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these two extremal cases. Accordingly, the outer automorphism group \( \text{Out}(A_F) \) is often seen as interpolating between the arithmetic group \( \text{GL}_n(\mathbb{Z}) = \text{Out}(\mathbb{Z}^n) \) and \( \text{Out}(F_n) \). Over the last years, this point of view has served both as a motivation for studying automorphism groups of RAAGs and as a source of techniques improving our understanding of these groups.

This article contributes to this programme by providing a new geometric structure generalising well-studied complexes associated to arithmetic groups and automorphism groups of free groups. On the arithmetic side, we have the Tits building associated to \( \text{GL}_n(\mathbb{Q}) \). It can be defined as the order complex of the poset of proper subspaces of \( \mathbb{Q}^n \), ordered by inclusion, and is homotopy equivalent to a wedge of \((n - 2)\)-spheres (this is the Solomon–Tits theorem). On the \( \text{Out}(F_n) \) side, there is the free factor complex, which is defined as the order complex of the poset of conjugacy classes of proper free factors of \( F_n \), ordered by inclusion of representatives. This complex is homotopy equivalent to a wedge of \((n - 2)\)-spheres as well (see the work of Hatcher and Vogtmann [30] and of Brück and Gupta [10]). In this article, we construct a (simplicial) complex interpolating between these two structures.

It should be mentioned that the free factor complex was introduced in [30] in order to obtain an analogue of the more classical Tits building for the setting of \( \text{Out}(F_n) \). The same is true for the curve complex \( \mathcal{C}(S) \) associated to a surface \( S \), which was defined by Harvey [28] and shown to be homotopy equivalent to a wedge of spheres by Harer [27] and Ivanov [32] — it is an analogue of a Tits building for the setting of mapping class groups. In this sense, the complex we construct can also be seen as an \( \text{Out}(A_F) \)-analogue of \( \mathcal{C}(S) \).

Instead of looking at \( \text{Out}(A_F) \) itself, we will throughout work with its finite index subgroup \( \text{Out}_0(A_F) \), called the pure outer automorphism group. The group \( \text{Out}_0(A_F) \) was first defined by Charney, Crisp and Vogtmann in [13] and has since become popular as it avoids certain technical difficulties coming from automorphisms of the graph \( A_F \); if \( A_F \) is free or free abelian, we have \( \text{Out}_0(A_F) = \text{Out}(A_F) \). From now on, let \( O := \text{Out}_0(A_F) \).

Above, we described the building associated to \( \text{GL}_n(\mathbb{Q}) \) in terms of flags of subspaces of \( \mathbb{Q}^n \). However, one can also describe the building associated to a group \( G \) with BN-pair in a more intrinsic way using the parabolic subgroups of \( G \). We use this definition as an inspiration for our construction: Given \( O \), we define a family of maximal standard parabolic subgroups \( P(O) \). Every element of \( P(O) \) is a proper, non-trivial subgroup of \( O \) and defined as the stabiliser \( \text{Stab}_O(A_\Delta) \) of some special subgroup \( A_\Delta \leq A_F \) (for the precise statement, see Definition 6.3). The complex we consider now is the coset complex of \( O \) with respect to the family \( P(O) \), that is, the simplicial complex whose vertices are given by cosets \( gP \), with \( g \in O \) and \( P \in P(O) \), and where a collection of such cosets forms a simplex if and only if its intersection is non-empty. This complex is denoted by \( \text{CC}(O, P(O)) \) and \( O \) acts on it by left multiplication. The main result of this paper is the following.

**Theorem A.** The complex \( \text{CC} := \text{CC}(O, P(O)) \) is homotopy equivalent to a wedge of spheres of dimension \( |P(O)| - 1 \), where we call \( \text{rk}(O) := |P(O)| \) the rank of \( O \).

The rank \( \text{rk}(O) \) seems to be an interesting invariant of the group \( O \), which has, to the best of the author’s knowledge, not been studied in the literature so far.

To prove Theorem A, we are led to study relative versions of it, namely, we have to consider the case where \( O \) is not given by all of \( \text{Out}_0(A_F) \), but rather by a relative outer automorphism group \( O = \text{Out}_0(A_F; G, H') \) as defined by Day and Wade [19] (for the definitions, see Sections 5.1 and 5.2). This is why we prove all of the results mentioned in this introduction in that more general setting. In particular, we define a set of maximal standard parabolic subgroups \( P(O) \) and the rank \( \text{rk}(O) \) for all such \( O \).
In addition to Theorem A, we show that \( CC \) has the following properties which indicate that it is a reasonable analogue of Tits buildings and free factor complexes.

**Properties of \( CC \)**

- **Building.** If \( O = \text{GL}_n(\mathbb{Z}) \), the complex \( CC \) is isomorphic to the building associated to \( \text{GL}_n(\mathbb{Q}) \) (Proposition 4.3).

- **Free factor complex.** If \( O = \text{Out}(F_n) \), the complex \( CC \) is isomorphic to the free factor complex associated to \( F_n \) (Proposition 4.6).

- **Cohen–Macaulayness.** \( CC \) is homotopy Cohen–Macaulay and in particular a chamber complex (Proposition 8.2).

- **Facet-transitivity.** Any maximal simplex of \( CC \) forms a fundamental domain for the action of \( O \) (Section 3.1.3).

- **Stabilisers.** The vertex stabilisers of this action are exactly the conjugates of the elements of \( \mathcal{P}(O) \). Stabilisers of higher dimensional simplices are given by the intersections of such conjugates and can be seen as parabolic subgroups of lower rank (Section 8.2).

- **Parabolics as relative automorphism groups.** Every maximal standard parabolic \( P \in \mathcal{P}(O) \) is itself a relative automorphism group of the form \( \text{Out}^0(A_\Gamma; \zeta, \mathcal{H}^t) \) and \( \text{rk}(P) = |\mathcal{P}(P)| = \text{rk}(O) - 1 \) (Proposition 8.4).

- **Rank via Weyl group.** Similar to a group with BN-pair, the rank \( \text{rk}(O) \) is equal to the rank of a naturally defined Coxeter subgroup \( \text{Aut}^0(\Gamma) \leq O \) (Corollary 8.9).

- **Direct and free products.** The construction is well-behaved under taking direct and free products of the underlying RAAGs, that is, under passing from \( \text{Out}^0(A_\Gamma) \) to \( \text{Out}^0(A_\Gamma \times A_\Gamma') \) or to \( \text{Out}^0(A_\Gamma \ast A_\Gamma') \) (Section 7.2.3).

**Remark 1.1.** This article focuses on topological properties of \( CC \). This is the perspective that makes the complexes it generalises look quite similar, reflecting the fact that \( \text{Out}(F_n), \text{GL}_n(\mathbb{Z}) \) and mapping class groups of surfaces share many homological properties (see, for example, [17] and references therein). It should be noted that from a geometric perspective, there are significant differences between the associated complexes: While the free factor complex [3] and the curve complex [34] are hyperbolic, spherical buildings such as the one associated to \( \text{GL}_n(\mathbb{Z}) \) have finite diameter. In general, \( \text{Out}(A_\Gamma) \) has many aspects of \( \text{GL}_n(\mathbb{Z}) \), so one should probably not expect associated complexes to have a ‘purely hyperbolic’ flavour (cf. the work of Haettel [25]).

As an application of Theorem A and the results of [9], we obtain higher generating families of subgroups of \( O \) in the sense of Abels–Holz (for the definition, see Section 3.1.2).

**Theorem B.** The family \( \mathcal{P}_m(O) \) of rank-\( m \) parabolic subgroups of \( O \) is \( m \)-generating.

Here, an element of \( \mathcal{P}_m(O) \) is given by the intersection of \( \text{rk}(O) - m \) distinct elements from \( \mathcal{P}(O) \) (see Section 8.2). Higher generation can be interpreted as an answer to the question ‘How much information about \( O \) is contained in the family of subgroups \( \mathcal{P}_m(O) \)?’ — as an immediate consequence of this theorem, we are able to give for every \( 2 \leq m \leq \text{rk}(O) - 1 \) a presentation of \( O \) in terms of the rank-\( m \) parabolic subgroups (Corollary 8.6).

The main ingredient in our proof of Theorem A is an inductive procedure developed by Day–Wade. In [19], they show how to decompose \( O \) using short exact sequences into basic building blocks which consist of free abelian groups, \( \text{GL}_n(\mathbb{Z}) \) and so-called Fouxe-Rabinovitch groups, which are groups of certain automorphisms of free products. We refine their induction in order to
get a better control on the induction steps that are needed and to get a more explicit description of the resulting base cases. An overview of this can be found in Section 7.1. To make use of this inductive procedure, we establish a theorem regarding the behaviour of coset posets and complexes under short exact sequences. This generalises results of Brown [8], Holz [31] and Welsch [46] and can be phrased as follows.

**Theorem C.** Let $G$ be a group, $\mathcal{H}$ a family of subgroups of $G$ and $N \triangleleft G$ a normal subgroup. If $\mathcal{H}$ is strongly divided by $N$, there is a homotopy equivalence

$$CC(G, \mathcal{H}) \simeq CC(G/N, \overline{\mathcal{H}}) \ast CC(N, \mathcal{H} \cap N).$$

Here, $\ast$ denotes the join on geometric realisations, $\overline{\mathcal{H}}$ and $\mathcal{H} \cap N$ are certain families of subgroups of $G/N$ and $N$, respectively, and being strongly divided by $N$ is a compatibility condition on the family $\mathcal{H}$. (For the definitions, see Section 3.2; for an explicitly stated special case of Theorem C that we will use in this article, see Corollary 3.19.)

Note that if two spaces $X$ and $Y$ are homotopy equivalent to wedges of spheres, then so is their join $X \ast Y$. Thus, combining Theorem C with the decomposition of Day–Wade, we are able to reduce the question of sphericity of $\mathbb{C}$ to the cases where $O$ is either isomorphic to $GL_n(\mathbb{Z})$ or a Fouxe-Rabinovitch group. In the former case, sphericity follows from the Solomon–Tits theorem (Proposition 4.3). In the latter case, we are led to study relative versions of free factor complexes (see Definition 4.4) and show the following.

**Theorem D.** Let $A = F_n \ast A_1 \ast \cdots \ast A_k$ be a finitely generated group. Then the complex of free factors of $A$ relative to $\{A_1, \ldots, A_k\}$ is homotopy equivalent to a wedge of spheres of dimension $n-2$.

In the case where the group $A$ is a RAAG, Theorem D is a special case of Theorem A. However, we prove it without making this assumption by using the techniques of [10].

**Structure of the article**

Many sections of this article can be read independently from the others. We start in Section 2 by recalling some well-known results from topology that we will use throughout the paper. Section 3 contains definitions and basic properties of coset complexes and higher generating subgroups as well as the proof of Theorem C; it can be read completely independently from the rest of this text. The reader not so much interested in details about coset complexes might just want to skim Section 3.1 and then have a look at Corollary 3.19, which summarises the results of this section in the way they will be used later on. In Section 4, we give a definition of the building associated to $GL_n(\mathbb{Z})$ and determine the homotopy type of relative free factor complexes (Theorem D). A reader willing to take this on faith may just have a look at the main results of this section, namely, Theorems 4.20 and 4.21; the general theory of automorphisms of RAAGs is still not needed for this. Section 5 contains background about (relative) automorphism groups of RAAGs. Section 6 is in some sense the core of this article: We define (maximal) parabolic subgroups and the rank of $O$ and combine the results of the previous sections in order to prove Theorem A. In Section 7, we summarise to which extent we can refine the inductive procedure of Day–Wade and give examples of our construction for specific graphs $\Gamma$. In Section 8.1, we show Cohen–Macaulayness of $CC$, define parabolic subgroups of lower rank and prove Theorem B. We then show how the dimension of our complex is related to the rank of a Coxeter subgroup of $O$ (see Corollary 8.9). We close with comments about the limitations of our construction and open questions in Section 9.
2 | PRELIMINARIES ON (POSET) TOPOLOGY

2.1 | Posets and their realisations

Let \( P = (P, \leq) \) be a poset (partially ordered set). If \( x \in P \), the sets \( P_{\leq x} \) and \( P_{\geq x} \) are defined by

\[
P_{\leq x} := \{ y \in P \mid y \leq x \}, \quad P_{\geq x} := \{ y \in P \mid y \geq x \}.
\]

Similarly, one defines \( P_{< x} \) and \( P_{> x} \). For \( x, y \in P \), the open interval between \( x \) and \( y \) is defined as

\[
(x, y) := \{ z \in P \mid x < z < y \}.
\]

A chain of length \( l \) in \( P \) is a totally ordered subset \( x_0 < x_1 < \cdots < x_l \). For each poset \( P = (P, \leq) \), one has an associated simplicial complex \( \Delta(P) \) called the order complex of \( P \). Its vertices are the elements of \( P \) and higher dimensional simplices are given by the chains of \( P \). When we speak about the realisation of the poset \( P \), we mean the geometric realisations of its order complex and denote this space by \( \|P\| := \|\Delta(P)\| \). By an abuse of notation, we will attribute topological properties (for example, homotopy groups and connectivity properties) to a poset when we mean that its realisation has these properties.

The join of two posets \( P \) and \( Q \), denoted \( P \ast Q \), is the poset whose elements are given by the disjoint union of \( P \) and \( Q \) equipped with the ordering extending the orders on \( P \) and \( Q \) and such that \( p < q \) for all \( p \in P, q \in Q \). The geometric realisation of the join of \( P \) and \( Q \) is homeomorphic to the topological join of their geometric realisations:

\[
\|P \ast Q\| \simeq \|P\| \ast \|Q\|.
\]

The direct product \( P \times Q \) of two posets \( P \) and \( Q \) is the poset whose underlying set is the Cartesian product \( \{(p, q) \mid p \in P, q \in Q\} \) and whose order relation is given by

\[
(p, q) \leq_{P \times Q} (p', q') \text{ if } p \leq_P p' \text{ and } q \leq_Q q'.
\]

A map \( f : P \to Q \) between two posets is called a poset map if \( x \leq y \) implies \( f(x) \leq f(y) \). Such a poset map induces a simplicial map from \( \Delta(P) \) to \( \Delta(Q) \) and hence a continuous map on the realisations of the posets. It will be denoted by \( \|f\| \) or just by \( f \) if what is meant is clear from the context.

2.2 | Fibre theorems

An important tool to study the topology of posets is given by so-called fibre lemmas comparing the connectivity properties of posets \( P \) and \( Q \) by analysing the fibres of a poset map between them. The first such fibre theorem appeared in [36, Theorem A] and is known as Quillen’s fibre lemma.

**Lemma 2.1** [37, Proposition 1.6]. Let \( f : P \to Q \) be a poset map such that the fibre \( f^{-1}(Q_{\leq x}) \) is contractible for all \( x \in Q \). Then \( f \) induces a homotopy equivalence on geometric realisations.
The following result shows that if one is given a poset map $f$ such that the fibres have only vanishing homotopy groups up to a certain degree, one can also transfer connectivity results between the domain and the image of $f$. Recall that for $n \in \mathbb{N}$, a space $X$ is $n$-connected if $\pi_i(X) = \{1\}$ for all $i \leq n$ and $X$ is $(-1)$-connected if it is non-empty.

**Lemma 2.2** [37, Proposition 7.6]. Let $f : P \to Q$ be a poset map such that the fibre $f^{-1}(Q \leq x)$ is $n$-connected for all $x \in Q$. Then $P$ is $n$-connected if and only if $Q$ is $n$-connected.

For a poset $P = (P, \leq)$, let $P^{op} = (P, \leq_{op})$ be the poset that is defined by $x \leq_{op} y : \iff y \leq x$. Using the fact that one has a natural identification $\Delta(P) \cong \Delta(P^{op})$, one can draw the same conclusion as in the previous lemmas if one shows that $f^{-1}(Q \geq x)$ is contractible or $n$-connected, respectively, for all $x \in Q$.

Another standard tool which is helpful for studying the topology of posets is as follows.

**Lemma 2.3** [37, 1.3]. If $f, g : P \to Q$ are poset maps that satisfy $f(x) \leq g(x)$ for all $x \in P$, then they induce homotopic maps on geometric realisations.

Later on, we will mostly use the following consequence of this lemma.

**Corollary 2.4.** Let $P'$ be a subposet of $P$ and $f : P \to P'$ a poset map such that $f|_{P'} = \text{id}_{P'}$. If $f$ is monotone, that is, $f(x) \leq x$ for all $x \in P$ or $f(x) \geq x$ for all $x \in P$, then it defines a deformation retraction $\|P\| \to \|P'\|$.

**Proof.** Without loss of generality, assume that $f(x) \leq x$ for all $x \in P$. Let $i : P' \hookrightarrow P$ denote the inclusion map. Then for all $x \in P$, we have $i \circ f(x) \leq x$, so by Lemma 2.3, this composition is homotopic to the identity. As $f \circ i = \text{id}_{P'}$, the inclusion $i$ is a homotopy equivalence and the claim follows from [29, Proposition 0.19].

### 2.3 Spherical complexes and their joins

Recall that a topological space is $n$-spherical if it is homotopy equivalent to a wedge of $n$-spheres; as a convention, we consider a contractible space to be homotopy equivalent to a (trivial) wedge of $n$-spheres for all $n$ and the empty set to be $(-1)$-spherical. By the Whitehead theorem, an $n$-dimensional CW-complex is $n$-spherical if and only if it is $(n-1)$-connected. Furthermore, sphericity is preserved under taking joins.

**Lemma 2.5.** Let $X$ and $Y$ be CW-complexes such that $X$ is $n$-spherical and $Y$ is $m$-spherical. Then the join $X \ast Y$ is $(n + m + 1)$-spherical.

### 2.4 The Cohen–Macaulay property

**Definition 2.6.** Let $X$ be a simplicial complex of dimension $d < \infty$. Then $X$ is homotopy Cohen–Macaulay if it is $(d-1)$-connected and the link of every $s$-simplex is $(d-s-2)$-connected.
The word ‘homotopy’ here refers to the original ‘homological’ notion of being ‘Cohen–Macaulay over a field $k$’. This homological condition is weaker than the homotopical one and came up in the study of finite simplicial complexes via their Stanley–Reisner rings (see [43]). For more details on Cohen–Macaulayness and its connections to other combinatorial properties of simplicial complexes, see [4].

3 | COSET POSETS AND COSET COMPLEXES

3.1 | Definitions and basic properties

Standing assumptions

Throughout this section, let $G$ be a group and let $H$ be a family of proper subgroups of $G$.

3.1.1 | Background and relation between poset and complex

Definition 3.1. Let $X$ be a set and $U$ be a collection of subsets of $X$ such that $U$ covers $X$. Then the nerve $N(U)$ of the covering $U$ is the simplicial complex that has vertex set $U$ and where the vertices $U_0, ..., U_k \in U$ form a simplex if and only if $U_0 \cap \cdots \cap U_k \neq \emptyset$.

Definition 3.2. Define $U := \bigsqcup_{H \in H} G/H$ to be the collection of cosets of the subgroups from $H$.

(1) The coset poset $CP(G, H) := (U, \subseteq)$ is the partially ordered set consisting of the elements of $U$, where $g_1 H_1 \leq g_2 H_2$ if and only if $g_1 H_1 \subseteq g_2 H_2$.
(2) The coset complex $CC(G, H)$ is the nerve $N(U)$ of the covering of $G$ given by $U$.

In this form, coset complexes were introduced by Abels and Holz in [1] but they appear with different names in several branches of group theory: The main motivation of Abels and Holz was to study finiteness properties of groups. Recent work in this direction can be found in the work of Bux, Fluch, Marschler, Witzel and Zaremsky [11] and Santos-Rego [39]. In [35], Meier, Meinert and VanWyk used these complexes to study the BNS invariants of right-angled Artin groups. Well-known examples of coset posets are given by Coxeter and Deligne complexes [14]. Brown [8] studied the coset poset of all subgroups of a finite group and its connection to zeta functions. Generalisations of his work can be found in the articles of Ramras [38] and Shareshian and Woodroofe [41]. However, the examples that are most important to the present work are given by Tits buildings and free factor complexes (see Section 4).

The order complex of the coset poset $CP(G, H)$ has the same vertices as the coset complex $CC(G, H)$ but the higher dimensional simplices do not have to agree (see Figure 1). However, if we assume that $H$ be closed under finite intersections, the topology of these complexes is the same.

Lemma 3.3. Suppose that $H_1, H_2 \in H$ implies $H_1 \cap H_2 \in H$. Then $CC(G, H)$ deformation retracts to $CP(G, H)$. In particular, we have

$$CP(G, H) \simeq CC(G, H).$$
The left-hand side shows $\text{CC}(\mathbb{Z}, \mathcal{H})$ and $\text{CP}(\mathbb{Z}, \mathcal{H})$, the right-hand side $\text{CC}(\mathbb{Z}, \tilde{\mathcal{H}})$ and $\text{CP}(\mathbb{Z}, \tilde{\mathcal{H}})$, where $\mathcal{H} = \{2\mathbb{Z}, 3\mathbb{Z}\}$ and $\tilde{\mathcal{H}} = \{2\mathbb{Z}, 3\mathbb{Z}, 6\mathbb{Z}\}$. In both pictures, the coset poset is drawn in black and the coset complex is obtained from it by adding the blue parts.

**Proof.** As $\mathcal{H}$ is closed under intersections, the intersection of two cosets from $\mathcal{U}'$ is either empty or also an element of $\mathcal{U}'$. Hence, we can define a map from the poset of simplices of $\text{CC}(G, \mathcal{H})$ to $\text{CP}(G, \mathcal{H})$ by sending $(g_0 H_0, \ldots, g_k H_k)$ to $\bigcap_i g_i H_i$. On the corresponding order complexes, this defines a deformation retraction from the barycentric subdivision of $\text{CC}(G, \mathcal{H})$ to $\Delta \text{CP}(G, \mathcal{H})$ (see [1, Theorem 1.4(b)]). See Figure 1 for an easy example. □

Let $\tilde{\mathcal{H}}$ denote the family consisting of all finite intersections of elements from $\mathcal{H}$. The following was proved by Holz in [31].

**Lemma 3.4.**

1. Let $\mathcal{H}'$ be a family of subgroups of $G$ with $\mathcal{H} \subseteq \mathcal{H}'$ and such that for all $H' \in \mathcal{H}'$, there is $H \in \mathcal{H}$ with $H' \subseteq H$. Then there is a homotopy equivalence $\text{CC}(G, \mathcal{H}) \simeq \text{CC}(G, \mathcal{H}')$.

2. There is a homotopy equivalence $\text{CC}(G, \mathcal{H}) \simeq \text{CP}(G, \tilde{\mathcal{H}})$.

**Proof.** The nerve $N(\mathcal{U}')$ of a collection $\mathcal{U}'$ of subsets of $X$ is homotopy equivalent to the simplicial complex whose simplices are the non-empty finite subsets of $X$ contained in some $U \in \mathcal{U}'$ (see [1, Theorem 1.4(a)]). This implies the first claim. The second statement is an immediate consequence of the first one. □

**Remark 3.5.** The preceding lemmas imply that for any family $\mathcal{H}$ of subgroups of $G$, we have

$$\text{CC}(G, \mathcal{H}) \simeq \text{CC}(G, \tilde{\mathcal{H}}) \simeq \text{CP}(G, \tilde{\mathcal{H}}).$$

It follows that we can always replace a coset complex by a coset poset. The advantage of this is that it allows us to apply the tools of poset topology, for example, the Quillen fibre lemma, to study the topology of these complexes. The trade-off, however, is that we have to increase the size of our family of subgroups.

