Existence and non-existence of Blow-up solutions for a non-autonomous problem with indefinite and gradient terms

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We deal with existence and non-existence of non-negative entire solutions that blow-up at infinity for a quasilinear problem depending on a non-negative real parameter. Our main objectives in this paper are to provide far more general conditions for existence and non-existence of solutions. To this end, we explore an associated µ-parameter convective ground state problem, sub and super solutions method combined and an approximation arguments to show existence of solutions. To show the result of non-existence of solutions, we follow an idea due to Mitidieri-Pohozaev.

2012 Mathematics Subject Classifications.
Key words: Quasilinear equations, Ground state solution, Existence and Non-existence, Large solutions, Gradient term.

1 Introduction

We consider the problem

\[
\begin{cases}
\Delta_p u = a(x)f(u) + \mu b(x)|\nabla u|^{\alpha} \text{ in } \mathbb{R}^N, \\
u \geq 0 \text{ on } \mathbb{R}^N, \quad u(x) \xrightarrow{|x| \to \infty} \infty,
\end{cases}
\]

(1.1)

where \( N \geq 1, \alpha \geq 0 \) and \( \mu \geq 0 \) is a real parameter, \( \Delta_p \) is the \( p \)-Laplacian operator with \( 1 < p < \infty \), \( f : [0, \infty) \rightarrow [0, \infty) \) is a continuous function such that \( f(t) > 0 \) for \( t > 0 \); \( a, b : \mathbb{R}^N \rightarrow \mathbb{R} \) are continuous functions with \( a \) being nonnegative and \( b \) can change of signal.

∗C.O. Alves was partially supported by CNPq/Brazil 303080/2009-4
†The author acknowledges the support of PROCAD/UFG/UnB and FAPDF under grant PRONEX 193.000.580/2009
‡The author acknowledges the support of CNPq/Brasil.
A solution of (1.1) is meant as a nonnegative function in $C^1(\mathbb{R}^N)$ that satisfies (1.1) in distributional sense. It is well-known as being a entire large (explosive or blow-up) solutions.

The research by conditions that lead to the existence, non-existence and behavior asymptotic of solutions for problem (1.1), in bounded domain, mainly without the dependence of the gradient term, has been much made recently. However for problem (1.1) in whole space, principally with dependance of the gradient term, there is a less expressive literature.

It is well-known in the mathematical literary that the issue of existence and non-existence of solution for problem (1.1), without dependance of gradient term, that is, $\mu = 0$ in (1.1), are very sensible to the behavior of the potential $a$ at the infinity. If $a = 1$ and $f \geq 0$, Keller [1] and Osserman [2] proved that problem (1.1), with $p = 2$, admits a positive solution if only if $f$ satisfies

$$\int_1^\infty F(t)^{-1/p} dt = \infty,$$

where $F(t) = \int_0^t f(s) ds$.

In 2000, Lair and Wood considered a such $f$, more specifically $f(u) = u^q$ with $0 < q < 1$, and showed in [3] that problem (1.1), with $p = 2$, $\mu = 0$ and $a$ is a radially-symmetric and nonnegative function, admits a solution if only if

$$\int_0^\infty r a(r) dr = \infty.$$ 

In this sense, that is, when the term $f$ satisfies the above condition, there are a lot of papers studying the issues about existence and non-existence of solution for (1.1) both in bounded and unbounded domains without or with dependance of gradient term. See for example, [4, 5, 6] and references therein.

In a similar way, when the term $f$ satisfies

$$\int_1^\infty F(t)^{-1/p} dt < \infty,$$

the looking for by existence of solutions should occurs by controlling the decaying fast of $a$ at infinity. The above condition is known as Keller-Osserman condition.

In this sense, Ye and Zhou [7] proved that a sufficient condition for existence of solutions for problem (1.1) with $p = 2$, $\mu = 0$, $f$ a increasing function satisfying $f(0) = 0$ and $(F)$ is that $a > 0$ be a continuous function such that the problem

$$\begin{cases}
-\Delta w = a(x) & \text{in } \mathbb{R}^N, \\
w > 0 & \text{on } \mathbb{R}^N, \quad w(x) \not\to 0 \quad \text{if } |x| \to \infty,
\end{cases}$$

admits a solution in $C^1(\mathbb{R}^N)$.

On the other hand, for the particular case $f(u) = u^q$ with $q > 1$ and $a$ being a radial continuous function, it was showed by Taliaferro in [8] that the existence of solution for (P) is also a necessary condition for the existence of solution for (1.1) with $p = 2$. These results show that the solvability of problem (P) is almost a optimal condition for existence of solution for problem (1.1) with $\mu = 0$, $p = 2$ and $f$ satisfying $(F)$.

For this class of problem, that is, (1.1) with $\mu = 0$ and $a$ be a non-negative continuous function, a natural approach to show existence of solution has been the sub and super solution technique using an argument of approximation by auxiliary problems defined in balls centered at origin of $\mathbb{R}^N$ with radius $k = 1, 2, \cdots$, namely $B_k$. So, sub and super solutions for (1.1) are constructed and some kind of comparison principle is used to order them.
Now, we are going to do a small overview about results related to problems like \( (\ref{eq:pos}) \) with \( \mu \neq 0 \) in bounded domain and whole space, which in the most have sign-defined potentials. In 1996, Bandle and Giarrusso \cite{Bandle1996} proved existence and studied behavior asymptotic of solutions for

\[
\begin{cases}
\Delta u = f(u) \pm |\nabla u|^\alpha & \text{in } \Omega, \\
u \geq 0 & \text{on } \Omega, \ u(x) \xrightarrow{d(x) \to 0} \infty,
\end{cases}
\]

where \( \Omega \subset \mathbb{R}^N \) is a smooth bounded domain, \( d(x) \) is the distance of \( x \) to the boundary of \( \Omega \), either \( f(u) = u^q \) or \( f(u) = e^u, q > 1 \) and \( \alpha > 0 \) is a fixe number.

Em 2006, Zhang \cite{Zhang2006} studied the problems

\[
\begin{cases}
\Delta u = a(x)f(u) \pm \lambda|\nabla u|^\alpha & \text{in } \Omega, \\
u \geq 0 & \text{on } \Omega, \ u(x) \xrightarrow{d(x) \to 0} \infty,
\end{cases}
\]

where range interval of \( \alpha > 0 \) depend on sign \( \pm \), \( a \) behavior like the unique solution of \(-\Delta u = 1\) in \( \Omega \) with \( u = 0 \) on the boundary of \( \Omega \) and \( f \) is like \( s^q \) at infinity for some appropriate \( q > 0 \).

In 2011, Huang, Li, Tian and Mu \cite{Huang2011} studied

\[
\begin{cases}
\Delta u = a(x)f(u) \pm b(x)|\nabla u|^\alpha & \text{in } \Omega, \\
u \geq 0 & \text{on } \Omega, \ u(x) \xrightarrow{d(x) \to 0} \infty,
\end{cases}
\]

where \( \alpha \geq 0, \ a,b \in C^\nu(\Omega) \) for some \( \nu \in (0,1) \) with \( a \) positive and \( b \) non-negative functions that can be singular or null in the boundary of \( \Omega \) and \( f \) positive is such \( f(s)/s \), \( s > 0 \) is increasing at infinity.

Recently, Hamydy in \cite{Hamydy2012} considered the \( p \)-Laplacian operator and showed the existence of solution for a problem like

\[
\begin{cases}
\Delta_p u = a(x)f(u) + b(x)|\nabla u|^{p-1} & \text{in } \Omega, \\
u \geq 0 & \text{on } \Omega, \ u(x) \xrightarrow{d(x) \to 0} \infty,
\end{cases}
\]

where \( b \in L^\infty(\Omega) \) can change of sign, \( p \geq 2 \), \( f \) is continuous and increasing with \( \inf_{s>0} f(s)/s \) is positive for some \( q > p - 1 \) and \( a(x) \geq a_{\infty} > 0, \ x \in \Omega \).

In the whole space, there exists a very few papers studying existence of solutions. In 1999, Lair and Wood \cite{Lair1999} showed the existence of solutions for the problem

\[
\begin{cases}
\Delta u = a(x)u^q \pm |\nabla u|^\alpha & \text{in } \mathbb{R}^N, \\
u \geq 0 & \text{in } \mathbb{R}^N, \ u(x) \xrightarrow{|x| \to \infty} \infty.
\end{cases}
\]

For the positive signal, they assumed for instance \( 0 \leq a(x) \leq M|x|^{-2-\beta} \) for big \( |x| \) and either \( q \leq 1 + \beta(1 - \alpha)/(2 - \alpha) \) with \( 0 < \alpha < 1 \) or \( \max\{q, \alpha\} > 2 \), if \( \alpha \geq 1 \) and for the negative problem they assumed \( a \geq 0 \) and with its zero points enclosed by a bounded surface of non-zero points satisfying

\[
\int_0^\infty \max a(x)dr < \infty
\]

for \( N \geq 3 \) and \( q > \max\{1, \alpha\} \).

Motivated by the above results, Hamydy, Massar and Tsouli \cite{Hamydy2011} in 2011 complemented this last result by considering a more general \( \mu \)-parameter problem

\[
\begin{cases}
\Delta_p u = a(x)f(u) + \mu|\nabla u|^{p-1} & \text{in } \mathbb{R}^N, \\
u \geq 0 & \text{in } \mathbb{R}^N, \ u(x) \xrightarrow{|x| \to \infty} \infty,
\end{cases}
\]
with \( \mu \neq 0 \), and they proved existence of solutions for \( p > 2 \) and \( a(x) \geq a_\infty, \ x \in \mathbb{R}^N \), for some \( a_\infty > 0 \). However, in this case \( f \) not satisfies the Keller-Osserman condition.

For the non-existence of solutions, there exists very few works. In 1999, Mitidieri and Pohozaev [14] introduced a test-function method to prove the non-existence of positive solution for

\[
\Delta_p u \geq |x|^{-\delta} u^q \quad \text{in} \quad \mathbb{R}^N,
\]

where \( 1 < p < N, \ q > p - 1 \) and \( p > \delta > 1 \). For related problems and by using different techniques, we quote Lair and Wood [3] in 2000, Ghergu and Radulescu [6] in 2004 and references therein.

In a recent paper, Felmer, Quaas and Sirakov [15] by using appropriate super solutions and comparison principles proved the non-existence of solutions for the autonomous inequality

\[
\Delta u \geq f(u) + g(|\nabla u|) \quad \text{in} \quad \mathbb{R}^N,
\]

where \( f \) and \( g \) are increasing continuous functions with \( f(0) = g(0) = 0 \) and either \( f \) does not satisfies Keller-Osserman condition or \( g \) satisfies \( \int_1^\infty ds/g(s) < \infty \).

In the above cases, when the potentials \( a \) and \( b \) are non-negative, the operator is elliptic uniform and its perturbations has \( C^1 \)-regularities, the classical standard comparison principles, like that in [16], have been used to compare the sub and super solution of (1.1), the solutions of these auxiliary problems each other and these solutions with the sub and super solutions. So, the solution is built by a diagonal process limit.

Since, our principal aim in this paper is to consider the \( p \)-Laplacian operator with \( 1 < p < \infty \) and to establish far more general conditions under potentials \( a \) and \( b \) (which can be non-constant and \( b \) can be indefinite potential) in the whole space, the existence and non-existence of solutions for (1.1) cannot obtained by standard comparison principles, at least in a direct way. The principal difficulty is when \( b^+ \neq 0 \).

To overcome this, we prove a comparison principle for this class of problem (see theorem 2.1). Besides this, in general the building of sub and super solution for problems, with dependance of gradient term, in whole space in general are not easy, principally because we need obtain the explosive behavior of the solution at infinity.

To get over these difficulties, we show the existence of a \( \mu \)-positive ground state solution for an associated \( \mu \)-parameter problem with dependance of gradient term which allows us constructing an super solution for the problem (1.1) whose \( L^\infty(\mathbb{R}^N) \)-norm is controlled by the parameter (see lemma 2.2).

Concerning to the non-existence of solutions for (1.1), a natural approach to do this is to construct some appropriate radial super solution for (1.1) and apply some comparison principle. However, this procedure does not work in our case because neither standard comparison principles nor our result can not be applied.

So, we exploit an idea, due to Mitidieri and Pohozaev [14], by constructing a test function that is null in the exterior of appropriate balls of \( \mathbb{R}^N \). By using this test function carefully constructed in \( C^\infty_0(\mathbb{R}^N) \) together the infinity-information on the nonlinearities we get our result after carefully calculations.

These improve and complement some the prior results of non-existence not only by it does not to require global information on the terms but also by it to permit a more class of the nonlinearities \( f \) and potentials \( a \) and \( b \). We quote the reader principally to [14], [17], [18] and [19] for whole space and [12] and [20] for bounded domain and references therein.

The main contribution of our work is related to the fact that we present some forms that the terms \( a \) and \( b \) should interact to produce existence or non-existence of solutions for (1.1) without
assuming \( f \) is monotonous. In a some sense, these results show that these interactions are connected with the solvability of a problem like

\[
(P_\rho) \quad \begin{cases} 
-\Delta_p w = \rho(x) & \text{in } \mathbb{R}^N, \\
w > 0 & \text{on } \mathbb{R}^N, \ w(x) \xrightarrow{|x| \to \infty} 0,
\end{cases}
\]

with \( \rho \) given by an appropriate combination of the potentials \( a \) and \( b \).

It is well-known that \( (P_\rho) \) has a \( C^1 \)-solution, if \( 1 < p < N \) and

\[
\int_1^\infty \left( t^{1-N} \int_0^t r^{N-1} \hat{\rho}(r) dr \right)^{\frac{1}{p-1}} dt < \infty
\]

holds, where \( \hat{\rho}(r) = \max_{|x|=r} \rho(x) \) and \( \rho \in C(\mathbb{R}^N) \) is a non-negative function. In fact, if \( p \geq N \), the problem \( (P_\rho) \) does not have solution for any function \( \rho \geq 0 \). See for example Serrin and Zou [21].

Now, we state our principal results. Before this, we need to consider the following condition.

\( (P_\rho) \): Problem \( (P_\rho) \), with \( \rho(x) = \max\{a(x), b^+(x)\}, x \in \mathbb{R}^N \), admits a super solution \( z \) belonging to

\[
(i) \quad C^1(\mathbb{R}^N), \text{ if } b^+ = 0 \text{ and } 0 \leq \alpha \leq p \quad (ii) \quad C^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N), \text{ if } b^+ \neq 0 \text{ and } \alpha = p-1.
\]

Throughout all this work we are going to denote by \( b^+(x) = \max\{b(x), 0\} \) and \( b^-(x) = \max\{-b(x), 0\} \), \( x \in \mathbb{R}^N \) as being the positive and negative parts of a function \( b \).

**Remark 1.1** In \( (P_\rho) \)-\( (ii) \), we note that the existence of a \( C^1(\mathbb{R}^N) \)-solution of \( (P_\rho) \) implies its \( W^{1,\infty}(\mathbb{R}^N) \) regularity, if \( \rho \in L^\infty(\mathbb{R}^N) \).

