Towards Funnel MPC for nonlinear systems with relative degree two

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Abstract: Funnel MPC, a novel Model Predictive Control (MPC) scheme, allows guaranteed output tracking of smooth reference signals with prescribed error bounds for nonlinear multi-input multi-output systems. To this end, the stage cost resembles the high-gain idea of funnel control. However, rigorous proofs for initial and recursive feasibility without incorporating additional output constraints in the Funnel MPC scheme are only available for systems with relative degree one and stable internal dynamics. In this paper, we extend these results to systems with relative degree two by incorporating also a term based on the idea of a derivative funnel in the stage cost.

Keywords: model predictive control, funnel control, output tracking, nonlinear systems

1. INTRODUCTION

Model Predictive Control (MPC) is a widely-used control technique for linear and nonlinear systems and has seen various applications, see e.g. Qin and Badgwell (2003). Key reasons for its success are its applicability to multi-input multi-output nonlinear systems and its ability to directly take control and state constraints into account. To this end, a finite-horizon Optimal Control Problem (OCP) is solved before the prediction horizon is shifted forward in time and the procedure is repeated ad infinitum, see e.g. Coron et al. (2020) or the textbook Grün and Pannek (2017); Rawlings et al. (2017) nicely illustrating the basic concept for discrete-time systems.

Recursive feasibility is essential for successfully applying MPC, see e.g. Esterhuizen et al. (2020). This means, solvability of the OCP at a particular time instant has to automatically imply solvability of the OCP at the successor time instant. In order to achieve this, often, suitably designed terminal conditions (cost and constraints) are incorporated in the OCP to be solved at each time instant, see Chen and Allgöwer (1998) or the textbook Rawlings et al. (2017) and the references therein. However, such (artificially introduced) terminal conditions increase the computational burden of solving the OCP and complicate the task of finding an initially-feasible solution. As a consequence, the domain of the MPC feedback controller might become significantly smaller, see e.g. Chen et al. (2003); González and Odloak (2009). This technique becomes considerably more involved in the presence of time-varying state constraints, see e.g. Manrique et al. (2014) and references therein.

To overcome these restrictions for a large system class, Funnel MPC (FMPC) was proposed in Berger et al. (2020). This allows output tracking such that the tracking error evolves in a pre-specified, potentially time-varying performance funnel. A “funnel-like” stage cost, which penalizes the tracking error and becomes infinite when approaching the funnel boundary, is used. By incorporating output constraints in the OCP and using properties of the system class in consideration, initial and recursive feasibility are shown – without imposing additional terminal conditions and independent of the length of the prediction horizon.

The novel stage cost used in FMPC is inspired by funnel control, a model-free output-error feedback controller first proposed in Ilchmann et al. (2002), see also the recent work by Berger et al. (2021b) for a comprehensive literature overview. The funnel controller is an adaptive controller which allows output tracking within a prescribed performance funnel for a fairly large class of systems solely invoking structural assumptions, i.e. stable internal dynamics, known relative degree, and a sign-definite high-frequency gain matrix.

It is shown in Berger et al. (2021a) that such funnel-inspired stage cost automatically ensure initial and recursive feasibility for a class of nonlinear systems with relative degree one and, in a certain sense, input-to-state stable internal dynamics. Since the requirement of a sign-definite gain matrix is omitted, the system class is larger than the one the original funnel controller is applicable to. Moreover, adding (artificial) output constraints to the OCP, as used in the prior work, is superfluous. In numerical simulations, FMPC shows superior performance compared to both MPC with quadratic stage cost and funnel control.

Based on these simulations, it was suspected that these results also hold true for systems with higher relative degree. We show that this is in fact true and that for the scalar case the findings in Berger et al. (2021a) can be generalized to systems with relative degree two. However, while previous results allow for an arbitrary
short prediction horizon, for this system class a sufficiently long horizon – depending on the funnel – is necessary. A further generalization of these results to MIMO systems with relative degree two can be found in Denneestadt (2022).