### 3.1.2 Higher generation

We now turn our attention to coset complexes.
Definition 3.6. The free product of $\mathcal{H}$ amalgamated along its intersections is the group given by the presentation $(X \mid R)$, where $X = \{x_g \mid g \in \bigcup \mathcal{H}\}$ and $R = \{x_gx_hx_{gh}^{-1} \mid \exists H \in \mathcal{H} : g, h \in H\}$.

Definition 3.7. We say that $\mathcal{H}$ is $n$-generating for $G$ if $CC(G, \mathcal{H})$ is $(n - 1)$-connected, that is, $\pi_i(CC(G, \mathcal{H})) = \{1\}$ for all $i < n$.

The term ‘higher generating subgroups’ was coined by Holz in [31] and is motivated by the following theorem.

Theorem 3.8. [1, Theorem 2.4]

1. $\mathcal{H}$ is 1-generating if and only if $\bigcup \mathcal{H}$ generates $G$.
2. $\mathcal{H}$ is 2-generating if and only if $G$ is the free product of $\mathcal{H}$ amalgamated along its intersections.

Roughly speaking, the latter means that the union of the subgroups in $\mathcal{H}$ generates $G$ and that all relations that hold in $G$ follow from relations in these subgroups. The concept of 3-generation has a similar interpretation using identities among relations (see [1, 2.8]).

3.1.3 | Group actions and detecting coset complexes

Coset complexes are endowed with a natural action of $G$ given by left multiplication. These complexes are highly symmetric in the sense that this action is facet transitive: Assume that $\mathcal{H}$ is finite. Then $CC(G, \mathcal{H})$ has dimension $|\mathcal{H}| - 1$ and $\mathcal{H}$ itself is the vertex set of a facet, that is, a maximal simplex, of the coset complex. This (and hence any other) facet is a strict fundamental domain for the action of $G$. The following converse of this observation is due to Zaremsky.

Proposition 3.9 (see [11, Proposition A.5]). Let $G$ be a group acting by simplicial automorphisms on a simplicial complex $X$, with a single facet $C$ as a strict fundamental domain. Let

$$P := \{\text{Stab}_G(v) \mid v \text{ is a vertex of } C\}.$$ 

Then the map

$$\psi : CC(G, P) \to X$$

$$g \text{Stab}_G(v) \mapsto g.v$$

is an isomorphism of simplicial $G$-complexes.

3.2 | Short exact sequences

We will later on study coset complexes in the setting where $G = \text{Out}(A_T)$, the outer automorphism group of a right-angled Artin group. For this, we want to use the decomposition sequences of $\text{Out}(A_T)$ developed in [19]. To do so, we need to study the following question: If $G$ fits into a short
exact sequence, can the coset complex $\text{CC}(G, \mathcal{H})$ be decomposed into ‘simpler’ complexes related to the image and kernel of the sequence? There is a special case where this question can easily be answered.

**Coset complexes and direct products**

Assume that we have a group factoring as a direct product $G = G_1 \times G_2$ and let $\mathcal{H}$ be a family of subgroups such that each $H \in \mathcal{H}$ contains either $\{1\} \times G_2$ or $G_1 \times \{1\}$; denote the set of those elements of $\mathcal{H}$ satisfying the former by $\mathcal{H}_1$ and the set of those satisfying the latter by $\mathcal{H}_2$. Now given $H_1, H'_1 \in \mathcal{H}_1$, we have

$$(g_1, g_2) \cdot H_1 \cap (g'_1, g'_2) \cdot H'_1 \neq \emptyset$$

$\iff (g_1, 1) \cdot H_1 \cap (g'_1, 1) \cdot H'_1 \neq \emptyset$

$\iff g_1 \cdot p_1(H_1) \cap g'_1 \cdot p_1(H'_1) \neq \emptyset,$

where $p_1$ is the projection map $G \to G_1$. The analogous statement holds for $H_2, H'_2 \in \mathcal{H}_2$. On the other hand, if we take $H_1 \in \mathcal{H}_1$ and $H_2 \in \mathcal{H}_2$, all of their cosets intersect non-trivially because

$$(g_1, g_2) \cdot H_1 = (g_1, g'_2) \cdot H_1 \quad \text{and} \quad (g'_1, g'_2) \cdot H_2 = (g_1, g'_2) \cdot H_2.$$ 

It follows that the coset complex $\text{CC}(G, \mathcal{H})$ decomposes as a join

$$\text{CC}(G, \mathcal{H}) \cong \text{CC}(G_1, p_1(\mathcal{H}_1)) \ast \text{CC}(G_2, p_2(\mathcal{H}_2)).$$

However, the situation becomes more complicated if we consider semi-direct products or general short exact sequences

$$1 \to N \to G \to Q \to 1.$$ 

References [31, Proposition 5.17; 46, Theorem 7.3; 8, Proposition 10] contain results in this direction for the cases where every $H \in \mathcal{H}$ is a complement of $N$, every $H \in \mathcal{H}$ contains $N$ and where $G$ is a finite group and $\mathcal{H}$ is the set of all subgroups of $G$, respectively. Our work in this section provides a common generalisation of all three of these results (see Theorem 3.18).

**Notation and standing assumptions**

From now on, we will fix a normal subgroup $N \triangleleft G$ and assume that $\mathcal{H}$ is a set of proper subgroups of $G$. In this situation, we can write $\mathcal{H}$ as a disjoint union $\mathcal{H} = \mathcal{H}_N \sqcup \mathcal{H}^N$, where

$$\mathcal{H}_N := \{ H \in \mathcal{H} \mid HN \neq G \} \quad \text{and} \quad \mathcal{H}^N := \{ K \in \mathcal{H} \mid KN = G \}.$$ 

For elements $g \in G$ and subgroups $H \trianglelefteq G$ of $G$, let $\bar{g}$ and $\bar{H}$ denote the image of $g$ and $H$ in the quotient $G/N$, respectively.

The family $\mathcal{H}_N$ gives rise to a family of proper subgroups of $G/N$, denoted by

$$\bar{\mathcal{H}} := \left\{ \bar{H} \mid H \in \mathcal{H}_N \right\}.$$
Similarly, $\mathcal{H}^N$ gives rise to a family of proper subgroups of $N$, denoted by

$$\mathcal{H} \cap N := \{ K \cap N \mid K \in \mathcal{H}^N \}.$$ 

### 3.2.1 Coset posets and short exact sequences

We start by considering the behaviour of coset posets under short exact sequences.

**Definition 3.10.** The family $\mathcal{H}$ of proper subgroups of $G$ is divided by $N$ if the following holds true.

1. For all $H \in \mathcal{H}_N$, one has $HN \in \mathcal{H}$.
2. For all $H \in \mathcal{H}_N$ and $K \in \mathcal{H}^N$, one has $HN \cap K \in \mathcal{H}$.

In what follows, we will use the following elementary observations.

**Lemma 3.11.** Let $H, K \leq G$ be two subgroups of $G$ and assume that $KN = G$. Then one has $(HN \cap K) \cdot N = HN$.

**Proof.** Obviously, $(HN \cap K) \cdot N$ is contained in $HN$. We claim that in fact, these sets are equal. Indeed, as $KN = G$, each $hn \in HN$ can be written as $hn = kn'$ with $k \in K$ and $n' \in N$. As $k = hnn'^{-1}$, it is contained in $HN \cap K$. Hence, $hn = kn' \in (HN \cap K) \cdot N$. $\square$

**Lemma 3.12.** Let $H \in \mathcal{H}_N$, $K \in \mathcal{H}^N$ and $g \in G$. If $\mathcal{H}$ is divided by $N$, then

$$(g \cdot HN) \cap K = k \cdot (HN \cap K)$$

for some $k \in K$. Furthermore, $(HN \cap K) \in \mathcal{H}_N$.

**Proof.** As $G = KN$, we can write $g = kn$ with $n \in N$ and $k \in K$. The intersection

$$(g \cdot HN) \cap K = (kn \cdot HN) \cap K = (k \cdot HN) \cap K$$

contains $k$, so it is equal to $k \cdot (HN \cap K)$. That $HN \cap K$ is contained in $\mathcal{H}$ is clear because $\mathcal{H}$ is divided by $N$; that it is contained in $\mathcal{H}_N$ is a consequence of Lemma 3.11. $\square$

The next proposition is a generalisation of [8, Proposition 10]. Our proof closely follows the ideas of Brown.

**Proposition 3.13.** If $\mathcal{H}$ is divided by $N$, then there is a homotopy equivalence

$$\text{CP}(G, \mathcal{H}) \simeq \text{CP}(G/N, \overline{\mathcal{H}}) \ast \text{CP}(G, \mathcal{H}^N).$$

**Proof.** Set $C := \text{CP}(G, \mathcal{H}), C_N := \text{CP}(G, \mathcal{H}_N)$ and $C^N := \text{CP}(G, \mathcal{H}^N)$. We define a map

$$f : C \rightarrow \text{CP}(G/N, \overline{\mathcal{H}}) \ast C^N$$
such that \( f \) restricts to the identity on \( C^N \) and \( f(gH) = \overline{gH} \) for all \( gH \in C_N \). As no coset from \( C^N \) can be contained in a coset from \( C_N \), this map is order-preserving, that is, a poset map. For \( x \in \text{CP}(G/N, \overline{H}) \ast C^N \), define
\[
F := f^{-1}(\langle \text{CP}(G/N, \overline{H}) \ast C^N \rangle_{\leq x})
\]
to be the fibre of \( x \) with respect to \( f \). We want to use Lemma 2.1 to show that \( f \) is a homotopy equivalence. For this, we need to show that \( F \) is contractible.

If \( x \in \text{CP}(G/N, \overline{H}) \), this is clear: Write \( x = \overline{gH} \) such that \( g \in G, H \in H_N \). As \( N \) divides \( H \), the subgroup \( HN \) is contained in \( H \) and \( g \cdot HN \) is the unique maximal element of \( F \). This immediately implies contractibility of \( F \).

Now assume \( x \in C^N \). Using the natural action of \( G \) on these posets, we can assume that \( x = K \in H^N \). By definition of the join, the poset \( F \) can as be written as \( F = C_N \cup C \leq K \). On the level of geometric realisations, it decomposes as

\[
\|F\| = \|C_N\| \cup \|C'\| \cup \|C \leq K\|,
\]
where \( C' := C_N \cap C \leq K \) is equal to \( (C_N) \leq K \). (To see this, note that no coset from \( C^N \) can be contained in a coset from \( C_N \) and that if \( gH \in C_N \) is contained in some \( g'H \in C \leq K \), we have \( gH \in C' \).) Next, we show that \( \|C'\| \) is a strong deformation retract of \( \|C_N\| \). This implies that \( F \) is homotopy equivalent to \( C \leq K \), which is contractible as it has \( K \) as unique maximal element.

The poset \( C' \) is given by all cosets \( gH \subseteq K \) such that \( H \in H_N \). Hence, Lemma 3.12 implies that for \( gH \in C_N \), the intersection \( (g \cdot HN) \cap K \) is an element of \( C' \). This allows us to define poset maps
\[
\phi : C_N \to C' \quad \text{and} \quad \psi : C' \to C_N
\]
\[
gH \mapsto (g \cdot HN) \cap K \quad \text{and} \quad gH \mapsto g \cdot HN.
\]
For \( gH \in C' \), we have \( gH \subseteq K \), hence
\[
\phi \circ \psi(gH) = (g \cdot HN) \cap K \supseteq gH \cap K = gH.
\]
If on the other hand \( gH \in C_N \), one has by Lemma 3.12
\[
\psi \circ \phi(gH) = ((g \cdot HN) \cap K) \cdot N
\]
\[
= k \cdot (HN \cap K) \cdot N
\]
for some \( k \in g \cdot HN \cap K \). By Lemma 3.11, we have \( (HN \cap K) \cdot N = HN \), so it follows that \( \psi \circ \phi(gH) = k \cdot HN \supseteq gH \). Lemma 2.3 now implies that \( \phi \) and \( \psi \) are homotopy equivalences which are inverse to each other. Furthermore, we have \( gH \subseteq \psi(gH) \) for all \( gH \in C' \), so again by Lemma 2.3, the map \( \psi \) is homotopic to the inclusion \( C' \hookrightarrow C_N \) which must hence be a homotopy equivalence as well. It follows that \( \|C'\| \) is a strong deformation retract of \( \|C_N\| \). \( \square \)
3.2.2 Coset complexes and short exact sequences

We will now translate the results obtained in the last section to coset complexes. The following observation follows from elementary group theory.

**Lemma 3.14.** Let $K_1 \neq K_2$ be subgroups of $G$ such that $G = (K_1 \cap K_2)N$. Then one has $K_1 \cap N \neq K_2 \cap N$.

We obtain the following relation between $CC(G, H^N)$ and $CC(N, H \cap N)$.

**Lemma 3.15.** Assume that for every finite collection $K_1, \ldots, K_m \in H^N$, one has $(K_1 \cap \ldots \cap K_m)N = G$. Then there is an isomorphism

$$CC(G, H^N) \cong CC(N, H \cap N).$$

**Proof.** As $G = KN = NK$ for all $K \in H^N$, each vertex of $CC(G, H^N)$ can be written as $nK$ with $n \in N$. Use this to define the map

$$\psi : CC(G, H^N) \to CC(N, H \cap N)$$

$$nK \mapsto n \cdot K \cap N,$$

which we claim is an isomorphism of simplicial complexes.

As $n \in N$, this map is well-defined on vertices. It also clearly is surjective on vertices. Now assume that for $n_1, n_2 \in N$ and $K_1, K_2 \in H^N$, one has $n_1 \cdot K_1 \cap N = n_2 \cdot K_2 \cap N$. As the two cosets coincide, so do the subgroups $K_1 \cap N = K_2 \cap N$. By Lemma 3.14, this implies that $K_1 = K_2$. It follows in particular that $n_1 K_1 = n_2 K_2$ which shows that $\psi$ defines a bijection between the vertex sets of the two coset complexes.

To see that $\psi$ is a simplicial map which defines a bijection between the set of simplices of the two complexes, take $n_1, \ldots, n_m \in N$ and $K_1, \ldots, K_m \in H^N$ and consider the following chain of equivalences:

$$\bigcap_i n_i K_i \neq \emptyset$$

$$\Leftrightarrow \exists g \in G : \bigcap_i n_i K_i = \bigcap_i gK_i = g \bigcap_i K_i$$

$$\Leftrightarrow \exists n \in N : \bigcap_i n_i K_i = n \bigcap_i K_i$$

$$\Leftrightarrow \emptyset \neq \left( \bigcap_i n_i K_i \right) \cap N = \bigcap_i n_i (K_i \cap N),$$

where * follows because $G = N(K_1 \cap \ldots \cap K_m)$.

This motivates the following definition.
Definition 3.16. The family $\mathcal{H}$ of proper subgroups of $G$ is \textit{strongly divided by} $N$ if the following holds true.

1. For all $H \in \mathcal{H}_N$, one has $N \subseteq H$.
2. For all $K_1, \ldots, K_m \in \mathcal{H}^N$, one has $(K_1 \cap \cdots \cap K_m)N = G$.

Using Lemma 3.11, it is easy to see that every family of subgroups which is strongly divided by $N$ is also divided by $N$. On top of that, given a family which is strongly divided, we can even produce a family which is closed under intersections and still divided by $N$ as the following lemma shows. Recall that $\mathcal{H}$ denotes the family of all finite intersections of elements from $\mathcal{H}$.

Lemma 3.17. If $\mathcal{H}$ is strongly divided by $N$, the family $\overline{\mathcal{H}}$ is divided by $N$. Furthermore, we have the following.

1. $\overline{\mathcal{H}}^N$ is equal to the family of all finite intersections of elements from $\mathcal{H}^N$, that is,
   $$\overline{\mathcal{H}}^N = \overline{\mathcal{H}}^N.$$

2. The image of $\overline{\mathcal{H}}_N$ in $G/N$ is equal to the family of finite intersections of elements from $\overline{\mathcal{H}}$, that is,
   $$\overline{\mathcal{H}} = \overline{\mathcal{H}}.$$

Proof. Every $\overline{H} \in \overline{\mathcal{H}}$ can be written as
   $$\overline{H} = H_1 \cap \cdots \cap H_n \cap K_1 \cap \cdots \cap K_m,$$
where for all $i$ and $j$, one has $N \subseteq H_i$ and $K_j \in \mathcal{H}^N$.

If $\overline{H} \in \overline{\mathcal{H}}^N$, we must have $n = 0$, that is, $\overline{H} = K_1 \cap \cdots \cap K_m$ is a finite intersection of elements from $\mathcal{H}^N$. On the other hand, every such finite intersection forms an element of $\overline{\mathcal{H}}^N$ because one has $(K_1 \cap \cdots \cap K_m)N = G$, which proves Item 1.

This also implies that if $\overline{H} \in \overline{\mathcal{H}}_N$, we have $n \geq 1$. It follows from Lemma 3.11 that $\overline{H}N$ is equal to $H_1 \cap \cdots \cap H_n$. This is a finite intersection of elements from $\mathcal{H}_N$ and hence contained in $\overline{H}$. Furthermore, this implies that the image $\overline{H}$ of $\overline{H}$ in $G/N$ is equal to $\overline{H} = \overline{H}_1 \cap \cdots \cap \overline{H}_n$, showing Item 2.

The last thing that remains to be checked is that $\overline{H}$ is divided by $N$, that is, that for all $\overline{H} \in \overline{\mathcal{H}}_N$ and $\overline{K} \in \overline{\mathcal{H}}^N$, one has $\overline{H}N \cap \overline{K} \in \overline{H}$. However, we already know that $\overline{H}N = H_1 \cap \cdots \cap H_n$, so $\overline{H}N \cap \overline{K}$ is itself a finite intersection of elements from $\mathcal{H}$. $\square$

We are now ready to prove Theorem C which we restate as follows.

Theorem 3.18. If $\mathcal{H}$ is strongly divided by $N$, there is a homotopy equivalence
   $$\text{CC}(G, \mathcal{H}) \simeq \text{CC}(G/N, \overline{\mathcal{H}}) \ast \text{CC}(N, \mathcal{H} \cap N).$$

Proof. It follows from Lemmas 3.3 and 3.4 that $\text{CC}(G, \mathcal{H})$ is homotopy equivalent to $\text{CP}(G, \overline{\mathcal{H}})$. Furthermore, Lemma 3.17 tells us that $\overline{\mathcal{H}}$ is divided by $N$. Hence, we can apply Proposition 3.13 to see that there is a homotopy equivalence
   $$\text{CP}(G, \overline{\mathcal{H}}) \simeq \text{CP}(G/N, \overline{\mathcal{H}}) \ast \text{CP}(G, \overline{\mathcal{H}}^N).$$
By Lemma 3.17, we have $\overline{H} = \overline{H}$. Hence, using Lemmas 3.3 and 3.4 again,

$$\text{CP}(G/N, \overline{H}) \simeq \text{CC}(G/N, \overline{H}) \simeq \text{CC}(G/N, \overline{H}).$$

On the other hand, Lemma 3.17 also tells us that $\overline{H}^N$ consists of all finite intersections of elements from $H^N$. It follows that

$$\text{CP}(G, \overline{H}^N) \simeq \text{CC}(G, \overline{H}^N) \simeq \text{CC}(G, H^N).$$

As $H$ is strongly divided by $N$, we can finally apply Lemma 3.15 and get that $\text{CC}(G, H^N) \simeq \text{CC}(N, H \cap N)$. \hfill \Box

### 3.2.3 Summary

We summarise the results of this section in the form that we will use later on.

**Corollary 3.19.** Let $G$ be a group and assume we have a short exact sequence

$$1 \to N \to G \xrightarrow{q} Q \to 1.$$  

Let $S$ be a set of generators for $G = \langle S \rangle$ and let $\mathcal{P}$ be a family of proper subgroups. Furthermore, assume that for all $P \in \mathcal{P}$, one of the following holds:

1. either $P$ contains the kernel $N = \ker q$, or
2. $P$ contains $S \setminus N$.

Then there is a homotopy equivalence

$$\text{CC}(G, \mathcal{P}) \simeq \text{CC}(Q, \overline{P}) \ast \text{CC}(N, \mathcal{P} \cap N),$$

where $\overline{P} = \{q(P) \mid P \in \mathcal{P}, N \subseteq P\}$ and $\mathcal{P} \cap N = \{P \cap N \mid P \in \mathcal{P}, S \setminus N \subseteq P\}$.

**Proof.** We stick with the notation and standing assumption defined in the paragraph before Section 3.2.1. If $P \in \mathcal{P}^N$, it cannot contain $N$. Hence, all such $P$ must contain the set $S \setminus N$ of elements from $S$ that are not contained in the kernel. It follows that for any $P_1, \ldots, P_m \in \mathcal{P}^N$, one has $(P_1 \cap \cdots \cap P_m)N = G$. On the other hand, for every $P \in \mathcal{P}_N$, our assumption implies that $N \subseteq P$. Hence, $P$ is strongly divided by $N$ and the claim follows from Theorem 3.18. \hfill \Box

### 4 THE BASE CASES: BUILDINGS AND RELATIVE FREE FACTOR COMPLEXES

In this section, we study complexes of parabolic subgroups associated to two particular families of (relative) automorphism groups: The first one is $GL_n(\mathbb{Z})$ (Section 4.1), the second one is given by so-called Fouxe-Rabinovitch groups (Section 4.2). On the one hand, these are special cases of the complexes we will consider in Section 6, on the other hand, they play a distinguished role.
because they appear as base cases of the inductive argument that we will use there. We show that in both situations, the complexes one obtains are spherical, but the methods for the two cases are quite different. In the first one, the result follows without much effort from the Solomon–Tits theorem while in the second one, we have to generalise the work of [10] to the ‘relative’ setting considered here.

4.1 The building associated to $\text{GL}_n(\mathbb{Z})$ and the Solomon–Tits theorem

The building associated to $\text{GL}_n(\mathbb{Q})$ is the order complex of the poset $Q$ of proper (that is, non-trivial and not equal to $\mathbb{Q}^n$) subspaces of $\mathbb{Q}^n$, ordered by inclusion.

This is a special case of a Tits building and a lot can be said about the structure of these simplicial complexes — we refer the reader to [2] for further details. However, the only non-trivial result about them that we need for this article is the following special case of the Solomon–Tits theorem.

**Theorem 4.1** [42]. The building associated to $\text{GL}_n(\mathbb{Q})$ is homotopy equivalent to a wedge of $(n - 2)$-spheres.

It is well-known that this building can equivalently be described as the coset complex of $\text{GL}_n(\mathbb{Q})$ with respect to the family of maximal standard parabolic subgroups. We will now show that it can also be described as a coset complex of $\text{GL}_n(\mathbb{Z}) = \text{Out}(\mathbb{Z}^n)$, an outer automorphism group of a RAAG.

A subgroup $A \leq \mathbb{Z}^n$ is called a direct summand if there is $B \leq \mathbb{Z}^n$ such that $\mathbb{Z}^n = A \oplus B$. We say that a direct summand $A$ is proper if it is neither trivial nor equal to $\mathbb{Z}^n$. Let $\mathcal{Z}$ be the poset of all proper direct summands of $\mathbb{Z}^n$, ordered by inclusion. The group $\text{GL}_n(\mathbb{Z})$ acts naturally on $\mathcal{Z}$.

Fix a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{Z}^n$ and for all $1 \leq i \leq n - 1$, set $S_i := \langle e_1, \ldots, e_i \rangle$. Note that $S_i \in \mathcal{Z}$ for all $i$ and define

$$P_i := \text{Stab}_{\text{GL}_n(\mathbb{Z})}(S_i)$$

to be the stabiliser of $S_i$ under the action of $\text{GL}_n(\mathbb{Z})$ on $\mathcal{Z}$. We define the set of maximal standard parabolic subgroups of $\text{GL}_n(\mathbb{Z})$ as

$$\mathcal{P} = \mathcal{P}(\text{GL}_n(\mathbb{Z})) := \{P_i \mid 1 \leq i \leq n - 1\}.$$

**Remark 4.2.** We called the elements of $\mathcal{P}$ the maximal standard parabolic subgroups of $\text{GL}_n(\mathbb{Z})$ to match the usual convention where an arbitrary parabolic subgroup is defined as the conjugate of a standard one. As we will, however, not work with non-standard parabolic subgroups in this article, we leave out this adjective from now on.

