Bayesian Inference using the Symmetric Monoidal Closed Category Structure

Kirk Sturtz

Abstract

Using the symmetric monoidal closed category structure of the category of measurable spaces, in conjunction with the Giry monad which we show is a strong monad, we analyze Bayesian inference maps and their construction in relation to the tensor product probability. This perspective permits the inference maps to be seen as a pullback construction.

1 Introduction

A theory for constructing Bayesian inference maps is developed by exploiting the symmetric monoidal closed category (SMCC) structure of the category of measurable spaces, $\text{Meas}$. Using this property the inference maps can be constructed as pullbacks.

While the construction of these maps requires the SMCC structure of $\text{Meas}$, the Bayesian inference problem itself is most naturally characterized within the Kleisli category of the Giry monad, $K(\mathcal{G})$, where $\mathcal{G}$ denote the Giry monad on the category of measurable spaces, $\text{Meas}$. In this introduction we provide a general overview of this perspective which permits one to quickly understand precisely what the Bayesian inference problem entails.

A Bayesian model is a pair $(P_X, P_{Y|X})$ consisting of a probability measure $P_X \in \mathcal{G}(X)$, and a (regular) conditional probability measure $P_{Y|X} : X \rightarrow \mathcal{G}(Y)$.[4] Thus at each point $x \in X$, $P_{Y|X}(\cdot|x)$ is a probability measure on $Y$, and this conditional probability $P_{Y|X}$ in turn determines the conditional probability $P_{X\times Y|X}$ defined as the probability measure on $X \times Y$ at every $x \in X$ as the composite

$1 \xrightarrow{\delta_x} X \xrightarrow{\delta_y} X \times Y \xrightarrow{P_{X\times Y|X}} Y$

$[1]*$A regular conditional probability is also commonly referred to as a kernel.
1 INTRODUCTION

giving the push forward probability measure $P_{X \times Y | X} (\cdot | x) = P_{Y | X} (\Gamma^{-1}_x (\cdot) , x)$, where

\[
\begin{array}{cccc}
Y & \xrightarrow{\Gamma_x} & X \times Y \\
\downarrow y & & & \downarrow (x, y)
\end{array}
\]

is the constant graph map. On the other hand, given $P_{X \times Y | X}$ we have

\[
P_{X | Y} (A | y) = (\delta_{\pi_X} \ast P_{X \times Y | Y}) (A | y) = P_{X \times Y | Y} (\pi_Y^{-1} (A) | y)
\]

where $\pi_X : X \times Y \to X$ is the coordinate projection map. These two processes are inverse to each other, and hence knowledge of either $P_{X | Y}$ or $P_{X \times Y | Y}$ solves the inference problem.

Given a Bayesian model $(P_X, P_{Y | X})$ we can construct the $K(G)$-diagram

\[
\begin{array}{c}
1 \\
\downarrow P \downarrow \\
X \times Y \\
\downarrow P_{X | Y} \downarrow \\
X \\
\downarrow P_{X \times Y | X} \downarrow \\
Y
\end{array}
\]

Diagram 1: The Bayesian inference problem.

where by definition $P = P_{X \times Y | X} \ast P_X$, and $P_Y = P_{\pi_Y^{-1}}$ denotes the marginal probability measure on $Y$ given $P$. An inference map for the Bayesian model is a conditional probability $P_{X \times Y | Y}$ such that one can use $P_Y$ and $P_{X \times Y | Y}$, in lieu of $P_X$ and $P_{X \times Y | X}$, to determine the same joint probability measure $P$ on $X \times Y$,

\[
P_{X \times Y | X} \ast P_X = P = P_{X \times Y | Y} \ast P_Y
\]

which is Bayes equation. Using the constant graph maps Bayes equation can be written

\[
\int_X P_{Y | X} (\Gamma_x^{-1} (\zeta) | x) \, dP_X = \int_Y P_{X | Y} (\Gamma_y^{-1} (\zeta) | y) \, dP_Y \quad \forall \zeta \in \Sigma_{X \times Y}
\]

