Bardasis-Schrieffer polaritons in excitonic insulators

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Bardasis-Schrieffer modes in superconductors are fluctuations in subdominant pairing channels, e.g., d-wave fluctuations in an s-wave superconductor. This paper generalizes the notion to excitonic insulators and shows that these modes generically occur. In s-wave excitonic insulators, a p-wave Bardasis-Schrieffer mode exists below the gap energy and has a non-vanishing optical matrix element with light. This mode hybridizes strongly with photons to form Bardasis-Schrieffer polaritons, which are observable in both far-field and near-field optical experiments.

Sixty years ago, Bardasis and Schrieffer [1] investigated ‘exciton-like’ sub gap collective modes in superconductors produced by fluctuations in channels different from the ground state, e.g., d-wave fluctuations in an s-wave superconductor. These modes are now referred to as Bardasis-Schrieffer modes (BaSh modes) [2, 3]. BaSh modes can be viewed as a cooperative wave of transitions between bound states of Cooper pairs, but in superconductors typically do not couple linearly to long-wavelength radiation because a uniform electric field couples only to the motion of the center of mass, not to the internal structure of a Cooper pair. A BaSh mode can couple to light in the presence of a supercurrent [3] or at nonzero momentum [4] in the near field optical response. Very recent Raman experiments have reported its signature in iron based superconductors [5–7].

The situation is different in excitonic insulators [8–12] where the Cooper pair is replaced by the electron hole pair which forms a condensate that does not break the U(1) charge conservation but in the idealized case does break the symmetry corresponding to conservation of the difference between the number of electrons and the number of holes. The opposite charge of the electron and hole means that a spatially uniform electric field may couple to the internal structure of a pair but (in simple cases) not to the center of mass motion, thus can excite, e.g., the p-wave BaSh mode in an s-wave condensate.

In this paper, we investigate the physics of Bardasis-Schrieffer modes in excitonic insulators, using the minimal model

$$H = \int dr \left[ \psi^\dagger \left( \xi (p - A) \sigma_3 + \phi \right) \psi \right] + \int dr dr' V(r - r') \psi^\dagger (r) \psi (r) \psi^\dagger (r') \psi (r').$$

(1)

Here $\psi^\dagger$ ($\psi^\dagger$, $\psi_2^\dagger$) is the two component electron creation operator corresponding to the electron and hole bands labeled 1 and 2, $p = -i\hbar \nabla$, $\xi (p) = e (p) - \mu$ is the kinetic energy, $(\phi, A)$ is the EM potential, $\sigma_i$ are the Pauli matrices in band space and we have set electron charge $e$ and speed of light $c$ to one. We have assumed that on the non-interacting level the numbers of electrons and holes are separately conserved and that the electron and hole bands have the same dispersion but with opposite sign. We assume a negative gap, so that the two dispersions cross at a wavevector $k_F$ with Fermi velocity $v_F$ as shown by the dashed lines in Fig. 1. In the two dimensional case of main interest here each band contributes a carrier density $n_i = k_F^2/(4\pi)$ and a density of state $\nu/2 = k_F^2/(2\pi\hbar v_F)$. This model omits many features of real solids including any asymmetry between electron and hole bands, coupling to phonons and the breaking of the idealized internal U(1) symmetry down to a discrete symmetry [13]. These complications are not relevant to the basic physics we wish to consider here.

The interaction is usually the screened Coulomb interaction. In the two dimensional, RPA-screened case, it reads $V(q) = 2\pi \alpha / q^2 \alpha^2 \approx 2\pi \alpha / q^2 \alpha^2$ in the static limit. Here the Thomas-Fermi wave vector $q_{TF} = e^2 m/\hbar^2$ does not depend on the carrier density. The dependence on $q$ means that higher an-
gular momentum channels generically exist, so BaSh modes are expected in all excitonic insulators.

