CONTROLLABILITY AND OBSERVABILITY OF STOCHASTIC IMPLICIT SYSTEMS AND STOCHASTIC GE-EVOLUTION OPERATOR

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Abstract. This paper discusses exact (approximate) controllability and exact (approximate) observability of stochastic implicit systems in Banach spaces. Firstly, we introduce the stochastic GE-evolution operator in Banach space and discuss existence and uniqueness of the mild solution to stochastic implicit systems by stochastic GE-evolution operator in Banach space. Secondly, we discuss conditions for exact (approximate) controllability and exact (approximate) observability of the systems considered in terms of stochastic GE-evolution operator and the dual principle. Finally, an illustrative example is given.

1. Introduction. Stochastic implicit systems play an important role in many application fields, for example, input-output economics, the problem of protein folding, the modelling of multi-body mechanisms and so on (e.g., [19], [23], [15], [16], [17]). Many results have been obtained concerning stochastic implicit systems in finite dimensional case (e.g., [3], [5], [4], [6], [7], [28], [30], [24], [26], [27], [29], [14], [25], [11], [12]), but there are few results in infinite dimensional case (see [19], [23], [15], [16], [17]). Classical and distributional solutions for a class of abstract stochastic implicit equations were studied by degenerate strongly continuous semigroup in Hilbert spaces in [19]. Stochastic optimal impulse control of parabolic systems of Sobolev type was discussed in [23]. The existence and uniqueness of the solution to a class of generalized stochastic evolution equations were considered which is based on a generalized Ito formular in [15]. The mild solution, strong solution and exact controllability for the following stochastic implicit equations were discussed in [16] and [17] by using $C_0$-semigroup in Hilbert spaces from 2015 to 2018:

$$Adx(t) = Bx(t)dt + Cv(t)dt + Ddw(t), t \geq 0, x(0) = x_0,$$

$$y(t) = Qx(t).$$

(1)
The main reason for the difficulties in studying stochastic implicit systems in infinite dimensional spaces lies in change of dimension and the unboundedness of state operator. Controllability and observability are very important for stochastic implicit systems in infinite dimensional spaces because they are the basic problems concerning the research of stochastic implicit systems in infinite dimensional spaces and have important application in the study of stability for stochastic implicit systems in infinite dimensional spaces (see [24]). The methods used in [16] and [17] are not suitable to study the controllability and observability of stochastic implicit systems in Banach spaces because their research process applies the orthogonality of Hilbert spaces. Therefore, the results obtained in [16] and [17] can not be extended to stochastic implicit systems in Banach spaces. The main reason for the difficulty is that Banach spaces have no orthogonality. Therefore, it is necessary to find a new way to study the controllability and observability of infinite dimensional stochastic implicit systems in Banach spaces. In this paper, we introduce the stochastic GE-evolution operator in Banach space and study exact (approximate) controllability and exact (approximate) observability concerning a class of stochastic implicit linear systems according to stochastic GE-evolution operator in Banach spaces, more specifically we discuss the following stochastic implicit linear system

\[
Adx(t) = Bx(t)dt + Cv(t)dt + Dx(t)dw(t), \quad t \geq 0, x(0) = x_0,
\]

\[
y(t) = Qx(t),
\]

where \(x(t) \in H\) is the state vector, \(v(t) \in U\) is the control vector, \(w(t)\) is the one-dimensional standard Wiener process, \(x_0 \in H\) is a given random variable, \(y(t) \in Y\) is the output vector, \(H, U, Y\) are Banach spaces. In section 2, the stochastic GE-evolution operator induced by bounded linear operator \(A\) is introduced, which is a generalization of random evolution system ([1]), and the expression for the mild solution to stochastic implicit linear system (2) is given by stochastic GE-evolution operator. In section 3, conditions for the exact (approximate) controllability of stochastic implicit linear system (2) are derived by the stochastic GE-evolution operator. In section 4, conditions for the exact (approximate) observability of stochastic implicit linear system (2) are derived according to stochastic GE-evolution operator, and the dual principle is given. In section 5, we give an example to illustrate the validity of the theoretical results obtained in this paper. At last, the conclusion is given in section 6.

As far as we know, there is no result for the controllability and observability of stochastic implicit system (2), controllability and observability are very important for system (2) because they are the basic problems concerning the research of system (2) and have important application in the study of stability for system (2), which inspires our current research. The main novelties of this paper consist of the following points: (i) The conditions for controllability and observability of system (2) are considered for the first time. Some necessary and sufficient conditions are obtained. (ii) The new concept of stochastic GE-evolution operator is introduced. Controllability and observability of system (2) are discussed by stochastic GE-evolution operator. (iii) System (2) is more complicated then system (1). We can not directly obtain conditions for controllability and observability of system (2) as system (1) by the methods as [16] and [17].

