Cosmology or Catastrophe? A non-minimally coupled scalar in an inhomogeneous universe

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Abstract

A non-minimally coupled scalar field can have, in principle, a negative effective Planck mass squared which depends on the scalar field. Surprisingly, an isotropic and homogeneous cosmological universe with a non-minimally coupled scalar field is perfectly smooth as the rolling scalar field causes the effective Planck mass to change sign and pass through zero. However, we show that any small deviations from homogeneity diverge as the effective Planck mass vanishes, with catastrophic consequences for the cosmology. The physical origin of the divergence is due to the presence of non-zero scalar anisotropic stress from the non-minimally coupled scalar field. Thus, while the homogeneous and isotropic cosmology appears surprisingly sensible when the effective Planck mass vanishes, inhomogeneities tell a different story.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The minimal coupling of a scalar field to gravity in curved space is through the curved metric replacement $\eta_{\mu\nu} \rightarrow g_{\mu\nu}$. However, scalar fields can also couple directly to the scalar curvature, as in the non-minimally coupled action

$$S = \int d^4x \sqrt{-g} \left( -\frac{m^2}{2} R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) + \frac{1}{2} \xi \phi^2 R \right). \quad (1)$$

The non-minimal coupling parameter $\xi$ is dimensionless and can be any value; for example, when $\xi = 1/6$ the gravity-scalar system has a well-known conformal symmetry, while $\xi \gg 1$
has been used in models of inflation [1–4] (although see [5, 6] for a discussion of problems with ξ ≫ 1). This type of direct coupling also appears in inflationary models as the eta problem [7]; see also [8] for a review. The latter effect underscores a more general property of non-minimal coupling: even if ξ = 0 at tree level, it is expected that ξ will be generated by RG flow (see [9] and discussion within).

The physical effect of the coupling in (1) is more transparent after rearranging terms,

\[ S = \int d^4x \sqrt{-g} \left( -\frac{1}{2} \left( m_p^2 - \xi \phi^2 \right) R + \frac{1}{2} g^{\mu\nu} \partial_{\mu}\phi \partial_{\nu}\phi - V(\phi) \right). \] (2)

Now the non-minimal coupling appears as a field-dependent ‘Effective Planck Mass’ \( m_{p,\text{eff}}^2 = m_p^2 - \xi \phi^2 \). For \( \xi > 0 \), there is a critical value of the scalar field \( \phi_* \equiv m_p/\sqrt{\xi} \) such that at \( \phi = \phi_* \) the effective Planck mass vanishes \( m_{p,\text{eff}}^2 = 0 \), and for \( \phi > \phi_* \), the effective (squared) Planck mass is negative \( m_{p,\text{eff}}^2 < 0 \). Certainly, there is reason to doubt that this region can actually be reached in the true quantum system; for example, as \( m_{p,\text{eff}} \to 0 \), one would expect that higher order Planck-suppressed operators like \( R^2 \) would become important, modifying the dynamics of the system considerably. However, in this paper we wish to take the classical action (2) at face value, and determine if there is something wrong with \( m_{p,\text{eff}}^2 \leq 0 \) classically.

In particular, we will consider (2) on a cosmological background in which a homogeneous scalar field \( \phi = \phi(t) \) begins above the critical value \( \phi > \phi_* \), so that \( m_{p,\text{eff}}^2 \to 0 \) initially. The scalar field then rolls down potential so that \( \phi(t) \) decreases with time; eventually \( \phi = \phi_* \), and the effective Planck mass vanishes \( m_{p,\text{eff}}^2 \to 0 \). Surprisingly, the homogeneous universe evolves smoothly through this critical point, as has been noted by other authors5 6 [12–14] (note, however, that closed cosmological models do not evolve smoothly through this critical point [15]). While a homogeneous and isotropic universe smoothly evolves through this critical point, deviations from homogeneity and isotropy would be expected to cause an obstruction at vanishing effective Planck mass. Indeed, earlier studies have shown that the presence of anisotropies cause a curvature singularity at the critical point7 [13, 14, 17].

The purpose of this study is to explore the behavior of inhomogeneous, but isotropic, perturbations, in contrast to the previous studies, as the effective Planck mass evolves through zero. In the process, we will derive the equations of cosmological perturbation theory [18] for the system (2) in terms of the gauge-invariant variables. Note that since \( m_{p,\text{eff}}^2 \) is changing sign (and going through zero), it is not possible to Weyl-rescale the cosmological solutions and study them in Einstein-frame, where much of the cosmology of non-minimally coupled scalar fields has been studied.

