ON THE LOCATING-CHROMATIC NUMBERS OF SUBDIVISIONS OF FRIENDSHIP GRAPH

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Abstract. Let \(c\) be a \(k\)-coloring of a connected graph \(G\) and let \(\pi = \{C_1, C_2, \ldots, C_k\}\) be the partition of \(V(G)\) induced by \(c\). For every vertex \(v\) of \(G\), let \(c_\pi(v)\) be the coordinate of \(v\) relative to \(\pi\), that is \(c_\pi(v) = (d(v, C_1), d(v, C_2), \ldots, d(v, C_k))\), where \(d(v, C_i) = \min\{d(v, x) | x \in C_i\}\). If every two vertices of \(G\) have different coordinates relative to \(\pi\), then \(c\) is said to be a locating \(k\)-coloring of \(G\). The locating-chromatic number of \(G\), denoted by \(\chi_L(G)\), is the least \(k\) such that there exists a locating \(k\)-coloring of \(G\). In this paper, we determine the locating-chromatic numbers of some subdivisions of the friendship graph \(F_{rt}\), that is the graph obtained by joining \(t\) copies of 3-cycle with a common vertex, and we give lower bounds to the locating-chromatic numbers of few other subdivisions of \(F_{rt}\).

Keywords: friendship graph, locating-chromatic number, locating coloring, subdivision

1. INTRODUCTION

The concept of locating-chromatic number was first studied by Chartrand et al. [1] by combining the concept of graph partition dimension and graph coloring. The locating-chromatic numbers of some classes of graphs were studied, especially recently for certain Barbell graphs in [2], Halin graphs in [3], and graphs resulting from certain operations of other graphs, such as join of graphs in [4] and Cartesian product of graphs in [5]. Trees with certain locating-chromatic number were also studied in [6] and [7]. Bounds for locating-chromatic numbers of trees and subdivisions of graph on one edge were also established in [8] and [9], respectively.

Suppose that \(G = (V, E)\) is a simple connected graph. Let \(c\) be a \(k\)-coloring on \(G\) and let \(\pi = \{C_1, C_2, \ldots, C_k\}\) be the partition of \(V = V(G)\) induced by \(c\). For every vertex \(v\) of \(G\), let \(c_\pi(v) = (d(v, C_1), d(v, C_2), \ldots, d(v, C_k))\) be the coordinate...
of $v$ relative to $\pi$, where $d(v, C_i) = \min\{d(v, x)|x \in C_i\}$ is the shortest distance between $v$ and vertices in $C_i$. If every two vertices of $G$ have different coordinates relative to $\pi$, then $c$ is said to be a locating $k$-coloring of $G$. The locating-chromatic number of $G$, denoted by $\chi_L(G)$, is the least $k$ such that there exists a locating $k$-coloring of $G$. As shown by Chartrand et al. in [1], if $u$ and $v$ are vertices of $G$ such that $d(u, v) = d(v, w)$ for every $w \in V - \{u, v\}$, then $c(u) \neq c(v)$.

In [4], Behroozi and Anbarloei studied the locating chromatic number of friendship graph $Fr_t$, which is the graph obtained by joining the complete graph $K_1$ to the $t$ disjoint copies of $K_2$. They showed that $\chi_L(Fr_t) = 1 + \min\{|k|t \leq \binom{n}{2}\}$ in this paper, we study some subdivisions of $Fr_t$ and their locating-chromatic numbers. In general, a subdivision of a graph $G$ is a graph obtained by replacing some edges of $G$, say $e_1, e_2, \ldots, e_r$, respectively with paths $P_1, P_2, \ldots, P_r$ of length one or greater, where these paths may differ in length. In particular, when we say a subdivision of a graph on some edges $l > 0$ times, we are specifying which or how many edges are replaced and ensuring the paths replacing the edges are all of length $l + 1$. Purwasih et al. [9] showed that $\chi_L(G) \leq 1 + \chi_L(H)$ if $G$ is a subdivision of a graph $H$ on one edge. We investigate the case where $H = Fr_t$ by determining the locating-chromatic number of any subdivision of $Fr_t$ on one edge and also the locating-chromatic number of any subdivision of $Fr_t$ once on each of its cycle. We also give a tight upper bound for any subdivision of $Fr_t$. Throughout this paper, for $t \geq 2$ we denote the center of $Fr_t$, that is the vertex with the largest degree, by $z$. For every natural number $n$, we also denote $[n] = \{1, 2, \ldots, n\}$.

2. MAIN RESULTS

In this section, we determine the locating-chromatic number of any subdivision of $Fr_t$ on one edge. We also determine the locating-chromatic number of any subdivision of $Fr_t$ once on each of its cycle.

