Stochastic lattice gases with symmetric hopping are described, on a coarse-grained level, by diffusion equation with density-dependent diffusion coefficient. Density fluctuations additionally depend on the local conductivity (which also describes the response to an infinitesimal applied field). A hydrodynamic description therefore requires the determination of these two transport coefficients. Generally for lattice gases even with rather simple hopping rules, analytic results are unattainable; however, when an additional feature, known as the gradient condition, is satisfied, the Green-Kubo formula takes a simple form and computations of the transport coefficients become feasible. For a number of lattice gases of gradient type, e.g., for the Katz-Lebowitz-Spohn model with symmetric hopping, for repulsion processes, for a lattice gas of leap-frogging particles, the diffusion coefficient has been rigorously computed. The gradient property is also true for the misanthrope process, a class of generalized exclusion processes.

For gradient type lattice gases, an exact expression for the diffusion coefficient can also be obtained by a perturbation approach: one writes the formula for the current at the discrete lattice level and then performs a continuous limit assuming that the density field is slowly varying. Generalized exclusion processes with multiple occupancies, in general, do not obey the gradient condition. However, we argued in [2] that the perturbation approach should, nevertheless, lead to an exact prediction for the diffusion coefficient. For the class of generalized exclusion processes which we studied [2], the diffusion coefficient is described, on a coarse-grained level, by diffusion equation with density-dependent diffusion coefficient. Density fluctuations additionally depend on the local conductivity (which also describes the response to an infinitesimal applied field). A hydrodynamic description therefore requires the determination of these two transport coefficients. Generally for lattice gases even with rather simple hopping rules, analytic results are unattainable; however, when an additional feature, known as the gradient condition, is satisfied, the Green-Kubo formula takes a simple form and computations of the transport coefficients become feasible. For a number of lattice gases of gradient type, e.g., for the Katz-Lebowitz-Spohn model with symmetric hopping, for repulsion processes, for a lattice gas of leap-frogging particles, the diffusion coefficient has been rigorously computed. The gradient property is also true for the misanthrope process, a class of generalized exclusion processes.

We emphasize that the perturbation approach is not a naive mean-field theory where correlations are obviously neglected as argued by Becker et al. In dense lattice gases, the equilibrium state itself is usually highly correlated; e.g., in the repulsion process, the occupation number of site $i$: the mean-field assumption is completely wrong. Yet, a careful use of the perturbation approach leads to the correct result.

The numerical results of Ref. [1] and our simulations (Fig. 1) show that the perturbation approach does not lead to the correct analytical results for the GEP(2). We emphasize that the perturbation approach is not a naive mean-field theory where correlations are obviously neglected as argued by Becker et al. In dense lattice gases, the equilibrium state itself is usually highly correlated; e.g., in the repulsion process, the occupation number of site $i$: the mean-field assumption is completely wrong. Yet, a careful use of the perturbation approach leads to the correct result.

The gradient condition is thus crucial for the applicability of the perturbation approach. For GEP($k$) with maximal occupancy $k$, the gradient condition is obeyed in extreme cases of $k = 1$ which reduces to the simple exclusion process and $k = \infty$ which reduces to random walks. Presumably because GEP($k$) is sandwiched between two extreme cases in which the perturbation approach works, this method provides a very good approximation when $1 < k < \infty$. 

**FIG. 1.** Stationary current multiplied by the system size: simulation results (dots) and the prediction from our previous approach. The latter holds for $L = \infty$, but is shown as a line.
We now clarify the underlying assumptions behind the perturbation approach and suggest some tracks to improve our results. For the GEP(2), the current reads

$$J_i = \langle \tau_i \rangle f(\tau_{i+1}) - f(\tau_i) \langle \tau_{i+1} \rangle,$$  \hspace{2cm} (1)

where $$\tau_i \in \{0, 1, 2\}$$ and $$f(n) = 1 - \frac{1}{2} n(n - 1)$$. In our computation of the diffusion coefficient \[2\], we used two assumptions. The first one concerns one-point functions. Let $$P[\tau_i = m]$$ be the probability of finding $$m$$ particles at site $$i$$. The density at $$i$$ is

$$\rho_i = \langle \tau_i \rangle = P[\tau_i = 1] + 2P[\tau_i = 2].$$  \hspace{2cm} (2)