The main contributions of this paper consist of the following points: (i) The new concept of stochastic GE-evolution operator is introduced for the first time, and the mild solution of system (2) is given by stochastic GE-evolution operator. (ii)
The controllability and observability operators of system (2) are given by stochastic GE-evolution operator, respectively. (iii) The necessary and sufficient conditions concerning exact (approximate) controllability and exact (approximate) observability of system (2) are given by stochastic GE-evolution operator in Banach spaces, respectively, and the dual principle is given.

In particular, the method used in this paper to study controllability and observability of system (2) in the sense of GE-semigroup is also suitable for studying the controllability and observability of system (1) which is studied in [16] and [17] but not vice versa.

**Notation.** In this paper, $H^*$ denotes the dual space of $H$; $A^*$ denotes the dual operator of $A$: $L(U,H)$ is the set of all bounded linear operators from $U$ to $H$. $A \in L(H,H)$, $C \in L(U,H)$, $D \in L(H,H)$, $Q \in L(H,Y)$, $B$ is a closed linear operator (possibly unbounded) from $D(B)$ to $H$ whose domain $D(B)$ need not be dense in $H$; $A, B, C, D, Q$ are deterministic and constant operators; $(\Omega, F, F_t, P)$ is a complete probability space with a filtration $F_t$ satisfying the usual condition (i.e., the filtration contains $P$-null sets and is right continuous); all processes are $F_t$-adapted; $w(t)$ is defined on $(\Omega, F, F_t, P)$; $E$ denotes the mathematical expectation; $\| \cdot \|_H$ denotes the norm in $H$; $L^2(\Omega, F_t, P, H)$ denotes the set of all random variables $x \in H$ such that $x$ is $F_t$-measurable and

$$\|x\|_{L^2(\Omega, F_t, P, H)} = (E(\|x\|^2_H))^1/2 < +\infty;$$

$L^2([0,T],\Omega, F_t, H)$ denotes the set of all stochastic processes $x(t) \in H$ defined on $(\Omega, F, F_t, P)$ such that the mean square norm

$$(E(\|x(t)\|^2_H))^1/2 < +\infty, \forall t \in [0,T];$$

$L^2([0,T],\Omega, H)$ denotes the set of all stochastic processes $x(t) \in L^2([0,T],\Omega, F_t, H)$ and

$$\|x\|_{L^2([0,T],\Omega, H)} = (\int_0^T E(\|x(t)\|^2_H)dt)^1/2 < +\infty.$$

2. Mild solution. First of all, we introduce the stochastic GE-evolution operator, and then discuss the mild solution of system (2).

**Definition 2.1.** Let $\Delta_T = \{(t,s) : 0 \leq s \leq t \leq T\}.S : \Delta_T \times \Omega \to L(H,H)$ is said to be a stochastic generalized evolution operator (i.e., stochastic GE-evolution operator) induced by $A$ on $[0,T]$ if it has the following properties:

(i) $S$ is strongly measurable;

(ii) $S(t,s)$ is strongly $F_t$-measurable for $t \geq s$;

(iii) $S(s,s) = S_0, 0 \leq s \leq T$, and $S(t,r)AS(r,s) = S(t,s)$ for any $0 \leq s \leq r \leq t \leq T$, where $S_0 \in L(H,H)$ is a steady operator;

(iv) For any $\xi \in H$, $(t,s) \to S(t,s)\xi$ is mean square continuous from $\Delta_T$ into $H$.

**Definition 2.2.** ([8]–[9]) Assume $\{U(t) : t \geq 0\}$ is one parameter family of bounded linear operators in Hilbert space $H$. If

$$U(t + s) = U(t)AU(s), t, s \geq 0,$$

then $\{U(t) : t \geq 0\}$ is called a generalized operator semigroup (i.e., GE-semigroup) induced by $A$.

If GE-semigroup $U(t)$ satisfy

$$\lim_{t \to 0} \|U(t)x - U(0)x\|_H = 0$$
for arbitrary \( x \in H \), then \( U(t) \) is called strongly continuous on \( H \).

**Lemma 2.3.** (\[8\]) Let GE-semigroup \( U(t) \) be strongly continuous on \( H \). Then existences \( M \geq 1 \) and \( \omega > 0 \) make

\[
\|U(t)\|_{L(H,H)} \leq Me^{\omega t}, \quad t \geq 0, \tag{5}
\]

which is called that \( U(t) \) is exponentially bounded.

**Definition 2.4.** (\[10\]) Let \( U(t) \) be a strongly continuous GE-semigroup induced by \( A \), and \( B \) be a closed linear operator in \( H \). If

\[
Bx = \lim_{h \to 0^+} \frac{AU(h)A - AU(0)A}{h} x, \tag{6}
\]

for any \( x \in D_1 \), where

\[
D_1 = \{ x \in D(B) \subseteq H, \forall x, \exists \lim_{h \to 0^+} \frac{AU(h)A - AU(0)A}{h} x \},
\]

then \( B \) is called a generator of GE-semigroup \( U(t) \) induced by \( A \).

In the following, we always suppose that \( B \) is a generator of GE-semigroup \( U(t) \) induced by \( A \).