**Theorem 1.1** Assume that \( \liminf_{t \to \infty} f(t)/t^q > 0 \) for some \( q > \max\{\alpha, p-1, 1\} \), \( (P_\rho) \) hold and \( a, b \in L^\infty_{loc}(\mathbb{R}^N) \) with a satisfying

\[
(a_\Omega) : \text{given a smooth bounded open set } \Omega \subset \mathbb{R}^N, \text{there exists } a_\Omega > 0 \text{ such that } a(x) \geq a_\Omega \text{ a.a. in } \Omega.
\]

Then there exists \( 0 < \mu^* \leq \infty \) such that the problem \( (1.1) \) admits at least one solution for each \( 0 \leq \mu < \mu^* \) given. Besides this, \( \mu^* = \infty \), if \( (P_\rho) \)-\( (i) \) holds.

In the sequel, we are interested in considering either \( \alpha > p - 1 \) or potentials \( a \) and \( b \) such that the problem \( (1.1) \) has no sub solution in \( C^1(\mathbb{R}^N) \). More specifically, we will consider the problem

\[
\begin{cases} 
\Delta_p u \geq a(x)f(u) + b(x)|\nabla u|^{\alpha}, & \text{in } \mathbb{R}^N, \\
u \geq 0 & \text{on } \mathbb{R}^N, \ u(x) \xrightarrow{|x| \to \infty} \infty,
\end{cases}
\]

where \( a, b : \mathbb{R}^N \to \mathbb{R} \) are \( L^\infty_{loc}(\mathbb{R}^N) \) nonnegative functions and \( f : (0, \infty) \to [0, \infty) \) is an appropriate function. We are going to denote by \( B_R \) the ball centered at origin of \( \mathbb{R}^N \) with radius \( R > 0 \).

**Theorem 1.2** Assume one of the below case holds for some \( R_0 > 0 \):

\[
(i) \quad a, b > 0 \text{ a.a. on } \mathbb{R}^N \setminus B_{R_0}, \liminf_{t \to \infty} f(t)/t^q > 0 \text{ for some } q > p - 1 \text{ and either}
\]

\[
(i) \quad \limsup_{R \to \infty} \frac{\int_{|x| \leq 2R} a(x) \frac{\theta}{|x|^q} dx}{R^{\frac{\theta}{q}} - R_0^{\frac{\theta}{q}}} < \infty \text{ for some } \theta \in (p - 1, q) \quad \text{or}
\]
that Proof. In what follows, we argue by contraction. Assume that $u$ then $C$.

Theorem 2.1 (A comparison Principle) Assume $\alpha$.

Lemma 2.1 Before proving our first result in this section, we state the below lemma, whose proof is easy.

\[ \int_{\Omega} |\nabla u|^p - 2 \nabla u \nabla \varphi \, dx + \int_{\Omega} [a(x) h(u) + \mu b(x)|\nabla u|^\alpha] \varphi \, dx \leq 0, \]  

(1.3) and

\[ \int_{\Omega} |\nabla v|^p - 2 \nabla v \nabla \varphi \, dx + \int_{\Omega} [a(x) h(v) + \mu b(x)|\nabla v|^\alpha] \varphi \, dx \geq 0, \]  

(1.4) for all $\varphi \geq 0$, $\varphi \in C^\infty_0(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\alpha \geq p - 1$, $h : \mathbb{R} \to \mathbb{R}$ is a increasing continuous function and $a, b \in L_\text{loc}^\infty(\Omega)$ with $a$ satisfying

(a): given a smooth open set $O \subset \subset \Omega$ there exists $a_0 > 0$ such that $a(x) \geq a_0$ a.a. in $O$.

Before proving our first result in this section, we state the below lemma, whose proof is easy.

Lemma 2.1 Assume $\alpha \geq 0$. Then for each $\tau > 1$ given, there exists a $\nu = \nu(\tau) > 0$ such that

(i) $t^\alpha - 1 \leq \nu(t - 1)^\alpha$, $t \geq \tau$ (ii) $|t^\alpha - 1| \leq \tau^\alpha - 1$, $\tau^{-1} < t < \tau$.

Theorem 2.1 (A comparison Principle) Assume $a, b$ and $h$ like above. If $u, v \in W_{\text{loc}}^{1,\infty}(\Omega) \cap C(\Omega)$ satisfy (1.3) and (1.4) respectively, and

\[ \lim_{x \to y} (u(x) - v(x)) \in [-\infty, 0], \text{ for each } y \in \partial \Omega, \]

then $u \leq v$ in $\Omega$.

Proof. In what follows, we argue by contraction. Assume that $\omega(x) = u(x) - v(x)$, $x \in \Omega$ is such that $\Omega = \text{sup}_x \omega(x) > 0$. So, for $\varepsilon \in (\Omega/2, \Omega)$ given, the function $\omega_{\varepsilon}$ defined by $\omega_{\varepsilon} = \max\{0, \omega - \varepsilon\}$ is not null precisely in

\[ \Omega_{\varepsilon} := \{ x \in \Omega, \ \varepsilon < \omega(x) \leq \varepsilon \}. \]

Besides this, we have

\[ \Omega_{\varepsilon_2} \subset \Omega_{\varepsilon_1} \subset \Omega_{\varepsilon/2}, \text{ for } \varepsilon/2 < \varepsilon_1 < \varepsilon_2 < \varepsilon \]  

(1.5) and

\[ \Omega_{\varepsilon} \subset \subset \Omega, \text{ that is, } \partial \Omega_{\varepsilon} \text{ is a compact set in } \Omega. \]
As \( \omega_\varepsilon \in W^{1,p}_0(\Omega) \) and \( \omega_\varepsilon \geq 0 \), we can use it as test function in \(|1.3|\) and \(|1.4|\) to obtain

\[
\int_{\Omega_\varepsilon} |\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v \nabla \omega_\varepsilon dx \leq \int_{\Omega_\varepsilon} \{a(x)[h(v) - h(u)] + \mu b(x)|\nabla v|^\alpha - |\nabla u|^\alpha\} \omega_\varepsilon dx.
\]

So, by a classical inequality,

\[
c_p M_\varepsilon \int_{\Omega_\varepsilon} |\nabla \omega_\varepsilon|^p dx \leq \int_{\Omega_\varepsilon} \{a(x)[h(v) - h(u)] + \mu b(x)|\nabla v|^\alpha - |\nabla u|^\alpha\} \omega_\varepsilon dx,
\]

where

\[
M_\varepsilon := \begin{cases} (|\nabla v|_{L^\infty(\Omega_\varepsilon)} + |\nabla u|_{L^\infty(\Omega_\varepsilon)} + 1)^{p-2}, & \text{if } 1 < p \leq 2, \\ 1, & \text{if } p \geq 2 \end{cases}
\]

and \( c_p \) is a positive constant that it does not depends on \( \varepsilon \). In particular, from \(|1.5|\),

\[
0 < M_{\varepsilon/2} \leq M_\varepsilon \leq 1, \text{ for all } \varepsilon \in (\varepsilon/2, \varepsilon).
\]

Hence,

\[
\int_{\Omega_\varepsilon} |\nabla \omega_\varepsilon|^p dx \leq \int_{\Omega_\varepsilon} \{a(x)[h(v) - h(u)] + \mu b(x)|\nabla v|^\alpha - |\nabla u|^\alpha\} \omega_\varepsilon dx.
\]

Now, given an \( \tau > 1 \), we shall consider the ensuing subsets of \( \Omega_\varepsilon \)

\[
G(\tau) = \{x \in \Omega_\varepsilon, \nabla u \neq \nabla v, |\nabla v| \geq \tau|\nabla u|\}
\]

\[
\tilde{G}(\tau) = \{x \in \Omega_\varepsilon, \nabla u \neq \nabla v, |\nabla v| \leq \frac{1}{\tau}|\nabla u|\},
\]

\[
L(\tau) = \{x \in \Omega_\varepsilon, \nabla u \neq \nabla v, \frac{1}{\tau}|\nabla u| < |\nabla v| < \tau|\nabla u|\},
\]

and

\[
I(\tau) = \{x \in \Omega_\varepsilon, \nabla u = \nabla v\}.
\]

Using \(|1.7|\) together with the monotonicity of \( h \) in \( I(\tau) \) and the above sets, we get

\[
c_p M_\varepsilon \int_{\Omega_\varepsilon} |\nabla \omega_\varepsilon|^p dx \leq \int_{\Omega_\varepsilon} \{a(x)[h(v) - h(u)] + \mu b(x)||\nabla v|^\alpha - |\nabla u|^\alpha\} \omega_\varepsilon dx \leq \int_{G(\tau)} \mu b(x)||\nabla v|^\alpha - |\nabla u|^\alpha\omega_\varepsilon dx \leq \int_{\tilde{G}(\tau)} \mu b(x)||\nabla v|^\alpha - |\nabla u|^\alpha\omega_\varepsilon dx \leq \int_{L(\tau)} a(x)[h(u) - h(v)] \omega_\varepsilon dx + \int_{G(\tau)} \mu b(x)||\nabla v|^\alpha - |\nabla u|^\alpha\omega_\varepsilon dx + \int_{\tilde{G}(\tau)} \mu b(x)||\nabla v|^\alpha - |\nabla u|^\alpha\omega_\varepsilon dx + \int_{L(\tau)} a(x)[h(u) - h(v)] \omega_\varepsilon dx.
\]

Now, by Lemma [2.4] and \(|1.3|\),

\[
c_p M_\varepsilon \int_{\Omega_\varepsilon} |\nabla \omega_\varepsilon|^p dx \leq \mu \nu \int_{G(\tau)} |b(x)||\nabla v| - |\nabla u|^\alpha \omega_\varepsilon dx + \mu \nu \int_{\tilde{G}(\tau)} |b(x)||\nabla v| - |\nabla u|^\alpha \omega_\varepsilon dx + \mu \nu \int_{L(\tau)} |b(x)|(\tau^\alpha - 1)|\nabla u|^\alpha \omega_\varepsilon dx + \mu \nu \int_{L(\tau)} a(x)[h(u) - h(v)] \omega_\varepsilon dx.
\]
Since, $h$ is increasing continuous, we have
\[ h(u(x)) - h(v(x)) \geq h(v(x) + \varepsilon/2) - h(v(x)) := \sigma_\varepsilon \text{ in } \Omega_\varepsilon, \]
where $\sigma_\varepsilon := \min_{\Omega_\varepsilon}[h(v(x) + \varepsilon/2) - h(v(x))] > 0$. Thus, by using the hypothesis $(a_\Omega)'$, there exists $\tau_\varepsilon > 1$, enough near of 1, such that
\[ \mu|b(x)|(\tau_\varepsilon^\alpha - 1)|\nabla u|^\alpha_{L^\infty(\Omega_\varepsilon)} - a(x)[h(u) - h(v)] \leq \mu|b|_{L^\infty(\Omega_\varepsilon)}(\tau_\varepsilon^\alpha - 1)|\nabla u|^\alpha_{L^\infty(\Omega_\varepsilon)} - a_\Omega \sigma_\varepsilon < 0 \]
in $L(\tau_\varepsilon)$. Hence,
\[ c_p M_\varepsilon \int_{\Omega_\varepsilon} |\nabla \omega_\varepsilon|^p dx \leq \mu \nu \int_{G(\tau_\varepsilon)} |b(x)||\nabla v| - |\nabla u|^{\alpha_\varepsilon} dx + \mu \nu \int_{G(\tau_\varepsilon)} |b(x)||\nabla v| - |\nabla u|^{\alpha_\varepsilon} dx \]
from where it follows that
\[ c_p M_\varepsilon \int_{\Omega_\varepsilon} |\nabla \omega_\varepsilon|^p dx \leq \mu \nu \int_{\Omega_\varepsilon} |b(x)||\nabla v| - |\nabla u|^{\alpha_\varepsilon} dx \leq \mu \nu \int_{\Omega_\varepsilon} |b(x)||\nabla \omega_\varepsilon|^{\alpha_\varepsilon} dx \]
and so,
\[ c_p M_\varepsilon \int_{\Omega_\varepsilon} |\nabla \omega_\varepsilon|^p dx \leq \mu \nu |b|_{L^\infty(\Omega_\varepsilon)} \int_{\Omega_\varepsilon} |\nabla \omega_\varepsilon|^{\alpha_\varepsilon} dx \]
\[ \leq \mu \nu |b|_{L^\infty(\Omega_\varepsilon)} d_\varepsilon \int_{\Omega_\varepsilon} |\nabla \omega_\varepsilon|^{\alpha_\varepsilon - 1} \omega_\varepsilon dx, \]
\[ \leq \mu \nu |b|_{L^\infty(\Omega_\varepsilon)} d_\varepsilon \left( \int_{\Omega_\varepsilon} |\nabla \omega_\varepsilon|^p \right)^{\frac{\alpha_\varepsilon - 1}{p}} \left( \int_{\Omega_\varepsilon} |\omega_\varepsilon|^p \right)^{\frac{1}{p}} dx \]
\[ \leq \mu \nu |b|_{L^\infty(\Omega_\varepsilon)} d_\varepsilon \left( \int_{\Omega_\varepsilon} |\nabla \omega_\varepsilon|^p \right)^{\frac{\alpha_\varepsilon - 1}{p}} \left( \int_{\Omega_\varepsilon} |\omega_\varepsilon|^p \right)^{\frac{1}{p}} med(\Omega_\varepsilon)^{\frac{1}{N}}, \]
where $d_\varepsilon = |\nabla \omega_\varepsilon|^{\alpha_\varepsilon - 1}_{L^\infty(\Omega_\varepsilon)}$ and $med(\Omega_\varepsilon)$ is the measure of Lebesgue of $\Omega_\varepsilon$. Again, from (1.5),
\[ 0 < d_\varepsilon \leq d_{\varepsilon/2} \text{ for } \varepsilon/2 < \varepsilon < \varepsilon. \]
(1.10)
Using the Sobolev imbedding, we know that
\[ \left( \int_{\Omega_\varepsilon} |\omega_\varepsilon|^p \right)^{\frac{1}{p}} \leq d \left( \int_{\Omega_\varepsilon} |\nabla \omega_\varepsilon|^p \right)^{\frac{1}{p}}, \]
where $d > 0$ is a constant not depending of $\varepsilon$. This combined with (1.9) gives
\[ c_p M_\varepsilon \int_{\Omega_\varepsilon} |\nabla \omega_\varepsilon|^p dx \leq \mu \nu |b|_{L^\infty(\Omega_\varepsilon)} d_\varepsilon d \left( \int_{\Omega_\varepsilon} |\nabla \omega_\varepsilon|^p \right)^{\frac{\alpha_\varepsilon - 1}{p}} \left( \int_{\Omega_\varepsilon} |\nabla \omega_\varepsilon|^p \right)^{\frac{1}{p}} med(\Omega_\varepsilon)^{\frac{1}{N}}, \]
that is,
\[ 1 \leq \frac{\mu \nu |b|_{L^\infty(\Omega_\varepsilon)} d_\varepsilon d}{c_p M_\varepsilon med(\Omega_\varepsilon)^{\frac{1}{N}}} \]
Thus, by (1.5), (1.6) and (1.10) together with $|b|_{L^\infty(\Omega_{\varepsilon})} \leq |b|_{L^\infty(\Omega_{\varepsilon}/2)}$, we get

$$1 \leq \frac{\mu \nu |b|_{L^\infty(\Omega_{\varepsilon}/2)} d_{\varepsilon/2} d}{c_p M_{\varepsilon/2} \text{med}(\Omega_{\varepsilon})^{\frac{1}{N}}}. $$

Once that $\text{med}(\Omega_{\varepsilon}) \to 0$ as $\varepsilon \to 0$, we obtain a contradiction. Therefore, this proves the theorem.

