Hydrodynamic equilibrium of a static star in the presence of a cosmological constant in $2 + 1$–dimensions

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Under the hydrodynamic equilibrium Buchdahl’s conditions on the behavior of the density and the pressure, for regular fluid static circularly symmetric star in $(2 + 1)$–dimensions in the presence of a cosmological constant, is established that there are no bounds from below on the pressure and also on the mass, except for their positiveness. The metric for a constant density distribution is derived and its matching with the external static solution with a negative cosmological constant is accomplished. Some mistakes of previous works on the topic are pointed out.

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I. INTRODUCTION

This work is devoted to the study of the equilibrium for static perfect fluid solutions with cosmological constant generalizing in this manner the Cruz–Zanelli analysis. The main objective of this contribution is to establish that in 2 + 1–dimensions there is no room for bounds of the mass distribution.

**Theorem**

If a perfect fluid distribution fulfills the conditions:

- it is described by a one-parameter state equation $p = p(\mu)$,
- the density is positive, $\mu > 0$, and monotonically decreasing, $\frac{d\mu}{dr} < 0$,
- it is microscopically stable, $\frac{dp}{d\mu} \geq 0 \rightarrow \frac{dp}{dr} \leq 0$,

then, in 2 + 1–dimensions, there is not a bound on the mass to the radius ratio.

II. STATIC CIRCULARLY SYMMETRIC PERFECT FLUID 2 + 1–SOLUTION WITH $\Lambda$

This section is devoted to the search of solution to the Einstein’s equations for a static 2 + 1 metric in curvature coordinates

$$g = -e^{2\nu(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2 d\theta^2.$$  (2.1)

for a perfect fluid in the presence of a cosmological constant. The fluid is described by the energy–momentum tensor $T_{\mu\nu} = (p(r) + \mu(r)) u_\mu u_\nu + p(r) g_{\mu\nu}, u_\alpha = e^{\nu(r)} \delta_t^\alpha$.

The Einstein equations for a perfect fluid with $\Lambda$ can be given as

$$E^\beta_\alpha := R^\beta_\alpha - \frac{1}{2} \delta^\beta_\alpha R - \kappa T^\beta_\alpha + \Lambda \delta^\beta_\alpha,$$

$$T^\beta_\alpha = -\rho(r) \delta^\beta_\alpha + p(r) \delta^\beta_r \delta^r_\alpha + p(r) \delta^\beta_\theta \delta^\theta_\alpha.$$  (2.2)

Explicitly, for the metric (2.1), one obtains

$$E^1_1 = 0 : \frac{d}{dr} \lambda = r(\kappa \mu + \Lambda) e^{2\lambda} \rightarrow e^{-2\lambda(r)} = C_0 - 2 \int^r (\kappa \mu + \Lambda) r dr = C_0 - \frac{\kappa}{\pi} m(r),$$

$$m(r) := 2\pi \int^r r(\mu + \Lambda/\kappa) dr = \int^r 2\pi r \mu dr + \frac{\pi}{\kappa} \Lambda r^2 = : M(r) + \frac{\pi}{\kappa} \Lambda r^2,$$  (2.3a)

$$E^2_2 = 0 : EQ_{\nu_1} := \frac{d\nu}{dr} - re^{2\lambda(r)} (\kappa p - \Lambda) = 0 \rightarrow \nu = \frac{\int^r (\kappa p - \Lambda) r dr}{C_0 - \frac{\kappa}{\pi} m(r)},$$  (2.3b)

$$E^3_3 = 0 : EQ_{\nu_2} := \frac{d^2\nu}{dr^2} + \left(\frac{d}{dr}\nu\right)^2 - \left(\frac{d}{dr}\nu\right) \frac{d}{dr} \lambda + (\Lambda - \kappa p) e^{2\lambda} = 0.$$  (2.3c)
The substitution of the derivative $\frac{d\nu}{dr}$ from (2.3b) into (2.3c) yields the same equation arising from the energy–momentum conservation law $T^{\alpha\beta}; \beta = 0$, namely the Tolman–Oppenheimer–Volkoff equation:

$$\frac{dp}{dr} = - (\mu + p) \frac{d\nu}{dr} = - e^{2\lambda} r (\kappa p - \Lambda) (\mu + p) \rightarrow \frac{dp}{dr} = - \frac{r (\kappa p - \Lambda) (\mu + p)}{C_0 - \kappa M(\tilde{r})/\pi - \Lambda r^2}. \quad (2.4)$$

