FINITE SYMMETRIC TENSOR CATEGORIES WITH THE CHEVALLY PROPERTY IN CHARACTERISTIC 2

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Dedicated to Nicolás Andruskiewitsch for his 60th birthday

Abstract. We prove an analog of Deligne’s theorem for finite symmetric tensor categories $C$ with the Chevalley property over an algebraically closed field $k$ of characteristic 2. Namely, we prove that every such category $C$ admits a symmetric fiber functor to the symmetric tensor category $D$ of representations of the triangular Hopf algebra $(k[d]/(d^2), 1 \otimes 1 + d \otimes d)$. Equivalently, we prove that there exists a unique finite group scheme $G$ in $D$ such that $C$ is symmetric tensor equivalent to $\text{Rep}_D(G)$. Finally, we compute the group $H^2_{\text{inv}}(A, K)$ of equivalence classes of twists for the group algebra $K[A]$ of a finite abelian $p$-group $A$ over an arbitrary field $K$ of characteristic $p > 0$, and the Sweedler cohomology groups $H^i_{\text{Sw}}(O(A), K)$, $i \geq 1$, of the function algebra $O(A)$ of $A$.

1. Introduction

The main objective of this paper is to classify finite symmetric tensor categories with the Chevalley property over an algebraically closed field $k$ of characteristic 2. This completes the classification of finite integral symmetric tensor categories with the Chevalley property over an algebraically closed field of characteristic $p > 0$, which for $p > 2$ was established in [EG2], since by [O, Theorem 1.5], integrality follows from the rest of the conditions for $p = 2, 3$.

Let $\alpha_2$ be the Frobenius kernel of the additive group $G_a$. Then $k\alpha_2 = k[d]/(d^2)$ with $d$ primitive. Let $D := \text{Rep}(\alpha_2, 1 \otimes 1 + d \otimes d)$ be the symmetric tensor category of finite dimensional representations of the triangular Hopf algebra $k[d]/(d^2)$ equipped with the $R$-matrix $1 \otimes 1 + d \otimes d$. Recall [V] that an object in $D$ is a finite dimensional $k$-vector space $V$ together with a linear map $d : V \to V$ satisfying $d^2 = 0$. In particular, $D$ has two indecomposable objects, namely, the unit object (i.e., the vector space $k$ with $d = 0$), and the two dimensional vector space $k^2$ with $d$ the strictly upper triangular matrix $E_{12}$.

Recall that a finite group scheme in $D$ is, by definition, a finite dimensional cocommutative Hopf algebra $H$ in $D$. In particular, this means that $d : H \to H$ is a derivation of $H$ satisfying $d^2 = 0$, and

$$\Delta(h) = (1 \otimes 1 + d \otimes d)(\Delta(h))_{21}, \ h \in H.$$ 

We can now state our main result (compare with [O, Conjecture 1.3]).
Theorem 1.1. Let $C$ be a finite symmetric tensor category with the Chevalley property over an algebraically closed field $k$ of characteristic 2. Then $C$ admits a symmetric fiber functor to $D$. Thus, there exists a unique finite group scheme $G$ in $D$ such that $C$ is symmetric tensor equivalent to the category $\text{Rep}_D(G)$ of finite dimensional representations of $G$ which are compatible with the action of $\pi_1(D)$.

Remark 1.2. Theorem 1.1 answers [BE, Question 1.2] for finite symmetric tensor categories with the Chevalley property over $k$, and we expect it to hold for every finite symmetric integral tensor category over $k$.

Finally, we note that the arguments used to prove [EG2, Theorem 1.1] in fact prove a stronger result (see Theorem 2.21).

The organization of the paper is as follows. Section 2 is devoted to the proof of Theorem 1.1. In Section 3 we compute the group $H^2_{\text{inv}}(A, K)$ of equivalence classes of twists for the group algebra $K[A]$ of a finite abelian $p$-group $A$ over an arbitrary field $K$ of characteristic $p > 0$ (see Theorem 3.5), and use it together with [EG2, Proposition 5.7] to compute the Sweedler cohomology groups $H^i_{Sw}(O(A), K)$ for every $i \geq 1$ (see Theorem 3.8).

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2. The proof of Theorem 1.1

All constructions in this section are done over an algebraically closed field $k$ of characteristic 2 unless otherwise is explicitly stated. To lighten notation, we sometimes write $1$ for $1 \otimes 1$ or $1 \otimes 1 \otimes 1$.

We refer the reader to [EGNO] for the general theory of finite tensor categories, to [Dr] for generalities on quasi-Hopf algebras (see also [EG2, 2.1]), and to [JW] for the general theory of finite group schemes (see also [EG2, 2.4]).

By [O, Theorem 1.5], any finite symmetric tensor category with the Chevalley property in characteristic 2 is integral (as $\text{Ver}_2 = \text{Vec}$). Thus by [EQ, Theorem 2.6], $C$ is symmetric tensor equivalent to $\text{Rep}(H, R, \Phi)$ for some finite dimensional triangular quasi-Hopf algebra $(H, R, \Phi)$ with the Chevalley property over $k$. Thus, we have to prove the following theorem.

Theorem 2.1. Let $(H, R, \Phi)$ be a finite dimensional triangular quasi-Hopf algebra with the Chevalley property over $k$. Then $(H, R, \Phi)$ is pseudotwist equivalent to a triangular Hopf algebra with $R$-matrix $1 + d \otimes d$ for some $d \in P(H)$ such that $d^2 = 0$.

We will prove Theorem 2.1 in several steps.

2.1. $\text{gr}(H)$. Let $(H, R, \Phi)$ be a finite dimensional triangular quasi-Hopf algebra with the Chevalley property over $k$. Let $I := \text{Rad}(H)$ be the Jacobson radical of $H$.

Since $I$ is a quasi-Hopf ideal of $H$, the associated graded algebra $\text{gr}(H) = \bigoplus_{r \geq 0} H[r]$ has a natural structure of a graded triangular quasi-Hopf algebra with some $R$-matrix $R_0 \in H[0] \otimes^2$ and associator $\Phi_0 \in H[0] \otimes^3$ (see, e.g., [EG2, 2.2]).

Proposition 2.2. [EG2, Proposition 3.2] The following hold:

(1) $H[0]$ is semisimple.

(2) $(H[0], R_0, \Phi_0)$ is a triangular quasi-Hopf subalgebra of $(\text{gr}(H), R_0, \Phi_0)$. 


(3) $\text{Rep}(H[0], R_0, \Phi_0)$ is symmetric tensor equivalent to $\text{Rep}(G)$ for some finite semisimple group scheme $G$ over $k$.

(4) $(\text{gr}(H), R_0, \Phi_0)$ is pseudotwist equivalent to a graded triangular Hopf algebra with $R$-matrix $1 \otimes 1$, whose degree $0$-component is $(kG, 1 \otimes 1)$. □

**Corollary 2.3.** [EG2, Corollary 3.3] Let $(H, R, \Phi)$ be a finite dimensional triangular quasi-Hopf algebra with the Chevalley property over $k$. Then $\text{gr}(H)$ is pseudotwist equivalent to $kG$ for some finite group scheme $G$ over $k$ containing $G$ as a closed subgroup scheme.

