A topological proof of the Shapiro–Shapiro conjecture

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Abstract We prove a generalization of the Shapiro–Shapiro conjecture on Wronskians of polynomials, allowing the Wronskian to have complex conjugate roots. We decompose the real Schubert cell according to the number of real roots of the Wronski map, and define an orientation of each connected component. For each part of this decomposition, we prove that the topological degree of the restricted Wronski map is given as an evaluation of a symmetric group character. In the case where all roots are real, this implies that the restricted Wronski map is a topologically trivial covering map; in particular, this gives a new proof of the Shapiro–Shapiro conjecture.

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1 Introduction

1.1 The Shapiro–Shapiro conjecture

In the mid-1990s, B. Shapiro and M. Shapiro formulated a striking conjecture about the real Schubert calculus. They considered the real 1-parameter family of flags osculating a rational normal curve, and postulated the following: for every 0-dimensional intersection of Grassmannian Schubert varieties defined relative to these osculating flags, all points of the intersection are real.

This conjecture is now a theorem of Mukhin, Tarasov, and Varchenko \[20, 21\], and it is notable for several reasons. For typical real flags, there is no reason to expect all points of a 0-dimensional Schubert intersection to be real, and in general they will not be. For most flag varieties (in type A, and beyond) it is not known whether it is even possible to choose flags such that the solutions to a Schubert intersection problem are all real. In the Grassmannian case, this was resolved by Vakil \[35\], but the Shapiro–Shapiro conjecture provides a radically different way of addressing this question.

It has long been known that solutions to Schubert problems can be counted using families of combinatorial objects, such as Young tableaux. It is, however, much more difficult to establish precise correspondences between these objects and individual solutions. A core obstruction to such a project is the presence of monodromy among the solutions. On one hand, standard geometric arguments involving Schubert varieties rely on Kleiman’s transversality theorem \[13\], which tells us that Schubert varieties defined relative to generic flags intersect transversely. This space of flags has an intractable fundamental group and, in general, Schubert problems with these flags have large monodromy groups \[1, 17, 34\]. The Shapiro–Shapiro conjecture, on the other hand, uses flags defined with respect to a tuple of distinct points on \(\mathbb{RP}_1\)—data with a small fundamental group. This affords the possibility of establishing an explicit mapping. Since the resolution of the Shapiro–Shapiro conjecture in 2005 (see below), this project has been carried out successfully \[24, 30\].

There are several equivalent formulations of the Shapiro–Shapiro conjecture, which connect it to other parts of algebraic geometry and representation theory. It can be stated in terms of limit linear series \[6\], parameterized rational curves \[14\], families (of Schubert problems and of representations) over the moduli space of stable curves \(\mathcal{M}_{0,n} \[10, 29, 30, 37\]\, and it has applications in combinatorics \[24, 26\], K-theory \[9, 15\], and control theory \[4, 5, 28\]. The geometry has also been generalized to other homogeneous spaces, to elliptic curves, and more \[2, 18, 22, 23, 25, 27\].

The most elementary formulation involves Wronskians of polynomials. If \(f_1, \ldots, f_d\) are polynomials with coefficients in a field of characteristic zero,
their Wronskian is the polynomial

$$\text{Wr}(f_1, \ldots, f_d) := \begin{vmatrix} f_1 & f'_1 & f''_1 & \cdots & f^{(d-1)}_1 \\ f_2 & f'_2 & f''_2 & \cdots & f^{(d-1)}_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_d & f'_d & f''_d & \cdots & f^{(d-1)}_d \end{vmatrix}.$$ 

Recall that a linear subspace $v \subset \mathbb{C}[z]$ is real if $v$ is invariant under complex conjugation. Equivalently, $v$ is real if $v$ has a basis in $\mathbb{R}[z]$.

**Theorem 1.1** (Shapiro–Shapiro conjecture) If $f_1(z), \ldots, f_d(z) \in \mathbb{C}[z]$ are linearly independent polynomials such that $\text{Wr}(f_1, \ldots, f_d)$ has only real roots, then the subspace of $\mathbb{C}[z]$ spanned by $(f_1, \ldots, f_d)$ is real.

Up to a scalar, the Wronskian depends only on the span of $f_1, \ldots, f_d$, and is zero if and only if $f_1, \ldots, f_d$ are linearly dependent; as such, Wr defines a map from a Grassmannian to projective space of the same dimension, called the Wronski map. Roughly, the connection to Schubert calculus comes from the fact that Schubert varieties in this Grassmannian map to linear spaces under the Wronski map.

Motivated by the Shapiro–Shapiro conjecture, Eremenko and Gabrielov [4] computed the topological degree of the Wronski map over $\mathbb{R}$. Their computation provides a lower bound for the number of real solutions to certain Schubert intersection problems. Unfortunately this result is not strong enough to deduce Theorem 1.1: the lower bound on the number of real solutions is always strictly less than the number of complex solutions (except in the trivial cases, when the Grassmannian is a projective space).

Mukhin, Tarasov, and Varchenko [20,21] have since given two proofs of Theorem 1.1. The second proof moreover establishes the transversality of Shapiro-type Schubert intersections. Both proofs are quite complicated: they use machinery from quantum integrable systems, Fuchsian differential equations, and representation theory. Their work establishes deep connections between these different areas and Schubert calculus. However, their approach also has two major drawbacks. First, it is heavily algebraic, using the Bethe Ansatz as a black box for solving systems of equations. As such, the proof offers little geometric insight into why the Shapiro–Shapiro conjecture is true. Second, there are several modifications and generalizations of the Shapiro–Shapiro conjecture which are still open problems (see [32]). It is not known how to adapt the Mukhin–Tarasov–Varchenko machinery to handle these related conjectures.

The purpose of this paper is to formulate and prove a generalization of Theorem 1.1, using geometric and topological methods. We address not only
the case where the roots of \( \text{Wr}(f_1, \ldots, f_d) \) are all real, but all cases where \( \text{Wr}(f_1, \ldots, f_d) \) has real coefficients. When the roots are all real, we obtain a new, independent, conceptually simpler proof of the Shapiro–Shapiro conjecture. Our main theorem is a degree computation, similar in some respects to that of Eremenko and Gabrielov, but with a significant twist. The statement is less obvious, more powerful, and involves a surprising connection to characters of the symmetric group.

1.2 Decomposition of real Schubert cells

The real Wronski map is a morphism from the real Grassmannian \( \text{Gr}(d, \mathbb{R}_{d+m-1}[z]) \), the space of \( d \)-planes inside the vector space of polynomials of degree at most \( d + m - 1 \), to projective space. In this paper, we will mainly focus on the restriction of this map to a Schubert cell.

Let \( \mathcal{P}_n(\mathbb{R}) \) denote the set of real monic polynomials of degree \( n \). For a partition \( \lambda \vdash n \) with \( d \) parts, the real Schubert cell \( \mathcal{X}_\lambda(\mathbb{R}) \) consists of all \( d \)-dimensional linear subspaces of \( \mathbb{R}[z] \) that have a basis of polynomials \( (f_1, \ldots, f_d) \), with \( \deg f_i = \lambda_i + d - i \). The Wronskian \( \text{Wr}(f_1, \ldots, f_d) \) is a polynomial of degree \( n \), which, up to a scalar multiple, is independent of the choice of basis. Rescaling so that \( \text{Wr}(f_1, \ldots, f_d) \) is monic, we obtain a map \( \text{Wr} : \mathcal{X}_\lambda(\mathbb{R}) \rightarrow \mathcal{P}_n(\mathbb{R}) \); this is the Wronski map restricted to \( \mathcal{X}_\lambda(\mathbb{R}) \).

Regarded as algebraic varieties \( \mathcal{P}_n(\mathbb{R}) \) and \( \mathcal{X}_\lambda(\mathbb{R}) \) are both isomorphic to affine space \( \mathbb{A}^n(\mathbb{R}) \), and the Wronski map is a finite morphism from \( \mathbb{A}^n(\mathbb{R}) \) to itself. The algebraic degree of this morphism is \( f_\lambda = \#\text{SYT}(\lambda) \), the number of standard Young tableaux of shape \( \lambda \). This comes from a standard calculation in the Schubert calculus: the intersection number of \( n \) Schubert divisors with an \( n \)-dimensional Schubert variety. Notably, we also have \( f_\lambda = \chi_\lambda(1^n) \), where \( \chi_\lambda \) is the irreducible symmetric group character associated to the partition \( \lambda \).

Regarded as a manifold, \( \mathcal{P}_n(\mathbb{R}) \) can be further decomposed according to the number of real and non-real roots of polynomials. Let \( \mu = 2^{n_2}1^{n_1} \) be a partition of \( n \) with parts of size at most 2. Let \( \mathcal{P}_n(\mu) \subseteq \mathcal{P}_n(\mathbb{R}) \) be the subset consisting of polynomials with \( n_1 \) distinct real roots, and \( n_2 \) conjugate pairs of non-real roots. (Real roots are required to be distinct, non-real roots are not.) The closure, \( \overline{\mathcal{P}_n(\mu)} \), consists of all real polynomials with at least \( n_1 \) real roots (not necessarily distinct), and at least \( n_2 \) conjugate pairs of roots (which are no longer required to be non-real).

Example 1.2 The polynomial \( z(z^2 + 1)^2 \) is in \( \mathcal{P}_5(2^21) \). The polynomial \( z^3(z^2 + 1) \) is in both \( \overline{\mathcal{P}_5(21^3)} \) and \( \mathcal{P}_5(22^1) \), but does not lie in any \( \mathcal{P}_5(\mu) \).

Note that \( \mathcal{P}_n(\mu) \) is a contractible open semi-algebraic subset of \( \mathcal{P}_n(\mathbb{R}) \); in particular it is a connected orientable (but as yet, not oriented) manifold. We orient all spaces \( \mathcal{P}_n(\mu) \) simultaneously, in the obvious way, by fixing
an orientation of $\mathcal{P}_n(\mathbb{R})$ and defining the orientation of $\mathcal{P}_n(\mu)$ to be the restriction of the orientation of the ambient space.

Define $\mathcal{X}^\lambda(\mu) := \text{Wr}^{-1}(\mathcal{P}_n(\mu))$. Then $\mathcal{X}^\lambda(\mu)$ is an open semi-algebraic subset of $\mathcal{X}^\lambda(\mathbb{R})$; hence $\mathcal{X}^\lambda(\mu)$ is an orientable manifold. However, unlike $\mathcal{P}_n(\mu)$, $\mathcal{X}^\lambda(\mu)$ typically has many components.

The restricted Wronski map $\text{Wr} : \mathcal{X}^\lambda(\mu) \to \mathcal{P}_n(\mu)$ is a proper map of $n$-dimensional manifolds over a connected base. Therefore, with the additional data of an orientation of $\mathcal{X}^\lambda(\mu)$, this map has a well-defined topological degree. However, there may be many possible choices of orientation for $\mathcal{X}^\lambda(\mu)$ (two choices for each component), and the degree depends on the choice of orientation.

This brings us to our main result. We will define an orientation of $\mathcal{X}^\lambda(\mu)$, called the character orientation.

**Theorem 1.3** With the character orientation, the topological degree of the restricted Wronski map $\text{Wr} : \mathcal{X}^\lambda(\mu) \to \mathcal{P}_n(\mu)$ is equal to $\chi^\lambda(\mu)$.

As an immediate consequence, we obtain new proofs of some previously known results. The first is a bound on the number of real points in the fibre of the Wronski map. The algebraic degree of a finite morphism gives an upper bound for the number of real preimages of any real point in the base; the topological degree gives a lower bound.

**Corollary 1.4** For $g \in \mathcal{P}_n(\mathbb{R})$, let $N_g$ be the number of real points in the fibre $\text{Wr}^{-1}(g)$, counted with algebraic multiplicity. If $g \in \mathcal{P}_n(\mu)$, then

$$|\chi^\lambda(\mu)| \leq N_g \leq f^\lambda.$$ 

Corollary 1.4 is equivalent to a special case of a theorem of Mukhin and Tarasov [19], which gives a lower bound for the number of real points in a Schubert intersection, with respect to (not-necessarily real) flags osculating a rational normal curve. Such Schubert problems were previously studied in [11, 12], where it was observed that there are non-trivial restrictions on the number of real intersection points. Mukhin and Tarasov’s theorem was the first to offer an explanation for some of these observations, using the machinery developed in [21]. Our approach can also be used to recover these inequalities. In Sect. 5.3, we sketch how to extend our argument to the general case using standard techniques. The two approaches are very different—Mukhin and Tarasov’s argument is predominantly algebraic, whereas ours is topological—and it is not clear why they produce exactly the same inequalities.

In the case where $\mu = 1^n$, the lower and upper bounds in Corollary 1.4 agree, and we deduce the following statement, which is a formulation of the Shapiro–Shapiro conjecture (equivalent to Theorem 1.1).
Corollary 1.5 \( \text{Wr} : \mathcal{X}^\lambda(1^n) \rightarrow \mathcal{P}_n(1^n) \) is a topologically trivial covering map of degree \( f^\lambda \).

Here, the statement about the degree is immediate from Corollary 1.4, and the fact that the map is topologically trivial follows from an argument of Eremenko and Gabrielov (see Sect. 6.2 for details). Theorem 1.3 and Corollaries 1.4 and 1.5 also have straightforward generalizations in which the Schubert cell \( \mathcal{X}^\lambda(\mathbb{R}) \) is replaced by an open Richardson variety. We discuss these in Sect. 6.1.

Although Theorem 1.3 does not imply the strong transversality statement proved by Mukhin, Tarasov, and Varchenko in [21], it does imply some important special cases of it. For example, when rephrased in terms of Schubert intersections, Corollary 1.5 asserts that a Shapiro-type intersection of \( n \) Schubert divisors with an \( n \)-dimensional Schubert variety is transverse. There are a few other similar corollaries, which we discuss in Sect. 6.2. A number of known applications of the Shapiro–Shapiro conjecture depend on these special cases, but do not require the full power of the Mukhin–Tarasov–Varchenko transversality theorem. For example, the geometric proof of the Littlewood–Richardson rule in [24] and the cyclic sieving results in [26] rely only on these special cases, and are thus consequences of Theorem 1.3.

1.3 Aspects of the character orientation

The key to Theorem 1.3 and the main novelty of this paper is the definition of the character orientation. In Sect. 1.4, we explain how this definition works in the special case where \( \lambda = (n - 1, 1) \); this case is particularly accessible and will serve as a recurring example throughout the paper. In the general case, the definition requires additional background on the Wronskian map, and will be given in Sect. 3. For now, we mention three features that are partially inferable from the statement of Theorem 1.3.

1. Although \( \mathcal{X}^\lambda(\mu) \) is an open submanifold of \( \mathcal{X}^\lambda(\mathbb{R}) \), in general the character orientation of \( \mathcal{X}^\lambda(\mu) \) is not a restriction of some orientation of the ambient space \( \mathcal{X}^\lambda(\mathbb{R}) \). If it were, then the degree of \( \text{Wr} : \mathcal{X}^\lambda(\mu) \rightarrow \mathcal{P}_n(\mu) \) would be equal to the degree of the total map \( \text{Wr} : \mathcal{X}^\lambda(\mathbb{R}) \rightarrow \mathcal{P}_n(\mathbb{R}) \), and hence would be independent of \( \mu \). Instead, the definition of the character orientation begins by choosing an orientation on ambient space \( \mathcal{X}^\lambda(\mathbb{R}) \); we then twist the orientation by a certain real regular function. As such, some components of \( \mathcal{X}^\lambda(\mu) \) will be oriented the same as the ambient orientation, and some will be oriented opposite to it.

2. The character orientation is defined globally on each component of \( \mathcal{X}^\lambda(\mu) \), rather than locally. This is more or less mandatory, because a priori, we do
not know enough about the topology of $\mathcal{X}^\lambda(\mu)$ to make any kind of local to global arguments.

3. The character orientation exhibits a kind of skew-symmetry with respect to Grassmann duality. If $\lambda^*$ denotes the conjugate partition to $\lambda$, there is an isomorphism $\delta : \mathcal{X}^{\lambda^*}(\mathbb{R}) \to \mathcal{X}^\lambda(\mathbb{R})$ such that $\text{Wr} \circ \delta$ is the Wronski map on $\mathcal{X}^{\lambda^*}(\mathbb{R})$. This restricts to a diffeomorphism $\delta : \mathcal{X}^{\lambda^*}(\mu) \to \mathcal{X}^\lambda(\mu)$. Algebraically, $\mathcal{X}^\lambda(\mu)$ and $\mathcal{X}^{\lambda^*}(\mu)$ are indistinguishable. Yet according to Theorem 1.3, the Wronski map can have different topological degrees on these two spaces. The difference is only a sign, since $\chi^{\lambda^*}(\mu) = (-1)^{n_2} \chi^\lambda(\mu)$, but it is not a global sign (it depends on $\mu$). This seems bizarre, as it implies that the orientation cannot depend only on the abstract geometry of the Wronski map. The situation is reconciled by the fact that there is a second orientation on $\mathcal{X}^\lambda(\mu)$ called the dual character orientation, which is interchanged with the character orientation under $\delta$. We show that the two orientations coincide on $\mathcal{X}^\lambda(\mu)$ if $n_2$ is even, and are opposites if $n_2$ is odd, which explains the signs.

Once we have formulated this definition, our proof of Theorem 1.3 proceeds along the following lines. For each $\mu$ we choose a polynomial $h_\mu(z) \in \mathcal{P}_n(\mu)$, such that we can identify all points in the fibre $\text{Wr}^{-1}(h_\mu)$. We label these points by tableaux. The topological degree of the map $\text{Wr} : \mathcal{X}^\lambda(\mu) \to \mathcal{P}_n(\mu)$ is then a signed count of points in the fibre (the sign is positive if the Wronski map is locally orientation preserving, and negative otherwise). To compute the signs, we connect up all of these points by a network of paths in $\mathcal{X}^\lambda(\mathbb{R})$, and count the number of sign changes along each path. Since the points are labelled by tableaux, we are left with a problem of counting certain tableaux with signs. We recognize this enumeration problem as a case of the Murnaghan–Nakayama rule, which gives us the answer as a character evaluation.

1.4 An example

We illustrate Theorem 1.3 with an elementary example: the case where $\lambda = (n-1, 1)$. Here, to compute $\text{Wr}^{-1}(g)$ for $g \in \mathcal{P}_n(\mathbb{R})$, we are looking for polynomials $(f_1, f_2)$ such that $\text{deg}(f_1) = n$, $\text{deg}(f_2) = 1$ and

$$\text{Wr}(f_1, f_2) = f_1 f_2' - f_1' f_2 = g.$$  \hspace{1cm} (1.1)

Two solutions to this equation represent the same point of $\mathcal{X}^\lambda(\mathbb{R})$, if they are linearly equivalent, i.e. they are bases for the same subspace of $\mathbb{R}[z]$.

We claim that $\text{Wr}^{-1}(g) \subset \mathcal{X}^\lambda(\mathbb{R})$ can be identified with the critical points of $g$. To see this, take derivatives of both sides of the equation (1.1), which gives $f_1 f_2'' - f_1'' f_2 = g'$. Since $f_2$ is a linear polynomial, $f_2'' = 0$, and we
obtain

\[-f_1'' f_2 = g'.\]

It follows that all solutions to (1.1) are of the form where \( f_2(z) = z - c, \)
g\((c) = 0\). Furthermore, it is easy to check that if \( g'(c) = 0 \), then up to linear
equivalence, there is a unique polynomial \( f_1 \) such that \( \text{Wr}(f_1(z), z-c) = g(z) \).
Thus for \( \lambda = (n-1, 1) \), we identify \( \mathcal{R}^\lambda(\mathbb{R}) \) with

\[ \{(g, c) \in \mathcal{P}_n(\mathbb{R}) \times \mathcal{A}^1(\mathbb{R}) \mid g'(c) = 0\} \] .

The Wronski map is identified with the projection onto the first factor.

From this description, we see immediately that Corollary 1.5 is true for
\( \lambda = (n-1, 1) \): if \( g \) is a degree \( n \) polynomial with \( n \) distinct real roots then
\( g \) has \( n-1 = f_1' \) distinct real critical points. There are \( n-1 \) components of
\( \mathcal{R}^\lambda(\mathbb{R}) \): the \( i \)th component is the set of pairs \((g, c)\) such that \( g \in \mathcal{P}_n(1^n) \),
and \( c \) is the unique critical point of \( g \) between the \( i \)th and \((i+1)\)th smallest
roots of \( g \); this clearly maps diffeomorphically to \( \mathcal{P}_n(1^n) \).

Before we consider Theorem 1.3, let us first compute the topological degree
of the full map \( \text{Wr} : \mathcal{R}^\lambda(\mathbb{R}) \to \mathcal{P}_n(\mathbb{R}) \). The Wronski map fails to be locally
one-to-one in a neighbourhood of \((g, c)\) when \( c \) is a double or higher order
critical point of \( g \), i.e. when \( g'(c) = g''(c) = 0 \). Thus we see that the Jacobian
of the Wronski map is, up to a scalar, the function \( \partial \text{Wr} : \mathcal{R}^\lambda(\mathbb{R}) \to \mathbb{R} \),
\( \partial \text{Wr}(g, c) = g''(c) \) (here, the partial derivatives in the Jacobian are computed
with respect to affine coordinates on \( \mathcal{R}^\lambda(\mathbb{R}) \) and \( \mathcal{P}_n(\mathbb{R}) \)). It follows
that we can find orientations of \( \mathcal{R}^\lambda(\mathbb{R}) \) and \( \mathcal{P}_n(\mathbb{R}) \) such that the Wronski
map is locally orientation preserving at \((g, c) \in \mathcal{R}^\lambda(\mathbb{R}) \) if and only if
\((-1)^n g''(c) > 0 \). We call these the ambient orientations. (The global sign
\((-1)^n \) is not necessary for this example in isolation, but ensures that the Wronski
map is locally orientation preserving at \((g, c_0) \), where \( c_0 \) is the smallest
critical point of \( g \); in accordance with conventions used throughout this paper.)

To compute the topological degree of the Wronski map with respect to the
ambient orientations, we can pick any \( g \in \mathcal{P}_n(\mathbb{R}) \) with distinct real critical
points, and count the critical points of \( g \) with signs: \( c \) is counted positively if
\((-1)^n g''(c) > 0 \), and negatively if \((-1)^n g''(c) < 0 \) (see Fig. 1, left). These
signs must alternate, so the degree is 1 if \( n \) is even, and 0 if \( n \) is odd. This
agrees with Eremenko and Gabrielov’s result (see Sect. 5.2).

