Expressing the power radiated by electric charged systems

C. Vrejoiu and D. Nicmorus
Faculty of Physics, University of Bucharest, 76900, Bucharest-Magurele, Romania
E-mail: cvrejoiu@yahoo.com, diananicmorus@hotmail.com

Abstract. After a systematic introduction of some formulae for the energy radiated by localized electric charges and currents distributions, one considers the multipole radiation and the reduction of the multipole tensors to the symmetric traceless ones.

1. Introduction

In the calculation of the energy radiated at large distances by a localised electric charged system it is not necessary to know the exact expressions of the electromagnetic fields $E$ and $B$ or of the potentials $A$ and $\Phi$. One may avoid the exact calculation, sometimes relatively complicate, in a simple way based on a formula for the power radiated by a charged system described by the charge $\rho$ and current $j$ densities with supports included in a finite domain $\mathcal{D}$ [1]:

$$\frac{dP}{d\Omega}(\nu, t) = \frac{r^2}{\mu_0 c} \left[ \nu \times \frac{\partial}{\partial t} \mathbf{A}_{\text{rad}}(r, t) \right]^2.$$  (1)

Here the origin $O$ of the coordinates is chosen in the domain $\mathcal{D}$, $\nu = r/r$, $dP/d\Omega$ is related to the flow of the energy detected in the observation point $r$ at large distance $r$ compared with the dimensions of the given charged system. The vector $\mathbf{A}_{\text{rad}}$ is obtained from the retarded potential

$$\mathbf{A}(r, t) = \frac{\mu_0}{4\pi} \int_{\mathcal{D}} \frac{1}{R} \mathbf{j}(r', t - \frac{R}{c})d^3x',$$  (2)

with $R = r - r'$, by retaining only the dominant terms at large distances. A first approximation for this vector is obtained by retaining only the dominant term $1/r$ from the series expansion of $1/R$,

$$\frac{1}{R} = \frac{1}{r} + r' \cdot \left( \nabla \frac{1}{R} \right)_{r'=0} + \ldots = \frac{1}{r} - r' \cdot \nabla \frac{1}{r} + \ldots = \frac{1}{r} + r' \cdot \frac{r}{r^3} + \ldots = \frac{1}{r} + O(1/r^2),$$

with a corresponding definition

$$\widetilde{\mathbf{A}}_{\text{rad}}(r, t) = \frac{\mu_0}{4\pi r} \int_{\mathcal{D}} \mathbf{j}(r', t - \frac{R}{c})d^3x'.$$  (2)
In these relations and in the following ones we denote by \( O(x^n) \) a series of powers of \( x \) beginning with \( x^n \).

Supposing \( r \gg \lambda \), where \( \lambda \) is an arbitrary wave length from the radiation spectrum, such that the observation point is in the wave region, and retaining in equation (1) only the terms having nonzero limits for \( r \to \infty \), one obtains an approximate expression related to the energy flow observed in the point \( r \) and moment \( t \). Rigorously, this is that part of the energy flowing in the neighbourhood of the observation point which contributes to the radiated energy. In the following we assume to work in this wave region.

2. The radiation field

In [1] the equation (1) is justified using the supposed plane wave behaviour of the radiated field but also in [1], in a footnote of the page 229, a rigorous proof is suggested for this. Indeed, this may be done by considering consistently only the terms from \( E \) and \( B \) contributing to the radiation [2].

Denoting by \( t' = t - R/c \) the retarded time, and by

\[
[r] = \rho(r', t'), \quad [j] = j(r', t')
\]

the retarded charge and current densities, we have

\[
B(r, t) = \nabla \times A(r, t) = \frac{\mu_0}{4\pi} \nabla \times \left[ \frac{1}{r} \int_D [j] d^3x' \right] + O(1/r^2)
\]

\[
= \frac{\mu_0}{4\pi} \left( \nabla \frac{1}{r} \right) \times \int_D [j] d^3x' - \frac{1}{r} \int_D \frac{\partial}{\partial t}[j] \times \nabla t' d^3x' \right] + O(1/r^2)
\]

\[
= \frac{\mu_0}{4\pi r} \int_D (\nabla t') \times \frac{\partial}{\partial t}[j] d^3x' + O(1/r^2).
\]

Considering the series expansion of \( R \),

\[
R = r + r' \cdot (\nabla' R)_{r' = 0} + \ldots = r - r' \cdot \nabla r + \ldots = r - \frac{r' \cdot r}{r} + O(1/r)
\]

we have

\[
t' = t - \frac{R}{c} = t - \frac{r}{c} + \frac{1}{c} \nu \cdot r' + O(1/r),
\]

and we may write

\[
\nabla t' = -\frac{1}{c} \nu + \frac{r \cdot r'}{c} \nabla \frac{1}{r} + \frac{1}{c} r' + \ldots = -\frac{1}{c} \nu + O(1/r).
\]

The vector \( B(r, t) \) may be written as

\[
B(r, t) = \frac{\mu_0}{4\pi cr} \int_D \frac{\partial}{\partial t}[j] \times \nu d^3x' + O(1/r^2) = \frac{1}{c} \left( \frac{\partial \tilde{A}_{\text{rad}}}{\partial t} \times \nu \right) + O(1/r^2)
\]
so that the part of \( \mathbf{B} \) contributing to the radiation is, in a first evaluation,

\[
\vec{\mathbf{B}}_{\text{rad}} = \frac{1}{c} \left( \frac{\partial \vec{\mathbf{A}}_{\text{rad}}}{\partial t} \times \nu \right).
\]  

(5)

The electric field \( \mathbf{E} = -\nabla \Phi - \partial \mathbf{A}/\partial t \) with the retarded scalar potential

\[
\Phi(r, t) = \frac{1}{4\pi \varepsilon_0} \int_{\mathcal{D}} \frac{\rho(r', t - R/c)}{R} \, d^3x'
\]

is given by

\[
\mathbf{E}(r, t) = -\frac{1}{4\pi \varepsilon_0} \nabla \left[ \frac{1}{r} \int_{\mathcal{D}} \rho \, d^3x' \right] - \frac{\mu_0}{4\pi} \frac{1}{r} \int_{\mathcal{D}} \frac{\partial [j]}{\partial t} d^3x' + O(1/r^2)
\]

\[
= \frac{1}{4\pi \varepsilon_0} \frac{1}{r} \int_{\mathcal{D}} \frac{\partial [\rho]}{\partial t} \nu d^3x' - \frac{\mu_0}{4\pi} \frac{1}{r} \int_{\mathcal{D}} \frac{\partial [j]}{\partial t} d^3x' + O(1/r^2)
\]

where the equation (3) is considered. Writing the continuity equation in the point \( r' \) at the retarded time \( t - R/c \),

\[
\frac{\partial}{\partial t} \rho(r', t - \frac{R}{c}) + [\nabla' \cdot j(r', \tau)]_{\tau = t - R/c} = 0,
\]

and the relations

\[
\nabla' j(r', t - \frac{R}{c}) = [\nabla' j(r', \tau)]_{\tau = t - R/c} + \frac{\partial}{\partial t} j(r', t - \frac{R}{c}) \cdot \nabla'(t - \frac{R}{c})
\]

\[
= [\nabla' j(r', \tau)]_{\tau = t - R/c} + \frac{1}{c} \nu \cdot \frac{\partial}{\partial t} j(r', t - \frac{R}{c}) + O(1/r),
\]

we have

\[
\mathbf{E}(r, t) = \frac{1}{4\pi \varepsilon_0} \frac{1}{r} \int_{\mathcal{D}} \nu \cdot \nabla'[j] d^3x' + \frac{1}{4\pi \varepsilon_0} \frac{1}{r^2} \int_{\mathcal{D}} \left( \nu \cdot \frac{\partial [j]}{\partial t} \right) d^3x' - \frac{\mu_0}{4\pi} \frac{1}{r} \int_{\mathcal{D}} \frac{\partial [j]}{\partial t} d^3x' + O(1/r^2).
\]

The first integral in the right hand side of the last equation is zero because \( j = 0 \) on the surface of \( \mathcal{D} \). Because \( (\nu \cdot \partial [j]/\partial t) - \partial [j]/\partial t = \nu \times (\nu \times [j]/\partial t) \), we obtain

\[
\mathbf{E}(r, t) = \frac{\mu_0}{4\pi} \frac{1}{r} \int_{\mathcal{D}} \nu \times \left( \nu \times \frac{\partial [j]}{\partial t} \right) d^3x' + O(1/r^2).
\]

As in the case of the magnetic field, we may write, in a first evaluation, the part of \( \mathbf{E} \) contributing to the radiation:

\[
\vec{\mathbf{E}}_{\text{rad}} = \nu \times \left( \nu \times \frac{\partial \vec{\mathbf{A}}_{\text{rad}}}{\partial t} \right).
\]  

(6)

The equations verified by the fields \( \vec{\mathbf{E}}_{\text{rad}} \) and \( \vec{\mathbf{B}}_{\text{rad}} \),

\[
\vec{\mathbf{E}}_{\text{rad}} = c \vec{\mathbf{B}}_{\text{rad}} \times \nu, \quad \vec{\mathbf{B}}_{\text{rad}} = \frac{1}{c} \nu \times \vec{\mathbf{E}}_{\text{rad}}, \quad \varepsilon_0 \vec{\mathbf{E}}_{\text{rad}}^2 = \frac{1}{\mu_0} \vec{\mathbf{B}}_{\text{rad}}^2.
\]
indicate the plane wave-like local structure of the radiation field corresponding to the radial direction $\mathbf{v}$ of propagation (these relations are valid also for the radiated fields $E_{\text{rad}}$ and $B_{\text{rad}}$ obtained from $E$ and $B$ by retaining only the terms with $1/r$).

Supposing that all the field variables are real functions, the Poynting vector of the radiated field is

$$
S_{\text{rad}} = \frac{1}{\mu_0} E_{\text{rad}} \times B_{\text{rad}} = \varepsilon_0 E_{\text{rad}}^2 = \frac{1}{\mu_0} B_{\text{rad}}^2
$$
or

$$
S_{\text{rad}} = \frac{1}{\mu_0 c} \left( \mathbf{v} \times \frac{\partial}{\partial t} A_{\text{rad}} \right)^2 \mathbf{v}.
$$

Let the sphere of the radius $r$ with the center in $O$. The radiated energy $\delta \Delta W_{\text{rad}}$ passing through the surface element $\Delta \sigma$, centered on the point $r$ and corresponding to the solid angle $\Delta \Omega$, in the time interval $(t, t + \delta t)$ is defined by

$$
\delta \Delta W_{\text{rad}} = \mathbf{v} \cdot S_{\text{rad}} r^2 \Delta \Omega \delta t.
$$

From the last equation one sees that the angular distribution of the radiation power is given by the equation (7).

3. The radiation of the point electric charge

Usually, one derives the angular distribution of the power radiated by a point electric charge $q$ using the results for the fields $E$ and $B$ obtained from the Liénard-Wiechert potentials and retaining from the corresponding expressions only the terms contributing to the radiation [1]-[3]. Here we illustrate the simplicity of the calculation using in this case the formula (1).