**Notation:** \(\mathbb{N}\) and \(\mathbb{R}\) denote natural and real numbers, resp. \(\mathbb{N}_0 := \mathbb{N} \cup \{0\}\) and \(\mathbb{R}_{\geq 0} := [0, \infty)\); \(\|\cdot\|\) denotes a norm in \(\mathbb{R}^n\). \(\text{GL}_n(\mathbb{R})\) is the group of invertible \(\mathbb{R}^{n \times n}\) matrices. \(C^p(V, \mathbb{R}^n)\) is the linear space of \(p\)-times continuously differentiable functions \(f : V \to \mathbb{R}^n\), where \(V \subset \mathbb{R}^m\) and \(p \in \mathbb{N}_0 \cup \{\infty\}\). We use the notation \(C(V, \mathbb{R}^n) := C^0(V, \mathbb{R}^n)\) to refer to the space of continuous functions. On an interval \(I \subset \mathbb{R}\), \(L^\infty(I, \mathbb{R}^n)\) denotes the space of measurable essentially bounded functions \(f : I \to \mathbb{R}^n\) and \(L^\infty_{\text{loc}}(I, \mathbb{R}^n)\) the space of locally bounded measurable functions. Further, \(W^{k,\infty}(I, \mathbb{R}^n)\) is the Sobolev space of all \(k\)-times weakly differentiable functions \(f : I \to \mathbb{R}^n\) such that \(f, \ldots, f^{(k)} \in L^\infty(I, \mathbb{R}^n)\).

2. **SYSTEM CLASS AND CONTROL OBJECTIVE**

In this section the problem statement is introduced. We present the considered system class and the control objective and recall some necessary definitions.

2.1 **System class**

We consider control affine multi-input multi-output systems

\[
\begin{align*}
\dot{x}(t) & = f(x(t)) + g(x(t))u(t), \quad x(t_0) = x^0, \\
y(t) & = h(x(t)),
\end{align*}
\]

with \(t_0 \in \mathbb{R}_{\geq 0}\), \(x^0 \in \mathbb{R}^n\), functions \(f \in C^2(\mathbb{R}^n, \mathbb{R}^n)\), \(g \in C^2(\mathbb{R}^n, \mathbb{R}^{n \times m})\), \(h \in C^2(\mathbb{R}^n, \mathbb{R}^m)\) and a control function \(u \in L^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)\). The system (1) has a solution in the sense of Carathéodory, that is a function \(x : [t_0^0, \omega) \to \mathbb{R}^n\), \(\omega > t_0^0\), with \(x(t_0^0) = x^0\) which is absolutely continuous and satisfies the ODE in (1) for almost all \(t \in [t_0^0, \omega)\). The response associated with \(u\) is any maximal solution of (1) and is denoted by \(x(\cdot; t_0^0, x_0^0, u)\). It is unique since the right-hand side of (1) is locally Lipschitz in \(x\).

We recall the notion of relative degree for system (1), see e.g. (Isidori, 1995, Sec. 5.1). Assuming that \(f, g, h\) are sufficiently smooth, the Lie derivative of \(h\) along \(f\) is defined by \((L_f h)(x) := \dot{h}(x)f(x)\). Lie derivatives of higher order are recursively defined by \((L^k_f h)(x) := L_f(L^{k-1}_f h)(x)\), for \(k \in \mathbb{N}\), with \(L^0_f h = h\). Furthermore, for the matrix-valued function \(g\) we have

\[
(L^k_f g)(x) := [(L^k_{g_1} g)(x), \ldots, (L^k_{g_m} g)(x)],
\]

where \(g_i\) denotes the \(i\)-th column of \(g\) for \(i = 1, \ldots, m\). Then system (1) is said to have (global and strict) relative degree \(r \in \mathbb{N}\), if

\[
\forall k \in \{1, \ldots, r - 1\} \quad \forall x \in \mathbb{R}^n : \quad (L^k_f L^{k-1}_f h)(x) = 0 \quad \text{and} \quad (L^r_f L^{r-1}_f h)(x) \in \text{GL}_m(\mathbb{R}).
\]