Now we consider the mild solution of stochastic implicit linear system (2).

**Definition 2.5.** If \( v(t) \in L^2([0,T],\Omega,U) \), \( x(t) \in L^2([0,T],\Omega,\mathcal{D}_1) \),

\[
x_0 \in L^2(\Omega,F_0,P,\mathcal{D}_1); Cv(t), \quad Dx(t) \in A(L^2([0,T],\Omega,\mathcal{D}_1)),
\]

then the mild solution \( x(t,x_0) \in L^2([0,T],\Omega,\mathcal{D}_1) \) to (2) is defined by

\[
x(t,x_0) = U(t)x_0 + \int_0^t U(t-\tau)Cv(\tau)d\tau + \int_0^t U(t-\tau)Dx(\tau,x_0)d\tau. \tag{7}
\]

**Lemma 2.6.** If \( v(t) \in L^2([0,T],\Omega,U) \), \( x(t) \in L^2([0,T],\Omega,\mathcal{D}_1) \),

\[
x_0 \in L^2(\Omega,F_0,P,\mathcal{D}_1); Cv(t), \quad Dx(t) \in A(L^2([0,T],\Omega,\mathcal{D}_1)),
\]

then system (2) has a unique mild solution \( x(t,x_0) \in L^2([0,T],\Omega,\mathcal{D}_1) \), which is given by (7).

**Proof.** Let \( H_1 \) denote the space of all \( \mathcal{D}_1 \) valued predictable processes \( \xi \) such that

\[
|\xi|_{H_1} = \sup_{t \in [0,T]} (E\|\xi(t)\|^2_{\mathcal{D}_1})^{1/2} < +\infty.
\]

For any \( \xi(t) \in H_1 \) define

\[
P(\xi)(t) = U(t)x_0 + \int_0^t U(t-s)Cv(s)ds + \int_0^t U(t-s)D\xi(s)dw(s), \quad t \in [0,T],
\]

and

\[
P(\xi)(t) = \int_0^t U(t-s)D\xi(s)dw(s), \quad t \in [0,T].
\]

Assume, see (5), that \( \|U(t)\|_{L(H,H)} \leq M_1, t \in [0,T] \), we have

\[
|P(\xi)|_{H_1} \leq \sup_{t \in [0,T]} (E\int_0^t \|U(t-s)D\xi(s)\|^2_{\mathcal{D}_1}ds)^{1/2}
\]

\[
\leq M_1 \|D\|_{L(H,H)}T^{1/2}|\xi|_{H_1}, \quad t \in [0,T].
\]

Therefore, if \( T \) is sufficient small, \( P \) is a contraction and it is easy to see that its unique fixed point can be identified as the mild solution to (2). The case of general \( T \) can be handled in a standard way.

\[\square\]
Definition 2.7. We say that stochastic GE-evolution operator $S(t, s)$ induced by $A$ is related to the linear homogeneous equation

$$Adx(t) = Bx(t)dt + Dx(t)dw(t), x(s) = x_0, 0 \leq s \leq t \leq T,$$  \hspace{1cm} (8)

if $x(t) = S(t, s)Ax_0$ is the mild solution to (8) with $x(s) = S(s, s)Ax_0 = x_0$ for arbitrary $x_0 \in L^2(\Omega, F_s, P, D_A)$.

In the following we suppose that there exists a stochastic GE-evolution operator $S(t, s)$ induced by $A$ related to (8) and Lemma 2.6 holds true. Furthermore, we suppose that the following estimates hold for any $0 \leq s \leq t \leq T$ and $\xi \in L^2(\Omega, F_s, P, D_A)$:

$$E\int_s^t \|S(r, s)\xi\|^2_{D_A}dr \leq c\|\xi\|^2_{L^2(\Omega, F_s, P, D_A)}; \hspace{1cm} (9)$$

$$\sup_{r \in [s, t]} E\|S(r, s)\xi\|^2_{D_A} \leq c\|\xi\|^2_{L^2(\Omega, F_s, P, D_A)}. \hspace{1cm} (10)$$

We can obtain the following theorem.

Theorem 2.8. The mild solution $x(t, x_0)$ to (2) can be written in the form

$$x(t, x_0) = S(t, 0)Ax_0 + \int_0^t S(t, s)Cv(s)ds. \hspace{1cm} (11)$$

Proof. Since $S(t, 0)Ax_0$ and $S(t, s)Cv(s)$ are mild solution of system (8) with $x(0) = x_0$ and $x(s) = S(s, s)Cv(s)$, respectively, we have that

$$S(t, 0)Ax_0 = U(t)Ax_0 + \int_0^t U(t - \tau)DS(\tau, 0)Ax_0dw(\tau),$$

$$S(t, s)Cv(s) = U(t - s)AS(s, s)Cv(s) + \int_s^t U(t - \tau)DS(\tau, s)Cv(s)dw(\tau) = U(t - s)Cv(s) + \int_s^t U(t - \tau)DS(\tau, s)Cv(s)dw(\tau).$$

