmetric second–order tensors of degree \( \leq 2 \). We illustrate the utility of the result by applying it to obtain a representation of the fluctuating velocity, pressure-gradient correlations of turbulence.

1 Introduction

Fourth-order tensor-valued functions are frequently encountered in applied sciences e.g. in computational mechanics where they play an important role in the formulation of constitutive relations [1]. In the study of turbulence, the importance of these functions becomes apparent when considering the state of the art in the closure of the Reynolds–averaged form of the Navier–Stokes equations. This level of closure requires the solution of a differential transport equation for each non–zero component of the Reynolds–stress tensor \( \tau_{ij} \) (see, e.g., [2],[3]). The equations governing the evolution of this tensor can be derived directly from the Navier–Stokes equations. For an incompressible
fluid of uniform properties, they may be written as:

$$\frac{\partial u_i u_j}{\partial t} + U_k \frac{\partial u_i u_j}{\partial x_k} = - \left( \frac{u_i u_k}{\partial x_k} + \frac{u_j u_k}{\partial x_k} \right) - 2\nu \left( \frac{\partial u_i \partial u_j}{\partial x_k \partial x_k} \right)$$

$$- \frac{\partial}{\partial x_k} \left( \frac{u_i u_j u_k}{\partial x_k} - \nu \frac{\partial u_i u_j}{\partial x_k} \right)$$

$$- \frac{1}{\rho} \left( \frac{u_j}{\partial x_i} + \frac{u_i}{\partial x_j} \right)$$

(1)

where $U_i$ is the time–averaged velocity, $\rho$ and $\nu$ are the fluid density and kinematic viscosity and $u_i$ and $p$ are the fluctuating velocity and pressure respectively. Repeated subscripts indicate summation.

An exact expression the last term in Eq. (1) was obtained by [4] by taking the divergence of the Navier–Stokes equations to obtain a Poisson equation for the fluctuating pressure. The general solution of this equation, after multiplication by the fluctuating velocity, is given as:

$$\frac{1}{\rho} \frac{\partial p}{\partial x_j} = \frac{1}{2\pi} \int \int \int \left( \frac{\partial U_k}{\partial x_i} \frac{\partial u_i'}{\partial x_j} \right) \frac{1}{r} dVol$$

$$+ \frac{1}{4\pi} \int \int \int \left( u_k' u_i' u_j \right)_{klj} \frac{1}{r} dVol$$

(2)

where the integrations extend over the whole moving fluid, d Vol is the volume element and terms with a prime relate to values at point which ranges over the region of the moving fluid separated by distance $r$ from the point where the pressure fluctuations are evaluated.

With the assumption of local homogeneity which is commonly invoked in turbulence studies, and by retaining only the part which gives rise to the fourth-order tensor, there results:

$$\frac{1}{\rho} \left( \frac{u_j}{\partial x_i} + \frac{u_i}{\partial x_j} \right) = (A_{ijkl} + A_{jikl}) \frac{\partial U_k}{\partial x_l}$$

(3)

where the tensor functions $A_{ijkl}, A_{jikl}$ (by permutation of $i$ and $j$) are defined by Eq. (2).
The ability to use Eq. (1) to model the effects of turbulence on the motion of a fluid thus depends on the ability to represent the fourth-order tensor functions $A_{ijkl}$ and $A_{jikl}$ in terms of second-order tensors. The purpose of this paper is to derive an appropriate representation for these functions, and to demonstrate the use of the result in this particular application.

2 Isotropic fourth-order tensor-valued functions

For convenience, we write:

$$A_{ijkl} = A_{ijkl}(\hat{T}), \quad \hat{T} = T_{ij}, \quad T_{ij} = u_i u_j \quad (4)$$

We proceed by assuming that $A_{ijkl}(\hat{T})$ is an isotropic fourth-order tensor-valued polynomial function of the components $T_{ij}$ of the symmetric second-order tensor $\hat{T}$ of degree $\leq 2$. Thus, we have

$$A_{ijkl}(\hat{T}) = \alpha_{ijklmn} T_{mn} + \beta_{ijklmnop} T_{mn} T_{pq}. \quad (5)$$

The tensors $\alpha_{ijklmn}$ and $\beta_{ijklmnop}$ are isotropic tensors and are expressible in terms of outer products of the Kronecker delta tensor $\delta_{ij}$.

