ON THE RIEMANN HYPOTHESIS FOR A CERTAIN FAMILY OF FORMAL WEIGHT ENUMERATORS

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Abstract. A formal weight enumerator is a homogeneous polynomial in two variables which behaves like the Hamming weight enumerator of a self-dual linear code except that the coefficients are not necessarily non-negative integers. Chinen discovered several families of formal weight enumerators and investigated the validity of the Riemann hypothesis analogue for them. In this paper, the zeta polynomial is computed for Chinen’s formal weight enumerators, and a simple criterion is given for the validity of the Riemann hypothesis analogue.

1. Introduction

The weight enumerator of an \([n, k, d]\)-linear code \(C\) over a finite field \(\mathbb{F}_q\) is defined as

\[A_C(x, y) = \sum_{i=0}^{n} A_i x^{n-i} y^i = x^n + \sum_{i=d}^{n} A_i x^{n-i} y^i,\]

where \(A_i\) denotes the number of codewords of Hamming weight \(i\) in \(C\). Duursma \([4, 5]\) defined the zeta function of \(C\) (or of \(A_C(x, y)\)), and showed that the MacWilliams identity yields a functional equation between the zeta functions of \(C\) and \(C^\perp\), the dual code of \(C\). In particular, the zeta function of a self-dual code satisfies a functional equation which has exactly the same shape as in the case of an algebraic curve over \(\mathbb{F}_q\). It is known that not all self-dual linear codes satisfy the Riemann hypothesis analogue (RH for short), and it is an open problem whether all extremal weight enumerators satisfy RH.

As has been observed by Chinen \([1–3]\), the definition of the zeta function works well for any homogeneous polynomial of the form

\[A(x, y) = x^n + \sum_{i=d}^{n} A_i x^{n-i} y^i \in \mathbb{C}[x, y]\]

and for any real number \(q > 0, q \neq 1\). Moreover, if \(A^\sigma(x, y) = \varepsilon A(x, y)\) holds for the MacWilliams transformation

\[\sigma = \frac{1}{\sqrt{q}} \begin{pmatrix} 1 & q - 1 \\ 1 & -1 \end{pmatrix}\]

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and for $\varepsilon \in \{\pm 1\}$, then the zeta function of $A(x, y)$ satisfies a functional equation similar to the case of self-dual linear codes. Such a homogeneous polynomial is called a twisted virtually self-dual weight enumerator in [7], and, with an additional divisibility condition, a formal weight enumerator in [1].

In [2, 3], Chinen discovered several families of formal weight enumerators and investigated the validity of RH for them. Some examples satisfy RH while others do not. We summarize his results in Table 1, together with the corresponding parameters in our notation.

In [8], the second named author succeeded in giving a simple expression for Chinen’s formal weight enumerators. To be precise, let $q > 1$ be a real number such that there exist integers $n$ and $j$ with

$$\frac{1}{\sqrt{q}} = \cos \frac{j\pi}{n}, \quad 0 < j < \frac{n}{2}, \quad \gcd(n, j) = 1. \quad (1.1)$$

Note that $q$ uniquely determines $n$ and $j$, but for a fixed $n \geq 3$ there are $\varphi(n)/2$ number of $q, \varphi$ being Euler’s totient function. The second named author showed in [8] that for such a $q$ there exists a unique homogeneous polynomial $\varphi_{n,q}(x, y)$ with the following properties:

- $\varphi_{n,q}^\sigma = (-1)^j \varphi_{n,q}, \varphi_{n,q}^\tau = \varphi_{n,q}$ for

$$\sigma = \frac{1}{\sqrt{q}} \begin{pmatrix} 1 & q - 1 \\ -1 & -1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where the action of

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

on $f(x, y)$ is defined by $f^g(x, y) = f(ax + by, cx + dy)$.
- $\varphi_{n,q}(x, 0) = x^n$.
- $\varphi_{n,q}$ is indecomposable in a suitable sense.

