Spaces of extremal magnitude

Tom Leinster* Mark Meckes†

Abstract
Magnitude is a numerical invariant of compact metric spaces. Its theory is most mature for spaces satisfying the classical condition of being of negative type, and the magnitude of such a space lies in the interval $[1, \infty]$. Until now, no example with magnitude $\infty$ was known. We construct some, thus answering a question open since 2010. We also give a sufficient condition for the magnitude of a space to converge to 1 as it is scaled down to a point, unifying and generalizing previously known conditions.

1 Introduction
Magnitude is an invariant defined in the wide generality of enriched categories and specializing to an invariant of metric spaces. (See [10], or [14] for a survey and [12] for a bibliography.) It carries abundant geometric information. For example, for compact $X \subseteq \mathbb{R}^N$, consider the function assigning to each $t > 0$ the magnitude of the rescaled space $tX$. The large-scale asymptotics of this function determine the Minkowski dimension of $X$, its volume, and, under hypotheses, its surface area (Corollary 7.4 of [18], Theorem 1 of [4], and Theorem 2(d) of [6]). Magnitude is also closely related to certain measures of biodiversity, which themselves are essentially entropies ([13], [15] and Chapter 6 of [11]).

The definitions are as follows. For a finite metric space $A$, write $Z_A$ for the matrix $(e^{-d(a,b)})_{a,b \in A}$. If $Z_A$ is invertible, the magnitude $|A|$ of $A$ is the sum of all the entries of $Z_A^{-1}$. A compact metric space $X$ is positive definite if $Z_A$ is positive definite for all finite $A \subseteq X$, and its magnitude $|X| \in [0, \infty]$ is then defined as $\sup\{|A| : \text{finite } A \subseteq X\}$. Positive definiteness ensures that this definition is consistent when $X$ is finite and, as shown in [17], allows the theory to be developed satisfactorily.

A stronger condition is that $tX$ is positive definite for all $t > 0$, where $tX$ is shorthand for $X$ equipped with the rescaled metric $td_X$; this is equivalent to the classical condition that $X$ is of negative type ([17], Theorem 3.3). When $X$ is a subset of a Banach space equipped with the subspace metric, we can equivalently instead consider $tX$ to be the usual dilatation of $X$ again equipped with the subspace metric.

*School of Mathematics, University of Edinburgh, Edinburgh EH9 3FD, Scotland; Tom.Leinster@ed.ac.uk.
†Department of Mathematics, Applied Mathematics, and Statistics, Case Western Reserve University, Cleveland, Ohio, U.S.A.; mark.meckes@case.edu.
Until now, no example was known of a compact positive definite space with magnitude $\infty$. The question of whether such a space exists was first raised in the paper [17] (text preceding Lemma 2.1) posted to the arXiv in 2010 and published in 2013. It was raised again in [18] (after Definition 3.3), and once again in [14] (as Open Problem 5(1)).

In section 2 we construct a family of such spaces $X$. They are moreover of negative type, and we prove not only that $|X| = \infty$, but also that $|tX| = \infty$ for all $t > 0$.

A complementary question involves the behavior of the magnitude when a space shrinks to a point. The magnitude of any nonempty positive definite space lies in the interval $[1, \infty]$, with the lower bound achieved only by the one-point space. We say that a compact metric space $X$ has the one-point property if $\lim_{t \to 0^+} |tX| = 1$. Example 2.2.8 of [10], due to Willerton, shows that even a finite space of negative type may fail to have this property. In section 3 we prove that a broad class of compact spaces of negative type do have the one-point property. Our result unifies and generalizes some previously known sufficient conditions, namely that $X$ is isometric to a subset of $\mathbb{R}^N$ equipped with either the Euclidean metric or the metric induced by the 1-norm.

The main tool used to prove both of our main results is Theorem 4.6 in [14], which provides an upper bound, and frequently an exact formula, for the magnitude of a compact, convex subset of $\ell_1^N$. Here $\ell_1^N$ denotes $\mathbb{R}^N$ equipped with the metric induced by the 1-norm. The new ingredient in both proofs is to combine the formula for magnitude in $\ell_1^N$ with finite-dimensional approximations in order to draw conclusions in the infinite-dimensional spaces $\ell_1$ and $L_1$.