Provided we restrict Meas to the category of Standard Spaces or Polish Spaces, the existence of inference maps is well known.\[3, 10\] It is possible to extend this class by using $D$-kernels rather than kernels which are regular conditional probabilities which we have been, and will continue, to call conditional probabilities. Using $D$-kernels requires an awkward category and hence we assume Standard Spaces. This issue is discussed further.
in the appendix. Our choice of Standard Spaces is philosophical based upon the idea probability theory should be independent of any topological properties. A good source covering the essential aspects of Standard Spaces is [5] Chapter 2. Faden [3, Proposition 5] shows that almost pre-standard spaces suffice.

The objective of this paper is to serve as a stepping stone to develop better computational methods for inference. Towards that end, we believe it is necessary to exploit the SMCC structure of Meas, in conjunction with the strong monad structure of G, to develop better computational schemes for inference. The basic structural maps, which are natural transformations, discussed herein provide the basis for such developments.

2 Some Background and Notation

A measurable function is called a map or a random variable. The σ-algebra associated with the product space $X \times Y$ is the product σ-algebra. If a probability on any space assumes only one of the two values $\{0, 1\}$ we call the probability deterministic. Every map $f : X \to Y$ induces a deterministic probability, defined for all $B \in \Sigma_Y$ and $x \in X$ by

$$\delta_f(B|x) = \begin{cases} 1 & \text{if } f(x) \in B \\ 0 & \text{otherwise} \end{cases}.$$ 

Using the graph of $f$, $\Gamma_f : X \to X \times Y$ which maps $x \mapsto (x, f(x))$ gives the deterministic measure

$$\delta_{\Gamma_f}(\zeta|x) = \begin{cases} 1 & \text{if } (x, f(x)) \in \zeta \\ 0 & \text{otherwise} \end{cases}$$

which plays an essential role in the theory.

If a Bayesian model $(P_X, P_{Y|X})$ has a deterministic probability $P_{Y|X}$ then we say we have a deterministic Bayesian model, and write it as $(P_X, \delta_f)$ where $f : X \to Y$ is the map giving rise to the deterministic probability. If $P_{Y|X}$ is not deterministic then we say the model $(P_X, P_{Y|X})$ is a nondeterministic Bayesian model.

The tensor product monad $(\text{Meas}, \otimes, 1)$ structure on Meas, defined on the objects by

$$\text{Meas} \times \text{Meas} \quad \otimes \quad \text{Meas}$$

$$X \times Y \quad \otimes \quad X \otimes Y$$


\[\text{In a countably generated measurable space very deterministic measure arises from a measurable function.} \] \[\text{Consequently, under the assumption of Standard Spaces, every deterministic measure arises from a map } f \text{ and vice versa.} \]
where $X \otimes Y$ is the set $X \times Y$ with the $\sigma$-algebra generated by the family of constant graph maps, $\Gamma_x$ and $\Gamma_y$,

$$
\begin{array}{c}
X \xrightarrow{\Gamma_y} X \otimes Y \xleftarrow{\Gamma_x} Y \\
x \mapsto (x, y) \mapsto y
\end{array}
$$

makes $\text{Meas}$ a symmetric monoidal closed category. Thus the evaluation maps

$$
\begin{array}{c}
X \xrightarrow{\Gamma'_f} X \otimes Y^X \xleftarrow{\Gamma_x} Y^X \\
f \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \jan

$$

where the function spaces $Y^X$ are defined such that the point evaluation maps are all measurable. Recall, the product $\sigma$-algebra is a sub $\sigma$-algebra of the tensor $\sigma$-algebra, and, typically we employ the tensor product $\sigma$-algebra only when we require the use of the evaluation maps. Without the finer $\sigma$-algebra the tensor product $\sigma$-algebra provides, the evaluation maps are not measurable.

