NATURAL PDE’S OF LINEAR FRACTIONAL WEINGARTEN SURFACES IN EUCLIDEAN SPACE

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ABSTRACT. We prove that the natural principal parameters on a given Weingarten surface are also natural principal parameters for the parallel surfaces of the given one. As a consequence of this result we obtain that the natural PDE of any Weingarten surface is the natural PDE of its parallel surfaces. We show that the linear fractional Weingarten surfaces are exactly the surfaces satisfying a linear relation between their three curvatures. Our main result is classification of the natural PDE’s of Weingarten surfaces with linear relation between their curvatures.

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1. Introduction

The relationship between the solutions of certain types of partial differential equations and the determination of various kinds of surfaces of constant curvature has generated many results which have applications to the areas of both pure and applied mathematics. This includes the determination of surfaces of either constant mean curvature or Gaussian curvature. It has long been known that there is a connection between surfaces of negative constant Gaussian curvature in Euclidean \( R^3 \) and the sine-Gordon equation. The fundamental equations of surface theory are found to yield a type of geometrically based Lax pair. For instance, given a particular solution of the sinh-Laplace equation, this Lax pair can be integrated to determine the three fundamental vector fields related to the surface. These are also used to determine the coordinate vector field of the surface.

In [4] Eisenhart considers three particular systems of lines on surfaces. He founds that the so-called distance-function and the radii of normal and geodesic curvature of the directions of the lines of all three systems are simple functions of the radii of principal curvature of the surfaces.

Further results are obtained based on the fundamental equations of surface theory, and it is shown how specific solutions of this sinh-Laplace equation can be used to obtain the coordinates of a surface in either Minkowski \( R_1^3 \) or Euclidean \( R^3 \) space [10, 11].

In [3] Bracken introduces some fundamental concepts and equations pertaining to the theory of surfaces in three-space, and, in particular, studies a class of sinh-Laplace equation which has the form \( \Delta u = \pm \sinh u \).

A surface \( S \) with principal curvatures \( \nu_1 \) and \( \nu_2 \) is a Weingarten surface (W-surface) [24, 25] if there exists a function \( \nu \) on \( S \) and two functions (Weingarten functions) \( f, g \) of one variable, such that

\[
\nu_1 = f(\nu), \quad \nu_2 = g(\nu).
\]

We proved in [8] that any W-surface admits locally special principal parameters - natural principal parameters. With respect to these natural principal parameters the functions

\[ Key \text{ } words \text{ } and \text{ } phrases. \] Natural principal parameters on W-surfaces, natural PDE of a W-surface, linear fractional W-surfaces, parallel surfaces.
\[ \sqrt{E} \exp \left( \int \frac{f' d\nu}{f-g} \right), \sqrt{G} \exp \left( \int \frac{g' d\nu}{g-f} \right) \] are constants, \( E, G \) being the coefficients of the first fundamental form on a W-surface.

With respect to natural principal parameters any W-surface \( S \) with Weingarten functions \( f, g \) is determined uniquely up to motions by the geometric function \( \nu \), which satisfies a non-linear partial differential equation - the natural PDE of the surface \( S \) [3]. This result solves the Lund-Regge reduction problem [6, 7, 12, 19] for W-surfaces in Euclidean space.

In Proposition 3.1 we prove that

*The natural principal parameters of a given W-surface \( S \) are natural principal parameters for all surfaces \( \tilde{S}(a) \), \( a = \text{const} \neq 0 \), which are parallel to \( S \).*

Theorem 3.2 states that

*The natural PDE of a given W-surface \( S \) is the natural PDE of any surface \( \tilde{S}(a) \), \( a = \text{const} \neq 0 \), which is parallel to \( S \).*

To motivate our investigations let us consider surfaces in Euclidean space, whose Gauss curvature \( K \) and mean curvature \( H \) satisfy the linear relation

\[(1.1) \quad \delta K = \alpha H + \gamma, \quad \alpha^2 + 4\gamma\delta \neq 0.\]

In [5] it was proved that all surfaces satisfying the linear relation (1.1) are integrable.

A similar relation has been studied in the three-dimensional Minkowski space [15].

There arises the following question: what are the natural PDE’s describing the surfaces, whose curvatures satisfy the relation (1.1)?

Any surface \( S \), whose invariants \( K \) and \( H \) satisfy the linear relation (1.1) is (locally) parallel to one of the following three types of surfaces: a minimal surface; a CMC-surface (or a surface with positive constant Gauss curvature); a surface with negative constant Gauss curvature.

- The surfaces, which are parallel to minimal surfaces, are described by the natural PDE

  \[ \lambda_{xx} + \lambda_{yy} = -e^{\lambda}. \]

- The surfaces, which are parallel to CMC-surfaces \( (H = \text{const}) \), are described by the one-parameter family of natural PDE’s

  \[ \lambda_{xx} + \lambda_{yy} = -2|H|\sinh \lambda. \]

  Up to similarity, the surfaces, which are parallel to CMC-surfaces, are described by the natural PDE of the surfaces with \( H = 1/2 \).

- The surfaces, which are parallel to pseudo-spherical surfaces \( (K = \text{const} < 0) \), are described by the one-parameter family of natural PDE’s

  \[ \lambda_{xx} - \lambda_{yy} = K^2 \sin \lambda. \]

  Up to similarity, the surfaces, which are parallel to pseudo-spherical surfaces, are described by the natural PDE of the surfaces with \( K = -1 \).

We call the surfaces with \( H = 0 \), \( H = 1/2 \) and \( K = -1 \) the basic classes of surfaces in the class of surfaces, determined by the relation (1.1).

Then we have:

*Up to similarity, the surfaces, whose curvatures satisfy the linear relation (1.1), are described by the natural PDE’s of the basic surfaces.*

In [14, 16] Milnor studies surface theory in Euclidean and Minkowski space, considering harmonic maps and various relations between the curvatures \( K, H \) and \( H' = \frac{\nu_1 - \nu_2}{2}. \)
The geometric quantity $\rho_1 - \rho_2$, where $\rho_1 := (\nu_1)^{-1}$, $\rho_2 := (\nu_2)^{-1}$ are the principal radii of curvature on a given surface $S$, has a definite physical meaning, being associated with the interval of Sturm [18], also known as the astigmatic interval, or the amplitude of astigmatism.

A. Ribaucour [17] has proved that a necessary condition for the curvature lines of the first and second focal surfaces (the first and second evolute surfaces) of a given surface $S$ to correspond to each other resp. to conjugate parametric lines on $S$ is $\rho_1 - \rho_2 = \text{const}$ resp. $\rho_1 \rho_2 = \text{const}$. These are the only W-surfaces whose focal sheets have corresponding lines of curvature [2, 4].