### Table 1. Chinen’s formal weight enumerators.

| Formal weight enumerator (Chinen’s notation) | $n$ | $j$ | $q$ | RH |
|---------------------------------------------|-----|-----|-----|----|
| $\varphi_4$                                | 4   | 1   | 2   | True |
| $\varphi_6$                                | 6   | 1   | 4/3 | True |
| $\varphi_{8}^-$                             | 8   | 1   | $4 - 2\sqrt{2}$ | True |
| $\varphi_{8}^+$                             | 8   | 3   | $4 + 2\sqrt{2}$ | False |
| $\varphi_{10}^-$                            | 10  | 1   | $2 - 2\sqrt{5}$ | True |
| $\varphi_{10}^+$                            | 10  | 3   | $2 + 2\sqrt{5}$ | False |
| $\varphi_{12}^-$                            | 12  | 1   | $8 - 4\sqrt{3}$ | True |
| $\varphi_{12}^+$                            | 12  | 5   | $8 + 4\sqrt{3}$ | False |
| $\varphi_3$                                | 3   | 1   | 4   | True |
| $\varphi_5$                                | 5   | 1   | $6 - 2\sqrt{5}$ | True |
| $\psi_5$                                   | 5   | 2   | $6 + 2\sqrt{5}$ | False |
Formal weight enumerators

| $j$ | 1  | 3  | 5  | 7  | 9  | 11 | 13 | 15 |
|-----|----|----|----|----|----|----|----|----|
| $C_j$ | 2  | 21.8649 | 61.3488 | 120.568 | 199.525 | 298.222 | 416.657 | 554.832 |

He also obtained an explicit expression of $\varphi_{n,q}$ using Dickson polynomials (essentially, Chebyshev polynomials).

Let $P_{n,q}$ denote the zeta polynomial of $\varphi_{n,q}$. (We work with zeta polynomials rather than zeta functions.) The Riemann hypothesis for $P_{n,q}$ claims that all zeros of this polynomial have absolute value $1/\sqrt{q}$. For each integer $j \geq 1$, let $C_j$ denote the unique solution $x \geq 2j$ of the equation

$$1 - \cos \frac{j\pi}{x} = \frac{2}{x}.$$ 

See Table 2 for numerical examples.

The main result of this paper is the following.

**Theorem 1.1.** Suppose $q$ satisfies (1.1).

1. The zeros of $P_{n,q}$ have absolute value $1/\sqrt{q}$, except possibly for a pair of real zeros $\alpha$ and $\beta$ with $\alpha, \beta > 0$, $\alpha \beta = 1/q$, $\alpha \neq \beta$.
2. $P_{n,q}$ has exceptional zeros described in (1) if and only if $(n, j) \equiv (1, 0) \pmod{2}$ or $n < C_j$.
3. $P_{n,q}$ satisfies the Riemann hypothesis if and only if $j$ is odd and $n \geq C_j$.

In view of Table 2, we see that this theorem implies Chinen’s results given in Table 1.

The organization of this paper is as follows. In Section 2 we compute the zeta polynomial $P_{n,q}$ using Duursma’s normalized weight enumerator [5]. The result is surprisingly simple:

$$P_{n,q}(t) = \frac{(-1)^{n-1}q^n t^{n-1}(t - 1) + qt - 1}{qt^2 - 2t + 1}.$$ 

The Chebyshev polynomials of the first kind play an important role in the computation of $P_{n,q}$. Then in Section 3 we shall prove Theorem 1.1. It turns out that, under a suitable change of variables, $P_{n,q}$ has an expression involving Chebyshev polynomials of the second, third, and fourth kinds, according to the parities of $n$ and $j$. The (well-known) zeros of the Chebyshev polynomials give sufficient information for us to locate the zeros of $P_{n,q}$. In the final Section 4, we observe that Chebyshev polynomials of the (missing) first kind appear in an expression for $P_{n,q}$ if we formally define $\varphi_{n,q}$ in the case when both $n$ and $j$ are even. We show that the Riemann hypothesis does not hold in this case.

We note that Duursma [6] utilized other orthogonal polynomials, ultraspherical polynomials, for the study of the zeta functions of extremal weight enumerators.

2. The zeta polynomial of $\varphi_{n,q}$

Henceforth we suppose that $q$ satisfies (1.1) and let $\varphi_{n,q}$ be as described in the Introduction. In this section, we give an explicit expression of the zeta polynomial of $\varphi_{n,q}$.
The second named author was not aware of the following simple expression when writing \[8\].