An integral quantity arises from the combination of (2.3b) and (2.3c), $\left( E\nu_2 - E\nu_1/r \right) e^{\nu(r)}$, namely

$$\frac{d}{dr} \left( \frac{e^{-\lambda} d\nu}{r} \right) = 0, \quad (2.5)$$

On the other hand, the substitution of $p(r)$ from (2.3b) into (2.3c) gives rise to

$$\frac{d^2\nu}{dr^2} + \left( \frac{d\nu}{dr} \right)^2 - \left( \frac{d\nu}{dr} \right) \frac{d\lambda}{dr} - \frac{1}{r} \frac{d\nu}{dr} = 0, \quad (2.6)$$

which is a first order equation for $N(r) := \frac{d\nu}{dr}$, which can be written in a very simple form by introducing the functions

$$\xi(r) := r e^{\lambda(r)}, \quad Z := N/\xi, \quad (2.7)$$

namely

$$\frac{d}{dr} \left( \frac{N}{\xi} \right) + \left( \frac{N}{\xi} \right)^2 = 0 \rightarrow dZ^{-1} = \xi \, dr \rightarrow Z^{-1} = C_1 + \int_0^r \xi(r) \, dr. \quad (2.8)$$

The equation for $\nu$ becomes

$$d\nu = Z \xi \, dr = \frac{\xi \, dr}{C_1 + \int_0^r \xi(\tilde{r}) \, d\tilde{r}} \rightarrow \nu(r) = \ln \left[ C_1 + \int_0^r \xi(r) \, dr \right] + \ln C_2/2,$$

$$\nu(r) = \ln \left( \int_0^r e^{\lambda(r)} r \, dr + C_1 \right) + \ln C_2/2, \quad (2.9)$$

the const $C_2/2 \rightarrow 1$ by scaling the time coordinate.

Substituting this integral in (2.3b) one obtains the pressure

$$p(r) = \frac{1}{\kappa e^{\lambda(r)} \left( \int_0^r e^{\lambda(r)} r \, dr + C_1 \right)} + \frac{\Lambda}{\kappa}, \quad (2.10)$$

Summarizing, the structural functions can be given as

$$e^{\lambda(r)} = 1/ \sqrt{C_0 - \frac{\kappa}{\pi} \int_0^r (2\pi \mu(\tilde{r}) \, \tilde{r} \, d\tilde{r}) - \Lambda r^2}, \quad (2.11)$$

$$e^{\nu(r)} = C_1 + \int_0^r r' \, dr' / \sqrt{C_0 - \frac{\kappa}{\pi} \int_0^r (2\pi \mu(\tilde{r}) \, \tilde{r} \, d\tilde{r}) - \Lambda r^2}. \quad (2.11)$$
Finally, the pressure results in
\[
\kappa p(r) = \frac{\sqrt{C_0 - \frac{2}{\pi} \int_0^r (2\pi \mu(\tilde{r}) \tilde{r} d\tilde{r}) - \Lambda r^2}}{C_1 + \int_0^r [r' d\tilde{r} / \sqrt{C_0 - \frac{2}{\pi} \int_0^{r'} (2\pi \mu(\tilde{r}) \tilde{r} d\tilde{r}) - \Lambda r^2}]} + \Lambda. \tag{2.12}
\]
This relation determines the pressure \( p \) through the energy density \( \mu \) in a functional manner, if \( p \) were expressed by a state equation of the form \( p = p(\mu) \), the equation (2.12) gives rise to an integral differential equation for the energy as function of the variable \( r \).

