Application of singular transformations method with regard to nonlinearities in the equations of dynamics and measurements

V. A. Tupysev1,2, A. V. Loparev1,3, Yu. A. Litvinenko1,3,

1 Concern CSRI Elektropribor, 30, Malaya Posadskaya str., St. Petersburg, Russia
2 State University of Aerospace Instrumentation, 67, Bol’shaya Morskaya str., St. Petersburg, Russia
3 ITMO University, 49, Kronverksky pr., St. Petersburg, Russia
avloparev@itmo.ru

Abstract. The possibility of state estimation for a dynamic system with nonlinearities in the equations of dynamics and measurements using the Singular Transformations Method is discussed. The features of the proposed approach application are considered. For a methodological example, the equations for the required estimates are specified and the estimation accuracy analysis is carried out.

1. Introduction
Currently, to solve a certain class of navigation information processing problems, it is proposed to use polynomial filters [1, 2], which allow calculating an optimal estimate in the class of linear estimates using Gaussian approximation of posterior density at the current step. However, the Gaussian approximation of posterior density leads to the fact that multistep estimation procedures turn out to be non-optimal [1-5]. In [6, 7], it was shown that under certain conditions, one can use an alternative approach based on the Singular Transformations Method. The essence of the method is to transform the original nonlinear system so that the nonlinear terms are eliminated. This approach was considered for the cases of nonlinearities either in the equations of measurements [6] or in the equations of dynamics [7], while the multistep problem becomes linear. Thereby Gaussian approximation of posterior density is not required, and thus the agreement of the calculated covariance matrix to the real estimation errors is ensured. This paper discusses the possibility and specificity of applying the Singular Transformations Method to the estimation problem for the case when both measurements and dynamics equations are nonlinear. In order to simplify the presentation, it is assumed that the equations of dynamics and measurements contain quadratic terms which determine the nonlinearity of the equations.

2. Problem statement
Let us consider the problem of estimating an \( n \)-dimensional vector Markov sequence described by a nonlinear difference stochastic equation
\[
\mathbf{x}_i(k) = \Phi_i(k)\mathbf{x}(k-1) + \mathbf{x}^T(k-1)\mathbf{B}_{w_i}(k)\mathbf{x}(k-1) + \mathbf{w}_i(k), \quad i \in \{1, \ldots, n\}, \quad \mathbf{x}_i(0) \in \mathcal{N} \{\mathbf{x}(0), \mathbf{P}(0)\},
\]
using \( m \)-dimensional nonlinear measurements

\[ y_j(k) = H_j(k)x(k) + x^T(k)B_{\psi}(k)x(k) + v_j(k), \quad j \in \{1, m\}, \]

where the second terms on the right-hand sides of (1) and (2) are quadratic forms of vectors \( x(k-1) \) and \( x(k) \), respectively; \( \Phi(k), B_{\psi}(k), B_{\psi}(k) \) are matrices of dimension \( 1 \times n, n \times n, n \times n \); \( w(k), v(k) \) are zero-mean Gaussian white noises, independent of each other and of the values \( x(0) \), with known covariance matrices \( Q(k) \) and \( R(k) \).

3. The essence of Singular Transformation Method (STM)

The idea of the STM in the considered problem statement is to transform the original system (1), (2) in such a way as to exclude the nonlinear terms contained in the equations. For this, following the approach outlined in [7], we will consider the equation (1) as a measurement model in the form

\[ 0 = -x(k) + \Phi(k)x(k-1) + B_{\psi}(k)u(k-1) + w(k), \]

where \( \Phi(k) = \left( \Phi^T_1(k) \Phi^T_2(k) \ldots \Phi^T_{\psi}(k) \right)^T \) is the matrix of dimension \( n \times n \); \( u(k) = \left( x_1^2(k) \ x_1(k)x_2(k) \ldots x_1(k)x_n(k) \ x_2^2(k) \ x_2(k)x_3(k) \ldots x_n^2(k) \right)^T \) is the vector of quadratic terms of dimension \( n(n+1)/2 \); \( B_{\psi}(k) \) is the matrix of dimension \( n \times n(n+1)/2 \).

In order to use STM, we represent the second term in the equation (2) as a function of the state vector at the previous step. Approximately (up to quadratic expansion terms), the measurements can be represented in the form

\[ y(k) \approx H(k)x(k) + B_{\psi}(k)u(k-1) + v(k), \]

where \( H(k) \) is the matrix of dimension \( m \times n \); \( B_{\psi}(k) \) is the matrix of dimension \( m \times n(n+1)/2 \).

