AdS spacetime in Lorentz covariant gauges

P. Valtancoli

Dipartimento di Fisica, Polo Scientifico Università di Firenze
and INFN, Sezione di Firenze (Italy)
Via G. Sansone 1, 50019 Sesto Fiorentino, Italy

Abstract

We show how to generate the AdS spacetime metric in general Lorentz covariant gauges. In particular we propose an iterative method for solving the Lorentz gauge.
1 Introduction

Recently there has been a renewed interest in the cosmological constant of general relativity, since it could be useful as a cheap explanation for the antigravity force which drives the accelerated expansion of the universe, better known as dark energy [1]-[2]. It is now widely accepted that the cosmological constant is a very small parameter, analogously to what happens for the neutrino mass, and related to the vacuum energy of the quantum fields [3].

The AdS space-time [4] is usually represented in a Lorentz non-covariant form, but there are physical applications like for example the study of cosmological gravitational waves [5], in which it is useful to reformulate it in a Lorentz covariant form. It is the purpose of this paper to generate all the Lorentz covariant solutions for AdS space-time. We start from solving perturbatively the Einstein equations up to the second order in $\Lambda$ in the Lorentz gauge, which turns out to be a tedious calculation.

To simplify the discussion we are going to present a general 5-dimensional representation of the AdS space-time in terms of an arbitrary function $f(\Lambda x^2)$, where $x^2 = \eta^{\mu\nu}x_\mu x_\nu$. We then show that the Lorentz gauge fixes this arbitrary function with a recursive method and the result agrees with the direct perturbative solution of the Einstein equations.

2 Perturbative calculation of the Einstein equations

We are going to discuss all the aspects of the AdS space-time in the Lorentz gauge. We start performing the perturbative calculation directly using the Einstein equations, and only afterwards we will introduce a faster method. We recall the known perturbative solution of the AdS space-time at the first order in $\Lambda$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{(1)\Lambda} + O(\Lambda^2)$$

$$h_{\mu\nu}^{(1)\Lambda} = -\frac{\Lambda}{9} ( x_\mu x_\nu + 2\eta_{\mu\nu}x^2 ) \quad (2.1)$$

It is straightforward to show that eq. (2.1) satisfies the Lorentz gauge:

$$\eta^{\mu\nu'}\partial_{\nu'}h_{\mu\nu}^{(1)\Lambda} = \frac{1}{2}\partial_\nu(\eta^{\mu\nu'}h_{\mu\nu}^{(1)\Lambda}) \quad (2.2)$$

The associated connection is, at this order, given by:

$$\Gamma_{\alpha\beta}^{(1)\Lambda} = -\frac{\Lambda}{9} ( 2\delta_\alpha^\mu x_\beta + 2\delta_\beta^\mu x_\alpha - \eta_{\alpha\beta}x^\mu ) \quad (2.3)$$
We can easily compute the curvature tensor, limited at the linear term in the connection

\[ R^{(1)\alpha}_{\mu\beta\nu} = \frac{\Lambda}{3} (\eta_{\beta\nu} \delta^\alpha_{\mu} - \eta_{\beta\mu} \delta^\alpha_{\nu}) \]  

(2.4)

The corresponding Ricci tensor and the curvature are then

\[ R^{(1)\beta\nu}_{\beta\nu} = \Lambda \eta_{\beta\nu} \quad R^{(1)\Lambda} = \eta^{\beta\nu} R^{(1)\beta\nu} = 4\Lambda \]  

(2.5)

solving perturbatively the Einstein equations.

