The IA-congruence kernel of high rank free Metabelian groups

David El-Chai Ben-Ezra

December 19, 2021

Abstract

The congruence subgroup problem for a finitely generated group \( \Gamma \) and \( G \leq \text{Aut}(\Gamma) \) asks whether the map \( \hat{G} \to \text{Aut}(\hat{\Gamma}) \) is injective, or more generally, what is its kernel \( C(G, \Gamma) \)? Here \( \hat{X} \) denotes the profinite completion of \( X \). In this paper we investigate \( C(IA(\Phi_n), \Phi_n) \), where \( \Phi_n \) is a free metabelian group on \( n \geq 4 \) generators, and \( IA(\Phi_n) = \ker(\text{Aut}(\Phi_n) \to GL_n(\mathbb{Z})) \).

We show that in this case \( C(IA(\Phi_n), \Phi_n) \) is abelian, but not trivial, and not even finitely generated. This behavior is very different from what happens for free metabelian group on \( n = 2, 3 \) generators, or for finitely generated nilpotent groups.

Mathematics Subject Classification (2010): Primary: 19B37, 20H05, Secondary: 20E36, 20E18.

Key words and phrases: congruence subgroup problem, automorphism groups, profinite groups, free metabelian groups.

Contents

1 Introduction 2
2 Some background in algebraic K-theory 7
3 \( IA(\Phi_n) \) and its subgroups 9
4 The subgroups \( C_i \) 15
5 The centrality of \( C_i \) 23
6 Some elementary elements of \( \langle IA(\Phi_n)^m \rangle \) 27
1 Introduction

The classical congruence subgroup problem (CSP) asks for, say, $G = SL_n(\mathbb{Z})$ or $G = GL_n(\mathbb{Z})$, whether every finite index subgroup of $G$ contains a principal congruence subgroup, i.e., a subgroup of the form $G(m) = \ker(G \to GL_n(\mathbb{Z}/m\mathbb{Z}))$ for some $0 \neq m \in \mathbb{Z}$. It is a classical 19th-century result that the answer is negative for $n = 2$. On the other hand, quite surprisingly, it was proved in the sixties by Mennicke [Men] and by Bass-Lazard-Serre [BaLS] that for $n \geq 3$ the answer to the CSP is affirmative. A rich theory of the CSP for more general arithmetic groups has been developed since then.

By the observation $GL_n(\mathbb{Z}) \cong Aut(\mathbb{Z}^n)$, the CSP can be generalized to automorphism groups as follows: Let $\Gamma$ be a group and $G \leq Aut(\Gamma)$. For a finite index characteristic subgroup $M \leq \Gamma$ denote $G(M) = \ker(G \to Aut(\Gamma/M))$.

A finite index subgroup of $G$ which contains $G(M)$ for some $M$ is called a “congruence subgroup”. The CSP for the pair $(G, \Gamma)$ asks whether every finite index subgroup of $G$ is a congruence subgroup.

One can easily see that the CSP is equivalent to the question: Is the congruence map $\hat{G} = \lim \leftarrow G/U \to \lim \leftarrow G/G(M)$ injective? Here, $U$ ranges over all finite index normal subgroups of $G$, and $M$ ranges over all finite index characteristic subgroups of $\Gamma$. When $\Gamma$ is finitely generated, it has only finitely many subgroups of given index $m$, and thus, the characteristic subgroups: $M_m = \cap \{ \Delta \leq \Gamma \mid [\Gamma : \Delta] | m \}$ are of finite index in $\Gamma$. Hence, one can write $\hat{\Gamma} = \lim \leftarrow M_m \subseteq \hat{\Gamma}$ and have

$$\lim \leftarrow G/G(M) = \lim \leftarrow M_m G/G(M_m) \leq \lim \leftarrow M_m Aut(\Gamma/M_m) \leq Aut(\lim \leftarrow M_m (\Gamma/M_m)) = Aut(\hat{\Gamma}).$$

Therefore, when $\Gamma$ is finitely generated, the CSP is equivalent to the question: Is the congruence map: $\hat{G} \to Aut(\hat{\Gamma})$ injective? More generally, the CSP asks what is the kernel $C(G, \Gamma)$ of this map. For $G = Aut(\Gamma)$ we will also use the

---

1 By the celebrated theorem of Nikolov and Segal which asserts that every finite index subgroup of a finitely generated profinite group is open [NS], the second inequality is actually an equality. However, we do not need it.
simpler notation $C(\Gamma) = C(G, \Gamma)$. The classical congruence subgroup result mentioned above can therefore be reformulated as $C(\mathbb{Z}^n) = \{e\}$ for $n \geq 3$, and it is also known that $C(\mathbb{Z}^2) = \hat{F}_\omega$, where $F_\omega$ is the free non-abelian profinite group on a countable number of generators (cf. [Mel], [L]).

Very few results are known when $\Gamma$ is non-abelian. Most of the results are related to $\Gamma = \pi(S_{g,n})$, the fundamental group of the closed surface of genus $g$ with $n$ punctures (see [DDH], [Mel], [A], [Bo1], [Bo2]). As observed in [BER], the result of Asada in [A] actually gives an affirmative solution to the case $\Gamma = F_2$, $G = \text{Aut}(F_2)$ (see also [BL]). Note that for every $n > 0$, one has $\pi(S_{g,n}) \cong F_{2g+n-1}$, the free group on $2g + n - 1$ generators. Hence, the aforementioned results relate to various subgroups of the automorphism group of finitely generated free groups. However, the CSP for the full $\text{Aut}(F_n)$ when $n \geq 3$ is still unsettled.

Denote now the free metabelian group on $n$ generators by $\Phi_n = F_n/F'_n$. Considering the metabelian case, it was shown in [BL] (see also [Be1]) that $C(\Phi_2) = \hat{F}_\omega$. In addition, it was proven there that $C(\Phi_3) \supseteq \hat{F}_\omega$. The basic motivation which led to this paper was to complete the picture in the free metabelian case and investigate $C(\Phi_n)$ for $n \geq 4$. Now, denote $IA(\Phi_n) = \ker(\text{Aut}(\Phi_n) \to GL_n(\mathbb{Z}))$. Then, the commutative exact diagram

$$1 \to IA(\Phi_n) \to \text{Aut}(\Phi_n) \to GL_n(\mathbb{Z}) \to 1$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\text{Aut}(\Phi_n) \to GL_n(\hat{\mathbb{Z}})$$

gives rise to the commutative exact diagram (see Lemma 2.1 in [BER])

$$IA(\Phi_n) \to \text{Aut}(\Phi_n) \to GL_n(\mathbb{Z}) \to 1$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\text{Aut}(\Phi_n) \to GL_n(\hat{\mathbb{Z}})$$.

Hence, by using the fact that $GL_n(\mathbb{Z}) \to GL_n(\hat{\mathbb{Z}})$ is injective for $n \geq 3$, one can obtain that $C(\Phi_n)$ is an image of $C(IA(\Phi_n), \Phi_n)$. Thus, for investigating $C(\Phi_n)$ it seems to be worthwhile to investigate $C(IA(\Phi_n), \Phi_n)$.

The first goal of the present paper is to prove the following theorem:

**Theorem 1.1.** For every $n \geq 4$, the group $C(IA(\Phi_n), \Phi_n)$ contains a subgroup $C$ which satisfies the following properties:

- $C$ is isomorphic to a product $C = \prod_{i=1}^{n} C_i$ of $n$ copies of $C_i \cong \ker(SL_{n-1}(\mathbb{Z}[x^{\pm 1}]) \to SL_{n-1}(\mathbb{Z}[x^{\pm 1}]))$.

- $C$ is a direct factor of $C(IA(\Phi_n), \Phi_n)$, i.e. there is a normal subgroup $N \triangleleft C(IA(\Phi_n), \Phi_n)$ such that $C(IA(\Phi_n), \Phi_n) = N \times C$.

Using techniques of Kassabov and Nikolov in [KN] one can show that the subgroups $C_i$ are not finitely generated. So as an immediate corollary, we obtain the following theorem:
Theorem 1.2. For every \( n \geq 4 \), the group \( C(IA(\Phi_n), \Phi_n) \) is not finitely generated.

It will be shown in an upcoming paper that when \( \Gamma \) is a finitely generated nilpotent group (of any class), then \( C(IA(\Gamma), \Gamma) = \{e\} \) is always trivial. So the free metabelian cases behave completely different from nilpotent cases. This result gives the impression that \( C(IA(\Phi_n), \Phi_n) \) is “big”. On the other hand, we have the following theorem (see [Be2]):

Theorem 1.3. For every \( n \geq 4 \), the group \( C(IA(\Phi_n), \Phi_n) \) is central in \( \widehat{IA}(\Phi_n) \).

We remark that in the case of arithmetic groups, the congruence kernel is known to have a dichotomous behavior: it is central if and only if it is finite (see [PR], Theorem 2). So in some sense, the congruence kernel \( C(IA(\Phi_n), \Phi_n) \) for \( n \geq 4 \) has an intermediate behavior: central, but not finite. The latter is similar to the behavior of the congruence kernel \( \ker(\widehat{SL_d(\mathbb{Z}[x])} \to SL_d(\mathbb{Z}[x])) \) for \( d \geq 3 \) that was investigated in [KN] (see Theorem 4.1).

Theorem 1.3 has already been stated in [Be2]. However, a substantial portion of the proof of Theorem 1.3 appears in this paper - this is the second goal of this present paper. To be more precise, all the steps of the proof of Theorem 1.3 that involve arguments in Algebraic K-theory are given in this paper, and in [Be2] we describe the structure of the proof, and present all the other steps. As will be presented in [5] the steps that are given in this present paper by themselves, will be sufficient for showing that the subgroup \( C \leq C(IA(\Phi_n), \Phi_n) \) that presented in Theorem 1.1 is contained in the center of \( \widehat{IA(\Phi_n)} \). We remark that the main results in this paper that are used in [Be2] in order to prove Theorem 1.3 are Lemma 7.1 and our work in [5] (see Remark 5.4 for a more precise description). The following problem is still open:

Problem 1.4. Is \( C(IA(\Phi_n), \Phi_n) = \prod_{i=1}^{n} C_i \) or does it contain more elements?

Remark 1.5. Considering the action of \( Aut(\Phi_n) \) on \( IA(\Phi_n) \) by conjugation, we have a natural map \( Aut(\Phi_n) \to Aut(IA(\Phi_n)) \) in which the copy of \( IA(\Phi_n) \) in \( Aut(\Phi_n) \) is mapped onto \( IA(\Phi_n) \to Inn(IA(\Phi_n)) \). Denote now \( IA_{n,m} = \cap\{N \leq IA(\Phi_n) \ | \ [IA(\Phi_n) : N] \mid m\} \). Then as for every \( n \geq 4 \), the group \( IA(\Phi_n) \) is finitely generated [BM], the characteristic subgroups \( IA_{n,m} \leq IA(\Phi_n) \) are of finite index. Hence \( IA(\Phi_n) = \lim_{m \in \mathbb{N}} (IA(\Phi_n)/IA_{n,m}) \) and therefore the action of \( Aut(\Phi_n) \) on \( IA(\Phi_n) \) induces an action of \( Aut(\Phi_n) \) on \( IA(\Phi_n) \) so we have a map \( Aut(\Phi_n) \to \lim_{m \in \mathbb{N}} Aut(IA(\Phi_n)/IA_{n,m}) \leq Aut(IA(\Phi_n)) \). The latter gives rise to a map

\[
Aut(\Phi_n) \to \lim_{m \in \mathbb{N}} Aut(IA(\Phi_n)/IA_{n,m}) \leq Aut(IA(\Phi_n))
\]
that actually gives an action of $Aut(\Phi_n)$ on $IA(\Phi_n)$, such that the closure $\overline{IA(\Phi_n)}$ of $IA(\Phi_n)$ in $Aut(\Phi_n)$ acts trivially on $Z(\overline{IA(\Phi_n)})$, the center of $IA(\Phi_n)$. Thus, as we have $Aut(\Phi_n)/\overline{IA(\Phi_n)} = GL_n(\mathbb{Z})$ we obtain a natural action of $GL_n(\mathbb{Z})$ on $Z(IA(\Phi_n))$. It will be clear from the description in the paper that the permutation matrices permute the copies $C_i$ through this natural action.

The aforementioned behavior of $C(IA(\Phi_n), \Phi_n)$ for $n \geq 4$ is also different from the behavior of $C(IA(\Phi_n), \Phi_n)$ for $n = 2, 3$. More precisely, as $C(\mathbb{Z}^3) = \{e\}$, similar arguments show that when $n = 3$ the group $C(\Phi_3)$ is an image of $C(IA(\Phi_3), \Phi_3)$. So as $C(\Phi_3) \supseteq F_2$ \cite{BL}, we obtain that $C(IA(\Phi_3), \Phi_3)$ is infinite non-abelian. On the other hand, regarding the case $n = 2$, it is known that $IA(\Phi_2) = Inn(\Phi_2)$ (see \cite{Bac}) and it is known that the center of $\Phi_2$ and $\hat{\Phi}_2$ is trivial (see \cite{Be1}). It follows that we have a canonical isomorphism

$$IA(\hat{\Phi}_2) = Inn(\Phi_2) \cong \hat{\Phi}_2 \cong Inn(\hat{\Phi}_2) \leq Aut(\hat{\Phi}_2)$$

so $C(IA(\Phi_2), \Phi_2) = \{e\}$ is trivial. Our results show that when $n \geq 4$, the behavior of $C(IA(\Phi_n), \Phi_n)$ stabilizes and it is abelian, but not trivial.

We also note that considering our basic motivation, as $C(\Phi_n)$ is an image of $C(IA(\Phi_n), \Phi_n)$ we actually obtain from Theorem 1.3 that when $n \geq 4$, the situation is dramatically different from the cases of $n = 2, 3$ described above, and:

**Theorem 1.6.** For every $n \geq 4$, the group $C(\Phi_n)$ is abelian.

We remark that despite the result of the latter theorem, we do not know whether $C(\Phi_n)$ is also not finitely generated. In fact we cannot even prove at this point that it is not trivial.

The paper is organized as follows. For a ring $R$, ideal $H < R$ and $d \in \mathbb{N}$ denote

$$GL_d(R, H) = \ker(GL_d(R) \rightarrow GL_d(R/H)).$$

For $n \in \mathbb{N}$ denote also the ring $R_n = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = \mathbb{Z}[[x]]$. Using the Magnus embedding of $IA(\Phi_n)$, in which $IA(\Phi_n)$ can be viewed as

$$IA(\Phi_n) = \left\{ A \in GL_n(R_n) \mid A \begin{pmatrix} x_1 - 1 \\ \vdots \\ x_n - 1 \end{pmatrix} = \begin{pmatrix} x_1 - 1 \\ \vdots \\ x_n - 1 \end{pmatrix} \right\}$$

we obtain for every $1 \leq i \leq n$, a natural embedding

$$GL_{n-1}(R_n, (x_i - 1)R_n) \hookrightarrow IA(\Phi_n)$$

and a surjective natural homomorphism

$$IA(\Phi_n) \xrightarrow{\rho_i} GL_{n-1}(\mathbb{Z}[x_i^{\pm 1}], (x_i - 1)\mathbb{Z}[x_i^{\pm 1}])$$
in which the obvious copy of the subgroup $GL_{n-1}(\mathbb{Z}[x_1^\pm 1], (x_i - 1)\mathbb{Z}[x_i^\pm 1])$ in $GL_{n-1}(R_n, (x_i - 1)R_n)$ is mapped onto itself via the composition map (Proposition 3.6). This description, combined with some classical notions and results from Algebraic K-theory presented in \[1\] enables us in \[2\] to show that for every $n \geq 4$ and $1 \leq i \leq n$, the group $C(IA(\Phi_n), \Phi_n)$ contains a copy of

$$C_i \cong \ker(GL_{n-1}(\mathbb{Z}[x_i^\pm 1], (x_i - 1)\mathbb{Z}[x_i^\pm 1]) \to GL_{n-1}(\mathbb{Z}[x_i^\pm 1]))$$

$$\cong \ker(SL_{n-1}(\mathbb{Z}[x_i^\pm 1]) \to SL_{n-1}(\mathbb{Z}[x_i^\pm 1]))$$

(1.1)

such that $C(IA(\Phi_n), \Phi_n)$ is mapped onto $C_i$ through the map $\hat{\rho}_i$ which is induced by $\rho_i$. The second isomorphism in Equation (1.1) is obtained by using a main lemma, Lemma 7.1, combined with some classical results from Algebraic K-theory (Propositions 4.5 and 4.6). The proof of Lemma 7.1 will be postponed until the end of the paper. In particular, we get that for every $1 \leq i \leq n$ one has

$$C(IA(\Phi_n), \Phi_n) = (C(IA(\Phi_n), \Phi_n) \cap \ker \hat{\rho}_i) \times C_i.$$ 

(see Proposition 3.3). In \[3\] we also show that the copies $C_i$ lie in $\ker \hat{\rho}_j$ whenever $j \neq i$ (Proposition 1.2). In particular we get that the copies $C_i$ intersect each other trivially. Then, following the techniques of Kassabov and Nikolov in \[4\] we show that $C_i$ is not finitely generated, and therefore deduce that $C(IA(\Phi_n), \Phi_n)$ is not finitely generated either, i.e. we prove Theorem 1.2 (see the end of \[4\]). Then, in \[5\] we show that the copies $C_i$ lie in the center of $IA(\Phi_n)$, using classical results from Algebraic K-theory and the main lemma, Lemma 7.1. In particular, using the aforementioned results, we obtain that

$$C(IA(\Phi_n), \Phi_n) = (C(IA(\Phi_n), \Phi_n) \cap \ker \hat{\rho}_i) \times \prod_{i=1}^{n} C_i.$$ 

This completes the proof of Theorem 1.1.

After that we turn to prove the main lemma, Lemma 7.1. In \[6\] we introduce some elements in $\langle IA(\Phi_n)^m \rangle$ which are needed for the proof of Lemma 7.1. In \[7\] using classical results from algebraic K-theory, we end the paper by proving Lemma 7.1 which asserts that for every $1 \leq i \leq n$, we have

$$GL_{n-1}(R_n, (x_i - 1)R_n) \cap E_{n-1}(R_n, H_{n,m^2}) \subseteq \langle IA(\Phi_n)^m \rangle$$

(1.2)

where:

- $GL_{n-1}(R_n, (x_i - 1)R_n)$ denotes its appropriate copy in $IA(\Phi_n)$ described above.
- $E_{n-1}(R_n, H_{n,m^2})$ is the subgroup of $E_{n-1}(R_n) = \langle I_{n-1} + rE_{i,j} \mid r \in R_n \rangle$ which is generated as a normal subgroup by the elementary matrices of the form $I_{n-1} + hE_{i,j}$ for $h \in H_{n,m^2} = \ker(R_n \to \mathbb{Z}_{m^2}[\mathbb{Z}_{m^2}], 1 \leq i \neq j \leq n)$. Here, $I_{n-1}$ is the $(n-1) \times (n-1)$ unit matrix and $E_{i,j}$ is the matrix which has 1 in the $(i,j)$-th entry and 0 elsewhere.
The intersection in Inclusion (1.2) is obtained by viewing the copy of $GL_{n-1}(R_n, (x_i - 1)R_n)$ in $IA(\Phi_n)$ as a subgroup of $GL_{n-1}(R_n)$.

We note that as described above, Lemma 7.1 is used in two places along the paper. It is used once to prove the second isomorphism in Equation (1.1). The second place is in the proof that the group $C$ lies in the center of $\hat{I}A(\Phi_n)$. We also note that almost all the work that we do in order to show that $C$ lies in the center of $\hat{I}A(\Phi_n)$, including Lemma 7.1 (but also most of §5), is used in [Be2] to prove Theorem 1.3 (see Remark 5.4).

Acknowledgements: I wish to offer my thanks to my supervisor during the research, Prof. Alexander Lubotzky, for his sensitive and devoted guidance. During the period of the research, I was supported by the Rudin foundation and, not concurrently, by NSF research training grant (RTG).

2 Some background in algebraic K-theory

In this section we fix some notations and recall some definitions and background in algebraic K-theory which will be used throughout the paper. One can find more general information in the references ([Ros], [Mi], [Bas]). In this section $R$ will always denote a commutative ring with identity. We start with recalling the following notations. Let $R$ be a commutative ring, $H \triangleleft R$ an ideal, and $d \in \mathbb{N}$. Then:

- $GL_d(R) = \{A \in M_n(R) \mid \det(A) \in R^*\}$.
- $SL_d(R) = \{A \in GL_d(R) \mid \det(A) = 1\}$.
- $E_d(R) = \langle I_d + rE_{i,j} \mid r \in R, 1 \leq i \neq j \leq d \rangle$.
- $GL_d(R, H) = \ker(GL_d(R) \to GL_d(R/H))$.
- $SL_d(R, H) = \ker(SL_d(R) \to SL_d(R/H))$.
- $E_d(R, H)$ is the normal subgroup of $E_d(R)$, which is generated as a normal subgroup by the elementary matrices of the form $I_d + hE_{i,j}$ for $h \in H$.