Von Lilienthal has proved in [20, 21, 22] that the first focal surface $\tilde{S}$ of a surface $S$ with $\rho_1 - \rho_2 = \text{const}$ is of constant negative Gauss curvature, and vice versa. The involute surfaces $\bar{S}(a)$, $a \in \mathbb{R}$ of $\tilde{S}$ are parallel surfaces of $S$ with the property $\rho_1 - \rho_2 = \text{const}$. This implies that the family $\tilde{S}(a)$ are integrable surfaces as a consequence of the integrability of $\tilde{S}$.

The curvatures of the above surfaces $S$ satisfy the relation $K = p H'$, $p = \text{const}$. On the Weingarten surfaces whose principal radii of curvature are bound by a relation of the form $\rho_1 = c \rho_2$, $c = \text{const}$, the characteristic lines [4] cut at constant angle and only in this case. Moreover, the conjugates of the mean orthogonal lines also cut under constant angle, so that the configuration of all three systems is the same at all points of such a surface [4]. The curvatures of these surfaces satisfy the relation $H = p H'$, $p = \text{const}$.

Obviously the surfaces with $K = p H'$, or $H = p H'$, $p = \text{const}$ are not included in the class characterized by (1.1).

These surfaces belong to the classes of W-surfaces, defined by the following more general linear relation

\begin{equation}
\delta K = \alpha H + \beta H' + \gamma, \quad \alpha, \beta, \gamma, \delta - \text{constants;} \quad \alpha^2 - \beta^2 + 4\gamma\delta \neq 0
\end{equation}

between the Gauss curvature $K$, the mean curvature $H$ and the curvature $H'$. We denote this class by $\mathcal{K}$.

We show that the class $\mathcal{K}$ is the class of linear fractional W-surfaces with respect to the principal curvatures. Furthermore, if $S$ is a surface in $\mathcal{K}$, then its parallel surfaces $\bar{S}(a)$, $a = \text{const}$, belong to $\mathcal{K}$ too.

We determine ten basic relations with respect to the constants in (1.2) and each of them generates a basic subclass of surfaces of $\mathcal{K}$. Any surface $S$, whose invariants $K$, $H$ and $H'$ satisfy the linear relation (1.2) is (locally) parallel to one of these basic surfaces.

According to Theorem 3.2, we find the natural PDE’s of all surfaces of the class $\mathcal{K}$.

It is essential to note that the natural PDE’s of the linear fractional W-surfaces are expressed by the following four operators:

$\Delta \lambda := \lambda_{xx} + \lambda_{yy}$, \quad $\bar{\Delta} \lambda := \lambda_{xx} - \lambda_{yy}$;

$\Delta^* \lambda := \lambda_{xx} + (\lambda^{-1})_{yy}$, \quad $\bar{\Delta}^* \lambda := \lambda_{xx} - (\lambda^{-1})_{yy}$.

The central theorem in this paper is the following

**Theorem A.** Up to similarity, the surfaces in Euclidean space free of umbilical points, whose curvatures $K$, $H$ and $H'$ satisfy the linear relation

\begin{equation}
\delta K = \alpha H + \beta H' + \gamma, \quad \alpha, \beta, \gamma, \delta - \text{constants;} \quad \alpha^2 - \beta^2 + 4\gamma\delta \neq 0,
\end{equation}

are described by the natural PDE’s of the following basic classes of surfaces:
The PDE’s with numbers 1, 2 and 8 are the classical case of integrable equations, which means that the corresponding classes of surfaces are also integrable.

The equation with number 9 is also integrable (cf. [1, 20, 21, 22, 23]).

The PDE’s with numbers 1, 2 and 8 exhaust those of them, which are expressed by the operators $\Delta$ and $\bar{\Delta}$, while the ninth equation is expressed by the operator $\Delta^*$.

For the remaining 6 types PDE’s it is not known if they are integrable, but all they are candidates to be investigated.

As an application we show that the W-surfaces with $\gamma_1 = 0$ are exactly the rotational Weingarten surfaces. As examples we construct the rotational surfaces in the classes (4) and (5) from the above theorem.

In [9] we study analogous problems for space-like surfaces in Minkowski 3-space.

2. Preliminaries

In this section we introduce the standard denotations and formulas in the theory of Weingarten surfaces in Euclidean space, which we use further.
Let $\mathbb{R}^3$ be the three dimensional Euclidean space with the standard flat metric $\langle \cdot, \cdot \rangle$. We assume that the following orthonormal coordinate system $Oe_1e_2e_3 : e_i^2 = 1, \ e_i e_j = 0, \ i \neq j$ is fixed and gives the orientation of the space.

Let $S : z = z(u, v)$, $(u, v) \in \mathcal{D}$ be a surface in $\mathbb{R}^3$ and $\nabla$ be the flat Levi-Civita connection of the metric $\langle \cdot, \cdot \rangle$. The unit normal vector field to $S$ is denoted by $l$ and $E, F, G; \ L, M, N$ stand for the coefficients of the first and the second fundamental forms, respectively.

We suppose that the surface has no umbilical points and the principal lines on $S$ form a parametric net, i.e.

$$F(u, v) = M(u, v) = 0, \ (u, v) \in \mathcal{D}.$$ 

The principal curvatures $\nu_1, \nu_2$ and the principal geodesic curvatures (geodesic curvatures of the principal lines) $\gamma_1, \gamma_2$ of $S$ are given by

$$\nu_1 = \frac{L}{E}, \ \nu_2 = \frac{N}{G}; \ \gamma_1 = -\frac{E_v}{2E\sqrt{G}}, \ \gamma_2 = \frac{G_u}{2G\sqrt{E}}.$$ 

We consider the tangential frame field $\{X, Y\}$ defined by

$$X := \frac{z_u}{\sqrt{E}}, \ Y := \frac{z_v}{\sqrt{G}}$$

and suppose that the moving frame $XYl$ is always positive oriented.

In what follows we consider surfaces with $\nu_1 - \nu_2 > 0$.