**Ginzburg-Landau action**—Absorbing the intraband ($\sigma_z$-portion) of the interaction into the band gap and making a Hubbard-Stratonovich transformation of the partition function $Z = \int D[\psi, \bar{\psi}] e^{-S} \Delta$ in the electron-hole pairing channel yields

$$Z = \int D[A] D[\bar{\psi}, \psi] D[\Delta, \bar{\Delta}] e^{-S(\psi, A, \bar{\Delta})}$$  \hspace{1cm} (2)

where the action

$$S = \int d\tau dr \left\{ \psi^\dagger G^{-1}_{\Delta A} \psi + \sum_{l} \frac{1}{2g_l} |\Delta_l|^2 \right\}$$  \hspace{1cm} (3)

describes coupled dynamics of the fermion field $\psi$, the EM field $A$ and the excitonic gap $\Delta_l$ with $l$ denoting the pairing channel (angular momentum in our simple case) and the coupling constants in the different channels are $g_l$ defined more precisely in the supplemental material.

The fermion kernel is

$$G^{-1}_{\Delta A} = \begin{pmatrix} \partial_\tau + \phi_1 + \xi (p - A_1) & \sum_i \Delta_i f_i(p) \\ \sum_i \Delta_i \bar{f}_i(p) & \partial_\tau + \phi_2 - \xi (p - A_2) \end{pmatrix}$$  \hspace{1cm} (4)

where $f_i(p)$ describes the momentum dependence of the pairing function in each pairing channel $i$. In any physical realization of an excitonic insulator, electrons and holes will be at different spatial positions, and in Eq. (4) we have therefore noted the possibility that they may feel different electromagnetic fields. In solid state realizations such as Ta$_2$NiSe$_5$ [12, 13] the electrons and holes are in such close spatial proximity that it is reasonable to assume that both feel the same electromagnetic fields but in electron hole bilayers [14–16] the separation can be large enough that (especially for the description of near field experiments) the difference between the fields on the two subsystems is important.

After integrating out the fermions, one obtains a Ginzburg-Landau action $\mathcal{S}(\Delta, A)$ in terms of order parameters and EM field. The saddle point of $\mathcal{S}(\Delta, A)$ gives the mean field gap. Assuming only one channel is relevant we obtain $|\Delta_l| = 2\Lambda_1 e^{-\frac{\lambda_l}{2g_l}}$ for $g_l \nu \ll 1$ where $\Lambda_1 = 1$ for $l = 0$, $\Lambda_l = d$ for $l > 0$ and $d$ is space dimension. The $l$ with the largest $g$ is then favored. The effective cutoff $\Lambda_l$ depends on types of interaction and the angular momentum channel and is at the order of $\nu F q_{TF}$ or screened Coulomb interaction [9, 17]. The collective modes are fluctuations around the mean field configuration.

We assume that the $l = 0$ component of the interaction is the strongest and thus the equilibrium state has $s$-wave pairing with pairing function $f_0 = 1$ and mean field gap $\Delta$ which without loss of generality we assume to be real. We focus here on the $p$-wave BaSh mode which couples to light already at zero momentum. In $d$ dimensions the $p$-wave order parameter is a vector that transforms as a $d$ dimensional representation of the symmetry group (here approximately $O(d)$ if lattice effects are weak). Denoting the components of the $p$-wave gap by $\tilde{\Delta}_j$ we have

$$\sum_l \Delta_l f_l(p) = \Delta + R + i\Delta \theta + \sum_j \tilde{\Delta}_j f_j(p)$$  \hspace{1cm} (5)

where $f_j = k_j/k_F$ are the $p$-wave pairing functions and $j = x, y, \ldots$. Here the fluctuations in the dominant order parameter has been explicitly separated into amplitude ($R$) and phase ($\theta$) degrees of freedom, while the $p$-wave fluctuations can be separated into real and imaginary parts as $\tilde{\Delta}_j = \Delta_j^{(1)} + i\Delta_j^{(2)}$.  