If (1) has relative degree \(r\), then, under the additional assumptions provided in (Byrnes and Isidori, 1991, Cor. 5.6), there exists a diffeomorphic coordinate transformation

\[
\Phi : \mathbb{R}^n \to \mathbb{R}^n, \quad \Phi(x(t)) = (y(t), \dot{y}(t), \ldots, y^{(r-1)}(t), \eta(t))
\]

which puts the system into Byrnes-Isidori form

\[
\begin{align*}
y^{(r-1)}(t) &= p(y(t), \dot{y}(t), \ldots, y^{(r-1)}(t), \eta(t)) + \gamma(y(t), \dot{y}(t), \ldots, y^{(r-1)}(t), \eta(t)) u(t), \\
\dot{\eta}(t) &= q(y(t), \dot{y}(t), \ldots, y^{(r-1)}(t), \eta(t))
\end{align*}
\]

where \(p \in C(\mathbb{R}^n, \mathbb{R}^m)\), \(q \in C(\mathbb{R}^n, \mathbb{R}^{n-m})\), 
\(\gamma = (L_g L^{r-1}_f h) \circ \Phi^{-1} \in C(\mathbb{R}^n, \mathbb{R}^{n \times m})\) and 
\((y(t), \dot{y}(t), y^{(r-1)}(t), \eta(t)) = \Phi(\varphi(t))\). Furthermore, we require the following bounded-input, bounded-state (BIBS) condition on the internal dynamics (3b):

\[
\forall \zeta \in L^\infty_{\text{loc}}([t_0, \infty), \mathbb{R}^m) : \quad \|\eta(\cdot, t_0^0, \eta_0^0, \zeta)\|_{\infty} \leq c_0
\]

Fig. 1. Error evolution in a funnel \(\mathcal{F}_{\psi_0}\) with boundary \(\psi_0(t)\).

These funnels are determined by the choice of the functions \(\psi_0, \psi_1\) belonging to

\[
\mathcal{G}^0 := \left\{ \psi \in W^2(\mathbb{R}_{\geq 0}, \mathbb{R}) : \inf_{t \geq 0} \psi(t) > 0 \right\}.
\]

Note that the funnel \(\psi_1\) is uniformly bounded away from zero; i.e. there exists a boundary \(\lambda > 0\) with \(\psi(t) > \lambda\) for all \(t \geq 0\). Thus, perfect or asymptotic tracking is not our control objective. However, \(\lambda\) can be arbitrarily small. Furthermore, the funnel boundary is not necessarily monotonically decreasing.

If the error \(e\) evolves within the funnel \(\mathcal{F}_{\psi_0}\) for some \(\psi_0 \in \mathcal{G}^0\), then the derivative \(\dot{e}\) has to satisfy at some point \(t \geq 0\)

\[
\dot{e}(t) < \dot{\psi_0}(t) \quad \text{or} \quad \dot{e}(t) > -\dot{\psi_0}(t).
\]
Thus, the derivative funnel must be large enough for the error $\varepsilon$ to follow the funnel boundary $\psi_0$ and we therefore assume that $\psi = (\psi_0, \psi_1)$ is an element of $G^1 := \{(\psi_0, \psi_1) \in G^0 \times G^0 \mid \exists \varepsilon > 0 \forall t \geq 0 : \psi_1(t) \geq \varepsilon - \psi_0(t)\}$. Typically, the specific application dictates constraints on the tracking error and thus indicates suitable choices for $\psi$.

3. FUNNEL MPC

In order to extend the results from Berger et al. (2021a) to systems of the form (1) with relative degree two, we define, for $y_{ref} \in W^{2,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, $\zeta = (\zeta^0, \zeta^1) \in \mathbb{R}^{2m}$, $e_i(t, \zeta) := \zeta^i - y_{ref}(t)$ for $i = 0, 1$. We propose for $\psi = (\psi_0, \psi_1) \in G^1$ and the design parameter $\lambda_u \geq 0$ the new stage cost function

$$\ell : \mathbb{R}_{\geq 0} \times \mathbb{R}^{2m} \times \mathbb{R}^m \rightarrow \mathbb{R} \cup \{\infty\},$$

$$(t, \zeta, u) \mapsto \sum_{i=0}^{1} \frac{1}{1-\|e_i(t, \zeta)\|^2/\psi_i(t)^2} + \lambda_u \|u\|^2,$$

for $i = 0, 1$ and else.

