Dynamic Intrusion Detection in Resource-Constrained Cyber Networks

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Abstract—We consider a large-scale cyber network with \( N \) components (e.g., paths, servers, subnets). Each component is either in a healthy state (0) or an abnormal state (1). Due to random intrusions, the state of each component transits from 0 to 1 over time according to certain stochastic process. At each time, a subset of \( K (K < N) \) components are checked and those observed in abnormal states are fixed. The objective is to design the optimal scheduling for intrusion detection such that the long-term network cost incurred by all abnormal components is minimized. We formulate the problem as a special class of Restless Multi-Armed Bandit (RMAB) process. A general RMAB suffers from the curse of dimensionality (PSPACE-hard) and numerical methods are often inapplicable. We show that, for this class of RMAB, Whittle index exists and can be obtained in closed form, leading to a low-complexity implementation of Whittle index policy with a strong performance. For homogeneous components, Whittle index policy is shown to have a simple structure that does not require any prior knowledge on the intrusion processes. Based on this structure, Whittle index policy is further shown to be optimal over a finite time horizon with an arbitrary length. Beyond intrusion detection, these results also find applications in queuing networks with finite-size buffers.

I. INTRODUCTION

The objective of Intrusion Detection Systems (IDS) is to locate malicious activities (e.g., denial of service attack, port scans, hackers) in the quickest way such that the infected parts can be timely fixed to minimize the overall damage to the network. With the increasing size, diversity, and interconnectivity of the cyber system, however, intrusion detection faces the challenge of scalability: how to rapidly locate intrusions and anomalies in a large dynamic network with limited resources. The two basic approaches to intrusion detection, namely, active probing and passive monitoring [1], [2], face stringent resource constraints when the network is large and dynamic. Specifically, active-probing based approaches need to choose judiciously which components of the network to probe to reduce overhead; passive-monitoring based approaches need to determine how to sample the network so that real-time processing of the resulting data is within the computational capacity of the IDS [3]. The problem is compounded by the fact that the adversarial behaviors are typically random and evolving.

In this paper, we address resource-constrained intrusion detection in large dynamic cyber networks. Specifically, we consider a network with \( N \) heterogeneous components which can be paths, routers, or subnets. At a given time, a component can be in a healthy state or an abnormal state. An abnormal component remains abnormal until the anomaly is detected and resolved. A healthy component may be attacked and become abnormal if the attack is successful. We consider a general attack model: the behavior of the intruder can be arbitrarily correlated in time and varies across components, and different attacks can be launched with different probabilities of successfully compromising the component under attack. As a consequence, the state of a component evolves according to an arbitrary stochastic process until it is probed/sampled. When a healthy component is probed/sampled, its state evolution (i.e., how likely it will become abnormal in each subsequent time instant) is reset. This models the scenario where proactive actions are taken (patches are installed, firewalls upgraded, etc.) by the IDS when probing/sampling a component to refresh its immunity to attacks. Note that this model is significantly different and more complicated than the SIS (susceptible-infected-susceptible) model and its variants (see, e.g., [4]).

For each component in an abnormal state, a cost (depending on the criticality of the component) per unit time is incurred. At each time, the IDS can choose a subset of \( K \) components to probe or sample (\( K \) is often much smaller than \( N \) due to resource constraints). The question here is how to dynamically probe or sample these \( N \) components to minimize the long-term cost over time. The key is to learn from past observations and decisions and dynamically adjust the probing/sampling actions.

A. Main Results

We formulate the dynamic intrusion detection problem as a special class of Restless Multi-Armed Bandit (RMAB) process, where each component is considered as an arm. While finding the optimal solution to a general RMAB problem is PSPACE-hard with exponential complexity in system size [5], we show that for this class of RMAB at hand, several structural properties exist that lead to simple robust solutions. Specifically, by exploring the reset nature of the problem, we first show that a sufficient statistic for choosing the optimal probing/sampling actions is given by a two-dimensional vector of each arm that can be easily updated at each time. This significantly reduces the state space for optimal decision making. Second, we show that this RMAB is indexable, thus an index policy—referred to as Whittle index policy—with strong performance and linear complexity in the size \( N \) of the cyber network can be constructed. Third, we show that the Whittle index can be obtained in closed form, leading
to negligible complexity of implementation. Fourth, we show that for homogeneous components, the low-complexity Whittle index policy has a simple robust structure that does not need any prior knowledge on the stochastic attack model and achieves the optimal performance.

In the context of RMAB, our results contribute to the study of the existence and optimality of Whittle index policy. In 1988, Whittle generalized the classic MAB to RMAB, a more powerful stochastic model to take into account system dynamics that cannot be directly controlled. Whittle proposed an index policy that has been shown to be asymptotically (when the system size approaches infinity) optimal under certain conditions. The difficulty of Whittle index policy lies in the complexity of establishing its existence (the so-called indexability) and computing the index. There is no general characterization regarding which class of RMAB is indexable, and little is known about the optimality of Whittle index (when it does exist) for finite-size systems. In this paper, we present a significant class of indexable RMAB with practical applications for which Whittle index policy is shown to be optimal for homogeneous arms. This result lends a strong justification for the existence and the optimality of linear complexity algorithms based on the Whittle index. Beyond intrusion detection, this special class of RMAB and the corresponding results can also be applied to the holding cost minimization problem in queuing networks with finite-size buffers, as elaborated in Sec. VII.

