Quantum evolution by discrete measurements

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Abstract. In this article we review two ways of driving a quantum system to a known pure state via a sequence discrete of von Neumann measurements. The first of them assumes that the initial state of the system is unknown, and the evolution is attained only with the help of two non-commuting observables. For this method, the overall success probability is maximized when the eigenstates of the involved observables constitute mutually unbiased bases. The second method assumes the initial state is known and it uses \(N\) observables which are consecutively measured to make the state of the system approach the target state. The probability of success of this procedure converges to 1 as the number of observables increases.

1. Introduction

The problem of preparing, manipulating and measuring a quantum system is a crucial problem in different contexts, including quantum computing and quantum communication. Quantum computing is based on the existence of a set of universal quantum gates which, concatenated, allow one to implement any unitary transformation within a fixed level of accuracy. These quantum gates are implemented through the controlled manipulation of the interactions among different physical systems. Quantum communication protocols, such as quantum teleportation [1], entanglement swapping [2], dense coding [3, 4], and quantum cloning [5, 6], also require the precise application of unitary transformations in a finite set of transformations.

Quantum measurement is another possible way to drive the evolution of a system toward a desired state. The control of the evolution of a system can be achieved by a continuous measurement process, as in the Anti-Zeno effect [7], or by a sequence of von Neumann measurements [8–10]. Quantum control via discrete measurements combined with decoherence processes is discussed in [11].

In this article we review two of the above proposals to map a mixed initial state onto a known pure state by using projective measurements as the only allowed resource. Unitary operations and ancillary systems are not used. The first of these proposals assumes that the initial state of the system is unknown, and the evolution is carried out only with the help of two non-commuting observables [9]. The overall success probability is maximum when the eigenstates of the involved observables constitute mutually unbiased bases, and this quickly converges to one as the number of measurement processes increases. The second proposal assumes the initial state is known and it uses \(N\) observables which are consecutively measured to make the state of the system approach the target state [10]. The probability of success of this procedure converges to 1 as the number of observables increases.
2. Unknown initial state

2.1. Two-dimensional Hilbert space

Let us consider a quantum system described by a two-dimensional Hilbert space. Let us assume the system is in a mixed state $\rho$, and our goal is to take it into a known target state $|\varphi\rangle$ only by means of von Neumann measurements. In order to accomplish this, we define a non-degenerate observable $\hat{\varphi}$, its spectral decomposition being

$$\hat{\varphi} = \lambda |\varphi\rangle \langle \varphi| + \lambda_\perp |\varphi_\perp\rangle \langle \varphi_\perp|,$$

where the $|\varphi\rangle$ and $|\varphi_\perp\rangle$ states are eigenstates of $\hat{\varphi}$ with eigenvalues $\lambda$ and $\lambda_\perp$ respectively. Thereby, the target state must belong to the spectral decomposition. A measurement of the observable $\hat{\varphi}$ onto the state $\rho$ projects the system to the target state $|\varphi\rangle$ with probability $p = \langle \varphi | \rho | \varphi \rangle$. In this case we succeed and no further action is required. However, the process fails with probability $1 - p$ when the measurement projects the system onto the state $|\varphi_\perp\rangle$. Since this state cannot be projected to $|\varphi\rangle$ by means of another measurement of $\hat{\varphi}$, it is necessary to introduce a second observable $\hat{\theta}$, with non-degenerate eigenstates $|0\rangle$ and $|1\rangle$ given by

$$|0\rangle = \cos \theta |\varphi\rangle + e^{i\phi} \sin \theta |\varphi_\perp\rangle,$$
$$|1\rangle = -e^{-i\phi} \sin \theta |\varphi\rangle + \cos \theta |\varphi_\perp\rangle,$$

with $\theta \neq n\pi, (2n+1)\pi/2$ ($n$ integer). A measurement of $\hat{\theta}$ projects the system from $|\varphi_\perp\rangle$ onto one of the states $|0\rangle$ or $|1\rangle$. Since both states have a component different from zero on $|\varphi\rangle$, a second measurement of $\hat{\varphi}$ allows us again to project, with a certain probability, to the target state $|\varphi\rangle$. The probability $p'$ that this procedure fails after a first measurement of $\hat{\varphi}$ but is successful after a consecutive measurement of the operators $\hat{\theta}$ and $\hat{\varphi}$ is

$$p' = \langle \varphi_\perp | \rho | \varphi_\perp \rangle (|\langle 0 | \varphi_\perp \rangle|^2 |\langle \varphi | 0 \rangle|^2 + |\langle 1 | \varphi_\perp \rangle|^2 |\langle \varphi | 1 \rangle|^2) = \frac{1}{2} \langle \varphi_\perp | \rho | \varphi_\perp \rangle \sin^2 2\theta. \hspace{2cm} (3)$$

