\textbf{\epsilon-EXPANSION IN QUANTUM FIELD THEORY IN CURVED SPACETIME}

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Abstract

We discuss \(\epsilon\)-expansion in curved space-time for asymptotically free and asymptotically non-free theories. The existence of stable and unstable fixed points is investigated for \(f\phi^4\) theory and \(SU(2)\) gauge theory. It is shown that \(\epsilon\)-expansion maybe compatible with asymptotic freedom on special solutions of the RG equations in a special case (supersymmetric theory). Using \(\epsilon\)-expansion RG technique the effective Lagrangian for covariantly constant gauge \(SU(2)\) field and effective potential for gauged NJL-model are found in \(4-\epsilon\)-dimensional curved space (in linear curvature approximation). The curvature-induced phase transitions from symmetric phase to asymmetric phase (chromomagnetic vacuum and chiral symmetry broken phase, respectively) are discussed for the above two models.
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1 Introduction

The knowledge of infrared structure of quantum field theory is very important in different applications. For example, the confining phase of SU(2) theory describes the infrared (IR) structure of the Standard Model. From another side, quantum field theory at non-zero temperature and its infrared structure are relevant for construction of the inflationary models of the universe.

To study IR properties of the theory, the well-known $\epsilon$-expansion techniques is often very useful, as it happens, for example, in the theory of critical phenomena. For renormalizable theories the $\epsilon$- expansion technique gives a standard way to investigate the critical points of the theory and their stability, the critical behavior and the nature of the phase transitions at non-zero temperature. Of course, $\epsilon$- expansion works normally well only near $D = 4$, i.e. for infinitesimal $\epsilon$. As usually in such cases, one employs standard assumptions that are not strictly correct mathematically (they are derived for small $\epsilon$) but continue to be valid at $\epsilon=1$. (The main motivation for such study is, of course, the search for finite-temperature phase transition or investigation of the Casimir effect at non-zero temperature). In different areas of physics there exists a variety of explicit examples (see [2]) where, having at our disposal both numerical and experimental data, we conclude that the comparison of both gives unexpectedly good agreement even for $\epsilon \approx 1$.

The present work is devoted to the extension of the $\epsilon$-expansion technique to curved space-time. Such method is clearly required for better understanding of quantum field theory in the early universe which is hot (non-zero temperature) and non-flat one (curvature). And that kind of theory is necessary for construction of the inflationary Universe.

We start in the next section from the simple discussion of RG equations modifications in 4-$\epsilon$- dimensional curved space, using $f \phi^4$ theory as an example. Working in the matter sector we describe the structure of fixed points of the theory and give the solutions of RG equations. The extension of this discussion for SU(2) gauge models is presented (in particular, the structure of fixed points is given again). We also show that asymptotic freedom on special solutions of RG maybe realized in 4-$\epsilon$- dimensions only for supersymmetric theory.

In section 3 we consider the calculation of the effective Lagrangian for a covariantly constant gauge field in 4- $\epsilon$-dimensions. Actually, that is a
particular application of the $\epsilon$-expansion. The possibility of phase transitions induced by the combined effect of curvature and temperature is discussed.

Section 4 is devoted to the study of chiral symmetry breaking in gauged NJL model in $4\epsilon$-dimensions, using $\epsilon$-expansion technique and equivalency with gauge-Higgs-Yukawa model (via Bardeen-Lindner-Hill compositeness conditions). The RG improved effective potential is found and critical curvature is defined. Some remarks and outlook are given in the concluding section.

2 Renormalization group equations in curved 4-$\epsilon$-dimensional space-time

In this section we will be interested in studying the RG behavior of matter theories in curved 4-$\epsilon$-dimensional space-time. The typical massless Lagrangian under discussion is written as follows [1]

$$\mathcal{L} = \mu^{-\epsilon}(a_1 R^2 + a_2 C_{\mu\nu\alpha\beta}^2 + a_3 G) + \frac{1}{2} \xi R \phi^2 + \mathcal{L}_m(\phi, \psi, A_\mu)$$

where $G$ is the Gauss-Bonnet term, $C_{\mu\nu\alpha\beta}$ is the Weyl tensor, $\phi$ is a scalar and $\mu$ is a mass parameter which is used to make coupling constant to be dimensionless in 4-$\epsilon$-dimensions. The symbolic form for the matter Lagrangian which includes scalars $\phi$, spinors $\psi$ and gauge fields $A_\mu$ is

$$\mathcal{L}_m(\phi, \psi, A_\mu) = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \frac{1}{2} g^{\mu\nu}(\nabla_\mu \phi)^a (\nabla_\nu \phi)^a - \frac{1}{4!} f \mu^c \phi^4 + \bar{\psi}^a i \gamma^\mu (x) \nabla^a_\mu \psi^b - h \mu \bar{\psi} \psi$$

Here $\phi^2 = \phi^a \phi_a$, $(\nabla_\mu \phi)^a$, $(\nabla_\mu \psi)^b$ are covariant derivatives of scalar and spinor, respectively. These covariant derivatives include the curved space covariant derivatives and standard gauge coupling term with $g$ changed in the way $g \to \mu \tilde{g}$. Note that explicit $\mu$-dependence is introduced in (2) in order to keep $h$, $f$ and $g$ to be dimensionless in D=4-$\epsilon$-dimensions. For a moment we consider the gauge group, the number of scalar and spinor multiplets, the features of Yukawa, gauge and scalar interactions to be arbitrary. However, the theory with the Lagrangian (1) is supposed to be multiplicatively renormalizable in four-dimensional curved spacetime (for details, see [1]).

