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| Citation       | Gopalakrishnan, Sarang, Michael Knap, and Eugene Demler. 2016. “Regimes of Heating and Dynamical Response in Driven Many-Body Localized Systems.” Physical Review B 94 (9). https://doi.org/10.1103/physrevb.94.094201. |
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Regimes of heating and dynamical response in driven many-body localized systems

Sarang Gopalakrishnan
Department of Physics and Walter Burke Institute, California Institute of Technology, Pasadena, California 91125, USA

Michael Knap
Department of Physics, Walter Schottky Institute, and Institute for Advanced Study, Technical University of Munich, 85748 Garching, Germany

Eugene Demler
Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA

We explore the response of many-body localized (MBL) systems to periodic driving of arbitrary amplitude, focusing on the rate at which they exchange energy with the drive. To this end, we introduce an infinite-temperature generalization of the effective “heating rate” in terms of the spread of a random walk in energy space. We compute this heating rate numerically and estimate it analytically in various regimes. When the drive amplitude is much smaller than the frequency, this effective heating rate is given by linear response theory with a coefficient that is proportional to the optical conductivity; in the opposite limit, the response is nonlinear and the heating rate is a nontrivial power law of time. We discuss the mechanisms underlying this crossover in the MBL phase. We comment on implications for the subdiffusive thermal phase near the MBL transition, and for response in imperfectly isolated MBL systems.

DOI: 10.1103/PhysRevB.94.094201

I. INTRODUCTION

Isolated quantum systems in the many-body localized (MBL) phase do not approach local thermal equilibrium starting from generic initial conditions [1–5]. Instead, in the MBL phase, transport and relaxation are absent, and a system retains memory of its initial conditions at arbitrarily late times. At present, there is strong evidence, from numerical studies [6–10], rigorous mathematical approaches [11], and experiments [12–16], that the MBL phase exists in strongly disordered one-dimensional spin and fermion systems. Moreover, a phenomenological description exists for systems deep in the MBL phase [17–23], and can be used to explore aspects of dynamics and response [24–31]. Recently, the transition between MBL and thermal phases has also been explored, using general arguments [32–36], mean-field theory [37], and renormalization-group schemes [38–40]. The nature of this transition, and the MBL phase, is of particular interest because equilibrium statistical mechanics fails at the transition and does not apply in the MBL phase. Thus, we might expect various features of dynamics and response in the MBL phase to differ dramatically from equilibrium expectations.

This work addresses one such exotic feature of MBL systems, namely, that in these systems, the dc limit for response functions is subtle and is highly sensitive to the sequence in which the zero-amplitude, zero-frequency, and isolated-system limits are taken. For concreteness, consider the conductivity of the system, i.e., its response to periodic driving by an electric field of amplitude $A$ and frequency $\omega$. In a typical thermalizing phase, this response is linear in $A$, for small enough $A$, regardless of the drive frequency: the linear response limit $A \to 0$ and the dc limit $\omega \to 0$ commute. However, in the MBL phase, these limits do not commute within a strictly isolated system [41]. Taking the limit $A \to 0$ at fixed finite frequency gives rise to the linear response optical conductivity $\sigma(\omega) \propto \omega^2$ discussed in Ref. [30], which vanishes as $\omega \to 0$. On the other hand, taking the $\omega \to 0$ limit at fixed $A$ gives rise to a drive-induced many-body delocalization transition [42,43], and therefore a breakdown of linear response theory [41]. If one includes a nonzero system-bath coupling $\gamma$, as in Refs. [2,44], then the $A \to 0$ and $\omega \to 0$ limits commute at nonzero $\gamma$; however, the breakdown of linear response is seen when $A \gtrsim \gamma, \omega$ (this is expected, as for $\gamma \neq 0$ there is no sharp distinction between MBL and thermal phases). Finally, we note that the $A > 0, \omega \to 0$ limit does not correspond to the dc conductivity as defined by the response to a static electric field, which is strictly zero for small enough $A$ [2]. This is because when $\omega > 0$ we are interested in the response at very long times compared with $1/\omega$. The response to a static field, by contrast, corresponds to times that are short compared with $1/\omega$.

The objective of this work is to study response in the MBL phase beyond these various limits, for general $A/\omega$ (but provided these are small compared with the intrinsic energy scales of the system; the opposite case is addressed in Refs. [45,46]). Our main results are as follows. We identify an observable (specifically, a generalized heating rate) that can be numerically extracted from the dynamics of the driven isolated system. This heating rate allows us to characterize dynamical response without relying on linear response theory (which breaks down as $\omega \to 0$). We then identify the processes that dominate heating and response for various regimes of $A/\omega$, arguing that linear response is due to absorption from resonant configuration pairs [30] and occurs in a time window $1/\omega \lesssim t \lesssim 1/A$ (starting from when the drive is turned on). These processes give rise to the expected Joule-heating behavior, in which the energy absorbed (or, equivalently, the dissipated power) $\sim A^2 \sigma(\omega)$. Linear response processes saturate on time scales $t \gtrsim 1/A$, but subleading processes still contribute slow dynamics. For stronger drive, we identify Landau-Zener transitions (and, potentially, thermal Griffiths
are interested in systems that are isolated from the environment on the time scales of interest, so there is no external source of dissipation to bring the system to its steady state. One must instead extract the conductivity from a transient: specifically, one can extract the conductivity from the dissipated power, or Joule-heating rate (given by $V^2G$, where $G$ is the conductance and $V$ the applied voltage), when a system is driven starting at some time $t = 0$. A practical challenge with computing heating rates, however, is that the regime of interest to us is one of high or even infinite temperature of the system. In this infinite-temperature limit, the amount of heating is necessarily small, which makes direct numerical extraction of heating rates challenging.\(^1\)

One can address this difficulty by thinking about the mechanics of the heating process. Suppose the system is initially in an eigenstate in the middle of the many-body spectrum. During a particular drive cycle, the system is equally likely to absorb or to emit a quantum of the drive. Thus, the energy of the system undergoes a random walk, with a step set by the drive frequency $\omega$. When the system is instead initialized near infinite temperature (i.e., at a temperature $T$ greater than the intrinsic system scales and drive frequency), then the initial occupation of an eigenstate (in the eigenbasis of the undriven Hamiltonian) is given by $\approx 1 - E/T$, where $E$ is the energy. As states with lower energy are slightly more likely to be initially occupied, on average the random walk causes energy to be gained and the system heats up; i.e., the energy space initially has a “concentration gradient” (proportional to $1/T$) and “heating” results from the dynamics relaxing this initial gradient (see Appendix A). Thus, it is plausible that, up to a factor of $T$, the heating rate in the high-temperature limit is related to the fictitious diffusion constant in energy space. Indeed, this connection is well understood for the thermal phase [47,48].