**Proposition 2.1.** One has

\[ \varphi_{n,q}(x, y) = \frac{1}{2} \left( (x + \sqrt{1 - q} \ y)^n + (x - \sqrt{1 - q} \ y)^n \right). \]

**Proof.** It is shown in \[8\] that

\[ \varphi_{n,q}(x, y) = \begin{cases} 
\frac{1}{2} D_n(2\sqrt{1 - q} \ y, -(x^2 + (q - 1)y^2)) & \text{if } n \text{ is even}, \\
xE_{n-1}(2\sqrt{1 - q} \ y, -(x^2 + (q - 1)y^2)) & \text{if } n \text{ is odd},
\end{cases} \]

where \( D_m \) and \( E_m \) are Dickson polynomials of the first and second kinds, respectively, of degree \( m \). Using the identities

\[ D_m(u_1 + u_1 u_2) = u_1^m + u_2^m, \]
\[ E_m(u_1 + u_1 u_2) = \frac{u_1^{m+1} - u_2^{m+1}}{u_1 - u_2}, \]

and taking \( u_1 = \sqrt{1 - q} \ y + x, \ u_2 = \sqrt{1 - q} \ y - x \), we obtain the claimed expression. \( \square \)

Let us recall the definition of the zeta polynomial. Let

\[ A(x, y) = x^n + A_d x^{n-d} y^d + \cdots + A_n y^n, \quad A_d \neq 0 \]

be a homogeneous polynomial and \( q \) a real number with \( q > 0, q \neq 1 \). We assume \( d > 1 \) and \( d^\perp > 1 \), where \( d^\perp \) denotes the ‘minimal distance’ of \( A^\sigma(x, y) \). Then the zeta polynomial of \( A(x, y) \) with respect to \( q \) is the unique polynomial \( P(t) \) of degree at most \( n - d \) such that the coefficient of \( t^{n-d} \) in the expansion of

\[ \frac{P(t)}{(1-t)(1-qt)}(y(1-t)+xt)^n \]

is equal to

\[ \frac{A(x, y) - x^n}{q - 1}. \]

For the existence and the uniqueness of \( P(t) \), see Duursma [5] or Chinen [1]. We use \( t \) as the variable instead of \( T \) adopted by Duursma and Chinen, in order to avoid confusion with the Chebyshev polynomials of the first kind.

Duursma [5] proved the following.

**Proposition 2.2.** The zeta polynomial \( P(t) \) and the normalized weight enumerator defined by

\[ a(t) = \frac{1}{q - 1} \sum_{i=d}^{n} \binom{n}{i} A_i t^{n-i-d} \]

are related by

\[ \frac{P(t)}{(1-t)(1-qt)}(1-t)^{d+1} \equiv a \left( \frac{t}{1-t} \right) \quad (\text{mod } t^{n-d+1}). \]
Suppose \( A(x, y) \) satisfies \( A^\sigma = \varepsilon A, \ \varepsilon \in \{\pm 1\} \), and \( d > 1 \). Then one can show that \( P(t) \) has degree \( n + 2 - 2d \) and satisfies the functional equation

\[
P(t) = \varepsilon P \left( \frac{1}{\sqrt{q}} t \right) (\sqrt{q} t)^{n+2-2d}.
\]

(2.1)

The proof of (2.1) is similar to that of (8) of Duursma [5, p. 59]. In this case, \( P(t) \) (or \( A(x, y) \)) is said to satisfy the Riemann hypothesis if all the roots of \( P(t) \) lie on the circle \( |t| = 1/\sqrt{q} \) in the complex plane.

Recall some basic facts on Chebyshev polynomials. The Chebyshev polynomials of the first, second, third, and fourth kinds, respectively, of degree \( k \) are characterized by

\[
T_k(\cos \theta) = \cos k \theta, \quad U_k(\cos \theta) = \frac{\sin(k + 1)\theta}{\sin \theta},
\]

\[
V_k(\cos \theta) = \frac{\cos(k + \frac{1}{2})\theta}{\cos \frac{1}{2}\theta}, \quad W_k(\cos \theta) = \frac{\sin(k + \frac{1}{2})\theta}{\sin \frac{1}{2}\theta},
\]

or equivalently by

\[
T_k \left( \frac{u + u^{-1}}{2} \right) = \frac{u^k + u^{-k}}{2},
\]

(2.2)

\[
U_k \left( \frac{u + u^{-1}}{2} \right) = \frac{u^{k+1} - u^{-k-1}}{u - u^{-1}},
\]

(2.3)

\[
V_k \left( \frac{u + u^{-1}}{2} \right) = \frac{u^{k+1/2} + u^{-k-1/2}}{u^{1/2} + u^{-1/2}},
\]

