Direct derivation of Liénard–Wiechert potentials, Maxwell’s equations and Lorentz force from Coulomb’s law

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Abstract

In 19th century Maxwell derived famous Maxwell equations from the knowledge of three experimental physical laws: the electrostatic Coulomb’s law, Ampere’s force law and Faraday’s law of induction. However, theoretical basis for Ampere’s force law and Faraday’s law remains unknown to this day. Furthermore, the Lorentz force is considered as summary of experimental phenomena, the theoretical foundation that explains generation of this force is still unknown.

To answer these fundamental theoretical questions, in this paper we derive relativistically correct Liénard – Wiechert potentials, Maxwell’s equations and Lorentz force from two simple postulates: (a) when all charges are at rest the Coulomb’s force acts between the charges, and (b) that disturbances caused by charge in motion propagate away from the source with finite velocity. The special relativity was not used in our derivations nor the Lorentz transformation. In effect, it was shown in this paper that all the electrodynamic laws, including the Lorentz force, can be derived from Coulomb’s law and time retardation.

This was accomplished by analysis of hypothetical experiment where test charge is at rest and where previously moving source charge stops at some time in the past. Then the generalized Helmholtz decomposition theorem, also derived in this paper, was applied to reformulate Coulomb’s force acting at present time as the function of positions of source charge at previous time when the source charge was moving. From this reformulation of Coulomb’s law the Liénard–Wiechert potentials and Maxwell’s equations were derived by careful mathematical manipulation.

In the second part of this paper, the energy conservation principle valid for moving charges is derived from the knowledge of electrostatic energy conservation principle valid for stationary charges. This again was accomplished by using generalized Helmholtz decomposition theorem. From this dynamic energy conservation principle the Lorentz force is finally derived.

Keywords: Coulomb’s law, Liénard–Wiechert potentials, Maxwell equations, Lorentz force
1. Introduction

In his famous *Treatise* [1, 2] Maxwell derived equations of electrodynamics based on the knowledge about the three experimental laws known at the time: the Coulomb’s law describing the electric force between charges at rest; Ampere’s law describing the force between current carrying wires, and the Faraday’s law of induction. Prior to Maxwell, magnetism and electricity were regarded to as separate phenomena. It was James Clerk Maxwell who unified these seemingly disparate phenomena into the set of equations collectively known today as Maxwell’s equations. In modern vector notation, the four Maxwell’s equations that govern the behavior of electromagnetic fields are written as:

\[ \nabla \cdot \mathbf{D} = \rho \] \hfill (1)
\[ \nabla \cdot \mathbf{B} = 0 \] \hfill (2)
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \] \hfill (3)
\[ \nabla \times \mathbf{B} = \mu \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \] \hfill (4)

where symbol \( \mathbf{D} \) denotes electric displacement vector, \( \mathbf{E} \) is electric field vector, \( \mathbf{B} \) is a vector called magnetic flux density, vector \( \mathbf{J} \) is called current density and scalar \( \rho \) is the charge density. Furthermore, there are two more important equations in electrodynamics that relate magnetic vector potential \( \mathbf{A} \) and scalar potential \( \phi \) to electromagnetic fields \( \mathbf{B} \) and \( \mathbf{E} \):

\[ \mathbf{B} = \nabla \times \mathbf{A} \] \hfill (5)
\[ \mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \] \hfill (6)

In standard electromagnetic theory, if a point charge \( q_s \) is moving with velocity \( \mathbf{v}_s(t) \) along arbitrary path \( r_s(t) \) the scalar potential \( \phi \) and vector potential \( \mathbf{A} \) caused by moving charge \( q_s \) are described by well known, relativistically correct, Liénard–Wiechert potentials [3, 4]:

\[ \phi = \phi(r, t) = \frac{1}{4\pi\epsilon} \frac{q_s}{(1 - \mathbf{n}_s(t_r) \cdot \beta_s(t_r)) |r - r_s(t_r)|} \] \hfill (7)
\[ \mathbf{A} = \mathbf{A}(r, t) = \frac{\mu c}{4\pi} \left( \frac{q_s \beta_s(t_r)}{(1 - \mathbf{n}_s(t_r) \cdot \beta_s(t_r)) |r - r_s(t_r)|} \right) \] \hfill (8)

where \( t_r \) is retarded time, \( r \) is the position vector of observer and vectors \( \mathbf{n}_s(t_r) \) and \( \beta_s(t_r) \) are:

\[ \beta(t_r) = \frac{\mathbf{v}_s(t_r)}{c} \] \hfill (9)
\[ \mathbf{n}(t_r) = \frac{r - r_s(t_r)}{|r - r_s(t_r)|} \] \hfill (10)
These equations were almost simultaneously discovered by Liénard and Wiechert around 1900’s and they represent explicit expressions for time-varying electromagnetic fields caused by charge in arbitrary motion. Nevertheless, Liénard-Wiechert potentials were derived from retarded potentials, which in turn, are derived from Maxwell equations.

Maxwell’s electrodynamic equations provide the complete description of electromagnetic fields, however, these equations say nothing about mechanical forces experienced by the charge moving in electromagnetic field. If the charge \( q \) is moving in electromagnetic field with velocity \( v \) then the force \( F \) experienced by the charge \( q \) is:

\[
F = q (E + v \times B) \tag{11}
\]

The force described by equation (11) is well known Lorentz force. Discovery of this electrodynamic force is historically credited to H.A. Lorentz [5], however, the similar expression for electromagnetic force can be found in Maxwell’s *Treatise*, article 598 [2]. The difference between the two is that Maxwell’s electromotive force acts on moving circuits and Lorentz force acts on moving charges.

However, it is not yet explained what causes the Lorentz force, Ampere’s force law and Faraday’s law. Maxwell derived his expression for electromotive force along moving circuit from the knowledge of experimental Faraday’s law. Later, Lorentz extended Maxwell’s reasoning to discover the force acting on charges moving in electromagnetic field [5]. Nevertheless, it would be impossible for Lorentz to derive his force law without the prior knowledge of Maxwell equations [6].

Nowadays, the Lorentz force (\( qv \times B \) term) is commonly viewed as an effect of Einstein’s special relativity. For example, an observer co-moving with source charge would not measure any magnetic field, while on the other hand, the stationary observer would measure the magnetic field caused by moving source charge. However, in this work, we demonstrate that the special relativity is not needed to derive the Lorentz force and Maxwell equations. In fact, we derive Maxwell’s equations and Lorentz force from more fundamental principles: the Coulomb’s law and time retardation.

There is another reason why the idea to derive Maxwell’s equations and Lorentz force from Coulomb’s law may seem plausible. Because of mathematical similarity between Coulomb’s law and Newton’s law of gravity many researchers thought that if Maxwell’s equations and Lorentz force could be derived from Coulomb’s law that this would be helpful in understanding of gravity. These two inverse-square physical laws are written:

\[
F_C = \frac{q_1 q_2}{4\pi \epsilon \|r_1 - r_2\|^3} \left( r_1 - r_2 \right) \tag{12}
\]

\[
F_G = -Gm_1 m_2 \frac{r_1 - r_2}{|r_1 - r_2|^3} \tag{13}
\]

The expressions for Coulomb’s force and Newton’s gravitational force are indeed similar, however, these two forces significantly differ in physical nature. The latter force is always attractive while the former can be either attractive or repulsive. Nevertheless, a number of researchers attempted to derive Maxwell’s equations from Coulomb’s law, and most of these attempts rely on Lorentz transformation of space-time coordinates between the rest frame of the moving charge and laboratory frame.
The first hint that Maxwell’s equations could be derived from Coulomb’s law and Lorentz transformation can be found in Einstein’s original 1905 paper on special relativity [7]. Einstein suggested that the Lorentz force term \( (v \times B) \) is to be attributed to Lorentz transformation of the electrostatic field from the rest frame of moving charge to the laboratory frame where the charge has constant velocity. Later, in 1912, Leigh Page derived Faraday’s law and Ampere’s law from Coulomb’s law using Lorentz transformation [8]. Frisch and Willets discussed the derivation of Lorentz force from Coulomb’s law using relativistic transformation of force [9]. Similar route to derivation of Maxwell’s equations and Lorentz force from Coulomb’s law was taken by Elliott in 1966 [10]. Kobe in 1986 derives Maxwell’s equations as the generalization of Coulomb’s law using special relativity [11]. Lorrain and Corson derive Lorentz force from Coulomb’s law, again, by using Lorentz transformation and special relativity [12]. Field in 2006 derives Lorentz force and magnetic field starting from Coulomb’s law by relating the electric field to electrostatic potential in a manner consistent with special relativity [13]. The most recent attempt comes from Singal [14] who attempted to derive electromagnetic fields of accelerated charge from Coulomb’s law and Lorentz transformation.

All of the mentioned attempts have in common that they attempt to derive Maxwell equations from Coulomb’s law by exploiting Lorentz transformation or Einstein’s special theory of relativity. However, historically the Lorentz transformation was derived from Maxwell’s equations [15], thus, the attempt to to derive Maxwell’s equations using Lorentz transformation seems to involve circular reasoning [16]. The strongest criticism came from Jackson who pointed out that it should be immediately obvious that, without additional assumptions, it is impossible to derive Maxwell’s equations from Coulomb’s law using theory of special relativity [17]. Schwartz addresses these additional assumptions and starting from Gauss’ law of electrostatics and by exploiting the Lorentz invariance and properties of Lorentz transformation he derives the Maxwell’s equations [18].

In addition to the criticism above, we point out that the derivations of Maxwell’s equations from Coulomb’s law using Lorentz transformation should only be considered valid for the special case of the charge moving along the straight line with constant velocity. This is because the Lorentz transformation is derived under the assumption that electron moves with constant velocity along straight line [15]. For example, if the particle moves with uniform acceleration along straight line the transformation of coordinates between the rest frame of the particle and the laboratory frame takes the different mathematical form than that of the Lorentz transformation [19]. If the particle is in uniform circular motion yet another coordinate transformation from the rest frame to laboratory frame, called Franklin transform, is valid [20]. None of the above cited papers consider the fact that Lorentz transformation is no longer valid when the charge is not moving along straight line with constant velocity.

To circumvent problems with special relativity and Lorentz transformation we take entirely different approach to derive Liénard–Wiechert potentials and Maxwell’s equations from Coulomb law. We start our derivation from the analysis of the following hypothetical experiment: consider two charges at rest at present time, one called the test charge, and the other called the source charge. The source charge was moving in the past but it is at rest at present time. Because both charges are at rest at present the force acting on test charge at present time is the Coulomb’s force.

However, in the past when the source charge was moving, we assume that the force
Figure 1: In (a) the source charge $q_s$ is at rest at present time $t_p$. Each point on closed contour $C$ is affected by Coulomb’s electrostatic field $E_c$. Energy conservation principle at present time is $\oint_C E_c \cdot dr = 0$. In (b) the source charge $q_s$ is moving along arbitrary path and it stops at past time $t_s < t_p$. Dynamic energy conservation principle valid in the past is assumed to be unknown when source charge was moving.

acting on test charge was not the Coulomb’s force. To discover the mathematical form of this "unknown" electrodynamic force acting in the past from the knowledge of known electrostatic force (Coulomb’s law) acting at present time the generalized Helmholtz decomposition theorem was applied. This theorem, derived in Appendix A, allowed us to relate Coulomb’s force acting at present time to the positions of source charge at past time. From here, Liénard–Wiechert potentials and Maxwell’s equations were derived by careful mathematical manipulation.

It should be emphasized that we did not resort to theory of special relativity nor to Lorentz transformation in our derivation of Maxwell’s equations. Not less importantly, the presented derivation of Maxwell’s equations from Coulomb’s law is valid for charges in arbitrary motion. In effect, we may say that more general physical law (Maxwell’s equations) acting at past time is derived from the knowledge of limited physical law (Coulomb’s law) acting at present time.

However, from Maxwell’s equations, it is very difficult, if not entirely impossible, to derive the Lorentz force without resorting to some form of energy conservation law. As shown in Fig. 1a, at present time, the single stationary charge creates Coulomb’s electrostatic field. Known energy conservation law valid at present time states that contour integral of Coulomb’s field along closed contour $C$ is equal to zero.

But, this electrostatic energy conservation law valid at present is not necessarily valid in the past when the source charge was moving. Thus, in the second part of this paper we derive this "unknown" dynamic energy conservation principle valid in the past from the knowledge of electrostatic energy conservation principle valid at present time. This was again achieved by the careful application of generalized Helmholtz decomposition theorem which allowed us to transform electrostatic energy conservation law valid at present to dynamic energy conservation law valid in the past.
This dynamic energy conservation law states that the work of non-conservative force along closed contour is equal to the time derivative of the flux of certain vector field through the surface bounded by this closed contour. From this dynamic energy conservation law the Lorentz force was finally derived.

