MULTIPLE SOLUTIONS FOR A CRITICAL QUASILINEAR EQUATION WITH HARDY POTENTIAL

FENGSHUANG GAO AND YUXIA GUO*

Department of Mathematical Science
Tsinghua University, Beijing, China

Abstract. In this paper, we investigate the following quasilinear equation involving a Hardy potential:

\[
\begin{cases}
- \sum_{i,j=1}^{N} D_j (a_{ij}(u) D_i u) + \frac{1}{2} \sum_{i,j=1}^{N} a_{ij}'(u) D_i u D_j u - \frac{\mu}{|x|^2} u = au + |u|^{2^*-2} u & \text{in } \Omega, \\
\quad u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \(2^* = \frac{2N}{N-2}\) is the Sobolev critical exponent for the embedding of \(H_0^1(\Omega)\) into \(L^p(\Omega)\), \(a > 0\) is a constant and \(\Omega \subset \mathbb{R}^N\) is an open bounded domain which contains the origin. We will prove that under some suitable assumptions on \(a_{ij}\), when \(N \geq 7\) and \(\mu \in [0, \mu^*)\) for some constant \(\mu^*\), problem (P) admits an unbounded sequence of solutions. To achieve this goal, we perform the subcritical approximation and the regularization perturbation.

1. Introduction. We consider the following critical quasilinear equation involving a Hardy potential:

\[
\begin{cases}
- \sum_{i,j=1}^{N} D_j (a_{ij}(u) D_i u) + \frac{1}{2} \sum_{i,j=1}^{N} a_{ij}'(u) D_i u D_j u - \frac{\mu}{|x|^2} u = au + |u|^{2^*-2} u & \text{in } \Omega, \\
\quad u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \(\Omega \subset \mathbb{R}^N\) is an open bounded domain which contains the origin, \(2^* = \frac{2N}{N-2}\) is the Sobolev critical exponent for the embedding of \(H_0^1(\Omega)\) into \(L^p(\Omega)\), \(a > 0\) is a constant and \(\mu \in [0, \mu^*)\) for some constant \(\mu^*\), which will be determined later. Note that \(\mu|x|^{-2} u\) is the Hardy potential.

When \(a_{ij}(t) = \delta_{ij}\) and \(\mu = 0\), problem (1) is reduced to the classical Brezis-Nirenberg problem:

\[
\begin{cases}
- \Delta u = au + |u|^{2^*-2} u & \text{in } \Omega, \\
\quad u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Since the pioneering work of Brezis and Nirenberg [4], lots of attention has been focused on the existence of nontrivial solutions for semilinear problem with critical exponent. See for example [4, 9, 10, 12, 15, 30] and references therein. In particular, Devillanova and Solimini [15] proved that if \(N \geq 7\), for any \(a > 0\), problem (2) admits

\[
2010 \text{ Mathematics Subject Classification.} \ 35B45, 35J25.
\]

\textit{Key words and phrases.} Quasilinear equations, critical exponents, Hardy potential, multiple bound state solutions.

The authors are supported by NSFC grant 11771235, 11331010, 11571040.

* Corresponding author.
admits infinitely many solutions. The work of [15] inspired a lot of efforts in the
last decade to generalize similar existence and multiplicity results to other classes
of equations. For instance, Cao, Peng and Yan [7] proved the existence of infinitely
many solutions for $p$-Laplacian equation, while Guo, Liu and Wang [17] investigated
a quasilinear equation. Further more, Cao and Yan [8] considered (1) with $a_{ij}(t) =
\delta_{ij}$ but $\mu \neq 0$. They proved that if $a_{ij}(t) = \delta_{ij}$, $N \geq 7, \mu \in [0, \frac{(N-2)^2}{4} - 4)$, problem
(1) has infinitely many solutions.

From the physical point of view, equations with Hardy potential arise from many
physical contexts, such as molecular physics [20], quantum cosmology [3] and lin-
erization of combustion models [16]. On the other hand, from the mathematical
point of view, the main reason of interest in Hardy potentials lies in their criticality.
More precisely, we see that although the Hardy term has the same homogeneity as
the Laplacian operator, it does not belong to the Kato’s class and hence can not be
regarded as a lower order perturbation term.

In the present paper, we will show the existence of infinitely many solutions for
a more general critical quasilinear equation with Hardy potential. Here the critical
exponent reads as $\frac{2N}{N-2}$. In general, for example, when $a_{ij}(t) = (1 + |t|^{2\alpha})\delta_{ij}$, $\alpha \geq 0$,
the corresponding critical exponent is $(1 + \alpha)\frac{2N}{N-2}$. Note that if $\int_{\mathbb{R}^N} u^{2\alpha} \nabla u \cdot \nabla u < +\infty$ for $\alpha > 0$, then the Hardy integral term $\int_{\mathbb{R}^N} \frac{u^2}{|x|^2}$ is compact. In this case, the Hardy
term has no effect to the equation. Therefore, we are only concerned with the case
when $\alpha = 0$, that is the case when $a_{ij}$ is bounded. For more general cases, such
as the case when $a_{ij}(t) = (1 + |t|^{2\alpha})\delta_{ij}$ with $\alpha > 0$, we could obtain the similar
existence results.

Formally, equation (1) has a variational structure and its corresponding func-
tional can be written as:

$$I(u) = \frac{1}{2} \int_{\Omega} \left( \sum_{i,j=1}^{N} a_{ij}(u)D_iuD_ju - \frac{\mu}{|x|^2} u^2 - au^2 \right)dx - \frac{1}{2} \int_{\Omega} |u|^2^\ast dx, \quad u \in H_0^1(\Omega).$$  \hspace{1cm} (3)

However the energy functional is continuous but is not of $C^1$ in $H_0^1(\Omega)$, even for
the subcritical problems (i.e., the critical exponent $2^\ast$ is replaced by a smaller exponent
$q < 2^\ast$). Hence the standard critical point theory can not be applied directly. To
overcome this difficulty, we apply a regularization approach proposed by Liu and
Wang (e.g., [21, 25, 26, 27]). On the other hand, since $2^\ast$ is the critical exponent for
the Sobolev embedding from $H_0^1(\Omega)$ to $L^p(\Omega)$, the second difficulty we are facing
is that the energy functional $I(u)$ does not satisfy the Palais.Smale condition (P.S.
condition in short) for large energy levels. The third difficulty is that, unlike in [17],
every nontrivial solution of equation (1) is singular at $x = 0$ if $\mu \neq 0$ (see [5]). So,
different techniques are needed to deal with the case $\mu \neq 0$.

We assume that $a_{ij}(s)$ satisfies the following conditions:

$(A_1)$ $a_{ij}, D_s a_{ij} \in C(\mathbb{R}, \mathbb{R}),$ $a_{ij} = a_{ji}$, and there exists a constant $C > 0$ such
that $|a_{ij}| \leq C, |D_s a_{ij}| \leq C$ for all $(x, s) \in \bar{\Omega} \times \mathbb{R}$.

$(A_2)$ there exist $\alpha, \beta > 0$ such that for all $\xi \in \mathbb{R}^N$, $s \in \mathbb{R}$, it holds

$$\alpha |\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(s)\xi_i\xi_j \leq \beta |\xi|^2.$$  \hspace{1cm} (A_4)

$(A_3)$ there exist $0 \leq \gamma_1 < \gamma_2 < 2^\ast - 2$ such that for all $\xi \in \mathbb{R}^N$, $s \in \mathbb{R}$, it holds
Suppose Theorem 1.1. are the following:

\[ (4) \]

solutions to the subcritical problems

Consider the following perturbed problem:

\[
\begin{align*}
- \sum_{i,j=1}^{N} D_{i}(a_{ij}(u)D_{i}u) + \frac{1}{2} \sum_{i,j=1}^{N} a'_{ij}(u)D_{i}uD_{j}u - \frac{\mu}{|x|^2} u &= |u|^{2*_{\varepsilon}} - \varepsilon u + au \text{ in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \varepsilon \to 0 \).

Let \( \bar{\mu} = \frac{(N-2)^2}{4}, \mu^* = \min\{\bar{\mu} - 1, \alpha(\bar{\mu} - 4)\} \), the main results of the present paper are the following:

**Theorem 1.1.** Suppose \((A_1) - (A_5)\) hold and \(N \geq 7\). Let \( \{u_n\} \) be a sequence of solutions to the subcritical problems \((4)\) with \(\varepsilon = \varepsilon_n \to 0\), and there exists a constant \(C\) independent of \(n\) such that \(\|u_n\| \leq C\). Then for any \(a > 0\), \(0 < \mu < \mu^*\), \(\{u_n\}\) converges strongly in \(H^1_0(\Omega)\) as \(n \to \infty\).

As a consequence of Theorem 1.1, we have the following multiplicity result.

**Theorem 1.2.** Suppose \((A_1) - (A_5)\) hold and \(N \geq 7\). Then for any \(a > 0\), \(0 < \mu < \mu^*\), \((1)\) has infinitely many solutions.

Before the end of this introduction, let us outline the methods of the proofs. By Theorem 1.1 in [14], \((4)\) admits a sequence of solutions \(\{u_{\varepsilon,l}\}\). Then we will study as \(\varepsilon \to 0\), the convergence of solutions \(\{u_{\varepsilon,l}\}\) to the solutions \(u_l\) of \((1)\). For this purpose, one of our main ingredients is to show that the weak solutions of the subcritical problem \((4)\) satisfy a variational inequality, and this allows us to adapt some of the arguments from [6] and [8]. For the convergence analysis, we first show a concentration result of solutions to the subcritical problems, then by using a decomposition result introduced by C. Tintarev and K. Fineseler [31, 32], we establish some integral estimates, which results in a finer control for the solutions near the blow up points. Moreover, as we mentioned before, due to the appearance of the Hardy term, the solution \(u_{\varepsilon,l}\) has singularity at the origin and is not in \(L^\infty\) anymore, so new ideas are needed to establish the local Pohozaev identity in small balls. Finally an application of the local Pohozaev identity rules out the possibility of bubble blow-ups which in turn gives the strong convergence result.

The paper is organized as follows. In Section 2, we discuss the concentration compactness of the solutions to the subcritical problems, while in section 3 and 4, according to the blow up locations given in Section 2, we investigate the bounds of the approximation solutions in the safe regions. In Section 5, we establish a local Pohozaev identity which together with the estimates in Sections 4 allows us to give the proofs of Theorem 1.1 and 1.2. Some essential integral estimates for a linear problem in divergent form are put in the Appendices.

Throughout this paper, without additional declarations, we will denote \(\alpha, \beta\) as in \((A_2)\), the norms of \(H^1_0(\Omega)\) and \(L^p(\Omega)\) by \(\| \cdot \|\) and \(\| \cdot \|_p\) respectively.
2. Concentration compactness analysis. In this section, we will consider the concentration behaviors for solutions of problem (4) as \( \varepsilon \to 0 \). The following result is important for us.

**Lemma 2.1.** Assume that \( u \) is a solution of (4) with \( \varepsilon \geq 0 \). Let \( v = |u| \). Then the following differential inequality holds:

\[
\int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(v) D_i v D_j \varphi \leq \int_{\Omega} (v^{2^*-1-\varepsilon} + \mu v) \varphi + \frac{\mu v \varphi}{|x|^2}, \quad \forall \varphi \in H_0^1(\Omega), \ \varphi \geq 0. \tag{5}
\]

**Proof.** Let \( v_\varepsilon = \sqrt{\varepsilon^2 + u^2} - \varepsilon, \ \varepsilon > 0 \). Then \( v_\varepsilon \in H_0^1(\Omega) \) and \( v_\varepsilon \to v \) strongly in \( H_0^1(\Omega) \) as \( \varepsilon \to 0 \). For any \( \varphi \in C_0^\infty(\Omega), \ \varphi \geq 0 \), we have \( \frac{\varphi v}{\sqrt{\varepsilon^2 + u^2}} \in H_0^1(\Omega) \), and

\[
\int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(v) D_i v_\varepsilon D_j \varphi
\]

\[
= \int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(v) \frac{u}{\sqrt{\varepsilon^2 + u^2}} D_i u D_j \varphi
\]

\[
= \int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(u) D_i u D_j \left( \frac{\varphi u}{\sqrt{\varepsilon^2 + u^2}} \right) - \int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(u) D_i u D_j u \left( \frac{\varphi u^2}{\varepsilon^2 + u^2} \right)
\]

\[
\leq \int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(u) D_i u D_j \left( \frac{\varphi u}{\varepsilon^2 + u^2} \right)
\]

\[
= -\frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(u) D_i u D_j u \left( \frac{\varphi u}{\varepsilon^2 + u^2} \right) + \int_{\Omega} \left( |u|^{2^*-2} - \varepsilon u + \frac{\mu u}{|x|^2} \right) \frac{\varphi u^2}{\varepsilon^2 + u^2}
\]

\[
\leq \int_{\Omega} \left( v^{2^*-1-\varepsilon} + \mu v \right) \frac{\varphi v}{\varepsilon^2 + u^2}.
\]

Therefore, for \( \forall \varphi \in C_0^\infty(\Omega), \ \varphi \geq 0 \), we obtain

\[
\int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(v) D_i v D_j \varphi \leq \int_{\Omega} \left( v^{2^*-1-\varepsilon} + \mu v \right) \frac{\varphi v}{\varepsilon^2 + u^2}. \tag{6}
\]

Notice that \( v_\varepsilon \to v \) strongly in \( H_0^1(\Omega) \) as \( \varepsilon \to 0 \). Letting \( \varepsilon \to 0 \) in (6), we obtain the desired result for \( \forall \varphi \in C_0^\infty(\Omega), \ \varphi \geq 0 \). Since \( C_0^\infty(\Omega) \) is dense in \( H_0^1(\Omega) \), using an approximation argument, we prove (5) for \( \forall \varphi \in H_0^1(\Omega), \ \varphi \geq 0 \).

The next lemma concerns the convergence of \( \{u_n\} \) to the solution of (1).

**Lemma 2.2.** Suppose \( \{u_n\} \) is a sequence of solutions of the problem (4) with \( \varepsilon = \varepsilon_n \to 0 \), as \( n \to \infty \). If \( u_n \to u \) in \( H_0^1(\Omega) \) as \( n \to \infty \), then \( u \) is a solution of (1).