The mean curvature and the Gauss curvature of $S$ are denoted as usual by $H$ and $K$, respectively. For our purposes we denote as the third curvature on $S$ the invariant function

$$H' := \frac{\nu_1 - \nu_2}{2} = \sqrt{H^2 - K}.$$

The moving frame field $XYl$ satisfies the following Frenet type formulas:

$$
\begin{align*}
\nabla_X X &= \gamma_1 Y + \nu_1 l, \\
\nabla_X Y &= -\gamma_1 X, \\
\nabla_X l &= -\nu_1 X;
\end{align*}
\begin{align*}
\nabla_Y X &= \gamma_2 Y, \\
\nabla_Y Y &= -\gamma_2 X + \nu_2 l, \\
\nabla_Y l &= -\nu_2 Y.
\end{align*}
$$

The integrability condition $\nabla_X \nabla_Y l - \nabla_Y \nabla_X l - \nabla_{[X,Y]}l = 0$ for this system is equivalent to the Codazzi equations

$$\gamma_1 = \frac{Y(\nu_1)}{\nu_1 - \nu_2} = \frac{(\nu_1)_v}{\sqrt{G}(\nu_1 - \nu_2)}, \quad \gamma_2 = \frac{X(\nu_2)}{\nu_1 - \nu_2} = \frac{(\nu_2)_u}{\sqrt{E}(\nu_1 - \nu_2)},$$

and the integrability condition $\nabla_X \nabla_Y Y - \nabla_Y \nabla_X Y - \nabla_{[X,Y]}Y = 0$ implies the Gauss equation

$$Y(\gamma_1) - X(\gamma_2) - (\gamma_1^2 + \gamma_2^2) = \nu_1 \nu_2 = K.$$ 

The Codazzi equations (2.1) imply the following equivalence

$$\gamma_1 \gamma_2 \neq 0 \iff (\nu_1)_v(\nu_2)_u \neq 0.$$

We consider two types of surfaces parameterized by principal parameters (cf [8]):

- **strongly regular** surfaces, determined by the condition

$$\gamma_1(u, v)\gamma_2(u, v) \neq 0, \ (u, v) \in \mathcal{D};$$

- surfaces, satisfying the conditions

$$\gamma_1(u, v) = 0, \ \gamma_2(u, v) \neq 0, \ (u, v) \in \mathcal{D}.$$ 

These surfaces are rotational surfaces [8] and their meridians are the first system of principal lines.
Because of (2.1), the coefficients of the first fundamental form for strongly regular surfaces can be expressed as functions of \( \nu_1, \nu_2, \gamma_1, \gamma_2 \) as follows:

\[
\sqrt{E} = \frac{(\nu_2)_u}{\gamma_2 (\nu_1 - \nu_2)} > 0, \quad \sqrt{G} = \frac{(\nu_1)_v}{\gamma_1 (\nu_1 - \nu_2)} > 0.
\]

A surface \( S : z = z(u, v), (u, v) \in \mathcal{D} \) is Weingarten if there exist two differentiable functions \( f(\nu), g(\nu), f(\nu) - g(\nu) \neq 0, f'(\nu)g'(\nu) \neq 0 \), \( \nu \in \mathcal{I} \subseteq \mathbb{R} \) such that the principal curvatures of \( S \) at every point are given by \( \nu_1 = f(\nu), \nu_2 = g(\nu), \nu = \nu(u, v), (\nu_u(u, v), \nu_v(u, v)) \neq (0, 0), (u, v) \in \mathcal{D} \).

Let \( S : z = z(u, v), (u, v) \in \mathcal{D} \) be a Weingarten surface parameterized by principal parameters. In [8] we proved that on \( S \) the function

\[
\lambda = \sqrt{E} \exp \left( \int \frac{f' d\nu}{f - g} \right)
\]
does not depend on \( v \), while the function

\[
\mu = \sqrt{G} \exp \left( \int \frac{g' d\nu}{g - f} \right)
\]
does not depend on \( u \).

The principal parameters \((u, v)\) are natural principal parameters [8] if

\[
\lambda(u) = \text{const}, \quad \mu(v) = \text{const}.
\]

Let \( a = \text{const} \neq 0, b = \text{const} \neq 0, (u_0, v_0) \) be a fixed point in \( \mathcal{D} \) and \( \nu_0 := \nu(u_0, v_0) \). The change of the parameters \((u, v) \in \mathcal{D}\) with \((\bar{u}, \bar{v}) \in \mathcal{D}\) by the formulas

\[
\bar{u} = a \int_{u_0}^{u} \sqrt{E} \exp \left( \int \frac{f' d\nu}{f - g} \right) du + \bar{u}_0, \quad \bar{u}_0 = \text{const},
\]

\[
\bar{v} = b \int_{v_0}^{v} \sqrt{G} \exp \left( \int \frac{g' d\nu}{g - f} \right) dv + \bar{v}_0, \quad \bar{v}_0 = \text{const}
\]
endows the surface \( S \) with natural principal parameters \((\bar{u}, \bar{v})\).

With respect to the natural principal parameters \((\bar{u}, \bar{v})\) we get

\[
E = \frac{1}{a^2} \exp \left( -2 \int_{\nu_0}^{\nu} \frac{f' d\nu}{f - g} \right), \quad G = \frac{1}{b^2} \exp \left( -2 \int_{v_0}^{v} \frac{g' d\nu}{g - f} \right)
\]
with

\[
a^2 E(u_0, v_0) = 1, \quad b^2 G(u_0, v_0) = 1.
\]

Let a Weingarten surface \( S : z = z(u, v), (u, v) \in \mathcal{D} \) be parameterized by principal parameters \((u, v)\). These parameters are natural principal if and only if [8]

\[
\sqrt{EG}(\nu_1 - \nu_2) = \text{const} \neq 0.
\]

The main theorem for Weingarten surfaces in [8, Theorems 5.8 and 5.13] is

**Theorem 2.1.** Given two differentiable functions \( f(\nu), g(\nu); \nu \in \mathcal{I}, \quad f(\nu) - g(\nu) \neq 0, \quad f'(\nu)g'(\nu) \neq 0 \) and a differentiable function \( \nu(u, v), (u, v) \in \mathcal{D} \) satisfying the condition \((\nu_u, \nu_v) \neq (0, 0), \quad \nu(u, v) \in \mathcal{I}\).
Let \( (u_0, v_0) \in \mathcal{D}, \nu_0 = \nu(u_0, v_0) \) and \( a \neq 0, b \neq 0 \) be two constants. If

\[
\begin{align*}
&b^2 \exp \left( 2 \int_{v_0}^{\nu} \frac{g' d\nu}{g - f} \right) \left[ f'\nu_{\nu
u} + \left( f'' - \frac{2f'^2}{f - g} \right) \nu_{\nu}^2 \right] \\
&- a^2 \exp \left( 2 \int_{v_0}^{\nu} \frac{f' d\nu}{f - g} \right) \left[ g'\nu_{\nu
u} + \left( g'' - \frac{2g'^2}{g - f} \right) \nu_{\nu}^2 \right] - fg(f - g) = 0,
\end{align*}
\]

then there exists a unique (up to a motion) Weingarten surface

\[ S: z = z(u, v), \ (u, v) \in \mathcal{D}_0 \subset \mathcal{D} \text{ with invariants} \]

\[
\nu_1 = f(\nu), \quad \nu_2 = g(\nu),
\]

\[
\gamma_1 = \exp \left( \int_{v_0}^{\nu} \frac{g' d\nu}{g - f} \right) \frac{bf'}{f - g} \nu_{\nu}, \quad \gamma_2 = - \exp \left( \int_{v_0}^{\nu} \frac{f' d\nu}{f - g} \right) \frac{ag'}{g - f} \nu_{\nu}.
\]

Furthermore, \( (u, v) \) are natural principal parameters for \( S \).

