Purely geometric path integral for spin-foams

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Abstract
Spin-foams are a proposal for defining the dynamics of loop quantum gravity via path integral. In order for a path integral to be at least formally equivalent to the corresponding canonical quantization, at each point in the space of histories it is important that the integrand have not only the correct phase—a topic of recent focus in spin-foams—but also the correct modulus, usually referred to as the measure factor. The correct measure factor descends from the Liouville measure on the reduced phase space, and its calculation is a task of canonical analysis.
The covariant formulation of gravity from which spin-foams are derived is the Plebanski–Holst formulation, in which the basic variables are a Lorentz connection and a Lorentz-algebra valued 2-form, called the Plebanski 2-form.
However, in the final spin-foam sum, one usually sums over only spins and intertwiners, which label eigenstates of the Plebanski 2-form alone. The spin-foam sum is therefore a discretized version of a Plebanski–Holst path integral in which only the Plebanski 2-form appears, and in which the connection degrees of freedom have been integrated out. We call this a purely geometric Plebanski–Holst path integral.
In prior work in which one of the authors was involved, the measure factor for the Plebanski–Holst path integral with both connection and 2-form variables was calculated. Before one discretizes this measure and incorporates it into a spin-foam sum, however, one must integrate out the connection in order to obtain the purely geometric version of the path integral. To calculate this purely geometric path integral is the principal task of the present paper, and it is done in two independent ways. Background independence of the resulting path integral is discussed in the final section, and gauge-fixing is discussed in appendix B.

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1. Introduction

In the path integral approach to constructing a quantum theory, the integrand of the path integral has two important parts: a phase part given by the exponential of $i$ times the classical action, and a measure factor. The form of the phase part in terms of the classical action ensures that solutions to the classical equations of motion dominate the path integral in the classical limit so that one recovers classical physics in the appropriate regime. The measure factor, however, arises from careful canonical analysis, and is important for the path integral to be equivalent to the corresponding canonical quantum theory. In most theories, this means that it is important, in particular, in order for the path integral theory to have such elementary properties as yielding a unitary $S$-matrix that preserves probabilities. The importance of having the correct measure factor is thus quite high.

Spin-foams are a path integral approach to quantum gravity in which one does not sum over classical gravitational histories, but rather quantum histories arising from canonical quantization. Specifically, in spin-foams, one sums over histories of canonical states of loop quantum gravity (LQG). These histories possess a natural four-dimensional space-time covariant interpretation, whence each can be thought of as a quantum space-time. This approach allows one to retain the understanding gained from LQG, such as the discreteness of area and volume spectra, while at the same time formulating the dynamics in a way that makes space-time symmetries more manifest [1].

The starting point for the derivation of spin-foams is the Plebanski–Holst formulation of gravity [2–6], in which the basic variables are a connection $\omega$ and what is called the Plebanski 2-form, $\Sigma$. However, in the final spin-foam sum, the connection $\omega$ is usually not present, and one sums over only spins and intertwiners, which determine certain eigenstates of $\Sigma$ alone. Because of this, the continuum path integral most directly related to the spin-foam sum is a Plebanski–Holst path integral in which only $\Sigma$ appears, and in which the connection has been integrated out. We call such a path integral purely geometric because $\Sigma$ directly determines the geometry of space-time.

Because of the quantum mechanical nature of the histories used in spin-foams, ensuring that the summand has the required phase part and measure factor is not completely trivial. Only within the last couple years has the correct phase part been achieved [7, 8]. Regarding the measure factor, a first step has been carried out in the work [5], where the correct measure factor is calculated for the Plebanski–Holst path integral with both $\omega$ and $\Sigma$ present. However, until now, the measure factor for the path integral with $\Sigma$ alone, necessary for spin-foams, had not yet been calculated. To carry out this calculation is the main purpose of the present paper.

We find that these two ways of calculating the measure factor exactly match, as must be the case, as the canonical measure factor ultimately descends from the Liouville measure on the reduced phase space, which is independent of the formulation of gravity used [5].

The path integral derived in this paper is ready to be discretized and translated into a spin-foam model, a task which will be carried out in forthcoming work. Furthermore, when this is accomplished, we would like to emphasize that, because both primary and secondary simplicity constraints are already incorporated in the continuum path integral [5, 9], they will be automatically incorporated in the resulting spin-foam model as well.

It should also be kept in mind that the raison d’être of the canonical path integral is to ensure formal equivalence with canonical quantization, and it may be that one can use such
equivalence as a more direct criterion for obtaining the correct ‘measure factor’ in a spin-foam model. The work [10] has begun to explore use of such alternative argumentation.

We begin the paper by reviewing some background on the path integral and how it is used. After that, we derive the purely geometric Plebanski–Holst path integral, first starting from [5], and then starting from the ADM path integral. Background independence of the resulting path integral is discussed in the final section, and gauge-fixing is discussed in appendix B.

2. Background

2.1. Path integral generalities

For an unconstrained field theory with canonically conjugate variables \( \phi, \pi \), the phase space path integral takes the form [11]

\[
Z(O) = \int D\phi D\pi O(\phi, \pi) \exp iS[\phi, \pi],
\]

where the integration is over histories of pairs \((\phi, \pi)\), \(S[\phi, \pi] = \int d^4x (\dot{\phi} \pi - H)\) with \(H\) the Hamiltonian density, and \(D\phi D\pi\) is a formal Cartesian product of Lesbesgue measures at each point in space-time—or, equivalently, a Cartesian product of Liouville measures on phase space at each moment of time. For a system with second class constraints, if \(\bar{\phi}, \bar{\pi}\) are taken to be canonically conjugate coordinates on the constrained phase space, then (1) still applies. However, if one uses coordinates on the unconstrained phase space, one obtains [12]

\[
Z(O) = \int D\phi D\pi \delta^n(C_i) |\det \{C_i, C_j\}| \frac{i}{2} O \exp iS[\phi, \pi],
\]

where the combined factor \(\delta^n(C_i) |\det \{C_i, C_j\}| \frac{i}{2}\) is independent of the way the constraint surface is represented by constraint functions \(C_i\) in which the action takes the form \(S[\phi, \pi, \lambda] = S[\phi, \pi] + \int \lambda \cdot C_i\), with \(\lambda\) Lagrange multipliers, one can write the path integral as [13]

\[
Z(O) = \int D\phi D\pi D\lambda O \exp iS[\phi, \pi, \lambda].
\]

Alternatively, one may introduce gauge fixing functions \(\xi_i\) so that \(\xi_i\) and \(C_i\) together form a second class set of constraints, in which case (2) again applies. If we assume that, e.g., the \(\xi_i\) are pure momentum or pure configuration, so that they all Poisson commute, the path integral then takes the form [14]

\[
Z(O) = \int D\phi D\pi D\lambda \delta^n(\xi) |\det \{C_i, \xi_j\}| O \exp iS[\phi, \pi, \lambda].
\]

In fact, under very general assumptions (which, however, have yet to be fully proven in the case of gravity) (3) and (4) are equal up to an infinite constant equal to the gauge volume (see [13, 15, 16], and appendix B), so that they determine the same physics in the manner to be reviewed below. In the rest of this paper for simplicity we will use expression (3). However, all derivations in this paper can just as well be done starting from expression (4)—one need only reinsert the omitted factor \(\delta^n(\xi) |\det \{C_i, \xi_j\}|\).