For every $d \geq 3$, the subgroup $E_d(R, H)$ is normal in $GL_d(R)$ (see Corollary 1.4 in [Sm]). Hence, we can consider the following groups:

$$
K_1(R; d) = GL_d(R) / E_d(R) \quad K_1(R, H; d) = GL_d(R, H) / E_d(R, H)
$$

$$
SK_1(R; d) = SL_d(R) / E_d(R) \quad SK_1(R, H; d) = SL_d(R, H) / E_d(R, H).
$$

We now go ahead with the following definition:

Definition 2.1. Let $R$ be a commutative ring, and $3 \leq d \in \mathbb{N}$. We define the “Steinberg group” $St_d(R)$ to be the group which generated by the elements $x_{i,j}(r)$ for $r \in R$ and $1 \leq i \neq j \leq d$, under the relations:

- $x_{i,j}(r_1) \cdot x_{i,j}(r_2) = x_{i,j}(r_1 + r_2)$. 

• \( [x_{i,j} \left( r_1 \right), x_{k,l} \left( r_2 \right)] = x_{i,k} \left( r_1 \cdot r_2 \right) \).

• \( [x_{i,j} \left( r_1 \right), x_{k,l} \left( r_2 \right)] = 1. \)

for every different \( 1 \leq i, j, k, l \leq d \) and every \( r_1, r_2 \in R \).

As the elementary matrices \( I_d + rE_{i,j} \) satisfy the relations which define \( St_d \left( R \right) \), the map \( x_{i,j} \left( r \right) \mapsto I_d + rE_{i,j} \) defines a natural homomorphism \( \phi_d : St_d \left( R \right) \rightarrow E_d (R) \). The kernel of this map is denoted by \( K_2 \left( R; d \right) = \ker \left( \phi_d \right) \).

Now, for two invertible elements \( u, v \in R^* \) and \( 1 \leq i \neq j \leq d \) define the “Steinberg symbol” by

\[
\{ u, v \}_{i,j} = h_{i,j} \left( uv \right) h_{i,j} \left( u \right)^{-1} h_{i,j} \left( v \right)^{-1} \in St_d \left( R \right)
\]

where \( h_{i,j} \left( u \right) = w_{i,j} \left( u \right) w_{i,j} \left( -1 \right) \)

and \( w_{i,j} \left( u \right) = x_{i,j} \left( u \right) x_{j,i} \left( -u^{-1} \right) x_{i,j} \left( u \right). \)

One can show that \( \{ u, v \}_{i,j} \in K_2 \left( R; d \right) \) and lie in the center of \( St_d \left( R \right) \). In addition, for every \( 3 \leq d \in \mathbb{N} \), the Steinberg symbols \( \{ u, v \}_{i,j} \) do not depend on the indices \( i, j \), so they can be denoted simply by \( \{ u, v \} \) (see [DS]). The Steinberg symbols satisfy many identities. For example

\[
\{ uv, w \} = \{ u, w \} \{ v, w \}, \quad \{ u, vw \} = \{ u, v \} \{ u, w \}.
\]

(2.1)

In the semi-local case we have the following [SD, Theorem 2.7]:

**Theorem 2.2.** Let \( R \) be a semi-local commutative ring and \( d \geq 3 \). Then, \( K_2 \left( R; d \right) \) is generated by the Steinberg symbols \( \{ u, v \} \) for \( u, v \in R^* \). In particular, \( K_2 \left( R; d \right) \) is central in \( St_d \left( R \right) \).

Let now \( R \) be a commutative ring, \( H \lhd R \) an ideal and \( d \geq 3 \). Denote \( \bar{R} = R/H \). Clearly, there is a natural map \( E_d \left( R \right) \rightarrow E_d \left( \bar{R} \right) \). It is clear that \( E_d \left( R, H \right) \) lies in the kernel of the latter map, so we have a map

\[
\pi_d : E_d \left( R \right) / E_d \left( R, H \right) \rightarrow E_d \left( \bar{R} \right).
\]

In addition, it is easy to see that we have a surjective map

\[
\psi_d : St_d \left( \bar{R} \right) \rightarrow E_d \left( R \right) / E_d \left( R, H \right)
\]

defined by: \( x_{i,j} \left( r \right) \mapsto I_d + rE_{i,j} \), such that \( \phi_d : St_d \left( \bar{R} \right) \rightarrow E_d \left( \bar{R} \right) \) satisfies: \( \phi_d = \pi_d \circ \psi_d \). Therefore, we obtain the surjective map

\[
K_2 \left( \bar{R}; d \right) = \ker \left( \phi_d \right) \xrightarrow{\psi_d} \ker \left( \pi_d \right) = \left( E_d \left( R \right) \cap SL_d \left( R, H \right) \right) / E_d \left( R, H \right)
\]

\[
\leq SK_1 \left( R, H; d \right).
\]

In particular, it implies that if \( E_d \left( R \right) = SL_d \left( R \right) \), then we have a natural surjective map

\[
K_2 \left( R/H; d \right) \rightarrow SK_1 \left( R, H; d \right).
\]

From this one can easily deduce the following corollary, which will be needed later in the paper.
Corollary 2.3. Let \( R \) be a commutative ring, \( H \triangleleft R \) ideal of finite index and \( d \geq 3 \). Assume also that \( E_d (R) = SL_d (R) \). Then:

1. \( SK_1 (R, H; d) \) is a finite group.
2. \( SK_1 (R, H; d) \) is central in \( GL_d (R) / E_d (R, H) \).
3. Every element of \( SK_1 (R, H; d) \) has a representative in \( SL_d (R, H) \) of the form \( \begin{pmatrix} A & 0 \\ 0 & I_{d-2} \end{pmatrix} \) such that \( A \in SL_2 (R, H) \).

Proof. The ring \( \bar{R} = R/H \) is finite. In particular, \( \bar{R} \) is Artinian and hence semi-local. Thus, by Theorem 2.2 \( K_2 (\bar{R}; d) \) is an abelian group which is generated by the Steinberg symbols \( \{ u, v \} \) for \( u, v \in \bar{R}^* \). As \( \bar{R} \) is finite, so is the number of Steinberg symbols. From Equation (2.1) we obtain that the order of any Steinberg symbol is finite. So \( K_2 (\bar{R}; d) \) is a finitely generated abelian group whose generators are of finite order. Thus, \( K_2 (\bar{R}; d) \) is finite. Moreover, as \( \bar{R} \) is semi-local, Theorem 2.2 implies that \( K_2 (\bar{R}; d) \) is central \( St_d (\bar{R}) \). Now, as we assume that \( E_d (R) = SL_d (R) \), we obtain that \( SK_1 (R, H; d) \) is the image of \( K_2 (\bar{R}; d) \) under the surjective map

\[ St_d (\bar{R}) \rightarrow E_d (R) / E_d (R, H) = SL_d (R) / E_d (R, H). \]

This implies Part 1 and that \( SK_1 (R, H; d) \) is central in \( SL_d (R) / E_d (R, H) \).

Moreover, as \( d \geq 3 \), we have \( \{ u, v \} = \{ u, v \}_{1,2} \) for every \( u, v \in \bar{R}^* \). Now, it is easy to check from the definition of the Steinberg symbols that the image of \( \{ u, v \}_{1,2} \) under the map \( St_d (\bar{R}) \rightarrow SL_d (R) / E_d (R, H) \) is of the form

\[ \begin{pmatrix} A & 0 \\ 0 & I_{d-2} \end{pmatrix} \cdot E_d (R, H) \]

for some \( A \in SL_2 (R, H) \). So as \( SK_1 (R, H; d) \) is generated by the images of the Steinberg symbols, the same holds for every element in \( SK_1 (R, H; d) \). So we obtain Part 3. Now, as \( d \geq 3 \) we can write

\[ GL_d (R) = SL_d (R) \cdot \{ I_d + (r-1) E_{3,3} \mid r \in \bar{R}^* \}. \]

Observe also that mod \( E_d (R, H) \), all the elements of the form \( I_d + (r-1) E_{3,3} \) for \( r \in \bar{R}^* \) commute with all the elements of the form (2.2). Hence, the centrality of \( SK_1 (R, H; d) \) in \( SL_d (R) / E_d (R, H) \) shows that actually \( SK_1 (R, H; d) \) is central in \( GL_d (R) / E_d (R, H) \), as required in Part 2.

3 IA(Φₙ) and its subgroups

We start our discussion of the IA-automorphism group of the free metabelian group, \( G = IA(\Phiₙ) = \ker (Aut(\Phiₙ) \rightarrow Aut(\Phiₙ/\Phiₙ')) = GL_n (\mathbb{Z}) \), by presenting some of its properties and subgroups. We begin with the following notations:
\( \Phi = \Phi_n = F_n/F_n'' \) is the free metabelian group on \( n \) elements. Here \( F_n'' \) denotes the second derivative of \( F_n \), the free group on \( n \) elements.

\( \Psi_m = \Phi/M_m \), where \( M_m = (\Phi'(\Phi^m))' (\Phi'(\Phi^m))^m \).

\( IG_m = G(M_m) = \ker (IA(\Phi) \to Aut(\Psi_m)) \).

\( IA_m = \cap \{ N \subset IA(\Phi) \mid [IA(\Phi):N]\mid m \} \)

\( R_n = \mathbb{Z}[\mathbb{Z}^n] = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) where \( x_1, \ldots, x_n \) are the generators of \( \mathbb{Z}^n \).

\( \mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z} \).

\( \sigma_i = x_i - 1 \) for \( 1 \leq i \leq n \).

\( \vec{\sigma} \) is the column vector which has \( \sigma_i \) in its \( i \)-th entry.

\( \Xi = \sum_{i=1}^n \sigma_i R_n \) is the augmentation ideal of \( R_n \).

\( H_m = \ker (R_n \to \mathbb{Z}_m[\mathbb{Z}_m^n]) = \sum_{i=1}^n (x_i^m - 1) R_n + mR_n \).

By the well known Magnus embedding (see [Bi], [RS], [Ma]), one can identify \( \Phi \) with the matrix group

\[
\Phi = \left\{ \begin{pmatrix} g & a_1 t_1 + \ldots + a_n t_n \\ 0 & 1 \end{pmatrix} \mid g \in \mathbb{Z}^n, a_i \in R_n, g - 1 = \sum_{i=1}^n a_i \sigma_i \right\}
\]

where \( t_i \) is a free basis for an \( R_n \)-module, under the identification of the generators of \( \Phi \) with the matrices

\[
\begin{pmatrix} x_i & t_i \\ 0 & 1 \end{pmatrix}
\]

for \( 1 \leq i \leq n \).

Moreover, for every \( \alpha \in IA(\Phi) \), one can describe \( \alpha \) by its action on the generators of \( \Phi \), by

\[
\alpha : \begin{pmatrix} x_i & t_i \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} x_i & a_{i,1} t_1 + \ldots + a_{i,n} t_n \\ 0 & 1 \end{pmatrix}.
\]

This description gives an injective homomorphism (see [Bac], [Bi])

\[
IA(\Phi) \leftrightarrow GL_n(R_n)
\]

defined by \( \alpha \mapsto \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} \)

which gives an identification of \( IA(\Phi) \) with the group

\[
IA(\Phi) = \{ A \in GL_n(R_n) \mid A\vec{\sigma} = \vec{\sigma} \} = \left\{ I_n + A \in GL_n(R_n) \mid A\vec{\sigma} = \vec{0} \right\}.
\]
Consider now the map

$$\Phi = \left\{ \left( \begin{array}{c} g \\ a_1 t_1 + \ldots + a_n t_n \\ 1 \end{array} \right) \mid g \in \mathbb{Z}^n, a_i \in R_n, g - 1 = \sum_{i=1}^n a_i \sigma_i \right\}$$

which is induced by the projections $\mathbb{Z}^n \to \mathbb{Z}_m^n$. Thus, for a given $1 \leq k, l \leq n$, Proposition 3.1.

**Proposition 3.1.** Let $I_n + A \in IA(\Phi)$ and denote the entries of $A$ by $a_{k,l}$ for $1 \leq k, l \leq n$. Then, for every $1 \leq k, l \leq n$, we have: $a_{k,l} \in \sum_{i \neq k, l} \sigma_i R_n \subseteq \mathfrak{A}$.

**Proof.** For a given $1 \leq k \leq n$, the condition $A \sigma_i = 0$ gives the equality

$$0 = a_{k,1} \sigma_1 + a_{k,2} \sigma_2 + \ldots + a_{k,n} \sigma_n.$$

Thus, for a given $1 \leq l \leq n$, the map $R_n \to S_l = \mathbb{Z}[x_i^{\pm 1}]$ which defined by $x_i \mapsto 1$ for every $i \neq l$, maps

$$0 = a_{k,1} \sigma_1 + a_{k,2} \sigma_2 + \ldots + a_{k,n} \sigma_n \mapsto \bar{a}_{k,l} \sigma_l \in \mathbb{Z}[x_i^{\pm 1}].$$

Hence, as $\mathbb{Z}[x_i^{\pm 1}]$ is a domain, $\bar{a}_{k,l} = 0 \in \mathbb{Z}[x_i^{\pm 1}]$. Thus: $a_{k,l} \in \sum_{i \neq k, l} \sigma_i R_n \subseteq \mathfrak{A}$, as required.

**Proposition 3.2.** Let $I_n + A \in IA(\Phi)$. Then $\det(I_n + A)$ is of the form:

$$\det(I_n + A) = \prod_{r=1}^n x_r^{s_r} \text{ for some } s_r \in \mathbb{Z}.$$

**Proof.** The invertible elements in $R_n$ are the elements of the form $\pm \prod_{r=1}^n x_r^{s_r}$ (see [CF], chapter 8). Thus, as $I_n + A \in GL_n(R_n)$ we have: $\det(I_n + A) = \pm \prod_{r=1}^n x_r^{s_r}$. However, according to Proposition 3.1, for every entry $a_{k,l}$ of $A$ we have: $a_{k,l} \in \mathfrak{A}$. Hence, under the projection $x_i \mapsto 1$ for every $1 \leq i \leq n$, one has $I_n + A \mapsto I_n$ and thus, $\pm \prod_{r=1}^n x_r^{s_r} = \det(I_n + A) \mapsto \det(I_n) = 1$. Therefore, the option $\det(I_n + A) = - \prod_{r=1}^n x_r^{s_r}$ is impossible, as required.

Let us step forward with the following definition:

**Definition 3.3.** Let $A \in GL_n(R_n)$, and for $1 \leq i \leq n$ denote by $A_{i,i}$ the minor which obtained from $A$ by erasing its $i$-th row and $i$-th column. Now, for every $1 \leq i \leq n$, define the subgroup $IGL_{n-1,i} \subseteq IA(\Phi)$, by

$$IGL_{n-1,i} = \left\{ I_n + A \in IA(\Phi) \mid \text{The } i \text{-th row of } A \text{ is 0, } I_n - A_{i,i} \in GL_{n-1}(R_n, \sigma, R_n) \right\}.$$
Proposition 3.4. For every $1 \leq i \leq n$ we have: $IGL_{n-1,i} \cong GL_{n-1}(R_n, \sigma_i R_n)$.

Proof. The definition of $IGL_{n-1,i}$ gives us a natural projection from $IGL_{n-1,i} \to GL_{n-1}(R_n, \sigma_i R_n)$ which maps an element $I_n + A \in IGL_{n-1,i}$ to $I_{n-1} + A_{i,i} \in GL_{n-1}(R_n, \sigma_i R_n)$. Thus, all we need is to explain why this map is injective and surjective.

Injectivity: Here, it is enough to show that given an element $I_n + A \in IA(\Phi)$, every entry in the $i$-th row is determined uniquely by the other entries in its row. Indeed, as $A$ satisfies the condition $Ad = 0$, for every $1 \leq k \leq n$ we have

$$a_{k,1}\sigma_1 + a_{k,2}\sigma_2 + \ldots + a_{k,n}\sigma_n = 0 \Rightarrow a_{k,i} = \frac{-\sum_{l=1}^{n} a_{k,l}\sigma_l}{\sigma_i}$$

(3.1)

i.e. we have a formula for $a_{k,i}$ in terms of the other entries in its row.

Surjectivity: Without loss of generality we assume $i = n$. Let $I_{n-1} + \sigma_n B \in GL_{n-1}(R_n, \sigma_n R_n)$, and denote by $\vec{b}_i$ the column vectors of $B$. Define

$$\begin{pmatrix} I_{n-1} + \sigma_n B & -\sum_{l=1}^{n-1} \sigma_l \vec{b}_i \\ 0 & 1 \end{pmatrix} \in IGL_{n-1,n}$$

and this is clearly a preimage of $I_{n-1} + \sigma_n B$. \hfill \Box

Under the above identification of $IGL_{n-1,i}$ with $GL_{n-1}(R_n, \sigma_i R_n)$, we will use throughout the paper the following notations:

Definition 3.5. Let $H \ll R_n$. We define

$$ISL_{n-1,i}(H) = IGL_{n-1,i} \cap SL_{n-1}(R_n, H)$$

$$IE_{n-1,i}(H) = IGL_{n-1,i} \cap E_{n-1}(R_n, H) \leq ISL_{n-1,i}(H)$$

Observe that as for every $1 \leq i \leq n$, we have

$$GL_{n-1}([x_i^+], \sigma_i \mathbb{Z}[x_i^+]) \leq GL_{n-1}(R_n, \sigma_i R_n)$$

the isomorphism $GL_{n-1}(R_n, \sigma_i R_n) \cong IGL_{n-1,i} \leq IA(\Phi)$ gives also a natural embedding of $GL_{n-1}([x_i^+], \sigma_i \mathbb{Z}[x_i^+])$ as a subgroup of $IA(\Phi)$. Actually:

Proposition 3.6. For every $1 \leq i \leq n$, there is a canonical surjective homomorphism

$$\rho_i : IA(\Phi) \to GL_{n-1}([x_i^+], \sigma_i \mathbb{Z}[x_i^+])$$

such that the following composition map is the identity:

$$GL_{n-1}([x_i^+], \sigma_i \mathbb{Z}[x_i^+]) \hookrightarrow IA(\Phi) \xrightarrow{\rho_i} GL_{n-1}([x_i^+], \sigma_i \mathbb{Z}[x_i^+])$$

Hence $IA(\Phi) = \ker \rho_i \times GL_{n-1}([x_i^+], \sigma_i \mathbb{Z}[x_i^+]$.  

12
isomorphically to itself by

$$I$$

Hence we obtain a map homomorphism

$$GL_n$$

Hence, the above map

$$Z$$

morphism

$$b$$

I

By the identification

Proof. Without loss of generality we assume $$i = n$$. First, consider the homomorphism

$$IA(\Phi) \rightarrow GL_n(Z[x_n^{\pm 1}])$$

which is induced by the projection

$$R_n \rightarrow Z[x_n^{\pm 1}]$$

defined by $$x_j \mapsto 1$$ for every $$j \neq n$$. By Proposition 3.1, given

$$I_n + A \in IA(\Phi)$$,

all the entries of the $$n$$-th column of $$A$$ are in $$\sum_{j=1}^{n-1} \sigma_j R_n$$.

Hence, the above map

$$IA(\Phi) \rightarrow GL_n(Z[x_n^{\pm 1}])$$

is actually a map

$$IA(\Phi) \rightarrow \left\{ I_n + \bar{A} \in GL_n(Z[x_n^{\pm 1}]) \mid \text{the } n \text{-th column of } \bar{A} \text{ is } 0 \right\}.$$ 

Observe now, that the right side of the above map is mapped naturally to

$$GL_{n-1}(Z[x_n^{\pm 1}])$$

by erasing the $$n$$-th column and the $$n$$-th row from every element.

Hence we obtain a map

$$IA(\Phi) \rightarrow GL_{n-1}(Z[x_n^{\pm 1}]).$$

Now, by Proposition 3.1, every entry of $$A$$ such that $$I_n + A \in IA(\Phi)$$, is in $$\mathfrak{A}$$. Thus, the entries of every $$\bar{A}$$ such that $$I_{n-1} + \bar{A} \in GL_{n-1}(Z[x_n^{\pm 1}])$$ is an image of $$I_n + A \in IA(\Phi)$$, are all in $$\sigma_n Z[x_n^{\pm 1}]$$. Hence, we actually obtain a homomorphism

$$\rho_n : IA(\Phi) \rightarrow GL_{n-1}(Z[x_n^{\pm 1}], \sigma_n Z[x_n^{\pm 1}]).$$

Observing that the copy of $$GL_{n-1}(Z[x_n^{\pm 1}], \sigma_n Z[x_n^{\pm 1}])$$ in $$IGL_{n-1,n}$$ is mapped isomorphically to itself by $$\rho_n$$, finishes the proof. 