The paper is organized as follows. In section 2, we review the homogeneous cosmological solutions of (2), and demonstrate that the solutions are perfectly smooth, even through the region where the effective Planck mass vanishes. In section 3.1, we review cosmological perturbation theory for a minimally coupled scalar field, and show that if \( m_p^2 \to 0 \) for minimal coupling, then all of the perturbations must be trivial. In section 3.2 we derive the equations of cosmological perturbation theory for (2), concentrating on the extra terms arising from the non-minimal coupling. We will show there that as the homogeneous background evolves through the critical point, the perturbations must either be trivial or divergent, signaling a cosmological ‘catastrophe’. In section 4, we conclude with a brief discussion. Some detailed calculations are relegated to the appendices.

5 We would like to thank Gary Felder for conversations on this point.
6 See also [10, 11] for discussion in the context of the gauge-flation models.
7 Perturbations of black hole solutions of (2) also have been shown to lead to instabilities [16].
2. Cosmology through the critical point

First, we will explore the homogeneous cosmology of the non-minimally coupled scalar field as it passes through the critical point. We will show that, surprisingly, the cosmology is completely smooth and non-trivial through the critical point where the effective Planck mass vanishes due the non-minimal coupling.

Starting from the action (2) for the non-minimally coupled scalar field, Einstein’s equations\(^8\) and the scalar field equation of motion take the form:

\[(m_p^2 - \xi \phi^2)G_{\mu}^\nu = \delta^{\mu}_\alpha \phi, \phi, \phi, - \delta^{\mu}_\alpha \left( \frac{1}{2} g^{\nu\gamma} \phi, \phi, \phi, - V(\phi) \right) + \xi \left( \delta^{\mu}_\alpha \Delta \phi^2 - \nabla^\alpha \nabla^\mu \phi^2 \right); \]

\[
\frac{1}{\sqrt{-g}} \partial_(\sqrt{-g} \delta^{\alpha\beta} \partial_\phi \phi) + \partial (V(\phi)) = \xi R \phi. \]

Notice that there are two places that a non-zero non-minimal coupling \(\xi\) leads to additional terms: First, in the coefficient of the Einstein tensor, modifying the effective Planck mass. Second, as additional ‘energy–momentum’ terms acting as sources on the right hand side of Einstein’s equation\(^9\).

Assuming a homogeneous and isotropic (and flat) universe, the metric and scalar field take the form,

\[ds^2 = dt^2 - a(t)^2 dx^2;\]

\[\varphi = \varphi_0(t).\]

Inserting these into the Einstein and scalar field equations above, we obtain the equations for the background scale factor and scalar field \(a(t), \varphi(t)\)

\[3(m_p^2 - \xi \varphi_0(t)^2)H^2 = \frac{1}{2} \dot{\varphi}_0^2 + V(\varphi_0) + 6\xi \varphi_0(t) \dot{\varphi}_0 H;\]

\[\dot{\varphi}_0 + 3H \dot{\varphi}_0 + V'(\varphi_0) + 6\xi (2H^2 + \dot{H}) \varphi_0 = 0.\]

where \(H(t) = \frac{\dot{a}}{a}\) is the Hubble expansion parameter.

Remarkably, it is possible to have a solution to the modified Friedmann equation (7) even when the effective Planck mass vanishes because of the additional term arising from the non-minimal coupling. In particular, the general solution of (7) for the Hubble parameter \(H\) in an expanding universe is

\[H = \frac{6\xi \varphi_0 \dot{\varphi}_0 + \sqrt{(6\xi \varphi_0 \dot{\varphi}_0)^2 + 12(m_p^2 - \xi \varphi_0^2)(\frac{1}{2} \dot{\varphi}_0^2 + V(\varphi_0))}}{6(m_p^2 - \xi \varphi_0^2)}.\]

When the scalar field is close to the critical point \(\varphi_0 \approx \varphi_* = m_p/\sqrt{\xi}\), the Hubble parameter becomes,

\[H \approx \frac{\xi (\varphi_0 \dot{\varphi}_0 + |\varphi_0 \dot{\varphi}_0|)}{m_p^2 - \xi \varphi_0^2} + \frac{\frac{1}{2} \dot{\varphi}_0^2 + V(\varphi_0)}{6\xi |\varphi_0 \dot{\varphi}_0|}.\]

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8 In principle, it is necessary to include a generalization of the Gibbons–Hawking–York term \(S_{GHY} = \int_M \sqrt{h} (m_p^2 - \xi \varphi^2) K\) following \([19]\), where \(h\) is the induced metric on the boundary \(\partial M\) of the manifold, and \(K\) is the trace of the extrinsic curvature, in order to cancel the boundary terms generated by variation of (2).