We have to prove that the process $x(t, x_0)$ in (11) is a solution to the integral equation (7). By the representation of $x(\tau, x_0)$, we have

$$\int_0^t U(t - \tau)Dx(\tau, x_0)dw(\tau) = \int_0^t U(t - \tau)DS(\tau, 0)Ax_0dw(\tau)$$

$$+ \int_0^t U(t - \tau)D(\int_0^\tau S(\tau, s)Cv(s)ds)dw(\tau)$$

$$= S(t, 0)Ax_0 - U(t)Ax_0 + \int_0^t ds \int_s^t U(t - \tau)DS(\tau, s)Cv(s)dw(\tau)$$

$$= S(t, 0)Ax_0 - U(t)Ax_0 + \int_0^t [S(t, s)Cv(s) - U(t - s)AS(s, s)Cv(s)]ds$$

$$= S(t, 0)Ax_0 - U(t)Ax_0 + \int_0^t S(t, s)Cv(s)ds - \int_0^t U(t - s)Cv(s)ds,$$

where the stochastic Fubini theorem is given in Theorem 4.33 of [22]. Therefore,

$$x(t, x_0) = S(t, 0)Ax_0 + \int_0^t S(t, s)Cv(s)ds$$

$$= U(t)Ax_0 + \int_0^t U(t - \tau)Cv(\tau)d\tau + \int_0^t U(t - \tau)Dx(\tau, x_0)dw(\tau),$$
which proves (7).

3. Controllability. In this section we discuss the exact and approximate controllability of stochastic implicit linear system (2) by stochastic GE-evolution operator theory, some criteria are obtained.

3.1. Exact controllability.

Definition 3.1. Stochastic implicit linear system (2) is called to be exactly controllable on \([0, T]\), if for all \(x_0 \in L^2(\Omega, F_0, P, D_1), x_T \in L^2(\Omega, F_T, P, D_1),\) there exists \(v(t) \in L^2([0, T], \Omega, U)\) such that the mild solution \(x(t, x_0)\) to stochastic implicit linear system (2) satisfies \(x(T, x_0) = x_T.\)

In order to obtain the criteria of exact controllability, the following concepts are introduced.

Controllability operator

\[
C^T_0 : L^2([0, T], \Omega, U) \to L^2(\Omega, F_T, P, \overline{D_1})
\]

associated with system (2) is defined as

\[
C^T_0 v = \int_0^T S(T, \tau)Cv(\tau)d\tau.
\]  

(12)

It is obvious that operators \(C^T_0\) is a bounded linear operator, and its dual

\[
C^T_0^* : L^2(\Omega, F_T, P, (\overline{D_1})^*) \to L^2([0, T], \Omega, U^*)
\]

is defined by

\[
C^T_0^* z^* = C^* S^*(T, \tau)E(z^*|F_\tau).
\]

where \(z^* \in L^2(\Omega, F_T, P, (\overline{D_1})^*)\).

From the Definition 3.1, we can get the following theorem immediately.

Theorem 3.2. Stochastic implicit system (2) is exactly controllable on \([0, T]\) if and only if \(\text{ran} C^T_0 = L^2(\Omega, F_T, P, \overline{D_1}).\)

Theorem 3.3. Assume that \(H\) and \(U\) are reflexive Banach spaces. Stochastic implicit system (2) is exactly controllable on \([0, T]\) if and only if one of the following conditions holds:

(a) \(\|C^T_0^* z^*\|_{L^2([0, T], \Omega, U^*)} \geq \gamma \|z^*\|_{L^2(\Omega, F_T, P, (\overline{D_1})^*)}\) for some \(\gamma > 0\) and all

\[z^* \in L^2(\Omega, F_T, P, (\overline{D_1})^*).\]

(b) \(\ker C^T_0 = \{0\}\) and \(\text{ran} C^T_0^*\) is closed.

Proof. (a) \(\Rightarrow\) (b) Notice that (a) implies that \(C^T_0^*\) is injective. To prove that \(C^T_0^*\) has closed range, assume that \(C^T_0^* z^*\) has a Cauchy sequence in \(L^2([0, T], \Omega, U^*),\) then (a) implies that \(z^*_n\) is a Cauchy sequence in \(L^2(\Omega, F_T, P, (\overline{D_1})^*).\) Since \(C^T_0^*\) is a bounded linear operator, if \(\lim_{n \to +\infty} z^*_n = z^*\), then \(\lim_{n \to +\infty} C^T_0^* z^*_n = C^T_0^* z^*\) and so \(C^T_0^*\) has closed range.

