Cosmological behavior in extended nonlinear massive gravity

Genly Leon\(^1\), Joel Saavedra\(^1\) and Emmanuel N Saridakis\(^{1,2}\)

\(^1\) Instituto de Física, Pontificia Universidad Católica de Valparaíso, Casilla 4950, Valparaíso, Chile
\(^2\) Physics Division, National Technical University of Athens, 15780 Zografou Campus, Athens, Greece

E-mail: genly.leon@ucv.cl, joel.saavedra@ucv.cl and Emmanuel_Saridakis@baylor.edu

Received 20 March 2013, in final form 1 May 2013
Published 3 June 2013
Online at stacks.iop.org/CQG/30/135001

Abstract

We perform a detailed dynamical analysis of various cosmological scenarios in extended (varying-mass) nonlinear massive gravity. Due to the enhanced freedom in choosing the involved free functions, this cosmological paradigm allows for a huge variety of solutions that can attract the universe at late times, compared to scalar-field cosmology or usual nonlinear massive gravity. Amongst others, it accepts quintessence, phantom or cosmological-constant-like late-time solutions, which moreover can alleviate the coincidence problem. These features seem to be general and non-sensitive to the imposed ansanțzes and model parameters, and thus extended nonlinear massive gravity can be a good candidate for the description of nature.

PACS numbers: 98.80.−k, 95.36.+x, 04.50.Kd

(Some figures may appear in colour only in the online journal)

1. Introduction

The idea of adding mass to the graviton is quite old [1], but the straightforward linear approach leads to the van Dam, Veltman, Zakharov discontinuity [2, 3], that is, the zero-mass limit of the obtained results does not provide the General Relativity results. This is due to the fact that not all the extra degrees of freedom, introduced by the graviton mass, decouple at the zero-mass limit, since the longitudinal graviton preserves a finite coupling to the trace of the energy–momentum tensor. This discontinuity can be removed if one incorporates nonlinear terms [4]; however, it was soon realized that these necessary nonlinear terms introduce the Boulware–Deser (BD) ghost degree of freedom [5], making the theory unstable.

However, recently, a specific nonlinear extension of massive gravity was formulated in [6, 7], requiring the BD ghost to be systematically removed (see [8] for a review). Such a construction is interesting at the theoretical level, since adding mass to a spin-2 particle is a well-defined problem by itself; however, it has an additional motivation, namely it is a
new class of (infra-red) gravity modification hoping to account for inflation and late-time acceleration. The theoretical and phenomenological advantages led to a significant amount of relevant research [9–70].

Despite the successes of nonlinear massive gravity, it was realized that the usual simple homogeneous and isotropic cosmological solutions are unstable at the perturbation level [71], which led to less symmetric models [72, 73]. However, in [74] a different approach was followed, namely to suitably extend the theory allowing for a varying graviton mass, driven by a scalar field. This extended (varying-mass) nonlinear massive gravity proves to exhibit interesting cosmological behavior, leading the universe to lie at the quintessence or phantom regime, experience the phantom-divide crossing [75] or exhibit bouncing and cyclic behavior [76].

Since extended (varying-mass) nonlinear massive gravity exhibits interesting phenomenological features when applied to cosmology, in this work we desire to perform a detailed dynamical analysis of such a scenario. In this way, we can bypass the complexities of the equations, which prevent any complete analytical treatment, and investigate in a systematic way the huge class of possible late-time cosmological behaviors, calculating various observable quantities, such as the dark-energy density and equation-of-state parameters and the deceleration parameter.

The plan of the work is as follows. In section 2, we briefly review the extended nonlinear massive gravity and its cosmological paradigm. In section 3, we perform a dynamical analysis of both flat and open geometries, and in section 4 we discuss the cosmological implications and the physical behavior of the scenario. Finally, section 5 is devoted to the summary of the obtained results.

2. Cosmology in extended nonlinear massive gravity

In this section, we briefly review cosmology in extended nonlinear massive gravity [74, 75]. In this gravitational framework, the graviton mass is generalized to be varying, driven by a scalar field. The total action is written as

$$ S = \int d^4x \sqrt{-g} \left[ \frac{M_p^2}{2} R + V(\psi)(U_2 + \alpha_3 U_3 + \alpha_4 U_4) - \frac{1}{2} \partial_\mu \psi \partial^\mu \psi - W(\psi) \right] + S_m, \quad (2.1) $$

where $M_p$ is the reduced Planck mass, $R$ is the Ricci scalar, $\psi$ is the extra canonical scalar field with $W(\psi)$ its usual potential and $V(\psi)$ a potential coupling to the graviton potentials, and $\alpha_3$ and $\alpha_4$ are dimensionless parameters. The graviton potentials are written as

$$ U_2 = K^\mu_{[\mu} K^\nu_{\nu]}, \quad U_3 = K^\mu_{[\mu} K^\nu_{\nu} K^\rho_3], \quad U_4 = K^\mu_{[\mu} K^\nu_{\nu} K^\rho_3 K^\sigma_{\sigma]}, \quad (2.2) $$

with

$$ K^\mu_{\nu} \equiv \delta^\mu_\nu - \sqrt{g^{\mu\rho} f_{AB} \partial_\nu \phi^A \partial_\rho \phi^B}, \quad (2.3) $$

where we use the notation

$$ K^\mu_{[\mu} K^\nu_{\nu]} \equiv \frac{1}{2} (K^\mu_{\nu} K^\nu_\mu - K^\mu_\mu K^\nu_\nu), \quad (2.4) $$

and similarly for the other antisymmetric expressions. Furthermore, $f_{AB}$ is a fiducial metric and $\phi^A(x)$ are the St"{u}ckelberg scalars introduced to restore general covariance [77]. The above extended scenario is still free of the BD ghost [74]. Finally, in order to obtain a realistic cosmology in (2.1) we have allowed for the standard matter action $S_m$, minimally coupled to the dynamical metric, corresponding to energy density $\rho_m$ and pressure $p_m$. 

2
2.1. Flat universe

In order to extract the cosmological equations, we need to consider specific ansätze for the two metrics. For the physical metric, we assume a flat Friedmann–Robertson–Walker (FRW) form

\[ d^2 s = -N(t)^2 \, dt^2 + a(t)^2 \, \delta_{ij} \, dx^i \, dx^j, \]

with \( a(t) \) being the scale factor and \( N(t) \) the lapse function, and for the Stückelberg fields we consider

\[ \phi^0 = b(t), \quad \phi^i = a_{\text{ref}} x^i, \]

with \( a_{\text{ref}} \) being a (constant) reference scale factor. We mention that contrary to standard massive gravity, where such a choice for the dynamical metric cannot be accompanied by a simple ansatz for the fiducial one [71], in the present extended scenario the extra freedom does allow for a simple Minkowski ansatz for the fiducial metric

\[ f_{AB} = \eta_{AB}. \]

Variation of the action with respect to \( N \) and \( a \) gives rise to the Friedmann equations

\[ 3M_{\text{Pl}}^2 H^2 = \rho_{\text{DE}} + \rho_m, \]  

\[ -2M_{\text{Pl}}^2 \dot{H} = \rho_{\text{DE}} + p_{\text{DE}} + \rho_m + p_m, \]

where we have defined the Hubble parameter \( H = \dot{a}/a \), with \( \dot{a} = da/(N \, dt) \), and finally we set \( N = 1 \). In the above expressions, we have defined the effective dark-energy density and pressure, incorporating the extra gravitational terms, as

\[ \rho_{\text{DE}} = \frac{1}{2} \psi^2 + W(\psi) + V(\psi) \, (u - 1) \, [f_3(u) + f_1(u)] \]

\[ p_{\text{DE}} = \frac{1}{2} \psi^2 - W(\psi) - V(\psi) \, f_4(u) - V(\psi) \bar{b} f_1(u), \]

where

\[ f_1(u) = 3 - 2u + \alpha_3 \, (3 - u) \, (1 - u) + \alpha_4 \, (1 - u)^2 \]

\[ f_2(u) = 1 - u + \alpha_3 \, (1 - u)^2 + \frac{\alpha_4}{3} \, (1 - u)^3 \]

\[ f_3(u) = 3 - u + \alpha_3 \, (1 - u) \]

\[ f_4(u) = -[6(1 - u) + u^2 + \alpha_3 \, (1 - u) \, (4 - 2u) + \alpha_4 \, (1 - u)^2]. \]

with

\[ u = \frac{a_{\text{ref}}}{a}. \]

These satisfy the usual conservation equation

\[ \dot{\rho}_{\text{DE}} + 3H(\rho_{\text{DE}} + p_{\text{DE}}) = 0, \]

and moreover we can define the dark-energy equation-of-state parameter as

\[ w_{\text{DE}} \equiv \frac{p_{\text{DE}}}{\rho_{\text{DE}}}. \]

Variation of the action (2.1) with respect to the scalar field \( \psi \) provides its evolution equation

\[ \ddot{\psi} + 3H \dot{\psi} + \frac{dW}{d\psi} + \frac{dV}{d\psi} \, [(u - 1) \, [f_3(u) + f_1(u)] + 3\bar{b} f_2(u)] = 0. \]

Additionally, variation of (2.1) with respect to \( b \) provides the constraint equation

\[ V(\psi) H f_1(u) + \dot{V}(\psi) f_2(u) = 0. \]
Finally, one must also consider the matter evolution equation \( \dot{\rho}_m + 3H(\rho_m + p_m) = 0 \). In the following, we assume matter to have a general equation-of-state parameter \( w_m = \gamma - 1 \equiv p_m/\rho_m \), where \( \gamma \) is the barotropic index, focusing on the usual dust case \( (\gamma = 1) \) only when necessary.

The above cosmological application in a flat universe, although it leads to interesting phenomenology, has significant theoretical disadvantages. These arise mainly from the constraint equation (2.17), which using (2.12) in general gives [75]

\[
V(\psi(t)) = V_0 e^{\int \frac{f_{(\psi)}}{m^2} da} \frac{V_0 a_{\text{ref}}^3}{(a - a_{\text{ref}})[\alpha_a a_{\text{ref}}^2 - (3\alpha_3 + 2\alpha_4)a_{\text{ref}} + (3 + 3\alpha_3 + \alpha_4)a^2]}.
\]

(2.18)

As we observe, this relation severely restricts the allowed coupling-potential \( V(\psi) \). Additionally, as we can see the varying graviton square mass \( V(\psi) \) diverges and changes sign at least for one finite scale factor (namely at \( a_{\text{ref}} \)), independently of the model parameters, and this would make the scenario unstable at the perturbation level. Although one can still choose \( a_{\text{ref}} \) at far past \( (a_{\text{ref}} \lesssim 10^{-3}) \) in order to be smaller than the Big Bang nucleosynthesis scale factor and not interfere with the standard thermal history of the universe, or at the far future, or even ‘shield’ \( a_{\text{ref}} \) with a cosmological bounce, case in which the universe is always away from it [76], such considerations can only cure the problem phenomenologically, since at the theoretical level it remains unsolved. Clearly, the scenario of a flat universe has a serious disadvantage and therefore one should try to construct generalizations in which these problems are absent. This will be performed in the following subsection, where the addition of curvature makes the graviton mass square always positive.

### 2.2. Open universe

Let us now consider an open\(^3\) FRW form for the physical metric [74]

\[
d^2s = -N(t)^2 dt^2 + a(t)^2 \delta_{ij} dx^i dx^j - a(t)^2 \frac{k^2(\delta_{ij} x^i x^j)}{1 + k^2(\delta_{ij} x^i x^j)},
\]

(2.19)

with \( N(t) \) being the lapse function and \( a(t) \) the scale factor, and \( K < 0 \) with \( k = \sqrt{|K|} \). For the St"uckelberg fields, we choose for simplicity the forms

\[
\phi^0 = b(t) \sqrt{1 + k^2(\delta_{ij} x^i x^j)}, \quad \phi' = kb(t) x^i.
\]

(2.20)

Note that in this case there is no need for the introduction of a reference scale factor \( a_{\text{ref}} \), since it has been absorbed in \( b(t) \). Lastly, similarly to the flat case for the fiducial we consider

\[
f_{AB} = \eta_{AB}.
\]

(2.21)

Variation of the action (2.1) with respect to \( N \) and \( a \) gives rise to the following Friedmann equations:

\[
3M_p^2 \left( H^2 - \frac{k^2}{a^2} \right) = \rho_{\text{DE}} + \rho_m,
\]

(2.22)

\[
-2M_p^2 \left( \dot{H} + \frac{k^2}{a^2} \right) = \rho_{\text{DE}} + p_{\text{DE}} + \rho_m + p_m.
\]

(2.23)

In the above expressions, we have defined the effective dark-energy density and pressure as

\[
\rho_{\text{DE}} = \frac{1}{3} \dot{\psi}^2 + W(\psi) + V(\psi)(X - 1)[f_1(X) + f_1(X)].
\]

(2.24)

\(^3\) Similarly to usual massive gravity, closed FRW solutions are not possible since the fiducial Minkowski metric cannot be foliated by closed slices [15, 74].
PDE = \frac{1}{2} \dot{\psi}^2 - W(\psi) - V(\psi)f_4(X) - V(\psi)\dot{b}f_1(X), \quad (2.25)

but now the relevant functions become

\begin{align*}
    f_1(X) &= 3 - 2X + \alpha_3(3 - X)(1 - X) + \alpha_4(1 - X)^2 \\
    f_2(X) &= 1 - X + \alpha_3(1 - X)^2 + \frac{\alpha_4}{3}(1 - X)^3 \\
    f_3(X) &= 3 - X + \alpha_3(1 - X) \\
    f_4(X) &= -[6 - 6X + X^2 + \alpha_3(1 - X)(4 - 2X) + \alpha_4(1 - X)^2], \quad (2.26)
\end{align*}

where

\begin{equation}
    X = \frac{kb}{a}. \quad (2.27)
\end{equation}

These verify the usual conservation equation

\begin{equation}
    \dot{\rho}_{\text{DE}} + 3H(\rho_{\text{DE}} + p_{\text{DE}}) = 0. \quad (2.28)
\end{equation}

Variation of (2.1) with respect to the scalar field \( \psi \) provides its evolution equation

\begin{equation}
    \ddot{\psi} + 3H\dot{\psi} + \frac{dW}{d\psi} + \frac{dV}{d\psi}\left[(X - 1)[f_3(X) + f_1(X)] + 3bf_2(X)\right] = 0. \quad (2.29)
\end{equation}

Furthermore, variation with respect to \( b \) provides the constraint equation

\begin{equation}
    V(\psi)\left(H - \frac{k}{a}\right)f_1(X) + V(\psi)f_2(X) = 0. \quad (2.30)
\end{equation}

Finally, we consider also the matter conservation equation \( \dot{\rho}_m + 3H(\rho_m + p_m) = 0. \)

3. Dynamical analysis

In order to investigate the cosmological behavior of the scenario of extended nonlinear massive gravity, we have to perform its dynamical analysis, and thus we have to transform the involved cosmological equations into the autonomous form \( X' = f(X) \) \cite{79–81, 78}, where \( X \) is the column vector of suitably introduced auxiliary variables, \( f(X) \) the corresponding column vector of the autonomous equations and a prime denotes the derivative with respect to \( t \). The critical points \( X_c \) are extracted through \( X' = 0 \), and in order to examine their stability properties we expand around \( X_c \) as \( X = X_c + U \), with \( U \) being the corresponding perturbations of the variables. Thus, at the linear perturbation level and for each critical point we find \( U' = Q \cdot U \), where the matrix \( Q \) contains the coefficients of the perturbation equations. Therefore, the eigenvalues of \( Q \) determine the type and stability of the specific critical point.