**Proof.** This result can be proved by following the same arguments as in Lemma 2.2 in [17] (where \( \mu = 0 \)).

By (A2) and (A5), we have

\[
\alpha |\xi|^2 \leq \sum_{i,j=1}^{N} A_{ij} \xi_i \xi_j \leq \beta |\xi|^2, \quad \forall \xi \in \mathbb{R}^N.
\]
Lemma 2.3. Assume \( \{u_n\} \) is a sequence of solutions of (4) with \( \varepsilon = \varepsilon_n \to 0 \) and \( u_n \to u \) in \( H_0^1(\Omega) \). Then

\[
\int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(u_n) D_i u_n D_j \varphi
= \int_{\Omega} \sum_{i,j=1}^{N} A_{ij} D_i u_n D_j \varphi + \int_{\Omega} \sum_{i,j=1}^{N} (a_{ij}(u) - A_{ij}) D_i u D_j \varphi + o_n(1) \|D\varphi\|_2. \tag{7}
\]

Proof. For any \( T > 0 \), denote \( u^T = u \), if \( |u| \leq T \) and \( u^T = \pm T \), if \( \pm u \geq T \). We first prove that \( u_n^T \to u^T \) in \( H_0^1(\Omega) \) as \( n \to \infty \). In fact, taking \( u^T \) and \( u_n^T \) as test functions in (1) and (4) respectively, we have

\[
\int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(u) D_i u D_j u^T + \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{N} a'_{ij}(u) D_i u D_j u u^T = \int_{\Omega} \left( |u|^2 - 2 u + a u + \frac{\mu u}{|x|^2} \right) u^T, 
\]

and

\[
\int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(u_n) D_i u_n D_j u_n^T + \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{N} a'_{ij}(u) D_i u_n D_j u_n u_n^T 
= \int_{\Omega} \left( |u_n|^2 - 2 - 2 u_n + a u_n + \frac{\mu u_n}{|x|^2} \right) u_n^T.
\]

Since \( u_n \to u \) in \( L^{2^* - 1} \), we have

\[
\int_{\Omega} \left( |u_n|^2 - 2 - 2 u_n + a u_n + \frac{\mu u_n}{|x|^2} \right) u_n^T \to \int_{\Omega} \left( |u|^2 - 2 u + a u + \frac{\mu u}{|x|^2} \right) u^T, \quad \text{as } n \to \infty.
\]

Thus by weakly lower semi-continuity, we have

\[
\int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(u_n) D_i u_n D_j u_n^T \to \int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(u) D_i u D_j u^T, \quad \text{as } n \to \infty,
\]

which implies that \( u_n^T \to u^T \) in \( H_0^1(\Omega) \).

Notice that \( u \in L^{\infty}(\Omega \ \setminus \ B_\delta(0)) \), for any \( \delta > 0 \). Taking \( T > \|u\|_{L^{\infty}(\Omega \ \setminus \ B_\delta(0))} \), we have \( u_n^T \to u \) in \( H_0^1(\Omega \ \setminus \ B_\delta(0)) \). Thus for \( \varphi \in C_0^\infty(\Omega) \) with \( \varphi(x) = 0 \) in \( |x| \leq \delta \), we have

\[
\int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(u_n) D_i u_n D_j \varphi
= \int_{|u_n| \geq T} \sum_{i,j=1}^{N} a_{ij}(u_n) D_i u_n D_j \varphi + \int_{|u_n| \leq T} \sum_{i,j=1}^{N} a_{ij}(u_n) D_i u_n D_j \varphi
= \int_{|u_n| \geq T} \sum_{i,j=1}^{N} A_{ij} D_i u_n D_j \varphi + o_T(1) \|D\varphi\|_2 + \int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(u_n^T) D_i u_n^T D_j \varphi
= \int_{\Omega} \sum_{i,j=1}^{N} A_{ij} D_i u_n^T D_j \varphi - \int_{|u_n| < T} \sum_{i,j=1}^{N} A_{ij} D_i u_n D_j \varphi + o_T(1) \|D\varphi\|_2
+ \int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(u) D_i u D_j \varphi + o_n(1) \|D\varphi\|_2.
\]
\begin{align}
&= \int_{\Omega} \sum_{i,j=1}^{N} A_{ij} D_i u_n D_j \varphi - \int_{\Omega} \sum_{i,j=1}^{N} A_{ij} D_i u D_j \varphi + \int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(u) D_i u D_j \varphi \\
&\quad + \mu r(1) \|D\varphi\|_2 + a_n(1) \|D\varphi\|_2.
\end{align}

Since $T$ is arbitrary, we obtain (7) for $\forall \varphi \in C^\infty_0(\Omega)$ with $\varphi(x) = 0$ in $|x| \leq \delta$.

Now given $\varphi \in C^\infty_0(\Omega)$. Choose $\varphi_\delta \in C^\infty_0(\Omega)$ such that $\varphi_\delta(x) = 0$ in $|x| \leq \delta$, $\varphi_\delta(x) = 1$ in $|x| \geq 2\delta$ and $|\nabla \varphi_\delta| \leq \frac{\delta}{2}$. Taking $\varphi \varphi_\delta$ as a test function in (7), we have

\begin{align}
&\int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(u_n) D_i u_n D_j \varphi_\delta \varphi + \int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(u_n) D_i u_n D_j \varphi \varphi_\delta \\
&= \int_{\Omega} \sum_{i,j=1}^{N} A_{ij} D_i u_n D_j \varphi_\delta \varphi + \int_{\Omega} \sum_{i,j=1}^{N} A_{ij} D_i u_n D_j \varphi \varphi_\delta + \int_{\Omega} \sum_{i,j=1}^{N} (a_{ij}(u) - A_{ij}) D_i u_n D_j \varphi_\delta \\
&\quad + \int_{\Omega} \sum_{i,j=1}^{N} (a_{ij}(u) - A_{ij}) D_i u_n D_j \varphi_\delta \varphi + a_n(1) \|D(\varphi \varphi_\delta)\|_2.
\end{align}

Notice that

\begin{align}
&\left| \int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(u_n) D_i u_n D_j \varphi_\delta \varphi \right| \leq \frac{C}{\delta} \int_{B_{2\delta}} |D u_n| |\varphi| \\
&\leq \frac{C}{\delta} \left( \int_{B_{2\delta}} |D u_n|^2 \right)^{\frac{1}{2}} \left( \int_{B_{2\delta}} |\varphi|^\frac{2N}{N-2} \right)^{\frac{N-2}{2N}} \left( \int_{B_{2\delta}} |dx| \right)^{\frac{1}{2}} \\
&\leq C \left( \int_{B_{2\delta}} |D u_n|^2 \right)^{\frac{1}{2}} \left( \int_{B_{2\delta}} |D \varphi|^2 \right)^{\frac{1}{2}} \to 0, \text{ as } \delta \to 0.
\end{align}

Similar estimates can be applied to prove $\int_{\Omega} \sum_{i,j=1}^{N} A_{ij} D_i u_n D_j \varphi_\delta \varphi \to 0$, $\int_{\Omega} \varphi^2 |D \varphi_\delta|^2 \to 0$, and $\int_{\Omega} \sum_{i,j=1}^{N} (a_{ij}(u) - A_{ij}) D_i u_n D_j \varphi_\delta \varphi \to 0$. Thus the result holds for $\forall \varphi \in C^\infty_0(\Omega)$. At last, applying a standard approximation argument, we can prove (7) holds for all $\varphi \in H^1_0(\Omega)$. \hfill \Box

**Lemma 2.4.** Assume \{\{u_n\}\} is a sequence of solutions of (4) with $\varepsilon = \varepsilon_n$ and $u_n \to u$ in $H^1_0(\Omega)$. Let $v_n = |u_n|$, $v = |u|$. Then $v_n \to v$ in $H^1_0(\Omega)$ and for any $T > 0$, $v_n^T \to v^T$ in $H^1_0(\Omega)$ as $n \to \infty$. Moreover,

\begin{align}
&\int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(v_n) D_i v_n D_j \varphi \\
&= \int_{\Omega} \sum_{i,j=1}^{N} A_{ij} D_i v_n D_j \varphi + \int_{\Omega} \sum_{i,j=1}^{N} (a_{ij}(v) - A_{ij}) D_i v D_j \varphi + a_n(1) \|D\varphi\|_2.
\end{align}

**Proof.** Without loss of generality, we may assume $v_n \to w$ in $H^1_0(\Omega)$, and $v_n(x) \to w(x)$, a.e. $x \in \Omega$. But $v_n(x) = |u_n(x)| \to |u(x)| = v(x)$, a.e. $x \in \Omega$, thus $w = v$. We obtain that $v_n^T = |u_n^T| \to |u^T| = v^T$ in $H^1_0(\Omega)$. And Lemma 2.4 can be proved by using the similar arguments as in the proof of Lemma 2.3. \hfill \Box

To proceed, let us recall some known facts from [31, 32]. For a function $u \in D^{1,2}(\mathbb{R}^N)$, we may define the dilation and translation of $u$ by $g_{\lambda,y}u = \lambda \frac{N-2}{4}\frac{u(\lambda \cdot - y)}{\lambda}$.
such that \( y \) for \( y \in \mathbb{R}^N \), \( \lambda \in \mathbb{R}_+ \). Consider the transformation group of dilations and translations:

\[
D = \{ g = g_{\lambda,y} | y \in \mathbb{R}^N, \lambda \in \mathbb{R}_+ \}.
\]

By Theorem 2.2 and 3.2 in [31], we have the following profile decomposition results:

\[
u_n = u + \sum_k g_n^k U_k + \sum r_n, g_n^k \in D, U_k \in D^{1,2}(\mathbb{R}^N),
\] (9)

such that

- (1) \( u_n \rightarrow u \) in \( H^1_0(\Omega) \), \((g_n^k)^{-1} u_n \rightarrow U_k \) in \( D^{1,2}(\mathbb{R}^N) \), as \( n \rightarrow \infty \);
- (2) If \( k \neq j \), \((g_n^k)^{-1} g_n^j u \rightarrow 0 \) in \( D^{1,2}(\mathbb{R}^N) \), as \( n \rightarrow \infty \), with \( g_n^0 = Id \);
- (3) \( \| u_n \|_{D^{1,2}(\mathbb{R}^N)}^2 = \| u \|_{D^{1,2}(\mathbb{R}^N)}^2 + \sum_k \| U_k \|_{D^{1,2}(\mathbb{R}^N)}^2 + \| r_n \|_{D^{1,2}(\mathbb{R}^N)}^2 + o(1) \);
- (4) \( \| r_n \|_{L^{2^*}(\mathbb{R}^N)} \rightarrow 0 \), as \( n \rightarrow \infty \).

Now we assume \( \varepsilon_n \rightarrow 0 \) and \( u_n \) is a weak solution of (4) with \( \varepsilon = \varepsilon_n \). We also suppose that \( u_n \rightarrow u \) in \( H^1_0(\Omega) \). Then by Lemma 2.2, we have that \( u \) is a solution of (1). In the following, we regard \( u_n, u \) as the elements of the space \( D^{1,2}(\mathbb{R}^N) \) by defining \( u_n, u \) to be zero in \( \mathbb{R}^N \setminus \Omega \). Then if \( \{u_n\} \) is bounded in \( H^1_0(\Omega) \), it is also bounded in \( D^{1,2}(\mathbb{R}^N) \) and has the profile decomposition (9), and if \( g_n^k = g_{\lambda_n,k,x_n,k} \), the assertion (2) in (9) is equivalent to

\[
\frac{\lambda_{n,k}}{\lambda_{n,j}} + \frac{\lambda_{n,j}}{\lambda_{n,k}} + \lambda_n k |x_n,k - x_n,j|^2 \rightarrow \infty, \text{ as } n \rightarrow \infty, \text{ for } k \neq j.
\]

Moreover, since \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^N) \), we also have (see [32])

\[
\lambda_{n,k} \rightarrow \infty, \text{ as } n \rightarrow \infty.
\]

**Lemma 2.5.** Suppose that \( u_n \) has a profile decomposition (9). Then \( V = |U_k| \) satisfies one of the following three differential inequalities:

(i)

\[
\int_{\mathbb{R}^N} \sum_{i,j=1}^N A_{ij} D_i V D_j \varphi \leq \int_{\mathbb{R}^N} V^{2^*-1} \varphi, \forall \varphi \in C_0^\infty(\mathbb{R}^N), \varphi \geq 0.
\] (11)

(ii)

\[
\int_{\mathbb{R}^N} \sum_{i,j=1}^N A_{ij} D_i V D_j \varphi \leq \int_{\mathbb{R}^N} V^{2^*-1} \varphi + \mu \int_{\mathbb{R}^N} \frac{V \varphi}{|x_0 + x|^2}, \forall \varphi \in C_0^\infty(\mathbb{R}^N), \varphi \geq 0,
\]

(12)

where \( x_0 \) is a point in \( \mathbb{R}^N \).

(iii) Up to an orthogonal transformation of the space \( \mathbb{R}^N \), there exists an \( L \geq 0 \) such that

\[
\int_{\mathbb{R}^N_L} \sum_{i,j=1}^N A_{ij} D_i V D_j \varphi \leq \int_{\mathbb{R}^N_L} V^{2^*-1} \varphi, \forall \varphi \in C_0^\infty(\mathbb{R}^N_L), \varphi \geq 0,
\] (13)

where \( \mathbb{R}^N_L = \{ x \in \mathbb{R}^N | x_N > -L \} \) and \( V = 0 \) outside of the domain \( \mathbb{R}^N_L \).

Moreover, in case (i) and (iii), there is a constant \( C \) such that \( V(x) \leq \frac{C}{(1 + |x|^2)^{\frac{m+2}{2}}} \). And in case (ii), \( V(x) \in L^p_{loc}(\mathbb{R}^N) \), \( \forall p < \frac{2^*}{\sqrt{p} - 2^*/p} \), and there exists a constant \( M \) such that \( V(x) \leq \frac{M}{1 + |x|^{\frac{2^*}{2}}}, \forall |x| \geq 1 \).
Proof. Set \( L = \lim_{n \to \infty} \text{dist}(x_{n,k}, \partial \Omega) \), then there are two cases.