Now, let \( C \) be a component of \( \mathcal{R}^\lambda(\mu) \). The character orientation of \( C \) is
defined to be consistent with the ambient orientation if \((-1)^{n-1} g(c) > 0 \) for
all \((g, c) \in C \), and opposite to the ambient orientation if \((-1)^{n-1} g(c) < 0 \)
for all \((g, c) \in C \). One of these two conditions must hold, because we cannot
have \( g(c) = 0 \) for \((g, c) \in C \), or else \( c \) would be a double real root of \( g \). Thus,
the Wronski map is locally orientation preserving at \((g, c)\) with respect to the character orientation if and only if 
\[-\frac{g''(c)}{g(c)} > 0.\]

To compute the topological degree with respect to the character orientation, we pick any \(g \in \mathcal{P}_n(\mu)\) with distinct critical points and count the critical points of \(g\) with signs. This time, \(c\) is counted positively if 
\[-\frac{g''(c)}{g(c)} > 0,\]
and negatively if 
\[-\frac{g''(c)}{g(c)} < 0\] (see Fig. 1, right). For example, if \(n_1 > 0\), we can take \(g\) to be a polynomial with \(n_1\) distinct real roots, and \(n_1-1\) distinct real critical points, which will all be counted with positive signs. If \(n_1 = 0\), we can take \(g\) to have no real roots and 1 real critical point, which is counted with a negative sign. In either case, we see that the topological degree is 
\[n_1 - 1 = \chi^\lambda(2^{n_2}1^{n_1}),\]
in agreement with Theorem 1.3.

We will revisit this example several times in Sects. 2 and 3, to illustrate other aspects of Theorem 1.3.

Remark 1.6 For every partition \(\lambda\) and \(g \in \mathcal{P}_n(\mathbb{R})\), there is a function \(\Theta^\lambda_g\) called the master function, whose critical points are identified with \(\text{Wr}^{-1}(g)\). This fact is the starting point for Mukhin, Tarasov and Varchenko’s proof of Theorem 1.1 in [20]. In general, \(\Theta^\lambda_g\) is rational function of several variables, and the analysis of its critical points cannot be carried out in an elementary way; \(\lambda = (n-1, 1)\) is exceptionally nice because it is the only case in which \(\Theta^\lambda_g\) is a univariate function. Our proof of Theorem 1.3 is not based on the master function, but instead generalizes this example in a different way.

1.5 Outline

Section 2 begins with an overview of the main properties of the Wronski map that will be needed throughout the paper. These include the connection with Schubert varieties, and an explicit formula for the Wronski map in
coordinates. We then use these properties to compute certain points in the fibre $W_{r^{-1}}(g)$, in two important special cases. We recall the statement of the Murnaghan–Nakayama rule, and state a lemma which labels the real points in certain “special fibres” of the Wronski map by Murnaghan–Nakayama tableaux (Lemma 2.15).

In Sect. 3, we define the ambient orientation of $\mathcal{P}_n(\mathbb{R})$ and $\mathcal{X}^\lambda(\mathbb{R})$, the character orientation of $\mathcal{X}^\lambda(\mu)$, and the dual character orientation. For each of these orientations, we consider how the sign of a point in $\mathcal{X}^\lambda(\mathbb{R})$ changes when traversing certain paths. We state two lemmas, which provide a collection of paths connecting up all of the real points in the aforementioned special fibres of the Wronski map (Lemmas 3.15 and 3.16). Since the sign changes along these paths are predictable, we can compute the signs of all points in each of our special fibres. Theorem 1.3 follows from this computation.

To complete the argument, we still need to prove Lemmas 2.15, 3.15 and 3.16. This is the goal of Sect. 4. We work with Speyer’s model of the Shapiro–Shapiro conjecture [30], in which the Wronski map is replaced by a related family over the moduli space of genus zero stable curves. The big advantage of this formulation is that it has a rich boundary structure, on which explicit fibre calculations can be carried out with relative ease. This allows us to analyze fibres and paths by degeneration arguments. We begin Sect. 4, by reviewing Speyer’s construction, and explaining how to compute fibres of this map over a stable curve that is a $\mathbb{P}^1$-chain. We prove Lemma 2.15 by replacing polynomials with appropriate curves, and allowing these curves to degenerate to $\mathbb{P}^1$-chains. The assertions of the lemma essentially translate into properties of this degeneration. The paths in Lemmas 3.15 and 3.16 are constructed and analyzed using similar ideas.

In Sect. 5 we discuss Corollary 1.4 and other related results. We show that in several cases, the lower bound in Corollary 1.4 is tight. We use the results of Sect. 3 to give a quick proof of the Eremenko–Gabrielov lower bound, and compare these results. We also discuss the relationship between Corollary 1.4 and the Mukhin–Tarasov bound in more detail.

We conclude with some generalizations of our main results and open questions, in Sect. 6.

Remark 1.7 Many of the geometric objects we are considering have a dual nature as algebraic varieties/schemes over $\mathbb{R}$ and differentiable manifolds, and we will often move back and forth between these points of view. For the most part, topological statements (involving orientations, paths, continuity, etc.) use the analytic topology. When we talk about fibres of a morphism over $\mathbb{R}$, we normally mean the scheme theoretic fibre. It should hopefully be clear from context how to interpret these types of statements.
2 Preliminaries

2.1 The Wronski map

We begin this section by recalling some of the fundamental properties of the Wronski map. We omit proofs of results that are fairly well established, and refer the reader to [24,31] or [32] for further details or additional background.

Let $\mathbb{F}$ be a field of characteristic zero. We denote the vector space of polynomials of degree at most $\ell$ over $\mathbb{F}$ by $\mathbb{F}_\ell[z]$. The Wronski map $\text{Wr} : \text{Gr}(d, \mathbb{F}_{d+m-1}[z]) \to \mathbb{P}(\mathbb{F}_{dm}[z])$, maps a $d$-plane spanned by polynomials $f_1, \ldots, f_d$ to the line in $\mathbb{F}_{dm}[z]$ spanned by $\text{Wr}(f_1, \ldots, f_d)$.

When the choice of field $\mathbb{F}$ is not relevant to discussion at hand, we will sometimes suppress it from our notation. In this case, we may also write the Wronski map as $\text{Wr} : \text{Gr}(d, d + m) \to \mathbb{P}^{dm}$.

The group $\text{GL}_2(\mathbb{F})$ acts on $\mathbb{F}_\ell[z]$ by Möbius transformations. If $\phi = (\phi_{11} \phi_{12}) \in \text{GL}_2(\mathbb{F})$ and $f(z) \in \mathbb{F}_\ell[z]$, the action is given by

$$\phi f(z) := (\phi_{21}z + \phi_{11})^\ell f\left(\frac{\phi_{22}z + \phi_{12}}{\phi_{21}z + \phi_{11}}\right).$$

This induces a $\text{PGL}_2(\mathbb{F})$ action on $\text{Gr}(d, \mathbb{F}_{d+m-1}[z])$ and $\mathbb{P}(\mathbb{F}_{dm}[z])$, and the Wronski map is $\text{PGL}_2(\mathbb{F})$-equivariant with respect to these actions.

We define a family of flags over $\mathbb{P}^1(\mathbb{F}) = \mathbb{F} \cup \{\infty\}$:

$$F_{\bullet}(a) : F_0(a) \subset F_1(a) \subset \cdots \subset F_{d+m}(a).$$

For $a \in \mathbb{F}$, $F_i(a) := (z + a)^{d+m-i}\mathbb{F}[z] \cap \mathbb{F}_{d+m-1}[z]$ is the subspace of polynomials in $\mathbb{F}_{d+m}[z]$ divisible by $(z + a)^{d+m-i}$, $i = 0, \ldots, d + m$. We also set $F_i(\infty) := \mathbb{F}_{i-1}[z] = \lim_{a \to \infty} F_i(a)$. We note that $\phi(F_{\bullet}(a)) = F_{\bullet}(\phi(a))$ for $\phi \in \text{PGL}_2(\mathbb{F})$.

Let $\lambda = (\lambda_1, \ldots, \lambda_d)$ be a partition, with $m \geq \lambda_1 \geq \cdots \geq \lambda_d \geq 0$. Then $\lambda$ indexes a Schubert cell relative to the flag $F_{\bullet}(a)$:

$$X^{\circ}_{\lambda}(a) := \{ x \in \text{Gr}(d, d+m) \mid \dim (x \cap F_j(a)) - \dim(x \cap F_{j-1}(a)) = \eta_j, \ j = 1, \ldots, d+m \},$$

where $\eta_j = 1$ if $j = m + i - \lambda_i$ for some $i$, and $\eta_j = 0$ otherwise. Its closure

$$X_{\lambda}(a) := X^{\circ}_{\lambda}(a)$$

is the Schubert variety. These conventions are such that $|\lambda|$ is the codimension of $X_{\lambda}(a)$ in $\text{Gr}(d, d + m)$.
We will often identify the partition \( \lambda \) with its diagram, \( \lambda = \{(i, j) \mid 1 \leq i \leq d, \; 1 \leq j \leq \lambda_i\} \), which is represented pictorially as an array of \(|\lambda|\) boxes, with \( \lambda_i \) boxes in row \( i \). We will write \( X_{\square}(a) \) to denote the Schubert variety associated to the partition \( \square = (1, 0, \ldots, 0) \), \( X_{\square}(a) \) for the partition \( \square = (2, 0, \ldots, 0) \), etc.

The relationship between these Schubert varieties and the Wronski map is given by the following lemma.

**Lemma 2.1** Let \( x \in \text{Gr}(d, d + m) \), and let \( g = \text{Wr}(x) \).

(i) For \( a \in \mathbb{F} \), \((z + a)^{\ell}\) divides \( g(z) \) if and only if \( x \in X_{\lambda}(a) \) for some partition \( \lambda \vdash \ell \). If \( (z + a)^{\ell}\) is the largest power of \((z + a)\) that divides \( g(z) \), then \( \lambda \) is unique and moreover \( x \in X_{\lambda}^{\infty}(a) \).

(ii) \( \deg(g) \leq dm - \ell \) if and only if \( x \in X_{\lambda}(\infty) \) for some \( \lambda \vdash \ell \). If \( \deg(g) = dm - \ell \), then in fact \( \lambda \) is unique and moreover \( x \in X_{\lambda}^{\infty}(\infty) \).

It follows that Schubert varieties of the form \( X_{\lambda}(a) \) intersect properly.

**Lemma 2.2** Let \( a_1, \ldots, a_k \in \mathbb{P}^1 \) be distinct, and let \( \alpha^1, \ldots, \alpha^k \) be partitions such that \( \sum_{i=1}^{k} |\alpha^i| = dm \). the intersection

\[
X_{\alpha^1}(a_1) \cap \cdots \cap X_{\alpha^k}(a_k)
\]

is proper and hence a finite scheme of length equal to the Schubert intersection number \( \int_{\text{Gr}(d, d+m)}[X_{\alpha^1}] \cdots [X_{\alpha^k}] \).

The Wronski map \( \text{Wr} : \text{Gr}(d, d + m) \to \mathbb{P}^{dm} \) is a finite morphism [6]. Its degree is the length of the finite scheme \( \text{Wr}^{-1}(g) \), for any \( g \in \mathbb{P}^{dm} \). This is independent of \( \mathbb{F} \). If \( \mathbb{F} \) is algebraically closed, then we can assume \( g(z) = \prod_{i=1}^{dm} (z + a_i) \), with distinct roots. Lemma 2.1 shows that the fibre is an intersection of \( dm \) Schubert divisors on \( \text{Gr}(d, d + m) \),

\[
\text{Wr}^{-1}(g) = X_{\square}(a_1) \cap \cdots \cap X_{\square}(a_{dm})
\]

and hence the degree is \( \#\text{SYT}(m^d) \).

The **Plücker coordinates** on the Grassmannian are homogeneous coordinates \([x_{\lambda}]\) indexed by the same partitions \( \lambda \) as the Schubert varieties. For \( x \in \text{Gr}(d, d + m) \), choose a basis \((f_1, \ldots, f_d)\), and let \( M \) be the \( d \times (m + d) \) matrix \( M_{ij} = f_i^{(j-1)}(0) \). Then \( x_{\lambda} \) is the maximal minor of \( M \), with columns \( 1 + \lambda_d, 2 + \lambda_{d-1}, \ldots, d + \lambda_1 \).

**Proposition 2.3** In terms of Plücker coordinates, the Wronski map is

\[
\text{Wr}(x; z) = \sum_{\ell=0}^{dm} \sum_{\lambda \vdash \ell} f^\ell x_{\lambda} \frac{z^\ell}{\ell!}.
\]
2.2 The Schubert cell $X^\lambda$

The *complementary partition* to $\lambda$ is $\lambda^\vee := (m - \lambda_d, \ldots, m - \lambda_1)$. The Schubert cell $X_{\lambda^\vee}(\infty) \subset \text{Gr}(d, d + m)$ will play a special role, and we denote it by $X^\lambda$. The use of a superscript is to indicate that $|\lambda|$ is the dimension of $X^\lambda$, rather than the codimension. Concretely, $X^\lambda(F)$ is the variety of $d$-planes in $F[z]$ that have a basis $(f_1, \ldots, f_d)$, with $\deg f_i = \lambda_i + d - i$. We note that this characterization is independent of $m$.

Now, fix $\lambda \vdash n$, and suppose $\kappa = (\kappa_1, \ldots, \kappa_d)$ is another partition. We write $\kappa \subset \lambda$ if $\kappa_i \leq \lambda_i$ for all $i$. For $a \in A^1$, the Schubert variety $X_\kappa(a)$ intersects the Schubert cell $X^\lambda$ non-trivially if and only if $\kappa \subset \lambda$. The intersection

$$X^\lambda_\kappa(a) := X_\kappa(a) \cap X^\lambda$$

is a *half-open Richardson variety*.

When we restrict the Wronski map to $X^\lambda$, the properties of the previous section translate into the following facts:

(i) The algebraic image of the Wronski map restricted to $X^\lambda(F)$ is the subvariety of $P(\text{P}d_m[z])$ of polynomials of degree exactly $n$. Rescaling so that the leading coefficient is 1, we identify this image with $P_n(F)$, the affine space of monic polynomials of degree $n$ in $F[z]$.

(ii) The subgroup $B_+ \subset \text{PGL}_2(F)$ of upper triangular matrices acts on $X^\lambda(F)$ and $P_n(F)$ by affine transformations, and the Wronski map is $B_+$-equivariant.

(iii) If $x \in X^\lambda(F)$, and $g = \text{Wr}(x)$, then $(z + a)^\ell$ divides $g(z)$ if and only if $x \in X^\lambda_\kappa(a)$ for some $\kappa \vdash \ell$.

(iv) The map $\text{Wr} : X^\lambda(F) \to P_n(F)$ is a finite morphism of affine varieties of degree $f^\lambda$. In this case, the degree computation follows from the fact that the fibre over $g(z) = \prod_{i=1}^n (z + a_i)$ is

$$\text{Wr}^{-1}(g) = X_{\square}(a_1) \cap \cdots \cap X_{\square}(a_n) \cap X_{\lambda^\vee}(\infty).$$

(v) For $x \in X^\lambda(F)$, the Plücker coordinates satisfy $x_\kappa = 0$ for $\kappa \not\subset \lambda$, and can be normalized so that $x_\lambda = 1$. In terms of normalized Plücker coordinates, the Wronski map $\text{Wr} : X^\lambda(F) \to P_n(F)$ is

$$\text{Wr}(x; z) = z^n + \left(\frac{n!}{n!}\right)^{-1} \sum_{\ell=0}^{n-1} \sum_{\kappa \vdash \ell} f_\kappa x_\kappa \frac{z^\ell}{\ell!}. \quad (2.1)$$
The Schubert cell $\mathcal{X}^\lambda(F)$ is isomorphic to affine space $A^n(F)$. Explicitly, a point $x \in \mathcal{X}^\lambda(F)$ has a unique basis of polynomials $(f_1, \ldots, f_d)$, of the form

$$f_i(z) = \frac{z^i + d - i}{(\lambda_i + d - i)!} + \sum_{j=1}^{\lambda_i} (-1)^{i+j_1} \frac{z^{j_1 - \lambda_j + d - 1}}{(j - \lambda_j + d - 1)!} \cdot x_{ij}, \quad (2.2)$$

where $\lambda^*$ denotes the conjugate partition. The coefficients $(x_{ij})_{(i, j) \in \lambda}$ of these polynomials give the affine coordinates of the point $x$. The coordinate ring of $\mathcal{X}^\lambda(F)$ is $F[x] := F[x_{ij}]_{(i, j) \in \lambda}$.

**Example 2.4** For $\lambda = 532$, here are the polynomials specified in (2.2).

$$f_1(z) = x_{11} + x_{12}z - x_{13} \frac{z^3}{3!} + x_{14} \frac{z^5}{5!} + x_{15} \frac{z^6}{6!} + \frac{z^7}{7!},$$

$$f_2(z) = -x_{21} - x_{22}z + x_{23} \frac{z^3}{3!} + \frac{z^4}{4!},$$

$$f_3(z) = x_{31} + x_{32}z + \frac{z^2}{2!}.$$

**Remark 2.5** The precise signs and constants in (2.2) are not too important for most practical purposes. They are chosen so that our affine coordinates are a subset of the normalized Plücker coordinates. This has the additional benefit that the coordinates are well-behaved under Grassmann duality. The duality isomorphism $\delta : \mathcal{X}^{\lambda^*} \to \mathcal{X}^\lambda$ is simply defined by $x_{ij} \mapsto x_{ji}$ in affine coordinates. Using Proposition 2.3, one can show that if $\text{Wr} : \mathcal{X}^\lambda \to \mathcal{P}_n$ is the Wronski map on $\mathcal{X}^\lambda$, then $\text{Wr} \circ \delta : \mathcal{X}^{\lambda^*} \to \mathcal{P}_n$ is the Wronski map on $\mathcal{X}^{\lambda^*}$.

**Proposition 2.6** Suppose $\lambda/\kappa$ has at most one box in each column, or at most one box in each row. Then $X^\lambda_{\kappa}(0)$ is the affine subspace of $\mathcal{X}^\lambda$ defined in affine coordinates by $x_{ij} = 0$ for $(i, j) \in \kappa$.

**Proof** $X^\lambda_{\kappa}(0)$ is defined in Plücker coordinates by $x_\alpha = 0$ for all $\alpha < \lambda$ such that $\kappa \not\subset \alpha$. In the case where $\lambda/\kappa$ has at most one box in each column (or each row), the equations $x_{ij} = 0$, $(i, j) \in \kappa$ are a subset of these defining equations. This subset cuts out an affine space $V$ of dimension $|\lambda/\kappa|$, and so $X^\lambda_{\kappa}(0)$ is a closed subvariety of $V$. Since dim $X^\lambda_{\kappa}(0) = |\lambda/\kappa|$, we have $X^\lambda_{\kappa}(0) = V$. \qed

By writing the Wronski map in affine coordinates, we can solve some specific instances of the inverse Wronskian problem by direct calculation. The most important examples of this are given in the next two lemmas.

**Lemma 2.7** Let $\kappa \subset \lambda$ be a partition such that $|\kappa| = n - 1$. If $g(z) = z^{n-1}(z + a) \in \mathcal{P}_n(\mathbb{R})$, then there is a unique (reduced) point $x \in \text{Wr}^{-1}(g) \cap X^\lambda_{\kappa}(0)$ and $x$ is real.
Proof Suppose the unique box of $\lambda/\kappa$ is in row $i_1$, and column $j_1$. By Proposition 2.6, $X^\lambda_\kappa(0)$ is defined by $x_{ij} = 0$, for $(i, j) \neq (i_1, j_1)$. Thus, (2.1) simplifies to

$$
\text{Wr}(\mathbf{x}; \ z) = z^n + \left(\frac{f^\lambda}{n!}\right)^{-1} \cdot f^\kappa x_{i_1 j_1} z^{n-1} = \frac{z^{n-1}}{(n-1)!}.
$$

Thus the unique solution to $\text{Wr}(\mathbf{x}) = g$ is given in affine coordinates by $x_{i_1 j_1} = \frac{af^\lambda}{n!}, x_{ij} = 0$ for $(i, j) \neq (i_1, j_1)$.

The *distance* between two boxes $(i_1, j_1)$ and $(i_2, j_2)$ in the diagram of $\lambda$ is defined to be $|i_1 - i_2| + |j_1 - j_2|$.

**Lemma 2.8** Let $\kappa \subset \lambda$ be a partition such that $|\kappa| = n - 2$. Let $L$ be the distance between the two boxes of the skew shape $\lambda/\kappa$. Let $g(z) = z^{n-2}(z + a_1)(z + a_2) \in \mathcal{P}_n(\mathbb{R})$ (hence, $a_1, a_2 \in \mathbb{R}$ or $a_1 = \overline{a_2} \in \mathbb{C}$).

(i) If $L = 1$, then there is a unique (reduced) point $\mathbf{x} \in \text{Wr}^{-1}(g) \cap X^\lambda_\kappa(0)$ and $\mathbf{x}$ is real.

(ii) If $L > 1$, then $\text{Wr}^{-1}(g) \cap X^\lambda_\kappa(0)$ is a finite scheme of length two. The two points $\mathbf{x}, \mathbf{x}' \in \text{Wr}^{-1}(g) \cap X^\lambda_\kappa(0)$ are identified with solutions to to a quadratic equation with discriminant $(a_1 + a_2)^2 - 4(1 - L^{-2})a_1a_2$. Hence:

- If $(a_1 + a_2)^2 - 4(1 - L^{-2})a_1a_2 > 0$, then $\mathbf{x}, \mathbf{x}'$ are distinct and real.
- If $(a_1 + a_2)^2 - 4(1 - L^{-2})a_1a_2 = 0$, then $\mathbf{x} = \mathbf{x}'$ is a double real point.
- If $(a_1 + a_2)^2 - 4(1 - L^{-2})a_1a_2 < 0$, then $\mathbf{x}, \mathbf{x}'$ are not real.

Proof The proof of (i) is similar to Lemma 2.7, and we omit it. For (ii), suppose that the positions of the two boxes of $\lambda/\kappa$ are $(i_1, j_1)$ and $(i_2, j_2)$. Since $L > 1$, these are both corners of $\lambda$. Let $\beta^1$ and $\beta^2$ denote the partitions of size $n - 1$, obtained by deleting corners $(i_1, j_1)$ and $(i_2, j_2)$ from $\lambda$ respectively. Proceeding as in the proof Lemma 2.7, $X^\lambda_\kappa(0)$ is defined by $x_{ij} = 0$ for $(i, j) \in \kappa$, and (2.1) simplifies to

$$
\text{ Wr}(\mathbf{x}; \ z) = z^n + \left(\frac{f^\lambda}{n!}\right)^{-1} \cdot \left[ f^{\beta^1} x_{i_1 j_1} z^{n-1} + f^{\beta^2} x_{i_2 j_2} z^{n-1} \right],
$$

Equating coefficients of $\text{Wr}(x) = g$, and solving for $x_{i_1 j_1}$ we obtain

$$
n \cdot \frac{f^{\beta^1}}{f^\lambda} x_{i_1 j_1}^2 + (a_1 + a_2)x_{i_1 j_1} + (n - 1)^{-1} \frac{f^{\beta^2}}{f^\kappa} a_1a_2 = 0.
$$
The discriminant of this quadratic equation is

\[(a_1 + a_2)^2 - 4n(n - 1)(f_{β_1}^β f_{β_2}^λ - f_{κ}^λ a_1 a_2).\]

Using the hook-length formula [7] for \( f^λ \), it is easy to check that
\[ n(n - 1)(f_{β_1}^β f_{β_2}^λ - f_{κ}^λ a_1 a_2) = (1 - L^{-2}), \]
from which the result follows.