The sixth-order isotropic tensor $\alpha_{ijklmn}$ is expressible as a linear combination of 15 distinct isomers of $\delta_{ij} \delta_{kl} \delta_{mn}$. Tensors arising from $\delta_{ij} \delta_{kl} \delta_{mn}$ upon permuting the subscripts $i j \ldots n$ are referred to as isomers of $\delta_{ij} \delta_{kl} \delta_{mn}$. We have:

$$\alpha_{ijklmn} = \alpha_1 \delta_{ij} \delta_{kl} \delta_{mn} + \alpha_2 \delta_{ij} \delta_{km} \delta_{ln} + \alpha_3 \delta_{ij} \delta_{kn} \delta_{lm} + \alpha_4 \delta_{ik} \delta_{jl} \delta_{mn} + \alpha_5 \delta_{ik} \delta_{jm} \delta_{ln} + \alpha_6 \delta_{ik} \delta_{jn} \delta_{lm} + \alpha_7 \delta_{ij} \delta_{jk} \delta_{mn} + \alpha_8 \delta_{ij} \delta_{jm} \delta_{kn} + \alpha_9 \delta_{ij} \delta_{jn} \delta_{km} + \alpha_{10} \delta_{im} \delta_{jk} \delta_{ln} + \alpha_{11} \delta_{im} \delta_{jl} \delta_{kn} + \alpha_{12} \delta_{im} \delta_{jn} \delta_{kl} + \alpha_{13} \delta_{in} \delta_{jk} \delta_{lm} + \alpha_{14} \delta_{in} \delta_{jl} \delta_{km} + \alpha_{15} \delta_{in} \delta_{jm} \delta_{kl} \quad (6)$$

The eighth-order isotropic tensor $\beta_{ijklmnop}$ is expressible as a linear combination of 105 distinct isomers of $\delta_{ij} \delta_{kl} \delta_{mn} \delta_{pq}$. We thus have

$$A_{ijkl} = (\alpha_1 \delta_{ij} \delta_{kl} \delta_{mn} + \ldots + \alpha_{15} \delta_{in} \delta_{jm} \delta_{kl}) T_{mn} + (\beta_1 \delta_{ij} \delta_{kl} \delta_{mn} \delta_{pq} + \ldots + \beta_{105} \delta_{iq} \delta_{jp} \delta_{km} \delta_{ln} T_{mn} T_{pq} \quad (7)$$

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Only 91 of the 105 distinct isomers of $\delta_{ij} \delta_{kl} \delta_{mn} \delta_{pq}$ are linearly independent. Consider tensors of the form:

$$\delta_{ikmp} = \begin{vmatrix} 
\delta_{ij} & \delta_{il} & \delta_{in} & \delta_{iq} \\
\delta_{kj} & \delta_{kl} & \delta_{kn} & \delta_{kq} \\
\delta_{mj} & \delta_{ml} & \delta_{mn} & \delta_{mq} \\
\delta_{pj} & \delta_{pl} & \delta_{pm} & \delta_{pq} 
\end{vmatrix}. \quad (8)$$