**Lemma 2.2** Suppose that $\eta < 0$, $\alpha \geq 0$ and $(P)_\rho-(ii)$ holds. Then, there exist $0 < \Lambda_* < \infty$ and $\omega = \omega_\mu \in C^1(\mathbb{R}^N)$ satisfying

$$\left\{ \begin{array}{l}
-\Delta_p \omega \geq a(x)[1 + (\omega(x) + 1)^\eta/2] + \mu b^+(x)|\nabla \omega|^\alpha, \quad \text{in} \quad \mathbb{R}^N,
\omega > 0 \quad \text{in} \quad \mathbb{R}^N, \quad \omega \big{|}_{x \to \infty} \to 0,
\end{array} \right.$$

for each $0 \leq \mu < \Lambda_*$ given. Besides this, if $0 \leq \alpha < p - 1$, then $\Lambda_* = \infty$.

**Proof.** First of all, we let $h(s) = 2 + s^\eta$ for $s \geq 0$ and

$$F(s) = s^2 / \int_0^s t h(t)^{1/(p-1)} dt, \quad s > 0.$$ 

We point out that $F(s)^{p-1} \geq h(s)$ and $F(s)/s$ is a non-increasing continuous function in $(0, +\infty)$.

So, we have well-defined the function

$$H(\tau) = \frac{1}{\tau} \int_0^\tau \frac{t}{F(t)} dt - \frac{1}{\tau} \int_0^1 \frac{t}{F(t)} dt, \quad \tau \geq 1$$

with $H(1) = 0$. Since,

$$\frac{1}{\tau} \int_0^\tau \frac{s}{F(s)} ds \geq \frac{1}{\tau} \int_{\tau/2}^\tau \frac{s}{F(s)} ds \geq \frac{1}{2} \frac{\tau^2}{F(\tau/2)}$$

$$\geq \frac{\tau^2}{8h(\tau/4)^{1/(p-1)}} \to +\infty \quad \text{as} \quad \tau \to \infty,$$

it follows that $\lim_{\tau \to \infty} H(\tau) = \infty$.

Thus, there exists a $\tau_\infty > 0$ such that

$$\frac{1}{\tau_\infty} \int_0^{\tau_\infty} \frac{t}{F(t)} dt > \|z\|_{\infty} + \frac{1}{\tau_\infty} \int_0^1 \frac{t}{F(t)} dt,$$

where $z \in C^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ is giving by the hypothesis $(P)_\rho-(ii)$.

After this, we can define a function $v \in C^1(\mathbb{R}^N)$ by

$$z(x) + \frac{1}{\tau_\infty} \int_0^1 \frac{t}{F(t)} dt = \frac{1}{\tau_\infty} \int_0^{v(x)+1} \frac{t}{F(t)} dt, \quad x \in \mathbb{R}^N \quad (1.11)$$
and infer that $1 \leq v(x) + 1 < \tau_{\infty}$ for all $x \in \mathbb{R}^N$ and $v(x) \to 0$ when $|x| \to \infty$. Moreover, by a direct computing, we also have

$$\int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \varphi dx = \int_{\mathbb{R}^N} \tau_\infty^{p-1}|\nabla v|^{p-2}\nabla v F'(v(x) + 1)\varphi \frac{1}{v(x) + 1}\varphi \frac{1}{v(x) + 1}\varphi dx$$

$$- \tau_\infty^{p-1}(p-1)\int_{\mathbb{R}^N} \frac{F(v(x) + 1)}{v(x) + 1} F'(s)\varphi|\nabla v|^{p-2}\nabla v \varphi dx$$

$$\geq \tau_\infty^{p-1}\int_{\mathbb{R}^N} \frac{F(v(x) + 1)}{v(x) + 1} \rho(x)\varphi dx$$

$$\geq \int_{\mathbb{R}^N} F(v(x) + 1)\rho(x)\varphi dx \geq \int_{\mathbb{R}^N} \rho(x)h(v(x) + 1)\varphi dx$$

$$\geq \int_{\mathbb{R}^N} a(x)(1 + (v(x) + 1)^\eta/2)\varphi dx + \frac{1}{2} \int_{\mathbb{R}^N} b^+(x)h(v(x) + 1)\varphi dx.$$  

Since,

$$\int_{\mathbb{R}^N} b^+(x)h(v(x) + 1)\varphi dx \geq \int_{\mathbb{R}^N} b^+(x)||\nabla v||^{-\alpha}_\infty||\nabla v||^{\alpha}_\infty\varphi dx$$

$$\geq \frac{1}{\tau_\infty^{\alpha}} \int_{\mathbb{R}^N} \left(\frac{v(x) + 1}{F(v(x) + 1)}\right)^{\alpha}||\nabla v||^{-\alpha}_\infty b^+(x)||\nabla v||^{\alpha}_\infty\varphi dx,$$

it follows by monotonicity of $F(s)/s$, $s \geq 0$ that

$$\int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \varphi dx \geq \int_{\mathbb{R}^N} a(x)[1 + (v(x) + 1)^\eta/2] + \mu b^+(x)||\nabla v||^{\alpha}_\infty\varphi dx$$

for all $0 \leq \varphi \in C_0^\infty(\mathbb{R}^N)$ and $0 \leq \mu \leq \Lambda_* := [\tau_{\infty}F(1)||\nabla z||_\infty]^{-\alpha} > 0$ given.

Beside this, if $0 \leq \alpha < p - 1$, then for each $0 \leq \mu < \infty$ given, we define $\varpi(x) = \theta v(x)$, $x \in \mathbb{R}^N$, where $v \in C^1(\mathbb{R}^N)$ is given by [111] and $\theta = \max\{1, (\mu/\Lambda_*)^{1/(p-1-\alpha)}\}$.

So, computing we have

$$\int_{\mathbb{R}^N} |\nabla \varpi|^{p-2}\nabla \varpi \varphi dx = \theta^{p-1}\int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \varphi dx$$

$$\geq \theta^{p-1}\int_{\mathbb{R}^N} a(x)(1 + \frac{1}{2}(v(x) + 1)^\eta) + \Lambda_* b^+(x)||\nabla v||^{\alpha}_\infty\varphi dx.$$  

Now, it follows from definition of $\theta$ and $\eta < 0$ that

$$1 + \frac{1}{2}(\varpi + 1)^\eta = 1 + \frac{1}{2}(\theta v + 1)^\eta \leq 1 + \frac{1}{2}(v + 1)^\eta \leq \theta^{p-1}(1 + \frac{1}{2}(v + 1)^\eta)$$

and

$$\mu b^+(x)||\nabla \varpi||^{\alpha}_\infty = \mu b^+(x)\theta^\eta||\nabla v||^{\alpha}_\infty \leq \theta^{p-1}\Lambda_* b^+(x)||\nabla v||^{\alpha}_\infty.$$

That is,

$$\int_{\mathbb{R}^N} |\nabla \varpi|^{p-2}\nabla \varphi dx \geq \int_{\mathbb{R}^N} a(x)(1 + \frac{1}{2}(\varpi(x) + 1)^\eta) + \mu b^+(x)||\nabla \varpi||^{\alpha}_\infty\varphi dx.$$  

This ends our proof.
3 Existence of solution for (1.1) in bounded domain

In this section, our main goal is proving the existence of solution for the problem

\[
\begin{aligned}
\Delta_p u &= a(x)f(u) + \mu b(x)|\nabla u|^\alpha \quad \text{in } \Omega, \\
u &\geq 0 \quad \text{in } \Omega, \quad u(x) \to \infty, \\
\end{aligned}
\]  

(1.12)

where $$\Omega \subset \mathbb{R}^N$$ is a smooth bounded domain, $$a, b : \Omega \to \mathbb{R}$$ are suitable functions with $$a \geq 0$$, $$f : [0, \infty) \to [0, \infty)$$ is a continuous function with $$f(0) = 0$$, $$0 \leq \alpha \leq p$$, $$\mu \geq 0$$ is a real parameter and $$N \geq 1$$.

To do this, we need to show the next result.

**Lemma 3.1** Assume that $$h \in L^\infty(\Omega)$$ is a nonnegative function and $$0 \leq \alpha \leq p$$ with $$p > 1$$. Then

\[
\begin{aligned}
\begin{cases}
-\text{div}((|\nabla u|^{p-2} + \varepsilon)\nabla u) = \mu h(x)(|\nabla u| + 1)^\alpha & \text{in } \Omega, \\
u &\geq 0 \quad \text{in } \Omega, \quad u(x) = 0 \quad \text{on } \partial \Omega
\end{cases}
\end{aligned}
\]  

(1.13)

admits a solution $$u = u_{\varepsilon, \mu} \in C^1(\Omega)$$ for each $$0 \leq \varepsilon < 1$$ and $$0 \leq \mu < \Lambda^*$$ given, for some $$\Lambda^* = \Lambda^*(\Omega) > 0$$. Besides this, $$\|u_{\varepsilon, \mu}\|_\infty \leq C$$ not depending on $$\varepsilon > 0$$.

**Proof** First, we note that for each $$h \in L^\infty(\Omega)$$, it follows by theorem of Browder-Minty that there exists a unique $$\omega_k \in W_0^{1,p}(\Omega)$$ solution of the problem

\[
\begin{aligned}
\begin{cases}
-\text{div}((|\nabla u|^{p-2} + \varepsilon)\nabla u) = h(x) & \text{in } \Omega, \\
u &\geq 0 \quad \text{in } \Omega, \quad u(x) = 0 \quad \text{on } \partial \Omega
\end{cases}
\end{aligned}
\]  

(1.14)

Besides this, taking $$-\omega_k^-$$ as a test function, we get $$\omega_k \geq 0$$, since $$h \geq 0$$.

**Claim:** $$\omega_k \in L^\infty(\Omega)$$ and $$\|\omega_k\|_\infty \leq C$$ for some $$C > 0$$, which does not depend of $$\varepsilon > 0$$.

In fact, first we note that using $$\varepsilon \geq 0$$, $$\omega_k$$ as test function and the Sobolev embedding, we have

\[
\|\omega_k\|_{1,p} \leq C\|h\|^{1/(p-1)} \quad \text{for some } C > 0.
\]  

(1.15)

So, if $$p \geq N$$, we get by using Sobolev embedding again that $$\|\omega_k\|_\infty \leq C\|h\|^{1/(p-1)}$$.

Now, if $$1 < p < N$$, we are going to denote by $$S > 0$$ the best constant of the inequality of Sobolev-Poincaré and let $$L = \|h\|_\infty^{1/p}S$$. Following the arguments in [22], we define the increasing sequence $$(\gamma_k)$$ with $$\gamma_1 > 1$$, $$\gamma_k \to \infty$$, $$\gamma_k^*$$ as

\[
\gamma_1 = p^*, \quad \gamma_k^* = \gamma_k - 1 + p, \quad \gamma_{k+1} = \gamma_k^* p^*/p
\]

and

\[
L_1 = \|\omega_k\|_{p^*} := \|\omega_k\|_{L^{p^*}(\Omega)}, \quad L_{k+1} = L_k^{\frac{p}{k}} \gamma_k \frac{1}{\gamma_k} \left(\frac{\gamma_k}{p}\right)^\frac{1}{k} L_k^{\gamma_k},
\]

where $$p^* = pN/(N - p)$$, if $$1 < p < N$$ and $$L_1 = \|\omega_k\|_{p^*} \leq C\|h\|^{1/(p-1)}$$ by using (1.15) together with Sobolev embedding.

As a consequence of this, we can prove, by a induction process, that

\[
\|\omega_k\|_{\gamma_k} \leq L_k \quad \text{for all } k, \quad \text{where } \|\cdot\|_{\gamma_k} := \|\cdot\|_{L^{\gamma_k}(\Omega)}.
\]  

(1.16)
To do this, we are going to consider a $\psi_n \in C^1([0, \infty))$ such that $0 \leq \psi'(t) \leq 1$, $\psi_n(t) = t$, $|t| \leq n$ and $\psi_n(t) = n + 2$, $|t| \geq n + 2$ for each $n \in \mathbb{N}$ and to define $u_n = \psi_n(\omega)$. So we have $0 \leq u_n \leq \omega_\epsilon$ in $\Omega$ and $u_n^l \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ for each $l \in [1, \infty)$.

Now, by induction hypothesis, we have

$$\int_\Omega h(x)u_n^{+k}dx \leq \|h\|_\infty \|u_n\|_{\gamma_k^*} \leq \|h\|_\infty \|\omega_\epsilon\|_{\gamma_k^*} \leq \|h\|_\infty L_k^{\gamma_k^*}$$

and by definitions of $\psi_n$, $(\gamma_k)$ and $(\gamma_k^*)$, we have

$$\gamma_k \int_\Omega (|\nabla \omega_\epsilon|^p + \epsilon)|\nabla \omega_\epsilon|^2|\nabla \psi'_n(\omega_\epsilon)u_n^{+k-1}dx \geq \gamma_k \int_\Omega |\nabla \omega_\epsilon|^p|\nabla \psi'_n(\omega_\epsilon)u_n^{+k-1}dx$$

$$\geq \gamma_k \int_\Omega |\nabla u_n|^p|u_n^{+k-1}dx = \gamma_k \left(\frac{p}{\gamma_k^*}\right)^p \int_\Omega |\nabla (u_n^p)|^pdx \geq S^{-p} \gamma_k \left(\frac{p}{\gamma_k^*}\right)^p \|u_n^p\|_{p^*}.$$ 

So, using $u_n^{+k}$ as a test function in (1.14), it follows

$$\left|\frac{\gamma_k}{\gamma_k^{*+}}\right|^p \leq S^{-p} \gamma_k^{-1}\left(\frac{\gamma_k}{p}\right)^p \|h\|_\infty L_k^{\gamma_k^*},$$

that is, by definition of $(\gamma_k)$ and $(\gamma_k^*)$, we have

$$\|u_n\|_{\gamma_k^{*+}} \leq S^{-p} \gamma_k^{-1}\left(\frac{\gamma_k}{p}\right)^p \|h\|_\infty L_k^{\gamma_k^*} = L_k^{\gamma_k^*}.$$ 

Now, doing $n \to \infty$, we get $\|\omega_\epsilon\|_{\gamma_k^{*+}} \leq L_k^{\gamma_k^*}$. This proves (1.16).

