We construct $Z_3$ vortex solutions in a model in which $SU(3)$ is spontaneously broken to $Z_3$. The model is truncated to one in which there are only two dimensionless free parameters and the interaction of vortices within this restricted set of models is studied numerically. We find that there is a curve in the two dimensional space of parameters for which the energy of two asymptotically separated vortices equals the energy of the vortices at vanishing separation. This suggests that the inter-vortex potential for $Z_3$ strings might be flat for these couplings, much like the case of $U(1)$ strings in the Bogomolnyi limit. However, we argue that the intervortex potential is attractive at short distances and repulsive at large separations leading to the possibility of unstable bound states of $Z_3$ vortices.

I. INTRODUCTION

Vortex solutions are well-studied in a wide variety of condensed matter systems. In superconductors, vortex solutions have been recognized since Abrikosov’s seminal work [1], while string solutions in relativistic field theory were first found by Nielsen and Olesen [2]. The interaction of vortices has also been a subject of continual investigation, starting from the work of Abrikosov who conjectured a vortex lattice due to the repulsive force between vortices. In relativistic models, early work on the interaction of vortices was carried out by Jacobs and Rebbi [3] in which they noted a transition from attractive to repulsive interaction as a certain parameter was varied. Furthermore, at a critical point in parameter space, the inter-vortex potential was found to be flat and this is the so called Bogomolnyi limit [4].

The investigations thus far have mostly considered the interaction of $U(1)$ vortices - the kind that commonly occur in superconductors. However, there is a much wider variety of vortices occurring in other condensed matter systems and there is a possibility that these may also exist in particle physics and cosmology. In particular there is a class of vortices called “$Z_N$ vortices” in which $N$ vortices are topologically equivalent to the vacuum. The simplest of these is the $Z_2$ (global) vortex that exists in nematic liquid crystals. $Z_4$ vortices can be found in the A-phase of He$^3$. To our knowledge, $Z_3$ vortices have not yet been observed but it is possible that these may be relevant to confinement in QCD. This is apparent in the dual standard model picture that one of us has proposed [5] and is indicated by ongoing work on supersymmetric dualities [6].

In this paper we study $Z_3$ vortices and their interaction. The symmetry breaking pattern we consider is

$$SU(3) \rightarrow Z_3$$

which can be accomplished by the vacuum expectation value (VEV) of three adjoint scalar fields. These details are provided in Sec. II. The vacuum manifold of the model is

$$\Sigma = \frac{SU(3)}{Z_3}$$

and since,

$$\pi_1(\Sigma) = Z_3,$$

the model admits $Z_3$ strings which we explicitly construct in Sec. III. The interaction of these strings is studied in Sec. IV by comparing the energy of infinitely separated (very distant) strings with the energy of the strings at vanishing separation.

An interesting result that we obtain is that there is a surface in parameter space such that the energy of two vortices at infinite and at vanishing separation are equal. This raises the possibility that perhaps the inter-vortex potential is flat for these values of the parameters, much like the Bogomolnyi case for $U(1)$ strings. However, we argue that the $Z_3$ inter-vortex potential is not flat but has a maximum at some finite vortex separation. This then indicates that there must exist an unstable, static, bound state of two separated $Z_3$ vortices.

In an early paper, de Vega and Schaposnik [7] investigated the properties of $Z_3$ strings. The group theoretic formalism developed there is very general and can be used to construct $Z_N$ strings for arbitrary $N$. The study of the properties of $Z_3$ strings was, however, restricted to a choice of parameters where the $Z_3$ strings are effectively identical to $U(1)$ strings. Furthermore, for this choice of parameters, the desired symmetry breaking pattern [8] is not uniquely picked out by the potential. That is, the $Z_3$ symmetric vacuum is degenerate with vacua having other symmetries and is no longer the unique ground state of the model. We explain these comments in more detail in Sec. V.

II. MODEL

The construction of a model exhibiting $SU(3) \rightarrow Z_3$ involves the following two steps. First, identify the nec-
essary ingredients of the model. In particular, determine the scalar field content of the model. Second, construct the most general scalar field potential and determine the range of parameters which lead to the desired symmetry breaking. We treat these steps in the next two subsections.