In terms of matrices, the maximal parabolic subgroups can be written in the form

$$P_i = \begin{pmatrix} \text{GL}_i(\mathbb{Z}) & M_{i,n-i}(\mathbb{Z}) \\ 0 & \text{GL}_{n-i}(\mathbb{Z}) \end{pmatrix} \leq \text{GL}_n(\mathbb{Z}).$$

**Proposition 4.3.** The building associated to $\text{GL}_n(\mathbb{Q})$ is $\text{GL}_n(\mathbb{Z})$-equivariantly isomorphic to the coset complex $\text{CC}(\text{GL}_n(\mathbb{Z}), \mathcal{P})$. 
Proof. Each $A \in \mathcal{Z}$ is isomorphic to $\mathbb{Z}^i$ for an integer $i := \text{rk}(A) \in \{1, \ldots, n - 1\}$, the rank of $A$. Furthermore, if $A \leq B$ in $\mathcal{Z}$, we have $\text{rk}(A) \leq \text{rk}(B)$ with equality if and only if $A$ and $B$ are equal. It follows that the maximal simplices of $\Delta(\mathcal{Z})$ are given by chains $A_1 \leq \cdots \leq A_{n-1}$, where $\text{rk}(A_i) = i$. The group $\text{GL}_n(\mathbb{Z})$ acts transitively on the set of all such chains and preserves the rank of each summand. Hence, the facet $S_1 \leq \cdots \leq S_{n-1}$ is a fundamental domain for this action and Proposition 3.9 implies that the order complex of $\mathcal{Z}$ is $\text{GL}_n(\mathbb{Z})$-equivariantly isomorphic to $\text{CC}(\text{GL}_n(\mathbb{Z}), \mathcal{P})$.

On the other hand, there is a poset map $f : \mathcal{Q} \to \mathcal{Z}$ defined by sending $V$ to $V \cap \mathbb{Z}^n$. This is a $\text{GL}_n(\mathbb{Z})$-equivariant isomorphism whose inverse is given by sending $A \leq \mathbb{Z}^n$ to its $\mathbb{Q}$-span $\langle A \rangle_{\mathbb{Q}}$ (see, for example, [18, Corollary 2.5]).

\[\square\]

4.2 Relative free factor complexes

The aim of this section is to generalise [10, Theorem A] which states that the complex of free factors of the free group $F_n$ is homotopy equivalent to a wedge of $(n-2)$-spheres. We want to extend this result to certain complexes of free factors of a free product $A = F_n \ast A_1 \ast \cdots \ast A_k$.

After adapting the definitions to this setting, the proofs of [10] largely go through without major changes. We still include most of them here in order to make this section as self-contained as possible.

4.2.1 Relative automorphism groups and relative Outer space

Relative automorphism groups

Let $A$ be a countable group. We will often use capital letters for elements from the outer automorphism group of $A$ and lowercase letters for the corresponding representatives from the automorphism group of $A$; that is, for $\Phi \in \text{Out}(A)$, we write $\Phi = [\phi]$, where $\phi \in \text{Aut}(A)$. Let $\Phi$ be an outer automorphism of a group $A$ and $H \leq A$ a subgroup. Then $\Phi$ stabilises $H$ or $H$ is invariant under $\Phi$ if there exists a representative $\phi \in \Phi$ such that $\phi(H) = H$. We say that $\Phi$ acts trivially on $H$ if there is $\phi \in \Phi$ restricting to the identity on $H$.

If $\mathcal{G}$ and $\mathcal{H}$ are families of subgroups of $A$, the relative outer automorphism group $\text{Out}(A; \mathcal{G}, \mathcal{H}^l)$ is the subgroup of $\text{Out}(A)$ consisting of all elements stabilising each $H \in \mathcal{G}$ and acting trivially on each $H \in \mathcal{H}$. If $\mathcal{G}$ or $\mathcal{H}$ are given by the empty set, we also write $\text{Out}(A; \mathcal{H}^l)$ or $\text{Out}(A; \mathcal{G})$ for this group.

If $O \leq \text{Out}(A)$ is a subgroup of the outer automorphism group of $A$ and $G \leq A$, we also write

$$\text{Stab}_{O}(G)$$

for the subgroup of $O$ consisting of all elements that stabilise $G$. In the case where $O$ is equal to $\text{Out}(A; \mathcal{G}, \mathcal{H}^l)$, we have $\text{Stab}_{O}(G) = \text{Out}(A; \mathcal{G} \cup \{G\}, \mathcal{H}^l)$.

Free splittings

A free splitting $S$ of $A$ is a non-trivial, minimal, simplicial $A$-tree with finitely many edge orbits and trivial edge stabilisers. The vertex group system of a free splitting $S$ is the (finite) set of conjugacy classes of its vertex stabilisers. Two free splittings $S$ and $S'$ are equivalent if they are equivariantly isomorphic. We say that $S'$ collapses to $S$ if there is a collapse map $S' \to S$ which
collapses an $A$-invariant set of edges. The poset of free splittings $\mathcal{FS}_n$ is given by the set of all equivalence classes of free splittings of $A$, where $S \leq S'$ if $S'$ collapses to $S$. The free splitting complex is the order complex $\Delta(\mathcal{FS}_n)$ of the poset of free splittings.

**Fouxe-Rabinovitch groups and relative Outer space**

Let $A$ be a finitely generated group that splits as a free product

$$A = F_n \ast A_1 \ast \cdots \ast A_k,$$

where $F_n$ denotes the free group on $n$ generators and $n + k \geq 2$. Define $A := \{A_1, \ldots, A_k\}$ and $O := \text{Out}(A; A')$. The group $O$ is also called a **Fouxe-Rabinovitch group** because of the work of Fouxe-Rabinovitch on automorphism groups of free products [21].

In [24], Guirardel and Levitt define a topological space called **relative Outer space** for such groups. This space contains a **spine**, which is denoted by $L = L(A, A)$. This spine is (the order complex of) the subposet of $\mathcal{FS}_n$ consisting of all free splittings whose vertex group system is equal to the set of conjugacy classes of elements from $A$. The poset $L$ is contractible and $O$ acts co-compactly on it.

### 4.2.2 Parabolic subgroups and relative free factor complexes

**Standing assumptions and notation**

From now on and until the end of Section 4.2, fix a finitely generated group $A = F_n \ast A_1 \ast \cdots \ast A_k$ with $n \geq 2$ and a basis $\{x_1, \ldots, x_n\}$ of $F_n$. As above, let $A := \{A_1, \ldots, A_k\}$ and $O := \text{Out}(A; A')$.

A free factor of $A$ is a subgroup $B \leq A$ such that $A$ splits as a free product $A = B \ast C$. There is a natural partial order on the set of conjugacy classes of free factors of $A$ given by $[B_1] \leq [B_2]$ if, up to conjugacy, $B_1$ is contained in $B_2$.

**Definition 4.4.** Let $\mathcal{P} = \mathcal{P}(A; A)$ denote the poset of all conjugacy classes of proper free factors $B \subset A$ such that there is a free factor $B'$ of $A$ with $[A_i] \leq [B']$ for all $i$ and $B'$ is a proper subgroup of $B$. (In particular, $[A_1 \ast \cdots \ast A_k] \not\in \mathcal{P}$.) We call the order complex of $\mathcal{P}$ the **free factor complex of $A$ relative to $A$**. It carries a natural, simplicial action of $O$.

**Remark 4.5.** If $k = 0$, the poset $\mathcal{P}$ consists of **all** conjugacy classes of proper free factors of $F_n$, so we recover the free factor complex of $F_n$. More generally, the free factor complex of $A$ relative to $A$ is a subcomplex of the **complex of free factor systems of $A$ relative to $A$** as defined by Handel and Mosher [26]. The ordering $\sqsubseteq$ of free factor systems defined there restricts to the ordering on $\mathcal{P}$ for free factor systems having only one component.

Our definition, however, differs from the one used by Guirardel and Horbez, for example, in [22]; in their definition, a proper free factor $B \leq A$ is relative to $A$ if $A = B \ast C$, where for all $i$, either $[A_i] \leq [B]$ or $[A_i] \leq [C]$ and $[A_i] \neq [B]$.

For studying geometric questions, the definition of the free factor complex and similar complexes is often adapted such that it becomes connected for low $n$ as well. This is not the case for the definition used in this article, where the free factor complex associated to $\text{Out}(F_2)$ is a disjoint union of points.
Corank

[26, Lemma 2.11] implies that the elements of \( F \) are conjugacy classes of groups of the form

\[
B = F * A_{a_1}^1 * ... * A_{a_k}^k,
\]

where \( a_j \in A \) and \( F \) is a free group with \( 1 \leq \text{rk}(F) \leq n-1 \). Furthermore, we can write \( A \) as a free product \( A = B * C \), where \( C \) is a free group of rank \( n - \text{rk}(F) \). The rank of \( C \) is an invariant of the conjugacy class \([B]\) (see [26, Section 2.3]). It is called the corank of \([B]\) and will be denoted by \( \text{crk}[B] \).

We study these relative free factor complexes because they can also be described as coset complexes of parabolic subgroups: Let

\[
S_i := \langle x_1, \ldots, x_i \rangle * A_1 * ... * A_k.
\]

Every \( S_i \) is a free factor of \( A \) because for all \( i \), we have \( A = S_i * \langle x_{i+1}, \ldots, x_n \rangle \). We set \( P_i := \text{Stab}_O(S_i) \) and define the set of maximal standard parabolic subgroups of \( O \) as

\[
P = P(O) := \{ P_i | 1 \leq i \leq n - 1 \}.
\]

As in the case of \( \text{GL}_n(\mathbb{Z}) \), we will usually leave out the adjective 'standard' (see Remark 4.2).

**Proposition 4.6.** The free factor complex of \( A \) relative to \( A \) is \( O \)-equivariantly isomorphic to the coset complex \( \text{CC}(O, P) \).

**Proof.** If \([B_1] \preceq [B_2]\), we know from [26, Proposition 2.10] that the corank of \([B_2]\) is smaller than or equal to the corank of \([B_1]\) and that equality holds if and only if \([B_1] = [B_2]\). Consequently, the simplices of \( \Delta(F) \) are given by chains of the form

\[
[B_1] \preceq [B_2] \preceq \cdots \preceq [B_m]
\]

with \( \text{crk}[B_1] < \text{crk}[B_2] < \cdots < \text{crk}[B_m] \). Let \( i_j := \text{crk}[B_j] \).

We claim that for each such chain, there exists \( \Phi \in O \) with \( [\phi(S_j)] = [B_j] \) for all \( j \). To see this, first observe that sending each \( A_i \) to a conjugate of itself and fixing all the other generators defines an automorphism of \( A \) that represents an element in \( O \). Hence, we can assume that \( A_1 * ... * A_k \preceq B_1 \). Now choose representatives such that \( B_j \preceq B_{j+1} \) for all \( j \). To use induction, assume that there is \( \Phi' \in O \) such that for some \( l \), we have \( \phi'(S_j) = B_j \) for all \( 0 \leq j \leq l \) — this is true for \( l = 0 \), where we define \( i_0 = 0 \) and \( B_0 = S_0 = A_1 * \cdots * A_k \). By assumption, \( \phi'(S_{i_l}) = B_l \preceq B_{l+1} \), so [26, Lemma 2.11] implies that

\[
A = \phi'(S_{i_l}) * C * D, \quad \text{where} \quad B_{l+1} = \phi'(S_{i_l}) * C
\]

and \( C \) and \( D \) are free groups of rank \((i_{l+1} - i_l)\) and \((n - i_{l+1})\), respectively. On the other hand, the group \( A \) also decomposes as a free product

\[
A = S_{i_l} * \langle x_{i_{l+1}}, \ldots, x_{i_{l+1}} \rangle * \langle x_{i_{l+1}+1}, \ldots, x_n \rangle.
\]
This allows us to define an automorphism \( \phi \) of \( A \) which agrees with \( \phi' \) on \( S_{ij} \), maps \( \langle x_{ij+1}, \ldots, x_{i+1} \rangle \) isomorphically to \( C \) and \( \langle x_{i+1}, \ldots, x_n \rangle \) to \( D \). As \( \phi \) agrees with \( \phi' \) on \( S_{ij} \), we know that \([\phi(S_{ij})] = [B_j] \) for all \( j \leq l \) and that \( \phi \) acts by conjugation on each \( A_i \), that is, \([\phi] \in \mathcal{O} \). Furthermore, we have

\[
\phi(S_{ij+1}) = \phi(S_{ij}) \cdot \phi(\langle x_{ij+1}, \ldots, x_{i+1} \rangle) = \phi'(S_{ij}) \cdot C = B_{i+1}.
\]

By induction, this proves the claim.

On the other hand, for each \([\phi] \in \mathcal{O} \), the chain

\[ [\phi(S_1)] \leq [\phi(S_2)] \leq \cdots \leq [\phi(S_{n-1})] \]

forms a facet in \( \Delta(F) \). Hence, every facet of \( \Delta(F) \) can be written in this form. It follows that the natural action of \( \mathcal{O} \) on \( \Delta(F) \) has a fundamental domain given by the simplex

\[ [S_1] \leq [S_2] \leq \cdots \leq [S_{n-1}] . \]

The result now follows from Proposition 3.9.

Note that the corank played in this proof the same role as the dimension and rank did in the proof of Proposition 4.3.

### 4.2.3 The associated complex of free splittings

To study the connectivity properties of relative free factor complexes, we will use yet another description of them; namely, we will show in this subsection that they are homotopy equivalent to certain posets of free splittings.

Let \( L := L(A, \mathcal{A}) \) be the spine of Outer space of \( A \) relative to \( \mathcal{A} \). Taking the quotient by the action of \( A \), each free splitting \( S \in L \) can equivalently be seen as a marked graph of groups \( G \). The edge groups of \( G \) are trivial and for all \( 1 \leq i \leq k \), there is exactly one vertex group which is conjugate to \( A_i \). All the other vertex groups are trivial. The marking is an isomorphism \( \pi_1(G) \to A \) that is well-defined up to composition with inner automorphisms. Using this description, the action of \( O \) on \( L \) is given by changing the marking. The underlying graph \( G \) of \( G \) is finite, has fundamental group of rank \( n \) and all of its vertices with valence one have non-trivial vertex group.

**Definition 4.7.**

1. A **labelled graph** is a pair \((G, l)\) consisting of a graph \( G \) and a map \( l : \{1, \ldots, k\} \to V(G) \) to its vertex set \( V(G) \). We call the image of \( l \) the **labelled vertices** of \( G \).
2. A connected labelled graph \((G, l)\) is called a **core graph** if it has non-trivial fundamental group and every vertex of valence one lies in the image of \( l \).

For the graph \( G \) associated to \( S \in L \) as above, there is a natural labelling \( l : \{1, \ldots, k\} \to V(G) \) of \( G \) given by defining \( l(i) \) as the vertex with vertex group conjugate to \( A_i \). It follows that \((G, l)\) is a core graph. If \( H \) is a connected subgraph of \( G \) that contains all the vertices with non-trivial
vertex group, then there is an induced structure of a marked graph of groups on $H$. We define the fundamental group $\pi_G(H)$ as the fundamental group of this graph of groups. It is a subgroup of $A$ that is well-defined up to conjugacy and has the form

$$\pi_G(H) = F * A_1^{a_1} * \cdots * A_k^{a_k},$$

where $a_i \in A$ and $F$ is a free group with rank equal to the rank of $\pi_1(H)$.

**Definition 4.8.** Let $S \in L$, let $G$ be the associated graph of groups and $(G, l)$ the underlying labelled graph. Let $B \leq A$ be a subgroup of $A$. We say that $S$ has a subgraph with fundamental group $[B]$ if there is a subgraph $H$ of $G$ such that $[\pi_G(H)] = [B]$.

If such a subgraph exists, there is also a unique core subgraph of $(G, l)$ with fundamental group $[B]$ which will be denoted by $B|S$. We then also say that $B|S$ is a subgraph of $S$.

**Notation**

To simplify notation, we will from now onwards not distinguish between a free splitting $S$ and the corresponding graph of groups. For example, we will talk about ‘(core) subgraphs of $S$’ and mean (core) subgraphs of the corresponding labelled graph $(G, l)$. Instead we will use the letter $G$ for elements in $L = L(A, \mathcal{B})$ and the letter $S$ for free splittings that have vertex group system different than $A$. If $G \in L$ and $H$ is a subgraph, let $G/H$ denote the free splitting obtained by collapsing $H$.

**Definition 4.9.** Let $\mathcal{F} S^1 = \mathcal{F} S^1(A; A)$ be the poset of all free splittings $S$ of $A$ that have exactly one conjugacy class $\mathcal{V}(S)$ of non-trivial vertex stabilisers and such that $\mathcal{V}(S) \in \mathcal{F}$.

For $[B] \in \mathcal{F}$, let $\mathcal{F} S^1(B)$ be the poset consisting of all $S \in \mathcal{F} S^1$ such that $[B] \leq \mathcal{V}(S)$.

**Proposition 4.10.** For all $[B] \in \mathcal{F}$, the poset $\mathcal{F} S^1(B)$ is contractible.

This proposition can be shown as [10, Theorem 5.8]; there, only the case where $A$ is a free group is considered, but the proof generalises to the present situation without any major changes. In what follows, we provide an outline of the main steps.

**Sketch of proof of Proposition 4.10.** For a chain of free factors of $A$ given by $B \subset B_1 \subset \cdots \subset B_l \subset C_0 \subset \cdots \subset C_m$, let $X(B, B_1, \ldots, B_l : C_0, \ldots, C_m)$ be the poset of all free splittings $S$ such that $\mathcal{V}(S) \in \{[B], [B_1], \ldots, [B_l]\}$ and $C_i|S$ is a subgraph of $S$ for every $0 \leq i \leq m$. We want to use induction on $l$ to show that this poset is contractible.

We start with the case $l = 0$. Let $D$ be the Outer space of $A$ relative to $\{[B]\}$ as defined in [24]. It can be seen as a subspace of the space of all non-trivial metric simplicial $A$-trees. In [23], Guirardel and Levitt show that its closure $\overline{D}$ in this space is contractible. To do so, they use Skora folding paths to define a map $\rho : \overline{D} \times [0, \infty] \to \overline{D}$. The map $\rho$ depends on the choice of a ‘base point’ $T_0 \in D$ and is defined such that for all $T$, one has $\rho(T, 0) = T$, whereas $\rho(T, \infty)$ is contained in a contractible subspace (a closed simplex of $\overline{D}$) containing $T_0$. In [10, Lemma 5.5], it is shown that for an appropriate choice of $T_0$ (namely, for a tree in $X(B : C_0, \ldots, C_m)$ with a minimal number of edge orbits), the map $\rho$ restricts to a continuous map on $\|X(B : C_0, \ldots, C_m)\|$. There, the argument is formulated for the case where $A$ a free group, but it applies verbatim in our setting as all the results in [23] are formulated in this more general situation anyway.
For \( l > 0 \), assume that by induction, we know that the posets

\[
X_{l-1} := X(B, B_1, ..., B_{l-1} : C_0, ..., C_m), \quad X_l := X(B_l : C_0, ..., C_m)
\]

and

\[
X_{l-1,l} := X(B, B_1, ..., B_{l-1} : B_l, C_0, ..., C_m)
\]

are contractible. The poset \( X(B, B_1, ..., B_l : C_0, ..., C_m) \) is the union of \( X_{l-1} \) and \( X_l \). Furthermore, an element \( S \in X_{l-1} \) collapses to some \( S' \in X_l \) if and only if \( S \in X_{l-1,l} \). It follows that

\[
\|X(B, B_1, ..., B_l : C_0, ..., C_m)\| = \|X_{l-1}\| \cup \|X_{l-1,l}\| \cup \|X_l\|
\]

is contractible. In particular, \( X(B, B_1, ..., B_l : A) \) is contractible.

Now fix \([B] \in F\). Each simplex \( \sigma \) in the order complex \( \Delta(FS^1(B)) \) is given by a sequence \( S_1, ..., S_l \), where every \( S_i \) is a free splitting in \( FS^1 \) that collapses to \( S_{i+1} \). It follows that the vertex groups of these splittings form a chain \( \mathcal{V}(S_1) \leq ... \leq \mathcal{V}(S_l) \) such that \([B] \leq \mathcal{V}(S_i)\) for all \( i \). Hence, the simplex \( \sigma \) is contained in the order complex of \( X(B, \mathcal{V}(S_1), ..., \mathcal{V}(S_l) : A) \). Consequently, the realisation \( \|FS^1(B)\| \) can be written as a union

\[
\|FS^1(B)\| = \bigcup_{B \subset B_1 \subset ... \subset B_l} \|X(B, B_1, ..., B_l : A)\|.
\]

By the arguments above, all of these sets are contractible. Also, one has

\[
\|X(B, B_1, ..., B_l : A)\| \cap \|X(B, C_1, ..., C_m : A)\| = \|X(B, D_1, ..., D_k : A)\|,
\]

where \([B] < [D_1] < ... < [D_k]\) is the longest common subchain of \([B] < [B_1] < ... < [B_l]\) and \([A] < [C_1] < ... < [C_m]\). This implies that finite intersections of the sets appearing on the right-hand side of Equation (1) are contractible. By the nerve lemma (see [4, Theorem 10.6]), \( \|FS^1(B)\| \) is homotopy equivalent to the nerve of this covering. This is contractible as all of these sets intersect non-trivially (they all contain \( \|X(B : A)\| \)).

**Proposition 4.11.** There is a homotopy equivalence \( FS^1 \simeq F \).

**Proof.** Assigning to each splitting \( S \in FS^1 \) the conjugacy class \( \mathcal{V}(S) \) of its non-trivial vertex stabiliser defines a poset map \( f : FS^1 \to FS^1_{op} \). As there is a natural isomorphism of the order complexes \( \Delta(FS^1_{op}) \cong \Delta(F) \), we will interpret \( f \) as an order-inverting map \( f : FS^1 \to F \). Now for any \( B \in F \), the fibre \( f^{-1}(F_{\geq B}) \) is equal to the poset \( FS^1(B) \) which is contractible by Proposition 4.10. The claim follows from Lemma 2.1. \( \square \)

### 4.2.4 Homotopy type of relative free factor complexes

To study the homotopy type of \( FS^1 \), we ‘thicken it up’ by elements from \( L = L(A, A) \), the spine of Outer space of \( A \) relative to \( A \).

**Definition 4.12.** Let \( Y \) be the subposet of the product \( L \times FS^1 \) consisting of all pairs \((G, S)\) such that \( S = G/H \) is obtained from \( G \) by collapsing a proper core subgraph \( H \) and let \( p_1 : Y \to L \) and \( p_2 : Y \to FS^1 \) be the natural projection maps.
By analysing the maps $p_1$ and $p_2$, we now want to show that $F$ is spherical. This closely follows [10, Section 7]. We first deformation retract the fibres of $p_2$ to a simpler subposet.