## 2.3 Tableaux

Suppose \( μ = (μ_1, \ldots, μ_k) \) is a composition of \( n \), i.e. an ordered list of positive integers summing to \( n \). There is a partition associated to \( μ \), obtained by sorting the parts of \( μ \) in decreasing order. We adopt the convention that whenever we use notation of the form “· (μ)” in a context where \( μ \) is supposed to be a partition, we will implicitly mean to use this associated partition. For example, \( χ^λ(μ) \) means \( χ^λ \) evaluated at the partition associated to \( μ \).

**Definition 2.9** A weakly increasing tableau of shape \( λ \) and content \( μ \) is a filling of the diagram of \( λ \) with positive integer entries, weakly increasing along rows and columns, such that \( μ_b \) of the entries are equal to \( b \), for \( b = 1, \ldots, k \). We denote the set of all such tableaux by \( \text{Tab}(λ; μ) \).

In particular, the set of standard Young tableaux of shape \( λ \) is \( \text{SYT}(λ) = \text{Tab}(λ; 1^n) \).

For \( T \in \text{Tab}(λ; μ) \) let \( T(i, j) \) denote the entry in row \( i \) and column \( j \). Let \( \text{shape}(T|_{≤b}) \) be the partition defined by the entries of \( T \) less than or equal to \( b \). Let \( \text{shape}(T|_b) := \text{shape}(T|_{≤b})/\text{shape}(T|_{≤b-1}) \) be the skew shape associated to the entries equal to \( b \). We note the following identity.

**Proposition 2.10**

\[ \sum_{T ∈ \text{Tab}(λ; μ)} \prod_{b=1}^{k} \#\text{SYT}(\text{shape}(T|_b)) = f^λ. \]

The Murnaghan–Nakayama rule computes characters of symmetric group representations in terms of tableaux. We recall the statement.

**Definition 2.11** \( T \in \text{Tab}(λ; μ) \) is a Murnaghan–Nakayama tableau if \( \text{shape}(T|_b) \) is a connected shape containing no \( 2 \times 2 \) square, for all \( b = 1, \ldots, k \). We denote the set of Murnaghan–Nakayama tableaux of shape \( λ \) and
content $\mu$ by $\text{MN}(\lambda; \mu)$. The sign of a Murnaghan–Nakayama tableau is

$$\text{sgn}(T) := \prod_{b=1}^{k} (-1)^{\text{rows}(T|_b)} - 1.$$ 

where $\text{rows}(T|_b)$ the number of non-empty rows in shape$(T|_b)$.

**Theorem 2.12** (Murnaghan–Nakayama rule)

$$\sum_{T \in \text{MN}(\lambda; \mu)} \text{sgn}(T) = \chi_\lambda(\mu).$$

We now specialize to the case where $\mu_i \in \{1, 2\}$, for all $i = 1, \ldots, k$. With this assumption, several things simplify. For a tableau $T \in \text{Tab}(\lambda; \mu)$, shape$(T|_b)$ consists of either

- a single box (iff $\mu_b = 1$);
- two boxes forming a horizontal domino (i.e. in the same row);
- two boxes forming a vertical domino (i.e. in the same column); or
- two boxes that are non-adjacent.

Denote the number of $b$ such that shape$(T|_b)$ falls into each of these cases by $\#_{\square}(T)$, $\#_{\mathfrak{m}}(T)$, $\#_{\mathfrak{b}}(T)$, and $\#_{\mathfrak{c}}(T)$ respectively. Proposition 2.10 reduces to the statement

$$2^{\#_{\mathfrak{c}}(T)} = f_\lambda.$$

$T$ is a Murnaghan–Nakayama tableau if and only if $\#_{\mathfrak{c}}(T) = 0$, in which case we have $\text{sgn}(T) = (-1)^{\#_{\mathfrak{b}}(T)}$.

**Example 2.13** For $\lambda = 543$, $\mu = (1, 1, 2, 2, 1, 1, 2, 1)$, consider the tableau

$$T = \begin{array}{cccccc}
1 & 2 & 4 & 6 & 7 \\
3 & 3 & 4 & 9 \\
5 & 8 & 8 \\
\end{array} \in \text{Tab}(\lambda; \mu).$$

We have $\#_{\square}(T) = 6$, $\#_{\mathfrak{m}}(T) = 2$, $\#_{\mathfrak{b}}(T) = 1$, and $\#_{\mathfrak{c}}(T) = 0$. Hence, $T$ is a Murnaghan–Nakayama tableau, with $\text{sgn}(T) = -1$.

**2.4 Special fibres of the Wronski map**

We continue to assume that $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ is a composition of $n$, with $\mu_i \in \{1, 2\}$ for $i = 1, \ldots, k$. We now assign points in $\mathcal{D}^\lambda$ to tableaux in $\text{Tab}(\lambda; \mu)$. 

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We begin working over the field \( \mathbb{C}(u) \) of formal Laurent series. Note that if \( x(u) \in \text{Gr}(d, \mathbb{C}(u)^{d+m}) \), we can always take \( \lim_{u \to 0} x(u) \) to obtain a point in \( \text{Gr}(d, \mathbb{C}^{d+m}) \). Define polynomials

\[
H_{\mu}(u, z) := \prod_{b=1}^{k} \left( z^{\mu_b} + \left( \frac{1}{2} u^{\mu_b} + \frac{1}{2} u^{\mu_b + \mu_b - 1} \right) \right),
\]

where \( \mu_b := n + 1 - \sum_{i=1}^{b} \mu_i \). Note that if we evaluate at any \( u \in (0, 1) \), \( H_{\mu}(u, z) \in \mathbb{P}^n(\mu) \). The roots of \( H_{\mu} \) are either of the form \(-u^j\) or \( \pm i(\frac{1}{2} u^j + u^{j+1}) \), where \( i \) denotes the imaginary unit.

**Example 2.14** For \( \mu = (2, 1, 2, 2, 1) \),

\[
H_{\mu}(u, z) = \left( z^2 + \left( \frac{1}{2} u^7 + \frac{1}{2} u^8 \right)^2 \right) \left( z + u^6 \right) \left( z^2 + \left( \frac{1}{2} u^4 + \frac{1}{2} u^5 \right)^2 \right) \cdot \left( z^2 + \left( \frac{1}{2} u^2 + \frac{1}{2} u^3 \right)^2 \right) \left( z + u^1 \right).
\]

**Lemma 2.15** Consider the Wronski map \( \text{Wr} : \mathcal{X}^\lambda(\mathbb{C}(u)) \to \mathcal{P}_n(\mathbb{C}(u)) \). The fibre \( \text{Wr}^{-1}(H_{\mu}) \) consists of \( \#T \) distinct points in \( \mathcal{X}^\lambda(\mathbb{C}(u)) \). For each point \( x(u) \in \text{Wr}^{-1}(H_{\mu}) \), the normalized Plücker coordinates \( \left( x_\kappa(u) \right)_{\kappa \subset \lambda} \) are power series with a positive radius of convergence. For each \( T \in \text{Tab}(\lambda; \mu) \), there is a set \( W_T \) consisting of \( 2^\#T \) distinct points in \( \mathcal{X}^\lambda(\mathbb{C}(u)) \), with the following properties:

(a) \( \bigcup_{T \in \text{Tab}(\lambda; \mu)} W_T = \text{Wr}^{-1}(H_{\mu}) \).

(b) For \( x(u) \in W_T \), and \( b = 1, \ldots, k \),

\[
\lim_{u \to 0} \begin{pmatrix} 1 & 0 \\ 0 & u^{\mu_b} \end{pmatrix} x(u) \in \mathcal{X}^\text{shape}(T|_{\leq b}),
\]

using the action of \( \text{PGL}_2 \) on the Grassmannian.

(c) For \( x(u) \in W_T \), \( x(u) \in \mathcal{X}^\lambda(\mathbb{R}(u)) \) if and only if \( T \) is a Murnaghan–Nakayama tableau.

We prove Lemma 2.15 in Sect. 4.4. It follows that for fixed \( \lambda \), and for all sufficiently small \( \epsilon > 0 \), the following are true:

- For every \( \mu \) and every point \( x \in \text{Wr}^{-1}(H_{\mu}) \), the series \( x_\kappa(\epsilon) \) converges for all \( \kappa \subset \lambda \).
- For \( x(u) \in W_T \), the point \( x(\epsilon) \in \mathcal{X}^\lambda(\mathbb{C}) \), defined by Plücker coordinates \( \left( x_\kappa(\epsilon) \right)_{\kappa \subset \lambda} \), is real if and only if \( T \) is a Murnaghan–Nakayama tableau.
- All of the points \( x(\epsilon) \) are distinct.
Pick such a suitable $\varepsilon$, and put $h_\mu(z) := H_\mu(\varepsilon, z) \in P_n(\mu)$. For $T \in MN(\lambda; \mu)$ put $w_T := x(\varepsilon) \in X^\lambda(\mu)$, where $x(u)$ is the unique point in $W_T$.

**Corollary 2.16** The fibre $W_r^{-1}(h_\mu)$ is reduced, and the set of real points in $W_r^{-1}(h_\mu)$ is

$$\{w_T \mid T \in MN(\lambda; \mu)\}.$$

**Remark 2.17** Lemma 2.15 could be stated much more generally: essentially, the proof uses only the asymptotic behaviour of the roots of $H_\mu$, as $u \to 0$. As such, the definition of $H_\mu$ is somewhat arbitrary, in that there are other choices that would work equally well. However, Lemma 2.15 is not the only consideration. The choices we have made here will be particularly convenient later on, for constructions involving paths in Sects. 4.5 and 4.6.

**Example 2.18** Consider $\lambda = \square\square$. Here, $\varepsilon = \frac{1}{2}$ is sufficiently small, and we have

$$h_{13}(z) = (z + \frac{1}{8})(z + \frac{1}{4})(z + \frac{1}{2})$$
$$h_{21}(z) = (z^2 + (\frac{3}{16})^2)(z + \frac{1}{2})$$
$$h_{12}(z) = (z + \frac{1}{8})(z + (\frac{3}{8})^2).$$

As explained in Sect. 1.4, when $\lambda$ is of the form $(n-1, 1)$, the points of the fibre $W_r^{-1}(h_\mu)$ correspond to the critical points of $h_\mu$. The polynomials $h_{13}$ and $h_{21}$ each have two real critical points, and $h_{12}$ has zero real critical points. According to Corollary 2.16, the real points of $W_r^{-1}(h_\mu)$ (hence the real critical points of $h_\mu$) are in bijection with tableaux in $MN(\lambda, \mu)$. Indeed, we have $\#MN(\square\square, 1^3) = \#MN(\square\square, 21) = 2$, and $\#MN(\square\square, 12) = 0$.

The precise identification between critical points of $h_\mu$ and Murnaghan-Nakayama tableaux is as shown in Fig. 2.

We can verify this by computing of $W_r^{-1}(H_\mu)$ directly. For example, for $\mu = 1^3$, $H_\mu(u, z) = (z + u^3)(z + u^2)(z + u)$, and the two points $x_0(u), x_1(u)$ of $W_r^{-1}(H_\mu)$ are spanned by the following polynomials in $\mathbb{R}[u][z]$:

$$x_0(u) = \langle z^3 + \frac{3}{2}u^2z^2 + 3u^5z + \cdots, z^2 + \frac{2}{3}u + \cdots \rangle$$
$$x_1(u) = \langle z^3 + 2uz^2 + 4u^4z + \cdots, z + \frac{1}{2}u^2 + \cdots \rangle.$$

Here we have only explicitly written the leading term in $u$ for each coefficient of $z$. Recall that the critical point associated to $x_i$ is the root of the linear polynomial in the basis for $x_i$; as $u \to 0$, these roots are asymptotically $-\frac{2}{3}u < -\frac{1}{2}u^2$, so $x_0$ corresponds to the smaller of the two critical points. To see which point corresponds to which tableau, we use part (b) of Lemma 2.15.
Fig. 2 Plots of the polynomials $h_\mu$, for $\lambda = \square$, and the Murnaghan–Nakayama tableaux corresponding to each critical point

\[
\begin{align*}
\lim_{u \to 0} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \begin{pmatrix} 1 & 0 \end{pmatrix} x_0(u) &= \left( \frac{3}{2} z^2 + 3z, \frac{3}{2} \right) & \lim_{u \to 0} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \begin{pmatrix} 1 & 0 \end{pmatrix} x_1(u) &= \left( 2z^2 + 4z, \frac{1}{2} \right) \\
\lim_{u \to 0} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \begin{pmatrix} 1 & 0 \end{pmatrix} x_0(u) &= \left( z^3 + \frac{3}{2} z^2 + \frac{1}{3} \right) & \lim_{u \to 0} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \begin{pmatrix} 1 & 0 \end{pmatrix} x_1(u) &= \left( 2z^2, z + \frac{1}{2} \right) \\
\lim_{u \to 0} \left( \begin{array}{c} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} x_0(u) &= \left( z^3, z + \frac{2}{3} \right) & \lim_{u \to 0} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \begin{pmatrix} 1 & 0 \end{pmatrix} x_1(u) &= \left( z^3 + 2z^2, z \right)
\end{align*}
\]

This shows that $x_0$ corresponds to $\begin{array}{c} 1 \\ 2 \\ 3 \end{array}$ and $x_1$ corresponds to $\begin{array}{c} 1 \\ 3 \\ 2 \end{array}$.

3 Orientations

3.1 Ambient orientations

We begin by fixing orientations on affine spaces $\mathcal{P}_n(\mathbb{R})$, and $\mathcal{X}^\lambda(\mathbb{R})$, which we will refer to as the ambient orientations. These will serve as a point of reference for defining orientations of $\mathcal{P}_n(\mu)$ and $\mathcal{X}^\lambda(\mu)$. Each component of these will either be oriented the same, or opposite to the ambient space in which it lies.

The degree of a map depends only on the relative orientations of the spaces: reversing the orientations of both the domain and codomain leaves the degree unchanged. Thus we can begin by making one choice without loss of generality. We select either orientation as the ambient orientation for $\mathcal{P}_n(\mathbb{R})$. Everything else will be defined relative to this choice.

Let $T_0 \in \text{SYT}(\lambda)$ be the standard Young tableau with entries $1, \ldots, n$ in order, from left to right and top to bottom (i.e. the unique tableau such that $T_0(i, j) < T_0(i', j')$ whenever $i < i'$). We define the ambient orientation of
\( \mathcal{X}^\lambda(\mathbb{R}) \) to be the orientation for which the Wronski map is locally orientation preserving in a neighbourhood of \( \mathcal{W}_0 \).

As stated in the introduction, the orientation on \( \mathcal{P}_n(\mu) \) will simply be the restriction of the ambient orientation. The character orientation of \( \mathcal{X}^\lambda(\mu) \) is the complicated one, and will be defined next.

### 3.2 The character orientation

Let \( \kappa \subset \lambda \) be a partition. The half-open Richardson varieties \( X^\lambda_\kappa(a), a \in A^1 \) define a flat family of affine subvarieties of \( \mathcal{X}^\lambda \) over \( A^1 \). Let \( E^\lambda_\kappa \subset \mathcal{X}^\lambda \times A^1 \) be the total space of this family. Let \( \pi_1 : \mathcal{X}^\lambda \times A^1 \to \mathcal{X}^\lambda \) be the projection onto the first factor, and define \( Z^\lambda_\kappa := \pi_1(\bar{E}^\lambda_\kappa) \) to be the algebraic image of \( E^\lambda_\kappa \) under this projection. Informally, \( Z^\lambda_\kappa \subset \mathcal{X}^\lambda \) is the union of all \( X^\lambda_\kappa(a), a \in A^1 \).

**Proposition 3.1** \( Z^\lambda_\kappa \) is a closed subvariety of \( \mathcal{X}^\lambda \).

**Proof** Let \( \bar{E} \) be the closure of \( E^\lambda_\kappa \) in \( \mathcal{X}^\lambda \times \mathbb{P}^1 \), where \( \mathcal{X}^\lambda = X^\lambda_\infty(\infty) \) is the Schubert variety. Note that \( \bar{E} \to \mathcal{X}^\lambda \) is proper and that

\[
\bar{E} \subset \bigcup_{a \in A^1} (X^\lambda_\square(a) \cap \mathcal{X}^\lambda) \times \{a\},
\]

so in particular, the fibre \( \bar{E}_\infty = \pi_1(\pi_2^{-1}(\infty)) \) is contained in the limiting fibre

\[
\lim_{a \to \infty} X^\lambda_\square(a) \cap \mathcal{X}^\lambda = \partial \mathcal{X}^\lambda.
\]

Therefore \( \bar{E}_\infty \) does not intersect the Schubert cell \( \mathcal{X}^\lambda \). Let \( \bar{E}|_{\mathcal{X}^\lambda} = \bar{E} \cap (\mathcal{X}^\lambda \times \mathbb{P}^1) \). Then \( \pi_1 : \bar{E}|_{\mathcal{X}^\lambda} \to \mathcal{X}^\lambda \) is again proper, so \( \pi_1(\bar{E}|_{\mathcal{X}^\lambda}) \) is closed. By the above,

\[
\pi_1(\bar{E}|_{\mathcal{X}^\lambda}) = \pi_1(\bar{E} \cap (\mathcal{X}^\lambda \times A^1)) = \pi_1(E^\lambda_\kappa) = Z^\lambda_\kappa. \quad \square
\]

**Proposition 3.2** \( Z^\lambda_\kappa \) is irreducible, and \( \dim Z^\lambda_\kappa = |\lambda| - |\kappa| + 1 \).

**Proof** The fibres of \( E^\lambda_\kappa \) over \( A^1 \) are all isomorphic to the irreducible variety \( X^\lambda_\kappa(0), \) so \( E^\lambda_\kappa \) is integral and has dimension \( |\lambda| - |\kappa| + 1 \). Therefore \( Z^\lambda_\kappa = \pi_1(E^\lambda_\kappa) \) is irreducible and has dimension at most \( |\lambda| - |\kappa| + 1 \). It has dimension at least \( |\lambda| - |\kappa| + 1 \) since a point of \( Z^\lambda_\kappa \) lies on only finitely-many varieties \( X^\lambda_\kappa(a), \) and each of these has dimension \( |\lambda| - |\kappa| \). \( \square \)

**Proposition 3.3** If \( x \in Z^\lambda_\kappa \), then \( \text{Wr}(x) \) has a root of multiplicity \( |\kappa| \). Conversely, if \( \text{Wr}(x) \) has a root of multiplicity \( \ell \), then \( x \in Z^\lambda_\kappa \) for some partition \( \kappa \vdash \ell \).
Proof This follows from Lemma 2.1. □

Now suppose that \(|\kappa| = 2\). In this case, by Proposition 3.2, \(Z^\lambda_\kappa\) is a closed hypersurface in \(\mathcal{X}^\lambda \cong \mathbb{A}^n\). Therefore the defining ideal of \(Z^\lambda_\kappa(\mathbb{F})\) is generated by a single polynomial \(\Phi^\lambda_\kappa(x) \in \mathbb{F}[x]\). Since \(Z^\lambda_\kappa\) is defined over \(\mathbb{Q}\), this polynomial has rational coefficients; in particular we can think of \(\Phi^\lambda_\kappa\) as a real valued function on \(\mathcal{X}^\lambda(\mathbb{R})\).

The **discriminant variety** \(\Delta_n \subset \mathcal{P}_n\) is the hypersurface defined the vanishing of the discriminant function \(g \mapsto \text{Disc}_z(g(z))\). We have \(g \in \Delta_n\) if and only if \(g\) has a repeated root. By Proposition 3.3,

\[
\text{Wr}^{-1}(\Delta_n) = Z^\lambda_{\Box} \cup Z^\lambda_{\Box}.
\]

Since the polynomials \(h_\mu\) are not in \(\Delta_n\), the points \(w_T, T \in \text{MN}(\lambda; \mu)\) are not in \(Z^\lambda_\kappa\) for either \(\kappa \vdash 2\). In particular, \(\Phi^\lambda_\kappa(w_{T_0}) \neq 0\), and we will assume that \(\Phi^\lambda_\kappa(w_{T_0}) > 0\).

We call the two functions \(\Phi^\lambda_{\Box}(x)\) and \(\Phi^\lambda_{\Box}(x)\) the **character orientation function** and the **dual character orientation function**, respectively.

**Lemma 3.4** Let \(C\) be a component of \(\mathcal{X}^\lambda(\mu)\), and \(\kappa \vdash 2\). Then exactly one of the following must be true:

(a) \(\Phi^\lambda_\kappa\) is globally non-negative on \(C\); or

(b) \(\Phi^\lambda_\kappa\) is globally non-positive on \(C\).

Proof Since \(\dim Z^\lambda_\kappa = n - 1\), and \(\dim C = n\), \(\Phi^\lambda_\kappa\) cannot be identically zero on \(C\). Therefore at most one of (a) and (b) holds.

To see that at least one of these holds, we show that \(C \setminus Z^\lambda_\kappa(\mathbb{R})\) is connected. Since \(\Phi^\lambda_\kappa\) is by definition non-vanishing on \(C \setminus Z^\lambda_\kappa(\mathbb{R})\) this implies either (a) or (b) holds.

Suppose that \(x \in Z^\lambda_\kappa(\mathbb{R}) \cap C\). Since \(x \in Z^\lambda_\kappa(\mathbb{R})\), \(\text{Wr}(x) \in \Delta_n\), i.e. \(\text{Wr}(x)\) must have a repeated root. On the other hand, by definition if \(x \in \mathcal{X}^\lambda(\mu)\) then \(\text{Wr}(x)\) cannot have a repeated real root. Therefore, \(\text{Wr}(x)\) must have a repeated non-real root.

Let \(K = \{g \in \mathcal{P}_n(\mathbb{R}) \mid g\) has a repeated non-real root\}. Then \(K\) has real codimension 2 in \(\mathcal{P}_n(\mathbb{R})\). Since \(\text{Wr}\) is a proper map with finite fibres, \(\text{Wr}^{-1}(K)\) has codimension 2 in \(\mathcal{X}^\lambda(\mathbb{R})\). Since we just showed \(Z^\lambda_\kappa(\mathbb{R}) \cap C \subset \text{Wr}^{-1}(K)\), it follows that \(Z^\lambda_\kappa(\mathbb{R}) \cap C\) has codimension at least 2, and hence \(C \setminus Z^\lambda_\kappa(\mathbb{R})\) is connected. □

**Definition 3.5** The **character orientation** of \(\mathcal{X}^\lambda(\mu)\) is defined as follows. For each component \(C\) of \(\mathcal{X}^\lambda(\mu)\) we orient according to the sign of the character orientation function.

(a) If \(\Phi^\lambda_{\Box}\) is globally non-negative on \(C\), then the character orientation of \(C\) is the same as the ambient orientation.
(b) If $\Phi^\lambda_B$ is globally non-positive on $C$, then the character orientation of $C$ is opposite to the ambient orientation.

The dual character orientation of $\mathcal{Z}^\lambda(\mu)$ is the orientation obtained similarly, using the dual character orientation function.

**Remark 3.6** A common way to define an orientation on a manifold $X$ is to specify a global non-vanishing volume form on $X$. The non-vanishing condition can be relaxed slightly to allow the form to vanish on a set of codimension 2. From this perspective, the character orientation on $X^\lambda(\mu)$ is simply the ambient orientation multiplied by the character orientation function.