**Remark 2.4.** By Nagata’s theorem (see, e.g., [A, p.223]), we have $G = \Gamma \ltimes P^D$, where $\Gamma$ is a finite group of odd order and $P$ is a finite abelian 2-group. Hence, we have $kG = k\Gamma \ltimes \mathcal{O}(P)$.

Let $\Gamma := \mathcal{G}/\mathcal{G}^\circ$. Then $\Gamma$ is a finite constant group of odd order, and we have $\mathcal{G} = \Gamma \ltimes \mathcal{G}^\circ$. Thus, we have $\mathcal{O}(\mathcal{G}) = \mathcal{O}(\Gamma) \otimes \mathcal{O}(\mathcal{G}^\circ)$ as algebras.

By the results of this subsection, we may assume without loss of generality in the proof of Theorem 2.1 that $R = 1+$ terms of higher degree.

**2.2. Trivializing $R$.** Let $V$ be a $k$-vector space, and let $\tau : V^{\otimes 2} \to V^{\otimes 2}$ be the flip map. Recall that

$$\wedge^2 V := \text{Im}(\text{id} + \tau) \subset \Gamma^2 V := \text{Ker}(\text{id} + \tau) \subset V^{\otimes 2},$$

$$S^2 V := V^{\otimes 2}/\wedge^2 V, \quad V^{(1)} := \Gamma^2 V/\wedge^2 V,$$

and that $V^{(1)}$ is called the Frobenius twist of $V$ and $\Gamma^2 V$ the divided second symmetric power of $V$. Note that $V^{(1)}$ is the image of the composition

$$\Gamma^2 V \hookrightarrow V^{\otimes 2} \twoheadrightarrow S^2 V.$$

Let $\pi : \Gamma^2 V \to V^{(1)}$ be the natural surjective map.

Let $(H, R, \Phi)$ be as in the end of Section 2.1.

**Proposition 2.5.** The following hold:

1. Suppose $R = 1 + d_{n-1} \otimes d_{n-1}$ modulo terms of degree $\geq n \geq 1$ such that $d_{n-1} \in \text{Rad}(H)$. Then $(H, R, \Phi)$ can be twisted to a form such that $R = 1 + d_n \otimes d_n$ modulo terms of degree $\geq n + 1$, where $d_n \in \text{Rad}(H)$ and $d_n - d_{n-1}$ has degree $\geq n/2$, by a pseudotwist $J_n$ such that $J_{n - 1}$ has degree $\geq n$ if $d_{n-1} = 0$, and degree $\geq \frac{n}{2} + p$ if $\deg(d_{n-1}) = p > 0$.

2. If $R \neq 1$ then $(H, R, \Phi)$ can be twisted to the form $R = 1 + d \otimes d$, where $d \in \text{Rad}(H)$ is an element of positive degree. Moreover, if $R = 1 + d' \otimes d'$ modulo terms of degree $\geq n$, where $d' \in \text{Rad}(H)$, then this can be achieved by a pseudotwist $J$ with $J - 1$ of degree $\geq n$ if $d' = 0$, and degree $\geq \frac{n}{2} + p$ if $d' \neq 0$ and has degree $p$, so that $d - d'$ has degree $\geq n/2$.

3. If $R = 1 + d \otimes d$ then $d^2 = 0$.

**Proof.**

1. Let $R = 1 + d_{n-1} \otimes d_{n-1}$ modulo terms of degree $\geq n$, and consider $R$ modulo terms of degree $\geq n + 1$. We have $R = 1 + d_{n-1} \otimes d_{n-1} + \tilde{s}$ modulo terms of degree $\geq n + 1$, where $\tilde{s} \in H^{\otimes 2}$ has degree $\geq n$. Let $s \in \text{gr}(H)^{\otimes 2}[n]$ be the leading part of $\tilde{s}$. Then $s$ is symmetric because $R_{21} R = 1 \otimes 1$, so $s \in \Gamma^2 \text{gr}(H)[n]$. Moreover, if $t \in \wedge^2 \text{gr}(H)[n]$ then we can replace $s$ by $s + t$ by twisting.

Let $v := \pi(s)$ be the image of $s$ in $\text{gr}(H)^{(1)}[n] = \text{gr}(H)[n/2]^{(1)}$ (note that this space can be nonzero only if $n$ is even). Then we can twist $s$ into the form $v \otimes v$ by a pseudotwist $J$ with $J - 1$ of degree $\geq n$. So we will get $R = 1 + d_{n-1} \otimes d_{n-1} + \tilde{v} \otimes \tilde{v}$
modulo terms of degree ≥ n + 1, where \( \bar{v} \) is a lift of v to H. If \( d_{n-1} = 0 \), this completes the proof (we can set \( d_n = \bar{v} \)). Thus, it remains to consider the case when \( d_{n-1} \neq 0 \) and has degree p; so we may assume that \( n > 2p \) (because for \( n \leq 2p \), we can set \( d_n = d_{n-1} \) and \( J = 1 \)). In this case, let us twist by \( J = 1 + d_{n-1} \otimes \bar{v} \) (note that \( \deg(J - 1) \geq 2 \)). Since \( R_{21} R = 1 \otimes 1 \), we have \( \deg(d_{n-1}^2) \geq n/2 \), hence

\[
\deg(d_{n-1}^2 \otimes d_{n-1} \bar{v}) \geq n/2 + n/2 + p = n + p \geq n + 1,
\]

so twisting by J brings R to the form \( R = 1 + (d_{n-1} + \bar{v}) \otimes (d_{n-1} + \bar{v}) \) modulo terms of degree ≥ n + 1, i.e., we may take \( d_n = d_{n-1} + \bar{v} \), as desired.

(2) Follows immediately from (1). Namely, for the first statement we take d to be the stable limit of the \( d_n \)'s and J to be the product of the \( J_m \)'s, and for the second statement we take \( d_{n-1} = d' \), d to be the stable limit of the \( d_m \)'s, and J to be the product of the \( J_m \)'s for \( m \geq n \).

(3) Follows from the identity \( R_{21} R = 1 \otimes 1 \). □

Thus, from now on we may assume that \( R = 1 + d \otimes d \) for some \( d \in \text{Rad}(H) \) with \( d^2 = 0 \) (but in general d is not a primitive element yet, as we have not made \( \Phi = 1 \)).

**Remark 2.6.** Proposition 2.5 implies that the degree p of d in Proposition 2.5(2) and its degree p part \( \delta \in \text{gr}(H)[p] \) (when \( d \neq 0 \)) are uniquely determined. Indeed, if \( (H, R, \Phi) \) is pseudotwist equivalent to \( (H', R', \Phi') \) where \( R = 1 + d \otimes d \) and \( R' = 1 + d' \otimes d' \) modulo terms of degree ≥ n, and if \( d \neq 0 \) and has degree \( p < n/2 \), then by Proposition 2.5(1) the pseudotwist J can be chosen so that \( J = 1 \) is of degree ≥ \( \frac{n}{2} + p > 2p \), so \( d' - d \) has degree ≥ \( p + 1 \), as desired. In particular, if \( R = J_{21}^{-1} J \) then whenever \( R \) is twisted to \( 1 + \delta \otimes d \), we must have \( d = 0 \). This is the case when \( \text{Rep}(H, R, \Phi) \) is Tannakian (as follows from Theorem 2.1). However, d itself is not unique (e.g., it can be conjugated by an invertible element \( x \) of \( 1 + \text{Rad}(H) \)), which results from applying the coboundary twist attached to \( x \).

