Abstract. In this paper it is proven that there is at most one way in which a connected, hyperbolic, orientable 3-manifold can fiber over the circle with monodromy in the Torelli group.

1. Introduction

One of Thurston’s contributions to 3-manifold theory was to provide a complete description of all the surprisingly many different ways in which a 3-manifold $M$ can fiber, [3]. This is done by defining a semi-norm on $H^1(M, \mathbb{R})$, similar to the Seifert genus of a knot, whose unit norm ball is a polyhedron. If an element of $H^1(M, \mathbb{R})$ is dual to a fiber, then every other integer point of $H^1(M, \mathbb{R})$ in the same cone over a facet of the polyhedron also determines a fibration.

The Torelli group of a surface $S$, $\mathcal{T}(S)$, is the subgroup of the mapping class group that acts trivially on homology.

Using homological arguments it is not hard to show that when the monodromy is in the Torelli group, the genus of the fiber is as small as possible. Conversely, when a fiber has this smallest possible genus, namely $(\text{rank}H_1(M, \mathbb{R}) - 1)/2$, the monodromy must be in the Torelli group.

This paper proves the following theorem, conjectured by Tom Church and Benson Farb, [2].

Theorem 1. Suppose that a connected, orientable 3-manifold can fiber over the circle with pseudo-Anosov monodromy in the Torelli group. Of all the infinite ways $M$ can fiber over the circle, the fibering with Torelli monodromy is unique up to isotopy.

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1This seems to be well known, perhaps first observed by Church-Farb
2Unpublished; communicated to the author by E. Hironaka
When it exists, the fibering with Torelli monodromy is therefore in some sense canonical, being the unique fibering with minimal genus.

An implicit assumption in Theorem 1 is that the genus of the fiber is at least two, otherwise the Torelli group is trivial.

By another classic theorem of Thurston, [4], $M$ is hyperbolic iff the monodromy is pseudo-Anosov. This assumption also ensures that a Thurston norm ball of any finite radius is compact. There can therefore be only finitely many integer points of $H^1(M, \mathbb{R})$ within or on the boundary of a ball. It follows that there are only finitely many different fibrations for any given genus. However, when the monodromy is not pseudo-Anosov, the Thurston semi-norm ball of any radius is not compact, and hence there can be infinitely many nonisotopic fibers with the same genus. An example of how to obtain countably many fibrations with nonisotopic fibers of the same genus is given in Example 4.

The basic outline of the proof Theorem 1 is as follows: Suppose there could be two fibrations of $M$, with fibers $S_1$ and $S_2$ and monodromies $\tau_1 \in T(S_1)$ and $\tau_2 \in T(S_2)$. A covering space of $M$ is found, that retracts onto a covering space $S_3$ of both $S_1$ and $S_2$. The assumption that the monodromies are in Torelli is used to show that algebraic intersection numbers from both $S_1$ and $S_2$ lift to $S_3$. This gives a contradiction.

Unlike with 3-manifolds, that fiber in many different ways if they fiber at all, in Theorem 1.1 of [2], it was shown that if $M$ is a surface bundle over a surface, whenever the monodromy is in the subgroup of the Torelli group generated by Dehn twists around separating curves (the “Johnson Kernel”), then the fibering is unique, unless it is a trivial product of two base spaces. A common theme is that, when the monodromy is in the Torelli group, the topology of the fiber determines the topology of the manifold to a large extent, which is used to obtain contradictions to the existence of fiberings. A survey of “fibering rigidity” results for 4-manifolds is given in [1].

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2. PROOF OF THEOREM

Before getting started on the proof of the theorem, some basic conventions and notations should be established. After that, some elementary properties of intersection numbers of curves in a covering space of both $S_1$ and $S_2$ are shown.

2.1. Conventions and notations. A fiber in $M$ determines a free homotopy class of surfaces in $M$, as well as an element of $H^1(M, \mathbb{Z})$ which is the class of the base space over which $M$ fibers. When it does not cause confusion, the same notation will be used for a surface embedded in $M$ representing the homotopy class and the fiber.

