Determination of the noise parameters in a one-dimensional open quantum system

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Abstract
We consider an electron magnetically interacting with a spin-1/2 impurity, embedded in an external environment whose noisy term acts only on the impurity’s spin, and we find expressions for the electron transmission and reflection probabilities in terms of the phenomenological noise parameters. Moreover, we give a simple example of the necessity of complete positivity for physical consistency, showing that a positive but not completely positive dissipative map can lead to negative transmission probabilities.

1 Introduction
In standard quantum mechanics the focus is mainly upon \textit{closed} physical systems, i.e. systems which can be considered isolated from the external environment and whose reversible time-evolution is described by one-parameter groups of unitary operators. On the other hand, when a system $S$ interacts with an environment $E$, it must be considered as an \textit{open} quantum system whose time-evolution is irreversible and exhibits dissipative and noisy effects. A standard way of obtaining a manageable dissipative time-evolution of the density matrix $\varrho_t$ describing the state of $S$ at time $t$ is to construct it as the solution of a Liouville-type Master equation $\partial_t \varrho_t = L[\varrho_t]$. This can be done by tracing away the environment degrees of freedom and by performing a Markovian approximation \cite{1, 2, 3}, i.e. by studying the evolution on a slow time-scale and neglecting fast decaying memory effects. Then the irreversible reduced dynamics of $S$ is described by one-parameter semigroups of linear maps obtained by exponentiating the generator $L$ of Lindblad type \cite{4, 5}: $\Gamma_t = e^{t L}$, $t \geq 0$, such
that \( \varrho_t \equiv \Gamma_t[\varrho] \). By means of standard \textit{weak} or \textit{singular coupling limit} techniques, the Master equation for \( S \) can be rewritten as follows [6, 7, 8]:

\[
\frac{\partial \varrho_t}{\partial t} = L_H[\varrho_t] + L_D[\varrho_t] = -\frac{i}{\hbar} [H, \varrho_t] + L_D[\varrho_t],
\]

(1)

where \( L_D[\varrho_t] \) is the Kossakowski-Lindblad term describing dissipation and noise. If the system \( S \) consists of a qubit, the Kossakowski-Lindblad term can be written explicitly as follows:

\[
L_D^{(2)}[\varrho(t)] = \sum_{i=1}^{3} C_{ij} \left[ \sigma_j \varrho \sigma_i - \frac{1}{2} \{ \sigma_i, \sigma_j, \varrho \} \right]
\]

(2)

with \( \sigma_i, i = 1, 2, 3 \), the Pauli matrices. The constants \( C_{ij} \) come from the Fourier transform of the bath correlation functions (see [9, 10, 8]) and form the so-called Kossakowski matrix \( C \equiv [C_{ij}] \). In order to guarantee full physical consistency, namely that \( \Gamma_t \otimes \text{id}_A \) be positivity preserving on all states of the compound system \( S + A \) for any inert ancilla \( A \), \( \Gamma_t \) must be completely positive [11] and this is equivalent to \( C \) being positive semidefinite [6, 12].

In this letter we propose a way of experimentally determining the Kossakowski matrix elements, i.e. the noise parameters, through the transmission and reflection probabilities of an electron, which can be measured. Moreover, in the simple case of a diagonal Kossakowski matrix, we explicitly show that, if one takes a positive but not completely positive dissipative map, one obtains negative transmission probabilities for certain states, which proves the necessity of complete positivity for physical consistency.

2 Determination of the Kossakowski matrix elements

We consider a system \( S \) in which an electron propagates in a one-dimensional wire interacting magnetically with a spin-1/2 impurity at \( x = 0 \), and we study the effects of a noisy environment on such a system. This is the first step in a framework analogous to that in [13]. There the authors considered an isolated system in which an electron propagates in a one-dimensional wire interacting magnetically with two spin-1/2 impurities at \( x = 0 \) and at \( x = x_0 \), and they analyzed the dependence of the electron’s transmittivity on the impurities’ states. Our idea was to study the effects of a noisy environment on such a system, starting by considering the simpler system of an electron interacting magnetically with only one spin-1/2 impurity. Therefore, firstly we will consider the case in which the system \( S \) of the electron and one impurity is isolated; then we will see what happens if \( S \) is embedded in an external environment which acts with a noisy term only on the spin degree of freedom of the impurity. This latter case will lead to expressions of the Kossakowski matrix elements in terms of the electron transmission coefficients.