For the sake of completeness we remind a concise introduction of the Liénard-Wiechert potentials [4]. Considering that the motion of the point charge $q$ is given by the law $t \rightarrow \xi(t)$, the charge and current densities are represented as

$$
\rho(r, t) = q \delta[r - \xi(t)], \quad j(r, t) = q \mathbf{v}(t) \delta[r - \xi(t)]
$$

where $\delta$ is the Dirac function. The retarded vector potential $A$ is represented by the integral

$$
A(r, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(r', t - |r - r'|/c) \, d^3 x'}{|r - r'|} = \frac{\mu_0}{4\pi} \int \, d^3 x' \int_{-\infty}^{+\infty} dt' \frac{\mathbf{j}(r', t') \, \delta[t' - t + |r - r'|/c]}{|r - r'|}
$$

$$
= \frac{\mu_0 q}{4\pi} \int_{-\infty}^{+\infty} dt' \frac{\mathbf{v}(t')}{R(t')} \delta \left[ t' - t + \frac{R(t')}{c} \right]
$$

where $R(t) = r - \xi(t)$ gives the position of the observation point with respect to the particle at the moment $t$. In the last equation we may use the relation

$$
\delta[f(x)] = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|f'(x_i)|}
$$
expressing the power radiated by electric charged systems

supposing that the equation \( f(x) = 0 \) has \( n \) roots \( x_1, \ldots, x_n \) with \( f'(x_i) = (df/dx)_{x=x_i} \).

Here

\[ f(t') = t' - t + \frac{R(t')}{c} \]  

(8)

and

\[ f'(t') = 1 - \beta(t') \cdot n(t') \]  

(9)

where \( \beta = v/c \) and \( n(t) = R(t)/R(t) \). Because \( v < c \) we have \( f'(t') > 0 \) so that the equation \( f(t') = 0 \) has only one root \( \tau \):

\[ t - \tau - \frac{1}{c} R(\tau) = 0. \]  

(10)

Denoting \( \eta(t) = 1 - \beta(t) \cdot n(t) \), \( s(t) = \eta(t)R(t) \), the vector potential \( A \) is given by the well-known Liénard-Wiechert expression

\[ A(r, t) = \frac{\mu_0 q}{4\pi} \left( \frac{v}{s} \right) \tau, \quad t - \tau + \frac{R(\tau)}{c} = 0, \]  

(11)

and, obviously now,

\[ \Phi(r, t) = \frac{q}{4\pi \varepsilon_0} \frac{1}{s(\tau)}. \]

Now let the potential \( \tilde{A}_{\text{rad}} \) corresponding to the field of the particle

\[ \tilde{A}_{\text{rad}}(r, t) = \frac{\mu_0 q}{4\pi} \int j(r', t - R/c) d^3x' = \frac{\mu_0 q}{4\pi} \int_{-\infty}^{+\infty} dt' v(t') \delta \left[ t' - t + \frac{R(t')}{c} \right] \]

such that

\[ \tilde{A}_{\text{rad}}(r, t) = \frac{\mu_0 q}{4\pi} \frac{v(\tau)}{\eta(\tau) r} \]  

(12)

with \( \tau \) defined by the equation \[11\]. The time derivative of \( \tilde{A}_{\text{rad}} \) is obtained by a simple calculation:

\[ \frac{\partial \tilde{A}_{\text{rad}}}{\partial t} = \frac{\mu_0 q}{4\pi r} \left[ \frac{a}{\eta} - \frac{v}{\eta^2} \frac{\partial \eta}{\partial t} \right] \frac{\partial \tau}{\partial t} \]

where \( a \) is the particle acceleration. To obtain \( \partial \tau / \partial t \) we consider the derivative of the equation \[11\]

\[ \frac{\partial \tau}{\partial t} = 1 + \frac{\partial \tau}{\partial t} \left( \frac{v \cdot R}{c R} \right) \]

so that we get

\[ \frac{\partial \tau}{\partial t} = \left( \frac{1}{1 - n \cdot \beta} \right) \tau = \frac{1}{\eta_0}, \]  

(13)

where \( \eta_0 = \eta(\tau) \). So

\[ \frac{\partial \tilde{A}_{\text{rad}}}{\partial t} = \frac{\mu_0 q}{4\pi r} \left[ \frac{a}{\eta} - \frac{v}{\eta^2} \left( -\frac{a \cdot R}{c R} + \frac{v^2}{c R} + \frac{v \cdot n}{R^2} \right) \right] \frac{1}{\eta_0} \]
and we may write
\[ \partial A_{\text{rad}}/\partial t = \mu_0 q \frac{1}{4\pi \eta_0^2} \left[ \eta a + (a \cdot n)\beta \right]_{\tau} + O(1/r^2). \] (14)

We have to calculate the expression \( \nu \times \partial A_{\text{rad}}/\partial t \) in order to introduce it in the equation (1). The observation point being given by its position vector \( r \), it is clear that in the approximation considered here the same result is obtained for any chosen origin \( O \) inside the domain \( D \). The unit vector \( \nu \) may be replaced by the unit vector \( n \) of the direction particle-observation point without changing the result for the angular distribution of the radiated power. This happens because
\[ n = \frac{R}{R} = \nu - (\xi \cdot \nabla) \frac{r}{r} + \ldots = \nu - \frac{\xi \cdot \nu}{r} + \ldots = \nu + O(1/r) \]
and
\[ \nu \times \tilde{A}_{\text{rad}}/\partial t = n \times \tilde{A}_{\text{rad}}/\partial t + O(1/r^2). \]

By a straightforward calculation one obtains
\[ \left( n \times \frac{\partial A_{\text{rad}}}{\partial t} \right)^2 = \left( \frac{\mu_0 q}{4\pi} \right)^2 \frac{1}{r^2 \eta_0^2} \left[ \eta a^2 + 2\eta(n \cdot a)(\beta \cdot a) - (1 - \beta^2)(n \cdot a)^2 \right]_{\tau} + O(1/r^3) \] (15)
and with the equation (15) one gets the equation (73,9) given in [1]:
\[ dP = \frac{q^2}{16\pi^2\varepsilon_0 c^3 \eta^6} \left[ \eta a^2 + 2\eta(n \cdot a)(\beta \cdot a) - (1 - \beta^2)(n \cdot a)^2 \right] d\Omega \] (16)
where all the particle parameters are considered at the retarded moment. Inserting the equation (15) in the equation (7), and casting the terms of \( O(1/r) \), we obtain that part of the energy passing through the surface element \( \Delta \sigma \), in the time interval \( \delta t \), which contributes to the radiation. This is just the radiated part of the energy flowing in the solid angle \( \Delta \Omega \) between the moments \( \tau \) and \( t \). But this energy was emitted by the particle in the time interval \( \delta \tau \) corresponding to the observation interval \( \delta t \). Therefore, writing
\[ \delta W_{\text{rad}} = S_{\text{rad}} \cdot nr^2 \Delta \Omega \frac{\partial t}{\partial \tau} \delta \tau, \]
the factor multiplying \( \delta \tau \) in the right hand side of the last equation represents in fact the radiated part of the energy emitted by the particle in the unit of time (at an arbitrary time) in the direction \( n \) and in the solid angle \( \Delta \Omega \). Because \( \partial t/\partial \tau = \eta(\tau) \) we may write the final result for the angular distribution of the radiation power emitted by the particle as an expression differing from the right hand side of the equation (15) by the factor \( 1/\eta^5 \) instead of \( 1/\eta^6 \) [1].

4. Multipolar expansion of \( A_{\text{rad}} \)

The series expansion of the integrand from the equation (2), by retaining only the \( 1/r \) terms contributing to the radiation, leads finally to the multipole expansion of the radiation field.
Expressing the power radiated by electric charged systems

Let us the Taylor series expansion of a function \(f(R)\),
\[
f(R) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x'_1 \cdots x'_n \partial_{i_1 \cdots i_n} f(R) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} r^n \cdot \nabla^n f(r)
\]
(17)
where
\[
\partial_{i_1 \cdots i_n} = \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_n}}
\]
and \(a^n\) is the \(n\)-fold tensorial product \((a \otimes \cdots \otimes a)_{i_1 \cdots i_n} = a_{i_1} \cdots a_{i_n}\). Denoting by \(T^{(n)}\) an \(n\)th order tensor, \(A^{(n)}||B^{(m)}\) is an \(|n - m|\)th order tensor with the components
\[
(A^{(n)}||B^{(m)})_{i_1 \cdots i_{|n-m|}} = \begin{cases} A_{i_1 \cdots i_{n-m} j_1 \cdots j_m} B_{j_1 \cdots j_m} & n > m \\ A_{j_1 \cdots j_n} B_{j_1 \cdots j_n} & n = m \\ A_{j_1 \cdots j_n} B_{j_1 \cdots j_n i_1 \cdots i_{m-n}} & n < m \end{cases}
\]
By introducing the expansion (17) into the equation (2), we obtain
\[
\tilde{A}_{\text{rad}}(r, t) = \frac{\mu_0}{4\pi r} e_i \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \partial_{i_1 \cdots i_n} \int_{D} x'_1 \cdots x'_n \cdot j_i (r', t - \frac{r}{c}) d^3 x'
\]
(18)
where
\[
a_i^{(n)} = \partial_{i_1 \cdots i_n} \int_{D} x'_1 \cdots x'_n \cdot j_i (r', t - \frac{r}{c}) d^3 x'
\]
(19)
and \(e_i\) are the orthogonal unit vectors along the axes.

In the following we use a generalisation to the dynamic case of a procedure given in [5] in the magnetostatic case. Let the identity
\[
\nabla [x_i j_i (r, t)] = j_i (r, t) + x_i \nabla j_i (r, t).
\]
Considering the continuity equation \(\nabla \cdot j + \partial \rho / \partial t = 0\) we may write
\[
j_i (r, t) = \nabla [x_i j_i (r, t)] + x_i \frac{\partial}{\partial t} \rho (r, t)
\]
(20)
and using this last equation in equation (19), we get
\[
a_i^{(n)} = \partial_{i_1 \cdots i_n} \int_{D} x'_1 \cdots x'_n \nabla' [x'_i j_i (r', t_0)] d^3 x' + \partial_{i_1 \cdots i_n} \int_{D} x'_1 \cdots x'_n x'_i \frac{\partial}{\partial t} \rho (r', t_0) d^3 x'
\]
\[
= - \partial_{i_1 \cdots i_n} \int_{D} x'_i j_i (r', t_0) \cdot \nabla' (x'_1 \cdots x'_n) d^3 x' + \partial_{i_1 \cdots i_n} \int_{D} x'_1 \cdots x'_n x'_i \frac{\partial}{\partial t} \rho (r', t_0) d^3 x'
\]
denoting \(t_0 = t - r/c\) and considering a null surface term because \(j = 0\) on \(\partial D\). Because of the symmetry of the derivative tensor and introducing the \(n\)th order electric multipole tensor
\[
P^{(n)} (t) = \int_{D} r^n \rho (r, t) d^3 x,
\]
(21)
we may write
\[ a_i^{(n)} = - n \partial_{i_1 \cdots i_n} \int_D x'_{i_1} \cdots x'_{i_{n-1}} x'_{i_n} j_{i_n}(r', t_0) d^3 x' + \left( \nabla^n || \frac{d}{dt} P^{(n+1)}(t_0) \right)_i \]
\[ = - n \partial_{i_1 \cdots i_n} \int_D x'_{i_1} \cdots x'_{i_{n-1}} (x'_{i_n} j_{i_n} - x'_{i_n} j_i) d^3 x' - n \partial_{i_1 \cdots i_n} \int_D x'_{i_1} \cdots x'_{i_n} j_i d^3 x' \]
\[ + \left( \nabla^n || \frac{d}{dt} P^{(n+1)}(t_0) \right)_i, \]
that is
\[ a_i^{(n)} = - \frac{n}{n + 1} \varepsilon_{k i n} \partial_{i_1 \cdots i_{n-1}} \int_D x'_{i_1} \cdots x'_{i_{n-1}} (r' \times j) k d^3 x' + \frac{1}{n + 1} \left( \nabla^n || \frac{d}{dt} P^{(n+1)}(t_0) \right)_i. \] (22)