**Lemma 4.13.** For all $S \in F S^1$, the fibre $p_2^{-1}(F S^1_{\geq S})$ deformation retracts to $p_2^{-1}(S)$.

**Proof.** Let $F := p_2^{-1}(F S^1_{\geq S})$ and define $f : F \to p_2^{-1}(S)$ as follows: If $(G', S')$ is an element of $F$, there are collapse maps $G' \to S'$ and $S' \to S$. Concatenating these maps, we see that $S$ is obtained from $G'$ by collapsing a subgraph $H' \subset G'$. The subgraph can be written as the union of a (possibly trivial) forest $T'$ and a unique maximal core graph $\hat{H}'$. We set $f(G', S') := (G'/T', S)$. As $S = (G'/T')/\hat{H}'$, this is indeed an element of $p_2^{-1}(S)$. Also if $(G'', S'') \geq (G', S')$ in $F$, we have $c(T'') \supseteq T'$ which implies $G''/T'' \geq G'/T'$. Consequently $f : F \to p_2^{-1}(S)$ is a well-defined, monotone poset map restricting to the identity on $p_2^{-1}(S)$. It follows from Corollary 2.4 that this defines a deformation retraction. □

Hence, instead of studying arbitrary fibres, it suffices to consider the preimages of single vertices.

**Lemma 4.14.** For all $S \in F S^1$, the preimage $p_2^{-1}(S)$ is contractible.

**Proof.** Let $[B] := \mathcal{V}(S)$. Every element in $p_2^{-1}(S)$ is given by a pair $(G, S)$ such that $H := B|G$ is a subgraph of $G$ and $S = G/H$. Forgetting the (constant) second coordinate, we can interpret these as elements of the Outer space of $A$ relative to $\mathcal{A}$. Let $X$ be the subspace of this Outer space that is given by all open simplices containing an element of $p_2^{-1}(S)$. Then $p_2^{-1}(S)$ is a deformation retract of $X$. In [10, Propositions 7.3.1 and 7.2], Skora folds are used to show that $X$ is contractible. The proof in [10] is formulated for the case where $A$ is free, but it applies here as well because it only uses the ideas of Guirardel and Levitt [23], which hold true in the generality needed in our setting. □

In particular, these fibres are all contractible, so by Lemma 2.1, we have the following.

**Corollary 4.15.** The map $p_2 : Y \to F S^1$ is a homotopy equivalence.

We now turn to $p_1$ and show that its fibres are highly connected as well.

**Definition 4.16.** For a labelled graph $(G, l)$, let $C(G, l)$ denote the poset of all proper core subgraphs of $(G, l)$, where the partial order is given by inclusion of subgraphs.

**Lemma 4.17.** Let $G \in L$, and let $(G, l)$ denote the induced structure of a labelled graph. Then the fibre $p_1^{-1}(L \leq G)$ is homotopy equivalent to $C(G, l)$.

**Proof.** Each element of $p_1^{-1}(L \leq G)$ consists of a pair $(G', S')$, where $G' \leq G$ in $L$ and $S' \in F S^1$ is obtained from $G'$ by collapsing a proper core subgraph $H'$. As $G'$ is obtained from $G$ by collapsing a forest, $H := \pi_{G'}(H')|G$ is a subgraph of $G$. 
The collapse $G \to G'$ induces a collapse $G/H \to G'/H' = S'$. Hence, we get a monotone poset map

$$f : p_1^{-1}(L_{\leq G}) \to p_1^{-1}(G)$$

$$(G', S') \mapsto (G, G/H)$$

restricting to the identity on $p_1^{-1}(G) \subseteq p_1^{-1}(L_{\leq G})$. Again Corollary 2.4 implies that $f$ defines a deformation retraction.

For every proper core subgraph $H$ of $G$, the pair $(G, G/H)$ forms an element of $p_1^{-1}(G)$. Also, if $H$ and $H'$ are proper core subgraphs of $G$, one has $G/H \succeq G/H'$ in $\mathcal{P}S^1$ if and only if $H \preceq H'$ in $C(G, l)$. Hence, the fibre $p_1^{-1}(G)$ can be identified with $C(G, l)^{op}$. Because $\|C(G, l)^{op}\| \cong \|C(G, l)\|$, this finishes the proof.

**Theorem 4.18.** The poset $C(G, l)$ is $(n - 2)$-spherical.

We postpone the proof of this result until Section 4.2.6 and first note the following corollary.

**Corollary 4.19.** The poset $Y$ is $(n - 3)$-connected.

**Proof.** The projection $p_1 : Y \to L$ is a map from $Y$ to the contractible poset $L$. By Lemma 4.17 and Theorem 4.18, the fibres of this map are $(n - 3)$-connected. Hence, the result follows from Lemma 2.2.

The main result of this section, which was stated as Theorem D in the introduction, is now an easy consequence of the last corollaries.

**Theorem 4.20.** The free factor complex of $A = F_n \ast A_1 \ast \cdots \ast A_k$ relative to $A = \{A_1, \ldots, A_k\}$ is homotopy equivalent to a wedge of $(n - 2)$-spheres.

**Proof.** By Corollary 4.15, there is a homotopy equivalence $F \simeq Y$. By Corollary 4.19, the poset $Y$ is $(n - 3)$-connected. As $F$ is $(n - 2)$-dimensional, the claim follows.

### 4.2.5 | Cohen–Macaulayness

The relative formulations allow us to deduce that $F$ or equivalently $CC(O, P)$ is even Cohen–Macaulay.

**Theorem 4.21.** The coset complex $CC(O, P)$ is homotopy Cohen–Macaulay.

**Proof.** By Proposition 4.6, we have to show that the link of every $s$-simplex $\sigma = [B_0] \leq \cdots \leq [B_s]$ in $\Delta(F)$ is $(n - s - 3)$-spherical. However, the link of this simplex is by definition given by the following join of posets

$$\text{lk}(\sigma) = F_{<[B_0]} \ast ([B_0], [B_1]) \ast \cdots \ast ([B_{s-1}], [B_s]) \ast F_{>[B_s]}.$$
As above, each $B_i$ can be written in the form

$$B_i = D_i \ast A_1^{a_1} \ast \cdots \ast A_k^{a_k},$$

where $D_i$ is a free group of rank $n - \text{crk}[B_i]$. Using malnormality of free factors, it follows that two subgroups of a free factor $B$ of $A$ are conjugate in $A$ if and only if they are conjugate in $B$ (see [26, Lemma 2.1]). It follows that there are isomorphisms

$$\mathcal{F}_{<[B_i]} \cong \mathcal{F}(B_0, \{A_1^{a_1}, \ldots, A_k^{a_k}\}),$$

$$([B_i], [B_{i+1}]) \cong \mathcal{F}(B_{i+1}, \{B_i\}),$$

$$\mathcal{F}_{>[B_i]} \cong \mathcal{F}(A, \{B_i\}).$$

The result now follows from Lemma 2.5 and Theorem 4.20. □

### 4.2.6 Posets of subgraphs

In this section, we prove Theorem 4.18 by studying posets of subgraphs of labelled graphs. The results we obtain generalise [10, Section 4.2] and we closely follow the structure of the proofs there.

In what follows, all graphs are allowed to have loops and multiple edges. For a graph $G$, let $V(G)$ denote the set of its vertices and $E(G)$ the set of its edges. If $e \in E(G)$ is an edge, then $G - e$ is defined to be the graph obtained from $G$ by removing $e$ and $G/e$ is obtained by collapsing $e$ and identifying its two endpoints to a new vertex $v_e$. For a labelled graph $(G, l)$ (see Definition 4.7), there are canonical labellings $\{1, \ldots, k\} \to G - e$ and $\{1, \ldots, k\} \to G/e$ that will be denoted by $l$ as well. A graph is called a **tree** if it is contractible.

**Definition 4.22.** For a labelled graph $(G, l)$, let $X(G, l)$ be the poset of all connected subgraphs of $G$ which are not trees, contain all the labelled vertices and whose fundamental group is strictly contained in $\pi_1(G)$.

The following lemma allows us to replace $C(G, l)$ with $X(G, l)$. This bigger poset will be easier to handle for the inductive arguments that we want to use.

**Lemma 4.23.** $X(G, l)$ deformation retracts to $C(G, l)$.

**Proof.** By restricting the labelling, every $H \in X(G, l)$ can be seen as a labelled graph $(H, l)$. It contains contains a unique maximal core subgraph $(\hat{H}, l)$. Also, if $H_1 \leq H_2$ in $X(G, l)$, one has $(\hat{H}_1, l) \subseteq (\hat{H}_2, l)$. Hence, sending $(H, l)$ to $(\hat{H}, l)$ defines a poset map $f : X(G, l) \to C(G, l)$ that restricts to the identity on $C(G, l)$. The claim now follows from Corollary 2.4. □

An edge $e \in E(G)$ is called **separating** if $G - e$ is disconnected; in particular, we consider edges adjacent to vertices of valence one to be separating.
Lemma 4.24. Let \((G, l)\) be a labelled graph where \(G\) is finite and connected. Let \(e \in E(G)\) be an edge that is not a loop and set

\[
Y_e := \{ H \in X(G, l) \mid H \cup \{e\} \text{ connected and } \pi_1(H \cup \{e\}) = \pi_1(G) \}.
\]

Then \(X(G, l) \setminus Y_e \simeq X(G/e, l)\).

Furthermore, if \(e\) is separating, then \(Y_e\) is empty, so \(X(G, l) \simeq X(G/e, l)\).

Proof. Whenever \(H \in X(G, l)\), the edges in \(E(H) \setminus \{e\}\) form a connected subgraph of \(G/e\) that will be denoted by \(H/e\). It contains all labelled vertices of \((G/e, l)\) and has non-trivial fundamental group.

If \(H\) is not in \(Y_e\), then either \(e\) is not adjacent to \(H\) and hence \(\pi_1(H/e) \cong \pi_1(H)\), or \(\pi_1(H/e) \leq \pi_1(H \cup \{e\}/e) \cong \pi_1(H \cup \{e\})\). In either case, \(\pi_1(H/e)\) is a proper subgroup of \(\pi_1(G/e) \cong \pi_1(G)\). Consequently, we get a poset map

\[
f : X(G, l) \setminus Y_e \to X(G/e, l)
\]

\[
H \mapsto H/e.
\]

On the other hand, if \(K \in X(G/e, l)\) contains the vertex \(v_e\) to which \(e\) was collapsed, it is easy to see that \(K \cup \{e\}\) is an element of \(X(G, l) \setminus Y_e\). This allows us to define a poset map

\[
g : X(G/e, l) \to X(G, l) \setminus Y_e
\]

\[
K \mapsto \begin{cases} K \cup \{e\} & , \ v_e \in V(K), \\ K & , \ \text{else.}
\end{cases}
\]

One has \(g \circ f(H) \supseteq H\) and \(f \circ g(K) = K\), so using Lemma 2.3, these two posets are homotopy equivalent, which proves the first part of the statement.

For the second part, note that if \(e\) is separating and \(H \in X(G, l)\) such that \(H \cup \{e\}\) is connected, then either \(e\) is contained in \(H\) or \(e\) has valence one in \(H \cup \{e\}\). In either case, we have \(\pi_1(H \cup \{e\}) = \pi_1(H) \neq \pi_1(G)\). □

To prove the following result, we apply an argument similar to the one used in [45, Proposition 2.2]. (Note that in [45], being \(n\)-spherical is only defined for \(n\)-dimensional posets.)

Proposition 4.25. Let \((G, l)\) be a labelled graph where \(G\) is finite, connected and has fundamental group of rank \(n \geq 2\). Then \(X(G, l)\) is \((n-2)\)-spherical.

Proof. If \(e \in E(G)\) is separating, then by Lemma 4.24, we have \(X(G, l) \simeq X(G/e, l)\). As \(G/e\) has one edge less than \(G\), we can apply induction to assume that \(G\) does not have any separating edges.

We do induction on \(n\) and start with the case \(n = 2\). By Lemma 4.23, it suffices to show that \(C(G, l)\) is homotopy equivalent to a wedge of 0-spheres, that is, a disjoint union of points. To see this, let \(H \in C(G, l)\). As \(1 < \pi_1(H) < \pi_1(G)\), the fundamental group of \(H\) is infinite cyclic. Let \(e \in H\) be an edge of \(H\). We distinguish between the two cases where \(e\) is non-separating or separating in \(H\). If \(e\) is non-separating, then \(H - e\) has trivial fundamental group while if \(e\) is separating, \(H - e\) has two connected components both of which either have non-trivial fundamental group
or contain at least one labelled vertex. In both cases, no \( K \in C(G, l) \) can be contained in \( H - e \). Hence, the order complex of \( C(G, l) \) does not contain any simplex of dimension greater than zero which proves the claim.

Now let \( n > 2 \). If every edge of \( G \) is a loop, \( G \) is a rose with \( n \) petals and every proper non-empty subset of \( E(G) \) forms an element of \( X(G, l) \). In this case, the order complex of \( X(G, l) \) is given by the set of all proper faces of a simplex of dimension \( n - 1 \) whose vertices are in one-to-one correspondence with the edges of \( G \) and hence is homotopy equivalent to an \((n - 2)\)-sphere.

So, assume that \( G \) has an edge \( e \) that is not a loop. As we assumed that \( e \) is non-separating, \( G - e \) is a connected graph having the same number of vertices as \( G \) and one edge less. This implies that \( \text{rk}(\pi_1(G - e)) = n - 1 \). Collapsing separating edges and using Lemma 4.24, we see that \( X(G - e, l) \simeq X(G', l) \) where \( G' \) has the same rank as \( G - e \), at most as many edges and no separating edges. Hence, \( X(G - e, l) \simeq X(G', l) \) is by induction homotopy equivalent to a wedge of \((n - 3)\)-spheres.

\[
\| X(G, l) \| \text{ is obtained from } \| X(G, l) \setminus \{G - e\} \| \text{ by attaching the star of } G - e \text{ along its link. The link of } G - e \text{ in } \| X(G, l) \| \text{ is isomorphic to } \| X(G - e, l) \| \text{ and its star is contractible. Gluing a contractible set to an } (n - 2)\text{-spherical complex along an } (n - 3)\text{-spherical subcomplex results in an } (n - 2)\text{-spherical complex, so the claim follows (see, for example, [12, Lemma 6.3]).}
\]

Proof of Theorem 4.18. That \( C(G, l) \) is \((n - 2)\)-spherical is an immediate consequence of Lemma 4.23 and Proposition 4.25.

\[
\square
\]

5 | RELATIVE AUTOMORPHISM GROUPS OF RAAGs

In this section, we examine relative automorphism groups of right-angled Artin groups. These groups were studied in detail in [19] and many of the results here are either taken from the work of Day–Wade or build on their ideas. For an overview about other literature on relative automorphism groups, see [19, Section 6.1]. In this article, such relative automorphism groups occur in two ways: On the one hand, they arise as the images and kernels of restriction and projection homomorphisms, which in turn play an important role for the inductive procedure of Day–Wade; on the other hand, the parabolic subgroups we will define in Section 6 are themselves relative automorphism groups of RAAGs. For the purpose of this text, this section mostly serves as a toolbox for the inductive proof of Theorem A in Section 6. Its main goals are to collect all the results from [19] that we will need afterwards, to adapt them to our purposes and, maybe most importantly, to set up the language we will use later on.

Standing assumption

From now on, all graphs that we consider will be finite and simplicial, that is, without loops or multiple edges. To emphasise this difference to Section 4, they will be denoted by Greek letters.

5.1 | RAAGs and their automorphism groups

Subgraphs, links and stars

In contrast to Section 4, if we talk about a subgraph \( \Delta \) of a graph \( \Gamma \), we will from now on always mean a full subgraph, that is, if two vertices \( v, w \in V(\Delta) \) are connected by an edge in \( \Gamma \), they are
connected in $\Delta$ as well. A full subgraph of $\Gamma$ can also be seen as a subset of the vertex set $V(\Gamma)$; we will often take this point of view, identify $\Delta$ with $V(\Delta)$ and write $\Delta \subseteq \Gamma$ or $\Delta < \Gamma$ if we want to emphasise that $\Delta$ is a proper subgraph of $\Gamma$.

Given a vertex $v \in V(\Gamma)$, the link $lk(v)$ of $v$ is the subgraph of $\Gamma$ consisting of all the vertices that are adjacent to $v$. The star $st(v)$ of $v$ is the subgraph of $\Gamma$ with vertex set $\{v\} \cup lk(v)$. We also write $lk_{\Gamma}(v)$ or $st_{\Gamma}(v)$ if we want to distinguish between links and stars in different graphs.

**RAAGs and special subgroups**

Given a graph $\Gamma$, the associated right-angled Artin group — abbreviated as RAAG — $A_\Gamma$ is defined to be the group generated by the set $V(\Gamma)$ subject to the relations $[v, w] = 1$ for all $v, w \in V(\Gamma)$ which are adjacent to each other.

Given any subgraph $\Delta \subseteq \Gamma$, the inclusion $V(\Delta) \to V(\Gamma)$ induces an injective homomorphism $A_\Delta \hookrightarrow A_\Gamma$. This allows us to interpret $A_\Delta$ as a subgroup of $A_\Gamma$. Subgroups of this type are called special subgroups of $A_\Gamma$.

**The standard ordering and its equivalence classes**

There is a so-called standard ordering on the vertex set $V(\Gamma)$ that is the partial pre-order given by $v \preceq w$ if and only if $lk(v) \subseteq st(w)$. The induced equivalence relation of this partial pre-order will be denoted by $\sim$, that is, $v \sim w$ if and only if $v \preceq w$ and $w \preceq v$. The equivalence class of $v$ will be denoted by $[v]$. The standard ordering induces a partial order on the equivalence classes where we say $[v] \preceq [w]$ if $v \preceq w$ (this does not depend on the choice of representatives). If two equivalent vertices $v \sim w$ are adjacent, it follows that the vertices from their equivalence class $[v]$ form a complete subgraph of $\Gamma$. In this case, the special subgroup $A_{[v]}$ is isomorphic to $\mathbb{Z}[|v|]$ and we call $[v]$ an abelian equivalence class. If on the other hand $[v]$ does not contain any pair of adjacent vertices, it can be seen as discrete subgraph of $\Gamma$. In this case, we call $[v]$ a free equivalence class because $A_{[v]}$ is isomorphic to the free group $F_{|v|}$. For more details about this ordering and the equivalence relation, see [16].

**Automorphisms of RAAGs**

Let $\text{Aut}(A_\Gamma)$ and $\text{Out}(A_\Gamma)$ denote the automorphism group and the group of outer automorphisms of $A_\Gamma$, respectively. By the work of Servatius [40] and Laurence [33], the group $\text{Aut}(A_\Gamma)$ is generated by the following automorphisms.

- **Graph automorphisms.** Any automorphism of the graph $\Gamma$ gives rise to an automorphism of $A_\Gamma$ by permuting the generators of the RAAG.
- **Inversions.** Let $v \in V(\Gamma)$. The map sending $v$ to $v^{-1}$ and fixing all the other generators induces an automorphism of $A_\Gamma$. It is called an inversion and denoted by $\iota_v$.
- **Transvections.** Let $v, w \in V(\Gamma)$ with $v \preceq w$. The transvection $\rho_{v}^{w}$ is the automorphism of $A_\Gamma$ induced by sending $v$ to $vw$ and fixing all the other generators. We call $w$ the acting letter of $\rho_{v}^{w}$.
- **Partial conjugations.** Let $v \in V(\Gamma)$ and $K$ a union of connected components of $\Gamma \setminus st(v)$. The map sending every vertex $w$ of $K$ to $vuwv^{-1}$ and fixing the remaining generators induces an automorphism $\pi_{K}^{v}$ of $A_\Gamma$ and is called a partial conjugation. We call $v$ the acting letter of $\pi_{K}^{v}$.

We will use the same notation to denote the images of these automorphisms in $\text{Out}(A_\Gamma)$ and call these (outer) automorphisms the Laurence generators of $\text{Aut}(A_\Gamma)$ or $\text{Out}(A_\Gamma)$, respectively.
The subgroup of \(\text{Out}(A_{\Gamma})\) generated by all inversions, transvections and partial conjugations is denoted by \(\text{Out}^0(A_{\Gamma})\). It is called the pure outer automorphism group of \(A_{\Gamma}\) and has finite index in \(\text{Out}(A_{\Gamma})\). If \(A_{\Gamma}\) is equal to \(\mathbb{Z}^n\) or \(F_n\), we have that \(\text{Out}^0(A_{\Gamma}) = \text{Out}(A_{\Gamma})\).

5.2 Generators of relative automorphism groups

Recall that for a group \(G\) and families of subgroups \(\mathcal{G}\) and \(\mathcal{H}\), the group \(\text{Out}(G; \mathcal{G}, \mathcal{H})\) is defined as the subgroup of \(\text{Out}(G)\) consisting of all elements stabilising each \(H \in \mathcal{G}\) and acting trivially on each \(H \in \mathcal{H}\) (see Section 4.2.1).

Given a pair \((\mathcal{G}, \mathcal{H})\) of families of special subgroups of \(A_{\Gamma}\), we define

\[
\text{Out}^0(A_{\Gamma}; \mathcal{G}, \mathcal{H}) := \text{Out}(A_{\Gamma}; \mathcal{G}, \mathcal{H}) \cap \text{Out}^0(A_{\Gamma})
\]

as the intersection of \(\text{Out}(A_{\Gamma}; \mathcal{G}, \mathcal{H})\) with \(\text{Out}^0(A_{\Gamma})\). Building on the work of Laurence, Day–Wade show the following.

**Theorem 5.1** [19, Theorem D]. If \(\mathcal{G}\) and \(\mathcal{H}\) are families of special subgroups of \(A_{\Gamma}\), the group \(\text{Out}^0(A_{\Gamma}; \mathcal{G}, \mathcal{H})\) is generated by the set of all inversions, transvections and partial conjugations of \(\text{Out}(A_{\Gamma})\) it contains.

To prove this, Day and Wade give a description of the Laurence generators contained in such a relative automorphism group. To state it, we first need to set up the terminology developed in their article.

**\(\mathcal{G}\)-components and \(\mathcal{G}\)-ordering**

Let \(\mathcal{G}\) be a family of proper special subgroups of \(A_{\Gamma}\). We say that \(v, w \in V(A_{\Gamma})\) are \(\mathcal{G}\)-adjacent if \(v\) is adjacent to \(w\) or if there is some \(A_\Delta \in \mathcal{G}\) such that \(v, w \in \Delta\). A subgraph \(\Delta \subset \Gamma\) is \(\mathcal{G}\)-connected if for all \(v, w \in \Delta\), there is a sequence of vertices in \(\Delta\) which starts with \(v\), ends with \(w\) and such that each of its vertices is \(\mathcal{G}\)-adjacent to the next one. A maximal \(\mathcal{G}\)-connected subgraph of \(\Gamma\) is called a \(\mathcal{G}\)-component.

The \(\mathcal{G}\)-ordering \(\preceq_{\mathcal{G}}\) on \(V(\Gamma)\) is the partial pre-order defined by saying that \(v \preceq_{\mathcal{G}} w\) if and only if \(v \preceq w\) and for all \(A_\Delta \in \mathcal{G}\), if \(v \in \Delta\), one has \(w \in \Delta\). The equivalence relation of this pre-order is denoted by \(\sim_{\mathcal{G}}\), its equivalence classes by \([\cdot]_{\mathcal{G}}\).

Note that in the case where \(\mathcal{G} = \emptyset\), a \(\mathcal{G}\)-component of \(\Gamma\) is just a connected component and the \(\mathcal{G}\)-ordering is the standard ordering on \(V(\Gamma)\).

For \(v \in V(\Gamma)\), let \(\mathcal{G}^v := \{A_\Delta \in \mathcal{G} \mid v \notin \Delta\}\). It is easy to see that every \(\mathcal{G}^v\)-component of \(\Gamma \setminus \text{st}(v)\) is a union of connected components of \(\Gamma \setminus \text{st}(v)\). Suppose that \(\mathcal{H}\) is a family of special subgroups of \(A_{\Gamma}\). The power set of \(\mathcal{H}\), denoted by \(P(\mathcal{H})\), is defined as the set of all special subgroups \(A_\Delta \leq A_{\Gamma}\) which are contained in some element of \(\mathcal{H}\).

