We study dynamical supersymmetry breaking and the transition point by non-perturbative lattice techniques in a class of two-dimensional $N = 1$ Wess-Zumino model. The method is based on the calculation of rigorous lower bounds on the ground state energy density in the infinite-lattice limit. Such bounds are useful in the discussion of supersymmetry phase transition. The transition point is determined with this method and then compared with recent results based on large-scale Green Function Monte Carlo simulations with good agreement.
1. Introduction

An important issue in the study of supersymmetric models is the occurrence of non-perturbative dynamical supersymmetry breaking  \(^\text{1}\). The problem can be studied in the \(N = 1\) Wess-Zumino model that does not involve gauge fields and is thus a simple theoretical laboratory. Since the breaking of supersymmetry is closely related to the symmetry properties of the ground state, we will adopt a Hamiltonian formulation of the model.

Let us remind the (continuum) \(N = 1\) super algebra, \(\mathcal{P}=Q_\alpha Q_\beta \mathcal{G} = 2 (\mathcal{P} \mathcal{C})_{\alpha \beta}\). Since \(P_i\) are not conserved on the lattice, the super algebra is explicitly broken by the lattice discretization. A very important advantage of the Hamiltonian formulation is the possibility of conserving exactly a subset of the full super algebra \([3]\). Specializing to \(1 + 1\) dimensions, in a Majorana basis \(\gamma_0 = C = \sigma_2, \gamma_1 = i \sigma_3\), the algebra becomes: \(Q^2_1 = Q^2_2 = P^0\) \(H\) and \(\mathcal{P} Q_1 Q_2 \mathcal{G} = 2 P^1 2 P\). On the lattice, since \(H\) is conserved but \(P\) is not, we can pick up one of the supercharges, say, \(Q_1\), build a discretized version \(Q_L\) and define the lattice Hamiltonian to be \(H = Q^2_L\). Notice that \(Q^2_1 = H\) is enough to guarantee \(E_0 = 0\). The explicit lattice model is built by considering a spatial lattice with \(L\) sites. On each site we place a real scalar field \(\phi_n\) together with its conjugate momentum \(p_n\) such that \([p_n, \phi_m] = i \delta_{m,n}\). The associated fermion is a Majorana fermion \(\psi_n\) with \(a = 1, 2\) and \(\mathcal{P} \psi_n \psi_m \mathcal{G} = \delta_{n,m}\). Notice that \(\psi^\dagger_n = \psi_{a,n}\).

The continuum 2-dimensional Wess-Zumino model is defined by the supersymmetric generators involving the superpotential \(V(\varphi)\),

\[
Q_{1,2} = \sum_{\varphi} dx \ p(\varphi) \psi_1(\varphi) \ \frac{\partial \varphi}{\partial x} \ V(\varphi(\chi)) \ \psi_2(\chi) ;
\]

where \(\varphi(\chi)\) is a real scalar field and \(\psi(\chi)\) is a Majorana fermion. The discretized supercharge is \([3, 5]\)

\[
Q_L = \sum_{n=1}^{L} p_n \psi_1(\chi) \ \frac{\varphi_{n+1}}{2} - \frac{\varphi_n}{2} + V(\varphi_n) \ \psi_2(\chi) ;
\]

and the Hamiltonian takes the form

\[
H = Q^2_L = \sum_{n=1}^{L} \frac{\pi^2_n}{2} + \frac{\varphi_{n+1}}{2} - \frac{\varphi_n}{2} + V(\varphi_n) \ \varphi^2_n \varphi_{n+1} + h \varphi_0 + V^0(\varphi_n) \ 2 \varphi^\dagger_n \varphi_n - 1
\]

where we replace the two Majorana fermion operators with a single Dirac operator \(\chi\) satisfying canonical anticommutation rules.