By introducing the “vectorial product” \( T^{(n)} \times a \) as the nth order tensor with components
\[ \left( T^{(n)} \times a \right)_{i_1 \cdots i_n} = \varepsilon_{i_1 i_2 \cdots i_n} T_{i_1 \cdots i_{n-1}} a_j \]
and observing that, particularly,
\[ (b^n \times a)_{i_1 \cdots i_n} = b_{i_1 \cdots i_{n-1}} (b \times a)_{i_n}, \]
we may use in the equation (22) the definition of the nth order magnetic multipolar momentum given in [3]
\[ M^{(n)}(t) = \frac{n}{n + 1} \int_D r^n \times j(r, t) d^3 x. \] (23)

So, the equation (22) may be written as
\[ a^{(n)} = - e_i \varepsilon_{i i_1 k} \partial_{i_1 \cdots i_{n-1}} M_{i_1 \cdots i_{n-1} k}(t_0) + \frac{1}{n + 1} \left( \nabla^n || \frac{d}{dt} P^{(n+1)}(t_0) \right)_i \]
\[ = - \nabla \times \left( \nabla^{n-1} || M^{(n)}(t_0) \right) + \frac{1}{n + 1} \left( \nabla^n || \frac{d}{dt} P^{(n+1)}(t_0) \right). \] (24)

Going back to the expansion (18),
\[ \widetilde{A}_{rad}(r, t) = \frac{\mu_0}{4 \pi r} \nabla \times \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \nabla^{n-1} || M^{(n)}(t - \frac{r}{c}) \]
\[ + \frac{\mu_0}{4 \pi r} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + 1)!} \nabla^n || \frac{\partial}{\partial t} P^{(n+1)}(t - \frac{r}{c}) \]
and, finally,
\[ \widetilde{A}_{rad}(r, t) = \frac{\mu_0}{4 \pi r} \nabla \times \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \nabla^{n-1} || M^{(n)}(t - \frac{r}{c}) \]
\[ + \frac{\mu_0}{4 \pi r} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \nabla^{n-1} || \frac{\partial}{\partial t} P^{(n)}(t - \frac{r}{c}). \] (25)

Now, we extract from the equation (23) the terms contributing to the radiation. Because
\[ \partial_{i_1 \cdots i_n} f(t - \frac{r}{c}) = \frac{(-1)^n}{c^n} \nu_1 \cdots \nu_n \frac{d^n}{dt^n} f(t - \frac{r}{c}) + O(\frac{1}{r}), \]
we have in the first sum of the equation (26):

\[ \nabla \times \left[ \nabla^{n-1} ||M^{(n)}(t - \frac{r}{c})|| \right] = e_i \varepsilon_{ijk} \partial_j \partial_{i_{n-1}...i_1} M_{i_1...i_{n-1}k}(t - \frac{r}{c}) \]

\[ = \frac{(-1)^n}{c^n} e_i \varepsilon_{ijk} \nu_{i_1} ... \nu_{i_{n-1}} \frac{d^n}{dt^n} M_{i_1...i_{n-1}k}(t - \frac{r}{c}) + O(\frac{1}{r}) \]

\[ = \frac{(-1)^{n-1}}{c^n} \left[ \nu^{n-1} || \frac{d^n}{dt^n} M^{(n)}(t - \frac{r}{c}) || \times \nu \right] + O(\frac{1}{r}) \]

obtaining, finally, the formula

\[ A_{rad}(r, t) = \frac{\mu_0}{4\pi c} \sum_{n=1}^{\infty} \frac{1}{n! c^n} \left[ \nu^{n-1} || \frac{d^n}{dt^n} M^{(n)}(t_0) || \right] \times \nu \]

\[ + \frac{\mu_0 c}{4\pi} r \sum_{n=1}^{\infty} \frac{1}{n! c^n} \nu^{n-1} || \frac{d^n}{dt^n} P^{(n)}(t_0) || \]  \hspace{1cm} (26)

This is obtained by the explicit contribution of each multipole to the radiation field.

5. Expressing the radiation by reduced multipolar tensors

For the following calculations it is suitable to write the expansion (26) in the more explicit form

\[ \frac{4\pi r}{\mu_0} A_{rad} = e_i \sum_{n \geq 1} \frac{1}{n! c^n} \varepsilon_{ikl} \left[ \nu^{n-1} || \frac{d^n}{dt^n} M^{(n)}(t_0) || \right] \nu_k + e_i \sum_{n \geq 1} \frac{1}{n! c^n} \left[ \nu^{n-1} || \frac{d^n}{dt^n} P^{(n)}(t_0) || \right] \]

\[ = e_i \sum_{n \geq 1} \frac{1}{n! c^n} \varepsilon_{ikl} \nu_{i_1} ... \nu_{i_{n-1}} \frac{d^n}{dt^n} M_{i_1...i_{n-1}k} + e_i \sum_{n \geq 1} \frac{1}{n! c^n} \nu_{i_1} ... \nu_{i_{n-1}} \frac{d^n}{dt^n} P_{i_1...i_{n-1}i} \]  \hspace{1cm} (27)

In [6, 7] was done a general procedure for the reduction of multipole tensors represented by Cartesian components to symmetric traceless ones in the static and dynamic cases. The transformations implied by this reduction are defined such that the electromagnetic potentials \( A \) and \( \Phi \) are modified only by gauge transformations implying a specific feature of the dynamic case: the redefinitions of the multipole tensors in the lower \( k < n \) orders induced by the reduction of tensors in a given order \( n \). In the present paper this procedure is applied to the radiation field and, obviously, only the vector potential is to be considered.

The reduction of multipole tensors beginning with a given order \( n \) is achieved by the following steps.

1. The reduction of the magnetic \( n \)-th order tensor \( M^{(n)} \), given by the equation (28), to a symmetric tensor \( M^{(n)}_{(sym)} \). Since the magnetic tensor \( M^{(n)} \) is symmetric only...
in the first \( n - 1 \) indices, the reduction to a symmetric one may be performed by the transformation [6]

\[
M_{i_1 \ldots i_n} \rightarrow M_{(\text{sym})i_1 \ldots i_n} = \frac{1}{n} \left[ M_{i_1 \ldots i_n} + M_{i_2 i_1 \ldots i_n} + \ldots + M_{i_n i_1 \ldots i_{n-1}} \right]
\]

\[
= M_{i_1 \ldots i_n} - \frac{1}{n} \sum_{\lambda=1}^{n-1} \left[ M_{i_1 \ldots i_n} - M_{i_{\lambda+1} \ldots i_{n-1}i_n} \right]
\]

\[
= M_{i_1 \ldots i_n} - \frac{1}{n} \sum_{\lambda=1}^{n-1} \left[ M_{i_{\lambda+1} \ldots i_n} - M_{i_1 \ldots i_n} \right]
\]

\[
= M_{i_1 \ldots i_n} - \frac{1}{n} \sum_{\lambda=1}^{n-1} \xi_{i_\lambda n} N_{i_1 \ldots i_{n-1}q}^{(\lambda)}
\]

(28)

where we use the notations

\[
N_{i_1 \ldots i_{n-1}} = \epsilon_{i_{n-1}ps} M_{i_1 \ldots i_{n-2}ps}, \quad f_{(\lambda)}^{(\lambda)} = f_{i_1 \ldots i_{\lambda+1} \ldots i_n}.
\]

(29)

If \( M^{(n)} \) is given by the original definition [28], the \( n - 1 \)th-order tensor \( N^{(n-1)} \) is given by

\[
N_{i_1 \ldots i_{n-1}} = \frac{n}{n+1} \int_D \xi_{i_1} \ldots \xi_{i_{n-2}} \epsilon_{i_{n-1}ps} \xi_p (\xi \times \dot{j})_s d^3 \xi
\]

\[
= \frac{n}{n+1} \int_D \xi_{i_1} \ldots \xi_{i_{n-2}} [\xi \times (\xi \times \dot{j})]_{i_{n-1}} d^3 \xi.
\]

(30)

We write explicitly the modification of the potential \( A_{\text{rad}} \) induced by the substitution [28]:

\[
\frac{4\pi r}{\mu_0} A_{\text{rad}} \rightarrow \frac{4\pi r}{\mu_0} A_{\text{rad}} - \frac{e_i}{n!nc^n} \sum_{\lambda=1}^{n-1} \epsilon_{ikl} \nu_{i_1} \ldots \nu_{i_{n-1}} N_{i_1 \ldots i_{n-1}q}^{(\lambda)}(t_0)
\]

\[
= \frac{4\pi r}{\mu_0} A_{\text{rad}} - \frac{\nu}{n!nc^n} \sum_{\lambda=1}^{n-1} \nu_{i_1 \ldots i_{n-1}} N_{i_1 \ldots i_{n-1}q}^{(\lambda)} + \frac{e_i}{n!nc^n} \sum_{\lambda=1}^{n-1} \nu_{i_1 \ldots i_{n-1}} N_{i_1 \ldots i_{n-1}q}^{(\lambda)}
\]

\[
= \frac{4\pi r}{\mu_0} A_{\text{rad}} + \frac{n-1}{n!nc^n} \left[ \nu^{n-2} \frac{d^n}{dt^n} N^{(n-1)}(t_0) \right] - \frac{n-1}{n!nc^n} \left[ \nu^{n-1} \frac{d^n}{dt^n} N^{(n-1)}(t_0) \right] \nu
\]

or

\[
\frac{4\pi r}{\mu_0} A_{\text{rad}} \rightarrow \frac{4\pi r}{\mu_0} A_{\text{rad}} + \frac{n-1}{n!nc^n} \left[ \nu^{n-2} \frac{d^n}{dt^n} N^{(n-1)}(t_0) \right] + \frac{4\pi r}{\mu_0} \psi(r, t) \nu
\]

(31)

where

\[
\psi(r, t) = -\frac{\mu_0}{4\pi r n!nc^n} \left[ \nu^{n-1} \frac{d^n}{dt^n} N^{(n-1)}(t_0) \right]
\]

(32)

and do not contribute to the fields \( E_{\text{rad}} \) and \( B_{\text{rad}} \) corresponding to a gauge transformation of the potential.
2. The extra-gauge alteration of the vector potential by the transformation (31) may be set off by the transformation of the electric multipolar tensor $P^{(n-1)}$:

$$P^{(n-1)} \rightarrow P^{(n-1)} = P^{(n-1)} - \frac{n - 1}{c^2 n^2} \frac{d}{dt} N^{(n-1)}$$

such that the final transformation of the potential is the gauge transformation

$$A_{\text{rad}} \rightarrow A_{\text{rad}} + \psi \nu.$$  (34)

3. After the reduction of the magnetic tensor $M^{(n)}$ to a symmetric one, we have to perform the reduction to a symmetric traceless tensor $\tilde{M}^{(n)}$. This reduction is achieved by the transformation

$$M_{(\text{sym})i_1...i_n} \rightarrow \tilde{M}_{i_1...i_n} = M_{(\text{sym})i_1...i_n} - \sum_{D(i)} \delta_{i_1i_2} \Lambda_{i_3...i_n}$$

(35)

where $\Lambda^{(n-2)}$ is a symmetric tensor and the sum over $D(i)$ is the sum over all permutations of the symbols $i_1, \ldots, i_n$ which give distinct terms. Applequist [8] has given an explicit formula for expressing the components of the symmetric traceless tensor $\tilde{M}^{(n)}$ in terms of the traces of the tensor $M^{(n)}$, \textit{(the detracer theorem, [8], equation (5.1))}:

$$\tilde{M}_{i_1...i_n} = M_{(\text{sym})i_1...i_n} - \frac{[n/2]}{2^{m(n-2m)!}m!} \sum_{D(i)} \delta_{i_1i_2} \cdots \delta_{i_{2m-1}i_{2m}} M^{(n,m)}_{(\text{sym})i_{2m+1}...i_n}$$