2. Generalized Helmholtz decomposition theorem

Because generalized Helmholtz decomposition theorem is central for deriving Maxwell equations and Lorentz force from Coulomb’s law, in this section, we briefly present this important theorem while the derivation itself is moved to Appendix A. There have been several previous attempts in the literature to generalize classical Helmholtz decomposition theorem to time dependent vector fields [21, 22, 23]. However, in none of the cited articles the Helmholtz theorem for functions of space and time is presented in the mathematical form usable for the mathematical developments described in this paper. This is probably caused by difficulties in stating such a theorem and this was clearly stated in [22]: ”There does not exist any simple generalization of this theorem for time-dependent vector fields”.

However, we show that there indeed exists the simple generalization of Helmholtz decomposition theorem for time-dependent vector fields and that it can derived from time-dependent inhomogeneous wave equation. To improve the clarity of this paper, the complete derivation of Helmholtz decomposition theorem for functions of space and time is moved to Appendix A, subsection A.2. As it was shown in Appendix A, the generalization of Helmholtz decomposition theorem for the vector function of space and time \( \mathbf{F}(r, t) \) can be written as:

\[
\mathbf{F}(r, t) = -\nabla \int_{\mathbb{R}} dt' \int_{\mathbb{R}^3} \left( \nabla' \cdot \mathbf{F}(r', t') \right) G(r, t; r', t') dV' \\
+ \frac{1}{c^2} \frac{\partial}{\partial t} \int_{\mathbb{R}} dt' \int_{\mathbb{R}^3} \frac{\partial}{\partial t'} \mathbf{F}(r', t') G(r, t; r', t') dV' \\
+ \nabla \times \int_{\mathbb{R}} dt' \int_{\mathbb{R}^3} \left( \nabla' \times \mathbf{F}(r', t') \right) G(r, t; r', t') dV'
\]

where scalar function \( G(r, t; r', t') \) is the fundamental solution of time-dependent inhomogeneous wave equation given as:

\[
\nabla^2 G(r, t; r', t') - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} G(r, t; r', t') = -\delta(r - r')\delta(t - t')
\]

In the equation above, \( \delta(r - r') = \delta(x - x')\delta(y - y')\delta(z - z') \) is 3D Dirac delta function, and \( \delta(t - t') \) is Dirac delta function in one dimension. Fundamental solution \( G(r, t; r', t') \), sometimes called Green’s function, represents the retarded in time solution of the inhomogeneous time dependent wave equation and it can be written as:

\[
G(r, t; r', t') = \frac{\delta \left( t' - t + \frac{|r - r'|}{c} \right)}{4\pi |r - r'|}
\]
where position vector $r'$ is the location of the source at time $t'$.

From equation (14) it is evident that the Helmholtz decomposition theorem for functions of space and time can be regarded to as a mathematical tool that allows us to rewrite any vector function that is function of present time $t$ and of present position $r$ as vector function of previous time $t'$ and of previous position $r'$. Furthermore, the generalized Helmholtz decomposition theorem (14) comes with additional limitation that it is valid if vector function $\mathbf{F}(r', t')$ approaches zero faster than $1/|r - r'|$ as $|r - r'| \to \infty$.

Very similar theorem was presented in article written by Heras [24]; the difference is that in Heras’ article the time integrals in equation (14) were a priori evaluated at retarded time $t' = t - |r - r'|/c$. As such, the generalized Helmholtz theorem presented in [24] is not suitable for the derivation of Maxwell equations and Lorentz force from the Coulomb’s law. Reason for this, as it will become evident later in this paper, is that if we immediately evaluate the time integrals in equation (14) the important information is lost from the equation.

3. Derivation of Maxwell Equations from Coulomb’s law

In this section we derive Maxwell equations from Coulomb’s law using generalized Helmholtz decomposition theorem represented by equation (14). To begin the discussion, we consider hypothetical experiment shown in Fig. 2, where source charge $q_s$ is moving along trajectory $r_s(t)$ and it stops at some past time $t_s$. The test charge $q$ is stationary at all times. We assume that the disturbances caused by moving source charge propagate outwardly from the source charge with finite velocity $c$. These disturbances originating from the source charge at past time manifest itself as the force acting on stationary test charge at present time. This means that there is a time delay $\Delta t$ between the past time $t_s$ when the source charge has stopped and the present time $t_p$ when this disturbance has propagated to the test charge:

$$\Delta t = t_p - t_s = \frac{|r - r_s(t_s)|}{c}$$

At precise moment in time $t_p$, that we call the present time, the force acting on stationary test charge $q$ is the Coulomb’s force because source charge and test charge are both at rest, and because the effect of source charge stopping at past time $t_s$ had enough time to propagate to test charge. The Coulomb’s force $\mathbf{F}_c$ experienced by the test charge $q$ at present time $t_p$ can be expressed by the following equation:

$$\mathbf{F}_c(r, t_p) = \frac{q q_s}{4\pi \varepsilon |r - r_s(t_s)|^3}$$

Let us now consider the time $t$ just one brief moment before the stopping time $t_s$:

$$t = t_s - \delta t$$

where $\delta t \to 0$ is very small time interval. This time interval $\delta t$ is so small that we might even call it infinitesimally small. Then at the moment in time infinitesimally before the
present time $t_p$ the force felt by test charge $q$ is still the Coulomb’s force if $\delta t \to 0$. Using these considerations, we can now rewrite equation (18) as:

$$F_c(r, t_p - \delta t) = \frac{q q_s}{4\pi\varepsilon} \frac{r - r_s(t)}{|r - r_s(t)|^3} \quad t = t_s - \delta t; \quad \delta t \to 0$$

Note that equation (20) is equivalent to equation (18) when $\delta t \to 0$. The reason why we have written the Coulomb’s law this way is to permit slight variation of time before stopping time $t_s$ so that we can exploit generalized Helmholtz decomposition theorem in order to derive Maxwell’s equations from Coulomb’s law. Had we not done this then the source charge position vector $r_s(t_s)$ would simply be the constant vector and generalized Helmholtz decomposition could not be used.

Because the right hand side of equation (20) is now the function of time $t$ and position $r$ we are allowed to use the generalized Helmholtz decomposition theorem to rewrite the right hand side of equation (20). This is because generalized Helmholtz decomposition theorem states that any vector function of time $t$ and position $r$ can be decomposed as described by this theorem if that function meets certain criteria. Thus, using generalized Helmholtz decomposition theorem we can rewrite the right hand side of equation (20) as:
\[ \mathbf{F}_c(\mathbf{r}, t_p - \delta t) = \frac{qq_s}{4\pi\epsilon} \frac{\mathbf{r} - \mathbf{r}_s(t)}{|\mathbf{r} - \mathbf{r}_s(t)|^3} = \]

\[
= -\nabla \int_{\mathbb{R}} dt' \int_{\mathbb{R}^3} \left( \nabla' \cdot \frac{qq_s}{4\pi\epsilon} \frac{\mathbf{r}' - \mathbf{r}_s(t')}{|\mathbf{r}' - \mathbf{r}_s(t')|^3} \right) G(\mathbf{r}, t; \mathbf{r}', t') dV'
\]

\[
+ \frac{1}{c^2} \frac{\partial}{\partial t} \int_{\mathbb{R}} dt' \int_{\mathbb{R}^3} \left( \frac{\partial}{\partial t'} \frac{qq_s}{4\pi\epsilon} \frac{\mathbf{r}' - \mathbf{r}_s(t')}{|\mathbf{r}' - \mathbf{r}_s(t')|^3} \right) G(\mathbf{r}, t; \mathbf{r}', t') dV'
\]

\[
+ \nabla \times \int_{\mathbb{R}} dt' \int_{\mathbb{R}^3} \left( \nabla' \times \frac{qq_s}{4\pi\epsilon} \frac{\mathbf{r}' - \mathbf{r}_s(t')}{|\mathbf{r}' - \mathbf{r}_s(t')|^3} \right) G(\mathbf{r}, t; \mathbf{r}', t') dV'
\]

To clarify the notation in equation above note that \( dV' = dx'dy'dz' \) represents the differential volume element of an infinite volume \( \mathbb{R}^3 \). As defined in Appendix A, subsection A.1, the primed position vector \( \mathbf{r}' \) is written in Cartesian coordinate system as:

\[ \mathbf{r}' = x'\hat{x} + y'\hat{y} + z'\hat{z} \]

where variables \( x', y', z' \in \mathbb{R} \). Vectors \( \hat{x}, \hat{y}, \hat{z} \) are orthogonal Cartesian unit basis vectors. Furthermore, in Cartesian coordinates, the primed del operator \( \nabla' \) that appears in equation (21) is defined as:

\[ \nabla' = \hat{x} \frac{\partial}{\partial x'} + \hat{y} \frac{\partial}{\partial y'} + \hat{z} \frac{\partial}{\partial z'} \]

From the definition above, it follows that primed del operator \( \nabla' \) acts only on functions of variables \( x', y', z' \), and consequently, on functions of primed position vector \( \mathbf{r}' = x'\hat{x} + y'\hat{y} + z'\hat{z} \). It does not act on functions of position vector of source charge \( \mathbf{r}_s(t') \) because this position vector is function of variable \( t' \). Using these definitions we can write the following simple relations:

\[ \frac{\mathbf{r}' - \mathbf{r}_s(t')}{|\mathbf{r}' - \mathbf{r}_s(t')|} = -\nabla' \frac{1}{|\mathbf{r}' - \mathbf{r}_s(t')|} \]

\[ \nabla', \frac{\mathbf{r}' - \mathbf{r}_s(t')}{|\mathbf{r}' - \mathbf{r}_s(t')|} = -\nabla'^2 \frac{1}{4\pi |\mathbf{r}' - \mathbf{r}_s(t')|} = \delta (\mathbf{r}' - \mathbf{r}_s(t')) \]

\[ \nabla' \times \frac{\mathbf{r}' - \mathbf{r}_s(t')}{|\mathbf{r}' - \mathbf{r}_s(t')|} = -\nabla' \times \nabla' \frac{1}{|\mathbf{r}' - \mathbf{r}_s(t')|} = 0 \]

where \( \delta (\mathbf{r}' - \mathbf{r}_s(t')) \) is 3D Dirac’s delta function. Inserting equations (25) and (26) into equation (21), and eliminating charge \( q \) from the equation, yields the following relation:

\[
\frac{qq_s}{4\pi\epsilon} \frac{\mathbf{r} - \mathbf{r}_s(t)}{|\mathbf{r} - \mathbf{r}_s(t)|^3} = -\nabla \int_{\mathbb{R}} dt' \int_{\mathbb{R}^3} \frac{qq_s}{\epsilon} \delta (\mathbf{r}' - \mathbf{r}_s(t')) G(\mathbf{r}, t; \mathbf{r}', t') dV'
\]

\[
+ \frac{1}{c^2} \frac{\partial}{\partial t} \int_{\mathbb{R}} dt' \int_{\mathbb{R}^3} \left( \frac{\partial}{\partial t'} \frac{qq_s}{4\pi\epsilon} \frac{\mathbf{r}' - \mathbf{r}_s(t')}{|\mathbf{r}' - \mathbf{r}_s(t')|^3} \right) G(\mathbf{r}, t; \mathbf{r}', t') dV'
\]
In Appendix C, subsection C.1, we have shown that the time derivative that appears in the second right hand side integral of equation (27) can be written as:

$$\frac{\partial}{\partial t'} \frac{q_s}{4\pi\epsilon} \frac{r' - r_s(t')}{|r' - r_s(t')|^3} = \frac{q_s}{4\pi\epsilon} \nabla' \times \nabla' \times \frac{v_s(t')}{|r' - r_s(t')|} - \frac{q_s}{\epsilon} v_s(t') \delta (r' - r_s(t'))$$

(28)

where $v_s(t')$ is the velocity of the source charge $q_s$ at time $t'$:

$$v_s(t') = \frac{\partial r_s(t')}{\partial t'}$$

(29)

By inserting equation (28) into equation (27) it is obtained that:

$$\frac{q_s}{4\pi\epsilon} \frac{r - r_s(t)}{|r - r_s(t)|^3} = -\nabla \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s}{\epsilon} \delta (r' - r_s(t')) G(r, t; r', t') dV'$$

$$- \frac{1}{c^2} \frac{\partial}{\partial t} \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s}{\epsilon} v_s(t') \delta (r' - r_s(t')) G(r, t; r', t') dV'$$

$$+ \frac{1}{c^2} \frac{\partial}{\partial t} \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s}{4\pi\epsilon} \left[ \nabla' \times \nabla' \times \frac{v_s(t')}{|r' - r_s(t')|} \right] G(r, t; r', t') dV'$$

(30)

We now make use of the following identity, also derived in Appendix C, subsection C.2, to rewrite the last right hand side term of equation (30) as:

$$\int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s}{4\pi\epsilon} \left[ \nabla' \times \nabla' \times \frac{v_s(t')}{|r' - r_s(t')|} \right] G(r, t; r', t') dV'$$

$$= \nabla \times \nabla \times \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s}{4\pi\epsilon} \frac{v_s(t')}{|r' - r_s(t')|} G(r, t; r', t') dV'$$

(31)