The scenario at hand, that was presented in the previous section, consists of equations (2.8), (2.9) or (2.16) and (2.18) for the flat geometry, and (2.22), (2.23) or (2.29) and (2.30) for the open geometry, with \( \alpha_3 \) and \( \alpha_4 \) being the model parameters. Although as we discussed the flat case has theoretical disadvantages, for completeness in the following we analyze it too, since it could still be cosmologically valid in suitable frameworks, for example embedded into bouncing evolution.

As we can see, there are three unknown functions involved, namely the usual scalar potential \( W(\psi) \), the varying graviton mass square \( V(\psi) \) and the St"uckelberg-field function \( b(t) \). However, due to the constraint equation, only two out of these three functions are free and can be considered as an input, while the third one is extracted from the equations of motion. As usual, \( W(\psi) \) is the one function that is always imposed by hand. Throughout the work, we will consider the usual scalar field potential to have the well-studied exponential form \cite{79–81, 78}

\begin{equation}
    W(\psi) = W_0 e^{-\lambda_0 \psi}. \quad (3.1)
\end{equation}
Thus, in the scenario at hand one could additionally impose \( V(\psi) \) at will and leave \( b(t) \) to be determined by the equations of motion in order to obtain a consistent solution, or impose \( b(t) \) as an input and leave \( V(\psi) \) to be determined by the equations. Definitely, the first approach is theoretically more robust, corresponding to the usual Lagrangian description where the potentials are imposed as inputs in the theory, and it is the one that is followed in all the works on the subject, that is, the St"uckelberg fields are always extracted by the equations \([70–74]\). Therefore, in the following subsection we will perform the phase-space analysis imposing \( V(\psi) \) as an input. However, for completeness, in a separate subsection we will also present the (theoretically less interesting) case where \( b(t) \) is considered as an input.

3.1. Imposing \( V(\psi) \) at will

For the graviton mass square, and in order to be phenomenologically consistent, without loss of generality, we assume an exponential form

\[
V(\psi) = V_0 e^{-\kappa \psi}. \tag{3.2}
\]

In this case, the graviton mass is small (at the order of the current Hubble parameter in order to drive the current acceleration \([8]\)) at late times, as required by observations, while it could play a significant role in the early universe. Additionally, note that in the special case where \( \lambda_V = 0 \), the scenario at hand in the open case corresponds to the usual (constant-mass) nonlinear massive gravity.

3.1.1. Flat universe. In order to transform the cosmological system (2.8), (2.9) or (2.16) and (2.18) into its autonomous form, we introduce the dimensionless variables

\[
u = \frac{a_{ref}}{a}, \quad Y = \frac{W(\psi)}{3H^2}, \quad Z = \frac{V(\psi)}{3H^2}. \tag{3.3}
\]

Taking the derivatives of (3.3) and using (2.8), (2.9) and (2.18), we obtain the evolution equations for \( u, Y \) and \( Z \), that is, the autonomous form of the cosmological system, as

\[
u' = -u
\]

\[
y' = Y \left[ 2(1 + q) - \frac{\lambda_W f_1(u)}{\lambda_V f_2(u)} \right]
\]

\[
z' = Z \left[ 2(1 + q) - \frac{f_1(u)}{f_2(u)} \right], \tag{3.4}
\]

where primes denote derivative with respect to \( \ln a \). In the above expressions, \( q = -1 - \frac{H}{\dot{H}} \) is the deceleration parameter and the involved \( \dot{H} \) can be expressed in terms of the auxiliary variables as

\[
\dot{H} = \frac{H^2 g_1(u, Y, Z)}{6\lambda_V^2 f_2^2 - f_1^2 f_2}. \tag{3.5}
\]

with

\[
g_1(u, Y, Z) = uf_1 \left( f_1 \frac{df_2}{du} - f_2 \frac{df_1}{du} \right) + 3\lambda_V^2 Zf_2 \left[ 3f_2 f_4 - (u - 1)(f_1 + f_3)[f_1 - 3(\gamma - 1)f_2] \right]
\]

\[+ 3f_1^2 f_2 + 3\lambda_V Y f_2^2 (3\gamma \lambda_V f_2 - \lambda_W f_1) - 9\gamma \lambda_V^2 f_2^3 + (\gamma - 2) \frac{3}{2} f_1^2 f_2, \tag{3.6}\]

as it arises from (2.9) and (2.16) through the elimination of \( \dot{b} \) (for simplicity we have omitted the argument \( u \) in \( f_1(u) \) and \( f_2(u) \)). On the other hand, \( \dot{H} \) elimination between (2.9) and (2.16) gives

\[
3zb = \frac{g_2(u, Y, Z)}{6\lambda_V^2 f_2^2 - f_1^2 f_2}, \tag{3.7}
\]
in terms of the auxiliary variables as

$$g_2(u, Y, Z) = -2u f_2 \frac{df_1}{du} + f_1 \left(2u \frac{df_2}{du} - 3Y f_2\right) + 6f_1 f_2 + 3Y f_2 (\gamma f_1 - 2\gamma \lambda w f_2)$$

$$+ 3Z f_2 \left\{(u - 1)(f_1 + f_3) \left[(\gamma - 1) f_1 - 2\lambda f_2\right] + f_1 f_4\right) + (\gamma - 2) \frac{f_1}{2\lambda f_2} f_1^3. \tag{3.8}$$

Furthermore, using (2.8) we can express the dark-energy density parameter

$$\Omega_{DE} = \frac{H^2}{3M^2}$$

in terms of the auxiliary variables as

$$\Omega_{DE} = \frac{f_1^2}{6\lambda^2 f_2^2} + (u - 1) Z [f_1(u) + f_3(u)] + Y, \tag{3.9}$$

while using (3.5) and (3.7) we can express the dark-energy equation-of-state parameter and the deceleration parameter respectively as

$$w_{DE} = -\frac{Z \left\{f_1(u) [z(u, Y, Z)] + f_4(u)\right\} + \frac{f_1(u)^2}{6\lambda^2 f_2(u)^2} - Y}{(X - 1) Z [f_1(u) + f_3(u)] + \frac{f_1(u)^2}{6\lambda^2 f_2(u)^2} + Y}, \tag{3.10}$$

$$q = -1 - \frac{g_1(u, Y, Z)}{6\lambda^2 f_2^2 - f_1 f_2}. \tag{3.11}$$

In summary, (3.4) accounts for an autonomous system defined in the phase space

$$\left\{(u, Y, Z) : 0 \leq \frac{f_1(u)^2}{6\lambda^2 f_2(u)^2} + (u - 1) Z [f_1(u) + f_3(u)] + Y \leq 1, u \geq 0, Y \geq 0, Z \geq 0\right\}, \tag{3.12}$$

as it arises from the physicality requirements $a \geq 0, V(\psi) \geq 0, W(\psi) \geq 0$ and $0 \leq \Omega_{DE} \leq 1$, and it is in general non-compact.

The real and physically meaningful critical points $(u_c, Y_c, Z_c)$ of the autonomous system (3.4) (that is corresponding to $0 \leq \Omega_{DE} \leq 1$) are obtained by setting the left-hand sides of these equations to zero, and they are presented in table 1, along with their existence conditions. For each critical point, we calculate the $3 \times 3$ matrix $Q$ of the linearized perturbation equations of the system (3.4), and examining the sign of the real part of the eigenvalues of $Q$ we determine the type and stability of this point. The details of the analysis and the various eigenvalues are presented in section A.1, and in table 1 we summarize the stability results (note that in the case of standard matter (\gamma = 1) points $P_1$ and $P_2$ belong to a curve of critical points). Finally, using (3.9), (3.10) and (3.11), for each critical point we calculate the corresponding values of $\Omega_{DE}, w_{DE}$ and $q$. 

### Table 1

| Cr. P. | $u_c$ | $Y_c$ | $Z_c$ | Existence | Stable for | $\Omega_{DE}$ | $w_{DE}$ | $q$ |
|--------|-------|-------|-------|-----------|------------|--------------|---------|-----|
| $P_1$  | 0     | 0     | 0     | for $\lambda^2 \geq \frac{3}{2}$ | $\gamma < \min \left\{1, \frac{g_1}{V}\right\}$, $\lambda^2 \geq \frac{3}{2}$ | $\frac{3}{2\lambda}$ | $\gamma - 1$ | $\frac{3}{2}$ |
| $P_2$  | 0     | 0     | $\frac{3 - 2\lambda}{2\lambda}$ | $\mu > 0$, $0 < \lambda^2 < \frac{3}{2}$ | $\gamma > 1$, $\frac{2w}{\lambda} > 1$ | 1 | 0 | $\frac{1}{2}$ |
| $P_3$  | 0     | 1     | $\frac{1}{2\lambda}$ | for $\lambda^2 \geq \frac{3}{2}$ | $\lambda^2 < \min \left\{1, \gamma\right\}$, $\lambda^2 \geq \frac{3}{2}$ | $\frac{2w}{\lambda} - 1$ | $\frac{3}{2\lambda} - 1$ | $\frac{3}{2\lambda} - 1$ |

saddle point otherwise

$\lambda^2 \geq \frac{3}{2}$ saddle point otherwise
3.1.2. Open universe. In order to transform the cosmological system \((22), (23)\) or \((29)\) into its autonomous form, we introduce the dimensionless variables

\[
X = \frac{kb}{a}, \quad Y = \frac{W(\psi)}{3H^2}, \quad Z = \frac{V(\psi)}{3H^2}, \quad U = \frac{\dot{\psi}}{\sqrt{6}H}, \quad \Omega_k = \frac{k}{aH^2}.
\tag{13.14}
\]

Differentiating with respect to \(\ln a\) we obtain the autonomous form of the cosmological system

\[
X' = -X + \Omega_k \dot{b}, \quad Y' = Y[2(q + 1) - \sqrt{6}\lambda_W U],
\]

\[
Z' = Z[2(q + 1) - \sqrt{6}\lambda_W U],
\]

\[
U' = 3\sqrt{\frac{2}{\Omega_1}}\frac{Zf_1}{Zf_2}b + \frac{1}{2}\{\sqrt{6}\lambda_W(X - 1)Z(f_1 + f_3) + \lambda_W Y\} + 2(q - 2)U,
\]

\[
\Omega_k' = \Omega_k,
\]}

with \(q = -1 - \frac{\dot{H}}{H^2}\), and where for simplicity we have omitted the argument \(X\) in \(f_1(X)\) and \(f_2(X)\). In the above expressions, \(H\) and \(\dot{b}\) are given by \((2.23)\) and \((2.29)\) as

\[
H = \frac{H^2g_1(Y, Z, U, \Omega_k, H^2)}{\lambda_V[2(\Omega_k - 1)\Omega_k f_1 \frac{df_2}{dX} + 2(\Omega_k - 1)\Omega_k f_1 \frac{df_2}{dX} + 3Zf_2^2[6(\Omega_k - 1)^2H^2 + 1]]},
\tag{13.15}
\]

\[
\dot{b} = \frac{g_2(Y, Z, U, \Omega_k, H^2)}{\lambda_V[2(\Omega_k - 1)\Omega_k f_1 \frac{df_2}{dX} + 2(\Omega_k - 1)\Omega_k f_1 \frac{df_2}{dX} + 3Zf_2^2[6(\Omega_k - 1)^2H^2 + 1]]},
\tag{13.16}
\]

with

\[
g_1(Y, Z, U, \Omega_k, H^2) = \frac{df_1}{dX} \left[ -3\lambda_V(\Omega_k - 1)Zf_1f_2 \left[ (\gamma - 1)(X - 1)\Omega_k + X \right] 
\right.
\]

\[
- \lambda_V(\Omega_k - 1)\Omega_k f_1 \left[ 3Z [(\gamma - 1)(X - 1)f_3 + f_4] + 3(\gamma - 2)U^2 \right] 
\]

\[
+ 3\gamma (Y + \Omega_k^2 + 1) - 2\Omega_k^2 \right] + \frac{df_2}{dX} \left[ 3\lambda_V(\Omega_k - 1)Zf_1 \left[ (\gamma - 1)(X - 1)\Omega_k + X \right] 
\right.
\]

\[
- \lambda_V(\Omega_k - 1)\Omega_k f_1 \left[ -3(\gamma - 1)(X - 1)Zf_3 - 3Zf_4 - 3(\gamma - 2)U^2 \right] 
\]

\[
- 3\gamma (Y + \Omega_k^2 + 1) + 2\Omega_k^2 \right] + f_1 \left[ 9\lambda_V(\Omega_k - 1)^2ZH^2 \left[ 3Z [(\gamma - 1)(X - 1)f_3 + f_4] 
\right.
\right.
\]

\[
+ 3(\gamma - 2)U^2 + 3\gamma (Y + \Omega_k^2 - 1) - 2\Omega_k^2 \right] - 3\lambda_V\Omega_k Z 
\]

\[
+ f_1 \left[ 27(\gamma - 1)\lambda_V(X - 1)\Omega_k - 1)^2Z^2f_2H^2 + 9(\Omega_k - 1)^2ZH^2 \left[ -3\lambda_V(X - 1)Zf_3 
\right.
\right]
\]

\[
+ \sqrt{6}U - \lambda_W Y \right] - 9\lambda_V(X - 1)(\Omega_k - 1)^2Z^2f_1^2H^2 \right]
\]

\[
g_2(Y, Z, U, \Omega_k, H^2) = -2\lambda_V X(\Omega_k - 1)f_2 \frac{df_1}{dX} + 2\lambda_V X(\Omega_k - 1)f_1 \frac{df_2}{dX} \]

\[
- \lambda_V f_1f_2 \left[ 3Z [(\gamma - 1)(X - 1)f_3 + f_4] + 3(\gamma - 2)U^2 + 3\gamma (Y + \Omega_k^2 - 1) \right] 
\]

\[
- 2(\Omega_k - 1)\Omega_k x + f_1 \left[ 6(\Omega_k - 1)^2H^2 \left[ -\lambda_V(X - 1)Zf_3 + \sqrt{6}U - \lambda_W Y \right] 
\right.
\]

\[
- 3(\gamma - 1)\lambda_V(X - 1)Zf_3 \right] - 6\lambda_V(X - 1)(\Omega_k - 1)^2Zf_1^2H^2, \]

where \(H^2\) is given from \((2.30)\) as

\[
H^2 = \left[ \frac{\lambda_V f_2(X)}{\sqrt{6}(1 - \Omega_k) f_1(Y)U} \right]^2. \tag{13.19}
\]

Furthermore, using \((2.22)\) we can express the dark-energy density parameter in terms of the auxiliary variables as

\[
\Omega_{DE} \equiv \frac{\rho_{DE}}{3H^2} = (X - 1)Z[f_1(X) + f_3(X)] + U^2 + Y, \tag{13.20}
\]
The real and physically meaningful curves of critical points, and individual critical points, of the autonomous system (3.14) and their existence conditions, for the case of dust matter ($\gamma = 1$), We have introduced the notations $\mu = (4x_3 + \alpha_4 + 6)$, $Y_4 = \sqrt{y_4 + U/[-2(x_3 + \alpha_4 + 6)][\sqrt{x_3} + \alpha_3] - 6}$ and $X = X_{11}$ is the unique real solution of the equation $-2x_3 (X^2 + X - 2) + \alpha_4 (X + 1)(X - 1)^2 + 6 = 0$.