(b) \(\Rightarrow\) (a). (b) shows that \(C^T_0^*\) has an algebraic inverse with domain equal to \(\text{ran} C^T_0^*\). Since \(\text{ran} C^T_0^*\) is closed, it is a Banach space under the norm of \(L^2([0, T], \Omega, U^*),\) i.e.,

\[
\|v^*\|_{\text{ran} C^T_0^*} = \|v^*\|_{L^2([0, T], \Omega, U^*)}, v^* \in \text{ran} C^T_0^*.
\]
By Corollary A.3.50 of [2], we have that \((C_0^T)^{-1}\) is bounded on this range, i.e., there exists a \(\gamma > 0\) such that
\[
\|(C_0^T)^{-1}v^*\|_{L^2(\Omega, F_T, \mathbf{P}, (\mathcal{D}_T)^*)} \leq \frac{1}{\gamma} \|v^*\|_{L^2([0, T], \Omega, U^*)}
\]
for every \(v^* \in \text{ran}C_0^T\). Substituting \(v^* = C_0^Tz^*\) proves (a).

It remains to show that (a) is equivalent to exact controllability of system (2).

Necesstiy. Assume that stochastic implicit system (2) is exactly controllable. By Theorem 3.2, we have ran\(C_0^T\) = \(L^2(\Omega, F_T, P, \mathcal{D}_T)\).

If \(C_0^T\) is a one to one operator, then \((C_0^T)^{-1}\) exists on \(L^2(\Omega, F_T, P, \mathcal{D}_T)\). According to the continuity of operator \(C_0^T\) we have that \((C_0^T)^{-1}\) is a closed operator. From the closed graph theorem, we obtain that \((C_0^T)^{-1}\) is a bounded linear operator on \(L^2(\Omega, F_T, P, \mathcal{D}_T)\), i.e.,
\[
((C_0^T)^{-1})^{-1} \in L(L^2(\Omega, F_T, P, \mathcal{D}_T), L^2([0, T], \Omega, U)).
\]
Therefore
\[
((C_0^T)^{-1})^{-1} \in L(L^2([0, T], \Omega, U^*), L^2(\Omega, F_T, P, (\mathcal{D}_T)^*)).
\]
This implies that there exists \(\gamma_T > 0\) such that
\[
\|((C_0^T)^{-1})^{-1}v^*\|_{L^2(\Omega, F_T, P, (\mathcal{D}_T)^*)} \leq \gamma_T \|v^*\|_{L^2([0, T], \Omega, U^*)}.
\]
Assume \(z^* \in L^2(\Omega, F_T, P, (\mathcal{D}_T)^*)\), then
\[
v^* = C_0^Tz^* \in L^2([0, T], \Omega, U^*).
\]
Therefore, for all \(z_0 \in L^2(\Omega, F_T, P, \mathcal{D}_T)\), we get that
\[
\langle z_0, ((C_0^T)^{-1})^{-1}v^* \rangle = \langle z_0, ((C_0^T)^{-1})^{-1}C_0^Tz^* \rangle = \langle (C_0^T)^{-1}z_0, C_0^Tz^* \rangle = \langle z_0, z^* \rangle,
\]
where \(\langle z_0, z^* \rangle = z^*(z_0)\). From (13), we obtain that
\[
\|z^*\|_{L^2(\Omega, F_T, P, (\mathcal{D}_T)^*)} = \sup_{\|v\|_{L^2(\Omega, F_T, P, (\mathcal{D}_T)^*)}=1} \langle z_0, z^* \rangle \leq \|((C_0^T)^{-1})^{-1}v^*\|_{L^2(\Omega, F_T, P, (\mathcal{D}_T)^*)} \leq \gamma_T \|v^*\|_{L^2([0, T], \Omega, U^*)},
\]
i.e.,
\[
\|C_0^Tv^*\|_{L^2([0, T], \Omega, U^*)} \geq \frac{1}{\gamma_T} \|z^*\|^2_{L^2(\Omega, F_T, P, (\mathcal{D}_T)^*)} = \gamma \|z^*\|_{L^2(\Omega, F_T, P, (\mathcal{D}_T)^*)},
\]
where \(\gamma = \frac{1}{\gamma_T}\). This implies that (a) holds.

If \(C_0^T\) is not a one to one operator, then
\[
\text{ker}C_0^T = \{v : v \in L^2([0, T], \Omega, U), C_0^Tv = 0\} \neq \{0\}.
\]
A factor space is defined as follows
\[
U_1 = L^2([0, T], \Omega, U)/\text{ker}C_0^T = \{v_1 : v_1 = \{v + u : u \in \text{ker}C_0^T\}\}.
\]
For \(v_1 \in U_1\),
\[
\|v_1\|_{U_1} = \inf_{u \in \text{ker}C_0^T} \|v + u\|_{L^2([0, T], \Omega, U)}.
\]
If we define operator
\[
C_1^T : U_1 \to L^2(\Omega, F_T, P, \mathcal{D}_T), C_1^Tv_1 = C_0^Tv,
\]
\[
\|((C_0^T)^{-1})^{-1}v^*\|_{L^2(\Omega, F_T, P, (\mathcal{D}_T)^*)} \leq \gamma_T \|v^*\|_{L^2([0, T], \Omega, U^*)},
\]
where
\[
\gamma = \frac{1}{\gamma_T}.\]
then
\[ C_1^T \in L(U_1, L^2(\Omega, F_T, P, \overline{D}_1)), \]
and \( C_1^T \) is an bijective operator. It can be seen from the above proof that
\[ \| C_1^{T*}z^*_{U_1^*} \| \geq \gamma \| z^* \|_{L^2(\Omega, F_T, P, (\overline{D}_1)^*)}. \]
According to the definition of \( U_1 \) and \( C_1^T \), we get
\[ \| C_1^{T*}z^*_{U_1^*} \| = \| C_0^{T*}z^*_{L^2([0,T],\Omega,U^*)} \|. \]
This implies that \( (a) \) holds.