**Lemma 5.2** [19, Proposition 3.9]. Let \(\mathcal{G}\) and \(\mathcal{H}\) be families of special subgroups of \(A_{\Gamma}\) such that \(\mathcal{G}\) contains \(P(\mathcal{H})\). Let \(v, w \in V(\Gamma)\) and let \(K\) be a union of connected components of \(\Gamma \setminus \text{st}(v)\). Then,

- the inversion \(i_v\) is contained in \(\text{Out}^0(A_{\Gamma}; \mathcal{G}, \mathcal{H})\) if and only if there is no subgroup \(A_\Delta \in \mathcal{H}\) with \(v \in \Delta\);
• the transvection \( \rho^v_w \) is contained in \( \text{Out}^0(\Gamma; \mathcal{G}, \mathcal{H}') \) if and only if \( v \leq_G w \);
• the partial conjugation \( \pi^v_K \) is contained in \( \text{Out}^0(\Gamma; \mathcal{G}, \mathcal{H}') \) if and only if \( K \) is a union of \( \mathcal{G}' \)-components of \( \Gamma \setminus \text{st}(v) \).

Note that it imposes no great restriction to assume that the power set of \( \mathcal{H} \) be contained in \( \mathcal{G} \) because for any families \( \mathcal{G} \) and \( \mathcal{H} \) of special subgroups, one has

\[
\text{Out}^0(\Gamma; \mathcal{G}, \mathcal{H}') = \text{Out}^0(\Gamma; \mathcal{G} \cup \mathcal{P}(\mathcal{H}), \mathcal{H}')
\]

(see [19, Lemma 3.8]).

The next result is the key ingredient for the proof of [19, Lemma 5.2]. We include it here because it will allow us a more convenient description of the parabolic subgroups that we will study later on.

**Lemma 5.3** [19, Lemma 2.2]. Let \( A_\Delta \) be a special subgroup of \( A_\Gamma \). Let \( v, w \in V(\Gamma) \) and let \( K \) be a union of connected components of \( \Gamma \setminus \text{st}(v) \). Then,

• the inversion \( \iota_v \) acts trivially on \( A_\Delta \) if and only if \( v \notin \Delta \); it always stabilises \( A_\Delta \);
• the transvection \( \rho^v_w \) acts trivially on \( A_\Delta \) if and only if \( v \notin \Delta \); it stabilises \( A_\Delta \) if it acts trivially on it or \( w \in \Delta \);
• the partial conjugation \( \pi^v_K \) acts trivially on \( A_\Delta \) if and only if

\[
K \cap \Delta = \emptyset \text{ or } \Delta \setminus \text{st}(x) \subseteq K;
\]

it stabilises \( A_\Delta \) if it acts trivially on it or \( w \in \Delta \).

### 5.3 Restriction and projection homomorphisms

Let \( O \) be a subgroup of \( \text{Out}(A_\Gamma) \). If the special subgroup \( A_\Delta \leq A_\Gamma \) is stabilised by \( O \), there is a restriction homomorphism

\[
R_\Delta : O \to \text{Out}(A_\Delta),
\]

where \( R_\Delta(\Phi) \) is the outer automorphism given by taking a representative \( \phi \in \Phi \) that sends \( A_\Delta \) to itself and restricting it to \( A_\Delta \). If the normal subgroup \( \langle A_\Delta \rangle \) generated by \( A_\Delta \) is stabilised by \( O \), there is a projection homomorphism

\[
P_{\Gamma \setminus \Delta} : O \to \text{Out}(A_{\Gamma \setminus \Delta}),
\]

which is induced by the quotient map

\[
A_\Gamma \to A_\Gamma/\langle A_\Delta \rangle \cong A_{\Gamma \setminus \Delta}.
\]

Restriction and projection maps were first defined in [13] and have since become an important tool for studying automorphism groups of RAAGs via inductive arguments.
5.3.1 Generators of image and kernel

Day–Wade obtained a complete description of the image and kernel of restriction homomorphisms. Again let $\mathcal{G}$ and $\mathcal{H}$ be families of special subgroups of $A_\Gamma$. We say that $\mathcal{G}$ is saturated with respect to $(\mathcal{G}, \mathcal{H})$, if it contains every proper special subgroup stabilised by $\text{Out}^0(A_\Gamma; \mathcal{G}, \mathcal{H}')$. Given a special subgroup $A_\Delta \leq A_\Gamma$, set

$$\mathcal{G}_\Delta := \{ A_{\Delta \cap \Theta} \mid A_\Theta \in \mathcal{G} \}.$$ 

We define $\mathcal{H}_\Delta$ analogously.

**Theorem 5.4** [19, Theorem E]. 
Let $\mathcal{G}$ be saturated with respect to $(\mathcal{G}, \mathcal{H})$ and let $A_\Delta \in \mathcal{G}$. The restriction homomorphism

$$R_\Delta : \text{Out}^0(A_\Gamma; \mathcal{G}, \mathcal{H}') \to \text{Out}(A_\Delta)$$

has image equal to

$$\text{im} R_\Delta = \text{Out}^0(A_\Delta; \mathcal{G}_\Delta, \mathcal{H}'_\Delta)$$

and kernel equal to

$$\ker R_\Delta = \text{Out}^0(A_\Gamma; \mathcal{G}, (\mathcal{H} \cup \{A_\Delta\})').$$

It is not hard to see that both restriction and projection maps send each Laurence generator either to the identity or to a Laurence generator of the same type. For the proof of Theorem 5.4, Day–Wade show that for restriction maps, a converse of this is true as well: Every Laurence generator in $\text{im} R_\Delta$ is given as the restriction of a Laurence generator of $\text{Out}^0(A_\Gamma; \mathcal{G}, \mathcal{H}')$.

In general, image and kernel of projection homomorphisms are more difficult to describe. However, we will only need to consider them in a special case: The centre $Z(A_\Gamma)$ of $A_\Gamma$ is generated by all vertices $z \in V(\Gamma)$ such that $\text{st}(z) = \Gamma$. If $Z(A_\Gamma)$ is non-trivial, these vertices form an abelian equivalence class $Z := [z]$ and $\Gamma$ can be written as a join $\Gamma = Z \ast \Delta$, where $\Delta = \Gamma \setminus Z$. If we have a graph of this form, the centre $Z(A_\Gamma) = A_Z$ is a normal subgroup which is stabilised by all of $\text{Out}(A_\Gamma)$. Hence, there is a projection map

$$P_\Delta : \text{Out}(A_\Gamma) \to \text{Out}(A_\Delta).$$

The image of this projection map can be described very similar to the one of a restriction map. In fact, the situation in this special case is even easier as we do not even need to assume any kind of saturation for our families of special subgroups.

**Lemma 5.5.** Assume that $\Gamma$ can be decomposed as a join $\Gamma = Z \ast \Delta$ where $Z$ is a complete graph. Let $\mathcal{G}$ and $\mathcal{H}$ be any two families of special subgroups of $A_\Gamma$ and let $A_Z \in \mathcal{G}$. The projection homomorphism

$$P_\Delta : \text{Out}^0(A_\Gamma; \mathcal{G}, \mathcal{H}') \to \text{Out}(A_\Delta)$$
has image equal to

\[ \text{im } P_{\Delta} = \text{Out}^0(A_{\Delta}; \mathcal{G}_{\Delta}, \mathcal{H}'_{\Delta}). \]

**Proof.** The inclusion ‘\( \subseteq \)’ follows immediately from the definitions.

For the other inclusion, we start by defining \( \tilde{\mathcal{G}} := \mathcal{G} \cup P(H) \) as the union of \( \mathcal{G} \) and the power set of \( H \). As observed above, we have

\[ O := \text{Out}^0(A_{\Gamma}; \mathcal{G}, \mathcal{H}') = \text{Out}^0(A_{\Gamma}; \tilde{\mathcal{G}}, \mathcal{H}'). \]

Furthermore, \( \tilde{\mathcal{G}}_{\Delta} = \mathcal{G}_{\Delta} \cup P(H_{\Delta}) \), so we also have

\[ O_{\Delta} := \text{Out}^0(A_{\Delta}; \mathcal{G}_{\Delta}, \mathcal{H}'_{\Delta}) = \text{Out}^0(A_{\Delta}; \tilde{\mathcal{G}}_{\Delta}, \mathcal{H}'_{\Delta}). \quad (2) \]

By Theorem 5.1, we know that \( O_{\Delta} \) is generated by the inversions, transvections and partial conjugations it contains. Hence, it suffices to find a preimage under \( P_{\Delta} \) for each of those generators. Combining Equation (2) with Lemma 5.2, we have a complete description of the generators in \( O_{\Delta} \).

In what follows, we will use this description to construct the preimages one generator at a time.

The inversion \( \iota_v \) is contained in \( O_{\Delta} \) if and only if \( v \in \Delta \) and there is no \( A_{\Delta'} \in H_{\Delta} \) such that \( v \in \Delta' \). However, this implies that there is no \( A_{\Delta'} \in H_{\Delta} \) with \( v \in \Delta' \), so the inversion at \( v \) is an element of \( O \). It will be denoted by \( \bar{\iota}_v \) and gets mapped to \( \iota_v \) under \( P_{\Delta} \).

If one has a transvection \( \rho_w^v \in O_{\Delta} \), Lemma 5.2 implies that \( v \leq \tilde{\mathcal{G}}_\Delta w \), that is, \( lk_{\Delta}(v) \subseteq st_{\Delta}(w) \) and for each \( A_{\Delta'} \in \tilde{\mathcal{G}}_{\Delta} \), one has that \( v \in \Delta' \) implies \( w \in \Delta' \). We want to show that \( v \leq \mathcal{G} \ w \). As \( \Gamma \) is a join \( \mathbb{Z} \ast \Delta \), the link and star of \( v \) and \( w \) in \( \Gamma \) are of the form

\[ lk_{\Gamma}(v) = lk_{\Delta}(v) \cup Z, \quad st_{\Gamma}(w) = st_{\Delta}(w) \cup Z. \]

In particular, \( lk_{\Gamma}(v) \subseteq st_{\Gamma}(w) \). The vertex \( v \) cannot be contained in any \( \Delta' \) with \( A_{\Delta'} \in P(H) \) as this would imply \( A_{\{v\}} \in P(H_{\Delta}) \subseteq \tilde{\mathcal{G}}_{\Delta} \), contradicting the assumption that \( v \leq \tilde{\mathcal{G}}_\Delta w \). Now take \( A_{\Delta'} \in \mathcal{G} \) such that \( v \in \Delta' \). If \( \Delta \subseteq \Delta' \), both \( v \) and \( w \) are contained in \( \Delta' \). If on the other hand \( \Delta \cap \Delta' \) is a proper subset of \( \Delta \), one has \( A_{\Delta \cap \Delta'} \in \mathcal{G}_{\Delta} \subseteq \tilde{\mathcal{G}}_{\Delta} \), so \( w \in \Delta' \). It follows that \( v \leq \mathcal{G} \ w \), so the transvection multiplying \( v \) by \( w \) defines an element of \( O \). It will be denoted by \( \bar{\rho}_w^v \in O \) and is a preimage of \( \rho_w^v \).

Again using Lemma 5.2, the partial conjugation \( \pi_v^w \) is contained in \( O_{\Delta} \) if and only if \( v \in \Delta \) and \( K \) is a union of \( \tilde{\mathcal{G}}_{\Delta} \)-components of \( \Delta \setminus st(v) \). We claim that every \( \tilde{\mathcal{G}}_{\Delta} \)-component \( C \) of \( \Delta \setminus st_{\Delta}(v) \) is also a \( \tilde{\mathcal{G}}_v \)-component of \( \Gamma \setminus st_{\Gamma}(v) \). To see this, first recall that each element of \( Z \) is connected to every vertex of \( \Gamma \), so \( \Gamma \setminus st_{\Gamma}(v) = \Delta \setminus st_{\Delta}(v) \). Furthermore, it follows right from the definitions that two vertices \( x, y \in \Delta \setminus st_{\Delta}(v) \) are \( \tilde{\mathcal{G}}_{\Delta} \)-adjacent in \( \Delta \setminus st_{\Delta}(v) \) if and only if they are \( \tilde{\mathcal{G}} \)-adjacent in \( \Gamma \setminus st_{\Gamma}(v) \). The claim follows and implies that the partial conjugation of \( K \) by \( v \) defines an element of \( O \). As above, it will be denoted by \( \bar{\pi}_v^w \) and we note that it is a preimage of \( \pi_v^w \).

\[ \square \]

5.3.2 | Relative orderings in image and kernel

**Standing assumptions and notation**

From now on and until the end of Section 5, let \( O := \text{Out}^0(A_{\Gamma}; \mathcal{G}, \mathcal{H}') \), where \( \mathcal{G} \) and \( H \) are families of special subgroups of \( A_{\Gamma} \) such that \( \mathcal{G} \) is saturated with respect to \( (\mathcal{G}, H) \); note that saturation implies that \( P(H) \subseteq \mathcal{G} \). Set \( \leq := \leq_{\mathcal{G}} \) to be the \( \mathcal{G} \)-ordering on \( V(\Gamma) \).
Remark 5.6. Given an arbitrary relative automorphism group, there might be several ways of ‘representing’ this group by families of subgroups that are stabilised or acted trivially upon, that is, we might have

$$\text{Out}^0(A_Γ; G_1, H_1^1) = \text{Out}^0(A_Γ; G_2, H_2^1)$$

with \((G_1, H_1) \neq (G_2, H_2)\). However, if in this situation, we have both \(P(H_1) \subseteq G_1\) and \(P(H_2) \subseteq G_2\), the orderings \(\leq_{G_1}\) and \(\leq_{G_2}\) agree: By Lemma 5.2, for every \(v, w \in V(Γ)\), there is a chain of equivalences

\[
v \leq_{G_1} w \iff \rho_v^w \in \text{Out}^0(A_Γ; G_1, H_1^1) = \text{Out}^0(A_Γ; G_2, H_2^1) \iff v \leq_{G_2} w.
\]

In particular, the ordering \(\leq_G\) of \(V(Γ)\) where \(G\) is saturated with respect to \((G, H)\) is an invariant of the group \(\text{Out}^0(A_Γ; G, H^1)\); it depends on the transvections contained in this group but not on any other choices.

As mentioned above, a restriction homomorphism maps every transvection that is not contained in its kernel to a transvection of the same type. The consequences for the relative ordering in image and kernel are as follows.

**Lemma 5.7.** Let \(A_Δ \subseteq G\) be a special subgroup that is stabilised by \(O\) and let \(R_Δ\) denote the corresponding restriction homomorphism. If we write

\[
im R_Δ = \text{Out}^0(A_Δ; G_{im}, H_{im}^1) \quad \text{and} \quad \ker R_Δ = \text{Out}^0(A_Γ; G_{ker}, H_{ker}^1)
\]

with \(G_{im}\) and \(G_{ker}\) saturated with respect to \((G_{im}, H_{im})\) and \((G_{ker}, H_{ker})\), respectively, the following holds true.

1. For \(v, w \in Δ\), one has \(v \leq_{G_{im}} w\) if and only if \(v \leq w\).
2. For \(v \neq w \in V(Γ)\), one has \(v \leq_{G_{ker}} w\) if and only if \(v \in V(Γ) \setminus Δ\) and \(v \leq w\).

**Proof.** As \(G\) is saturated, we know that \(\im R_Δ = \text{Out}^0(A_Δ; G_Δ, H_Δ^1)\). For \(v, w \in Δ\), [19, Proposition 4.1] shows that \(v \leq_{G_Δ} w\) if and only if \(v \leq w\). Again because of the saturation of \(G\), we have \(P(H) \subseteq G\). Hence, \(P(H_Δ) \subseteq G_Δ\). As in Remark 5.6, it follows that \(v \leq_{G_Δ} w\) if and only if \(v \leq_{G_{im}} w\) for \(G_{im}\) saturated with respect to \((G_{im}, H_{im})\).

For the second point, we have \(v \leq_{G_{ker}} w\) if and only if \(\rho_v^w \in \ker R_Δ\). This is the case if and only if \(\rho_v^w\) is contained in \(O\) and acts trivially on \(A_Δ\). The claim now follows from Lemmas 5.2 and 5.3. 

**5.3.3 | Stabilisers in image and kernel**

Theorem 5.4 gives us a complete description of the image and kernel of a restriction map \(R_Δ: \text{Out}^0(A_Γ; G, H^1) \to \text{Out}(A_Δ)\) in the case where \(G\) is saturated with respect to \((G, H)\). However, if we consider a subgroup of the form

\[
\text{Stab}_G(A_Δ) = \text{Out}^0(A_Γ; G \cup \{A_Δ\}, H^1).
\]
the family $\mathcal{G} \cup \{A_\Delta\}$ is not necessarily saturated with respect to $(\mathcal{G} \cup \{A_\Delta\}, H)$ and its image under $R_\Delta$ is more difficult to describe. However, the parabolic subgroups we will consider in Section 6 are exactly of this form. The next two lemmas show that in special cases, we can describe their images under $R_\Delta$ without passing to saturated pairs.

**Lemma 5.8.** Assume that $O$ stabilises a special subgroup $A_\Delta \leq A_\Gamma$ and let $R_\Delta : O \to \text{Out}(A_\Delta)$ denote the corresponding restriction homomorphism. Take $\Theta \subseteq \Gamma$. Then,

1. $\text{Stab}_O(A_\Theta) \cap \ker R_\Delta = \text{Stab}_{\ker R_\Delta}(A_\Theta)$;
2. if $\Theta \subseteq \Delta$, one has $R_\Delta(\text{Stab}_O(A_\Theta)) = \text{Stab}_{\text{im} R_\Delta}(A_\Theta)$.

**Proof.** The first point becomes tautological after spelling out the definitions.

For the second point, the inclusion ‘$\subseteq$’ is clear. On the other hand, each $\Phi \in \text{im} R_\Delta$ can by definition be written as $\Phi = [\psi]_{A_\Delta}$, where $[\psi] \in O$ and $\psi(A_\Delta) = A_\Delta$. If $\Phi$ stabilises $A_\Theta$, we know that $\psi$ conjugates $A_\Theta$ to a subgroup of $A_\Delta$. Hence, $[\psi] \in \text{Stab}_O(A_\Theta)$ and the second claim follows.

**Lemma 5.9.** Assume that $\Gamma$ can be decomposed as a join $\Gamma = Z \ast \Delta$ where $Z$ is a complete graph and $A_Z \in \mathcal{G}$. Let $P_\Delta$ denote the projection map $O \to \text{Out}^0(A_\Delta)$. Then for every $\Theta \subseteq \Gamma$ one has

$$P_\Delta(\text{Stab}_O(A_\Theta)) = \text{Stab}_{\text{im} P_\Delta}(A_{\Theta \cap \Delta}).$$

**Proof.** The stabiliser $\text{Stab}_O(A_\Theta)$ is the same as the relative automorphism group $\text{Out}^0(A_\Theta; \mathcal{G} \cup \{A_\Theta\}, H^t)$. By Lemma 5.5, the image of this group is equal to

$$P_\Delta(\text{Stab}_O(A_\Theta)) = \text{Out}^0(A_\Theta; \mathcal{G}_\Delta \cup \{A_{\Theta \cap \Delta}\}, H^t_\Delta).$$

(3)

On the other hand, we have $\text{im} P_\Delta = \text{Out}^0(A_\Delta; \mathcal{G}_\Delta, H^t_\Delta)$, so the right-hand side of Equation (3) is also equal to $\text{Stab}_{\text{im} P_\Delta}(A_{\Theta \cap \Delta})$ and the claim follows.

### 5.4 Restrictions to conical subgroups

In this section, we define a family of special subgroups that will play an important role in our inductive arguments later on and study some properties of these special subgroups.

For a vertex $v \in V(\Gamma)$, define the following subgraphs of $\Gamma$:

$$\Gamma_{\geq v} := \{w \in V(\Gamma) \mid v \leq w\} \text{ and } \Gamma_{> v} := \{w \in V(\Gamma) \mid v < w\},$$

where $v < w$ if $v \leq w$ and $w \sim_\mathcal{G} v$. We define

$$A_{\geq v} := A_{\Gamma_{\geq v}} \text{ and } A_{> v} := A_{\Gamma_{> v}}$$

as the special subgroups of $A_\Gamma$ corresponding to these subgraphs. Note that these special subgroups only depend on the $\sim_\mathcal{G}$-equivalence class of $v$, that is, if $v \sim_\mathcal{G} w$, we have $A_{\geq v} = A_{\geq w}$ and $A_{> v} = A_{> w}$.

In the ‘absolute setting’ where $\mathcal{G}$ and $H$ are trivial and $\leq$ is equal to the standard ordering of $V(\Gamma)$, these special subgroups appear as admissible subgroups in the work of Duncan and Remeslennikov [20]. We will also refer to them as conical subgroups of $A_\Gamma$ as they are generated by elements corresponding to an upwards-closed cone in the Hasse diagram of the partial order that $\leq$ induces on the equivalence classes of $\sim_\mathcal{G}$ (see Figure 2).
The elements of $\text{Out}^0(A_\Gamma)$ are characterised among all elements of $\text{Out}(A_\Gamma)$ by the property that they stabilise these special subgroups. Namely, the following holds true.

**Lemma 5.10** [19, Proposition 3.3]. Let $\mathcal{G}_\geq$ be the set of special subgroups of $A_\Gamma$ of the form $A_{\geq v}$. Then

$$\text{Out}^0(A_\Gamma) = \text{Out}(A_\Gamma; \mathcal{G}_\geq).$$

In particular, each of these special subgroups is stabilised by all of $\text{Out}^0(A_\Gamma)$. We will need a relative version of this statement.

**Lemma 5.11.** Let $v, x \in V(\Gamma)$, let $K$ be a union of $\mathcal{G}_x$-components of $\Gamma \setminus \text{st}(x)$ and let $\pi^x_K \in O$ denote the corresponding partial conjugation. If $v \not\leq x$, the partial conjugation $\pi^x_K$ acts trivially on $A_{\geq v}$.

**Proof.** As $v$ is not smaller than $x$ with respect to $\leq$, there either is an element in $\text{lk}(v)$ which is not contained in $\text{st}(x)$ or there is $A_\Delta \in \mathcal{G}$ such that $\Delta$ contains $v$ but does not contain $x$. We claim that in both cases, $\Gamma_{\geq v}$ intersects at most one $\mathcal{G}_x$-component of $\Gamma \setminus \text{st}(x)$.

Indeed, if there is $y \in \text{lk}(v) \setminus \text{st}(x)$, one has $y \in \text{st}(w) \setminus \text{st}(x)$ for all $w \in \Gamma_{\geq v}$. Hence, all elements of $\Gamma_{\geq v}$ are adjacent to $x$ and $\Gamma_{\geq v} \setminus \text{st}(x)$ is contained in a single $\mathcal{G}_x$-component of $\Gamma \setminus \text{st}(x)$. If on the other hand for some $A_\Delta \in \mathcal{G}$, one has $v \in \Delta$, it follows that $w \in \Delta$ for all $w \in \Gamma_{\geq v}$. Now if $x \not\in \Delta$, this implies that all elements of $\Gamma_{\geq v}$ are $\mathcal{G}_x$-adjacent, so they in particular lie in the same $\mathcal{G}_x$-component. Either way, Lemma 5.3 implies that $\pi^x_K$ acts trivially on $A_{\geq v}$. \hfill \Box

**Proposition 5.12.** For every vertex $v \in V(\Gamma)$, the special subgroup $A_{\geq v}$ is stabilised by every element from $O$.

**Proof.** As $O$ is generated by the inversions, transvections and partial conjugation it contains, it suffices to prove the statement for each such element. As above, this can be done using Lemmas 5.2 and 5.3.