In the first case the eigenvalue equation for the energy is the following:

\[
H |E\rangle \equiv \left( \frac{p^2}{2m} + \delta(x) J \vec{\sigma} \cdot \vec{\sigma} \right) |E\rangle = E |E\rangle
\]

(3)
where \( p = -i\hbar \nabla \), \( m \) is the electron mass, \( J \) is the magnetic coupling constant between electron and impurity, and \( \vec{\sigma}, \vec{s} \) is the electron, respectively impurity, spin operator.\(^4\)

Having defined the total spin \( \vec{S} = \vec{\sigma} + \vec{s} \), we can rewrite the eigenvalue equation (3) as follows:

\[
\left( \frac{p^2}{2m} + \delta(x) \frac{J}{2} \left( S^2 - \frac{3}{2} \right) \right) |E\rangle = E|E\rangle. \tag{4}
\]

\( S^2 \) and \( S_z \) are the constants of motion with eigenvalues \( s \in m = -s, \ldots, s \) respectively. In our case we are considering two spin-1/2 systems, so the possible values of the total spin eigenvalues \( s \) are 1 and 0.

Given an energy eigenstate \( |E\rangle \), it is always possible to expand it in terms of the spatial and total spin eigenfunctions: \(|E\rangle = \sum_{i=1}^{4} |\psi_{S_i}\rangle \otimes |S_i\rangle\), where we have taken \( \{|S_i\rangle\}_{i=1}^{4} = \{|S_1\rangle := \frac{|01\rangle - |10\rangle}{\sqrt{2}}, |S_2\rangle := |00\rangle, |S_3\rangle := \frac{|01\rangle + |10\rangle}{\sqrt{2}}, |S_4\rangle := |11\rangle \} \) as the total spin basis.

If we project equation (3) onto the electron position eigenstates \( \{|x\rangle\} \), for a fixed spin \( S_i \), we get the differential equation for the wave function \( \psi_{k;S_i}(x) \):

\[
-\frac{\hbar^2}{2m} \psi''_{k;S_i}(x) + \delta(x) \frac{J}{2} \left( S^2 - \frac{3}{2} \right) \psi_{k;S_i}(x) = E \psi_{k;S_i}(x). \tag{5}
\]

For positive energies \( E = \frac{\hbar^2 k^2}{2m} > 0 \), the solution of equation (5) is

\[
\psi_{k;S_i}(x) = \begin{cases} 
  e^{ikx} + r_{S_i}^E e^{-ikx} & \text{if } x < 0 \\
  t_{S_i}^E e^{ikx} & \text{if } x > 0
\end{cases}
\]

where \( r_{S_i}^E \) and \( t_{S_i}^E \) are the electron reflection and transmission coefficients respectively.

The explicit expressions for these coefficients are found by imposing the continuity condition in \( x = 0 \) and integrating the Schrödinger equation around \( x = 0 \):

\[
r_{S_i}^E = -\frac{1}{1 + \frac{4\pi}{\hbar} J \varrho(E)(S_i^2 - \frac{3}{2})}, \quad t_{S_i}^E = -\frac{1}{1 + \frac{4\pi}{\hbar} J \varrho(E)(S_i^2 - \frac{3}{2})}, \tag{6}
\]

with \( \varrho(E) = \frac{1}{\pi\hbar} \sqrt{\frac{2m}{E}} \) the linear density of states in the wire.

For the positive energies solutions, the eigenstates can be written as follows:

\[
|E\rangle = \int dx \sum_{i=1}^{4} \left[ (e^{ikx} + r_{S_i}^E e^{-ikx}) \chi_L(x) + t_{S_i}^E e^{ikx} \chi_R(x) \right] |x\rangle \otimes |S_i\rangle, \tag{7}
\]

where \( |S_i\rangle, i = 1, \ldots, 4 \), are the total spin basis elements listed above, \( r_{S_i}^E \) and \( t_{S_i}^E \) are the reflection and transmission coefficients from (6), and \( \chi_L(R) \) is the characteristic function for \( x \geq 0 \) (\( x \leq 0 \)).