### 2.3. Trivializing \( \Phi \)

Let \( (H, R, \Phi) \) be a finite dimensional triangular quasi-Hopf algebra with the Chevalley property over k, where \( R = 1 + d \otimes d \) for some element \( d \in \text{Rad}(H) \) with \( d^2 = 0 \).

By Corollary 2.3 \( \text{gr}(H) = k[G] = \bigoplus_{i \geq 0} k[G][i] \), as graded Hopf algebras, for some finite group scheme \( G \) over k. We let \( m, \varepsilon \) denote the multiplication and counit maps of \( G \).

If \( \Phi = 1 \) then \( d^2 = 0 \) and \( \Delta(d) = d \otimes 1 + 1 \otimes d \), so we are done. Thus we may assume that \( \Phi \neq 1 \). Consider \( \Phi - 1 \). If it has degree \( \ell \) then let \( \hat{\phi} \) be its projection to \( \text{gr}(H) \otimes^3 k \).

For every permutation \( (i_1 i_2 i_3) \) of \( (123) \), we will use \( \phi_{i_1 i_2 i_3} \) to denote the 3-tensor obtained by permuting the components of \( \phi \) accordingly.

**Lemma 2.7.** The following hold:

1. \( \phi \in Z^3(\mathcal{O}(G), k) \) is a normalized Hochschild 3-cocycle of \( \mathcal{O}(G) \) with coefficients in the trivial module k, i.e.,
   \[
   \phi \circ (id \otimes id \otimes m) + \phi \circ (m \otimes id \otimes id) = \varepsilon \otimes \phi + \phi \circ (id \otimes m \otimes id) + \phi \otimes \varepsilon
   \]
   and
   \[
   \phi \circ (id \otimes id \otimes 1) = \phi \circ (1 \otimes id \otimes id) = \phi \circ (id \otimes 1 \otimes id) = \varepsilon \otimes \varepsilon.
   \]
2. \( \text{Alt}(\phi) := \phi_{312} + \phi_{132} + \phi_{123} + \phi_{231} + \phi_{213} + \phi_{321} = 0 \).
Proof. (1) Follows from [EG2 (2.1)-(2.2)] in a straightforward manner.
(2) Follows from [EG2 (2.8)] in a straightforward manner. □

2.3.1. The case $R = 1 \otimes 1$. In this subsection we will assume that $R = 1 \otimes 1$, i.e., $d = 0$.

Lemma 2.8. The following hold:

(1) $\phi_{312} + \phi_{132} + \phi_{123} = 0 = \phi_{231} + \phi_{213} + \phi_{123}$.
(2) $\phi_{123} = \phi_{321}$.
(3) $\text{Cyc}(\phi) := \phi_{312} + \phi_{231} + \phi_{123} = 0$.

Proof. (1) Follows from [EG2 (2.6)-(2.7)] in a straightforward manner.
(2) Using (1) and Lemma 2.7(2), we get
$$0 = \phi_{312} + \phi_{132} + \phi_{123} + (\phi_{231} + \phi_{213} + \phi_{123}) = \text{Alt}(\phi) + \phi_{321} + \phi_{123} = \phi_{123} + \phi_{321},$$
as claimed.
(3) By (2), we have $\phi_{132} = \phi_{321}$. Thus the claim follows from (1). □

Following [EG2 2.8]\textsuperscript{2}, we set $y_t := x_t^*$ and $y_t^{(l)} := (x_t^l)^*$, $1 \leq t \leq n$, $1 \leq l \leq r_t - 1$ (so, $y_t^{(1)} = y_t$), and for every $1 \leq i, j \leq n$, let

$$(2.1) \quad \beta_j := \sum_{i=1}^{2^{j-1}} y_j^{(i)} \otimes y_j^{(2^{j-1} - i)}.$$

Proposition 2.9. The 3-cocycle $\phi$ is a coboundary.

Proof. By Lemma 2.7(1), $\phi \in Z^3(O(\mathcal{G}), k)$ so we can express it in the following form:
$$\phi = \sum_{1 \leq i < j < l \leq n} b_{ijl}(y_i \otimes y_j \otimes y_l) + \sum_{i,j} a_{ij} \beta_i \otimes y_j + df,$$
for some $b_{ijl}, a_{ij} \in k$ and $f \in k[\mathcal{G}]^\otimes 2$.

Thus by Lemma 2.8(3), we have
$$\text{Cyc}(df) = \sum_{1 \leq i < j < l \leq n} b_{ijl} \text{Cyc}(y_i \otimes y_j \otimes y_l) + \sum_{i,j} a_{ij} \text{Cyc}(\beta_i \otimes y_j).$$

Also, since $\text{Alt}(df) = \text{Alt}(\beta_i \otimes y_j) = 0$, it follows from Lemma 2.7(2) and the above that we have
$$0 = \text{Alt}(\phi) = \sum_{1 \leq i < j < l \leq n} b_{ijl} \text{Alt}(y_i \otimes y_j \otimes y_l).$$

Therefore $b_{ijl} = 0$ for every $i < j < l$, and we have
$$\text{Cyc}(df) = \sum_{i,j} a_{ij} \text{Cyc}(\beta_i \otimes y_j).$$

It is also straightforward to verify that we have
$$(2.2) \quad \text{Cyc}(df) = \text{Cyc}((\Delta \otimes \text{id})(f) + (\text{id} \otimes \Delta)(f)).$$

Consider the surjective homomorphism
$$\Psi := \pi \otimes \text{id} : \Gamma^2 k[\mathcal{G}] \otimes k[\mathcal{G}] \rightarrow k[\mathcal{G}]^{(1)} \otimes k[\mathcal{G}],$$

\textsuperscript{2}In [EG2 2.8], $y_t^{(l)}$ was denoted by $(x_t^l)^*$.\]
where \( \pi : \Gamma^2 k[\mathcal{G}] \to k[\mathcal{G}]^{(1)} \) is the natural surjective homomorphism. Observe that we have \( \pi(\Delta(u)) = 0 \) for every \( u \in \text{Rad}(H) \). Indeed this holds for \( u = u_1 \cdots u_m \), where \( u_1, \ldots, u_m \) are primitive, and each element of \( \text{Rad}(H) \) is a linear combination of such with coefficients in \( G \).