Replacing the last right hand side integral in equation (30) with equation (31) and differentiating the resulting equation with respect to time $t$ yields:

$$\frac{q_s}{4\pi\epsilon} \frac{\partial}{\partial t} \frac{r - r_s(t)}{|r - r_s(t)|^3} = -\frac{\partial}{\partial t} \nabla \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s}{\epsilon} \delta (r' - r_s(t')) G(r, t; r', t') dV'$$

$$- \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s}{\epsilon} v_s(t') \delta (r' - r_s(t')) G(r, t; r', t') dV'$$

$$+ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \nabla \times \nabla \times \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s}{4\pi\epsilon} \frac{v_s(t')}{|r' - r_s(t')|} G(r, t; r', t') dV'$$

(32)

In the physical setting shown in Fig. 2 the coordinates of the test charge $q$ are fixed, hence, order in which we apply operator $\nabla \times \nabla \times$ and second order time derivative $\frac{\partial^2}{\partial t^2}$ can be swapped (because operator $\nabla$ does not affect variable $t$). Furthermore, because variables $t'$, $x'$, $y'$ and $z'$ are independent of time $t$ we can move the double time derivative under the integral sign in the last right hand side integral of above equation:
\[
\frac{q_s}{4\pi\epsilon} \frac{\partial}{\partial t} \frac{r - r_s(t)}{|r - r_s(t)|^3} = - \frac{\partial}{\partial t} \nabla \int_{\mathbb{R}} dt' \int_{\mathbb{R}^3} \frac{q_s}{\epsilon} \delta (r' - r_s(t')) G(r, t; r', t') dV' \tag{33}
\]
\[
- \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \int_{\mathbb{R}} dt' \int_{\mathbb{R}^3} \frac{q_s}{\epsilon} \nabla_s(t') \delta (r' - r_s(t')) G(r, t; r', t') dV' \\
+ \nabla \times \nabla \times \int_{\mathbb{R}} dt' \int_{\mathbb{R}^3} \frac{q_s}{4\pi\epsilon} \frac{v_s(t')}{|r' - r_s(t')|} \frac{1}{c^2} \frac{\partial^2}{\partial t^2} G(r, t; r', t') dV'
\]

The second order time derivative of \(G(r, t; r', t')\) in the last term of equation (33) can be replaced with equation (15) to obtain:

\[
\frac{q_s}{4\pi\epsilon} \frac{\partial}{\partial t} \frac{r - r_s(t)}{|r - r_s(t)|^3} = - \frac{\partial}{\partial t} \nabla \int_{\mathbb{R}} dt' \int_{\mathbb{R}^3} \frac{q_s}{\epsilon} \delta (r' - r_s(t')) G(r, t; r', t') dV' \tag{34}
\]
\[
- \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \int_{\mathbb{R}} dt' \int_{\mathbb{R}^3} \frac{q_s}{\epsilon} \nabla_s(t') \delta (r' - r_s(t')) G(r, t; r', t') dV' \\
+ \nabla \times \nabla \times \int_{\mathbb{R}} dt' \int_{\mathbb{R}^3} \frac{q_s}{4\pi\epsilon} \frac{v_s(t')}{|r' - r_s(t')|} \delta (r - r') \delta (t - t') dV' \\
+ \nabla \times \nabla \times \int_{\mathbb{R}} dt' \int_{\mathbb{R}^3} \frac{q_s}{4\pi\epsilon} \frac{v_s(t')}{|r' - r_s(t')|} \nabla^2 G(r, t; r', t') dV'
\]

Using sifting property of Dirac’s delta function allows us to rewrite the third right hand side term of equation (34) as:

\[
\nabla \times \nabla \times \int_{\mathbb{R}} dt' \int_{\mathbb{R}^3} \frac{q_s}{4\pi\epsilon} \frac{v_s(t')}{|r' - r_s(t')|} \delta (r - r') \delta (t - t') dV' = \tag{35}
\]
\[
= \nabla \times \nabla \times \int_{\mathbb{R}} dt' \frac{q_s}{4\pi\epsilon} \frac{v_s(t')}{|r - r_s(t)|} \delta (t - t') = \\
= \nabla \times \nabla \times \frac{q_s}{4\pi\epsilon} \frac{v_s(t)}{|r - r_s(t)|}
\]

To continue the derivation of Maxwell’s equations from Coulomb’s law we should note that operator \(\nabla\) does not affect vector \(v_s(t)\) because \(v_s(t)\) is a function of variable \(t\). Hence, the application of standard vector calculus identity \(\nabla \times \nabla \times \mathbf{P} = \nabla (\nabla \cdot \mathbf{P}) - \nabla^2 \mathbf{P}\) yields:

\[
\nabla \times \nabla \times \frac{q_s}{4\pi\epsilon} \frac{v_s(t)}{|r - r_s(t)|} = \frac{q_s}{4\pi\epsilon} \nabla \left( \nabla \cdot \frac{v_s(t)}{|r - r_s(t)|} \right) - \frac{q_s}{4\pi\epsilon} \nabla^2 \frac{v_s(t)}{|r - r_s(t)|} = \tag{36}
\]
\[
= \frac{q_s}{4\pi\epsilon} \nabla \left( v_s(t) \cdot \frac{1}{|r - r_s(t)|} \right) - \frac{q_s}{4\pi\epsilon} \nabla_s(t) \nabla^2 \frac{1}{|r - r_s(t)|} = \\
= - \frac{q_s}{4\pi\epsilon} \nabla \frac{\partial}{\partial t} \frac{1}{|r - r_s(t)|} + \frac{q_s}{\epsilon} v_s(t) \delta (r - r_s(t)) = \\
= \frac{q_s}{4\pi\epsilon} \frac{\partial}{\partial t} \frac{r - r_s(t)}{|r - r_s(t)|} + \frac{q_s}{\epsilon} v_s(t) \delta (r - r_s(t))
\]
Combining equations (34), (35) and (36), after cancellation of appropriate terms, yields:

\[
0 = -\frac{\partial}{\partial t} \nabla \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s}{\epsilon} \delta (\mathbf{r}' - \mathbf{r}_s(t')) G(\mathbf{r}, t; \mathbf{r}', t') dV' \\
- \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s}{\epsilon} \mathbf{v}_s(t') \delta (\mathbf{r}' - \mathbf{r}_s(t')) G(\mathbf{r}, t; \mathbf{r}', t') dV' \\
+ \frac{q_s}{\epsilon} \mathbf{v}_s(t) \delta (\mathbf{r} - \mathbf{r}_s(t)) \\
+ \nabla \times \nabla \times \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s}{4\pi\epsilon |\mathbf{r}' - \mathbf{r}_s(t')|} \nabla^2 G(\mathbf{r}, t; \mathbf{r}', t') dV'
\]  

(37)

In Appendix Appendix C, subsection C.3, we have derived the following mathematical identity:

\[
\int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s}{4\pi\epsilon |\mathbf{r}' - \mathbf{r}_s(t')|} \nabla^2 G(\mathbf{r}, t; \mathbf{r}', t') dV' = - \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s}{\epsilon} \mathbf{v}_s(t') \delta (\mathbf{r}' - \mathbf{r}_s(t')) G(\mathbf{r}, t; \mathbf{r}', t') dV' 
\]  

(38)

By inserting equation (38) into equation (37) it is obtained that:

\[
0 = -\frac{\partial}{\partial t} \nabla \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s}{\epsilon} \delta (\mathbf{r}' - \mathbf{r}_s(t')) G(\mathbf{r}, t; \mathbf{r}', t') dV' \\
- \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s}{\epsilon} \mathbf{v}_s(t') \delta (\mathbf{r}' - \mathbf{r}_s(t')) G(\mathbf{r}, t; \mathbf{r}', t') dV' \\
+ \frac{q_s}{\epsilon} \mathbf{v}_s(t) \delta (\mathbf{r} - \mathbf{r}_s(t)) \\
- \nabla \times \nabla \times \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s}{\epsilon} \mathbf{v}_s(t') \delta (\mathbf{r}' - \mathbf{r}_s(t')) G(\mathbf{r}, t; \mathbf{r}', t') dV'
\]  

(39)

If we now introduce new constant \( \mu = \frac{1}{c^2} \) and divide whole equation (39) by \( c^2 \) we obtain:

\[
0 = -\frac{1}{c^2} \frac{\partial}{\partial t} \nabla \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s}{\epsilon} \delta (\mathbf{r}' - \mathbf{r}_s(t')) G(\mathbf{r}, t; \mathbf{r}', t') dV' \\
- \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s\mu}{\epsilon} \mathbf{v}_s(t') \delta (\mathbf{r}' - \mathbf{r}_s(t')) G(\mathbf{r}, t; \mathbf{r}', t') dV' \\
+ q_s\mu \mathbf{v}_s(t) \delta (\mathbf{r} - \mathbf{r}_s(t)) \\
- \nabla \times \nabla \times \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s\mu}{\epsilon} \mathbf{v}_s(t') \delta (\mathbf{r}' - \mathbf{r}_s(t')) G(\mathbf{r}, t; \mathbf{r}', t') dV'
\]  

(40)

Although it is perhaps not yet apparent, equation (39) is Maxwell-Ampere equation given in introductory part of this paper as equation (4). To evaluate right hand side integrals in equation (40) we use sifting property of Dirac’s delta function \( \int_{\mathbb{R}^3} \delta (\mathbf{r}' - \mathbf{r}_s(t')) f(\mathbf{r}') dV' = \)
The right hand side integrals in equations (43) and (44) can be evaluated by making use of the identity (45) it follows that the time when the disturbance created by moving source charge \( r_s(t_r) \) was created. This disturbance moves through the space with the velocity \( v_s(t_r) \), where \( v_s(t_r) = \beta (t_r) \cdot n(t_r) \) are given by equations (9) and (10), respectively. From equation (45) it follows that the time \( t_r \) is the solution to the following equation:

\[
t - t_r - \frac{|r - r_s(t_r)|}{c} = 0
\]
We now define scalar function $v_s(t)$ and reaches the position $r$ of the test charge at time $t$. In the electromagnetic literature this time $t_r$ is commonly known as retarded time.

To proceed with derivation of Maxwell equations, we now insert equation (46) into equations (43) and (44), and evaluate the integrals over $t'$ using the sifting property of Dirac’s delta function to obtain:

\[
\int_{\mathbb{R}} dt' \int_{\mathbb{R}^3} \frac{q_s}{c} \delta (r' - r_s(t')) G(r, t; r', t') dV' = \frac{1}{4\pi \varepsilon} \frac{q_s}{|r - r_s(t_r)| (1 - \beta(t_r) \cdot n(t_r))}
\]

\[
\int_{\mathbb{R}} dt' \int_{\mathbb{R}^3} \frac{q_s \mu c \nabla \delta}{c} (r' - r_s(t')) G(r, t; r', t') dV' = \frac{\mu c}{4\pi} \frac{q_s \beta_s(t_r)}{|r - r_s(t_r)| (1 - \beta(t_r) \cdot n(t_r))}
\]

By inserting equations (48) and (49) into equation (40), and rearranging, it is obtained:

\[
\nabla \times \nabla \times \frac{\mu c}{4\pi} q_s v_s(t_r) G(r, t; r, t_r) = q_s \mu v_s(t) \delta (r - r_s(t))
\]

\[
= q_s \mu v_s(t) \delta (r - r_s(t)) - \frac{1}{\varepsilon} \frac{1}{c^2} \nabla^2 \frac{1}{|r - r_s(t_r)| (1 - \beta(t_r) \cdot n(t_r))} q_s
\]

\[
= q_s \mu v_s(t) \delta (r - r_s(t)) - \frac{1}{c^2} \frac{\mu c}{4\pi} q_s \beta_s(t_r)
\]

The first right hand side term of the equation above can be identified as the current $J$ of the point charge distribution moving with velocity $v_s(t)$ multiplied by constant $\mu$:

\[
\mu J = \mu q_s v_s(t) \delta (r - r_s(t))
\]

We now define scalar function $\theta(r, t)$ and vector function $Q(r, t)$ as:

\[
\theta(r, t) = \frac{1}{4\pi \varepsilon} \frac{q_s}{|r - r_s(t_r)| (1 - \beta(t_r) \cdot n(t_r))}
\]

\[
Q(r, t) = \frac{\mu c}{4\pi} \frac{q_s \beta_s(t_r)}{|r - r_s(t_r)| (1 - \beta(t_r) \cdot n(t_r))}
\]

With the aid of scalar function $\theta(r, t)$, vector function $Q(r, t)$, and expression $\mu J$ given by equation (51) the equation (50) can be written as:

\[
\nabla \times \nabla \times Q(r, t) = \mu J - \frac{1}{c^2} \frac{\partial}{\partial t} \left( -\nabla \theta(r, t) - \frac{\partial}{\partial t} Q(r, t) \right)
\]

Furthermore, we now define two vector functions $M$ and $N$ as:

\[
M = \nabla \times Q(r, t)
\]

\[
N = -\nabla \theta(r, t) - \frac{\partial}{\partial t} Q(r, t)
\]
Using definitions of vector functions $\mathbf{M}$ and $\mathbf{N}$ given by equations (55) and (56) we can rewrite equation (54) as:

$$\nabla \times \mathbf{M} = \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{N}}{\partial t} \tag{57}$$

We shall now investigate the mathematical properties of vector fields $\mathbf{M}$ and $\mathbf{N}$. Note that because for any differentiable vector field $\mathbf{P}$ we can write $\nabla \cdot \nabla \times \mathbf{P} = 0$, from equation (55) it follows that:

$$\nabla \cdot \mathbf{M} = 0 \tag{58}$$

The curl of the gradient of any differentiable scalar function $\psi$ is equal to zero, i.e. $\nabla \times \nabla \psi = 0$. Thus, taking the curl of equation (56) yields:

$$\nabla \times \mathbf{N} = -\nabla \times \frac{\partial}{\partial t} \mathbf{Q}(\mathbf{r}, t) = -\frac{\partial}{\partial t} \nabla \times \mathbf{Q}(\mathbf{r}, t) = -\frac{\partial \mathbf{M}}{\partial t} \tag{59}$$

Finally, in Appendix Appendix C, subsection C.4, we have shown that the divergence of vector field $\mathbf{N}$ is:

$$\nabla \cdot \mathbf{N} = \frac{q_s}{\epsilon} \delta (\mathbf{r} - \mathbf{r}_s(t)) = \frac{\rho(\mathbf{r}, t)}{\epsilon} \tag{60}$$

which completes the derivation of electrodynamic equations from Coulomb’s law.

