Additively weighted Hamming index of graphs

Mardiningsih and Saib Suwilo
Department of Mathematics, Universitas Sumatera Utara, Medan-Indonesia 20155
E-mail: *saib@usu.ac.id

Abstract. Hamming distance between two vertices of a finite simple graph is defined to be the Hamming distance of rows of (0,1)-matrix of the graph correspond to the two vertices. Hamming index of a graph is the sum of Hamming distances between all pairs of vertices in the graph. In this paper, we introduce the notion of additively weighted Hamming index of graph where the Hamming distance between two vertices is weighted by the sum of the degrees of the two vertices. We discuss the additively weighted Hamming index of a graph with respect to adjacency and incidence matrix of the graph. We relate the additively weighted Hamming index of a graph to the order of the graph, the size of the graph and the degree of each vertex in the graph. We then use this relationship to determine the additively weighted Hamming index of graph whose vertices have almost uniform degree.

1. Introduction
Throughout of this paper we let $G$ to be finite graph on $s$ vertices $V(G) = \{v_1, v_2, \ldots, v_s\}$ and $t$ edges $E(G) = \{e_1, e_2, \ldots, e_t\}$. We say $G$ is of order $s$ and size $t$ and is denoted by $(s, t)$-graph. We assume that $G$ is simple, that is $G$ has no loops and has no multiple edges. Notation $v_k \sim v_\ell$ means that the vertex $v_k$ is adjacent to the vertex $v_\ell$. The notation $d(v_i)$ means the degree of a vertex $v_i$, that is the number of vertices adjacent to $v_i$. The neighbor a vertex $v_i$ is the set $N(v_i) = \{v_k \in V(G) : v_k \sim v_i\}$. The common neighbor of vertices $v_i$ and $v_j$ is the set $N(v_i, v_j) = \{v_k \in V(G) : v_k \sim v_i$ and $v_k \sim v_j\}$. A graph $G$ is said to be $r$-regular if each vertex of $G$ is of degree $r$. The distance between distinct vertices $v_k$ and $v_\ell$, denoted $d_G(v_k, v_\ell)$, is the length of a shortest path connecting $v_k$ and $v_\ell$ [1].

The Wiener index of a graph $G$ is defined to be

$$W(G) = \sum_{v_k, v_\ell \in V(G)} d_G(v_k, v_\ell).$$

Weighted Wiener index has also been discussed in the literatures. The additively weighted Wiener index of a graph $G$ is defined to be

$$W_S(G) = \sum_{v_k, v_\ell \in V(G)} (d(v_k) + d(v_\ell))d_G(v_k, v_\ell).$$

Discussion on Wiener index and weighted Wiener index of simple graphs can be found in [2–4].

Consider a $(0,1)$-matrix $Z$ of size $s$ by $t$. For each $k = 1, 2, \ldots, s$, the symbol $Z(k, :)$ means the $k$th row of $Z$. Notice that $Z(k, :)$ is a binary digit string of length $t$. By wt($Z(k, :)$) we mean the weight of $Z(k, :)$. That is the number of digit 1 that appears on $Z(k, :)$. For any two rows
Z(k,:) and Z(ℓ,:) of Z, by \( H_d(Z(k,:), Z(ℓ,:)) \) we mean the Hamming distance between Z(k,:) and Z(ℓ,:), that is the number of position \( p \) such that the digit of \( Z(k,p) \) is different from the digit of \( Z(ℓ,p) \). We note that \( H_d(Z(k,:), Z(ℓ,:)) = wt(Z(k,:) + Z(ℓ,:)) \) where the addition is taken to be the addition modulo 2.

For a graph \( G \), we let \( Z \) to be a \((0,1)\)-matrix obtained from \( G \). The matrix \( Z \) could be an \( s \) by \( s \) adjacency matrix of \( G \) or \( Z \) could also be an \( s \) by \( t \) incidence matrix of \( G \). Let \( Z(k,:) \) be the row of \( Z \) corresponds to the vertex \( v_k \) of \( G \). The Hamming distance between two vertices \( v_k \) and \( v_ℓ \) with respect to the \((0,1)\)-matrix \( Z \) of \( G \), denoted by \( H_d(v_k, v_ℓ : Z) \), is the Hamming distance between rows \( Z(k,:) \) and \( Z(ℓ,:) \) of the \((0,1)\)-matrix \( Z \) of \( G \). That is

\[
H_d(v_k, v_ℓ : Z) = H_d(Z(k,:), Z(ℓ,:)).
\]

In a similar fashion to Wiener index, the Hamming index of a finite simple graph \( G \) with respect to the \((0,1)\)-matrix \( Z \) of \( G \) is defined to be

\[
H(G : Z) = \sum_{v_k,v_ℓ \in V(G)} H_d(v_k, v_ℓ : Z) = \sum_{k≠ℓ} H_d(Z(k,:), Z(ℓ,:)).
\]

Discussion on Hamming distance between two vertices and Hamming index of a graph can be found on [5–11].