Proof. By the identification

$$IG_m = \left\{ I_n + A \in GL_n(R_n, H_m) \mid A\sigma = 0 \right\}$$

and by applying the formula of Equation (3.1) to the $$i$$-th column of elements in $$IGL_{n-1,i}$$, it is easy to see that the elements of $$IGL_{n-1,i}$$ which correspond to the elements of $$GL_{n-1}(S_i, \sigma_i, J_{i,m})$$ are clearly in $$\text{Im}(\rho_i) \cap IG_m$$. For the opposite inclusion, without loss of generality assume that $$i = n$$, and let $$I_n + A \in \text{Im}\rho_n \cap IG_m$$. Then $$I_n + A$$ has the form

$$\begin{pmatrix} 1 & 0 \\ \sum_{l=1}^{n-1} \sigma_l b_l \\ 0 & 1 \end{pmatrix} \in IGL_{n-1,n}$$

where the entries of $$B$$ satisfy $$b_{k,l} \in S_n$$ and $$\sum_{j=1}^{n-1} \sigma_j b_{k,j} \in H_m$$. Notice now that for every $$l \neq n$$, by projecting $$\sigma_j \mapsto 0$$ for $$j \neq l, n$$, we see that actually $$\sigma_l b_{k,l} \in H_m$$. From here it is easy to see that we necessarily have $$b_{k,l} \in H_m$$. I.e. $$b_{k,l} \in H_m \cap S_n = (x_n^m - 1)S_n + mS_n = J_{n,m}$$, and the claim follows.
Proposition 3.8. For every $1 \leq i \leq n$ and $m \in \mathbb{N}$ one has

$$\rho_i(IG_{m^2}) \subseteq \text{Im}(\rho_i) \cap IG_m \subseteq \rho_i(IG_m).$$

Proof. As every element in $\text{Im}\rho_i$ is mapped to itself via $\rho_i$ we clearly have $\text{Im}\rho_i \cap IG_m = \rho_i(\text{Im}\rho_i \cap IG_m) \subseteq \rho_i(IG_m)$. On the other hand, if $I_n + A \in IG_{m^2}$ then viewing $\text{Im}\rho_i \cong GL_{n-1}(S_i, \sigma_iS_i)$ for $S_i = \mathbb{Z}[x_i^{\pm 1}]$, the entries of $\rho_i(I_n + A) = I_{n-1} + B$ belong to $(x_i^{m^2} - 1)S_i + m^2\sigma_iS_i$. Observe now that we have: $\sum_{r=0}^{m-1} x_i^{mr} \leq (x_i^m - 1)S_i + mS_i = J_{i,m}$. Hence

$$x_i^{m^2} - 1 = \sigma_i \sum_{r=1}^{m^2-1} x_i^r = \sigma_i \sum_{r=0}^{m-1} x_i^r \sum_{r=0}^{m-1} x_i^{mr} \in \sigma_iJ_{i,m} \quad (3.2)$$

So by proposition 3.7 $\rho_i(I_n + A) \in \text{Im}\rho_i \cap IG_m$ as required. \qed

Proposition 3.9. For every $m \in \mathbb{N}$ and $1 \leq i \leq n$ one has

$$\rho_i(IA_m) = \text{Im}(\rho_i) \cap IA_m$$

where $IA_m = \cap \{N \triangleleft IA(\Phi) \mid [IA(\Phi) : N] \mid m\}$.

Proof. As every element in $\text{Im}\rho_i$ is mapped to itself via $\rho_i$ we clearly have $\text{Im}\rho_i \cap IA_m = \rho_i(\text{Im}\rho_i \cap IA_m) \subseteq \rho_i(IA_m)$. For the opposite, assume that $\alpha \in IA_m$, and denote $\rho_i(\alpha) = \beta \in \text{Im}\rho_i$. We want to show that $\beta \in IA_m$. So let $N \triangleleft IA(\Phi)$ such that $[IA(\Phi) : N] \mid m$. Then obviously $[\text{Im}\rho_i : (N \cap \text{Im}\rho_i)] \mid m$. Thus, as $\rho_i$ is surjective $[IA(\Phi) : \rho_i^{-1}(N \cap \text{Im}\rho_i)] \mid m$ so $\alpha \in \rho_i^{-1}(N \cap \text{Im}\rho_i)$ and hence $\beta = \rho_i(\alpha) \in N \cap \text{Im}\rho_i \leq N$. As this is valid for every such $N$, we have: $\beta \in IA_m$, as required. \qed

We close this section with the following definition:

Definition 3.10. For every $1 \leq i \leq n$, denote

$$IGL'_{n-1,i} = \{I_n + A \in IA(\Phi) \mid \text{The } i\text{-th row of } A \text{ is 0}\}.$$

Obviously, $IGL_{n-1,i} \leq IGL'_{n-1,i}$, and by the same injectivity argument as in the proof of Proposition 3.6 one can deduce that:

Proposition 3.11. The subgroup $IGL'_{n-1,i} \leq IA(\Phi)$ is canonically embedded in $GL_{n-1}(R_n)$, by the map: $I_n + A \mapsto I_{n-1} + A_{i,i}$.

Remark 3.12. Note that in general $IGL_{n-1,i} \subseteq IGL'_{n-1,i}$. For example, $I_4 + \sigma_3E_{1,2} - \sigma_2E_{1,3} \in IGL'_{3,4} \setminus IGL_{3,4}$.  

14
4 The subgroups $C_i$

In this section we define the subgroups $C_i \leq C(IA(\Phi_n), \Phi_n)$, and we show that for each $i$ we can view $C(IA(\Phi_n), \Phi_n)$ as a semi-direct product of $C_i$ with another subgroup. We also show that when $n \geq 4$

$$C_i \cong \ker(SL_{n-1}(\mathbb{Z}[x^\pm 1]) \to SL_{n-1}(\mathbb{Z}[x^\pm 1]))$$

and use it to show that $C(IA(\Phi_n), \Phi_n)$ is not finitely generated. We recall the notations:

- $\Phi = \Phi_n$.
- $\Psi_m = \Phi/M_m$, where $M_m = (\Phi^m)(\Phi^m)^m$.
- $IG_m = G(M_m) = \ker(IA(\Phi) \to Aut(\Psi_m))$.
- $IA_m = \cap \{N < IA(\Phi) \mid [IA(\Phi) : N] \mid m\}$.

It is proven in [Be1] that $\hat{\Phi} = \varprojlim \Psi_m$. So, as for every $m \in \mathbb{N}$ the group $\ker(\Phi \to \Psi_m)$ is characteristic in $\Phi$, we can write explicitly

$$C(IA(\Phi), \Phi) = \ker(IA(\Phi) \to Aut(\Phi))$$

$$= \ker(IA(\Phi) \to \varprojlim Aut(\Psi_m))$$

$$= \ker(IA(\Phi) \to \varprojlim(IA(\Phi)/IG_m))$$

Now, as for every $n \geq 4$ we know that $IA(\Phi)$ is finitely generated (see [BM]), as explained in Remark [1], we have $IA(\Phi) = \varprojlim(IA(\Phi)/IA_m)$. Hence

$$C(IA(\Phi), \Phi) = \ker(IA(\Phi)/IA_m) \to \varprojlim(IA(\Phi)/IG_m)$$

$$= \ker(IA(\Phi)/IA_m) \to \varprojlim(IA(\Phi)/IG_m \cdot IA_m)$$

$$= \varprojlim(IA_m \cdot IG_m/IA_m).$$

Similarly, we can write $C(IA(\Phi), \Phi) = \varprojlim(IA_m \cdot IG_m/IA_m)$.

Remember now that for every $1 \leq i \leq n$ the composition map

$$GL_{n-1}(\mathbb{Z}[x_i^\pm 1], \sigma_i\mathbb{Z}[x_i^\pm 1]) \to IA(\Phi) \overset{\rho_i}{\to} GL_{n-1}(\mathbb{Z}[x_i^\pm 1], \sigma_i\mathbb{Z}[x_i^\pm 1])$$

is the identity on $GL_{n-1}(\mathbb{Z}[x_i^\pm 1], \sigma_i\mathbb{Z}[x_i^\pm 1])$. Hence, the induced composition map of the profinite completions

$$GL_{n-1}(\mathbb{Z}[x_i^\pm 1], \sigma_i\mathbb{Z}[x_i^\pm 1]) \overset{\hat{\rho_i}}{\to} IA(\Phi) \overset{\hat{\rho_i}}{\to} GL_{n-1}(\mathbb{Z}[x_i^\pm 1], \sigma_i\mathbb{Z}[x_i^\pm 1])$$

is the identity on $GL_{n-1}(\mathbb{Z}[x_i^\pm 1], \sigma_i\mathbb{Z}[x_i^\pm 1])$. In particular, the map $\hat{\rho}$ is injective, so we can write

$$GL_{n-1}(\mathbb{Z}[x_i^\pm 1], \sigma_i\mathbb{Z}[x_i^\pm 1]) \overset{\hat{\rho_i}}{\to} IA(\Phi) \overset{\hat{\rho_i}}{\to} GL_{n-1}(\mathbb{Z}[x_i^\pm 1], \sigma_i\mathbb{Z}[x_i^\pm 1]).$$

This enables us to write: $IA(\Phi) = \ker \rho_i \times \text{Im} \rho_i$ and $IA(\Phi) = \ker \hat{\rho_i} \times \text{Im} \hat{\rho_i}$. 

15
Definition 4.1. We define

\[ C_i = C(IA(\Phi), \Phi) \cap \text{Im} \hat{\rho}_i = \ker(\text{Im} \hat{\rho}_i \rightarrow \text{Aut}(\hat{\Phi})). \]

Proposition 4.2. If \(1 \leq i \neq j \leq n\), then \(C_i \subseteq \ker \hat{\rho}_j\). In particular, for every \(i \neq j\) we have: \(C_i \cap C_j = \{e\}\).

Proof. By the explicit description \(\widehat{IA(\Phi)} = \varprojlim(IA(\Phi)/IA_m)\), one can write

\[
C_i = \ker(\text{Im} \hat{\rho}_i \rightarrow \text{Aut}(\hat{\Phi}))
= \ker(\text{Im}(IA_m \cdot \text{Im} \rho_i / IA_m) \rightarrow \frac{\text{Im}(IA(\Phi)/IG_m)}{IA_m})
= \ker(\text{Im}(IA_m \cdot \text{Im} \rho_i / IA_m) \rightarrow \frac{\text{Im}(IA(\Phi)/IG_m \cdot IA_m)}{IA_m})
= \frac{\text{Im}(IA_m \cdot \text{Im} \rho_i) \cap (IA_m \cdot IG_m)}{IA_m}
\]

and similarly \(C_i = \frac{\text{Im}(IA_m \cdot \text{Im} \rho_i) \cap (IA_m \cdot IG_m)}{IA_m}\). We claim now that

\[
(IA_m \cdot \text{Im} \rho_i) \cap (IA_m \cdot IG_m^2) \\
\subseteq IA_m \cdot (\text{Im} \rho_i \cap IG_m) \\
\subseteq (IA_m \cdot \text{Im} \rho_i) \cap (IA_m \cdot IG_m).
\]

The second inclusion is obvious. For the first one, we have to show that if \(ar = bs\) such that \(a, b \in IA_m\), \(r \in \text{Im} \rho_i\) and \(s \in IG_m^2\), then there exist \(c \in IA_m\) and \(t \in \text{Im} \rho_i \cap IG_m\) such that \(ar = bs = ct\). Indeed, write: \(\text{Im} \rho_i \ni r = a^{-1}bs\). Then: \(r = \rho_i(r) = \rho_i(a^{-1}b) \rho_i(s)\), and by Propositions 3.8 and 3.9

\[
\rho_i(a^{-1}b) \in \rho_i(IA_m) = \text{Im} \rho_i \cap IA_m \\
\rho_i(s) \in \rho_i(IG_m^2) \subseteq \text{Im} \rho_i \cap IG_m.
\]

Therefore, by defining \(c = a \cdot \rho_i(a^{-1}b)\), and \(t = \rho_i(s)\) we get the required inclusion. Thus, for \(j \neq i\) we have

\[
C_i = \frac{\text{Im}(IA_m \cdot (\text{Im} \rho_i \cap IG_m)/IA_m)}{\hat{\rho}_j \rightarrow \frac{\text{Im} \rho_j(IA_m) \cdot \rho_j(\text{Im} \rho_i \cap IG_m)/\rho_j(IA_m)}{IA_m}}.
\]

Using the definition of \(\rho_j\) it is not difficult to show that

\[
\rho_j(\text{Im} \rho_i \cap IG_m) = \langle I_n + m(\sigma_i E_{k,j} - \sigma_j E_{k,i}) \mid k \neq i, j \rangle \\
= \rho_j(\langle I_n + m(\sigma_i E_{k,j} - \sigma_j E_{k,i}) \mid k \neq i, j \rangle) \\
= \rho_j(\langle I_n + \sigma_i E_{k,j} - \sigma_j E_{k,i} \mid k \neq i, j \rangle^m) \subseteq \rho_j(IA_m).
\]

Hence, \(C_i \subseteq \ker \hat{\rho}_j\), as required.

We can now prove the following proposition:
Proposition 4.3. For every $1 \leq i \leq n$ we have

$$C_i \hookrightarrow C(IA(\Phi), \Phi) \xrightarrow{\hat{\rho}_i} C_i.$$  

In particular: $C(IA(\Phi), \Phi) = (\ker \hat{\rho}_i \cap C(IA(\Phi), \Phi)) \rtimes C_i.$

Proof. In the proof of Proposition 4.2 we saw that

$$C_i = \varprojlim (IA_m \cdot (\im \rho_i \cap IG_m) / IA_m).$$

Similarly $C_i = \varprojlim (IA_m \cdot (\im \rho_i \cap IG_m^2) / IA_m).$ We remind that by Propositions 3.8 and 3.9 we have

$$\rho_i(IG_m^2) \subseteq \im \rho_i \cap IG_m \subseteq \rho_i(IG_m),$$

$$\rho_i(IA_m) = \im \rho_i \cap IA_m.$$

Therefore, we have

$$C_i = \varprojlim IA_m \cdot (\im \rho_i \cap IG_m) / IA_m = \varprojlim IA_m \cdot (\im \rho_i \cap IG_m^2) / IA_m = C(IA(\Phi), \Phi)$$

$$\xrightarrow{\hat{\rho}_i} \varprojlim \im \rho_i(IA_m) \cdot \rho_i(IG_m) / \rho_i(IA_m) = \varprojlim \im \rho_i(IA_m) \cdot \rho_i(IG_m^2) / \rho_i(IA_m)$$

$$= \varprojlim (\im \rho_i \cap IA_m) \cdot (\im \rho_i \cap IG_m) / (\im \rho_i \cap IA_m)$$

$$= \varprojlim IA_m \cdot (\im \rho_i \cap IG_m) / IA_m = C_i.$$

The latter equality follows from the inclusion $\im \rho_i \cap IG_m \subseteq \im \rho_i.$ \qed

Computing $C_i$

We turn now to the computation of $C_i.$ We are going to show that $C_i$ are canonically isomorphic to

$$\ker(SL_{n-1}(\mathbb{Z}[x^\pm 1]) \to SL_{n-1}(\mathbb{Z}[x^\pm 1]))$$

and going to use it in order to show that $C(IA(\Phi), \Phi)$ is not finitely generated. So fix $n \geq 4, 1 \leq i_0 \leq n,$ and denote:

- $x = x_{i_0}.$
- $\sigma = \sigma_{i_0} = x_{i_0} - 1.$
- $IGL_{n-1} = IGL_{n-1,i_0}.$
- $IE_{n-1}(H) = IE_{n-1,i_0}(H).$
- $S = \mathbb{Z}[x^\pm 1] = \mathbb{Z}[x_{i_0}^\pm 1].$
- $J_m = (x^m - 1) S + mS$ for $m \in \mathbb{N}.$
\[ \bullet \rho = \rho_\circ : IA(\Phi) \to GL_{n-1}(S,\sigma S). \]
\[ \bullet \hat{\rho} = \hat{\rho}_\circ : IA(\Phi) \to GL_{n-1}(\hat{S},\sigma S). \]

Now, write (the last equality is by Proposition 3.7)

\[
C_{i_0} = \ker(\text{Im} \hat{\rho} \to Aut(\hat{\Phi})) \\
= \ker(GL_{n-1}(\hat{S},\sigma S) \to Aut(\hat{\Phi})) \\
= \ker(GL_{n-1}(\hat{S},\sigma S) \to \lim (IA(\Phi)/IG_m)) \\
= \ker(GL_{n-1}(\hat{S},\sigma S) \to \lim \left(GL_{n-1}(S,\sigma S) \cdot IG_m / IG_m\right)) \\
= \ker(GL_{n-1}(\hat{S},\sigma S) \to \lim GL_{n-1}(S,\sigma S) / (GL_{n-1}(S,\sigma S) \cap IG_m)) \\
= \ker(GL_{n-1}(\hat{S},\sigma S) \to \lim GL_{n-1}(S,\sigma S) / GL_{n-1}(S,\sigma J_m)).
\]

Now, by the same computation as in Proposition 3.8 one can show that for every \( m \in \mathbb{N} \) we have \((J_m \cap \sigma S) \subseteq \sigma J_m \subseteq (J_m \cap \sigma S)\), so the latter is equal to

\[
\ker(GL_{n-1}(\hat{S},\sigma S) \to \lim GL_{n-1}(S,\sigma S) / (GL_{n-1}(S,\sigma S) \cap GL_{n-1}(S,J_m))) \\
= \ker(GL_{n-1}(\hat{S},\sigma S) \to \lim GL_{n-1}(S,\sigma S) \cdot GL_{n-1}(S,J_m) / GL_{n-1}(S,J_m)) \\
= \ker(GL_{n-1}(\hat{S},\sigma S) \to \lim GL_{n-1}(S) / GL_{n-1}(S,J_m)) \\
= \ker(GL_{n-1}(\hat{S},\sigma S) \to \lim GL_{n-1}(S/J_m)).
\]

Now, if \( \hat{S} \) is a finite quotient of \( S \), then as \( x \) is invertible in \( S \), its image \( \bar{x} \in \hat{S} \) is invertible in \( \hat{S} \). Thus, there exists \( r \in \mathbb{N} \) such that \( \bar{x}^r = 1_{\hat{S}} \). In addition, there exists \( t \in \mathbb{N} \) such that \( 1_{\hat{S}} + \ldots + 1_{\hat{S}} = 0_{\hat{S}} \). Therefore, for \( m = r \cdot t \) the map \( S \to \hat{S} \) factorizes through \( Z_n[z_m] \cong S/J_m \). Thus, we have \( \hat{S} = \lim \) \( S/J_m \), which implies that: \( GL_{n-1}(\hat{S}) \cong \lim GL_{n-1}(S/J_m) \). Therefore

\[ C_{i_0} = \ker(GL_{n-1}(\hat{S},\sigma S) \to GL_{n-1}(\hat{S})). \]

Now, the short exact sequence

\[ 1 \to GL_{n-1}(S,\sigma S) \to GL_{n-1}(S) \to GL_{n-1}(\mathbb{Z}) \to 1 \]

gives rise to the exact sequence (see [BER], Lemma 2.1)

\[ GL_{n-1}(\hat{S},\sigma S) \to GL_{n-1}(S) \to GL_{n-1}(\mathbb{Z}) \to 1 \]

which gives rise to the commutative diagram

\[ \begin{array}{ccc}
GL_{n-1}(S,\sigma S) & \to & GL_{n-1}(S) \\
\downarrow & & \downarrow \\
\hat{GL}_{n-1}(S) & \to & GL_{n-1}(\mathbb{Z}) \\
\end{array} \]

\[ 1 \]

\[ \begin{array}{ccc}
GL_{n-1}(\hat{S}) & \to & GL_{n-1}(\hat{\mathbb{Z}}) \\
\end{array} \]

\[ 1 \]
Assuming $n \geq 4$ and using the affirmative answer to the classical congruence subgroup problem ([Men, BaLS]), the map: $GL_{n-1}(\mathbb{Z}) \to GL_{n-1}(\overline{\mathbb{Z}})$ is injective. Thus, by diagram chasing we obtain that the kernel ker($GL_{n-1}(S, \sigma S) \to GL_{n-1}(\hat{S})$) is mapped onto ker($GL_{n-1}(S) \to GL_{n-1}(\hat{S})$). In order to proceed from here we need the following lemma:

**Lemma 4.4.** Let $d \geq 3$ and denote: $D_m = \{ I_d + (x^{k,m} - 1) E_{1,1} \mid k \in \mathbb{Z} \}$ for $m \in \mathbb{N}$. Then

$$
\widehat{GL_d}(S) = \varprojlim (GL_d(S) / (D_m E_d(S, J_m)))
$$

$$
\widehat{SL_d}(S) = \varprojlim (SL_d(S) / E_d(S, J_m)).
$$

**Proof.** We will prove the first part and the second is similar but easier. We first claim that $D_m E_d(S, J_m)$ is a finite index normal subgroup of $GL_d(S)$. Indeed, by a well-known result of Suslin [Su], $SL_d(S) = E_d(S)$. Thus, by Corollary 2.3 $SK_1(S, J_m; d) = SL_d(S, J_m) / E_d(S, J_m)$ is finite. As the subgroup $SL_d(S, J_m)$ is of finite index in $SL_d(S)$, so is $E_d(S, J_m)$. Now, it is not difficult to see that the group of invertible elements of $S$ is equal to $S^* = \{ \pm x^k \mid k \in \mathbb{Z} \}$ (see [CE], chapter 8). So as $\{ x^{k,m} \mid k \in \mathbb{Z} \}$ is of finite index in $S^*$, the subgroup $D_m SL_d(S)$ is of finite index in $GL_d(S)$. We deduce that also $D_m E_d(S, J_m)$ is of finite index in $GL_d(S)$. It remains to show that $D_m E_d(S, J_m)$ is normal in $GL_d(S)$.