2.1. Subdivision of $Fr_t$ on one edge. Throughout this subsection, let $t \geq 2$ and $l \geq 1$ be two natural numbers and let $G$ be a subdivision of $Fr_t$ on one edge $l$ times. For each $n \geq 3$, we define $d_n = \binom{n-1}{2} + 1$. Observe that if $t \geq 3$, we have $d_k - 1 < t \leq d_k$ for some $k \in \{4, 5, 6, \ldots\}$. We begin with the following lemmas.

Lemma 1. If $3 \leq t = d_k$ and $l = 2$, then $\chi_L(G) = k + 1$.

Proof. By definition of $G$, there are exactly $d_k - 1$ number of 3-cycles and a 5-cycle in $G$. Consider the collection of 2-subsets of $[k - 1]$, denoted by $[k - 1]^2$. Since $|[k - 1]^2| = \binom{k-1}{2}$, we can denote the elements of $[k - 1]^2$ by $u_1, u_2, \ldots, u_{\binom{k-1}{2}}$.

Now, we start by assigning colors to the vertices of $G$. We immediately assign the color $k + 1$ to the vertex $z$. To assign colors to other vertices, observe that since there are $d_k - 1 = \binom{k-1}{2}$ number of 3-cycles, there are $2\binom{k-1}{2}$ vertices other than $z$ that lie on a 3-cycle. We denote these vertices by $v_1, v_2, \ldots, v_{2\binom{k-1}{2}}$, where $v_{2i-1}$ and $v_{2i}$ are on the same 3-cycle for each $i = 1, 2, \ldots, \binom{k-1}{2}$. If we write
To finish the color assignment, let the 5-cycle in $G$ be $zw_1w_2w_3w_4z$. Assign the colors 1, $k+1$, $k$, and 1 respectively to $w_1$, $w_2$, $w_3$, and $w_4$. Let $c$ be the obtained coloring. Clearly, $c$ is a well-defined graph coloring since no two adjacent vertices are assigned the same color.

We show that $c$ is a locating coloring. Let $x$ and $y$ be two vertices with the same color. If $x$ and $y$ are in the same cycle, then the only possibilities are, without loss of generality, either $(x,y) = (z, w_2)$ or $(x,y) = (w_1, w_4)$. However, in both of these cases, the $k$-th component of the coordinate of $x$ and $y$ differ since $2 = d(x, w_3) \neq d(y, w_3) = 1$ and $w_3$ is the only vertex colored $k$.

Let us now assume that $x$ and $y$ are in different cycles. If both are in different 3-cycles, clearly their coordinates differ since their neighbors other than $z$ have different colors by definition of $u_1, u_2, \ldots, u_{k-1}$. If, without loss of generality, $x$ is in a 5-cycle and $y$ is in a 3-cycle, then either $x = w_1$ or $x = w_4$ since the colors $k$ and $k+1$ are not assigned to $y$. In both cases, however, their neighbors other than $z$ have different colors. Hence, their coordinates differ. Thus, we have shown that $c$ is a locating coloring and that $\chi_L(G) \leq k + 1$.

We now show that $\chi_L(G) > k$ by contradiction. Suppose that there exists a locating $k$-coloring $c'$ for $G$. Suppose that $c'(z) = k$. Hence, without loss of generality, the pair of vertices $\{v_{2i-1}, v_{2i}\}$ has to be assigned by the pair of colors $u_i = \{a_i, b_i\}$ for each $i = 1, 2, \ldots, (k-1)$. Moreover, without loss of generality, let $c'(w_1) = 1$. If $c'(w_2) \neq k$, then we let $c'(w_2) = m \in \{2, 3, \ldots, k-1\}$. However, there are two vertices $v_p$ and $v_{p+1}$ such that $(c'(v_p), c'(v_{p+1})) = (1, m)$. Observe that $d(w_1, w) = d(v_p, w)$ for any vertex $w$ that is assigned by any color other than 1. Hence, $w_1$ and $v_p$ have the same coordinate, contradicting the definition of a locating coloring. Thus, we must have $c'(w_2) = k$. However, by the same argument, we must also have $c'(w_3) = k$, which contradicts the definition of coloring. Thus, we have shown that $\chi_L(G) > k$ and we conclude that $\chi_L(G) = k + 1$. \hfill \blacksquare

**Lemma 2.** If $3 \leq t = d_k$ and $l \neq 2$, then $\chi_L(G) = k$.

**Proof.** Since $t = d_k$ and $l \neq 2$, there are exactly $d_k - 1$ number of 3-cycles and an $(l + 3)$-cycle in $G$. We start by coloring $G$. Assign the color $k$ to the vertex $z$ and assign colors to the vertices lying in 3-cycles other than $z$ by using the same way used in the proof of the previous lemma. Consider these cases.