**The BaSh mode action**—Expanding to quadratic order in the fluctuations around the mean field configuration, working in the gauge $\phi = 0$, one obtains the effective action for the order parameter and EM field:

$$\mathcal{S}(\Delta, A) = \frac{1}{2} \left\{ G^{-1}_R(q) R(-q) R(q) + G^{-1}_0(q) \theta(-q) \theta(q) + G^{-1}_{BS}(q) \Delta_0^{(2)}(q) A_j(q) \Delta_j^{(2)}(q) + 2C_{ij}(q) \Delta_0^{(2)}(q) A_j(q) \right\}$$  \hspace{1cm} (6)

where $q$ means both momentum and frequency and summation over $q$ and repeated indices is assumed. In the weak coupling (BCS) regime only the ‘imaginary’ $p$-wave fluctuations $\Delta_j^{(2)}$ give rise to collective modes [4] so we have not written the terms involving the real components of the $p$-wave order parameter. These are briefly treated in our discussion of the strong coupling (BEC) regime at the end and in the supplemental material.

$G_R$ and $G_0$ are the familiar amplitude and phase mode propagators. They are identical to that of a BCS superconductor [4] due to the formal analogy of the action Eq. (3) to the BCS action, with the electron and hole band index being mapped to the spin index in the superconductor. The BaSh mode propagator

$$G_{BS}^{-1}_{ij} = \frac{1}{g_p} \delta_{ij} + \chi_{\sigma z \sigma z} f_i f_j (\omega) = \left( \frac{1}{g_p} - \frac{1}{d} - \frac{d}{d^2} \omega^2 F(\omega) \right) \delta_{ij}$$  \hspace{1cm} (7)

is also identical to the superconducting case. The function $F$ describes the physics of quasiparticle excitations and is

$$F(\omega) = \sum_k \frac{1}{E_k} \left( \frac{1}{4E_k^2 - \omega^2} \right) = \frac{\nu}{4\Delta^2} \frac{\sin^2 \frac{\omega}{2\Delta}}{\sqrt{1 - \left( \frac{\omega}{2\Delta} \right)^2}}$$  \hspace{1cm} (8)

which diverges as $1/\sqrt{2\Delta - \omega}$ as the frequency approaches the quasi particle excitation edge.

The key difference from superconductivity is the coupling to the electromagnetic field: the superconducting phase mode couples as $\partial_\mu \theta_s \rightarrow \partial_\mu \theta_s + A_\mu$, but in the excitonic case the neutrality of the particle-hole pair means there is no such coupling at long wavelength. On the other hand, the allowed dipole matrix element leads to the photon kernel

$$K_{ij}(\omega) = \frac{n}{m} \delta_{ij} + \chi_{\sigma_3 \nu_i \sigma_3 \nu_j} = \left( \frac{n}{m} - \frac{4}{d} \nu_F^2 \Delta^2 F(\omega) \right) \delta_{ij}$$  \hspace{1cm} (9)
which contains pair breaking excitations even without assistance of disorder. Moreover, there is a linear coupling between the BaSh mode and the EM vector potential:

\[ C_{ij}(\omega) = \chi_{\alpha \beta, \gamma \delta} \delta_{ij} \omega \frac{v_F}{d} F(\omega) \delta_{ij}. \]  

The simple model of Eq. (1) has U(1) symmetries associated with the separate conservation of electrons and holes in the high temperature phase, implying a gapless phase mode [10] with \( G^{-1}_{\theta} = v_{\theta}^2 + v_{\eta}^2 q^2 l/d \) in the low energy limit. Lattice effects may reduce the U(1) symmetry to a discrete one [13] and open a gap to the phase mode dispersion. The amplitude mode has the gap \( 2\Delta \) and does not couple to light linearly at zero momentum. However, the BaSh mode couples to the electric field even at zero momentum because the latter exerts opposite forces on the electron and hole in an exciton, and excites it from the s bound state to p bound state. This induces a BaSh mode pole in the optical conductivity even at zero momentum as we will show later.