**Case 1.** \( L = \infty \). Note that for any \( \varphi \in C_0^\infty(\mathbb{R}^N) \), we have \( \psi_n = g_{n,k}^k \varphi = \lambda_{n,k}^{-1} \varphi(\lambda_{n,k}(\cdot - x_{n,k})) \in C_0^\infty(\Omega) \) for \( n \) large, and \( \| \psi_n \|_{D^{1,2}(\mathbb{R}^N)} = \| \varphi \|_{D^{1,2}(\mathbb{R}^N)} \), \( \psi_n \to 0 \) in \( D^{1,2}_0(\Omega) \). By Lemma 2.1, \( v_n = |u_n| \) satisfies the inequality (5). Substituting \( \psi_n \) into (5), one gets

\[
\int_\Omega \sum_{i,j=1}^N a_{ij}(v_n) D_i v_n D_j \psi_n \leq \int_\Omega \left( 2v_n^{2^*-1} + A + \frac{\mu v_n}{|x|^2} \right) \psi_n,
\]

where \( A \) is a constant. Note that \( (g_{n,k}^k)^{-1} u_n \to U_k \) in \( D^{1,2}(\mathbb{R}^N) \) implies that \( (g_{n,k}^k)^{-1} v_n \to V \) in \( D^{1,2}(\mathbb{R}^N) \). By Lemma 2.4, for the left hand side of (14), we have

\[
\int_\Omega \sum_{i,j=1}^N a_{ij}(v_n) D_i v_n D_j \psi_n = \int_\Omega \sum_{i,j=1}^N A_{ij} D_i v_n D_j g_{n,k}^k \varphi + \int_\Omega \sum_{i,j=1}^N (a_{ij}(v) - A_{ij}) D_i v_n D_j g_{n,k}^k \varphi + o_n(1) \| \varphi \|_{D^{1,2}(\mathbb{R}^N)}
\]

\[
= \int_{\mathbb{R}^N} \sum_{i,j=1}^N A_{ij} D_i (g_{n,k}^k)^{-1} v_n D_j \varphi + o_n(1) \| \varphi \|_{D^{1,2}(\mathbb{R}^N)}
\]

\[
\to \int_{\mathbb{R}^N} \sum_{i,j=1}^N A_{ij} D_i V D_j \varphi.
\]

On the other hand, we may estimate the right hand side of (14) as follows:

\[
\int_\Omega \left( 2v_n^{2^*-1} + A + \frac{\mu v_n}{|x|^2} \right) \psi_n = \int_\Omega \left( 2v_n^{2^*-1} + A + \frac{\mu v_n}{|x|^2} \right) g_{n,k}^k \varphi = \int_{\mathbb{R}^N} \left( (g_{n,k}^k)^{-1} \left( 2v_n^{2^*-1} + A + \frac{\mu v_n}{|x|^2} \right) \right) \varphi + o_n(1).
\]

Since \( (g_{n,k}^k)^{-1} v_n \to V \) in \( D^{1,2}(\mathbb{R}^N) \), we have \( (g_{n,k}^k)^{-1} v_n \to V \) in \( L^{2^*_\text{loc}}(\mathbb{R}^N) \), and

\[
\mu \int_{\mathbb{R}^N} \frac{(g_{n,k}^k)^{-1}(v_n)}{|x|^2} \varphi = \mu \int_{\mathbb{R}^N} \frac{v_n(\lambda_{n,k}^{-1} t + x_{n,k})^\frac{N+2}{2}}{|x_{n,k} + \lambda_{n,k}^{-1} t|^2} \varphi(t)
\]

\[
= \mu \int_{\mathbb{R}^N} \frac{(g_{n,k}^k)^{-1} v_n \varphi(t)}{|\lambda_{n,k} t + x_{n,k}|^2} \to \mu \int_{\mathbb{R}^N} \frac{V \varphi}{|x_0 + x|^2},
\]

where \( x_0 \) denotes the limitation of \( \{\lambda_{n,k} x_{n,k}\} \) when it is finite, otherwise the above term goes to 0.

Combining the above arguments, we deduce that

\[
\int_{\mathbb{R}^N} \sum_{i,j=1}^N A_{ij} D_i V D_j \varphi \leq \int_{\mathbb{R}^N} 2V^{2^*-1} \varphi, \forall \varphi \in C_0^\infty(\mathbb{R}^N), \varphi \geq 0,
\]

or

\[
\int_{\mathbb{R}^N} \sum_{i,j=1}^N A_{ij} D_i V D_j \varphi \leq \int_{\mathbb{R}^N} 2V^{2^*-1} \varphi + \mu \int_{\mathbb{R}^N} V \varphi, \forall \varphi \in C_0^\infty(\mathbb{R}^N), \varphi \geq 0.
\]


Using Moser iteration, we can prove that
\[ s^{2^*-1-\varepsilon_n} + \alpha s \leq (1 + \delta)s^{2^*-1} + A_\delta, \]
we obtain the desired result (11) and (12) for \( V = |U_k| \).

**Case 2.** \( L < \infty \). Without loss of generality, we assume \( x_{n,k} \to x^* \in \partial \Omega \) as \( n \to \infty \) and the inner normal at \( x^* \) is the \( O_{x_N} \) axis. We take \( \varphi \in C_0^\infty(\mathbb{R}^N) \). Then for \( n \) large enough, we still have \( \psi = g_{n,k}^c \varphi = \lambda_{n,k}^{N-2} \varphi(\lambda_{n,k}(\cdot - x_{n,k})) \in C_0^\infty(\Omega) \). Therefore, we can use the similar arguments as in the **Case 1** to complete the proof of (13).

Moreover, if (11) holds, let \( y = xQ \), where \( Q^T AQ = Id \), then \( V(x) \to \tilde{V}(y) \) and \( \tilde{V} \) satisfies
\[
\int_{\mathbb{R}^N} D\tilde{V}D\varphi \leq \int_{\mathbb{R}^N} \tilde{V}^{2^*-1} \varphi, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N), \quad \varphi \geq 0.
\]  
(18)
Take Kelvin transform \( \tilde{v}(y) = \frac{1}{|y|^{N-2}}\tilde{V}\left(\frac{y}{|y|^{N-2}}\right) \), then \( \tilde{v} \) satisfies
\[
\int_{\mathbb{R}^N} D\tilde{v}D\varphi \leq \int_{\mathbb{R}^N} \tilde{v}^{2^*-1} \varphi, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N \setminus \{0\}), \quad \varphi \geq 0.
\]  
(19)
Using Moser iteration, we can show that \( \tilde{v}(y) \leq C \) for all \( |y| \leq 1 \). Since \( \tilde{V} \in L^\infty_{loc}(\mathbb{R}^N) \), we deduce that there exists a constant \( C > 0 \) such that \( \tilde{V}(y) \leq \frac{C}{(1+|y|^2)^{\frac{N-2}{2}}} \), thus the result holds for \( V \). And if (13) holds, we can obtain the desired results by using a similar argument.

If (12) holds, let \( y = (x + x_0)Q \), where \( Q^T AQ = Id \), then \( V(x) \to \tilde{V}(y) \) and \( \tilde{V} \) satisfies
\[
\int_{\mathbb{R}^N} DV D\varphi \leq \int_{\mathbb{R}^N} V^{2^*-1} \varphi + \mu \int_{\mathbb{R}^N} \frac{V^2}{|y|^2}, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N), \quad \varphi \geq 0.
\]  
(20)
Taking the Kelvin transform \( \tilde{v}(y) = \frac{1}{|y|^{N-2}}\tilde{V}\left(\frac{y}{|y|^{N-2}}\right) \), then \( \tilde{v} \) satisfies that
\[
\int_{\mathbb{R}^N} D\tilde{v}D\varphi \leq \int_{\mathbb{R}^N} \tilde{v}^{2^*-1} \varphi + \frac{\mu \tilde{v}^2}{|y|^2}, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N \setminus \{0\}), \quad \varphi \geq 0.
\]  
(21)
Set \( v(y) = |y|^\frac{N-2}{2} \tilde{V}(y) \), by a direct calculation, we deduce that for all \( \varphi \in C_0^\infty(\mathbb{R}^N \setminus \{0\}) \), \( \varphi \geq 0 \)
\[
\int_{\mathbb{R}^N} |y|^{-2(\sqrt{\mu} - \sqrt{M})} DV D\varphi \leq \int_{\mathbb{R}^N} |y|^{-2(\sqrt{\mu} - \sqrt{M})} v^{2^*-1} \varphi.
\]  
(22)
Using Moser iteration, we can prove that \( v(y) \leq C \) for any \( |y| \leq \frac{1}{2} \). Which implies that \( V(x) \leq \frac{1}{1+|x|^2} \), \( \forall |x| \geq 1 \).

**Corollary 1.** The sum \( \sum_k \) in the formula (9) has only a finite terms.

**Proof.** Indeed, under the assumptions that \( V = |U_k| \) satisfies one of (11), (12) and (13), we can always deduce that either
\[
\int_{\mathbb{R}^N} \sum_{i,j=1}^N A_{ij} D_i V D_j V - \frac{\mu V^2}{|x+x_0|^2} \leq \int_{\mathbb{R}^N} V^{2^*},
\]
or
\[
\int_{\mathbb{R}^N} \sum_{i,j=1}^N A_{ij} D_i V D_j V \leq \int_{\mathbb{R}^N} V^{2^*}.
\]
In the first situation, by assumption (A2) and Hardy inequality, we have
\[
\left(\alpha - \frac{\mu}{\rho}\right) \int_{\mathbb{R}^N} |DV|^2 \leq \int_{\mathbb{R}^N} V^{2^*} \leq S^{-1} \left( \int_{\mathbb{R}^N} |DV|^2 \right)^{\frac{N}{2}}.
\]

Thus, there exists a constant $C > 0$ such that $\int_{\mathbb{R}^N} |DV|^2 \geq C$. We obtain the desired result from (3) in (9). And in the second case one can prove it similarly. \[\square\]

3. Integral estimate. By Lemma 2.1, we consider the following differential inequality:
\[
\int_{\Omega} \sum_{i,j=1}^N a_{ij}(v) D_i v D_j \varphi \leq \int_{\Omega} (v^{2^* - 1} + av) \varphi + \frac{\mu v^2}{|x|^2}, \quad \forall \varphi \in H_0^1(\Omega), \quad \varphi \geq 0. \tag{23}
\]

We let $b_j(x) = a_{ij}(u)$. Then, for any $\xi \in \mathbb{R}^N$, it holds $\alpha |\xi|^2 \leq \sum_{i,j=1}^N b_{ij} \xi_i \xi_j \leq \beta |\xi|^2$.

For any $p_2 < 2^* < p_1$, $\rho > 0$ and $\lambda > 0$, consider the following relation:
\[
\begin{cases}
\|u_1\|_{p_1} \leq \rho, \\
\|u_2\|_{p_2} \leq \rho \lambda^\frac{\beta}{2} \|\frac{\alpha}{\rho}\|^{-\frac{\beta}{2}}. 
\end{cases} \tag{24}
\]

We define the norm $\|\cdot\|_{p_1, p_2, \lambda}$ by:
\[
\|u\|_{p_1, p_2, \lambda} = \inf\{\rho > 0 | \text{there are } u_1 \text{ and } u_2 \text{ such that (24) holds and } |u| \leq u_1 + u_2\}.
\]

Similar as in [8], we introduce the following norm:

**Definition 3.1.** For any $p_2 < 2^* < p_1$, $\lambda > 0$, and $\rho > 0$, we consider the following relation:
\[
\begin{cases}
\|u_1\|_{p_1} \leq \rho, \\
\|u_2\|_{p_2} \leq \rho \lambda^\frac{\beta}{2} \|\frac{\alpha}{\rho}\|^{-\frac{\beta}{2}}. 
\end{cases} \tag{25}
\]

where
\[
\|u\|_{p_1, p_2, \lambda} = \|u\|_p + \left( \int_{\Omega} \frac{\mu |u|^{2^*}}{|x|^2} \right)^\frac{2}{2^*}.
\]

We define the norm $\|u\|_{p_1, p_2, \lambda}$ by
\[
\|u\|_{p_1, p_2, \lambda} = \inf\{\rho > 0 | \text{there are } u_1, u_2 \text{ satisfying (25) and } |u| \leq u_1 + u_2\}.
\]

In the following, we will establish some integral estimates for the approximation solution $u_n$ under the norm $\|\cdot\|_{p_1, p_2, \lambda}$. As we mentioned before, due to the appearance of the Hardy potential, the solutions are not in $L^\infty(\Omega)$, fortunately, by using Morse iteration, we could have some $L^p$ (for some $p$) estimates, which turned out to be enough for our proof.

**Proposition 1.** Suppose $u_n$ is a solution of (4) with $\varepsilon = \varepsilon_n \to 0$, and has a decomposition in form of (9). Denote $\lambda_n = \inf\{\lambda_n, k\}$, for any $p_1, p_2 \in \left( \frac{2^* \sqrt{\alpha \mu}}{\sqrt{\alpha \mu} + \sqrt{\alpha \mu} - \rho}, \frac{2^* \sqrt{\alpha \mu}}{\sqrt{\alpha \mu} + \sqrt{\alpha \mu} - \rho} \right)$, $p_2 < 2^* < p_1$, there is a constant $C$, depending on $p_1, p_2$, such that $\|u_n\|_{p_1, p_2, \lambda_n} \leq C$.

In order to prove Proposition 1, we need a few lemmas. Indeed with the aid of Lemma 2.1, we only need to prove these lemmas in a simple form.
Lemma 3.2. Let \( u \in H^1_0(\Omega) \), \( u \geq 0 \) satisfies that
\[
\int_{\Omega} \sum_{i,j=1}^{N} b_{ij} D_i D_j \varphi - \frac{\mu u \varphi}{|x|^2} \leq \int_{\Omega} u^2 - 1 \varphi + A \varphi, \quad \forall \varphi \in H^1_0(\Omega), \ \varphi \geq 0,
\]
for any \( p_1, p_2 \in \left( \frac{N+2}{N-2} \frac{2N\sqrt{\mu}}{(N+2)\sqrt{\mu}+(N-2)\sqrt{\alpha_\mu - \mu}}, \frac{N+2}{N-2} \frac{2N\sqrt{\alpha_\mu}}{(N+2)\sqrt{\alpha_\mu}-(N-2)\sqrt{\mu}} \right) \), with \( p_2 < 2^* < p_1 \), let \( q_i \) given by
\[
1 = \frac{N + 2}{(N - 2)p_i} - \frac{2}{N}.
\]
Then there is a constant \( C = C(p_1, p_2) \) such that for any \( \lambda > 0 \),
\[
\|u\|_{*,q_1,q_2,\lambda} \leq C\|u\|_{p_1,p_2,\lambda}^{\frac{N+2}{N-2}} + C.
\]

Proof. For any small \( \theta > 0 \), let \( v_1 \geq 0, v_2 \geq 0 \) be the functions such that \( u \leq v_1 + v_2 \), and (24) holds with \( \rho = \|u\|_{p_1,p_2,\lambda} + \theta \). Consider
\[
\int_{\Omega} \sum_{i,j=1}^{N} b_{ij} D_i u_1 D_j \varphi - \frac{\mu u_1 \varphi}{|x|^2} = C \int_{\Omega} v_1^{\frac{N+2}{N-2}} \varphi + A \varphi,
\]
and
\[
\int_{\Omega} \sum_{i,j=1}^{N} b_{ij} D_i u_2 D_j \varphi - \frac{\mu u_2 \varphi}{|x|^2} = C \int_{\Omega} v_2^{\frac{N+2}{N-2}} \varphi,
\]
then \( u \leq u_1 + u_2 \). Let \( \bar{p}_i = p_i \frac{N-2}{N_2} \), then \( q_i = \frac{N\bar{p}_i}{N-2\bar{p}_i} \). Besides, for \( p_i \in \left( \frac{N+2}{N-2} \frac{2N\sqrt{\mu}}{(N+2)\sqrt{\mu}+(N-2)\sqrt{\alpha_\mu - \mu}}, \frac{N+2}{N-2} \frac{2N\sqrt{\alpha_\mu}}{(N+2)\sqrt{\alpha_\mu}-(N-2)\sqrt{\mu}} \right) \), \( \bar{p}_i \in \left( \frac{N+2}{N-2} \frac{2N\sqrt{\mu}}{(N+2)\sqrt{\mu}+(N-2)\sqrt{\alpha_\mu - \mu}}, \frac{N+2}{N-2} \frac{2N\sqrt{\alpha_\mu}}{(N+2)\sqrt{\alpha_\mu}-(N-2)\sqrt{\mu}} \right) \). By Lemma B.1, we have
\[
\|u_1\|_{*,q_1} \leq C\|v_1\|_{p_1}^{\frac{N+2}{N-2}} + A\|\bar{p}_1 \leq C\|v_1\|_{p_1}^{\frac{N+2}{N-2}} + 1 \leq C\left(\|u\|_{p_1,p_2,\lambda} + \theta \right)^{\frac{N+2}{N-2}} + 1,
\]
and
\[
\|u_2\|_{*,q_2} \leq C\|v_2\|_{p_2}^{\frac{N+2}{N-2}} \leq C\left(\|u\|_{p_1,p_2,\lambda} + \theta \right)^{\frac{N+2}{N-2}} \frac{N+2}{N-2} \frac{N}{N-2} \frac{N+2}{N-2} \frac{N}{N-2} \frac{N}{N-2} \frac{N}{N-2},
\]
then the result follows. \( \square \)