The problem of predicting the pattern of supersymmetry breaking is not easy. In principle, the form of \(V(\varphi(\chi))\) is enough to determine whether supersymmetry is broken or not. At least at tree level supersymmetry is broken if and only if \(V\) has no zeros. The Witten index \([3]\) can help in the analysis: If \(V(\varphi)\) has an odd number of zeroes then \(I \neq 0\) and supersymmetry is unbroken. If \(V(\varphi)\) has an even number of zeroes, when \(I = 0\) we can not conclude anything. An alternative non-perturbative analysis for the case \(I = 0\) is thus welcome. The simplest way to analyze the pattern of supersymmetry breaking for a given \(V\) is to compute the ground state energy \(E_0\) through

\(^1\)See \([3]\) for recent reviews and a complete list of references.
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numerical simulations and/or strong coupling expansion. On the lattice, accurate numerical results are available \([5, 6]\), although a clean determination of the supersymmetry breaking transition remains rather elusive. All the predictions for the model with cubic prepotential, \(V = \phi^3\), indicated unbroken supersymmetry. Dynamical supersymmetry breaking in the model with quadratic prepotential \(V = \lambda_2 \phi^2 + \lambda_0\) was studied performing numerical simulations \([5]\) along a line of constant \(\lambda_2\), confirming the existence of two phases: a phase of broken supersymmetry with unbroken discrete \(Z_2\) at high \(\lambda_0\) and a phase of unbroken supersymmetry with broken \(Z_2\) at low \(\lambda_0\), separated by a single phase transition.

On the other hand, from the strong coupling analysis what comes out is the following: for odd \(q\), strong coupling and weak coupling expansion results agree and supersymmetry is expected to be unbroken \([5]\). This conclusion gains further support from the non vanishing value of the Witten index \([4]\). For even \(q\) in strong coupling, the ground state has a positive energy density also for \(L \to \infty\) and supersymmetry appears to be broken. In particular, for \(V = \lambda_2 \phi^2 + \lambda_0\), weak coupling predicts unbroken supersymmetry for \(\lambda_0 < 0\), whereas strong coupling prediction gives broken supersymmetry for all \(\lambda_0\).

2. Numerical Simulations and Discussion

We used two different approaches to investigate the pattern of dynamical supersymmetry breaking. In the first one, \([5, 6]\), the numerical simulations were performed using the Green Function Monte Carlo (GFMC) algorithm and strong coupling expansion. The GFMC is a method that computes a numerical representation of the ground state wave function on a finite lattice with \(L\) sites in terms of the states carried by an ensemble of \(K\) walkers. Numerical results using the GFMC algorithm for the odd prepotential confirm unbroken supersymmetry.

A more interesting case is the even prepotential. When \(V = \lambda_2 \phi^2 + \lambda_0\) and for fixed \(\lambda_2 = 0\), we may expect (in the \(L \to \infty\) limit) a phase transition at \(\lambda_0 = \lambda_0^{(c)}(\lambda_2)\) separating a phase of broken supersymmetry and unbroken \(Z_2\) (high \(\lambda_0\)) from a phase of unbroken supersymmetry and broken \(Z_2\) (low \(\lambda_0\)).

The usual technique for the study of a phase transition is the crossing method applied to the Binder cumulant, \(B\). The crossing method consists in plotting \(B\) vs. \(\lambda_0\) for several values of \(L\). The crossing point \(\lambda_0^{(c)}(L_1, L_2)\), determined by the condition \(B(\lambda_0^{(c)}; L_1) = B(\lambda_0^{(c)}; L_2)\) is an estimator of \(\lambda_0^{(c)}\). The value obtained is showed in Fig. 1 and corresponds to \(\lambda_0^{(c)} = 0.4801\) \([5]\). The main source of systematic errors in this method is the need to extrapolate to infinity both \(K\) and \(L\). For this reason, an independent method to test the numerical results of \([5]\) is welcome.

The second method is based on the calculation of rigorous lower bounds on the ground state energy density in the infinite-lattice limit \([4, 5]\). Such bounds are useful in the discussion of supersymmetry breaking as follows: The lattice version of the Wess-Zumino model conserves enough supersymmetry to prove that the ground state has a non negative energy density \(\rho \geq 0\), as its continuum limit. Moreover the ground state is supersymmetric if and only if \(\rho = 0\), whereas it breaks (dynamically) supersymmetry if \(\rho > 0\). Therefore, if an exact positive lower bound \(\rho_{LB}\) is found with \(0 < \rho_{LB} < \rho\), we can claim that supersymmetry is broken.