Hence, with respect to natural principal parameters each Weingarten surface possesses a natural PDE (2.5).

We show briefly that the condition \( \gamma_1 = 0 \), i.e. \( \nu = \nu(u) \), in Theorem 2.1 characterizes the class of rotational W-surfaces and the natural PDE of any rotational W-surface reduces to an ODE.

In [8, Section 3] we described locally in a geometric (constructive) way the class of surfaces whose first family \( \mathcal{F}_1 \) of principal lines consists of geodesics.

Let \( c_2: x = x(v), \ v \in J_2, \) be a smooth regular curve in \( \mathbb{R}^3 \) parameterized by a natural parameter \( v \) with vector invariants \( t(v), n(v), b(v) \), curvature \( \kappa(v) > 0 \) and torsion \( \tau(v) \).

A unit normal vector field \( y(v) \) along the curve \( c_2 \) is said to be \textit{torse-forming} if \( \nabla_t y = \alpha t \) for a certain function \( \alpha(v), \ v \in J_2 \) on the curve \( c_2 \).

We consider an orthonormal pair \( \{y_1(v), y_2(v)\} \) of torse-forming normals along the curve \( c_2 \) and denote by \( \theta = \angle(n(v), y_1(v)) \). The vector pair

\[
\begin{align*}
y_1 &= \cos \theta n + \sin \theta b, \\
y_2 &= - \sin \theta n + \cos \theta b,
\end{align*}
\]

is determined uniquely up to a constant angle \( \theta_0 \) by the condition \( \theta(v) = - \int_{0}^{v} \tau(v) dv + \theta_0 \).

We choose \( \theta_0 = 0 \), i.e. the pair \( \{y_1(v), y_2(v)\} \) satisfies the initial conditions \( y_1(0) = n(0), \ y_2(0) = b(0) \).

The orthonormal frame field \( t(v)y_1(v)y_2(v) \) satisfies the Frenet type formulas

\[
\begin{align*}
t' &= \kappa \cos \theta y_1 - \kappa \sin \theta y_2, \\
y_1' &= - \kappa \cos \theta t, \\
y_2' &= \kappa \sin \theta t.
\end{align*}
\]

For any \( v \in J_2 \) we consider the regular plane curve

\[ c_1: z(s_1) = x(v) + \lambda(s_1)y_1(v) + \mu(s_1)y_2(v) \quad s_1 \in J_1, \]

rigidly connected with every Cartesian coordinate system \( x(v)y_1(v)y_2(v) \). We suppose that the parameter \( s_1 \) is natural \( (\dot{\lambda}^2 + \dot{\mu}^2 = 1) \) for \( c_1 \) and the functions \( \lambda, \mu \) satisfy the initial conditions \( \lambda(0) = \mu(0) = 0; \ \dot{\lambda}(0) = 0, \ \dot{\mu}(0) = 1 \). Then the (plane) curvature \( \kappa_1 = \kappa_1(s_1) > 0 \).
of $c_1$ completely determines the functions $\lambda$ and $\mu$:

$$
\lambda(s_1) = \int_0^{s_1} \sin \left( \int_0^{s_1} \kappa_1(s_1) ds_1 \right) ds_1, \quad \mu(s_1) = \int_0^{s_1} \cos \left( \int_0^{s_1} \kappa_1(s_1) ds_1 \right) ds_1.
$$

Now, let us consider the surface

$$(2.6) \quad S: Z(s_1, v) = x(v) + \lambda(s_1) y_1(v) + \mu(s_1) y_2(v); \quad s_1 \in J_1, \quad v \in J_2.$$ Computing

$$Z_{s_1} = \dot{\lambda} y_1 + \dot{\mu} y_2, \quad Z_v = [1 - \kappa(\lambda \cos \theta - \mu \sin \theta)] t,$$

$$Z_{s_1} \times Z_v = -[1 - \kappa(\lambda \cos \theta - \mu \sin \theta)](-\dot{\mu} y_1 + \dot{\lambda} y_2),$$

we get that the surface $S$ is smooth at the points, where

$$\lambda \cos \theta - \mu \sin \theta \neq \frac{1}{\kappa}.$$ We orientate the surface $S$ by choosing $l = -\dot{\mu} y_1 + \dot{\lambda} y_2$, i.e. the normal to $S$ is the plane normal to $c_1$, and put $X = Z_{s_1}, \ Y = t$.

Let $\Gamma$ be the class of surfaces (2.6) under the smoothness condition.

*Any surface $S$ of the class $\Gamma$ has the following properties:
  1) the parametric lines are principal;
  2) the family $\mathcal{F}_1$ consists of geodesics.*

The invariants of any surface from the class $\Gamma$ are:

$$\nu_1 = \kappa_1(s_1) > 0, \quad \gamma_1 = \tau_1 = 0;$$

$$\nu_2 = \frac{-\kappa(\lambda \sin \theta + \mu' \cos \theta)}{1 - \kappa(\lambda \cos \theta - \mu \sin \theta)}; \quad \gamma_2 = \frac{-\kappa(\lambda' \cos \theta - \mu' \sin \theta)}{1 - \kappa(\lambda \cos \theta - \mu \sin \theta)};$$

$$\kappa_2 = \frac{\kappa^2}{(1 - \kappa(\lambda \cos \theta - \mu \sin \theta))^2}, \quad \tau_2 = \frac{\gamma_2 Y(\nu_2) - \nu_2 Y(\gamma_2)}{\gamma_2^2 + \nu_2^2}.$$ Theorem 3.2 states that:

*Let $S$ be a surface parameterized by principal parameters. If the family $\mathcal{F}_1$ of principal lines consists of regular geodesics, then $S$ is locally part of a surface from the class $\Gamma$.*

Let now $S$ be a W-surface, parameterized by natural principal parameters $(u, v)$ and let its family $\mathcal{F}_1$ of principal lines consists of regular geodesics, i.e. $\gamma_1 = 0$. The geometric function $\nu(u, v)$ in Theorem 2.1 depends in this case on $u$ only. What is more, the invariants $\nu_1, \nu_2, \gamma_2$ of $S$ are also functions of $u$ only and

$$\frac{ds_1}{du} = \sqrt{E} = \frac{1}{a^2} \exp \left( - \int_0^v \frac{f'(v) dv}{f(v) - g(v)} \right).$$

Since $S$ is at the same time a surface from the class $\Gamma$, then the principal curvature $\nu_2$ in (2.7) does not depend of $v$. This is equivalent to the conditions

$$(\kappa(v) \sin \theta(v))_v = (\kappa(v) \cos \theta(v))_v = 0.$$ Hence $\kappa = \text{const} > 0, \ \theta = \text{const} = 0, \ \tau_2 = 0$ and the curve $c_2 : x = x(v)$ is a circle.

