A Haar Wavelet Series Solution of Heat Equation with Involution

Burma Saparova¹, Roza Mamytova¹, Nurjamal Kurbanbaeva², Anvarjon Akhatjonovich Ahmedov³,*

¹ Uzgen Institute of Technology and Education, Uzgen, Kyrgyzstan
² Osh State University, Osh, Kyrgyzstan
³ Pusat Sains Matematik, Universiti Malaysia Pahang, Lebuhraya Tun Razak, 26300 Gambang, Kuantan, Pahang, Malaysia

It is well known that the wavelets have widely applied to solve mathematical problems connected with the differential and integral equations. The application of the wavelets possesses several important properties, such as orthogonality, compact support, exact representation of polynomials at certain degree and the ability to represent functions on different levels of resolution. In this paper, new methods based on wavelet expansion are considered to solve problems arising in approximation of the solution of heat equation with involution. We have developed new numerical techniques to solve heat equation with involution and obtained new approximative representation for solution of heat equations.

1. Introduction

The wavelets have wide application in engineering sciences and can be applied to solve mathematical problems connected with the differential and integral equations. The application of the wavelets possesses several important properties, such as orthogonality, compact support, exact representation of polynomials at certain degree and the ability to represent functions on different levels of resolution. Haar wavelets are orthonormal wavelets with a compact support. Haar wavelet method is an effective tool for solving ordinary and partial differential equations. There are many researchers who uses Haar wavelets to solve differential and integral equations notably papers [1-3]. A computational method for solving Poisson equations and biharmonic equations which are based on the use of Haar wavelets is investigated in Shi and Cao [4]. The method of two-dimensional Haar wavelets has been applied to obtain the solution of the partial differential equations in Lepik [5]. A collocation method based on Haar wavelet and Kronecker tensor product for solving three-dimensional partial differential equations is presented in Lepik [6], where the proposed method is originated from the idea of approximating a sixth-order mixed derivative by a series of Haar wavelet basis functions, which is suitable for numerical solution of all kinds of three-dimensional Poisson and Helmholtz equations.

* Corresponding author.
E-mail address: anvarjon@ump.edu.my

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In this paper, new methods based on wavelet expansion are considered to approximate the solution of heat equation with involution. The output is a new approximative representation of solution for heat equations.

2. Haar Wavelets

There are two functions that play a primary role in wavelet analysis, i.e., the scaling function given by
\[
\phi(x) = \begin{cases} 
1, & \text{if } 0 \leq x \leq 1, \\
0, & \text{elsewhere,} 
\end{cases}
\]
and the wavelet
\[
\psi = \begin{cases} 
1, & 0 \leq x \leq \frac{1}{2}, \\
-1, & \frac{1}{2} \leq x \leq 1, \\
0, & \text{otherwise.}
\end{cases}
\]

These two functions generate a family of functions that can be used to reconstruct signals. Let $M$ be any positive integer. We divide the interval $[0, 1]$ into $2M$ equal subintervals. We intend to do $J$ levels of resolutions, hence we let $2M = 2^{J+1}$.

The Haar wavelet is system of functions defined by the following
\[
h_n = \psi_{j,k}(x) = 2^j \psi(2^j x - k) = \begin{cases} 
1, & \frac{k}{2^j} \leq x < \frac{k+0.5}{2^j}, \\
-1, & \frac{k+0.5}{2^j} \leq x \leq \frac{k+1}{2^j}, \\
0, & \text{otherwise,}
\end{cases}
\]

where $n = 2^j + k + 1, j = 0, 1, 2 \ldots J, k = 0, 1, 2 \ldots 2^j - 1$. The Haar wavelet system (3) form an orthonormal basis for $L_2(\mathbb{R})$. The wavelet expansion of $f \in L_2[0,1]$ is given by
\[
f(x) = \sum_{n=0}^{\infty} a_n h_n(x),
\]
where
\[
a_n = \int_0^1 f(x)h_n(x)dx.
\]
The integration of Haar wavelets is given by
\[
P_{\alpha,i}(x) = \frac{1}{(\alpha-1)!} \int_A (x - \tau)^{\alpha-1} h_i(\tau) d\tau, \alpha = 1, 2, \ldots n, i = 1, 2, \ldots 2M.
\]