Thus, the success probability in the sequence of measurements $M(\hat{\varphi}) M(\hat{\theta}) M(\hat{\varphi})$ is

$$p + p' = 1 - \langle \varphi_\perp | \rho | \varphi_\perp \rangle (1 - \frac{1}{2} \sin^2 2\theta). \hspace{2cm} (4)$$

Similarly, the probability of success $p_s$ of mapping the initial state $\rho$ onto the target state $|\varphi\rangle$ after applying the consecutive measurement processes $[M(\hat{\varphi}) M(\hat{\theta})]N M(\hat{\varphi})$ is given by

$$p_{s,N} = 1 - \langle \varphi_\perp | \rho | \varphi_\perp \rangle \left( 1 - \sum_{j=0}^{1} |\langle j | \varphi_\perp \rangle|^2 |\langle \varphi | j \rangle|^2 \right)^N = 1 - \langle \varphi_\perp | \rho | \varphi_\perp \rangle \left( 1 - \frac{1}{2} \sin^2 2\theta \right)^N. \hspace{2cm} (5)$$

As follows from (5), $p_{s,N}$ reaches its maximum value

$$p_{s,\text{max}} = 1 - \frac{\langle \varphi_\perp | \rho | \varphi_\perp \rangle}{2^N}, \hspace{2cm} (6)$$

when $\theta = \pi/4$, that is, when the eigenvectors of the observables $\hat{\theta}$ and $\hat{\varphi}$ form two mutually unbiased bases of the two-dimensional Hilbert space. Fig. 1a illustrates the behavior of the maximum success probability $p_{s,\text{max}}$ as a function of $N$ for different values of $\langle \varphi_\perp | \rho | \varphi_\perp \rangle$. We observe that $p_{s,\text{max}}$ quickly converges to 1 almost independently of the $\langle \varphi_\perp | \rho | \varphi_\perp \rangle$ even if the initial state $\rho$ belongs to a subspace orthogonal to $|\varphi\rangle$. Fig. 1b shows $p_{s,N}$ versus $N$ for $\theta$ equal to $\pi/12$ (circle), $\pi/8$ (square), and $\pi/4$ (triangle). Since mutually unbiased bases give the optimal process for each $N$, $p_{s,N}$ approaches to 1 faster than in the other cases. As is apparent from Fig. 1b, the convergence of the success probability strongly depends on the relation between the involved bases.
Figure 1. (a) $p_{s,\text{max}}$ as a function of $N$ for three values of $\langle \varphi_\perp \mid \rho \mid \varphi_\perp \rangle$: 1/3 (triangle), 2/3 (square), 1 (circle), (b) $p_{s,N}$ as a function of $N$ for three values of $\theta$: $\pi/4$ (triangle), $\pi/8$ (square), $\pi/12$ (circle), with $\langle \varphi_\perp \mid \rho \mid \varphi_\perp \rangle = 1$.

2.2. Generalization to $d$ dimensions

What happen if the target state $|\varphi_1\rangle$ belongs to a $d$-dimensional Hilbert space? Let $\rho$ be the initial state of the system, and $\{ |\varphi_1\rangle, |\varphi_2\rangle, \ldots, |\varphi_d\rangle \}$ and $\{ |1\rangle, |2\rangle, \ldots, |d\rangle \}$ orthonormal bases defined by the spectral decompositions of the non-degenerate observables $\hat{\varphi}$ and $\hat{\theta}$, respectively. The success probability of taking the system to the target state after the sequence of measurements $[M(\hat{\varphi})M(\hat{\theta})]^N M(\hat{\varphi})$ is given by

$$p_{s,N} = \langle \varphi_1 \mid \rho \mid \varphi_1 \rangle + \sum_{i=2}^{d} \langle \varphi_i \mid \rho \mid \varphi_i \rangle \sum_{k=1}^{N} \prod_{n=1}^{d} \left( \sum_{j_n=2}^{d} p_{i,j_1,j_2,\ldots,j_n} \right),$$

(7)

where

$$p_{k,j} = \sum_{i=1}^{d} |\langle i \mid \varphi_k \rangle|^2 |\langle \varphi_j \mid i \rangle|^2.$$ 

(8)

It is shown also for this case that the success probability given by Eq. (7) is maximum when the observables $\theta$ and $\varphi$ define two mutually unbiased bases [9], for which

$$p_{s,\text{max}} = 1 - (1 - \langle \varphi_1 \mid \rho \mid \varphi_1 \rangle) \left( 1 - \frac{1}{d} \right)^N,$$