This fictitious diffusion constant has a nonzero limit at infinite temperature, and is easy to measure numerically, by initializing the system in an eigenstate (or a wave packet with narrow energy spread) and measuring the energy spread of the wave packet as a function of time. Specifically, we introduce the energy spread ($\Delta E$) as

$$\Delta E \equiv \langle m(t) | \hat{H}^2 | m(t) \rangle - \langle m(t) | \hat{H} | m(t) \rangle^2. \quad (1)$$

Here, $|m(t)\rangle = \hat{U}(t)|m\rangle$ is the time-evolved state, with $|m\rangle$ being an eigenstate of the unperturbed Hamiltonian $\hat{H}$ and $\hat{U}(t)$ the unitary time-evolution operator generated by $\hat{H} + \hat{H}_{int}(t)$. We emphasize that this “fictitious” diffusion constant is distinct from, and not directly related to, the “true” energy diffusion constant of the undriven system: the “fictitious” energy diffusion constant measures the spread of probability in Fock space, whereas the “true” energy diffusion constant measures the spread of energy in real space. For a driven system, energy is not conserved (and thus the true energy

\(^1\)Heating from the ground state is considered in Refs. [45,46]. Note that the linear response conductivity at zero temperature is the same for MBL and noninteracting systems [30], and that the discussion of Ref. [41] is also greatly modified in this limit. Thus, the results of Refs. [45,46] do not directly address the conceptual issues that are relevant to this work.
The effective degrees of freedom \( \tau_i^+ \) are related to the microscopic ones (denoted \( \hat{S}_i^+ \)) by a finite-depth unitary transformation \([50]\), up to exponential tails. For notational simplicity (and to make contact with numerics), we shall work in one dimension, with open boundary conditions; none of our considerations rely crucially on these assumptions. Then, the time-varying electric field can be written as \( \hat{H}_{\text{dc}} = A \sin \omega t \sum_i x_i \hat{S}_i^+ \). We emphasize that the existence of such a description in terms of local integrals of motion has been rigorously established for certain one-dimensional spin chains \([11]\).

The expansion of a particular \( \hat{S}_i \) operator, e.g., \( \hat{S}_i^+ \), in terms of \( \tau \) operators, has the form \( \hat{S}_i^+ = \sum F_{ij} \tau_j^+ + \sum F_{ijk} \tau_j^+ \tau_k^+ + \ldots \). The \( F \) coefficients fall off exponentially with the furthest distance between the \( \tau \) spins involved, and also fall off exponentially with the number of \( \tau \) operators (i.e., \( \tau^\alpha \) or \( \tau^\beta \)) involved \([30,37]\). For example, the coefficient of a term of the form \( \prod_{i=1}^m \tau_i^+ \prod_{m+1}^p \tau_i^+ \) would fall off as \( \exp(-x/\xi - m/\zeta) \), where \( x \equiv \max(|i_p - i_m|) \). Stability of the MBL phase at infinite temperature requires that \( s \zeta < 1 \) \([2,30,37]\) as the available phase space for \( m \) spin flips grows as \( \exp(sm) \). At infinite temperature, the entropic factor \( s \sim \ln 2 \).

### IV. REGIMES OF HEATING AND RELEVANT SCALES

In this section, we qualitatively introduce the two primary heating mechanisms: resonant transitions and Landau-Zener transitions. We then identify regimes in which each mechanism is dominant, and explore the implications for heating in those regimes.

#### A. Resonant transitions

We first consider what happens when one drives the Hamiltonian \( (2) \) very weakly at relatively high frequency, \( A/\omega \ll 1 \). We assume that the drive is turned on instantaneously at time \( t = 0 \), and that the system is initialized in a many-body eigenstate, i.e., in a product state of the effective spins \( \tau_i \). The drive is diagonal in the physical-spin basis; thus, in the effective-spin basis, the drive generically has off-diagonal matrix elements for rearranging multiple effective spins. These typically fall off exponentially with order and interspin distance (as discussed in the previous section). However, there are rare pairs of effective-spin configurations between which the drive has a large matrix element. For an illustrative example, consider a well-localized Anderson insulator. Most (single-particle) eigenstates in the Anderson insulator are localized on single sites; however, rare eigenstates are delocalized across a resonant pair of accidentally degenerate sites \([51]\). The eigenstates in this resonant pair of sites consist of symmetric and antisymmetric combinations of the single-site orbitals, i.e., \( |\psi_+\rangle = |a\rangle \pm |b\rangle \) where \( a \) and \( b \) are the two orbitals. The electric field (which in this basis is \( \sim |a\rangle \langle a| - |b\rangle \langle b| \)) has matrix elements between \( |\psi_+\rangle \) and \( |\psi_-\rangle \) that grow with the distance between the two sites. Such resonant pairs exist at all scales, and dominate the linear response conductivity in both single-particle \([51]\) and MBL systems \([30,52,53]\). However, the number of resonances at scale \( x \) falls off exponentially with \( x \) whenever the MBL phase is stable. In the MBL phase, such resonant pairs exist not just between different sites, but also between different pairs of configurations \([30,52,53]\); thus, the number of resonant pairs is parametrically larger, but their qualitative physics is not greatly modified.

In the initial eigenstate of the undriven system, each of these resonant pairs is either symmetric or antisymmetric state. When the drive is turned on, it induces transitions between these two states provided the transition is resonant with the drive frequency \( \omega \); the associated Rabi frequency is set by \( A x_{\omega} \), where \( x_{\omega} \) is the size of the resonant pair (i.e., the dipole moment of the transition). Thus, on a time scale set by the drive amplitude \( A \), these resonant pairs are saturated (i.e., each resonant pair is precessing), and beyond this point there is very little absorption. Note that the saturation of resonant pairs is analogous to the phenomenon of spectral hole burning in glasses \([34]\). To make contact with the effective-spin language of Sec. III, the occupations of the symmetric and antisymmetric orbitals count as conserved quantities \( \tau_+^\alpha \). The drive mixes the two orbitals, and therefore has an off-diagonal matrix element of the form \( \langle \tau_+^\alpha \tau_-^\beta | H | \tau_+^\alpha \tau_-^\beta \rangle \).
We now briefly review the counting [30] of these resonant pairs in the MBL phase, when the system is driven at frequency $\omega$. For brevity, we shall quote and use the result of Ref. [30] that (at high temperature) the most common resonances at low frequencies involve flipping a substantial fraction ($\sim \frac{1}{3}$ at infinite temperature) of the effective spins within a region of length $x$. Thus, we shall take $n \sim x$ in what follows. Now, resonances that flip $n$ effective spins have matrix elements $M \sim W \exp(-n/\xi)$. Thus, such resonances also have splittings (owing to hybridization) $\delta \sim W \exp(-n/\xi)$, and do not contribute at lower frequencies. By contrast, when two configurations are separated by an energy $\omega$, but the matrix element is $W \exp(-n/\xi) \ll \omega$, then these configurations will not be resonant. Consequently, resonant transitions at frequency $\omega$ are predominantly those for which $W \exp(-n/\xi) \approx \omega$. Taking $n \sim x$, this sets a length scale for resonant transitions

$$x_{\text{Mott}} \sim \xi \ln(W/\omega). \quad (3)$$

One might intuitively expect linear response theory to hold when resonant transitions are dominant because the transition rate is proportional to $A^2$ owing to the golden rule. We shall see below that this is indeed the case.