We already stated previously (see [2]) that $E_d(S, J_m)$ is normal in $GL_d(S)$. Thus, noticing the group identity

$$
gbeg^{-1} = h(h^{-1}gbg^{-1})(gbg^{-1})
$$

it is enough to show that the commutators of the elements of $D_m$ with any set of generators of $GL_d(S)$, are in $E_d(S, J_m)$. By the aforementioned result of Suslin and as $S^* = \{ \pm x^r \mid r \in \mathbb{Z} \}$, the group $GL_d(S)$ is generated by the elements of the forms

1. $I_d + (\pm x - 1) E_{1,1}$
2. $I_d + r E_{i,j}$ \quad $r \in S$, $2 \leq i \neq j \leq d$
3. $I_d + r E_{1,j}$ \quad $r \in S$, $2 \leq j \leq d$
4. $I_d + r E_{i,1}$ \quad $r \in S$, $2 \leq i \leq d$.

Now, obviously, the elements of $D_m$ commute with the elements of the forms 1 and 2. In addition, for the elements of the forms 3 and 4, one can easily compute that

$$
[I_d + (x^{k,m} - 1) E_{1,1}, I_d + r E_{1,j}] = I_d + r (x^{k,m} - 1) E_{1,j} \in E_d(S, J_m)
$$

$$
[I_d + (x^{k,m} - 1) E_{1,1}, I_d + r E_{i,1}] = I_d + r (x^{k,m} - 1) E_{i,1} \in E_d(S, J_m)
$$

for every $2 \leq i, j \leq d$, as required.
Now, clearly, every finite index normal subgroup of $GL_d(S)$ contains $D_m$ for some $m \in \mathbb{N}$. In addition, it is not hard to show that when $d \geq 3$, every finite index normal subgroup $N \triangleleft GL_d(S)$ contains $E_d(S,J)$ for some finite index ideal $J \triangleleft S$ (see [KN], Section 1). Thus, as we saw previously that every finite index normal subgroup $J \triangleleft S_n$ contains $J_m$ for some $m$, we obtain that $GL_d(S) = \varprojlim \left( GL_d(S) / (D_m E_d(S,J_m)) \right)$, as required. \hfill $\square$

In order to prove the following proposition, we are going to use Lemma 7.1 that its proof is left to the last section of the paper.

**Proposition 4.5.** Let $n \geq 4$. Then, the map $\widehat{GL_{n-1}(S,\sigma S)} \rightarrow \widehat{GL_{n-1}(S)}$ is injective. Hence, the surjective map

$$C_\iota = \text{ker}(\widehat{GL_{n-1}(S,\sigma S)} \rightarrow \widehat{GL_{n-1}(S)}) \rightarrow \text{ker}(\widehat{GL_{n-1}(S)} \rightarrow \widehat{GL_{n-1}(S)})$$

is an isomorphism.

**Proof.** We showed in the previous lemma that

$$\widehat{GL_{n-1}(S)} = \varprojlim \widehat{GL_{n-1}(S) / (D_m E_{n-1}(S,J_m))}$$

where: $D_m = \{ I_{n-1} + (x^{k-m} - 1) E_{1,1} | k \in \mathbb{Z} \}$ and $J_m = (x^{m-1} - 1) S + mS$.

Hence, the image of $\widehat{GL_{n-1}(S,\sigma S)}$ in $\widehat{GL_{n-1}(S)}$ is

$$\varprojlim (GL_{n-1}(S,\sigma S) \cdot D_m E_{n-1}(S,J_m)) / (D_m E_{n-1}(S,J_m))$$

$$= \varprojlim GL_{n-1}(S,\sigma S) / (GL_{n-1}(S,\sigma S) \cap D_m E_{n-1}(S,J_m)).$$

Using that $D_m \subseteq GL_{n-1}(S,\sigma S)$, one can see that the latter equals to

$$\varprojlim GL_{n-1}(S,\sigma S) / (D_m (GL_{n-1}(S,\sigma S) \cap E_{n-1}(S,J_m))).$$

Recall now the following notations:

- $R_n = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.
- $H_m = \sum_{i=1}^{n} (x_i^m - 1) R_n + m R_n < R_n$.
- $IE_{n-1}(H_m) = IGL_{n-1} \cap E_{n-1}(R_n, H_m)$ under the identification of $IGL_{n-1}$ with $GL_{n-1}(R_n, \sigma R_n)$.

Then, following the definition of the map $\rho : IA(\Phi) \rightarrow GL_{n-1}(S,\sigma S)$ we have

$$\langle IA(\Phi)^m \rangle \xrightarrow{\rho} \langle GL_{n-1}(S,\sigma S)^m \rangle$$

$$IE_{n-1}(H_m) \xrightarrow{\rho} GL_{n-1}(S,\sigma S) \cap E_{n-1}(S,J_m).$$

So as by the main Lemma (Lemma 7.1) we have $IE_{n-1}(H_m) \subseteq \langle IA(\Phi)^m \rangle$, we have also

$$GL_{n-1}(S,\sigma S) \cap E_{n-1}(S,J_m) \subseteq \langle GL_{n-1}(S,\sigma S)^m \rangle.$$
As obviously $D_m^2 \subseteq \langle GL_{n-1}(S,\sigma S)^m \rangle$, we deduce the following natural surjective maps

\[
\lim_{\leftarrow} GL_{n-1}(S,\sigma S) / (D_m(GL_{n-1}(S,\sigma S) \cap E_{n-1}(S,J_m)))
= \lim_{\leftarrow} GL_{n-1}(S,\sigma S) / (D_m^2(GL_{n-1}(S,\sigma S) \cap E_{n-1}(S,J_m^2)))
\rightarrow \lim_{\leftarrow} GL_{n-1}(S,\sigma S) / (GL_{n-1}(S,\sigma S)^m)
\rightarrow GL_{n-1}(S,\sigma S)
\rightarrow \lim_{\leftarrow} GL_{n-1}(S,\sigma S) / (D_m(GL_{n-1}(S,\sigma S) \cap E_{n-1}(S,J_m)))
\]

such that the composition gives the identity map. Hence, these maps are also injective, and in particular, the map

\[
\lim_{\leftarrow} GL_{n-1}(S,\sigma S) \rightarrow \lim_{\leftarrow} GL_{n-1}(S,\sigma S) / (D_m(GL_{n-1}(S,\sigma S) \cap E_{n-1}(S,J_m)))
\]

is injective, as required.

**Proposition 4.6.** Let $d \geq 3$. Then, the natural embedding $SL_d(S) \leq GL_d(S)$ induces a natural isomorphism

\[
\ker(GL_{\hat{d}}(S) \rightarrow GL_{\hat{d}}(\hat{S})) \cong \ker(SL_{\hat{d}}(S) \rightarrow SL_{\hat{d}}(\hat{S})).
\]

**Proof.** By Lemma 4.4 we have

\[
\ker(GL_{\hat{d}}(S) \rightarrow GL_{\hat{d}}(\hat{S})) = \lim_{\leftarrow} GL_{\hat{d}}(S/J_m)
= \ker(\lim_{\leftarrow} GL_{\hat{d}}(S) / D_mE_d(S,J_m) \rightarrow \lim_{\leftarrow} GL_{\hat{d}}(S) / GL_d(S,J_m))
= \lim_{\leftarrow} GL_{\hat{d}}(S,J_m) / D_mE_d(S,J_m)
\]

where $D_m = \{I_{n-1} + (x^{k-m} - 1)E_{1,1} \mid k \in \mathbb{Z}\}$. We claim now that when $m > 2$ then $GL_d(S,J_m) = D_mSL_d(S,J_m)$. Indeed, for every $A \in GL_d(S,J_m)$ we have $\det(A) = \pm x^k$ for some $k \in \mathbb{Z}$. However, as under the map $S \rightarrow \mathbb{Z}_m[\mathbb{Z}_m]$ we have $A \mapsto I_d$, the map $S \rightarrow \mathbb{Z}_m[\mathbb{Z}_m]$ also implies $\det(A) \mapsto 1$. Hence $\det(A) = \pm x^k$ for some $k \in \mathbb{Z}$, and when $m > 2$ we even get $\det(A) = x^{k-m}$ for some $k \in \mathbb{Z}$. It follows that $GL_d(S,J_m) = D_mSL_d(S,J_m)$. Therefore, since $D_m \cap SL_d(S,J_m) = \{I_d\}$, we deduce that

\[
\ker(GL_{\hat{d}}(S) \rightarrow GL_{\hat{d}}(\hat{S})) = \lim_{\leftarrow} D_mE_d(S,J_m) / D_mE_d(S,J_m)
= \lim_{\leftarrow} SL_d(S,J_m) / E_d(S,J_m)
= \lim_{\leftarrow} \ker(SL_{\hat{d}}(S) \rightarrow SL_{\hat{d}}(\hat{S})).
\]

The immediate corollary from Propositions 4.3 and 4.6 is:

**Corollary 4.7.** For every $n \geq 4$, we have $C_{i_0} \cong \ker(SL_{n-1}(S) \rightarrow SL_{n-1}(\hat{S}))$. 

21
We close the section by showing that \( \ker \left( \SL_{n-1}(S) \to \SL_{n-1}(\hat{S}) \right) \) is not finitely generated, using the techniques in [KN]. It is known that the group ring \( S = \mathbb{Z}[x^{\pm 1}] = \mathbb{Z}[\hat{y}] \) is Noetherian (see [I], [BrLS]). In addition, it is known that the Krull dimension of \( \mathbb{Z} \) is \( \dim(\mathbb{Z}) = 1 \) and thus \( \dim(S) = \dim(\mathbb{Z}[\hat{y}]) = 2 \) (see [Sm]). Therefore, by Proposition 1.6 in [Su], as \( n - 1 \geq 3 \), for every \( J < S \), the canonical map

\[
SK_1(S; J; n - 1) \to SK_1(S; J) := \lim_{d \in \mathbb{N}} SK_1(S; J; d)
\]

is surjective. Hence, the canonical map (when \( J < S \) ranges over all finite index ideals of \( S \))

\[
\ker(\SL_{n-1}(S) \to \SL_{n-1}(\hat{S})) = \lim_{d \in \mathbb{N}} (\SL_{n-1}(S, J)/E_{n-1}(S, J)) = \lim_{d \in \mathbb{N}} SK_1(S, J; n - 1) \to \lim_{d \in \mathbb{N}} SK_1(S, J)
\]

is surjective, so it is enough to show that \( \lim_{d \in \mathbb{N}} SK_1(S, J) \) is not finitely generated. By a result of Bass (see [Bas], chapter 5, Corollary 9.3), for every \( J < K < S \) of finite index in \( S \), the map: \( SK_1(S, J) \to SK_1(S, K) \) is surjective. Hence, it is enough to show that for every \( l \in \mathbb{N} \) there exists a finite index ideal \( J < S \) such that \( SK_1(S, J) \) is generated by at least \( l \) elements. Now, as \( SK_1(S) = 1 \) (Su), we obtain the following exact sequence for every \( J < S \) (see Theorem 6.2 in [Mi])

\[
K_2(S) \to K_2(S/J) \to SK_1(S, J) \to SK_1(S) = 1.
\]

In addition, by a classical result of Quillen ([Q], [Ros] Theorem 5.3.30), we have

\[
K_2(S) = K_2(\mathbb{Z}[x^{\pm 1}]) = K_2(\mathbb{Z}) \oplus K_1(\mathbb{Z})
\]

so by the classical facts \( K_2(\mathbb{Z}) = K_1(\mathbb{Z}) = \{ \pm 1 \} \) (see [Mi] chapters 3 and 10) we deduce that \( K_2(S) \) is of order 4. Hence, it is enough to prove that for every \( l \in \mathbb{N} \) there exists a finite index ideal \( J < S \) such that \( K_2(S/J) \) is generated by at least \( l \) elements. Following [KN], we state the following proposition (which holds by the proof of Theorem 2.8 in [SD]):

**Proposition 4.8.** Let \( p \) be a prime, \( l \in \mathbb{N} \) and denote by \( P < \mathbb{Z}[y] \) the ideal which is generated by \( y^2 \) and \( y^3 \). Then, for \( \hat{S} = \mathbb{Z}[y]/P \), the group \( K_2(\hat{S}) \) is an elementary abelian \( p \)-group of rank \( \geq l \).

Observe now that for every \( l \geq 0 \)

\[
(y + 1)^{p^{l+1}} = (y^p + 1 + p \cdot a(y))^p = 1 \mod P
\]

so \( y + 1 \) is invertible in \( \hat{S} \). Therefore we have a well defined surjective homomorphism \( S \to \hat{S} \) which is defined by sending \( x \to y + 1 \). In particular, \( J = \ker(S \to \hat{S}) \) is a finite index ideal of \( S \) which satisfies the above requirements. This shows that indeed \( C_{\text{in}} = \ker(SL_{n-1}(S) \to SL_{n-1}(\hat{S})) \) is not finitely generated, and by the description in Proposition 4.8 it follows that \( C(IA(\Phi) \cdot \Phi) \) is not finitely generated either.
5 The centrality of $C_i$

In this section we will prove that for every $n \geq 4$, the copies $C_i$ lie in the center of $\hat{IA}(\Phi)$. Along the section we will assume that $n \geq 4$ is constant, and will show it for $i = n$. Symmetrically, it will be valid for every $i$. We recall:

- $R_n = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.
- $H_m = \sum_{i=1}^{n} (x_i^m - 1) R_n + m R_n$.
- $IG_m = \{ I_n + A \in GL_n (R_n, H_m) \mid A \sigma = 0 \}$.
- $IA_m = \cap \{ N \triangleleft IA(\Phi) \mid [IA(\Phi) : N] \mid m \}$.
- $S = S_n = \mathbb{Z}[x_n^{\pm 1}]$.
- $Im \rho \cap IG_m = Im \rho_n \cap IG_m \simeq GL_{n-1} (S, \sigma_n H_m \cap S)$ (see Proposition 3.7).

We saw in Section 4 that we can write

$$C_n = \lim_{\leftarrow} (IA_m \cdot (Im \rho \cap IG_m) / IA_m)$$

$$= \lim_{\leftarrow} (IA_m \cdot (Im \rho \cap IG_m^4) / IA_m)$$

$$\leq \lim_{\leftarrow} (IA(\Phi) / IA_m) = \hat{IA}(\Phi).$$

Hence, if we want to show that $C_n$ lies in the center of $\hat{IA}(\Phi)$, it suffices to show that for every $m \in \mathbb{N}$, the group $IA_m \cdot (Im \rho \cap IG_m^4) / IA_m$ lies in the center of $IA(\Phi) / IA_m$.

We first claim that under the isomorphism $Im \rho \cap IG_m^4 \simeq GL_{n-1} (S, \sigma_n H_m \cap S)$ one has

$$IA_m \cdot (Im \rho \cap IG_m^4) / IA_m \subseteq IA_m \cdot SL_{n-1} (S, \sigma_n H_m \cap S) / IA_m. \quad (5.1)$$

Indeed, if $\alpha \in Im \rho \cap IG_m^4$ then $det(\alpha) \in 1 + \sigma_n H_m \cap S \subseteq 1 + H_m \cap S$. Combining it with Proposition 3.2, $det(\alpha)$ has the form $det(\alpha) = x_n^{mt}$ for some $t \in \mathbb{Z}$. Hence

$$det((I_n + \sigma_n E_{1,1} - \sigma_1 E_{1,n})^{-mt} \cdot \alpha) = 1.$$ 

Now, as we have (see the computation in the proof of Proposition 3.8)

$$x_n^{mt} = 1 + (x_n^{mt} - 1) = 1 + \sigma_n \sum_{i=1}^{m-1} (x_n^i)^t$$

$$= 1 + \sigma_n ((x_n^{mt} - 1) S + m^2 S) \in 1 + \sigma_n H_m \cap S$$

we obtain that

$$(I_n + \sigma_n E_{1,1} - \sigma_1 E_{1,n})^{mt} \in \langle IA(\Phi)^m \rangle \cap GL_{n-1} (S, \sigma_n H_m \cap S)$$

$$\subseteq IA_m \cap GL_{n-1} (S, \sigma_n H_m \cap S).$$
Therefore, writing \( \alpha = (I_n + \sigma_n E_{1,1} - \sigma_1 E_{1,n})^{m+1} \cdot ((I_n + \sigma_n E_{1,1} - \sigma_1 E_{1,n})^{-m+1} \cdot \alpha) \) we deduce that

\[
\text{Im} \rho \cap IG_m \subseteq IA_m \cdot SL_{n-1} (S, \sigma_n H_{m^2} \cap S)
\]

and we get Inclusion (5.1). It follows that if we want to show that \( C_n \) lies in the center of \( IA(\Phi) \) it suffices to show that \( IA_m \cdot SL_{n-1} (S, \sigma_n H_{m^2} \cap S) \) lies in the center of \( IA(\Phi) / IA_m \). However, we are going to show even more. We will show that:

**Proposition 5.1.** For every \( m \in \mathbb{N} \), the group

\[
IA_m \cdot ISL_{n-1,m} (\sigma_n H_{m^2}) / IA_m
\]

lies in the center of \( IA(\Phi) / IA_m \).

Let \( F \) be the free group on \( f_1, \ldots, f_n \). It is a classical result by Magnus ([MKS], Chapter 3, Theorem N4) that \( IA(F) \) is generated by the automorphisms of the form

\[
\alpha_{r,s,t} = \begin{cases} f_r \rightarrow [f_t, f_s] f_r & u \neq r \\ f_u \rightarrow f_u & u = r \end{cases}
\]

where \([f_t, f_s] = f_t f_s f_t^{-1} f_s^{-1}\) and \( 1 \leq r, s \neq t \leq n \) (notice that we may have \( r = s \)). In their paper [BM], Bachmuth and Mochizuki show that when \( n \geq 4 \), the group \( IA(\Phi) \) is generated by the images of these generators under the natural map \( Aut(F) \rightarrow Aut(\Phi) \). I.e. \( IA(\Phi) \) is generated by the elements of the form

\[
E_{r,s,t} = I_n + \sigma_n E_{r,s} - \sigma_s E_{r,t} \quad 1 \leq r, s \neq t \leq n.
\]

Therefore, for showing the centrality of \( C_n \), it is enough to show that given:

- an element: \( \tilde{\lambda} \in IA_m \cdot ISL_{n-1,m} (\sigma_n H_{m^2}) / IA_m \),
- and one of generators: \( E_{r,s,t} = I_n + \sigma_n E_{r,s} - \sigma_s E_{r,t} \) for \( 1 \leq r, s \neq t \leq n \),

there exists \( \lambda \in ISL_{n-1,m} (\sigma_n H_{m^2}) \), a representative of \( \tilde{\lambda} \), such that \([E_{r,s,t}, \lambda] \in IA_m \). So assume that we have an element: \( \lambda \in IA_m \cdot ISL_{n-1,m} (\sigma_n H_{m^2}) / IA_m \). Then, a representative for \( \lambda \) has the form

\[
\lambda = \left( \begin{array}{c}
I_{n-1} + \sigma_n B - \sum_{i=1}^{n-1} \sigma_i b_i \\
0
\end{array} \right) \in ISL_{n-1,m} (\sigma_n H_{m^2})
\]

for some \((n-1) \times (n-1)\) matrix \( B \) which its entries \( b_{i,j} \) admit \( b_{i,j} \in H_{m^2} \) and its column vectors denoted by \( b_i \).