9 As pointed out in \([20]\) these extra contributions create an ambiguity in the definition of the energy–momentum tensor \(T^{\mu\nu}_{\text{m}}\). If \(T^{\mu\nu}_{\text{m}}\) is defined formally as the functional derivative of the matter action \(T_{\mu\nu} = \frac{1}{2} \delta \mathcal{L}_{\text{m}}}{\delta g^{\mu\nu}}\), it will not be divergence free. However, if we define \(T^{\mu\nu}_{\text{m}}\) to be equal to the source of the Einstein tensor in the Einstein equations, \(m_p^2 G^{\mu\nu} = T^{\mu\nu}_{\text{m}}\), then the divergence will vanish but it will not be the same as that obtained by variation of the matter action. Our organization of the Einstein equations (3) reflects the structure of the latter approach, although the two definitions are just related by algebra.
When $\phi_0$ or $\dot{\phi}_0$ are separately negative, the first term vanishes and the Hubble parameter is finite even when the effective Planck mass vanishes. If, however, the product $\phi_0\dot{\phi}_0$ is positive, the first term does not vanish and the Hubble parameter becomes infinite when the effective Planck mass vanishes. Similarly, in regions where the effective Planck mass is negative, real solutions only exist when $\xi\phi_0\dot{\phi}_0$ is sufficiently large, otherwise the Hubble parameter is imaginary, signaling the existence of AdS solutions instead of cosmological solutions. This structure is shown in the phase-space plot of figure 1.

Remarkably, it is possible to start in a region where the effective Planck mass is negative with real and positive Hubble parameter, and smoothly evolve through the critical point, as seen by the highlighted trajectory in figure 1. Figure 2 displays the value of the scalar field for this trajectory as a function of time. Notice again that the system smoothly passes through the critical point $\phi = \phi_*$ where the effective Planck mass vanishes. Thus, the cosmology of the non-minimal scalar field is smooth and continuous, even through the critical point!

3. Catastrophe—perturbations through the critical point

3.1. Perturbations for minimally coupled scalar field

Let us review cosmological perturbation theory for a minimally coupled ($\xi = 0$) scalar field $\phi$, as can be found in [18]. In longitudinal gauge, the metric and scalar perturbations are

$$d^2 = a^2[(1+2\phi)d\eta^2 - (1-2\psi)\delta_{ij}dx^i dx^j];$$

$$\psi = \phi_0(\eta) + \delta\psi(\mathbf{x}, \eta).$$
Figure 2. Top: the evolution of the scalar field $\phi(t)$ versus time for the highlighted trajectory from figure 1 is shown. Note that the evolution is completely smooth through the critical point $\phi^* = m_p / \sqrt{\xi}$. Bottom: the evolution of the Hubble parameter $H(t)$ versus time for the same trajectory. Note that the Hubble parameter $H(t)$ is finite and smooth as the scalar field passes through the critical point.

Note that we have switched to conformal time $\eta$, defined by $\eta = \int a(t)^{-1} dt$, which is a more convenient time variable for cosmological perturbations. The Einstein equations, to first order in the perturbations, are then,

$$a^2 m_p^2 \delta G_0 = 2 m_p^2 \left[ \Delta \Psi - 3 \dot{H} \Psi' - (\dot{H}' + 2H^2) \Phi \right] = \phi_0' \delta \Phi + \delta \Phi a^2 V_{\psi};$$  \hspace{1cm} (13)

$$a^2 m_p^2 \delta G_i = 2 m_p^2 \left[ \Psi' + \dot{H} \Phi \right]_i = (\phi_0' \delta \Phi)_i;$$  \hspace{1cm} (14)

$$a^2 m_p^2 \delta G_j = 2 m_p^2 \left[ \Psi'' + \dot{H} (2 \Psi' + \dot{\Phi})' + (\dot{H}' + 2H^2) \Phi + \frac{1}{2} \Delta (\Phi - \Psi) \right] \delta_j - m_p^2 \left[ \Phi - \Psi \right]_j = [\phi_0' \delta \Phi' - \delta \Phi a^2 V_{\psi}] \delta_j;$$  \hspace{1cm} (15)