### B. Landau-Zener crossings

In addition to resonant pairs, a second class of processes that contribute to heating are Landau-Zener (LZ) transitions [55], which we now discuss. Suppose the system begins in a many-body eigenstate, i.e., a product state, or particular configuration, of the effective $\tau$ spins. The drive has matrix elements that are diagonal in the effective-spin basis, and thus change the energies of the various configurations (Sec. III); in addition, it has off-diagonal matrix elements that can cause transitions between $\tau$-spin eigenstates. During a typical drive cycle, various configurations cross each other in energy. When such a crossing occurs, there is some probability of an adiabatic transition, i.e., one in which the system switches between configurations (as opposed to a diabatic transition, in which the system maintains its initial configuration). The matrix element for an adiabatic transition depends on the real-space and configuration-space distances between the configurations (Sec. III). At longer distances, there are more crossings, but they are less likely to be adiabatic (because the matrix element decreases). Quantitatively, the probability of an adiabatic transition at distance $x$ in the many-body case is given by

$$P_{\text{ad}}(x) \sim 1 - \exp[-M^2/(A\omega)]$$

where $M \sim W \exp(-x/\xi)$. This is the matrix element between the configurations

$$P_{\text{ad}}(x) \sim 1 - \exp[-W^2e^{-2x/\xi}/(A\omega)]. \quad (4)$$

To get the contribution of these LZ crossings to the heating rate, we must identify the conditions under which they cause heating. During each drive cycle, a given LZ crossing occurs twice. If it is crossed adiabatically or diabatically on both attempts, the system deterministically returns to its initial configuration at the end of a drive cycle. This does not cause heating. Rather, the rate at which a particular transition causes heating is given by $P_{\text{ad}}(1 - P_{\text{ad}})$ [41,43]; thus, transitions that cause heating are those that have an appreciable probability of happening diabatically and also an appreciable probability of happening adiabatically.\(^2\)

We now estimate $P_{\text{ad}}$ for the crossings that typically occur when the system is driven with amplitude $A$. Let us consider a segment of size $x$. An electric field of amplitude $A$ lifts energy levels by an amount $\sim Ax$. The number of configurations of the effective spins in this segment is $\exp(x/A)$ (specifically, \(^3\) at infinite temperature), and their energy bandwidth is $Wx$. Thus, if the initial configuration covers an energy window $Ax$, it will typically cross $\exp(x/A)/W$ configurations. Thus, in order for at least one LZ transition to typically occur, one needs to look at segments of size

$$x_{\text{LZ}} \sim (1/s) \ln(W/A). \quad (5)$$

There are two regimes of behavior depending on whether $P_{\text{ad}}(x_{\text{LZ}}) \ll 1$ (i.e., most LZ crossings are diabatic) or not. In the limit that $P_{\text{ad}}(x_{\text{LZ}}) \ll \frac{1}{2}$, the density of adiabatic LZ transitions per cycle is low. In this case, LZ transitions do not destabilize the MBL phase, but simply provide an additional heating channel in addition to resonant transitions. In the opposite limit, $P_{\text{ad}}(x_{\text{LZ}}) \sim O(1)$, adiabatic LZ transitions become dense; thus, delocalization takes place through a series of LZ hops. This corresponds to a drive-induced many-body delocalization phase transition [43]. The resulting delocalized phase is presumably thermal (in the sense that it heats up to infinite temperature), but its properties (such as response functions) are not adiabatically connected to those of the undriven system.

### C. Length scales and regimes of response

The previous discussion suggests that there are three separate length scales governing the response of the system. One of these is the “Mott” length scale $x_{\text{Mott}} \sim \xi \ln(W/\omega)$, which is the length scale on which resonant transitions take place [Eq. (3)]. The second is the Landau-Zener crossing scale $x_{\text{LZ}} \sim (1/s) \ln(W/A)$, which is the distance (in real and/or configuration space) to the nearest Landau-Zener crossing [Eq. (5)]. Finally, there is a length scale, which we call the “adiabatic” scale $x_{\text{ad}}$, determined by the condition that $P_{\text{ad}}(x_{\text{ad}}) \sim \frac{1}{2}$. An approximate formula for this scale is

$$x_{\text{ad}} \sim (\xi / 2) \ln[W^2/(A\omega)], \quad (6)$$

which is obtained by inverting Eq. (4). The arguments of Ref. [43] can be rephrased as saying that when $x_{\text{ad}} \approx x_{\text{LZ}}$, a drive-induced delocalization transition takes place. The behavior of these three length scales is shown in Fig. 2. Note that if $A$ is increased at fixed $\omega$ in the MBL phase, the first crossing that occurs is $x_{\text{ad}} \approx x_{\text{Mott}}$ when $\omega = A$. At this drive amplitude, $x_{\text{LZ}} \geq x_{\text{Mott}}, x_{\text{ad}}$ because of the above definitions combined with the condition $x_{\text{LZ}} < 1$, which is required for the stability of the MBL phase, as discussed in Sec. III.

Even before the drive causes delocalization, it causes the breakdown of linear response. The crossover between

\(^2\)Here, we have assumed that only two levels are involved, as this is the case of most interest to us. However, one can generalize the principle that dissipation is governed by the probability of the system not returning to its original configuration after a drive cycle.
linear and nonlinear response can be understood as follows (see Fig. 3, top): when the drive amplitude is very small $x_{\text{Mott}} \lesssim x_{\text{ad}}$. In that regime, resonant transitions (whose density is set by $\omega$) dominate the response. But as $A$ is ramped up, eventually the phase space for LZ crossings (whose density is set by $A$) dominates that for resonant transitions (even though these LZ crossings have relatively small adiabatic rates). This corresponds to a breakdown of linear response, which is accompanied by a breakdown of the rotating-wave approximation for the driven resonant pairs (cf. top-left and top-right illustrations in Fig. 3). In addition to the crossover in the transient dynamical response, a steady-state transition from localized to thermal effective Floquet Hamiltonians can be introduced, which is solely set by the density of TLS and not by their character.

There are thus in total three distinct regimes (Fig. 3): (i) linear response due to isolated resonant TLS’s with Floquet steady states that are many-body localized; (ii) nonlinear response due Landau-Zener TLS’s, which are nevertheless isolated from one another and hence the steady state remains many-body localized as well (intermediate regime in Figs. 2 and 3); (iii) nonlinear response due to percolation between TLS’s accompanied with thermal steady states induced by strong drive.

V. TRANSIENT LINEAR RESPONSE

This section focuses on regimes in which linear response behavior emerges. There are two such regimes: in the MBL phase, for sufficiently small $A/\omega$, and in the thermal phase, for general $A/\omega$. We shall consider these in turn. Although our primary concern is with the behavior of the MBL phase, the thermal behavior is instructive and helps to set up our discussion of Griffiths effects in Sec. VIC.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Behavior of the length scales $x_{\text{Mott}}, x_{\text{LZ}}, x_{\text{ad}}$ (defined in text) as the drive amplitude $A$ is varied at fixed frequency $\omega$. We assume $A \omega \ll W$ (where $W$ is the single-particle bandwidth) and set $s \zeta = \frac{1}{2}$, so the system is relatively deep in the MBL phase. Thus, there are two separate crossovers as $A$ is increased: linear response fails when $x_{\text{ad}} \simeq x_{\text{Mott}}$, and Landau-Zener transitions percolate when $x_{\text{ad}} \simeq x_{\text{LZ}}$. These crossovers are separated by an intermediate regime (shaded region) in which rare Landau-Zener transitions dominate the response. As $s \zeta$ increases, the intermediate regime shrinks and disappears when $s \zeta = 1$ (i.e., at the MBL transition).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{Regimes of transient (top) and steady-state behavior (bottom) in driven MBL systems. As the drive strength $A$ is increased at constant frequency $\omega$, there is a crossover between linear and nonlinear response in the transient dynamics, set by the failure of the rotating-wave approximation to the driven two-level systems (TLS’s) that govern the response of the system. This crossover happens when these TLS’s transition from a drive-resonant regime (top left) to a Landau-Zener regime (top right). There is a separate steady-state phase transition between MBL and thermal Floquet Hamiltonians (bottom). This is determined not by the nature of TLS’s, but rather by the density of the dominant type. When TLS’s percolate, the steady state is thermal; otherwise it is MBL. The steady-state transition is set by the condition $A^{1-\zeta/2} \sim \omega^{\zeta/2} W^{1-\zeta}$. In summary, there are three steady-state regimes for a driven MBL system: (i) MBL long-time behavior with isolated resonant transitions; (ii) MBL long-time behavior with isolated Landau-Zener transitions; (iii) thermal long-time behavior because of percolating Landau-Zener transitions.

