On the modular classes of Poisson-Nijenhuis manifolds

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Abstract We prove a property of the Poisson-Nijenhuis manifolds which yields new proofs of the bihamiltonian properties of the hierarchy of modular vector fields defined by Damianou and Fernandes.

Introduction

In [2], Damianou and Fernandes defined the modular vector field and the modular class of a Poisson-Nijenhuis manifold, and they proved that the hierarchy generated by the modular vector field coincides with the canonical hierarchy of bihamiltonian vector fields already defined in [5]. A theorem of Beltrán and Monterde [1] states that, in a PN-manifold, the derived bracket (see e.g. [3]) of the interior products by $N$ and $P$ acting on forms is the interior product by the hamiltonian vector field with hamiltonian $-\frac{1}{2}\text{Tr}N$. In this Letter, we give an elementary proof of a particular case of this theorem, a simple consequence of which, stated in Corollary [11], enables us to give new proofs of the hamiltonian properties of the hierarchy of modular vector fields of PN-manifolds. These can be extended to the case of arbitrary PN-algebroids in a straightforward manner.

1 Poisson-Nijenhuis structures

There are many ways of expressing the compatibility of a pair $(P, N)$, where $N$ is a Nijenhuis tensor and $P$ is a Poisson bivector on a manifold $M$ satisfying the condition that $NP$ be skew symmetric, in order to ensure that $NP, N^2P, \ldots, N^kP, \ldots$ be a sequence of pairwise-compatible Poisson brackets. Let $d_N = [i_N, d]$ be the differential on forms associated with the deformed bracket of vector fields, $[\cdot, \cdot]_N$, and let $[\cdot, \cdot]_P$ be the graded bracket of forms defined by $P$. When no confusion is possible, we denote by $N$ both the Nijenhuis tensor and its transpose, and by $P$ both the Poisson bivector and the map from 1-forms to vectors it defines, with the convention $P\alpha = i_\alpha P$. Let $H^P_f = Pf$ be the hamiltonian vector field with hamiltonian $f \in C^\infty(M)$ in the Poisson structure $P$. The derived bracket $[[i_N, d], i_P] = [d_N, i_P]$ is denoted by $[i_N, i_P]_d$. 
Proposition 1.1. The following conditions on $N$ and $P$ are equivalent:

- (i) $NP = PN$ and (ii) $C(P,N) = 0$, where, for all $\alpha, \beta \in \Gamma(T^*M)$,

$$C(P,N)(\alpha, \beta) = [\alpha, \beta]_N - ([N\alpha, \beta]_P + [\alpha, N\beta]_P - N[\alpha, \beta]_P) .$$

- $d_N$ is a derivation of bracket $[\cdot, \cdot]_P$.
- $d_P = [P, \cdot]$ is a derivation of the deformed bracket $[\cdot, \cdot]_N$.
- Let $\{\cdot, \cdot\}_NP$ be the Poisson bracket of functions with respect to $NP$.

(i) $NP = PN$ and (ii) $d\{f, g\}_NP = L_{H_f}^P d_N g - L_{H_g}^P d_N f - d_N(H_f^P(g))$, for all $f, g \in C^\infty(M)$.

This last condition follows from $C(P,N)(df, dg) = 0$, for all functions $f$, $g \in C^\infty(M)$, using the relation $[\alpha, df]_P = -i_{H_f}^P d\alpha$.

Definition 1.1. When any one of the above conditions is satisfied, $N$ and $P$ are called compatible. The pair $(P, N)$ is a Poisson-Nijenhuis structure (or PN-structure) if $N$ and $P$ are compatible. A manifold with a Poisson-Nijenhuis structure is called a Poisson-Nijenhuis manifold (or PN-manifold).

The compatibility of $P$ and $N$ can also be stated in terms of the morphism properties of maps $P, N^k P, N^k$ and $(t N)^k$, $k \geq 1$, relating the various Lie algebroid structures on $T M$ and $T^*M$.

Proposition 1.2. Let $P$ be a Poisson bivector and $N$ a Nijenhuis tensor on $M$ such that $PN = NP$. Then, for all $\alpha \in \Gamma(T^*M)$,

$$\frac{1}{2} \text{Tr}(C(P,N)\alpha) = \frac{1}{2} < Pd\text{Tr} N, \alpha > + [i_N, i_P]d\alpha , \quad (1.1)$$

where $[\cdot, \cdot]_d$ denotes the derived bracket.