where we have replaced the perturbations with their corresponding gauge-invariant variables $\phi \rightarrow \Phi, \psi \rightarrow \Psi, \delta \phi \rightarrow \delta \Psi$ (see [18] for more details on the gauge-invariant variables $\Phi, \Psi, \delta \Phi$).
Notice that the off-diagonal terms of (15) require $\Phi = \Psi$, implying the absence of anisotropic stress due to the perturbations of the scalar field. As we will see later, the surprising presence of a source of anisotropic stress for the non-minimal scalar field leads to some significant consequences for the behavior of the perturbations.

In order to develop some physical intuition, let’s consider the equations (13)–(15) in the limit where $m_p^2 \to 0$. Certainly, the non-minimal coupling introduces many additional terms, so the non-minimal coupling is more than just controlling the size and sign of $m_p^2$. However, analyzing this limit will allow us to learn about how the equations behave under this limit, and what is ‘normal’ and what is due to the non-minimal coupling.

In the limit $m_p^2 \to 0$, the equations (13)–(15) (as well as the minimal version of the background equation of motion (7) and the perturbed version of the scalar field equation of motion (4)) become

$$0 = \delta \bar{\psi}' + 2H \delta \bar{\psi} - \Delta \delta \bar{\psi} + a^2 \nabla^2 \psi \delta \bar{\psi} - \psi_0 (3\Psi + \Phi)' + 2a^2 V_\phi \Phi = 0; \quad (16)$$

$$0 = \psi_0 \delta \bar{\psi} + \delta \bar{\psi} a^2 V_\phi; \quad (17)$$

$$0 = (\psi_0' \delta \bar{\psi})_i; \quad (18)$$

$$0 = \psi_0' \delta \bar{\psi}' - \delta \bar{\psi} a^2 V_\phi; \quad (19)$$

$$0 = \psi_0^2. \quad (20)$$

These equations require trivial solutions

$$\delta \bar{\psi} \to 0; \quad \psi_0' \to 0; \quad \Psi \to 0; \quad \Phi \to 0; \quad (21)$$

so that for a minimally coupled scalar field, a vanishing Planck mass only allows trivial solutions.

### 3.2. Perturbations for non-minimally coupled scalar field

For the non-minimally coupled scalar field, the Einstein equations (3) in longitudinal gauge (11), (12) become at linear order

$$(m_p^2 - \xi \psi_0^2) a^2 \delta G_i^0 - 6\xi \mathcal{H}^2 \psi_0 \delta \bar{\psi} = -\Phi (\psi_0')^2 + \psi_0' \delta \bar{\psi}' + \delta \bar{\psi} a^2 V_\phi$$

$$-2\xi (\psi_0 \Delta \delta \bar{\psi} - 3H (\psi_0 \delta \bar{\psi}')^2 + 3\psi_0 \psi_0' (\Psi' + 2H \Phi)); \quad (22)$$

$$(m_p^2 - \xi \psi_0^2) a^2 \delta G^0_i = \psi_0 \delta \bar{\psi}_i - 2\xi ((\psi_0 \delta \bar{\psi}')^2 - \psi_0 (H \delta \bar{\psi} + \Phi \psi_0'))_i; \quad (23)$$

$$(m_p^2 - \xi \psi_0^2) a^2 \delta G^j_i = 2\xi (\mathcal{H}^2 + 2H') \psi_0 \delta \bar{\psi} \delta_j^i = 2\xi \psi_0 \delta \bar{\psi}_{ij} + \delta_j^i (\Phi (\psi_0')^2 - \psi_0' \delta \bar{\psi}' + \delta \bar{\psi} a^2 V_\phi$$

$$-2\xi (\psi_0 \Delta \delta \bar{\psi} + 2\Phi (H \psi_0 \psi_0' + (\psi_0')^2 + \psi_0 \psi_0') + (\Phi + 2\Psi)' \psi_0 \psi_0'$$

$$-\psi_0' \delta \bar{\psi}' - \mathcal{H} (\psi_0 \delta \bar{\psi})'))); \quad (24)$$

with the $\delta G^0_i$ given by (13)–(15). Consider (24) for $i \neq j$:

$$(m_p^2 - \xi \psi_0^2) (\Phi - \Psi)_i,_{ij} = -2\xi (\psi_0 \delta \bar{\psi})_i,_{ij}. \quad (25)$$