Choosing for the curve $c_2$ the initial conditions $\kappa(0) = 1$ and $b(0) = e_3$, with respect to the cartesian coordinate system $Oe_1e_2e_3$ in $\mathbb{R}^3$ we get the meridian of the rotational W-surface $S$ to be the curve

$$x_1 = 1 - \lambda(u), \quad x_2 = 0, \quad x_3 = \mu(u),$$
and its rotational axis is the coordinate axis Ox3.

The natural ODE of S is
\[
a^2 \exp \left( 2 \int_{\nu_0}^{\nu} \frac{f'd\nu}{f-g} \right) \left[ g'\nu_{uu} + \left( g'' - \frac{2g'^2}{g-f} \right) \nu_u^2 \right] + fg(f-g) = 0.
\]

3. Parallel surfaces and their natural PDE’s

Let \( S : z = z(u, v) \), \((u, v) \in D\) be a surface, parameterized by principal parameters and \( l(u, v) \) be the unit normal vector field of \( S \). The parallel surfaces of \( S \) are given by
\[
\bar{S}(a) : \bar{z}(u, v) = z(u, v) + a l(u, v), \quad a = \text{const} \neq 0, \quad (u, v) \in D.
\]

We call the family \( \{ \bar{S}(a), a = \text{const} \neq 0 \} \) the parallel family of \( S \).

In this section we prove that a W-surface and its parallel family have the same natural PDE.

Taking into account (3.1), we find
\[
(3.2) \quad \bar{z}_u = (1 - a \nu_1) z_u, \quad \bar{z}_v = (1 - a \nu_2) z_v.
\]

Excluding the points, where \( (1 - a \nu_1)(1 - a \nu_2) = 0 \), we obtain that the corresponding unit normal vector fields \( \bar{l} \) to \( S(a) \) and \( l \) to \( S \) satisfy the equality
\[
\bar{l} = \varepsilon l, \quad \text{where} \quad \varepsilon := \text{sign} (1 - a \nu_1)(1 - a \nu_2).
\]

Then the relations between the principal curvatures \( \nu_1(u, v), \nu_2(u, v) \) of \( S \) and \( \bar{\nu}_1(u, v), \bar{\nu}_2(u, v) \) of its parallel surface \( \bar{S}(a) \) are
\[
(3.3) \quad \bar{\nu}_1 = \varepsilon \frac{\nu_1}{1 - a \nu_1}, \quad \bar{\nu}_2 = \varepsilon \frac{\nu_2}{1 - a \nu_2}; \quad \nu_1 = \frac{\varepsilon \bar{\nu}_1}{1 + a \varepsilon \bar{\nu}_1}, \quad \nu_2 = \frac{\varepsilon \bar{\nu}_2}{1 + a \varepsilon \bar{\nu}_2}.
\]

Let \( K, H, H' \) be the three invariants of the surface \( S \). The equalities (3.3) imply the relations between the invariants \( \bar{K}, \bar{H} \) and \( \bar{H}' \) of \( \bar{S}(a) \) and the corresponding invariants of \( S \):
\[
(3.4) \quad K = \frac{\bar{K}}{1 + 2a \varepsilon \bar{H} + a^2 \bar{K}}, \quad H = \frac{\varepsilon \bar{H} + a \bar{K}}{1 + 2a \varepsilon \bar{H} + a^2 \bar{K}}, \quad H' = \frac{\varepsilon \bar{H}'}{1 + 2a \varepsilon \bar{H} + a^2 \bar{K}}.
\]

Now let \( S : z = z(u, v) \), \((u, v) \in D\) be a Weingarten surface with Weingarten functions \( f(\nu) \) and \( g(\nu) \). We suppose that \((u, v)\) are natural principal parameters for \( S \). We show that \((u, v)\) are also natural principal parameters for any parallel surface \( \bar{S}(a) \).

**Proposition 3.1.** The natural principal parameters \((u, v)\) of a given W-surface \( S \) are natural principal parameters for all parallel surfaces \( \bar{S}(a) \), \( a = \text{const} \neq 0 \) of \( S \).

**Proof:** Let \((u, v) \in D\) be natural principal parameters for \( S \), \((u_0, v_0)\) be a fixed point in \( D \) and \( \nu_0 = \nu(u_0, v_0) \). The coefficients \( E \) and \( G \) of the first fundamental form of \( S \) are given by (2.3). The corresponding coefficients \( \bar{E} \) and \( \bar{G} \) of \( \bar{S}(a) \) in view of (3.2) are
\[
(3.5) \quad \bar{E} = (1 - a \nu_1)^2 E, \quad \bar{G} = (1 - a \nu_2)^2 G.
\]

Equalities (3.3) imply that \( \bar{S}(a) \) is again a Weingarten surface with Weingarten functions
\[
(3.6) \quad \bar{\nu}_1(u, v) = \bar{f}(\nu) = \frac{\varepsilon f(\nu)}{1 - a f(\nu)}, \quad \bar{\nu}_2(u, v) = \bar{g}(\nu) = \frac{\varepsilon g(\nu)}{1 - a g(\nu)}.
\]

Using (3.6), we compute
\[
\bar{f} - \bar{g} = \frac{\varepsilon(f - g)}{(1 - a f)(1 - a g)},
\]
which shows that \( \text{sign} (\bar{f} - \bar{g}) = \text{sign} (f - g) \).
Further, we denote by \( f_0 := f(\nu_0), \ g_0 := g(\nu_0) \) and taking into account (2.4) and (3.5), we compute
\[
\sqrt{E \ G}(\bar{f} - \bar{g}) = \sqrt{E \ G}(f - g) = \text{const},
\]
which proves the assertion.

Using the above statement, we prove the following theorem.

**Theorem 3.2.** The natural PDE of a given \( W \)-surface \( S \) is the natural PDE of any parallel surface \( \bar{S}(a) \), \( a = \text{const} \neq 0 \), of \( S \).