(36)

where $[n/2]$ denotes the integer part of $n/2$ and $M^{(n,m)}_{(\text{sym})i_{2m+1}...i_n}$ the components of the $(n-2m)$th-order tensor obtained from $M_{(\text{sym})}$ by the contractions of $m$ pairs of symbols $i$, and obviously

$$\sum_{D(i)} \delta_{i_1i_2} \cdots \delta_{i_{2m-1}i_{2m}} M^{(n,m)}_{(\text{sym})i_{2m+1}...i_n} = \frac{1}{2(m(n-2m)!)} \sum_{P(i)} \delta_{i_1i_2} \cdots \delta_{i_{2m-1}i_{2m}} M^{(n,m)}_{(\text{sym})i_{2m+1}...i_n}$$

(37)

where the sum over $P(i)$ is the sum over all the permutations of the symbols $i$. Using equation (36) we may give explicitly the components of the tensor $\Lambda^{(n-2)}$:

$$\Lambda_{i_1...i_n} = \sum_{m=1}^{[n/2]} \frac{(-1)^{m-1}}{(2n-1)!m!} \sum_{D(i)} \delta_{i_1i_4} \cdots \delta_{i_{2m-1}i_{2m}} M^{(n,m)}_{(\text{sym})i_{2m+1}...i_n}.$$  (38)

In terms of the tensor $\Lambda$ the modification of $A_{\text{rad}}$ induced by the substitutions (28), (33) and (35) is obtained by a straightforward calculation:

$$\frac{4\pi r}{\mu_0} A_{\text{rad}} \rightarrow \frac{4\pi r}{\mu_0} A_{\text{rad}} + \frac{4\pi r}{\mu_0} \psi \nu - \frac{e_i}{n!c^n} \varepsilon_{ikl} \nu_{i_1} \cdots \nu_{i_{n-1}} \sum_{D(i)} \delta_{i_1i_2} \frac{d^n}{dt^n} \Lambda_{i_3...i_{n-1}k}$$

$$= \frac{4\pi r}{\mu_0} A_{\text{rad}} + \frac{4\pi r}{\mu_0} \psi \nu - \frac{e_i}{n!c^n} \varepsilon_{ikl} \nu_{i_1} \cdots \nu_{i_{n-2}} \frac{d^n}{dt^n} \Lambda_{i_3...i_{n-2}k}$$

$$- \frac{e_i}{2n!c^n} \varepsilon_{ikl} \nu_{i_1} \cdots \nu_{i_{n-3}} \frac{d^n}{dt^n} \Lambda_{i_3...i_{n-3}k}$$

$$= \frac{4\pi r}{\mu_0} A_{\text{rad}} + \frac{4\pi r}{\mu_0} \psi \nu + e_i \frac{(n-1)(n-2)}{2n!c^n} \varepsilon_{ikl} \nu_l \left[ \nu^{n-3} \frac{d^n}{dt^n} \Lambda^{(n-2)} \right]_k.$$
So, the transformation of the potential may be written as
\[
\frac{4\pi r}{\mu_0} \mathbf{A}_{\text{rad}} \rightarrow \frac{4\pi r}{\mu_0} \mathbf{A}_{\text{rad}} + \frac{4\pi r}{\mu_0} \psi \nu + \frac{(n-1)(n-2)}{2n!c^n} \nu \times \left[ \nu^{n-3} \left\| \frac{d^n}{dt^n} \Lambda^{(n-2)} \right\| \right]
\]  
(39)

4. It is a simple matter to see that the last extra-gauge term may be set off by the transformation
\[
\mathbf{M}^{(n-2)} \rightarrow \mathbf{M}'^{(n-2)} = \mathbf{M}^{(n-2)} + \frac{n-2}{2c^2n} \frac{d^2}{dt^2} \Lambda^{(n-2)}.
\]  
(40)

5. This step consists in the reduction of the symmetric nth-order electric multipolar tensor \(\mathbf{P}^{(n)}\) to a symmetric and traceless one by a transformation of the type [33]:
\[
\mathbf{P}_{i_1\ldots i_n} \rightarrow \tilde{\mathbf{P}}_{i_1\ldots i_n} = \mathbf{P}_{i_1\ldots i_n} - \sum_{D(i)} \delta_{i_1i_2} \Pi_{i_3\ldots i_n}
\]  
(41)

where the symmetric tensor \(\Pi^{(n-2)}\) is defined in terms of the traces of the tensor \(\mathbf{P}^{(n)}\) by a relation similar to equation (38). The resulting transformation of \(\mathbf{A}_{\text{rad}}\) is
\[
\frac{4\pi r}{\mu_0} \mathbf{A}_{\text{rad}} \rightarrow \frac{4\pi r}{\mu_0} \mathbf{A}_{\text{rad}} + \frac{4\pi r}{\mu_0} \psi \nu - \frac{e_i}{n!c^{n-1}} \nu_i \ldots \nu_{i_{n-1}} \frac{d^n}{dt^n} \delta_{i_1i_2} \Pi_{i_3\ldots i_{n-1}}
\]  
(42)

In this equation we have \((n-1)\) terms with \(\delta_{ik}\), \(k = 1, \ldots, n - 1\) and \((n-1)(n-2)/2\) terms with \(\delta_{ij}\), \(j, k = 1, 2, \ldots, n - 1\) so that
\[
\frac{4\pi r}{\mu_0} \mathbf{A}_{\text{rad}} \rightarrow \frac{4\pi r}{\mu_0} \mathbf{A}_{\text{rad}} + \frac{4\pi r}{\mu_0} \psi \nu - \frac{n-1}{n!c^{n-1}} \nu_i \ldots \nu_{i_{n-1}} \frac{d^n}{dt^n} \Pi_{i_2\ldots i_{n-1}}
\]  
(43)

that is
\[
\frac{4\pi r}{\mu_0} \mathbf{A}_{\text{rad}} \rightarrow \frac{4\pi r}{\mu_0} \mathbf{A}_{\text{rad}} + \frac{4\pi r}{\mu_0} (\psi + \psi') \nu - \frac{(n-1)(n-2)}{2n!c^{n-1}} \left[ \nu^{n-3} \left\| \frac{d^n}{dt^n} \Pi^{(n-2)} \right\| \right]
\]  
(43)

where
\[
\psi' = -\frac{4\pi r (n-1)}{\mu_0 n!c^{n-1}} \nu^{n-2} \left\| \frac{d^n}{dt^n} \Pi^{(n-2)} \right\|.
\]  
(44)

6. The alteration of the potential represented by the last term in equation (43) is set off by the transformation
\[
\mathbf{P}^{(n-2)} \rightarrow \mathbf{P}'^{(n-2)} + \frac{n-2}{2nc^2} \frac{d^2}{dt^2} \Pi^{(n-2)}
\]  
(45)

which preserves the symmetry properties of \(\mathbf{P}^{(n-2)}\).

By this last transformation (45), the reduction of the multipolar tensors in the given nth-order is achieved. Now, to carry out this procedure to the \(n-1\)th order, we must realize that in this order some tensors was been already modified in order to set off the alterations of the electromagnetic field by the reductions in the nth-order. So, the transformation (43) alters the symmetry properties of the \((n-1)\)th order electric multipole tensor because
\[
\delta \mathbf{P}^{(n-1)} = -\frac{n-1}{c^2n^2} \frac{d}{dt} \mathbf{N}^{(n-1)}(t_0)
\]  
(46)
Expressing the power radiated by electric charged systems

is symmetric only in the first \( n - 2 \) indices. To restore the full symmetry of the \((n-1)\)th-order electric moment, we perform the reduction of \( N^{(n-1)} \) to a symmetric tensor by the transformation

\[
N_{i_1 \ldots i_{n-1}} \rightarrow N_{i_1 \ldots i_{n-1}} - \frac{1}{n-1} \sum_{\lambda=1}^{(n-2)} \left[ N_{i_1 \ldots i_{n-1}} N^{(\lambda)}_{i_1 \ldots i_{n-1} i_{n-1} \lambda} \right]
\]  \hspace{1cm} (47)

By introducing the tensor \( N^{(n-2)} \) with the components

\[
N_{i_1 \ldots i_{n-2}} = \varepsilon_{i_{n-2} p q} N_{i_1 \ldots i_{n-3} p q}
\]  \hspace{1cm} (48)

the transformation (47) may be written as

\[
N_{i_1 \ldots i_{n-1}} \rightarrow N_{i_1 \ldots i_{n-1}} - \frac{1}{n-1} \sum_{\lambda=1}^{(n-2)} \varepsilon_{i_{n_{n-1}} \lambda} N^{(\lambda)}_{i_1 \ldots i_{n-2} q}
\]  \hspace{1cm} (49)

If \( M^{(n)} \) is given by the original definition (23), then we can write

\[
N_{i_1 \ldots i_{n-2}} = \frac{n}{n+1} \int_{D} \xi_{i_{1} \ldots i_{n-3}} (\xi \times j)_{i_{n-2}} d^3 \xi.
\]  \hspace{1cm} (50)

The alteration of the vector potential \( A \) by the transformation (49) is given by

\[
A \rightarrow A - \frac{n-2}{n! n c^2} \nu \times \left[ \nu^{n-3} \frac{d}{dt} N^{(n-2)} (t_0) \right].
\]  \hspace{1cm} (51)

This alteration of \( A \) is set off by the transformation of \( M^{(n-2)} \), given by equation (40),

\[
M^{(n-2)} \rightarrow \frac{d^2}{dt^2} M^{(n-2)} - \frac{n-2}{n^2 (n-1) c^2} \frac{d}{dt} N^{(n-2)}.
\]  \hspace{1cm} (52)

By this transformation the symmetry properties of \( M^{(n-2)} \) are preserved. Particularly, by reducing the \((n-2)\)th-order multipolar tensors, in the case of \( M^{(n-2)} \) we have to achieve only the symmetrisation of the supplementary term from the equation (40).

6. Concluding remarks

The equation (11) is a basic formula in the investigation of the radiation of electric charges distributions. In a course on Electrodynamics the radiation chapter is one of the mains goals of the introduction of Maxwell’s equations. Sometimes it is necessary to give quickly some results illustrating the properties of the electromagnetic radiation. For the economy of this course it is a benefit to avoid some unnecessary intermediate results such as the expressions of the fields \( E \) and \( B \). By exposing this problem in the sections 2 and 3, the present paper is not claimed as an original one but would be warranted at least by concerns of methodological completeness.
We summarize the results of the sections 4 and 5 by the following statements.

(1) The reduction of the magnetic \( n \)th-order multipole tensor to a symmetric traceless one by the transformation (equations (28) and (35))

\[
\tilde{M}_{i_1...i_n} \rightarrow \tilde{M}_{i_1...i_n} = M_{i_1...i_n} - \frac{1}{n} \sum_{\lambda=1}^{n-1} \varepsilon_{i_1...i_n\lambda} M^{(\lambda)}_{i_1...i_{n-1}q} - \sum_{D(i)} \delta_{i_1i_2} \Lambda_{i_3...i_n}
\]  

(53)

together the modifications of the electric \((n-1)\)th-order and of the magnetic \((n-2)\)th-order multipole tensors,

\[
P_{i_1...i_n} \rightarrow \tilde{P}_{i_1...i_n} = \frac{n-1}{c^2 n^2} \tilde{N}_{i_1...i_{n-1}},
\]  

(54)

where we use the super dot notation for the time derivatives and,

\[
M_{i_1...i_{n-2}} \rightarrow M_{i_1...i_{n-2}} + \frac{n-2}{2 c^2 n} \tilde{\Lambda}_{i_1...i_{n-2}},
\]  

(55)

leads to a gauge transformation of the potential \(A_{\text{rad}}\).