Now we are going to compute the perturbative metric at the second order in \( \Lambda \)

\[ g_{\mu\nu} = \eta_{\mu\nu} + h^{(1)\Lambda}_{\mu\nu} + h^{(2)\Lambda}_{\mu\nu} + O(\Lambda^3) \]  

(2.6)

We can again impose the Lorentz gauge

\[ \eta^{\mu\mu'} \partial_{\mu'} h^{(2)\Lambda}_{\mu\nu} = \frac{1}{2} \partial_{\nu}(\eta^{\mu\mu'} h^{(2)\Lambda}_{\mu\mu'}) \]  

(2.7)

We must be careful that the connection is made by several pieces

\[ \Gamma^{(2)\alpha}_{\alpha\beta} = \Gamma^{(2)\alpha}_{\alpha\beta} \quad + \Gamma^{(2)\alpha}_{\alpha\beta} \quad II \]  

\[ \Gamma^{(2)\alpha}_{\alpha\beta} \quad I = \frac{1}{2} \eta^{\mu\mu'} \left( \partial_{\alpha} h^{(2)\Lambda}_{\mu\beta} + \partial_{\beta} h^{(2)\Lambda}_{\mu\alpha} - \partial_{\mu} h^{(2)\Lambda}_{\alpha\beta} \right) \]  

\[ \Gamma^{(2)\alpha}_{\alpha\beta} \quad II = -\frac{1}{2} h^{(1)\Lambda}_{\beta\nu} \left( \partial_{\alpha} h^{(1)\Lambda}_{\mu\beta} + \partial_{\beta} h^{(1)\Lambda}_{\mu\alpha} - \partial_{\mu} h^{(1)\Lambda}_{\alpha\beta} \right) \]  

(2.8)

where \( h^{(1)\Lambda}_{\mu\nu} = \eta^{\mu\mu'} \eta^{\nu\nu'} h^{(1)\Lambda}_{\mu\nu} \).

Let us work out the second term

\[ \Gamma^{(2)\alpha}_{\beta\nu} \quad II = -\frac{\Lambda^2}{81} \left( 4x^\alpha x_\beta x_\nu - 3\eta_{\beta\nu} x^\alpha x^2 + 4\delta^\alpha_{\beta} x_\nu x^2 + 4\delta^\alpha_{\nu} x_\beta x^2 \right) \]  

(2.9)

Its contribution to the curvature tensor is given by

\[ R^{(2)\alpha}_{\beta\mu\nu} \quad II = \partial_{\mu} \Gamma^{(2)\alpha}_{\beta\nu} \quad II - (\mu \leftrightarrow \nu) = \]  

\[ = -\frac{\Lambda^2}{81} \left[ 10\eta_{\beta\mu} x_\nu x^\alpha - 10\eta_{\beta\nu} x_\mu x^\alpha + 4\delta^\alpha_{\beta} x_\nu x^2 \right. \]  

\[ -4\delta^\alpha_{\mu} x_\nu x^2 + 7\delta^\alpha_{\nu} \eta_{\beta\mu} x^2 - 7\delta^\alpha_{\mu} \eta_{\beta\nu} x^2 \]  

(2.10)
The Ricci tensor

\[ R^{(2)\Lambda \Gamma}_{\beta \nu} = \delta^\mu_\alpha R^{(2)\Lambda \mu \Gamma}_{\beta \alpha \nu} = \frac{\Lambda^2}{81} \left( 2x_\beta x_\nu + 31\eta_{\beta \nu}x^2 \right) \]  

(2.11)

and its contribution to the Einstein equation is

\[ R^{(2)\Lambda \Gamma}_{\beta \nu} - \frac{1}{2} R^{(2)\Lambda \Gamma}_{\alpha \mu \nu} \eta^\mu_{\beta \nu} = \frac{2\Lambda^2}{81} \left( x_\beta x_\nu - 16\eta_{\beta \nu}x^2 \right) \]  

(2.12)

The first term of the connection produces the following curvature tensor \( \Gamma^\mu_{\alpha \beta} R^{(2)\Lambda \mu \Gamma}_{\alpha \beta \nu} = -\frac{1}{2} \eta^{\rho \sigma} \partial_\rho \partial_\sigma h^{(2)\Lambda}_{\beta \nu} \) giving rise to the another contribution to the Einstein equations

\[ R^{(2)\Lambda \Gamma}_{\beta \nu} - \frac{1}{2} R^{(2)\Lambda \Gamma}_{\alpha \mu \nu} \eta^\mu_{\beta \nu} = -\frac{1}{2} \eta^{\rho \sigma} \partial_\rho \partial_\sigma \left( h^{(2)\Lambda}_{\beta \nu} - \frac{1}{2} \eta_{\beta \nu} h^{(2)\Lambda} \right) \]  