By setting $\zeta = (y(t), \dot{y}(t))$, the terms $\frac{1}{1-\|e_i(t, \zeta)\|^2/\psi_i(t)^2}$ penalize the distance of the tracking error $\varepsilon(t) = y(t) - y_{ref}(t)$ and its derivative $\dot{e}(t)$ to their respective funnel boundaries $\psi_i(t)$. The parameter $\lambda_u$ allows to adjust a suitable trade off between tracking performance and required control effort. Note that we allow $\lambda_u = 0$. The stage cost $\ell$ is motivated by the construction of the funnel controller in Hackl et al. (2013) which also introduces an additional funnel for the derivative in order to generalize the results from Ilichmann et al. (2002) to systems with relative degree two.

Based on the stage cost (5), we may define the Funnel MPC (FMPC) algorithm as follows.

**Algorithm 1. (FMPC).**

**Given:** System (1), reference signal $y_{ref} \in W^{2,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, funnel function $\psi = (\psi_0, \psi_1) \in G^1$, stage cost function $\ell$ as in (5), $M > 0$, $t^0 \in \mathbb{R}_{\geq 0}$, and $x^0 \in \mathbb{R}^n$ with

$$\Phi(x^0) := \{(\zeta, \eta) \in \mathbb{R}^{2m} \times \mathbb{R}^{n-2m} \mid \|e_i(t_0, \zeta)\|^2/\psi_i(t_0)^2 < \eta \}$$

for $i = 0, 1$ (6)

where $\Phi$ is the diffeomorphism from (2).

**Set** the time shift $\delta > 0$, the prediction horizon $T > \delta$ and initialize the current time $t := t^0$.

**Steps:**

(a) Obtain a measurement of the state $x = \Phi^{-1}(y, \dot{y}, \eta)$ at time $t$ and set $\dot{x} := x(t)$.

(b) Compute a solution $u^* \in L^\infty([\hat{t}, \hat{t} + T], \mathbb{R}^m)$ of the Optimal Control Problem (OCP)

$$\begin{align*}
\text{minimize} & \quad \int_{\hat{t}}^{\hat{t} + T} \ell(t, y(t), \dot{y}(t), u(t)) dt \\
\text{subject to} & \quad (1), \quad x(t) = x(t), \\
& \quad \|u(t)\| \leq M \quad \text{for } t \in [\hat{t}, \hat{t} + T] (7)
\end{align*}$$

(c) Apply the feedback law

$$\mu : [\hat{t}, \hat{t} + \delta] \times \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad \mu(t, \dot{x}) = u^*(t)$$

in system (1). Increase $\hat{t}$ by $\delta$ and go to Step (a).

In practical application there usually is a limitation $M > 0$ on the maximal control that can be applied to the system 1. To ensure that the control signal meets this bound, the constraint $\|u(t)\| \leq M$ is added to the OCP (7) of the FMPC Algorithm 1.

4. MAIN RESULT

Our main results is to show that for scalar systems the Funnel MPC Algorithm 1 is, given a sufficiently long prediction horizon $T > 0$ and large enough control constraint $M > 0$, initially and recursively feasible and that it guarantees the evolution of the tracking error $\varepsilon$ and its derivative $\dot{\varepsilon}$ within their respective performance funnels $F_{\psi_0}$.