Sufficiency. Assume \( (a) \). It is need to prove that if \( z \in L^2(\Omega, F_T, P, D_1) \), then \( z \in \text{ran} C_0^T \). From
\[ C_0^T \in L(L^2([0,T], \Omega, U), L^2(\Omega, F_T, P, \overline{D}_1)), \]
we have that
\[ C_0^{T*} \in L(L^2(\Omega, F_T, P, (\overline{D}_1)^*), L^2([0,T], \Omega, U^*)). \]
For \( z \in L^2(\Omega, F_T, P, \overline{D}_1) \), we can define a functional \( g \) on \( \text{ran} C_0^{T*} \) satisfying
\[ g(C_0^{T*}h^*) = \langle z, h^* \rangle, h^* \in L^2(\Omega, F_T, P, (\overline{D}_1)^*). \] (14)
This implies that \( g \) is linear for \( C_0^{T*}h^* \). According to \( (a) \), if
\[ \lim_{n \to \infty} C_0^{T*}h_n^* = 0, \]
then
\[ \lim_{n \to \infty} h_n^* = 0 \]
and
\[ \lim_{n \to \infty} g(C_0^{T*}h_n^*) = \lim_{n \to \infty} \langle z, h_n^* \rangle = 0. \]
Therefore, \( g \) is continuous linear functional on
\[ \text{ran} C_0^{T*} \subset L^2([0,T], \Omega, U^*). \]
By Hahn-Banach theorem, we have that \( g \) can be extended as a continuous linear functional on \( L^2([0,T], \Omega, U^*) \). According to \( U^{**} = U \), the existence of
\[ v \in L^2([0,T], \Omega, U) = L^2([0,T], \Omega, U^{**}) \]
makes
\[ g(C_0^{T*}h^*) = \langle v, C_0^{T*}h^* \rangle, h^* \in L^2(\Omega, F_T, P, (\overline{D}_1)^*). \] (15)
According to (14) and (15), we obtain that for every \( h^* \in L^2(\Omega, F_T, P, (\overline{D}_1)^*) \)
\[ \langle z, h^* \rangle = \langle C_0^T v, h^* \rangle. \]
Hence \( z = C_0^T v \), i.e.,
\[ \text{ran} C_0^T = L^2(\Omega, F_T, P, \overline{D}_1). \]
From Theorem 3.2, stochastic implicit system (2) is exactly controllable. \( \square \)
3.2. Approximate controllability.

**Definition 3.4.** Stochastic implicit linear system (2) is called to be approximately controllable on $[0, T]$, if for any state $x_T \in L^2(\Omega, F_T, P, \mathcal{D}_1)$, any initial state $x_0 \in L^2(\Omega, F_0, P, \mathcal{D}_1)$, and any $\epsilon > 0$, existence $v \in L^2([0, T], \Omega, U)$ makes that the mild solution $x(t, x_0)$ satisfies

$$
\|x(T, x_0) - x_T\|_{L^2(\Omega, F_T, P, \mathcal{D}_1)} < \epsilon.
$$

It is obvious that the necessary and sufficient condition for the stochastic implicit linear system (2) to be approximately controllable on $[0, T]$ is

$$
\text{ran} C_T^0 = L^2(\Omega, F_T, P, \mathcal{D}_1).
$$

**Theorem 3.5.** Stochastic implicit systems (2) is approximate controllable on $[0, T]$ if and only if one of the following conditions holds:

(a) $\|C_T^0 z^*\|_{L^2([0, T], \Omega, U^*)} > 0$ for all $z^* \in L^2(\Omega, F_T, P, (\mathcal{D}_1)^*)$, $z^* \neq 0$.

(b) $\ker C_T^0 = \{0\}$.

**Proof.** It is obvious that (a) is equivalent to (b). We only need to prove that (b) is equivalent to approximate controllability of (2).

If

$$
\text{ran} C_T^0 = L^2(\Omega, F_T, P, \mathcal{D}_1),
$$

i.e., $C_T^0 z^* = 0$, then

$$
<v, C_T^0 z^*> = <C_T^0 v, z^* >, v \in L^2([0, T], \Omega, U).
$$

Since $\text{ran} C_T^0 = L^2(\Omega, F_T, P, \mathcal{D}_1)$, we have

$$
<v, z^*> = 0, z \in L^2(\Omega, F_T, P, \mathcal{D}_1).
$$

Therefore $z^* = 0$, i.e., $\ker C_T^0 = \{0\}$.