To study the critical behavior of the system under discussion we will employ the $\epsilon$-expansion technique (for an introduction to $\epsilon$-expansion technique
and theory of critical phenomena, see \cite{2, 3}). We will be mainly interested in $\epsilon = 1$, i.e. in finite temperature systems. It is well-known \cite{4, 5} that to study the critical behavior of a system it is enough to consider only the massless subset of the theory. That is why we do not take into account masses in \cite{1}.

Let us start from the trivial example the $f\phi^4$-theory in curved $D=4-\epsilon$-dimensional spacetime. We discuss only the matter sector of such a theory. RG flows are generated by the following RG equations:

\[
\frac{df}{dt} = -\epsilon f + \frac{3f^2}{(4\pi)^2},
\]
\[
\frac{d\xi}{dt} = f\left(\xi - \frac{1}{6}\right) \frac{1}{(4\pi)^2}.
\]

As one can easily see the Eqs. (3) have the standard form \cite{4, 5}: classical scaling dimension in $4-\epsilon$-dimensions plus one-loop $\beta$-function. Eq. (3) for $\xi$ has again the same form as in four dimensions \cite{1}.

The IR behavior of the system defines the critical phenomena. There are two fixed points of Eqs. (3):

1) Unstable fixed point: $f^* = 0$, $\xi^*$ is arbitrary

2) Stable fixed point: $f^* = \frac{(4\pi)^2\epsilon}{3}$, $\xi^* = \frac{1}{6}$.

The last fixed point for $\xi^* = \frac{1}{6}$ indicates the phenomenon of asymptotic conformal invariance which was found in \cite{6}. Note that we discuss here only matter sector. Phase space is more extended as the complete fixed point is given by $(f^*, \xi^*, a_1^*, a_2^*, a_3^*)$, (see also Appendix).

It is quite well-known \cite{2, 3, 4} that near the stable fixed point of the $\epsilon$-expansion the system experiences second-order phase transitions. At the same time, first order phase transitions which are typically common in quantum field theory are predicted \cite{5} if the theory possesses no stable fixed point. Moreover, as a rule they are only weakly of first order.

One can also note the explicit solution of the RG equations:
$$f(t) = \frac{f \epsilon (4\pi)^2}{3f - (3f - \epsilon (4\pi)^2)e^{\epsilon t}}$$

(5)

$$\xi(t) - \frac{1}{6} = (\xi - \frac{1}{6})(e^{\epsilon t} f(t))^\frac{1}{3}$$

When $t \to -\infty$, $f(t) \to f^*$, $\xi(t) \to \xi^*$ corresponding to the IR stable fixed point (4). For a discussion of the equivalence between the IR behavior of finite-temperature systems ($\epsilon = 1$) and three-dimensional theories, see [7].

Let us consider now the more interesting example of $SU(2)$ gauge theory with scalars and spinors. Such a model is of importance because the IR behavior of SM at finite temperature is described by the confining phase of some $SU(2)$ gauge theory with matter. We will be interested in the theories which are asymptotically free [14] on special solutions of RG (4). The Lagrangian of such a model is given by eq. (3.116) of ref [1] with the evident replacement $f \to f^\mu$, $h \to h^\mu$, $g \to g^\mu$. The theory contains two scalar multiplets and one spinor multiplet, four scalar couplings: $f_1, f_2, f_3, f_4$ and two Yukawa couplings: $h_1, h_2$. Using results of [8, 6] one can explicitly write RGEs for all coupling constants in 4-$\epsilon$-dimensions. After that, the fixed points of the $\epsilon$-expansion can be easily found as numerical solutions of algebraic equations, i.e. zeros of the r.h.s. of RG equations. However, from the very beginning, one finds that the corresponding system of fixed points does not have stable solutions. Only the unstable fixed point is possible for gauge coupling constant $g_2^* = 0$. Hence only first order phase transitions may be expected.

We are interested later in the behavior of the effective potential for the system under discussion. To calculate it we need the explicit solution of RG equations. However, even in exactly four dimensions the explicit solution (so-called special solution) of RG equations for the theory under discussion can be found only in two regimes [8]:

$$h_i^2(t) = k^2_i g^2(t), \quad f_j(t) = k^2_j g^2(t)$$

(6)

where numerical constants $k_i^2$, $i = 1, 2$ and $k_j^2$, $j = 1, \cdots, 4$ are different in the above two regimes. It is clear that in 4-$\epsilon$-dimensions the r.h.s. of the RG equations are modified by classical scaling of coupling constants. The special solution of the type (3) may survive in exceptional cases. The theory under consideration with Lagrangian (3.116) of [1] belongs to this class. In one of the regimes of asymptotic freedom where the model corresponds to
N=2 supersymmetric theory

\[ f_1 = f_2 = f_3 = 0, \quad h_1 = h_2 = g, \quad f_4 = g^2 \]  \hspace{1cm} (7)

One can see that this asymptotically free solution is not spoiled by \( \epsilon \)-dependent terms of RG equations. In other words, in this regime 4 different RG equations for coupling 1s \( g^2, h_1^2, h_2^2, f_4 \) reduce to the same RG equation:

\[ \frac{dg^2}{dt} = -\epsilon g^2 - \frac{8g^4}{(4\pi)^2} \]  \hspace{1cm} (8)

with the following explicit solution:

\[ g^2(t) = \frac{g^2}{-\frac{8g^2}{(4\pi)^2\epsilon} + e^{\epsilon t}(1 + \frac{8g^2}{(4\pi)^2\epsilon})} \]  \hspace{1cm} (9)

For \( t \to \infty \), \( g^2(t) \to 0 \) (asymptotic freedom).

In IR (\( t \to -\infty \)), \( g^2(t) \to g^*_2 = -\frac{(4\pi)^2\epsilon}{8} \). Hence, from the analysis of the general solution for coupling constant we also see the appearance of fixed points in IR and UV regions.