Notice that the non-minimal coupling introduces non-zero scalar anisotropic stress. Normally, scalar anisotropic stress is generated by quadrupole radiation fields. For instance, in the

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10 Again, we have replaced $\phi, \psi, \phi$ with their corresponding gauge-invariant quantities $\Phi, \Psi, \delta \bar{\psi}$, which are defined in the same way as for minimal coupling.
evolution of scalar perturbations in our Universe, the only appreciable amount of scalar
anisotropic stress comes from the quadrupole moment of the neutrino background radiation
[21].

Let’s now explore the behavior of the equations as the background passes through the
critical point, where the effective Planck mass vanishes. Since the homogeneous background
is smooth and continuous through the critical point, the vanishing of the effective Planck mass
only occurs at a specific instant of time η∗, where ϕ_{0}(η∗) = ϕ_{*} = m_{p}/√ξ. Further, because the
equations are linear we can study the behavior of individual Fourier modes of the perturbations
\( \Phi(x, η) = Φ_{k}(η) e^{i\vec{k} \cdot \vec{x}}, \quad Ψ(x, η) = Ψ_{k}(η) e^{i\vec{k} \cdot \vec{x}}; \) (26)
since different Fourier modes decouple. The off-diagonal Einstein equation (25) then becomes
\( (m_{p}^{2} - ξ ϕ_{0}^{2})(Φ_{k} - Ψ_{k}) = -2ξ ϕ_{0} \tilde{ϕ}_{k}. \) (27)

As the critical point is approached, the coefficient of the left-hand-side of (27) vanishes, but
the right-hand-side does not necessarily vanish. Thus, if \( \tilde{ϕ}_{k}(η∗) \neq 0 \) at the critical point, then
(27) requires that the perturbations diverge \( Φ_{k}, Ψ_{k} \rightarrow \infty \), and thus we have a ‘catastrophe’ at
the critical point! As we argue in the appendix, the vanishing of the scalar field fluctuation at
the critical point \( δϕ_{k}(η∗) = 0 \) can only occur if all of the fluctuations are trivial for all time.
Thus, any non-zero inhomogeneous (but isotropic) perturbations to the smooth cosmological
backgrounds discussed in section 2 lead to a ‘catastrophe’ as the effective Planck mass
vanishes.

4. Discussion

We have seen that the homogeneous and isotropic cosmology of a non-minimally coupled
scalar field can smoothly evolve from negative effective Planck mass squared \( m_{p, eff}^{2} < 0, \)
through the ‘critical point’ where \( m_{p, eff}^{2} = 0, \) into the region where \( m_{p, eff}^{2} > 0. \) The absence of
any divergence near the critical point is due to the presence of additional ‘energy–momentum’
source terms in the Einstein equations.

However, we have also shown that the presence of these additional terms cause small
isotropic inhomogeneous perturbations to this cosmological background to diverge at the
critical point. Thus, while the homogeneous and isotropic cosmology appears to be sensible
as it evolves through the critical point, any perturbation away from homogeneity will cause
a divergence at the critical point. This result is complementary to earlier results [13, 14, 17]
that found that any deviation from isotropy causes a divergence at the critical point. Thus, we
have found that it is not just deviations from isotropy, but also deviations from homogeneity,
that prevents a cosmological universe from passing through the critical point. This classical
obstruction to \( m_{p, eff}^{2} = 0 \) is in addition to any additional quantum effects that are expected to
be important in this regime.

As discussed in section 3.2, the origin of the divergence in the inhomogeneities is easy
to see. Of the additional energy–momentum source terms, there is a contribution to the scalar
anisotropic stress that is non-vanishing at the critical point. Recall that scalar anisotropic stress
refers to the difference between the Newtonian potential \( Φ \) and the perturbation to the spatial
curvature \( Ψ. \) In the absence of anisotropic stress, these two potentials are equal. Normally,
scalar anisotropic stress comes from the quadrupole moment of a cosmic fluid; for a minimally
coupled scalar field, the scalar anisotropic stress vanishes at linear order in the perturbations.
Its presence here for the non-minimally coupled scalar field is a surprise. Since the scalar
anisotropic stress is non-zero, even at the critical point, the gravitational potentials \( Φ, Ψ \) must
diverge there.
Of course, our analysis only studies the behavior of linear perturbations; it would be interesting to study the full nonlinear behavior in future work. It would also be interesting to study how these additional energy–momentum terms modify the usual cosmological perturbation theory and spectrum of inflationary perturbations. We hope to explore this in the future.

