Remarks on Phase Transitions in Matrix Models
and $\mathcal{N} = 1$ Supersymmetric Gauge Theory

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Abstract

A hermitian one-matrix model with an even quartic potential exhibits a third-order phase transition when the cuts of the matrix model curve coalesce. We use the known solutions of this matrix model to compute effective superpotentials of an $\mathcal{N} = 1$, $SU(N)$ supersymmetric Yang-Mills theory coupled to an adjoint superfield, following the techniques developed by Dijkgraaf and Vafa. These solutions automatically satisfy the quantum tracelessness condition and describe a breaking to $SU(N/2) \times SU(N/2) \times U(1)$. We show that the value of the effective superpotential is smooth at the transition point, and that the two-cut (broken) phase is more favored than the one-cut (unbroken) phase below the critical scale. The $U(1)$ coupling constant diverges due to the massless monopole, thereby demonstrating Ferrari’s general formula. We also briefly discuss the implication of the Painlevé II equation arising in the double scaling limit.

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1 Introduction

Over the recent couple of years, much progress has been made in computing effective superpotentials in $\mathcal{N} = 1$ supersymmetric gauge theories. The idea for this technique was motivated by geometric considerations of dualities in string theory \[1, 2\], and then it was recognized that the computation was closely related to that in the old matrix models \[3, 4, 5\]. Later the conjecture was proved under some certain conditions \[6, 7\], and the validity of this approach has been extensively tested in a variety of situations.

This approach provides direct connections between the computations in the old matrix models to those in supersymmetric gauge theories, enabling us to ‘recycle’ \[8\] the old matrix model results to extract interesting information on gauge theory dynamics from them, without doing many new computations. In this paper, we use some known solutions of a matrix model that exhibits a phase transition, and examine what it implies to the corresponding gauge theory.

Our model is a hermitian one-matrix model with a familiar symmetric quartic potential

$$W_{\text{tree}}(M) = \frac{m}{2} M^2 + \frac{g_1}{4} M^4.$$ \(1\)

The one-cut solution of this model is well-known \[9\], and a two-cut solution was also obtained in \[10\] (See also \[11\].) Many things are known on this model; for example, it exhibits a third-order one-cut/two-cut phase transition \[10\], and in the double scaling limit the critical behavior is described by the Painlevé II equation \[12\]. The same equation was found to appear \[13\] near the Gross-Witten transition point \[14\] of a unitary matrix model.

Following the techniques recently developed by Dijkgraaf and Vafa, we will use these solutions to compute low-energy effective superpotentials for $\mathcal{N} = 1$, $SU(N)$ supersymmetric gauge theories, for both the phase with the maximally unbroken gauge group and that broken to $SU(N/2) \times SU(N/2) \times U(1)$ for an even $N$. Since the solutions are all $Z_2$ symmetric, the matrix models automatically satisfy the quantum tracelessness condition \[11\], and hence describes this particular pattern of gauge symmetry breaking.

We will show that the values of the effective superpotentials are smoothly connected at the transition point; the two-cut value of the superpotential is lower than that of the one-cut case below the critical scale, being consistent with what one would naively expect from the renormalization group argument. We will also confirm Ferrari’s general formula \[15\] for the critical behavior of the $U(1)$ coupling constant. At the transition point, the $U(1)$ coupling constant diverges, signaling the effect of other degrees of freedom (the monopole), and the
The one-cut versus two-cut solutions in a hermitian one-matrix model

We consider a hermitian one-matrix model with an even quartic tree-level potential (1) with an \( \hat{N} \times \hat{N} \) hermitian matrix \( M \). The free energy is given by

\[
F = - \log \int dM \exp \left( - \frac{\hat{N}}{\mu} \text{Tr}W_{\text{tree}}(M) \right) = \sum_{g=0}^{\infty} \hat{N}^{-2g+2} F_g,
\]

where we have included the matrix model ’t Hooft coupling \( \mu \). Due to the redundancy of parameters, we will set \( g_4 = 1 \) in the following; clearly, the \( g_4 \)-dependence can be recovered by replacing \( \mu \rightarrow \frac{\mu}{g_4} \) and \( m \rightarrow \frac{m}{g_4} \). \( \mu > 0 \) is also assumed in this paper.
2.1 The one-cut solution