- a. Suppose that the $(l + 3)$-cycle is $zs_{1}s_{2} \ldots s_{4q-2}z$ for some $q$. Assign the color $k$ to $s_2, s_4, \ldots, s_{4q-2}$. Assign the color 1 to $s_1, s_3, \ldots, s_{2q-1}$. Assign the color 2 to $s_{2q+1}, s_{2q+3}, \ldots, s_{4q-1}$. Let $c$ be the resulting coloring. Clearly, $c$ is a well-defined coloring. We show that $c$ is a locating coloring. Let $x$ and $y$ be two vertices with the same color. If $x$ and $y$ are in the same cycle, then both have to be in the $(l + 3)$-cycle. If both are colored $k$ and their neighbors are only vertices of color 1, then their coordinates differ by their distances to a vertex colored 3 since $t = 3$, or other colors other than 1 and 2. It is also the case when their neighbors are only vertices of color 2. If
their neighbors are vertices of color 1 and 2, one of them is $z$ and the other one is $s_{2q}$. In this case, their coordinates also differ by their distances to a vertex colored other than 1 and 2. The same argument also applies if the color of $x$ and $y$ are 1 or 2. Moreover, if $x$ is in the $(l+3)$-cycle and $y$ is in a 3-cycle without loss of generality, then $x$ and $y$ are not colored $k$.

However, the neighbors of $x$ are only vertices colored $k$, while some of the neighbors of $y$ are not colored $k$. Hence, their coordinates differ. Thus, $c$ is a locating $k$-coloring.

d. Suppose that the $(l+3)$-cycle is $zs_1s_2 \ldots s_{4q-3}z$ for some $q$. Assign the color $k$ to $s_2, s_4, \ldots, s_{4q-4}$. Assign the color 1 to $s_1, s_3, \ldots, s_{2q-1}$. Assign the color 2 to $s_{2q+1}, s_{2q+3}, \ldots, s_{4q-3}$. Let $c$ be the resulting coloring. Clearly, $c$ is a well-defined coloring. We show that $c$ is a locating coloring. Let $x$ and $y$ be two vertices with the same color. By using the same argument as in part a, we see that $c$ is indeed a locating $k$-coloring.

c. Suppose that the $(l+3)$-cycle is a $(2q-1)$-cycle $zs_1s_2 \ldots s_{2q-2}z$. Assign the color $k$ to $s_2, s_4, \ldots, s_{2q-6}$ and $s_{2q-3}$. Assign the color 1 to $s_1, s_3, \ldots, s_{2q-5}$. Assign the color 2 to $s_{2q-2}$ and $s_{2q-4}$. Let $c$ be the resulting coloring. Clearly, $c$ is well-defined. The argument to show that $c$ is indeed a locating $k$-coloring is similar to part a or part b with minor difference, that is if $x$ and $y$ are two vertices of color $k$ other than $z$, then their neighbors are either only vertices of color 1 or only vertices of color 2.

Thus, we have $\chi_L(G) \leq k$. We now show that $\chi_L(G) > k-1$ by contradiction. Suppose that there exists a $(k-1)$-coloring $c'$ for $G$. Let $c'(z) = k-1$. Since $z$ is adjacent to all vertices in the 3-cycles, the colors of those vertices have to be in $[k-2]$. However, there are more than $\binom{k-2}{2}$ number of 3-cycles in $G$, while the cardinality of $[k-2]^2$ is $\binom{k-2}{2}$. Hence, by the pigeon-hole principle, there exist two pairs of vertices lying in 3-cycle, say $\{a_1, b_1\}$ and $\{a_2, b_2\}$, where their elements are different, such that $\{c'(a_1), c'(b_1)\} = \{c'(a_2), c'(b_2)\}$. Let $a_1$ and $b_1$ are colored the same as $a_2$ and $b_2$, respectively. However, the distances of $a_1$ and $a_2$ to a vertex colored other than $c'(a_1) = c'(a_2)$ is equal. This means their coordinates are equal, contradicting the definition of locating coloring. Thus, we have $\chi_L(G) > k-1$ so that $\chi_L(G) = k$.

Lemma 3. If $3 \leq t < d_k$, then $\chi_L(G) = k$.

Proof. We start by coloring the graph $G$. Assign the color $k$ to $z$. Assign colors to vertices lying in 3-cycles other than $z$ by using the same way used in the proof of the first lemma, that is by taking different elements in the set $[k-1]^2$ as pairs of colors for pairs of vertices in each 3-cycle. Since $t < d_k$, there are less than $\binom{k-1}{2}$ number of 3-cycles, so that there exist elements in $[k-1]^2$ that are not used as a pair of color in any 3-cycle. Denote this element by $(g_1, g_2)$.