Lemma 3.3. Let \( v_n(x) = |u_n(x)| \) in \( \Omega \), then there are constants \( C > 0 \), and \( p_1, p_2 \in \left( \frac{2^*\sqrt{\mu}}{\sqrt{\alpha_\mu}+\sqrt{\alpha_\mu - \mu}}, \infty \right) \) with \( p_2 < 2^* < p_1 \), such that
\[
\|v_n\|_{*,p_1,p_2,\lambda} \leq C.
\]

Proof. We decompose \( u_n \) as \( u_n = u_0 + u_{n,1} + u_{n,2} \), where
\[
u_{n,1} = \sum_{j=1}^{m} g_{x_{n,j},\lambda_{n,j}}(U_j) + \sum_{j=m+1}^{k} g_{x_{n,j},\lambda_{n,j}}(U_j),
\]
among them \( U_j \) satisfies (12), \( j = 1, 2, \cdots, m \), else \( U_j \) satisfies (11) or (13), and \( u_{n,2} = r_n \). Let \( a_i = C|u_{n,i}|^{\frac{N+2}{N-2}} \), \( i = 0, 1, 2 \), where \( C > 0 \) is a large enough constant.
Then, we have
\[
\int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(v_n) D_i v_n D_j \phi - \frac{\mu v_n \phi}{|x|^2} \leq \int_{\Omega} \left( (a_0 + a_1 + a_2) v_n + A \right) \phi,
\]
thus \(v_n \leq G(a_0 v_n + A) + G(a_1 v_n) + G(a_2 v_n)\), where \(u = G(v)\) denotes the solution of
\[
\int_{\Omega} \sum_{i,j=1}^{N} a_{ij}(v_n) D_i u D_j \phi = \int_{\Omega} v \phi.
\]
Let \(p > \frac{2N}{N+2}\) be a constant such that \(p - \frac{2N}{N+2}\) is so small that \(p_1 := \frac{Np}{N-2p} \in (2^*, 2^* - 2\sqrt{p})\). Then by Lemma B.1 and Hölder inequality
\[
\|G(a_0 v_n + A)\|_{s,p_1} \leq C + C \left( \int_{\Omega} |a_0|^{2^*_p} \right)^{1/2}.
\]
By Remark 1, \(u_0 \in L^{\frac{2N}{N-2\sqrt{p}-2}}\), and for \(p - \frac{2N}{N+2}\) small enough, we have
\[
|U_0|^{2^*_p} \leq u_0^{(2^*+\theta)},
\]
where \(\theta > 0\) is small if \(p - \frac{2N}{N+2} > 0\) is small. As a result,
\[
\int_{\Omega} |a_0|^{2^*_p} \leq C,
\]
thus \(\|G(a_0 v_n + A)\|_{s,p_1} \leq C\).

Next, we estimate the term \(G(a_1 v_n)\). Let \(p_2 \in \left( \frac{2^* \sqrt{\mu}}{\sqrt{\mu} + \sqrt{\mu - \mu}}, 2^* \right)\) be a constant, by Lemma B.3,
\[
\|G(a_1 v_n)\|_{s,p_2} \leq C \|a_1\| \|v_n\|_{2^*} \leq C \|a_1\|_{r},
\]
where \(r\) is determined by \(\frac{1}{p_2} = \frac{1}{r} + \frac{1}{2^*} - \frac{2}{N}\). But
\[
\int_{\Omega} |g_{x_n,\lambda}(U_j)|^{\frac{2^*_p}{r-2}} = \lambda_{n,j}^{N+2r} \int_{\Omega_{x_n,\lambda}} |U_j|^{\frac{2^*_p}{r-2}},
\]
where \(\Omega_{x,\lambda} = \{ y \mid x + \lambda^{-1}y \in \Omega \}\).

For \(j = m + 1, \cdots, k\), we have
\[
|U_j| \leq \frac{C}{1 + |x|^N},
\]
Hence, for any \(r \in (\frac{N}{2}, \frac{N}{2})\),
\[
\int_{\Omega_{x_n,\lambda}} |U_j|^{\frac{2^*_p}{r-2}} \leq C, \ j = m + 1, \cdots, k.
\]
For \(j = 1, \cdots, m\), we have
\[
U_j \in L_{\text{loc}}^p(\mathbb{R}^N), \forall p < \frac{2^* \sqrt{\mu}}{\sqrt{\mu} - \sqrt{\mu - \mu}}
\]
and by Lemma 2.5,
\[
|U_j| \leq \frac{C}{1 + |x|^r}, \ |x| \geq 1.
\]
Note that \( r \to \frac{N}{2} \) as \( p_2 \to 2^* \). We choose \( p_2 \) close to \( 2^* \) so that
\[
\frac{4r}{N-2} \left( \sqrt{\mu} + \sqrt{\mu - \mu} \right) > N,
\]
and
\[
\frac{4r}{N-2} < \frac{2^* \sqrt{\mu}}{\sqrt{\mu} - \sqrt{\mu - \mu}}
\]
Then we have
\[
\int_{\Omega_{x,n,j,\lambda_n,j}} |U_j|^{\frac{4r}{N-2}} \leq C, \ j = 1, \cdots, m.
\]
Thus, we have proved that there is a \( p_2 < 2^* \) close to \( 2^* \), such that
\[
\|G(a_1 v_n)\|_{s,p_2} \leq C \lambda_n^{\frac{2^*}{N}} \lambda_n^{\frac{2^*}{p_2}}.
\]
Finally, we deal with the term \( G(a_2 v_n) \). It follows from Lemma B.2 that
\[
\|G(a_2 v_n)\|_{s,p_1,p_2,\lambda_n} \leq C \|a_2\|_{\frac{N}{2}} \|v_n\|_{s,p_1,p_2,\lambda_n} \leq \frac{1}{2} \|v_n\|_{s,p_1,p_2,\lambda_n},
\]
since \( \|a_2\|_{\frac{N}{2}} = \|u_{n,2}\|_{\frac{N}{2}} \to 0 \) as \( n \to \infty \).

Combining the above estimates, we get
\[
\|v_n\|_{s,p_1,p_2,\lambda_n} \leq \|G(a_0 v_n + A)\|_{s,p_1,p_2,\lambda_n} + \|G(a_1 v_n)\|_{s,p_1,p_2,\lambda_n} + \|G(a_2 v_n)\|_{s,p_1,p_2,\lambda_n}
\]
\[
\leq \|G(a_0 v_n + A)\|_{s,p_1} + \|G(a_1 v_n)\|_{s,p_2,\lambda_n} \lambda_n^{\frac{N}{p_2}} \lambda_n^{\frac{N}{2}} + \|G(a_2 v_n)\|_{s,p_1,p_2,\lambda_n}
\]
\[
\leq C + \frac{1}{2} \|v_n\|_{s,p_1,p_2,\lambda_n}.
\]
And the result follows.

**Proof of Proposition 1.** By Lemma 3.3, the result is true for some \( p_1, p_2 \) close to \( 2^* \) with \( p_2 < 2^* < p_1 \).

Using the bootstrap Lemma 3.2, we can prove Proposition 1 for some pairs of \( \{p_1, p_2\} \). Generally, for any \( (p_1, p_2) \in \left( \frac{2^* \sqrt{\mu} \sqrt{\mu - \mu}}{\sqrt{\mu} + \sqrt{\mu - \mu}}, \frac{2^* \sqrt{\mu} \sqrt{\mu - \mu}}{\sqrt{\mu} + \sqrt{\mu - \mu}} \right) \), \( \|u_1\|_{s,p_1} \leq C \), assume \( p_2 = \kappa p_2^* + (1 - \kappa) p_2 \), then
\[
\left( \int_{\Omega} (u_2)^{p_2} \right)^\frac{1}{p_2} \leq \left( \int_{\Omega} \left( \int (u_2)^{p_2} \right)^\frac{1}{p_2} \right)^\frac{1}{p_2} + \left( \int_{\Omega} \frac{\mu |u_2|}{|x|^2} \frac{2p_2}{p_2} \right)^\frac{1}{p_2} \left( \int_{\Omega} \frac{|u_2|^2}{|x|^2} \right) \frac{2^* \sqrt{\mu} \sqrt{\mu - \mu}}{\sqrt{\mu} + \sqrt{\mu - \mu}}
\]
\[
\leq C \left( \lambda_n^{\frac{N}{p_2}} \right)^\frac{1}{p_2} \left( \lambda_n^{\frac{N}{p_2}} \right)^\frac{(1-\kappa)}{p_2} \leq C \lambda_n^{\frac{N}{p_2}}.
\]
We complete the proof.

4. **The estimates in the safe domains.** Suppose \( \{u_n\} \) have decomposition forms of (9), and assume \( \lambda_n = \lambda_n, x_n = x_n, \) By Corollary 1, the number of the bubbles of \( u_n \) is finite, we can find a constant \( \tilde{c} \) independent of \( n \), such that the region
\[
A_n^1 = \left( B_{(\tilde{c} + 2)\lambda_n} \right) \left( B_{\tilde{c} \lambda_n} \right) \cap \Omega
\]
does not contain any concentration points of \( u_n \), for any \( n \). We call this region a safe region for \( u_n \). Let
\[
A^2_n = \left( B_{(c+4)\lambda_n^{-\frac{1}{2}}}(x_n) \right) \setminus B_{(c+1)\lambda_n^{-\frac{1}{2}}}(x_n) \right) \cap \Omega.
\]

**Proposition 2.** Suppose \( \{u_n\} \) is a sequence of solutions for the subcritical problem (4) with \( \varepsilon = \varepsilon_n \), and \( u_n \to u \) in \( H^1_0(\Omega) \) as \( \varepsilon_n \to 0 \). Then there exist constants \( r_0 \) and \( C_{r_0} \) with \( 0 < r_0 < 1 \) and \( r_0 = C(r_0) > 0 \) such that
\[
\left( \int_{D-r_n^{-\frac{1}{2}}(x)} |u_n|^p \right) \leq C_{r_0} \lambda_n^{\frac{N}{2} + \frac{pN}{2}},
\]
where \( x \in A^2_n \), \( 0 < r \leq r_0 \), and \( p_1, p \) are constants satisfying \( \frac{2^* \sqrt{\mu_1}}{\sqrt{\mu + \sqrt{\alpha \mu}} - \mu} > p_1 > 2^* \).

**Proof.** Since \( D_{\lambda_n^{-\frac{1}{2}}}(x) \in A^2_n \), does not contain any concentration points of \( u_n \), we can deduce that
\[
\int_{D-r_n^{-\frac{1}{2}}(x)} |u_n|^{\frac{2N}{N-2}} = o(1), \text{ as } n \to \infty.
\]

By Lemma 2.1, we see that \( v_n = |u_n| \) satisfies the following differential inequality
\[
\int_{\Omega} \sum_{i,j=1}^N a_{ij}(v_n) D_i v_n D_j \varphi \varphi - \frac{\mu v_n \varphi}{|x|^2} \leq \int_{\Omega} \left( v_n^{\frac{1}{2}} + A \right) v_n \varphi, \forall \varphi \in H^1_0(\Omega), \varphi \geq 0. \tag{26}
\]

For \( \forall x \in A^2_n \), \( 0 < r < R \leq 1 \), let \( y = \lambda_n^{-\frac{1}{2}} x \), and \( \tilde{v}_n(z) = v_n(\lambda_n^{-\frac{1}{2}} z), z \in \Omega_n \), where \( \Omega_n = \{ z | \lambda_n^{-\frac{1}{2}} z \in \Omega \} \). We **claim** that if \( p \) is a constant satisfying \( \frac{2 \sqrt{\alpha \mu}}{\sqrt{\alpha \mu} + \sqrt{\alpha \mu - \mu}} < p < \frac{2 \sqrt{\alpha \mu}}{\sqrt{\alpha \mu - \sqrt{\alpha \mu - \mu}}} \), then
\[
\|\tilde{v}_n\|_{L^p(D_n(y))} \leq C \|\tilde{v}_n\|_{L^1(D_n(y))}, \forall 1 \geq R > r > 0.
\]

Indeed, take \( \eta \in C_0^\infty(\mathbb{R}^N) \), such that \( \eta(y) = 1 \) if \( |y - x| \leq r \lambda_n^{-\frac{1}{2}} \), \( \eta(y) = 0 \) if \( |y - x| \geq R \lambda_n^{-\frac{1}{2}} \), \( |D\eta| \leq \frac{2}{R - r} \lambda_n^{-\frac{1}{2}} \).