Now since \( k[\mathcal{G}] \) is cocommutative, it follows from (2.2) that \( \text{Cyc}(df) \) is symmetric, hence we have

\[
\Psi(\text{Cyc}(df)) = 0.
\]

We also have

\[
\Psi(\beta_i \otimes y_j) = y_i^{(2^{r_i-1})} \otimes y_j.
\]

Thus

\[
\sum_{i,j} a_{ij} y_i^{(2^{r_i-1})} \otimes y_j = 0,
\]

which implies that \( a_{ij} = 0 \) for all \( i, j \). Thus \( \phi = df \) is a coboundary, as claimed. \( \square \)

**Lemma 2.10.** In Proposition 2.4 we can choose \( f \in \Gamma^2 k[\mathcal{G}] \), i.e., we can choose \( f \) to be symmetric.

**Proof.** Since \( q_{123} = \phi_{321} \) by Lemma 2.8(2), we have \( df = d(f_{21}) \). This implies that \( f + f_{21} \in Z^2(\mathcal{O}(\mathcal{G}), k) \) is a 2-cocycle, so it follows from [EG2, Proposition 2.4(2)] that we have

\[
f + f_{21} = \sum_i a_i \beta_i + \sum_{i < j} b_{ij} (y_i \otimes y_j) + z \otimes 1 + 1 \otimes z + \Delta(z)
\]

for some \( a_i, b_{ij} \in k \) and \( z \in k[\mathcal{G}] \). Since the left hand side is symmetric and \( \Delta = \Delta^{\cop} \), we must have \( b_{ij} = 0 \) for every \( i < j \). Applying the map \( \Psi \) then yields \( a_i = 0 \) for every \( i \). Thus, we have

\[
f + f_{21} = z \otimes 1 + 1 \otimes z + \Delta(z).
\]

Hence, applying the operator \( y \mapsto y \otimes 1 + 1 \otimes y + \Delta(y) \) to the first tensorand, we get

\[
f_{12,3} + f_{1,3} + f_{2,3} + f_{3,12} + f_{3,1} + f_{3,2} = z_{123} + z_{12} + z_{23} + z_{13} + z_1 + z_2 + z_3.
\]

Hence, the left hand side is symmetric, so

\[
\text{Cyc}(df) = \text{Cyc}(f_{12,3}+f_{1,3}+f_{2,3}+f_{3,12}+f_{3,1}+f_{3,2}) = z_{123} + z_{12} + z_{23} + z_{13} + z_1 + z_2 + z_3.
\]

Since \( \text{Cyc}(df) = 0 \), this implies that

\[
(2.5)\quad z_{123} + z_{12} + z_{23} + z_{13} + z_1 + z_2 + z_3 = 0.
\]

Let \( w := z_{1,2} + z_1 + z_2 \). Then Equation (2.5) implies

\[
w_{12,3} + w_{1,3} + w_{2,3} = 0.
\]

This means that the tensorands of \( w \) are primitive, hence \( w = \sum_{i,j} c_{ij} p_i \otimes p_j \), where \( p_i \) is a basis of primitive elements, with \( c_{ij} = c_{ji} \). Moreover, \( \pi(w) = 0 \), which implies that \( c_{ii} = 0 \) for all \( i \). Now replacing \( f \) with \( f + \sum_{i < j} c_{ij} p_i \otimes p_j \) (which is possible since this sum is a 2-cocycle) we come to a situation where \( f \) is symmetric, as desired. \( \square \)
Choose \( f \in (\mathcal{O}(\mathcal{G}))^\otimes \mathbb{Z}^2 \) symmetric with the same degree \( \ell \) as \( \phi \) such that \( \phi = df \), which is possible by Lemma 2.10. Let \( \tilde{f} \) be a symmetric lift of \( f \) to \( H \). Then the pseudotwist \( F := 1 + \tilde{f} \) is symmetric, which implies that \((H, 1, \Phi)^F = (H^F, 1, \Phi^F)\), and the pseudotwisted associator \( \Phi^F \) is equal to 1+ terms of degree \( \geq \ell + 1 \). By continuing this procedure, we will come to a situation where \((H, 1, \Phi)^F = (H^F, 1, 1)\) for some pseudotwist \( F \in H^\otimes \), as desired. This concludes the proof of Theorem 2.1 in the case where \( R = 1 \).

2.3.2. The case \( R = 1 + d \otimes d \) with \( d \neq 0 \). In this subsection we will assume that \( R = 1 + d \otimes d \) with \( d \neq 0 \). Suppose \( d \) has degree \( p \), and let \( \delta \) be its projection to \( \text{gr}(H)[p] \).

The following lemma is the analogue of Lemma 2.8 in this case.

**Lemma 2.11.** The following hold:

(1) \( \Delta(\delta) = \delta \otimes 1 + 1 \otimes \delta \).

(2) The degree of \( \Delta(d) - d \otimes 1 - 1 \otimes d \) is \( \geq \ell - p \).

(3) Let \( T \in \text{gr}(H)^{\otimes 2}[\ell - p] \) be the part of \( \Delta(d) - d \otimes 1 - 1 \otimes d \) of degree exactly \( \ell - p \) (so \( T = 0 \) if \( \ell \leq 2p \)). Then we have

\[
\begin{align*}
(\alpha) & \quad T \otimes \delta + \phi_{312} + \phi_{132} + \phi_{123} = 0. \\
(\beta) & \quad \delta \otimes T + \phi_{231} + \phi_{213} + \phi_{123} = 0. \\
(\gamma) & \quad \phi_{123} + \phi_{321} = T \otimes \delta + \delta \otimes T. \\
(\delta) & \quad \text{Cyc}(\phi) = \text{Cyc}(T \otimes \delta).
\end{align*}
\]

(4) \( T \) is a symmetric 2-cocycle.

**Proof.** (1) is clear. (2) and (3) follow immediately from the hexagon relations \([\text{EG2}](2.6)-(2.7)\) ((3)(d) is obtained by applying Cyc to (3)(a) and using that \( \text{Alt}(\phi) = 0 \)). Also, let \( Q := T + T_{21} \). By (3)(c), we have \( Q \otimes \delta = \delta \otimes Q \). Thus both left and right tensorands of \( Q \) can only be multiples of \( \delta \), i.e., \( Q \) is a multiple of \( \delta \otimes \delta \). But \( \pi(Q) = 0 \), hence \( Q = 0 \), proving (4). \( \square \)

**Proposition 2.12.** The 3-cocycle \( \phi \) has the form

\[ \phi = T \otimes \delta + df \]

for some \( f \in k[\mathcal{G}]^\otimes \mathbb{Z}^2[\ell] \).

**Proof.** By Lemma 2.7(1), \( \phi \in Z^3(\mathcal{O}(\mathcal{G}), k) \) and we can express it in the following form:

\[ \phi = \sum_{1 \leq i < j < l \leq n} b_{ijl}(y_i \otimes y_j \otimes y_l) + \sum_{i,j} a_{ij} \beta_i \otimes y_j + df', \]

for some \( b_{ijl}, a_{ij} \in k \) and \( f' \in k[\mathcal{G}]^\otimes \mathbb{Z}^2 \).