The $\ell_1$ intrinsic volumes of a compact, convex set $A \subseteq \ell_1^N$ are defined by

$$V'_k(A) = \sum_{1 \leq i_1 < \cdots < i_k \leq N} \text{Vol}_k(\pi_{i_1, \ldots, i_k} A)$$

where $\pi_{i_1, \ldots, i_k}$ is orthogonal projection onto the subspace spanned by the standard basis vectors $e_{i_1}, \ldots, e_{i_k}$. These quantities were introduced in [9], where it was shown that there exists a version of integral geometry adapted to the 1-norm, with the $\ell_1$ intrinsic volumes playing the role of the classical intrinsic volumes $V_k$ in Euclidean integral geometry (see e.g. [8]). In fact, this $\ell_1$ integral geometry is valid for the wider class of $\ell_1$-convex sets (defined in [9]), as are some of the results in section 3 below; but for simplicity, we state our results for ordinary convex sets only.

The aforementioned Theorem 4.6 in [14] is the following.

**Theorem 1.1** If $A \subseteq \ell_1^N$ is compact and convex, then

$$|A| \leq \sum_{i=0}^N \frac{1}{2^i} V'_i(A) = \sum_{i=0}^N V'_i\left(\frac{1}{2} A\right),$$

(1)

with equality if $A$ has nonempty interior.

We note that $\sum_{i=0}^N V'_i$ can also be considered an $\ell_1$ analogue of the Wills functional $W = \sum_{i=0}^N V_i$ (see e.g. [2]).
2 Spaces with infinite magnitude

As usual, $\ell_1$ denotes the space of real sequences $(x_i)$ whose 1-norm $\sum |x_i|$ is finite, with the metric induced by the 1-norm. Write $e_i$ for the $i$th standard basis vector $(0, \ldots, 0, 1, 0, \ldots)$ of $\ell_1$ or $\ell_1^N$.

Let $(a_i)$ be a sequence of positive reals converging to 0, with $\sum a_i = \infty$. Denote by $X$ the closed convex hull in $\ell_1$ of $\{a_1 e_1, a_2 e_2, \ldots\}$, with the subspace metric. Equivalently,

$$X = \{(x_1, x_2, \ldots) : x_i \geq 0, \sum x_i/a_i \leq 1\}.$$

**Theorem 2.1** The metric space $X$ is compact and of negative type, and $|tX| = \infty$ for all $t > 0$.

Spaces similar to $X$, but with the $\ell_2$ metric, have been studied in the geometry of Banach spaces (e.g. by Ball and Pajor [3]).

**Proof** First note that $\lim_{i \to \infty} a_i e_i = 0$, which implies that $\{a_i e_i : i \geq 1\} \cup \{0\}$ is compact and that its closed convex hull is $X$. But in a Banach space, the closed convex hull of a compact set is compact (Theorem 5.35 of [1]), so $X$ is compact.

That $X$ is of negative type is immediate, since $\ell_1$ is of negative type (Theorem 3.6(2) of [17]).

It remains to prove that $|tX| = \infty$ for all $t > 0$. Since $tX$ is of the same form as $X$, we may assume that $t = 1$.

For $N \geq 1$, write $X_N$ for the convex hull of $\{a_1 e_1, \ldots, a_N e_N, 0\}$ in $\ell_1^N$, with the subspace metric. Theorem 1.1 implies that

$$|X_N| \geq \frac{1}{2} V'_1(X_N) = \frac{1}{2} \sum_{i=1}^{N} a_i.$$

Now $\sum_{i=1}^{\infty} a_i = \infty$, so $|X_N| \to \infty$ as $N \to \infty$.