The path integral was originally discovered as a way to write transition amplitudes between states in quantum mechanics. Let \(\{O_i\}\) denote a set of phase space functions whose quantum analogues \(\hat{O}_i\) form a complete commuting set. Let \(\Psi_{(i, \nu_i)}\) denote the simultaneous eigenstate of the operators \(\{\hat{O}_i\}\) with eigenvalues \(\{\nu_i\}\). Then the path integral determines the
transition amplitude between two such states via
\[
\langle \Psi_{(\sigma_1, \psi_1)} | U(T' - T) \Psi_{(\sigma_0, \psi_0)} \rangle = Z \left( \prod_i \delta (O_i((\varphi, \pi)(T')) - v_i') \delta (O_i((\varphi, \pi)(T)) - v_i) \right) = \int O_i((\varphi, \pi)(T')) \text{det} \eta(\xi) \text{det} \lambda(\xi) \exp i S[\varphi, \pi, \lambda] \]  

where \( U(T' - T) \) is the time evolution operator from \( t = T \) to \( t = T' \). For a theory with first class constraints of which the Hamiltonian is a linear combination—i.e., a time reparametrization invariant theory—there is no time evolution operator, and, instead of (5), the interpretation of the path integral involves a ‘rigging map’ or ‘group averaging map’ \( \eta \) [17] from kinematical (unconstrained) states to physical states (states annihilated by the constraints) [1, 18, 19]:

\[
\langle \eta(\Psi_{(\sigma_1, \psi_1)}) , \eta(\Psi_{(\sigma_0, \psi_0)}) \rangle_{\text{phys}} = \int O_i((\varphi, \pi)(T')) \text{det} \eta(\xi) \text{det} \lambda(\xi) \exp i S[\varphi, \pi, \lambda]. \]  

This is the case relevant for us. Note the physical inner product, and therefore \( Z(\mathcal{O}) \), is only physically meaningful up to rescaling by a constant. In the following we will use the notation \( \hat{=} \) to denote ‘equality up to rescaling by a constant’.

By setting \( \mathcal{O} = 1 \), one obtains the so-called partition function. Often in the literature one says that the partition function defines the quantum theory. Of course, what is meant is that the mathematical form—or more precisely, the integrand—of the partition function defines the quantum theory. One then uses the integrand to integrate against general functions \( \mathcal{O} \), the results of which are used as above. In this paper, we are explicit about this fact for clarity.

The transition amplitude (5) or (6) defines the dynamics of the theory. This, together with a kinematical quantum framework of states and relevant operators, is enough to allow one to calculate any quantity of physical interest. In the case of general relativity (GR), canonical LQG provides the kinematics. In this case, by choosing the observables \( O_i \) appropriately, the canonical states \( \Psi_{(\sigma_0, \eta)} \) can be made to be spin-network states [20], generalized spin-network states [21], or Livine–Speziale coherent states [22]. In all of these cases, the relevant observables \( O_i \) are purely geometric, depending only on the pull-back of \( \Sigma_{\mu \nu} \) to the spatial slice.

3. Derivation of a purely geometric path integral from Plebanski–Holst

We will consider both Euclidean and Lorentzian gravity simultaneously, defining \( s := +1 \) and \( s := -1 \) respectively in these two cases. In addition to space-time manifold indices, denoted here by lower case Greek letters, the Plebanski and Plebanski–Holst formulations of gravity make use of mixed tensors with ‘internal’ indices \( I, J, K, \ldots \in \{0, 1, 2, 3\} \) which are raised and lowered with a fixed ‘internal’ metric \( \eta_{IJ} := \text{diag}(s, 1, 1, 1) \). The basic variables are an \( \mathfrak{so}(\eta) \)-valued connection \( \omega^I_{\mu} \) and an \( \mathfrak{so}(\eta) \)-valued 2-form \( \chi^{IJ}_{\mu \nu} \), the Plebanski 2-form. In the theory, \( \chi^{IJ}_{\mu \nu} \) is constrained to satisfy the so-called simplicity constraint

\[
C_{\mu \nu \rho \sigma} := \epsilon_{IJKL} \chi^{IJ}_{\mu \nu} \chi^{KL}_{\rho \sigma} = \frac{s}{4!} \epsilon_{\mu \nu \rho \sigma} \epsilon^{\alpha \beta \gamma \delta} \epsilon_{IJKL} \chi^{IJ}_{\alpha \beta} \chi^{KL}_{\gamma \delta} \approx 0 \]  

where \( \epsilon_{IJKL} \) denotes the ‘internal’ Levi-Civita tensor with \( \epsilon_{0123} = 1 \), and \( \epsilon_{\mu \nu \rho \sigma} \), \( \epsilon^{\alpha \beta \gamma \delta} \) denote the Levi-Civita tensor densities of weight \(-1\) and \(1\), respectively. This constraint has 20 independent components per point and restricts \( \chi^{IJ}_{\mu \nu} \) to belong to one of five sectors. The
first is the degenerate sector, in which \( \epsilon^{\mu\nu\rho\sigma} \epsilon_{IJKL}X_{\mu\nu}^{II}X_{\rho\sigma}^{KL} = 0 \). The other four sectors each correspond to \( X_{\mu\nu}^{II} \) taking one of the following 4-forms

\[
X_{\mu\nu}^{II} = \begin{cases} 
\pm \frac{1}{\kappa} \epsilon^\ell \wedge e^\ell & \text{(I±)} \\
\pm \frac{1}{2\kappa} \epsilon_{IJ}^{\ell K } e^K \wedge e^\ell & \text{(II±)}
\end{cases}
\]

for some non-degenerate tetrad \( e_\mu^I \) (see \([5, 23]\)), where we have chosen to include the factor \( \kappa := 16\pi G \) here instead of explicitly in front of the action (given below). Given an element \( Y_{IJ} \in so(\eta) \), it is useful to define \( Y^{IJ} := (Y + \frac{1}{\gamma} Y)^{IJ} \), where \( \gamma \in \mathbb{R}^+ \) is the Barbero–Immirzi parameter, and \( (Y)^{IJ} := \frac{1}{2} \epsilon^{IJ} KYKL \). Thus, in particular, \( Y^{IJ} := (X + \frac{1}{\gamma} X)^{IJ} \). Using these variables and notation, we start from the Plebanski–Holst path integral derived in \([5]\)

\[
Z(O) = \int_{\{\mathcal{M}\}} D\omega_{IJ}^I D\gamma_{IJ} \delta(C) V_\gamma \exp i \int X_{IJ} \wedge F^{IJ} \tag{8}
\]

where the action is an integral over the space-time manifold \( \mathcal{M} \), and the ‘(II±)’ subscript indicates that we are restricting the integration to the (II+) and (II−) sectors, in which

\[
X_{\mu\nu}^{IJ} = \pm \frac{1}{2\kappa} \epsilon_{IJ}^{\ell K } e^K \wedge e^\ell . \tag{9}
\]

The \( V \) appearing in (8) is the space-time volume density and \( V_\gamma \) is the spatial volume density determined by \( e_\mu^I \). For brevity, from now on we will omit the ‘(II±)’ subscript on the path integral (8), leaving it understood. In the end, to ensure that solutions to Einstein’s equations dominate in the classical limit of the spin-foam sum, it is necessary to restrict to a particular combination of sectors (II+) and (II−) called the Einstein–Hilbert sector \([6–8]\). However, this is not relevant for the work of the present paper. Without the simplicity constraint, the action appearing in (8), with \( B = \bar{X} \), is the so-called BF action. Restriction to the (II±) sectors reduces this action to the Holst action \([24]\)

\[
S_{BF} = \int \left( X_{IJ} + \frac{1}{\gamma} X_{IJ} \right) \wedge F^{IJ} = \pm \frac{1}{2\kappa} \int \left( e_I \wedge e_J + \frac{8}{2\gamma} \epsilon_{IJKL} e^K \wedge e^L \right) \wedge F^{IJ} = \pm S_{\text{Holst}} \tag{10}
\]

The equations of motion derived by varying this action are

\[
d_\omega X_{\rho\sigma}^{II} = 0 \Leftrightarrow d_\omega X_{\rho\sigma}^{II} = 0 \Leftrightarrow d_\gamma e^\ell = 0, \quad \text{and} \quad \epsilon_I^\ell \epsilon_J^\ell F_{\mu\nu}^{IJ} = 0,
\]

the first being the torsion-free condition on \( \omega \), and the second being equivalent to the Einstein equations.