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Appendix. Trivial solution

In section 3.2, our argument that inhomogeneities diverge when the background effective Planck mass vanishes was based on \( \delta \psi_k(\eta^*) \neq 0 \) at the critical point. But it is possible that the evolution of \( \delta \psi_k(\eta) \) be such that \( \delta \psi_k(\eta^*) = 0 \) precisely at the critical point, avoiding the ‘catastrophe’. In this appendix, we argue that this is not the case—the only way that \( \delta \psi_k(\eta^*) = 0 \) is if all of the fluctuations vanish for all time, thus implying that the only non-trivial solution at the critical point is the ‘catastrophe’.

The equations (22)–(25) at the critical point become a set of equations for \( \delta \psi_k(\eta^*) \), \( \Phi_k(\eta^*) \) and \( \Psi_k(\eta^*) \) (where we drop the functional dependence on \( \eta^* \) for notational simplicity):

\[
6\xi H^2 \varphi_{\kappa} \delta \psi_k - \Phi_k (\varphi_k')^2 - \varphi_k' \delta \psi_k - \delta \psi_k a^3 V_{,\varphi} - 2\xi [-k^2 \varphi_k \delta \psi_k - 3H(\varphi_k \delta \psi_k)'] + 3\varphi_{\kappa} \varphi_k' (\Psi_k' + 2H\Phi_k)] = 0; \tag{A.1}
\]

\[
k_{\kappa} \psi_{\kappa} \delta \psi_k - 2\xi ((\varphi_{\kappa} \delta \psi_k)' - \varphi_k (H \delta \psi_k + \Phi_k \varphi_k')) = 0; \tag{A.2}
\]

\[
k_{\kappa} k_\kappa 2\xi \varphi_{\kappa} \delta \psi_k = 0; \tag{A.3}
\]

\[
2\xi (H^2 + 2H') \varphi_{\kappa} \delta \psi_k - \varphi_k' \delta \psi_k + \Phi_k (\varphi_k')^2 + \delta \psi_k a^3 V_{,\varphi} + \frac{1}{2} k^2 \xi \varphi_{\kappa} \delta \psi_k - 2\xi [-k^2 \varphi_k \delta \psi_k - (\varphi_k \delta \psi_k)'' - H(\varphi_k \delta \psi_k)'] + 2\Phi_k (H \varphi_k \varphi_k') + (\varphi_k')^2 + \varphi_{\kappa} \varphi_k'' + (\Phi_k + 2\Psi_k) \varphi_k \varphi_k' = 0. \tag{A.4}
\]

Clearly, (A.3) is solved by \( \delta \psi_k(\eta^*) = 0 \). Combining and simplifying (A.1), (A.2) and (A.4), we obtain the three simplified equations:

\[
\delta \psi_k - \varphi_0' \Phi_k = 0; \tag{A.5}
\]

\[
\Psi_k' + H(\Phi_k) = 0; \tag{A.6}
\]

\[
\varphi_0(\Phi_k \varphi_0' - \delta \psi_k')' + \frac{(\varphi_0 \varphi_0')'}{\varphi_0' - H\varphi_0} \delta \psi_k = 0. \tag{A.7}
\]

Using (23), we can write

\[
(\delta \psi_k - \Phi_k \varphi_0')' = \left( - \frac{\varphi_0'}{2\xi \varphi_0} - \frac{\varphi_0'}{\varphi_0} + H \right) \delta \psi_k. \tag{A.8}
\]
Inserting this into (A.7), we obtain
\[ \left[ (1 + \frac{1}{2\epsilon}) \phi_0'' + \frac{\psi_0''}{\phi_0'} - 2\mathcal{H}\phi_0 \right] \delta\phi_k = 0. \]  
(A.9)