Suppose that the $(l+3)$-cycle in $G$ is $zs_1s_2 \ldots s_{t+2}z$. If $l+3$ is odd, use the colors $g_1, g_2, g_1, g_2, \ldots, g_1, g_2$ respectively to color $s_1, s_2, s_3, s_4, \ldots, s_{t+2}$. Otherwise, assign the color $k$ to the vertex $s_{\frac{t+3}{2}}$ and use the colors $g_1, g_2, g_1, g_2, \ldots$ respectively.
to color $s_1, s_2, s_3, \ldots, s_j$, where $j = \frac{l+3}{2} - 1$, and use the colors $g_2, g_1, g_2, g_1, \ldots$ respectively to color $s_{l+2}, s_{l+1}, s_l, s_{l-1}, \ldots, s_{j+2}$.

It is easy to see, by using the same argument as in the proof of previous lemma, that all vertices colored the same have different coordinates. In this case, vertices colored $g_1$ and $g_2$ create the differences. Hence, $\chi_{\ell}(G) \leq k$. The proof showing that $\chi_{\ell}(G) > k - 1$ is similar to the last paragraph of the proof of the last lemma. Thus, we have $\chi_{\ell}(G) = k$.

From previous three lemmas, we have the following theorem.

**Theorem 1.** Let $t \geq 3$ and $l \geq 1$ be two natural numbers. Let $G$ be a subdivision of $Fr_t$ on one edge $l$ times. For each $n \geq 3$, let $d_n = (n-1)/2 + 1$ and $d_{k-1} < t \leq d_k$ for some $k$. Hence, we have $\chi_{\ell}(G) = k + 1$ if $t = d_k$ and $l = 2$, and $\chi_{\ell}(G) = k$ otherwise.

We treat the case $t = 2$ separately in the next proposition.

**Proposition 1.** Let $l \geq 1$ be a natural number. If $G$ is a subdivision of $Fr_2$ on one edge $l$ times, then $\chi_{\ell}(G) = 4$.

**Proof.** We start by coloring $G$. Assign the color 4 to the vertex $z$. Assign the color 1 and 2 to the two vertices lying in the only existing 3-cycle. Now, denote the $(l + 3)$-cycle in $G$ by $zu_1u_2 \ldots u_{l+2}u_z$.

Assume first that $l$ is even. Assign the colors 1, 3, 1, 3, 1, 3 respectively to the vertices $u_1, u_2, u_3, u_4, \ldots, u_{l+2}$. By doing this, the vertices colored 3 have their coordinates differ by their distances to the vertex colored 4, and the vertices colored 1 have their coordinates differ by their distances to the vertex colored 2. Thus, we obtain a locating 4-coloring.

Assume now that $l$ is odd. Assign the color 4 to the vertex $u_{l+2}$. Assign the color 1 to each vertex of the form $u_1, u_3, \ldots, u_{l_1}$, where $l_1 < \frac{l+3}{2}$, and vertex of the form $u_{l_2}, \ldots, u_{l-1}, u_{l+1}$, where $l_2 > \frac{l+3}{2}$. Assign the color 3 to other remaining vertices. By doing this, the vertices colored 3 have their coordinates differ by their distances to the vertex colored 2, and so do the vertices colored 1. Thus, we obtain a locating 4-coloring.

We have shown that $\chi_{\ell}(G) \leq 4$. We now show that $\chi_{\ell}(G) > 3$. Suppose that there exists a locating 3-coloring on $G$. Without loss of generality, assume that the 3-cycle in $G$ is colored by 1, 2, and 3, where $z$ is assigned the color 3. Suppose that there exists a vertex colored by 2 in the $(l + 3)$-cycle in $G$. Let $j$ be the least index such that $u_j$ is colored by 2. If $j$ is odd, then the vertices $u_1, u_3, \ldots, u_{j-2}$ have to be colored by 1 since the color of $z$ is 3, and we must also have the vertices $u_2, u_4, \ldots, u_{j-1}$ colored by 3. However, the coordinate of $u_{j-1}$ is equal to the coordinate of $z$, contradicting the definition of locating coloring. If $j$ is even instead, then the vertices $u_1, u_3, \ldots, u_{j-1}$ have to be colored 1 since the color of $z$ is 3, and we must also have the vertices $u_2, u_4, \ldots, u_{j-2}$ colored by 3.
However, the coordinate of $u_{t-1}$ is equal to the coordinate of the vertex colored by 1 on the 3-cycle, contradicting again the definition of locating coloring. Hence, there must not be any vertex colored by 2 on the $(l + 3)$-cycle. This means that, since $z$ is colored by 3, $u_1, u_3, \ldots, u_{t+2}$ have to be colored by 1 and $u_2, u_4, \ldots, u_{t+1}$ have to be colored by 3. However, the vertices $u_1$ and $u_{t+2}$ have the same color and the same coordinate, contradicting the definition of the locating coloring. Thus, we have $\chi_L(G) = 4$.