Let us take into consideration the extended state vector \( x_\epsilon(k) = \left( x^T(k) \ x^T(k-1) \right)^T \). Taking into account (3) and (4), measurements in the extended state space can be written as

\[ \begin{pmatrix} 0 \\ y(k) \end{pmatrix} = \begin{pmatrix} -E & \Phi(k) \\ H(k) & 0 \end{pmatrix} \begin{pmatrix} x(k) \\ x(k-1) \end{pmatrix} + \begin{pmatrix} B_{\psi}(k) \\ B_{\psi}(k) \end{pmatrix} u(k-1) + \begin{pmatrix} w(k) \\ v(k) \end{pmatrix}. \]

To eliminate the nonlinear terms on the right-hand side of the expression (5), we multiply both sides of the equation (5) on the left by a singular transformation matrix \( A(k) = \left( A_{\psi}(k) \ A_\psi(k) \right) \), which is a matrix of the maximum possible rank such that

\[ A_{\psi}(k)B_{\psi}(k) + A_\psi(k)B_{\psi}(k) = 0. \]

As a result, the equation (5) is transformed to a linear measurement equation in the form

\[ A_\psi(k)y(k) = H_1(k)x(k) + H_2(k)x(k-1) + A_\psi(k)w(k) + A_\psi(k)v(k), \]

where \( H_1(k) = -A_{\psi}(k) + A_\psi(k)H(k) \), \( H_2(k) = A_{\psi}(k)\Phi(k) \). Thus, the nonlinear filtering problem (1,2) is reduced to the linear Gaussian problem of estimating the extended vector \( x_\epsilon(k) \) by the measurements (6), the solution to which, within the Bayesian approach [4], is sought taking into account the a priori mathematical expectation \( \tilde{x}_\epsilon(k) \) and the covariance matrix \( L(k) \) of the vector \( x_\epsilon(k) \) such that

\[ \tilde{x}_\epsilon(k) = \left( 0 \ x^T(k-1) \right)^T, \quad L^{-1}(k) = \begin{pmatrix} 0 & 0 \\ 0 & P^{-1}(k-1) \end{pmatrix}. \]

Here we have appreciated that only the measurement model (6) could be taken into consideration now, so a priori covariance of the vector \( x(k) \) should be “infinite”, and the upper left block of the matrix \( L^{-1}(k) \) becomes zero.

Based on the obtained measurement model, it is possible to introduce (within the framework of the Bayesian approach) an algorithm for estimating the required vector \( x(k) \) in the form
\[ \hat{x}(k) = \left( H_1^T(k) \left( H_2(k) P(k-1) H_2^T(k) + \tilde{R}(k) \right)^{-1} H_1(k) \right)^{-1} H_1^T(k) \left( H_2(k) P(k-1) H_2^T(k) + \tilde{R}(k) \right)^{-1} \times \]
\[ \hat{y}(k) - H_2(k) \hat{x}(k-1) \],
\[ P(k) = \left( H_1^T(k) \left( H_2(k) P(k-1) H_2^T(k) + \tilde{R}(k) \right)^{-1} H_1(k) \right)^{-1}, \]
where \( \hat{y}(k) = A_y(k) y(k) \), \( \tilde{R}(k) = A_y(k) Q(k) A_y^T(k) + A_y(k) R(k) A_y^T(k) \).

Note that the transformation of the original nonlinear estimation problem into a linear one is carried out here by using an approximate equation (4), so the agreement between the calculated and real covariances of the estimation errors does not guarantee. Further, we will consider an alternative algorithm that is based on the singular transformation as well, but does not require an approximate representation of the measurement vector.

4. Extrapolated Singular Transformation Method (ESTM)

We continue considering the dynamic equation in the form (3). Similarly to (3), the measurement equation can be written in the form
\[ y(k) = H(k)x(k) + \tilde{B}_{wu}(k)u(k) + v(k), \]
where \( \tilde{B}_{wu}(k) \) is the matrix of dimension \( m \times n(n+1)/2 \). Equation (9) for the \( (k-1) \)-th step has the form
\[ y(k-1) = H(k-1)x(k-1) + \tilde{B}_{wu}(k-1)u(k-1) + v(k-1). \]