Below, we are going to show that $(L_k)$ is bounded. To do this we are going to define $(E_k)$ as $E_k = \gamma_k \ln L_k$. So,

$$E_{k+1} = \frac{\gamma_k^{*+}}{p} \left[\frac{p}{\gamma_k^*} \ln L - \frac{1}{\gamma_k} \ln \gamma_k + \frac{p}{\gamma_k^*} \ln \gamma_k^* - \frac{p}{\gamma_k} \ln p + \frac{p}{\gamma_k} \ln L_k\right]$$

$$\leq \frac{\gamma_k^{*+}}{p} \left[\frac{p}{\gamma_k^*} \ln L + \frac{p}{\gamma_k^*} \ln \gamma_k^* + \frac{p}{\gamma_k} \ln L_k\right]$$

$$= p^{*+} \ln (L\gamma_k^*) + L_k^{*+} E_k := r_k + aE_k,$$

where $r_k = p^{*+} \ln (L\gamma_k^*)$ and $a = p^{*+}/p > 1$.

As a consequence of this, we have

$$E_k \leq r_{k-1} + ar_{k-2} + \ldots + a^{k-2}r_1 + a^{k-1}E_1.$$  \hspace{1cm} (1.17)

Besides this,

$$\gamma_k = \gamma_{k-1}a = (\gamma_{k-1} - 1 + p)a = \gamma_{k-2}a^2 + (p - 1)a$$

$$= \gamma_k a^2 + (p - 1)a^2 + (p - 1)a = \ldots +$$

$$= \gamma_1 a^{k-1} + (p - 1)a^{k-1} + (p - 1)a^{k-2} + \ldots + (p - 1)a$$

$$= a^{k-1}(p^{*+} - \theta) + \theta,$$

where $\theta = a(p - 1)/(1 - a) = p^{*+}(1 - p)/(p^{*+} - p) < 0$. Hence,

$$r_k = p^{*+} \ln (L\gamma_k^*) = p^{*+} \ln L[a^{k-1}(p^{*+} - \theta) + \theta - 1 + p]$$

with $\theta - 1 + p < 0$. 

where \( b := p^* \ln [L(p^* - \theta)] \).

Now, as a consequence of this in (1.17), we have

\[
E_k \leq a^{k-1}E_1 + \sum_{i=1}^{k-1} a^{i-1} r_{k-i}
\]

\[
\leq a^{k-1}E_1 + p^* \ln a \sum_{i=1}^{k-1} (k - i - 1)a^{i-1} + b \sum_{i=1}^{k-1} a^{i-1}
\]

\[
\leq a^{k-1}E_1 + p^* \ln a \left( \frac{a^{k-1} - 1}{(a-1)^2} \right) + b \left( \frac{a^{k-1} - 1}{a-1} \right),
\]

because we used in last inequality

\[
\sum_{i=1}^{k-1} (k - i - 1)a^{i-1} \leq \frac{a^{k-1} - 1}{(a-1)^2}
\]

and

\[
\sum_{i=1}^{k-1} a^{i-1} = \frac{a^{k-1} - 1}{a-1}.
\]

Therefore, we have

\[
E_k \leq a^{k-1}E_1 + \frac{\{b(a - 1) + p^* \ln a\}(a^{k-1} - 1)}{(a-1)^2}.
\]

That is,

\[
L_k \leq e^{\frac{a^{k-1}E_1 + \{b(a - 1) + p^* \ln a\}(a^{k-1} - 1)}{(a-1)^2} + b} \quad \text{for each } k \in \mathbb{N}.
\]

Hence,

\[
\|\omega_\varepsilon\|_{\infty} \leq \lim \sup_{k \to \infty} \|\omega_\varepsilon\|_{\gamma_k} \leq \lim \sup_{k \to \infty} L_k \leq e^d,
\]

where \( d = \frac{E_1 + \{b(a - 1) + p^* \ln a\}(a^{k-1} - 1)}{(a-1)^2} + b \) is bounded above by a constant not depending on \( \varepsilon \), because \( E_1 = \gamma_1 \ln L_1 \) and \( L_1 \) is bounded above by a constant independent of \( \varepsilon \). This proves the claim.

As a consequence of this claim, we have by Lieberman \([23]\) that \( \omega_\varepsilon \in C^{1, \nu}(\overline{\Omega}) \) for some \( 0 < \nu < 1 \) and \( \|\omega_\varepsilon\|_{C^{1, \nu}(\Omega)} \leq C \), where \( C \) does not depend on \( \varepsilon > 0 \). So, we can define

\[
\Lambda^* := \Lambda^*(\Omega) := \sup \{ (\|\nabla \omega_\varepsilon\|_{\infty} + 1)^{-\alpha} / 0 < \varepsilon < 1 \} > 0.
\]

Now, given \( 0 \leq \mu < \Lambda^* \), we have

\[
- \text{div}(\|\nabla \omega_\varepsilon\|^{p-2} + \varepsilon) \nabla \omega_\varepsilon) = h(x) \geq \mu h(x)(\|\nabla \omega_\varepsilon\| + 1)^\alpha \quad \text{in } \Omega,
\]

that is, \( \omega_\varepsilon \) is a super solution of (1.13).\( z \) is 0 \( \omega_\varepsilon \) is a sub solution of (1.13), it follows by sub and super solution theorem in \([24]\) and regularities results in \([23]\) the proof of lemma.

From now on, let us say that \( a \) is a \emph{\( c_\Omega \)-positive function}, if the following property holds:

If \( a(x_0) = 0 \) for some \( x_0 \in \Omega \), then there exists \( \Theta \subset \subset \Omega \) such that \( x_0 \in \Theta \) and \( a(x) > 0 \) on \( \partial \Theta \).

The below theorem complements the principal results in Bandle and Giarrusso \([9]\) by permitting \( p \neq 2 \) and non-autonomous potentials \( a \) and \( b \) and Hamydy \([12]\) (and works quoted therein), because it permits \( 1 < p < \infty, \alpha \neq p - 1 \), non-monotonous term \( f \) and more general terms \( a \)
**Theorem 3.1** Suppose $1 < p < \infty$, $0 \leq \alpha \leq p$, $\liminf_{t \to \infty} f(t)/t^q > 0$ for some $q > \max\{\alpha, p - 1, 1\}$, $b \in L^\infty(\Omega)$ and either

(a1) $a \in C(\Omega) \cap L^\infty(\Omega)$ is a $c_\Omega$-positive function or (a2) $a \in L^\infty(\Omega)$ is such that $(a_\Omega)'$ holds.

Then there exists $0 < \mu_* \leq \infty$ such that the problem (1.12) has at least a solution $u = u_\mu \in C^1(\Omega)$ for each $0 \leq \mu < \mu_*$ given. Besides this, $\mu_* = \infty$, if (a2) holds.

In the proof of the above result, we need of the following technical lemma

**Lemma 3.2** Assume $h : [0, \infty) \to [0, \infty)$ is a continuous function such that $h(t) > 0$ for $t > 0$, $h(0) = 0$ and

$$\liminf_{s \to \infty} \frac{h(s)}{s^q} > 0, \text{ for some } q > 0.$$  

Then there exist increasing functions $\underline{h}, \overline{h} : [0, \infty) \to [0, \infty)$ in $C^1(0, \infty) \cap C[0, \infty)$ satisfying $\underline{h}(0) = \overline{h}(0) = 0, \underline{h}(t) \leq h(t) \leq \overline{h}(t), t > 0,$

$$\liminf_{s \to \infty} \frac{\underline{h}(s)}{s^q} > 0 \text{ and } \liminf_{s \to \infty} \frac{\overline{h}(s)}{s^q} > 0.$$  

**Proof.** At first, we are going to prove the existence of $\overline{h}$. Defining $l(t) = \max_{s \in [0, t]} h(s)$, it is to check that $l$ is continuous and

$$l(t) \geq h(t), \ t \geq 0, \ l(0) = 0 \text{ and } l \text{ is nondecreasing.}$$

To the regularity, we are going to define $\tilde{l} : [0, \infty) \to [0, \infty)$ by $\tilde{l}(0) = 0$ and

$$\tilde{l}(t) = \frac{1}{t} \int_t^{2t} l(s)ds, \ t > 0.$$  

So, it is immediate that

$$(\tilde{l}')(t) \geq 0, \ \text{and} \ h(t) \leq l(t) \leq \tilde{l}(t) \leq l(2t), \ \forall \ t \geq 0$$

and defining

$$\overline{h}(s) = \tilde{l}(s) + \int_0^s h(\zeta)d\zeta, \ s \geq 0,$$

we have the claimed.

Now, let us prove the existence of $\underline{h}$. Since $\liminf_{s \to \infty} h(s)/s^q > 0$, for some $q > 0$, then there exist positive constants $M$ and $C$ such that $h(s) \geq Cs^q, s \geq M$. Set $\eta(s) = \min\{\min_{t \geq s} h(t), Cs^q\}$ for $s \in [0, M]$, and define

$$\underline{h}(t) = \begin{cases} \frac{1}{M} \int_0^t \eta(s)ds, & t \in [0, M], \\ \int_0^M \eta(s)ds - \frac{1}{t^{q+1}} \int_M^t \eta(s)ds, & t \in [M, \infty). \end{cases}$$

Finally, defining the $C^1(0, \infty) \cap C[0, \infty)$ function $\underline{h} : [0, \infty) \to [0, \infty)$ by $\underline{h}(0) = 0$ and

$$\underline{h}(t) = \frac{1}{t} \int_{\frac{t}{2}}^t \overline{h}(s)ds, \ t > 0,$$
we have proved the claiming.

**Proof of Theorem 3.1** Due to the lack of ellipticity of the operator $\Delta_p$, we cannot apply standard comparison principle. So, we are going to consider a modified problem by $0 < \varepsilon < 1$ given by

$$
\begin{aligned}
\{ & \div((|\nabla u|^{p-2} + \varepsilon)\nabla u) = a(x)f(u) + \mu b(x)|\nabla u|^\alpha \text{ in } \Omega, \\
& u \geq 0 \text{ in } \Omega, \quad u(x) = 1 \text{ on } \partial \Omega.
\end{aligned}
$$

(1.18)

Since 0 and 1 are sub and super solutions of (1.18) respectively, it follows by a theorem in Kura [12], that (1.18) admits a solution $\zeta_1^\varepsilon \in C^{1,\nu}(\overline{\Omega})$ for some $\nu \in (0, 1]$, not depending on $\varepsilon$, such that $0 \leq \zeta_1^\varepsilon \leq 1$ in $\overline{\Omega}$.

Now, inductively repeating this process, using $\zeta_{k-1}^\varepsilon$ as a sub solution and $k$ as a super solution, we get a sequence $\zeta_k^\varepsilon \in C^{1,\nu}(\overline{\Omega})$ (the same $\nu$ as before) that satisfies

$$
0 \leq \zeta_1^\varepsilon \leq \zeta_2^\varepsilon \leq \cdots \leq \zeta_{k-1}^\varepsilon \leq \zeta_k^\varepsilon \leq k \text{ in } \overline{\Omega}
$$

(1.19)

and

$$
\begin{aligned}
\{ & \div((|\nabla u|^{p-2} + \varepsilon)\nabla u) = a(x)f(u) + \mu b(x)|\nabla u|^\alpha \text{ in } \Omega, \\
& u \geq 0 \text{ in } \Omega, \quad u(x) = k \text{ on } \partial \Omega.
\end{aligned}
$$

(1.20)

As a consequence of this and Lemma 3.2 with $h = f$, we have $\zeta_k^\varepsilon \in C^{1,\nu}(\overline{\Omega})$ satisfies

$$
\begin{aligned}
\{ & \div((|\nabla u|^{p-2} + \varepsilon)\nabla u) \geq a(x)f(u) - \mu b^{-}(x)(|\nabla u| + 1)^\alpha \text{ in } \Omega, \\
& u \geq 0 \text{ in } \Omega, \quad u(x) = k \text{ on } \partial \Omega.
\end{aligned}
$$

(1.21)

Now, we are going to assume $(a_1)$.

**Claim:** For each $x \in \Omega$, there exist a open $V_x \subset \subset \Omega$ and a function $\zeta_x \in C^2(V_x)$ satisfying

$$
0 \leq \zeta_1^x \leq \zeta_2^x \leq \cdots \leq \zeta_{k-1}^x \leq \zeta_k^x \leq \cdots \leq \zeta_x \text{ in } V_x, \text{ for all } 0 < \varepsilon < 1, \ k \in \mathbb{N}
$$

and $0 \leq \mu < \Lambda^x(\Omega)$ given, where $\Lambda^x(\Omega) > 0$ was defined in Lemma 3.1.

In fact, given a $x_0 \in \Omega$, we are going to consider two cases:

**Case 1:** $a(x_0) > 0$. In this case, consider $V_{x_0} \subset \Omega$ a smooth open domain such that $a(x) \geq a_0 > 0$ for all $x \in \overline{V_{x_0}}$, $u \in C^2(V_{x_0})$ the solution of problem

$$
\begin{aligned}
\{ & -\Delta u = 1 \text{ in } V_{x_0}, \\
& u > 0 \text{ in } V_{x_0}, \quad u(x) = 0 \text{ on } \partial V_{x_0}
\end{aligned}
$$

(1.22)

and denote by $g(x) = -\Delta_p v(x)$, $x \in V_{x_0}$. So, $g \in L^\infty(V_{x_0})$.