Calculating the transmission and reflection coefficients for the spin basis elements \( \{S_i\}_{i=1}^{4} \)

\(^4\)The spin operators are such that the eigenvalues of \( \sigma_z \) and \( s_z \) are \( \pm 1/2 \).
explicitly, we find $t_{S_2}^E = t_{S_4}^E = t_{S_3}^E := t^E$, $t_{S_1}^E := t^E$, and $r_{S_3}^E = r_{S_4}^E := r_2^E$, $r_{S_1}^E := r_1^E$. Thus, having redefined the spin states as

$$|\phi_0^{\text{spin}}\rangle := |S_1\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}, \quad |\phi_1^{\text{spin}}\rangle := \frac{1}{\sqrt{3}} \sum_{i=2}^4 |S_i\rangle = \frac{1}{\sqrt{3}} \left(|00\rangle + \frac{|01\rangle + |10\rangle}{\sqrt{2}} + |11\rangle\right),$$

the energy eigenstates can be rewritten as $|E\rangle = |\phi_0^E\rangle \otimes |\phi_0^{\text{spin}}\rangle + |\phi_1^E\rangle \otimes |\phi_1^{\text{spin}}\rangle$, where the vectors $|\phi_0^E\rangle$, $|\phi_1^E\rangle$ are such that

$$\phi_0^E(x) \equiv \langle x|\phi_0^E\rangle := \left(e^{ikx} + r_0^E e^{-ikx}\right)\chi_L(x) + t_0^E e^{ikx}\chi_R(x),$$

$$\phi_1^E(x) \equiv \langle x|\phi_1^E\rangle := \sqrt{3}\left((e^{ikx} + r_1^E e^{-ikx})\chi_L(x) + t_1^E e^{ikx}\chi_R(x)\right).$$

Finally, considering the basis of maximally entangled Bell states, $|\psi_0\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$, $|\psi_1\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$, $|\psi_2\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$, $|\psi_3\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$, the spin states (8) can be rewritten as

$$|\phi_0^{\text{spin}}\rangle \equiv |\psi_2\rangle, \quad |\phi_1^{\text{spin}}\rangle \equiv \frac{1}{\sqrt{3}} \left(|\psi_1\rangle + \sqrt{2}|\psi_0\rangle\right).$$

If the system $S$ is embedded in an external environment to which it is weakly coupled, the evolution of the system eigenstates is described by the Master equation (11) with solution $\dot{\rho}(t) \equiv \Gamma_t[\rho] = \exp(tL)[\rho] = \exp(t(L_H + L_D))[\rho]$, where $\rho \equiv |E\rangle\langle E|$. We will consider a dissipative map $\Gamma_t$ whose noisy effects act only on the spin degree of freedom of the impurity and leave the electron spin unchanged: therefore $L_D \equiv I \otimes L_D^{(2)}$, with $I$ the identity operator acting on the space and electron spin degrees of freedom, and $L_D^{(2)}$ the Kossakowski-Lindblad term (2) corresponding to the dissipative map $\gamma_t$ acting on the impurity’s spin. It is precisely the elements of the Kossakowski matrix relative to $\gamma_t$ that we want to write in terms of the electron transmission and reflection coefficients, and thus determine operatively. Since the total evolution of the system eigenstates is governed by the map $\Gamma_t$ which cannot be factorized between space and spin degrees of freedom, it is sufficient to consider an expansion for small times and stop at the first order in $t$:

$$\Gamma_t \langle E\rangle\langle E\rangle = \exp(tL)\langle E\rangle\langle E\rangle = (I + t(L_H + L_D))\langle E\rangle\langle E\rangle + O(t^2).$$

Since we want to find explicit expressions for the Kossakowski matrix elements, we will isolate the contributions from the dissipative part of the map $\Gamma_t$, due to $L_D$. In order to do so, we will consider the spin state $|\psi_3\rangle = (I_2 \otimes \sigma_3)|\psi_0\rangle$, which is orthogonal to the spin states in the eigenstate expansion, and then we will calculate the probability of finding the system at a point $x = x_0$ with total spin state $|\psi_3\rangle$. Therefore we will evaluate the mean value of (12) with respect to the state $|x_0\rangle \otimes |\psi_3\rangle \equiv |x_0; \psi_3\rangle$:

$$P_t(x = x_0; |\psi_3\rangle\langle \psi_3|) = \langle x_0; \psi_3| (I + t(L_H + L_D))\langle E\rangle\langle E\rangle |x_0; \psi_3\rangle + O(t^2).$$