We point out that if \(M^{(n)}\) is given by the equation (23), the symmetric traceless tensor \(\tilde{M}^{(n)}\) may be identified with the tensor \(\mathcal{M}^{(n)}\) given by (6)

\[
\mathcal{M}_{i_1...i_n}(t) = \frac{(-1)^n}{(n+1)(2n-1)!!} \sum_{\lambda=1}^{n} \int _D r^{2n+1} [\mathbf{j}(r,t) \times \nabla] \partial_{i_1...i_n}^{(\lambda)} \frac{1}{r} d^3x.
\]  

(56)

(2) The reduction of the electric \( n \)th-order multipole tensor to a symmetric traceless one,

\[
P_{i_1...i_n} \rightarrow \tilde{P} = P_{i_1...i_n} - \sum_{D(i)} \delta_{i_1i_2} \Pi_{i_3...i_n},
\]  

(57)

together with the transformation of the electric \((n-2)\)th-order tensor,

\[
P_{i_1...i_{n-2}} \rightarrow \tilde{P}_{i_1...i_{n-2}} = \frac{n-2}{2 c^2 n} \tilde{\Pi}_{i_1...i_{n-2}},
\]  

(58)

leads also to a gauge transformation of the vector potential \(A_{\text{rad}}\).

We point out also that if \(P^{(n)}\) is given by the equation (24), the symmetric traceless tensor \(\tilde{P}^{(n)}\) may be identified with the tensor \(\mathcal{P}^{(n)}\) given by (9)

\[
\mathcal{P}_{i_1...i_n} = \frac{(-1)^n}{(2n-1)!!} \int _D \rho(r,t) r^{2n+1} \nabla^n \frac{1}{r} d^3x.
\]  

(59)

If we begin the reduction from a given order \( n \), then the results of the reductions of \(P^{(n)}\) and \(P^{(k)}\) are the tensors \(\mathcal{P}^{(n)}\) and \(\mathcal{M}^{(n)}\) given by the equations (59) and (56) but for \( k < n \) the \( k \)th-order reduced multipole tensors may differ from \(\mathcal{P}^{(k)}\) and \(\mathcal{M}^{(k)}\) by terms induced by the procedure of the reductions from the previous steps. These last terms give contributions to the potentials and fields expressed by toroidal moments and mean radii of various orders.

We give here some simple examples of such reductions and we will see how naturally the toroidal moments appear as a result of such an approach.

Let us the reduction of the magnetic and electric multipole tensors begins from the \( \mu \)th and \( \varepsilon \)th orders respectively (generally, considering the multipole’s contributions of the same orders, \( \mu = \varepsilon - 1 \)).
Expressing the power radiated by electric charged systems

For \((\mu, \varepsilon) = (1, 2)\), we have \(\mathbf{M}^{(1)} \rightarrow \tilde{\mathbf{M}}^{(1)} = \mathbf{M}^{(1)}, \mathbf{P}^{(1)} \rightarrow \tilde{\mathbf{P}}^{(1)} = \mathbf{P}^{(1)}, \mathbf{P}^{(2)} \rightarrow \tilde{\mathbf{P}}^{(2)} = \mathbf{P}^{(2)}\). These transformations produce only a gauge transformation of \(\mathbf{A}\).

For \((\mu, \varepsilon) = (2, 3), (3, 4), (4, 5), (5, 6)\) the reductions are given in Appendix. For arbitrary \(\mu\) and \(\varepsilon\), we think this is possible to find a general rule or, at least, to elaborate symbolic computer programs.

In the case \((\mu, \varepsilon) = (2, 3)\),
\[
\mathbf{M}^{(2)} \rightarrow \tilde{\mathbf{M}}^{(2)} = \mathbf{M}^{(2)}, \mathbf{M}^{(1)} \rightarrow \tilde{\mathbf{M}}^{(1)} = \mathbf{M}^{(1)}, \mathbf{P}^{(3)} \rightarrow \tilde{\mathbf{P}}^{(3)} = \mathbf{P}^{(3)}, \mathbf{P}^{(2)} \rightarrow \tilde{\mathbf{P}}^{(2)} = \mathbf{P}^{(2)}
\]
\[
\mathbf{p}^{(1)} \rightarrow \tilde{\mathbf{p}}^{(1)} = \mathbf{p}^{(1)} - \frac{1}{4c^2} \tilde{\mathbf{N}}^{(1)} + \frac{1}{6c^2} \tilde{\mathbf{\Pi}}^{(1)}.
\]
Here, \(\mathbf{N}^{(1)}\) and \(\mathbf{\Pi}^{(1)}\) are given by the equation \((A.10)\) for \(P_{qqi} = 0\), \(N_{qqi} = 0\), \(P_{qppqi} = 0\) that is eliminating the contributions from the orders \(n_\mu > 2\) of the magnetic multipole tensors and from the orders \(n_\varepsilon > 3\) for the electric ones.

Taking into account the continuity equation verified by \(\rho\) and \(\mathbf{j}\), we obtain
\[
\tilde{\mathbf{P}}_i = \mathbf{p}_i - \frac{1}{c^2} \tilde{T}_i,
\]
where
\[
\tilde{T}_i = \frac{1}{10} \int_D \left[ \left( \mathbf{\xi} \cdot \mathbf{j} \right) \xi_i - 2\xi^2 j_i \right] d^3 \xi,
\]
is the toroid dipole tensor \([10-12]\).

In the case \((\mu, \varepsilon) = (3, 4)\) we have the changes
\[
\mathbf{M}^{(3)} \rightarrow \tilde{\mathbf{M}}^{(3)} = \mathbf{M}^{(3)}, \mathbf{M}^{(2)} \rightarrow \tilde{\mathbf{M}}^{(2)} = \mathbf{M}^{(2)}, \mathbf{M}^{(1)} \rightarrow \tilde{\mathbf{M}}^{(1)} = \mathbf{M}^{(1)} + \frac{1}{c^2} \tilde{\Lambda}^{(1)}
\]
\[
\mathbf{p}^{(4)} \rightarrow \tilde{\mathbf{p}}^{(4)} = \mathbf{p}^{(4)}, \mathbf{p}^{(3)} \rightarrow \tilde{\mathbf{p}}^{(3)} = \mathbf{p}^{(3)}, \mathbf{p}^{(2)} \rightarrow \tilde{\mathbf{p}}^{(2)} = \mathbf{p}^{(2)} - \frac{2}{9c^2} \tilde{\mathbf{N}}^{(2)} + \frac{1}{4c^2} \tilde{\mathbf{\Pi}}^{(2)}
\]
\[
\mathbf{p}^{(1)} \rightarrow \tilde{\mathbf{p}}^{(1)} = \mathbf{p}^{(1)} - \frac{1}{4c^2} \tilde{\mathbf{N}}^{(1)} + \frac{1}{6c^2} \tilde{\mathbf{\Pi}}^{(1)}
\]
where
\[
\tilde{\mathbf{N}}_{ij} = \frac{1}{2} (\mathbf{N}_{ij} + \mathbf{N}_{ji})
\]
and \(\Lambda_i, \mathbf{N}_{ij}\) and \(\mathbf{\Pi}_{ij}\) are given by the equations \((A.3), (A.4), (A.7)\) and \((A.8)\) by eliminating the contributions from the orders \(n_\mu > 3\) and \(n_\varepsilon > 4\). In this case one obtains the contribution of the toroidal quadrupol tensor \(T^{(2)}, [10-13]\):
\[
T_{ik} = \frac{1}{42} \int_D \left[ 4(\mathbf{\xi} \cdot \mathbf{j}) \xi_i \xi_k - 5\xi^2 (\xi_i j_k + \xi_k j_i) + 2\xi^2 (\mathbf{\xi} \cdot \mathbf{j}) \delta_{ik} \right] d^3 \xi
\]
having, beside the equation \((61)\),
\[
\tilde{\mathbf{P}}_{ik} = \mathbf{p}_{ik} - \frac{1}{c^2} \tilde{T}_{ik}
\]
and the dipolar magnetic moment modified by a mean-square current radius:
\[
\tilde{\mathbf{M}}_i = \mathbf{M}_i + \frac{1}{c^2} \frac{1}{20} \int_D \xi^2 (\mathbf{\xi} \times \mathbf{j}) d^3 \xi.
\]
In the case \((\mu, \varepsilon) = (4, 5)\) we obtain the following results of the reductions:

\[ \tilde{M}^{(4)} = \tilde{M}^{(4)}, \quad \tilde{M}^{(3)} = \tilde{M}^{(3)}, \]

\[ \tilde{M}^{(2)} = \tilde{M}^{(2)} + \frac{1}{4c^2} \tilde{\Lambda}^{(2)} - \frac{1}{24c^2} \tilde{\nabla}^{(2)} : \]

\[ \tilde{M}_{ik} = M_{ik} + \frac{1}{42c^2} \int_D \xi^2 [\xi_i (\xi \times j)_k + \xi_k (\xi \times j)_i] d^3 \xi, \]

\[ \tilde{M}_i = M_i + \frac{1}{20c^2} \int_D \xi^2 (\xi \times j)_i d^3 \xi; \quad (67) \]

\[ \tilde{P}^{(5)} = \mathcal{P}^{(5)}, \quad \tilde{P}^{(4)} = \mathcal{P}^{(4)}, \]

\[ \tilde{P}^{(3)} = \mathcal{P}^{(3)} - \frac{3}{10c^2} \frac{\tilde{\Xi}^{(3)}}{D} + \frac{3}{10c^2} \frac{\tilde{\Pi}^{(3)}}{D} : \tilde{P}_{ijk} = \mathcal{P}_{ijk} - \frac{1}{c^2} \tilde{T}_{ijk}, \]

\[ \mathcal{T}_{ij} = \frac{1}{60} \int_D \left[ \xi_i^2 \sum_{D(i,j,k)} \delta_{ij} j_k + \xi^2 (\xi \cdot j) \sum_{D(i,j,k)} \delta_{ij} \xi_k + 5 (\xi \cdot j) \xi_i \xi_j \xi_k - 5 \xi^2 \sum_{D(i,j,k)} \xi_i \xi_j \xi_k \right] d^3 \xi, \]

\[ \tilde{P}^{(2)} = \mathcal{P}^{(2)} - \frac{2}{9c^2} \frac{\tilde{\Xi}^{(2)}}{D} + \frac{1}{4c^2} \frac{\tilde{\Pi}^{(2)}}{D} : \mathcal{P}_{ij} = \mathcal{P}_{ij} - \frac{1}{c^2} \tilde{T}_{ij}, \]

\[ \tilde{P}^{(1)} = \mathcal{P}^{(1)} - \frac{1}{4c^2} \frac{\tilde{\Xi}^{(1)}}{D} + \frac{1}{96c^4} \frac{\tilde{\Pi}^{(1)}}{D} : \tilde{P}_i = \mathcal{P}_i - \frac{1}{c^2} \tilde{T}_i - \frac{1}{c^4} \tilde{\Delta}_i, \]

\[ \Delta_i = \frac{1}{1400} \int_D \left[ 10 \xi^2 (\xi \cdot j) \xi_i - 15 \xi^4 j_i \right] d^3 \xi. \]

These results show that one may obtain from the formula \((26)\) the correct representation of the electromagnetic field by the reduced multipolar tensors but introducing these tensors up to a given order \(n\), we obtain separate contributions from some electric toroidal moments and mean \(2n\)—power radii. This was pointed out firstly by Dubovik et al [10-12]. In the present paper we point out that considering the contributions to the electromagnetic field of some toroidal moments, one suppose the reduction of the multipole tensors up to a well defined maximal order \(n\).

We illustrate also this statement by calculating the total power radiated by a system of electric charges and currents.