| $X_{p}$ | $X_{c}$ | $Y_{c}$ | $Z_{c}$ | $U_{c}$ | $\Omega_{0}$ | Exists for |
|---------|---------|---------|---------|---------|-------------|-----------|
| $Q_1$   | 0       | $\frac{2x}{y + 4y}$ | $X_{c}$ | 0       | 0           | $0 \leq \frac{x}{y + 4y} - \mu Z_{c} \leq 1$ |
| $Q_2$   | 0       | 0       | $Z_{c}$ | $\frac{\sqrt{2}}{y + 4y}$ | 0           | $0 \leq \frac{1}{y} - \mu Z_{c} \leq 1$ |
| $Q_3$   | 0       | 0       | $Z_{c}$ | $\frac{2x}{y}$ | 0           | $0 \leq \frac{x}{y} - \mu Z_{c} \leq 1$ |
| $Q_4$   | 0       | 0       | 0       | $U_{c}$ | 0           | $0 \leq U_{c}^2 \leq 1$ |
| $Q_5$   | 0       | $Y_{15}$ | 0       | $U_{c}$ | 0           | $0 \leq \frac{X_{15} + \alpha_4}{Y_{15}} (2x_3 + \alpha_3 + 3) \leq 1$ |
| $Q_6^+$ | $X_{c}$ | $\frac{4}{y + 4y}$ | $\frac{\sqrt{2}}{y + 4y}$ | $\pm \sqrt{1 - \frac{1}{y}}$ | $\lambda_{0}^2 \geq 2$ |
| $Q_7^+$ | 0       | $-\frac{4}{3\sqrt{y}}$ | $\frac{\sqrt{2}}{y + 4y}$ | $\pm \sqrt{1 - \frac{1}{y}}$ | $\lambda_{0}^2 \geq 2$ |
| $Q_9^-$ | 0       | 0       | 0       | 0       | 1           | Always |
| $Q_{10}$ | $\frac{2x_3 + \alpha_4 - \sqrt{4x_3 - 6x_3}}{\alpha_4}$ | 0       | 0       | 0       | 1           | $x_{11}^2 \geq \frac{3}{3\alpha_4}$ |
| $Q_{11}$ | $X_{11}$ | 0       | 0       | 0       | $-1$       | $X_{11} \in \reals$ |

while using (3.16) we can express the dark-energy equation-of-state parameter as

$$w_{DE} = \frac{-3[f(X)b + f_4(X)] + U^2 - Y}{(X - 1)[f(X) + f_3(X)] + U^2 + Y},$$

and finally using (3.15) and (3.19) the deceleration parameter is expressed as

$$q = -1 - \frac{g_1(X, Y, Z, U, \Omega_k, H^2)}{\lambda_{V} \left[ -2(\Omega_k - 1)\Omega_k f_2 \frac{df}{d\lambda} + 2(\Omega_k - 1)\Omega_k f_4 \frac{df_4}{d\lambda} + 3Z f_1^2 f_2 [6(\Omega_k - 1)^2 H^2 + 1] \right]}.$$

(3.21)

(3.22)

| $X, Y, Z, U, \Omega_k$ : $0 \leq (X - 1)Z [f(X) + f_3(X)] + U^2 + Y$ $+ \Omega_k^2 \leq 1, X > 0, Y > 0, Z > 0$, |

which is in general non-compact.

Let us extract the critical points of the autonomous system (3.14), setting the left-hand sides of these equations to zero. From the last equation of (3.14) it follows that either $q = 0$ or $\Omega_k = 0$, and therefore we can simplify the investigation and examine these two cases separately. The details of the analysis, the critical points and critical curves, the various eigenvalues and the stability conditions are presented in section A.2, and in the table we display the real and physically meaningful critical points and their existence conditions for the most interesting case of dust matter ($\gamma = 1$), while in table 3 we present their stability conditions and the values of the observables $\Omega_{DE}$, $w_{DE}$ and $q$ using (3.20), (3.21) and (3.22).

We mention here that the variable choice (3.13) allows for an easy, partial classification of expanding and contracting solutions. In particular, solutions with $\Omega_k = k/(aH) > 0$ correspond to $H > 0$ and thus to expansion, while those with $\Omega_k < 0$ correspond to $H < 0$ and therefore to contraction ($k = \sqrt{|K|}$ throughout this work). That is why points with $\Omega_k > 0$ are denoted with the subscript ‘+’, while those with $\Omega_k < 0$ are denoted with the subscript ‘−’.
Table 3. The stability conditions and the values of the observables \( \Omega_{DE} \), \( w_{DE} \) and \( q \), for the real and physically meaningful curves of critical points, and individual critical points, of the autonomous system (3.14), for the case of dust matter (\( \gamma = 1 \)). The notations are the same with table 2. Additionally, we have defined \( w_{DE1} = \frac{2 \lambda c \Omega_{DE} (2 \lambda c + 3)}{(U c + \lambda c)^2 - \sqrt{U c (\lambda c + 3)} + 3} \) and \( w(X) = \frac{\lambda c \Omega_{DE} (2 \lambda c + 3)}{(U c + \lambda c)^2 - \sqrt{U c (\lambda c + 3)} + 3} \).

The symbol \( \Re[z] \) denotes the real part of the complex number \( z \).

| Cr. P. | Stability | \( \Omega_{DE} \) | \( w_{DE} \) | \( q \) |
|--------|-----------|-----------------|--------------|--------|
| \( Q_1 \) | Non-hyperbolic, 3D stable manifold | \( \frac{\lambda c}{\mu c} = Z_c \) | \( w_{DE1} \) | \( -1 \) |
| \( Q_2 \) | Saddle point | \( \frac{\lambda c}{\mu c} = Z_c \) | 0 | \( \frac{1}{3} \) |
| \( Q_3 \) | Non-hyperbolic, 3D stable manifold for \( \lambda c / \mu c < 2, \lambda c (\lambda c - \mu c) < 0 \) saddle otherwise | \( U c \) | \( \frac{\lambda c \Omega_{DE}}{2 \lambda c + 3} \) | \( \frac{-\lambda c \Omega_{DE}}{2 \lambda c + 3} \) |
| \( Q_4 \) | Non-hyperbolic, 3D stable manifold for \( -1 < U c < -1/ \sqrt{\lambda c - \mu c} \) or \( \lambda c (\lambda c - \mu c) > 0 \) saddle otherwise | \( \Omega_{DE5} \) | \( w_{DE5} \) | \( \sqrt{\frac{2 \lambda c U c}{\lambda c + 3}} - 1 \) |
| \( Q_5 \) | Non-hyperbolic, 2D stable manifold for \( 0 < \Omega_{DE5} < 1 \) and \( U c \lambda c < \min \left( \frac{\lambda c}{U c}, \frac{\mu c}{\lambda c} \right) \) saddle otherwise | \( \Omega_{DE5} \) | \( w_{DE5} \) | \( \sqrt{\frac{2 \lambda c U c}{\lambda c + 3}} - 1 \) |
| \( Q_6^c \) | Non-hyperbolic, 4D stable manifold for \( 2 \leq \lambda c / \mu c < \frac{4}{3}, \Re[z] > 1 \) or \( \lambda c / \mu c > \frac{4}{3}, \Re[z] > 1 \) saddle otherwise | \( \frac{1}{\mu c} \) | \( \frac{1}{\lambda c} \) | 0 |
| \( Q_7^c \) | Non-hyperbolic, 4D stable manifold for \( \Re[z] > 1, \lambda c \Re[z] > 0 \), \( \Re[z] \) \( \frac{\lambda c \Re[z]}{\lambda c - \Re[z]} > 0 \), \( \Re[z] \) \( \frac{\lambda c \Re[z]}{\lambda c - \Re[z]} < 0 \) saddle otherwise | \( \frac{1}{\mu c} \) | \( \frac{1}{\lambda c} \) | 0 |
| \( Q_8 \) | Saddle point | 0 | 0 | \( \frac{1}{3} \) |
| \( Q_9 \) | Saddle point | 0 | -1 | 0 |
| \( Q_{10} \) | Saddle point | 0 | 0 | 0 |
| \( Q_{11} \) | Saddle point | 0 | \( \frac{w(X_{11})}{3} \) | 0 |
However, this is only a partial classification, since it cannot work for solutions with $\Omega_k = 0$, which can be either expanding or contracting. Furthermore, note that although our model admits expanding and contracting solutions, from the fifth equation of (3.14) we deduce that the sign of $\Omega_k$ is invariant, and thus transitions from contracting to expanding solutions or vice versa do not exist. Nevertheless, since such transitions do exist in the flat geometry [76], they could still exist in the non-flat scenario at hand too, but at the edge of the phase space, which could be revealed only through application of Poincaré central projection method [82–84]. This analysis lies beyond the scope of this work and it is left for future investigation.

Finally, we stress that the curve of critical points $Q_5$ contains many interesting individual points, and for that reason we display them separately in table 4, along with their existence and stability conditions and the corresponding values of the observables. Note that these points contain the standard quintessence points [79, 85]; however, the stability conditions are slightly different, due to the presence of extra phase-space dimensions, namely curvature and graviton mass.

3.2. Imposing $b(t)$ at will

In the previous subsection, we performed the dynamical analysis following the theoretically robust approach in Lagrangian descriptions, that is, imposing the potential $V(\psi)$ (graviton varying square mass) as an input and letting the Stückelberg field function $b(t)$ to be determined by the equations. However, for completeness, and in order to compare with similar studies in the literature [86], in this section we follow the theoretically less justified, alternative approach, that is, to impose $b(t)$ at will and let $V(\psi)$ be determined by the equations. Similarly to the previous subsection, we will consider the flat and open geometry separately, using different $b(t)$ ansätze in the two cases for convenience.

3.2.1. Flat universe. In this case, we impose $b(t) = Bt$ with $B > 0$, since this leads to $\dot{b} = B$ which simplifies significantly the cosmological equations. In the following, we focus on the dust matter case ($\gamma = 1$); however, the analysis can be straightforwardly extended to the general $\gamma$ case too. In order to transform the cosmological system (2.8), (2.9) or (2.16) and (2.18) into its autonomous form, we introduce the dimensionless variables

$$x = \frac{\dot{\psi}}{\sqrt{6}H}, \quad y = \frac{\sqrt{W(\psi)}}{\sqrt{3}H}, \quad u = \frac{a_{ref}}{\dot{a}}, \quad v = \frac{V(\psi)}{H^2}. \quad (3.24)$$

Taking derivatives with respect to $\ln a$, we obtain the autonomous form of the cosmological system as

$$x' = (q - 2)x + \sqrt{\frac{3}{2}}y^2 + \frac{3x[3\alpha_3 + \alpha_4 + u^2(\alpha_3 + \alpha_4) - 2u(2\alpha_3 + \alpha_4 + 1) + 3][4\alpha_3 + \alpha_4 + u(\alpha_3(u - 5) + \alpha_4(u - 2) - 3) + 6]}{2x[-3\alpha_3(u - 1) + \alpha_4(u - 1)^2 + 3]}$$

$$-\frac{3Bv[3\alpha_3 + \alpha_4 + u^2(\alpha_3 + \alpha_4) - 2u(2\alpha_3 + \alpha_4 + 1) + 3]}{2x}, \quad (3.25)$$

$$y' = y\left(q - \sqrt{\frac{3}{2}}\lambda x + 1\right), \quad (3.26)$$

$$u' = -u, \quad (3.27)$$
Table 4. The interesting individual critical points of the curve of critical points $Q_3$ of table 3, their existence and stability conditions, and the corresponding values of the observables $\Omega_{\text{DE}}, w_{\text{DE}}$ and $q$.

| Cr. P. | $X_e$ | $Y_e$ | $Z_e$ | $U_e$ | $\Omega_{\text{DE}}$ | Exists for | Stability | $\Omega_{\text{DE}}$ | $w_{\text{DE}}$ | $q$ |
|--------|-------|-------|-------|-------|----------------|------------|----------|----------------|-------------|----|
| $Q_{12}$ | 0 | 0 | 0 | $-1$ | 0 | Always | Unstable | 1 | 1 | 2 |
| $Q_{13}$ | 0 | 0 | 0 | 1 | 0 | Always | Unstable | 1 | 1 | 1 |
| $Q_{14}$ | 0 | $1 - \frac{i \lambda W}{2}$ | 0 | $\frac{i \lambda W}{\sqrt{2}}$ | 0 | $\lambda W < 6$ | Stable node for $-\sqrt{3} < \lambda W < 0, \lambda V < \lambda W$ or $0 < \lambda W < \sqrt{3}, \lambda V > \lambda W$ | 1 | $-1 + \frac{i \lambda W}{2}$ | $-1 + \frac{i \lambda W}{2}$ |
| $Q_{15}$ | 0 | $\frac{1}{\sqrt{\lambda W}}$ | 0 | $\frac{1}{\sqrt{\lambda W}}$ | 0 | $\lambda W > 3$ | Saddle point | $\frac{1}{\sqrt{\lambda W}}$ | 0 | $\frac{1}{7}$ |
| $Q_{16}$ | 0 | 0 | 0 | 0 | 0 | Always | Saddle point | 0 | 0 | $\frac{1}{7}$ |
| $Q_{17}$ | 0 | 0 | 0 | 0 | $-1$ | Always | Saddle point | 0 | 0 | 0 |
| $Q_{18}$ | 0 | 0 | 0 | 0 | 1 | Always | Saddle point | 0 | 0 | 0 |
v' = [(u - 1)[−3α3(u - 1) + α4(u - 1)2 + 3] − 1\{3v(3α3 + α4 + α4u2 - 3α3u - 2α4u + 3) × [3α3 + α3 + u2(α3 + α4) - 2u(2α3 + α4 + 1) + 3]\} + 2(q + 1)v]. \quad \text{(3.28)}

Furthermore, using (2.8), (2.9) and (2.15) we can express the dark-energy density parameter, the dark-energy equation-of-state parameter and the deceleration parameter in terms of the auxiliary variables respectively as

$$\Omega_{\text{DE}} = \frac{1}{3}(u - 1)v[u(\alpha_3 + \alpha_3 + (u - 2)\alpha_4 - 3) + 4\alpha_3 + \alpha_4 + 6] + 3(x^2 + y^2)], \quad \text{(3.29)}$$

$$w_{\text{DE}} = \frac{v[4\alpha_3 + \alpha_4 + u^2(2\alpha_3 + \alpha_4 + 1) - 2u(3\alpha_3 + \alpha_4 + 3) + 6] + x^2 - y^2}{(u - 1)v[4\alpha_3 + \alpha_4 + u[\alpha_3(u - 5) + \alpha_4(u - 2) - 3] + 6] + x^2 + y^2}$$

$$- \frac{Bv[3\alpha_3 + \alpha_4 + u^2(\alpha_3 + \alpha_4) - 2u(2\alpha_3 + \alpha_4 + 1) + 3]}{(u - 1)v[4\alpha_3 + \alpha_4 + u^2(\alpha_3 + \alpha_4) - 2u(2\alpha_3 + \alpha_4 + 1) + 3]}, \quad \text{(3.30)}$$

$$q = \frac{1}{2}(3v[4\alpha_3 + \alpha_4 + u^2(2\alpha_3 + \alpha_4 + 1) - 2u(3\alpha_3 + \alpha_4 + 3) + 6] + 3x^2 - 3y^2 + 1)$$

$$- \frac{3Bv[3\alpha_3 + \alpha_4 + u^2(\alpha_3 + \alpha_4) - 2u(2\alpha_3 + \alpha_4 + 1) + 3]. \quad \text{(3.31)}$$

In summary, (3.25)–(3.28) account for an autonomous system defined in the physical phase space given by

$$\left\{ (x, y, u, v) : 0 \leq \frac{1}{3}(u - 1)v[u(\alpha_3 + \alpha_3 + (u - 2)\alpha_4 - 3) + 4\alpha_3 + \alpha_4 + 6] + 3(x^2 + y^2)] < 0, \ u \geq 0, \ v \geq 0 \right\}, \quad \text{(3.32)}$$

where the first inequality follows from the physical condition $0 \leq \Omega_{\text{DE}} \leq 1$ and the second inequality follows from the requirement the graviton mass square $V(\psi)$ to remain positive.