It should be noted that the path integral (8) is valid only when \( O \) is purely geometric (i.e., a function of \( X_{\mu\nu}^{II} \) only). For, in the derivation in \([5]\), when the Henneaux–Slavnov trick \([9]\) is used, a change of variables involving \( \omega_{IJ}^I \) is performed. If the presence of \( O \) is made explicit throughout the derivation, one sees that if \( O \) depends explicitly on \( \omega_{IJ}^I \), then the argument leading to (8) will change, as will the final path integral expression\(^1\). However, the restriction of \( O \) to be purely geometric is not a real restriction, as, on-shell, \( \omega \) is uniquely determined by \( X \). Furthermore, the principal application of (8) will be to computing the physical inner product between spin-network states, which are eigenstates of \( X \) only (see equation (6)).

The goal of this section is to integrate out the connection in the expression (8). We do this by first casting the integral explicitly in Gaussian form and integrating out the connection, giving the result in terms of the determinant of the appropriate matrix. The determinant of this matrix is then calculated.

\(^1\) Specifically, in the final path integral, if \( \gamma_{HS} \) denotes the Henneaux–Slavnov canonical transformation, then the \( \omega \) argument of \( O \) will need to be replaced by \( \gamma_{HS} \cdot \omega \).
3.1. Integrating out the connection

The next step is to evaluate the Gaussian integral:
\[
I(X) := \int D\omega^I \exp i \int X_{IJ} \wedge F^{IJ} = \int D\omega^I \exp i \int X_{IJ} \wedge (d\omega^I + \omega^K \wedge \omega_K^{IJ}).
\]
Because, in the path integral (8), \(X\) is constrained to be of the form (9), we assume \(X\) to be of this form throughout the section. We begin by noting that if we define
\[
d^\alpha_{IJ}[X,\omega] := \frac{1}{2} e^{a_{\beta\gamma\rho}} X_{\beta\gamma\rho}[K][|\eta J|][L],
\]
where the square brackets denote anti-symmetrization, and \(b^\alpha_{IJ} := \frac{1}{3!} e^{a_{\beta\gamma\rho}} (dX_{IJ})_{\beta\gamma\rho}\), then the action becomes
\[
S_{BF}[X,\omega] = \int d^4x \left( d^\alpha_{IJ}[X,\omega] \omega^K_{IJ} + b^\alpha_{IJ} \omega^K_{IJ} \right),
\]
so that
\[
I(X) = \int D\omega \exp i \int d^4x \left( d^\alpha_{IJ}[X,\omega] \omega^K_{IJ} + b^\alpha_{IJ} \omega^K_{IJ} \right).
\]
This casts the integral in the explicit Gaussian form in equations (A.1) and (A.2). Using the result (A.4), one has that, modulo an overall \(X\)-independent constant,
\[
I(X) \equiv (\det a)^{-1} \exp i S[X],
\]
where \(S[X] := S_{BF}[X,\omega[X]]\) with \(\omega[X]\) the unique connection determined by \(X\) via the equation of motion of BF theory found by varying \(\omega\),
\[
d_{\omega[X]}X^I = dX^I + 2\omega[X]^I_K \wedge X^K_{IJ} = 0.
\]
Because \(X\) is of the form (9), \(\omega[X]\) is furthermore the unique Lorentz spin-connection determined by the tetrad \(e\), i.e., such that
\[
d_{\omega[e]}e^I = 0,
\]
so that the action reduces to the Holst and Palatini actions
\[
S[X] \equiv S_{BF}[X,\omega[X]] \approx \pm S_{Holst}[e,\omega[e]] = \pm S_{Palatini}[e,\omega[e]],
\]
where \(\approx\) denotes equality when \(X\) is of the form (9), and the last equality holds because, when \(\omega = \omega[X]\), the extra ‘topological’ term added to the Palatini action to give the Holst action vanishes due to the Bianchi identity [24].

In order to use equation (12) in the path integral, it remains to calculate the determinant of \(a\).

3.2. The determinant of \(a\)

We have
\[
d^\alpha_{IJ}[X,\omega] = \frac{1}{2} e^{a_{\beta\gamma\rho}} X_{\beta\gamma\rho}[K][|\eta J|][L]
\]
\[
= \frac{1}{2\kappa} e^{a_{\beta\gamma\rho}} \left( e^\gamma_y e^\gamma_y e_{MN}[K][I] + \frac{2}{\gamma} e^\gamma_y e_{[K]y}[I] \eta_{J|L} \right)
\]
\[
= \frac{\gamma}{2\kappa} \left( e^\gamma_y e^\gamma_y e_{POMN} e_{MN}[K][I] + \frac{2\gamma}{\gamma} e^\gamma_y e^\gamma_y e_{PO}[K][I] \eta_{J|L} \right)
\]
\[
= \frac{\gamma}{\kappa} \left( e^\gamma_y e^\gamma_y (2\delta^{[Q}_y \eta_{J|[K]} + \frac{s}{\gamma} e^\gamma_y e_{[K]y}[I] \eta_{J|L} \right) e^\gamma_y
\]
\[
= \frac{\gamma}{\kappa} \left( e^\gamma_y e^\gamma_y (2\delta^{[Q}_y \eta_{J|[K]} + \frac{s}{\gamma} e^\gamma_y e_{[K]y}[I] \eta_{J|L} \right) e^\gamma_y
\]
\[
= \frac{\gamma}{\kappa} \left( e^\gamma_y e^\gamma_y (2\delta^{[Q}_y \eta_{J|[K]} + \frac{s}{\gamma} e^\gamma_y e_{[K]y}[I] \eta_{J|L} \right) e^\gamma_y
\]
\[
= \frac{\gamma}{\kappa} \left( e^\gamma_y e^\gamma_y (2\delta^{[Q}_y \eta_{J|[K]} + \frac{s}{\gamma} e^\gamma_y e_{[K]y}[I] \eta_{J|L} \right) e^\gamma_y
\]
so that if we define the $24 \times 24$ matrices

$$E_{\alpha[IJ][M^N]}^{[P]} := e_\alpha^P \delta^M_I \delta^N_J,$$

and

$$\tilde{a}_{[MN]}^{[P]}[R] := 2 \delta^P_Q \eta_{[N]S} \eta^R_S + \frac{s}{\sqrt{\gamma}} \epsilon^{PQ} \eta_{[N]S} \eta^R_S,$$

the above becomes

$$a_{\alpha[IJ]}^{[P]}[\beta [KL] := \frac{\gamma}{k} E_{[\alpha [MN]}^{[P]} \cdot \tilde{a}_{[MN]}^{[P]}[R] \cdot E_{[KL]}^{R[S]}].$$

Thus,

$$\det a = \left(\frac{\gamma}{k}\right)^{24} (\det E)^2 (\det \tilde{a})$$

$$\cong \gamma^{24} \det E^2$$

where the fact that $\det \tilde{a}$ is constant on the space of histories has been used\(^2\). Finally, because $E$ is block diagonal with six copies of the block $e_\alpha^P$, one has

$$\det E = (\det e_\alpha^P)^6 = \gamma^{-6}$$

whence

$$\det a \cong \gamma^{12}.$$  

### 3.3. Final continuum path integral

Using this expression in (12) and substituting that into (8) yields the final pure geometric continuum path integral

$$Z(O) \cong \int DU^{13} \delta(C) V^{3} V_{\gamma} \exp i \int \Xi_{IJ} \wedge F[\omega[X]]^{IJ}. \quad (13)$$

We wish to note that this expression derives from the full canonical path integral, with primary as well as secondary simplicity constraints imposed, and with the full requisite determinant factors. The only assumption necessary in the derivation is that $O$ does not explicitly depend on $\omega$. In spin-foam sums, each spin-foam is labeled by quantum numbers which are a discrete analogue of precisely the 2-form $X$, making the above expression ideally suited for use with spin-foams.