Besides this, by Lemma 3.2, there exist $s_0 > 0$ such that $f(s) \geq cs^q$ for $s \geq s_0$, where $d = \liminf_{s \to \infty} f(s)/s^q > 0$ for some $q > \max\{\alpha, p-1, 1\}$ and $c = d/2$. Now, defining $\omega = Mv^{-\beta} \in$
$C^2(V_\omega)$, where $M$ and $\beta$ are positive real parameters, we have, for each $0 \leq \varphi \in C^\infty_0(V_\omega)$, that 

$$
\int_{V_{\omega}} ((\nabla \omega)^{p-2} + \epsilon) \nabla \omega \nabla \varphi dx + \int_{V_{\omega}} [ca(x) \omega^q - \mu b^-(x)(|\nabla \omega| + 1)^\alpha] \varphi dx = 
$$

$$
- \int_{V_{\omega}} \beta^{p-1} M^{p-1} v^{(-\beta-1)(p-1)} |\nabla v|^{p-2} \nabla v \nabla \varphi dx - \int_{V_{\omega}} \epsilon \beta M v^{-\beta} \nabla \varphi dx + 
$$

$$
\int_{V_{\omega}} [ca(x) M^q v^{-\beta q} - \mu b^-(x)(\beta M v^{-\beta-1} |\nabla v| + 1)^\alpha] \varphi dx = 
$$

$$
- \int_{V_{\omega}} \beta^{p-1} M^{p-1} |\nabla v|^{p-2} \nabla v \nabla (v^{(-\beta-1)(p-1)} \varphi) dx - \int_{V_{\omega}} \beta^{p-1} M^{p-1} (\beta + 1)(p-1) v^{(-\beta-1)(p-1)-1} |\nabla v|^p \varphi dx 
$$

$$
- \int_{V_{\omega}} \epsilon \beta M \nabla \nabla (v^{-\beta-1} \varphi) dx - \int_{V_{\omega}} \epsilon \beta M (\beta + 1) v^{-\beta-2} |\nabla v|^2 \varphi dx + 
$$

$$
\int_{V_{\omega}} [ca(x) M^q v^{-\beta q} - \mu b^-(x)(\beta M v^{-\beta-1} |\nabla v| + 1)^\alpha] \varphi dx. 
$$

So, from \ref{[122]} and $g = -\Delta_p v$, we get 

$$
\int_{V_{\omega}} ((\nabla \omega)^{p-2} + \epsilon) \nabla \omega \nabla \varphi dx + \int_{V_{\omega}} [ca(x) \omega^q - \mu b^-(x)(|\nabla \omega| + 1)^\alpha] \varphi dx \geq 
$$

$$
- \int_{V_{\omega}} \beta^{p-1} M^{p-1} g(x) v^{(-\beta-1)(p-1)} \varphi dx - \int_{V_{\omega}} \beta^{p-1} M^{p-1} (\beta + 1)(p-1) v^{(-\beta-1)(p-1)-1} |\nabla v|^p \varphi dx 
$$

$$
- \int_{V_{\omega}} \epsilon \beta M v^{-\beta-1} \varphi dx - \int_{V_{\omega}} \epsilon \beta M (\beta + 1) v^{-\beta-2} |\nabla v|^2 \varphi dx 
$$

$$
+ \int_{V_{\omega}} [ca(x) M^q v^{-\beta q} - \mu b^-(x)2^\alpha (\beta^\alpha M^\alpha v^{-\beta-1} |\nabla v|^\alpha + 1)] \varphi dx. 
$$

Now, fixing

$$
\beta = \max \left\{ \frac{\alpha}{q - \alpha}, \frac{p}{q - p + 1}, \frac{2}{q - 1} \right\},
$$

we have

$$
\min\{-\beta - 1, p - 1\} - 1 + \beta q, (-\beta - 1)\alpha + \beta q, (-\beta - 2) + \beta q \geq 0.
$$

and as a consequence of this and $0 \leq \epsilon < 1$, we have 

$$
\int_{V_{\omega}} ((\nabla \omega)^{p-2} + \epsilon) \nabla \omega \nabla \varphi dx + \int_{V_{\omega}} [ca(x) \omega^q - \mu b^-(x)(|\nabla \omega| + 1)^\alpha] \varphi dx \geq 
$$

$$
\int_{V_{\omega}} M^{p-1} v^{-\beta q} \left[ - \beta^{p-1} \|g\|_\infty \|v\|^{(\beta-1)(p-1)+\beta q} - \beta^{p-1} (\beta + 1)(p-1) \|v\|^{(\beta-1)(p-1)-1+\beta q} \|\nabla v\|_\infty^p 
$$

$$
- \beta M^{2-p} \|v\|^{\beta + 1-\beta q} - \beta M^{2-p} (\beta + 1) \|v\|^{-2-\beta q} \|\nabla v\|_\infty^2 
$$

$$
- \mu b\|b\|^{2^\alpha} (\beta^\alpha M^\alpha v^{-p+1}\|v\|^{-\beta-1-\beta q} \|\nabla v\|_\infty + M^{1-p} \|v\|_\infty^\beta) + c M^{q-p+1} a_0 \right] \varphi dx.
$$
Now, since $q > \max\{\alpha, p - 1, 1\}$, we can choose a constant $M = M_{\mu, V_{x_0}} > 0$ (not depending on $\epsilon$) large enough such that
\[
\int_{V_{x_0}} (|\nabla \omega|^{p-2} + \epsilon) \nabla \omega \nabla \varphi dx + \int_{V_{x_0}} [ca(x)\omega^q - \mu b^-(x)(|\nabla \omega| + 1)^{\alpha}] \varphi dx \geq 0
\]
and defining $\zeta_{x_0}(x) = \omega(x) + s_0$ (not depending on $\epsilon$), we have that $\zeta_{x_0} \in C^2(V_{x_0})$ and satisfies
\[
\begin{cases}
\text{div}((|\nabla u|^{p-2} + \epsilon) \nabla u) \leq a(x)f(u) - \mu b^-(x)(|\nabla u| + 1)^{\alpha} & \text{in } V_{x_0}, \\
u \geq s_0 & \text{in } V_{x_0}, \quad u(x) \xrightarrow{d(x) \to 0} \infty
\end{cases}
\]
for each $0 \leq \epsilon < 1$ and $\mu \geq 0$ given.

Besides this, for $0 < \epsilon < 1$ (that is, $\epsilon \neq 0$) given, it follows from (1.24), that there exist subsequences of $(\zeta_{x_0})_{x_0}$ and defining $\zeta_{x_0}(x) = \omega(x) + s_0$ (not depending on $\epsilon$), we have that $\zeta_{x_0} \in C^2(V_{x_0})$ and satisfies
\[
\begin{cases}
\text{div}((|\nabla u|^{p-2} + \epsilon) \nabla u) \leq a(x)f(u) - \mu b^-(x)(|\nabla u| + 1)^{\alpha} & \text{in } V_{x_0}, \\
u \geq s_0 & \text{in } V_{x_0}, \quad u(x) \xrightarrow{d(x) \to 0} \infty
\end{cases}
\]
for each $0 \leq \epsilon < 1$ and $\mu \geq 0$ given.

**Case 2:** $a(x_0) = 0$. Since $a$ is a $c_\Omega$–positive function, there exists an open $V_{x_0} \subset \Omega$ such that
\[
x_0 \in V_{x_0} \quad \text{and} \quad a(x) > 0 \quad \text{for all } x \in \partial V_{x_0}.
\]
Taking a finite cover of $\partial V_{x_0}$, namely $V_i, \ i = 1, \ldots, n$, such that
\[
\partial V_{x_0} \subset \bigcup_{i=1}^n V_i \quad \text{and} \quad a(x) \geq a_i > 0, \ x \in V_i,
\]
it follows from the argument of the case 1 that there exists $\zeta_{x_0}^k \in C^2(V_i)$ such that $0 \leq \zeta_{x_0}^k \leq \zeta_{x_0}^1$ in $V_i$ for all $0 < \epsilon < 1$ and $k \in \mathbb{N}$. In particular, there exists a positive real constant $A = A_{x_0} > 0$ such that $\zeta_{x_0}^k \leq A$ on $\partial V_{x_0}$, $\forall \ k \in \mathbb{N}$ and $0 < \epsilon < 1$.

Now, taking $u = u_{\epsilon, \mu} \in C^1(\overline{\Omega})$ for $0 \leq \mu < A^*$ a solution of problem (1.13), given by Lemma 3.1, we have that $A + u_{\epsilon, \mu}$ satisfies
\[
\begin{cases}
\text{div}((|\nabla u|^{p-2} + \epsilon) \nabla u) \leq a(x)f(u) - \mu b^-(x)(|\nabla u| + 1)^{\alpha} & \text{in } V_{x_0}, \\
u \geq A & \text{in } V_{x_0}, \quad u(x) \geq A & \text{on } \partial V_{x_0}
\end{cases}
\]
and $\zeta_{x_0}^k \leq A \leq A + u_{\epsilon, \mu}$ on $\partial V_{x_0}$. So, it follows of a comparison principle in [16] that $\zeta_{x_0}^k \leq A + u_{\epsilon, \mu}$ in $V_{x_0}$. Since by Lemma 3.1, we have $\|u_{\epsilon, \mu}\|_\infty \leq C$, with $C > 0$ not depending on $\epsilon$, the claim follows by taking $\zeta_{x_0} = A + C$.

As a consequence of the both prior cases, it follows that given a compact set $K \subset \Omega$ there exists a constant $C_K > 0$ such that
\[
0 \leq \zeta_1^k \leq \zeta_2^k \leq \cdots \leq \zeta_k^k \leq \cdots \leq C_K \text{ in } K \text{ and } \zeta_k^k \in C^{1,\alpha}(\overline{K}) \text{ for all } \epsilon \in (0, 1) \text{ and } k \in \mathbb{N}.
\]
That is, taking $\epsilon_n \in (0, 1)$ with $\epsilon_n \to 0$ and $\Omega_j \subset \subset \Omega$ smooth open sets such that
\[
\Omega_j \subset \subset \Omega_{j+1} \text{ and } \Omega = \bigcup_{j=1}^{\infty} \Omega_j,
\]
it follows from [1.24], that there exist subsequences of $(\epsilon_n)$, denoted by $(\epsilon_{n_{j_i}})$, where
\[
\cdots \subseteq N_j \subseteq N_{j-1} \subseteq \cdots \subseteq N_1 \subseteq N \text{ with } N_j = \{n_{j_1}, n_{j_2}, n_{j_3}, \cdots\},
\]
such that $\zeta_{k_i n_{j_i}} \xrightarrow{i \to \infty} \zeta_{k_i}^j \in C^{1,\theta}(\overline{\Omega_j})$ for some $0 < \theta < \nu \leq 1$, with $\theta$ does not depend on $\epsilon$, and $\zeta_{k_i n_{j_i}} = \zeta_{k_i}^j$ for each $k, j \in \mathbb{N}$. 

Now, defining \( \zeta_k = \zeta_k^j \) for \( x \in \overline{\Omega}_j \), it follows that \( \zeta_k^{n,j} \xrightarrow{j \to \infty} \zeta_k \) in \( C^1_{\text{loc}}(\Omega) \) for some \( 0 < \vartheta < \theta < 1 \), with \( \vartheta \) does not depending on \( \epsilon \) with \( \zeta_k \) satisfying

\[
0 \leq \zeta_1 \leq \zeta_2 \leq \ldots \leq \zeta_k \leq \ldots \leq C_{\overline{\Omega}_j} \text{ in } \overline{\Omega}_j \text{ for each } j \in \mathbb{N}
\]

(1.26)

and

\[
\begin{cases}
\Delta \rho \nu = a(x)f(u) + \mu b(x)|\nabla u|^\alpha & \text{in } \Omega, \\
u \geq 0 & \text{in } \Omega, \quad u(x) = k \text{ on } \partial \Omega
\end{cases}
\]

(1.27)

for each \( k \in \mathbb{N} \) given.

Hence, applying the diagonal process again, now in \( k \), it follows from (1.26) and (1.27) that there exists a \( \zeta \in C^1(\Omega) \) solution of (1.12).

Now, we are going to assume \( (a_2) \).

In what follows, we will take \( \Omega_j \subset \subset \Omega \) smooth open sets satisfying (1.25) again. Then, it follows from hypothesis \( (a_0)' \) that there exists a \( a_{0n} > 0 \) such that \( a(x) \geq a_{0n} \) in \( \Omega_n \). This permit us, in a similar way to Case 1, to build a function \( \tilde{u}_n \in C^2(\Omega_n) \) (\( \tilde{u}_n \) independent of \( \epsilon \)) satisfying

\[
\begin{cases}
div((|\nabla \omega|^{p-2} + \epsilon)\nabla \omega) \leq a(x)f(\omega) - \mu b^-(x)(|\nabla \omega| + 1)^\alpha & \text{in } \Omega_n, \\
\omega \geq 0 \text{ in } \Omega_n, \quad \omega(x) \xrightarrow{d(x) \to 0} \infty
\end{cases}
\]

(1.28)

for each \( 0 \leq \epsilon < 1 \) and \( \mu \geq 0 \) given.

Beside this, for each \( 0 < \epsilon < 1 \), we have \( 0 \leq \zeta_k \leq \tilde{u}_n \) in \( \Omega_n \) for all \( k \in \mathbb{N} \), where \( \zeta_k^i \in C^{1,\nu}(\overline{\Omega}) \) satisfies (1.19) and (1.20). So, given a compact set \( K \subset \Omega \) there exists a \( n_K \in \mathbb{N} \) such that \( K \subset \Omega_{n_K} \). Thus, there exists a constant \( C_K > 0 \) such that (1.21) holds again.

That is, under the notations of last diagonal process, we obtain \( \zeta_k^{n,j} \xrightarrow{i \to \infty} \zeta_k^j \) in \( C^{1,\theta}(\overline{\Omega}_j) \) for some \( 0 < \theta < \nu \leq 1 \), with \( \theta \) does not depend on \( \epsilon \), \( 0 \leq \zeta_k^j \leq \tilde{u}_{j+1} \) in \( \overline{\Omega}_j \) and \( \zeta_k^j_{|\overline{\Omega}_{j-1}} = \zeta_k^{j-1} \) for each \( k, j \in \mathbb{N} \). So, repeating the argument as before, we get a that is a solution of (1.12). These end the proof of Theorem 3.1.

\section{4 Proof of Theorem 1.1}

First, we are going to consider the case \((P)_{\rho^-}(ii)\), because in the proof of \((P)_{\rho^-}(i)\) we let us use the proof of the first case with \( \mu = 0 \).

**Case 1**: Assume \((P)_{\rho^-}(ii)\), that is, \( b^+ \neq 0 \).

At first, we are going to build a nonnegative sub solution \( u \) of (1.1) by proving the existence of a solution for the problem

\[
\begin{cases}
\Delta \rho u = a(x)f(u) + \mu b(x)|\nabla u|^{p-1} & \text{in } \mathbb{R}^N, \\
u \geq 0 & \text{in } \mathbb{R}^N, \quad u(x) \xrightarrow{|x| \to +\infty} +\infty,
\end{cases}
\]

(1.29)

where \( f \) was built as in Lemma 3.2.

To do this, first we note that of Theorem 3.1 we get a \( u_n \in C^1(B_n) \) solution of problem

\[
\begin{cases}
\Delta \rho u = a(x)f(u) + \mu b(x)|\nabla u|^{p-1} & \text{in } B_n, \\
u \geq 0 & \text{in } B_n, \quad u(x) = +\infty, \quad \text{on } \partial B_n
\end{cases}
\]
and as a consequence of Theorem 2.1, we have \( u_n \geq u_{n+1} \geq 0 \) in \( B_n \). In this case, \( \mu_* = \mu_*(B_n) = \infty \), since \( (a_2) \) holds for each \( n \in \mathbb{N} \).