B. Related Work

In [9], the problem of intrusion recognition by classifying system patterns was addressed based on data mining. Without resource constraint, the focus is on the best selection of system features to detect intrusion from the accessible system data statistics. Similar problems of statistical modeling of data and detection algorithms under various scenarios were considered in a number of papers, e.g., [10]–[13]. These studies mainly address the intrusion detection problem from a machine learning or pattern recognition perspective and do not consider the constraint on the system monitoring capacity. Our work is a stochastic control approach for intrusion detection in large networks with resource constraints, where the problem of how to adaptively allocate the limited detecting and repair power for performance optimization is of great interest. In [14], a set of heuristic classification, path selection and link anomaly localization algorithms were proposed based on the active probe-enabled network measurements. In [15], the intrusion detection problem was formulated as a zero-sum game with two players (the intruder and the IDS), where the game evolutions and outcomes were studied through numerical examples based on Markovian decision processes and Q-learning. The previous algorithm designs mainly take into account the static or Markovian dynamics of the networks. The results in this paper thus represent a step forward over the previous work by addressing the general non-Markovian network dynamics.

In the literature of RMAB, the indexability was studied in [16], where efficient algorithms were constructed to numerically test indexability and compute Whittle index for finite-state systems. For the problem at hand, the system state space is infinite, and thus numerical methods are generally infeasible, even for a fixed realization of system parameters. We show that, however, indexability holds regardless of the system parameters and Whittle index can be solved in closed-form. The optimality of Whittle index policy was subsequently established for homogeneous arms. For a special class of RMAB as detailed in the next paragraph, the optimality of Whittle index policy was established for homogeneous arms under certain conditions. In general, the optimality of Whittle index policy has rarely been established. Nevertheless, numerical studies have demonstrated the near-optimality of Whittle index policy for numerous RMAB models (see, e.g., [17]–[20]).

In the context of dynamic spectrum access and multi-agent tracking systems, a class of RMAB modeled by a two-state Markovian model was considered in [21], [22]. The indexability was established and Whittle index was solved in closed form. The Markovian model yields special structural properties of the system dynamic equations that significantly simplify the establishment of the indexability and Whittle index. However, these structural properties no longer hold for the RMAB considered here that deals with arbitrary underlying random processes, and the approaches in [21], [22] do not apply. In this paper, we propose a new approach for establishing the indexability and the closed-form Whittle index based on a comparing argument on the optimal stopping times. Besides the RMAB model at hand, this approach is extendable to general two-state reset processes with partially observable states. In [23], Whittle index policy was shown to be equivalent to the myopic policy for homogeneous arms, which leads to its optimality under certain conditions based on the previous results on the myopic policy established in [23]–[25]. Again, the approaches in [23]–[25] are based on the special properties, e.g., the linearity of the value function, of the myopic policy under the Markovian model. For the problem at hand, although the equivalence between Whittle index policy and the myopic policy is preserved for homogeneous arms, the properties under the Markovian model no longer hold. To show the optimality, we take a different approach by establishing the monotonicity of the value function, as detailed in Sec. VII.

II. NETWORK MODEL

Consider a cyber network with \( N \) inhomogeneous components that are subject to random attacks over time. At each discrete time, each component is either in the healthy state (0) or the abnormal state (1). If an attack to a healthy component is successful, the component enters the abnormal state until it is probed and fixed. We assume that different components experience statistically independent but not necessarily identical attack processes.

Each attack process can be arbitrarily correlated over time. Consequently, the state evolution of a component is given by an arbitrary probability sequence \( \{p_n(t)\}_{t \geq 0} \), where \( p_n(t) \) is the probability that component \( n \) enters state 1 after \( t \) steps since the last time it was probed. Specifically, if a component...
(say, component $n$) is probed and observed in state 0, a simple maintenance action is taken which resets its state evolution according to $\{p_n(t)\}_{t \geq 0}$. If component $n$ is observed in state 1, a sophisticated repair action is taken, and the component will be back to the normal state in the next time instant and then evolve according to $\{p_n(t)\}_{t \geq 0}$. Note that $\{p_n(t)\}_{t \geq 0}$ is a monotonically increasing sequence since state 1 is absorbing when the component is unobserved. A simple example is given by the i.i.d. attack process, where component $n$ is compromised with a constant probability $q_n \in (0,1)$ at each time. For this example, the state of component $n$ transits as a Markov chain shown in Fig. 1 and we have

$$p_n(t) = 1 - (1-q_n)^t,$$

which monotonically converges to 1 at the geometric rate $(1-q_n)$ as $t$ increases. In general, we do not require any specific form of $\{p_n(t)\}_{t \geq 0}$.

For each abnormal component (say, component $n$), a cost $c_n$ is incurred per unit time. With limited resource, only a subset of $K$ ($K < N$) components can be probed for maintenance/repair. The objective is to minimize the long-term average network cost by designing the optimal sequential component probing policy.

### III. RMAB FORMULATION

In this section, we formulate the intrusion detection problem as a special class of Restless Multi-Armed Bandit (RMAB) process. The concepts of indexability and Whittle index are also introduced.

#### A. RMAB and Sufficient Statistics

In a general RMAB, a player chooses $K$ out of $N$ independent arms to activate at each time based on the current states of all arms. At each time, the state of each arm transits according to two potentially different Markovian rules depending on whether it is made active or passive. Each arm contributes an immediate reward depending on its current state and the imposed action. The objective is to maximize the long-term reward by optimally selecting arms to activate over time based on the arm state evolutions.