### 4. ADM path integral

As a secondary derivation of (13) we start from the ADM formalism. Because one has no second class constraints in the ADM formalism, one does not need to use the Henneaux–Slavnov trick [9]. As a consequence, this derivation can also be thought of as a ‘check’ on the use of the Henneaux–Slavnov trick in this case.

\(^2\) $\det \tilde{a}$ is furthermore non-zero. This is clear from the fact that when $e_\alpha^P$ is non-degenerate, so is $\alpha$, which is equivalent to the usual fact that the equations of motion obtained by varying the action (10), (11) with respect to $\omega$ determines $\omega$ uniquely.
The non-gauge-fixed canonical path integral in the ADM formalism in terms of the canonical variables \((h_{ab}, \pi^{ab})\) is

\[
Z_{\text{ADM}}(\mathcal{O}) = \int D h_{ab} D \pi^{ab} D N D N^a \exp i \int d^4 x (\pi^{ab} \dot{h}_{ab} - \mathcal{H}_{\text{ADM}}[h, \pi, N, \vec{N}])
\]

where recall \(a, b, c, \ldots\), denote spatial indices. Here \(N\) and \(N^a\) are lapse and shift lagrange multipliers and \(\mathcal{H}_{\text{ADM}}\) is the Hamiltonian density given by [25]

\[
\mathcal{H}_{\text{ADM}} = h^{1/2} N (-^{(3)} R + h^{-1} \pi^{ab} \pi_{ab} - h^{-1} \pi^{2}) + 2\pi^{ab} D_a N_b,
\]

where \(h := \det h_{ab}, \pi := \pi^a_a\), and \(^{(3)} R\) denotes the scalar curvature of \(h_{ab}\). The integral (14) can be written

\[
Z_{\text{ADM}}(\mathcal{O}) = \int D h_{ab} D \pi^{ab} D N D N^a \exp i \int d^4 x (A_{ab,cd} \pi^{ab} \pi^{cd} + B_{ab} \pi^{ab} + C)
\]

where \(A, B\) and \(C\) are

\[
A_{ab,cd} := -Nh^{-\frac{1}{2}} (h_{ab} h_{cd} + h_{ad} h_{bc} - h_{ab} h_{cd}),
\]

\[
B_{ab} := \dot{h}_{ab} - 2D_a N_b,
\]

and

\[
C := Nh^{-\frac{1}{2}} (^{(3)} R).
\]

We assume \(\mathcal{O}\) does not depend explicitly on \(\pi\), so that (15) is again of the form (A.1 and A.2), and the integration over \(\pi^{ab}\) yields

\[
Z_{\text{ADM}}(\mathcal{O}) \equiv \int D h_{ab} D N D N^a |\det A|^{-1/2} \mathcal{O} \exp i \int d^4 x (\pi^{ab}[h, N, \vec{N}] \dot{h}_{ab}
\]

\[
- \mathcal{H}_{\text{ADM}}[h, \pi[h, N, \vec{N}]]
\]

where \(\pi^{ab}[h, N, \vec{N}]\) is the value of \(\pi^{ab}\) determined by \(h_{ab}, N,\) and \(N^a\) via the appropriate equation of motion of the Hamiltonian theory. Substituting the explicit expression for \(\pi^{ab}[h, N, \vec{N}]\) into (16) yields

\[
Z_{\text{ADM}}(\mathcal{O}) \equiv \int D h_{ab} D N D N^a |\det A|^{-1/2} \mathcal{O} \exp i \int d^4 x \mathcal{L}_{\text{ADM}}
\]

where

\[
\mathcal{L}_{\text{ADM}} := N\sqrt{h}(K_{ab} K_{ab} - K^2 + ^{(3)} \mathcal{R})
\]

is the usual ADM Lagrangian with

\[
K_{ab} := \frac{1}{2} L_a h_{ab} = \frac{1}{2N}(\dot{h}_{ab} - D_a N_b - D_b N_a)
\]

the usual extrinsic curvature.

It remains to find the determinant of \(A\).

4.2. The determinant of \(A\)

By the symmetry of \(h_{ab}\), there exists an orthogonal matrix \(O_{a}^{\varepsilon}\) such that \(O_{a}^{\varepsilon} O_{b}^{\gamma} h_{ef} = \lambda_{\alpha} \delta_{ab}\) for some \(\lambda_{a}\), so that

\[
A_{ab,cd}^\prime := O_{a}^{\varepsilon} O_{b}^{\gamma} O_{c}^{\delta} O_{d}^{\epsilon} A_{ef,gh} = \frac{N}{2\sqrt{h}} [\lambda_{a} \lambda_{b} \eta_{ac} \delta_{bd} + \lambda_{a} \lambda_{b} \eta_{ad} \delta_{bc} - \lambda_{a} \lambda_{c} \delta_{ab} \delta_{cd}].
\]
Now because rows \((ab)\) = \((12), (13)\) and \((23)\) in \(A'\) each have just one non zero element \((A'_{12,12}, A'_{13,13}\) and \(A'_{23,23},\) respectively) the determinant of the above equation yields

\[
A'_{12,12} A'_{13,13} A'_{23,23} \det Q = \det A
\]

where \(Q_{ab} := A'_{ab,ab}\). Using \(A'_{12,12} = \frac{N}{\sqrt{h}} \lambda_1 \lambda_2\), \(A'_{13,13} = \frac{N}{\sqrt{h}} \lambda_1 \lambda_3\) and \(A'_{23,23} = \frac{N}{\sqrt{h}} \lambda_2 \lambda_3\) gives

\[
\det A = \left(\frac{N}{2\sqrt{h}}\right)^3 (\lambda_1 \lambda_2 \lambda_3)^2 \det Q.
\]

Furthermore,

\[
\det Q = \det \begin{pmatrix} A_{11,11} & A_{11,22} & A_{11,33} \\ A_{22,11} & A_{22,22} & A_{22,33} \\ A_{33,11} & A_{33,22} & A_{33,33} \end{pmatrix} = \left(\frac{N}{2\sqrt{h}}\right)^3 \det \begin{pmatrix} \lambda_1^2 & -\lambda_1 \lambda_2 & \lambda_1 \lambda_3 \\ -\lambda_1 \lambda_2 & \lambda_2^2 & -\lambda_2 \lambda_3 \\ -\lambda_1 \lambda_3 & -\lambda_2 \lambda_3 & \lambda_3^2 \end{pmatrix}
\]

\[
= -4 \left(\frac{N}{2\sqrt{h}}\right)^3 (\lambda_1 \lambda_2 \lambda_3)^2.
\]