We assumed that one-site probabilities satisfy

$$P[\tau_i = m] \simeq X_m(\rho_i)$$  \hspace{2cm} (3)

where the $$X_m$$’s represent the single-site weights in an infinite lattice or on a ring:

$$X_0(\rho) = \frac{1}{Z}, \quad X_1(\rho) = \frac{\lambda}{Z}, \quad X_2(\rho) = \frac{\lambda^2}{2Z}$$  \hspace{2cm} (4)

with the fugacity $$\lambda$$ and the normalization $$Z$$

$$\lambda(\rho) = \sqrt{1 + 2\rho - \rho^2 + \rho - 1}, \quad Z = 1 + \lambda + \frac{1}{2} \lambda^2.$$  \hspace{2cm} (5)

The second assumption was to rewrite the current as

$$J_i \simeq \langle \tau_i \rangle \langle f(\tau_{i+1}) \rangle - \langle f(\tau_i) \rangle \langle \tau_{i+1} \rangle.$$  \hspace{2cm} (6)

This, indeed, is a mean-field type assumption \[1\]. The assumptions \[3\], \[4\] are asymptotically true in the stationary state of a large system ($$L \to \infty$$): We have checked these facts by performing additional simulations.

Our numerical results suggest more precise expressions for \[3\] and \[6\] with some scaling functions $$\kappa$$ and $$\mu$$:

$$P[\tau_i = m] = X_m(\rho_i) + \frac{1}{L} \kappa_m \left( \frac{i}{L} \right),$$  \hspace{2cm} (7)

$$J_i = \langle \tau_i \rangle \langle f(\tau_{i+1}) \rangle - \langle f(\tau_i) \rangle \langle \tau_{i+1} \rangle + \frac{1}{L} \mu \left( \frac{i}{L} \right),$$  \hspace{2cm} (8)

where we omitted $$o(L^{-1})$$ terms. Performing the perturbation approach with the refined expressions \[7\], \[8\], we obtain

$$J = -\frac{1}{L} \frac{d\rho}{dx} \left( 1 - X_2(\rho) + \rho \frac{dX_2(\rho)}{d\rho} \right) + \frac{1}{L} \mu(x)$$  \hspace{2cm} (9)

where we have switched from the discrete variable $$i$$ to $$x = i/L$$. The functions $$\kappa_m$$ do not appear in \[9\], but $$\mu(x)$$ does, and it was missing in our paper \[2\] leading to the wrong expressions for the current and for the stationary density profile. In order to calculate $$\mu(x)$$, we are presently examining nearest-neighbor correlation functions for the GEP(2). Numerically at least, these nearest-neighbor correlations exhibit a neat scaling behavior and simple patterns; detailed results will be reported in \[14\].

[1] T. Becker, K. Nelissen, B. Cleuren, B. Partoens, and C. Van den Broeck, Phys. Rev. E 93, 046101 (2016).
[2] C. Arita, P. L. Krapivsky, and K. Mallick, Phys. Rev. E 90, 052108 (2014).
[3] H. Spohn, Large Scale Dynamics of Interacting Particles (New York: Springer-Verlag, 1991).
[4] S. Katz, J. L. Lebowitz, and H. Spohn, J. Stat. Phys. 34, 497 (1984).
[5] P. L. Krapivsky, J. Stat. Mech. P06012 (2013).
[6] J. M. Carlson, J. T. Chayes, E. R. Grannan, and G. H. Swindle, Phys. Rev. Lett. 65, 2547 (1990).
[7] D. Gabrielli and P. L. Krapivsky, in preparation.
[8] C. Cocozza-Thivent, Z. Wahrscheinlichkeitstheorie verw. Gebiete 70, 509 (1985).
[9] C. Arita and C. Matsui, arXiv:1605.00917.
[10] C. Kipnis, C. Landim, and S. Olla, Commun. Pure Appl. Math. 47, 1475 (1994).
[11] C. Kipnis, C. Landim, and S. Olla, Ann. Inst. H. Poincaré 31, 191 (1995).
[12] T. Seppäläinen, Ann. Prob. 27, 361 (1999).
[13] T. Becker, K. Nelissen, B. Cleuren, B. Partoens, and C. Van den Broeck, Phys. Rev. Lett. 111, 110601 (2013).
[14] C. Arita, P. L. Krapivsky, and K. Mallick, in preparation.