We need to note that the states of all arms are assumed to be completely observable and obey Markovian transition rules in an RMAB. However, for the intrusion detection problem at hand, the state (0/1) of each component is not observable unless it is probed, and the state transition rules are non-Markovian in general. It is thus not suitable to model the component state as the arm state. By exploring the reset nature of the problem, we show in the next lemma that a sufficient statistic for optimal decision making is given by the two-dimensional vector set $\{(i_n, t_n)\}_{n=1}^N$, where $i_n \in \{0,1\}$ is the last observed state of component $n$ and $t_n$ the time lapsed since the last observation. As a consequence, we can treat $(i_n, t_n)$ as the arm state of component $n$, which is complete observable but with an infinite dimension. In the rest of paper, we refer to $(i_n, t_n)$ as the arm state of component $n$ to distinguish it from the component state $S_n \in \{0/1\}$. We also let $a_n \in \{\text{active/probe (1)}, \text{passive/not probe (0)}\}$ denote the probing action on arm $n$.

**Lemma 1:** For the intrusion detection problem, the vector set $\{(i_n, t_n)\}_{n=1}^N$ is a sufficient statistics for optimal decision making. Furthermore, given the current probing actions and observations, the arm state $(i_n, t_n)$ of component $n$ transits according to the following Markovian rules.

$$\Gamma(i_n, t_n) = \begin{cases} (0,1), & \text{if } a_n = 1, S_n = 0 \\ (1,1), & \text{if } a_n = 1, S_n = 1 \\ (i_n, t_n + 1), & \text{if } a_n = 0 \end{cases}$$

where $\Gamma(\cdot)$ denotes the one-step transition of the arm state given the current arm state and action.

**Proof:** Recall that each active action on each component (say, component $n$) resets its state evolution according to the probability sequence $\{p_n(t)\}_{t \geq 0}$ (see Sec. III). Given $(i_n, t_n)$, the future state statistics of component $n$ is independent of previous actions and observations. The vector set $\{(i_n, t_n)\}_{n=1}^N$ is thus a sufficient statistic. The one-step update of $\{(i_n, t_n)\}_{n=1}^N$ is straightforward.

Now we complete the RMAB formulation of the intrusion detection problem by observing that the immediate reward $R_n(S_n)$ offered by component $n$ can be modeled by $-c_n$ if it is currently in the abnormal state and 0 otherwise. Consequently, the reward maximization is equivalent to the cost minimization. In the rest of the paper, we use RMAB-IDS to denote this class of RMAB.

#### B. The Optimality Equation

In this subsection, we establish the optimality equation for RMAB-IDS. We consider the following strong average-reward criterion under which not only the steady-state average reward but also the transient reward starting from an arbitrary initial arm state is maximized, leading to the maximum long-term total reward growth rate.

$$G + F(\{(i_n, t_n)\}_{n=1}^N) = \max_{\mathcal{A}} \mathbb{E}_\mathcal{A} \left[ \sum_{n=1}^N R_n(S_n) \right]$$

(1)

$$+ F(\{\Gamma(i_n, t_n|a_n, S_n)\}_{n=1}^N),$$

where $\mathcal{A} = \{a_n\}_{n=1}^N$ with $\sum_{n=1}^N a_n = K$ denotes the current probing actions, $G$ the maximum steady-state average reward over the infinite horizon, $F(\cdot)$ the transient reward starting from the initial arm states, and $\mathbb{E}_\mathcal{A} [\cdot]$ the expectation operator.
given \( A \). Solving the optimality equation (1) suffers from the curse of dimension and has an exponential complexity for dynamic programming. In Sec. IV, we show that for RMAB-IDS, the linear-complexity Whittle index policy exists and can be obtained in closed form with a near-optimal performance.

C. Definition of Whittle Index Policy

The key idea of Whittle index policy is to provide a subsidy for passivity to measure the attractiveness of activating an arm based on its current state. Based on the strong decomposability of Whittle index, it is sufficient to focus on each single arm [6].

1) Single-Armed Bandit with Subsidy: Consider the single-armed bandit for the intrusion detection problem with only one arm/component. At each time instant, we decide whether to activate the arm or make it passive. Assume that a subsidy for passivity, denoted by \( \lambda \), is gained whenever the arm is made passive. We have the following optimality equations. For simplicity of presentation, we will drop the component index from the notations.

\[
g + f(0, t) = \max \{ \lambda - p(t)c + f(0, t + 1), \\
- p(t)c + p(t)f(1, 1) + (1 - p(t))f(0, 1) \} \\
g + f(1, t) = \max \{ \lambda + f(0, t + 1), \\
p(t)f(1, 1) + (1 - p(t))f(0, 1) \}
\]

where \( g \) and \( f(\cdot) \) denote, respectively, the maximum steady-state average reward and the transient reward by playing the single arm. The optimal policy for this single-arm problem is essentially given by an optimal partition of the arm state space \( \bigcup_{i=0,1} \{(i, t)\}_{t \geq 1} \) into a passive set

\[
\mathcal{P}(\lambda) = \{(i, t) : a^*(i, t, \lambda) = 0\}
\]

and its complement, an active set \( \mathcal{A}(\lambda) = \{(i, t) : a^*(i, t, \lambda) = 1\} \), where \( a^*(i, t, \lambda) \) denotes the optimal action at arm state \( (i, t) \) under subsidy \( \lambda \).

2) Indexability and Whittle Index: To define Whittle index policy, it is required that the RMAB is indexable [6].

**Definition 1:** An RMAB is indexable if for each arm, the passive set \( \mathcal{P}(\lambda) \) increases monotonically from the empty set \( \phi \) to the entire state space \( \bigcup_{i=1,2} \{(i, t)\}_{t \geq 1} \) as the subsidy \( \lambda \) increases from \(-\infty\) to \( +\infty \). An RMAB is strictly indexable if the states join the passive set one by one (instead of as groups) as \( \lambda \) continuously increases.