So that,

\[
\det A = \left(\frac{N}{2\sqrt{h}}\right)^6 4(\lambda_1 \lambda_2 \lambda_3)^4 = \frac{N^6}{2^6 (\sqrt{h})^6} \hbar^4 = \frac{N^6}{2^4 \hbar}
\]

where \(\lambda_1 \lambda_2 \lambda_3 = \det h_{ab}\) has been used. Substituting this into (17) gives

\[
Z_{ADM}(\mathcal{O}) \equiv \int D h_{ab} D N^a D N^b N^{-3} h^{-\frac{1}{2}} \mathcal{O} \exp i \int d^4 x L_{ADM}.
\]  

### 4.3. Space-time covariant variables

It remains to change from the variables \((h_{ab}, N, N^a)\) to the space-time covariant variable \(g_{\mu\nu}\), we have

\[
Dg_{\mu\nu} = Dg_{ab} Dg_{00} Dg_{0a} = |\det J| D h_{ab} D N^a
\]

where

\[
J := \frac{\partial (g_{ab}, g_{00}, g_{0a})}{\partial (h_{cd}, N, N^c)}.
\]

Using the relations

\[
g_{ab} = h_{ab} \quad g_{00} = -N^2 + h_{ab} N^a N^b \quad g_{0a} = h_{ab} N^b
\]

it is easy to check that

\[
\det J = -2hN
\]

so that

\[
Dg_{\mu\nu} = 2hN^D h_{ab} D N^a.
\]
Equation (18) thus yields
\[ Z_{\text{ADM}}(O) \cong \int Dg_{\mu\nu} N^{-4} h^{-\frac{1}{2}} O \exp iS_{\text{ADM}}. \]  
\[ (19) \]

### 4.4. Comparison with the result of the last section

Our goal in this section is to compare the ADM measure in (19) to the final continuum path integral (13) derived in section 3
\[ Z(O) \cong \int DX^{IJ}_{\mu\nu} \delta(C) V^3 V_4 \exp i \int X_{IJ} \wedge F[\omega[X]]^{IJ}. \]

Since these two expressions have different variables, again a change of variables is necessary. As discussed in [5], one can perform a change of variables \[ X^{IJ}_{\mu\nu} \rightarrow (e^I_{\mu}, C), \]
from \[ X^{IJ}_{\mu\nu} \] to the tetrad \[ e^I_{\mu} \] together with the simplicity constraints. The resulting change in measure is given by [5]
\[ DX^{IJ}_{\mu\nu} \rightarrow V^{-6} D e^I_{\mu} D C. \]

so that
\[ Z(O) \cong \int De^I_{\mu} D C \delta(C) V^{-3} V_4 \exp i \int X_{IJ} \wedge F[\omega[X]]^{IJ}. \]

Next, we change from tetrad variables to metric variables. This can be done by fixing an arbitrary reference tetrad \[ \hat{e}^I_{\mu}[g] \] for each metric \[ g_{\mu\nu}, \] and defining a local gauge transformation \[ \Lambda^I_J \] via
\[ e^I_{\mu} = \Lambda^I_J e^J_{\mu}[g]. \]

At each space-time point, \[ \Lambda^I_J \] is an element of either the four-dimensional Euclidean group or the Lorentz group, depending on \( s \). The change of variables \( (e^I_{\mu}) \rightarrow (g_{\mu\nu}, \Lambda^I_J) \) yields the change of measure [27]
\[ DG_{\mu\nu} D \Lambda^I_J \cong \sqrt{g} De^I_{\mu}, \]
where \( D \Lambda^I_J \) is the measure defined in [27]. Because the integral \( \int D \Lambda^I_J \) is independent of the metric, it can be dropped from the path integral, leaving
\[ Z(O) \cong \int Dg_{\mu\nu} D C \delta(C) V^{-4} V_4 \exp i \int X_{IJ} \wedge F[\omega[X]]^{IJ}. \]

Using the relations \( V_4 = \hat{h}^{-\frac{1}{2}}, V = g^{-\frac{3}{2}} = NV_4, \) and the fact that \( \int D C \delta(C) = 1 \) gives
\[ Z(O) \cong \int Dg_{\mu\nu} N^{-4} h^{-\frac{1}{2}} O \exp iS_{\text{ADM}} \]
which is the same as (19).

3 Superficially the measure in (19) seems to differ from that in equations (1.3), (2.23) of [26] by a single power of the lapse. However, in order to compare (19) with the path-integral (1.3) in [26], one must include gauge fixing in (19). In doing this, one inserts the canonical [14] Faddeev–Popov determinant, appearing in equation (4), appropriate for the ADM formulation (note that this determinant depends on the scaling of the constraints in the particular canonical formulation used). By contrast, the measure (2.23) in [26] is intended to be used in (1.3) of [26] with a different, covariantly defined Faddeev–Popov determinant \( J^I_J[g] \). These two determinants differ by precisely that single power of the lapse needed to make the total measure here and in [26] the same, as can be deduced from the following equations in [26] (1.5) and those between (2.28) and (2.31) inclusive.

4 The indices for \( C \) are suppressed here for simplicity. In this paper \( C \) is defined as (7) whereas in [5] \( C \) is defined in terms of three parts \( (C^a_{\mu}, \xi_{\beta\gamma}, \xi_{\mu\nu}) \). The formal Lebesgue measure and Dirac delta functions of these two formulations of the simplicity constraint are related by an irrelevant constant Jacobian.
5. Background independence of the path integral

In this last section, we show in what sense the path integral derived in the foregoing sections is background independent.

The path integral was originally discovered as a way to write transition amplitudes in quantum mechanics. In the case of a time reparametrization invariant theory where the Hamiltonian is a linear combination of the constraints (such as GR), the path integral more precisely provides the projector onto physical states together with the physical inner product [1, 19]. As noted in section 2, this physical inner product, together with the rest of the existing canonical LQG framework, is enough to calculate all physically relevant quantities. Let \( \Sigma \) denote the spatial manifold of the canonical theory, and let \( \mathcal{M} \) denote a region of space-time with past and future boundary \( \Sigma_1, \Sigma_2 \), each diffeomorphic to \( \Sigma \). Select diffeomorphisms \( \tau_1 : \Sigma \to \Sigma_1 \subset \mathcal{M} \) and \( \tau_2 : \Sigma \to \Sigma_2 \subset \mathcal{M} \). Select a set of observables \( \{ O_i \} \) which are functions of (the pull-back to \( \Sigma \) of) \( X_{ij}^{\mu \nu} \) only, and whose quantum analogues \( \{ \hat{O}_i \} \) form a complete commuting set. Let \( \Psi_{(0, \nu_1)} \) denote the simultaneous eigenstate of \( \{ \hat{O}_i \} \) satisfying \( \hat{O}_i \Psi_{(0, \nu_1)} = v_i \Psi_{(0, \nu_1)} \). As mentioned in section 2, for different choices of \( O_i \), these states could be spin networks [20], generalized spin networks [21], or Livine–Speziale coherent states [22]. From (6), the path integral (13) then determines the physical inner product in LQG via

\[
\langle \eta(\Psi_{(0, \nu_1)}), \eta(\Psi_{(0, \nu_1)}) \rangle_{\text{phys}} = \int_{\Omega_{\nu_1}[\alpha]} D\alpha_{ij}^{\mu \nu} V_j \delta(C) e^{iS[X]} = Z[\Psi_{(0, \nu_1)}], \Psi_{(0, \nu_1)}]
\]

(20)

where \( \eta \) is the ‘rigging map’ or ‘group averaging map’ from kinematical (unconstrained) states to physical states (states annihilated by the constraints)\(^5\), and \( S[X] := S_{\text{BF}}[X, \omega[X]] \) is the purely geometric action descending from BF theory.

