Asymptotic Analysis of Self-Adjusting Contraction Trees

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In this report, we analyze the asymptotic efficiency of self-adjusting contraction trees proposed as part of the Slider project [2,3]. Self-adjusting contraction trees are used for incremental computation [1,4,5,8]. Our analysis extends the asymptotic efficiency analysis of Incoop [6,7]. We consider two different runs: the initial run of an Slider computation, where we perform a computation with some input \( I \), and a second run for dynamic update where we change the input from \( I \) to \( I' \) and perform the same computation with the new input. In the common case, we perform a single initial run followed by many dynamic updates.

For the initial run, we define the overhead as the slowdown of Slider compared to a conventional implementation of MapReduce such as with Hadoop. We show that the overhead depends on communication costs and, if these are independent of the input size, which they often are, then it is also constant. Our experiment evaluation confirms that the overhead is relatively small. We show that dynamic updates are dominated by the time it takes to execute fresh tasks that are affected by the changes to the input data, which, for a certain class of computations and small changes, is logarithmic in the size of the input.

In the analysis, we use the following terminology to refer to the three different types of computational tasks that form an Slider computation: Map tasks, Self-adjusting balanced tree (applications of the Combiner function for three different modes of operation for sliding-window computations), and Reduce tasks.

Our bounds depend on the total number of map tasks, written \( N_M \), and the total number of reduce tasks written \( N_R \). In addition, we also take in account the total number of stages in self-adjusting balanced tree, denoted as \( N_C \). We write \( n_i \) and \( n_o \) to denote the total size of the input and output respectively, \( n_m \) to denote the total number of key-value pairs output by the Map phase, and \( n_{mk} \) to denote the set of distinct keys emitted by the Map phase. The number of stages in self-adjusting balanced tree is a property of sliding-window computation mode: append-only (\( N_{CA} = O(n_{mk}) \)), fixed-width window slides (\( N_{CF} = O(n_{mk} \cdot \lceil \log_2(\text{buckets}) \rceil) \)), and variable-width window slides (\( N_{CV} = O(n_{mk} \cdot \log_2(N_M)) \)).

For our time bounds, we will additionally assume that each Map, Combine, and Reduce function performs work that is asymptotically linear in the size of their inputs. Furthermore, we will assume that the Combine function is monotonic, i.e., it produces an output that is no larger than its input. This assumption is satisfied in most applications, because Combiners often reduce the size of the data (e.g., a Combine function to compute the sum of values takes multiple values and outputs a single value).

**Theorem 1** (Initial Run: Time and Overhead).
Assuming that Map, Combine, and Reduce functions take time asymptotically linear in their input size and that Combine functions are monotonic, total time for performing an incremental MapReduce computation in Slider with an input of size \( n_i \), where \( n_{mk} \) key-value pairs are emitted by the Map phase is \( O(N_M + (N_R + N_C)) = O(n_i + n_m) \). This results in an overhead of \( O(N_C) = O(N_{CA}||N_{CF}||N_{CV}) \) over conventional MapReduce.

**Proof.** The number of Map and Reduce tasks in a particular job can be derived from the input size and the number of distinct keys that are emitted by the Map function: the Map function is applied to splits that consist of one or more input chunks, and each application of the Map function is performed by one Map task. Hence, the number of Map tasks \( N_M \) is in the order of input size \( O(n_i) \). In the Reduce phase, each Reduce task processes all previously emitted key-value pairs for at least one key, which results in at most \( N_R = n_{mk} \) reduce tasks. To bound the number of self-adjusting balanced tree, we note that the tree leaves are the output data chunks of the Map phase, whose internal nodes each has at least two children. Since there are at most \( n_m \) pairs output by the Map phase, the total number of reduce tasks is bounded by \( n_m \).
Hence the total number of stages in self-adjusting balanced tree is bounded by $N_C \in O(n)$. Since the number of reduce tasks is bounded by $n_{mk} \leq n_m$, the total number of tasks is $O(n_i + n_m)$.

**Theorem 2 (Initial Run: Space).** Total storage space for performing an Slider computation with an input of size $n_i$, where $n_{mk}$ key-value pairs are emitted by the Map phase, and where Combine is monotonic is $O(n_m)$.

**Proof.** Slider requires additional storage space for storing the intermediary output of the self-adjusting balanced tree. Since Slider only keeps data from the most recent run (initial or dynamic run), we use storage for remembering only the task output from the most recent run. The output size of the map tasks is bounded by $n_{mk}$. With monotonic Combine functions, the size of the output of Combine tasks is bounded by $O(n_m)$.

**Theorem 3 (Dynamic Update: Space and Time).** In Slider, a dynamic update requires time, where $F$ where $F$ is the set of changed or new (fresh) Map, Combiner, and Reduce tasks, is

$$O \left( \sum_{a \in F} t(a) \right).$$

The total storage requirement is the same as an initial run.

**Proof.** Consider Slider performing an initial run with input $I$ and changing the input to $I'$ and then performing a subsequent run (dynamic update). During the dynamic update, tasks with the same type and input data will re-use the memoized result of the previous runs, avoiding recomputation. Thus, only the fresh tasks need to be executed, which takes $O \left( \sum_{a \in F} t(a) \right)$, where $F$ is the set of changed or new (fresh) Map, Contract and Reduce tasks, respectively, and $t(\cdot)$ denotes the processing time for a given task.

In the common case, we expect the execution of fresh tasks to dominate the time for dynamic updates. The time for dynamic update is therefore likely to be determined by the number of fresh tasks that are created as a result of a dynamic change. It is in general difficult to bound the number of fresh tasks, because it depends on the specifics of the application. As a trivial example, consider, inserting a single key-value pair into the input. In principle, the new pair can force the Map function to generate a very large number of new key-value pairs, which can then require performing many new reduce tasks. In many cases, however, small changes to the input lead only to small changes in the output of the Map, Combine, and Reduce functions, e.g., the Map function can use one key-value pair to generate several new pairs, and the Combine function will typically combine these, resulting in a relatively small number of fresh tasks. As a specific case, assume that the Map function generates $k$ key-value pairs from a single input record, and that the Combine function monotonically reduces the number of key-value pairs.

**Theorem 4 (Number of Fresh Tasks).** If the Map function generates $k$ key-value pairs from a single input record, and the Combine function is monotonic, then the number of fresh tasks is at most $O(k \log n_m + k)$.

**Proof.** At most $k$ combine at each level of the self-adjusting tree will be fresh, and $k$ fresh reduce tasks will be needed. Since the depth of the contraction tree is $n_m$, the total number of fresh tasks will therefore be $O(k \log n_m + k) = O(k \log n_m)$.

Taken together the last two theorems suggest that small changes to data will lead to the execution of only a small number of fresh tasks, and based on the tradeoff between the memoization costs and the cost of executing fresh tasks, speedups can be achieved in practice.

**References**

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