**Example 3.7** Recall, from Sect. 1.4, that for $\lambda = (n - 1, 1)$, $\mathcal{Z}^\lambda$ is identified with the set of pairs $(g, c)$, where $g \in P_n$ and $g'(c) = 0$. On $P_n$, the variety defined by the discriminant is irreducible, but working on $\mathcal{Z}^\lambda$, we have the additional data of a specified critical point, and we can identify two components of $\text{Wr}^{-1}(\Delta_n)$. Specifically $Z^\lambda_Z$ consists pairs $(g, c)$ such that $c$ is a double (or higher order) root of $g$; this is cut out by the additional equation $g(c) = 0$. $Z^\lambda_{\overline{Z}}$ is the closure of the set of pairs $(g, c)$ such $g$ has a double root at some point other than $c$. In this case, the character orientation function is $\Phi^\lambda_B(g, c) = (-1)^{n-1}g(c)$. (The sign is explained by the fact that the point $w_T$ corresponds to the pair $(h^n_1, c_0)$, where $c_0$ is the smallest critical point of $h^n_1$, and $(-1)^{n-1}h_1^n(c_0) > 0$.) As noted in Sect. 1.4, this function is in fact globally positive or globally negative on any component of $\mathcal{Z}^\lambda(\mu)$. The dual character orientation function is $\Phi^\lambda_{\overline{B}}(g, c) = (-1)^{n-1}\text{Disc}_z(g(z))/g(c)$.

### 3.3 Signs of points in $\mathcal{Z}^\lambda(\mathbb{R})$

Let $\partial \text{Wr} : \mathcal{Z}^\lambda(\mathbb{R}) \to \mathbb{F}$ denote the Jacobian of the Wronski map $\text{Wr} : \mathcal{Z}^\lambda(\mathbb{R}) \to P_n(\mathbb{F})$ (with respect to affine coordinates). The ramification divisor of $\text{Wr} : \mathcal{Z}^\lambda \to P_n$, is the hypersurface in $\mathcal{Z}^\lambda$ on which $\partial \text{Wr}$ vanishes. We denote it by $R^\lambda$.

For $x \in \mathcal{Z}^\lambda(\mu) \setminus R^\lambda(\mathbb{R})$, define the sign of $x$ to be $\text{sgn}(x) = 1$ if the Wronski map is orientation preserving with respect to the character orientation in a neighbourhood of $x$, and $\text{sgn}(x) = -1$ otherwise. Similarly, define the dual sign of $x$, denoted $\text{sgn}^\ast(x)$, using the dual character orientation. Define the ambient sign of $x$, denoted $\text{asgn}(x)$, using the ambient orientation.

Our main goal is to prove the following theorem.

**Theorem 3.8** For every $T \in \text{MN}(\lambda; \mu)$,

$$\text{sgn}(w_T) = \text{sgn}(T).$$
Our strategy is to join the points \( w_T \) together by paths, and count the number of times the sign changes along a path. Let \( \gamma : [0, 1] \rightarrow \mathcal{X}^\lambda(\mathbb{R}), t \mapsto \gamma_t \) be a path, and assume the following:

- \( \gamma_0, \gamma_1 \notin R^\lambda(\mathbb{R}) \cup Z^\lambda_{\text{g0}}(\mathbb{R}) \cup Z^\lambda_{\text{g1}}(\mathbb{R}) \);
- \( \gamma_t \in R^\lambda(\mathbb{R}) \cup Z^\lambda_{\text{g0}}(\mathbb{R}) \cup Z^\lambda_{\text{g1}}(\mathbb{R}) \) for only finitely many values of \( t \).

These conditions ensure that \( \text{sgn}(\gamma_t), \text{sgn}^*(\gamma_t) \) and \( \text{asgn}(\gamma_t) \) are defined at all but finitely many points, including \( \gamma_0 \) and \( \gamma_1 \).

We first establish the main properties of interest in detecting sign changes along paths in \( \mathcal{X}^\lambda(\mathbb{R}) \). In Sect. 3.4, we then discuss how the existence of sufficiently nice paths allows us to prove Theorem 3.8.

**Proposition 3.9** If \( \text{sgn}(\gamma_t) \) changes at the point \( t \), then either \( \gamma_t \in R^\lambda(\mathbb{R}) \), or \( \gamma_t \in Z^\lambda_{\text{g1}}(\mathbb{R}) \). If \( \text{sgn}^*(\gamma_t) \) changes at the point \( t \), then either \( \gamma_t \in R^\lambda(\mathbb{R}) \), or \( \gamma_t \in Z^\lambda_{\text{g0}}(\mathbb{R}) \). If \( \text{asgn}(\gamma_t) \) changes at the point \( t \), then \( \gamma_t \in R^\lambda(\mathbb{R}) \).

**Proof** If \( \text{sgn}(\gamma_t) \) changes, then either the sign of the Jacobian of \( W_T \) reverses, or the orientation of the space reverses. The former can happen only when \( \partial W_T(\gamma_t) = 0 \), i.e. the path crosses \( R^\lambda(\mathbb{R}) \), and the latter can happen only when \( \Phi^\lambda_{\text{g1}}(\gamma_t) = 0 \), i.e. the path crosses \( Z^\lambda_{\text{g1}}(\mathbb{R}) \). The other statements are similar. \( \square \)

For the converse, we need a slightly stronger condition.

**Definition 3.10** Let \( V \subset \mathcal{X}^\lambda \) be an algebraic hypersurface defined over \( \mathbb{R} \), and let \( \gamma : [0, 1] \rightarrow \mathcal{X}^\lambda(\mathbb{R}) \) be a path such that \( \gamma_t \in V(\mathbb{R}) \) for only finitely many values of \( t \). Let \( t \in (0, 1) \) be such a value. We say \( \gamma \) has a **simple crossing** of \( V(\mathbb{R}) \) at \( t \), if \( \gamma_t \) is an algebraically smooth point of \( V(\mathbb{R}) \), and \( \gamma \) crosses \( V(\mathbb{R}) \) at \( t \). (Formally, “crossing” means the following: There exists an open neighbourhood \( U \subset \mathcal{X}^\lambda(\mathbb{R}) \) of \( \gamma_t \) such that for every sufficiently small \( \epsilon > 0 \), \( \gamma_t - \epsilon \) and \( \gamma_t + \epsilon \) are in different components of \( U \setminus V(\mathbb{R}) \).)

**Remark 3.11** In the above definition, the path \( \gamma \) itself need not be a smooth function of \( t \) at the crossing point—only the hypersurface it crosses needs to be smooth. In fact, the paths we consider will be constructed piecewise, in such a way that they are non-differentiable at precisely the points where they cross one of the hypersurfaces of interest.

**Proposition 3.12** If \( \gamma \) has a simple crossing of \( R^\lambda(\mathbb{R}) \) or \( Z^\lambda_{\text{g1}}(\mathbb{R}) \) at \( t \), then \( \text{sgn}(\gamma_t) \) changes at \( t \). If \( \gamma \) has a simple crossing of \( R^\lambda(\mathbb{R}) \) or \( Z^\lambda_{\text{g0}}(\mathbb{R}) \) at \( t \), then \( \text{sgn}^*(\gamma_t) \) changes at \( t \). If \( \gamma \) has a simple crossing of \( R^\lambda(\mathbb{R}) \) at \( t \), then \( \text{asgn}(\gamma_t) \) changes at \( t \).

**Proof** All three statements follow immediately from the following more general statement. Let \( V \subset \mathcal{X}^\lambda \) be a hypersurface, defined as the zero locus of
a polynomial $\Phi$. If $\gamma$ has a simple crossing of $V(\mathbb{R})$ at $t$, then $\Phi(\gamma_t)$ changes sign at $t$.

To prove this, note that since $V$ is smooth at $\gamma_t$, we may, by perturbing $\gamma$, assume that $\gamma$ is smooth and transverse to $V(\mathbb{R})$. The function $\Phi(\gamma_t)$ vanishes at $t$, and $\frac{d}{dt} \Phi(\gamma_t)$ cannot vanish at $t$, since (by transversality) the tangent vector to $\gamma$ at $t$ is not in the tangent space of $V$ at $\gamma_t$. Therefore $\Phi(\gamma_t)$ changes signs at $t$.

**Remark 3.13** For any point $x \in \mathcal{X}^{-\lambda}(\mu) \setminus R^\lambda(\mathbb{R})$, $\mu = 2^n 2^{n-1}$, there exists a path $\gamma$ from $w_{T_0}$ to $x$ such that all crossings are simple. Let $g_t = \text{Wr}(\gamma_t)$. We note that whenever the number of real roots of $g$ changes, $\gamma_t$ must have a simple crossing of $Z^\lambda_{\mathbb{R}}$ at $t$, or a simple crossing of $Z^\lambda_{\mathbb{C}}$ at $t$, but not both. We therefore have a relationship between $\text{sgn}(x)$ and $\text{sgn}^*(x)$:

$$\text{sgn}(x) \cdot \text{sgn}^*(x) = (-1)^{n_2}.$$ 

Equivalently this shows that the dual character orientation of $\mathcal{X}^{-\lambda}(\mu)$ is the same as the character orientation if $n_2$ is even, and opposite if $n_2$ is odd.

The following technical lemma will help to identify simple crossings of $R^\lambda(\mathbb{R})$.

**Lemma 3.14** Let $\psi : X \to Y$ be a quasifinite map of smooth varieties of the same dimension. Let $R \subset X$ be the ramification divisor (defined locally by the vanishing of the Jacobian determinant). Then the smooth locus of $R$ is

$$R^\text{sm} = \{ x \in R : \text{the ramification degree at } x \text{ is exactly 2} \}.$$ 

**Proof** The claim is local on $X$ and $Y$, so we may assume $X = \text{Spec}(A, m_x)$ and $Y = \text{Spec}(B, m_y)$ are spectra of regular local rings, and $\psi$ is induced by a map of local rings $\psi^* : B \to A$, with $\psi^*(m_y) \subset m_x$. We have a short exact sequence of modules of differentials

$$0 \to \psi^* \Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0,$$

and the ramification locus $R$ is defined by the vanishing of the determinant of the map of free $A$-modules $\psi^* \Omega_Y \to \Omega_X$.

First assume the ramification degree is exactly 2, that is, the scheme-theoretic fibre has length 2:

$$\text{vdim}_{\mathbb{C}}(B/m_y \otimes_B A) = \text{vdim}_{\mathbb{C}}(A/\psi^*(m_y)) = 2.$$ 

In particular $\psi^*(m_y)$ must contain $m_x^2$ (otherwise the quotient will be too large). In fact, $\psi^*(m_y)$ has the form $m_x^2 + L$, where $L \subset m_x/m_x^2$ is some codimension-1 vector subspace.
Consider the map $m_y/m_y^2 \rightarrow m_x/m_x^2$ induced by $\psi$. By the above, the image of this map is $L$. Choosing and lifting bases to obtain minimal generators of the ideals, we may assume $m_x = (x_1, \ldots, x_n)$ and $m_y = (y_1, \ldots, y_n)$ where

$$\psi^*(y_1) = 0 \mod m_x \quad \text{and} \quad \psi^*(y_i) = x_i \quad \text{for} \quad 2 \leq i \leq n. \quad (3.1)$$

In particular, $\psi^*(dy_i) = dx_i$ for $2 \leq i \leq n$.

As for $y_1$, we have $\psi^*(y_1) \in m_x^2$. By Cohen–Macaulayness $\text{Sym}^2(m_x/m_x^2) = m_x^2/m_x^3$. So, write $\psi^*(y_1)$ mod $m_x^3$ as a quadratic polynomial $q(x_1, \ldots, x_n)$. Taking differentials we get

$$\psi^*(dy_1) = \sum_{i=1}^{n} \frac{\partial q}{\partial x_i} dx_i \mod m_x^2(dx_1, \ldots, dx_n).$$

Note that $\frac{\partial q}{\partial x_1}$ is a linear form and is non-zero, since $\frac{\partial q}{\partial x_1} = 0$ would imply $x_1^2 \notin \psi^*(m_y)$. Thus the matrix for the map $\psi^*\Omega_Y \rightarrow \Omega_X$ has the form

$$
\begin{pmatrix}
\frac{\partial q}{\partial x_1} + s_1 & 0 & \ldots & 0 \\
\frac{\partial q}{\partial x_2} + s_2 & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
\frac{\partial q}{\partial x_n} + s_n & 0 & \ldots & 1
\end{pmatrix}
$$

with $s_i \in m_x^2$ for all $i$, and the determinant is $\partial \psi = \frac{\partial q}{\partial x_1} + s_1$. This cuts out a smooth divisor since $\partial \psi$ is part of a minimal generating set for $m_x$.

We have shown that $R^{sm}$ contains the points of ramification index 2. For the reverse direction, the proof is similar in spirit. Since we will not need it, we omit it.

\[ \square \]

3.4 Paths for the proof of Theorem 3.8

To prove Theorem 3.8, we will establish the following two lemmas.

**Lemma 3.15** Let $T \in MN(\lambda; \mu)$, $\mu = (\mu_1, \ldots, \mu_k)$, and suppose $\mu_b = 2$. Let $T'$ be the tableau obtained from $T$ by incrementing entries $b+1, \ldots, k$ by 1, and changing the lower-right $b$ to $b+1$. There exists a path $\gamma : [0, 1] \rightarrow X^\lambda$ from $wT'$ to $wT$, such that:

(a) $\gamma_t \notin R^\lambda(\mathbb{R})$ for all $t \in (0, 1)$.

(b) If shape$(T|_b)$ is a horizontal domino, then $\gamma_t \notin Z^\lambda_B(\mathbb{R})$ for all $t \in (0, 1)$, and there is exactly one $t \in (0, 1)$ such $\gamma_t \in Z^\lambda(\mathbb{R})$, and this is a simple crossing.
(c) If shape(T|b) is a vertical domino, then γt \notin Z^λ_{\square}(\mathbb{R}) for all t \in (0, 1), and there is exactly one t \in (0, 1) such that γt \in Z^λ_\Box(\mathbb{R}), and this is a simple crossing.

**Lemma 3.16** Let T' ∈ MN(λ; μ'), μ' = (μ'_1, ..., μ'_{k+1}). Suppose μ'_b = μ'_{b+1} = 1, and the entries b and b + 1 of T' are non-adjacent. Let T'' be the tableau obtained by swapping these two entries. There exists a path γ : [0, 1] → \mathcal{R}^λ from wT' to wT'', such that:

(a) There is exactly one t \in (0, 1) such that γt \in R^λ(\mathbb{R}), and this is a simple crossing.

(b) There is exactly one t \in (0, 1) such that γt \in Z^λ_\Box(\mathbb{R}), and this is a simple crossing.

(c) There is exactly one t \in (0, 1) such that γt \in Z^λ_\Box(\mathbb{R}), and this is a simple crossing.

**Example 3.17** We now continue Example 2.18, to illustrate Lemmas 3.15 and 3.16 in the case where λ = □□. Here, the four relevant tableaux are

\[
T_0 = \begin{array}{cc}
1 & 2 \\
3 &
\end{array}, \quad
T_1 = \begin{array}{cc}
1 & 3 \\
2 &
\end{array}, \quad
T_2 = \begin{array}{cc}
1 & 1 \\
2 &
\end{array}, \quad
T_3 = \begin{array}{cc}
1 & 2 \\
1 &
\end{array}.
\]

It is clear from Fig. 2 that we can find a path g : [0, 1] → \mathcal{R}_3(\mathbb{R}) from h_{13} to h_{21} such that g_t has two distinct real critical points, c_{0,t} < c_{1,t}, for all t \in [0, 1].

Lifting this path to \mathcal{R}^λ(\mathbb{R}), we obtain two paths: γ₀, γ₁ : [0, 1] → \mathcal{R}^λ(\mathbb{R}), γ₁,t = (g_t, c_{1,t}) for i = 1, 2; γ₀ connects wT₀ to wT₂, and γ₁ connects wT₁ to wT₃. Note that the larger critical point c_{1,t} must change sign along the path g_t, and so for some t, g_t(c_{1,t}) = 0. As explained in Example 3.7, this means that γ₀ crosses Z^λ_\Box(\mathbb{R}) and γ₁ crosses Z^λ_\Box(\mathbb{R}). Since g_t has two distinct critical points for all t, neither path crosses R^λ(\mathbb{R}). These are the two types of paths described in Lemma 3.15.

To connect wT₀ to wT₁, we need a path γ : [0, 1] → \mathcal{R}^λ(\mathbb{R}), from (h_{13}, c₀) to (h_{13}, c₁), where c₀ < c₁ are the two critical points of h_{13}. To do this, we begin with a different path g : [0, \frac{1}{2}] → \mathcal{R}_n(\mathbb{R}) such that g₀ = h_{13}, g_t has a two real critical points, c_{0,t} < c_{1,t}, for t \in [0, \frac{1}{2}), and g_{1/2} has a double critical point. Again, we can lift this path to \mathcal{R}^λ(\mathbb{R}) in two ways, but this time the two lifted paths meet at the fibre over g_{1/2}. We can then combine these two lifts into a single path:

\[
γ_t = \begin{cases}
(g_t, c_{0,t}) & \text{for } t \in [0, \frac{1}{2}] \\
(g_{1-t}, c_{1,1-t}) & \text{for } t \in [\frac{1}{2}, 1].
\end{cases}
\]

This is illustrated in Fig. 3. There are three special points along this path. Since g_{1/2} has a double critical point, we have γ_{1/2} ∈ R^λ(\mathbb{R}). There is also

\[\square\]
Fig. 3  Connecting $w_{T_0}$ to $w_{T_1}$, for $\lambda = \square$

Fig. 4  The points $w_{T_0}, w_{T_1} \in \mathcal{P}_\lambda(1^3)$ (shaded), $w_{T_2}, w_{T_3} \in \mathcal{P}_\lambda(21)$ (unshaded), and the three paths connecting them. The paths cross $Z_{BB}^\lambda$, $Z_{BB}^\lambda$, and $R^\lambda$ as described in Lemmas 3.15 and 3.16.

a point $t \in (0, \frac{1}{2})$ such that $g_t$ has a double real root at $c_{0,t}$. This means that $\gamma_t \in Z_{BB}^\lambda(\mathbb{R})$, and $\gamma_{1-t} \in Z_{BB}^\lambda(\mathbb{R})$. This is the type of path described in Lemma 3.16.

Figure 4 shows (a two-dimensional projection of) the Schubert cell $\mathcal{P}_\lambda(\mathbb{R})$, along with $Z_{BB}^\lambda(\mathbb{R})$, $Z_{BB}^\lambda(\mathbb{R})$, and $R^\lambda(\mathbb{R})$, and the three paths described in this example.

Lemmas 3.15 and 3.16 will be proved in Sect. 4.6. Modulo these, we can now prove Theorems 3.8 and 1.3.

Proof of Theorem 3.8  By the definition of the character orientation, $\text{sgn}(w_{T_0}) = 1$. By Lemma 3.16 we can connect all the points $w_T, T \in \text{SYT}(\lambda)$ by a network of paths $\gamma_t$ such that $\text{sgn}(\gamma_t)$ changes twice along each path. It follows that $\text{sgn}(w_T) = \text{sgn}(w_{T_0}) = 1 = \text{sgn}(T)$, for all $T \in \text{SYT}(\lambda)$.

For $T \in \text{MN}(\lambda; \mu), \mu \neq 1^n$, we can connect $T$ to some $T'$ as in Lemma 3.15. Then $\text{sgn}(T) = \text{sgn}(T')$ if shape($T|_b$) is a horizontal domino, $\text{sgn}(T) = -\text{sgn}(T')$ if shape($T|_b$) is a vertical domino. Following the path $\gamma$ connecting $w_T$ to $w_{T'}$, we see that $\text{sgn}(\gamma_t)$ does not change in the horizontal domino case, and $\text{sgn}(\gamma_t)$ changes once in the vertical domino case. The result now follows by a simple induction.  

\[\square\]
Proof of Theorem 1.3 Since \( h_\mu \in \mathcal{P}_n(\mu) \), the topological degree of \( \text{Wr} : \mathcal{R}^\lambda(\mu) \rightarrow \mathcal{P}_n(\mu) \) is

\[
\sum_{x \in \text{Wr}^{-1}(h_\mu)} \text{sgn}(x) = \sum_{T \in \text{MN}(\lambda; \mu)} \text{sgn}(w_T) = \sum_{T \in \text{MN}(\lambda; \mu)} \text{sgn}(T) = \chi^\lambda(\mu),
\]

where the equalities above are by Corollary 2.16, Theorem 3.8, and Theorem 2.12, respectively. \( \square \)

4 Stable curves

4.1 Families over \( \overline{M}_{0,n+3} \)

In order to connect up points \( w_T \in \mathcal{R}^\lambda(\mathbb{R}) \), we work with a different model of the Shapiro–Shapiro picture, in which \( \text{Wr} : \mathcal{R}^\lambda \rightarrow \mathcal{P}_n \) is replaced by a related finite flat map \( \Psi : \overline{\mathcal{M}}^0 \rightarrow \overline{\mathcal{M}}_{0,n+3} \), over the moduli space of genus 0 stable curves with \( n + 3 \) marked points. This model was described by Speyer [30].

Consider the set \( A_1 = \{0, 1, \infty, a_1, \ldots, a_n\} \), where each element is regarded purely as a formal symbol. For a stable curve \( C \in \overline{\mathcal{M}}_{0,n+3} \), we will label the marked points with the \( n + 3 \) symbols from \( A_1 \). \( \overline{\mathcal{M}}_{0,n+3} \subset \overline{\mathcal{M}}_{0,n+3} \) is the open subvariety of smooth curves, isomorphic to \( \mathbb{P}^1 \) with distinct marked points. If \( C \in \overline{\mathcal{M}}_{0,n+3} \), we will choose coordinates on \( C \) such that the marked points labelled 0, 1, \( \infty \) are placed at 0, 1, \( \infty \in \mathbb{P}^1 \) respectively. Abusing notation somewhat, we also write \( a_i \) to denote the coordinate of the point labelled \( a_i \) in \( C \), whenever it appears in a formula. More generally, if \( C \in \overline{\mathcal{M}}_{0,n+3} \), there is a unique morphism \( C \rightarrow \mathbb{P}^1 \) such that the points labelled 0, 1, \( \infty \) in \( C \) map to 0, 1, \( \infty \in \mathbb{P}^1 \). In this case \( a_i \) will denote the coordinate of the image of the point labelled \( a_i \) in \( \mathbb{P}^1 \). This allows us to associate a polynomial

\[
\text{pol}(C) = \text{pol}(C, z) := \prod_{a_i \neq \infty} (z + a_i)
\]

to every curve \( C \in \overline{\mathcal{M}}_{0,n+3} \).
Let $A_\ell$ denote the set of all ordered $\ell$-tuples of distinct elements of $A_1$. For $p, q, r$ distinct points of $\mathbb{P}^1$, define $\phi_{p,q,r}(s) = \frac{(p-s)(q-r)}{(p-q)(s-r)}$ for $s \in \mathbb{P}^1$; this is the unique transformation such that

$$
\phi_{p,q,r}(p) = 0, \quad \phi_{p,q,r}(q) = 1, \quad \phi_{p,q,r}(r) = \infty.
$$

Given a curve $C \in \mathcal{M}_{0,n+3}$ and $(p, q, r) \in A_3$, we interpret $\phi_{p,q,r} \in \text{PGL}_2$ to be the unique projective linear transformation of $\mathbb{P}^1$ that sends marked points $(p, q, r)$ to $(0, 1, \infty)$ respectively. We have an injection $\text{Gr}(d, d + m) \times \mathcal{M}_{0,n+3} \rightarrow \text{Gr}(d, d + m)^{A_3} \times \mathcal{M}_{0,n+3}$,

$$(x, C) \mapsto \left(\left(\phi_{p,q,r}(x)\right)_{(p,q,r) \in A_3}, C\right).$$

Define $\widetilde{\text{Gr}}(d, d + m)$ to be the closure of the image of this map. The projection

$$
\Psi : \widetilde{\text{Gr}}(d, d + m) \rightarrow \mathcal{M}_{0,n+3}
$$

defines a family over $\mathcal{M}_{0,n+3}$. We also have a projection map onto $\text{Gr}(d, d + m)$ for each $(p, q, r) \in A_3$; In particular, let

$$
\pi : \widetilde{\text{Gr}}(d, d + m) \rightarrow \text{Gr}(d, d + m)
$$

onto the Grassmannian factor corresponding to $(p, q, r) = (0, 1, \infty) \in A_3$.