If \( \frac{2 \sqrt{\alpha \mu}}{\sqrt{\alpha \mu - \sqrt{\alpha \mu - \mu}}} \geq q \geq 2 \), let \( a_n = v_n^{\frac{4}{N-2}} + A \), we take \( \varphi = \eta^q \eta^{q-1} \in H^1_0(\Omega) \) as a test function, then
\[
\left( \int_{\Omega} v_n^q \eta^2 \right)^{\frac{2N}{N-2}} \leq C \int_{\Omega} |D(v_n^q \eta)|^2
\]
\[
\leq \frac{C}{\left( \frac{4 \alpha (q-1)}{q^2} - \frac{\mu}{\mu} \right)} \int_{\Omega} \sum_{i,j=1}^N a_{ij}(v_n) D_i v_n D_j (v_n^q - 1) \eta^2 - \frac{\mu v_n^q \eta^2}{|x|^2} + C \int_{\Omega} v_n^q |D\eta|^2
\]
\[
\leq \frac{C}{\left( \frac{4 \alpha (q-1)}{q^2} - \frac{\mu}{\mu} \right)} \int_{D_{\lambda_n^{-\frac{1}{2}}}(x)} a_n v_n^q \eta^2 + C \int_{\Omega} v_n^q |D\eta|^2
\]
\[
\leq \frac{C}{\left( \frac{4 \alpha (q-1)}{q^2} - \frac{\mu}{\mu} \right)} \|a_n\|_{L^\infty(D_{\lambda_n^{-\frac{1}{2}}}(x))} \left( \int_{\Omega} (v_n^q \eta^2)^{\frac{N}{N-2}} \right)^{\frac{N-2}{N}} + C \lambda_n \int_{D_{\lambda_n^{-\frac{1}{2}}}(x)} v_n^q \eta^2.
\]
By the estimates above, we may assume \( \|a_n\|_{L^\frac{N}{N-2}(D_{r\lambda_n^{-1}}(x))} \leq \frac{1}{2} \). Then we deduce that
\[
\int_D \frac{(v_n^q)^{\frac{N}{N-2}}}{r^{\lambda_n^{-1} \frac{1}{2}}(x)} \leq \int_\Omega \frac{(v_n^q)^{\frac{N}{N-2}}}{r^{\lambda_n^{-1} \frac{1}{2}}(x)} \leq \left( C\lambda_n \int_D \frac{v_n^q}{r^{\lambda_n^{-1} \frac{1}{2}}(x)} \right)^{\frac{N}{N-2}},
\]
thus
\[
\int_{D_r(y)} \frac{\lambda_n^N v_n^{\frac{N}{N-2}}}{R^{\lambda_n^{-1} \frac{1}{2}}(x)} \leq C\lambda_n \left( \lambda_n \int_{D_{r\lambda_n^{-1}}(y)} \frac{v_n^{\frac{N}{N-2}}}{R^{\lambda_n^{-1} \frac{1}{2}}(x)} \right)^{\frac{N}{N-2}} = C\lambda_n \left( \int_{D_{R}(y)} \frac{v_n^{\frac{N}{N-2}}}{R^{\lambda_n^{-1} \frac{1}{2}}(x)} \right)^{\frac{N}{N-2}},
\]
that is
\[
\|\tilde{v}_n\|_{L^\frac{N}{N-2}(D_r(y))} \leq C\|\tilde{v}_n\|_{L^2(D_r(y))}.
\]
By iteration, we have for \( 2 \leq p < \frac{2\sqrt{N\eta}}{\sqrt{\alpha_\mu+\sqrt{\alpha_\mu-\mu}}} \)
\[
\|\tilde{v}_n\|_{L^p(D_r(y))} \leq C\|\tilde{v}_n\|_{L^2(D_r(y))}.
\]
Notice that
\[
\|\tilde{v}_n\|_{L^2(D_r(y))} \leq \|\tilde{v}_n\|_{L^1(D_r(y))}^{\frac{1-\kappa}{\kappa}} \|\tilde{v}_n\|_{L^p(D_r(y))}^{\frac{\kappa}{1-\kappa}} \leq \frac{1}{2} \|\tilde{v}_n\|_{L^p(D_r(y))} + C\|\tilde{v}_n\|_{L^1(D_r(y))},
\]
where \( \kappa = \frac{p-2}{p-1} \). So,
\[
\|\tilde{v}_n\|_{L^p(D_r(y))} \leq \frac{1}{2} \|\tilde{v}_n\|_{L^p(D_r(y))} + C\|\tilde{v}_n\|_{L^1(D_r(y))}, \quad \forall 1 \geq R > r > 0.
\]
This yields
\[
\|\tilde{v}_n\|_{L^p(D_r(y))} \leq C\|\tilde{v}_n\|_{L^1(D_r(y))}, \quad \forall 1 \geq R > r > 0.
\]
For the case \( \frac{2\sqrt{N\eta}}{\sqrt{\alpha_\mu+\sqrt{\alpha_\mu-\mu}}} < q < 2 \), take \( \varphi = (v_n + \theta)^{q-1} - \theta^{q-1} \) in (26), then by using the similar arguments, we can prove the desired results.

All in all, for \( \frac{2\sqrt{N\eta}}{\sqrt{\alpha_\mu+\sqrt{\alpha_\mu-\mu}}} < p < \frac{2\sqrt{N\eta}}{\sqrt{\alpha_\mu+\sqrt{\alpha_\mu-\mu}}} \), we have
\[
\|\tilde{v}_n\|_{L^p(D_r(y))} \leq C\|\tilde{v}_n\|_{L^1(D_r(y))}, \quad \forall 1 \geq R > r > 0.
\]
Thus
\[
\left( \int_D \frac{v_n^p \lambda_n^N}{r^{\lambda_n^{-1} \frac{1}{2}}(x)} \right)^{\frac{1}{p}} \leq C\lambda_n \left( \int_D \frac{v_n^q}{r^{\lambda_n^{-1} \frac{1}{2}}(x)} \right)^{\frac{N}{N-2}}, \quad (27)
\]
On the other hand, by Lemma C.1, for any \( x \in A_n^2, \), \( r = \lambda_n^{-\frac{1}{2}} \), \( q \in (1, \frac{N}{N-1}) \), and \( p_1 \in (2', \frac{2\sqrt{N\eta}}{\sqrt{\alpha_\mu+\sqrt{\alpha_\mu-\mu}}}) \), we have
\[
\int_{\lambda_n^{-\frac{1}{2}}(x)} v_n^q \leq C\lambda_n^{\frac{N}{p_1}-\frac{N}{q}}.
\]
By H"older inequality, we have
\[
\int_{D_{\lambda_n}^{\frac{1}{2}}(x)} v_n \leq \left( \int_{D_{\lambda_n}^{\frac{1}{2}}(x)} v_n^\theta \right)^{\frac{1}{\theta}} \left( \lambda_n^{-\frac{N}{2}} \right)^{\frac{2-\theta}{4}} \leq C\lambda_n^{\frac{N}{2p^*} - \frac{N}{2}}.
\]  
(28)

Combining (27) and (28), we have for \( \frac{2\sqrt{\alpha p}}{\sqrt{\alpha p} - \sqrt{\alpha p} - \mu} < p < \frac{2\sqrt{\alpha p}}{\sqrt{\alpha p} - \sqrt{\alpha p} - \mu} \), and \( 2^* < p_1 < \frac{2\sqrt{\alpha p}}{\sqrt{\alpha p} - \sqrt{\alpha p} - \mu} \),
\[
\int_{D_{r\lambda_n}^{\frac{1}{2}}(x)} v_n^p \leq C\lambda_n^{\frac{N}{2p^*} - \frac{N}{2}}.
\]
We complete the proof. \( \square \)

**Corollary 2.** For \( \frac{2\sqrt{\alpha p}}{\sqrt{\alpha p} - \sqrt{\alpha p} - \mu} > p_1 > 2^* \), we have
\[
\int_{A_n^2} |Du_n|^2 \leq C\lambda_n^{\frac{N}{2p^*} + \frac{N}{p^*}}.
\]

**Proof.** Applying Proposition 2, there exists an \( r_0 \) such that for \( r \leq r_0 \)
\[
\left( \int_{D_{r\lambda_n}^{\frac{1}{2}}(x)} |u_n|^2 \right)^{\frac{1}{2}} \leq C\lambda_n^{\frac{N}{2p^*} + \frac{2N}{2p^*}}.
\]
let \( \eta \in C^\infty_0(\mathbb{R}^N) \) such that \( \eta(y) = 1 \) if \( |y - x| \leq \frac{1}{2}r_0\lambda_0^{-\frac{1}{2}} \), \( \eta(y) = 0 \) if \( |y - x| \geq r_0\lambda_0^{-\frac{1}{2}} \), \( |D\eta| \leq \frac{4}{r_0} \lambda_0^{-\frac{1}{2}} \). We choose \( \varphi = \eta^2 v_n \) as a test function in (5), then
\[
\int_{\Omega} |Du_n|^2 \eta^2 \leq C \int_{\Omega} u_n^2 |D\eta|^2 + C \int_{\Omega} (|u_n|^{2^* - \epsilon_n} + au_n)u_n\eta^2.
\]
Since \( p_1 > 2^* \), we see \( -\frac{N}{2} + \frac{2N}{2p^*} < 1 - \frac{N}{2} + \frac{N}{p^*} \), thus
\[
\int_{D_{\lambda_0^{-\frac{1}{2}}}(x)} |Du_n|^2 \leq C \int_{D_{r\lambda_0^{-\frac{1}{2}}}(x)} (au_n + |u_n|^{2^* - \epsilon_n}) + C \int_{D_{r\lambda_0^{-\frac{1}{2}}}(x)} u_n^2 |D\eta|^2
\]
\[
\leq C\lambda_n^{\frac{N}{2p^*} + \frac{2N}{2p^*}} + C\lambda_n^{1 - \frac{N}{2} + \frac{N}{p^*}} \leq C\lambda_n^{\frac{2N}{2p^*} + \frac{N}{p^*}},
\]
which implies that
\[
\int_{A_n^2} |Du_n|^2 \leq C\lambda_n^{\frac{2N}{2p^*} + \frac{N}{p^*}},
\]
and the result follows from Hardy inequality. \( \square \)

5. Local Pohozaev identity, the proof of Theorems. Let \( D_n = D_{(\epsilon + 4)\lambda_n^{-\frac{1}{2}}} \), \( \partial_n D_n = \partial D_n \cap \partial \Omega \). We define a cut-off function \( \eta \) such that \( \eta(x) = 1 \) if \( |x - x_n| \leq (\epsilon + 1)\lambda_n^{-\frac{1}{2}} \), \( \eta(x) = 0 \) if \( |x - x_n| \geq (\epsilon + 4)\lambda_n^{-\frac{1}{2}} \), \( |\nabla\eta| \leq 4\lambda_n^{-\frac{1}{2}} \). Denote \( B_n = B_{(\epsilon + 4)\lambda_n^{-\frac{1}{2}}}(x_n), n = (n_1, \ldots, n_N) \) the unit outward normal vector on \( \partial \Omega \).

We have three cases to consider: (i) \( B_n \cap \Omega^c \neq \emptyset \); (ii) \( B_n \subset \Omega \; \text{and} \; 0 \notin B_n \); (iii) \( B_n \subset \Omega \; \text{and} \; 0 \in B_n \). In each case, the point \( x^* \) is chosen as follows: In case (i), we take \( x^* \in \Omega^c \) with \( |x^* - x_n| \leq (\epsilon + 7)\lambda_n^{-\frac{1}{2}} \) and \( n(x - x^*) \leq 0 \) on \( \partial_n D_n \). With this \( x^* \), we can check that \( x \cdot x^* \geq 0 \) in \( B_n \). In case (ii), we take a point \( x^* = x_n \), then \( x \cdot x^* \geq 0 \) in \( B_n \). In case (iii), we take a point \( x^* = 0 \). Thus, in each case, we have \( x \cdot x^* \geq 0 \) in \( B_n \).
Lemma 5.1. Let $p_n = 2^* - \varepsilon_n$, then $u_n$ satisfies the following local Pohozaev identity:
\[
\left( \frac{N}{p_n} - \frac{N-2}{2} \right) \int_{D_n} |u_p|^{p_n} \eta + a \int_{D_n} |u_n|^2 \eta + \frac{N-2}{4} \int_{D_n} \sum a_{ij}(u_n) D_i u_n D_j u_n \eta \\
+ \mu \int_{D_n} \frac{x \cdot x |u_n|^2 \eta}{|x|^4} + \frac{\mu}{2} \int_{D_n} \frac{(x-x^*) \cdot D \eta u_n^2}{|x|^2} \\
= \frac{1}{2} \int_{D_n} \sum_{i,j,k=1}^N a_{ij}(u_n) D_i u_n D_j u_n (x_k - x^*_k) D_k \eta \\
- \frac{N-2}{2} \int_{D_n} \sum_{i,j,k=1}^N a_{ij}(u_n) D_i u_n D_j u_n \eta - \int_{D_n} \left( \frac{1}{p_n} |u_n|^{p_n} + \frac{a}{2} u_n^2 \right) \sum_{k=1}^N (x_k - x^*_k) D_k \eta \\
+ \frac{1}{2} \int_{\partial D_n} \sum_{i,j,k=1}^N a_{ij}(u_n) D_i u_n D_j u_n (x_k - x^*_k) n_k \eta d\sigma. \tag{29}
\]

Proof. Let $f(s) = |s|^{p_n-2} s + a s$, $F(s) = \frac{1}{p_n} |s|^{p_n} + \frac{a}{2} s^2$. Multiplying the equation (4) by $\sum_{k=1}^N (x_k - x^*_k) D_k u_n \eta$ and integrating by parts over $D_n$, we have
\[
- \frac{N-2}{2} \int_{D_n} \sum_{i,j,k=1}^N a_{ij}(u_n) D_i u_n D_j u_n \eta \quad \frac{1}{2} \int_{D_n} \sum_{i,j,k=1}^N a_{ij}(u_n) D_i u_n D_j u_n (x_k - x^*_k) D_k \eta \\
+ \frac{1}{2} \int_{D_n} \sum_{i,j,k=1}^N a_{ij}(u_n) D_i u_n D_j u_n (x_k - x^*_k) D_k \eta d\sigma \\
+ \frac{\mu}{2} \int_{D_n} \frac{(N-2) u_n^2 \eta}{|x|^2} + 2 \sum_{k} x_k x^*_k u_n^2 \eta |x|^2 + \sum_{k} (x_k - x^*_k) D_k \eta u_n^2 |x|^2 \\
= - \int_{D_n} F(u_n) \eta - \int_{D_n} F(u_n) \sum_{k=1}^N (x_k - x^*_k) D_k \eta + \int_{\partial D_n} F(u_n) \sum_{k=1}^N (x_k - x^*_k) n_k \eta. \tag{30}
\]

Taking $\varphi = u_n \eta$ as the test function in (4), we have
\[
\int_{D_n} \sum_{i,j=1}^N a_{ij}(u_n) D_i u_n D_j u_n \eta + \int_{D_n} \sum_{i,j=1}^N a_{ij}(u_n) D_i u_n D_j \eta u_n \\
- \mu \int_{D_n} \frac{u_n^2 \eta}{|x|^2} + \frac{1}{2} \int_{D_n} \sum_{i,j=1}^N a_{ij}(u_n) D_i u_n D_j u_n \eta \\
= \int_{D_n} f(u_n) u_n \eta. \tag{31}
\]