Conversely, if $\ker C_T^0 = \{0\}$ but

$$
\text{ran} C_T^0 \neq L^2(\Omega, F_T, P, \mathcal{D}_1),
$$

then $\text{ran} C_T^0$ is the proper subspace of $L^2(\Omega, F_T, P, \mathcal{D}_1)$. According to Hahn-Banach theorem, there exists

$$
z^* \in L^2(\Omega, F_T, P, (\mathcal{D}_1)^*), z^* \neq 0
$$

such that

$$
<v, C_T^0 z^*> = 0, v \in L^2([0, T], \Omega, U).
$$

Thus $<v, C_T^0 z^*> = 0$, i.e., $C_T^0 z^* = 0$. By $\ker C_T^0 = \{0\}$, we get that $z^* = 0$. This contradicts $z^* \neq 0$. Therefore,

$$
\text{ran} C_T^0 = L^2(\Omega, F_T, P, \mathcal{D}_1).
$$

Hence (16) is true if and only if (b) holds, i.e., stochastic implicit system (2) is approximately controllable on $[0, T]$ if and only if (b) holds. \qed
4. **Observability.** Consider the following stochastic implicit system

\[ Adx(t) = Bx(t)dt + Dx(t)dw(t), \quad t \geq 0, \quad x(0) = x_0, \]
\[ y(t) = Qx(t), \]
and its dual stochastic implicit system

\[ A^*dx^*(t) = B^*x^*(t)dt + Q^*v^*(t)dt + D^*x^*(t)dw(t), \]
\[ t \geq 0, \quad x^*(0) = x_0^*. \]

For the stochastic implicit system (17), the following concepts are defined.

**Definition 4.1.** Stochastic implicit system (17) is said to be exactly observable on \([0, T]\) if and only if its dual stochastic implicit system (18) is exactly (approximately) controllable on \([0, T]\).

**Theorem 4.3.** Assume that \(H\) and \(Y\) are reflexive. Stochastic implicit system (17) is exactly (approximately) observable on \([0, T]\) if and only if its dual stochastic implicit system (18) is exactly (approximately) controllable on \([0, T]\).

**Proof.** Here we only prove the case of exact observability. Since

\[ O_0^T : L^2(\Omega, F_T, P, D_1) \rightarrow L^2([0, T], \Omega, Y) \]

defined by \( O_0^T z = QS(T,t)E(z|F_t) \), its dual operator

\[ O_0^{T*} : L^2([0, T], \Omega, Y^*) \rightarrow L^2(\Omega, F_T, P, (D_1)^*) \]

is

\[ O_0^{T*}y^* = \int_0^T S^*(T,t)Q^*y^*(t)dt. \]

**Definition 4.1.** Stochastic implicit system (17) is said to be exactly observable on \([0, T]\) if \( O_0^T \) is injective and its inverse is bounded on \( \text{ran} O_0^T \), i.e., the state \( x_0 \) can be uniquely and continuously constructed from the knowledge of the output \( y(t) \) in \( L^2([0, T], \Omega, Y) \).

**Definition 4.2.** Stochastic implicit system (17) is said to be approximately observable on \([0, T]\) if \( O_0^T \) is injective, i.e., the state \( x_0 \) can be uniquely constructed from the knowledge of the output \( y(t) \) in \( L^2([0, T], \Omega, Y) \).

We can obtain the following dual principle.

**Theorem 4.3.** Assume that \(H\) and \(Y\) are reflexive. Stochastic implicit system (17) is exactly (approximately) observable on \([0, T]\) if and only if its dual stochastic implicit system (18) is exactly (approximately) controllable on \([0, T]\).

**Proof.** Here we only prove the case of exact observability. Since

\[ O_0^{T*}y^* = \int_0^T S^*(T,t)Q^*y^*(t)dt \]

happens to be the controllability operator \( C_0^T \) of stochastic implicit system (18), so \( C_0^{T*} = O_0^{T*} \).

If the stochastic implicit system (17) is exactly observable, then there exists \( 1/\gamma > 0 \) such that

\[ \|(O_0^T)^{-1}y\|_{L^2(\Omega, F_T, P, D_1)} \leq \frac{1}{\gamma} \|y\|_{L^2([0, T], \Omega, Y)} \]

for all \( y \in \text{ran} O_0^T \). This implies that

\[ \gamma \|z\|_{L^2(\Omega, F_T, P, D_1)} = \gamma \|(O_0^T)^{-1}O_0^Tz\|_{L^2(\Omega, F_T, P, D_1)} \]
\[ \leq \|O_0^Tz\|_{L^2([0, T], \Omega, Y)} = \|C_0^{T*}z\|_{L^2([0, T], \Omega, Y)}, \]

where

\[ y = O_0^Tz, \quad z \in L^2(\Omega, F_T, P, D_1). \]

According to Theorem 3.3 (a), we have that (18) is exactly controllable.
Assume next that the stochastic implicit system (18) is exactly controllable. From Theorem 3.3 (b), we have that $O^T_0$ is injective and has closed range. According to closed graph theorem $(O^T_0)^{-1}$ is bounded on $\text{ran} O^T_0$.