Equations (3) and (10) in the extended state space take the form
\[ \begin{pmatrix} 0 \\ y(k-1) \end{pmatrix} = \begin{pmatrix} -E & \Phi(k) \\ 0 & H(k-1) \end{pmatrix} \begin{pmatrix} x(k) \\ x(k-1) \end{pmatrix} + \begin{pmatrix} B_{wu}(k) \\ \tilde{B}_{wu}(k-1) \end{pmatrix} u(k-1) + \begin{pmatrix} w(k) \\ v(k-1) \end{pmatrix}. \]

By analogy with the previous case, we introduce a singular transformation matrix \( A(k) = \left( A_y(k) \quad A_y(k) \right) \) such that
\[ A_y(k) B_{wu}(k) + A_y(k) \tilde{B}_{wu}(k-1) = 0, \]
and multiply both sides of the equation (11) on the left by \( A(k) \)
\[ A_y(k) y(k-1) = H_1(k)x(k) + H_2(k)x(k-1) + A_y(k)w(k) + A_y(k)v(k-1), \]
where \( H_1(k) = -A_y(k), \quad H_2(k) = A_y(k) \Phi(k) + A_y(k) H(k-1) \).

Obviously, in this case the estimation algorithm will also have the form (8) where the matrices \( H_1(k) \) and \( H_2(k) \) are defined as stated above, and \( \hat{y}(k) = A_y(k) y(k-1), \quad \tilde{R}(k) = A_y(k) Q(k) A_y^T(k) + A_y(k) R(k-1) A_y^T(k) \).

Note that within the framework of the approach under consideration, the current estimate is obtained using the measurements up to the previous step. In this sense, this approach can be interpreted as an Extrapolated Singular Transformation Method (ESTM). Despite the fact that in this case the agreement between the calculated and real covariances of the estimation errors is ensured, ignoring the current measurements should be considered as a disadvantage of the ESTM.

5. Example

Let us consider the problem of estimating a scalar sequence
\[ x(k) = a_1 x(k-1) + a_2 x^2(k-1) + w(k), \quad w(k) \in N \{ 0, Q \}, \quad x(0) \in N \{ 0, P_o \}, \]
by measurements
\[ y(k) = b_1 x(k) + b_2 x^2(k) + v(k), \quad v(k) \in N \{ 0, R \}. \]

In accordance with (5), we can approximately write the measurement equation in the expanded state space:
\[
\begin{bmatrix}
0 \\
y(k)
\end{bmatrix} \approx \begin{bmatrix}
-1 & a_1 \\
b_1 & 0
\end{bmatrix} \begin{bmatrix}
x(k) \\
x(k-1)
\end{bmatrix} + \begin{bmatrix}
a_2 \\
a_1^2 b_2
\end{bmatrix} u(k-1) + \begin{bmatrix}
w(k) \\
v(k)
\end{bmatrix}.
\]

The singular transformation matrix satisfies the equation \( A_y(k)a_z + A_y(k)a_1^2 b_2 = 0 \). Thus, the following parameters of this matrix can be chosen: \( A_y(k) = a_1^2 b_2 \), \( A_y(k) = -a_z \).

The Bayesian estimate equation in accordance with (8) has the form
\[
\hat{x}(k) = \frac{a_1^2 b_2 \hat{x}(k-1) + a_z y(k)}{a_2 b_2 + a_1^2 b_2^2};
\]
in this case, the variance of the estimation error changes in accordance with the equation
\[
P(k) = \frac{a_1^4 b_2^4 P(k-1) + a_2^2 R + a_1^4 b_2^2 Q}{(a_2 b_2 + a_1^2 b_2^2)^2}.
\]

When using the ESTM, the equation in the extended state space can be written as
\[
\begin{bmatrix}
0 \\
y(k-1)
\end{bmatrix} \approx \begin{bmatrix}
-1 & a_1 \\
b_1 & 0
\end{bmatrix} \begin{bmatrix}
x(k) \\
x(k-1)
\end{bmatrix} + \begin{bmatrix}
a_2 \\
b_2
\end{bmatrix} u(k-1) + \begin{bmatrix}
w(k) \\
v(k-1)
\end{bmatrix}.
\]

The singular transformation matrix, in this case, contains the elements \( A_y(k) = b_2 \), \( A_y(k) = -a_z \); the equations for the estimate and variance of the estimation error take the form:
\[
\hat{x}(k) = \left( a_1 - \frac{a_z b_1}{b_2} \right) \hat{x}(k-1) + \frac{a_2}{b_2} y(k-1),
\]
\[
P(k) = \left( a_1 - \frac{a_z b_1}{b_2} \right)^2 P(k-1) + \frac{a_2^2}{b_2^2} R + Q.
\]