(9)

or in the limit $d \gg 1$

$$p_{s,\text{max}} = 1 - (1 - \langle \varphi_1 \mid \rho \mid \varphi_1 \rangle) e^{-N/d}.$$ 

(10)

Thus, in order to get a success probability close to one, the number $N$ of measurement processes of the type $M(\hat{\varphi})M(\hat{\theta})$ must be larger than the dimension $d$ of the Hilbert space.
The above ideas can be applied to the preparation of an entangled state of a bipartite system, where each of the components is described by a \(d\)-dimensional Hilbert space. Assuming a factorized initial state of the form
\[
\rho_i = \rho \otimes \rho,
\] (11)
let us suppose we want to drive the system to the target state
\[
|\varphi_1\rangle = \alpha|\psi\rangle|\psi\rangle + \beta|\psi\rangle|\psi\rangle_{\perp}.
\] (12)
The success probability of the process, after carrying out the sequence of measurements \([M(\hat{\varphi})M(\hat{\theta})]^N]\), is given by
\[
p_{s,N} = 1 - [1 - \langle \psi|\rho|\psi\rangle(1 - \langle \psi|\rho|\psi\rangle)(1 + 2\Re(\alpha\beta^*|\gamma|^2))(1 - \frac{1}{d^2})^N],
\] (13)
where the coefficient \(\gamma\) gives account of the initial decoherence process affecting the state \(\rho\), defined by
\[
\langle \psi|\rho|\psi\rangle_{\perp} = \gamma\sqrt{\langle \psi|\rho|\psi\rangle\langle \psi\perp|\rho|\psi\perp\rangle},
\] (14)
with \(0 \leq |\gamma| \leq 1\). It can be shown that
\[
p_{s,N} \leq 1 - [1 - \frac{1}{4}(1 + 2\Re(\alpha\beta^*|\gamma|^2))(1 - \frac{1}{d^2})^N].
\] (15)
Thereby, when \(\Re(\alpha\beta^*) > 0\), the maximum probability for fixed \(N\) is achieved under the condition \(|\gamma|^2 = 1\), that is, for a pure initial state. However, if \(\Re(\alpha\beta^*) < 0\), the probability is maximum when \(|\gamma|^2 = 0\). This means that, for states fulfilling the condition \(\Re(\alpha\beta^*) < 0\), such as the singlet state, the success probability is higher in the case of total initial decoherence than in any other case, corresponding the smallest probability to an initially pure state.

This scheme can also be connected to the application of quantum erasure. If we fix the target state, and consequently the operators \(\hat{\theta}\) and \(\hat{\varphi}\), then the sequence of measurements will map any initial state onto that same target state. Thereby, the overall effect will correspond to probabilistically erasing the information content of the initial state.

3. Known initial state
An important goal of quantum information theory is understanding the resources necessary and sufficient for processing of quantum states. In quantum teleportation [1] an unknown state is transmitted from a sender to a receiver using classical communication and prior entanglement. Two bits of forward classical communication and one ebit (a maximally entangled pair of qubits) per teleported qubit are both necessary and sufficient, and neither resource can be traded off against the other. In remote state preparation (RSP) [13] the goal is the same, but the sender starts with complete classical knowledge of the state. They showed that the asymptotic classical communication cost of RSP is one bit per qubit, half that of teleportation, and even less when transmitting part of a known entangled state. In this context it can be designed a measurement-driving scheme which take advantage of knowing the initial state of the system.

3.1. Driving the evolution by two observables
Let us consider a quantum system described by a two dimensional Hilbert space. The system is prepared in some known initial state \(\rho\), and the objective is to drive it to the target state \(|\zeta\rangle\) only by von Neumann measurements. Let \(\hat{\zeta}\) be an observable with non-degenerate eigenstates \(|\zeta\rangle, |\zeta\rangle_{\perp}\}. The success probability in projecting the system into \(|\zeta\rangle\) as result of a measurement of \(\hat{\zeta}\) is \(p_d = \langle \zeta|\rho|\zeta\rangle\). Is it
possible to increase the probability of success by making an intermediate measurement of an observable \( \hat{\theta}_1 \)? And if so, what conditions must be fulfilled by \( \rho, \zeta \) and \( \hat{\theta}_1 \) in order to maximize it?