The root of Eq. (7) gives the BaSh mode frequency [1, 3, 4, 18] which decreases from \( 2\Delta \) to zero as \( g_p \) grows from zero to \( d_g \), as shown in Fig. 3. In the weak and strong \( p \)-wave pairing limits, the BaSh mode frequencies are

\[ \omega_{\text{BaSh}} = 2\Delta \sqrt{1 - \frac{v_{\pi}^2 (v_{\theta} g_p)^2}{\left( \frac{d}{g_p v} \right)^2 - 1}} \left( g_p < d_g \right), \]

\[ \omega_{\text{BaSh}} = 2\Delta \left( \frac{n}{m} - \frac{4}{d} v_F^2 \Delta^2 F(\omega) \right) + \sigma_{\text{BaSh}}. \]  

Let’s first look at it without the BaSh mode contribution \( \sigma_{\text{BaSh}} \). In the zero frequency limit, the second term exactly cancels the first term such that the Drude spectra weight is zero, i.e., the system is an insulator [10]. The second term has zero total spectra weight since it decays faster than \( 1/\omega \) at large frequency. Thus it works to transfer the Drude weight \( D = \pi n \sigma_{\text{BaSh}} \) in the electron hole fermi liquid phase to the pair breaking excitations above the gap in the excitonic insulating phase, as shown by the dashed line in Fig. 2.

The BaSh mode contribution to the optical conductivity is

\[ \sigma_{\text{BaSh}}(\omega) = \frac{4}{d^2} v_F^2 \Delta^2 F(\omega) F(-\omega) i\omega G_{BS}(\omega) \]

which contains the BaSh mode pole below \( 2\Delta \). This term transfers spectra weight from the pair breaking excitations to the BaSh mode pole as shown by the red solid line in Fig. 2. In the weak coupling limit, the spectra weight of BaSh mode,

\[ A_{\text{BaSh}}(t)/D = \frac{g(t)^2}{2 g(t) + i d t g(t)} \]

is a scaling function of \( t = \omega_{\text{BaSh}} (\omega_{\text{BaSh}} < 2\Delta) \) where \( g(t) = F(2\Delta t) t \). As \( g_p \) becomes larger, the BaSh mode frequency \( \omega_{\text{BaSh}} \) becomes lower and approaches zero as \( g_p \) approaches \( d_g \). At the same time, the spectra weight of the BaSh mode pole grows until it reaches the total spectra weight of the system at \( g_p = d_g \), as shown by Fig. 3. Note that in order for the BaSh mode frequency to be significantly below the gap, \( g_p \) needs to be quite large which also implies a very large BaSh mode spectra weight.

**BaSh polariton**—There are two types of BaSh modes, which may be characterized as longitudinal (polarization parallel to momentum) and transverse (polarization perpendicular to momentum) and \( d - 1 \) fold degenerate. The longitudinal mode couples strongly to electromagnetic fluctuations; the resulting hybridized mode is a BaSh polariton. In 2D, the polariton dispersion in the near field limit \( \omega \ll c q \) can be found from the zeros of the 2D dielectric function:

\[ \epsilon_{2D} = 1 + \frac{2 \pi q t}{\omega} \sigma(\omega) = 0. \]  

Around zero momentum, the polariton frequency starts from \( \omega_{\text{BaSh}} \) and shifts up linearly with momentum due to the Coulomb potential associated with the dipolar fluctuation. In the weak BaSh case, the polariton dispersion is

\[ q = \frac{2 \omega}{v_F^2 D g_p v} \left( \omega - \omega_{\text{BaSh}} \right) \omega \rightarrow \omega_{\text{BaSh}}, \]

\[ q = \frac{2 \Delta^2}{D} \frac{1}{2 \pi (g_p v)^2 + 1} \omega \rightarrow 2\Delta. \]
Around zero momentum, the velocity of the polariton is determined by the spectra weight of the \( g_B \) mode pole: 
\[ v_B = \frac{\pi^2 g_B v D}{(2 \Delta)^2} = \frac{\pi^2}{2 \varepsilon_F} \frac{\omega_B}{\Delta} \rho \] which is at the order of or larger than the Fermi velocity if the Fermi energy \( \varepsilon_F \gg \Delta \). In the strong \( g_B \) mode case, the optical conductivity Eq. (12) becomes that of a Lorentzian oscillator: 
\[ \sigma = \frac{D}{\pi \omega^2 - \omega_B^2} \] and the \( g_B \) polariton dispersion is simply 
\[ q = \frac{1}{2 D} (\omega^2 - \omega_B^2) \] just like the longitudinal phonon polaritons in 2D polar insulators \([19]\) and the exciton polaritons in 2D semiconductors in the near field regime (without a cavity).