The standard isometry $\ell_1^N \hookrightarrow \ell_1$ restricts to an isometry $X_N \to X$ for every $N$. For compact positive definite spaces, magnitude is monotone with respect to inclusion, so $|X_N| \leq |X|$ for all $N$. Hence $|X| = \infty$. $\Box$

**Remark 2.2** If $(a_i)$ is a sequence of positive reals such that $a_i \to 0$ but $\sum a_i < \infty$, then $|X| < \infty$. Indeed, $X$ is a subspace of the infinite-dimensional box $Y = \prod_{i=1}^{\infty} [0, a_i]$ in $\ell_1$, so

$$|X| \leq |Y| = \prod_{i=1}^{\infty} (1 + \frac{a_i}{2}) \leq e^{\sum a_i/2} < \infty,$$

as observed in Open Problem 5(1) of [14]. Thus, for 0-convergent sequences $(a_i)$, the space $X$ has finite magnitude if and only if the sum $\sum a_i$ is finite.
Spaces $X$ of the class considered above clearly have the property that if $X$ has finite magnitude, then its magnitude function is finite for every $t > 0$. This latter phenomenon holds in greater generality, as the following results show.

**Proposition 2.3** If $A$ is a positive definite compact metric space and $n \in \mathbb{N}$, then $nA$ is positive definite and $|nA| \leq |A|^n$.

**Proof** The map $x \mapsto (x, \ldots, x)$ is an isometric embedding $nA \hookrightarrow A^n$, where $A^n$ is given the $\ell_1$-sum metric. Therefore $nA$ is positive definite and $|nA| \leq |A|^n$, by Lemma 3.1.3 and Proposition 3.1.4 of [10]. □

**Corollary 2.4** Suppose that $A$ is a compact and convex subset of a Banach space and is positive definite. Then $A$ is of negative type, and $|A| < \infty$ if and only if $|tA| < \infty$ for every $t > 0$.

**Proof** By translation we may assume that $0 \in A$, so by convexity $t_1A \subseteq t_2A$ whenever $0 \leq t_1 \leq t_2$, and in particular $tA \subseteq [t]A$ for every $t > 0$. By Proposition 2.3, $[t]A$ is positive definite and therefore $tA$ is as well, and furthermore $|tA| \leq |[t]A| \leq |A|^{[t]} < \infty$. □

## 3 The one-point property

Write $L_1 = L_1[0,1]$ for the Banach space of measurable functions $f : [0,1] \to \mathbb{R}$ whose integral 1-norm $\int |f|$ is finite, with the metric induced by the 1-norm. We note that a separable Banach space is a positive definite metric space (equivalently, of negative type), with the metric induced by its norm, if and only if it is isometrically isomorphic to a subspace of $L_1$ (Corollary 3.5 in [17]). Examples include both $\ell_1^N$ and $\mathbb{R}^N$ with the Euclidean metric.

Our second main theorem is the following.

**Theorem 3.1** Suppose $A$ is a nonempty compact subset of a finite-dimensional subspace of $L_1$. Then $|A| < \infty$ and $A$ has the one-point property.

The rest of this section is devoted to the proof.

The finiteness statement in Theorem 3.1 was previously proved (in a less elementary way) in Proposition 4.13 of [14], following special cases proved earlier in [10] and [17]. The one-point property for compact subsets of $\ell_1^N$ was first explicitly noted in Proposition 4.4 of [14] (but follows easily from results in [10]). Independent proofs of the one-point property for subsets of $\mathbb{R}^N$ were given in [4, 21, 19]. Theorem 3.1 simultaneously generalizes these facts. (In [19], it was further proved that so-called GB-bodies in a Hilbert space have finite magnitude and the one-point property, with a proof closely related to the proof of Theorem 3.1.)

The proof of Theorem 3.1 has three main ingredients: a classical approximation procedure that allows us to reduce consideration to subspaces of $\ell_1^N$, a bound on magnitude in terms of $V_1^t$ which follows from Theorem 1.1, and a dimension-independent bound on $V_1^t$ for polytopes. Here, a **polytope** is the convex hull of a finite set.
Lemma 3.2  If \( A \subseteq \ell_1^N \) is convex, then
\[
(j + k)!V'_{j+k}(A) \leq (j!V'_j(A))(k!V'_k(A))
\]
for each \( j, k \geq 0 \).