The BIPZ large-$\hat{N}$ one-cut free energy is given [9] in our notation

$$F_0 = -\frac{1}{24}(A - 1)(A - 9) - \frac{1}{2}\log \frac{\mu A}{m\Lambda_0^2} + \frac{3}{4}$$  \hspace{1cm} (3)

with

$$A = \frac{mb_1^2}{4\mu}, \quad b_1^2 = \frac{2}{3} \left( \sqrt{12\mu - m^2} - m \right).$$  \hspace{1cm} (4)

$\Lambda_0$ is an arbitrary integration constant and can be identified as the cutoff parameter in the corresponding gauge theory (See next section.). In the original BIPZ formula, a constant

$$F_0(g_4=0) = -\frac{1}{2}\log \frac{\mu}{m\Lambda_0^2} + \frac{3}{4}$$  \hspace{1cm} (5)

is subtracted from the free energy ($g_4=0$ implies $A=1$).

The resolvent $\omega(z) = \frac{1}{N} \text{Tr} \frac{1}{z-M}$ and the spectral density $\rho(\lambda) = -\frac{1}{2\pi i}(\omega(\lambda+i0) - \omega(\lambda-i0))$ are

$$\omega(z) = \frac{1}{2\mu} \left( mz + z^3 - (m + \frac{b_1^2}{2} + z^2)(z^2 - b_1^2)\right),$$  \hspace{1cm} (6)

$$\rho(\lambda) = \frac{1}{2\pi \mu} (m + \frac{b_1^2}{2} + \lambda^2) \sqrt{b_1^2 - \lambda^2}. $$  \hspace{1cm} (7)

$\lambda$, the eigenvalue of $M$, is distributed only on the interval between $-b_1$ and $b_1$. $\rho(\lambda) = 0$ otherwise.

If $m > 0$, $\rho(\lambda)$ is always positive on the interval $[-b_1, b_1]$, while if $m$ is negative and $m < -2\sqrt{\mu}$, $\rho(\lambda)$ takes a negative value in a region near $\lambda = 0$. This negative eigenvalue density is unacceptable as a matrix model, indicating a split of the cut.

2.2 The two-cut solution

The symmetric two-cut solution of this matrix model was obtained by Shimamune [10] (See also [11]). The free energy is

$$F_0 = -\frac{1}{4} \log \frac{\mu}{A^4} - \frac{m^2}{4\mu} + \frac{3}{8}. $$  \hspace{1cm} (8)
Again, to compare with the literature, the constant (5) needs to be subtracted. The resolvent and spectral density are

$$\omega(z) = \frac{1}{2\mu} \left( mz + z^3 - z(z^2 - a^2)^{\frac{1}{2}}(z^2 - b^2)^{\frac{1}{2}} \right),$$

$$\rho(\lambda) = \frac{1}{2\pi\mu} |\lambda| \sqrt{(\lambda^2 - a^2)(b^2 - \lambda^2)},$$

(9)  (10)

where

$$a^2 = -m - 2\sqrt{\mu}, \quad b^2 = -m + 2\sqrt{\mu}. \quad (11)$$

The eigenvalues are symmetrically distributed on the two intervals \([-b, -a]\) and \([a, b]\).