For $C \in \mathcal{M}_{0,n+3}$, the fibre $\Psi^{-1}(C)$ of this family is isomorphic to $\text{Gr}(d, d + m)$. Specifically, $\pi : \Psi^{-1}(C) \rightarrow \text{Gr}(d, d + m)$ is an isomorphism (and the same is true for if we replace $\pi$ by the projection onto any other Grassmannian factor) [30, Proposition 3.3]. If on the other hand $C \in \mathcal{M}_{0,n+3}$ is a nodal curve, the fibre is a flat degeneration of the Grassmannian [30, Theorem 3.2].

Speyer’s construction also allows arbitrary Schubert conditions placed at the marked points of the curve. For $(p, q, r) \in A_3, s \in A_1$, and a partition $\lambda$, let

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Shapiro–Shapiro conjecture

\[ U_\lambda(p, q, r; s) := \begin{cases} 
X_\lambda(0) & \text{if } p = s \\
X_\lambda(1) & \text{if } q = s \\
X_\lambda(\infty) & \text{if } r = s \\
\text{Gr}(d, d + m) & \text{otherwise.}
\end{cases} \]

Define

\[ \tilde{X}_\lambda(s) := \left( \prod_{(p, q, r) \in A_3} U_\lambda(p, q, r; s) \times \overline{\mathcal{M}_{0, n+3}} \right) \cap \tilde{\text{Gr}}(d, d + m). \]

The restricted map \( \Psi : \tilde{X}_\lambda(s) \to \overline{\mathcal{M}_{0, n+3}} \), defines a family in which the fibre over a smooth curve \( C \in \mathcal{M}_{0, n+3} \) is identified with the Schubert variety \( X_\lambda(s) \). Specifically, if \( C \in \mathcal{M}_{0, n+3} \), \( \pi : \Psi^{-1}(C) \to X_\lambda(s) \) is an isomorphism [30, Proposition 3.3]. If \( C \in \mathcal{M}_{0, n+3} \) is a nodal curve, the fibre is a degeneration of the Schubert variety. Speyer describes these degenerate fibres explicitly [30, Theorem 1.2], and in Sect. 4.2, we will state this result for the special case we need.

Speyer also proves that intersections of the varieties \( \tilde{X}_\lambda(s) \) are well-behaved [30, Theorem 1.1].

**Theorem 4.1** For any partitions \( \alpha^s \subset m^d \), \( s \in A_1 \),

\[ \Psi : \bigcap_{s \in A_1} \tilde{X}_{\alpha^s}(s) \to \overline{\mathcal{M}_{0, n+3}} \]

defines a flat, proper, Cohen–Macaulay family over \( \overline{\mathcal{M}_{0, n+3}} \) of relative dimension \( dm - \sum_{s \in A_1} |\alpha^s| \).

We will be primarily interested in the family defined by the following intersection.

\[ \overline{\mathcal{Y}}^\lambda := \tilde{X}_{\Box}(a_1) \cap \cdots \cap \tilde{X}_{\Box}(a_n) \cap \tilde{X}_\lambda(\infty). \quad (4.1) \]

In this case, \( \Psi : \overline{\mathcal{Y}}^\lambda \to \overline{\mathcal{M}_{0, n+3}} \) is a finite morphism. Let \( \mathcal{Y}^\lambda := \Psi^{-1}(\mathcal{M}_{0, n+3}) \) denote the restriction of this family to \( \mathcal{M}_{0, n+3} \).

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Remark 4.2 The marked points 0 and 1 do not appear in the definition of $Y^\lambda$, and it is possible to define a similar finite family over $M_{0,n+1}$, with only marked points $\infty, a_1, \ldots, a_n$. The reason for including the two additional marked points is to provide the following connection to the Wronski map.

**Proposition 4.3** The diagrams below commute.

\[
\begin{array}{ccc}
Y^\lambda & \xrightarrow{\pi} & X^\lambda \\
\det & \downarrow Wr & \det \\
M_{0,n+3} & \xrightarrow{\text{pol}} & \mathcal{P}_n \\
\end{array}
\quad
\begin{array}{ccc}
Y^\lambda & \xrightarrow{\pi} & \mathcal{P}^\lambda \\
\det & \downarrow Wr & \\
M_{0,n+3} & \xrightarrow{\text{pol}} & \mathcal{P}_n \\
\end{array}
\]

In the first diagram, $X^\lambda = X_\lambda(\infty)$ is the closure of $X^\lambda$ in $\text{Gr}(d, d+m)$, and $\mathcal{P}_n = P^n$ is the closure of $\mathcal{P}_n$ in $P^m$. The second diagram is a fibre product: $Y^\lambda = M_{0,n+3} \times \mathcal{P}_n X^\lambda$.

**Proof** Over a smooth curve $C \in M_{0,n+3}$, $\pi$ restricts to an isomorphism $\Psi^{-1}(C) \to X_\Omega(a_1) \cap \cdots \cap X_\Omega(a_n) \cap X_\lambda(\infty)$ [30, Proposition 3.3]. By Lemma 2.1, $X_\Omega(a_1) \cap \cdots \cap X_\Omega(a_n) \cap X_\lambda(\infty) = \text{Wr}^{-1}(\text{pol}(C))$; hence the second diagram is a fibre product. Taking closures everywhere in the second diagram gives the first diagram, which therefore also commutes. \qed

### 4.2 $\mathbb{P}^1$-chains

The families $\mathcal{X}_\lambda(\infty)$ and $Y^\lambda$ are defined as subvarieties of $\text{Gr}(d, d+m)A_3 \times M_{0,n+3}$, but this encoding is highly redundant—locally, we only need a small number of factors to get a faithful image. The only nodal curves we will need in this paper are curves that are $\mathbb{P}^1$-chains, and if we restrict to these, the families $\mathcal{X}_\lambda(\infty)$ and $Y^\lambda$ have a simpler description.

Suppose $C \in M_{0,n+3}$ has components $C_1, C_2, \ldots, C_{k+1}$. Choose an isomorphism $C_i \to \mathbb{P}^1$, and for each point $p \in C$, let $p^{(i)} \in \mathbb{P}^1$ denote the image of $p$ under the contraction map $C \to C_i \to \mathbb{P}^1$; we call $(p^{(1)}, \ldots, p^{(k+1)}) \in (\mathbb{P}^1)^{k+1}$ the $C$-coordinates of $p \in C$. We thereby associate a polynomial to each component:

$$\text{pol}^{(i)}(C) = \text{pol}^{(i)}(C, z) := \prod_{a_j^{(i)} \neq \infty} (z + a_j^{(i)}).$$

By a $\mathbb{P}^1$-chain, we mean that the nodes and marked points are arranged as follows:

1. There is a node $o_i \in C$, joining $C_i$ to $C_{i+1}$, $i = 1, \ldots, k$. We assume our coordinates are such that $o_i^{(i)} = \infty$ and $o_i^{(i+1)} = 0$. 

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(2) The marked points 0, 1, ∞ are on $C_1, C_{b_0}, C_{k+1}$ respectively, where $1 \leq b_0 \leq k + 1$. We assume our coordinates are such that $0^{(1)} = 0, 1^{(b_0)} = 1$, and $\infty^{(k+1)} = \infty$.

(3) For $j = 1, \ldots, n$, put $b_j := i$ if $a_j \in C_i$. For $i = 1, \ldots, k + 1$, let $\mu_i := \# \{j \mid b_j = i \}$. We require $\mu_i \geq 1$ for $i \neq b_0$, but $\mu_{b_0} = 0$ is allowed. We write $\mu = (\mu_1, \ldots, \mu_{k+1})$. (This is equivalent to the stability condition on the curve.)

(4) We allow the possibility a **double marked point**, where two of the marked points $a_1, \ldots, a_n$ are at the same point of $C$. In the usual way of thinking about $\overline{M}_{0,n+3}$, when two marked points collide, the curve gains a new $\mathbb{P}^1$-component (sometimes called a “bubble”) containing the two marked points, attached to the original curve at the collision point. But since this resolution is canonical, it is also fine to think of this as a double marked point on $C$. We will take this perspective, and we will not count these extra bubbles when counting the components of $C$. The marked points must still be distinct from the nodes $o_1, \ldots, o_k$, and we do not allow triple marked points, or any other configuration normally forbidden by the definition of a stable curve.

**Remark 4.4** We can specify a $\mathbb{P}^1$-chain by specifying values for $k, b_0, b_1, \ldots, b_n$, and $a_1^{(b_1)}, \ldots, a_n^{(b_n)}$. Note, however, that this is also specifying $C$-coordinate isomorphisms $C_i \to \mathbb{P}^1$, which are not part of the actual data of the curve, and are not canonical if $k \geq 1$. Each of the coordinate maps $C_i \to \mathbb{P}^1$, $i \neq b_0$ can be rescaled by any non-zero scalar, without changing the curve. (The map $C_{b_0} \to \mathbb{P}^1$, however, is pinned down by conditions 1 and 2.) In particular, there is more than one way to specify the same nodal curve in $\overline{M}_{0,n+3}$. For example, there is a unique curve with $n + 1$ components and $b_j = j + 1$ for all $j$; the values of $a_j^{(b_j)}$ specify coordinates on this curve, but do not help specify the curve itself.

**Example 4.5** An example of $\mathbb{P}^1$-chain is shown in Fig. 5.

In this example we have $b_5 = 1, b_2 = b_3 = b_4 = b_8 = 2, b_0 = b_1 = 3$, and $b_6 = b_7 = 4$. The dashed circle on each $\mathbb{P}^1$ represents points with real $C$-coordinates, and the solid outer circle represents points with pure imaginary
\( \mathbf{C} \)-coordinates. In this case,

\[
\begin{align*}
    a_1^{(3)} &= -\frac{1}{2}, & a_2^{(2)} &= 1, & a_3^{(2)} &= a_4^{(2)} = \iota, & a_5^{(1)} &= -1, \\
    a_6^{(4)} &= \iota, & a_7^{(4)} &= -\frac{i}{2}, & a_8^{(2)} &= \frac{1}{2}.
\end{align*}
\]

Note that \( a_3 = a_4 \) is a double marked point. The associated polynomials are:

\[
\begin{align*}
    \text{pol}^{(1)}(z) &= (z - 1), \\
    \text{pol}^{(2)}(z) &= z(z + 1)(z + \frac{i}{2})(z + \iota)^2, \\
    \text{pol}^{(3)}(z) &= z^5(z - \frac{i}{2}), \\
    \text{pol}^{(4)}(z) &= z^6(z + \frac{i}{2})(z - \frac{i}{2}),
\end{align*}
\]

and since \( b_0 = 3 \), \( \text{pol}(z) = \text{pol}^{(3)}(z) \).

For \( j = 1, \ldots, n \), let \( \pi_j : \mathring{\mathrm{Gr}}(d, d + m) \to \mathrm{Gr}(d, d + m) \) denote the projection onto the Grassmannian factor corresponding to \( (0, a_j, \infty) \in A_3 \). When we restrict our base to the open subvariety of \( \mathbf{P}^1 \)-chains in \( \mathcal{M}_{0, n+3} \),

\[
\mathring{X}_{\lambda \vee}(\infty) \xrightarrow{(\pi_1, \ldots, \pi_n)} \mathrm{Gr}(d, d + m)^n \times \mathcal{M}_{0, n+3}
\]

restricts to an injective map. Since all calculations in this paper take place within this restriction, we will henceforth identify \( \mathring{X}_{\lambda \vee}(\infty) \) with its image under this map.

Let \( C \) be a \( \mathbf{P}^1 \)-chain, and let \( Q(C) \) denote the fibre of the map \( \Psi : \mathring{X}_{\lambda \vee}(\infty) \to \mathcal{M}_{0, n+3} \) over \( C \) (regarded as a subvariety of \( \mathrm{Gr}(d, d + m)^n \)). If \( C \) has more than one component, then \( Q(C) \) is a reducible scheme, and its components are indexed by \( \text{Tab}(\lambda; \mu) \), where \( \mu = (\mu_1, \ldots, \mu_{k+1}) \) is as above. Write \( Q_T(C) \) for the component indexed by \( T \in \text{Tab}(\lambda; \mu) \). Write \( Q_T^{(b)} \) for the closure of \( X_{\text{shape}(T|_{z=b})} \).

**Theorem 4.6** We have an isomorphism \( Q_T(C) \to \prod_{b=1}^{k+1} Q_T^{(b)} \),

\[
y \mapsto (y^{(1)}, \ldots, y^{(k+1)}),
\]

characterized by

\[
\pi_j(y) = \phi_{0, a_j^{(b_j)}, \infty}(y^{(b_j)}), \quad \text{for } j = 1, \ldots, n.
\]

**Proof** The existence of such an isomorphism is the content of [30, Theorem 1.2], in the case where the curve is \( \mathbf{P}^1 \)-chain, and there is only one Schubert
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condition. The details of the isomorphism, in general, are described in [30, Section 3].

We call \((y^{(1)}, \ldots, y^{(k+1)}) \in \prod_{b=1}^{k+1} Q_T^{(b)}\) the \textbf{C-coordinates} of the corresponding point \(y \in Q_T(C)\).

Let \(Y(C)\) denote the fibre of the map \(\Psi : \overline{\mathcal{M}}^k \to \overline{\mathcal{M}}_{0,n+3}\) over \(C\). Then \(Y(C) \subset Q(C)\), so every point in \(Y(C)\) is in some component \(Q_T(C)\). Write \(Y_T(C) := Y(C) \cap Q_T(C)\).

\textbf{Theorem 4.7} Every point of \(Y(C)\) is in \(Y_T(C)\) for some unique \(T \in \text{Tab}(\lambda; \mu)\). Under the isomorphism (4.2), \(Y_T(C)\) is identified with \(\prod_{b=1}^{k+1} Y_T^{(b)}(C)\), where

\[ Y_T^{(b)}(C) = X_{\text{shape}(T_{|\leq b})}(0) \cap \text{Wr}^{-1}(\text{pol}^{(b)}(C)). \]

\textit{Proof} First, suppose \(C\) has no double marked points. Then by definition,

\[ Y(C) = Q(C) \cap \bigcap_{j=1}^n \pi_j^{-1} X_{\square}(1). \]

By (4.3), intersecting \(Q_T(C)\) with \(\pi_j^{-1} X_{\square}(1)\) corresponds to intersecting the factor \(Q_T^{(b_j)}\) with \(X_{\square}(a_j^{(b_j)})\). Thus \(Y_T(C)\) is identified with

\[ \prod_{b=1}^{k+1} \left( Q_T^{(b)} \cap \bigcap X_{\square}(a_j^{(b_j)}) \right), \]

where the intersection is taken over all \(j\) such that \(b_j = b\). The result now follows from Lemma 2.1. In the case where \(C\) has a double marked point, we deduce the result by taking limits and observing that

\[
\lim_{a' \to a} X_{\square}(a) \cap X_{\square}(a') = X_{\square}(a) \cup X_{\square}(a). \]

In the case where the curve has a double marked point, we distinguish the following two cases.

\textbf{Corollary 4.8} If \(C\) has a double marked point, \(a_j = a_j'\) on component \(C_b\), and \(y \in Y_T(C)\), then we have either \(\pi_j(y) \in X_{\square}(1) \cap Q_T^{(b)}\) or \(\pi_j(y) \in X_{\square}(1) \cap Q_T^{(b)}\) but not both.

\textit{Proof} By definition \(y^{(b)} \in Q_T^{(b)}\), and by Theorem 4.7, \(y^{(b)} \in \text{Wr}^{-1}(\text{pol}^{(b)}(C))\). Since \(\text{pol}^{(b)}(C)\) has a double root at \(a_j^{(b)}\), by Lemma 2.1 we have \(y^{(b)} \in X_{\square}(a_j^{(b)})\) or \(y^{(b)} \in X_{\square}(a_j^{(b)})\) but not both. Applying (4.3) gives the result. \(\square\)
Remark 4.9 For many of the nodal curves we consider, we will have \( b_0 = k + 1 \) and \( \mu_{k+1} = 0 \), i.e. 1 and \( \infty \) are the only marked points \( C_{k+1} \). When this happens, \( Q^{(k+1)}_T \) is a point, and \( y^{(k+1)} \) does not appear on the right side of (4.3). We will therefore sometimes drop the \((k + 1)\)-term from our notation, e.g. writing \( \mu = (\mu_1, \ldots, \mu_k) \), or \( Y_T(C) = \prod_{b=1}^k Y_T^{(b)}(C) \), etc.

4.3 Real structures

For each quadruple \((p, q, r, s) \in A_4\) of distinct elements of \( A_1 \), there is a natural map \( \theta_{p,q,r,s} : \mathcal{M}_{0,n+3} \to \mathcal{M}_{0,4} \), which forgets all marked points except for \( p, q, r, s \), and contracts any unstable components of the curve. Using the canonical identification of \( \mathcal{M}_{0,4} \) with \( \mathbb{P}^1 \) (in which \((p, q, r) \mapsto (0, 1, \infty)) \), we rewrite this as \( \theta_{p,q,r,s} : \mathcal{M}_{0,n+3} \to \mathbb{P}^1 \),

\[
\theta_{p,q,r,s}(C) = \frac{(\hat{p} - \hat{s})(\hat{q} - \hat{r})}{(\hat{p} - \hat{q})(\hat{r} - \hat{s})}
\]

where \( \hat{p}, \hat{q}, \hat{r}, \hat{s} \) denote the images of points \( p, q, r, s \) in \( \mathcal{M}_{0,4} \), in any coordinates. The product of all such maps gives an embedding

\[
\mathcal{M}_{0,n+3} \hookrightarrow (\mathbb{P}^1)^{A_4}.
\] (4.4)

From this construction we obtain the standard real structure on \( \mathcal{M}_{0,n+3} \), which is inherited from the standard real structure on \( \mathbb{P}^1 \): the complex conjugate of a stable curve \( C \in \mathcal{M}_{0,n+3} \) is obtained by conjugating each of the nodes and marked points. Similarly, there is a standard real structure on \( \text{Gr}(d, d+m) \), defined via its embedding in \( \text{Gr}(d, d+m)^{A_3} \times \mathcal{M}_{0,n+3} \). For either of these spaces, we will denote this complex conjugation map by \( \xi \).

The standard notion of a real point of \( \mathcal{M}_{0,n+3} \) and \( \mathcal{M}^k \) does not precisely correspond to the notion of a real point of \( \mathcal{P}_n \) or \( \mathcal{V}^\lambda \). Informally a point of \( \mathcal{M}_{0,n+3} \) is real iff all nodes and marked points are real, whereas a point of \( \mathcal{P}_n \) is real whenever its roots are a mixture of real points and complex conjugate pairs. Thus the real points of \( \mathcal{M}_{0,n+3} \) map to the closure of \( \mathcal{P}_n(1^n) \subset \mathcal{P}_n(\mathbb{R}) \). To study curves such that \( \text{pol}(C) \in \mathcal{P}_n(\mu) \) for \( \mu \neq 1^n \), we need to consider other real structures on \( \mathcal{M}_{0,n+3} \), in which specified pairs of marked points are required to be complex conjugates of each other.

Let \( \mathcal{S}_n \) denote the symmetric group of permutations of \{1, \ldots, n\}. \( \mathcal{S}_n \) acts on \( \mathcal{M}_{0,n+3} \) by permuting the marked points \( a_1, \ldots, a_n \), and on \( \text{Gr}(d, d+m) \) by also permuting the corresponding factors in \( \text{Gr}(d, d+m)^{A_3} \). If \( \sigma \in \mathcal{S}_n \) is an involution, we define \( \xi^\sigma := \sigma \circ \xi \) acting on either \( \mathcal{M}_{0,n+3} \) or \( \text{Gr}(d, d+m) \) (or any of its \( \xi \)- and \( \mathcal{S}_n \)-invariant subvarieties, e.g. \( \bar{X}_{\lambda,\nu}(\infty) \) or \( \mathcal{V}^\lambda \)). If \( \sigma \in \mathcal{S}_n \) is the identity element, then \( \xi^\sigma = \xi \); otherwise, \( \xi^\sigma \) is the complex conjugation.
for a different real structure on these spaces. The real points with respect to this real structure are the $\xi^\sigma$-fixed points, and for any $\xi$- and $\mathfrak{S}_n$-invariant subvariety $V$ of $\widetilde{\text{Gr}}(d, d + m)$ or $\mathcal{M}_{0,n+3}$, we denote the $\xi^\sigma$-fixed points of $V$ by $V(\mathbb{R}^\sigma)$.

**Proposition 4.10** Let $\sigma \in \mathfrak{S}_n$ be an involution. We have the following commutative diagram.

\[
\begin{array}{c}
\mathcal{Y}^\lambda(\mathbb{R}^\sigma) & \xrightarrow{\pi} & \mathcal{X}^\lambda(\mathbb{R}) \\
\downarrow \Psi & & \downarrow \text{Wr} \\
\mathcal{M}_{0,n+3}(\mathbb{R}^\sigma) & \xrightarrow{\text{pol}} & \mathcal{P}(\mathbb{R})
\end{array}
\]

If $C \in \mathcal{M}_{0,n+3}$ and $\text{pol}(C) \in \mathcal{P}(\mathbb{R})$, then there is a unique involution $\sigma \in \mathfrak{S}_n$ such that $C \in \mathcal{M}_{0,n+3}(\mathbb{R}^\sigma)$. If $y \in Y(C)$ and $\pi(y) \in \mathcal{X}^\lambda(\mathbb{R})$, then $\sigma$ is also the unique involution such that $y \in \mathcal{Y}^\lambda(\mathbb{R}^\sigma)$.

Note that, when $\sigma$ has cycle type $\mu$, the images of $\mathcal{Y}^\lambda(\mathbb{R}^\sigma)$ and $\mathcal{M}_{0,n+3}(\mathbb{R}^\sigma)$ are the closures of $\mathcal{X}^\lambda(\mu) \subset \mathcal{X}^\lambda(\mathbb{R})$ and $\mathcal{P}(\mu) \subset \mathcal{P}(\mathbb{R})$.

**Proof** We have $\text{pol} \circ \sigma = \text{pol}$, so for $C \in \mathcal{M}_{0,n+3}$, $\xi^\sigma C = C$ implies that $\text{pol}(C)$ is real. Since $\sigma$ does not permute $0, 1, \infty$, it fixes the $(0, 1, \infty)$-factor in $\text{Gr}(d, d + m)^{A_3}$, so $\pi \circ \sigma = \pi$, and so a similar argument applies for the map $\pi$.