We may also ask what happens with the scalar-gravitational coupling constants \( \xi_1, \xi_2 \). The corresponding RG equations are written in the book [1] (see Eqs (3.117)) and they are the same in 4 or in 4-\( \epsilon \)-dimensions. Taking into account conditions [7] we get

\[ (4\pi)^2 \frac{d\xi_1(t)}{dt} = -4g^2(t)(\xi_1(t) - \frac{1}{6}) + 4g^2(t)(\xi_2(t) - \frac{1}{6}), \]
\[ (4\pi)^2 \frac{d\xi_2(t)}{dt} = 4g^2(t)(\xi_1(t) - \frac{1}{6}) - 4g^2(t)(\xi_1(t) - \frac{1}{6}) \]  \hspace{1cm} (10)

Analyzing the r.h.s. of (10) we see that for the UV stable fixed point \( g^*_2 = 0 \) the fixed points \( \xi^*_1, \xi^*_2 \) are given by arbitrary values in agreement with the results of [3].

For the IR fixed points \( g^*_2 \) we get the following: \( \xi^*_1 - \xi^*_2 = 0 \), otherwise \( \xi^*_1 \) and \( \xi^*_2 \) have arbitrary values. In particular, one can choose \( \xi^*_1 = \xi^*_2 = \frac{1}{6} \), to realize the asymptotic conformal invariance in the IR region.

The explicit solution of Eq. (10) maybe written as follows:
\[ \xi_1(t) = \frac{1}{2} (\xi_1 + \xi_2) + \frac{1}{2} (\xi_1 - \xi_2) \left( e^{\epsilon t} g^2(t) \right)^{\frac{3}{2}} \]
\[ \xi_2(t) = \frac{1}{2} (\xi_1 + \xi_2) - \frac{1}{2} (\xi_1 - \xi_2) \left( e^{\epsilon t} g^2(t) \right)^{\frac{3}{2}} \]

(11)

So far, we discussed the behavior of SU(2) gauge model with matter in D=4-\( \epsilon \)-dimensional curved spacetime. In the same manner one can analyze other gauge models, where unfortunately one can not obtain the explicit solutions of RGE's. For example, let us consider another SU(2) gauge theory with \( m \) spinor triplets, one scalar triplet, one Yukawa coupling and one scalar coupling (the Lagrangian is given by Eq.(3.97) from [1]). The RG equations maybe easily written

\[ \frac{dg(t)^2}{dt} = -\epsilon g(t)^2 - \frac{(14 - \frac{16}{3}m)g(t)^4}{(4\pi)^2}, \]
\[ \frac{dh(t)^2}{dt} = -\epsilon h(t)^2 + \frac{16h(t)^4 - 24h(t)^2g(t)^2}{(4\pi)^2}, \]
\[ \frac{df(t)}{dt} = -\epsilon f(t) + \frac{11f(t)^2 - 24g(t)^2f(t) + 72g(t)^4}{(4\pi)^2} \]
\[ + \frac{16f(t)h(t)^2 - 96h(t)^4}{(4\pi)^2}, \]
\[ \frac{d\xi(t)}{dt} = \frac{1}{(4\pi)^2} \left( \xi(t) - \frac{1}{6} \right) \left( \frac{5}{3} f(t) + 8 h(t)^2 - 12 g(t)^2 \right) \]

(12)

This system of RG equations for \( m=1,2 \) has only one UV stable point at \( g = 0, h=0, f=0 \) and \( \xi \) arbitrary. For \( m \geq 3 \) we have also an IR stable point at

\[ g^2 = -\frac{\epsilon(4\pi)^2}{14 - \frac{16}{3}m}, \quad h^2 = -\frac{1}{16} \epsilon(4\pi)^2 \left( \frac{10 - \frac{16}{3}m}{14 - \frac{16}{3}m} \right), \]
\[ f = -\frac{\epsilon(4\pi)^2}{14 - \frac{16}{3}m} \left[ \frac{3}{11} \left( \frac{3}{8} \left( 10 + \frac{16}{3}m \right)^2 - 72 \right) \right]^{1/2}, \quad \xi = \frac{1}{6}. \]

Note finally that by considering interaction of the above models with renormalizable higher-derivative quantum gravity one can estimate the in-
fluence of quantum gravity effects on fixed points of RG in the $\epsilon$-expansion.

Another interesting point is related to the possibility of calculating the RG improved effective potential in the above theories in 4-$\epsilon$-dimensions. Let us start from the RG equations for the effective potential (this equation is satisfied due to multiplicative renormalizability):

$$ (\mu \frac{\partial}{\partial \mu} + \beta_{p_i} \frac{\partial}{\partial \beta_{p_i}} - \gamma \phi \frac{\partial}{\partial \phi}) V(\phi) = 0 \quad (13) $$

where $p_i$ are all coupling constants, $\beta_{p_i}$ are corresponding $\beta$-functions and $\gamma$ is the anomalous scaling dimension of scalar. Solving (13) by the method of characteristics, using tree level potential as boundary condition we obtain the RG improved effective potential [10] (for a recent discussion of RG improvement in flat and curved space, see [11, 12], respectively). For example, for scalar self-interacting theory one gets

$$ V(\phi) = \mu^2 f(t) \phi^4 - \frac{1}{2} \gamma(t) R \phi^2 \quad (14) $$

where the running coupling constants are given by Eq. (5). Taking $\epsilon = 1$ we get leading-log behavior of the effective potential in three dimensions. Note that $t = \frac{1}{2} \log \frac{\phi^2}{\mu^2}$ where $\mu$ should be identified with the temperature [13]. Similarly, one can find the RG improved effective potential for a SU(2) gauge model or any other theory.