A. MBL phase: Linear response through resonances

In this section we analyze a simplified version of the effective-spin model in Sec. III, in which we neglect all degrees of freedom that are not resonant pairs. The two states of each resonant pair can be treated as a two-level system. Note that these resonant two-level systems (RTLS’s) are not by their character, we shall denote the RTLS’s as $T_0$. Because of their sparseness, we neglect interactions among RTLS’s.

We now discuss the dynamics of this ensemble of non-interacting RTLS’s. We work in the effective spin representation of the undriven system; in the associated natural eigenbasis, each TLS points along $z$ in the absence of drive. The full Hamiltonian of the driven RTLS $\alpha$ can be written as

$$H_{\text{RTLS}}(\alpha) = \varepsilon_\alpha T_\alpha^z + 2 A \zeta \ln(W/\varepsilon_\alpha) \cos(2 \omega t) \Theta(t) T_\alpha^z. \quad (7)$$

Here, we have used the result (from Sec. IV A) that a RTLS with splitting $\varepsilon$ is typically one of size $x(\varepsilon) \sim \zeta \ln(W/\varepsilon)$, and that the corresponding dipole matrix element of the electric field is $A x \sim A \zeta \ln(W/\varepsilon)$. The density of these RTLS’s is also given by similar reasoning. The number of available

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3Operationally, a distant or many-spin “resonant pair” is a pair of effective spin configurations that are far apart in either real or configuration space, but have a large matrix element of the electric field.
states at scale $x$ goes as $\exp(s x)$, and the corresponding many-body level spacing is $W x \exp(-s x)$ [since $W x$ is the energy bandwidth of a region of size $x$]. Substituting $x(x)$ into this expression, we immediately arrive at the result

$$\rho(x) \sim e^{-s x}.$$  \hfill (8)

Note that this is the density of states of RTLS’s, not necessarily that of effective $\tau$ spins.

With these assumptions, we can apply the rotating-wave approximation to Eq. (7) and the dynamics of RTLS’s becomes exactly solvable. In what follows, we shall further simplify by neglecting the logarithmic correction due to the dipole moment. Now, one can use the Rabi formula to find that at time $t$, the energy variance of a single RTLS is given by

$$\langle \Delta E^2 \rangle = \frac{4 A^2 \epsilon^2 e^2}{\Omega^2} \sin^2(\Omega x t),$$  \hfill (9)

where $\Omega = \sqrt{(A \epsilon)^2 + (|E_a| - |\omega|)^2}$ is the Rabi frequency of RTLS $\alpha$.

To get the response of the full system, one ensemble averages the response of the RTLS. This gives the expression

$$\langle \Delta E^2 \rangle = W^{s x - 1} \int d\epsilon \left[ \frac{4 + A^2 \epsilon^2 e^2}{\Omega^2} \sin^2(\Omega x t) \right].$$  \hfill (10)

This integral has four regimes. At short times compared with $1/W$ it goes as $A^2 x^2$. At long times compared with $t \gtrsim 1/A \gtrsim 1/\Omega$, it saturates. There are two intermediate regimes: $1/W \ll t \ll 1/\omega$ and $1/\omega \ll t \ll 1/A$. The former regime is not of interest to us: at these time scales, the frequency $\omega$ cannot be resolved. Thus, we can specialize to $1/\omega \ll t \ll 1/A$. Here, the integral (10) splits into three parts: from $0$ to $\omega - 1/t$, from $\omega - 1/t$ to $\omega + 1/t$, and from $\omega + 1/t$ to $W$. In the “outer” regimes, we can approximate $\sin^2 x \approx \frac{1}{2}$, and in the “inner” regime, we can expand it as $\sin^2 x \approx x^2$. Using these results, we find that the leading $t$ dependence in this regime is given by

$$\langle \Delta E^2 \rangle_{LR}(t) \sim W^{s x - 1} A^2 \omega^2 e^2 \sin^2(\Omega x t).$$  \hfill (11)

This is, as expected, proportional to the linear response conductivity $\sigma(\omega) \sim \omega^2 e^2 \equiv \omega^2$ [30] (cf. Fig. 4).

B. Linear response in the thermal phase

We now turn to the thermal phase, and briefly consider how linear response emerges there. As we shall eventually be interested in finite-size thermal blocks in the insulating phase, we focus on a finite thermal system of size $L$, with an intrinsic thermalization time $\sim 1/W \ll 1/\omega$. Using the “off-diagonal” part of the eigenstate thermalization hypothesis [56], one can estimate the matrix elements of the electric field between many-body eigenstates of the thermal system as $M(L) \sim A \exp(-s L/2)$. (Here, we are ignoring subleading factors of $L$ that are not in the exponent.) When $L$ is large enough, such matrix elements are always much smaller than $\omega$; hence, resonant transitions always dominate Landau-Zener transitions, and the golden rule is appropriate. Furthermore, the time scale at which linear response breaks down due to saturation in a large, deeply thermal inclusion is not determined by $t \sim 1/A$, as in the previous section. Instead, it is set by the shorter of the following two time scales. (1) The time scale on which the occupation of the initial eigenstate is appreciably depleted. This time scale is set by the golden-rule rate $\sim A^2/W$, which is independent of $L$ but is parametrically longer than in the localized phase, since $A/W \ll 1$ for our purposes. (2) The time scale on which a particular final state has an appreciable chance of being populated. This is set by the matrix element $M(L) \sim A \exp(-s L/2)$. For very large thermal systems, process (1) governs saturation, whereas for small thermal systems (such as some of the Griffiths regions we will consider below), process (2) governs saturation. A crossover between these processes takes place when $L \sim (2/s) \ln(W/A)$.

VI. REGIMES OF NONLINEAR RESPONSE

The perturbative resonances discussed in the previous section saturate on a time scale $\sim 1/A$. When $A \gtrsim \omega$, these resonances are essentially saturated within the first drive cycle. Thus, any heating that occurs after the first drive cycle is due to slower, more collective processes. We now consider various types of such processes: (a) perturbative resonances that are slow compared with the main Mott transitions, and thus give rise to slower heating; (b) perturbative resonances that are higher order in the drive amplitude; (c) thermal Griffiths inclusions; and (d) Landau-Zener transitions.
A. Anomalously distant resonant pairs

First, we extend the analysis of Sec. V to times that are long compared with 1/A; at these times, the dominant Mott resonances have saturated. However, rare Mott pairs with anomalously small Rabi frequency still exist, as do pairs of states with splitting \( \omega \) at larger scales than \( x_{\text{Mott}} \). We expect the latter to dominate, as they are more abundant, so we shall focus on them. Unlike the Mott pairs, these subleading resonances are induced by the drive, i.e., although the pairs are split by \( \omega \), this splitting is due to detuning rather than hybridization. Thus, they are hybridized by the off-diagonal matrix elements of the drive. The hybridization is given (at distances \( x \gg x_{\text{Mott}} \)) by \( \tilde{A}(x) \sim A x \exp(-x/\zeta) \). Moreover the number of these resonances increases with distance as \( \exp(x/\zeta) \). Now, let us consider the dynamics on a time scale \( t \). On this time scale, resonances with \( \tilde{A}(x) \gtrsim 1/t \) will have saturated and do not contribute any further to heating. However, further-out drive-induced resonant pairs will still be absorbing linearly. The absorption at time \( t \) is thus dominated by resonances with \( \tilde{A}(x) \gtrsim 1/t \). Plugging this into Eq. (11), the contribution from these resonances to heating goes as

\[
(\Delta E)^2_{\text{anom-Mott}}(t) \sim \tilde{A}(x)^2 e^{x/\zeta} \omega^2 t/W \sim A^{x/\zeta} \omega^2 t^{x-1}/W. \tag{12}
\]

Stability of the MBL phase entails \( x/\zeta < 1 \), so that these processes give rise to a slow, power-law approach to saturation on time scales \( t \gg 1/A \) even when the drive is weak.