If $C \in \mathcal{M}_{0,n+3}$ and $\text{pol}(C)$ is real, then $\text{pol}(\xi C) = \overline{\text{pol}(C)} = \text{pol}(C)$. It follows that $\xi$ is just permuting the marked points of $C$, i.e. $\xi C = \sigma C$ for some $\sigma \in \mathfrak{S}_n$, or equivalently $C \in \mathcal{M}_{0,n+3}(\mathbb{R}^\sigma)$ ($\sigma$ must therefore be an involution, since $\xi$ is an involution). Finally, suppose $y \in Y(C)$ and $\pi(y)$ is real. Then

$$\pi(\sigma y) = \pi(y) = \overline{\pi(y)} = \pi(\xi y)$$

and

$$\Psi(\sigma y) = \sigma C = \xi C = \Psi(\xi y),$$

so by the last part of Proposition 4.3, it follows that $\sigma y = \xi y$. \qed

If $C$ is a $\mathbb{P}^1$-chain, then we have the following characterization. We say that the $C$-coordinates are $\xi^\sigma$-compatible, if $a^{(i)}_j$ is the complex conjugate of $a^{(i)}_{\sigma(j)}$ for all $i = 1, \ldots, k + 1$, $j = 1, \ldots, n$ (this requires $a_j$ and $a_{\sigma(j)}$ to be on the same component of $C$). Note that this implies that $C \in \mathcal{M}_{0,n+3}(\mathbb{R}^\sigma)$ is $\xi^\sigma$-fixed. Conversely if $C$ is $\xi^\sigma$-fixed there exists a choice of $\xi^\sigma$-compatible $C$-coordinates (though not every choice of $C$-coordinates is $\xi^\sigma$-compatible).
Proposition 4.11 Let $\sigma \in S_n$ be an involution, and let $C$ be a $\mathbb{P}^1$-chain with $\xi^\sigma$-compatible $C$-coordinates. For $y \in Y(C)$, we have $\xi^\sigma y = y$ if and only if $(y^{(1)}, \ldots, y^{(k+1)})$ are real.

Proof First suppose $\xi^\sigma y = y$. This means that

$$\pi_j(y) = \xi \pi_{\sigma(j)}(\sigma y).$$

for $j = 1, \ldots, n$. Let $b \in \{1, \ldots, k\}$. If the component $C_b$ in the chain does not contain any of the marked points $a_1, \ldots, a_n$, then $Q_T^{(b)}$ consists of a single real point, so $y^{(b)}$ is real. Otherwise, let $a_j$ be a marked point on $C_b$, and let $a = a^{(b)}_j$ be its coordinate. Then $a_{\sigma(j)} \in C_b$ and since the coordinates are $\xi^\sigma$-compatible, $a^{(b)}_{\sigma(j)} = \bar{a}$. By (4.3), we have

$$\pi_j(y) = \phi_{0,a,\infty}(y^{(b)})$$

and

$$\xi \pi_{\sigma(j)}(\sigma y) = \xi \phi_{0,\bar{a},\infty}(\sigma y^{(b)}) = \xi \phi_{0,\bar{a},\infty}(y^{(b)}) = \phi_{0,a,\infty}(\xi y^{(b)}),$$

which implies that $y^{(b)}$ is real.

Conversely, if $y^{(b)}$ is real, the calculation above shows that (4.5) holds for all $j$ such that $a_j$ is on $C_b$. If $y^{(b)}$ is real all $b$, then (4.5) holds for all $j$, so $\xi^\sigma y = y$. \qed

Remark 4.12 There is a well-known CW-complex description of $\mathcal{M}_{0,n}(\mathbb{R})$ in terms of associahedra [3]. It would be interesting to see an analogous description of $\mathcal{M}_{0,n}(\mathbb{R}^\sigma)$ and of the attachments between the twisted structures for each $\sigma$.

4.4 Special fibres

Let $\mu = (\mu_1, \ldots, \mu_k)$, be a composition with $\mu_i \in \{1, 2\}$. Recall that $\bar{\mu}_b = n + 1 - \sum_{i=1}^{b} \mu_i$.

Working over $\mathbb{C}((u))$, we define $C_\mu(u) \in \mathcal{M}_{0,n+3}(\mathbb{C}((u)))$ to be the curve whose marked points $a_1, \ldots, a_n$ are specified as follows:

$$a_j = \begin{cases} u^j & \text{if } j = \bar{\mu}_i \text{ for some } i, \mu_i = 1 \\ \frac{i}{2}(u^j + u^{j+1}) & \text{if } j = \bar{\mu}_i \text{ for some } i, \mu_i = 2 \\ -\frac{i}{2}(u^{j-1} + u^j) & \text{if } j - 1 = \bar{\mu}_i \text{ for some } i, \mu_i = 2 \end{cases}$$

(4.6)

Note that $\text{pol}(C_\mu(u), z) = H_\mu(u, z)$. 

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Proposition 4.13 The limit curve \( \lim_{u \to 0} C_{\mu}(u) \) is the \( \mathbb{P}^1 \)-chain \( C_{\mu}(0) \), with \( k+1 \) components, \( b_0 = k+1 \), specified by the following coordinate data:

- if \( j = \bar{\mu}_i, \mu_i = 1 \), then \( b_j = i \) and \( a_j^{(b_j)} = 1 \);
- if \( j = \bar{\mu}_i, \mu_i = 2 \), then \( b_j = b_{j+1} = i \) and \( (a_j^{(b_j)}, a_j^{(b_{j+1})}) = (\frac{1}{2}, -\frac{1}{2}) \).

Proof Using the embedding (4.4), it suffices to show that for all \( (p, q, r, s) \), we have \( \lim_{u \to 0} \theta_{p,q,r,s}(C_{\mu}(u)) = \theta_{p,q,r,s}(C_{\mu}(0)) \). There are several cases, but this is straightforward. Note that \( a_j \)’s from distinct \( \mu_i \)’s have distinct leading orders as \( u \to 0 \), which greatly simplifies the calculation. \( \square \)

Example 4.14 For \( \mu = (2, 1, 2, 2, 1) \), the curve \( C_{\mu}(u) \) has marked points 0, 1, \( \infty \) and

\[
\begin{align*}
a_7 &= \frac{1}{2}(u^7 + u^8), & a_6 &= u^6,
 & a_4 &= \frac{1}{2}(u^4 + u^5), & a_2 &= \frac{1}{2}(u^2 + u^3), \\
a_8 &= -\frac{1}{2}(u^7 + u^8), & a_5 &= -\frac{1}{2}(u^4 + u^5), & a_3 &= -\frac{1}{2}(u^2 + u^3), & a_1 &= u^1.
\end{align*}
\]

The polynomial \( \text{pol}(C_{\mu}(u), z) = H_{\mu}(u, z) \) is given in Example 2.14. The limit curve \( C_{\mu}(0) \) is shown in Fig. 6.

Let \( \sigma \in \mathfrak{S}_n \) be the involution

\[
\sigma(j) = \begin{cases} 
  j & \text{if } j = \bar{\mu}_i, \mu_i = 1 \\
  j + 1 & \text{if } j = \bar{\mu}_i, \mu_i = 2 \\
  j - 1 & \text{if } j - 1 = \bar{\mu}_i, \mu_i = 2.
\end{cases}
\]  

(4.7)

This is the unique involution such that the curve \( C_{\mu}(u) \) is \( \xi^\sigma \)-fixed. Moreover, the \( C_{\mu}(0) \)-coordinates in Proposition 4.13 are \( \xi^\sigma \)-compatible.

We can now prove Lemma 2.15. First, we establish the analogous result for the limit fibre \( Y(C_{\mu}(0)) \).

Lemma 4.15 For \( T \in \text{Tab}(\lambda; \mu) \), \( Y_T(C_{\mu}(0)) \) consists of \( 2^{\#\mathbf{A}(T)} \) reduced points. If \( \#\mathbf{A}(T) = 0 \), the unique point in \( Y_T(C_{\mu}(0)) \) is \( \xi^\sigma \)-fixed. If \( \#\mathbf{A}(T) > 0 \), then none of the points in \( Y_T(C_{\mu}(0)) \) are \( \xi^\sigma \)-fixed.
Proof By Theorem 4.7, this is identified with \( \prod_{b=1}^{n} Y_T^{(b)}(C_{\mu}(0)) \), where

\[
Y_T^{(b)}(C_{\mu}(0)) = X_{\text{shape}(T|_{b-1})}^{(b)} \cap \text{Wr}^{-1}(\text{pol}^{(b)}(C)).
\]

Now,

\[
\text{pol}^{(b)}(C) = \begin{cases} 
  z^{\mu_1+\cdots+\mu_b-1}(z+1) & \text{if } \mu_b = 1 \\
  z^{\mu_1+\cdots+\mu_b-1}(z-i)(z+i) & \text{if } \mu_b = 2.
\end{cases}
\]

By Lemmas 2.7 and 2.8, there is a unique point in \( Y_T^{(b)}(C_{\mu}(0)) \) when shape\( (T|_{b}) \) is either a single box, or two adjacent boxes. If shape\( (T|_{b}) \) has two boxes that are non-adjacent, of distance \( L > 1 \) apart, then the discriminant in Lemma 2.8 is \(-1 - L^{-2} < 0\), so both solutions are non-real. Thus we see that \( \prod_{b=1}^{n} Y_T^{(b)}(C_{\mu}(0)) \) consists of \( 2\#_{\text{pol}}^{(T)} \) points, which are all non-real, unless \( \#_{\text{pol}}^{(T)} = 0 \) (in which case \( \prod_{b=1}^{k} Y_T^{(b)}(C_{\mu}(0)) \) consists of a single real point).

By Proposition 4.11, it follows that \( Y_T(C_{\mu}(0)) \) consists of \( 2\#_{\text{pol}}^{(T)} \) points; none are \( \xi^\alpha \)-fixed, unless \( \#_{\text{pol}}^{(T)} = 0 \).

Note that by Lemma 4.15, \( Y(C_{\mu}(0)) = \bigcup_{T \in \text{Tab}(\lambda,\mu)} Y_T(C_{\mu}(0)) \) consists of \( f^{\lambda} \) distinct (reduced) points. Abusing notation slightly, we define

\[
Y_T(C_{\mu}(u)) := \{ y(u) \in Y(C_{\mu}(u)) \mid \lim_{u \to 0} y(u) \in Y_T(C_{\mu}(0)) \}.
\]

Proof of Lemma 2.15 The curves \( C_{\mu}(u) \) are actually defined over \( \mathbb{C}(u) \), and hence the points of \( Y(C_{\mu}(u)) \) are defined over some algebraic extension of \( \mathbb{C}(u) \). This, together with the fact the limit points over \( C_{\mu}(0) \) are distinct implies that the \( f^{\lambda} \) points of \( Y(C_{\mu}(u)) \) are distinct and their coordinates are defined by power series with a positive radius of convergence.

If \( y(u) \in \overline{Y}_{\lambda}(\mathbb{C}(u)) \) is \( \xi^\alpha \)-fixed, then \( \lim_{u \to 0} y(u) \) is \( \xi^\alpha \)-fixed. Conversely, if \( y(u) \) is not \( \xi^\alpha \)-fixed, then either \( \lim_{u \to 0} y(u) \) is also not \( \xi^\alpha \)-fixed, or \( \lim_{u \to 0} y(u) = \lim_{u \to 0} \xi^\alpha y(u) \). The latter does not occur for points in \( Y(C_{\mu}(u)) \), since the points of \( Y(C_{\mu}(0)) \) are distinct. Therefore \( y(u) \in Y(C_{\mu}(u)) \) is \( \xi^\alpha \)-fixed if and only if \( \lim_{u \to 0} y(u) \in Y(C) \) is \( \xi^\alpha \)-fixed.

Now, define

\[
W_T := \{ \pi(y(u)) \mid y(u) \in Y_T(C_{\mu}(u)) \}.
\]

Since \( \pi : Y(C) \to \text{Wr}^{-1}(H_{\mu}) \) is an isomorphism, \( W_T \) consists of \( 2\#_{\text{pol}}^{(T)} \) points, defined by power series with a positive radius of convergence. To see that the normalized Plücker coordinates are power series, note that
lim_{u \to 0} x(u) \in \mathcal{X}^\lambda for x(u) \in W_T^{-1}(H_\mu), which would be false if some normalized Plücker coordinate involved a negative power of u. Moreover, this isomorphism establishes property (a). To see that property (b) holds, let x(u) \in W_T, and write x(u) = \pi(y(u)), y(u) \in Y_T(C_\mu(u)). Then
\lim_{u \to 0} \begin{pmatrix} 1 & 0 \\ 0 & u^{-b} \end{pmatrix} x(u) = y(0)^{(b)},
which is in \mathcal{X}^{\text{shape}(T) \leq b} by Theorem 4.7. By Proposition 4.10, none of the points of W_T are real except when \#_{\mathcal{X}}(T) = 0, so property (c) holds.

4.5 Paths in \overline{\mathcal{M}}_{0,n+3}

Fix a composition \mu = (\mu_1, \ldots, \mu_k), with \mu_i \in \{1, 2\}, and an index b such that \mu_b = 2. Let \mu' = (\mu_1, \ldots, \mu_{b-1}, 1, 1, \mu_{b+1}, \ldots, \mu_k). We now define curves \Gamma_t(u) \in \overline{\mathcal{M}}_{0,n+3}, for each t \in [0, 1], u \in [0, \epsilon], where \epsilon is a (sufficiently small) positive real number. These curves have the property that \Gamma_0(u) = C_{\mu'}(u) and \Gamma_1(t) = C_\mu(u), so for fixed u, t \mapsto \Gamma_t(u) is a path from C_{\mu'}(u) to C_\mu(u) in \overline{\mathcal{M}}_{0,n+3}.

First, suppose u > 0. In this case, we can specify \Gamma_t(u) by specifying the marked points a_1, \ldots, a_n. Let c := \overline{\mu}_b. For j \notin \{c, c+1\}, a_j is independent of t, and is defined by (4.6). The marked points a_c and a_{c+1} depend on t, and are defined as follows:

\begin{align*}
a_c &= \begin{cases} (1-t)u^c + tu^{c+1} & \text{if } 0 \leq t \leq \frac{1}{2} \\
e^{\pi i(t-\frac{1}{2})}(\frac{1}{2}u^c + \frac{1}{2}u^{c+1}) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \\
a_{c+1} &= \begin{cases} tu^c + (1-t)u^{c+1} & \text{if } 0 \leq t \leq \frac{1}{2} \\
e^{-\pi i(t-\frac{1}{2})}(\frac{1}{2}u^c + \frac{1}{2}u^{c+1}) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}
\end{align*}

For t \neq \frac{1}{2}, a_1, \ldots, a_n are distinct and distinct from \{0, 1, \infty\} so this uniquely specifies a curve in \overline{\mathcal{M}}_{0,n+3}. For t = \frac{1}{2}, we have a double marked point, \text{a}_c = a_{c-1} but all other marked points are distinct, so this is still a \text{P}^1\text{-chain (though not in } \overline{\mathcal{M}}_{0,n+3}).

Note that the marked points a_c and a_{c+1} begin at u^c and u^{c+1} respectively (when t = 0). They come together along the real axis and collide (when t = \frac{1}{2}) to produce a double marked point at \frac{1}{2}(u^c + u^{c+1}). Then they move apart as a conjugate pair along a circle in the complex plane, to end up at \pm\frac{1}{2}(u^c + u^{c+1}) (when t = 1). See Fig. 7.

For u = 0 we define the curve \Gamma_t(0) := \lim_{u \to 0} \Gamma_t(u), for every t \in [0, 1]. For t = 0, 1 we have \Gamma_0(0) = C_{\mu'}(0) and \Gamma_1(0) = C_\mu(0), as described in...
Fig. 7 The path $G_t(u)$ in $\mathcal{M}_{0,n+3}$, for $u > 0$

Fig. 8 The limiting path $G_t(0)$ in $\mathcal{M}_{0,n+3}$

Proposition 4.13. These curves have $k + 2$ and $k + 1$ components respectively. For $t \in (0, 1)$, $G_t(0)$ is a $\mathbb{P}^1$-chain with $k + 1$ components, $b_0 = k + 1$, specified by the following coordinate data:

- for $j \notin \{c, c + 1\}$, $b_j$ and $a_j^{(b_j)}$ are the same as for $C_\mu(0)$;
- $b_c = b_{c+1} = b$, and

\[
a_c^{(b)} = \begin{cases} 
1 - t & \text{if } 0 < t \leq \frac{1}{2}, \\
\frac{1}{2}e^{\pi i(t-\frac{1}{2})} & \text{if } \frac{1}{2} \leq t \leq 1,
\end{cases}
\]
\[
a_{c+1}^{(b)} = \begin{cases} 
1 - t & \text{if } 0 < t \leq \frac{1}{2}, \\
\frac{1}{2}e^{\pi i(t-\frac{1}{2})} & \text{if } \frac{1}{2} \leq t \leq 1,
\end{cases}
\]

See Fig. 8. Note that as $t \to 0$, $a_{c+1}$ approaches a node; hence at $t = 0$, a new component forms such that $a_c$ and $a_{c+1}$ are on different components.
Proposition 4.16  The map
\[ G : [0, 1] \times [0, \varepsilon] \to \mathcal{M}_{0,n+3} \]
\[ (t, u) \mapsto G_t(u) \]
is continuous.

Proof  Using the embedding (4.4), we must show that for each \((p, q, r, s)\), the map \((t, u) \mapsto \theta_{p,q,r,s}(G_t(u))\) is continuous. Since \(G_t(u)\) is a \(\mathbb{P}^1\)-chain for all \((t, u)\), it suffices to show this for \((p, q, r, s) = (0, a_{j_1}, \infty, a_{j_2})\), \(j_1 < j_2\). This is straightforward. For example, in the case \((p, q, r, s) = (0, a_c, \infty, a_{c+1})\), we find that \(\theta_{p,q,r,s}(G_t(u)) = \frac{a_{c+1}}{a_c}\), which is a ratio of two continuous functions, and the denominator is never zero. \(\Box\)

All of the curves \(G_t(u)\) are real, but not all with respect to the same real structure. Define \(\sigma \in \mathbb{S}_n\) as in (4.7), and define \(\sigma' \in \mathbb{S}_n\) analogously, with \(\mu'\) in place of \(\mu\). For \((t, u) \in [0, \frac{1}{2}] \times [0, \varepsilon]\) we have \(G_t(u) \in \mathcal{M}_{0,n+3}(\mathbb{R}^\sigma)\), and for \((t, u) \in [\frac{1}{2}, 1] \times [0, \varepsilon]\) we have \(G_t(u) \in \mathcal{M}_{0,n+3}(\mathbb{R}^\sigma)\). If \(t = \frac{1}{2}\), the curve is real with respect to both real structures, which is possible because \(G_{1/2}(u)\) has a double marked point, and is therefore not in \(\mathcal{M}_{0,n+3}\).

4.6 Paths in \(\mathcal{Y}^\lambda\)

We now lift the family of paths \(G_t(u)\) in \(\mathcal{M}_{0,n+3}\) to a family of paths in \(\mathcal{Y}^\lambda\). Projecting to \(\mathcal{X}^\lambda\) will give the paths we need for Lemmas 3.16 and 3.15, thereby allowing us to prove these statements.

Let \(\mu, \mu'\) be as in the previous section. Suppose \(T' \in \text{MN}(\lambda; \mu')\). Let \(T \in \text{Tab}(\lambda; \mu)\) be the tableau obtained by decrementing all entries \(b+1, \ldots, k+1\). Then \(T\) may or may not be a Murnaghan–Nakayama tableau: we have \(T \in \text{MN}(\lambda; \mu)\) if and only if \(b\) and \(b+1\) are adjacent in \(T'\). We consider these two cases separately.

First, suppose \(T \in \text{MN}(\lambda; \mu)\). Then \(T\) and \(T'\) are as in the statement of Lemma 3.15. In this case, \(G_t(u)\) lifts isomorphically to a family of curves in \(\mathcal{Y}^\lambda\), as shown in Fig. 9.

Lemma 4.17  (Path lifting, \(\square/\Box\) case) For sufficiently small \(\varepsilon > 0\), there exists a continuous map
\[ \Gamma : [0, 1] \times [0, \varepsilon] \to \mathcal{Y}^\lambda \]
\[ (t, u) \mapsto \Gamma_t(u), \]
with the following properties:

(a)  (Path lifting) For all \((t, u)\), \(\Psi(\Gamma_t(u)) = G_t(u)\).
The lift $\Gamma$ of $G_t(u)$ constructed in Lemma 4.17. The endpoints of $G_t(u)$ are $C_{\mu'}(u)$ (at $t = 0$) and $C_\mu(u)$ (at $t = 1$), which lift to $Y_{T'}(C_{\mu'}(u))$ and $Y_T(C_\mu(u))$ respectively. The fibre over $t = \frac{1}{2}$ maps to $Z_\Delta^\lambda$ or $Z_B^\lambda$ in $\mathcal{M}^\lambda(\mathbb{R})$, depending on shape(T)_{b}.

(b) (Connecting T to T') $\Gamma_0(u) \in Y_{T'}(C_{\mu'}(u))$ and $\Gamma_1(u) \in Y_T(C_\mu(u))$.

(c) (No ramification) For all $(t,u)$, $\Gamma_t(u)$ is a reduced point of the fibre $Y(G_t(u))$.

(d) (Compatibility with real structure) $\Gamma_t(u) \in \mathcal{M}^\lambda(\mathbb{R}^r)$ for $t \in [0, \frac{1}{2}]$, and $\Gamma_t(u) \in \mathcal{M}^\lambda(\mathbb{R}^r)$ for $t \in [\frac{1}{2}, 1]$.

(e) (Crossing $Z_\Delta^\lambda$ or $Z_B^\lambda$) For $u > 0$, $\pi(\Gamma_{1/2}(u)) \in Z_{\Delta}^\lambda$ if shape(T)_{b} is a horizontal domino, and $\pi(\Gamma_{1/2}(u)) \in Z_B^\lambda$ if shape(T)_{b} is a vertical domino.

Proof By Lemma 4.15, there is a unique reduced point $\Gamma_0(0) \in Y_{T'}(G_0(0))$ and by a similar argument, there is a unique reduced point $\Gamma_t(0) \in Y_T(G_0(0))$ for all $t \in (0, 1]$. This defines a continuous path $\Gamma_t(0)$, $t \in [0, 1]$. As in the proof of Lemma 4.15 we see that $\Gamma_t(0)$ is $\xi^{\sigma'}$-fixed for $0 \leq t \leq \frac{1}{2}$, and $\xi^{\sigma}$-fixed for $\frac{1}{2} \leq t \leq 1$. When $t = \frac{1}{2}$, the curve $G_{1/2}(0)$ has a double marked point $a_c = a_{c+1}$, $c = \overline{b}_1$. By Corollary 4.8, $\pi_c(\Gamma_{1/2}(0)) \in X_{\Delta} \cap Q_T^{(b)}$ or $\pi_c(\Gamma_{1/2}(0)) \in X_{\Delta} \cap Q_T^{(b)}$. But by Lemma 2.2, if shape(T)_{b} is a vertical domino then $X_{\Delta} \cap Q_T^{(b)}$ is empty, and if shape(T)_{b} is a vertical domino, then $X_{\Delta} \cap Q_T^{(b)}$ is empty. Thus we must have $\pi_c(\Gamma_{1/2}(0)) \in X_{\Delta}$ if shape(T)_{b} is a horizontal domino, and $\pi_c(\Gamma_{1/2}(0)) \in X_{\Delta}$ if shape(T)_{b} is a vertical domino.