**Theorem 2.** Consider scalar system (3) with strict relative degree $r = 2$ and $m = 1$. Assume that there exists a diffeomorphism $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that the coordination transformation in (2) puts the system (1) in the Byrnes-Isidori form (3) satisfying (4). Let $\psi = (\psi_0, \psi_1) \in G^1$, $y_{ref} \in W^{2,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R})$, $t^0 \in \mathbb{R}_{\geq 0}$, $\delta > 0$, and $B \subset \mathcal{D}_{\psi_0}$ a compact set. Then there exist $T > \delta$ and $M > 0$ such that the FMPC Algorithm 1 is initially and recursively feasible for every $x^0 \in B$, i.e. at time $t = t^0$ and at each successor time $t = t^0 + \delta$ and the OCP (7) has a solution. In particular, the closed-loop system consisting of (1) and the FMPC feedback (8) has a (not necessarily unique) global solution $x : [t^0, \infty) \rightarrow \mathbb{R}^n$ and the corresponding input is given by $u_{FMPC}(t) = \mu(t, x(t))$, $t \in [t^0, t^0 + \delta)$, $i \in [0, 1)$.

Furthermore, each global solution $x$ with corresponding input $u_{FMPC}$ satisfies:

(i) $\forall t \geq t^0 : \|u_{FMPC}(t)\| \leq M$.

(ii) $\forall t \geq t^0 : \|\dot{\varepsilon}(t)\| < \psi_1(t)$ for $\dot{\varepsilon}$ in particular the error $\varepsilon = y - y_{ref}$ evolves within the funnel $F_{\psi_0}$ and $\dot{\varepsilon}$ within $F_{\psi_1}$.

**Proof:** We provide a sketch of the proof. For more details we refer to Demnstädt (2022). For $T > 0$, $M > 0$, $i^0 \geq t_0$, $\dot{x} \in \mathcal{D}_i$ as in (6), and the interval $I_T^i := [\hat{t}, \hat{t} + T]$, we denote by $u^*_{M}(i^0, \dot{x})$ the set

$$\{u \in L^\infty(I_T^i, \mathbb{R}) \mid \|u\| \leq M, \ x(t; \hat{t}, \dot{x}, u) \text{ satisfies (1) and} \ \Phi(x(t; \hat{t}, \dot{x}, u)) \in \mathcal{D}_i \text{ for all } t \in I_T^i \}.$$

This is the set of all $L^\infty$-controls $u$ bounded by $M > 0$ which, if applied to system (1), guarantee that the error $\varepsilon(t) = y(t) - y_{ref}(t)$ and its derivative $\dot{\varepsilon}(t)$ evolve within their respective funnels. A straightforward adaption of Theorem 4.3 and Theorem 4.5 from Berger et al. (2021a) to the current setting yields: if the set $U^M_{i}(i^0, \dot{x})$ is non-empty, then the OCP (7) has a solution $u^* \in U^M_{i}(i^0, \dot{x})$. Therefore, if $u^*$ is applied to system (1), then $\|\dot{\varepsilon}(t)\|^2/\psi_1(t)^2 < \psi_1(t)$ for all $t \in I_T^i$ and $i = 0, 1$. In particular $\Phi(x(t; \hat{t}, \dot{x}, u^*)) \in \mathcal{D}_i$. Thus, it is sufficient to show that there exist $T > 0$ and $M > 0$ such that $\Phi(\dot{x}) \in \mathcal{D}_i$ implies the non-emptiness of set $U^M_{i}(i^0, \dot{x})$ at time $t = t^0$ and at each successor time $t \in t^0 + \delta$ during the Funnel MPC Algorithm 1.