B. Resonances from higher-order processes in the drive

In the previous sections we considered one way in which the drive can induce \( n \)-particle rearrangements, namely, that the expansion of the electric field in terms of effective spins has matrix elements for rearranging \( n \) spins. For large \( n \) such a process is suppressed because it falls off as \( \exp(-n/\zeta) \) (see Sec. III). Nevertheless, it is still leading order in the drive amplitude \( A \). When the drive amplitude is large, one must also consider resonant \( n \)-particle rearrangements that are higher order in the drive amplitude. For instance, one can rearrange \( n \) effective spins by going to \( n \)th order in the drive. The amplitude for such a process can be estimated in perturbation theory as \( \tilde{A}_n \sim A^n/W^{n-1} \) (up to a combinatorial factor) because the typical energy change upon flipping an effective spin is \( W \). To see which type of \( n \) spin rearrangement is more important, one must compare \( \zeta \) with \( 1/\ln(W/A) \); the bigger of these will dominate. We have considered the former type (first order in \( A \)) above; now we consider the latter (high order in \( A \)).

The resonances that go as \( A^n \) saturate only on time scales \( t \sim 1/\tilde{A}_n \); thus, \( n \)-th order processes can dominate response once all lower-order processes have saturated. At a time \( t \), the dominant unsaturated resonances are of order \( n \) such that \( \tilde{A}_n \sim 1/t \), and thus \( n(t) \sim \ln(t/\ln(W/A)) \). These \( n \)-th order processes can be analyzed in terms of the Rabi formula, precisely as in Sec. V but replacing \( A \) with the renormalized Rabi frequency \( \tilde{A}_n \sim 1/t \). Thus, \( (\Delta E)^2_{\text{hi-res}} \sim \tilde{A}_n^2 e^{x/\zeta} t \). Substituting for \( \tilde{A}_n \) and \( n(t) \), we arrive at the result

\[
(\Delta E)^2_{\text{hi-res}} \sim t^{-1+\text{const.} x/\ln(W/A)}. \tag{13}
\]

up to an overall constant due to the combinatorics of \( n \)-th order processes. Thus, higher-order processes give rise to a power law that is (a) sensitive to the drive amplitude \( A \), and (b) can be either positive or negative. In the limit \( A,\omega \to 0 \), we expect these processes to be subleading (since \( A/W \to 0 \)) but for numerically accessible frequencies, it is plausible that these processes will be relevant for the late-time dynamics.

C. Thermal Griffiths inclusions

So far, we have focused on heating processes involving isolated two-level systems inside the MBL phase. A separate channel for response and heating comes from thermal “inclusions,” or thermalizing islands embedded in a localized bulk. We expect this channel to be particularly important near the delocalization transition. To explore it, we first discuss the response due to a single deeply thermal segment of length \( L \), with linear response conductivity \( \sigma_{\text{th}}(\omega) \). As discussed in Sec. VB, the heating rate of this inclusion is given by the linear response result \( \sim A^2/W\sigma_{\text{th}}(\omega) \), and saturates on a time scale \( t_s \sim 1/A \min(e^{L^2/2W}/A) \). We are interested in relatively small islands, and in the \( A/W \to 0 \) limit, so we shall consider the first case \( t_s \sim (Ae^{-L^2/2}/\omega)^{-1} \). Moreover, the probability of having a thermal inclusion of length \( L \) goes as \( p_L \), where \( p \) is some probability per unit length that vanishes deep in the localized phase, and presumably approaches unity at the delocalization transition [30,34].

Let us now consider the response at time \( t \), such that \( 1/A \lesssim t \lesssim W/A^2 \). At this time, the smallest Griffiths regions that have not saturated have size \( L \sim (2/s) \ln(At) \); the density of such rare regions decreases as \( t^{-2\ln(1/p)s} \), and each region contributes \( (A^2/W)\sigma_{\text{th}}(\omega)t \) to the energy spread. Combining these results, we find that the Griffiths contribution (from strongly thermal inclusions) to the heating rate is given by

\[
(\Delta E)^2_{\text{Griff}} \sim A^{2−g} t^{1−g}, \tag{14}
\]

where \( g \equiv 2 \ln(1/p)/s \) is expected to be generically small, and thus the overall exponent is expected to be generically positive, near the MBL transition.

The above estimate is for the contribution from thermal inclusions. However, it is possible that fractal critical inclusions could give an even faster heating rate: in particular, it seems that the probability of critical inclusions might vanish as \( \exp(-g/\ell^d_f) \), where \( d_f < 1 \) [40]. This might lead to a parametrically faster energy spread than the simple thermal inclusions we are considering; however, at present the heating behavior of such critical inclusions is unclear.

D. Landau-Zener transitions

In addition to the perturbative resonances discussed above, one expects that absorption due to Landau-Zener processes should also be important in the low-frequency limit. The Landau-Zener contribution has two regimes, depending on the scale \( x_{\text{ad}} \), which separates mostly adiabatic resonances from mostly diabatic ones: when \( x_{\text{ad}} \ll x_{\text{LZ}} \) [i.e., \( A^{1−x/\zeta}/\omega^{x/\zeta} W^{1−x/\zeta} \)] the “active” Landau-Zener transitions, i.e., those that have an appreciable probability of being both diabatic and adiabatic, are rare and isolated, and can thus be treated individually. In the opposite limit \( x_{\text{ad}} \gg x_{\text{LZ}} \), the Landau-
Zener transitions form a percolating network, and the system delocalizes.

1. Isolated Landau-Zener transitions

We now estimate the heating rate due to isolated Landau-Zener transitions. In general, a transition that is always adiabatic or always diabatic does not contribute to energy spread (see Sec. IV B); rather, the time scale on which a given Landau-Zener transition acts dissipatively (or, equivalently, loses memory of its initial state) is given by

$$T(x) \simeq 1/|\omega P_{ad}(x)(1 - P_{ad}(x))|. \quad (15)$$

In the regime we are considering, Landau-Zener transitions are isolated. Thus, on time scales long compared with $T(x)$, all Landau-Zener transitions at a length scale $x$ are saturated and do not contribute to heating. Let us consider the response at time $t$. Then, the leading contribution to heating will be from transitions with $T(x) \simeq t$. At long times, this means the transitions that have not yet saturated are mostly adiabatic or mostly diabatic. The phase space for mostly diabatic transitions, approximating Landau-Zener transitions at a length scale $x \gtrsim x_{ad}$, is larger (because these correspond to larger-scale rearrangements, of which there are more) so we focus on those. For such transitions, $P_{ad} \ll 1$, so we can simplify Eq. (15) by approximating $P_{ad} \sim W^2 \exp(-2x/\zeta)/(Ax\omega)$ to write

$$x(t) \simeq (\zeta/2)\ln[W^2 t/(Ax)]. \quad (16)$$

At a length scale $x(t)$, there are $(A/W)\exp[ sx(t) ] \sim A^{1-s/2}/t^{s/2}$ Landau-Zener crossings. Each of these contributes $\sim A x(t)$ of energy. Thus, up to logarithmic factors, the total Landau-Zener contribution to energy spread is

$$(\Delta E)^2_{LZ} \sim A^{2-s/2}t^{s/2}. \quad (17)$$

This analysis is incomplete because it ignores interference between subsequent Landau-Zener transitions. Thus, it would naively suggest that any degree of freedom always delocalizes at sufficiently long times because a Landau-Zener crossing inevitably takes place. Interference effects qualitatively modify this picture at long distances, as discussed in Appendix B, ensuring the stability of the MBL phase. In a simple model where the Landau-Zener transitions can be treated as entirely isolated, this approach gives us the late-time asymptotic result is $\sim A^{2-s/2}/t^{s/2}$. It is not clear, however, that this result is correct for the setup we have in mind, in which the drive is suddenly turned on at time $t = 0$. In this setup, even the typical effective spins (which are not involved in resonant or Landau-Zener transitions) nevertheless exhibit weak precessional dynamics and only undergo quantum revivals at very long times (as discussed, e.g., in Ref. [29])

Thus, the environment of a given Landau-Zener transition is never exactly $\omega$ periodic, which complicates a full analysis of interference between Landau-Zener transitions.