In deriving (8), some care must be taken because there is no eigenvalue at \(\lambda = 0\) for the two-cut solution, and therefore the integrated saddle point equation

$$\int d\lambda' \frac{\rho(\lambda')}{\lambda - \lambda'} = \frac{1}{2\mu} W'(\lambda) \quad (12)$$

does not hold at \(\lambda = 0\). Instead, taking \(\lambda\) to be an end of a cut, one obtains

$$F_0 = \int_a^b d\lambda \rho(\lambda) \left( \frac{1}{\mu} W(\lambda) - \log |\lambda + a| - \log |\lambda - a| \right) + \frac{1}{2\mu} W(a),$$

(13)

which replaces the free-energy formula of [9].

For \(\mu\) fixed, the two cuts get closer as \(|m|\) decreases from a large negative value, until \(m\) reaches to \(-2\sqrt{\mu}\) when the two end points coalesce. One sees that the third derivative of the free energy with respect to the couplings is discontinuous at this point, exhibiting a third-order phase transition [10, 11].

### 3 Phase Transition in Gauge Theory

We will now use the results in the previous section and study what they mean in the corresponding 4d \(\mathcal{N} = 1\) supersymmetric gauge theory coupled to an adjoint chiral superfield \(\Phi\) with the same (super)potential

$$W_{\text{tree}}(\Phi) = \frac{m}{2} \Phi^2 + \frac{1}{4} \Phi^4. \quad (14)$$

As we mentioned in the introduction, the solutions in the previous section describe gauge theories with gauge groups \(SU(N)\) and \(SU(N/2) \times SU(N/2) \times U(1)\) in the classical vacua (Therefore when we talk about the transition, \(N\) must be even.), respectively.
The procedure to compute the effective superpotential is summarized as follows [1, 3, 4, 5, 7]. Each branch cut of a resolvent $\omega(z)$ corresponds to a non-abelian factor of the unbroken gauge group. Let $n$ be the number of cuts, and (initially) the gauge group $U(N)$ be partially broken to $\prod_{i=1}^{n} SU(N_i) \times U(1)^n$ ($\sum_{i=1}^{n} N_i = N$). One then computes the low-energy effective superpotential $W_{\text{eff}}$ as

$$W_{\text{eff}} = 2\pi \sum_{i=1}^{n} (i\tau s_i + N_i \pi_i) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial \pi_i}{\partial s_j} w_{\alpha i} w_{\alpha j},$$

where $s_i$ and $w_{\alpha i}$ are the glueball and the $U(1)$ superfields

$$s_i = -\frac{1}{32\pi^2} \text{Tr} W_{i\alpha} W_{i\alpha}^\alpha, \quad w_{\alpha i} = \frac{1}{4\pi} \text{Tr} W_{i\alpha}$$

for each $U(N_i)$ factor. $s_i$ and $\pi_i$ can be computed as a period of a complex curve

$$y = 2\mu \omega(z) - W_{\text{tree}}(z).$$

The contour for $s_i$ surrounds the $i$-th cut, while for $\pi_i$ starts from the cutoff $\Lambda_0$ on the second sheet, goes through the cut and back to $\Lambda_0$ on the first sheet. $\tau$ is the $\Lambda_0$-dependent bare coupling constant. The formula (15) is for $U(N)$ gauge theories, but separating the $SU(N_i)$ piece from $s_i$ as

$$s_i = \tilde{s}_i - \frac{1}{2N_i} w_{\alpha i} w_{\alpha i}^\alpha,$$

the overall $U(1)$ contributions cancel out, leaving only the bare coupling term [7]. Discarding this, one obtains an $SU(N)$ effective superpotential.

### 3.1 The one-cut case

First, we consider the one-cut solution. This corresponds to the case when the full gauge group ($SU(N)$) is unbroken and maximally confined at low energies, and was already studied in e.g. [24, 25, 7]. The matrix model curve