**Lemma 5.2.** Let \( \tilde{\lambda} \in IA_m \cdot ISL_{n-1,m} (\sigma_n H_{m^2}) / IA_m \). Then, for every \( 1 \leq l < k \leq n - 1 \), \( \lambda \) has a representative in \( ISL_{n-1,m} (\sigma_n H_{m^2}) \), of the following form
Observe that
\[ \sigma \]
Then
\[ I \]
x \( \in \)
representative of \( \bar{\sigma} \) and
\[ H \]
images of \( D \) observe that
\[ \text{for some} \]
\[ A, \]
respectively. Observe that obviously, \( \bar{\sigma} \), so
\[ \lambda = \left( I_{n-1} + \sigma_n B - \sum_{i=1}^{n-1} \sigma_i \delta_i \right) \in ISL_{n-1,n} (\sigma_n H_{m^2}) . \]
Then \( I_{n-1} + \sigma_n B \in SL_{n-1} (R_n, \sigma_n H_{m^2}) \). Consider now the ideal
\[ R_n \triangleright H'_{m^2} = \sum_{r=1}^{n-1} (x_r^{m^2} - 1) R_n + \sigma_n (x_n^{m^2} - 1) R_n + m^2 R_n . \]
Observe that \( \sigma_n H_{m^2} \triangleleft H'_{m^2} \triangleleft H_{m^2} \triangleleft R_n \) and that \( H'_{m^2} \cap \sigma_n R_n = \sigma_n H_{m^2} \). In addition, by similar computations as in the proof of proposition \[ \text{for every} \]
x \( \in \) \( R_n \) we have \( x^{m^2} - 1 \in (x - 1)(x^{m^2} - 1) R_n + (x - 1)m^2 R_n \), and thus \( H_{m^2} \triangleleft H_{m^2} \), so \( H'_{m^2} \) is of finite index in \( R_n \).
Now, \( I_{n-1} + \sigma_n B \in SL_{n-1} (R_n, \sigma_n H_{m^2}) \subseteq SL_{n-1} (R_n, H'_{m^2}) \). Thus, by the third part of Corollary \[ \text{as} \]
\[ H'_{m^2} \triangleleft R_n \] an ideal of finite index, \( n - 1 \geq 3 \) and \( E_{n-1} (R_n) = SL_{n-1} (R_n) \) \[ \text{one can write the matrix} \]
\[ I_{n-1} + \sigma_n B = AD \text{ when } A = \left( \begin{array}{cc} A' & 0 \\ 0 & I_{n-3} \end{array} \right) \]
for some \( A' \in SL_2 (R_n, H'_{m^2}) \) and \( D \in E_{n-1} (R_n, H'_{m^2}) \). Now, consider the images of \( D \) and \( A \) under the projection \( \sigma_n \rightarrow 0 \), which we denote by \( \bar{D} \) and \( \bar{A} \), respectively. Observe that obviously, \( \bar{D} \in E_{n-1} (R_n, H'_{m^2}) \). In addition, observe that
\[ AD \in GL_{n-1} (R_n, \sigma_n R_n) \Rightarrow \bar{A} \bar{D} = I_{n-1} . \]
Thus, we have \( I_{n-1} + \sigma_n B = A \bar{A}^{-1} \bar{D}^{-1} D \). Therefore, by replacing \( D \) by \( \bar{D}^{-1} D \) and \( A \) by \( A \bar{A}^{-1} \) we can assume that
\[ I_{n-1} + \sigma_n B = AD \text{ for } A = \left( \begin{array}{cc} A' & 0 \\ 0 & I_{n-3} \end{array} \right) \]
Thus, we obtain from Lemma 7.1 that
\[ D \subseteq E_{n-1}(R_n, H_{m^2}) \cap GL_{n-1}(R_n, \sigma_n R_n) := IE_{n-1,n}(H_{m^2}). \]

Now, as we prove in the main lemma (Lemma 7.1) that \( IE_{n-1,n}(H_{m^2}) \subseteq \langle IA(\Phi)^m \rangle \subseteq IA_m \), this argument shows that \( \lambda \) can be replaced by a representative of the form \( \langle \rangle \).

We now return to our initial mission. Let \( \lambda \in IA_m \cdot ISL_{n-1,n}(\sigma_n H_{m^2}) \subseteq IA_m \), and let \( E_{r,s,t} = I_n + \sigma_n E_{r,s} - \sigma_n E_{r,t} \) for \( 1 \leq r, s \neq t \leq n \) be one of the above generators for \( IA(\Phi) \). We want to show that there exists \( \lambda \in ISL_{n-1,n}(\sigma_n H_{m^2}) \), a representative of \( \lambda \), such that \( [E_{r,s,t}, \lambda] \in IA_m \). We separate the treatment to two cases. We note that Lemma 7.2 is needed only for the second case, which is a bit more delicate.

The first case is: \( 1 \leq r \leq n - 1 \). In this case one can take an arbitrary representative \( \lambda \in ISL_{n-1,n}(\sigma_n H_{m^2}) \cong SL_{n-1}(R_n, \sigma_n H_{m^2}) \). Considering the embedding of \( IGL'_{n-1,n} \) in \( GL_{n-1}(R_n) \), we have: \( E_{r,s,t} \in IGL'_{n-1,n} \subseteq GL_{n-1}(R_n) \) (see Definition 3.10 and Proposition 3.11). Thus, as by Corollary 2.6
\[ SK_1(R_n, H_{m^2}; n-1) = SL_{n-1}(R_n, H_{m^2})/E_{n-1}(R_n, H_{m^2}) \]
is central in \( GL_{n-1}(R_n)/E_{n-1}(R_n, H_{m^2}) \), we have
\[ [E_{r,s,t}, \lambda] \in [GL_{n-1}(R_n), SL_{n-1}(R_n, \sigma_n H_{m^2})] \subseteq E_{n-1}(R_n, H_{m^2}). \]
In addition, as \( SL_{n-1}(R_n, \sigma_n H_{m^2}) \subseteq GL_{n-1}(R_n, \sigma_n R_n) \) and \( GL_{n-1}(R_n, \sigma_n R_n) \) is normal in \( GL_{n-1}(R_n) \), we have
\[ [E_{r,s,t}, \lambda] \in [GL_{n-1}(R_n), GL_{n-1}(R_n, \sigma_n R_n)] \subseteq GL_{n-1}(R_n, \sigma_n R_n). \]
Thus, we obtain from Lemma 7.1 that
\[ [E_{r,s,t}, \lambda] \in E_{n-1}(R_n, H_{m^2}) \cap GL_{n-1}(R_n, \sigma_n R_n) \subseteq IE_{n-1,n}(H_{m^2}) \subseteq \langle IA(\Phi)^m \rangle \subseteq IA_m. \]

The second case is: \( r = n \). This case is a bit more complicated than the previous one, as \( E_{r,s,t} \) is not in \( IGL'_{n-1,n} \). Here, by Lemma 7.2 one can choose \( \lambda \in ISL_{n-1,n}(\sigma_n H_{m^2}) \) whose \( t \)-th row equals to the standard vector \( \vec{e}_t \). As \( t \neq r = n \), we obtain thus that both \( \lambda, E_{r,s,t} \in IGL'_{n-1,t} \). Considering the embedding \( IGL'_{n-1,t} \rightarrow GL_{n-1}(R_n) \), we have \( E_{r,s,t} \in GL_{n-1}(R_n, \sigma_t R_n) \). In addition, remember that \( \lambda \) has the form
\[ \lambda = \begin{pmatrix} I_{n-1} + \sigma_n B & -\sum_{i=1}^{n-1} \sigma_i \vec{b}_i \\ 0 & 1 \end{pmatrix} \]
for \( I_{n-1} + \sigma_n B \in SL_{n-1}(R_n, \sigma_n H_{m^2}) \), so that the entries of \( \vec{b}_i \) are in \( H_{m^2} \). It follows that regarding the embedding \( IGL'_{n-1,t} \rightarrow GL_{n-1}(R_n) \) we have \( \lambda \in SL_{n-1}(R_n, H_{m^2}) \).
Remark 5.3. Note that when considering \( \lambda \in IGL_{n-1,n} \hookrightarrow GL_{n-1}(R_n) \), i.e. when considering \( \lambda \in GL_{n-1}(R_n) \) through the embedding of \( IGL_{n-1,n} \) in \( GL_{n-1}(R_n) \), we have \( \lambda \in GL_{n-1}(R_n, \sigma_n H_m) \leq GL_{n-1}(R_n) \). However, when we consider \( \lambda \in IGL_{n-1,t} \hookrightarrow GL_{n-1}(R_n) \) we do not necessarily have: \( \lambda \in GL_{n-1}(R_n, \sigma_n H_m) \), but we still have \( \lambda \in GL_{n-1}(R_n, H_m^2) \).

Thus, by similar arguments as in the first case

\[
[E_{r,s,t}, \lambda] \in [GL_{n-1}(R_n, \sigma_t R_n), SL_{n-1}(R_n, H_m^2)] \\
\subseteq E_{n-1}(R_n, H_m^2) \cap GL_{n-1}(R_n, \sigma_t R_n) \\
= I E_{n-1,t}(H_m^2) \subseteq (IA(\Phi)^m) \subseteq IA_n.
\]

This finishes the argument which shows that \( C_i \) are central in \( \widehat{IA(\Phi)} \).

Remark 5.4. One can follow and see that completely similar arguments gives that the group

\[
\langle IA(\Phi)^m \rangle \cdot ISL_{n-1,n}(\sigma_n H_m^2) / \langle IA(\Phi)^m \rangle
\]

lies in the center of \( IA(\Phi) / \langle IA(\Phi)^m \rangle \). The reason is that the only property of \( IA_n \) that we used here was that \( \langle IA(\Phi)^m \rangle \subseteq IA_n \). This claim is used in [Be2] to prove Theorem 1.3. We note that in this paper we were careful not to use the subgroups \( \langle IA(\Phi)^m \rangle \) directly as we still didn’t show that they are of finite index in \( IA(\Phi) \), and therefore we cannot write \( IA(\Phi) = \varprojlim (IA(\Phi) / \langle IA(\Phi)^m \rangle) \).

However, on the way of proving Theorem 1.3 we do show that \( \langle IA(\Phi)^m \rangle \) are of finite index in \( IA(\Phi) \) (provided \( n \geq 4 \)).

6 Some elementary elements of \( \langle IA(\Phi_n)^m \rangle \)

In this section we introduce some elements in \( \langle IA(\Phi_n)^m \rangle \), which are needed for the proof of Lemma 7.1. In [Be2] we introduce a list of elements in \( \langle IA(\Phi_n)^m \rangle \) that contains the list below (see Propositions 4.1 and 4.2 therein). However, we will not need here the whole list of [Be2], and also do not need all the notations that are used in [Be2]. Hence, for the convenience of the reader we include here only the list that is needed for the proof of Lemma 7.1 and repeat the arguments that are related to this shorter list.

**Proposition 6.1.** Let \( n \geq 4 \), \( 1 \leq u \leq n \) and \( m \in \mathbb{N} \). Denote by \( \vec{e}_i \) the \( i \)-th row standard vector. Then, the elements of \( IA(\Phi_n) \) of the following form, lie in \( \langle IA(\Phi_n)^m \rangle \):

\[
\begin{pmatrix}
I_{u-1} & & & & \\
0 & a_{u,1} & \cdots & a_{u,u-1} & 1 & a_{u,u+1} & \cdots & a_{u,n} \\
0 & 0 & \cdots & 0 & I_{n-u}
\end{pmatrix}
\]

where \((a_{u,1}, \ldots, a_{u,u-1}, 0, a_{u,u+1}, \ldots, a_{u,n})\) is a linear combination of the vectors

1. \( \{m(\sigma_j \vec{e}_j - \sigma_i \vec{e}_i) \mid i,j \not= u, i \not= j \} \)
2. \( \{(x_k^m - 1)(\sigma_i \vec{e}_j - \sigma_j \vec{e}_i) \mid i,j,k \not= u, i \not= j \} \)
3. \( \{\sigma_u (x_u^m - 1)(\sigma_i \vec{e}_j - \sigma_j \vec{e}_i) \mid i,j \not= u, i \not= j \} \)

27
with coefficients in \( R_n \). Notation \( (6.1) \) means that the matrix is similar to the identity matrix, except the entries in the \( u \)-th row.

**Proof.** Without loss of generality, we assume that \( u = 1 \). Observe now that for every \( a_i, b_i \in R_n \) for \( 2 \leq i \leq n \) one has

\[
\begin{pmatrix}
1 & a_2 & \cdots & a_n \\
0 & I_{n-1}
\end{pmatrix}
\begin{pmatrix}
1 & b_2 & \cdots & b_n \\
0 & I_{n-1}
\end{pmatrix} = \begin{pmatrix}
1 & a_2 + b_2 & \cdots & a_n + b_n \\
0 & I_{n-1}
\end{pmatrix}.
\]

Hence, it is enough to prove that the elements of the following forms belong to \( \langle IA(\Phi_n)^m \rangle \) (when we write \( a\vec{e}_i \) we mean that the entry of the \( i \)-th column in the first row is \( a \)):

1. \( \begin{pmatrix}
1 & mf(\sigma_i\vec{e}_j - \sigma_j\vec{e}_i) \\
0 & I_{n-1}
\end{pmatrix} \quad i, j \neq 1, i \neq j, f \in R_n \)

2. \( \begin{pmatrix}
1 & (x_k^m - 1)f(\sigma_i\vec{e}_j - \sigma_j\vec{e}_i) \\
0 & I_{n-1}
\end{pmatrix} \quad i, j, k \neq 1, i \neq j, f \in R_n \)

3. \( \begin{pmatrix}
1 & \sigma_1(x_1^m - 1)f(\sigma_i\vec{e}_j - \sigma_j\vec{e}_i) \\
0 & I_{n-1}
\end{pmatrix} \quad i, j \neq 1, i \neq j, f \in R_n. \)

We start with the elements of Form 1. Here we have

\[
\begin{pmatrix}
1 & mf(\sigma_i\vec{e}_j - \sigma_j\vec{e}_i) \\
0 & I_{n-1}
\end{pmatrix} = \begin{pmatrix}
1 & f(\sigma_i\vec{e}_j - \sigma_j\vec{e}_i) \\
0 & I_{n-1}
\end{pmatrix}^m \in \langle IA(\Phi_n)^m \rangle.
\]

We pass to the elements of Form 2. In this case we have

\[
\langle IA(\Phi_n)^m \rangle \ni \left[ \begin{pmatrix}
1 & f(\sigma_i\vec{e}_j - \sigma_j\vec{e}_i) \\
0 & I_{n-1}
\end{pmatrix}^{-1} \cdot \begin{pmatrix}
x_k & -\sigma_1\vec{e}_k \\
0 & I_{n-1}
\end{pmatrix}^m \right]
\]

\[
= \begin{pmatrix}
1 & (x_k^m - 1)f(\sigma_i\vec{e}_j - \sigma_j\vec{e}_i) \\
0 & I_{n-1}
\end{pmatrix}^m.
\]

We finish with the elements of Form 3. The computation here is more complicated than in the previous cases, so we will demonstrate it for the special case: \( n = 4, i = 2, j = 3 \). It is clear that symmetrically, with similar arguments, the same holds in general when \( n \geq 4 \) for every \( i, j \neq 1, i \neq j \). By similar arguments as in the previous case we get

\[
\langle IA(\Phi_4)^m \rangle \ni \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & \sigma_3(x_1^m - 1)f & -\sigma_2(x_1^m - 1)f & 1
\end{pmatrix}.
\]
Therefore, we also have
\[
\langle IA(\Phi_4)^m \rangle \ni \begin{pmatrix}
\begin{pmatrix} x_4 & 0 & 0 & -\sigma_1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\end{pmatrix} \cdot \begin{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\end{pmatrix} \cdot \begin{pmatrix}
\begin{pmatrix} 1 & -\sigma_3 \sigma_1(x_1^m - 1)f & \sigma_2 \sigma_1(x_1^m - 1)f & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\end{pmatrix},
\]
\[
= \begin{pmatrix}
\begin{pmatrix} x_4 & 0 & 0 & -\sigma_1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\end{pmatrix} \cdot \begin{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\end{pmatrix} \cdot \begin{pmatrix}
\begin{pmatrix} 1 & -\sigma_3 \sigma_1(x_1^m - 1)f & \sigma_2 \sigma_1(x_1^m - 1)f & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\end{pmatrix}.
\]

\section{A main lemma}

We recall and present some new notations that will be used in this section:

- \( IA^m = \langle IA(\Phi)^m \rangle \), where \( \Phi = \Phi_n \).
- \( R_n = \mathbb{Z}[\mathbb{Z}^n] = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) where \( x_1, \ldots, x_n \) are the generators of \( \mathbb{Z}^n \).
- \( \sigma_r = x_r - 1 \) for \( 1 \leq r \leq n \).
- \( U_{r,m} = (x_r^m - 1)R_n \) for \( 1 \leq r \leq n \) and \( m \in \mathbb{N} \).
- \( O_m = mR_n \).
- \( H_m = \sum_{r=1}^{n} (x_r^m - 1)R_n + mR_n = \sum_{r=1}^{n} U_{r,m} + O_m \).
- \( IE_{n-1,i}(H) = IGL_{n-1,i} \cap E_{n-1}(R_n, H) \leq ISL_{n-1,i}(H) \) for \( H < R_n \) under the identification of \( IGL_{n-1,i} \leq IA(\Phi) \) with \( GL_{n-1}(R_n, \sigma_i R_n) \) (see Proposition \ref{prop:1} and Definition \ref{def:1}).

In this section we prove the main lemma which asserts that:

\textbf{Lemma 7.1. (Main lemma)} For every \( n \geq 4 \), \( m \in \mathbb{N} \) and \( 1 \leq i \leq n \) we have

\[ IE_{n-1,i}(H_{m^2}) \subseteq IA^m. \]

For simplifying the proof and the notations, we will prove the lemma for the special case \( i = n \), and symmetrically, all the arguments are valid for every \( 1 \leq i \leq n \).

In addition, using the identification \( IGL_{n-1,n} \cong GL_{n-1}(R_n, \sigma_n R_n) \), we will identify \( IGL_{n-1,n} \) with \( GL_{n-1}(R_n, \sigma_n R_n) \), and the group \( IE_{n-1,n}(H_m) \) with \( GL_{n-1}(R_n, \sigma_n R_n) \cap E_{n-1}(R_n, H_m) \). So the goal of this section is proving that

\[ GL_{n-1}(R_n, \sigma_n R_n) \cap E_{n-1}(R_n, H_{m^2}) \subseteq IA^m. \]
Throughout the proof we use also elements of $\text{IGL}_{n-1,n}$ (see Definition 3.10). We remind that

$$IE_{n-1,n}(H_m) \subseteq IGL_{n-1,n} \subseteq IGL'_{n-1,n} \hookrightarrow GL_{n-1}(R_n)$$

(Proposition 3.11), so all the elements that are being used throughout the section are naturally embedded in $GL_{n-1}(R_n)$. Using this embedding we will do all the computations in $GL_{n-1}(R_n)$, and make the notations simpler by omitting the $n$-th row and column from each matrix.

We note that many ideas in the proof of Lemma 7.1 below are based on ideas of the proof of the “Main Lemma” in [BM] (See Section 4 therein). However, our arguments do not rely directly on the arguments in [BM], so on the whole we cannot make a formal reference to [BM] throughout the proof of Lemma 7.1.

### 7.1 Decomposing the proof

In this subsection we will start the proof of Lemma 7.1. At the end of the subsection, there will be a few tasks left, which will be accomplished in the forthcoming subsections. We start with the following definition:

**Definition 7.2.** For every $m \in \mathbb{N}$, define the following ideal of $R_n$:

$$T_m = \sum_{r=1}^{n} \sigma_r^2 U_{r,m} + \sum_{r=1}^{n} \sigma_r O_{m} + O_m^2.$$

Observe that as for every $x \in R_n$ we have $\sum_{j=0}^{m-1} x^j \in (x-1)R_n + mR_n$, one has

$$x^{m^2} - 1 = (x - 1) \sum_{j=0}^{m^2-1} x^j = (x - 1) \sum_{j=0}^{m-1} x^j \sum_{j=0}^{m-1} x^{jm} \subseteq (x - 1) ((x-1)R_n + mR_n) ((x^m - 1)R_n + mR_n) \subseteq (x - 1)^2 (x^m - 1)R_n + (x - 1)^2 mR_n + (x - 1)m^2 R_n.$$

It follows that $H_{m^2} \subseteq T_m$. Hence, it is enough to prove that

$$GL_{n-1}(R_n, \sigma_n R_n) \cap E_{n-1}(R_n, T_m) \subseteq IA^m.$$

Equivalently, it is enough to prove that the group

$$(GL_{n-1}(R_n, \sigma_n R_n) \cap E_{n-1}(R_n, T_m)) \cdot IA^m / IA^m$$

is trivial. We continue with the following proposition, which is actually a proposition of Suslin (Corollary 1.4 in [Su]) with some elaborations of [BM] (see the remark that follows Proposition 3.5 in [BM] and the beginning of the proof of the “Main Lemma” in Section 4 therein).
Proposition 7.3. Let \( R \) be a commutative ring, \( d \geq 3 \), and \( H \triangleleft R \) ideal. Then, \( E_d(R, H) \) is generated by the matrices of the form

\[
(I_d - fE_{i,j}) (I_d + hE_{i,j}) (I_d + fE_{i,j})
\]

(7.1)

for \( h \in H, f \in R \) and \( 1 \leq i \neq j \leq d \).