2. Percolating network of Landau-Zener transitions

When the drive amplitude is large enough that $x_{ad} \approx x_{LZ}$, then a chain of Landau-Zener transitions percolates through the system. This leads to a delocalized steady state in which the system heats up to infinite temperature. It seems plausible (as discussed below) that the delocalized state near the percolation transition exhibits anomalous transport.\(^4\) Even in such an anomalous-transport phase, however, the long-time heating behavior, for finite-frequency driving, is expected to be linear in time, i.e., the finite-frequency linear response coefficients are well defined in this phase in the high-temperature limit (see Ref. [57], Sec. 5.4). Nevertheless, close to the drive-induced delocalization transition, the typical relaxation time scales are very long; absorption on much shorter time scales is dominated by single Landau-Zener transitions, as discussed in the previous section. We emphasize that this "physical" charge diffusion is not to be confused with the "fictitious" diffusion process discussed in Sec. II.

E. Summary and Floquet perspective

In this section, we have discussed various mechanisms that cause heating on time scales $t \gtrsim 1/A$: anomalously large-scale (and therefore slow) Mott resonances, higher-order processes in the drive amplitude, thermal Griffiths inclusions, and Landau-Zener transitions. We have argued that all these effects give rise to nonlinear heating characterized by continuously varying power laws in time (owing to a wide distribution of saturation time scales), but the exponent can be negative, e.g., with anomalously large Mott resonances, or positive, as with Griffiths inclusions and Landau-Zener crossings (in the intermediate-time window where interference effects are not important).

These results are relevant for intermediate times. However, at asymptotically late times, these behaviors all reduce to two types: power-law approach to a saturated value of $\Delta E^2 \sim t^{-\phi}$, or linear growth in case the Landau-Zener transitions percolate. These can be understood from the following complementary perspective. One can regard the protocol we have discussed as being a quantum quench into an effective Floquet Hamiltonian $\hat{H}_F$, defined via $\exp[-i\Delta \hat{H}_F/\omega] = \hat{U}(2\pi/\omega)$, which is itself either MBL or delocalized. The late-time behavior after such quenches is well understood in both the MBL and thermal phases. When the Floquet Hamiltonian is itself localized, local operators approach their eventual expectation values with a slow power law [28,30]. On the other hand, when the Floquet Hamiltonian is deep in its thermal phase, one naively expects essentially linear heating at times $\gg 1/A$.

This Floquet perspective also suggests that near the drive-induced delocalization transition, the system should be in a Griffiths phase with anomalous charge diffusion and associated slow dynamics. The delocalization transition point depends on $\Delta$, which is spatially fluctuating. Thus, in the delocalized phase near the transition, there will be regions of the system, e.g., with anomalously small $\Delta$, that are locally still in the Floquet-MBL phase, and these will presumably act as transport bottlenecks. Late-time dynamics after a quench into such a Floquet Hamiltonian with anomalous charge diffusion (note that charge, unlike energy, is conserved by the drive) has

\(^4\)Note that although energy is not conserved in the driven system, there might be other conserved quantities (such as $\sigma^z$ in the XXZ model we studied numerically), so that it is meaningful to discuss transport.
not been explored in detail. A simple estimate is that the heating at late times $t$ is governed by the density of locally insulating regions at time $t$ (as these take a long time to heat up). This would then suggest \[ 36 \] that the late-time approach to saturation should go as $t^{-1/2}$, and thus should go logarithmically at the critical point. This is consistent with what is seen numerically (see below, and Ref. [46]).

VII. NUMERICAL RESULTS

To support our analytic estimates, we perform numerical simulations on the random-field XXZ chain, described by the Hamiltonian

$$
\hat{H} = \frac{J}{2} \sum_{\langle ij \rangle} (\hat{S}_i^+ \hat{S}_j^- + \text{H.c.}) + J_z \sum_i \hat{S}_i^z \hat{S}_{i+1}^z + \sum_i h_i \hat{S}_i^z,
$$

(18)

where $h_i$ is a local random field drawn from a uniform distribution of range $[-W, W]$, $J$ is the spin exchange scale, and $J_z$ the spin-spin coupling strength, which we set equal and use as energy unit throughout this work. The monochromatic drive

$$
\hat{H}_{\text{drv}}(t) = A \sin \omega t \sum_i x_i \hat{S}_i^z
$$

(19)

is switched on for $t \geq 0$. For our purposes, it is necessary to use a monochromatic drive, instead of the square-wave drives in Refs. [42,58]. While implementing a square-wave drive is numerically simpler, it complicates the extraction of frequency-dependent response because the higher harmonics of the drive (corresponding to larger $\omega$) have higher conductivity and thus dominate the heating at short to intermediate times. We initialize the dynamics by an eigenstate of $\hat{H}$ and propagate it in time by discretizing the time-evolution operator $\hat{U}(t) = \mathcal{T}_e \exp[-i \int_0^t dt' [\hat{H} + \hat{H}_{\text{drv}}(t')]] \approx \prod_{n=1}^N \exp[-i \Delta t (\hat{H} + \hat{H}_{\text{drv}}(n \Delta t))], \text{ where } \Delta t = t/N$. The stepwise propagation is performed by Lanczos time evolution which allows us to efficiently update the instantaneous Hamiltonian $\hat{H} + \hat{H}_{\text{drv}}(n \Delta t)$. All data are taken for systems with open boundary conditions in order to avoid the jump of the electric field in space.

A. Linear response regime

First, we check for the validity of linear response theory, which should apply for any fixed frequency when the amplitude goes to zero. We find, indeed, that for small-amplitude driving there is a considerable regime where the energy spread is linear (as linear response theory would predict) (see Fig. 4). In this regime, increasing the drive strength does not change the exponent, but causes saturation to set in sooner.

In order to further benchmark this dynamical regime against linear response theory, we extract the rate of energy spread (i.e., the prefactor of the linear regime) as a function of frequency, and compare it with the linear response exponents obtained in Ref. [30] for systems with open boundary conditions. We find that the two sets of exponents are largely consistent (see inset of Fig. 4). Near the MBL transition and on the ergodic side a direct comparison gets complicated by finite-size effects which are different for the Kubo conductivity and the energy spread. However, deep in the localized phase, the exponents agree reasonably.