The path integral measure in (20) (as well as the path integral measures thus far derived in the literature for all other formulations of gravity [23, 26, 30], including those in equations (8) and (18)) depends on (1) a choice of foliation \( \Xi \) of \( \mathcal{M} \) (because of the presence of three-dimensional volume factors) and (2) a choice of coordinate system \( \Phi \) compatible with \( \Xi \) (because the 4- and 3-volume factors are densities). That is, the measure depends on background structures. However, what matters physically is the physical inner product in (20), or more precisely, the physical inner product modulo constant rescalings. One can then ask: does the physical inner product (modulo rescalings) determined by the above equation retain this dependence on background? In this section, we show that the physical inner product is in fact background independent modulo rescalings, thus respecting an important guiding principle from general relativity.

To begin the argument, we first note that any function can always be made diffeomorphism covariant by making all background structure an additional explicit argument. Thus, if we express the transition amplitude as a function of an initial state \( \Psi_{(0, \nu_1)} \), a final state \( \Psi_{(0, \nu_1')} \), and as a function of the choice of foliation \( \Xi \) and compatible coordinate system \( \Phi \),

\[
Z[\Psi_{(0, \nu_1)}, \Psi_{(0, \nu_1')}, \Xi, \Phi],
\]

then it is by construction diffeomorphism covariant. As \( Z \) is a scalar-valued function, that means it is diffeomorphism invariant:

\[
Z[\alpha \cdot \Psi_{(0, \nu_1)}, \alpha \cdot \Psi_{(0, \nu_1')}, \alpha \cdot \Xi, \alpha \cdot \Phi] = Z[\Psi_{(0, \nu_1)}, \Psi_{(0, \nu_1')}, \Xi, \Phi]
\]

(21)

for all diffeomorphisms \( \alpha \) of \( \mathcal{M} \).

\(^5\) In order to construct such a map which also projects onto solutions of the Hamiltonian constraint, one makes use of the Master constraint [28, 29].
Now, suppose \((\Xi, \Phi), (\tilde{\Xi}, \tilde{\Phi})\) are two possible choices of foliation and coordinate system. Because the foliation arises from Feynman’s procedure of skeletonization in time [31], the initial and final slices \(\Sigma_1\) and \(\Sigma_2\) will always be leaves of \(\Xi\) and \(\tilde{\Xi}\). Because of this, there always exists a 4-diffeomorphism \(\alpha\) such that \(\alpha \cdot \Xi = \tilde{\Xi}\), and such that \(\alpha\) is the identity on the initial and final hypersurfaces \(\Sigma_1\) and \(\Sigma_2\). Because of the latter property, \(\alpha \cdot \Psi_i(\alpha, \nu) = \Psi_i(\alpha, \nu)\) and \(\alpha \cdot \Psi_i((\alpha, \nu')) = \Psi_i((\alpha, \nu'))\), so that (21) becomes

\[
Z[\Psi_i((\alpha, \nu)), \Psi_i((\alpha, \nu')), \tilde{\Xi}, \tilde{\Phi}] = Z[\Psi_i((\alpha, \nu)), \Psi_i((\alpha, \nu')), \Xi, \Phi].
\]

Next, note that, under a change of coordinate system, the Lesbesgue measure, 3-volume and 4-volume densities, and Dirac delta function in (20) change only by Jacobian factors which are constant on the space of histories. Because of this, the left-hand side of the above equation is equal to \(Z[\Psi_i((\alpha, \nu)), \Psi_i((\alpha, \nu')), \tilde{\Xi}, \tilde{\Phi}]\) up to a constant, so that

\[
Z[\Psi_i((\alpha, \nu)), \Psi_i((\alpha, \nu')), \tilde{\Xi}, \tilde{\Phi}] = (\text{const.}) Z[\Psi_i((\alpha, \nu)), \Psi_i((\alpha, \nu')), \Xi, \Phi] \quad (22)
\]

where the constant is independent of \(\Psi_i((\alpha, \nu))\) and \(\Psi_i((\alpha, \nu'))\). Let \([\eta(\cdot), \eta(\cdot)]_{\text{phys}}\) denote the physical inner product, as determined by the path integral using \(\Xi, \Phi\), modulo constant rescalings. (22) then tells us that

\[
[\eta(\cdot), \eta(\cdot)]_{\Xi, \Phi} = [\eta(\cdot), \eta(\cdot)]_{\text{phys}}\Phi, \Xi.
\]

Consequently \([\eta(\cdot), \eta(\cdot)]_{\text{phys}}\) is independent of \(\Xi\) and \(\Phi\), and hence background independent, as claimed.

6. Conclusions

Spin-foams are a path integral approach to quantum gravity based on the Plebanski–Holst formulation of general relativity. The basic variables of the Plebanski–Holst formulation are a Lorentz connection and the Plebanski 2-form, the pull-backs of which to any Cauchy surface are conjugate to each other. The Plebanski 2-form by itself completely determines the space-time geometry. In spin-foams, one sums over histories of spins and intertwiners which label eigenstates of the Plebanski 2-form. Because of this, the spin-foam sum may be understood as a discretization of a Plebanski–Holst path integral in which the connection degrees of freedom have been integrated out—that is, it is a discretization of what we have called a purely geometric Plebanski–Holst path integral.

In order to ensure that a path integral quantization be equivalent to canonical quantization, it is important that the correct canonical path integral measure be used. The path integral measure for Plebanski–Holst, with both connection and Plebanski 2-form variables present, was calculated in the earlier work [5]. In the present work, we have calculated the pure geometric form of this path integral, whose discretization will yield the necessary measure factor for spin-foams. We have calculated the measure for this path integral in two independent ways: (1) by integrating out the connection from the path integral derived in [5], and (2) by ensuring consistency with the canonical ADM path integral. Both methods lead to the same final measure factor, providing a check on the detailed powers of the space-time and spatial volume elements present. We have furthermore shown in what sense the resulting path integral is background independent. The next step is to discretize this measure on a spin-foam cell complex, expressing it directly in terms of spins and intertwiners. This will involve non-trivial choices which will in part be fixed by considerations of gauge-invariance. This will be discussed in a later, complementary paper.
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Appendix A. Integration of a path integral with quadratic action

Let \((\cdot, \cdot)\) be a symmetric, non-degenerate, but not necessarily positive definite, inner product on some real vector space \(V\). Let \(\hat{a}\) be an invertible operator on \(V\) symmetric with respect to \((\cdot, \cdot)\), \(b\) an element of \(V\), and \(c\) a real number. Consider the action
\[
S[v] := (v, \hat{a}v) + (v, b) + c \tag{A.1}
\]
an

and the path integral
\[
\int Dv \, e^{iS[v]} \tag{A.2}
\]
where \(Dv\) is any translation-invariant measure on \(V\) (unique up to rescaling). If one fixes a basis on \(V\) and uses components with respect to this basis as coordinates on \(V\), \(Dv\) will just be a real number times the Lebesgue measure in the chosen coordinates. From the complex analytic continuation of the usual formula for the Gaussian integral, one obtains
\[
\int Dv \, e^{iS[v]} = C (\det \hat{a})^{-1/2} \exp \left( -\frac{i}{4} \left( b, \hat{a}^{-1}b \right) + c \right) \tag{A.3}
\]
where the constant \(C\) depends exclusively on the relative scaling of the inner product \((\cdot, \cdot)\) and the measure \(Dv\). Variation of the action gives
\[
\delta S|_{v=v_0} = (\delta v, \hat{a}v) + (v, \hat{a}\delta v) + (\delta v, b) = (\delta v, 2\hat{a}v + b).
\]
Setting this equal to zero for all \(\delta v\) yields the following expression for the extremum \(v_0\) of \(S[v]\):
\[
v_0 = -\frac{1}{2} \hat{a}^{-1} b.
\]
Substituting this into (A.1) directly gives \(S[v_0] = -\frac{i}{4} \left( b, \hat{a}^{-1}b \right) + c\), so that (A.3) can be written
\[
\int Dv e^{iS[v]} = C (\det \hat{a})^{-1/2} e^{iS[v_0]} \tag{A.4}
\]
with \(v_0\) the unique extremum of \(S[v]\) and \(C\) depending exclusively on the relative scaling of \((\cdot, \cdot)\) and \(Dv\).

