THE UNPOLARIZED GLUON ANOMALOUS DIMENSION AT SMALL $x^a$

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We discuss the quantitative consequences of the resummation of the small-$x$ contributions to the anomalous dimensions beyond next-to-leading order in $\alpha_s$ and up to next order in $\ln(1/x)$ (NL$_x$) in a framework based on the renormalization group equations. We find large and negative effects leading to negative values for the total splitting function $P_{gg}(x, \alpha_s)$ already for $x \lesssim 0.01$ at $Q^2 \simeq 20$ GeV$^2$. Terms less singular than those under consideration turn out to be quantitatively as important and need to be included. We derive the effects of the conformal part of the NL$_x$ contributions to the anomalous dimensions and discuss the exponent $\omega$ describing the $s^{-\omega}$ behavior of inclusive cross sections.

1 Introduction

The scaling violations of deep-inelastic structure functions are described by renormalization group equations (RGE’s) stemming from the ultraviolet singularities of the corresponding local operators of a given twist. For the twist-2 contributions, these RGE’s are equivalent to the ones derived from mass factorization. In the following we will consider only these twist-2 terms, and study the impact of their resummed small-$x$ contributions on the evolution of structure functions. The non-perturbative input distributions at a scale $Q_0^2$ factorize. Furthermore, both the coefficient functions $c_{k,l}(x, Q^2/\mu^2)$ and the splitting functions $P_{ij}(x, \mu^2)$ are completely known up to two-loop order. Therefore they can be accounted for in complete form, whereas the resummed small-$x$ terms are considered beyond this order in addition.

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The generating functional for the leading small-\(x\) order (Lx) has been known for a long time\(^1\). More recently the resummed NLx coefficient functions and quarkonic anomalous dimensions were calculated\(^2\). Their structure and their numerical impact on the evolution of the deep-inelastic structure functions, as well as the consequences of the NLx anomalous dimension \(P_{gg}\), have been discussed in great detail, cf. ref.\(^3\). As the derivation of \(P_{gg}\) at NLx order\(^4\) was completed recently\(^5\),\(^6\), the present paper provides an update of previous results\(^7\) the quantitative changes with respect to that analysis being rather minor. We first describe the structure of \(P_{gg}\) as emerging from the NLx resummation, then present some numerical results, and finally discuss consequences on the intercept \(\omega(\alpha_s)\) describing the \(s^\omega\) behavior of inclusive cross sections, which sometimes is qualitatively interpreted also as the ‘rising’ power of structure functions at small \(x\). As already pointed out several years ago\(^7\), and in various more recent numerical studies of a large variety of small-\(x\) resummations\(^8\),\(^3\), we find that, also in the present case, meaningful results can only be obtained including the less singular orders which turn out to be quantitatively as important as the known ones.

### 2 The structure of \(P_{gg}^{NLx}\)

In ref.\(^9\) a Bethe–Salpeter equation with an infrared finite kernel at \(O(\alpha_s^2)\) was derived summing the leading pole \(\propto 1/(N-1)\) contributions. Its diagonalization can be performed by Mellin transform in the transverse–momentum space,

\[
\int dq_2^2 (q_2^2/q_1^2)\gamma^{-1}K(q_1, q_2) = \frac{N_c \alpha_s(q_1^2)}{\pi} \left[ \chi_0(\gamma) + \frac{N_c \alpha_s(q_1^2)}{\pi} \delta(\gamma) \right].
\]

The r.h.s. of eq. (1) can be related to the resummed anomalous dimension \(\gamma_{gg}(N, \alpha_s)\) in NLx order, see, e.g., ref.\(^3\). This can be easily seen in the conformal limit\(^10\) \(m = 0, \beta(\alpha_s) = 0\), see ref.\(^10\) for details, where the coupling constant \(\alpha_s\) is a fixed number. In Lx order the Bethe–Salpeter equation\(^9\) obeys this criterion. Therefore the solution \(\gamma\) of the resulting eigenvalue equation,

\[
1 = \overline{\alpha_s} \chi_0(\gamma), \quad \chi_0(\gamma) = 2\psi(1) - \psi(1 - \gamma), \quad \overline{\alpha_s} = \frac{N_c}{\pi} \alpha_s,
\]

represents the corresponding part of the anomalous dimension at all orders of the coupling constant.

In NLx order the situation is more complicated. Instead of eq. (2) one obtains

\[
1 = \frac{\overline{\alpha_s}}{N - 1} \left[ \chi_0(\gamma^+) - \frac{\alpha_s}{4} \delta(\gamma^+, q_1^2, \mu^2) \right],
\]

2
with \( \gamma_+ \) the larger eigenvalue of the singlet anomalous dimension matrix and

\[
\delta(\gamma, q_1^2, \mu^2) = \frac{\beta_0}{3} \chi_0(\gamma) \ln \left( \frac{q_1^2}{\mu^2} \right) + \left[ \frac{\beta_0}{6} + \frac{d}{d\gamma} \right] \left[ \chi_0^2(\gamma) + \chi_0'(\gamma) \right] + \chi_1^{\text{symm}}(\gamma). \tag{4}
\]

All terms but the last one are related to the breaking of conformal invariance in NLx order, cf. also ref. The second term changes for a different scale choice. Thus only \( \chi_1^{\text{symm}} \) can deal with as in the Lx case. Note that the NLx quarkonic anomalous dimensions result from a conformally invariant kernel in the \( Q_0 \)-scheme, they are therefore free of this problem.