In this paper we introduce a new invariant of a graph \( G \) called the additively weighted Hamming index of a graph. We define the additively weighted Hamming index of a graph \( G \) with respect to the \((0,1)\)-matrix \( Z \) of \( G \) to be

\[
H_S(G : Z) = \sum_{v_k,v_ℓ \in V(G)} (d(v_k) + d(v_ℓ))H_d(v_k, v_ℓ : Z) \tag{1}
\]

We express the \( H_S(H : Z) \) in terms of the order of \( G \), the size of \( G \) and the degree of each vertex of \( G \). In Section 2, we briefly review the Hamming distance between two vertices in a graph with respect to a \((0,1)\)-matrix of the graph. In Section 3, we discuss the additively weighted Hamming index of a graph.

2. Necessary background

We review some basic properties of adjacency and incidence matrix of a graph. We then discuss an algorithm to calculate the additively weighted Hamming index of a graph. Finally, we present formulae for Hamming distance between two vertices in a graph.

An adjacency matrix \( A = (a_{kℓ}) \) of a \((s,t)\)-graph \( G \) is an \( s \) by \( s \) \((0,1)\)-matrix defined by

\[
a_{kℓ} = \begin{cases} 1, & \text{if } v_k \sim v_ℓ \\ 0, & \text{if } v_k \not\sim v_ℓ. \end{cases}
\]

We note that for each \( k = 1, 2, \ldots, s \), we have \( wt(A(k,:)) = d(v_k) \).

An incidence matrix \( B = (b_{kℓ}) \) of a \((s,t)\)-graph \( G \) is an \( s \) by \( t \) \((0,1)\)-matrix defined by

\[
b_{kℓ} = \begin{cases} 1, & \text{if } v_k \text{ is incident to } e_ℓ \\ 0, & \text{otherwise}. \end{cases}
\]

Notice that for each \( k = 1, 2, \ldots, s \), \( wt(B(k,:)) = d(v_k) \).

In the following we present an algorithm for calculating the weighted Hamming index of a graph \( G \). The input of the algorithm is the \((0,1)\)-matrix \( Z \) of \( G \), that is \( Z \) is an adjacency matrix \( A \) of \( G \) or an incidence matrix \( B \) of \( G \).
Algorithm 1. Additively weighted Hamming index

| Input: (0,1)-matrix $Z$ of $G$ |
|---|
| Output: Additively weighted Hamming index $HS$ |
| 1 $HS = 0$ |
| 2 for $k = 1$ to $s - 1$ |
| 3 for $\ell = k + 1$ to $s$ |
| 4 $S = Z(k,:) + Z(\ell,:)$ |
| 5 $HS = HS + (\text{wt}(Z(k,:)) + \text{wt}(Z(\ell,:)))\text{wt}(S)$ |
| 6 end |
| 7 end |

We note that the addition on Line 4 of Algorithm 3 is done by using addition modulo 2. An implementation of Algorithm 3 in MATLAB programming can be found on Example 4.

Example 2. MATLAB code for algorithm 1

```matlab
function hs=hamming(matrix01)
[s,t]=size(matrix01);
hs=0;
for i=1:s-1
    for j=i+1:s
        degreei=sum(matrix01(i,:));
        degreej=sum(matrix01(j,:));
        summod2=mod(matrix01(i,:)+matrix01(j,:),2);
        distance=sum(summod2);
        weights=(degreei+degreej)*distance;
        hs=hs+weights;
    end
end
```

The following two lemmas, that relate the Hamming distance between two vertices to the degree of the corresponding vertices, are fundamental for our result.

Lemma 3. [7] If $A$ is an adjacency matrix of a graph $G$, then for any pair of distinct vertices $v_k$ and $v_\ell$ we have $H_d(v_k,v_\ell : A) = d(v_k) + d(v_\ell) - 2|N(v_k,v_\ell)|$.