A. Ingredients

A scalar field, $\Phi$, in the adjoint representation of $SU(3)$ can be written as the $3 \times 3$ Hermitian matrix

$$\Phi = \sum_{a=1}^{8} \Phi^a \lambda_a$$

where, $\lambda_a$ are the Gell-Mann matrices [8] and $\Phi^a$ are 8 real scalar fields. Under the action of $g \in SU(3)$, $\Phi$ transforms as:

$$\Phi \rightarrow \Phi' = g\Phi g^{-1},$$

where, $g$ may be written as: $\exp(i\alpha^a \lambda_a)$ for any set of $\alpha^a$.

Now the center of $SU(3)$ is $Z_3$ and the elements of $Z_3$ are of the form:

$$e^{i2\pi n/3} \mathbf{1}, \ n = \text{integer},$$

where $\mathbf{1}$ is the identity matrix. Hence, $\Phi$ is left invariant under transformations belonging to the center. And so the VEVs of any number of adjoint scalar fields cannot break the $Z_3$ center. Then, one way to achieve would be to give VEVs to as many adjoint fields as necessary to break the $SU(3)$ maximally, that is, only leaving the center unbroken. Indeed, one can check that two adjoints $\Phi_1$ and $\Phi_2$ are sufficient because if,

$$\Phi_1 = \lambda_1, \ \Phi_2 = \lambda_4$$

(2)

then the only group elements that commute with both $\Phi_1$ and $\Phi_2$ are the ones proportional to $\mathbf{1}$, that is, the elements of the $Z_3$ center. For completeness, in Table I we show the various possible VEVs for $(\Phi_1, \Phi_2)$ and the resulting symmetry breaking pattern.

In the next subsection, we will consider an $SU(3)$ invariant potential for two adjoint scalar fields. As the full potential has a lot of parameters, we restrict our attention to a certain region of parameter space. Within this truncated model, we will find that it is not possible to construct a potential that will lead to the VEVs in (2) at a unique, global minimum. Then it will be necessary to introduce a third adjoint field $\Phi_3$. In Table II we show the symmetry breaking patterns for different directions of $(\Phi_1, \Phi_2, \Phi_3)$.

In what follows, we will construct a potential that will have a global minimum when $(\Phi_1, \Phi_2, \Phi_3)$ acquire VEVs in the following direction:

$$(\lambda_4, \lambda_6, \lambda_8).$$

(SU(3) rotations of these VEVs will yield the manifold of global minima.) As shown in Table II, the residual symmetry group will be $Z_3$ in this case.

| 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|----|----|----|----|----|----|----|----|
| 1  | U1U1 | U(1) | U(1) | Z_3 | Z_3 | Z_3 | Z_3 |
| 2  | U(1) | U1U1 | U(1) | Z_3 | Z_3 | Z_3 | Z_3 |
| 3  | U(1) | U1U1 | U(1) | U(1) | U(1) | U(1) | U(1) |
| 4  | Z_3 | Z_3 | U(1) | U(1) | U1U1 | Z_3 | Z_3 |
| 5  | Z_3 | Z_3 | U(1) | U(1) | U1U1 | Z_3 | U(1) |
| 6  | Z_3 | Z_3 | U(1) | Z_3 | Z_3 | U1U1 | U(1) |
| 7  | Z_3 | Z_3 | U(1) | Z_3 | U(1) | U1U1 | U(1) |
| 8  | U1U1 | U1U1 | U(1) | U(1) | U(1) | U(1) | SU2U1 |

TABLE I. $SU(3)$ breaking with two adjoint Higgs fields. The rows and columns label the direction of the VEVs of each of the two fields while the table entry gives the residual symmetry. For convenience of notation we have defined: $U1U1 = U(1) \times U(1)$, and $SU2U1 = SU(2) \times U(1)$

| $U(1)$ | $(\lambda_1, \lambda_2, \lambda_3)$, $(\lambda_1, \lambda_2, \lambda_8)$, $(\lambda_1, \lambda_4, \lambda_8)$, |
|        | $(\lambda_2, \lambda_3, \lambda_8)$, $(\lambda_3, \lambda_8, \lambda_8)$, $(\lambda_3, \lambda_5, \lambda_8)$, |
|        | $(\lambda_3, \lambda_6, \lambda_8)$, $(\lambda_3, \lambda_7, \lambda_8)$, $(\lambda_4, \lambda_5, \lambda_8)$, |
|        | $(\lambda_6, \lambda_7, \lambda_8)$, $Z_3$ all other $(\lambda_i, \lambda_j, \lambda_k)$ with distinct $i, j, k$ |

TABLE II. $SU(3)$ symmetry breaking with three distinct adjoint scalar fields. The first column shows the residual symmetry group if the fields get VEVs in the directions shown in the second column.