\section*{2.2. Subdivision of $Fr_t$, once on one edge of each cycle.}

We now determine the locating-chromatic number of the subdivision of $Fr_t$ once on one edge of each cycle. This means that each cycle of the graph is a 4-cycle. Let $G$ be such graph, where $t \geq 2$.

\begin{theorem}
For each $n \geq 3$, let $e_n = \left\lceil \frac{n-1}{2} \right\rceil + (n - 1) \left\lceil \frac{n-2}{2} \right\rceil$ and $e_{k-1} < t \leq e_k$ for some $k$. Hence, we have $\chi_L(G) = k$.
\end{theorem}

\begin{proof}
We define a locating $\chi_L(G)$-coloring $c : V \to [k]$ on $G$. We first set $c(z) := k$.

Assume that $C(1), C(2), \ldots, C(t)$ are all of the 4-cycles in $G$ and denote $C(i)$ by $zu_1u_2u_3z$ for each $i$. Clearly, $k \geq 4$ since $2 \leq t \leq e_k$.

Define a 3-tuple $W'_i := (w'_i, w'_i, w'_i, w'_i)$ with $w'_i := 2i - 1, w'_{i+2} := k, w'_{i+3} := 2i$ for $i = 1, 2, \ldots, \lfloor \frac{k-1}{2} \rfloor$. Next, for $j = 1, 2, \ldots, (n - 1) \lfloor \frac{k-2}{2} \rfloor$, define a 3-tuple $W''_j := (w''_j, w''_j, w''_j, w''_j)$ with

\begin{align*}
& (w''_{i+1}, w''_{i+2}, w''_{i+3}) := (i + 1, i + 2, i + 3), \\
& (w''_{i+1}, w''_{i+2}, w''_{i+3}) := (i, i + 1, i + 2), \\
& (w''_{i+1}, w''_{i+2}, w''_{i+3}) := (i, i + 1, i + 2).
\end{align*}

for $i = 1, 2, \ldots, k - 1$, by noting that the components are calculated under modulo $k - 1$. Observe that in $W''_j$, there is no entry that is equal to $k$. Observe also that $W'_i$ and $W''_j$ never equal to each other since their second entries differ for any $i$ and $j$. By definition, we also see that $W'_{i+1}$ and $W''_{i+2}$ differ for any different $i$ and $j$, and that $W'_{i+1}$ and $W''_{j+2}$ differ for any different $j$ and $j$. Hence, if we write $W := \{W'_i | i = 1, 2, \ldots, \lfloor \frac{k-1}{2} \rfloor \} \cup \{W''_j | j = 1, 2, \ldots, (k - 1) \lfloor \frac{k-2}{2} \rfloor \}$, we have $|W| = e_k$. We can then write $W = \{W_1, W_2, \ldots, W_{e_k} \}$. Now, for each $i \in [t]$, define the coloring $c(C(i)) := (c(u_{i,1}), c(u_{i,2}), c(u_{i,3})) := W_i$.

From the definition of $W$, clearly $c$ is a $k$-coloring. We now show that $c$ is a locating coloring. Let $x$ and $y$ be two different vertices with the same color in $G$. If $x = z$, then $y = u_{t-1}$ for some $i$ such that $c(u_{t-1}) = k$, if it exists. However, since $t \geq 2$ and by definition of $c$, the vertex $x$ is adjacent to vertices with colors other than $c(u_{i,1})$ and $c(u_{i,3})$, while $y$ is only adjacent to vertices with these colors. Hence, the coordinates of $x$ and $y$ differ, so we assume that $x$ and $y$ are not $z$.

Now, let our $x$ and $y$ be in the 4-cycles $C(i_1)$ and $C(i_2)$, respectively, where $i_1$ and $i_2$ are two different elements of $[t]$. 

Let $c[C(i_1)]$ and $c[C(i_2)]$ both be in $\{W''_{j} | j = 1, 2, \ldots, \lfloor \frac{k-2}{2} \rfloor \}$. Hence, we must have $x = u_{i_1,2}$ and $y = u_{i_2,2}$, or vice-versa. However, the colors of the neighbors of $x$ clearly differ from the colors of the neighbors of $y$. Thus, their coordinates differ.

Let $c[C(i_1)]$ and $c[C(i_2)]$ both be in $\{W''_{j} | j = 1, 2, \ldots, (k-1) \lfloor \frac{k-2}{2} \rfloor \}$. There are some cases to consider. For the first case, if $x = u_{i_1,2}$ and $y = u_{i_2,2}$, then clearly they have different coordinates by looking at the colors of their neighbors. For the next case, if $x = u_{i_1,2}$ and $y = u_{i_2,3}$ (or $y = u_{i_3,2}$), then it is adjacent to $z$, which is a vertex colored $k$, but $x$ is not adjacent to any vertex colored $k$, so we know that their coordinates differ. For the last case, if $x = u_{i_1,1}$ and $y = u_{i_2,1}$ (without loss of generality), then, by definition of $c$, the colors of $u_{i_1,1}$ and $u_{i_2,1}$ differ, so that the colors of the neighbors of $x$ and $y$ also differ, and hence $x$ and $y$ have different coordinates.