The real and physically meaningful critical points $(x_c, y_c, u_c, v_c)$ of the autonomous system (3.25)-(3.28), along with their existence conditions, are presented in Table 5. For each critical point, we calculate the $4 \times 4$ matrix $Q$ of the linearized perturbation equations, and we determine its type and stability by examining the sign of the real part of the eigenvalues of $Q$. The details of the analysis and the various eigenvalues are presented in section B.1, and

---

Table 5. The real and physically meaningful critical points of the autonomous system (3.25)-(3.28) and their existence conditions. We have introduced the notations $\mu_1 = [4\alpha_3 + \alpha_4 - B(3\alpha_3 + \alpha_4 + 3) + 6]$ and $\mu_2 = [4\alpha_3 + \alpha_4 - B(3\alpha_3 + \alpha_4 + 3) + 6]$. 

| Cr. P. | $x_c$ | $y_c$ | $u_c$ | $v_c$ | Exists for |
|-------|-------|-------|-------|-------|------------|
| $R_1$ | 0     | 0     | 0     | 0     | All $\lambda_W$ |
| $R_2$ | 1     | 0     | 0     | 0     | All $\lambda_W$ |
| $R_3$ | -1    | 0     | 0     | 0     | All $\lambda_W$ |
| $R_4^\pm$ | $\pm \sqrt{1 - \frac{4}{3\mu_1}}$ | 0     | 0     | 0     | $0 < \lambda_W \leq 6$ |
| $R_5^\pm$ | $\pm \sqrt{2\mu_1}$ | 0     | 0     | $\lambda_W > 3$ |
| $R_6^\pm$ | 0     | $\pm 1$ | 0     | 0     | $\lambda_W = 0$ |
| $R_7$ | $\pm \sqrt{2\mu_1}$ | $y_c$ | 0     | $\lambda_W \neq 0, \frac{2\alpha_3 - 1}{\mu_1} \geq 0$ |
| $R_8$ | $x_c$ | 0     | 0     | $\frac{-\sqrt{3\mu_1}}{2}$ | $0 \leq 2(1 - \mu_2)\lambda_W^2 + 3(1 + \mu_2) < 2\lambda_W^2$ |

---
in table 6 we display the stability conditions and the corresponding values of the observables \( \Omega_{\text{DE}}, w_{\text{DE}} \) and \( q \).

We mention here that the variable choice (3.24) allows for an easy classification of expanding and contracting solutions. In particular, solutions with \( y > 0 \) correspond to \( H > 0 \) and thus to expansion, while those with \( y < 0 \) correspond to \( H < 0 \) and therefore to contraction. That is why points with \( y > 0 \) are denoted with the subscript ‘+’, while those with \( y < 0 \) are denoted with the subscript ‘−’. However, from (3.26) it is implied that the sign of \( y \) is invariant, and thus transitions from contracting to expanding solutions or vice versa do not exist (there could still exist at the edge of the phase space, which could be revealed only through application of Poincaré central projection method [82–84], but such an analysis lies beyond the scope of this work and it is left for future investigation).

### 3.2.2. Open universe

In this case, it proves convenient to impose the ansatz \( b(t) = b_0 a(t) \), since this leads to \( \dot{b} = b_0 \dot{a} \), which simplifies significantly the cosmological equations. In the following, we focus on the dust matter (\( \gamma = 1 \)); however, the analysis can be straightforwardly extended to the general \( \gamma \) case too. In order to transform the cosmological system (2.22), (2.23) or (2.29) and (2.30) into its autonomous form, we introduce the dimensionless variables

\[
\begin{align*}
\phi &= \frac{\dot{\psi}}{\sqrt{6}H}, & y &= \frac{\sqrt{W(\psi)}}{\sqrt{3}H}, & u &= \frac{k}{a}, & v &= \frac{V(\psi)}{H^2}, & \Omega_k &= \frac{k}{aH},
\end{align*}
\]

with \( \Omega_k = \sqrt{|K|} \). Taking derivatives with respect to \( \ln a \) we obtain the autonomous form of the cosmological system as

\[
\begin{align*}
\dot{x} &= \frac{3u^2 \nu x \beta \delta}{\Omega_k^2} - \frac{3u^2 \nu x \delta [2 \alpha_3 (\beta^2 + \beta - 2) - \alpha_4 (\beta + 1) (\beta - 1)^2 - 6]}{\Omega_k^2 [3 \alpha_3 (\beta - 1) - \alpha_4 (\beta - 1)^2 - 3]} - \frac{\Omega_{\text{DE}}^2 x}{2} + \frac{3x^3}{2} \\
&+ \frac{\sqrt{3}}{2} \nu x^2 \lambda_w + \frac{\nu x \delta}{2 \Omega_k} \left\{ - \frac{6u^2 [2 \alpha_3 (\beta - 4) (\beta - 1) + \alpha_4 (\beta - 1)^2 - 3 \beta + 6]}{3 \alpha_3 (\beta - 1) - \alpha_4 (\beta - 1)^2 - 3} - \beta \right\} \\
&+ \frac{1}{2} \left\{ v [\beta^2 (2 \alpha_3 + \alpha_4 + 1) - 2 \beta (3 \alpha_3 + \alpha_4 + 3) + 4 \alpha_3 + \alpha_4 + 6] - 3 (\gamma^2 + 1) \right\},
\end{align*}
\]

### Table 6

The stability conditions and the values of the observables \( \Omega_{\text{DE}}, w_{\text{DE}} \) and \( q \), for the real and physically meaningful critical points of the autonomous system (3.25)–(3.28). The notations are the same as table 5.

| Cr. P. | Stability | \( \Omega_{\text{DE}} \) | \( w_{\text{DE}} \) | \( q \) |
|--------|-----------|----------------|----------------|-----|
| \( R_1 \) | Saddle point | 0 | Arbitrary | \( \frac{1}{2} \) |
| \( R_2 \) | Saddle point | 1 | 1 | - |
| \( R_3 \) | Saddle point | 1 | 1 | - |
| \( R_4 \) | Stable node for \( 0 < \lambda_w^2 < 3 \) | 1 | \(-1 + \frac{\lambda_w^2}{4}\) | \(-1 + \frac{\lambda_w^2}{4}\) |
| \( R_5 \) | Saddle point for \( 3 < \lambda_w^2 < 6 \) | 1 | \(-1 + \frac{\lambda_w^2}{4}\) | \(-1 + \frac{\lambda_w^2}{4}\) |
| \( R_6 \) | Non-hyperbolic | 1 | \( \frac{1}{\lambda_w^2} \) | 1 |
| \( R_7 \) | Stable | 1 | \( \frac{1}{\lambda_w^2} \) | 1 |
| \( R_8 \) | Stable for \( x_c \lambda_w > \sqrt{\frac{3}{2}} \) | 1 | \( \frac{1}{\lambda_w^2} (1 + \mu_2) \) | 1 |

saddle point otherwise.
\[ y' = \frac{1}{2}v[\beta^2(2\alpha_3 + \alpha_4 + 1) - 2\beta(3\alpha_3 + \alpha_4 + 3) + 4\alpha_3 + \alpha_4 + 6] - 3y^2 + 3 \]
\[ - \frac{\nu y \beta \delta}{2 \Omega_k} - \frac{\Omega_k^4 y}{2} + \frac{3x^2 y}{2} - \sqrt{\frac{3}{2} x y \lambda_y}, \]  
(3.35)

\[ u' = -u, \]  
(3.36)

\[ v' = -v^2 \beta \delta - v \Omega_k^2 + 3vx^2 + \frac{18u^2 vx^2 \delta}{\Omega_k^2 (\beta - 1)[-3\alpha_3 (\beta - 1) + \alpha_4 (\beta - 1)^2 + 3]} + v[v\beta^2(2\alpha_3 + \alpha_4 + 1) - 2\beta(3\alpha_3 + \alpha_4 + 3) + 4\alpha_3 + \alpha_4 + 6] - 3y^2 + 3 \]
\[ - \frac{18u^2 vx^2 \delta}{\Omega_k^2 (\beta - 1)[-3\alpha_3 (\beta - 1) + \alpha_4 (\beta - 1)^2 + 3]}, \]  
(3.37)

\[ \Omega_k' = \Omega_k \left[ \frac{1}{2}v[\beta^2(2\alpha_3 + \alpha_4 + 1) - 2\beta(3\alpha_3 + \alpha_4 + 3) + 4\alpha_3 + \alpha_4 + 6] - 3y^2 + 1 \right] + \frac{3x^2}{2}, \]  
(3.38)

where \( \beta = \frac{b_0 k}{c} \) and \( \delta = \beta^2(\alpha_3 + \alpha_4) - 2\beta(3\alpha_3 + \alpha_4 + 1) + 3\alpha_3 + \alpha_4 + 3 \).

Furthermore, using (2.22) and (2.23) we can express the dark-energy density parameter, the dark-energy equation-of-state parameter and the deceleration parameter, in terms of the auxiliary variables respectively as:

\[ \Omega_{\text{DE}} = \frac{1}{3}v(\beta - 1)[\alpha_3(\beta - 4)(\beta - 1) + \alpha_4(\beta - 1)^2 - 3\beta + 6] + 3(x^2 + y^2)], \]

\[ w_{\text{DE}} = \frac{v[\beta^2(2\alpha_3 + \alpha_4 + 1) - 2\beta(3\alpha_3 + \alpha_4 + 3) + 4\alpha_3 + \alpha_4 + 6] + 3(x^2 + y^2)}{(\beta - 1)v[\alpha_3(\beta - 4)(\beta - 1) + \alpha_4(\beta - 1)^2 - 3\beta + 6] + 3(x^2 + y^2)} - \frac{\beta v[\beta^2(2\alpha_3 + \alpha_4 + 1) + 3\alpha_3 + \alpha_4 + 3]}{\Omega_k^2 (\beta - 1)v[\alpha_3(\beta - 4)(\beta - 1) + \alpha_4(\beta - 1)^2 - 3\beta + 6] + 3(x^2 + y^2)}, \]

\[ q = -\frac{\beta v[\beta^2(2\alpha_3 + \alpha_4 + 1) - 2\beta(3\alpha_3 + \alpha_4 + 3) + 4\alpha_3 + \alpha_4 + 6] + 3(x^2 \beta + y^2)]}{2\Omega_k} + \frac{1}{2}v[\beta^2(2\alpha_3 + \alpha_4 + 1) - 2\beta(3\alpha_3 + \alpha_4 + 3) + 4\alpha_3 + \alpha_4 + 6] + 3(x^2 - y^2) + 1]. \]  
(3.39)

In summary, the autonomous system (3.34)–(3.38) defines a flow in the physical phase space given by

\[ \{x, y, u, v, \Omega_k \} : 0 \leq \frac{1}{4}v(\beta - 1)[\alpha_3(\beta - 4)(\beta - 1) + \alpha_4(\beta - 1)^2 - 3\beta + 6] + 3(x^2 + y^2)], \]
\[ + \Omega_k^2 \leq 1, u \geq 0, v \geq 0, \]  
(3.40)

as it arises from the physicality requirements \( a \geq 0, V(\psi) \geq 0, W(\psi) \geq 0 \) and \( 0 \leq \Omega_{\text{DE}} + \Omega_k^2 \leq 1 \).

The real and physically meaningful critical points \( (x_c, y_c, u_c, v_c, \Omega_{k_c}) \) of the autonomous system (3.34)–(3.38), along with their existence conditions, are displayed in table 7. For each critical point, we calculate the \( 5 \times 5 \) matrix \( Q \) of the linearized perturbation equations, and we determine its type and stability by examining the sign of the real part of the eigenvalues of \( Q \). The details of the analysis and the various eigenvalues are presented in section B.1, and in table 8 we display the stability conditions and the corresponding values of the observables \( \Omega_{\text{DE}}, w_{\text{DE}} \) and \( q \).

15
Table 7. The real and physically meaningful critical points of the autonomous system (3.34)–(3.38) and their existence conditions.

| Cr. P. | $x_c$ | $y_c$ | $u_c$ | $v_c$ | $\Omega_{k_c}$ | Exists for $\Omega_1$ |
|--------|-------|-------|-------|-------|----------------|---------------------|
| $S_1$  | 0     | 0     | 0     | 0     | 0              | Always              |
| $S_2$  | 1     | 0     | 0     | 0     | 0              | Always              |
| $S_3$  | -1    | 0     | 0     | 0     | 0              | Always              |
| $S_4^+$| $\frac{\sqrt{3}}{\sqrt{2}}$ | $\pm \sqrt{1 - \frac{\lambda}{\sqrt{6}}}$ | 0 | 0 | 0 | $\lambda_{W} \leq 6$ |
| $S_5^\pm$ | $\frac{1}{\sqrt{2}}$ | $\pm \sqrt{\frac{1}{3} - \frac{\lambda^2}{2\Omega_1}}$ | 0 | 0 | 0 | $\lambda_{W} \geq 3$ |
| $S_5^\pm$ | 0 | $\pm 1$ | 0 | 0 | 0 | $\lambda_{W} = 0$ |
| $S_7^\pm$ | 0 | 0 | 0 | 0 | $\pm 1$ | Always |
| $S_8^\pm$ | $\sqrt{\frac{1}{2} + \frac{\lambda^2}{2\Omega_1}}$ | $\pm \frac{3}{\sqrt{2\Omega_1}}$ | 0 | 0 | $\pm \sqrt{1 - \frac{\lambda^2}{\sqrt{6}}} \lambda_{W} \geq 2$ |

Table 8. The stability conditions and the values of the observables $\Omega_{DE}$, $w_{DE}$, and $q$, for the real and physically meaningful critical points of the autonomous system (3.34)–(3.38).

| Cr. P. | Stability | $\Omega_{DE}$ | $w_{DE}$ | $q$ |
|--------|-----------|--------------|--------|-----|
| $S_1$  | Saddle point | 0 | Arbitrary | $\frac{1}{2}$ |
| $S_2$  | Saddle point | 1 | 1 | 2 |
| $S_3$  | Saddle point | 1 | 1 | 2 |
| $S'_4$ | Saddle point | $1 - 1 + \frac{1}{2} \frac{\lambda^2}{\Omega_1}$ | $-1 + \frac{1}{2} \frac{\lambda^2}{\Omega_1}$ |
| $S_5^\pm$ | Saddle point | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ |
| $S_5^\pm$ | Non-hyperbolic (4D stable manifold) | 1 | -1 | -1 |
| $S_6^\pm$ | Saddle point | 0 | Arbitrary | 0 |
| $S_7^\pm$ | Saddle point | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 |

Note that the variable choice (3.33) allows for an easy classification of expanding and contracting solutions. In particular, solutions with $\Omega_k = k/(aH) > 0$ or $y > 0$ correspond to $H > 0$ and thus to expansion, while those with $\Omega_k < 0$ or $y < 0$ correspond to $H < 0$ and therefore to contraction. However, from equations (3.35) and (3.38) we deduce that the sign of $y$ and the sign of $\Omega_k$ are invariant and thus transitions from contracting to expanding solutions or vice versa do not exist. Nevertheless, since such transitions do exist in the flat geometry [76], there could still exist in the non-flat scenario too, at the edge of the phase space, which could be revealed only through the application of the Poincaré central projection method [82–84]. This analysis lies beyond the scope of this work and it is left for a future project.