(2.13)

(2.14)

Till now we have worked out in the curvature tensor only linear terms in the connections. Let us add now the non linear ones:

\[ R^{(2)\Lambda \Gamma}_{\alpha \beta \mu \nu} = \Gamma^{(1)\Lambda}_{\mu \alpha} \Gamma^{(1)\Gamma}_{\beta \nu} - (\mu \leftrightarrow \nu) = \]  

\[ \frac{\Lambda^2}{81} \left[ 4\delta^\beta_\mu x_\nu x_\beta - 2\delta^\beta_\mu \eta_{\beta \nu}x^2 - 4\delta^\beta_\nu x_\mu x_\beta + 
+ 2\delta^\beta_\nu \eta_{\beta \mu} x^2 - \eta_{\beta \mu} x^\alpha x_\nu + \eta_{\beta \nu} x^\alpha x_\mu \right] \]  

(2.15)

The associated Ricci tensor is

\[ R^{(2)\Lambda \Gamma}_{\beta \nu} = \frac{\Lambda^2}{81} \left( 11x_\nu x_\beta - 5\eta_{\beta \nu}x^2 \right) \]  

(2.16)

giving rise to the third contribution to the Einstein equations

\[ R^{(2)\Lambda \Gamma}_{\beta \nu} - \frac{1}{2} R^{(2)\Lambda \Gamma}_{\alpha \mu \nu} \eta^\mu_{\beta \nu} = \frac{\Lambda^2}{81} \left( 11x_\nu x_\beta - \frac{1}{2} \eta_{\beta \nu}x^2 \right) \]  

(2.17)

We have not yet finished. We must add some residual extra contributions, given by

\[ \frac{1}{2} R^{(2)\Lambda \Gamma}_{\alpha \beta \mu \nu} = h^{(1)\Lambda \beta \nu} R^{(1)\Lambda}_{\beta \alpha \nu} \rightarrow -\frac{1}{2} R^{(2)\Lambda \Gamma}_{\beta \nu} \eta_{\beta \nu} = -\frac{\Lambda^2 x^2}{2} \eta_{\beta \nu} \]  

(2.18)
and finally
\[-\frac{1}{2} R_{\beta\nu}^{(1)A} h_{\beta\nu}^{(1)A} + \Lambda h_{\beta\nu}^{(1)A} = -\Lambda h_{\beta\nu}^{(1)A}\] (2.19)

By collecting all the various (five) terms
\[-\frac{1}{2} \eta^{\rho\rho'} \partial_\rho \partial_{\rho'} \left( h_{\beta\nu}^{(2)A} - \frac{1}{2} \eta_{\beta\nu} h_{\beta\nu}^{(2)A} \right) + \frac{2\Lambda^2}{81} \left( x_\beta x_\nu - 16\eta_{\beta\nu} x^2 \right) + \frac{\Lambda^2}{81} \left( 11x_\nu x_\beta - \frac{1}{2} \eta_{\beta\nu} x^2 \right) - \frac{\Lambda^2 x^2}{2} \eta_{\beta\nu} + \frac{\Lambda^2}{9} (x_\beta x_\nu + 2\eta_{\beta\nu} x^2) = 0\] (2.20)

we arrive at the final equation
\[\eta^{\rho\rho'} \partial_\rho \partial_{\rho'} \left( h_{\beta\nu}^{(2)A} - \frac{1}{2} \eta_{\beta\nu} h_{\beta\nu}^{(2)A} \right) = \frac{22}{81} (2x_\beta x_\nu - 5\eta_{\beta\nu} x^2)\] (2.21)

The compatibility between this equation and the Lorentz gauge is straightforward since applying the operator \(\partial_\beta\) (or \(\partial_\nu\)) to both sides of this equation we simply get zero.