The idea is to construct a sequence \(\rho^{(k)}\) of exact lower bounds representing the ground state energy densities of modified lattice Hamiltonians describing a cluster of \(L\) sites and converging to
\( \rho \) in the limit \( L ! \rightarrow \infty \). The bounds \( \rho^{(L)} \) can be computed numerically on a finite lattice with \( L \) sites. The relevant quantity for our analysis is the ground state energy density \( \rho \) evaluated on the infinite lattice limit \( \rho = \lim_{L \rightarrow \infty} \rho^{(L)} \). It can be used to tell between the two phases of the model: supersymmetric with \( \rho = 0 \) or broken with \( \rho > 0 \).

In Ref. [7] we presented how to build a sequence of bounds \( \rho^{(L)} \) which are the ground state energy density of the Hamiltonian \( H \) with modified couplings on a cluster of \( L \) sites: given a translation-invariant Hamiltonian \( H \) on a regular lattice it is possible to obtain a lower bounds on its ground state energy density from a cluster decomposition of \( H \), i.e., given a suitable finite sublattice \( \Lambda \), it is possible to introduce a modified Hamiltonian \( H \) restricted to \( \Lambda \) such that its energy density \( \rho_\Lambda \) bounds \( \rho \) from below. The difference between \( H \) and \( H \) amounts to a simple rescaling of its coupling constants. The only restriction on \( H \) being that its interactions must have a finite range [7].

We compute numerically \( \rho^{(L)} \) at various values of the cluster \( L \): if we find \( \rho^{(L)} > 0 \) for some \( L \) we conclude that we are in the broken phase. We know that \( \rho^{(L)} ! = \rho \) for \( L ! \rightarrow \infty \) and the study of \( \rho^{(L)} \) as a function of both \( L \) and the coupling constants permit the identification of the phase in all cases. The calculation of \( \rho^{(L)} \) is numerically feasible because it requires to determine the ground state energy of a Hamiltonian quite similar to \( H \) and defined on a finite lattice with \( L \) sites.

To test the effectiveness of the proposed bound and its relevance to the problem of locating the supersymmetry transition in the Wess-Zumino model we study in detail the case of a quadratic prepotential \( V = \lambda_2 \phi^2 + \lambda_0 \) at a fixed value \( \lambda_2 = 0 \) [7]. An argument by Witten [4] suggest the existence of a negative number \( \lambda_0 \) such that \( \rho (\lambda_0) \) is positive when \( \lambda_0 > \lambda_0 \) and it vanishes for \( \lambda_0 < \lambda_0 \). \( \lambda_0 \) is the value of \( \lambda_0 \) in which dynamical supersymmetry breaking occurs. In Fig. 2 we show a qualitative pattern of the curves representing \( \rho^{(L)} (\lambda_0) \). We see that a single zero is expected in \( \rho^{(L)} (\lambda_0) \) at some \( \lambda_0 = \lambda_0 (L) \). Since \( \lim_{L \rightarrow \infty} \rho^{(L)} = \rho \), we expect that \( \lambda_0 (L) ! = \lambda_0 \) for \( L ! \rightarrow \infty \) allowing for a determination of the critical coupling \( \lambda_0 \). The continuum limit of the model is obtained by following a Renormalization Group trajectory that, in particular, requires the limit \( \lambda_2 ! \rightarrow 0 \) [5].

The properties of the bound \( \rho^{(L)} (\lambda_0) \) guarantee that for \( L \) large enough it must have a single zero \( \lambda_0 (L) \) converging to \( \lambda_0 \) as \( L ! \rightarrow \infty \). In any case for each \( L \) we can claim that \( \lambda_0 > \lambda_0 (L) \). To obtain the numerical estimate of \( \rho^{(L)} (\lambda_0) \) we used the world line path integral (WLPI) algorithm.
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\[ \lambda_0 \]

\[ \rho \]

\[ \rho (L) \]

\[ \rho (L_1) \]

\[ \rho (L_2) \]

\[ \rho (L_3) \]

\[ L > L > L_1 > L_2 > L_3 \]

\[ T = 50 \]

\[ T = 100 \]

\[ T = 150 \]

\[ \beta = 14.0 \]

\[ \beta = 10.0 \]

\[ \beta = 12.0 \]

\[ \text{Cluster size } L = 14 \]

\[ \text{Cluster size } L = 18 \]

\[ \text{Fig. 2: Qualitative plot of the functions } \rho (\lambda_0) \text{ and } \rho (L) (\lambda_0) \].

\[ \text{Fig. 3: Plot of the energy lower bound } \rho (L) (\beta, T) \text{ at } L = 14 \text{ and } L = 18. \]