So, by a diagonal process, we can show that \( \underline{u}_n \to \underline{u} \) in \( C^1(\mathbb{R}^N) \) that satisfies
\[
\int_{\mathbb{R}^N} |\nabla u|^p - 2 \nabla u \nabla \phi \, dx + \int_{\mathbb{R}^N} [a(x) \overline{f}(u) + \mu b(x) |\nabla u|^{p-1}] \phi \, dx = 0, \quad \phi \in C_0^\infty(\mathbb{R}^N).
\]

To complete the building of \( \underline{u} \), just remain to prove that \( \underline{u}(x) \to +\infty \) when \( |x| \to +\infty \). To do this, defining \( \omega^n \in C^1(B_n) \) by
\[
\omega^n(x) = \int_{u^*_n(x)}^{\infty} (\overline{f}(t) + 1)^{-\frac{1}{p-1}} \, dt, \quad x \in B_n,
\]
we have \( \omega^n > 0 \) in \( B_n \), \( \omega^n(x) = 0 \) on \( \partial B_n \) and
\[
\int_{B_n} |\nabla \omega^n|^p - 2 \nabla \omega^n \nabla \varphi \, dx = - \int_{B_n} \overline{f}(u_n) + 1 |\nabla u_n|^p - 2 \nabla u_n \nabla \varphi \, dx
\leq \int_{B_n} \overline{f}(u_n) + 1 |a(x) \overline{f}(u_n) + \mu b(x) |\nabla u_n|^{p-1}| \varphi \, dx
\leq \int_{B_n} [a(x) + \mu b^+(x)] |\nabla \omega^n|^{p-1}| \varphi \, dx.
\]
That is,
\[
\int_{B_n} |\nabla \omega^n|^p - 2 \nabla \omega^n \nabla \varphi \, dx \leq \int_{B_n} [a(x) (1 + (\omega + 1)^p/2) + \mu b^+(x)] |\nabla \omega^n|^{p-1}| \varphi \, dx,
\]
for every \( \varphi \in C^\infty(\mathbb{R}^N) \) with \( \varphi \geq 0 \).

So, given \( 0 \leq \mu < \Lambda_* \), it follows from Theorem 2.1 that \( \omega^n \leq \omega_{\mu} \) in \( B_n \) for all \( n \), where \( \Lambda_* \) and \( \omega_{\mu} \) were given in Lemma 2.1. Since \( u_n \to \underline{u} \) in \( C^1(\mathbb{R}^N) \), it follows from (1.30) that there exists a \( \omega_0 \in C^1(\mathbb{R}^N) \) with \( \omega_0 \leq \omega_{\mu} \) and \( \omega_0(x) \to 0 \) as \( |x| \to \infty \) such that \( \omega^n \to \omega_0 \) in \( C^1(\mathbb{R}^N) \) and
\[
\omega_0(x) = \int_{u^*_0(x)}^{\infty} (\overline{f}(t) + 1)^{-\frac{1}{p-1}} \, dt, \quad x \in \mathbb{R}^N.
\]
As a consequence of this, we have \( \underline{u}(x) \to \infty \) as \( |x| \to \infty \). This shows that \( \underline{u} \) is a solution of (1.29), that is, \( \underline{u} \) is a sub solution of (1.1).

Now, considering the problem
\[
\begin{cases}
\Delta_p u = a(x) f(u) + \mu b(x) |\nabla u|^{p-1} & \text{in } B_n, \\
u \geq 0 & \text{in } B_n, \\
u(x) = \underline{u}(x) & \text{on } \partial B_n
\end{cases}
\tag{1.31}
\]
we have that \( \underline{u} \) and \( \overline{u}_n \) are sub and super of (1.31) and \( \underline{u} \leq \overline{u}_n \) em \( B_n \), where \( \overline{u}_n \) satisfies (1.28) with \( \Omega_n = B_n \) and \( \epsilon = 0 \). Then, by sub and super solution method and regularity theory, the problem (1.31) has a solution \( u_n \in C^1(B_n) \) with \( \underline{u} \leq u_n \leq \overline{u}_n \) for all \( n \in \mathbb{N} \).

So, applying the Theorem 2.1 again, we have \( \underline{u} \leq u_m \leq \overline{u}_n \) in \( B_n \) for all \( m, n \in \mathbb{N} \) such that \( m \geq n \) and as a consequence of this, by a diagonal process, there is a function \( u \in C^1(\mathbb{R}^N) \) and a subsequence of \( u_n \), denoted by itself, such that \( u_n \to u \) with \( u \geq \underline{u} \) in \( \mathbb{R}^N \) and \( u \) a solution of (1.1).

**Case 2:** Suppose \((P)_{\rho^- (i)}\).
At first, given $\epsilon \in (0, 1)$ and $n \in \mathbb{N}$, we are going to consider $\zeta^{\epsilon, n}_1 \in C^1(B_n)$ and $\varpi^n \in C^1(B_n)$ solutions of the problems (1.18) and (1.28), respectively in $B_n$. So, $\zeta^{\epsilon, n}_1$ and $\varpi^n$ are sub and super solutions of the problem

\[
\begin{align*}
\text{div}(|\nabla u|^{p-2} + \epsilon)\nabla u &= a(x)f(u) - \mu b^-(x)(|\nabla u| + 1)^\alpha \quad \text{in } B_n, \\
\epsilon &> 0 \quad \text{in } B_{n-1/k}, \quad u(x) = 1 \quad \text{on } \partial B_{n-1/k}
\end{align*}
\tag{1.32}
\]

and, by standard principle comparison, we have $\zeta^{\epsilon, n}_1 \leq \varpi^n$ in $B_n$. We remember that $\varpi^n$ does not depend of $\epsilon \in (0, 1)$.

Now, taking $B_{n-1/k} \subset B_n$, where $k \in \mathbb{N}$, it follows by a sub and super solution of (24) and a result of regularities in (23) that the problem

\[
\begin{align*}
\text{div}(|\nabla u|^{p-2} + \epsilon)\nabla u &= a(x)f(u) - \mu b^-(x)(|\nabla u| + 1)^\alpha \quad \text{in } B_{n-1/k}, \\
\epsilon &> 0 \quad \text{in } B_{n-1/k}, \quad u(x) = \zeta^{\epsilon, n}_1 \mid_{B_{n-1/k}} \quad \text{on } \partial B_{n-1/k}
\end{align*}
\]

admits a solution $u^{\epsilon, n}_{1,k} \in C^{1,\nu}(\overline{B_{n-1/k}})$, for some $0 < \nu \leq 1$. After this, applying a diagonal process in $k$, we show that the problem (1.32) admits a solution $u^{\epsilon, n}_{1} \in C^{1,\theta}(\overline{B_n})$, for some $0 < \theta < \nu$, such that $\zeta^{\epsilon, n}_1 \leq u^{\epsilon, n}_{1} \leq \varpi^n$ in $B_n$.

Repeating this process, by using $u^{\epsilon, n}_{k-1}$ as a sub solution and $\varpi^n$ as a super solution, we obtain a sequence $\{u^{\epsilon, n}_{k}\}^\infty_{k=1} \subset C^{1,\theta}(\overline{B_n})$ satisfying

\[
0 \leq u^{\epsilon, n}_{1} \leq u^{\epsilon, n}_{2} \leq \ldots \leq u^{\epsilon, n}_{k-1} \leq u^{\epsilon, n}_{k} \leq \ldots \leq \varpi^n \quad \text{in } B_n
\tag{1.33}
\]

and

\[
\begin{align*}
\text{div}(|\nabla u^{\epsilon, n}_{k}|^{p-2} + \epsilon)\nabla u^{\epsilon, n}_{k} &= a(x)f(u^{\epsilon, n}_{k}) - \mu b^-(x)(|\nabla u^{\epsilon, n}_{k}| + 1)^\alpha \quad \text{in } B_n, \\
u^{\epsilon, n}_{k} &> 0 \quad \text{in } B_n, \quad u^{\epsilon, n}_{k}(x) = k \quad \text{on } \partial B_n.
\end{align*}
\]

Now, by a diagonal process, it follows from (1.33) that there exists a function $u^{\epsilon, n} \in C^{1,\vartheta}(B_n)$, for some $0 < \vartheta < \theta$, such that

\[
\begin{align*}
\text{div}(|\nabla u^{\epsilon, n}|^{p-2} + \epsilon)\nabla u^{\epsilon, n} &= a(x)f(u^{\epsilon, n}) - \mu b^-(x)(|\nabla u^{\epsilon, n}| + 1)^\alpha \quad \text{in } B_n, \\
u^{\epsilon, n} &> 0 \quad \text{in } B_n, \quad u^{\epsilon, n}(x) = \infty \quad \text{on } \partial B_n
\end{align*}
\]

and, by comparison principle in (16),

\[
0 \leq u^{\epsilon, n+1} \leq u^{\epsilon, n} \leq \varpi^n \quad \text{in } B_n.
\]

So, following the same argument as in the proof of Case 2 of Theorem 3.1, we show that there exists a $u^n \in C^1(B_n)$ solution of the problem

\[
\begin{align*}
\Delta_p u &= a(x)f(u) - \mu b^-(x)(|\nabla u| + 1)^\alpha \quad \text{in } B_n, \\
u &> 0 \quad \text{in } B_n, \quad u(x) \to \infty \quad \text{as } x \to \partial B_n
\end{align*}
\]

satisfying

\[
0 \leq \cdots \leq u^{n+1} \leq u^n \leq \varpi^n \quad \text{in } \overline{B_n}.
\]

On the other hand, it follows from the case 1, with $\mu = 0$. (In this case, in the proof of Lemma 2.2, it is necessary just that the solution of $(P_\rho)$ belongs to $C^1(\mathbb{R}^N)$) that there exists a $v \in C^1(\mathbb{R}^N)$ satisfying

\[
\begin{align*}
\Delta_p v &= a(x)f(v) \geq a(x)f(v) \quad \text{in } \mathbb{R}^N, \\
v &> 0 \quad \text{in } \mathbb{R}^N, \quad v(x) \to \infty \quad \text{as } x \to \partial \mathbb{R}^N.
\end{align*}
\]
Beside this, by comparison principle \cite{25}, we have $v \leq u^n$ in $B_n$ for all $n \in \mathbb{N}$.

So, by a diagonal process, there exists a $\overline{u} \in C^1(\mathbb{R}^N)$ such that $v \leq \overline{u}$ in $\mathbb{R}^N$, $u^n \to \overline{u}$ in $C^1(\mathbb{R}^N)$ and $\overline{u}$ is a solution of the problem

$$
\begin{aligned}
\Delta_p u &= a(x)f(u) - \mu b^-(x)(|\nabla u| + 1)^\alpha \quad \text{in } \mathbb{R}^N, \\
u &\geq 0 \text{ in } \mathbb{R}^N, \quad u(x) \to +\infty.
\end{aligned}
$$

Thus, since $v$ and $\overline{u}$ are sub and super solutions of the problem

$$
\begin{aligned}
\Delta_p u &= a(x)f(u) - \mu b^-(x)|\nabla u|^\alpha \quad \text{in } \mathbb{R}^N, \\
u &\geq 0 \text{ in } \mathbb{R}^N, \quad u(x) \to +\infty
\end{aligned}
\tag{1.34}
$$

it follows by a theorem of sub and super solution in \cite{24}, that there exists a solution $u \in C^1(\mathbb{R}^N)$ for the problem \textbf{(1.34)}. This finishes the proof.

As an immediate consequence of the arguments used in the proof of last theorem, we have

\begin{corollary}
Assume that $\Omega \subset \mathbb{R}^N$ is smooth bounded domain, $\liminf_{t \to \infty} f(t)/t^q > 0$ for some $q > \max\{\alpha, p - 1, 1\}$ and $a, b \in L^\infty_{\text{loc}}(\Omega)$ with $a$ satisfying $(a_\Omega)'$ and

$$
\begin{aligned}
\begin{cases}
-\Delta_p w = \rho(x) &\text{in } \Omega, \\
w > 0 &\text{on } \Omega, \quad w(x) = 0 \text{ at } \partial \Omega,
\end{cases}
\end{aligned}
\tag{1.35}
$$

has a solution in $C^1(\Omega)$, where $\rho(x) = \max\{a(x), b^+(x)\}$, $x \in \Omega$ with either

(i) $b^+ = 0$ and $0 \leq \alpha \leq p$ or (ii) $b^+ \neq 0$ and $\alpha = p - 1$.

Then there exists $\mu^* \in (0, +\infty]$ such that the problem

$$
\begin{aligned}
\begin{cases}
\Delta_p u = a(x)f(u) + \mu b(x)|\nabla u|^\alpha &\text{in } \Omega, \\
u \geq 0 &\text{on } \Omega, \quad u(x) \to +\infty
\end{cases}
\tag{1.36}
$$

has a solution in $C^1(\Omega)$, for each $0 \leq \mu < \mu^*$ given. In additional, if (i) holds, then $\mu^* = +\infty$.
\end{corollary}

This Corollary complements some above quoted results principally by permitting the oscillatory and explosive behavior of potentials $a$ and $b$ on boundary of $\Omega$. In particular, it complements a result by Liu e Yang \cite{27} that considered in \textbf{(1.36)} the nonlinearity $f$ as a non-decreasing function satisfying $f(s) \leq C_1 s^{p_1(p-1)}$ for $s \in (0, \infty)$, $f(s) \geq C_2 s^{p_2(p-1)}$ for $s > > 0$, where $p_1 \geq p_2$, $b(x) = \pm 1$ and $a$ satisfying $C_3(d(x))^{\gamma_2} \leq a(x) \leq C_4(d(x))^{\gamma_1}$ for all $x \in \Omega$ with $-p < \gamma_1 \leq \gamma_2$ and $C_i$ positive constants.

As examples of non-null and non-negative potentials $\rho$ satisfying \textbf{(1.35)}, we have:

(i) $a, b^+ \in L^q(\Omega)$ for some $q > N > 1$. For details, see \cite{28},

(ii) $a, b \in C(\Omega)$ such that $a(x), b^+(x) \leq C_0 d(x)^{-\gamma(x)}$, $x \in \Omega$, where $\gamma \in C(\overline{\Omega})$ and $\gamma(x) < 1/N$ for $x \in \partial \Omega$, for some positive constant $C_0$. This situation permits singular behaviors for the potential $a$ in the sense that $a(x) \to +\infty$ and $a(x) \to a_o$ for $x_o \neq x_1$. The same can occur for $b$ too. For more details, see \cite{29}.
5 Proof of Theorem 1.2

The proof of Theorem 1.2 consists principally of delicate and sensible estimates involving the operator and the nonlinearities. In this result, we are mainly interested in showing nonexistence of entire solutions that blow-up at infinity. In the literature there are some results that prove nonexistence of either subsolutions, supersolutions or solutions without requiring their behavior at infinity and demanding strongest conditions under the nonlinearities.