Proof. We shall use the expression of the components of $C(P,N)$ in local coordinates $[4]$.

$$C^{kj}_m = P^{lj}_m \partial_l N^k_j + P^{kl}_m \partial_l N^j_k - N^l_m \partial_l P^{kj} + N^l_i \partial_m P^{kl} - P^{lj}_m \partial_m N^k_i ,$$

whence

$$C^{kj}_k = P^{lj}_k \partial_l N^k_j + P^{kl}_k \partial_l N^j_k - N^l_k \partial_l P^{kj} + N^l_i \partial_k P^{kl} - P^{lj}_k \partial_k N^k_i .$$

From the assumption $NP = PN$, i.e., $P^{lj}_k N^k_j + P^{lk}_j N^j_k = 0$, we obtain

$$N^j_k \partial_m P^{lj} + P^{lj}_m \partial_m N^k_j + N^l_j \partial_m P^{lk} + P^{lk}_m \partial_m N^j_i = 0 ,$$

whence

$$N^j_k \partial_k P^{lj} + P^{lj} \partial_k N^k_j + N^l_j \partial_k P^{lk} + P^{lk} \partial_k N^j_i = 0 .$$
This identity implies that
\[ \frac{1}{2} C^{kj}_k = \frac{1}{2} P^{lj}_j \partial_l N^k_k + P^{lk}_k \partial_k N^i_i. \]

Thus, for any 1-form \( \alpha \),
\[ \frac{1}{2} \text{Tr}(C(P, N)\alpha) = \frac{1}{2} P^{lj}_j \partial_l N^k_k \alpha_j + P^{lk}_k \partial_k N^i_i \alpha_j. \]

Since \( i_{NP} = i_{PN} = i_P i_N \),
\[ (i_{PD} i_N - i_{PN})\alpha = [i_P, [d, i_N]]\alpha = [[i_N, d], i_P] \alpha = [i_N, i_P] d \alpha. \]

These equalities imply (1.1). \( \square \)

The following corollary, a consequence of the compatibility, will be used in Section 2.

**Corollary 1.1.** Let \((P, N)\) be a Poisson-Nijenhuis structure on a manifold. For all \( f \in C^{\infty}(M)\),
\[ i_P (d_N df) = -\frac{1}{2} H^P_{I_1}(f), \]
where \( H^P_{I_1} = P d \text{Tr} N \) is the Hamiltonian vector field with Hamiltonian \( I_1 = \text{Tr} N \) in the Poisson structure \( P \).

**Proof.** When \( C(P, N) = 0 \), formula (1.1) for \( \alpha = df \) yields (1.2). \( \square \)

**Remark 1.1.** When \( P \) and \( N \) are compatible, the derived bracket \([i_N, i_P]_d\)
is a derivation of degree \(-1\) of the algebra of forms equal to the interior product by the vector field \(-\frac{1}{2} P d \text{Tr} N\). A proof of this property can be found in [1].

## 2 The hierarchy of modular classes of a Poisson-Nijenhuis manifold

### 2.1 The modular class of \((TM, N, [\cdot, \cdot]_N)\).

Let \( N \) be a Nijenhuis tensor on manifold \( M \). Given \( \lambda \otimes \mu \), where \( \lambda \) is a nowhere vanishing multivector of top degree and \( \mu \) a volume element on \( M \), the modular class of the Lie algebroid \((TM, N, [\cdot, \cdot]_N)\) is the class in the \( d_N \)-cohomology of the 1-form \( \xi^{(N)} \) such that, for all \( X \in \Gamma(TM) \),
\[ < \xi^{(N)}, X > \lambda \otimes \mu = [X, \lambda]_N \otimes \mu + \lambda \otimes L_{NX} \mu. \]
If \( e_1, \ldots, e_n \) is a local basis of \( \Gamma(TM) \) such that \( \lambda = e_1 \wedge \ldots \wedge e_n \), then

\[
[X, \lambda]_N = \sum_{j=1}^{n} (-1)^j [X, e_j]_N e_1 \wedge \ldots \wedge \hat{e}_j \wedge \ldots \wedge e_n .
\]

Since \([X, Y]_N = [NX, Y] + (L_X N)Y\), we obtain

\[
[X, \lambda]_N = L_{NX} \lambda + \sum_{j=1}^{n} (L_X N)^j e_1 \wedge \ldots \wedge \hat{e}_j \wedge \ldots \wedge e_n.
\]

Choosing \( \lambda \) and \( \mu \) such that \( < \lambda, \mu > = 1 \) which implies that \( L_{NX} \lambda \otimes \mu + \lambda \otimes L_{NX} \mu = 0 \), and using the relation \( \sum_{j=1}^{n} (L_X N)^j = \sum_{j=1}^{n} L_X (N^j) \), we obtain

\[
< \xi^{(N)}, X > \lambda \otimes \mu = i_X (dTrN) \lambda \otimes \mu .
\]

Thus we have recovered the result of [2]:

**Proposition 2.1.** The modular class in the \( d_N \)-cohomology of the Lie algebroid \((TM, N, [\cdot, \cdot])_N\) is the class of the 1-form \( dTrN \).