$$y(z) = -(z^2 + m + \frac{b_1^2}{2})(z^2 - \frac{b_1^2}{2})^{\frac{1}{2}}$$

has a branch cut at $[-b_1, b_1]$ and two zeroes at $z = \pm \sqrt{-m - \frac{b_1^2}{2}}$. If $m$ is positive and very large compared to a fixed $\mu$, these zeroes are pure imaginary and far apart from each other, and from the real axis. They move toward $z = 0$ as $m$ decreases, but they are still off the real axis when $m = 0$ (where the tree-level potential develops a quadruple zero), until they meet at $z = 0$ when $m$ reaches $-2\sqrt{\mu}$. This is the critical point discussed in section 2. $m < -2\sqrt{\mu}$ is the region where the two-cut solution is well-defined, but formally a one-cut solution still exists, although the spectral density $\rho(\lambda)$ becomes not positive definite. In this region the two zeroes are apart, located on the real axis, and $\rho(\lambda)$ is negative between them (Figure 1).
Figure 1: The tree-level superpotentials $W_{\text{tree}}$ and the cuts and zeroes on the matrix model curves. (a) $W_{\text{tree}}$ for (from inside) $m = -1, -2$ and $-3$ with $\mu = 1$. (b) One-cut, $m = -1$. (c) One-cut, $m = -2$ (critical). (d) One-cut, $m = -3$. (e) Two-cut, $m = -3$.

Let $S$ and $\Pi$ denote the unique pair of periods. They are calculated as

$$S = \frac{1}{2\pi i} \int_{-b_i}^{b_i} dz y(z) = \mu,$$  \hspace{1cm} (20)  

$$\Pi = \frac{1}{2\pi} \int_{b_i}^{\Lambda_0} dz y(z)$$

$$= -\frac{1}{2\pi} \left( W_{\text{tree}}(\Lambda_0) + 2\mu \log \frac{b_1}{2\Lambda_0} - \frac{\mu}{2} + \frac{mb_1^2}{8} \right).$$  \hspace{1cm} (21)
Λ₀ is the cutoff parameter of dimension 3/4. One can easily verify the special geometry relation (with the \(\hat{N}^2\) factor of the genus-0 free energy taken into account) \(\Pi = \partial(\hat{S}^2F_0)\) up to a \(\Lambda_0\)-dependent constant \(W_{\text{tree}}(\Lambda_0)\). Plugging (20) (21) into (15) and discarding the overall \(U(1)\) term, we obtain [25]

\[
W_{\text{eff}}(\hat{S}) = N \left(-2\hat{S} \log \frac{\hat{b}}{2\Lambda} + \hat{S} + \frac{m\hat{b}^2}{8}\right) \quad \text{(one-cut),} \tag{22}
\]

where \(\hat{b} = b_1|_{\mu = \hat{S}}\). Λ is the physical scale defined through the ‘renormalization’ of the gauge coupling constant [1]

\[
\log \Lambda = \log \Lambda_0 + \frac{\pi i}{N} r. \tag{23}
\]

\(W_{\text{eff}}(\hat{S})\) is minimized with respect to \(\hat{S}\) at \(\hat{b} = 2\Lambda\) \(^1\). Evaluating \(W_{\text{eff}}(\hat{S})\) at this point reproduces the known \(SU(N)\) effective superpotential [24, 7]

\[
W_{\text{low}} = N \left(\frac{3\Lambda^4}{2} + m\Lambda^2\right) \quad \text{(one-cut).} \tag{24}
\]