Finally we obtain the solution of this equation at the second order in \(\Lambda\):
\[
h_{\beta\nu}^{(2)A} = \frac{11}{16 \cdot 81} \Lambda^2 (4x_\beta x_\nu x^2 + 5\eta_{\beta\nu} x^4)
\]
\[h_{\beta\nu}^{(2)A} = \eta^{\beta\nu} h_{\beta\nu}^{(2)A} = \frac{11}{54} \Lambda^2 x^4 \equiv (x^2)^2\] (2.22)

We have just learned that the Lorentz gauge imposes strong constraints on the general solution. Let us suppose to make the perturbative calculation at the generic order \(n\):
\[g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{(1)A} + \ldots + h_{\mu\nu}^{(n)A} + \ldots\] (2.23)

the final equation, similar to eq. (2.21), at the order \(n\) is of the type
\[\eta^{\rho\rho'} \partial_\rho \partial_{\rho'} \left( h_{\beta\nu}^{(n)A} - \frac{1}{2} \eta_{\beta\nu} h_{\beta\nu}^{(n)A} \right) = \left( A_n x_\beta x_\nu x^{2(n-2)} + B_n \eta_{\beta\nu} x^{2(n-1)} \right)\] (2.24)

Let us apply the operator \(\partial_\beta\) (or \(\partial_\nu\)) to both sides of this equation
\[\partial_\beta (\ldots) = 0 \rightarrow B_n = -\frac{2n + 1}{2(n - 1)} A_n\] (2.25)
the Lorentz gauge implies the constraint

$$\eta^\rho\partial_\rho \partial_\rho' \left( h^{(n)\Lambda}_{\beta\nu} - \frac{1}{2} \eta_{\beta\nu} h^{(n)\Lambda}_\mu \right) = c_n \Lambda^n \left[ 2(n-1)x_\beta x_\nu x^{2(n-2)} - (2n+1)\eta_{\beta\nu} x^{2(n-1)} \right]$$  (2.26)

Therefore the solution at the order $n$ is of the type

$$h^{(n)\Lambda}_{\beta\nu} = c_n \Lambda^n \left[ \frac{x_\beta x_\nu x^{2(n-1)}}{2(n+2)} + \frac{(n+3)\eta_{\beta\nu} x^{2n}}{4n(n+2)} \right]$$

$$h^{(n)\Lambda} = \eta^{\beta\nu} h^{(n)\Lambda}_{\beta\nu} = \frac{3}{2n} c_n \Lambda^n x^{2n} \quad x^{2n} \equiv (x^2)^n$$  (2.27)

Unfortunately the $c_n$ coefficients cannot be computed only from the Lorentz gauge and must be determined by solving the Einstein equation.

### 3 General solution of the Einstein equations

To solve the Einstein equation, we are going to embed the $AdS$ space-time in the following 5$d$ space

$$ds^2 = dX_0^2 + dX_1^2 - dX_2^2 - dX_3^2 - dX_4^2$$  (3.1)

subject to the constraint

$$X_0^2 + \eta^{ij} X_i X_j = \frac{3}{\Lambda} \quad i = 1, ..., 4$$  (3.2)

We can automatically generate solutions of the Einstein equations simply determining the mapping $X_0 = X_0(x_\mu), X_i = X_i(x_\mu)$. In this choice we must respect the Lorentz covariance of the 4$d$ space-time and we are forced to choose this mapping

$$X_0 = \sqrt{\frac{3}{\Lambda}} \sqrt{1 - \frac{\Lambda}{3} x^2 f^2(\Lambda x^2)}$$

$$X_i = x_i f(\Lambda x^2)$$  (3.3)

with the 4$d$ metric defined as usual by the formula
\[ g_{\mu\nu} = \eta^{ab} \frac{dX_a}{dx^\mu} \frac{dX_b}{dx^\nu} \]  \hspace{1cm} (3.4)