*The center of a group consists of elements that commute with all other elements.
B. Construction of the potential

The SU(3) invariant potential for three adjoint fields can be written as follows:

$$V(\{\Phi_l\}) = V_1(\{\Phi_l\}) + V_2(\{\Phi_l\})$$

where

$$V_1(\{\Phi_l\}) = \sum_{l=1}^{3} [-m_l^2(\text{Tr}\Phi_l^2) + a_l(\text{Tr}\Phi_l^2)^2 + b_l\text{Tr}\Phi_l^2] \quad (5)$$

and

$$V_2(\{\Phi_l\}) = \sum_{l=1}^{3} [c_l\text{Tr}(\Phi_m\Phi_n) + d_l(\text{Tr}(\Phi_m\Phi_n))^2$$

$$+ e_l(\text{Tr}(\Phi_m\Phi_n)^2 + f_l\text{Tr}(\Phi_m^2\Phi_n^2) + g_l(\text{Tr}(\Phi_m^2\Phi_n^2) + h_l(\text{Tr}(\Phi_m^2\Phi_n^2))]$$

with, \(l, m, n\) taking cyclic values over 1, 2, 3. (In writing the potential we have omitted cubic terms for simplicity.)

The full Lagrangian can now be written:

$$L = \sum_l \text{Tr}(D_{\mu}\Phi_l D^{\mu}\Phi_l) \frac{1}{2} \text{Tr}(G_{\mu\nu}G^{\mu\nu}) - V(\{\Phi_l\})$$

where,

$$D_{\mu}\Phi_l \equiv \partial_{\mu}\Phi_l + g[A_{\mu}, \Phi_l]$$,

\(A_{\mu}\) is the matrix-valued gauge field,

$$G_{\mu\nu} = G^{\alpha}_{\mu\nu}\lambda_\alpha = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + g[A_{\mu}, A_{\nu}]$$.

The general potential (5) has 27 parameters and is far too complicated for us to handle. We will assume certain relationships among the parameters to enable us to proceed further. These are:

$$a_l = a, \quad b_l = b, \quad m_l = \frac{m}{2},$$

and

$$c_l = 0 = d_l = g_l = h_l.$$

Now to check if two scalar fields would have been sufficient for the symmetry breaking (6), we eliminate all terms containing \(\Phi_3\) in (5), restrict our attention to the truncated region in parameter space, then feed in the various possible VEVs for \(\Phi_1\) and \(\Phi_2\) from Table 3. We find that the VEVs leading to a \(Z_3\) residual symmetry give a higher (or equal) energy than the other symmetry breaking patterns. So the \(Z_3\) vacuum cannot be a unique global minimum. Hence, it is necessary for us to include the third scalar field \(\Phi_3\).

With three scalar fields, we introduce

$$\Lambda = 16a + 8b,$$

and, make the further choice of parameters:

$$\epsilon = f_l = e_1 = e_2 = -e_3$$

where, \(\epsilon\) is a new free parameter. The minus sign in front of \(e_3\) is a crucial feature that ensures that the global minimum of the potential has \(Z_3\) symmetry.

The potential for this restricted set of parameters is

$$V(\{\Phi_l\}) = \sum_{l=1}^{3} \left[ -\frac{m^2}{4} (\text{Tr}\Phi_l^2) + a(\text{Tr}\Phi_l^2)^2 + b\text{Tr}\Phi_l^2 \right]$$

$$+ \epsilon \left[ \text{Tr}(\Phi_2\Phi_3)^2 + \text{Tr}(\Phi_3\Phi_1)^2 - \text{Tr}(\Phi_1\Phi_2)^2 \right.$$

$$\left. + \text{Tr}(\Phi_2^2\Phi_3^2) + \text{Tr}(\Phi_3^2\Phi_1^2) + \text{Tr}(\Phi_1^2\Phi_2^2) \right] \quad (8)$$