### 3.2 The two-cut case

We next consider the two-cut case. For general two-cut solutions, [15] computes a superpotential for the gauge group \(U(N)\) broken to \(SU(N_+) \times SU(N_-) \times U(1)^2\) for arbitrary \(N_+\) and \(N_- = N - N_+\). On the other hand, our solution [9] is \(Z_2\) symmetric and the two periods \(S_\pm\) are not independent (We will denote the period whose contour surrounds the cut \([a, b][(-b, -a)]\) by \(S_+(S_-)\) and its dual period by \(\Pi_+ (\Pi_-)\)). This constraint can be thought of as a consequence of the quantum tracelessness condition [1]: Suppose that we perturb \(W_{\text{tree}}(\Phi)\) by a small linear potential \(\delta W_{\text{tree}}(\Phi) = \sigma \text{Tr} \Phi\). Clearly, \(\delta \Pi_+ = -\delta \Pi_- = O(\sigma)\), and therefore the symmetric solution satisfies the quantum tracelessness condition \(\frac{\partial W_{\text{eff}}}{\partial \sigma} = 0\) if \(N_+ = N_-\). Conversely, if \(N_+ = N_-\), the equation \(\frac{\partial W_{\text{eff}}}{\partial \sigma} = 0\) imposes a constraint that \(S_+ = S_-\) and \(\Pi_+ = \Pi_-\). Thus our symmetric two-cut solution corresponds to a gauge theory with a gauge group \(SU(N)\) broken to \(SU(N/2) \times SU(N/2) \times U(1)\) for some even \(N\)\(^2\).

To compute the \(U(1)\) coupling in [15] for this phase, we still need to have two independent \(S_\prime\)s. For this purpose we slightly relax the \(Z_2\) symmetry, and seek for a slightly asymmetric

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\(^1\)with a possible \(Z_N\) chiral-symmetry phase factor, which we omit in this paper. Other vacua may be obtained by rotating the complex plane on which the matrix model curve is defined.

\(^2\)Therefore, if \(N\) is odd, the gauge theory in the broken phase is not described by this symmetric family of matrix model solutions.
solution by a perturbation. Let the locations of two cuts be $[b_-,a_-]$ and $[a_+,b_+]$, then the resolvent $\omega(z)$ is given by

$$
\omega(z) = \frac{1}{4\pi i\mu}(z - b_-)^{1/2}(z - a_-)^{1/2}(z - a_+)^{1/2}(z - b_+)^{1/2} 
\cdot \oint d\lambda \frac{W'_{\text{tree}}(\lambda)}{(z - \lambda)(\lambda - b_-)^{1/2}(\lambda - a_-)^{1/2}(\lambda - a_+)^{1/2}(\lambda - b_+)^{1/2}}.
$$

(25)

The locations of the end points are determined so that $\omega(z)$ behaves like $\sim z^{-1}$ as $|z| \to \infty$. Defining

$$Q = \frac{a_+ + b_+ + a_- + b_-}{4},
$$

(26)

we compute the deviations of the end-point locations from $a_\pm = \pm a$, $b_\pm = \pm b$ to first order in $Q$. The result is

$$
a_\pm = \pm a + \left(1 - \frac{m}{2\sqrt{\mu}}\right)Q,
$$

$$b_\pm = \pm b + \left(1 + \frac{m}{2\sqrt{\mu}}\right)Q,
$$

(27)

where we have omitted the terms of $O(Q^2)$ (and will also do in the equations below). $\omega(z)$ is modified from (9) to

$$\omega(z) = \frac{1}{2\mu} \left(mz + z^3 - (z + 2Q)(z - b_-)^{1/2}(z - a_-)^{1/2}(z - a_+)^{1/2}(a - b_+)^{1/2}\right),
$$

(28)

and hence the matrix model curve

$$g(z) = -(z + 2Q)(z - b_-)^{1/2}(z - a_-)^{1/2}(z - a_+)^{1/2}(a - b_+)^{1/2}.
$$

(29)

Thus we find the $Q$-dependence of the periods of slightly asymmetric solutions as

$$S_\pm = \frac{\mu}{2} \mp \frac{a^2bQ}{\pi}K\left(\sqrt{1 - \frac{a^2}{b^2}}\right),
$$

(30)

$$\Pi_\pm = -\frac{1}{2\pi} \left(W_{\text{tree}}(\Lambda_0) + \frac{\mu}{2} \log \frac{\mu}{\Lambda_0^4} - \frac{\mu}{2} + \frac{m^2}{4}\right) \mp \frac{a^2bQ}{\pi}K\left(\frac{a}{b}\right)
$$