All these mappings produce solutions of the Einstein equation in a covariant gauge and depend on an arbitrary function \( f(\Lambda x^2) \). First let us study the basic solution with \( f(\Lambda x^2) = 1 \):

\[ X_0 = \sqrt{\frac{3}{\Lambda}} \sqrt{1 - \frac{\Lambda}{3} x^2} \quad X_i = x_i \]

\[ g^{(0)}_{\mu\nu} = \eta_{\mu\nu} + \frac{\Lambda}{3} \frac{x_\mu x_\nu}{1 - \frac{\Lambda}{3} x^2} \]  \hspace{1cm} (3.5)

We can easily verify that it solves the Einstein equations exactly. The inverse metric is given by

\[ g^{\mu\nu(0)} = \eta^{\mu\nu} - \frac{\Lambda}{3} x^\mu x^\nu \]  \hspace{1cm} (3.6)

We can compute the associated connection

\[ \Gamma_{\alpha\beta}^{\mu(0)} = \frac{\Lambda}{3} x^\mu g_{\alpha\beta}^{(0)} \]  \hspace{1cm} (3.7)

and the complete curvature tensor

\[ R_{\beta\mu\nu}^{(0)} = \partial_\mu \Gamma_{\beta\nu}^{\alpha(0)} + \Gamma_{\sigma\mu}^{\alpha(0)} \Gamma_{\beta\nu}^{\sigma(0)} - (\mu \leftrightarrow \nu) = \frac{\Lambda}{3} \left( \delta_\mu^{\alpha} g_{\beta\nu}^{(0)} - \delta_\nu^{\alpha} g_{\beta\mu}^{(0)} \right) \]  \hspace{1cm} (3.8)

Analogously the Ricci tensor and the curvature have simple forms

\[ R_{\beta\nu}^{(0)} = \Lambda g_{\beta\nu}^{(0)} \quad R^{(0)} = 4\Lambda \]  \hspace{1cm} (3.9)

solving exactly the Einstein equations. Let us note that this solution ( similarly to those with an arbitrary function \( f(\Lambda x^2) \) ) is singular for large events of the order

\[ x^2 \simeq \frac{1}{\Lambda} \]  \hspace{1cm} (3.10)

This singularity is present both for positive and negative cosmological constant.
It is possible fixing the arbitrary function $f(\Lambda x^2)$ adding into this scheme the Lorentz gauge condition

$$\eta^{\mu\nu'} \partial_{\nu'} h_{\mu\nu} = \frac{1}{2} \partial_{\nu}(\eta^{\mu\nu'} h_{\mu\nu'})$$

(3.11)

By substituting the definition of the metric in terms of the mapping $X^a = X^a(x_i)$ we get

$$\eta^{ab} \eta^{\mu\nu'} \partial_{\mu} \partial_{\nu'} X_a \cdot \partial_{\nu} X_b = 0$$

(3.12)

The variables $X_a$ are constrained leading to the property $\eta^{ab} X_a \cdot \partial_{\nu} X_b = 0$, therefore we must impose that

$$\eta^{\mu\nu'} \partial_{\mu}\partial_{\mu'} X_a = \lambda(\Lambda x^2) X_a$$

(3.13)

where we have introduced a second function $\lambda(\Lambda x^2)$. We are going to show that this equation (3.13) determines univocally both functions $f(\Lambda x^2)$ and $\lambda(\Lambda x^2)$.