Proof  By the definition of the \( \ell_1 \) intrinsic volumes,
\[
(j + k)!V'_{j+k}(A) = \sum_{i_1, \ldots, i_{j+k}=1}^N \text{Vol}_{j+k}(\pi_{i_1, \ldots, i_j}(A) \times \pi_{i_{j+1}, \ldots, i_{j+k}}(A)),
\]
noting that if \( i_1, \ldots, i_{j+k} \) are not all distinct then the corresponding summand vanishes. Hence
\[
(j + k)!V'_{j+k}(A) \leq \sum_{i_1, \ldots, i_{j+k}=1}^N \text{Vol}_{j+k}(\pi_{i_1, \ldots, i_j}(A) \times \pi_{i_{j+1}, \ldots, i_{j+k}}(A)) = (j!V'_j(A))(k!V'_k(A)).
\]

The \( j = 1 \) case of Lemma 3.2 implies the following result by induction.

Proposition 3.3  If \( A \subseteq \ell_1^N \) is compact and convex, then
\[
V_k'(A) \leq \frac{1}{k!}V'_1(A)^k
\]
for each \( 0 \leq k \leq N \).

Combining Proposition 3.3 and Theorem 1.1 we obtain the following.

Corollary 3.4  If \( A \subseteq \ell_1^N \) is compact and convex, then
\[
|A| \leq \exp(V'_1(\frac{1}{2}A)).
\]

Remark 3.5  For the classical intrinsic volumes \( V_k \), the estimate \( V_k \leq \frac{1}{k!}V_1^k \) analogous to Proposition 3.3 was independently derived by Chevet (Lemma 4.2 in [5]) and McMullen (Theorem 2 in [16]) from the Alexandrov–Fenchel inequalities. As noted by McMullen, this implies the bound \( W \leq \exp(V_1) \) on the Wills functional \( W = \sum V_k \), analogous to Corollary 3.4.

Lemma 3.6  Suppose that \( P \subseteq \ell_1^N \) is a polytope with \( m \) vertices. Then
\[
V_1'(P) \leq 2(m - 1) \text{diam}(P),
\]
where \( \text{diam}(P) \) is the diameter of \( P \) in the \( \ell_1 \) metric.

Proof  By translation, we may assume that one of the vertices of \( P \) is at the origin. We write \( P = \text{conv}\{v_1, \ldots, v_{m-1}, 0\} \), set \( v_m = 0 \), and denote \( v_k = (v_k(1), \ldots, v_k(N)) \). Then
\[
V_1'(P) = \sum_{i=1}^N (\max_k v_k(i) - \min_k v_k(i)) \leq \sum_{i=1}^N 2 \max_k |v_k(i)|
\]
\[
\leq 2 \sum_{i=1}^N \sum_{k=1}^{m-1} |v_k(i)| = 2 \sum_{k=1}^{m-1} \|v_k\|_1 \leq 2(m - 1) \text{diam}(P). \]
Corollary 3.4 and Lemma 3.6 immediately imply the following.

**Corollary 3.7** If $P \subseteq \ell_1^N$ is a polytope with $m$ vertices, then

$$|P| \leq \exp((m - 1) \text{diam}(P)).$$

**Remark 3.8** Theorem 6.2 of [9] and Proposition 3.3 together imply that

$$\text{Vol}_N(P + [0, 1]^N) = \sum_{k=0}^{N} V_k'(P) \leq \exp(V_1'(P))$$

for every polytope $P \subseteq \ell_1^N$. It follows from Lemma 3.6 that

$$\text{Vol}_N(P + [0, 1]^N) \leq \exp(2(m - 1) \text{diam}(P)),$$

where $m$ is the number of vertices and the diameter is in the $\ell_1$ metric. Despite the classical flavor of this estimate we have not seen it stated elsewhere.

**Corollary 3.9** If $P \subseteq L_1$ is the convex hull of $m$ points, then

$$|P| \leq \exp((m - 1) \text{diam}(P)).$$

**Proof** Let $E \subseteq L_1$ be the linear span of $P$. It is well known (e.g. [20], section 1) that $E$ can be approximated in the Banach–Mazur distance by a sequence of subspaces $E_n \subseteq \ell_1^{N_n}$. It follows (as in section 3.A of [7]) that $P$ is the limit, in the Gromov–Hausdorff distance, of a sequence of polytopes $P_n \subseteq \ell_1^{N_n}$, each with at most $m$ vertices.