### 3 Some structural maps in $(\text{Meas}, \otimes, 1)$

The monoidal closed structure of $\text{Meas}$, in conjunction with the strong monad structure of $\mathcal{G}$, provides several natural transformation which are indispensable tools. These basic “structural maps” are due to A. Kock [7, 8]. As we need to show $\mathcal{G}$ is a strong monad it is necessary to prove

**Theorem 3.1.** There is a natural transformation between the two functors

$$
\cdot \otimes \mathcal{G}(: \cdot \cdot) : \text{Meas} \times \text{Meas} \to \text{Meas}
$$

defined at component $(X, Y)$ by

$$
X \otimes \mathcal{G}(Y) \xrightarrow{\tau_{X,Y}^\mu} \mathcal{G}(X \times Y).
$$

\begin{equation}
(x, Q) \mapsto Q \Gamma_x^{-1}
\end{equation}

---

3 Further details concerning the SMCC structure of $\text{Meas}$ can be found in [9]. The necessary proofs follow readily from the definition of the final $\sigma$-algebra on the set $X \times Y$ via the constant graph maps.
Proof. To verify $\tau''$ is measurable consider the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\Gamma_Q} & X \otimes G(Y) \xrightarrow{\Gamma_x} G(Y) \\
\downarrow{\tau''} & & \downarrow{ev_{\Gamma_x^1(\zeta)}} \\
G(X \times Y) & \xrightarrow{ev_{\zeta}} & [0,1] \\
\downarrow{ev_{\zeta}} & & \downarrow{ev_{\zeta}} \\
Q & & Q \Gamma^{-1}_x(\zeta)
\end{array}
\]

For a fixed $Q \in G(Y)$, the composite mapping $x \mapsto Q \Gamma^{-1}_x(\zeta)$ is well known to be measurable\[1\), Proposition 5.1.2, p156\]. On the other hand, for a fixed $x \in X$, the composite map on the right hand triangle in the above diagram is precisely the evaluation map $ev_{\Gamma_x^1(\zeta)}$, and these maps generate the $\sigma$-algebra on $G(Y)$. Thus, for every $\zeta \in \Sigma_{X \times Y}$,

\[
\Gamma^{-1}_x(\tau''_{X,Y}^{-1}(ev^{-1}_\zeta(U))) \in G(Y) \quad \forall U \in \Sigma_I
\]

Since these are all measurable in $G(Y)$, and $X \otimes G(Y)$ has the largest $\sigma$-algebra such that all the constant graph maps are measurable it follows the argument of $\Gamma^{-1}_x$ in the above equation is measurable. Thus $\tau''_{X,Y}^{-1}(ev^{-1}_\zeta(U)) \in \Sigma_{X \otimes G(Y)}$. Now, using the fact $G(X \times Y)$ is generated by the evaluation maps $ev_{\zeta}$, the measurability of $\tau''_{X,Y}$ follows.

To prove naturality in the first argument note that for $X \xrightarrow{f} X'$ we obtain the $\text{Meas}$ arrow $G(X) \xrightarrow{G(f)} G(X')$ mapping $P \mapsto Pf^{-1}$. This gives the commutative square

\[
\begin{array}{ccc}
X' \otimes G(Y) & \xrightarrow{\tau''_{X',Y}} & G(X' \otimes Y) & \xrightarrow{G(id_{X',Y})} & G(X' \times Y) \\
f \otimes 1_{G(Y)} & & G(f \otimes 1_Y) & & G(f \times 1_Y) \\
X \otimes G(Y) & \xrightarrow{\tau''_{X,Y}} & G(X \otimes Y) & \xrightarrow{G(id_{X \times Y})} & G(X \times Y)
\end{array}
\]

which maps elements according to

\[
(f(x), Q) \xrightarrow{Q \Gamma^{-1}_{f(x)}} Q \Gamma^{-1}_x(f \otimes 1_Y)^{-1} \xrightarrow{Q \Gamma^{-1}_x} (x, Q)
\]
where equality in the upper right hand corner follows from the observation that

\[
\begin{array}{ccc}
X \times Y & \xrightarrow{f \times 1_Y} & X' \times Y \\
\downarrow{id_{X \times Y}} & & \downarrow{id_{X' \times Y}} \\
X \otimes Y & \xrightarrow{f \otimes 1_Y} & X' \otimes Y \\
\downarrow{\Gamma_x} & & \downarrow{\Gamma_{f(x)}} \\
Y & & .
\end{array}
\]