By curve is meant here a free homotopy class of maps from the 1-sphere into a surface or 3-manifold that is not homotopic to a point. Curves are not necessarily simple, and will sometimes be assumed to pass through a basepoint. This basepoint in $M$ is chosen to be in the intersection of $S_1$ and $S_2$. A curve will often be confused with the image in a surface or 3-manifold of a chosen representative of the homotopy class. A multicurve is a disjoint union of embedded curves in a surface.

All curves, surfaces and 3-manifolds are assumed to be oriented.

Let $\tilde{M}_{\tau_1}$ and $\tilde{M}_{\tau_2}$ be the infinite cyclic coverings $\tilde{M}_{\tau_1} \simeq S_1 \times \mathbb{R}$ and $\tilde{M}_{\tau_2} \simeq S_2 \times \mathbb{R}$. Denote by $\tilde{M}_T$ the smallest covering space of both $\tilde{M}_{\tau_1}$ and $\tilde{M}_{\tau_2}$; in other words, $M_T$ is the universal cover $\tilde{M}$ of $M$ modulo the subgroup $\pi_1(\tilde{M}_{\tau_1}) \cap \pi_1(\tilde{M}_{\tau_2}) \subset \pi_1(M)$. The group $\pi_1(\tilde{M}_T)$ is not trivial because there are curves in the intersection of $S_1$ and $S_2$ in $M$.

Informally, curves in $M_T$ represent curves that are both in $S_1$ and $S_2$. These curves should have a well defined algebraic intersection number, in some appropriate space, which will be defined in Subsection 2.2.

Remark 2. The reader should be warned that in what follows, we will be very relaxed about where a curve lives. Any two curves $c_1$ and $c_2$ in $M$ that lift to $\tilde{M}_T$ can be homotoped onto both of the fibers $S_1$ and $S_2$, and the same notation will be used for these curves in the fibers. Similarly, when a particular fundamental domain in $S_2$, $\tilde{M}_T$, $\tilde{M}_{\tau_1}$ or $\tilde{M}_{\tau_2}$ is assumed, the same notation will be used for a curve in the fundamental
domain and its projection to $M$ or one of the fibers. It will always be stated in what space we are working, and all curves mentioned should be assumed to be in that space.

2.2. Intersection Theory. In any 3-manifold that retracts onto a surface, e.g. $\tilde{M}_{\tau_1}$, algebraic intersection number is defined by projecting curves onto the surface.

The 3-manifold $\tilde{M}_{T}$ is a covering space of a space that retracts onto a surface, and therefore also retracts onto a surface, call it $S_3$.

Maps between homology. The surface $S_3$ is a covering space of $S_1$. If $\alpha$ is a curve in $S_3$ homologous to $\beta$, the simplicial complex $S$ in $S_3$ bounded by $\alpha - \beta$ projects down to a simplicial complex in $S_1$ with boundary $\pi(\alpha) - \pi(\beta)$ in $\tilde{M}_{\tau_1}$. A map on homology, from $H_1(\tilde{M}_{T})$ into $H_1(\tilde{M}_{\tau_1})$ is therefore obtained. Similarly with $S_1$ replaced by $S_2$. Since the monodromies are assumed to be in the Torelli group, the embeddings of the fibers in $M$ induce isomorphisms of $H_1(S_1; \mathbb{Z})$ and $H_1(S_2; \mathbb{Z})$ with their images in $H_1(M; \mathbb{Z})$.

The next little argument will be used throughout the paper, so has been made into a lemma.

Lemma 3. The diagram in Figure 1 commutes.

Proof. Two curves in $\tilde{M}_{T}$ that project to homologous curves in $\tilde{M}_{\tau_1}$ are mapped to homologous curves in $M$ by the inclusion map of $S_1$ into $M$. Since the curves came from $\tilde{M}_{T}$, the images in $M$ represent curves in the image of the inclusion map of $S_2$ into $M$, whose pullback are also homologous in $S_2$, because the inclusion map induces an isomorphism onto its image. $\square$
Since $\tilde{M}_T \cong S_3 \times \mathbb{R}$, $\hat{i}([c_1], [c_2])$ is defined to be the algebraic intersection number in $S_3$. Algebraic intersection number of $[c_1]$ and $[c_2]$ in $S_1$ is denoted by $\hat{i}_1([c_1], [c_2])$ and in $S_2$ by $\hat{i}_2([c_1], [c_2])$.