### 3 Effective Lagrangian for a covariantly constant gauge field in 4-$\epsilon$-dimensions

The $\epsilon$-expansion discussed in the previous section maybe well applied also to study the effective potential for a covariantly constant gauge field in 4-$\epsilon$-dimensional curved space-time. In flat D=4 space such an effective potential (chromomagnetic potential) has been discussed some time ago in [13]. It has been shown that the chromomagnetic potential may have a nonzero minimum, i.e. there is the possibility of chromomagnetic vacuum in an electroweak theory. Unfortunately, such a state could not be the true vacuum of the theory because the effective potential has an imaginary part [13]. There have been various proposals about how this vacuum maybe stabilized.
It is quite possible to expect the presence of large (chromo)-magnetic fields in early hot universe. There are indications that non-zero curvature may lead to some stabilization of the chromomagnetic vacuum \[16\]. It could be that the combined effect of non-zero temperature and non-zero curvature may result in the vanishing of the imaginary part of the chromomagnetic potential and stabilization of the corresponding vacuum. The $\epsilon$-expansion technique gives the way to calculate the approximate effective potential under such circumstances.

We will consider pure SU(2) gauge theory without matter in a weakly curved constant curvature space-time $R_{\mu\nu} = g_{\mu\nu}R$ where the expansion of the effective potential over curvature maybe used. Taking into account that the theory is renormalizable, and only the RG equation for gauge coupling is changing due to classical scaling dimension one can repeat the arguments given in \[16\] to obtain the RG improved effective Lagrangian:

$$\mathcal{L} = \frac{1}{4} g^2(t) G_{\mu\nu}^a G^{a,\mu\nu} + a_1(t) R^2 + a_2(t) C_{\mu\nu\alpha\beta}^2 + a_3(t) G$$ \hspace{1cm} (15)

here $g^2(t)$ has the form similar to \(9\) with $\epsilon = 1$. For SU(2) gauge theory the condition of the covariantly constant background is

$$\hat{\nabla}^\mu ab G^b_{\mu\nu} = 0 \hspace{1cm} (16)$$

This equation should be understood as a normal coordinates expansion where the first term gives the flat space equation and the remaining ones give curvature corrections. Limiting ourselves to the case of a covariantly constant magnetic field we get

$$\frac{1}{4} G_{\mu\nu}^a G^{a,\mu\nu} = \frac{1}{2} H^2 + O(R) \hspace{1cm} (17)$$

Now, we have to define RG parameter $t$. For flat space \[15\] $t = \frac{1}{2} \ln \frac{gH}{\mu^2}$ where $\mu$ should be identified with temperature $T$.

At the same time in curved space-time with vanishing gauge field the effective mass of the theory is given by curvature \[1, 12\]. Hence a possible choice for the RG parameter $t$ is

$$t = \frac{1}{2} \ln \frac{R + gH}{\mu^2} \hspace{1cm} (18)$$
Note that this is very rough estimation as actually we have a mass matrix which should be properly diagonalized. As a result we will get few effective masses and a procedure of RG improvement with few effective masses should be used [17]. However, for a first, rough estimate the RG parameter $t$ [18] maybe employed. Then we get ($\epsilon = 1$)

$$\mathcal{L} = -\frac{1}{2} \frac{g^2}{g^2(t)} H^2 + a_1(t) R^2 + a_2(t) C^2_{\mu\nu\alpha\beta} + a_3(t) G$$

(19)

where

$$g^2(t) = \frac{g^2}{12\pi^2 g^2} + e^t (1 + \frac{11}{128\pi^2} g^2) , \quad g^2 < 1$$

Due to choice of background $G = \frac{R^2}{6}, \quad C_{\mu\nu\alpha\beta} = 0$ and Eq. (19) maybe rewritten as

$$\mathcal{L} = -\frac{1}{2} \frac{g^2}{g^2(t)} H^2 + (a_1 + \frac{1}{6} a_3 - \frac{62 t}{720(4\pi)^2}) R^2$$

(20)

The approximate minimum of the effective potential is given by analysis of the equation

$$\frac{\partial V}{\partial H} = \frac{\partial}{\partial H} \left( \frac{1}{2} \frac{g^2}{g^2(t)} H^2 \right) = 0$$

(21)

Numerical analysis of the effective potential maybe done. For example, in $D=4$ ($\epsilon = 0$) one gets [19]

$$g H_{\text{min}} = \mu^2 \exp \frac{-24\pi^2}{11 g^2} - \frac{R}{4}$$

(22)

Curvature slightly modifies the (unstable) chromomagnetic vacuum of [16], and may even act in the direction of its stabilization [17].

At non-zero temperature ($\epsilon = 1$) and curvature one obtains:

a) At zero curvature

$$g H_{\text{min}} = T^2 \left( \frac{11 g^2}{15\pi^2} \right)^\frac{1}{2}$$

(23)

So we find the possibility of temperature-induced phase transitions. However, the non-zero vacuum state is unstable [15] what can not be seen in our approximation. The qualitative form of the potential is given in Fig. 1.
b) At non-zero curvature \( (R \ll gH) \) we get

\[
\left( \frac{gH_{\text{min}} + \frac{R}{4}}{T^2} \right)^{1/2} = \frac{11g^2}{15\pi^2} + \sqrt{\left( \frac{11g^2}{15\pi^2} \right)^2 + \frac{R}{8\pi^2}} \tag{24}
\]

Hence we find here the possibility of phase transitions induced by the combined effect of curvature and temperature. The schematic behavior of the effective potential is again the same as above. The critical line on the phase plane \( R-T \) is given by Eq. (24) at \( H_{\text{min}} = 0 \).