The zeroth order term in (13) vanishes because of the orthogonality of $|\psi_3\rangle$ to the spin states of the eigenstate $|E\rangle$, whereas the Hamiltonian term is always zero on the eigenstates, so we
are left with
\[ P_t(x = x_0; |\psi_3\rangle\langle\psi_3|) = t\langle x_0; \psi_3 | L_D [ |E\rangle\langle E| ] | x_0; \psi_3 \rangle + O(t^2). \]  
(14)

Making use of (9) and (10), expression (14) can be conveniently rewritten as follows:
\[ P_t(x = x_0; |\psi_3\rangle\langle\psi_3|) = t\langle x_0; \psi_3 | L_D [ |\psi_3\rangle\langle\psi_3| ] D |E\rangle\langle E| | x_0; \psi_3 \rangle + O(t^2), \]  
(15)

where \(|\phi^E(x)| = \left( \begin{array}{c} \phi_0^E(x) \\ \phi_1^E(x) \end{array} \right)\)
and
\[ \tilde{D} = \left( \begin{array}{cc} \langle \psi_3 | L_D [ |\phi_0^{spin}\rangle\langle\phi_0^{spin}| ] | \psi_3 \rangle & \langle \psi_3 | L_D [ |\phi_1^{spin}\rangle\langle\phi_1^{spin}| ] | \psi_3 \rangle \\ \langle \psi_3 | L_D [ |\phi_1^{spin}\rangle\langle\phi_1^{spin}| ] | \psi_3 \rangle & \langle \psi_3 | L_D [ |\phi_0^{spin}\rangle\langle\phi_0^{spin}| ] | \psi_3 \rangle \end{array} \right). \]  
(16)

The matrix \(\tilde{D}\) has the following explicit expression:
\[ \tilde{D} = \left( \begin{array}{cc} C_{11} & \frac{1}{\sqrt{3}} \left( -iC_{21} + C_{31} \right) \\ \frac{1}{\sqrt{3}} (iC_{12} + C_{13}) & \frac{1}{3} \left( C_{22} + 2C_{33} + 2\sqrt{2} Im(C_{23}) \right) \end{array} \right), \]  
(17)

where \(C_{ij}, i, j = 1, 2, 3,\) are the elements of the Kossakowski matrix \(C\) which corresponds to the map \(\gamma_t\) acting on the spin degree of freedom of the impurity.

In this paper we consider only entropy-increasing maps, which describe many interesting situations in different areas of physics (stochastic magnetic fields, quantum baker’s map [14, 15, 16], XY spin-1/2 chain with quenching of the transverse field [17]): in this case the Kossakowski matrix \(C\) is symmetric and real [1, 18], and therefore has only six different elements. So, in order to explicitly find the Kossakowski matrix elements, we need six independent linear equations for the \(C_{ij}\)’s.

The first two linear equations are given by the explicit evaluation of (15) for \(x > 0\) and for \(x < 0\), in which the transmission and reflection coefficients appear respectively; whereas the other four can be obtained by calculating the analog of (15) for \(x > 0\) and for \(x < 0\) rotating the spin basis. A simple choice for the rotations is to exchange two Pauli matrices while keeping the third fixed, thus rearranging the elements \(C_{ij}\) in (17). We can take a rotation
\[ R^{(k)} \] that keeps \(\sigma_k (k = 1, 2, 3)\) fixed while changing \(\sigma_l \) in \(\pm \sigma_m (l, m = 1, 2, 3, l, m \neq k)\): this will exchange the elements \(C_{lm}\) and \(C_{ml}\) while leaving those with \(i, j = k\) unchanged. In particular we chose
\[ R^{(1)}(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}} \begin{array}{cc} 1 & -i \\ -i & 1 \end{array} \]  
and
\[ R^{(2)}(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}} \begin{array}{cc} 1 & 1 \\ -1 & 1 \end{array} \]  
(18)

and
(19)
which lead to the following rotated spin bases respectively:

\[ |\tilde{\psi}^{\text{spin}}_i\rangle := \mathbb{I}_2 \otimes R^{(1)} |\psi^{\text{spin}}_i\rangle \quad \text{and} \quad |\hat{\psi}^{\text{spin}}_i\rangle := \mathbb{I}_2 \otimes R^{(2)} |\psi^{\text{spin}}_i\rangle. \] (20)