Notice that $\eta = 0$ on $\partial D_n \cap \partial B_n$, $u_n = 0$ and $D_k u_n = \frac{\partial u_n}{\partial n} n_k$ on $\partial \Omega$, $D_k u_n = \frac{\partial u_n}{\partial n} n_k$ on $\partial D_n$, $D_k u_n = \partial \Omega_k$, we have
\[
\int_{D_n} \sum_{i,j,k=1}^N a_{ij}(u_n) D_i u_n n_j (x_k - x^*_k) D_k u_n \eta d\sigma + \frac{\mu}{2} \int_{D_n} \frac{u_n^2 \sum_{k} (x_k - x^*_k) n_k \eta}{|x|^2} d\sigma
\]
Thus $$\int_{\partial D_n} F(u_n) \sum_{k=1}^{N} (x_k - x_k^*) n_k \eta d\sigma = 0.$$ We have

$$\frac{1}{2} \int_{D_n} \sum_{i,j,k=1}^{N} a_{ij}(u_n) D_i u_n D_j u_n (x_k - x_k^*) D_k \eta - \int_{D_n} \sum_{i,j,k=1}^{N} a_{ij}(u_n) D_i u_n D_j \eta (x_k - x_k^*) D_k u_n$$

$$- \frac{N - 2}{2} \int_{D_n} \sum_{i,j=1}^{N} a_{ij}(u_n) D_i u_n D_j u_n \eta - \frac{N - 2}{4} \int_{D_n} \sum_{i,j=1}^{N} a'_{ij}(u_n) D_i u_n D_j u_n u_n \eta$$

$$+ \frac{1}{2} \int_{D_n} \sum_{i,j,k=1}^{N} a_{ij}(u_n) D_i u_n D_j u_n (x_k - x_k^*) n_k \eta d\sigma - \mu \int_{D_n} \sum_{k=1}^{N} \frac{x_k x_k^* \eta^2 |n_k|^2}{|x|^4}$$

$$= \left( \frac{N}{p_n} - \frac{N - 2}{2} \right) \int_{D_n} |u_n|^p \eta + a \int_{D_n} |u_n|^2 \eta + \int_{D_n} \left( \frac{1}{p_n} |u_n|^p + \frac{1}{2} |u_n|^2 \right) \sum_{k=1}^{N} (x_k - x_k^*) D_k \eta.$$ Thus

$$\left( \frac{N}{p_n} - \frac{N - 2}{2} \right) \int_{D_n} |u_n|^p \eta + a \int_{D_n} |u_n|^2 \eta + \frac{N - 2}{4} \int_{D_n} \sum_{i,j,k=1}^{N} a'_{ij}(u_n) D_i u_n D_j u_n u_n \eta$$

$$+ \mu \int_{D_n} \frac{x \cdot x^* u_n^2 \eta}{|x|^4} + \mu \int_{D_n} \frac{(x - x^*) \cdot D \eta u_n^2}{|x|^2}$$

$$= \frac{1}{2} \int_{D_n} \sum_{i,j,k=1}^{N} a_{ij}(u_n) D_i u_n D_j u_n (x_k - x_k^*) D_k \eta - \int_{D_n} \sum_{i,j,k=1}^{N} a_{ij}(u_n) D_i u_n D_j \eta (x_k - x_k^*) D_k u_n$$

$$- \frac{N - 2}{2} \int_{D_n} \sum_{i,j=1}^{N} a_{ij}(u_n) D_i u_n D_j u_n \eta - \int_{D_n} \left( \frac{1}{p_n} |u_n|^p + \frac{1}{2} |u_n|^2 \right) \sum_{k=1}^{N} (x_k - x_k^*) D_k \eta$$

$$+ \frac{1}{2} \int_{D_n} \sum_{i,j,k=1}^{N} a_{ij}(u_n) D_i u_n D_j u_n (x_k - x_k^*) \eta d\sigma.$$ In case (i) and (ii), $$u_n \in C^2(D_n),$$ thus (29) is the usual local Pohozaev identity. Now we prove that (29) holds in case (iii). Let $$D_n, \theta = D_n \setminus B_\theta(0),$$ then we have

$$\left( \frac{N}{p_n} - \frac{N - 2}{2} \right) \int_{D_n, \theta} |u_n|^p \eta + a \int_{D_n, \theta} |u_n|^2 \eta + \frac{N - 2}{4} \int_{D_n, \theta} \sum_{i,j,k=1}^{N} a'_{ij}(u_n) D_i u_n D_j u_n u_n \eta$$
\[ \frac{1}{2} \int_{D_n, \theta} \sum_{i,j,k=1}^N a_{ij}(u_n) D_i u_n D_j u_n x_k D_k \eta - \mu \frac{1}{2} \int_{D_n, \theta} \sum_{k=1}^N \frac{x_k D_k \eta u_n^2}{|x|^2} \]

\[ - \int_{D_n, \theta} \sum_{i,j,k=1}^N a_{ij}(u_n) D_i u_n D_j x_k D_k u_n - \frac{N-2}{2} \int_{D_n, \theta} \sum_{i,j=1}^N a_{i,j}(u_n) D_i u_n D_j \eta u_n \]

\[ - \int_{D_n, \theta} \left( \frac{1}{p_n} |u_n|^{p_n} + \frac{1}{2} a |u_n|^2 \right) \sum_{k=1}^N x_k D_k \eta + \frac{1}{2} \int_{\partial \Omega \cap D_n, \theta} \sum_{i,j,k=1}^N a_{ij}(u_n) D_i u_n D_j x_k n_k \eta d \sigma. \]

Note that

\[ \frac{1}{2} \int_{\partial \Omega \cap D_n, \theta} \sum_{i,j,k=1}^N a_{ij}(u_n) D_i u_n D_j x_k D_k \eta + \mu \frac{1}{2} \int_{\partial \Omega \cap D_n, \theta} \sum_{k=1}^N \frac{x_k D_k \eta u_n^2}{|x|^2} \]

\[ + \int_{\partial \Omega \cap D_n, \theta} \sum_{i,j,k=1}^N a_{ij}(u_n) D_i u_n D_j x_k D_k u_n + \frac{N-2}{2} \int_{\partial \Omega \cap D_n, \theta} \sum_{i,j=1}^N a_{i,j}(u_n) D_i u_n D_j \eta u_n \]

\[ + \int_{\partial \Omega \cap D_n, \theta} \left( \frac{1}{p_n} |u_n|^{p_n} + \frac{1}{2} a |u_n|^2 \right) \sum_{k=1}^N x_k D_k \eta + \frac{1}{2} \int_{\partial \Omega \cap D_n, \theta} \sum_{i,j,k=1}^N a_{ij}(u_n) D_i u_n D_j x_k n_k \eta d \sigma \]

\[ \leq C \left( \int_{\partial \Omega \cap D_n} |D u_n|^2 + |D u_n||u_n|^{\frac{1}{2}} + |u_n|^{p_n} + |u_n|^2 \right) + C \theta \int_{\partial \Omega \cap D_n} |D u_n|^2 \to 0, \text{ as } \theta \to 0, \]

we obtain (29).

**Proof of Theorem 1.1.** Using Lemma 5.1, we obtain

\[ a \int_{D_n} u_n^2 \eta \leq \frac{1}{2} \int_{D_n} \sum_{i,j,k=1}^N a_{ij}(u_n) D_i u_n D_j u_n (x_k - x_k^*) D_k \eta \]

\[ - \frac{\mu}{2} \int_{D_n} \sum_{k=1}^N \frac{(x_k - x_k^*) D_k \eta u_n^2}{|x|^2} = \int_{D_n} \sum_{i,j,k=1}^N a_{ij}(u_n) D_i u_n D_j \eta (x_k - x_k^*) D_k u_n \]

\[ - \frac{N-2}{2} \int_{D_n} \sum_{i,j=1}^N a_{i,j}(u_n) D_i u_n D_j \eta u_n - \int_{D_n} \left( \frac{1}{p_n} |u_n|^{p_n} + \frac{a}{2} |u_n|^2 \right) \sum_{k=1}^N (x_k - x_k^*) D_k \eta. \]

By the choice of \( \eta \), the integral are indeed over \( A_n^2 \), thus

\[ a \int_{A_n^2} u_n^2 \eta \leq C \int_{A_n^2} \left( |D u_n|^2 + |D u_n||u_n|^{\frac{3}{2}} + |u_n|^{p_n} + |u_n|^2 \right), \]

from Proposition 2 and Corollary 2, for \( p_1 > 2^* \), we have that

\[ RHS \leq C \left( \frac{2}{\lambda_n + \frac{\mu}{2}} + \lambda_n \left( \frac{2}{\lambda_n + \frac{\mu}{2}} + \frac{1}{2} \left( \frac{2}{\lambda_n + \frac{\mu}{2}} + \frac{1}{2} \left( \frac{2}{\lambda_n + \frac{\mu}{2}} + \frac{1}{2} \left( \frac{2}{\lambda_n + \frac{\mu}{2}} + \frac{1}{2} \left( \frac{2}{\lambda_n + \frac{\mu}{2}} + \frac{1}{2} \right) \right) \right) \right) \right) \]

\[ \leq C \lambda_n^{1 - \frac{2}{p_1} + \frac{\mu}{2}}. \]

On the other hand, let \( D_{n}^\prime = D_{\lambda_n^{-1}}(x_n) \), recall the decomposition of \( u_n, u_n = u_0 + u_{n,1} + u_{n,2} \), with \( \|u_{n,2}\| \to 0 \), as \( n \to \infty \). For \( n \) large, we have

\[ \int_{D_n} u_n^2 \geq \int_{D_{n}^\prime} u_0^2 \geq \frac{1}{2} \int_{D_{n}^\prime} |u_{n,1}|^2 - 2 \int_{D_{n}^\prime} |u_0|^2 - 2 \int_{D_{n}^\prime} |u_{n,2}|^2. \]

But

\[ \int_{D_{n}^\prime} |u_0|^2 \leq C \left( \int_{D_{n}^\prime} |u_0|^{2^*} \right)^{\frac{2}{2^*}} \lambda_n^{-2} = o(1) \lambda_n^{-2}. \]
and
\[ \int_{\Omega} |u_n,2|^2 \leq C \left( |u_n,2|^{2^*} \right)^\frac{2}{2^*} \lambda_n^{-2} = o(1)\lambda_n^{-2}. \]
Moreover, for \( \mu > \bar{\mu} - 1 \), we have \( 2(\sqrt{\mu} + \sqrt{\bar{\mu} - \mu}) > N \), so \( U_j \in L^2(\mathbb{R}^N) \), if \( \mu < \bar{\mu} - 1 \). By the definition of \( u_n,1 \), and since \( \lambda_n \) is the slowest concentration rate of blow-up, there is a constant \( C > 0 \) such that
\[ \int_{\Omega} |u_{n,1}|^2 \geq C\lambda_n^{-2}. \]
Thus
\[ \text{LHS} \geq C\lambda_n^{-2}. \]
And
\[ \lambda_n^{-2} \leq C\lambda_n^{-\frac{2+2^*}{2^*} + \frac{N}{p_1}}, \]
for any \( 2^* < p_1 < \frac{2^* \sqrt{\alpha}}{\sqrt{\alpha} - \sqrt{\alpha - \mu}} \), and \( \mu < \bar{\mu} - 1 \).

Choose \( p_1 = \frac{2N}{N - 0} + \sigma \) with \( p_1 < \frac{2^* \sqrt{\alpha}}{\sqrt{\alpha} - \sqrt{\alpha - \mu}} \), where \( \sigma \) is a small constant. This can be achieved if \( \frac{N}{\alpha} < \bar{\mu} - 4 \), then with this \( p_1 \), \( 2 < \frac{N-2}{2} - \frac{N}{p_1} \), we obtain a contradiction.

In order to prove Theorem 1.2, we first look at the existence of solutions to the subcritical problem (4), under assumptions \((A_1) - (A_5)\). Following the idea of [14] (see also [17]), we apply the method of regularity approximation. For reader’s convenience, we give the sketch of the proof.

Indeed, the functional corresponding to (4) is defined as:
\[ I_\varepsilon(u) = \frac{1}{2} \int_{\Omega} \left( \sum_{i,j=1}^{N} a_{ij}(u) D_i u D_j u - \frac{\mu}{|x|^2} u^2 - a u^2 \right) - \frac{1}{2^* - \varepsilon} \int_{\Omega} |u|^{2^* - \varepsilon}, \quad u \in H_0^1(\Omega). \] (32)
For fixed \( q > N \) and any \( \theta \in (0,1] \), we define the perturbation functional \( I_{\varepsilon_n, \theta} \) by:
\[ I_{\varepsilon_n, \theta} = \frac{\theta}{2} \left( \int_{\Omega} |Du|^q \right)^\frac{2}{q} + I_{\varepsilon_n}(u). \] (33)
Then \( I_{\varepsilon_n, \theta} \) is a \( C^1 \) functional defined on \( W_0^{1,q}(\Omega) \), and as in [14], we can prove that \( I_{\varepsilon_n, \theta} \) satisfies Palais-Smale condition.

Next, similar as in [17], we may choose two functionals \( J \) and \( \tilde{J} \) independent of \( \varepsilon_n \) and \( \theta \) such that
\[ J(u) \leq I_{\varepsilon_n, \theta}(u) \leq \tilde{J}(u), \quad \forall u \in W_0^{1,q}(\Omega). \] (34)
This is possible, for example, we may take
\[ J(u) = \frac{\alpha - \frac{\mu}{2}}{2} \int_{\Omega} |Du|^2 - \frac{a + \sigma}{2} \int_{\Omega} u^2 - \frac{c_\sigma}{2^*} \int_{\Omega} |u|^{2^*}, \] (35)
\[ \tilde{J}(u) = \frac{1}{2} \left( \int_{\Omega} |Du|^q \right)^\frac{2}{q} + \frac{\beta}{2} \int_{\Omega} |Du|^2 - \frac{a}{2} \int_{\Omega} u^2 - \frac{1}{2^*} \int_{\Omega} |u|^p + \tilde{C}, \] (36)
where \( \alpha, \beta \) are the constants in the condition \((A_2)\), \( \sigma \) is any positive constant, \( c_\sigma \) and \( \tilde{C} \) are some positive constants, and \( p \in (2,2^*) \) is a fixed constant.
Without loss of generality, we may assume \( \alpha - \frac{\mu}{2} = 1 \). Let \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \) be the eigenvalue of Laplacian with Dirichlet boundary condition on \( \partial \Omega \). Assume \( m \) is the least integer such that \( \lambda_{m+1} > a \). Let \( E_m \) be the direct sum of the eigenspaces
critical value by:
\[ c_k^1(\theta) = \inf_{\varphi \in E_k} \sup_{t \in B_k} I_{\varepsilon_n, \theta}(\varphi(t)), \quad (37) \]

where
\[ E_k = \{ \varphi \in C(B_k, W_0^{1,q}(\Omega)), \varphi \text{ is odd, } \tilde{J}(\varphi(t)) < -1 \text{ if } t \in \partial B_k \}. \quad (38) \]

Then we deduce similarly as Lemma 6.2 in [17], there exists a constant \( \sigma_0 > 0 \) such that
\[ \sigma_0 < c_k^n(\theta) \leq b_k, \quad (39) \]

where \( b_k = \sup_{t \in B_k} \tilde{J}(\phi(t)), \phi \in E_k \).

By the symmetry mountain theorem, we see that \( c_k^n(\theta) \) is a critical value of \( I_{\varepsilon_n, \theta} \).

Suppose \( u_k^n(\theta) \) is the corresponding critical point, by (34) and (39), \( I_{\varepsilon_n, \theta}(u_k^n(\theta)) \leq b_k \) and \( \| u_k^n(\theta) \| \leq M_k \), for some constant \( M_k \). Let \( c_k^n = \lim_{\theta \to 0} c_k^n(\theta) \), then \( \sigma_0 \leq c_k^n \leq b_k \).

By the convergence theorem in [14], we have \( u_k^n(\theta) \to u^n \) in \( H_0^1(\Omega), I_{\varepsilon_n, \theta}(u_k^n(\theta)) \to I_{\varepsilon_n}(u^n) = c_k^n \), as \( \theta \to 0 \). And \( u_k^n \) is a weak solution of (4) with \( \varepsilon = \varepsilon_n \) with \( \| u_k^n \| \leq M_k \).

**Proof of Theorem 1.2.** It is sufficient to prove the existence of infinitely many solutions of \( u \) to the problem (1). Indeed, for any \( k \) and \( n \), suppose that \( c_k^n \) is the critical value of the subcritical problem with \( \varepsilon = \varepsilon_n \). By (39), we see that \( \sigma_0 \leq c_k^n \leq b_k \). Set \( c_k = \lim_{n \to \infty} c_k^n \), then \( \sigma_0 \leq c_k \leq b_k \).

By Theorem 1.1, we have \( u_k^n \to u_k \) in \( H_0^1(\Omega), I_{\varepsilon_n}(u_k^n) \to I(u_k) = c_k \), as \( n \to \infty \). And \( u_k \) is a weak solution of (1).