Proof. In the proof of Corollary 1.4 in \([Su]\), Suslin shows that whenever \( d \geq 3 \), \( E_d(R, H) \) is generated by the elements of the form

\[
I_d + h\vec{u}(u_j\vec{e}_i - u_i\vec{e}_j)
\]

where \( h \in H, i \neq j \), and \( \vec{u} = (u_1, u_2, ..., u_d) \in R^d \) such that \( \vec{u} \cdot \vec{v} = 1 \) for some \( \vec{v} \in R^d \). In the remark which follows Proposition 3.5 in \([BM]\), Bachmuth and Mochizuki observe that

\[
I_d + h\vec{u}(u_j\vec{e}_i - u_i\vec{e}_j) = (I_d + h(u_i\vec{e}_i + u_j\vec{e}_j)^t(u_j\vec{e}_i - u_i\vec{e}_j))
\cdot \prod_{l \neq i,j} (I_d + h(u_l\vec{e}_l)^tu_j\vec{e}_i) \cdot \prod_{l \neq i,j} (I_d - h(u_l\vec{e}_l)^tu_i\vec{e}_j).
\]

Hence, by observing that all the factors in the above expression are all of the form

\[
I_d + h(f_1\vec{e}_i + f_2\vec{e}_j)^t(f_2\vec{e}_i - f_1\vec{e}_j)
\]

(7.2)

for some \( f_1, f_2 \in R, h \in H \) and \( 1 \leq i \neq j \leq d \), it is enough to show that the matrices of the form (7.2) are generated by the matrices of the form (7.1). We will show it for the case \( i, j, d = 1, 2, 3 \) and it will be clear that the general argument is similar. So we have the matrix

\[
I_d + h(f_1\vec{e}_1 + f_2\vec{e}_2)^t(f_2\vec{e}_1 - f_1\vec{e}_2) = \begin{pmatrix}
1 + hf_1f_2 & -hf_2^2 & 0 \\
hf_2^2 & 1 - hf_1f_2 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

for some \( f_1, f_2 \in R \) and \( h \in H \), which is equal to

\[
= \begin{pmatrix}
1 & 0 & -hf_1 \\
0 & 1 & -hf_2 \\
f_2 & -f_1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & hf_1 \\
0 & 1 & hf_2 \\
-f_2 & f_1 & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
f_2 & -f_1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & -hf_1 \\
0 & 1 & -hf_2 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-f_2 & f_1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & hf_1 \\
0 & 1 & hf_2 \\
0 & 0 & 1
\end{pmatrix}.
\]

As the matrix

\[
\begin{pmatrix}
1 & 0 & hf_1 \\
0 & 1 & hf_2 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

31
is generated by the matrices of the form (7.1), it remains to show that
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
f_2 - f_1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & -hf_1 \\
0 & 1 & -hf_2 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-f_2 & f_1 & 1
\end{pmatrix}
\]
\[
= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
f_2 - f_1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & -hf_1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-f_2 & f_1 & 1
\end{pmatrix}
\]
is generated by the matrices of the form (7.1). Now
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
f_2 - f_1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & -hf_1 \\
0 & 1 & -hf_2 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-f_2 & f_1 & 1
\end{pmatrix}
\]
\[
= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-hf_1^2 f_2 & 1
\end{pmatrix}
\begin{pmatrix}
1 & -hf_1 \\
0 & 1 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-f_2 & 0 & 1
\end{pmatrix}
\]
is generated by the matrices of the form (7.1), and by similar computation
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
f_2 - f_1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -hf_2 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-f_2 & f_1 & 1
\end{pmatrix}
\]
is generated by these matrices as well.

We proceed with the following lemma. Some of the ideas in its proof are based on the proof of Proposition 3.5 in [BM].

**Lemma 7.4.** Let \( n \geq 4 \). Recall \( U_{r,m} = (x^m r - 1)R_n, O_m = mR_n \), and denote the corresponding ideals of \( R_{n-1} = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \subseteq R_n \) by
\[
\tilde{O}_m = mR_{n-1} \subseteq O_m, \quad \tilde{U}_{r,m} = (x^m r - 1)R_{n-1} \subseteq U_{r,m}
\]
for \( 1 \leq r \leq n-1 \).

Then, every element of \( GL_{n-1}(R_n, \sigma_n R_n) \cap E_{n-1}(R_n, T_m) \) can be decomposed as a product of elements of the following four forms:
|   |   |   |
|---|---|---|
| 1 | $A^{-1}(I_{n-1} + hE_{i,j}) A$ | $h \in \sigma_n O_m$ |
| 2 | $A^{-1}(I_{n-1} + hE_{i,j}) A$ | $h \in \sigma^2_n U_{n,m}, \sigma_n \sigma^2_r U_{r,m}$ for $1 \leq r \leq n - 1$ |
| 3 | $A^{-1} [(I_{n-1} + hE_{i,j}), (I_{n-1} + fE_{j,i})] A$ | $h \in O^2_m, f \in \sigma_n R_n$ |
| 4 | $A^{-1} [(I_{n-1} + hE_{i,j}), (I_{n-1} + fE_{j,i})] A$ | $h \in \sigma^2_r U_{r,m}, \sigma_r O_m$ for $1 \leq r \leq n - 1$, $f \in \sigma_n R_n$ |

where $A \in GL_{n-1}(R_n)$ and $i \neq j$.

**Remark 7.5.** Notice that as $GL_{n-1}(R_n, \sigma_n R_n)$ is normal in $GL_{n-1}(R_n)$, every element of the above forms is an element of $GL_{n-1}(R_n, \sigma_n R_n) \cong IGL_{n-1,n} \cong IA(\Phi)$.

**Proof.** (of Lemma 7.4) Let $B \in GL_{n-1}(R_n, \sigma_n R_n) \cap E_{n-1}(R_n, T_m)$. We first claim that for proving the lemma, it is enough to show that $B$ can be decomposed as a product of the elements in the lemma (Lemma 7.4), and arbitrary elements in $GL_{n-1}(R_{n-1})$. Indeed, assume that we can write $B = A_1 D_1 \cdots A_n D_n$ for some $D_i$ of the forms in the lemma and $A_i \in GL_{n-1}(R_{n-1})$ (notice that $A_1$ or $D_n$ might be equal to $I_{n-1}$). Observe now that we can therefore write

$$B = A_1 D_1 A_1^{-1} \cdots (A_1 \cdots A_n) D_n (A_1 \cdots A_n)^{-1} (A_1 \cdots A_n)$$

and by definition, the conjugations of the $D_i$-s can also be considered as of the forms in the lemma. On the other hand, we have

$$(A_1 \cdots A_n) D_n^{-1} (A_1 \cdots A_n)^{-1} \cdots A_1 D_1^{-1} A_1^{-1} B = A_1 \cdots A_n$$

and as the matrices of the forms in the lemma are all in $GL_{n-1}(R_n, \sigma_n R_n)$ (by Remark 7.5) we deduce that

$$A_1 \cdots A_n \in GL_{n-1}(R_n, \sigma_n R_n) \cap GL_{n-1}(R_{n-1}) = \{I_{n-1}\}$$

i.e. $A_1 \cdots A_n = I_{n-1}$. Hence

$$B = A_1 D_1 A_1^{-1} \cdots (A_1 \cdots A_n) D_n (A_1 \cdots A_n)^{-1}$$

i.e. $B$ is a product of matrices of the forms in the lemma, as required.

So let $B \in GL_{n-1}(R_n, \sigma_n R_n) \cap E_{n-1}(R_n, T_m)$. According to Proposition 7.3, as $B \in E_{n-1}(R_n, T_m)$ and $n - 1 \geq 3$, we can write $B$ as a product of elements of the form

$$(I_{n-1} - fE_{i,j}) (I_{n-1} + hE_{i,j}) (I_{n-1} + fE_{j,i})$$

for some $f \in R_n$, $h \in T_m = \sum_{r=1}^n \sigma_r^2 U_{r,m} + \sum_{r=1}^n \sigma_r O_m + O^2_m$ and $1 \leq i \neq j \leq n - 1$. We will show now that every element of the above form can be written as a product of the elements of the forms in the lemma and elements of $GL_{n-1}(R_{n-1})$. 

33
So let \( h \in T \) and \( f \in R_n \). Observe first that by division by \( \sigma_n \) (with residue) one has

\[
T_m = \sum_{r=1}^{n} \sigma_r^2 U_{r,m} + \sum_{r=1}^{n} \sigma_r O_m + O_m^2
\]

\[
\subseteq \sigma_n \left( \sum_{r=1}^{n-1} \sigma_r^2 U_{r,m} + \sigma_n U_{n,m} + O_m \right) + \sum_{r=1}^{n-1} \sigma_r^2 \hat{U}_{r,m} + \sum_{r=1}^{n-1} \sigma_r \hat{O}_m + \hat{O}_m^2.
\]

Hence, we can decompose \( h = \sigma_n h_1 + h_2 \) for some: \( h_1 \in \sum_{r=1}^{n-1} \sigma_r^2 U_{r,m} + \sigma_n U_{n,m} + O_m \) and \( h_2 \in \sum_{r=1}^{n-1} \sigma_r^2 \hat{U}_{r,m} + \sum_{r=1}^{n-1} \sigma_r \hat{O}_m + \hat{O}_m^2 \). Therefore, we can write

\[
(I_{n-1} - fE_{i,j}) (I_{n-1} + hE_{j,i}) (I_{n-1} + fE_{i,j})
\]

\[
= (I_{n-1} - fE_{i,j}) (I_{n-1} + \sigma_n h_1 E_{j,i}) (I_{n-1} + fE_{i,j})
\]

\[
\cdot (I_{n-1} - fE_{i,j}) (I_{n-1} + h_2 E_{j,i}) (I_{n-1} + fE_{i,j}).
\]

Thus, as the matrix \( (I_{n-1} - fE_{i,j}) (I_{n-1} + \sigma_n h_1 E_{j,i}) (I_{n-1} + fE_{i,j}) \) is clearly a product of elements of Forms 1 and 2 in the lemma, it is enough to deal with the matrix

\[
(I_{n-1} - fE_{i,j}) (I_{n-1} + h_2 E_{j,i}) (I_{n-1} + fE_{i,j})
\]

when \( h_2 \in \sum_{r=1}^{n-1} \sigma_r^2 \hat{U}_{r,m} + \sum_{r=1}^{n-1} \sigma_r \hat{O}_m + \hat{O}_m^2 \). Let us now write: \( f = \sigma_n f_1 + f_2 \) for some \( f_1 \in R_n \) and \( f_2 \in R_{n-1} \), and write

\[
(I_{n-1} - fE_{i,j}) (I_{n-1} + h_2 E_{j,i}) (I_{n-1} + fE_{i,j})
\]

\[
= (I_{n-1} - f_2 E_{i,j}) (I_{n-1} - \sigma_n f_1 E_{i,j})
\]

\[
\cdot (I_{n-1} + h_2 E_{j,i}) (I_{n-1} + \sigma_n f_1 E_{i,j}) (I_{n-1} + f_2 E_{i,j}).
\]

Now, as \( (I_{n-1} \pm f_2 E_{i,j}) \in GL_{n-1} (R_{n-1}) \), it is enough to deal with the element

\[
(I_{n-1} - \sigma_n f_1 E_{i,j}) (I_{n-1} + h_2 E_{j,i}) (I_{n-1} + \sigma_n f_1 E_{i,j})
\]

which can be written as a product of elements of the form

\[
(I_{n-1} - \sigma_n f_1 E_{i,j}) (I_{n-1} + kE_{j,i}) (I_{n-1} + \sigma_n f_1 E_{i,j})
\]

\[
k \in \hat{O}_m, \sigma_r \hat{O}_m, \sigma_r \hat{O}_m, \text{ for } 1 \leq r \leq n - 1.
\]

Finally, as for every such \( k \) one can write

\[
(I_{n-1} - \sigma_n f_1 E_{i,j}) (I_{n-1} + kE_{j,i}) (I_{n-1} + \sigma_n f_1 E_{i,j})
\]

\[
= (I_{n-1} + kE_{j,i}) [(I_{n-1} - kE_{j,i}), (I_{n-1} - \sigma_n f_1 E_{i,j})]
\]

and \( (I_{n-1} + kE_{j,i}) \in GL_{n-1} (R_{n-1}) \), we actually finished. \( \square \)

**Corollary 7.6.** For proving Lemma [7.1] it is enough to show that every element of the forms in Lemma [7.4] is in \( IA^m \).

34
We start here by dealing with the elements of Form 1:

**Proposition 7.7.** Recall $O_m = mR_n$. The elements of the following form are in $IA^m$:

$$A^{-1}(I_{n-1} + hE_{i,j}) A, \quad \text{for } A \in GL_{n-1}(R_n), \ h \in \sigma_n O_m \text{ and } i \neq j.$$

**Proof.** In this case we can write $h = \sigma_n mh'$ for some $h' \in R_n$. So as

$$A^{-1}(I_{n-1} + \sigma_n h'E_{i,j}) A \in GL_{n-1}(R_n, \sigma_n R_n) \leq IA(\Phi)$$

we obtain that

$$A^{-1}(I_{n-1} + hE_{i,j}) A = A^{-1}(I_{n-1} + \sigma_n mh'E_{i,j}) A = (A^{-1}(I_{n-1} + \sigma_n h'E_{i,j}) A)^m \in IA^m$$

as required. \qed

We will devote the remaining sections to deal with the elements of the other three forms. In these cases the proof will be more difficult, and we will need the help of the following computations.

### 7.2 Some auxiliary computations

**Proposition 7.8.** For every $f, g \in R_n$ we have the following equalities:

$$
\begin{pmatrix}
1 - fg & -fg & 0 \\
fg & 1 + fg & 0 \\
0 & 0 & 1
\end{pmatrix}

= \begin{pmatrix}
1 & 0 & 0 \\
fg & 1 & 0 \\
fg^2 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 + fg & -f \\
0 & fg^2 & 1 - fg
\end{pmatrix} \cdot
\begin{pmatrix}
1 & -fg & 0 \\
0 & 1 & 0 \\
0 & -fg^2 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & -f \\
0 & 1 & f \\
0 & 0 & 1
\end{pmatrix}

(7.3)

$$
\begin{pmatrix}
1 & 0 & 0 \\
fg & 1 & 0 \\
-fg^2 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 + fg & f \\
0 & -fg^2 & 1 - fg
\end{pmatrix} \cdot
\begin{pmatrix}
1 & -fg & 0 \\
0 & 1 & 0 \\
0 & fg^2 & 1 + fg
\end{pmatrix}
\begin{pmatrix}
1 & 0 & f \\
0 & 1 & -f \\
0 & 0 & 1
\end{pmatrix}

(7.4)

$$
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
f & f & 1
\end{bmatrix}
\begin{bmatrix}
1 - fg & 0 & fg^2 \\
0 & 1 & 0 \\
f g & 1 & -f g^2
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
fg & 1 & -f g^2 \\
0 & 0 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 + fg & fg^2 \\
0 & -f & 1 - f g
\end{bmatrix}
\begin{bmatrix}
1 - f g & 0 & -f g^2 \\
0 & 1 & 0 \\
f g & 0 & 1 + f g
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
f & f & 1
\end{bmatrix}
\begin{bmatrix}
1 - f g & 0 & -f g^2 \\
0 & 1 & 0 \\
f g & 1 & f g^2
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 + f g & -f g^2 \\
0 & f & 1 - f g
\end{bmatrix}
\begin{bmatrix}
1 - f g & 0 & -f g^2 \\
0 & 1 & 0 \\
f g & 0 & 1 + f g
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 + f g & -f g^2 \\
0 & f & 1 - f g
\end{bmatrix}
\begin{bmatrix}
1 - f g & 0 & -f g^2 \\
0 & 1 & 0 \\
f g & 0 & 1 + f g
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
f g & 1 & 0 + f g
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
f g & 1 & 0 + f g
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 + f g & -f g^2 \\
0 & f & 1 - f g
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
f g & 0 & 1 + f g
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
f g & 0 & 1 + f g
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
f g & 0 & 1 + f g
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

Proof. We use square brackets to help the reader follow the steps of the computation. Here is the computation for Equation (7.3):
Corollary 7.9. Let \( f g \in R_n \) simultaneously. Here is the computation for Equation (7.5):

\[
\begin{pmatrix}
1 - fg & -fg & 0 \\
fg & 1 + fg & 0 \\
0 & 0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & g \\
0 & 1 & -g \\
fg & f & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & -g \\
0 & 1 & g \\
-f & -f & 1
\end{pmatrix}
\]

\[
= 
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
f & f & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & g \\
0 & 1 & -g \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-f & -f & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & -g \\
0 & 1 & g \\
0 & 0 & 1
\end{pmatrix}
\]

\[
= 
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
f & f & 1
\end{pmatrix}
\begin{pmatrix}
1 - fg & 0 & fg^2 \\
fg & 1 & -fg^2 \\
-f & 0 & 1 + fg
\end{pmatrix}
\begin{pmatrix}
1 & -fg & -fg^2 \\
0 & 1 + fg & fg^2 \\
0 & -f & 1 - fg
\end{pmatrix}
\]

\[
= 
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
f & f & 1
\end{pmatrix}
\begin{pmatrix}
1 - fg & 0 & fg^2 \\
0 & 1 & 0 \\
-f & 0 & 1 + fg
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
fg & 1 & -fg^2 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
\cdot 
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 + fg & fg^2 \\
0 & -f & 1 - fg
\end{pmatrix}
\begin{pmatrix}
1 & -fg & -fg^2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

and Equation (7.6) is obtained similarly by changing the signs of \( f \) and \( g \) simultaneously.

In the following corollary, a \( 3 \times 3 \) matrix \( B \in GL_3(R_n) \) denotes the block matrix

\[
\begin{pmatrix}
B & 0 \\
0 & I_{n-4}
\end{pmatrix}
\in GL_{n-1}(R_n).
\]

Corollary 7.9. Let \( n \geq 4, f \in \sigma_n(\sum_{r=1}^{n-1} \sigma_r U_r m + U_{n,m} + O_m) \) and \( g \in R_n \). Then, mod \( I A^m \) we have the following equalities (the indices are intended to help us later to recognize forms of matrices: form 7, form 12 etc.):

\[
\begin{pmatrix}
1 - fg & -fg & 0 \\
fg & 1 + fg & 0 \\
0 & 0 & 1
\end{pmatrix}
\equiv 
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 + fg & -f \\
0 & fg^2 & 1 - fg
\end{pmatrix}
\begin{pmatrix}
1 - fg & 0 & f \\
0 & 1 & 0 \\
-fg^2 & 0 & 1 + fg
\end{pmatrix}
\]

37
\begin{eqnarray*}
&\equiv& \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+fg & f \\ 0 & -fg^2 & 1-fg \end{pmatrix} \begin{pmatrix} 1-fg & 0 & -f \\ 0 & 1 & 0 \\ fg^2 & 0 & 1+fg \end{pmatrix}
\begin{pmatrix} 1-fg & 0 & -f \\ 0 & 1 & 0 \\ fg^2 & 0 & 1+fg \end{pmatrix} \\
&\equiv& \begin{pmatrix} 1-fg & 0 & fg^2 \\ 0 & 1 & 0 \\ -f & 0 & 1+fg \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+fg & fg^2 \\ 0 & -f & 1-fg \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+fg & fg^2 \\ 0 & -f & 1-fg \end{pmatrix} \\
&\equiv& \begin{pmatrix} 1-fg & 0 & -fg^2 \\ 0 & 1 & 0 \\ f & 0 & 1+fg \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+fg & -fg^2 \\ 0 & f & 1-fg \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+fg & -fg^2 \\ 0 & f & 1-fg \end{pmatrix}.
\end{eqnarray*}

Moreover (the inverse of a matrix is denoted by the same index - one can observe that the inverse of each matrix in these equations is obtained by changing the sign of $f$)
\begin{eqnarray*}
&\equiv& \begin{pmatrix} 1-fg & 0 & \scriptstyle{-fg^2} \\ -fg & 1+fg & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&\equiv& \begin{pmatrix} 1-fg & 0 & \scriptstyle{-fg^2} \\ 0 & 1 & 0 \\ f & 0 & 1+fg \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+fg & \scriptstyle{-fg^2} \\ 0 & f & 1-fg \end{pmatrix}.
\end{eqnarray*}
and
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 - fg & -fg \\
0 & fg & 1 + fg
\end{pmatrix}_{17}
\]
\[
\begin{pmatrix}
1 - fg & 0 & fg^2 \\
0 & 1 & 0 \\
-f & 0 & 1 + fg
\end{pmatrix}_{11}
\]
\[
\equiv
\begin{pmatrix}
1 + fg & -fg^2 & 0 \\
0 & f & 1 - fg \\
0 & 0 & 0
\end{pmatrix}^{(7.11)}
\]
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 - fg & fg \\
0 & -fg & 1 + fg
\end{pmatrix}_{18}
\]
\[
\begin{pmatrix}
1 + fg & f & 0 \\
-fg^2 & 1 - fg & 0 \\
0 & 0 & 1
\end{pmatrix}_{10}
\]
\[
\equiv
\begin{pmatrix}
1 - fg & 0 & -f \\
0 & 1 & 0 \\
fg^2 & 0 & 1 + fg
\end{pmatrix}^{(7.12)}
\]

Remark 7.10. We remark that as \( f \in \sigma_n R_n \), then every matrix which takes part in the above equalities is indeed in \( \text{GL}_{n-1}(R_n, \sigma_n R_n) \). Hence, Equation (7.7) is obtained by applying Proposition 7.8 combined with Proposition 6.1. Equation (7.8) is obtained similarly by transposing all the computations which led to the first part of Equation (7.7). Similarly, by switching the roles of the second row and column with the third row and column, one obtains Equations (7.9) and (7.10). By switching one more time the roles of the first row and column with the second row and column, we obtain Equations (7.11) and (7.12) as well. \( \square \)

### 7.3 Elements of Form 2

**Proposition 7.11.** Recall \( U_{r,m} = (x_r^m - 1)R_n \). The elements of the following form, belong to \( IA^m \):
\[
A^{-1} (I_{n-1} + hE_{i,j}) A
\]
where \( A \in \text{GL}_{n-1}(R_n) \), \( h \in \sigma_n \sigma_r^2 U_{r,m}, \sigma_n^2 U_{n,m} \) for \( 1 \leq r \leq n - 1 \) and \( i \neq j \).