Since all points of these fibres are reduced and the map $\Psi : \mathcal{M}^\lambda \to \mathcal{M}_{0,n+3}$ is finite, this extends uniquely to a continuous family $\Gamma : [0, 1] \times [0, \varepsilon] \rightarrow \mathcal{M}^\lambda$, satisfying (a) and (c), for some sufficiently small $\varepsilon$. By construction, (b) is also satisfied. For fixed $t \in [0, \frac{1}{2}]$, $G_t(u)$ is $\xi^\sigma$-fixed for all $u$. Thus, $\Gamma_t(u)$ can only cease to be $\xi^{\sigma'}$-fixed at a double point of the fibre $Y(G_t(u))$. Since property (c) ensures that there are no such points for $u \in [0, \varepsilon]$, $\Gamma_t(u)$ is $\xi^{\sigma'}$-fixed for all such $u$. A similar argument holds for $t \in [\frac{1}{2}, 1]$, which establishes (d). When $t = \frac{1}{2}$, $a_c = a_{c+1}$ is a double marked point of the curve $G_{1/2}(u)$; by
continuity, \( \pi_{c}(\Gamma_{1/2}(u)) \in X_{\o}(1) \) if shape \((T|_{b})\) is a horizontal domino, and \( \pi_{c}(\Gamma_{1/2}(u)) \in X_{\o}(1) \) if shape \((T|_{b})\) is a vertical domino. But on \( \mathcal{Y}^\lambda \), \( \pi_{c} \) and \( \pi \) are related by a transformation \( \phi \in B_{+} \subset \text{PGL}_{2}(\mathbb{C}) \), so the previous statement implies (e).

\( \square \)

\textbf{Proof of Lemma 3.15} Let \( \Gamma_{1}(u) \) be as in Lemma 4.17. Consider the path \( \gamma : [0, 1] \to \mathcal{P}_{n}^{\lambda}(\mathbb{C}) \), defined by \( \gamma_{t} := \pi(\Gamma_{1}(\varepsilon)) \), \( t \in [0, 1] \). First note that \( \gamma_{t} \) is in fact a path in \( \mathcal{P}_{n}^{\lambda}(\mathbb{R}) \), by property (d) and Proposition 4.10. Next note that if follows from property (b) and the definition of \( W_{T} \) (see Proof of Lemma 2.15), \( \pi(\Gamma_{0}(u)) \in W_{T} \); therefore \( \gamma_{0} = \mathbf{w}_{T} \). Similarly \( \gamma_{1} = \mathbf{w}_{T} \).

Let \( g_{t} := \text{Wr}(\gamma_{t}) \) the image of the path \( \gamma_{t} \) in \( \mathcal{P}_{n}^{\lambda}(\mathbb{R}) \). Note that we also have \( g_{t} = \text{pol}(G_{t}(\varepsilon)) \). By property (c) and Proposition 4.3, \( \gamma_{t} \) is a reduced point of \( \text{Wr}^{-1}(g_{t}) \). In particular, this means \( \gamma_{t} \notin R^{\lambda}(\mathbb{R}) \), for all \( t \in (0, 1) \).

By property (e), \( \gamma_{1/2} \in Z_{\o}^{\lambda}(\mathbb{R}) \) if shape \((T|_{b})\) is a horizontal domino, and \( \gamma_{1/2} \in Z_{\o}^{\lambda}(\mathbb{R}) \) if shape \((T|_{b})\) is a vertical domino. Since \( t = \frac{1}{2} \) is the only value of \( t \) for which \( g_{t} \) has a repeated root, there are no other crossings of either of these varieties. To see that the crossing at \( t = \frac{1}{2} \) is a simple crossing, note that since \( \gamma_{1/2} \notin R^{\lambda}(\mathbb{R}) \), \( \text{Wr} : \mathcal{P}_{n}^{\lambda}(\mathbb{R}) \to \mathcal{P}_{n}(\mathbb{R}) \) is a diffeomorphism in a neighbourhood of \( \gamma_{1/2} \). Thus \( \gamma_{t} \) has a simple crossing of \( Z_{\o}^{\lambda}(\mathbb{R}) \) or \( Z_{\o}^{\lambda}(\mathbb{R}) \) at \( t = \frac{1}{2} \) if and only if \( g_{t} \) has a a simple crossing of the discriminant variety \( \Delta_{n}(\mathbb{R}) \). Since \( g_{t} = \text{pol}(G_{t}(\varepsilon)) \) is given completely explicitly, it is straightforward to check that this is a simple crossing.

Now, suppose \( T \notin \text{MN}(\lambda; \mu) \). This means that \( b \) and \( b+1 \) are non-adjacent in \( T' \). Therefore switching the positions of these entries results in another tableau \( T'' \). Note that both \( T', T'' \in \text{MN}(\lambda; \mu') \), and are related as in the statement of Lemma 3.16. Moreover, this relation is symmetrical, and both are related to \( T \) in the same way.

Note that since \( T \) is not a Murnaghan–Nakayama tableau, by Lemma 4.15 the fibre \( Y_{T}(G_{1}(0)) = Y_{T}(C_{\mu}(0)) \) has no \( \xi^{\sigma} \)-fixed points. We will therefore not be able to lift all \( G_{t}(u) \) to \( \mathcal{Y}^{\lambda} \), in a way that satisfies properties (a)–(d) of Lemma 4.17, because we already know (d) must be false at \( (t, u) = (1, 0) \). Instead, we restrict the domain from \( [0, 1] \times [0, \varepsilon] \) to the subset over which (d) will hold. We then obtain two different lifts of \( G_{t}(u) \) over this domain, which are associated to the two tableaux \( T' \) and \( T'' \). See Fig. 10.

\textbf{Lemma 4.18} (Path lifting, \( \square \) case) For sufficiently small \( \varepsilon > 0 \), there exists a subset \( K \subset [0, 1] \times [0, \varepsilon] \) and continuous maps

\[
\begin{align*}
\Gamma' : K \to \mathcal{Y}^{\lambda} & \quad \Gamma'' : K \to \mathcal{Y}^{\lambda} \\
(t, u) \mapsto \Gamma'(u) & \quad (t, u) \mapsto \Gamma''(u),
\end{align*}
\]

with the following properties:

\( \square \) Springer
Fig. 10 The two lifts $\Gamma'\prime$, $\Gamma''\prime$ of $G_t(u)$ constructed in Lemma 4.18. The fibres over $t = \frac{1}{2}$ map to $Z^{\lambda}_B$ and $Z^{\lambda}_B$ in $\mathcal{X}^{\lambda}(\mathbb{R})$, while the fibre over $t = t_{\text{max}}(u)$ maps to $R^{\lambda}(\mathbb{R})$

(a) **(Path lifting)** For all $(t, u) \in K$, $\Psi(\Gamma_t'(u)) = \Psi(\Gamma_t''(u)) = G_t(u)$.

(b) **(Shape of $K$)** $K$ is of the form

$$K = \{(t, u) \mid u \in [0, \varepsilon], t \in [0, t_{\text{max}}(u)]\}.$$ 

where $t_{\text{max}} : [0, \varepsilon] \to (\frac{1}{2}, 1)$ is a function, in particular $t_{\text{max}} > \frac{1}{2}$.

(c) **(Starting from $T'$ and $T''$)** $\Gamma_0'(u) \in Y_{T'}(C_{\mu'}(u))$ and $\Gamma_0''(u) \in Y_{T''}(C_{\mu'}(u))$.

(d) **(No ramification until $t_{\text{max}}$)** For $(t, u) \in K$ with $t < t_{\text{max}}(u)$, $\Gamma_t'(u)$ and $\Gamma_t''(u)$ are reduced points of the fibre $Y(G_t(u))$.

(e) **(Paths join at $t_{\text{max}}$)** For $t = t_{\text{max}}(u)$, $\Gamma_t'(u) = \Gamma_t''(u)$ is a double point of the fibre $Y(G_t(u))$.

(f) **(Compatibility with real structure)** $\Gamma_t'(u) \in \overline{\mathcal{W}}^{\lambda}(\mathbb{R}^{\sigma'})$ for $t \in [0, \frac{1}{2}]$, and $\Gamma_t'(u) \in \overline{\mathcal{W}}^{\lambda}(\mathbb{R}^{\sigma})$ for $t \in [\frac{1}{2}, t_{\text{max}}(u)]$. The same holds for $\Gamma''$.

(g) **(Crossing $Z^{\lambda}_B$ and $Z^{\lambda}_B$)** Either $\pi(\Gamma_{1/2}'(u)) \in Z^{\lambda}_B$ and $\pi(\Gamma_{1/2}''(u)) \in Z^{\lambda}_B$ for all $u > 0$, or vice-versa.

**Proof** First, we compute $Y_T(G_t(0))$, for $t \in (0, 1]$. Proceeding as in Lemma 4.15, we find that $Y_T(G_t(0))$ consists of two distinct $\xi^{\sigma'}$-fixed points if $t \in (0, \frac{1}{2}]$. The limit fibre $\lim_{t \to 0} Y_T(G_t(0)) = Y_{T'}(C_{\mu'}(0)) \cup Y_{T''}(C_{\mu'}(0))$; again we have two distinct $\xi^{\sigma'}$-fixed points. For $t \in [\frac{1}{2}, 1]$, the discriminant in Lemma 2.8 for computing $Y_T^{(b)}(G_t(0))$ is equal to $L^{-2} - \sin^2 \left(\pi(t - \frac{1}{2})\right)$; hence $Y_T(G_t(0))$ consists of:

- two distinct $\xi^{\sigma}$-fixed points if $t < \frac{1}{2} + \frac{1}{\pi} \arcsin(L^{-1})$;
- a double $\xi^{\sigma}$-fixed point if $t = \frac{1}{2} + \frac{1}{\pi} \arcsin(L^{-1})$;
- two distinct $\xi^{\sigma}$-conjugate points if $t > \frac{1}{2} + \frac{1}{\pi} \arcsin(L^{-1})$.

\[\square\] Springer
where \( L \) is the distance between the entries \( b \) and \( b + 1 \) in \( T' \).

Let \( J \subset \mathcal{M}_{0,n+3} \) denote the image of the map \([0, 1] \rightarrow \mathcal{M}_{0,n+3}, t \mapsto G_t(0)\). Then \( \Psi^{-1}(J) \cap Q_T \) is the union of all \( Y_T(G_t(0)), t \in (0, 1) \), together with \( \lim_{t \to 0} Y_T(G_t(0)) = Y_T(C_{\mu}(0)) \cup Y_T(C_{\mu'}(0)) \). Since \( \Psi^{-1}(J) \cap Q_T \) is a closed subset of \( \Psi^{-1}(J) \), we can find an open subset \( \mathcal{U} \subset \overline{\mathcal{Y}} \), containing \( \Psi^{-1}(J) \cap Q_T \) such that \( \mathcal{U} \) is a component on \( \Psi(\Psi^{-1}(\mathcal{U})) \).

Assuming \( \varepsilon > 0 \) is sufficiently small, \( Y(G_t(u)) \cap \mathcal{U} \) is then a finite scheme of length 2, for all \( (t, u) \in [0, 1] \times [0, \varepsilon] \). Hence this is either two distinct points, or a (non-reduced) double point. By the same argument as in Lemma 4.17, \( Y(G_t(u)) \cap \mathcal{U} \) consists of two distinct points in \( \overline{\mathcal{Y}}(\mathbb{R}^{\sigma'}) \) for \( t \in [0, \frac{1}{2}] \), \( u > 0 \). Moreover, for \( t = \frac{1}{2} \), \( u > 0 \), the two points of \( Y(G_t(u)) \cap \mathcal{U} \) are also in \( \overline{\mathcal{Y}}(\mathbb{R}^{\sigma'}) \), and for \( t = 1 \) they are not in \( \overline{\mathcal{Y}}(\mathbb{R}^{\sigma'}) \). This implies that for all \( u \) there exists a \( t \in (\frac{1}{2}, 1) \) such that \( Y(G_t(u)) \cap \mathcal{U} \) is a double point. We put

\[
t_{\text{max}}(u) := \min\{ t \in [0, 1] \mid Y(G_t(u)) \cap \mathcal{U} \text{ is a double point} \},
\]

and take (b) to be the definition of \( K \). Note that \( K \) is simply connected, and therefore \( \Psi : \Psi^{-1}(K) \cap \mathcal{U} \rightarrow K \) is a topologically trivial two-to-one covering map away from \( t = t_{\text{max}}(u) \).

We can therefore define \( \Gamma' : K \rightarrow \overline{\mathcal{Y}} \) to be the unique map such that \( \Gamma'_0(0) \) is the unique point in \( Y_T(C_{\mu'}(0)) \), and \( \Gamma'_1(u) \in P(G_t(u)) \cap \mathcal{U} \) for all \( (t, u) \in K \). Similarly we define \( \Gamma'' : K \rightarrow \overline{\mathcal{Y}} \), with \( T'' \) in place of \( T' \). Properties (a), (c), (d), (e) are immediate, and the proof of (f) is identical to the proof of (d) in Lemma 4.17.

When \( t = \frac{1}{2} \), the curve \( G_{1/2}(0) \) has a double marked point \( a_c = a_{c+1}, c = \mu_b \). By Corollary 4.8, either \( \pi_c(\Gamma'_{1/2}(0)) \in X_{\mathcal{M}}(1) \cap Q_T^{(b)} \) or \( \pi_c(\Gamma''_{1/2}(0)) \in X_{\mathcal{M}}(1) \cap Q_T^{(b)} \), and the same is true for \( \Gamma'' \). By Lemma 2.2, both \( X_{\mathcal{M}}(1) \cap Q_T^{(b)} \) and \( X_{\mathcal{M}}(1) \cap Q_T^{(b)} \) consist of a single point. Recall that \( Y_T(G_{1/2}(0)) \) is identified with \( \prod_{i=1}^k Y_{\sigma_i}^{(i)}(G_{1/2}(0)) \). By Lemmas 2.7 and 2.8, every term in this product consists of a single point, except \( Y_T^{(b)}(G_{1/2}(0)) \) which has two points. \( \Gamma'_{1/2}(0) \) and \( \Gamma''_{1/2}(0) \) are distinct points of \( Y_T(G_{1/2}(0)) \), and the only coordinate on which they can differ is \( \Gamma'_{1/2}(0)^{(b)} \neq \Gamma''_{1/2}(0)^{(b)} \). Equivalently \( \pi_c(\Gamma'_{1/2}(0)) \neq \pi_c(\Gamma''_{1/2}(0)) \). It follows that we have either \( \pi_{\mu_b}(\Gamma'_{1/2}(0)) \in X_{\mathcal{M}}(1) \cap Q_T^{(b)} \) and \( \pi_{\mu_b}(\Gamma''_{1/2}(0)) \in X_{\mathcal{M}}(1) \cap Q_T^{(b)} \), or the other way around. Arguing now as in the proof of Lemma 4.17, we deduce (g).

\[\square\]

\textbf{Remark 4.19} With a more thorough analysis of \( Y_T(G_t(0)), t \in [0, \frac{1}{2}] \), property (g) in Lemma 4.17 can be replaced by the following stronger statement. If \( b \) is left of \( b + 1 \) in \( T' \) then \( \pi(\Gamma'_{1/2}(u)) \in Z_\mathcal{C}^1 \) and \( \pi(\Gamma''_{1/2}(u)) \in Z_\mathcal{C}^1 \). If \( b \) is right of \( b + 1 \), the identification is the other way around. We omit the proof,
since we do not actually need this fact here. Note that in each case, the two boxes $b, b+1$ rectify to the corresponding domino shape (compare with [24, Theorem 6.4] and [30, Theorem 4.4]).

**Proof of Lemma 3.16** Let $K, t_{\text{max}}, \Gamma_t'(u)$ and $\Gamma_t''(u)$ be as in Lemma 4.18. We define the path $\gamma : [0, 1] \to \mathcal{X}(\mathbb{C})$ by joining together the paths $\pi(\Gamma_t'(u))$ and $\pi(\Gamma_t''(u))$:

$$
\gamma_t := \begin{cases} 
\pi(\Gamma_{2t}(u)) & \text{for } t \in [0, \frac{1}{4}] \\
\pi(\Gamma'_{1/2 + (t_{\text{max}} \epsilon - 1/2)(4t-1)}(u)) & \text{for } t \in [\frac{1}{4}, \frac{1}{2}] \\
\pi(\Gamma''_{1/2 + (t_{\text{max}} \epsilon - 1/2)(3-4t)}(u)) & \text{for } t \in [\frac{1}{2}, \frac{3}{4}] \\
\pi(\Gamma_{2-2t}(u)) & \text{for } t \in [\frac{3}{4}, 1].
\end{cases}
$$

Let $g_t = \text{Wr}(\gamma_t)$; note that $g_t = g_{1-t}$.

Many pieces of the argument are essentially the same as in the proof of Lemma 3.15. By property (f), $\gamma_t \in \mathcal{Z}_\lambda(\mathbb{R})$ for all $t \in [0, 1]$. By property (c), $\gamma_0 = \pi(\Gamma_0'(u)) = w_T$ and $\gamma_1 = \pi(\Gamma_0''(u)) = w_T'$. By property (g), one of $\gamma_{1/4} = \pi(\Gamma_{1/2}(u))$ and $\gamma_{3/4} = \pi(\Gamma_{1/2}(u))$ is in $Z_\mathcal{A}(\mathbb{R})$ and the other is in $Z_\mathcal{B}(\mathbb{R})$, and these are the only points $\gamma_t$ in either of these varieties. These are simple crossings of $Z_\mathcal{A}(\mathbb{R})$ and $Z_\mathcal{B}(\mathbb{R})$, by the same reasoning as in the proof of Lemma 3.15.

We now consider crossings of $R^\lambda(\mathbb{R})$. By property (e), $\gamma_{1/2} = \pi(\Gamma_{1/2}(u)) = \pi(\Gamma_{1/2}(u))$ is a double point of $\text{Wr}^{-1}(g_{1/2})$; thus $\gamma_{1/2} \notin R^\lambda(\mathbb{R})$. This is certainly a crossing, because $\gamma_{1/2-\epsilon}$ and $\gamma_{1/2+\epsilon}$ are two different points of the fibre over $g_{1/2-\epsilon} = g_{1/2+\epsilon}$, and therefore in different components of some neighbourhood of $\text{Wr}^{-1}(g_{1/2}) \setminus R^\lambda(\mathbb{R})$. By Lemma 3.14, $\gamma_{1/2}$ is a smooth point of $R^\lambda(\mathbb{R})$, and therefore this is a simple crossing. By property (d), $\gamma_t \notin R^\lambda$ for $t \neq \frac{1}{2}$. $\square$

5 Upper and lower bounds

5.1 Bounds from Theorem 1.3

For $g \in \mathcal{P}_n(\mathbb{R})$, let $N_g$ denote the number of real points in the fibre $\text{Wr}^{-1}(g)$, counted with algebraic multiplicity. One consequence of Theorem 1.3 is that we obtain bounds on $N_g$. When the lower bound is non-zero, this gives a proof of the existence of real solutions to the equation $\text{Wr}(f_1, \ldots, f_d) = g$.

**Corollary 1.4** If $g \in \mathcal{P}_n(\mu)$, then

$$|\chi^\lambda(\mu)| \leq N_g \leq f^\lambda.$$
Proof If \( g \in \mathcal{P}_n(\mu) \) and \( g \) is a regular value of the Wronski map, the the multiplicities of the points in \( \text{Wr}^{-1}(g) \) are all 1, and the topological degree of the map \( \text{Wr} : \mathcal{R}^\lambda(\mu) \to \mathcal{P}_n(\mu) \) is a signed count of the points in \( \text{Wr}^{-1}(g) \). Thus in this case, the lower bound \( N_g \geq |\chi^\lambda(\mu)| \) holds. Since \( N_g \) is an upper semi-continuous function of \( g \), the result remains true if we pass to the closure.

For the upper bound, note that \( f^\lambda \) is the number of complex points in any fibre of the map \( \text{Wr} : \mathcal{R}^\lambda(\mathbb{C}) \to \mathcal{P}_n(\mathbb{C}) \), and the real points in the fibre are subset of the complex points.

If \( g \in \mathcal{P}_n(\mu) \) for more than one \( \mu \) (which occurs when \( g \) has repeated real roots), then we get more than one lower bound. In general, there does not appear to be any simple rule as to which \( \mu \) will give the best lower bound. Of course, if all the roots of \( g \) are all real, the best lower bound comes from \( \mu = 1^n \), and \( N_g = f^\lambda \), regardless of whether \( g \) has repeated roots. In particular, the upper bound in Corollary 1.4 is tight for all \( \mu \), since it is achieved when all roots of \( g \) are real, and such \( g \) exist in \( \mathcal{P}_n(\mu) \). Moreover, we can find such \( g \) such that \( \text{Wr}^{-1}(g) \) is reduced, which implies that the upper bound can also be achieved by polynomials properly in \( \mathcal{P}_n(\mu) \), by starting with a polynomial with only real roots in the closure, and perturbing.

In some cases, we can also see that the lower bound is tight.

**Theorem 5.1** Let \( \mu = (\mu_1, \ldots, \mu_k) \) be a composition of \( n \), with \( \mu_i \in \{1, 2\} \). If all tableaux in \( \text{MN}(\lambda; \mu) \) have the same sign, then the lower bound in Corollary 1.4 is tight. That is, there exists a polynomial \( g \in \mathcal{P}_n(\mu) \) such that \( N_g = |\chi^\lambda(\mu)| \).

Proof Consider \( h_\mu \in \mathcal{P}_n(\mu) \). By Corollary 2.16, \( N_{h_\mu} = \# \text{MN}(\lambda; \mu) \). But if all tableaux in \( \text{MN}(\lambda; \mu) \) have the same sign, then \( \# \text{MN}(\lambda; \mu) = |\chi^\lambda(\mu)| \).

Note that for a given partition, there are many compositions with that associated partition. If any one of these compositions satisfies the hypothesis of Theorem 5.1, then we conclude the lower bound is tight.

**Example 5.2** Here a few noteworthy cases where all tableaux in \( \text{MN}(\lambda; \mu) \) have the same sign, and hence Theorem 5.1 tells us that the lower bound is tight.

(i) If \( \lambda \vdash n \) is any partition, and \( \mu = (1^n) \). This is the all real roots case, where the lower bound is exact.

(ii) If \( |\lambda| \) is even, and \( \mu = (2^n) \).

(iii) If \( |\lambda| \) is odd, and \( \mu = (1, 2^n) \). Cases (ii) and (iii) are where we have the maximal number of complex roots.

(iv) If \( \lambda = (\lambda_1, \lambda_2) \vdash n \) is any 2-part partition, and \( \mu = (1, 2^l, 1^{n-2l-1}) \), \( 0 \leq l < n/2 \). This, together with (ii), shows the lower bound is always tight if \( \lambda \) has two parts.
(v) If $\lambda = m^d$ is a rectangle with $dm$ even, and $\mu = (1, 2^{dm/2-1}, 1)$.