The magnitude of compact positive definite metric spaces is lower semicontinuous with respect to the Gromov–Hausdorff distance [17, Theorem 2.6], and diameter is continuous. By Corollary 3.7 we therefore have

$$|P| \leq \liminf_{n \to \infty} |P_n| \leq \liminf_{n \to \infty} e^{(m-1)\text{diam}(P_n)} = e^{(m-1)\text{diam}(P)}. \qed$$

**Proof of Theorem 3.1** Let $P$ be a polytope lying in the linear span of $A$ and containing $A$. Then for each $t > 0$,

$$1 \leq |tA| \leq |tP| \leq \exp((m - 1)t \text{diam}(P)),$$

where $m$ is the number of vertices of $P$. The theorem follows. \qed

**Acknowledgements** TL was supported in part by a Leverhulme Research Fellowship. MM’s research was supported in part by Collaboration Grant 315593 from the Simons Foundation. This work was partly done while MM was visiting the Mathematical Institute of the University of Oxford, partially supported by ERC Advanced Grant 740900 (LogCorRM) to Prof. Jon Keating and Simons Fellowship 678148 to Elizabeth Meckes. MM thanks the Institute and Prof. Keating for their hospitality.
References

[1] C. D. Aliprantis and K. C. Border. *Infinite Dimensional Analysis: A Hitchhiker’s Guide*. Springer, Berlin, 3rd edition, 2006.

[2] D. Alonso-Gutiérrez, M. Hernández Cifre, and J. Yepes Nicolás. Further inequalities for the (generalized) Wills functional. *Communications in Contemporary Mathematics*, 23(3):2050011, 2021.

[3] K. Ball and A. Pajor. The entropy of convex bodies with ‘few’ extreme points. In P. F. X. Müller and W. Schachermayer, editors, *Geometry of Banach Spaces*, volume 158 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1990.

[4] J. A. Barceló and A. Carbery. On the magnitudes of compact sets in Euclidean spaces. *American Journal of Mathematics*, 140(2):449–494, 2018.

[5] S. Chevet. Processus Gaussiens et volumes mixtes. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 36(1):47–65, 1976.

[6] H. Gimpel and M. Goffeng. On the magnitude function of domains in Euclidean space. *American Journal of Mathematics*, 143(3):939–967, 2021.

[7] M. Gromov. *Metric Structures for Riemannian and Non-Riemannian Spaces*. Birkhäuser, Boston, 2001.

[8] D. A. Klain and G.-C. Rota. *Introduction to Geometric Probability*. Lezioni Lincee. Cambridge University Press, Cambridge, 1997.

[9] T. Leinster. Integral geometry for the 1-norm. *Advances in Applied Mathematics*, 49:81–96, 2012.

[10] T. Leinster. The magnitude of metric spaces. *Documenta Mathematica*, 18:857–905, 2013.

[11] T. Leinster. *Entropy and Diversity: The Axiomatic Approach*. Cambridge University Press, Cambridge, 2021.

[12] T. Leinster. Magnitude: a bibliography. Available at www.maths.ed.ac.uk/~tl/magbib, 2021.

[13] T. Leinster and M. Meckes. Maximizing diversity in biology and beyond. *Entropy*, 18(88), 2016.

[14] T. Leinster and M. Meckes. The magnitude of a metric space: from category theory to geometric measure theory. In N. Gigli, editor, *Measure Theory in Non-Smooth Spaces*, pages 156–193. De Gruyter Open, Warsaw, 2017.

[15] T. Leinster and E. Roff. The maximum entropy of a metric space. *Quarterly Journal of Mathematics*, to appear, 2021.

[16] P. McMullen. Inequalities between intrinsic volumes. *Monatshefte für Mathematik*, 111:47–53, 1991.

[17] M. W. Meckes. Positive definite metric spaces. *Positivity*, 17:733–757, 2013.

[18] M. W. Meckes. Magnitude, diversity, capacities, and dimensions of metric spaces. *Potential Analysis*, 42:549–572, 2015.

[19] M. W. Meckes. On the magnitude and intrinsic volumes of a convex body in Euclidean space. *Mathematika*, 66:343–355, 2020.

[20] M. Talagrand. Embedding subspaces of $L_1$ into $l^n_1$. *Proceedings of the American Mathematical Society*, 108(2):363–369, 1990.

[21] S. Willerton. The magnitude of odd balls via Hankel determinants of reverse Bessel polynomials. *Discrete Analysis*, 5:1–42, 2020.