**Proof.** We have to express the equation (2.5) in terms of the Weingarten functions of the parallel surface \( \bar{S}(a) \) of \( S \). Using (3.6), we compute successively
\[
\exp \left( 2 \int_{\nu_0}^{\nu} \frac{\dot{g}}{\bar{g} - \bar{f}} \right) \left( \dot{f} \nu_{\nu\nu} + \left( \bar{f} - \frac{2 \dot{f}^2}{\bar{g} - \bar{f}} \right) \nu_{\nu}^2 \right)
\]
\[
= \varepsilon \frac{(1 - a \ g_0)^2}{(1 - a \ f)^2(1 - a \ g)^2} \exp \left( 2 \int_{\nu_0}^{\nu} \frac{g'}{\bar{g} - \bar{f}} \right) \left( f' \nu_{\nu\nu} + \left( f'' - \frac{2 f'^2}{\bar{g} - \bar{f}} \right) \nu_{\nu}^2 \right)
\]
and
\[
\exp \left( 2 \int_{\nu_0}^{\nu} \frac{\dot{g}}{\bar{f} - \bar{g}} \right) \left( \dot{g} \nu_{\nu\nu} + \left( \bar{g} - \frac{2 \dot{g}^2}{\bar{f} - \bar{g}} \right) \nu_{\nu}^2 \right)
\]
\[
= \varepsilon \frac{(1 - a \ f_0)^2}{(1 - a \ f)^2(1 - a \ g)^2} \exp \left( 2 \int_{\nu_0}^{\nu} \frac{f'}{\bar{f} - \bar{g}} \right) \left( g' \nu_{\nu\nu} + \left( g'' - \frac{2 g'^2}{\bar{f} - \bar{g}} \right) \nu_{\nu}^2 \right).
\]

We have also
\[
\bar{f} \bar{g} (\bar{f} - \bar{g}) = \frac{\varepsilon}{(1 - a \ f)^2(1 - a \ g)^2} \ f \ g (f - g).
\]

Putting
\[
\bar{E}_0 = (1 - a \nu_1(u_0, \nu_0))^2 \ E_0 = a^{-2} (1 - a \ f_0)^2 =: \bar{a}^{-2},
\]
\[
\bar{G}_0 = (1 - a \nu_2(u_0, \nu_0))^2 \ G_0 = b^{-2} (1 - a \ g_0)^2 =: \bar{b}^{-2},
\]
we compute the left hand side of (2.5) to be
\[
\bar{b}^2 \exp \left( 2 \int_{\nu_0}^{\nu} \frac{\dot{g}}{\bar{g} - \bar{f}} \right) \left( \dot{f} \nu_{\nu\nu} + \left( \bar{f} - \frac{2 \dot{f}^2}{\bar{g} - \bar{f}} \right) \nu_{\nu}^2 \right)
\]
\[
- \bar{a}^2 \exp \left( 2 \int_{\nu_0}^{\nu} \frac{\dot{f}}{\bar{f} - \bar{g}} \right) \left( \dot{g} \nu_{\nu\nu} + \left( \bar{g} - \frac{2 \dot{g}^2}{\bar{f} - \bar{g}} \right) \nu_{\nu}^2 \right)
\]
\[
- \bar{f} \bar{g} (\bar{f} - \bar{g})
\]
\[
= \bar{b}^2 \exp \left( 2 \int_{\nu_0}^{\nu} \frac{g' \ d\nu}{\bar{g} - \bar{f}} \right) \left( f' \nu_{\nu\nu} + \left( f'' - \frac{2 f'^2}{\bar{g} - \bar{f}} \right) \nu_{\nu}^2 \right)
\]
\[
- \bar{a}^2 \exp \left( 2 \int_{\nu_0}^{\nu} \frac{f' \ d\nu}{\bar{f} - \bar{g}} \right) \left( g' \nu_{\nu\nu} + \left( g'' - \frac{2 g'^2}{\bar{f} - \bar{g}} \right) \nu_{\nu}^2 \right)
\]
\[
- \bar{f} \bar{g} (f - g).
\]

Hence, the natural PDE of \( \bar{S}(a) \) in terms of the Weingarten functions \( \bar{f}(\nu), \bar{g}(\nu) \) coincides with the natural PDE of \( S \) in terms of the Weingarten functions \( f(\nu) \) and \( g(\nu) \).

\[\square\]
4. Surfaces whose curvatures satisfy a linear relation and proof of Theorem A

We now consider surfaces, whose three invariants $K$, $H$ and $H'$ satisfy a linear relation:

$$\delta K = \alpha H + \beta H' + \gamma, \quad \alpha, \beta, \gamma, \delta - \text{constants}, \quad \alpha^2 - \beta^2 + 4\gamma\delta \neq 0.$$  

In [8] we introduced linear fractional Weingarten surfaces as Weingarten surfaces whose principal curvature functions $\nu_1$ and $\nu_2$ are related as follows

$$\nu_1 = \frac{A\nu_2 + B}{C\nu_2 + D}, \quad A, B, C, D - \text{constants}, \quad BC - AD \neq 0.$$  

We show that the classes of surfaces with characterizing conditions (4.1) and (4.2), respectively, coincide.

**Lemma 4.1.** Any surface whose invariants $K, H, H'$ satisfy a linear relation (4.1) is a linear fractional Weingarten surface determined by the relation (4.2), and vice versa.

The relations between the constants $\alpha, \beta, \gamma, \delta$ and $A, B, C, D$ are given by the equalities:

$$\alpha = A - D, \quad \beta = -(A + D), \quad \gamma = B, \quad \delta = C.$$  

We denote by $\mathcal{K}$ the class of all surfaces, free of umbilical points, whose curvatures satisfy (4.1) or equivalently (4.2).

The aim of our study is to classify all natural PDE’s of the surfaces from the class $\mathcal{K}$.

The scheme of our investigations is the following:

The parallelism between two surfaces given by (3.1) is an equivalence relation. On the other hand, Theorem 3.2 shows that the surfaces from an equivalence class have one and the same natural PDE. Hence, it is sufficient to find the natural PDE’s of the equivalence classes. For any equivalence class, we use a special representative, which we call a basic class. Thus the classification of the natural PDE’s of the surfaces in the class $\mathcal{K}$ reduces to the classification of the natural PDE’s of the basic classes.

There are two important classes of linear fractional Weingarten surfaces: the class of linear W-surfaces and the general class of linear fractional W-surfaces.