(vi) If $\lambda = (d, d-1, \ldots, 2, 1)$ is a staircase, and $\mu = (\mu_1, \ldots, \mu_k)$ is any composition such that $\mu_k = 2$. This, together with (i), shows that the lower bound is always tight if $\lambda$ is a staircase.

(vii) If $\lambda = (\lambda_1, \ldots, \lambda_d) \vdash n$ is any partition such that $\lambda_i - \lambda_{i+1}$ is odd for all $i = 1, \ldots, d-1$, and $\mu = (1^{n_1}, 2^{n_2})$, where $n_1 + 2n_2 = n$. This generalizes (vi): the lower bound is tight for all such partitions $\lambda$.

In case (vi), $\chi_\lambda(\mu) = MN_\lambda(\lambda; \mu) = 0$, and Theorem 5.1 is asserting that the Wronski map $\text{Wr} : \mathcal{X}^{-\lambda}(\mu) \to \mathcal{P}_n(\mu)$ is not surjective. This also happens in (v), in the case where $m$ and $d$ are both even, as was first shown in [5].

5.2 Eremenko and Gabrielov’s lower bound

Eremenko and Gabrielov computed degrees of real Wronski maps in [4]. Their result is stated for the projective real Wronski map $\text{Wr} : \text{Gr}(d, \mathbb{R}^{d+m}) \to \mathbb{P}^{dm}(\mathbb{R})$, but as is pointed out in [33], the computation can be done on any Schubert cell and we state the result as such here. We include a concise proof, based on Lemma 3.16.

Definition 5.3 Let $T \in \text{SYT}(\lambda)$. An inversion in $T$ is a pair of cells $(i, j), (i', j') \in \lambda$ such that $i < i'$ and $T(i, j) > T(i', j')$. Let $\text{inv}(T)$ denote the number of inversions in $T$.

Theorem 5.4 (Eremenko–Gabrielov) With respect to the ambient orientation, the topological degree of the Wronski map $\text{Wr} : \mathcal{X}^{-\lambda}(\mu) \to \mathcal{P}_n(\mu)$ is

$$I_\lambda := \sum_{T \in \text{SYT}(\lambda)} (-1)^{\text{inv}(T)}.$$ 

Proof We claim that for every tableau $T \in \text{SYT}(\lambda)$, $\text{asgn}(w_T) = (-1)^{\text{inv}(T)}$. If $T = T_0$, this is certainly true. Suppose $T, T' \in \text{SYT}(\lambda)$ are two tableaux related as in Lemma 3.16. There is path joining $w_T$ to $w_{T'}$, along which the ambient sign changes exactly once, at the simple crossing of $R^\lambda(\mathbb{R})$. Therefore $\text{asgn}(w_T) = -\text{asgn}(w_{T'})$. We also have $\text{inv}(T) = \text{inv}(T') \pm 1$, and the claim follows. Therefore, the topological degree of the Wronski map is

$$\sum_{x \in \text{Wr}^{-1}(n, \mu)} \text{asgn}(x) = \sum_{T \in \text{SYT}(\lambda)} \text{asgn}(w_T) = I_\lambda.$$

This also yields a lower bound for the number of real points in the fibre of the Wronski map.
Corollary 5.5 (Eremenko–Gabrielov) For every \( g \in \mathcal{P}_n(\mathbb{R}) \), we have

\[
|I_{\lambda}| \leq N_g.
\]

In the case of a rectangle, \( \lambda = m^d \), the Schubert cell \( \mathcal{P}^{-\lambda}(\mathbb{R}) \) is an open dense subset of \( \text{Gr}(d, \mathbb{R}^{d+m}) \). If \( m + d \) is odd, then both \( \text{Gr}(d, \mathbb{R}^{d+m}) \) and \( \mathbb{P}^{dm}(\mathbb{R}) \) are non-orientable. Eremenko and Gabrielov showed that the real projective Wronski map lifts to a map between the oriented double covers of these spaces, and hence has a well-defined degree up to sign, which is \( \pm I_{\lambda} \). In this case, \( I_{\lambda} \neq 0 \) [36]. However, if \( m + d \) is even, then both spaces are orientable, and the degree of the real projective Wronski map cannot be defined. Example 5.2 also shows that the real Wronski map is not surjective in this case.

Example 5.6 Let \( \lambda = 3^5 \). Then \( |\chi_{\lambda}(\mu)| \geq 6 \) for every partition \( \mu \) of the form \( 2^n1^{n1} \). (The minimum value of \( |\chi_{\lambda}(\mu)| \) occurs for \( \mu = 2^41^7 \).) Thus, \( N_g \geq 6 \) for every \( g \in \mathcal{P}_{15}(\mathbb{R}) \). As noted above, \( I_{\lambda} = 0 \) in this case, which shows that the Eremenko–Gabrielov lower bound is not tight.

Example 5.7 Let \( \lambda = 3^6 \). Then \( I_{\lambda} = 12 \), which means \( N_g \geq 12 \) for all \( g \in \mathcal{P}_{18}(\mathbb{R}) \). On the other hand, if \( \mu = 2^61^6 \) or \( \mu = 2^71^4 \), then \( \chi_{\lambda}(\mu) = 0 \), so the bound from Corollary 1.4 is not tight.

5.3 Mukhin and Tarasov’s lower bound

As mentioned in the introduction, the lower bound in Corollary 1.4 is a special case of a more general inequality. Let \( a_1, \ldots, a_k \) be distinct real numbers, and let \( b_1, \bar{b}_1, \ldots, b_l, \bar{b}_l \) be distinct complex numbers. Consider a 0-dimensional intersection of the form

\[
X_{\alpha^1}(a_1) \cap \cdots \cap X_{\alpha^k}(a_k) \cap X_{\beta^1}(b_1) \cap \cdots \cap X_{\beta^l}(b_l) \\
\cap X_{\beta^1}(\bar{b}_1) \cap \cdots \cap X_{\beta^l}(\bar{b}_l)
\]

inside the Grassmannian \( \text{Gr}(d, \mathbb{C}_{d+m-1}[z]) \). (Here, \( \alpha^1, \ldots, \alpha^k, \beta^1, \ldots, \beta^l \) are partitions.) In [19], Mukhin and Tarasov gave a lower bound for the number of real points of such an intersection, which depends only on the discrete data \( (k, l, \alpha^1, \ldots, \alpha^k, \beta^1, \ldots, \beta^l) \).
In the case where, $\alpha_1 = \lambda \lor, \alpha_2 = \cdots = \alpha_k = \beta_1 = \cdots = \beta_l = 0$, and $a_1 = \infty$, the intersection (5.1) is precisely $W_{r-1}(g)$, where

$$g(z) = \prod_{i=2}^{k}(z + a_i) \cdot \prod_{j=1}^{l}(z + b_j)(z + \overline{b_j}).$$

The lower bound in Corollary 1.4 coincides with the Mukhin–Tarasov lower bound in this special case. (This equivalence is not completely obvious the way things are stated in [19], but it is not hard to show using standard results in symmetric function theory.) The two proofs, however, are completely different. Mukhin and Tarasov obtain their inequality using the fact that the number of real eigenvalues of an operator that is self-adjoint with respect to an indefinite Hermitian form is at least the absolute value of the signature of the form. The machinery in [21] identifies the points of intersection (5.1) with one-dimensional eigenspaces of an algebra commuting self-adjoint operators; the point is real if and only if the associated eigenvalues are real. Hence, they obtain a lower bound by computing the signature of the associated form. Since the signature of the form depends only on the discrete data, it is an invariant for the problem. We do not know why this invariant coincides with the topological degree of the restricted Wronski map with the character orientation. It is fairly natural to conjecture that the sign of a real intersection point is equal to the signature of the Hermitian form restricted to the associated eigenspace.

In the general case, by Lemma 2.1, intersection (5.1) is contained in the fibre $W_{r-1}(g)$

$$g(z) = \prod_{i=1}^{k}(z + a_i)^{\lceil \alpha_i \rceil} \cdot \prod_{j=1}^{l}((z + b_j)(z + \overline{b_j}))^{\lceil \beta_j \rceil},$$

and we might try to use this to count the points of (5.1) with signs. Unfortunately, these signs will not always be defined, either because $g \notin P_n(\mu)$ for any $\mu$, or because $g$ is a critical value of the Wronski map. It is nevertheless possible to give a topological proof of the Mukhin–Tarasov lower bound in its full generality, using Theorem 1.3 in combination with results from [24, 26]. Although the argument is too long to include here in full detail, we give a brief sketch of how this can be done.

Let $\lambda = m^d$, so $P^\lambda(R)$ is an open dense subset of $Gr(d, \mathbb{R}^{d+m})$. The real points of the intersection (5.1) will be in $P^\lambda(R)$, and as noted above, specifically contained in the fibre $W_{r-1}(g)$. Perturb $g \in P_n(R)$ to a nearby polynomial $g'$ which is a regular value of the Wronski map with distinct roots, in such a way that the number of real roots does not change. For any point $x$ in the intersection (5.1), we can consider all points $x' \in W_{r-1}(g)$ that
perturbations of $x$. By properties of $g'$ each such $x'$ has a well-defined sign. We define the **weight** of $x$ to be the average of the signs of all $x' \in \text{Wr}^{-1}(g')$ that are perturbations of $x$. One can show the sum of the weights of the points in the intersection (5.1) is an invariant (i.e. it depends only on the discrete data), by relating to the topological degree of a restriction of the Wronski map. Moreover, since the weight of $x$ is a rational number in the interval $[-1, 1]$, this invariant gives a lower bound on the number of points in the intersection.

It remains to compute the invariant. It suffices to compute it for a single tuple of parameters $(a_1, \ldots, a_k, b_1, \ldots, b_l)$, and a specific perturbation $g$. We take $a_i = u^i$, and $b_j = iu^{k+j}$, where (at the end of the day) $u$ will be evaluated at some small parameter $\varepsilon > 0$, and we perturb the roots of $g$ in such a way that all roots remain either real or pure imaginary. Using [26, Theorem 3.15] and degeneration arguments similar to Sect. 4, we can label the points of the intersection by certain equivalence classes of tableaux, and we can say which of these points are real. The tableaux themselves label the perturbed points, and the equivalence classes tell us which points of $\text{Wr}^{-1}(g')$ are perturbations of of the same point in $\text{Wr}^{-1}(g)$. As in Sect. 3, we can compute the signs of the signs of the points in $\text{Wr}^{-1}(g')$ from the associated tableaux, and hence compute the weights of the points in (5.1). (It turns out that, for this fibre, all perturbations of the same point have the same sign, so the weights are in fact $\pm 1$.) This gives a combinatorial formula for the sum of the weights of the points in the intersection.

We note that when there are no complex points ($l = 0$), this argument reproduces the proof of the Littlewood-Richardson rule in [24]. Finally, using known results from symmetric function theory, one can identify this combinatorial formula as equivalent to the formula given by Mukhin and Tarasov.

We note that proof of [26, Theorem 3.15] referenced above uses Theorem 1.1 in an essential way. As such, to prove the stronger theorem independently of Mukhin–Tarasov–Varchenko’s work, one needs to establish the weaker Theorem 1.3 first.

### 6 Concluding remarks

#### 6.1 Generalization to Richardson varieties

With a few small changes, Theorem 1.3 generalizes to the open Richardson variety.

Let $\lambda / \lambda'$ be a skew shape such that $|\lambda / \lambda'| = n$. The **open Richardson variety** is the intersection of Schubert cells $\mathcal{P}^{\lambda / \lambda'} := X_{\lambda'}^\circ(\infty) \cap X_{\lambda'}^\circ(0)$. This is a smooth affine variety. Let $\mathcal{P}_n^\circ := \{ g \in \mathcal{P}_n \mid g(0) \neq 0 \}$. The Wronski map
induces a finite proper map

\[ \text{Wr}_{\lambda'/\lambda'} : \mathcal{R}^{\lambda'/\lambda'} \to \mathcal{P}_n^0 \]

\[ x \mapsto z^{-|\lambda'|} \text{ Wr}(x, z). \]

For a partition \( \mu = 2^{n_2}1^{n_1} \vdash n \), we define \( \mathcal{P}_n^0(\mu) := \mathcal{P}_n(\mu) \cap \mathcal{P}_n^0(\mathbb{R}) \), and \( \mathcal{X}^{\lambda'/\lambda'}(\mu) := \text{Wr}^{-1}_{\lambda'/\lambda'}(\mathcal{P}_n^0(\mu)) \). As before, \( \text{Wr}_{\lambda'/\lambda'} \) restricts to a proper map \( \text{Wr}_{\lambda'/\lambda'} : \mathcal{X}^{\lambda'/\lambda'}(\mu) \to \mathcal{P}_n^0(\mu) \) for each \( \mu \).

To proceed further, we use the following technical fact about \( \mathcal{X}^{\lambda'/\lambda'} \) (see [16]).

**Lemma 6.1** The divisor class group of \( \mathcal{X}^{\lambda'/\lambda'}(\mathbb{R}) \) is trivial.

We can now define a character orientation function on \( \mathcal{X}^{\lambda'/\lambda'} \), with the following modifications to the discussion in Sect. 3. We work over \( T^1 = \mathbb{A}^1 \setminus \{0\} \) instead of \( \mathbb{A}^1 \). For a partition \( \kappa \), let \( Y^{\lambda'/\lambda'}(a) \subset \mathcal{X}^{\lambda'/\lambda'} \times T^1 \) be the total space of the family of \( X_{\kappa(a)} \cap \mathcal{X}^{\lambda'/\lambda'}, a \in T^1 \). Then define \( Z^{\lambda'/\lambda'}(a) \subset \mathcal{X}^{\lambda'/\lambda'} \) to the image of the projection onto the first factor. For \( |\kappa| = 2 \), \( Z^{\lambda'/\lambda'}(a) \subset \mathcal{X}^{\lambda'/\lambda'} \) is a hypersurface. By Lemma 6.1, it is the zero locus of a single real polynomial function \( \Phi^{\lambda'/\lambda'}_B \). From here, we can proceed exactly as in Sect. 3, using \( \Phi^{\lambda'/\lambda'}_B \) to define the character orientation of \( \mathcal{X}^{\lambda'/\lambda'}(\mu) \).

One can then compute the topological degree of \( \text{Wr}_{\lambda'/\lambda'} : \mathcal{X}^{\lambda'/\lambda'}(\mu) \to \mathcal{P}_n^0(\mu) \) with respect to the character orientation. The entire proof of Theorem 1.3 works almost verbatim if we simply replace \( \lambda \) by \( \lambda'/\lambda \) throughout. The biggest modification is in Sect. 4.1, where we need to replace the family (4.1) by

\[ \overline{Y}^{\lambda'/\lambda'} := \tilde{X}_\mathcal{D}(a_1) \cap \cdots \cap \tilde{X}_\mathcal{D}(a_n) \cap \tilde{X}_\lambda(\infty) \cap \tilde{X}_{\lambda'}(0). \]

In this way, we obtain the following theorem.

**Theorem 6.2** With the character orientation, the topological degree of the map \( \text{Wr}_{\lambda'/\lambda'} : \mathcal{X}^{\lambda'/\lambda'}(\mu) \to \mathcal{P}_n^0(\mu) \) is equal to \( \chi^{\lambda'/\lambda'}(\mu) \).

Here \( \chi^{\lambda'/\lambda} \) denotes the skew symmetric group character (see [8]). We therefore also obtain analogues of Corollaries 1.4 and 1.5.

**Corollary 6.3** For \( g \in \mathcal{P}_n^0(\mathbb{R}) \), let \( N_g \) be the number of real points in the fibre \( \text{Wr}^{-1}_{\lambda'/\lambda'}(g) \), counted with algebraic multiplicity. If \( g \in \mathcal{P}_n^0(\mu) \), then

\[ |\chi^{\lambda'/\lambda'}(\mu)| \leq N_g \leq f^{\lambda'/\lambda'}. \]
Corollary 6.4 \( \text{Wr}_{\lambda/\lambda'} : \mathcal{X}^{\lambda/\lambda'}(1^n) \to \mathcal{P}_n^\circ(1^n) \) is a topologically trivial covering map of degree \( f/\lambda/\lambda' \).

Corollary 6.4 can also be deduced without Theorem 6.2, as we explain next, in Sect. 6.2.

Remark 6.5 It is important that \( Z_{\mathcal{B}}^{\lambda} \subset \mathcal{X}^{\lambda/\lambda'} \) is a principal divisor over \( \mathbb{R} \). To see why, recall that our goal is to construct an orientation of \( X(\mathbb{R}) \setminus V(\mathbb{R}) \), where \( X \) is a smooth affine variety over \( \mathbb{R} \) and \( V \subset X \) is a hypersurface, such that the orientation reverses along \( V(\mathbb{R}) \). If \( X(\mathbb{R}) \) is orientable, this implies the real line bundle associated to \( V(\mathbb{R}) \) must also be orientable, which, in our case, is guaranteed by Lemma 6.1. As an example of what could go wrong, consider the circle \( X = \text{Spec } \mathbb{F}[x, y]/(x^2 + y^2 - 1) \), and let \( V \) be the point \((1, 0)\). If \( \mathbb{F} = \mathbb{R} \), \( V(\mathbb{R}) \) is not a principal divisor on \( X(\mathbb{R}) \), and clearly we cannot orient \( X(\mathbb{R}) \setminus V(\mathbb{R}) \) in such a way that the orientation reverses along \( V(\mathbb{R}) \). Nevertheless \( V(\mathbb{R}) \) is locally principal, and its complexification \( V(\mathbb{C}) \) is a principal divisor of \( X(\mathbb{C}) \), so neither of these criteria is sufficient.

6.2 Transversality

Since some of the known applications of the Shapiro–Shapiro conjecture rely on some form of transversality or reducedness property, such as Corollary 1.5, we give a brief account of those that are known to follow from Theorem 1.1. The strongest known transversality theorem for Shapiro-type Schubert intersections is Mukhin, Tarasov and Varchenko’s theorem in [21], which does not appear to follow easily from Theorem 1.1. However, we do get some important special cases. These are deduced via the following lemma (see [32, Theorem 13.2]).

Lemma 6.6 Let \( \psi : X \to Y \) be a finite morphism of smooth varieties defined over \( \mathbb{R} \). If \( \mathcal{U} \subset Y(\mathbb{R}) \) is an analytic open subset such that every point of \( \psi^{-1}(\mathcal{U}) \) is real, then \( \psi \) is unramified over \( \mathcal{U} \).

For example, taking \( \psi \) to be the Wronski map \( \text{Wr} : \mathcal{X}^\lambda \to \mathcal{P}_n \), and \( \mathcal{U} = \mathcal{P}_n^\circ(1^n) \) yields Corollary 1.5.

We can also take \( \psi \) to be the map \( \text{Wr}_{\lambda/\lambda'} : \mathcal{X}^{\lambda/\lambda'} \to \mathcal{P}_n^\circ \), and \( \mathcal{U} = \mathcal{P}_n^\circ(1^n) \), which gives Corollary 6.4. Restated in terms of Schubert intersections, this says the following.

Corollary 6.7 Let \( \alpha, \beta \) be partitions and let \( a_1, a_2, \ldots, a_{n+2} \in \mathbb{P}^1(\mathbb{R}) \) be distinct, where \( |\alpha| + |\beta| + n = dm \). Then the intersection

\[
X_\square(a_1) \cap \cdots \cap X_\square(a_n) \cap X_{\alpha}(a_{n+1}) \cap X_{\beta}(a_{n+2}) \quad \text{in } \text{Gr}(d, d + m)
\]

is transverse.
For the orthogonal Grassmannian $\text{OG}(n, 2n+1)$, there is a finite map to projective space which is an analogue of the Wronski map. In [25], the second author showed that Theorem 1.1 implies an analogous reality theorem for $\text{OG}(n, 2n+1)$, and thus Lemma 6.6 can also be applied to $\text{OG}(n, 2n+1)$. In particular, Theorem 1.1 implies the analogue of Corollary 6.7 for $\text{OG}(n, 2n+1)$. It is interesting to note that we do not have an obvious analogue of Theorem 1.3 for $\text{OG}(n, 2n+1)$, as there is only a single codimension 2 Schubert variety in $\text{OG}(n, 2n+1)$.

6.3 Open questions

Quite a few generalizations of Theorem 1.1 have been conjectured, supported by large quantities of computational evidence. We will not give a complete overview of these here, but instead refer the reader to the discussion in [32, Ch. 13 & 14]. Although it is not immediately obvious how to generalize Theorem 1.3 to these other settings, we hope that it will lead to new ideas toward solving these problems.

Since a large part of the proof of Theorem 1.3 involved the moduli space of stable curves, it is natural to wonder if there is an a statement analogous to Theorem 1.3 involving the finite family $\Psi : \mathcal{M}^{\lambda} \to \mathcal{M}_{0,n+1}$, instead of the Wronski map. The semialgebraic sets $\mathcal{X}^{\lambda}(\mu)$ and $\mathcal{P}_n(\mu)$ would be replaced by actual real loci $\mathcal{M}^{\lambda}(\mathbb{R}^\sigma)$ and $\mathcal{M}_{0,n+1}(\mathbb{R}^\sigma)$ respectively. One would then expect the character orientation function to be replaced by a section of the anticanonical bundle, which naturally defines an orientation. This would be very nice, but there are some difficulties with this idea, not the least of which is the fact that $\mathcal{M}_{0,n+1}(\mathbb{R}^\sigma)$ is not always orientable.

Another possible approach to proving Theorem 1.3 would be to determine the character orientation function $\Phi^\lambda_B$ explicitly. With this, it should be possible to compute the signs of the points $w_T$ directly, i.e. without using Lemmas 3.15 and 3.16. Conceivably, this could yield a more concise proof. It would also be interesting to see if the ideas in [19] can be used to obtain an alternate proof of Theorem 1.3.

Finally, the obvious question: is there a similar interpretation of $\chi^\lambda(\mu)$ if $\mu$ is an arbitrary partition of $n$? We propose that the general answer should take the following form. If $\sigma \in \mathfrak{S}_n$ is a permutation of cycle type $\mu$, and $C \in \mathcal{M}_{0,n+1}^{\sigma}$ is a $\sigma$-fixed curve, then

$$\chi^\lambda(\mu) = \sum_{y \in \Psi^{-1}(C)^\sigma} \text{sgn}(y)$$

where $\Psi^{-1}(C)^\sigma$ denotes the $\sigma$-fixed subscheme of the fibre of the map $\Psi : \mathcal{M}^{\lambda} \to \mathcal{M}_{0,n+1}$, and $\text{sgn}(y) \in \{ \pm 1 \}$ is some geometrically meaningful sign.
assigned to \( y \). If \( C \) is a \( \mathbb{P}^1 \)-chain without double marked points, it is not hard to show that \( \# \Psi^{-1}(C)\sigma = \# MN(\lambda; \mu') \) for some composition \( \mu' \) with associated partition \( \mu \), so in this case, there is some way to assign signs so that this equation is true. However, not all \( \sigma \)-fixed curves are of this form, and without some sort of geometric interpretation for the signs, this is not particularly deep. The crux of the question is to explain the geometric meaning of the signs in the Murnaghan–Nakayama rule.

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