The present picture indicates that for some values of curvature and temperature the imaginary part is canceled by the combined effect of non-zero curvature and temperature (stabilization of chromomagnetic vacuum). However to check this conjecture one would have to consider the above problem not in \( \varepsilon \)-expansion but exactly (making explicit calculation of effective Lagrangian at non-zero \( T \) and some fixed gravitational background).

4 Gauged NJL model in 4-\( \varepsilon \)-dimensional curved space-time

In the present section we will consider gauge-Higgs-Yukawa theory of the type (2). We work in modified \( 1/N_c \)-expansion in 4-\( \varepsilon \)-dimensional curved space-time and we use the equivalence of the above model with gauged NJL model (what was shown in [22]). That gives the way to study gauged NJL-model in 4-\( \varepsilon \)-dimensions.

We start from the \( SU(N_c) \) gauge theory with scalars and spinors in 4-\( \varepsilon \)-dimensional curved space-time:

\[
\mathcal{L}_m + \frac{1}{2} \xi R G^2 = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - \frac{1}{2} m^2 \sigma^2 - \frac{1}{2} \lambda \mu \sigma^4 \\
+ \frac{1}{2} \xi R \sigma^2 + \sum_{i=1}^{N_f} \bar{\psi}_i i\gamma^\mu \nabla_\mu \psi_i - \sum_{i=1}^{n_f} y_{\mu} \tilde{\sigma} \psi_i \bar{\psi}_i \tag{25}
\]

where \( \sigma \) is a single scalar, \( \lambda \) is scalar coupling, \( N_f \) fermions belong to the \( R \) representation of \( SU(N_c) \) and we denote the Yukawa constant by \( y \) here as in [23].
The modified $\frac{1}{N_c}$-approximation of [22] consists in choosing small gauge coupling (first non-trivial order in $g^2$), $N_f \sim N_c$ but $n_f \ll N_f$, and dropping of scalar loop contributions (leading order of $\frac{1}{N_c}$). Within such an approach and taking also the classical scaling dimensions into account we get

\[
\begin{align*}
\frac{dg}{dt} &= -\frac{\epsilon}{2}g(t) - \frac{b g^3(t)}{(4\pi)^2}, \\
\frac{dy}{dt} &= y(t) \left[ \frac{a y^2(t)}{(4\pi)^2} - \frac{c g^2(t)}{(4\pi)^2} - \frac{\epsilon}{2} \right], \\
\frac{d\lambda}{dt} &= u y^2(t) \left[ \frac{\lambda(t)}{(4\pi)^2} - \frac{y^2}{(4\pi)^2} \right] - \epsilon \lambda, \\
\frac{d\xi}{dt} &= \frac{1}{(4\pi)^2} \frac{2 a y^2(t)}{(4\pi)} - \frac{1}{6} \lambda(t) = \frac{1}{2} a_c - \frac{1}{6} y^2(t) g^2(t) \left[ 1 + h_0(e^{\epsilon t} \eta(t))^{1-\frac{c}{b}} \right]^{-1} - \frac{1}{6} \left( \xi(t) - \frac{1}{6} \right) y^2(t) g^2(t) \left( 1 + k_0(e^{c t} \eta(t))^{1-\frac{a}{c}} \right)
\end{align*}
\]

where [22]: $b = \frac{11 N_c - 4 T(R) N_f}{3}, c = 6 C_2(R), a = \frac{n_f}{4} = 2 n_f N_c$. Here $t = \ln \frac{\mu}{\mu_0}$, and for the fundamental representation $T(R) = \frac{1}{2}, C_2(R) = \frac{N_f^2 - 1}{2 N_c}$. Only the case $c > b$ will be considered. Note that the extension of above model to the case of external gravitational and magnetic fields has been done in [23, 24] respectively. Solving the RG system (26) one can find that this system of RG equations only leads to an UV stationary point at $g=0, y=0, \lambda = 0, \xi$ arbitrary.

One can easily obtain the solution of RG equations in terms of RG invariants:

\[
\begin{align*}
\eta(t) &= \frac{\alpha(t)}{\alpha} = \frac{\epsilon}{-\frac{2 b y^3}{(4\pi)^2} + (\epsilon + \frac{2 b y^2}{(4\pi)} e^{\epsilon t})}, \\
y^2(t) &= \frac{c - b}{a} g^2(t) [1 + h_0(e^{\epsilon t} \eta(t))^{1-\frac{c}{b}}]^{-1}, \\
\lambda(t) &= \frac{2 a}{2 c} y^2(t) [1 + k_0(e^{c t} \eta(t))^{1-\frac{a}{c}}], \\
\xi(t) - \frac{1}{6} &= \left( \xi - \frac{1}{6} \right) y^2(t) g^2(t) (e^{1-\frac{c}{b} t} \eta^{1-\frac{a}{c}}(t)),
\end{align*}
\]

where the RG invariants $h_0, k_0$ are defined as:
\[ h(t) = -(e^{et} \eta(t))^{-1 + \frac{1}{2}} \left[ 1 - \frac{c - b g^2(t)}{a y^2(t)} \right], \]
\[ k(t) = -(e^{et} \eta(t))^{-1 + \frac{1}{2}} \left[ 1 - \frac{2c - b \lambda(t) g^2(t)}{2a y^2(t) y^2(t)} \right]. \] (28)

(They don’t depend on \( t \) so we may add subscript zero).