Since \( \text{Alt}(df') = 0 \), using Lemma 2.7(2) this implies that

\[ 0 = \text{Alt}(\phi) = \sum_{1 \leq i < j < l \leq n} b_{ijl} \text{Alt}(y_i \otimes y_j \otimes y_l). \]

Therefore \( b_{ijl} = 0 \) for every \( i < j < l \). Thus by Lemma 2.11(2)(c), we have

\[ \text{Cyc}(df') = \sum_{i,j} a_{ij} \text{Cyc}(\beta_i \otimes y_j) + \text{Cyc}(T \otimes \delta). \]

Now by 2.3, we have \( \Psi(\text{Cyc}(df')) = 0 \).
We also have
\[ \pi(\beta_i) = y_i^{(2^{r_i} - 1)}. \]
Thus,
\[ \sum_{i,j} a_{ij} y_i^{(2^{r_i} - 1)} \otimes y_j + \pi(T) \otimes \delta = 0, \]
which implies that
\[ \sum_j a_{ij} y_j = a_i \delta, \]
and
\[ \sum_i a_i y_i^{(2^{r_i} - 1)} = \pi(T) \]
for some \( a_i \in k \). Hence,
\[ \sum a_i \beta_i + T = dh \]
for some \( h \in k[G] \) (as the left hand side is a symmetric 2-cocycle killed by \( \pi \), hence a coboundary). So,
\[ \sum a_i \beta_i \otimes \delta = T \otimes \delta + df, \]
where \( f := f' + h \otimes \delta \), as desired. \( \square \)

**Lemma 2.13.** In Proposition 2.12 we can choose \( f \in \Gamma^2 k[G] \), i.e., we can choose \( f \) to be symmetric.

**Proof.** Since \( \phi_{123} = \phi_{321} + T \otimes \delta + \delta \otimes T \) by Lemma 2.11(3)(c), we have \( df = d(f_{21}) \).
Thus, \( f + f_{21} \in Z^2(O(G), k) \) is a 2-cocycle, and we can proceed in exactly the same way as in the proof of Lemma 2.10 to get to a situation where \( f \) is symmetric. \( \square \)

**Proposition 2.14.** The 4-cocycle \( T \otimes T \) is a coboundary.

**Proof.** Let \( f \) be a symmetric element provided by Lemma 2.13 and let \( \tilde{f} \) be a symmetric lift of \( f \) to \( H \). Then the pseudotwist \( F := 1 + \tilde{f} \) is symmetric. Thus, \( (H, R, \Phi)^F = (H^F, R^F, \Phi^F) \), and \( \Phi^F - 1 \) has degree \( \geq \ell \) with degree \( \ell \) part \( T \otimes \delta \). Thus, we have
\[ \Phi^F = 1 + (\Delta(d) - d \otimes 1 - 1 \otimes d) \otimes \delta + U, \]
where \( U \in H \otimes U \) has degree \( \geq \ell + 1 \).

The pentagon equation \([EG2, (2.3)]\) for \( \Phi^F \) yields that \( dU \) has degree \( \geq 2\ell - 2p \), and its part of degree \( 2\ell - 2p \) is \( T \otimes T \). This means that \( U \) has degree \( s \leq 2\ell - 2p \). Let \( u \) be the leading part of \( U \). If \( s < 2\ell - 2p \) then the pentagon equation \([EG2, (2.3)]\) yields that \( du = 0 \), and arguing as above we see that \( u = df \), where \( f \) is symmetric. Thus, by a gauge transformation, we can make sure that \( u = 0 \). Thus, we may assume that \( s = 2\ell - 2p \). In this case \([EG2, (2.3)]\) yields \( du = T \otimes T \), i.e., \( T \otimes T \) is a coboundary, as claimed. \( \square \)

**Proposition 2.15.** The 3-cocycle \( \phi \) is a coboundary.

**Proof.** By \([EG2, Proposition 2.4(2)]\) on the structure of cohomology, \( \pi(T) = 0 \). Thus by \([2.6], a_{ij} = 0 \) for all \( i, j \), so \( \phi \) is a coboundary. \( \square \)

We can now proceed as in the case \( R = 1 \). Namely, by Proposition 2.15 we have \( \phi = df \) for some \( f \in (O(G)^*)^{\otimes 2} \) with the same degree \( \ell \) as \( \phi \), and by Lemma 2.13 we can choose \( f \) to be symmetric. Then letting \( \tilde{f} \) be a symmetric lift of \( f \) to \( H \), we get the symmetric pseudotwist \( F := 1 + \tilde{f} \), and by this pseudotwist we come to
the situation where $\Phi - 1$ has degree $\geq \ell + 1$. Thus $\Delta(d) - d \otimes 1 - 1 \otimes d$ also has degree $\geq \ell + 1$.

However, unlike in the case $R = 1$, we are not done yet since the pseudotwist $F$ spoils the $R$-matrix. Namely, since $f$ is symmetric, $R$ has been brought to the form

$$R = 1 + d \otimes d + [d \otimes d, f] + \text{terms of degree } > 2\ell - 2p.$$  

Thus, we need the following lemma.

**Lemma 2.16.** We can twist further to make sure that $R = 1 + d \otimes d$ and still $\Phi - 1$ has degree $\geq \ell + 1$.

**Proof.** Let $v := \pi(f)$. Then $f = v \otimes v + h_{21}$ for some $h \in k[G] \otimes 2$. Thus, by twisting by the pseudotwist $J := 1 + [d \otimes d, h] + dv \otimes vd$, we come to the situation where $\Phi - 1$ still has degree $\geq \ell + 1$, but

$$R = 1 + d \otimes d + [d, v] \otimes [d, v] + \text{terms of degree } > 2\ell - 2p.$$  

Now, if $\ell < 4p$ then $\ell/2 + 2p > \ell$, so twisting by $J := 1 + d \otimes [d, v]$, we get to a situation when $\Phi - 1$ is of degree $\geq \ell + 1$ and

$$R = 1 + d \otimes d + \text{terms of degree } > 2\ell - 2p.$$  

Now Proposition 2.5 implies that using twists $J$ with $J - 1$ of degree $\geq \ell + 1$ we can come to a situation where $\Phi = 1$ modulo degree $\geq \ell + 1$ and $R = 1 + d \otimes d$ on the nose, providing the desired induction step.