To prove naturality in the second argument let \( Y \xrightarrow{g} Y' \). (As the restriction to the subspace \( \sigma \)-algebra, \( \Sigma_{X \times Y} \subseteq \Sigma_{X \otimes Y} \), should now be evident, it is not included in the following diagrams.) This gives us the commutative square

\[
\begin{array}{ccc}
X \otimes G(Y') & \xrightarrow{\tau''_{X,Y'}} & G(X \otimes Y') \\
\downarrow{1_X \otimes G(g)} & & \downarrow{G(1_X \otimes g)} \\
X \otimes G(Y) & \xrightarrow{\tau''_{X,Y}} & G(X \otimes Y)
\end{array}
\]

which maps elements according to

\[
(x, Qg^{-1}) \xrightarrow{Q^{-1}\Gamma_x^{-1}} Q^{-1}\Gamma_x^{-1} = Q\Gamma_x^{-1}(1 \otimes g)^{-1}
\]

where equality in the upper right hand corner follows from the observation that

\[
\begin{array}{ccc}
Y' & \xrightarrow{\Gamma_x} & X \otimes Y' \\
\downarrow{g} & & \downarrow{1_X \otimes g} \\
Y & \xrightarrow{\Gamma_x} & X \otimes Y
\end{array}
\]

\[\square\]
Using the fact that the monoidal structure is symmetric we also have a natural transformation defined on components by

\[
\begin{array}{ccc}
\mathcal{G}(X) \otimes Y & \overset{\tau'_{X,Y}}{\longrightarrow} & \mathcal{G}(X \times Y) \\
(P, y) & \mapsto & P \Gamma_y^{-1}
\end{array}
\]

The superscript on \(\tau\) is used to denote which coordinate the probability measures are given.

The natural transformation \(st\)

\[
\begin{array}{ccc}
X^Y & \overset{st_{X,Y}}{\longrightarrow} & \mathcal{G}(X)^{\mathcal{G}(Y)} \\
g & \mapsto & \mathcal{G}(g)
\end{array}
\]

can now be constructed using the natural transformation \(\tau''\) via

\[
\begin{array}{ccc}
Y^X & \overset{st_{X,Y}}{\longrightarrow} & \mathcal{G}(Y)^{\mathcal{G}(X)} \\
\Gamma_{Y^X} & \downarrow & \downarrow \\
(Y^X \otimes \mathcal{G}(X))^{\mathcal{G}(X)} & \overset{\tau''_{Y^X,X}}{\longrightarrow} & \mathcal{G}(Y^X \otimes X)^{\mathcal{G}(X)} \\
\mathcal{G}(ev_{Y^X,X})^{\mathcal{G}(X)} & \mathcal{G}(X)^{\mathcal{G}(Y)}
\end{array}
\]

where \(\Gamma_{Y^X}\) is the unit of the adjunction \(\bullet \otimes \mathcal{G}(X) \dashv \mathcal{G}(\bullet^{\mathcal{G}(X)})\),

\[
\begin{array}{ccc}
Y^X & \overset{\Gamma_{Y^X}^{-1}}{\longrightarrow} & (Y^X \otimes \mathcal{G}(X))^{\mathcal{G}(X)} \\
f & \mapsto & \Gamma_f
\end{array}
\]