(2.4)

\[
W_k \left( \frac{u + u^{-1}}{2} \right) = \frac{u^{k+1/2} - u^{-k-1/2}}{u^{1/2} - u^{-1/2}}.
\]

(2.5)

The following identities easily follow from (2.2):

\[
\sum_{k=0}^{n} T_k(x) z^k = \frac{1 - xz - T_{n+1}(x) z^{n+1} + T_n(x) z^{n+2}}{1 - 2xz + z^2},
\]

(2.6)

\[
\sum_{k=0}^{\infty} T_k(x) z^k = \frac{1 - xz}{1 - 2xz + z^2}.
\]

(2.7)

We are now ready to compute the zeta polynomial of \( \varphi_{n,q} \).

**Theorem 2.3.** Suppose \( q \) satisfies (1.1) and let \( P_{n,q} \) denote the zeta polynomial of \( \varphi_{n,q} \). We have

\[
P_{n,q}(t) = -\sum_{k=0}^{n-2} T_{k+1} \left( \frac{1}{\sqrt{q}} \right) \sqrt{q}^{k+1} t^k
\]

\[
= \frac{(-1)^{j-1} \sqrt{q}^{-1} t^{n-1}(t-1) + qt - 1}{qt^2 - 2t + 1},
\]

(2.8)

where \( j \) is as in (1.1).
Proof. First note that \( d = d^\perp = 2 \) for \( \varphi_{n,q} \). Using Proposition 2.1 and the definition of the normalized weight enumerator, we easily see that

\[
a(t) = - \sum_{k=1}^{[n/2]} ((1 - q)t^2)^{k-1}.
\]

So by Proposition 2.2 we have, modulo \( t^{n-1} \),

\[
\frac{P_{n,q}(t)}{1 - qt} (1 - t)^2 \equiv - \sum_{k=1}^{[n/2]} (1 - q) \left( \frac{t}{1 - t} \right)^2 k^{-1}
\equiv - \sum_{k=1}^{\infty} (1 - q) \left( \frac{t}{1 - t} \right)^2 k^{-1}
= - \frac{(1 - t)^2}{1 - 2t + qt^2},
\]

and therefore

\[
P_{n,q}(t) \equiv - \frac{1 - qt}{1 - 2t + qt^2}
= - \frac{1}{t} \left( \frac{1 - t}{1 - 2t + qt^2} - 1 \right)
= - \frac{1}{t} \sum_{k=1}^{\infty} T_k \left( \frac{1}{\sqrt{q}} \right) (\sqrt{q} t)^k
= - \sum_{k=0}^{n-2} T_{k+1} \left( \frac{1}{\sqrt{q}} \right) \sqrt{q}^{k+1} t^k.
\]

Since \( P_{n,q}(t) \) has degree at most \( n - 2 \) and is unique, this proves the first equality.

Then using (2.6) and noting that

\[
T_n \left( \frac{1}{\sqrt{q}} \right) = T_n \left( \cos \frac{j\pi}{n} \right) = \cos j\pi = (-1)^j,
\]

\[
T_{n-1} \left( \frac{1}{\sqrt{q}} \right) = \cos \left( j\pi - \frac{j\pi}{n} \right) = \frac{(-1)^j}{\sqrt{q}},
\]

we obtain the second equality.

Example 2.4. Applying (2.8) for \( n = 3, 4, 5, 6, 8 \) (see Table 1 for the corresponding \( q \)), we recover Chinen’s computation in [2, 3]. For example,

\[
P_{3,4}(t) = 2t - 1,
\]

\[
P_{4,2}(t) = 2t^2 - 1,
\]

\[
P_{6,4/3}(t) = \frac{1}{9} (4t^2 - 3)(4t^2 + 2t + 3).
\]
3. Proof of Theorem 1.1

First, it follows from the functional equation (2.1) that \( P_{n,q}(1/\sqrt{q}) = 0 \) if \( j \) is odd and that \( P_{n,q}(-1/\sqrt{q}) = 0 \) if \( n + j \equiv 1 \pmod{2} \). We call these zeros of \( P_{n,q} \) the trivial zeros.

Substituting \( u = \sqrt{q} \) and \( w = (u + u^{-1})/2 \) into (2.8), and using (2.3), (2.4), and (2.5), we easily obtain the following expressions.