For the potential to have a global minimum at finite VEVs of the fields, we need

$$\Lambda > 0.$$ 

The requirement that the VEVs of the fields be non-vanishing gives the constraint:

$$\frac{\epsilon}{\Lambda} < \frac{3}{2}.$$

Within this parameter range, we have inserted all possible choices of directions of \(\Phi_l\) (\(l = 1, 2, 3\)) in the potential and find that it has a global minimum when the residual symmetry group is \(Z_3\), provided

$$\epsilon > 0.$$

Further, the VEVs yielding the global minimum are in the \((\lambda_4, \lambda_6, \lambda_8)\) directions (and \(SU(3)\) rotations of these directions):

$$\Phi_1 = \eta_1\lambda_4$$

$$\Phi_2 = \eta_2\lambda_6$$

$$\Phi_3 = \eta_3\lambda_8$$

where

$$\eta_1 = \eta_2 = m \sqrt{\frac{\Lambda - 2\epsilon/3}{\Lambda^2 + 2\epsilon\Lambda - 8\epsilon^2/9}}$$

$$\eta_3 = m \sqrt{\frac{\Lambda + 2\epsilon/3}{\Lambda^2 + 2\epsilon\Lambda - 8\epsilon^2/9}}.$$ 

Let us now define

$$v_l^2 \equiv \frac{1}{2}\text{Tr}(\Phi_l^2).$$

Then, for a vortex, \(v_l\) will vary in space and the relevant potential is:

$$V(\{v_l\}) = \sum_l \left[ -\frac{m^2}{2} v_l^2 + \frac{\Lambda}{4} v_l^4 \right.$$ \n
$$\left. + \frac{\epsilon}{3}(v_1^2 + v_2^2) v_3^2 + \epsilon v_1^2 v_2^2 \right].$$  

(12)
III. \textit{Z}_3 \textit{STRING ANSatz AND SOLUTION}

Having truncated the full model to one which is simple enough to analyze, we now write down the ansatz for \textit{Z}_3 strings, insert it into the field equations and then find the string solutions.

To write a string ansatz, we must first specify a closed path on the vacuum manifold, \( P(\theta) \), parametrized by \( \theta \in [0, 2\pi] \), which is incontractable. This is given by:

\[
P(\theta) = e^{in\lambda_0 \theta/\sqrt{3}}.
\]

This path is incontractable since \( P(2\pi) \) is a non-trivial element of the discrete residual group \( \text{Z}_3 \) for \( n = 1, 2 \).

We now identify \( \theta \) with the spatial polar coordinate. Then the scalar field ansatz is:

\[
\Phi_1(r \to \infty, \theta) = P(\theta)^4 \Phi_1(r \to \infty, \theta = 0) P(\theta).
\]

With \( \Phi_1(\theta = 0) \) given by eq. (10), this leads to:

\[
\Phi_1(r, \theta) = v_1(r)(\cos n\theta \lambda_4 + \sin n\theta \lambda_5),
\]

\[
\Phi_2(r, \theta) = v_2(r)(\cos n\theta \lambda_6 + \sin n\theta \lambda_7),
\]

\[
\Phi_3(r, \theta) = v_3(r)\lambda_8
\]

\[
A^8_\theta = -\frac{n}{\sqrt{6g}}\frac{\alpha(r)}{r}
\]

where we have included scalar field profile functions \( v_i(r) \) and given the gauge field ansatz with its profile function \( \alpha(r) \). All other components of the gauge field are taken to vanish.

We now insert this ansatz into the field equations to get:

\[
v_1'' + \frac{1}{r} v_1' - \frac{n^2}{r^2} (1 - \alpha)^2 v_1 - \Lambda v_1^4 + m^2 v_1 \\
-2\epsilon v_1^2 v_1^2 v_1 = 0
\]

\[
v_2'' + \frac{1}{r} v_2' - \frac{n^2}{r^2} (1 - \alpha)^2 v_2 - \Lambda v_2^4 + m^2 v_2 \\
-2\epsilon v_2^2 v_2^2 v_2 = 0
\]

\[
v_3'' + \frac{1}{r} v_3' - \Lambda v_3^4 + m^2 v_3 - \frac{2}{3}\epsilon (v_1^2 + v_2^2) v_3 = 0
\]

\[
\alpha'' - \frac{1}{r} \alpha' + 3g^2 (v_1^2 + v_2^2)(1 - \alpha) = 0
\]

where a prime denotes differentiation with respect to \( r \).