It remains to consider the situation $\ell \geq 4p$. By twisting by $J := 1 + d \otimes [d, v]$, we will get to a situation where $\Phi - 1 = d \otimes W + \text{terms of degree } > \ell + 1$ and

$$R = 1 + d \otimes d + \text{terms of degree } > 2\ell - 2p,$$  

where

$$W := \Delta([d, v]) + [d, v] \otimes 1 + 1 \otimes [d, v].$$  

If $\deg(W) > \ell - p$ then we are done with the induction step, so it remains to consider the case $\deg(W) \leq \ell - p$. In this case the hexagon relations [2.6]-[2.7] yield $W = 0$. Thus we come to a situation where $\Phi - 1$ has degree $\geq \ell + 1$ and $R - 1 - d \otimes d$ has degree $> 2\ell - 2p$. So by Proposition 2.5, by applying twists of degree $> \ell$, we can make sure that $R = 1 + d \otimes d$ and still $\Phi - 1$ has degree $\geq \ell + 1$, as desired. □

Thus it follows from the above that by continuing this procedure, we will come to a situation where

$$(H, 1 + d \otimes d, \Phi)^F = (H^F, 1 + d \otimes d, 1)$$  

for some pseudotwist $F \in H^{\otimes 2}$, as desired. This concludes the proof of Theorem 2.1 in the case where $R = 1 + d \otimes d$. The proofs of Theorems 2.1 and 1.1 are complete. □

**Remark 2.17.** Here is another short proof of the case when $R$ is twist equivalent to 1, which uses the result of Coulembier. If $R = 1$ then the symmetric square of a representation $V$ is the usual one, so for any injection $k \to V$ the induced map $k \to S^2V$ is injective. By [C] Theorem C, this implies that the category $\text{Rep}(H, 1, \Phi)$ is locally semisimple. Hence by [C] Proposition 6.2.2, the maximal Tannakian subcategory of $\text{Rep}(H, 1, \Phi)$ is a Serre subcategory. Since the subcategory of $\text{Rep}(H, 1, \Phi)$ generated by simple objects is Tannakian, we see that the whole category $\text{Rep}(H, 1, \Phi)$ is Tannakian, which implies the desired statement.
Remark 2.18. The case when \( R \neq 1 \) is more subtle, as it is not captured by first order deformation theory. Indeed, the category \( \mathcal{D} = \text{Rep}(k[d]/(d^2), 1 + d \otimes d) \) has a nontrivial first order deformation over \( k[h]/(h^2) \), with the same \( R \)-matrix \( R \), but with \( \Delta(d) = d \otimes 1 + 1 \otimes d + h d \otimes d \) and associator \( \Phi := 1 + h d \otimes d \otimes d \). This deformation is nontrivial because \( \phi := d \otimes d \otimes d \) is a nontrivial 3-cocycle. However, it does not lift to \( k[h]/(h^3) \), as the difference between the left hand side and the right hand side of the pentagon equation \([EG2, (2.3)]\) is \( h^2 d \otimes d \).

The existence of such deformations is typical. For example, consider the category \( \text{Vec}(\mathbb{Z}/p\mathbb{Z}) \) in characteristic \( p > 0 \). Clearly, it has no nontrivial formal deformations, since \( H^3(\mathbb{Z}/p\mathbb{Z}, k^\times) \) is trivial. However, it has a nontrivial first order deformation, since \( H^3(\mathbb{Z}/p\mathbb{Z}, k) = k \). This deformation in fact lifts modulo \( h^i \) for any \( i \leq p \), but does not lift modulo \( h^{p+1} \). This is because \( \mu_p \) and \( \alpha_p \) are “the same” up to order \( p - 1 \) inclusively, but differ in order \( p \).

Corollary 2.19. Let \((H, R)\) be a finite dimensional triangular Hopf algebra with the Chevalley property over \( k \). Then \((H, R)\) is twist equivalent to a triangular Hopf algebra with \( R \)-matrix \( 1 + d \otimes d \) for some \( d \in P(H) \) such that \( d^2 = 0 \).

Proof. Applying Theorem 2.1 to \((H, R, 1)\) yields the existence of a pseudotwist \( J \) for \( H \) such that \((H, R, 1)^J = (H^J, 1 + d \otimes d, 1) \). In particular, we have \( 1^J = 1 \), which is equivalent to \( J \) being a twist. \( \square \)

Corollary 2.20. Let \( C \) be a finite symmetric tensor category over \( k \) such that \( \text{FPdim}(C) = 2 \). Then \( C \) is symmetric tensor equivalent to either \( \text{Vec}(\mathbb{Z}/2\mathbb{Z}) \), \( \text{Rep}(\mathbb{Z}/2\mathbb{Z}) \), \( \text{Rep}(\alpha_2) \) or \( \mathcal{D} \).

Proof. Follows immediately from Theorem 1.1. \( \square \)

2.4. Strengthening of \([EG2, \text{Theorem 1.1}]\) and Theorem 1.1 The arguments used in this section and \([EG2, \text{Section 3}]\) in fact prove a stronger result. Namely, we have the following theorem.

Theorem 2.21. Let \( \mathcal{E} \subset C \) be finite symmetric tensor categories over an algebraically closed field \( k \) with characteristic \( p > 0 \), such that \( \mathcal{E} \) contains all the simples of \( C \). The following hold:

1. Suppose \( p > 2 \). If \( \mathcal{E} \) has a fiber functor to \( s\text{Vec} \), then so does \( C \).
2. Suppose \( p = 2 \). If \( \mathcal{E} \) has a fiber functor to \( \text{Vec} \), then \( C \) has a fiber functor to \( \mathcal{D} \).

Indeed, in both cases it follows that \( C \) is integral, so we have \( C = \text{Rep}(H, R, \Phi) \) for some finite dimensional triangular quasi-Hopf algebra over \( k \). Now the arguments are exactly the same, except the radical of \( H \) should be replaced by the annihilator of \( \mathcal{E} \) inside \( C \), which is a nilpotent quasi-Hopf ideal of \( H \) since \( \mathcal{E} \) contains all the simples of \( C \).

3. Twists and Sweedler cohomology for finite abelian \( p \)-groups

In this section we let \( K \) be an arbitrary field of characteristic \( p > 0 \), and \( \mathbb{F}_q \) be a finite field of characteristic \( p > 0 \).
3.1. **Truncated Witt vectors.** Let \( W_n(K) \) be the ring of truncated Witt vectors of length \( n \) with coefficients in \( K \). Recall that \( W_n(K) = K^n \) as a set, with nontrivial addition and multiplication given, e.g., in [L, VI, p.330-332].

**Example 3.1.** We have the following:

1. \( W_1(K) = K \) as rings.
2. The addition and multiplication in \( W_2(K) \) are given as follows
   \[
   (x_0, x_1) + (y_0, y_1) = \left( x_0 + y_0, x_1 + y_1 + \sum_{i=1}^{p-1} \frac{1}{i} (p-1) x_0^i y_0^{p-i} \right)
   \]
   and
   \[
   (x_0, x_1)(y_0, y_1) = (x_0 y_0, x_0^p y_0 + y_1 x_0^p).
   \]

3. \( W_n(\mathbb{F}_p) = \mathbb{Z}/p^n\mathbb{Z} \) for every \( n \geq 1 \).

For \( x := (x_0, \ldots, x_{n-1}) \in W_n(K) \), let \( F(x) = (x_0^p, \ldots, x_{n-1}^p) \). (Note that if \( n > 1 \) then \( F(x) \neq x^p \).) Recall that \( F : W_n(K) \to W_n(K) \) is a ring homomorphism, and we have an additive homomorphism
\[
\mathcal{P} : W_n(K) \to W_n(K), \ x \mapsto F(x) - x.
\]
The kernel of \( \mathcal{P} \) is the cyclic group \( W_n(\mathbb{F}_p) = \mathbb{Z}/p^n\mathbb{Z} \).