Thus we calculated (15) for \( x > 0 \) and for \( x < 0 \) for the three different spin bases, i.e. we evaluated the probability of finding the evolved state at a point \( x \equiv x_0 \) in the three different spin states \( |\psi_3\rangle, |\tilde{\psi}_3\rangle, |\hat{\psi}_3\rangle\). We then wrote the reflection coefficient as \( r_{S_i}^E = 1 - t_{S_i}^E \), and finally we obtained six independent linear equations for the elements \( C_{ij} \). This linear system can be written in vector form as follows:

\[ |P(t)\alpha\rangle = M_{\alpha\beta} |C_{\beta}\rangle, \quad \alpha, \beta = 1, \ldots, 6, \] (21)

with \( M \equiv [M_{\alpha\beta}] \) a 6 \times 6 matrix, and vectors

\[ |C_{\beta}\rangle := \begin{pmatrix} C_{11} \\ C_{12} \\ C_{13} \\ C_{22} \\ C_{23} \\ C_{33} \end{pmatrix}, \quad |P(t)\alpha\rangle := \begin{pmatrix} P_{0}^T \\ P_{1}^T \\ P_{2}^T \\ P_{0}^R \\ P_{1}^R \\ P_{2}^R \end{pmatrix}, \]

where \( P_{a}^T, P_{a}^R, a = 0, 1, 2, \) are the transmission, respectively reflection, probabilities for the three different spin bases.

The linear system of equations can be inverted if \( \text{Det}(M) \neq 0 \). This is indeed the case, and therefore one can explicitly write the Kossakowski matrix elements in terms of the transmission and reflection probabilities, and thus determine the \( C_{ij} \)'s experimentally.

Moreover, the experimental determination of the \( C_{ij} \)'s leads to the possibility of actually verifying whether the Kossakowski matrix is positive semidefinite\(^3\) and thus whether the evolution is completely positive. Further, this could also be a test for the Markovian approximation used: since physically consistent Markovian approximations for the Master equation \( (1) \) must lead to completely positive dynamical semigroups, if the results obtained for the \( C_{ij} \)'s yield a Kossakowski matrix which is not positive semidefinite, this could imply that the particular Markovian approximation chosen to describe the dynamics is not appropriate.

**Remark 1** Our proposal for an operational determination of the Kossakowski matrix elements through transmission probabilities, which can be measured, could also be viewed in the context of experimental characterization of the dynamical evolution of an open quantum system. A well-studied procedure with this aim is known as quantum process tomography (QPT) \[19, 20, 21\], where a quantum system \( A \) is subjected to an unknown quantum process \( \mathcal{E} \). In order to determine \( \mathcal{E} \), one prepares a fixed set of states \( \{ \hat{\varrho}_j \} \) that form a basis for the

\(^2\)Details are given in the Appendix.

\(^3\)Notice that, in order for the Kossakowski matrix to be positive semidefinite, the transmission and reflection probabilities must be such that the following positivity conditions for \( C \) are fulfilled:

\[
C_{11} \geq 0, C_{22} \geq 0, C_{33} \geq 0, C_{11}C_{22} - C_{12}^2 \geq 0, C_{11}C_{33} - C_{13}^2 \geq 0, C_{22}C_{33} - C_{23}^2 \geq 0, \text{Det}(C) \geq 0.
\]
set of operators acting on the state space of $A$ and applies the process $\mathcal{E}$ to each input state $\varrho_j$; then $\mathcal{E}(\varrho_j)$ can be experimentally determined through quantum state tomography [22, 23] on the outputs; finally the process $\mathcal{E}$ can be fully characterized through the operation elements $E_k$ in its operator sum representation $\mathcal{E}(\varrho) = \sum_k E_k \varrho E_k^\dagger$. The physical systems and detailed procedures used in quantum process tomography (see, for example, [24, 25, 26]) differ from those used in this work, and in our case the quantum operation describing the evolution of the system is not supposed to be unknown; nevertheless both methods could be viewed in the context of experimental characterization of a quantum process on an open quantum system. Moreover, the analysis of results in quantum process tomography leading to a non-completely positive evolution may be useful in better understanding the implications, in our case, of experimentally obtaining a Kossakowski matrix which is not positive semidefinite. In [26], for example, it is shown that experimental errors made in the QPT procedure can yield results which lead to a non-completely positive quantum operation and that this unphysical result can be corrected. Therefore, it might be possible also in our case, that experimental results leading to a non-positive semidefinite Kossakowski matrix be due to experimental errors in the measuring procedure. Further discussion about this hypothesis, however, would involve taking into account the exact experimental situation, and is therefore outside the scope of this paper.