The WLPI algorithm computes numerically the quantity \( \rho (L) (\beta, T) = \frac{1}{L} \text{Tr} \psi^\dagger \frac{\bar{H}_1}{\text{Tr} \psi^\dagger \frac{\bar{H}_1}{\text{Tr} \psi^\dagger \frac{\bar{H}_2}{T}} \psi^\dagger \frac{\bar{H}_2}{T}} \) where the Hamiltonian for a cluster of \( L \) sites is written as \( H = H_1 + H_2 \), by separating in a convenient way the various bosonic and fermionic operators in the subhamiltonians \( H_1 \) and \( H_2 \). The desired lower bound is obtained by the double extrapolation \( \rho (L) = \lim_{\beta \to \infty} \lim_{T \to \infty} \rho (L) (\beta, T) \), with polynomial convergence \( 1=\beta \) in \( T \) and exponential in \( \beta \). Numerically, we determined \( \rho (L) (\beta, T) \) for various values of \( \beta \) and \( T \) and a set of \( \lambda_0 \) that should include the transition point, at least according to the GFMC results. In Fig. 3 we plot the function \( \rho (L) (\beta, T) \) for the cluster sizes \( L = 14 \) and \( L = 18 \), various \( \beta \) and \( T = 50, 100, 150 \). Here we see that the energy lower bound behaves as expected: it is positive around \( \lambda_0 = 0 \) and decreases as \( \lambda_0 \) moves to the left. At a certain unique point \( \lambda_0 (L) \), the bound vanishes and remains negative for \( \lambda_0 < \lambda_0 (L) \). This means that supersymmetry breaking can

\[ \text{we do not report } H_1 \text{ and } H_2 \text{ here, see Ref. } [7] \text{ for details.} \]
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Figure 4: Plot of $\lambda_0(L)$ vs. $1/L$ for $L = 6; 10; 14; 18$. The best fit with a quadratic polynomial in $1/L$ gives $\lambda_0 = -0.49 \pm 0.06$ that should be compared with the best GFMC result obtained with $K = 500$ walkers.

be excluded for $\lambda_0 > \min_L \lambda_0(L)$. Also, consistency of the bound means that $\lambda_0(L)$ must converge to the infinite-volume critical point as $L \to \infty$. Since the difference between the exact Hamiltonian and the one used to derive the bound is $O(1/L)$, we can fit $\lambda_0(L)$ with a polynomial in $1/L$. This is shown in Fig. 4 where we also show the GFMC result. The best fit with a parabolic function gives $\lambda_0 = -0.49 \pm 0.06$ [7] quite in agreement with the previous $\lambda_{0,\text{GFMC}} = -0.48 \pm 0.01$ [5]. In conclusion, both methods reported here are quite in agreement and confirm the existence of two phases separated by a single phase transition at $\lambda_0$ for the quadratic prepotential.

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References

[1] A. Feo, Mod. Phys. Lett. A19 (2004) 2387 [hep-lat/0410012]; A. Feo, Nucl. Phys. Proc. Suppl. 119 (2003) 198 [hep-lat/0210015].

[2] S. Elitzur, E. Rabinovici, A. Schwimmer, Phys. Lett. B119 (1982) 165.

[3] J. Ranft, A. Schiller, Phys. Lett. B138 (1984) 166.

[4] E. Witten, Nucl. Phys. B188 (1981) 513; E. Witten, Nucl. Phys. B202 (1982) 253.

[5] M. Beccaria, M. Campostrini and A. Feo, Phys. Rev. D69 095010 (2004) [hep-lat/0402007]; M. Beccaria, M. Campostrini and A. Feo, Nucl. Phys. Proc. Suppl. 129 (2004) [hep-lat/0309054]; M. Beccaria, M. Campostrini and A. Feo, Nucl. Phys. Proc. Suppl. 119 891 (2003) [hep-lat/0209010].

[6] M. Beccaria and C. Rampino, Phys. Rev. D67, 127701 (2003) [hep-lat/0303021].

[7] M. Beccaria, G. De Angelis, M. Campostrini and A. Feo, Phys. Rev. D70 035011 (2004) [hep-lat/0405016].

[8] M. Beccaria, G. De Angelis, M. Campostrini and A. Feo, AIP Conf. Proc. 756 (2005) 451 [hep-lat/0412020].