To compare these equations to Maxwell’s equations, in Table 1 we have summarized governing equations for scalar potential $\theta(\mathbf{r}, t)$, vector potential $\mathbf{Q}(\mathbf{r}, t)$, vector field $\mathbf{M}$ and vector field $\mathbf{N}$ which are all derived from Coulomb’s law. By comparison with Liénard–Wiechert potentials given in Table 2, we see that scalar potential $\theta(\mathbf{r}, t)$ is identical to Liénard–Wiechert scalar potential $\phi(\mathbf{r}, t)$ and vector potential $\mathbf{Q}(\mathbf{r}, t)$ is identical to Liénard–Wiechert magnetic vector potential $\mathbf{A}(\mathbf{r}, t)$. Furthermore, by comparing Table 1 and Table 2 we find that vector field $\mathbf{M}$ is identical to magnetic flux density $\mathbf{B}$ and that vector field $\mathbf{N}$ is identical to electric field $\mathbf{E}$.

In Table 3 we have compared Maxwell’s equations governing fields $\mathbf{B}$ and $\mathbf{E}$ with differential equations governing vector fields $\mathbf{M}$ and $\mathbf{N}$. Clearly, left hand side of Table 3 is identical in the mathematical form to the right hand side of the same table, hence, differential equations governing vector fields $\mathbf{M}$ and $\mathbf{N}$ are identical to those governing vector fields $\mathbf{B}$ and $\mathbf{E}$. This is expected, because we already know that vector field $\mathbf{N} = \mathbf{E}$ and vector field $\mathbf{M} = \mathbf{B}$.

Thus, it should be evident by now that we have derived Maxwell equations and Liénard–Wiechert potentials directly from Coulomb’s law. This was achieved by mathematically relating known electrostatic Coulomb’s law acting on test charge at present time to “unknown” electrodynamic fields acting at past. The mathematical link between the static case in the present and dynamic case in the past was provided by generalized Helmholtz theorem. The derived equations are valid for arbitrarily moving source charge and these equations are not confined to motions along straight line. Furthermore, it should be noted that we have derived the Maxwell equations and Liénard–Wiechert potentials directly from Coulomb’s law without resorting to special relativity or Lorentz transformation.
Table 1: Potentials and vector fields derived from Coulomb’s law.

| symbol | equation | description |
|--------|----------|-------------|
| $\theta(r,t)$ | $\frac{1}{4\pi\epsilon} \frac{q_s}{|r-r_s(t_r)|} (1 - \beta(t_r) \cdot \mathbf{n}(t_r))$ | scalar potential derived from Coulomb’s law |
| $Q(r,t)$ | $\frac{\mu c}{4\pi} \frac{q_s}{|r-r_s(t_r)|} (1 - \beta(t_r) \cdot \mathbf{n}(t_r))$ | vector potential derived from Coulomb’s law |
| $\mathbf{M}$ | $\nabla \times Q(r,t)$ | vector field $\mathbf{M}$ derived from Coulomb’s law |
| $\mathbf{N}$ | $-\nabla \phi(r,t) - \frac{\partial}{\partial t} Q(r,t)$ | vector field $\mathbf{N}$ derived from Coulomb’s law |

Table 2: Standard electromagnetic theory expressions for Liénard–Wiechert scalar potential $\phi(r,t)$, Liénard–Wiechert vector potential $\mathbf{A}(r,t)$, magnetic flux density $\mathbf{B}$ and electric field $\mathbf{E}$.

| symbol | equation | description |
|--------|----------|-------------|
| $\phi(r,t)$ | $\frac{1}{4\pi\epsilon} \frac{q_s}{|r-r_s(t_r)|} (1 - \beta(t_r) \cdot \mathbf{n}(t_r))$ | Liénard–Wiechert scalar potential |
| $\mathbf{A}(r,t)$ | $\frac{\mu c}{4\pi} \frac{q_s}{|r-r_s(t_r)|} (1 - \beta(t_r) \cdot \mathbf{n}(t_r))$ | Liénard–Wiechert vector potential |
| $\mathbf{B}$ | $\nabla \times \mathbf{A}(r,t)$ | magnetic flux density |
| $\mathbf{E}$ | $-\nabla \phi(r,t) - \frac{\partial}{\partial t} \mathbf{A}(r,t)$ | electric field |

Table 3: Maxwell equations for electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ compared with differential equations for vector fields $\mathbf{M}$ and $\mathbf{N}$ derived from Coulomb’s law.

| Maxwell equation | description | equations derived from Coulomb’s law |
|------------------|-------------|--------------------------------------|
| $\nabla \times \mathbf{B} = \mathbf{J} + \frac{1}{\epsilon} \frac{\partial \mathbf{E}}{\partial t}$ | Maxwell-Ampere equation | $\nabla \times \mathbf{M} = \mathbf{J} + \frac{1}{\epsilon} \frac{\partial \mathbf{N}}{\partial t}$ |
| $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ | Faraday’s law | $\nabla \times \mathbf{N} = -\frac{\partial \mathbf{M}}{\partial t}$ |
| $\nabla \cdot \mathbf{E} = \xi$ | Gauss’ law for electric field | $\nabla \cdot \mathbf{N} = \xi$ |
| $\nabla \cdot \mathbf{B} = 0$ | Gauss’ law for magnetic field | $\nabla \cdot \mathbf{M} = 0$ |
4. Derivation of Electrodynamical Energy Conservation Law and Lorentz Force

To derive the electrodynamical energy conservation law from Coulomb’s law we first consider a hypothetical physical setting shown in Fig. 3 where the source charge $q_s$ is moving along arbitrary trajectory $\mathbf{r}_s(t)$. Then the source charge $q_s$ stops at some time in the past $t_s$. In this physical setting, closed contour $C$ is at rest at all times. At present time $t_p > t_s$ all the points inside the sphere of radius $R = c(t_p - t_s)$ are affected only by electrostatic Coulomb’s field. The known energy conservation law valid at present dictates that contour integral of electrostatic field along any closed contour immersed inside the sphere of radius $R$ equals to zero:

$$\oint_C \mathbf{E}_c(r, t_p) \cdot d\mathbf{r} = \int_C \frac{q_s}{4\pi\epsilon} \frac{\mathbf{r} - \mathbf{r}_s(t_s)}{|\mathbf{r} - \mathbf{r}_s(t_s)|^3} \cdot d\mathbf{r} = 0$$  \hspace{1cm} (61)$$

where $\mathbf{E}_c$ is Coulomb’s electrostatic field, $\mathbf{r}_s(t_s)$ is the position vector of source charge when it stopped moving, and vector $\mathbf{r}$ is the position vector of the point on contour $C$. This electrostatic energy conservation law, valid at present time $t_p$, states that no net work is done in transporting the unit charge along any closed contour immersed in electrostatic field.

To proceed, we assume that in the past, when the source charge was moving, the energy conservation law is unknown. However, generalized Helmholtz decomposition
theorem allows us to derive this "unknown" electrodynamic energy conservation law valid in the past from the knowledge of electrostatic energy conservation law valid at present. To derive this unknown electrodynamic conservation law we consider the contour integral \( C \) at the moment \( t_{\text{inf}} \) infinitesimally before the time when the source charge stopped:

\[
t = t_{\text{s}} - \delta t
\]  

(62)

where \( \delta t \) is infinitesimally small time interval. If time interval \( \delta t \) approaches zero \((\delta t \to 0)\) we can rewrite the contour integral (61) as the function of time \( t \):

\[
\oint_{C} E_c(r, t_p - \delta t) \cdot dr = \oint_{C} \frac{q_s}{4\pi\epsilon} \frac{r - r_s(t)}{|r - r_s(t)|^3} \cdot dr = 0
\]  

(63)

Because the integrand on the right hand side of equation (63) is the function of varying time \( t \) and position vector \( r \) the generalized Helmholtz decomposition theorem can be applied to rewrite this integrand as the function of past positions and velocities of the source charge. In fact, such expression is already derived in previous section as equation (33), repeated here for clarity:

\[
\frac{q_s}{4\pi\epsilon} \frac{r - r_s(t)}{|r - r_s(t)|^3} = -\nabla \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s}{\epsilon} \delta(r' - r_s(t')) G(r, t; r', t') dV'
\]

\[
- \frac{1}{c^2} \frac{\partial}{\partial t} \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s}{\epsilon} v_s(t') \delta(r' - r_s(t')) G(r, t; r', t') dV'
\]

\[
+ \frac{1}{c^2} \frac{\partial}{\partial t} \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s}{4\pi\epsilon} \left[ \nabla' \times \nabla' \times \frac{v_s(t')}{|r' - r_s(t')|} \right] G(r, t; r', t') dV'
\]  

(64)

Substituting the first two right hand side terms of equation (64) with equations (48) and (49) and combining the result with equations (52) and (53), and using \( c^2 = 1/\mu\epsilon \) yields:

\[
\frac{q_s}{4\pi\epsilon} \frac{r - r_s(t)}{|r - r_s(t)|^3} = -\nabla \theta(r, t) - \frac{\partial}{\partial t} Q(r, t) + K(r, t)
\]  

(65)

where vector function \( K(r, t) \) is equal to the last right hand side term of equation (64):

\[
K(r, t) = \frac{1}{c^2} \frac{\partial}{\partial t} \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s}{4\pi\epsilon} \left[ \nabla' \times \nabla' \times \frac{v_s(t')}{|r' - r_s(t')|} \right] G(r, t; r', t') dV'
\]  

(66)

Replacing the first two terms on the right hand side of equation (65) with equation (56) yields:

\[
\frac{q_s}{4\pi\epsilon} \frac{r - r_s(t)}{|r - r_s(t)|^3} = N(r, t) + K(r, t)
\]  

(67)

Then, by inserting equation (67) into right hand side of equation (63) it is obtained that:

\[
0 = \oint_{C} E_c(r, t_p - \delta t) \cdot dr = \oint_{C} (N(r, t) + K(r, t)) \cdot dr
\]  

(68)
The space-time integral on the right hand side of equation (66) is very difficult to evaluate. However, we can eliminate vector field \( \mathbf{K}(\mathbf{r},t) \) from the right hand side of equation (68) by the application of Stokes’ theorem:

\[
0 = \oint_C \mathbf{E}_c(\mathbf{r},t_p - \delta t) \cdot d\mathbf{r} = \oint_C \mathbf{N}(\mathbf{r},t) \cdot d\mathbf{r} + \int_S \nabla \times \mathbf{K}(\mathbf{r},t) \cdot d\mathbf{S} \tag{69}
\]

From here, we take the curl of both sides of equation (67) and by combining with equation (59) it is obtained that:

\[
\nabla \times \mathbf{K}(\mathbf{r},t) = -\nabla \times \mathbf{N}(\mathbf{r},t) = \frac{\partial}{\partial t} \mathbf{M}(\mathbf{r},t) \tag{70}
\]

Because surface \( S \) and contour \( C \) are stationary we can write that \( \frac{\partial}{\partial t} \mathbf{M}(\mathbf{r},t) = \frac{d}{dt} \int_S \mathbf{M}(\mathbf{r},t) \cdot d\mathbf{S} \).

Inserting equation (70) into equation (69) and taking into account that surface \( S \) and contour \( C \) are not moving yields:

\[
0 = \oint_C \mathbf{E}_c(\mathbf{r},t_p - \delta t) \cdot d\mathbf{r} = \oint_C \mathbf{N}(\mathbf{r},t) \cdot d\mathbf{r} + \frac{d}{dt} \int_S \mathbf{M}(\mathbf{r},t) \cdot d\mathbf{S} \tag{71}
\]

The right hand side of equation (71) is unknown energy conservation principle valid for varying in time dynamic fields \( \mathbf{N}(\mathbf{r},t) \) and \( \mathbf{M}(\mathbf{r},t) \) and it is derived from electrostatic energy conservation principle valid at present time. If \( \mathbf{N} \) is replaced by \( \mathbf{E} \) and if \( \mathbf{M} \) is replaced by \( \mathbf{B} \) it can be seen that we have just obtained the physical law known in electrodynamics as Faraday’s law.