(31)

to $O(Q)$, where

$$K(k) = \int_0^1 \frac{dt}{\sqrt{1 - t^2}(1 - k^2t^2)}
$$

(32)
is the complete elliptic integral of the first kind. Using (30) and (31), we may readily find in the symmetric limit ($Q = 0$)

$$
\frac{\partial \Pi_\pm}{\partial S_\pm} = - \frac{1}{4\pi} \left( \ln \frac{\mu}{\Lambda_0^2} + \frac{K\left(\frac{\mu}{\Lambda_0^2}\right)}{K\left(\frac{\Lambda_0^2}{\sqrt{1-a^2b^2}}\right)} \right).
$$

Thus (15) computes the effective superpotential as

$$
W_{\text{eff}}(\hat{S}, w_\perp^\alpha) = N \left( -\frac{\hat{S}}{2} \log \frac{\hat{S}}{\Lambda^4} + \frac{\hat{S}}{2} - \frac{m^2}{4} \right)
$$

$$
+ \frac{1}{4} \left( \log \frac{\mu}{\Lambda^4} + \frac{K\left(\frac{\mu}{\Lambda^4}\right)}{K\left(\frac{\Lambda^2}{\sqrt{1-a^2b^2}}\right)} \right) w_{\perp\perp} w_\perp^\alpha |w_{\perp\perp} = w_\perp^\alpha |_{\mu = \hat{S}}.
$$

where the $U(1)$ gauge superfield $w_\perp^\alpha$ has been defined by

$$
S_\pm = \frac{\hat{S}}{2} - \frac{1}{N} w_\pm^\alpha w_\pm^\alpha, \quad w_\pm^\alpha = \frac{1}{2} (w_0^\alpha \pm w_\perp^\alpha)
$$

and the bare coupling term of the overall $U(1)$ $w_0^\alpha$ has been discarded in (34). Similarly to the one-cut case, we have also defined the physical scale $\Lambda$ by the same equation as (23). Minimizing the first term with respect to $\hat{S}$, we find that this occurs when $\hat{S} = \Lambda^4$. Plugging this into (34), we finally obtain

$$
W_{\text{low}} = N \left( \frac{\Lambda^4}{2} - \frac{m^2}{4} \right) + \frac{1}{4} \frac{K\left(\frac{\mu}{\Lambda^4}\right)}{K\left(\frac{\Lambda^2}{\sqrt{1-a^2b^2}}\right)} w_{\perp\perp} w_\perp^\alpha \quad (\text{two-cut}),
$$

where $a$ and $b$ are given by the equations (11) with $\mu$ replaced by $\Lambda^4$. (36) agrees with the on-shell analysis of [15, 23] derived using the matrix model curves with the double-zero factors removed.

### 3.3 The behavior near the transition point

Let us now compare (24) and (36) near the critical scale $\Lambda = \sqrt{\frac{|m|}{2}}$. Figure 2 shows comparison of the minimum values of the effective superpotentials $W_{\text{low}}$. No two-cut solution exists for $\Lambda > \sqrt{\frac{|m|}{2}}$ and there are only one-cut solutions. The values of the broken and the unbroken phases are smoothly connected at $\Lambda = \sqrt{\frac{|m|}{2}}$. Below the critical scale $\Lambda = \sqrt{\frac{|m|}{2}}$, the broken phase (two-cut solution, dashed line) is more favored than the unbroken phase (one-cut solution, solid line). This is what one would naively expect from the renormalization group argument: At high energies the Tr$\Phi^4$ term is more relevant than the Tr$\Phi^2$ term,
enforcing the gauge group to be unbroken. At lower energies, the effect of the indentation of the potential becomes more relevant, and the gauge group is broken due to the Higgs mechanism. Since the effective superpotential in our model is proportional to a derivative of the free energy, the smoothness agrees with the fact that the transition is third order in the matrix model.