The pressure has to vanish at boundary circumference \( r_b \) of the circle, \( p(r_b) = 0 \), where the mass function \( M(r) \) determines the total mass of the fluid \( M(r_b) \) on the circle. Because the metric signature has to be preserved throughout the whole space–time, the positiveness of \( g_{rr} \) imposes an upper bound on the value of the total mass, namely
\[
M(r_b) \leq \frac{\pi \kappa (C_0 - \Lambda r_b^2)}{\kappa}. \tag{2.13}
\]

A. Cotton tensor types

The Cotton tensor for this perfect fluid occurs to be
\[
C_{\mu \nu} = -\kappa \frac{e^{-\lambda}}{4} \frac{d\mu(r)}{dr} (r e^{-\nu} \delta^\mu_t \delta^\nu_r - e^\nu \delta^\mu_r \delta^\nu_t). \tag{2.14}
\]

Therefore, the solution with constant density \( \mu_0 \) is conformally flat, vanishing Cotton tensor. Moreover, the search for its eigenvectors yields
\[
\lambda_1 = 0; \ V1 = [0, V^2, 0], \ V1_\mu = V^2 g_{rr} \delta^\nu_\mu, \ V1^\mu V1_\mu = (V^2)^2 g_{rr}, \ V1 = S1, \\
\lambda_2 = i\kappa \frac{e^{-\lambda}}{4} \frac{d\mu(r)}{dr} ; \ V2 = [V^1, 0, V^3 = -\frac{ie^\nu V^1}{r}], \ V2 = Z, \\
\lambda_3 = -i\kappa \frac{e^{-\lambda}}{4} \frac{d\mu(r)}{dr} ; \ V3 = [V^1, 0, V^3 = \frac{ie^\nu V^1}{r}], \ V3 = \bar{Z}, \tag{2.15}
\]

consequently the corresponding tensor type is
\[
\text{Type} : \{S, Z, \bar{Z}\}.
\]

III. CRUZ–ZANELLI EXISTENCE OF HYDROSTATIC EQUILIBRIUM FOR \( \Lambda \leq 0 \)

In the work [2] it is established that: A perfect fluid in hydrostatic equilibrium,
\[
(\mu(r \leq r_b) > 0, \ p(r \leq r_b) > 0, \ \left. \frac{dp}{dr}\right|_{r \leq r_b} \text{ pressure monotonically decreasing}),
\]
is only possible for \( \Lambda \leq 0 \).

The condition on \( \Lambda \) follows from the energy–momentum conservation equation (2.4)
\[
\frac{d}{dr} p(r) = -\frac{r (\kappa p - \Lambda) (\mu + p)}{C_0 - \kappa M(r) / \pi - \Lambda r^2},
\]
which evaluated at the boundary yields
\[ \frac{dp}{dr}\bigg|_{r=r_b} = \frac{r_b \Lambda}{C_0 - \Lambda r_b^2 - \kappa M(r_b) / \mu(r_b)}, \]
which for \( \mu(r) \geq 0 \), and \( M(r_b) \) fulfilling (2.13) is non–positive only if \( \Lambda \leq 0 \). Moreover, since for \( \Lambda \leq 0 \) the right–hand side of
\[ \frac{d}{dr}p(r) = -r(\kappa p - \Lambda)(p + \mu) g_{rr} \]
is always negative, then \( p(r) \) is a decreasing function such that
\[ p(r = 0) = p_c > p(r_b) = 0. \]

**IV. POSITIVENESS OF THE HYDROSTATIC PRESSURE**

Form (2.3a) one has
\[ \frac{dM}{dr} = 2 \pi r \mu(r) \]
which combined with (2.3d) gives
\[ \frac{d}{dr}(p - A M(r)) = -(\kappa p + \frac{1}{l^2})(p + \mu(r))g_{rr} - 2 A \pi r \mu(r) \quad (4.1) \]
since the right–hand side of this equation, for positive \( A = \text{const} > 0 \), is always negative, then
\[ R(r) = p - A M(r) \]
is monotonically decreasing, and \( R(0) > R(r_b) \rightarrow p_c - A M(0) > -A M \), and consequently \( p_c > -A (M - M(0)) \), which means that the hydrostatic pressure cannot be negative.