with \(\Gamma_f\) is the constant graph function with value \(f\). The equality

\[
\mathcal{G}(f) = \mathcal{G}(ev_{X,Y}) \circ \tau''_{Y^X,X} \circ \Gamma_f
\]

then follows from

\[
\mathcal{G}(f)("P") = (\mathcal{G}(ev_{X,Y}) \circ \tau''_{Y^X,X} \circ \Gamma_f)("P") = \mathcal{G}(ev_{Y^X,X})(\tau''_{Y^X,X}(f, "P")) = \mathcal{G}(ev_{Y^X,X})(P \Gamma_f^{-1}) \text{ note } \Gamma_f \neq \tilde{\Gamma}_f = \tilde{P}^{-f^{-1}}e_{Y^X,X} = P f^{-1}
\]
where the last line follows from the fact the constant graph map $\tilde{\Gamma}_f$ satisfies the equation expressed by the commutativity of the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\tilde{\Gamma}_f} & Y^X \otimes X \\
& \searrow_f & \downarrow_{ev_{X,Y}} \\
& & Y \\
\end{array}
$$

4 The construction of Bayesian inference maps

The construction of Bayesian inference maps using the symmetric monoidal closed category structure of $\text{Meas}$ applies to both the deterministic and nondeterministic Bayesian models. For simplicity we first present the deterministic case and subsequently show the minor changes required to address the nondeterministic Bayesian model.

However, it is worth noting that in constructing Bayesian inference maps it suffices (for standard spaces) to consider the special case where the given conditional probability $P_{Y|X}$ is a deterministic probability. The general problem, as given in diagram (1) can then be solved using the same approach on the Bayesian model $(P_{X\times Y}, \delta_{\pi_Y})$, expressed by the $K(\mathcal{G})$-diagram

$$
\begin{array}{ccc}
P_{X\times Y} & P_{X\times Y} \Gamma_{\pi_Y}^{-1} & P_Y = P_{X\times Y} \pi_Y^{-1} \\
& \delta_{\Gamma_{\pi_Y}} & \\
X \times Y \times Y & \delta_{\pi_Y} & \\
& \delta_{\pi_Y} & \\
X \times Y & P_{X\times Y|Y} & Y
\end{array}
$$

(1)

to construct the conditional probability $P_{X\times Y|Y}$, which in turn specifies $P_{X|Y}$.

4.1 Inference maps for deterministic models

Our generic deterministic Bayesian problem (base model) is represented by the diagram
Diagram 2: The generic deterministic Bayesian model.

where the dashed arrows are the inference maps to be determined.

The inference maps can be constructed by taking the pullback of the Meas-diagram

where the map $ev_{P_Y}$ is evaluation at $P_Y$, i.e., given any map $^rT \in \mathcal{G}(X \times Y)^{G(Y)}$ precompose it with $^rP_Y$. The map $m$ in this pullback diagram is the subset of all $P_{X \times Y | Y} \in \mathcal{G}(X \times Y)^{Y}$, which upon application of the vertical arrows on the right hand side of the diagram, then yield the composite

---

4The notation $^rP_X \Gamma_f^{-1}$ is used to emphasize that the diagram is in Meas, rather than $K(\mathcal{G})$ where probabilities are actual arrows.
which coincides with $P_X\Gamma_f^{-1}$ since the diagram is a pullback. The commutativity of this diagram is Bayes equation for the deterministic model. The fact the diagram is a pullback is trivial as Meas has all limits and the point $'P_X\Gamma_f^{-1}$ is monic so the limit corresponds to a subset of $G(X \times Y)^Y$.

The limit, $\text{Lim}$, in the pullback can be constructed, and characterized, using the tensor product probability $P_X \otimes P_Y$, or alternatively, it can be constructed “pointwise” using the family of tensor probabilities $P_X \otimes \delta_y = P_X \Gamma_y^{-1}$.