**I.** Let $S$ be a linear W-surface, i.e. $C = 0$ in (4.2). Then the equality (4.1) gets the form

$$\alpha H + \beta H' + \gamma = 0, \quad (\alpha, \gamma) \neq (0, 0), \quad \alpha^2 - \beta^2 \neq 0.$$  

In this case for the invariants of the parallel surface $\tilde{S}(a)$ of $S$, because of (3.4), we get the relation

$$\varepsilon (\alpha + 2a \gamma) \tilde{H} + \varepsilon \beta \tilde{H}' + \gamma = -a (\alpha + a \gamma) \tilde{K}.$$  

**II.** Let $S$ be an essential linear fractional W-surface, i.e. $C \neq 0$ ($C = 1$) in (4.2). Then the equality (4.1) gets the form

$$K = \alpha H + \beta H' + \gamma.$$  

The corresponding relation between the invariants of the parallel surface $\tilde{S}(a)$ of $S$ is

$$\varepsilon(\alpha + 2a \gamma) \tilde{H} + \varepsilon \beta \tilde{H}' + \gamma = (1 - a \alpha - a^2 \gamma) \tilde{K}.$$  

**Proof of Theorem A.**

Let $S$ be a W-surface from the class $\mathcal{K}$. 

Each time choosing appropriate values for the constants $a$, $b$ and $\nu_0$ in (2.5) we get the following subclasses of W-surfaces of the class $\mathfrak{A}$ and their natural PDE’s.

I. The class of linear W-surfaces.

Let $\eta := \text{sign}(\alpha^2 - \beta^2)$. In this case we have the following subclasses:

1) $\alpha = 0$, $\beta \neq 0$, $\gamma \neq 0$. Assuming that $\gamma = 1$, the relation (4.4) becomes

$$\beta H' + 1 = 0.$$  

Choosing $a^2 = e^{-\beta \nu_0}$ and $b^2 = e^{\beta \nu_0}$, the natural PDE for these W-surfaces becomes

$$(e^{\beta \nu})_{\nu\nu} + (e^{-\beta \nu})_{\nu\nu} = \frac{2}{\beta} \nu (\beta \nu - 2).$$

Up to similarities these W-surfaces are generated by the basic class $H' = 1$ ($\beta = -1$) with the natural PDE

$$\Delta^*(e^\nu) = -2 \nu (\nu + 2),$$

which is the class (3) in the statement of the theorem.

2) $\alpha \neq 0$, $\gamma = 0$. Assuming that $\alpha = 1$, the relation (4.4) becomes

$$H + \beta H' = 0.$$  

2.1) $\beta \neq 0$, $\eta = -1$ ($\beta^2 - 1 > 0$). Choosing $b^2 \frac{\beta - 1}{\beta + 1} \nu_0^{-(\beta + 1)} = 1$, $a^2 \nu_0^{\beta - 1} = 1$, the natural PDE for these W-surfaces becomes

$$(\nu^\beta)_{\nu\nu} + (\nu^{-\beta})_{\nu\nu} = -2 \frac{\beta(\beta - 1)}{(\beta + 1)^2} \nu.$$  

Putting $p = -\beta$, we get the natural PDE

$$\Delta^*(\nu^p) = -2 \frac{p(p + 1)}{(p - 1)^2} \nu$$

for the basic class (4) in the statement of the theorem.

2.2) $\beta \neq 0$, $\eta = 1$ ($\beta^2 - 1 < 0$). Choosing $b^2 \frac{\beta - 1}{\beta + 1} \nu_0^{-(\beta + 1)} = -1$, $a^2 \nu_0^{\beta - 1} = 1$, the natural PDE for these W-surfaces becomes

$$(\nu^\beta)_{\nu\nu} - (\nu^{-\beta})_{\nu\nu} = 2 \frac{\beta(\beta - 1)}{(\beta + 1)^2} \nu.$$  

Putting $p = -\beta$, we get the natural PDE

$$\bar{\Delta}^*(\nu^p) = -2 \frac{p(p + 1)}{(p - 1)^2} \nu$$

for the basic class (5) in the statement of the theorem.

2.3) $\beta = 0$. Putting $\nu = -e^\lambda$, we get the natural PDE for minimal surfaces (the elliptic Liouville equation)

$$\Delta \lambda + e^\lambda = 0,$$

which is the basic class (1) in the statement of the theorem.
3) $\alpha \neq 0$, $\beta = 0$, $\gamma \neq 0$. Assuming that $\alpha = 1$, the relation (4.4) becomes

$$H + \gamma = 0.$$ 

Putting $|H| e^\lambda := H - \nu = H' > 0$, we get the one-parameter system of natural PDE's for CMC surfaces with $H = -\gamma$:

$$\Delta \lambda = -2 |H| \sinh \lambda.$$ 

Up to similarities these W-surfaces are generated by the basic class $|H| = \frac{1}{2}$ with the natural PDE

$$\Delta \lambda = - \sinh \lambda,$$

which is the class (2) in the statement of the theorem.

4) $\alpha \neq 0$, $\beta \neq 0$, $\gamma \neq 0$. Assuming that $\alpha = 1$ we have

$$H + \beta H' + \gamma = 0, \quad \beta^2 - 1 \neq 0.$$ 

Let $\lambda := 2 H' = \frac{-2}{\beta + 1} (\nu + \gamma) > 0$.

4.1) If $\eta = -1$ ($\beta^2 - 1 > 0$), we choose

$$b^2 = \frac{\beta + 1}{\beta - 1} \left( \frac{-2}{\beta + 1} (\nu_0 + \gamma) \right)^{\beta + 1}, \quad a^2 = \left( \frac{-2}{\beta + 1} (\nu_0 + \gamma) \right)^{-(\beta - 1)}.$$

The natural PDE becomes

$$\left( \lambda^\beta \right)_{vv} - \left( \lambda^{-\beta} \right)_{uu} = \frac{-\beta}{2 (\beta + 1)} \frac{((\beta + 1)\lambda + 2 \gamma)((\beta - 1)\lambda + 2 \gamma)}{\lambda}.$$ 

Up to similarities these W-surfaces are generated by the basic class $H = p H' + 1$, $p^2 > 1$ ($p = -\beta$, $\gamma = -1$) with the natural PDE

$$\Delta^*(\lambda^p) = -p \frac{(p - 1)\lambda + 2)((p + 1)\lambda + 2)}{2 (p - 1) \lambda},$$

which is the class (6) in the statement of the theorem.