Since the right–hand side of (4.1) for \( A = 0 \) is always negative, then \( p(r) \) is a monotonically decreasing function, and consequently \( p(r = 0) = p_c > p(r_b) = 0 \rightarrow p_c > 0 \).

In no way for the choice of \( A = -(2 \pi l^2 C_0)^{-1} < 0 \), equivalent to the one done in CZ [2] (13), one could establish the bound 1CZ (10).

**V. BUCHDAHL’S THEOREM IN 2 + 1 HYDROSTATICS**

If a perfect fluid distribution fulfills the conditions:

1. it is described by a one-parameter state equation \( p = p(\mu) \),
2. the density is positive \( \mu > 0 \) and monotonically decreasing \( \frac{dp}{d\mu} < 0 \),
3. it is microscopically stable, \( \frac{dp}{d\mu} \geq 0 \), \( \frac{dp}{d\mu} \leq 0 \),
then there is not a bound on the density.

The Lemma on no existence of a bound for the density is based on Einstein equations (2.3b) and (2.3c), which yield

\[
(EQ_{\nu_2} - EQ_{\nu_1}/r) e^{\nu(r)} = \frac{r}{\sqrt{C_0 - \frac{\kappa}{\pi} m(r)}} \frac{d}{dr} \left[ \frac{1}{r} \sqrt{C_0 - \frac{\kappa}{\pi} m(r)} \frac{d}{dr} e^{\nu(r)} \right] = 0, \tag{5.1}
\]

because \( EQ_{\nu_1} = 0 = EQ_{\nu_2} \), therefore

\[
\frac{1}{r} \sqrt{C_0 - \frac{\kappa}{\pi} m(r)} \frac{d}{dr} e^{\nu(r)} = C_\nu = \text{const.}, \quad d e^{\nu(r)} = C_\nu \sqrt{C_0 - \frac{\kappa}{\pi} m(r)} \frac{d}{dr} = C_\nu \xi(r) dr,
\]

\[
\rightarrow e^{\nu(r)} = C_\nu \left( C_1 + \int_0^r r e^{\lambda(r)} dr \right), \quad (C_\nu \rightarrow 1, \ t \rightarrow t/\sqrt{C_\nu}), \tag{5.2}
\]

which are no others than the equations for \( \nu \) and its differential given in (2.8). Thus one cannot establish a condition on the energy density in contrast with the statement of [2] CZ (15), which is based on a miss–interpretation of equations [2] CZ (A4) and [2] CZ (A5).

The equation CZ [2] (A1):

\[
\frac{d}{dr} \left[ \frac{d\nu}{dr} \frac{2\pi C_0}{\kappa} - 2m(r)/r \right] + \frac{d\nu}{dr} \frac{2\pi C_0}{\kappa} - 2m(r)/r + \frac{d m}{dr} \frac{d\nu}{dr} = 0.
\]

is equivalent to (5.1), which by introducing \( \gamma = \int_0^r \frac{2\pi C_0}{\kappa} - 2m(r)^{-1/2} r \frac{d}{dr} \) can be written as

\[
\frac{d^2}{d\gamma^2} e^{\nu(r)} = 0 \rightarrow \frac{d}{d\gamma} e^{\nu(r)} = C_\nu = \text{const.}, \quad \rightarrow e^{\nu(r)} = C_\nu \gamma + e^{\nu(0)} \neq e^{\nu(r)} = \gamma \frac{d e^{\nu(r)}}{d\gamma} + e^{\nu(0)} \tag{A5}.
\]

The equation [2] CZ (A5) is the source of the mistake.