4. Cosmological implications

In the previous section, we performed a complete dynamical analysis of the scenario of extended (varying-mass) nonlinear massive gravity for both flat and open FRW geometries, extracted the late-time stable solutions and calculated the corresponding observables. In this section, we discuss the cosmological implications of the various scenarios case by case.

4.1. Imposing $V(\psi)$ at will

4.1.1. Flat universe. First of all we mention that the scenario at hand coincides with standard quintessence if the graviton mass square $V(\psi) = V_0 e^{-\lambda \psi}$ is identically zero. If this is not the case, then standard quintessence can be obtained only asymptotically. Additionally, if $V_0$ is not
zero, then \( \lambda_V \) cannot be zero, since the constraint (2.17) cannot be satisfied in general. Thus, we conclude that in general this scenario has \( \lambda_V \neq 0 \), and therefore there are no parameter values that make it coincide completely with usual (constant mass) massive gravity, as discussed in [75].

As we observe from table 1, there exist three critical points and all of them can be stable according to the parameter values. Point \( P_1 \) in the case of standard matter (\( \gamma = 1 \)) corresponds to a non-accelerating universe, with a dark energy behaving as dust. Although it has the advantage that \( 0 < \Omega_{DE} < 1 \), that is, it can alleviate the coincidence problem since dark-energy and dark matter density parameters can be of the same order, the above features disfavor it. Lastly, note that the corresponding graviton mass has become zero.

Point \( P_2 \) corresponds to a dark-energy dominated, non-accelerating, universe, with dark energy behaving as dust, and thus it is also disfavored by observations. Additionally, the graviton mass remains finite.

Point \( P_3 \) is the most interesting solution that can attract the universe at late times. It corresponds to a dark-energy dominated universe, which can be accelerating (for \( \frac{\lambda_W}{\lambda_V} < \frac{1}{3} \)) or non-accelerating according to the parameter values, and where dark energy can lie either in the quintessence [87, 88] or in the phantom regime [89] (for \( \lambda_W \lambda_V < 0 \)). Moreover, the graviton mass dynamically becomes zero. These features make this point a very good candidate for the description of late-time universe, in agreement with observations. Furthermore, note that if the universe starts from the quintessence regime, then the attraction to \( P_3 \) implies the phantom divide crossing [90]. The realization of the phantom regime and/or of the phantom-divide crossing is a great advantage of extended nonlinear massive gravity, as was analyzed in detail in [75, 76].

We mention here that naively it looks strange that \( P_3 \) can be a phantom solution although the graviton mass tends to zero and the model should look like quintessence. However, this is easily explained since, as we show in section A.1, in the case \( \lambda_W \lambda_V < 0 \) where \( P_3 \) is phantom, \( V \) tends to zero but \( W \) and \( H \) tend to infinity, which is a Big-Rip-type behavior (realized at infinity and not at a finite scale factor) [91–97] that is a typical fate of phantom scenarios. In other words, the graviton mass does tend asymptotically to zero, but its previous effect has already led the universe to a phantom regime without exit (although not so strongly in order to exhibit a Big Rip at a finite scale factor).

In order to present the above behavior in a more transparent way, we evolve numerically the autonomous system (3.4) in the invariant set \( u = 0 \), for the parameters \( \gamma = 1 \) (dust matter), \( \lambda_V = 2, \lambda_W = -1 \), \( \alpha_1 = \alpha_4 = 0.1 \), and in figure 1 we depict the corresponding phase-space behavior in the \( Y-Z \) plane. The physical part of the phase space is marked by the shadowed region limited by the red lines. As we observe, in this specific example the universe results in the phantom stable late-time solution \( P_3 \).

4.1.2. Open universe. As we show in detail in section A.2, and as we have depicted in tables 2–4, the scenario at hand admits many stable late-time solutions, and this reveals its advantages and capabilities, comparing to standard quintessence, as well as to standard (constant-mass) nonlinear massive gravity. Note that this scenario also admits curves of solutions apart from individual points, which is an additional indication of its generalized features.

In particular, the first interesting solution, that can attract the universe at late times, is the curve of critical points \( Q_1 \). It corresponds to an accelerating universe, in which dark energy lies always in the phantom regime. Moreover, it has the advantage that \( 0 < \Omega_{DE} < 1 \), that is, it can alleviate the coincidence problem, and the graviton mass can be zero or not according to the parameter values.
Figure 1. Trajectories in the $Y$–$Z$ plane of the cosmological scenario (3.4), where the varying graviton mass square $V(\psi)$ is imposed at will, in a flat universe. We use $\gamma = 1, \lambda_V = 2, \lambda_W = -1, \alpha_3 = \alpha_4 = 0.1$. The physical part of the phase space is marked by the shadowed region limited by the red lines. In this specific example the universe is led to the phantom stable late-time solution $P_3$.

The curves of critical points $Q_3$ and $Q_5^\pm$, as well as the individual critical points $Q_4^\pm$, can be stable and thus attract the universe at late times, however, since they correspond to zero acceleration are not favored by observations (although they have $0 < \Omega_{DE} < 1$ and thus they can solve the coincidence problem).

The curves of critical points $Q_4$ and $Q_5$ can be stable (although their stable manifold has smaller dimensionality and thus the stability is weaker), that is, they can be the late-time state of the universe, corresponding to an accelerating or non-accelerating universe according to the parameter values. Furthermore, note that according to the parameter values they can lie in the quintessence or phantom regime, and they possess $0 < \Omega_{DE} < 1$. Additionally, the graviton mass becomes zero. These features make $Q_4$ and $Q_5$ good candidates for the description of the universe.

In particular, as we discussed in section 3.1.2, the curve $Q_5$ contains the quintessence-like critical points presented in table 4, which are obtained in standard quintessence too in flat [79] or non-flat geometries [85]. Note however that the stability properties are slightly different, since now we have the additional direction of the graviton mass. Amongst these points, $Q_{14}$ is stable, corresponding to a dark-energy dominated, quintessence universe, which can be accelerating or non-accelerating according to the parameter values, and thus it is a good candidate for the description of the universe. On the other hand, point $Q_{15}$, which is stable in standard quintessence, in the present case it becomes saddle and therefore it cannot be the late-time state of the universe. Let us present the above results more transparently. In figure 2, we show the corresponding phase-space behavior in the $X$–$Y$ plane as it arises from numerical elaboration of the autonomous system (3.14). We focus on the invariant set $\Omega_k = U = Z = 0$ and we choose $\gamma = 1, \lambda_V = -2, \lambda_W = 1, \alpha_3 = \alpha_4 = 0.1$. In this specific example, the stable late-time state of the universe is the phantom solution $Q_1$. Similarly, in figure 3 we depict
Figure 2. Trajectories in the $X$–$Y$ plane of the cosmological scenario (3.14), where the varying graviton mass square $V(\psi)$ is imposed at will, in an open universe. We focus on the invariant set $\Omega_k = U = Z = 0$ and we choose $\gamma = 1$, $\lambda_V = -2$, $\lambda_W = 1$, $\alpha_3 = \alpha_4 = 0.1$. In this specific example, the stable late-time state of the universe is the phantom solution $Q_1$.

Figure 3. Trajectories in the $Y$–$Z$ plane of the cosmological scenario (3.14), where the varying graviton mass square $V(\psi)$ is imposed at will, in an open universe. We focus on the invariant set $\Omega_k = X = Z = 0$ and we choose $\gamma = 1$, $\lambda_V = 2$, $\lambda_W = 1$, $\alpha_3 = \alpha_4 = 0.1$ and $U_c = \frac{\lambda_v}{2 \pi}$. In this specific example, the stable late-time state of the universe is the quintessence-like point $Q_{14}$. 
Figure 4. Trajectories of the cosmological scenario (3.14), where the varying graviton mass square $V(\psi)$ is imposed at will, in an open universe, in the subset $X = Z = 0$, which is invariant provided $3 + 3\alpha_3 + \alpha_4 = 0$, $\alpha_3 \neq -2$, $\alpha_4 \neq 3$. We use the parameters values $\gamma = 1$, $\lambda_w = 3$. In this specific example, the stable late-time solutions of the universe are the expanding, non-accelerating $Q^+_6$ (its basin of attraction is the half-subspace $\Omega_k > 0$) and the contracting $Q^-_6$ (its basin of attraction is the half-subspace $\Omega_k < 0$). Additionally, we can see the saddle points $Q_{12}$ (non-accelerating, matter-dominated), $Q_{17}$ (curvature-dominated, contracting) and $Q_{18}$ (non-accelerating, curvature-dominated, expanding), as well as the unstable points $Q_{12}$ and $Q_{13}$ (non-accelerating, dark-energy dominated, with stiff $w_{DE}$).

the corresponding phase-space behavior of the autonomous system (3.14), but restricted to the invariant set $\Omega_k = X = Z = 0$, and using $\gamma = 1$, $\lambda_V = 2$, $\lambda_w = 1$, $\alpha_3 = \alpha_4 = 0.1$ and $U_c = \frac{\lambda w}{\rho}$. In this case, the late-time stable solution of the universe is the quintessence-like point $Q_{14}$.

Finally, in figure 4 we present the phase-space behavior of the autonomous system (3.14), in the subset $X = Z = 0$, which is invariant provided $3 + 3\alpha_3 + \alpha_4 = 0$. In this case, the universe can be attracted by two stable late-time solutions, namely the expanding, non-accelerating $Q^+_6$ (its basin of attraction is the half-subspace $\Omega_k > 0$), and the contracting $Q^-_6$ (its basin of attraction is the half-subspace $\Omega_k < 0$). Finally, as we discussed in the end of section 3.1.2 and in section A.2, we mention that in the scenario at hand the sign of $\Omega_k$ is invariant. Thus, although our model admits expanding (lower half of figure 4) and contracting evolution (upper half of figure 4), there is no transition from contracting to expanding solutions or vice versa, that is, a cosmological bounce or turnaround is not possible.

In summary, as we can see, the scenario of extended nonlinear massive gravity in open geometry has a great variety of stable late-time solutions, as was shown in [75, 76] through specific examples.

4.2. Imposing $b(t)$ at will

4.2.1. Flat universe. In this case, the scenario at hand admits a variety of stable late-time solutions. In particular, point $R^+_4$ corresponds to an expanding dark-energy dominated universe, with dark energy lying in the quintessence regime, which can be accelerating or non-accelerating according to the usual potential exponent, and the graviton mass is zero. This
Figure 5. Trajectories of the cosmological scenario (3.25)–(3.28), where the St¨uckelberg field function \( b(t) \) is imposed at will, in a flat universe, using \( \gamma = 1, \lambda_W = 1, \alpha_3 = \alpha_4 = 0.5, B = 1.7 \).

In this specific example, the stable late-time state of the universe is the expanding, dark-energy dominated, quintessence-like point \( R_4^+ \). Additionally, we depict the saddle points \( R_1 \) (non-accelerating, matter-dominated), and \( R_2, R_3 \) (non-accelerating, dark-energy dominated).

point exists in standard quintessence too [79], and it is quite important since it possesses \( \Omega_{DE} \) and \( q \) compatible with observations.

Point \( R_5^+ \) has the advantage that \( 0 < \Omega_{DE} < 1 \), that is, it can alleviate the coincidence problem, and moreover the graviton mass is zero; however, it has the disadvantage that it is not accelerating and possesses \( w_{DE} = 0 \), which are not favored by observations. This point exists in standard quintessence too [79]; however, note that in the present case it is non-hyperbolic, and thus its stability is weaker (due to the existence of an extra dimension in the phase space, namely the graviton mass).

Point \( R_6^+ \) exists for \( \lambda_W = 0 \) and it is always stable. Although at first sight it seems to be the \( \lambda_W \to 0 \) limit of \( R_4^+ \) this is not the case since the complete equations are different. It corresponds to an accelerating, dark-energy dominated universe, in which dark energy behaves like a cosmological constant, and moreover the graviton mass is zero.

The curves of critical points \( R_7 \) and \( R_8 \) can also be the late-time state of the universe (they are non-hyperbolic and thus their stability is weaker). They correspond to non-accelerating solutions, where the dark energy behaves like dust and where \( 0 < \Omega_{DE} < 1 \), and additionally they possess a non-zero value for the graviton mass. These features disfavor these curves of critical points. Finally, we mention here that although the aforementioned individual points were obtained in [86] too, these curves of critical points were missed, due to the fact that in the analysis one of the phase-space directions was frozen for simplicity.

In figure 5, we depict orbits of the autonomous system (3.25)–(3.28), restricting to the invariant set \( u = 0 \), and using \( \gamma = 1, \lambda_W = 1, \alpha_3 = \alpha_4 = 0.5, B = 1.7 \). In this specific example, the stable late-time state of the universe is the expanding, dark-energy dominated, quintessence-like point \( R_4^+ \).

4.2.2. Open universe. This scenario possesses only one stable solution that can attract the universe at late times, namely \( S_6^+ \) (although at first sight it seems to be the \( \lambda_W \to 0 \) limit of \( S_4^+ \) this is not the case since the complete equations are different). This point corresponds to a dark-energy dominated, accelerating universe, with zero graviton mass, and where the dark energy behaves like cosmological constant. This solution is the global attractor of this cosmological system, that is, the universe will be always led there, for every initial conditions.
These features make this point a good candidate for the description of the universe. However, we mention that it exists only for $\lambda_W = 0$, that is, a form of parameter-tuning is needed. On the other hand, for $\lambda_W \neq 0$ the system does not accept any stable solutions, due to the fact that there are unstable directions related to both curvature and graviton mass. In summary, this implies that in general the scenario at hand has disadvantages, unless one tunes the model parameters. Finally, note that since the sign of $\Omega_k$ is invariant, although the model admits expanding and contracting evolution, a cosmological bounce or a turnaround is not possible.

In figure 6, we present orbits of the autonomous system (3.34)–(3.38), restricting to the invariant set $u = v = 0$, and using $\gamma = 1$, $\lambda_W = 0$. Note that the evolution is independent of the values of $\alpha_3$, $\alpha_4$ and $b_0$, since they do not appear explicitly in the equations governing the dynamics in this invariant set. In this specific example, the stable late-time state of the universe is the cosmological-constant-like solution, $S_{6+}$. Additionally, we depict the saddle points $S_1$ (non-accelerating, matter-dominated), and $S_2$, $S_3$ (non-accelerating, dark-energy dominated).