If the tensor is three–dimensional, it is a null tensor. For any of the $3^8$ possible choices of values which $i, \ldots, q$ may assume, at least two rows of the determinant will be the same and the component will be zero. With (8), we have:

$$\delta_{ikmp} T_{mn} T_{pq} = 2(T_{ij} T_{kl} - T_{il} T_{jk}) + (\delta_{ij} \delta_{kl} - \delta_{il} \delta_{jk}) \left( (tr \hat{T})^2 - tr \hat{T}^2 \right)$$

$$+ 2(\delta_{ij} T_{kl}^2 + \delta_{kl} T_{ij}^2 - \delta_{il} T_{jk}^2 - \delta_{jk} T_{il}^2)$$

$$+ 2(\delta_{il} T_{jk} + \delta_{jk} T_{il} - \delta_{ij} T_{kl} - \delta_{kl} T_{ij}) tr \hat{T}$$

$$= 0. \quad (9)$$

There are (see [6, p.204]) 14 independent expressions of the form (8). Applying these to $T_{mn} T_{pq}$ will yield one additional equation of the form (9) which is given by:

$$\delta_{ikmp} T_{mn} T_{pq} = 2(T_{ik} T_{jl} - T_{il} T_{jk}) + (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) \left( (tr \hat{T})^2 - tr \hat{T}^2 \right)$$

$$+ 2(\delta_{ik} T_{jl}^2 + \delta_{jl} T_{ik}^2 - \delta_{il} T_{jk}^2 - \delta_{jk} T_{il}^2)$$

$$+ 2(\delta_{il} T_{jk} + \delta_{jk} T_{il} + \delta_{ij} T_{kl} + \delta_{kl} T_{ij}) tr \hat{T}$$

$$= 0. \quad (10)$$

We now require expressions for the function $A_{ijkl}(\hat{T})$ given by Eq. (7). The isotropic fourth–order tensor–valued function $A_{ijkl}(\hat{T})$ given by Eq. (7) is expressible as a linear combination of 9 linear and 19 quadratic terms which are given by:

$$\delta_{ij} \delta_{kl} tr \hat{T}, \delta_{ik} \delta_{jl} tr \hat{T}, \delta_{il} \delta_{jk} tr \hat{T}$$

$$\delta_{ij} T_{kl}, \delta_{ik} T_{jl}, \delta_{il} T_{jk}, \delta_{jk} T_{il}, \delta_{ij} T_{il}, \delta_{kl} T_{ij} \quad (11)$$

and

$$\delta_{ij} \delta_{kl} (tr \hat{T})^2, \delta_{ik} \delta_{jl} (tr \hat{T})^2, \delta_{il} \delta_{jk} (tr \hat{T})^2,$$
\[ \delta_{ij}\delta_{kl}tr\hat{T}^{2}, \delta_{ik}\delta_{jl}tr\hat{T}, \delta_{il}\delta_{jk}tr\hat{T}, \delta_{kl}\delta_{ij}tr\hat{T}, \]
\[ \delta_{ij}\delta_{kl}tr\hat{T}, \delta_{ik}\delta_{jl}tr\hat{T}, \delta_{il}\delta_{jk}tr\hat{T}, \delta_{kl}\delta_{ij}tr\hat{T}, \]
\[ T_{ij}T_{kl} + T_{ik}T_{jl} + T_{il}T_{jk} \tag{12} \]

where

\[ tr\hat{T} = T_{ii} = T_{11} + T_{22} + T_{33}, \]
\[ T_{ij}^{2} = T_{ik}T_{kj}, tr\hat{T}^{2} = T_{ij}T_{ji}. \tag{13} \]

The results given by Eq. (11) are obtained from listing the distinct terms found upon applying the 15 distinct isomers of \( \delta_{ij}\delta_{kl}\delta_{mn} \) to \( T_{mn} \). If we apply the 105 distinct isomers of \( \delta_{ij}\delta_{kl}\delta_{mn}\delta_{pq} \) to \( T_{mn}T_{pq} \), we obtain the first 18 terms listed in (12) together with the terms \( T_{ij}T_{kl}, T_{ik}T_{jl}, T_{il}T_{jk} \). The two identities (9) and (10) enable us to replace the three terms \( T_{ij}T_{kl}, T_{ik}T_{jl}, T_{il}T_{jk} \) by the single term \( T_{ij}T_{kl} + T_{ik}T_{jl} + T_{il}T_{jk} \). It would be permissible to employ any one of the terms \( T_{ij}T_{kl}, T_{ik}T_{jl}, T_{il}T_{jk} \) in place of \( T_{ij}T_{kl} + T_{ik}T_{jl} + T_{il}T_{jk} \) in (12).