From equation (71) the conclusion can be drawn about the nature of Faraday’s law. It represents the energy conservation principle valid for non-conservative dynamic fields and it is dynamic equivalent of electrostatic energy conservation principle valid for Coulomb’s electrostatic field.

However, even the Faraday’s law itself can be considered as consequence of something else. To see this, consider simply connected volume \( V \) bounded by surface \( \partial V \) as shown in Fig. 4. The surface \( \partial V \) is union of two surfaces \( S \) and \( S_1 \) bounded by respective contours \( C \) and \( C_1 \). Contours \( C \) and \( C_1 \) consist of exactly the same spatial points, however, the Stokes’ orientation of these contours is opposite \( C = -C_1 \). Then, using \( \nabla \times \mathbf{N}(\mathbf{r},t) = -\frac{\partial}{\partial t} \mathbf{M}(\mathbf{r},t) \) the first right hand side contour integral of equation (71) can be written as:

\[
\oint_C \mathbf{N}(\mathbf{r},t) \cdot d\mathbf{r} = -\oint_{C_1} \mathbf{N}(\mathbf{r},t) \cdot d\mathbf{r} = -\int_{S_1} \nabla \times \mathbf{N}(\mathbf{r},t) \cdot d\mathbf{S} = \frac{d}{dt} \int_{S_1} \mathbf{M}(\mathbf{r},t) \cdot d\mathbf{S} \tag{72}
\]

Replacing the first right hand side term of equation (71) with equation (72) yields different form of dynamic energy conservation law:

\[
0 = \oint_C \mathbf{E}_c(\mathbf{r},t_p - \delta t) \cdot d\mathbf{r} = \frac{d}{dt} \int_{\partial V} \mathbf{M}(\mathbf{r},t) \cdot d\mathbf{S} \tag{73}
\]

If we replace \( \mathbf{M}(\mathbf{r},t) \) with \( \mathbf{B} \) we see that right hand side of equation (73) is time derivative of Gauss’ law for magnetic fields. The standard interpretation of Gauss’ law for magnetic fields is that magnetic monopoles do not exist. However, from equation (73) we conclude that alternative interpretation of this law is that its time derivative represents
the dynamic energy conservation law. From the derivations presented, we might even say that Faraday’s law is the consequence of Gauss’ law for magnetic fields. It should be noted that these energy-conservation equations were all derived from simple electrostatic Coulomb’s law.

From dynamic energy conservation law the derivation of Lorentz force is straightforward: we now assume that all the points on surface \( \partial V \) shown in Fig. 4 have some definite velocity \( \mathbf{v} \) such that \( |\mathbf{v}| < c \). Then the surface \( \partial V \) is the function of time, hence, \( C = C(t) \) and \( S = S(t) \). Hence, we can rewrite equation (73) as the sum of two surface integrals:

\[
\frac{d}{dt} \oint_{\partial V} \mathbf{M}(\mathbf{r}, t) \cdot d\mathbf{S} = \frac{d}{dt} \oint_{S(t)} \mathbf{M}(\mathbf{r}, t) \cdot d\mathbf{S} + \frac{d}{dt} \oint_{S_1(t)} \mathbf{M}(\mathbf{r}, t) \cdot d\mathbf{S} = 0 \tag{74}
\]

where \( \partial V = S(t) \cup S_1(t) \). The Leibniz identity [25] for moving surfaces states that for any differentiable vector field \( \mathbf{P} \) we can write:

\[
\frac{d}{dt} \int_{S(t)} \mathbf{P} \cdot d\mathbf{S} = \int_{S(t)} \left[ \frac{\partial}{\partial t} \mathbf{P} + (\nabla \cdot \mathbf{P}) \mathbf{v} \right] \cdot d\mathbf{S} - \oint_{C(t)} \mathbf{v} \times \mathbf{P} \cdot d\mathbf{r} \tag{75}
\]

Applying the Leibniz identity to the surface integral over surface \( S_1(t) \) in equation (74) and using \( \nabla \cdot \mathbf{M}(\mathbf{r}, t) = 0 \) yields:

\[
\frac{d}{dt} \oint_{S(t)} \mathbf{M}(\mathbf{r}, t) \cdot d\mathbf{S} + \oint_{S_1(t)} \frac{\partial}{\partial t} \mathbf{M}(\mathbf{r}, t) \cdot d\mathbf{S} - \oint_{C_1(t)} \mathbf{v} \times \mathbf{M}(\mathbf{r}, t) \cdot d\mathbf{r} = 0 \tag{76}
\]

Using the result from previous section, i.e. \( \nabla \times \mathbf{N}(\mathbf{r}, t) = -\frac{\partial}{\partial t} \mathbf{M}(\mathbf{r}, t) \), and applying the Stokes’ theorem yields:

\[
\frac{d}{dt} \oint_{S(t)} \mathbf{M}(\mathbf{r}, t) \cdot d\mathbf{S} - \oint_{C_1(t)} \mathbf{N}(\mathbf{r}, t) \cdot d\mathbf{r} - \oint_{C_1(t)} \mathbf{v} \times \mathbf{M}(\mathbf{r}, t) \cdot d\mathbf{r} = 0 \tag{77}
\]

Because curves \( C_1(t) \) and \( C(t) \) comprise of same points, however, Stokes’ orientation of curves \( C_1(t) \) and \( C(t) \) is opposite, i.e. \( C_1(t) = -C(t) \), we can rewrite equation (77) as:

\[
\oint_{C(t)} [\mathbf{N}(\mathbf{r}, t) + \mathbf{v} \times \mathbf{M}(\mathbf{r}, t)] \cdot d\mathbf{r} + \frac{d}{dt} \oint_{S(t)} \mathbf{M}(\mathbf{r}, t) \cdot d\mathbf{S} = 0 \tag{78}
\]

![Figure 4: Closed surface \( \partial V \) that bounds volume \( V \) is union of two surfaces \( S \) and \( S_1 \). Contour \( C \) bounds surface \( S \) and contour \( C_1 \) bounds surface \( S_1 \). Contours \( C \) and \( C_1 \) are identical, however they have different Stokes’ orientation.](image)
Note that equation (78) could not be derived from the right hand side of equation (71), i.e. from Faraday’s law, even with Leibniz rule. For that reason, we might take that the energy conservation law on the right hand side of equation (73) is perhaps more general than the one given by equation (71).

Furthermore, note that the time derivative of the surface integral in equation (78) does not represent the work of any force. However, from equation (73) we know that the terms in equation (78) have dimensions of the work done by electrodynamic force in moving the unit charge along contour $C(t)$. Because the first term in equation (78) is contour integral of vector field we can conclude that this term represents the non-zero work done by non-conservative electrodynamic force in transporting the unit charge along contour $C(t)$.

Hence, just as the left hand side of equation (73) represents the work done by conservative electrostatic force in transporting the unit charge along contour $C$, the contour integral on the left of equation (78) represents the work done by non-conservative electrodynamic force in transporting the unit charge along the same contour. The purpose of surface integral on the left hand side of equation (78) is to balance non-zero work of non-conservative electrodynamic force along contour $C$. Thus, it can be concluded that the electrodynamic force $F_D$ on charge $q$ moving with velocity $v$ along contour $C$ is:

$$F_D = qN(r, t) + qv \times M(r, t) \tag{79}$$

Finally, in previous section we have shown that $N = E$ and that $M = B$. Thus, by replacing $N$ with $E$ and $M$ with $B$ it is obtained that:

$$F_D = q(E + v \times B) \tag{80}$$

which is expression for well known Lorentz force. It was derived theoretically from the knowledge of electrostatic energy conservation law which, in turn, can be derived from Coulomb’s law. Thus, we may say that we have just derived the Lorentz force from simple electrostatic Coulomb’s law.

5. Conclusion

In this paper we have presented the theoretical framework that explains Maxwell equations and the Lorentz force on more fundamental level than it was previously done. Maxwell derived Maxwell equations from experimental Ampere’s force law and experimental Faraday’s law, and Lorentz continued work on Maxwell’s theory to discover the Lorentz force. In last 150 years, no successful theory was presented that would explain Maxwell’s equations and Lorentz force on more fundamental level.

To accomplish this, relativistically correct Liénard–Wiechert potentials, Maxwell equations and the Lorentz force were derived directly from electrostatic Coulomb’s law. In contrast to frequently criticized previous attempts to derive Maxwell’s equations from Coulomb’s law using special relativity and Lorentz transformation, the Lorentz transformation was not used in our derivations nor the theory of special relativity. In fact, in this work, dynamic Liénard–Wiechert potentials, Maxwell equations and Lorentz force were derived from Coulomb’s law using the following two simple postulates:
(a) when charges are at rest the Coulomb’s law describes the force acting between charges

(b) disturbances caused by moving charges propagate outwardly from moving charge with finite velocity

The derivation of these dynamic physical laws from electrostatic Coulomb’s law would not be possible without generalized Helmholtz decomposition theorem also derived in this paper. This theorem allows the vector function of present position and present time to be written as space-time integral of positions and velocities at previous time. In contrast, standard Helmholtz decomposition theorem is valid for functions of space only and it ignores time.

To derive the Lorentz force from Coulomb’s law, in section 4, the “unknown” dynamic energy conservation law valid in the past was derived from the knowledge of electrostatic energy conservation law valid at present. The link between the present and the past was again provided by generalized Helmholtz decomposition theorem. This “unknown” dynamic energy conservation principle turned out to be Faraday’s law of induction. Additionally, it was shown that Faraday’s law of induction can be considered equivalent to time derivative of Gauss’ law for magnetic field. From these energy conservation considerations the Lorentz force was derived.

From the presented analysis one important question naturally arises: are Maxwell’s equations and Lorentz force the consequence of electrostatic Coulomb’s law? They are most probably not. It is rather the opposite, Coulomb’s law is the limiting case of Lorentz force when the source charge becomes stationary. However, as it was shown in this paper, it is entirely possible to deduce dynamic Maxwell equations and Lorentz force from the knowledge of simple electrostatic Coulomb’s law.

Finally, this paper attempts to answer another important question: how can we deduce more general dynamic physical laws from the limited knowledge provided by static physical law? The significance of answering this question is that in the future it will perhaps become possible that similar reasoning could deepen the understanding of physical laws other than Maxwell equations and Lorentz force.

Appendix A. Derivation of generalized Helmholtz decomposition theorem

In this appendix, we derive the generalized Helmholtz decomposition theorem for vector functions of space and time. However, in effort to enhance the readability of this work, we first start by considering some basic identities given in section A.1 of this appendix.

A.1. Preliminary considerations

To clarify notation used throughout this paper we first define position vectors \( \mathbf{r} \) and \( \mathbf{r}' \) as:

\[
\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z} \quad \text{(A.1.1)}
\]

\[
\mathbf{r}' = x'\hat{x} + y'\hat{y} + z'\hat{z} \quad \text{(A.1.2)}
\]
where \( \hat{x}, \hat{y} \) and \( \hat{z} \) are Cartesian, mutually orthogonal, unit basis vectors. Variables \( x, y, z \in \mathbb{R} \) and \( x', y', z' \in \mathbb{R} \) are linearly independent variables. Furthermore, throughout this paper we use position vector \( \mathbf{r}_s(t') \) to indicate the position of the source charge. This position vector \( \mathbf{r}_s(t') \) is defined as:

\[
\mathbf{r}_s(t') = x_s(t') \hat{x} + y_s(t') \hat{y} + z_s(t') \hat{z}
\]

(A.1.3)

where \( x_s(t'), y_s(t') \) and \( z_s(t') \) are all functions of real variable \( t' \in \mathbb{R} \) which is independent of variables \( x, y, z \in \mathbb{R} \) and \( x', y', z' \in \mathbb{R} \). The time derivative of position vector \( \mathbf{r}_s(t') \) is velocity \( \mathbf{v}_s(t') \) of the source charge:

\[
\mathbf{v}_s(t') = \frac{\partial \mathbf{r}_s(t')}{\partial t'}
\]

(A.1.4)

On many occasions in this paper we have used differential operators \( \nabla \) and \( \nabla' \) defined as:

\[
\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}
\]

\[
\nabla' = \hat{x}'\frac{\partial}{\partial x'} + \hat{y}'\frac{\partial}{\partial y'} + \hat{z}'\frac{\partial}{\partial z'}
\]

(A.1.5)

(A.1.6)

Operator \( \nabla \) acts only on functions of variables \( x, y, z \), hence, on functions of position vector \( \mathbf{r} \). On the other hand, operator \( \nabla' \) acts only on functions of variables \( x', y', z' \), thus, it acts on functions of position vector \( \mathbf{r}' \). For example, if function \( f \) is the function of position vector \( \mathbf{r} \), that is \( f = f(\mathbf{r}) \) we can generally write:

\[
\nabla f(\mathbf{r}) \neq 0 \quad \nabla' f(\mathbf{r}) = 0
\]