Proof. Given $R > 0$ define $\xi_R \in C^1(\mathbb{R}^N, \mathbb{R})$ such that $\xi_R(x) = 1$, $0 \leq |x| \leq R$ and $\xi_R(x) = 0$, $|x| \geq 2R$ satisfying

$$0 \leq \xi_R(x) \leq 1, \quad |\nabla \xi_R(x)| \leq \frac{1}{R}, \quad x \in \mathbb{R}^N.$$ 

Now, considering the $C^1$-functions $\chi = \xi_R^\mu$ and $u^\beta \chi$, where $\mu, \beta > 1$ are real parameters, and using the last one as a test function in (1.2), we get

$$\int_{\mathbb{R}^N} a(x)f(u)u^\beta \chi dx + \int_{\mathbb{R}^N} b(x)|\nabla u|^\alpha u^\beta \chi dx + \int_{\mathbb{R}^N} \beta u^{\beta-1}|\nabla u|^p \chi dx \leq \int_{\mathbb{R}^N} |\nabla u|^{p-1} u^\beta |\nabla \chi| dx.$$

By the hypothesis under $f$, there exists a $R_0 > 0$ (we can consider this $R_0 > 0$ such that $a, b > 0$ on $\mathbb{R}^N \setminus B_{R_0}$) such that $f(u(x)) \geq Cu^q(x)$ and $u(x) \geq 1$ for all $|x| \geq R_0$ for some $C > 0$, since $u(x) \to \infty$ as $|x| \to \infty$. That is,

$$C \int_{R_0 \leq |x| \leq 2R} a(x)u^{\beta+q} \chi dx + \int_{R_0 \leq |x| \leq 2R} b(x)|\nabla u|^\alpha u^\beta \chi dx + \beta \int_{R_0 \leq |x| \leq 2R} u^{\beta-1}|\nabla u|^p \chi dx \leq \int_{R_0 \leq |x| \leq 2R} |\nabla u|^{p-1} u^\beta |\nabla \chi| dx, \quad R > R_0,$$

for some $C > 0$. Now, we can rewrite the above inequality as

$$\int_{R_0 \leq |x| \leq 2R} a(x)u^{\beta+q} \chi dx + \int_{R_0 \leq |x| \leq 2R} b(x)|\nabla u|^\alpha u^\beta \chi dx + \int_{R_0 \leq |x| \leq 2R} u^{\beta-1}|\nabla u|^p \chi dx \leq \tilde{C} \int_{R_0 \leq |x| \leq 2R} |\nabla u|^{p-1} u^\beta |\nabla \chi| dx,$$

(1.37)

where $\tilde{C} > 0$ is a real constant depending of $C$ and $\beta$.

From now on, we going to consider two cases:

Case 1: $(i)$ holds. First, we note that $q > p - 1$. So, given $\tau \in (1 + 1/q, 1 + 1/(p - 1))$ and considering $\tau' > 1$ satisfying $1/\tau' + 1/\tau = 1$, we can use the inequality of Young, to obtain

$$\int_{R_0 \leq |x| \leq 2R} a(x)\frac{1}{\tau'} u^{\beta+q - \frac{1}{\tau'}} |\nabla u|^q \chi dx = \int_{R_0 \leq |x| \leq 2R} a(x)u^{\beta+q} \chi dx \leq \tau \int_{R_0 \leq |x| \leq 2R} a(x)u^{\beta+q} \chi dx + \frac{1}{\tau} \int_{R_0 \leq |x| \leq 2R} u^{\beta-1}|\nabla u|^p \chi dx.$$

(1.38)
So, from (1.37) and (1.38), we get

\[
\int_{R_0 \leq |x| \leq 2R} a(x) \frac{1}{
\beta \tau \tau' + q \tau - \tau'} |\nabla u|^{\frac{p}{m}} \chi \, dx + \int_{R_0 \leq |x| \leq 2R} b(x) |\nabla u|^{\alpha} u^\beta \chi \, dx \leq \tilde{C} \int_{R_0 \leq |x| \leq 2R} |\nabla u|^{p-1} u^\beta |\nabla \chi| \, dx.
\]

and so, we have

(I) \[ \int_{R_0 \leq |x| \leq 2R} a(x) \frac{1}{
\beta \tau \tau' + q \tau - \tau'} |\nabla u|^{\frac{p}{m}} \chi \, dx \leq \tilde{C} \int_{R_0 \leq |x| \leq 2R} |\nabla u|^{p-1} u^\beta |\nabla \chi| \, dx \]

and

(II) \[ \int_{R_0 \leq |x| \leq 2R} b(x) |\nabla u|^{\alpha} u^\beta \chi \, dx \leq \tilde{C} \int_{R_0 \leq |x| \leq 2R} |\nabla u|^{p-1} u^\beta |\nabla \chi| \, dx. \]

Assume (I) holds. So, letting

\[ m = \frac{\beta \tau \tau' + q \tau - \tau'}{\beta \tau \tau'} > 1 \]

and

\[ n = \frac{\beta \tau \tau' + q \tau - \tau'}{q \tau - \tau'} > 1, \]

we have \(1/m + 1/n = 1\) and by H"{o}lder inequality, it follows that

\[
\int_{R_0 \leq |x| \leq 2R} |\nabla u|^{p-1} u^\beta |\nabla \chi| \, dx = \int_{R_0 \leq |x| \leq 2R} a(x) \frac{1}{m} u^\beta |\nabla u|^{\frac{p}{m}} \chi a(x) - \frac{1}{m} \chi \, dx \leq \tilde{C} \int_{R_0 \leq |x| \leq 2R} \left( \int_{R \leq |x| \leq 2R} a(x) - \frac{n}{m} \chi \, dx \right) \frac{1}{n}. \tag{1.39}
\]

Now, choosing

\[ \beta = \beta(\tau) = \frac{(p-1)(q \tau - \tau')}{\tau'(p - \tau p + \tau)}, \tau > 1 \]

and noticing that \((p-1)m = p/\tau\), we get

\[
\int_{R_0 \leq |x| \leq 2R} a(x) \frac{1}{\tau} u^{\frac{p}{m}} \chi \, dx \leq \tilde{C} \int_{R_0 \leq |x| \leq 2R} \left( \int_{R \leq |x| \leq 2R} a(x) - \frac{n}{m} \chi \, dx \right) \frac{1}{n}. \tag{1.40}
\]

That is

\[
\int_{R_0 \leq |x| \leq 2R} a(x) \frac{1}{\tau} u^{\frac{p}{m}} \chi \, dx \leq \tilde{C} \int_{R \leq |x| \leq 2R} a(x) - \frac{n}{m} \chi \, dx. \tag{1.41}
\]

Besides this, we noting that

\[
\chi^{-\frac{\beta \tau \tau'}{q \tau - \tau'}} = \xi^{-\frac{\beta \tau \tau'}{q \tau - \tau'}} \text{ and } |\nabla \chi| = |\mu \xi^{-1} |\nabla \xi| \leq \mu \xi^{-1} R^{-1}, \quad x \in \mathbb{R}^N,
\]

we get for \(\mu\) large enough, that

\[
\chi^{-\frac{\beta \tau \tau'}{q \tau - \tau'}} |\nabla \chi|^n \leq \mu^n \xi^{(\mu-1)n} - \frac{\beta \tau \tau'}{q \tau - \tau'} R^{-n} \leq C \mu R^{-n}, \quad \text{for all } x \in \mathbb{R}^N,
\]

for some \(C_\mu > 0\).
Now, fixing a such $\mu > 1$, we have that
\[
\int_{R \leq |x| \leq 2R} a(x)^{-\frac{p}{q - r - r'}} \chi \frac{\partial x}{\partial x'} |\nabla \chi|^n dx \leq C \mu R^{-\frac{p}{q - r + r'}} \int_{R \leq |x| \leq 2R} a(x)^{-\frac{(p-1)(q-1)}{r + r'}} dx. \tag{1.41}
\]

Now, given $\theta \in (p - 1, q)$ we can take a $\tau = \tau_0 \in (1 + 1/q, 1 + 1/(p - 1))$ such that $\theta = q(p - 1)/(\tau - 1)$. So, from (1.41), we have
\[
\int_{R \leq |x| \leq 2R} a(x)^{-\frac{p}{q - r - r'}} \chi \frac{\partial x}{\partial x'} |\nabla \chi|^n dx \leq R^{-\frac{r}{q - q'}} \int_{R \leq |x| \leq 2R} a(x)^{-\frac{p}{q - q'}} dx, \quad R > R_0. \tag{1.42}
\]

Hence, it follows from (1.40), (1.42) and the hypothesis (i), that
\[
\int_{|x| \geq R_0} a(x)^{\frac{1}{m}} u^{(\beta + \frac{q}{r} - \frac{1}{r'})} |\nabla u|^\frac{n}{r'} dx < \infty. \tag{1.43}
\]

Now returning in (1.39), rewriting its last integrals with the domain $R_0 \leq |x| \leq 2R$ instead of $R \leq |x| \leq 2R$ and using (1.42), we get
\[
\int_{R_0 \leq |x| \leq 2R} a(x)^{\frac{1}{m}} u^{(\beta + \frac{q}{r} - \frac{1}{r'})} |\nabla u|^\frac{n}{r'} dx \leq
\]
\[
C \left( \int_{R \leq |x| \leq 2R} a(x)^{\frac{1}{m}} u^{(\beta + \frac{q}{r} - \frac{1}{r'})} |\nabla u|^\frac{n}{r'} \chi dx \right)^\frac{1}{m} \left( \int_{R \leq |x| \leq 2R} a(x)^{-\frac{p}{q - q'}} dx \right)^\frac{1}{n}.
\]

Now, it follows from (1.43) and the hypothesis, that
\[
\int_{|x| \geq R_0} a(x)^{\frac{1}{m}} u^{(\beta + \frac{q}{r} - \frac{1}{r'})} |\nabla u|^\frac{n}{r'} dx = 0,
\]
that is, $u(x) = c$, for all $x \in \mathbb{R}^N \setminus B_{R_0}$, for some real constant $c > 0$. This is impossible, because $u(x) \to \infty$ as $|x| \to \infty$.

Assume (II) holds. First, we note that we can take $\beta = 0$ (return at the beginning of the proof and taking just $\chi$ as a test function in place of $u^\beta \chi$). Now, letting $m = \alpha/(p - 1) > 1$ and $n = \alpha/(\alpha - p + 1) > 1$, we have $1/m + 1/n = 1$ and
\[
\int_{R_0 \leq |x| \leq 2R} b(x) |\nabla u|^\alpha \chi dx \leq C \int_{R_0 \leq |x| \leq 2R} b(x)(b(x) \chi)^{-\frac{1}{m}} |\nabla u|^{p-1} |\nabla \chi| dx
\]
\[
\leq C \left( \int_{R_0 \leq |x| \leq 2R} b(x) \chi |\nabla u|^\alpha dx \right)^\frac{1}{m} \left( \int_{R_0 \leq |x| \leq 2R} b(x)(b(x) \chi)^{-\frac{1}{m}} |\nabla \chi| dx \right)^\frac{1}{n}.
\]

That is,
\[
\int_{R_0 \leq |x| \leq 2R} b(x) |\nabla u|^\alpha \chi dx \leq C \int_{R_0 \leq |x| \leq 2R} b(x)(b(x) \chi)^{-\frac{m}{n}} |\nabla \chi|^n dx. \tag{1.44}
\]

Since,
\[
|\nabla \chi|^n \chi^{-\frac{m}{n}} = \mu^n \xi^{\frac{m}{n}} |\nabla \xi|^n \leq \mu^n R^{-n} \quad \text{for all } \mu \geq n
\]

it follows from (1.44), that
\[
\int_{R_0 \leq |x| \leq 2R} b(x) |\nabla u|^\alpha \chi dx \leq C R^{-\frac{\alpha}{p - 1}} \int_{R_0 \leq |x| \leq 2R} b(x)(b(x) \chi)^{-\frac{m}{n}} |\nabla \chi| dx.
\]
Thus, it follows from the hypothesis, that
\[ \int_{|x| \geq R_0} b(x)|\nabla u|^\alpha dx < \infty \]
and the rest of the proof follows like the last case.

Case 2: \((ii)\) holds. Given \(\tau \in (1, \alpha/(p-1))\), take \(\tau' > 1\) such that \(1/\tau + 1/\tau' = 1\). So, we have
\[ \int_{R_0 \leq |x| \leq 2R} \left( a(x)u^\beta + q \chi \right)^{1/\tau'} \left( b(x)u^{1/\tau} |\nabla u|^\alpha \chi \right)^{1/\tau'} dx \]
\[ \leq \frac{1}{\tau} \int_{R_0 \leq |x| \leq 2R} a(x)u^\beta + q \chi dx + \frac{1}{\tau} \int_{R_0 \leq |x| \leq 2R} b(x)u^{1/\tau} |\nabla u|^\alpha \chi dx \]
\[ \leq \int_{R_0 \leq |x| \leq 2R} a(x)u^\beta + q \chi dx + \int_{R_0 \leq |x| \leq 2R} b(x)u^{1/\tau} |\nabla u|^\alpha \chi dx. \]

Hence, it follows from \([1.37]\) that
\[ \int_{R_0 \leq |x| \leq 2R} a(x)^{1/\tau'} b(x)^{1/\tau} u^{(\beta + q)/\tau} \chi |\nabla u|^{\alpha/\tau} dx \leq \tilde{C} \int_{R_0 \leq |x| \leq 2R} |\nabla u|^{p-1} u^\beta |\nabla \chi| dx. \tag{1.45} \]

Now, defining
\[ \beta = \beta(\tau) = \frac{q(p-1)}{(\tau'-1)(\alpha - \tau p + \tau)}, \quad m = \frac{\beta \tau' + q}{\beta \tau'} \quad \text{and} \quad n = \frac{\beta \tau' + q}{q}, \]
we have \(1/m + 1/n = 1\) and
\[ \int_{R_0 \leq |x| \leq 2R} |\nabla u|^{p-1} u^\beta |\nabla \chi| dx = \]
\[ \int_{R_0 \leq |x| \leq 2R} a(x)^{\beta \tau' + q} b(x)^{\beta \tau'} u^{\beta \tau'} |\nabla u|^{p-1} a(x)^{-\beta \tau' - q} b(x)^{-\beta \tau'} |\nabla \chi| dx \leq \]
\[ C \left( \int_{R_0 \leq |x| \leq 2R} a(x)^{1/\tau} b(x)^{1/\tau} u^{(\beta + q)/\tau} \chi |\nabla u|^{p-1} dx \right)^{1/m} \left( \int_{R_0 \leq |x| \leq 2R} a(x)^{-\beta \tau' - q} b(x)^{-\beta \tau'} |\nabla \chi|^{n} dx \right)^{1/n}. \]

Since that \((p-1)m = \alpha/\tau\), it follows that
\[ \int_{R_0 \leq |x| \leq 2R} |\nabla u|^{p-1} u^\beta |\nabla \chi| dx \leq \]
\[ C \left( \int_{R_0 \leq |x| \leq 2R} a(x)^{1/\tau} b(x)^{1/\tau} u^{(\beta + q)/\tau} \chi |\nabla u|^{p} dx \right)^{1/m} \left( \int_{R_0 \leq |x| \leq 2R} a(x)^{-\beta \tau' - q} b(x)^{-\beta \tau'} |\nabla \chi|^{n} dx \right)^{1/n}. \]