Given the indexability, the Whittle index \( W(i, t) \) of an arm state \( (i, t) \) is defined as the infimum subsidy \( \lambda \) that makes the passive action optimal at \( (i, t) \):

\[
W(i, t) = \inf \{ \lambda : a^*(i, t, \lambda) = 0 \} \\
= \inf \{ \lambda : \lambda + f(i, t + 1) \geq p(t-i)f(1,1) + (1-p(t-i))f(0,1) \}
\]

Whittle index essentially measures how attractive it is to activate an arm based on subsidy \( \lambda \). The minimum subsidy \( \lambda \) that is needed to move an arm state from the active set to the passive set under the optimal partition thus measures how attractive this arm state is.

Whittle index policy is naturally given by playing the \( K \) arms with the largest Whittle indexes.

IV. INDEXABILITY AND THE CLOSED-FORM WHITTLE INDEX FOR RMAB-IDS

In this section, we establish the indexability of RMAB-IDS and solve for Whittle index in closed form. Based on the indexability and Whittle index, we study the optimal policy for RMAB-IDS under a relaxed constraint.

A. Indexability

**Theorem 1:** RMAB-IDS is indexable.

**Proof:** Consider the single-armed bandit with subsidy. Without loss of generality, we assume that the cost \( c = 1 \). Define stopping time \( t_i \) as the number of steps until the first activation after observing the arm in component state \( i \in \{0,1\} \). We can rewrite the dynamic equations (2) and (3) as follows.

\[
f(0) = \max_{t_0 \geq 1} \{-gt_0 + \lambda(t_0 - 1) - \sum_{k=1}^{t_0} p(k) + p(t_0)f(1) + (1 - p(t_0))f(0)\},
\]

\[
f(1) = \max_{t_1 \geq 1} \{-gt_1 + \lambda(t_1 - 1) - \sum_{k=1}^{t_1} p(k - 1) + p(t_1 - 1)f(1) + (1 - p(t_1 - 1))f(0)\},
\]

where \( f(i) (i \in \{0,1\}) \) is the transient reward starting from arm state \( (i,0) \). Note that we can set \( f(0) = 0 \) since only \( f(1) - f(0) \) is determined by the above equations. We thus have

\[
0 = \max_{t_0 \geq 1} \{-gt_0 + \lambda(t_0 - 1) - \sum_{k=1}^{t_0} p(k) + p(t_0)f(1)\},
\]

\[
f(1) = \max_{t_1 \geq 1} \{-gt_1 + \lambda(t_1 - 1) - \sum_{k=1}^{t_1} p(k - 1) + p(t_1 - 1)f(1)\}.
\]

To prove indexability, it is equivalent to prove that the optimal \( \{t_i^*\}_{i=0,1} \) in (4) and (5) are nondecreasing with \( \lambda \). For the case that \( \lambda < 0 \), all states are in the active set, i.e., \( t_i^* = 1 \) for \( i \in \{0,1\} \). This is because that both the time portion of the occurrence of the abnormal component state and the passive time are minimized by always activating the arm.

Consider the case that \( \lambda \geq 0 \). We should always make the arm passive if the observation of the component state in the previous slot is 1, since the current component state is guaranteed to be 0 after repair and there is no benefit to observe it again. Consequently, \( t_i^* \geq 1 \). Combined with (4) and (5), we further observe that \( t_i^* = t_0^* + 1 \). Note that this
holds not only for the optimal stopping times \( \{t_i^*\}_{i=0,1} \) but also for all stationary policies with \( t_i > 1 \). By considering \( t_i^* \) in \((4)\) and \((5)\), we can solve for \( f(1) \) and \( g \) and obtain
\[
g = \frac{\lambda(t_0^* - 1 + p(t_0^*)) - \sum_{k=1}^{t_0^*} p(k)}{t_0^* + p(t_0^*)}.
\]

Now suppose it is better to activate the arm at the \( t \)th step instead of any earlier step after observing component state \( s \). The Whittle index function is subsequently obtained.

\[
C_1, \text{ RMAB-IDS is strictly indexable. The closed-form Whittle index is also for all stationary policies with} \ \text{in} \ (4) \text{ and } (5), \ \text{we can solve for} \ f(1) \ \text{and} \ g \ \text{and obtain}
\]
\[
g = \frac{\lambda(t_0^* - 1 + p(t_0^*)) - \sum_{k=1}^{t_0^*} p(k)}{t_0^* + p(t_0^*)}.
\]

We can further simplify \( f(7) \) and obtain for all \( s \in \{1, \ldots, t\} \),
\[
\lambda(t - s + p(t) - p(s)) \geq \sum_{k=1}^{t} p(k)(s + p(s)) - \sum_{k=1}^{s} p(k)(t + p(t)).
\]

Based on the monotone property of \( \{p(t)\}_{t \geq 0} \), we have \( t - s + p(t) - p(s) \geq 0 \) and hence \( \lambda(s) \geq 0 \). Equivalently, the set of \( t \) for which \( f(7) \) and \( f(8) \) are true is nondecreasing in \( \lambda \). We thus conclude that \( \{t_i^*(\lambda)\}_{i=0,1} \) are nondecreasing in \( \lambda \). Since this further implies that \( \mathcal{P}(\lambda) \) is nondecreasing in \( \lambda \), we proved the indexability.