3 Complete positivity

In order to guarantee full physical consistency, namely that $\gamma_t \otimes \text{id}_A$ be positivity preserving on all states of the compound system $S + A$ for any inert ancilla $A$, $\gamma_t$ must be completely positive [11] and this is equivalent to $C$ being positive semidefinite [6, 12].

The necessity of complete positivity arises from the existence of entanglement, since in general entangled bipartite states may become non-positive under the action of positive but not completely positive transformations [27].

In this section we will show that, if we take a positive but not completely positive dissipative map $\gamma_t$ acting on the impurity’s spin and calculate the probability (15) for certain entangled states, we find negative values for the transmission probability.

In order to give an explicit example of this fact, we will use a specific dissipative map $\gamma_t$ and for simplicity we will consider a diagonal Kossakowski matrix. In this case the matrix $\tilde{D}$ from (17) reduces to

$$\tilde{D} = \begin{pmatrix} C_{11} & 0 \\ 0 & \frac{1}{3} (C_{22} + 2C_{33}) \end{pmatrix}$$

(22)

and the probability (15), to first order in $t$, is therefore

$$P_t(x = x_0; |\psi_3\rangle \langle \psi_3|) = t \left( C_{11} |t_0^E|^2 + (C_{22} + 2C_{33}) |t_1^E|^2 \right)$$

$$= t \left( C_{11} \frac{16}{16 + 9(\frac{\pi}{2} J \rho(E))^2} + (C_{22} + 2C_{33}) \frac{16}{16 + (\frac{\pi}{2} J \rho(E))^2} \right)$$

(23)
having inserted the explicit expressions for the transmission coefficients in the second line. In particular, we consider a positive but not completely positive map \( \gamma_t \) acting on the impurity’s spin with Kossakowski matrix \( C = \text{diag}(1,1,-1) \). Thus the Kossakowski-Lindblad term (2) explicitly reads

\[
L^{(2)}_D [\varrho_{\text{spin}}] = \sigma_1 \varrho_{\text{spin}} \sigma_1 + \sigma_2 \varrho_{\text{spin}} \sigma_2 - \sigma_3 \varrho_{\text{spin}} \sigma_3 - \varrho_{\text{spin}}
\]

with \( \varrho_{\text{spin}} \) the impurity’s spin state. Since the spin-1/2 impurity consists of a qubit, its state can be written in Bloch vector form: \( \varrho_{\text{spin}} = \frac{1 + \hat{g} \cdot \sigma}{2} \), where \( 1_2 \) is the identity in \( \mathbb{C}^2 \), \( \hat{g} = (g_1, g_2, g_3) \) and \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \), with \( \sigma_i, \ i = 1, 2, 3 \), the Pauli matrices. Thus the evolved state will be

\[
\varrho_{\text{spin}}(t) \equiv \gamma_t [\varrho_{\text{spin}}] = \frac{1 + \varrho_1 \sigma_1 + \varrho_2 \sigma_2 + e^{-it} \varrho_3 \sigma_3}{2}.
\]

\( \varrho_{\text{spin}}(t) \) is such that \( ||\varrho_{\text{spin}}(t)||^2 < ||\varrho_{\text{spin}}||^2 \leq 1 \) and therefore the map \( \gamma_t \) gives rise to a positive evolution. On the other hand, though, the Kossakowski matrix \( C = \text{diag}(1,1,-1) \) is not positive semidefinite, thus the corresponding dissipative map \( \gamma_t \) is not completely positive, and evaluating (23) explicitly it is straightforward to see that we obtain a negative transmission probability.