Notice that for every \( n \geq 4 \), the groups \( E_{n-1}(\sigma_n^2 U_{n,m}) \) and \( E_{n-1}(\sigma_n^2 U_{r,m}) \) for \( 1 \leq r \leq n - 1 \) are normal in \( \text{GL}_{n-1}(R_n) \), and thus, all the above elements are in \( E_{n-1}(\sigma_n^2 U_{n,m}) \) and \( E_{n-1}(\sigma_n^2 U_{r,m}) \). Hence, for proving Proposition 7.11 it is enough to show that for every \( 1 \leq r \leq n - 1 \) we have, \( E_{n-1}(\sigma_n^2 U_{r,m}), E_{n-1}(\sigma_n^2 U_{r,m}) \subseteq IA^m \). Therefore, by Proposition 7.3 for proving Proposition 7.11 it is enough to show that the elements of the following form belong to \( IA^m \):
\[
(I_{n-1} - fE_{j,i})(I_{n-1} + hE_{i,j})(I_{n-1} + fE_{j,i})
\]
when \( h \in \sigma_n \sigma_r^2 U_{r,m}, \sigma_r^2 U_{m,n} \) for \( 1 \leq r \leq n - 1 \), \( f \in R_n \) and \( i \neq j \). We will prove it in a few stages, starting with the following lemma.
Lemma 7.12. Let \( h \in \sigma_n \sigma_r U_{r,m}, \sigma_n U_{n,m} \) for \( 1 \leq r \leq n-1 \) and \( f_1, f_2 \in R_n \). Assume that the elements of the forms

\[
(I_{n-1} + f_1 E_{j,i}) (I_{n-1} + h E_{i,j}) (I_{n-1} \mp f_1 E_{j,i}) \\
(I_{n-1} + f_2 E_{j,i}) (I_{n-1} + h E_{i,j}) (I_{n-1} \mp f_2 E_{j,i})
\]

for every \( 1 \leq i \neq j \leq n-1 \), belong to \( IA^m \). Then, the elements of the form

\[
(I_{n-1} \pm (f_1 + f_2) E_{j,i}) (I_{n-1} + h E_{i,j}) (I_{n-1} \mp (f_1 + f_2) E_{j,i})
\]

for \( 1 \leq i \neq j \leq n-1 \), also belong to \( IA^m \).

Proof. Observe first that by Proposition 6.11 all the matrices of the form \( I_{n-1} + h E_{i,j} \) for \( h \in \sigma_n \sigma_r U_{r,m}, \sigma_n U_{n,m} \) belong to \( IA^m \). We will use it in the following computations. Without loss of generality, under the assumptions of the proposition, we will show that for \( i, j = 2, 1 \) we have

\[
(I_{n-1} - (f_1 + f_2) E_{1,2}) (I_{n-1} + h E_{2,1}) (I_{n-1} + (f_1 + f_2) E_{1,2}) \in IA^m
\]

and the general argument is similar. In the following computation we use the following notations:

- A matrix \( \begin{pmatrix} B & 0 \\ 0 & I_{n-4} \end{pmatrix} \in GL_{n-1}(R_n) \) is denoted by \( B \in GL_3(R_n) \).
- “\( = \)” denotes an equality between matrices in \( GL_{n-1}(R_n) \).
- “\( \equiv \)” denotes an equality in \( IA(\Phi)/IA^m \).
- We use square brackets to help the reader follow the steps of the computation.

So let’s compute:

\[
(I_{n-1} - (f_1 + f_2) E_{1,2}) (I_{n-1} + h E_{2,1}) (I_{n-1} + (f_1 + f_2) E_{1,2}) \\
= \begin{pmatrix} 1 - h (f_1 + f_2) & -h (f_1 + f_2)^2 & 0 \\ h & 1 + h (f_1 + f_2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
= \begin{pmatrix} 1 & 0 & -(f_1 + f_2) \\ 0 & 1 & 1 \\ -h & -h (f_1 + f_2) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & (f_1 + f_2) \\ 0 & 1 & -1 \\ h & h (f_1 + f_2) & 1 \end{pmatrix} \\
= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -h & -h (f_1 + f_2) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -(f_1 + f_2) \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & (f_1 + f_2) \\ 0 & 1 & -1 \\ h & h (f_1 + f_2) & 1 \end{pmatrix}
\]

40
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-h & -h (f_1 + f_2) & 1
\end{pmatrix}
\]
\[
\cdot \begin{pmatrix}
1 & 0 & -f_2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\cdot \begin{pmatrix}
1 & 0 & f_2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\cdot \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-h & -hf_1 & 1
\end{pmatrix}
\cdot \begin{pmatrix}
1 & 0 & f_1 \\
0 & 1 & 0 \\
h & hf_1 & 1
\end{pmatrix}
\cdot \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
h & hf_1 & 1
\end{pmatrix}
\cdot \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
h & hf_1 & 1
\end{pmatrix}
\cdot \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
h & hf_1 & 1
\end{pmatrix}
\cdot \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
h & hf_1 & 1
\end{pmatrix}
\cdot \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
h & hf_1 & 1
\end{pmatrix}
\cdot \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
h & hf_1 & 1
\end{pmatrix}
\cdot \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
h & hf_1 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-h & -h (f_1 + f_2) & 1
\end{pmatrix}
\cdot \begin{pmatrix}
1 & 0 & (f_1 + f_2) f_2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\cdot \begin{pmatrix}
1 & 0 & f_2 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\cdot \begin{pmatrix}
1 & 0 & f_1 \\
0 & 1 & 0 \\
h & hf_1 & 1
\end{pmatrix}
\cdot \begin{pmatrix}
1 & 0 & f_2 \\
0 & 1 & 0 \\
h & hf_1 & 1
\end{pmatrix}
\cdot \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
h & hf_1 & 1
\end{pmatrix}
\cdot \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
h & hf_1 & 1
\end{pmatrix}
\cdot \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
h & hf_1 & 1
\end{pmatrix}
\cdot \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
h & hf_1 & 1
\end{pmatrix}
\cdot \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
h & hf_1 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 + h f_2 & -h f_2 \\
0 & h f_2 & 1 - h f_2
\end{pmatrix}
\end{pmatrix}
\]

Notice now that by assumption, and by the remark at the beginning of the proof, mod IA^m, the latter expression is congruent to
\[
\equiv \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 + h f_2 & -h f_2 \\
0 & h f_2 & 1 - h f_2
\end{pmatrix}
\cdot
\]

Consider now Equation \((7.12)\) in Corollary \([4.9]\) and switch the roles of \(f, g\) by \(-h, f_2\) respectively. Using this identity we deduce that, mod IA^m, the latter expression is congruent to
\[
\equiv \begin{pmatrix}
1 - h f_2 & -h & 0 \\
h f_2 & 1 + h f_2 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 + h f_2 & 0 & h \\
0 & 1 & 0 \\
-h f_2 & 0 & 1 - h f_2
\end{pmatrix}
\]
that is \(\equiv I_{n-1}\) by assumption. This finishes the proof of the lemma.
We pass to the next stage:

**Proposition 7.13.** The elements of the following form belong to $IA^m$:

$$(I_{n-1} - fE_{j,i}) (I_{n-1} + hE_{i,j}) (I_{n-1} + fE_{j,i})$$

where $h \in \sigma_n \sigma_r^2 U_{r,m}, \sigma_n^2 U_{n,m}$ for $1 \leq r \leq n-1, f \in \mathbb{Z}$ and $i \neq j$.

**Remark 7.14.** We note that some of the matrices that we use in the following computations lie in $IGL'_{n-1,n} \hookrightarrow GL_{n-1}(\mathbb{R}^n)$ and not necessarily in $IGL_{n-1,n}$ (see Definition 3.10 and Proposition 3.11).

**Proof.** (of Proposition 7.13) According to Lemma 7.12, it is enough to prove the proposition for $f = \pm 1$. Without loss of generality, we will prove the proposition for $r = 1$, i.e. $h \in \sigma_n \sigma_1 U_{1,m}$, and symmetrically, the same is valid for every $1 \leq r \leq n-1$. The case $h \in \sigma_n^2 U_{n,m}$ will be considered separately.

So let $h \in \sigma_n \sigma_1^2 U_{1,m}$ and write: $h = \sigma_1 u$ for some $u \in \sigma_n \sigma_1 U_{1,m}$. We will prove the proposition for $i \neq j \in \{1, 2, 3\}$. As one can see below, we will do it simultaneously for all the options for $i \neq j \in \{1, 2, 3\}$. The treatment in the other cases in which $i \neq j \in \{1, k, l\}$ such that $1 < k \neq l \leq n-1$ is obtained symmetrically, so we get that the proposition is valid for every $1 \leq i \neq j \leq n-1$.

As before, we denote a block matrix of the form

$$\begin{pmatrix} B & 0 & 0 \\ 0 & I_{n-4} \end{pmatrix} \in GL_{n-1}(\mathbb{R}^n)$$

by $B \in GL_3(\mathbb{R}^n)$. In the following computations, the indices of the matrices are intended to help the reader recognize the corresponding matrix type in Corollary 7.12 as will be explained below. We remind that the inverse of a matrix is denoted by the same index, and one can observe that the inverse of each indexed matrix is obtained by changing the sign of $u$. We also remind that $u \in \sigma_n \sigma_1 U_{1,m} \subseteq \sigma_n R_n$. Thus, by Proposition 6.14 we have

$$\begin{pmatrix} 1 - \sigma_1 u & -\sigma_2^2 u & 0 \\ u & 1 + \sigma_1 u & 0 \\ 0 & 0 & 1 \end{pmatrix}_{12} = \begin{pmatrix} x_2 & -\sigma_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in IA^m$$

$$\begin{pmatrix} x_2^{-1} & x_2^{-1} & \sigma_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in IA^m$$

$$\begin{pmatrix} 1 - \sigma_1 u & 0 & -\sigma_2^2 u \\ 0 & 1 & 0 \\ u & 0 & 1 + \sigma_1 u \end{pmatrix}_7 = \begin{pmatrix} x_3 & 0 & -\sigma_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in IA^m$$

$$\begin{pmatrix} x_3^{-1} & 0 & -x_3^{-1} \sigma_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in IA^m$$

42
Proposition 7.15. The elements of the following form belong to $IA^m$:

$$
(I_{n-1} - fE_{j,i}) (I_{n-1} + hE_{i,j}) (I_{n-1} + fE_{j,i})
$$
where \( h \in \sigma_r^2 U_{r,m}, \sigma_n^2 U_{r,m} \) for \( 1 \leq r \leq n-1, \ f \in \sigma_s R_n \) for \( 1 \leq s \leq n \) and \( i \neq j \).

**Proof.** We will prove it for \( s = 1, i \neq j \in \{1, 2, 3\} \), and denote a block matrix of the form

\[
\begin{pmatrix}
B & 0 \\
0 & I_{n-4}
\end{pmatrix} \in GL_{n-1}(R_n)
\]

by \( B \in GL_3(R_n) \). We will use again the result of Corollary 7.9 when we switch the roles of \( f, g \) in the corollary by \( h, \sigma_1 u \) respectively for some \( u \in R_n \).

As \( h \in \sigma_r^2 U_{r,m}, \sigma_n^2 U_{r,m} \), we have also \( \sigma_1 u h \in \sigma_r^2 U_{r,m}, \sigma_n^2 U_{n,m} \). Hence, we obtain from the previous proposition, that the matrices of Forms 13–18 belong to \( I A^m \). In addition

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 - u \sigma_1 h & h \\
0 & -u^2 \sigma_1^2 h & 1 + u \sigma_1 h
\end{pmatrix}_1 = \begin{pmatrix}
1 & 0 & 0 \\
-uh \sigma_2 & 1 & 0 \\
-hu^2 \sigma_1 \sigma_2 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-uh_2 \sigma_1 \sigma_2 & 0 & 1
\end{pmatrix} \in I A^m
\]

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 - u \sigma_1 h & -u^2 \sigma_1^2 h \\
0 & h & 1 + u \sigma_1 h
\end{pmatrix}_6 = \begin{pmatrix}
1 & 0 & 0 \\
-hu^2 \sigma_1 \sigma_3 & 1 & 0 \\
hu \sigma_3 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-hu_3 \sigma_1 \sigma_3 & 0 & 1
\end{pmatrix} \in I A^m
\]

and by switching the signs of \( u \) and \( h \) simultaneously, we get also Forms 3 and 8. So we easily conclude from Corollary 7.9 (Equations 7.7 and 7.9) that also the matrices of the other eight forms are in \( I A^m \). In particular, the matrices of the form

\[
(I_{n-1} - \sigma_1 u E_{j,i}) (I_{n-1} + h E_{i,j}) (I_{n-1} + \sigma_1 u E_{j,i}), \ i \neq j \in \{1, 2, 3\}
\]

belong to \( I A^m \). The treatment for every \( i \neq j \) and \( 1 \leq s \leq n-1 \) is similar, and the treatment in the case \( s = n \) is obtained by replacing \( \sigma_1 \) by \( \sigma_n \) and \( \sigma_2, \sigma_3 \) by 0 in the above equations.

**Corollary 7.16.** As every \( f \in R_n \) can be decomposed as \( f = \sum_{s=1}^n \sigma_s f_s + f_0 \) for some \( f_0 \in \mathbb{Z} \) and \( f_i \in R_n \), we obtain from Lemma 7.12 and from the above two propositions that we actually finish the proof of Proposition 7.11.

### 7.4 Elements of Form 3

**Proposition 7.17.** Recall \( \Omega_m = m R_{n-1} \) where \( R_{n-1} = \mathbb{Z}[x_{-1}^{\pm 1}, \ldots, x_{n-1}^{\pm 1}] \subseteq R_n \). Then, The elements of the following form, belong to \( I A^m \):

\[
A^{-1} [(I_{n-1} + h E_{i,j}), (I_{n-1} + f E_{j,i})] A
\]
where \( A \in GL_{n-1}(R_n), f \in \sigma_n R_n, h \in \bar{O}_n^2 \) and \( i \neq j \).

We will prove the proposition in the case \( i, j = 2, 1 \), and the same arguments are valid for arbitrary \( i \neq j \). In this case one can write: \( h = m^2 h' \) for some \( h' \in R_{n-1} \), and thus, our element is of the form

\[
A^{-1} \begin{pmatrix}
1 - fm^2 h' & f & 0 \\
-f \left( m^2 h' \right)^2 & 1 + fm^2 h' & 0 \\
0 & 0 & I_{n-3}
\end{pmatrix} A
\]

for some \( A \in GL_{n-1}(R_n), f \in \sigma_n R_n \) and \( h' \in R_{n-1} \). The proposition will follow easily from the following lemma:

**Lemma 7.18.** Let \( h_1, h_2 \in R_n, f \in \sigma_n R_n \) and denote a block matrix of the form \( \begin{pmatrix} B & 0 \\ 0 & I_{n-4} \end{pmatrix} \in GL_{n-1}(R_n) \) by \( B \in GL_3(R_n) \). Then

\[
A^{-1} \begin{pmatrix}
1 - fm (h_1 + h_2) & f & 0 \\
-f \left( m (h_1 + h_2) \right)^2 & 1 + fm (h_1 + h_2) & 0 \\
0 & 0 & 1
\end{pmatrix} \equiv \begin{pmatrix}
1 - fm h_1 & f & 0 \\
-f \left( mh_1 \right)^2 & 1 + fm h_1 & 0 \\
0 & 0 & 1
\end{pmatrix} \cdot \begin{pmatrix}
1 - f & 0 \\
0 & 1 \\
0 & 0
\end{pmatrix} \mod IA^m
\]

Now, if Lemma 7.18 is proved, one can deduce that for \( f \in \sigma_n R_n \) and \( h = m^2 h', h' \in R_n \), we have

\[
A^{-1} \begin{pmatrix}
1 - fm^2 h' & f & 0 \\
-f \left( m^2 h' \right)^2 & 1 + fm^2 h' & 0 \\
0 & 0 & I_{n-3}
\end{pmatrix} A
\]

\[
\equiv \left[ A^{-1} \begin{pmatrix}
1 - fm h' & f & 0 \\
-f \left( mh' \right)^2 & 1 + fm h' & 0 \\
0 & 0 & I_{n-3}
\end{pmatrix} \right]^m \mod IA^m
\]

and as the latter element is obviously belong to \( IA^m \), Proposition 7.17 follows. So it is enough to prove Lemma 7.18.

**Proof.** (of Lemma 7.18) Throughout the computation we will use the observation that as \( GL_{n-1}(R_n, \sigma_n R_n) \) is normal in \( GL_{n-1}(R_n) \), every conjugate of an element of \( GL_{n-1}(R_n, \sigma_n R_n) \leq IA(\Phi) \) by an element of \( GL_{n-1}(R_n) \), belongs to \( GL_{n-1}(R_n, \sigma_n R_n) \leq IA(\Phi) \) (as was mentioned in Remark 7.5) - even though \( GL_{n-1}(R_n) \not\leq IA(\Phi) \). Throughout the computation, we will use the notations which we used in the proof of Lemma 7.12.
A matrix \(
\begin{pmatrix}
B & 0 \\
0 & I_{n-4}
\end{pmatrix}
\) \(\in GL_{n-1}(R_n)\) is denoted by \(B \in GL_3(R_n)\).

• “=” denotes an equality between matrices in \(GL_{n-1}(R_n)\).

• “≡” denotes an equality in \(IA(Φ)/IA^m\).

• We use square brackets to help the reader follow the steps of the computation. Whenever square brackets are used, it is recommended to concentrate in the expression inside them separately in order to follow the transition to the next step.

So let’s compute:

\[
\begin{align*}
A^{-1} & \begin{pmatrix}
1 - fm(h_1 + h_2) & f & 0 \\
-f(m(h_1 + h_2))^2 & 1 + fm(h_1 + h_2) & 0 \\
0 & 0 & 1
\end{pmatrix} A \\
= & \ A^{-1} \begin{pmatrix}
1 & 0 & -f \\
0 & 1 & -fm(h_1 + h_2) \\
-m(h_1 + h_2) & 1 & 1
\end{pmatrix} \\
& \cdot \begin{pmatrix}
1 & 0 & f \\
0 & 1 & fm(h_1 + h_2) \\
m(h_1 + h_2) & -1 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-f & fmh_1 & 0
\end{pmatrix} A \\
= & \ A^{-1} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-mh_2 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & -f \\
0 & 1 & 0 \\
-mh_1 & 1 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-m(h_1 + h_2) & 1 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-mh_2 & 0 & 1
\end{pmatrix} A \\
\cdot \ A^{-1} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-m(h_1 + h_2) & 1 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-mh_2 & 0 & 1
\end{pmatrix} A \\
\cdot \ A^{-1} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & f(h_1 + h_2) \\
0 & 0 & 1
\end{pmatrix} A = \ A^{-1} \begin{pmatrix}
1 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} A
\end{align*}
\]
\[
\begin{align*}
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -mh_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - fmh_1 & f \\ -f (mh_1)^2 & 1 + fmh_1 \\ 0 & 0 & 1 \end{pmatrix} \\
&\cdot \begin{pmatrix} f & 0 \\ 0 & 1 \\ -f mh_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ mh_2 & 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -m (h_1 + h_2) & 1 & 1 \end{pmatrix}
\end{align*}
\]
\[ \begin{align*}
A &= A^{-1} \left( \begin{array}{ccc}
1 - f mh_1 & f & 0 \\
-f (mh_1)^2 & 1 + f mh_1 & 0 \\
0 & 0 & 1
\end{array} \right) \left( \begin{array}{ccc}
1 - f & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) \\
\cdot \left( \begin{array}{ccc}
1 & f & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) A \\
\cdot \left( \begin{array}{ccc}
1 & 0 & f \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) A \\
\equiv A^{-1} \left( \begin{array}{ccc}
1 - f mh_1 & f & 0 \\
-f (mh_1)^2 & 1 + f mh_1 & 0 \\
0 & 0 & 1
\end{array} \right) \left( \begin{array}{ccc}
1 - f & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) A \\
\cdot \left( \begin{array}{ccc}
1 - f mh_2 & 0 & -f \\
0 & 1 & 0 \\
f (mh_2)^2 & 0 & 1 + f mh_2
\end{array} \right) \left( \begin{array}{ccc}
1 & 0 & f \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) A.
\end{align*} \]

So it remains to show that
\[ \begin{align*}
A^{-1} \left( \begin{array}{ccc}
1 - f mh_2 & 0 & -f \\
0 & 1 & 0 \\
f (mh_2)^2 & 0 & 1 + f mh_2
\end{array} \right) \left( \begin{array}{ccc}
1 & 0 & f \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) A \\
\equiv A^{-1} \left( \begin{array}{ccc}
1 - f mh_2 & f & 0 \\
-f (mh_2)^2 & 1 + f mh_2 & 0 \\
0 & 0 & 1
\end{array} \right) \left( \begin{array}{ccc}
1 - f & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right) A.
\end{align*} \]

By a similar computation as for Equation (7.10), we have (switch the roles of \(f, g\) in the equation by \(f, mh_2\) respectively, and then switch the roles of the first row and column with the third row and column)
\[ \begin{align*}
\left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 + f mh_2 & f mh_2 \\
0 & -f mh_2 & 1 - f mh_2
\end{array} \right) = \left( \begin{array}{ccc}
1 - f & -f \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array} \right)
\end{align*} \]

48
Therefore, using Proposition 7.7 and the observation

\[
A^{-1} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 + fmh_2 & 0 \\
0 & -fmh_2 & 1 - fmh_2
\end{pmatrix}
A = \left[ A^{-1} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 + fh_2 & fh_2 \\
0 & -fh_2 & 1 - fh_2
\end{pmatrix} A \right] m \in IA^m.
\]

we obtain that mod \( IA^m \) we have

\[
A^{-1} \begin{pmatrix}
1 + fmh_2 & 0 & f \\
0 & 1 & 0 \\
-f(mh_2)^2 & 0 & 1 - fmh_2
\end{pmatrix}
A \equiv A^{-1} \begin{pmatrix}
1 & f & f \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} A.
\]

From here, we easily get Equation (7.13) by noticing that the inverse of every matrix in Equation (7.14) is obtained by replacing \( f \) by \(-f\). This finishes the proof of the lemma, and hence, also the proof of Proposition 7.17.