**B. The Closed-Form Whittle Index**

Given the indexability established in Sec III-A, we proceed to solve for the closed-form Whittle index of RMAB-IDS. For simplicity of presentation, we focus on the case that the bandit is strictly indexable (see Definition 1), i.e., there is no tie among the Whittle indexes. A simple condition in the following is adopted to guarantee the strict indexability.

**Cl: p(t + 1) − p(t) \text{ is strictly decreasing with} \ t.**

Note that Cl is always satisfied under the Markovian state model (see Sec III). As shown in the following theorem, under Cl, RMAB-IDS is strictly indexable. The closed-form Whittle index function is subsequently obtained.

**Theorem 2:** Under Cl, RMAB-IDS is strictly indexable and the Whittle index \( W(s) \) is given below.

\[
W(0, t) = \frac{p(t + 1)(t + p(t))}{1 + p(t + 1) - p(t)} - t \sum_{k=1}^{t} p(k),
\]

\[
W(1, t) = W(0, t - 1), \quad W(0, 0) \triangleq 0.
\]

**Proof:** We first prove the following lemma that establishes a sufficient and necessary condition for strict indexability and the associated Whittle index.

**Lemma 2:** Define \( W(0, t) \) as in \((9)\). RMAB-IDS is strictly indexable if and only if \( W(0, t) \) is strictly increasing with \( t \). In this case, the Whittle index of state \((i, t) \ (i \in \{0, 1\}) \) is given by \((9)\) and \((10)\).

**Proof:** Without loss of generality, we assume that the cost \( c = 1 \). We first prove the necessity. If the bandit is strictly indexable, the states \( \{(0, t)\}_{t \geq 1} \) join the passive set one by one as \( \lambda \) continuously increases. From the proof of Theorem 1 after observing component state \( 0 \), it is optimal to activate the arm at the \( t \)-th step under subsidy \( \lambda \) if and only if
\[
\lambda \geq \frac{d(t, s)}{c(t, s)}, \forall s < t, \quad (11)
\]
\[
\lambda \leq \frac{d(u, t)}{c(u, t)}, \forall u > t, \quad (12)
\]

Consider an arbitrary \( v \geq 1 \). If both \((11)\) and \((12)\) hold with equality by letting \((u, t, s) = (v + 2, v + 1, v) \) and \( \lambda = W(0, v) \), then Whittle indexes for states \((0, v)\) and \((0, v + 1)\) would be the same. This contradicts the strict indexability. We thus have that \( d(v + 1, v)/c(v + 1, v) \) is strictly increasing at \( v \).

Now we prove the sufficiency. Assume that \( W(0, t) \) is strictly increasing with \( t \). This implies that \( W(0, t) \) is positive for all \( t \) since
\[
W(0, 1) = p(2) - p(1) + p^2(1) > 0.
\]

For an arbitrary \( v \geq 1 \), there must exist a subsidy \( \lambda > 0 \) such that both \((11)\) and \((12)\) hold with strict inequality by letting \((u, t, s) = (v + 2, v + 1, v) \). So the Whittle index for state \((0, v)\) is smaller than this \( \lambda \) while the Whittle index for state \((0, v + 1)\) is larger than it. This proves the strict indexability.

Under the strict indexability, if we set the subsidy \( \lambda_c \) as the Whittle index of state \((0, t)\), then it is optimal to either activate on \((0, t)\) or wait one more step to activate on \((0, t + 1)\). We thus have
\[
\lambda_c c(t + 1, t) = d(t + 1, t),
\]

which leads to the Whittle index of state \((0, t)\) as given in \((9)\). Recall that for any nonnegative subsidy, the optimal activation time after observing component state 1 is one step later compared to that after observing component state 0. We arrive at \( W(1, t) = W(0, t - 1) \) for \( t \geq 2 \). Based on the proof of Theorem 1 it is not hard to see that \( W(1, 1) = 0 \). We thus proved the lemma.

Based on Lemma 2 we only need to prove that Cl implies the strict monotonicity increasing property of \( W(0, t) \). Equivalent, for any \( t \geq 1 \), we need to prove
\[
\frac{d(t + 2, t + 1)}{c(t + 2, t + 1)} > \frac{d(t + 1, t)}{c(t + 1, t)}.
\]

Define \( \delta(t) \triangleq p(t + 1) - p(t) \) which is positive under Cl. By simplifying \((14)\), it is equivalent to prove
\[
p(t + 1)\delta(t) + p^2(t + 1)\delta(t) + \delta(t)\delta(t + 1) + p(t + 1) + p^2(t + 1) + \delta(t + 1)(t + 1) > p(t)\delta(t + 1) + p^2(t)\delta(t + 1) + \delta(t)\delta(t + 1)p(t) + p(t) + \delta(t)p(t) + \delta(t). \]

(15)
Since \( p(t) \) is increasing and \( \delta(t) \) is strictly decreasing with \( t \) (under C1), we have
\[
\begin{align*}
p(t+1)\delta(t) + p^2(t+1)\delta(t) + \delta(t)\delta(t+1) & > p(t)\delta(t+1) + p^2(t)\delta(t+1) + \delta(t)\delta(t+1)p(t).
\end{align*}
\] (16)
To prove (15), it is sufficient to prove
\[
\begin{align*}
p(t+1)t + p^2(t+1) + \delta(t+1)(t+1) & > p(t)t + p^2(t) + \delta(t)p(t) + \delta(t)t.
\end{align*}
\] (17)
After some simplifications of (17), we need to prove
\[
\delta(t)p(t+1) + \delta(t+1)(t+1) > 0,
\] which is always true under C1. We thus proved Theorem 2.