**Lemma 3.2.** The following hold:

1. If \( K \) is perfect then \( W_n(K)/\mathcal{P}(W_n(K)) \) is a free \( \mathbb{Z}/p^n\mathbb{Z} \)-module.
2. \( W_n(\mathbb{F}_q)/\mathcal{P}(W_n(\mathbb{F}_q)) \cong \mathbb{Z}/p^n\mathbb{Z} \).

**Proof.** (1) First note that since \( K \) is perfect, we have \( W_n(K)/p^a W_n(K) \cong W_n(K) \) for every \( 0 \leq a \leq n \).

Secondly, let \( a \in W_n(K) \) be an element such that its image \( a_0 \) in \( K \) is not in \( \mathcal{P}(K) \). We claim that the order of \( a \) in \( W_n(K)/\mathcal{P}(W_n(K)) \) is \( p^a \). Indeed, suppose \( s < n \) is such that \( p^a = 0 \) in \( W_n(K)/\mathcal{P}(W_n(K)) \), i.e., \( p^a = \mathcal{P}(y) \) for some \( y \in W_n(K) \). Then \( \mathcal{P}(y) = 0 \) in \( W_n(K)/p^a W_n(K) = W_s(K) \). Thus \( y = k \in \mathbb{Z}/p^a \mathbb{Z} \subseteq W_n(K)/p^a W_n(K) \) (as \( \ker(\mathcal{P}) = \mathbb{Z}/p^n\mathbb{Z} \)), so \( y = k + p^s z \) for some integer \( k \) and \( z \in W_n(K) \). But then \( p^a = \mathcal{P}(y) = \mathcal{P}(p^s z) \), so \( z \) is the image of \( z \) in \( K \) then \( a_0 = \mathcal{P}(z_0) \), which is a contradiction.

Finally, take \( a \in W_n(K) \) such that \( p^{n-1} a = 0 \) in \( W_n(K)/\mathcal{P}(W_n(K)) \), and consider its image \( a_0 \) in \( K \). We have shown that \( a_0 = 0 \) must be in \( \mathcal{P}(K) \), i.e., \( a_0 = x_0^p - x_0 \) for some \( x_0 \) in \( K \). Let \( x := (x_0, 0, \ldots, 0) \in W_n(K) \). We have \( a - \mathcal{P}(x) = py \) for some \( y \in W_n(K) \) (again using that \( K \) is perfect). Thus \( a = py \) in \( W_n(K)/\mathcal{P}(W_n(K)) \), proving freeness.

(2) Since the kernel of \( \mathcal{P} : W_n(\mathbb{F}_q) \to W_n(\mathbb{F}_q) \) is \( \mathbb{Z}/p^n\mathbb{Z} \), it follows that the cokernel of \( \mathcal{P} \) has order \( p^n \). Thus \( W_n(\mathbb{F}_q)/\mathcal{P}(W_n(\mathbb{F}_q)) \) is abelian of order \( p^n \), so the claim follows from Part (1).

**Remark 3.3.** If \( K \) is not perfect then for instance \( W_2(K) \) is not a free \( \mathbb{Z}/p^2\mathbb{Z} \)-module. Indeed, take an element \((0, a)\) in \( W_2(K) \), where \( a \in K \) is not a \( p \)-th power. Then \( p(0, a) = (0, 1)(0, a) = 0 \), but \((0, a) \neq p(x, y)\) for any \( x, y \), since \( p(x, y) = (0, x^p) \).
3.2. Twists for abelian groups and torsors. Recall that an interesting invariant of a tensor category \(\mathcal{C}\) over \(K\) is the group of tensor structures on the identity functor of \(\mathcal{C}\) (i.e., the group of isomorphism classes of tensor autoequivalences of \(\mathcal{C}\) which act trivially on the underlying abelian category) up to an isomorphism [Dal] [BC]. This group is called the second invariant (or lazy) cohomology group of \(\mathcal{C}\) and denoted by \(H^2_{\text{inv}}(\mathcal{C}, K)\).

In particular, if \(\mathcal{C} := \text{Rep}_K(A)\) is the representation category of a finite abelian group \(A\) then \(H^2_{\text{inv}}(A, K) := H^2_{\text{inv}}(\mathcal{C}, K)\) is the group of gauge equivalence classes of twists for the Hopf algebra \(K[A]\) [EG1].

**Lemma 3.4.** Let \(A\) be a finite abelian group. We have a canonical group isomorphism \(H^2_{\text{inv}}(A, K) \cong \text{Hom}(G, A)\), where \(G := \text{Aut}(\overline{K}/K) = \text{Gal}(K^*/K)\).  

**Proof.** Let \(J\) be a twist for \(K[A]\), and consider the twisted \(K\)-algebra \((K[A]_J)^*\). Observe that (up to \(K\)-algebra isomorphism) this algebra depends only on \([J]\). Since by [AEGN] Theorem 6.5] every twist for \(\overline{K}[A]\) is trivial, it follows that \((K[A]_J)^* \otimes_K \overline{K}\) and \(\text{Fun}(A, \overline{K})\) are isomorphic as \(\overline{K}\)-algebras. Thus, \((K[A]_J)^*\) is a semisimple commutative \(K\)-algebra. Furthermore, \((K[A]_J)^*\) is an \(A\)-algebra, which is isomorphic to the regular representation of \(A\) as an \(A\)-module. Thus \((K[A]_J)^* \cong \text{A-torsor}\).

Conversely, suppose \(B\) is an \(A\)-torsor, i.e., a commutative semisimple \(K\)-algebra with an \(A\)-action such that \(B \otimes_K \overline{K} \cong \text{Fun}(A, \overline{K})\). By Wedderburn theorem, \(B\) decomposes uniquely into a direct sum of field extensions \(L_i\) of \(K\): \(B = \bigoplus_i L_i\). Since the space of \(A\)-invariants in \(B\) is 1-dimensional, \(A\) acts transitively on the set of fields \(L_i\). Let \(H \subseteq A\) be the stabilizer of \(L := L_1\). Clearly \(L\) is a cyclic extension of \(K\) with Galois group \(H\). Then it is well known that \(L \cong (K[H]^*)^J\) for a unique (up to gauge equivalence) Hopf 2-cocycle \(J\) for \(K[H]^*\). Viewing \(J\) as a twist for \(K[H]^*\) (hence for \(K[A]\)), it is easy to see that the class \([J]\) is uniquely determined by the isomorphism class of the \(A\)-torsor \(B\).

Finally we note that \(A\)-torsors form an abelian group under the product rule \((B_1, B_2) \mapsto (B_1 \otimes B_2)^A\), where \(a \in A\) acts on \(B_1\) by \(a\) and on \(B_2\) by \(a^{-1}\), and that \((K[A]_J)^* \cong ((K[A])^* \otimes (K[A]_J)^*)^A\) (see, e.g., [AEGN] Remark 3.12]).