Theorems 3.3 and Definitions 4.1 and 4.2 yield the following conditions for observability of stochastic implicit system (17).

**Corollary 1.** Stochastic implicit system (17) is exactly observable on $[0,T]$ if and only if one of the following conditions holds for some $\gamma > 0$ and for all $z \in L^2(\Omega, F_T, P, D_1)$:

(a) $\|O^T_0 z\|_{L^2([0,T], \Omega, Y)} \geq \gamma \|z\|_{L^2(\Omega, F_T, P, D_1)}$.

(b) $\ker O^T_0 = \{0\}$ and $\text{ran} O^T_0$ is closed.

**Corollary 2.** Stochastic implicit system (17) is approximately observable on $[0,T]$ if and only if $\ker O^T_0 = \{0\}$.

5. **An illustrative example.** In this section we give an example to illustrate the effectiveness of the obtained results.

According to [16], in input-output economics, many models were established to describe the real economics. The economics Leontief dynamic input-output model can be extended as an ordinary differential equation of the form:

$$A \frac{dx(t)}{dt} = Bx(t) + Cv(t)$$

(19)

in Banach space $H$, where $A \in L(H, H)$ and $B : D(B) \to H$ is a linear and possibly unbounded operator, while $x(t), v(t) \in H$ for $t \geq 0$. However, in reality, there many unpredicted parameters and different type of uncertainties that have not been implemented in the mathematical modelling process of this system. Nonetheless, according to [21]- [18], we can consider a stochastic version of the implicit system (19) with the standard Wiener process $w(t)$ used to model the uncertainties of the form:

$$Adx(t) = Bx(t)dt + Cv(t)dt + Dx(t)dw(t).$$

(20)

This stochastic version of the input-output model is a stochastic implicit system in Banach space $H$ of the form (2).

We consider the following unforced stochastic implicit system, i.e., $v(t) = 0$ in stochastic implicit system (20):

$$Adx(t) = Bx(t)dt + Dx(t)dw(t),$$

$$y(t) = Qx(t).$$

(21)

System (21) is the form of system (17). In what follows, we will verify the effectiveness of Corollary 2.

If for some concrete engineering practice, the following data are taken in (21):

$$A = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 2I_1 & 0 \\ 0 & 5I_2 \end{bmatrix}, D = \begin{bmatrix} I_1 & 0 \\ 0 & 3I_2 \end{bmatrix}, Q = \begin{bmatrix} 7I_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where $I_1, I_2$ are identical operators in Banach spaces $H_1, H_2$ respectively. System (21) can be written as

$$\begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} dx_1(t) \\ dx_2(t) \end{bmatrix} = \begin{bmatrix} 2I_1 & 0 \\ 0 & 5I_2 \end{bmatrix} \begin{bmatrix} x_1(t)dt \\ x_2(t)dt \end{bmatrix}$$

$$+ \begin{bmatrix} I_1 & 0 \\ 0 & 3I_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} dw(t)$$
\[ y(t) = \begin{bmatrix} 7I_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad (22) \]

where \[ \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in H_1 \oplus H_2 = H = Y. \] By Theorem 1.5.4 of [20], we can obtain that \( \mathcal{D}_1 = H_1 \). According to [13], we can obtain
\[
S(T,t) = \begin{bmatrix} \exp \left[ \frac{3}{2} (T-t) + w(T) - w(t) \right] & 0 \\ 0 & I_2 \end{bmatrix}.
\]

If \[ \begin{bmatrix} z \\ 0 \end{bmatrix} \in L^2(\Omega,F_T,P,\mathcal{D}_1), \]
and
\[
QO_0^T \begin{bmatrix} z \\ 0 \end{bmatrix} = QS(T,t)E \left( \begin{bmatrix} z \\ 0 \end{bmatrix} | F_t \right) = 0 \text{ for } t \in [0,T],
\]
then
\[
QS(T,T)E \left( \begin{bmatrix} z \\ 0 \end{bmatrix} | F_T \right) = 0.
\]

This implies that \( \ker O_0^T = \{0\} \). Therefore system (22) is approximately observable by Corollary 2.

6. Conclusion. In this paper, we have discussed exact (approximate) controllability and exact (approximate) observability for stochastic implicit linear systems by stochastic GE-evolution operator theory in Banach spaces. The necessary and sufficient conditions have been proposed for exact (approximate) controllability and exact (approximate) observability of stochastic implicit linear systems. These results are very convenient and effective for judging the exact (approximate) controllability and exact (approximate) observability of stochastic implicit linear systems. If the stochastic implicit linear systems are time varying, it will be discussed in another paper.

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