3. Heat Equation

Consider a uniform rod of length 1 with non-uniform temperature lying on the $x$-axis from $x = 0$ to $x = 1$. By uniform rod, we mean the density $\rho$, specific heat $c$, thermal conductivity $K_0$ are constant. Assume the sides of the rod are insulated and only the ends may be exposed. Let denote
by \( u(t, x) \) a temperature throughout the slice of the rod at the point \( x \) and time \( t \). Further we assume that the heating process is restricted to the interval \( 0 \leq t \leq T \), where \( T \) is sufficiently big positive number. We consider the second order linear heat equation with involution of the following form

\[
\frac{\partial u}{\partial t} (T-t, x) = \frac{\partial^2 u(t, x)}{\partial x^2} + f(t, x), \quad 0 < t < T, 0 < x < 1
\]  

(5)

\[ u(0, x) = 0, \quad x \in [0,1], \]  

(6)

\[ u(t, 0) = u(t, 1) = 0, \quad t \in [0,1]. \]  

(7)

To solve the problem (5)-(7), we use the method of approximation by wavelets. Assume that

\[ u(t, x) = v(t) \cdot w(x). \]

We have

\[ u(t, x)|_{t=0} = v(0) \cdot w(x) = 0 \Rightarrow v(0) = 0; \]

\[ u(t, x)|_{x=0} = v(t) \cdot w(0) = 0 \Rightarrow w(0) = 0; \]

\[ u(t, x)|_{x=1} = v(t) \cdot w(1) = 0 \Rightarrow w(1) = 0; \]

By substituting a function \( u(t, x) = v(t) \cdot w(x) \), into the Eq. (5), we obtain

\[
\frac{dv(T-t)}{dt} \cdot w(x) + v(t) \frac{d^2 w(x)}{dx^2} = \lambda v(t) \cdot w(x). 
\]

From here

\[
\frac{1}{v(t)} \frac{dv(T-t)}{dt} = \frac{1}{w(x)} \left( \lambda w(x) - \frac{d^2 w(x)}{dx^2} \right) \equiv \mu,
\]

where \( \mu \) - certain constant. Thus, if the solution of the problem (5)-(7) is represented as \( u(t, x) = v(t) \cdot w(x) \), then the functions \( v(t) \) and \( w(x) \) are solutions to the following spectral problems:

\[
\frac{dv(T-t)}{dt} = \mu v(t), \quad v(0) = 0,
\]

and

\[
\frac{d^2 w(x)}{dx^2} = -\gamma w(x), \quad x \in [0,1]
\]
\[ w(0) = w(1) = 0, \]

where \( \gamma = \mu - \lambda \). The spectral problem

\[
\frac{dv(T-t)}{dt} = \mu v(t), \quad v(0) = 0,
\]

has an infinite set of eigenvalues

\[
\mu_n = (-1)^n \left( n + \frac{1}{2} \right) \frac{\pi}{T}, \quad n = 0, 1, 2, \ldots,
\]

and their corresponding eigenfunctions

\[
v_n(t) = \sqrt{\frac{2}{T}} \sin \left( n + \frac{1}{2} \right) \frac{\pi t}{T}, \quad n = 0, 1, \ldots,
\]

which form an orthonormal basis of the space \( L_2(0, T) \). We use expansions of the functions \( u(t, x) \) and \( f(t, x) \) in the system \( \left\{ \sin \left( n + \frac{1}{2} \right) \frac{\pi t}{T} \right\}_{n=0}^{\infty} \) for fixed \( x \in [0, 1] \), as follows

\[
u(t, x) = \sum_{n=1}^{\infty} u_n(x) \cdot \sin \left( n + \frac{1}{2} \right) \frac{\pi t}{T} \quad (8)
\]

\[f(t, x) = \sum_{n=1}^{\infty} f_n(x) \cdot \sin \left( n + \frac{1}{2} \right) \frac{\pi t}{T}. \quad (9)
\]