Since the set $B$ is compact, there exists $\varepsilon > 0$ such that for all initial values $x^0$ of system (1) with $\Phi(x^0) \in B$
the initial tracking error \( e^0 = y(t^0) - y_{\text{ref}}(t^0) \) satisfies 
\[ |e^0| < \psi_0(t^0) - \varepsilon \] and \( \varepsilon \leq \inf_{t \geq t^0} \psi_0(t) \). Using the Byrnes-Isidori form (3) and (BIBS) condition (4), one can show that there exists \( M > 0 \) such that for every initial value \( x^0 \) with \( \Phi(x^0) \in B \) there exists a control \( \ddot{u} \in L^\infty([t^0, \bar{t}], \mathbb{R}) \) bounded by \( M \) for which the following holds. If \( \ddot{u} \) is applied to (1), then there exists \( \ddot{t} > t^0 \) with \( |e(\ddot{t})| < \psi_0(\ddot{t}) \) for all \( t \in [t^0, \bar{t}] \), \( \psi_0(\ddot{t}) - \varepsilon < |e(\ddot{t})| < \psi_0(\ddot{t}) \), and \( \dot{e}(\ddot{t}) - \psi_0(\ddot{t}) = 0 \) for all \( t \geq \ddot{t} \). Thus, the distance of tracking error \( e \) to the upper (lower) funnel boundary remains constant from time \( \ddot{t} \) onwards. Hence, \( |e(\ddot{t})| < \psi_0(\ddot{t}) \) for all \( t \geq \ddot{t} \). Since there exists \( \varepsilon > 0 \) with \( \psi_1(t) \geq \varepsilon - \psi_0(t) \) for all \( t \geq 0 \), the derivative \( \dot{e} \) also stays within the funnel boundary \( \psi_1 \) from time \( \ddot{t} \) onwards. Choosing \( M > 0 \) large enough, this can also be achieved up to \( t \). Thus, \( \ddot{u} \in U^M_T(\ddot{t}, \ddot{x}) \) for all \( T > 0 \). The bound \( M \) depends on \( \psi_0, \psi_1, \varepsilon, \varepsilon \), the set \( B \), and the functions \( p, \gamma, q \) from (3). An explicit construction of \( M > 0 \) and \( \ddot{u} \) can be found in (2022).

If the tracking error \( e \) satisfies 
\[ |e(t)| < \psi_0(t) - \varepsilon \] at a time \( t \in \mathbb{R}^+ \), then \( U^M_T(t, \dot{x}) \neq \emptyset \). Otherwise, \( U^M_T(t, \dot{x}) \) is non-empty, since the solution \( u^* \) of the OCP (7) from the previous time step is an element of \( U^M_T(t, \dot{x}) \). Choosing \( T := T - \delta \) large enough, it is possible to prove that there exists a control \( \dddot{u} \in U^M_T(t, \dot{x}) \) for which the following holds. If \( \dddot{u} \) is applied to the system (1), there exists a \( t > \ddot{t} \) with either \( |e(t)| < \psi_0(t) - \varepsilon \) or \( |e(t)| = \psi_0(t) \) = 0. Details on the construction of \( \dddot{u} \) depending on a large enough horizon \( T - \delta \) can be found in (2022).

In the latter case, one can change the control \( \dddot{u} \) bounded by \( M \) for which the following holds. \( \dddot{u}(t) \) is applied to the system (1), \( |\dot{e}(t)| - \psi_0(t) = 0 \) for all \( t \geq \ddot{t} \). The application of \( \dddot{u} \) guarantees that the tracking error \( e \) remain within the funnel boundary from time \( \ddot{t} \) onwards. Thus, \( |e(t)| < \psi_0(t) \) for all \( t \geq \ddot{t} \). Since there exists \( \varepsilon > 0 \) with \( \psi_1(t) \geq \varepsilon - \psi_0(t) \) for all \( t \geq 0 \), the derivative \( \dot{e} \) also stays within the funnel boundary \( \psi_1 \). Hence, \( |\dot{e}(t)| < \psi_0(t) \) for all \( t \geq \ddot{t} \). Therefore, \( \dddot{u} \in U^M_T(t, \dot{x}) \). The horizon \( T \) depends on the funnel boundaries \( \psi_0, \psi_1 \), and the bound \( M > 0 \). This completes the proof.