It now follows from the above that the group \(H^2_{\text{inv}}(A, K)\) is canonically isomorphic to the group of \(A\)-torsors over \(K\). Since the latter is canonically isomorphic to the Galois cohomology group \(H^1(G, A) = \text{Hom}(G, A)\), the claim follows. \(\square\)

3.3. Invariant cohomology of abelian groups. Let \(A\) be a finite abelian group of exponent dividing \(p^n\). Let \(G\) be as in Section 3.2, and let \(G_n\) be its maximal abelian quotient of exponent dividing \(p^n\). Then \(\text{Hom}(G, A) = \text{Hom}(G_n, A)\). Thus by Lemma 3.3 we have a canonical group isomorphism

\[
H^2_{\text{inv}}(A, K) \cong \text{Hom}(G_n, A).
\]

**Theorem 3.5.** Let \(A\) be a finite abelian group of exponent dividing \(p^n\). Then the following hold:

1. We have a canonical group isomorphism

\[
H^2_{\text{inv}}(A, K) \cong \text{Hom}(A^\vee, W_n(K)/\mathcal{P}(W_n(K)));
\]

where \(A^\vee := \text{Hom}(A, \mathbb{Z}/p^n\mathbb{Z})\).

\[\text{When considering } \text{Hom} \text{ from a profinite group, as usual it means continuous homomorphisms.}\]

\[\text{\(K^*\) is the separable closure of } K.\]
(2) If moreover $K$ is perfect then we have a canonical group isomorphism

$$H^2_{\text{inv}}(A, K) \cong A \otimes_{\mathbb{Z}/p^n \mathbb{Z}} (W_n(K)/\mathcal{P}(W_n(K))).$$

Proof. (1) Recall that Artin-Schreier-Witt theory provides a canonical group isomorphism

$$G_n \cong \text{Hom}(W_n(K)/\mathcal{P}(W_n(K)), \mathbb{Z}/p^n \mathbb{Z})$$

(see, e.g., [L, VI, p.330–332]). Thus we get from (3.1) a canonical group isomorphism

$$H^2_{\text{inv}}(A, K) \cong \text{Hom}(\text{Hom}(W_n(K)/\mathcal{P}(W_n(K)), \mathbb{Z}/p^n \mathbb{Z}), A).$$

The claim follows now from the fact that $\text{Hom}(A^\vee, A) = \text{Hom}(A^\vee, B)$ for every $B$.

(2) By Lemma 3.2(1), $W_n(K)/\mathcal{P}(W_n(K))$ is a free $\mathbb{Z}/p^n \mathbb{Z}$-module. Therefore the group

$$\text{Hom}(A^\vee, W_n(K)/\mathcal{P}(W_n(K))) \cong \text{Hom}(\text{Hom}(W_n(K)/\mathcal{P}(W_n(K)), \mathbb{Z}/p^n \mathbb{Z}), A)$$

is the same as the group $A \otimes_{\mathbb{Z}/p^n \mathbb{Z}} (W_n(K)/\mathcal{P}(W_n(K)))$, as desired. \hfill $\square$

Corollary 3.6. We have a group isomorphism

$$H^2_{\text{inv}}(\mathbb{Z}/p^n \mathbb{Z}, K) \cong W_n(K)/\mathcal{P}(W_n(K)).$$

In particular, we have a group isomorphism

$$H^2_{\text{inv}}(\mathbb{Z}/p^n \mathbb{Z}, F_q) \cong \mathbb{Z}/p^n \mathbb{Z}.$$

Proof. By Theorem 3.5(1), $H^2_{\text{inv}}(\mathbb{Z}/p^n \mathbb{Z}, F_q) \cong W_n(F_q)/\mathcal{P}(W_n(F_q))$, so the second claim follows from Lemma 3.2(2).

Remark 3.7. (1) Theorem 3.5(1) implies that if $K$ is algebraically closed then $H^2_{\text{inv}}(A, K) = 0$, which agrees with [EG2] Proposition 5.7 for $i = 2$.

(2) Theorem 3.5(1) was obtained by Guillot [G] for $p = 2$ and $n = 1$.

3.4. Sweedler cohomology of algebras of functions on abelian groups. Let $A$ be a finite abelian group, and let $\mathcal{O}(A)$ be the Hopf algebra of functions on $A$ with values in $K$. Recall that $H^2_{\text{sw}}(A, K)$ coincides with the second Sweedler cohomology group $H^2_{\text{sw}}(\mathcal{O}(A), K)$ with coefficients in $K$.

Theorem 3.8. Let $A$ be a finite abelian group of exponent dividing $p^n$. Then the Sweedler cohomology of $\mathcal{O}(A)$ with coefficients in $K$ is as follows:

1. $H^1_{\text{sw}}(\mathcal{O}(A), K) = A$.
2. $H^1_{\text{sw}}(\mathcal{O}(A), K) = \text{Hom}(A^\vee, W_n(K)/\mathcal{P}(W_n(K)))$.
3. $H^1_{\text{sw}}(\mathcal{O}(A), K) = 0$ for every $i \geq 3$.

Proof. (1) is clear and (2) is Theorem 3.5(1). To prove (3) consider the normalized complex computing $H^i_{\text{sw}}(\mathcal{O}(A), K)$:

$$C^0(K) \rightarrow C^1(K) \rightarrow C^2(K) \rightarrow \cdots,$$

where $C^i$ is the algebraic group such that for any field $L$, $C^i(L) = (L[A]^{\otimes i})^\times$ is the group of invertible elements $a$ in $L[A]^{\otimes i}$ with $\varepsilon(a) = 1$. Then $C^i$ is a connected commutative unipotent algebraic group over $K$ (i.e., an iterated extension of $\mathbb{G}_a$).

Now fix $n \geq 2$. Since by [EG2] Proposition 5.7,

$$H^n_{\text{sw}}(\mathcal{O}(A), K) = H^{n+1}_{\text{sw}}(\mathcal{O}(A), K) = 0,$$

we have a short exact sequence

$$0 \rightarrow C^{n-1}/D^{n-1} \rightarrow C^n \rightarrow D^{n+1} \rightarrow 0,$$
where $D^i \subseteq C^i$ is the kernel of the differential map $d : C^i \rightarrow C^{i+1}$. Thus we have an exact sequence

$$0 \rightarrow (C^{n-1}/D^{n-1})(K) \rightarrow C^n(K) \rightarrow D^{n+1}(K) \rightarrow H^1(K,C^{n-1}/D^{n-1}),$$

where $H^1(K,C^{n-1}/D^{n-1}) := H^1(\text{Gal}(\overline{K}/K),(C^{n-1}/D^{n-1})(K))$ is the Galois cohomology group. But since $C^{n-1}/D^{n-1}$ is an iterated extension of $\mathbb{G}_a$, and $H^1(K,\mathbb{G}_a) = 0$, the Galois cohomology group $H^1(K,C^{n-1}/D^{n-1})$ vanishes. Thus we have a short exact sequence

$$0 \rightarrow (C^{n-1}/D^{n-1})(K) \rightarrow C^n(K) \rightarrow D^{n+1}(K) \rightarrow 0,$$

which implies that $H^1_{SW}(\mathcal{O}(A), K) = D^{n+1}(K)/d(C^n(K)) = 0$, as claimed. \qed

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