In order to prove \( c_k \to \infty \), as \( k \to \infty \), we will compare \( c_k \) with the corresponding minimax values of the functional \( J \). Note that the functional \( J \) is associated to the semilinear problem. Let
\[ G_k = \{ \varphi \in C(B_k, H_0^1(\Omega)), \varphi \text{ is odd and } J(\varphi(t)) < -1 \text{ if } t \in \partial B_k \}, \quad (40) \]
\[ d_k = \inf_{\varphi \in G_k} \sup_{t \in \partial B_k} J(\varphi(t)). \quad (41) \]

Since \( J \leq I_{\varepsilon_n, \theta} \leq J, E_k \subset G_k \), we have
\[ c_k^n(\theta) \geq d_k, \quad (42) \]

hence
\[ c_k \geq d_k. \]

But \( d_k \to \infty \) (see Lemma 6.3 in [17]). We finish the proof of Theorem 1.2. \( \square \)

**Appendix A. \( L^p \) estimates.**

**Lemma A.1.** Let the functions \( b_{ij}(x) \) satisfy
\[ \alpha |\xi|^2 \leq \sum_{i,j=1}^N b_{ij}(x) \xi_i \xi_j \leq \beta |\xi|^2, \text{ for all } x \in \Omega, \xi \in \mathbb{R}^n. \]

If \( u \in H^1(\mathbb{R}^N) \) satisfies that
\[ \int_{\mathbb{R}^N} \sum_{i,j=1}^N b_{ij}(x) D_i u D_j \varphi - \frac{\mu}{|x|^2} u \varphi \leq \int_{\mathbb{R}^N} g u \varphi, \forall \varphi \in H^1(\mathbb{R}^N), \]

then...
There exists a small $\delta > 0$, such that if $\int_{B_r(x_0)} |g|^\frac{N}{2} \leq \delta$, then

$$u \in \bigcap_{p < p_{lim}} L^p(B_{r/2}(x_0)) \text{ with } p_{lim} = \frac{2^* \sqrt{\alpha \mu}}{\sqrt{\alpha \mu} - \sqrt{\alpha \mu} - \mu}.$$ 

Proof. The proof is similar to that of the Lemma A.4 in [8]. Thus we omit it. \hfill \square

Remark 1. Let $u \in H^1_0(\Omega)$ be a solution of (1). In view of Lemma A.1, we have

$$u \in L^p(\Omega), \forall p < \frac{2^* \sqrt{\mu}}{\sqrt{\mu} - \sqrt{\mu} - \mu}.$$ 

Appendix B. Estimates for the quasilinear problem with Hardy potential. In the sequel, we assume that for any $x \in \Omega$, $\xi \in \mathbb{R}^N$, $b_{ij}(x)$ satisfy that

$$\alpha |\xi|^2 \leq \sum_{i,j=1}^N b_{ij}(x)\xi_i\xi_j \leq \beta |\xi|^2.$$ 

Lemma B.1. Suppose $f \in C^1(\Omega; B_2(0))$ for any small $\delta > 0$, $\frac{2N^\gamma}{(N+2)\sqrt{\alpha \mu} + (N-2)\sqrt{\mu} - \mu} < p < \frac{2N^\gamma}{(N+2)\sqrt{\alpha \mu} + (N-2)\sqrt{\mu} - \mu}$, and $u \in H^1_0(\Omega)$, $u \geq 0$ satisfies

$$\int_{\Omega} \sum_{i,j=1}^N b_{ij} D_i u D_j \varphi - \frac{\mu}{|x|^2} u \varphi \leq \int_{\Omega} f \varphi, \forall \varphi \in H^1_0(\Omega), \varphi \geq 0. \quad (43)$$

Then $\|u\|_{L^p_{\Omega}} \leq C\|f\|_{L^p_{\Omega}}$.

Proof. We may assume $f \geq 0$, this is because if (43) is true, then it certainly holds for $|f|$. Let $\gamma = \frac{p(N-2)}{4(N-2p)}$. Then $\gamma > \frac{1}{2}$ and $2^*\gamma = (2\gamma - 1)\frac{p}{p-1} = \frac{Np}{N-2p}$.

For $\theta > 0$, take $\varphi = (u + \theta)^{2\gamma - 1} - \theta^{2\gamma - 1} \in H^1_0(\Omega)$ as a test function in (43), we have

$$\int_{\Omega} (2\gamma - 1)\sum_{i,j=1}^N b_{ij} D_i u D_j u (u + \theta)^{2\gamma - 2} - \int_{\Omega} \frac{\mu u((u + \theta)^{2\gamma - 1} - \theta^{2\gamma - 1})}{|x|^2} \leq \int_{\Omega} f((u + \theta)^{2\gamma - 1} - \theta^{2\gamma - 1}).$$

Notice that $u((u + \theta)^{2\gamma - 1} - \theta^{2\gamma - 1}) \leq (u + \theta)^{2\gamma}$, we obtain

$$\frac{\alpha(2\gamma - 1)}{\gamma^2} \int_{\Omega} |D(u + \theta)|^2 \leq \frac{\mu}{\mu} \int_{\Omega} |D(u + \theta)|^2 \leq \int_{\Omega} f(u + \theta)^{2\gamma - 1},$$

thus

$$(\frac{\alpha(2\gamma - 1)}{\gamma^2} - \frac{\mu}{\mu}) \int_{\Omega} |D(u + \theta)|^2 \leq \int_{\Omega} f(u + \theta)^{2\gamma - 1}.$$ 

By Sobolev imbedding, we have

$$C\left(\frac{\alpha(2\gamma - 1)}{\gamma^2} - \frac{\mu}{\mu}\right) \left(\int_{\Omega} f(u + \theta)^2\right)^\frac{N-2}{2} \leq \left(\int_{\Omega} f^p\right)^\frac{1}{p} \left(\int_{\Omega} (u + \theta)^{(2\gamma - 1)}\right)^\frac{p-1}{p},$$

and by Hardy inequality, we get

$$C\int_{\Omega} \frac{\mu}{|x|^2} (u + \theta)^{2\gamma} \leq \left(\int_{\Omega} f^p\right)^\frac{1}{p} \left(\int_{\Omega} (u + \theta)^{(2\gamma - 1)}\right)^\frac{p-1}{p}.$$ 

Note that $\frac{N-2}{N} - \frac{p-1}{p} = \frac{N-2p}{Np}$, we obtain $\|u\|_{L^p_{\Omega}} \leq C\|f\|_{L^p_{\Omega}}$. \hfill \square
Lemma B.2. Assume \( f, v \in C^2(\Omega \setminus B_\delta(0)) \) for any \( \delta > 0 \) small. Then for \( p_i, i = 1, 2, \) with \( \frac{2^* \sqrt{\alpha}}{\sqrt{\alpha} + \sqrt{\alpha} - \mu} < p_i < \frac{2^* \sqrt{\alpha}}{\sqrt{\alpha} + \sqrt{\alpha} - \mu} \), if \( u \in H^1_0(\Omega) \), \( u \geq 0 \) satisfies

\[
\int_{\Omega} \sum_{i,j=1}^{N} b_{ij} D_i u D_j \varphi - \int_{\Omega} \frac{\mu u \varphi}{|x|^2} \leq \int_{\Omega} f v \varphi, \quad \forall \varphi \in H^1_0(\Omega), \quad \varphi \geq 0,
\]

then

\[
\|u\|_{p_1, p_2, \lambda} \leq C \|f\|_q \|v\|_{p_1, p_2, \lambda}.
\]

Proof. Let \( q = \frac{Np}{N + 2p} \), then for \( \frac{2^* \sqrt{\alpha}}{\sqrt{\alpha} + \sqrt{\alpha} - \mu} < p \leq \frac{2^* \sqrt{\alpha}}{\sqrt{\alpha} + \sqrt{\alpha} - \mu} \), \( \frac{2N \sqrt{\alpha}}{(N + 2) \sqrt{\alpha} + (N - 2) \sqrt{\alpha} - \mu} < q < \frac{2N \sqrt{\alpha}}{(N + 2) \sqrt{\alpha} + (N - 2) \sqrt{\alpha} - \mu} \). And \( p = \frac{Nq}{N - q} \), \( \frac{1}{p} = \frac{1}{q} - \frac{2}{\mu} \). By Lemma B.1 and Hölder inequality

\[
\|u\|_{*, p} = \|u\|_{*, \frac{Nq}{N - q}} \leq C \|f\|_q \|v\|_p.
\]

For any \( \rho > \|v\|_{p_1, p_2, \lambda}, \exists v_1, v_2, \|v_1\|_{p_1} \leq \rho, \|v_2\|_{p_2} \leq \rho \lambda^{\frac{N + 2}{2} - \frac{N}{2}} \). We take \( u_k \in H^1_0(\Omega) \), \( k = 1, 2 \), such that

\[
\int_{\Omega} \sum_{i,j=1}^{N} b_{ij} D_i u_k D_j \varphi - \int_{\Omega} \frac{\mu u_k \varphi}{|x|^2} = \int_{\Omega} f v_k \varphi,
\]

then \( \|u_k\|_{*, p_k} \leq C \|f\|_q \|v_k\|_{p_k} \). On the other hand, since \( u \leq u_1 + u_2 \), we have \( \|u\|_{*, p_1, p_2, \lambda} \leq C \rho \|f\|_q \|v\|_{p_1, p_2, \lambda} \). By the arbitrary choice of \( \rho > \|v\|_{p_1, p_2, \lambda} \), we obtain \( \|u\|_{*, p_1, p_2, \lambda} \leq C \|f\|_q \|v\|_{p_1, p_2, \lambda} \).

Lemma B.3. Let \( u \in H^1_0(\Omega) \), \( u \geq 0 \) satisfies that

\[
\int_{\Omega} \sum_{i,j=1}^{N} b_{ij} D_i u D_j \varphi - \frac{\mu u \varphi}{|x|^2} \leq \int_{\Omega} a v \varphi, \quad \forall \varphi \in H^1_0(\Omega), \quad \varphi \geq 0,
\]

where \( a \geq 0, v \geq 0 \) are functions in \( C^2(\Omega \setminus B_\delta(0)) \) for any \( \delta > 0 \) small. Then for any \( \frac{2^* \sqrt{\alpha}}{\sqrt{\alpha} + \sqrt{\alpha} - \mu} < p_2 < 2^* \), there is a constant \( C = C(p_2) \) such that

\[
\|u\|_{*, p_2} \leq C \|a\|_r \|v\|_{2^*},
\]

where \( r \) is determined by \( \frac{1}{p_2} = \frac{1}{r} + \frac{1}{2} - \frac{2}{\mu} \).

Proof. Let \( q = \frac{2p_2}{2^*} \), since \( p_2 \in (\frac{2^* \sqrt{\alpha}}{\sqrt{\alpha} + \sqrt{\alpha} - \mu}, 2^*), \) we see \( \frac{\sqrt{\alpha}}{\sqrt{\alpha} + \sqrt{\alpha} - \mu} > \frac{q}{2} < 1 < \frac{\sqrt{\alpha}}{\sqrt{\alpha} + \sqrt{\alpha} - \mu} \). Let \( t = \frac{2N}{N + 2} \), deducing similarly as in Lemma B.1, we obtain

\[
C \|u\|^q_{p_2} = C \|u\|^{q/2}_{2^*} \leq \int_{\Omega} a u v^{q-1} \leq \|v\|_{2^*} \left( \int_{\Omega} (a u^{q-1})^r \right)^{\frac{1}{r}} \leq \|v\|_{2^*} \cdot \|a\|_r \left( \int_{\Omega} u^{\frac{(q-1)r}{r}} \right)^{\frac{1}{r} - \frac{1}{q}},
\]

and

\[
C \int_{\Omega} \frac{|u| q}{|x|^2} \leq \|v\|_{2^*} \cdot \|a\|_r \left( \int_{\Omega} u^{\frac{(q-1)r}{r}} \right)^{\frac{1}{r} - \frac{1}{q}}.
\]

By the definition of \( q \), we have

\[
\frac{(q - 1)r}{r - t} = \frac{2p_2}{2^*} - 1 = p_2,
\]

\[
\frac{1}{r - t} = \frac{1}{r} - \frac{2}{\mu}.
\]
that is
\[ \|u\|_p^q \leq C\|v\|_2 \|a\|_r \|u\|_p^{2(\frac{1}{r} - \frac{1}{2})}. \]

On the other hand, one can check
\[ p_2 \left( \frac{1}{t} - \frac{1}{r} \right) = \frac{2p_2}{2^*} - 1 = q - 1. \]

The result follows. 

\[ \square \]

Appendix C. Integral estimates for subcritical solution $u_n$.

**Lemma C.1.** Suppose $u_n$ satisfies (4) with $\varepsilon = \varepsilon_n$, $q \in (1, \frac{N}{N-1})$. Then there exists a constant $C = C(\varepsilon)$ independent of $n$ such that
\[ r^{-N} \int_{D_r(x)} |u_n|^q \leq C\lambda_n^{\frac{N}{N-1}} \lambda_n^{-\frac{1}{2}}, \quad \forall r \geq \lambda_n^{-\frac{1}{2}}, \]
where $x \in A^2_n$, $D_r(x) = B_r(x) \cap \Omega$, $p_1$ is some constant satisfying $2^* < p_1 < \frac{2^*}{\sqrt{\alpha \mu} - \sqrt{\alpha \mu - \mu}}$.

In order to prove Lemma C.1, we need the following two lemmas, since their proofs are similar to those of Lemma 4.4 and Lemma 4.5 in [17], we omit them.

**Lemma C.2.** Suppose $v \geq 0$ satisfies the differential inequality (5), $a \geq 0$, $d > 0$, $\gamma \in (1, \frac{N}{N-1})$, $q = \frac{2^*}{\alpha}$. Then there is a constant $C = C(N, \gamma)$ such that if
\[ |D_{2r} \cap \{v - a > 0\}| < \frac{1}{2}d^{-\gamma} \int_{D_r \cap \{v - a > 0\}} (v - a)^{\gamma}, \]
then
\[ \left( d^{-\gamma}r^{-N} \int_{D_r \cap \{v - a > 0\}} (v - a)^{\gamma} \right)^{\frac{2}{\gamma}} \leq C d^{-\gamma}r^{-N} \int_{D_{2r} \cap \{v - a > 0\}} (v - a)^{\gamma} + C a^{1-\gamma}r^{-2-N} \int_{D_{2r} \cap \{v - a > 0\}} f, \]
where $f = v^{2^* - \varepsilon - 1} + av + \frac{\mu v}{|x|^2}$, $D_r = D_r(x)$.

Let $r_0 = diam(\Omega)$, $a_0 = 0$, $r_j = 2^{j-1}r_0$, $j = 1, 2, \cdots$, and
\[ a_{j+1} = a_j + \frac{1}{\delta} \left( r_j^{-N} \int_{D_{r_{j+1}} \cap \{v(y) - a_j > 0\}} (v(y) - a_j)^{\gamma} \right)^{\frac{1}{\gamma}}, \quad (44) \]
where $\gamma \in (1, \frac{N}{N-1})$, $\delta > 0$ is a fixed constant.