3 Example of Application

We apply the result of the previous section to obtain the expression for the fluctuating velocity, pressure–gradient correlations as given by Eq. (3). For convenience, and following the convention in turbulence modeling (e.g. [3]), these expressions are given in terms of the symmetric and skew–symmetric parts of the tensor \( U_{i,j} \). We employ matrix notation. Thus,

\[ U = ||U_{i,j}||, \quad U^{T} = ||U_{j,i}||, \quad C = ||C_{ij}||, \]
\[ T = ||T_{ij}||, \quad T^{2} = ||T_{ik}T_{kj}|| \tag{14} \]

where \( U^{T} \) denotes the transpose of \( U \). We have

\[ S = \frac{1}{2}(U + U^{T}), \quad W = \frac{1}{2}(U - U^{T}), \]
\[ U = S + W, \quad U^{T} = S - W. \tag{15} \]
With Eqs. (11), (12), (14) and (15), we have:

\[
(A_{ijkl} + A_{jikl}) \frac{\partial U_k}{\partial x_l} = a_1 \text{Str}T + a_2 \delta_{ij} trTS + a_3 (TS + ST) \\
+ a_4 (TW - WT) + a_5 (trT)^2 + a_6 \text{Str}T^2 \\
+ a_7 \delta_{ij} (trT)^2 S + a_8 (TS + ST) trT + a_9 (TW - WT) trT \\
+ a_{10} \delta_{ij} trT^2 S + a_{11} (T^2 S + ST^2) + a_{12} (T^2 W - WT^2) \\
+ a_{13} T(trTS).
\] (16)

In obtaining Eq. (16), it should be noted that the term \(T_{ij} T_{kl} + T_{ik} T_{jl} + T_{il} T_{jk}\) appears in Eq. (12), and \((T_{ij} T_{kl} + T_{ik} T_{jl} + T_{il} T_{jk}) U_{k,l}\) yields \(T trTS + 2TST\). In obtaining Eq. (16), it should also be noted that the following identity was employed (see reference [6], p. 207):

\[
2 \ TST + 2(T^2 S + ST^2) - 2(TS + ST) trT - 2T trTS \\
- \text{Str}T^2 + S(trT)^2 + 2\delta_{ij} trT TtrTS - 2E_3 trT^2 S = 0.
\] (17)

4 Closing Remarks

Fourth-order tensor-valued functions appear in a number of equations whose solution is required to model the behavior of a solid undergoing deformation or a fluid in turbulent motion. The present method for representing these functions in terms of second-order tensors that can more easily be obtained employs the assumption that these functions are expressible as polynomials in the components of the second-order tensors. This produces a representation that is expressible as a linear combination of a number of linear and quadratic terms. The utility of this approach was demonstrated with respect to the fluctuating velocity, pressure-gradient correlations of turbulence.

References

[1] M. Itskov, ‘On the theory of fourth-order tensors and their applications in computational mechanics’, Comput. Methods Appl. Mech. Engrg. 189, 419-438 (2000).

[2] B. E. Launder, G. J. Reece & W. Rodi, ‘Progress in the development of a Reynolds stress turbulence closure’, J. Fluid Mech. 68, 537–566 (1975).
[3] C. G. Speziale, S. Sarkar, T. B. Gatski, 'Modelling the pressure-strain correlations of turbulence', J. Fluid Mech. 227, 245-272 (1991).

[4] P. Y. Chou, ‘On velocity correlations and the solutions of the equations of turbulent fluctuation’, Quart. Appl. Maths. 3, 38–54 (1945).

[5] G. F. Smith, 'The Crystallographic property tensors of order 1 to 8', Annals of the New York Academy of Sciences 172, 57–106 (1970).

[6] G. F. Smith, Constitutive Equations for Anisotropic and Isotropic Materials, (North-Holland, Amsterdam 1994).