Appendix B. Equivalence of gauge-fixed and non-gauge-fixed path integrals

In this appendix, we address the equivalence of the path integrals (3) and (4). The argument used here is based on the proof for the Yang–Mills case given by Faddeev and Popov [13]. A version of this argument is also given in [16] and [15]. However, here we keep the argument more general and give more details.
B.1. The argument

Consider a system with first class constraints $C_i$, generating a gauge group $G$ on shell, and an action of the form $S[\xi, \lambda] = S_0[\xi] + \int \lambda^i C_i$, where $\lambda$ is short hand for a set of canonically conjugate variables $(\varphi, \pi)$, and $\lambda$ are Lagrange multipliers. We start with (3)

$$Z(O) = \int D\xi D\lambda O[\xi] \exp i S[\xi, \lambda] = \int D\xi \delta(C(\xi)) O[\xi] \exp i S_0[\xi].$$

where $D\xi := D\varphi D\pi$, and where, for this appendix, we assume $O$ is gauge invariant. Faddeev and Popov in their original paper [13] almost start with this same path integral, the only difference being that here we use phase space variables, which is the more fundamental starting point from the canonical perspective [12, 14]. The Faddeev–Popov strategy [13] is to factor out the divergences in the path integral due to the integration over the gauge group. We here adapt their argument to the general path integral (B.1) as follows. First choose gauge-fixing functions $\xi_j = \xi_j(\xi)$ which are regular, that is, have non-vanishing gradient, at $\xi_j \equiv 0$. The $\xi_j$ then form a good set of coordinates on each gauge-orbit in a neighborhood of $\xi_j \equiv 0$. Furthermore, given any coordinates $\alpha_i \mapsto g(\alpha) \in G$ on the gauge group $G$, and any phase space point $\xi = (\varphi, \pi)$, one can define another set of coordinates on the gauge orbit containing $\xi$ via $\alpha_i \mapsto g(\alpha) \cdot \xi$. One then has

$$\int D\alpha \delta(\xi(g(\alpha) \cdot \xi)) \left| \det \frac{\partial \xi^j(g(\alpha) \cdot \xi)}{\partial \alpha^i} \right| = \int D\xi \delta(\xi) = 1$$

which can be inserted into the path integral (B.1) to obtain

$$Z(O) = \int D\xi D\alpha \delta(C(\xi)) \delta(\xi(g(\alpha) \cdot \xi)) \left| \det \frac{\partial \xi^j(g(\alpha) \cdot \xi)}{\partial \alpha^i} \right| O[\xi] \exp S_0[\xi].$$

We next perform the change of variables $\xi \mapsto \xi' := g(\alpha) \cdot \xi$. As $\xi \mapsto g(\alpha) \cdot \xi$ is a canonical transformation, and $d\xi = d\varphi d\pi$ is the Liouville measure, we have $d\xi = d\xi'$. Furthermore, $S_0[\xi] = S_0[\xi']$ and $O[\xi] = O[\xi']$ as the action (and hence the action with constraints satisfied, $S_0$) and $O[\xi]$ are gauge-invariant. It remains to consider the constraint factor $\delta(C(\xi))$, and the determinant factor $|\det \frac{\partial \xi^j(g(\alpha) \cdot \xi)}{\partial \alpha^i}|$. In order to facilitate calculation, we now make a specific choice for $g(\alpha)$: For each $\alpha'$, we define $g(\alpha) : \Gamma \to \Gamma$ to be the Hamiltonian flow generated by $\alpha' C_i$, evaluated at unit parameter time. Equivalently, $g(\alpha)$ may be defined by the equations

$$[\alpha \cdot C, f]_{\tilde{\xi}} = \left. \frac{df(g(-t\alpha) \cdot \xi)}{dt} \right|_{t=0},$$

$$g(0) = \text{id}$$

If we let $X_{\alpha \cdot C}$ denote the derivative operator $X_{\alpha \cdot C} f := [\alpha \cdot C, f]$, then the equations above implies the explicit expression

$$f(g(-\alpha) \cdot \xi) = \sum_{n=0}^{\infty} \frac{1}{n!} (X_{\alpha \cdot C})^n f|_{\tilde{\xi}} \equiv \exp (X_{\alpha \cdot C} f)|_{\tilde{\xi}}.$$

The constraint factor $\delta(C(\xi))$ can be rewritten $\delta(g(-\alpha)^* C(\xi'))$, where * denotes pull-back. Because the flow generated by first class constraints is always tangent to the constraint surface, $g(-\alpha)^* C_i$ will again be a linear combination of the constraints, so that

$$g(-\alpha)^* C_i =: \mu(\alpha)/C_j$$

for some matrix $\mu(\alpha)/C_j$ of functions on phase space, whence

$$\delta(g(-\alpha)^* C(\xi')) = |\det \mu(\alpha)|^{-1} \delta(C(\xi')).$$

(B.4)
Turning now to the last remaining factor, we have
\[
\frac{\partial \xi_j (g(\alpha + t \delta_i ) \cdot \xi)}{\partial \alpha^i} = \frac{d}{dt} \left. \xi_j (g(\alpha + t \delta_i ) \cdot \xi) \right|_{t=0} = \frac{d}{dt} \left. \xi_j (g(\alpha + t \delta_i )g(- \alpha) \cdot \xi') \right|_{t=0}
\]
\[
= \int_0^1 ds \frac{d}{dt} \left. \xi_j (g(s \alpha \delta_j )g(-s \alpha) \cdot \xi') \right|_{t=0} = \int_0^1 ds \left[ C_i \cdot g(s \alpha \delta_j )g(-s \alpha) \cdot \xi' \right] |_{s=0} = \int_0^1 ds \left[ C_i \cdot g(s \alpha \delta_j ) \right] |_{s=0} \cdot \xi'
\]
\[
= \int_0^1 ds \left[ g(-s \alpha) \cdot C_i \cdot \xi_j \cdot \xi' \right] |_{s=0} = \left( \int_0^1 ds \mu(s \alpha \delta) \right)^2 \left[ C_i \cdot \xi_j \right] |_{s=0} \tag{B.5}
\]
where \((\delta_i )^j := \delta_i ^j\). Here the definition of the partial derivative has been used in the first line, equation (B.6) from the next subsection has been used in the second line, equation (B.3) has been used in the third line, and the invariance of the Poisson bracket under the gauge transformation \(g(s \alpha)\) has been used in the last line. Define \(M(\alpha) = \int_0^1 ds \mu(s \alpha \delta) \). Taking the determinant of (B.5) then yields
\[
\det \left( \frac{\partial \xi_j (g(\alpha \cdot \xi))}{\partial \alpha^i} \right) = \det M(\alpha) |_{\xi'} \cdot \det [C_i \cdot \xi_j ] |_{\xi'}.
\]
Using this and equation (B.4) in equation (B.2) gives us
\[
\mathcal{Z}(O) = \int D\zeta' \left( \int D\alpha \left[ \frac{\det M(\alpha)}{\det \mu(\alpha)} \right] |_{\xi'} \delta(C(\zeta')) \delta(\xi(\zeta')) | \det [C_i \cdot \xi_j ] |_{\xi'} \exp i S_\alpha[\zeta'] \right)
\]
\[
= \int D\zeta''D\lambda \left( \int D\alpha \left[ \frac{\det M(\alpha)}{\det \mu(\alpha)} \right] |_{\xi'} \delta(\xi(\zeta')) \delta(\xi(\zeta'')) | \det [C_i \cdot \xi_j ] |_{\xi'} \exp i S[\zeta', \lambda] \right).
\]
At this point, all dependence on \(\alpha\) is restricted to the inner integral in parentheses, which can be thought of as a ‘gauge orbit volume,’ with \(\frac{\det M(\alpha)}{\det \mu(\alpha)} \) acting as a ‘volume element’. If we can show that this gauge orbit volume is independent of \(\zeta'\), then we can drop it as an overall constant, thereby proving the equivalence of the non-gauge-fixed (B.1, 3) and gauge-fixed (4) path integrals.