The near-optimal performance of Whittle index policy is observed through numerical examples (see Sec. VI). In Sec. V, we show that when all components are homogeneous, Whittle index policy is equivalent to the myopic policy and achieves the optimal performance.

C. The Optimal Policy under a Relaxed Constraint

In this subsection, we consider the scenario with a relaxed resource constraint, where we only require the average number of activated arms to be no more than \( K \). This scenario often arises in systems where the resource constraint is more strict on the average value rather than the peak value, e.g., the energy-saving systems. Under the relaxed constraint, the indexability and the Whittle index leads to a simple optimal policy for RMAB-IDS.

As explained by Whittle in [6], the subsidy \( \lambda \) for passivity is essentially the Lagrangian multiplier for the general RMAB with the following relaxed constraint
\[
\mathbb{E}_{\pi}\left[\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} K(t)\right] \leq K,
\] (19)
where \( K(t) \) is the number of activated arms at time \( t \).

Specifically, the subsidy \( \lambda \) controls the expected time portion, i.e., the stead-state probability \( \pi_n(\lambda) \), that arm \( n \) (\( 1 \leq n \leq N \)) is made active under the corresponding single-arm optimal policy. For RMAB-IDS, under the optimal subsidy \( \lambda^* \), we have
\[
\mathbb{E}_{\pi^*}\left[\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} K(t)\right] = \sum_{n=1}^{N} \pi_n(\lambda^*) = K
\] (20)
and (19) is satisfied with equality.

Given the optimal subsidy \( \lambda^* \), the optimal policy under the relaxed constraint is simply given by the composition of \( N \) independent single-arm optimal policies (applied on the \( N \) arm respectively) under the common subsidy \( \lambda^* \). Specifically, at each time, if the Whittle index of an arm is larger than \( \lambda^* \) then we activate the arm; otherwise we make the arm passive. Note that if the Whittle index of an arm is equal to \( \lambda^* \), randomizing between the active and passive actions would be necessary to satisfy (20) as detailed in [7]. Given the closed-form Whittle index established in Theorem 2, it remains to solve for the optimal subsidy \( \lambda^* \). Note that based on the Lagrangian multiplier theorem [6], we have
\[
\lambda^* = \arg \min_{\lambda} \left\{ \sum_{n=1}^{N} g_n(\lambda) - (N - K)\lambda \right\},
\] (21)
where \( g_n(\lambda) \) is the maximum average reward of arm \( n \) under the single-arm policy for subsidy \( \lambda \) and is convex in \( \lambda \). From the closed-form Whittle index, it is not hard to solve for the optimal stopping times \( \{t_1^*(\lambda)\}_{\lambda=0.1} \) (see [6]) and the maximum average reward \( g(\lambda) \) for each \( \lambda \). We can then obtain the optimal \( \lambda^* \) from (21) by any classic algorithm for finding the minimum of a convex function.

V. Optimality in Homogeneous Networks

In this section, we study the performance of Whittle index policy in homogeneous networks, i.e., all components have the same parameters: the probability sequence \( \{p(t)\}_{t \geq 0} \) and the per-unit cost \( c \) for being abnormal.

We first establish the equivalence of Whittle index policy with the myopic policy for homogeneous components. In general, the myopic policy chooses the \( K \) components to solely minimize the expected cost in the next slot. It is not hard to show that for homogeneous components, the myopic policy is reduced to choosing the \( K \) components with the largest probabilities of being in the abnormal state. The myopic action \( A(\cdot) \) as a function of the current states of all arms is thus given below.
\[
\hat{A}(\{i_n, t_n\}_{n=1}^{N}) = \arg \max_{A} \left\{ \sum_{n=1}^{N} \Pr(S_n = 1 | (i_n, t_n)) \right\}
\] (22)

Lemma 3: For homogeneous components, Whittle index policy is equivalent to the myopic policy and has the following simple structure: initialize a queue in which components are ordered according to the descending order of their initial probabilities of being in the abnormal state. Each time we probe the \( K \) components at the head of the queue. In the next slot, these \( K \) components will be moved to the bottom of the queue while keeping those observed in state 1 a higher position than those observed in state 0.

Proof: Based on the proof of Theorem 1, the Whittle index \( W(i, t) \) of an arm is monotonically increasing with \( t \) for fixed \( i \in \{0, 1\} \) and \( W(1, t) = W(0, t-1) \) with \( W(0, 0) = 0 \). Based on the monotonic increasing property of \( \{p(t)\}_{t \geq 0} \), it is not hard to see that the Whittle index \( W(i, t) \) is monotonically increasing with \( \Pr(S = 1 | (i, t)) \). Whittle index policy is thus equivalent to the myopic policy for homogeneous arms.

From the equivalence of Whittle index policy with the myopic policy, its structure is straightforward since based on the current observations, all components observed in state 1 will have zero probability of being abnormal and those observed in state 0 will have the second smallest probability \( p(1) \) of being abnormal, while those unobserved arms will...
have the same rank in the probability of being abnormal in the next slot based on the monotonicity of \( \{p(t)\}_{t \geq 0} \).

From Lemma 5 Whittle index policy can be implemented without knowing the system parameters \( \{p(t)\}_{t \geq 0} \) and \( c \). Furthermore, Whittle index policy is optimal, as given in the following theorem.

**Theorem 3:** For homogeneous components, Whittle index policy minimizes the expected total cost over a finite time horizon of an arbitrary length \( T \) \( (T \geq 1) \). It is thus also optimal under the strong average-reward criterion over the infinite time horizon.