In the high frequency regime \( \omega \gg 2\Delta \), the exciton physics becomes irrelevant and the optical conductivity approaches the Drude form \( \sigma \rightarrow i n/(\hbar m a) \), meaning that the \( g_B \) polariton crosses over to the high energy Drude plasmons. The consequences for near field probes \([4, 20, 21]\) can be illustrated by the near field reflection coefficient 
\[ R_B(\omega, q) = \frac{1}{e(\omega, q)} - 1 \] which is plotted in Fig. 4.

The transverse \( g_B \) mode does not couple to the coulomb interaction and is weakly dispersive: \( \omega_q = \omega_{q_B} + O(\omega_q^2 q^2/\Delta) \). The separation of the transverse and longitudinal modes is similar to infrared active polar phonons \([19–21]\). If the 2D excitonic insulator is placed in an optical cavity similar to that studied in Ref. [3], the transverse \( g_B \) mode can be red shifted due to coupling to a higher energy transverse photon. The combined photon/transverse \( g_B \) mode is also referred to as a polariton.

In 3D, the bulk \( g_B \) polariton frequency is determined by root of the 3D dielectric function \( \varepsilon_{3D} = 1 + \frac{4\pi i}{\omega} \sigma(\omega) = 0 \) which is typically too high in energy to be relevant. However, the transverse \( g_B \) mode still has the dispersion shown in Fig. 1(b) and at zero momentum can be measured by far field optics.

**BEC**—We briefly discuss the \( g_B \) polariton in the strong coupling case, in which the excitons are very strongly bound pairs and the transition to the excitonic condensate is essentially a Bose-Einstein condensation (BEC) of these pre-formed pairs. In the BEC state, the \( g_B \) mode corresponds exactly to the atomic excitation of an \( s \) bound state to a \( p \) bound state, just as in a Hydrogen atom. The \( 1s \rightarrow 2p \) transition from light induced \( s \) excitons has been observed by Merkl et al \([22]\). In this excitation, the ‘imaginary’ and ‘real’ \( p \)-wave order parameter fluctuations both appear, corresponding to the interconversion of the dipole moment and current of an oscillating electric dipole. The \( g_B \) mode frequency at zero momentum is thus the energy difference of the two bound states, i.e., \( \hbar \omega_{g_B} = (1-1/4)E_B \) in the case of Coulomb interaction where \( E_B = \frac{1}{2} m e^4/h^2 \) is the \( s \) state binding energy. Its spectra weight in the optical conductivity becomes \( A_{g_B} = \frac{\Delta^2}{\pi^2} \hbar n_{exciton} c_R^2/\rho \) where \( n_{exciton} \) is the number of excitons in the condensate. The dimensionless number \( c_R \sim 1 \) is defined as \( \langle s(x)p \rangle = c_R a_b \) where \( a_b = 2\hbar^2/(me^2) \) is the Bohr radius. This examination of the very strong coupling limit thus shows that in general the \( g_B \) exciton has both imaginary and real components.

**Electron hole bilayer**—In an electron hole bilayer, due to the non-negligible distance \( a \) between the electron layer and the hole layer, the acoustic phase mode also couples to light since it is an exciton density fluctuation which induces local accumulation of \( z \) direction dipole moment. This dipole moment costs Coulomb potential which can be estimated as the charging energy of a capacitor. This in turn shifts...
up the velocity of this ‘superfluid’ sound. This Coulomb coupling effect can also be understood as a repulsive interaction between the excitons, each carrying a z direction dipole. To describe this mode, one needs to assume $A_1 \neq A_2$ in Eq. (4) to account for the difference of the EM field on the two layers. Performing a local gauge transformation $(\psi_1, \psi_2) \rightarrow (\psi_1 e^{i \theta}, \psi_2 e^{-i \theta})$ where $2 \theta$ is equal to the local phase of the s-wave gap, one effectively moves the phase terms in the off-diagonal components to the diagonal terms [4] in Eq. (4). After integrating out the fermions, one obtains the low energy effective action