In the case of gravity, or any other theory with non-compact gauge orbits, this gauge orbit volume is infinite, so it is not clear what it means to be constant on phase space. In a moment, we will show that, in the case when the algebra of constraints under consideration has structure constants, then at least the gauge orbit volume element is constant on phase space. Assuming that the ranges of the coordinates \(\alpha^i\) on each gauge orbit are then also constant, this implies that the fully integrated gauge orbit volume itself is also constant, as required for equivalence.

The proof of the constancy of the gauge orbit volume element, \(\frac{\det M(\alpha)}{\det \mu(\alpha)} \), when there are structure constants, is straightforward. We have
\[
\mu(s \alpha) C_j := g(-s \alpha) C_j = \sum_{n=0}^\infty \frac{s^n}{n!} [\alpha \cdot C, [\alpha \cdot C, \ldots [\alpha \cdot C, C_j ] \ldots ],]
\]
where in each term there are \(n\) nested Poisson brackets, with the \(n = 0\) term being just \(C_i\). If one has structure constants, each Poisson bracket introduces multiplication by a matrix which is constant on phase space. The product of these matrices, summed over \(n\), is then equal to the matrix \(\mu(s \alpha) \) on the left-hand side, which is therefore also constant on phase space, leading to \(M(\alpha) \) being constant on phase space.

To handle the case of structure functions, which is in the case relevant for gravity, one must look not only at the ‘gauge orbit volume element’, but also at the fully integrated ‘gauge
orbit volume’. As this volume is infinite, as already mentioned, it is not so clear what it means for it to be constant on the phase space. A better understanding of how to handle this infinite volume through an appropriate regulator, or experimentation with toy models with structure functions in which the gauge volume is finite, could shed light on how to extend the last step of this proof of equivalence to systems of interest with structure functions.

B.2. A result for non-Abelian gauge groups

In this subsection we prove equation (B.6) below, which has been key in the above subsection. We begin by proving a general result for linear operators, which we then apply to the relevant case at hand.

**Lemma 1.** For any two linear operators A and B on any vector space V,

\[
\frac{d}{dt} e^{tB - A} \bigg|_{t=0} = \int_0^1 ds e^{sA} (-B) e^{-sA}.
\]

**Proof.** Let \( I := \frac{d}{dt} \{ \exp A \exp(-tB - A) \} \bigg|_{t=0} \). We then have

\[
I = \int_0^1 ds \left( \frac{\partial^2}{\partial s \partial t} \{ s \exp(sA) \exp(-tB - sA) \} \bigg|_{t=0} \right)
\]

Using the Leibniz rule to evaluate the derivative with respect to \( t \), one obtains

\[
I = \int_0^1 ds \left[ s \exp(sA) \sum_{n=0}^\infty \sum_{m=0}^n \frac{1}{n!} (-sA)^{n-m} (-B)^{-m+1} \right]
\]

\[
= \int_0^1 ds \left[ \exp(sA) \sum_{n=0}^\infty \sum_{m=0}^n \frac{(-s)^n}{n!} A^{-m} B A^{m+1} \right]
\]

\[
= \int_0^1 ds \left[ \exp(sA) \sum_{m=0}^\infty \sum_{n=m}^\infty \frac{(-s)^n}{n!} A^{-m} A B A^{m-1} + \frac{(-s)^m}{(n-1)!} A^{m-1} B A^{m-1} \right].
\]

Notice that we have switched the order of summation in the last line and used the fact that \( A \exp(sA) = \exp(sA) A \). Simplifying further,

\[
I = \int_0^1 ds \left[ \exp(sA) \sum_{m=0}^\infty \sum_{n=m}^\infty \frac{(-s)^n}{n!} A^{-m+1} A B^{m-1} + \frac{(-s)^m}{(n-1)!} A^{-m} B A^{m-1} \right].
\]

In this expression, the sum over \( n \) telescopes, so that all terms in the sum cancel except the second term in the \( n = m \) case. One is thus left with

\[
I = \int_0^1 ds \left[ \exp(sA) \sum_{m=1}^\infty \frac{(-s)^{m-1}}{(m-1)!} B A^{m-1} \right] = - \int_0^1 ds \exp(sA) B \exp(-sA) ds.
\]

**Proposition 2.**

\[
\frac{d}{dt} f(g(\alpha + t\beta)g(-\alpha)\zeta) \bigg|_{t=0} = \int_0^1 ds \frac{d}{dt} f(g(s\alpha)g(t\beta)g(-s\alpha)\zeta) \bigg|_{t=0}.
\]  

(B.6)
Proof.

\[
\frac{d}{dt} f(g(\alpha + t\beta)g(-\alpha)) \bigg|_{t=0} = \frac{d}{dt} [\exp(X_{\alpha - t\beta}C)f(g(-\alpha))]_{t=0} \\
= \frac{d}{dt} [\exp(X_{\alpha}C) \exp(X_{\alpha - t\beta}C)f(g(\xi))]_{t=0} \\
= \frac{d}{dt} [\exp(X_{\alpha}C) \exp(-X_{\alpha}C - tX_{\beta}C)]_{t=0} f|_{\Gamma} \\
= \int_0^1 ds [\exp(sX_{\alpha}C)(-X_{\beta}C) \exp(-sX_{\alpha}C)]f|_{\Gamma} \\
= \int_0^1 ds \frac{d}{dt} [\exp(sX_{\alpha}C) \exp(-tX_{\beta}C) \exp(-sX_{\alpha}C)]|_{t=0} f|_{\Gamma} \\
= \int_0^1 ds \frac{d}{dt} [\exp(sX_{\alpha}C) \exp(X_{-t\beta}C) \exp(X_{-s\alpha}C)f(g(\xi))]_{t=0} \\
= \int_0^1 ds \frac{d}{dt} f(g(\alpha)g(t\beta)g(-\alpha))|_{t=0} \\
\]

where, in the fourth line, lemma 1 was used in the case where \( V \) is the space of functions on \( \Gamma, A = X_{\alpha}C, \) and \( B = X_{\beta}C. \)

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