**7.5 Elements of Form 4**

**Proposition 7.19.** Recall \( \bar{O}_m = mR_{n-1} \) and \( \bar{U}_{r,m} = (x_i^m - 1)R_{n-1} \) for \( 1 \leq r \leq n - 1 \), where \( R_{n-1} = \mathbb{Z}[x_1^{1}, \ldots, x_{n-1}^{1}] \subseteq R_n \). Then, the elements of the following form, belong to \( IA^m \):

\[
A^{-1} [(I_{n-1} + hE_{i,j}), (I_{n-1} + fE_{i,j})] A
\]

where \( A \in GL_{n-1}(R_n), f \in \sigma_r R_n, h \in \sigma_r^2 \bar{U}_{r,m}, \sigma_r \bar{O}_m \) for \( 1 \leq r \leq n - 1 \) and \( i \neq j \).

As before, throughout the subsection we denote a block matrix of the form

\[
\begin{pmatrix}
B & 0 \\
0 & I_{n-4}
\end{pmatrix}
\]

by \( B \in GL_3(R_n) \). We start the proof of this proposition with the following lemma.

**Lemma 7.20.** Let \( f, h \in R_n \) and \( A \in GL_{n-1}(R_n) \). Then

\[
A^{-1} \begin{pmatrix}
1 - fh & -fh & 0 \\
fh & 1 + fh & 0 \\
0 & 0 & 1
\end{pmatrix} A
\]
and I.e. we have the following corollary (notice that we switched the sign of Corollary 7.21.

For every $GL_n$ with $n$,

The lemma follows from Proposition 7.8, Equation (7.3), by substituting $g$ with $h$, combined with verifying the identity

$$
\begin{pmatrix}
1 & 0 & 0 \\
fh & 1 & 0 \\
fh^2 & 0 & 1 \\
\end{pmatrix}
A
$$

Proof. The lemma follows from Proposition 7.8, Equation (7.3), by substituting

Therefore, by Propositions 7.7, 7.11 and the previous lemma, for every $A \in GL_{n-1}(R_n)$ we have the following equality mod $IA^m$:

$$
A^{-1}
\begin{pmatrix}
1 & 0 & 0 \\
fh & 1 & 0 \\
fh^2 & 0 & 1 \\
\end{pmatrix}
A
$$

Observe now that if we have $f \in \sigma_n R_n$ and $h \in \sigma_r U_{r,m}, \sigma_r O_m$ for $1 \leq r \leq n - 1$, then by Propositions 7.7 and 7.11 we have

$$
A^{-1}
\begin{pmatrix}
1 & 0 & 0 \\
fh & 1 & 0 \\
fh^2 & 0 & 1 \\
\end{pmatrix}
A \in IA^m.
$$

Therefore, by Propositions 7.7, 7.11 and the previous lemma, for every $A \in GL_{n-1}(R_n)$ we have the following equality mod $IA^m$:

$$
A^{-1}
\begin{pmatrix}
1 & 0 & 0 \\
fh & 1 & 0 \\
fh^2 & 0 & 1 \\
\end{pmatrix}
A
$$

I.e. we have the following corollary (notice that we switched the sign of $f$):

**Corollary 7.21.** For every $h \in \sigma_r U_{r,m}, \sigma_r O_m$ for $1 \leq r \leq n - 1$, $f \in \sigma_n R_n$ and $A \in GL_{n-1}(R_n)$, the following elements are congruent mod $IA^m$

$$
A^{-1}
\left[
\begin{pmatrix}
1 & 0 & 0 \\
fh & 1 & 0 \\
fh^2 & 0 & 1 \\
\end{pmatrix}
\right] A \equiv A^{-1}
\left[
\begin{pmatrix}
1 & 0 & 0 \\
fh & 1 & 0 \\
fh^2 & 0 & 1 \\
\end{pmatrix}
\right]^{-1} A.
$$
We proceed with the following proposition:

**Proposition 7.22.** Let $h \in \sigma^2 \bar{U}_{1,m}, \sigma_1 \bar{O}_m$ and $f \in \sigma_n R_n$. Then
\[
[(I_{n-1} + hE_{3,2}), (I_{n-1} + fE_{2,3})] \in I A^m.
\]

**Proof.** Denote $h = \sigma_1 u$ for some $u \in \sigma_1 \bar{U}_{1,m}, \bar{O}_m$. By Proposition 6.1 we have
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-\sigma_2 u & \sigma_1 u & 1
\end{pmatrix}
\in I A^m
\]
and hence
\[
I A^m \ni \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-\sigma_2 u & \sigma_1 u & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & f \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
\sigma_2 u & -\sigma_1 u & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -f \\
0 & 0 & 1
\end{pmatrix}
\]
\[
= \begin{pmatrix}
1 & 0 & 0 \\
f \sigma_2 u & 0 & 0 \\
\sigma_1 \sigma_2 u^2 f & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & f \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -\sigma_1 u & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -f \\
0 & 0 & 1
\end{pmatrix}
\]
As by Proposition 6.1 the first matrix in the right hand side is also in $I A^m$, we obtain that
\[
[(I_{n-1} + hE_{3,2}), (I_{n-1} + fE_{2,3})] = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & h & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & f \\
0 & 0 & 1
\end{pmatrix}
\in I A^m
\]
as required. \(\square\)

We can now pass to the following proposition.

**Proposition 7.23.** Let $h \in \sigma^2 \bar{U}_{1,m}, \sigma_1 \bar{O}_m$, $f \in \sigma_n R_n$ and $A \in GL_{n-1}(R_n)$. Then
\[
\begin{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & h & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 \\
0 & h & 1
\end{pmatrix}
\end{pmatrix} A^{-1} [(I_{n-1} + hE_{3,2}), (I_{n-1} + fE_{2,3})] A \in I A^m.
\]

**Proof.** We will prove the proposition by induction. By a result of Suslin [Su], as $n - 1 \geq 3$, the group $SL_{n-1}(R_n)$ is generated by the elementary matrices of the form
\[
I_{n-1} + rE_{i,k} \text{ for } r \in R_n, \text{ and } 1 \leq l \neq k \leq n - 1.
\]
So as the invertible elements of $R_n$ are the elements of the form $\pm \prod_{i=1}^{n} x_i^{s_i}$ for $s_i \in \mathbb{Z}$ (see [CF], chapter 8), $GL_{n-1}(R_n)$ is generated by the elementary matrices and the matrices of the form: $I_{n-1} + (\pm x_i - 1)E_{1,1}$ for $1 \leq i \leq n$. Therefore, by the previous proposition it is enough to show that if
\[
\begin{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & h & 1
\end{pmatrix}
\end{pmatrix} A^{-1} [(I_{n-1} + hE_{3,2}), (I_{n-1} + fE_{2,3})] A \in I A^m
\]

51
and $E$ is one of the above generators, then mod $IA^m$ we have
\[
A^{-1}E^{-1} [(I_{n-1} + hE_{3,2}), (I_{n-1} + fE_{2,3})] EA \equiv A^{-1} [(I_{n-1} + hE_{3,2}), (I_{n-1} + fE_{2,3})] A.
\]
(7.15)

So if $E$ is of the form $I_{n-1} + (\pm x_i-1)E_{1,1}$, we obviously have Property (7.15). If $E$ is an elementary matrix of the form $I_{n-1} + rE_{i,k}$ such that $l, k \notin \{2,3\}$ then we also have Property (7.15) in an obvious way. Consider now the case $l, k = 2,3$. In this case, by Corollary 7.21 we have the following mod $IA^m$
\[
A^{-1}E^{-1} \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & h & 1 \end{array} \right], \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & f \\ 0 & 0 & 1 \end{array} \right] \right] EA \equiv A^{-1} \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & -r \\ 0 & 0 & 1 \end{array} \right) \left[ \begin{array}{ccc} 1 & 0 & -f \\ 0 & 1 & 0 \\ h & 0 & 1 \end{array} \right], \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) A
\]
\[
= A^{-1} \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & -r \\ 0 & 0 & 1 \end{array} \right) \left[ \begin{array}{ccc} 1 & -hf + h^2 f^2 & 0 \\ 0 & 1 & 0 \\ -h^2 f & 0 & 1 + hf \end{array} \right] AA^{-1} \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) A
\]
\[
= A^{-1} \left[ \begin{array}{ccc} 1 & 0 & -f \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right], \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h & 0 & 1 \end{array} \right] \right] AA^{-1} \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) A.
\]

So by applying Propositions 7.7, 7.11 and Corollary 7.21 once again on the opposite way, we obtain Property (7.15). The other cases for $l, k$ are treated by similar arguments: if $l, k = 3,2$ we do exactly the same, and if $l$ or $k$ are different from 2 and 3, then the situation is easier - we use similar arguments, but without passing to $[(I_{n-1} - fE_{1,3}), (I_{n-1} + hE_{3,1})]$ through Corollary 7.21.

**Corollary 7.24.** Let $h \in \sigma^2_iU_{1,m}, \sigma_iO_m, f \in \sigma_nR$ and $A \in GL_{n-1} (R_n)$. Then, for every $i \neq j$ we have
\[
A^{-1} [(I_{n-1} + hE_{i,j}), (I_{n-1} + fE_{j,i})] A \in IA^m.
\]

**Proof.** Denote a permutation matrix, which its action on $GL_{n-1} (R_n)$ by conjugation, moves $2 \rightarrow j$ and $3 \rightarrow i$, by $P$. Then, by the previous proposition, we have
\[
A^{-1} [(I_{n-1} + hE_{i,j}), (I_{n-1} + fE_{j,i})] A
\]
\[
= A^{-1} P^{-1} [(I_{n-1} + hE_{3,2}), (I_{n-1} + fE_{2,3})] PA \in IA^m.
\]

Now, as one can see that symmetrically, the above corollary is valid for every $h \in \sigma^2_iU_{r,m}, \sigma_rO_m$ for $1 \leq r \leq n - 1$, we actually finished the proof of Proposition 7.19.
8 Index of notations

For the convenient of the reader, we gathered here some notations that play role along the paper, and mention the section in the body of the paper when they appear for the first time.

- $F_n =$ the free group on $n$ elements, Section 3
- $\Phi = \Phi_n = F_n/F'_n =$ the free metabelian group on $n$ elements, Section 3
- $\Psi_m = \Phi/M_m$, where $M_m = (\Phi'\Phi^m)'(\Phi'\Phi^m)^m$, Section 3
- $IA(\Phi) = \ker (\text{Aut} (\Phi) \to \text{Aut} (\Phi/\Phi'))$, Section 3
- $IG_m = G(M_m) = \ker(IA(\Phi) \to \text{Aut}(\Psi_m))$, Section 3
- $IA^m = \langle IA(\Phi)^m \rangle$, Section 7
- $IA_m = \cap \{ N \triangleleft IA(\Phi) \mid |IA(\Phi) : N| \leq m \}$, Section 3
- $R_n = Z[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ where $x_1, \ldots, x_n$ are free commutative variables, Section 3
- $R_{n-1} = Z[x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}]$, Section 7
- $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$, Section 3
- $\sigma_i = x_i - 1$ for $1 \leq i \leq n$, Section 3
- $\bar{\sigma} =$ the column vector which has $\sigma_i$ in its $i$-th entry, Section 3
- $\mathbb{A} = \sum_{i=1}^n \sigma_i R_n \triangleleft R_n =$ the augmentation ideal of $R_n$, Section 3
- $O_m = mR_n \triangleleft R_n$, Section 7
- $\bar{O}_m = mR_{n-1} \triangleleft R_{n-1}$, Section 7
- $U_{r,m} = (x_r^{-m} - 1)R_n \triangleleft R_n$, for $1 \leq r \leq n$, Section 7
- $\bar{U}_{r,m} = (x_r^{-m} - 1)R_{n-1} \triangleleft R_{n-1}$, for $1 \leq r \leq n$, Section 7
- $H_m = \sum_{i=1}^n (x_i^{-m} - 1)R_n + mR_n \triangleleft R_n$, Section 3
- $S = \mathbb{Z}[x_1^{\pm 1}]$, Section 4
- $J_m = (x^{-m} - 1) S + mS \triangleleft S$, Section 4
- $E_d (R) = \{ I_d + rE_{i,j} \mid r \in R, 1 \leq i \neq j \leq d \} \leq SL_d (R)$, where $R$ is a ring and $E_{i,j}$ is the matrix that has 1 in its $(i,j)$-th entry and 0 elsewhere, Section 4
- $SL_d (R,H) = \ker (SL_d (R) \to SL_d (R/H))$, where $R$ is a ring and $H \triangleleft R$, Section 4
\[ GL_d(R, H) = \ker(GL_d(R) \to GL_d(R/H)), \text{ where } R \text{ is a ring and } H \triangleleft R, \text{ Section 2} \]

\[ E_d(R, H) = \text{the normal subgroup of } E_d(R), \text{ generated as a normal subgroup by the matrices of the form } I_d + hE_{i,j} \text{ for } h \in H, \text{ Section 2} \]

\[ IGL_{n-1,i} = \left\{ I_n + A \in IA(\Phi) \mid \begin{array}{c}
\text{The } i\text{-th row of } A \text{ is 0,} \\
I_{n-1} + A_{i,i} \in GL_{n-1}(R_n, \sigma_i R_n)
\end{array} \right\}, \text{ for } 1 \leq i \leq n, \text{ Section 3} \]

\[ ISL_{n-1,i}(H) = IGL_{n-1,i} \cap SL_{n-1}(R_n, H), \text{ under the identification of } IGL_{n-1,i} \text{ with } GL_{n-1}(R_n, \sigma_i R_n), \text{ Section 3} \]

\[ IE_{n-1,i}(H) = IGL_{n-1,i} \cap E_{n-1}(R_n, H), \text{ under the identification of the group } IGL_{n-1,i} \text{ with } GL_{n-1}(R_n, \sigma_i R_n), \text{ Section 3} \]

\[ IGL'_{n-1,i} = \{ I_n + A \in IA(\Phi) \mid \text{The } i\text{-th row of } A \text{ is 0} \}, \text{ for } 1 \leq i \leq n, \text{ Section 3} \]

References

[A] M. Asada, The faithfulness of the monodromy representations associated with certain families of algebraic curves, J. Pure Appl. Algebra 159 (2001), 123–147.

[Bac] S. Bachmuth, Automorphisms of free metabelian groups, Trans. Amer. Math. Soc. 118 (1965) 93-104.

[Bas] H. Bass, Algebraic K-theory, W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[Be1] D. E-C. Ben-Ezra, The congruence subgroup problem for the free metabelian group on two generators. Groups Geom. Dyn. 10 (2016), 583–599.

[Be2] D. E-C. Ben-Ezra, The congruence subgroup problem for the free metabelian group on \( n \geq 4 \) generators, [arXiv:1701.02459], accepted for publication in Groups Geom. Dyn.

[Bi] J. S. Birman, Braids, links, and mapping class groups, Princeton University Press, Princeton, NJ, University of Tokyo Press, Toyko, 1975.

[Bo1] M. Boggi, The congruence subgroup property for the hyperelliptic modular group: the open surface case, Hiroshima Math. J. 39 (2009), 351–362.

[Bo2] M. Boggi, A generalized congruence subgroup property for the hyperelliptic modular group, [arXiv:0803.3841v5].
[BER] K-U. Bux, M. V. Ershov, A. S. Rapinchuk, The congruence subgroup property for $\text{Aut}(F_2)$: a group-theoretic proof of Asada’s theorem, Groups Geom. Dyn. 5 (2011), 327–353.

[BL] D. E-C. Ben-Ezra, A. Lubotzky, The congruence subgroup problem for low rank free and free metabelian groups, J. Algebra 500 (2018), 171–192.

[BaLS] H. Bass, M. Lazard, J.-P. Serre, Sous-groupes d’indice fini dans $SL_n(Z)$, (French) Bull. Amer. Math. Soc. 70 (1964) 385–392.

[BrLS] K. A. Brown, T. H. Lenagan, J. T. Stafford, K-theory and stable structure of some Noetherian group rings, Proc. London Math. Soc. 42 (1981), 193–230.

[BM] S. Bachmuth, H. Y. Mochizuki, $\text{Aut}(F) \to \text{Aut}(F/F'')$ is surjective for free group $F$ of rank $\geq 4$, Trans. Amer. Math. Soc. 292 (1985), 81–101.

[CF] R. H. Crowell, R. H. Fox, Introduction to knot theory, Ginn and Co., Boston, Mass., 1963.

[DDH] S. Diaz, R. Donagi, D. Harbater, Every curve is a Hurwitz space, Duke Math. J. 59 (1989), 737–746.

[DS] R. K. Dennis, M. R. Stein, The functor $K_2$: a survey of computations and problems, Algebraic K-theory, II: ”Classical” algebraic K-theory and connections with arithmetic, 243–280, Lecture Notes in Math., Vol. 342, Springer, Berlin, 1973.

[I] S. V. Ivanov, Group rings of Noetherian groups, (Russian) Mat. Zametki 46 (1989), 61–66, 127; translation in Math. Notes 46 (1989), 929–933 (1990).

[KN] M. Kassabov, M. Nikolov, Universal lattices and property tau, Invent. Math. 165 (2006), 209–224.

[L] A. Lubotzky, Free quotients and the congruence kernel of $SL_2$, J. Algebra 77 (1982), 411–418.

[Ma] W. Magnus, On a theorem of Marshall Hall, Ann. of Math. 40 (1939), 764–768.

[Mc] D. B. McReynolds, The congruence subgroup problem for pure braid groups: Thurston’s proof, New York J. Math. 18 (2012), 925–942.

[Mel] O. V. Mel’nikov, Congruence kernel of the group $SL_2(Z)$, (Russian) Dokl. Akad. Nauk SSSR 228 (1976), 1034–1036.

[Men] J. L. Mennicke, Finite factor groups of the unimodular group, Ann. of Math. 81 (1965), 31–37.
[Mi] J. Milnor, Introduction to algebraic K-theory, Annals of Mathematics Studies, No. 72, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1971.

[MKS] W. Magnus, A. Karrass, D. Solitar, Combinatorial group theory: Presentations of groups in terms of generators and relations, Interscience Publishers, New York-London-Sydney, 1966.

[NS] N. Nikolov, D. Segal, Finite index subgroups in profinite groups, C. R. Math. Acad. Sci. Paris 337 (2003), 303–308.

[PR] G. Prasad, A. S. Rapinchuk, Developments on the congruence subgroup problem after the work of Bass, Milnor and Serre, Collected papers of John Milnor. V: Algebra, Amer. Math. Soc., Providence, RI, 2010.

[Q] D. Quillen, Higher algebraic K-theory. I, Algebraic K-theory, I: Higher K-theories, 85–147, Lecture Notes in Math., Vol. 341, Springer, Berlin, 1973.

[Rom] N. S. Romanovskiĭ, On Shmel’kin embeddings for abstract and profinite groups, (Russian) Algebra Log. 38 (1999), 326–334.

[Ros] J. Rosenberg, Algebraic K-theory and its applications, Graduate Texts in Mathematics 147, Springer-Verlag, New York, 1994.

[RS] V. N. Remeslennikov, V. G. Sokolov, Some properties of a Magnus embedding, (Russian) Algebra i Logika 9 (1970), 566–578, translation in Algebra and Logic 9 (1970), 342–349.

[Sm] P. F. Smith, On the dimension of group rings, Proc. London Math. Soc. 25 (1972), 288–302.

[Su] A. A. Suslin, The structure of the special linear group over rings of polynomials, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 41 (1977), 235–252, 477.

[SD] M. R. Stein, R. K. Dennis, $K_2$ of radical ideals and semi-local rings revisited, Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic, 281–303, Lecture Notes in Math. Vol. 342, Springer, Berlin, 1973.

Department of Mathematics
University of California in San-Diego
San-Diego, California, 92093
davidel-chai.ben-ezra@mail.huji.ac.il