The surface $S_3$ can be complicated. Calculating algebraic intersection number directly from the definition is not practical, so it will be related to intersection numbers in $S_1$ and $S_2$ by using local maps from $S_3$ into $S_1$ or $S_2$.

**Diamonds.** To understand the covering space $\tilde{M}_T$ of $M$, note that a “diamond”, as shown in Figure 2, lifts rigidly to $\tilde{M}_T$. A diamond is the closure of a connected component of $M\setminus(S_1 \cup S_2)$. It is not really a polytope, but a union of 3-manifolds of the form $f \times I$, where $f$ is a subsurface of one of the fibers. The monodromies $\tau_1$ and $\tau_2$ determine deck transformations on $\tilde{M}_T$, and map diamonds to diamonds.

![Figure 2. A “diamond”, shown in grey lines.](image)

2.3. **Pseudo-Anosov monodromies.** Before actually calculating intersection numbers in $S_3$, this subsection is devoted to some consequences of the assumption that $\tau_1$ and $\tau_2$ are pseudo-Anosov.

The next example illustrates a phenomenon that we would like to use the pseudo-Anosov assumption to rule out.

**Example 4** (Fibrations with Torelli monodromy that are not pseudo-Anosov). Let $\tau_1$ be a mapping class in the Torelli group that is not pseudo-Anosov and leaves a simple curve $c$ on $S_1$ invariant. There is then a torus $T$ in $M$ that intersects $S_1$ along the curve $c$. Suppose $S_2$ is obtained from $S_1$ by Dehn twisting around $T$ in a direction transverse
to $S_1$. The fibers are clearly non-isotopic, and since they have the same genus, both monodromies must be in the Torelli group.

In Example 3, we would like to argue that the fiberings are not really distinct in some sense, despite having non-isotopic fibers. The following formulation was suggested by N. Salter: In Example 3, $S_3$ is an infinite cyclic covering space of both $S_1$ and $S_2$ obtained by cutting $S_1$ or $S_2$ along $c$ and gluing together infinitely many copies. The fundamental group of $S_3$ is a free group normally generated by the elements of $\pi_1(S_1 \setminus c)$ or $\pi_1(S_2 \setminus c)$. The mapping class $\tau_1$ determines a deck transformation that acts on $\pi_1(S_3)$ by the same outer automorphism class as $\tau_2$. The translations on $\tilde{M}_T$ coming from the deck transformations are not linearly independent.

Assuming once again that $\tau_1$ and $\tau_2$ are pseudo-Anosov, $S_3$ is the covering space of $S_1$ obtained as follows: let $D$ be a subsurface of $S_1$ on the boundary of a diamond. Then $S_3$ is obtained by gluing together the orbit of subsurfaces $\ldots \tau_2^{-1}(D), D, \tau_2(S), \tau_2^2(D), \ldots$. In Figure 4, $\tau_1$ orbits of these covering spaces are depicted by horizontal lines.

Figure 3. Orbits of the intersections $\{P_i\}$ of $S_2$ with $M \setminus S_1$.

Suppose $S_1$ and $S_2$ are in minimal position in $M$, and let $\{P_i\}$ be the set of connected components of $S_2 \cap (M \setminus S_1)$, as shown in Figure 3. The surface $P_1$ has boundary consisting of a multicurve $m_1^0 \subset S_1 \times \{0\}$ and a multicurve $m_1^1 \subset S_1 \times \{1\}$. If $\tau_2$ could determine the same deck transformation as $\tau_1$, then it must be that $\tau_1(m_1^0) = m_1^1$ is freely homotopic in $S_1 \times I$ to $m_1^0$, as illustrated in Figure 3. Similarly, $\tau_1(m_2^0)$ is freely homotopic to $m_3^0$, etc. However, this is a contradiction, because there are only finitely many elements of the set $\{P_i\}$, so this means there is some finite closed orbit of curves under the pseudo-Anosov monodromy $\tau_1$. It follows that the translations on $\tilde{M}_T$ corresponding to the deck transformations induced by $\tau_1$ and $\tau_2$ must be linearly independent for
pseudo-Anosov monodromies.