Now, it follows from \([1.45]\), that
\[ \int_{R_0 \leq |x| \leq 2R} a(x)^{1/\tau} b(x)^{1/\tau} u^{(\beta + q)/\tau} \chi |\nabla u|^{p} dx \leq C \int_{R_0 \leq |x| \leq 2R} a(x)^{-\beta \tau' - q} b(x)^{-\beta \tau'} |\nabla \chi|^{n} dx, \]
that is,
\[
\int_{R_0 \leq |x| \leq 2R} a(x)^{\frac{1}{p}} b(x)^{\frac{1}{q}} u^{(\beta + \frac{\alpha}{q})} |\nabla u|^\frac{\alpha}{q} dx \leq C \int_{R \leq |x| \leq 2R} a(x)^{-\frac{p-1}{(\tau-1)(\alpha - \tau p + \tau)}} b(x)^{-\frac{p-1}{\alpha - \tau p + \tau}} \chi^{-\frac{\beta \tau}{q}} |\nabla \chi|^n dx.
\]

Since
\[
\chi^{\frac{\beta \tau}{q}} |\nabla \chi|^n \leq C R^{-n} = C R^{-\frac{\alpha}{\alpha - \tau p + \tau}},
\]

it follows that
\[
\int_{R_0 \leq |x| \leq 2R} a(x)^{\frac{1}{p}} b(x)^{\frac{1}{q}} u^{(\beta + \frac{\alpha}{q})} |\nabla u|^\frac{\alpha}{q} dx \leq C R^{-\frac{\alpha}{\alpha - \tau p + \tau}} \int_{R \leq |x| \leq 2R} [a(x)^{(\tau - 1)} b(x)]^{-\frac{p-1}{\alpha - \tau p + \tau}} dx.
\]

Now, given \( \theta \in (p - 1, \alpha) \), we take \( \tau = \tau_0 \in (1, \alpha/(p - 1)) \) such that \( \theta = (p - 1)\tau \). Thus,
\[
R^{-\frac{\alpha}{\alpha - \tau p + \tau}} \int_{R \leq |x| \leq 2R} [a(x)^{(\tau - 1)} b(x)]^{-\frac{p-1}{\alpha - \tau p + \tau}} dx = R^\frac{\alpha}{\alpha - \tau p + \tau} \int_{R \leq |x| \leq 2R} [a(x)^{\theta - 1} b(x)]^{-\frac{p-1}{\alpha - \tau p + \tau}} dx.
\]

So, making \( R \to \infty \) in (1.46), it follows from the hypothesis (ii) and the last equality, that
\[
\int_{|x| \geq R_0} a(x)^{\frac{1}{p}} b(x)^{\frac{1}{q}} u^{(\beta + \frac{\alpha}{q})} |\nabla u|^\frac{\alpha}{q} dx < \infty.
\]

Now, making the same arguments after (1.43) we arrive in a contradiction. So, we have finished the proof of theorem 1.2.

The next result, is a byproduct of Theorems 1.1 and 1.2. As a novelty there, in the existence issue, we have the presence of the term \( b \) that can change of sign. The case \( b = 0 \) is very studied. See for instance [7] with \( p = 2 \) and [30] for \( 1 < p < \infty \) and references therein.

**Corollary 5.1** Assume that \( a, b \in L^\infty_{\text{loc}}(\mathbb{R}^N) \) with \( a \geq 0 \) and \( \liminf_{t \to \infty} f(t)/t^q > 0 \) for some \( q > p - 1 \). Then the problem

\[
\begin{align*}
\Delta_p u &= a(x) f(u) + b(x), \quad \text{in} \ \mathbb{R}^N, \\
u &\geq 0 \ \text{on} \ \mathbb{R}^N, \ \ u(x) \xrightarrow{|x| \to \infty} \infty,
\end{align*}
\]

admits:

(i) at least one solution, if \( \int_1^\infty (r^{1-N} \int_0^r s^{N-1} \max\{a(x), b^+(x)\} ds)^{\frac{1}{p-1}} dr < \infty \) holds, and a satisfies (\( a_\Omega \)),

(ii) no solution, if \( b \geq 0, p < N, \int_1^\infty (r^{1-N} \int_0^r s^{N-1-\varepsilon} \min\{a(x)\} ds)^{\frac{1}{p-1}} dr = \infty \) and there exists \( \lim_{|x| \to \infty} |x|^{p-\varepsilon} a(x), \) for some \( \varepsilon \in (0, N - 1) \).

**Proof.** Assume (i) holds. In this case, we know that problem \( (P_\rho) \), with \( \rho(x) = \max\{a(x), b^+(x)\} \), \( x \in \mathbb{R}^N \), has a solution in \( C^1(\mathbb{R}^N) \). So, we adapt the proof of Theorem 3.1 taking \( \epsilon = \alpha = 0 \) and \( \mu = 1 \), for the problem

\[
\begin{align*}
\Delta_p u &= a(x) f(u) + b(x) \quad \text{in} \ B_n, \\
u &\geq 0 \ \text{in} \ B_n, \ \ u(x) \xrightarrow{|x| \to \partial B_n} \infty.
\end{align*}
\]

We point out that in this case, we can use a comparison principle of Tolksdorf [24], instead of that in [16]. Now, we apply the proof of Case 1 of Theorem 1.1 taking again \( \mu = 1 \). So following the same proceedings, we get the problem (1.47) has a solution.
Suppose (ii) holds. Since

\[ \int_1^\infty \left( r^{1-N} \int_0^r s^{N-1-\varepsilon} \min_{|x|=s} a(x) ds \right)^{\frac{1}{p-1}} dr = \infty \]

holds, it follows by a direct computation that \( \lim_{|x| \to \infty} |x|^{\delta} a(x) = +\infty \), for all \( p - \varepsilon < \delta < p \). Otherwise, there would be a \( p - \varepsilon < \delta_0 \leq p \) such that

\[ \int_1^\infty \left( r^{1-N} \int_0^r s^{N-1-\varepsilon} \min_{|x|=s} a(x) ds \right)^{\frac{1}{p-1}} dr \leq C \int_1^\infty s^{(1-\varepsilon-\delta_0)/(p-1)} ds < \infty \]

for some \( C > 0 \). That is impossible by hypothesis.

So, there exists a \( R_0 > 0 \) and \( \theta = \theta_\varepsilon \in (p-1,q) \) such that \( N + (pq - \delta\theta)/(\theta - q) < 0 \) and \( a(x)^{\theta/(\theta-q)} \leq C|x|^{-\alpha\theta/(\theta-q)} \) for \( |x| > R_0 \).

Now, just computing, we get

\[ \limsup_{R \to \infty} R^{\frac{q}{p-q}} \int_{R \leq |x| \leq 2R} a(x)^{\frac{q}{p-q}} dx = 0 < \infty. \]

So, from Theorem 1.2 we have the claimed. \( \square \)

**Corollary 5.2** Assume \( \rho \in C(\mathbb{R}^N) \) with \( \rho > 0 \) on \( \mathbb{R}^N \) is such that

\[ \limsup_{R \to \infty} R^{\frac{q}{p-q}} \int_{R \leq |x| \leq 2R} \rho(x)^{\frac{q}{p-q}} dx < \infty, \]

for some \( \theta \in (0,1) \) (1.48) holds. Then \( (P_\rho) \) has no solution in \( C^1(\mathbb{R}^N) \).

**Proof** Consider a \( \theta \in (0,1) \) satisfying (1.48). So, taking a \( q > (p-1)/\theta > p-1 \) and admitting that problem \( (P_\rho) \) admits a solution in \( C^1(\mathbb{R}^N) \), it follows from the proof of Corollary 5.1 with \( b = 0 \) and \( f(t) = t^2 \) that the problem

\[ \begin{cases} 
\Delta_p v = \rho(x)v^q & \text{in } \mathbb{R}^N, \\
v > 0 & \text{in } \mathbb{R}^N, \ u(x) \stackrel{|x| \to \infty}{\to} \infty,
\end{cases} \]

has a solution in \( C^1(\mathbb{R}^N) \).

Now, defining \( \tilde{\theta} = q/\theta \), we have \( \tilde{\theta} \in (p-1,q) \) and

\[ \limsup_{R \to \infty} R^{\frac{q}{\tilde{\theta}-q}} \int_{R \leq |x| \leq 2R} \rho(x)^{\frac{q}{\tilde{\theta}-q}} dx = \limsup_{R \to \infty} R^{\frac{q}{p-q}} \int_{R \leq |x| \leq 2R} \rho(x)^{\frac{q}{p-q}} dx < \infty. \]

So, by Theorem 1.2, it follows that problem (1.49) does not have solution in \( C^1(\mathbb{R}^N) \), but this a contradiction. \( \square \)

As examples of non-null and non-negative potentials \( \rho \) and \( b \) satisfying Theorem 1.2 and Corollary 5.2 \( (\rho = a \text{ in Theorem 1.2}) \), we have (the first two cases below satisfy Theorem 1.2.1 \( (i_1) \) and Corollary 5.2 and third case satisfies Theorem 1.2 \( (i_2) \)).

(i) \( \rho \in L^\infty_{loc}(\mathbb{R}^N) \) such that \( \liminf_{|x| \to \infty} |x|^\delta \rho(x) > 0 \) for some either \( \delta < p \) or \( \delta = p \geq N \),

(ii) \( \rho : \mathbb{R}^N \to [0,\infty) \) is continuous function with \( \lim_{|x| \to \infty} |x|^{p-\varepsilon} \tilde{\rho}(|x|) \geq 0 \) satisfying

\[ \int_1^\infty \left( r^{1-N} \int_0^r s^{N-1-\varepsilon} \tilde{\rho}(s) ds \right)^{\frac{1}{p-1}} dr = \infty, \]

for some \( \varepsilon \in (0,N-1) \), where \( \tilde{\rho}(r) = \min_{|x|=r} \rho(x) \) and \( 1 < p < N \),

(iii) \( b \in L^\infty_{loc}(\mathbb{R}^N) \) is such that \( \liminf_{|x| \to \infty} |x|^\delta b(x) > 0 \) for some \( -\infty < \delta < N - \alpha(N-1)/(p-1) \).
Referências

[1] Keller J. B., On solution of $\Delta u = f(u)$, Communication of Pure and Applied Mathematics 10 503-510 (1957)

[2] Osserman R., On the inequality $\Delta u \geq f(u)$, Pacific Journal of Mathematics 7 1641-1647 (1957)

[3] Lair A. V.; Wood A. W., Large solutions of sublinear elliptic equations, Nonlinear Analysis 39 745-753 (2000)

[4] Hamydy A.; Massar M. and Tsouli N., Existence of blow-up solutions for a non-linear equation with gradient term in $\mathbb{R}^N$, Journal of Mathematical Analysis and Applications 377, 161-169 (2011)

[5] Bachar I; Zeddini R., On the existence of positive solutions for a classe of semilinear elliptic equations, Nonlinear Analysis 52 1239-1247 (2003)

[6] Ghergu M. and Radulescu V., Nonradial blow-up solutions of sublinear elliptic equations with gradient term, Communication of Pure and Applied Mathematics 3 no.3,465-474 (2004)

[7] Ye D.; Zhou F., Existence and nonexistence of entire large solutions for some semilinear elliptic equations, J. Partial Differential Equations 21, 253-262 (2008)

[8] Taliaferro S. D., Radial symmetry of large solutions of nonlinear elliptic equations, Proceedings of the American Mathematical Society 124 no. 2,447-455 (1996)

[9] Bandle C.; Giarrusso E., Boundary blow up for semilinear elliptic equations with nonlinear gradient terms, Advances in Differential Equations 1, 133-150 (1996)

[10] Zhang Z., Existence of large solutions for a semilinear elliptic problem via explosive sub-supersolutions, Electronic Journal of Differential Equations, No. 02, 1-8 (2006)

[11] Huang S.; Li W-T.; Tian Q.; Mu C., Large solution to nonlinear elliptic equation with nonlinear gradient terms, Journal of Differential Equations 25, 3297-3328 (2011)

[12] Hamydy A., Existence and Uniqueness of Nonnegative Solutions for a Boundary Blow-up Problem, Journal of Mathematical Analysis and Applications, 534-545 (2010)

[13] Lair A.; Wood A. W., Large Solutions of Semilinear Elliptic Equations with Nonlinear Gradient Terms, International Journal of Mathematics and Mathematical Sciences, Vol. 22 no.4, 869-883 (1999)

[14] Mitidieri E.; Pohozaev S. I., A Priori Estimates and The Absence of Solutions of Nonlinear Partial Differential Equations and Inequalities, Proc. Steklov Inst. Math. 234 1-362 (2001)

[15] Felmer P., Quaas A. and Sirakov B., Solvability of Nonlinear Elliptic Equations With Gradient Terms, Journal of Differential Equations, no. 11, 4327-4347 (2013)

[16] Pucci P.; Serrin J., The Maximum Principle, Progress in Nonlinear Differential Equations and Their Applications (2007)

[17] Dai Q.; Peng L., Necessary and sufficient conditions for the existence of nonnegative solutions of inhomogeneous $p$-Laplace equation, Acta Mathematica Scientia, 27 B(1): 34-56 (2007)
[18] Li X.; Li F., Nonexistence of solutions for singular quasilinear differential inequalities with a gradient nonlinearity, Nonlinear Analysis 75, 2812-2822 (2012)

[19] Armstrong S. N.; Sirakov B., Nonexistence of positive supersolutions of elliptic equations via the maximum principle, Communications in Partial Differential Equations 36 no.11, 2011-2047 (2011)

[20] Alarcón S.; García-Melian J.; Quaas A., Keller-Osserman type conditions for some elliptic problems with gradient terms, Journal of Differential Equations 252, 886-914 (2012).

[21] Serrin J.; Zou H., Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities, Acta Math., 189, 79-142 (2002)

[22] Stavrouakis N.M.; Zographopoulos N.B., Multiplicity and regularity results for some quasilinear elliptic systems on $\mathbb{R}^N$, Nonlinear Analysis 50, 55-69 (2002)

[23] Lieberman G., Boundary Regularity for Solutions of Degenerate Elliptic Equations, Nonlinear Analysis 12, no. 11, 1203-1219 (1988)

[24] Kura T., The Week Supersolution-Subsolution Method for Second Order Quasilinear Elliptic Equations, Hiroshima Math. J., 19, 1-36 (1989)

[25] Tolksdrof P., On the Dirichlet problem for quasilinear equations in domains with conical boundary points. Commu. Partial Differential Equations 8, 773-817 (1983)

[26] Mohammed A., Existence and asymptotic behavior of blow-up solutions to weighted quasilinear equations. Journal of Mathematical Analysis and Aplications, 621-637 (2004)

[27] Liu C.; Yang Z., Existence of large solutions for a quasilinear elliptic problem via explosive sub-supersolutions. Applied Mathematics and Computation, 199 414-424 (2008)

[28] Perera K. Perera; Zhang Z., Multiple Positive Solutions of Singular $p$-Laplacian Problems by Variacional Methods, Boundary Value Problems 3, 377-382 (2005)

[29] García-Mélián J., Large Solutions for Equations Involving the $p$-Laplacian and Singular Weights, Zeitschrift für angewandte Mathematik und Physik ZAMP 60, 594-607 (2009)

[30] Goncalves J. V.; Zhou J., Remarks on existence of large solutions for $p$—Laplacian equations with strongly nonlinear terms satisfying the Keller-Osserman condition. Advanced Nonlinear Studies, vol. 10, no. 04, 757-770 (2010)