For inversions, there is nothing to show as they always stabilise every special subgroup. If we have a transvection $\rho^y_x \in O$, we must have $x \leq y$. The set $\Gamma_{\geq v}$ is upwards-closed with respect to $\leq$, hence $x \in \Gamma_{\geq v}$ implies $y \in \Gamma_{\geq v}$. It follows that $\rho^y_x$ stabilises $A_{\geq v}$. Given a partial conjugation $\pi^x_K \in O$, we either have $v \not\leq x$, in which case Lemma 5.11 implies that $\pi^x_K$ even acts trivially on $A_{\geq v}$, or we have $x \in \Gamma_{\geq v}$ which implies that $\pi^x_K$ stabilises $A_{\geq v}$. \hfill \Box
A consequence of this is that for every equivalence class $[v]_G$ of vertices of $\Gamma$, we have a restriction map

$$R_{\geq v} = R_{A_{\geq v}} : O \to \text{Out}^0(A_{\geq v}).$$

These maps are crucial for the line of argument in the following section. We will study some of their properties in Lemma 6.6.

### 6 | A SPHERICAL COMPLEX FOR $\text{Out}(A_\Gamma)$

In this section, we define maximal parabolic subgroups of $\text{Out}^0(A_\Gamma)$ in the general case. We then prove Theorem A which states that the coset complex associated to these parabolic subgroups is homotopy equivalent to a wedge of spheres.

**Notation and standing assumptions**

As before, let $\Gamma$ be a graph, $G$ and $H$ families of special subgroups of $A_\Gamma$ such that $G$ is saturated with respect to $(G, H)$, define $O := \text{Out}^0(A_\Gamma; G, H')$ and set $\leq := \leq_G$ to be the $G$-ordering on $V(\Gamma)$. Let $T_G$ denote the set of $\sim_G$-equivalence classes of vertices of $\Gamma$.

#### 6.1 Rank and maximal parabolic subgroups

**Definition 6.1.** We define the rank of $O$ as

$$\text{rk}(O) := \sum_{[v]_G \in T_G} (|[v]_G| - 1) = |V(\Gamma)| - |T_G|.$$

Now fix an ordering $[v]_G = \{v_1, \ldots, v_n\}$ on each equivalence class $[v]_G \in T_G$. For all $[v]_G \in T_G$ and $1 \leq j \leq n - 1$, let $\Delta^j_v \subset \Gamma$ be the full subgraph of $\Gamma$ with vertex set $\{v_1, \ldots, v_j\} \cup \Gamma_{\geq v}$.

**Lemma 6.2.** For all $[v]_G \in T_G$ and $1 \leq j \leq n - 1$, the stabiliser $\text{Stab}_O(A_{\Delta^j_v})$ is a proper subgroup of $O$.

**Proof.** Again, we use Lemmas 5.2 and 5.3: As all vertices of $[v]_G$ are equivalent with respect to $\leq_G$, the transvection $\rho_{\varepsilon v}^{v_n}$ is an element of $O$. However, this transvection does not stabilise $A_{\Delta^j_G}$ because $v_1$ is contained in $\Delta^j_G$ while $v_n$ is not.

For $[v]_G \in T_G$, let

$$P_{[v]_G} := \left\{\text{Stab}_O(A_{\Delta^j_v}) \mid 1 \leq j \leq n - 1\right\},$$

where if $|[v]_G| = 1$, this is to be understood as $P_{[v]_G} = \emptyset$.

**Definition 6.3.** We define the set of maximal standard parabolic subgroups of $O$ as the union

$$P(O) := \bigcup_{[v]_G \in T_G} P_{[v]_G}.$$
The reader might at this point want to verify that for the graph \( \Gamma \) depicted in Figure 2, one has \(|P(\text{Out}^0(A_\Gamma))| = 4\). The term ‘maximal’ parabolic will become clear in Section 8.1 where we will define and study parabolic subgroups of lower rank. As before, we will usually leave out the adjective ‘standard’ (see Remark 4.2).

**Remark 6.4.** We note the following properties of \( P(O) \) and \( \text{rk}(O) \).

1. \( \text{rk}(O) = |P(O)| \). We will also give an alternative interpretation of \( \text{rk}(O) \) in Section 8.3.
2. By Lemma 6.2, every element of \( P(O) \) is a proper subgroup of \( O \).
3. Following Remark 5.6, the definition of parabolic subgroups depends on the ordering chosen for each equivalence class, but not on the pair \((G, H)\) we chose to represent \( O \).
4. If \( O \) is equal to \( \text{GL}_n(Z) \) or a Foux-Rabinovitch group, we recover the definitions of parabolic subgroups in these groups as defined in Sections 4.1 and 4.2. Furthermore, \( \text{rk}(\text{GL}_n(Z)) = \text{rk}(\text{Out}(F_n)) = n - 1 \).

Note that it is possible that there is no \( G \)-equivalence class of size bigger than one. In this case, the rank of \( O \) is zero and \( P(O) \) is empty. For further comments on this, see Section 9.

### 6.2 The parabolic sieve

In this subsection, we explain the idea of the inductive argument that we will use to show sphericity of the coset complexes \( \text{CC}(O, P(O)) \).

**Outline of proof**

Whenever \( \Delta \subset \Gamma \) is stabilised by \( O \), the restriction map \( R_\Delta \) gives rise to a short exact sequence

\[
1 \to N \to O \xrightarrow{R_\Delta} Q \to 1
\]

and by Theorem 5.4, both \( N \) and \( Q \) are relative automorphism groups of RAAGs again. Using the considerations of Section 5, we will show that for the correct choice of \( \Delta \), every \( P \in P(O) \) satisfies the following dichotomy: Either \( R_\Delta(P) \) is contained in \( P(Q) \) or \( P \cap N \) forms an element of \( P(N) \). Applying a restriction homomorphism hence has the effect of a sieve on \( P(O) \) — some of the parabolic subgroups pass through and form parabolics of the quotient \( Q \) while others remain in the sieve and form parabolics of the subgroup \( N \). Now using the results of Section 3, this allows us to describe the homotopy type of \( \text{CC}(O, P(O)) \) in terms of the topology of the lower dimensional coset complexes \( \text{CC}(Q, P(Q)) \) and \( \text{CC}(N, P(N)) \). This is used for an inductive argument with two phases: We first apply restriction maps to conical subgroups (Section 6.2.1) and then analyse the homotopy type of coset complexes in the conical setting (Section 6.2.2). Concrete examples of this induction will be given in Section 7.

#### 6.2.1 Conical restrictions

**Lemma 6.5** (Induction step). Let \( v \in V(\Gamma) \) and let \( R := R_{\geq v} \) denote the corresponding restriction map to \( A_{\geq v} \). Then there is a homotopy equivalence

\[
\text{CC}(O, P(O)) \cong \text{CC}(\text{im } R, P(\text{im } R)) \ast \text{CC}(\text{ker } R, P(\text{ker } R)).
\]
We want to apply Corollary 3.19 to prove this statement. To do so, we have to show that for each $P \in \mathcal{P}(O)$, either $\ker R \subseteq P$ or $P$ contains all inversions, transvections and partial conjugations of $O$ that are not contained in $\ker R$. This is the content of the following lemma.

**Lemma 6.6.** The restriction map $R = R_{\geq v}$ has the following properties.

1. For all $\Delta \subseteq \Gamma_{\geq v}$, one has $\ker R \subseteq \text{Stab}_O(\Delta)$.
2. For all $w \in V(\Gamma)$, the following holds: If $\Delta \subseteq \Gamma_{\geq w}$, then

$$\Gamma_{\geq v} \cap \Gamma_{\geq w} \subseteq \Delta,$$

the stabiliser $\text{Stab}_O(\Delta)$ contains all inversions, transvections and partial conjugations of $O$ that are not contained in $\ker R$.

**Proof.** By Theorem 5.4, the kernel of $R$ consists of all elements from $O$ that act trivially on the special subgroup $A_{\geq v}$. This immediately implies the first claim.

For the second one, we again use Lemmas 5.2 and 5.3. First note that $\text{Stab}_O(\Delta)$ contains all inversions of $O$.

Next assume we have a transvection $\rho^y_x \in O$. If $x \notin \Delta$, the transvection is contained in $\ker R$. If on the other hand $x \in \Delta$, the transvection $\rho^y_x$ acts trivially on $A_{\Delta}$ and hence is contained in $\text{Stab}_O(\Delta)$. Now observe that the assumption that $\Gamma_{\geq v} \cap \Gamma_{\geq w} \subseteq \Delta$ implies that $\Delta \cap \Gamma_{\geq v}$ is equal to $\Gamma_{\geq v} \cap \Gamma_{\geq w}$, a set which is upwards-closed with respect to $\leq$. So if $x \in \Delta \cap \Gamma_{\geq v}$, we also have $y \in \Delta \cap \Gamma_{\geq v}$. Again it follows that $\rho^y_x \in \text{Stab}_O(\Delta)$.

Finally, consider a partial conjugation $\pi^x_K \in O$. If $v \not\leq x$, Lemma 5.11 implies that $\pi^x_K$ is contained in $\ker R$. This lemma also show that if $w \not\leq x$, the partial conjugation $\pi^x_K$ acts trivially on $A_{\geq w}$, and hence is contained in $\text{Stab}_O(\Delta)$. The only case that remains is that $x$ is greater than both $v$ and $w$, that is, $x \in \Gamma_{\geq v} \cap \Gamma_{\geq w}$. As we assumed that $\Gamma_{\geq v} \cap \Gamma_{\geq w} \subseteq \Delta$, this implies that $x \in \Delta$, so again $\pi^x_K \in \text{Stab}_O(\Delta)$. □

**Proof of Lemma 6.5.** Set $\mathcal{P} := \mathcal{P}(O)$.

Take $[w]_G \in T_G$ and $P = \text{Stab}_O(\Delta) = P_{[w]_G}$ with $\Delta = \Delta_{w}^v$ as above. If $v \leq w$, we have $\Delta \subseteq \Gamma_{\geq v}$. Hence by the first point of Lemma 6.6, we know that $\ker R \subseteq P$. If on the other hand $w < v$, one has $\Gamma_{\geq v} \cap \Gamma_{\geq w} = \Gamma_{\geq v} \not\subseteq \Delta$. Similarly if $v$ and $w$ are incomparable, one has $\Gamma_{\geq v} \cap \Gamma_{\geq w} \subseteq \Gamma_{\geq w} \not\subseteq \Delta$. In both cases, the second point of Lemma 6.6 tells us that $P$ contains all inversions, transvections and partial conjugations of $O$ which are not contained in $\ker R$. From this, it follows that

$$\mathcal{P}_{\ker R} = \left\{ P \in P_{[w]_G} \mid v \leq w \right\} \quad \text{and} \quad \mathcal{P}^{\ker R} = \left\{ P \in P_{[w]_G} \mid v \not\leq w \right\}, \quad (4)$$

with notation as defined in the paragraph before Section 3.2.1. Corollary 3.19 now shows that there is a homotopy equivalence

$$\text{CC}(O, \mathcal{P}) \approx \text{CC}(\text{im} R, \mathcal{P}) \ast \text{CC}(\ker R, \mathcal{P} \cap \ker R),$$

where $\mathcal{P} = \{ R(P) \mid P \in \mathcal{P}_{\ker R} \}$ and $\mathcal{P} \cap \ker R = \{ P \cap \ker R \mid P \in \mathcal{P}^{\ker \mathcal{R}} \}$. 

If we have $P \in \mathcal{P}_{ker R}$, there is $\Delta \subset \Gamma_{\geq v}$ such that $P = \text{Stab}_O(A_{\Delta})$. Using Lemma 5.8, it follows that $R(P) = \text{Stab}_{im R}(A_{\Delta})$. Lemma 5.7 implies that one has
\[
\bar{P} = \mathcal{P}(\text{im } R).
\]

For every $P = \text{Stab}_O(A_{\Delta}) \in \mathcal{P}_{ker R}$, we know by Lemma 5.8 that
\[
P \cap \ker R = \text{Stab}_{ker R}(A_{\Delta}).
\]

Write $\ker R = \text{Out}_0(A_{\Gamma}; \mathcal{G}_{ker}, \mathcal{H}_{ker})$ where $\mathcal{G}_{ker}$ is saturated with respect to $(\mathcal{G}_{ker}, \mathcal{H}_{ker})$. Then by Lemma 5.7, for $x, y \in V(\Gamma)$, we have $x \preceq \mathcal{G}_{ker} y$ if and only if $v \not\preceq x$ and $x \preceq y$. Combining this with Equation (4), it follows that
\[
P \cap \ker R = \mathcal{P}(\text{ker } R).
\]

This finishes the proof. □

For the first phase of our induction, we now use this iteratively in order to obtain the following proposition.

**Proposition 6.7.** There is a homotopy equivalence
\[
\text{CC}(O, \mathcal{P}(O)) \simeq^{*}_{[v]_G} \mathcal{C}(O_v, \mathcal{P}(O_v)),
\]

where for all $[v]_G \in T_\mathcal{G}$, one has $O_v = \text{Out}^0(A_{\geq v}; \mathcal{G}_v, \mathcal{H}_v)$ such that:

1. $\mathcal{G}_v$ is saturated with respect to $(\mathcal{G}_v, \mathcal{H}_v)$;
2. $\mathcal{H}_v = \mathcal{H}_{\geq v} \cup \{A_{\geq w} \mid v < w\}$;
3. for $x \neq y \in \Gamma_{\geq v}$, one has $x \preceq \mathcal{G}_v y$ if and only if $x \in [v]_\mathcal{G}$ and $x \preceq y$.

**Proof.** We want to inductively use the restriction maps $R_{\geq w}$. To do this, assume that we have shown that $\text{CC}(\text{Out}^0(A_\Theta), \mathcal{P}(O))$ is homotopy equivalent to a join of coset complexes of the form
\[
\text{CC}(U, \mathcal{P}(U)),
\]

where $U = \text{Out}^0(A_\Theta; \mathcal{E}, \mathcal{F}^*)$ with $\Theta = \Gamma_{\geq v}$ and such that the following hypotheses hold:

1. $\mathcal{E}$ is saturated with respect to $(\mathcal{E}, \mathcal{F}^*)$;
2. $\mathcal{F} = \mathcal{H}_{\Theta} \cup \mathcal{F}^*$, where $\mathcal{F}^* \subseteq \{A_{\geq w} \mid v < w\}$;
3. for all $w \in \Theta$, either $U$ acts trivially on $A_{\geq w}$ or $\Theta_{\geq w} = \Theta_{\geq w}$.

Note that *a priori*, there is slight ambiguity in writing $A_{\geq w}$ without specifying the ambient graph. Here we can, however, ignore this issue because for all $w > v$, we have $\Gamma_{\geq w} = \Theta_{\geq w}$. For technical reasons we allow $v$ to be a formal element 0 with $\Gamma_{\geq 0} := \Gamma$. These three hypotheses hold in particular for the initial case, where $v = 0$, $\mathcal{E} = \mathcal{G}$ and $\mathcal{F} = \mathcal{H}$.

Now assume that there is $w \in \Theta$ such that $U$ does not act trivially on $A_{\geq w}$. In this case, we have $\Theta_{\geq w} = \Theta_{\geq w}$, so by Proposition 5.12, the special subgroup $A_{\geq w} \leq A_\Theta$ is stabilised by $U$ and we can consider the restriction map $R : U \rightarrow \text{Out}^0(A_{\geq w})$. By Lemma 6.5, this yields a homotopy
equivalence

\[ CC(U, P(U)) \simeq CC(\text{im } R, P(\text{im } R)) \ast CC(\ker R, P(\ker R)). \]

By Theorem 5.4, we can write

\[ \ker R = \text{Out}_0(A_\Theta; E_{\ker}, (F \cup \{A_{\geq w}\}^f)), \]

where \( E_{\ker} \) is saturated with respect to the pair \( (E_{\ker}, F \cup \{A_{\geq w}\}) \). Furthermore, Lemma 5.7 together with the third hypothesis of our induction imply that for all \( w \in \Theta \), either \( \ker R \) acts trivially on \( A_{\geq w} \) or \( \Theta_{\geq_{\ker} w} = \Theta_{\geq w} \). It follows that \( CC(\ker R, P(\ker R)) \) satisfies the hypotheses of our induction.

Again using Theorem 5.4, we can write

\[ \text{im } R = \text{Out}_0(A_{\geq w}; E_{\text{im}}, F_{\text{im}}^f), \]

where \( E_{\text{im}} \) is saturated with respect to \( (E_{\text{im}}, F_{\text{im}}) \) and the elements of \( F_{\text{im}} \) are the special subgroups generated by the vertices of \( \Delta \cap \Gamma_{\geq w} \) for some \( \Delta \in F \). This implies that \( CC(\text{im } R, P(\text{im } R)) \) satisfies the first hypotheses of our induction. The third one is an immediate consequence of Lemma 5.7.

Now apply induction to these coset complexes. This process ends if we arrive at a case where for all \( w > v \), the group \( U \) acts trivially on \( A_{\geq w} \). But then we can set \( F = H_\Theta \cup \{A_{\geq w} \mid v < w\} \) and for \( x \neq y \), \( x \leq_{E} y \) is only possible if \( x \in [v]_\Gamma \). If \( v = 0 \), this means that the relative ordering on \( \Gamma_{\leq 0} = \Gamma \) is trivial, so \( P(U) = \emptyset \). If \( v \neq 0 \), the group \( O_v := U \) satisfies all conditions of the claim.

6.2.2 Coset complexes of conical RAAGs

We now want to deal with the coset complexes \( CC(O_v, P(O_v)) \) of conical RAAGs that we obtained in Proposition 6.7. This is why in this subsection, we impose the following assumptions.

**Standing assumptions**

Until the end of Section 6.2.2, we assume that:

1. there is a vertex \( v \in V(\Gamma) \) such that \( \Gamma = \Gamma_{\geq v} \), that is, every vertex of \( \Gamma \) is greater than or equal to \( v \) with respect to \( \leq \);
2. for all \( w > v \), the group \( O \) acts trivially on the special subgroup \( A_{\{w\}} \leq A_\Gamma \).

Observe that Item 1 implies that

\[ \text{for all } A_\Delta \in \mathcal{G}, \text{ we have } \Delta \subseteq \Gamma_{\geq v} : \] \hfill (5)

By Lemmas 5.2 and 5.3, every \( \Delta \subseteq \Gamma \) such that \( O \) stabilises \( A_\Delta \) must be upwards-closed with respect to \( \leq \). Hence, if \( \Delta \) intersects \( [v]_\mathcal{G} \) non-trivially, it follows that \( \Delta = \Gamma \). Furthermore, Item 2 implies that for all \( w > v \), the equivalence class \( [w]_\mathcal{G} \) is a singleton. Hence, we have

\[ P(O) = P_{[v]_\mathcal{G}}. \] \hfill (6)
In this situation, let

\[ Z := \{ w \in V(\Gamma) \mid v < w \text{ and } w \text{ is adjacent to } v \}. \]

We define the **group of twists by elements in** \( \Gamma_{>v} \) as the subgroup \( T \leq O \) generated by the transvections \( \rho_x^z \) with \( x \in [v]_G \) and \( z \in Z \).

**Lemma 6.8.** \( T \) is a free abelian group. Furthermore, \( \Gamma \) can be decomposed as a join \( \Gamma = Z \star \Delta \) and there is a short exact sequence

\[ 1 \to T \to O \xrightarrow{P_\Delta} \text{Out}^0(A_\Delta; G_\Delta, H^1_\Delta) \to 1. \]

**Proof.** If \( Z = \emptyset \), the statement is trivial, so we can assume that \( Z \) contains at least one element. By definition, we have \( Z \subseteq \text{lk}(v) \setminus [v]_G \). As every vertex of \( \Gamma \) is greater than or equal to \( v \) with respect to \( \preceq \), this implies that \( Z \) is a complete graph and we can write \( \Gamma = Z \star \Delta \).

Using the assumption that \( O \) acts trivially on \( A_{[w]} \) for all \( w > v \), Lemmas 5.2 and 5.3 imply that \( O \) acts trivially on the normal subgroup \( A_Z \lhd A_\Gamma \). Consequently, we have a well-defined projection map \( P_\Delta : O \to \text{Out}(A_\Delta) \). By Lemma 5.5, the image of this map is equal to \( \text{Out}^0(A_\Delta; G_\Delta, H^1_\Delta) \).

The description of the kernel \( \ker P_\Delta \) as the free abelian group \( T \) generated by the transvection \( \rho_x^z \) with \( x \in [v]_G \) and \( z \in Z \) follows from [16, Proposition 4.4] because \( O \) acts trivially on \( A_Z \) (see also [19, 5.1.4]). \( \square \)

**Lemma 6.9.** Let \( \Delta := \Gamma \setminus Z \) and let \( P_\Delta \) denote the corresponding projection map. There is a homotopy equivalence

\[ \text{CC}(O, P(O)) \simeq \text{CC}(\text{im} P_\Delta, P(\text{im} P_\Delta)). \]

**Proof.** By Lemma 6.8, we have a short exact sequence

\[ 1 \to T \to O \xrightarrow{P_\Delta} \text{im} P_\Delta \to 1, \]

where \( \text{im} P_\Delta = \text{Out}^0(A_\Delta; G_\Delta, H^1_\Delta) \). We first claim that every parabolic subgroup \( P \in P(O) \) contains \( T \). Indeed, we observed above that \( P(O) = P_{[v]_G} \) (see Equation (6)). By definition, every \( P \in P_{[v]_G} \) is of the form \( P = \text{Stab}_O(A_\Theta) \) for some \( \Theta \) containing \( \Gamma_{>v} \). The claim now follows from Lemma 5.3.

Hence, Corollary 3.19 yields a homotopy equivalence

\[ \text{CC}(O, P(O)) \simeq \text{CC}(\text{im} P_\Delta, \overline{P(O)}) * \emptyset = \text{CC}(\text{im} P_\Delta, \overline{P(O)}). \]

The ordering \( \leq_{\gamma_\Delta} \) is just the restriction of \( \leq \) to \( \Delta \), so Lemma 5.9 implies that \( \overline{P(O)} = P(\text{im} P_\Delta) \). \( \square \)

We now distinguish between the case where \( [v]_G \) is an abelian and the case where it is a free equivalence class.
Lemma 6.10. Assume that $\Gamma = \Gamma_{\geq v}$, where $[v]_G$ is an abelian equivalence class of size $n := |[v]_G| \geq 2$. Then the coset complex

$$CC(O, \mathcal{P}(O))$$

is homotopy equivalent to the Tits building associated to $GL_n(\mathbb{Q})$.

Proof. By Lemma 6.9, we have a homotopy equivalence

$$CC(O, \mathcal{P}(O)) \simeq CC(im P_\Delta, \mathcal{P}(im P_\Delta)),$$

where $\Delta = \Gamma \setminus Z$ and $im P_\Delta = \text{Out}^0(A_\Delta; G_\Delta, \mathcal{H}^I_\Delta)$.

By assumption, the abelian equivalence class $[v]_G$ contains at least two elements which are adjacent to each other. As every vertex of $\Gamma$ is greater than or equal to $v$ with respect to $\preceq$, this implies that every vertex of $\Gamma_{\geq v}$ must be adjacent to $v$. Hence, $Z = \Gamma_{\geq v}$ and $\Delta = [v]_G$. As observed above (Equation (5)), every $\Theta \subseteq \Gamma$ with $A_\Theta \in G$ is entirely contained in $\Gamma_{\geq v}$. Consequently, we have $G_\Delta = H_\Delta = \emptyset$ and

$$\text{Out}^0(A_\Delta; G_\Delta, \mathcal{H}^I_\Delta) = GL_n(\mathbb{Z}).$$

This means that $CC(O, \mathcal{P}(O)) \simeq CC(GL_n(\mathbb{Z}), \mathcal{P}(GL_n(\mathbb{Z})))$ and this coset complex is isomorphic to the Tits building associated to $GL_n(\mathbb{Q})$ by Proposition 4.3. \qed

In the setting of a free equivalence class, the situation is slightly more complicated: As before, we start by projecting away from $Z$, but we then might have to apply further restriction maps.