Notice that this physical inconsistency arises from dealing with a positive but not completely positive map and an entangled state. Indeed, using duality, the matrix \( \tilde{D} \) that appears in expression (13) for the transmission probability can be rewritten as

\[
\tilde{D} = \begin{pmatrix}
\langle \varphi_0^\text{spin} | L_D | \psi_3 \rangle \langle \psi_3 | \varphi_0^\text{spin} \\
\langle \varphi_1^\text{spin} | L_D | \psi_3 \rangle \langle \psi_3 | \varphi_1^\text{spin}
\end{pmatrix},
\]

where the spin state \( |\psi_3\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} \) is maximally entangled for the electron and impurity spins. Therefore, evaluating the probability (23) of finding the system at a point \( x = x_0 \) with total spin state \( |\psi_3\rangle \) is equivalent to applying the generator \( L_D \equiv I \otimes L^{(2)}_D \), relative to the positive but not completely positive map \( \gamma_t \), to the entangled state \( |\psi_3\rangle \), and this leads to the physical inconsistency of a negative transmission probability.

Remark 2 If, instead, the map \( \gamma_t \) is completely positive, expression (13) is always positive. Indeed, from the choice of the spin state \( |\psi_3\rangle \), the only contribution to (14) is given by the noise term \( N[\varrho] \), which can be written as follows:

\[
N[\varrho] = \sum_{i,j} C_{ij} \sigma_j \varrho_i^\dagger = \sum_l c_l \left( \sum_j \bar{\psi}_l^{(j)} \sigma_j \right) \varrho \left( \sum_i \bar{\psi}_l^{(i)} \sigma_i^\dagger \right) = \sum_l c_l \left( \sum \bar{\psi}_l^{(j)} \sigma_j \right) \varrho \left( \sum \bar{\psi}_l^{(i)} \sigma_i \right)^\dagger = \sum_l \sqrt{c_l} \sqrt{\tilde{W}_l} \varrho \sqrt{\tilde{W}_l}^\dagger = \sum_l W_l \varrho W_l^\dagger,
\]

where \( \{ |\psi_l\rangle \} \) is a basis of eigenstates of the Kossakowski matrix \( C \equiv \sum_l c_l |\psi_l\rangle \langle \psi_l| \) such that \( C_{ij} = \sum_l c_l \bar{\psi}_l^{(i)} \bar{\psi}_l^{(j)} \). \( N[\varrho] \) is in Kraus-Stinespring form and thus completely positive.
Therefore, rewriting the action of the dissipative generator $L_D$ in (14), we have:

$$P_t(x = x_0; |\psi_3\rangle\langle\psi_3|) = t\langle x_0; \psi_3 | I \otimes N^{\text{spin}} || E \rangle \langle E || | x_0; \psi_3 \rangle + O(t^2),$$

where $I$ is the identity operator for the position and electron spin degrees of freedom, while $N^{\text{spin}}$ is the noise term acting on the impurity’s spin.

The complete positivity of the latter guarantees the positivity of $I \otimes N^{\text{spin}}$, and thus $P_t(x = x_0; |\psi_3\rangle\langle\psi_3|)$ is always positive.

4 Conclusions

In this paper we proposed a way of experimentally determining the elements $C_{ij}$ of a Kossakowski matrix, i.e. the noise parameters, in terms of the transmission and reflection probabilities of an electron, which can be measured. We considered a system in which an electron propagates in a one-dimensional wire interacting magnetically with a spin-1/2 impurity, embedded in an external environment which acts with a noisy term only on the spin degree of freedom of the impurity. We calculated the electron’s transmission and reflection coefficients, and found expressions for the transmission and reflection probabilities in terms of the Kossakowski matrix elements. This leads to the possibility of having experimental access to the noise parameters and of actually verifying whether the Kossakowski matrix is positive semidefinite, and thus whether the evolution is completely positive. Further, it could also be a test for the Markovian approximation used: if the results obtained for the $C_{ij}$’s lead to a Kossakowski matrix which is not positive semidefinite, this could imply that the particular Markovian approximation used to describe the dynamics is not appropriate. Moreover, we gave a concrete example of the necessity of complete positivity for physical consistency, showing that a positive but not completely positive dissipative map acting on the impurity’s spin can yield negative transmission probabilities for certain entangled states.
Appendix  The first two equations for the transmission and reflection probabilities in terms of the Kossakowski matrix elements and transmission coefficients, computed to the first order in $t$, are as follows:

$$P^T_0(t) = C_{11}|t_0^E|^2 + C_{12}[-2Im(t_0^E)Re(t_1^E) + 2IIm(t_0^E)Re(t_0^E)]$$

$$+ C_{13}[2Re(t_0^E)Re(t_1^E) + 2Im(t_1^E)Im(t_0^E)] + C_{22}|t_1^E|^2 + 2C_{33}|t_1^E|^2,$$

$$\frac{P^R_0(t)}{t} = C_{11}[2 - 2\cos(2kx) + |t_0^E|^2 - 2Re(t_0) + 2Re(t_0)\cos(2kx) + 2Im(t_0)\sin(2kx)]$$

$$+ C_{12}[-2Im(t_0^E)Re(t_1^E) + 2Im(t_0^E)Re(t_1^E) + 2Im(t_0^E) - 2Im(t_1^E)]$$

$$+ 2Re(t_0)\sin(2kx) - Im(t_0)\cos(2kx) - 2Re(t_1)\sin(2kx) + Im(t_1)\cos(2kx)]$$

$$+ C_{13}[4 - 4\cos(2kx) + 2Re(t_0^E)Re(t_1^E) + 2Im(t_1^E)Im(t_0^E) + 2Re(t_1)\cos(2kx)]$$

$$+ 2Im(t_1)\sin(2kx) - 2Re(t_1) - 2Re(t_0) + 2Re(t_0)\cos(2kx) + 2Im(t_0)\sin(2kx)]$$

$$+ C_{22}[2 - 2\cos(2kx) + |t_1^E|^2 - 2Re(t_1) + 2Re(t_1)\cos(2kx) + 2Im(t_1)\sin(2kx)]$$

$$+ 2C_{33}[2 - 2\cos(2kx) + |t_1^E|^2 - 2Re(t_1) + 2Re(t_1)\cos(2kx) + 2Im(t_1)\sin(2kx)].$$

The other four equations for $P^{T,R}_{a=1,2}$, have the same form just with a different ordering of the elements $C_{ij}$ in accordance with the rotation of the spin bases, as previously explained, using (18), (19), (20). In particular, taking $x = \frac{na}{2\pi}$ with $n = 4m + 1$ and $m = 0, 1, 2, \ldots$, we get $\sin(2kx) = 1$, $\cos(2kx) = 0$: this leads to six distinct equations where the expressions for the reflection probabilities are somewhat simplified whereas those for the transmission probabilities remain unchanged. If we then write this linear system in vector form, we get (21), where $M$ is a matrix whose entries are the constants multiplying the elements $C_{ij}$ in the system of equations. This matrix explicitly reads:

$$M = \begin{pmatrix}
 a_0 & b & c & a_1 & 0 & 2a_1 \\
 a_0 & -c & b & 2a_1 & 0 & a_1 \\
 2a_1 & 0 & -c & a_1 & b & a_0 \\
 d_0 & e & f & d_1 & 0 & 2d_1 \\
 d_0 & -f & e & 2d_1 & 0 & d_1 \\
 2d_1 & 0 & -f & d_1 & e & d_0
\end{pmatrix},$$

with

$$a_i = Re(t_i^E) = |t_i^E|^2, \quad i = 0, 1,$$

$$b = 2[-Im(t_0^E)Re(t_1^E) + Im(t_0^E)Re(t_0^E)],$$

$$c = 2[Re(t_0^E)Re(t_1^E) + Im(t_1^E)Im(t_0^E)],$$

$$d_i = 2 - |t_i^E|^2 + 2Im(t_i^E), \quad i = 0, 1,$$

$$e = 2[Im(t_0^E)Re(t_1^E) + Im(t_0^E)Re(t_0^E) + Re(t_0^E) - Re(t_0^E) + Im(t_0^E) - Im(t_0^E)],$$

$$f = 2[2 + Re(t_0^E)Re(t_1^E) + Im(t_1^E)Im(t_0^E) - Re(t_0^E) - Re(t_0^E) + Im(t_0^E) + Im(t_0^E)].$$
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