Let us introduce the following power series

$$\lambda(\Lambda x^2) = \sum_{n=1}^{\infty} \lambda_n \Lambda^n x^{2(n-1)}$$

$$f(\Lambda x^2) = 1 + \sum_{n=1}^{\infty} f_n \Lambda^n x^{2n}$$

(3.14)

The coefficients $\lambda_n$ are determined by the eq. for $X_0$ while the coefficients $f_n$ are determined by the eq. for $X_i$ in a recursive scheme. We start from the lowest order

$$X_0 \simeq \sqrt{\frac{3}{\Lambda}} \left(1 - \frac{\Lambda}{6} x^2\right) \rightarrow \lambda_1 = -\frac{4}{3}$$

(3.15)

The coefficient $\lambda_1$ allows to compute the coefficient $f_1$ by using the equation for $X_i$

$$\eta^{\rho\rho'} \partial_{\rho} \partial_{\rho'} X_i \simeq \eta^{\rho\rho'} \partial_{\rho} \partial_{\rho'} (x_i \cdot f_1 \Lambda x^2) = -\frac{4}{3} \Lambda x_i$$

(3.16)

from which we obtain

$$f_1 = -\frac{1}{9} \rightarrow f(\Lambda x^2) = 1 - \frac{\Lambda}{9} x^2 + O(\Lambda^2 x^4)$$

(3.17)
Going back to the eq. for $X_0$ at the next order in $\Lambda$, we can use $f_1$ to determine $\lambda_2$ as follows

$$X_0 \simeq \sqrt{\frac{3}{\Lambda}} \left(1 - \frac{\Lambda}{6} x^2 + \frac{5}{9 \cdot 24} \Lambda^2 x^4 + O(\Lambda^3 x^6)\right)$$  \quad (3.18)$$

from which we compute $\lambda_2 = \frac{1}{3}$ and therefore

$$\lambda(\Lambda x^2) = -\frac{4}{3} \Lambda + \frac{1}{3} \Lambda^2 x^2 + O(\Lambda^3 x^4)$$  \quad (3.19)$$

Repeating the iteration, the coefficient $\lambda_2$ determines $f_2$ at the next order in $\Lambda$ by using the equation for $X_i$

$$f_2 = \frac{13}{27 \cdot 32}$$  \quad (3.20)$$

from which we know $X_i$ up to the second order in $\Lambda$

$$X_i \simeq x_i \left(1 - \frac{\Lambda}{9} x^2 + \frac{13}{27 \cdot 32} \Lambda^2 x^4 + O(\Lambda^3 x^6)\right)$$  \quad (3.21)$$

By substituting these findings in the equation (3.4), defining the metric in terms of the mapping $X_a(x_i)$

$$g_{\mu\nu} = \eta^{ab} \frac{dX_a}{dx^\mu} \frac{dX_b}{dx^\nu} = \eta_{\mu\nu} + h^{(1)\Lambda}_{\mu\nu} + h^{(2)\Lambda}_{\mu\nu} + ...$$  \quad (3.22)$$

we recover the perturbative solutions (2.1) e (2.22)

$$h^{(1)\Lambda}_{\mu\nu} = -\frac{\Lambda}{9} (x_\mu x_\nu + 2 \eta_{\mu\nu} x^2)$$

$$h^{(2)\Lambda}_{\mu\nu} = \frac{11}{16 \cdot 81} \Lambda^2 (4 x_\mu x_\nu x^2 + 5 \eta_{\mu\nu} x^4)$$  \quad (3.23)$$

obtained with a very laborious calculation from the Einstein equations.

4 Conclusion

The $AdS$ space-time is an example of curved space-time in the absence of matter. In this paper, we have searched how to describe it with a Lorentz covariant coordinate system,
which is better suited for discussing the propagation of gravitational waves in such a curved space.

We first built the perturbative solution of the Einstein equations with a cosmological constant $\Lambda$ in the Lorentz gauge, and then introduced a more systematic method to construct the solution by embedding the $AdS$ space-time into a $5d$ space with a quadratic constraint between the coordinates.

We have succeeded to identify the right mapping $X_a = X_a(x_i)$ which allows us to reach the Lorentz gauge and constructed an iterative method to calculate the $AdS$ solution to the various perturbative orders in $\Lambda$. This method is the easiest way to solve the Lorentz gauge condition if compared to the direct perturbative calculation of the Einstein equations.

Once the structure of the $AdS$ space-time has been accurately identified in a Lorentz covariant coordinate system, we believe that it will be easier to study the propagation of a gravitational wave in the presence of such a background.

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