$$\mathcal{L} = \frac{V}{2} (\psi^2 \theta + \phi^2) + \frac{n}{2m} (\nabla \theta - A_0)^2 \quad (19)$$

for the phase fluctuation where $(\phi, A_0, A_2) = (\phi_1 - \phi_2, A_1 - A_2)$ are the anti symmetric components of the EM field. The latter couples to the phase which is accompanied by anti symmetric charge fluctuations in the two layers. The symmetric component of the EM field does not couple to the phase mode. In the quasi static limit $\omega \ll cq$, the kinetic action of the anti symmetric EM field is just its electric field energy which reads $S_a = \sum_q \phi_a(q)^2 / (2 V_{\text{eff}}(q))$ in the gauge $A_a = 0$. The effective interaction kernel is $V_a(q) = (1 - e^{-\mu q}) 2 \pi / q$. Adding the kinetic EM action to Eq. (19) and solving the equation of motion, one obtains the dispersion of the phase mode

$$\omega_{\text{phase}}(q) = q \sqrt{\frac{1}{V} + V_{\text{eff}}(q)} \frac{n}{m} \approx q \sqrt{\frac{\pi \nu^2}{d + 2Da}} \quad (20)$$

which is the same as the anti symmetric plasmon mode of double layer superconductors [4].

A nonzero tunneling between the layers induces a Josephson effect in the electron hole bilayer system [23] which also gives a nonzero gap to the phase mode. However, we don’t consider this physics here.

The response of the phase mode to near field probe can be represented by its contribution to the near field reflection coefficient

$$R_{\text{phase}}(\omega, q) = \frac{1}{4\pi} \frac{n}{m} q^3 V_{\text{eff}}(q)^2 (2\pi)^{3/2} / \omega^2 - \omega_{\text{phase}}(q)^2 . \quad (21)$$

In the case of $2Da \gg v_F^2$ (which is true for layer separation in typical devices [4–16]), the phase mode shows up in the near field response with a spectra weight of $(aq)^2 \sqrt{\pi / (m \nu)(2\pi)^{3/2}}$ which is smaller than that of the BaSh polaron by roughly the factor $(aq)^{3/2} \ll 1$.

Discussion—In electron hole bilayer made of transition metal dichalcogenides (TMD) [14, 15], semiconductor quantum wells [16] or double bilayer graphene [24], the BaSh polaron is the only observable collective mode at energies close to the gap. In natural crystals suspected to exhibit excitonic insulating phase, such as Ta$_2$NiSe$_5$ [12, 13] and 1T-TiSe$_2$ [11], there are complications due to the lattice, but we expect the qualitative feature of the BaSh polaron to apply there. Therefore, far field optics can probe the transverse BaSh mode and near field optics [20, 21, 25–27] is the ideal tool to probe BaSh polaritons. However, if the p-wave coupling is weak or modest, the BaSh mode frequency may be very close to the 2$\Delta$ gap. In Ta$_2$NiSe$_5$, far field measurements [28, 29] have not identified an obvious subgap BaSh mode peak in the optical conductivity. This may be because the BaSh mode frequency is too close to the 2$\Delta$ gap. It is also possible that one of the subgap phonons [29] is actually the BaSh mode.

The damping rate of the BaSh mode is not calculated here. A zero temperature, the BaSh mode is inside the gap which suppresses dissipation. There are still two possible damping pathways for the BaSh polaritons. The first is disorder/phonon induced life time [17] of the electrons/holes and smooth disorder induced inhomogeneous broadening of the BaSh mode. The second is the scattering between the BaSh polaritons or between BaSh modes and phase modes due to nonlinear coupling terms. At nonzero temperature, thermally excited carriers will damp the BaSh modes by contributing a Drude optical conductivity that dissipates electric field energy. Investigation of the damping rate of the BaSh modes is a meaningful future direction of research.