Thus, we have shown that $c$ is a locating $k$-coloring, so that $\chi_L(G) \leq k$.

We now show that $\chi_L(G) > k-1$. Suppose that there exists a locating $(k-1)$-coloring on $G$, which we denote by $c'$. Assume that $c'(z) = k - 1$. We divide all of the 4-cycles into $k - 1$ types. Type $a$ consists of all 4-cycles $C(i) = x_{u_{i,1}} u_{i,2} u_{i,3} z$ with $c'(u_{i,2}) = a$. Observe that if there exist two 4-cycles of type $k-1$, say $C(i)$ and $C(j)$, where $c'(u_{i,1}) = c'(u_{j,1})$ without loss of generality, then the coordinates of $u_{i,1}$ and $u_{j,1}$ must be the same since both are adjacent only to two vertices colored $k-1$ and $d(u_{i,1}, u_{i,3}) = d(u_{j,1}, z) + d(z, u_{i,3}) = d(u_{j,1}, z) + d(u_{i,1}, u_{i,3}) = 2$. Moreover, $2 = d(u_{j,1}, u_{j,3}) = d(u_{i,1}, u_{j,3})$ and $d(u_{i,1}, x) = d(u_{j,1}, z) + d(z, x) = d(u_{j,1}, z) + d(z, x)$ for each vertex $x$ that is not $u_{i,1}, u_{i,2}, u_{i,3}, u_{j,1}, u_{j,2}, u_{j,3}$. Hence, since $u_{i,1}$ and $u_{j,1}$ must not be colored $k-1$, there are at most $\lfloor \frac{k-2}{2} \rfloor$ number of 4-cycles of type $k-1$ by the pigeon-hole principle.

Let $C(i)$ be a 4-cycle of type $b$ where $c'(u_{i,2}) = b \in [k - 2]$. By the similar observation to the previous paragraph, there are at most $\lfloor \frac{k-2}{2} \rfloor$ number of 4-cycles of type $b$. Hence, there are at most $(k-2) \lfloor \frac{k-2}{2} \rfloor$ number of 4-cycles of type other than $k-1$. Thus, by combining with the previous paragraph, there are at most $\epsilon_{k-1}$ number of 4-cycles in $G$. This contradicts the assumption on $t$. We conclude that $\chi_L(G) > k - 1$, so that $\chi_L(G) = k$.

2.3. Upper bound for arbitrary subdivision of Fr$_t$. We now study the upper bound for arbitrary subdivision of Fr$_t$. It is known that $\chi_L(G) \leq 1 + \chi_L(H)$ if $G$ is a subdivision of a graph $H$ on one edge. For $H = Fr_t$, this bound is strengthened.

**Theorem 3.** If $G$ is a subdivision of Fr$_t$ where $t \geq 2$, then we have $\chi_L(G) \leq \chi_L(Fr_t)$. Precisely, if $(k-2)/t < t \leq (k-1)/2$, we have $\chi_L(G) \leq k$. 
Proof. Clearly, \( k \geq 4 \). We construct a locating \( k \)-coloring \( c : V(G) \to [k] \) on \( G \). We start by setting \( c(z) := k \). Let \( C(1), C(2), \ldots, C(t) \) denote all the cycles in \( G \). We start with the first case where \( t = \binom{k-1}{2} \).

Suppose that there is no 4-cycle in \( G \). Write \( C(i) = w_{i,1}w_{i,2} \ldots w_{i,s(i)}w_{i,1} \) where \( s(i) \neq 4 \) and \( w_{i,1} = z \) for each \( i \). Assume that \( [k-1]^2 = \{u_1,u_2, \ldots, u_t\} \).

If \( s(i) \) is odd, then set \( c(w_{i,2}) := c(w_{i,4}) := \cdots := c(w_{i,s(i)-1}) := a_i \) and \( c(w_{i,3}) := c(w_{i,5}) := \cdots := c(w_{i,s(i)-2}) := b_i \), where \( \{a_i,b_i\} := u_i \). If \( s(i) \) is even, then set \( c(w_{i,2}) := c(w_{i,4}) := \cdots := c(w_{i,s(i)-1}) := a_i \), \( c(w_{i,s(i)-3}) := \cdots := c(w_{i,s(i)}) := a_i \), \( c(w_{i,5}) := \cdots := c(w_{i,s(i)}) := b_i \), \( c(w_{i,s(i)-2}) := b_i \), and \( c(w_{i,s(i)+1}) := k \), where \( \{a_i,b_i\} := u_i,s_1(i) < \frac{s(i)}{2} + 1, s_2(i) > \frac{s(i)}{2} + 1, s_3(i) < \frac{s(i)}{2} + 1, \) and \( s_4(i) > \frac{s(i)}{2} + 1 \). Observe that two adjacent vertices in \( G \) lie in a \( C(i) \). By definition of \( c \), those two vertices have different colors. Thus, \( c \) is a \( k \)-coloring.