The \( d_N \)-cocycle \( \xi^{(N)} = dTrN \) is in fact independent of the choice of \( \lambda \otimes \mu \). The class it defines can also be considered to be the class of the morphism of Lie algebroids \( N: (TM, N, [\cdot, \cdot])_N \rightarrow (TM, \text{id}, [\cdot, \cdot]) \).

Similarly, the modular classes associated to the Nijenhuis tensors \( N^k, k \in \mathbb{N}, k \geq 2 \), are the \( d_{N^k} \)-classes of the 1-forms \( dTr(N^k) \).

### 2.2 The modular class of a Poisson-Nijenhuis manifold

We shall now consider the case of a manifold \( M \) with a PN-structure. Let \( P_0 = P \) and \( P_1 = NP, \ldots, P_k = N^kP, \ldots \).

For each Poisson structure \( P_k \) on \( M \), \( k \geq 0 \), the modular vector field associated to a volume form \( \mu \) on \( M \) is, by definition, the \( d_{P_k} \)-cocycle \( X^k_\mu \) satisfying

\[
< X^k_\mu, df > \mu = L_{H^k f} \mu , \quad (2.1)
\]

for all \( f \in C^\infty(M) \), that is \( < X^k_\mu, df > \mu = dP_k df \mu \). It follows that, for all 1-forms \( \alpha \),

\[
< X^k_\mu, \alpha > \mu = dP_k \alpha \mu - (i_{P_k} d\alpha)\mu . \quad (2.2)
\]

We now consider the vector fields

\[
X^{(k)} = X^k_\mu - N X^{k-1}_\mu , \quad (2.3)
\]

for \( k \geq 1 \), and we list their basic properties:

- For each \( k \), \( X^{(k)} \) is independent of \( \mu \). It is called the \( k \)-th modular vector field of \((M, P, N)\).
• $X^{(k)}$ is a $d_P$-cocycle. Its class is called the $k$-th modular class of the PN-manifold. In particular, the $d_{NP}$-class of $X^{(1)}$ is called the modular class of $(M,P,N)$.

• The $k$-th modular class of $(M,P,N)$ is one-half the relative modular class of the morphism of Lie algebroids $\iota^N : (T^*M, P_k, [\ ,\ ]_{P_k}) \to (T^*M, P_{k-1}, [\ ,\ ]_{P_{k-1}})$.

2.3 Properties of the hierarchy of modular vector fields

Proposition 2.2. The modular vector fields of a PN-manifold $(M,P,N)$ satisfy

$$X^{(k)} = -\frac{1}{2}H^P_{I_k}, \ k \geq 1,$$

where $I_k = \text{Tr}_{N^k}^P$, $k \geq 1$, is the sequence of fundamental functions in involution.

Proof. For clarity, we first prove the case $k = 1$. It follows from formula (2.2) and Corollary 1.1 that, for all $f \in C^\infty(M)$,

$$< NX^0_\mu, df > \mu = < X^0_\mu, Ndf > \mu = d_{NP}df \mu + \frac{1}{2} P dTrN, df > \mu,$$

while

$$< X^1_\mu, df > \mu = d_{NP}df \mu.$$

Therefore $X^{(1)} = X^1_\mu - NX^0_\mu = -\frac{1}{2} P dTrN = -\frac{1}{2} H^P_{I_1}$.

The case $k \geq 2$ is similar. Applying Corollary 1.1 to the compatible pair $(N^{k-1}P, N)$, we obtain

$$< X^{(k)}_\mu, df > = i_{N^{k-1}P}dfNdf = i_{N^{k-1}P}dNdf = -\frac{1}{2} < N^{k-1}P dTrN, df > .$$

The result follows from $N^{k-1}P dTrN = PN^{k-1}dTrN = P dTr^k$.

Remark 2.1. The sequence of modular vector fields $X^{(k)}$, $k \geq 1$, coincides with the well-known sequence [5] of bihamiltonian vector fields of a PN-manifold. It follows that $X^{(k)} = N X^{(k-1)}$.