Let us the total radiation power obtained by integrating the equation \((\Pi)\):

\[ \mathcal{T}_{\mu, \varepsilon} = \frac{1}{\mu_0 c} \int \left( \nu \times \tilde{A} \right)^2_{\mu, \varepsilon} r^2 \, d\Omega(\nu), \quad (68) \]

considering only the contributions of the magnetic and electric multipoles up to the \(\mu\)th and \(\varepsilon\)th orders respectively. Usig equation \((26)\) we may write

\[
\left( \frac{4\pi r}{\mu_0} \right)^2 \left( \nu \times \tilde{A} \right)^2_{\mu, \varepsilon} \\
= \sum_{n=1}^{\mu} \sum_{m=1}^{\mu} \frac{1}{n!m!c^{n+m}} \left[ \left( \nu^{n-1} || \tilde{M}_{n,m}^{(n)} \right) \cdot \left( \nu^{m-1} || \tilde{M}_{n,m}^{(m)} \right) - \left( \nu^n || \tilde{M}_{n,m}^{(n)} \right) \left( \nu^m || \tilde{M}_{n,m}^{(m)} \right) \right] \\
+ \sum_{n=1}^{\varepsilon} \sum_{m=1}^{\varepsilon} \frac{1}{n!m!c^{n+m-2}} \left[ \left( \nu^{n-1} || \tilde{P}_{n,m}^{(n)} \right) \cdot \left( \nu^{m-1} || \tilde{P}_{n,m}^{(m)} \right) - \left( \nu^n || \tilde{P}_{n,m}^{(n)} \right) \left( \nu^m || \tilde{P}_{n,m}^{(m)} \right) \right] \\
+ 2 \sum_{n=1}^{\mu} \sum_{m=1}^{\varepsilon} \frac{1}{n!m!c^{n+m-1}} \left\{ \left( \nu^{n-1} || \tilde{M}_{n,m}^{(n)} \right) \cdot \left( \nu \times \left( \nu^{m-1} || \tilde{P}_{n,m}^{(m)} \right) \right) \right\}. \quad (69)
\]
where
\[ T_{\nu}^{(n)} = \frac{d^k}{dt^k} T^{(n)}. \]

The calculation of the integrals in equation (68) is reduced to the calculation of
\[ <\nu_1 \ldots \nu_n \nu > \nu, \quad n = 0, 1, \ldots \text{ with} \]
\[ f(\nu) > \nu = \frac{1}{4\pi} \int f(\nu) d\Omega(\nu), \]
\[ <\nu_1 \ldots \nu_{2n+1} > \nu = 0, \]
\[ <\nu_1 \ldots \nu_{2n} > \nu = C_n \sum_{D(i)} \delta_{i1i2} \ldots \delta_{i2n-1i2n}, \quad C_n = \frac{1}{(2n + 1)!!} \] (70)

Let us the symmetric traceless tensors \( A^{(n)} \) and \( B^{(m)} \) and the averaged contraction
\[ \left\langle \left( \nu^k || A^{(n)} \right) \left( \nu^{k'} || B^{(m)} \right) \right\rangle_{\nu} = \left\langle \nu_1 \ldots \nu_k \nu_{j1} \ldots \nu_{j_k} A_{i1} \ldots A_{i_k + 1} \ldots i_{n} B_{j1} \ldots j_{k'k} \ldots j_{m} \right\rangle_{\nu}. \]

This is non zero only for the products of \( \delta_{i,j} \) with \( p = 1, \ldots k, \quad q = 1, \ldots k' \) and it is easy to demonstrate the relation
\[ \left\langle \left( \nu^k || A^{(n)} \right) \left( \nu^{k'} || B^{(m)} \right) \right\rangle_{\nu} = \frac{k!}{(2k + 1)!!} \left[ A^{(n)} || B^{(m)} \right] \delta_{k,k'}. \] (71)

The terms of the last sum from the equation (68) give contributions to the total radiated power of the form
\[ \left\langle \nu_1 \ldots \nu_{i_n-1} \nu_{j1} \ldots \nu_{j_{m-1}} \nu_p \right\rangle \varepsilon_{i_n p q} A_{i1} \ldots A_{i_n} B_{j1} \ldots j_{m-1} q \]
but all the terms from the sum of \( \delta - \)products representing the averaged products of \( \nu \)’s contain either \( \delta_{i,j} \) or \( \delta_{j,i} \), \( k = 1, \ldots, n - 1, \quad l = 1, \ldots, m - 1 \) such that, because of \( \varepsilon_{i_n p q} \) and of the traceless character of \( A \) and \( B \), the result is zero. Using these results in equations (68) and (69) we obtain
\[ I_{\mu,\varepsilon} = \frac{1}{4\pi \varepsilon_0 c^2} \left[ \sum_{n=1}^{\mu} \frac{n + 1}{n!} \frac{1}{2^n} \left[ \tilde{M}^{(n)}_{n+1} || \tilde{M}^{(n)}_{n+1} \right] + \sum_{n=1}^{\varepsilon} \frac{n + 1}{n!} \frac{1}{2^n} \left[ \tilde{P}^{(n)}_{n+1} || \tilde{P}^{(n)}_{n+1} \right] \right] \] (72)

For comparison with results existing in literature [1,10-12] we write here the results in the following cases. The case \( (\mu, \varepsilon) = (1, 2) \) is given in [11]:
\[ I_{1,2} = \frac{1}{4\pi \varepsilon_0} \left[ \frac{2}{3} \tilde{P}^{(1)}_{2} || \tilde{P}^{(1)}_{2} + \frac{1}{20c^2} \tilde{P}^{(2)}_{3} || \tilde{P}^{(2)}_{3} + \frac{2}{3c^2} \tilde{M}^{(1)}_{2} || \tilde{M}^{(1)}_{2} \right] \]
\[ = \frac{1}{4\pi \varepsilon_0 c^2} \left[ \frac{2}{3} P^{2} + \frac{2}{3c^2} m^2 + \frac{1}{20c^2} P^{(2)} || P^{(2)} \right], \] (73)
this result being justified by the invariance of the radiation field to the transformation \( P^{(2)} \rightarrow P^{(2)} \).

In the case \( (\mu, \varepsilon) = (2, 3) \) we obtain
\[ I_{2,3} = \frac{1}{4\pi \varepsilon_0} \left[ \frac{2}{3} \tilde{P}^{(1)}_{2} || \tilde{P}^{(1)}_{2} + \frac{1}{20c^2} \tilde{P}^{(2)}_{3} || \tilde{P}^{(2)}_{3} + \frac{2}{945c^4} \tilde{P}^{(3)}_{4} || \tilde{P}^{(3)}_{4} + \frac{2}{3c^2} \tilde{M}^{(1)}_{2} || \tilde{M}^{(1)}_{2} + \frac{1}{20c^2} \tilde{M}^{(2)}_{3} || \tilde{M}^{(2)}_{3} \right] \]
Expressing the power radiated by electric charged systems

\[
I = \frac{1}{4\pi \varepsilon_0 c^4} \left[ \frac{2}{3} \tilde{p} - \frac{1}{c^2} \tilde{T} \right]^2 + \frac{2}{3c^2} \tilde{m}^2 + \frac{1}{20c^2} \tilde{P}^{(2)} \parallel \tilde{P}^{(2)} \\
+ \frac{1}{20c^4} \tilde{M}^{(2)} \parallel \tilde{M}^{(2)} + \frac{2}{945c^4} \tilde{P}^{(3)} || \tilde{P}^{(3)} \right] \tag{74}
\]

In the case \((\mu, \varepsilon) = (3, 4), \)

\[
I_{3,4} = \frac{1}{4\pi \varepsilon_0 c^4} \left[ \frac{2}{3} \tilde{P}^{(1)} || \tilde{P}^{(1)} + \frac{1}{20c^2} \tilde{P}^{(3)} || \tilde{P}^{(3)} + \frac{2}{945c^4} \tilde{P}^{(4)} || \tilde{P}^{(4)} \right] \\
+ \frac{2}{3c^4} \tilde{M}^{(1)} || \tilde{M}^{(1)} + \frac{1}{20c^4} \tilde{M}^{(3)} || \tilde{M}^{(3)} + \frac{2}{945c^4} \tilde{M}^{(4)} || \tilde{M}^{(4)} \right] \\
= \frac{1}{4\pi \varepsilon_0 c^4} \left[ \frac{2}{3} \left( \tilde{p} - \frac{1}{c^2} \tilde{T} \right)^2 + \frac{2}{3c^2} \left( \tilde{m} + \frac{1}{c^2} \frac{d^4}{dt^4} \Lambda \right)^2 \\
+ \frac{1}{20} \left( \tilde{P}^{(2)} - \frac{1}{c^2} \frac{d^4}{dt^4} \tilde{T}^{(2)} \right) || \left( \tilde{P}^{(2)} - \frac{1}{c^2} \frac{d^4}{dt^4} \tilde{T}^{(2)} \right) + \frac{1}{20} \tilde{M}^{(2)} || \tilde{M}^{(2)} \\
+ \frac{2}{945c^4} \left( \frac{d^4}{dt^4} \tilde{P}^{(3)} || \frac{d^4}{dt^4} \tilde{P}^{(3)} + \frac{1}{c^2} \frac{d^4}{dt^4} \tilde{M}^{(3)} || \frac{d^4}{dt^4} \tilde{M}^{(3)} \right) \\
+ \frac{1}{18144 c^6} \frac{d^5}{dt^5} \tilde{P}^{(4)} || \frac{d^5}{dt^5} \tilde{P}^{(4)} \right] \tag{75}
\]

Appendix A. Reduction of multipole tensors

In this Appendix we give in a diagram form the reductions of multipole tensors for the cases \((\mu, \varepsilon) = (2, 3), (3, 4), (4, 5), (5, 6). \)

\((\mu, \varepsilon) = (2, 3)\)

\[\begin{array}{c}
(\mu, \varepsilon) = (2, 3) \\
\downarrow \\
\begin{array}{c}
M^{(2)} \\
\downarrow \\
N^{(1)} \rightarrow \text{Eq. 39, } n=2 \Rightarrow \tilde{p} \rightarrow \tilde{p} - \frac{1}{4c^2} \tilde{N}^{(1)}
\end{array}
\end{array} \]

\[\begin{array}{c}
\begin{array}{c}
M^{(2)} = M^{(2)}_{\text{(sym)}} \\
\downarrow \\
\Pi^{(1)} \rightarrow \text{Eq. 40, } n=3 \Rightarrow \tilde{p}^{(1)} - \frac{1}{4c^2} \tilde{N}^{(1)} \rightarrow \tilde{p}^{(1)} - \frac{1}{4c^2} \tilde{N}^{(1)} + \frac{1}{6c^2} \tilde{\Pi}^{(1)}
\end{array}
\end{array} \]

\[\begin{array}{c}
P^{(3)} \\
\downarrow \\
\Pi^{(1)} \rightarrow \text{Eq. 41, } n=4 \Rightarrow \tilde{p}^{(1)} - \frac{1}{4c^2} \tilde{N}^{(1)} \rightarrow \tilde{p}^{(1)} - \frac{1}{4c^2} \tilde{N}^{(1)} + \frac{1}{6c^2} \tilde{\Pi}^{(1)}
\end{array} \]
Expressing the power radiated by electric charged systems

\[(\mu, \varepsilon) = (3, 4)\]