Lemma 6.11. Assume that $\Gamma = \Gamma_{\geq v}$, where $[v]_G$ is a free equivalence class of size $n := |[v]_G| \geq 2$. Then there is a special subgroup $A \leq A_\Gamma$ such that

$$A = F \ast A_1 \ast \cdots \ast A_k,$$

where $F$ is the free group of rank $n$ generated by $[v]_G$ and the coset complex

$$CC(O, \mathcal{P}(O))$$

is homotopy equivalent to the free factor complex of $A$ relative to $\{[A_1], \ldots, [A_k]\}$.

Proof. Again by Lemma 6.9, we have a homotopy equivalence

$$CC(O, \mathcal{P}(O)) \simeq CC(im P_\Delta, \mathcal{P}(im P_\Delta)),$$

where $\Delta = \Gamma \setminus Z$ and $im P_\Delta = \text{Out}^0(A_\Delta; G_\Delta, \mathcal{H}^I_\Delta)$. As noted above, the $G_\Delta$-ordering on $\Delta$ is just the restriction of $\leq$ to $\Delta$; in particular we have $[v]_{G_\Delta} = [v]_G$ and $\Delta = \Delta_{\geq v}$.

As no two vertices from $[v]_G$ are adjacent to each other, the link $\text{lk}_{\Gamma}(v)$ is entirely contained in $Z$, so every element of $[v]_G$ forms an isolated vertex of $\Delta$. This implies that $\Delta$ decomposes as a
disjoint union \( \Delta = [v]_\mathcal{G} \cup \bigsqcup \Delta_i \), where each \( \Delta_i \) is a \( G_\Delta \)-component of \( \Delta \). In particular, we have \( A_\Delta = A_{[v]_\mathcal{G}} \ast A_{\Delta_1} \ast \cdots \ast A_{\Delta_k} \).

Moreover, for all \( i \), the group \( \text{im} P_\Delta \) stabilises \( A_{\Delta_i} \); if \( \Delta_i \) contains at least two vertices, this is \([19, \text{Lemma 3.13.1}] \) and if \( \Delta_i \) is a singleton, the action on \( A_{\Delta_i} \) is trivial by assumption.

If there is an \( i \) such that \( \text{im} P_\Delta \) acts non-trivially on \( \Delta_i \), there is a non-trivial restriction map \( R : \text{im} P_\Delta \to \text{Out}(A_{\Delta_i}) \). Its kernel can be written as \( \ker R = \text{Out}^0(A_\Delta; G_{\ker}, (H_\Delta \cup \{A_{\Delta_i}\})^\perp) \), where \( G_{\ker} \) is saturated with respect to \((G_{\ker}, H_\Delta \cup \{A_{\Delta_i}\})\). One can easily check that each \( P \in \mathcal{P}(\text{im} P_\Delta) \) contains all the inversions, transvections and partial conjugations not contained in \( \ker R \): The kernel contains all inversions and transvections from \( \text{im} P_\Delta \) as well as the partial conjugations that have acting letter contained in \( [v]_\mathcal{G} \). The remaining partial conjugations are contained in all of the parabolic subgroups.

Hence, by Corollary 3.19, we have a homotopy equivalence

\[
\text{CC} \big( \text{im} P_\Delta, \mathcal{P}(\text{im} P_\Delta) \big) \simeq \emptyset \ast \text{CC}(\ker R, \mathcal{P} \cap \ker R) \simeq \text{CC}(\ker R, \mathcal{P} \cap \ker R).
\]

Lemma 5.7 implies that the ordering \( \leq_{G_{\ker}} \) agrees with \( \leq \) on \( \Delta \); hence using Lemma 5.8, we obtain \( \mathcal{P}(\ker R) = \mathcal{P} \cap \ker R \). All the \( A_{\Delta_i} \) are stabilised by \( \ker R \), so we can use induction and apply restriction maps until we reach the group \( \text{Out}^0(A_\Delta; \{A_{\Delta_i}\}^\perp) \). This group is equal to \( \text{Out}(A_\Delta, \{A_{\Delta_i}\}^\perp) \) and hence a Fouxe-Rabinovitch group. The claim now follows from Proposition 4.6.

Using the results of Section 4, the last two lemmas can be summarised as follows.

**Corollary 6.12.** Assume that \( \Gamma = \Gamma \geq v \), where \([v]_\mathcal{G} \) is an equivalence class of size \( n := |[v]_\mathcal{G}| \). Then \( \text{CC}(O, \mathcal{P}(O)) \) is \( (n - 2) \)-spherical.

**Proof.** If \( n = 1 \), the statement is trivial as in this case, the set \( \mathcal{P}(O) = P_{[v]_\mathcal{G}} \) is empty. Hence, the complex \( \text{CC}(O, \mathcal{P}(O)) \) is the empty set which we consider to be \((-1)\)-spherical (see Section 2.3). Now let \( n \geq 2 \). If \([v]_\mathcal{G} \) is abelian, Lemma 6.10 implies that the coset complex is homotopy equivalent to the Tits building associated to \( \text{GL}_n(Q) \) which is \((n - 2)\)-spherical by the Solomon–Tits theorem. If on the other hand \([v]_\mathcal{G} \) is free, it is by Lemma 6.11 homotopy equivalent to a relative free factor complex which is by Theorem 4.20 \((n - 2)\)-spherical as well.

---

### 6.3 Proof of Theorem A

We return to the general situation where \( \Gamma \) is any graph and \( \mathcal{G} \) and \( \mathcal{H} \) are any families of special subgroups of \( A_\Gamma \) such that \( \mathcal{G} \) is saturated with respect to \((\mathcal{G}, \mathcal{H})\). Recall that \( \leq \) denotes the \( \mathcal{G} \)-ordering of \( V(\Gamma) \) and \( T_\mathcal{G} \) denotes the set of associated \( \sim_\mathcal{G} \)-equivalence classes.

The only thing that is left to be done for the proof of Theorem A, which we restate below, is to collect the results obtained in Section 6.2.
Theorem 6.13. Let $O := \text{Out}^0(\mathcal{A}_\Gamma; \mathcal{G}, \mathcal{H}^i)$. The coset complex $\text{CC}(O, \mathcal{P}(O))$ is homotopy equivalent to a wedge of spheres of dimension $\text{rk}(O) - 1$.

Proof. By Proposition 6.7, we know that there is a homotopy equivalence

$$\text{CC}(O, \mathcal{P}(O)) \simeq \bigwedge_{[v]_G \in T_G} \text{CC}(O_v, \mathcal{P}(O_v)),$$

where for all $[v]_G \in T_G$, one has $O_v = \text{Out}^0(A_{\geq v}; \mathcal{G}_v, \mathcal{H}_v^i)$ such that:

1. $\mathcal{G}_v$ is saturated with respect to $(\mathcal{G}_v, \mathcal{H}_v^i)$;
2. $\mathcal{H}_v = H_{\geq v} \cup \{A_{\geq w} \mid v < w\}$;
3. for $x \neq y \in \Gamma_{\geq v}$, one has $x \leq_G y$ if and only if $x \in [v]_G$ and $x \leq y$.

Let $[v]_G \in T_G$. Condition 3 implies that the $\mathcal{G}_v$-equivalence class of $v$ in $\Gamma_{\geq v}$ is equal to $[v]_G$ and that all other $w \in \Gamma_{\geq v}$ are greater than $v$ with respect to $\leq_G$. Now Condition 2 implies that for all $w$ with $w > \mathcal{G}_v v$, the group $O_v$ acts trivially on $A_{[w]} \leq A_{\geq v}$. Hence, the assumptions of Section 6.2.2 are fulfilled and Corollary 6.12 implies that $\text{CC}(O_v, \mathcal{P}(O_v))$ is spherical of dimension $|\mathcal{G}_v| - 2$. It follows from Lemma 2.5 that the join of these complexes is spherical of dimension $\sum_{[v]_G \in T_G} (|\mathcal{G}_v| - 1) - 1 = \text{rk}(O) - 1$. □

7 | SUMMARY OF THE INDUCTIVE PROCEDURE AND EXAMPLES

7.1 | Consequences for the induction of Day–Wade

The proof of Theorem 6.13 relies on the inductive procedure defined in [19]: The authors there show that for every graph $\Gamma$, the group $\text{Out}^0(\mathcal{A}_\Gamma)$ has a subnormal series

$$1 = N_0 \leq N_1 \leq \cdots \leq N_k = \text{Out}^0(\mathcal{A}_\Gamma)$$

such that for all $i$, the quotient $N_{i+1}/N_i$ is isomorphic to either a free abelian group, to $\text{GL}_n(\mathbb{Z})$ or to a Fouxes-Rabinovitch group (see [19, Theorem A]). The methods we use in Section 6.2 provide more detailed information about this inductive procedure which decomposes $\text{Out}^0(\mathcal{A}_\Gamma)$ in terms of short exact sequences related to restriction and projection homomorphisms: We are able to give an explicit description of the restriction and projection maps that one has to use during the induction and of the base cases one obtains this way. In what follows, we will give a summary of these results (see also Figure 3).

To simplify notation, we will describe the decomposition of $O = \text{Out}^0(\mathcal{A}_\Gamma)$, however, all of this can also be stated in the more general case where $O$ is any relative automorphism group of a RAAG.

Step 1: First one iteratively restricts to conical subgroups $A_{\geq v}$ until one is left with relative automorphism groups that act trivially on all of their proper conical subgroups — for this, one needs to apply exactly one restriction map for every (standard) equivalence class of $V(\Gamma)$ and the order in which one applies the corresponding restriction maps does not change the base cases of this first induction step. One of these base cases is given by the intersection of the kernels of all the conical restriction maps; it is the group $\text{Out}^0(\mathcal{A}_\Gamma; \{A_{\geq v} \mid v \in V(\Gamma)\}^i)$ which does not contain any inversions or transvections. The other base cases are all of the form $\text{Out}^0(A_{\geq v}; \mathcal{G}; \{A_{\geq w} \mid w \in \Gamma_{>v}\}^i)$ for
BETWEEN BUILDINGS AND FREE FACTOR COMPLEXES

Figure 3 Decomposition of $\text{Out}^0(A_\Gamma)$. Step 1 is coloured in blue, Step 2 in green and Step 3 in magenta.

some $v \in V(\Gamma)$ and some family $G$ of special subgroups of $A_{\geq v}$. There is exactly one such base case for every equivalence class $[v]$ of $V(\Gamma)$ and it is generated by all the restrictions to $A_{\geq w}$ of inversions, transvections and partial conjugations of $\text{Out}^0(A_\Gamma)$ that act trivially on $A_{\geq w}$ for every $w > v$.

**Step 2:** Now for each of these groups, one applies the (possibly trivial) projection map $P_\Delta$, where $\Delta := \Gamma_{\geq v} \setminus Z$ and $Z$ is the full subgraph of $\Gamma_{\geq v}$ consisting of all those vertices of $\Gamma$ which are adjacent to $v$ and strictly greater than $v$ with respect to the standard ordering on $V(\Gamma)$. The kernel of this projection map is given by the free abelian group $T$ generated by all twist of elements in $[v]$ by elements in $\Gamma_{> v}$. We now have to distinguish two cases: If $[v]$ is an abelian equivalence class of size $n \geq 2$, then the image of $P_\Delta$ is given by $\text{Out}(A[v] \ast A_{\Delta_1} \ast \ldots \ast A_{\Delta_k}; G_\Delta, H_\Delta^i), [v]$ free.

**Step 3:** If $[v]$ is a free equivalence class, the graph $\Delta$ decomposes as a disjoint union $\Delta = [v] \sqcup \bigsqcup \Delta_i$, where each $\Delta_i$ is a relative connected component of $\text{im}(P_\Delta)$. One can show that the $\Delta_i$ are precisely the non-empty intersections $\Delta_i = (\Delta \setminus [v]) \cap \Gamma_i$, where $\Gamma_i$ is a connected component of $\Gamma \setminus \text{lk}(v)$. We now iteratively apply the restriction maps $R_{\Delta_i}$. This yields two kinds of base cases: The first one is by the intersection of the kernels of all the $R_{\Delta_i}$ and can be described as the Fouxe-Rabinovitch group $\text{Out}(A_\Delta; \{A_{\Delta_i}\}_i^\Gamma)$. The second one is given by the images of the restriction maps. For each $i$, this is a relative automorphism group of $A_{\Delta_i}$; as $\Delta_i \subseteq \Gamma_{> v}$, this group contains no inversions or transvections and is generated by partial conjugations.

**The base cases**

In summary, our induction yields the following base cases:

1. the ‘left-most’ kernel $\text{Out}^0(A_\Gamma; \{A_{\geq v} \mid v \in V(\Gamma)\}^\Gamma)$;
2. for every abelian equivalence class $[v]$ of size $n \geq 2$:
   (a) the free abelian group $T$ generated by all twist of elements in $[v]$ by elements in $\Gamma_{>v}$, which has rank $n \cdot |\Gamma_{>v} \cap \text{lk}(v)|$;
   (b) $\text{Out}(A_{[v]}) \cong \text{GL}_n(\mathbb{Z})$;
   
3. for every free equivalence class $[v]$:
   (a) the free abelian group $T$ generated by all twist of elements in $[v]$ by elements in $\Gamma_{>v}$, which has rank $n \cdot |\Gamma_{>v} \cap \text{lk}(v)|$;
   (b) for every connected component $\Gamma_i$ of $\Gamma \setminus \text{lk}(v)$ such that
   \[
   \Delta_i := (\Gamma_{>v} \setminus \text{lk}(v)) \cap \Gamma_i \neq \emptyset:
   \]
   a subgroup of $\text{Out}^0(A_{\Delta_i})$ generated by partial conjugations;
   (c) a Fouxe-Rabinovitch group $\text{Out}(A_{\Delta}, \{A_{\Delta_i}\}_{i})$, where $\Delta = \Gamma_{>v} \setminus \text{lk}(v)$ and the $\Delta_i$ are as in Item 3(b).

The base cases having a non-empty set of parabolic subgroups are Items (2)b and 3(c) if $|[v]| \geq 2$. Note that we allow $|v| = 1$ in Item 3(c) which results in $\text{Out}([v]) \cong \mathbb{Z}/2\mathbb{Z}$ if $v$ is maximal. One should mention that Items (1) and 3(b) are not necessarily base cases of the induction of Day–Wade: There might be further non-trivial restriction and projection maps and after applying them one can decompose these groups into Fouxe-Rabinovitch groups and free abelian groups generated by partial conjugations.

7.2 | Examples

7.2.1 | String of diamonds

Let $\Gamma$ be the string of $d$ diamonds (see Figure 4), as considered in [15, Section 5.3; 19, Section 6.3.1]. Assume $d \geq 2$. The standard equivalence classes of $\Gamma$ are given by

$$[a_i] = [b_i] = \{a_i, b_i\}, \quad 1 \leq i \leq d \quad \text{and} \quad [c_i] = \{c_i\}, \quad 0 \leq i \leq d.$$ 

The conical subgroups here are

$$\Gamma_{\geq a_i} = [a_i] = \{a_i, b_i\}, \quad 1 \leq i \leq d,$$

$$\Gamma_{\geq c_i} = [c_i], \quad 1 \leq i \leq d - 1,$$

$$\Gamma_{\geq c_0} = [c_0] \cup [c_1] \cup [a_1] \quad \text{and} \quad \Gamma_{\geq c_d} = [c_d] \cup [c_{d-1}] \cup [a_d].$$
If we order \([a_i, b_i]\) as \((a_i, b_i)\), the family of maximal parabolic subgroups of the group \(O := \text{Out}^0(A_\Gamma)\) is given as
\[
P(O) = \{\text{Stab}_O((a_i)) \mid 1 \leq i \leq d\}
\]
and we have \(\text{rk}(O) = d\), that is, \(\text{CC}(O, P(O))\) is \((d - 1)\)-spherical.

After restricting to these conical subgroups (Step 1 of our induction), we are left with the following base cases.

1. The left-most kernel \(\text{Out}^0(A_\Gamma; \{ A_{\geq v} \mid v \in V(\Gamma) \}^\ell)\) which here is generated by all the partial conjugations of \(O\).
2. For all \(1 \leq i \leq d - 1\): the group \(\text{Out}((a_i, b_i)) \cong \mathbb{Z}/2\mathbb{Z}\).
3. \(\text{Out}^0(\langle c_0, c_1, a_1, b_1 \rangle; \{\langle c_1 \rangle, \langle a_1, b_1 \rangle\}^\ell) = \text{Out}^0(\langle c_0, c_1, a_1, b_1 \rangle; \{\langle c_1 \rangle, \langle a_1, b_1 \rangle\}^\ell)\).
4. \(\text{Out}^0(\langle c_d, c_{d-1}, a_d, b_d \rangle; \{\langle c_{d-1}, c_d, a_d, b_d \rangle; \{\langle c_{d-1}, c_d, a_d, b_d \rangle\}^\ell)\).
5. For all \(1 \leq i \leq d\): the group \(\text{Out}((a_i, b_i)) \cong \text{Out}(\mathbb{F}_2)\).

Only the groups of the last item have a non-empty set of parabolic subgroups (each given by the singleton \(\{\text{Stab}_\text{Out}((a_i, b_i))((a_i))\}\)). All items but the first one describe Fouxe-Rabinovitch groups, so the induction already ends here and we do not have to apply Steps 2 and 3.

The following direct argument gives a more explicit description of the coset complex: For all \(i\), we have a surjective restriction map \(O \rightarrow \text{Out}((a_i, b_i))\). These can be amalgamated to a map \(R : O \rightarrow \bigoplus_{i=1}^n \text{Out}(<a_i, b_i>)\). We have \(\ker(R) \subseteq P_i := \text{Stab}_O((a_i))\) for all \(i\), so by Corollary 3.19,
\[
\text{CC}(O, P(O)) \cong \text{CC}(\bigoplus_{i=1}^n \text{Out}(<a_i, b_i>), P(O)) \cong \ast_i \text{CC}(\text{Out}(<a_i, b_i>), \text{Stab}(<a_i>)).
\]
(For the second isomorphism, we used that \(P_i\) contains \(\text{Out}((a_j, b_j))\) for \(j \neq i\).) Each factor in this join is a copy of the free factor complex associated to \(\text{Out}(F_2)\) and \(O\) acts on their join in the obvious way.

### 7.2.2 Trees

Let \(\Gamma\) be a tree, define \(O := \text{Out}^0(A_\Gamma)\) and, to simplify notation, assume that \(|V(\Gamma)| \geq 3\). Let \(L\) denote the set of leaves of \(\Gamma\). For each leaf \(l\), let \(z_l\) denote the (unique) vertex adjacent to \(l\) and let \(Z = \{z_l \mid l \in L\}\) be the set of vertices of \(\Gamma\) that are adjacent to some leaf. Then we have
\[
[v] = \begin{cases} 
\{v\}, & v \in V(\Gamma) \setminus L; \\
\text{lk}(z_v) \cap L, & v \in L.
\end{cases}
\]

The conical subgroups are given by
\[
\Gamma_{\geq v} = \{v\}, v \in V(\Gamma) \setminus L \quad \text{and} \quad \Gamma_{\geq z} = \text{st}(z_l), l \in L.
\]

Now for each \(z \in Z\), let \(\{v_1^z, ..., v_{k_z}^z\} = \text{lk}(z) \setminus L\) be the non-leaf vertices adjacent to \(z\) and \(\{l_1^z, ..., l_{n_z}^z\} = \text{lk}(z) \cap L\) the leaves adjacent to \(z\) (see Figure 5). Then
\[
P(O) = \bigcup_{z \in Z} \left\{\text{Stab}_O \left(\left\langle l_1^z, ..., l_{n_z}^z, v_1^z, ..., v_{k_z}^z, z\right\rangle\right) \mid 1 \leq i \leq n_z - 1\right\}
\]
and \( \text{rk}(O) = \sum_{z \in Z}(n_z - 1) = |L| - |Z| \), which implies that \( CC(O, P(O)) \) is \((|L| - |Z| - 1)\)-spherical.

Step 1 of our induction leads to the following base cases:

1. the left-most kernel \( \text{Out}^0(A_{\Gamma}; \{ A_{\geq v} | v \in V(\Gamma) \}) \) generated by all the partial conjugations of \( O \) acting trivially on \( \text{st}(z) \) for all \( z \in Z \);
2. for all \( v \in V(\Gamma) \setminus L \): the group \( \text{Out}(\langle v \rangle) \cong \mathbb{Z} / 2\mathbb{Z} \);
3. for all \( z \in Z \): the group \( \text{Out}^0(A_{\text{st}(z)}; \{ \langle v_1^z \rangle, \ldots, \langle v_{k_z}^z \rangle \}) \).

In Step 2, we apply for each group of the last item the projection map \( \mathcal{P}_\Delta \), where \( \Delta = \text{lk}(z) \). Its kernel is a free abelian group of rank \( n_z \), generated by the twists of the leafs adjacent to \( z \). The image of this projection map is the Fouxe-Rabinovitch group

\[
\text{Out}^0 \left( A_{\text{lk}(z)}; \{ \langle v_1^z \rangle, \ldots, \langle v_{k_z}^z \rangle \} \right).
\]

It contains \( n_z - 1 \) maximal parabolic subgroups, given by the stabilisers of \( \langle l_1^z, \ldots, l_i^z, v_1^z, \ldots, v_{k_z}^z \rangle \), \( 1 \leq i \leq n_z - 1 \). Again, we do not have to apply Step 3.

### 7.2.3 Constructions

This section does not really contain examples but rather shows how to obtain new examples from known ones.

#### Direct product

Let \( \Gamma_1 \) and \( \Gamma_2 \) be graphs and let \( \Gamma := \Gamma_1 * \Gamma_2 \) be their join. On the level of RAAGs, this means that \( A_{\Gamma} = A_{\Gamma_1} \times A_{\Gamma_2} \). Let \( \leq, \leq_1, \leq_2 \) be the standard orderings and \([ \cdot ], [ \cdot ]_1, [ \cdot ]_2 \) the corresponding equivalence classes of \( \Gamma, \Gamma_1, \Gamma_2 \), respectively. Let \( Z, Z_1, Z_2 \) be the (possibly trivial) subgraphs of \( \Gamma, \Gamma_1, \Gamma_2 \) consisting of all vertices that are adjacent to every vertex of the corresponding graph; clearly, \( Z = Z_1 * Z_2 \). It is easy to see that for \( v_i \in \Gamma_i \), one has

\[
[v_i] = \begin{cases} 
Z, & v_i \in Z_i; \\
[v_i], & v_i \notin Z_i; 
\end{cases} \quad \text{and} \quad \Gamma_{\geq v_i} = (\Gamma_i)_{\geq v_i} \cup Z_j.
\]
We do not spell out the consequences for all of the induction, but would like to point out the following implication for the ranks of the corresponding automorphism groups and hence the dimensions of the associated coset complexes

$$\text{rk}(\text{Out}^0(A_\Gamma)) = \begin{cases} \text{rk}(\text{Out}^0(A_{\Gamma_1})) + \text{rk}(\text{Out}^0(A_{\Gamma_2})), & Z_i = \emptyset \text{ for some } i; \\ \text{rk}(\text{Out}^0(A_{\Gamma_1})) + \text{rk}(\text{Out}^0(A_{\Gamma_2})) + 1, & \text{otherwise}. \end{cases}$$

Note that $Z_i = \emptyset$ is equivalent to saying that the centre $Z(A_{\Gamma_i})$ is trivial.