**Lemma C.3.** For $\delta > 0$ small enough, there exists a constant $C = C(N, \gamma)$ such that for all $k$,
\[ a_k \leq 2a_1 + C \sum_{j=1}^{k} \frac{1}{r_j^{N-1}} \int_{D_{r_j}(x)} f, \]
where $f$ denotes the function in Lemma C.2.

**Proof of Lemma C.1.** We define $a_k$, $k = 1, 2, \cdots$, as in (44). By Lemma C.3, we have
\[ a_k \leq C \int_{r_k}^{r_0} \left( \frac{1}{t^{N-1}} \int_{D_t(x_0)} f \right). \]
By the definition of $a_k$, we have
\[ \left( r_k^{-N} \int_{D_{r_k}(x_0) \cap \{v - a_k > 0\}} (v - a_k)^{\gamma} \right)^{\frac{1}{\gamma}} \leq 2^N \delta a_k \leq C \int_{r_k}^{r_0} \left( \frac{1}{t^{N-1}} \int_{D_t(x_0)} f \right). \]
Thus
\[
\left(r_k^{-N} \int_{D_{r_k}(x_0)} v \right) \leq C\left(r_k^{-N} \int_{D_{r_k}(x_0) \cap \{v < -a_k\}} v \right)^{\frac{1}{2}} + C\left(r_k^{-N} \int_{D_{r_k}(x_0) \cap \{v < -a_k\}} v \right)^{\frac{1}{2}} + C \int_{r_k}^{r_0} \left( \frac{1}{t^{N-1}} \int_{D_t(x_0)} f \right).
\]

For \( r \in [\lambda_n^{-\frac{1}{2}}, r_k] \), take \( r_{k+1} < r \leq r_k \), then we have
\[
\left(r^{-N} \int_{D_r(x_0)} v \right) \leq C\left(r_k^{-N} \int_{D_{r_k}(x_0)} v \right)^{\frac{1}{2}} \leq C + C \int_{r_k}^{r_0} \left( \frac{1}{t^{N-1}} \int_{D_t(x_0)} f \right). \tag{45}
\]

Now let \( v = v_n, f = v_n^{2* - 1 - \varepsilon_n} + \alpha v_n + \frac{\mu v_n}{|x|^2} \). By Proposition 1, \( \|v_n\|_{\ast, p_1, p_2, \lambda_n} \leq C \) for any \( p_1, p_2 \in (\frac{2* \sqrt{|\alpha|}}{\sqrt{\alpha + \sqrt{\alpha \mu - \mu}}, \frac{2* \sqrt{|\alpha|}}{\sqrt{\alpha + \sqrt{\alpha \mu - \mu}}}) \), \( p_2 < 2* < p_1 \). Let \( p_1 \) be a constant satisfying \( 2* < p_1 < \frac{2* \sqrt{|\alpha|}}{\sqrt{\alpha + \sqrt{\alpha \mu - \mu}}} \), let \( p_2 \) close to \( 2* \frac{\sqrt{|\alpha|}}{\sqrt{\alpha + \sqrt{\alpha \mu - \mu}}} \). Then we can choose \( v_{1,n} \) and \( v_{2,n} \) such that \( v_n \leq v_{1,n} + v_{2,n} \), and \( \|v_{1,n}\|_{\ast, p_1} \leq C, \|v_{2,n}\|_{\ast, p_2} \leq C \lambda_n^{\frac{N}{2p_1} - \frac{N}{2p_2}} \).

So
\[
\int_{r_k}^{r_0} \left( \frac{1}{t^{N-1}} \int_{D_t(x_0)} \frac{|v_{1,n}|}{|x|^2} \right) \leq C \int_{r_k}^{r_0} \left( \frac{1}{t^{N-1}} \int_{D_t(x_0)} \frac{\mu |v_{1,n}|}{|x|^2} \right)^{\frac{1}{2}} \left( \int_{D_t(x_0)} \frac{\mu}{|x|^2} \right)^{\frac{1}{2}} \leq C \int_{r_k}^{r_0} t^{-N+1+(N-2)(1-\frac{2*}{p_1})} \leq C r^{-\frac{N-2*}{2p_1}} \leq C \lambda_n^{\frac{N}{2p_1} - \frac{N}{2p_2}} \tag{46}
\]

and
\[
\int_{r_k}^{r_0} \left( \frac{1}{t^{N-1}} \int_{D_t(x_0)} \frac{|v_{2,n}|}{|x|^2} \right) \leq C \int_{r_k}^{r_0} t^{-N+1+(N-2)(1-\frac{2*}{p_2})} \leq C \lambda_n^{\frac{N}{2p_1} - \frac{N}{2p_2}} \tag{47}
\]

thus by (46) and (47), we obtain
\[
\int_{r_k}^{r_0} \left( \frac{1}{t^{N-1}} \int_{D_t(x_0)} \frac{|v_{n}|}{|x|^2} \right) \leq C \lambda_n^{\frac{N}{2p_1} - \frac{N}{2p_2}}. \tag{48}
\]

For any \( p_1 \in (2*, \frac{2* \sqrt{|\alpha|}}{\sqrt{\alpha + \sqrt{\alpha \mu - \mu}}}) \), if \( \frac{2* \sqrt{|\alpha|}}{\sqrt{\alpha + \sqrt{\alpha \mu - \mu}}} \leq \frac{N+2}{N-2} < 2* \), let \( \tilde{p_2} = \frac{N+2}{N-2} \), else choose \( \tilde{p_2} \) sufficiently close to \( \frac{2* \sqrt{|\alpha|}}{\sqrt{\alpha + \sqrt{\alpha \mu - \mu}}} \). Then we can choose \( \tilde{v}_{1,n} \) and \( \tilde{v}_{2,n} \) such that \( v_n \leq \tilde{v}_{1,n} + \tilde{v}_{2,n} \), and \( \|\tilde{v}_{1,n}\|_{\ast, p_1} \leq C, \|\tilde{v}_{2,n}\|_{\ast, p_2} \leq C \lambda_n^{\frac{N}{2p_1} - \frac{N}{2p_2}} \).

Since \( p_1 > 2* \), we know \( \frac{N(N+2)}{2p_1(N-2)} - 1 < \frac{N}{2p_1} \), we obtain
\[
\int_{r_k}^{r_0} \left( \frac{1}{t^{N-1}} \int_{D_t(x_0)} \left( \left( \tilde{v}_{1,n} \right)^{\frac{N+2}{N-2}} + A \right) \right)
\]
Thus in either case, we have
\[ \int_{r}^{r_0} \frac{1}{t^{N-1}} \left( \int_{D_t(x_0)} |\bar{v}_{1,n}|^{\frac{N+2}{N-2}} \right) \leq C \int_{r}^{r_0} t^{-1} N^{\frac{N(N+2)}{2(N-2)}} + C \leq C \lambda_n^{\frac{N}{2(N-2)}}. \]

and if \( \frac{2^* \sqrt{\alpha p}}{\sqrt{\alpha p} + \sqrt{\mu - \rho}} \leq N+2 \), \( \bar{p}_2 = \frac{N+2}{N-2} \), then we have
\[ \int_{r}^{r_0} \frac{1}{t^{N-1}} \left( \int_{D_t(x_0)} (\bar{v}_{2,n})^{\frac{N+2}{N-2}} \right) \leq C \int_{r}^{r_0} \frac{1}{t^{N-1}} \left( \int_{D_t(x_0)} (\bar{v}_{2,n}) \bar{p}_2 \right) \leq C \lambda_n^{\frac{N}{2(N-2)}}. \]

else if \( \frac{2^* \sqrt{\alpha p}}{\sqrt{\alpha p} + \sqrt{\mu - \rho}} > N+2 \), notice that \( 1 - \frac{N+2}{(N-2)\bar{p}_2} \frac{1}{\bar{p}_2} \) when \( \bar{p}_2 \) is sufficiently close to \( \frac{2^* \sqrt{\alpha p}}{\sqrt{\alpha p} + \sqrt{\mu - \rho}} \), we have
\[ \int_{r}^{r_0} \frac{1}{t^{N-1}} \left( \int_{D_t(x_0)} (\bar{v}_{2,n})^{\frac{N+2}{N-2}} \right) \leq C \lambda_n^{\frac{N}{2(N-2)}} \int_{r}^{r_0} t^{-1} N^{\frac{N(N+2)}{2(N-2)}} \leq C \lambda_n^{\frac{N}{2(N-2)}}, \]

thus in either case, we have
\[ \int_{r}^{r_0} \frac{1}{t^{N-1}} \left( \int_{D_t(x_0)} (v_n)^{\frac{N+2}{N-2}} \right) \leq C \int_{r}^{r_0} \frac{1}{t^{N-1}} \left( \int_{D_t(x_0)} (\bar{v}_{1,n})^{\frac{N+2}{N-2}} \right) + C \int_{r}^{r_0} \frac{1}{t^{N-1}} \left( \int_{D_t(x_0)} (\bar{v}_{2,n})^{\frac{N+2}{N-2}} \right) \]
\[ \leq C \lambda_n^{\frac{N}{2(N-2)}}. \]

Combining inequalities (45), (48) and (52), we obtain
\[ r^{-N} \int_{D_r(x)} |u_n|^\gamma \leq C \lambda_n^{\frac{N}{2(N-2)}}, \]
that is
\[ r^{-N} \int_{D_r(x)} |u_n|^\gamma \leq C \lambda_n^{\frac{N}{2(N-2)}}. \]

We conclude the proof. \( \square \)

Acknowledgment. This work is supported by NSFC (11771235, 11331010, and 11571040).

REFERENCES
[1] A. Ambrosetti and P. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal., 14 (1973), 349–381.
[2] A. Ambrosetti and Z. Q. Wang, Positive solutions to a class of quasilinear elliptic equations on R, Discrete Contin. Dyn. Syst., 9 (2003), 55–68.
[3] H. Berestycki and M. Esteban, Existence and bifurcation of solutions for an elliptic degenerate problem, J. Differential Equations, 134 (1997), 1–25.
[4] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Commun. Pure Appl. Math., 36 (1983), 437–477.
[5] D. Cao and P. Han, Solutions to critical elliptic equations with multi-singular inverse sequare potentials, J. Differential Equations, 224 (2006), 332–372.
[6] D. Cao, S. Peng and S. Yan, Multiplicity of solutions for the plasma problem in two dimensions, Adv. Math., 225 (2010), 2741–2785.
[7] D. Cao, S. Peng and S. Yan, Infinitely many solutions for $p-$Laplacian equation involving critical Sobolev growth, *J. Funct. Anal.*, 262 (2012), 2861–2902.

[8] D. Cao and S. Yan, Infinitely many solutions for an elliptic problem involving critical Sobolev growth and Hardy potential, *Calc. Var. Part. Diff. Equ.*, 38 (2010), 471–501.

[9] A. Capozzi, D. Fortunato and G. Palmieri, An existence result for nonlinear elliptic problems involving critical Sobolev exponents, *Ann. Inst. H. Poincare and Non Lineaire*, 2 (1985), 463–470.

[10] G. Cerami, S. Solimini and M. Struwe, Some existence results for superlinear elliptic problem involving critical exponents, *J. Funct. Anal.*, 69 (1986), 289–306.

[11] M. Colin and L. Jeanjean, Solutions for a quasilinear Schrödinger equation: A dual approach, *Nonlinear. Anal. Theor. Meth. App.*, 56 (2004), 213–226.

[12] J. M. Coron, Topologie et cas limite des injections de Sobolev (Topology and limit case of Sobolev embeddings), *C. R. Acad. Sci. Paris Ser. I Math.*, 199 (1984), 209–212.

[13] A. de Bouard, N. Hayashi and J. C. Saut, Global existence of small solutions to a relativistic nonlinear Schrödinger equation, *Comm. Math. Phys.*, 189 (1997), 73–105.

[14] Y. Deng, Y. Guo and J. Liu, Existence of solutions for quasilinear elliptic equations with Hardy potential, *J. Math. Phys.*, 57 (2016), 031503, 15 pp.

[15] G. Divillanova and S. Solimini, Concentration estimates and multiple solutions to elliptic problems at critical growth, *Adv. Differential Equations*, 7 (2002), 1257–1280.

[16] J. P. García Azorero and I. Peral Alonso, Hardy inequalities and some critical elliptic and parabolic problems, *J. Differential Equations*, 144 (1998), 441–476.

[17] Y. Guo, J. Liu and Z. Wang, On a Brezis– Nirenberg type quasilinear problem, *J. Fixed Point Theory Appl.*, 19 (2017), 719–753.

[18] T. Kilpeläinen and J. Malý, The Winer test and potential estimates for quasilinear elliptic equations, *Acta Math.*, 172 (1994), 137–161.

[19] A. M. Kosevich, B. A. Ivanov and A. S. Kovalev, Magnetic solitons in superfluid films, *J. Phys. Soc. Japan.*, 50 (1981), 3262–3267.

[20] L. Leblond and J. Marc, Electron capture by polar molecules, *Phys. Rev.*, 153 (1967), 1–4.

[21] J. Liu, X. Liu and Z.-Q. Wang, Multiple sign-changing solutions for quasilinear elliptic equations via perturbation method, *Comm. PDE*, 39 (2014), 2216–2239.

[22] J. Liu and Z.-Q. Wang, Soliton solutions for quasilinear Schrödinger equation I, *Proc. Amer. Math. Soc.*, 131 (2003), 441–448.

[23] J. Liu, Y. Wang and Z.-Q. Wang, Soliton solutions for quasilinear Schrödinger equation II, *J. Differential Equations*, 187 (2003), 473–493.

[24] J. Liu, Y. Wang and Z.-Q. Wang, Solutions for quasilinear Schrödinger equation via the Nehari method, *Comm. PDE*, 29 (2004), 879–901.

[25] X. Liu, J. Liu and Z.-Q. Wang, Quasilinear elliptic equations via perturbation method, *Proc. Amer. Math. Soc.*, 141 (2013), 253–263.

[26] X. Liu, J. Liu and Z.-Q. Wang, Quasilinear elliptic equations with critical growth via perturbation method, *J. Differential Equations.*, 254 (2013), 102–124.

[27] X. Liu, J. Liu and Z. Wang, Quasilinear equations via elliptic regularization method, *Adv. Non. Sta.*, 13 (2013), 517–531.

[28] M. Maris, Profile decomposition for sequences of Borel measures, https://arxiv.org/abs/1406.6125.

[29] M. Poppenberg, K. Schmitt and Z.-Q. Wang, On the existence of soliton solutions to quasilinear Schrödinger equations, *Calc. Var. Part. Diff. Equ.*, 14 (2002), 329–344.

[30] M. Struwe, A global compactness result for elliptic boundary value problems involving limiting nonlinearities, *M. Math Z.*, 187 (1984), 511–517.

[31] C. Tintarev, Concentration analysis and cocompactness, *Concentration Analysis and Applications to PDE*, 117–141, Trends Math., Birkhäuser/Springer, Basel, 2013.

[32] C. Tintarev and K. H. Fineseler, *Concentration Compactness, Functional Analytic Grounds and Applications*, Imperial College Press, London, 2007.

Received December 2017; revised April 2018.

E-mail address: gfs16@mails.tsinghua.edu.cn
E-mail address: yguo@math.tsinghua.edu.cn