**Method 1: Using Radon Nikodym derivatives.** Note that $P_X\Gamma_f^{-1} \ll P_X \otimes P_Y$. On the basic rectangles, $A \times B \in \Sigma_{X \times Y}$ we have $P_X\Gamma_f^{-1}(A \times B) = P_X(A \cap f^{-1}(B)) \leq \min(P_X(A), P_Y(B))$.\footnote{In the standard theoretical approach to computing inference maps the Radon Nikodym (RN) derivatives are computed with respect to measures on the space $Y$ rather than the product space $X \times Y$. Using RN derivatives, either directly or indirectly, on the product space is preferable as subsequent arguments illustrate. (One need not "glue together" a family of RN derivatives.)}

Let $h$ be a Radon Nikodym derivative for this absolute continuity condition,

$$P_X\Gamma_f^{-1}(\zeta) = \int \zeta h \, d(P_X \otimes P_Y) = \int_Y \int \zeta h \, d(P_X \Gamma_y^{-1}) \, dP_Y = \int_Y (\int \zeta h \, dP_X) \, dP_Y \quad (2)$$

As $P_X\Gamma_f^{-1}(X \times Y) = 1$ it follows that $\int_X h(\cdot, y) \, dP_X = 1 \, P_Y - a.e.$. Suppose $V \in \Sigma_Y$ such that $\int_X h(\cdot, y) \, dP_X = 1$ for all $y \in V$.

Then

$$P_{X \times Y|Y}(\zeta|y) \triangleq \left\{ \begin{array}{ll} \int \zeta h(\cdot, \cdot) \, d(P_X \Gamma_f^{-1}) & y \in V \\ P_X\Gamma_f^{-1} & y \notin V \end{array} \right.$$ defines a conditional probability, the inference map for the deterministic model, which is absolutely continuous with respect to $P_X\Gamma_f^{-1}$, and by equation \footnote{The probability $P_X\Gamma_f^{-1}$ is the push forward of the tensor product probability $P_X \otimes P_Y$ by the idempotent map $\Gamma_f \circ \pi_X$, $(P_X \otimes P_Y)(\Gamma_f \circ \pi_X)^{-1} = P_X\Gamma_f^{-1}$.} (2), the weighted sum

$$P_X\Gamma_f^{-1} = \int_Y P_{X \times Y}(\cdot|y) \, dP_Y. \quad (3)$$

The sequence of vertical arrows in the pullback diagram show that this expression make sense without requiring evaluation on any measurable set in $X \times Y$, as the multiplication
natural transformation \( \mu_{X \times Y} \) and \( ev_{P_Y} \) allows us to “integrate” without evaluation on any measurable set.\[7\]

**Method 2: Using the constant graph maps.** For each point \( y \in Y \), consider the \( K(\mathcal{G})\)-diagram

![Diagram](image)

Provided that the two measures \( P_{X \Gamma^{-1}_y}, P_{X \Gamma^{-1}_f} \) are not singular with respect to each other, \( P_{X \Gamma^{-1}_y} \perp P_{X \Gamma^{-1}_f} \), Lebesgue Decomposition theorem allows us to uniquely write \( P_{X \Gamma^{-1}_f} \) as a convex sum of two probability measures,

\[
P_{X \Gamma^{-1}_f} = \alpha P_y^a + (1 - \alpha) P_y^b.
\]

If we define\[8\]

\[
P_{X \times Y | Y}(\cdot | y) = \begin{cases} P_y^a & y \in V \\ P_{X \Gamma^{-1}_f} & y \notin V \end{cases}
\]

then the two characterizations are equivalent. The limit of the pullback is the subset \( \text{Lim} \subset \mathcal{G}(X \times Y)^Y \) consisting of all possible inference maps. However, these inference maps are \( P_Y - a.e. \) equal, which is just the statement in equation \[3\]. The following results are now obvious.