5. Conclusions

In this work, we investigated the dynamical behavior of extended (varying-mass) nonlinear massive gravity, which is an extension of the usual nonlinear massive gravity [6, 7] where the graviton mass is promoted to a scalar-field potential [74]. This scenario has a lot of freedom due to the involved free functions, and thus its cosmological implications are significant.

In order to extract the basic features of the above paradigm, we performed a detailed dynamical analysis in the case of an open geometry, adding for completeness the flat case, although it proves to have disadvantages that can be cured only at the phenomenological level. In both analyses we followed two approaches, namely the theoretically robust one to impose the graviton mass square at will and let the equations determine suitably the St"uckelberg field function, or the theoretically less-justified one to impose the St"uckelberg field function at will and let the equations to determine the graviton mass square. In all cases, we extracted the late-time solutions and calculated the corresponding observables, such as the dark-energy
equation-of-state parameter, the deceleration parameter and the dark-energy and matter density parameter.

One basic feature of the scenario at hand is that it can lead to an accelerating universe, with an effective dark energy lying in the quintessence or in the phantom regime, or experience the phantom-divide crossing during the evolution. This is a great advantage since the model at hand utilizes only a canonical field. Additionally, and more interestingly, the universe cannot only be phantom at one stage of its evolution, but also at its final late-time solutions it can be quintessence or phantom like. This is not the case in other modified-gravity scenarios, where the universe results to quintessence-like solutions even if it has passed through the phantom regime [98]. The above features were discussed in [75, 76] using specific solutions, but in this work they arise from a general dynamical analysis.

An additional advantage of extended nonlinear massive gravity is that the graviton mass goes asymptotically to zero at late times, without fine-tuning, which is in agreement with observations. Note that this is not the case in usual massive gravity, where ones needs to fine-tune the graviton mass to a very small value by hand.

Finally, another advantage of the present scenario is that the dark-energy density parameter at the late-time solutions can be between zero and one, which can alleviate the coincidence problem since dark-energy and dark matter density parameters can be of the same order.

In the above analysis, we used the exponential ansatz for the usual scalar-field potential, and then we used an exponential form for the graviton square mass, in order to be phenomenologically consistent. One could ask whether the above behaviors are a result of these specific ansatze, or they have a general character. Although this would need an explicit investigation from the beginning, the details of our analysis indicate that the results are qualitatively robust for many phenomenologically consistent varying graviton mass choices too. However, in the alternative and less-justified approach where the St"uckelberg field is imposed at will, our results are quite sensitive to the input ansatz, and therefore a detailed analysis is required for every new choice. The fact that the results are very sensitive in the St"uckelberg field ansatz, is known to happen in the usual nonlinear massive gravity too [71–73].

In summary, the scenario of extended (varying-mass) nonlinear massive gravity exhibits a larger variety and a richer structure of interesting cosmological late-time solutions, comparing to usual quintessence, phantom, and quintom cosmology, and also to usual (constant-mass) massive gravity. These features are in agreement with observations and thus they make this paradigm a good candidate for the description of nature. However, an additional requirement for the validity of this scenario is to behave consistently beyond the background level too. Since the theory at hand is based on the usual massive gravity formalism in order to become Boulware–Deser ghost free, the perturbation analysis could reveal interesting issues too [99, 100]. Although such a perturbation investigation is therefore necessary, it lies beyond the scope of this work and it is left for a future project.

Acknowledgments

The authors would like to thank S Lepe for reading the original manuscript and making helpful suggestions. GL was supported by MECESUP FSM0806 from Ministerio de Educació, Chile and by PUCV through Proyecto DI Postdoctorado 2013. JS has been supported by Comisión Nacional de Ciencias y Tecnología through FONDECYT Grant 1110076, 1090613 and 1110230 and also by PUCV grant no. 123.713/2012. The research project is implemented within the framework of the Action “Supporting Postdoctoral Researchers” of the Operational Program ‘Education and Lifelong Learning’ (Actions Beneficiary: General
Table A1. The eigenvalues of matrix Q of the perturbation equations of the autonomous system (3.4), and the corresponding stability conditions.

| Cr. P | Eigenvalues | Stability |
|-------|-------------|-----------|
| P₁    | -1, 3(γ - 1), 3(γ - \(\frac{1}{1 + \gamma}\)) | Stable for \(γ < \min\{1, \frac{1}{1 + \gamma}\}\), \(\lambda_2^V \geq \frac{1}{2}\) saddle point otherwise |
| P₂    | -1, -3(γ - 1), 3(1 - \(\frac{1}{1 + \gamma}\)) | Stable for \(γ > 1, \frac{1}{1 + \gamma} > 1\) saddle point otherwise |
| P₃    | -1, -3(γ - \(\frac{1}{1 + \gamma}\)), -3(1 - \(\frac{1}{1 + \gamma}\)) | \(\frac{1}{1 + \gamma} < \min\{1, \gamma\}\), \(\lambda_2^V \geq \frac{1}{2}\) saddle point otherwise |

Appendix A. Stability when \(V(ψ)\) is imposed at will

A.1. Flat universe

For the critical points \((u_c, Y_c, Z_c)\) of the autonomous system (3.4), the coefficients of the perturbation equations form a \(3 \times 3\) matrix \(Q\); however, since they are quite complicated expressions we do not display them explicitly. Despite this complicated form, using the specific critical points presented in table 1, the matrix \(Q\) obtains a simple form that allows for an easy calculation of its eigenvalues. The corresponding eigenvalues and the stability conditions for each critical point are presented in table A1.

Since in the special case \(γ = 1\) (dust matter) one eigenvalue of \(P_1\) and \(P_2\) becomes zero, we need to examine this case separately. For \(γ = 1\), the system (3.4) is restricted to the invariant set \(u = 0\) and it admits the general solution

\[
Y(τ) = \frac{3 - 2λ_2^V}{e^{(2λ_2^V - 3)\frac{1}{1 + \gamma} \tau} - 2λ_2^V}
\]

\[
Z(τ) = \frac{c_2 e^{τ/2}}{\sqrt{e^{2λ_1^V λ_2^V + τ} - 2λ_2^V e^{3λ_2^V + \frac{1}{1 + \gamma}}}}, \tag{A.1}
\]

where \(c_1\) and \(c_2\) are integration constants. In this case, the system (3.4) admits two classes of critical points: the point \(P_3\) for which the stability conditions reduce to \(\frac{1}{1 + \gamma} < 1, \lambda_2^V \geq \frac{1}{2}\) and the \(Z\)-axis which is a curve of equilibrium points containing the points \(P_1\) and \(P_2\). The center direction of the curve is tangent to the \(Z\)-axis, and therefore this curve of critical points is normally hyperbolic [101] (a set of non-isolated singular points is called normally hyperbolic if the only eigenvalues with zero real parts are those whose corresponding eigenvectors are tangent to the set), and since by definition any point on a set of non-isolated singular points will have at least one eigenvalue which is zero, all points in the set are non-hyperbolic. The stability of a set which is normally hyperbolic can be completely classified by considering the signs of the eigenvalues in the remaining directions [101]. In conclusion, in the special case \(γ = 1\), the curve of critical points that contains \(P_1\) and \(P_2\) is stable for \(\frac{1}{1 + \gamma} > 1\).

Finally, let us comment on the asymptotic behavior of \(P_3\). From the constraint equation (2.17), it follows that

\[
\frac{dψ}{dτ} = \frac{f_1(a_{ref} e^{-τ})}{λ_V f_2(a_{ref} e^{-τ})}, \tag{A.2}
\]
which has the solution
\[
\lambda_Y (\psi - \psi_0) = \int_0^\tau \frac{f_1 (a_{\text{ref}} e^{-\lambda_Y})}{f_1 (a_{\text{ref}} e^{-\lambda_Y})} \, d\eta,
\]
where the current scale factor is set to 1 and \(\psi_0\) denotes the current \(\psi\)-value. Hence,
\[
V \propto e^{\lambda_Y (\psi - \psi_0)} = \frac{(1 - a_{\text{ref}})[-3\alpha_3 (a_{\text{ref}} - 1) + \alpha_4 (a_{\text{ref}} - 1)^2 + 3]}{(e^{\tau} - a_{\text{ref}})[e^{2\tau} (3\alpha_3 + \alpha_4 + 3) + \alpha_4 a_{\text{ref}}^2 - a_{\text{ref}} e^{\tau} (3\alpha_3 + 2\alpha_4)]},
\]
while
\[
W \propto e^{-\lambda_Y (\psi - \psi_0)} = \left\{ \frac{(1 - a_{\text{ref}})[-3\alpha_3 (a_{\text{ref}} - 1) + \alpha_4 (a_{\text{ref}} - 1)^2 + 3]}{(e^{\tau} - a_{\text{ref}})[e^{2\tau} (3\alpha_3 + \alpha_4 + 3) + \alpha_4 a_{\text{ref}}^2 - a_{\text{ref}} e^{\tau} (3\alpha_3 + 2\alpha_4)]} \right\}^{\frac{\lambda_Y}{\lambda_Y}}.
\]

Therefore, since \(a_{\text{ref}} \lesssim 10^{-6}\), if \(\frac{\lambda_Y}{\lambda_Y} > 0\) both \(V\) and \(W\) tend to zero as \(\tau \to +\infty\). However, for \(\frac{\lambda_Y}{\lambda_Y} < 0\), \(V\) tends to zero but \(W\) tends to infinity as \(\tau \to \infty\), and since \(Y \neq 0\) we deduce that \(H \to \infty\) as \(\tau \to \infty\). This is a Big-Rip-type behavior; however, it is realized at infinity and not at a finite scale factor [91–97].

A.2. Open universe

Let us discuss the critical points of the autonomous system (3.14) and their stability conditions. From the last equation of (3.14) it follows that either \(q = 0\) or \(\Omega_k = 0\), and therefore we can simplify the investigation and examine these two cases separately.

Note that the variable choice (3.13) allows for an easy, partial classification of expanding and contracting solutions. In particular, solutions with \(\Omega_k = k/(aH) > 0\) correspond to \(H > 0\) and thus to expansion, while those with \(\Omega_k < 0\) correspond to \(H < 0\) and therefore to contraction (\(k = \sqrt{|K|}\) throughout this work). That is why points with \(\Omega_k > 0\) are denoted with the subscript ‘+’, while those with \(\Omega_k < 0\) are denoted with the subscript ‘−’. However, this is only a partial classification, since it cannot work for solutions with \(\Omega_k = 0\), which can be either expanding or contracting. Finally, we mention that although our model admits expanding and contracting solutions, since the sign of \(\Omega_k\) is invariant, there is no transition from contracting to expanding solutions or vice versa. Nevertheless, there could still exist at the edge of the phase space, and in such a case they could be revealed only through the application of the Poincaré central projection method [82–84]. This analysis lies beyond the scope of this work and it is left for future investigation.

Analysis in the invariant set \(\Omega_k = 0\)

In this case, from the first equation of (3.14) it follows that \(X = 0\). Thus, the curvatureless equilibrium solutions must satisfy
\[
Y[\sqrt{6}U^2[(\gamma - 2)\lambda_Y - \lambda_W] + 2U(\lambda_Y \lambda_W + 3)]
- \sqrt{6}[\lambda_W Y + \lambda_W [Y - \gamma Y + (\gamma - 1)Z(4\alpha_3 + \alpha_4 + 6)]] = 0,
\]
\[
Z[-\sqrt{6}(\gamma - 3)\lambda_Y U^2 - 2(\lambda_Y^2 + 3)U]
+ \sqrt{6}[\lambda_W Y + \lambda_Y [Y - \gamma Y + (\gamma - 1)Z(4\alpha_3 + \alpha_4 + 6)]] = 0,
\]
\[
U \neq -\frac{\sqrt{6}}{3} \lambda_Y.
\]
Note that in this case the evolution equation for \(U\) reduces to \(U' = 0\), which implies that in the former expressions \(U\) behaves as a parameter (a constant).

Thus, in the case of \(\Omega_k = 0\) we have the following curves of critical points.

25
the following three possibilities:

- **Curve Q₁**: \( X_{c1} = 0, Y_{c1} = \frac{\lambda_c}{\gamma} \), \( Z_{c1} = Z_e, U_{c1} = 0, \Omega_{c1} = 0, \) with eigenvalues
  \[ \{-1, -1, 0, 0, -3\gamma\} \].

- **Curve Q₂**: \( X_{c2} = 0, Y_{c2} = 0, Z_{c2} = Z_e, U_{c2} = \frac{-\lambda_c + \sqrt{\delta(\gamma - 3)\lambda_c}}{\delta}, \Omega_{c2} = 0, \) with eigenvalues
  \[ \left\{ 0, -1, -\frac{2\gamma + \lambda_c + \sqrt{\delta(\gamma - 3)\gamma + (\gamma - 3)(\gamma - 1)Z_e(4\lambda_c + \alpha_4 + 6) + 1) + 9 + 3}{2\gamma + 3}, \frac{6(\gamma - 3)(\gamma - 1)\lambda_c Z_e(4\lambda_c + \alpha_4 + 6) + 1 + 9 + 3}{(2\gamma - 5)\lambda_c + \sqrt{\delta(\gamma - 3)\gamma + (\gamma - 3)(\gamma - 1)Z_e(4\lambda_c + \alpha_4 + 6) + 1) + 9 + 3}, \right\} \].

- **Curve Q₃**: \( X_{c3} = 0, Y_{c3} = 0, Z_{c3} = Z_e, U_{c3} = \frac{-\lambda_c + \sqrt{\delta(\gamma - 3)\gamma + (\gamma - 3)(\gamma - 1)Z_e(4\lambda_c + \alpha_4 + 6) + 1) + 9 + 3}{2\gamma + 3}, \Omega_{c3} = 0, \) with eigenvalues
  \[ \left\{ 0, -1, -\frac{2\gamma - \lambda_c + \sqrt{\delta(\gamma - 3)\gamma + (\gamma - 3)(\gamma - 1)Z_e(4\lambda_c + \alpha_4 + 6) + 1) + 9 + 3}{2\gamma - 5} \right\} \].

- **Curve Q₄**: \( X_{c4} = 0, Y_{c4} = 0, Z_{c4} = 0, U_{c4} = U_c, \Omega_{c4} = 0, \) with eigenvalues
  \[ \left\{ 0, -1, \frac{\sqrt{\delta(2 - 3)\gamma + 3\sqrt{\delta(\gamma - 3)\gamma + 2}\gamma U_c^2 + 2(2\gamma + 3)U_c^2}}{6\gamma - 2\sqrt{\delta\gamma + 2}}, \frac{3 - \sqrt{\delta(2 - 3)\gamma + 3\sqrt{\delta(\gamma - 3)\gamma + 2}\gamma U_c^2 + 2(2\gamma + 3)U_c^2}}{3\gamma - \sqrt{\delta\gamma + 2}}, \right\} \].