2.4. Lifts of curves are homologous in $S_3$. This subsection relates intersection numbers in the covering space $S_3$ with intersection numbers in the fibers $S_1$ and $S_2$. Informally speaking, the contradiction to the existence of multiple fiberings with Torelli monodromy comes from the observation that intersection numbers in $S_3$ lift from both $S_1$ and $S_2$.

**Lemma 5.** Suppose $\tau_1$ and $\tau_2$ are pseudo-Anosov monodromies in the Torelli group, and $c_1$ and $c_2$ are simple curves in $S_1$ that lift to $\tilde{M}_T$. The algebraic intersection number, $\hat{i}([c_1],[c_2])$, can be computed by projecting the homology classes $[c_1]$ and $[c_2]$ into either $H_1(\tilde{M}_{\tau_1};\mathbb{Z})$ or $H_1(\tilde{M}_{\tau_2};\mathbb{Z})$, because

$$\hat{i}_1([c_1],[c_2]) = \hat{i}_2([c_1],[c_2]) = \hat{i}([c_1],[c_2])$$

**Proof.** Let $d$ be a diamond in $M$. To simplify notation, the same symbol will be used for a diamond in $M$ or its lift to $\tilde{M}_T$. By construction, $S_3 \times \{i\} \cap d \subset \tilde{M}_T$ for any fixed $i$ projects one to one onto each of the subsurfaces of $S_1$ and $S_2$ on $\partial d \subset M$. So if $c_1$ and $c_2$ are both contained in $d$, the algebraic intersection number can be calculated within $d$, and is independent of whether the curves are both projected to $S_3 \times \{i\}$, or a subsurface of $S_1$ or $S_2$ on $\partial d$.

Let $c_1$ and $c_2$ be fixed representatives of their free homotopy classes in $S_1$ or $S_2$. If $c_1$ and $c_2$ are not contained within a diamond, they can each be written as a sum in the chain complex of $\tilde{M}_T$ of curves and arcs that are. Algebraic intersection number of $c_1$ and $c_2$ is obtained by summing the algebraic intersection numbers of these curves and arcs.

It remains to be shown that intersection number in $S_3$ is independent of the choice of lifts from $S_1$ and $S_2$. This will be done by showing that different lifts are homologous in $S_3$; a fact that relies heavily on the assumption that $\tau_1$ and $\tau_2$ are both in the Torelli group.

Suppose $c_1$ is in $S_1$ and lifts to $\tilde{M}_T$, where $l_1$ and $l_2$ are different lifts to $S_3$. The projection to $S_1$ of $l_1$ and $l_2$ are homologous by construction. Since the monodromies are Torelli, it follows from Lemma [3] that the projection of $l_1$ and $l_2$ to $S_2$ are homologous in $S_2$. The same is true with $S_1$ and $S_2$ interchanged. Informally speaking, the simplicial
complex in \( \tilde{M}_T \) bounded by \( l_1 - l_2 \) is obtained by splicing together the simplicial complexes bounded by the projections of \( l_1 \) and \( l_2 \) to each of \( \tilde{M}_{\tau_1} \) and \( \tilde{M}_{\tau_2} \). This is illustrated in Figure 4, where the surface drawn in faint grey lines with boundary \( l_2 - \tau_2^{-1}(l_2) \) is what we are trying to show the existence of.

**Remark 6.** When the monodromies are not pseudo-Anosov, as in Example 4, \( l_1 \) and \( l_2 \) can project to the same curves in both of the fibers. The fact that the projections are homologous in both of the fibers then does not need the fact that \( \tau_2 \) is Torelli.