By substitution to the Eq. (5) we obtain

\[
\sum_{n=1}^{\infty} \left( (-1)^n \left( n + \frac{1}{2} \right) \frac{\pi}{T} u_n(x) - \frac{d^2 u_n(x)}{dx^2} \right) \sin \left( n + \frac{1}{2} \right) \frac{\pi t}{T} = \sum_{n=1}^{\infty} f_n(x) \sin \left( n + \frac{1}{2} \right) \frac{\pi t}{T} \quad (10)
\]

By comparing the terms on the left and right sides we obtain

\[
(-1)^n \left( n + \frac{1}{2} \right) \frac{\pi}{T} u_n(x) - \frac{d^2 u_n(x)}{dx^2} = f_n(x)
\]

\[u_n(0) = \frac{d u_n(0)}{dx} = 0. \]

We assume that the solution is represented by Haar wavelets (3) as follows

\[
\frac{d^2 u_n(x)}{dx^2} = \sum_{i=1}^{2M} a_i h_i(x) \quad (11)
\]

By integrating the Eq. (11) in the interval \([0, x]\) we have

\[
\frac{d u_n(x)}{dx} = \sum_{i=1}^{2M} a_i P_{i1}(x) + \frac{d u_n(0)}{dx}
\]

\[u_n(x) = \sum_{i=1}^{2M} a_i P_{i2}(x) + u_n(0)\]
where \( P_{i1}(x) \) and \( P_{i2}(x) \) are defined as in Eq. (4). Substitute to ordinary differential equations

\[
(-1)^n \left( n + \frac{1}{2} \right) \frac{\pi}{T} \sum_{i=1}^{2M} a_i P_{i2}(x) - \sum_{i=1}^{2M} a_i P_{i1}(x) = f_n(x)
\]

After we simplify

\[
\sum_{i=1}^{2M} a_i \left( (-1)^n \left( n + \frac{1}{2} \right) \frac{\pi}{T} P_{i2}(x) - P_{i1}(x) \right) = f_n(x)
\]

We evaluate the latter in the following collocation points \( x_l = \frac{2l-1}{4M}, l = 1, 2, ..., 2M \).

\[
\sum_{i=1}^{2M} a_i \left( (-1)^n \left( n + \frac{1}{2} \right) \frac{\pi}{T} P_{i2}(x_l) - P_{i1}(x_l) \right) = f_n(x_l)
\]

The evaluation at collocation points leads to

\[
U = Pa
\]

\[
U = (u_n(x_1), u_n(x_2), ..., u_n(x_l))^T,
\]

\[
a = (a_1, a_2, ..., a_n)^T,
\]

\[
P = \begin{pmatrix}
p_{21}(x_1) & \cdots & p_{2l}(x_1) \\
\vdots & \ddots & \vdots \\
p_{21}(x_l) & \cdots & p_{2l}(x_l)
\end{pmatrix}.
\]

By solving the system, we obtain

\[
a = P^{-1}U
\]

which substitute into the Eq. (11) yields the following

\[
\frac{d^2u_n(x)}{dx^2} = HP^{-1}U,
\]

where we denoted

\[
H = \begin{pmatrix}
h_1(x_1) & \cdots & h_l(x_1) \\
\vdots & \ddots & \vdots \\
h_1(x_l) & \cdots & h_l(x_l)
\end{pmatrix}.
\]

4. Conclusion

We have developed new methods based on wavelet expansion are considered to solve problems arising in approximation of the solution of heat equation with involution. New numerical techniques are modified to solve heat equation with involution and obtained new approximative representation for solution of heat equations.
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