Let \( \{ |0_1 \rangle, |1_1 \rangle \} \) be the non-degenerate eigenstates of \( \hat{\theta}_1 \). The probability of reaching the target state \( |\zeta \rangle \) by a measurement of \( \hat{\theta}_1 \) followed by a measurement of \( \hat{\zeta} \) can be written as

\[
p_{1,s} = |\langle 1_1 | \zeta \rangle|^2 \left( \langle \theta_1 | \rho | \zeta \rangle - \langle \zeta | \rho | \zeta \rangle \right) + \frac{2}{(1 - (\langle \theta_1 | \rho | \zeta \rangle)^2)} \left( |\langle 0_1 | \zeta \rangle|^2 \right)
\]

(16)

It is shown that, in general, given an initial state \( \rho \) there exist an interval of values of \( |\langle 0_1 | \zeta \rangle|^2 \) for which \( p_{1,s} \) exceeds \( p_d \), and that \( p_{1,s} \) reaches a maximum value, say \( p_{\text{max}} \), in that interval [10]. Optimizing Eq. (16) with respect to the variable \( |\langle 0_1 | \zeta \rangle|^2 \), one finds that \( p_{\text{max}} \) is

\[
p_{\text{max}} = \frac{\langle \zeta | \rho | \zeta \rangle}{2} + \frac{1}{4} (1 + R) ,
\]

(17)

with

\[
R = \sqrt{(1 - \gamma^2)(2|\langle \zeta | \rho | \zeta \rangle - 1|)^2 + \gamma^2},
\]

(18)

where \( \gamma \) is the decoherence parameter of the initial state,

\[
|\langle \zeta | \rho | \zeta \rangle| = \gamma \sqrt{|\langle \zeta | \rho | \zeta \rangle|^2}, \quad 0 \leq \gamma \leq 1.
\]

(19)

Fig. 2a shows \( p_{\text{max}} \) as a function of \( \gamma \) for different values of \( \gamma \). The diagonal solid line corresponds to \( p_d \). Notice that for all \( \gamma \neq 0 \) the optimal probability \( p_{\text{max}} \) exceeds \( p_d \) for all values of \( \gamma \). Also, larger values of \( \gamma \) result in larger values of \( p_{\text{max}} \).

The eigenstate \( |0_1 \rangle \) of \( \hat{\theta}_1 \) which optimizes \( p_{1,s} \) has a component on the target state \( |\zeta \rangle \) given by

\[
|\langle 0_1 | \zeta \rangle|^2 = \frac{1}{2} \left( 1 - \frac{1}{\sqrt{2}} \sqrt{1 + 4(\gamma^2)} \right).
\]

(20)

As evident from the above expression, the observable \( \hat{\theta}_1 \) which maximizes \( p_{1,s} \) depends explicitly on the initial state \( \rho \). Fig. 2b shows \( |\langle 0_1 | \zeta \rangle|^2 \), as a function of the initial probability of the state \( |\zeta \rangle \) for different values of \( \gamma \).

4. Driving the evolution by N+1 observables

Now, let us suppose that before measuring the observable \( \hat{\zeta} \), one measure \( N \) observables, say \( \hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_N \), each one defining an orthonormal basis \( \{ |0_j \rangle, |1_j \rangle \} \), \( j = 1, 2, \ldots, N \). The probability of driving the known initial state \( \rho \) towards the target state \( |\zeta \rangle \), by means of the sequence of measurements \( M(\hat{\zeta})M(\hat{\theta}_N) \ldots M(\hat{\theta}_2)M(\hat{\theta}_1) \) can be calculated recursively as

\[
p_{N,s} = \langle \zeta | \rho_N | \zeta \rangle,
\]

(21)

where \( \rho_N \) is given by

\[
\rho_N = \sum_{i_N=0}^{1} |\langle i_N | \rho_{N-1} | i_N \rangle| i_N \rangle |i_N \rangle.
\]

(22)

One would like that adding a new observable to the process made the probability of success increase, that is, that the difference \( \Delta = p_{N+1,s} - p_{N,s} \) was larger than 0. This difference reads

\[
\Delta = \sum_{i_N=0}^{1} |\langle i_N | \rho_{N-1} | i_N \rangle| \times \left( \sum_{i_{N+1}=0}^{1} |\langle i_N | i_{N+1} \rangle|^2 |\langle i_{N+1} | \zeta \rangle|^2 - |\langle i_N | \zeta \rangle|^2 \right).
\]

(23)
Figure 2. (a) maximum probability of success $p_{max}$ as a function of $\langle \zeta |\rho|\zeta \rangle$ for different values of $\gamma$: $\gamma = 1$ (dot-dash), $\gamma = 0.7$ (dot), $\gamma = 0.4$ (dash), and $\gamma = 0$ (solid). (b) $|\langle 0|\zeta \rangle|^2$ as a function of $\langle \zeta |\rho|\zeta \rangle$ for different values of $\gamma$: $\gamma = 1$ (dot-dash), $\gamma = 0.7$ (dot), $\gamma = 0.4$ (dash), and $\gamma = 0$ (solid).