- **Curve Q₅**: \( X_{c5} = 0, Y_{c5} = 1 - U_c^2 + \frac{(\sqrt{\delta(2 - 3)\gamma + 3\sqrt{\delta(\gamma - 3)\gamma + 2}\gamma U_c^2 + 2(2\gamma + 3)U_c^2})}{2\gamma - \sqrt{\delta\gamma + 2}}, Z_{c4} = 0, U_{c5} = U_c, \Omega_{c5} = 0. \)

In order to determine the stability of this curve of critical points, we need to resort to numerical inspection.

The examination of the sign of the above eigenvalues is straightforward for the general case \( \gamma \neq 1 \); however, in the special case \( \gamma = 1 \), which is the most interesting in physical terms since it corresponds to dust matter, some eigenvalues become zero and thus the corresponding curves of critical points become non-hyperbolic. In this case, if the remaining eigenvalues have different signs then the curve of critical points behaves like saddle, while if they are of the same sign then the non-hyperbolic curve of critical points has a stable or unstable manifold of smaller dimensionality (in principle one must apply the center manifold theorem [101]). The curves of critical points \( Q₁-Q₅ \) for the special case \( \gamma = 1 \) are summarized in table 2, while their stability conditions are displayed in table 3.

**Analysis in the invariant set q = 0**

In the case \( q = 0 \), from (3.14) we deduce that the equilibrium solutions must satisfy one of the following three possibilities:

- \( Y_c \neq 0, Z_e = 0, U_c = \frac{\sqrt{\gamma}}{3\lambda_c} \).
- \( Y_c = 0, Z_e \neq 0, U_c = \frac{\sqrt{\gamma}}{3\lambda_c} \).
- \( Y_c = 0, Z_e = 0. \)

In the first case, substituting the values of \( Z_e = 0, U_c = \frac{\sqrt{\gamma}}{3\lambda_c} \) into the fourth equation of (3.14) we conclude that the equilibrium solution satisfies \( Y_c = \frac{4}{3\gamma \lambda_c} \). Inserting this into the expression for \( q \) we obtain the additional constraint \( \frac{(3\gamma - 2)U_c^4}{2\sqrt{\delta + 2}} = 0 \), which leads
to $\Omega_{kc} = \pm \sqrt{1 - \frac{2}{\Lambda_1}}$ (corresponding to expanding and contracting universe respectively).

Finally, inserting these expressions in the relation for $\dot{b}$ (3.16) we find that $\Omega_{k} \dot{b} = X$, and thus the first equation of (3.14) is satisfied identically, irrespectively the value of X. In summary, in this case we obtain two curves of critical points, namely

$$Q_0^+: X_{c6}^+ = X_c, Y_{c6}^+ = \frac{4}{3\lambda_w^2}, Z_{c6}^+ = 0, U_{c6}^+ = \frac{\sqrt{6}}{3\lambda^2}, \Omega_{kc6}^+ = \sqrt{1 - \frac{2}{\lambda_w^2}},$$

and

$$Q_0^-: X_{c6}^- = X_c, Y_{c6}^- = \frac{4}{3\lambda_w^2}, Z_{c6}^- = 0, U_{c6}^- = \frac{\sqrt{6}}{3\lambda^2}, \Omega_{kc6}^- = -\sqrt{1 - \frac{2}{\lambda_w^2}}.$$  

In the second case, the system admits two critical points, namely

$$Q_\gamma^+: X_{c7}^+ = 0, Y_{c7}^+ = 0, Z_{c7}^+ = -\frac{4}{3\lambda_w^2(4\alpha_3 + \alpha_4 + 6)}, U_{c7}^+ = \frac{\sqrt{6}}{3\lambda^2}, \Omega_{kc7}^+ = \sqrt{1 - \frac{2}{\lambda_w^2}},$$

and

$$Q_\gamma^-: X_{c7}^- = 0, Y_{c7}^- = 0, Z_{c7}^- = -\frac{4}{3\lambda_w^2(4\alpha_3 + \alpha_4 + 6)}, U_{c7}^- = \frac{\sqrt{6}}{3\lambda^2}, \Omega_{kc7}^- = -\sqrt{1 - \frac{2}{\lambda_w^2}}.$$  

Finally, in the third case, from the fourth equation of (3.14) it follows that $U_c = 0$. Thus, substituting $Y_c = 0, Z_c = 0, U_c = 0$ in the rest of the equations, and assuming that $\gamma \neq 2$, we obtain $\Omega_{kc} = \pm 1$. Therefore, for $\Omega_{kc} = + 1$ the first equation of (3.14) gives

$$Q_8: X_{c8} = 1, Y_{c8} = 0, Z_{c8} = 0, U_{c8} = 0, \Omega_{kc8} = 1$$

$$Q_9: X_{c9} = \frac{4\alpha_3^2 - 6\alpha_4 + 2\alpha_3 + \alpha_4}{\alpha_4^2}, Y_{c9} = 0, Z_{c9} = 0, U_{c9} = 0, \Omega_{kc9} = 1$$

$$Q_{10}: X_{c10} = \frac{4\alpha_3^2 - 6\alpha_4 + 2\alpha_3 + \alpha_4}{\alpha_4}, Y_{c10} = 0, Z_{c10} = 0, U_{c10} = 0, \Omega_{kc10} = 1,$$

while for $\Omega_{kc} = -1$, we obtain that $X = \tilde{X}_{c11}$, where $\tilde{X}_{c11}$ is the unique real solution of the equation $-2\alpha_3 (X^2 + X - 2) + \alpha_4 (X + 1)(X - 1)^2 + 6 = 0$.

Let us now examine the eigenvalues associated with the critical points or curves $Q_0^+ - Q_{11}$. The eigenvalues of $Q_0^+$ are $[0, 2 - 3\gamma, 2 - \frac{2\alpha_3}{\alpha_4}, -\sqrt{\lambda_1^2 - 3\lambda_2^2} - 1, \sqrt{\lambda_1^2 - 3\lambda_2^2} - 1]$. The eigenvalues of $Q_\gamma^+$ (for $\gamma = 1$) are $[0, -1, 2 - \frac{2\alpha_3}{\alpha_4}, \Delta_1(\alpha_3, \alpha_4, \lambda_V, \lambda_W), \Delta_2(\alpha_3, \alpha_4, \lambda_V, \lambda_W)]$, where $\Delta_1, \Delta_2(\alpha_3, \alpha_4, \lambda_V, \lambda_W)$ are complicated functions of their arguments that can be obtained explicitly only by numerical elaboration. The eigenvalues associated with $Q_8$ are $[0, 2, 2, -2, 2 - 3\gamma]$. The eigenvalues associated with $Q_9$ and $Q_{10}$ are $[-2, 2, 2, 0, \Delta_3(\gamma, \alpha_3, \alpha_4)]$ and finally for $Q_{11}$ they are $[-2, 2, 2, 0, \Delta_4(\gamma, \alpha_3, \alpha_4)]$, where $\Delta_3, \Delta_4(\gamma, \alpha_3, \alpha_4)$ are complicated expressions of their arguments. Thus, $Q_8 - Q_{11}$ are always saddle since at least two eigenvalues have different signs.

The individual critical points $Q_\gamma^+ - Q_{10}$ and the curves of critical points $Q_0^+$ and $Q_{11}$, for the special case $\gamma = 1$, are summarized in table 2, while their stability conditions are displayed in table 3.

Quintessence-like solutions

We close this appendix by mentioning that the curve of critical points $Q_5$ analyzed above includes many interesting cosmological solutions, and in particular the points of standard
Table A2. The eigenvalues of matrix $Q$ of the perturbation equations of the autonomous system (3.14), and the corresponding stability conditions, for the quintessence-like solutions presented in table 4.

| Cr. P. | Eigenvalues | Stability |
|--------|-------------|-----------|
| $Q_{12}$ | $2, -1, 0$, $\sqrt{6}\lambda_{Y} + 6$, $\sqrt{6}\lambda_{W} + 6$ | Saddle point |
| $Q_{13}$ | $2, -1, 0$, $6 - \sqrt{6}\lambda_{Y}$, $6 - \sqrt{6}\lambda_{W}$ | Saddle point |
| $Q_{14}$ | $-1, 0$, $\lambda_{Y}^2$, $\frac{2}{3}(\lambda_{Y}^2 - 2)$, $\lambda_{W}(\lambda_{W} - \lambda_{Y})$ | Stable node for $-\sqrt{2} < \lambda_{W} < 0$, $\lambda_{Y} < \lambda_{W}$ or $0 < \lambda_{W} < \sqrt{2}$, $\lambda_{Y} > \lambda_{W}$ saddle point otherwise |
| $Q_{15}$ | $-1, \frac{1}{2}, 0$, $-\frac{2}{3}(\lambda_{Y} - \lambda_{W})$, $-\frac{2}{6}\lambda_{Y}(\lambda_{Y} - \lambda_{W})$ | Saddle point |
| $Q_{16}$ | $3, 3$, $-1$, $\frac{1}{2}$, $0$ | Saddle point |
| $Q_{17}$ | $2, 2$, $-1$, $0$, $2$ | Saddle point |
| $Q_{18}$ | $2, 2$, $-1$, $-1$, $-4$ | Saddle point |

quintessence [79, 85]. Focusing for simplicity on the case $\gamma = 1$, these points were presented in table 4. However, the stability conditions are different than the usual conditions in [79, 85] due to the presence of extra phase-space directions, namely those of curvature and graviton mass.

In particular, for the critical points $Q_{12}$ to $Q_{18}$ of table 4, the coefficients of the perturbation equations form a $5 \times 5$ matrix $Q$ that allows for an easy calculation of its eigenvalues. The corresponding eigenvalues and the stability conditions for each critical point are displayed in table A2. Finally, some of these points possess one zero eigenvalue and are thus non-hyperbolic. In the case of normally-hyperbolic curves of critical points (that is the only eigenvalues with zero real parts are those whose corresponding eigenvectors are tangent to the set), the stability is extracted considering the signs of the rest eigenvalues [101]. For isolated non-hyperbolic critical points, we can determine the dimensionality of their stable manifold using the linearization technique [101].

Appendix B. Stability when $b(t)$ is imposed at will

B.1. Flat universe

For the critical points $(x_c, y_c, u_c, v_c)$ of the autonomous system (3.25)–(3.28), the coefficients of the perturbation equations form a $4 \times 4$ matrix $Q$, which using the specific critical points presented in table 5 acquires a simple form that allows for an easy calculation of its eigenvalues. The corresponding eigenvalues and the stability conditions for each critical point are presented in table B1. We mention that point $R_1$ is non-hyperbolic, but since it has eigenvalues with different signs, and using the center manifold analysis [101], we can straightforwardly show that it behaves as a saddle point. Moreover, the non-hyperbolic curve of critical points $R_7$ has a central direction normal to the set and therefore it behaves as stable. Finally, note that although $R_4^\pm$ at first sight seems to be the $\lambda_{W} \to 0$ limit of $R_4^\pm$ this is not the case since the complete equations are different.

B.2. Open universe

For the critical points $(x_c, y_c, u_c, v_c, \Omega_{k_c})$ of the autonomous system (3.34)–(3.38), the coefficients of the perturbation equations form a $5 \times 5$ matrix $Q$, which using the specific critical points presented in table 7 acquires a simple form that allows for an easy calculation of
provided its eigenvalues. The corresponding eigenvalues for each critical point are presented in Table B2.

Finally, note that although the system in the invariant set \( v = 0 \) at first sight seems to be the \( \lambda_w \to 0 \) limit of \( S^4 \), this is not the case since the complete equations are different.

In order to examine the corresponding stability conditions we have to examine the signs of these eigenvalues. An interesting observation from (3.37) is that the sign of \( v \) (which according to (3.33) is the auxiliary variable proportional to the graviton mass square) is invariant. Therefore, \( v \) remains zero if initially it is zero, and in this case we can examine the system in the invariant set \( v = 0 \). In this case, the possible late-time solutions are either \( S^4 \) provided \( \lambda_w^2 < 2 \) or either \( S^6 \) for \( \lambda_w^2 > 2 \). In the particular case of \( 2 < \lambda_w^2 \leq \frac{8}{3} \), the points
$S^c_n$ are spiral attractors in a 2D sub-manifold (two negative real eigenvalues and two complex conjugated eigenvalues with negative real part). Finally, points $S^c_6$ are non-hyperbolic, with a 4D stable manifold.

However, in the case where $v \neq 0$ only points $S^c_5$ behave as stable, since all the other become saddle points. In particular, introducing the local coordinates \(\{x - x_c, y - y_c, v, \Omega_k\} = \epsilon \{\tilde{x}, y, \tilde{v}, \tilde{\Omega}_k\} + \mathcal{O}(\epsilon^2)\) where $\epsilon$ is a constant satisfying $\epsilon \ll 1$, we deduce that

$$\tilde{v}' = 3\tilde{v} (x_c^2 - y_c^2 + 1) + \text{h.o.t}$$

$$\tilde{\Omega}_k' = \frac{1}{2} \tilde{\Omega}_k \left( 3x_c^2 - 3y_c^2 + 1 \right) - \frac{1}{2} \tilde{v} \beta [\beta^2 (\alpha_3 + \alpha_4) - 2\beta (2\alpha_3 + \alpha_4 + 1) + 3\alpha_3 + \alpha_4 + 3] + \text{h.o.t},$$

where $x_c$ and $y_c$ are the coordinates of the critical points $S_1$ to $S_5$ and h.o.t denoting ‘higher order terms’. These equations admit the general solutions

$$\tilde{v} = c_1 e^{3x} (x_c^2-y_c^2+1),$$

$$\tilde{\Omega}_k = c_2 e^{2y} (3x_c^2-3y_c^2+1) - e^{4x} (\alpha_3 \beta^2 - 4\alpha_3 \beta + 3\alpha_3 + \alpha_4 \beta^2 - 2\alpha_4 \beta + \alpha_4 - 2\beta + 3)$$

$$+ c_2 e^{2x} (3x_c^2-3y_c^2+1),$$

which implies that the system is unstable in the $v$ and $\Omega_k$ directions.

In the special case of point $S^c_6$, using a similar approach we extract that the perturbations $\tilde{v}$ and $\tilde{\Omega}_k$ satisfy the equations

$$\tilde{v}' = - \frac{\partial \tilde{v}^2}{\Omega_k}, \quad \tilde{\Omega}_k' = \frac{1}{2} (-\tilde{v} \theta - 2\tilde{\Omega}_k),$$

where $\theta = \beta [\beta^2 (\alpha_3 + \alpha_4) - 2\beta (2\alpha_3 + \alpha_4 + 1) + 3\alpha_3 + \alpha_4 + 3]$. The system (B.3) admits two general solutions

$$\tilde{v} = \frac{4 e^{2y}}{(e^x - e^{2y} \theta)^2}, \quad \tilde{\Omega}_k = \frac{2 \theta e^{2x-\tau}}{e^x - e^{2y} \theta}$$

and

$$\tilde{v} = \frac{4 e^{2y} c_1}{(e^{2y} \theta + e^x)^2}, \quad \tilde{\Omega}_k = \frac{2 e^{2y} \theta}{(e^{2y} \theta + e^x) (2 e^{2y} \theta + e^x)},$$

where $c_1$ and $c_2$ are integration constants. In both cases, the $v$-perturbations and $\Omega_k$-perturbations decay to zero in the limit $\tau \to +\infty$, and thus points $S^c_6$ are stable.