**Proof:** We prove the theorem based on a backward induction on the time horizon. Any policy, including Whittle index policy, is optimal at the last time instant \( t = T \) since the current action affects only the future cost but not the immediate cost. Now assume that Whittle index policy is optimal at time instants \( t+1, t+2, \ldots, T \). We need to prove that it is optimal at time \( t \). Without loss of generality, we set \( c = 1 \). Let \( \Omega(t) = (\omega_1, \omega_2, \ldots, \omega_N) \) with \( \omega_n \in \{p(t)\}_{t \geq 0} \) denote an unordered set consisting of probabilities that the \( N \) components are in state 1 at time \( t \). Define the value function \( V_t(\Omega(t)) \) of Whittle index policy as the expected total cost from time \( t \) up to \( T \). Consider a policy that activate the \( K \) components with probabilities \( (\omega_1, \omega_2, \ldots, \omega_K) \) of being in state 1 at time \( t \) and follows Whittle index policy in the future time instants up to \( T \). The value function \( \hat{V}_t(\omega_1, \omega_2, \ldots, \omega_N) \), i.e., the expected total cost from time \( t \) up to \( T \), of this policy is given by

\[
\hat{V}_t(\omega_1, \omega_2, \ldots, \omega_N) = \sum_{k=1}^{N} \omega_k + \mathbb{E}[V_{t+1}(0, \ldots, 0, p(1), \ldots, p(1), \ldots, \tau(\omega_N))],
\]

where the expectation is taken over the random variables \( \{k_i\}_{i=1}^{K} (k_i + k_0 = K) \) that denote respectively the number of components observed in state 1 and state 0, and \( \tau(\cdot) \) denote the one-step update of the abnormal probability for unobserved components based on \( \{p(t)\}_{t \geq 0} \). Note that if \( \omega_1 \geq \omega_2 \geq \cdots \omega_N \), then \( \hat{V}_t = V_t \).

To prove that Whittle index policy, i.e., the myopic policy, is optimal at time \( t \), it is sufficient to prove that for any \( y \geq x, x, y \in \{p(t)\}_{t \geq 0} \),

\[
\hat{V}_t(\omega_1, \ldots, y, \ldots, x, \ldots, \omega_N) \leq \hat{V}_t(\omega_1, \ldots, x, \ldots, y, \ldots, \omega_N).
\]

This means that a component with higher probability of being in state 1 should be given a higher priority. To show (23), we first present the following lemma that establishes the monotonicity of the value function of Whittle index policy.

**Lemma 4:** The value function \( V_t(\omega_1, \omega_2, \ldots, \omega_N) \) of Whittle index policy is an increasing function at each entry \( \omega_n \) \( (n \in \{1, 2, \ldots, N\}) \).

**Proof:** Without loss of generality, we assume that all probabilities within \( V_t(\cdot) \) are in a descending order. The proof is based on a backward induction on time \( t \). If \( t = T \), the claim is clearly true. Assume that the lemma holds for \( s = t + 1, t + 2, \ldots, T \). Consider time \( t \). We need to show

\[
V_t(\omega_1', y, \omega_2') \geq V_t(\omega_1, x, \omega_2), \quad \forall \ y \geq x, \ x, y \in \{p(t)\}_{t \geq 0}, \quad (24)
\]

where \( \omega_1', \omega_2' \) are arbitrary (possibly empty) probability vectors with \( |\omega_1'| + |\omega_2'| = N - 1 \).

Define \( t_1 \geq 1 \) as the first stopping time that the component denoted by \( y/x \) in (23) is probed under Whittle index policy. Based on the structure of Whittle index policy, \( t_1 \) is deterministic. We have

\[
V_t(\omega_1', y, \omega_2') = A(\omega_1', \omega_2') + \sum_{k=1}^{t_1} \tau^{k-1}(y) + \mathbb{E}[\tau^{t_1-1}(y)V_{t+1}(\omega_1', 0, \omega_2')]
\]

where \( A(\omega_1', \omega_2') \) is the expected total cost up to \( t_1 \) determined by components other than that denoted by \( y/x \), vectors \( \omega_1', \omega_2' \) are stochastically determined by \( \omega_1, \omega_2 \) based on the observations between time \( t \) and \( t + t_1 - 1 \), and \( \tau^{k}(\cdot) \) denotes the \( k \)-th iteration of operator \( \tau(\cdot) \). We point out that based on the structure of Whittle index policy, the total cost \( A(\omega_1', \omega_2') \) does not depend on the state of the component denoted by \( y/x \). From (23) and (26), we have that (24) holds if

\[
\sum_{k=1}^{t_1-1} (\tau^{k-1}(y) - \tau^{k-1}(x)) + \tau^{t_1-1}(y) - \tau^{t_1-1}(x) \mathbb{E}[1 + V_{t+1}(\omega_1', 0, \omega_2') - V_{t+1}(\omega_1', p(1), \omega_2')] \geq 0. \quad (27)
\]

From the monotonic increasing property of \( \{p(t)\}_{t \geq 0} \), we have

\[
\tau^{k}(y) - \tau^{k}(x) \geq 0, \quad \forall \ y \geq x, \ x, y \in \{p(t)\}_{t \geq 0}, \quad k \geq 0.
\]

To show (27), it is sufficient to show

\[
\mathbb{E}[1 + V_{t+1}(\omega_1', 0, \omega_2') - V_{t+1}(\omega_1', p(1), \omega_2')] \geq 0. \quad (28)
\]