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In the BCS regime of two dimensional excitonic insulators, assuming that the excitonic effects occur near a high symmetry point so lattice effects are unimportant, we can choose \( f_l = \cos(l\theta_k) \) or \( \sin(l\theta_k) \) and the corresponding pairing interaction is \( g_l = \frac{1}{2\pi} \int d\theta \cos(l\theta) \left( g(2k_F \sin(\theta/2)) \right) \). Note that for \( l = 0 \), the \( 1/2\pi \) factor should be changed to \( 1/4\pi \). For Thomas-Fermi screened interaction \( V(q) = \frac{2\pi}{\epsilon(q^2+q^2)} \) in 2D where \( q_{TF}/(2k_F) = \alpha = e^2/((\hbar v_F)) \) and \( \epsilon \) is the dielectric constant of the environment, the s-wave pairing strength is

\[
v_{gs} = \frac{\alpha}{4\pi} \int d\theta \frac{2\pi}{2k_F|\sin(\theta/2)| + q_{TF}} = \frac{\alpha}{\sqrt{1 - \alpha^2}} \frac{1}{\pi} \text{Tanh}^{-1} \left( \sqrt{1 - \alpha^2} \right) \tag{S1}\]

and the p-wave one is

\[
v_{gp} = \frac{\alpha}{2\pi} \int d\theta \frac{2\pi \cos \theta}{2k_F|\sin(\theta/2)| + q_{TF}} = \alpha \left[ \frac{4}{\pi} - 2\alpha + \frac{4}{\pi} \frac{(1 - 2\alpha^2)}{\sqrt{1 - \alpha^2}} \frac{1}{\sqrt{1 - \alpha^2}} \left( \text{Tanh}^{-1} \left( \sqrt{1 - \alpha^2} \right) - \text{Tanh}^{-1} \left( \sqrt{1 - \alpha^2} \right) \right) \right] \tag{S2}\]

where \( \nu = k_F/(\pi\hbar v_F) \) is the normal state density of state without spin degeneracy and \( \alpha = e^2/((\hbar v_F)) \) is the ‘fine structure constant’ in this system.

The pairing interactions are shown in Fig. S1 for the screened Coulomb interaction in 2D. To obtain a substantial \( g_p/(2g_s) \), one needs the high density case where the fermi velocity is large so that the Thomas Fermi wave vector is smaller than the fermi momentum: \( q_{TF}/(2k_F) = \alpha = e^2/((\hbar v_F)) \ll 1 \). Dielectric screening of the environment can further reduce \( \alpha \) and increase \( g_p/(2g_s) \).

### CORRELATION FUNCTIONS

The correlation function \( \chi_{\sigma_1,\sigma_2} \) is defined as

\[
\chi_{\sigma_1,\sigma_2}(q) = \langle \hat{T} \left( \psi^{\dagger \sigma_1} \psi \right)_{r,t} \left( \psi^{\dagger \sigma_2} \psi \right)_{0} \rangle_q = \sum_{\omega_n,k} \text{Tr} \left[ G(k, i\omega_n) \sigma_1 G(k + q, i(\omega_n + \Omega)) \sigma_2 \right] \tag{S3}\]

where \( \hat{T} \) is the time order symbol, \( x = (r, t) \), \( q = (q, i\Omega) \) and

\[
G(k, i\omega_n) = G_\Delta(k, i\omega_n) = \langle \hat{T} \psi(x) \psi^{\dagger}(0) \rangle_{k,i\omega_n} = \frac{1}{i\omega_n - \xi_k \sigma_3 - \Delta \sigma_1} \tag{S4}\]
The electron Green’s function. The BaSh mode propagator is
\[
G_{BSSx}^{-1} = \frac{1}{g_p} + \chi_{\sigma f_1, \sigma f_2}(\omega) = \frac{1}{g_p} + \sum_k \frac{4 \cos^2(\theta_k) E_k}{\omega^2 - 4 E_k^2} = \frac{1}{g_p} - \frac{1}{\alpha} \left( \frac{1}{g_S} + \omega^2 F(\omega) \right),
\] (S5)

where the last equality comes from the gap equation \( \frac{1}{g_S} = \sum_k \frac{1}{E_k} \). The photon kernel is
\[
K_{xx}(\omega) = \frac{n}{m} + \chi_{\sigma v_1, \sigma v_2}(\omega) = \frac{n}{m} + \frac{1}{d} \sum_k \frac{\Delta^2}{\omega E_k - 2 E_k^2} - \frac{4 v_F^2}{\omega^2} = \frac{n}{m} - \frac{4}{d} v_F^2 \Delta^2 F(\omega).
\] (S6)