The stability conditions for the critical points $S_1$-$S^c_6$ are summarized in table B2.

References

[1] Fierz M and Pauli W 1939 On relativistic wave equations for particles of arbitrary spin in an electromagnetic field Proc. R. Soc. Lond. A 173 211
[2] van Dam H and Veltman M J G 1970 Massive and massless Yang–Mills and gravitational fields Nucl. Phys. B 22 397
[3] Zakharov V I 1970 Linearized gravitation theory and the graviton mass JETP Lett. 12 312
Zakharov V I 1970 Pis. Zh. Eksp. Teor. Fiz. 12 447
[4] Vainshtein A I 1972 To the problem of nonvanishing gravitation mass Phys. Lett. B 39 393
[5] Boulware D G and Deser S 1972 Can gravitation have a finite range? Phys. Rev. D 6 3368
[6] de Rham C and Gabadadze G 2010 Generalization of the Fierz–Pauli action Phys. Rev. D 82 044020 (arXiv:1007.0443)
[7] de Rham C, Gabadadze G and Tolley A J 2011 Resummation of massive gravity Phys. Rev. Lett. 106 231101 (arXiv:1011.1232)
[8] Hinterbichler K 2012 Theoretical aspects of massive gravity Rev. Mod. Phys. 84 671 (arXiv:1105.3735)
[9] Koyama K, Niz G and Tasinato G 2011 Strong interactions and exact solutions in non-linear massive gravity Phys. Rev. D 84 064033 (arXiv:1104.2143)
[10] Hassan S F and Rosen R A 2012 Resolving the ghost problem in non-linear massive gravity Phys. Rev. Lett. 108 041101 (arXiv:1106.3344)
[11] de Rham C, Gabadadze G and Tolley A 2012 Ghost free massive gravity in the Stückelberg language Phys. Lett. B 711 190 (arXiv:1107.3820)
[12] Cuadros-Melgar B, Papantonopoulos E, Tsoukalas M and Zamarias V 2012 Massive gravity with anisotropic scaling Phys. Rev. D 85 124035 (arXiv:1108.3771)
[13] Hassan S F and Rosen R A 2012 Bimetric gravity from ghost-free massive gravity J. High Energy Phys. JHEP02(2012)126 arXiv:1109.3515
[14] Kluson J 2012 Note about Hamiltonian structure of non-linear massive gravity J. High Energy Phys. JHEP01(2012)013 arXiv:1109.3052
[15] Gumrukcuoglu A E, Lin C and Mukohyama S 2011 Open FRW universes and self-acceleration from nonlinear massive gravity J. Cosmol. Astropart. Phys. JCAP11(2011)030 (arXiv:1109.3845)
[16] Volkov M S 2012 Cosmological solutions with massive gravitons in the bigravity theory J. High Energy Phys. JHEP01(2012)035 (arXiv:1110.6153)
[17] von Strauss M, Schmidt-May A, Enander J, Mortsell E and Hassan S F 2012 Cosmological solutions in bimetric gravity and their observational tests J. Cosmol. Astropart. Phys. JCAP03(2012)042 (arXiv:1111.1655)
[18] Comelli D, Crisostomi M, Nesti F and Pilo L 2012 FRW cosmology in ghost free massive gravity J. High Energy Phys. JHEP03(2012)067 (arXiv:1111.1983)
[19] Comelli D, Crisostomi M, Nesti F and Pilo L 2012 J. High Energy Phys. JHEP06(2012)020 (erratum)
[20] Hassan S F and Rosen R A 2012 Confirmation of the secondary constraint and absence of ghost in massive gravity and bimetric gravity J. High Energy Phys. JHEP04(2012)123 (arXiv:1111.2070)
[21] Berezhiani L, Chkareuli G, de Rham C, Gabadadze G and Tolley A J 2012 On black holes in massive gravity Phys. Rev. D 85 044024 (arXiv:1111.3613)
[22] Gumrukcuoglu A E, Lin C and Mukohyama S 2012 Cosmological perturbations of self-accelerating universe in nonlinear massive gravity J. Cosmol. Astropart. Phys. JCAP03(2012)006 (arXiv:1111.4107)
[23] Khosravi N, Rahmanpour N, Sepangi H R and Shahidi S 2012 Multi-metric gravity via massive gravity Phys. Rev. D 85 024049 (arXiv:1111.5346)
[24] Brihaye Y and Verbin Y 2012 Perfect fluid spherically-symmetric solutions in massive gravity Phys. Rev. D 86 024031 (arXiv:1112.1901)
[25] Buchbinder I L, Pereira D D and Shapiro I L 2012 One-loop divergences in massive gravity theory Phys. Lett. B 712 104 (arXiv:1201.3145)
[26] Ahmedov H and Aliyev A N 2012 Type N spacetimes as solutions of extended new massive gravity Phys. Lett. B 711 117 (arXiv:1201.5724)
[27] Bergshoeff E A, Fernandez-Melgarejo J J, Rosseel J and Townsend P K 2012 On ‘new massive’ 4D gravity J. High Energy Phys. JHEP04(2012)070 (arXiv:1202.1501)
[28] Comelli D, Crisostomi M and Pilo L 2012 Perturbations in massive gravity cosmology J. High Energy Phys. JHEP06(2012)085 (arXiv:1202.1986)
[29] Paulos M F and Tolley A J 2012 Massive gravity theories and limits of ghost-free bigravity models J. High Energy Phys. JHEP09(2012)002 (arXiv:1203.4268)
[30] Hassan S F, Schmidt-May A and von Strauss M 2012 Proof of consistency of nonlinear massive gravity in the Stückelberg formulation Phys. Lett. B 715 335 (arXiv:1203.5283)
[31] Comelli D, Crisostomi M, Nesti F and Pilo L 2012 Degrees of freedom in massive gravity Phys. Rev. D 86 015004 (arXiv:1202.1027)
[32] Sbisa F, Niz G, Koyama K and Tasinato G 2012 Characterising Vainshtein solutions in massive gravity Phys. Rev. D 86 024033 (arXiv:1204.1193)
[33] Kluson J 2012 Non-linear massive gravity with additional primary constraint and absence of ghosts Phys. Rev. D 86 044024 (arXiv:1204.2957)
[34] Tasinato G, Koyama K and Niz G 2012 New symmetries in Fierz–Pauli massive gravity J. High Energy Phys. JHEP07(2012)062 (arXiv:1204.5880)
[35] Morand K and Solodukhin S N 2012 Dual massive gravity Phys. Lett. B 715 260 (arXiv:1204.6224)
[36] Cardone V F, Radicella N and Parisi L 2012 Constraining massive gravity with recent cosmological data Phys. Rev. D 85 124005 (arXiv:1205.1613)
[37] Baccetti V, Martin-Moruno P and Visser M 2013 Massive gravity from bimetric gravity Class. Quantum Grav. 30 015004 (arXiv:1205.2158)
[37] Gratia P, Hu W and Wyman M 2012 Self-accelerating massive gravity: exact solutions for any isotropic matter distribution Phys. Rev. D 86 061504 (arXiv:1205.4241)
[38] Volkov M S 2012 Exact self-accelerating cosmologies in the ghost-free bigravity and massive gravity Phys. Rev. D 86 061502 (arXiv:1205.5713)
[39] de Rham C and Renaux-Petel S 2012 Massive gravity on de Sitter and unique candidate for partially massless gravity arXiv:1206.3482
[40] Berg M, Buchberger I, Enander J, Mortsell E and Sjors S 2012 Growth histories in bimetric massive gravity J. Cosmol. Astropart. Phys. JCAP12(2012)021 (arXiv:1206.3496)
[41] D’Amico G 2012 Cosmology and perturbations in massive gravity Phys. Rev. D 86 124019 (arXiv:1206.3617)
[42] Fasiello M and Tolley A J 2012 Cosmological perturbations in massive gravity and the Higuchi bound J. Cosmol. Astropart. Phys. JCAP11(2012)035 (arXiv:1206.3852)
[43] D’Amico G, Gabadadze G, Hui L and Pirtskhalava D 2012 Quasi-dilaton: theory and cosmology arXiv:1206.4253
[44] Baccetti V, Martin-Moruno P and Visser M 2012 Gordon and Kerr–Schild ansatze in massive and bimetric gravity arXiv:1206.4720
[45] Gong Y 2012 Cosmology in massive gravity arXiv:1207.2726
[46] Volkov M S 2012 Exact self-accelerating cosmologies in the ghost-free bigravity and massive gravity— the detailed derivation Phys. Rev. D 86 104022 (arXiv:1207.3723)
[47] Nojiri S and Odintsov S D 2012 Ghost-free $F(R)$ bigravity and accelerating cosmology Phys. Lett. B 716 377 (arXiv:1207.5106)
[48] Deffayet C, Mourad J and Zahariade G 2012 Covariant constraints in ghost free massive gravity arXiv:1207.6338
[49] Chiang C-I, Izumi K and Chen P 2012 Spherically symmetric analysis on open FLRW solution in non-linear massive gravity J. Cosmol. Astropart. Phys. JCAP12(2012)025 (arXiv:1208.1222)
[50] Hassan S F, Schmidt-May A and von Strauss M 2012 On consistent theories of massive spin-2 fields coupled to gravity arXiv:1208.1515
[51] Kuhnel F 2012 On instability of certain bi-metric and massive-gravity theories arXiv:1208.1764
[52] Motohashi H and Suyama T 2012 Self-accelerating solutions in massive gravity on isotropic reference metric Phys. Rev. D 86 081502 (arXiv:1208.3019)
[53] Deffayet C, Mourad J and Zahariade G 2012 A note on ‘symmetric’ vielbeins in bimetric, massive, perturbative and non perturbative gravities arXiv:1208.4493
[54] Lambiase G 2012 Constraints on massive gravity theory from big bang nucleosynthesis J. Cosmol. Astropart. Phys. JCAP10(2012)028 (arXiv:1208.5512)
[55] Gumrukcuoglu A E, Kuroyanagi S, Lin C, Mukohyama S and Tanahashi N 2012 Gravitational wave signal from massive gravity Class. Quantum Grav. 29 235026 (arXiv:1208.5975)
[56] Gabadadze G, Hinterbichler K, Khoury J, Pirtskhalava D and Trodden M 2012 A covariant master theory for novel Galilean invariant models and massive gravity Phys. Rev. D 86 124004 (arXiv:1208.5773)
[57] Kluson J 2012 Note about hamiltonian formalism for general non-linear massive gravity action in Stuckelberg formalism arXiv:1209.3612
[58] Tasinato G, Koyama K and Niz G 2012 Vector instabilities and self-acceleration in the decoupling limit of massive gravity arXiv:1209.3627
[59] Gong Y 2012 Observational constraints on massive gravity arXiv:1210.5396
[60] Zhang Y-I, Saito R and Sasaki M 2012 Hawking–Moss instanton in nonlinear massive gravity arXiv:1210.6224
[61] Park M and Sorbo L 2012 Massive gravity from higher derivative gravity with boundary conditions J. High Energy Phys. JHEP01(2013)043 (arXiv:1210.7733)
[62] Cai Y-F, Esson D A, Gao C and Saridakis E N 2012 Charged black holes in nonlinear massive gravity arXiv:1211.0563
[63] Wyman M, Hu W and Gratia P 2012 Self-accelerating massive gravity: time for field fluctuations arXiv:1211.4576
[64] Burrage C, Kaloper N and Padilla A 2012 Strong coupling and bounds on the graviton mass in massive gravity arXiv:1211.6001
[65] Nojiri S, Odintsov S D and Shirai N 2012 Variety of cosmic acceleration models from massive $F(R)$ bigravity arXiv:1212.2079
[66] Park M and Sorbo L 2012 Vacua and instantons of ghost-free massive gravity arXiv:1212.2691
[67] Alexandrov S, Krasnov K and Speziale S 2012 Chiral description of ghost-free massive gravity arXiv:1212.3614
[68] de Rham C, Gabadadze G, Heisenberg L and Pirtskhalava D 2012 Non-renormalization and naturalness in a class of scalar-tensor theories arXiv:1212.4128
[69] Hinterbichler K, Stokes J and Trodden M 2013 Cosmologies of extended massive gravity arXiv:1301.4993
[70] Langlois D and Naruko A 2012 Cosmological solutions of massive gravity on de Sitter arXiv:1206.6810
[71] De Felice A, Gumrukcuoglu A E and Mukohyama S 2012 Massive gravity: nonlinear instability of the homogeneous and isotropic universe arXiv:1206.2080
[72] D’Amico G, de Rham C, Dubovsky S, Gabadadze G, Pirtskhalava D and Tolley A J 2011 Massive cosmologies arXiv:1108.5231
[73] Gumrukcuoglu A E, Lin C and Mukohyama S 2012 Anisotropic Friedmann–Robertson–Walker universe from nonlinear massive gravity arXiv:1206.2725
[74] Huang Q-G, Piao Y-S and Zhou S-Y 2012 Mass-varying massive gravity Phys. Rev. D 86 124014 (arXiv:1206.5678)
[75] Saridakis E N 2012 Phantom crossing and quintessence limit in extended nonlinear massive gravity arXiv:1207.1800
[76] Cai Y-F, Gao C and Saridakis E N 2012 Bounce and cyclic cosmology in extended nonlinear massive gravity J. Cosmol. Astropart. Phys. JCAP10(2012)048 (arXiv:1207.3786)
[77] Arkani-Hamed N, Georgi H and Schwartz M D 2003 Effective field theory for massive gravitons and gravity in theory space Ann. Phys. 305 96 (arXiv:hep-th/0210184)
[78] Chen X-m, Gong Y-g and Saridakis E N 2009 Phase-space analysis of interacting phantom cosmology J. Cosmol. Astropart. Phys. JCAP04(2009)001 (arXiv:0812.1117)
[79] Leon G and Fadragas C R 2011 Cosmological Dynamical Systems (Saarbrücken: Lambert Academic Publishing)
[80] Lynch S 2007 Dynamical Systems with Applications using Mathematica (Boston, MA: Birkhauser)
[81] Sami M and Toporensky A 2004 Phantom field and the fate of universe Mod. Phys. Lett. A 19 1509 (arXiv:gr-qc/0310209)
[82] Nojiri S, Odintsov S D and Tsujikawa S 2005 Properties of singularities in (phantom) dark energy universe Phys. Rev. D 71 063004 (arXiv:hep-th/0501025)
[83] Capoziello S, De Laurentis M, Nojiri S and Odintsov S D 2009 Classifying and avoiding singularities in the alternative gravity dark energy models Phys. Rev. D 79 124007 (arXiv:0903.2753)
[84] Saridakis E N and Weller J M 2010 A quintom scenario with mixed kinetic terms Phys. Rev. D 81 123523 (arXiv:1012.5304)
[85] Xu C, Saridakis E N and Leon G 2012 Phase-space analysis of teleparallel dark energy J. Cosmol. Astropart. Phys. JCAP07(2012)005 (arXiv:1202.3781)
[86] Deser S and Waldron A 2012 Acasualty of massive gravity arXiv:1212.5835
[87] Deser S, Sandora M and Waldron A 2013 Nonlinear partially massless from massive gravity? arXiv:1301.5621
[88] Aulbach B 1984 Continuous and Discrete Dynamics Near Manifolds of Equilibria (Lecture Notes in Mathematics vol 1058) (Berlin: Springer)