Figure 3 shows scale dependence of the inverse square of the $U(1)$ coupling constant $\tau_{\perp} \equiv \frac{1}{4} \frac{K(\theta)}{K(\sqrt{1 - \frac{a^2}{b^2}})}$ (the coefficient of $w_{\perp} w_{\perp}^\alpha$) in the broken phase. It rapidly goes to zero like $(\log(\sqrt{|m|} - \Lambda))^{-1}$ near $\Lambda = \sqrt{|m|}/2$, and the kinetic term vanishes at the transition point; this is consistent with the fact that above $\Lambda = \sqrt{|m|}/2$ is the unbroken phase and no $U(1)$ gauge field is there. Although the coupling constant grows very large near the transition point, it does not diverge until the parameters reaches the values where the one-cut solution starts. Therefore, one may say that the matrix model approach is still valid near the phase transition point, except right at the singularity. Note that the $U(1)$ kinetic term does not vanish off shell at the transition point. These observations qualitatively agree with [23] for the cubic-potential case. At very low energies, $\tau_{\perp}$ is logarithmically divergent, reproducing the one-loop running as expected.
As we mentioned in the introduction, the singular behavior of the all-genus free energy, near the critical point we have considered in this paper, is known to be governed by the Painlevé II equation \[12\]:

\[
\frac{d^2 O}{dx^2} + \frac{1}{2} x O - O^3 = 0,
\]

where \(x\) is a variable related to the degree of polynomials in the orthogonal polynomial method, and \(O(x)\) is related to the difference of the smooth limit of the even and odd recursion coefficients (See \[12\] for more detail.).

In general, the orthogonal polynomial method extracts the leading critical singularity from the free energy, giving its expansion in terms of \(\hat{N}^2 (g - g_c)^{2-\gamma}\). The singular behavior of the free energy is controlled by \(O(x)\) with large \(x\). The solution of (37) is expanded around the infinity as

\[
O(x) = \sqrt{\frac{x}{2}} \left( 1 - \frac{1}{4x^3} - \frac{73}{32x^6} - \frac{10657}{128x^9} - \cdots \right) .
\]

The leading singular behavior of the free energy is given by \(\frac{d^F}{dz} \sim -(O(z))^2\) \[12\], where \(z^3 = \hat{N}^2 (1 - 4S/m^2)^3\). Thus we find

\[
F \sim \hat{N}^2 F_0 + F_1 + \left( \frac{3\hat{N}^{-2}}{16(1 - 4S/m^2)^3} + \frac{63\hat{N}^{-4}}{32(1 - 4S/m^2)^6} + \cdots \right) .
\]
Since the string susceptibility $\gamma$ is $-1$, the genus-0 and -1 terms are non-singular in the expansion and hence not reliable in this analysis, while the terms higher than $z^{-3}$ indicate large gravitational corrections [5, 26, 27] near the transition point. Note that, although the matrix model size $\hat{N}$ is sent to infinity, the rank of the gauge group $N$ is not necessarily large in this double scaling limit.

5 Conclusions

We have used the classic solutions of a hermitian one-matrix model with an even quartic potential to compute low-energy effective superpotentials for $\mathcal{N} = 1$, $SU(N)$ supersymmetric gauge theories. Since the solutions are all $Z_2$ symmetric, the matrix models automatically satisfy the quantum tracelessness condition and describe a phase with the gauge group $SU(N/2) \times SU(N/2) \times U(1)$ (for an even $N$).

We have shown that the values of the effective superpotentials are smoothly connected at the transition point, and the two-cut value of the superpotential is lower than that of the one-cut case below the critical scale. The latter indicates that the broken phase is more favored at low energies, as naively expected. At the transition point, the $U(1)$ coupling constant diverges, signaling the effect the light monopole, and the $U(1)$ kinetic term consistently disappears there from the effective action, thereby confirming Ferrari’s general formula for the critical behavior of the $U(1)$ coupling constant. We have also discussed that the Painlevé II equation of the double-scaled matrix model indicates large gravitational corrections near the transition point for the broken side, if the gauge theory is coupled to a gravitational background.