- **If** \( n = 2m \) and \( j \) is odd, then
  \[
P_{n,q}(t) = \frac{1}{2} u^{m-2} \left( u^2 - 1 \right) \frac{U_{m-1}(w) - \sqrt{q} U_{m-2}(w)}{w - 1/\sqrt{q}}.
  \]

- **If** \( n = 2m + 1 \) and \( j \) is even, then
  \[
P_{n,q}(t) = -\frac{1}{2} u^{m-1} \left( u + 1 \right) \frac{V_{m}(w) - \sqrt{q} V_{m-1}(w)}{w - 1/\sqrt{q}}.
  \]

- **If** \( n = 2m + 1 \) and \( j \) is odd, then
  \[
P_{n,q}(t) = \frac{1}{2} u^{m-1} \left( u - 1 \right) \frac{W_{m}(w) - \sqrt{q} W_{m-1}(w)}{w - 1/\sqrt{q}}.
  \]

The factors \( u \pm 1 \) correspond to the trivial zeros. Since \( P_{n,q}(0) = -1 \) by (2.8), we may ignore the factor \( u \).

Let

\[
F_{n,q}(w) = \begin{cases} 
U_{m-1}(w) - \sqrt{q} U_{m-2}(w) & \text{if } n = 2m \text{ and } j \text{ is odd}, \\
V_{m}(w) - \sqrt{q} V_{m-1}(w) & \text{if } n = 2m + 1 \text{ and } j \text{ is even}, \\
W_{m}(w) - \sqrt{q} W_{m-1}(w) & \text{if } n = 2m + 1 \text{ and } j \text{ is odd}.
\end{cases}
\]

We can verify \( F_{n,q}(1/\sqrt{q}) = 0 \) as follows. If \( n = 2m \) and \( j \) is odd, then, writing \( 1/\sqrt{q} = (u + u^{-1})/2 \), we have

\[
F_{n,q}(1/\sqrt{q}) = \frac{u + u^{-1}}{2} \left( \frac{u^m - u^{-m}}{u - u^{-1}} - \frac{u^{m-1} - u^{-m+1}}{u - u^{-1}} \right)
= \frac{u^m + u^{-m}}{2} = T_m \left( \frac{u + u^{-1}}{2} \right) = T_m \left( \cos \frac{j\pi}{n} \right) = \cos \frac{j\pi}{2} = 0.
\]

The remaining cases can be treated similarly.

**Lemma 3.1.** One has \( F_{n,q}(\pm 1) \neq 0 \).

**Proof.** By (2.3), (2.4), and (2.5), we have

\[
U_k(1) = k + 1, \quad U_k(-1) = (-1)^k (k + 1), \\
V_k(1) = 1, \quad V_k(-1) = (-1)^k (2k + 1), \\
W_k(1) = 2k + 1, \quad W_k(-1) = (-1)^k.
\]
Moreover, \( f \) has a root \( w > 1 \) of the first kind in the next lemma.)

Verified as follows. (For later use, we also include the case of Chebyshev polynomials of the first kind in the next lemma.)

**Proof.** Let \( k \) be a positive integer, let \( C \) be a positive real number, and let \( f(w) \) be either one of \( T_k(w) - CT_{k-1}(w) \), \( U_k(w) - CU_{k-1}(w) \), \( V_k(w) - CV_{k-1}(w) \), or \( W_k(w) - CW_{k-1}(w) \). Then the roots of \( f \) are all real, \( w > -1 \), and, possibly with one exception, \( w < 1 \). These can be verified as follows. (For later use, we also include the case of Chebyshev polynomials of the first kind in the next lemma.)

**Lemma 3.2.** Let \( k \) be a positive integer, let \( C \) be a positive real number, and let \( f(w) \) be either one of \( T_k(w) - CT_{k-1}(w) \), \( U_k(w) - CU_{k-1}(w) \), \( V_k(w) - CV_{k-1}(w) \), or \( W_k(w) - CW_{k-1}(w) \). Then the roots of \( f \) are all real, \( w > -1 \), and at least \( k - 1 \) of them satisfy \( w < 1 \). Moreover, \( f \) has a root \( w > 1 \) if and only if \( f(1) < 0 \).