Below, we only consider the case of fixed gauge coupling \( b \to +0 \):

\[ (e^{et} \eta(t))^{-1 \frac{1}{2}} \to \exp \left\{ \frac{\alpha}{\alpha_c} \left( \frac{1 - e^{-et}}{\epsilon} \right) \right\} \] (29)

where \( \alpha_c^{-1} = \frac{3c_4(R)}{\pi}, \alpha = \frac{g^2}{4\pi} \) and

\[ y^2(t) = \frac{c - b}{a} g^2(t) \left[ 1 + h_0 \exp \left\{ \frac{\alpha}{\alpha_c} \left( \frac{1 - e^{-et}}{\epsilon} \right) \right\} \right]^{-1} \]
\[ \lambda(t) = \frac{2a}{(4\pi)^2} \frac{\alpha_c}{\alpha} y^4(t) \left[ 1 + k_0 \exp \left\{ \frac{2\alpha}{\alpha_c} \left( \frac{1 - e^{-et}}{\epsilon} \right) \right\} \right] \] (30)

The analysis of the behavior of the above coupling constants at \( \epsilon = 0 \) has been done in [22] where phase structure has been studied with all details.

Now we turn to the \( SU(N_c) \) gauged NJL model with four-fermion coupling constant \( \tilde{G} \) in curved space-time:

\[ \mathcal{L} = -\frac{1}{4} G^a_{\mu \nu} + \sum_{i=1}^{N_f} \bar{\psi}_i \gamma^\mu(x) \nabla_\mu \psi_i + \tilde{G} \sum_{i=1}^{n_f} (\bar{\psi}_i \psi_i)^2, \] (31)

\( \tilde{G} \) let denote four-fermion coupling.

The easiest way to study such a model is to introduce an auxiliary field \( \sigma \), in order to identify the NJL- model with the Higgs-Yukawa model. As has been shown by Bardeen-Hill-Lindner [25] one can put a set of boundary conditions (compositeness condition) for the effective couplings of the gauge-Higgs-Yukawa model at \( t_\Lambda = \ln \frac{\Lambda}{\mu_0} \) (where \( \Lambda \) is UV cutoff of the gauged NJL model) in order to prove the equivalence of the gauged NJL model with the gauge-Higgs-Yukawa model [22]. Then one can study gauged NJL model using the RG method as for usual renormalizable theories but the RG method is more easy than the complicated Schwinger-Dyson equation [27].
In our discussion using compositeness conditions for coupling constant we may actually define RG invariants. Then, omitting the details which are very similar to ones given in [25, 22] (without \(\epsilon\)-dependence), one may use Eqs. (27), (30) and find the running Yukawa and scalar couplings in the gauged NJL model (\(g(t)\) is not changing). We write them below for \(b \to +0:\)

\[
y^2(t) = y^2_\Lambda(t) = \frac{(4\pi)^2 \alpha}{2a} \frac{\alpha_c}{\alpha_c} \left[ 1 - \left( \frac{\exp \left( \frac{1-(\mu_0/\mu)^\epsilon}{\epsilon} \right)}{\exp \left( \frac{1-(\mu_0/\Lambda)^\epsilon}{\epsilon} \right)} \right)^{2\alpha_c} \right]^{-1}
\]

\[
\lambda(t) \quad \frac{\lambda_\Lambda(t)}{\lambda^4(t)} = \frac{2a}{(4\pi)^2} \frac{\alpha_c}{\alpha_c} \left[ 1 - \left( \frac{\exp \left( \frac{1-(\mu_0/\mu)^\epsilon}{\epsilon} \right)}{\exp \left( \frac{1-(\mu_0/\Lambda)^\epsilon}{\epsilon} \right)} \right)^{2\alpha_c} \right]^{-1}
\]

where \(t < t_\Lambda\). In the limit \(\epsilon \to 0\) Eqs. (32) coincide with the corresponding Eqs of [22].

It is interesting to note that in the limit \(\Lambda \to \infty\) we get from (32):

\[
y^2_\Lambda(t) \quad \lambda_\Lambda(t) \quad \frac{\lambda_\Lambda(t)}{\lambda^4(t)} \to \frac{(4\pi)^2 \alpha}{2a} \frac{\alpha_c}{\alpha_c} \left[ 1 - \left\{ \exp \left( -\frac{(\mu_0/\mu)^\epsilon}{\epsilon} \right) \right\} a^{\alpha_c/\alpha_c} \right]^{-1}
\]

In addition to compositeness conditions for scalar and Yukawa couplings, in order to prove the equivalence between gauge-Higgs-Yukawa model (25) and gauged NJL model (31) one should also have the compositeness condition for mass and for scalar- gravitational coupling constant.

The analysis of the compositeness condition for mass is completely similar to the one given in [22]. So we present only the result in the limit \(b \to +0:\)

\[
m^2(t) = \frac{2a}{(4\pi)^2} y^2_\Lambda(t) \left( \exp \left\{ \frac{(\mu_0/\Lambda)^\epsilon - (\mu_0/\mu)^\epsilon}{\epsilon} \right\} \right)^{\alpha_c/\alpha_c} \Lambda^2 \left[ \frac{1}{g_4(\Lambda)} - \frac{1}{w} \right]^{\alpha_c/\alpha_c}
\]

where \(g_4(\Lambda)\) is a dimensionless constant defined by \(\tilde{G} \equiv \left( \frac{(4\pi)^2}{\alpha} \right) \frac{g_4(\Lambda)}{\Lambda^2}\) and \(w = 1 - \frac{\alpha}{\alpha_c}, y^2_\Lambda(t)\) is given by Eq (32).
The compositness condition for $\xi(t)$ is the same as in non-gauged NJL model \cite{26} (see also discussion in \cite{23} for gauged NJL model):

$$\xi(t) = \frac{1}{6}$$ (35)

Thus using the equivalence with the gauge-Higgs-Yukawa theory we get the description of the gauged NJL model via running coupling constants.