As another alternative, let us consider the Polynomial Filtering (PF) algorithm [1] represented for the example given as follows:
\[
\hat{x}(k \mid k-1) = a_1 \hat{x}(k-1) + a_2 \hat{x}^2(k-1) + a_z P(k-1),
\]
\[
P(k \mid k-1) = (a_1 + 2a_z \hat{x}(k-1))^2 P(k-1) + 2a_z^2 P^2(k-1) + Q,
\]
\[
\hat{x}(k) = \frac{(b_1 + 2b_2 \hat{x}(k \mid k-1) + y(k)) - b_1 b_2 P(k \mid k-1) + \hat{x}(k \mid k-1) R/P(k \mid k-1)}{(b_1 + 2b_2 \hat{x}(k \mid k-1))^2 + 2b_2^2 P(k \mid k-1) + R/P(k \mid k-1)},
\]
\[
P(k) = (b_1 + 2b_2 \hat{x}(k \mid k-1))^2 P(k \mid k-1) + 2b_2^2 P^2(k \mid k-1) + R.
\]

Here, \( \hat{x}(k \mid k-1) \), \( P(k \mid k-1) \) are, respectively, the prediction estimate of \( x(k) \) and the calculated variance of the prediction errors.

![Fig 1. Plot of real (solid lines) and calculated (dashed lines) RMS errors of the estimation (enlarged scale on the right).](image-url)
Figure 1 shows the real (solid lines) and calculated (dashed lines) root-mean-square errors (RMS) of the estimation by the three considered algorithms, obtained by averaging $10^4$ realizations. In this case, the following values of the model (13), (14) parameters were set: $a_1 = a_2 = b_2 = 0.1; b_1 = 1; Q = R = 10^{-4}; P_0 = 1$.

Note that in this case, the real RMS errors of the estimates obtained by STM practically coincided with the calculated values, despite the use of the approximation (4). The application of a polynomial filter makes it possible to obtain more accurate estimates; however, the calculated variance is not consistent with the actual error. The algorithm based on the ESTM, despite the coincidence of the real and calculated variances, is significantly (by an order of magnitude) inferior to the other two algorithms both in the steady time and in the value of the steady-state RMS.

6. Conclusion
The paper demonstrates the basic possibility of using the Singular Transformations Method to solve estimation problems when nonlinearities both in the dynamic and measurement equations occur. Due to the loss of a part of the measurement information owing to a singular transformation, the method allows reducing the original nonlinear problem statement to a linear one. This approach ensures the consistency of the calculated covariance matrix with real estimation errors and does not require Gaussian approximation of prediction and posterior density.

This work was supported by the Russian Foundation for Basic Research, project no. 18-08-01261а, and the Government of the Russian Federation (Grant 08-08).

References
[1] Hernandez-Gonzalez, M., Basin, M., and Stepanov, O., Discrete-time state estimation for stochastic polynomial systems over polynomial observations, International Journal of General Systems, 2018, vol. 47, pp. 512–528.
[2] Stepanov, O.A., Vasiliev, V.A., Toropov, A.B., Loparev, A.V., and Basin, M.V., Efficiency analysis of a filtering algorithm for discrete-time linear stochastic systems with polynomial measurements, Journal of the Franklin Institute, 2019, vol. 356, no. 10, pp. 5573–5591.
[3] Stepanov O.A., Linear optimal algorithm for nonlinear navigation problems, Giroskopiya i navigatsiya, 2006, vol. 55, no. 4, pp. 11–20.
[4] Stepanov, O.A., and Toropov, A.B., A comparison of linear and nonlinear optimal estimators in nonlinear navigation problems, Gyroscopy and Navigation, 2010, vol. 1, no. 3, pp. 183–190.
[5] Zhao, Zh., Li, T., and Jilkov, V. P., Best linear unbiased filtering with nonlinear measurements for target tracking, IEEE Transactions on Aerospace and electronic Systems, 2004, vol. 40, no. 4, pp. 1324–1336.
[6] Loparev, A.V., and Tupysev, V.A., Comparative analysis of efficiency of the second-order nonlinear filters and method of singular transformations, Proc. 31st Conference in Memory of N.N. Ostryakov, St. Petersburg: CSRI Elektropribor, 2018, pp. 163–170.
[7] Tupysev, V.A., and Loparev, A.V., Performance analysis for the method of singular transformations in respect to estimation problem for nonlinear stochastic systems, Proc. First International Scientific Conference on Aerospace Instrumentation and Operation Technologies, St. Petersburg, GUAP, 2020, pp. 94–98.