---

\[7\] This statement requires the SMCC structure where the evaluation maps 

\[
\mathcal{G}(X \times Y)^Y \circ \mathcal{G}(Y) \xrightarrow{ev} \mathcal{G}(X \times Y)
\]

are measurable. Formally, equation \[3\] is the messy expression

\[
(ev_{P_X} \circ \mu_{X \times Y}^{\mathcal{G}(Y)} \circ st_{Y, \mathcal{G}(X \times Y)} \circ \eta_{\mathcal{G}(X \times Y)^Y})(P_{X \times Y | Y}) = P_X \Gamma^{-1}_f.
\]

We feel justified in using the integral sign, because viewing probability measures as weakly averaging affine functionals which preserve limits, integrals \( \int \) correspond precisely to evaluation maps.\[9\]

\[8\] It would be more elegant to define

\[
P_{X \times Y | Y}(\cdot | y) = \begin{cases} P_y^a & \text{whenever } P_{X \Gamma^{-1}_f} \perp P_{X \Gamma^{-1}_y} \\ P_{X \Gamma^{-1}_f} & \text{otherwise} \end{cases}
\]

However, we do not know how to prove the set \( \{ y \in Y | P_{X \Gamma^{-1}_f} \perp P_{X \Gamma^{-1}_y} \} \) is measurable.
Lemma 4.1. The function

\[ Y \xrightarrow{P_{X \times Y|Y}} \mathcal{G}(X \times Y) \]

\[ y \xleftarrow{P_{X \times Y|Y}} P_{X \times Y|Y}(\bullet|y) = \int h \, d(P_X \Gamma_y^{-1}) \]

is measurable.

Theorem 4.2. The conditional probability \( P_{X \times Y|Y} \) satisfies the condition

\[ P_{X \times Y|X} \ast P_X = P_X \Gamma_f^{-1} = P_{X \times Y|Y} \ast P_Y \]

making \( P_{X \times Y|Y} \) and \( P_{X|Y} = \delta_{\tau_x} \ast P_{X \times Y|Y} \) inference maps for the deterministic Bayesian model \( (P_X, \delta_f) \).

Proof. This follows directly from equation (2) as

\[ P_{X \times Y|Y} (\zeta|y) = \int \zeta \, h \, d(P_X \Gamma_y^{-1}). \]

4.2 Inference maps for nondeterministic models

The only change required from the deterministic construction is to replace the probability measure \( P_X \Gamma_f^{-1} : 1 \to \mathcal{G}(X \times Y) \) used in the pullback construction with the joint probability measure specified by the composite

\[ 1 \xrightarrow{\tau''_{X,Y}} \mathcal{G}(X) \xrightarrow{\mathcal{G}(\Gamma_{P_{Y|X}})} \mathcal{G}(X \otimes \mathcal{G}(Y)) \xrightarrow{\mathcal{G}(\tau''_{X,Y})} \mathcal{G}(\mathcal{G}(X \otimes Y)) \xrightarrow{\mu_{X \times Y}} \mathcal{G}(X \otimes Y) \]

\[ \mathcal{G}(id_{X \times Y}) \xrightarrow{\mathcal{G}(id_{X \times Y})} \mathcal{G}(X \times Y) \]

5 Appendix

The inadequacy of kernels.

Example Let \( \mathcal{L} \) denote the Lebesque measure. Take the identity map \([0,1] \to [0,1]\) where the domain space has the \( \sigma \)-algebra of Lebesque measurable sets, and the codomain space has the Borel \( \sigma \)-algebra. The deterministic Bayesian model \( (\mathcal{L}, \delta_{id}) \) apparently has no corresponding inference map. The logical choice for the inference map, the identity map \( \delta_{id_{[0,1]}} \) is not an option because for any non Borel measurable set \( F \) the map \( \delta_{id_{[0,1]}}(F|\cdot) \) is not measurable.

This elementary problem captures the difficulty arising in constructing inference maps. Namely, problems can arise on sets of measure 0. Restricting Meas to Standard Spaces
or Polish Spaces circumvents the problem by not allowing the use of Lebesgue measurable
sets, or any other “unpleasant” $\sigma$-algebra where the same type of difficulty with sets of
measure 0 arises.