Next, to show that \( c \) is locating coloring, let \( z_1 \) and \( z_2 \) be two different vertices having the same color in \( G \), that is \( c(z_1) = c(z_2) = a_0 \). If one of \( z_1 \) or \( z_2 \) is \( z \), then we know that, by definition of \( c \) and the fact that \( t \geq 2 \), \( z_1 \) is adjacent to at least 4 vertices which are two vertices in the cycle where \( z_2 \) belongs and two other vertices in another cycle, and that three of these four vertices have different colors. However, \( z_2 \) is only adjacent to at most two vertices with different colors.

Hence, the coordinates of \( z_1 \) and \( z_2 \) differ. Now, let \( z_1 \) and \( z_2 \) be vertices other than \( z \). Assume that both are in different cycles, say \( C(i_1) \) and \( C(i_2) \), respectively. If \( a_0 = k \), then \( C(i_1) \) and \( C(i_2) \) are cycles of even length by definition of \( c \). Again, by definition of \( c \), \( z_1 \) and \( z_2 \) are vertices that have their distances to \( z \) the greatest in the cycles containing them, so that \( z_1 \) is adjacent to two vertices colored \( a_{i_1} \) and \( b_{i_1} \), and \( z_2 \) is adjacent to two vertices colored \( a_{i_2} \) and \( b_{i_2} \), but \( u_{i_1} \neq u_{i_2} \). Thus, the coordinates of \( z_1 \) and \( z_2 \) are different. If \( a_0 \neq k \), then, since no cycle is a 4-cycle and each of \( z_1 \) and \( z_2 \) has a neighbor \( z'_{i_1} \) and \( z'_{i_2} \), respectively, that \( c(z'_{i_1}) \neq c(z'_{i_2}) \) by definition, the coordinates of \( z_1 \) and \( z_2 \) are different.

Now, let \( z_1 \) and \( z_2 \) be in the same cycle \( C(i) \), and both are not \( z \). By definition of \( c \), we have \( a_0 \neq k \). Again by definition of \( c \) and the fact that \( t \geq 2 \), there exists a vertex \( z' \) colored \( a'_0 \) outside of \( C(i) \) and no vertex in \( C(i) \) is colored \( a'_0 \). By the numbering of \( C(i) \), we have \( d(z_1,z') = d(z_2,z') = d(z_2,z) = d(z,z') = d(z_1,z') \). Hence, the coordinates of \( z_1 \) and \( z_2 \) differ. Thus, we have shown that \( c \) is a locating coloring and that \( \chi_L(G) \leq k \).

For the next case, suppose that there are \( q \) number of 4-cycles in \( G \). We show that \( \chi_L(G) \leq k \). Write \( q = (k-1)m + r \) where \( r \) and \( m \) are unique integers satisfying \( 0 \leq r < k-1 \) and \( m \geq 0 \) by the division algorithm. Let the 4-cycles be denoted by \( Q_1, Q_2, \ldots, Q_q \). Consider the complete graph \( H \) on the set \([k-1]\).

Assume that \( k \) is odd. We must have \( m \leq \frac{k-1}{2} - 1 \), which is a contradiction. Since \( k-1 \) is even, by decomposing \( H \) to obtain its Hamiltonian cycles and its 1-factors, there exist subgraphs \( H_1, H_2, \ldots, H_{\frac{k-1}{2}-1}, E_1, E_2, \ldots, E_{\frac{k+1}{2}} \) of \( H \). We continue by noting that \( H_i \) is
a Hamiltonian cycle and $H_i$ and $H_j$ are edge-disjoint subgraphs for each different $i$ and $j$, and that $E_i$ is a complete graph on two vertices and $E_i$ and $E_j$ are edge-disjoint subgraphs for each different $i$ and $j$.