4.2) If $\eta = 1$ ($\beta^2 - 1 < 0$), we choose

$$b^2 = \frac{\beta + 1}{\beta - 1} \left( \frac{-2}{\beta + 1} (\nu_0 + \gamma) \right)^{\beta + 1}, \quad a^2 = \left( \frac{-2}{\beta + 1} (\nu_0 + \gamma) \right)^{-(\beta - 1)}.$$

The natural PDE becomes

$$\left( \lambda^\beta \right)_{vv} + \left( \lambda^{-\beta} \right)_{uu} = \frac{\beta}{2 (\beta + 1)} \frac{((\beta + 1)\lambda + 2 \gamma)((\beta - 1)\lambda + 2 \gamma)}{\lambda}.$$ 

Up to similarities these W-surfaces are generated by the basic class $H = p H' + 1$, $p^2 < 1$ ($p = -\beta$, $\gamma = -1$) with the natural PDE

$$\Delta^*(\lambda^p) = -p \frac{(p - 1)\lambda + 2)((p + 1)\lambda + 2)}{2 (p - 1) \lambda},$$

which is the class (7) in the statement of the theorem.

II. The general class of linear fractional W-surfaces.

We consider the subclasses:
5) $\alpha = \gamma = 0$, $\beta \neq 0$. The relation (4.6) becomes

$$K = \beta H' \iff \rho_1 - \rho_2 = -2 \beta^{-1},$$

where $\rho_1 = (\nu_1)^{-1}$, $\rho_2 = (\nu_2)^{-1}$ are the principal radii of curvature of $S$.

Putting $\lambda := 4 \frac{\nu - \beta}{2 \nu - \beta}$ and choosing $\nu_0 = \beta$, the natural PDE of these surfaces gets the form

$$(e^\lambda)_{uu} + (e^{-\lambda})_{vv} + \frac{\beta^4}{8} = 0.$$

Up to similarities these W-surfaces are generated by the basic class $K = 2 H'$ with the natural PDE

$$\Delta^*(e^\lambda) = -2,$$

which is the class (9) in the statement of the theorem.

6) $(\alpha, \gamma) \neq (0, 0)$, $\alpha^2 + 4 \gamma \geq 0$. The relation (4.7) implies that there exists a constant $a$, such that $\gamma a^2 + \alpha a - 1 = 0$, so that the surface $\bar{S}(a)$, parallel to $S$, has curvatures satisfying the relation (4.4). Hence the natural PDE of this surface $\bar{S}(a)$ is one of the PDE’s in the linear case.

7) $\alpha^2 + 4 \gamma < 0$. It follows that $\gamma < 0$. The relation (4.7) implies that there does not exist a constant $a$, such that $\gamma a^2 + \alpha a - 1 = 0$, but for $a = -\frac{\alpha}{2 \gamma}$ the surface $\bar{S}(a)$, parallel to $S$, has curvatures satisfying a relation

$$(4.8) \quad K = \beta H' + \gamma.$$

7.1) $\beta = 0$. The relation (4.8) becomes $K = \gamma < 0$, i.e. $S$ is of constant negative sectional curvature $\gamma$. Putting $\lambda := 2 \arctan \frac{\nu}{\sqrt{-\gamma}}$, we get the natural PDE of the surface $S$

$$\tilde{\Delta} \lambda = K^2 \sin \lambda.$$

Up to similarities these W-surfaces are generated by the basic class $K = -1$ with the natural PDE

$$\tilde{\Delta} \lambda = \sin \lambda,$$

which is the class (8) in the statement of the theorem.

7.2) $\beta \neq 0$, $\gamma < 0$. The natural PDE of $S$ is

$$(\exp(\beta I))_{uu} + (\exp(-\beta I))_{vv} = \frac{\beta \gamma}{2} \frac{\lambda (\beta \lambda + 2 \gamma)}{\lambda^2 - \gamma},$$

where

$$I = \frac{1}{\sqrt{-\gamma}} \arctan \frac{\lambda}{\sqrt{-\gamma}}, \quad \lambda := \nu - \frac{\beta}{2},$$

which is the class (10) in the statement of the theorem. \qed

Finally we note that any of the ten basic classes of W-surfaces contains an important subclass consisting of rotational surfaces.

As an example we describe and construct the rotational surfaces in the basic classes (4) and (5) from Theorem A.
The principal curvatures of any surface $S$ in the class (4) or (5) satisfy the relation $\nu_1 = \frac{\beta + 1}{\beta - 1} \nu_2$ with $\beta \neq 0, \pm 1$. Choosing $\nu_2 = \nu(u, v)$, the natural parameter $s_1$ and the curvature $\kappa_1$ of any curve of the first family of principal lines of $S$ are respectively \cite{8} \cite{9}

$$\frac{ds_1}{du} = \sqrt{E} = \nu^{-\frac{\beta+1}{2}}, \quad \kappa_1^2 = \nu_1^2 + \gamma_1^2 = \left(\frac{\beta + 1}{\beta - 1}\right)^2 \nu^2 \left[1 + \left(\frac{\beta - 1}{\beta + 3}\right) \nu^{-(-\beta - 3)} \nu_1^2\right].$$

Let now $S$ be a rotational surface and the meridians of $S$ be the curves of the first family of principal lines. From Theorem 2.1 we get $\gamma_1 \equiv 0$ and $\nu = \nu(u)$. Thus any meridian of $S$ is given by the parametric equations

$$\nu_1 = \left|\frac{\beta + 1}{\beta - 1}\right|, \quad s_1(u) = \int_{u_0}^{u} \nu^{-\frac{\beta+1}{2}} du.$$

The natural ODE of $S$ is

$$(\nu^\beta)_{uu} = -2 \frac{\beta(\beta + 1)}{(\beta - 1)^2} \nu,$$

and the natural equation $\kappa_1 = \kappa_1(s_1)$ of any meridian of $S$ is a solution of the ODE

$$\kappa_1'' + \frac{2}{\beta + 1} \kappa_1^3 = 0,$$

where the derivatives of $\kappa_1$ are taken with respect to the natural parameter $s_1$.

We recall that the Mylar Balloon (Latex Balloon) is constructed by taking two circular disks of Mylar, sewing them along their boundaries and then inflating with gas. The shape of the balloon, when it is fully inflated, is a rotational surface with principal curvatures satisfying the equality $\nu_1 = 2 \nu_2$, i.e. the Mylar Balloon is a rotational surface from the class (4) \cite{13}.

Using the construction in section 2, it is easy to be shown that the surface $S$ with

$$\lambda(u) = \int \nu^{-\frac{1+\beta}{2}} \sin \left(\frac{\beta + 1}{\beta - 1} \int \nu^{-\frac{1+\beta}{2}} du\right) du, \quad \mu(u) = \int \nu^{-\frac{1+\beta}{2}} \cos \left(\frac{\beta + 1}{\beta - 1} \int \nu^{-\frac{1+\beta}{2}} du\right) du,$$

is a rotational surface in the basic class (4) ($\beta^2 > 1$), or (5) ($\beta^2 < 1$).

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