(A.1.7)

On the other hand, if function \( f \) is the function of position vector \( \mathbf{r}' \), that is if \( f = f(\mathbf{r}') \) we can write:

\[
\nabla f(\mathbf{r}') = 0 \quad \nabla' f(\mathbf{r}') \neq 0
\]

(A.1.8)

Furthermore, because variable \( t' \) is independent of variables \( x, y, z \) and \( x', y', z' \) neither operator \( \nabla \) nor \( \nabla' \) acts on position vector \( \mathbf{r}_s(t') \) and velocity vector \( \mathbf{v}_s(t') \). Using these considerations we see that the following equations are correct:

\[
\nabla \cdot \mathbf{r}_s(t') = 0 \quad \nabla \cdot \mathbf{v}_s(t') = 0
\]

\[
\nabla' \cdot \mathbf{r}_s(t') = 0 \quad \nabla' \cdot \mathbf{v}_s(t') = 0
\]

(A.1.9)

However, both operators \( \nabla \) and \( \nabla' \) act on Green’s function \( G(\mathbf{r}, t; \mathbf{r}', t') \) given by equation (16). In fact, one can easily verify that the following equations hold:

\[
\frac{\partial \nabla G(\mathbf{r}, t; \mathbf{r}', t')}{\partial t} = -\nabla' G(\mathbf{r}, t; \mathbf{r}', t')
\]

\[
\frac{\partial^2 G(\mathbf{r}, t; \mathbf{r}', t')}{\partial t^2} = -\frac{\partial}{\partial t} G(\mathbf{r}, t; \mathbf{r}', t')
\]

\[
\frac{\partial \nabla^2 G(\mathbf{r}, t; \mathbf{r}', t')}{\partial t} = -\nabla'^2 G(\mathbf{r}, t; \mathbf{r}', t')
\]

\[
\frac{\partial^2 G(\mathbf{r}, t; \mathbf{r}', t')}{\partial t^2} = -\frac{\partial}{\partial t} G(\mathbf{r}, t; \mathbf{r}', t')
\]

(A.1.10)
A.2. Generalized Helmholtz decomposition theorem

To start deriving generalized Helmholtz decomposition theorem for vector functions of space and time we first consider inhomogeneous transient wave equation:

\[ \nabla^2 G(\mathbf{r}, t; \mathbf{r}', t') - \frac{1}{c^2} \frac{\partial^2}{\partial t'^2} G(\mathbf{r}, t; \mathbf{r}', t') = -\delta(\mathbf{r} - \mathbf{r}') \delta(t - t') \] (A.2.11)

where \( G(\mathbf{r}, t; \mathbf{r}', t') \) is the function called fundamental solution or Green’s function and \( \delta \) is Dirac’s delta function. The Green’s function for inhomogeneous wave equation is well known and it represents an outgoing diverging spherical wave:

\[ G(\mathbf{r}, t; \mathbf{r}', t') = \frac{\delta \left( t - t' - \frac{|\mathbf{r} - \mathbf{r}'|}{c} \right)}{4\pi |\mathbf{r} - \mathbf{r}'|} \] (A.2.12)

Let us now suppose that vector field \( \mathbf{F} \) is the function of both space \( \mathbf{r} \) and time \( t \), i.e. \( \mathbf{F} = \mathbf{F}(\mathbf{r}, t) \). Using sifting property of Dirac delta function we can write vector function \( \mathbf{F}(\mathbf{r}, t) \) as the volume integral over infinite volume \( \mathbb{R}^3 \) and over all the time \( \mathbb{R} \) as:

\[ \mathbf{F}(\mathbf{r}, t) = \int_{\mathbb{R}} dt' \int_{\mathbb{R}^3} \mathbf{F}(\mathbf{r}', t') \delta(\mathbf{r} - \mathbf{r}') \delta(t - t') dV' \] (A.2.13)

where differential volume element \( dV' \) is \( dV' = dx'dy'dz' \). We now replace \( \delta(\mathbf{r} - \mathbf{r}') \delta(t - t') \) in equation above with left hand side of equation (A.2.11) to obtain:

\[ \mathbf{F}(\mathbf{r}, t) = -\int_{\mathbb{R}} dt' \int_{\mathbb{R}^3} \mathbf{F}(\mathbf{r}', t') \left[ \nabla^2 G(\mathbf{r}, t; \mathbf{r}', t') - \frac{1}{c^2} \frac{\partial^2}{\partial t'^2} G(\mathbf{r}, t; \mathbf{r}', t') \right] dV' \] (A.2.14)

From the discussion presented in section A.1 of this appendix, we know that D’Alambert operator \( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t'^2} \) does not act on variables \( x', y', z' \) and \( t' \) nor does it act on vector function \( \mathbf{F}(\mathbf{r}', t') \). Hence, we can write the D’Alambert operator \( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t'^2} \) in front of the integral:

\[ \mathbf{F}(\mathbf{r}, t) = - \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t'^2} \right) \int_{\mathbb{R}} dt' \int_{\mathbb{R}^3} \mathbf{F}(\mathbf{r}', t') G(\mathbf{r}, t; \mathbf{r}', t') dV' \] (A.2.15)

Using standard vector calculus identity \( \nabla \times \nabla \times \mathbf{P} = \nabla(\nabla \cdot \mathbf{P}) - \nabla^2 \mathbf{P} \) we can rewrite equation (A.2.15) as:

\[ \mathbf{F}(\mathbf{r}, t) = \nabla \times \nabla \times \int_{\mathbb{R}} dt' \int_{\mathbb{R}^3} \mathbf{F}(\mathbf{r}', t') G(\mathbf{r}, t; \mathbf{r}', t') dV' \] (A.2.16)

\[ - \nabla \left( \nabla \cdot \int_{\mathbb{R}} dt' \int_{\mathbb{R}^3} \mathbf{F}(\mathbf{r}', t') G(\mathbf{r}, t; \mathbf{r}', t') dV' \right) \]

\[ + \frac{1}{c^2} \frac{\partial^2}{\partial t'^2} \int_{\mathbb{R}} dt' \int_{\mathbb{R}^3} \mathbf{F}(\mathbf{r}', t') G(\mathbf{r}, t; \mathbf{r}', t') dV' \]

Because operators \( \nabla \) and \( \frac{\partial}{\partial t'} \) do not act on variables \( x', y', z' \) and \( t' \) we can move operator \( \nabla \) and partial derivative \( \frac{\partial}{\partial t'} \) under right hand side integrals in equation (A.2.16).
Then using standard vector calculus identities \( \nabla \times (\psi \mathbf{P}) = \nabla \psi \times \mathbf{P} + \psi \nabla \times \mathbf{P} \) and \( \nabla \cdot (\psi \mathbf{P}) = \nabla \psi \cdot \mathbf{P} + \psi \nabla \cdot \mathbf{P} \), and noting that \( \nabla \times \mathbf{F}(r', t') = 0 \) and \( \nabla \cdot \mathbf{F}(r', t') = 0 \) we can rewrite equation (A.2.16) as:

\[
\mathbf{F}(r, t) = \nabla \times \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \nabla' G(r, t; r', t') \times \mathbf{F}(r', t') dV' \quad (A.2.17)
\]

We now use identities \( \nabla G(r, t; r', t') = -\nabla' G(r, t; r', t') \) and \( \frac{\partial}{\partial t'} G(r, t; r', t') = -\frac{\partial}{\partial t} G(r, t; r', t') \) to rewrite the right hand side integrals in equation (A.2.17) as:

\[
\mathbf{F}(r, t) = -\nabla \times \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \nabla' G(r, t; r', t') \times \mathbf{F}(r', t') dV' \quad (A.2.18)
\]

Using vector calculus identity \( \nabla \times (\psi \mathbf{P}) = \nabla \psi \times \mathbf{P} + \psi \nabla \times \mathbf{P} \) and the the form of divergence theorem \( \int_V \nabla \times \mathbf{P} dV = \oint_{\partial V} \mathbf{P} \times dS \) we rewrite the first right hand side integral over \( \mathbb{R}^3 \) as:

\[
\int_{\mathbb{R}^3} \nabla' G(r, t; r', t') \times \mathbf{F}(r', t') dV' = \int_{\partial \mathbb{R}^3} G(r, t; r', t') \mathbf{F}(r', t') \times dS' \quad (A.2.19)
\]

Note that the surface \( \partial \mathbb{R}^3 \) is an infinite surface that bounds an infinite volume \( \mathbb{R}^3 \). Furthermore, for the surface integral in the equation above, position vector \( r' \) is located on infinite surface \( \partial \mathbb{R}^3 \), i.e. \( r' \in \partial \mathbb{R}^3 \). Hence, if vector function \( \mathbf{F}(r', t') \) decreases faster than \( 1/|r-r'| \) as \( |r-r'| \to \infty \) the surface integral in equation (A.2.19) vanishes. In that case, we can write:

\[
\int_{\mathbb{R}^3} \nabla' G(r, t; r', t') \times \mathbf{F}(r', t') dV' = -\int_{\mathbb{R}^3} G(r, t; r', t') \nabla' \times \mathbf{F}(r', t') dV' \quad (A.2.20)
\]

Using similar considerations, vector calculus identity \( \nabla \cdot (\psi \mathbf{P}) = \nabla \psi \cdot \mathbf{P} + \psi \nabla \cdot \mathbf{P} \) and standard divergence theorem \( \int_V \nabla \cdot \mathbf{P} dV = \oint_{\partial V} \mathbf{P} \cdot dS \) it is obtained that:

\[
\int_{\mathbb{R}^3} \mathbf{F}(r', t') \cdot \nabla' G(r, t; r', t') dV' = -\int_{\mathbb{R}^3} G(r, t; r', t') \nabla' \cdot \mathbf{F}(r', t') dV' \quad (A.2.21)
\]
To treat the last integral on the right hand side of equation (A.2.18) we use the following identity:

\[ F(r', t') \frac{\partial}{\partial t'} G(r, t; r, t') = \frac{\partial}{\partial t'} (F(r', t')G(r, t; r, t')) - G(r, t; r, t') \frac{\partial}{\partial t'} F(r', t') \]  \hspace{1cm} (A.2.22)

Using the identity above and noting that \( t' \) is independent of \( x', y' \) and \( z' \) we can rewrite the last right hand side integral of equation (A.2.18) as:

\[ \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} F(r', t') \frac{\partial}{\partial t'} G(r, t; r', t') dV' = \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} F(r', t')G(r, t; r', t') dV' \]  
\[ - \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} G(r, t; r', t') \frac{\partial}{\partial t'} F(r', t') dV' \]  \hspace{1cm} (A.2.23)

By integrating over \( t' \), it can be shown that the first right hand side integral in equation (A.2.23) vanishes:

\[ \left[ \int_{\mathbb{R}^3} F(r', t')G(r, t; r', t') dV' \right]_{t' \to \infty} \]  
\[ = \left[ \int_{\mathbb{R}^3} F(r', t')G(r, t; r', t') dV' \right]_{t' \to -\infty} \]  \hspace{1cm} (A.2.24)

If \( t' \to \pm \infty \), and if \( t \) is finite, then from equation (A.2.12) follows that \( G(r, t; r', t') = 0 \), thus, the right hand side of equation (A.2.24) is equal to zero. Inserting this result into equation (A.2.23) yields:

\[ \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} F(r', t') \frac{\partial}{\partial t'} G(r, t; r', t') dV' = - \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} G(r, t; r', t') \frac{\partial}{\partial t'} F(r', t') dV' \]  \hspace{1cm} (A.2.25)

By inserting equations (A.2.20), (A.2.21) and (A.2.25) into equation (A.2.18) we obtain the generalized Helmholtz theorem for vector functions of space and time:

\[ F(r, t) = -\nabla \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \left( \nabla' \cdot F(r', t') \right) G(r, t; r', t') dV' \]
\[ + \frac{1}{c^2} \frac{\partial}{\partial t} \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \left( \frac{\partial}{\partial t'} F(r', t') \right) G(r, t; r', t') dV' \]
\[ + \nabla \times \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \left( \nabla' \times F(r', t') \right) G(r, t; r', t') dV' \]  \hspace{1cm} (A.2.26)

The theorem is valid for functions \( F(r', t') \) that decrease faster than \( 1/|r - r'| \) as \( |r - r'| \to \infty \).