For each $p \in [m]$, consider the $k-1$ number of 4-cycles $Q_{(k-1)(p-1)+1}$, $Q_{(k-1)(p-1)+2}$, \ldots, $Q_{(k-1)p}$. We define the coloring $c$ on these cycles that is associated with the subgraph $H_p$. Let us write $H_p = h_{p,1}, h_{p,2}, \ldots, h_{p,k-1}, h_{p,1}$. There exist three vertices $v_{p,j,1}, v_{p,j,2}, v_{p,j,3}$ that are not $z$ on $Q_{(k-1)(p-1)+j}$ where $j \in [k-1]$. Set $c(v_{p,j,1}) := h_{p,j}, c(v_{p,j,2}) := h_{p,j+1}, c(v_{p,j,3}) := h_{p,j+2}$, where $j$, $j+1$, and $j+2$ are calculated under modulo $k-1$. By this definition, adjacent vertices on these cycles have different colors.

For the case $r > \frac{k-1}{2} + 1$, we have $m < \frac{k-1}{2} - 1$, so that there exist a Hamiltonian cycle $H_{m+1}$ that have not yet been associated with the 4-cycles on the previous paragraph. We define the coloring $c$ on the 4-cycles $Q_{(k-1)m+1}, Q_{(k-1)m+2}, \ldots, Q_{(k-1)m+r}$ associated with the subgraph $H_{m+1}$. Similar to the previous paragraph, by changing the role of $p$ with $m+1$ and $j \in [r]$, and when $j = r$, we set $c(v_{m+1,j,2}) := k$, we see that adjacent vertices on these cycles have different colors.

For the case $r \leq \frac{k-1}{2}$, consider the $r$ number of 4-cycles $Q_{(k-1)m+1}, Q_{(k-1)m+2}, \ldots, Q_{(k-1)m+r}$. We define a coloring $c$ on the cycle $Q_{(k-1)m+p}$ for each $p \in [r]$ associated with the subgraph $E_p$. Assume that $E_p = e_{p,1}e_{p,2}$. There are three vertices $x_{p,1}, x_{p,2}, x_{p,3}$ that are not $z$ on $Q_{(k-1)m+p}$. Set $c(x_{p,1}) := e_{p,1}, c(x_{p,2}) := k, c(x_{p,3}) := e_{p,2}$. Hence, adjacent vertices on these cycles have different colors.

Note that for the case that $k$ is even, we obtain Hamilton cycles $H_1, H_2, \ldots, H_{\frac{k-2}{2}}$ of $H$. Again, this time we must have $m \leq \frac{k-2}{2}$. The coloring is done by using the similar way to the case that $k$ is odd, except that the case $r > \frac{k-1}{2} + 1$ is replaced with the case $r \neq 0$, and the case $r \leq \frac{k-1}{2}$ is not needed. Next, color the remaining $t-q$ cycles by using the similar coloring used to color cycles before there was any 4-cycle, by noting that the pair $\{a_i, b_i\}$ that is used is the label of two adjacent vertices that have not been used to color the 4-cycles on the above decomposition. Observe that there are exactly $t-q$ pairs of such labels. Thus, we have shown that $c$ is a $k$-coloring.

We show that $c$ is indeed a locating coloring. Let $x$ and $y$ be two different vertices in $G$ with $c(x) = c(y)$. The cases for $x$ and $y$ that must be considered are:

1. One of them is $z$
2. Both are not $z$ and are in the same 4-cycle
3. Both are not $z$ and are in the same cycle that is not a 4-cycle
4. Both are not $z$, $x$ is in a 4-cycle, and $y$ is in a cycle that is not a 4-cycle
5. Both are not $z$ and are in different 4-cycles
6. Both are not $z$ and are in different cycles, but these cycles are not 4-cycles.

The first four cases are easily verified. For the fifth case, if both $x$ and $y$ are colored $k$, then, since both are not $z$, their neighbors have to be only two vertices that have the color pair from some labels $E_i$ and $E_j$, respectively, that are edge-disjoint subgraphs. Hence, the colors of the neighbors of $x$ and $y$ differ. If both are not colored $k$, then the colors of the neighbors of $x$ and $y$ also differ since each two
4-cycles have their vertices colored based on the labels of the subgraphs of $H$ that are edge-disjoint subgraphs and since $x$ and $y$ are different vertices. In fact, if the colors of the neighbors of $x$ and $y$ are only $k$, then, since they belong to different 4-cycles, the colors in both of these cycles must be based on $E_i$ and $E_j$ that are edge-disjoint subgraphs. This is impossible if $x$ and $y$ are different vertices.

For the sixth case, the same argument also applies by observing the possibilities of the position of $x$ and $y$ and the labels used. Thus, we have shown that $c$ is a locating $k$-coloring so that $\chi_L(G) \leq k$.

Lastly, for the case $t < \left(\frac{k-1}{2}\right)$, the $(t+1)$-th, $(t+2)$-th, ... and so on that have been colored from the coloring on the case $t = \left(\frac{k-1}{2}\right)$ before is removed so that there are $t < \left(\frac{k-1}{2}\right)$ cycles remaining, and the above cases can be verified again the similar way. Thus, the theorem is proved. ■

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