The equations for \( v_1 \) and \( v_2 \) are identical and so are the asymptotic boundary conditions as is seen from (10). Therefore we set

\[
v_1(r) = v_2(r).
\]

It is now convenient to define rescaled coordinates, parameters and fields as follows:

\[
\lambda^2 = \frac{\Lambda}{3g^2}, \quad \beta = \frac{\epsilon}{3g^2}
\]

\[
v_1 = v_2 = \frac{m}{\sqrt{\Lambda}} f(x), \quad v_3 = \frac{m}{\sqrt{\Lambda}} h(x)
\]

Primes will now denote differentiation with respect to the rescaled coordinate \( x \).

The rescaled equations are:

\[
f'' + \frac{f'}{x} - \frac{n^2}{x^2} (1 - \alpha)^2 f - \frac{\lambda^2}{2} (f^2 - 1) f - \beta (f^2 + \frac{\Lambda}{3}) f = 0
\]

\[
h'' + \frac{h'}{x} - \frac{\lambda^2}{2} (h^2 - 1) h - \frac{2\beta}{3} f^2 h = 0
\]

\[
\alpha'' - \frac{\alpha'}{x} + f^2 (1 - \alpha) = 0.
\]

The boundary conditions on the functions \( f, h \) and \( \alpha \) follow by requiring single-valuedness and regularity of the fields at the origin,

\[
\alpha(0) = f(0) = h'(0) = 0.
\]

At infinity, the fields should go to their vacuum expectation values:

\[
\alpha(x \to \infty) = 1, \quad f(x \to \infty) = F_0, \quad h(x \to \infty) = H_0
\]

with,

\[
F_0 = \sqrt{\frac{1 - 2\sigma/3}{1 + 2\sigma - 8\sigma^2/9}}, \quad H_0 = \sqrt{\frac{1 + 2\sigma/3}{1 + 2\sigma - 8\sigma^2/9}}
\]

where

\[
\sigma = \frac{\beta}{\lambda^2}.
\]

If we set \( \beta = 0 \), the \( h \) equation is solved by \( h = 1 \) and the \( f \) and \( \alpha \) equations are exactly the Nielson-Olesen equations for the Abelian-Higgs vortex. These vortices have been studied extensively and this is also the case discussed by de Vega and Schaposhnik [7]. The interaction of Abelian-Higgs vortices is characterized by the single ratio of length scales entering the problem - namely, the ratio of the (single) scalar and vector masses. With \( \beta \neq 0 \), however, the picture is more complicated since
we have three length scales corresponding to the masses of the two independent scalar fields in our truncated model, and the gauge fields. Therefore there are two independent dimensionless ratios we can construct, and a correspondingly richer structure to the interaction of $Z_3$ vortices.

We have solved the equations of motion (15)-(17) by a numerical shooting routine. In Fig. 1 we show the behaviour of the $f$, $h$ and $v$ fields for a particular choice of parameters and for $n = 1$.

FIG. 1. The profile functions $f$, $h$ and $α$ for the unit winding $Z_3$ vortex for $λ = 1.13$ and $β = 0.42$.

The equations of motion for the $n = 2$ vortex can also be solved. So the $n = 2$ vortex is indeed a solution, though it will be unstable to decay into the topologically equivalent $n = -1$ solution which has lower energy. This instability of the $n = 2$ vortex is not relevant for us since all that we are interested in is the energy of two overlapping $n = 1$ vortices which is the same as the energy of the $n = 2$ vortex solution.

IV. ENERGY FUNCTIONAL AND NUMERICAL EVALUATION

The set of field equations (15)-(17) can also be obtained by extremizing the energy functional

$$E = \int d^3x \left[ \frac{1}{4} G_{ij}^a G^{aj} + \frac{1}{2} (D_l \Phi_i^a)(D^l \Phi_i^a) + V(\{\Phi_i\}) \right]$$

(21)

(The index $a = 1, ..., 8$ is the group index and $l = 1, 2, 3$ labels the different adjoint fields.) Here we will obtain the extremum values of the energy for the $n = 1, 2$ topological configurations directly, that is, without solving the field equations of motion. This is the technique used by Jacobs and Rebbi \[3\] to study the interaction of $U(1)$ vortices, and we will employ it to study the interaction of $Z_3$ vortices.