Lemma 4. [8] If $B$ is an incidence matrix of a graph $G$, then

$$H_d(v_k,v_\ell : B) = \begin{cases} d(v_k) + d(v_\ell) - 2 & \text{if } v_k \sim v_\ell \\ d(v_k) + d(v_\ell) & \text{if } v_k \not\sim v_\ell \end{cases}$$

for all distinct vertices $v_k$ and $v_\ell$ in $G$.

3. Weighted Hamming Index

We recall that for a $(s,t)$-graph the definition of additively weighted Hamming index given in equation (1) requires $(\binom{s}{2})$ summand. In this section we present formulae to calculate the additively weighted Hamming index that requires only $s$ summand. We first discuss the additively weighted Hamming index with respect to adjacency matrix.

Theorem 5. If $A$ is an adjacency matrix of a $(s,t)$-graph $G$, then

$$H_S(G : A) = 4t^2 + s \sum_{k=1}^{s} d(v_k)^2 - 2 \sum_{k=1}^{s} d(v_k)d(N(v_k)).$$
Proof. From the definition in equation (1) we have $H_S(G : A) = \sum_{i \neq j} (d(v_i) + d(v_j))H_d(v_i, v_j : A)$. By lemma 3 we have $H_d(v_i, v_j : A) = d(v_i) + d(v_j) - 2|N(v_i, v_j)|$. This implies

$$H_S(G : A) = \sum_{i \neq j} (d(v_i) + d(v_j))H_d(v_i, v_j : A)$$

$$= \sum_{i \neq j} (d(v_i) + d(v_j))(d(v_i) + d(v_j) - 2|N(v_i, v_j)|)$$

$$= \sum_{i \neq j} (d(v_i) + d(v_j))^2 - 2\sum_{i \neq j} (d(v_i) + d(v_j))|N(v_i, v_j)|$$

$$= \sum_{i \neq j} (d(v_i))^2 + d(v_j)^2 + 2\sum_{i \neq j} d(v_i) d(v_j)$$

$$- 2\sum_{k=1}^{s} \left( d(v_k) \sum_{\ell \neq k} |N(v_k, v_\ell)| \right). \quad (2)$$

We first simplify the expression $\sum_{i \neq j} (d(v_i))^2 + d(v_j)^2$. Since for each $k = 1, 2, \ldots, s$, the expression $d(v_k)^2$ appears $(s - 1)$ times on $\sum_{i \neq j} (d(v_i))^2 + d(v_j)^2$, we have

$$\sum_{i \neq j} (d(v_i))^2 + d(v_j)^2 = \sum_{k=1}^{s} (s - 1)d(v_k)^2 = (s - 1)\sum_{k=1}^{s} d(v_k)^2. \quad (3)$$

Notice that

$$2\sum_{i \neq j} d(v_i) d(v_j) = \sum_{k=1}^{s} \left( d(v_k) \sum_{\ell \neq k} d(v_\ell) \right).$$

Since $\sum_{\ell \neq k} d(v_\ell) = 2t - d(v_k)$, we now have

$$2\sum_{i \neq j} d(v_i) d(v_j) = \sum_{k=1}^{s} d(v_k)(2t - d(v_k))$$

$$= 2t\sum_{k=1}^{s} d(v_k) - s\sum_{k=1}^{s} d(v_k)^2$$

$$= 4t^2 - \sum_{k=1}^{s} d(v_k)^2. \quad (4)$$

Let $Q_k = \sum_{\ell \neq k} |N(v_k, v_\ell)|$. We note that $|N(v_k, v_\ell)| \neq 0$ if and only if there is a path of length 2 connecting vertices $v_k$ and $v_\ell$. Therefore, if $v_p \sim v_k$, then the contribution of $v_p$ to $Q_k$ is $d(v_p) - 1$. This implies $Q_k = d(N(v_k)) - d(v_k)$. We now have that

$$2\sum_{k=1}^{s} \left( d(v_k) \sum_{\ell \neq k} |N(v_k, v_\ell)| \right) = 2\sum_{k=1}^{s} d(v_k)(d(N(v_k)) - d(v_k))$$

$$= 2\sum_{k=1}^{s} d(v_k)d(N(v_k)) - 2\sum_{k=1}^{s} d(v_k)^2. \quad (5)$$
Substitute equations (3), (4) and (5) to equation (2) we get

\[ H_S(G : A) = 4t^2 + s \sum_{i=1}^{s} d(v_i)^2 - 2 \sum_{i=1}^{s} d(v_i)d(N(v_k)), \]

and hence we have proved the theorem.