![Figure 4](image)

**Figure 4.** The upper part of the figure depicts the two fundamental domains \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) in \( \tilde{M}_T \). The thick striated line is a copy of \( S_3 \). The lower part of the diagram is a blowup of several diamonds showing \( h_1, h_2, l_2 \) and \( \tau_2^{-1}(l_2) \). The thick grey line is \( \tau_2^{-1}(S_3) \).

As shown in Figure 4, \( \tilde{M}_{\tau_1} \), when cut open along a union of cylinders, is identified with a fundamental domain \( \mathcal{F}_1 \) of \( \tilde{M}_T \), which is translated by the action of \( \tau_2 \). The fundamental domain \( \mathcal{F}_2 \) is defined similarly.
Recall that $l_1$ is a lift of a simple curve, so it can be assumed without loss of generality that $l_1$ is contained within $F_1$, because otherwise $[l_1]$ can be written as a sum of homology classes of curves in $\tilde{M}_T$ that are each contained within a fundamental domain, and the lemma proven for each of these curves.

Since $l_1 \subset F_1$, the curve $l_2$ is contained within a fundamental domain $\tau_m F_1$ for some integer $m$. The fundamental domain $F_1$ can be collapsed in the $R$ direction to obtain a fundamental domain for the covering space $\tilde{S}_1$ of $S_1$ in $\tilde{M}_T$, as shown in Figure 4. Similarly for $F_2$ and the covering space $\tilde{S}_2$.

Choose $F_1$ and $F_2$ as shown in Figure 4. For simplicity, the same symbols, $l_1$ and $\tau^{-m}(l_2)$, will be used for curves in $\tilde{M}_{T_1}$ and $F_1$. Similarly for curves in $\tilde{M}_{T_2}$, $F_2$ and $S_2$. The curve $\tau^{-m}(l_2) \subset F_1$ is freely homotopic in $F_1$ to a curve $h_1$ on $\tilde{S}_2 \times \{i\} \cap F_2 \subset M_T$. The curve $h_2 = \tau^{-1}_2 h_1$ on $\tilde{S}_2 \times \{i-1\} \cap F_2$ or $h_2 = \tau_2^1 h_1$ (as in Figure 4) on $\tilde{S}_2 \times \{i+1\} \cap F_2$, is homologous to $h_1$ in $S_2$, because $\tau_2$ is Torelli.

Although $l_1$ and $l_2$ are in different fundamental domains of the deck transformation coming from $\tau_1$, a sequence of curves, $l_1, \tau_2(l_1), \tau_2^2(l_1), \ldots l_2$ (or $l_1, \tau_2(l_1), \tau_2^2(l_1), \ldots l_2$) has just been constructed. This sequence has the property that a curve in the sequence is contained in the boundary of the same fundamental domain of the action of $\tau_2$ as the preceding curve. The curves will be shown to be homologous within these fundamental domains, from which it then follows that $l_1$ is homologous to $l_2$ in $\tilde{M}_T$. This is where the assumption that the monodromies are pseudo-Anosov is used, because it was shown in Subsection 2.3 that the pseudo-Anosov assumption ensures the fundamental domains of the induced deck transformations are different. Informally, the fact that the fundamental domains overlap but do not coincide is used to keep extending the subsurface of $S_3$ on which intersection numbers lift from both $S_1$ and $S_2$, until it includes both $l_1$ and $l_2$.

Homotope both $h_1$ and $h_2$ within $F_2$ onto $\tilde{S}_2 \times \{i\}$. If the resulting curves are not disjoint, resolve the points of intersection of $h_1 - h_2$ in the usual way, as shown in Figure 5. It is not hard to see that this does not change the homology class of $h_1 - h_2$ in $\tilde{M}_T$. A multicurve in $F_2$ is obtained, whose projection to $S_2$ is null homologous in $S_2$. Let $N$ be a submulticurve whose projection to $S_2$ is null homologous without any proper null homologous submulticurves. Since $N$ is in $\tilde{M}_T$, it also
projects to a multicurve in $M$ and $S_1$, and is null in $S_1$ by Lemma 3. The aim is to show that all possible choices of $N$ are null homologous in $\tilde{M}_T$.