The positivity of $\Delta$ is guaranteed under the conditions

$$\langle 0_N |\rho_{N-1}|0_N \rangle > \langle 1_N |\rho_{N-1}|1_N \rangle,$$  \hspace{1cm} (24)

and

$$\sum_{i_{N+1}=0}^{1} |\langle 0_N |i_{N+1}\rangle|^2 |\langle i_{N+1}|\zeta \rangle|^2 > |\langle 0_N |\zeta \rangle|^2.$$  \hspace{1cm} (25)

The latter condition indicates that the basis $\{|0_{N+1}\rangle, |1_{N+1}\rangle\}$ must be chosen in such a way that the probability of taking the state $|0_N\rangle$ to the state $|\zeta\rangle$ by means of the sequence of measurements $M(\zeta) M(\hat{\theta}_{N+1})$ should be higher than the probability obtained by means of the process $M(\zeta)$. In the previous section it was shown that this can be always achieved. Thus, it is always possible to increase the probability of success by adding to the sequence of measurements a new observable, suitably chosen. Therefore, when $N \to \infty$ then the probability of success tends to 1.

Fig. 3 shows the results of a numerical simulation which finds the probability $p_{N,s}$, Eq. (21), for a given set of bases $\{|0_j\rangle, |1_j\rangle\}$ ($j = 1, 2, \ldots, N$) with $N$ and $\rho$ fixed. On it, it is plotted the maximum value of $p_{N,s}$ as a function of $\langle \zeta |\rho|\zeta \rangle$, the initial probability, for different values of $N$: $N = 1$ (solid line), $N = 2$ (dashed line), $N = 3$ (dotted line), when (a) $\gamma = 1$ and (b) $\gamma = 0$. In Fig. 3a, which corresponds to an initial pure state, it is observed that $p_{N,s}$ increases with respect to $p_d$ (the diagonal line) for all values of the initial probability $\langle \zeta |\rho|\zeta \rangle$. In Fig. 3b, which corresponds to an initial mixed state (diagonal in the $\{|\zeta\rangle, |\zeta_{\perp}\rangle\}$ basis), it can bee seen that $p_{N,s}$ increases with respect to $p_d$ (the diagonal) only for $\langle \zeta |\rho|\zeta \rangle < 1/2$. 


Figure 3. Maximum value of success probability \( p_{N,s} \) as a function of the direct probability \( \langle \zeta | \rho | \zeta \rangle \) for different \( N \) values: \( N = 1 \) (solid line), \( N = 2 \) (dash line), \( N = 3 \) (dotted line), for: (a) an initial pure state \( \gamma = 1 \) and (b) a mixed initial state \( \gamma = 0 \). The diagonal solid line corresponds to \( p_d \).

Figure 4. Comparison of maximum value of the \( p_{2,s} \) probability (\( \gamma = 1 \) solid) and \( \gamma = 0 \) (dash)) with its counterpart \( p_{s,max} \) with \( N = 1 \) (dot).

5. Unknown versus known initial state
How much probability it can gained by knowing the initial state versus not to know it? One can compare the two reviewed schemes by examining the maximal success probabilities for processes with the same
number of measurements. Namely, one can compare $M(\hat{\zeta})M(\hat{\theta}_2)M(\hat{\theta}_1)$ in the second scheme with $M(\hat{\zeta})M(\hat{\theta})M(\hat{\zeta})$ in the first scheme. In Fig. 4, the solid line corresponds to the maximum success probability $p_{2,s}$ when the initial state is pure ($\gamma = 1$). The dashed line correspond to the maximum success probability $p_{2,s}$ when the initial state is totally incoherent ($\gamma = 0$). The maximum probability $p_{2,s}$ for any other value of the partial decoherence, characterized by $\gamma$, is inside the area contained between these two curves. The dotted line corresponds to the maximal probability $p_{s,max}$, Eq. (6), with $N = 1$. We observe that, for pure initial states, the scheme for known initial states leads to a larger success probability than the scheme for unknown initial states. It is also clear from this picture that a set of totally incoherent initial states has a higher success probability when the protocol for an unknown initial state is applied. However, for partially incoherent states ($0 < \gamma < 1$) the knowledge of the initial state does not necessarily lead to an increment of the success probability. Thus, in some cases the knowledge of the initial state indicates that it is more advantageous the second scheme than the first.

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