Remark 2.2. The sequence of modular vector fields of a Poisson-Nijenhuis manifold introduced by Damianou and Fernandes in [2] is $X_k$, $k \geq 1$, defined by the recursion $X_1 = X_N = X^1_\mu - N X^0_\mu$ and $X_k = N X_{k-1}$, for $k \geq 2$. They proved that $X_k = -\frac{1}{2} P dTr^k$, for $k \geq 1$. Though the definition of the hierarchy $X^{(k)}$ that we have considered differs from theirs, the two hierarchies still coincide.
If we denote the modular vector field of the PN-structure \((N, P)\) by \(X_{N,P}\), then \(X^{(k)} = X_{N,N^{k-1}P}\), while \(X_k = N^{k-1}X_{N,P}\). The vector fields \(X_{N,P}\) satisfy
\[
X_{N,NP} + N X_{N,P} = X_{N^2,P} ,
\]
and, more generally,
\[
X_{N,N^kP} + N X_{N,N^{k-1}P} = X_{N^2,N^{k-1}P} .
\]
This relation is immediate from the definition. Each term is a hamiltonian vector field with respect to \(N^kP\); each of the terms on the left-hand side is equal to \(-\frac{1}{2}PN^k d\text{Tr}N\), while the right-hand side is \(-\frac{1}{2}PN^{k-1} d\text{Tr}N^2 = -PN^k d\text{Tr}N\).

**Remark 2.3.** It follows from the morphism properties of \(P, NP\) and \(tN\) that the relative modular classes of \(P: (T^*M, P[, ]) \rightarrow (TM, Id[, ])\), \(NP: (T^*M, NP[, ]_{NP}) \rightarrow (TM, Id[, ])\), and \(tN: (T^*M, NP[, ]_{NP}) \rightarrow (T^*M, P[, ]_P)\) are defined and satisfy
\[
\text{Mod}^{NP} - N\text{Mod}^P = \text{Mod}^{1N} .
\] (2.5)

A representative of this \(d_{NP}\)-cohomology class is \(-P d\text{Tr}N = 2X^{(1)}\).

More generally, a representative of the modular class of the morphism \(tN^k\) from \((T^*M, P_k[, ]_{P_k})\) to \((T^*M, P[, ]_P)\) is \(-P d\text{Tr}N^k = 2kX^{(k)}\).

**Remark 2.4.** The modular classes of the morphisms \(N: (TM, N[, ]_N) \rightarrow (TM, Id[, ]\)) and \(tN: (T^*M, NP[, ]_{NP}) \rightarrow (T^*M, P[, ]_P)\) are related by
\[
\text{Mod}^{tN} = -P\text{Mod}^N .
\] (2.6)

Relation (2.6) can be generalized in two ways.

**Proposition 2.3.** (i) The modular classes of the morphisms
\[
N^k: (TM, N^k[, ]_{N^k}) \rightarrow (TM, Id[, ])
\]
and
\[
tN^k: (T^*M, P_k[, ]_{P_k}) \rightarrow (T^*M, P[, ]_P)
\]
are related by
\[
\text{Mod}^{tN^k} = -P\text{Mod}^{N^k} .
\]

(ii) The modular classes of the morphisms
\[
N^{[k]}: (TM, N^k[, ]_{N^k}) \rightarrow (TM, N^{k-1}[ , ]_{N^{k-1}})
\]
and
\[
tN^{[k]}: (T^*M, P_k[, ]_{P_k}) \rightarrow (T^*M, P_{k-1}[ , ]_{P_{k-1}})
\]
are related by
\[
\text{Mod}^{tN^{[k]}} = -P\text{Mod}^{N^{[k]}} ,
\]
and a representative of the modular class of the morphism \(tN^{[k]}\) is \(2X^{(k)}\).
Proof. (i) follows from Proposition 2.1 and Remark 2.3. To prove (ii), we compute a representative of the modular class of $N^k$,
\[ d\text{Tr}N^k - tN d\text{Tr}N^{k-1} = d\text{Tr}\frac{N^k}{k}, \]
and a representative of the modular class of $tN^k$,
\[ 2(X^k - N X_{k-1}^k) = 2X^{(k)} = -P d\text{Tr}\frac{N^k}{k}. \]

\[ \square \]

Remark 2.5. Computations of a representative of $\text{Mod}^t N^k$ either from the equality $2(X^k - N X_{k}^0) = 2 \sum_{\ell=1}^{k} N^{k-\ell} X^{(\ell)}$ or from the equality $\text{Mod}^t N^k = \sum_{\ell=1}^{k} N^{k-\ell} \text{Mod}^{t}[N^{(\ell)}]$ both recover the fact, stated in Remark 2.3 that a representative of $\text{Mod}^t N^k$ is equal to $-\sum_{\ell=1}^{k} N^{k-\ell} P d\text{Tr}\frac{N^{(\ell)}}{\ell} = -P d\text{Tr}N^k = 2kX^{(k)}.$

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