**Appendix B. Novel vector calculus identities**

In this appendix we prove two novel vector calculus identities, without which, it would be very difficult, perhaps even not possible, to derive Maxwell’s equations from
Coulomb’s law. These two novel vector calculus identities are given by the following two equations:

\[
\int_V \psi \nabla^2 P \, dV = \oint_{\partial V} \psi (dS \cdot \nabla) \, P \quad \text{(B.1)}
\]

\[
\int_V P \nabla^2 \psi \, dV = \oint_{\partial V} P (\nabla \psi \cdot dS) - \int_V (\nabla \psi \cdot \nabla) \, P \, dV \quad \text{(B.2)}
\]

where \( P \) is differentiable vector field, \( \psi \) is differentiable scalar function, volume \( V \subset \mathbb{R}^3 \) is simply connected volume, \( \partial V \) is the bounding surface of volume \( V \) and \( dS \) is differential surface element of \( \partial V \) such that \( dS = n \, dS \). Vector \( n \) is an outward unit normal to the surface \( \partial V \). In Cartesian coordinate system the product \( \psi \nabla^2 P \) can be written in terms of Cartesian components as:

\[
\psi \nabla^2 P = \hat{x} \psi \nabla^2 P_x + \hat{y} \psi \nabla^2 P_y + \hat{z} \psi \nabla^2 P_z \quad \text{(B.3)}
\]

where \( P_x, P_y \) and \( P_z \) are Cartesian components of vector \( P \) and vectors \( \hat{x}, \hat{y} \) and \( \hat{z} \) are Cartesian unit basis vectors. Using standard vector calculus identity \( \nabla \cdot f T = \nabla f \cdot T + f \nabla \cdot T \), valid for some scalar function \( f \) and for some vector function \( T \), we can rewrite equation (B.3) as:

\[
\psi \nabla^2 P = \hat{x} \nabla \cdot (\psi \nabla P_x) + \hat{y} \nabla \cdot (\psi \nabla P_y) + \hat{z} \nabla \cdot (\psi \nabla P_z) \quad \text{(B.4)}
\]

To proceed, we now expand the identity \( (\nabla \psi \cdot \nabla) \, P \) in terms of its Cartesian components as:

\[
(\nabla \psi \cdot \nabla) \, P = \left( \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial z} \right) \cdot \left( \hat{x} P_x + \hat{y} P_y + \hat{z} P_z \right) = \hat{x} \nabla \psi \cdot \nabla P_x + \hat{y} \nabla \psi \cdot \nabla P_y + \hat{z} \nabla \psi \cdot \nabla P_z \quad \text{(B.5)}
\]

Inserting equation (B.5) into (B.4) it is obtained that:

\[
\psi \nabla^2 P = \hat{x} \nabla \cdot (\psi \nabla P_x) + \hat{y} \nabla \cdot (\psi \nabla P_y) + \hat{z} \nabla \cdot (\psi \nabla P_z) - (\nabla \psi \cdot \nabla) \, P \quad \text{(B.6)}
\]

We now integrate equation (B.6) over volume \( V \) and apply the divergence theorem \( \int_V \nabla \cdot T \, dV = \oint_{\partial V} T \cdot dS \) to obtain:

\[
\int_V \psi \nabla^2 P \, dV = \hat{x} \oint_{\partial V} \psi \nabla P_x \cdot dS + \hat{y} \oint_{\partial V} \psi \nabla P_y \cdot dS + \hat{z} \oint_{\partial V} \psi \nabla P_z \cdot dS - \int_V (\nabla \psi \cdot \nabla) \, P \, dV \quad \text{(B.7)}
\]
The first three right hand side terms of equation (B.7) can be rewritten as:

\[
\hat{x} \oint_{\partial V} \psi \nabla P_x \cdot dS + \hat{y} \oint_{\partial V} \psi \nabla P_y \cdot dS + \hat{z} \oint_{\partial V} \psi \nabla P_z \cdot dS = \oint_{\partial V} \psi (dS \cdot \nabla) P
\] (B.8)

Inserting equation (B.8) into (B.7) yields:

\[
\int_V \psi \nabla^2 P dV = \oint_{\partial V} \psi (dS \cdot \nabla) P - \int_V (\nabla \psi \cdot \nabla) P dV
\] (B.9)

which we intended to prove. To prove equation (B.2) we rewrite \(P \nabla^2 \psi\) in terms of Cartesian components as:

\[
P \nabla^2 \psi = \hat{x} P_x \nabla^2 \psi + \hat{y} P_y \nabla^2 \psi + \hat{z} P_z \nabla^2 \psi
\] (B.10)

By using standard differential calculus identity \(f \nabla^2 h = \nabla \cdot (f \nabla h) - \nabla f \cdot \nabla h\), where \(f\) and \(h\) are differentiable scalar functions, equation (B.10) can be written as:

\[
P \nabla^2 \psi = \hat{x} \nabla \cdot (P_x \nabla \psi) - \hat{x} \nabla P_x \cdot \nabla \psi + \hat{y} \nabla \cdot (P_y \nabla \psi) - \hat{y} \nabla P_y \cdot \nabla \psi + \hat{z} \nabla \cdot (P_z \nabla \psi) - \hat{z} \nabla P_z \cdot \nabla \psi
\] (B.11)

Inserting equation (B.5) into equation (B.11) yields:

\[
P \nabla^2 \psi = \hat{x} \nabla \cdot (P_x \nabla \psi) + \hat{y} \nabla \cdot (P_y \nabla \psi) + \hat{z} \nabla \cdot (P_z \nabla \psi) - (\nabla \psi \cdot \nabla) P
\] (B.12)

Integrating equation (B.12) over volume \(V\) and applying divergence theorem \(\int_V \nabla \cdot T dV = \oint_{\partial V} T \cdot dS\) it is obtained that:

\[
\int_V P \nabla^2 \psi dV = \oint_{\partial V} P (\nabla \psi \cdot dS) - \int_V (\nabla \psi \cdot \nabla) P dV
\] (B.13)

which we intended to prove.

Appendix C. Derivation of auxiliary mathematical identities

In this appendix we derive auxiliary mathematical identities that we find useful for the derivation of Maxwell equations from Coulomb’s law.

C.1. Derivation of equation (28)

Equation (24) allows us to rewrite the time derivative in the second right hand side integral in equation (27) as:

\[
\frac{\partial}{\partial t'} \frac{q_s}{4\pi \epsilon} \frac{r' - r_s(t')}{|r' - r_s(t')|^3} = -\frac{q_s}{4\pi \epsilon} \frac{\partial}{\partial t'} \frac{1}{|r' - r_s(t')|}
\] (C.1.1)
Because coordinates $x'$, $y'$ and $z'$ are independent of time $t'$ we can swap operator $\nabla'$ and time derivative with respect to time $t'$ as:

$$\frac{\partial}{\partial t'} \frac{q_s}{4\pi \epsilon} \frac{r' - r_s(t')}{|r' - r_s(t')|^3} = -\frac{q_s}{4\pi \epsilon} \nabla' \frac{1}{|r' - r_s(t')|} \tag{C.1.2}$$

Furthermore, because coordinates $x'$, $y'$ and $z'$ are independent of time $t'$, the time derivative of $r'$ is equal to zero $\frac{\partial r'}{\partial t'} = 0$. The inner time derivative in the equation (C.1.2) can now be written as:

$$\frac{\partial}{\partial t'} \frac{1}{|r' - r_s(t')|} = \frac{v_s(t') \cdot (r' - r_s(t'))}{|r' - r_s(t')|^3} = -\nabla' \cdot \frac{v_s(t')}{|r' - r_s(t')|} \tag{C.1.3}$$

where $v_s(t')$ is the velocity of the source charge $q_s$ at time $t'$ given by equation (A.1.4). Inserting equation (C.1.3) into equation (C.1.2) yields:

$$\frac{\partial}{\partial t'} \frac{q_s}{4\pi \epsilon} \frac{r' - r_s(t')}{|r' - r_s(t')|^3} = \frac{q_s}{4\pi \epsilon} \nabla' \left( \nabla' \cdot \frac{v_s(t')}{|r' - r_s(t')|} \right) \tag{C.1.4}$$

To proceed with derivation, we now make use of standard vector calculus identity $\nabla \times \nabla \times \mathbf{P} = \nabla (\nabla \cdot \mathbf{P}) - \nabla^2 \mathbf{P}$, valid for any differentiable vector function $\mathbf{P}$. This identity allows us to rewrite the equation (C.1.4) as:

$$\frac{\partial}{\partial t'} \frac{q_s}{4\pi \epsilon} \frac{r' - r_s(t')}{|r' - r_s(t')|^3} = \frac{q_s}{4\pi \epsilon} \nabla' \times \nabla' \times \frac{v_s(t')}{|r' - r_s(t')|} + \frac{q_s}{4\pi \epsilon} \nabla'^2 \frac{v_s(t')}{|r' - r_s(t')|} \tag{C.1.5}$$

Since Laplacian operator $\nabla'^2$ does not have effect on velocity vector $v_s(t')$ the last right hand side term in equation (C.1.5) can be written using 3D Dirac's delta function as:

$$\frac{q_s}{4\pi \epsilon} \nabla'^2 \frac{v_s(t')}{|r' - r_s(t')|} = \frac{q_s}{4\pi \epsilon} v_s(t') \nabla'^2 \frac{1}{|r' - r_s(t')|} = -\frac{q_s}{4\pi \epsilon} v_s(t') \delta (r' - r_s(t')) \tag{C.1.6}$$

Hence, replacing the last right hand side term of equation (C.1.5) with equation (C.1.6) yields:

$$\frac{\partial}{\partial t'} \frac{q_s}{4\pi \epsilon} \frac{r' - r_s(t')}{|r' - r_s(t')|^3} = \frac{q_s}{4\pi \epsilon} \nabla' \times \nabla' \times \frac{v_s(t')}{|r' - r_s(t')|} - \frac{q_s}{4\pi \epsilon} v_s(t') \delta (r' - r_s(t')) \tag{C.1.7}$$

which proves equation (28).

C.2. Derivation of equation (31)

To derive equation (31) we make use of standard vector calculus identity $\nabla \times (\psi \mathbf{P}) = \nabla \psi \times \mathbf{P} + \psi \nabla \times \mathbf{P}$, where $\psi$ is a scalar function and $\mathbf{P}$ is a vector function, to rewrite the integrand in the last right hand side term of equation (30) as:
\[
\n\left[ \nabla' \times \nabla' \times \frac{\mathbf{v}_s(t')}{|\mathbf{r}' - \mathbf{r}_s(t')|} \right] G(\mathbf{r}, t; \mathbf{r}', t') = \nabla' \times \left[ \nabla' \times \frac{\mathbf{v}_s(t')}{|\mathbf{r}' - \mathbf{r}_s(t')|} \right] G(\mathbf{r}, t; \mathbf{r}', t')
\]

(C.2.8)

Integrating equation (C.2.8) with respect to variables \(x', y', z'\) and \(t'\), and making use of a standard form of divergence theorem \(\int_V \nabla \cdot \mathbf{P} \, dV = \int_{\partial V} \mathbf{P} \cdot d\mathbf{S}\) it is obtained that:

\[
\int_\mathbb{R} dt' \int_{\mathbb{R}^3} \left[ \nabla' \times \nabla' \times \frac{\mathbf{v}_s(t')}{|\mathbf{r}' - \mathbf{r}_s(t')|} \right] G(\mathbf{r}, t; \mathbf{r}', t') \, dV' = \nabla' \times \left[ \nabla' \times \frac{\mathbf{v}_s(t')}{|\mathbf{r}' - \mathbf{r}_s(t')|} \right] G(\mathbf{r}, t; \mathbf{r}', t') \int_{\mathbb{R}^3} d\mathbf{S}' \times \left( \nabla' \times \frac{\mathbf{v}_s(t')}{|\mathbf{r}' - \mathbf{r}_s(t')|} \right) G(\mathbf{r}, t; \mathbf{r}', t')
\]

(C.2.9)

where \(d\mathbf{V} = dx'dy'dz'\), \(\partial \mathbb{R}^3\) is an infinite surface that bounds \(\mathbb{R}^3\), and \(d\mathbf{S}\) is differential surface element of the surface \(\partial \mathbb{R}^3\). Because \(\partial \mathbb{R}^3\) is an infinite surface, the first right hand side integral vanishes. To see this, we can use standard vector identity \(\nabla \times (\psi \mathbf{P}) = \nabla \psi \times \mathbf{P} + \psi \nabla \times \mathbf{P}\) and using \(\nabla' \times \mathbf{v}_s(t') = 0\) to rewrite the first term in the first right hand side integrand as:

\[
\nabla' \times \frac{\mathbf{v}_s(t')}{|\mathbf{r}' - \mathbf{r}_s(t')|} = -\frac{\mathbf{r}' - \mathbf{r}_s(t')}{|\mathbf{r}' - \mathbf{r}_s(t')|^3} \times \mathbf{v}_s(t')
\]

(C.2.10)

Because \(\mathbf{r}' \in \partial \mathbb{R}^3\) this means that \(|\mathbf{r}'| \to \infty\). Provided that charge \(q_\infty\) is moving with finite velocity \(\mathbf{v}_s(t')\) it is clear that right hand side term of equation (C.2.10) vanishes as \(|\mathbf{r}'| \to \infty\). Thus, we can now write equation (C.2.9) as:

\[
\int_\mathbb{R} dt' \int_{\mathbb{R}^3} \left[ \nabla' \times \nabla' \times \frac{\mathbf{v}_s(t')}{|\mathbf{r}' - \mathbf{r}_s(t')|} \right] G(\mathbf{r}, t; \mathbf{r}', t') \, dV' = -\int_\mathbb{R} dt' \int_{\mathbb{R}^3} \nabla' G(\mathbf{r}, t; \mathbf{r}', t') \times \left[ \nabla' \times \frac{\mathbf{v}_s(t')}{|\mathbf{r}' - \mathbf{r}_s(t')|} \right] \, dV'
\]