Remark: Note that proving the recursive feasibility of the OCP is not trivial. Similar to the proof of the recursive feasibility of the FMPC Algorithm in (2021a), the main challenge is to guarantee that the set of controls \( U^M_T(t, \dot{x}) \) as in (9) is non-empty at each time step \( t \) of the Funnel MPC Algorithm 1. While for systems with relative degree one it is possible to find \( M > 0 \) such that \( U^M_T(t, \dot{x}) \) is non-empty at time step \( t \) independent of the horizon \( T > 0 \), for systems with relative degree two this seems not to be possible. An explicit construction of \( M \) and \( T \) such that \( U^M_T(t, \dot{x}) \neq \emptyset \) is guaranteed can be found in (2021a) together with a generalization of Theorem 2 to MIMO systems, i.e. \( m > 1 \).

Note further that while the proof of Theorem 2 makes extensive use of the diffeomorphism \( \Phi \) as in (1) and the Byrnes-Isidori form (3), their computation is not necessary for the application of the FMPC Algorithm 1. Condition (6) requires the initial tracking error \( e(t^0) \) and its derivative \( \dot{e}(t^0) \) to be within their respective funnel boundaries. This can easily be verified by different means.

5. SIMULATION

To demonstrate the application of the FMPC Algorithm 1, we consider the example of a mass-spring system mounted on a car from Seifried and Blajer (2013). The mass \( m_2 \) moves on a ramp inclined by the angle \( \vartheta \in [0, 2\pi] \) and mounted on a car with mass \( m_1 \) by a spring-damper system, see Figure 2. It is possible to control the force \( F \).
\[ \ell : \mathbb{R}_{\geq 0} \times \mathbb{R}^{2n} \times \mathbb{R}^m \to \mathbb{R} \cup \{ \infty \}, \]
\[ (t, \zeta, u) \mapsto \begin{cases} 
\frac{1}{1-\|e_0(t, \zeta)\|^2/\psi_0(t)^2} + \lambda_u \|u\|^2, & \|e_0(t, \zeta)\| \neq \psi_0(t) \\
\infty, & \text{else}. 
\end{cases} \]

The function \( \ell \) penalizes the distance of the tracking error \( e(t) = y(t) - \hat{y}_{\text{ref}}(t) \) to the funnel boundary \( \psi_0 \), but, contrary to the stage cost function \( (5) \), not the distance of the derivative \( \dot{e}(t) = \dot{y}(t) - \dot{\hat{y}}_{\text{ref}}(t) \) to the boundary \( \psi_1 \). In both cases we choose for the FMPC Algorithm 1 the prediction horizon \( T = 0.6 \) and time shift \( \delta = 0.04 \). Due to discretisation, only step functions with constant step length 0.04 are considered for the OCP \( (7) \). We further choose for both stage cost functions the parameter \( \lambda_u = 5 \cdot 10^{-3} \) and allow a maximal control value of \( M = 30 \). All simulations are performed on the time interval \([0, 7]\) with the MATLAB routine \texttt{ode45} and are depicted in Figure 3. Figure 3a shows the tracking error of the two different FMPC schemes evolving within the funnel boundaries given by \( \psi_0 \), while Figure 3b displays the derivative of the error within the boundaries given by \( \psi_1 \). The respective control signals generated by the controllers is displayed in Figure 3c.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3}
\caption{Simulation of system \((10)\) with output \((11)\) under FMPC Algorithm 1 and FMPC from Berger et al. (2021a)}
\end{figure}

Moreover, the FMPC Algorithm 1 with stage cost function \( (5) \) exhibits a smaller range of employed control values as the FMPC scheme from Berger et al. (2021a).

6. CONCLUSION

In this note we outline a conceptual framework to extend the FMPC scheme proposed in Berger et al. (2021a), which solves the problem of tracking a reference signal within a prescribed performance funnel, to systems with relative degree two. By exploiting concepts from funnel control and using a “funnel-like” stage cost, feasibility is achieved without the need for additional terminal or explicit output constraints while also being restricted to (a priori) bounded control values. In particular, additional output constraints in the OCP of FMPC as considered in Berger et al. (2020) are not required to infer the feasibility results. However, contrary to previous results the prediction horizon has to be sufficiently long in order to guarantee recursive feasibility of the Funnel MPC algorithm.

Extending these results to systems with arbitrary relative degree \( r > 2 \) is subject of future work.

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