B. Nonlinear response and amplitude dependence

We now turn to the nonlinear response of larger drive amplitudes. In Fig. 5, we show the energy spread for different values of disorder strength $W$, ranging from the ergodic to the localized phase for fixed driving frequency $\omega = 0.1J$ and drive strength $A$ that is weak in (a) $A = 0.001J$ and strong in (b) $A = 0.1J$. In the weak drive limit the response is linear in a wide time window irrespective of the disorder strength $W$ (cf. inset which shows the exponent as a function of disorder strength). By contrast, for strong drive, the power-law exponent of the energy spread decreases significantly with disorder strength (inset), indicating that the sublinear regime has been entered. This behavior has been predicted by all the mechanisms discussed in Sec. VI. For even stronger drive, a percolating network of Landau-Zener transition forms,
FIG. 6. Response of percolating Landau-Zener transitions. In the strong drive limit, Landau-Zener transitions form a percolating network and the effective Floquet Hamiltonian is delocalized. In the crossover to that regime, we find that the energy spread grows slower than naively expected as a logarithm in time (in contrast to the power-law growth which we predict in the linear response and the intermediate nonlinear regime), consistent with the findings of Refs. [45,46]. The data are shown for driving frequency \( \omega = 0.1 J \), drive amplitude \( A = J \), and system size \( L = 12 \), for different values of the disorder strength \( W \) as stated in the legend.

and the energy spread changes from power-law slow heating to logarithmically slow heating (Fig. 6) consistent with the findings of Refs. [45,46]. This logarithmic growth of the energy spread in time is characteristic for the crossover regime to the thermal phase [46] and is slower than the naively expected linear growth for a Floquet Hamiltonian being deep in the thermal phase.

We now study the drive amplitude dependence at strong disorder \( W = 6 J \) [Fig. 7(a)]. For intermediate driving amplitudes \( A \gtrsim 0.01 J \), the energy spread grows sublinearly in time, with a power law that increases weakly with the amplitude. We conjecture that this dependence on the drive amplitude arises from higher-order resonances as discussed in Sec. VI B.

The frequency dependence of the energy spread \( \langle (\Delta E)^2 \rangle \) for intermediate driving amplitude \( A = 0.01 J \) transitions from sublinear growth at low driving frequency to an intermediate linear growth at higher frequency [see Fig. 7(b)], in agreement with the picture of saturating two-level systems.

C. Additional probe: Edwards-Anderson parameter and von Neumann entanglement entropy

A complementary perspective to switching on the periodic modulation is to regard it as a quantum quench from the original Hamiltonian to the Floquet Hamiltonian. From this perspective, a key question is whether the corresponding Floquet Hamiltonian is localized or delocalized. We have explored this issue by looking at the evolution of the Edwards-Anderson parameter (or Hamming distance [59])

\[
\chi(t) = \frac{4}{L} \sum_i \langle p | \hat{U}^\dagger(t) S_i^z \hat{U}(t) S_i^z | p \rangle,
\]

where \( | p \rangle \) is an arbitrary product state which we take as a random initial state. A special case of the Edwards-Anderson order parameter is the decay of contrast of an initial staggered magnetization, which has been used as an order parameter in recent experiments [14–16]. In the MBL phase and for a drive in linear response regime \( A \ll \omega \), the Edwards-Anderson order parameter saturates in the infinite-time limit to a finite value, since at weak drive the effective Floquet Hamiltonian remains to be localized [Fig. 8(a), top]. By contrast, in the strong drive limit \( A \gg \omega \), it decays to zero since the effective Floquet Hamiltonian is thermal [Fig. 8(b), top], which confirms that for the strong drive considered in Fig. 6 a percolating network of Landau-Zener transitions has been formed.

In addition, we have computed the von Neumann entanglement entropy growth due to the drive (Fig. 8, bottom row). Well in the localized regime, \( W = 6 J \) and for weak driving amplitude \( A = 0.001 J \), the entanglement entropy does not
FIG. 8. Edwards-Anderson order parameter and von Neumann entanglement entropy. The Edwards-Anderson order parameter (or Hamming distance), top row, and entropy, bottom row, for (a) weak drive $A = 0.001 J$ and (b) strong drive $A = J$, driving frequency $\omega = 0.1 J$, and three values of disorder strength $W = \{4, 5, 6\} J$. The system size is $L = 12$ (solid lines) and $L = 16$ (dashed lines). For weak drive, the system and hence the effective Floquet Hamiltonian remain localized, while for strong drive, it delocalizes manifesting in a decay in the Edwards-Anderson parameter and a strong increase of the entanglement entropy.

VIII. DISCUSSION

Our objective in this work was to identify regimes for which linear response theory correctly predicts the dynamics of a driven MBL system, and those for which the response is essentially nonlinear. Our key results are that heating in the finite-frequency, weak-drive regime is essentially conventional (corresponding to linear response theory with the appropriate conductivity), whereas the behavior at larger drive amplitudes (or lower frequencies) is not. It seems that in this regime neither the amplitude dependence nor the time dependence of the heating correspond to linear response predictions. Rather, as we discussed, both are characterized by continuously varying power laws. The predicted nonlinear behavior in time is clearly seen in numerical simulations; these simulations also suggest nonlinear dependence on the amplitude, although we could not extract the precise form of the amplitude dependence. A feature that is distinctive to the MBL phase is the existence of a broad parameter regime in which linear response theory breaks down, i.e., the transient response to driving changes its character, although the eigenstates of the Floquet Hamiltonian remain localized. This intermediate regime shrinks to a point as the MBL transition is approached (Fig. 2, inset): there, the breakdown of linear response coincides with the breakdown of the Floquet-MBL steady state. (We are assuming here that $\xi < \zeta$ at the MBL transition, as conjectured in Ref. [30]. It is also possible that the transition occurs for $\xi > \zeta$, in which case a small intermediate regime would persist at the transition.)

Although, for reasons of numerical tractability, we worked in the infinite-temperature limit and with one-dimensional systems, we expect that the same regimes of heating should exist throughout the MBL phase regardless of temperature or of dimensionality. We emphasize that since most of our discussion has concerned the dynamics relatively deep in the MBL phase, it is not expected to be sensitive to finite-size effects until very late times [specifically, times on the order of $\exp(L/\zeta)$ where $L$ is the linear dimension of the system]. Thus, in experiments finite-time effects, such as dissipation, are likely to pose a challenge for our schemes than finite-size effects. Because the distinction between the Landau-Zener and Mott regimes is a generic feature of response in MBL systems, we expect that alternative time-dependent probes, such as modulation spectroscopy [60].
will also be able to see the differences between the various regimes.

It is natural to ask about the fate of this linear-to-nonlinear response crossover beyond the MBL transition, i.e., in the subdiffusive thermal Griffiths phase. We now briefly discuss this at a qualitative level. Suppose the undriven system is in its Griffiths phase. Then, its transport is bottlenecked by rare regions that are locally “in the MBL phase.” However, when one drives the system at large $A/\omega$, some fraction of these rare regions become delocalized by the drive (because they locally satisfy the condition that $A^{1-\xi/2} \sim \omega^{\xi/2} W^{1-\xi}$). Thus, they cease to act as bottlenecks unless their local $s\xi$ is sufficiently small. As one continues to increase $A/\omega$, an increasing fraction of rare regions delocalize, until eventually the remaining bottlenecks become too sparse to prevent regular diffusion. Thus, our results directly imply that the $A \to 0$ and $\omega \to 0$ limits fail to commute in the thermal Griffiths phase as well as the MBL phase: taking $A \to 0$ first gives anomalous diffusion whereas taking $\omega \to 0$ first gives regular diffusion. Our findings thus suggest the schematic phase diagram of Fig. 9, which shows the linear and nonlinear response regimes as a function of disorder strength and the ratio of the drive amplitude and frequency. Driving a system in the MBL phase with increasingly strong fields leads to a transient crossover from linear to nonlinear response (dashed lines), which need not coincide with the dynamical steady-state transitions of the Floquet Hamiltonian from a localized phase, to a subdiffusive Griffiths phase, and finally a diffusive phase (solid lines). Up to logarithmic corrections, our analysis suggests that the crossover from linear to nonlinear response should occur at $A \sim \omega$ throughout the MBL phase, including at the critical point and in the thermal Griffiths phase. This result for the critical behavior is natural [61] if we take the critical point to be an infinite-randomness one, as suggested in Refs. [9,38,39]:

\[ \sigma_{ss}(A) \sim \frac{\omega_s}{\sigma}(A/\gamma) \]

\[ \sim \frac{\omega_s}{\gamma} \left( \frac{A}{\gamma} \right) \]

Thus, for strong drives or near the transition, our arguments suggest that the steady-state nonlinear conductivity is a continuously varying power law of the system-bath coupling. Confirming this conjecture numerically would, however, require detailed master-equation simulations [64] that are outside the scope of this work. Beyond these quantitative features, we expect that the steady state of the driven dissipative system will have a highly inhomogeneous temperature profile in the linear response regime (with hot spots near resonances), but become relatively homogeneous at strong temperature when the Floquet Hamiltonian is thermal. Understanding these crossovers is an important step for a full dynamical characterization of the MBL phase.