Let us first define

$$\mu = \lambda \sqrt{1 - \frac{2}{3} \sigma}, \quad ν = \lambda \sqrt{\frac{1 + 2\sigma/3}{1 + 2\sigma - 8\sigma^2/9}}.$$  \hspace{1cm} (22)

Then the solution to eq. (15)-(17) may be written as:

$$f = F_0 \left(1 + \sum_{j=0}^{\infty} f_j x^j e^{-\mu x}\right)$$

(23)

$$h = H_0 \left(1 + \sum_{j=0}^{\infty} h_j x^j e^{-\nu x}\right)$$

(24)

$$α = 1 + \sum_{j=0}^{\infty} \alpha_j x^j e^{-x}.$$  \hspace{1cm} (25)

In this form, the functions automatically satisfy the desired boundary at infinity (eq. (19)). The boundary conditions at the origin (eq. (18)) require

$$α_0 = -1, \quad f_0 = -1, \quad h_1 = νh_0,$$

for both $n = 1$ and $n = 2$ vortices. In addition, regularity of the gauge fields at the origin requires

$$α'(0) = 0$$

and,

$$f'(0) = 0, \quad \text{for } n = 2.$$  \hspace{1cm}

These conditions give

$$α_1 = -1$$

and

$$f_1 = -\mu, \quad \text{for } n = 2.$$

In this scheme, since we are numerically evaluating the energy functional, it is necessary to ensure that the potential vanishes in the true vacuum. This requires that we shift the potential in (12) by a constant $v_0$:

$$v_0 = -(2F_0^4 + H_0^4) + 2(2F_0^2 + H_0^2) - \frac{8}{3} σ F_0^2 H_0^2 + 4σ F_0^4.$$  \hspace{1cm}

The potential can now be written in terms of the fields $f$ and $h$:  \hspace{1cm}
\[ V(f, h) = \frac{\chi^2}{8} \left[ (2f^4 + h^4) - 2(2f^2 + h^2) + \frac{8}{3} \sigma f^2 h^2 + 4\sigma f^4 + v_0 \right] \]

In terms of the fields \( f \), \( h \) and \( \alpha \), the energy is:

\[ E = \frac{2\pi m}{\sqrt{6A_0}} \int dz \, dx \, E[f, h, \alpha] \]

where \( z \) is the rescaled (as in eq. (14)) dimensionless coordinate along the string, and

\[ E[f, h, \alpha] = f^2 + \frac{1}{2} h^2 + n^2 \left( \frac{\alpha'}{x} \right)^2 + \frac{n^2}{2} (1 - \alpha)^2 f^2 + V(f, h) \]

We now have to evaluate the energy functional in terms of the various coefficients \( f_i, h_i \) and \( \alpha_i \) (infinite in number), for different choices of \( \{\lambda, \beta\} \) and winding number \( n \). On inserting the expansions for the fields in the energy functional, we end up with integrals that can be evaluated in terms of Gamma functions. These form the energy of the various coefficients \( f, h, \alpha \). This gives \( E = E(\{f_i\}, \{h_i\}, \{\alpha_i\}) \) where \( i = 0, \ldots, \infty \). We then truncate the expansion by retaining only the first 7 terms in each of the expansions \( (i = 0, \ldots, 6) \), making a total of 21 parameters that need to be varied to minimize \( E \). We finally find the global minimum of \( E \) with respect to the variation of the coefficients for \( n = 1, 2 \) and \( \lambda \in (0, 10) \) and \( \beta \in (0, 6) \).

Sample results for the dependence of the energy on the \( \beta \) parameter are shown in Fig. 2.

FIG. 2. The dependence of the energy of \( n = 1 \) and \( n = 2 \) vortices on the parameter \( \beta \) for \( \lambda = 2.0 \). Also shown is twice the energy of the \( n = 1 \) vortex.

V. CRITICAL COUPLINGS AND INTER-VORTEX FORCES

In the \( \beta = 0 \) limit, the equation of motion (16) is simply solved by

\[ h(x; \beta = 0) = 1 \]

and the remaining equations (15) and (17) are identical to the \( U(1) \) equations. So, in this case, the structure and interaction of \( Z_3 \) vortices is identical to those of \( U(1) \) vortices. In particular, for \( \lambda = 1 \), the inter-vortex potential vanishes \( \hat{6} \). Indeed, the VEVs of the three different fields can point in the same direction, say in the \( \lambda_0 \) direction, leading to an \( SU(2) \times U(1) \) residual symmetry. So to pick out the desired symmetry breaking we must necessarily consider \( \beta \neq 0 \).