A particularly simple instance of this is the situation where $A_{\Gamma} = F_n \times F_m$ with $n, m > 1$. Then, $\text{Out}^0(A_\Gamma) = \text{Out}(F_n) \times \text{Out}(F_m)$ and the coset complex $\text{CC}(\text{Out}^0(A_\Gamma), \mathcal{P}(\text{Out}^0(A_\Gamma)))$ is isomorphic to the the join of the free factor complexes associated to $\text{Out}(F_n)$ and $\text{Out}(F_m)$.

**Free product**

Let $\Gamma := \Gamma_1 \sqcup \Gamma_2$ be the disjoint union of the graphs $\Gamma_1$ and $\Gamma_2$, that is, $A_{\Gamma} = A_{\Gamma_1} \ast A_{\Gamma_2}$, and keep the notation of the prior paragraph otherwise. Let $D, D_1, D_2$ be the (possibly trivial) subgraphs of $\Gamma, \Gamma_1, \Gamma_2$ consisting of all their isolated vertices; we have $D = D_1 \sqcup D_2$. For $v_i \in \Gamma_i$, one has

$$[v_i] = \begin{cases} D_i, & v_i \in D_i; \\ [v_i], & v_i \notin D_i; \end{cases} \quad \text{and} \quad \Gamma \geq v_i = \begin{cases} \Gamma, & v_i \in D_i; \\ (\Gamma_i) \geq \Gamma, & v_i \notin D_i. \end{cases}$$

Similar to the case of direct products, this implies

$$\text{rk}(\text{Out}^0(A_\Gamma)) = \begin{cases} \text{rk}(\text{Out}^0(A_{\Gamma_1})), & D_i = \emptyset \text{ for some } i; \\ \text{rk}(\text{Out}^0(A_{\Gamma_1})), & \text{otherwise}. \end{cases}$$

This allows for example to generalise the example of tree-RAAGs given above to the setting of forests.

**Complement graph**

For a graph $\Gamma$, let $\Gamma^c$ denote its complement, that is, the graph with vertex set $V(\Gamma)$ where $u$ and $w$ form an edge if and only if they do not form an edge in $\Gamma$. Let $\leq_c$ and $[\cdot]_c$ denote the standard ordering and its equivalence classes on $\Gamma^c$. Then it is easy to see that

$$[v] = [v]_c \quad \text{and} \quad v \leq_c w \iff w \leq v.$$ 

In particular, one has $\text{rk}(\text{Out}^0(A_{\Gamma^c})) = \text{rk}(\text{Out}^0(A_\Gamma))$. This also explains the analogy between the settings of direct and free products considered above.

### 8 COHEN–MACAULAYNESS, HIGHER GENERATION AND RANK

In this section, we generalise the results of Section 6: We show that the coset complex of parabolic subgroups of a relative automorphism group $O$ of a RAAG is not only spherical, but even Cohen–
Macaulay. This is used to determine the degree of generation that families of (possibly non-maximal) parabolic subgroups provide. We also give an interpretation of the rank in terms of a ‘Weyl group’ of $O$.

**Notation and standing assumptions**

As before, let $\Gamma$ be a graph, $\mathcal{G}$ and $\mathcal{H}$ families of special subgroups of $A_{\Gamma}$ such that $\mathcal{G}$ is saturated with respect to $(\mathcal{G}, \mathcal{H})$, define $O := \text{Out}^0(A_{\Gamma}; \mathcal{G}, \mathcal{H'})$ and set $\leq := \leq_{\mathcal{G}}$ to be the $\mathcal{G}$-ordering on $V(\Gamma)$. Let $T_{\mathcal{G}}$ denote the set of $\sim_{\mathcal{G}}$-equivalence classes of vertices of $\Gamma$.

### 8.1 Cohen–Macaulayness

For coset complexes, the Cohen–Macaulay property can be characterised as follows.

**Theorem 8.1** [9, Theorem 2.11]. Let $G$ be a group and $\mathcal{H}$ be a finite family of subgroups of $G$. Then $CC(G, \mathcal{H})$ is homotopy Cohen–Macaulay if and only if every $\mathcal{H}' \subseteq \mathcal{H}$ is $(|\mathcal{H}'| - 1)$-generating for $G$.

This allows us to generalise Theorem 6.13 in the following way.

**Theorem 8.2.** Let $O := \text{Out}^0(A_{\Gamma}; \mathcal{G}, \mathcal{H'})$. The coset complex $CC(O, P(O))$ is homotopy Cohen–Macaulay.

**Proof.** By Theorem 8.1, it suffices to show that for all $P' \subseteq P$, the coset complex $CC(O, P')$ is $(|P'| - 1)$-spherical. This can be done following the induction of Section 6.2: We first iteratively apply restriction maps to conical subgroups as in Section 6.2.1. In each step, the parabolic subgroups in $P'$ satisfy a dichotomy that allows us to apply Corollary 3.19. We get an analogue of Proposition 6.7: The coset complex $CC(O, P')$ is homotopy equivalent to the join $\ast_{[v]_{\mathcal{G}} \in T_{\mathcal{G}}} CC(O_v, P_v)$, where $O_v$ is exactly as in Proposition 6.7 and $P_v \subseteq P(O_v)$. There is a one-to-one correspondence between the parabolic subgroups occurring in the join and the elements of $P'$; in particular, $\sum_{[v]_{\mathcal{G}} \in T_{\mathcal{G}}} |P_v| = |P'|$. One now follows the arguments of Section 6.2.2 to show that if $[v]_{\mathcal{G}}$ is an abelian equivalence class of size $n$, we have $CC(O_v, P_v) \simeq CC(GL_n(\mathbb{Z}), Q)$ with $Q \subseteq P(GL_n(\mathbb{Z}))$ and that if $[v]_{\mathcal{G}}$ is a free equivalence class, we have

$$CC(O_v, P_v) \simeq CC(\text{Out}(A; A', Q),$$

where $A = F_n \ast A_1 \ast \cdots \ast A_k$, $A = \{A_1, \ldots, A_k\}$ and $Q \subseteq P(\text{Out}(A; A'))$. Both $CC(GL_n(\mathbb{Z}), P(GL_n(\mathbb{Z})))$ and $CC(\text{Out}(A; A'), P(\text{Out}(A; A')))$ are homotopy Cohen–Macaulay: In the first case, this holds because the coset complex is isomorphic to the building associated to $GL_n(\mathbb{Q})$ (see Proposition 4.3), in the second case this is Theorem 4.21. Hence, Theorem 8.1 implies that $CC(O_v, P_v)$ is spherical of dimension $|P_v| - 1$. It now follows from Lemma 2.5 that $CC(O, P')$ is spherical of dimension $(\sum_{[v]_{\mathcal{G}} \in T_{\mathcal{G}}} |P_v|) - 1 = |P'| - 1$.

An immediate consequence of this is that $CC(O, P(O))$ is a chamber complex, that is, that each pair $\sigma, \tau$ of facets of $CC(O, P(O))$ can be connected by a sequence of facets $\sigma = \tau_1, \ldots, \tau_k = \tau$ such that for all $1 \leq i \leq k$, the intersection of $\tau_i$ and $\tau_{i+1}$ is a face of codimension 1 (see [4, Proposition 11.7; 9, Remark 2.8]).
8.2 Parabolic subgroups of lower rank

Definition 8.3. Let \( r := \text{rk}(O) \) and \( 1 \leq m \leq r - 1 \). We define the family of rank-\( m \) standard parabolic subgroups of \( O \) as the set of all intersections of \( (r - m) \) distinct maximal standard parabolic subgroups,

\[
P_m(O) := \{ P_1 \cap \cdots \cap P_{r-m} \mid P_1, \ldots, P_{r-m} \text{ distinct elements of } \mathcal{P}(O) \}.
\]

In particular, we have \( \mathcal{P}(O) = P_{r-1}(O) \).

Every parabolic subgroup of \( O \) is itself a relative automorphism group of \( A_{\Gamma} \). The term ‘rank-\( m \)’ parabolic subgroup is justified by the following proposition.

Proposition 8.4. For all \( P \in \mathcal{P}_m(O) \), we have \( \text{rk}(P) = m \).

Proof. For every \( P \in \mathcal{P}_m(O) \), there is a \( V \subset V(\Gamma) \) and for every \( v \in V \) a subset \( J_v \subset \{1, \ldots, |[v]|_\mathcal{G} \} \) such that

\[
P = \text{Out}^0(\mathcal{A}_\Gamma; \mathcal{G}', H'), \text{ where } \mathcal{G}' = \mathcal{G} \cup \{ A_{\Delta_j^v} \mid v \in V, j \in J_v \}, \quad \Delta_j^v = \{ v_1, \ldots, v_j \} \cup \Gamma_{\succ v} \text{ and } \sum_{v \in V} |J_v| = r - m.
\]

As \( \mathcal{G} \) contains \( P(H) \), so does \( \mathcal{G}' \). It is easy to check that if \( v \in V \), the \( \mathcal{G} \)-equivalence class \( [v]_\mathcal{G} \) can be written as the disjoint union of \( (|J_v| + 1) \)-many \( \mathcal{G}' \)-equivalence classes and that otherwise, one has \( [v]_\mathcal{G} = [v]_{\mathcal{G}'} \). From this, the claim follows immediately. \( \square \)

Theorem B is now an easy corollary of Cohen–Macaulayness of \( \text{CC}(O, \mathcal{P}(O)) \) and the results of [9].

Corollary 8.5. The family \( \mathcal{P}_m(O) \) of rank-\( m \) parabolic subgroups of \( O \) is \( m \)-generating, the corresponding coset complex \( \text{CC}(O, \mathcal{P}_m(O)) \) is \( m \)-spherical.

Proof. As the coset complex is homotopy Cohen–Macaulay by Theorem 8.2, this is an immediate consequence of [9, Theorem 2.9]. \( \square \)

In the case where \( O = \text{GL}_n(\mathbb{F}) \), this is [1, Theorem 3.3]; for this case, 2-generation was also already shown by Tits in [44, Section 13]. Observe that although \( \text{CC}(O, \mathcal{P}_m(O)) \) is \( m \)-spherical, it is a priori a complex of dimension

\[
|\mathcal{P}_m(O)| = \binom{r}{m} - 1.
\]

Presentations for \( O \)

A consequence of higher generation is that one can obtain presentations of \( O \) from presentations of the parabolic subgroups as follows: Write \( \mathcal{P}_m(O) = \{ P_1, \ldots, P_{\binom{r}{m}} \} \). For each \( i \), let \( L_i \) be the set of all inversions, transvections and partial conjugations of \( O \) that are contained in \( P_i \). By Theorem 5.1,
the set $L_i$ generates $P_i$. Let $P_i = \langle L_i \mid R_i \rangle$ be a presentation for $P_i$. Then we have the following corollary.

**Corollary 8.6.** Let $1 \leq m \leq r - 1$ and $k := \binom{r}{m}$. Then,

1. the union $\bigcup_{i=1}^{k} L_i$ is a generating set for $O$;
2. if $m \geq 2$, a presentation for $O$ is given by $O = \langle \bigcup_{i=1}^{k} L_i \mid \bigcup_{i=1}^{k} R_i \rangle$.

**Proof.** This follows from Corollary 8.5 and Theorem 3.8.

### 8.3 Interpretation of rank in terms of Coxeter groups

The rank of a group with a $BN$-pair is given by the rank of the associated Weyl group $W$, which is a Coxeter group. This is also true in the setting of relative automorphism groups of RAAGs as we will see in what follows.

**Definition 8.7.** Let $\text{Aut}(\Gamma)$ denote the group of graph automorphisms of $\Gamma$. This group embeds in $\text{Out}(A_\Gamma)$ and we define $\text{Aut}^0(\Gamma)$ as the intersection $\text{Aut}(\Gamma) \cap O$.

If $O$ is equal to $\text{Out}(F_n)$ or $\text{GL}_n(\mathbb{Z})$, we have $\text{Aut}^0(\Gamma) = \text{Aut}(\Gamma) = \text{Sym}(n)$, the Weyl group associated to $\text{GL}_n(\mathbb{Q})$, which has rank $n - 1$. In general, $\text{Aut}^0(\Gamma)$ can be seen as the group of ‘algebraic’ graph automorphisms of $\Gamma$. It appears as $\text{Sym}^0(\Gamma)'$ in [13, Section 3.2] where it is studied under the additional assumption that $\Gamma$ be connected and triangle-free.

**Lemma 8.8.** The group $\text{Aut}^0(\Gamma)$ is naturally isomorphic to the direct product

$$\text{Aut}^0(\Gamma) \cong \bigoplus_{[v]_C \in T_C} \text{Sym}([v]_C).$$

**Proof.** If $|[v]_C| > 1$, the group $O$ contains for all $x, y \in [v]_C$ the transvection $t_y^x$ and the inversion $t_y$. It follows that the full group $\text{Sym}([v]_C)$ of permutations of $[v]_C$ is contained in $\text{Aut}^0(\Gamma)$, so the direct product $\bigoplus_{[v]_C \in T_C} \text{Sym}([v]_C)$ is a subgroup of $\text{Aut}^0(\Gamma)$.

It remains to show that this group does not contain any other elements, that is, that every element of $\text{Aut}^0(\Gamma)$ preserves all the $C$-equivalence classes of $V(\Gamma)$. To see this, assume that $\phi \in \text{Aut}(A_\Gamma)$ represents an element of $O$ such that $\phi(v) = v'$ for some $v, v' \in V(\Gamma)$. We will show that $v \sim_C v'$:

For $x \in V(\Gamma)$ and a word $w$ in the alphabet $V(\Gamma)^{\pm 1}$, let $\#_x(w) \in \mathbb{Z}$ denote the number of occurrences of $x$ in $w$, counted with sign. For $g \in A_\Gamma$, let $\#_x(g) \in \mathbb{Z}/2\mathbb{Z}$ be the image of $\#_x(w)$ in $\mathbb{Z}/2\mathbb{Z}$, where $w$ is a word representing $g$ — this number only depends on $g$ and not on the chosen representative $w$. Now assume that $v \neq v'$. Then clearly, $\#_v(v) = 0$ and $\#_{v'}(\phi(v)) = \#_{v'}(v') = 1$. Writing $\phi$ as a product of inversions, transvections and partial conjugations, it follows that there must be such a Laurence generator $[\psi] \in O$ with $\#_{v'}(\phi(v)) = 1$. This is only possible if $\psi$ is given by the transvection $t_{v'}$. However, if this is contained in $O$, we know that $v \leq v'$. As $\phi^{-1}$ sends $v'$ to $v$, we also have $v' \leq v$, hence $v \sim_C v'$.

Recall that the **rank of a Coxeter system** $(W, S)$ is given by $\text{rk}(W, S) = |S|$. 

**Corollary 8.9.** There is a subset $S \subset \text{Aut}^0(\Gamma) \leq O$ such that $(\text{Aut}^0(\Gamma), S)$ is a Coxeter system of rank equal to $\text{rk}(O)$.

**Proof.** The symmetric group on a set of $n$ elements is the Coxeter group of type $A_{n-1}$, so the claim follows from Lemma 8.8. □

Additional comments on this can be found in the ‘BN-pairs’ paragraph of Section 9.

# 9 | CLOSING COMMENTS AND OPEN QUESTIONS

We conclude with comments on the limitations of our constructions and on open questions related to the complex $CC = CC(O, \mathcal{P}(O))$.

**Description as a subgroup poset in $A_{\Gamma}$**

Both in the setting of $\text{GL}_n(\mathbb{Z})$ and of Fouxe-Rabinovitch groups $\text{Out}(A; A')$, we studied the coset complex of parabolic subgroups by finding an isomorphic poset of subgroups of $A_{\Gamma}$ and then determined its homotopy type. These were the poset of direct summands of $\mathbb{Z}^n$ and the relative free factor complex $\mathcal{P}(A, A)$, respectively. In general, however, the author is not aware of a natural description of $CC$ which looks similar.

It is not hard to see that if $P = \text{Stab}_O(A_{\Delta})$ and $P' = \text{Stab}_O(A_{\Delta'})$ are distinct parabolic subgroups, then the $O$-orbits of $[A_{\Delta}]$ and $[A_{\Delta'}]$ intersect trivially. Hence, the map $g \text{Stab}_O(A_{\Delta}) \mapsto [g(A_{\Delta})]$ defines a bijection between the vertices of $CC$ and the union of the $O$-orbits of conjugacy classes of special subgroups $A_{\Delta^j_v}$ that we used for the definition of parabolic subgroups. However, it is not true that the adjacency relation in $CC$ is given by containment of corresponding subgroups of $A_{\Gamma}$.

These orbits can be described more explicitly: Let $\Delta^j_v \subseteq \Gamma$ be the full subgraph of $\Gamma$ with vertex set ${v_1, \ldots, v_j} \cup \Gamma > v$. As $A_{\geq v}$ is stabilised by $O$, it is clear that $[g(A_{\Delta^j_v})] \leq [A_{\geq v}]$. If $[v]$ is abelian, $\Gamma_{\geq v} \subseteq \text{st}([v])$. This implies that the $O$-orbit of $[A_{\Delta^j_v}]$ is given by

$$\left\{ [g(A_{\{v_1, \ldots, v_j\}}) \times A_{\geq v}] \mid g \in \text{Out}(A_{[v]}) \cong \text{GL}_{|v|}(\mathbb{Z}) \right\}.$$

If on the other hand $[v]$ is a free equivalence class, one has $\Gamma_{\geq v} = \Delta \ast Z$, where $Z := \text{lk}(v) \cap \Gamma_{\geq v}$ is a complete graph and $\Delta = [v] \cup \Gamma_1 \cup \cdots \cup \Gamma_k$ with $\Gamma_1, \ldots, \Gamma_k$ the connected components of $\Gamma_{> v} \setminus Z$. Hence, $A_{\Gamma}$ decomposes as a direct product $A_\Delta \times A_Z$ and every element of $O$ preserves this product structure. It follows that the $O$-orbit of $[A_{\Delta^j_v}]$ is equal to

$$\left\{ [g(A_{\{v_1, \ldots, v_j\}}) \ast A_{\Gamma_1} \ast \cdots \ast A_{\Gamma_k} \times A_Z] \mid g \in O \cap \text{Out}(A_{\Delta}) \right\}.$$

Using [26, Lemma 2.11] (see Section 4.2.2), every element in this orbit is of the form $F \ast A_{\Gamma_1}^{a_1} \ast \cdots \ast A_{\Gamma_k}^{a_k}$, where $a_j \in A_{\Delta}$ and $F$ is a free group of rank $j$.

**Limitations of our construction**

It seems that our definition of parabolic subgroups and the corresponding coset complex capture well the aspects of $\text{Out}(A_{\Gamma})$ that come from similarities of this group with $\text{GL}_n(\mathbb{Z})$ and $\text{Out}(F_n)$: Firstly, our definitions recover the Tits building as $CC(\text{GL}_n(\mathbb{Z}), \mathcal{P}(\text{GL}_n(\mathbb{Z})))$ and the free factor
complex as $\text{CC}(\text{Out}(F_n), \mathcal{P}(\text{Out}(F_n)))$. Secondly, the results we obtain show strong similarities in behaviour between the general situation of $\text{Out}(A_G)$ and these special cases: The associated coset complex is spherical, even Cohen–Macaulay (Theorem 8.2) and families of parabolic subgroups are highly generating with the degree of generation depending on the rank of these subgroups (Corollary 8.5). Another strong indication which suggests a certain optimality of our definitions is the description of $\text{rk}(\text{Out}^0(A_G))$ in terms of a Coxeter subgroup (Corollary 8.9). Furthermore, our induction leads to well-suited families of parabolic subgroups in all those ‘components’ of $\text{Out}^0(A_G)$ that closely resemble general linear groups and automorphism groups of free groups; that is, the base cases that are given by $\text{GL}_n(\mathbb{Z})$, $n \geq 2$, and Fouxe-Rabinovitch groups containing transvections (Items 2(b) and 3(c) in Section 7.1).

However, our construction is rather transvection-based in the sense that the standard ordering of $V(\Gamma)$ — which is used to define the parabolic subgroups — is entirely determined by the transvections that $\text{Out}(A_G)$ contains. This makes our definition of parabolic subgroups quite local: Whether or not $v \leq w$ can be read off from the one-balls around these vertices. This is also reflected by the fact that the conical subgraphs $\Gamma_{\geq v}$, which play a central role in our induction, are contained in the two-balls around $v$ if $v$ is not an isolated vertex. In contrast, certain aspects of $\text{Out}(A_G)$ seem not to be mere generalisations of phenomena in arithmetic groups and automorphism groups of free groups. For example, $\text{Out}(A_G)$ contains partial conjugations which cannot be written as a product of transvections. The existence of these partial conjugations is a global phenomenon in the sense that the shape of the connected components of $\Gamma \setminus \text{st}(v)$ is not determined by local conditions on $\Gamma$. These aspects are not very well-represented in $\text{CC}$: The base cases of our induction that correspond to them do not contain any parabolic subgroups. In the extremal case where there is no equivalence class of $V(\Gamma)$ that has size greater than one, $\mathcal{P}(\text{Out}^0(A_G))$ is even empty. For specific applications, one might try to overcome this by introducing further parabolic subgroups that capture these global aspects. However, the author currently does not see a canonical way to do this.

**BN-pairs**

The existence of a ‘Weyl group’ $\text{Aut}^0(\Gamma)$ as described in Section 8.3 suggests that one might be able to transfer additional notions from the theory of groups with BN-pair to automorphism groups of RAAGs. It does, for instance, seem reasonable to define a ‘Borel-subgroup’ by taking the intersection of all standard parabolic subgroups or to use the Weyl group to define apartments in $\text{CC}$. For this, it might be helpful to use the standard representation $\text{Out}(A_G) \to \text{GL}_{|V(\Gamma)|}(\mathbb{Z})$ induced by the abelianisation. The question that has yet to be clarified is to what extent this point of view might be fruitful for studying automorphism groups of RAAGs; one should keep in mind that all this can also be done for $O = \text{Out}(F_n)$ which is far away from having a BN-pair.

**Boundary structures**

Both buildings and free factor complexes can be seen as boundary structures of classifying spaces — in the first case, this is due to Borel–Serre who constructed a bordification of symmetric spaces whose boundary can be described by rational Tits buildings [5]; in the second case, it was shown in [10] that the free factor complex can be seen as a subspace of the simplicial boundary of Culler–Vogtmann Outer space. In the RAAG-setting, one may ask whether a similar statement holds and $\text{CC}$ can be seen as a boundary structure of the RAAG Outer space defined in [6, 15] or a similar space. However, without further changes, this will not work for arbitrary $O$. In particular, if $O$ does not contain any transvection, the complex $\text{CC}$ is trivial, while this need not be the case for the RAAG Outer space and its boundary. This, for example, occurs for RAAGs defined by focused...
graphs that appear in the work of Bregman and Fullarton [7], if the standard ordering on the graph $\Gamma$ is trivial. In that case, $\text{Out}^0(A_\Gamma)$ is a semi-direct product of a free abelian group generated by partial conjugations and the (finite) group of inversions.

**Geometric aspects**

This text focuses on the topology of $\mathbb{C}$. It also seems very reasonable, however, to ask what can be said about the geometry of this complex. Motivated by the work of Masur and Minsky [34], who showed that the curve complex $\mathbb{C}(S)$ is hyperbolic, Bestvina and Feighn [3] proved that the free factor complex is hyperbolic as well. This is only one of many results in the study of $\text{Out}(F_n)$ from a geometric point of view, which has become popular in recent years. On the other hand, there is also a rich theory concerning metric aspects of buildings (for an overview, see [2, Section 12]). Combining these two theories should be an interesting topic for further investigations.

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