\[
\begin{align*}
\mathbf{M}^{(3)} & \rightarrow \mathbf{P}^{(2)} - \frac{2}{9c^2} \dot{\mathbf{N}}^{(2)} + \frac{1}{4c^2} \ddot{\mathbf{\Pi}}^{(2)} \quad \text{Eq. (33), } n=3 \\
\mathbf{M}_{(\text{sym})}^{(3)} & \rightarrow \mathbf{M}^{(1)} + \frac{1}{6c^2} \dddot{\Lambda}^{(1)} \quad \text{Eq. (40), } n=3 \\
\mathbf{M}^{(2)} & \rightarrow \mathbf{M}^{(1)} + \frac{1}{6c^2} \dddot{\Lambda}^{(1)} - \frac{1}{18c^2} \dddot{\mathbf{N}}^{(1)} \quad \text{Eq. (52), } n=3 \\
\mathbf{M}^{(2)} & \rightarrow \mathbf{P}^{(1)} - \frac{1}{4c^2} \dddot{\mathbf{N}}^{(1)} + \frac{1}{6c^2} \dddot{\mathbf{\Pi}}^{(1)} \quad \text{Eq. (45), } n=4 \\
\mathbf{P}^{(1)} & \rightarrow \mathbf{P}^{(1)} - \frac{1}{4c^2} \dddot{\mathbf{N}}^{(1)} + \frac{1}{6c^2} \dddot{\mathbf{\Pi}}^{(1)} \quad \text{Eq. (45), } n=3 \\
\mathbf{P}^{(2)} & \rightarrow \mathbf{P}^{(2)} - \frac{2}{9c^2} \dot{\mathbf{N}}^{(2)} + \frac{1}{4c^2} \ddot{\mathbf{\Pi}}^{(2)} \quad \text{Eq. (33), } n=2 \\
\mathbf{M}^{(1)} & \rightarrow \mathbf{M}^{(1)} + \frac{1}{6c^2} \dddot{\Lambda}^{(1)} + \frac{1}{18c^2} \dddot{\mathbf{N}}^{(1)} \quad \text{Eq. (52), } n=2
\end{align*}
\]
Expressing the power radiated by electric charged systems

\[(\mu, \varepsilon) = (4, 5)\]

\[\mathbf{M}^{(4)} \downarrow\]

\[\mathbf{N}^{(3)} \downarrow\]

\[\mathbf{M}^{(4)}_{(\text{sym})} \downarrow\]

\[\Lambda^{(2)} \downarrow\]

\[\mathbf{M}^{(4)} \downarrow\]

\[\mathbf{M}^{(3)} \downarrow\]

\[\mathbf{N}^{(2)} \downarrow\]

\[\mathbf{M}^{(3)}_{(\text{sym})} \downarrow\]

\[\Lambda^{(1)} \downarrow\]

\[\mathbf{M}^{(2)} + \frac{1}{4c^2} \tilde{\Lambda}^{(2)} \downarrow\]

\[\mathbf{N}^{(1)} \downarrow\]

\[(\mathbf{M}^{(2)}_{(\text{sym})} + \frac{1}{4c^2} \tilde{\Lambda}^{(2)}) \downarrow\]

\[\mathbf{P}^{(5)} \downarrow\]

\[\mathbf{P}^{(5)} \downarrow\]

\[\mathbf{P}^{(4)} \downarrow\]

\[\mathbf{P}^{(4)} \downarrow\]

\[\mathbf{P}^{(3)} - \frac{3}{16c^2} \dot{\mathbf{N}}^{(3)} + \frac{3}{10c^2} \ddot{\Pi}^{(3)} \downarrow\]

\[\Lambda^{(2)} \downarrow\]

\[\mathbf{P}^{(3)} - \frac{3}{16c^2} \dot{\mathbf{N}}^{(3)}_{\text{sym}} + \frac{3}{10c^2} \ddot{\Pi}^{(3)} \downarrow\]

\[\Pi^{(1)} \downarrow\]

\[\mathbf{P}^{(3)} - \frac{3}{16c^2} \dot{\mathbf{N}}^{(3)} + \frac{3}{10c^2} \ddot{\Pi}^{(3)} \downarrow\]

\[\mathbf{M}^{(2)} \downarrow\]

\[\mathbf{M}^{(2)} \downarrow\]

\[\mathbf{M}^{(3)} \downarrow\]

\[\mathbf{M}^{(4)} \downarrow\]

\[\mathbf{M}^{(4)} \downarrow\]

\[\Lambda^{(2)} \downarrow\]

\[\mathbf{M}^{(4)} \downarrow\]

\[\mathbf{M}^{(3)} \downarrow\]

\[\mathbf{M}^{(3)}_{(\text{sym})} \downarrow\]

\[\Lambda^{(1)} \downarrow\]

\[\mathbf{M}^{(2)} + \frac{1}{4c^2} \tilde{\Lambda}^{(2)} \downarrow\]

\[\mathbf{N}^{(1)} \downarrow\]

\[(\mathbf{M}^{(2)}_{(\text{sym})} + \frac{1}{4c^2} \tilde{\Lambda}^{(2)}) \downarrow\]

\[\mathbf{P}^{(5)} \downarrow\]

\[\mathbf{P}^{(5)} \downarrow\]

\[\mathbf{P}^{(4)} \downarrow\]

\[\mathbf{P}^{(4)} \downarrow\]

\[\mathbf{P}^{(3)} - \frac{3}{16c^2} \dot{\mathbf{N}}^{(3)} + \frac{3}{10c^2} \ddot{\Pi}^{(3)} \downarrow\]

\[\mathbf{M}^{(2)} \downarrow\]

\[\mathbf{M}^{(2)} \downarrow\]

\[\mathbf{M}^{(3)} \downarrow\]

\[\mathbf{M}^{(4)} \downarrow\]

\[\mathbf{M}^{(4)} \downarrow\]

\[\Lambda^{(2)} \downarrow\]

\[\mathbf{M}^{(4)} \downarrow\]

\[\mathbf{M}^{(3)} \downarrow\]

\[\mathbf{M}^{(3)}_{(\text{sym})} \downarrow\]

\[\Lambda^{(1)} \downarrow\]

\[\mathbf{M}^{(2)} + \frac{1}{4c^2} \tilde{\Lambda}^{(2)} \downarrow\]

\[\mathbf{N}^{(1)} \downarrow\]

\[(\mathbf{M}^{(2)}_{(\text{sym})} + \frac{1}{4c^2} \tilde{\Lambda}^{(2)}) \downarrow\]

\[\mathbf{P}^{(5)} \downarrow\]

\[\mathbf{P}^{(5)} \downarrow\]

\[\mathbf{P}^{(4)} \downarrow\]

\[\mathbf{P}^{(4)} \downarrow\]

\[\mathbf{P}^{(3)} - \frac{3}{16c^2} \dot{\mathbf{N}}^{(3)} + \frac{3}{10c^2} \ddot{\Pi}^{(3)} \downarrow\]

\[\mathbf{M}^{(2)} + \frac{1}{4c^2} \tilde{\Lambda}^{(2)} \downarrow\]

\[\mathbf{M}^{(2)} \downarrow\]

\[\mathbf{M}^{(3)} \downarrow\]

\[\mathbf{M}^{(4)} \downarrow\]

\[\mathbf{M}^{(4)} \downarrow\]

\[\Lambda^{(2)} \downarrow\]

\[\mathbf{M}^{(4)} \downarrow\]

\[\mathbf{M}^{(3)} \downarrow\]

\[\mathbf{M}^{(3)}_{(\text{sym})} \downarrow\]

\[\Lambda^{(1)} \downarrow\]

\[\mathbf{M}^{(2)} + \frac{1}{4c^2} \tilde{\Lambda}^{(2)} \downarrow\]

\[\mathbf{N}^{(1)} \downarrow\]

\[(\mathbf{M}^{(2)}_{(\text{sym})} + \frac{1}{4c^2} \tilde{\Lambda}^{(2)}) \downarrow\]

\[\mathbf{P}^{(5)} \downarrow\]

\[\mathbf{P}^{(5)} \downarrow\]

\[\mathbf{P}^{(4)} \downarrow\]

\[\mathbf{P}^{(4)} \downarrow\]

\[\mathbf{P}^{(3)} - \frac{3}{16c^2} \dot{\mathbf{N}}^{(3)} + \frac{3}{10c^2} \ddot{\Pi}^{(3)} \downarrow\]

\[\mathbf{M}^{(2)} \downarrow\]

\[\mathbf{M}^{(2)} \downarrow\]

\[\mathbf{M}^{(3)} \downarrow\]

\[\mathbf{M}^{(4)} \downarrow\]

\[\mathbf{M}^{(4)} \downarrow\]

\[\Lambda^{(2)} \downarrow\]

\[\mathbf{M}^{(4)} \downarrow\]

\[\mathbf{M}^{(3)} \downarrow\]

\[\mathbf{M}^{(3)}_{(\text{sym})} \downarrow\]

\[\Lambda^{(1)} \downarrow\]

\[\mathbf{M}^{(2)} + \frac{1}{4c^2} \tilde{\Lambda}^{(2)} \downarrow\]

\[\mathbf{N}^{(1)} \downarrow\]

\[(\mathbf{M}^{(2)}_{(\text{sym})} + \frac{1}{4c^2} \tilde{\Lambda}^{(2)}) \downarrow\]

\[\mathbf{P}^{(5)} \downarrow\]

\[\mathbf{P}^{(5)} \downarrow\]

\[\mathbf{P}^{(4)} \downarrow\]

\[\mathbf{P}^{(4)} \downarrow\]

\[\mathbf{P}^{(3)} - \frac{3}{16c^2} \dot{\mathbf{N}}^{(3)} + \frac{3}{10c^2} \ddot{\Pi}^{(3)} \downarrow\]

\[\mathbf{M}^{(2)} \downarrow\]

\[\mathbf{M}^{(2)} \downarrow\]

\[\mathbf{M}^{(3)} \downarrow\]

\[\mathbf{M}^{(4)} \downarrow\]

\[\mathbf{M}^{(4)} \downarrow\]

\[\Lambda^{(2)} \downarrow\]

\[\mathbf{M}^{(4)} \downarrow\]

\[\mathbf{M}^{(3)} \downarrow\]

\[\mathbf{M}^{(3)}_{(\text{sym})} \downarrow\]

\[\Lambda^{(1)} \downarrow\]

\[\mathbf{M}^{(2)} + \frac{1}{4c^2} \tilde{\Lambda}^{(2)} \downarrow\]

\[\mathbf{N}^{(1)} \downarrow\]

\[(\mathbf{M}^{(2)}_{(\text{sym})} + \frac{1}{4c^2} \tilde{\Lambda}^{(2)}) \downarrow\]

\[\mathbf{P}^{(5)} \downarrow\]

\[\mathbf{P}^{(5)} \downarrow\]

\[\mathbf{P}^{(4)} \downarrow\]

\[\mathbf{P}^{(4)} \downarrow\]

\[\mathbf{P}^{(3)} - \frac{3}{16c^2} \dot{\mathbf{N}}^{(3)} + \frac{3}{10c^2} \ddot{\Pi}^{(3)} \downarrow\]
Expressing the power radiated by electric charged systems

\[ P(2) - \frac{2}{3} c^2 \dot{N}(2) + \frac{1}{144} c^2 \ddot{\Pi}(2) \]

Eq. (52), \( n = 3 \)

\[ M(1) \rightarrow P(1) \]

\[ M(1) + \frac{1}{6} \dot{N}(1) + \frac{1}{144} c^2 \ddot{\Pi}(1) + \text{term} \]