Having this description of the gauged NJL model via renormalizable SU($N_c$) gauge theory one can apply RG improvement technique to calculate the effective potential (taking account of leading logarithms from the whole set of diagrams). Following the method of \cite{22} (and its extension to curved space-time \cite{23}) one can solve the RG equation for effective potential in following form:

$$V(g, y, \lambda, m^2, \xi, \sigma, \mu) = V(\bar{g}(t), \ldots, \bar{\sigma}(t), \mu e^t)$$ (36)

Here the effective coupling constants $\bar{g}(t), \ldots, \bar{\sigma}(t)$ are defined by RG equation (26) (at scale $\mu e^t$) and RG eqs. for $\bar{m}^2(t), \bar{\sigma}(t)$ are given in \cite{22}. As the boundary condition it is convenient to use the one-loop effective potential \cite{23}.

For gauged NJL model we have to substitute to RG improved potential the effective couplings which fulfill the compositeness conditions (see discussion above). Evaluating the running parameters at $t$ determined by

$$\bar{M}_F(t) \equiv \bar{g}(t)\bar{\sigma}(t) = e^t\mu$$ (37)

one should insert (37) into the effective potential (36). From the RG invariant (at $b \rightarrow +0$) and (37)

$$\exp\left\{\frac{\alpha}{\alpha_c} \frac{1 - e^{-\epsilon t}}{\epsilon}\right\} M_F(t)$$ (38)

one defines the connection between $t$ and $M_F$:

$$\exp\left\{\frac{\alpha}{\alpha_c} \frac{1 - e^{-\epsilon t}}{\epsilon}\right\} e^{2t} = \frac{M_F^2(\mu)}{\mu^2}$$ (39)

Using the fact that $\bar{m}^2(t)\bar{\sigma}^2(t)$ is also RG invariant, applying (38), (34) and (34), (37) we can find the quadratic part of the RG improved effective potential (for related discussion, see \cite{24, 23, 24})
Here, $\bar{y}_2(t) = y_2(e^t \mu)$ is the rescaling of $y_2(t)$. It is obtained from the (33) where one takes $e^t \mu$ instead of $\mu$. The factor $e^t$ is fixed by the relation (39).

In a similar way one can find the four-scalar part of the effective potential which is not important for us. That is the quadratic part which defines the symmetry breaking:

$$V_2 < 0$$  \hspace{1cm} (41)

For $V_2 < 0$ the symmetry is broken. Even in flat space there is the possibility for chiral symmetry breaking. In curved space-time, the condition $V_2 = 0$ defines the critical curvature which gives the critical line between the symmetric phase and the chiral symmetry broken phase. Hence, we again showed the principal possibility of curvature-induced phase transition in gauged NJL model in curved $4-\epsilon$- dimensional space-time.

5 Discussion

In the present work we have developed the $\epsilon$-expansion technique for quantum field theory in curved spacetime. Unlike the case of flat space, the phase space of the present theory is extended. One has additional RG equation for the scalar-gravitational coupling constant $\xi$ and for vacuum coupling constants and the corresponding fixed points.

We discussed the possibility for survival of asymptotic freedom of all coupling constants as special solution of the RG equations in $4-\epsilon$-dimensions. This phenomenon may happen only with supersymmetric theory (in flat space), unlike to the case of four dimensions. As applications of the $\epsilon$-expansion technique we found the RG improved effective Lagrangian for co-variantly constant gauge field in SU(2) theory and the RG improved effective potential for gauged NJL-model in curved space with non-zero temperature. The phase structure of both models was discussed. It is very interesting to
note that such calculation of the RG improved effective action in curved space with non-zero temperature may find cosmological applications (for example, in the calculation of the wave function of the universe).

Another interesting line of research is related with the application of the $\epsilon$-expansion in the study of Casimir effect at non-zero temperature in solid state physics. Here, one again has to find some RG improved effective action. (We give few remarks on Casimir effect in media in Appendix B).

Finally, let us note that there exists a long-standing proposal of Weinberg [18] to formulate consistent, asymptotically-safe quantum gravity in $2 + \epsilon$-dimensions. The subsequent analytic continuation to $\epsilon = 1$ or $\epsilon = 2$ is then necessary. (Note that it is quite probable that the program of Weinberg may be consistently realized in dilatonic gravity [19, 21], at least up to the point before analytic continuation). From this point of view, further study of the $\epsilon$-expansion in 4-\(\epsilon\)-dimensional curved spacetime (and subsequently, in 4-\(\epsilon\)-dimensional quantum gravity) may provide us with new ideas, as some synthesis of 2+\(\epsilon\)-dimensional approach with 4-\(\epsilon\)-dimensional approach could lead to coinciding results for $\epsilon = 1$. At least, such analysis may enrich both approaches.

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6 Appendix A: Running vacuum couplings

In the present Appendix we describe the behavior of vacuum effective couplings in 4-\(\epsilon\)-dimensions.

For vacuum (or external fields) effective couplings in $f \phi^4$-theory we get (compare with [28])

$$
\begin{align*}
\frac{da_1}{dt} &= \epsilon a_1 + \frac{1}{2(4\pi)^2}(\zeta - \frac{1}{6})^2 \\
\frac{da_2}{dt} &= \epsilon a_2 + \frac{1}{120(4\pi)^2} \\
\frac{da_3}{dt} &= \epsilon a_3 - \frac{1}{360(4\pi)^2}
\end{align*}
$$

(42)

The general solution of Eqs. (42) is:
\[ a_1(t) = \left[ a_1 + \frac{(\xi - \frac{1}{6})^2}{6f} \left( \ln(4\pi)^2\epsilon + et + \ln f(t)f(4\pi)^2\epsilon^2 f \right) \right] e^{et} \]

\[ a_2(t) = -\frac{1}{120(4\pi)^2\epsilon} + \left[ a_2 + \frac{1}{120(4\pi)^2\epsilon} \right] e^{et} \]

\[ a_3(t) = \frac{1}{360(4\pi)^2\epsilon} + \left[ a_3 + \frac{1}{360(4\pi)^2\epsilon} \right] e^{et} \]

(43)

At the IR fixed point (when \( t \to -\infty \)) we get

\[ a_1(t) \to 0, \quad a_2(t) \to -\frac{1}{120(4\pi)^2\epsilon}, \quad a_3(t) \to \frac{1}{360(4\pi)^2\epsilon}, \quad (44) \]

Similarly, one can find surface coupling constants [28, 31] in 4-\( \epsilon \)-dimensions. It is also easy to generalize the above RG equations for more complicated theories, the qualitative structure of solutions (43) will be the same.