We note that theorem 5 is most easily applied when the vertices of the graph \( G \) have almost uniform degree. For \( p \geq 3 \), the wheel \( W_p \) is a graph on \( p + 1 \) vertices with vertex set \( V(G) = \{v_1, v_2, \ldots, v_{p+1}\} \) and edge set \( E(G) = \{(v_i, v_{i+1}) : i = 1, 2, \ldots, p - 1\} \cup \{(v_1, v_p)\} \cup \{(v_i, v_{p+1}) : i = 1, 2, \ldots, p\}. \)

Hence \( W_p \) is a \((p + 1, 2p)\)-graph. Notice that in \( W_p \), for \( k = 1, 2, \ldots, p \), we have \( d(v_k) = 3 \) and \( d(N(v_k)) = (6 + p) \). Furthermore, \( d(v_{p+1}) = p \) and \( d(N(v_{p+1})) = 3p \).

**Corollary 6.** If \( A \) is an adjacency matrix of a wheel \( W_p \), then \( H_S(W_p : A) = p^3 + 14p^2 - 27p \).

**Proof.** Since \( W_p \) is a \((p + 1, 2p)\)-graph, by theorem 5 we find that

\[
H_S(W_p : A) = 4(2p)^2 + (p + 1) \sum_{k=1}^{p+1} d(v_k)^2 - 2 \sum_{k=1}^{p+1} d(v_k)d(N(v_k))
\]

\[
= 16p^2 + (p + 1) \left( \sum_{k=1}^{p} 3^2 + p^2 \right) - 2 \left( \sum_{k=1}^{p} 3(6 + p) + p(3p) \right)
\]

\[
= 16p^2 + (p + 1)(9p + p^2) - 2(18p + 6p^2)
\]

\[
= p^3 + 14p^2 - 27p.
\]

**Corollary 7.** If \( A \) is an adjacency matrix of a \( r \)-regular \((s, t)\)-graph \( G \), then \( H_S(G : A) = 2s^2r^2 - 2sr^3 \).

**Proof.** Since \( G \) is \( r \)-regular, for each \( k = 1, 2, \ldots, s \) we have \( d(v_k) = r \) and \( d(N(v_k)) = r^2 \). Moreover, \( 2t = sr \). By theorem 5 we get

\[
H_S(G : A) = 4t^2 + s \sum_{k=1}^{s} r^2 - 2 \sum_{k=1}^{s} r(r^2)
\]

\[
= (2t)^2 + s^2r^2 - 2sr^3 = (rs)^2 + s^2r^2 - 2sr^3
\]

\[
= s^2r^2 - 2sr^3.
\]

So the result has been proven.

**Example 8.** Notice that a cycle of length \( s \), \( C_s \), is a \( 2 \)-regular graph and the complete graph \( K_s \) is a \((s - 1)\)-regular graph. Then for \( s \geq 3 \) we have

- \( H_S(C_s : A) = 8s(s - 2) \)
- \( H_S(K_s : A) = 2s(s - 1)^2 \).

We now discuss the additively weighted Hamming index of a graph \( G \) with respect to an incidence matrix \( B \) of the graph \( G \).

**Theorem 9.** If \( B \) is an incidence matrix of an \((s, t)\)-graph \( G \), then the additively weighted Hamming index \( H_S(G : B) = 4t^2 + (s - 4) \sum_{k=1}^{s} d(v_k)^2 \).
Proof. From definition we have

\[ H_S(G : B) = \sum_{i \neq j} (d(v_i) + d(v_j))H_d(v_i, v_j) \]

\[ = \sum_{v_i \sim v_j} (d(v_i) + d(v_j))H_d(v_i, v_j) + \sum_{v_i \not\sim v_j} (d(v_i) + d(v_j))H_d(v_i, v_j). \]

Considering lemma 4 we find that

\[ H_S(G : B) = \sum_{v_i \sim v_j} (d(v_i) + d(v_j))(d(v_i) + d(v_j) - 2) \]

\[ + \sum_{v_i \not\sim v_j} (d(v_i) + d(v_j))(d(v_i) + d(v_j)) \]

\[ = \sum_{i \neq j} (d(v_i) + d(v_j))^2 - 2 \sum_{v_i \sim v_j} (d(v_i) + d(v_j)) \]