By choosing representatives of free homotopy classes accordingly, it can be assumed without loss of generality, that for any diamond $d_i$ in $\tilde{M}_T$, $(N \times \mathbb{R}) \cap d_i$ is not contained in $\partial d_i$. It is clear that $N \times \mathbb{R}$ is locally separating in $\tilde{M}_T$, i.e. $(N \times \mathbb{R}) \cap d_i$ is either empty or is separating in $d_i$. This is because the boundary of any diamond can be identified with a subsurface of $S_1$ or $S_2$. A separating multicurve in a surface cuts any subsurface it intersects into two or more pieces. It remains to be shown that $N \times \mathbb{R}$ is globally separating in $\tilde{M}_T$.

Let $S$, $\mathcal{F}_2 \subset S \subset \tilde{M}_T$, be a connected union of diamonds in which $N \times \mathbb{R}$ is separating in $S$ and such that no further diamond can be
attached to $S$ without destroying this property. Since $N$ and $S$ are assumed to be oriented, it is possible to define the left side of $S \setminus (N \times \mathbb{R})$ and the right side of $S \setminus (N \times \mathbb{R})$. By assumption, if a diamond $d_j$ is attached to $S$ along its boundary, it is attached both to the left side of $S$ and to the right side, as illustrated in Figure 6.

As a consequence, there is a curve $c$ in $S \cup d_j$ such that:

- $c$ intersects $N \times \mathbb{R}$ once.
- $c$ is a union of two arcs; one of which passes through $a_1$ diamonds and is contained in the lift of the fiber $S_1$, and the other passes through $a_2$ diamonds and is contained in the lift of the fiber $S_2$.

By construction, the curve $c$ is the lift of a bouquet of curves in $M$, one of which, call it $b_1$, is contained in $S_1$, and the other, call it $b_2$, is contained in $S_2$. Also by construction, $b_1$ intersects $S_2$ $a_1$ times, and $b_2$ intersects $S_1$ $a_2$ times, where $a_1$ and $a_2$ can not both be zero.

In order to lift to $\tilde{M}_T$, it must be possible to homotope $b_1 b_2$ into both $S_1$ and $S_2$. However, $b_1 b_2$ can not be homotoped onto $S_1$ because it has nonzero intersection number with $S_1$, nor can it be homotoped into $S_2$, for the same reason. The curve $b_1 b_2$ therefore could not possibly lift to a curve in $\tilde{M}_T$. It follows that $S$ is all of $\tilde{M}_T$, and hence $h_1$ is homologous to $h_2$, and $l_1$ is homologous in $S_3$ to $l_2$. Since algebraic intersection number is an invariant of homology classes, the algebraic intersection number is independent of the choice of lift from $S_1$. The argument is identical with $S_1$ and $S_2$ interchanged, from which the lemma follows.

It will now be shown that Theorem 1 is a corollary of Lemma 5.

**Proof.** Let $k_1$ and $k_2$ be two simple curves in $S_3$, both contained in a fundamental domain coming from the covering of $S_1$. Suppose $k_1$ and $k_2$ intersect once; this is possible because $S_1$ is assumed to have genus at least two. Let $k_3$ be the curve freely homotopic to the commutator of $k_1$ and $k_2$, i.e. $k_3$ is the boundary of a genus one subsurface of $S_3$ containing $k_1$ and $k_2$. Let $c_1$, $c_2$ and $c_3$ be the projections to $S_1$ of $k_1$, $k_2$ and $k_3$, and $k'_1$, $k'_2$, and $k'_3$ be different lifts of $c_1$, $c_2$ and $c_3$. The curve $k'_1$ is necessarily disjoint from $k'_3$; contradicting Lemma 5, which states that $k'_1$ has to intersect $k_2$. This is the promised contradiction to the
assumption of two different fibrations with pseudo-Anosov monodromy in the Torelli group.

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