The construction used to relate the solution of eq. (3) to the NLx anomalous dimension beyond the conformally invariant part is to absorb the term \( \propto \ln(q_1^2/\mu^2) \) into the coupling constant and to keep all the other terms. The implicit equation (3) is then solved representing \( \gamma \) as an infinite series in \( \alpha_s/(N - 1) \) for complex values of \( N \). All details for this solution as well a discussion of the mixing problem, which has to be solved order by order in \( \alpha_s \) in the singlet case, have been given in ref. Here we would like to add only a few remarks on the various contributions to

\[
\chi_1(\gamma) = \delta(\gamma, q_1^2, \mu^2) - \frac{\beta_0}{3} \chi_0(\gamma) \ln \left( \frac{q_1^2}{\mu^2} \right) = \chi_1^{(1)}(\gamma) + \chi_1^{(2)}(\gamma) + \chi_1^{(3)}(\gamma),
\]

\[
\chi_1^{(1)}(\gamma) = \frac{\pi^2}{\sin^2(\pi\gamma)} \cos(\pi\gamma) \ln(22 - \beta_0),
\]

\[
\chi_1^{(2)}(\gamma) = \frac{\pi^2}{\sin^2(\pi\gamma)} \cos(\pi\gamma) \ln \left( \frac{\gamma(1 - \gamma)}{(1 + 2\gamma)(3 - 2\gamma)} \right) \left( 1 + \frac{N_f}{3} \right) - \left( \frac{67}{9} - 2\zeta(2) - \frac{10}{27}N_f \right) \chi_0(\gamma) + 4\Phi(\gamma) - \frac{\pi^3}{\sin^2(\pi\gamma)},
\]

\[
\chi_1^{(3)}(\gamma) = \left[ \frac{\beta_0}{6} + \frac{d}{d\gamma} \left[ \chi_0^2(\gamma) + \chi_0'(\gamma) \right] \right] - 6\zeta_3.
\tag{5}
\]

The terms \( \chi_1^{(i)}(\gamma) \) contribute for the first time in \( i \)-loop order. Note that the \( \beta_0 \)-term \( \chi_1^{(1)} \) is due to the gluon self–energy, and has nothing to do with running coupling effects, unlike the one in \( \chi_1^{(3)} \). Therefore in the first two orders in \( \alpha_s \) only conformally invariant terms contribute to the gluon anomalous dimension in the small-\( x \) limit even in NLx order. These terms are unique in the class of DIS schemes. Let us note that the function \( \Phi(\gamma) \) obeys the representation

\[
\Phi(\gamma) = \frac{1}{\gamma} \sum_{l=2}^{\infty} (-1)^l \zeta(l) \gamma^{l-2} + \sum_{k=0}^{\infty} \left[ \frac{2\pi^2}{3} \eta(2k + 2) + c_{2k+1} \right] \gamma^{2k+1} \tag{6}
\]
with \( \eta(k) = \zeta(k) \left[ 1 - 2^{1-k} \right] \) and \( c_k = \frac{-2}{k!} \int_0^1 dz \ln^k(1/z) \text{Li}_2(z)/(1+z) \), which are new transcendentals. At 3-loop order the running coupling effects contribute for the first time. It should be mentioned that concerning the \( \zeta(3) \)-term differing results exist in the literature. The gluon Regge-trajectory was recently recalculated and agreement with the results of ref. was found. The numerical effect on both the anomalous dimension \( \gamma_{gg} \) in 3-loop order and the intercept \( \omega \) arising from the difference of the results is not negligible, see ref. Finally, starting with 4-loop order, scheme–specific terms by which, e.g., the \( Q_0 \) and the usual DIS scheme differ, and other effects due to the running of \( \alpha_s \), which are not discussed in ref. contribute to the anomalous dimension (for details see ref.). The latter effects are numerical very large.