As one can see in Fig. 3, there are points in parameter space where

\[ E(\beta, \lambda; n = 2) = 2 \times E(\beta, \lambda; n = 1) \] \hspace{1cm} (26)

In this case, the energy of two infinitely separated vortices is equal to the energy of two overlapping vortices. We shall call the parameters for which (26) holds, to be “critical”. In Fig. 3 we plot the critical curve in \((\beta, \lambda)\) space.

FIG. 3. The curve in parameter space for which the energy of two infinitely separated vortices have the same energy as two vortices at zero separation, that is, the \( n = 2 \) vortex. The region in which the \( n = 2 \) vortex is more energetic than two \( n = 1 \) vortices is the “repulsive” region while that in which the \( n = 2 \) vortex is less energetic is the “attractive” region.

The question we now address is whether the critically coupled vortices have a mutual repulsion or attraction at intermediate separations. In the critically coupled \( U(1) \) case \((\beta = 0, \lambda = 1)\), the intervortex potential is flat and hence the intervortex forces vanish at all separations. It is useful to think of this in terms of the repulsive gauge field.
and attractive scalar field interactions. The exchange of spin one gauge particles between identical vortices leads to a repulsive force while the exchange of spin zero particles leads to an attractive force. So the gauge field repulsion is balanced by the scalar field attraction in the critical U(1) case.

In the case of the Z₃ string, the crucial observation is that the curve of critical couplings lies in the region λ ≥ 1 and small σ = β/Λ². Hence, along the critical curve, the gauge field mass is smaller than either of the scalar field masses μ and ν given in (23). So the gauge field interaction is of longer range than the scalar field interaction. Therefore, if we bring in two infinitely separated vortices, they will first experience the repulsive gauge field interaction. When they come in closer, the attractive interaction due to the scalar fields will turn on. Since there are two scalar fields in our model, the attraction is stronger than in the U(1) case and is more effective in cancelling out the gauge field repulsion. So the intervortex potential is expected to turn over, as schematically depicted in Fig. 4. For the critically coupled case, the turn over is such that the energy at zero vortex separation equals that at infinite vortex separation. The presence of a turning point in the intervortex potential at some vortex separation s₀ means that two vortices separated by this distance can be in relative equilibrium. In the present case, this is an unstable equilibrium because the potential is a maximum.

FIG. 4. A schematic depiction of the expected intervortex potential (U) as a function of vortex separation (s) in the critically coupled case where the gauge field mass is less than the scalar field masses. Being lighter, the gauge field provides a longer range interaction than the scalar fields and leads to a repulsive force between vortices in the asymptotic region. In a U(1) model, the single scalar field is unable to overcome the repulsive potential at short distances. However, in the Z₃ string case, the two scalar fields successfully turn the potential over at short distances leading to a maximum at s = s₀.

Note that this argument is based on the fact that the curve of critical couplings lies in the λ ≥ 1 region. This feature can be understood by realizing that the presence of two scalar fields instead of one implies an enhanced attractive force between vortices at short distances. So if one imagines starting out with repulsive U(1) vortices, and adding a second scalar field as is present in the Z₃ vortices, this can turn over the intervortex potential at small separations and make the energy at s = 0 the same as that at s = ∞. But since one needs to start out with repulsive U(1) vortices, this means that critically coupled Z₃ vortices should lie in the λ ≥ 1 region. (If one started out with attractive (λ < 1) U(1) vortices, the addition of a second scalar field would simply make the vortices even more attractive at short distances and eq. (23) could never be satisfied.)

VI. CONCLUSIONS

We have constructed field theory solutions for Z₃ strings and have studied their interaction within a range of model parameters. The solution for the structure of the vortex is shown in Fig. 4 while the dependence of the energy on the new parameter in the model, that is, the parameter not present in the U(1) case, is shown in Fig. 5. Infinitely separated Z₃ strings have the same energy as strings at zero separation along a curve in our two dimensional space of parameters (Fig. 6). However, we have given general arguments to show that, unlike the U(1) case, the intervortex potential is not trivial for these critically coupled Z₃ strings. In fact, the intervortex potential is expected to have a maximum value at some non-vanishing vortex separation (Fig. 7). This suggests that an unstable bound state of two Z₃ vortices should exist.

Although we have worked in detail within a specific range of parameters, we have understood the intervortex forces based on the number of scalar and vector fields present in the model and their masses. This reasoning (described in Sec. IV) is expected to be valid quite generally and should apply to the full range of parameters in this model as well as to other models.

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