**Proof.** Let \( f(w) = T_k(w) - CT_{k-1}(w) \). Recall that the roots of \( T_k \) are

\[
\alpha_l = \cos \frac{2l - 1}{2k} \pi \quad (l = 1, 2, \ldots, k).
\]

We have

\[
T_{k-1}(\alpha_l) = \cos \frac{(k-1)(2l-1)}{2k} \pi = (-1)^{l+1} \sin \frac{2l - 1}{2k} \pi,
\]

so that \( f(\alpha_l) \) has the same sign as \((-1)^l\). We have also \( f(\infty) = \infty \). So \( f \) has a real root in each of the open intervals \((\alpha_k, \alpha_{k-1})\), \((\alpha_{k-1}, \alpha_{k-2})\), \ldots, and \((\alpha_1, \infty)\). Since the degree of \( f \) is \( k \), this proves the assertions.

The proof is similar for the other kinds of polynomials; the roots of \( U_k \) are \( \alpha_l = \cos(l\pi/(k+1)) \) \((l = 1, 2, \ldots, k)\), for which we have \( U_{k-1}(\alpha_l) = (-1)^{l+1} \), the roots of \( V_k \) are \( \alpha_l = \cos((2l-1)/(2k+1))\pi \) \((l = 1, 2, \ldots, k)\), for which we have

\[
V_{k-1}(\alpha_l) = (-1)^{l+1} 2 \sin \frac{2l - 1}{2k + 1} \pi,
\]

and the roots of \( W_k \) are \( \alpha_l = \cos(2l/(2k+1))\pi \) \((l = 1, 2, \ldots, k)\), for which we have

\[
W_{k-1}(\alpha_l) = (-1)^{l+1} 2 \cos \frac{l}{2k + 1} \pi.
\]

\( \square \)
For the proof of part (2) of Theorem 1.1, recall that $P_{n,q}(t)$ has exceptional zeros if and only if $F_{n,q}(w)$ has a root outside $(-1, 1)$. By Lemmas 3.1 and 3.2, the possible exceptional root of $F_{n,q}(w)$ must be in the range $w > 1$. By Lemma 3.2, this happens when and only when $F_{n,q}(1) < 0$. In view of (3.1), this leads to the conclusion of part (2).

Part (3) of Theorem 1.1 is clear from parts (1) and (2). This completes the proof.

4. Chebyshev polynomials of the first kind

We have seen in Section 3 that, according to the parities of $n$ and $j$, Chebyshev polynomials of the second, third, and fourth kinds appear in an expression of the zeta polynomial, but not the first kind. In this section, we consider the case when $n$ and $j$ are both even, and observe that Chebyshev polynomials of the first kind do appear.

Let $q > 1$ be a real number, $n \geq 4$ an even integer, and consider the following conditions on a homogeneous polynomial $\varphi(x, y)$:

- $\varphi^\sigma = \varphi^\tau = \varphi$ for
  \[ \sigma = \frac{1}{\sqrt{q}} \begin{pmatrix} 1 & q-1 \\ 1 & -1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

- $\varphi(x, 0) = x^n$.

- $\varphi$ is indecomposable in a suitable sense.

It is shown in [8, Theorem 2] that a homogeneous polynomial satisfying these three conditions exists if and only if

$$ \frac{1}{\sqrt{q}} = \cos \frac{j \pi}{n}, \quad 0 < j < \frac{n}{2}, \quad \text{gcd}(n/2, j/2) = 1 \quad (4.1) $$

holds for some even $j$, and that there are infinitely many $\varphi$ for such fixed $n$ and $q$.

In view of this result, we start with even integers $n$ and $j$ and a real number $q$ satisfying (4.1), and formally define $\varphi_{n,q}(x, y)$ as in Proposition 2.1. It is readily verified that $\varphi_{n,q}$ satisfies the three conditions given above. The computation of the zeta polynomial carried out in Section 2 is valid in this case, and we have the same conclusion as Theorem 2.3 if we replace (1.1) by (4.1).

A simple calculation then shows that

$$ P_{n,q}(t) = -u^{m-1} \frac{T_m(w) - \sqrt{q} T_{m-1}(w)}{w - 1/\sqrt{q}}, $$

where $u = \sqrt{q} t$, $w = (u + u^{-1})/2$, and $m = n/2$. Using the same reasoning as in Section 3 for

$$ F_{n,q}(w) = T_m(w) - \sqrt{q} T_{m-1}(w) $$

and noting that $F_{n,q}(1) = 1 - \sqrt{q} < 0$, we arrive at the following result.

**Proposition 4.1.** Suppose $q$ satisfies (4.1) with even integers $n$ and $j$. Then, $P_{n,q}$ does not satisfy the Riemann hypothesis.
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