VI. STATIC STAR WITH CONSTANT DENSITY \( \mu_0 \) AND \( \Lambda = -1/\ell^2 \leq 0 \)

The static star with uniform density \( \mu_0 \) is characterized by mass and pressure given respectively as:

\[
M(r) = \pi \mu_0 r^2, \quad m(r) = \pi \mu_0 r^2 + \frac{\pi}{\kappa} \Lambda r^2
\]

\[
p(r) = \mu_0 \frac{\sqrt{C_0 - \frac{\kappa}{\pi} m(r)} - \sqrt{C_0 - \frac{\kappa}{\pi} m(r_b)}}{-\sqrt{C_0 - \frac{\kappa}{\pi} m(r)} - \frac{\kappa}{\pi} \mu_0 \sqrt{C_0 - \frac{\kappa}{\pi} m(r_b)}}. \tag{6.1}
\]

A star of uniform density in hydrostatic equilibrium (\( \Lambda = -1/\ell^2 \)) possesses central mass and pressure of the form

\[
M(0) = 0, m(0) = 0,
\]

\[
p_e = \mu_0 \frac{\sqrt{C_0} - \sqrt{C_0 - \frac{\kappa}{\pi} m(r_b)}}{-\sqrt{C_0} + \kappa \ell^2 \mu_0 \sqrt{C_0 - \frac{\kappa}{\pi} m(r_b)}}, \tag{6.2}
\]
at the boundary

\[ M(r_b) = \pi r_b^2 \mu_0 = \mu_0 S \Phi, \quad p(r_b) = 0, \]

while for \( r_b = \sqrt{\frac{\kappa C_0}{\kappa \mu_0} \sqrt{1 + \frac{1}{\kappa^2 l^2}} \mu_0 = \mu_0} \), the pressure becomes infinity, \( p \to \infty \), and the mass equates to

\[ M = \frac{\kappa C_0}{\pi} (1 + \frac{1}{\kappa^2 l^2}). \]

The evaluation of \( e^{2\nu} \) or the metric component \( g_{tt} = -e^{2\nu} \) yields

\[ e^{2\nu} = \left( \frac{\kappa \mu_0 l^2 \sqrt{C_0 - \frac{S}{\pi} M(r_b)} - \sqrt{C_0 - \frac{S}{\pi} m(r)}}{\kappa \mu_0 l^2 - 1} \right)^2, \tag{6.3} \]

which is a slightly different expression compared with the one of CZ \cite{2}(23) which is given with an extra multiplicative factor \( (r^2/l^2 - M_0)/\sqrt{C_0 - \frac{S}{\pi} m(r_b)} \).

The external solution to which the uniform fluid solution can be matched is the static anti–de Sitter metric with parameter \( M_0 \) known also as the static BTZ solution

\[ g = -(-M_0 + \frac{r^2}{l^2}) dt^2 + \frac{dr^2}{(-M_0 + \frac{r^2}{l^2})} + r^2 d\phi^2 \tag{6.4} \]

the continuity at the boundary \( r_b \) of the metric for the fluid is achieved

\[ e^{2\nu(r_b)} = e^{-2\lambda(r_b)} = C_0 + \frac{r_b^2}{l^2} - \frac{\kappa}{\pi} M = -M_0 + \frac{r_b^2}{l^2} \to \kappa = \pi, \quad M(r_b) = M = M_0 - C_0. \tag{6.5} \]

The static perfect fluid solution with \( \Lambda = -1/l^2 \) could exhibit an event horizon at \( r_h = \frac{C_1}{\kappa \mu_0 l^2 - 1}. \)

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\[ \text{[1] H.A. Buchdahl, “General relativistic fluid spheres,” Phys. Rev. 116, 1027 (1959).} \]

\[ \text{[2] N. Cruz and J. Zanelli, “Stellar equilibrium in 2 + 1 dimensions,” Class. Quantum Grav. 12, 975 (1995).} \]