On the other hand, this problem can be avoided if instead of using kernels, which is
what we have referred to as (regular) conditional probabilities, we use $D$-kernels.

Suppose $f : X \to Y$, and $P_X \in \mathcal{G}(X)$. If $\nu : \Sigma_Y \times X \to [0, 1]$ then we say $\nu$ is a $D$-kernel provided

1. for each $x \in X$, $Q(\cdot | x)$ is a probability measure on $Y$, and
2. for each $A \in \Sigma_X$, $Q(\cdot | A)$ is a $\Sigma_X^* - \mathcal{B}$ measurable function, where $\Sigma_X^*$ is the completion
   of $\Sigma_X$ with respect to $P_X$.

The difference between kernels and $D$-kernels is the kernels requires measurability with
respect to $\Sigma_X$ rather than the completion with respect to the measure $P_X$.

Using $D$-kernels the identity map $id_{[0, 1]}$ is an inference map for the example. So why
not use these instead of kernels? The difficulty with using $D$-kernels arises from the
awkward category it imposes upon us due to the fact we need to be able to compose such
maps.

For example, we could take the objects to be a triple $(X, \Sigma_X, D)$ where $D \subseteq \mathcal{G}(X)$, and an arrow $Q$

$$(X, \Sigma_X, D) \xrightarrow{Q} (Y, \Sigma_Y, E)$$

is a $D$-kernel for every $P_X \in D$, and $Q(\cdot | x) \in E$ for every $x \in X$. The set $E$ plays no role
whatsoever in the requirement of the arrow $Q$. But upon precomposition with any other
arrow

$$(W, \Sigma_W, F) \xrightarrow{R} (X, \Sigma_X, D) \xrightarrow{Q} (Y, \Sigma_Y, E)$$

the composition requires the integrand in

$$\int_X Q(\zeta | \cdot) dR(\cdot | w)$$

to be $R(\cdot | w)^* \text{-measurable}$. Hence the requirement that each $R(\cdot | w) \in D$ for every $w \in W$.

Changing $R$ to another arrow shows that for such a scheme to work in defining a category
requires that with every space $(X, \Sigma_X)$ in $\text{Meas}$, we need to make a lot of copies of it, so
we can proceed to have every possible composition we had within the $K(\mathcal{G})$ framework
using kernels.

References

[1] D. L. Cohn, Measure Theory. Birkhauser, 1980.
[2] J. Culbertson and K. Sturtz. A categorical foundation for Bayesian Probability. Applied Categorical Structures, August 2014, Volume 22, Issue 4, pp 647-662.

[3] A. M. Faden, The existence of regular conditional probabilities: necessary and sufficient conditions, The Annals of Probability, Vol. 13, No. 1, p288-298.

[4] M. Giry, A categorical approach to probability theory, in Categorical Aspects of Topology and Analysis, Vol. 915, 68-85, Springer-Verlag, 1982.

[5] Robert M. Gray, Probability, Random Processes, and Ergodic Properties, Springer-Verlag, 2006. http://www.csee.wvu.edu/~xinl/library/books/arp.pdf

[6] P. Halmos, Measure Theory. Springer-Verlag, 1978.

[7] A. Kock, Strong Functors and Monoidal Monads, Archiv der Math. 23 (1972), 113-120.

[8] A. Kock, Monads on Symmetric Monoidal Closed Categories, Archiv der Mathematik XXI, Vol. XXI, 1970.

[9] K. Sturtz, Categorical Probability Theory, http://arxiv.org/pdf/1406.6030.pdf

[10] B. B. Winter, An alternate development of conditioning, Statistica Neerlandica Vol. 33, Issue 4, 1979, 197-212.

Kirk Sturtz
Universal Mathematics
kirksturtz@UniversalMath.com