Starting from time \( t + t_1 \), define \( t_2 \) as the first stopping time that the component denoted by \( 0/p(1) \) is probed under Whittle index policy. Between time \( t + t_1 \) to \( t + t_1 + t_2 \), the difference in the expected total cost incurred by this component when its abnormal probabilities are respectively given by \( 0 \) and \( p(1) \) is equal to \( p(t_2) \). This is because that the update of the abnormal probability when staring from 0 is one step lagged of that from \( p(1) \). Again, based on the structure of Whittle index policy, the expected total cost incurred by other components is independent of the state of this component. By expanding the value function in (28) at time \( t + t_1 + t_2 \) and after some simplifications, it is equivalent to show

\[
\mathbb{E}[1 - p(t_2) + (p(t_2) - 1 - p(t_2))] V_{t+t_1+t_2}(\omega_1''', 0, \omega_2'') - V_{t+t_1+t_2}(\omega_1', p(1), \omega_2'') \geq 0, \quad (29)
\]
where vectors $\vec{\omega}_1', \vec{\omega}_2'$ are stochastically determined by $\vec{\omega}_1, \vec{\omega}_2$ based on observations between time $t + t_1$ and $t + t_1 + t_2 - 1$. By induction, for any $\vec{\omega}_1'', \vec{\omega}_2''$,

$$V_{t+t_1+t_2}(\vec{\omega}_1'', 0, \vec{\omega}_2'') - V_{t+t_1+t_2}(\vec{\omega}_1', p(1), \vec{\omega}_2') \leq 0.$$ 

It is thus not hard to see that (29) holds. Note that for the realizations of $t_1$ and $t_2$ such that $t + t_1 > T$ and/or $t + t_1 + t_2 > T$, the monotonicity of the conditional value function is straightforward to prove. We thus proved the lemma. 

Now we are ready to prove (23). If the positions of $y$ and $x$ are both in top $K$ or both after top $K$, then the inequality holds with equality. Consider the case that $y$ is in top $K$ but $x$ not. We have for any probability vectors $\{\vec{\omega}_i\}_{i=1,2,3}$,

$$\hat{V}_t(\vec{\omega}_1, y, \vec{\omega}_2, x, \vec{\omega}_3) = \mathbb{E}[yV_{t+1}(\vec{\omega}_1', \vec{\omega}_3', 0) + (1 - y)V_{t+1}(\vec{\omega}_2', \vec{\omega}_3', p(1))] \leq \mathbb{E}[\max(xV_{t+1}(\vec{\omega}_1', \vec{\omega}_2', \vec{\omega}_3', 0), (1 - x)V_{t+1}(\vec{\omega}_2', \vec{\omega}_3', p(1))] = \hat{V}_t(\vec{\omega}_1, x, \vec{\omega}_2, y, \vec{\omega}_3),$$

where $\vec{\omega}_1', \vec{\omega}_2', \vec{\omega}_3'$ are stochastically determined by $\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3$ based on the observation at time $t$, and the two inequalities are due to Lemma 4. We thus proved the optimality of Whittle index policy over a finite horizon of an arbitrary length $T$. By contradiction, if Whittle index policy is not optimal under the strong average-reward criterion, there must exist a $T_0$ such that Whittle index policy performs worse than the optimal policy over the horizon of length $T_0$. Consequently, Whittle index policy is also optimal under the strong average-reward criterion over the infinite time horizon.

### VI. Numerical Examples

In this section, we present some numerical examples and evaluate the performance of Whittle index policy for nonhomogeneous components.

In Fig. 2 we illustrate the Whittle index as a function of the arm state. The monotonicity and concavity of the Whittle index are observed. In Fig. 3 we compare the performance of Whittle index policy versus the optimal policy. Due to the complexity of the dynamic programming problem given in (1), we only computed the optimal cost over a short time horizon. Note that the cost under the non-stationary optimal policy over a finite time horizon is a lower bound on that achieved by the stationary optimal policy over the infinite time horizon. We observe that Whittle index policy achieves a near-optimal performance.

In Fig. 4 we compare Whittle index policy with the myopic policy over a long time horizon. We observe that for inhomogeneous components, Whittle index policy outperforms the myopic policy, and the performance improvement becomes significant as time goes.

Numerical results similar to the above have been observed through extensive examples with randomly generated system parameters.

### VII. Applications to Queueing Networks

Another application of the RMAB model considered in this paper is on holding cost minimization in queueing networks. Consider a queueing network where customers randomly arrive at $K$ servers. As shown in Fig. 5 all servers share a set of $N$ finite-size buffers (for $N$ different classes of customers) that are either empty or full based on the batch arrivals. We assume that new customers of a class do not arrive if the corresponding server is full. At each time, each server chooses one buffer to serve and clear its packets. The objective is to minimize the holding cost ($e.g.$, delay) of the customers. By likening a customer arrival to an attack, it is not hard to see that the problem can be modeled as the RMAB at hand under certain conditions, $e.g.$, when the arrival process of each class is i.i.d. or Markovian over time (given the buffer is empty). Such a queueing network often arises in backorder control systems and peer-to-peer communication networks. For example, in a backorder control system, random orders for $N$ commodities arrive at a seller and the seller needs
In this paper, we studied the intrusion detection problem in large cyber networks under general attack processes. By adopting a reset model of the network dynamics, we formulated the problem as a class of RMAB under a strong average-reward criterion. We showed that this class of RMAB is indexable and Whittle index can be solved in closed-form. This result leads to a low-complexity implementation of Whittle index policy that achieves a near-optimal performance. We further showed that for homogeneous components, Whittle index policy can be implemented without knowing the system parameters and is optimal over both finite and infinite time horizons.

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