(6)

The proof of theorem 5 shows that

\[ \sum_{i \neq j} (d(v_i) + d(v_j))^2 = 4t^2 + (s - 2) \sum_{k=1}^{s} d(v_k)^2. \]

(7)

We now consider the sum \( Q = \sum_{v_i \sim v_j} (d(v_i) + d(v_j)) \). We note that for each \( k = 1, 2, \ldots, v_k \), the expression \( d(v_k) \) appears in \( Q \) exactly \( d(v_k) \) times. Hence

\[ Q = \sum_{v_i \sim v_j} (d(v_i) + d(v_j)) = \sum_{k=1}^{s} d(v_k)^2. \]

(8)

Substitute equations (7) and (8) into equation (6) we find \( H_S(G : B) = 4t^2 + (s-4) \sum_{k=1}^{s} d(v_k)^2 \).

We now have completed the proof. \( \square \)

We next discuss the value of \( H_S(G : B) \) when \( G \) is a wheel and when \( G \) is a \( r \)-regular graph.

**Corollary 10.** If \( p \geq 3 \), then \( H_S(W_p : B) = p^3 + 22p^2 - 27p \).

Proof. Recall that \( W_p \) is a \( (p + 1, 2p) \)-graph, \( d(v_k) = 3 \) for \( k = 1, 2, \ldots, p \), and \( d(v_{p+1}) = p \). Hence, theorem 9 implies that

\[ H_S(W_p : B) = 4(2p)^2 + (p + 1 - 4) \sum_{k=1}^{p+1} d(v_k)^2 \]

\[ = 16p^2 + (p - 3) \left( \sum_{k=1}^{p} 3^2 + p^2 \right) \]

\[ = 16p^2 + (p - 3)(9p) + (p - 3)p^2 = p^3 + 22p^2 - 27p. \]

We now have completed the proof. \( \square \)

**Corollary 11.** If \( B \) is an incidence matrix of a \( r \)-regular \( (s, t) \)-graph \( G \), then \( H_S(G : B) = 2str^2(s - 2) \).
Proof. By theorem 9 we have \( H_S(G : B) = 4t^2 + (s - 4) \sum_{k=1}^{s} d(v_k)^2 \). Since \( G \) is \( r \)-regular, then \( 2t = sr \) and for each \( k = 1, 2, \ldots, s \) we have \( d(v_k) = r \). This implies

\[
H_S(G : B) = 4t^2 + (s - 4) \sum_{k=1}^{s} d(v_k)^2 = (sr)^2 + (s - 4)sr^2 = 2sr^2(s - 2).
\]

The proof is completed. \( \square \)

Example 12. Since \( C_s \) is a 2-regular graph and and \( K_s \) is a \((s - 1)\)-regular graph, we have from Corollary 11 that

- \( H_S(C_s : B) = 8s(s - 2) \)
- \( H_S(K_s : B) = 2s(s - 1)^2(s - 2) \).

References

[1] Diestel R 2000 *Graph Theory*, (New York: Springer).
[2] Wiener H 1947 *J. Am. Chem. Soc.* 69(1), 17–20.
[3] Dobrynin A and Kochetova A A 1994 *J. Chem. Inf. Comput. Sci.* 34(5), 1082–1086.
[4] Gutman I 1994 *J. Chem. Inf. Comput. Sci.* 34(5), 1087–1089.
[5] Ramane H S and Ganagi A B 2013 *International Journal of Current Engineering and Technology*, Special Edition 1, 205–208.
[6] Ganagi A B and Ramane H S 2016 *Algebra and Discrete Mathematics*, 22 (1), 82–93.
[7] Pasaribu R L, Mardiningshih and Suwilo S 2018 *Bulletin of Mathematics* 10(1), 25–32.
[8] Ramane H S, Yalnaik A and Gudodagi G A 2016 *International Journal of Mathematical Archive* 7(8), 7–12.
[9] Ramane H S, Joshi V B, Jummannaver R B, Manjalapur V V, Patil S C, Shindhe S D, Hadimani V S, Kyalkonda V K and Baddi B C 2015 *International Journal of Mathematics, Science and Engineering Application*, 9, No. I, 93–103.
[10] Sitorus W, Suwilo S and Mardiningsih 2018 *Proceedings of MICoMS 2017*, 621–628
[11] Ali S, Suwilo S and Mardiningsih 2019 *Journal of Physics: Conf. Series* 1225 012044.