In Ref.[21], some evidence for a structure of the $N$-reduced KP hierarchy has been found in some $\mathcal{N} = 1$ analogue of the Argyres-Douglas singularities. Our singular curve, on the other hand, does not belong to this class; nevertheless the all-genus free energy is described by the Painlevé II equation, which can be obtained as a similarity reduction of the modified KdV equation. It would be interesting to investigate if a structure of the 2-reduced KP (= KdV) hierarchy underlies our system in the sense of [21].
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References

[1] F. Cachazo, K. A. Intriligator and C. Vafa, Nucl. Phys. B 603 (2001) 3 [arXiv:hep-th/0103067].

[2] F. Cachazo and C. Vafa, arXiv:hep-th/0206017.

[3] R. Dijkgraaf and C. Vafa, Nucl. Phys. B 644 (2002) 3 [arXiv:hep-th/0206255].

[4] R. Dijkgraaf and C. Vafa, Nucl. Phys. B 644 (2002) 21 [arXiv:hep-th/0207106].

[5] R. Dijkgraaf and C. Vafa, arXiv:hep-th/0208048.

[6] R. Dijkgraaf, M. T. Grisaru, C. S. Lam, C. Vafa and D. Zanon, arXiv:hep-th/0211017.

[7] F. Cachazo, M. R. Douglas, N. Seiberg and E. Witten, JHEP 0212 (2002) 071 [arXiv:hep-th/0211170].

[8] N. Seiberg, talk given at Strings 2003, Kyoto.

[9] E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber, Commun. Math. Phys. 59 (1978) 35.

[10] Y. Shimamune, Phys. Lett. B 108 (1982) 407.

[11] G. M. Cicuta, L. Molinari and E. Montaldi, Mod. Phys. Lett. A 1 (1986) 125.

[12] M. R. Douglas, N. Seiberg and S. H. Shenker, Phys. Lett. B 244 (1990) 381.

[13] V. Periwal and D. Shevits, Phys. Lett. 64B (1990) 1326.

[14] D. J. Gross and E. Witten, Phys. Rev. D 21 (1980) 446.

[15] F. Ferrari, Phys. Rev. D 67 (2003) 085013 [arXiv:hep-th/0211069].

[16] N. Seiberg and E. Witten, Nucl. Phys. B 426 (1994) 19 [Erratum-ibid. B 430 (1994) 485] [arXiv:hep-th/9407087].
[17] A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, Phys. Lett. B 344 (1995) 169
\texttt{arXiv:hep-th/9411048}.

[18] P. C. Argyres and A. E. Faraggi, Phys. Rev. Lett. 74 (1995) 3931
\texttt{arXiv:hep-th/9411057}.

[19] P. C. Argyres and M. R. Douglas, Nucl. Phys. B 448 (1995) 93 \texttt{arXiv:hep-th/9505062}.

[20] F. Cachazo, N. Seiberg and E. Witten, JHEP 0302 (2003) 042 \texttt{arXiv:hep-th/0301006}.

[21] T. Eguchi and Y. Sugawara, JHEP 0305 (2003) 063 \texttt{arXiv:hep-th/0305050}.

[22] G. Bertoldi, JHEP 0306 (2003) 027 \texttt{arXiv:hep-th/0305058}.

[23] D. Shih, \texttt{arXiv:hep-th/0308001}

[24] F. Ferrari, Nucl. Phys. B 648 (2003) 161 \texttt{arXiv:hep-th/0210135}.

[25] H. Fuji and Y. Ookouchi, JHEP 0212 (2002) 067 \texttt{arXiv:hep-th/0210148}.

[26] H. Ooguri and C. Vafa, \texttt{arXiv:hep-th/0302109}

[27] H. Ooguri and C. Vafa, \texttt{arXiv:hep-th/0303063}