One can directly extend this idea to estimate the steady-state conductivity for weakly dissipative systems in the nonlinear regime, by substituting $\gamma \sim 1/t$ in our results for the time-dependent energy spread $(\Delta E)^2(t)$, and then dividing this steady-state energy absorption by $A^2$. Thus, for strong drives or near the transition, our arguments suggest that the steady-state nonlinear conductivity is a continuously varying power law of the system-bath coupling. Confirming this conjecture numerically would, however, require detailed master-equation simulations [64] that are outside the scope of this work. Beyond these quantitative features, we expect that the steady state of the driven dissipative system will have a highly inhomogeneous temperature profile in the linear response regime (with hot spots near resonances), but become relatively homogeneous at strong temperature when the Floquet Hamiltonian is thermal. Understanding these crossovers is an important step for a full dynamical characterization of the MBL phase.

**Note added.** Recently, we became aware of other numerical studies of the dynamical response in strongly driven many-body localized systems [45,46], as well as a related, as yet unpublished, study of the response “phase diagram” of driven MBL systems [65].

**ACKNOWLEDGMENTS**

We thank D. Abanin, I. Bloch, B. Bordia, B. DeMarco, M. Heyl, D. Huse, H. Lüschen, I. Martin, R. Nandkishore, V. Oganesyan, S. Parameswaran, F. Pollmann, and U. Schneider for helpful discussions. We thank D. Huse for a critical reading of the manuscript. S.G. acknowledges financial support from the Walter Burke Institute at Caltech and from the National Science Foundation under Grant No. NSF PHY11-25915. M.K. acknowledges financial support from Technical University of Munich-Institute for Advanced Study, funded by the German Excellence Initiative and the European Union FP7 under Grant Agreement No. 291763. E.D. acknowledges support from the Harvard-MIT CUA, NSF Grant No. DMR-1308435, AFOSR Quantum Simulation MURI, the ARO-MURI on Atomtronics, ARO MURI Quism program, the Simons foundation, the
Humboldt Foundation, Dr. M. Rössler, the Walter Haefner Foundation, and the ETH Foundation.

APPENDIX A: DIFFUSION ACROSS CONCENTRATION GRADIENTS

The most commonly considered case of biased diffusion is that in which particles are subjected to noise (which causes diffusion) as well as a deterministic force, such as an electric field (which causes drift). The situation considered here is somewhat different. We are concerned with the random walk of a “particle” (i.e., an initial configuration) in a high-dimensional configuration space. The random walk itself is unbiased, in the sense that the rates for energy-increasing and energy-decreasing transitions mediated by the drive are identical.

However, the gradient comes in via the initial conditions: lower-energy configurations are slightly more likely to be occupied at \( t = 0 \), when the drive is switched on. Since the driven dynamics itself is “unbiased” it is equally likely to heat or cool the system on any cycle; thus, over time the driven system tends to “forget” its initial gradient. (When the Floquet Hamiltonian is thermal this causes heating to infinite temperature; when the Floquet Hamiltonian is localized, most degrees of freedom are unaffected by the drive, but the few responsive degrees of freedom precess with random phases.)

To make this idea more concrete, we assume that heating occurs via local processes, and that each region of the system (above a certain characteristic size \( L \)) heats up independently. This assumption is manifestly valid in the MBL phase; we also believe it to be valid deep in the thermal phase. We take the temperature \( T \) to be greater than \( L W \), where \( W \) is the single-particle bandwidth. This allows us to linearize the Boltzmann factors for the various states in the system as \( \exp(-E_n/T) \sim 1 - E_n/T \). This linear energy dependence of Boltzmann factors maps onto a linear concentration gradient in the energy-space diffusion problem. Note that the boundedness of the energy spectrum maps on to the finite extent of space over which the concentration gradient is present.

A straightforward application of these ideas is to a generic two-state system, with states labeled 1 and 2 (having energies \( E_1 \) and \( E_2 \) and occupation probabilities \( P_1 \) and \( P_2 \)). The master equation for \( P_1 \) reads as \( \dot{P}_1 = \Gamma_{21} P_2 - \Gamma_{12} P_1 \), where the \( \Gamma \)'s are intrinsic transition rates. Since these rates are unbiased (as discussed above), we have \( \dot{P}_1 = \Gamma (P_2 - P_1) \), and similarly \( \dot{P}_2 = \Gamma (P_1 - P_2) \). Subtracting these rates, we have that

\[
\frac{d(P_1 - P_2)}{dt} \approx -\Gamma (P_1 - P_2),
\]

so the initial concentration gradient decays at a rate \( \Gamma \), which is also evidently the rate of “energy spread” in this two-site example, as it is the rate at which the system undergoes transitions between configurations (“sites”) of definite energy.

APPENDIX B: THEORY OF MOSTLY DIABATIC LANDAU-ZENER CROSSINGS

For a given crossing one can rewrite the time-dependent Hamiltonian in a rotating frame in the form (see Appendix C of Ref. [55])

\[
H' = \sum_n \Delta \sqrt{\omega/\mathcal{A}} \left[ \exp(-i\omega t)\sigma_+ + \text{h.c.} \right] + \epsilon_0 \sigma_z. \tag{B1}
\]

The sum over \( n \) is cut off on a scale \( n \sim A/\omega \). The matrix element \( A \) is the bare hopping at the scale of the particular TLS, \( \Delta \sim \mathcal{W} \exp(-n/\xi) \). We have assumed \( \omega \ll A \) as Landau-Zener transitions are important chiefly in this regime. At a large distance \( x \), the first term in Eq. (B1) can be treated perturbatively in the spirit of the rotating-wave approximation.

The bandwidth of states involved in LZ transitions at this distance \( \sim Ax \), and there are \( n(x) \sim Ax/\omega \) harmonics within this window. States that lie within \( \Delta(x)\sqrt{\omega/\mathcal{A}} \) of one of these \( n(x) \) harmonics of the drive frequency are resonant in the Floquet picture, and cause transport. When \( x \) is relatively small, \( \Delta(x)\sqrt{\omega/\mathcal{A}} \gtrsim Ax/n(x) = \omega \). Thus, different harmonics overlap, and any transition within the drive bandwidth \( Ax \) occurs (as the LZ picture would predict). However, when \( x \) is large and \( \Delta(x) \) is correspondingly small, the inequality is flipped, and most transitions that are “allowed” on a naive LZ analysis are in fact off resonant and do not contribute to transport. Thus, the MBL phase is stable against extremely long-distance LZ transitions. These transitions can instead be treated using a straightforward generalization of the Rabi-formula approach in the main text, with the matrix element \( \sim A \) replaced by \( \Delta \sqrt{\omega/\mathcal{A}} \).

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