This can solved by either \( \delta\phi_k (\eta^*) = 0 \) or with the term in parentheses vanishing. In general, the term in parentheses does not vanish, but it could be possible to construct a very special solution that does so, similar to the special anisotropic cases found in [17]. Ignoring this very special solution, we will take the more general case \( \delta\phi_k (\eta^*) = 0 \). Inserting \( \delta\phi_k = 0 \) back into (A.5)–(A.7), we find that if \( \delta\phi_k (\eta^*) = 0 \), then not only do the other perturbations vanish there as well \( \Phi_k (\eta^*) = 0 = \Psi_k (\eta^*) \), but derivatives of the fluctuations also vanish:
\[ \begin{align*}
\delta\phi_k (\eta^*) &= 0, \\
\Phi_k (\eta^*) &= 0, \\
\Psi_k (\eta^*) &= 0.
\end{align*} \]  
(A.10)

Since the equations of motion for \( \delta\phi_k (\eta) \), \( \Psi_k (\eta) \), \( \Phi_k (\eta) \) are linear first and second order differential equations, then the vanishing of the perturbations and their derivatives at the critical point implies that the perturbations themselves vanish for all time \( \delta\phi_k (\eta) = 0 = \Psi_k (\eta) = \Phi_k (\eta) \). In particular, the general form of the equations of motion for \( \delta\phi_k \), \( \Phi_k \), \( \Psi_k \) are three independent equations of motion
\[ \begin{align*}
\delta\phi_k'' + A_1(\eta)\delta\phi_k' + A_2(\eta)\delta\phi_k + A_3(\eta)\Psi_k' + A_4(\eta)\Psi_k + A_5(\eta)\Phi_k' + A_6(\eta)\Phi_k = 0; \quad (A.11)
\end{align*} \]
\[ \begin{align*}
\Phi_k' + B_1(\eta)\Phi_k + B_2(\eta)\delta\phi_k + B_3(\eta)\delta\phi_k + B_4(\eta)\Psi_k = 0; \quad (A.12)
\end{align*} \]
\[ \begin{align*}
C_0(\eta)\Psi_k'' + C_1(\eta)\Psi_k' + C_2(\eta)\Psi_k + C_3(\eta)\delta\phi_k' + C_4(\eta)\delta\phi_k + C_5(\eta)\Phi_k = 0; \quad (A.13)
\end{align*} \]

where the \( A_i(\eta) \), \( B_i(\eta) \), \( C_i(\eta) \) are some generally non-vanishing functions of \( \eta \), except for \( C_0(\eta) \), which we can see from (15), (24) is proportional to \( m_{\epsilon,\text{eff}}^2 \), which vanishes at \( \eta = \eta^* \). These equations are all solved at \( \eta = \eta^* \) by (A.10). Taking a derivative of (A.11)–(A.13), we obtain, equations of the form,
\[ \begin{align*}
\delta\phi_k'' + (A_1(\eta)\delta\phi_k')' + (A_2(\eta)\delta\phi_k)' + (A_3(\eta)\Psi_k')' + (A_4(\eta)\Psi_k')' + (A_5(\eta)\Phi_k')' + (A_6(\eta)\Phi_k')' &= 0; \quad (A.14)
\end{align*} \]
\[ \begin{align*}
\Phi_k' + (B_1(\eta)\Phi_k)' + (B_2(\eta)\delta\phi_k')' + (B_3(\eta)\delta\phi_k)' + (B_4(\eta)\Psi_k)' &= 0; \quad (A.15)
\end{align*} \]
\[ \begin{align*}
(C_0(\eta)\Psi_k')' + (C_1(\eta)\Psi_k')' + (C_2(\eta)\Psi_k')' + (C_3(\eta)\delta\phi_k')' + (C_4(\eta)\delta\phi_k')' + (C_5(\eta)\Phi_k')' &= 0. \quad (A.16)
\end{align*} \]

Evaluating (A.14)–(A.16) at \( \eta = \eta^* \) by imposing (A.10), we then find that the next order in derivatives must vanish,
\[ \begin{align*}
\delta\phi_k'' (\eta^*) = 0 = \Phi_k'' (\eta^*) = \Psi_k'' (\eta^*). \quad (A.17)
\end{align*} \]

Extending this argument to an arbitrary number of derivatives shows that all derivatives of \( \delta\phi_k \), \( \Phi_k \), \( \Psi_k \) must vanish at the critical point. If all derivatives vanish at \( \eta = \eta^* \), then the perturbations themselves must vanish for all time \( \delta\phi_k (\eta) = 0 = \Psi_k (\eta) = \Phi_k (\eta) \). Thus, if \( \delta\phi_k (\eta^*) = 0 \), then all of the perturbations must be trivial.
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