One can apply the RG equations (43) to find RG improved effective action \( \Gamma \) on quasi-De Sitter background \( S_3 \times S_1: R_{\mu\nu} = \Lambda g_{\mu\nu} \).

To this end one can write the RG improved effective action on \( S_4 \) [28] (at zero background scalar)

\[ \Gamma = 24\pi^2 \left\{ 16a_1(t) + \frac{8}{3} a_3(t) \right\} \quad (45) \]

Now, in order to translate this expression to \( \Gamma \) on \( S_3 \times S_1 \) one should use \( a_1(t) \) and \( a_2(t) \) given by Eqs. (43) (with \( \epsilon = 1 \) and \( t = \frac{1}{2} \ln \frac{4\Lambda^2}{\mu^2} \). Note that \( \mu \) should be identified with the temperature [13]. Of course, such approximation is rather qualitative as we consider \( \epsilon = 1 \).

Note also that the above result (45) provides a good example of the calculation of gravitational Casimir effect (for a good introduction to the theory of Casimir effect, see [29, 30]).

7 Appendix B: On the calculation of the Casimir energy in realistic media

We close our work by making some remarks on the Casimir energy under realistic physical conditions emphasizing the electromagnetic case. Note that
this Appendix is somehow outside of the general line of discussion.

First of all, one must bear in mind that any boundary surface is made up of molecules of a real medium. The medium possesses in general both dispersive and absorptive properties. The boundary conditions that are conventionally adopted in scalar field theory, namely the Dirichlet or the Neumann conditions, can never be regarded as anything else than purely idealized conditions that lose their validity at extreme frequencies. The discussion is most conveniently carried out in the electromagnetic case, where the simplest boundary conditions are those of perfectly conducting surfaces. As long as one confines oneself to a "normal" frequency interval, it is in most cases quite legitimate to take the surfaces to be perfectly conducting. However, once the frequency \( \omega \) becomes much higher than the absorption frequency \( \omega_p \) (we consider only one absorption frequency, for simplicity), then the medium behaves like a plasma. For a normal metal such as copper, \( \omega_p = 5 \times 10^{16} \text{ s}^{-1} \). For \( \omega \gg \omega_p \) the material gradually becomes more "soft", and finally ceases to have any influence on the photons. In this way, nature itself provides us with a dispersive, soft UV cutoff. Analogous considerations apply to the scalar case: Dirichlet or Neumann conditions cease to exist at sufficiently high frequencies. The material, and thus also boundaries, gradually become soft and fade out of recognition for the field quanta.

As typical example in electrodynamics, let us consider the calculation of Casimir energy \( E \) for a spherical vacuum cavity of radius \( a \) in a dielectric medium (this example is closely related to the phenomenon of sonoluminescence). It is natural in this context to calculate \( E \) via the Casimir surface for \( F \) which acts at the boundary surface \( r=a \). This force, in turn, is equal to the difference between the radial diagonal components of the Maxwell stress tensor at \( r = a \pm \). Now, there has been some discussion in the recent literature about how to regularize the expression for \( F \). There is actually a need for two kinds of regularization here: first, we have to subtract from the Green function the pure volume contributions. That is at \( r=a- \) we subtract the Green function corresponding to the inner region (vacuum) filling all space, whereas at \( r=a+ \) we subtract the Green function corresponding to the outer region (dielectric) filling all space. This implies that the calculated expression for \( F \) goes to zero when \( r \rightarrow \infty \), and makes it accordingly for us to derive the Casimir energy via the formula
\[ F = -\frac{1}{4\pi a^2} \frac{\partial E}{\partial a}. \] (46)

This is the method of calculation given, for instance, by Milton and Ng \[32\] -cf. also \[33\].

The need of a second kind of regularization stems from the fact that the expression for \(E\) diverges when summed over angular momenta. It has turned out a very convenient way of obtaining the nondispersive part if the energy (or force) is to make use of the Riemann zeta function. This approach was followed in \[32, 33\].

So what is the role of dimensional regularization in the context? Dimensional regularization, generally speaking, is a calculations device in which singularities in calculated expressions can be isolated. It therefore becomes clear that this kind of regularization may serve as a substitute for the second step above, i. e. , a substitute for the Riemann zeta function method \[34\]. Although it would be of interest to calculate the Casimir energy for the cavity by means of dimensional regularization, we are not aware that such a calculation has been done.

So far, we have ignored gravitation. If a gravitational field is present, the analysis goes through in the same manner as above, provided that the gravitational radius is much larger than the intermolecular spacing in the material. Then all phenomenological parameters, such as permitivity or permeability, can still be used. By contrast, if the field becomes so strong that gravitational radius is of the same order as the intermolecular spacing, then we have leave the simplest macroscopic picture of the material and resort to ma much more complicated many-body picture. It would be very interesting to include the temperature to above consideration (for example, via the \(\epsilon\)-expansion technique).

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