In the table above an update is given of the coefficients \( \Delta_{gg} \) of table 1 in ref. is given, based on the recent results in refs.. The various contributions to the splitting function \( P_{gg}(x, \alpha_s) \) are illustrated for the kinematic

| \( k \) | \( \Delta_{gg} \) |
|---|---|
| 0 | -1.65000 E+1 |
| 1 | 0.00000 E+0 |
| 2 | -2.78734 E+1 |
| 3 | -2.25279 E+2 |
| 4 | -1.65853 E+2 |
| 5 | -7.24788 E+2 |
| 6 | -3.14501 E+3 |
| 7 | -3.49585 E+3 |
| 8 | -1.51028 E+4 |
| 9 | -4.91970 E+4 |
| 10 | -7.46877 E+4 |
| 11 | -2.99245 E+5 |
| 12 | -8.31843 E+5 |
| 13 | -1.59528 E+6 |
| 14 | -5.82155 E+6 |
| 15 | -1.49497 E+7 |
| 16 | -3.7088 E+7 |
| 17 | -1.12828 E+8 |
| 18 | -2.81522 E+8 |
| 19 | -7.03719 E+8 |

Figure 1: The \( x \)-dependence of the the splitting function \( P_{gg}(x, \alpha_s) \) at a typical value of \( \alpha_s \). The cumulative effect of the different orders is shown by consecutively adding the LO, NLO, Lx, NLx,qgq, and NLx contributions.
range at HERA in Figure 1. The NLx contributions are very large, leading to negative values for \( P_{gg}(x, \alpha_s) \) for \( x \approx 0.01 \) and \( Q^2 \approx 20 \text{ GeV}^2 \), thus destroying the probabilistic interpretation of \( P_{gg}(x) \) in LO. As demonstrated in ref.\(^4\), yet unknown less singular terms are expected to appear at the same size but with different sign. They are therefore quantitatively as important for the anomalous dimension in the small-\( x \) range as the Lx and NLx terms.

3 The ‘rising’ power \( \omega(\alpha_s) \)

The leading \( s \approx \ln(1/x) \) behavior of the scattering cross section can be described by

\[
\sigma(s) \propto s^{\omega(\alpha_s)}. \tag{7}
\]

In general the \( s \)-dependence of the cross section is not only due to perturbative contributions but also determined by the impact factors. If one neglects non-perturbative effects and discusses, in an informal way, only \( \omega(\alpha_s) \) as a perturbative, idealized value, this exponent is obtained evaluating \( \chi(\gamma) \) for the branch point \( \gamma = 1/2 \) for \( N_f = 4 \) resulting in

\[
\omega(\alpha_s) = 2.65 \alpha_s [1 - 6.36 \alpha_s]. \tag{8}
\]

The quadratic relation (8) leads to a maximum \( \omega_{\text{max}} = 0.10 \) at \( Q^2 \approx 8.7 \times 10^6 \text{ GeV}^2 \) which is comparable to the phenomenological value \( \omega_{\text{DL}} = 0.0808 \) for the soft pomeron \(^5\). Moreover, in most of the small-\( x \) kinematic range at HERA it takes negative values (for \( Q^2 \lesssim 600 \text{ GeV}^2 \)), e.g., \( \omega(20 \text{ GeV}^2) \approx -0.35 \), while the leading order value amounts to \( \omega(20 \text{ GeV}^2) \approx +0.64 \). Since second order correction yield such a drastic modification of \( \omega \), yet unknown less singular terms are to be expected to change this result significantly once again.

In the above exponentiation, both the conformally invariant parts in \( \chi(\gamma) \) and those which are due to the breaking terms of conformal invariance have been included. The latter contributions are related to the terms \( \chi_0(\gamma) + \chi'(\gamma) \), cf. eq. (4). It is known that the conformally invariant terms exponentiate \(^6\), however, the effect of the latter terms is less clear. If, as an example, one completely discards all terms which are due to the breaking of conformal invariance (see above) one obtains, (again for \( N_f = 4 \))

\[
\omega_{\text{conf}}(\alpha_s) = 2.65 \alpha_s [1 - 2.55 \alpha_s]. \tag{9}
\]

with a maximum value of \( \omega_{\text{conf}}^\text{max} = 0.26 \) at \( Q^2_{\text{max}} \approx 90 \text{ GeV}^2 \). Furthermore \( \omega_{\text{conf}} \) is positive for \( Q^2 \approx 2 \text{ GeV}^2 \). Note that both values above are idealized ones,

\(^c\) Note a numerical error in eq. (16) of ref.\(^5\).
and clearly the effects of the less singular terms, those due to the running of $\alpha_s$, and the non-perturbative input do need much further understanding for this quantity.

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Note added.

After completion of the present paper related work concerning the value of $\omega(\alpha_s)$ appeared. Higher order contributions in the saddle point method to determine $\omega$ were discussed in [A], starting with the full expression for $\chi(\gamma)$. By this method values of $\omega \approx 0.2$ are obtained. In ref. [B] the exponentiation of the $\beta$-independent contribution of $\chi(\gamma)$ was considered for the intercept. However, not all of these terms are scale independent. The conformal part of is obtained by discarding as well the other terms $\propto [\chi_0^2(\gamma) + \chi_0'(\gamma)]$ from $\chi(\gamma)$. The corresponding values for the intercept $\omega$ are therefore different.

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