The linear coupling between BaSh mode and the EM vector potential is
\[
C_{ij}(\omega) = \chi_{\sigma f_1, \sigma f_2}(\omega) = i \Delta \omega \sum_k \frac{2}{E_k} f_i(k) v_j(k) = -2 i \Delta \omega \frac{v_F}{d} F(\omega) \delta_{ij}.
\] (S7)

**STRONG COUPLING CASE**

The full quadratic action for the two components of the \( p \)-wave fluctuations is
\[
S_j(\Delta) = \frac{1}{2} \sum_q \begin{pmatrix} \Delta_{j}^{(1)} \\ \Delta_{j}^{(2)} \end{pmatrix}_{-\omega} M \begin{pmatrix} \Delta_{j}^{(1)} \\ \Delta_{j}^{(2)} \end{pmatrix}_q = \frac{1}{2} \sum_q \begin{pmatrix} \Delta_{j}^{(1)} \\ \Delta_{j}^{(2)} \end{pmatrix}_{-\omega} \begin{pmatrix} \frac{1}{8g_p} + \chi_{\sigma f_1, \sigma f_1}(q) & \chi_{\sigma f_1, \sigma f_2}(q) \\ \chi_{\sigma f_2, \sigma f_1}(q) & \frac{1}{8g_p} + \chi_{\sigma f_2, \sigma f_2}(q) \end{pmatrix}_q \begin{pmatrix} \Delta_{j}^{(1)} \\ \Delta_{j}^{(2)} \end{pmatrix}_q \right)
\] (S8)

which when restricted to zero momentum fluctuations simplifies to
\[
S_j(\Delta) = \frac{1}{2} \sum_{\omega} \begin{pmatrix} \Delta_{j}^{(1)} \\ \Delta_{j}^{(2)} \end{pmatrix}_{-\omega} \begin{pmatrix} \frac{1}{8g_p} \sum_k f_i(k)^2 & f_i(k)^2 \\ f_i(k)^2 & \frac{1}{8g_p} \sum_k f_i(k)^2 \end{pmatrix}_{\omega} -2 i \omega \sum_k \frac{f_i(k)^2}{4 E_k^2 - \omega^2} \xi_k + \frac{1}{8g_p} \sum_k f_i(k)^2 \xi_k \begin{pmatrix} \Delta_{j}^{(1)} \\ \Delta_{j}^{(2)} \end{pmatrix}_{\omega} \right)
\] (S9)

The collective mode frequencies are determined by the zeros of the determinant of the matrix in Eq. S9.

In the weak coupling (BCS) limit studied in the main text, the factor \( \xi_k / E_k \) changes sign as \( k \) crosses \( k_F \) so that in the off-diagonal term the sum of \( k \) gives a small value, \( \theta \Delta / \mu \) relative to the diagonal terms, and in the \( \Delta^{(1)} - \Delta^{(1)} \) term the
factor $\xi_k^2/E_k^2$ ensures that the $\sum_k f_j(\xi_k^2)^4 E_k^2$ does not diverge as $|\omega| \to 2\Delta$, so there is no zero of the inverse response function associated with the real part of $\Delta_j$. The $\Delta^{(2)} - \Delta^{(2)}$ term is just the BaSh kernel studied in the main text.

Away from the weak coupling BCS regime, the off-diagonal terms become non-negligible which means the real and imaginary fluctuations are mixed together in the BaSh mode and some details of the structure of the individual terms change. However, we find that the determinant of the BaSh mode matrix still has one root at frequencies less than the gap; this root is at a frequency lower than the $\omega_{\text{BaSh}}$ defined in Eq. (7), meaning the BaSh mode frequency is pushed down by this cross coupling, and the eigenvector of this mode is thus of mixed imaginary-real characters. The appearance of the BaSh mode in optical conductivity stays qualitatively the same.