(C.2.11)

There is another useful property of Green’s function \(G(\mathbf{r}, t; \mathbf{r}', t')\) which enables us to proceed with the derivation of equation (31). This property can be written as follows:

\[
\nabla' G(\mathbf{r}, t; \mathbf{r}', t') = -\nabla G(\mathbf{r}, t; \mathbf{r}', t')
\]

(C.2.12)

Using this property and standard vector calculus identity \(\nabla \times (\psi \mathbf{P}) = \nabla \psi \times \mathbf{P} + \psi \nabla \times \mathbf{P}\) allows us to rewrite the integrand in equation (C.2.9) as:

30
\[
\n\nabla' G(r, t; r', t') \times \left[ \nabla' \times \frac{v_s(t')}{|r' - r_s(t')|} \right] = (C.2.13)
\]

\[
= -\nabla G(r, t; r', t') \times \left[ \nabla' \times \frac{v_s(t')}{|r' - r_s(t')|} \right] =
\]

\[
= -\nabla \times \left[ G(r, t; r', t') \nabla' \times \frac{v_s(t')}{|r' - r_s(t')|} \right]
\]

Equation above is valid because operator \(\nabla\) does not act on velocity vector \(v_s(t')\), nor does it act on position vectors \(r'\) and \(r_s(t')\). It only acts on Green’s function \(G(r, t; r', t')\) because it is a function of position vector \(r\). Inserting equation (C.2.13) into equation (C.2.15) yields:

\[
\int_\mathbb{R} dt' \int_\mathbb{R}^3 \left[ \nabla' \times \nabla' \times \frac{v_s(t')}{|r' - r_s(t')|} \right] G(r, t; r', t') dV' = (C.2.14)
\]

\[
= \int_\mathbb{R} dt' \int_\mathbb{R}^3 \nabla \times \left[ G(r, t; r', t') \nabla' \times \frac{v_s(t')}{|r' - r_s(t')|} \right] dV'
\]

Because differential volume element is \(dV' = dx'dy'dz'\) and because operator \(\nabla\) does not act on variables \(x', y', z'\) and \(t'\) we can write operator \(\nabla\) in front of the integral:

\[
\int_\mathbb{R} dt' \int_\mathbb{R}^3 \left[ \nabla' \times \nabla' \times \frac{v_s(t')}{|r' - r_s(t')|} \right] G(r, t; r', t') dV' = (C.2.15)
\]

\[
= \nabla \times \int_\mathbb{R} dt' \int_\mathbb{R}^3 G(r, t; r', t') \nabla' \times \frac{v_s(t')}{|r' - r_s(t')|} dV'
\]

Using the same trick again, that is, by using standard vector calculus identity \(\nabla \times (\psi P) = \nabla \psi \times P + \psi \nabla \times P\), using \(\nabla' G(r, t; r', t') = -\nabla G(r, t; r', t')\) and noting that operator \(\nabla\) does not act on variables \(x', y'\) and \(z'\) we can rewrite the integrand in equation (C.2.15) as:

\[
G(r, t; r', t') \nabla' \times \frac{v_s(t')}{|r' - r_s(t')|} = (C.2.16)
\]

\[
= \nabla' \times \left[ G(r, t; r', t') \frac{v_s(t')}{|r' - r_s(t')|} \right] + \nabla \times \left[ G(r, t; r', t') \frac{v_s(t')}{|r' - r_s(t')|} \right]
\]

Then, by inserting equation (C.2.16) into equation (C.2.15) and using a form of standard divergence theorem \(\int_V \nabla \times P dV = \int_{\partial V} dS \times P\) we obtain that:

\[
\int_\mathbb{R} dt' \int_\mathbb{R}^3 \left[ \nabla' \times \nabla' \times \frac{v_s(t')}{|r' - r_s(t')|} \right] G(r, t; r', t') dV' = (C.2.17)
\]

\[
= \nabla \times \int_\mathbb{R} dt' \int_{\partial \mathbb{R}^3} dS' \times \frac{v_s(t')}{|r' - r_s(t')|} + \nabla \times \int_\mathbb{R} dt' \int_{\partial \mathbb{R}^3} dS' \times \frac{v_s(t')}{|r' - r_s(t')|} dV'
\]
Because surface \( \partial \mathbb{R}^3 \) is an infinite surface the magnitude of position vector \( \mathbf{r}' \) is infinite, hence the surface integral in the first right hand side term of equation (C.2.17) vanishes. Furthermore, because operator \( \nabla \) does not act on variables \( x', y', z' \) and \( t' \) we can write operator \( \nabla \) in front of the second right hand side space-time integral. Hence, equation (C.2.17) can be written as:

\[
\int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \left[ \nabla \times \nabla' \times \frac{v_s(t')}{|\mathbf{r}' - \mathbf{r}_s(t')|} \right] G(\mathbf{r}, t; \mathbf{r}', t') dV' = \quad (C.2.18)
\]

\[
= \nabla \times \nabla \times \int_{\partial \mathbb{R}^3} G(\mathbf{r}, t; \mathbf{r}', t') \frac{v_s(t')}{|\mathbf{r}' - \mathbf{r}_s(t')|} dV'
\]

thus, proving the equation (31).

C.3. Derivation of equation (38)

Using the mathematical identity \( \nabla^2 G(\mathbf{r}, t; \mathbf{r}', t) = \nabla'^2 G(\mathbf{r}, t; \mathbf{r}', t) \), valid for Green’s function \( G(\mathbf{r}, t; \mathbf{r}', t) \), one can rewrite the left hand side integral in equation (38) as:

\[
\int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s}{4\pi \varepsilon |\mathbf{r}' - \mathbf{r}_s(t')|} \nabla^2 G(\mathbf{r}, t; \mathbf{r}', t') dV' = \quad (C.3.19)
\]

\[
= \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s}{4\pi \varepsilon |\mathbf{r}' - \mathbf{r}_s(t')|} \nabla'^2 G(\mathbf{r}, t; \mathbf{r}', t') dV'
\]

In Appendix Appendix B, we have derived two novel vector calculus identities. Subtracting vector identity (B.1) from vector identity (B.2) yields:

\[
\int_V \mathbf{P} \nabla^2 \psi dV = \int_V \psi \nabla^2 \mathbf{P} dV + \oint_{\partial V} \mathbf{P} (\nabla \psi \cdot d\mathbf{S}) - \oint_{\partial V} \psi (d\mathbf{S} \cdot \nabla) \mathbf{P} \quad (C.3.20)
\]

Using vector identity (C.3.20) we can rewrite equation (C.3.19) as:

\[
\int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s}{4\pi \varepsilon |\mathbf{r}' - \mathbf{r}_s(t')|} \nabla^2 G(\mathbf{r}, t; \mathbf{r}', t') dV' = \quad (C.3.21)
\]

\[
= \int_{\mathbb{R}^3} dt' \int_{\mathbb{R}^3} \frac{q_s}{4\pi \varepsilon |\mathbf{r}' - \mathbf{r}_s(t')|} \nabla'^2 G(\mathbf{r}, t; \mathbf{r}', t') dV' + \frac{q_s}{4\pi \varepsilon |\mathbf{r}' - \mathbf{r}_s(t')|} \nabla' \cdot \mathbf{v_s}(t') - \frac{q_s}{4\pi \varepsilon |\mathbf{r}' - \mathbf{r}_s(t')|} \nabla' \cdot \mathbf{v_s}(t')
\]

Because surface \( \partial \mathbb{R}^3 \) is an infinite surface the magnitude of position vector \( \mathbf{r}' \in \partial \mathbb{R}^3 \) has an infinite magnitude, \( |\mathbf{r}'| \to \infty \). In that case, both right hand side surface integrals over surface \( \partial \mathbb{R}^3 \) vanish in equation (C.3.21). Hence, equation (C.3.21) becomes:
\[
\int \frac{d^3 q_s}{4\pi \epsilon} \frac{\mathbf{v}_s(t')}{|\mathbf{r}' - \mathbf{r}_s(t')|} \nabla^2 G(\mathbf{r}, t; \mathbf{r}', t') dV' = \int \frac{d^3 q_s}{4\pi \epsilon} G(\mathbf{r}, t; \mathbf{r}', t') \nabla^2 \frac{\mathbf{v}_s(t')}{|\mathbf{r}' - \mathbf{r}_s(t')|} dV' \tag{C.3.22}
\]

The operator \(\nabla'\) does not affect vector \(\mathbf{v}_s(t')\) which is a function of variable \(t'\). Thus, we can rewrite the Laplacian in the equation above as:

\[
\nabla'^2 \frac{\mathbf{v}_s(t')}{4\pi |\mathbf{r}' - \mathbf{r}_s(t')|} = \mathbf{v}_s(t') \nabla'^2 \frac{1}{4\pi |\mathbf{r}' - \mathbf{r}_s(t')|} = -\mathbf{v}_s(t') \delta (\mathbf{r}' - \mathbf{r}_s(t')) \tag{C.3.23}
\]

Inserting equation (C.3.23) into right hand side of equation (C.3.22) yields:

\[
\int \int \frac{d^3 q_s}{4\pi \epsilon} \frac{\mathbf{v}_s(t')}{|\mathbf{r}' - \mathbf{r}_s(t')|} \nabla^2 G(\mathbf{r}, t; \mathbf{r}', t') dV' = \int \int \frac{d^3 q_s}{\epsilon} \mathbf{v}_s(t') \delta (\mathbf{r}' - \mathbf{r}_s(t')) G(\mathbf{r}, t; \mathbf{r}', t') dV' \tag{C.3.24}
\]

which proves equation (38).

\section*{C.4. Derivation of equation (60)}

Equation (60) can be derived from equation (67) by taking the divergence of both sides of this equation to obtain:

\[
\nabla \cdot \mathbf{N}(\mathbf{r}, t) + \nabla \cdot \mathbf{K}(\mathbf{r}, t) = \nabla \cdot \frac{q_s}{4\pi \epsilon} \frac{\mathbf{r} - \mathbf{r}_s(t)}{|\mathbf{r} - \mathbf{r}_s(t)|} = \frac{q_s}{\epsilon} \delta (\mathbf{r} - \mathbf{r}_s(t)) \tag{C.4.25}
\]

To find \(\nabla \cdot \mathbf{K}(\mathbf{r}, t)\) we can write operator \(\nabla\) under the right hand side integral of equation (66) and then apply identity \(\nabla G(\mathbf{r}, t; \mathbf{r}', t) = -\nabla' G(\mathbf{r}, t; \mathbf{r}', t)\) to obtain:

\[
\nabla \cdot \mathbf{K}(\mathbf{r}, t) = -\frac{1}{c^2} \frac{\partial}{\partial t} \int \int \frac{d^3 q_s}{4\pi \epsilon} \left[ \nabla' \times \nabla' \times \frac{\mathbf{v}_s(t')}{|\mathbf{r}' - \mathbf{r}_s(t')|} \right] \cdot \nabla' G(\mathbf{r}, t; \mathbf{r}', t') dV' \tag{C.4.26}
\]

From here, using standard vector identity \(\nabla \cdot (\psi \mathbf{P}) = \nabla \psi \cdot \mathbf{P} + \psi \nabla \cdot \mathbf{P}\) and divergence theorem it is obtained that:

\[
\nabla \cdot \mathbf{K}(\mathbf{r}, t) = \frac{-1}{c^2} \frac{\partial}{\partial t} \int \int \frac{d^3 q_s}{4\pi \epsilon} \left[ \nabla' \times \nabla' \times \frac{\mathbf{v}_s(t')}{|\mathbf{r}' - \mathbf{r}_s(t')|} \right] G(\mathbf{r}, t; \mathbf{r}', t') \cdot d\mathbf{S}' + \frac{1}{c^2} \frac{\partial}{\partial t} \int \int \frac{d^3 q_s}{4\pi \epsilon} \left[ \nabla' \cdot \nabla' \times \nabla' \times \frac{\mathbf{v}_s(t')}{|\mathbf{r}' - \mathbf{r}_s(t')|} \right] G(\mathbf{r}, t; \mathbf{r}', t') dV' \tag{C.4.27}
\]

33
Clearly, the surface integral on the right hand side of equation (C.4.27) vanishes as $|\mathbf{r}'| \rightarrow 0$. Furthermore, because $\nabla \cdot \nabla \times \mathbf{P} = 0$ the second right hand side term vanishes as well. Hence, we can write:

$$\nabla \cdot \mathbf{K}(\mathbf{r}, t) = 0 \quad \text{(C.4.28)}$$

Inserting equation (C.4.28) into equation (C.4.25) yields:

$$\nabla \cdot \mathbf{N}(\mathbf{r}, t) = \frac{\partial}{\partial t} \frac{\epsilon}{\epsilon} (\mathbf{r} - \mathbf{r}_s(t)) \quad \text{(C.4.29)}$$

thus, proving the equation (60).

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