Eq. (33), \( n = 2 \)
Expressing the power radiated by electric charged systems

\[(\mu, \varepsilon) = (5, 6)\]

\[\begin{align*}
M(5) & \quad \text{Eq. (33), } n = 5 \\
N(4) & \quad P(4) \rightarrow P(4) - \frac{4}{25c^2} \dot{N}(4)
\end{align*}\]

\[\begin{align*}
M_{(sym)}(5) & \quad \text{Eq. (40), } n = 5 \\
\Lambda(3) & \quad M(3) \rightarrow M(3) + \frac{3}{10c^2} \dot{\Lambda}(3)
\end{align*}\]

\[\begin{align*}
\mathcal{M}(5) & \quad \text{Eq. (40), } n = 5 \\
M(4) & \quad M(2) \rightarrow M(2) + \frac{1}{4c^2} \dot{\Lambda}(2)
\end{align*}\]

\[\begin{align*}
N(4) & \quad \text{Eq. (33), } n = 4 \\
M_{(sym)}(4) & \quad P(3) \rightarrow P(3) - \frac{3}{16c^2} \dot{N}(3)
\end{align*}\]

\[\begin{align*}
\Lambda(2) & \quad \text{Eq. (40), } n = 4 \\
\mathcal{M}(4) & \quad M(1) \rightarrow M(1) + \frac{1}{6c^2} \dot{\Lambda}(1)
\end{align*}\]

\[\begin{align*}
N(3) & \quad \text{Eq. (33), } n = 3 \\
M_{(sym)}(3) & \quad P(2) \rightarrow P(2) - \frac{2}{9c^2} \dot{N}(2)
\end{align*}\]

\[\begin{align*}
\Lambda(1) & \quad \text{Eq. (40), } n = 3 \\
\mathcal{M}(3) & \quad M(1) \rightarrow M(1) + \frac{1}{6c^2} \dot{\Lambda}(1)
\end{align*}\]

\[\begin{align*}
N(2) & \quad \text{Eq. (33), } n = 2 \\
M_{(sym)}(2) & \quad P(1) \rightarrow P(1) - \frac{1}{4c^2} \dot{N}(1)
\end{align*}\]

\[\begin{align*}
\Lambda(0) & \quad \text{Eq. (40), } n = 2 \\
\mathcal{M}(2) & \quad M(1) \rightarrow M(1) + \frac{1}{6c^2} \dot{\Lambda}(1)
\end{align*}\]

\[\begin{align*}
N(1) & \quad \text{Eq. (33), } n = 1 \\
M_{(sym)}(1) & \quad P(1) \rightarrow P(1) - \frac{1}{4c^2} \dot{N}(1)
\end{align*}\]

\[\begin{align*}
\Lambda(0) & \quad \text{Eq. (40), } n = 1 \\
\mathcal{M}(1) & \quad M(1) \rightarrow M(1) + \frac{1}{6c^2} \dot{\Lambda}(1)
\end{align*}\]

\[\begin{align*}
N(0) & \quad \text{Eq. (33), } n = 0 \\
M_{(sym)}(0) & \quad P(1) \rightarrow P(1) - \frac{1}{4c^2} \dot{N}(1)
\end{align*}\]
Expressing the power radiated by electric charged systems

\[ P^{(4)} - \frac{1}{25c^2} \dot{N}^{(4)} + \frac{1}{3c^2} \ddot{\Pi}^{(4)} \]

\[ \mathcal{M}^{(n=5)} \rightarrow \mathcal{M}^{(3)} \rightarrow \mathcal{M}^{(3)} + \frac{3}{10c^2} \dot{\Lambda}^{(3)} - \frac{3}{100c^2} \dot{\dot{\Lambda}}^{(3)} \]

\[ \mathcal{P}^{(4)} = \frac{4}{25c^2} \dot{N}^{(4)} + \frac{1}{3c^2} \ddot{\Pi}^{(4)} \]

\[ \mathcal{P}^{(2)} - \frac{2}{9c^2} \dot{N}^{(2)} \rightarrow \mathcal{P}^{(2)} - \frac{2}{9c^2} \dot{N}^{(2)} + \frac{1}{4c^2} \ddot{\Pi}^{(2)} \]

\[ \mathcal{P}^{(4)} - \frac{4}{25c^2} \dot{N}^{(4)} + \frac{1}{3c^2} \ddot{\Pi}^{(4)} \]

\[ \mathcal{P}^{(2)} \rightarrow \mathcal{P}^{(2)} - \frac{2}{9c^2} \dot{N}^{(2)} \rightarrow \mathcal{P}^{(2)} - \frac{2}{9c^2} \dot{N}^{(2)} + \frac{1}{4c^2} \ddot{\Pi}^{(2)} \]

\[ \mathcal{P}^{(4)} - \frac{4}{25c^2} \dot{N}^{(4)} + \frac{1}{3c^2} \ddot{\Pi}^{(4)} \]

\[ \mathcal{P}^{(2)} \rightarrow \mathcal{P}^{(2)} - \frac{2}{9c^2} \dot{N}^{(2)} \rightarrow \mathcal{P}^{(2)} - \frac{2}{9c^2} \dot{N}^{(2)} + \frac{1}{4c^2} \ddot{\Pi}^{(2)} \]

\[ \mathcal{P}^{(4)} - \frac{4}{25c^2} \dot{N}^{(4)} + \frac{1}{3c^2} \ddot{\Pi}^{(4)} \]

\[ \mathcal{P}^{(2)} \rightarrow \mathcal{P}^{(2)} - \frac{2}{9c^2} \dot{N}^{(2)} \rightarrow \mathcal{P}^{(2)} - \frac{2}{9c^2} \dot{N}^{(2)} + \frac{1}{4c^2} \ddot{\Pi}^{(2)} \]

\[ \mathcal{P}^{(4)} - \frac{4}{25c^2} \dot{N}^{(4)} + \frac{1}{3c^2} \ddot{\Pi}^{(4)} \]

\[ \mathcal{P}^{(2)} \rightarrow \mathcal{P}^{(2)} - \frac{2}{9c^2} \dot{N}^{(2)} \rightarrow \mathcal{P}^{(2)} - \frac{2}{9c^2} \dot{N}^{(2)} + \frac{1}{4c^2} \ddot{\Pi}^{(2)} \]
Expressing the power radiated by electric charged systems

\[ \mathcal{P}^{(2)} = \frac{2}{9c^2} \mathcal{N}^{(2)} + \frac{1}{4c^2} \mathcal{\Pi}^{(2)} + \frac{1}{150c^2} \dot{\mathcal{N}}^{(2)} \]

In these diagrams are used the following notations:

\[ \Lambda_{ijk} = \frac{1}{15} \sum_{D(i)} M_{qqijk} - \frac{1}{14 \times 45} \sum_{D(i)} \delta_{ij} M_{qqppk} \]

\[ \Lambda_i = \frac{1}{15} M_{qqi} + \frac{3}{5 \times 700} \dot{M}_{qqppi}, \quad N_i = \frac{2}{3} \int [\boldsymbol{\xi} \times (\boldsymbol{\xi} \times \boldsymbol{j})]_i d^3 \xi, \]

\[ \Pi_{ijkl} = \frac{1}{11} P_{qqijkl} - \frac{1}{18 \times 11} \sum_{D(i)} \delta_{ij} P_{qqppkl} + \frac{1}{21 \times 99} \sum_{D(i)} \delta_{ij} \delta_{kl} P_{qqpprr} \]

\[ \Pi_{ijk} = \frac{1}{9} P_{qqijk} - \frac{1}{9 \times 14} \sum_{D(i)} \delta_{ij} P_{qqppk}, \quad N_{ijk} = -\frac{5}{6} \int \xi^2 \xi_i \xi_j (\boldsymbol{\xi} \times \boldsymbol{j})_k d^3 \xi \]

\[ \Pi_{ij} = \left[ \frac{1}{7} P_{qqij} - \frac{1}{70} \delta_{ij} P_{qqpp} \right] - \frac{4}{25c^2} \left[ \frac{1}{7} \dot{N}_{qqij} + \dot{\mathcal{N}}_{qqjj} \right] - \frac{1}{3c^2} \left[ \frac{1}{7} \dot{M}_{qqij} - \frac{1}{70} \delta_{ij} \dot{M}_{qqpp} \right] \]

\[ \mathcal{N}_{ik} = -\frac{4}{5} \int \xi^2 \xi_i (\boldsymbol{\xi} \times \boldsymbol{j})_k d^3 \xi, \quad \mathcal{N}_i = -\frac{3}{4} \int \xi^2 (\boldsymbol{\xi} \times \boldsymbol{j})_i d^3 \xi, \quad \tilde{\Pi}_{ij} = \Pi_{ij} - \frac{1}{3} \delta_{ij} \Pi_{kk} \]

\[ \mathcal{N}'_{ik} = -\frac{5}{6} \int \xi^2 \xi_i [\boldsymbol{\xi} \times (\boldsymbol{\xi} \times \dot{\boldsymbol{j}})]_k d^3 \xi, \quad \Lambda'_k = \frac{1}{15} \mathcal{N}'_{qqk} = -\frac{1}{18} \int \xi^4 (\boldsymbol{\xi} \times \dot{\boldsymbol{j}})_k d^3 \xi \]

\[ \mathcal{N}'_k = -\frac{4}{5} \int \xi^2 [\boldsymbol{\xi} \times (\boldsymbol{\xi} \times \dot{\boldsymbol{j}})]_k d^3 \xi, \quad \Pi_i = \frac{1}{5} P_{qqi} - \frac{1}{80c^2} \dot{N}_{qqi} + \frac{3}{700c^2} \dot{P}_{qqpi} \]

and,

\[ \tilde{\Lambda}_{ijk} = \Lambda_{ijk} - \frac{1}{5} \sum_{D(i)} \delta_{ij} \Lambda_{qqk}, \]

\[ \tilde{\mathcal{N}}_{ijkl} = \mathcal{N}_{(\text{sym})ijkl} - \frac{1}{28} \sum_{D(i)} \delta_{ij} (N_{qqkl} + \mathcal{N}_{qqkl}), \]

\[ \tilde{\Pi}_{ijkl} = \Pi_{ijkl} - \sum_{D(i)} \delta_{ij} \left( \frac{1}{7} \Pi_{qqkl} - \frac{1}{70} \sum_{D(i)} \delta_{kl} \Pi_{qqpp} \right) \]
Expressing the power radiated by electric charged systems

\[ \tilde{N}_{ijk} = N_{(sym)}^{ijk} - \frac{1}{15} \sum_{D(i)} \delta_{ij} N_{qqk}, \]

\[ \tilde{\Pi}_{ijk} = \Pi_{ijk} - \frac{1}{5} \sum_{D(i)} \delta_{ij} \Pi_{qqk}, \]

\[ \tilde{\Pi}_{ij} = \frac{1}{2} (N_{ij} + N_{ji}), \quad \tilde{\Pi}_{ij} = \Pi_{ij} - \frac{1}{3} \delta_{ij} \Pi_{qq}, \]

For \( \mu < 5 \) and \( \varepsilon < 6 \), the quantities \( N, \Lambda, ... \) are obtained from the above expressions by eliminating the contributions of the magnetic multipolar tensors of orders \( n_m > \mu \) and of the the electric ones for orders \( n_e > \varepsilon \).

We point out that the reduction diagrams described in this appendix are valid also in the case of an arbitrary electromagnetic field as may be seen from \[7\].

References
[1] Landau L, Lifchitz E 1970 Théorie des Champs (Éditions MIR Moscou) p. 228-9
[2] Vrejoiu C 1993 Electrodynamics and Relativity Theory(in romanian) (E.D.P. Bucharest) p. 433-5
[3] Jackson J D 1975 Classical Electrodynamics (Wiley New York) p.755-8
[4] Becker R 1982 Electromagnetic Fields and Interactions (Dover Publications,INC, New York)
[5] Castellanos A, Panizo M, Rivas J 1978 Am.J.Phys., 46 1116-17
[6] Vrejoiu C. 1984 St. Cercet Fiz. 36 863
[7] Vrejoiu C 2002 J. Phys. A: Math. Gen., 35 9911-22
[8] Applequist J. 1989 J. Phys. A: Math. Gen., 22 4303-4330
[9] Jansen L 1957 Physica 23 599
[10] Dubovik V. M. and Cheshkov A. A. 1974 Fiz. Elem. Chastits At. Yadra, 5 791-836
[11] Dubovik V. M. and Tosunyan L. A. 1983 Fiz. Elem. Chastits At. Yadra, 14 1193-1228
[12] Dubovik V. M. and Tugushev V. V. 1990 Phys. Rep 187 145-202
[13] Porsev S. G. 1994 Phys. Rev. A 49 5105